Atom interferometer phase in the presence of proof mass

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This is presented the justification of the expression for the atom interferometer phase in the presence of proof mass used in http://arxiv.org/abs/1407.7287. Quantum corrections to that expression are also derived. The corrections allow one to calculate numerically atom interferometer phase with accuracy 1ppm or better.

I. MAIN RELATIONS

For times between the Raman pulses the atomic density matrix evolves as

\[ i\hbar \dot{\rho} = [H, \rho], \]

(1)

where

\[ H = \frac{\hat{p}^2}{2M_a} + U(\vec{x}, t) \]

(2)

is Hamiltonian, \( \hat{p}, M_a \) and \( U \) are, respectively, atomic momentum, mass and gravity potential. In the Wigner representation

\[ \rho(\vec{x}, \vec{p}, t) = \frac{1}{(2\pi\hbar)^3} \int d\vec{s} \rho(\vec{x} + \frac{1}{2}\vec{s}, \vec{x} - \frac{1}{2}\vec{s}, t) \exp(-i\vec{p} \cdot \vec{s}/\hbar) \]

(3)

one finds [1], Sec. III

\[ \begin{cases} \partial_t + \frac{\vec{p}}{M_a} \partial_{\vec{x}} - \partial_{\vec{x}} U(\vec{x}, t) \partial_{\vec{p}} + Q \end{cases} \rho(\vec{x}, \vec{p}, t) = 0, \]

(4a)

\[ Q = - (i\hbar)^{-1} \left[ U \left( \vec{x} + \frac{1}{2}i\hbar\partial_{\vec{p}}, t \right) - U \left( \vec{x} - \frac{1}{2}i\hbar\partial_{\vec{p}}, t \right) \right] + \partial_{\vec{x}} U(\vec{x}, t) \partial_{\vec{p}}. \]

(4b)

When the size of the gravity potential

\[ L \sim U(\vec{x}) / |\partial_{\vec{x}} U(\vec{x})| \]

(5)

is sufficiently large so that

\[ \frac{\hbar}{\Delta p L} \ll 1 \]

(6)

where \( \Delta p \) is a size of the momentum dependence of the density matrix, one expand over the \( \partial_{\vec{p}} \) term and gets approximately

\[ Q \approx - \frac{\hbar^2}{24} \chi'_{ikl}(\vec{x}, t) \partial_{\vec{p}_i} \partial_{\vec{p}_k} \partial_{\vec{p}_l} \]

(7a)

\[ \chi'_{ikl}(\vec{x}, t) = - \partial_{\vec{x}_i} \partial_{\vec{x}_k} \partial_{\vec{x}_l} U(\vec{x}, t) \]

(7b)

An implicit summation convention of Eq. [7] will be used in all subsequent equations. Repeated indices and symbols appearing on the right-hand-side (rhs) of an equation are to be summed over, unless they also appear on the left-hand side (lhs) of that equation. For the relative weight of \( Q \)-term one finds

\[ \frac{Q}{\partial_{\vec{x}} U(\vec{x}, t) \partial_{\vec{p}}} \sim \frac{\hbar^2}{24 (L\Delta p)^2} \ll 1, \]

(8)

so that one can consider \( Q \)-term as a small perturbation and accept that

\[ \rho(\vec{x}, \vec{p}, t) = \rho_0(\vec{x}, \vec{p}, t) + \rho_Q(\vec{x}, \vec{p}, t), \]

(9)
where \( \rho_0(\vec{x}, \vec{p}, t) \) is unperturbed density matrix obeying equation

\[
\left\{ \partial_t + \frac{\vec{p}}{M_a} \partial_\vec{x} - \partial_\vec{x} U(\vec{x}, t) \partial_\vec{p} \right\} \rho_0(\vec{x}, \vec{p}, t) = 0
\]  

(10)

and \( \rho_Q(\vec{x}, \vec{p}, t) \) is a perturbation evolving as

\[
\left\{ \partial_t + \frac{\vec{p}}{M_a} \partial_\vec{x} - \partial_\vec{x} U(\vec{x}, t) \partial_\vec{p} \right\} \rho_Q(\vec{x}, \vec{p}, t) = -Q \rho_0(\vec{x}, \vec{p}, t)
\]  

(11)

We assume that density matrix is known at some preceding time \( t' \), i.e. Eqs. (10, 11) are subject to initial conditions

\[
\begin{align*}
\rho_0(\vec{x}, \vec{p}, t') &= \rho(\vec{x}, \vec{p}, t'), \\
\rho_Q(\vec{x}, \vec{p}, t') &= 0.
\end{align*}
\]  

(12a) (12b)

Solution of the homogeneous Eq. (10) is given by [1], Sec III

\[
\rho_0(\vec{x}, \vec{p}, t) = \rho_0 \left( \vec{R}(\vec{x}, \vec{p}, t'), \vec{P}(\vec{x}, \vec{p}, t') \right),
\]  

(13)

where \( \{ \vec{R}(\vec{x}, \vec{p}, t_1, t_2), \vec{P}(\vec{x}, \vec{p}, t_1, t_2) \} \) are atomic classical position and momentum at time \( t_1 \) subject to initial conditions \( \{ \vec{x}, \vec{p} \} \) at time \( t_2 \). Atomic classical trajectory, evidently, obeys the multiplication law,

\[
\left\{ \frac{\vec{R}}{\vec{P}} \right\} \left( \vec{R}(\vec{x}, \vec{p}, t', t''), \vec{P}(\vec{x}, \vec{p}, t', t''), t', t'' \right) = \left\{ \frac{\vec{R}}{\vec{P}} \right\} (\vec{x}, \vec{p}, t, t'') .
\]  

(14)

Consider now solution of the Eq. (11). Since \( Q \)-term is initially equal 0, solution of the inhomogeneous Eq. (11) contains only the stimulated part consisting of contributions produced at different times \( t' < t'' < t \). Since operator inside the curly brackets in Eq. (11) is a full time derivative,

\[
\frac{d}{dt} = \partial_t + \frac{\vec{p}}{M_a} \partial_\vec{x} - \partial_\vec{x} U(\vec{x}, t) \partial_\vec{p}
\]

in the time interval \([t'', t'' + dt'']\) one produces contributions to the \( Q \)-term

\[
d\rho_Q(\vec{x}, \vec{p}, t'' + dt'') = -Q \rho_0(\vec{x}, \vec{p}, t'') \, dt''.
\]  

(15)

Later on this contribution evolves freely [i.e. satisfies Eq.(11) subject to the initial condition (15) at time \( t'' + dt'' \)] and at time \( t \) one finds for this contribution

\[
d\rho_Q(\vec{x}, \vec{p}, t) = \frac{\hbar^2}{24} dt'' \left[ \chi_{ikl} \left( \vec{\xi}, t'' \right) \right]_{\vec{x}=\vec{R}(\vec{x}, \vec{p}, t'), \vec{p}=\vec{P}(\vec{x}, \vec{p}, t'), \vec{x}, \vec{p}, t''}.
\]  

(16)

Total solution is a sum of all these contributions

\[
\rho_Q(\vec{x}, \vec{p}, t) = \frac{\hbar^2}{24} \int_{t''}^{t} dt'' \left[ \chi_{ikl} \left( \vec{\xi}, t'' \right) \right]_{\vec{x}=\vec{R}(\vec{x}, \vec{p}, t'), \vec{p}=\vec{P}(\vec{x}, \vec{p}, t'), \vec{x}, \vec{p}, t''}.
\]  

(17)

Substituting here solution (13) for \( \rho_0 \) one finds

\[
\rho_Q(\vec{x}, \vec{p}, t) = \frac{\hbar^2}{24} \int_{t''}^{t} dt'' \left[ \chi_{ikl} \left( \vec{\xi}, t'' \right) \right]_{\vec{x}=\vec{R}(\vec{x}, \vec{p}, t'), \vec{p}=\vec{P}(\vec{x}, \vec{p}, t'), \vec{x}, \vec{p}, t''}.
\]  

(18)

II. AI PHASE

Consider now an atom cloud launched at time \( t_0 \), and interacting with \( \frac{\pi}{2} - \pi - \frac{\pi}{2} \) sequence of Raman pulses applied at times

\[
\tau = \{t_0 + t_1, t_0 + t_1 + T, t_0 + t_1 + 2T\},
\]  

(19)
where \( t_1 \) is time delay between cloud launch and 1st Raman pulse. \( T \) is time delay between pulses. We assume that Raman pulses produce coherence between two hyperfine sublevels \( g \) and \( e \) of the atomic ground state manifold, and that initially atomic density matrix \( \rho \) is given by

\[
\rho_{gg}(\vec{x}, \vec{p}, t_0) = f(\vec{x}, \vec{p}), \quad (20a)
\]
\[
\rho_{eg}(\vec{x}, \vec{p}, t_0) = \rho_{ee}(\vec{x}, \vec{p}, t_0) = 0. \quad (20b)
\]

One finds, for example in [1], that after \( \frac{\pi}{2} \)-pulse applied at time \( t \), density matrix elements jump to the values

\[
\rho_{ee}(\vec{x}, \vec{p}, t + 0) = \frac{1}{2} \left[ \rho_{ee}(\vec{x}, \vec{p}, t - 0) + \rho_{gg}(\vec{x}, \vec{p} - h\vec{k}, t - 0) \right] + \text{Re} \left\{ i \exp \left[ -i \left( \vec{k} \cdot \vec{x} - \delta_{12} t - \phi \right) \right] \rho_{eg}(\vec{x}, \vec{p} - \frac{\hbar \vec{k}}{2}, t - 0) \right\}, \quad (21a)
\]
\[
\rho_{gg}(\vec{x}, \vec{p}, t + 0) = \frac{1}{2} \left[ \rho_{ee}(\vec{x}, \vec{p} + h\vec{k}, t - 0) + \rho_{gg}(\vec{x}, \vec{p}, t - 0) \right] - \text{Re} \left\{ i \exp \left[ -i \left( \vec{k} \cdot \vec{x} - \delta_{12} t - \phi \right) \right] \rho_{eg}(\vec{x}, \vec{p} + \frac{\hbar \vec{k}}{2}, t - 0) \right\}, \quad (21b)
\]
\[
\rho_{eg}(\vec{x}, \vec{p}, t + 0) = \frac{i}{2} \exp \left[ i \left( \vec{k} \cdot \vec{x} - \delta_{12} t - \phi \right) \right] \rho_{ee}(\vec{x}, \vec{p} + \frac{\hbar \vec{k}}{2}, t - 0) - \rho_{gg}(\vec{x}, \vec{p} - \frac{\hbar \vec{k}}{2}, t - 0) + \frac{1}{2} \rho_{eg}(\vec{x}, \vec{p}, t - 0) + \exp \left[ 2i \left( \vec{k} \cdot \vec{x} - \delta_{12} t - \phi \right) \right] \rho_{ge}(\vec{x}, \vec{p}, t - 0), \quad (21c)
\]

and after \( \pi \)-pulse, the density matrix elements jump to the values

\[
\rho_{ee}(\vec{x}, \vec{p}, t + 0) = \rho_{gg}(\vec{x}, \vec{p} - h\vec{k}, t - 0), \quad (22a)
\]
\[
\rho_{gg}(\vec{x}, \vec{p}, t + 0) = \rho_{ee}(\vec{x}, \vec{p} + h\vec{k}, t - 0), \quad (22b)
\]
\[
\rho_{eg}(\vec{x}, \vec{p}, t + 0) = \exp \left[ 2i \left( \vec{k} \cdot \vec{x} - \delta_{12} t - \phi \right) \right] \rho_{ge}(\vec{x}, \vec{p}, t - 0), \quad (22c)
\]

where \( \vec{k} \) is effective wave vector, \( \delta_{12} \) is detuning between fields’ frequency difference and hyperfine transition frequency, \( \phi \) is phase difference between traveling components of the Raman field. We allow pulses to have different detunings and phases \( \delta_{12}^{(i)}, \phi_i \) (\( i = 1, 2, 3 \)).

Our purpose is to obtain the atomic density matrix of the excited state \( \rho_{ee}(\vec{x}, \vec{p}, \tau_3 + 0) \). We will achieve this by applying consequently Eqs. (13-18) for density matrix evolution between Raman pulses and before the 1st Raman pulse, and Eqs. (24-27) for density matrix jumps after the pulse.

Before the first pulse, the density matrix becomes

\[
\rho_{gg}(\vec{x}, \vec{p}, \tau_1 - 0) = f \left( \vec{R}(\vec{x}, \vec{p}, t_0, \tau_1), \vec{P}(\vec{x}, \vec{p}, t_0, \tau_1) \right) \quad (23)
\]

After this \( \frac{\pi}{2} \)-pulse

\[
\rho_{ee}(\vec{x}, \vec{p}, \tau_1 + 0) = \frac{1}{2} f \left( \vec{R}(\vec{x}, \vec{p} - h\vec{k}, t_0, \tau_1), \vec{P}(\vec{x}, \vec{p} - h\vec{k}, t_0, \tau_1) \right), \quad (24a)
\]
\[
\rho_{gg}(\vec{x}, \vec{p}, \tau_1 + 0) = \frac{1}{2} f \left( \vec{R}(\vec{x}, \vec{p}, t_0, \tau_1), \vec{P}(\vec{x}, \vec{p}, t_0, \tau_1) \right), \quad (24b)
\]
\[
\rho_{eg}(\vec{x}, \vec{p}, \tau_1 + 0) = \frac{i}{2} \exp \left[ i \left( \vec{k} \cdot \vec{x} - \delta_{12}^{(1)} \tau_1 - \phi_1 \right) \right] f \left( \vec{R}(\vec{x}, \vec{p} - \frac{h\vec{k}}{2}, t_0, \tau_1), \vec{P}(\vec{x}, \vec{p} - \frac{h\vec{k}}{2}, t_0, \tau_1) \right) \quad (24c)
\]

One uses this density matrix as an initial value at moment \( \tau_1 \) for the free evolution between 1st and 2nd pulses of the unperturbed density matrix

\[
\rho_0(\vec{x}, \vec{p}, \tau_1 + 0) = \rho(\vec{x}, \vec{p}, \tau_1 + 0) \quad (25)
\]
We now consider $Q$–term before the second pulse action. From Eqs. (18, 24c, 25) one obtains

$$
\rho_{Qeg}(\vec{x}, \vec{p}, \tau_2 - 0) = -i \frac{\hbar^2}{48} \int_{\tau_1}^{\tau_2} dt \exp \left[ i \left( \hat{k} \cdot \hat{R}(\vec{\xi}, \vec{\pi}, \tau_1, t) - \delta^{(1)}_{12} \tau_1 - \phi_1 \right) \right] \times f \left( \hat{R}(\vec{\xi}, \vec{\pi}, \tau_1, t), \hat{P}(\vec{\xi}, \vec{\pi}, \tau_1, t) - \frac{\hbar \hat{k}}{2}, t_0, \tau_1 \right) \times \left( \frac{\vec{\xi}}{\vec{\pi}} \right) = \left( \hat{R}(\vec{x}, \vec{p}, \tau_2) \right)
$$

In the Eq. (26), the derivative on the momentum $\vec{\pi}$ is of the order of $\Delta p^{-1}$, where $\Delta p$ is the momentum size of the factor in brackets. This factor has two terms, phase-factor and atom distribution $f$. Initially, for $t = \tau_1$ phase-factor is $\vec{\pi}$–independent (because $\hat{R}(\vec{\xi}, \vec{\pi}, \tau_1, \tau_1) = \vec{\xi}$) and derivative is of the order of

$$
\partial_{\vec{\pi}, \text{Thermal}} \sim p_0^{-1},
$$

where

$$
p_0 = (2M_a k_B T_c)^{1/2}(29)
$$

is thermal momentum, the $T_c$ is cloud temperature.

Consider now the phase factor $\exp \left[ i \left( \hat{k} \cdot \hat{R}(\vec{\xi}, \vec{\pi}, \tau_1, t) - \delta^{(1)}_{12} \tau_1 - \phi_1 \right) \right]$ at $t - \tau_1 \sim T$. For the purpose of the estimates lets "turn off" the gravity field. Then atom trajectory evidently equal to

$$
\hat{R}(\vec{\xi}, \vec{\pi}, \tau_1, t) = \vec{\xi} - (t - \tau_1) \frac{\vec{\pi}}{M_a}
$$

and the phase factor acquires Doppler phase $\hat{k} \cdot \frac{\vec{\pi}}{M_a} (t - \tau_1)$. It becomes a rapidly oscillating function of momentum $\vec{\pi}$ having period of the order of

$$
p_D \sim M_a / kT
$$

(30)

and phase factor derivative is of magnitude

$$
\partial_{\vec{\pi}, \text{Doppler}} \sim p_D^{-1}.
$$

(31)

For the $^{133}\text{Cs}$ at temperature $T_C \approx 3\mu K$, $k = 1.4743261770 \times 10^7 \text{m}^{-1}$, $T = 160\text{ms}$ one finds

$$
\frac{\partial_{\vec{\pi}, \text{Thermal}}}{\partial_{\vec{\pi}, \text{Doppler}}} \sim \frac{1}{k v_0 T} \approx 2 \times 10^{-5} \ll 1,
$$

(32)

where $v_0 = p_0 / M_a$ is thermal velocity. Condition (32) means that time separation between pulses $T$ is sufficiently large to be in the Doppler limiting case, when Doppler phase $k v_0 T \gg 1$. In this limit atomic distribution over momentum is sufficiently smooth to neglect its derivation, and throughout the text we include only contribution to the $Q$–term arising from the derivative of phase factor. For this reason we did not consider above the $Q$–term for cloud evolution before 1st pulse. As we will show, the atomic levels' populations ($\rho_{ee}$ and $\rho_{gg}$) have no phase factor at $t_0 < t < \tau_3$, and therefore $Q$–term arises only from atomic coherence $\rho_{eq}$. Using the Doppler limiting case (32) results in Al phase pretty much independent of the atomic momentum and spatial distribution.
Calculating derivatives, one obtains

\[
\rho_{Qeg}(\vec{x}, \vec{p}, \tau_2 - 0) = \frac{\hbar^2}{48} \int_{\tau_1}^{\tau_2} dt \times \left[ k_u k_v k_w \chi_{ijkl}(\vec{u}, t) \partial_{\vec{u}} \vec{R}_u(\vec{u}, \vec{v}, \tau_1, t) \partial_{\vec{v}} \vec{R}_v(\vec{u}, \vec{v}, \tau_1, t) \partial_{\vec{u}} \vec{R}_u(\vec{u}, \vec{v}, \tau_1, t) \right] \left\{ \frac{\vec{u}}{\vec{p}} \right\}(\vec{x}, \vec{p}, \tau_2) \]

The expression inside the curly brackets of Eq. (33) becomes \(t\)-independent and the \(Q\)-term in front of the 2nd pulse is then given by

\[
\rho_{Qeg}(\vec{x}, \vec{p}, \tau_2 - 0) = \frac{\hbar^2}{48} \left\{ \exp \left[ i \left( \vec{\kappa} \cdot \vec{R}(\vec{x}, \vec{p}, \tau_1) - \delta_{12}^{(1)} \tau_1 - \phi_1 \right) \right] \right\} \left\{ \frac{\vec{x}}{\vec{p}} \right\}(\vec{x}, \vec{p}, \tau_1, \tau_2) \]

From Eqs. (13, 24, 14)

\[
\rho_{ee}(\vec{x}, \vec{p}, \tau_2 - 0) = \frac{1}{2} \left\{ \vec{R}(\vec{x}, \vec{p}, t_0, \tau_1) \right\} \left\{ \vec{P}(\vec{x}, \vec{p}, \tau_1, t_0) \right\} \tau_2
\]

From Eqs. (22) after the 2nd pulse density matrix jumps to the value
\[
\rho_{ee}(\bar{x}, \bar{p}, \tau_{2} + 0) = \frac{1}{2} f \left( \hat{R} \left( \bar{x}, \bar{p} - h\hat{k}, t_0, \tau_2 \right), \hat{P} \left( \bar{x}, \bar{p} - h\hat{k}, t_0, \tau_2 \right) \right),
\]

\[
\rho_{gg}(\bar{x}, \bar{p}, \tau_{2} + 0) = \frac{1}{2} f \left( \hat{R} \left( \hat{\xi}, \hat{\pi}, t_0, \tau_1 \right), \hat{P} \left( \hat{\xi}, \hat{\pi}, t_0, \tau_1 \right) \right) \xi = \hat{R}(\bar{x}, \bar{p} + h\hat{k}, \tau_{12}) = \hat{R}(\bar{x}, \bar{p} + h\hat{k}, \tau_{12}) - h\hat{k}
\]

\[
\rho_{Qeg}(\bar{x}, \bar{p}, \tau_{2} + 0) = \frac{i}{2} \left\{ \exp \left\{ i \left[ \hat{k} \cdot (2\bar{x} - \hat{\xi}) \right] - 2\delta^{(2)}_{12} \tau_2 + \delta^{(1)}_{12} \tau_1 - 2\phi_2 + \phi_1 \right\} \right\}
\times f \left( \hat{R} \left( \hat{\xi}, \hat{\pi}, t_0, \tau_1 \right), \hat{P} \left( \hat{\xi}, \hat{\pi}, t_0, \tau_1 \right) \right) \xi = \hat{R}(\bar{x}, \bar{p}, \tau_{12}), \pi = \hat{P}(\bar{x}, \bar{p}, \tau_{12}) - h\hat{k}/2,
\]

\[
\rho_{Qeg}(\bar{x}, \bar{p}, \tau_{2} + 0) = \frac{i}{2} \left\{ \exp \left\{ i \left[ \hat{k} \cdot (2\bar{x} - \hat{\xi}) \right] - 2\delta^{(2)}_{12} \tau_2 + \delta^{(1)}_{12} \tau_1 - 2\phi_2 + \phi_1 \right\} \right\}
\times f \left( \hat{R} \left( \hat{\xi}, \hat{\pi}, t_0, \tau_1 \right), \hat{P} \left( \hat{\xi}, \hat{\pi}, t_0, \tau_1 \right) \right) \xi = \hat{R}(\bar{x}, \bar{p}, \tau_{12}), \pi = \hat{P}(\bar{x}, \bar{p}, \tau_{12}) - h\hat{k}/2
\]

Consider now $Q$-term produced during free evolution inside the time interval $[\tau_2, \tau_3]$. Each density matrix element \(37\) produces $Q$-term. However, since diagonal matrix elements \(37a, 37b\) contain no rapidly oscillating in momentum space phase factors and we neglect their $Q - terms$. Term \(37d\) is already linear in $Q$ term and can produce only higher order non-linear in $Q$ contributions, which we are not including yet. So one has to consider only the $Q$-term produced by coherence \(37c\), which we denote as $\rho_{Qeg}$. From Eq. (38) one finds

\[
\rho_{Qeg}(\bar{x}, \bar{p}, \tau_{3} + 0) = \frac{i}{48} \int_{\tau_2}^{\tau_3} dt \left\{ \text{exp} \left\{ i \left[ \hat{k} \cdot (2\bar{x} - \hat{\xi}) \right] - 2\delta^{(2)}_{12} \tau_2 + \delta^{(1)}_{12} \tau_1 - 2\phi_2 + \phi_1 \right\} \right\}
\times f \left( \hat{R} \left( \hat{\xi}, \hat{\pi}, t_0, \tau_1 \right), \hat{P} \left( \hat{\xi}, \hat{\pi}, t_0, \tau_1 \right) \right) \xi = \hat{R}(\bar{x}, \bar{p}, \tau_{12}), \pi = \hat{P}(\bar{x}, \bar{p}, \tau_{12}) - h\hat{k}/2
\]

where we used the multiplication law \(14\).

\[
\hat{R} \left( \hat{\xi}, \hat{\pi}, \tau_2, t \right), \hat{P} \left( \hat{\xi}, \hat{\pi}, \tau_2, t \right) = \hat{R} \left( \hat{\xi}, \hat{\pi}, \tau_1, t \right).
\]

In Eq. (38) we differentiate over momentum $\hat{\pi}$ only the rapidly oscillating exponent. After differentiation, we apply the multiplication law two more times to the phase factor and distribution $f$, namely

\[
\left\{ \hat{R} \left( \hat{\xi}, \hat{\pi}, \tau_2, t \right), \hat{P} \left( \hat{\xi}, \hat{\pi}, \tau_2, t \right) \right\} \xi = \hat{R} \left( \hat{\xi}, \hat{\pi}, \tau_3, t \right) = \hat{R} \left( \hat{\xi}, \hat{\pi}, \tau_3, t \right),
\]

and therefore

\[
\left\{ \hat{R} \left( \hat{\xi}, \hat{\pi}, \tau_2, t \right), \hat{P} \left( \hat{\xi}, \hat{\pi}, \tau_2, t \right) \right\} \xi = \hat{R} \left( \hat{\xi}, \hat{\pi}, \tau_3, t \right) = \hat{R} \left( \hat{\xi}, \hat{\pi}, \tau_3, t \right)
\]

\[
\left\{ \hat{R} \left( \hat{\xi}, \hat{\pi}, \tau_2, t \right), \hat{P} \left( \hat{\xi}, \hat{\pi}, \tau_2, t \right) \right\} \xi = \hat{R} \left( \hat{\xi}, \hat{\pi}, \tau_3, t \right) = \hat{R} \left( \hat{\xi}, \hat{\pi}, \tau_3, t \right)
\]
to find out that these terms become $t$–independent. As a result one gets for $Q$–term $\rho_{Qeg}$ before the 3-rd pulse

$$
\rho_{Qeg}(\vec{x},\vec{p},\tau_3 - 0) = \frac{\hbar^2}{48} \left\{ \exp \left\{ i \left[ \vec{R} \cdot \left( 2\vec{R} (\vec{x},\vec{p},\tau_2,\tau_3) - \vec{\xi} \right) - 2\delta_{12}^{(2)} \tau_2 + \delta_{12}^{(1)} \tau_1 - 2\phi_2 + \phi_1 \right] \right\} 
\times f \left( \vec{R} (\vec{x},\vec{p},\tau_0,\tau_1) , \vec{P} (\vec{x},\vec{p},\tau_0,\tau_1) \right) \right\} \left\{ \vec{\xi} \right\} = \{ \vec{R} \left( \vec{x},\vec{p},\tau_3,\tau_3 \right) , \vec{\pi} = \vec{P} (\vec{x},\vec{p},\tau_1,\tau_3) - \hbar \vec{k} / 2 \}
\times \int_{\tau_2}^{\tau_3} dt \left\{ \chi_{ikl} (\vec{\xi},t) \vec{k}_w \vec{k}_w \left[ \partial_{\vec{\xi} k} \left( \vec{R}_w (\vec{\xi},\vec{\pi},\tau_1, t) - 2\vec{R}_w (\vec{\xi},\vec{\pi},\tau_2, t) \right) \right] \right\} \{ \vec{\xi} \} = \{ \vec{R} \left( \vec{x},\vec{p},t,\tau_3 \right) \} .
\right.
$$

(42)

Applying Eq. (13) one finds that other matrix elements in front of the 3rd pulse become

$$
\rho_{ee}(\vec{x},\vec{p},\tau_3 - 0) = \frac{1}{2} \left\{ \exp \left\{ i \left[ 2kR (\vec{x},\vec{p},\tau_2,\tau_3) - \vec{k} \cdot \vec{\xi} - 2\delta_{12}^{(2)} \tau_2 + \delta_{12}^{(1)} \tau_1 - 2\phi_2 + \phi_1 \right] \right\} 
\times f \left( \vec{R} (\vec{x},\vec{p},\tau_0,\tau_1) , \vec{P} (\vec{x},\vec{p},\tau_0,\tau_1) \right) \right\} \left\{ \vec{\xi} \right\} = \{ \vec{R} (\vec{x},\vec{p},\tau_3,\tau_3) , \vec{\pi} = \vec{P} (\vec{x},\vec{p},\tau_1,\tau_3) - \hbar \vec{k} / 2 \}
\times \int_{\tau_2}^{\tau_3} dt \left\{ \vec{k}_w \vec{k}_w \chi_{ikl} (\vec{\xi},t) \partial_{\vec{\xi},k} \vec{R}_w (\vec{\xi},\vec{\pi},\tau_1, t) \right\} \{ \vec{\xi} \} = \{ \vec{R} (\vec{x},\vec{p},t,\tau_3) \} .
\right.
$$

(43a)

$$
\rho_{gg}(\vec{x},\vec{p},\tau_3 - 0) = \frac{1}{2} \left\{ \exp \left\{ i \left[ 2kR (\vec{x},\vec{p},\tau_2,\tau_3) - \vec{k} \cdot \vec{\xi} - 2\delta_{12}^{(2)} \tau_2 + \delta_{12}^{(1)} \tau_1 - 2\phi_2 + \phi_1 \right] \right\} 
\times f \left( \vec{R} (\vec{x},\vec{p},\tau_0,\tau_1) , \vec{P} (\vec{x},\vec{p},\tau_0,\tau_1) \right) \right\} \left\{ \vec{\xi} \right\} = \{ \vec{R} (\vec{x},\vec{p},\tau_3,\tau_3) , \vec{\pi} = \vec{P} (\vec{x},\vec{p},\tau_1,\tau_3) - \hbar \vec{k} / 2 \}
\times \int_{\tau_2}^{\tau_3} dt \left\{ \vec{k}_w \vec{k}_w \chi_{ikl} (\vec{\xi},t) \partial_{\vec{\xi},k} \vec{R}_w (\vec{\xi},\vec{\pi},\tau_1, t) \right\} \{ \vec{\xi} \} = \{ \vec{R} (\vec{x},\vec{p},t,\tau_3) \} .
\right.
$$

(43b)

$$
\rho_{Qeg}(\vec{x},\vec{p},\tau_3 - 0) = -\frac{\hbar^2}{48} \left\{ \exp \left\{ i \left[ 2kR (\vec{x},\vec{p},\tau_2,\tau_3) - \vec{k} \cdot \vec{\xi} - 2\delta_{12}^{(2)} \tau_2 + \delta_{12}^{(1)} \tau_1 - 2\phi_2 + \phi_1 \right] \right\} 
\times f \left( \vec{R} (\vec{x},\vec{p},\tau_0,\tau_1) , \vec{P} (\vec{x},\vec{p},\tau_0,\tau_1) \right) \right\} \left\{ \vec{\xi} \right\} = \{ \vec{R} (\vec{x},\vec{p},\tau_3,\tau_3) , \vec{\pi} = \vec{P} (\vec{x},\vec{p},\tau_1,\tau_3) - \hbar \vec{k} / 2 \}
\times \int_{\tau_2}^{\tau_3} dt \left\{ \vec{k}_w \vec{k}_w \chi_{ikl} (\vec{\xi},t) \partial_{\vec{\xi},k} \vec{R}_w (\vec{\xi},\vec{\pi},\tau_1, t) \right\} \{ \vec{\xi} \} = \{ \vec{R} (\vec{x},\vec{p},t,\tau_3) \} .
\right.
$$

(43c)

Aggregating different parts of the nondiagonal density matrix elements in Eq. (13), one concludes that in the linear in $Q$–term approximation this term acts only on the atomic coherence phase:

$$
\rho_{eg}(\vec{x},\vec{p},\tau_3 - 0) \approx \rho_{eg}(\vec{x},\vec{p},\tau_3 - 0) + \rho_{Qeg}(\vec{x},\vec{p},\tau_3 - 0) + \rho_{Qeg}(\vec{x},\vec{p},\tau_3 - 0)
\times \frac{1}{2} \left\{ \exp \left\{ i \left[ 2kR (\vec{x},\vec{p},\tau_2,\tau_3) - \vec{k} \cdot \vec{\xi} - 2\delta_{12}^{(2)} \tau_2 + \delta_{12}^{(1)} \tau_1 - 2\phi_2 + \phi_1 \right] \right\} 
\times f \left( \vec{R} (\vec{x},\vec{p},\tau_0,\tau_1) , \vec{P} (\vec{x},\vec{p},\tau_0,\tau_1) \right) \right\} \left\{ \vec{\xi} \right\} = \{ \vec{R} (\vec{x},\vec{p},\tau_3,\tau_3) , \vec{\pi} = \vec{P} (\vec{x},\vec{p},\tau_1,\tau_3) - \hbar \vec{k} / 2 \}
\times \int_{\tau_2}^{\tau_3} dt \left\{ \chi_{ikl} (\vec{\xi},t) \partial_{\vec{\xi},k} \vec{R}_w (\vec{\xi},\vec{\pi},\tau_1, t) \partial_{\vec{\xi},k} \vec{R}_w (\vec{\xi},\vec{\pi},\tau_1, t) \right\} \{ \vec{\xi} \} = \{ \vec{R} (\vec{x},\vec{p},t,\tau_3) \} .
\right.
$$

(44a)

$$
\tilde{\phi}_Q (\vec{x},\vec{p}) = -\frac{\hbar^2}{24} \int_{\tau_2}^{\tau_3} dt \chi^{ikl} (\vec{\xi},t) \partial_{\vec{\xi},k} \vec{R}_w (\vec{\xi},\vec{\pi},\tau_1, t) \partial_{\vec{\xi},k} \vec{R}_w (\vec{\xi},\vec{\pi},\tau_1, t)
\times \partial_{\vec{\xi},k} \vec{R}_w (\vec{\xi},\vec{\pi},\tau_1, t) - 2\vec{R}_w (\vec{\xi},\vec{\pi},\tau_2, t) \right\} \{ \vec{\xi} \} = \{ \vec{R} (\vec{x},\vec{p},t,\tau_3) \} .
\right.
$$

(44b)
For the $\frac{\pi}{2} - \pi - \frac{\pi}{2}$ AI, after the 3rd pulse, one should calculate only the atomic distribution in the excited sublevel. One finds from the Eqs. (21a, 43a, 43b, 44)

\[
\rho_{ee}(\vec{x}, \vec{p}, \tau_3 + 0) = \frac{1}{4} f \left( \tilde{R} \left( \vec{\xi}, \vec{\pi}, t_0, \tau_2 \right), \tilde{P} \left( \vec{\xi}, \vec{\pi}, t_0, \tau_2 \right) \right) \xi = \tilde{R}(\vec{x}, \vec{p}, \tau_2, \tau_3), \vec{\pi} = \tilde{P}(\vec{x}, \vec{p}, \tau_2, \tau_3) - \hbar \vec{k}
\]

\[+
\frac{1}{4} f \left( \tilde{R} \left( \vec{\xi}, \vec{\pi}, t_0, \tau_1 \right), \tilde{P} \left( \vec{\xi}, \vec{\pi}, t_0, \tau_1 \right) \right) \left\{ \begin{array}{l}
\xi = \tilde{R}(\vec{x}, \vec{p}, \tau_2, \tau_3), \vec{\pi} = \tilde{P}(\vec{x}, \vec{p}, \tau_2, \tau_3) + \hbar \vec{k}, \tau_1, \tau_3 \\
\tilde{R}(\vec{x}, \vec{p}, \tau_2, \tau_3), \tilde{P}(\vec{x}, \vec{p}, \tau_2, \tau_3) + \hbar \vec{k}, \tau_1, \tau_3 \\
\end{array} \right\}
\]

\[-\frac{1}{2} \left\{ \cos \left[ \tilde{k}_x - 2\tilde{k}_R \left( \vec{x}, \vec{p} - \frac{\hbar \vec{k}}{2}, \tau_2, \tau_3 \right) + \tilde{k} \cdot \tilde{\xi} + \tilde{\phi}_Q \left( \vec{x}, \vec{p} - \frac{\hbar \vec{k}}{2} \right) \right] - \delta_{12}^{(3)} \tau_3 + 2\delta_{12}^{(2)} \tau_2 - \delta_{12}^{(1)} \tau_1 - \phi_3 + 2\phi_2 - \phi_1 \right\}
\]

\[\times f \left( \tilde{R} \left( \vec{\xi}, \vec{\pi}, t_0, \tau_1 \right), \tilde{P} \left( \vec{\xi}, \vec{\pi}, t_0, \tau_1 \right) \right) \right\} \xi = \tilde{R}(\vec{x}, \vec{p}, -\hbar \vec{k}/2, \tau_1, \tau_3), \vec{\pi} = \tilde{P}(\vec{x}, \vec{p}, -\hbar \vec{k}/2, \tau_1, \tau_3) - \hbar \vec{k}/2 \right.
\]

(45)

This density matrix should be used to calculate any response associated with atoms on the excited state. We are using it to get the probability of atom cloud excitation defined as

\[
w = \int d\vec{x} d\vec{p} \rho_{ee}(\vec{x}, \vec{p}, \tau_3 + 0).
\]

First two terms in Eq. (45) are responsible for background. Since phase space stays invariant under the atom free motion and recoil during interactions with Raman pulses, the background equals to $\frac{1}{2}$ and

\[
w = \frac{1}{2} (1 - \tilde{w}),
\]

(46)

where interferometric term is given by

\[
\tilde{w} = \int d\vec{x} d\vec{p} \left\{ \cos \left[ \tilde{k}_x - 2\tilde{k}_R \left( \vec{x}, \vec{p} - \frac{\hbar \vec{k}}{2}, \tau_2, \tau_3 \right) + \tilde{k} \cdot \tilde{\xi} + \tilde{\phi}_Q \left( \vec{x}, \vec{p} - \frac{\hbar \vec{k}}{2} \right) \right] - \delta_{12}^{(3)} \tau_3 + 2\delta_{12}^{(2)} \tau_2 - \delta_{12}^{(1)} \tau_1 - \phi_3 + 2\phi_2 - \phi_1 \right\}
\]

\[\times f \left( \tilde{R} \left( \vec{\xi}, \vec{\pi}, t_0, \tau_1 \right), \tilde{P} \left( \vec{\xi}, \vec{\pi}, t_0, \tau_1 \right) \right) \right\} \xi = \tilde{R}(\vec{x}, \vec{p}, -\hbar \vec{k}/2, \tau_1, \tau_3), \vec{\pi} = \tilde{P}(\vec{x}, \vec{p}, -\hbar \vec{k}/2, \tau_1, \tau_3) - \hbar \vec{k}/2 \right.
\]

(48)

Selecting as integration variables atomic initial position and momentum at the moment of atom launching $t_0$,

\[
\{ \vec{x}', \vec{p}' \} = \left\{ \tilde{R} \left( \vec{\xi}, \vec{\pi}, t_0, \tau_1 \right), \tilde{P} \left( \vec{\xi}, \vec{\pi}, t_0, \tau_1 \right) \right\}
\]

(49)

one obtains

\[
\left\{ \vec{\xi}, \vec{\pi} \right\} = \left\{ \tilde{R} \left( \vec{x}, \vec{p}, \tau_1, t_0 \right), \tilde{P} \left( \vec{x}, \vec{p}, \tau_1, t_0 \right) \right\},
\]

(50a)

\[
\{ \vec{x}, \vec{p} \} = \left\{ \tilde{R} \left( \vec{x}', \vec{p}', \tau_1, t_0 \right), \tilde{P} \left( \vec{x}', \vec{p}', \tau_1, t_0 \right) + \hbar \vec{k}/2, \tau_3, \tau_1 \right\},
\]

(50b)

\[
\tilde{P} \left( \vec{x}, \vec{p}, \tau_1, t_0 \right) \right\} \xi = \tilde{R}(\vec{x}, \vec{p}, \tau_1, t_0) + \hbar \vec{k}/2, \tau_2, \tau_1 \right.
\]

(50c)

\[
\tilde{R} \left( \vec{x}, \vec{p} - \frac{\hbar \vec{k}}{2}, \tau_2, \tau_3 \right) = \tilde{R} \left( \vec{x}', \vec{p}', \tau_1, t_0 \right) + \hbar \vec{k}/2, \tau_2, \tau_1 \right.
\]

(50d)

After that, replacing $\{ \vec{x}', \vec{p}' \} \to \{ \vec{x}, \vec{p} \}$, one obtains

\[
\tilde{w} = \int d\vec{x} d\vec{p} \cos \left[ \phi \left( \vec{x}, \vec{p} \right) - \delta_{12}^{(3)} \tau_3 + 2\delta_{12}^{(2)} \tau_2 - \delta_{12}^{(1)} \tau_1 - \phi_3 + 2\phi_2 - \phi_1 \right] f \left( \vec{x}, \vec{p} \right),
\]

(51)
where the phase of the AI is given by

\begin{align}
\phi (\vec{x}, \vec{p}) &= \phi_r (\vec{x}, \vec{p}) + \phi_Q (\vec{x}, \vec{p}), \\
\phi_r (\vec{x}, \vec{p}) &= \vec{k} \cdot \left[ \vec{R} \left( \vec{x}, \vec{p}, \tau_3, \tau_1 \right) - 2 \vec{R} \left( \vec{x}, \vec{p}, \tau_2, \tau_1 \right) + \vec{\xi} \right] \left\{ \vec{\xi} = \vec{R}(\vec{x}, \vec{p}, \tau_1, t_0), \vec{\pi} = \vec{P}(\vec{x}, \vec{p}, \tau_1, t_0) + \hbar \vec{k}/2 \right\}, \\
\phi_Q (\vec{x}, \vec{p}) &= \tilde{\phi}_Q \left[ \vec{R} \left( \vec{x}, \vec{p}, \tau_3, \tau_1 \right), \vec{P} \left( \vec{x}, \vec{p}, \tau_3, \tau_1 \right) \right] \left\{ \vec{\xi} = \vec{R}(\vec{x}, \vec{p}, \tau_1, t_0), \vec{\pi} = \vec{P}(\vec{x}, \vec{p}, \tau_1, t_0) + \hbar \vec{k}/2 \right\},
\end{align}

where \( \tilde{\phi}_Q \) is given by Eq. [44b].

Part \( \phi_r \) includes “classical” part of the phase (erasing in the limit \( \hbar \to 0 \)) and recoil effect during interacion with Raman pulses. For the rotating spherical Earth this part had been calculated in [1]. We calculate here additions to \( \phi_r \) caused by proof mass field.

Part \( \phi_Q \) is originated from quantum correction \( Q \) to the density matrix in Wigner representation equation in time between pulses. If atom trajectory is smaler than the size of the gravity field, one can expand gravity field in the vicinity of atoms launching point. Holding only gravity-gradient terms in that expansion one gets \( \phi_Q = 0 \), because tensor \( \tilde{\phi}_Q \) disappers. For this reason we did not consider \( Q \)-term in [1]. Part \( \phi_Q \) one has to know for precise measurements of the Newtonian gravitational constant [2] [3]. Keeping in mind this application we will calculate \( \phi_Q \) below.

### A. Part \( \phi_r \)

Propagation functions \( \{ \vec{R}(\vec{x}, \vec{p}, t, t'), \vec{P}(\vec{x}, \vec{p}, t, t') \} \), i.e. atomic position and momentum at time \( t \) subject to initial value \( \{ \vec{x}, \vec{p} \} \) at moment \( t' \), evolve as

\begin{align}
\vec{R}(\vec{x}, \vec{p}, t, t') &= \frac{\vec{P}(\vec{x}, \vec{p}, t, t')}{M_a}, \\
\vec{P}(\vec{x}, \vec{p}, t, t') &= M_a \left\{ \vec{g} + \delta \vec{g} \left[ \vec{R}(\vec{x}, \vec{p}, t, t') \right], t \right\},
\end{align}

where \( \vec{g} \) is Earth gravity field, \( \delta \vec{g}(\vec{x}, t) \) is proof mass gravity field. We neglect in Eq. [53] the gravity-gradient, centrifugal and Coriolis forces caused by the rotating Earth. When \( \delta \vec{g}(\vec{x}, t) \) is a perturbation, approximate solutions of the Eq. [11] are [3]

\begin{align}
\vec{R}(\vec{x}, \vec{p}, t, t') &\approx \vec{R}^{(0)}(\vec{x}, \vec{p}, t, t') + \delta \vec{R}(\vec{x}, \vec{p}, t, t'), \\
\vec{P}(\vec{x}, \vec{p}, t, t') &\approx \vec{P}^{(0)}(\vec{x}, \vec{p}, t, t') + \delta \vec{P}(\vec{x}, \vec{p}, t, t'), \\
\vec{R}^{(0)}(\vec{x}, \vec{p}, t, t') &= \vec{x} + \frac{\vec{p}}{M_a} (t - t') + \frac{\vec{g} (t - t')^2}{2}, \\
\vec{P}^{(0)}(\vec{x}, \vec{p}, t, t') &= \vec{p} + M_a \vec{g} (t - t'); \\
\delta \vec{R}(\vec{x}, \vec{p}, t, t') &= \int_{t'}^{t} dt'' (t - t'') \delta \vec{g} \left[ \vec{R}^{(0)}(\vec{x}, \vec{p}, t', t'') \right], t'' \\
\delta \vec{P}(\vec{x}, \vec{p}, t, t') &= M_a \int_{t'}^{t} dt'' \delta \vec{g} \left[ \vec{R}^{(0)}(\vec{x}, \vec{p}, t', t'') \right], t''.
\end{align}

Functions \( \{ \vec{R}^{(0)}, \vec{P}^{(0)} \}, \delta \vec{R}, \delta \vec{P} \) obey multiplication laws

\begin{align}
\begin{cases}
\vec{R}^{(0)}(\vec{x}, \vec{p}, t', t''), \vec{P}^{(0)}(\vec{x}, \vec{p}, t', t''), t', t'' \end{cases}
\end{align}

\begin{align}
\begin{cases}
\delta \vec{R}, \delta \vec{P} \end{cases}
\end{align}

To get phase one needs aproximate expression for propagator

\begin{align}
\vec{R} \left( \vec{R}(\vec{x}, \vec{p}, \tau_1, t_0), \vec{P}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar \vec{k}}{2}, \tau_s, \tau_1 \right).
\end{align}

Using Eqs. [54] [55] one consequently finds
\[
\tilde{R} \left( \tilde{R}(\vec{x}, \vec{p}, \tau_1, t_0), \tilde{\tilde{R}}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar \tilde{k}}{2}, \tau_s, \tau_1 \right) \approx \tilde{\tilde{R}}^{(0)} \left( \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) + \delta \tilde{\tilde{R}}(\vec{x}, \vec{p}, \tau_1, t_0), \tilde{\tilde{\tilde{R}}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) + \delta \tilde{\tilde{\tilde{R}}}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar \tilde{k}}{2}, \tau_s, \tau_1 \right) \\
+ \delta \tilde{\tilde{R}} \left( \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0), \tilde{\tilde{\tilde{R}}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar \tilde{k}}{2}, \tau_s, \tau_1 \right) \\
= \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) + \delta \tilde{\tilde{R}}(\vec{x}, \vec{p}, \tau_1, t_0) + \\
\frac{1}{M_a} \left[ \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) + M_a \int_{t_0}^{\tau_1} dt' \delta \tilde{g} \left[ \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, t', t_0), t' \right] + \frac{\hbar \tilde{k}}{2} \right] \left( \tau_s - \tau_1 \right) + \frac{1}{2} \delta \tilde{g}(\tau_s - \tau_1)^2 + \\
+ \int_{\tau_1}^{\tau_s} dt' (\tau_s - t') \delta \tilde{g} \left[ \tilde{\tilde{R}}^{(0)} \left( \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0), \tilde{\tilde{\tilde{R}}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar \tilde{k}}{2}, t', \tau_1 \right), t' \right], \quad (56)
\]

Since Eqs. (56, 57),
\[
\tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_s, t_0) = \tilde{R}^{(0)} \left( \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0), \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0), \tau_s, \tau_1 \right) \\
= \tilde{R}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{1}{M_a} \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) (\tau_s - \tau_1) + \frac{1}{2} \delta \tilde{g}(\tau_s - \tau_1)^2 \quad (57)
\]
one rewrites
\[
\tilde{R} \left( \tilde{R}(\vec{x}, \vec{p}, \tau_1, t_0), \tilde{\tilde{R}}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar \tilde{k}}{2}, \tau_s, \tau_1 \right) \approx \tilde{R}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar \tilde{k}}{2M_a} (\tau_s - \tau_1) + \\
(\tau_s - \tau_1) \int_{t_0}^{\tau_1} dt'' \delta \tilde{g} \left[ \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, t'', t_0), t'' \right] + \int_{t_0}^{\tau_1} dt'' (\tau_1 - t'') \delta \tilde{g} \left[ \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, t', t_0), t' \right] + \\
+ \int_{\tau_1}^{\tau_s} dt' (\tau_s - t') \delta \tilde{g} \left[ \tilde{\tilde{R}}^{(0)} \left( \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0), \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar \tilde{k}}{2}, t', \tau_1 \right), t' \right], \quad (58)
\]

One uses now that
\[
\tilde{R}^{(0)} \left( \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0), \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar \tilde{k}}{2}, t', \tau_1 \right) = \frac{\hbar \tilde{k}}{2M_a} (t' - \tau_1) \\
+ \tilde{R}^{(0)} \left( \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0), \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0), t', \tau_1 \right) \\
= \tilde{R}^{(0)}(\vec{x}, \vec{p}, t', t_0) + \frac{\hbar \tilde{k}}{2M_a} (t' - \tau_1), \quad (59)
\]
and therefore
\[
\tilde{R} \left( \tilde{R}(\vec{x}, \vec{p}, \tau_1, t_0), \tilde{\tilde{R}}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar \tilde{k}}{2}, \tau_s, \tau_1 \right) \approx \tilde{R}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar \tilde{k}}{2M_a} (\tau_s - \tau_1) + \\
(\tau_s - \tau_1) \int_{t_0}^{\tau_1} dt'' \delta \tilde{g} \left[ \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, t', t_0), t'' \right] + \int_{t_0}^{\tau_1} dt'' (\tau_1 - t'') \delta \tilde{g} \left[ \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, t', t_0), t'' \right] + \\
+ \int_{\tau_1}^{\tau_s} dt' (\tau_s - t') \delta \tilde{g} \left[ \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, t', t_0), t' \right] \\
+ \int_{\tau_1}^{\tau_s} dt' (\tau_s - t') \left\{ \delta \tilde{g} \left[ \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, t', t_0) + \frac{\hbar \tilde{k}}{2M_a} (t' - \tau_1), t' \right] - \delta \tilde{g} \left[ \tilde{\tilde{R}}^{(0)}(\vec{x}, \vec{p}, t', t_0), t' \right] \right\}, \quad (60)
\]
The propagator \((54e)\) can be rewritten as

\[
\delta \bar{R}(\vec{x}, \vec{p}, \tau_s, t_0) = \int_{t_0}^{\tau_s} dt' (\tau_s - t') \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0), t' \right] 
= \int_{\tau_1}^{\tau_s} dt' (\tau_s - t') \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0), t' \right] + \int_{t_0}^{\tau_1} dt' (\tau_s - t') \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0), t' \right] 
= \int_{\tau_1}^{\tau_s} dt' (\tau_s - t') \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0), t' \right] + \int_{t_0}^{\tau_1} dt' (\tau_1 - t') \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0), t' \right] 
+ (\tau_s - \tau_1) \int_{t_0}^{\tau_1} dt' \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0), t' \right],
\]

which coincides with the sum of 3rd, 4th, and 5th terms on the right-hand-side (rhs) of Eq. \((60)\). Finally

\[
\bar{R} \left( \bar{R}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0), \bar{R}(\vec{x}, \vec{p}, \tau_1, t_0) + \frac{\hbar k}{2}, \tau_s, \tau_1 \right) \approx \bar{R}^{(0)}(\vec{x}, \vec{p}, \tau_s, t_0) + \delta \bar{R}(\vec{x}, \vec{p}, \tau_s, t_0) + \frac{\hbar k}{2M_a} (\tau_s - \tau_1) 
+ \int_{\tau_1}^{\tau_s} dt' (\tau_s - t') \left\{ \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0) + \frac{\hbar k}{2M_a} (t' - \tau_1), t' \right] - \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0), t' \right] \right\}. \tag{61}
\]

Substituting this result in the brackets of Eq. \((52b)\) for the 1st \((s = 3)\) and 2nd \((s = 2)\) terms, one finds that the phase of the atom interferometer consist of the terms corresponding to Earth gravity and the proof mass gravity field, and quantum correction,

\[
\phi_s(\vec{x}, \vec{p}) = \phi_0(\vec{x}, \vec{p}) + \frac{\hbar}{2} \delta \bar{g}, \tag{63a}
\]

\[
\phi_0(\vec{x}, \vec{p}) = \bar{R}^{(0)}(\vec{x}, \vec{p}, \tau_3, t_0) - 2 \bar{R}^{(0)}(\vec{x}, \vec{p}, \tau_2, t_0) + \bar{R}^{(0)}(\vec{x}, \vec{p}, \tau_1, t_0) = \bar{k} \bar{g} T^2, \tag{63b}
\]

\[
\delta \bar{g}(\vec{x}, \vec{p}) = 2 \int_{\tau_2}^{\tau_3} dt' \delta \bar{g}(\vec{x}, \vec{p}, t_0) + \frac{\hbar k}{2M_a} (t' - \tau_1), t' \left\{ \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0) + \frac{\hbar k}{2M_a} (t' - \tau_1), t' \right] - \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0), t' \right] \right\}. \tag{63c}
\]

Using Eq. \((54e)\) and piecing together terms corresponding to the integrations in time intervals \([t_0, \tau_1], [\tau_1, \tau_2], \) and \([\tau_2, \tau_3]\) one gets \((3)\)

\[
\delta \phi = \bar{k} \cdot (\tau_3 \vec{u}_{30} - t_1 \vec{u}_{20} + \vec{u}_{21} - \vec{u}_{31}), \tag{64a}
\]

\[
\vec{u}_{\alpha \beta} = \int_{\tau_{\alpha - 1}}^{\tau_{\alpha}} dt \bar{g}(\vec{u} + \vec{v} t + \bar{g} E t^2, t) \tag{64b}
\]

Consider now quantum correction \((63e)\), vector \((63e)\) can be rewritten as

\[
\bar{v}_q = \int_{\tau_2}^{\tau_3} dt' (\tau_3 - t') \left\{ \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0) + \frac{\hbar k}{2M_a} (t' - \tau_1), t' \right] - \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0), t' \right] \right\} 
+ \int_{\tau_1}^{\tau_2} dt' (t' - \tau_1) \left\{ \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0) + \frac{\hbar k}{2M_a} (t' - \tau_1), t' \right] - \delta \bar{g} \left[ \bar{R}^{(0)}(\vec{x}, \vec{p}, t_0), t' \right] \right\}. \tag{65}
\]

Substituting \(t' = \tau_2 + \theta\) for the 1st term of Eq. \((27)\), and \(t' = \tau_1 + \theta\) for the 2nd term one finds

\[
\bar{v}_q = \int_0^{T} d\theta \left\{ (T - \theta) \left[ \delta \bar{g} \left( \bar{R}^{(0)}(\vec{x}, \vec{p}, T + \tau_2 + \theta, t_0) + \frac{\hbar k}{2M_a} (T + \tau_2 + \theta) \right) - \delta \bar{g} \left( \bar{R}^{(0)}(\vec{x}, \vec{p}, T + \tau_2 + \theta, t_0), \tau_2 + \theta \right) \right] 
+ \theta \left[ \delta \bar{g} \left( \bar{R}^{(0)}(\vec{x}, \vec{p}, \tau_1 + \theta, t_0) + \frac{\hbar k}{2M_a} \theta, \tau_1 + \theta \right) - \delta \bar{g} \left( \bar{R}^{(0)}(\vec{x}, \vec{p}, \tau_1 + \theta, t_0), \tau_1 + \theta \right) \right] \right\}. \tag{66}
\]
B. Part $\phi_Q$

Consider now $Q$-term (52d). Since

$$\left\{ \begin{array}{c} \vec{R} \\ \vec{P} \end{array} \right\} \left( \vec{R}(\vec{x},\vec{p},\tau_1,t_0), \vec{P}(\vec{x},\vec{p},\tau_1,t_0) + \hbar \vec{k}/2, \tau_3, \tau_1 \right),$$

this term becomes

$$\phi_Q(\vec{x},\vec{p}) = -\frac{\hbar^2}{24} k_u k_v k_w \left\{ \int_{\tau_1}^{T_2} dt \chi'_{ikl}(\vec{\xi},t) \partial_{\vec{z}_i}(\vec{R}_u(\vec{\xi},\vec{\pi},\tau_1,t)) \partial_{\vec{z}_k}(\vec{R}_w(\vec{\xi},\vec{\pi},\tau_1,t)) \right\},$$

$$\phi_Q(\vec{x},\vec{p}) = \delta_{ij} \frac{\delta g(\vec{x},t)}{M_a}.$$

When atoms move between pulses under the action of the homogeneous Earth gravity field $\vec{g}$ and small but inhomogeneous perturbation $\delta \vec{g}(\vec{x},t)$ caused by the proof mass tensor (7b) is caused only by the proof mass,

$$\chi_{ikl} = M_a \chi_{ikl},$$

$$\chi_{ikl} = 0.$$ (69b)

While we calculate the AI phase in the linear in proof mass gravity approximation, it is sufficient to calculate the atom trajectory in Eq. (67) in the absence of the proof mass, when

$$\left\{ \begin{array}{c} \vec{R} \\ \vec{P} \end{array} \right\}(\vec{x},\vec{p},t,t') = \left\{ \begin{array}{c} \vec{R}(0) \\ \vec{P}(0) \end{array} \right\}(\vec{x},\vec{p},t,t')$$

and therefore

$$\partial_{\vec{P}} R_j(\vec{x},\vec{p},t,t') = \frac{\delta_{ij}}{M_a} (t-t').$$

Using this result and Eqs. (19, 68)

$$\phi_Q(\vec{x},\vec{p}) = \frac{\hbar^2}{24 M_a^2} k_i k_j k_l \left\{ \int_{\tau_1}^{T_2} dt \chi_{ijl}(\vec{\xi},t)(t-\tau_1)^3 \right\}$$

$$\phi_Q(\vec{x},\vec{p}) = \delta_{ij} \frac{\delta g(\vec{x},t)}{M_a}.$$

Since we are interested in calculating of the $Q$-term to the 2nd order in recoil momentum $\hbar \vec{k}$, we then can neglect recoil in the brackets of Eq. (72). We then apply the multiplication law (14) and obtain

$$\phi_Q(\vec{x},\vec{p}) = \frac{\hbar^2}{24 M_a^2} k_i k_j k_l \left\{ \int_{\tau_1}^{T_2} dt \chi_{ijl}(\vec{R}(\vec{x},\vec{p},t,t_0),t)(t-\tau_1)^3 + \int_{\tau_2}^{T_3} dt \chi_{ijl}(\vec{R}(\vec{x},\vec{p},t,t_0),t)(\tau_3-t)^3 \right\}$$

$$
\phi_Q(\vec{x},\vec{p}) = \frac{\hbar^2}{24 M_a^2} k_i k_j k_l \int_0^T d\theta \left\{ \theta^3 \chi_{ijl}(\vec{R}(\vec{x},\vec{p},\tau_1+\theta,t_0),\tau_1+\theta) + (T-\theta)^3 \chi_{ijl}(\vec{R}(\vec{x},\vec{p},\tau_2+\theta,t_0),\tau_2+\theta) \right\}.

$$
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