EXAMPLES OF AREA MINIMIZING SURFACES IN 3-MANIFOLDS

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ABSTRACT. In this paper, we give some examples of area minimizing surfaces to clarify some well-known features of these surfaces in more general settings. The first example is about Meeks-Yau’s result on the embeddedness of the solution to the Plateau problem. We construct an example of a simple closed curve in $\mathbb{R}^3$ which lies in the boundary of a mean convex domain in $\mathbb{R}^3$, but the area minimizing disk in $\mathbb{R}^3$ bounding this curve is not embedded. Our second example shows that B. White’s boundary decomposition theorem does not extend when the ambient space has nontrivial homology. Our last examples show that there are properly embedded absolutely area minimizing surfaces in a mean convex 3-manifold $M$ such that, while their boundaries are disjoint, they intersect each other nontrivially, unlike the area minimizing disks case.

1. INTRODUCTION

In this paper, we give examples of some area minimizing surfaces related with some well-known theorems. The first example is on Meeks-Yau’s result on the embeddedness of solutions of the Plateau problem. The Plateau problem asks the existence of an area minimizing disk for a given curve in the ambient manifold $M$. Meeks-Yau’s famous result (Theorem 3.1) says that for any simple closed curve in the boundary of a mean convex 3-manifold $M$, the solution to the Plateau problem in $M$ must be embedded. Since any convex body $C$ in $\mathbb{R}^3$ is mean convex, and any solution to the Plateau problem for a simple closed curve in $C$ must belong to $C$ because of the convexity, the result automatically implies that for any extreme curve in $\mathbb{R}^3$, the solution to the Plateau problem is embedded. However, our example shows that this is not the case for an $H$-extreme curve $\Gamma$ in $\mathbb{R}^3$, i.e. $\Gamma \subset \partial \Omega$ where $\Omega$ is mean convex in $\mathbb{R}^3$. We construct an $H$-extreme curve $\Gamma$ in $\mathbb{R}^3$ where the area minimizing disk $\Sigma$ in $\mathbb{R}^3$ with $\partial \Sigma = \Gamma$ is not embedded (See Figure 2).

Our second example shows that White’s decomposition theorem for the absolutely area minimizing surfaces bounding curves with multiplicity does not generalize to manifolds with nontrivial second homology. It follows from [Wh1] that if $\Gamma$ is a simple closed curve in $\mathbb{R}^3$, and $T$ is an absolutely area minimizing surface with $\partial T = k\Gamma$ where $k$ is an integer greater than 1, then $T = \sum_{i=1}^{k} T_i$ where each

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This theorem naturally extends to higher dimensions, and to the case of orientable Riemannian manifolds with trivial second homology. Trivial homology plays crucial role in the proof, and we give an example which shows that the decomposition theorem does not generalize to 3-manifolds with nontrivial second homology (See Figure 6).

Finally, in the last set of examples, we address the issue of intersections of absolutely area minimizing surfaces in mean convex 3-manifolds. It is known that if $\Gamma_1$ and $\Gamma_2$ are two disjoint simple closed curves in the boundary of a mean convex 3-manifold $M$, then the area minimizing disks they bound in $M$ must be disjoint, too [MY2]. Our examples shows that the same statement is not true for absolutely area minimizing surfaces. In particular, it is known that if $\Sigma_1$ and $\Sigma_2$ are two absolutely area minimizing surfaces with disjoint boundaries, then they must also be disjoint, provided that $\Sigma_1$ and $\Sigma_2$ are homologous rel. $\partial M$ (cf. [Co], [Ha]). However, in the case when they are not homologous, we construct absolutely area minimizing surfaces $\Sigma_1$ and $\Sigma_2$ in a mean convex 3-manifold such that $\partial \Sigma_i = \Gamma_i \subset \partial M$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, but $\Sigma_1 \cap \Sigma_2 \neq \emptyset$.

In particular, we can restate the result mentioned above ([Co], [Ha]) in the following form: Let $M$ be a strictly mean convex 3-manifold, and let $\Sigma_1$ and $\Sigma_2$ be two absolutely area minimizing surfaces in $M$ with $\partial \Sigma_i = \Gamma_i \subset \partial M$. Let $\Gamma_1 \cap \Gamma_2 = \emptyset$, but $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. Then, either $\Gamma_1$ is not homologous to $\Gamma_2$ in $\partial M$, or $H_2(M)$ is not trivial. Our examples show that both situations are possible (See Figures 8 and 11).

The organization of the paper is as follows: In the next section, we give basic definitions and results which will be used in the following sections. In Section 3, we describe the example about Meeks-Yau’s embeddedness result. In Section 4, we give the example on White’s decomposition theorem. In the last section, we construct intersecting absolutely area minimizing surfaces with disjoint boundary.

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2. Preliminaries

In this section, we will give the basic definitions for the following sections.

Definition 2.1. An area minimizing disk is a disk which has the smallest area among the disks with the same boundary. An absolutely area minimizing surface is a surface which has the smallest area among all orientable surfaces (with no topological restriction) with the same boundary.

Definition 2.2. Let $M$ be a compact Riemannian 3-manifold with boundary. Then $M$ is mean convex (or sufficiently convex) if the following conditions hold.
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• $\partial M$ is piecewise smooth.
• Each smooth subsurface of $\partial M$ has nonnegative curvature with respect to inward normal.
• There exists a Riemannian manifold $N$ such that $M$ is isometric to a submanifold of $N$ and each smooth subsurface $S$ of $\partial M$ extends to a smooth embedded surface $S'$ in $N$ such that $S' \cap M = S$.

Definition 2.3. A simple closed curve is an extreme curve if it is on the boundary of its convex hull. A simple closed curve is called as $H$-extreme curve if it is a curve in the boundary of a mean convex manifold $M$.

3. EXAMPLE I: MEEKS-YAU EMBEDDEDNESS RESULT

The Plateau problem asks the existence of an area minimizing disk bounding a given curve in $\mathbb{R}^3$. In other words, for a given simple closed curve $\Gamma$ in $\mathbb{R}^3$, does there exist a disk $\Sigma$ with the smallest area among the disks with boundary $\Gamma$, i.e. for any disk $D \subset \mathbb{R}^3$ with $\partial D = \Gamma$, $|D| \geq |\Sigma|$. This problem was solved by Douglas [Do], and Rado [Ra] in early 1930s. Later, it was generalized to homogeneously regular 3-manifolds by Morrey [Mo]. Then, the regularity (nonexistence of branch points) of these solutions was shown by Osserman [Os], Gulliver [Gu] and Alt [Al].

In the following decades, another version of the Plateau problem was investigated: Without any restriction on the genus of surface, does there exist a smallest area surface bounding a given curve in $\mathbb{R}^3$? In 1960s, the geometric measure theory techniques proved to be quite powerful, and De Georgi, Federer-Fleming showed that for any simple closed curve $\Gamma$ in $\mathbb{R}^3$, there exists an absolute area minimizing surface $S$ which minimizes area among the surfaces (no topological restriction) with boundary $\Gamma$ [Fe]. Moreover, any such surface is embedded in $\mathbb{R}^3$.

When we go back to the original Plateau problem, in the 1970s, the question of embeddedness of the area minimizing disk was studied: For which curves in $\mathbb{R}^3$, the area minimizing disks are embedded? It was conjectured that if the simple closed curve in $\mathbb{R}^3$ is extreme (lies in the boundary of its convex hull), then the area minimizing disk spanning the curve must be embedded. After several partial results, Meeks and Yau proved the conjecture: Any solution to the Plateau problem for an extreme curve must be embedded [MY1]. Indeed, they proved more:

Theorem 3.1. [MY2], [MY3] Let $M$ be a compact, mean convex 3-manifold, and $\Gamma \subset \partial M$ be a nullhomotopic simple closed curve. Then, there exists an area minimizing disk $D \subset M$ with $\partial D = \Gamma$. Moreover, all such disks are properly embedded in $M$.

This theorem automatically implies the conjecture for extreme curves in $\mathbb{R}^3$. If $\Gamma$ is an extreme curve in $\mathbb{R}^3$, then let $M$ be the convex hull of $\Gamma$, i.e. $M = CH(\Gamma)$. Since $CH(\Gamma)$ is a convex set in $\mathbb{R}^3$, it automatically satisfies the mean convexity condition. The theorem says that there is an area minimizing disk $\Sigma$ in $M$ with
$\partial \Sigma = \Gamma$. Now, since $M$ is not just mean convex, but also convex in $\mathbb{R}^3$, the area minimizing disk $\Sigma$ in $M$ is also area minimizing in $\mathbb{R}^3$. In other words, $\Sigma$ has the smallest area among the disks with boundary $\Gamma$ not only in $M$, but also in $\mathbb{R}^3$. This shows that any solution to the Plateau problem in $\mathbb{R}^3$ must belong to $M$, hence by the theorem above, it must be embedded.

In this theorem, there is a subtle point. The theorem says that for a given simple closed nullhomotopic curve $\Gamma$ in $\partial M$, the area minimizing disk in $M$ is embedded. This does not say that if $M$ is a mean convex domain in $\mathbb{R}^3$, and $\Gamma$ is a simple closed curve in $\partial M$, the area minimizing disk $D$ in $\mathbb{R}^3$ with $\partial D = \Gamma$ is embedded. In other words, the theorem gives the embeddedness of the minimizer in $M$, not the minimizer in $\mathbb{R}^3$. In this paper, we will construct an explicit example for this difference. We note that Spadaro has recently constructed an example of a simple closed curve $\Gamma$ on the boundary of a mean convex domain $\Omega$ in $\mathbb{R}^3$ where the area minimizing disk in $\mathbb{R}^3$ with boundary $\Gamma$ does not belong to $\Omega$ [Sp]. While the minimizers in $\mathbb{R}^3$ are still embedded in his examples, here we construct examples where the minimizers in $\mathbb{R}^3$ are not embedded.

Now, we describe the example, a simple closed curve $\Gamma$ in $\mathbb{R}^3$ which lies in the boundary of a mean convex domain $M \subset \mathbb{R}^3$, but the area minimizing disk $\Sigma$ in $\mathbb{R}^3$ with $\partial \Sigma = \Gamma$ is not embedded.

Consider $\mathbb{R}^3$ with $xyz$ coordinate system. Define $\Gamma_1$ as follows:

Let $\alpha_1$ be the line segment in $xy$-plane connecting the points $p_1 = (1, -1, 0)$ and $p_2 = (1, 1, 0)$. Let $\alpha_2$ be the line segment in $xy$-plane connecting the points $p_2$ and $p_3 = (\epsilon, 1, 0)$. Let $\alpha_3$ be the line segment in $\{x = \epsilon\}$-plane connecting the points $p_3$ and $p_4 = (\epsilon, 1, C)$. Let $\alpha_4$ be the line segment in $\{x = \epsilon\}$-plane connecting the points $p_4$ and $p_5 = (\epsilon, -1, -C)$. Let $\alpha_5$ be the line segment in $\{x = \epsilon\}$-plane connecting the points $p_5$ and $p_6 = (\epsilon, -1, 0)$. Let $\alpha_6$ be the line...
Let choosing

\[ \{ \Gamma \} \]

be the square in the \( \mathbb{R}^3 \)-plane connecting the points \( p_6 \) and \( p_1 \). Then, let \( \Gamma_1 = \bigcup_{i=1}^{6} \alpha_i \) (See Figure [1]).

Similarly, define \( \beta_i \) and \( q_i \) as the reflection of \( \alpha_i \) and \( p_i \) with respect to \( yz \)-plane, and define \( \Gamma_2 = \bigcup_{i=1}^{6} \beta_i \), i.e. \( \Gamma_2 \) is the reflection of \( \Gamma_1 \) with respect to \( yz \)-plane.

Now, \( \Gamma_1 \) is an extreme curve as it is in boundary of a convex box \( B_1 \) with corners:

\[ (1, -1, -C), (1, 1, C), p_4, (\epsilon, -1, C), (1, -1, -C), (1, 1, -C), (\epsilon, 1, -C) \]

and \( \Gamma_2 \) is in the boundary of the box \( B_2 \), the reflection of \( B_1 \) with respect to \( yz \)-plane, and hence \( \Gamma_2 \) is an extreme curve, too. Without loss of generality, we can smooth out the corners of \( \Gamma_1 \) and \( \Gamma_2 \) so that \( \Gamma_1 \) and \( \Gamma_2 \) are smooth curves with \( \Gamma_1 \subset \partial B_i \).

Let \( \Sigma_1 \) and \( \Sigma_2 \) be the area minimizing disks in \( \mathbb{R}^3 \) with \( \partial \Sigma_i = \Gamma_i \). Since the boxes are convex, \( \Sigma_i \subset B_i \). Let \( \gamma \) be the line segment between the points \( p_5 \) and \( q_5 \). Now, recall the bridge principle for stable minimal surfaces.

**Lemma 3.2.** \([\text{MY3} \text{ Theorem 7}]) \text{(Bridge Principle)} \text{ Let } \Sigma_1 \text{ and } \Sigma_2 \text{ be two stable orientable compact minimally immersed surfaces in } \mathbb{R}^3 \text{ and } \gamma \text{ be a Jordan curve joining } \partial \Sigma_1 \text{ and } \partial \Sigma_2 \text{. Then for any tubular neighborhood of } \gamma, \text{ we can find a bridge pair joining } \partial \Sigma_1 \text{ and } \partial \Sigma_2 \text{ such that the new configuration is the boundary of a compact minimal surface which is close to the union of } \Sigma_1 \text{ and } \Sigma_2 \text{ joint by a strip in the tubular neighborhood of } \gamma. \)**

Then by applying the lemma above to \( \Sigma_1, \Sigma_2 \) and \( \gamma \), we get a stable minimal surface \( \tilde{\Sigma} \sim \Sigma_1 \sharp \gamma \Sigma_2 \) with the boundary \( \tilde{\Gamma} = \Gamma_1 \sharp \gamma \Gamma_2 \). In other words, \( \tilde{\Sigma} \) is the stable minimal surface close to the union of \( \Sigma_1 \) and \( \Sigma_2 \) joint by a strip near \( \gamma \), and \( \tilde{\Gamma} \) is its boundary.

Now, \( \tilde{\Sigma} \) is an embedded stable minimal disk with smooth boundary \( \tilde{\Gamma} \). By \([\text{MY3} \text{ Corollary 1 at page 159}]) \text{, there is a mean convex neighborhood of } \tilde{\Sigma}, \text{ say } M, \text{ in } \mathbb{R}^3 \text{ such that } \partial \Sigma = \tilde{\Gamma} \subset \partial M. \text{ Hence, } \tilde{\Gamma} \text{ is a simple closed curve in } \mathbb{R}^3 \text{ which lies in the boundary of a mean convex domain } M. \text{ Now, we claim that the area minimizing disk } D \text{ in } \mathbb{R}^3 \text{ with } \partial D = \tilde{\Gamma} \text{ is not embedded.}

Let \( \tau \) be the square in the \( xy \)-plane with corners \( p_1, p_2, q_1, q_2 \). Let \( E \) be the disk in \( xy \)-plane with boundary \( \tau \). Clearly, \( E \) is the unique area minimizing disk with boundary \( \tau \). Here, uniqueness comes from the foliation of \( \mathbb{R}^3 \) by planes \( \{ z = t \mid t \in \mathbb{R} \} \). If there was another such area minimizing disk, it would have a tangential intersection with one of the planes at the maximum (or minimum) height, which contradicts to the maximum principle.

First, note that by taking \( C \) accordingly, we can make \( |\tilde{\Sigma}| \sim |\Sigma_1| + |\Sigma_2| \sim 2(C + \sqrt{C^2 + 1}) \) as large as we want, where \( |.| \) represents the area. Hence, by choosing \( C \) sufficiently large, we can assume \( |\tilde{\Sigma}| \gg |E| = 4 \).

Now, by modifying \( E \) slightly, we will get another disk \( \tilde{E} \) in \( \mathbb{R}^3 \) with \( \partial \tilde{E} = \tilde{\Gamma} \). Let \( \eta_{1a} \) be the line segment in the \( yz \)-plane connecting the points \( (0, 1, 0) \) and \( (0, 1, C) \). Let \( \eta_{1b} \) be the line segment in the \( yz \)-plane connecting the points \( (0, 1, C) \)
Figure 2. The H-extreme curve $\Gamma^\varepsilon$ bounds a non-embedded area minimizing disk $\hat{E}^\varepsilon$. 

and $(0, -1, -C)$. Let $\eta_1 = \eta_{1a} \cup \eta_{1b}$. Let $\eta_2$ be the line segment in the $yz$-plane connecting the points $(0, -1, -C)$ and $(0, -1, 0)$.

Let $S_{1a} = \{(x, 1, z) \mid x \in [-\varepsilon, \varepsilon], z \in [0, C]\}$ be the strip near $\eta_{1a}$. Let $S_{1b} = \{(x, y, Cy) \mid x \in [-\varepsilon, \varepsilon], y \in [-1, 1]\}$ be the strip near $\eta_{1b}$. Finally, let $S_2 = \{(x, -1, z) \mid x \in [-\varepsilon, \varepsilon], z \in [-1, 0]\}$. In other words, $S_{1a}$ is strip near $\eta_{1a}$ in $\{y = 1\}$ plane, $S_{1b}$ is strip near $\eta_{1a}$ in $\{z = Cy\}$ plane, and $S_2$ is the strip in $\{y = -1\}$ plane with thicknesses $2\varepsilon$. Let $S_1 = S_{1a} \cup S_{1b}$.

To get $\hat{E}$, modify/trim the tips of the strips $S_1$ and $S_2$, say $\tilde{S}_1$ and $\tilde{S}_2$, so that $\partial \hat{E} = \hat{\Gamma}$ where $\hat{E} = E \cup \tilde{S}_1 \cup \tilde{S}_2$ (See Figure 2). Clearly, $|\hat{E}| \sim |E| + |S_{1a}| + |S_2| = 4 + 2\varepsilon(C + \sqrt{C^2 + 1})$. Hence, by fixing sufficiently large $C_0$, there exists $\varepsilon_0 > 0$ such that $|\hat{\Sigma}^\varepsilon| > |\hat{E}^\varepsilon|$ for any $\varepsilon < \varepsilon_0$. This shows that the area minimizing disk in $\mathbb{R}^3$ bounding $\hat{\Gamma}^\varepsilon$ does not lie in $M$.

From now on, we fix a sufficiently large $C_0$ as above, and $\hat{\Sigma}^\varepsilon$, $\hat{E}^\varepsilon$ and $\hat{\Gamma}^\varepsilon$ represents the corresponding disks and curve for $\varepsilon < \varepsilon_0$. Hence, $\partial \hat{\Sigma}^\varepsilon = \partial \hat{E}^\varepsilon = \hat{\Gamma}^\varepsilon$, and $|\hat{\Sigma}^\varepsilon| \gg |\hat{E}^\varepsilon|$ for any $\varepsilon < \varepsilon_0$.

Claim: There is a sufficiently small $\varepsilon < \varepsilon_0$ such that the area minimizing disk $D^\varepsilon$ in $\mathbb{R}^3$ with $\partial D^\varepsilon = \hat{\Gamma}^\varepsilon$ is not embedded.

Proof: First, we will show that $D^\varepsilon$ is very close to $\hat{E}^\varepsilon$ for small $\varepsilon > 0$. In particular, we claim that for any $\rho > 0$, there exists $\varepsilon < \varepsilon_0$ such that $d(D^\varepsilon, \hat{E}^\varepsilon) < \rho$ where the distance $d$ represents the Hausdorff distance, and $D^\varepsilon$ is the area minimizing disk in $\mathbb{R}^3$ with $\partial D^\varepsilon = \hat{\Gamma}^\varepsilon$. 
Assume on the contrary that there is a $\rho_0 > 0$ such that $d(D^\epsilon, \hat{E}^\epsilon) \geq \rho_0$ for any $\epsilon > 0$. Let $\epsilon_i \searrow 0$ be a sequence converging to 0. Let $T_i = [D^{\epsilon_i}]$ be the corresponding currents. Then, $\partial T_i = Y_i = [\hat{\Gamma}^{\epsilon_i}]$. Let $T_0 = [E]$ and $Y_0 = [\tau]$.

Let $M$ represent the mass of a current. Then, $M(T_i) = |D^{\epsilon_i}| \leq |\hat{E}^{\epsilon_i}| = 4 + 2\epsilon(C_0 + \sqrt{C_0^2 + 1})$ for any $i$ as $D^{\epsilon_i}$ is the area minimizing disk, and $\hat{E}^{\epsilon_i}$ is a disk with the same boundary $\hat{\Gamma}^{\epsilon_i}$. Hence, the currents $T_i$ have uniformly bounded masses, and thus by the Federer-Fleming compactness theorem [Fe], after passing to a subsequence, $T_i \rightarrow T_0$ in the sense of currents, where $T$ is an integral current.

We claim that $T = T_0 = [E]$. As $M(T_i) \rightarrow 4$, we get $M(T) = 4$. By construction $Y_i \rightarrow Y_0$ in the sense of currents, and hence $\partial T = Y_0$. Note that $E$ is the unique absolutely area minimizing surface in $\mathbb{R}^3$ with $\partial E = \tau$. As $M(T) = |E|$ with $\partial T = Y_0$, this implies $T = T_0$. This proves that $T_i \rightarrow T_0$ in the sense of currents.

Recall that by assumption, there is $\rho_0 > 0$ such that $d(D^{\epsilon_i}, \hat{E}^{\epsilon_i}) \geq \rho_0$ for any $i$. In other words, there exists $x_i \in D^{\epsilon_i}$ such that $d(x_i, \hat{E}^{\epsilon_i}) \geq \rho_0$ for any $i$. After passing to a subsequence, we get $x_i \rightarrow x_0 \in \mathbb{R}^3$, and by construction $d(x_0, E \cup \eta_1 \cup \eta_2) \geq \rho_0$ as $E \cup \eta_1 \cup \eta_2 \subset \hat{E}^{\epsilon_i}$ for any $i$. Since $x_i \rightarrow x_0$, there exists $i_0$, such that for any $i \geq i_0$ we have $B_{\rho_0/4}(x_i) \subset B_{\rho_0/2}(x_0) \subset B_{\rho_0}(x_i)$ where $B_{\rho}(x)$ denotes the ball of radius $\rho$ and centered at $x$ in $\mathbb{R}^3$.

Hence, using the monotonicity formula [Si] and the fact that $x_i \in \text{spt } T_i \setminus \text{spt } \partial T_i$, we get $\mu_{T_i}(B_{\rho_0/4}(x_i)) \geq \mu_{T_i}(B_{\rho_0/2}(x_i)) \geq \pi \left(\frac{\rho_0}{4}\right)^2$

Now, by using the measure convergence, we have that

$$\mu_{T_0}(B_{\rho_0/4}(x_0)) = \lim_{i} \mu_{T_i}(B_{\rho_0/4}(x_i)) \geq \pi \left(\frac{\rho_0}{4}\right)^2$$

and since $B_{\rho_0/4}(x_0) \subset B_{\rho_0}(x_i)$

$$\mu_{T_0}(B_{\rho_0}(x_i)) \geq \mu_{T_0}(B_{\rho_0/2}(x_0)) \geq \pi \left(\frac{\rho_0}{4}\right)^2$$

which implies that $\text{spt } T_0 \cap B_{\rho_0}(x_i) \neq \emptyset$. However, by assumption $d(x_i, E) \geq \rho_0$, and this is contradiction.

Hence, for any $\rho > 0$, there is a sufficiently small $\epsilon > 0$ such that the area minimizing disk $D^\epsilon$ is $\rho$-close to $\hat{E}^\epsilon$. Since $\hat{E}^\epsilon$ has transversal self-intersection, this implies $D^\epsilon$ is not embedded for sufficiently small $\epsilon$. This finishes the proof of the claim.

Hence, the claim above implies the area minimizing disk $D^\epsilon$ in $\mathbb{R}^3$ which $\Gamma^\epsilon$ bounds is not embedded, even though $\Gamma^\epsilon$ lies in the boundary of a mean convex domain $M^\epsilon$ in $\mathbb{R}^3$. In other words, Meeks-Yau’s result (Theorem [5.1]) says that for an $H$-extremal curve $\Gamma \subset \partial M$, the solution to the Plateau problem in the mean convex manifold $M$ is embedded, and this implies the solution to the Plateau problem for an extreme
curve in $\mathbb{R}^3$ is embedded. However, it does not say that the solution to the Plateau problem for an $H$-extreme curve in $\mathbb{R}^3$ is embedded, and the example above shows that this is not true.

**Remark 3.1.** Let $\hat{\Sigma}^\epsilon$ and $D^\epsilon$ be the area minimizing disks in the example above with $\partial\hat{\Sigma}^\epsilon = \partial D^\epsilon = \hat{\Gamma}^\epsilon$ ($\hat{\Sigma}^\epsilon$ is the minimizer in $M^\epsilon$ while $D^\epsilon$ is minimizer in $\mathbb{R}^3$). Notice that by choosing $C$ and $\epsilon$ accordingly in the construction, the ratio $|\hat{\Sigma}^\epsilon|/|D^\epsilon|$ between the area minimizing disks $\hat{\Sigma}^\epsilon$ and $D^\epsilon$ can be made as large as we want. In other words, while the area of the minimizer in the ambient space $\mathbb{R}^3$ is very small, the minimizer in the mean convex manifold $M^\epsilon \subset \mathbb{R}^3$ can have very large area.

4. **Example II: White’s Decomposition Theorem**

White’s decomposition theorem for absolutely area-minimizing hypersurfaces at boundaries with multiplicity states the following:

**Theorem 4.1.** [Wh1] Let $\Gamma$ be a codimension-2 smooth submanifold in $\mathbb{R}^{n+1}$, and $T$ is an absolutely area minimizing hypersurface with $\partial T = k\Gamma$ where $k > 1$. Then, $T = \sum_i^k T_i$ where each $T_i$ is an absolutely area minimizing hypersurface with $\partial T_i = \Gamma$.

In particular, this theorem extends naturally to the orientable Riemannian manifolds with trivial second homology. In this paper, by constructing an explicit example, we will show that the decomposition theorem does not generalize to the orientable manifolds with nontrivial second homology. Note that our example is 3-dimensional, but by using similar arguments, it can be extended to higher dimensions.

An important observation about this theorem is the following: If this decomposition theorem was true in general setting, it would imply that the absolutely area minimizing surfaces with the same boundary in such a manifold would be disjoint. This is because if $\Sigma_1$ and $\Sigma_2$ are two absolutely area minimizing surfaces with $\partial \Sigma_i = \Gamma$, and $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, then we can make a surgery along the intersection curve $\alpha$ (See Figure 3). First, note that $\Sigma_1$ and $\Sigma_2$ cannot be separating each other. This is because if the two surfaces are separating each other, then we get a contradiction by a swaping argument as follows: If $\Sigma_1$ and $\Sigma_2$ are separating each other, then $\Sigma_1 - \Sigma_2 = S^+_1 \cup S^-_1$ and $\Sigma_2 - \Sigma_1 = S^+_2 \cup S^-_2$ where $S^+_i$ be the component in $\Sigma_i$ containing the boundary $\Gamma$, and $S^-_i$ be the other components in $\Sigma_i$. Note that $|S^+_1| = |S^+_2|$ as they are also absolutely area minimizing surfaces with the same boundary. Then define $\Sigma'_1 = S^+_1 \cup S^-_2$ and $\Sigma'_2 = S^+_2 \cup S^-_1$. Notice that the new surfaces $\Sigma'_1$ and $\Sigma'_2$ have the same area with $|\Sigma'_1| = |\Sigma'_2|$. However, new surfaces contain folding curves $\Sigma_1 \cap \Sigma_2$. We get a contradiction as we can get a smaller area surfaces by pushing the surfaces to the convex side along folding curve [MY2]. Note that the same
contradiction can be obtained by using the regularity theorem for absolutely area minimizing surfaces as the absolutely area minimizing surfaces must be smooth \cite{Fe}. This implies that the surfaces can not separate each other.

Note that if two surfaces were homologous, either the intersection would be empty, or the surfaces would separate each other. Hence, since $\Sigma_1 \cap \Sigma = \alpha$ is nonempty, $\Sigma_1$ and $\Sigma_2$ are not homologous. This implies by doing surgery (choosing the correct sides to match the orientation) along $\alpha$, we get a connected oriented surface $\Sigma$ with the area $|\Sigma_1| + |\Sigma_2|$ and with $\partial \Sigma = 2\Gamma$ (See Figure 3). $\Sigma$ is an absolutely area minimizing surface for the boundary $2\Gamma$ as the decomposition theorem implies that the smallest area surface with boundary $2\Gamma$ has area $|\Sigma_1| + |\Sigma_2| = 2|\Sigma_i|$. This is a contradiction as before, since we have a folding curve along $\alpha$ in the absolutely area minimizing surface $\Sigma$. This shows that if there are such surfaces $\Sigma_1$ and $\Sigma_2$ intersecting each other, the decomposition theorem does not generalize. As we will see in the following section, this is not the case (See Remark 5.2). Hence, the examples constructed in the next section also show (indirectly) that the decomposition theorem is not true for the oriented 3-manifolds with nontrivial second homology.

In this section, we will construct an example of a connected, oriented absolutely area minimizing surface $\Sigma$ with $\partial \Sigma = 2\Gamma$ with less area than $|\Sigma_1| + |\Sigma_2|$, which shows directly that the decomposition theorem does not extend to manifolds with nontrivial homology.

Now, we construct the example. First, to give the basic idea, we will start the topological construction where the area minimizing surface is not smooth. Then,
we will modify the original example to make the ambient space strictly mean convex and the surfaces are smooth (See Remark 5.1).

Let $T^3$ be the 3-torus obtained by identifying the opposite faces of the rectangular box of dimensions $[0,1] \times [0,1] \times [0,h]$. Take the induced flat metric on $T^3$. Let $B^3$ be the 3-ball in the shape of a parallelepiped in the $T^3$ with square base of dimensions $[\delta,1-\delta] \times [\delta,1-\delta]$ and height $\frac{h}{3}$. We also assume that the parallelepiped has slope $\theta_0$ in $x$-direction where $\tan \theta_0 < 1/6$. In particular, the corners of the parallelepiped is as follows: The vertices of the bottom square are $a_1 = (\delta, \delta, \frac{h}{3})$, $a_2 = (\delta, 1-\delta, \frac{h}{3})$, $a_3 = (1-\delta, 1-\delta, \frac{h}{3})$, $a_4 = (1-\delta, \delta, \frac{h}{3})$. Let $\sigma = \frac{h}{3} \cot \theta_0 > 2h$. Then, by shifting all the bottom vertices $\sigma$ (assume $\sigma < \delta$) in the negative $x$-direction, we get the vertices of the top square: $b_1 = (\delta - \sigma, \delta, \frac{2h}{3})$, $b_2 = (\delta - \sigma, 1-\delta, \frac{2h}{3})$, $b_3 = (1-\delta - \sigma, 1-\delta, \frac{2h}{3})$, $b_4 = (1-\delta - \sigma, \delta, \frac{2h}{3})$.

Now, define the ambient space $M$ as $T^3 - B^3$ where $h$ and $\sigma$ are to be determined later.

For $\frac{h}{3} < c < \frac{2h}{3}$, let $\Gamma_c$ be the simple closed curve in $\partial M$ with $\Gamma_c = \partial M \cap T_c$ where $T_c$ is the 2-torus corresponding to $\{z = c\}$ square in $T^3$ before identification. In other words, $\Gamma_c$ is the union of four line segments, i.e. $\Gamma = \tau_1 \cup \tau_2 \cup \tau_3 \cup \tau_4$ where $\tau_1 = \{\delta\} \times [\delta, 1-\delta] \times \{c\}$, $\tau_2 = [\delta, 1-\delta] \times \{1-\delta\} \times \{c\}$, $\tau_3 = \{1-\delta\} \times [\delta, 1-\delta] \times \{c\}$, and $\tau_4 = [\delta, 1-\delta] \times \{\delta\} \times \{c\}$.

Let $\Sigma_c$ be the surface obtained by the intersecting the 2-torus $T_c$ with $M$. Hence, $\Sigma_c$ is a 2-torus where a disk $(T_c \cap B^3)$ is removed and $\partial \Sigma_c = \Gamma_c$ (See Figure 5). Observe that either $\Sigma_c$ is the absolutely area minimizing surface in $M$ with $\partial \Sigma_c = \Gamma_c$ or the absolutely area minimizing surface in $M$ with boundary $\Gamma_c$ completely lies in $\partial M$. This is because the family of horizontal 2-tori $\{T_c\}$ foliates the flat torus $T^3$ by minimal surfaces. If there was another area minimizing surface $S$ in $M$ with $\partial S = \Gamma_c$, which does not completely lie in $\partial M$, $S \subset T^3$ must have a tangential intersection with one of horizontal 2-tori $T_c\prime$ by lying in one side, which contradicts to the maximum principle.
Assuming $c < \frac{h}{2}$, let $D_c$ be the smaller disk in $\partial M$ with $\partial D_c = \Gamma_c$. Then, the area of $D_c$ would be $x^2 + 4x \frac{c-h/3}{\sin \theta_0}$ where $x = 1 - 2\delta$, the side of the square base of $\partial M$. The area of $\Sigma_c$ is $1 - x^2$ for any $c \in \left[ \frac{h}{3}, \frac{2h}{3} \right]$. Hence, when $x^2 + 4x \frac{c-h/3}{\sin \theta_0} > 1 - x^2$, $\Sigma_c$ is the absolutely area minimizing surface, and when $x^2 + 4x \frac{c-h/3}{\sin \theta_0} < 1 - x^2$, $D_c$ is the absolutely area minimizing surface in $M$ with boundary $\Gamma_c$ by the discussion above. Also, for $c_o \in \left[ \frac{h}{3}, \frac{2h}{3} \right]$ with $x^2 + 4x \frac{c_o-h/3}{\sin \theta_0} = 1 - x^2$, both $\Sigma_{c_o}$ and $D_{c_o}$ would be absolutely area minimizing surfaces with boundary $\Gamma_{c_o}$. Notice that by choosing $x = 1 - 2\delta$ sufficiently close to $1/\sqrt{2}$ from below, we can choose $h$ as small as we want, and we would have a solution $c_o$ to the equation $x^2 + 4x \frac{c_o-h/3}{\sin \theta_0} = 1 - x^2$ in $\left[ \frac{h}{3}, \frac{2h}{3} \right]$. Now, we will show that there is a connected absolutely area minimizing surface $\Sigma$ with $\partial \Sigma = 2\Gamma_{c_o}$.

Now, as in the figure, remove a disk $O_1$ from $\Sigma_{c_o}$ and remove a disk $O_2$ from $D_{c_o}$ where both $O_1$ and $O_2$ are disks with radius $\epsilon$ where $h < \epsilon < \frac{h}{2} = \frac{h}{6} \cot \theta_0$. In

**Figure 5.** $\Sigma_c$ is the absolutely area minimizing surface in $M$ with $\partial \Sigma_c = \Gamma_c$.

**Figure 6.** The green dots represents $\Gamma_{c_o}$. The red dots represents the identification in the torus. $\Sigma_{c_o}$ (blue surface) and $D_{c_o}$ (purple surface) are both absolutely area minimizing surfaces bounding $\Gamma_{c_o}$ in $M$. In the middle figure, the tube $T$ connecting $D_{c_o}$ and $\Sigma_{c_o}$ gives the connected surface $\Sigma$ with $\partial \Sigma = 2\Gamma_{c_o}$ with less area. In the figure right, we see that if we place the green tube in that way, the surface we are going to have will have boundary $0. \Gamma_{c_o} = \emptyset$ as $D_{c_o}$ and $\Sigma_{c_o}$ must be oppositely oriented at the beginning to have an oriented surface after surgery in this configuration.
particular, let $O_1$ have center $(1 - \delta - \frac{\sigma}{2}, \frac{1}{2}, c_o)$, and $O_2$ have center $(1 - \delta - \frac{\sigma}{2}, \frac{1}{2}, h \frac{\sigma}{3})$. Let $T$ be the cylinder in $M$ with boundary $\partial O_1 \cup \partial O_2$ as in the Figure 6 (middle). Then, the area of the tube $T$ would be $2.\pi.e.(h - (c_o - \frac{h}{3})) = 2.\pi.e.\sigma, e^2$. Hence, for $\frac{4h}{3} - c_o < h < \epsilon$, $|T| < |O_1| + |O_2|$, and the new surface $\Sigma' = (\Sigma_{c_o} - O_1) \cup (D_{c_o} - O_2) \cup T$ would have less area than the $\Sigma_{c_o} \cup D_{c_o}$. Moreover, by the choice of $T$, $\Sigma'$ is an orientable surface with $\partial \Sigma' = 2\Gamma_{c_o}$ (See Remark 4.1). Recall that if the decomposition theorem was true in this case, then the oriented absolutely area minimizing surface $S$ with $\partial S = 2\Gamma_{c_o}$ would decompose as $S = S_1 + S_2$ where $S_i$ is an absolutely area minimizing surface with $\partial S_i = \Gamma_{c_o}$. This would imply $|S| = |S_1| + |S_2| = 2|S_1|$. However, there is an oriented surface $\Sigma'$ with $\partial \Sigma' = 2\Gamma_{c_o}$ and $|\Sigma'| < |\Sigma_{c_o}| + |D_{c_o}| = 2|\Sigma_{c_o}| = 2|D_{c_o}|$. This proves that the decomposition theorem is not valid for ambient manifold $M$.

Remark 4.1. Note that here the choice of the handle, the cylinder $T$, in the construction is very important to get an oriented surface with boundary $2\Gamma_{c_o}$. The other choice of the handle $T'$ as in the Figure 6 (right) would give us another oriented surface with boundary $0.\Gamma_{c_o} = \emptyset$ as we need to reverse the orientation on $D_{c_o}$ or $\Sigma_{c_o}$ to have oriented $T'$.

Now, in order to get smooth examples, we will modify the metric on $M$. To do this, we will change the metric $g$ on $M$ to $\tilde{g}$ so that $M$ with this new metric $\tilde{g}$, say $\tilde{M}$, will be strictly mean convex (See Remark 5.1). Then, the absolutely area minimizing surfaces in $\tilde{M}$ would be smoothly embedded ([ASS], [Wh2]). Clearly, $M$ above with the induced flat metric is not mean convex as the dihedral angles at the boundary are greater than $\pi$ (See 2.2 condition 3).

Note that as the mean convexity is a local condition, it will suffice to change the metric only near the boundary $\partial M$. Hence, the new metric $\tilde{g}$ will be same with $g$ everywhere on $M$ except a small neighborhood of $\partial M$, say $N_{\tilde{g}}(\partial M)$. First, change the metric in $N^{\frac{1}{2}}(\partial M)$ so that it is isometric to $N^{\frac{1}{2}}(\partial B_{r_o})$ in $B_{r_o}$ where $B_{r_o}$ is the closed ball of radius $r_o$ in $R^3$ with $|\partial B_{r_o}| = |\partial M|$. Here, we chose $r_o$ with $|\partial B_{r_o}| = |\partial M|$, since after modification of the metric, we want the curve $\Gamma_c \subset \partial \tilde{M}$ constructed above to have similar features as before. In particular, in order to employ the construction above, after the modification of the metric, we would like to have a $c_o$ with for $c < c_o$, the absolutely area minimizing surface bounding $\Gamma_c$ is the disk $D_{c_o}$ near boundary, and for $c > c_o$, the absolutely area minimizing surface bounding $\Gamma_c$ is the punctured torus $\Sigma_c$.

After making the metric isometric to $N^{\frac{1}{2}}(\partial B_{r_o})$ in $N^{\frac{1}{2}}(\partial M)$, then to make the new metric smooth on $M$, use partition of unity on the part $N^{\frac{1}{2}}(\partial M) - N^{\frac{1}{2}}(\partial M)$, which is homeomorphic to $S^2 \times I$ so that it has small cross-sectional area (cross-sectional annuli $\gamma \times I$ have small area $\sim \xi, |\gamma|$). We ask this to keep the areas of $\Sigma_c$
small as before. Hence, we get a smooth metric on $M$ so that $M$ is strictly mean convex. Note that we can construct $\hat{g}$ as rotationally symmetric in the $xy$-direction because of the setting.

We will call $M$ with the new metric $\hat{g}$ as $\hat{M}$ for short. Note that we just changed the original metric in a very small neighborhood of the boundary, $N_\epsilon(\partial M)$. We will use the same coordinates as before. Let $\{\Gamma_c\}$ be the family of simple closed curves in $\partial \hat{M}$ as above. Consider the absolutely area minimizing surfaces in $\hat{M}$ with boundary $\Gamma_c$. By the construction of the metric, these absolutely area minimizing surfaces would be close to $\Sigma_c$ or $D_c$, depending on $c$ as before. Call the absolutely area minimizing surfaces in $\hat{M}$ close to $\Sigma_c$ and $D_c$ as $\hat{\Sigma}_c$ and $\hat{D}_c$ respectively.

Notice that the area of $\hat{\Sigma}_c$ and $\hat{D}_c$ changes continuously with respect to $c$, as $\Gamma_i \rightarrow \Gamma$ the area of the annulus $A_i$ between $\Gamma_i$ and $\Gamma$ goes to $0$, i.e. $|A_i| \rightarrow 0$. Hence, for some $c_o$, $|\hat{\Sigma}_{c_o}| = |\hat{D}_{c_o}|$. Now, we will imitate the construction above, but this time we will get a smooth, oriented connected surface at the end.

Like before, remove a disk $\hat{O}_1$ from $\hat{\Sigma}_{c_o}$ and remove a disk $\hat{O}_2$ from $\hat{D}_{c_o}$ where $\hat{O}_1$ and $\hat{O}_2$ are both disks with radius $\epsilon$. As before, we can choose the disks close in horizontal direction. Then, there is an area minimizing cylinder $\hat{T}$ with the boundary $\partial \hat{O}_1 \cup \partial \hat{O}_2$ by [MY2], as $|\hat{T}| \sim |T| = 2.\pi.\epsilon.(h - c)$ and $|\hat{O}_1| + |\hat{O}_2| \sim |O_1| + |O_2| = 2.\pi.\epsilon^2$. Let $\hat{\Sigma}' = (\hat{\Sigma}_{c_o} - \hat{O}_1) \cup (\hat{D}_{c_o} - \hat{O}_2) \cup \hat{T}$. Then, like before $|\hat{\Sigma}'| < 2|\hat{D}_{c_o}| = |\hat{\Sigma}_{c_o}| + |\hat{D}_{c_o}|$ and $\partial \hat{\Sigma}' = 2\Gamma_{c_o}$.

**Remark 4.2.** If we took $\Sigma_1$ and $\Sigma_2$ in the example as homologous surfaces, the oriented surface we would get after the surgery would have boundary $0\Gamma = \emptyset$ not $2\Gamma$. This is because if $\Sigma_1$ and $\Sigma_2$ are homologous in $M$, then $\Sigma_1 \cup \Sigma_2$ would separate $\hat{M}$ into $\Omega^+ \cup \Omega^-$. Hence, for any choice of the handle $T (T \subset \Omega^+$ or $T \subset \Omega^-)$, if $T$ is oriented surface with $\partial T = \partial O_1 \cup \partial O_2$ where $O_i$ is a small disk in $\Sigma_i$ as above, then $T$ would be homologous to $(\Sigma_1 - O_1) + (\Sigma_2 - O_2)$. This would force $\Sigma_1$ and $\Sigma_2$ to be oppositely oriented, and we get an oriented connected surface $\Sigma' = (\Sigma_1 - O_1) \cup (\Sigma_2 - O_2) \cup T$ with less area. However, in this situation, $\partial \Sigma' = 0, \Gamma_{c_o} = \emptyset$.

Intuitively, when $\Sigma_1$ and $\Sigma_2$ are both oriented so that $\partial \Sigma_1 = \partial \Sigma_2 = +\Gamma$, the "upper sides" of $\Sigma_1$ and $\Sigma_2$ look into the same direction. To keep this orientations after surgery, the surgery tube $T$ must start from the "upper side" of $\Sigma_1$ and end in the "upper side" of $\Sigma_2$. However, If $\Sigma_1 \cup \Sigma_2$ bounds the domain $\Omega^+$, and $T \subset \Omega^+$, this is not possible as $T$ starts from the upper side of $\Sigma_1$ but ends in "bottom side" of $\Sigma_2$. Hence, to have $\Sigma'$ oriented, we must reverse the orientation on $\Sigma_2$, which forces $\partial \Sigma' = \Gamma - \Gamma = \emptyset$ (See Figure [7](left)).

However, if $\Sigma_1$ and $\Sigma_2$ are not homologous, then $\Sigma_1 \cup \Sigma_2$ does not separate $\hat{M}$, and the tube $T$ is free to land in the "upper side" $\Sigma_2$, which gives the surface $\Sigma'$ with the desired orientation (See Figure [7](right)).
If the surfaces $\Sigma_1$ and $\Sigma_2$ are homologous, we may not make a surgery (green "tube") consisting with the original orientations given on $\Sigma_1$ and $\Sigma_2$ (left). However, if $\Sigma_1$ and $\Sigma_2$ are not homologous, there is a surgery consistent with the original orientations for any given orientations on $\Sigma_1$ and $\Sigma_2$ so that $\partial \Sigma_1 \# \Sigma_2 = 2\Gamma$ (right).

5. Example III: Intersections of Absolutely Area Minimizing Surfaces

In this section, we will give explicit examples describing when the two absolutely area minimizing surfaces with disjoint boundaries can intersect. Let $M$ be a mean convex manifold. Let $\Sigma_1$ and $\Sigma_2$ be two absolutely area minimizing surfaces in $M$ with $\partial \Sigma_i = \Gamma_i$, where $\Gamma_1$ and $\Gamma_2$ are disjoint simple closed curves in $\partial M$. Then, it is known that if $\Sigma_1$ and $\Sigma_2$ are homologous in $M$ (relative to $\partial M$), then they must be disjoint ([Co], Lemma 4.1), ([Ha], Theorem 2.3).

Hence, if two such absolutely area minimizing surfaces with disjoint boundaries intersect each other, they cannot be homologous. This gives us two situations about the ambient space and boundary curves. The first situation is $H_2(M)$ is not trivial, and $\Sigma_1$ and $\Sigma_2$ belongs to different homology classes in $H_2(M, \partial M)$. The second situation is $H_2(M)$ is trivial, but $\Gamma_1$ and $\Gamma_2$ are not homologous in $\partial M$. In this paper, we will construct explicit examples in both situations. In particular, we will show the following:

**Theorem 5.1.** Let $M$ be a mean convex 3-manifold. Let $\Gamma_1$ and $\Gamma_2$ be two simple closed curves in $\partial M$ and let $\Sigma_1$ and $\Sigma_2$ be two absolutely area minimizing surfaces in $M$ with $\partial \Sigma_i = \Gamma_i$. Let $\Gamma_1 \cap \Gamma_2 = \emptyset$, but $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. Then, either $\Gamma_1$ is not homologous to $\Gamma_2$ in $\partial M$, or $H_2(M)$ is not trivial. Also, there are examples in both cases.
Note that the statement of the theorem is equivalent to the result mentioned in the first paragraph [Co], [Ha]. In particular, in this setting the statement "\(\Sigma_1\) and \(\Sigma_2\) are not homologous in \(M\) (relative to \(\partial M\))" is equivalent to say that either \(\partial \Sigma_1\) is not homologous to \(\partial \Sigma_2\), or \(\Sigma_1\) and \(\Sigma_2\) differ by a homology class of \(H_2(M)\). Our examples show that both situations are possible.

5.1. Example III-A: \(H_2(M)\) is trivial. In this part, we will describe an example when the ambient space \(M\) has trivial second homology. Let \(\Sigma_i\) and \(\Gamma_i\) be as in the theorem, and let \(H_2(M)\) be trivial. Then, \(\Sigma_1\) and \(\Sigma_2\) are homologous in \(M\) (rel. \(\partial M\)) if and only if \(\Gamma_1\) and \(\Gamma_2\) are homologous in \(\partial M\). This is because if \(S\) is a subsurface in \(\partial M\) with \(\partial S = \Gamma_1 \cup \Gamma_2\), then \(\Sigma_1 \cup \Sigma_2 \cup S\) would be a closed surface in \(M\). As \(H_2(M)\) is trivial, it separates \(M\), and hence \(\Sigma_1\) and \(\Sigma_2\) are homologous.

In the other direction, if \(\Sigma_1\) and \(\Sigma_2\) are homologous, then they separate a piece \(\Omega\) from \(M\). Then, \(\partial \Omega\) contains \(\Sigma_1\) and \(\Sigma_2\). Let \(S = \partial \Omega \cap \partial M\). Then, \(\partial S = \Gamma_1 \cup \Gamma_2\) which shows that \(\Gamma_1\) and \(\Gamma_2\) are homologous in \(\partial M\). Hence, we can construct such an example in the trivial homology case only if the disjoint simple closed curves \(\Gamma_1\) and \(\Gamma_2\) in \(\partial M\) are not homologous in \(\partial M\).

Now, we construct the example. Let \(B\) be the closed unit ball in \(\mathbb{R}^3\). Let \(\gamma_i^+\) be the curve \(\partial B \cap \{z = \frac{1}{i}\}\), i.e. \(\gamma_i^+ = \{(x, y, \frac{1}{i}) \in \mathbb{R}^3 \mid x^2 + y^2 = \frac{24}{25}\}\). Let \(\gamma_i^-\) be the curve \(\partial B \cap \{z = -\frac{1}{i}\}\). Similarly, let \(\gamma_2^+\) be the curve \(\partial B \cap \{z = \frac{1}{10}\}\), and let \(\gamma_2^-\) be the curve \(\partial B \cap \{z = -\frac{1}{10}\}\). Since the total area of the disks \(D_1^+\) and \(D_1^-\) with \(\partial D_1^+ = \gamma_1^+ (\sim 2\pi)\) is greater than the annulus with boundary \(\gamma_1^+ \cup \gamma_1^- (\sim \frac{2}{5}\pi)\), the absolutely area minimizing surface \(A_1\) bounding \(\gamma_1^+ \cup \gamma_1^-\) is an annulus, which is a segment of a catenoid. Similarly, let \(A_2\) be the absolutely area minimizing surface with boundary \(\gamma_2^+ \cup \gamma_2^-\). Clearly, \(A_1 \cap A_2 \neq \emptyset\). Indeed, since \(A_2\) is just reflection of \(A_1\) with respect to the \(xy\)-plane, \(A_1\) intersects \(A_2\) on the \(xy\)-plane.

Since \(\partial A_i = \gamma_i^+ \cup \gamma_i^-\) is not connected for each \(i\), we cannot use \(A_1\) and \(A_2\) as counterexamples. Our aim is to add a bridge \(S_i\) to \(\partial A_i\) so that new surfaces \(\Sigma_i \sim A_i \cup S_i\) has connected boundaries. However, since \(\partial B\) is a sphere and adding a bridge connecting \(\gamma_1^+\) and \(\gamma_1^-\) without intersecting \(\partial A_2\) is impossible. As we want the resulting boundaries to be disjoint, too, we need to modify the ambient space by adding handles to \(B\) to bypass this problem.

Now, we will add two 1-handles to \(B\). Let \(\alpha_1\) be the circular arc with endpoints \(x = (0, \sqrt{1 - (3/20)^2}, 3/20)\) and \(y = (0, \sqrt{1 - (1/20)^2}, -1/20)\), and perpendicular to the the unit sphere \(\partial B\) (See Figure 3(left)). Let \(\alpha_2\) be the reflection of \(\alpha_1\) with respect to the origin. Let \(T_1 = N_c(\alpha_1)\), and \(T_2 = N_c(\alpha_2)\) be the 1-handles which we attach to \(B\). Notice that the 3-manifold \(M' = B \cup T_1 \cup T_2\) is not mean convex because of the intersections of handles with \(\partial B\), i.e. \(\partial T_1 \cap \partial B\).

Now, we will modify \(M'\) to get a mean convex 3-manifold. Consider the line \(l\) through the point \(x\) and perpendicular to the unit sphere \(\partial B\). Parametrize \(l\) by arclength such that \(l(0) = x\). Then, let \(l(\delta) = x^+\) and \(l(-\delta) = x^-\). Let \(P^+\)
Figure 8. $M$ is a mean convex 3-manifold in $\mathbb{R}^3$. $\Gamma_1$ and $\Gamma_2$ are disjoint simple closed curves in $\partial M$ (left). Homeomorphic images of $M$, $\Gamma_1$ and $\Gamma_2$ gives the picture in the right, where $\hat{M}$ is a genus 2 handlebody, and $\Gamma_1$ and $\Gamma_2$ are non homologous curves in $\partial \hat{M}$.

be the plane through $x^+$ and perpendicular to $l$ and similarly, let $P^-$ be the plane through $x^-$ and perpendicular to $l$. Let $\beta^+$ be the round circle $P^+ \cap \partial M'$, and let $\beta^-$ be the round circle $P^- \cap \partial M'$. Now consider the catenoid $C$ with axis $l$ and containing the circles $\beta^+$ and $\beta^-$. By choosing $\delta$ sufficiently small, we can make sure that the segment $\hat{C}$ of the catenoid $C$ between the circles $\beta^+$ and $\beta^-$ lies completely in $M'$. Then, by removing the region between $\hat{C}$ and $\partial M'$ to $M'$, and do the same operation at all other 3 basepoints of the 1-handles $T_1$ and $T_2$, we get a mean convex manifold $M$. By replacing $\hat{C}$ in the construction, we can assume $M$ is strictly mean convex, too.

Let $\tau_1$ be a path in $\partial M$ connecting the curves $\gamma_1^+$ and $\gamma_1^-$ through the handle $T_1$. In particular, let $\tau_1$ be the shorter arc in $\partial M \cap yz$-plane between $\gamma_1^+$ and $\gamma_1^-$, i.e. the endpoints of $\tau_1$ are $(0, \frac{\sqrt{23}}{5}, \frac{1}{5})$ and $(0, \frac{\sqrt{29}}{10}, -\frac{1}{10})$. Similarly, define $\tau_2$ to be the shorter arc in $\partial M \cap yz$-plane between $\gamma_2^+$ and $\gamma_2^-$ going through the handle $T_2$. Then by using the bridge principle for absolutely area minimizing surfaces ([BC], Lemma 3.8b), we get the simple closed curve $\Gamma_1$ obtained by putting a thin bridge along $\tau_1$ between $\gamma_1^+$ and $\gamma_1^-$ such that the absolutely area minimizing surface $\Sigma_1$ in $M$ with $\partial \Sigma_1 = \Gamma_1$ would be the surface obtained from $A_1$ by attaching a thin strip in $M$ near $\tau_1$, i.e. $\Sigma_1 \sim A_1 \cup S_{\tau_1}$ where $S_{\tau_1}$ is a thin strip along $\tau_1$. Similarly, we get a simple closed curve $\Gamma_2$ bounding the absolutely area minimizing surface $\Sigma_2$ with $\Sigma_2 \sim A_2 \cup S_{\tau_2}$. By the construction, $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. This finishes the example in the trivial homology case.

Notice that the simple closed curves $\Gamma_1$ and $\Gamma_2$ are not homologous in $\partial M$ as mentioned at the beginning (See Figure 8 (right)). The discussion above shows that when $H_2(M)$ is trivial, if $\Gamma_1$ and $\Gamma_2$ are homologous disjoint simple closed curves
in \( \partial M \), then the surfaces \( \Sigma_1 \) and \( \Sigma_2 \) with \( \partial \Sigma_i = \Gamma_i \) must be homologous in \( M \). In other words, if \( \Gamma_1 \cup \Gamma_2 \) separates \( \partial M \), then \( \Sigma_1 \cup \Sigma_2 \) must separate \( M \). Then, by [Co], [Ha], this implies the absolutely area minimizing surfaces \( \Sigma_1 \) and \( \Sigma_2 \) must be disjoint by a simple swapping argument. We rephrase this statement in this context as follows.

**Theorem 5.2.** [Ha, Co] Let \( M \) be a strictly mean convex 3-manifold with trivial \( H_2(M) \). Let \( \Gamma_1 \) and \( \Gamma_2 \) be two homologous disjoint simple closed curves in \( \partial M \). Then, the absolutely area minimizing surfaces \( \Sigma_1 \) and \( \Sigma_2 \) in \( M \) with \( \partial \Sigma_i = \Gamma_i \) are disjoint, too.

**Remark 5.1.** In a non-mean convex ambient space \( N \), the interaction of the absolutely area minimizing surface and the boundary of the manifold \( \partial N \) can get very complicated. However, for a strictly mean convex 3-manifold \( M \), the absolutely area minimizing surface in \( M \) must be away from the boundary \( \partial M \) because of the maximum principle. In particular, as it is seen in Figure 9, the absolutely area minimizing surfaces in such a manifold may not be smooth. On the other hand, one can easily construct trivial examples of intersecting absolutely area minimizing surfaces \( \Sigma_1 \) and \( \Sigma_2 \) with disjoint boundaries like in the figure in a non-mean convex 3-manifold \( N \). To avoid these trivial situations, we construct strictly mean convex examples.

![Figure 9](image)

**Figure 9.** In the figure, \( N \) represents a non-mean convex 3-manifold, obtained by removing a large solid cone from a 3-ball. \( \Gamma_1 \) and \( \Gamma_2 \) are two disjoint simple closed curves in \( \partial N \). \( \Sigma_1 \) and \( \Sigma_2 \) are the absolutely area minimizing surfaces in \( N \) with \( \partial \Sigma_i = \Gamma_i \). Even though \( H_2(N) \) is trivial, \( \Gamma_1 \) and \( \Gamma_2 \) are homologous in \( \partial N \), \( \Sigma_1 \cap \Sigma_2 \neq \emptyset \).
5.2. Example III-B: $H_2(M)$ is nontrivial. In this part, we will give another example of intersecting absolutely area minimizing surfaces in $M$ with disjoint boundaries in $\partial M$. In this example, $H_2(M)$ will be nontrivial, and $\partial M \simeq S^2$. So, any two simple closed curves in $\partial M$ will be homotopic, and homologous in $\partial M$. Hence, the examples we are going to construct in this part will be very different from the one in previous part in an essential way.

First we describe the ambient manifold $M$. The ambient manifold $M$ we are going to construct in this part will be very similar to the ambient manifold in Section 3. Let $T^3$ be the 3-torus obtained by identifying the opposite faces of the cube with dimensions $[0, 1] \times [0, 1] \times [0, 1]$. Take the induced flat metric on $T^3$. Let $B^3$ be the cube in the $T^3$ with dimensions $[\delta, 1-\delta] \times [\delta, 1-\delta] \times [\delta, 1-\delta]$. Define the ambient space $M$ as $T^3 - B^3$ where $\delta$ are to be declared later.

Let $\hat{\Sigma}_c$ be the be 2-torus corresponding to the $\{ z = c \}$ square in $T^3$ before the identification (See Figure 10(left)). Since the family of minimal tori $\{ \hat{\Sigma}_c \}$ foliates $T^3$, $\hat{\Sigma}_c$ is an absolutely area minimizing surface for any $c$. Let $\Gamma_c$ be the simple closed curve in $\partial M$ with $\Gamma_c = \partial M \cap \hat{\Sigma}_c$ where $\delta < c < 1 - \delta$. Let $\Sigma_c$ be the surface obtained by intersecting $\hat{\Sigma}_c$ with $M$ (similar to Figure 5). Because of the foliation, $\Sigma_c$ is an absolutely area minimizing surface in $M$ with boundary $\Gamma_c$ unless the competitor disk in $\partial M$ with the same boundary has smaller area. Hence, an easy computation shows that $\Sigma_c$ is the absolutely area minimizing surface in $M$ with $\partial \Sigma_c = \Gamma_c$ if we choose $\delta$ such that $\delta < \frac{\sqrt{2}}{4} \sim 0.14$, i.e. $|D| = (1 - 2\delta)^2 > 1 - (1 - 2\delta)^2 = |\Sigma_c|$ where $D$ is the bottom (or top) disk in $\partial M$.

Now, we will define the second surface. Let $\hat{S}_d$ be the surface in $T^3$ which is the projection of $y + z = d$-plane in $\mathbb{R}^3$ (the universal cover of $T^3$) to $T^3$ via the covering map (See Figure 10(right)). In other words, $\hat{S}_d$ is the 2-torus

Figure 10. The absolutely area minimizing surfaces $\hat{\Sigma}_c$ and $\hat{S}_d$ in $T^3$. 

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corresponding to \( \{ y + z = d \} \)-plane in \( T^3 \) before identification (in this notation, some piece of \( \hat{S}_d \) lives in \( \{ y + z = d \pm 1 \} \)-plane).

Notice that each \( \hat{S}_d \) is a minimal surface in \( T^3 \), as the cover \( \{ y + z = d \} \)-plane is minimal in the universal cover. Since the family of 2-tori \( \{ \hat{S}_d \mid d \in [0, 1) \} \) foliates \( T^3 \), each \( \hat{S}_d \) is an absolutely area minimizing surface in \( T^3 \). Let \( S_d \) be the surface which is the intersection of \( \hat{S}_d \) and \( M \), and \( \alpha_d = \hat{S}_d \cap \partial M \) be the simple closed curve(s) in \( \partial M \) (See Figure 11 (left)). Notice that \( \alpha_d \) is one component for \( 0 \leq d < 2\delta \) and \( 1 - 2\delta < d < 1 \), and \( \alpha_d \) has two components in \( \partial M \) for \( 2\delta < d < 1 - 2\delta \). From now on, we will assume \( 0 < d < 2\delta \). Now, by maximum principle, \( S_d \) would be absolutely area minimizing surface in \( M \) with \( \partial S_d = \alpha_d \) unless the absolutely area minimizing surface in \( M \) with boundary \( \alpha_d \) completely lies in \( \partial M \). Since by our assumption \( 0 \leq d < 2\delta \), a simple computation shows that the absolutely area minimizing surface with boundary \( \alpha_d \) cannot lie in \( \partial M \). Hence, \( S_d \) is an absolutely area minimizing surface in \( M \) for \( 0 \leq d < 2\delta \).

![Figure 11](image-url)

**Figure 11.** In the left, the disjoint simple closed curves \( \alpha_d \) (blue curve) and \( \Gamma_c \) (green curve) in \( \partial M \) are pictured. In the right, the absolutely area minimizing surfaces \( S_d \) and \( \Sigma_c \) (right) are intersecting each other along the simple closed curve \( \beta \) even though they have disjoint boundaries \( \alpha_d \) and \( \Gamma_c \) in \( \partial M \) where \( M \) is a mean convex 3-manifold.

Now, choose \( d = \delta \) and choose \( c = \frac{3\delta}{2} \). Then, \( \Gamma_c \) and \( \alpha_d \) would be disjoint simple closed curves in \( \partial M \) (See Figure 11 (left)). Moreover, the absolutely area minimizing surfaces \( \Sigma_c \) and \( S_d \) intersects in a simple closed curve \( \beta \) (See Figure 11 (right)). Here, \( \beta \) corresponds to the line segment between the points \( (0, 1 - \delta, \frac{3\delta}{2}) \) and \( (1, 1 - \delta, \frac{3\delta}{2}) \) in \( T^3 \) before identification. This shows the existence of absolutely area minimizing surfaces \( \Sigma_c \) and \( S_d \) with nontrivial intersection even though they have disjoint boundaries \( \Gamma_c \) and \( \alpha_d \) in \( \partial M \).
To get a mean convex example $\hat{M}$, one can follow the steps in Section 4 by modifying the metric on $M$ near $\partial M$. Then suitable modifications give corresponding absolutely area minimizing surfaces $\hat{\Sigma}_c$ and $\hat{S}_d$ in $\hat{M}$ with the desired properties. This finishes the second example.

The main difference between the two examples of this section is that in the first example the reason for intersection is not the topological complexity of the manifold, but the topological difference of the boundaries. In the second example, even though the boundary curves are topologically "same", the surfaces are in different homological classes which forces the intersection.

**Remark 5.2.** Another interesting question might be "what if the surfaces have the same boundary?". In other words, if $\Sigma_1$ and $\Sigma_2$ are two absolutely area minimizing surfaces in a mean convex 3-manifold $M$ with $\partial \Sigma_1 = \partial \Sigma_2 = \Gamma \subset \partial M$. Then, must $\Sigma_1$ and $\Sigma_2$ be disjoint or not? The answer to this question is clearly "Yes" when $H_2(M)$ is trivial by the discussion at the beginning of the Section 5.1. However, the answer is "No" when $H_2(M)$ is not trivial.

One can take $\Gamma_c$ and $\Sigma_c$ in the example above. It is possible to construct another absolutely area minimizing surface $T$ with $\partial T = \Gamma_c$ as follows: Let $T$ be an area minimizing surface in the homology class of $S_d$ (the example above) with $\partial T = \Gamma_c$. Of course, $T$ is not an absolutely area minimizing surface in $M$ as it is just area minimizing in its homology class. Also, $T \cap \Sigma_c \neq \emptyset$ by homological reasons. Indeed, the intersection must be in the same homology class with the simple closed curve $\beta$. Now, we can modify the metric on $M$ near $T$ and away from $\Sigma_c$ so that both $\Sigma_c$ and $T$ are absolutely area minimizing surface in $M$ with the new metric. In particular, take sufficiently large disk $D$ in $T$ away from the boundary and the intersection, and change the metric smoothly on a very small neighborhood $N_\varepsilon(D)$ of $D$ so that $\Sigma_c$ and $T$ have the same area. Then, we get two intersecting absolutely area minimizing surfaces with the same boundary.

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