A NOTE ON THE ROBUSTNESS OF PAIR SEPARATIONS
METHODS IN COSMIC TOPOLOGY

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It is well known that general relativity is a local metrical theory and therefore the corresponding field equations do not fix the global topology of spacetime. This freedom has fuelled a great deal of interest in the possibility that the universe may possess spatial sections with non-trivial topology (see for example Refs. 1 and 2).

An immediate observational consequence of a nontrivial topology (multiple-connectedness) of the 3-space $M$ is that the sky may show multiple images of radiating sources. We are assuming here and in what follow that the universe has a detectable (nontrivial) topology (for details on this point see Ref. 3).

However, the direct identification of multiple images is a formidable observational task to carry out because it involves a number of problems. This has motivated the development of methods in which the cosmic images are statistically treated in the search for a sign of a possible nontrivial topology of the universe. In a universe with nontrivial topology the 3-D positions of the multiple images are correlated, and these correlations can be couched in terms of pair-separation correlations. The first statistical method (cosmic crystallography) looks for these correlations by using pair separations histograms (PSH). But the only significant (measurable) sign of a nontrivial topology in PSH’s was shown to be spikes, and they can be used merely to reveal a possible nontrivial topology of universes that admit Clifford translations (for details see, e.g. Ref. 5).

The determination of the positions of cosmic sources, however, involves inevitable uncertainties, some of which have been discussed by Lehoucq et al. Here we briefly report our results concerning the sensitivity of the topological spikes in the presence of the uncertainties in the positions of sources, which arise from uncertainties in the values of the density parameters.

For brevity, we shall consider only flat universes ($\Omega_m + \Omega_{\Lambda} = 1$), but a similar

\*By topology of the universe we mean the topology of the space-like section $M$. 

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analysis can be carried out for spherical universes, with qualitatively the same results. The redshift-distance relation for the flat case reads

\[ r(z, \Omega_{m0}) = \frac{c}{H_0} \int_1^{1+z} \left[ (x^3 - 1)\Omega_{m0} + 1 \right]^{-1/2} dx. \tag{1} \]

Consider now two cosmic sources at distances \( r_1 \) and \( r_2 \) from the observer \( O \), with their lines of sight forming an angle \( \theta \). The law of cosines gives the pair-separation \( s = d^2 \) of these objects

\[ s = r_1^2 + r_2^2 - 2 r_1 r_2 \cos \theta \quad (r_1 \leq r_2). \tag{2} \]

The uncertainty in \( s \) which arises from the uncertainties in \( \Omega_{m0} \) is

\[ \Delta s = \frac{ds}{d\Omega_{m0}} \Delta \Omega_{m0} = 2 \left[ (r_1 - r_2 \cos \theta) \frac{\partial r_1}{\partial \Omega_{m0}} + (r_2 - r_1 \cos \theta) \frac{\partial r_2}{\partial \Omega_{m0}} \right] \Delta \Omega_{m0}. \tag{3} \]

For a fixed pair separation \( s \) the uncertainties \( \Delta s^{(1)} \) and \( \Delta s^{(2)} \) at, respectively, \( r_1 \) and \( r_2 \) are such that \( |\Delta s^{(2)}| \geq |\Delta s^{(1)}| \). For \( r_1 = r_2 = r \) the relative error \( \sigma \) is

\[ \frac{\Delta s}{s} = 2 \sigma \Delta \Omega_{m0} \quad \text{with} \quad \sigma(z, \Omega_{m0}) = \frac{\partial \ln r}{\partial \Omega_{m0}}. \tag{4} \]

Figure 1 shows the behaviour of the relative error for different values of \( \Omega_{m0} \). It makes apparent that the error \( |\sigma| \) grows with \( z \) and is lower in universes with smaller \( \Omega_{\Lambda0} \). The smallest values for the error \( |\sigma| \) correspond to is for Einstein-de-Sitter model \( (\Omega_{\Lambda0} = 0). These curves also suggest \( |\sigma| \) is upper bounded, and in fact, it can be shown that \( \lim_{z \to \infty} \frac{\partial \sigma}{\partial z} = 0. \)

From (1) and (4) it is clear that the error \( |\sigma| \) in the determination of the positions grows with \( r \). In practice (real world) equal separations \( s \) of correlated pairs (used in PSH) change. This can have basically the following effects: (i) spread the pair separations enough to destroy the spikes; (ii) spread the pair separations and move the spikes, without destroying them. These possibilities depend on both the error and also on the bin size. A suitable compromise between these variables (error \( |\sigma| \) and bin size \( \delta s = 4 s_{\text{max}}^2/m \)) can, however, be found.
Figure 2. PSH’s $\Phi(s)$ of a flat universe, whose corresponding topology is a cubic torus, for ‘exact’ (left) and ‘approximate’ values (right) values of density parameters.

Figure 2 exhibits two PSH cubic torus of side $L = 2\sqrt{2}/5$ for ‘exact’ values of density parameters $\Omega_{m}^{(e)} = 0.3, \Omega_{\Lambda}^{(e)} = 0.7$ (left) and for ‘approximate’ values $\Omega_{m} = 0.28, \Omega_{\Lambda} = 0.72$ (right). The bin size $\delta s$ is fixed by $s_{\text{max}} = 0.8$ and $m = 300$. A close inspection of this figures makes clear that, for a fixed suitable bin size, when one considers uncertainties in the density parameters: (i) the spikes are preserved but the amplitudes of the spikes decrease (this effect is clearer for large values of the pair-separation, for which the errors are larger); (ii) the spikes are spread (consistent with the decreasing of their amplitudes) and moves to the right. In brief, the uncertainties in the density parameters break the degeneracies in pair separations due to translations, and put limits on the bin size $\delta s$ of PSH’s: it has to be chosen large enough for not to resolve pair separations differences that arises from these uncertainties, but small enough not to include uncorrelated pair separations. However, it is always possible to conveniently choose the bin size so that the spikes are robust with respect to uncertainties in the density parameter.$^7$

To close this work, we mention that in the collected correlated pairs (CCP) method,$^8$ an indicator of a detectable nontrivial topology of the spatial section $M$ of the universe is the CCP index

$$R_{\epsilon} = \frac{\mathcal{N}_{\epsilon}}{P - 1}, \tag{5}$$

where $\mathcal{N}_{\epsilon} = \text{Card}\{i : \Delta_{i} \leq \epsilon\}$, and $\epsilon > 0$ is a parameter that deal with the uncertainties in the determination of the pairs separations. As in the PSH case, the CCP method also rely on the knowledge of the 3-D positions of the cosmic sources, which again involve uncertainties. Therefore, the sensitivity of the CCP index in the presence of the uncertainties in the positions of the sources, which arise from uncertainties the density parameters, can similarly be made and is under development.$^7$

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