On the $q$-log-convexity conjecture of Sun

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Abstract. In his study of Ramanujan-Sato type series for $1/\pi$, Sun introduced a sequence of polynomials $S_n(q)$ as given by

$$S_n(q) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n - k)}{n - k} q^k,$$

and he conjectured that the polynomials $S_n(q)$ are $q$-log-convex. By imitating a result of Liu and Wang on generating new $q$-log-convex sequences of polynomials from old ones, we obtain a sufficient condition for determining the $q$-log-convexity of self-reciprocal polynomials. Based on this criterion, we then give an affirmative answer to Sun’s conjecture.

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1 Introduction

The main objective of this paper is to prove a conjecture of Sun [12] on the $q$-log-convexity of the polynomials $S_n(q)$, which are given by

$$S_n(q) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n - k)}{n - k} q^k.$$  \hspace{1cm} (1.1)

These polynomials $S_n(q)$ were introduced by Sun [12] in his study of the Ramanujan-Sato type series for $1/\pi$. 

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Let us first review some definitions. Recall that a nonnegative sequence \( \{a_n\}_{n \geq 0} \) is said to be log-concave if, for any \( n \geq 1 \),

\[
a_n^2 \geq a_{n-1}a_{n+1};
\]

and is said to be log-convex if, for any \( n \geq 1 \),

\[
a_{n-1}a_{n+1} \geq a_n^2.
\]

Many sequences arising in combinatorics, algebra and geometry, turn out to be log-concave or log-convex, see Brenti \[1\] or Stanley \[11\].

For a sequence of polynomials with real coefficients, Stanley introduced the notion of \( q \)-log-concavity. Throughout we are concerned only with polynomials with real coefficients. A polynomial sequence \( \{f_n(q)\}_{n \geq 0} \) is said to be \( q \)-log-concave if, for any \( n \geq 1 \), the difference

\[
f_n^2(q) - f_{n+1}(q)f_{n-1}(q)
\]

has nonnegative coefficients. The \( q \)-log-concavity of polynomial sequences has been extensively studied, see Bulter \[2\], Krattenthaler \[7\], Leroux \[8\] and Sagan \[10\]. Similarly, a polynomial sequence \( \{f_n(q)\}_{n \geq 0} \) is said to be \( q \)-log-convex if, for any \( n \geq 1 \), the difference

\[
f_{n+1}(q)f_{n-1}(q) - f_n^2(q)
\]

has nonnegative coefficients. Liu and Wang \[9\] showed that many classical combinatorial polynomials are \( q \)-log-convex, see also \[4\], \[5\], \[6\]. It should be noted that Butler and Flanigan \[3\] introduced a different kind of \( q \)-log-convexity.

Sun posed six conjectures on the expansions of \( 1/\pi \) in terms of \( S_n(q) \), one of which reads

\[
\sum_{n=0}^{\infty} \frac{140n + 19}{4624^n} \binom{2n}{n} S_n(64) = \frac{289}{3\pi}.
\]

He also conjectured that the polynomials \( S_n(q) \) are \( q \)-log-convex. It is easy to see that the coefficients of \( S_n(q) \) are symmetric. Such polynomials are also said to be self-reciprocal. More precisely, a polynomial

\[
f(q) = a_0 + a_1q + \cdots + a_nq^n
\]

is called a self-reciprocal polynomial of degree \( n \) if \( f(q) = q^n f(1/q) \).

In this paper, we shall give a proof of the \( q \)-log-convexity conjecture of \( S_n(q) \). Our proof is closely related to a result of Liu and Wang, which provides a mechanism of generating new \( q \)-log-convex sequences of polynomials from certain log-convex sequences of positive numbers and \( q \)-log-convex sequences.
of polynomials. The critical point of their result is to determine the sign of some statistic arising from the difference $f_{n+1}(q)f_{n-1}(q) - f_n^2(q)$ for a given $q$-log-convex sequence $\{f_n(q)\}_{n \geq 0}$. Assume that $f_n(q)$ has the following form:

$$f_n(q) = \sum_{k=0}^{n} a(n, k) q^k.$$  \hspace{1cm} (1.2)

Write the difference $f_{n+1}(q)f_{n-1}(q) - f_n^2(q)$ as

$$2n \sum_{t=0}^{2n} \left[ \sum_{k=0}^{\lfloor t/2 \rfloor} \tilde{L}_t(a(n, k)) \right] q^t,$$

where

$$\tilde{L}_t(a(n, k)) = \begin{cases} 
  a(n+1, k)a(n-1, t-k) + a(n-1, k)a(n+1, t-k) & \text{if } 0 \leq k < \frac{t}{2}, \\
  -2a(n, k)a(n, t-k) & \text{if } t \text{ is even and } k = \frac{t}{2}, \\
  a(n+1, k)a(n-1, k) - a^2(n, k) & \text{if } t \text{ is even and } k = \frac{t}{2}. 
\end{cases}$$

Liu and Wang’s criterion to determine the $q$-log-convexity of a polynomial sequence is as follows.

**Theorem 1.1** ([9, Theorem 4.8]) *Let $\{u_k\}_{k \geq 0}$ be a log-convex sequence and let $\{f_n(q)\}_{n \geq 0}$ be a $q$-log-convex sequence as defined in (1.2). Given $n \geq 1$ and $0 \leq t \leq 2n$, if there exists an index $k'$ associated with $n, t$ such that

$$\tilde{L}_t(a(n, k)) \begin{cases} 
  \geq 0, & \text{if } 0 \leq k \leq k', \\
  \leq 0, & \text{if } k' < k \leq \frac{t}{2},
\end{cases}$$

then, the polynomial sequence $\{g_n(q)\}_{n \geq 0}$ defined by

$$g_n(q) = \sum_{k=0}^{n} a(n, k) u_k q^k$$  \hspace{1cm} (1.3)

is $q$-log-convex.*

We attempted to use the above result to prove the $q$-log-convexity of $\{S_n(q)\}_{n \geq 0}$ by taking

$$u_k = \binom{2k}{k}, \quad a(n, k) = \binom{n}{k} \binom{2n-2k}{n-k}. \hspace{1cm} (1.4)$$

Experimental evidence suggests that $\tilde{L}_t(a(n, k))$ meets Liu and Wang’s criterion. The determination of the sign of $\tilde{L}_t(a(n, k))$ relies on the relative
position of two polynomials in $t$ on the interval $[0, 2n]$. While it is easier to determine their relative position on the interval $[0, n]$ than on $[0, 2n]$. This forces us to consider the symmetry of the coefficients of the self-reciprocal polynomials to circumvent the above difficulty. As a result, we obtain a criterion for the $q$-log-convexity of self-reciprocal polynomials in the spirit of Theorem 1.1 which shall be given in Section 2. By using this criterion, we then confirm the $q$-log-convexity conjecture of Sun in Section 3.

2 A criterion for $q$-log-convexity

The aim of this section is to present a criterion for proving a sequence of self-reciprocal polynomials to be $q$-log-convex.

Noting that for $\tilde{L}_t(a(n, t/2))$ in Theorem 1.1 only its sign should be considered, we make the following modification to $\tilde{L}_t(a(n, k))$ for convenience:

$$L_t(a(n, k)) = a(n + 1, k)a(n - 1, t - k) + a(n - 1, k)a(n + 1, t - k) - 2a(n, k)a(n, t - k), \quad \text{if } 0 \leq k \leq \frac{t}{2}. \quad (2.1)$$

Then we give the following criterion which is applicable to $\{S_n(q)\}_{n \geq 0}$.

**Theorem 2.1** Given a log-convex sequence $\{u_k\}_{k \geq 0}$ and a $q$-log-convex sequence $\{f_n(q)\}_{n \geq 0}$ as defined in (1.2), let $\{g_n(q)\}_{n \geq 0}$ be the polynomial sequence defined by (1.3). Assume that the following two conditions are satisfied:

$(C1)$ for each $n \geq 0$, the polynomial $g_n(q)$ is a self-reciprocal polynomial of degree $n$; and

$(C2)$ for given $n \geq 1$ and $0 \leq t \leq n$, there exists an index $k'$ associated with $n, t$ such that

$$L_t(a(n, k)) \begin{cases} \geq 0, & \text{if } 0 \leq k \leq k', \\ \leq 0, & \text{if } k' < k \leq \frac{t}{2}. \end{cases}$$

Then, the polynomial sequence $\{g_n(q)\}_{n \geq 0}$ is $q$-log-convex.

**Proof of Theorem 2.1** Since each $g_n(q)$ is a self-reciprocal polynomial of degree $n$, we have

$$g_{n-1}(q) = q^{n-1}g_{n-1}(q^{-1}),$$

$$g_n(q) = q^n g_n(q^{-1}),$$

$$g_{n+1}(q) = q^{n+1} g_{n+1}(q^{-1}).$$

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Therefore,
\[ g_n^2(q) = q^{2n}g_n^2(q^{-1}), \]
\[ g_{n-1}(q)g_{n+1}(q) = q^{2n}g_{n-1}(q^{-1})g_{n+1}(q^{-1}), \]
i.e., both \( g_{n-1}(q)g_{n+1}(q) \) and \( g_n^2(q) \) are self-reciprocal polynomials of degree 2n.

Writing the difference \( g_{n-1}(q)g_{n+1}(q) - g_n^2(q) \) as
\[ \sum_{t=0}^{2n} B(n, t)q^t, \]
we obtain that, for \( 0 \leq t \leq 2n \),
\[ B(n, t) = B(n, 2n - t) \]
due to reciprocity. Accordingly, to prove the \( q \)-log-convexity of \( \{g_n(q)\}_{n \geq 0} \), it suffices to show that \( B(n, t) \) is nonnegative for any \( 0 \leq t \leq n \).

It is ready to see that
\[ B(n, t) = \begin{cases} \sum_{k=0}^s L_t(a(n, k))u_ku_{t-k}, & \text{if } t = 2s + 1, \\ \sum_{k=0}^{s-1} L_t(a(n, k))u_ku_{t-k} + \frac{L_t(a(n,s))}{2}u_s^2, & \text{if } t = 2s. \end{cases} \]

To prove the nonnegativity of \( B(n, t) \), we further need to use the log-convexity of \( \{u_k\}_{k \geq 0} \) and the \( q \)-log-convexity of \( \{f_n(q)\}_{n \geq 0} \).

On one hand, by the log-convexity of \( \{u_k\}_{k \geq 0} \), we have
\[ u_0u_t \geq u_1u_{t-1} \geq \cdots. \tag{2.2} \]

On the other hand, if we write
\[ f_{n-1}(q)f_{n+1}(q) - f_n^2(q) = \sum_{t=0}^{2n} A(n, t)q^t, \]
then
\[ A(n, t) = \begin{cases} \sum_{k=0}^s L_t(a(n, k)), & \text{if } t = 2s + 1, \\ \sum_{k=0}^{s-1} L_t(a(n, k)) + \frac{L_t(a(n,s))}{2}, & \text{if } t = 2s. \end{cases} \]

Since \( \{f_n(q)\}_{n \geq 0} \) is \( q \)-log-convex, for any \( 0 \leq t \leq 2n \), it holds that \( A(n, t) \geq 0 \).

Now we proceed to prove that \( B(n, t) \geq 0 \) for \( 0 \leq t \leq n \). We first consider the case when \( t \) is odd, namely, \( t = 2s + 1 \) for some \( s \in \mathbb{N} \). By (2.2) and the
condition (C2), we obtain
\[ B(n, t) = \sum_{k=0}^{s} \mathcal{L}_t(a(n, k))u_ku_{t-k} \]
\[ \geq \sum_{k=0}^{s} \mathcal{L}_t(a(n, k))u_k'u_{t-k'} \]
\[ = A(n, t)u_k'u_{t-k'} \geq 0. \]
By the same arguments, if \( t = 2s \) for some \( s \in \mathbb{N} \), then
\[ B(n, t) = s - 1 \sum_{k=0}^{s-1} \mathcal{L}_t(a(n, k))u_ku_{t-k} + \mathcal{L}_t(a(n, s))u_s^2 \]
\[ \geq \sum_{k=0}^{s-1} \mathcal{L}_t(a(n, k))u_k'u_{t-k'} + \mathcal{L}_t(a(n, s))u_k'u_{t-k'} \]
\[ = A(n, t)u_k'u_{t-k'} \geq 0. \]
This completes the proof.

3 The \( q \)-log-convexity of \( S_n(q) \)

In this section we wish to use Theorem 2.1 to prove Sun’s \( q \)-log-convexity conjecture of \( S_n(q) \). The main result of this section is the following theorem.

**Theorem 3.1** The polynomials \( S_n(q) \) given by (1.1) form a \( q \)-log-convex sequence.

To this end, take
\[ u_k = \binom{2k}{k}, \quad a(n, k) = \binom{n}{k} \binom{2n-2k}{n-k}, \quad (3.1) \]
and hence, by (1.2) and (1.3), we have
\[ f_n(q) = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-2k}{n-k} q^k, \quad g_n(q) = S_n(q). \]
(3.2)

It is routine to verify that \( \{u_k\}_{k \geq 0} \) is a log-convex sequence. It is also clear that \( S_n(q) \) is a self-reciprocal polynomial of degree \( n \). There remains to show that the sequence \( \{f_n(q)\}_{n \geq 0} \) is \( q \)-log-convex, and the triangular array \( \{a(n, k)\}_{0 \leq k \leq n} \) satisfies the condition (C2) of Theorem 2.1. For the former, we have the following result, and for the latter, see Theorem 3.3.
Theorem 3.2 For \( n \geq 0 \), let \( f_n(q) \) be polynomials given by (3.2). Then the sequence \( \{f_n(q)\}_{n \geq 0} \) is \( q \)-log-convex.

Proof. It suffices to show that the polynomials

\[
q^n f_n(q^{-1}) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} q^k
\]

form a \( q \)-log-convex sequence. In view of the \( q \)-log-convexity of \( \{(1 + q)^n\}_{n \geq 0} \) and the log-convexity of \( \{(2k\binom{n}{k})\}_{k \geq 0} \), it is natural to consider whether the triangular array \( \{\binom{n}{k}\}_{0 \leq k \leq n} \) satisfies the condition of Theorem 1.1.

Note that, for \( n \geq 1 \), \( 0 \leq t \leq 2n \) and \( 0 \leq k \leq t/2 \), we have

\[
L_t \left( \binom{n}{k} \right) = \left( \frac{n+1}{k} \right) \left( \frac{n-1}{t-k} \right) + \left( \frac{n+1}{t-k} \right) \left( \frac{n-1}{k} \right) - 2 \left( \frac{n}{t-k} \right) \left( \frac{n}{k} \right)
\]

where

\[
\varphi^{(n,t)}(x) = (n+1)(n-x)(n-x+1) + (n+1)(n-t+x)(n-t+x+1) - 2n(n-x+1)(n-t+x+1).
\]

Thus, the sign of \( L_t \left( \binom{n}{k} \right) \) depends on that of \( \varphi^{(n,t)}(k) \) for \( 0 \leq k \leq t/2 \).

To see the sign changes of \( \varphi^{(n,t)}(k) \) as \( k \) varies from 0 to \( t/2 \), we consider the values of \( \varphi^{(n,t)}(x) \) as \( x \) varies over the interval \([0, t/2]\). Taking the derivative of \( \varphi^{(n,t)}(x) \) with respect to \( x \), we obtain that

\[
(\varphi^{(n,t)}(x))' = (4n+2)(2x-t) \leq 0, \quad \text{for } x \leq t/2.
\]

Thus \( \varphi^{(n,t)}(x) \) is decreasing on the interval \([0, t/2]\). With \( \varphi^{(n,t)}(0) = (n+1)(t^2-t) \geq 0 \), for given \( n \) and \( t \), there exists \( k' \) such that

\[
\varphi^{(n,t)}(k) \begin{cases} 
\geq 0, & \text{if } 0 \leq k \leq k', \\
\leq 0, & \text{if } k' < k \leq \frac{t}{2},
\end{cases}
\]

and hence

\[
L_t \left( \binom{n}{k} \right) \begin{cases} 
\geq 0, & \text{if } 0 \leq k \leq k', \\
\leq 0, & \text{if } k' < k \leq \frac{t}{2}.
\end{cases}
\]

By Theorem 1.1, we obtain the desired \( q \)-log-convexity of \( \{q^n f_n(q^{-1})\}_{n \geq 0} \). 

The remaining part of this section is to prove the following result.
Theorem 3.3 Let \( \{a(n, k)\}_{0 \leq k \leq n} \) be the triangular array defined by (3.1). Then, for any \( n \geq 1 \) and \( 0 \leq t \leq n \), there exists an index \( k' \) with respect to \( n, t \) such that

\[
\mathcal{L}_t(a(n, k)) \begin{cases} 
\geq 0, & \text{if } 0 \leq k \leq k', \\
\leq 0, & \text{if } k' < k \leq \frac{t}{2}.
\end{cases}
\]

Before proving Theorem 3.3, let us make some observations. For \( n \geq 1 \), \( 0 \leq t \leq n \) and \( 0 \leq k \leq t/2 \), we have

\[
\mathcal{L}_t(a(n, k)) = \binom{n+1}{k} \binom{2n-2k+2}{n-k+1} \binom{n-1}{t-k} \binom{2n-2t+2k-2}{n-t+k-1} \\
+ \binom{n-1}{k} \binom{2n-2k-2}{n-k-1} \binom{n+1}{t-k} \binom{2n-2t+2k+2}{n-t+k+1} \\
- 2 \binom{n}{k} \binom{2n-2k}{n-k} \binom{n}{t-k} \binom{2n-2t+2k}{n-t+k}.
\]

By factorization, we obtain

\[
\mathcal{L}_t(a(n, k)) = \frac{1}{(n-k+1)^2(n-t+k+1)^2(2n-2k-1)(2n-2t+2k-1)} \times \frac{1}{n} \binom{n}{k} \binom{2n-2k}{n-k} \binom{n}{t-k} \binom{2n-2t+2k}{n-t+k} \psi^{(n,t)}(k),
\]

where

\[
\psi^{(n,t)}(x) = (n+1)(n-x)^2(n-x+1)^2(2n-2t+2x+1)(2n-2t+2x-1) \\
+ (n+1)(n-t+x)^2(n-t+x+1)^2(2n-2x-1)(2n-2x+1) \\
- 2n(n-x+1)^2(n-t+x+1)^2(2n-2x-1)(2n-2t+2x-1).
\]

Clearly, the sign of \( \mathcal{L}_t(a(n, k)) \) coincides with that of \( \psi^{(n,t)}(k) \) unless \( t = n \) and \( k = 0 \). Based on this observation, we divide the proof of Theorem 3.3 into the following three steps:

(S1) For \( n \geq 1 \) and \( 0 \leq t \leq n \), prove that \( \mathcal{L}_t(a(n, 0)) \geq 0 \), see Proposition 3.4.

(S2) For \( n \geq 2 \) and \( 0 \leq t \leq n-1 \), prove that there exists \( k' \) such that

\[
\psi^{(n,t)}(k) \begin{cases} 
\geq 0, & \text{if } 1 \leq k \leq k', \\
\leq 0, & \text{if } k' < k \leq \frac{t}{2},
\end{cases}
\]

see Proposition 3.5.
(S3) For \( n \geq 2 \) and \( t = n \), prove that there exists \( k' \) such that
\[
\psi^{(n,n)}(k) \begin{cases} 
\geq 0, & \text{if } 1 \leq k \leq k', \\
\leq 0, & \text{if } k' < k \leq \frac{n}{2}, 
\end{cases}
\]
see Proposition 3.6.

Let us first prove the nonnegativity of \( \mathcal{L}_t(a(n,0)) \).

**Proposition 3.4** For any \( n \geq 1 \) and \( 0 \leq t \leq n \), we have \( \mathcal{L}_t(a(n,0)) \geq 0 \).

**Proof.** For \( 1 \leq n \leq 4 \), the nonnegativity of \( \mathcal{L}_t(a(n,0)) \) can be proved directly as follows:
\[
\begin{align*}
\mathcal{L}_0(a(1,0)) &= 4, \quad \mathcal{L}_1(a(1,0)) = 0, \\
\mathcal{L}_0(a(2,0)) &= 8, \quad \mathcal{L}_1(a(2,0)) = 8, \quad \mathcal{L}_2(a(2,0)) = 0, \\
\mathcal{L}_0(a(3,0)) &= 40, \quad \mathcal{L}_1(a(3,0)) = 40, \quad \mathcal{L}_2(a(3,0)) = 46, \quad \mathcal{L}_3(a(3,0)) = 8, \\
\mathcal{L}_0(a(4,0)) &= 280, \quad \mathcal{L}_1(a(4,0)) = 336, \\
\mathcal{L}_2(a(4,0)) &= 472, \quad \mathcal{L}_3(a(4,0)) = 332, \quad \mathcal{L}_4(a(4,0)) = 60.
\end{align*}
\]

For the remainder of the proof, we assume that \( n \geq 5 \). It is routine to compute that the sign of \( \mathcal{L}_t(a(n,0)) \) coincides with that of
\[
\frac{\binom{2n}{n} \binom{2n-2}{n-t} \theta(t)}{n(n+1)(2n-1)(n-t+1)^2(2n-2t-1)},
\]
where
\[
\theta(x) = (4n^2 - 1)x^4 - 2(2n - 1)(2n^2 + 2n + 1)x^3 + (4n^4 + 8n^3 + 8n^2 - 1)x^2 \\
- 2n(n + 1)(2n^2 + 4n - 1)x + 2n(2n - 1)(n + 1)^2. \tag{3.6}
\]

To prove that \( \mathcal{L}_t(a(n,0)) \geq 0 \), there are two cases to consider:

(i) \( t = n \). In this case it suffices to show that \( \theta(n) \leq 0 \). For \( n \geq 5 \), one can readily check that
\[
\theta(n) = -n(n - 1)(n - 2)(n + 1) < 0.
\]

(ii) \( 0 \leq t < n \). In this case it suffices to show that \( \theta(t) \geq 0 \). To this end, we consider the monotonicity of \( \theta(x) \), regarded as a function of \( x \), over the
interval \([0, n - 1]\). By (3.6), taking the derivative of \(\theta(x)\) with respect to \(x\), we have

\[
\theta'(x) = 2(n - x)\theta_1(x),
\]

where

\[
\theta_1(x) = 2(1 - 4n^2)x^2 + (2n - 1)(2n^2 + 4n + 3)x - (2n^3 + 6n^2 + 3n - 1).
\]

We further need the derivative of \(\theta_1(x)\):

\[
\theta_1'(x) = (2n - 1)\theta_2(x),
\]

where

\[
\theta_2(x) = -4(2n + 1)x + (2n^2 + 4n + 3).
\]

Note that, for \(n \geq 5\),

\[
\theta_2(0) = 2n^2 + 4n + 3 > 0, \quad \theta_2(n - 1) = -6n^2 + 8n + 7 < 0.
\]

Therefore, \(\theta_2(x)\) decreases from a positive value to a negative value as \(x\) increases from 0 to \(n - 1\). Hence, \(\theta_1(x)\) first increases and then decreases as \(x\) increases from 0 to \(n - 1\). Since, for \(n \geq 5\),

\[
\begin{align*}
\theta_1(0) & = 1 - 2n^3 - 6n^2 - 3n < 0, \\
\theta_1(1) & = n(2n^2 - 2 - 9) > 0, \\
\theta_1(n - 1) & = -4n^4 + 16n^3 - 16n^2 - 12n + 6 < 0,
\end{align*}
\]

there exist \(0 < x_1 < x_2 < n - 1\) such that

\[
\theta_1(x) \begin{cases} < 0, & \text{if } x \in [0, x_1), \\ \geq 0, & \text{if } x \in [x_1, x_2], \\ < 0, & \text{if } x \in (x_2, n - 1].
\end{cases}
\]

Thus, \(\theta(x)\) is decreasing on the interval \([0, x_1]\), increasing on \([x_1, x_2]\), and decreasing on \((x_2, n - 1]\).

Note that, for \(n \geq 5\), we have

\[
\begin{align*}
\theta(0) & = 2n(2n - 1)(n + 1)^2 > 0, \\
\theta(1) & = 2n^2(2n - 1)(n - 1) > 0, \\
\theta(2) & = 2(n - 2)(6n^3 - 13n^2 + 1) > 0, \\
\theta(n - 1) & = -4 + 8n + 3n^4 - 10n^3 + 11n^2 > 0.
\end{align*}
\]
It is easy to check that
\[ \theta(0) > \theta(1) < \theta(2) > \theta(n-1). \]

By virtue of the monotonicity of \( \theta(x) \) on the interval \([0, n-1]\), we must have \( x_1 \leq 2 \). If \( x_2 > 2 \), then \( \theta(x) \) is increasing on \([2, x_2]\), and decreasing on \((x_2, n-1]\). If \( x_2 \leq 2 \), then \( \theta(x) \) decreases on \((2, n-1]\). In both cases, we obtain that \( \theta(x) > 0 \) for \( x \in [2, n-1] \). In view of \( \theta(0) > 0 \) and \( \theta(1) > 0 \), it is clear that \( \theta(t) > 0 \) for any integer \( 0 \leq t \leq n-1 \).

Combining (i) and (ii), we obtain the desired result. \( \square \)

Now we proceed to determine the sign of \( \psi^{(n,t)}(k) \) for \( n \geq 2 \) and \( 0 \leq t \leq n-1 \).

**Proposition 3.5** Given \( n \geq 2 \) and \( 0 \leq t \leq n-1 \), there exists \( k' \) with respect to \( n, t \) such that

\[
\psi^{(n,t)}(k) \begin{cases} 
\geq 0, & \text{if } 1 \leq k \leq k', \\
\leq 0, & \text{if } k' < k \leq \frac{t}{2}.
\end{cases}
\]

**Proof.** By (3.4) and Proposition 3.4, we know that \( \psi^{(n,t)}(0) \geq 0 \). Therefore, it suffices to prove that there exists \( 0 \leq t_0 \leq t/2 \) such that \( \psi^{(n,t)}(x) \), regarded as a function of \( x \), is increasing on the interval \([0, t_0]\) and decreasing on the interval \([t_0, t/2]\). To this end, we need to determine the sign changes of the derivative of \( \psi^{(n,t)}(x) \) with respect to \( x \) on the interval \([0, t/2]\).

Taking the derivative of \( \psi^{(n,t)}(x) \), we obtain that

\[
(\psi^{(n,t)}(x))' = 2(2x-t)\psi_1^{(n,t)}(x),
\]

where

\[
\psi_1^{(n,t)}(x) = 12(2n+1)x^4 - 24t(2n+1)x^3 \\
- 2(16n^3 - 8(2t-1)n^2 - 2(7t+1)n - (8t^2 - 4t + 3))x^2 \\
+ 2t(16n^3 - 8(2t-1)n^2 - 2(t^2 + 3t + 1)n - (2t^2 - 4t + 3))x \\
+ (8n^5 - 4(2t-1)n^4 + 4(t^2 - t - 3)n^3 + 4(-t^2 + 5t + t^3 - 2)n^2 \\
+ (4t^3 - 10t^2 - 1 + 11t)n - (2t^2 - 3t + 1)).
\]

We further need to consider the derivative of \( \psi_1^{(n,t)}(x) \):

\[
(\psi_1^{(n,t)}(x))' = 2(2x-t)\psi_2^{(n,t)}(x),
\]
where
\[
\psi_2^{(n,t)}(x) = 12(2n + 1)x^2 - 12t(2n + 1)x - 16n^3 + 8(2t - 1)n^2
+ 2(t^2 + 3t + 1)n + (2t^2 - 4t + 3).
\]

Note that the axis of symmetry of the quadratic function \(\psi_2^{(n,t)}(x)\) is \(x = t/2\). Hence, \(\psi_2^{(n,t)}(x)\) decreases as \(x\) increases from 0 to \(t/2\). It is routine to verify that, for \(n \geq 1\) and \(0 \leq t < n\),
\[
\psi_2^{(n,t)} \left( \frac{t}{2} \right) = -4n(2n - t)^2 - (4n - t - 1)(2n - t) - 3(t - 1) < 0.
\]

Let \(x_0\) be the zero of \(\psi_2^{(n,t)}(x)\) to the left of the axis of symmetry. Then we have
\[
\psi_2^{(n,t)}(x) \begin{cases} > 0, & \text{if } 0 \leq x < x_0, \\ < 0, & \text{if } x_0 < x < t/2. \end{cases}
\]

By (3.7), we have
\[
(\psi_1^{(n,t)}(x))' \begin{cases} < 0, & \text{if } 0 \leq x < x_0, \\ > 0, & \text{if } x_0 < x < t/2. \end{cases}
\]

If \(x_0 \leq 0\), this means that \(\psi_1^{(n,t)}(x)\) is increasing on \([0, t/2]\). If \(x_0 > 0\), this means that \(\psi_1^{(n,t)}(x)\) is decreasing on \([0, x_0]\) and increasing on \([x_0, t/2]\).

We proceed to determine the sign changes of \((\psi^{(n,t)}(x))'\) based on the above monotonicity of \(\psi_1^{(n,t)}(x)\). For our purpose, the values of \(\psi_1^{(n,t)}(x)\) at the two endpoints of the interval \([0, t/2]\) are to be examined.

We claim that, for any integers \(n \geq 2\) and \(0 \leq t < n\), it holds \(\psi_1^{(n,t)}(t/2) > 0\). Using Maple, we find that
\[
\psi_1^{(n,t)} \left( \frac{t}{2} \right) = 8n^5 - 16n^4t + 12n^3t^2 - 4n^2t^3 + \frac{1}{2}nt^4 + 4n^4 - 4n^3t + nt^3 - \frac{1}{4}t^4
- 12n^3 + 20n^2t - 11nt^2 + 2t^3 - 8n^2 + 11nt - \frac{7}{2}t^2 - n + 3t - 1
= \left( \frac{1}{2}n - \frac{1}{4} \right)(2n - t)^4 + (n - 2)(2n - t)^3 + \left( n - \frac{7}{2} \right)(2n - t)^2
+ 3(n - 1)(2n - t) + 5n - 1,
\]
which is greater than 0 whenever \(n \geq 4\) and \(0 \leq t < n\). It remains to check the validity of \(\psi_1^{(n,t)}(t/2) > 0\) for \(n = 2, 3\). In fact, for \(n = 2\), we have \(0 \leq t < 2\) and hence
\[
\psi_1^{(2,t)} \left( \frac{t}{2} \right) = \frac{3}{4} \left( (4 - t)^2 - 1 \right)^2 + 3(4 - t) + \frac{33}{4} > 0.
\]
For $n = 3$, we have $0 \leq t < 3$ and hence
\[
\psi_{1}^{(3,t)} \left( \frac{t}{2} \right) = \frac{5}{4} (6 - t)^4 + \left( \frac{11}{2} - t \right) (6 - t)^2 + 6(6 - t) + 14 > 0.
\]

As we see, the value of $\psi_{1}^{(n,t)}(t/2)$ must be positive. By further taking into account the value of $x_0$ and the sign of $\psi_{1}^{(n,t)}(0)$, there are three cases to determine the monotonicity of $\psi_{1}^{(n,t)}(x)$:

(i) $x_0 \leq 0$ and $\psi_{1}^{(n,t)}(0) \geq 0$. In this case, $\psi_{1}^{(n,t)}(x)$ increases from a nonnegative value to a positive value as $x$ increases from 0 to $t/2$. Thus, $(\psi_{1}^{(n,t)}(x))'$ takes only nonpositive values on $[0, t/2]$. That is to say, $\psi_{1}^{(n,t)}(x)$ is decreasing on the interval $[0, t/2]$.

(ii) $x_0 \leq 0$ and $\psi_{1}^{(n,t)}(0) < 0$. In this case, $\psi_{1}^{(n,t)}(x)$ increases from a negative value to a positive value as $x$ increases from 0 to $t/2$. Therefore, there exists $0 < t_0 < t/2$ such that
\[
\psi_{1}^{(n,t)}(x) \begin{cases} 
\leq 0, & \text{if } 0 \leq x \leq t_0, \\
\geq 0, & \text{if } t_0 < x \leq t/2.
\end{cases}
\]

Hence, we have
\[
(\psi_{1}^{(n,t)}(x))' \begin{cases} 
\geq 0, & \text{if } 0 \leq x \leq t_0, \\
\leq 0, & \text{if } t_0 < x \leq t/2.
\end{cases}
\]

That is to say, $\psi_{1}^{(n,t)}(x)$ is increasing on $[0, t_0]$ and decreasing on $[t_0, t/2]$.

(iii) $0 < x_0 < t/2$. In this case, we must have $\psi_{1}^{(n,t)}(0) < 0$. Once this assertion is proved, we obtain the desired monotonicity of $\psi_{1}^{(n,t)}(x)$ on $[0, t/2]$, by using similar arguments as in case (ii). Note that the condition $0 < x_0 < t/2$ implies that $\psi_{2}^{(n,t)}(0) > 0$.

Now we are to deduce $\psi_{1}^{(n,t)}(0) < 0$ from the positivity of $\psi_{2}^{(n,t)}(0)$. Using maple, we find that
\[
\psi_{1}^{(n,t)}(0) = (n + 1) \left( 4nt^3 + 2(2n^2 - 4n - 1)t^2 - (16n^3 - 12n^2 - 8n - 3)t \\
+ (8n^4 - 4n^3 - 8n^2 - 1) \right),
\]
\[
\psi_{2}^{(n,t)}(0) = 2(n + 1)t^2 + 2(8n^2 + 3n - 2)t - (2n - 1)(8n^2 + 8n + 3).
\]

Recall that $0 \leq t \leq n-1$ by the hypothesis. We may regard $\psi_{1}^{(n,t)}(0)/(n+1)$ as a polynomial in the variable $t$ over the interval $[0, n-1]$, denoted
by $\xi(t)$, and similarly, regard $\psi^{(n,t)}_2(0)$ as a polynomial $\eta(t)$. Now we can divide the proof of $\psi^{(n,t)}_1(0) < 0$ into the following three statements:

**Claim 1.** If $\psi^{(n,t)}_2(0) > 0$, then $n \neq 2, 3$.

*Proof of Claim 1.* In fact, it is routine to check that $\psi^{(n,t)}_2(0) < 0$ if $(n, t) \in \{(2, 0), (2, 1), (3, 0), (3, 1), (3, 2)\}$, contradicting the positivity of $\psi^{(n,t)}_2(0)$.

**Claim 2.** For any integer $n \geq 4$, the polynomial $\xi(t)$ takes only negative values on the interval $[\frac{3}{4}n, n - 1]$.

*Proof of Claim 2.* Note that, for $n \geq 4$, it is routine to check that

$$\xi\left(\frac{3}{4}n\right) = -\frac{1}{64} \left(4n^2(n - 4)^2 + 136 \left(n - \frac{9}{17}\right)^2 + \frac{440}{17}\right) < 0,$$

$$\xi(n - 1) = -(4n - 18)n^2 - 13n - 6 < 0.$$

We further need to consider the first order derivative and the second order derivative of $\xi(t)$ with respect to $t$:

$$\xi'(t) = 12nt^2 + (8n^2 - 16n - 4)t + (12n^2 - 16n^3 + 8n + 3),$$

$$\xi''(t) = 24nt + (8n^2 - 16n - 4).$$

Note that, for $n \geq 4$, we have

$$\xi''(0) = 8(n - 1)^2 - 12 > 0.$$

Hence, $\xi''(t) > 0$ for any $0 \leq t \leq n - 1$. That is to say, $\xi'(t)$ is strictly increasing on the interval $[0, n - 1]$. It is clear that, for $n \geq 4$,

$$\xi'\left(\frac{3}{4}n\right) = -\frac{13}{4}n^3 + 5n + 3 < 0.$$

Thus, there exists $3n/4 \leq t_1 \leq n - 1$ such that

$$\xi'(t) \begin{cases} 
\leq 0, & \text{if } \frac{3}{4}n \leq t \leq t_1, \\
> 0, & \text{if } t_1 < t \leq n - 1.
\end{cases}$$

In view of $\xi(3n/4) < 0$ and $\xi(n - 1) < 0$, we obtain $\xi(t) < 0$ for any $t \in [\frac{3}{4}n, n - 1]$. This completes the proof of Claim 2.

**Claim 3.** For any integer $n \geq 2$, the polynomial $\eta(t)$ takes only negative values on the interval $[0, \frac{3}{4}n]$. 

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Proof of Claim 3. For \( n \geq 2 \), a straightforward computation shows that

\[
\eta(0) = -16n^3 - 8n^2 + 2n + 3 < 0,
\]

\[
\eta \left( \frac{3}{4}n \right) = -\frac{23}{8}n^3 - \frac{19}{8}n^2 - n + 3 < 0.
\]

Note that the axis of symmetry of the quadratic function \( \eta(t) \) is

\[
t = \frac{-8n^2 + 3n - 2}{2(n + 1)},
\]

which lies strictly to the left of y-axis. Therefore, \( \eta(t) < 0 \) for any \( t \in [0, 3n/4] \) since both \( \eta(0) \) and \( \eta(3n/4) \) are negative. This completes the proof of Claim 3.

Now we can prove the negativity of \( \psi_1^{(n,t)}(0) \). From \( \psi_2^{(n,t)}(0) > 0 \) it follows \( \eta(t) > 0 \). By Claim 3, we must have \( t > 3n/4 \). Then by Claim 1 and Claim 2, we get \( \xi(t) < 0 \), and hence \( \psi_1^{(n,t)}(0) < 0 \), as desired.

Combining (i), (ii) and (iii), we complete the proof.

The above proposition is the key step for the proof of Theorem 3.3. Finally, we need to establish the following result.

**Proposition 3.6** Given \( n \geq 2 \), there exists \( k' \) with respect to \( n \) such that

\[
\psi^{(n,n)}(k) \begin{cases} 
\geq 0, & \text{if } 1 \leq k \leq k', \\
\leq 0, & \text{if } k' < k \leq \frac{n}{2}.
\end{cases}
\]

**Proof.** By (3.5), we obtain that

\[
\psi^{(n,n)}(x) = 8(2n + 1)x^6 - 24n(2n + 1)x^5 + 2(26n^3 - 2n + 12n^2 + 3)x^4 \\
- 4n(3 + 6n^3 + 2n^2 - 2n)x^3 + 2(4n^2 + 2n - 1 - 4n^3 + 2n^5)x^2 \\
+ 2n(n - 1)(2n - 1)(n + 1)x - n(n - 1)(n - 2)(n + 1)^2.
\]

It is easy to check that \( \psi^{(2,2)}(1) = 8 \). Hence, the proposition holds for \( n = 2 \). For the remainder of the proof, assume that \( n \geq 3 \). To determine the sign of \( \psi^{(n,n)}(k) \), let us consider the derivative of \( \psi^{(n,n)}(x) \) with respect to \( x \). Using Maple, we get

\[
(\psi^{(n,n)}(x))' = 2(2x - n)\psi_1^{(n,n)}(x),
\]
where
\[ \psi_1^{(n,n)}(x) = 12(1 + 2n)x^4 - 24n(1 + 2n)x^3 + 2(6n^2 - 2n + 3 + 14n^3)x^2 
- 2n(2n^3 + 3 - 2n)x - (n - 1)(2n - 1)(n + 1). \]

We also need to consider the derivative of \( \psi_1^{(n,n)}(x) \) with respect to \( x \):
\[
(\psi_1^{(n,n)}(x))' = 2(2x - n)\psi_2^{(n,n)}(x),
\]
where
\[ \psi_2^{(n,n)}(x) = 12(1 + 2n)x^2 - 12n(1 + 2n)x + 2n^3 + 3 - 2n. \]

Note that the axis of symmetry of the quadratic function \( \psi_2^{(n,n)}(x) \) is \( x = n/2 \), and, for \( n \geq 3 \),
\[
\psi_2^{(n,n)}(0) = 2n^3 - 2n + 3 > 0,
\]
\[
\psi_2^{(n,n)}(n/2) = -4n^3 - 3n^2 - 2n + 3 < 0.
\]

Thus, \( \psi_2^{(n,n)}(x) \) decreases from a positive value to a negative value as \( k \) increases from 0 to \( n/2 \). Hence, there exists \( 0 < x_0 < n/2 \) such that
\[
(\psi_1^{(n,n)}(x))' \begin{cases} 
\leq 0, & \text{if } 0 \leq x \leq x_0, \\
\geq 0, & \text{if } x_0 < x \leq n/2.
\end{cases}
\]

In view of that, for \( n \geq 3 \),
\[
\psi_1^{(n,n)}(0) = -n^2(n - 1) - n(n^2 - 2) - 1 < 0,
\]
\[
\psi_1^{(n,n)}(n/2) = \frac{1}{4}(2n^3(n^2 - 2) + n^2(3n^2 - 2) + 4(2n - 1)) > 0,
\]
there exists \( 0 < x_1 < n/2 \) such that
\[
\psi_1^{(n,n)}(x) \begin{cases} 
\leq 0, & \text{if } 0 \leq x \leq x_1, \\
\geq 0, & \text{if } x_1 < x \leq n/2.
\end{cases}
\]

Therefore,
\[
(\psi_1^{(n,n)}(x))' \begin{cases} 
\geq 0, & \text{if } 0 \leq x \leq x_1, \\
\leq 0, & \text{if } x_1 < x \leq n/2.
\end{cases}
\]

Moreover, it is easy to verify that, for \( n \geq 3 \),
\[
\psi^{(n,n)}(1) = (n - 1)((3n - 16)n^3 + (21n^2 + 8n - 12)) > 0,
\]

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\[ \psi^{(n,n)}(n/2) = -\frac{1}{8}n(n - 1)(n^2 - n - 4)(n + 2)^2 < 0. \]

Thus, there exists \( 1 < x_2 < n/2 \) such that

\[ \psi^{(n,n)}(x) \begin{cases} 
  \geq 0, & \text{if } 1 \leq x \leq x_2, \\
  \leq 0, & \text{if } x_2 < x \leq n/2.
\end{cases} \]

Thus, there exists an index \( k' = k'(n,n) \) such that \( \psi^{(n,n)}(k) \geq 0 \) for \( 1 \leq k \leq k' \) and \( \psi^{(n,n)}(k) \leq 0 \) for \( k' < k \leq n/2 \), as desired. This completes the proof.

We now come to the proof of Theorem 3.3.

**Proof of Theorem 3.3** By Proposition 3.4, for any \( n \geq 1 \) and \( 0 \leq t \leq n \), we have \( \mathcal{L}_t(a(n,0)) \geq 0 \). Given \( n \geq 1 \), it suffices to show that, for \( 0 \leq t \leq n \), there exists \( k' \) such that \( \mathcal{L}_t(a(n,k)) \geq 0 \) for \( 1 \leq k \leq k' \) and \( \mathcal{L}_t(a(n,k)) \leq 0 \) for \( k' < k \leq t/2 \). By (3.4), for \( k \geq 1 \), the sign of \( \mathcal{L}_t(a(n,k)) \) coincides with that of \( \psi^{(n,t)}(k) \). Combining Propositions 3.5 and 3.6 we obtain the desired result.

Finally, we can prove the \( q \)-log-convexity of \( \{S_n(q)\}_{n \geq 0} \).

**Proof of Theorem 3.4** This immediately follows from Theorems 2.1, 3.2 and 3.3.

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