Abstract

Let $A$ and $B$ be rings, $U$ a $(B, A)$-bimodule and $T = \left( \begin{array}{cc} A & 0 \\ U & B \end{array} \right)$ the triangular matrix ring. In this paper, several notions in relative Gorenstein algebra over a triangular matrix ring are investigated. We first study how to construct $w$-tilting (tilting, semidualizing) over $T$ using the corresponding ones over $A$ and $B$. We show that when $U$ is relative (weakly) compatible we are able to describe the structure of $G_C$-projective modules over $T$. As an application, we study when a morphism in $T$-$\text{Mod}$ has a special $G_C P(T)$-precover and when the class $G_C P(T)$ is a special precovering class. In addition, we study the relative global dimension of $T$. In some cases, we show that it can be computed from the relative global dimensions of $A$ and $B$. We end the paper with a counterexample to a result that characterizes when a $T$-module has a finite projective dimension.

2020 Mathematics Subject Classification. Primary: 16D90, 18G25

Keywords: Triangular matrix ring, weakly Wakamatsu tilting modules, relative Gorenstein dimensions.
modules. Motivated (in part) by Enochs and Jenda’s extensions of the classical G-dimension given in \[11\]. Holm and Jørgensen, extended in \[18\] this notion to arbitrary modules. After that, several generalizations of semidualizing and $G_C$-dimension have been made by several authors (\[29\], \[23\], \[3\]).

As the authors mentioned in \[6\], to study the Gorenstein projective modules and dimension relative to a semidualizing $(R, S)$-bimodule $C$, the condition $\text{End}_S(C) \cong R$, seems to be too restrictive and in some cases unnecessary. So the authors introduced weakly Wakamatsu tilting as a weakly notion of semidualizing which made the theory of relative Gorenstein homological algebra wider and less restrictive but still consistent. Weakly Wakamatsu tilting modules were subject of many publications which showed how important these modules could become in developing the theory of relative (Gorenstein) homological algebra (\[6\], \[7\], \[5\]).

Let $A$ and $B$ be rings and $U$ be a $(B, A)$-bimodule. The ring $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ is known as the formal triangular matrix ring with usual matrix addition and multiplication. Such rings play an important role in the representation theory of algebras. The modules over such rings can be described in a very concrete fashion. So, formal triangular matrix rings and modules over them have proved to be a rich source of examples and counterexamples. Some important Gorenstein notions over formal triangular matrix rings have been studied by many authors (see \[30\], \[10\], \[26\]). For example, Zhang \[30\] introduced compatible bimodules and explicitly described the Gorenstein projective modules over triangular matrix Artin algebra. Enochs, Izurdiaga, and Torrecillas \[10\] characterized when a left module over a triangular matrix ring is Gorenstein projective or Gorenstein injective under the "Gorenstein regular" condition. Under the same condition, Zhu, Liu, and Wang \[26\] investigated Gorenstein homological dimensions of modules over triangular matrix rings. Mao \[25\] studied Gorenstein flat modules over $T$ (without the "Gorenstein regular" condition) and gave an estimate of the weak global Gorenstein dimension of $T$.

The main objective of the present paper is to study relative Gorenstein homological notions (w-tilting, relative Gorenstein projective modules, relative Gorenstein projective dimensions and relative global projective dimension) over triangular matrix rings.

This article is organized as follows:

In Section 2, we give some preliminary results.

In Section 3, we study how to construct w-tilting (tilting, semidualizing) over
1 INTRODUCTION

Using w-tilting (tilting, semidualizing) over $A$ and $B$ under the condition that $U$ is relative (weakly) compatible. We introduce (weakly) $C$-compatible $(B, A)$-bimodules for a $T$-module $C$ (Definition 3.2). Given two w-tilting modules $A C_1$ and $B C_2$, we prove in Proposition 3.6 that $C = \left( U \otimes_A C_1 \right) \oplus C_2$ is a w-tilting $T$-module when $U$ is weakly $C$-compatible.

In Section 4, we first describe relative Gorenstein projective modules over $T$. Let $C = \left( U \otimes_A C_1 \right) \oplus C_2$ be a $T$-module. We prove in Theorem 4.3, that if $U$ is $C$-compatible then a $T$-module $M = \left( M_1 \right)$ is $G_C$-projective if and only if $M_1$ is $G_{C_1}$-projective $A$-module, $\text{Coker } \varphi^M$ is $G_{C_2}$-projective $B$-module and $\varphi^M : U \otimes_A M_1 \to M_2$ is injective. As an application, we prove that the converse of Proposition 3.6 and refine in relative setting (Proposition 4.9), a result of when $T$ is left (strongly) CM-free due to Enochs, Izurdiaga, and Torrecillas in [10]. Also when $C$ is w-tilting, we characterize when a $T$-morphism is a special precover (see Proposition 4.10). Then in Theorem 4.11, we prove that class of $G_{C_2}$-projective $T$-modules is special precovering if and only if so are the classes of $G_{C_1}$-projective $A$-modules and $G_{C_2}$-projective $B$-modules, respectively.

Finally, in Section 5, we give an estimate of $G_C$-projective dimension of a left $T$-module and the left $G_C$-projective global dimension of $T$. First, it is proven that, given a $T$-module $M = \left( M_1 \right)$, if $C = \mathfrak{p}(C_1, C_2) := \left( U \otimes_A C_1 \right) \oplus C_2$ is w-tilting, $U$ is $C$-compatible and

$$SG_{C_2} - PD(B) := \sup\{G_{C_2} - \text{pd}(U \otimes_A G) \mid G \in G_{C_1} P(A)\} < \infty,$$

then

$$\max\{G_{C_1} - \text{pd}(M_1), (G_{C_2} - \text{pd}(M_2)) - (SG_{C_2} - PD(B))\} \leq G_C - \text{pd}(M) \leq \max\{(G_{C_1} - \text{pd}(M_1)) + (SG_{C_2} - PD(B)) + 1, G_{C_2} - \text{pd}(M_2)\}.$$

As an application, we prove that, if $C = \mathfrak{p}(C_1, C_2)$ is w-tilting and $U$ is $C$-compatible then

$$\max\{G_{C_1} - PD(A), G_{C_2} - PD(B)\} \leq G_C - PD(T) \leq \max\{G_{C_1} - PD(A) + SG_{C_2} - PD(B) + 1, G_{C_2} - PD(B)\}.$$

Some cases when this estimation becomes an exact formula are also given.
The authors in [2] establish a relationship between the projective dimension of modules over $T$ and modules over $A$ and $B$. Given an integer $n \geq 0$ and $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ a $T$-module, they proved that $\text{pd}_T(M) \leq n$ if and only if $\text{pd}_A(M_1) \leq n$, $\text{pd}_B(M_2) \leq n$ and the map related to the $n$-th syzygy of $M$ is injective. We end the paper by giving a counterexample to this result (Example 5.11).

2 Preliminaries

Throughout this paper, all rings will be associative (not necessarily commutative) with identity, and all modules will be, unless otherwise specified, unitary left modules. For a ring $R$, we use $\text{Proj}(R)$ (resp., $\text{Inj}(R)$) to denote the class of all projective (resp., injective) $R$-modules. The category of all left $R$-modules will be denoted by $R\text{-Mod}$. For an $R$-module $C$, we use $\text{Add}_R(C)$ to denote the class of all $R$-modules which are isomorphic to direct summands of direct sums of copies of $C$, and $\text{Prod}_R(C)$ will denote the class of all $R$-modules which are isomorphic to direct summands of direct products of copies of $C$.

Given a class of modules $\mathcal{F}$ (which will always be considered closed under isomorphisms), an $\mathcal{F}$-precover of $M \in R\text{-Mod}$ is a morphism $\varphi : F \to M$ ($F \in \mathcal{F}$) such that $\text{Hom}_R(F', \varphi)$ is surjective for every $F' \in \mathcal{F}$. If, in addition, any solution of the equation $\text{Hom}_R(F, \varphi)(g) = \varphi$ is an automorphism of $F$, then $\varphi$ is said to be an $\mathcal{F}$-cover. The $\mathcal{F}$-precover $\varphi$ is said to be special if it is surjective and $\text{Ext}^1(F, \text{ker} \varphi) = 0$ for every $F \in \mathcal{F}$. The class $\mathcal{F}$ is said to be special (pre)covering if every module has a special $\mathcal{F}$-(pre)cover.

Given the class $\mathcal{F}$, the class of all modules $N$ such that $\text{Ext}_R^i(F, N) = 0$ for every $F \in \mathcal{F}$ will be denoted by $\mathcal{F}^\perp$ (similarly, $\mathcal{F}^\perp = \{ N ; \text{Ext}_R^i(N, F) = 0 \ \forall F \in \mathcal{F} \}$). The right and left orthogonal classes $\mathcal{F}^\perp$ and $^\perp \mathcal{F}$ are defined as follows:

$$\mathcal{F}^\perp = \cap_{i \geq 1} \mathcal{F}^\perp_i \text{ and } ^\perp \mathcal{F} = \cap_{i \geq 1} ^\perp \mathcal{F}$$

It is immediate to see that if $C$ is any module then $\text{Add}_R(C)^\perp = \{ C \}^\perp$ and $^\perp \text{Prod}_R(C) = ^\perp \{ C \}$.

Given a class of $R$-modules $\mathcal{F}$, an exact sequence of $R$-modules

$$\cdots \to X^1 \to X^0 \to X_0 \to X_1 \to \cdots$$

is called $\mathcal{F}$-exact (resp., $\mathcal{F}$-$\mathcal{F}$-exact) if the functor $\text{Hom}_R(\mathcal{F}, -)$ (resp., $\text{Hom}_R(\mathcal{F}, -)$) leaves the sequence exact whenever $F \in \mathcal{F}$. If $\mathcal{F} = \{ F \}$, we
simply say $\text{Hom}_R(-,F)$-exact. Similarly, we can define $\mathcal{F} \otimes_R -$exact sequences when $\mathcal{F}$ is a class of right $R$-modules.

We now recall some concepts needed throughout the paper.

**Definition 2.1** 1. ([17, Definition 2.1]) A semidualizing bimodule is an $(R, S)$-bimodule $C$ satisfying the following properties:

(a) $RC$ and $CS$ both admit a degreewise finite projective resolution in the corresponding module categories ($R$-Mod and Mod-$S$).

(b) $\text{Ext}_R^{\geq 1}(C, C) = \text{Ext}_S^{\geq 1}(C, C) = 0$.

(c) The natural homothety maps $R \xrightarrow{\gamma} \text{Hom}_S(C, C)$ and $S \xrightarrow{\gamma} \text{Hom}_R(C, C)$ both are ring isomorphisms.

2. ([28, Section 3]) A Wakamatsu tilting module, simply tilting, is an $R$-module $C$ satisfying the following properties:

(a) $RC$ admits a degreewise finite projective resolution.

(b) $\text{Ext}_R^{\geq 1}(C, C) = 0$

(c) There exists a $\text{Hom}_R(-, C)$-exact exact sequence of $R$-modules

\[ X = 0 \to R \to C^0 \to C^1 \to \cdots \]

where $C^i \in \text{add}_R(C)$ for every $i \in \mathbb{N}$.

It was proved in [28, Corollary 3.2], that an $(R, S)$-bimodule $C$ is semidualizing if and only if $RC$ is tilting with $S = \text{End}_R(C)$. So the following notion, which will be crucial in this paper, generalizes both concepts.

**Definition 2.2** ([6], Definition 2.1) An $R$-module $C$ is weakly Wakamatsu tilting (w-tilting for short) if it has the following two properties:

1. $\text{Ext}_R^{i \geq 1}(C, C^{(I)}) = 0$ for every set $I$.

2. There exists a $\text{Hom}_R(-, \text{Add}_R(C))$-exact exact sequence of $R$-modules

\[ X = 0 \to R \to A^0 \to A^1 \to \cdots \]

where $A^i \in \text{Add}_R(C)$ for every $i \in \mathbb{N}$.

If $C$ satisfies 1. but perhaps not 2. then $C$ will be said to be $\Sigma$-self-orthogonal.
Definition 2.3 ([6], Definition 2.2) Given any $C \in R$-$\text{Mod}$, an $R$-module $M$ is said to be $G_C$-projective if there exists a $\text{Hom}_R(-, \text{Add}_R(C))$-exact exact sequence of $R$-modules

$$X = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots$$

where the $P_i$'s are all projective, $A^i \in \text{Add}_R(C)$ for every $i \in \mathbb{N}$, $M \cong \text{Im}(P_0 \rightarrow A^0)$.

We use $G_C P(R)$ to denote the class of all $G_C$-projective $R$-modules.

It is immediate from the definitions that $w$-tilting modules can be characterized as follows.

Lemma 2.4 An $R$-module $C$ is $w$-tilting if and only if both $C$ and $R$ are $G_C$-projective modules.

Now we recall some facts about triangular matrix rings. Let $A$ and $B$ be rings and $U$ a $(B, A)$-bimodule. We shall denote by $T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix}$ the generalized triangular matrix ring. By [15, Theorem 1.5], the category $T$-$\text{Mod}$ of left $T$-modules is equivalent to the category $\Omega$ whose objects are triples $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi_M}$, where $M_1 \in A$-$\text{Mod}$, $M_2 \in B$-$\text{Mod}$ and $\varphi_M : U \otimes_A M_1 \rightarrow M_2$ is a $B$-morphism, and whose morphisms from $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi_M}$ to $\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi_N}$ are pairs $(f_1, f_2)$ such that $f_1 \in \text{Hom}_A(M_1, N_1)$, $f_2 \in \text{Hom}_B(M_2, N_2)$ satisfying that the following diagram is commutative.

$$\begin{array}{c}
U \otimes_A M_1 \xrightarrow{\varphi_M} M_2 \\
1_U \otimes f_1 \downarrow \quad \quad \downarrow f_2 \\
U \otimes_A N_1 \xrightarrow{\varphi_N} N_2
\end{array}$$

Since we have the natural isomorphism

$$\text{Hom}_B(U \otimes_A M_1, M_2) \cong \text{Hom}_A(M_1, \text{Hom}_B(U, M_2)),$$

there is an alternative way of defining $T$-modules and $T$-homomorphisms in terms of maps $\varphi^M : M_1 \rightarrow \text{Hom}_B(U, M_2)$ given by $\varphi^M(u)(x) = \varphi_M(u \otimes x)$ for each $u \in U$ and $x \in M_1$. 
Analogously, the category \( \text{Mod}-T \) of right \( T \)-modules is equivalent to the category \( \Omega_T \) whose objects are triples \( M = (M_1, M_2)_{\varphi^M} \), where \( M_1 \in \text{Mod}-A \), \( M_2 \in \text{Mod}-B \) and \( \varphi^M : M_2 \otimes_B U \to M_1 \) is an \( A \)-morphism, and whose morphisms from \( (M_1, M_2)_{\varphi^M} \) to \( (N_1, N_2)_{\varphi^N} \) are pairs \( (f_1, f_2) \) such that \( f_1 \in \text{Hom}_A(M_1, N_1) \), \( f_2 \in \text{Hom}_B(M_2, N_2) \) satisfying that the following diagram

\[ \begin{array}{ccc}
M_2 \otimes B U & \xrightarrow{\varphi^M} & M_1 \\
\downarrow f_2 \otimes 1_U & & \downarrow f_1 \\
M_2 \otimes B U & \xrightarrow{\varphi^N} & N_1
\end{array} \]

is commutative.

In the rest of the paper we shall identify \( T\text{-Mod} \) (resp. \( \text{Mod}-T \)) with \( T\Omega \) (resp. \( \Omega_T \)). Consequently, through the paper, a left (resp. right) \( T \)-module will be a triple \( M = (M_1, M_2)_{\varphi^M} \) (resp. \( M = (M_1, M_2)_{\varphi^M} \)) and, whenever there is no possible confusion, we shall omit the morphisms \( \varphi^M \) and \( \varphi^N \). For example, \( T_T \) is identified with \( (A U \oplus B) \) and \( T_T \) is identified with \( (A \oplus U, B) \).

A sequence of left \( T \)-modules \( 0 \to (M_1, M_2) \to (M_1', M_2') \to (M_1'', M_2'') \to 0 \) is exact if and only if both sequences \( 0 \to M_1 \to M_1' \to M_1'' \to 0 \) and \( 0 \to M_2 \to M_2' \to M_2'' \to 0 \) are exact.

Throughout this paper, \( T = \begin{pmatrix} A & 0 \\ U & B \end{pmatrix} \) will be a generalized triangular matrix ring. Given a \( T \)-module \( M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \), the \( B \)-module \( \text{Coker}_{\varphi^M} \) will be denoted as \( \overline{M}_2 \) and the \( A \)-module \( \text{Ker}_{\varphi^M} \) as \( \overline{M}_1 \). A \( T \)-module \( N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N} \) is a submodule of \( M \) if \( N_1 \) is a submodule of \( M_1 \), \( N_2 \) is a submodule of \( M_2 \) and \( \varphi^M |_{U \otimes_A N_1} = \varphi^N \).

As an interesting and special case of triangular matrix rings, we recall that the \( T_2 \)-extension of a ring \( R \) is given by

\[ T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix} \]

and the modules over \( T(R) \) are triples \( M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M} \), where \( M_1 \) and \( M_2 \) are \( R \)-modules and \( \varphi^M : M_1 \to M_2 \) is an \( R \)-homomorphism.

There are some pairs of adjoint functors \( (p, q), (q, h) \) and \( (s, r) \) between the category \( T\text{-Mod} \) and the product category \( A\text{-Mod} \times B\text{-Mod} \) which are defined.
as follows:

1. \( p : A\text{-Mod} \times B\text{-Mod} \to T\text{-Mod} \) is defined as follows: for each object \((M_1, M_2)\) of \(A\text{-Mod} \times B\text{-Mod}\), let \( p(M_1, M_2) = \left( M_1 (U \otimes_A M_1) \oplus M_2 \right) \) with the obvious map and for any morphism \((f_1, f_2)\) in \(A\text{-Mod} \times B\text{-Mod}\), let \( p(f_1, f_2) = \left( f_1 (1_U \otimes_A f_1) \oplus f_2 \right) \).

2. \( q : T\text{-Mod} \to A\text{-Mod} \times B\text{-Mod} \) is defined, for each left \(T\text{-module} \) \( (M_1, M_2) \) as \( q \left( \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \right) = (M_1, M_2) \), and for each morphism \( (f_1, f_2) \) in \(T\text{-Mod} \) as \( q(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}) = (f_1, f_2) \).

3. \( h : A\text{-Mod} \times B\text{-Mod} \to T\text{-Mod} \) is defined as follows: for each object \((M_1, M_2)\) of \(A\text{-Mod} \times B\text{-Mod}\), let \( h(M_1, M_2) = \left( M_1 \oplus \text{Hom}_B(U, M_2) \right) \) with the obvious map and for any morphism \((f_1, f_2)\) in \(A\text{-Mod} \times B\text{-Mod}\), let \( h(f_1, f_2) = \left( f_1 \oplus \text{Hom}_B(U, f_2) \right) \).

4. \( r : A\text{-Mod} \times B\text{-Mod} \to T\text{-Mod} \) is defined as follows: for each object \((M_1, M_2)\) of \(A\text{-Mod} \times B\text{-Mod}\), let \( r(M_1, M_2) = \left( M_1 \right) \) with the zero map and for any morphism \((f_1, f_2)\) in \(A\text{-Mod} \times B\text{-Mod}\), let \( r(f_1, f_2) = \left( f_1 \right) \).

5. \( s : T\text{-Mod} \to A\text{-Mod} \times B\text{-Mod} \) is defined, for each left \(T\text{-module} \) \( (M_1, M_2) \) as \( s(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}) = (M_1, \overline{M}_2) \), and for each morphism \( (f_1, f_2) \) in \(T\text{-Mod} \) as \( s(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}) = (f_1, \overline{f}_2) \), where \( \overline{f}_2 \) is the induced map.

It is easy to see that \( q \) is exact. In particular, \( p \) preserves projective objects and \( h \) preserves injective objects. Note that the pairs of adjoint functors \((p, q)\) and \((q, h)\) are defined in [10]. In general, the three pairs of adjoint functors defined above can be found in [13].

For a future reference, we list these adjointness isomorphisms:

\[
\text{Hom}_T \left( \begin{pmatrix} M_1 \\ (U \otimes_A M_1) \oplus M_2 \end{pmatrix}, N \right) \cong \text{Hom}_A(M_1, N_1) \oplus \text{Hom}_B(M_2, N_2).
\]

\[
\text{Hom}_T(N, \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}) \cong \text{Hom}_A(N_1, M_1) \oplus \text{Hom}_B(N_2, M_2).
\]
\[
\text{Hom}_T(M, \left( \frac{N_1 \oplus \text{Hom}_B(U, N_2)}{N_2} \right)) \cong \text{Hom}_A(M_1, N_1) \oplus \text{Hom}_B(M_2, N_2).
\]

Now we recall the characterizations of projective, injective and finitely generated \(T\)-modules.

**Lemma 2.5** Let \(M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}\) be a \(T\)-module.

1. ([19, Theorem 3.1]) \(M\) is projective if and only if \(M_1\) is projective in \(A\)-Mod, \(\overline{M}_2 = \text{Coker} \varphi^M\) is projective in \(B\)-Mod and \(\varphi^M\) is injective.
2. ([20, Proposition 5.1]) \(M\) is injective if and only if \(M_1 = \text{Ker} \tilde{\varphi}^M\) is injective in \(A\)-Mod, \(M_2\) is injective in \(B\)-Mod and \(\tilde{\varphi}^M\) is surjective.
3. ([16]) \(M\) is finitely generated if and only if \(M_1\) and \(M_2\) are finitely generated.

The following Lemma improves [24, Lemma 3.2].

**Lemma 2.6** Let \(M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_{\varphi^M}\) and \(N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{\varphi^N}\) be two \(T\)-modules and \(n \geq 1\) be an integer number. Then we have the following natural isomorphisms:

1. If \(\text{Tor}_i^A(U, M_1) = 0, 1 \leq i \leq n\), then \(\text{Ext}_T^n\left( \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}, N \right) \cong \text{Ext}_A^n(M_1, N_1)\).
2. \(\text{Ext}_T^n\left( \begin{pmatrix} 0 \\ M_2 \end{pmatrix}, N \right) \cong \text{Ext}_B^n(M_2, N_2)\).
3. \(\text{Ext}_T^n(M, \begin{pmatrix} N_1 \\ 0 \end{pmatrix}) \cong \text{Ext}_A^n(M_1, N_1)\).
4. If \(\text{Ext}_B^i(U, N_2) = 0, 1 \leq i \leq n\), then \(\text{Ext}_T^n(M, \begin{pmatrix} \text{Hom}_B(U, N_2) \\ N_2 \end{pmatrix}) \cong \text{Ext}_B^n(M_2, N_2)\).

**Proof.** We prove only 1., since 2. is similar and 3. and 4. are dual. Assume that \(\text{Tor}_i^A(U, M_1) = 0\) and consider an exact sequence of \(A\)-modules

\[
0 \rightarrow K_1 \rightarrow P_1 \rightarrow M_1 \rightarrow 0
\]

where \(P_1\) is projective. So, there exists an exact sequence of \(T\)-modules

\[
0 \rightarrow \begin{pmatrix} K_1 \\ U \otimes_A K_1 \end{pmatrix} \rightarrow \begin{pmatrix} P_1 \\ U \otimes_A P_1 \end{pmatrix} \rightarrow \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \rightarrow 0
\]

where \(\begin{pmatrix} P_1 \\ U \otimes_A P_1 \end{pmatrix}\) is projective by Lemma 2.5.
Let $n = 1$. By applying the functor $\text{Hom}_T(-, N)$ to the above short exact sequence and since $\left(\frac{P_1}{U \otimes_A P_1}\right)$ and $P_1$ are projectives, we get a commutative diagram with exact rows

$$
\begin{array}{cccccc}
\text{Hom}_T\left(\left(\frac{P_1}{U \otimes_A P_1}\right), N\right) & \rightarrow & \text{Hom}_T\left(\left(\frac{K_1}{U \otimes_A K_1}\right), N\right) & \rightarrow & \text{Ext}_T^1\left(\left(\frac{M_1}{U \otimes_A M_1}\right), N\right) \\
\odot \downarrow \quad \cong \quad \downarrow \cong & & \downarrow \quad \cong \\
\text{Hom}_A(P_1, N_1) & \rightarrow & \text{Hom}_A(K_1, N_1) & \rightarrow & \text{Ext}_A^1(M_1, N_1)
\end{array}
$$

where the first two columns are just the natural isomorphisms given by adjointeness and the last two horizontal rows are epimorphisms. Thus, the induced map

$$\text{Ext}_T^1\left(\left(\frac{M_1}{U \otimes_A M_1}\right), N\right) \rightarrow \text{Ext}_A^1(M_1, N_1)$$

is an isomorphism such that the above diagram is commutative.

Assume that $n > 1$. Using the long exact sequence, we get a commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \rightarrow & \text{Ext}_T^{n-1}\left(\left(\frac{K_1}{U \otimes_A K_1}\right), N\right) & \rightarrow & \text{Ext}_T^n\left(\left(\frac{M_1}{U \otimes_A M_1}\right), N\right) & \rightarrow & 0 \\
\sigma \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 & \rightarrow & \text{Ext}_A^{n-1}(K_1, N_1) & \rightarrow & \text{Ext}_A^n(M_1, N_1) & \rightarrow & 0
\end{array}
$$

where $\sigma$ is a natural isomorphism by induction, since $\text{Tor}_k^A(U, K_1) = 0$ for every $k \in \{1, \ldots, n-1\}$ because of the exactness of the following sequence

$$0 = \text{Tor}_k^A(U, M_1) \rightarrow \text{Tor}_k^A(U, K_1) \rightarrow \text{Tor}_k^A(U, P_1) = 0.$$

Thus, the composite map

$$g\sigma f^{-1} : \text{Ext}_T^n\left(\left(\frac{M_1}{U \otimes_A M_1}\right), N\right) \rightarrow \text{Ext}_A^n(M_1, N_1)$$

is a natural isomorphism, as desired.

Since $T$ can be viewed as a trivial extension (see [13] and [4] for more details), the following Lemma can be easily deduced from ([4] Theorem 3.1 and Theorem 3.4). For the convenience of the reader, we give a proof.
Lemma 2.7 Let $X = \left( \begin{array}{c} X_1 \\ X_2 \end{array} \right)$ be a $T$-module and $(C_1, C_2) \in A\text{-Mod} \times B\text{-Mod}$.

1. $X \in \text{Add}_T(p(C_1, C_2))$ if and only if
   (i) $X \cong p(X_1, X_2)$,
   (ii) $X_1 \in \text{Add}_A(C_1)$ and $X_2 \in \text{Add}_B(C_2)$.

   In this case, $\varphi^X$ is injective.

2. $X \in \text{Prod}_T(h(C_1, C_2))$ if and only if
   (i) $X \cong h(X_1, X_2)$,
   (ii) $X_1 \in \text{Prod}_A(C_1)$ and $X_2 \in \text{Prod}_B(C_2)$.

   In this case, $\tilde{\varphi}^X$ is surjective.

Proof. We only need to prove (1), since (2) is dual.

For the "if" part. If $X_1 \in \text{Add}_A(C_1)$ and $X_2 \in \text{Add}_B(C_2)$, then $X_1 \oplus Y_1 = C_1^{(I_1)}$ and $X_2 \oplus Y_2 = C_2^{(I_2)}$ for some $(Y_1, Y_2) \in A\text{-Mod} \times B\text{-Mod}$ and some sets $I_1$ and $I_2$. Without loss of generality, we may assume that $I = I_1 = I_2$. Then

$$X \oplus p(Y_1, Y_2) \cong p(X_1, X_2) \oplus p(Y_1, Y_2)$$

$$= \left( (U \otimes_A X_1) \oplus X_2 \right) \oplus \left( (U \otimes_A Y_1) \oplus Y_2 \right)$$

$$\cong \left( C_1^{(I_1)} \right) \oplus \left( C_2^{(I_2)} \right)$$

$$\cong \left( (U \otimes_A C_1) \oplus C_2 \right)^{(I)}$$

$$= p(C_1, C_2)^{(I)}.$$

Hence $X \in \text{Add}_T(p(C_1, C_2))$.

Conversely, let $X \in \text{Add}_T(p(C_1, C_2))$ and $Y = \left( \begin{array}{c} Y_1 \\ Y_2 \end{array} \right)$ be a $T$-module such that $X \oplus Y = p(C_1, C_2)^{(I)}$ for some set $I$. Then $\varphi^X$ is injective, as $X$ is a submodule of $C := p(C_1, C_2)^{(I)}$ and $\varphi_C$ is injective. Consider now the split exact sequence

$$0 \to Y \xrightarrow{\lambda_1} C \xrightarrow{(p_1, p_2)} X \to 0$$
which induces the following commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & \rightarrow & U \otimes_A Y_1 & \xrightarrow{1_U \otimes \lambda_1} & U \otimes_A C_1^{(I)} & \xrightarrow{1_U \otimes p_1} & U \otimes_A X_1 & \rightarrow & 0 \\
\downarrow{\phi^Y} & & \downarrow{\phi^C} & & \downarrow{\phi^X} & & \downarrow{\phi^X} & & \\
0 & \rightarrow & Y_2 & \xrightarrow{\lambda_2} & U \otimes_A C_1^{(I)} \oplus C_2^{(I)} & \xrightarrow{p_2} & X_2 & \rightarrow & 0 \\
\downarrow{\phi^Y} & & \downarrow{\phi^C} & & \downarrow{\phi^X} & & \downarrow{\phi^X} & & \\
0 & \rightarrow & Y_2 & \xrightarrow{\lambda_2} & C_2^{(I)} & \xrightarrow{p_2} & X_2 & \rightarrow & 0 \\
\downarrow{0} & & \downarrow{0} & & \downarrow{0} & & \downarrow{0} & & \\
0 & & 0 & & 0 & & 0 & & 
\end{array}
\]

where $\phi^Y$, $\phi^C$ and $\phi^X$ are the canonical projections. Clearly, $p_1 : C_1^{(I)} \rightarrow X_1$ and $p_2 : C_2^{(I)} \rightarrow X_2$ are split epimorphisms. Then $X_1 \in \text{Add}_A(C_1)$ and $X_2 \in \text{Add}_B(C_2)$. It remains to prove that $X \cong p(X_1, X_2)$. For this, it suffices to prove that the short exact sequence

\[
0 \rightarrow U \otimes_A X_1 \xrightarrow{i} X_2 \xrightarrow{\iota} X_2 \rightarrow 0
\]
splits. Let $r_2$ be the retraction of $p_2$. If $i : C_2^{(I)} \rightarrow (U \otimes_A C_1^{(I)}) \oplus C_2^{(I)}$ denotes the canonical injection, then $\phi^X p_2 i r_2 = p_2 \phi^C i r_2 = p_2 r_2 = 1_{X_2}$ and the proof is finished.

**Remark 2.8**

1. Since the class of projective modules over $T$ is nothing but the class $\text{Add}_R(T)$, when we take $C_1 = A$ and $C_2 = B$ in Lemma 2.7, we recover the characterization of projective and injective $T$-modules.

2. Let $(C_1, C_2) \in A\text{-Mod} \times B\text{-Mod}$. By Lemma 2.7, every module in $\text{Add}_R(p(C_1, C_2))$ has the form $p(X_1, X_2)$ for some $X_1 \in \text{Add}_A(C_1)$ and $X_2 \in \text{Add}_B(C_2)$.

**3 w-Tilting modules**

In this section, we study when the functor $p$ preserves w-tilting modules.
3 W-tilting Modules

It is well known that the functor $p$ preserves projective modules. However, the functor $p$ does not preserve w-tilting modules in general, as the following example shows.

**Example 3.1** Let $Q$ be the quiver

$$e_1 \xrightarrow{\alpha} e_2,$$

and let $R = kQ$ be the path algebra over an algebraic closed field $k$. Put $P_1 = Re_1$, $P_2 = Re_2$, $I_1 = \text{Hom}_k(e_1 R, k)$ and $I_2 = \text{Hom}_k(e_2 R, k)$. Note that, $C_1$ and $C_2$ are projective and injective $R$-modules, respectively. By [3, Example 2.3],

$$C_1 = P_1 \oplus P_2(= R) \quad \text{and} \quad C_2 = I_1 \oplus I_2.$$

are semidualizing $(R, R)$-bimodules and then w-tilting $R$-modules. Now, consider the triangular matrix ring

$$T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}.$$

We claim that $p(C_1, C_2)$ is not a w-tilting $T(R)$-module. Note that $I_1$ is not projective. Since $R$ is left hereditary by [1, Proposition 1.4], $\text{pd}_R(I_1) = 1$. Hence $\text{Ext}^1_R(I_1, R) \neq 0$. Using Lemma 2.6, we get that $\text{Ext}^1_{T(R)}(p(C_1, C_2), p(C_1, C_2)) \cong \text{Ext}^1_R(C_1, C_1) \oplus \text{Ext}^1_R(C_2, C_1) \oplus \text{Ext}^1_R(C_2, C_2) \cong \text{Ext}^1_R(I_1, R) \neq 0$. Thus, $p(C_1, C_2)$ is not a w-tilting $T(R)$-module.

Motivated by the definition of compatible bimodules in [30, Definition 1.1], we introduce the following definition which will be crucial throughout the rest of this paper.

**Definition 3.2** Let $(C_1, C_2) \in A\text{-Mod} \times B\text{-Mod}$ and $C = p(C_1, C_2)$. The bimodule $B_UA$ is said to be $C$-compatible if the following two conditions hold:

(a) The complex $U \otimes_A X_1$ is exact for every exact sequence in $A\text{-Mod}$

$$X_1 : \cdots \rightarrow P^1_i \rightarrow P^0_i \rightarrow C^0_i \rightarrow C^1_i \rightarrow \cdots$$

where the $P^0_i$'s are all projective and $C^1_i \in \text{Add}_A(C_1) \forall i$.

(b) The complex $\text{Hom}_B(X_2, U \otimes_A \text{Add}_A(C_1))$ is exact for every $\text{Hom}_B(\cdots, \text{Add}_B(C_2))$-exact sequence in $B\text{-Mod}$

$$X_2 : \cdots \rightarrow P^1_i \rightarrow P^0_i \rightarrow C^0_i \rightarrow C^1_i \rightarrow \cdots$$

where the $P^0_i$'s are all projective and $C^1_i \in \text{Add}_B(C_2) \forall i$. 

13
Moreover, $U$ is called weakly $C$-compatible if it satisfies (b) and the following condition

(a') The complex $U \otimes_A X_1$ is exact for every $\text{Hom}_A(\cdot, \text{Add}_A(C_1))$-exact exact sequence in $A$-Mod

$$X_1 : \cdots \to P_1^1 \to P_0^1 \to C_0^1 \to C_1^1 \to \cdots$$

where the $P_i^j$'s are all projective and $C_i^j \in \text{Add}_A(C_1) \forall i$.

When $C = \tau T = p(A, B)$, the bimodule $U$ will be called simply (weakly) compatible.

**Remark 3.3**

1. It is clear by the definition that every $C$-compatible is weakly $C$-compatible.

2. The $(B, A)$-bimodule $U$ is weakly compatible if and only if the functor $U \otimes_A - : A$-Mod $\to B$-Mod is weak compatible (see [21]).

3. If $A$ and $B$ are Artin algebras and since $\tau T = \begin{pmatrix} A & U \\ U \otimes B & B \end{pmatrix} = p(A, B)$, it is easy to see that $\tau T$-compatible bimodules are nothing but compatible $(B, A)$-bimodules as defined in [30].

The following can be applied to produce examples of (weakly) $C$-compatible bimodules later on.

**Lemma 3.4** Let $C = p(C_1, C_2) = \tau T$ be a $T$-module.

1. Assume that $\text{Tor}_1^A(U, C_1) = 0$. If $\text{fd}_A(U) < \infty$, then $U$ satisfies (a).

2. Assume that $\text{Ext}_B^1(C_2, U \otimes_A C_1^{(I)}) = 0$ for every set $I$. If $\text{id}_B(U \otimes_A C_1) < \infty$, then $U$ satisfies (b).

3. If $U \otimes_A C_1 \in \text{Add}_B(C_2)$, then $U$ satisfies (b).

**Proof.** (3) is clear. We only prove (1), as (2) is similar. Consider an exact sequence of $A$-modules

$$X_1 : \cdots \to P_1^1 \to P_0^1 \to C_0^1 \to C_1^1 \to \cdots$$

where the $P_i^j$'s are all projective and $C_i^j \in \text{Add}_A(C_1) \forall i$. We use induction on $\text{fd}_A U$. If $\text{fd}_A U = 0$, then the result is trivial. Now suppose that $\text{fd}_A U = n \geq 1$. Then, there exists an exact sequence of right $A$-modules

$$0 \to L \to F \to U \to 0$$
where \( \text{fd}_A L = n - 1 \) and \( F \) is flat. Applying the functor \( - \otimes X_1 \) to the above short exact sequence, we get the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & L \otimes P_1^0 & \rightarrow & F \otimes A P_1^0 & \rightarrow & U \otimes A P_1^0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{Tor}_1^A(U, C_1^0) & \rightarrow & L \otimes A C_1^0 & \rightarrow & F \otimes A C_1^0 & \rightarrow & U \otimes A C_1^0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{Tor}_1^A(U, C_1^1) & \rightarrow & L \otimes A C_1^1 & \rightarrow & F \otimes A C_1^1 & \rightarrow & U \otimes A C_1^1 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
\end{array}
\]

Since \( \text{Tor}_1^A(U, C_1) = 0 \), the above diagram induces an exact sequence of complexes

\[
0 \rightarrow L \otimes_A X_1 \rightarrow F \otimes_A X_1 \rightarrow U \otimes_A X_1 \rightarrow 0.
\]

By induction hypothesis, the complexes \( L \otimes_A X_1 \) and \( F \otimes_A X_1 \) are exact. Thus \( U \otimes_A X_1 \) is exact as well. \( \blacksquare \)

Given a \( T \)-module \( C = p(C_1, C_2) \), we have simple characterizations of conditions (a') and (b) if \( C_1 \) and \( C_2 \) are w-tilting.

**Proposition 3.5** Let \( C = p(C_1, C_2) \) be a \( T \)-module.

1. If \( C_1 \) is w-tilting, then the following assertions are equivalent:
   
   (i) \( U \) satisfies (a').
   
   (ii) \( \text{Tor}_1^A(U, G_1) = 0, \forall G_1 \in G_{C_1} P(A) \).
   
   (iii) \( \text{Tor}_{i \geq 1}^A(U, G_1) = 0, \forall G_1 \in G_{C_1} P(A) \).

   In this case, \( \text{Tor}_{i \geq 1}^A(U, C_1) = 0 \).

2. If \( C_2 \) is w-tilting, then the following assertions are equivalent:

   (i) \( U \) satisfies (b).
   
   (ii) \( \text{Ext}_B^1(G_2, U \otimes_A X_1) = 0, \forall G_2 \in G_{C_2} P(B), \forall X_1 \in \text{Add}_A(C_1) \).
(iii) \( \text{Ext}_B^{i \geq 1}(G_2, U \otimes_A X_1) = 0, \forall G_2 \in G_{C_2}P(B), \forall X_1 \in \text{Add}_A(C_1). \)

In this case, \( \text{Ext}_B^{i \geq 1}(C_2, U \otimes_A X_1) = 0, \forall X_1 \in \text{Add}_A(C_1). \)

Proof. We only prove (1), since (2) is similar.

(i) \( \Rightarrow \) (iii) Let \( G_1 \in G_{C_1}P(R). \) There exists a \( \text{Hom}_A(\cdot, \text{Add}_A(C_1))-\text{exact} \)
exact sequence in \( A\text{-Mod} \)

\[
X_1 : \cdots \to P_1 \to P_0 \to C_0 \to C_1 \to \cdots
\]

where the \( P_i \)'s are all projective, \( G_1 \cong \text{Im}(P_0 \to C_0) \) and \( C_i \in \text{Add}_A(C_1) \)
\( \forall i. \) By condition \( (a'), U \otimes_A X_1 \) is exact, which means in particular that
\( \text{Tor}_A^{i \geq 1}(U, G_1) = 0. \)

(iii) \( \Rightarrow \) (ii) Clear.

(ii) \( \Rightarrow \) (i) Follows by [6, Corollary 2.13].

Finally, to prove that \( \text{Tor}_A^{i \geq 1}(U, C_1) = 0, \) note that \( C_1 \in G_{C_1}P(A) \) by [6, Theorem 2.12].

In the following proposition, we study when \( p \) preserves w-tilting (tilting) modules.

Proposition 3.6 Let \( C = p(C_1, C_2) \) be a \( T \)-module and assume that \( U \) is weakly \( C \)-compatible. If \( C_1 \) and \( C_2 \) are w-tilting (tilting), then \( p(C_1, C_2) \) is w-tilting (tilting).

Proof. By Lemma 2.5, the functor \( p \) preserves finitely generated modules, so we only need prove the statement for w-tilting. Assume that \( C_1 \) and \( C_2 \) are w-tilting and let \( I \) be a set. Then \( \text{Ext}_A^{i \geq 1}(C_1, C_1(I)) = 0 \) and \( \text{Ext}_B^{i \geq 1}(C_2, C_2(I)) = 0. \) By Proposition above, we have \( \text{Ext}_B^{i \geq 1}(C_2, U \otimes_A C_1(I)) = 0 \) and \( \text{Tor}_A^{i \geq 1}(U, C_1) = 0. \) Using Lemma 2.6 for every \( n \geq 1 \) we get that

\[
\text{Ext}_T^n(C, C_1(I)) = \text{Ext}_T^n(p(C_1, C_2), p(C_1, C_2)(I))
\]

\[
\cong \text{Ext}_A^n(C_1, C_1(I)) \oplus \text{Ext}_B^n(C_2, U \otimes_A C_1(I)) \oplus \text{Ext}_B^n(C_2, C_2(I))
\]

\[
= 0.
\]

Moreover, there exist exact sequences

\[
X_1 : 0 \to A \to C_0^1 \to C_1^1 \to \cdots
\]

and

\[
X_2 : 0 \to B \to C_0^2 \to C_1^2 \to \cdots
\]
which are \( \text{Hom}_A(-, \text{Add}_A(C_1)) \)-exact and \( \text{Hom}_B(-, \text{Add}_B(C_2)) \)-exact, respectively, and such that \( C_1^i \in \text{Add}_A(C_1) \) and \( C_2^i \in \text{Add}_B(C_2) \) for every \( i \in \mathbb{N} \). Since \( U \) is weakly \( C \)-compatible, the complex \( U \otimes_A X_1 \) is exact. So we construct in \( T \)-\text{Mod} the exact sequence

\[
p(X_1, X_2) : 0 \rightarrow T \rightarrow p(C_1^0, C_2^0) \rightarrow p(C_1^1, C_2^1) \rightarrow \cdots
\]

where \( p(C_1^i, C_2^i) = \left( (U \otimes_A C_1^i) \oplus C_2^i \right) \in \text{Add}_T(p(C_1, C_2)), \forall i \in \mathbb{N}, \) by Lemma \[2.7\].

Let \( X \in \text{Add}_T(p(C_1, C_2)) \). As a consequence of Lemma \[2.7\], \( X = p(X_1, X_2) \) where \( X_1 \in \text{Add}_A(C_1) \) and \( X_2 \in \text{Add}_B(C_2) \). Using the adjoinness \( (p, q) \), we get an isomorphism of complexes

\[
\text{Hom}_T(p(X_1, X_2), X) \cong \text{Hom}_A(X_1, X_1) \oplus \text{Hom}_B(X_2, U \otimes X_1) \oplus \text{Hom}_B(X_2, X_2).
\]

But \( \text{Hom}_A(X_1, X_1) \) and \( \text{Hom}_B(X_2, X_2) \) are exact and the complex \( \text{Hom}_B(X_2, U \otimes X_1) \) is also exact since \( U \) is weakly \( C \)-compatible. Then, \( \text{Hom}_T(p(X_1, X_2), X) \) is exact as well and the proof is finished. \( \blacksquare \)

Now, we illustrate Proposition \[3.6\] with two applications.

**Corollary 3.7** Let \( C = p(C_1, C_2) \) be a \( T \)-module, \( A' \) and \( B' \) be two rings such that \( A' C_{A'} \) and \( A C_{B'} \) are bimodules and assume that \( U \) is weakly \( C \)-compatible. If \( A' C_{A'} \) and \( A C_{B'} \) are semidualizing bimodules, then \( p(C_1, C_2) \) is a semidualizing \( (T, \text{End}_T(C)) \)-bimodule.

**Proof.** Follows by Proposition \[3.6\] and [28 Corollary 3.2]. \( \blacksquare \)

**Corollary 3.8** Let \( R \) and \( S \) be rings, \( \theta : R \rightarrow S \) be a homomorphism with \( S_R \) flat, and \( T = T(\theta) =: \left( \begin{array}{cc} R & 0 \\ S & S \end{array} \right) \). Let \( C_1 \) be an \( R \)-module such that \( S \otimes_R C_1 \in \text{Add}_R(C_1) \) (for instance, if \( R \) is commutative or \( R = S \)). If \( R C_1 \) is \( w \)-tilting, then

1. \( S \otimes_R C_1 \) is a \( w \)-tilting \( S \)-module.
2. \( C = \left( (S \otimes_R C_1) \oplus (S \otimes_R C_1) \right) \) is a \( w \)-tilting \( T(\theta) \)-module.

**Proof.** 1. Let \( C_2 = S \otimes_R C_1 \) and note that \( C = p(C_1, C_2) \) and that \( S_S \) is \( C \)-compatible. So, by Proposition \[3.6\] we only need to prove that \( C_2 \) is \( w \)-tilting \( S \)-module.
Since $R C_1$ is w-tilting, there exist $\text{Hom}_R(-, \text{Add}_R(C_1))$-exact exact sequences

$$P : \cdots \to P_1 \to P_0 \to C \to 0$$

and

$$X : 0 \to R \to C_0 \to C_1 \to \cdots$$

with each $R P_i$ projective and $R C_i \in \text{Add}_R(C_1)$. Since $S_R$ is flat, we get an exact sequence

$$S \otimes_R P : \cdots \to S \otimes_R P_1 \to S \otimes_R P_0 \to S \otimes_R C \to 0$$

and

$$S \otimes_R X : 0 \to S \to S \otimes_R C_0 \to S \otimes_R C_1 \to \cdots$$

with each $S \otimes_R P_i$ a projective $S$-module and $S \otimes_R C_i \in \text{Add}_R(C_2)$.

We prove now that $S \otimes_R P$ and $S \otimes_R X$ are $\text{Hom}_S(-, \text{Add}_S(C_2))$-exact. Let $I$ be a set. Then, the complex $\text{Hom}_S(S \otimes_R P, S \otimes_R C_1^{(I)}) \cong \text{Hom}_R(P, \text{Hom}_S(S, S \otimes_R C_1^{(I)})) \cong \text{Hom}_R(P, S \otimes_R C_1^{(I)})$ is exact since $S \otimes_R C_1^{(I)} \in \text{Add}_R(C_1)$. Similarly, $S \otimes_R X$ is $\text{Hom}_S(-, \text{Add}_S(C_2))$-exact.

2. This assertion follows from Proposition 3.6. We only need to note that $S$ is weakly $C$-compatible since $S_R$ is flat and $S \otimes_R C_1 \in \text{Add}_R(C_2)$.

We end this section with an example of a w-tilting module that is neither projective nor injective.

**Example 3.9** Take $R$ and $C_2$ as in example 3.1. So, by Corollary 3.8, $C = \begin{pmatrix} C_2 \\ C_2 + C_2 \end{pmatrix}$ is a w-tilting $T(R)$-module. By Lemma 2.5, $C$ is not projective since $C_2$ is not and it is not injective since the map $\tilde{\varphi}^C : C_2 \to C_2 \oplus C_2$ is not surjective.

Moreover, by [1, Proposition 2.6], $\text{gl.dim}(T(R)) = \text{gl.dim}(R) + 1 \leq 2$. So, if $0 \to T(R) \to E^0 \to E^1 \to E^2 \to 0$ is an injective resolution of $T(R)$, then $C^1 = E^0 \oplus E^1 \oplus E^2$ is a w-tilting $T(R)$-module. Note that $T(R)$ has at least three w-tilting modules, $C^1$, $C^2 = T(R)$ and $C^3 = C$.

### 4 Relative Gorenstein projective modules

In this section, we describe $G_C$-projective modules over $T$. Then we use this description to study when the class of $G_C$-projective $T$-modules is a special precovering class.
Clearly the functor $p$ preserves projective module. So we start by studying when the functor $p$ also preserves relative Gorenstein projective modules. But first we need the following

**Lemma 4.1** Let $C = p(C_1, C_2)$ be a $T$-module and $U$ be weakly $C$-compatible.

1. If $M_1 \in G_{C_1}P(A)$, then $\left( \begin{array}{c} M_1 \\ U \otimes_A M_1 \end{array} \right) \in G_CP(T)$.

2. If $M_2 \in G_{C_2}P(B)$, then $\left( \begin{array}{c} 0 \\ M_2 \end{array} \right) \in G_CP(T)$.

**Proof.** 1. Suppose that $M_1 \in G_{C_1}P(A)$. There exists a $\text{Hom}_A(-, \text{Add}_A(C_1))$-exact exact sequence

$$X_1: \cdots \rightarrow P_1^1 \rightarrow P_1^0 \rightarrow C_1^0 \rightarrow C_1^1 \rightarrow \cdots$$

where the $P_i^i$s are all projective, $C_1^i \in \text{Add}_A(C_1) \forall i$ and $M_1 \cong \text{Im}(P_1^0 \rightarrow C_1^0)$. Using the fact that $U$ is weakly $C$-compatible, we get that the complex $U \otimes_A X_1$ is exact in $B$-$\text{Mod}$, which implies that the complex $p(X_1, 0)$:

$$\cdots \rightarrow \left( \begin{array}{c} P_1^1 \\ U \otimes_A P_1^1 \end{array} \right) \rightarrow \left( \begin{array}{c} P_1^0 \\ U \otimes_A P_1^0 \end{array} \right) \rightarrow \left( \begin{array}{c} C_1^0 \\ U \otimes_A C_1^0 \end{array} \right) \rightarrow \left( \begin{array}{c} C_1^1 \\ U \otimes_A C_1^1 \end{array} \right) \rightarrow \cdots$$

is exact with $\left( \begin{array}{c} M_1 \\ U \otimes_A M_1 \end{array} \right) \cong \text{Im}(\left( \begin{array}{c} P_1^0 \\ U \otimes_A P_1^0 \end{array} \right) \rightarrow \left( \begin{array}{c} C_1^0 \\ U \otimes_A C_1^0 \end{array} \right))$. Clearly, $p(P_1^i, 0) = \left( \begin{array}{c} P_1^i \\ U \otimes_A P_1^i \end{array} \right) \in \text{Proj}(T)$ and $p(C_1^i, 0) = \left( \begin{array}{c} C_1^i \\ U \otimes_A C_1^i \end{array} \right) \in \text{Add}_T(C) \forall i \in \mathbb{N}$ by Lemma 2.3(1) and Lemma 2.7(1). If $X \in \text{Add}_T(C)$, then $X_1 \in \text{Add}_A(C_1)$ by Lemma 2.7(1) and using the adjointness, we get that the complex $\text{Hom}_T(p(X_1, 0), X) \cong \text{Hom}_A(X_1, X_1)$ is exact. Hence $\left( \begin{array}{c} M_1 \\ U \otimes_A M_1 \end{array} \right)$ is $G_C$-projective.

2. Suppose that $M_2$ is $G_{C_2}$-projective. There exists a $\text{Hom}_B(-, \text{Add}_B(C_2))$-exact exact sequence

$$X_2: \cdots \rightarrow P_2^1 \rightarrow P_2^0 \rightarrow C_2^0 \rightarrow C_2^1 \rightarrow \cdots$$

where the $P_i^i$s are all projective, $C_2^i \in \text{Add}_B(C_2) \forall i$ and $M_2 \cong \text{Im}(P_2^0 \rightarrow C_2^0)$. Clearly the complex

$$p(0, X_2) : \cdots \rightarrow \left( \begin{array}{c} 0 \\ P_2^1 \end{array} \right) \rightarrow \left( \begin{array}{c} 0 \\ P_2^0 \end{array} \right) \rightarrow \left( \begin{array}{c} 0 \\ C_2^0 \end{array} \right) \rightarrow \left( \begin{array}{c} 0 \\ C_2^1 \end{array} \right) \rightarrow \cdots$$

is exact with $\left( \begin{array}{c} 0 \\ M_2 \end{array} \right) \equiv \text{Im}(\left( \begin{array}{c} 0 \\ P_2^0 \end{array} \right) \rightarrow \left( \begin{array}{c} 0 \\ C_2^0 \end{array} \right))$, $p(0, P_2^i) = \left( \begin{array}{c} 0 \\ P_2^i \end{array} \right) \in \text{Proj}(T)$ and $p(0, C_2^i) = \left( \begin{array}{c} 0 \\ C_2^i \end{array} \right) \in \text{Add}_T(C) \forall i$, by Lemma 2.3(1) and Lemma 2.7(1). Let...
$X \in \text{Add}_T(C)$. Then, by Lemma 2.7(1), \(X = p(X_1, X_2)\) where \(X_1 \in \text{Add}_A(C_1)\) and \(X_2 \in \text{Add}_B(C_2)\). Using adjointness, we get that

\[
\text{Hom}_T(p(0, X_2), X) \cong \text{Hom}_B(X_2, U \otimes A X_1) \oplus \text{Hom}_B(X_2, X_2)
\]

The complex \(\text{Hom}_B(X_2, X_2)\) is exact and since \(U\) is weakly \(C\)-compatible, the complex \(\text{Hom}_B(X_2, U \otimes A X_1)\) is also exact. This means that \(\text{Hom}_T(p(0, X_2), X)\) is exact as well and \(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}\) is \(G_C\)-projective.

**Proposition 4.2** Let \(C = p(C_1, C_2)\) be a \(T\)-module. If \(U\) is weakly \(C\)-compatible, then the functor \(p\) sends \(G_{(C_1, C_2)}\)-projectives to \(G_C\)-projectives. The converse holds provided that \(C_1\) and \(C_2\) are \(w\)-tilting.

**Proof.** Note that \(p(M_1, M_2) = \begin{pmatrix} M_1 \\ U \otimes A M_1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ M_2 \end{pmatrix}\).

So this direction follows from Lemma 4.1 and [6, Proposition 2.5].

Conversely, assume that \(C_1\) and \(C_2\) are \(w\)-tilting. By Proposition 3.5, it suffices to prove that \(\text{Tor}_A^1(U, G_{C_1} P(A)) = 0 = \text{Ext}^1_B(G_{C_2} P(B), U \otimes A \text{Add}_A(C_1))\).

Let \(G_1 \in G_{C_1} P(A)\). By [6, Corollary 2.13], there exists a \(\text{Hom}_A(-, \text{Add}_A(C_1))\)-exact exact sequence \(0 \to L_1 \xrightarrow{i} P_1 \to G_1 \to 0\), where \(A P_1\) is projective and \(L_1\) is \(G_{C_1}\)-projective. Note that \(A, C_1 \in G_{C_1} P(A)\) and \(B, C_2 \in G_{C_2} P(B)\) by Lemma 2.4. Then \(rT = p(A, B)\) and \(C = p(C_1, C_2)\) are \(G_C\)-projective, which imply by Lemma 2.4 that \(C\) is \(w\)-tilting. Moreover \(\begin{pmatrix} L_1 \\ U \otimes A L_1 \end{pmatrix} = p(L_1, 0)\) is also \(G_C\)-projective and by [6, Corollary 2.13] there exists a short exact sequence

\[
0 \to \begin{pmatrix} L_1 \\ U \otimes A L_1 \end{pmatrix} \to \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \xrightarrow{\varphi_X} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \xrightarrow{\varphi_H} 0
\]

where \(X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \text{Add}_T(C)\) and \(H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \) is \(G_C\)-projective.

Since \(X_1 \in \text{Add}_A(C_1)\), we have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & L_1 & \xrightarrow{i} & P_1 & \xrightarrow{=} & G_1 & \to & 0 \\
0 & \to & L_1 & \xrightarrow{=} & X_1 & \xrightarrow{=} & H_1 & \to & 0
\end{array}
\]
So if we apply the functor $U \otimes_A -$ to the above diagram, we get the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
U \otimes_A L_1 & \xrightarrow{1_U \otimes i} & U \otimes_A P_1 & \xrightarrow{U \otimes \varphi} & U \otimes_A G_1 & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
U \otimes_A L_1 & \rightarrow & U \otimes_A X_1 & \rightarrow & U \otimes_A H_1 & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & U \otimes_A L_1 & \rightarrow & U \otimes_A X_2 & \rightarrow & H_2 & \rightarrow 0
\end{array}
\]

The commutativity of this diagram implies that the map $1_U \otimes i$ is injective, and since $P_1$ is projective, $\text{Tor}_1^A(U, G_1) = 0$.

Now let $G_2 \in G_{C_2} P(B)$ and $Y_2 \in \text{Add}_A(C_1)$. By hypothesis, \[\begin{pmatrix} 0 \\ G_2 \end{pmatrix} = \text{p}(0, G_2) \] is $G_{C_1}$-projective and by Lemma 2.7, $\text{Ext}_1^B(G_2, U \otimes Y_1) = \text{Ext}_1^T(\begin{pmatrix} 0 \\ G_2 \end{pmatrix}, U \otimes Y_1) = 0$ by Lemma 2.6 and 6 Proposition 2.4.

**Theorem 4.3** Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \varphi^M$ and $C = \text{p}(C_1, C_2)$ be two $T$-modules. If $U$ is $C$-compatible, then the following assertions are equivalent:

1. $M$ is $G_C$-projective.
2. (i) $\varphi^M$ is injective.
   (ii) $M_1$ is $G_{C_1}$-projective and $M_2 := \text{Coker} \varphi^M$ is $G_{C_2}$-projective.

In this case, if $C_2$ is $\Sigma$-self-orthogonal, then $U \otimes_A M_1$ is $G_{C_2}$-projective if and only if $M_2$ is $G_{C_2}$-projective.

**Proof.** 2. $\Rightarrow$ 1. Since $\varphi^M$ is a injective, there exists an exact sequence in $T$-Mod

\[0 \rightarrow \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} \rightarrow M \rightarrow \begin{pmatrix} 0 \\ M_2 \end{pmatrix} \rightarrow 0\]

Note that $\begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ M_2 \end{pmatrix}$ are $G_C$-projective $T$-module by Lemma 4.1.

So, $M$ is $G_C$-projective by [6 Proposition 2.5].

1. $\Rightarrow$ 2. There exists a $\text{Hom}_T(\_, \text{Add}_T(C))$-exact sequence in $T$-Mod

\[X = \cdots \rightarrow \begin{pmatrix} P_1^1 \\ P_2^1 \end{pmatrix} \varphi^{P_1} \rightarrow \begin{pmatrix} P_1^0 \\ P_2^0 \end{pmatrix} \varphi^{P_0} \rightarrow \begin{pmatrix} C_1^1 \\ C_2^1 \end{pmatrix} \varphi^{C_1} \rightarrow \begin{pmatrix} C_1^0 \\ C_2^0 \end{pmatrix} \varphi^{C_0} \rightarrow \cdots\]
where \( C^i = \left( \begin{array}{l} C^i_1 \\ C^i_2 \end{array} \right) \in \text{Add}_T(C) \), \( P^i = \left( \begin{array}{l} P^i_1 \\ P^i_2 \end{array} \right) \in \text{Proj}(T) \) \( \forall i \in \mathbb{N} \), and such that \( M \cong \text{Im}(P^0 \to C^0) \). Then, we get the exact sequence

\[
\mathbf{X}_1 = \cdots \to P^1_1 \to P^0_1 \to C^0_1 \to C^1_1 \to \cdots
\]

where \( C^i_1 \in \text{Add}_A(C_1) \), \( P^i_1 \in \text{Proj}(A) \) \( \forall i \in \mathbb{N} \) by Lemma 2.7(1) and Lemma 2.5(1), and such that \( M_1 \cong \text{Im}(P^0_1 \to C^0_1) \). Since \( U \) is \( C \)-compatible, the complex \( U \otimes_A \mathbf{X}_1 \) is exact with\( U \otimes_A M_1 \cong \text{Im}(U \otimes_A P^0_1 \to U \otimes_A C^0_1) \). If \( \iota_1 : M_1 \to C^0_1 \) and \( \iota_2 : M_2 \to C^0_2 \) are the inclusions, then \( 1_U \otimes \iota_1 \) is injective and the following diagram commutes:

\[
\begin{array}{cccc}
U \otimes_A M_1 & \xrightarrow{\iota_1 \otimes \iota_1} & U \otimes_A C^0_1 \\
\varphi^M & \downarrow & \varphi^{C^0} \\
M_2 & \xrightarrow{\iota_2} & C^0_2
\end{array}
\]

By Lemma 2.7(1), \( \varphi^{C^0} \) is injective, then \( \varphi^M \) is also injective. For every \( i \in \mathbb{N} \), \( \varphi^{P^i} \) and \( \varphi^{C^i} \) are injective by Lemma 2.5 and Lemma 2.7(1). Then the following diagram with exact columns

\[
\begin{array}{cccccccc}
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
\cdots & \xrightarrow{\varphi^{P^1}} & U \otimes_A P^1_1 & \xrightarrow{\varphi^{P^0}} & U \otimes_A P^0_1 & \xrightarrow{\varphi^{C^0}} & U \otimes_A C^0_1 & \xrightarrow{\varphi^{C^1}} & C^0_1 \\
\cdots & \xrightarrow{\varphi^{P^2}} & P^1_2 & \xrightarrow{\varphi^{P^1}} & P^0_2 & \xrightarrow{\varphi^{C^0}} & C^0_2 & \xrightarrow{\varphi^{C^1}} & C^1_2 \\
\cdots & \xrightarrow{\varphi^{P^3}} & \cdots & \xrightarrow{\varphi^{P^2}} & \cdots & \xrightarrow{\varphi^{C^1}} & \cdots & \xrightarrow{\varphi^{C^0}} & \cdots \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

is commutative. Since the first row and the second row are exact, we get the exact sequence of \( B \)-modules

\[
\mathbf{X}_2 : \cdots \to P^1_2 \to P^0_2 \to C^0_2 \to C^1_2 \to \cdots
\]

where \( P^i_2 \in \text{Proj}(B) \), \( C^i_2 \in \text{Add}_B(C_2) \) by Lemma 2.5 and Lemma 2.7(1), and such that \( M_2 = \text{Im}(P^0_2 \to C^0_2) \). It remains to see that \( \mathbf{X}_1 \) and \( \mathbf{X}_2 \)
are \( \text{Hom}_A(\cdot, \text{Add}(C_1)) \)-exact and \( \text{Hom}_B(\cdot, \text{Add}_B(C_2)) \)-exact, respectively. Let \( X_1 \in \text{Add}_A(C_1) \) and \( X_2 \in \text{Add}_B(C_2) \). Then \( p(X_1, 0) = \left( \begin{array}{c} X_1 \\ U \otimes_A X_1 \end{array} \right) \in \text{Add}_T(C) \) and \( p(0, X_2) = \left( \begin{array}{c} 0 \\ X_2 \end{array} \right) \in \text{Add}_T(C) \) by Lemma 2.7(1). So, by using adjointness, we get that \( \text{Hom}_B(X_2, X_2) \cong \text{Hom}_T(X, \left( \begin{array}{c} 0 \\ X_2 \end{array} \right)) \) is exact. Using adjointness again we get that
\[
\text{Hom}_T(X, \left( \begin{array}{c} 0 \\ U \otimes_A X_1 \end{array} \right)) \cong \text{Hom}_B(X_2, U \otimes_A X_1)
\]
and
\[
\text{Hom}_T(X, \left( \begin{array}{c} X_1 \\ 0 \end{array} \right)) \cong \text{Hom}_A(X_1, X_1).
\]
Note that \( C^i \cong p(C_1^i, \overline{C}_2^i) \) by Lemma 2.7(1). Hence \( \text{Ext}_T^1(C^i, \left( \begin{array}{c} 0 \\ U \otimes_A X_1 \end{array} \right)) \cong \text{Ext}_T^1(\overline{C}_2^i, U \otimes_A X_1) = 0 \) by Lemma 2.6. So, if we apply the functor \( \text{Hom}_T(X, \cdot) \) to the sequence
\[
0 \rightarrow \left( \begin{array}{c} 0 \\ U \otimes_A X_1 \end{array} \right) \rightarrow \left( \begin{array}{c} X_1 \\ U \otimes_A X_1 \end{array} \right) \rightarrow \left( \begin{array}{c} X_1 \\ 0 \end{array} \right) \rightarrow 0,
\]
we get the following exact sequence of complexes
\[
0 \rightarrow \text{Hom}_B(\overline{X}_2, U \otimes_A X_1) \rightarrow \text{Hom}_T(X, \left( \begin{array}{c} X_1 \\ U \otimes_A X_1 \end{array} \right)) \rightarrow \text{Hom}_A(X_1, X_1) \rightarrow 0.
\]
Since \( U \) is \( C \)-compatible, it follows that \( \text{Hom}_B(\overline{X}_2, U \otimes_A X_1) \) is exact and since \( C \) is \( w \)-tilting, \( \text{Hom}_T(X, \left( \begin{array}{c} X_1 \\ U \otimes_A X_1 \end{array} \right)) \) is also exact. Thus \( \text{Hom}_A(X_1, X_1) \) is exact and the proof is finished.

The following consequence of the above theorem gives the converse of Proposition 3.6.

**Corollary 4.4** Let \( C = p(C_1, C_2) \) and assume that \( U \) is \( C \)-compatible. Then \( C \) is \( w \)-tilting if and only if \( C_1 \) and \( C_2 \) are \( w \)-tilting.

**Proof.** An easy application of Proposition 2.4 and Theorem 4.3 on the \( T \)-modules \( C = \left( \begin{array}{c} C_1 \\ (U \otimes_A C_1) \oplus C_2 \end{array} \right) \) and \( T = \left( \begin{array}{c} A \\ U \oplus B \end{array} \right) \).

One would like to know if every \( w \)-tilting \( T \)-module has the form \( p(C_1, C_2) \) where \( C_1 \) and \( C_2 \) are \( w \)-tilting. The following example gives a negative answer to this question.
Example 4.5 Let $R$ be a quasi-Frobenius ring and $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$. Consider the exact sequence of $T$-modules

$$0 \to T \to \begin{pmatrix} R \oplus R \\ R \oplus R \end{pmatrix} \to \begin{pmatrix} R \\ 0 \end{pmatrix} \to 0.$$ 

By Lemma 2.5, $I^0 = \begin{pmatrix} R \oplus R \\ R \oplus R \end{pmatrix}$ and $I^1 = \begin{pmatrix} R \\ 0 \end{pmatrix}$ are both injective $T(R)$-modules. Note that $T(R)$ is noetherian ([17, Proposition 1.7]) and then we can see that $C := I^0 \oplus I^1$ is a $w$-tilting $T(R)$-module but does not have the form $p(C_1, C_2)$ where $C_1$ and $C_2$ are $w$-tilting by Lemma 2.7 since $I^1 \in \text{Add}_{T(R)}(C)$ and $\varphi^{I^1}$ is not injective.

As an immediate consequence of Theorem 4.3, we have the following.

Corollary 4.6 Let $R$ be a ring and $T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$. If $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ and $C = p(C_1, C_1)$ are two $T(R)$-modules with $C_1$ $\Sigma$-self-orthogonal, then the following assertions are equivalent:

1. $M$ is $G_C$-projective $T(R)$-module
2. $M_1$ and $\overline{M}_2$ are $G_{C_1}$-projective $R$-modules and $\varphi^M$ is injective
3. $M_1$ and $M_2$ are $G_{C_1}$-projective $R$-modules and $\varphi^M$ is injective

An Artin algebra $\Lambda$ is called Cohen-Macaulay free, or simply, CM-free if any finitely generated Gorenstein projective module is projective. The authors in [10], extended this definition to arbitrary rings and defined strongly CM-free as rings over which every Gorenstein projective module is projective. Now, we introduce a relative notion of these rings and give a characterization of when $T$ is such rings.

Definition 4.7 Let $R$ be a ring. Given an $R$-module $C$, $R$ is called CM-free (relative to $C$) if $G_C P(R) \cap R$-$\text{mod} = \text{add}_R(C)$ and it is called strongly CM-free (relative to $C$) if $G_C P(R) = \text{Add}_R(C)$.

Remark 4.8 Let $R$ be a ring and $C$ a $\Sigma$-self-orthogonal $R$-module. Then $\text{Add}_R(C) \subseteq G_C P(R)$ and $\text{add}_R(C) \subseteq G_C P(R) \cap R$-$\text{mod}$ by [10] Proposition 2.5, 2.6 and Corollary 2.10, then $R$ is CM-free (relative to $C$) if and only if every finitely generated $G_C$-projective is in $\text{add}_R(C)$ and it is strongly CM-free (relative to $C$) if every $G_C$-projective is in $\text{Add}_R(C)$. 

24
Using the above results we refine and extend \cite[Theorem 4.1]{10} to our setting. Note that the condition $B$ is left Gorenstein regular is not needed.

**Proposition 4.9** Let $A C_1$ and $B C_2$ be $\Sigma$-self-orthogonal, and $C = p(C_1, C_2)$. Assume that $U$ is weakly $C$-compatible and consider the following assertions:

1. $T$ is (strongly) CM-free relative to $C$.

2. $A$ and $B$ are (strongly) CM-free relative to $C_1$ and $C_2$, respectively.

Then $1. \Rightarrow 2.$ If $U$ is $C$-compatible, then $1. \Leftrightarrow 2.$

**Proof.** We only prove the the result for relative strongly CM-free, since the the case of relative CM-free is similar.

1. $\Rightarrow 2.$ By the remark above, we only need to prove that $G_{C_1} P(A) \subseteq \text{Add}_A(C_1)$ and $G_{C_2} P(B) \subseteq \text{Add}_B(C_2)$. Let $M_1$ be a $G_{C_1}$-projective $A$-module and $B M_2$ a $G_{C_2}$-projective $B$-module. By the assumption and Proposition 4.2, $p(M_1, M_2) \in G_{C_1} P(T) = \text{Add}_C(T)$. Hence $M_1 \in \text{Add}_A(C_1)$ and $M_2 \in \text{Add}_B(C_2)$ by Lemma 2.7.

2. $\Rightarrow 1.$ Assume $U$ is $C$-compatible. Clearly, $C$ is $\Sigma$-self-orthogonal, then by Remark above, we only need to prove that $G_{C_1} P(T) \subseteq \text{Add}_T(C)$. Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \varphi^M$ be a $G_C$-projective $T$-module. By the assumption and Theorem 4.3, $M_1 \in G_{C_1} P(A) = \text{Add}_A(C_1)$ and $\overline{M_2} \in G_{C_2} P(B) = \text{Add}_B(C_2)$ and the map $\varphi^M$ is injective. By the assumption, we can easily see that $\text{Ext}^{1 \geq 1}_B (U \otimes_A M_1, \overline{M_2}) = 0$. So the map $0 \rightarrow U \otimes_A M_1 \varphi^M \rightarrow M_2 \rightarrow \overline{M_2} \rightarrow 0$ splits. Hence $M \cong p(M_1, M_2) \in \text{Add}_T(C)$ by Lemma 2.7.

Our aim now is to study special $G_{C} P(T)$-precovers in $T\text{-Mod}$. We start with the following result.

**Proposition 4.10** Let $C = p(C_1, C_2)$ be $w$-tilting, $U$ $C$-compatible, $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \varphi^M$ and $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \varphi^G$ two $T$-modules with $G$ $G_C$-projective. Then

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : G \rightarrow M$$

is a special $G_{C_1} P(T)$-precover if and only if

(i) $G_1 \xrightarrow{f_1} M_1$ is a special $G_{C_1} P(A)$-precover.

(ii) $G_2 \xrightarrow{f_2} M_2$ is surjective with its kernel lies in $G_{C_2} P(B)^{\perp_1}$.
In this case, if $G_2 \in G_{C_2}P(B)$, then $G_2 \xrightarrow{f_2} M_2$ is a special $G_{C_2}P(B)$-precover.

**Proof.** First of all, let $K = \text{Ker}f = \left( \begin{array}{c} K_1 \\ K_2 \end{array} \right)$ and note that, since $C_1$ is w-tilting, $\text{Tor}_1^A(U, H_1) = 0$ for every $H_1 \in G_{C_1}P(A)$ by Proposition 3.5(1).

$\Rightarrow$ Since $f$ is surjective, so are $f_1$ and $f_2$. Let $H_1 \in G_{C_1}P(A)$ and $H_2 \in G_{C_2}P(B)$. Then $\left( \begin{array}{c} H_1 \\ U \otimes_A H_1 \end{array} \right) \cdot \left( \begin{array}{c} 0 \\ H_2 \end{array} \right) \in GCP(T)$ by Theorem 4.3. Using Lemma 2.6 and the fact that $K$ lies in $G_{C_1}P(R)^{⊥1}$, we get that

$$\text{Ext}_1^A(H_1, K_1) \cong \text{Ext}_T^1( \left( \begin{array}{c} H_1 \\ U \otimes_A H_1 \end{array} \right), K) = 0$$

and

$$\text{Ext}_B^1(H_2, K_2) \cong \text{Ext}_T^1( \left( \begin{array}{c} 0 \\ H_2 \end{array} \right), K) = 0.$$

It remains to see that $G_1 \in G_{C_1}P(A)$, which is true by Theorem 4.3 since $G$ is $G_C$-projective.

$\Leftarrow$ The morphism $f$ is surjective since $f_1$ and $f_2$ are. So we only need to prove that $K$ lies in $G_{C_1}P(R)^{⊥1}$. Let $H \in G_{C_1}P(R)$. By Theorem 4.3 we have the short exact sequence of $T$-modules

$$0 \to \left( \begin{array}{c} H_1 \\ U \otimes_A H_1 \end{array} \right) \to H \to \left( \begin{array}{c} 0 \\ \mathcal{P}_2 \end{array} \right) \to 0$$

where $H_1$ is $G_{C_1}$-projective and $\mathcal{P}_2$ is $G_{C_2}$-projective. So by hypothesis and Lemma 2.6 we get that $\text{Ext}_T^1( \left( \begin{array}{c} H_1 \\ U \otimes_A H_1 \end{array} \right), K) = 0$ and $\text{Ext}_T^1( \left( \begin{array}{c} 0 \\ \mathcal{P}_2 \end{array} \right), K) = 0$. Then, the exactness of this sequence

$$\text{Ext}_T^1( \left( \begin{array}{c} H_1 \\ U \otimes_A H_1 \end{array} \right), K) \to \text{Ext}_T^1(H, K) \to \text{Ext}_T^1( \left( \begin{array}{c} 0 \\ \mathcal{P}_2 \end{array} \right), K)$$

implies that $\text{Ext}_T^1(H, K) = 0$. ■

**Theorem 4.11** Let $C = p(C_1, C_2)$ be w-tilting and $U$ $C$-compatible. Then the class $G_{C_1}P(T)$ is special precovering in $T$-Mod if and only if the classes $G_{C_1}P(A)$ and $G_{C_2}P(B)$ are special precovering in $A$-Mod and $B$-Mod, respectively.

**Proof.** $\Rightarrow$ Let $M_1$ be an $A$-module and $\left( \begin{array}{c} G_1 \\ G_2 \end{array} \right) \xrightarrow{\varrho_G} \left( \begin{array}{c} M_1 \\ 0 \end{array} \right)$ be a special $G_{C_1}P(T)$-precover in $T$-Mod. Then by Proposition 4.10 $G_1 \to M_1$ is a special $G_{C_1}P(A)$-precover in $A$-Mod.
Let $M_2$ be a $B$-module and \( \begin{pmatrix} 0 \\ f_2 \end{pmatrix} : \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ M_2 \end{pmatrix} \) be a special $G_C P(T)$-precover in $T$-Mod. By Proposition 4.10, $G_1 \rightarrow 0$ is a special $G_{C_1} P(A)$-precover.

Then $\text{Ext}^1_A(G_{C_1} P(A), G_1) = 0$. On the other hand, by [6, Proposition 2.8], there exists an exact sequence of $A$-modules

$$0 \rightarrow G_1 \rightarrow X_1 \rightarrow H_1 \rightarrow 0$$

where $X_1 \in \text{Add}_A(C_1)$ and $H_1$ is $G_{C_1}$-projective. But this sequence splits, since $\text{Ext}^1_A(H_1, G_1) = 0$, which implies that $G_1 \in \text{Add}_A(C_1)$. Let $K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \varphi^K$ be the kernel of $\begin{pmatrix} 0 \\ f_2 \end{pmatrix}$. Note that $K_1 = G_1$. So, there exists a commutative diagram

Using the snake lemma, there exists an exact sequence of $B$-modules

$$0 \rightarrow K_2 \rightarrow G_2 \rightarrow M_2 \rightarrow 0$$

where $G_2$ is $G_{C_2}$-projective by Theorem 4.3. It remains to see that $K_2$ lies in $G_{C_2} P(B)^{1.1}$. Let $H_2 \in G_{C_2} P(B)$. Then $\text{Ext}^1_B(H_2, K_2) = 0$ by Proposition 4.10 and $\text{Ext}^2_B(H_2, U \otimes_A G_1) = 0$ by Proposition 3.5(2). From the above diagram, $\varphi^K$ is injective. So, if we apply the functor $\text{Hom}_B(H_2, -)$ to the short exact sequence

$$0 \rightarrow U \otimes_A G_1 \rightarrow K_2 \rightarrow K_2 \rightarrow 0,$$

we get an exact sequence

$$\text{Ext}^1_B(H_2, K_2) \rightarrow \text{Ext}^1_B(H_2, K_2) \rightarrow \text{Ext}^2_B(H_2, U \otimes_A G_1)$$
5 RELATIVE GLOBAL GORENSTEIN DIMENSION

which implies that \( \text{Ext}^1_B(H_2, K_2) = 0 \).

\( \Leftarrow \) Note that the functor \( U \otimes_A - : A\text{-Mod} \to B\text{-Mod} \) is \( GC_1 P(A) \)-exact since \( \text{Tor}^A_i(U, GC_1 P(A)) = 0 \) by Proposition 3.3. So this direction follows by [21, Theorem 1.1.] since \( GC_1 P(T) = \{ M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \in T\text{-Mod} | M_1 \in GC_1 P(A), M_2 \in GC_2 P(B) \text{ and } \varphi^M \text{ is injective} \} \) by Theorem 4.3. 

Corollary 4.12 Let \( R \) be a ring, \( T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix} \) and \( C = p(C_1, C_1) \) a w-tilting \( T(R) \)-module. Then \( GC_1 P(T(R)) \) is a special precovering class if and only if \( GC_1 P(R) \) is a special precovering class.

5 Relative global Gorenstein dimension

In this section, we investigate \( GC \)-projective dimension of \( T \)-modules and the left \( GC \)-projective global dimension of \( T \).

Let \( R \) be a ring. Recall ([34]) that a module \( M \) is said to have \( GC \)-projective dimension less than or equal to \( n \), \( GC - \text{pd}(M) \leq n \), if there is an exact sequence

\[ 0 \to G_n \to \cdots \to G_0 \to M \to 0 \]

with \( G_i \in GC P(R) \) for every \( i \in \{0, \cdots, n\} \). If \( n \) is the least nonnegative integer for which such a sequence exists then \( GC - \text{pd}(M) = n \), and if there is no such \( n \) then \( GC - \text{pd}(M) = \infty \).

The left \( GC \)-projective global dimension of \( R \) is defined as:

\[ GC - PD(R) = \sup \{ GC - \text{pd}(M) | M \text{ is an } R\text{-module} \} \]

Lemma 5.1 Let \( C = p(C_1, C_2) \) be w-tilting and \( U \) \( C \)-compatible.

1. \( GC_2 - \text{pd}(M_2) = GC - \text{pd}( \begin{pmatrix} 0 \\ M_2 \end{pmatrix} ) \).

2. \( GC_1 - \text{pd}(M_1) \leq GC - \text{pd}( \begin{pmatrix} M_1 \\ U \otimes_A M_1 \end{pmatrix} ) \), and the equality holds if \( \text{Tor}^A_{r \geq 1}(U, M_1) = 0 \).

Proof. 1. Let \( n \in \mathbb{N} \) and consider an exact sequence of \( B \)-modules

\[ 0 \to K^n_2 \to G^{n-1}_2 \to \cdots \to G^0_2 \to M_2 \to 0 \]
where each $G^2_i$ is $G_{C_2}$-projective. Thus, there exists an exact sequence of $T$-modules

$$0 \rightarrow \left( \begin{array}{c} 0 \\ K^2_n \end{array} \right) \rightarrow \left( \begin{array}{c} 0 \\ G_{C_2}^{n-1} \end{array} \right) \rightarrow \cdots \rightarrow \left( \begin{array}{c} 0 \\ G^2_2 \end{array} \right) \rightarrow \left( \begin{array}{c} 0 \\ M_2 \end{array} \right) \rightarrow 0$$

where each $\left( \begin{array}{c} 0 \\ G^2_i \end{array} \right)$ is $G_C$-projective by Theorem 4.3. Again, by Theorem 4.3, $\left( \begin{array}{c} 0 \\ K^2_n \end{array} \right)$ is $G_C$-projective if and only if $K^2_n$ is $G_{C_1}$-projective which means that $G_C - \text{pd}(\left( \begin{array}{c} 0 \\ M_2 \end{array} \right)) \leq n$ if and only if $G_{C_2} - \text{pd}(M_2) \leq n$ by [9, Theorem 3.8]. Hence $G_C - \text{pd}(\left( \begin{array}{c} 0 \\ M_2 \end{array} \right)) = G_{C_2} - \text{pd}(M_2)$.

2. We may assume that $n = G_{C_1} - \text{pd}(\left( \begin{array}{c} M_1 \\ U \otimes_A M_1 \end{array} \right)) < \infty$. By Definition, there exists an exact sequence of $T$-modules

$$0 \rightarrow G^n \rightarrow G^{n-1} \rightarrow \cdots \rightarrow G^0 \rightarrow \left( \begin{array}{c} M_1 \\ U \otimes_A M_1 \end{array} \right) \rightarrow 0$$

where each $G^i = \left( \begin{array}{c} G^1_i \\ G^2_i \end{array} \right)_{\phi^i}$ is $G_C$-projective. Thus, there exists an exact sequence of $A$-modules

$$0 \rightarrow G^n_1 \rightarrow G^{n-1}_1 \rightarrow \cdots \rightarrow G^0_1 \rightarrow M_1 \rightarrow 0$$

where each $G^1_i$ is $G_{C_1}$-projective by Theorem 4.3. So, $G_{C_1} - \text{pd}(M_1) \leq n$. Conversely, we prove that $G_{C_1} - \text{pd}(\left( \begin{array}{c} M_1 \\ U \otimes_A M_1 \end{array} \right)) \leq G_{C_1} - \text{pd}(M_1)$. We may assume that $m := G_{C_1} - \text{pd}(M_1) < \infty$. The hypothesis means that if

$$X_1 : 0 \rightarrow K^m_1 \rightarrow P^{m-1}_1 \rightarrow \cdots \rightarrow P^0_1 \rightarrow M_1 \rightarrow 0$$

is an exact sequence of $A$-modules where each $P^i_1$ is projective, then the complex $U \otimes_A X_1$ is exact. Since $C_1$ is $w$-tilting, each $P_i$ is $G_{C_1}$-projective by [6, Proposition 2.11] and then $K^m$ is $G_{C_1}$-projective by [6, Theorem 3.8]. Thus, there exists an exact sequence of $T$-modules

$$0 \rightarrow \left( \begin{array}{c} K^m_1 \\ U \otimes_A K^m_1 \end{array} \right) \rightarrow \left( \begin{array}{c} P^{m-1}_1 \\ U \otimes_A P^{m-1}_1 \end{array} \right) \rightarrow \cdots \rightarrow \left( \begin{array}{c} P^0_1 \\ U \otimes_A P^0_1 \end{array} \right) \rightarrow \left( \begin{array}{c} M_1 \\ U \otimes_A M_1 \end{array} \right) \rightarrow 0$$

where $\left( \begin{array}{c} K^m_1 \\ U \otimes_A K^m_1 \end{array} \right)$ and all $\left( \begin{array}{c} P^i_1 \\ U \otimes_A P^i_1 \end{array} \right)$ are $G_C$-projectives by Theorem 4.3. Therefore, $G_C - \text{pd}(\left( \begin{array}{c} M_1 \\ U \otimes_A M_1 \end{array} \right)) \leq m = G_{C_1} - \text{pd}(M_1)$. ■
Given a $T$-module $C = p(C_1, C_2)$, we introduce a strong notion of $G_{C_2}$-projective global dimension of $B$, which will be crucial when we estimate the $G_C$-projective of a $T$-module and the left global $G_C$-projective dimension of $T$. Set

$$SG_{C_2} - PD(B) = \sup \{G_{C_2} - \text{pd}_B(U \otimes_A G) \mid G \in G_{C_1}P(A)\}.$$

**Remark 5.2**
1. Clearly, $SG_{C_2} - PD(B) \leq G_{C_2} - PD(B)$.

2. Note that $\text{pd}_B(U) = \sup \{\text{pd}_B(U \otimes_A P) \mid A_P \text{ is projective} \}$. Therefore, in the classical case, the strong left global dimension of $B$ is nothing but the projective dimension of $BU$.

**Theorem 5.3** Let $C = p(C_1, C_2)$ be w-tilting, $U C$-compatible, $M = \left( \begin{array}{c} M_1 \\ M_2 \end{array} \right) \varphi^M$ a $T$-module and $SG_{C_2} - PD(B) < \infty$. Then

$$\max \{G_{C_1} - \text{pd}_A(M_1), (G_{C_2} - \text{pd}_B(M_2)) - (SG_{C_2} - PD(B))\}$$

$$\leq G_C - \text{pd}(M) \leq$$

$$\max \{(G_{C_1} - \text{pd}_A(M_1)) + (SG_{C_2} - PD(B)) + 1, G_{C_2} - \text{pd}_B(M_2)\}$$

**Proof.** First of all, note that $C_1$ and $C_2$ are w-tilting by Proposition 4.4 and let $k := SG_{C_2} - PD(B)$.

Let us first prove that

$$\max \{G_{C_1} - \text{pd}(M_1), G_{C_2} - \text{pd}(M_2) - k\} \leq G_C - \text{pd}(M).$$

We may assume that $n := G_C - \text{pd}(M) < \infty$. Then, there exists an exact sequence of $T$-modules

$$0 \to G^n \to G^{n-1} \to \cdots \to G^0 \to M \to 0$$

where each $G^i = \left( \begin{array}{c} G^i_1 \\ G^i_2 \end{array} \right) \varphi^{\psi^i}$ is $G_C$-projective. Thus, there exists an exact sequence of $A$-modules

$$0 \to G^n_1 \to G^{n-1}_1 \to \cdots \to G^0_1 \to M_1 \to 0$$

where each $G^i_1$ is $G_{C_1}$-projective by Theorem 4.3. So, $G_{C_1} - \text{pd}(M_1) \leq n$. By Theorem 4.3 for each $i$, there exists an exact sequence of $B$-modules

$$0 \to U \otimes_A G^i_1 \to G^i_2 \to G^i_2 \to 0$$

30
where $G_2$ is $G_{C_1}$-projective. Then $G_{C_2} - \text{pd}(G_2) = G_{C_2} - \text{pd}(U \otimes_A G_1) \leq k$ by [6, Proposition 3.11]. So, using the exact sequence of $B$-modules

$$0 \to G_2^n \to G_2^{n-1} \to \cdots \to G_2^0 \to M_2 \to 0$$

and [6, Proposition 3.11(4)], we get that $G_{C_2} - \text{pd}(M_2) \leq n + k$.

Next we prove that

$$G_C - \text{pd}(M) \leq \max\{G_{C_1} - \text{pd}(M_1) + k + 1, G_{C_2} - \text{pd}(M_2)\}$$

We may assume that

$$m := \max\{G_{C_1} - \text{pd}(M_1) + k + 1, G_{C_2} - \text{pd}(M_2)\} < \infty.$$ 

Then $n_1 := G_{C_1} - \text{pd}(M_1) < \infty$ and $n_2 := G_{C_2} - \text{pd}(M_2) < \infty$. Since $G_{C_1} - \text{pd}(M_1) = n_1 \leq m - k - 1$, there exists an exact sequence of $A$-modules

$$0 \to G_1^{m-k-1} \to \cdots \to G_1^{n_2-k} \to \cdots \to f_1^{0} f_2^{0} M_1 \to 0$$

where each $G_1^{i}$ is $G_{C_1}$-projective. Since $C_2$ is w-tilting, there exists an exact sequence of $B$-modules $G_2^0 \xrightarrow{f_2^0} M_2 \to 0$ where $G_2^0$ is $G_{C_2}$-projective by [6, Corollary 2.14]. Let $K_1^{i} = \text{Ker} f_1^{i}$ and define the map $f_2^{0} : U \otimes_A G_1^{0} \oplus G_2^0 \to M_2$ to be $(\varphi^M(1_U \otimes f_1^{0})) \oplus g_2^{0}$. Then, we get an exact sequence of $T$-modules

$$0 \to \left( \begin{array}{c} K_1^{1} \\ K_2^{1} \end{array} \right) \xrightarrow{\varphi^{K_1}} \left( \begin{array}{c} G_1^{0} \\ G_2^{0} \end{array} \right) \xrightarrow{\left( \begin{array}{c} f_1^{0} \\ f_2^{0} \end{array} \right)} M \to 0.$$

Similarly, there exists an exact sequence of $B$-modules $G_2^1 \xrightarrow{g_2^{1}} K_2^{1} \to 0$ where $G_2^{1}$ is $G_{C_2}$-projective and then, we get an exact sequence of $T$-modules

$$0 \to \left( \begin{array}{c} K_1^{2} \\ K_2^{2} \end{array} \right) \xrightarrow{\varphi^{K_2}} \left( \begin{array}{c} G_1^{1} \\ G_2^{0} \end{array} \right) \to \left( \begin{array}{c} K_1^{1} \\ K_2^{1} \end{array} \right) \xrightarrow{\varphi^{K_1}} 0.$$

repeat this process, we get the exact sequence of $T$-modules

$$0 \to \left( \begin{array}{c} 0 \\ K_2^{m-k} \end{array} \right) \xrightarrow{G_2^{m-k-1}} \left( \begin{array}{c} G_1^{m-k-1} \\ G_2^{m-k-1} \end{array} \right) \xrightarrow{\left( \begin{array}{c} f_1^{m-k-1} \\ f_2^{m-k-1} \end{array} \right)} \cdots \to \left( \begin{array}{c} G_1^{3} \\ G_2^{2} \end{array} \right) \xrightarrow{\left( \begin{array}{c} f_1^{1} \\ f_2^{1} \end{array} \right)} \left( \begin{array}{c} G_1^{2} \\ G_2^{0} \end{array} \right) \xrightarrow{\left( \begin{array}{c} f_1^{0} \\ f_2^{0} \end{array} \right)} M \to 0.$$
Note that $G_C^2 - \text{pd}((U \otimes A G^1_i) \oplus G^2_i) = G_C^2 - \text{pd}(U \otimes A G^1_i) \leq k$, for every $i \in \{0, \ldots, m - k - 1\}$. So, by \[6, \text{Proposition 3.11(2)}\] and the exact sequence

$$0 \to K_2^{m-k} \to (U \otimes_A G_1^{m-k-1}) \oplus G_2^{m-k-1} \overset{f_2^{m-k-1}}{\to} \cdots \to (U \otimes_A G^1_0) \oplus G^2_0 \overset{f_0^2}{\to} M_2 \to 0$$

we get that $G_C^2 - \text{pd}(K_2^{m-k}) \leq k$. This means that, there exists an exact sequence of $B$-modules

$$0 \to G^m_2 \to \cdots \to G^{m-k+1}_2 \to G^{-k}_2 \to K_2^{-k} \to 0.$$ 

Thus, there exists an exact sequence of $T$-modules

$$0 \to \begin{pmatrix} 0 \\ G^m_2 \end{pmatrix} \to \cdots \to \begin{pmatrix} 0 \\ G^{m-k+1}_2 \end{pmatrix} \to$$

$$\begin{pmatrix} 0 \\ G^{m-k}_2 \end{pmatrix} \to \begin{pmatrix} G^{m-k-1}_1 \\ (U \otimes_A G^{m-k-1}_1) \oplus G^{m-k-1}_2 \end{pmatrix} \overset{\begin{pmatrix} f_1^{m-k-1} \\ f_2^{m-k-1} \end{pmatrix}}{\to} \cdots \to \begin{pmatrix} G^1_i \\ (U \otimes_A G^1_i) \oplus G^2_i \end{pmatrix} \overset{\begin{pmatrix} f_1^0 \\ f_2^0 \end{pmatrix}}{\to} M \to 0$$

By Theorem 4.3 all $\begin{pmatrix} G^1_i \\ (U \otimes_A G^1_i) \oplus G^2_i \end{pmatrix}$ and all $\begin{pmatrix} 0 \\ G^2_0 \end{pmatrix}$ are $G_C$-projectives. Thus, $G_C - \text{pd}(M) \leq m$. 

The following consequence of Theorem 5.3 extends \[10, \text{Proposition 2.8(1)}\] and \[26, \text{Theorem 2.7(1)}\] to the relative setting.

**Corollary 5.4** Let $C = p(C_1, C_2)$ be w-tilting, $U$ $C$-compatible and $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}_T$ a $T$-module. If $SG_{C_2} - PD(B) < \infty$, then $G_C - \text{pd}(M) < \infty$ if and only if $G_{C_1} - \text{pd}(M_1) < \infty$ and $G_{C_2} - \text{pd}(M_2) < \infty$.

The following theorem gives an estimate of the left $G_C$-projective global dimension of $T$.

**Theorem 5.5** Let $C = p(C_1, C_2)$ be w-tilting and $U$ $C$-compatible. Then

$$\max\{G_{C_1} - PD(A), G_{C_2} - PD(B)\}$$

$$\leq G_C - PD(T) \leq$$

$$\max\{G_{C_1} - PD(A) + SG_{C_2} - PD(B) + 1, G_{C_2} - PD(B)\}.$$
Proof. We prove first that \( \max\{G_{C_1} - PD(A), G_{C_2} - PD(B)\} \leq G_C - PD(T) \).
We may assume that \( n := G_C - PD(T) < \infty \). Let \( M_1 \) be an \( A \)-module and \( M_2 \) be a \( B \)-module. Since \( G_C - pd(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}) \leq n \) and \( G_C - pd(\begin{pmatrix} 0 \\ M_2 \end{pmatrix}) \leq n \),
\( G_{C_1} - pd(M_1) \leq n \) and \( G_{C_2} - pd(M_2) \leq n \) by Lemma 5.1. Thus \( G_{C_1} - PD(A) \leq n \) and \( G_{C_2} - PD(B) \leq n \).

Next we prove that
\[ G_C - PD(T) \leq \max\{G_{C_1} - PD(A) + 1 + SG_{C_2} - PD(B), G_{C_2} - PD(B)\} . \]
We may assume that \( m := \max\{G_{C_1} - PD(A) + 1 + SG_{C_2} - PD(B), G_{C_2} - PD(B)\} < \infty \).

Then \( n_1 := G_{C_1} - PD(A) < \infty \) and \( k := SG_{C_2} - PD(B) \leq n_2 := G_{C_2} - PD(B) < \infty \).

Let \( M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \) be a \( T \)-module. By Theorem 5.3,
\[ G_C - pd(M) \leq \max\{n_1 + k + 1, n_2\} \leq m. \]

\[ \Box \]

Corollary 5.6 Let \( C = p(C_1, C_2) \) be \( w \)-tilting and \( U \) \( C \)-compatible. Then
\( G_C - PD(T) < \infty \) if and only if \( G_{C_1} - PD(A) < \infty \) and \( G_{C_2} - PD(B) < \infty \).

Recall that a ring \( R \) is called left Gorenstein regular if the category \( R\)-Mod
is Gorenstein (\cite[Definition 2.1]{10} and \cite[Definition 2.18]{9}).

We know by \cite[Theorem 1.1]{8}, that the following equality holds:
\[ \sup\{Gpd_R(M) \mid M \in R\text{-Mod}\} = \sup\{Gid_R(M) \mid M \in R\text{-Mod}\}. \]

and this common value is called the left global Gorenstein dimension of \( R \),
de-noted by \( l.Ggldim(R) \). As a consequence of \cite[Theorem 2.28]{9}, a ring \( R \) is left
Gorenstein regular if and only if the global Gorenstein dimension of \( R \) is finite.

We shall say that a ring \( R \) is left \( n \)-Gorenstein regular if \( n = l.Ggldim(R) < \infty \).

Enochs, Izurdiaga and Torrecillas, characterized in \cite[Theorem 3.1]{10} when
\( T \) is left Gorenstein regular under the conditions that \( B_U \) has finite projective
dimension and \( U_A \) has finite flat dimension. As a direct consequence of Corollary 5.6
we refine this result.
Corollary 5.7 Assume that $U$ is compatible. Then $T$ is left Gorenstein regular if and only if so are $A$ and $B$.

There are some cases when the estimate in Theorem 5.5 becomes an exact formula which computes left $G_C$-projective global dimension of $T$.

Recall that an injective cogenerator $E$ in $R$-Mod is said to be strong if any $R$-module embeds in a direct sum of copies of $E$.

Corollary 5.8 Let $C = p(C_1, C_2)$ be $w$-tilting and $U C$-compatible.

1. If $U = 0$ then
   $$G_C - PD(T) = \max\{G_{C_1} - PD(A), G_{C_2} - PD(B)\}$$

2. If $A$ is left noetherian and $AC_1$ is a strong injective cogenerator, then
   $$G_C - PD(T) = \begin{cases} G_{C_2} - PD(B) & \text{if } U = 0 \\ \max\{SG_{C_2} - PD(B) + 1, G_{C_2} - PD(B)\} & \text{if } U \neq 0 \end{cases}$$

Proof. 1. Using a similar way as we do in the proof of Theorem 5.3 and 5.5, we can prove this statement. We only need to notice that if $U = 0$, then a $T$-module $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ is $G_C$-projective if and only if $M_1$ is $G_{C_1}$-projective and $M_2$ is $G_{C_2}$-projective (since $\varphi^M$ is always injective and $M_2 = \overline{M_2}$) by Theorem 4.3.

2. Note first that $G_{C_1} - PD(A) = 0$ by [7 Corollary 2.3]. Then the case $U = 0$ follows by 1. Assume that $U \neq 0$. Note that by Theorem 4.3 $\begin{pmatrix} A \\ 0 \end{pmatrix}$ is not $G_C$-projective since $U \neq 0$. Hence $G_{C_2} - PD(B) \geq G_C - pd_T(\begin{pmatrix} A \\ 0 \end{pmatrix}) \geq 1$.

By Theorem 5.5 we have the inequality
   $$G_{C_2} - PD(B) \leq G_C - PD(T) \leq \max\{SG_{C_2} - PD(B) + 1, G_{C_2} - PD(B)\}.$$

So, the case $SG_{C_2} - PD(B) + 1 \leq G_{C_2} - PD(B)$ is clear and we only need to prove the result when $SG_{C_2} - PD(B) + 1 > n := G_{C_2} - PD(B)$. Since $G_{C_2} - pd(U \otimes_A G) \leq G_{C_2} - PD(B) = n$ for every $G \in G_{C_1}P(A), SG_{C_2} - PD(B) = n$. Let $G_1$ be a $G_{C_1}$-projective $A$-module with $G_{C_2} - pd(U \otimes_A G_1) = n$ and consider the following short exact sequence
   $$0 \to \begin{pmatrix} U \otimes_A G_1 \\ 0 \end{pmatrix} \to \begin{pmatrix} G_1 \\ U \otimes_A G_1 \end{pmatrix} \to \begin{pmatrix} G_1 \\ 0 \end{pmatrix} \to 0.$$
5 RELATIVE GLOBAL GORENSTEIN DIMENSION

By Theorem 4.3, \((G_1 \otimes_A G_1)\) is \(G\)-projective and by Lemma 5.1

\[ G_{C} - \text{pd}(G_1 \otimes_A G_1) = G_{C_2} - \text{pd}(U \otimes_A G) = n. \]

Thus by [6, Proposition 3.11(4)]

\[ G_{C} - \text{pd}(G_1) = G_{C} - \text{pd}(U \otimes_A G_1) + 1 = n + 1 = SG_{C_2} - PD(B) + 1. \]

This shows that \(G_{C} - PD(T) = SG_{C_2} - PD(B) + 1\) and the proof is finished.

Corollary 5.9 Let \(R\) be a ring, \(T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}\) and \(C = p(C_1, C_1)\) where \(C_1\) is w-tilting. Then

\[ G_{C} - PD(T(R)) = G_{C_1} - PD(R) + 1. \]

Proof. Note first that \(C\) is w-tilting \(T(R)\)-module, \(R\) is \(C\)-compatible and \(SG_{C_1} - PD(R) = 0\). So, by Theorem 5.5

\[ G_{C_1} - PD(R) \leq G_{C} - PD(T(R)) \leq G_{C_1} - PD(R) + 1. \]

The case \(G_{C_1} - PD(R) = \infty\) is clear. Assume that \(n := G_{C_1} - PD(R) = \infty\).

There exists an \(R\)-module \(M\) with \(G_{C_1} - \text{pd}(M) = n\) and \(\Ext^n_R(M, X) \neq 0\) for some \(X \in \text{Add}_R(C_1)\) by [6, Theorem 3.8]. If we apply the functor \(\text{Hom}_{T(R)}(-, \begin{pmatrix} 0 \\ X \end{pmatrix})\) to the exact sequence of \(T(R)\)-modules

\[ 0 \to \begin{pmatrix} 0 \\ M \end{pmatrix} \to \begin{pmatrix} M \\ M \end{pmatrix}_{1M} \to \begin{pmatrix} M \\ 0 \end{pmatrix} \to 0 \]

we get an exact sequence

\[ \cdots \to \Ext^n_{T(R)}(\begin{pmatrix} M \\ M \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \to \Ext^n_{T(R)}(\begin{pmatrix} 0 \\ M \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \to \]

\[ \Ext^{n+1}_{T(R)}(\begin{pmatrix} M \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \to \Ext^{n+1}_{T(R)}(\begin{pmatrix} M \\ M \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \to \cdots \]

By Lemma 2.6 \(\Ext^{i+1}_{T(R)}(\begin{pmatrix} M \\ M \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \cong \Ext^{i+1}_{R}(M, 0) = 0\). Again by Lemma 2.6 and the above exact sequence,

\[ \Ext^{n+1}_{T(R)}(\begin{pmatrix} M \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \cong \Ext^n_{T(R)}(\begin{pmatrix} 0 \\ M \end{pmatrix}, \begin{pmatrix} 0 \\ X \end{pmatrix}) \cong \Ext^n_R(M, X) \neq 0. \]
since \( \begin{pmatrix} 0 \\ X \end{pmatrix} \in \text{Add}_{T(R)}(C) \) by Lemma \( 2.7(1) \), it follows that \( n < G_{C - pd(\begin{pmatrix} M \\ 0 \end{pmatrix})} \) by \( \text{[6, Theorem 3.8]} \). But \( G_{C - pd(\begin{pmatrix} M \\ 0 \end{pmatrix})} \leq G_{C - PD(T(R))} \leq n + 1 \). Thus \( G_{C - pd(\begin{pmatrix} M \\ 0 \end{pmatrix})} = n + 1 \). which means that \( G_{C - PD(T(R))} = n + 1 \).

**Corollary 5.10** Let \( R \) be a ring, \( T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix} \) and \( n \geq 0 \) an integer. Then \( T(R) \) is left \((n + 1)\)-Gorenstein regular if and only if \( R \) left \( n \)-Gorenstein regular.

The authors in \([2]\) establish a relationship between the projective dimension of modules over \( T \) and modules over \( A \) and \( B \). Given an integer \( n \geq 0 \) and \( M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \phi^M \) a \( T \)-module, they proved that \( pd_T(M) \leq n \) if and only if \( pd_A(M_1) \leq n, \) \( pd_B(M_2) \leq n \) and the map related to the \( n \)-th syzygy of \( M \) is injective. The following example shows that this is not true in general.

**Example 5.11** Let \( R \) be a left hereditary ring which not semisimple and let \( T(R) = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix} \). Then \( lD(T(R)) = lD(R) + 1 = 2 \) by \([27, \text{corollary 3.4(3)}]\].

This means that there exists a \( T(R) \)-module \( M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \phi^M \) with \( pd_{T(R)}(M) = 2 \). If \( K^1 = \begin{pmatrix} K_1^1 \\ K_1^2 \end{pmatrix} \phi^{K^1} \) is the first syzygy of \( M \), then there exists an exact sequence of \( T(R) \)-modules

\[
0 \to K^1 \to P \to M \to 0
\]

where \( P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \phi^P \) is projective. Then we get the following commutative diagram
By Snake Lemma $\varphi^{K^1}$ is injective. On the other hand, Since $lD(R) = 1,$ $\text{pd}_R(M_1) \leq 1$ and $\text{pd}_R(M_2) \leq 1.$ But $\text{pd}_{T(R)}(M) = 2 > 1.$

Acknowledgement. The third and forth authors were partially supported by Ministerio de Economía y Competitividad, grant reference 2017MTM2017-86987-P, and Junta de Andalucía, grant reference P20-00770. The authors would like to thank Professor Javad Asadollahi for the discussion on the Example 5.11.

References

[1] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Stud. Adv. Math., vol. 36, Cambridge Univ. Press, 1995.

[2] J. Asadollahi, S. Salarian, On the vanishing of Ext over formal triangular matrix rings, Forum Math. 18 (2006) 951-966.

[3] T. Araya, R. Takahashi, Y. Yoshino: Homological invariants associated to semi-dualizing bimodules. J. Math. Kyoto Univ. 45 (2005), 287-306.

[4] D. Benkhadra, D. Bennis, J. R. García Rozas, The category of modules on an u-trivial extension: basic properties, to appear in , Algebra Colloquium (no. AC20190094). https://arxiv.org/pdf/1911.09364.pdf.

[5] D. Bennis, E. Duarte, J.R. García Rozas and L. Oyonarte. The role of w-tilting modules in relative Gorenstein (co)homology. Submitted.
REFERENCES

[6] D. Bennis, J.R. García Rozas and L. Oyonarte, \textit{Relative Gorenstein dimensions}, Mediterr. J. Math. 13 (2016), 65-91.

[7] D. Bennis, J.R. García Rozas and L. Oyonarte, \textit{Relative Gorenstein global dimension}, Int. J. Algebra Comput. 26 (2016), 1597-1615.

[8] D. Bennis and N. Mahdou, \textit{Global Gorenstein Dimensions}, Proc. Amer. Math. Soc. 138 (2010) 461-465.

[9] Enochs EE, Estrada S, García-Rozas JR. \textit{Gorenstein categories and Tate cohomology on projective schemes}, Math Nachr 2008; 281: 525-540.

[10] E.E. Enochs, M.C. Izurdiaga, B. Torrecillas, \textit{Gorenstein conditions over triangular matrix rings}, J. Pure Appl. Algebra 218 (2014) 1544-1554.

[11] Enochs, E.E., Jenda, O.M.G.: Gorenstein injective and projective modules. Math. Z. 220, 611-633 (1995)

[12] Foxby, H.B.: \textit{Gorenstein modules and related modules}. Math. Scand. 31, 276-284 (1972).

[13] R.M. Fossum, P. Griffith, I. Reiten, \textit{Trivial extensions of Abelian categories}, in: \textit{Homological Algebra of Trivial Extensions of Abelian Categories with Applications to Ring Theory}, in: Lect. Notes in Math., vol.456, Springer-Verlag, 1975.

[14] Golod, E.S.: \textit{G-dimension and generalized perfect ideals. Algebraic geometry and its applications}. Collection of articles. Trudy Mat. Inst. Steklov. 165, 62-66 (1984)

[15] E.L. Green. \textit{On the representation theory of rings in matrix form}, Pac. J. Math. 100 (1982), 123-138.

[16] K.R. Goodearl, R.B. Warfield, \textit{An introduction to Non-commutative Noetherian Rings}, London Math. Soc. Student Texts, vol. 16, 1989.

[17] Holm, H., White, D.: \textit{Foxby equivalence over associative rings}. J. Math. Kyoto Univ. 47, 781-808 (2007).

[18] Holm, H., Jørgensen, P.: \textit{Semi-dualizing modules and related Gorenstein homological dimensions}. J. Pure Appl. Algebra 205, 423-445 (2006)
REFERENCES

[19] A. Haghany, K. Varadarajan, Study of modules over formal triangular matrix rings, J. Pure Appl. Algebra 147 (2000), 41-58.

[20] A. Haghany and K. Varadarajan, Study of formal triangular matrix rings, Comm. Algebra 27 (1999), 5507-5525.

[21] J. Hu and H. Zhu, Special precovering classes in comma categories, Sci. China Math., https://arxiv.org/pdf/1911.03345.pdf

[22] P. Krylov, A. Tuganbaev, Formal Matrices, Springer International Publishing AG, Gewerbestrasse 11, 6330 Cham, Switzerland, 2017.

[23] Liu, Z., Huang, Z., Xu, A.: Gorenstein projective dimension relative to a semidualizing bimodule. Commun. Algebra 41, 1-18 (2013)

[24] L. Mao, Cotorsion pairs and approximation classes over formal triangular matrix rings, J. Pure Appl. Algebra (2019), 106271, doi: https://doi.org/10.1016/j.jpaa.2019.106271.

[25] L. Mao Gorenstein flat modules and dimensions over triangular matrix rings, J. Pure Appl. Algebra 224 (2020) 1-10.

[26] R.M. Zhu, Z.K. Liu, Z.P. Wang, Gorenstein homological dimensions of modules over triangular matrix rings, Turk. J. Math. 40 (2016) 146-150.

[27] Vasconcelos, W.V.: Divisor theory in module categories. In: North-Holland Mathematics Studies, No. 14. Notas de Matemática 53. North-Holland, Amsterdam (1974)

[28] Wakamatsu, T.: Tilting modules and Auslander’s Gorenstein property. J. Algebra 275, 3-39 (2004)

[29] White, D.: Gorenstein projective dimension with respect to a semidualizing module. J. Commun. Algebra 2, 111-137 (2010)

[30] P. Zhang, Gorenstein-projective modules and symmetric recollements, J. Algebra 388 (2013) 65-80.