THE MODIFIED SCATTERING FOR DIRAC EQUATIONS OF SCATTERING-CRITICAL NONLINEARITY

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Abstract. In this paper, we consider the Maxwell-Dirac system in 3 dimension under zero magnetic field. We prove the global well-posedness and modified scattering for small solutions in the weighted Sobolev class. Imposing the Lorenz gauge condition, (and taking the Dirac projection operator), it becomes a system of Dirac equations with Hartree type nonlinearity with a long range potential as $|x|^{-1}$. We perform the weighted energy estimates. In this procedure, we have to deal with various resonance functions that stem from the Dirac projections. We use the spacetime resonance argument of Germain-Masmoudi-Shatah ([13, 14, 15]), as well as the spinorial null-structure. On the way, we recognize a long range interaction which is responsible for a logarithmic phase correction in the modified scattering statement.

1. Introduction

Consider the Maxwell-Dirac equations (MD) in $\mathbb{R} \times \mathbb{R}^3$:

\[
\begin{cases}
(\partial_t + igA_0)\Psi = \sum_{j=1}^{3} \alpha^j(\partial_j + igA_j)\Psi - im\beta\Psi & \text{in } \mathbb{R}^{1+3}, \\
-\Box A_0 = g(\Psi, \Psi), \\
\Box A_j = g(\Psi, \alpha^j\Psi).
\end{cases}
\]

(MD)

Here the unknown $\Psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ is the spinor field, $A = (A_0, A_1, A_2, A_3)$ is given real-valued 1-form, and the $4 \times 4$ matrices $\alpha^j$’s and $\beta$ are Dirac matrices as follows:

\[
\alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},
\]

where $\sigma^j$’s are Pauli matrices given by

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The constant $m > 0$ is a mass and $g > 0$ is a coupling constant. The d’Alembert operator $\Box$ has the form $\Box := \partial_t^2 - \Delta$ and the inner product is defined by $\langle \psi, \phi \rangle = \psi^i\phi.$

The equations (MD) model an electron in electromagnetic field and form a fundamental system in quantum electrodynamics. (MD) is invariant under the gauge transformation: $(\Psi, A) \to (e^{i\chi}\Psi, A - d\chi)$ for a real-valued function $\chi$ on $\mathbb{R} \times \mathbb{R}^3$.

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For concreteness of our discussion, let us choose Lorenz gauge, which is defined by the condition

\[-\partial_t A_0 + \sum_{j=1}^{3} \partial_j A_j = 0.\]

By assuming vanishing magnetic field

\[
\text{curl}(A_1, A_2, A_3) = 0
\]

one can find a function \( \varphi \) such that \( \nabla \varphi = (A_1, A_2, A_3) \). The gauge transformation \( \psi(t, x) := e^{i\varphi(t, x)} \mathcal{F}(\psi(t, x)) \) uncouples the equations (MD) to satisfy the Dirac equations (DE)

\[
\begin{cases}
\partial_t + \sum_{j=1}^{3} \alpha_j \partial_j + im \beta \psi = ic_1 (|x|^{-1} \ast |\psi|^2) \psi \text{ in } \mathbb{R}^{1+3}, \\
\psi(0) = \psi_0,
\end{cases}
\]

where \( c_1 = \frac{q^2}{2} \) and \( |\psi|^2 = \langle \psi, \psi \rangle \). We refer to [4] for rigorous derivation. Any smooth solution to (DE) satisfies the \( L^2 \) conservation laws:

\[
\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}.
\]

By a scaling in \( L^2 \) we set \( m = 1 \) hereafter. The purpose of this paper is to develop a modified scattering theory of global solutions to (DE) for small initial data.

As observed in [9], the square of linear operator is diagonalized

\[
(\sum_{j=1}^{3} \alpha_j \cdot \xi_j + \beta)^2 = \langle \xi \rangle^2 I_4,
\]

where \( \langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}} \) for frequency vector \( \xi \in \mathbb{R}^3 \). We introduce the Dirac projection operators \( \Pi_{\pm}(D) \) defined by

\[
\Pi_{\pm}(D) := \frac{1}{2} \left( I_4 \pm \frac{1}{(D)} \left[ \sum_{j=1}^{3} \alpha_j D_j + \beta \right] \right),
\]

where \( D = (D_1, D_2, D_3) \), \( D_j = -i \partial_j \), and \( \mathcal{F}(i(D)_f) = \langle \xi \rangle \hat{f} \). Here \( \mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) \, dx \). Then using (1.2), we get

\[
i \left( -i \sum_{j=1}^{3} \alpha_j \partial_j + \beta \right) = \langle D \rangle (\Pi_{\pm}(D) - \Pi_{\mp}(D)).
\]

By definition of projections we further get

\[
\Pi_{\pm}(D) + \Pi_{\mp}(D) = I_4, \quad \Pi_{\pm}(D) \Pi_{\mp}(D) = \Pi_{\pm}(D), \quad \Pi_{\pm}(D) \Pi_{\mp}(D) = 0.
\]

By \( \psi_{\pm} \) we denote \( \Pi_{\pm}(D) \psi \). Then, the Dirac equation (DE) can be rewritten as:

\[
\begin{cases}
(i\partial_t - \langle D \rangle) \psi_+ = -c_1 \Pi_{\pm}(D) \left( |x|^{-1} \ast |\psi|^2 \right) \psi, \\
(i\partial_t + \langle D \rangle) \psi_- = -c_1 \Pi_{\pm}(D) \left( |x|^{-1} \ast |\psi|^2 \right) \psi,
\end{cases}
\]

\[
\psi_+(0) = \psi_{0,+} := \Pi_{\pm}(D) \psi_0 \quad \text{and} \quad \psi_-(0) = \psi_{0,-} := \Pi_{\mp}(D) \psi_0.
\]
which is a system of half Klein-Gordon equations coupled by Hartree nonlinearity. Denoting by \( e^{\mp i t(D)} \psi_{0,\pm} \) the free solutions to (1.5)

\[
e^{\mp i t(D)} \psi_{0,\pm}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{i(x \cdot \xi \mp it(\xi))} \widetilde{\psi}_{0,\pm}(\xi) \, d\xi,
\]

we can express the free solution to (DE) as

\[
U(t)\psi_0 := e^{-i t(D)} \Pi_+(D) \psi_0 + e^{i t(D)} \Pi_-(D) \psi_0.
\]

By Duhamel’s principle, the solutions to (1.5) satisfy the following integral equation

\[
\begin{align*}
\psi_+(t) &= e^{-i t(D)} \psi_{0,+} + i \int_0^t e^{-i (t-t') (D)} \Pi_+(D) \left[ (|x|^{-1} * |\psi|^2) \psi \right](t') \, dt', \\
\psi_-(t) &= e^{i t(D)} \psi_{0,-} + i \int_0^t e^{i (t-t') (D)} \Pi_-(D) \left[ (|x|^{-1} * |\psi|^2) \psi \right](t') \, dt'.
\end{align*}
\]

There has been considerable mathematical interest in the Cauchy problem for (MD). The local well-posedness (LWP) for small smooth initial data was first proved by Groos [17]. Later, Bournaveas [3] considered rough initial data and obtained (LWP) result in \( H^s(\mathbb{R}^3) \) for \( s > \frac{1}{2} \). D’Ancona, Foschi and Selberg [10] proved almost optimal (LWP) in \( H^s(\mathbb{R}^3) \) with \( s > 0 \) in the sense that (MD) is \( L^2 \) critical, which means that it leaves the \( L^2 \)-mass invariant by a scaling. (MD) on other dimensions are also widely studied. Huh [21] considered (MD) on \( \mathbb{R}^{1+1} \) and proved global well-posedness (GWP) in \( L^2_x(\mathbb{R}) \) and asymptotic behavior of the solutions. D’Ancona and Selberg [11] considered (MD) on \( \mathbb{R}^{1+2} \) and established (GWP) in \( L^2_x(\mathbb{R}^2) \). Recently, Gavrus and Oh [12] established a linear scattering theory of (MD) on \( \mathbb{R}^{1+d} \) for \( d \geq 4 \) under the Coulomb gauge condition.

The Cauchy problem for (DE) also has been extensively studied by several authors. Chadam and Glasssey [4] showed the existence of a unique global solution for smooth initial data with compact support when \( d = 2 \). Herr and Tesfahun [20] proved a small data scattering of (DE) with a potential replaced by Yukawa potential \( e^{-|x|/x} \) in \( H^s(\mathbb{R}^3) \) for \( s > \frac{1}{2} \). As explained in [20, Remark 1.2], (LWP) for (DE) in \( H^s(\mathbb{R}^3) \) for \( s > \frac{1}{4} \) can be derived by a slight modification of the argument in Herr and Lenzmann [19].

In this paper, we are interested in asymptotic behaviors of small global solutions to (DE). The equation is scaling-critical in the sense that the Duhamel terms decay as fast as the linear solutions and so it cannot be seen purely perturbative term. A heuristic computation shows that a long range potential in the nonlinearity gives \(|x|^{-1} * [e^{-\theta t(D)} \psi \psi] \approx t^{-1} \) if \( \psi \neq 0 \) in \( L^2(\mathbb{R}^3) \). Indeed, in [5, 7], the authors showed that the linear scattering is not possible in \( L^2 \) space. Hence, one can anticipate a modified scattering, which means solutions decay like linear solutions and converge to linear-like solutions after logarithmic phase correction. Here, we state our main result of global existence and modified scattering.

**Theorem 1.1.** Let \( k \geq 1000 \). There exists \( \varepsilon_0 > 0 \) such that:

(i) Suppose that \( \psi_0 : \mathbb{R}^3 \to \mathbb{C} \) is small in weighted spaces as follows:

\[
\sum_{\theta \in \{+,+\}} \left[ \|\psi_{0,\theta}\|_{H^s} + \|\langle x\rangle^2 \psi_{0,\theta}\|_{H^2} + \|\langle \xi\rangle^{10} \widetilde{\psi}_{0,\theta}\|_{L^\infty_{\xi}} \right] < \varepsilon_0
\]

1In [5, 7], the authors proved the nonexistence of linear scattering for the equations with \( |\psi|^2 \) rather than \( |\psi|^4 \). But almost the same argument can be applied to (1.5).
for $\varepsilon_0$ with $0 < \varepsilon_0 < \frac{1}{2}$. Then the Cauchy problem (DE) with initial data $\psi_0$ has a unique global solution $\psi$ to (DE) decaying as

\begin{equation}
\|\psi(t)\|_{L^\infty_x} \lesssim \varepsilon_0(t)^{-\frac{3}{2}}.
\end{equation}

(ii) Moreover, there exists a free solution $U(t)\phi$ such that

\begin{equation}
\left\| (\xi)^{10} \mathcal{F} \left[ \psi(t) - U_B(t)U(t)\phi \right] (\xi) \right\|_{L_x^\infty} \lesssim \langle t \rangle^{-\delta_0} \varepsilon_0,
\end{equation}

for some $0 < \delta_0 < \frac{1}{100}$, where the phase corrections are given by

$$
U_B(t) = \sum_{\theta \in \{\pm\}} e^{-iB(t, \theta)D} \Pi_\theta(D)
$$

and

\begin{equation}
\begin{cases}
B(t, \xi) = B_+(t, \xi) + B_-(t, \xi), \\
B_\theta(t, \xi) = \frac{c_1}{(2\pi)^3} \int_0^t \int_{\mathbb{R}^3} \left\| \xi \theta - \langle \sigma \rangle \right\|^{-1} \left\| \tilde{\psi}_\theta(\sigma) \right\|^2 ds \rho(s^{-\frac{n}{2n}}\xi) ds,
\end{cases}
\end{equation}

for $t > 0$, a real constant $c_1$ and $\rho \in C_0^\infty(B(0, 2))$. A similar result holds for $t < 0$ by time reversal symmetry.

**Remark 1.2.** In Theorem 1.1, the scattering sense (1.9) is described in the Fourier space. This is because the energy estimates are carried out in $L_x^\infty$. (See Propositions 4.2.) But it can also be expressed in the physical space by the following observation

$$
\left\| \psi(t) - U_B(t)U(t)\phi \right\|_{L^\infty_x} \lesssim \left\| (\xi)^{10} \mathcal{F} \left[ \psi(t) - U_B(t)U(t)\phi \right] \right\|_{L^\infty_x} \xrightarrow{t \to \infty} 0.
$$

The modified scattering of small solutions often occurs when the nonlinearity contains long-range interactions. This topic has been studied by many authors. Ozawa [24] showed the modified scattering for 1D cubic nonlinear Schrödinger equations (NLS). Hayashi and Naumkin [18] showed it for high dimensional NLS. Later, it turns out that the spacetime-resonance argument by Germain, Masmoudi, and Shatah is efficient for these problems [16, 22, 23, 25]. Among others, the most relevant reference to this work is the modified scattering result by Pusateri [25] for Boson star equation:

\begin{equation}
(-i\partial_t + \langle D \rangle) u = (|x|^{-1} * |u|^2) u.
\end{equation}

Using the Dirac projection, (DE) is written in (1.5), which has a similar linear structure to (1.11). But the scalar equation (1.11) has a single resonance function $p = \langle \xi \rangle - \langle \xi - \eta \rangle - \langle \eta + \sigma \rangle + \langle \sigma \rangle$ in the interaction representation. In particular, since $\nabla_\xi p$ gives a null-structure near $\eta = 0$, a possible loss of time factor can be recovered when weighted energy estimates\(^2\) are concerned. However, in (1.5) each nonlinear term contains all signs of Dirac projections, and it gives rise to various phases of interaction,

\[ p_\Theta = \theta_0 \langle \xi \rangle - \theta_1 \langle \xi - \eta \rangle - \theta_2 \langle \eta + \sigma \rangle + \theta_3 \langle \sigma \rangle, \quad \Theta = (\theta_0, \theta_1, \theta_2, \theta_3), \]

with $\theta_j \in \{+, -\}$ (see (5.1)). Hence the aforementioned null-structures arising from $\nabla_\xi p_\Theta$ may be lost for some combinations of signs. This causes a significant obstacle in the course of weighted energy estimates. To overcome this obstacle, for some cases, we utilize the spinorial null-structure occurring from the Dirac projections.
\(\Pi_\theta\), which is successful except for the cases \(\theta_0 \neq \theta_1, \theta_2 = \theta_3\). The remaining cases \(\theta_0 \neq \theta_1, \theta_2 \neq \theta_3\) can be treated by the usual time non-resonance (integration by parts in time) (see case b in Section 5).

The proof is based on the bootstrap argument. We construct a function space adapted to time decay estimates below (2.2). By assuming the norm is a priori small, we deduce that the solutions decay like a solution to a linear equation. To achieve this, we first establish the weighted energy estimates, which require to control \(x^2 \psi_\theta\) and \(x^2 \psi_\theta\) in \(H^2\). Here, several null conditions, as well as time non-resonance, play an important role in the course of estimates (see Section 5.1). And then, for asymptotic analysis, we approximate the interaction function \(f_\pm = e^{\pm it(D)} \psi_\pm\) in the Fourier space by using the Taylor expansion to figure out the leading contribution where the phase logarithm correction is required. For rigorous analysis, we decompose the singular potential in the Fourier space, \(F(|x|^{-1}) \sim |\eta|^{-2}\), with a suitable scale in \(\eta\) depending on time, say \(t^{1/2}\). Then, \(|\eta| \lesssim t^{1/2}\) corresponds to the main term, and, at the same time, remaining contributions on \(|\eta| \gtrsim t^{1/2}\) are shown to be integrable \(O(t^{-1/2})\) via the integration by parts in space variables.

The paper is organized as follows: In Section 2, we introduce a function space adapted to time decay estimates. Under a priori assumption, we establish several frequency localized multilinear estimates. In Section 3, we investigate the null structures which improve the estimates obtained in Section 2. The improved estimates play a crucial role in the course of the weighted energy estimates. In Section 4, we provide the proof of the main theorem by assuming the weighted energy estimates and \(L^\infty\) estimates, followed by the proof of these two estimates in the last two sections, respectively.

1.1. Notations. • (Mixed-normed spaces) For a Banach space \(X\) and an interval \(I\), \(u \in L^p_I X\) if \(u(t) \in X\) for a.e. \(t \in I\) and \(\|u\|_{L^p_I X} := \|u(t)\|_X\) for all \(t \in I\). Especially, we denote \(L^p_{\infty} = L^p_{\infty}(I; L^\infty_{\infty}(\mathbb{R}^3))\), \(L^p_{\infty} = L^p_{\infty}(I; L^p_{\infty}(\mathbb{R}^3))\), \(L^p_{\infty} = L^p_{\infty}(I; L^p_{\infty}(\mathbb{R}^3))\), \(L^p_{\infty} = L^p_{\infty}(I; L^p_{\infty}(\mathbb{R}^3))\), \(L^p_{\infty} = L^p_{\infty}(I; L^p_{\infty}(\mathbb{R}^3))\).

• For \(\alpha \in \mathbb{N}\), \(\|f\|_{W^s,\alpha} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty}\), where \(\alpha\) denotes the multi-index and \(\|f\|_{W^s} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty}\), with \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\).

• (Dyadic decomposition) Let \(\rho\) be a bump function such that \(\rho \in C^\infty_0(B(0, 2))\) with \(\rho(\xi) = 1\) for \(|\xi| \leq 1\) and define \(\rho_N(\xi) := \rho(\xi) - \rho(\xi/2)\) for \(N \in \mathbb{Z}\). Then we define the frequency dyadic projection operator \(P_N\) by \(F(P_N f)(\xi) = \rho_N(\xi) \hat{f}(\xi)\). Similarly, we define \(P_{\leq N}\) and \(P_{< N}\) as the fourier multipliers with symbols \(\rho_{\leq N}(\xi) := 1 - \sum_{N > N_0} \rho_N\) and \(\rho_{< N} := \rho_{< N} + \rho_{N} + \rho_{< 2N}\), respectively. We observe that \(P_N P_N = P_N\). In addition, we denote \(P_{N_1 \leq \cdots \leq N_2} := \sum_{N_1 \leq N \leq N_2} P_N\). Especially, we denote \(P_N f\) simply by \(f_N\) for any measurable function \(f\).

• Let \(A = (A_i, B = (B_j) \in \mathbb{R}^n\). Then \(A \otimes B\) denotes the usual tensor product such that \((A \otimes B)_{ij} = A_i B_j\). We also denote a tensor product of \(A \in \mathbb{C}^n\) and \(B \in \mathbb{C}^m\) by a matrix \(A \otimes B = (A_i B_j)_{i=1,\ldots,n, j=1,\ldots,m}\). We often denote \(A \otimes B\) simply by

\[ \sum_{i=1}^{n} \sum_{j=1}^{m} A_i B_j. \]
2. Function space and Time decay

2.1. Time decay of Dirac equations. For given small initial data as in (1.7), by a standard local theory in weighted energy spaces, we have a small local solution $\psi_0(t)$ on $[0,T_0]$ for $\theta \in \{+,-\}$. Our goal is to show the solutions are global and decay in time. Let us begin with introducing a spacetime norm incorporating time decay. Let $k \geq 300$ and $0 < \delta_0 < \frac{1}{100}$. For $\varepsilon_1 > 0$ to be chosen later, we assume a priori smallness of solutions: for a large time $T > 0$,

$$\|\psi\|_{\Sigma_T} := \|\Pi_+(D)\psi\|_{\Sigma^+_T} + \|\Pi_-(D)\psi\|_{\Sigma^-_T} \lesssim \varepsilon_1,$$

where

$$\|\phi\|_{\Sigma^\pm_T} := \sup_{t \in [0,T]} \left\{ \langle t \rangle^{-\delta_0} \|\phi(t)\|_{H^k} + \|\langle x \rangle^{1/2} e^{\pm i t(D)} \phi(t)\|_{H^2} + \|\langle \xi \rangle^{10} \widetilde{\phi(t)}\|_{L^\infty_x} \right\}.$$

One may relax regularity assumptions, e.g., $H^k$, $\langle \xi \rangle^{10}$, or $\delta_0$. We do not pursue to optimize those parameters.

In order to obtain pointwise decay of solutions from the above a priori bound, we use the following linear estimates.

**Proposition 2.1** (Linear decay estimates). Let $f : \mathbb{R}^3 \to \mathbb{C}^4$. For any $t \in \mathbb{R}$ one has

$$\left\| e^{\pm i t(D)} f \right\|_{L^\infty_x(\mathbb{R}^3)} \lesssim \frac{1}{(1 + |t|)^{\frac{7}{2}}} \left\| (1 + |\xi|)^6 \hat{f}(\xi) \right\|_{L_x^\infty(\mathbb{R}^3)} + \frac{1}{(1 + |t|)^{\frac{3}{2}}} \left\{ \left\| \langle x \rangle^{2} f \right\|_{L^2_x(\mathbb{R}^3)} + \| f \|_{H^5(\mathbb{R}^3)} \right\}.$$

We refer to [25] for the proof of (2.2). As a consequence of Proposition 2.1 and a priori assumption (2.1), there exists $C$ such that

$$\|\psi(t)\|_{W^{2,\infty}} \leq \|\Pi_+(D)\psi(t)\|_{W^{2,\infty}} + \|\Pi_-(D)\psi(t)\|_{W^{2,\infty}} \leq C\varepsilon_1 (1 + t)^{-\frac{7}{2}}$$

for any $0 \leq t \leq T$.

2.2. Nonlinear term estimates. Let us denote the Hartree nonlinear term by

$$\mathcal{N}(\psi_1, \psi_2, \psi_3)(s) := (|x|^{-1} * (\psi_3(s), \psi_2(s))) \psi_1(s).$$

In the next lemma, we introduce several inequalities for the Hartree nonlinear term. The proof is an immediate consequence of the Hardy-Littlewood-Sobolev inequality, so we omit it.

**Lemma 2.2** (Hartree nonlinear term estimates). Let $\psi_i : \mathbb{R}^{1+3} \to \mathbb{C}^4$ for $i = 1, 2, 3$. Then,

$$\left\| \mathcal{N}(\psi, \psi, \psi) \right\|_{H^m(\mathbb{R}^3)} \lesssim \|\psi\|_{L^2} \|\psi\|_{H^m} \|\psi\|_{L^6}, \quad \text{for } m \in \mathbb{N} \cup \{0\},$$

$$\left\| \mathcal{N}(\psi_1, \psi_2, \psi_3) \right\|_{H^2(\mathbb{R}^3)} \lesssim \|\psi_1\|_{H^2} \left( \|\psi_2\|_{H^2} \|\psi_3\|_{L^6} + \|\psi_2\|_{L^6} \|\psi_3\|_{H^2} \right),$$

$$\left\| \mathcal{N}(\psi_1, \psi_2, \psi_3) \right\|_{W^{2,\infty}(\mathbb{R}^3)} \lesssim \|\psi_1\|_{W^{2,\infty}} \left( \|\psi_2\|_{H^1} \|\psi_3\|_{L^6} + \|\psi_2\|_{L^6} \|\psi_3\|_{H^1} \right).$$
As a consequence, we have the following time decay estimates for the nonlinear term when $\psi$ is in $\mathcal{S}_T$.

**Corollary 2.3.** Let $\psi$ be solution to (DE) satisfying the a priori assumption (2.1). Then,
\[
\|N(\psi, \psi, \psi)(s)\|_{L^6(R^3)} \lesssim \langle s \rangle^{-1+\delta_0} \epsilon_1^3,
\]
(2.5)
\[
\|N(\psi, \psi, \psi)(s)\|_{W^{2, \infty}(R^3)} \lesssim \langle s \rangle^{-\frac{2}{3}} \epsilon_1^3.
\]

**Proof.** By the previous Lemma, we only suffice to bound $L^6$ norm. Interpolating a priori decay assumption (2.3) and the conservation law (1.1), we have
\[
\|\psi(s)\|_{L^6(R^3)} \lesssim \langle s \rangle^{-1} \epsilon_1.
\]
Therefore Lemma 2.2 finishes the proof of Corollary 2.3.

Next, we establish frequency localized estimates, which will be used several times in the course of the energy estimates. The proof is quite standard (see [6, 25]), but we provide the proof for self-containment.

**Lemma 2.4.** Let $\psi_i : R^{1+3} \rightarrow C^4$ for $i = 1, 2$ and $N \in 2^Z$ be a dyadic number. Then, we get
\[
\|P_N \psi_1, \psi_2\|_{L^2(R^3)} \lesssim N^2 \langle N \rangle^{-5} \left\| \left( \hat{D} \right)^{10} \psi_1 \right\|_{L^2} \left\| \left( \hat{D} \right)^{10} \psi_2 \right\|_{L^\infty},
\]
\[
\|P_N \psi_1, \psi_2\|_{L^2(R^3)} \lesssim \langle N \rangle^{-2} \left\| \psi_1 \right\|_{W^{2, \infty}} \left\| \psi_2 \right\|_{W^{2, \infty}},
\]
\[
\|P_N \psi_1, \psi_2\|_{L^2(R^3)} \lesssim N^3 \left\| \psi_1 \right\|_{L^2} \left\| \psi_2 \right\|_{L^2}.
\]

**Proof.** By Plancherel’s theorem and Hölder inequality, we see that
\[
\|P_N \psi_1, \psi_2\|_{L^2} \lesssim \int_{R^3} \rho_N^2(\xi) |\mathcal{F}(\psi_1, \psi_2)(\xi)|^2 \, d\xi
\]
\[
\lesssim \int_{R^3} \rho_N^2(\xi) \left| \int_{R^3} \langle \eta \rangle^{-10} (\xi - \eta)^{-10} \left\langle \left( \hat{D} \right)^{10} \psi_1(\eta), \left( \hat{D} \right)^{10} \psi_2(\xi - \eta) \right\rangle \, d\eta \right|^2 \, d\xi
\]
\[
\lesssim \left\| \left( \hat{D} \right)^{10} \psi_1 \right\|_{L^\infty} \left\| \left( \hat{D} \right)^{10} \psi_2 \right\|_{L^2} \int_{R^3} \rho_N^2(\xi) \left| \int_{R^3} \langle \eta \rangle^{-10} (\xi - \eta)^{-10} \, d\eta \right|^2 \, d\xi
\]
\[
\lesssim N^3 \langle N \rangle^{-10} \left\| \left( \hat{D} \right)^{10} \psi_1 \right\|_{L^\infty} \left\| \left( \hat{D} \right)^{10} \psi_2 \right\|_{L^2}.
\]

Next, we estimate
\[
\|P_N \psi_1, \psi_2\|_{L^\infty} = \sup_{x \in R^3} \int_{R^3} e^{ix \cdot \xi} \rho_N(\xi) \mathcal{F}(\psi_1, \psi_2)(\xi) \, d\xi
\]
\[
= \sup_{x \in R^3} \int_{R^3} e^{ix \cdot \xi} \rho_N(\xi) (\xi)^{-2} \mathcal{F} \left[ (1 - \Delta) \psi_1, \psi_2 \right] (\xi) \, d\xi
\]
\[
\lesssim \|\mathcal{F}^{-1}(\rho_N(\xi)^{-2}) \ast (1 - \Delta) \psi_1, \psi_2\|_{L^\infty}
\]
\[
\lesssim \|\mathcal{F}^{-1}(\rho_N(\xi)^{-2})\|_{L^1} \| (1 - \Delta) \psi_1, \psi_2\|_{L^\infty}
\]
\[
\lesssim \langle N \rangle^{-2} \left\| \psi_1 \right\|_{W^{2, \infty}} \left\| \psi_2 \right\|_{W^{2, \infty}}.
\]

We also have
\[
\|P_N \psi_1, \psi_2\|_{L^\infty} \lesssim \|\rho_N \mathcal{F}(\psi_1, \psi_2)\|_{L^1} \lesssim \|\rho_N\|_{L^1} \|\mathcal{F}(\psi_1, \psi_2)\|_{L^\infty}
\]
\[
\lesssim N^3 \left\| \psi_1 \right\|_{L^2} \left\| \psi_2 \right\|_{L^2}.
\]
As a direct consequence of the Lemma 2.4, we find time decay of localized bilinear term with \( \psi \) satisfying the \( a \) priori assumption (2.1).

**Corollary 2.5.** Let \( \theta_j \in \{ +, - \} \) and \( \psi_j \) satisfy \( \| \psi_j \|_{L^2_x} \lesssim \varepsilon_1 \) for \( j = 1, 2 \). For a dyadic number \( \mathcal{N} \in 2^\mathbb{Z} \), we have

\[
\| P_N \langle \psi_1, \psi_2 \rangle (s) \|_{L^2_x} \lesssim \mathcal{N}^2 \varepsilon^2_1,
\]

\[
\| P_N \langle \psi_1, \psi_2 \rangle (s) \|_{L^\infty_x} \lesssim \min(\langle \mathcal{N} \rangle^{-2}(s)^{-3}, \mathcal{N}^2 \varepsilon^2_1).
\]

Also, the time decay estimates for the frequency localized Hartree term can be easily obtained from Lemma 2.4 and Corollary 2.3.

**Corollary 2.6.** Let \( \psi \) and \( \phi \) satisfy (2.1). For a dyadic number \( \mathcal{N} \in 2^\mathbb{Z} \), we have

\[
\| P_N \mathcal{N} \langle \psi, \psi, \phi \rangle (s) \|_{L^2_x} \lesssim \min \left( \mathcal{N}^2 \langle s \rangle^{-2}, \langle \mathcal{N} \rangle^{-10} \langle s \rangle^{-1} \right) \varepsilon^3_1,
\]

\[
\| P_N \mathcal{N} \langle \psi, \psi, \phi \rangle (s) \|_{L^\infty_x} \lesssim \min \left( \langle \mathcal{N} \rangle^{-2}(s)^{-4}, \mathcal{N}^3 \langle s \rangle^{-1} \right) \varepsilon^4_1.
\]

We close this section by introducing a useful lemma about pseudo-product operators which will repeatedly used in our analysis.

**Lemma 2.7** (Coifman-Meyer operator estimates). Assume that a multiplier \( \mathbf{m} \) satisfies

\[
\| \mathbf{m} \|_{CM} := \left\| \int_{\mathbb{R}^3} \mathbf{m}(\xi, \eta) e^{i\xi \cdot \xi} e^{i\eta \cdot \eta} d\xi d\eta \right\|_{L^1_{\xi,\eta}(\mathbb{R}^3 \times \mathbb{R}^3)} < \infty.
\]

Then, for \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \),

\[
\left\| \int_{\mathbb{R}^3} \mathbf{m}(\xi, \eta) \hat{\psi}(\xi \mp \eta) \hat{\phi}(\eta) d\eta \right\|_{L^2_i} \lesssim \| \mathbf{m} \|_{CM} \| \hat{\psi} \|_{L^p} \| \hat{\phi} \|_{L^q},
\]

and for \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \),

\[
\left| \int_{\mathbb{R}^3} \mathbf{m}(\eta, \sigma) \hat{\psi}_1(\eta \mp \sigma) \hat{\psi}_2(\eta) \hat{\psi}_3(\sigma) d\eta d\sigma \right| \lesssim \| \mathbf{m} \|_{CM} \| \hat{\psi}_1 \|_{L^p} \| \hat{\psi}_2 \|_{L^q} \| \hat{\psi}_3 \|_{L^r}.
\]

3. Null-structure

To prove Theorem 1.1, we have to use structure of the equation obtained from the Dirac projection operator (1.3) and the resonance function. Lemmas in this section will be used throughout the proofs of Section 5 and 6.

From (1.3), the symbol of Dirac projection operators are given by

\[
\Pi_{\pm}(\xi) = \frac{1}{2} \left( I \pm \frac{\alpha \cdot \xi + \beta}{\langle \xi \rangle} \right).
\]

The symbol are bounded in the sense that

\[
|\nabla^n \Pi_{\pm}(\xi)| \lesssim \frac{1}{\langle \xi \rangle^{n-1}},
\]

which implies the boundedness of Dirac projection operators in the Sobolev spaces: For \( 1 < p < \infty \) and \( s \geq 0 \),

\[
\| \Pi_{\pm}(D) \psi \|_{W^{s,p}} \lesssim \| \psi \|_{W^{s,p}}.
\]
Furthermore, one can easily verify the null structure
\begin{equation}
\Pi_{\pm}(\xi)\Pi_{\pm}(\xi) = 0,
\end{equation}
which proved very useful especially when we deal with bilinear operators in the nonlinear term \((1, 2, 9, 10)\).

**Lemma 3.1.** For \(\xi, \eta \in \mathbb{R}^3\) satisfying \(|\eta| \ll |\xi|\), we have
\begin{equation}
\left| \nabla^{m}_{\eta} \nabla^{p}_{\xi} |\Pi_{\pm}(\xi)\Pi_{\pm}(\xi - \eta)| \right| \lesssim |\eta|^{1-m} \max \left( \frac{1}{|\xi|^{n+1}}, \frac{1}{(|\xi| - |\eta|)^{n+1}} \right).
\end{equation}

**Proof.** We only consider \(\pm = +\). We compute
\begin{equation}
4\Pi_{+}(\xi)\Pi_{-}(\xi - \eta) = \left( I + \frac{\alpha \cdot \xi + \beta}{\langle \xi \rangle} \right) \left( I - \frac{\alpha \cdot (\xi - \eta) + \beta}{\langle \xi - \eta \rangle} \right) = \alpha \cdot \left( \frac{\xi}{\langle \xi \rangle} - \frac{\xi - \eta}{\langle \xi - \eta \rangle} \right) + \beta \left( \frac{\alpha \cdot \xi}{\langle \xi \rangle} - \frac{1}{\langle \xi - \eta \rangle} \right) - \frac{\alpha \cdot \xi}{\langle \xi \rangle} \frac{(\alpha \cdot \xi)(\alpha \cdot \eta)}{\langle \xi \rangle \langle \xi - \eta \rangle} + \frac{(\alpha \cdot \eta)\beta}{\langle \xi \rangle} \frac{(\eta - 2\xi) \cdot \eta I}{\langle \xi - \eta \rangle (\langle \xi \rangle + \langle \xi - \eta \rangle)}.
\end{equation}

For the first term, we apply the mean value theorem
\begin{equation}
\frac{\xi}{\langle \xi \rangle} - \frac{\xi - \eta}{\langle \xi - \eta \rangle} = \int_{0}^{1} \frac{1}{\xi_{\lambda}} \left( I - \frac{\xi_{\lambda} \otimes \xi_{\lambda}}{\langle \xi_{\lambda} \rangle^2} \right) \eta d\lambda,
\end{equation}
where \(\xi_{\lambda} = \xi + \lambda(\xi - \eta)\). Then, a direct computation gives that
\begin{equation}
\left| \nabla^{p}_{\xi} \left( \frac{\xi}{\langle \xi \rangle} - \frac{\xi - \eta}{\langle \xi - \eta \rangle} \right) \right| \lesssim \frac{|\eta|}{\langle \xi \rangle^{n+1}}.
\end{equation}
The second term can be estimated similarly. The estimates for the remaining three terms are easily dealt with thanks to (3.1).

Using the Lemma 3.1, we get the following bilinear estimates.

**Lemma 3.2.** Let \(N \in 2^{2}\) and \(\theta \in \{+, -\}\). Assume that \(\|\psi_1\|_{\Sigma^{-\theta}_2}, \|\psi_2\|_{\Sigma^{-\theta}_2} \lesssim \epsilon_1\). Then we get
\begin{equation}
\|P_N(\Pi_\theta(D)\psi_1(t), \Pi_{-\theta}(D)\psi_2(t))\|_{L^\infty_x} \lesssim N(N)^{-2}(t)^{-3+\frac{4\theta}{M}}\epsilon_1^2.
\end{equation}

**Remark 3.3.** As a direct consequence of (2.8) and the boundedness of Dirac projections (3.1), one has
\begin{equation}
\|P_N(\Pi_\theta(D)\psi_1(t), \Pi_{-\theta}(D)\psi_2(t))\|_{L^\infty_x} \lesssim (N)^{-2}(t)^{-3+\frac{4\theta}{M}}\epsilon_1^2.
\end{equation}

Thus, by exploiting the spinorial null structure in (3.4), we obtain an extra \(N\) factor which is useful when \(N \leq 1\).

**Proof.** As in (2.6), one can obtain
\begin{equation}
\|P_N(\Pi_\theta(D)\psi_1(t), \Pi_{-\theta}(D)\psi_2(t))\|_{L^\infty_x} \lesssim AN\|D^{-\varepsilon}(D)^2\psi_1(t)\|_{L^\infty_x} \|D^{-\varepsilon}(D)^2\psi_2(t)\|_{L^\infty_x},
\end{equation}
where
\[ A_N = \iint_{\mathbb{R}^{3+3}} e^{iy\xi} e^{iz\sigma} \langle \xi + \sigma \rangle^{-2} \langle \sigma \rangle^{-2} \xi + \sigma \rangle^{-2} |\sigma|^2 \Pi_{\theta}(\sigma) \Pi_{-\theta}(\xi + \sigma) d\xi d\sigma \]
for \( \varepsilon \) satisfying \( 0 < \left( 2 + \frac{2}{3} \delta_0 \right) \varepsilon < \frac{\delta_0}{10} \). Since
\[
|||D||^\gamma \phi||_{L^\gamma_x} \lesssim ||\phi||_{L^2_x}^{1-\frac{2}{N}} ||\phi||_{L^\infty_x}^{\frac{2}{N}} \quad \text{for} \quad \phi \in L^2_x \cap L^\infty_x \quad \text{and} \quad 0 < \varepsilon < 1,
\]
we have by (3.5) and the a priori assumption (2.1) that
\[
|||D||^{-\varepsilon} \langle D \rangle^2 \psi_j(t)||_{L^\infty_x} \lesssim (t)^{-\frac{3}{4} + \frac{1}{2} \varepsilon + 3 \varepsilon}, \quad \text{for} \quad j = 1, 2.
\]
Thus, we suffice to prove that
\[
(3.6) \quad A_N \lesssim N(N)^{-2}.
\]
We apply the dyadic decomposition to obtain
\[
A_N \leq \sum_{N_1, N_2 \in 2^N} A_{(N,N_1,N_2)},
\]
where
\[
A_{(N,N_1,N_2)} = \iint_{|\xi| \leq N_1} e^{iy\xi} e^{iz\sigma} K_{(N,N_1,N_2)}(\xi, \sigma) d\xi d\sigma,
\]
with
\[
(3.7) \quad K_{(N,N_1,N_2)}(\xi, \sigma) := \rho_N(\xi) \rho_{N_1}(\xi + \sigma) \rho_{N_2}(\xi + \sigma)^{-2} \langle \sigma \rangle^{-2} |\sigma|^2 \Pi_{\theta}(\sigma) \Pi_{-\theta}(\xi + \sigma).
\]
A direct computation using Lemma 3.1 gives that for \( N \lesssim N_1 \),
\[
|\nabla_\sigma^m \nabla_\xi^n K_{(N,N_1,N_2)}(\xi, \sigma)| \lesssim (N_1)^{-2} (N_2)^{-2} N_1 N_2 N^{-n} N^{-m}.
\]
Integrating by parts and using the differential inequalities, we estimate
\[
(3.8) \quad \sum_{N_{1} \leq N_1} A_{(N,N_1,N_2)} \lesssim \sum_{N \leq N_1} \left( \left| \iint_{|\xi| \leq N_1} e^{iy\xi} \Delta_\sigma K_{(N,N_1,N_2)}(\xi, \sigma) d\xi \right|_{L^1_x} \right.
\]
\[
+ \left. \left| \iint_{|\xi| \leq N_1} e^{iy\xi} \Delta_\sigma^2 K_{(N,N_1,N_2)}(\xi, \sigma) d\xi \right|_{L^1_x} \right) \lesssim \sum_{N \leq N_1} \left( \left| \iint_{|y| \leq N_1} \Delta_\xi (N_2^{-1} \Delta_\sigma + N_2 \Delta_\sigma^2) K_{(N,N_1,N_2)}(\xi, \sigma) d\xi \right|_{L^1_x} \right.
\]
\[
+ \left. \left| \iint_{|y| \leq N_1} \Delta_\xi^2 (N_2^{-1} \Delta_\sigma + N_2 \Delta_\sigma^2) K_{(N,N_1,N_2)}(\xi, \sigma) d\xi \right|_{L^1_x} \right) \lesssim \sum_{N \leq N_1} (N_1)^{-2} (N_2)^{-2} N_1^2 N_2^2 \lesssim N(N)^{-2}.
\]
For the remaining contribution where \( N_1 \ll N \), we can obtain the desired bound via the change of variables by the symmetry of \( N_1 \) and \( N_2 \). Indeed, by defining
\[
\bar{K}_{(N,N_1,N_2)}(\xi, \sigma) := K_{(N,N_1,N_2)}(\xi, -\xi + \sigma),
\]
we have
\[ \sum_{N_1 \ll N} A_{(N,N_1,N_2)} = \sum_{N_1 \ll N} \iint \left| \iint e^{iy\cdot\xi} e^{iz\cdot\sigma} \tilde{K}_{(N,N_1,N_2)}(\xi,\sigma)d\xi d\sigma \right| dz dy'. \]

Now, one can verify that \( \tilde{K}_{(N,N_1,N_2)} \) satisfies the differential inequalities
\[ \left| \nabla_\sigma \nabla_\xi \tilde{K}_{(N,N_1,N_2)}(\xi,\sigma) \right| \lesssim \langle N \rangle^{-2} \langle N_2 \rangle^{-2} N N_1^\epsilon N^{-n} N_1^{-m}. \]

Similar computation as in (3.8) yields that \( \sum_{N_1 \ll N} A_{(N,N_1,N_2)} A_N \lesssim N\langle N \rangle^{-2}. \]

Following lemma concerns another null-structure.

**Lemma 3.4.** Let \( N \in 2^\mathbb{Z} \). Assume that \( \psi_1, \psi_2 \) satisfy a priori assumption (2.1). Then,
\[ \| P_N \left( \langle D \rangle \psi_1(t), \psi_2(t) \right) - \langle \psi_1(t), \langle D \rangle \psi_2(t) \rangle \|_{L^\infty_x(\mathbb{R}^3)} \lesssim N \min \left( \langle N \rangle^{-2} \langle t \rangle^{-3+\frac{4\epsilon}{11}}, N^3 \right) \epsilon^2. \]

**Remark 3.5.** A slight modification of the proof of (2.8) gives that
\[ \| P_N \langle \langle D \rangle \psi(t), \psi(t) \rangle \|_{L^\infty_x(\mathbb{R}^3)} \lesssim \min \left( \langle N \rangle^{-1} \langle t \rangle^{-3+\frac{4\epsilon}{11}}, N^3 \right). \]

Thus, by exploiting the structure in the left side of (3.9), we indeed obtained an extra \( N \) gain when \( N \leq 1 \), compared to the one obtained just by applying the triangle inequality and (3.10).

**Proof.** The proof of the first bounds proceeds similarly to the proof of Lemma 3.2. We estimate by using (3.5)
\[ \| P_N \left( \langle D \rangle \psi_1(t), \psi_2(t) \right) - \langle \psi_1(t), \langle D \rangle \psi_2(t) \rangle \|_{L^\infty_x(\mathbb{R}^3)} \lesssim B_N \left\| D^{-\epsilon} \langle D \rangle^2 \psi_1(t) \right\|_{L^\infty_x} \left\| D^{-\epsilon} \langle D \rangle^2 \psi_2(t) \right\|_{L^\infty_x} \lesssim B_N \langle t \rangle^{-3+2\epsilon+\frac{4\epsilon}{11} \epsilon^2}, \]
where
\[ B_N := \iint \left| \iint e^{iy\cdot\xi} e^{iz\cdot\sigma} \langle \xi + \sigma \rangle^{-2} \langle \sigma \rangle^{-2} |\xi + \sigma|^\epsilon |\sigma|^{\epsilon \left( \langle \sigma \rangle - \langle \xi + \sigma \rangle \right)} d\xi d\sigma \right| dz dy \]
for \( \epsilon \) satisfying \( 0 < \left( 2 + \frac{4}{11} \delta_0 \right) \epsilon \ll \frac{4\epsilon}{11} \). In the integrand of \( B_N \), the cancellation term \( \left( \langle \sigma \rangle - \langle \xi + \sigma \rangle \right) \) plays a alternative role to the spinorial null-structure \( \Pi_\theta(\sigma) \Pi_{-\theta}(\xi + \sigma) \) in (3.7). Thus, in a similar way to the proof of (3.6), one can obtain
\[ B_N \lesssim N\langle N \rangle^{-2}. \]
Next, we show the bounds in (3.9). By the mean value theorem and Hölder inequality, we estimate
\[ \left\| P_N \left( \left\langle (D) \psi_1(t), \psi_2(t) \right\rangle - \left\langle \psi_1(t), (D) \psi_2(t) \right\rangle \right) \right\|_{L^\infty_t(R^3)} \]
\[ \lesssim \left\| \int \rho_N(\xi) \left( \left\langle (\sigma) - \langle \xi + \sigma \rangle \right\rangle \left\langle \psi_1(t, \sigma), \psi_2(t, \xi + \sigma) \right\rangle d\sigma \right\|_{L^1_t} \]
\[ \lesssim \left\| \rho_N(\xi) \right\|_{L^1_t} \left\| \psi_1 \right\|_{L^2_t} \left\| \psi_2 \right\|_{L^2_t} \lesssim N^4 \varepsilon_1^2. \]

4. Proof of Theorem 1.1

For any \( \psi_0 \) satisfying (1.7), it is quite standard to show the existence of local solution \( \psi(t) \) in \( \Sigma_T \) for some \( T \) (for instance see [6, 8, 19]). Then the solution to (1.5) can be extended globally by a continuity argument. To achieve this we have only to prove the following: Given any \( \psi \) satisfying (4.2) will be done if we prove Propositions 4.1 and 4.2 below. Recall that for simplicity we denote \( \Pi(D) \psi \) by \( \psi_b \) for \( \theta \in \{+,-\} \), so
\[ \psi = \psi_+ + \psi_. \]

**Proposition 4.1 (Weighted energy estimate).** Assume that \( \psi \in C([0,T], H^k) \) satisfies the condition (2.1). Then we obtain the following estimates: For \( \theta_0 \in \{+,-\} \),
\[ \sup_{t \in [0,T]} \left\| \left( t \right)^{-\delta_0} \left\| \psi_{\theta_0}(t) \right\|_{H^2} \right\| \leq \varepsilon_0 + C \varepsilon_1^3, \]
\[ \sup_{t \in [0,T]} \left\| \left( t \right)^{-\delta_0} \left\| (x) e^{i \theta_0 it(D)} \psi_{\theta_0}(t) \right\|_{H^2} \right\| \leq \varepsilon_0 + C \varepsilon_1^3, \]
\[ \sup_{t \in [0,T]} \left\| \left( t \right)^{-2\delta_0} \left\| (x)^2 e^{i \theta_0 it(D)} \psi_{\theta_0}(t) \right\|_{H^2} \right\| \leq \varepsilon_0 + C \varepsilon_1^3 \]
for some \( \delta_0 \) with \( 0 < \delta_0 < \frac{1}{100} \).

The proof of this proposition constitutes the main part of our analysis. It will be given in Section 5.

To complete the proof of (4.1), we need to control the \( L^\infty_\xi \)-norm. In order to estimate \( L^\infty_\xi \)-norm, we introduce a profile modification as follows: For \( \theta_0 \in \{+,-\} \),
\[ g_{\theta_0}(t, \xi) := e^{iB(t, \theta_0 \xi)} e^{i \theta_0 it(\xi)} \psi_{\theta_0}(t, \xi), \]
where \( B(t, \theta_0 \xi) \) is given in (1.10).

**Proposition 4.2 (\( L^\infty_\xi \)-estimates).** Let \( \psi \in C([0,T]; H^k) \) satisfy a priori assumption (2.1). Then we get
\[ \sup_{t_1 \leq t_2 \in [0,T]} \left\langle t_2 \right\rangle^{\delta_0} \left\| \left( \xi \right)^{10} \left( g_{\theta_0}(t_2) - g_{\theta_0}(t_1) \right) \right\|_{L^\infty_\xi} \leq \varepsilon_1 \]
for sufficiently small $\delta_0 > 0$ and $\theta_0 \in \{+, -\}$. 

This proposition will be proved in Section 6. The bound (4.5) implies that the global solution $\psi_{\theta_0} = \Pi_{\theta_0}(D)\psi$ converges to a scattering profile $\phi_{\theta_0}$ defined by

$$\phi_{\theta_0}(\xi) := \mathcal{F}^{-1}\left(\lim_{t \to \infty} g_{\theta_0}(t, \xi)\right).$$

Then, (4.5) leads us to

$$(4.6) \quad \left\| \langle \xi \rangle^{10} \mathcal{F} \left[ \psi_{\theta_0}(t) - e^{-i\theta_0 t B(t, \beta_0 D)} e^{-i\theta_0 t(D)} \phi_{\theta_0} \right] \right\|_{L^\infty_t} \lesssim (t)^{-\delta_0} \varepsilon_0.$$ 

Setting $\phi := \phi_+ + \phi_-$, (4.6) implies the modified scattering (1.9). This completes the proof of Theorem 1.1.

5. PROOF OF PROPOSITION 4.1

In this section, we prove the estimates (4.2)-(4.4), with a priori assumption (2.1). We write (1.6) by decomposing the nonlinear term

$$\psi_{\theta_0}(t) = e^{-i\theta_0 t(D)} \psi_{0, \theta_0} + i \int_t^0 e^{-i\theta_0 (t-s)(D)} \Pi_{\theta_0}(D) \left( |x|^{-1} s |\psi|^2 \psi \right)(s) \, ds,$$

$$= e^{-i\theta_0 t(D)} \psi_{0, \theta_0} + i \sum_{\theta_j \in \{\pm\}^4, j=1,2,3} \int_0^t e^{-i\theta_0 (t-s)} \Pi_{\theta_0}(D) N(\psi_{\theta_1}, \psi_{\theta_2}, \psi_{\theta_3})(s) \, ds,$$

where $\theta_0 \in \{+, -\}$ and $\psi_{\theta_j} = \Pi_{\theta_j}(D)\psi$ for $j = 1, 2, 3$. In order to prove Proposition 4.1, we prove

$$\|\psi_{\theta_0}\|_{\sum_{t=1}^\infty} \leq \|\psi_{\theta_0}\|_{\sum_{t=1}^\infty} + C \sum_{\theta_j \in \{\pm\}^4, j=1,2,3} \|\psi_{\theta_1}\|_{\sum_{t=1}^\infty} \|\psi_{\theta_2}\|_{\sum_{t=1}^\infty} \|\psi_{\theta_3}\|_{\sum_{t=1}^\infty}.$$ 

The proof is based on the energy method in weighted energy space.

5.1. Idea of proof. For the solution $\psi_{\theta}$ to (1.5) with $\theta \in \{+, -\}$, we use the interaction representation of $\psi_{\theta}(t)$ so as to track the scattering states

$$f_{\theta}(t, x) := e^{i\theta t(D)} \psi_{\theta}(t, x).$$

We also use a method of space-time resonance developed in the works [13, 14, 15, 22, 25]. In addition, we exploit the spinorial null-structures intrinsic in the our main equation (1.5). Then, we can rewrite (1.6) in terms of the Fourier transform of $f_{\theta_0}$. Then,

$$\widehat{f}_{\theta_0}(t, \xi) = \widehat{\psi}_{\theta_0}(\xi) + ic_1 \sum_{\theta_j \in \{\pm\}^4, j=1,2,3} \Im(\Phi)(t, \xi),$$

$$(5.1) \quad \Im(\Phi)(t, \xi) = \int_0^t \int_{\mathbb{R}^3} \Pi_{\theta_0}(\xi) e^{i\exp(\theta_0 \xi_1)(\xi, \eta)} |\eta|^{-2} \mathcal{F} \langle \psi_{\theta_1}, \psi_{\theta_2} \rangle (s, \eta) \widehat{f}_{\theta_1}(s, \xi - \eta) \, d\eta \, ds,$$

where we denote 4-tuple of signs by $\Theta = (\theta_0, \theta_1, \theta_2, \theta_3)$ and the resonance function is given by

$$(5.2) \quad p_{(\theta_0, \theta_1)}(\xi, \eta) = \theta_0 \langle \xi \rangle - \theta_1 \langle \xi - \eta \rangle.$$ 

We have to deal with 4 cases of $\Theta$. Each $p_{(\theta_0, \theta_1)}$ has their own features, so need to be dealt with separately.
Let us take a closer look on the structure of nonlinear terms. In the course of energy estimates, we take $\nabla_\xi$ and $\nabla_\xi^2$ to $I_\Theta$. When the derivatives fall on the phase $e^{ip_0(\theta_0,\theta_1)}$, we have

$$\int_0^t \int_{\mathbb{R}^3} \Pi_\theta_0(\xi) sm(\xi,\eta)e^{ip_0(\theta_0,\theta_1)(\xi,\eta)}|\eta|^{-2}F(\psi_{\theta_1},\psi_{\theta_2})(s,\eta)\times \hat{f}_{\theta_1}(s,\xi-\eta)d\eta ds,$$

(5.3) $$\int_0^t \int_{\mathbb{R}^3} \Pi_\theta_0(\xi)s^2[m(\xi,\eta)]^2e^{ip_0(\theta_0,\theta_1)(\xi,\eta)}|\eta|^{-2}F(\psi_{\theta_1},\psi_{\theta_2})(s,\eta)\times \hat{f}_{\theta_1}(s,\xi-\eta)d\eta ds,$$

where

$$m(\xi,\eta) = \nabla_\xi p_{(\theta_0,\theta_1)}(\xi,\eta) = \theta_0 \frac{\xi}{\langle \xi \rangle} - \theta_1 \frac{\xi - \eta}{\langle \xi - \eta \rangle},$$

$$[m(\xi,\eta)]^2 = m(\xi,\eta) \otimes m(\xi,\eta).$$

Here, we have to bound the singularity $|\eta|^{-2}$ and recover the loss of $s$. To achieve this, we use two null structures suited for given $\Theta$. The first one is that a smooth function $m$ satisfies $m(\xi,0) = 0$ if $\theta_0 = \theta_1$, more concretely,

$$|\nabla_\xi^n \nabla_\eta^m m(\xi,\eta)| \sim \frac{|\eta|^{1-m}}{|\xi|^{n+1}}, \quad \text{when } |\eta| \ll |\xi|,$$

(5.4) Thanks to (5.4), we can think that the multiplier $sm(\xi,\eta)|\eta|^{-2}$ behaves like $|\eta|^{-2}$, as far as the estimates are concerned. The other is the spinorial null structure (3.2) which we can apply when $\theta_0 \neq \theta_1$ or $\theta_2 \neq \theta_3$. Here, (3.3) corresponds to (5.4).

On the other hand, we have to recover the loss of time growth $s^2$ in (5.3). It can be compensated thanks to the null structures expect for the case when $\theta_0 \neq \theta_1$ and $\theta_2 = \theta_3$ where only the spinorial null structure from $\theta_0 \neq \theta_1$ can be applied to recover one loss of time growth $s$. A key observation is that when $\theta_0 = \theta_1$ the resonance functions

$$p_{(\theta_0,\theta_1)}(\xi,\eta) = \theta_0(\xi + \langle \xi - \eta \rangle)$$

are non-degenerate, which enables us to use the time non-resonance via integration by parts in time variable to recover the remaining loss of $s$. Moreover, an additional factor $|\eta|$ essential to close the estimates can be obtained as a by product, which we regard as another null structure, see Lemma 3.4.

5.2. Estimates for (4.2) and (4.3). It follows from (2.5) that

$$\|\psi_{\theta_0}(t)\|_{H^s} \leq \epsilon_0 + C\epsilon_1 \int_0^t (1 + s)^{-1}ds,$$

which completes the proof of (4.2). Let us consider (4.3). Note that

$$\|xe^{itD}\psi_{\theta_0}\|_{H^2} \sim \|\langle \xi \rangle^2 F(xe^{itD}\psi_{\theta_0})\|_{L^2_x} \sim \|\langle \xi \rangle^2 \nabla_x \hat{f}_{\theta_0}\|_{L^2_x}.$$

From the Duhamel’s formula (5.1), $\nabla_\xi \hat{f}_{\theta_0}$ satisfies that

$$\nabla_\xi \hat{f}_{\theta_0}(t,\xi) = \nabla_\xi \hat{\psi}_{\theta_0,\theta_0}(\xi) + ic_1 \sum_{\theta_j \in \{\pm\}, j=1,2,3} \left[I_{\Theta_0}(t,\xi) + I_{\Theta_0}(t,\xi) + I_{\Theta_0}(t,\xi) \right].$$
where
\[
I^1_\Theta(t, \xi) = \int_0^t \int_{\mathbb{R}^3} \Pi_\theta_0(\xi) \Pi_{\theta_1}(\xi - \eta) e^{is p_{\theta_0, \theta_1}(\xi, \eta)} |\eta|^{-2} \nabla (\psi_{\theta_0}, \psi_{\theta_1})(\eta) \nabla \tilde{f}_{\theta_1}(\xi - \eta) d\eta ds
\]
\[
I^2_\Theta(t, \xi) = \int_0^t \int_{\mathbb{R}^3} \nabla \xi [\Pi_\theta_0(\xi) \Pi_{\theta_1}(\xi - \eta)] e^{is p_{\theta_0, \theta_1}(\xi, \eta)} |\eta|^{-2} \nabla (\psi_{\theta_0}, \psi_{\theta_1})(\eta) \tilde{f}_{\theta_1}(\xi - \eta) d\eta ds
\]
\[
I^3_\Theta(t, \xi) = i \int_0^t s \int_{\mathbb{R}^3} \Pi_\theta_0(\xi) \Pi_{\theta_1}(\xi - \eta) m(\xi, \eta) e^{is p_{\theta_0, \theta_1}(\xi, \eta)} |\eta|^{-2} \nabla (\psi_{\theta_0}, \psi_{\theta_1})(\eta) \tilde{f}_{\theta_1}(\xi - \eta) d\eta ds,
\]
and
\[
p_{\theta_0, \theta_1}(\xi, \eta) = \theta_0(\xi) - \theta_1(\xi - \eta), \quad m_{\theta_0, \theta_1}(\xi, \eta) = \nabla \xi p_{\theta_0, \theta_1} = \theta_0(\xi) - \theta_1(\xi - \eta).
\]

It suffices to show the following estimates: for any \(\Theta = (\theta_0, \theta_1, \theta_2, \theta_3)\),
\[
\|\langle \xi \rangle^2 I^1_\Theta(t, \xi)\|_{L^2_\xi} + \|\langle \xi \rangle^2 I^2_\Theta(t, \xi)\|_{L^2_\xi} + \|\langle \xi \rangle^2 I^3_\Theta(t, \xi)\|_{L^2_\xi} \lesssim (t)^{\delta_0} e_1^3.
\]

**Estimate for** \(I^1_\Theta\) **and** \(I^2_\Theta\). Using (2.4) and the a priori assumption (2.1), we estimate
\[
\|\langle \xi \rangle^2 I^1_\Theta(t, \xi)\|_{L^2_\xi} 
\lesssim \int_0^t \|N(e^{-\theta_1 i t (D)} \nabla \Pi_{\theta_1}(D) e^{\theta_1 i t (D)} \psi_{\theta_1}(s), \psi_{\theta_2}(s), \psi_{\theta_3}(s))\|_{H^2} ds
\lesssim \int_0^t \|x e^{\theta_1 i t (D)} \psi_{\theta_1}(s)\|_{H^2} (\|\psi_{\theta_2}(s)\|_{L^6} \|\psi_{\theta_3}(s)\|_{H^2} + \|\psi_{\theta_2}(s)\|_{H^2} \|\psi_{\theta_3}(s)\|_{L^6} ) ds
\lesssim \int_0^t \langle s \rangle^{-1+\delta_0} e_1^3 ds \lesssim (t)^{\delta_0} e_1^3.
\]
Since the Dirac projection operator is bounded (3.1), we estimate similarly that
\[
\|\langle \xi \rangle^2 I^2_\Theta(t, \xi)\|_{L^2_\xi} 
\lesssim \int_0^t \left\|N \left( e^{-\theta_1 i t (D)} \nabla \Pi_{\theta_1}(D) e^{\theta_1 i t (D)} \psi_{\theta_1}(s), \psi_{\theta_2}(s), \psi_{\theta_3}(s) \right) \right\|_{H^2}
+ \|\nabla \Pi_{\theta_0}(D) N(\psi_{\theta_1}(s), \psi_{\theta_2}(s), \psi_{\theta_3}(s))\|_{H^2} ds
\lesssim \int_0^t \|\psi_{\theta_1}(s)\|_{H^2} (\|\psi_{\theta_2}(s)\|_{L^6} \|\psi_{\theta_3}(s)\|_{H^2} + \|\psi_{\theta_2}(s)\|_{H^2} \|\psi_{\theta_3}(s)\|_{L^6} ) ds
\lesssim \int_0^t \langle s \rangle^{-1+\delta_0} e_1^3 ds \lesssim (t)^{\delta_0} e_1^3.
\]

**Estimate for** \(I^3_\Theta\). We decompose \(|\xi|, |\xi - \eta|, |\eta|\) into dyadic pieces \(N_0, N_1, N_2\), respectively. Then, by the triangle inequality, we get
\[
\|\langle \xi \rangle^2 I^3_\Theta(t, \xi)\|_{L^2_\xi} \lesssim \sum_{N \in 2^j, j=0,1,2} (N_0)^2 \|I^3_{\Theta, N}(t, \xi)\|_{L^2_\xi},
\]
where
\[ I_{\Theta,N}^{3}(t,\xi) := \int_0^t \int_{\mathbb{R}^3} m(x_0,t_0,1)N(\xi,\eta)F(\psi_{t_0},\psi_{t_1})(\eta)e^{-\theta_{t_0}x(\xi-\eta)P_{N_1}f_{\theta_1}(\xi-\eta)}d\eta ds \]
and
\[ N := (N_0, N_1, N_2), \]
\[ m(x_0,t_0,1)(\xi,\eta) := \Pi_{\theta_0}(\xi)\Pi_{\theta_1}(\xi-\eta)m(x_0,t_0,1)(\xi,\eta)|\eta|^{-2} \times \rho_{N_0}(\xi)\rho_{N_1}(\xi-\eta)\rho_{N_2}(\eta). \]

It suffices to show that
\[ \sum_{N_j \in 2^j, j=0,1,2} \langle N_0 \rangle^2 \| I_{\Theta,N}^{3}(t,\xi) \|_{L^2} \lesssim \langle t \rangle^{10}\varepsilon_1^{\gamma}. \]

Using Hölder inequality with the pointwise bound \(|m(x_0,t_0,1)(\xi,\eta)| \lesssim N_2^{-1}\), we see that
\[ \langle N_0 \rangle^2 \| I_{\Theta,N}^{3}(t,\xi) \|_{L^2} \lesssim \int_0^t \sum_{\{N:N_0 \leq \langle s \rangle^{-2}\}} \sum_{\{N:N_0 \geq \langle s \rangle^{-2}\}} \langle N_0 \rangle^2 \| P_{N_2}(\psi_{s_1},\psi_{s_2})(s) \|_{L^2} \| P_{N_1}f_{\theta_1}(s) \|_{L^2} ds. \]

From the a priori assumption \((2.1)\), we have
\[ \| P_{N_1}f_{\theta_1}(s) \|_{L^2} \lesssim N_1^{\frac{5}{2}}(N_1)^{-\delta_1}. \]

We consider the sum over those \(N\) such that \(N_0 \leq \langle s \rangle^{-2}\) and \(N_0 \geq \langle s \rangle^{-2}\) in the integrand of \(I_{\Theta,N}^{3}\) as follows:
\[ I_{\Theta,N}^{3}(t,\xi) = \int_0^t \left[ \sum_{\{N:N_0 \leq \langle s \rangle^{-2}\}} \sum_{\{N:N_0 \geq \langle s \rangle^{-2}\}} \cdots \right] ds \]
\[ =: I_{\Theta,N}^{3,1}(t,\xi) + I_{\Theta,N}^{3,2}(t,\xi). \]

Thus, applying \((5.7)\) and \((2.7)\) to \((5.6)\), we estimate
\[ \langle N_0 \rangle^2 \| I_{\Theta,N}^{3,1}(t,\xi) \|_{L^2} \lesssim \int_0^t s \| \sum_{\{N:N_0 \leq \langle s \rangle^{-2}\}} \langle N_0 \rangle^2 N_2^{1/2} N_1^{3/2} (N_1)^{-\delta_1} N_2^{1/2} (N_2)^{-\delta_1} ds \]
\[ \lesssim \int_0^t s \langle s \rangle^{-3}\varepsilon_1^2 ds \lesssim \langle t \rangle^{10}\varepsilon_1^{\gamma}. \]

Besides the pointwise bound, one can verify that
\[ \| m(x_0,t_0,1) \|_{CM} = \left\| \int_{\mathbb{R}^{3+3}} m(x_0,t_0,1)(\xi,\eta)e^{ix\cdot\xi}e^{iy\cdot\eta}d\xi d\eta \right\|_{L_{x,y}^{1,\infty}} \lesssim N_2^{-1}. \]

Indeed, we observe that \(m(x_0,t_0,1)\) satisfies the differential inequalities
\[ \left| \nabla_\xi \nabla_\eta m(x_0,t_0,1)(\xi,\eta) \right| \lesssim N_0^{-m} N_2^{1-m}, \text{ for } N_0 \lesssim N_1. \]

Here, when \(N_2 \ll N_0 \sim N_1\), we used \((5.4)\) for \(\theta_0 = \theta_1\) or \((3.3)\) for \(\theta_0 \neq \theta_1\). For \(N_0 \gg N_1\), we first change variables
\[ \left\| \int_{\mathbb{R}^{3+3}} m(x_0,t_0,1)(\xi,\eta)e^{ix\cdot\xi}e^{iy\cdot\eta}d\xi d\eta \right\|_{L_{x,y}^{1,\infty}} \]
\[ = \left\| \int_{\mathbb{R}^{3+3}} m(x_0,t_0,1)(\xi,\xi-\eta)e^{ix\cdot\xi}e^{iy\cdot\eta}d\xi d\eta \right\|_{L_{x,y}^{1,\infty}} \]
and find the differential inequalities

\[(5.10) \quad \left| \nabla_\xi \nabla_\eta^m m_{(\theta_0, \theta_1), N}(\xi, \xi - \eta) \right| \lesssim N_2^{-1} N_0^{-n} N_1^{-m}. \]

Then, a standard calculation together with (5.9) and (5.10), as we did in (3.8), yields (5.8). Applying the Coifman-Meyer operator estimates, Lemma 2.7, with (5.8) we obtain

\[(5.11) \quad \langle N_0 \rangle^2 \left\| I_{\theta_0, N}(t, \xi) \right\|_{L^2_\xi} \lesssim \int_0^t \sum_{\{N: N_0 \geq (s)^{-2}\}} \langle N_0 \rangle^2 N_2^{-1} \left\| P_{N_2}(\psi_{\theta_1}, \psi_{\theta_2}) \right\|_{L^\infty_\xi} \left\| P_{N_1}f_{\theta_1} \right\|_{L^2_\xi} ds. \]

Then, using the frequency localized estimates (2.8) and (5.7), we bound the sum in integrand of (5.11) for \( N_2 \leq \langle s \rangle^{-1} \) by

\[
\int_0^t se_1^3 \sum_{\{N: N_0 \geq (s)^{-2}, N_2 \leq (s)^{-1}\}} \langle N_0 \rangle^2 N_2^2 N_1^2 (N_1)^{-10} \lesssim \int_0^t s\langle s \rangle^{-2+\delta_0} e_1^3 ds)
\]

and the sum in integrand of (5.11) for \( N_2 \geq \langle s \rangle^{-1} \) by

\[
\int_0^t se_1^3 \sum_{\{N: N_0 \geq (s)^{-2}, N_2 \geq (s)^{-1}\}} \langle N_0 \rangle^2 N_2^{-1} N_2^2 N_1^2 (N_1)^{-10} \langle s \rangle^{-3} ds \lesssim \int_0^t s\langle s \rangle^{-2+\delta_0} e_1^3 ds.
\]

5.3. Estimates for (4.4). In this subsection, we devote to establish (4.4). By Plancherel’s theorem, \( \| |x|^2 e^{i\theta_0 t(D)} \psi_{\theta_0} \|_{H^2} \sim \| \langle \xi \rangle^2 \nabla_\xi^2 \hat{f}_{\theta_0} \|_{L^2_\xi}. \) If we take \( \nabla_\xi^2 \) to (5.1), derivatives may fall on \( \hat{f}_{\theta_0}(\xi - \eta), \Pi_{\theta_0}(\xi) \Pi_{\theta_1}(\xi - \eta), \) or \( e^{i\theta_0 t(D)} e^{i\theta_1 t(D)} \) (see Case D below). Since the other cases are estimated similarly as in the proof of (4.3), we will omit the details.

Case A: At least one derivative \( \nabla_\xi \) falls on \( \hat{f}_{\theta_0}. \)

The proof is essentially same as the proof of (4.3). We have only to redo the estimates for \( I_{\theta_0}^1, I_{\phi}^2 \) and \( I_{\theta_1}^3 \) in subsection 5.2 replacing \( \hat{f}_{\theta_1} \) with \( \nabla_\xi \hat{f}_{\theta_1}. \)

Case B: \( \nabla_\xi^2 \) falls on \( \Pi_{\theta_0}(\xi) \Pi_{\theta_1}(\xi - \eta). \)

We simply use the rough bound

\[ \nabla_\xi^2 [\Pi_{\theta_0}(\xi) \Pi_{\theta_1}(\xi - \eta)] \lesssim 1 \]

and follow the argument for \( I_{\theta_0}^2 \) in subsection 5.2.

Case C: Only one derivative \( \nabla_\xi \) falls on \( e^{i\theta_0 t(D)} e^{i\theta_1 t(D)}. \) Then, the multipliers in integrals correspond to one of the followings:

1. \( sm_{(\theta_0, \theta_1)}(\xi, \eta) \nabla_\xi (\Pi_{\theta_0}(\xi) \Pi_{\theta_1}(\xi - \eta)). \)
2. \( s \nabla_\xi m_{(\theta_0, \theta_1)}(\xi, \eta) \Pi_{\theta_0}(\xi) \Pi_{\theta_1}(\xi - \eta). \)

Both can be dealt with by repeating the argument in the proof of \( I_{\theta_0}^2 \) in subsection 5.2. Actually, the additional \( \xi \)-derivative, \( \nabla_\xi, \) compared to (5.5) makes a bound in (5.8) better by giving a \( \langle N_0 \rangle^{-1} \) factor.
Case D: $\nabla_\xi^2$ falls on $e^{isp(\theta_0, s_1)}$. Then, we consider the following integral

$$\langle \xi \rangle^2 J_\Theta(t, \xi) := \int_0^t \int_{\mathbb{R}^3} \langle \xi \rangle^2 s^2 \left[ m_{(\theta_0, \theta_1)}(\xi, \eta) \otimes m_{(\theta_0, \theta_1)}(\xi, \eta) \right](\xi, \eta) \eta^2 \times \Pi_{\theta_0}(\xi)\Pi_{\theta_1}(\xi - \eta) e^{isp(\theta_0, s_1)}(\xi, \eta) f_{\theta_1}(\xi - \eta) d\eta ds$$

and prove that

$$\left\| \langle \xi \rangle^2 J_\Theta(s, \xi) \right\|_{L^2_\xi(\mathbb{R}^3)} \lesssim C(t)^{2\delta_0} \varepsilon_1^3. \quad (5.12)$$

Most of all, one needs to handle an extra time growth $s^2$. To compensate the time growth, we need to scrutinize the inner product $\langle \psi_{\theta_3}, \psi_{\theta_2} \rangle$. By applying the dyadic decomposition, we may write

$$J_\Theta(t, \xi) = \sum_{N_j \in 2^j, j = 0, 1, 2} J_{\Theta, N}(t, \xi),$$

where

$$J_{\Theta, N}(t, \xi) = \int_0^t \int_{\mathbb{R}^3} m_{(\theta_0, \theta_1), N}(\xi, \eta) e^{isp(\theta_0, s_1)}(\xi, \eta) \times P_{N_2}(\psi_{\theta_3}, \psi_{\theta_2})(\eta) P_{N_1} f_{\theta_1}(\xi - \eta) d\eta ds,$$

and

$$m_{(\theta_0, \theta_1), N}(\xi, \eta) = \left( \theta_0 \frac{\xi}{(\xi)} - \theta_1 \frac{\xi - \eta}{(\xi - \eta)} \right)^2 \Pi_{\theta_0}(\xi)\Pi_{\theta_1}(\xi - \eta) |\eta|^{-2} \times \rho_{N_0}(\xi)\rho_{N_1}(\xi - \eta)\rho_{N_2}(\eta).$$

We deal with the following three cases separately:

(i) $\theta_0 = \theta_1$,
(ii) $\theta_0 \neq \theta_1$ and $\theta_2 \neq \theta_3$,
(iii) $\theta_0 \neq \theta_1$ and $\theta_2 = \theta_3$.

Estimates for (i). As for (5.12), it suffices to show that

$$\sum_{N_j \in 2^j, j = 0, 1, 2} (N_0)^2 \left\| J_{\Theta, N}(t) \right\|_{L^2_\xi} \lesssim \langle t \rangle^{2\delta_0} \varepsilon_1^3. \quad (5.13)$$

For $N_0$ with $N_0 \leq \langle s \rangle^{-2}$ in the integrand of $J_{\Theta, N}$, the desired bound can be obtained as in (5.6). For the remaining contribution, we apply the Coifman-Meyer multiplier estimates. Using the differential inequalities (5.4) and (3.3), one can show as in (5.8) that

$$\left\| m_{(\theta_0, \theta_0), N} \right\|_{CM} \lesssim 1. \quad (5.14)$$
Applying Lemma 2.7 with (5.14) and using (2.8), we estimate that

$$
\langle N_0 \rangle^2 \left\| \mathcal{J}_{\Theta, N}(t) \right\|_{L^2_\xi}
\lesssim \int_0^t \int_{N_0 \geq (s)^{-2}} \frac{N_0^2}{\langle s \rangle} \, \left\| P_{N_2} \langle \psi_{\theta_1} \rangle \right\| \left\| P_{N_1} f_{\theta_1} \right\|_{L^2_\xi} \, ds
\lesssim \int_0^t \int_{N_0 \geq (s)^{-2}} \frac{N_0^2}{\langle s \rangle} \, \left\| P_{N_2} \langle \psi_{\theta_2} \rangle \right\| \left\| P_{N_1} f_{\theta_1} \right\|_{L^2_\xi} \, ds
\lesssim \int_0^t \int_{N_0 \geq (s)^{-2}} \frac{N_0^2}{\langle s \rangle} \, \left\| P_{N_2} \langle \psi_{\theta_2} \rangle \right\| \left\| P_{N_1} f_{\theta_1} \right\|_{L^2_\xi} \, ds
\lesssim \int_0^t \langle s \rangle^{-1+2\delta_0} \frac{1}{\varepsilon_1^3} \, ds \lesssim \langle t \rangle^{2\delta_0} \frac{1}{\varepsilon_1^3}.
$$

Estimates for (ii). We proceed as in the previous case. The sum in (5.13) for $N_0 \lesssim \langle s \rangle^{-1}$ can be obtained as before. In the multiplier estimates, since $\theta_0 \neq \theta_1$, we can only use the spinorial null structure from $\Pi_{\theta_0}(\xi)\Pi_{\theta_1}(\xi - \eta)$ to obtain

$$
\| \mathcal{M}_{(\theta_0, -\theta_0)} \|_{CM} \lesssim N_2^{-1},
$$

which is worse than (5.14) when $N_2 \ll 1$. This can be compensated, however, by the null structure in the bilinear form $P_{N_2} \langle \psi_{\theta_2}, \psi_{\theta_2} \rangle$. Indeed, applying Lemma 2.7 with (5.15) and then using (3.4) instead of (2.8), we estimate

$$
\langle N_0 \rangle^2 \left\| \mathcal{J}_{\Theta, N}(t) \right\|_{L^2_\xi}
\lesssim \int_0^t \int_{N_0 \geq (s)^{-2}} \frac{N_0^2}{\langle s \rangle} \, \left\| P_{N_2} \langle \psi_{\theta_2} \rangle \right\| \left\| P_{N_1} f_{\theta_1} \right\|_{L^2_\xi} \, ds
\lesssim \int_0^t \int_{N_0 \geq (s)^{-2}} \frac{N_0^2}{\langle s \rangle} \, \left\| P_{N_2} \langle \psi_{\theta_2} \rangle \right\| \left\| P_{N_1} f_{\theta_1} \right\|_{L^2_\xi} \, ds
\lesssim \int_0^t \langle s \rangle^{-1+2\delta_0} \frac{1}{\varepsilon_1^3} \, ds \lesssim \langle t \rangle^{2\delta_0} \frac{1}{\varepsilon_1^3}.
$$

Estimates for (iii). It remains to handle the case $\theta_0 \neq \theta_1$ and $\theta_2 = \theta_3$. A key observation is that the resonance function

$$
P_{(\theta_0, \theta_1)}(\xi, \eta) = \theta_0 \left( \langle \xi \rangle + \langle \xi - \eta \rangle \right)
$$

is non-degenerate which enables to perform an integration by parts in time variables or exploit the time non-resonance. In this procedure, the condition $\theta_2 = \theta_3$ plays an important role to obtain a factor $N_2$ essential to bound the singularity. We begin with writing the integral $\mathcal{J}_{\Theta, N}$ as

$$
\mathcal{J}_{\Theta, N}(t, \xi)
= \int_0^t s^2 \int_{\mathbb{R}^3} \mathcal{M}_{(\theta_0, -\theta_0)}(\xi, \eta) e^{i s p_{(\theta_0, \theta_1)}(\xi, \eta)} P_{N_2} \langle \psi_{\theta_3}, \psi_{\theta_2} \rangle (s, \eta) P_{N_1} f_{\theta_1} (s, \xi - \eta) \, dy \, ds
= -\theta_0 i \int_0^t s^2 \int_{\mathbb{R}^3} \mathcal{M}_{(\theta_0, -\theta_0)}(\xi, \eta) \frac{\partial_\xi e^{i s p_{(\theta_0, \theta_1)}(\xi, \eta)}}{\langle \xi \rangle + \langle \xi - \eta \rangle} P_{N_2} \langle \psi_{\theta_3}, \psi_{\theta_2} \rangle (s, \eta)
\times P_{N_1} f_{\theta_1} (s, \xi - \eta) \, dy \, ds.
$$
From now, we fix $\theta_0 = \theta_2 = +$ for simplicity and denote $p_{(+,-)}$ by $p$. Let us define a multiplier $\tilde{M}_N$ by

$$\tilde{M}_N(\xi, \eta, \sigma) = \left( \frac{\xi}{\langle \xi \rangle} + \frac{\xi - \eta}{\langle \xi - \eta \rangle} \right)^2 \Pi_+(\xi) \Pi_-(\xi - \eta) \frac{|\eta|^2}{\langle \xi \rangle + \langle \xi - \eta \rangle} \times \rho_{N_0}(\xi) \rho_{N_1}(\xi - \eta) \rho_{N_2}(\eta).$$

Performing the integration by parts in time, we obtain

$$\mathcal{J}_{\Theta, N}(t, \xi) = t^2 \int_{\mathbb{R}^3} \tilde{M}_N(\xi, \eta)e^{it(\xi + (\xi - \eta))} \mathcal{F}(P_{N_2}(\psi_+, \psi_+))(t, \eta) \widehat{P_{N_1}f}(t, \xi - \eta)d\eta$$

$$+ \int_0^t 2s \int_{\mathbb{R}^3} \tilde{M}_N(\xi, \eta)e^{is(\xi + (\xi - \eta))} \mathcal{F}(P_{N_2}(\psi_+, \psi_+))(s, \eta) \widehat{P_{N_1}f}(s, \xi - \eta)d\eta ds$$

$$+ \int_0^t s^2 \int_{\mathbb{R}^3} \tilde{M}_N(\xi, \eta)e^{is(\xi + (\xi - \eta))}\partial_s (P_{N_2}(\psi_+, \psi_+))(s, \eta) \widehat{P_{N_1}f}(s, \xi - \eta)d\eta ds$$

$$+ \int_0^t s^2 \int_{\mathbb{R}^3} \tilde{M}_N(\xi, \eta)e^{is(\xi + (\xi - \eta))} \mathcal{F}(P_{N_2}(\psi_+, \psi_+))(s, \eta) \partial_s \widehat{P_{N_1}f}(s, \xi - \eta)d\eta ds$$

$$=: \mathcal{L}_{N_1}(t, \xi) + \mathcal{L}_{N_2}^2(t, \xi) + \mathcal{L}_{N_3}(t, \xi) + \mathcal{L}_{N_4}(t, \xi).$$

The first two terms can be dealt with analogously to the estimates for $\mathcal{I}_{\Theta}^3$ in the previous subsection by using the following multipliers bounds:

$$\mathcal{L}_{N_1}^3(t, \xi) \lesssim \min \left( \langle N_1 \rangle^{-1}, \langle N_2 \rangle^{-1} \right) N_2^{-1}.$$
estimates with (5.16). For $\mathcal{L}_N^{3,1}$, we estimate
\begin{align}
\langle N_0 \rangle^2 \| \mathcal{L}_N^{3,1}(t, \xi) \|_{L^2} & \lesssim \langle N_0 \rangle^2 \int_0^t s^2 \sum_{\{N: N_0 \geq \langle s \rangle^{-1}\}} N_2^{-1} \| P_{N_1} \psi_-(s) \|_{L^2} \\
& \times \| P_{N_2} \left( \langle (D) \psi_+(s), \psi_+(s) \rangle - \langle \psi_+(s), (D) \psi_+(s) \rangle \right) \|_{L^\infty} ds
\end{align}
(5.17)

Here, we take advantage of the structure in $L^\infty$-norm. Indeed, by using (3.9), we bound the sum in the above by
\begin{align}
(5.17) & \lesssim \varepsilon_1^3 \int_0^t s^2 \sum_{\{N: N_0 \geq \langle s \rangle^{-1}\}} \langle N_0 \rangle^2 N_1^{\frac{2}{3}} \langle N_1 \rangle^{-10} \min \left( \langle N_2 \rangle^{-2} \langle s \rangle^{-3 + \frac{4}{10}}, N_2^3 \right) ds \\
& \lesssim \varepsilon_1^3 \int_0^t s^2 \left( \sum_{N_2 \leq \langle s \rangle^{-1}} N_2^3 + \sum_{N_2 \geq \langle s \rangle^{-1}} \langle N_2 \rangle^{-2} \langle s \rangle^{-3 + \frac{4}{10}} \right) ds \\
& \lesssim \langle t \rangle^{2\delta_0} \varepsilon_1^3.
\end{align}

For $\mathcal{L}_N^{3,2}$, using (2.10) we estimate
\begin{align}
\langle N_0 \rangle^2 \| \mathcal{L}_N^{3,2}(t, \xi) \|_{L^2} & \lesssim \int_0^t s^2 \sum_{\{N: N_0 \geq \langle s \rangle^{-2}\}} \langle N_0 \rangle^2 \| P_{N_1} \psi_-(s) \|_{L^2} \| P_{N_2} \left( \psi_+, \Pi_+(D) \mathcal{N} \psi_+, \psi \right)(s) \|_{L^\infty} ds \\
& \lesssim \varepsilon_1^5 \int_0^t s^2 \sum_{\{N: N_0 \geq \langle s \rangle^{-2}\}} \langle N_0 \rangle^2 N_1^{\frac{2}{3}} \langle N_1 \rangle^{-10} \min \left( N_2^3 \langle s \rangle^{-1}, \langle N_2 \rangle^{-2} \langle s \rangle^{-3 + \frac{4}{10}} \right) ds \\
& \lesssim \varepsilon_1^3 \int_0^t s^2 \left( \sum_{N_2 \leq \langle s \rangle^{-\frac{4}{10}}} N_2^{-1} \langle s \rangle^{-1} + \sum_{N_2 \geq \langle s \rangle^{-\frac{4}{10}}} N_2^{-1} \langle N_2 \rangle^{-2} \langle s \rangle^{-4 + \frac{4}{10}} \right) ds \\
& \lesssim \langle t \rangle^{2\delta_0} \varepsilon_1^3.
\end{align}

Finally, consider $\mathcal{L}_N^4$. The time derivative of $P_{N_1} f_-$ is given by
\[
\partial_s P_{N_1} f_-(s) = e^{-is(D)} P_{N_1} \Pi_-(D) \mathcal{N}(\psi, \psi, \psi)(s).
\]
Since the Dirac projection operator is bounded in $L^2$, using (2.9), we have
\[\| \partial_s P_{N_1} f_-(s) \|_{L^2} \lesssim \min \left( N_1^2 \langle s \rangle^{-\frac{2}{3}}, \langle N_1 \rangle^{-10} \langle s \rangle^{-1} \right) \varepsilon_1^4.\]

With the help of an additional time decay $\langle s \rangle^{-1}$ compared to (5.7), we can show as before that
\[
\sum_{\mathcal{N}} \langle N_0 \rangle^2 \| \mathcal{L}_N^4(t, \xi) \|_{L^2} \lesssim \langle t \rangle^{2\delta_0} \varepsilon_1^4.
\]

Indeed, for the sum over $N_0 \leq \langle s \rangle^{-2}$ in the integrand can be estimated by using the Hölder inequality, while for $N_0 \geq \langle s \rangle^{-2}$ applying the Coifman-Meyer estimates, Lemma 2.7, with (5.16).
6. Modified asymptotic states

This section is devoted to proving Proposition 4.2. We assume the a priori bound (2.1) and we will justify the phase correction for modified scattering profile

\[
\begin{align*}
B(t, \xi) &= B_+(t, \xi) + B_-(t, \xi), \\
B_\theta(t, \xi) &= \frac{c_1}{(2\pi)^3} \int_0^t \int_{\mathbb{R}^3} \left| \frac{\xi}{(\xi + \theta_0 \sigma)} \right|^{-1} \left| \psi_0(\sigma) \right|^2 d\sigma \rho(s - \frac{2}{3} \xi) ds,
\end{align*}
\]

and

\[
g_{\theta_0}(t, \xi) = e^{itB(t, \theta_0 \xi)} e^{tB_0(t, \xi)} \psi_{\theta_0}(t, \xi)
\]

where \( \theta, \theta_0 \in \{ \pm \} \). The phase correction \( B(t, \xi) \) is required to remedy the nonlinear interaction when the resonance function is degenerate, specifically, the combinations of sign satisfies (6.4). The explicit form of \( B(t, \xi) \) is derived from the computation via the Taylor expansion in the Fourier space.

To prove (4.5), we will follow the argument in [25]. We show that if \( t_1 \leq t_2 \in [M - 2, 2M] \cap [0, T] \) for a dyadic number \( M \in 2\mathbb{N} \), then for some \( \delta_0 > 0 \),

\[
(6.1) \quad \left\| (\xi)^{10} (g_{\theta_0}(t_2, \xi) - g_{\theta_0}(t_1, \xi)) \right\|_{L^\infty_{\xi}} \lesssim M^{-\delta_0} \varepsilon^3.
\]

Unlike the representation \( \mathcal{I}_\Theta \) of (5.1), we abbreviate the integrand of time integral in the nonlinear part of (5.1) by

\[
\mathcal{I}_\Theta(s, \xi) := \int_{\mathbb{R}^3} \Pi_{\theta_0}(\xi) e^{isp(\theta_0, \theta_1)(\xi, \eta)} |\eta|^{-2} \mathcal{F}(\psi_{\theta_1}, \psi_{\theta_2})(s, \eta) \hat{\mathcal{F}}_{\theta_1}(s, \xi - \eta) d\eta
\]

\[
\mathcal{I}_\Theta(s, \xi) = \int_{\mathbb{R}^{3+3}} \Pi_{\theta_0}(\xi) e^{is\Theta(\xi, \eta, \sigma)} |\eta|^{-2} \mathcal{F}_{\theta_1}(s, \xi + \eta)
\]

\[
\times \left\langle \hat{\mathcal{F}}_{\theta_1}(s, \xi + \eta), \hat{\mathcal{F}}_{\theta_2}(s, \xi + \eta) \right\rangle d\eta d\sigma.
\]

By the change of variables as

\[
\sigma \mapsto \xi - \eta + \sigma, \quad \eta \mapsto -\eta,
\]

we obtain

\[
\mathcal{I}_\Theta(s, \xi) = \int_{\mathbb{R}^{3+3}} e^{is\Theta(\xi, \eta, \sigma)} \Pi_{\theta_0}(\xi) |\eta|^{-2} \mathcal{F}_{\theta_1}(s, \xi + \eta)
\]

\[
\times \left\langle \hat{\mathcal{F}}_{\theta_1}(s, \xi + \eta), \hat{\mathcal{F}}_{\theta_2}(s, \xi + \eta) \right\rangle d\eta d\sigma.
\]

where, in the view of (5.2), the resonance function is given

\[
\rho_{\Theta} := \theta_0(\xi) - \theta_1(\xi + \eta) - \theta_2(\xi + \sigma) + \theta_3(\xi + \eta + \sigma), \quad (\Theta = (\theta_0, \theta_1, \theta_2, \theta_3)).
\]

Among the contribution \( \mathcal{I}_\Theta \)'s, the cases when \( \theta_0 = \theta_1 \) and \( \theta_2 = \theta_3 \) are critical in the sense of scattering where the phase correction \( e^{itB(t, \theta_0 \xi)} \) is required to the asymptotic formula.
We first decompose \( I(\Theta) \) dyadically in terms of \(|\eta| \sim L\) into
\[
I(s, \xi) = ic_1 \int_{\mathbb{R}^{3+3}} e^{i\theta(s, \xi)} \Pi_{\theta,0}(\xi) |\eta|^{-2} \mathfrak{f}_{\theta}(s, \xi + \eta) \times \left( \mathfrak{f}_{\theta}(s, \xi + \eta + \sigma), \mathfrak{f}_{\theta}(s, \xi + \sigma) \right) d\eta d\sigma
\]
\[
= I(s, \xi) + \sum_{L \in 2^L : L > L_0} I_{s, L}(s, \xi),
\]
where
\[
I(s, \xi) := ic_1 \int_{\mathbb{R}^{3+3}} e^{i\theta(s, \xi)} \Pi_{\theta,0}(\xi) |\eta|^{-2} \rho_{L_0}(\eta) \mathfrak{f}_{\theta}(s, \xi + \eta) \times \left( \mathfrak{f}_{\theta}(s, \xi + \eta + \sigma), \mathfrak{f}_{\theta}(s, \xi + \sigma) \right) d\eta d\sigma,
\]
\[
I_{s, L}(s, \xi) := ic_1 \int_{\mathbb{R}^{3+3}} e^{i\theta(s, \xi)} \Pi_{\theta,0}(\xi) |\eta|^{-2} \rho_{L}(\eta) \mathfrak{f}_{\theta}(s, \xi + \eta) \times \left( \mathfrak{f}_{\theta}(s, \xi + \eta + \sigma), \mathfrak{f}_{\theta}(s, \xi + \sigma) \right) d\eta d\sigma,
\]
and \( L_0 \in 2^\mathbb{Z} \) such that
\[
(6.2) \quad L_0 \sim M\left(\frac{-2}{3} + \frac{1}{3M} \right).
\]
Plugging the above decomposition of \( I(\Theta) \) into \( g_{\theta,0} \), we have for \( \theta_0 \in \{\pm\} \),
\[
(6.3) \quad \partial_s g_{\theta,0}(s, \xi) = e^{-iB(s, \xi)}
\]
\[
\times \left[ \sum_{\theta_1, \theta_2, \theta_3 \in \{\pm\}} \left( I_{s, L_0}(s, \xi) + \sum_{L > L_0} I_{s, L}(s, \xi) \right) - i \left[ \partial_s B(s, \xi) \right] \mathfrak{f}_{\theta,0}(s, \xi) \right].
\]
In the estimates of (6.3), the cancellation effect from \( \partial_s B(t, \theta_0 \xi) \) play a role only when \( \theta_0 = \theta_1 \) and \( \theta_2 = \theta_3 \) denoted by
\[
(6.4) \quad \Xi := (\theta_0, \theta_0, \theta_2, \theta_2)
\]
so, we divide the cases into \( \Theta = \Xi \) and \( \Theta \neq \Xi \). Thus, to prove (6.1), it suffices to show that for \( \xi \) with \(|\xi| \sim N_0 \in 2^\mathbb{Z} \),
\[
(6.5) \quad \sum_{\theta_1, \theta_2, \theta_3 \in \{\pm\}} \int_{t_1}^{t_2} e^{-iB(s, \theta_0 \xi)} I_{s, L_0}(s, \xi) \left( 1 - \rho(s^{-\frac{3}{2}} \xi) \right) ds \lesssim \varepsilon_1^3 M^{-\delta_0}(N_0)^{-10},
\]
\[
(6.6) \quad \left| \int_{t_1}^{t_2} e^{-iB(s, \theta_0 \xi)} \left[ I_{s, L_0}(s, \xi) \rho(s^{-\frac{3}{2}} \xi) - i \partial_s B(s, \theta_0 \xi) \mathfrak{f}_{\theta,0}(s, \xi) \right] ds \right| \lesssim \varepsilon_1^3 M^{-\delta_0}(N_0)^{-10},
\]
\[
(6.7) \quad \left| \int_{t_1}^{t_2} e^{-iB(s, \theta_0 \xi)} \sum_{\theta_1, \theta_2, \theta_3 \in \{\pm\}, \theta_0 \neq \Xi} I_{s, L_0}(s, \xi) \rho(s^{-\frac{3}{2}} \xi) ds \right| \lesssim \varepsilon_1^3 M^{-\delta_0}(N_0)^{-10},
\]
and

\[
\sum_{\theta_1, \theta_2, \theta_3 \in \{\pm\}} \left| \int_{t_1}^{t_2} e^{-iB(s, \theta_0 \xi)} \sum_{L > L_0} \mathcal{I}_{\Theta, L}(s, \xi) ds \right| \lesssim \varepsilon_1^3 M^{-\delta_0} \langle N_0 \rangle^{-10}.
\]

The phase modification term in (6.6) will cancel a possible resonance in \( \mathcal{I}_{\Theta, L_0} \) later.

**Proof of (6.5).** When \( N \ll M^\frac{2}{3} \), the integrand in (6.5) vanishes, so we may assume \( M^\frac{2}{3} \lesssim N \). It suffices to show the following bound

\[
\left| \mathcal{I}_{\Theta, L_0}(s, \xi) \right| \lesssim \varepsilon_1^3 M^{-(1+\delta_0)} \langle N \rangle^{-10}.
\]

We further decompose \( \mathcal{I}_{\Theta, L_0} \) dyadically in \( \sigma \) variable as follows:

\[
\mathcal{I}_{\Theta, L_0}(s, \xi) = ic_1 \sum_{L_1 \leq L_0 + 10} \mathcal{J}_{L_1}(s, \xi),
\]

where

\[
\mathcal{J}_{L_1}(s, \xi) = \iiint_{\mathbb{R}^{3+3}} e^{is\rho \Phi(\xi, \eta, \sigma)} \Pi_{\theta_0}(\xi) |\eta|^{-2} \rho_{L_1}(\sigma) \rho_{L_0}(\eta) \hat{f}_1(s, \xi + \eta)
\times \left( \hat{f}_3(s, \xi + \eta + \sigma), \hat{f}_2(s, \xi + \sigma) \right) d\eta d\sigma.
\]

Then, by a priori assumption (2.1), we estimate

\[
|\mathcal{J}_{L_1}(s, \xi)| \lesssim L_1^{-2}(\xi)^{-10} \int_{\mathbb{R}^{3+3}} \left( \langle N \rangle^{10} \Pi_{\theta_0}(\xi) \rho_{L_1}(\sigma) \rho_{L_0}(\eta) \hat{f}_1(s, \xi + \eta)
\times \left( \hat{\psi}_3(s, \xi + \eta + \sigma), \hat{\psi}_2(s, \xi + \sigma) \right) \right) d\eta d\sigma
\]

\[
\lesssim L_1^{-2}(\xi)^{-10} L_1^3 \left( \hat{f}_1 \right)^{10} L_1^3 \left( \hat{\psi}_3 \right)^{10} \left( \hat{\psi}_2 \right)^{10} \|\psi_{\theta_3}\|_{H^{10}} \|\psi_{\theta_2}\|_{H^{10}}
\lesssim L_1 M^{2\delta_0} \langle N \rangle^{-10} \varepsilon_1^{-3}.
\]

On the other hand, by Hölder inequality, we get

\[
|\mathcal{J}_{L_1}(s, \xi)| \lesssim L_1^{-\frac{7}{2}} N^{-k} \prod_{j=1}^{3} \|f_{\theta_j}\|_{H^{k}} \lesssim L_1^{-\frac{7}{2}} M^{-3} M^{3\delta_0} \varepsilon_1^{-3}.
\]

Above two estimates induce that

\[
\sum_{L_1 \leq L_0 + 10} |\mathcal{J}_{L_1}(s, \xi)|
\lesssim \sum_{L_1 \leq M^{-(1+3\delta_0)}} L_1 M^{2\delta_0} \langle N_0 \rangle^{-10} \varepsilon_1^{-3} + \sum_{L_1 > M^{-(1+3\delta_0)}} L_1^{-\frac{7}{2}} M^{-3+3\delta_0} \varepsilon_1^{-3}
\lesssim M^{-(1+\delta_0)} \langle N_0 \rangle^{-10} \varepsilon_1^{-3}.
\]

This completes the proof of (6.5).

**Proof of (6.6).** Due to the cut-off \( \rho(s^{-\frac{2}{3}}\xi) \), we may assume \( N_0 \leq M^\frac{2}{3} \). It suffices to bound the integrand by

\[
|\mathcal{I}_{\Theta, L_0}(s, \xi) - i\partial_s B(s, \theta_0 \xi) \hat{f}_0(s, \xi)| \lesssim \varepsilon_1^3 M^{-1-\delta_0} \langle N_0 \rangle^{-10}.
\]
for $\Xi = (\theta_0, \theta_0, \theta_2, \theta_2)$. Let us observe that
\[
\begin{align*}
p_{\Xi}(\xi, \eta, \sigma) &= \theta_0 \left(\langle \xi \rangle - \langle \xi + \eta \rangle\right) - \theta_2 \left(\langle \xi + \sigma \rangle - \langle \xi + \eta + \sigma \rangle\right) \\
&= \left(-\theta_0 \frac{|\eta|^2 + 2 \eta \cdot \xi}{\langle \xi \rangle + \langle \xi + \eta \rangle} - \theta_2 \frac{-|\eta|^2 - 2 \eta \cdot (\xi + \sigma)}{\langle \xi + \sigma \rangle + \langle \xi + \eta + \sigma \rangle}\right) \\
&= -\eta \cdot \left(\theta_0 \frac{\xi}{\langle \xi \rangle} - \theta_2 \frac{\xi + \sigma}{\langle \xi + \sigma \rangle}\right) + O(|\eta|^2)
\end{align*}
\]
We now set
\[
\begin{align*}
\mathcal{I}_{(\theta_0, \theta_2)}(s, \xi) &= ic_1 \int_{\mathbb{R}^{3+1}} e^{isq(\theta_0, \theta_2)(\xi, \eta, \sigma)} \Pi_{\theta_0}(\xi) \rho_{\leq L_0}(\eta) |\eta|^{-2} \\
&\quad \times f_{\theta_0}(\xi + \eta) \left(\hat{f}_{\theta_2}(\xi + \eta + \sigma), \hat{f}_{\theta_2}(\xi + \sigma)\right) \, d\eta d\sigma
\end{align*}
\]
Then we estimate
\[
\begin{align*}
&|\mathcal{I}_{\Xi, L_0}(s, \xi) - \mathcal{I}_{(\theta_0, \theta_2)}(s, \xi)| \\
&\lesssim \int_{\mathbb{R}^{3+1}} |p_{\Xi}(\xi, \eta, \sigma) - q_{(\theta_0, \theta_2)}(\xi, \eta, \sigma)| \\
&\quad \times |\Pi_{\theta_0}(\xi) \rho_{\leq L_0}(\eta) |\eta|^{-2} \hat{f}_{\theta_0}(\xi + \eta) \left(\hat{f}_{\theta_2}(\xi + \eta + \sigma), \hat{f}_{\theta_2}(\xi + \sigma)\right)| \, d\eta d\sigma \\
&\lesssim M \int_{\mathbb{R}^{3+1}} \left|\Pi_{\theta_0}(\xi) \rho_{\leq L_0}(\eta) \hat{f}_{\theta_0}(\xi + \eta) \left(\hat{f}_{\theta_2}(\xi + \eta + \sigma), \hat{f}_{\theta_2}(\xi + \sigma)\right)\right| \, d\eta d\sigma \\
&\lesssim M L_0^3 \left\|f_{\theta_2}\right\|_{L^2}^2 \left\|\hat{f}_{\theta_0}\right\|_{L^\infty} \\
&\lesssim M \left(-\frac{4}{3} + \frac{1}{\theta_0} + 2\theta_0\right) \xi^2,
\end{align*}
\]
where we used (6.2) in the last inequality. Next, we approximate $\mathcal{I}_{(\theta_0, \theta_2)}(s, \xi)$ by $\tilde{\mathcal{I}}_{(\theta_0, \theta_2)}(s, \xi)$ which is defined by
\[
\begin{align*}
\tilde{\mathcal{I}}_{(\theta_0, \theta_2)}(s, \xi) &= ic_1 \int_{\mathbb{R}^{3+1}} e^{isq(\theta_0, \theta_2)(\xi, \eta, \sigma)} \Pi_{\theta_0}(\xi) \rho_{\leq L_0}(\eta) |\eta|^{-2} \\
&\quad \times f_{\theta_0}(\xi) \left|\hat{f}_{\theta_0}(\xi + \sigma)\right|^2 \, d\eta d\sigma
\end{align*}
\]
Set $R = L_0^{-1}$. Then we estimate for $\theta = \theta_0$ or $\theta_2$,
\[
\begin{align*}
&\left|\hat{f}_{\theta}(\xi + \eta) - \hat{f}_{\theta}(\zeta)\right| \\
&\lesssim \left|\hat{\rho}_{R} f_{\theta}(\xi + \eta) - \hat{\rho}_{R} f_{\theta}(\zeta)\right| + \left|\hat{\rho}_{\leq R} f_{\theta}(\xi + \eta) - \hat{\rho}_{\leq R} f_{\theta}(\zeta)\right| \\
&\lesssim \left\|\hat{\rho}_{R} f_{\theta}\right\|_{L^\infty} + \left\|\hat{\rho}_{\leq R} f_{\theta}\right\|_{L^\infty} \\
&\lesssim R^{-\frac{1}{2}} \left\|\langle x \rangle^2 f_{\theta}\right\|_{L^2} + R^2 L_0 \left\|\langle x \rangle^2 f_{\theta}\right\|_{L^2} \\
&\lesssim L_0^4 M^{2\theta_0},
\end{align*}
\]
where the spatial cut-off functions $\rho_{\leq R}(x)$ and $\rho_{> R}(x)$ are introduced. From this and (6.2), we have
\[
\left| I_{(\theta_0, \theta_2)}(s, \xi) - \widetilde{I}_{(\theta_0, \theta_2)}(s, \xi) \right| \leq \int_{\mathbb{R}^{3+3}} \rho_{\leq L_0}(\eta) |\eta|^{-2} \times \left| \hat{f}_{\theta_0}(\xi + \eta + \sigma, \hat{f}_{\theta_2}(\xi + \sigma) - \hat{f}_{\theta_0}(\xi) |\hat{f}_{\theta_2}(\xi + \sigma)|^2 \right| d\eta d\sigma \\
\leq L_0^3 M^{2 \delta_0} \varepsilon_1^3 \lesssim M^{-\frac{3}{4} + \frac{1}{4} \varepsilon_1^2}.
\]
To conclude the proof, it remains to show
\[
\left| \sum_{\theta_2 \in \{\pm\}} I_{(\theta_0, \theta_2)}(s, \xi) - i \partial_s B(s, \theta_0 \xi) \hat{f}_{\theta_0}(s, \xi) \right| \lesssim M^{-(1 + \delta_0)} |N_0|^{-10} \varepsilon_1^3.
\]
Setting
\[
\zeta := \left( \frac{\theta_0 - \xi}{\xi} - \frac{\theta_2}{\sigma} \frac{\sigma}{\sigma} \right),
\]
we estimate (6.9)
\[
\left| \sum_{\theta_2 \in \{\pm\}} I_{(\theta_0, \theta_2)}(s, \xi) - i \partial_s B(s, \theta_0 \xi) \hat{f}_{\theta_0}(s, \xi) \right| \leq \left| \hat{f}_{\theta_0}(\xi) \right| \left| \int_{\mathbb{R}^{3+3}} e^{i \eta \zeta} \rho_{\leq L_0}(\eta) |\eta|^{-2} \left| \hat{f}_{\theta_2}(\sigma) \right|^2 d\eta d\sigma - \frac{(2 \pi)^3}{s} \left| \int_{\mathbb{R}^3} |\zeta|^{-1} \left| \hat{f}_{\theta_2}(\sigma) \right|^2 d\sigma \right| \right| \\
\leq \lim_{A \to \infty} \left| \hat{f}_{\theta_0}(\xi) \right| \left| \int_{\mathbb{R}^3} e^{i \eta \zeta} \rho_{\leq L_0}(\eta) - \rho_{\leq A}(\eta) \right| \left| \hat{f}_{\theta_2}(\sigma) \right|^2 d\sigma \right| \\
\lesssim (N)^{-10} M^{-\frac{3}{4} + \frac{1}{2} \varepsilon_1} \left| \int_{\mathbb{R}^3} |\zeta|^{-2} \left| \hat{f}_{\theta_2}(\sigma) \right|^2 d\sigma \right| .
\]
Here we used the formula
\[
\frac{(2 \pi)^3}{|s|} \frac{1}{A} \rightarrow \infty \int_{\mathbb{R}^3} e^{i \eta \zeta} \rho_{\leq A}(\eta) d\eta ,
\]
which gives for $L_0 \ll A$,
\[
\left| \int_{\mathbb{R}^3} e^{i \eta \zeta} |\eta|^{-2} (\rho_{\leq L_0}(\eta) - \rho_{\leq A}(\eta)) d\eta \right| = |s|^{-2} \left| \int_{\mathbb{R}^3} (\nabla_\eta^2 e^{i \eta \zeta}) |\eta|^{-2} \rho_{\leq L_0}(\eta) - \rho_{\leq A}(\eta) d\eta \right| \\
\lesssim M^{-2} L_0^{-1} |\zeta|^{-2}.
\]
Since $|\zeta| \gtrsim \min \left\{ 1, |\sigma|, \frac{|\theta_0 - \theta_2|}{|\sigma|} \right\}$, a priori assumption (2.1) yields that
\[
\left| \int_{\mathbb{R}^3} |\zeta|^{-2} \left| \hat{f}_{\theta_2}(\sigma) \right|^2 d\sigma \right| \lesssim \varepsilon_1^2.
\]
Plugging this bound into (6.9), we complete the proof of (6.6).
Proof of (6.7). When we estimate the low frequency parts $|\xi| \leq s^{\frac{3}{4}}$ with $\Theta \neq \Xi$, where $\theta_0 = -\theta_1$ or $\theta_2 = -\theta_3$, the cancellation effect from the oscillatory term $e^{iB(s, \theta_0 \xi)}$ could be ignored. Instead, the spinorial structure from $\Pi_{\theta_0}(D)\Pi_{\theta_1}(D)$ or $\Pi_{\theta_2}(D)\Pi_{\theta_3}(D)$ plays an essential role to bound the singularity. For instance, if $\theta_0 = -\theta_1$, we estimate using (3.3)

$$
\left| I_{\Theta, L_0}(s, \xi) \right| \lesssim \int_{\mathbb{R}^{3+3}} \left| \check{\Pi}_{\theta_0}(\xi) \Pi_{\theta_0}(\xi + \eta) \right| \left| \rho_{\leq L_0}(\eta) \right| |\eta|^{-2} \times \left| \check{f}_{-\theta_0}(\xi + \eta) \left( \check{f}_{\theta_1}(\xi + \eta + \sigma), \check{f}_{\theta_2}(\xi + \sigma) \right) \right| d\eta d\sigma
$$

$$
\lesssim L^2 \left\| f_{\theta_0} \right\|_{L^2}^2 \left\| f_{\theta_2} \right\|_{L^2} \left\| \check{f}_{-\theta_0} \right\|_{L^2}^2
$$

$$
\lesssim M \left( \frac{1}{2} + \frac{1}{10} \right) \varepsilon_1^3.
$$

The other cases can be estimated similarly.

Proof of (6.8). We localize the frequencies of $f_{\theta_i}$ as follows:

$$
I_{\Theta, L}(s, \xi) = i\varepsilon_1 \sum_{N \in (2^3)^3} I^N_{\Theta, L}(s, \xi),
$$

$$
I^N_{\Theta, L}(s, \xi) = \int_{\mathbb{R}^{3+3}} e^{i p_{\Theta}(\xi, \eta, \sigma) \Pi_{\theta_0}(\xi)} |\eta|^{-2} \check{f}_{\theta_1, N_1}(s, \xi + \eta) \times \left( \check{f}_{\theta_1, N_1}(s, \xi + \eta + \sigma), \check{f}_{\theta_2, N_2}(s, \xi + \sigma) \right) d\eta d\sigma,
$$

where $f_{\theta_j, N_j} = P_{N_j} f_{\theta_j}$, $N = (N_1, N_2, N_3)$ is 3-tuple of dyadic numbers, and

$$
p_{\Theta}(\xi, \eta, \sigma) = \theta_0(\xi) - \theta_1(\xi + \eta) - \theta_2(\xi + \sigma) + \theta_3(\xi + \eta + \sigma).
$$

Then we have only to prove

$$
(6.10) \quad \sum_{L > L_0, N \in (2^3)^3} \left| I^N_{\Theta, L}(s, \xi) \right| \lesssim \varepsilon_1^3 M^{-1 - \delta_0} \langle N \rangle^{-3}.
$$

By Hölder inequality, we readily have

$$
\left| I^N_{\Theta, L}(s, \xi) \right| \lesssim L^{-\frac{3}{4}} \prod_{j=1}^{3} \left\| \check{f}_{\theta_j, N_j}(s) \right\|_{L^2},
$$

and the a priori assumption implies the following bounds

$$
\left\| \check{f}_{\theta_j, N_j}(s) \right\|_{L^2} \lesssim \min \left( N_j^{\frac{3}{4}} \langle N_j \rangle^{-10}, \langle N_j \rangle^{-3} M^{\delta_0} \right) \varepsilon_1 \quad \text{for} \quad j = 1, 2, 3.
$$

The last two estimates above are sufficient to show (6.10) for the sum over those indices $N$ satisfying $N_{\max} \geq M^{\frac{3}{4}}$ or $N_{\min} \leq M^{-1}$. Here we denoted $\max(N_1, N_2, N_3)$ and $\min(N_1, N_2, N_3)$ by $N_{\max}$ and $N_{\min}$, respectively. Then, it remains to bound the sum in (6.10) over $L > L_0$ and $N$ satisfying

$$
M^{-1} \leq N_1, N_2, N_3 \leq M^{\frac{3}{4}}.
$$
Let us further localize \( \sigma \) variable by \( L' \in 2\mathbb{Z} \) to write
\[
\mathcal{T}_{0,L}^N(s, \xi) = \sum_{L' \in 2\mathbb{Z}} \mathcal{T}_{0,L}^N(s, \xi),
\]
\[
\mathcal{T}_{0,L}^N(s, \xi) = \int_{\mathbb{R}^{3+3}} e^{i s p \theta (\xi, \eta, \sigma)} \Pi_{\theta_0}(\xi) \left| \eta \right|^{-2} \rho_L(\eta) \rho_L'(\sigma) \mathcal{F}_{\theta_1, N_1}(s, \xi + \eta) \times \mathcal{F}_{\theta_2, N_2}(s, \xi + \sigma) \, dy \, d\sigma,
\]
where \( L = (L, L') \). Then we suffice to prove that
\[
\sum_{(N, L) \in \mathcal{A}} \left| \mathcal{T}_{0,L}^N(s, \xi) \right| \lesssim \varepsilon_1^3 M^{-1 - \delta_0} \langle N \rangle^{-10},
\]
where the summation runs over
\[
(6.11) \quad \mathcal{A} = \left\{ (N, L) \in (2\mathbb{Z})^3 \times (2\mathbb{Z})^2 : M^{-1} \leq N_1, N_2, N_3 \leq M^{\frac{2}{7}}, L_0 \leq L \right\}.
\]
We observe that the other dyadic numbers also satisfy \( N, L, L' \lesssim M^\frac{2}{7} \). We divide the summation over \( \mathcal{A} \) into
\[
\sum_{(N, L) \in \mathcal{A}} = \sum_{(N, L) \in \mathcal{A}_1} + \sum_{(N, L) \in \mathcal{A}_2} + \sum_{(N, L) \in \mathcal{A}_3} + \sum_{(N, L) \in \mathcal{A}_4},
\]
where
- **Case 1:** \( \mathcal{A}_1 = \mathcal{A} \cap \{ L' \leq L^{-\frac{1}{2}} M^{-\frac{1}{2} - \frac{1}{5}} \} \),
- **Case 2:** \( \mathcal{A}_2 = \mathcal{A} \cap \{ L' \geq L^{-\frac{1}{2}} M^{-\frac{1}{2} - \frac{1}{5}} \text{ and } \max(N_1, N_3) \lesssim L \} \),
- **Case 3:** \( \mathcal{A}_3 = \mathcal{A} \cap \{ L' \geq L^{-\frac{1}{2}} M^{-\frac{1}{2} - \frac{1}{5}}, \max(N_1, N_3) \gg L, \text{ and } N_1 \sim N_3 \} \),
- **Case 4:** \( \mathcal{A}_4 = \mathcal{A} \cap \{ L' \geq L^{-\frac{1}{2}} M^{-\frac{1}{2} - \frac{1}{5}}, \max(N_1, N_3) \gg L, \text{ and } N_1 \sim N_3 \} \).

In the estimates for the first three cases, the signs \( \Theta \) will play no role.

**Estimates for Case 1.** The low frequency part with respect to \( \sigma \) variable can be estimated by the trivial estimates
\[
\sum_{(N, L) \in \mathcal{A}_1} \left| \mathcal{T}_{0,L}^N(s, \xi) \right| \lesssim \sum_{(N, L) \in \mathcal{A}_1} \left\| L(L')^3 \langle N_{\max} \rangle^{-10} \prod_{j=1}^3 \left\| \langle \xi \rangle^{10} \mathcal{F}_{\theta_j, N_j}(s, \xi) \right\|_{L^\infty} \right. \]
\[
\lesssim \varepsilon_1^3 M^{-(1 + \delta_0)} \langle N \rangle^{-10}.
\]

**Estimates for Case 2.** We apply Lemma 2.7 to \( \mathcal{T}_{0,L}^N \) with
\[
m_{N,L}(\xi, \eta, \sigma) := \left| \eta \right|^{-2} \rho_L(\eta) \rho_L'(\sigma) \rho_{N_1}(\xi + \eta) \rho_{N_3}(\xi + \eta + \sigma)
\]
and \( \|m_{N,L}(\xi, \eta, \sigma)\|_{CM} \lesssim L^{-2} \) to estimate
\[
\sum_{(N, L) \in \mathcal{A}_2} \left| \mathcal{T}_{0,L}^N(s, \xi) \right| \lesssim \sum_{(N, L) \in \mathcal{A}_2} L^{-2} \| \mathcal{F}_{\theta_1, N_1} \|_{L^2} \| \psi_{\theta_2, N_2} \|_{L^\infty} \| \mathcal{F}_{\theta_3, N_3} \|_{L^2}
\]
\[
\lesssim \varepsilon_1^3 M^{-\frac{2}{7}} \sum_{(N, L) \in \mathcal{A}_2} L^{-2} N_1^{\frac{2}{7}} \langle N_1 \rangle^{-10} L^{-2} N_2^{\frac{2}{7}} \langle N_2 \rangle^{-10} N_3^{\frac{2}{7}} \langle N_3 \rangle^{-10}
\]
\[
\lesssim \varepsilon_1^3 M^{-\frac{2}{7} + \delta_0} \langle N \rangle^{-10}.
\]
Estimates for Case 3. In this case we extract an extra time decay via the normal form approach. To achieve this, using the relation
\[ e^{ip\theta} = -\frac{1}{s} \left( \frac{\nabla_q p\theta \cdot \nabla_q e^{ip\theta}}{|\nabla_q p\theta|^2} \right), \]
we perform an integration by parts in $\eta$ to obtain
\[ I_{\Theta, L}^N(s, \xi) = J_1(s, \xi) + J_2(s, \xi), \]
where
\[ J_1(s, \xi) := \frac{i}{s} \Pi_0(\xi) \int_{\mathbb{R}^{3+3}} e^{ip\theta(\xi, \eta, \sigma)} m_1(\xi, \eta, \sigma) \times \nabla_\eta \left( f_{\theta_1, N_1}(s, \xi + \eta) \left( f_{\theta_2, N_2}(s, \xi + \eta), f_{\theta_3, N_3}(s, \xi + \sigma) \right) \right) d\eta d\sigma, \]
and
\[ J_2(s, \xi) := \frac{i}{s} \Pi_0(\xi) \int_{\mathbb{R}^{3+3}} e^{ip\theta(\xi, \eta, \sigma)} m_2(\xi, \eta, \sigma) \times f_{\theta_1, N_1}(s, \xi + \eta) \left( f_{\theta_2, N_2}(s, \xi + \eta), f_{\theta_3, N_3}(s, \xi + \sigma) \right) d\eta d\sigma, \]
where
\[ m_1(\xi, \eta, \sigma) = \frac{\nabla_q p\theta(\xi, \eta, \sigma)}{|\nabla_q p\theta(\xi, \eta, \sigma)|^2} |\eta|^{-2} \rho_L(\eta) \rho_L(\sigma) \rho_{N_1}(\xi + \eta) \rho_{N_3}(\xi + \sigma), \]
\[ m_2(\xi, \eta, \sigma) = \nabla_\eta \left( \frac{\nabla_q p\theta(\xi, \eta, \sigma)}{|\nabla_q p\theta(\xi, \eta, \sigma)|^2} |\eta|^{-2} \rho_L(\eta) \rho_L(\sigma) \rho_{N_1}(\xi + \eta) \rho_{N_3}(\xi + \sigma) \right). \]
We first claim that for fixed $\xi \in \mathbb{R}^3$,
\[ \| m_1 \|_{CM} := \left\| \int_{\mathbb{R}^{3+3}} m_1(\xi, \eta, \sigma) e^{ix\cdot\eta e^{iy\cdot\sigma}} d\eta d\sigma \right\|_{L^1_{x,y}(\mathbb{R}^3 \times \mathbb{R}^3)} \lesssim \max(N_1, N_3)^{-1} \min((N_1), (N_3))^{15} L^{-2}. \]
Indeed, by changing variables, we have
\[ \| m_1 \|_{CM} = \| \tilde{m}_1 \|_{CM}, \]
where
\[ \tilde{m}_1(\xi, \eta, \sigma) = \frac{\frac{q(\theta_1, \theta_2)}{q(\theta_1, \theta_3)}(\eta, \sigma)}{|\frac{q(\theta_1, \theta_2)}{q(\theta_1, \theta_3)}(\eta, \sigma)|^2} |\eta - \sigma|^{-2} \rho_L(\xi - \eta) \rho_L(\sigma - \eta) \rho_{N_1}(\eta) \rho_{N_3}(\sigma), \]
and
\[ q(\theta_1, \theta_3)(\eta, \sigma) = \theta_1 \frac{\eta}{\langle \eta \rangle} - \theta_3 \frac{\sigma}{\langle \sigma \rangle}. \]
Using the inequalities
\[ \left| \frac{\eta}{\langle \eta \rangle} \pm \frac{\sigma}{\langle \sigma \rangle} \right| \geq \frac{|\eta| - |\sigma|}{\min(\langle \eta \rangle, \langle \sigma \rangle) \max(\langle \eta \rangle, \langle \sigma \rangle)}^2, \]
one can show that $\tilde{m}_1$ satisfies the following differential inequalities
\[ \left| \nabla_\eta \nabla_\sigma \tilde{m}_1(\xi, \eta, \sigma) \right| \leq C(N, L) L^{-m} N_3^{-n} \rho_L(\eta - \xi) \rho_{N_1}(\sigma), \]
where
\[ C(N, L) = \begin{cases} L^{-2} \min(N_1, N_3)^{2+2(m+n)} & \text{if } \min(N_1, N_3) \geq 1, \\ L^{-2} \max(N_1, N_3)^{-1} \max((N_1), (N_3)) & \text{if } \min(N_1, N_3) \leq 1, \end{cases} \]
which implies (6.13) by a straightforward computation. Hence, applying Lemma 2.7 with (6.13) to $\mathcal{J}_3$, we estimate

$$
\sum_{(N,L) \in A_3} |\mathcal{J}_3(s, \xi)| 
\lesssim M^{-2+\frac{24}{23}} \langle N \rangle^{-10} \sum_{(N,L) \in A_3} L^{-3} \max(N_1, N_3)^{-2} \min((N_1), (N_3))^{15} L^{-2} \times \left( \| \nabla \tilde{f}_{\theta_1, N_1} \|_{L^2} \| \tilde{f}_{\tilde{\theta}_3, N_3} \|_{L^2} + \| \tilde{f}_{\theta_1, N_1} \|_{L^2} \| \nabla \tilde{f}_{\tilde{\theta}_1, N_3} \|_{L^2} \right) \| \psi_{\theta_2, N_2} \|_{L^\infty}
\lesssim \varepsilon_1^3 M^{-2+\frac{24}{23}} \langle N \rangle^{-10} M^{2-\frac{4}{23}} M^{\delta_0} M^{-\frac{2}{23}} M^{4\frac{2}{23}} \lesssim \varepsilon_1^3 M^{-(1+\delta_0)} \langle N \rangle^{-10},
$$

where we used $M^{-\frac{4}{23}} + \frac{24}{23} \sim L_0 \leq L$. Next, in order to estimate $\mathcal{J}_2(s, \xi)$ we perform an integration by parts in $\eta$ once again to obtain

$$
\mathcal{J}_2(s, \xi) = \mathcal{J}_3(s, \xi) + \mathcal{J}_4(s, \xi),
$$

\begin{align*}
\mathcal{J}_3(s, \xi) &:= -\frac{1}{s^2} \Pi_{\theta_0}(\xi) \int \int_{\mathbb{R}^{3+3}} e^{\im \varphi \theta_0(\xi, \eta, \sigma)} m_3(\xi, \eta, \sigma) \\
&\times \nabla_n \left( \langle \tilde{f}_{\theta_1, N_1}(s, \xi + \eta), \tilde{f}_{\tilde{\theta}_3, N_3}(s, \xi + \sigma), \tilde{f}_{\theta_2, N_2}(s, \xi + \sigma) \rangle \right) \, d\eta d\sigma,
\end{align*}

\begin{align*}
\mathcal{J}_4(s, \xi) &:= -\frac{1}{s^2} \Pi_{\theta_0}(\xi) \int \int_{\mathbb{R}^{3+3}} e^{\im \varphi \theta_0(\xi, \eta, \sigma)} m_4(\xi, \eta, \sigma) \\
&\times \langle \tilde{f}_{\theta_1, N_1}(s, \xi + \eta), \tilde{f}_{\tilde{\theta}_3, N_3}(s, \xi + \sigma), \tilde{f}_{\theta_2, N_2}(s, \xi + \sigma) \rangle \, d\eta d\sigma,
\end{align*}

where

$$
m_3(\xi, \eta, \sigma) = \frac{\nabla_n \varphi \theta_0(\xi, \eta, \sigma)}{\| \nabla_n \varphi \theta_0(\xi, \eta, \sigma) \|^2} m_2(\xi, \eta, \sigma),
$$

$$
m_4(\xi, \eta, \sigma) = \nabla_n \left( \frac{\nabla_n \varphi \theta_0(\xi, \eta, \sigma)}{\| \nabla_n \varphi \theta_0(\xi, \eta, \sigma) \|^2} m_2(\xi, \eta, \sigma) \right).
$$

An analogous computation as in the proof of (6.13) yields that

$$
\| m_3 \|_{CM} \lesssim \max(N_1, N_3)^{-2} \min((N_1), (N_3))^{20} L^{-3}.
$$

Applying Lemma 2.7 with the bound, we estimate

$$
\sum_{(N,L) \in A_3} |\mathcal{J}_3(s, \xi)|
\lesssim M^{-2+\frac{24}{23}} \langle N \rangle^{-10} \sum_{(N,L) \in A_3} L^{-3} \max(N_1, N_3)^{-2} \min((N_1), (N_3))^{20} \left( \| \nabla \tilde{f}_{\theta_1, N_1} \|_{L^2} \| \tilde{f}_{\tilde{\theta}_3, N_3} \|_{L^2} + \| \tilde{f}_{\theta_1, N_1} \|_{L^2} \| \nabla \tilde{f}_{\tilde{\theta}_1, N_3} \|_{L^2} \right) \| \psi_{\theta_2, N_2} \|_{L^\infty}
\lesssim \varepsilon_1^3 M^{-2+\frac{24}{23}} \langle N \rangle^{-10} M^{2-\frac{4}{23}} M^{\delta_0} M^{-\frac{2}{23}} \sum_{L \geq M^{1-\frac{2}{23}}} L^{-3} \sum_{L \geq L^\prime \geq M^{1-\frac{2}{23}}} (L')^{-\frac{4}{23}}
\lesssim \varepsilon_1^3 M^{-\frac{2}{23}} + \frac{24}{23} M^{\frac{1-\frac{4}{23}}{2\delta_0}} M^{-\frac{2}{23}} M^{1-\frac{2}{23}} \langle N \rangle^{-10} \lesssim \varepsilon_1^3 M^{-(1+\delta_0)} \langle N \rangle^{-10}.
$$

Next, from (6.14), one can find the pointwise bound of $m_4$

$$
| m_4(\xi, \eta, \sigma) | \lesssim \max(N_1, N_3)^{-2} \min((N_1), (N_3))^{10} L^{-4}.
$$
Using this and Hölder inequality, we estimate
\[
\sum_{(N, L) \in A_3} |J_4(s, \xi)| \lesssim M^{-2 + \frac{22}{37}} \langle \eta \rangle^{-10} \sum_{(N, L) \in A_3} \max(N_1, N_3)^{-2} \min(\langle N_1 \rangle, \langle N_3 \rangle)^{10} L^{-1}
\]
\[
\times \left\| \hat{f}_{\theta_1, N_1} \right\|_{L^2} \left\| \hat{f}_{\theta_2, N_2} \right\|_{L^\infty} \left\| \hat{f}_{\theta_3, N_3} \right\|_{L^2}
\]
\[
\lesssim \varepsilon_1^3 M^{-2 + \frac{22}{37} + \delta_0} \langle \eta \rangle^{-10} \sum_{L \geq M^{-\frac{2}{5} + \frac{22}{37}}} L^{-1} \sum_{L' \geq M^{-\frac{2}{5}}} (L')^{-\frac{1}{4}} \lesssim \varepsilon_1^3 M^{-3(1 + \delta_0)} \langle \eta \rangle^{-10},
\]
where we used
\[
M^{-\frac{2}{5}} \lesssim L', \quad L' \sim \sigma \lesssim \max(N_1, N_3),
\]
which easily follows from the condition on dyadic pieces (6.11).

**Estimates for Case 4.** As in Case 3, we perform an integration by parts to gain a time decay. The strategy depends on combinations of sign \(\Theta\).

1. If \(\theta_1 = \theta_3\), we perform an integration by parts with respect to \(\eta\) variable using the following
\[
|\nabla_\eta p_{\Theta}(\xi, \eta, \sigma)| = \left| \theta_1 \left( \frac{\xi + \eta}{\xi + \sigma} - \frac{\xi + \eta + \sigma}{\xi + \eta + \sigma} \right) \right| \gtrsim L' \langle N_1 \rangle^{-3}.
\]

2. If \(\theta_1 \neq \theta_3\), we perform an integration by parts with respect to \(\sigma\) variable using the following
\[
|\nabla_\sigma p_{\Theta}(\xi, \eta, \sigma)| \gtrsim \begin{cases} 
\left| \frac{\xi + \sigma}{\xi + \sigma} - \frac{\xi + \eta + \sigma}{\xi + \eta + \sigma} \right| \gtrsim L \langle N_2 \rangle^{-3}, & \text{if } \theta_2 = \theta_3, \\
\left| \frac{\xi + \sigma}{\xi + \sigma} + \frac{\xi + \eta + \sigma}{\xi + \eta + \sigma} \right| \gtrsim N_2 \langle N_2 \rangle^{-1}, & \text{if } \theta_2 \neq \theta_3.
\end{cases}
\]

Here we make an observation for (6.16) when \(\theta_2 \neq \theta_3\). If \(L \sim N_2 \sim N_3\), \(|\nabla_\sigma p_{\Theta}|\) does not have lower bound. However, in this case, we only consider the case \(L \ll N_3\) which makes the lower bound as in (6.16) when \(\theta_2 \neq \theta_3\).

1. **Estimates when \(\theta_1 = \theta_3\).** We perform an integration by parts twice in \(\eta\) variable to obtain
\[
\mathcal{I}_{3, \Theta}^{\mathbf{SL}}(s, \xi) = \mathcal{K}_1(s, \xi) + \mathcal{K}_2(s, \xi) + \mathcal{K}_3(s, \xi),
\]
\[
\mathcal{K}_1(s, \xi) := -\frac{1}{s^2} \int_{\mathbb{R}^{3+3}} e^{is\phi_{\Theta}(\xi, \eta, \sigma)} k_1(\xi, \eta, \sigma) \nabla_\eta \left[ \hat{f}_{\theta_1, N_1}(s, \xi + \eta) \right. \\
\times \left. \left\langle \hat{f}_{\theta_1, N_1}(s, \xi + \eta + \sigma), \hat{f}_{\theta_2, N_2}(s, \xi + \sigma) \right\rangle \right] d\eta d\sigma,
\]
\[
\mathcal{K}_2(s, \xi) := -\frac{2}{s^2} \int_{\mathbb{R}^{3+3}} e^{is\phi_{\Theta}(\xi, \eta, \sigma)} k_2(\xi, \eta, \sigma) \nabla_\eta \left[ \hat{f}_{\theta_3, N_1}(s, \xi + \eta) \right. \\
\times \left. \left\langle \hat{f}_{\theta_3, N_1}(s, \xi + \eta + \sigma), \hat{f}_{\theta_2, N_2}(s, \xi + \sigma) \right\rangle \right] d\eta d\sigma,
\]
\[
\mathcal{K}_3(s, \xi) := -\frac{1}{s^2} \int_{\mathbb{R}^{3+3}} e^{is\phi_{\Theta}(\xi, \eta, \sigma)} k_3(\xi, \eta, \sigma) \hat{f}_{\theta_1, N_1}(s, \xi + \eta) \\
\times \left\langle \hat{f}_{\theta_3, N_1}(s, \xi + \eta + \sigma), \hat{f}_{\theta_2, N_2}(s, \xi + \sigma) \right\rangle d\eta d\sigma,
\]
where

$$k_1(\xi, \eta, \sigma) := \frac{\nabla p(\xi, \eta, \sigma)}{|\nabla p(\xi, \eta, \sigma)|^2}m_1(\xi, \eta, \sigma),$$

$$k_2(\xi, \eta, \sigma) := \frac{\nabla p(\xi, \eta, \sigma)}{|\nabla p(\xi, \eta, \sigma)|^2}m_2(\xi, \eta, \sigma),$$

$$k_3(\xi, \eta, \sigma) := m_3(\xi, \eta, \sigma).$$

Here $m_1$, $m_2$, and $m_3$ are defined in (6.12) and (6.15), respectively. Since $\theta_1 = \theta_3$, by the mean value theorem we have

$$|\nabla p(\xi, \eta, \sigma)| = \left| \frac{\xi + \eta}{\xi + \eta} - \frac{\xi + \eta + \sigma}{\xi + \eta + \sigma} \right| \lesssim \frac{L'}{(N_1)^3},$$

which leads us to the differential inequalities

$$|\nabla^m \nabla^\ell k_1(\xi, \eta, \sigma)| \lesssim L^{-2}(L')^{-2}(N_1)^6(N_1)^2(m+\ell)L^{-m}(L')^{-\ell},$$

$$|\nabla^m \nabla^\ell k_2(\xi, \eta, \sigma)| \lesssim L^{-3}(L')^{-2}(N_1)^6(N_1)^2(m+\ell)L^{-m}(L')^{-\ell},$$

$$|\nabla^m \nabla^\ell k_3(\xi, \eta, \sigma)| \lesssim L^{-4}(L')^{-2}(N_1)^6.$$

for $0 \leq m, \ell \leq 4$, and the pointwise bound

$$(6.17) \quad |k_3(\xi, \eta, \sigma)| \lesssim L^{-4}(L')^{-2}(N_1)^6.$$

Applying Lemma 2.7 with $||k_1||_{CM} = L^{-2}(L')^{-2}(N_1)^6 M^{\frac{4}{m}}$, we estimate

$$\sum_{(N,L) \in A_4} |K_1(s, \xi)| \lesssim M^{-2+\frac{4}{m}+\frac{4}{\alpha}}(N)^{-10} \sum_{(N,L) \in A_4} L^{-2}(L')^{-2}(N_1)^6 \|\psi_{t_0, N_2}\|_{L^\infty} \times (N_1^{-2} \|x^2 f_0\|_{L^2} \|f_{t_0, N_2}\|_{L^2} + N_3^{-2} \|f_{t_0, N_1}\|_{L^2} \|x^2 f_{t_0}\|_{L^2}) \lesssim \varepsilon_1^3 M^{-2+\frac{4}{m}+2\delta_0} \sum_{(N,L) \in A_4} L^{-2}(L')^{-2}(N_2)^{-2} N_1^{-\frac{2}{m}}(N_1)^{-4} \lesssim \varepsilon_1^3 M^{-\frac{2}{m}+\frac{4}{m}+2\delta_0}(N)^{-10} M^{-\frac{2}{m}+\frac{4}{m}+\frac{2}{m}+\delta_0} \lesssim \varepsilon_1^3 M^{-1+\delta_0}(N)^{-10},$$

where we used that

$$L^{-\frac{2}{m}}(L')^{-2} \leq M^{\frac{2}{m}+\frac{2}{m}}$$

and

$$M^{-\frac{2}{m}+\frac{2}{m}} \lesssim L \ll N_1.$$

Similarly, we estimate by using $||k_2||_{CM} = L^{-3}(L')^{-2}(N_1)^6 M^{\frac{4}{m}}$

$$\sum_{(N,L) \in A_4} |K_2(s, \xi)| \lesssim M^{-2+\frac{4}{m}+\frac{4}{\alpha}}(N)^{-10} \sum_{(N,L) \in A_4} L^{-3}(L')^{-2}(N_1)^6 \|\psi_{t_0, N_2}\|_{L^\infty} \times (N_1^{-1} \|x f_{t_0}\|_{L^2} \|f_{t_0, N_2}\|_{L^2} + N_3^{-1} \|f_{t_0, N_1}\|_{L^2} \|x f_{t_0}\|_{L^2}) \lesssim \varepsilon_1^3 M^{-2+\frac{4}{m}}(N)^{-10} M^{-\frac{2}{m}+\frac{4}{m}+\frac{2}{m}+\delta_0} \lesssim \varepsilon_1^3 M^{-1+\delta_0}(N)^{-10},$$

where we used that

$$L^{-\frac{2}{m}} \leq M^{-\frac{2}{m}+\frac{2}{m}}$$

and

$$L^{-\frac{2}{m}}(L')^{-2} \leq M^{\frac{2}{m}+\frac{2}{m}}.$$
Using Hölder inequality and (6.17), we see that
\[
\sum_{(N,L) \in A_4} |\mathcal{K}_3(s, \xi)| 
\lesssim M^{-2} \sum_{(N,L) \in A_4} L^{-1} L' \langle N_1 \rangle^6 \left\| \hat{f}_{\theta_1, N_1} \right\|_{L^\infty} \left\| \hat{f}_{\theta_2, N_2} \right\|_{L^\infty} \left\| \hat{f}_{\theta_3, N_3} \right\|_{L^\infty} 
\lesssim \varepsilon_1^3 M^{-2} M^{\frac{2}{3}(1 + \delta_0)} \langle N \rangle^{-10} \lesssim \varepsilon_1^3 M^{-(1 + \delta_0)} \langle N \rangle^{-10}.
\]

(2) Estimates when \( \theta_1 \neq \theta_3 \). The proof proceeds similarly to the previous case. We integrate by parts twice in \( \sigma \) variable
\[
I_{\mathbf{N}, \mathbf{L}}(s, \xi) = \tilde{K}_4(s, \xi) + \tilde{K}_2(s, \xi) + \tilde{K}_3(s, \xi),
\]
where
\[
\tilde{K}_1(s, \xi) := \frac{1}{s^2} \int_{\mathbb{R}^{1+3}} e^{isp_{\theta}(\xi, \eta, \sigma)} \tilde{k}_1(\xi, \eta, \sigma) \nabla^2_{\sigma} \hat{f}_{\theta_1, N_1}(s, \xi + \eta) 
\times \left( \hat{f}_{\theta_3, N_3}(s, \xi + \eta + \sigma), \hat{f}_{\theta_2, N_2}(s, \xi + \sigma) \right) \, d\eta d\sigma,
\]
\[
\tilde{K}_2(s, \xi) := \frac{2}{s^2} \int_{\mathbb{R}^{1+3}} e^{isp_{\theta}(\xi, \eta, \sigma)} \tilde{k}_2(\xi, \eta, \sigma) \nabla_{\sigma} \hat{f}_{\theta_1, N_1}(s, \xi + \eta) 
\times \left( \hat{f}_{\theta_3, N_3}(s, \xi + \eta + \sigma), \hat{f}_{\theta_2, N_2}(s, \xi + \sigma) \right) \, d\eta d\sigma,
\]
\[
\tilde{K}_3(s, \xi) := \frac{1}{s^2} \int_{\mathbb{R}^{1+3}} e^{isp_{\theta}(\xi, \eta, \sigma)} \tilde{k}_3(\xi, \eta, \sigma) \hat{f}_{\theta_1, N_1}(s, \xi + \eta) 
\times \left( \hat{f}_{\theta_3, N_3}(s, \xi + \eta + \sigma), \hat{f}_{\theta_2, N_2}(s, \xi + \sigma) \right) \, d\eta d\sigma,
\]

As observed in (6.16), \(|\nabla_{\sigma} p_{\theta}|\) has different lower bounds depending on the sign of \( \theta_2 \) and \( \theta_3 \). One can verify that, however, if \((\mathbf{N}, \mathbf{L}) \in A_1\), the lower bounds of \(|\nabla_{\sigma} p_{\theta}|\) when \( \theta_2 \neq \theta_3 \) are always greater than those when \( \theta_2 = \theta_3 \)
\[
N_2 \langle N_2 \rangle^{-1} \gtrsim L \langle N_2 \rangle^{-3}.
\]
Hence, we suffice to treat the latter case, when \( \theta_2 = \theta_3 \). Using (6.16), we can show the pointwise bounds of the multipliers
\[
|\tilde{k}_1(\xi, \eta, \sigma)| \lesssim L^{-4} \langle N_2 \rangle^6,
\]
\[
|\tilde{k}_2(\xi, \eta, \sigma)| \lesssim L^{-4} (L')^{-1} M^{\frac{2}{3}} \langle N_2 \rangle^6,
\]
\[
|\tilde{k}_3(\xi, \eta, \sigma)| \lesssim L^{-4} (L')^{-2} M^{\frac{2}{3}} \langle N_2 \rangle^6.
\]
Then, we estimate by the H"older inequality
\[
\sum_{(N,L)\in A_4} |\tilde{K}_1(s,\xi)| \\
\lesssim M^{-2} \sum_{(N,L)\in A_4} L^{-4} (N_2)^6 L^3 \left| \widehat{f_{\theta_1,N_1}} \right|_{L^\infty} \\
\times (N_2^{-2} \left\| x^2 f_{\theta_2} \right\|_{L^2} \left\| f_{\theta_3,N_3} \right\|_{L^2} + N_3^{-2} \left\| f_{\theta_3,N_2} \right\|_{L^2} \left\| x^2 f_{\theta_3} \right\|_{L^2}) \\
\lesssim \varepsilon_1^3 M^{-2+2\delta_0} \sum_{(N,L)\in A_4} L^{-1} (L')^{-\frac{1}{2}} (N_3)^{-14} \\
\lesssim \varepsilon_1^3 M^{-2+\delta_0} M^{\frac{12}{3}-\frac{1}{2\sigma}} M^{\frac{4}{3}+\frac{2}{\sigma}} (N)^{-10} \lesssim \varepsilon_1^3 M^{-(1+\delta_0)} (N)^{-10},
\]
and
\[
\sum_{(N,L)\in A_4} |\tilde{K}_2(s,\xi)| \\
\lesssim M^{-2+\frac{1}{\sigma}} \sum_{(N,L)\in A_4} L^{-4} (L')^{-1} (N_2)^6 L^3 \left| \widehat{f_{\theta_1,N_1}} \right|_{L^\infty} \\
\times (N_2^{-1} \left\| x f_{\theta_2} \right\|_{L^2} \left\| f_{\theta_3,N_3} \right\|_{L^2} + N_3^{-1} \left\| f_{\theta_3,N_2} \right\|_{L^2} \left\| x f_{\theta_3} \right\|_{L^2}) \\
\lesssim \varepsilon_1^3 M^{-2+\frac{1}{\sigma}} M^{\frac{12}{3}-\frac{1}{2\sigma}} M^{\frac{4}{3}+\frac{2}{\sigma}} M^{\delta_0} (N)^{-10} \lesssim \varepsilon_1^3 M^{-(1+\delta_0)} (N)^{-10},
\]
where we used that
\[
L' \ll N_2 \sim N_3, \quad L^{-1} \leq M^{\frac{2}{3}-\frac{1}{\sigma}}, \quad \text{and} \quad L^{-\frac{1}{2}} (L')^{-1} \leq M^{\frac{4}{3}+\frac{2}{\sigma}}.
\]
In a similar way, we get
\[
\sum_{(N,L)\in A_4} |\tilde{K}_3(s,\xi)| \\
\lesssim M^{-2+\frac{1}{\sigma}} \sum_{(N,L)\in A_4} L^{-4} (L')^{-2} (N_2)^3 (L')^3 \left| \widehat{f_{\theta_1,N_1}} \right|_{L^\infty} \left| \widehat{f_{\theta_2,N_2}} \right|_{L^\infty} \left| \widehat{f_{\theta_3,N_3}} \right|_{L^\infty} \\
\lesssim \varepsilon_1^3 M^{-2+\frac{1}{\sigma}} M^{\frac{4}{3}-\frac{2}{\sigma}} (N)^{-10} \lesssim \varepsilon_1^3 M^{-(1+\delta_0)} (N)^{-10}.
\]
This concludes the proof of \((6.8)\), and thus of Proposition 4.2.

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