Comment on “Helical MRI in magnetized Taylor-Couette flow”

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Liu et al. [Phys. Rev. E 74, 056302 (2006)] have presented a WKB analysis of the helical magnetorotational instability (HMRI), and claim that it does not exist for Keplerian rotation profiles. We show that if radial boundary conditions are included, the HMRI can exist even for rotation profiles as flat as Keplerian, provided only that at least one of the boundaries is sufficiently conducting.

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The helical magnetorotational instability (HMRI) [1, 2, 3, 4, 5], is similar to the standard magnetorotational instability (SMRI) [6, 7, 8], in the sense that both are mechanisms whereby hydrodynamically stable differential rotation profiles may be destabilized by the addition of magnetic fields. However, the way in which they behave in the limit of small magnetic Prandtl number $Pm$ is very different. Unlike the SMRI, which ceases to exist for zero $Pm$, the HMRI continues to exist, with the instability (SMRI) [6, 7, 8], in the sense that both are essentially the same as in [1], except that we restrict attention to the $Pm \to 0$ limit, ensuring that any instabilities we obtain will necessarily be the HMRI. (Priede et al. [10] have very recently also considered this $Pm \to 0$ limit; some of their results are quite relevant here, and will be discussed below.)

The boundary conditions associated with $v$ and $\psi$ are no-slip, just as in [1]. The boundary conditions associated with $b$ are

$$b = \epsilon (rb)' \quad \text{at} \quad r_1 = 1, \quad b = 0 \quad \text{at} \quad r_o = 2.$$

The outer boundary is therefore insulating, whereas the nature of the inner boundary depends on $\epsilon$: $\epsilon = 0$ is insulating, $\epsilon = \infty$ is perfectly conducting. Intermediate values correspond to a boundary consisting of a thin layer of relative conductance $\epsilon$ (see for example [11, 12] for this thin boundary layer approximation in other contexts).

$\Omega(r)$ is the rotation profile whose stability is to be investigated. In [1, 2] we considered Taylor-Couette profiles of the form $\Omega = c_1 + c_2/r^2$. To facilitate comparison with Liu et al., here we will primarily consider profiles of the form $\Omega = r^n$. As we will see though, the two choices yield almost identical behavior.

Figure 1 shows contour plots of the critical Reynolds number for the onset of the HMRI, as a function of $n$ and $\beta$, and optimized over the Hartmann number $Ha$ and the axial wavenumber $k$. Turning first to the $\epsilon = 0$ plot on the left, we note how increasing $\beta$ from 1 to 5 facilitates the instability, that is, allows it to exist increasingly far beyond the Rayleigh line at $n = -2$. Beyond $\beta \approx 5$ though the $Re_c$ curves become largely independent of $\beta$. And crucially, even the $Re_c = 10^8$ curve asymptotes just to the left of the Liu et al. line at $n = -1.66$. These $\epsilon = 0$ results are therefore in excellent agreement with their prediction that the HMRI only exists to the left of this line.

However, if we now turn to the $\epsilon = \infty$ plot on the right, we note that $Re_c = 10^4$ already extends beyond the Liu et al. line, and $Re_c = 10^5$ extends beyond the Kepler line $n = -1.5$. Simply switching the inner boundary from insulating to conducting is sufficient to allow the HMRI to operate even for Keplerian rotation profiles. Similar
results are obtained if instead it is the outer boundary that is switched from insulating to conducting. Why the electromagnetic boundary conditions should have such a dramatic effect is not clear, but it is certainly well known in many other contexts, e.g. [13, 14], that they can play a crucial role.

Note also that exactly the same phenomenon illustrated in Fig. 1 is already implicit in Fig. 2 of [10], where the HMRI exists up to \( \mu \approx 0.32 \) for insulating boundaries, but up to \( \mu \approx 0.45 \) for conducting boundaries, where \( \mu = \Omega_o/\Omega_i \). Translating from \( \mu \) to \( n \) via \( \mu = 2^n \), their results become \( n \approx -1.64 \) for insulating boundaries, versus \( n \approx -1.15 \) for conducting boundaries. That is, the Keplerian value \( n = -1.5 \) is accessible with conducting boundaries, but not with insulating ones.

At first sight it would appear that their value of \( n \approx -1.64 \) for insulating boundaries is already in conflict with the Liu et al. limit \( n < -1.66 \). In fact, this slight discrepancy is due to the difference between the \( \Omega = c_1 + c_2/r^2 \) profile used by [10], and the \( \Omega = r^n \) profile considered by Liu et al. Even if \( c_1 \) and \( c_2 \) are chosen to match \( \Omega_i = 1 \) and \( \Omega_o = 2^n \), a Taylor-Couette profile will be somewhat steeper near \( r_i \), and corresponding somewhat shallower near \( r_o \). By concentrating near the inner boundary, the instability can then operate at somewhat larger \( \Omega_o/\Omega_i \) values than for an \( r^n \) profile. The dotted lines in Fig. 1 quantify this effect, showing the \( \text{Re}_c = 10^3 \) curves if the \( r^n \) profile is replaced by a Taylor-Couette profile, with \( c_1 \) and \( c_2 \) chosen as indicated above. We see that qualitatively the two profiles yield exactly the same behavior, but that the Taylor-Couette profile extends to slightly larger values of \( n \). It was precisely to avoid this effect, and thereby allow a direct comparison with Liu et al., that we chose here to concentrate on the \( r^n \) profiles.

The dashed lines in Fig. 1 show the locations along which \( \text{Re}_c \) is optimized not only over \( k \) and \( \text{Ha} \), but over \( \beta \) as well. Fig 2 shows \( \text{Re}_c \) as a function of \( n \) along these lines, now including not just \( \epsilon = 0 \) and \( \infty \), but also \( \epsilon = 0.5 \) and 1. We see that \( \epsilon = 0.5 \) is already large enough to reach the Kepler value \( n = -1.5 \). That is, if the conductance of the inner boundary is only 1/2 that of the fluid region, the HMRI already exists for rotation profiles as flat as Keplerian.

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