Relational evolution of the degrees of freedom of generally covariant quantum theories

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We study the classical and quantum dynamics of generally covariant theories with vanishing Hamiltonian and with a finite number of degrees of freedom. In particular, the geometric meaning of the full solution of the relational evolution of the degrees of freedom is displayed, which means the determination of the total number of evolving constants of motion required. Also a method to find evolving constants is proposed. The generalized Heisenberg picture needs $M$ time variables, as opposed to the Heisenberg picture of standard quantum mechanics where one time variable $t$ is enough. As an application, we study the parameterized harmonic oscillator and the $SL(2,R)$ model with one physical degree of freedom that mimics the constraint structure of general relativity where a Schrödinger equation emerges in its quantum dynamics.

Key words: Evolving constants of motion.

INTRODUCTION

Describing evolution of the degrees of freedom of generally covariant theories is an unsolved puzzle, and constitutes one of the challenges of the human thinking of our time. The study of generally covariant theories has been motivated by general relativity, which has this peculiar property (see for instance [1]). In gravity, general covariance

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means the theory is diffeomorphism invariant, and this symmetry of gravity implies at the Hamiltonian level that the theory has not a genuine Hamiltonian for describing the evolution of the degrees of freedom of the gravitational field, rather, dynamics is gauge; generated by the first class constraints of the theory. This is the so-called problem of time in classical general relativity [2].

On the other hand, if the classical regime of general relativity is only a limit, which emerges in a suitable way from its fundamental quantum behavior, then the theory is in trouble. Our standard methods of quantization crash and do not apply to the particular physical situation raised by general relativity. Standard quantization methods in field theory are background-dependent, quantum gravity needs a background independent procedure in its quantization. So, how to make compatible the symmetry of general relativity with the quantum theory? what is the meaning of evolution in quantum gravity? Loop quantum gravity answers the first question, because the quantization of the gravitational field is carried out in a background independent way [3]. With respect to the second, it remains as an open question. Among the several proposals for describing the evolution of the degrees of freedom of generally covariant theories, I find Rovelli’s proposal as one of the most creative ones [4–6].

Here, following the spirit of relationism, which is the heart in Rovelli’s point of view, we analyse the ‘problem of time’ in generally covariant theories with vanishing Hamiltonian and with a finite number $D$ of degrees of freedom. To obtain the relations involving the coordinates and momenta of the unreduced phase space $\gamma_{\text{ex}}$ with the physical states that label the points of the physical phase space $\gamma_{\text{ph}}$, we need to start from the embedding equations which give the dependence of the coordinates and momenta with respect to the $M$ time variables $t^m$ as well as the $2D$ physical states ($\tilde{q}^a, \tilde{p}_a$). These equations constitute the classical version of the generalized Heisenberg picture, which arises when these equations are promoted to quantum operators in the reduced Hilbert space of the theory. By plugging the expressions of the time variables $t^m$ in terms the original canonical variables into the expressions of coordinates and momenta, we get the full relational evolution of the phase space degrees of freedom for any physical state of $\gamma_{\text{ph}}$. This way of expressing the full solution of the dynamics of generally covariant theories constitutes the full set of evolving constants of motion required in their dynamics, and is displayed in Sect. I. In addition, an alternative mechanism which generates also the evolving constants of motion is proposed. Of course, we study also the quantum version of the evolving constants of motion. In sect. II, we analyze the parameterized harmonic oscillator (as an example of parameterized systems). In Sect. III, we continue the study of the $SL(2, R)$ model which constraint algebra mimics the algebra structure of general relativity. Due to the fact a Schrödinger equation emerges in its quantum dynamics, we compare the generalized Heisenberg picture (related with the evolving constants of motion) which needs $M$ time variables with the Schrödinger picture which singles out one time variable only. We also especulate on the classical limit generally covariant theories and its possible relation with the full set of evolving constants of motion. Our conclusions are summarized in Sect. IV.
I. RELATIONAL CLASSICAL AND QUANTUM DYNAMICS

Classical dynamics. The classical dynamics of a constrained theory with a finite number of degrees of freedom characterized by first class constraints is as follows \[7\]. The theory is obtained from the Hamiltonian form of the action

\[ S[q^i, p_i, \lambda^m] = \int_{\tau_1}^{\tau_2} d\tau \left\{ \frac{dq^i}{d\tau} p_i - \lambda^m C_m(q^i, p_i) \right\}, \]

which is invariant under arbitrary reparameterizations of the parameter \( \tau \). Hence, \( \tau \) is a non physical coordinate time. The unreduced phase space \( \gamma_{ex} \) is coordinated by the canonical pairs \((q^i, p_i)\); \( i = 1, 2, ..., N \). The canonical 2-form on \( \gamma_{ex} \) is \( \omega_{ex} = dp_i \wedge dq^i \). Thus, \((\gamma, \omega)\) is a symplectic space. The variation of the action \( S[q^i, p_i, \lambda^m] \) with respect to the canonical coordinates \( q^i, p_i \) gives the equations of motion

\[ \frac{dq^i}{d\tau} = \lambda^m \frac{\partial C_m(q^i, p_i)}{\partial p_i}, \]
\[ \frac{dp_i}{d\tau} = -\lambda^m \frac{\partial C_m(q^i, p_i)}{\partial q^i}, \]

while the variation of the action with respect to the Lagrange multipliers \( \lambda^m \) gives the constraint equations

\[ C_m = C_m(q^i, p_i) = 0, \quad m = 1, 2, ..., M. \]

The variation of the action has been done under the standard boundary conditions \( q^i(\tau_s) = q^i_s; \ s = 1, 2 \), namely, the allowed paths are those with fixed values for the configuration variables at the boundary points \( \tau_s \). The boundary conditions can be changed and thus to modify the action by suitable boundary terms to allow the gauge symmetry generated by the constraints \[8\]. The constraints generate Hamiltonian vector fields \( X_{dC_m} \), which are tangent vectors to the constraint surface, given by

\[ X_{dC_m} = -\frac{\partial C_m}{\partial p_i} \frac{\partial}{\partial q^i} + \frac{\partial C_m}{\partial q^i} \frac{\partial}{\partial p_i}. \]

More important, the integral curves of these Hamiltonian vectors fields constitute the gauge submanifold or the orbits of the constraint surface, and the dynamics of the system with respect to \( \tau \) is the unfolding of this gauge symmetry, i.e., dynamics is gauge.

The first class constraints satisfy, in general, a non Lie algebra

\[ \{ C_m, C_n \} = C_{mn} \overline{i}(q^i, p_i) C_l, \]
and the number of independent physical degrees of freedom of the theory is \( D = N - M \).

The constraint surface defined by the constraint equations (3) is a \((2D + M)\)-dimensional manifold. The constraint surface can be parameterized by the set of independent coordinates \((\tilde{q}^a, \tilde{p}_a, t^m)\), where \((\tilde{q}^a, \tilde{p}_a), a = 1, 2, ..., D\) are (local) canonical variables which coordinatize the open sets of the physical phase space \(\gamma_{ph}\) of the theory, and \(t^m, m = 1, 2, ..., M\) coordinatize the orbits, i.e., the gauge submanifold of the constraint surface generated by the first class constraints. Notice that the canonical coordinates \(q_0^\alpha\) and \(p_0^\alpha\) are the physical observables of the system. Of course, they satisfy \(\{\tilde{q}^a, \tilde{p}_b\} = \delta^a_b\) on the physical phase space, and the symplectic form on \(\gamma_{ph}\) in these coordinates is \(\omega_{ph} = d\tilde{p}_a \wedge d\tilde{q}^a\).

Therefore, the general solution of the dynamics of the constrained theory can be expressed as

\[
q^i = q^i(t^m; \tilde{q}^0, \tilde{p}_0),
\]

\[
p_i = p_i(t^m; \tilde{q}^0, \tilde{p}_0).
\]

It is important to emphasize that that such dependence is local. For instance, in the case in which the physical phase space \(\gamma_{ph}\) is compact, a finite set of physical observables \((\tilde{q}^a, \tilde{p}_a)\) is needed to coordinate the open sets of \(\gamma_{ph}\) due to its compactness. So, the constraint surface looks like a ‘fibre bundle’ \(P(\gamma_{ph}, \text{Orbits})\), the constraint surface being the total space \(P\), the physical phase space \(\gamma_{ph}\) being the base space, and the orbits being the fibers of the bundle. In the generic case, \(P(\gamma_{ph}, \text{Orbits})\) is locally trivial. This means that non global gauge condition is allowed in general, and that local gauge conditions associated with each open set of the physical phase space can be specified only.

At the same time, the full solution of the theory implies to give the dependence of the physical observables \(\tilde{q}^a\), and \(\tilde{p}_a\) of the system in terms of the coordinates of the unreduced phase space

\[
\tilde{q}^a = \tilde{q}^a(q^i, p_i),
\]

\[
\tilde{p}_a = \tilde{p}_a(q^i, p_i),
\]

as well as the orbit coordinates \(t^m\)

\[
t^m = t^m(q^i, p_i).
\]

What these equations tell us is that one single internal time variable is not enough to describe the evolution of the system, rather, \(M\) internal time variables are needed. In the way the full solution has been expressed in (4), and (5), these \(M\) time variables are \(t^m\), \(m = 1, 2, ..., M\). Notice also that these \(M\) time variables are internal clocks, given by (14). One of the properties of these internal clocks \(t^m\) is that they do not run taking increasing values of \(t^m\) when time goes on. In fact, they can run ‘forward’ and ‘backward’ depending on the values of the coordinates and momenta the system is reaching through Eq. (14). Other property is that these clocks can run with different ‘speeds’ for the same reason.
So, the meaning of time that arise in generally covariant theories is completely different with respect to the monotonous function we are familiarized with. In [9] was showed that Eqs. (6) and (7) can be obtained by a combination of a canonical transformation plus Hamilton-Jacobi techniques. That approach implies the modification of the original set of first class constraints. This can be done, but it is not necessary in principle. Moreover, the full solution requires (10) (missing in Ref. [9]) as we have seen, and more important, it is the combination of Eqs. (6), and (7) with Eq. (10) which leads to the relational description of the dynamics of the system, as we will see it later on.

It is worth to mention the relationship between the time variables $t^m$ and the full gauge transformation generated by the first class constraints $C_m$. Assuming that the full gauge transformation of the original canonical variables is given by

$$q_i' = q_i'(q^i, p_i, \alpha^m(\tau)), \quad p_i' = p_i'(q^i, p_i, \alpha^m(\tau)), \quad (11)$$

with $\alpha^m$ the $m$ gauge parameters involved in the gauge transformation. Then, by plugging (6) and (7) into the right hand side of (11), we get

$$q_i' = q_i'(t^m; \tilde{q}^a, \tilde{p}_a), \quad (12)$$

$$p_i' = p_i'(t^m; \tilde{q}^a, \tilde{p}_a), \quad (13)$$

where the functional dependence in the right-hand side of the above equations is exactly the same as that given by (6)-(7) but with $t^m = t^m(\tilde{q}^a, \tilde{p}_a, t_m)$,

$$t^m = t^m(\tilde{q}^a, \tilde{p}_a), \quad (14)$$

which relates the time variables $t^m$ after and before of any finite gauge transformation of the canonical variables (11).

The map $\phi : P(\gamma_{ph}, \text{Orbits}) \rightarrow \gamma_{ex}, \phi(\tilde{q}^a, \tilde{p}_a, t^m) \mapsto (q^i(t^m; \tilde{q}^a, \tilde{p}_a), p_i(t^m; \tilde{q}^a, \tilde{p}_a))$ allows us to define on the constraint surface $P(\gamma_{ph}, \text{Orbits})$ the pull back $\Omega = \phi^*\omega_{ex}$ of the canonical form $\omega_{ex} = dp_i \wedge dq^i$ on $\gamma_{ex}$, which is degenerate. Thus, the geometry of constrained systems involves three spaces: the unconstrained phase space ($\gamma_{ex}, \omega_{ex}$), the constraint surface ($P(\gamma_{ph}, \text{Orbits}), \Omega = \phi^*\omega_{ex}$), and the physical phase space ($\gamma_{ph}, \omega_{ph}$). The map that connects the constraint surface and the physical phase space is the projection $\pi : P(\gamma_{ph}, \text{Orbits}) \rightarrow \gamma_{ph}, \pi(\tilde{q}^a, \tilde{p}_a, t^m) \mapsto (\tilde{q}^a, \tilde{p}_a)$. Due to the fact that the Hamiltonian vector fields (4) are tangent vectors to the orbits, they can be expressed in terms of the local coordinates $\tilde{q}^a, \tilde{p}_a, t^m$ of the constraint surface as

$$X_{dc_m} = \{C_m, t^m\}(\tilde{q}^a, \tilde{p}_a, t^m) \frac{\partial}{\partial t^m}. \quad (15)$$

The observables (8), (9), and the orbits (11) also generate Hamiltonian vectors fields, their restriction on the constraint surface are
where it is understood that all the quantities are evaluated in the point \((\vec{q}^a, \vec{p}_a, t^m)\). Thus, \(\{X_{dC_m}, X_{\vec{d}q^a}, X_{\vec{d}p_a}\}\) is a basis, naturally adapted to the involved geometry, of the tangent space of the constraint surface. The vectors \(X_{dC_m}\) play the role of vertical vectors because they have vanishing projection on the tangent space of \(\gamma_{ph}, d\pi (X_{dC_m}) = 0\). \(X_{\vec{d}q^a}\), and \(X_{\vec{d}p_a}\) are the horizontal lifts on the constraint surface of the coordinate basis on \(\gamma_{ph}, d\pi (X_{\vec{d}q^a}) = \frac{\partial}{\partial \vec{q}^a}, d\pi (X_{\vec{d}p_a}) = -\frac{\partial}{\partial \vec{q}^a}\). In summary, the solution of the dynamics of the constrained system means to specify Eqs. (3)-(10). This fact, rises a new problem: the problem of the meaning of physical time of generally covariant theories, i.e. the specification of an internal clock in the framework of the theory with respect to which to describe the evolution of the degrees of freedom of the theory in a gauge invariant way. Let us explain, the dynamics with respect to \(\tau\) is given by

\[
q^i(\tau) = q^i(t^m(\tau); \vec{q}^a, \vec{p}_a),
\]

\[
p_i(\tau) = p_i(t^m(\tau); \vec{q}^a, \vec{p}_a),
\]

for any physical state \((\vec{q}^a, \vec{p}_a)\) of the system. So, this dynamics is non gauge-invariant, i.e., it depends on \(\tau\). The question is, can we describe evolution of the system in a gauge invariant way? The answer is yes. At first sight, this sounds strange or impossible in a system with gauge freedom. To see how this can be achieved, let us plug the time variables (14) into the full solution (3), and (4)

\[
q^i = q^i(t^m(q^i, p_i); \vec{q}^a, \vec{p}_a),
\]

\[
p_i = p_i(t^m(q^i, p_i); \vec{q}^a, \vec{p}_a).
\]

Last equations are very important, they relate the original phase space variables \(q^i\), and \(p_i\) with the physical states of the physical phase space \((\vec{q}^a, \vec{p}_a)\). These equations admit two, related, interpretations. First, they give the relational evolution of the coordinates \(q^i\) and the momenta \(p_i\) for any fixed point \((\vec{q}^a, \vec{p}_a)\) of the physical phase space, i.e., it is possible to choose \(M\) coordinates denoted by \(q^m\) (or momenta \(p_m\); or a combination of both) as ‘clocks’ and describe the evolution of the remaining set of coordinates and momenta as functions of the \(q^m\) for any physical state \((\vec{q}^a, \vec{p}_a)\) of the system. Second, if we fix the values of this \(M\) coordinates, say \(q^m = q'^m\) then, the before mentioned expressions of coordinates and momenta give \(M\)-parameter families of physical observables defined on \(\gamma_{ph}, q'^m\) being the parameters. Eqs. (19), and (20) are evolving constants of motion in the sense of Rovelli [1, 3]. This concept captures the essence that the before mentioned
observables (defined on $\gamma_{ph}$) describe the relational evolution of the coordinates $q^i$ and momenta $p_i$, and at the same time they are physical observables.

Let us consider particular cases of (10), say

$$t^m = q^m, \quad m = 1, 2, ..., M,$$

then (19), and (20) acquire the form

$$q^m = q^m, \quad m = 1, 2, ..., M, \quad (22)$$

$$q^i = q^i(q^m; \bar{q}^a, \bar{p}_a), \quad i = M + 1, ..., N, \quad (23)$$

$$p_i = p_i(q^m; \bar{q}^a, \bar{p}_a), \quad i = 1, ..., N. \quad (24)$$

Thus, the ‘clocks’ are given by $q^m$ and last two pairs of equations are the evolving constants of motion involved. Other particular case is given by

$$t^m = p_m, \quad m = 1, 2, ..., M,$$

and (19), and (20) acquire the form

$$q^i = q^i(p_m; \bar{q}^a, \bar{p}_a), \quad i = 1, 2, ..., N, \quad (26)$$

$$p_m = p_m, \quad m = 1, 2, ..., M, \quad (27)$$

$$p_i = p_i(p_m; \bar{q}^a, \bar{p}_a), \quad i = M + 1, ..., N. \quad (28)$$

In this case, the ‘clocks’ are $p_m$ and last two pairs of equations are the evolving constants of motion required.

As we have seen, the general relations involving the coordinates $q^i$ and momenta $p_i$ with the physical states $(\bar{q}^a, \bar{p}_a)$ is given by (19), and (20). The explicit form of (19), and (20) could be complicated for particular theories, but this fact would rise technical rather than conceptual difficulties (see [10,11] for the opposite viewpoint where the authors rise questions on interpretation, consistency, and the degree to which the resulting quantum theory emerging from the before classical dynamics coincide with, or generalizes, the usual non-relativistic theory). Thus, Eqs. (19), and (20) constitute the full set of evolving constants needed in the relational description of the dynamics of generally covariant theories with a finite number $D$ of degrees of freedom. The solution (13), and (24) sits in the spirit that in covariant theories there is non privileged observable with respect to which to describe evolution, and that only relational evolution makes sense. From this point of view, general covariance forces us to use relational evolution, namely, to describe the change of some variables of the system with respect to the others. This is the essence of relationism, which appears to be the natural language for describing the evolution of the degrees of freedom of generally covariant theories [2,4–6].
In addition, in this paper, we propose an alternative mechanism to generate the evolving constants. This mechanism is essentially to compute the action of the Hamiltonian vector fields $X_{dC_m}$ on some evolving constant $E^1$

$$X_{dC_m}(E) =: E^m.$$  \hspace{1cm} (29)

The evolving function $E^1$ depends on the canonical coordinates of the unconstrained phase space $\tilde{q}^a$, and $\tilde{p}_a$ as well as on the canonical coordinates $\tilde{q}^a$, and $\tilde{p}_a$ of the physical phase space. Therefore, in the computation of the action of the Hamiltonian vector fields (4) on the evolving function we can proceed in two ways. First, taking the observables $\tilde{q}^a$, and $\tilde{p}_a$ constants in the dependence of the evolving function $E$. This can be done because $\tilde{q}^a$, and $\tilde{p}_a$ are constant along the orbits. b) Taking the explicit dependence of the physical observables in terms of the canonical variables of the unconstrained phase space given by (8), and (9). Of course, both approaches lead to the same results. The repeated application of the Hamiltonian vector fields on the new evolving constants $E^2$, $E^3$, ..., gives another evolving constants, and so on until no new evolving constants are obtained, and the process ends. From the knowledge of the evolving constants and the expressions of the physical observables, the full solution of the dynamics of the system encoded in Eqs. (3)-(10) is obtained.

Quantum dynamics. Let us begin with the quantum description of the system. We use the Dirac method. In the same way as in the classical dynamics we have three spaces $(\gamma_{ex}, \omega_{ex}), (\mathcal{P}(\gamma_{ph}, \text{Orbits}), \Omega)$, and $(\gamma_{ph}, \omega_{ph})$. In the quantum theory we have three Hilbert spaces; the unconstrained Hilbert space $\mathcal{H}$ or a suitable extension of it if the constraints have continuum spectrum, the physical Hilbert space $\mathcal{H}_{phys}$, and the reduced Hilbert space $\mathcal{H}_r$ obtained by projecting $\mathcal{H}_{phys}$. Suppose we have solved the quantum theory in a full way, i.e., we have the physical Hilbert space $\mathcal{H}_{phys}$ of the theory. A general physical state $|\phi\rangle$ of the system is killed by all the constraints of the theory $\hat{C}_m |\psi\rangle = |0\rangle$, and it is given by

$$|\psi\rangle = \sum_{n_1, n_2, ..., n_D} c_{n_1, n_2, ..., n_D} |n_1, n_2, ..., n_D\rangle.$$  \hspace{1cm} (30)

in Dirac notation. Here, the physical states are labelled by the quantum numbers $n_a$, $a = 1, 2, ..., D$ which come from a complete set of commuting physical observables $\hat{O}_a$, $a = 1, 2, ..., D$ of the system

$$\hat{O}_a |n_1, n_2, ..., n_D\rangle = O(n_a) |n_1, n_2, ..., n_D\rangle.$$  \hspace{1cm} (31)

Of course, these quantum observables are combinations of the physical observables $\tilde{q}^a$, and $\tilde{p}_a$. We have come to the heart of the problem, how to describe relational evolution in the quantum theory.

Quantum evolving constants. Let us see how the quantum version of the classical evolving constants looks. The idea is to search for a representation of the physical states...
in the reduced Hilbert space associated with the physical phase space of the system. Explicitly

\[
\psi(q^a) = \langle \bar{q}^a | \psi \rangle = \sum_{n_1,n_2,\ldots,n_D} c_{n_1,n_2,\ldots,n_D} \langle \bar{q}^a | n_1, n_2, \ldots, n_D \rangle.
\] (31)

The inner product in the Hilbert space

\[
\langle \psi | \phi \rangle = \int d\mu(\bar{q}^a) \psi^*(\bar{q}^a)\phi(\bar{q}^a),
\] (32)

can be determined with the condition that the operators \( \hat{\bar{q}}^a \) and \( \hat{\bar{p}}_a \) be hermitian operators and with the implementation of the action of the operators \( \hat{\bar{q}}^a \), \( \hat{\bar{p}}_a \) on this Hilbert space. Notice also that is always possible to build creation and annihilation operators \( \hat{a}_a = \hat{\bar{q}}^a + i\hat{\bar{p}}_a \), \( \hat{a}_a^\dagger = \hat{\bar{q}}^a - i\hat{\bar{p}}_a \) for each pair of canonical operators \( \hat{\bar{q}}^a \), and \( \hat{\bar{p}}_a \) because the number of these operators is even. \( \hat{a}_a, \hat{a}_a^\dagger \) can help in the construction of \( \mathcal{H}_f \).

With the before machinery, the quantum version of the evolving constants is

\[
\hat{q}^i = q^i(t^m(q^i,p_i); \hat{\bar{q}}^a, \hat{\bar{p}}_a),
\] (33)
\[
\hat{p}_i = p_i(t^m(q^i,p_i); \hat{\bar{q}}^a, \hat{\bar{p}}_a),
\] (34)

or, equivalently,

\[
\langle \psi | \hat{q}^i | \psi \rangle = \langle \psi | q^i(t^m(q^i,p_i); \hat{\bar{q}}^a, \hat{\bar{p}}_a) | \psi \rangle,
\] (35)
\[
\langle \psi | \hat{p}_i | \psi \rangle = \langle \psi | p_i(t^m(q^i,p_i); \hat{\bar{q}}^a, \hat{\bar{p}}_a) | \psi \rangle,
\] (36)

where the mean values are computed with the inner product (32). In the case of parameterized systems, last equations reduce to the standard ones which describe the evolution of the position and momenta operators as well as the evolution of the mean values of the position and momenta operators in the Heisenberg picture. Of course, the well-known ordering problems for the operators might appear here too.

II. PARAMETERIZED HARMONIC OSCILLATOR

A. Classical dynamics

In order to make these ideas concrete, let us consider a familiar example: the parameterized harmonic oscillator, which action is
\[ S = \int d\tau \left[ \frac{dx}{d\tau} p + \frac{dt}{d\tau} p_t - \lambda \left( p_t + \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \right) \right]. \] (37)

The unconstrained classical space \( \Gamma \) is coordinatized by the canonical pairs \((x, p)\), and \((t, p_t)\). By doing the variation of the action with respect to \(x\), \(p\), \(t\), and \(p_t\) we find the equations of motion
\[
\frac{dp}{d\tau} = -\lambda m\omega^2 x, \quad \frac{dx}{d\tau} = \lambda \frac{p}{m}, \quad \frac{dp_t}{d\tau} = 0, \quad \frac{dt}{d\tau} = \lambda. \] (38)

The variation of the action with respect to the Lagrange multiplier \(\lambda\) gives the first class constraint
\[
C = p_t + \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2. \] (39)

The classical dynamics is the unfolding of the gauge symmetry of the system generated by the first class constraint \(C\). The gauge orbit on the constrained surface \(C = 0\) is the integral curve of the Hamiltonian vector field
\[
X_{dC} = -\frac{\partial}{\partial t} - \frac{p}{m} \frac{\partial}{\partial x} + m\omega^2 x \frac{\partial}{\partial p}. \] (40)

If we have a solution \(x(\tau), p(\tau), t(\tau)\), and \(p_t(\tau)\) of the equations of motion (38), any other solution \(x'(\tau), p'(\tau), t'(\tau)\), and \(p_t'(\tau)\) can be found through the relations
\[
x'(\tau) = \cos (\theta(\tau)) x(\tau) + \frac{1}{m\omega} \sin (\theta(\tau)) p(\tau),
\]
\[
p'(\tau) = -m\omega \sin (\theta(\tau)) x(\tau) + \cos (\theta(\tau)) p(\tau),
\]
\[
t'(\tau) = \frac{\theta(\tau)}{\omega} + t(\tau),
\]
\[
p_t' = p_t,
\] (41)

that connect all the solutions, while the Lagrange multiplier transforms as
\[
\lambda'(\tau) = \lambda(\tau) + \frac{1}{\omega} \dot{\theta}(\tau), \] (42)

in order to leave the action invariant, here \(\dot{\theta}(\tau) = \frac{d\theta(\tau)}{d\tau}\).

Let us construct the general solution in a given gauge. We choose the gauge \(\lambda = 1\). We still have one gauge fixing to impose at \(\tau = 0\). We choose \(t(0) = 0\). Using the constraint equation and the solution of (38), we obtain
\[ x(\tau) = A \cos(\omega \tau) + B \sin(\omega \tau), \]
\[ t(\tau) = \tau, \]
\[ p(\tau) = -m\omega A \sin(\omega \tau) + m\omega B \cos(\omega \tau), \]
\[ p_t(\tau) = -\frac{1}{2}m\omega^2(A^2 + B^2), \]  
(43)

where \((A, B)\) are the physical observables that coordinatize the physical phase space of the system, which is \(R^2\). It is clear that \(x, t, p\) are non-observables (they depend on \(\tau\)).

The two physical observables \((A, B)\) can be expressed in terms of the phase space variables as

\[ A = \cos(\omega t)x - \frac{1}{m\omega} \sin(\omega t)p, \]
\[ B = \frac{1}{m\omega} \cos(\omega t)p + \sin(\omega t)x \]  
(44)

Notice that \(A = x(t = 0) \equiv x_0\) and \(B = \frac{p(t = 0)}{m\omega} \equiv \frac{p_0}{m\omega}\), i.e., the position \(x_0\) of the harmonic oscillator when the internal clock measures \(t = 0\), and the momentum \(p_0\) when the internal clock measures \(t = 0\) are (physical) observables. Moreover, Eq. (44) means that the precise combination of the position \(x = X\) and the momentum \(p = P\) of the harmonic oscillator, when the internal clock indicates \(t = T\) in the form expressed by the formula (44) is an observable of the (composed) system: harmonic oscillator + internal clock. These observables have vanishing Poisson brackets with the first class constraint \(C\) as required by the formalism of constrained systems. Actually, the Dirac method requires observables to have weakly vanishing Poisson brackets with the first class constraints. Here, the observables \(A, B\) have strong vanishing Poisson brackets with the constraint \(C\). The Poisson brackets between \(A\) and \(B\) in the physical phase space reads

\[ \{A, B\} = \frac{1}{m\omega}. \]  
(45)

**Classical evolving constants.** From (43), we obtain the evolving constant

\[ x = x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t), \]  
(46)

of the system. As before mentioned, last equation admits two, related, interpretations. First, for any fixed point \((A, B)\) (equivalently \((x_0, p_0)\)) of the physical phase space, (46) gives the relative evolution of the configuration variables \(x\), and \(t\) of the system

\[ x = X(t; x_0, p_0) = x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t). \]  
(47)
Second, for any fixed $t$, it gives a one-parameter family of physical observables, $t$ being the parameter, on the physical phase space.

*Generation of evolving constants.* We define the function $E^1$ on $\Gamma$

$$E^1(x, t, p, p_t) := x - x_0 \cos (\omega t) - \frac{p_0}{m\omega} \sin (\omega t). \quad (48)$$

The restriction of this function on the constraint surface is $E^1|_C = 0$. The action of the Hamiltonian vector field $X_{dC}$ on $E^1$ is

$$X_{dC}(E^1) =: E^2 = -\omega x_0 \sin (\omega t) + \frac{p_0}{m} \cos (\omega t) - \frac{p}{m}, \quad (49)$$

and the restriction of $E^2$ on the constraint surface is $E^2|_C = 0$, and more important, the equation $E^2|_C = 0$ is precisely an evolving constant

$$p = -m\omega x_0 \sin (\omega t) + p_0 \cos (\omega t). \quad (50)$$

Note also that the action of $X_{dC}$ on $E^2$ gives again $E^1$, and the process ends. In other words, the evolving constant (50) was obtained from the application of the Hamiltonian vector field $X_{dC}$ on $E^1$, and vice versa.

*The full solution.* In the present case the constraint surface is coordinatized by the coordinates of the physical phase space $(x_0, p_0)$ and by the internal time $t$. Therefore, Eqs. (5) and (7) acquire the form

$$x = X(t; x_0, p_0) = x_0 \cos (\omega t) + \frac{p_0}{m\omega} \sin (\omega t),$$

$$t = T(t; x_0, p_0) = t,$$

$$p = P(t; x_0, p_0) = -m\omega x_0 \sin (\omega t) + p_0 \cos (\omega t),$$

$$p_t = P_T(t; x_0, p_0) = -\frac{p_0^2}{2m} - \frac{1}{2} m\omega^2 x_0^2. \quad (51)$$

Of course, last equations are also (19), and (20). Notice that Eqs. (8) and (9) acquire the form

$$x_0 = \cos (\omega t) x - \frac{1}{m\omega} \sin (\omega t) p,$$

$$p_0 = \cos (\omega t) p + m\omega \sin (\omega t) x, \quad (52)$$

and the dependence of the orbit coordinate $x^1 = t$, see (10), is

$$t = T(x, t, p, p_t) = t. \quad (53)$$
Last equations constitute the full solution of the classical dynamics of the system.

Notice that the internal time variable $x_1 = t = T(x, t, p, p_t) = t$ is not a physical observable because the Poisson bracket with the first class constraint does not vanish. Nevertheless, when we take the full solution into account we can express $t = \tilde{T}(x, p, p_0, x_0)$, given by

$$\cos \omega t = \frac{\left(\frac{1}{2}m\omega xx_0 + \frac{1}{2m}pp_0\right)}{H_0}, \quad (54)$$

with $H_0 = \frac{1}{2m}p_0^2 + \frac{1}{2}m\omega x^2$. The above expression is an evolving constant. From this point of view, the internal clock $t$ defines a two-parameter family of physical observables on the physical phase space; $x$, and $p$ being the parameters. So, the internal clock $t$ becomes a physical clock $t(x, p)$, namely, a physical observable when the full solution is considered. We restrict the analysis to a branch of the above multivalued function to compute the time $t(x, p)$ at which the particle reaches the position $x$ and the momentum $p$ evolving from an initial position $x_0$ and momentum $p_0$

$$t(x, p) = \tilde{T}(x, p; x_0, p_0) = \frac{1}{\omega} \arccos \left(\frac{\frac{1}{2}m\omega xx_0 + \frac{1}{2m}pp_0}{H_0}\right). \quad (55)$$

Or in terms of $x$ only

$$t_{\pm}(x) = \frac{1}{\omega} \arccos \left(\frac{\frac{1}{2}m\omega xx_0 \pm \sqrt{\frac{1}{2m} \left(H_0 - \frac{1}{2}m\omega x^2\right)p_0}}{H_0}\right). \quad (56)$$

These classical expressions have a quantum version as we will see later.

**B. Quantum dynamics**

At quantum level, as Dirac showed, the physical states are those killed by the first class constraint. We associate abstract operators with the classical coordinates and momenta, given by

$$x \rightarrow \hat{X}, \quad t \rightarrow \hat{T}, \quad p \rightarrow \hat{P}, \quad p_t \rightarrow \hat{P}_T, \quad (57)$$

which satisfy the Dirac rule
\[ [\hat{X}, \hat{P}] = i\hbar \quad [\hat{T}, \hat{P}_T] = i\hbar, \quad (58) \]

and by inserting these operators in the quantum constraint \( \hat{C} | \psi \rangle \), this equation becomes
\[
\left( \hat{P}_T + \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 \right) | \psi \rangle = 0. \quad (59)
\]

Any physical state can be expressed in terms of the single quantum number of the harmonic oscillator, in abstract Dirac notation
\[
| \psi \rangle = \sum_n C_n | n \rangle,
\]
\[
\hat{I} = \sum_n | n \rangle \langle n | . \quad (60)
\]

In last expression, the physical states \(| \psi \rangle\) are 'frozen' (i.e. they are abstract vectors), the complex coefficients \(C_n\) are constants. Notice that we have not chosen the coordinate basis yet. Taking a 'coordinate representation' \(| x,t \rangle\) where the operators acquire the form
\[
\langle x,t | \hat{X} \psi \rangle = x \langle x,t | \psi \rangle, \\
\langle x,t | \hat{T} \psi \rangle = t \langle x,t | \psi \rangle, \\
\langle x,t | \hat{P} \psi \rangle = \frac{i}{\hbar} \frac{\partial}{\partial x} \langle x,t | \psi \rangle, \\
\langle x,t | \hat{P}_T \psi \rangle = \frac{i}{\hbar} \frac{\partial}{\partial t} \langle x,t | \psi \rangle, \quad (61)
\]

any physical state vector \(| \psi \rangle\) is expanded in the coordinate basis \(| x,t \rangle\) as
\[
\langle x,t | \psi \rangle = \psi(x,t) = \sum_n C_n \langle x,t | n \rangle, \\
\langle x',t' | x,t \rangle = \sum_n \langle x',t' | n \rangle \langle n | x,t \rangle, \quad (62)
\]

with \(\langle x,t | n \rangle = e^{-\frac{i}{\hbar}E_n t} f_n(x), \quad E_n = \hbar \omega \left(n + \frac{1}{2}\right)\). Thus, in the Dirac framework, the coordinate representation is nothing but the ‘Heisenberg picture’ of the standard quantum mechanics, where the coordinate basis \(| x,t \rangle\) is ‘rotating’ and the physical state \(| \psi \rangle\) is fixed (see Eq. (60) where the coefficients \(C_n\) are constant complex numbers).

**Schrödinger equation.** In addition, we can build a ‘Schrödinger basis’ from the ‘Heisenberg basis’ \(| x,t \rangle\). In this ‘Schrödinger basis’, which we denote by \(| x \rangle\), the state vector is ‘moving around’ the ‘fixed basis’ \(| x \rangle\). Explicitly,
\[
\psi(x,t) = \langle x | \psi(t) \rangle \quad \text{Schrödinger basis } | x \rangle, \quad (63)
\]
\[ | \psi(t) \rangle = \sum_n \tilde{C}_n(t) | \tilde{n} \rangle = \sum_n C_n e^{-i \frac{\bar{\hbar} E_n t}{\bar{\hbar}}} | \tilde{n} \rangle , \]
\[ \langle x | \tilde{n} \rangle = f_n(x) . \]  

(64)

Taking the derivative with respect to the coordinate \( t \) of \( | \psi(t) \rangle \), the familiar Schrödinger equation emerges in the formalism \[12\]
\[ i \bar{\hbar} \frac{d}{dt} | \psi(t) \rangle = \hat{H} | \psi(t) \rangle , \]

(65)

with \( \hat{H} = \frac{1}{2m} \hat{P}^2 + \frac{1}{2} m \omega^2 \hat{X}^2 \). As usual, the physical vector \( | \psi(t) \rangle \) evolves in \( t \) while the coordinate basis \( | x \rangle \) is fixed. In other words, if we consider the system composed of the harmonic oscillator plus the clock together, we are describing the evolution of the degrees of freedom of the harmonic oscillator with respect to the internal clock itself, that is to say, the evolution of one part of the system with respect to the rest of it. In the next section, we will carry out the same procedure we applied here in order to analyze the meaning of evolution in generally covariant quantum theories.

**Quantum evolving constants.** Let us now go to the quantum version of the evolving constants. The Hilbert space is built with the implementation of the physical state vectors \( | \psi \rangle = \sum_n C_n | n \rangle \) in the reduced Hilbert space \( \mathcal{H}_r \) associated with the physical phase space of the harmonic oscillator. In the present case
\[ \psi(x_0) = \langle x_0 | \psi \rangle = \sum_n C_n f_n(x_0) . \]  

(66)

The inner product in \( \mathcal{H}_r \)
\[ \langle \psi | \phi \rangle = \int d\mu(x_0) \psi^*(x_0) \phi(x_0) \]

(67)

can be determined with the condition that the operators \( \hat{x}_0, \) and \( \hat{p}_0 \) be hermitian operators.

Thus, in the classical expression
\[ x = X(t; x_0, p_0) = x_0 \cos(\omega t) + \frac{p_0}{m \omega} \sin(\omega t) , \]

(68)

\( x_0, \) and \( p_0 \) are physical observables given by the Eq. (44) and they become operators acting on \( \mathcal{H}_r, \) so the quantum version of the classical evolving constant is
\[ \hat{x}(t) = x(t; \hat{x}_0, \hat{p}_0) = \hat{x}_0 \cos(\omega t) + \frac{\hat{p}_0}{m \omega} \sin(\omega t) , \]  

(69)
which is the well-known evolution equation for the position operator $\hat{X}$ in the Heisenberg picture.

In addition, the classical expression

$$p = P(t; x_0, p_0) = -m\omega x_0 \sin (\omega t) + p_0 \cos (\omega t),$$  

has its quantum analog

$$\hat{p}(t) = p(t; \hat{x}_0, \hat{p}_0) = -m\omega \hat{x}_0 \sin (\omega t) + \hat{p}_0 \cos (\omega t),$$  

and finally

$$\hat{p}_t = p_t(t; \hat{x}_0, \hat{p}_0) = -\frac{\hat{p}_0^2}{2m} - \frac{1}{2} m\omega^2 \hat{x}_0^2.$$  

In summary, for the case of parameterized systems, the quantum version of the evolving constants equations constitutes the Heisenberg equations for the physical operators involved in each particular theory. In the case of the harmonic oscillator, Eqs. (69) and (71).

**Time operator.** The classical expression (55) becomes an operator $\hat{T}(X, P) = t(X, P; \hat{x}_0, \hat{p}_0)$ which is defined on the reduced Hilbert space $\mathcal{H}_r$. Taken arbitrarily the order of the operators, we have

$$\hat{T}(X, P) = \frac{1}{\omega} \arccos \left( \frac{\frac{1}{2} m\omega X \hat{x}_0 + \frac{1}{2} mP \hat{p}_0}{\hat{H}_0} \right),$$  

with $\hat{H}_0 = \frac{1}{2m} \hat{p}_0^2 + \frac{1}{2} m\omega \hat{x}_0^2$. From this operator, we can compute the ‘time of arrival’ operator $\hat{T}(X)$

$$\hat{T}_\pm(X) = \frac{1}{\omega} \arccos \left( \frac{\frac{1}{2} m\omega X \hat{x}_0 \pm \sqrt{\frac{1}{2m} (\hat{H}_0 - \frac{1}{2} m\omega X^2) \hat{p}_0}}{\hat{H}_0} \right),$$  

associated with the time at which the harmonic oscillator is detected with an apparatus located in $x = X$. The ‘time of arrival’ operator for a free particle has been studied in [13]. The analysis of the ‘time of arrival’ operator for the harmonic oscillator deserves to be studied.
III. $SL(2, R)$ MODEL WITH TWO HAMILTONIAN CONSTRAINTS

A. Classical dynamics

Let us see how the relative evolution looks in a non familiar generally covariant model. A nonlinear generally covariant system with two Hamiltonian constraints and with one physical degree of freedom was introduced in [14]. This model mimics the constraint structure of general relativity. Here, we continue the study of this model. In particular, we display the full set of evolving constants required in its classical and quantum dynamics. Moreover, for a Schrödinger-like equation of motion arises in its quantum dynamics, we compare the meaning of time (evolution) in both, evolving constants and Schrödinger-like equation, viewpoints.

First, a brief summary of its classical dynamics, for more details and its physical interpretation see Ref. [14]. The model is defined by the action

$$S[\vec{u}, \vec{v}, N, M, \lambda] = \frac{1}{2} \int dt \left[ N (D\vec{u}^2 + \vec{v}^2) + M (D\vec{v}^2 + \vec{u}^2) \right],$$  

(75)

where

$$D\vec{u} = \frac{1}{N} (\dot{\vec{u}} - \lambda \vec{u}), \quad D\vec{v} = \frac{1}{M} (\dot{\vec{v}} + \lambda \vec{v});$$  

(76)

the two Lagrangian dynamical variables $\vec{u} = (u^1, u^2)$ and $\vec{v} = (v^1, v^2)$ are two-dimensional real vectors; $N, M$ and $\lambda$ are Lagrange multipliers. The squares are taken in $R^2$: $\vec{u}^2 = \vec{u} \cdot \vec{u} = (u^1)^2 + (u^2)^2$. The action can be put in the Hamiltonian form

$$S[\vec{u}, \vec{v}, \vec{p}, \vec{\pi}, \lambda^m] = \int d\tau \left[ \vec{u} \cdot \vec{p} + \vec{v} \cdot \vec{\pi} - \lambda^m C_m \right].$$  

(77)

The canonical pairs that coordinatize the unconstrained classical phase space are $(u^1, p^1)$, $(u^2, p^2)$, $(v^1, \pi^1)$, and $(v^2, \pi^2)$. Also $\lambda^1 = N$, $\lambda^2 = M$, and $\lambda^3 = \lambda$. The first class constraints have the form

$$C_1 = \frac{1}{2} \left( \vec{p}^2 - \vec{v}^2 \right),$$

$$C_2 = \frac{1}{2} \left( \vec{\pi}^2 - \vec{u}^2 \right),$$

$$C_3 = \vec{u} \cdot \vec{p} - \vec{v} \cdot \vec{\pi},$$

(78)

which algebra is isomorphic to the $sl(2, R)$ Lie algebra.
\{C_1, C_2\} = C_3 \\
\{C_1, C_3\} = -2C_1 \\
\{C_2, C_3\} = 2C_2. \quad (79)

The classical dynamics is the unfolding of the gauge symmetry generated by the Hamiltonian vector fields

\begin{align*}
X_{dC_1} &= -\vec{p} \cdot \vec{\nabla} u - \vec{v} \cdot \vec{\nabla} \pi, \\
X_{dC_2} &= -\vec{\pi} \cdot \vec{\nabla} v - \vec{u} \cdot \vec{\nabla} p, \\
X_{dC_3} &= -\vec{u} \cdot \vec{\nabla} u + \vec{v} \cdot \vec{\nabla} v + \vec{p} \cdot \vec{\nabla} p - \vec{\pi} \cdot \vec{\nabla} \pi, \quad (80)
\end{align*}

associated with the first class constraints of the model.

The physical phase space can be coordinated by the points \((J, \phi, \epsilon, \epsilon')\), and these physical observables have the following form

\begin{align*}
\epsilon &= \frac{u^1 p^2 - p^1 u^2}{|u^1 p^2 - p^1 u^2|}, \\
\epsilon' &= \frac{\pi^1 v^2 - v^1 \pi^2}{|\pi^1 v^2 - v^1 \pi^2|}, \\
J &= |u^1 p^2 - p^1 u^2|, \\
\phi &= \arctan \frac{u^1 v^2 - p^1 \pi^2}{u^1 v^1 - p^1 \pi^1}. \quad (81)
\end{align*}

The Poisson brackets between \(J\) and \(\phi\) in the reduced phase space reads

\[\{J, \phi\} = \epsilon \epsilon'. \quad (82)\]

**Classical evolving constants.** Finally, the relation between the Lagrangian variables \((\vec{u}, \vec{v})\) and the physical states \((J, \phi, \epsilon, \epsilon')\)

\[\left[u^1 v^1 + \epsilon' u^2 v^2\right] \cos \phi + \left[u^1 v^2 - \epsilon' u^2 v^1\right] \sin \phi = J, \quad (83)\]

which leads to the notion of evolving constants of the system \([1,3]\). The evolving constants give the evolution of the Lagrangian variables of the system in a gauge invariant way, i.e., for any fixed physical state of the system \((J, \phi, \epsilon, \epsilon')\), Eq. (83) gives the change of one of the four coordinates as a function of the other three coordinates, say

\[U^1(x, y, z; J, \phi, \epsilon, \epsilon') = \frac{-\epsilon' x(z \cos \phi - y \sin \phi) + \epsilon J}{\epsilon(y \cos \phi + z \sin \phi)}. \quad (84)\]
This relative evolution among the coordinates is gauge invariant. In addition, for any fixed \( x, y, z \) last equation gives a three-parameter family of physical observables, the three parameters are the three coordinates \( x, y, z \), on the physical phase space.

**Generation of evolving constants.** We start with the evolving constant (83), and define the evolving function \( E^1 \)

\[
E^1(u, v, p, \pi) := \left[ u^1 v^1 + \epsilon \epsilon' u^2 v^2 \right] \cos \phi + \left[ u^1 v^2 - \epsilon \epsilon' u^2 v^1 \right] \sin \phi - J.
\]

The restriction of \( E^1 \) on the constraint surface vanishes, \( E^1 \mid_C = 0 \). The action of the Hamiltonian vector field \( X_{dH_1} \) on \( E^1 \) is

\[
X_{dH_1}(E^1) =: E^2 = - \left[ p_1 v^1 + \epsilon \epsilon' p_2 v^2 \right] \cos \phi - \left[ p_1 v^2 - \epsilon \epsilon' p_2 v^1 \right] \sin \phi,
\]

and the restriction of \( E^2 \) on the constraint surface vanishes, so \( E^2 \mid_C = 0 \) gives the evolving constant

\[
\left[ p_1 v^1 + \epsilon \epsilon' p_2 v^2 \right] \cos \phi + \left[ p_1 v^2 - \epsilon \epsilon' p_2 v^1 \right] \sin \phi = 0.
\]

The action of \( X_{dH_1} \) on \( E^2 \) gives zero, so the process ends. Now, we compute the action of the Hamiltonian vector field \( X_{dH_2} \) on \( E^1 \)

\[
X_{dH_2}(E^1) =: E^3 = \left[ u^1 \pi_1 + \epsilon \epsilon' u^2 \pi_2 \right] \cos \phi + \left[ u^1 \pi_2 - \epsilon \epsilon' u^2 \pi_1 \right] \sin \phi,
\]

and the restriction of \( E^3 \) on the constraint surfaces vanishes, so \( E^3 \mid_C = 0 \) gives the evolving constant

\[
\left[ u^1 \pi_1 + \epsilon \epsilon' u^2 \pi_2 \right] \cos \phi + \left[ u^1 \pi_2 - \epsilon \epsilon' u^2 \pi_1 \right] \sin \phi = 0.
\]

The action of \( X_{dH_2} \) on \( E^3 \) gives zero, so the process ends. Finally, the computation of the action of the Hamiltonian vector field \( X_{dD} \) on \( E^1 \)

\[
X_{dD}(E^1) =: E^4 = -E^1 - J,
\]

so we recover the original evolving constant we start with, and no more evolving can be obtained from (83).

**The full solution.** Eqs. (19), and (20) acquire the form

\[
u^1 = U^1(u^2, v^1, v^2, J, \phi, \epsilon, \epsilon') = -\epsilon' u^2 (v^2 \cos \phi - v^2 \sin \phi) + \epsilon J \over \epsilon (v^1 \cos \phi + v^2 \sin \phi),
\]

19
\[ u^2 = U^2(u^2, v^1, v^2; J, \phi, \epsilon, \epsilon') = u^2, \]
\[ v^1 = V^1(u^2, v^1, v^2; J, \phi, \epsilon, \epsilon') = v^1, \]
\[ v^2 = V^2(u^2, v^1, v^2; J, \phi, \epsilon, \epsilon') = v^2, \]
\[ p_1 = P_1(u^2, v^1, v^2; J, \phi, \epsilon, \epsilon') = \epsilon' \left( v^1 \sin \phi - v^2 \cos \phi \right), \]
\[ p_2 = P_2(u^2, v^1, v^2; J, \phi, \epsilon, \epsilon') = \epsilon \left( v^1 \cos \phi + v^2 \sin \phi \right), \]
\[ \pi_1 = \Pi_1(u^2, v^1, v^2; J, \phi, \epsilon, \epsilon') = \frac{\epsilon u^2 v^1 + \epsilon' J \sin \phi}{(v^1 \cos \phi + v^2 \sin \phi)}, \]
\[ \pi_2 = \Pi_2(u^2, v^1, v^2; J, \phi, \epsilon, \epsilon') = \frac{\epsilon u^2 v^2 - \epsilon' J \cos \phi}{(v^1 \cos \phi + v^2 \sin \phi)}, \]
(91)

and the Eqs. (8), and (9) are precisely the Eqs. (81) while the Eqs. (10) acquire the form

\[ u^2 = U^2(u^i, v^i, p_i, \pi_i) = u^2, \]
\[ v^1 = V^1(u^i, v^i, p_i, \pi_i) = v^1, \]
\[ v^2 = V^2(u^i, v^i, p_i, \pi_i) = v^2. \]
(92)

So, the dynamics of this model can be described in a relational fashion way. The difference with respect to parameterized systems, as the example of the harmonic oscillator previously analyzed, is that in the present case a single internal time variable is not enough, rather, we need three internal time variables. In the way we have expressed the full solution (91), \( u^2, v^1, v^2 \) are clocks, i.e., once the component \( u^2 \) of the position of the first particle, and the position \( (v^1, v^2) \) of the second particle are known, the change of the component \( u^1 \) of the first particle and the change of the momenta of both particles \( \vec{p}, \vec{\pi} \) are also known when the system is an particular physical state \( (J, \phi, \epsilon, \epsilon') \). Therefore, the full relational evolution of the system is expressed in terms of three internal clocks \( u^2, v^1, \) and \( v^2 \).

**B. Quantum dynamics**

At quantum level, the model is characterized by the following set of observables

\[ \hat{J} | m, \epsilon, \epsilon' \rangle = m\hbar | m, \epsilon, \epsilon' \rangle, \]
\[ \hat{\epsilon} | m, \epsilon, \epsilon' \rangle = \epsilon | m, \epsilon, \epsilon' \rangle, \]
\[ \hat{\epsilon}' | m, \epsilon, \epsilon' \rangle = \epsilon' | m, \epsilon, \epsilon' \rangle, \]
(93)

and the physical states are given by

\[ | \psi \rangle = \sum_{m, \epsilon, \epsilon'} C_{m, \epsilon, \epsilon'} | m, \epsilon, \epsilon' \rangle, \]
(94)
in abstract Dirac notation. In the ‘coordinate representation’ \(| u, v, \alpha, \beta \rangle\), which is nothing but the Heisenberg picture in standard quantum mechanics because all the coordinates \((u, v, \alpha, \beta)\) are put at the same level, the state reads

\[
\psi(u, v, \alpha, \beta) = \langle u, v, \alpha, \beta | \psi \rangle = \sum_{m, \epsilon, \epsilon'} C_{m, \epsilon, \epsilon'} \langle u, v, \alpha, \beta | m, \epsilon, \epsilon' \rangle,
\]

with \(\langle u, v, \alpha, \beta | m, \epsilon, \epsilon' \rangle = e^{im(\alpha - \epsilon' \beta)} J_m u e^{\alpha \epsilon \bar{\epsilon}'}.\) Thus the basis \(| u, v, \alpha, \beta \rangle\) is ‘rotating’ and the state \(| \psi \rangle\) is fixed, i.e., the coefficients \(C_{m, \epsilon, \epsilon'}\) are constant complex numbers. The ‘coordinate representation’ appears as the most ‘democratic’ basis because it does not prefer one coordinate more than the others.

**Schrödinger equation.** In the same sense that in parameterized systems we were able to build a ‘Schrödinger basis’ from the Heisenberg basis, we can do the same here, and rewrite the physical state (94). In the present example, we can build two Schrödinger bases \(| u, v, \beta \rangle\), and \(| u, v, \alpha \rangle\). In the first one, the physical state vector (94) is expressed as

\[
\psi(u, v, \alpha, \beta) = \langle u, v, \beta | \psi(\alpha) \rangle,
\]

with

\[
| \psi(\alpha) \rangle = \sum_{m, \epsilon, \epsilon'} \tilde{C}_{m, \epsilon, \epsilon'}(\alpha) | m, \epsilon, \epsilon' \rangle = \sum_{m, \epsilon, \epsilon'} C_{m, \epsilon, \epsilon'} e^{im\epsilon \bar{\epsilon}} | m, \epsilon, \epsilon' \rangle,
\]

\[
\langle u, v, \beta | m, \epsilon, \epsilon' \rangle = e^{-im\epsilon' \beta} J_m \left(\frac{uv}{\hbar}\right).
\]

Taking the derivative with respect to the coordinate \(\alpha\) of \(| \psi(\alpha) \rangle\), a Schrödinger equation emerges in the formalism

\[
i\hbar \frac{d}{d\alpha} | \psi(\alpha) \rangle = -\frac{\epsilon}{\epsilon'} \hat{O}_{34} | \psi(\alpha) \rangle,
\]

and the physical observable \(\hat{O}_{34}\) has the form

\[
\langle u, v, \beta | \hat{O}_{34} | \psi \rangle = -\frac{\hbar}{i} \frac{\partial}{\partial \beta} \langle u, v, \beta | \psi \rangle,
\]

in the ‘Schrödinger basis’ \(| u, v, \beta \rangle\). As expected, in the Schrödinger basis \(| u, v, \beta \rangle\), the state \(| \psi(\alpha) \rangle\) evolves while the basis \(| u, v, \beta \rangle\) is fixed with respect to \(\alpha\). This is not a matter of terminology, in fact the evolution equation (98) is well defined, and we are really able of describing evolution under this picture, namely, to describe the change of the some part of the whole state with respect to rest of it, in complete agreement with the spirit of relationism.
In the second Schrödinger basis $|u,v,\alpha\rangle$, the physical state vector (94) is expressed as
\[ \psi(u, v, \alpha, \beta) = \langle u, v, \alpha | \psi(\beta) \rangle, \tag{100} \]
with
\[ |\psi(\beta)\rangle = \sum_{m, \epsilon, \epsilon'} \tilde{C}_{m, \epsilon, \epsilon'}(\beta) |m, \epsilon, \epsilon'\rangle, \]
\[ \langle u, v, \alpha | m, \epsilon, \epsilon' \rangle = e^{im\alpha} J_m \left( \frac{uv}{\hbar} \right). \tag{101} \]

Taking the derivative with respect to the coordinate $\beta$ of $|\psi(\beta)\rangle$, a Schrödinger equation emerges in the formalism
\[ i\hbar \frac{d}{d\beta} |\psi(\beta)\rangle = \frac{\epsilon'}{\epsilon} \hat{O}_{12} |\psi(\beta)\rangle, \tag{102} \]
and the physical observable $\hat{O}_{12}$ has the form
\[ \langle u, v, \alpha | \hat{O}_{12} | \psi \rangle = \frac{\hbar}{i} \frac{\partial}{\partial \alpha} \langle u, v, \alpha | \psi \rangle, \tag{103} \]
in the ‘Schrödinger basis’ $|u, v, \alpha\rangle$.

Quantum evolving constants. The quantum version of the evolving constants is as follows. More precisely, the quantum version of the full classical solution (91) is expressed as
\[ \hat{u}^1 = u^1(u^2, v^1, v^2; \hat{J}, \sin \phi, \cos \phi, \hat{\epsilon}, \hat{\epsilon}') = \frac{-\hat{\epsilon}'(v^2 \cos \phi - v^1 \sin \phi) + \hat{J}}{v^1 \cos \phi - v^2 \sin \phi}, \]
\[ \hat{p}_1 = p_1(u^2, v^1, v^2; \hat{J}, \sin \phi, \cos \phi, \hat{\epsilon}, \hat{\epsilon}') = \hat{\epsilon}' \left( v^1 \sin \phi - v^2 \cos \phi \right), \]
\[ \hat{p}_2 = p_2(u^2, v^1, v^2; \hat{J}, \sin \phi, \cos \phi, \hat{\epsilon}, \hat{\epsilon}') = \hat{\epsilon} \left( v^1 \cos \phi + v^2 \sin \phi \right), \]
\[ \hat{\pi}_1 = \pi_1(u^2, v^1, v^2; \hat{J}, \sin \phi, \cos \phi, \hat{\epsilon}, \hat{\epsilon}') = \frac{\hat{\epsilon} u^2 v^1 + \hat{\epsilon}' \hat{J} \sin \phi}{v^1 \cos \phi + v^2 \sin \phi}, \]
\[ \hat{\pi}_2 = \pi_2(u^2, v^1, v^2; \hat{J}, \sin \phi, \cos \phi, \hat{\epsilon}, \hat{\epsilon}') = \frac{\hat{\epsilon} u^2 v^2 - \hat{\epsilon}' \hat{J} \cos \phi}{v^1 \cos \phi + v^2 \sin \phi}. \tag{104} \]

The meaning of the first equation in (104) is the following: we have to take the mean value of the operator $\hat{u}^1$ with respect to generic states $|\psi\rangle$ of the reduced Hilbert space $\mathcal{H}_r$ of the model [14].
In summary, the quantum dynamics of parameterized systems can be described in terms of a ‘Schrödinger equation’ or in terms of the ‘Heisenberg picture’. The Schrödinger equation arises as a consequence of the Dirac quantization, as we have seen for the case of the harmonic oscillator. On the other hand, in the $SL(2, R)$ model, we were able to build two (dependent) Schrödinger equations, and thus to identify two (dependent) internal time variables $\alpha$ and $\beta$ with respect to which the physical states of the $SL(2, R)$ model evolve. This does not mean that is always possible to single out in general an internal time variable, given by a Schrödinger equation, in generally covariant theories once the Dirac quantization has been performed. Therefore, in general, a Schrödinger equation does not arise in the formalism. The Schrödinger picture, when this picture emerges in the formalism as a consequence of the Dirac quantization, singles out one internal clock only. More important, the quantization of generally covariant theories based on the reduced Hilbert space (generalized Heisenberg picture) need $M$ internal clocks, where $M$ is the number of first class constraints. In the case of the $SL(2, R)$ model, the clocks are $u^2$, $v^1$, and $v^2$ in the generalized Heisenberg picture. In the Schrödinger picture, the internal clock is given by $\alpha$ (or $\beta$).

Classical limit. Now, we compare the quantum evolving constants of the $SL(2, R)$ model with those of the harmonic oscillator in order to get insights on the classical limit of generally covariant theories, and in particular of the $SL(2, R)$ model. We expect that the classical limit of generally covariant theories should be attached to the concept of coherent states as it happens in standard quantum mechanics (parameterized systems). In the case of the harmonic oscillator, the coherent states are roughly those states $|\psi\rangle$ in the reduced Hilbert space $\mathcal{H}_r$ such that the mean values $\langle\psi|\hat{x}(t)|\psi\rangle$, and $\langle\psi|\hat{p}(t)|\psi\rangle$ reproduce the classical behavior of the system. Of course this condition is not enough to single out the coherent states of the system. In addition, those states have also to minimize the uncertainty relations of position and momentum. Of course, these two conditions are still not enough to identify the coherent states due to the fact that both conditions are satisfied by both squeezed and coherent states. In the case of the parameterized harmonic oscillator a mechanism that identifies the coherent states is available following standard methods. It is natural to expect that a combination of the coherent states approach to the quantization of generally covariant theories [15] with the full set of evolving constants of motion required in their quantum dynamics displayed here could bring the classical limit of constrained systems.

IV. CONCLUDING REMARKS

We have displayed the full solution of the relational evolution of the degrees of freedom of fully constrained theories with a finite number of degrees of freedom (see Eqs. (19), and (20)). Our procedure follows from the embedding equations of the coordinates and momenta in the unconstrained phase space (see Eqs. (8), and (7)) plus the expressions of the $M$ internal time variables (see Eq. (10)). The form of the solution contains
all the evolving constants of motion needed in the description of the classical dynamics of fully constrained theories, i.e., we have given the full mathematical solution to the Rovelli’s point of view on the ‘problem of time’ pioneered in Refs. [4–6]. Of course, the physical (and philosopical) interpretation is due to Rovelli. Also, we have explored a method to generate those evolving constants. This method consists in the repeated application of the Hamiltonian vector fields associated with the first class constraints on some initial evolving constant. Combining the expressions of this evolving constants with the expressions of the physical observables the full relational evolution of the coordinates and momenta is obtained. Finally, we have also analysed on a general setting the quantum version of the relational evolution of the degrees of freedom of fully constrained theories.

To find the full solution of the relational evolution of the degrees of freedom for gravity, matter fields coupled to gravity (see [16] for the first steps), topological quantum field theories, or for a background-independent string theory constitutes one of the challenges of the new millenium.

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