Finite Codimensionality Method for Infinite-Dimensional Optimization Problems

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Abstract

This paper is devoted to establishing an enhanced Fritz John condition for a general constrained nonlinear infinite-dimensional optimization problem. Unlike traditional constraint qualifications in optimization theory, a condition of finite codimensionality is employed to ensure the existence of nontrivial Lagrange multipliers. As applications, first-order necessary conditions for optimal control problems of some deterministic/stochastic control systems are derived in a unified way. Our finite codimensionality condition, which is equivalent to some a priori estimate, can offer a straightforward and analytical verification process in each of these applications.

Key Words. Infinite-dimensional optimization problem, Fritz John condition, KKT condition, finite codimensionality, a priori estimate.

AMS subject classifications. 49K27, 93C25, 49K15, 35Q93, 93E20.

1 Introduction

Let $V$ be a complete metric space, $X$ be a Banach space with its dual space $X'$, and $E$ be a nonempty closed convex subset of $X$. Given continuous functions $f_0 : V \to \mathbb{R}$ and $f : V \to X$ and closed subset $K \subset V$, we consider the following optimization problem:

$$
\begin{align*}
\text{(P)} & \quad \text{Minimize } f_0(u), \quad \text{subject to } f(u) \in E \text{ and } u \in K.
\end{align*}
$$

When $V = \mathbb{R}^n$, $X = \mathbb{R}^m$ (for $n, m \in \mathbb{N}$), $f = (g_1, g_2, \cdots, g_m, h_{m_1+1}, \cdots, h_m)^T$ with $m_1 \in \mathbb{N}$ and $m_1 \leq m$, $g_i : \mathbb{R}^n \to \mathbb{R}$ ($i = 1, 2, \cdots, m_1$) and $h_j : \mathbb{R}^n \to \mathbb{R}$ ($j = m_1 + 1, m_1 + 2, \cdots, m$), $K = V$, and

$$
E = \{(x_1, x_2, \cdots, x_{m_1}, 0, \cdots, 0)^T \in \mathbb{R}^m \mid x_i \leq 0, i = 1, 2, \cdots, m_1\},
$$

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the problem \((P)\) can be rewritten in the classical finite-dimensional form:

\[
\begin{align*}
\begin{cases}
\text{Minimize } f_0(u), & u \in \mathbb{R}^n, \\
\text{subject to } g_i(u) \leq 0, & i = 1, 2, \ldots, m_1; \\
& \text{and} \\
\end{cases}
\end{align*}
\]

\[
\begin{align*}
h_j(u) = 0, & \quad j = m_1 + 1, m_1 + 2, \ldots, m.
\end{align*}
\]

Under the assumptions that \(f_0, g_i (i = 1, 2, \ldots, m_1)\) and \(h_j (j = m_1 + 1, m_1 + 2, \ldots, m)\) are sufficiently smooth, a well-known first-order necessary optimality condition for the problem \((P)\) (e.g., [3, Page 199]) concludes that, if \(\bar{u} \in \mathbb{R}^n\) solves \((P)\), then there exist \(z_i \geq 0 (i = 0, 1, 2, \ldots, m_1)\) and \(z_j \in \mathbb{R} (j = m_1 + 1, m_1 + 2, \ldots, m)\), not vanishing simultaneously, with the complementary slackness condition that \(z_i g_i(\bar{u}) = 0 (i = 1, 2, \ldots, m_1)\), such that

\[
z_0 f_0'(\bar{u}) + \sum_{i=1}^{m_1} z_i g_i'(\bar{u}) + \sum_{j=m_1+1}^{m} z_j h_j'(\bar{u}) = 0. \tag{1.1}
\]

Here, \(\varphi'(\bar{u})\) is the gradient of the function \(\varphi : \mathbb{R}^n \to \mathbb{R}\) at \(\bar{u}\) for \(\varphi = f_0, g_i\) and \(h_j\), where \(i = 1, 2, \ldots, m_1\) and \(j = m_1 + 1, m_1 + 2, \ldots, m\). The first-order necessary condition (1.1), known as the Fritz John condition, is a commonly used condition in optimization theory. It was first introduced by F. John in [18], initially for the case where \(m_1 = m\). In the condition (1.1), the non-zero vector

\[
z = (z_0, z_1, \ldots, z_{m_1}, z_{m_1+1}, \ldots, z_m)^\top
\]

is referred to as a nontrivial Lagrange multiplier. It is essential to require \(z \neq 0\) in order to ensure the validity of the first-order necessary condition; otherwise the condition (1.1) reduces to \("0=0"\), which does not provide any useful information on the solution \(\bar{u}\).

If \(z_0 = 0\), then the above \(z\) is referred to as a singular Lagrange multiplier. If \(z_0 \neq 0\), then, without loss of generality, we may assume that \(z_0 = 1\); in this case, \(z\) is called a normal Lagrange multiplier, and (1.1) is known as the KKT condition, which was independently derived by W. Karush in [19], and by H. W. Kuhn and A. W. Tucker in [22].

In the general situation with \(V\), \(X\), \(E\), \(f_0\), \(f\) and \(K\) being given at the very beginning of this paper, for simplicity, if similar first-order necessary conditions as above hold, then they are also called the Fritz John condition and the KKT condition, respectively.

The KKT condition is stronger than the Fritz John one because it not only concludes that \(z_0 = 1\), but also contains some information about the function \(f_0\). However, in general, the KKT condition may not hold for the problem \((P)\) (see [3, Page 185] for example). It can be shown that the KKT condition holds true for the problem \((P)\) if and only if

\[
-f_0'(\bar{u}) \in f'(\bar{u})^* E_1 \quad \text{with } E_1 = \{(z_1, z_2, \ldots, z_m)^\top \in \mathbb{R}^m \mid z_i \geq 0, i = 1, 2, \ldots, m_1\}. \tag{1.2}
\]

Here and henceforth, for a bounded linear operator \(F\), we write \(F^*\) for its conjugate operator and therefore, in (1.2), \(f'(\bar{u})^* = (g_1'(\bar{u}), \ldots, g_{m_1}'(\bar{u}), h'_{m_1+1}(\bar{u}), \ldots, h'_m(\bar{u}))\). This means that some constraint qualifications for equality and inequality constraints, such as the Mangasarian-Fromovitz constraint qualification in [30], are required to ensure the KKT condition or, equivalently, the existence of a normal Lagrange multiplier. On the other hand, for the Fritz John condition, no extra condition is required when \(X\) is finite-dimensional, even if \(V\) is an infinite-dimensional Banach space (see [10]).

Generally speaking, there are two main methods to derive the Fritz John condition in optimization theory:
• The first one is known as the separation method, which utilizes the separation theorem in the
range space $X$ of $f$ or the space $\mathbb{R} \times X$ to find the Lagrange multipliers. This method has
been explored in various works such as [6, 23, 34].

• The second one is the penalty function method, which involves introducing proper penalty
functions and constructing a sequence of approximating optimization problems without con-
straints. By taking the limit in the first-order necessary conditions for solutions to these
approximating problems, the desired Lagrange multipliers may be obtained. This method
has been used in the papers [7, 10, 14, 35] and so on.

Compared with the separation method, the penalty function method has its own advantages.
Indeed, the separation method only confirms the existence of Lagrange multipliers, while the penalty
function method provides a way to find a special Lagrange multiplier as a limit of a certain sequence.
As a result, the penalty function method typically offers more information about the Lagrange
multiplier. Further enhanced results of the Fritz John condition can be found in [4, Proposition
5.2.1, Page 284], [5] and [16].

Both of the above methods work well when $X$ is finite-dimensional. Indeed, recall that, by
the separation theorem, for any nonempty convex set in a finite-dimensional space, there always
exists a hyperplane which supports this set at any of its boundary points. Meanwhile, in the
setting of finite dimensions, any sequence on the unit sphere has a subsequence converging to a
non-zero vector. These two facts ensure the existence of nontrivial Lagrange multipliers by the
separation method and the penalty function method, respectively, provided that $X$ is a finite-
dimensional space. Nevertheless, things will become very much different when the range space $X$
of the constraint map $f$ in the problem (P) is infinite-dimensional.

In the context of general infinite-dimensional optimization problem (P), the existence of a
nontrivial Lagrange multiplier cannot be guaranteed without additional assumptions (see [9]). This
arises from the complexity of infinite-dimensional spaces:

• In an infinite-dimensional space, the existence of the support hyperplane for a nonempty con-
 vex set is not guaranteed unless this set satisfies certain additional conditions. For instance,
such a set might be required to have a nonempty relative interior to ensure the aforementioned
existence (see [21]).

• A sequence residing on the unit sphere of an infinite-dimensional space could exhibit a weak
or weak* converging subsequence, potentially culminating in a limit of zero.

To obtain the first-order necessary condition for the constrained infinite-dimensional optimization
problem (P), certain constraint qualifications have been proposed in the previous literatures:

• In order to apply the separation method when the space $X$ is a Banach space, specific sepa-
ration properties of the following set

$$Z(\vec{u}) \triangleq f'(\vec{u})(\mathcal{R}_K(\vec{u})) - \mathcal{R}_E(f(\vec{u}))$$ (1.3)

are proposed, e.g., $Z(\vec{u})$ has a nonempty relative interior or, the closure of the set $Z(\vec{u})$ is
not the whole space $X$. Here and henceforth, $f'(\vec{u})$ is the Fréchet derivative of $f$ at $\vec{u}$, and
$\mathcal{R}_D(v)$ is the radial cone (see Definition 2.2 in the next section) of a set $D \subseteq X$ at $v \in D$.
For more details in this respect we refer to [6, 23, 34, 37] and the references therein.

• By the penalty function method, the existence of nontrivial Lagrange multipliers for solutions
to the problem (P) was proved in [14] under a constraint qualification described in terms of
some property of variation sets on a sequence of approximate optimal solutions (see [14,  
Corollary 6.4.5, Page 267] or Condition (H4) in Remark 2.3 of this paper).
Enhancing the above constraint qualifications may ensure the existence of nontrivial Lagrange multipliers for the infinite-dimensional optimization problem \((P)\). Nevertheless, how to verify these conditions directly poses challenges, especially in optimal control problems, which can be regarded as typical applications of the infinite-dimensional optimization problem \((P)\). For example, consider a special case where \(V\) is a Banach space, \(K = V\) and \(E = \{0\}\). In this scenario, \(Z(\bar{u}) = \text{Im}(f'(\bar{u}))\), where \(\text{Im}(f'(\bar{u}))\) stands for the range of the bounded linear operator \(f'(\bar{u}) : V \to X\). Since \(\text{Im}(f'(\bar{u}))\) is a subspace of \(X\), its relative interior is nonempty if and only if \(\text{Im}(f'(\bar{u}))\) is closed. Despite alternative characterizations for the closedness of \(\text{Im}(f'(\bar{u}))\) exist (e.g., [20, Theorem 5.2, Chapter IV] or Proposition 4.1 in Section 4 of this paper), confirming such a closedness directly in infinite-dimensional spaces is generally quite difficult. In Sections 2–3, we shall show that in some optimal control problems, \(\text{Im}(f'(\bar{u}))\) coincides with the reachable set of a linearized control system, and such a set is sometimes hard to be precisely characterized. We note that, directly verifying the constraint qualification in [14, Corollary 6.4.5, Page 267] poses even a greater challenge, because it involves properties of a sequence of approximate optimal solutions. From an application standpoint, it is valuable to introduce new verifiable sufficient conditions for the existence of nontrivial Lagrange multipliers in the infinite-dimensional optimization problem \((P)\), even if this requires some stronger (but verifiable) conditions than the existing constraint qualifications.

In this paper, we introduce a finite codimensionality condition (which is rooted in functional analysis) to establish an enhanced Fritz John (first-order necessary) condition for the problem \((P)\). Notably, our finite codimensionality condition is equivalent to suitable \textit{a priori} estimate, making it more practical to be verified than the conventional constraint qualifications (see Section 3 for a more detailed explanation). Unlike typical nonlinear optimization problems on topological vector spaces, \(V\) in the problem \((P)\) is assumed to be a complete metric space, and hence our results can be applied to optimal control problems, which are usually lack of linear structural constraints on the control regions.

It is worth noting that some sorts of finite codimensionality conditions have been utilized in [13, 25, 26, 27, 28] and the extensive references therein for establishing the Pontryagin maximum principles for various infinite-dimensional deterministic optimal control problems with state constraints. However, in these references, the corresponding finite codimensionality condition was either assumed directly without any verification process or verified only for some special deterministic infinite-dimensional control systems (as seen in [28]). This paper aims to derive some equivalent characterizations of a finite codimensionality condition for a general setting of infinite-dimensional optimization and provides a unified framework for establishing the first-order necessary conditions for various optimal control problems with state constraints. We notice that the constraint qualification used in [14, Corollary 6.4.5, Page 267] is closely related to our finite codimensionality condition. What sets our condition apart from the one in [14] is that we focus solely on a solution to the original optimization problem, rather than [14, Corollary 6.4.5, Page 267], which is on a sequence of approximate optimal solutions. Moreover, under suitable conditions, verifying the constraint qualification employed in [14] is sufficient by checking the finite codimensionality condition in this paper (see Remark 2.3 for more details).

The rest of this paper is organized as follows. In Section 2, we introduce a finite codimensionality condition to establish an enhanced Fritz John (first-order necessary) condition for the problem \((P)\), with an application to optimal control problems for infinite-dimensional evolution equations. In Section 3, various equivalent characterizations of our finite codimensionality condition are provided. Then, these general results are applied to study three optimal control problems: a deterministic evolution equation with an end-point constraint, an elliptic control system with a pointwise state constraint, and a stochastic control system with an end-point constraint. In Section 4, we present some equivalent \textit{a priori} estimates on closed-range operators and provide some additional perspec-
tives. Finally, proofs of several technical results that appear in the context are given in Appendixes A–C.

2 First-order necessary condition in optimization problems

In this section, we derive a Fritz John (first-order necessary) condition for the problem \((P)\) under a finite codimensionality condition. As an immediate application, this necessary condition is then utilized to analyze optimal control problems associated with certain infinite-dimensional evolution equations.

2.1 Some preliminaries

First, we introduce some notations. Denote by \(d_X(\cdot,\cdot)\) the metric on \(V\). We use \(|\cdot|_X\) to represent the norm of the Banach space \(X\) and \(\langle \cdot, \cdot \rangle_{X',X}\) to represent the dual product between \(X'\) and \(X\).

Especially, when \(X\) is a Hilbert space, we simply write \(\langle \cdot, \cdot \rangle_X\) for an inner product of \(X\). For another Banach space \(Z\), \(\mathcal{L}(X;Z)\) represents the set of all bounded linear operators from \(X\) to \(Z\). For simplicity, we use \(\mathcal{L}(X)\) as a short form for \(\mathcal{L}(X;X)\).

Let \(D\) be a subset of \(X\). Denote by \(\overline{D}\) the closure of \(D\), by \(\text{co}D\) the closed convex hull of \(D\), and by \(\text{Int}D\) the interior of \(D\). For any \(\alpha \in \mathbb{R}\), \(\alpha D \overset{\Delta}{=} \{\alpha x \mid x \in D\}\). The subspace spanned by \(D\) is defined as

\[
\text{span}D \overset{\Delta}{=} \left\{ \sum_{i=1}^{n} \alpha_i x_i \in X \mid \alpha_i \in \mathbb{R}, x_i \in D, i = 1, 2, \cdots, n, n \in \mathbb{N} \right\}.
\]

Write \(\text{span}D\) for the closure of \(\text{span}D\).

For any two subsets \(D_1\) and \(D_2\) of \(X\), put

\[
D_1 \pm D_2 \overset{\Delta}{=} \{x_1 \pm x_2 \in X \mid x_1 \in D_1, x_2 \in D_2\}
\]

and

\[
D_1 \setminus D_2 \overset{\Delta}{=} \{x \in X \mid x \in D_1 \text{ and } x \notin D_2\}.
\]

**Definition 2.1** Let \(X_1\) be a closed subspace of \(X\). A subspace \(X_2\) of \(X\) is said to be a topological complement of \(X_1\), if \(X_1\) is closed, \(X_1 \cap X_2 = \{0\}\) and \(X_1 + X_2 = X\).

Note that if \(X_1\) and \(X_2\) are complementary subspaces of \(X\), then every \(x \in X\) is uniquely written as \(x = x_1 + x_2\) with \(x_1 \in X_1\) and \(x_2 \in X_2\). We define the projection operator \(\Pi_{X_1}\) from \(X\) to the closed subspace \(X_1\) by \(\Pi_{X_1}(x) = x_1\) for any \(x = x_1 + x_2 \in X\) with \(x_1 \in X_1\) and \(x_2 \in X_2\). \(\Pi_{X_2}\) can be defined in a similar way. Especially, when \(X_1\) is a finite-dimensional subspace of \(X\), \(X_1\) has a topological complement and the projection operator \(\Pi_{X_1}\) is well-defined. When \(X\) is a Hilbert space, we denote by \(X_1^\perp\) the orthogonal complement for a subspace \(X_1\) of \(X\).

Denote by \(X \times Z\) the product space of two Banach spaces \(X\) and \(Z\). For a set \(S \subseteq X \times Z\), we set

\[
\pi_{X}(S) \overset{\Delta}{=} \{x \in X \mid (x, z) \in S \text{ for some } z \in Z\}.
\]

\(\pi_{Z}(S)\) can be defined in a similar way.

For \(x_0 \in X\) and \(\rho > 0\), let

\[
B_X(x_0, \rho) \overset{\Delta}{=} \{x \in X \mid |x - x_0|_X \leq \rho\}.
\]
Similarly, we can define $B_{X'}(x'_0, \rho)$ for $x'_0 \in X'$ and set $B_{V}(v_0, \rho) \overset{\Delta}{=} \{ v \in V \mid d_{V}(v, v_0) \leq \rho \}$ for $v_0 \in V$.

Recall that a Banach space $X$ is strictly convex, if any $x, y \in X$ with $|x|_X = |y|_X = 1$ and $x \neq y$ satisfies
\[ \frac{|x + y|}{2} < 1. \]  
(2.1)

Clearly, any Hilbert space is strictly convex.

For two Banach spaces $X$ and $Z$, a map $\varphi : X \to Z$ is said to be directional differentiable at $x$ along a direction $v \in X$, if the following limit exists. Then, the above limit is denoted by $\varphi'(x; v)$ and called the directional derivative of $\varphi$ at $x$ along the direction $v$. In addition, if $\varphi$ is directional differentiable at $x$ along every direction $v \in X$, $\varphi$ is said to be directionally differentiable at $x$. Furthermore, if there is a bounded linear operator $\varphi'(x) \in L(X; Z)$, such that
\[ \lim_{y \to x} \frac{\varphi(y) - \varphi(x) - \varphi'(x)(y - x)}{|y - x|_X} = 0, \]
then $\varphi$ is said to be Fréchet differentiable at $x$ and $\varphi'(x)$ is called the Fréchet derivative of $\varphi$ at $x$. Clearly, when $\varphi$ is Fréchet differentiable at $x$, $\varphi$ is directionally differentiable at $x$ and $\varphi'(x; v) = \varphi'(x)v$ for any $v \in X$. If $\varphi$ is Fréchet differentiable on a neighborhood of $x$ and its Fréchet derivative is continuous (with the operator norm topology) at $x$, $\varphi$ is called continuously differentiable at $x$. When $\varphi$ is directionally differentiable (Fréchet differentiable, continuously differentiable) at any $x \in X$, we simply call $\varphi$ directionally differentiable (Fréchet differentiable, continuously differentiable).

Next, we recall some notions in variational analysis (More details can be found in [1, 6, 14]).

**Definition 2.2** For a nonempty closed convex subset $E$ of $X$ and $e \in E$, the set
\[ \mathcal{R}_E(e) = \bigcup_{\alpha \in (0, \infty)} \alpha(E - e) \]
is called the radial cone of $E$ at $e$. The closure of $\mathcal{R}_E(e)$, denoted by $\mathcal{T}_E(e)$, is called the tangent cone of $E$ at $e$.

**Definition 2.3** A set $\mathcal{N}_E(e) \subseteq X'$ is called the normal cone of $E$ at $e \in E$, if
\[ \mathcal{N}_E(e) = \{ w \in X' \mid \langle w, \bar{e} - e \rangle_{X', X} \leq 0, \ \forall \ \bar{e} \in E \}. \]

It is easy to show that $\mathcal{N}_E(e)$ can be defined equivalently as
\[ \mathcal{N}_E(e) = \{ w \in X' \mid \langle w, v \rangle_{X', X} \leq 0, \ \forall \ v \in \mathcal{T}_E(e) \}. \]

Let $K$ be a subset of $X$. The distance function $\text{dist}(\cdot, K) : X \to \mathbb{R}$ is defined as
\[ \text{dist}(x, K) \overset{\Delta}{=} \inf_{y \in K} |y - x|_X, \ \forall \ x \in X. \]
For any \( x \in X \), we define the metric projection of \( x \) onto \( K \) as

\[
P_K(x) = \{ \bar{e} \in K \mid \text{dist}(x, K) = |x - \bar{e}|_X \}.
\] (2.2)

When \( X \) is a reflexive Banach space and \( K \) is a nonempty closed convex subset of \( X \), \( P_K(x) \) is nonempty for any \( x \in X \). In addition, when \( X \) is strictly convex, by (2.1), \( P_K(x) \) is a singleton.

For a nonempty closed convex subset \( E \) of \( X \), the distance function \( \text{dist}(\cdot, E) \) is Lipschitz continuous and convex on \( X \), and the subdifferential of \( \text{dist}(\cdot, E) \) at \( x \in X \) is defined by

\[
\partial \text{dist}(x, E) = \{ w \in X' \mid \text{dist}(\bar{x}, E) - \text{dist}(x, E) \geq \langle w, \bar{x} - x \rangle_{X'}, \quad \forall \bar{x} \in X \}.
\]

When \( X \) is a reflexive Banach space,

\[
\partial \text{dist}(x, E) = N_E(\bar{e}) \bigcap \{ w \in B_{X'}(0,1) \mid \langle w, x - \bar{e} \rangle_{X',X} = |x - \bar{e}|_X \}, \quad \forall \bar{e} \in P_E(x).
\] (2.3)

Here, \( P_E(x) \) is the metric projection of \( x \) onto \( E \) defined by (2.2). Note that when the set \( P_E(x) \) might not be a singleton, the set on the right side of (2.3) remains the same for different \( \bar{e} \in P_E(x) \). When \( X' \) is strictly convex and \( x \notin E \), \( \partial \text{dist}(x, E) \) is a singleton. In this case, for the unique element \( w \) in \( \partial \text{dist}(x, E) \), \( |w|_X = 1 \). Especially, when \( X \) is a Hilbert space and \( x \notin E \),

\[
\partial \text{dist}(x, E) = \left\{ \frac{x - P_E(x)}{|x - P_E(x)|_X} \right\}.
\] (2.4)

We refer to [6, Example 2.130, Page 89] for more details of \( \partial \text{dist}(\cdot, E) \).

Let us recall the following result for \( \partial \text{dist}(\cdot, E) \).

**Lemma 2.1** [27, Proposition 3.11, Page 146] For any \( x \in X \), the set \( \partial \text{dist}(x, E) \) is a nonempty convex weak\(^*\)-compact subset of \( X' \) and \( \partial \text{dist}(x, E) \subseteq B_{X'}(0,1) \). Moreover, \( \text{dist}(\cdot, E) \) is directional differentiable, and for any \( x, v \in X \),

\[
\text{dist}'(x, E; v) = \max \{ \langle \zeta, v \rangle_{X',X} \in \mathbb{R} \mid \zeta \in \partial \text{dist}(x, E) \},
\] (2.5)

where \( \text{dist}'(x, E; v) \) is the directional derivative of \( \text{dist}(\cdot, E) \) at \( x \) along the direction \( v \).

The following concept of variation is originally from [14, Page 94] and [15, Page 43]).

**Definition 2.4** We call \( \xi \in X \) a variation of \( f : V \to X \) with respect to subset \( K \subseteq V \) at \( e \in K \), if there exists a sequence \( \{ h_k \}_{k=1}^\infty \subseteq (0, +\infty) \) with \( \lim_{k \to \infty} h_k = 0 \) and a sequence \( \{ e_k \}_{k=1}^\infty \subseteq K \), such that

\[
d_V(e_k, e) \leq h_k \quad \text{and} \quad \lim_{k \to \infty} \frac{f(e_k) - f(e)}{h_k} = \xi.
\]

The set of all variations of \( f \) with respect to subset \( K \subseteq V \) at \( e \in K \) is denoted by \( \text{Var}_K f(e) \).

In the definition of the variation in [14, Page 94] and [15, Page 43]), the domain \( D(f) \) of \( f \) may not be the whole metric space \( V \). It is easy to see that the definition of the variation of \( f \) at \( e \in D(f) \) in [14, Page 94] and [15, Page 43]) coincides with Definition 2.4 by taking \( K = D(f) \).

The following example shows that, in some special cases, the variation of \( f \) is closely related to its directional derivative.
Example 2.1 If $V$ is a reflexive Banach space, $f : V \to X$ is locally Lipschitz continuous and directionally differentiable, and $K \subseteq V$ is a closed convex subset, then for any $e \in K$,

$$
\{ f'(e; v) \in X \mid v \in \mathcal{T}_K(e) \cap B_V(0, 1) \} \subseteq \text{Var}_K f(e).
$$

(2.6)

In addition, if $f$ is Fréchet differentiable or $V$ is finite-dimensional, then

$$
\text{Var}_K f(e) = \{ f'(e)v \in X \mid v \in \mathcal{T}_K(e) \cap B_V(0, 1) \}.
$$

(2.7)

As a special case, when $f$ is Fréchet differentiable and $e \in \text{Int} K$,

$$
\text{Var}_K f(e) = \{ f'(e)v \in X \mid v \in B_V(0, 1) \}.
$$




For the convenience of readers, we shall give detailed proofs of (2.6) and (2.7) in Appendix A.

A set-valued map $F : V \rightsquigarrow X$ is characterized by its graph $\text{Gph}(F)$, the subset of the product space $V \times X$ defined by $\text{Gph}(F) := \{ (e, \xi) \in V \times X \mid \xi \in F(e) \}$. We will use the symbol $F(e)$ to denote the value of $F$ at $e$, which is a subset of $X$. The domain of $F$ is the subset of elements $e \in V$ such that $F(e)$ is nonempty, i.e., $\mathcal{D}(F) := \{ e \in V \mid F(e) \neq \emptyset \}$.

Definition 2.5 A set-valued map $F : V \rightsquigarrow X$ is called locally bounded at $e_0 \in \mathcal{D}(F)$, if there are two positive constants $\delta$ and $C$, such that for any $e \in \mathcal{D}(F)$ with $d_V(e, e_0) \leq \delta$ and any $\xi \in F(e)$, it holds that $|\xi|_X \leq C$.

Remark 2.1 If $f : V \to X$ is locally Lipschitz continuous at $e_0 \in K$ in the sense that

$$
|f(e_1) - f(e_2)|_X \leq L d_V(e_1, e_2), \quad \forall e_1, e_2 \in V \text{ with } d_V(e_1, e_0) \leq \delta \text{ and } d_V(e_2, e_0) \leq \delta,
$$

for two positive constants $L$ and $\delta$, then the set-valued map $\text{Var}_K f(\cdot)$ is locally bounded at $e_0$.

Indeed, for any $e \in K$ with $d_V(e, e_0) \leq \frac{\delta}{2}$ and any $\xi \in \text{Var}_K f(e)$, it follows from Definition 2.4 that, there exists a sequence $\{ h_k \}_{k=1}^{\infty} \subseteq (0, +\infty)$ with $\lim_{k \to \infty} h_k = 0$ and a sequence $\{ e_k \}_{k=1}^{\infty} \subseteq K$ with

$$
d_V(e_k, e) \leq h_k,
$$

such that

$$
\xi = \lim_{k \to \infty} \frac{f(e_k) - f(e)}{h_k}.
$$

When $k$ is large enough, we have $d_V(e_k, e_0) \leq d_V(e_k, e) + d_V(e, e_0) \leq \delta$. Then, by the locally Lipschitz continuity of $f$ at $e_0$,

$$
|\xi|_X = \left| \lim_{k \to \infty} \frac{f(e_k) - f(e)}{h_k} \right|_X \leq \lim_{k \to \infty} \frac{L d_V(e_k, e)}{h_k} \leq L.
$$

At the end of this subsection, we recall some concepts concerning the finite codimensionality (See, e.g., [8, 27]).

Definition 2.6 A linear subspace $X_1$ of $X$ is called finite codimensional\footnote{In [8, Page 351], a linear subspace $X_1$ is called finite codimensional, if there exists $m \in \mathbb{N}$ and linearly independent $x_1, x_2, \cdots, x_m \in X \setminus X_1$, such that $X_1 + \text{span}\{ x_1, x_2, \cdots, x_m \} = X$. Clearly, the definition in [8] is a pure algebraic concept and it only involves linear operations. Throughout this paper, we use Definition 2.6 as the definition of finite codimensionality for the linear subspace $X_1$, which involves the topological closure of $X_1$. When $X_1$ is closed, Definition 2.6 coincides with that in [8].}, if there exists $m \in \mathbb{N}$ and linearly independent $x_1, x_2, \cdots, x_m \in X \setminus X_1$, such that

$$
\overline{X_1} + \text{span}\{ x_1, x_2, \cdots, x_m \} = X.
$$

\[8\]
Definition 2.7 A subset $D$ of $X$ is called finite codimensional in $X$, if there exists $x_0 \in \overline{D}$ such that

(i) $\overline{\text{span}}\{D - x_0\}$ is a finite codimensional subspace of $X$; and

(ii) $\overline{\text{co}}\{D - x_0\}$ has at least one interior point in $\overline{\text{span}}\{D - x_0\}$.

The following result shows that the finite codimensionality of the subset $D$ is independent of the choice of $x_0 \in \overline{D}$.

Proposition 2.1 If the conditions (i) and (ii) stated in Definition 2.7 hold for some $x_0 \in \overline{D}$, then these conditions hold for all $x_0 \in \overline{D}$.

Proposition 2.1 can be deduced from [27, Proposition 3.1, Page 137]. For the readers’ convenience, we shall give a proof of Proposition 2.1 in Appendix B.

The following result is crucial for the main result of this paper.

Lemma 2.2 ([27, Lemma 3.6, Page 142]) Let $D$ be a finite codimensional subset in $X$ and $\{\varepsilon_k\}_{k=1}^{\infty}$ be a nonnegative sequence such that $\lim_{k \to \infty} \varepsilon_k = 0$. Assume that $\{\Lambda_k\}_{k=1}^{\infty} \subseteq X'$ and $\Lambda_k$ converges weakly* to $\Lambda \in X'$ as $k \to \infty$. If

(i) $|\Lambda_k|_{X'} \geq \delta$ for some $\delta > 0$, $\forall k \in \mathbb{N}$; and

(ii) $\langle \Lambda_k, x \rangle_{X',X} \geq -\varepsilon_k$, $\forall (x,k) \in D \times \mathbb{N}$,

then $\Lambda \neq 0$.

For the readers’ convenience, a detailed proof of Lemma 2.2 will be given in Appendix C.

2.2 First-order necessary conditions for the problem (P)

In this subsection, we shall discuss the first-order necessary conditions for the problem (P). We call $\bar{u} \in \mathcal{K}$ a solution to the problem (P), if

$$f_0(\bar{u}) = \min \{ f_0(u) \in \mathbb{R} \mid f(u) \in E \text{ and } u \in \mathcal{K} \}.$$ 

In what follows, we assume that there exists a set-valued map $\mathcal{V} : \mathcal{K} \rightrightarrows \mathbb{R} \times X$ satisfying $\mathcal{V}(u) \subseteq \text{Var}_{\mathcal{K}}(f_0,f)(u)$ for any $u \in \mathcal{K}$ and a modulus of continuity $\ell : [0,\infty) \to [0,\infty)$ with $\lim_{s \to 0^+} \ell(s) = 0$, such that

(H$_1$) For any $(\xi_1, \xi_2)^\top \in \mathcal{V}(\bar{u})$ and $u^\varepsilon \in \mathcal{K}$ with $\mathbf{d}_V(\bar{u}, u^\varepsilon) \to 0$ as $\varepsilon \to 0^+$, there is $(\xi_1^\varepsilon, \xi_2^\varepsilon)^\top \in \mathcal{V}(u^\varepsilon)$, such that $\xi_1^\varepsilon \to \xi_1$ in $\mathbb{R}$ as $\varepsilon \to 0^+$, and for small enough $\varepsilon$,

$$|\xi_2 - \xi_2^\varepsilon|_X \leq \ell(\mathbf{d}_V(\bar{u}, u^\varepsilon));$$

(2.8)

(H$_2$) $\pi_{\mathbb{R}}(\mathcal{V}(\cdot))$ is locally bounded at $\bar{u}$;

(H$_3$) $\pi_X(\mathcal{V}(\bar{u})) - E$ is finite codimensional in $X$.

The main result of this subsection is the following necessary optimality condition for the problem (P).
Theorem 2.1 Let $X$ be a reflexive Banach space with $X'$ being strictly convex, $\bar{u} \in \mathbb{K}$ be a solution to the problem $(P)$ with $\bar{e} = f(\bar{u})$. If the conditions $(H_1)$-$(H_3)$ are satisfied, then there is a non-zero pair $(z_0, z) \in [0, +\infty) \times X'$ with $z \in N_E(\bar{e})$, such that the following Fritz John condition holds:

$$z_0\xi_1 + \langle z, \xi_2 \rangle_{X', X} \geq 0, \quad \forall \ (\xi_1, \xi_2)^\top \in \mathcal{V}(\bar{u}).$$

Moreover, the following two enhanced results hold true:

(i) If $z_0 \neq 0$, then one has the following KKT condition

$$\xi_1 + \langle \bar{z}, \xi_2 \rangle_{X', X} \geq 0, \quad \forall \ (\xi_1, \xi_2)^\top \in \mathcal{V}(\bar{u})$$

with $\bar{z} = z/z_0$;

(ii) If $z \neq 0$, then there exists a sequence $\{u^k\}_{k=1}^\infty \subseteq \mathbb{K}$, converging to $\bar{u}$ as $k \to \infty$, such that

$$f(u^k) \not\in E \text{ for } k \text{ being large enough},$$

$$\lim_{k \to \infty} \text{dist}(f(u^k), E) = 0,$$

$$\lim_{k \to \infty} f_0(u^k) = f_0(\bar{u}).$$

Furthermore, when $X$ is a Hilbert space, the sequence $\{P_E(f(u^k))\}_{k=1}^\infty$ satisfies that $P_E(f(u^k)) \to f(\bar{u})$ as $k \to \infty$ and

$$(\bar{z}, f(u^k) - P_E(f(u^k)))_X > 0, \quad \forall \ k \in \mathbb{N},$$

where $\bar{z} \in X$ is the element corresponding to $z \in X'$ by the Riesz-Frèchet isomorphism.

Proof. The proof is divided into four steps.

Step 1. For any $\varepsilon \in (0, 1)$ and $u \in \mathbb{K}$, set

$$\Phi_{\varepsilon}(u) = \left\{ \left[ \text{dist}(f(u), E) \right]^2 + \left[ (f_0(u) - f_0(\bar{u}) + \varepsilon)^+ \right]^2 \right\}^{1/2}$$

and $F(u) = (f_0(u), f(u))^\top$, where $s^+ = s$ if $s \geq 0$ and $s^+ = 0$ if $s < 0$. Then $\Phi_{\varepsilon}(\cdot)$ is continuous on $\mathbb{K}$, $\Phi_{\varepsilon}(u) > 0$ for any $u \in \mathbb{K}$ and

$$\Phi_{\varepsilon}(\bar{u}) = \varepsilon \leq \inf_{u \in \mathbb{K}} \Phi_{\varepsilon}(u) + \varepsilon.$$

Since $\mathbb{K}$ is a complete metric space with the metric $d_Y(\cdot, \cdot)$, by the Ekeland variational principle, there exists $u^\varepsilon \in \mathbb{K}$ such that

$$\Phi_{\varepsilon}(u^\varepsilon) \leq \Phi_{\varepsilon}(\bar{u}), \quad d_Y(\bar{u}, u^\varepsilon) \leq \sqrt{\varepsilon},$$

and

$$-\sqrt{\varepsilon}d_Y(u^\varepsilon, u) \leq \Phi_{\varepsilon}(u) - \Phi_{\varepsilon}(u^\varepsilon), \quad \forall \ u \in \mathbb{K}.$$ (2.12)

Step 2. For any $\xi^\varepsilon = (\xi_1^\varepsilon, \xi_2^\varepsilon)^\top \in \mathcal{V}(u^\varepsilon)$, by Definition 2.4, there exists a sequence $\{h_k^\varepsilon\}_{k=1}^\infty \subseteq (0, +\infty)$ with $\lim_{k \to \infty} h_k^\varepsilon = 0$ and a sequence $\{u_k^\varepsilon\}_{k=1}^\infty \subseteq \mathbb{K}$, such that

$$d_Y(u_k^\varepsilon, u^\varepsilon) \leq h_k^\varepsilon$$ (2.13)

and

$$F(u_k^\varepsilon) = F(u^\varepsilon) + h_k^\varepsilon \xi^\varepsilon + o(h_k^\varepsilon), \quad \text{as } k \to \infty,$$ (2.14)
where \( o(h_k^\varepsilon) = \left( o_1(h_k^\varepsilon), o_2(h_k^\varepsilon) \right)^T \) denotes a higher order infinitesimal of \( h_k^\varepsilon \). It follows from (2.13) and (2.14) that

\[
\Phi_\varepsilon(u_k^\varepsilon) - \Phi_\varepsilon(u^\varepsilon) = \left[ \left( f_0(u_k^\varepsilon) - f_0(u) + \varepsilon \right)^+ \right]^2 - \left[ \left( f_0(u^\varepsilon) - f_0(u) + \varepsilon \right)^+ \right]^2
\]

\[
= \frac{\left[ \left( f_0(u^\varepsilon) + h_k^\varepsilon \xi_1^\varepsilon + o_1(h_k^\varepsilon) - f_0(u) + \varepsilon \right)^+ \right]^2 - \left[ \left( f_0(u^\varepsilon) - f_0(u) + \varepsilon \right)^+ \right]^2}{\Phi_\varepsilon(u_k^\varepsilon) + \Phi_\varepsilon(u^\varepsilon)} + \frac{\left[ \text{dist}(f(u_k^\varepsilon), E) \right]^2 - \left[ \text{dist}(f(u^\varepsilon), E) \right]^2}{\Phi_\varepsilon(u_k^\varepsilon) + \Phi_\varepsilon(u^\varepsilon)}.
\]

Clearly,

\[
\lim_{k \to \infty} \frac{\left[ \left( f_0(u^\varepsilon) + h_k^\varepsilon \xi_1^\varepsilon + o_1(h_k^\varepsilon) - f_0(u) + \varepsilon \right)^+ \right]^2 - \left[ \left( f_0(u^\varepsilon) - f_0(u) + \varepsilon \right)^+ \right]^2}{\Phi_\varepsilon(u_k^\varepsilon) + \Phi_\varepsilon(u^\varepsilon)} = \left( f_0(u^\varepsilon) - f_0(u) + \varepsilon \right)^+ \xi_1^\varepsilon.
\]

In addition, by the Lipschitz continuity of distance function and Lemma 2.1, we get that

\[
\lim_{k \to \infty} \frac{\text{dist}(f(u^\varepsilon) + h_k^\varepsilon \xi_2^\varepsilon + o_2(h_k^\varepsilon), E) - \text{dist}(f(u^\varepsilon), E)}{h_k^\varepsilon} = \lim_{k \to \infty} \frac{\text{dist}(f(u^\varepsilon) + h_k^\varepsilon \xi_2^\varepsilon, E) - \text{dist}(f(u^\varepsilon), E)}{h_k^\varepsilon} = \max \left\{ \langle \psi, \xi_2^\varepsilon \rangle \mid \psi \in \partial \text{dist}(f(u^\varepsilon), E) \right\}.
\]

Then, by the weak* compactness of \( \partial \text{dist}(f(u^\varepsilon), E) \) (the subdifferential of \( \text{dist}(\cdot, E) \) at \( f(u^\varepsilon) \)), there is \( \psi_2^\varepsilon \in \partial \text{dist}(f(u^\varepsilon), E) \), such that

\[
\lim_{k \to \infty} \frac{\text{dist}(f(u^\varepsilon) + h_k^\varepsilon \xi_2^\varepsilon + o_2(h_k^\varepsilon), E) - \text{dist}(f(u^\varepsilon), E)}{h_k^\varepsilon} = \langle \psi_2^\varepsilon, \xi_2^\varepsilon \rangle.
\]

Therefore,

\[
\lim_{k \to \infty} \frac{\left[ \text{dist}(f(u^\varepsilon) + h_k^\varepsilon \xi_2^\varepsilon + o_2(h_k^\varepsilon), E) \right]^2 - \left[ \text{dist}(f(u^\varepsilon), E) \right]^2}{h_k^\varepsilon \left[ \Phi_\varepsilon(u_k^\varepsilon) + \Phi_\varepsilon(u^\varepsilon) \right]} = \lim_{k \to \infty} \left[ \frac{\text{dist}(f(u^\varepsilon) + h_k^\varepsilon \xi_2^\varepsilon + o_2(h_k^\varepsilon), E) + \text{dist}(f(u^\varepsilon), E)}{\Phi_\varepsilon(u_k^\varepsilon) + \Phi_\varepsilon(u^\varepsilon)} \times \frac{\text{dist}(f(u^\varepsilon) + h_k^\varepsilon \xi_2^\varepsilon + o_2(h_k^\varepsilon), E) - \text{dist}(f(u^\varepsilon), E)}{h_k^\varepsilon} \right]
\]

\[
= \frac{\text{dist}(f(u^\varepsilon), E) \langle \psi_2^\varepsilon, \xi_2^\varepsilon \rangle}{\Phi_\varepsilon(u^\varepsilon)}.
\]
On the other hand, by (2.12),
\[ \Phi_\varepsilon(u_k^\varepsilon) - \Phi_\varepsilon(u^\varepsilon) \geq -\sqrt{\varepsilon} h_k^\varepsilon. \]
Therefore,
\[ \lim_{k \to \infty} \frac{\Phi_\varepsilon(u_k^\varepsilon) - \Phi_\varepsilon(u^\varepsilon)}{h_k^\varepsilon} = \frac{(f_0(u^\varepsilon) - f_0(\bar{u}) + \varepsilon) + \text{dist}(f(u^\varepsilon), E)\langle \psi_2^\varepsilon, \xi_2^\varepsilon \rangle_{X',X}}{\Phi_\varepsilon(u^\varepsilon)} \geq -\sqrt{\varepsilon}, \] (2.15)
for some \( \psi_2^\varepsilon \in \partial\text{dist}(f(u^\varepsilon), E) \).

If \( f(u^\varepsilon) \in E \), then \( \text{dist}(f(u^\varepsilon), E) = 0 \). Otherwise, if \( f(u^\varepsilon) \not\in E \), by the strict convexity of \( X' \), \( \partial\text{dist}(f(u^\varepsilon), E) \) is a singleton and as the unique element \( \psi_2^\varepsilon \) in \( \partial\text{dist}(f(u^\varepsilon), E) \), \( |\psi_2^\varepsilon|_{X'} = 1 \). Therefore, by (2.15), in both cases, we can always find a \( \psi_\varepsilon \in \partial\text{dist}(f(u^\varepsilon), E) \) with \( |\psi_\varepsilon|_{X'} = 1 \) such that
\[ \frac{(f_0(u^\varepsilon) - f_0(\bar{u}) + \varepsilon) + \text{dist}(f(u^\varepsilon), E)\langle \psi_\varepsilon, \xi_\varepsilon^\varepsilon \rangle_{X',X}}{\Phi_\varepsilon(u^\varepsilon)} \geq -\sqrt{\varepsilon}, \quad \forall (\xi_1^\varepsilon, \xi_2^\varepsilon)^\top \in \mathcal{V}(u^\varepsilon). \]

Write
\[ a_\varepsilon = \frac{(f_0(u^\varepsilon) - f_0(\bar{u}) + \varepsilon) + \text{dist}(f(u^\varepsilon), E)\psi_\varepsilon}{\Phi_\varepsilon(u^\varepsilon)} \quad \text{and} \quad b_\varepsilon = \frac{\text{dist}(f(u^\varepsilon), E)\psi_\varepsilon}{\Phi_\varepsilon(u^\varepsilon)}. \] (2.16)

Then,
\[ -\sqrt{\varepsilon} \leq a_\varepsilon \xi_1^\varepsilon + \langle b_\varepsilon, \xi_\varepsilon^\varepsilon \rangle_{X',X}, \quad \forall (\xi_1^\varepsilon, \xi_2^\varepsilon)^\top \in \mathcal{V}(u^\varepsilon). \] (2.17)

Noting that \( \psi_\varepsilon \in \partial\text{dist}(f(u^\varepsilon), E) \), we have
\[ \langle b_\varepsilon, x - f(u^\varepsilon) \rangle_{X',X} \leq 0, \quad \forall x \in E. \] (2.18)

Moreover, by the definition of \( \Phi_\varepsilon(u^\varepsilon) \), \( a_\varepsilon \) and \( b_\varepsilon \), and the fact that \( |\psi_\varepsilon|_{X'} = 1 \), we have that
\[ a_\varepsilon^2 + |b_\varepsilon|^2_{X'} = 1. \]
Consequently, there exists a subsequence \( \{a_{\varepsilon_k}, b_{\varepsilon_k}\}_{k=1}^{\infty} \) of \( \{a_\varepsilon, b_\varepsilon\}_{\varepsilon > 0} \) and a pair \( (z_0, z) \in [0, +\infty) \times X' \), such that \( a_{\varepsilon_k} \to z_0 \) in \( \mathbb{R} \) and \( b_{\varepsilon_k} \to z \) weakly* in \( X' \), as \( k \to \infty \).

Furthermore, by \((H_1)\), for any \( (\xi_1, \xi_2)^\top \in \mathcal{V}(\bar{u}) \), there exists a pair \( (\xi_1^\varepsilon_k, \xi_2^\varepsilon_k)^\top \in \mathcal{V}(u^\varepsilon_k) \) such that \( \xi_1^\varepsilon_k \to \xi_1 \) in \( \mathbb{R} \) as \( k \to \infty \) and for sufficiently large \( k \),
\[ |\xi_2 - \xi_2^\varepsilon_k|_X \leq \ell(d_{\mathcal{V}}(\bar{u}, u^\varepsilon_k)). \] (2.19)

Then, by (2.17) and (2.18),
\[ z_0 \xi_1 + \langle z, \xi_2 \rangle_{X',X} \geq 0, \quad \forall (\xi_1, \xi_2)^\top \in \mathcal{V}(\bar{u}) \]
and
\[ \langle z, x - \varepsilon \rangle_{X',X} \leq 0, \quad \forall x \in E. \]

This, together with Definition 2.3, implies that \( z \in \mathcal{N}_E(\bar{e}) \).

**Step 3.** Now, we prove that the pair \( (z_0, z) \) is not zero. If \( z_0 = 0 \), by (2.18) and (2.17), for any \( x \in E, \xi_2 \in \pi_X(\mathcal{V}(\bar{u})) \) and \( (\xi_1^\varepsilon_k, \xi_2^\varepsilon_k)^\top \in \mathcal{V}(u^\varepsilon_k) \), it follows that
\[ \langle b_{\varepsilon_k}, \xi_2 - (x - \varepsilon) \rangle_{X',X} \]
\[ = \langle b_{\varepsilon_k}, \xi_2 \rangle_{X',X} + \langle b_{\varepsilon_k}, f(u^\varepsilon_k) - x \rangle_{X',X} + \langle b_{\varepsilon_k}, f(u^\varepsilon_k) - f(\bar{u}) \rangle_{X',X} \]
\[ \geq \langle b_{\varepsilon_k}, \xi_2^\varepsilon_k \rangle_{X',X} + \langle b_{\varepsilon_k}, \xi_2 - \xi_2^\varepsilon_k \rangle_{X',X} + \langle b_{\varepsilon_k}, f(u^\varepsilon_k) - f(\bar{u}) \rangle_{X',X} \]
\[ \geq -\sqrt{\varepsilon} - a_{\varepsilon_k} \xi_1^\varepsilon_k + \langle b_{\varepsilon_k}, \xi_2 - \xi_2^\varepsilon_k \rangle_{X',X} + \langle b_{\varepsilon_k}, f(u^\varepsilon_k) - f(\bar{u}) \rangle_{X',X}, \] (2.20)
where
\[ a_{\varepsilon k} = \frac{[f_0(u^{\varepsilon k}) - f_0(\bar{u}) + \varepsilon_k]}{\Phi_{\varepsilon k}(u^{\varepsilon k})} \quad \text{and} \quad b_{\varepsilon k} = \frac{\text{dist}(f(u^{\varepsilon k}), E) \psi_{\varepsilon k}}{\Phi_{\varepsilon k}(u^{\varepsilon k})}. \]

Further, by (H₂) and (H₁), there exists \((\xi_1^{\varepsilon k}, \xi_2^{\varepsilon k})^T \in V(u^{\varepsilon k})\) such that, for sufficiently large \(k\),
\[ |\xi_1^{\varepsilon k}| \leq C \quad \text{and} \quad |\xi_2 - \xi_2^{\varepsilon k}|_X \leq \ell(V(\bar{u}, u^{\varepsilon k})). \]

Here, \(C\) is a positive constant independent of \(k\), and \(\ell\) is the modulus of continuity in (H₁).

Then, by (2.20),
\[
\begin{align*}
\langle b_{\varepsilon k}, \xi_2 - (x - \bar{e}) \rangle_{X', X} \\
\geq -\sqrt{\varepsilon_k} - C a_{\varepsilon k} - |\xi_2 - \xi_2^{\varepsilon k}|_X - |f(u^{\varepsilon k}) - f(\bar{u})|_X \\
\geq -\sqrt{\varepsilon_k} - C a_{\varepsilon k} - \ell(V(\bar{u}, u^{\varepsilon k})) - |f(u^{\varepsilon k}) - f(\bar{u})|_X.
\end{align*}
\]  

Define \(\delta_{\varepsilon k} = \sqrt{\varepsilon_k} + C a_{\varepsilon k} + \ell(V(\bar{u}, u^{\varepsilon k})) + |f(u^{\varepsilon k}) - f(\bar{u})|_X\). Then, \(\delta_{\varepsilon k} > 0\) and \(\lim_{k \to \infty} \delta_{\varepsilon k} = 0\). By Lemma 2.2 and the finite codimensionality of \(\pi_X(V(\bar{u})) - E\) in (H₃), we obtain \(z = 0\).

**Step 4.** Finally, we prove the enhanced results (i) and (ii). Clearly, \(z_0 \geq 0\). When \(z_0 \neq 0\), the conclusion (i) follows from dividing by \(z_0\) on both sides of the inequality (2.9).

Now, consider the case that \(z \neq 0\). Assume that \(\{u^{\varepsilon k}\}_{k=1}^{\infty}, \{a_{\varepsilon k}\}_{k=1}^{\infty}, \{b_{\varepsilon k}\}_{k=1}^{\infty}\) and \(\{\varepsilon_k\}_{k=1}^{\infty}\) are, respectively, the subsequences given in Steps 1 and 2, such that
\[ d_V(\bar{u}, u^{\varepsilon k}) \leq \sqrt{\varepsilon_k}, \]
and as \(k \to \infty\),
\[
\begin{align*}
\{a_{\varepsilon k}\} &\to z_0 \quad \text{in} \quad \mathbb{R}, \\
\{b_{\varepsilon k}\} &\to z \quad \text{weakly* in} \quad X', \\
\varepsilon_k &\to 0 \quad \text{in} \quad \mathbb{R}.
\end{align*}
\]

By the continuity of \(f_0\) and \(f\), we find that, \(f_0(u^{\varepsilon k}) \to f_0(\bar{u})\) in \(\mathbb{R}\) and \(f(u^{\varepsilon k}) \to f(\bar{u})\) in \(X\) as \(k \to \infty\). Since \(z \neq 0\), by the weakly* lower semicontinuity of \(|\cdot|_{X'}\), we have
\[
\lim_{k \to \infty} |b_{\varepsilon k}|_{X'} = \lim_{k \to \infty} \left| \frac{\text{dist}(f(u^{\varepsilon k}), E) \psi_{\varepsilon k}}{\Phi_{\varepsilon k}(u^{\varepsilon k})} \right|_{X'} \geq |z|_{X'} > 0,
\]
which implies that \(\text{dist}(f(u^{\varepsilon k}), E) > 0\) for sufficiently large \(k \in \mathbb{N}\). Therefore, \(f(u^{\varepsilon k}) \notin E\) when \(k\) is large enough.

Furthermore, assume that \(X\) is a Hilbert space. Then, \(\mathcal{P}_E(f(u^{\varepsilon k}))\) is a singleton and
\[
\lim_{k \to \infty} |\mathcal{P}_E(f(u^{\varepsilon k})) - f(\bar{u})|_X \leq \lim_{k \to \infty} |\mathcal{P}_E(f(u^{\varepsilon k})) - f(u^{\varepsilon k})|_X + \lim_{k \to \infty} |f(u^{\varepsilon k}) - f(\bar{u})|_X
\]
\[
= \lim_{k \to \infty} \text{dist}(f(u^{\varepsilon k}), E) + \lim_{k \to \infty} |f(u^{\varepsilon k}) - f(\bar{u})|_X = 0.
\]

By (2.4), when \(k\) is sufficiently large,
\[
b_{\varepsilon k} = \frac{\text{dist}(f(u^{\varepsilon k}), E)}{\Phi_{\varepsilon k}(u^{\varepsilon k})} \times \frac{f(u^{\varepsilon k}) - \mathcal{P}_E(f(u^{\varepsilon k}))}{|f(u^{\varepsilon k}) - \mathcal{P}_E(f(u^{\varepsilon k}))|_X} = \frac{f(u^{\varepsilon k}) - \mathcal{P}_E(f(u^{\varepsilon k}))}{\Phi_{\varepsilon k}(u^{\varepsilon k})},
\]

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where \( \hat{b}_{\varepsilon_k} \in X \) is the element corresponding to \( b_{\varepsilon_k} \in X' \) by the Riesz-Frédéch isomorphism. Then,

\[
\hat{b}_{\varepsilon_k} \to \hat{z} \text{ weakly in } X \text{ as } k \to \infty,
\]

and hence,

\[
\lim_{k \to \infty} \left\langle z, \frac{f(u^{\varepsilon_k}) - P_E(f(u^{\varepsilon_k}))}{\Phi_{\varepsilon_k}(u^{\varepsilon_k})} \right\rangle_{X',X} = \lim_{k \to \infty} \left\langle \hat{z}, \frac{f(u^{\varepsilon_k}) - P_E(f(u^{\varepsilon_k}))}{\Phi_{\varepsilon_k}(u^{\varepsilon_k})} \right\rangle_X = \lim_{k \to \infty} (\hat{z}, \hat{b}_{\varepsilon_k})_X = |\hat{z}|_X^2 = |\hat{z}|_{X'}^2 > 0,
\]

where \( \hat{z} \in X \) is the element corresponding to \( z \in X' \) by the Riesz-Frédéch isomorphism. Then, we conclude that for \( k \in \mathbb{N} \) large enough,

\[
\left( \hat{z}, \frac{f(u^{\varepsilon_k}) - P_E(f(u^{\varepsilon_k}))}{\Phi_{\varepsilon_k}(u^{\varepsilon_k})} \right)_X > 0.
\]

Hence, the conclusion (2.11) follows from that \( \Phi_{\varepsilon_k}(u^{\varepsilon_k}) > 0 \) for any \( k \in \mathbb{N} \). This completes the proof of Theorem 2.1.

In the following remark, we give a comparison of the conclusions in Theorem 2.1 with some known results for the problem (P).

**Remark 2.2** If \( V \) is a reflexive Banach space, \( K \) is a nonempty closed convex subset of \( V \), and \( f_0 \) and \( f \) are continuously differentiable, then by Example 2.1, for any \( u \in K \), we have

\[
\text{Var}_K(f_0, f)(u) = \left\{ (f_0'(u)v, f'(u)v) \in \mathbb{R} \times X \mid v \in T_K(u) \cap B_V(0,1) \right\}.
\]

Let us choose \( \mathcal{V}(\cdot) = \text{Var}_K(f_0, f)(\cdot) \) and assume that the conditions in Theorem 2.1 hold true. Then, the first-order necessary condition in Theorem 2.1 can be expressed as

\[
\langle z_0 f_0'(\bar{u}) + f'((\bar{u})^* z), v \rangle_{\mathcal{V}', \mathcal{V}} \geq 0, \quad \forall v \in T_K(\bar{u}) \cap B_V(0,1),
\]

or equivalently,

\[
-z_0 f_0'(\bar{u}) - f'((\bar{u})^* z) \in \mathcal{N}_K(\bar{u}).
\]

If \( K = V \), then \( T_K(u) = V \) and \( \mathcal{N}_K(u) = \{0\} \) for any \( u \in V \). In this case, the first-order necessary condition for \( \bar{u} \) is specialized as

\[
z_0 f_0'(\bar{u}) + f'((\bar{u})^* z) = 0.
\]

Hence, the first-order necessary condition in Theorem 2.1 extends the classical Fritz John condition (see [6, Page 153]) for optimization problems to the setting of complete metric spaces. Furthermore, the enhanced results (i) and (ii) of Theorem 2.1 extend the known enhanced Fritz John conditions (e.g., [5, 16]) to the case that the range of the constraint map \( f \) belongs to an infinite-dimensional space.

In many optimal control problems, control regions are genuinely some metric spaces. It is impossible to define the directional derivatives or some more general directional derivatives in nonsmooth analysis for \( f_0 \) and \( f \) with respect to the control variable. However, variations of \( f_0 \) and \( f \) make sense in these scenarios. The broader applicability of first-order necessary conditions represented by variations, compared to those represented by directional derivatives, is an advantage for the problem (P).
On the other hand, characterizing the set-valued map $\text{Var}_K(f_0, f)(\cdot)$ precisely might be quite difficult in some concrete problems. In such cases, having another set-valued map $\mathcal{V}(\cdot)$ so that $\mathcal{V}(\cdot) \subseteq \text{Var}_K(f_0, f)(\cdot)$ may offer more flexibility and convenience. This approach is particularly useful in the situations where obtaining $\mathcal{V}(\cdot)$ is more straightforward than determining completely $\text{Var}_K(f_0, f)(\cdot)$. An illustrative example demonstrating the application of Theorem 2.1 to an optimal control problem with end-point constraints is presented in Subsection 2.3.

In what follows, several examples and remarks are given for the conditions (H_1)-(H_3) in Theorem 2.1.

First, the condition (H_1) in Theorem 2.1 plays a crucial role. As we shall see below, [7, Example 3.5] may serve as a counterexample, illustrating that the Fritz John condition may fail without (H_1), even if both (H_2) and (H_3) are satisfied.

**Example 2.2** Let us recall [7, Example 3.5]. Set $V = \mathbb{R} \times L^2(0,1)$, $X = L^2(0,1)$, $E = \{0\}$, $K = \mathbb{R} \times \{v \in L^2(0,1) \mid -1 \leq v(t) \leq 1, \text{ a.e. } t \in (0,1)\}$,

$$f_0(\alpha, v) = -\alpha \quad \text{and} \quad f(\alpha, v) = \alpha q - v, \quad \forall (\alpha, v) \in V,$$

where $q \in L^2(0,1) \setminus L^\infty(0,1)$ is a given function. Clearly, $(0, 0) \in \mathbb{R} \times L^2(0,1)$ is the unique feasible point, that is, $f(\alpha, v) \in E$ if and only if $(\alpha, v) = (0, 0)$. Therefore, $\bar{u} = (0, 0)$ is the unique solution to the corresponding optimization problem $(P)$.

In this example, $f_0$ and $f$ are continuously differentiable. Then, by Example 2.1, for any $(\alpha, v) \in K$,

$$\pi_X(\text{Var}_K(f_0, f)(\alpha, v)) = \{ -\beta \in \mathbb{R} \mid (\beta, w) \in \mathcal{T}_K(\alpha, v), |\beta| + |w|_{L^2(0,1)} \leq 1 \}$$

and

$$\pi_X(\text{Var}_K(f_0, f)(\alpha, v)) = \{ \beta q - w \in L^2(0,1) \mid (\beta, w) \in \mathcal{T}_K(\alpha, v), |\beta| + |w|_{L^2(0,1)} \leq 1 \}.$$  

(2.23)

By Definition 2.2, it can be observed that $\mathcal{T}_K(0, 0) = V = \mathbb{R} \times L^2(0,1)$. Therefore,

$$\pi_X(\text{Var}_K(f_0, f)(0, 0)) = \{ \beta q - w \in L^2(0,1) \mid |\beta| + |w|_{L^2(0,1)} \leq 1 \}.$$  

(2.24)

In Theorem 2.1, if we choose $\mathcal{V}(\cdot) = \text{Var}_K(f_0, f)(\cdot)$, (2.22) implies the local boundedness of $\pi_X(\text{Var}_K(f_0, f)(\cdot))$ at $\bar{u}$ and (2.24) implies the finite codimensionality of the set $\pi_X(\text{Var}_K(f_0, f)(\bar{u}))$. Hence, the conditions (H_2) and (H_3) are satisfied. However, it has been shown in [7] that the Fritz John condition in this example fails.

In what follows, we prove that (2.8) in the condition (H_1) fails in this example. It suffices to show that, for any small enough $\epsilon > 0$ and any modulus of continuity $\ell : [0, +\infty) \rightarrow [0, +\infty)$ with $\lim_{s \rightarrow 0^+} \ell(s) = 0$, there exists $(\alpha_\epsilon, v_\epsilon) \in B_V(\bar{u}, \epsilon) \cap K$ and $w_\epsilon \in \pi_X(\text{Var}_K(f_0, f)(\bar{u}))$, such that

$$w_\epsilon \notin \pi_X(\text{Var}_K(f_0, f)(\alpha_\epsilon, v_\epsilon)) + \ell(|\alpha_\epsilon| + |v_\epsilon|_{L^2(0,1)}) B_{L^2(0,1)}(0, 1).$$

(2.25)

For this aim, we choose $\alpha_\epsilon = 0$ and define

$$v_\epsilon(t) = \begin{cases} 0, & t \in [0, 1] \setminus E_\epsilon, \\ 1, & t \in E_\epsilon, \end{cases}$$

(2.26)
where $E_\epsilon \subseteq [0, 1]$ is a measurable set with the Lebesgue measure $\text{mes}(E_\epsilon) = \epsilon^2$. It follows that $(\alpha_\epsilon, v_\epsilon) = (0, v_\epsilon) \in B_V(\bar{u}, \epsilon) \cap \mathcal{K}$, and

\[ |(0, v_\epsilon) - (0, 0)|_V = |v_\epsilon|_{L^2(0, 1)} = \epsilon \to 0, \text{ as } \epsilon \to 0^+. \]

Further, by Definition 2.2,

\[
\mathcal{T}_K(0, v_\epsilon) = \mathbb{R} \times \bigcup_{h \geq 0} \left\{ h(v - v_\epsilon) \in L^2(0, 1) \mid v(\cdot) \in L^2(0, 1) \text{ with } v(t) \in [-1, 1] \right\}
\]

\[
= \mathbb{R} \times \left\{ w \in L^\infty(0, 1) \mid w(t) \leq 0, \text{ a.e. } t \in E_\epsilon \right\}
\]

\[
= \mathbb{R} \times \left\{ w \in L^2(0, 1) \mid w(t) \leq 0, \text{ a.e. } t \in E_\epsilon \right\}.
\]

Therefore, by (2.23) and the above equality,

\[
\pi_X(\text{Var}_K(f_0, f)(\alpha_\epsilon, v_\epsilon)) = \pi_X(\text{Var}_K(f_0, f)(0, v_\epsilon))
\]

\[
= \left\{ \beta q - w \in L^2(0, 1) \mid |\beta| + |w|_{L^2(0, 1)} \leq 1, w(\cdot) \in L^2(0, 1) \text{ with } w(t) \leq 0, \text{ a.e. } t \in E_\epsilon \right\}.
\]

On the other hand, we take

\[
w_\epsilon(t) = \begin{cases} 
0, & t \in [0, 1] \setminus E_\epsilon, \\
\frac{1}{\epsilon}, & t \in E_\epsilon.
\end{cases} \tag{2.27}
\]

It is obvious that $|w_\epsilon|_{L^2(0, 1)} = 1$ and hence, by (2.24), $w_\epsilon \in \pi_X(\text{Var}_K(f_0, f)(0, 0))$. For any $\bar{u}_\epsilon \in \pi_X(\text{Var}_K(f_0, f)(0, v_\epsilon))$, there exists a $\beta \in \mathbb{R}$ and $w \in L^2(0, 1)$, such that $|\beta| + |w|_{L^2(0, 1)} \leq 1$, $w(t) \leq 0$ a.e. in $E_\epsilon$ and $\bar{u}_\epsilon = \beta q - w$. Then,

\[
|w_\epsilon - \bar{u}_\epsilon|_{L^2(0, 1)}^2 = \int_0^1 |w_\epsilon(t) - \bar{u}_\epsilon(t)|^2 dt
\]

\[
= \int_{E_\epsilon} |w_\epsilon(t) - \bar{u}_\epsilon(t)|^2 dt + \int_{[0, 1] \setminus E_\epsilon} |w_\epsilon(t) - \bar{u}_\epsilon(t)|^2 dt
\]

\[
\geq \int_{E_\epsilon} |w_\epsilon(t) - [\beta q(t) - w(t)]|^2 dt
\]

\[
\geq \frac{1}{2} \int_{E_\epsilon} |w_\epsilon(t) + w(t)|^2 dt - \int_{E_\epsilon} |q(t)|^2 dt.
\]

Notice that $w(t) \leq 0$ and $w_\epsilon(t) = -1/\epsilon$ in $E_\epsilon$. Hence, the above inequality implies

\[
|w_\epsilon - \bar{u}_\epsilon|_{L^2(0, 1)}^2 \geq \frac{1}{2} \int_{E_\epsilon} \frac{1}{\epsilon^2} dt - \int_{E_\epsilon} |q(t)|^2 dt = \frac{1}{2} - \int_{E_\epsilon} |q(t)|^2 dt.
\]

This shows that $|w_\epsilon - \bar{u}_\epsilon|_{L^2(0, 1)}^2 \geq 1/4$ for sufficiently small $\epsilon > 0$. However, since $\bar{u} = (0, 0)$, for any modulus of continuity $\ell(\cdot)$ in (H1),

\[
\ell(|\alpha_\epsilon| + |v_\epsilon|_{L^2(0, 1)}) = \ell(|v_\epsilon|_{L^2(0, 1)}) \to 0, \text{ as } \epsilon \to 0^+.
\]

Consequently, (2.25) holds and the condition (H1) in this example fails.
Though \((H_1)\) may fail for some general optimization problems, the following example demonstrates that, under appropriate conditions, both \((H_1)\) and \((H_2)\) in Theorem 2.1 are satisfied for \(V(\cdot) = \mathrm{Var}_K(f_0, f)(\cdot)\) in the optimization problem \((P)\).

**Example 2.3** Let \(V\) be a reflexive Banach space, \(K\) be a nonempty closed convex subset of \(V\) and, \(\bar{u} \in K\) be a solution to the problem \((P)\). Assume that \(f_0\) and \(f\) are continuously differentiable and there exists a constant \(\delta > 0\) and a modulus of continuity \(\ell_i : [0, +\infty) \to [0, +\infty)\) with \(\lim_{s \to 0^+} \ell_i(s) = 0\) \((i = 1, 2)\), such that

\[
\mathcal{T}_K(\bar{u}) \cap B_V(0, 1) \\
\subseteq \mathcal{T}_K(u) \cap B_V(0, 1) + \ell_1(|\bar{u} - u|) B_V(0, 1), \quad \forall u \in B_V(\bar{u}, \delta) \cap K,
\]

and

\[
|f'(u_1) - f'(u_2)|_{\mathcal{L}(V;X)} \leq \ell_2(|u_1 - u_2|) , \quad \forall u_1, u_2 \in B_V(\bar{u}, \delta) \cap K.
\]

Then the conditions \((H_1)\)-(\(H_2)\) in Theorem 2.1 are satisfied for \(V(\cdot) = \mathrm{Var}_K(f_0, f)(\cdot)\). Furthermore, if \(\bar{u}\) is an interior point of \(K\), then the condition \((2.28)\) can be removed and only the condition \((2.29)\) is required.

Indeed, since \(f_0\) and \(f\) are continuous differentiable, \(f'_0(\cdot)\) and \(f'(\cdot)\) are locally bounded at \(\bar{u}\), and \(f_0(\cdot)\) and \(f(\cdot)\) are locally Lipschitz continuous at \(\bar{u}\).

For any \((\xi_1, \xi_2)^T \in \mathrm{Var}_K(f_0, f)(\bar{u})\), by Example 2.1, there is a \(v \in \mathcal{T}_K(\bar{u}) \cap B_V(0, 1)\), such that

\[
(\xi_1, \xi_2)^T = (f'_0(\bar{u})v, f'(\bar{u})v)^T.
\]

For any \(u^\varepsilon \in K\) with \(u^\varepsilon \to \bar{u}\) in \(V\) as \(\varepsilon \to 0^+\), by \((2.28)\), when \(\varepsilon\) is small enough,

\[
v \in \mathcal{T}_K(u^\varepsilon) \cap B_V(0, 1) + \ell_1(|\bar{u} - u^\varepsilon|) B_V(0, 1).
\]

This implies that there exists a \(v^\varepsilon \in \mathcal{T}_K(u^\varepsilon) \cap B_V(0, 1)\), such that

\[
|v - v^\varepsilon|_V \leq \ell_1(|\bar{u} - u^\varepsilon|).
\]

Next, define

\[
(\xi_1^\varepsilon, \xi_2^\varepsilon)^T \triangleq (f'_0(u^\varepsilon)v^\varepsilon, f'(u^\varepsilon)v^\varepsilon)^T.
\]

Then, \((\xi_1^\varepsilon, \xi_2^\varepsilon)^T \in \mathrm{Var}_K(f_0, f)(u^\varepsilon)\). By the local boundedness of \(f'_0(\cdot)\) and \(f'(\cdot)\), there exists an \(L > 0\), such that

\[
|f'(u^\varepsilon)|_{\mathcal{L}(V;X)} + |f'_0(u^\varepsilon)|_{V'} \leq L.
\]

Further,

\[
|\xi_1 - \xi_1^\varepsilon| = |f'_0(\bar{u})v - f'_0(u^\varepsilon)v^\varepsilon| \\
\leq |f'_0(\bar{u}) - f'_0(u^\varepsilon)|_{V'}|v|_V + |f'_0(u^\varepsilon)|_{V'}|v^\varepsilon - v|_V \\
\leq |f'_0(\bar{u}) - f'_0(u^\varepsilon)|_{V'} + L|v^\varepsilon - v|_V.
\]

This shows that \(\xi_1^\varepsilon \to \xi_1\) in \(X\) as \(\varepsilon \to 0^+\). By \((2.29)\) and \((2.30)\),

\[
|\xi_2 - \xi_2^\varepsilon|_X = |f'(\bar{u})v - f'(u^\varepsilon)v^\varepsilon|_X \\
\leq |f'(\bar{u}) - f'(u^\varepsilon)|_{\mathcal{L}(V;X)}|v|_V + |f'(u^\varepsilon)|_{\mathcal{L}(V;X)}|v^\varepsilon - v|_V \\
\leq \ell_2(|\bar{u} - u^\varepsilon|) + L\ell_1(|\bar{u} - u^\varepsilon|) \triangleq \ell(|\bar{u} - u^\varepsilon|).
\]
This indicates that \( (H_1) \) holds. Clearly,

\[
\pi_\mathbb{R}(\text{Var}_K(f_0,f)(u)) = \{ f_0'(u)v \in X \mid v \in T_K(u) \cap B_V(0,1) \}.
\]

Then, \( (H_2) \) follows from the local boundedness of \( f_0' \).

If \( \bar{u} \) is an interior point of the set \( K \), (2.28) is satisfied trivially, and in the above derivation, it suffices to choose \( v^\varepsilon = v \in B_V(0,1) \).

Besides, in the following two typical cases, the condition (2.28) is also satisfied.

1) \( K \) is a polyhedral set of \( V \) represented by

\[
K = \{ u \in V \mid \langle a_i, u \rangle_{V',V} \leq b_i, \ i = 1, 2, \cdots, m \},
\]

where \( a_i \in V' \) and \( b_i \in \mathbb{R} \) (\( i = 1, 2, \cdots, m \)) for some \( m \in \mathbb{N} \).

In this situation,

\[
T_K(u) = \{ v \in V \mid \langle a_i, v \rangle_{V',V} \leq 0, \ i = I(u) \},
\]

where \( I(u) = \{ i = 1, 2, \cdots, m \mid \langle a_i, u \rangle_{V',V} = b_i \} \). Clearly, there exists a \( \delta > 0 \) such that \( I(u) \subseteq I(\bar{u}) \) for any \( u \in B_V(\bar{u}, \delta) \cap K \). Consequently,

\[
T_K(\bar{u}) \subseteq T_K(u), \quad \forall \ u \in B_V(\bar{u}, \delta) \cap K
\]

and the condition (2.28) holds true.

2) \( V \) is a Hilbert space and there is a continuously differentiable convex function \( g : V \to \mathbb{R} \), such that

\[
K = \{ u \in V \mid g(u) \leq 0 \},
\]

and \( g'(u) \neq 0 \) for any \( u \in V \) satisfying \( g(u) = 0 \).

In this case, as shown in [1, Proposition 4.3.7],

\[
T_K(u) = \{ v \in V \mid g'(u)v \leq 0 \}, \quad \forall \ u \in K \text{ with } g(u) = 0.
\]

Consequently, when the derivative of \( g \) satisfies, for some \( \delta > 0 \) and a modulus of continuity \( \ell : [0, \infty) \to [0, \infty) \),

\[
|g'(\bar{u}) - g'(u)|_V \leq \ell(|\bar{u} - u|_V), \quad \forall \ u \in B_V(\bar{u}, \delta),
\]

the condition (2.28) is satisfied.

The above example shows that, when \( f_0 \) and \( f \) are continuously differentiable, and the tangent cone for \( K \) has appropriate continuity near the solution \( \bar{u} \), the conditions \( (H_1)-(H_2) \) are satisfied. In Section 3, we shall see that the conditions \( (H_1)-(H_2) \) are satisfied for specific optimal control problems under some mild assumptions.

In [14, Corollary 6.4.5, Page 267], the penalty function method is also adopted to prove the Fritz John condition for the problem \( (P) \). To ensure the nontriviality of the multiplier pair \( (z_0, z) \in \mathbb{R} \times X' \), a constraint qualification closely correlated with the condition \( (H_3) \) of this paper was introduced. In the following remark, we make a comparison between the condition \( (H_3) \) and the one in [14].
Remark 2.3 In [14, Corollary 6.4.5, Page 267], the Fritz John type first-order necessary condition for the problem (P) is studied under the setting that the domains $\mathcal{D}(f^0)$ and $\mathcal{D}(f)$ of $f^0$ and $f$ are subsets of $V$, respectively, and $\mathcal{K} = \mathcal{D}(f^0) \cap \mathcal{D}(f)$.

In order to guarantee the nontriviality of multiplier pair, the following condition was imposed:

(H₄) There exist $k_0 \in \mathbb{N}$, $\rho > 0$ and a precompact sequence $\{Q_k\}_{k=1}^\infty \subseteq X$ (i.e., any sequence $\{q_k\}_{k=1}^\infty \subseteq X$ with $q_k \in Q_k$ for any $k \in \mathbb{N}$ has a convergent subsequence), such that

$$\text{Int} \left( \bigcap_{k=k_0}^\infty (\Delta_k + Q_k) \right) \neq \emptyset,$$

(2.32)

where $\Delta_k = \pi_X \left( \text{Var}_K(f_0,f)(u_k) \right) - \mathcal{T}_E(e_k) \cap B_X(0,\rho)$, $\{u_k\}_{k=1}^\infty \subseteq K$ (with $K = \mathcal{D}(f^0) \cap \mathcal{D}(f)$) and $\{e_k\}_{k=1}^\infty \subseteq E$ satisfy certain conditions in [14, Theorem 6.4.2, Page 265].

In general, it is challenging to verify the condition (H₄) directly, because this condition depends on the choice of sequences $\{u_k\}_{k=1}^\infty$, $\{e_k\}_{k=1}^\infty$ and $\{Q_k\}_{k=1}^\infty$. On the other hand, the condition (H₃) is only related to the solution $\bar{u}$, and we shall see that in Section 3, condition (H₃) can be characterized by some equivalent a priori estimates, which make it much easier to be verified in some concrete problems.

In the following, we will show that, under suitable conditions, the condition (H₃) implies the condition (H₄).

Let $\bar{u}$ be the solution to the problem (P). Assume that $V$ is a reflexive Banach space, $K$ is a nonempty closed convex subset of $V$ and $E = \{0\}$. Suppose that $f_0$ and $f$ are continuously differentiable and there exists a $\delta > 0$ and a modulus of continuity $\ell_1 : [0, +\infty) \to [0, +\infty)$ with $\lim_{s \to 0^+} \ell_1(s) = 0$, such that (2.28) in Example 2.3 is satisfied. Then, for $V(\cdot) = \text{Var}_K(f_0,f)(\cdot)$, the condition (H₃) implies the condition (H₄), for any $\{u_k\}_{k=1}^\infty \subseteq K$ satisfying that $u_k \to \bar{u}$ in $V$ as $k \to \infty$.

Indeed, under the above conditions, for any $u \in K$, it follows from Example 2.1 that

$$\pi_X \left( \text{Var}_K(f_0,f)(u) \right) = \{ f'(u)v \in X \mid v \in \mathcal{T}_K(u) \cap B_V(0,1) \},$$

(2.33)

and, the condition (H₃) with $V(\cdot) = \text{Var}_K(f_0,f)(\cdot)$ becomes that $\pi_X \left( \text{Var}_K(f_0,f)(\bar{u}) \right)$ is finite codimensional in $X$.

Clearly, $\pi_X \left( \text{Var}_K(f_0,f)(\bar{u}) \right)$ is a closed convex set in $X$. By [27, Corollary 3.3, Page 140], there exists a compact subset $Q \subseteq X$, such that

$$\text{Int} \left( \pi_X \left( \text{Var}_K(f_0,f)(\bar{u}) \right) + Q \right) \neq \emptyset.$$

Therefore, there exists an $x_0 \in X$ and $\gamma_0 > 0$, such that

$$B_X(x_0,\gamma_0) \subseteq \pi_X \left( \text{Var}_K(f_0,f)(\bar{u}) \right) + Q.$$

It follows that for any $x \in B_X(0,\gamma_0)$, we can find $\tilde{v} \in \mathcal{T}_K(u) \cap B_V(0,1)$ and $q \in Q$, so that

$$x_0 + x = f'(\bar{u})\tilde{v} + q.$$

(2.34)

Furthermore, for any $\epsilon \in (0,\gamma_0)$, there exist constants $\eta \in (0,\delta)$ and $M_1 \geq 0$, such that for any $u \in B_V(\bar{u},\eta) \cap K$,

$$|f'(\bar{u}) - f'(u)|_{L(V,X)} \leq \frac{\epsilon}{2}, \quad |f'(u)|_{L(V,X)} \leq M_1 \quad \text{and} \quad \ell_1(\|\bar{u} - u\|_V) \leq \frac{\epsilon}{2M_1}.$$
By (2.28), for any \( u \in B_V(\bar{u}, \eta) \cap \mathcal{K} \), there is a \( v \in T_K(u) \cap B_V(0, 1) \), such that
\[
|v - \bar{v}|_V \leq \ell_1(|\bar{u} - u|_V).
\]
Consequently,
\[
|f'(\bar{u})\bar{v} - f'(u)v|_X \leq |f'(\bar{u}) - f'(u)|_{\mathcal{L}(V; X)}|\bar{v}|_V + |f'(u)|_{\mathcal{L}(V; X)}|\bar{v} - v|_V \\
\leq |f'(\bar{u}) - f'(u)|_{\mathcal{L}(V; X)} + M_1\ell_1(|\bar{u} - u|_V)
\tag{2.35}
\]
By (2.34),
\[
x = -x_0 + f'(u)v + [f'(\bar{u})\bar{v} - f'(u)v] + q.
\tag{2.36}
\]
Then, by (2.33), (2.35) and (2.36),
\[
B_X(0, \gamma_0) \subseteq -x_0 + \pi_X(\text{Var}_K(f_0, f)(u)) + B_X(0, \epsilon) + Q \\
\subseteq -x_0 + \pi_X(\text{Var}_K(f_0, f)(u)) + B_X(0, \epsilon) + \overline{\partial Q}. 
\tag{2.37}
\]
Since \( Q \) is compact, \( \overline{\partial Q} \) is also compact, and \( \pi_X(\text{Var}_K(f_0, f)(u)) + \overline{\partial Q} \) is closed and convex. This implies that
\[
B_X(0, \gamma_0 - \epsilon) \subseteq -x_0 + \pi_X(\text{Var}_K(f_0, f)(u)) + \overline{\partial Q}.
\]

By the arbitrariness of \( u \) in \( B_V(\bar{u}, \eta) \cap \mathcal{K} \), we obtain that
\[
B_X(x_0, \gamma_0 - \epsilon) \subseteq \bigcap_{u \in B_V(\bar{u}, \eta) \cap \mathcal{K}} \left[ \pi_X(\text{Var}_K(f_0, f)(u)) + \overline{\partial Q} \right].
\]

By the conditions on \( E \), \( f_0 \) and \( f \), in (H4), choosing \( \Delta_k = \pi_X(\text{Var}_K(f_0, f)(u_k)) \), \( e_k = 0 \) and \( Q_k = \overline{\partial Q} \), we have that the condition (H4) holds true for sufficiently large \( k_0 \in \mathbb{N} \) so that \( |\bar{u} - u_k|_V \leq \eta \) for any \( k \geq k_0 \). This proves that in some special circumstances, we can verify (H4) by checking (H3).

The following results is a consequence of Theorem 2.1.

**Corollary 2.1** Let \( X \) be a reflexive Banach space with \( X' \) being strictly convex, \( \bar{u} \in \mathcal{K} \) be a solution to the problem (P) with \( \bar{e} = f(\bar{u}) \). If, in addition to (H1)-(H2), the condition
\[
0 \in \text{Int}(\pi_X(V(\bar{u})) - [E - f(\bar{u})])
\tag{2.38}
\]
holds true, then there exists a \( z \in X' \) with \( z \in \mathcal{N}_E(\bar{e}) \), such that
\[
\xi_1 + \langle z, \xi_2 \rangle_{X', X} \geq 0, \quad \forall (\xi_1, \xi_2)^\top \in V(\bar{u}).
\tag{2.39}
\]
Moreover, if \( z \neq 0 \), then there exists a sequence \( \{u^k\}_{k=1}^\infty \subseteq \mathcal{K} \), converging to \( \bar{u} \) as \( k \to \infty \), such that the conclusions in (2.10) hold. Especially, when \( X \) is a Hilbert space, the sequence \( \{\mathcal{P}_E(f(u^k))\}_{k=1}^\infty \) satisfies that \( \mathcal{P}_E(f(u^k)) \to f(\bar{u}) \) as \( k \to \infty \), and the condition (2.11) holds true.

**Proof.** Clearly, (2.38) implies the condition (H3). Then, by Theorem 2.1, there exists a non-zero pair \( (z_0, z) \in [0, +\infty) \times X' \) with \( z \in \mathcal{N}_E(\bar{e}) \) such that
\[
z_0\xi_1 + \langle z, \xi_2 \rangle_{X', X} \geq 0, \quad \forall (\xi_1, \xi_2)^\top \in V(\bar{u}).
Next, we use the contradiction argument to prove that \( z_0 \neq 0 \). Assume that \( z_0 = 0 \). By (2.21) in Step 3 of the proof in Theorem 2.1, we would have
\[
\langle z, \xi_2 - (x - f(\bar{u})) \rangle_{X', X} \geq 0, \quad \forall \xi_2 \in \pi_X(\mathcal{V}(\bar{u})) \text{ and } x \in E.
\]
From the condition (2.38), it follows that \( z = 0 \). This contradicts with the conclusion that \((z_0, z) \neq (0, 0)\). Hence, \( z_0 \neq 0 \).

The rest conclusions of this corollary follow from Theorem 2.1 immediately. \( \square \)

The first-order necessary condition (2.39) in Corollary 2.1 is a KKT-type necessary condition for problem (P). In the following remark, we give a comparison between condition (2.38) and the Robinson constraint qualification, which is a classical and frequently used condition for investigating the KKT condition in optimization.

**Remark 2.4** If \( V \) is a reflexive Banach space, \( \mathcal{K} \) is a nonempty closed convex subset of \( V \), \( f_0 \) and \( f \) are continuously differentiable and, \( \mathcal{V}(\cdot) = \text{Var}_K(f_0, f)(\cdot) \), then we can express the condition (2.38) as:
\[
0 \in \text{Int} \{ f'(\bar{u})v - [e - f(\bar{u})] \in X \mid v \in \mathcal{T}_K(\bar{u}) \cap B_V(0, 1) \text{ and } e \in E \}. \tag{2.40}
\]

First, the Robinson constraint qualification
\[
0 \in \text{Int} \{ f'(\bar{u})(u - \bar{u}) - [e - f(\bar{u})] \in X \mid u \in \mathcal{K} \text{ and } e \in E \} \tag{2.41}
\]
implies the condition (2.40). Indeed, (2.41) implies the fact that
\[
0 \in \text{Int} \mathcal{Z}(\bar{u}) = \text{Int} \left( f'(\bar{u})(\mathcal{R}_K(\bar{u})) - \mathcal{R}_E(f(\bar{u})) \right).
\]
Since \( \mathcal{Z}(\bar{u}) \) is a cone, we have \( X = \mathcal{Z}(\bar{u}) \). By [37, Theorem 2.1], it follows that
\[
0 \in \text{Int} \{ f'(\bar{u})\bar{v} - \bar{w} \in X \mid \bar{v} \in (\mathcal{K} - \bar{u}) \cap B_V(0, 1) \text{ and } \bar{w} \in (E - f(\bar{u})) \cap B_X(0, 1) \}.
\]
Since \( \mathcal{K} - \bar{u} \subseteq \mathcal{T}_K(\bar{u}) \) and \( (E - f(\bar{u})) \cap B_X(0, 1) \subseteq E - f(\bar{u}) \), we have that (2.40) holds.

On the other hand, if \( E = \{0\} \) and (2.28) in Example 2.3 is satisfied, then (2.40) also implies the Robinson constraint qualification (2.41). Indeed, in this special case, (2.40) is reduced to
\[
0 \in \text{Int} \{ f'(\bar{u})v \in X \mid v \in \mathcal{T}_K(\bar{u}) \cap B_V(0, 1) \}. \tag{2.42}
\]
Similarly to the proof of (2.37) in Remark 2.3, we can find positive constants \( \gamma_0, \epsilon_0 \) and \( \eta_0 \) with \( \epsilon_0 < \gamma_0 \), such that for any \( u \in B_V(\bar{u}, \eta_0) \cap \mathcal{K} \), it holds that
\[
B_X(0, \gamma_0) \subseteq \{ f'(u)v \in X \mid v \in \mathcal{T}_K(u) \cap B_V(0, 1) \} + B_X(0, \epsilon_0).
\]
By [1, Theorem 3.4.5, Page 96], we conclude that the system
\[
f(u) = 0 \text{ with } u \in \mathcal{K}
\]
enjoys the following metric regularity at \( \bar{u} \in f^{-1}(0) \cap \mathcal{K} \):
\[
\text{dist} \left( u, f^{-1}(x) \cap \mathcal{K} \right) \leq M_2 |f(u) - x|_V, \quad \forall (u, x) \in (B_V(\bar{u}, \eta_1) \cap \mathcal{K}) \times B_X(0, \gamma_1)
\]
for some positive constants \( \gamma_1, \eta_1 \) and \( M_2 \). Consequently, as shown in [11, Corollary 2.2], the above metric regularity and the Robinson constraint qualification (2.41) are equivalent, which gives the desired conclusion.

From the above arguments, generally speaking, the condition (2.38) is weaker than the Robinson constraint qualification (2.41). Furthermore, the first-order necessary optimality condition in Corollary 2.1 serves as a generalization of the classical KKT condition and the enhanced KKT condition for constrained optimization problems in metric spaces, with which the range of the constraint map \( f \) being in an infinite-dimensional space.
To conclude this subsection, inspired by the work of [22], we present an example illustrating the effectiveness of the Fritz John condition and the limitation of the KKT condition. Moreover, this example highlights how the enhanced Fritz John condition offers more precise information, compared to the Fritz John condition derived by the separation method.

**Example 2.4** Let \( V = X = \ell^2 = \{(u_1, u_2, \ldots)\} \) \( i \in \mathbb{N} \) and \( \sum_{i=1}^{\infty} u_i^2 < +\infty \). Define a map \( f : V \to X \) as follows:

\[
f(u) = (u_2 + (u_1 - 1)^3, -u_2 + (u_1 - 1)^3, 0, u_4, u_5, \ldots)^\top,
\]
and consider the nonlinear optimization problem:

\[
\text{Minimize } f_0(u) = u_1 \quad \text{subject to } f(u) = 0 \text{ for } u = (u_1, u_2, \ldots)^\top \in \ell^2.
\] (2.43)

Clearly, \( K = \ell^2 \), \( E = \{0\} \) and, for any \( u_3 \in \mathbb{R} \),

\[
\bar{u} = (1, 0, u_3, 0, 0, \ldots)^\top
\] (2.44)

solves the optimization problem (2.43).

Note that \( f_0 \) and \( f \) are continuously differentiable on \( \ell^2 \),

\[
\begin{align*}
f'_0(\bar{u})v &= v_1, \\
f'(\bar{u})v &= (v_2, -v_2, 0, v_4, v_5, \ldots)^\top, \\
\forall v &= (v_1, v_2, \ldots)^\top \in \ell^2,
\end{align*}
\]

\[
f(\bar{u})^\top z = (0, z_1 - z_2, 0, z_4, z_5, \ldots)^\top, \quad \forall z = (z_1, z_2, \ldots)^\top \in \ell^2,
\]

and for any \( u = (u_1, u_2, \ldots)^\top \in \ell^2 \) and \( v = (v_1, v_2, \ldots)^\top \in \ell^2 \),

\[
f'(u)v = (v_2 + 3(u_1 - 1)^2v_1, -v_2 + 3(u_1 - 1)^2v_1, 0, v_4, v_5, \ldots)^\top.
\]

Choose \( \mathcal{V}(\cdot) = \text{Var}_K(f_0, f)(\cdot) \). By Example 2.3, it is easy to show that (2.29) holds and \( \bar{u} \) is an interior point of \( V \). Therefore, (H1) and (H2) are satisfied. Further, it is easy to show that

\[
|\phi|_{\ell^2} \leq |f'(\bar{u})^\top \phi|_{\ell^2}, \quad \forall \phi = (0, 0, 0, z_4, z_5, \ldots) \in \ell^2.
\]

By Theorem 3.1 in the next section, \( \text{Var}_K f(\bar{u}) \) is finite codimensional in \( X \) and (H3) holds. Hence, any solution \( \bar{u} \) defined by (2.44) satisfies the following classical Fritz John condition:

\[
z_0 f'_0(\bar{u}) + f'(\bar{u})^\top z = 0,
\]

with \( z_0 = 0 \) and \( z = (\alpha, \alpha, \beta, 0, 0, \ldots)^\top \) for any \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha \neq 0 \), or \( \alpha = 0 \) and \( \beta \neq 0 \).

Since \( f'_0(\bar{u}) \notin \text{Im}(f'(\bar{u})^\top) \), the KKT condition for \( \bar{u} \) is not satisfied.

Moreover, for any \( z = (\alpha, \alpha, \beta, 0, 0, \ldots)^\top \) with \( \alpha, \beta \in \mathbb{R} \) and any \( u \in \ell^2 \), by the fact that \( E = \{0\} \), we have that

\[
(z, f(u) - \mathcal{P}_E(f(u)))_{\ell^2} = (z, f(u))_{\ell^2} = 2\alpha(u_1 - 1)^3.
\]

Consequently, \( z_0 = 0 \) and \( \hat{z} = (0, 0, \beta, 0, 0, \ldots)^\top \) with \( \beta \neq 0 \) do not satisfy the enhanced Fritz John condition (2.11).

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From the proof of Theorem 2.1, \( \{u^k\}_{k=1}^{\infty} \) in (2.11) should satisfy that

\[
\left\{ \left| f(u^k) \right|_{L^2} + \left[ (f_0(u^k) - f_0(\bar{u}) + \varepsilon)^+ \right]^{1/2} \right\} \leq \varepsilon, \quad \forall k \in \mathbb{N}.
\]

It implies that \( f_0(u^k) - f_0(\bar{u}) = u^k - 1 \leq 0 \). Then, \( z_0 = 0, \ \hat{z} = (\alpha, \alpha, \beta, 0, 0, \ldots)^\top \) with \( \alpha > 0 \) and \( \beta \in \mathbb{R} \) also do not satisfy the enhanced Fritz John condition (2.11). Therefore, the multiplier pair in the enhanced Fritz John condition should be

\[
\begin{aligned}
z_0 &= 0, \\
z &= (\alpha, \alpha, \beta, 0, 0, \ldots)^\top \quad \text{with } \alpha < 0, \beta \in \mathbb{R}.
\end{aligned}
\]

### 2.3 Application to an optimal control problem in infinite dimension

In this subsection, by Theorem 2.1, we derive a first-order necessary condition for optimal controls of an infinite-dimensional optimal control problem with end-point state constraints.

Let \( T > 0, \ U \) be a separable complete metric space and \( X \) be a Hilbert space. Assume that \( A : \mathcal{D}(A) \subseteq X \rightarrow X \) generates a \( C_0 \)-semigroup on \( X \). Set

\[
U_1(0, T) \triangleq \{ u : (0, T) \rightarrow U \mid u(\cdot) \text{ is a measurable function} \}.
\]

Clearly, \( U_1(0, T) \) is a complete metric space with the Ekeland metric (see [27, Proposition 3.10]):

\[
d_{U_1(0, T)}(u, v) \triangleq \text{mes}\left\{ t \in (0, T) \mid u(t) \neq v(t) \right\}, \quad \forall u, v \in U_1(0, T).
\]

Let \( y_0 \in X, \ F : [0, T] \times X \times U \rightarrow X \) and \( g_0 : [0, T] \times X \times U \rightarrow \mathbb{R} \) be given functions. Consider the following controlled evolution equation:

\[
\begin{aligned}
y_t(t) &= Ay(t) + F(t, y(t), u(t)), \quad t \in [0, T], \\
y(0) &= y_0,
\end{aligned}
\]

(2.46)

with a cost functional

\[
J(u(\cdot)) = \int_0^T g_0(t, y(t), u(t))dt.
\]

(2.47)

Here \( F \) and \( g_0 \) satisfy suitable conditions to be stated later so that for any control \( u(\cdot) \in U_1(0, T) \) the equation (2.46) admits a unique mild solution. Also, for any \( u(\cdot) \in U_1(0, T) \) and the corresponding solution \( y(\cdot) = y(\cdot; u) \) to (2.46), the function \( g_0(\cdot, y(\cdot; u), u(\cdot)) \in L^1(0, T) \).

Let \( E \) be a closed convex subset of \( X \). Set

\[
U_{ad} = \{ u(\cdot) \in U_1(0, T) \mid y(T; u(\cdot)) \in E \}
\]

and consider the following optimal control problem:

\[
\text{Minimize } J(u(\cdot)) \quad \text{subject to } u(\cdot) \in U_{ad}.
\]

(2.48)

Let

\[
V = U_1(0, T), \quad X = X, \quad f_0(u) = J(u(\cdot)), \quad f(u) = y(T; u) \quad \text{and } \quad K = U_1(0, T).
\]

Then the optimal control problem (2.48) is indeed a special case of the optimization problem (P). The solution \( \bar{u}(\cdot) \in U_{ad} \) to the problem (P) is the optimal control of (2.48) and the corresponding
solution \( \bar{y}(\cdot) = y(\cdot; \bar{u}) \) to (2.46) is the optimal trajectory. To obtain the first-order necessary condition for the optimal control \( \bar{u}(\cdot) \) by Theorem 2.1, we need to find the set-valued map \( \mathcal{V}(\cdot) \) satisfying (\( H_1 \))-\( (H_3) \). It should be remarked that the requirement of \( U \) being a complete separable metric space comes from some engineering applications, and a typical case is \( U = \{-1, 1\} \), which corresponds to the classical “bang-bang” controls.

Let us begin with the following hypotheses on \( F \) and \( g_0 \):

(\( A_1 \)) Let \( F : [0, T] \times \mathbb{X} \times U \to \mathbb{X} \) and \( g_0 : [0, T] \times \mathbb{X} \to \mathbb{R} \) be strongly measurable with respect to \( t \) in \( (0, T) \), and continuously differentiable with respect to \( y \) in \( \mathbb{X} \). \( F(t, \cdot, \cdot), F_y(t, \cdot, \cdot), g(t, \cdot, \cdot) \) and \( g_0(t, \cdot, \cdot) \) are continuous, and there exist positive constants \( \delta^*_1, L_1 \) and \( L_2 \), such that

\[
|F_y(t, y, u)|_{L(\mathbb{X})} + |g_0(t, y, u)|_{\mathbb{X}'} + |F(t, 0, u)|_{\mathbb{X}} + |g_0(t, 0, u)| \leq L_1, \\
\forall (t, y, u) \in [0, T] \times \mathbb{X} \times U,
\]

(2.49)

and

\[
|F_y(t, y_1, u) - F_y(t, y_2, u)|_{L(\mathbb{X})} \leq L_2|y_1 - y_2|_{\mathbb{X}}, \\
\forall t \in [0, T], u \in U \text{ and } y_1, y_2 \in \mathbb{X} \text{ with } |y_1 - y_2|_{\mathbb{X}} \leq \delta^*_1.
\]

(2.50)

Under the condition (\( A_1 \)), the system (2.46) is well-posed for any \( u(\cdot) \in U_1(0, T) \). Denote by \( y(\cdot; u) \) the solution to (2.46) corresponding to the control \( u(\cdot) \in U_1(0, T) \).

For any \( v(\cdot) \in U_1(0, T) \), we consider the following control system:

\[
\begin{aligned}
\dot{\xi}_t(t) &= A_1(t) + F_y(t, y(t; u), u(t))\xi(t) + F(t, y(t; u), v(t)) - F(t, y(t; u), u(t)), \quad t \in [0, T], \\
\xi(0) &= 0,
\end{aligned}
\]

(2.51)

and define a set-valued map as

\[
\mathcal{V}(u) = \left\{ \frac{1}{T}(\xi^0, \xi(T; v)) \right\} \in \mathbb{R} \times \mathbb{X} \bigm| \xi(\cdot; v) \text{ is the solution to (2.51) and}
\xi^0 = \int_0^T \left( g_0(t, y(t; u), u(t)), \xi(t; v) \right)_{\mathbb{X}' \times \mathbb{X}} + g_0(t, y(t; u), v(t)) - g_0(t, y(t; u), u(t)) \right) dt
\]

for some \( v(\cdot) \in U_1(0, T) \).

By the spike variation technique, one can prove that (e.g., [28])

\[
\mathcal{V}(u) \subseteq \text{Var}_{U_1(0, T)}(f_0, f)(u), \quad \forall u \in U_1(0, T).
\]

Particularly, for the optimal pair \( (\bar{u}(\cdot), \bar{y}(\cdot)) \), letting \( \hat{\xi}(\cdot; v) \) be the solution to the equation

\[
\begin{aligned}
\dot{\hat{\xi}}_t(t) &= A_1\hat{\xi}(t) + F_y(t, \bar{y}(t), \bar{u}(t))\hat{\xi}(t) + F(t, \bar{y}(t), v(t)) - F(t, \bar{y}(t), \bar{u}(t)), \quad t \in [0, T], \\
\hat{\xi}(0) &= 0,
\end{aligned}
\]

(2.52)

we have

\[
\mathcal{V}(\bar{u}) = \left\{ \frac{1}{T}(\hat{\xi}^0, \hat{\xi}(T; v)) \right\} \in \mathbb{R} \times \mathbb{X} \bigm| \hat{\xi}(\cdot; v) \text{ is the solution to (2.52) and}
\hat{\xi}^0 = \int_0^T \left( g_0(t, \bar{y}(t), \bar{u}(t)), \hat{\xi}(t; v) \right)_{\mathbb{X}' \times \mathbb{X}} + g_0(t, \bar{y}(t), v(t)) - g_0(t, \bar{y}(t), \bar{u}(t)) \right) dt
\]

(2.53)

for some \( v(\cdot) \in U_1(0, T) \).
Then,
\[ \pi_X(\mathcal{V}(\bar{u})) = \left\{ \frac{1}{T} \hat{\xi}(T; v) \in \mathbb{X} \mid \hat{\xi}(\cdot; v) \text{ solves (2.52) for } v(\cdot) \in \mathcal{U}_1(0, T) \right\} \]
and
\[ \pi_R(\mathcal{V}(\bar{u})) = \left\{ \frac{1}{T} \hat{\xi}^0 \in \mathbb{R} \mid \hat{\xi}^0 = \int_0^T \left( \langle g_0,y(t), \bar{u}(t) \rangle, \hat{\xi}(t; v) \rangle_{\mathcal{X}', \mathcal{X}} + g_0(t, \bar{y}(t), v(t)) - g_0(t, \bar{y}(t), \bar{u}(t)) \right) dt \text{ for some } v(\cdot) \in \mathcal{U}_1(0, T) \right\}. \]

Now, we prove that (\(H_1\)) in Theorem 2.1 holds.
Let \(\xi(\cdot; v)\) and \(\hat{\xi}(\cdot; v)\) be respectively the solutions to (2.51) and (2.52) with respect to \(v(\cdot) \in \mathcal{U}_1(0, T)\). Set \(\eta(\cdot) = \xi(\cdot; v) - \hat{\xi}(\cdot; v)\). Then, \(\eta(\cdot)\) satisfies
\[
\left\{ \begin{array}{ll}
\eta_t = A\eta + F_y(t, y(t; u), u(t))\eta + [F_y(t, y(t; u), u(t)) - F_y(t, \bar{y}(t), \bar{u}(t))] \hat{\xi} \\
&+ [F(t, y(t; u), v(t)) - F(t, \bar{y}(t), v(t))] \\
&- [F(t, y(t; u), u(t)) - F(t, \bar{y}(t), \bar{u}(t))], \quad t \in [0, T], \\
\eta(0) = 0.
\end{array} \right.
\tag{2.54}
\]

In the rest of this section, we denote by \(C\) a positive constant, independent of any \(u(\cdot) \in \mathcal{U}_1(0, T)\), which may vary from line to line. When we want to emphasize a special constant which also independent of any \(u(\cdot) \in \mathcal{U}_1(0, T)\), we use the notation \(C_1, C_2\), etc. By the classical estimates for solutions to evolution equations (see [27, Lemma 4.1, Page 151] for example), we get that
\[ |y(\cdot; u)|_{C([0,T]; \mathcal{X})} \leq C, \tag{2.55} \]
\[ |y(\cdot; u) - \bar{y}(\cdot)|_{C([0,T]; \mathcal{X})} \leq C_1 \text{d}_{\mathcal{U}_1(0,T)}(\bar{u}, u), \tag{2.56} \]
and
\[ |\hat{\xi}(\cdot; v)|_{C([0,T]; \mathcal{X})} \leq C. \tag{2.57} \]

By (2.49), for the mild solution \(\eta(\cdot)\) to (2.54), we have that
\[
|\eta(t)|_{\mathcal{X}} \leq C \int_0^t \left| F_y(s, y(s; u), u(s)) \right|_{\mathcal{X}} |\eta(s)|_{\mathcal{X}} ds + C \int_0^t \left| F_y(s, y(s; u), u(s)) - F_y(s, \bar{y}(s), u(s)) \right|_{\mathcal{X}} |\xi(s)|_{\mathcal{X}} ds
+ C \int_{\{s \in [0,t] \mid u(s) \neq \bar{u}(s)\}} \left| F_y(s, \bar{y}(s), u(s)) - F_y(s, \bar{y}(s), \bar{u}(s)) \right|_{\mathcal{X}} |\hat{\xi}(s)|_{\mathcal{X}} ds 
+ C \int_0^t 2L_1 |y(s; u) - \bar{y}(s)|_{\mathcal{X}} ds
+ C \int_{\{s \in [0,t] \mid u(s) \neq \bar{u}(s)\}} |F(s, \bar{y}(s), u(s)) - F(s, \bar{y}(s), \bar{u}(s))|_{\mathcal{X}} ds.
\tag{2.58}
\]
By (2.56), when \(\text{d}_{\mathcal{U}_1(0,T)}(\bar{u}, u) \leq \frac{\delta^*_1}{C_1}\), it holds that \(\sup_{t \in [0,T]} |y(t; u) - \bar{y}(t)|_{\mathcal{X}} \leq \delta^*_1\). Then, by (2.50), for any \(s \in [0,t]\),
\[
|F_y(s, y(s; u), u(s)) - F_y(s, \bar{y}(s), u(s))|_{\mathcal{X}} \leq L_2 |y(s; u) - \bar{y}(s)|_{\mathcal{X}}. \tag{2.59}
\]
Combining (2.58)-(2.59) with (2.55), (2.57) and (2.49), we obtain that
\[
|\eta(t)|_X \leq C \int_0^t |\eta(s)|_X ds + C d \xi_{I_1(0,T)}(\bar{u}, u)
+ C \int_{\{s \in [0,T] \mid u(s) \neq \bar{u}(s)\}} |F(s, \bar{y}(s), u(s)) - F(s, 0, u(s))|_X ds
+ C \int_{\{s \in [0,T] \mid u(s) \neq \bar{u}(s)\}} |F(s, \bar{y}(s), \bar{u}(s)) - F(s, 0, \bar{u}(s))|_X ds
+ C \int_{\{s \in [0,T] \mid u(s) \neq \bar{u}(s)\}} |F(s, 0, u(s)) - F(s, 0, \bar{u}(s))|_X ds
\leq C \int_0^t |\eta(s)|_X ds + C d \xi_{I_1(0,T)}(\bar{u}, u).
\]

This, together with the Gronwall inequality, implies that, when \(d \xi_{I_1(0,T)}(\bar{u}, u) \leq \frac{\delta^2}{c^2 t}\), there is a constant \(L_1^* > 0\), such that
\[
\sup_{t \in [0,T]} |\xi(t; v) - \hat{\xi}(t; v)|_X
\leq L_1^* d \xi_{I_1(0,T)}(\bar{u}, u) = L_1^* \times \text{mes}\left(\left\{ t \in [0, T] \mid \bar{u}(t) \neq u(t) \right\}\right), \quad \forall \ v(\cdot) \in \mathcal{U}_1(0, T).
\] (2.60)

For any \(\frac{1}{T} \xi^0, \hat{\xi}(T; v) \) \(\in \mathcal{V}(\bar{u})\) (recall (2.53) for \(\mathcal{V}(\bar{u})\)), there is a \(v(\cdot) \in \mathcal{U}_1(0, T)\), such that \(\hat{\xi}(T; v)\) is the terminal value of the solution \(\xi(\cdot; v)\) to (2.52) with respect to \(v(\cdot)\) and
\[
\xi^0 = \int_0^T \left( \langle g_0, y(t; y(t), u(t)) \rangle + g_0(t, y(t), v(t)) - g_0(t, \bar{y}(t), \bar{u}(t)) \right) dt.
\]
Further, let \(\xi(T; v)\) be the terminal value of the solution \(\xi(\cdot; v)\) to (2.51) with respect to the above \(v(\cdot)\) and let
\[
\xi^0 = \int_0^T \left( \langle g_0, y(t; y(t), u(t)) \rangle + g_0(t, y(t), v(t)) - g_0(t, \bar{y}(t), \bar{u}(t)) \right) dt.
\]
Then, \(\frac{1}{T} (\xi^0, \xi(T; v)) \) \(\in \mathcal{V}(u)\). By (2.56), (2.60) and Lebesgue’s dominated convergence theorem, we have that
\[
\xi^0 \rightarrow \hat{\xi}(T; v), \quad \text{as} \quad d \xi_{I_1(0,T)}(\bar{u}, u) \rightarrow 0,
\]
and when \(d \xi_{I_1(0,T)}(\bar{u}, u) \leq \frac{\delta^2}{c^2 t}\),
\[
|\xi(T; v) - \hat{\xi}(T; v)|_X \leq L_1^* d \xi_{I_1(0,T)}(\bar{u}, u).
\]
Consequently, the condition (\(H_1\)) is satisfied.

By the condition (\(A_1\)), it is easy to check that \(\pi_{\mathcal{X}}(\mathcal{V}(\cdot))\) is locally bounded at \(\bar{u}\), i.e., the condition (\(H_2\)) in Theorem 2.1 holds.

Define \(M_1 \overset{\Delta}{=} \pi_{\mathcal{X}}(\mathcal{V}(\bar{u}))\). As a consequence of Theorem 2.1, we have the following first-order necessary optimality condition for \(\bar{u}\).

**Corollary 2.2** Assume that (\(A_1\)) holds. Let \(\bar{u}\) be the optimal control of (2.48) and \(\bar{y}\) be the corresponding optimal state. If \(M_1 - E\) is finite codimensional in \(\mathcal{X}\), then there exists a non-zero pair \((z_0, z) \in [0, +\infty) \times \mathcal{X}'\), such that \(z \in \mathcal{N}_{E}(\bar{y}(T))\) and
\[
H(t, \bar{y}(t), \bar{u}(t), z_0, \psi(t)) = \max_{u \in U} H(t, \bar{y}(t), u, z_0, \psi(t)), \quad \text{a.e.} \ t \in (0, T),
\] (2.61)
where
\[ H(t, y, u, z_0, \psi) = \langle \psi, F(t, y, u) \rangle_{X'} - z_0 g_0(t, y, u), \quad \forall (t, y, u, z_0, \psi) \in (0, T) \times X \times U \times \mathbb{R} \times X' \] (2.62)
and
\[
\begin{align*}
\psi_t(t) &= -A^* \psi(t) - F_y(t, \bar{y}(t), \bar{u}(t))^* \psi + z_0 g_0, \quad t \in [0, T], \\
\psi(T) &= -z.
\end{align*}
\] (2.63)

Moreover,
(i) If \( z_0 \neq 0 \), then the above results hold with \( (z_0, z) \) replaced by \( (1, \frac{z}{z_0}) \).
(ii) If \( z \neq 0 \), then there exists a sequence \( \{u^k(\cdot)\}_{k=1}^{\infty} \subseteq \mathcal{U}_1(0, T) \), which converges to \( \bar{u}(\cdot) \) (in the Ekeland metric) as \( k \to \infty \), such that
\[
\begin{align*}
y(T; u^k) \notin E & \text{ for } k \text{ being large enough}, \\
\lim_{k \to \infty} \text{dist}(y(T; u^k), E) &= 0, \\
\lim_{k \to \infty} J(u^k) &= J(\bar{u}).
\end{align*}
\]
Furthermore,
\[ \lim_{k \to \infty} \mathcal{P}_E(y(T; u^k)) = y(T; \bar{u}) \quad \text{in } X \]
and
\[ (\dot{z}, y(T; u^k) - \mathcal{P}_E(y(T; u^k)))_{X'} \geq 0, \quad \forall k \in \mathbb{N}, \]
where \( \dot{z} \in X \) is the element corresponding to \( z \in X' \) by the Riesz-Fréchet isomorphism.

**Proof.** By Theorem 2.1, there exists a non-zero pair \( (z_0, z) \in [0, +\infty) \times X' \) with \( z \in \mathcal{N}_E(\bar{y}(T)) \), such that
\[ z_0 \xi^0 + \langle z, \dot{\bar{y}}(T; v) \rangle_{X'} \geq 0, \quad \forall (\xi^0, \dot{\bar{y}}(T; v))^\top \in \mathcal{V}(\bar{u}). \] (2.64)
Then, by the duality relationship between the system (2.52) and the adjoint system (2.63), we obtain from (2.64) that
\[
\int_0^T [H(t, \bar{y}(t), \bar{u}(t), z_0, \psi(t)) - H(t, \bar{y}(t), v(t), z_0, \psi(t))] dt \geq 0, \quad \forall v(\cdot) \in \mathcal{U}_1(0, T),
\]
which implies (2.61). The rest of Corollary 2.2 can be obtained by Theorem 2.1 immediately.

The first-order necessary condition (2.61), also known as the Pontryagin maximum principle, is a known result (see [27, Chapter 4, Theorem 1.6]). As pointed out in [27, Chapter 4], if the finite codimensionality condition does not hold, then it may happen that one cannot find a non-zero pair \( (z_0, z) \) so that (2.61) holds.

**Remark 2.5** In [27, Page 130] (as well as [28, Page 185]), only the condition (2.49) was assumed to study infinite-dimensional optimal control problems for evolution equations with endpoint constraints. However, the condition (2.49) alone is not enough to guarantee the condition (H1) in Theorem 2.1 to hold true. Note that the condition (H1) has been used essentially in Step 3 of the proof of Theorem 2.1 and its importance has also been shown in Example 2.2. Hence, in the present work in order to ensure the validity of the condition (H1) in Theorem 2.1, both (2.49) and (2.50) are assumed.
3 Verification of finite codimensionality condition in optimal control problems

In this section, we shall discuss infinite-dimensional optimal control problems with state constraints. We will consider different types of control systems, including deterministic evolution equations, elliptic equations and stochastic differential equations. Our main goal is to understand how to verify the corresponding finite codimensionality conditions in these optimal control problems.

It should be noted that verifying a set to satisfy the finite codimensionality condition by Definition 2.7 directly for optimal control problems is not an easy task, even when the control set is the whole space. In the following, we will provide some equivalent criteria for determining finite codimensionality in an abstract framework.

Assume that \( V \) and \( X \) are reflexive Banach spaces, and \( F \in L(V; X) \). Write \( M = \{ F(v) \in X : |v|_V \leq 1 \} \). Then it is easy to check that

\[
\begin{align*}
\text{span} M &= \text{Im}(F), \\
\text{co} M &= M.
\end{align*}
\]

We have the following results.

Theorem 3.1 The following assertions are equivalent:

(1) The set \( M \) is finite codimensional in \( X \).

(2) The subspace \( \text{Im}(F) \) is a finite codimensional closed subspace in \( X \).

(3) There exists a finite codimensional closed subspace \( \bar{X} \) of \( X' \) and a constant \( C > 0 \) (independent of \( \phi \in \bar{X} \)), such that

\[
|\phi|_{X'} \leq C |F^*(\phi)|_{V'}, \quad \forall \phi \in \bar{X}.
\] (3.2)

(4) There exists a Banach space \( W \), a compact operator \( G : X' \rightarrow W \) and a constant \( C > 0 \) (independent of \( \phi \in X' \)), such that

\[
|\phi|_{X'} \leq C (|F^*(\phi)|_{V'} + |G(\phi)|_W), \quad \forall \phi \in X'.
\] (3.3)

(5) \( \ker(F^*) \) is finite-dimensional in \( X' \) and \( \text{Im}(F^*) \) is closed in \( V' \).

Proof. First, by [12, Lemma 2.1 and Remark 4.2] and [33, Lemma 3], the equivalence among (3), (4) and (5) can be derived directly. One only needs to take the continuous linear mapping in the above known results as the bounded linear operator \( F^* \). Furthermore, \( \text{Im}(F^*) \) is closed if and only if \( \text{Im}(F) \) is closed. Then, \( \ker(F^*) \) is finite-dimensional and \( \text{Im}(F^*) \) is closed, if and only if \( \text{Im}(F) \) is a closed finite codimensional subspace. This implies the equivalence between (2) and (5).

Next, we prove that (2) implies (1).

For any \( k \in \mathbb{N} \), set \( N_k = \{ F(v) \in X : |v|_V \leq k \} \). Then \( N_1 = \text{co} M \) and \( \bigcup_{n \in \mathbb{N}} N_k = \text{Im}(F) \). By (2) and (3.1), \( \text{Im}(F) = \overline{\text{span} M} = \overline{\text{co} M} \) is a finite codimensional subspace of \( X \). By the Baire category theorem, there exists \( \tilde{k} \in \mathbb{N} \) such that \( N_{\tilde{k}} = \overline{N_{\tilde{k}}} \) has at least one interior point \( x_0 \) in \( \overline{\text{co} M} \). Then \( x_0/\tilde{k} \) is an interior point of \( \overline{\text{co} M} \) in \( \overline{\text{span} M} \). Hence, \( M \) is finite codimensional in \( X \).

Finally, we prove that (1) implies (2).

Notice that \( \overline{\text{co} M} \subseteq \text{Im}(F) \). By Proposition 2.1, we may choose \( x_0 = 0 \) in Definition 2.7. By (ii) in Definition 2.7, \( \overline{\text{co} M} \) has at least one interior point in the subspace \( \overline{\text{span} M} = \text{Im}(F) \). Hence,
Im (F) also has an interior point in \( \overline{\text{Im} (F)} \). Since \( \text{Im} (F) \) and \( \overline{\text{Im} (F)} \) are two linear subspaces of \( X \), and \( \text{Im} (F) \) is dense in \( \overline{\text{Im} (F)} \), we have that \( \text{Im} (F) = \overline{\text{Im} (F)} \). By (i) in Definition 2.7, \( \text{Im} (F) = \text{span} \mathbf{M} \) is a closed finite codimensional subspace of \( X \). Hence, (2) holds.

This completes the proof of Theorem 3.1. \( \square \)

In the following, we will give some applications of Theorem 3.1 to optimal control problems for different control systems with state constraints.

### 3.1 An optimal control problem for an evolution equation with end-point state constraint

Let \( \mathbb{V} \) and \( \mathbb{X} \) be two Hilbert spaces, \( \mathcal{U}_2(0, T) = L^2(0, T; \mathbb{V}) \) and \( E = \{y_1\} \) for \( y_1 \in \mathbb{X} \). Given \( F : [0, T] \times \mathbb{X} \times \mathbb{V} \to \mathbb{X} \) and \( g_0 : [0, T] \times \mathbb{X} \times \mathbb{V} \to \mathbb{R} \), consider the optimal control problem:

\[
\text{Minimize } J(u(\cdot)) \text{ subject to } u(\cdot) \in \mathcal{U}_2(0, T) \text{ and } y(T; u) = y_1, \tag{3.4}
\]

where \( J \) is defined by (2.47) and \( y(\cdot; u) \) is the mild solution to (2.46) with respect to the control \( u(\cdot) \in \mathcal{U}_2(0, T) \).

By choosing \( V = \mathcal{U}_2(0, T) \), \( f_0(u) = J(u(\cdot)) \), \( f(u) = y(T; u) \) and \( \mathcal{K} = \mathcal{U}_2(0, T) \), the optimal control problem (3.4) can be regarded as the optimization problem (P).

We assume that

\[
(F) \quad F : [0, T] \times \mathbb{X} \times \mathbb{V} \to \mathbb{X} \text{ and } g_0 : [0, T] \times \mathbb{X} \times \mathbb{V} \to \mathbb{R} \text{ are strongly measurable with respect to } t \text{ in } (0, T), \text{ and continuously differentiable with respect to } (y, u) \text{ in } \mathbb{X} \times \mathbb{V}. \text{ Moreover, there exists a positive constant } L_3, \text{ such that for any } (t, y, u) \in [0, T] \times \mathbb{X} \times \mathbb{V},
\]

\[
\begin{align*}
\|F(t, 0, u)\|_{\mathbb{X}} &\leq L_3|u|_{\mathbb{V}}, \\
\|F_y(t, y, u)\|_{L(\mathbb{X})} + \|F_u(t, y, u)\|_{L(\mathbb{V}; \mathbb{X})} &\leq L_3, \\
|g_0(t, y, u)| &\leq L_3(1 + |y|_{\mathbb{X}}^2 + |u|_{\mathbb{V}}^2), \\
|g_{0,y}(t, y, u)|_{\mathbb{X}} &\leq L_3(1 + |y|_{\mathbb{X}} + |u|_{\mathbb{V}}).
\end{align*}
\tag{3.5}
\]

In addition, there exist positive constants \( \delta_2^*, \delta_3^*, L_4, L_5 \text{ and } L_6 \), such that

\[
\begin{align*}
\|F_y(t, y_1, u) - F_y(t, y_2, u)\|_{L(\mathbb{X})} + \|F_u(t, y_1, u) - F_u(t, y_2, u)\|_{L(\mathbb{V}; \mathbb{X})} &\leq L_4|y_1 - y_2|_{\mathbb{X}}, \\
&\forall t \in [0, T], u \in \mathbb{V} \text{ and } y_1, y_2 \in \mathbb{X} \text{ with } |y_1 - y_2|_{\mathbb{X}} \leq \delta_2^*,
\end{align*}
\tag{3.6}
\]

and

\[
\begin{align*}
\|F_y(t, y, u_1) - F_y(t, y, u_2)\|_{L(\mathbb{X})} + \|F_u(t, y, u_1) - F_u(t, y, u_2)\|_{L(\mathbb{V}; \mathbb{X})} &\leq L_5|u_1 - u_2|_{\mathbb{V}}, \\
&\forall t \in [0, T], y \in \mathbb{X} \text{ and } u_1, u_2 \in \mathbb{V},
\end{align*}
\tag{3.7}
\]

and

\[
\begin{align*}
|g_{0,y}(t, y_1, u_1) - g_{0,y}(t, y_2, u_2)|_{\mathbb{X}} + |g_{0,u}(t, y_1, u_1) - g_{0,u}(t, y_2, u_2)|_{\mathbb{V}} &\leq L_6(|y_1 - y_2|_{\mathbb{X}} + |u_1 - u_2|_{\mathbb{V}}), \\
&\forall t \in [0, T], y_1, y_2 \in \mathbb{X} \text{ with } |y_1 - y_2|_{\mathbb{X}} \leq \delta_3^* \text{ and } u_1, u_2 \in \mathbb{V}.
\end{align*}
\tag{3.8}
\]

By (A_2), (2.46) is well-posed and \( J(u(\cdot)) < +\infty \) for any \( u(\cdot) \in \mathcal{U}_2(0, T) \).
For any \( u(\cdot), v(\cdot) \in \mathcal{U}_2(0, T) \), consider the following system:
\[
\begin{align*}
\xi_t(t) &= A\xi(t) + F_y(t, y(t; u), u(t))\xi(t) + F_u(t, y(t; u), u(t))v(t), \quad t \in [0, T], \\
\xi(0) &= 0.
\end{align*}
\] (3.9)

By (A_2) and Example 2.1, for any \( u(\cdot) \in \mathcal{U}_2(0, T) \),
\[
\text{Var}_{\mathcal{U}_2(0,T)}(f_0, f)(u) = \left\{ \xi^0 \in \mathbb{R} \times \mathbb{X} \mid \xi(\cdot; v) \text{ is the solution to (3.9) and} \right. \\
\xi^0 = \int_0^T \left( \langle g_0, y(t, y(t; u), u(t))\xi(t; v)\rangle_{\mathcal{X}', \mathcal{X}} + \langle g_0, u(t, y(t; u), u(t))\rangle_{\mathcal{V}', \mathcal{V}} \right) dt,
\]
for some \( v(\cdot) \in \mathcal{U}_2(0, T) \) with \( |v|_{L^2(0,T;\mathcal{V})} \leq 1 \).

Then we have that
\[
\pi^\mathcal{X}(\text{Var}_{\mathcal{U}_2(0,T)}(f_0, f)(u)) = \text{Var}_{\mathcal{U}_2(0,T)}(f_0)(u) \quad \text{and} \quad \pi^\mathcal{V}(\text{Var}_{\mathcal{U}_2(0,T)}(f_0, f)(u)) = \text{Var}_{\mathcal{U}_2(0,T)}(f)(u),
\]
with
\[
\text{Var}_{\mathcal{U}_2(0,T)}(f_0)(u) = \left\{ \xi^0 \in \mathbb{R} \mid \xi^0 = \int_0^T \left( \langle g_0, y(t, y(t; u), u(t))\xi(t; v)\rangle_{\mathcal{X}', \mathcal{X}} \\
\quad + \langle g_0, u(t, y(t; u), u(t))\rangle_{\mathcal{V}', \mathcal{V}} \right) dt,
\]
\( \xi(\cdot; v) \) solves (3.9) for \( v(\cdot) \in \mathcal{U}_2(0, T) \) with \( |v|_{L^2(0,T;\mathcal{V})} \leq 1 \).

And
\[
\text{Var}_{\mathcal{U}_2(0,T)}(f)(u) = \left\{ \xi(T; v) \in \mathbb{X} \mid \xi(\cdot; v) \text{ solves (3.9) for } v(\cdot) \in \mathcal{U}_2(0, T) \text{ with } |v|_{L^2(0,T;\mathcal{V})} \leq 1 \right\}.
\]

Particularly, for the optimal pair \((\bar{u}, \bar{y})\), letting \( \hat{\xi}(\cdot; v) \) be the solution to the equation
\[
\begin{align*}
\hat{\xi}_t(t) &= A\hat{\xi}(t) + F_y(t, \bar{y}(t), \bar{u}(t))\hat{\xi}(t) + F_u(t, \bar{y}(t), \bar{u}(t))v(t), \quad t \in [0, T], \\
\hat{\xi}(0) &= 0,
\end{align*}
\] (3.10)

we have
\[
\text{Var}_{\mathcal{U}_2(0,T)}(f_0)(\bar{u}) = \left\{ \hat{\xi}^0 \in \mathbb{R} \mid \hat{\xi}^0 = \int_0^T \left( \langle g_0, y(t, \bar{y}(t), \bar{u}(t))\hat{\xi}(t; v)\rangle_{\mathcal{X}', \mathcal{X}} \\
\quad + \langle g_0, u(t, \bar{y}(t), \bar{u}(t))\rangle_{\mathcal{V}', \mathcal{V}} \right) dt,
\]
\( \hat{\xi}(\cdot; v) \) solves (3.10) for \( v(\cdot) \in \mathcal{U}_2(0, T) \) with \( |v|_{L^2(0,T;\mathcal{V})} \leq 1 \).

And
\[
\text{Var}_{\mathcal{U}_2(0,T)}(f)(\bar{u}) = \{ \hat{\xi}(T; v) \in \mathbb{X} \mid \hat{\xi}(\cdot; v) \text{ solves (3.10) for } v(\cdot) \in \mathcal{U}_2(0, T) \text{ with } |v|_{L^2(0,T;\mathcal{V})} \leq 1 \}. \quad (3.12)
\]

Let \( \mathcal{V}(u) = \text{Var}_{\mathcal{U}_2(0,T)}(f_0, f)(u) \), and \( \xi(\cdot; v) \) and \( \hat{\xi}(\cdot; v) \) be respectively the solutions to (3.9) and (3.10) with respect to \( v(\cdot) \in \mathcal{U}_2(0, T) \). By (3.5), we have the following estimates:
\[
\begin{align*}
|y(\cdot; u)|_{C([0, T]; \mathcal{X})} &\leq C|u|_{L^2(0,T;\mathcal{V})}, \\
|y(\cdot; u) - \bar{y}(\cdot)|_{C([0, T]; \mathcal{X})} &\leq C|u - \bar{u}|_{L^2(0,T;\mathcal{V})}, \\
|\hat{\xi}(\cdot; v)|_{C([0, T]; \mathcal{X})} &\leq C|v|_{L^2(0,T;\mathcal{V})}.
\end{align*}
\] (3.13)
Further, by (3.6)-(3.8), for any \( u, v \in L^2(0, T; \mathcal{V}) \),
\[
|\xi^0 - \hat{\xi}^0| + |\xi(\cdot ; v) - \hat{\xi}(\cdot ; v)|_{C([0, T]; \mathcal{X})} \leq C|u - \bar{u}|_{L^2(0, T; \mathcal{V})}v|_{L^2(0, T; \mathcal{V})}.
\]
(3.14)

Here \( C \) is a positive constant depending on \( \delta_N^2, \delta_N^3, L_3, L_4, L_5, L_6 \) and \( \bar{u} \), but independent of \( u(\cdot) \) and \( v(\cdot) \) in \( L^2(0, T; \mathcal{V}) \). Note that \( |v|_{L^2(0, T; \mathcal{V})} \leq 1 \) in (3.13) and (3.14). It follows from (3.14) that \((H_1)\) and \((H_2)\) in Theorem 2.1 hold true. Then we have the following necessary condition on \( \bar{u}(\cdot) \) by Theorem 2.1.

**Corollary 3.1** Assume that \((A_2)\) holds. Let \( \bar{u} \) be the optimal control of (3.4) and \( \bar{y} \) be the corresponding optimal state. If \( \text{Var}_{U_2(0, T)} f(\bar{u}) \) defined by (3.12) is finite codimensional in \( \mathcal{X} \), then there exists a non-zero pair \( (z_0, z) \in [0, +\infty) \times \mathcal{X}' \), such that
\[
H_u(t, \bar{y}(t), \bar{u}(t), z_0, \psi(t)) = 0, \quad \text{a.e. } t \in (0, T),
\]
where \( H \) is the Hamiltonian function defined by (2.62) and \( \psi \) is the solution to (2.63).

Moreover,
(i) If \( z_0 \neq 0 \), then the above results hold with \( (z_0, z) \) replaced by \( (1, \frac{z_0}{z_0}) \).
(ii) if \( z \neq 0 \), then there exists a sequence \( \{u^k(\cdot)\}_{k=1}^\infty \subseteq U_2(0, T) \), which converges to \( \bar{u}(\cdot) \) in \( L^2(0, T; \mathcal{V}) \) as \( k \to \infty \), such that
\[
\begin{align*}
g(T; u^k) &\neq y_1, \\
\lim_{k \to \infty} g(T; u^k) &= y_1, \\
\lim_{k \to \infty} J(u^k) &= J(\bar{u}).
\end{align*}
\]
Furthermore,
\[
(z, \hat{\xi}(T; v) - y_1)_\mathcal{X} > 0, \quad \forall \ k \in \mathbb{N},
\]
where \( \hat{\xi} \in \mathcal{X} \) is the element corresponding to \( z \in \mathcal{X}' \) by the Riesz-Fréchet isomorphism.

**Proof.** By Theorem 2.1, there exists a non-zero pair \( (z_0, z) \in [0, +\infty) \times \mathcal{X}' \) with \( z \in \mathcal{N}_E(\bar{y}(T)) \), such that
\[
0 = \xi^0 + (z, \hat{\xi}(T; v))_{\mathcal{X}' \times \mathcal{X}} \geq 0, \quad \forall \ (\xi^0, \hat{\xi}(T; v))^{\top} \in \text{Var}_{U_2(0, T)}(f(0), f)(\bar{u})
\]
(3.16)
with \( \text{Var}_{U_2(0, T)}(f(0), f)(\bar{u}) \) defined by (3.11). Then, by the duality relationship between the system (3.10) and the adjoint system (2.63), we obtain from (3.16) that
\[
\int_0^T \langle F_u(t, \bar{y}(t), \bar{u}(t))^{\top} \psi(t) + z_0 g(t, \bar{y}(t), \bar{u}(t)), v(t) \rangle_{\mathcal{V}' \times \mathcal{V}} \, dt \geq 0, \quad \forall \ v(\cdot) \in U_2(0, T),
\]
which implies (3.15). The rest of Corollary 3.1 follows from Theorem 2.1 directly.

**Remark 3.1** The conditions (3.6)-(3.8) are used to ensure the validity of \((H_1)\) in Theorem 2.1. The condition \((A_{11})\) in [28, Page 185] was employed to study an optimal control problem for evolution equations with endpoint constraints. The condition \((A_{11})\) is weaker than the conditions in \((A_2)\) of this paper, and it is not sufficient for \((H_1)\) in Theorem 2.1.

For any \( \phi_T \in \mathcal{X}' \), consider the following equation corresponding to (3.10):
\[
\begin{align*}
\phi_1(t) &= -A^* \phi(t) - F_y(t, \bar{y}(t), \bar{u}(t))^{\top} \phi(t), \quad t \in [0, T], \\
\phi(T) &= \phi_T.
\end{align*}
\]
(3.17)
It follows from the duality relationship between (3.10) and (3.17) that
\[ f'(\bar{u})^* \phi_T = F_u(t, \bar{y}(t), \bar{u}(t))^* \phi \quad \forall \phi_T \in \mathbb{X}'. \]

By Theorem 3.1, we have the following results.

**Corollary 3.2** The following assertions are equivalent:

1. The set \( \text{Var}_{t \in (0,T)} f(\bar{u}) \) is finite codimensional in \( \mathbb{X} \).
2. There is a finite codimensional subspace \( \mathbb{K} \subseteq \mathbb{X}' \), such that for any \( \phi_T \in \mathbb{K} \), the solution \( \phi(\cdot) \) to (3.17) satisfies that
   \[ |\phi_T|_{\mathbb{K}} \leq C F_u(\cdot, \bar{y}(\cdot), \bar{u}(\cdot))^* \phi_{L^2(0,T;\mathbb{Y})}. \]
3. There is a Banach space \( \mathbb{W} \) and a compact operator \( \mathcal{G} \) from \( \mathbb{X}' \) to \( \mathbb{W} \), such that for any \( \phi_T \in \mathbb{K} \), the solution \( \phi(\cdot) \) to (3.17) satisfies that
   \[ |\phi_T|_{\mathbb{K}} \leq C (|F_u(\cdot, \bar{y}(\cdot), \bar{u}(\cdot))^* \phi_{L^2(0,T;\mathbb{Y})} + |\mathcal{G}\phi_T|_{\mathbb{W}}). \]

This result has been applied to study the finite codimensionality condition of optimal control problems for certain wave/heat equations, as described in [28]. The associated \( \text{a priori} \) estimates have been easily verified to be true or false.

### 3.2 An optimal control problem for an elliptic equation with pointwise state constraint

In this subsection, an elliptic optimal control problem with a pointwise state constraint is studied. The purpose of this subsection is to show that different control and state spaces may lead to different results on finite codimensionality by using a simple example.

Assume that \( G \subseteq \mathbb{R}^n (n \in \mathbb{N}) \) is a bounded domain with a smooth boundary \( \Gamma \), and \( \hat{X} \) and \( \hat{U} \) are two Hilbert spaces. In what follows, we consider two different cases:

(I) \( \hat{X} = \hat{U} = L^2(G) \);

(II) \( \hat{X} = H^1_0(G) \) and \( \hat{U} = H^{-1}(G) \).

Assume that \( (a^{ij}(\cdot))_{1 \leq i,j \leq n} \in W^{1,\infty}(G; \mathbb{R}^{n \times n}) \) is uniformly positive definite in \( G \). Consider the following elliptic equation:

\[
\begin{cases}
- \sum_{i,j=1}^n (a^{ij}(x)y_{x_i})_{x_j} = F(x)y + u & \text{in } G, \\
y = 0 & \text{on } \Gamma,
\end{cases}
\]

(3.18)

where \( u(\cdot) \in \hat{U} \) is the control variable, \( y(\cdot) \in \hat{X} \) is the state variable and \( F(\cdot) \in L^\infty(G) \) with \( F(x) \leq 0 \) in \( G \). For any given \( y_d(\cdot) \in \hat{X} \), set

\[ J(u(\cdot)) = \frac{1}{2} |y(\cdot) - y_d(\cdot)|^2_{\hat{X}} + \frac{1}{2} |u(\cdot)|^2_{\hat{U}} \]

and

\[ E = \{ y(\cdot) \in \hat{X} \mid y(x) \geq 0 \text{ a.e. in } G \}. \]

In both cases (I) and (II), (3.18) is well-posed. The optimal control problem considered in this subsection is the following one:

\[ \text{Minimize } J(u(\cdot)) \text{ subject to } y(\cdot; u) \in E \text{ and } u(\cdot) \in \hat{U}. \]

(3.19)
Assume that \( \bar{u}(\cdot) \) is an optimal control of the problem (3.19) and denote by \( \bar{y}(\cdot) \) the corresponding optimal state.

This optimal control problem is a special case of the optimization problem (P) by choosing

\[
V = \hat{U}, \quad X = \hat{X}, \quad f_0(u) = J(u(\cdot)), \quad f(u) = y(\cdot; u) \quad \text{and} \quad K = \hat{U},
\]

where \( y(\cdot; u) \) is the solution to (3.18) corresponding to \( u(\cdot) \in \hat{U} \).

Consider the following elliptic equation:

\[
\left\{ \begin{array}{l}
- \sum_{i,j=1}^{n} (a^{ij}(x_\xi))_x = F(x)\xi + v \quad \text{in } G, \\
\xi = 0 \quad \text{on } \Gamma.
\end{array} \right. \tag{3.20}
\]

We have that, for any \( u(\cdot) \in \hat{U} \) with the corresponding state \( y(\cdot; u) \) to (3.18),

\[
\text{Var}_{\hat{U}}(f_0, f)(u) = \left\{ (\xi^0, \xi(\cdot; v))^T \in \mathbb{R} \times \hat{X} \mid \xi(\cdot; v) \text{ solves (3.20) and} \right. \\
\xi^0 = (y(\cdot; u) - y_d, \xi(\cdot; v))_{\hat{X}} + (u, v)_{\hat{U}}, \text{ for } v(\cdot) \in \hat{U} \text{ with } |v|_{\hat{U}} \leq 1 \right\},
\]

where \( (\cdot, \cdot)_{\hat{X}} \) and \( (\cdot, \cdot)_{\hat{U}} \), respectively, denote the inner products on the Hilbert space \( \hat{X} \) and \( \hat{U} \). For the optimal control \( \bar{u}(\cdot) \) with the corresponding optimal state \( \bar{y}(\cdot) \), it is easy to show that

\[
\text{Var}_{\hat{U}}(f_0, f)(\bar{u}) = \left\{ (\xi^0, \xi(\cdot; v))^T \in \mathbb{R} \times \hat{X} \mid \xi(\cdot; v) \text{ solves (3.20) and} \right. \\
\xi^0 = (\bar{y} - y_d, \xi(\cdot; v))_{\hat{X}} + (\bar{u}, v)_{\hat{U}}, \text{ for some } v \in \hat{U} \text{ with } |v|_{\hat{U}} \leq 1 \right\},
\]

and for any \( u \in \hat{U}, f'(u)v = \xi(\cdot; v) \), where \( \xi(\cdot; v) \) is the solution to (3.20) corresponding to \( v(\cdot) \).

In addition, for any \( h \in \hat{X}' = \hat{U}' = \hat{X}' \) is the corresponding element of the solution \( \varphi \in \hat{X} \) to the following elliptic equation by the canonical isomorphism:

\[
\left\{ \begin{array}{l}
- \sum_{i,j=1}^{n} (a^{ij}(x)\varphi_{x_i})_x = F(x)\varphi + h \quad \text{in } G, \\
\varphi = 0 \quad \text{on } \Gamma.
\end{array} \right. \tag{3.21}
\]

Choose \( \mathcal{V}(\cdot) = \text{Var}_{\hat{U}}(f_0, f)(\cdot) \). It is easy to show that the conditions \( (H_1) \) and \( (H_2) \) in Theorem 2.1 hold. Consider the following elliptic equation:

\[
\left\{ \begin{array}{l}
- \sum_{i,j=1}^{n} (a^{ij}(x)\psi_{x_i})_x = F(x)\psi + z + \bar{w} \quad \text{in } G, \\
\psi = 0 \quad \text{on } \Gamma,
\end{array} \right. \]

where \( \bar{w} \in \hat{X}' \) is the corresponding element of \( z_0(\bar{y} - y_d) \in \hat{X} \) by the Riesz-Fréchet isomorphism with \( z_0 \in \mathbb{R} \).

As a corollary of Theorem 2.1, we have that if \( \text{Var}_{\hat{U}}(f)(\bar{u}) \) is finite codimensional in \( \hat{X} \), then there exists a non-zero pair \((z_0, z) \in \mathbb{R} \times \hat{X}'\), such that \( \psi \in \hat{X} \) is the corresponding element of \( -z_0\bar{u} \in \hat{U} = X' \) by the Riesz-Fréchet isomorphism.

By Theorem 3.1, the finite codimensionality of the set \( \text{Var}_{\hat{U}}(f)(\bar{u}) \) in \( \hat{X} \) is reduced to the following estimate:

\[
|h|_{\hat{X}'} \leq C(|\bar{\varphi}|_{\hat{U}'} + |\mathcal{G}h|_{W}) = C(|\varphi|_{\hat{X}} + |\mathcal{G}h|_{W}), \quad \forall \ h \in \hat{X}', \tag{3.22}
\]
where $G$ is a compact operator from $\hat{X}'$ to a Banach space $W$ and $\varphi(\cdot)$ is the solution to (3.21).

In the following, we choose different spaces $\hat{X}$ and $\hat{U}$ to study finite codimensionality of $\text{Var}_{\hat{G}} f(\hat{u})$, based on the \textit{a priori} estimate (3.22).

1) Let $\hat{X} = \hat{U} = L^2(G)$. We claim that $\text{Var}_{\hat{G}} f(\hat{u})$ is not finite codimensional in $\hat{X}$.

Otherwise, by (3.22), we have that

$$|h|_{L^2(G)} \leq C(|\varphi|_{L^2(G)} + |Gh|_W), \quad \forall \ h \in L^2(G). \tag{3.23}$$

Write

$$S = \{ \varphi \in H^2(G) \cap H^1_0(G) \mid \varphi \text{ is the solution to (3.21) for some } h \in L^2(G) \}. \tag{3.24}$$

By the classical $L^2$ estimate for elliptic equations and (3.23), $S$ is an infinite-dimensional subspace of $H^2(G) \cap H^1_0(G)$ and

$$|\varphi|_{H^2(G)} \leq C|h|_{L^2(G)} \leq C(|\varphi|_{L^2(G)} + |Gh|_W), \quad \forall \ h \in L^2(G). \tag{3.25}$$

Let $\{\varphi_k\}_{k=1}^{\infty} \subseteq S$ with $|\varphi_k|_{H^2(G)} = 1$. Set

$$h_k = -\sum_{i,j=1}^{n} (a^{ij}(x)\varphi_{k,x_i})x_j - F(x)\varphi_k.$$ 

Then we have

$$|h_k|_{L^2(G)} \leq C|\varphi_k|_{H^2(G)} + |F|_{L^\infty(G)}|\varphi_k|_{L^2(G)} \leq C.$$ 

This implies that $\{h_k\}_{k=1}^{\infty}$ has a weakly convergent subsequence $\{h_{k_j}\}_{j=1}^{\infty}$ in $L^2(G)$. Since $G$ is a compact operator,

$$\{Gh_{k_j}\}_{j=1}^{\infty} \text{ is a Cauchy sequence in } W. \tag{3.26}$$

From (3.24) and (3.25), $\{\varphi_{k_j}\}_{j=1}^{\infty}$ is a Cauchy sequence in $H^2(G) \cap H^1_0(G)$. This contradicts with the fact that $S$ is an infinite-dimensional subspace of $H^2(G) \cap H^1_0(G)$. Therefore, $\text{Var}_{\hat{G}} f(\hat{u})$ is not finite codimensional in $L^2(G)$.

2) Choose $\hat{X} = H^1_0(G)$ and $\hat{U} = H^{-1}(G)$. Then, the set $\text{Var}_{\hat{G}} f(\hat{u})$ is finite codimensional in $\hat{X}$.

By the classical $L^2$ theory for elliptic equations, we have

$$|h|_{H^{-1}(G)} \leq C|\varphi|_{H^1_0(G)}, \quad \forall \ h \in H^{-1}(G).$$ 

Then, (3.22) holds true with $G = 0$. Consequently, by Theorem 3.1, $\text{Var}_{\hat{G}} f(\hat{u})$ is finite codimensional in $H^1_0(G)$.

\textbf{Remark 3.2} This example presents us with a method of choosing a suitable control space and state space to guarantee the finite codimensionality. In fact, by this method, it is also easy to show that if $\hat{X} = H^2(G) \cap H^1_0(G)$ and $\hat{U} = L^2(G)$, then $\text{Var}_{\hat{G}} f(\hat{u})$ is finite codimensional in $\hat{X}$. But if $\hat{X} = H^1_0(G)$ and $\hat{U} = L^2(G)$, then $\text{Var}_{\hat{G}} f(\hat{u})$ is not finite codimensional in $\hat{X}$.
3.3 An optimal control problem for stochastic differential equations with endpoint state constraint

3.3.1 Formulation of the problem and a necessary condition for optimal controls

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space, on which a one-dimensional standard Brownian motion \(\{B(t)\}_{t \geq 0}\) is defined, such that \(\{\mathcal{F}_t\}_{t \geq 0}\) is the natural filtration generated by \(B(\cdot)\), augmented by all the \(\mathbb{P}\)-null sets in \(\mathcal{F}\). Denote by \(\mathbb{F}\) the progressive \(\sigma\)-field (in \([0, T] \times \Omega\)) with respect to \(\{\mathcal{F}_t\}_{t \geq 0}\). For any \(\alpha, \beta \geq 1\) and \(t \in [0, T]\), denote by \(L^\beta_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)\) the set of all \(\mathbb{R}^n\)-valued, \(\mathcal{F}_t\)-measurable random variables \(\zeta\) with \(\mathbb{E}|\zeta|^{\beta}_{\mathbb{R}^n} < \infty\), by \(L^\beta_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^n))\) the space of \(\mathbb{R}^n\)-valued, \(\mathbb{F}\)-progressively measurable continuous stochastic processes \(\eta\) such that \(\mathbb{E}(\sup_{0 \leq t \leq T} |\eta(t)|^{\beta}_{\mathbb{R}^n}) < \infty\), and by \(L^\beta_{\mathcal{F}}(\Omega; L^\alpha(0, T; \mathbb{R}^n))\) the set of all \(\mathbb{R}^n\)-valued, \(\mathbb{F}\)-progressively measurable stochastic processes \(\eta\) with \(\mathbb{E}\left(\int_0^T |\eta(t, \omega)|^{\alpha}_{\mathbb{R}^n} dt\right)^{\beta/\alpha} < \infty\). When \(\alpha = \beta\), it is simply denoted by \(L^\beta_{\mathbb{F}}(0, T; \mathbb{R}^n)\). As usual, when the context is clear, we omit the argument \(\omega \in \Omega\) in the functions.

Consider the following controlled stochastic differential equation:

\[
\begin{align*}
    dy(t) &= a(t, y(t), u(t))dt + b(t, y(t), u(t))dB(t), \quad t \in [0, T], \\
y(0) &= y_0,
\end{align*}
\]

where \(u(\cdot)\) is the control variable and \(y(\cdot)\) is the state variable. Set

\[
J(u(\cdot)) = \mathbb{E} \int_0^T g_0(t, y(t), u(t))dt, \quad \forall u(\cdot) \in L^\beta_{\mathbb{F}}(0, T; \mathbb{R}^m).
\]

Assume that \(a, b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}^n\) and \(g_0 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}\) satisfy the following conditions:

\((A_3)\) For any \((y, u) \in \mathbb{R}^n \times \mathbb{R}^m\), \(a(\cdot, y, u, \cdot), b(\cdot, y, u, \cdot) : [0, T] \times \Omega \to \mathbb{R}^n\) and \(g_0(\cdot, y, u, \cdot) : [0, T] \times \Omega \to \mathbb{R}\) are \(\mathbb{F}\)-progressively measurable. For a.e. \((t, \omega) \in [0, T] \times \Omega\), \(a(t, \cdot, \cdot, \omega), b(t, \cdot, \cdot, \omega)\) and \(g_0(t, \cdot, \cdot, \omega)\) are continuously differentiable. Moreover, there is a positive constant \(L_7\) and \(\eta(\cdot) \in L^\infty_{\mathbb{F}}(0, T; \mathbb{R})\), such that, for a.e. \((t, \omega) \in [0, T] \times \Omega\), \(y \in \mathbb{R}^n\) and \(u \in \mathbb{R}^m\),

\[
    \begin{align*}
    |a(t, y, u)|_{\mathbb{R}^{n \times n}} + |a_0(t, y, u)|_{\mathbb{R}^{n \times m}} &\leq L_7, \\
    |b(t, y, u)|_{\mathbb{R}^{n \times n}} + |b_0(t, y, u)|_{\mathbb{R}^{n \times m}} &\leq L_7, \\
    |a(t, 0, 0)|_{\mathbb{R}^n} + |b(t, 0, 0)|_{\mathbb{R}^n} &\leq \eta(t),
    \end{align*}
\]

and

\[
    \begin{align*}
    |g_0(t, y, u)| &\leq L_7(1 + |y|_{\mathbb{R}^n}^2 + |u|_{\mathbb{R}^m}^2), \\
    |g_{0,y}(t, y, u)|_{\mathbb{R}^n} &\leq L_7(1 + |y|_{\mathbb{R}^n} + |u|_{\mathbb{R}^m}).
    \end{align*}
\]

In addition, there exist positive constants \(L_8, L_9,\) and \(L_{10}\), such that

\[
    \begin{align*}
    |a(t, y_1, u) - a(t, y_2, u)|_{\mathbb{R}^{n \times n}} + |b(t, y_1, u) - b(t, y_2, u)|_{\mathbb{R}^{n \times n}} + |a_0(t, y_1, u) - a_0(t, y_2, u)|_{\mathbb{R}^{n \times m}} + |b_0(t, y_1, u) - b_0(t, y_2, u)|_{\mathbb{R}^{n \times m}} &\leq L_8|y_1 - y_2|_{\mathbb{R}^n}, \\
    &\text{a.e. } (t, \omega) \in [0, T] \times \Omega, \forall u \in \mathbb{R}^m \text{ and } \forall y_1, y_2 \in \mathbb{R}^n,
    \end{align*}
\]

\[
    \begin{align*}
    |a_y(t, y_1, u) - a_y(t, y_2, u)|_{\mathbb{R}^{n \times n}} + |b_y(t, y_1, u) - b_y(t, y_2, u)|_{\mathbb{R}^{n \times n}} + |a_0(t, y_1, u) - a_0(t, y_2, u)|_{\mathbb{R}^{n \times m}} + |b_0(t, y_1, u) - b_0(t, y_2, u)|_{\mathbb{R}^{n \times m}} &\leq L_9|u_1 - u_2|_{\mathbb{R}^m}, \\
    &\text{a.e. } (t, \omega) \in [0, T] \times \Omega, \forall y \in \mathbb{R}^n \text{ and } \forall u_1, u_2 \in \mathbb{R}^m,
    \end{align*}
\]

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and
\[ |g_{0,y}(t, y_1, u_1) - g_{0,y}(t, y_2, u_2)| \in \mathbb{R}^n + |g_{0,u}(t, y_1, u_1) - g_{0,u}(t, y_2, u_2)| \in \mathbb{R}^m \]
\[ \leq L_1(|y_1 - y_2| \in \mathbb{R}^n + |u_1 - u_2| \in \mathbb{R}^m), \]
a.e. \((t, \omega) \in [0, T] \times \Omega, \forall y_1, y_2 \in \mathbb{R}^n\) and \(\forall u_1, u_2 \in \mathbb{R}^m\).

Under the condition \((A_3)\), the equation (3.26) is well-posed (see, e.g., [29, Section 3.1]) and \(J(u(\cdot)) < +\infty\) for any \(u(\cdot) \in L^2(0, T; \mathbb{R}^m)\).

For a given \(y_T \in L^2_{F_T}(\Omega; \mathbb{R}^n)\), consider the following stochastic optimal control problem:

\[
\text{Minimize } J(u(\cdot)) \text{ subject to } y(t, u) = y_T \text{ and } u(\cdot) \in L^2(0, T; \mathbb{R}^m). \tag{3.27}
\]

Suppose that \(\tilde{u}(\cdot)\) solves the optimal control problem (3.27) and \(\bar{y}(\cdot)\) is the corresponding state of (3.26). Let us give a necessary condition of \(\tilde{u}\) by Theorem 2.1. In the optimization problem (P), choose

\[
\begin{cases}
V = L^2(0, T; \mathbb{R}^m), & X = L^2(\Omega; \mathbb{R}^n), & E = \{y_T\}, \\
f_0(u) = J(u(\cdot)), & f(u) = y(T; u), & \mathcal{K} = L^2(0, T; \mathbb{R}^m),
\end{cases}
\]

where \(y(\cdot) = y(\cdot; u)\) is the solution to (3.26) corresponding to \(u(\cdot) \in L^2(0, T; \mathbb{R}^m)\).

For any \(u, v \in L^2(0, T; \mathbb{R}^m)\), consider the stochastic differential equation:

\[
\begin{align*}
d\xi(t) & = [a_y(t, y(t; u), u(t)) \xi(t) + a_u(t, y(t; u), u(t))v(t)]dt \\
& \quad + [b_y(t, y(t; u), u(t)) \xi(t) + b_u(t, y(t; u), u(t))v(t)]dB(t), \quad t \in [0, T], \\
\xi(0) & = 0.
\end{align*}
\tag{3.28}
\]

Under the condition \((A_3)\), the equation (3.28) is also well-posed. Moreover, we have

\[
\text{Var}_{L^2(0, T; \mathbb{R}^m)}(f_0, f)(u) = \left\{ \left( \xi^0, \xi(T; u) \right)^\top \in \mathbb{R} \times L^2_{F_T}(\Omega; \mathbb{R}^n) \bigg| \xi(\cdot; v) \text{ solves (3.28)} \right\},
\]

\[
\xi^0 = E^\mathbb{F} \int_0^T \left[ g_{0,y}(t, y(t; u), u(t))^\top \xi(t; v) + g_{0,u}(t, y(t; u), u(t))^\top v(t) \right] dt,
\]

for \(v(\cdot) \in L^2(0, T; \mathbb{R}^m)\) with \(|v|_{L^2(0, T; \mathbb{R}^m)} \leq 1\)

and \(f'(u)v = \xi(T; v)\).

For simplicity of notations, we write

\[
\begin{align*}
A_1(t) & \triangleq a_y(t, \bar{y}(t), \bar{u}(t)), & C_1(t) & \triangleq a_u(t, \bar{y}(t), \bar{u}(t)), \\
A_2(t) & \triangleq b_y(t, \bar{y}(t), \bar{u}(t)), & C_2(t) & \triangleq b_u(t, \bar{y}(t), \bar{u}(t)), \\
A_3(t) & \triangleq g_{0,y}(t, \bar{y}(t), \bar{u}(t)), & C_3(t) & \triangleq g_{0,u}(t, \bar{y}(t), \bar{u}(t)),
\end{align*}
\]

and consider the stochastic differential equation:

\[
\begin{cases}
d\hat{\xi}(t) = [A_1(t)\hat{\xi}(t) + C_1(t)v(t)]dt + [A_2(t)\hat{\xi}(t) + C_2(t)v(t)]dB(t), \quad t \in [0, T], \\
\hat{\xi}(0) = 0.
\end{cases}
\tag{3.29}
\]

Then

\[
\text{Var}_{L^2(0, T; \mathbb{R}^m)}(f_0, f)(\bar{u}) = \left\{ \left( \xi^0, \hat{\xi}(T; v) \right)^\top \in \mathbb{R} \times L^2_{F_T}(\Omega; \mathbb{R}^n) \bigg| \hat{\xi}(\cdot; v) \text{ solves (3.29)} \right\},
\]

\[
\xi^0 = E^\mathbb{F} \int_0^T \left[ A_3(t)^\top \hat{\xi}(t; v) + C_3(t)^\top v(t) \right] dt,
\tag{3.30}
\]

for \(v(\cdot) \in L^2(0, T; \mathbb{R}^m)\) with \(|v|_{L^2(0, T; \mathbb{R}^m)} \leq 1\).
and
\[
\text{Var}_{L^2_p}(0,T;\mathbb{R}^m)f(\bar{u}) = \left\{ \hat{\xi}(T; v) \in L^2_p(\Omega; \mathbb{R}^n) \mid \hat{\xi}(\cdot; v) \text{ solves (3.29)}, \right. \\
\left. \text{for } v(\cdot) \in L^2_p(0,T; \mathbb{R}^m) \text{ with } |v|_{L^2_p(0,T;\mathbb{R}^m)} \leq 1 \right\}. \quad (3.31)
\]

Choose \( \mathcal{V}(\cdot) = \text{Var}_{L^2_p}(0,T;\mathbb{R}^m)(f^0, f)(\cdot) \). By the condition (A_3), it is easy to show that the conditions (H_1) and (H_2) in Theorem 2.1 hold. As a corollary of Theorem 2.1, one gets the following necessary condition for the optimal control \( \bar{u}(\cdot) \).

**Corollary 3.3** Assume that (A_3) holds. Let \( \bar{u} \) be the optimal control of (3.27) and \( \bar{y} \) be the corresponding optimal state. If \( \text{Var}_{L^2_p}(0,T;\mathbb{R}^m) f(\bar{u}) \) defined as (3.31) is finite codimensional in \( L^2_p(\Omega; \mathbb{R}^n) \), then there exists a non-zero pair \( (z, z) \in \mathbb{R} \times L^2_p(\Omega; \mathbb{R}^n) \), such that
\[
\mathcal{H}_u(t, \bar{y}(t), \bar{u}(t), z_0, \bar{v}(t), \bar{\Psi}(t)) = 0, \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega, \quad (3.32)
\]
where
\[
\mathcal{H}(t, y, u, z_0, \psi, \bar{\Psi}) = \Delta(\psi, a(t, y, u))_{\mathbb{R}^n} + (\bar{\Psi}, b(t, y, u))_{\mathbb{R}^n} - z_0 g_0(t, y, u),
\]
\[
\forall (t, y, u, z_0, \psi, \bar{\Psi}) \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times \Omega, \quad (3.33)
\]
and \((\psi, \bar{\Psi})\) solves the backward stochastic differential equation
\[
\begin{aligned}
d\psi(t) &= -[z_0 A_3(t) + A_1(t)^{\top} \psi(t) + A_2(t)^{\top} \bar{\Psi}(t)] dt + \bar{\Psi}(t) dB(t), \quad t \in [0, T], \\
\psi(T) &= z.
\end{aligned} \quad (3.34)
\]

**Proof.** By Theorem 2.1, there exists a non-zero pair \( (z_0, z) \in \mathbb{R} \times L^2_p(\Omega; \mathbb{R}^n) \), such that
\[
z_0 \hat{\xi}^0 + \mathbb{E}(z^{\top} \hat{\xi}(T; v)) \geq 0, \quad \forall (\hat{\xi}^0, \hat{\xi}(T; v))^{\top} \in \text{Var}_{L^2_p}(0,T;\mathbb{R}^m)(f, f_0)(\bar{u}), \quad (3.35)
\]
where \( \text{Var}_{L^2_p}(0,T;\mathbb{R}^m)(f, f_0)(\bar{u}) \) is defined by (3.30). By the classical well-posedness result for backward stochastic differential equations (e.g., [29, Section 4.1]), (3.34) admits a unique solution \((\psi(\cdot), \bar{\Psi}(\cdot)) \in L^2_p(\Omega; C([0, T]; \mathbb{R}^n)) \times L^2_p(0, T; \mathbb{R}^n) \). Further, by the duality relationship between (3.29) and (3.34), it is easy to show that (3.35) implies (3.32). \( \square \)

### 3.3.2 The finite codimensionality of \( \text{Var}_{L^2_p}(0,T;\mathbb{R}^m)f(\bar{u}) \)

To begin with, we introduce the following backward stochastic differential equation:
\[
\begin{aligned}
d\phi(t) &= -[A_1(t)^{\top} \phi(t) + A_2(t)^{\top} \bar{\Phi}(t)] dt + \bar{\Phi}(t) dB(t), \quad t \in [0, T], \\
\phi(T) &= \phi_T.
\end{aligned} \quad (3.36)
\]

By the duality relationship between (3.29) and (3.36), and the definition of \( f'(\bar{u}) \), it is easy to show that
\[
f'(\bar{u})^{\top} (\phi_T) = C_1(\cdot)^{\top} \phi(\cdot) + C_2(\cdot)^{\top} \bar{\Phi}(\cdot), \quad \forall \phi_T \in L^2_p(\Omega; \mathbb{R}^n),
\]
where \((\phi(\cdot), \bar{\Phi}(\cdot))\) is the solution to (3.36) with the terminal datum \( \phi_T \). Then, by Theorem 3.1, one has the following results.
Theorem 3.2

The assertion (3.36) (e.g., [29, Section 4.1]), where the process $\phi$ This implies that $G$ only if $C$ constant.

First, we assume that $\text{rank}(G) = n$. By the well-posedness result for stochastic differential equations (e.g., [29, Section 3.1]), there is a Banach space $W$ and a compact operator $G$ from $L^2_{T_T}(\Omega; \mathbb{R}^n)$ to $W$, such that for any $\phi_T \in L^2_{T_T}(\Omega; \mathbb{R}^n)$, the solution $(\phi(\cdot), \Phi(\cdot))$ to (3.36) satisfies

$$E|\phi_T|_{\mathbb{R}^n}^2 \leq C E \int_0^T |C_1(t)^T \phi(t) + C_2(t)^T \Phi(t)|_{\mathbb{R}^n}^2 dt. \quad (3.37)$$

Next, we consider a special case, i.e., $C_2(\cdot) = C_2 \in \mathbb{R}^{n \times m}$. This holds when $b(\cdot, \cdot, \cdot)$ (in (3.26)) has the form of $b(t, y, u) = b_1(t, y) + C_2 u$. Define a linear operator $G : L^2_{T_T}(\Omega; \mathbb{R}^n) \to \mathbb{R}^n$ by

$$G\phi_T = \phi(0), \quad \forall \phi_T \in L^2_{T_T}(\Omega; \mathbb{R}^n), \quad (3.38)$$

where the process $\phi(\cdot)$ is the solution to (3.36) with the terminal datum $\phi_T$. By the well-posedness of (3.36) (e.g., [29, Section 4.1]), $G$ is a bounded linear operator. Then we have the following result.

Theorem 3.2 The assertion (3) in Corollary 3.4 holds for the operator $G$ defined by (3.38), if and only if $\text{rank}(C_2) = n$.

Proof. First, we assume that $\text{rank}(C_2) = n$. By the well-posedness of (3.36), there is a positive constant $C$, such that for any $\phi_T \in L^2_{T_T}(\Omega; \mathbb{R}^n)$,

$$|\phi(0)|_{\mathbb{R}^n} \leq C |\phi_T|_{L^2_{T_T}(\Omega; \mathbb{R}^n)}. \quad (3.37)$$

This implies that $G(\mathcal{M})$ is a bounded set in $\mathbb{R}^n$ whenever $\mathcal{M} \subseteq L^2_{T_T}(\Omega; \mathbb{R}^n)$ is a bounded set. Therefore, $G$ is compact.

Set

$$\hat{\ell}(t) = C_1(t)^T \phi(t) + C_2(t)^T \Phi(t), \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega. \quad (3.39)$$

Since $\text{rank}(C_2) = n$, there is a matrix $\tilde{C}_2 \in \mathbb{R}^{n \times m}$, such that $\tilde{C}_2 C_2^T \Phi(t) = \Phi(t)$. Then we have

$$\Phi(t) = \tilde{C}_2 \hat{\ell}(t) - \tilde{C}_2 C_1(t)^T \phi(t), \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega. \quad (3.40)$$

If a pair $(\phi(\cdot), \Phi(\cdot))$ solves (3.36), by (3.40), we obtain

$$\begin{cases} d\phi(t) = -[A_1^T(t) \phi(t) + A_2^T(t) \tilde{C}_2 \hat{\ell}(t) - A_2^T(t) \tilde{C}_2 C_1^T \phi(t)] dt + [\tilde{C}_2 \hat{\ell}(t) - \tilde{C}_2 C_1(t)^T \phi(t)] dB(t), & t \in [0, T], \\ \phi(0) = \phi(0). \end{cases} \quad (3.41)$$

By the well-posedness result for stochastic differential equations (e.g., [29, Section 3.1]), there is a positive constant $C$, such that

$$E|\phi(T)|_{\mathbb{R}^n}^2 \leq C \left( E \int_0^T |\hat{\ell}(t)|_{\mathbb{R}^n}^2 dt + |\phi(0)|_{\mathbb{R}^n}^2 \right).$$
This, together with (3.39), implies (3.37).

On the other hand, we assume that the assertion (3) in Corollary 3.4 holds for the operator $G$ defined by (3.38). Consider the following stochastic differential equation:

$$
\begin{align*}
\left\{ \begin{array}{l}
d\psi(t) = -[A_1(t)^\top \psi(t) + A_2(t)^\top r(t)]dt + r(t)dB(t), \quad t \in [0,T], \\
\psi(0) = \psi_0,
\end{array} \right.
\end{align*}
$$

(3.42)

where $\psi_0 \in \mathbb{R}^n$ and $r(\cdot) \in L^2_E(0,T;\mathbb{R}^n)$. Clearly, if $\psi(\cdot)$ is a solution to (3.42) with the initial datum $\psi_0$, then $(\phi(\cdot), \Phi(\cdot)) \hat{=} (\psi(\cdot), r(\cdot))$ is a solution to (3.36) with the terminal datum $\phi(T) = \psi(T)$. By (3.37) and the definition of $G$, we have the following estimate for any solution to (3.42):

$$
\mathbb{E}|\psi(T)|^2_{\mathbb{R}^n} \leq C(\mathbb{E} \int_0^T |C_1(t)^\top \psi(t) + C_2^\top r(t)|^2_{\mathbb{R}^n} dt + |\psi_0|^2_{\mathbb{R}^n}),
$$

\(\forall \psi_0 \in \mathbb{R}^n\) and \(r(\cdot) \in L^2_E(0,T;\mathbb{R}^n)\).

(3.43)

If rank($C_2$) < $n$, we can find a $\hat{r} \in \mathbb{R}^n \setminus \{0\}$, such that $C_2^\top \hat{r} = 0$. For any $k \in \mathbb{N}$, let $r_k(\cdot) = \chi_{[T-1/k,T]}(\cdot)\sqrt{k}\hat{r}$, and $\psi_k(\cdot)$ be the corresponding solution to (3.42) for $r(\cdot) = r_k(\cdot)$ and $\psi_0 = 0$, where $\chi_{[T-1/k,T]}(\cdot)$ denotes the characteristic function on $[T-1/k, T]$. Then, by the well-posedness of the equation (3.42) (e.g., [29, Section 3.1]), $\psi(t) = 0$ in $[0, T-1/k]$. It follows from (3.43) that

$$
\mathbb{E}|\psi_k(T)|^2_{\mathbb{R}^n} \leq C \mathbb{E} \int_{T-1/k}^T |C_1(t)^\top \psi_k(t)|^2_{\mathbb{R}^n} dt.
$$

(3.44)

By the well-posedness of (3.42) again, we have that

$$
\mathbb{E} \sup_{0 \leq t \leq T} |\psi_k(t)|^2_{\mathbb{R}^n} \leq C \mathbb{E} \int_0^T |r_k(t)|^2_{\mathbb{R}^n} dt = C|\hat{r}|^2_{\mathbb{R}^n},
$$

which, together with (3.44), implies that

$$
\mathbb{E}|\psi_k(T)|^2_{\mathbb{R}^n} \leq \frac{C}{k}|\hat{r}|^2_{\mathbb{R}^n},
$$

(3.45)

with $C$ independent of $k \in \mathbb{N}$. Meanwhile, $(\psi_k(\cdot), r_k(\cdot))$ can be regarded as a solution to (3.36). By the well-posedness of (3.36), there is a positive constant $C$, such that

$$
|r_k|^2_{\mathbb{R}^n} = \mathbb{E} \int_{T-1/k}^T |r_k(t)|^2_{\mathbb{R}^n} dt \leq C \mathbb{E}|\psi_k(T)|^2_{\mathbb{R}^n}.
$$

This contradicts with (3.45) for a sufficiently large $k \in \mathbb{N}$. This indicates that rank($C_2$) = $n$.

**Remark 3.3** It is well known that when $C_2$ is an invertible matrix, the following controlled system is not exactly controllable (e.g., [29, Proposition 6.3]):

$$
\begin{align*}
\left\{ \begin{array}{l}
d\xi(t) = A_1(t)\xi dt + [A_2(t)\xi + C_2u(t)]dB(t), \quad t \in [0,T], \\
\xi(0) = \xi_0,
\end{array} \right.
\end{align*}
$$

(3.46)

where $u$ is the control variable and $\xi$ is the state variable. It implies that

$$
\mathcal{R}_T \overset{\Delta}{=} \{ \xi(T; u) \in L^2_{\mathbb{F}}(\Omega; \mathbb{R}^n) \mid \xi(\cdot; u) \text{ is the solution to (3.46) for some } u(\cdot) \in L^2_\mathbb{F}(0,T; \mathbb{R}^m) \} \\
\subsetneq L^2_{\mathbb{F}}(\Omega; \mathbb{R}^n).
$$
Hence, the Robinson constraint qualification does not hold and the following estimate fails for solutions to (3.36):

$$E|\phi_T|^2_{L^2_F} \leq CE \int_0^T |C_2^T \Phi(t)|^2_{R^m} dt, \quad \forall \phi_T \in L^2_{F_T}(\Omega; R^n).$$

But Theorem 3.2 implies that when $C_2$ is an invertible matrix, the above inequality will hold, if an extra term with a compact operator $G$ is added to it. This weaker estimate is enough to obtain a Fritz John condition for the optimal control problem (3.27).

Example 3.1 Let us consider the following controlled stochastic differential equation:

$$\begin{cases}
\text{dy}(t) = [A(t)y(t) + D_1(t)u(t)] dt + [C(t)y(t) + D_2u(t)] dB(t), & t \in [0, T],
\text{y}(0) = y_0,
\end{cases}$$

where $A(\cdot), C(\cdot) \in L^\infty([0, T]; R^{n \times n})$, $D_1(\cdot) \in L^\infty([0, T]; R^{n \times m})$ and $D_2 \in R^{n \times m}$. The system (3.47) is indeed a special case of (3.26). It is a model of investment in a financial market (e.g., [31]). The constant matrix $D_2$ means that volatilities of the stocks are constant. This is reasonable if $T$ is not very large (e.g., [32]). By Theorem 3.2, to guarantee the existence of a non-zero pair $(z_0, z) \in R \times L^2_{F_T}(\Omega; R^n)$ as the Lagrange multiplier, a sufficient condition is $\text{rank}(D_2) = n$. This condition means that there are enough numbers of stocks in the market, which holds for a modern financial market.

4 Further comments

This paper focuses on establishing an enhanced Fritz John (first-order necessary) condition for optimization problems in infinite-dimensional spaces using a finite codimensionality method. As applications, the paper provides first-order necessary conditions for various types of optimal control problems and proposes verification techniques for finite codimensionality conditions in these problems.

In the field of infinite-dimensional optimization theory, various constraint qualifications have been introduced to ensure the existence of nontrivial or normal Lagrange multipliers for optimal solutions (e.g., see [6] and [7]). However, validating these constraint qualifications is generally challenging.

To illustrate the difficulty, let’s consider a special case for the problem (P), in which

$$K = V \quad \text{and} \quad E = \{0\},$$

with $V$ being a Banach space, and both $f_0$ and $f$ continuously differentiable. In this case, $\mathcal{Z}(\bar{u}) = \text{Im} (f'(\bar{u}))$ (see (1.3) for the definition of $\mathcal{Z}(\bar{u})$). In [23, Theorem 4.2], it was proved that for an affine $f$, the existence of a nontrivial Lagrange multiplier is equivalent to the closeness of $\text{Im} (f'(\bar{u}))$. In [6], it was required that $\text{Im} (f'(\bar{u}))$ has a nonempty relative interior, which is equivalent to the condition that $\text{Im} (f'(\bar{u}))$ is closed in $X$. However, verifying the closeness of a subspace in an infinite-dimensional space may be difficult in practice.

In the following proposition, it will be demonstrated that, similar to the finite codimensionality condition discussed in Section 3, the closed range condition can also be established through certain a priori estimates. However, validating these estimates is not any easier than proving the associated estimates for finite codimensionality.

Let $V$ be a reflexive Banach space, $X$ be a Hilbert space and $F \in \mathcal{L}(V; X)$. We have the following result.
**Proposition 4.1** The following assertions are equivalent:

1. \( \text{Im} (F) \) is closed in \( X \).
2. There exists a positive constant \( C \), such that
   \[
   |h|_{\text{Im}(F)^{\perp}} \leq C|F^{\ast}(h)|_{V^{\perp}}, \quad \forall h \in (\text{Im}(F))^{\perp}.
   \] (4.2)
3. There exists a positive constant \( C \), such that
   \[
   |h|_{X^{\perp}} \leq C\left( |F^{\ast}(h)|_{V^{\perp}} + \left| \Pi_{(\text{Im}(F))^{\perp}} h \right|_{X^{\perp}} \right), \quad \forall h \in X^{\perp}.
   \] (4.3)

**Proof.** First, we prove the equivalence between (1) and (2). Set \( \tilde{X}_0 = \overline{\text{Im}(F)} \). Then, \( \tilde{X}_0 \) is a Banach space by the norm \( | \cdot |_X \). Consider the mapping \( F : V \to \tilde{X}_0 \) and the identity mapping \( \mathbb{I}_{\tilde{X}_0} : \tilde{X}_0 \to \tilde{X}_0 \). Then, it is easy to check that \( F \in \mathcal{L}(V; \tilde{X}_0) \) and \( \mathbb{I}_{\tilde{X}_0} \in \mathcal{L}(\tilde{X}_0) \). If \( \text{Im}(F) \) is closed in \( X \), it follows that \( \text{Im}(\mathbb{I}_{\tilde{X}_0}) \subseteq \text{Im}(F) \). By the range comparison theorem, there exists a positive constant \( C \), such that
   \[
   |h|_{\tilde{X}_0} \leq C|F^{\ast}(h)|_{V^{\perp}}, \quad \forall h \in \tilde{X}_0.
   \]
   This means that (2) holds.

   Conversely, if (2) is true, one has that \( \text{Im}(\mathbb{I}_{\tilde{X}_0}) \subseteq \text{Im}(F) \), which implies (1).

Next, we prove that (2) implies (3). Since \( \tilde{X}_0 = \overline{\text{Im}(F)} \) is a closed subspace of the Hilbert space \( X \), we have \( X^{\perp} = \tilde{X}_0 \oplus (\tilde{X}_0)^{\perp} \). Hence, for any \( h \in X^{\perp} \),
   \[
   h = \Pi_{\tilde{X}_0} h + \Pi_{(\tilde{X}_0)^{\perp}} h.
   \]
   By (2), it holds that
   \[
   |h|_{X^{\perp}} \leq |\Pi_{\tilde{X}_0} h|_{X^{\perp}} + |\Pi_{(\tilde{X}_0)^{\perp}} h|_{X^{\perp}} \leq C|F^{\ast}(\Pi_{\tilde{X}_0} h)|_{V^{\perp}} + |\Pi_{(\tilde{X}_0)^{\perp}} h|_{X^{\perp}}
   \]
   \[
   \leq C( |F^{\ast}(h)|_{V^{\perp}} + |F^{\ast}(\Pi_{(\tilde{X}_0)^{\perp}} h)|_{V^{\perp}} + |\Pi_{(\tilde{X}_0)^{\perp}} h|_{X^{\perp}})
   \]
   Conversely, it is obvious that (3) implies (2). \( \square \)

**Remark 4.1** From the proof of Proposition 4.1, the equivalence between (1) and (2) still holds for a Banach space \( X \).

**Remark 4.2** By [17, Theorems 1.1.1 and 1.1.2] and [33, Lemma 4], if \( V \) and \( X \) are two Hilbert spaces, and \( F : \mathcal{D}(F) \subseteq V \to X \) is a linear closed densely defined operator, then the following assertions are equivalent:

(a) \( \text{Im} (F) \) is closed in \( X \).

(b) There exists a positive constant \( C \), such that
   \[
   |h|_{V} \leq C|F(h)|_{X}, \quad \forall h \in \mathcal{D}(F) \cap \text{Im}(F^{\ast}),
   \]
   where \( F^{\ast} \) denotes the adjoint operator of \( F \).

(c) \( \text{Im} (F^{\ast}) \) is closed in \( V \).
(d) There exists a positive constant $C$, such that
\[ |h|_X \leq C|F^*(h)|_V, \quad \forall h \in D(F^*) \cap \text{Im} (F). \]

(e) There exists a positive constant $C$, such that
\[ |h|_X \leq C(|F^*(h)|_V + |\Pi_{\text{ker}(F^*)}h|_X), \quad \forall h \in X, \]
where $\text{ker}(F^*)$ denotes the null space of $F^*$.

It is easy to show that when $V$ and $X$ are Hilbert spaces, and $F \in \mathcal{L}(V; X)$, every assertion in Proposition 4.1 is equivalent to one of the above (a), (b), (c), (d) and (e).

In the infinite-dimensional optimization problem $(P)$, if $V$ is a reflexive Banach space, $X$ is a Hilbert space, $K = V$, $E = \{0\}$, and $f$ and $f^0$ are continuously differentiable, by Proposition 4.1, the assertion that
\[ \text{Im} (f'(\bar{u})) \text{ is closed in } X \] (4.4)
is equivalent to the following estimate:
\[ |h|_{\text{Im}(f'(\bar{u}))} \leq C|f'(\bar{u})^*(h)|_{V'}, \quad \forall h \in \left(\text{Im} (f'(\bar{u}))\right)' \] (4.5)
or the estimate
\[ |h|_{X'} \leq C\left(|f'(\bar{u})^*h|_{V'} + |\Pi_{\text{Im}(f'(\bar{u}))}h|_{X'}\right), \quad \forall h \in X'. \] (4.6)

Though the condition of nonempty relative interior for $Z(\bar{u})$ is in general weaker than the finite codimensionality condition, it is difficult to verify directly the existence of a nonempty relative interior for $Z(\bar{u})$. Even in the special case (4.1), proving the closeness of $\text{Im} (f'(\bar{u}))$ in $X$ sometimes requires stronger sufficient conditions, such as the surjectivity of $f'(\bar{u})$ or the finite codimensionality of $\text{Im} (f'(\bar{u}))$. This is because it is often challenging to characterize precisely $\left(\text{Im} (f'(\bar{u}))\right)'$ or $\left(\text{Im} (f'(\bar{u}))\right) \perp$, which makes the conclusions in Proposition 4.1 (i.e. (4.5) and (4.6)) difficult to apply in practice. We will provide an illustrative example below for further explanation.

Let $G \subseteq \mathbb{R}^n$ be a convex bounded domain with a smooth boundary $\Gamma$ and $T > 2 \sup_{x,y \in G} |x - y|_{\mathbb{R}^n}$. Put $Q = G \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. Assume that $G_0$ is a nonempty open subset of $G$. Denote by $\chi_{G_0}$ the characteristic function of $G_0$. Consider the following controlled wave equation:
\[
\begin{aligned}
&\begin{cases}
y_{tt} - \Delta y + ay = \chi_{G_0}u & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(0) = y_0, \ y_t(0) = y_1 & \text{in } G,
\end{cases}
\end{aligned}
\] (4.7)
with the cost functional
\[ J(u(\cdot)) = \frac{1}{2} \int_0^T \left[q(x, t)|y(x, t)|^2 + r(x, t)|u(x, t)|^2\right] dx dt, \] (4.8)
where $u \in L^2(Q)$ is the control variable, $(y, y_t)$ is the state variable, $(y_0, y_1) \in H^1_0(G) \times L^2(G)$ is an initial value, and $a, q, r \in L^\infty(Q)$.

For a given $(y^0_d, y^1_d) \in H^1_0(G) \times L^2(G)$, set
\[ U_{ad} = \{ u(\cdot) \in L^2(Q) \mid \text{the solution } y(\cdot; u) \text{ to (4.7) satisfies that } (y(T; u), y_t(T; u)) = (y^0_d, y^1_d)\}. \]
Assume that $\bar{u}(\cdot) \in \mathcal{U}_{ad}$ is an optimal control of the following optimal control problem:

$$
\text{Minimize } J(u(\cdot)) \quad \text{subject to } u(\cdot) \in \mathcal{U}_{ad}.
$$

This optimal control problem is a special case of the optimization problem (P) by taking

$$
\begin{align*}
V &= L^2(Q), & X &= H^1_0(G) \times L^2(G), & E &= \{(y^0, y^1)\}, \\
f_0(u) &= J(u(\cdot)), & f(u) &= (y(\cdot, T; u), y_t(\cdot, T; u)), & K &= L^2(Q),
\end{align*}
$$

where $y(\cdot; u)$ is the solution to (4.7) associated to $u(\cdot)$. Further, it is easy to show that

$$
\text{Im}(f'(\bar{u})) = \{(y(\cdot, T; u), y_t(\cdot, T; u)) \in H^1_0(G) \times L^2(G) \mid y(\cdot; u) \text{ is the solution to (4.7)} \}
$$

By Proposition 4.1, the closeness of $\text{Im}(f'(\bar{u}))$ is equivalent to the following estimate:

$$
|\langle \phi_1, \phi_2 \rangle|_{L^2(G) \times H^{-1}(G)} \leq C \left| \langle \phi | L^2(G_0 \times (0,T)) \rangle + \left| \Pi_{\text{Im}(f'(\bar{u}))} \phi_2 \right|_{L^2(G) \times H^{-1}(G)} \right|, \\
\forall (\phi_1, \phi_2) \in L^2(G) \times H^{-1}(G);
$$

and by Theorem 3.1, the finite codimensionality of $\text{Im}(f'(\bar{u}))$ is equivalent to the following estimate:

$$
|\langle \phi_1, \phi_2 \rangle|_{L^2(G) \times H^{-1}(G)} \leq C |\phi|_{L^2(G_0 \times (0,T))}, \quad \forall (\phi_1, \phi_2) \in H_0,
$$

where $H_0$ is a finite codimensional closed subspace of $L^2(G) \times H^{-1}(G)$ and $\phi$ is the solution to

$$
\begin{align*}
\phi_{tt} - \Delta \phi + a \phi &= 0 & \text{in } Q, \\
\phi &= 0 & \text{on } \Sigma, \\
\phi(T) &= \phi_1, & \phi_t(T) &= \phi_2 & \text{in } G.
\end{align*}
$$

We have the following results.

(1) In the case of $a = 0$, if $(G, T, G_0)$ fulfills the geometric control condition (e.g., [2]), then $f'(\bar{u})$ is surjective, which implies the closeness of $\text{Im}(f'(\bar{u}))$.

Indeed, under the geometric control condition, the system (4.7) (with $a = 0$) is exactly controllable. Hence, any solution $\phi$ to (4.11) (with $a = 0$) satisfies the following estimate:

$$
|\langle \phi_1, \phi_2 \rangle|_{L^2(G) \times H^{-1}(G)} \leq C |\phi|_{L^2(G_0 \times (0,T))}, \quad \forall (\phi_1, \phi_2) \in L^2(G) \times H^{-1}(G).
$$

This implies the estimate (4.9).

Meanwhile, if $(G, T, G_0)$ does not fulfill the geometric control condition, (4.9) will be untrue. Indeed, since the system (4.7) (with $a = 0$) is approximately controllable at $T$ (e.g., [24]), $\text{Im}(f'(\bar{u}))$ is dense in $X$. On the other hand, since the geometric control condition does not hold, the system (4.7) (with $a = 0$) is not exactly controllable and therefore, $\text{Im}(f'(\bar{u}))$ is not closed. This shows that in the case of $a = 0$, when the time $T$ is sufficiently large, the closedness of $\text{Im}(f'(\bar{u}))$ is equivalent to the geometric control condition. In this particular instance, the conditions that guarantee the closedness and the finite codimensionality of $\text{Im}(f'(\bar{u}))$ are identical.

(2) In the general case of $a \in L^\infty(Q)$, if $(G, T, G_0)$ fulfills the geometric control condition, $\text{Im}(f'(\bar{u}))$ is finite codimensional in $X$, which implies the closedness of $\text{Im}(f'(\bar{u}))$.

As proved in [28, Proposition 6.2], the finite codimensionality condition for $\text{Im}(f'(\bar{u}))$ holds whenever $(G, T, G_0)$ fulfills the geometric control condition. However, under the same condition, how to prove the surjectivity of $f'(\bar{u})$ or the estimate (4.12) remains open. In addition, at present, how to verify the estimate (4.9) or the closedness of $\text{Im}(f'(\bar{u}))$ directly seems difficult.

Similar examples can be provided for optimal control problems of other types of partial differential equations, such as heat equations, Schrödinger equations, plate equations, etc.
A Proofs of (2.6) and (2.7)

In this appendix, we prove (2.6) and (2.7) in Example 2.1.

Proof of (2.6). We first prove that for any \( e \in \mathcal{K} \),

\[
\{ f'(e; v) \in X \mid v \in \mathcal{R}_\mathcal{K}(e) \cap B_V(0, 1) \} \subseteq \text{Var}_\mathcal{K} f(e). 
\]  

(A.1)

Indeed, for any \( v \in \mathcal{R}_\mathcal{K}(e) \cap B_V(0, 1) \), there exist \( \alpha > 0 \) and \( y \in \mathcal{K} \), such that \( v = \alpha(y - e) \). Let \( \ell = \frac{1}{\alpha} \). Then

\[
e + \ell v = e + \ell \alpha(y - e) = \ell \alpha y + (1 - \ell \alpha)e \in \mathcal{K}.
\]

Take a sequence \( \{ h_k \}_{k=1}^{\infty} \subseteq (0, \ell) \) with \( \lim_{k \to \infty} h_k = 0 \). Set \( e_k = e + h_k v \) for \( k \in \mathbb{N} \). Then \( \{ e_k \}_{k=1}^{\infty} \subseteq \mathcal{K} \) and \( |e_k - e|_V = h_k |v|_V \leq h_k \). Consequently,

\[
f'(e; v) = \lim_{k \to \infty} \frac{f(e + h_k v) - f(e)}{h_k} = \lim_{k \to \infty} \frac{f(e_k) - f(e)}{h_k} \in \text{Var}_\mathcal{K} f(e).
\]

Thus, (A.1) holds.

For any \( v \in \mathcal{T}_\mathcal{K}(e) \cap B_V(0, 1) \), there exists a sequence \( \{ v_n \}_{n=1}^{\infty} \subseteq \mathcal{R}_\mathcal{K}(e) \) such that \( \lim_{n \to \infty} v_n = v \). Without loss of generality, we may assume that \( |v_n|_V \leq 1 \) for every \( n \in \mathbb{N} \). Indeed, if \( |v|_V < 1 \), then \( |v_n|_V < 1 \) for sufficiently large \( n \in \mathbb{N} \). On the other hand, if \( |v|_V = 1 \), we set \( \tilde{v}_n = \frac{v_n}{|v_n|_V} \). It is easy to show that \( \tilde{v}_n \in \mathcal{R}_\mathcal{K}(e) \cap B_V(0, 1) \) and \( \lim_{n \to \infty} \tilde{v}_n = v \).

By the locally Lipschitz continuity of \( f \), we get that

\[
\lim_{n \to \infty} f'(e; v_n) = f'(e; v).
\]

Since \( f'(e; v_n) \in \text{Var}_\mathcal{K} f(e) \), there exists a sequence \( \{ h^n_k \}_{k=1}^{\infty} \subseteq (0, +\infty) \) with \( \lim_{k \to \infty} h^n_k = 0 \) and a sequence \( \{ e^n_k \}_{k=1}^{\infty} \subseteq \mathcal{K} \), such that

\[
|e^n_k - e|_V \leq h^n_k \quad \text{and} \quad f'(e; v_n) = \lim_{k \to \infty} \frac{f(e^n_k) - f(e)}{h^n_k}.
\]

Therefore, we may find a sequence \( \{ h_k \}_{k=1}^{\infty} \subseteq (0, +\infty) \) with \( \lim_{k \to \infty} h_k = 0 \) and a sequence \( \{ e_k \}_{k=1}^{\infty} \subseteq \mathcal{K} \), such that

\[
|e_k - e|_V \leq h_k \quad \text{and} \quad f'(e; v) = \lim_{k \to \infty} \frac{f(e_k) - f(e)}{h_k}.
\]

This implies that

\[
f'(e; v) \in \text{Var}_\mathcal{K} f(e).
\]

This, together with the arbitrariness of \( v \in \mathcal{T}_\mathcal{K}(e) \cap B_V(0, 1) \), implies (2.6). \( \square \)

Proof of (2.7). By (2.6), it suffices to prove that for any \( e \in \mathcal{K} \),

\[
\text{Var}_\mathcal{K} f(e) \subseteq \{ f'(e; v) \in X \mid v \in \mathcal{T}_\mathcal{K}(e) \cap B_V(0, 1) \}.
\]  

(A.2)

We first handle the case where \( f \) is Fréchet differentiable. For any \( e \in \mathcal{K} \) and \( \xi \in \text{Var}_\mathcal{K} f(e) \), there exists a sequence \( \{ h_k \}_{k=1}^{\infty} \subseteq (0, +\infty) \) with \( \lim_{k \to \infty} h_k = 0 \) and a sequence \( \{ e_k \}_{k=1}^{\infty} \subseteq \mathcal{K} \), such that

\[
|e_k - e|_V \leq h_k \quad \text{and} \quad \xi = \lim_{k \to \infty} \frac{f(e_k) - f(e)}{h_k}.
\]
Since \( \frac{|e_k - e|}{h_k} \mid_{V} \leq 1 \), there are subsequences of \( \{h_k\}_{k=1}^{\infty} \) and \( \{e_k\}_{k=1}^{\infty} \) (denoted still by themselves for simplicity), and \( v \in B_{V}(0, 1) \), such that
\[
\frac{e_k - e}{h_k} \text{ converges weakly to } v \text{ in } V, \quad \text{as } k \to \infty. \tag{A.3}
\]
Since \( \mathcal{T}_K(e) = \bigcup_{\alpha \geq 0} \alpha(K - e) \), we have that \( \frac{e_k - e}{h_k} \in \mathcal{T}_K(e) \). Noting that \( \mathcal{T}_K(e) \) is closed and convex, we find \( v \in \mathcal{T}_K(e) \). Moreover,
\[
\xi = \lim_{k \to \infty} \frac{f(e_k) - f(e)}{h_k} = \lim_{k \to \infty} \frac{f'(e)(e_k - e) + o(|e_k - e|)}{h_k} = f'(e)v = f'(e; v),
\]
which implies (A.2).

Next, we deal with the case that \( V \) is finite-dimensional. Note that the convergence in (A.3) becomes strong convergence, and by the locally Lipschitz continuity of \( f \),
\[
\xi = \lim_{k \to \infty} \frac{f(e_k) - f(e)}{h_k} = \lim_{k \to \infty} \frac{f(e + h_k v) - f(e)}{h_k} + \lim_{k \to \infty} \frac{f(e + h_k \frac{e_k - e}{h_k}) - f(e + h_k v)}{h_k} = f'(e; v).
\]
This proves (A.2).

\[\square\]

## B Proofs of Proposition 2.1

In this appendix, we give a proof of Proposition 2.1. The whole proof is divided into two parts.

### Step 1
In this step, we prove that
\[
\text{span}\{d - x_0 \in X \mid d \in D\} = \text{span}\{d_1 - d_2 \in X \mid d_1, d_2 \in D\}, \quad \forall \ x_0 \in \text{co} D. \tag{B.1}
\]
Indeed, for any \( \xi \in \text{span}\{d_1 - d_2 \in X \mid d_1, d_2 \in D\} \), there exists a \( k \in \mathbb{N}, \alpha_1, \cdots, \alpha_k \in \mathbb{R} \) and \( d_{11}, \cdots, d_{1k}, d_{21}, \cdots, d_{2k} \in D \), such that
\[
\xi = \sum_{i=1}^{k} \alpha_i (d_{1i} - d_{2i}) = \sum_{i=1}^{k} \alpha_i (d_{1i} - x_0) - \sum_{i=1}^{k} \alpha_i (d_{2i} - x_0)
\]
\[
\in \text{span}\{d - x_0 \in X \mid d \in D\}.
\]
Consequently,
\[
\text{span}\{d_1 - d_2 \in X \mid d_1, d_2 \in D\} \subseteq \text{span}\{d - x_0 \in X \mid d \in D\}. \tag{B.2}
\]

On the other hand, for any \( \xi \in \text{span}\{d - x_0 \in X \mid d \in D\} \), there exists a \( k \in \mathbb{N}, \alpha_1, \cdots, \alpha_k \in \mathbb{R} \) and \( d_1, \cdots, d_k \in D \), such that
\[
\xi = \sum_{i=1}^{k} \alpha_i (d_i - x_0).
\]
Notice that \( x_0 \in \text{co} D \). We can find a sequence \( \{x_j\}_{j=1}^{\infty} \subset \text{co} D \) such that \( x_j \to x_0 \) as \( j \to \infty \). From the definition of the convex hull, for any \( j \in \mathbb{N} \), there exists a \( k_j \in \mathbb{N}, \beta_{j1}, \cdots, \beta_{jk_j} \in [0, 1] \) and \( d_{j1}, \cdots, d_{jk_j} \in D \), such that
\[
\sum_{\ell=1}^{k_j} \beta_{j\ell} = 1 \quad \text{and} \quad x_j = \sum_{\ell=1}^{k_j} \beta_{j\ell} d_{j\ell}.
\]
Then,

$$\xi = \lim_{j \to \infty} \sum_{i=1}^{k} \alpha_i (d_i - x_j) = \lim_{j \to \infty} \sum_{i=1}^{k} \alpha_i \left( d_i - \sum_{\ell=1}^{k_j} \beta_{j\ell} d_{j\ell} \right)$$

$$= \lim_{j \to \infty} \sum_{i=1}^{k} \sum_{\ell=1}^{k_j} \alpha_i \beta_{j\ell} (d_i - d_{j\ell}) \in \overline{\text{span}} \left\{ d_1 - d_2 : d_1, d_2 \in D \right\}.$$  

This yields

$$\overline{\text{span}} \left\{ d - x_0 : d \in D \right\} \subseteq \overline{\text{span}} \left\{ d_1 - d_2 : d_1, d_2 \in D \right\}. \tag{B.3}$$

Combining (B.2) and (B.3), we obtain (B.1).

**Step 2.** In this step, we prove that if for some $x_0^1 \in \overline{\text{co}} D$ and $\delta > 0$, there exists a $w_1 \in \overline{\text{co}} (D - x_0^1)$ such that

$$B_X (w_1, \delta) \cap \overline{\text{span}} \left\{ D - x_0^1 \right\} \subseteq \overline{\text{co}} (D - x_0^1), \tag{B.4}$$

then for any $x_0^2 \in \overline{\text{co}} D$,

$$B_X (w_2, \delta) \cap \overline{\text{span}} \left\{ D - x_0^2 \right\} \subseteq \overline{\text{co}} (D - x_0^2) \tag{B.5}$$

with $w_2 = w_1 + x_0^1 - x_0^2 \in \overline{\text{co}} (D - x_0^2)$.

Indeed, for any $\eta \in B_X (w_2, \delta) \cap \overline{\text{span}} \left\{ D - x_0^2 \right\}$, it holds that $|\eta - w_2|_X \leq \delta$. Hence, for $\xi = \eta - x_0^1 + x_0^2$,

$$|\xi - w_1|_X = |\eta - x_0^1 + x_0^2 - w_1|_X = |\eta - w_2|_X \leq \delta.$$ 

This implies

$$\xi \in B_X (w_1, \delta). \tag{B.6}$$

Meanwhile, since $\eta \in \overline{\text{span}} \left\{ D - x_0^2 \right\}$, there is $\{ \eta_j \}_{j=1}^{\infty} \subseteq \overline{\text{span}} \left\{ D - x_0^1 \right\}$ such that $\eta_j \to \eta$ as $j \to \infty$. Then, for any $j \in \mathbb{N}$, there exists a $k_j \in \mathbb{N}$, $\beta_{j1}, \ldots, \beta_{jk_j} \in \mathbb{R}$ and $d_{j1}, \ldots, d_{jk_j} \in D$, such that

$$\eta_j = \sum_{\ell=1}^{k_j} \beta_{j\ell} (d_{j\ell} - x_0^1) \in \overline{\text{span}} \left\{ D - x_0^2 \right\} \quad \text{and} \quad \eta = \lim_{j \to \infty} \eta_j.$$  

Consequently,

$$\xi = \eta - x_0^1 + x_0^2 = \lim_{j \to \infty} \left[ \sum_{\ell=1}^{k_j} \beta_{j\ell} (d_{j\ell} - x_0^1) - x_0^1 + x_0^2 \right]$$

$$= \lim_{j \to \infty} \left[ \sum_{\ell=1}^{k_j} \beta_{j\ell} (d_{j\ell} - x_0^1) - \sum_{\ell=1}^{k_j} \beta_{j\ell} (x_0^2 - x_0^1) + (x_0^2 - x_0^1) \right].$$

Since $x_0^2 \in \overline{\text{co}} D$, we get that $x_0^2 - x_0^1 \in \overline{\text{span}} \left\{ D - x_0^1 \right\}$. Then, the above equality implies that $\xi \in \overline{\text{span}} \left\{ D - x_0^1 \right\}$. This, together with (B.6), implies that

$$\xi \in B_X (w_1, \delta) \cap \overline{\text{span}} \left\{ D - x_0^1 \right\}. \tag{B.7}$$

Combining (B.4) and (B.7), we find that

$$\xi \in \overline{\text{co}} (D - x_0^1) = \overline{\text{co}} D - x_0^1,$$

which implies that

$$\eta \in \overline{\text{co}} D - x_0^2 = \overline{\text{co}} (D - x_0^2).$$

This, together with the arbitrariness of $\eta \in B_X (w_2, \delta) \cap \overline{\text{span}} \left\{ D - x_0^2 \right\}$, yields (B.5). \qed

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C Proof of Lemma 2.2

In this appendix, we give a proof of Lemma 2.2. The whole proof is divided into four parts.

Step 1. First, by Definition 2.7 and finite codimensionality of $D$, there exists $z_0 \in \overline{co}D$ such that $\overline{co}(D - z_0)$ has at least an interior point $w_0$ in the subspace $\text{span}\{D - z_0\}$. Hence, for some $\sigma > 0$, it holds that

$$B_X(w_0, \sigma) \cap \text{span}\{D - z_0\} \subseteq \overline{co}(D - z_0).$$

This implies that

$$B_X(0, \sigma) \cap (\text{span}\{D - z_0\} - w_0) \subseteq \overline{co}(D - z_0) - w_0. \quad (C.1)$$

On the other hand, for any $z \in \text{span}\{D - z_0 - w_0\}$, there exists a sequence $\{z_j\}_{j=1}^\infty \subseteq \text{span}\{D - z_0 - w_0\}$ such that

$$z_j \to z \quad \text{in} \quad X, \quad \text{as} \quad j \to \infty.$$ 

For any $j \in \mathbb{N}$, there exists a positive integer $k \in \mathbb{N}$, $w_j^\ell \in D - z_0$ and $\alpha_j^\ell \in \mathbb{R}$ ($\ell = 1, 2, \cdots, k$), such that

$$z_j = \sum_{\ell=1}^k \alpha_j^\ell (w_j^\ell - w_0) = \sum_{\ell=1}^k \alpha_j^\ell w_j^\ell + (1 - \sum_{\ell=1}^k \alpha_j^\ell) w_0 - w_0 \in \text{span}\{D - z_0\} - w_0.$$

Then, $z \in \text{span}\{D - z_0\} - w_0$, and therefore,

$$\text{span}\{D - z_0 - w_0\} \subseteq \text{span}\{D - z_0\} - w_0.$$

This, together with (C.1), means that

$$B_X(0, \sigma) \cap \text{span}\{D - z_0 - w_0\} \subseteq \overline{co}(D - z_0) - w_0. \quad (C.2)$$

Step 2. By the finite codimensionality of $D$, we have that $\text{span}\{D - z_0\}$ is a finite codimensional subspace of $X$. Then, by Proposition 2.1, the subspace $\text{span}\{D - z_0 - w_0\}$ is also finite codimensional, since $z_0 + w_0 \in \overline{co}D$.

Hence, there exists a $\nu \in \mathbb{N}$ and linearly independent $\tilde{x}_1, \cdots, \tilde{x}_\nu \in X \setminus \text{span}\{D - z_0 - w_0\}$, such that

$$X = \text{span}\{D - z_0 - w_0\} + \text{span}\{\tilde{x}_1, \cdots, \tilde{x}_\nu\}.$$ 

Set

$$L_j = \text{span}\{D - z_0 - w_0\} + \text{span}\{\tilde{x}_1, \cdots, \tilde{x}_{j-1}, \tilde{x}_{j+1}, \cdots, \tilde{x}_\nu\}, \quad 1 \leq j \leq \nu.$$ 

Then, $\tilde{x}_j \notin L_j$ and by the Hahn-Banach theorem, there exist $f_1, \cdots, f_\nu \in X'$, such that

$$f_j(\tilde{x}_j) = 1, \quad \text{and} \quad L_j \subseteq \mathcal{N}(f_j) = \left\{ z \in X \mid f_j(z) = 0 \right\}, \quad \forall \ j = 1, \cdots, \nu.$$ 

It follows that

$$\text{span}\{D - z_0 - w_0\} \subseteq \bigcap_{j=1}^\nu \mathcal{N}(f_j).$$

On the other hand, for any $x \in \bigcap_{j=1}^\nu \mathcal{N}(f_j)$, there exists an $x_0 \in \text{span}\{D - z_0 - w_0\}$ and $\beta_1, \cdots, \beta_\nu \in \mathbb{R}$, such that

$$x = x_0 + \sum_{j=1}^\nu \beta_j \tilde{x}_j.$$
Then for any $j = 1, \cdots , \nu$,

$$0 = f_j(x) = f_j(x_0) + \sum_{i=1}^{\nu} \beta_i f_j(\bar{x}_i) = \beta_j.$$  

Hence, $x = x_0 \in \text{span}\{D - z_0 - w_0\}$. This implies that

$$\text{span}\{D - z_0 - w_0\} = \bigcap_{j=1}^{\nu} N(f_j).$$

**Step 3.** Set $\kappa = \max \{ |f_j|_{X'} \mid j = 1, \cdots , \nu \}$ and

$$K = \left\{ \sum_{i=1}^{\nu} \alpha_i \bar{x}_i \in X \mid |\alpha_i| \leq \kappa, i = 1, \cdots , \nu \right\}.$$  

Then $K$ is a compact set in $X$.

Let $x \in B_X(0, 1)$. For any $j = 1, \cdots , \nu$, by the definition of $f_j$, $f_j(x - \sum_{i=1}^{\nu} f_i(x) \bar{x}_i) = 0$. Hence,

$$x - \sum_{i=1}^{\nu} f_i(x) \bar{x}_i \in \bigcap_{j=1}^{\nu} N(f_j) = \text{span}\{D - z_0 - w_0\}.$$  

In addition, since $|f_j(x)| \leq |f_j|_{X'}|x|_{X} \leq \kappa$, we have $\sum_{i=1}^{\nu} f_i(x) \bar{x}_i \in K$. Therefore,

$$x = \left[ x - \sum_{i=1}^{\nu} f_i(x) \bar{x}_i \right] + \sum_{i=1}^{\nu} f_i(x) \bar{x}_i \in \text{span}\{D - z_0 - w_0\} + K.$$  

Note that $K$ is a compact set. There is a $\gamma > 0$, such that

$$\left| x - \sum_{i=1}^{\nu} f_i(x) \bar{x}_i \right|_{X} \leq |x|_{X} + \left| \sum_{i=1}^{\nu} f_i(x) \bar{x}_i \right|_{X} \leq 1 + \gamma.$$  

It implies that

$$B_X(0, 1) \subseteq \text{span}\{D - z_0 - w_0\} \cap B_X(0, 1 + \gamma) + K.$$  

Letting $0 < \rho \leq \min\{\frac{\sigma}{1+\gamma}, 1\}$, we have

$$B_X(0, \rho) \subseteq \text{span}\{D - z_0 - w_0\} \cap B_X(0, \sigma) + K.$$  

By (C.2),

$$B_X(0, \rho) \subseteq \overline{\text{co}}(D - z_0) - w_0 + K = \overline{\text{co}}(D - z_0 - w_0) + K.$$  

**Step 4.** For any $y \in B_X(0, \rho)$,

$$y = x + z, \quad \text{with } x \in \overline{\text{co}}(D - z_0 - w_0) \text{ and } z \in K.$$  

By the assumptions on $\Lambda_k$,

$$\langle \Lambda_k, y \rangle_{X', X} = \langle \Lambda_k, x \rangle_{X', X} + \langle \Lambda_k, z \rangle_{X', X} \geq -\varepsilon_k - \langle \Lambda_k, z_0 + w_0 \rangle_{X', X} + \langle \Lambda_k, z \rangle_{X', X}.$$  

Therefore,

$$\left| \langle \Lambda_k \rangle_{X'} \right| \leq \frac{1}{\rho} \varepsilon_k + \frac{1}{\rho} \left| \langle \Lambda_k, z_0 + w_0 \rangle_{X', X} \right| + \frac{1}{\rho} \sup_{z \in K} \left| \langle \Lambda_k, z \rangle_{X', X} \right|$$

$$\leq \frac{1}{\rho} \varepsilon_k + \frac{1}{\rho} \left| \langle \Lambda_k, z_0 + w_0 \rangle_{X', X} \right| + \frac{\kappa}{\rho} \sum_{i=1}^{\nu} \left| \langle \Lambda_k, \bar{x}_i \rangle_{X', X} \right|.$$  

If $\Lambda = 0$, then $\lim_{k \to \infty} \left| \langle \Lambda_k \rangle_{X'} \right| = 0$. This contradicts with the fact that $\left| \langle \Lambda_k \rangle_{X'} \right| \geq \delta > 0$.  

}$\Box$
Acknowledgements

The authors gratefully acknowledge an anonymous referee for pointing out a serious mistake in an early version of this paper.

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