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A Factor Model for the Calculation of Portfolio Credit VaR

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Abstract

In nowadays financial institutions pay more and more attention to credit risk. The default correlation between credit instruments has a fatal influence on credit risk. In this paper, we use the factor copula model to calculate the credit VaR (value at risk) for the target portfolio. We first use the asset value obtained from Merton model to substitute the equity price which is used by most researchers. Then we employ the principal component analysis method to construct a virtual variable representing the macro factor, and estimate the VaR for a credit portfolio.

Keywords: Factor copula; Credit VaR; Principal component analysis

1. Introduction

Credit risk refers to the risk that the counterparty failing to fulfill its obligations. As long as financial institutions have contractual agreements, credit risk exists. So institutions are paying more and more attention on how to model and control the credit risk. These models can characterize the potential loss distribution due to credit risk. Loss distributions are used to measure the profitability of transactions related to the risk. The loss distribution can also be used to determine the level of capital that a bank needs in order to protect itself.

When estimating the portfolio credit loss distribution, both the individual default rates of each firm and joint probabilities across all obligors need to be considered. The main and hard problem is how to estimate the joint default probabilities. Li [1] extended Sklar’s theorem [2] that a copula function can be applied to solve default correlation problems. He inferred that if the dependence structure was assumed to be normally distributed, the joint default probability would be consistent with the result obtained from using a Gaussian copula function. Despite its very strong assumption, his one factor Gaussian Copula has become a market standard for credit risk management. Gregory & Laurent [3], Andersen, Sidenius & Basu [4], Hull & White [5], and Laurent & Gregory [6] have used the copula approach to price CDOs within a semi-analytical framework.

In this paper, we will use the factor copula model together with the principal component analysis method to calculate the credit VaR for a portfolio.

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2. Methodology

2.1. Estimation of the Dependence Structure

Consider a portfolio consisting of m credits. The marginal distribution of each individual credit risk can be constructed by using either the historical approach or the market implicit approach. But the question is how to describe the joint distribution or co-movement between these risks (default correlation). The simplest way is assuming the credit risks to be mutual independent. However, the independent assumption is not realistic. Undoubtedly, the default rates for a group of credits tend to be higher when the economy is in recession period and lower in booming period. This implies that each credit is subject to the same factors from macroeconomic environment, and there exists some form of dependence among these credits.

When given individual distribution of each asset in a portfolio, we can obtain the joint distribution by using a copula. But if there are more assets (N) in the portfolio, we need to construct N*(N-1)/2 number of correlations, which will result in the crisis of dimensionality. In this paper, we will introduce the factor model to solve this problem.

The main idea of factor model is: under a certain macro environment, credit default events are independent to each other. And the main factors that affect default events come from potential market economic conditions. This model provides one way to avoid dealing with high dimensional simulation problem.

One-Factor Model for Firm Value

According to Vasicek [7], the values of the obligors follow the following model:

\[
\ln(V_{t+1}) = \ln(V_t) + \xi_{t+1}
\]

\[\xi_t \sim \text{i.i.d. } N(\mu, \sigma^2)\]  \hspace{1cm} (1)

where \( V \) denotes the theoretical value and \( \mu \) is a constant variable.

We can also rearrange \( \xi_t \) as:

\[
\xi_t = \tau X_t + \sigma \epsilon_t
\] \hspace{1cm} (2)

where \( X_t \) and \( \epsilon_t \) follow standard normal distribution. \( X_t \) is the macroeconomic factor which is a systematic factor affecting the value of all assets in the portfolio. \( \tau X \) denotes the asset exposure to systematic factor and \( \sigma \epsilon \) is the asset’s idiosyncratic risk.

If the macroeconomic factor is known, the unconditional default probability is as follows:

\[
P(V_t < V_{def} | X_t = X) = P(\ln \frac{V_t}{V_0} < \ln \frac{V_{def}}{V_0}) = P(\tau X_t + \sigma \epsilon_t < \ln \frac{V_{def}}{V_0}) = N(\frac{\ln \frac{V_{def}}{V_0} - \tau X_t}{\sigma})
\]  \hspace{1cm} (3)

where \( V_{def} \) is the default threshold which means that if the company’s theoretical asset value is less than \( V_{def} \), the company will choose to default.
The great importance of this model is that conditional on the realization of the systematic factor $X$, the firm’s values and the defaults are independent. So the correlations between the assets’ movement are

$$\text{Cov}(V_i, V_j) = \tau_i \tau_j$$  

(4)

2.2. Asset Valuation from Merton’s Model

In most papers, researchers usually use equity price to substitute asset value. Now we will suggest a way to calculate the obligor’s asset value.

According to Merton [8], the default of each obligor is triggered by the change of firms’ asset value. We denote the asset value of obligor $i$ at time $t$ by $V_i(t)$, and $V_{\text{def}}$ is assumed to be the exposure when default. By thinking the company’s stock price $E$ as the call option on the asset value, we can derive the following results according to Black-Scholes option pricing model:

$$E = V_i(t) N(d_1) - V_{\text{def}} e^{-rt} N(d_2)$$  

(5)

$$d_2 = d_1 - \sigma_v \sqrt{T}$$  

(6)

$$\sigma_E = \sigma_v N(d_1) \frac{V_i(t)}{E}$$  

(7)

By combining these equations, we can get $V_i(t)$ and $\sigma_v$.

2.3. Generating a Macro Factor Using Principal Component Analysis

Now we know that the key problem of factor model is how to select a proper factor to represent so many macro factors. We can use principal component analysis to solve the problem.

In statistics, researchers always use principal component analysis to reduce the multivariable problems. If someone wants to explain one phenomenon he has to gather interpreting variables relating to the movements of the phenomenon. But if there are too many interpreting variables, it will become hard to explain those random variables. Principal component analysis provides a way to extract main interpreting variable to cover maximum variance of variables. Those representative variables may not be “real” variables. We will not talk about the process of deriving the virtual factor $Y$, but we can expect that the virtual factor can represent more useful information than just using the GDP or stock index.

3. Empirical calculation of portfolio’s credit VaR

In this section we will give the empirical result. First we will calculate the company’s theoretical value and test whether the return follow normal distribution. Then we will try to generate a virtual macroeconomic variable by using principal component analysis. And then we will forecast $X_{t+1}$ by using first-order autoregressive model. Finally we will calculate the credit VaR of a portfolio which contains two assets using Monte Carlo simulation.
### 3.1. Calculating the Theoretical Value of Assets

We choose three companies which are listed on Chinese Stock Exchange, and their codes are 000005, 600048 and 600572.

The time period is from the end of July, 2010 to the end of July, 2012. We choose every month’s closing price and because one company (000005) is specially treated, there are 13 time series.

The calculation steps are as follows:

1. Calculate stocks’ logarithmic rate of return and the variance
   \[ \mu_t = \ln \frac{S_t}{S_{t-1}} \]  \hspace{1cm} \text{(8)}
   where S is the closing price of every month.

2. Calculate each asset’s theoretical value:
   \[ E = E_c + E_{nc} \]  \hspace{1cm} \text{(9)}
   \[ E_c = P \times S_c \]  \hspace{1cm} \text{(10)}
   \[ E_{nc} = A \times S_{nc} \]  \hspace{1cm} \text{(11)}
   where \( E_c \) is the price of the stock which is at circulation and \( E_{nc} \) is the price of the stock which is not at circulation. \( A \) is the book value of the stock.

   \[ V_{\text{def}} = CL + 0.75LL \]  \hspace{1cm} \text{(12)}
   where CL is the current liability and LL is the long-term debt.

The following Table 1 gives the logarithmic rate of return for each asset.

| Stock Code | 000005 | 600048 | 600572 |
|------------|--------|--------|--------|
| 1          | 1.20E-02 | 2.08E-02 | 1.49E-02 |
| 2          | 3.68E-02 | 3.83E-02 | 3.03E-02 |
| 3          | -9.57E-02 | 10.4E-02 | 4.33E-02 |
| 4          | -7.98E-03 | 5.65E-02 | 3.23E-02 |
| 5          | -1.01E-01 | -6.54E-02 | -5.94E-02 |
| 6          | -4.23E-02 | 9.47E-02 | 4.84E-02 |
| 7          | -4.01E-02 | 0.69E-02 | -3.88E-03 |
Because the factor model assumes that the asset’s value follows the geometric Brown motion, the return of the asset’s value has the normal distribution. We must make the normality test before we use the time series data in the factor model.

Table 2. Normality test of 000005’s return

| Kolmogorov-Smirnova | Shapiro-Wilk |
|---------------------|--------------|
| Statistic           | df | Sig. | Statistic | df | Sig. |
| .168                | 13 | .200*| .894      | 13 | .109 |

Table 3. Normality test of 600048’s return

| Kolmogorov-Smirnova | Shapiro-Wilk |
|---------------------|--------------|
| Statistic           | df | Sig. | Statistic | df | Sig. |
| .132                | 13 | .126 | .610      | 13 | .082 |

Table 4. Normality test of 600572’s return

| Kolmogorov-Smirnova | Shapiro-Wilk |
|---------------------|--------------|
| Statistic           | df | Sig. | Statistic | df | Sig. |
| .127                | 13 | .123 | .633      | 13 | .086 |

We know that the K-S and S-W test’s null hypothesis is that the distribution is normal distribution and the three sigs are all more than 0.05, so we have no reason to reject that the three series follow the normal distribution.

And we can see from the table that 000005’s sig are bigger than the other two’s sig values. We can understand this result because 000005 has been specially treated. So it is mainly affected by its idiosyncratic factors.
3.2 Generating the virtual macroeconomic factor

We use macro economy boom index (leading index) and the Shanghai Composite Index to compose a virtual factor. The reasons we select the two factors are as follows: first, the stock price reflects the market’s overall behavior, but because the index components often has the particularity, the index is not universal; second, macro economy boom index is the best surrogate variable of GDP, and it solves the GDP’s lagging problems; finally, the two variables are updated monthly, and this can satisfy the requirements of credit risk management.

Table 5. Component score coefficient matrix

| components          |          |
|---------------------|----------|
| Return of boom index| .617     |
| Return of Shanghai  |          |
| Composite index     | .617     |

From above table we can conclude that the virtual macro factor X equals to

\[ 0.617 \times \text{the return of boom index} + 0.617 \times \text{the return of Shanghai Composite index}. \]  

So we can get the series of virtual macro factor using this new model.

Before we apply this series to the factor model, we also need to make a normality test. The sig value is 0.2, thus we have no reason to reject that this series follow normal distribution.

3.3 Calculating the portfolio’s credit VaR using factor model

The steps to calculate the credit VaR are as follows:

1. Calculate factor model’s parameters \( \tau \) and \( \sigma \).
   
   We can use the regression model to derive the two parameters, and the results are given in Table 6.

Table 6. Parameters estimation

|          | \( \tau \) | \( \sigma \) |
|----------|-----------|-------------|
| 600048   | 0.857     | 0.36        |
| 600572   | 0.594     | 0.189       |

2. Forecast the future virtual factor
   
   We use the first order autoregressive model to forecast the future virtual factor, and the results are given in Table 7.
Table 7. Forecasting the future virtual factor

| First order autoregressive | Coefficient | t   | Sig. |
|----------------------------|-------------|-----|------|
|                            | B           |     |      |
| constant                   | .012        | .179| .570 | .574 |
|                            | .468        | .187| 2.508| .020 |

Because the sig value 0.02 is less than 0.05, the coefficient is significantly not equal to zero. The model is

$$X_{t+1} = 0.468 \times X_t + 0.012.$$  \hspace{1cm} (13)

(3) Calculating the portfolio’s credit VaR using Monte Carlo simulation

We first generate a set of mutually independent random numbers by using MATLAB software (heterogeneous factors simulation), then we can get the distribution for each company’s asset value through factor model. Then transform it into the joint distribution through the Cholesky decomposition technology.

Comparing the asset value and default value with given recovery rate, we can get the loss distribution. Then after sort the loss, we can derive the credit VaR for a certain significance level.

Without loss of generality, we construct the following portfolio. On June 30, 2012, we hold the company bonds of 600048 and 600572. Their weights are both 0.5. Now calculate the portfolio’s credit VaR. The period is one year and significance level is 5%. Assume the recovery rates of the two corporate bonds are both 60%. Then the portfolio’s credit VaR is 0.2.

People often use back test to check the accuracy of obtained VaR. But the key premise to make back test is that there exists historical data. For the market VaR or the integrated VaR, the test is feasible because of the accessibility of historical data. But because there is no accessible default data, we cannot do back test for credit VaR.

4. Conclusion

In this paper, we use the asset value obtained from Merton’s model to replace the equity price which is used by most researchers. And we employ the principal component analysis to construct a virtual variable representing the macro factor. Thus we provide a method for estimating credit portfolio VaR.

There is also some weakness in this paper. The main problem is that our method assumes the return of asset value and virtual factor follow normal distribution. This assumption is too strong and far from reality. This is what we will do further in the future research.

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