Semi-Implicit and Explicit Runge Kutta Methods for Stiff Ordinary Differential Equations

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Abstract. In this work, we study the stability of the additive methods of Runge-Kutta kind of orders ranging from 2 up to 4 that will be applied for determining some stiff nonlinear system of the ODEs. Moreover, we find the stability function for the additive Runge-Kutta method and some methods of this type of order 2, 3, and 4. Where the method (A) is A-stable and semi-implicit and method (B) is explicit. Furthermore, the stiff term is managed by the semi-implicit Runge-Kutta method while no stiff term is treated by the explicit Runge Kutta method. Those methods are suitable for solving chemical reactions problems that include stiff and non-stiff terms.

Keywords: Split method, IMEX-RK methods, SDIMEX-RK methods.

1. Introduction
Several stiff systems of differential equations can be written in the scheme

\[ x' = f(x) + g(t, x), \]

where \( f = f(x) \in \mathbb{R}^n \) and \( f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n \). If \( f \) describes the stiff part concerning the linear system then utilising semi-implicit Runge-Kutta scheme and \( g \) expresses the non-stiff part of the nonlinear system implementing an explicit Runge-Kutta scheme. Combing semi-implicit and explicit Runge-Kutta scheme is identified as additive Runge-Kutta technique. The first additive Runge-Kutta techniques to the stiff ordinary differential equation was investigated by A. Sayfay and G. J. Copper [1] the determined additive Runge-Kutta method is applied to determine the subsequent systems of differential equations \( x' = f(t)x + g(t, x) \), wherever one method, which is \( A \) – stable and semi explicit is used to the linear (stiff) part. Because of their difficulties, most of these problems do not have exact analytic. Furthermore, these problems have very different time scales occurring simultaneously. Therefore, many of researches have been attracted much interest and many numerical schemes have been proposed over the years [1-19].

Nicolette Rattenbury [4] considered a general linear processes solution of the stiff equation and no stiff and offers a class of methods it was called almost Runge-Kutta methods. Those individual class of general linear methods which retains several of the characteristics of traditional Runge-Kutta methods, including some advantages. Qing and Yong [6] investigated schemes that are utilised to stiff the reaction-diffusion equation, the stability constraint on the time step terms: the diffusion and reaction semi-implicitly schemes deal with the linear diffusions exactly and explicitly, and for the nonlinear reactions implicitly. Thor Gjesdal [8] analysed implicit-explicit schemes approach for the solution of time-dependent differential equation which has the structure \( y' = f(t, y) + g(t, y) \), where \( f \) and \( g \) are the terms while another character so that one assume that \( g \) orders implicit approach
whereas \( y' = f(t, y) \) can efficiently via an explicit.

Sabawi [14, 18] investigated some modern Implicit-Explicit schemes of the order 5 that will be applied for determining fascinating nonlinear system of the ODEs, particularly for ephemeral discrimination of fascinating nonlinear system of the PDEs having the constraints.

The rest of this work is organized as. In Section 2, some necessary definitions are given. Section 3, Conditions for A-stability are introduced. Order conditions are defined in Section 4. The derivation of the stability function is proven in Section 5. Numerical Experiments is shown in Section 6, Finally, conclusions are given in Section 6.

2. Preliminaries

**Definition 2.1** [3]: The stability function to the \( N \) –additive schemes are analysed utilising equation
\[
F(U) = \sum_{i=1}^{N} \frac{d^{[i]}{U}}{dt^{[i]}}
\]
form that it is determined that the stability function will be as follow
\[
R(z^{[1]}, z^{[2]}, ..., z^{[N]}) = \frac{P(z^{[1]}, z^{[2]}, ..., z^{[N]})}{Q(z^{[1]}, z^{[2]}, ..., z^{[N]})} = \frac{\text{Det}[I - \sum_{i=1}^{N} z^{[i]} A^{[i]}] + \sum_{i=1}^{N} z^{[i]} e \otimes b^{[i]}]}{\text{Det}[I - \sum_{i=1}^{N} (z^{[i]} A^{[i]})]}
\]

**Definition 2.2** [7]: An \( A(\alpha) \) – stable domain of the additive methods is complex plane for \( z_{f} = h\lambda_{f} \) is
\[
S_{\alpha} = \{ z_{g} \in \mathbb{C} : |R(z_{f}, z_{g})| \leq 1, \forall z_{f} \in S_{\alpha} \}
\]
where \( S_{\alpha} = \{ z_{f} \in \mathbb{C} : |\arg(-z_{f})| \leq \alpha, z_{f} \neq 0 \} \cup \{0\} \), where \( R(z_{f}, z_{g}) \) is the stability function of what such that that \( y_{1} = R(z_{f}, z_{g}) a z_{f} = h\lambda_{f}, z_{g} = h\lambda_{g} \) an additive scheme is called L – stable whenever it is A – stable ( i.e. \( A(\pi/2) \) – stable
\[
\lim_{z_{f} \to \infty} |R(z_{f}, z_{g})| = 0).
\]

3. Conditions for A-stability [1]

A method \( (A, B_{1}) \) from type semi-implicit Runge Kutta which have stages defined by
\[
\begin{array}{cccccc}
0 & 0 & \cdots & 1 & b_{11} & 0 \\
0 & 0 & \cdots & 1 & b_{21} & b_{22} \\
0 & 0 & \cdots & 1 & b_{31} & b_{32} & b_{33} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & b_{s1} & b_{s2} & b_{s3} & \cdots & b_{ss} & c_{s}
\end{array}
\]
to applicant this method on test problem
\[
x' = \lambda x, x(0) = 1, \text{ where } Re(\lambda) < 0 \text{ we get } y_{s}^{(m)} = \frac{p(z)}{q(z)} y_{s}^{(m-1)}, \text{ } z = \lambda h, m = 1,2, \ldots g(z) \neq 0. \text{ Then }
\]
\[
y_{s}^{(m)} = \left( \frac{p(z)}{q(z)} \right)^{(m)} y_{s}^{(0)}, \text{ } m = 1,2, \ldots
\]
where \( p \) is a polynomial having the degree \( (s - 1) \) and \( q \) is a polynomial having the degree \( (s) \).

Now, \( p(z) = 1 - \alpha_{1} z + \alpha_{2} z^{2} - \cdots + (-1)^{s-1} \alpha_{s-1} z^{s-1} \) and \( q(z) = \beta_{0} - \beta_{1} z + \beta_{2} z^{2} - \cdots + (-1)^{s} \beta_{s} z^{s} \). Then, we have
\[
\begin{align*}
\alpha_{1} &= \beta_{1} - 1, \\
\alpha_{2} &= \beta_{2} - \beta_{1} + \frac{1}{2}, \\
\alpha_{3} &= \beta_{3} - \beta_{2} + \frac{1}{2} \beta_{1} - \frac{1}{6} \\
\vdots & \quad \vdots \\
\alpha_{s} &= \beta_{p} - \beta_{p-1} + \frac{1}{2} \beta_{p-2} - \frac{1}{6} \beta_{p-3} + \cdots + \frac{(-1)^{p}}{p!}, \\
\beta_{0} &= 0 \quad \text{ and } \beta_{s} = b_{11} b_{22} b_{33} \cdots b_{ss},
\end{align*}
\]
and $\alpha_{rr} > 0, r = 1,2,\ldots,s,$

$$
\sum_{r=\left[\frac{p}{2}\right]+1}^{s} y^r \sum_{j=0}^{r} (-1)^{r+j} (p_{2r-j}p_j - q_{2r-j}q_j) \geq 0,
$$

where $r$ is the integer part of $\left(\frac{p}{2} + 1\right)$ and the asterisk * denote that term $j = r$ is midpoint.

4. Order conditions

In the following, the order conditions for an $(A,B)$ method of Runge Kutta type with $p_s \leq 4$ are given in group. Then, the Taylor expansion provides the order conditions will be concerning the general

$$
\begin{array}{|c|c|}
\hline
p_s & \text{Conditions} \\
\hline
1 & b_i(0) = 0, \ i = 1,2,\ldots,s. \\
\hline
2 & c_s(\sigma) = 0, b_s(\sigma) = 0, \sigma = 1,2,\ldots,p_s - 1 \\
\hline
3 & \sum_{i=1}^{s} a_{sl} t^{\delta-1} a_i(\sigma) = 0, \sum_{i=1}^{s} a_{sl} t^{\delta-1} b_l(\sigma) = 0, \sigma + \delta \leq p \\
\hline
& \sum_{i=1}^{s} b_{sl} t^{\delta-1} a_i(\sigma) = 0, \sum_{i=1}^{s} b_{sl} t^{\delta-1} b_l(\sigma) = 0, \sigma + \delta \leq p \\
\hline
4 & \sum_{i=1}^{s} a_{si} \sum_{j=1}^{s} c_{ij} a_j(2) = 0, \sum_{i=1}^{s} a_{si} \sum_{j=1}^{s} c_{ij} b_j(2) = 0, \\
& \sum_{i=1}^{s} a_{si} \sum_{j=1}^{s} b_{ij} a_j(2) = 0, \sum_{i=1}^{s} a_{si} \sum_{j=1}^{s} b_{ij} b_j(2) = 0, \\
& \sum_{i=1}^{s} b_{si} \sum_{j=1}^{s} c_{ij} a_j(2) = 0, \sum_{i=1}^{s} b_{si} \sum_{j=1}^{s} c_{ij} b_j(2) = 0, \\
& \sum_{i=1}^{s} b_{si} \sum_{j=1}^{s} b_{ij} a_j(2) = 0, \sum_{i=1}^{s} b_{si} \sum_{j=1}^{s} b_{ij} b_j(2) = 0, \\
\hline
\end{array}
$$

are satisfied $\sigma$ and $\delta$ take all possible positive integer values.

where subscripts are $c_i(\sigma) = t_i^\sigma - \sigma \sum_{j=1}^{s} c_{ij} t_i^{\sigma-1}, c_i(\sigma) = t_i^\sigma - \sigma \sum_{j=1}^{s} b_{ij} t_i^{\sigma-1}.$

5. Semi-Implicit and Explicit Runge Kutta (SIMEXRK) Method

We can simply see that, additive RK methods of general structure are entirely implicit also will become very valuable whenever utilised for determining ODEs [3]. Moreover, explicit schemes that are easy to achieve have poor stability limits. On the other hand, implicit schemes possess sufficient stability limits but need the solution of an implicit system of equations [7]. Certain schemes (ARK) propose to bring together both handle and approaches the stiff term in semi-implicit (SIM) while giving the rest of the terms to stay integrated explicitly. Then, the first coefficient matrix $A$ will be a lower triangular matrix whenever the second coefficient matrix $B$ will be strictly lower triangular method. The purpose is to get $A[\alpha] - \text{stable additive techniques of Runge-Kutta type whenever}$ $0 \leq \alpha \leq \lambda, 0 \ll \lambda < 1$ and the methods $(A,B_1)$ will be semi-implicit also the method $(A,B_2)$ is explicit such a scheme with $(s - 1) - \text{stages and order } p$ represented by the latter Butcher tableau:

$$
\begin{array}{cccccccc}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{array}
$$
where $E_1 = (c_{i,j}), E_2 = (b_{i,j})$.

### (5.1) Method from order $p = 2$ and stage $s = 3$ defined by the following

|   | $c_{s,1}$ | $c_{s,2}$ | ... | $a_s$ | $b_{s,1}$ | $b_{s,2}$ | ... | $b_{s,s-1}$ | 0 |
|---|---|---|---|---|---|---|---|---|---|
| 1 |   |   |   |   | $b_{s,1}$ | $b_{s,2}$ | ... | $b_{s,s-1}$ | 0 |

where $E_1 = [c_{i,j}]$ and $E_2 = [b_{i,j}]$.

The stability function for the method (5.1) defined by:

\[
R(z_f, z_g) = \frac{\text{Det}[p(z_f, z_g)]}{\text{Det}[q(z_f, z_g)]} = \frac{1 + r_1 z_f + r_2 z_g + r_3 z_f z_g + r_4 z_f^2 + r_5 z_g^2}{1 - r_6 z_f + r_7 z_f^2},
\]

\[
p(z_f, z_g) = \begin{pmatrix} 1 + b_{31} z_f + c_{31} z_g \\ b_{31} z_f - b_{32} z_f + c_{31} z_g - c_{21} z_g \\ 0 \\ 1 \\ -b_{31} z_f - c_{31} z_g \\ -b_{32} z_f - c_{32} z_g \\ 1 - b_{33} z_f \end{pmatrix}
\]

\[
q(z_f, z_g) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 - b_{33} z_f \end{pmatrix}
\]

where $r_1 = b_{32} - b_{22} + b_{31}$, $r_2 = c_{32} + c_{31}$, $r_3 = b_{32} c_{21} - b_{22} c_{31} + b_{21} c_{31}$, $r_4 = b_{22} c_{31} + b_{21} c_{32}$, $r_5 = c_{32} c_{31}$, $r_6 = b_{22} + b_{22}$, $r_7 = b_{22} + b_{33}$.

Since $E(\alpha) = (1 - \alpha)E_1 + \alpha E_2$, and

\[
B(\alpha) = \begin{pmatrix} c_{21}(1 - \alpha) + b_{31} \alpha & 0 & 0 \\ c_{31}(1 - \alpha) + b_{31} \alpha & 0 & 0 \\ c_{32}(1 - \alpha) + b_{32} \alpha & 0 & 0 \end{pmatrix}.
\]

It follows that the methods $(A, B(\alpha))$ is also of order 2. Therefore, $A$ – stable conditions gives the sufficient conditions for $A$ – stability defined in conditions for $A$ – stability and the choice $c_{22} = c_{33} = \mu$. So that for a given $\mu$ the $(A, B(\alpha))$ method is $A$ – stable for $0 \leq \alpha \leq 1 - 1/4\mu$, $\mu \neq 0$. By solving the order conditions we get the class of additive techniques in two parameters, i.e.

\[
A(\alpha) = \begin{pmatrix} 0 \\ \lambda - \mu \\ \frac{-2\lambda \mu + 2(\lambda + \mu) - 1}{2\lambda} \mu \\ \frac{2\lambda - 1}{2\lambda} \frac{1}{\lambda} \mu \\ 0 \end{pmatrix}
\]

which are $A(\alpha)$ – stable for $0 \leq \alpha \leq 1 - 1/4\mu$, $\mu \neq 0$. Two appropriate methods are produced by the following array

|   | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|
| 0 |   | 0 | 0 | 0 | $\lambda$ | 0 | 0 |
| 1 |   | $\lambda - \mu$ | $\mu$ | 0 | $\lambda$ | 0 | 0 |

which are $A(\alpha)$ – stable for $0 \leq \alpha \leq 1 - 1/4\mu$, $\mu \neq 0$. Two appropriate methods are produced by the following array

|   | 0 | 0 | 0 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|
| 0 |   | 0 | 0 | 0 | $\lambda$ | 0 | 0 |
The method (3) is $A(\alpha) -$ stable for $0 \leq \alpha \leq \frac{1}{2}$ and the method (4) is $A(\alpha) -$ stable for $0 \leq \alpha \leq \frac{3}{4}$.

(5.2) Method from order $p = 3$ and stages $s = 4$ defined by the Butcher tableau:

\[
\begin{array}{cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & -1 & 3 & 1 & 0 \\
1/4 & 4 & 4 & 4 & 4 & 0 \\
\end{array}
\]

The stability function of the method (5.2) defined by:

\[
R(z_f, z_g) = \frac{\text{Det}[p(z_f, z_g)]}{\text{Det}[q(z_f, z_g)]} = \frac{r_1z_f + r_2z_g + r_3z_fz_g + r_4z_f^2 + r_5z_g^2 + r_6z_gz_f^2 + r_7z_fz_g^2 + r_8z_f^3 + r_9z_g^3}{r_{10}z_f + r_{11}z_f^2 + r_{12}z_f^3},
\]

\[
q(z_f, z_g) = \begin{pmatrix}
0 & 0 & 1 - b_{22}z_f & b_{42}z_f + c_{43}z_g & 0 \\
-2b_{21}z_f - c_{21}z_g & 0 & 0 & 0 & 0 \\
-b_{31}z_f + c_{31}z_g & -b_{32}z_f - c_{32}z_g & 1 - b_{33}z_f & 0 \\
-b_{41}z_f + c_{41}z_g & -b_{42}z_f - c_{42}z_g & -b_{43}z_f - c_{43}z_g & 1 - b_{44}z_f
\end{pmatrix}
\]

$r_1 = b_{43} - b_{33}, r_2 = c_{32}, r_3 = c_{32}b_{43} + c_{31}b_{43} - b_{22}c_{43} - c_{42}b_{33} + c_{42}c_{43} + c_{32}b_{43},

r_4 = b_{22}b_{33} - b_{22}b_{43} - b_{42}b_{33} + b_{32}b_{43} + b_{31}b_{43}, r_5 = c_{43}c_{32} + c_{31}c_{43},

r_6 = c_{41}b_{22}b_{33} - b_{21}c_{42}b_{33} + b_{21}b_{32}c_{43} - c_{31}c_{42}b_{33} + b_{31}b_{22}c_{43} + b_{21}c_{32}b_{43} - c_{31}b_{42}b_{33},

r_7 = c_{31}c_{42}b_{33} + c_{31}c_{43}b_{32} + c_{31}b_{43}c_{32} + c_{31}c_{43}b_{44}, r_8 = b_{41}b_{22}b_{33} - b_{23}b_{42}b_{21} + b_{21}b_{32}b_{43} - b_{31}b_{22}b_{43} + b_{31}c_{42}c_{33},

r_9 = c_{31}c_{42}c_{33}, r_{10} = 1 - b_{22} - b_{33} + b_{22}b_{33} - b_{44}, r_{11} = b_{44}b_{33} + b_{22}b_{44}.

r_{12} = -b_{22}b_{44}.

Since $E(\alpha) = (1 - \alpha)E_1 + \alpha E_2$, we get

\[
B(\alpha) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & (1 - \alpha)c_{22} \\
c_{31}(1 - \alpha) + b_{31} \alpha & c_{32}(1 - \alpha) + b_{32} \alpha & (1 - \alpha)c_{33} & 0 \\
c_{41}(1 - \alpha) + b_{41} \alpha & c_{42}(1 - \alpha) + b_{42} \alpha & c_{43}(1 - \alpha) + b_{43} \alpha & (1 - \alpha)c_{44}
\end{pmatrix}
\]

It follows that the methods $(A, B(\alpha))$ is also of order 3, hence $A -$ stable conditions gives the sufficient conditions for $A -$ stability defined in conditions for $A -$ stability. The choice $c_{22} = c_{33} = \mu$.

Now determining the order conditions, we get that the class of additive techniques in two parameters, i.e.
The method (6) and (7) are stable for \( \mu = \frac{3+\sqrt{3}}{6} \) and the method (3.6) is stable for \( \mu = \frac{3+\sqrt{3}}{6}, \lambda = \frac{2}{3} \).

(5.3) Method from order \( p = 4 \) and stages \( s = 5 \) defined by the following Butcher tableau:

\[
\begin{array}{cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & \lambda - \mu & \mu & 0 & 0 & \lambda & 0 & 0 \\
\frac{2}{3} & 0 & \frac{2}{3} - \mu & \mu & 0 & \frac{2}{9} & \frac{4}{9} & 0 \\
1 & 0 & \frac{9\beta - 6}{12\beta - 8} & 1 & \frac{1}{9} & 0 & \frac{3}{9} & 0 \\
\end{array}
\]

\[
\begin{array}{cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & 0 & \frac{2}{3} - \mu & \mu & 0 & \lambda & 0 & 0 \\
\frac{2}{3} & 0 & \frac{5 - 3\mu}{12} & \frac{1 - 3\mu}{4} & \mu & 0 & \frac{1}{6} & \frac{1}{2} \\
1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\
\end{array}
\]

The stability function of the method (5.2) defined by:

\[
R(z_f, z_g) = \frac{\text{Det}[p(z_f, z_g)]}{\text{Det}[q(z_f, z_g)]} = \frac{1 + r_1z_f^4 + r_2z_g + r_3z_f^2z_g + r_4z_g + r_5z_f^2z_g^2 + r_6z_fz_g^2 + r_7z_g^2 + r_8z_f^3 + r_9z_g^3}{1 - r_{10} + r_{11}z_f^2 - r_{12}z_g^3 + r_{13}z_f^4},
\]
It follows that the methods \((A, B(\alpha))\) is also of order 4, hence \(A\) – stable conditions gives the sufficient conditions for \(A\) – stability defined in conditions gives for \(A\) – stability. Now solving the order conditions, we get the class of additive methods in one parameter.

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1 - 2\mu}{2} & \mu & 0 & 0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1 - 6\mu + 8\mu^2}{2} & 2\mu(1 - 2\mu) & \mu & 0 & 0 & \frac{1}{2} & 0 \\
1 & \frac{1 - 2\mu}{4} & \frac{1 - 6\mu}{4} & \mu & 0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
\end{array}
\]

which is \(A\) –stable for \(\mu = 1.06857902130\).

6. Test problems [5]:
In this section we will find the solution for stiff differential equation by using the additive Runge-Kutta methods which have \(A\) – stable with the step size \((1 \times 10^{-1})\). Then the results of these experiments are compared with the numerical solution obtained by using the classical fourth order Runge-Kutta method with step size \((1 \times 10^{-4})\) for example 1 and example 2. Thus the solution are considering to be the “true” solutions of the two examples.

**Example 1:** Consider this problem from Electrical physics problems
\[
\begin{align*}
x_1' &= x_2 \\
x_2' &= 10(1 - x^2)x_2 - x_1 \\
\end{align*}
; \(x_1(0) = 2\), \(x_2(0) = 0\).
Integration range: $0 \leq t \leq 1$ and step size $h = 0.1$

**Table 1**: By using the semi-implicit and explicit Runge Kutta method, result the solution of example 1 is defined by (3)

| $t$ | $x_1$ | $x_2$ | $x_1$ | $x_2$ |
|-----|-------|-------|-------|-------|
| 0   | 2.00000000 | 0.00000000 | 2.00000000 | 0.00000000 |
| 0.1 | 1.99381992 | -0.13849122 | 1.99380956 | -0.12371833 |
| 0.2 | 1.99391992 | -0.11504231 | 1.99390857 | -0.12371732 |
| 0.3 | 1.99384662 | -0.11371733 | 1.99381782 | -0.12371631 |
| 0.4 | 1.99386889 | -0.11371721 | 1.99385721 | -0.12371523 |
| 0.5 | 1.99389992 | -0.12371611 | 1.99388312 | -0.12371433 |
| 0.6 | 1.99398992 | -0.12371599 | 1.99397882 | -0.12371522 |
| 0.7 | 1.99411225 | -0.12371421 | 1.99409999 | -0.12371411 |
| 0.8 | 1.99421392 | -0.12371211 | 1.99428976 | -0.12371200 |
| 0.9 | 1.99461562 | -0.12371166 | 1.99466122 | -0.12371199 |
| 1   | 1.99481997 | -0.12371055 | 1.99480776 | -0.12371088 |

**Example 2**: Consider this problem from Kinetic of chemical reactions problem

$y_1' = -0.013y_1 - 1000y_1y_3 \quad ; y_1(0) = 1,$

$y_2' = -2500y_2y_3 \quad ; y_2(0) = 1,$

$y_3' = -0.013y_1 - 1000y_1y_3 - 2500y_2y_3 \quad ; y_3(0) = 0,$

Integration range: $0 \leq x \leq 50$ and step size $h = 0.1$.

**Table 2**: By using the semi-implicit and explicit Runge Kutta method, result the solution of example 2 is defined by (6).

| $t$ | $y_1$ | $y_2$ | $y_3$ | $y_1$ | $y_2$ | $y_3$ |
|-----|-------|-------|-------|-------|-------|-------|
| 0   | 1.00000000 | 1.00000000 | 0.00000000 | 1.00000000 | 1.00000000 | 0.00000000 |
| 5   | 0.99070782 | 1.00896172 | -0.00055070 | 0.99073192 | 1.00926441 | -0.00000360 |
| 10  | 0.99070712 | 1.00926223 | -0.00018870 | 0.99072192 | 1.00926341 | -0.00000350 |
| 15  | 0.99070624 | 1.00925172 | -0.00009667 | 0.99071192 | 1.00926421 | -0.00000340 |
| 20  | 0.99070582 | 1.00925172 | -0.00004227 | 0.99070181 | 1.00926401 | -0.00000320 |
| 25  | 0.99070570 | 1.00925172 | -0.00002207 | 0.99070171 | 1.00922521 | -0.00000310 |
| 30  | 0.99070560 | 1.00925172 | -0.00000227 | 0.99070161 | 1.00926421 | -0.00000290 |
| 35  | 0.99070542 | 1.00925172 | -0.00000240 | 0.99070140 | 1.00926231 | -0.00000260 |
| 40  | 0.99070522 | 1.00925172 | -0.00000240 | 0.99070140 | 1.00922980 | -0.00000250 |

7. Conclusion

The main result of this work is that we study the $A[\alpha] -$stability of the Additive methods of Runge Kutta type. This stability is equivalent to A-stability of methods where $E(\alpha) = (1 - \alpha)E_1 + \alpha E_2$. Also, we find the stability function of the additive Runge Kutta methods and some methods of this type of order 2, 3 and 4. A classification of this species is characterized by couple of methods.
(A, B) wherever the method A is semi-implicit that have A – stabilite and method B is the explicit. Those methods are suitable for stiff system of the form \( x' = f(x) + g(t, y) \), since involve only the solution of linear differential equations. We notice the semi-implicit explicit Runge Kutta scheme with same diagonal elements in the method \( E_1 \) provides a computational advantage over several implementation schemes for methods for the modified Newton or any iteration method is applied. The test results show that these schemes are stable and accurate for calculation. This work can be extended to stochastic equations with growth model [20, 21]. Another interesting of this work is to use discontinuous Galerkin methods for estimating this type of problem in terms of \( L_\infty(L_2) + L_2(H^1) \) and \( L_\infty(L_2) \) [22, 23, 24].

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