UPPER BOUNDS FOR THE REGULARITY OF POWERS OF EDGE IDEALS OF GRAPHS

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Abstract. Let $G$ be a finite simple graph and $I(G)$ denote the corresponding edge ideal. In this paper, we obtain an upper bound for $\text{reg}(I(G)^q)$ in terms of certain invariants associated with $G$. We also prove a weaker version of a conjecture by Alilooee, Banerjee, Bayerlslan and H`a on an upper bound for the regularity of $I(G)^q$ and we prove the conjectured upper bound for the class of vertex decomposable graphs.

1. Introduction

Let $I$ be a homogeneous ideal of a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$ with usual grading. In [4], Bertram, Ein and Lazarsfeld have initiated the study of the Castelnuovo-Mumford regularity of $I^q$ as a function of $q$ by proving that if $I$ is the defining ideal of a smooth complex projective variety, then $\text{reg}(I^q)$ is bounded by a linear function of $q$. Then, Chandler [8] and Geramita, Gimigliano and Pitteloud [13] proved that if $\dim(R/I) \leq 1$, then $\text{reg}(I^q) \leq q \cdot \text{reg}(I)$ for all $q \geq 1$. However, Swanson [29] proved that there exists $k \geq 1$ such that for all $q \geq 1$, $\text{reg}(I^q) \leq kq$. Thereafter, Cutkosky, Herzog and Trung, [10], and independently Kodiyalam [24], proved that for a homogeneous ideal $I$ in a polynomial ring, $\text{reg}(I^q)$ is a linear function for $q \gg 0$ i.e., there exist non negative integers $a$ and $b$ depending on $I$ such that $\text{reg}(I^q) = aq + b$ for all $q \gg 0$. While the coefficient $a$ is well-understood ([10], [24], [30]), the free constant $b$ and the stabilization index $q_0 = \min\{q' \mid \text{reg}(I^{q'}) = aq + b, \text{ for all } q \geq q'\}$ are quite mysterious. Therefore, the attention has been to identify classes for which the linear polynomial can be computed or bounded using invariants associated to $I$. There have been some attempts on computing the free constant and stabilization index for several class of ideals. For instance, if $I$ is an equigenerated homogeneous ideal, then $b$ is related to the regularity of fibers of certain projection map (see for example, [28]). If $I$ is $(x_1, \ldots, x_n)$-primary, then $q_0$ can be related to partial regularity of the Rees algebra of $I$ (see for example, [3]). In this paper, we study the regularity of powers of edge ideals associated to finite simple graphs.

Let $G$ be a finite simple graph on the vertex set $\{x_1, \ldots, x_n\}$ and $I(G) := (\{x_ix_j \mid \{i,j\} \in E(G)\}) \subset \mathbb{K}[x_1, \ldots, x_n]$ be the edge ideal corresponding to the graph $G$. It is known that $\text{reg}(I(G)^q) = 2q + b$ for some $b$ and $q \geq q_0$. There are very few classes of graphs for which $b$ and $q_0$ are known. We refer the reader to [2] and the references cited there for a review of results in the literature in this direction. While the aim is to obtain the linear polynomial corresponding to $\text{reg}(I(G)^q)$, it seems unlikely that a single combinatorial invariant will represent the constant term for all graphs. This naturally give

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rise to two directions of research. One, to obtain linear polynomials for particular classes of graphs. Two, to obtain upper and lower bounds for \( \text{reg}(I(G)^q) \) using combinatorial invariants associated to the graph \( G \). It was proved by Beyarslan, Hà and Trung that

\[
2q + \nu(G) - 1 \leq \text{reg}(I(G)^q) \quad \text{for all } q \geq 1,
\]

where \( \nu(G) \) denotes the induced matching number of \( G \). In [21], the authors along with Narayanan proved that for a bipartite graph \( G \),

\[
\text{reg}(I(G)^q) \leq 2q + \text{co-chord}(G) - 1
\]

for all \( q \geq 1 \), where \( \text{co-chord}(G) \) denotes the co-chordal cover number of \( G \). There is no general upper bound known for powers of edge ideals of arbitrary graphs. Therefore, one may ask:

Q1. Does there exist a graph invariant of \( G \), say \( \rho(G) \), such that \( \text{reg}(I(G)^q) \leq 2q + \rho(G) \) for all \( q \geq 1 \)?

Q2. Can one obtain the linear polynomial corresponding to \( \text{reg}(I(G)^q) \) for various classes of graphs?

This paper evolves around these two questions.

The first main result of the paper answers Question Q1. Hà and Woodroofe [18] defined an invariant in terms of star packing, denoted by \( \zeta(G) \) (see Section 4 for definition), and proved that \( \text{reg}(I(G)) \leq \zeta(G) + 1 \). In this paper, we extend Hà and Woodroofe’s bound to include all powers of \( I(G) \). We prove:

**Theorem 4.5.** If \( G \) is a graph, then for all \( q \geq 1 \),

\[
\text{reg}(I(G)^q) \leq 2q + \zeta(G) - 1.
\]

So far, in the literature, for the classes of graphs for which the regularity of powers of edge ideals have been computed, they satisfy either \( \text{reg}(I(G)^q) = 2q + \nu(G) - 1 \) or \( \text{reg}(I(G)^q) = 2q + \text{co-chord}(G) - 1 \), for all \( q \geq 2 \). In [21], the authors raised the question whether there exists a graph \( G \) with

\[
2q + \nu(G) - 1 < \text{reg}(I(G)^q) < 2q + \text{co-chord}(G) - 1, \quad \text{for } q \gg 0.
\]

As a consequence of our investigation, we obtain a class of graphs which attain the upper bound in Theorem 4.5 and the above strict inequalities are satisfied.

Another way of bounding the function \( \text{reg}(I(G)^q) \), than using combinatorial invariants, is to relate it to the regularity of \( G \) itself. It was conjectured by Alilooee, Banerjee, Beyarslan and Hà, [2, Conjecture 7.11(2)]:

**Conjecture 1.1.** If \( G \) is a graph, then for all \( q \geq 1 \), \( \text{reg}(I(G)^q) \leq 2q + \text{reg}(I(G)) - 2 \).

There are some classes of graphs for which this conjecture is known to be true, see [2]. As a consequence of the techniques that we have developed, we prove the conjecture with an additional hypothesis:

**Theorem 4.7.** Let \( G \) be a graph. If every induced subgraph \( H \) of \( G \) has a vertex \( x \) with \( \text{reg}(I(H \setminus N_H[x])) + 1 \leq \text{reg}(I(H)) \), then for all \( q \geq 1 \),

\[
\text{reg}(I(G)^q) \leq 2q + \text{reg}(I(G)) - 2.
\]

We then move on to study regularity of powers of vertex decomposable graphs. A graph \( G \) is said to be vertex decomposable if \( \Delta(G) \) is a vertex decomposable, where \( \Delta(G) \) denotes the independence complex of \( G \) (see Section 5 for definition). Vertex decomposability was first introduced by Provan and Billera [27], in the case when all the maximal faces are of...
equal cardinality, and extended to the arbitrary case by Björner and Wachs [7]. We have the chain of implications:

vertex decomposable $\implies$ shellable $\implies$ sequentially Cohen-Macaulay,

where a graph $G$ is shellable if $\Delta(G)$ is a shellable simplicial complex and $G$ is sequentially Cohen-Macaulay if $R/I(G)$ is sequentially Cohen-Macaulay. Both the above implications are known to be strict. Recently, a number of authors have been interested in classifying or identifying vertex decomposable graphs $G$ in terms of the combinatorial properties of $G$, [6, 12, 32]. We prove the Conjecture 1.1 for vertex decomposable graphs.

**Theorem 5.3.** Let $G$ be a vertex decomposable graph. Then for all $q \geq 1$,

$$\text{reg}(I(G)^q) \leq 2q + \text{reg}(I(G)) - 2.$$  

As a consequence, we obtain the linear polynomial corresponding to $\text{reg}(I(G)^q)$ for several classes of graphs such as $C_5$-free vertex decomposable, chordal, sequentially Cohen-Macaulay bipartite graphs and certain whiskered graphs.

Our paper is organized as follows. In the Section 2, we collect the terminology and preliminary results that are essential for the rest of the paper. We prove, in Section 3 several technical lemmas which are needed for the proof of our main results which appear in Sections 4 and 5.

2. Preliminaries

Throughout this article, $G$ denotes a finite simple graph without isolated vertices. For a graph $G$, $V(G)$ and $E(G)$ denote the set of all vertices and the set of all edges of $G$ respectively. The *degree* of a vertex $x \in V(G)$, denoted by $\deg_G(x)$, is the number of edges incident to $x$. A subgraph $H \subseteq G$ is called *induced* if for $u, v \in V(H)$, $\{u, v\} \in E(H)$ if and only if $\{u, v\} \in E(G)$. For $\{u_1, \ldots, u_r\} \subseteq V(G)$, let $N_G(u_1, \ldots, u_r) = \{v \in V(G) \mid \{u_i, v\} \in E(G) \text{ for some } 1 \leq i \leq r\}$ and $N_G[u_1, \ldots, u_r] = N_G(u_1, \ldots, u_r) \cup \{u_1, \ldots, u_r\}$. For $U \subseteq V(G)$, denote by $G \setminus U$ the induced subgraph of $G$ on the vertex set $V(G) \setminus U$.

A subset $X$ of $V(G)$ is called *independent* if there is no edge $\{x, y\} \in E(G)$ for $x, y \in X$. A *matching* in a graph $G$ is a subgraph consisting of pairwise disjoint edges. The largest size of a matching in $G$ is called its matching number. If the subgraph is an induced subgraph, the matching is an *induced matching*. The largest size of an induced matching in $G$ is called its induced matching number and denoted by $\nu(G)$.

The *complement* of a graph $G$, denoted by $G^c$, is the graph on the same vertex set in which $\{u, v\}$ is an edge of $G^c$ if and only if it is not an edge of $G$. A graph $G$ is *chordal* if every induced cycle in $G$ has length 3, and is co-chordal if the complement graph $G^c$ is chordal. The co-chordal cover number, denoted co-chord($G$), is the minimum number $n$ such that there exist co-chordal subgraphs $H_1, \ldots, H_n$ of $G$ with $E(G) = \bigcup_{i=1}^{n} E(H_i)$.

One important tool in the study of regularity of powers of edge ideals is even-connections. We recall the concept of *even-connectedness* from [1].

**Definition 2.1.** Let $G$ be a graph. Two vertices $u$ and $v$ ($u$ may be same as $v$) are said to be even-connected with respect to an $s$-fold products $e_1 \cdots e_s$, where $e_i$’s are edges of $G$, not necessarily distinct, if there is a path $p_0p_1 \cdots p_{2k+1}$, $k \geq 1$ in $G$ such that:

1. $p_0 = u, p_{2k+1} = v$. 

(2) For all \(0 \leq l \leq k - 1\), \(p_{2l+1}p_{2l+2} = e_i\) for some \(i\).
(3) For all \(i\), \(|\{l \geq 0 \mid p_{2l+1}p_{2l+2} = e_i\}| \leq |\{j \mid e_j = e_i\}|\).
(4) For all \(0 \leq r \leq 2k\), \(p_r \cdot p_{r+1}\) is an edge in \(G\).

**Remark 2.2.** For convenience, we set an edge to be trivially even-connected, i.e., we get \(k\) \([1, \text{Theorem 6.1 and Theorem 6.7}]\) as a minimal generator of \(M\) \([19, \text{Corollary 1.6.3(a)}]\).

**Theorem 2.3.** \([2]\) Let \(G\) be a graph with edge ideal \(I = \mathcal{I}(G)\), and let \(s \geq 1\) be an integer. Let \(M\) be a minimal generator of \(I^s\). Then \((I^{s+1} : M)\) is minimally generated by monomials of degree 2, and \(uv\) (and \(v\) may be the same) is a minimal generator of \((I^{s+1} : M)\) if and only if either \(u, v \in E(G)\) or \(u\) and \(v\) are even-connected with respect to \(M\).

Polarization is a process that creates a squarefree monomial ideal (in a possibly different polynomial ring) from a given monomial ideal, \([19, \text{Section 1.6}]\). In this paper, we repeatedly use one of the important properties of the polarization, namely:

**Corollary 2.4.** \([19, \text{Corollary 1.6.3(a)}]\) Let \(I\) be a monomial ideal in \(\mathbb{K}[x_1, \ldots, x_n]\). Then \(\text{reg}(I) = \text{reg}(\mathcal{I})\).

### 3. Technical Lemmas

In this section, we prove several technical results concerning the graph associated with \((\mathcal{I}(G)^{s+1} : e_1 \cdots e_s)\) and some of its induced subgraphs. We begin by fixing the notation for the most of our results.

**Notation 3.1.** Let \(G\) be a graph with \(V(G) = \{x_1, \ldots, x_n\}\) and \(e_1, \ldots, e_s, s \geq 1\), be some edges of \(G\) which are not necessarily distinct. By Theorem 2.3 and Corollary 2.4, \((\mathcal{I}(G)^{s+1} : e_1 \cdots e_s)\) is a quadratic squarefree monomial ideal in an appropriate polynomial ring. Let \(G'\) be the graph associated to \((\mathcal{I}(G)^{s+1} : e_1 \cdots e_s)\).

One of the key ingredients in the proof of the main results is a new graph, \(G'\), obtained from a given graph \(G\) by joining even-connected vertices by an edge. Our main aim in this section is to get an upper bound for regularity of certain induced subgraphs of \(G'\) which in turn will help us in bounding \(\text{reg}(\mathcal{I}(G'))\). For this purpose, we need to understand the structure of the graph \(G'\) in more detail. First we show that whiskers can be ignored while taking even-connections.

**Lemma 3.2.** Let \(G\) be a graph with \(e_1, \ldots, e_s \in E(G)\), \(s \geq 1\) and \(N_G(x) = \{y\}\). If \(e_i = \{x, y\}\), for some \(1 \leq i \leq s\), then \((\mathcal{I}(G)^{s+1} : e_1 \cdots e_s) = (\mathcal{I}(G)^s : \prod_{j \neq i} e_j)\).

**Proof.** Clearly \((\mathcal{I}(G)^s : \prod_{j \neq i} e_j) \subseteq (\mathcal{I}(G)^{s+1} : e_1 \cdots e_s)\). Let \(uv \in (\mathcal{I}(G)^{s+1} : e_1 \cdots e_s)\). For \(k \geq 0\), let \((u = p_0) \cdots (p_{2k+1} = v)\) be an even-connection in \(G\) with respect to \(e_1 \cdots e_s\). For \(0 \leq r \leq k - 1\), set \(e_{r+1} = \{p_{2r+1}, p_{2r+2}\}\). If \(k = 0\), then \(uv \in (\mathcal{I}(G)^s : \prod_{j \neq i} e_j)\). Assume \(k \geq 1\). If \(e_i \neq e_{r+1}\), for all \(0 \leq r \leq k - 1\), then \(u\) is an even-connected to \(v\) in \(G\) with respect
Lemma 3.4. Let the notation be as in [3.1]. Suppose \( e_i = e_{i+1} \), for some \( 0 \leq r \leq k - 1 \). Since \( N_G(x) = \{y\} \), \( p_{2r+1} = p_{2r+3} \). Then \( (u = p_0)p_1 \cdots (p_{2r+1} = p_{2r+3})p_{2r+4} \cdots (p_{2k+1} = v) \) gives an even-connection in \( G \) with respect to \( e_1 \cdots e_i \cdots e_{i+1} \cdots e_s \). Hence \( uv \in (I(G)^s : \prod_{j \neq i} e_j) \). □

The following result shows that if a vertex has no intersection with a set of edges, then removing such a vertex and taking even-connection with respect to the set of those edges commute with each other.

Lemma 3.3. Let the notation be as in [3.1]. If for \( x \in V(G) \), \( \{x\} \cap e_i = \emptyset \), for all \( 1 \leq i \leq s \), then

\[
I(G' \setminus x) = (I(G \setminus x)^{s+1} : e_1 \cdots e_s).
\]

Proof. Clearly \((I(G \setminus x)^{s+1} : e_1 \cdots e_s) \subseteq I(G' \setminus x)\). Suppose \( u, v \in E(G' \setminus x) \). Let \((u = p_0)p_1 \cdots p_{2k}(p_{2k+1} = v)\) be an even-connection in \( G \) with respect to \( e_1 \cdots e_s \). Since \( e_i \cap \{x\} = \emptyset \), for all \( 1 \leq i \leq s \), \( u \) is even-connected to \( v \) in \( G \setminus x \) with respect to \( e_1 \cdots e_s \). □

The next result talks about new even-connections made out of a given even-connection and the neighbors of some of the vertices in the even-connection.

Lemma 3.4. Let the notation be as in [3.1]. Suppose \((u = p_0)p_1 \cdots p_{2k}(p_{2k+1} = v)\) is an even-connection in \( G \) with respect to \( e_1 \cdots e_s \), for some \( k \geq 1 \). If \( \{w, p_i\} \in E(G') \), for some \( 0 \leq i \leq 2k + 1 \), then either \( \{u, w\} \in E(G') \) or \( \{v, w\} \in E(G') \).

Proof. If \( i = 0, 2k + 1 \), then we are done. Assume that \( i = 2j + 1 \), for some \( j \geq 0 \). Let \((w = q_0)q_1 \cdots (q_{2j+1} = p_i)\) be an even-connection with respect to \( e_1 \cdots e_s \) in \( G \). If \( \{q_{2\alpha+1}, q_{2\alpha+2}\} \) and \( \{p_{2\beta+1}, p_{2\beta+2}\} \) do not have a common vertex, for all \( 0 \leq \alpha \leq j - 1 \), \( j \leq \beta \leq k - 1 \), then \((w = q_0)q_1 \cdots (q_{2j+1} = p_i)p_i+1 \cdots (p_{2k+1} = v)\) is an even-connection with respect to \( e_1 \cdots e_s \) in \( G \). Therefore, \( uv \in I(G') \). If \( \{q_{2\alpha+1}, q_{2\alpha+2}\} \) and \( \{p_{2\beta+1}, p_{2\beta+2}\} \) have a common vertex, for some \( 0 \leq \alpha \leq j - 1 \), \( j \leq \beta \leq k - 1 \), then by [1, Lemma 6.13], \( w \) is even-connected either to \( u \) or to \( v \) in \( G \). Therefore either \( wu \in I(G') \) or \( \{u, v\} \in E(G') \).

If \( i = 2j + 2 \), then proof is similar. □

The following lemma, which throws more light into the structure of \( G' \), is very useful for the induction process.

Lemma 3.5. Let the notation be as in [3.1]. Let \( y \in V(G) \) and \( H = G \setminus N_G[y] \). If \( \{e_1, \ldots, e_s\} \cap E(H) = \{e_i, \ldots, e_i\} \) and \( H' \) is the graph associated to \((I(H)^{t+1} : e_i \cdots e_i) \), then \( G' \setminus N_{G'}[y] \) is an induced subgraph of \( H' \). In particular,

\[
\text{reg}(I(G' \setminus N_{G'}[y])) \leq \text{reg}(I(H')).
\]

Proof. Let \( \{u, v\} \in E(G' \setminus N_{G'}[y]) \). Let \((u = p_0)p_1 \cdots (p_{2k+1} = v)\) be an even-connection in \( G \) with respect to \( e_1 \cdots e_s \), for some \( k \geq 0 \). If \( p_i \in N_{G'}[y] \), for some \( 0 \leq i \leq 2k + 1 \), then by Lemma 3.4, \( y \) is even-connected either to \( u \) or to \( v \). This contradicts the assumption that \( \{u, v\} \in G' \setminus N_{G'}[y] \). Therefore, for each \( 0 \leq i \leq 2k + 1 \), \( p_i \notin N_{G'}[y] \). Hence \( \{u, v\} \in E(H') \), which proves that \( G' \setminus N_{G'}[y] \) is a subgraph of \( H' \). If \( a, b \in V(G' \setminus N_{G'}[y]) \) is such that \( \{a, b\} \in E(H) \), then \( \{a, b\} \in E(G' \setminus N_{G'}[y]) \). Hence \( G' \setminus N_{G'}[y] \) is an induced subgraph of \( H \). The assertion on the regularity follows from [20, Proposition 4.1.1]. □
In the following results, we show that the even-connections in a parent graph with
respect to edges coming from an induced subgraph, induces an even-connection in the
induced subgraph.

**Lemma 3.6.** Let $G$ be a graph and $H$ be an induced subgraph of $G$. If for any $e_1, \ldots, e_s \in E(H)$, $s \geq 1$, then $H'$ is an induced subgraph of $G'$, where $H'$ and $G'$ are the graph
associated to $(I(H)^{s+1}: e_1 \cdots e_s)$ and $(I(G)^{s+1}: e_1 \cdots e_s)$ respectively. In particular,

$$\text{reg}(I(H')) \leq \text{reg}(I(G')).$$

**Proof.** Let $\{a, b\} \in E(H')$. For some $k \geq 0$, let $(a = p_0)p_1 \cdots p_{2k}(p_{2k+1} = b)$ be an
even-connection in $G$ with respect to $e_1 \cdots e_s$. Since $H$ is an induced subgraph of $G$ and
$\{p_{2r+1}, p_{2r+2}\} \in E(H)$, for all $0 \leq r \leq k - 1$, $(a = p_0)p_1 \cdots p_{2k}(p_{2k+1} = b)$ is an even-
connection in $G$ with respect to $e_1 \cdots e_s$. Therefore, $\{a, b\} \in E(G')$. Hence $H'$ is the
subgraph of $G'$. As in the previous lemma, it can be seen that the subgraph is an induced
subgraph. \qed

Let the notation be as in 3.1. For some $1 \leq i \leq s$, set $e_i = \{x, y\}$. We further explore
the even-connections between $N_{G'}[y]$ and $N_{G'}(x)$. If $(u = p_0)p_1 \cdots p_{2k}(p_{2k+1} = y)$ ($u$ may
be equal to $y$) be an even-connection in $G$ with respect to $e_1 \cdots e_s$, for some $k \geq 0$, then
there are three possibilities:

1. $\{p_{2\lambda+1}, p_{2\lambda+2}\} \neq e_i$ for any $0 \leq \lambda \leq k - 1$;
2. There exists $0 \leq \lambda \leq k - 1$ with $\{p_{2\lambda+1}, p_{2\lambda+2}\} = e_i$ and $p_{2\lambda+1} = y, p_{2\lambda+2} = x$;
3. For some $0 \leq \lambda \leq k - 1$, $p_{2\lambda+1} = x$ and $p_{2\lambda+2} = y$ whenever $\{p_{2\lambda+1}, p_{2\lambda+2}\} = e_i$;

Note that, fixing an even-connection between $u$ and $y$, (1), (2) and (3) are mutually
exclusive. Let

$$X_1 = \left\{ u \in N_{G'}[y] | (u = p_0)p_1 \cdots p_{2k}(p_{2k+1} = y) \text{ satisfies either (1) or (2)} \right\};$$

$$X_2 = \left\{ u \in (N_{G'}[y]) \setminus X_1 | (u = p_0)p_1 \cdots p_{2k}(p_{2k+1} = y) \text{ satisfies (3)} \right\}. \tag{3.1}$$

**Lemma 3.7.** Following the notation set above, let $E(G \setminus N_{G}[u, x]) \cap \{e_1, \ldots, e_s\} = \{e_{j_1}, \ldots, e_{j_t}\}$ and $(G' \setminus N_{G'}[u, x])'$ denote the graph associated to $(I(G' \setminus N_{G'}[u, x])^{t+1}: e_{j_1} \cdots e_{j_t})$.

1. If $u \in X_1$, then $G' \setminus N_{G'}[u]$ is an induced subgraph of $(G \setminus N_{G}[u, x])'$. In particular,

$$\text{reg}(I(G' \setminus N_{G'}[u])) \leq \text{reg}(I((G \setminus N_{G}[u, x])')).$$

2. Set $G'_1 = G' \setminus X_1$. If $u \in X_2$, then $G'_1 \setminus N_{G'_1}[u]$ is an induced subgraph of $(G \setminus N_{G}[u, x])'$.

$$\text{reg}(I(G'_1 \setminus N_{G'_1}[u])) \leq \text{reg}(I((G \setminus N_{G}[u, x])')).$$

**Proof.** (1) Set $H = G' \setminus N_{G'}[u]$ and $K = (G \setminus N_{G}[u, x])'$. Let $\{a, b\} \in E(H)$ and $(a = q_0)q_1 \cdots (q_{2l+1} = b)$ be an even-connection in $G$ with respect to $e_1 \cdots e_s$, for some $l \geq 0$.

We show that $\{a, b\} \in E(K)$. Note that by Lemma 3.1, $q_i \notin N_{G'}[u]$, for all $0 \leq i \leq 2l+1$. Suppose $q_i \in N_{G}[x]$, for some $0 \leq i \leq 2k+1$. Since $u \in X_1$, then $\{u, q_i\} \in E(G')$. By Lemma 3.1 $u$ is an even-connected either to $a$ or to $b$ in $G$ with respect to $e_1 \cdots e_s$. This is a contradiction to $\{a, b\} \in E(H)$. Therefore, $q_i \notin N_{G}[x]$, for all $0 \leq i \leq 2l+1$. Hence $a$ is even-connected to $b$ in $G \setminus N_{G}[u, x]$ with respect to $e_{j_1} \cdots e_{j_t}$. Therefore, $\{a, b\} \in E(K)$.
(2) Set \( H = G_1' \setminus N_{G_1'}[u] \) and \( K = (G \setminus N_G[u,x])' \). Let \( \{a,b\} \in E(H) \) and \( (a = q_0)q_1 \cdots (q_{2l+1} = b) \) be an even-connection in \( G \) with respect to \( e_1 \cdots e_s \), for some \( l \geq 0 \). If \( \{p_{2a+1},p_{2a+2}\} \) and \( \{q_{2\beta+1},q_{2\beta+2}\} \) have a common vertex, for some \( 0 \leq \alpha \leq k-1, \ 0 \leq \beta \leq l-1 \), then by \([1, \text{Lemma 6.13}]\), \( u \) is even-connected either to \( a \) or to \( b \) in \( G \) with respect to \( e_1 \cdots e_s \). This is a contradiction to our assumption that \( \{a,b\} \in E(H) \). Therefore, for all \( 0 \leq \alpha \leq k-1 \) and \( 0 \leq \beta \leq l-1 \), \( \{p_{2a+1},p_{2a+2}\} \) and \( \{q_{2\beta+1},q_{2\beta+2}\} \) do not have a common vertex. Note that by \( \text{Lemma 3.3} \) \( q_i \notin N_G[u], \) for all \( 0 \leq i \leq 2l+1 \). Suppose \( q_i \in N_G[x], \) for some \( 0 \leq i \leq 2l+1 \). Since \( u \in X_2 \), there exists \( 0 \leq \lambda \leq k-1 \) with \( p_{2\lambda+1} = x \) and \( p_{2\lambda+2} = y \). Therefore, \( (p_{2k+1} = y)p_{2k} \cdots p_{2\lambda+1}p_{2\lambda+2}q_k \) is an even-connection in \( G \) with respect to \( e_1 \cdots e_s \). By \( \text{Lemma 3.3} \) \( y \) is even-connected either to \( a \) or to \( b \) in \( G \) with respect to \( e_1 \cdots e_s \). Since for all \( 0 \leq \alpha \leq k-1 \) and \( 0 \leq \beta \leq l-1 \), \( \{p_{2a+1},p_{2a+2}\} \) and \( \{q_{2\beta+1},q_{2\beta+2}\} \) do not have a common vertex, either \( a \in X_1 \) or \( b \in X_1 \). This is a contradiction to our assumption that \( \{a,b\} \in E(H) \). Hence \( q_i \notin N_G[x], \) for all \( 0 \leq i \leq 2l+1 \) so that \( \{a,b\} \) is an even-connection in \( G \setminus N_G[u,x] \) with respect to \( e_j, \cdots e_j \). Therefore \( \{a,b\} \in E(K) \).

As in the previous lemma, it can be seen that the subgraphs considered in (1) and (2) are induced subgraphs. The assertion on the regularity in (1) and (2) follows from \([20, \text{Proposition 4.1.1}]\). \( \square \)

4. Regularity powers of graphs

In this section, we obtain a general upper bound for the regularity powers of edge ideals of graphs. We first recall the definition of the invariant \( \zeta(G) \), introduced in \([18]\).

Let \( v_1 \in V(G) \) be a vertex of degree at least 2. If \( G \setminus N_G[v_1] \) does not have any vertex of degree at least 2, then set \( \sigma = \{v_1\} \) and \( H_1 \) be the graph obtained by removing all the isolated vertices of \( G \setminus N_G[v_1] \). If there exists \( v_2 \in V(H) \) such that \( \deg_{H_1}(v_2) \geq 2 \), then set \( \sigma = \{v_1,v_2\} \) and \( H_2 \) be the graph obtained by removing the isolated vertices of \( H_1 \setminus N_{H_1}[v_2] \). Continuing like this, we obtain \( \sigma = \{v_1, \ldots, v_k\} \) such that \( H_k \setminus N_{H_k}[v_k] \) is a collection of disconnected edges, say \( \{w_1, \ldots, w_r\} \) and isolated vertices. Let \( \mathcal{P} = \{N_G[v_1], \ldots, N_G[v_k]\} \cup \{w_1, \ldots, w_r\} \). We call this \( \mathcal{P} \) a star packing of \( G \) and set \( \zeta_P(G) = k + r \). Define

\[
\zeta(G) := \max \left\{ \zeta_P(G) \mid \mathcal{P} \text{ is a star packing of } G \right\}.
\]

For example, if \( G = C_n \), cycle on \( n \) vertices, then for any \( x \in V(G) \), \( N_G[x] \) is a path on \( 3 \) vertices and hence \( G \setminus N_G[x] \) is a path on \( n - 3 \) vertices. A maximal star packing can be obtained by successively taking out \( N_G[x] \), where \( x \) is the neighbour of a degree 1 vertex. Therefore, if \( n \equiv \{0,1\}(\text{mod } 3) \), then \( \zeta(G) = \left\lceil \frac{n}{2} \right\rceil = \nu(G) \) and if \( n \equiv 2(\text{mod } 3) \), then \( \zeta(G) = \left\lceil \frac{n}{2} \right\rceil + 1 = \nu(G) + 1 \).

It may be noted that for a graph \( G \), \( \nu(G) \leq \zeta(G) \), \([18]\). Hà and Woodroofe proved:

**Theorem 4.1.** \([18, \text{Theorem 1.6}]\) Let \( G \) be a graph. Then

\[
\text{reg}(I(G)) \leq \zeta(G) + 1.
\]

It is easy to see that \( \zeta(G) \) is at most the matching number of \( G \). There are two general upper bounds known for the class of edge ideals, namely, \( \text{reg}(I(G)) \leq \text{co-chord}(G) + 1, \) \([33, \text{Theorem 1}]\) and \( \text{reg}(I(G)) \leq \text{min-max}(G) + 1, \) \([17, 33]\), where \( \text{min-max}(G) \) denotes the minimum number of a maximal matching in \( G \). We would like to note here that the
invariants $\zeta(G)$, co-chord($G$) and min-max($G$) are not comparable in general, as can be seen from the following examples.

**Example 4.2.** Let $G = C_7$. Then one can see that the co-chordal subgraphs of $G$ are paths with at most 3 edges so that co-chord($G$) = 3. Also, one seen that $\zeta(G) = 2$. Let $H$ be obtained from $C_4$ by attaching a pendant to any of the vertices. It is easy to see that co-chord($H$) = 1 and $\zeta(H) = 2$.

Let $G$ be the graph obtained by adding a pendant vertex each to two vertices having a common neighbor in $C_6$. Then min-max($G$) = 2 and $\zeta(G) = 3$. If $H = C_4$, then it can be seen that min-max($G$) = 2 while $\zeta(G) = 1$.

We now make an observation about the behaviour of the invariant which is crucial in our inductive arguments.

**Observation 4.3.** Let $x$ be a vertex of $G$ of degree at least 2, then $\zeta(G \setminus N_G[x]) + 1 \leq \zeta(G)$.

**Proof.** Let $H = G \setminus N_G[x]$. Let $\mathcal{P}'$ be a star packing of $H$ such that $\zeta_{\mathcal{P}'}(H) = \zeta(H)$. Then $\mathcal{P} = \mathcal{P}' \cup \{N_G[x]\}$ is a star packing of $G$. Thus, $\zeta(H) + 1 = \zeta_{\mathcal{P}'}(H) + 1 = \zeta_{\mathcal{P}}(G) \leq \zeta(G)$. □

We first prove that for a given graph $G$ and edges $e_1, \ldots, e_s$, the regularity of $G'$ is bounded above by one more than $\zeta(G)$.

**Theorem 4.4.** If $G$ is a graph, then for any $e_1, \ldots, e_s \in E(G)$, $s \geq 1$

\[
\text{reg}(I(G)^{s+1} : e_1 \cdots e_s) \leq \zeta(G) + 1.
\]

**Proof.** Let $G'$ be the graph associated to $(I(G)^{s+1} : e_1 \cdots e_s)$ contained in an appropriate polynomial ring $R_1$. We prove the assertion by induction on $s$. Suppose $s = 1$. Set $e_1 = \{x, y\}$. If either $\deg_G(x) = 1$ or $\deg_G(y) = 1$, then by Lemma 3.2 $I(G') = I(G)$. Therefore by Theorem 4.3 $\text{reg}(I(G')) = \text{reg}(I(G)) \leq \zeta(G) + 1$.

Suppose $\deg_G(x) > 1$ and $\deg_G(y) > 1$. Setting $U = N_{G'}(y) = \{y_1, \ldots, y_r\}$ and $J = I(G')$, consider the exact sequences:

\[
0 \rightarrow \frac{R_1}{(J : y_1)}(-1) \xrightarrow{y_1} \frac{R_1}{J} \rightarrow \frac{R_1}{(J, y_1)} \rightarrow 0;
\]

\[
\vdots \rightarrow \vdots \rightarrow \frac{R_1}{(J, y_1, \ldots, y_{r-1})}(-1) \xrightarrow{y_r} \frac{R_1}{(J, y_1, \ldots, y_{r-1})} \rightarrow \frac{R_1}{(J, U)} \rightarrow 0.
\]

It follows from these exact sequences that

\[
\text{reg}(R_1/J) \leq \max \left\{ \text{reg} \left( \frac{R_1}{(J : y_1)} \right) + 1, \ldots, \text{reg} \left( \frac{R_1}{(J, y_1, \ldots, y_{r-1})} \right) + 1, \text{reg} \left( \frac{R_1}{(J, U)} \right) \right\}.
\]

We now prove that each of the regularities appearing on the right hand side of the above inequality is bounded above by $\zeta(G)$.

Since $(J, U)$ corresponds to an induced subgraph of $G$, by [20, Proposition 4.1.1] and Theorem 4.3,

\[
\text{reg} \left( \frac{R_1}{(J, U)} \right) \leq \text{reg} \left( \frac{R}{I(G)} \right) \leq \zeta(G).
\]
Let the notation be as in Equation \((3.1)\). Since \(N_{G'}(y) = N_G(y),\ U \subseteq X_1\). Note that \(x \in N_G(y)\) and \(e_1 \cap E(G \setminus N_G[y_j, x]) = \emptyset,\) for any \(y_j \in U\). Therefore, we have

\[
\text{reg}((J : y_j)) = \text{reg}(I(G' \setminus N_{G'}[y_j])) \leq \text{reg}(I(G \setminus N_G[y_j, x])) \leq \text{reg}(I(G \setminus N_G[x])) \leq \zeta(G \setminus N_G[x]) + 1 \leq \zeta(G). \quad \text{(by Lemma 3.4)}
\]

Since \(((J, y_1, \ldots, y_{j-1}) : y_j)\) corresponds to an induced subgraph of \((J : y_j)\), it follows that

\[
\text{reg} \left( \frac{R_1}{((J, y_1, \ldots, y_{j-1}) : y_j)} \right) + 1 \leq \text{reg} \left( \frac{R_1}{(J : y_j)} \right) + 1 \leq \zeta(G).
\]

Therefore, \(\text{reg} \left( \frac{R_1}{J} \right) \leq \zeta(G)\).

Suppose \(s > 1\). Assume by induction that for any graph \(G\) and for any \(e_1, \ldots, e_{s-1} \in E(G),\) \(\text{reg}(I(G)^s : e_1 \cdots e_{s-1}) \leq \zeta(G) + 1\).

Let \(G\) be a graph, \(e_1, \ldots, e_s \in E(G),\ s \geq 2\). For \(1 \leq i \leq s,\) let \(e_i = \{a_i, b_i\}\). If for some \(1 \leq j \leq s,\) either \(\text{deg}_{G_e}(a_j) = 1\) or \(\text{deg}_{G_e}(b_j) = 1,\) then by Lemma 3.2

\[
\text{reg}((I(G)^{s+1} : e_1 \cdots e_s)) = \text{reg}((I(G)^s : e_1 \cdots e_{j-1}e_{j+1} \cdots e_s)) \leq \zeta(G) + 1,
\]

where the last inequality follows by induction hypothesis on \(s\).

Suppose \(\text{deg}_{G_e}(a_i) > 1\) and \(\text{deg}_{G_e}(b_i) > 1,\) for all \(1 \leq i \leq s.\) Let \(N_{G'}(b_s) = \{y_1, \ldots, y_p, z_1, \ldots, z_q\}.\)

Following the notation in Equation \((3.1)\), set \(X_1 = \{y_1, \ldots, y_p\},\ X_2 = \{z_1, \ldots, z_q\}\) and \(J = I(G').\) It follows from set of short exact sequences, similar to Equation \((4.1)\), that

\[
\text{reg}(R_1 / J) \leq \max \left\{ \text{reg} \left( \frac{R_1}{(J:y_1)} \right) + 1, \ldots, \text{reg} \left( \frac{R_1}{(J:z_t)} \right) + 1, \ldots, \text{reg} \left( \frac{R_1}{((J,X_1):z_1)} \right) + 1, \ldots, \right\}.
\]

Now,

\[
\text{reg}((J, X_1, X_2)) = \text{reg}(I(G' \setminus N_{G'}[b_s])) \quad \text{(by [3] Remark 2.5)}
\]

\[
\leq \text{reg}(I((G \setminus N_G[b_s])) \quad \text{(by Lemma 3.3)}
\]

where \((G \setminus N_G[b_s]) \cap \{e_1, \ldots, e_s\} = \{e_j, \ldots, e_j\}\) and \((G \setminus N_G[b_s])'\) is the graph associated to \((I(G \setminus N_G[b_s]))^{t+1} : e_j \cdots e_j)\). Since \(a_s \in N_G(b_s), e_s \notin E(G \setminus N_G[b_s])\) so that \(t < s.\) Hence by induction hypothesis on \(s\) and Observation 4.3 we get

\[
\text{reg}(I(G' \setminus N_{G'}[b_s])) \leq \text{reg}(I((G \setminus N_G[b_s])) \leq \zeta(G \setminus N_G[b_s]) + 1 \leq \zeta(G).
\]

Let \(E(G \setminus N_G[y_i, a_s]) \cap \{e_1, \ldots, e_s\} = \{e_j, \ldots, e_j\}\) and \((G \setminus N_G[y_i, a_s])'\) be the graph associated to \((I(G \setminus N_G[y_i, a_s]))^{t+1} : e_j \cdots e_j).\) For any \(y_j \in X_1,\) we have

\[
\text{reg}((J : y_i)) = \text{reg}(I(G' \setminus N_{G'}[y_i])) \leq \text{reg}(I((G \setminus N_G[y_i, a_s]))') \leq \zeta(G \setminus N_G[y_i, a_s]) + 1 \leq \zeta(G).\quad \text{(by Lemma 3.4)}
\]

Since \(((J, y_1, \ldots, y_{i-1}) : y_i)\) corresponds to an induced subgraph of \((J : y_i)\), it follows that

\[
\text{reg}(((J, y_1, \ldots, y_{i-1}) : y_i)) \leq \text{reg}(J : y_i) \leq \zeta(G).
\]

Let \(G'_1 = G' \setminus X_1.\) For any \(z_i \in X_2,\) we may conclude, as done earlier, that

\[
\text{reg}((J, X_1 : z_i)) \leq \text{reg}(I((G \setminus N_G[z_i, a_s]))') \leq \zeta(G \setminus N_G[z_i, a_s]) + 1 \leq \zeta(G).
\]
Since \(((J, X_1, z_1, \ldots, z_{i-1}) : z_i)\) corresponds to an induced subgraph of \((J, X_1 : z_i)\), it follows that

\[
\operatorname{reg}((J, X_1, z_1, \ldots, z_{i-1}) : z_i) \leq \operatorname{reg}(J, X_1 : z_i) \leq \zeta(G).
\]

Therefore \(\operatorname{reg}(J) \leq \zeta(G) + 1\). Hence, for any \(e_1, \ldots, e_s \in E(G), s \geq 1\),

\[
\operatorname{reg}((I(G)^s + 1 : e_1 \cdots e_s)) \leq \zeta(G) + 1.
\]

\(\square\)

Now we prove an upper bound for the regularity of powers of edge ideals of graphs.

**Theorem 4.5.** If \(G\) is a graph, then for all \(q \geq 1\),

\[
\operatorname{reg}(I(G)^q) \leq 2q + \zeta(G) - 1
\]

**Proof.** We prove by induction on \(q\). If \(q = 1\), then the assertion follows from Theorem 4.1. Assume that \(q > 1\). By applying [20, Theorem 7.6.28] and using induction, it is enough to prove that for edges \(e_1, \ldots, e_q\) of \(G, \operatorname{reg}(I(G)^{q+1} : e_1 \cdots e_q) \leq \zeta(G) + 1\) for all \(q > 1\). This follows from Theorem 4.3. \(\square\)

If \(G_1\) and \(G_2\) are graphs for which the linearity of \(\operatorname{reg}(I(G_1)^s)\) and \(\operatorname{reg}(I(G_2)^s)\) are known for \(s \geq s_1\) and \(s \geq s_2\) respectively, then by [26, Theorem 5.7], it is known that \(\operatorname{reg}(I(G_1 \coprod G_2)^s)\) is linear for \(s \geq s_1 + s_2\). Using this result, we obtain a class of graphs for which the upper bound in Theorem 4.5 is attained.

**Proposition 4.6.** For \(p \geq 0, r > p,\) let \(H = (\coprod_{i=1}^p C_{n_i}) \coprod (\coprod_{j=p+1}^r C_{n_j})\), where \(n_1, \ldots, n_p \equiv 2(\text{mod } 3), n_{p+1}, \ldots, n_r \equiv \{0, 1\}(\text{mod } 3)\). Then for all \(q \geq 1\),

\[
\operatorname{reg}(I(H)^q) = 2q + \zeta(H) - 1.
\]

**Proof.** It follows from [20, Theorem 7.6.28] and [5, Theorem 5.2] that

\[
\operatorname{reg}(I(C_n)) = \begin{cases} 
\nu(C_n) + 1 & \text{if } n \equiv \{0, 1\}(\text{mod } 3), \\
\nu(C_n) + 2 & \text{if } n \equiv 2(\text{mod } 3),
\end{cases}
\]

and that for all \(q \geq 2\), \(\operatorname{reg}(I(C_n)^q) = 2q + \nu(C_n) - 1\). If \(q = 1\), then by [33, Lemma 8], we get \(\operatorname{reg}(I(H)) = \zeta(H) + 1\). If \(p = 0\), then \(H = (\coprod_{j=p+1}^r C_{n_j})\). By [26, Theorem 5.7], \(\operatorname{reg}(I(H)^q) = 2q + \zeta(H) - 1\), for all \(q \geq 1\).

Suppose \(p > 0\). Set \(H' = (\coprod_{j=p+1}^r C_{n_j})\). Let \(L_1 = H' \coprod C_{n_1}, \) where \(n_1 \equiv 2(\text{mod } 3)\). Then by [16, Proposition 2.7], \(\operatorname{reg}(I(L_1)^2) = \zeta(L_1) + 3\). By [26, Theorem 5.7], for \(q \geq 3\), we have \(\operatorname{reg}(I(L_1)^q) = 2q + \zeta(L_1) - 1\). For \(i \geq 2\), let \(L_i = L_{i-1} \coprod C_{n_i}, \) where \(n_i \equiv 2(\text{mod } 3)\). Recursively applying [16, Proposition 2.7] and [26, Theorem 5.7], to the graphs \(L_i\), we get for all \(q \geq 2\), \(\operatorname{reg}(I(H)^q) = 2q + \zeta(H) - 1\). \(\square\)

In [21], the authors asked if there exists a graph \(G\) with \(2q + \nu(G) - 1 < \operatorname{reg}(I(G)^q) < 2q + \operatorname{co-chord}(G) - 1\) for all \(q \gg 0\). [21, Question 5.8]. We show that some of the graphs considered in Proposition 4.6 satisfy this inequality. Let \(H\) be a graph as in Proposition 4.6 with \(n_j \equiv 1(\text{mod } 3)\) for \(j = p+1, \ldots, q\). Then \(\nu(H) = \sum_{i=1}^q \left\lfloor \frac{n_i}{3} \right\rfloor, \zeta(H) = p + \sum_{i=1}^q \left\lfloor \frac{n_i}{3} \right\rfloor\) and \(\operatorname{co-chord}(G) = q + \sum_{i=1}^q \left\lfloor \frac{n_i}{3} \right\rfloor\). Therefore, we get

\[
2q + \nu(H) - 1 < \operatorname{reg}(I(H)^q) = 2q + \zeta(H) - 1 < 2q + \operatorname{co-chord}(H) - 1.
\]
Using techniques very similar to the ones used in the proof of Theorem 4.4, we prove a weaker version of Conjecture 1.1.

**Theorem 4.7.** Let $G$ be a graph. If every induced subgraph $H$ of $G$ has a vertex $x$ with $\text{reg}(I(H \setminus N_H[x])) + 1 \leq \text{reg}(I(H))$, then for all $q \geq 1$,

$$\text{reg}(I(G)^q) \leq 2q + \text{reg}(I(G)) - 2.$$

**Proof.** Let $G$ be a graph satisfying the given hypothesis. We prove the assertion by induction on $q$. If $q = 1$, then we are done. Assume that $q > 1$. For any graph $K$, set

$$\mathcal{P}(K) = \{x \in V(K) \mid \text{reg}(I(K)) \geq \text{reg}(I(K \setminus N_K[x])) + 1\}.$$

By hypothesis, $\mathcal{P}(G) \neq \emptyset$. By applying \[1\] Theorem 5.2 and using induction, it is enough to prove that for edges $e_1, \ldots, e_s$ of $G$, $\text{reg}((I(G)^{s+1} : e_1 \cdots e_s)) \leq \text{reg}(I(G))$ for all $s \geq 1$. We prove this by induction on $s$.

Let $G'$ be the graph associated to the ideal $(I(G)^{s+1} : e_1 \cdots e_s)$ which is contained in an appropriate polynomial ring $R_1$. Suppose $s = 1$.

**Case 1:** Suppose $e_1 \cap \mathcal{P}(G) \neq \emptyset$.

Let $e_1 = \{x, y\}$ with $x \in \mathcal{P}(G)$. We proceed as in the proof of Theorem 4.4. If either $\deg_G(x) = 1$ or $\deg_G(y) = 1$, then by Lemma 3.2, $I(G') = I(G)$. Therefore, $\text{reg}(I(G')) = \text{reg}(I(G))$.

Suppose $\deg_G(x) > 1$ and $\deg_G(y) > 1$. Setting $U = N_G(y) = \{y_1, \ldots, y_r\}$ and $J = I(G')$ and considering the exact sequences as in Equation (4.1), we get

$$\text{reg}(R_1/J) \leq \max \left\{ \text{reg} \left( \frac{R_1}{(J : y_1)} \right) + 1, \ldots, \text{reg} \left( \frac{R_1}{(J : y_r)} \right) + 1 \right\}.$$

Proceeding as in that proof, we conclude that $\text{reg}(R_1/(J, U)) \leq \text{reg}(R/I(G))$ and $\text{reg}(J : y_j) \leq \text{reg}(I(G \setminus N_G(x))) < \text{reg}(I(G))$, where the last inequality follows from the hypothesis. Since $(J, y_1, \ldots, y_{j-1}) : (y_j)$ corresponds to an induced subgraph of $(J : y_j)$, we get

$$\text{reg} \left( \frac{R_1}{((J, y_1, \ldots, y_{j-1}) : y_j)} \right) + 1 \leq \text{reg} \left( \frac{R_1}{(J : y_j)} \right) + 1 \leq \text{reg} \left( \frac{R}{I(G)} \right).$$

Therefore, $\text{reg} \left( \frac{R_1}{J} \right) \leq \text{reg} \left( \frac{R}{I(G)} \right)$.

**Case 2:** Suppose $e_1 \cap \mathcal{P}(G) = \emptyset$.

Let $|V(G)| = n$. We proceed by induction on $n$. If $n \leq 4$, then it can be seen that $e \cap \mathcal{P}(G) \neq \emptyset$, for all $e \in E(G)$ so that Case 2 does not occur. Therefore $n \geq 5$.

Suppose $n = 5$. There are 23 simple graphs (without isolated vertices) on 5 vertices. Among these graphs, it can be verified, manually or using computational packages such as Macaulay 2 \[14\] or SAGE \[11\], that all, except the graph given on the left, satisfy the property that $G \setminus \mathcal{P}(G)$ is a set of isolated vertices.

For this graph $G$, $\text{reg}(I(G)) = 2$ and $\mathcal{P}(G) = \{t_2, t_5\}$. Hence $\{t_3, t_4\} \in E(G \setminus \mathcal{P}(G))$ and $(I(G)^2 : t_3t_4) = I(G)$ so that $G' = G$. Therefore, the assertion holds true.

Now assume that $n > 5$. By induction, assume that if $K$ is a graph with $|V(K)| < n$ and for every induced subgraph $K'$ of $K$, there exists $z \in V(K')$ such that $\text{reg}(I(K' \setminus$}
\(N_{K'}[z]) + 1 \leq \text{reg}(I(K'))\), then \(\text{reg}(I(K)^2 : e) \leq \text{reg}(I(K))\) for every \(e \in E(K)\) such that \(e \cap P(K) = \emptyset\).

Let \(x \in P(G)\). Then by [15, Theorem 3.4],
\[
\text{reg}(I(G')) \leq \max \left\{ \text{reg}(I(G' \setminus \{x\})), \text{reg}(I(G' \setminus N_{G'}[x]) + 1 \right\}.
\]
Since \(e_1 \cap \{x\} = \emptyset\), by Lemma 3.3
\[
\text{reg}(I(G' \setminus \{x\})) = \text{reg}(\{ (I(G \setminus x)^2 : e_1) \}.
\]
Note that \(G \setminus x\) is a graph with \(|V(G \setminus x)| < n\) and every induced subgraph of \(G \setminus x\) is an induced subgraph of \(G\). If \(e_1 \cap P(G \setminus x) \neq \emptyset\), then by Case 1, or if \(e_1 \cap P(G \setminus x) = \emptyset\), then by induction on the number of vertices, we get
\[
\text{reg}(I(G \setminus x)^2 : e_1) \leq \text{reg}(I(G \setminus x)) \leq \text{reg}(I(G)).
\]
Now we prove that \(\text{reg}(I(G') \setminus N_{G'}[x]) + 1 \leq \text{reg}(I(G))\). Note that \(G \setminus N_G[x]\) is a graph with \(|V(G \setminus N_G[x])| < n\). If \(e_1 \cap P(G \setminus N_G[x]) \neq \emptyset\), then by Case 1, or if \(e_1 \cap P(G \setminus N_G[x]) = \emptyset\), then by induction on the number of vertices, we get
\[
\text{reg}(I(G' \setminus N_{G'}[x])) + 1 \leq \text{reg}(I(G \setminus N_G[x])^2 : e_1) + 1 \quad \text{(by Lemma 3.3)}
\]
\[
\leq \text{reg}(I(G \setminus N_G[x])) + 1
\]
\[
\leq \text{reg}(I(G)).
\]
Hence \(\text{reg}(I(G')) \leq \text{reg}(I(G))\). This proves the case \(s = 1\).

Suppose \(s > 1\). We now show that \(\text{reg}(I(G)^{s+1} : e_1 \cdots e_s) \leq \text{reg}(I(G))\). Let \(e_i = \{a_i, b_i\}\) for \(1 \leq i \leq s\). If \(\deg_G(a_i) = 1\) for all \(i\), then by Lemma 3.2 it follows that
\[
\text{reg}(I(G)^{s+1} : e_1 \cdots e_s) = \text{reg}(I(G)^s : e_1 \cdots e_{i-1}e_{i+1} \cdots e_s) \leq \text{reg}(I(G)),
\]
where the last inequality follows from induction on \(s\).

Assume now that \(\deg_G(a_i) \geq 2\) and \(\deg_G(b_i) \geq 2\) for all \(1 \leq i \leq s\).

**CASE 3:** Suppose \(e_i \cap P(G) \neq \emptyset\), for some \(1 \leq i \leq s\).

Without loss of generality, assume that \(e_s \cap P(G) \neq \emptyset\) and \(a_s \in P(G)\). Proceeding as in the proof Theorem 4.4 following the same notation, one gets
\[
\text{reg}(J, X_1, X_2) \leq \text{reg}(I(G' \setminus N_{G'}[b_s])) \leq \text{reg}(I(G \setminus N_G[b_s])) \leq \text{reg}(I(G));
\]
\[
\text{reg}(J : y_t) \leq \text{reg}(I((G \setminus N_G[y_i, a_s])')) \leq \text{reg}(I((G \setminus N_G[a_s])'))
\]
\[
\leq \text{reg}(I(G \setminus N_G[a_s])) \leq \text{reg}(I(G));
\]
\[
\text{reg}(J, X_1 : z_i) \leq \text{reg}(I((G \setminus N_G[y_i, a_s])')) \leq \text{reg}(I((G \setminus N_G[a_s])'))
\]
\[
\leq \text{reg}(I(G \setminus N_G[a_s])) < \text{reg}(I(G)).
\]

For the above conclusions, we use, in a similar manner as in the proof of Theorem 4.4, Lemmas 3.5, 3.6, 3.7 and induction on \(s\). Using these inequalities, we conclude that \(\text{reg}(J) \leq \text{reg}(I(G))\).

**CASE 4:** Suppose \(e_i \cap P(G) = \emptyset\), for all \(1 \leq i \leq s\).

If \(|V(G)| \leq 4\), then one can see that the case \(e_i \cap P(G) = \emptyset\) does not occur. If \(|V(G)| = 5\), then as remarked in Case 2, one can see that there is only one graph \(G\), which is given there, such that \(G \setminus P(G)\) has an edge. In this case, the only possibility is \(e_i = \{t_i, t_4\}\) for all \(i = 1, \ldots, s\). Hence \(G' = G\) and hence the assertion holds true.
Now assume that $|V(G)| = n > 5$. Let $x \in P(G)$. Then by by [15, Theorem 3.4],
\[ \text{reg}(I(G')) \leq \max \left\{ \text{reg}(I(G' \setminus \{x\})), \text{reg}(I(G' \setminus N_{G'}[x])) + 1 \right\}. \]

By Lemma 3.3, $\text{reg}(I(G' \setminus x)) = \text{reg}(I(G \setminus x)) + 1$ and by Lemma 3.5, $\text{reg}(I(G' \setminus N_{G'}[x])) \leq \text{reg}(I(G \setminus N_{G}[x])) + 1$, where $E(G \setminus N_{G}[x]) \cap \{e_1, \ldots, e_s\} = \{e_{i_1}, \ldots, e_{i_t}\}$. Proceeding as in Case 2, using the above inequalities and induction on $|V(G)|$ as well as on $s$, one can conclude that $\text{reg}(I(G')) \leq \text{reg}(I(G))$. Therefore $\text{reg}(I(G') : e_1 \cdots e_s) \leq \text{reg}(I(G))$. \qed

In view of the above theorem, we would like to ask:

**Question 4.8.** Does every finite simple graph $G$, having no isolated vertices, have a vertex $x$ such that $\text{reg}(I(G) : x) + 1 \leq \text{reg}(I(G))$?

We note here that if the answer to the above question is positive, then it follows from Theorem 4.7 that the Conjecture 1.1 is true.

### 5. Vertex decomposable graphs

In this section, we prove Conjecture 1.1 for vertex decomposable graphs. We first recall the definition of simplicial complex and vertex decomposable graph.

A **simplicial complex** $\Delta$ on $V = \{x_1, \ldots, x_n\}$ is a collection of subsets of $V$ such that:

1. $\{x_i\} \in \Delta$ for $i = 1, \ldots, n$, and
2. if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

Elements of $\Delta$ are called the **faces** of $\Delta$, and the maximal elements, with respect to inclusion, are called the facets. The **link** of a face $F$ in $\Delta$ is $\text{link}_{\Delta}(F) = \{F' \mid F' \cup F \text{ is a face in } \Delta, F' \cap F = \emptyset\}$.

A simplicial complex $\Delta$ is recursively defined to be **vertex decomposable** if it is either a simplex or else has some vertex $v$ so that

1. both $\Delta \setminus v$ and $\text{link}_{\Delta} v$ are vertex decomposable, and
2. no face of $\text{link}_{\Delta} v$ is a facet of $\Delta \setminus v$.

The **independence complex** of $G$, denoted $\Delta(G)$, is the simplicial complex on $V(G)$ with face set
\[ \Delta(G) = \left\{ F \subseteq V(G) \mid F \text{ is an independent set of } G \right\}. \]

A graph $G$ is said to be **vertex decomposable** if $\Delta(G)$ is a vertex decomposable simplicial complex. In [32], Woodroofe translated the notion of vertex decomposable for graphs as follows.

**Definition 5.1.** [32, Lemma 4] A graph $G$ is recursively defined to be vertex decomposable if $G$ is totally disconnected (with no edges) or if

1. there is a vertex $x$ in $G$ such that $G \setminus x$ and $G \setminus N_G[x]$ are both vertex decomposable, and
2. no independent set in $G \setminus N_G[x]$ is a maximal independent set in $G \setminus x$. 


A vertex $x$ which satisfies the second condition is called a shedding vertex of $G$. If $G$ is a vertex decomposable graph, then by [3] Theorem 2.5, $G \setminus N_G[x]$ is a vertex decomposable graph, for any $x \in V(G)$. For any vertex decomposable graph $K$, set

$$\mathcal{S}(K) = \{ x \in V(K) \mid x \text{ is a shedding vertex and } K \setminus x \text{ is a vertex decomposable graph} \}.$$ 

Note that if $K$ is vertex decomposable, then $\mathcal{S}(K) \neq \emptyset$. 

**Observation 5.2.** Let $G$ be a vertex decomposable graph and $x \in \mathcal{S}(G)$. By [18] Theorem 4.2,

$$\text{reg}(I(G)) = \max \left\{ \text{reg}(I(G \setminus x)), \text{reg}(I(G \setminus N_G[x])) + 1 \right\}.$$ 

Therefore, $\text{reg}(I(G \setminus N_G[x])) + 1 \leq \text{reg}(I(G))$.

We prove Conjecture [11] for the class of vertex decomposable graphs.

**Theorem 5.3.** Let $G$ be a vertex decomposable graph. Then for all $q \geq 1$,

$$\text{reg}(I(G)^q) \leq 2q + \text{reg}(I(G)) - 2.$$ 

**Proof.** This proof is also very similar to the proof of Theorem [4.7]. We give a sketch of the proof here. The proof is by induction on $q$. If $q = 1$, then we are done. Assume that $q > 1$. By applying [11] Theorem 5.2 and using induction, it is enough to prove that for edges $e_1, \ldots, e_s$ of $G$ (not necessarily distinct), $\text{reg}((I(G)^{s+1}: e_1 \cdots e_s)) \leq \text{reg}(I(G))$ for all $s \geq 1$. We prove this by induction on $s$. Let $G'$ be the graph associated to $(I(G)^{s+1}: e_1 \cdots e_s)$, for any $e_1, \ldots, e_s \in E(G)$.

Let $s = 1$. As in the proof of Theorem [4.7] we split the proof into two cases.

**CASE 1:** Suppose $e_1 \cap \mathcal{S}(G) \neq \emptyset$.

The proof is identical to **CASE 1** in Theorem [4.7]. The only difference in the proof is that while we used the hypothesis in Theorem [4.7] to conclude that $\text{reg}(J : y_i) < \text{reg}(I(G))$, here we use **Observation 5.2** for that conclusion.

**CASE 2:** Suppose $e_1 \cap \mathcal{S}(G) = \emptyset$.

Let $|V(G)| = n$. We proceed by induction on $n$. If $G$ is a vertex decomposable with $n \leq 4$, then it can be seen that $e \cap \mathcal{S}(G) \neq \emptyset$, for all $e \in E(G)$. Therefore $n \geq 5$.

Suppose $n = 5$. It can verified, manually or using computational packages such as Macaulay 2 [12], SimplicialDecomposability [9] or SAGE [11], that there are 20 vertex decomposable graphs (without isolated vertices) on 5 vertices. Among these graphs, all except two graphs satisfy the property that $G \setminus \mathcal{S}(G)$ is a set of isolated vertices. The two graphs for which $G \setminus \mathcal{S}(G)$ contain edges are given in the figure on the left. Then for any choice of $e \in E(G \setminus \mathcal{S}(G))$, $(I(G)^2 : e) = I(G)$ so that $G' = G$.

Now assume that $n > 5$. By induction, assume that if $H$ is a vertex decomposable graph with $|V(H)| < n$ and $e \in E(H)$ such that $e \cap \mathcal{S}(H) = \emptyset$, then $\text{reg}(I(H)^2 : e) \leq \nu(H) + 1$. 

![Vertex decomposable graphs with $|V(G)| = 5$ and $\bullet$ denote shedding vertices.](image-url)
Let \( x \in S(G) \). By [15, Theorem 3.4],
\[
\reg(I(G')) \leq \max \left\{ \reg(I(G' \setminus \{x\})), \reg(I(G' \setminus N_G[x]) + 1 \right\}.
\]

Now the proof is identical to that of the proof of Case 2 in Theorem 4.7. In this case also, we use Observation 5.2 to conclude \( \reg(I(G \setminus N_G[x])) < \reg(I(G)) \).

Suppose \( s > 1 \). Assume by induction that for any vertex decomposable graph \( G \) and edges \( e_1, \ldots, e_{s-1} \), \( \reg(I(G)^s : e_1 \cdots e_{s-1}) \leq \reg(I(G)) \). We now prove that for edges \( e_1, \ldots, e_s \), \( \reg(I(G)^{s+1} : e_1 \cdots e_s) \leq \reg(I(G)) \). Let \( e_i = \{a_i, b_i\} \). If \( \deg_G(a_i) = 1 \) or \( \deg_G(b_i) = 1 \) for some \( i \), then the assertion follows as in the proof of Theorem 4.7. Therefore, we may assume that \( \deg_G(a_i) \geq 2 \) and \( \deg_G(b_i) \geq 2 \) for all \( 1 \leq i \leq s \).

**Case 3:** Suppose \( e_i \cap S(G) \neq \emptyset \), for some \( 1 \leq i \leq s \).

Without loss of generality, assume that \( e_s \cap S(G) \neq \emptyset \) and \( a_s \in S(G) \). Proceeding as in the proof Theorem 4.7 following the same notation, one gets
\[
\reg(J, X_1, X_2) \leq \reg(I(G' \setminus N_G[b_s])) \leq \reg(I(G \setminus N_G[b_s])) \leq \reg(I(G));
\]
\[
\reg(J : y_i) \leq \reg(I((G \setminus N_G[y_i, a_s])) \leq \reg(I((G \setminus N_G[a_s])))
\]
\[
\leq \reg(I(G \setminus N_G[a_s])) < \reg(I(G));
\]
\[
\reg(J, X_1 : z_i) \leq \reg(I((G \setminus N_G[y_i, a_s])) \leq \reg(I((G \setminus N_G[a_s])))
\]
\[
\leq \reg(I(G \setminus N_G[a_s])) < \reg(I(G)).
\]

Here also, we use Observation 5.2 along with Lemmas 3.5, 3.6, 3.7 and induction on \( s \) for the above conclusions. Using these inequalities, we conclude, as in the proof of Theorem 4.7 that \( \reg(J) \leq \reg(I(G)) \).

**Case 4:** Suppose \( e_i \cap S(G) = \emptyset \), for all \( 1 \leq i \leq s \).

Let \( x \in S(G) \). By [15, Theorem 3.4],
\[
\reg(I(G')) \leq \max \left\{ \reg(I(G' \setminus \{x\})), \reg(I(G' \setminus N_G[x]) + 1 \right\}.
\]

Here too, the proof is identical to Case 4 of Theorem 4.7.

Finally, we obtain \( \reg(I(G)^{s+1} : e_1 \cdots e_s) \leq \reg(I(G)) \). Therefore, for all \( q \geq 1 \),
\[
\reg(I(G)^q) \leq 2q + \reg(I(G)) - 2.
\]

As an immediate consequence of the above result, we obtain the linear polynomial corresponding to \( \reg(I(G)^q) \) for several classes of graphs.

**Corollary 5.4.** If

1. \( G \) is \( C_5 \)-free vertex decomposable;
2. \( G \) is chordal;
3. \( G \) is sequentially Cohen-Macaulay bipartite, or
4. \( G = H \cup W(S) \), where \( S \subseteq V(H) \), \( H \setminus S \) is a chordal graph and \( H \cup W(S) \) denotes the graph obtained from \( H \) by adding a whisker to each vertex in \( S \),

then for all \( q \geq 1 \),
\[
\reg(I(G)^q) = 2q + \nu(G) - 1.
\]
Proof. (1) Follows from Theorem [5.3][5] Theorem 4.5 and [23] Lemma 2.3.

(2) Since \( G \) is chordal, it is \( C_5 \)-free and by \([32]\) \( G \) is vertex decomposable. By (1), \( \text{reg}(I(G)^q) = 2q + \nu(G) - 1 \).

(3) By \([31]\) Theorem 2.10, \( G \) is vertex decomposable. Since a bipartite graph is \( C_5 \)-free, the assertion follows from (1).

(4) First we show that \( \text{reg}(I(G)) = \nu(G) + 1 \). By \([22]\) Lemma 2.2, we have \( \nu(G) + 1 \leq \text{reg}(I(G)) \). Therefore, it is enough to prove that \( \text{reg}(I(G)) \leq \nu(G) + 1 \). Set \(|S| = m \). If \( m = 0 \), then by \([17]\) Corollary 6.9 \( \text{reg}(I(G)) = \nu(G) + 1 \). Suppose \( m \geq 1 \). There is a vertex \( x \) in \( S \) such that \( N_G(x) = \{x\} \). By \([15]\) Theorem 3.4,

\[
\text{reg}(I(G)) \leq \max\{\text{reg}(I(G\setminus x)), \text{reg}(I(G\setminus N_G(x))) + 1\}.
\]

By induction hypothesis on \( m \), \( \text{reg}(I(G\setminus x)) \leq \nu(G) + 1 \) and \( \text{reg}(I(G\setminus N_G(x))) + 1 \leq \nu(G) + 1 \). If \( \{f_1, \ldots, f_t\} \) is an induced matching of \( G \setminus N_G(x) \), then \( \{f_1, \ldots, f_t, \{x, z\}\} \) is an induced matching of \( G \). Therefore \( \nu(G \setminus N_G(x)) + 1 \leq \nu(G) \). Hence \( \text{reg}(I(G)) = \nu(G) + 1 \). By \([6]\) Corollary 4.6, \( G \) is a vertex decomposable graph. Therefore, by Theorem \([5.3]\) and \([5]\) Theorem 4.5, \( \text{reg}(I(G)^q) = 2q + \nu(G) - 1 \). \( \square \)

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