Isosinglet Scalar Mesons Below 2 GeV and the Scalar Glueball Mass

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A collective treatment of the \(I = 0\) scalar mesons below 2 GeV \([\sigma(550), f_0(980), f_0(1370), f_0(1500)\) and \(f_0(1710)\)] in a non-linear chiral Lagrangian framework that is constrained by the mass and the partial decay widths of the \(I = 1/2, 1\) scalars \([\kappa(900), K^*_0(1430), a_0(980)\) and \(a_0(1450)\)] is presented. The sub-structure of these states in terms of two and four quark components, as well as a glueball component is explored, and its correlation with the mass of \(f_0(1370)\) is studied. Consistency with the available experimental data suggests that the \(\sigma(550)\) is dominantly a non-strange four-quark state, whereas the sub-structure of other \(I = 0\) states is sensitive to the input mass of \(f_0(1370)\). This investigation estimates the scalar glueball mass in the range 1.47–1.64 GeV.

I. INTRODUCTION

Scalar mesons play important roles in low-energy QCD, and are at the focus of many theoretical and experimental investigations. Scalars are important from the theoretical point of view because they are Higgs bosons of QCD and induce chiral symmetry breaking, and therefore, are probes of the QCD vacuum. Scalars are also important from a phenomenological point of view, as they are very important intermediate states in Goldstone boson interactions away from threshold, where chiral perturbation theory is not applicable. There are 9 candidates for the lowest-lying scalar mesons \((m < 1 \text{ GeV}): f_0(980) \ [I = 0]\) and \(a_0(980) \ [I = 1]\) which are well established experimentally \([1]\); \(\sigma(560)\) or \(f_0(600) \ [I = 0]\) with uncertain mass and decay width \([1]\); and \(\kappa(900) \ [I = 1/2]\) which is not listed but mentioned in PDG \([1]\). The \(\kappa(900)\) is observed in some theoretical models \([2, 3, 4]\), as well as in some experimental investigations \([5, 6]\). It is known that a simple \(q\bar{q}\) picture does not explain the properties of these mesons. Different theoretical models that go beyond a simple \(q\bar{q}\) picture have been developed, including: MIT bag model \([6]\), \(K\bar{K}\) molecule \([6]\), unitarized quark model \([2, 3, 8]\), QCD sum-rules \([6]\), and chiral Lagrangians \([2, 10, 11, 12, 13, 15]\).

The next-to-lowest scalars \((1 \text{ GeV} < m < 2 \text{ GeV})\) are: \(K^*_0(1430) \ [I = 1/2]\); \(a_0(1450) \ [I = 1]\); \(f_0(1370), f_0(1500), f_0(1710) \ [I = 0]\), and are all listed in \([1]\). The \(f_0(1500)\) is believed to contain a large glue component and therefore a good candidate for the lowest scalar glueball state. These states, are generally believed to be closer to \(q\bar{q}\) objects; however, some of their properties cannot be explained based on a pure \(q\bar{q}\) structure.

Chiral Lagrangians, provide a powerful framework for studying the lowest and the next-to-lowest scalar states probed in different Goldstone boson interactions \((\pi\pi, \pi K, \pi\eta, \ldots)\) away from threshold \([2, 10, 11, 12, 13, 15]\). In this approach, a description of the scattering amplitudes which are, to a good approximation, both crossing symmetric and unitary is possible. To construct scattering amplitudes, all contributing intermediate resonances up to the energy of interest are considered, and only tree diagrams (motivated by large \(N_c\) approximation) are taken into account. In this way, crossing symmetry is satisfied, but the constructed amplitudes should be regularized. Regularization procedure in turn unitarizes the scattering amplitude. By fitting the resulting scattering amplitude to experimental data, the unknown physical properties (mass, decay width, ...) of the light scalar mesons can be extracted. It is shown in \([10]\) that there is a need for a \(\sigma\) meson with a mass around 550 MeV in order to describe the experimental data on \(\pi\pi\) scattering amplitude. Similarly, in \([2]\) the need for a \(\kappa\) meson with a mass around 900 MeV for describing the available data on \(\pi K\) scattering amplitude is presented. Motivated by the evidence for a \(\sigma\) and a \(\kappa\), and taking into account the experimentally well-established scalars, the \(f_0(980)\) and the \(a_0(980)\), a possible classification of these states (all below 1 GeV) into a lowest-lying scalar meson nonet is investigated in \([11]\). In this approach, the non-linear chiral Lagrangian is expressed in terms of this nonet, and it is shown \([11]\) that the consistency with several low energy processes \((\pi\pi\) and \(\pi K\) scatterings, and \(\Gamma[f_0(980) \to \pi\pi])\) requires a scalar mixing angle which is more consistent with a four-quark assignment for the lowest-lying scalar states. This model also well describes the experimental measurements for the \(\eta' \to \eta\pi\pi\) decay \([12]\), and estimates the total decay width of \(a_0(980)\) to be around 70 MeV consistent with a recent

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experimental confirmation \[18\]. This model has also been employed to describe the radiative decays $\phi \to \pi\eta\gamma$ and $\phi \to \pi\pi\gamma$ in ref. \[19\].

In order to study properties of the $a_0(1450)$ and the $K_0^*(1430)$, in the framework of a non-linear chiral, a next-to-lowest lying scalar nonet is introduced in \[14\]. Similar two-nonet treatment of the scalar mesons in the context of linear sigma model are investigated in \[15, 20\]. It is shown in \[14\] that a mixing between the lowest nonet (a four-quark nonet) and the next-to-lowest nonet (a two-quark nonet) provides a description of the mass spectrum and the partial decay widths of the $I = 1/2$ and $I = 1$ scalars $|\kappa(900), a_0(980), K^*_0(1430)\rangle$ and $a_0(1450)$. The case of $I = 0$ states is not studied in \[14\] and is the goal of the present work. Specifically, we would like to see if the $I = 0$ scalar states below 2 GeV can be described within a framework that is constrained by the mixing scenario of ref. \[14\], and in addition includes terms that are relevant to $I = 0$ states only, such as mixing with a scalar glueball. The mixing of scalar glueballs with $\bar{q}q$ scalar mesons has been studied in the literature \[21, 22\]. However, the importance of mixing between the $\bar{q}q$ and $\bar{q}q\bar{q}q$ nonets in ref. \[14\] suggests that the mixing of glueballs with the four-quark nonet should also be examined. Therefore, in the present work we investigate a description of the $I = 0$ states in terms of two and four quark components, and a glueball component. Moreover, we investigate the sensitivity of the results on the input data.

After a brief review of the mixing mechanism of ref. \[14\] in Sec. II, we study the mass spectrum and the substructure of the isosinglet states in Sec. III. A description of the two-pseudoscalar decay widths of these states is investigated in Sec. IV, followed by a summary and conclusion in Sec. V. The basic formulas are listed in the appendixes.

II. MIXING MECHANISM FOR I=1/2 AND I=1 SCALAR STATES

In ref. \[14\] the properties of the $I = 1/2$ and $I = 1$ scalar mesons, $\kappa(900), K^*_0(1430), a_0(980)$ and $a_0(1450)$, in a non-linear chiral Lagrangian framework is studied in detail. In this approach, a $\bar{q}q\bar{q}q$ nonet $N'$ mixes with a $\bar{q}q$ nonet $N$ and provides a description of the mass spectrum and decay widths of these scalars. This mixing provides an explanation for some unexpected properties of the $K^*_0(1430)$ and the $a_0(1450)$, which are generally believed to be good candidates for a $\bar{q}q$ nonet \[1\], but some of their properties do not quite follow this scenario. For example, in a $\bar{q}q$ nonet, isotriplet is expected to be lighter than the isodoublet, but for these two states \[1\]:

$$m_{a_0(1450)} = 1474 \pm 19 \text{ MeV} > m_{K^*_0(1430)} = 1412 \pm 6 \text{ MeV}$$

(1)

Also their decay ratios given in PDG \[1\] do not closely follow a pattern expected from an SU(3) symmetry (given in parenthesis):

$$\frac{\Gamma[a_0(1450) \rightarrow \pi K]}{\Gamma[K^*_0(1430) \rightarrow \pi K]} = 0.92 \pm 0.12 \quad (1.51), \quad \frac{\Gamma[a_0(980) \rightarrow K\pi]}{\Gamma[a_0(980) \rightarrow \pi\pi]} = 0.88 \pm 0.23 \quad (0.55), \quad \frac{\Gamma[a_0(1450) \rightarrow \pi\pi]}{\Gamma[a_0(1450) \rightarrow \pi\pi]} = 0.35 \pm 0.16 \quad (0.16)$$

(2)

These properties of the $K^*_0(1430)$ and the $a_0(1450)$ are naturally explained by the mixing mechanism of ref. \[14\]. The general mass terms contributing to the $I = 1/2$ and the $I = 1$ states can be written as

$$\mathcal{L}^{I=1/2,1} = -a\text{Tr}(NN) - b\text{Tr}(NNM) - a'\text{Tr}(N'N') - b'\text{Tr}(N'N'M)$$

(3)

where $M = \text{diag}(1,1,x)$ with $x$ being the ratio of the strange to non-strange quark masses, and $a, b, a'$ and $b'$ are unknown parameters fixed by the “bare” masses:

$$m^2[a_0] = 2(a + b) \quad m^2[K^*_0] = 2a + (1 + x)b \quad m^2[a_0] = 2(a' + b') \quad m^2[K_0^*] = 2a' + (1 + x)b'$$

(4)

where the subscript “0” denotes the “bare” states (i.e. before the mixing between $N$ and $N'$ is taken into account). Therefore, as $N$ is a four-quark nonet and $N'$ a two-quark nonet, we expect:

$$m^2[K_0] < m^2[a_0] \leq m^2[a_0] < m^2[K^*_0]$$

(5)

Introducing a simple mixing

$$\mathcal{L}^{I=1/2,1}_{\text{mix}} = -\gamma\text{Tr}(NN')$$

(6)

it is shown in \[14\] that for $0.51 < \gamma < 0.62$ GeV$^2$, it is possible to recover the physical masses such that the “bare” masses have the expected ordering of \[14\]. Therefore in this mechanism, the “bare” isotriplet states split more than
the isodoublets, and consequently, the physical isovector state $a_0(1450)$ becomes heavier than the isodoublet state $K_0^*(1430)$ as observed in [4]. The light isovector and isodoublet states are the $a_0(980)$ and the $\kappa(900)$. With the physical masses $m[a_0(980)] = 0.9835$ GeV, $m[\kappa(900)] = 0.875$ GeV, $m[a_0(1450)] = 1.455$ GeV and $m[K_0^*(1430)] = 1.435$ GeV, the best values of $\gamma$ and the “bare” masses are found in [4]

$$m_{a_0} = m_{a_0}' = 1.24 \text{ GeV}, \quad m_{K_0} = 1.06 \text{ GeV}, \quad m_{K_0'} = 1.31 \text{ GeV}, \quad \gamma = 0.58 \text{ GeV}^2$$ (7)

The decay ratios [2] are also investigated in [4] in which the scalar-pseudoscalar-pseudoscalar interaction part relevant to the $I = 1/2$ and $I = 1$ scalar states is given as

$$L_{\text{int.}}^{I=1/2,1} = A\epsilon^{abc}\epsilon_{def}N_d^a\partial_\mu\phi_2\partial_\nu\phi_3^c + C\text{Tr}(N\partial_\mu\phi)\text{Tr}(\partial_\nu\phi) + A'\epsilon^{abc}\epsilon_{def}N_d^a\partial_\mu\phi_2^c\partial_\nu\phi_3^f + C'\text{Tr}(N\partial_\mu\phi)\text{Tr}(\partial_\nu\phi)$$ (8)

where $A, A', C$ and $C'$ are unknown parameters fixed by decay properties of the scalars, and $\phi$ is the conventional pseudoscalar meson nonet. It is shown in [4] that with parameters

$$A = 1.19 \pm 0.16 \text{ GeV}^{-1}, \quad A' = -3.37 \pm 0.16 \text{ GeV}^{-1}, \quad C = 1.05 \pm 0.49 \text{ GeV}^{-1}, \quad C' = -6.87 \pm 0.50 \text{ GeV}^{-1},$$ (9)

a reasonable agreement on the decay ratios of the $K_0^*(1430)$ and the $a_0(1450)$ consistent with [2], as well as the expected decay widths of the $a_0(980)$ and the $\kappa(900)$ can be obtained. In next section, we include the Lagrangians [3] and [4] together with parameters [7] and [9] as part of the mass and interaction Lagrangian of the $I = 0$ scalar states.

### III. ISOSINGLET STATES

The general mass terms for nonets $N$ and $N'$, and a scalar glueball $G$ can be written as:

$$L_{\text{mass}}^{I=0} = L_{\text{mass}}^{I=1/2,1} - c\text{Tr}(N)\text{Tr}(N) - d\text{Tr}(N)\text{Tr}(N,M) - c'\text{Tr}(N')\text{Tr}(N') - d'\text{Tr}(N')\text{Tr}(N'M) - gG^2$$ (10)

The a priori unknown parameters $c$ and $d$ induce “internal” mixing between the two $I = 0$ flavor combinations $[(N_1^1 + N_2^2)/\sqrt{2}$ and $N_3^2]$ of nonet $N$. Similarly, $c'$ and $d'$ play the same role in nonet $N'$. Parameters $c, d, c'$ and $d'$ do not contribute to the mass spectrum of the $I = 1/2$ and $I = 1$ states. The last term represents the glueball mass term. The term $L_{\text{mass}}^{I=1/2,1}$ is imported from Eq. [3] together with its parameters from Eq. [4].

The mixing between $N$ and $N'$, and the mixing of these two nonets with the scalar glueball $G$ can be written as

$$L_{\text{mix}}^{I=0} = L_{\text{mix}}^{I=1/2,1} - \rho\text{Tr}(N)\text{Tr}(N') - \epsilon\text{G} \text{Tr}(N) - f\text{G} \text{Tr}(N')$$ (11)

where the first term is given in [4] with $\gamma$ from [4]. The second term does not contribute to the $I = 1/2,1$ mixing, and in special limit of $\rho \rightarrow -\gamma$:

$$-\gamma\text{Tr}(N'N') - \rho\text{Tr}(N)\text{Tr}(N') = \gamma\epsilon^{abc}\epsilon_{ade}N_a^dN_c^e$$ (12)

This particular mixing is more consistent with the OZI rule than the individual $\gamma$ and $\rho$ terms and is studied in [22]. Here we do not restrict the mixing to this particular combination, and instead, examine a range of $\rho$ values. Terms with unknown couplings $\epsilon$ and $f$ describe mixing with the scalar glueball $G$. As a result, the five isosinglet below 2 GeV, become a mixture of five different flavor combinations, and their masses can be organized as

$$L_{\text{mass}}^{I=0} = L_{\text{mix}}^{I=0} = \frac{1}{2} F_0 M_0^2 F_0 = \frac{1}{2} F M_{\text{diag}}^2 F$$ (13)

with

$$F_0 = \left( \begin{array}{c} N_3^3 \\ (N_1^1 + N_2^2)/\sqrt{2} \\ N_3^3 \\ (N_1^1 + N_2^2)/\sqrt{2} \\ G \end{array} \right) = \left( \begin{array}{c} ud \bar{d} u \bar{d} \\ (\bar{s}d + \bar{u}s)/\sqrt{2} \\ \bar{s}s \\ (\bar{u}u + \bar{d}d)/\sqrt{2} \\ G \end{array} \right) = \left( \begin{array}{c} f_0^{NS} \\ f_0^S \\ f_0^S \\ f_0^{NS} \end{array} \right)$$ (14)

where the superscript $NS$ and $S$ respectively represent the non-strange and strange combinations. $F$ contains the physical fields

$$F = \left( \begin{array}{c} \sigma(550) \\ f_0(980) \\ f_0(1370) \\ f_0(1500) \\ f_0(1710) \end{array} \right) = K^{-1} F_0$$ (15)
where $K^{-1}$ is the transformation matrix. The mass squared matrix is

$$
M^2 = \begin{bmatrix}
2m_{K^0}^2 - m_{a_0}^2 + 2(c + d\chi) & \sqrt{2}[2c + (1 + x)d] & \gamma + \rho & \sqrt{2}\rho & e \\
\sqrt{2}[2c + (1 + x)d] & m_{a_0}^2 + 4(c + d) & \sqrt{2}\rho & \gamma + 2\rho & \sqrt{2}\rho \\
\gamma + \rho & \sqrt{2}\rho & 2m_{a_0}^2 - m_{a_0}'^2 + 2(c' + d'\chi) & \sqrt{2}[2c' + (1 + x)d'] & f \\
\sqrt{2}\rho & \gamma + 2\rho & \sqrt{2}[2c' + (1 + x)d'] & m_{a_0}'^2 + 4(c' + d') & \sqrt{2}f \\
e & \sqrt{2}e & f & \sqrt{2}f & 2g \\
\end{bmatrix}
$$

(16)

in which the value of the unmixed $I = 1/2, 1$ masses, and the mixing parameter $\gamma$ are substituted in from (17). We search for the unknown parameters $c, c', d', e, f, g$ and $\rho$ in (10) by fitting its eigenvalues to the mass of the physical states. We take $m_\pi = 550 \pm 50$ MeV expected from chiral Lagrangian treatment of the $\pi\pi$ scattering in [10], as well as the following experimental values from PDG [1]:

$$
\begin{align*}
m[f_0(980)] &= 980 \pm 10 \text{ MeV} \\
m[f_0(1370)] &= 1200 \rightarrow 1500 \text{ MeV} \\
m[f_0(1500)] &= 1507 \pm 5 \text{ MeV} \\
m[f_0(1710)] &= 1713 \pm 6 \text{ MeV}
\end{align*}
$$

(17)

The largest experimental uncertainty is on the mass of $f_0(1370)$, and for our initial study we take its central value of 1350 MeV. The sensitivity of the results on the mass of $f_0(1370)$ turns out to be the main source of error and will be discussed later. Figure 1 shows the dependency of $\chi^2$ on parameter $\rho$. We see that for $-0.4 \text{GeV}^2 < \rho < 0$ the $\chi^2$ is very small, but significantly increases outside this interval. For five values of $\rho$ the result of fits are given in table I, in which the sensitivity of the fitted parameters on $\rho$ can be seen. As far as fitting the isosinglet masses is concerned, all chosen values of $\rho$ in table I give more or less the same description. Parameters $d$ and $d'$ induce SU(3) symmetry breaking and are at least an order of magnitude smaller than $c$ and $c'$. The fitted parameters determine the rotation matrix (13) which in turn probes the quark substructure of the scalars. For $\rho = 0$ the rotation matrix is given in (18) in which the errors reflect the experimental uncertainties in (17). For other values of $\rho$ in table I the corresponding rotation matrices are given in Appendix A. The overall results show that the sub-structure of $\sigma(550)$ is not sensitive to $\rho$ and is dominantly a non-strange four quark combination $\bar{u}ud$, consistent with the investigation of ref. [11].

The sub-structure of the $f_0(980)$ is more sensitive to $\rho$; for $\rho < -0.2 \text{GeV}^2$ it has a dominant $ss$ component, whereas for $\rho \geq -0.2 \text{GeV}^2$ the non-strange $(\bar{u}u + \bar{d}d)/\sqrt{2}$ component dominates. The later case has some support in QCD sum-rules [3]. The $f_0(1370)$ has substantial two-quark components with some glueball admixture, and the $f_0(1500)$ and $f_0(1710)$ have large glueball components. For each value of $\rho$, the corresponding glueball mass is also given in table I showing a variation in the range 1.47-1.60 GeV. As $\rho \rightarrow -\gamma$ the result indicates that the glueball component of the $f_0(1500)$ significantly increases, but in this limit the $\chi^2$ is relatively large, and therefore, the result is not as accurate as the cases given in table I. The rotation matrix is given in Eq. (A1), and is consistent with the result of ref. [22].

| Fitted Parameters $\rho = -0.5\text{GeV}^2$ | $\rho = -0.4\text{GeV}^2$ | $\rho = -0.2\text{GeV}^2$ | $\rho = 0$ | $\rho = 0.075\text{GeV}^2$ |
|-----------------|-----------------|-----------------|-------------|-----------------|
| $c$ (GeV$^2$)   | 0.141           | 0.181           | 0.281       | 0.157 ± 0.009   | 0.201           |
| $d$ (GeV$^2$)   | -0.00494        | -0.0106         | -0.0175     | -0.00779 ± 0.00187 | -0.00840       |
| $c'$ (GeV$^2$)  | -0.0389         | -0.0475         | -0.0356     | -0.0128 ± 0.0052 | -0.00184       |
| $d'$ (GeV$^2$)  | -0.00138        | -0.00407        | -0.00452    | -0.00927 ± 0.00116 | -0.00825       |
| $e$ (GeV$^2$)   | -0.169          | -0.165          | -0.242      | -0.323 ± 0.018   | -0.248          |
| $f$ (GeV$^2$)   | 0.0941          | 0.234           | 0.323       | 0.115 ± 0.0568   | 0.0874          |
| $g$ (GeV$^2$)   | 1.211           | 1.249           | 1.080       | 1.272 ± 0.021    | 1.159           |
| $m_G$ (GeV)     | 1.556           | 1.581           | 1.470       | 1.595 ± 0.013    | 1.523           |
| $\chi^2$       | 0.0740          | ≈ 0             | ≈ 0         | ≈ 0             | 0.0932          |

TABLE I: Best numerical values for the unknown parameters in Lagrangian (13), with $m[f_0(1370)] = 1.35$ GeV and several values of $\rho$. For $\rho = 0$, the errors on the fitted parameters are also given, and reflect the experimental uncertainties in (17).
sensitivity of the glueball content of the $f_{1500}$ and show a strong dependence. For example, we see that below the central value of $m[f_0(1370)]$ the fitted parameters are given in table III and the corresponding rotation matrices are given in Appendix A. The resulting glueball mass is in the range $1.2$ and $54$ GeV. Of course this periodic-like behavior is not surprising as the correlation between the mixing parameter $\rho$ and $m[f_0(1370)]$ = 1.35 GeV. The first column corresponds to the general case $f_{1370}$ which has the largest experimental uncertainty in (17). We find that very good $\chi^2$ fits can be obtained for $1.31$ GeV $\leq m[f_0(1370)] \leq 1.45$ GeV. Outside this interval, however, the goodness of the fits significantly reduces. This observation further restricts the $m[f_0(1370)]$ in (17). For several values of $m[f_0(1370)]$ the fitted parameters are given in table III and the corresponding rotation matrices are given in Appendix A. The resulting glueball mass is in the range $1.54$ GeV $\leq m_G \leq 1.61$ GeV. For this range of $m[f_0(1370)]$, the sensitivity of the glueball content of the $f_{0}(1370)$, $f_{0}(1500)$ and $f_{0}(1710)$ on the mass of $f_{0}(1370)$ are examined in figure 2 and show a strong dependence. For example, we see that below the central value of $m[f_0(1370)]$ = 1.35 GeV, the glueball component of the $f_{0}(1710)$ dominates those of the $f_{0}(1500)$ and $f_{0}(1370)$. However, above $m[f_0(1370)] \approx 1.4$ GeV, the $f_{0}(1500)$ acquires the largest glue component followed by the components of the $f_{0}(1710)$ and $f_{0}(1370)$. The order again changes around $m[f_0(1370)] \approx 1.42$ GeV. Of course this periodic-like behavior is not surprising as the rotation matrix, which depends on ten mixing angles, goes through periodic variations as we tune $m[f_0(1370)]$. Therefore, due to this sensitivity on the mass of $f_{0}(1370)$, a precise extraction of the glueball content of these states requires an accurate knowledge of the $m[f_0(1370)]$.

The correlation between the mixing parameter $\rho$ and the input mass for the $f_{0}(1370)$ are also examined. Although the texture of the rotation matrix varies with these two parameters, the glueball mass remains more or less within the same intervals obtained by uncorrelated variations of $\rho$ and $m[f_0(1370)]$ (see above). The overall numerical work shows

\[ 1.47 \text{ GeV} \leq m_G \leq 1.64 \text{ GeV} \]  

in agreement with the lattice QCD estimates [23]. There are values of $\rho$ and $m[f_0(1370)]$, for which the glueball masses are as low as 1.32 GeV. However, for these cases the $f_{0}(980)$ acquires a large glueball component which is not consistent with either the molecule or the four-quark description of this state.

It is important to note that the coupling of the glueball to both nonets $N$ and $N'$ should be taken into account. This is examined in table III for $\rho = 0$ and $m[f_0(1370)]=1.35$ GeV. The first column corresponds to the general case

\[
K^{-1} = \begin{pmatrix}
0.853 \pm 0.015 & -0.011 \pm 0.024 & -0.476 \pm 0.018 & -0.148 \pm 0.024 & 0.156 \pm 0.017 \\
0.278 \pm 0.040 & -0.497 \pm 0.009 & 0.211 \pm 0.019 & 0.774 \pm 0.013 & -0.181 \pm 0.035 \\
0.394 \pm 0.020 & 0.311 \pm 0.026 & 0.817 \pm 0.025 & -0.102 \pm 0.022 & 0.266 \pm 0.048 \\
-0.125 \pm 0.021 & 0.550 \pm 0.024 & -0.247 \pm 0.055 & 0.588 \pm 0.015 & 0.525 \pm 0.020 \\
-0.157 \pm 0.023 & -0.595 \pm 0.017 & 0.032 \pm 0.019 & -0.153 \pm 0.017 & 0.773 \pm 0.016
\end{pmatrix}
\]  

\[ \text{FIG. 1: } \chi^2 \text{ vs } \rho \]
where the coupling of the glueball to both nonets $N$ and $N'$ is taken into account. We see that scalar glueball coupling to $N$ is larger than its coupling to $N'$. In a recent work [22] a similar analysis is given in which the coupling to the lowest-lying nonet is neglected. To illustrate the importance of the glueball coupling to both nonets, in the second and third columns of Table III the glueball coupling to $N$ and $N'$ is respectively suppressed. We see that the result is sensitive to the glueball coupling to $N$ and $N'$. For example, when $e = 0$ the glueball mass significantly increases to 1.45 GeV, whereas when $f = 0$ the glueball mass decreases to 1.22 GeV. (We see that the result seems to be more sensitive to $e$.) Other fitted parameters in Table III also show a similar sensitivity to the suppression of the couplings $e$ and $f$. Therefore, it is important to consider a general treatment in which both couplings are taken into account.

### IV. INTERACTION LAGRANGIAN

To further examine the present model for the scalar mesons, we need to investigate the interaction Lagrangian for these states and study their partial decay widths to several two-pseudoscalar channels. We saw in previous section that the experimental input mass of $f_0(1370)$ is the main source of uncertainty on the rotation matrix which in
TABLE III: Numerical values for the unknown parameters in Lagrangian (13). The first column corresponds to the case where the scalar glueball couples to both the light scalar meson nonet \( N \) as well as to the heavy scalar meson nonet \( N' \). The second and third columns represent cases where the scalar glueball only couples to the nonet \( N' \), and to the nonet \( N \), respectively.

| Parameter | Fit 1  | Fit 2  | Fit 3  |
|-----------|-------|-------|-------|
| \( c \) (GeV) | 0.16  | 0.081 | 0.21  |
| \( d \) (GeV) | −0.0078 | −0.0067 | −0.0098 |
| \( c' \) (GeV) | −0.013 | −0.027 | −0.013 |
| \( d' \) (GeV) | −0.0093 | −0.0058 | −0.013 |
| \( e \) (GeV) | −0.32 | 0 | −0.25 |
| \( f \) (GeV) | 0.12 | 0.13 | 0 |
| \( g \) (GeV) | 1.27 | 1.45 | 1.22 |

The scalar-pseudoscalar-pseudoscalar interaction takes the general form:

\[
\mathcal{L}_{int.}^{I=0} = \mathcal{L}_{int.}^{I=1/2,1} + B \text{Tr} (N) \text{Tr} (\partial_\mu \phi \partial_\mu \phi) + D \text{Tr} (N) \text{Tr} (\partial_\mu \phi) \text{Tr} (\partial_\mu \phi) + B' \text{Tr} (N') \text{Tr} (\partial_\mu \phi \partial_\mu \phi) + D' \text{Tr} (N') \text{Tr} (\partial_\mu \phi) \text{Tr} (\partial_\mu \phi) + E \text{GTr} (\partial_\mu \phi \partial_\mu \phi) + F \text{GTr} (\partial_\mu \phi) \text{Tr} (\partial_\mu \phi)
\]

where \( B \) and \( D \) are unknown coupling constants describing the coupling of the four-quark nonet \( N \) to the pseudoscalars. Similarly, \( B' \) and \( D' \) are couplings of \( N' \) to the pseudoscalars. \( E \) and \( F \) describe the coupling of a scalar glueball to the pseudoscalar mesons, and \( \mathcal{L}_{int.}^{I=1/2,1} \) is the interaction Lagrangian for \( I = 1/2,1 \) states given in (8) together with parameters in (9).

The pseudoscalar part of the Lagrangian can be found in ref. [2]. The interaction Lagrangian (20) can be rewritten as:

\[
-\mathcal{L}_{int} = \frac{1}{\sqrt{2}} \gamma_{s\pi}^i F_i \partial_\mu \pi \partial_\mu \pi + \frac{1}{\sqrt{2}} \gamma_{K\pi}^i F_i \partial_\mu K \partial_\mu K + \gamma_{\eta\pi}^i F_i \partial_\mu \eta \partial_\mu \eta + \gamma_{\eta'\eta}^i F_i \partial_\mu \eta \partial_\mu \eta' + \gamma_{\eta'\eta'}^i F_i \partial_\mu \eta' \partial_\mu \eta'
\]

where \( \gamma_{ss'}^i \) is the coupling of the \( i \)-th scalar [see Eq. (15)] to pseudoscalars \( s \) and \( s' \), and is given by

\[
\gamma_{ss'} = \sum_j (\gamma_{ss'}^j K)_{ji}
\]

with \( K \) defined in (15) and

\[
\gamma_{ss'} = \begin{pmatrix}
\gamma_{ss}^{NS} & \gamma_{ss}^S & \gamma_{ss}^{SNS} \\
\gamma_{ss'}^{NS} & \gamma_{ss'}^S & \gamma_{ss'}^{SNS} \\
\gamma_{ss'}^{NS} & \gamma_{ss'}^S & \gamma_{ss'}^{NS}
\end{pmatrix}
\]

in which the diagonal elements are the couplings of the pseudoscalars \( s \) and \( s' \) to the \( f_0^{NS}, f_0^S, f_0^{SNS} \) [defined in (14)] and the scalar glueball \( G \), respectively. The diagonal elements for all decay channels \( ss' \) are listed in the Appendix B.

To determine the unknown couplings we need to fit the prediction of this Lagrangian to experimental data. Here we use the estimates of the decay ratios by the WA102 collaboration (24) in Table IV. We should note, however, that the decay ratios alone are not sufficient to determine the free parameters and need to be supplemented by more data.
such as the individual partial decay widths, or the total decay widths [1]:

\[
\begin{align*}
\Gamma_{\text{total}}[f_0(980)] &= 40 \rightarrow 100 \text{ MeV} \\
\Gamma_{\text{total}}[f_0(1370)] &= 200 \rightarrow 500 \text{ MeV} \\
\Gamma_{\text{total}}[f_0(1500)] &= 109 \pm 7 \text{ MeV} \\
\Gamma_{\text{total}}[f_0(1710)] &= 125 \pm 10 \text{ MeV}
\end{align*}
\]

For example, with \( \rho = 0 \) and \( m[f_0(1370)] = 1.35 \text{ GeV} \), a numerical study of the decay ratios is given in table IV together with the predicted decay widths in table V. The result is compared with estimates extracted from other works, and shows an overall qualitative agreement, even though some of the predicted decay widths such as \( \Gamma[\sigma(550) \rightarrow \pi\pi] \) or \( \Gamma[f_0(1500) \rightarrow \pi\pi] \) do not quite agree with other investigations [1]. However we should again note that the current experimental status of the decay widths is not quite established, and that together with the large uncertainty on the mass of the \( f_0(1370) \) are the main sources of error in estimates given in tables IV and V. Certainly more work is needed to investigate the parameters of the interaction Lagrangian in more detail, and study its correlation with the mass of \( f_0(1370) \). We postpone this goal for future works.

| Fitted Parameters | This fit | WA102 Collaboration |
|-------------------|---------|---------------------|
| \( B \) (GeV\(^{-2}\)) | \(-1.0 \rightarrow -0.7\) | \(2.17 \pm 0.9\) |
| \( B' \) (GeV\(^{-2}\)) | \(0.8 \rightarrow 1.0\) | \(0.35 \pm 0.21\) |
| \( D(\text{GeV}\(^{-2}\))\) | \(-0.1 \rightarrow 2.5\) | \(0.32 \pm 0.07\) |
| \( D' \) (GeV\(^{-2}\)) | \(-3.5 \rightarrow 0.5\) | \(5.5 \pm 0.84\) |
| \( E \) (GeV\(^{-2}\)) | \(-1.6 \rightarrow -0.3\) | |
| \( F \) (GeV\(^{-2}\)) | \(-2.2 \rightarrow 2.4\) | |

**TABLE IV:** Numerical values for the parameters in the scalar-pseudoscalar-pseudoscalar Lagrangian [Eq. (20)], obtained by fitting its prediction for decay ratios (with the specific choice of \( m[f_0(1370)] = 1.35 \text{ GeV} \) and \( \rho = 0 \)) to the experimental data by WA102 collaboration.

V. SUMMARY AND CONCLUSION

In this work we studied the \( I = 0 \) scalar mesons below 2 GeV \([\sigma(550), f_0(980), f_0(1370), f_0(1500), \text{ and } f_0(1710)]\) using a non-linear chiral Lagrangian which is constrained by the mass and the decay properties of the \( I = 1/2 \) and \( I = 1 \) scalar meson below 2 GeV \([\kappa(900), K_0^*(1430), a_0(980) \text{ and } a_0(1450)]\). In this framework the lowest-lying four-quark scalar meson nonet \( N \) mixes with the next-to-lowest lying two-quark nonet \( N' \) and a scalar glueball \( G \). We showed that this model can describe the mass spectrum of the scalars, and studied the correlation between the mass of \( f_0(1370) \) and the substructure of these states. We showed that consistency of this model with the experimental mass spectrum favors \( 1.31 \text{ GeV} \leq m[f_0(1370)] \leq 1.45 \text{ GeV} \), and sets a bound on the scalar glueball mass in the 1.47 GeV to 1.64 GeV range. We also showed that it is important to take into account the coupling of the scalar

---

1 Note that the \( \Gamma[\sigma \rightarrow \pi\pi] \), which is proportional to the \( \gamma_2^{\sigma}_{\pi\pi} \), is not the same as the Breit-Wigner width that appears in the denominator of the \( \sigma \) propagator [10]. This deviation from a Breit-Wigner shape, is also a characteristic of the \( \kappa(900) \) meson [11].
complex mixing terms between two and four quark nonets. It is also interesting to examine this model with higher order effects, such as higher derivative terms or more scalar mesons are expected to play important roles [12, 28], and therefore are interesting directions for future work. The author wishes to thank A. Abdel-Rahim, D. Black and J. Schechter for very helpful discussions. This work has been supported by 2003 grant from the State of New York/UUP Professional Development Committee; 2003 Grant from the School of Arts and Sciences, SUNY Institute of Technology.

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APPENDIX A: THE ROTATION MATRICES

1. \( m[f_0(1370)] = 1.35 \text{ GeV} \) and \( \rho \) a variable:

\[
K^{-1}_{(\rho=-0.5 \text{ GeV}^2)} = \begin{bmatrix}
0.829 & 0.033 & 0.030 & 0.556 & 0.034 \\
-0.289 & 0.605 & 0.648 & 0.362 & -0.009 \\
-0.431 & -0.099 & -0.491 & 0.692 & -0.292 \\
-0.066 & 0.411 & -0.430 & 0.049 & 0.800 \\
0.197 & 0.674 & -0.393 & -0.280 & -0.524
\end{bmatrix}
\]

\[
K^{-1}_{(\rho=-0.4 \text{ GeV}^2)} = \begin{bmatrix}
0.884 & -0.047 & -0.047 & 0.463 & -0.003 \\
-0.224 & 0.402 & 0.685 & 0.535 & -0.183 \\
-0.368 & -0.362 & -0.482 & 0.616 & -0.349 \\
-0.128 & 0.602 & -0.445 & 0.266 & 0.594 \\
0.129 & 0.586 & -0.315 & -0.223 & -0.701
\end{bmatrix}
\]

TABLE V: Partial decay widths of \( I = 0 \) scalars predicted by the fit in Table IV

| Decay Widths (MeV) | This Model | Extracted from other works |
|-------------------|------------|---------------------------|
| \( \Gamma[\sigma(550) \rightarrow \pi\pi] \) | 53 \rightarrow 61 | \( \approx 100 \) [10] |
| \( \Gamma[f_0(980) \rightarrow \pi\pi] \) | 107 \rightarrow 116 | \( \approx 65 \) [10] |
| \( \Gamma[f_0(1370) \rightarrow \pi\pi] \) | 121 \rightarrow 238 | 34 \rightarrow 175 [25] |
| \( \Gamma[f_0(1370) \rightarrow K\bar{K}] \) | 36 \rightarrow 85 | 44 \rightarrow 240 [25] |
| \( \Gamma[f_0(1370) \rightarrow \eta\eta] \) | < 20 | - |
| \( \Gamma[f_0(1500) \rightarrow \pi\pi] \) | 103 \rightarrow 329 | 36 \rightarrow 65 [25] |
| \( \Gamma[f_0(1500) \rightarrow K\bar{K}] \) | 2 \rightarrow 17 | 2.4 \rightarrow 7.5 [25] |
| \( \Gamma[f_0(1500) \rightarrow \eta\eta] \) | < 62 | - |
| \( \Gamma[f_0(1500) \rightarrow \eta\eta'] \) | < 23 | large [26] |
| \( \Gamma[f_0(1710) \rightarrow \pi\pi] \) | < 92 | 1.7 \rightarrow 5.5 [27] |
| \( \Gamma[f_0(1710) \rightarrow K\bar{K}] \) | 3 \rightarrow 36 | 22 \rightarrow 64 [27] |
| \( \Gamma[f_0(1710) \rightarrow \eta\eta] \) | < 8 | 6 \rightarrow 24 [27] |
| \( \Gamma[f_0(1710) \rightarrow \eta\eta'] \) | < 1.5 | - |

glueball to \( N \) as well as to \( N' \). We found that the \( \sigma(550) \) is mainly a non-strange four quark state, whereas the substructure of other \( I = 0 \) states is sensitive to the mass of the \( f_0(1370) \). The numerical results show that the \( f_0(1500) \) and \( f_0(1710) \) have significant glueball admixtures. We also investigated the interaction Lagrangian and gave a preliminary study of the decay widths of the \( I = 0 \) scalars into various pseudoscalar-pseudoscalar channels. Probing scalar-pseudoscalar-pseudoscalar couplings is important for low energy processes such as \( \eta'/3\pi, \eta' \rightarrow \eta\pi\pi \), in which the scalar mesons are expected to play important roles [12, 28], and therefore are interesting directions for future works. It is also interesting to examine this model with higher order effects, such as higher derivative terms or more complex mixing terms between two and four quark nonets.
\[ K^{-1}_{(\rho=-0.2\text{GeV}^2)} = \begin{bmatrix} 0.927 & -0.138 & -0.292 & 0.155 & 0.108 \\ 0.039 & -0.107 & 0.429 & 0.783 & -0.436 \\ 0.329 & 0.342 & 0.804 & -0.293 & 0.210 \\ -0.141 & 0.455 & -0.126 & 0.524 & 0.694 \\ 0.104 & 0.803 & -0.264 & -0.042 & -0.522 \end{bmatrix} \]  \hspace{1cm} (A3)

\[ K^{-1}_{(\rho=0.075\text{GeV}^2)} = \begin{bmatrix} 0.842 & -0.016 & -0.490 & -0.183 & 0.133 \\ 0.274 & -0.531 & 0.156 & 0.769 & -0.167 \\ 0.414 & 0.062 & 0.852 & -0.232 & 0.212 \\ -0.077 & 0.302 & -0.075 & 0.435 & 0.842 \\ 0.199 & 0.789 & 0.057 & 0.365 & -0.448 \end{bmatrix} \]  \hspace{1cm} (A4)

2. \( \rho = 0 \) and \( m[f_0(1370)] \) a variable:

\[ K^{-1}_{(m[f_0(1370)]=1.34\text{GeV})} = \begin{bmatrix} 0.844 & -0.003 & -0.490 & -0.159 & 0.149 \\ 0.298 & -0.506 & 0.220 & 0.762 & -0.163 \\ 0.398 & 0.348 & 0.798 & -0.096 & 0.274 \\ -0.129 & 0.558 & -0.273 & 0.602 & 0.485 \\ -0.155 & -0.559 & 0.028 & -0.148 & 0.801 \end{bmatrix} \]  \hspace{1cm} (A5)

\[ K^{-1}_{(m[f_0(1370)]=1.36\text{GeV})} = \begin{bmatrix} 0.853 & -0.034 & -0.476 & -0.142 & 0.159 \\ 0.277 & -0.467 & 0.230 & 0.784 & -0.196 \\ 0.391 & 0.267 & 0.821 & -0.149 & 0.283 \\ -0.126 & 0.486 & -0.215 & 0.554 & 0.629 \\ 0.167 & 0.688 & -0.035 & 0.192 & -0.679 \end{bmatrix} \]  \hspace{1cm} (A6)

\[ K^{-1}_{(m[f_0(1370)]=1.38\text{GeV})} = \begin{bmatrix} 0.867 & -0.030 & -0.447 & -0.124 & 0.181 \\ 0.241 & -0.481 & 0.180 & 0.785 & -0.251 \\ 0.389 & 0.204 & 0.862 & -0.122 & 0.220 \\ -0.110 & 0.490 & -0.148 & 0.570 & 0.633 \\ 0.167 & 0.698 & -0.044 & 0.170 & -0.674 \end{bmatrix} \]  \hspace{1cm} (A7)

\[ K^{-1}_{(m[f_0(1370)]=1.40\text{GeV})} = \begin{bmatrix} 0.883 & -0.031 & -0.417 & -0.100 & 0.189 \\ 0.197 & -0.483 & 0.136 & 0.793 & -0.283 \\ 0.384 & 0.132 & 0.895 & -0.120 & 0.138 \\ -0.085 & 0.502 & -0.061 & 0.567 & 0.645 \\ 0.163 & 0.705 & -0.049 & 0.158 & -0.671 \end{bmatrix} \]  \hspace{1cm} (A8)

**APPENDIX B: THE COUPLING CONSTANTS AND DECAY WIDTHS**

The coupling of \( I = 0 \) states \( f_0^{NS}, f_0^{S}, f_0^{S}, f_0^{NS} \) and \( G \) to different two-pseudoscalar channels are:
The two-body partial decay widths of physical states \( F \) is the pseudoscalar mixing angle defined as

\[
\eta = \sin \theta_p - \sqrt{2} \sin \theta_p \cos \theta_p - \sqrt{2} \sin \theta_p \cos \theta_p - \sqrt{2} \sin \theta_p \cos \theta_p
\]

\[
\frac{(\phi_1^* + \phi_2^*)}{\sqrt{2}}
\]

where \( \theta_p \) is the pseudoscalar mixing angle defined as

\[
\left( \begin{array}{c} \eta \\ \eta' \end{array} \right) = \left( \begin{array}{cc} \cos \theta_p & -\sin \theta_p \\ \sin \theta_p & \cos \theta_p \end{array} \right) \left( \begin{array}{c} \phi_1^* + \phi_2^* \\phi_3^3 \end{array} \right) / \sqrt{2}
\]

with \( \theta_p \approx 37^\circ \). The two-body partial decay widths of physical states \( F \) are:

\[
\Gamma[F_i \to \pi\pi] = 3 \left( \frac{\eta^2}{32\pi M_{F_i}^2} \right) \left[ M_{F_i}^2 - 2m_\pi^2 \right]^2
\]

\[
\Gamma[F_i \to K\bar{K}] = \left( \frac{\eta^2}{32\pi M_{F_i}^2} \right) \left[ M_{F_i}^2 - 2m_K^2 \right]^2
\]
\[ \Gamma[F_i \to \eta \eta] = 2 \left( \frac{q^2}{32\pi M_{F_i}} \right) \left[ M_{F_i}^2 - 2m_\eta^2 \right]^2 \] (B29)

\[ \Gamma[F_i \to \eta \eta'] = \left( \frac{q^2}{32\pi M_{F_i}} \right) \left[ M_{F_i}^2 - (m_\eta^2 + m_{\eta'}^2) \right]^2 \] (B30)

where \( q \) is the center of mass momentum of the final state mesons, and for a general two-body decay \( A \to BC \) is given as:

\[ q = \sqrt{[m_A^2 - (m_B + m_C)^2]} \left[ \frac{m_A^2 - (m_B - m_C)^2}{2m_A} \right] \] (B31)

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