On a spacetime duality in \(2 + 1\) gravity

Alejandro Corichi\(^{1,2}\) and Andrés Gomberoff\(^{1,2,3,4}\)

1. Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México
   A. Postal 70-543, México D.F. 04510, México
2. Institute for Theoretical Physics, University of California
   Santa Barbara, CA 93106, USA
3. Physics Department, Syracuse University
   Syracuse, NY 13244, USA
4. Centro de Estudios Científicos de Santiago
   Casilla 16433, Santiago 9, Chile

(June, 1999)

Abstract

We consider \(2 + 1\) dimensional gravity with a cosmological constant, and explore a duality that exists between space-times that have the De Sitter group \(SO(3,1)\) as its local isometry group. In particular, the Lorentzian theory with a positive cosmological constant is dual to the Euclidean theory with a negative cosmological constant. We use this duality to construct a mapping between apparently unrelated space-times. More precisely, we exhibit a relation between the Euclidean BTZ family and some \(T^2\)-cosmological solutions, and between De-Sitter point particle space-times and the analytic continuations of Anti-De Sitter point particles. We discuss some possible applications for BH and \(AdS\) thermodynamics.

Pacs No: 04.20.Fy, 04.60.Kz

*Electronic Address: corichi@nuclecu.unam.mx
†Electronic Address: agombero@phy.syr.edu
I. INTRODUCTION

In order to capture some qualitative features of $3 + 1$ dimensional gravity, it is always useful to consider simpler toy models that are free of many of the technical difficulties of the full theory. This is particularly true of $2 + 1$ gravity. Before 1984, roughly speaking, the theory was considered to be “too trivial” to deserve any attention. It was with the work of Deser, Jackiw and t’Hooft [1], [2], who considered point particle solutions, and later with the papers by Achucarro and Townsend [3] and Witten [4], that an avalanche of papers on $2 + 1$ gravity followed. A very wide spectrum of issues have been addressed in this years, from the “problem of time” to the “loop representation”, and more recently, black hole solutions [5]. For recent reviews see [6] and particularly [7]. For a review of the $2 + 1$ black hole see [8].

$(2 + 1)$–dimensional gravity has been successful in providing simplified models in the study of black hole physics. In particular, the case when a negative cosmological constant is added (anti–de Sitter space-times) has been extensively studied in the last years. On the one hand it has been shown [4] that making certain identifications in anti–de Sitter (AdS) geometry, one obtains a Black Hole solution (BTZ black hole). On the other hand, in string theories it has been found that there is a whole family of supersymmetric black brane solutions whose “near horizon” geometry is the product of spheres and the three dimensional AdS spacetime [10]. The study of the quantum properties of the BTZ black hole has been studied using many approaches, all of them giving a quantum mechanical derivation of the black hole entropy [11], [12], [13].

The case of a positive cosmological constant has also received recent attention. Maldacena and Strominger [14] have given a CFT derivation of the entropy associated to the cosmological horizon, following earlier results by Carlip [15]. More recently, similar derivations in the Euclidean continuation have given an alternative explanation of the Hawking-Gibbons entropy of De Sitter space-time [16,17].

In this note we consider three dimensional gravity with a cosmological constant. The purpose of the paper is to explore a little-known duality that exists between Lorentzian $2 + 1$ gravity with a positive cosmological constant and Euclidean gravity with a negative constant [18]. As first shown in [4], $2 + 1$ gravity can be formulated as a theory of connections. There are two ways of writing the 3-dimensional action that involve connections. The Einstein-Hilbert-Palatini action has as independent fields a “spin” connection $\omega$ and a (non-degenerate) co-triad field $e$. The “pure-connection” formulation considers a Chern-Simons action where $\omega$ and $e$ are combined into a single connection $A$ living in a larger gauge group. This later description simplifies the structure of the theory and it is possible to completely characterize the state space. When one works in the so called “frozen formalism”, i.e., in the reduced phase space of the theory, one is dealing with the “true degrees of freedom” of the theory. That is, roughly speaking, an equivalence class of classical solutions where one does not distinguish between configurations that are related by a “gauge transformation”. However, in the cases of interest for this note ((− + +) signature with $\Lambda > 0$ and (+ + +) with $\Lambda < 0$), both the space of histories and the reduced phase spaces have been shown to coincide [18]. Thus, we have a mathematical identical description of the two systems. This is quite surprising and disturbing at first. But, as it is the case in many physical situations, the “physics” is in the interpretation of the formalism. As we shall see, there is a precise way to reconstruct two space-times, each one with a different signature, for each point of
the (reduced) phase space. This construct also provides an explicit mapping between this space-times, or using a “Wheelerism”, a *Wick rotation without Wick rotation*.

Let us illustrate this duality using some heuristic arguments. Take the Euclidean BTZ solution with positive mass $M$ and no angular momentum $l$,

$$ds^2 = \frac{(r^2 - Ml^2)}{l^2}d\tau^2 + \frac{l^2}{r^2 - Ml^2}dr^2 + r^2d\phi^2, \quad (1.1)$$

and make the following identifications:

$$l \rightarrow il \quad ; \quad \tau \rightarrow i\tau. \quad (1.2)$$

We obtain,

$$ds^2 = -\frac{l^2}{T^2 + Ml^2}dT^2 + \frac{(T^2 + Ml^2)}{l^2}dR^2 + T^2d\phi^2. \quad (1.3)$$

Here we have changed the names of the variables $r \rightarrow T$ and $\tau \rightarrow R$.

This solution looks like a cosmological “big–bang” solution, with a singularity at $T = 0$. In the next sections we will see that this is indeed the case, and that this space-time is in a precise sense, dual to the BTZ solution.

Now take (1.1) with $M = -\alpha^2 \in [-1,0]$. These are “particle” solutions, with no horizon and naked, conical singularities at $r = 0$. The angle deficit of this cones is $\Omega = 2\pi(1 - \alpha)$. For $\Omega = 0$ we get the singular free global AdS spacetime. Proceeding with (1.2) in this solution we get

$$ds^2 = -\frac{(l^2\alpha^2 - r^2)}{l^2}d\tau^2 + \frac{l^2}{l^2\alpha^2 - r^2}dr^2 + r^2d\phi^2. \quad (1.4)$$

This represents a spacetime with a cosmological horizon at $r = r_c = l\alpha$ and a conical singularity at $r = 0$ with deficit angle $\Omega$. Near the horizon, defining $\rho^2 = l^2 - r^2/\alpha^2$, we find,

$$ds^2 \approx -\frac{\alpha^2\rho^2}{l^2}d\tau^2 + d\rho^2 + \alpha^2l^2d\phi^2. \quad (1.5)$$

Note that in order that the Euclidean continuation of this solution is singularity free in the $(\rho, \tau)$ plane, we must identify the time $\tau$ with period $\beta = 2\pi l/\alpha$. This period can be associated to the inverse temperature of the solution, which in this case is $[16]$, $T = \frac{\alpha\hbar}{2\pi l} = \frac{r_c\hbar}{2\pi l^2} \quad (1.6)$

It is worth noting that we could formally associate a temperature to the dual of the particle case, which had no horizon and thus no natural notion of temperature. Reciprocally,

\footnote{We are using units such that $8G = 1$. In three spacetime dimensions this means that mass is dimensionless and $\hbar$ has units of length.}
the originally hot BTZ black hole is continued to a cosmological solution with no natural
temperature, for which one could again assign some formal temperature.

The structure of the paper is as follows. In Sec. II we recall the gauge theory formulation
of 2 + 1 gravity with a cosmological constant, and the equivalence between the choices of
signature and cosmological constant $\Lambda$ that we are interested in. In Sec. III we explore in
detail this duality, and apply it to the interesting case of the BTZ black hole ‘family’. In
particular, we give a rigorous justification for the ‘dualities’ presented above. We end with a
discussion in Sec. IV. In the Appendix, we consider the case of non-zero angular momentum.

II. PRELIMINARIES

In this section we recall some basic features of 2 + 1 gravity in the presence of a cosmo-
llogical constant. The Einstein-Hilbert action for 3-dimensional gravity,

$$S_{EH}[g] := \frac{1}{2\pi} \int_M d^3x \sqrt{|g|} (R - 2\Lambda), \quad (2.1)$$
can be rewritten in terms of a 1–form connection $\omega$ and a 1-form triad field $e$ as follows

$$S_{P}[\omega, e] := \frac{1}{2\pi} \int_M \epsilon_{IJK} \left( F(\omega)^{IJ} \wedge e^K - \frac{\Lambda}{3} e^I \wedge e^J \wedge e^K \right), \quad (2.2)$$

where the capital indices denote internal vectors, that is, vectors in a fiducial 3 dimensional
vector space $W$. The field $e^I$ is the soldering form that maps internal vectors to tangent vec-
tors to the 3 manifold $M$: $v^I := e^I_{\mu} v^\mu$. In the internal vector space $W$ there is a fixed fiducial
metric $g_{IJ}$ whose signature coincides with the one of the space-time metric. Therefore, in
the case of Euclidean gravity $g_{IJ} = \delta_{IJ}$ and for Lorentzian signature it is the 3 dimensional
Minkowski metric $\eta_{IJ}$ with signature $(-,+,+)$. There is an Lorentz invariant (or rotation
invariant in the Euclidean case) volume element $\epsilon_{IJK}$ in $W$ defined from $g_{IJ}$. The $e$ with
“upstairs” indices is defined from $\epsilon_{IJK}$ in $W$ defined from $g_{IJ}$. The $e$ with “upstairs” indices is defined from $\epsilon_{IJK}$ by “raising the indices” with $g^{IJ}$. We can recover
the space-time metric $g_{\mu\nu}$ from the soldering form $e^I_{\mu}$,

$$g_{\mu\nu} := e^I_{\mu} e^J_{\nu} g_{IJ}. \quad (2.3)$$

Finally, $F_{\mu\nu}$ is the curvature 2–form of the connection $\omega^{IJ}_{\mu}$, i.e., $F^{IJ} = d\omega^{IJ} + \omega^J_K \wedge \omega^{KJ}$. The variation of (2.2) with respect to $\omega^{IJ}_{\mu}$ yields,

$$de^I + \omega^I_J \wedge e^J = 0, \quad (2.4)$$

and the equation from $e$ is,

$$F^{IJ} - \Lambda e^I \wedge e^J = 0. \quad (2.5)$$

The first equation states that the torsion of $\omega$ vanishes, and implies that the covariant
derivative defined by $\omega^I_{\mu\nu}$ coincides with the one compatible with the co-triad, that is,
$\omega^I_{\mu\nu} = \Gamma^I_{\mu\nu}$. We can therefore replace $F$ by $R$ in (2.3) and we arrive at Einstein equations:
\[ G^{\mu\nu} + \Lambda g^{\mu\nu} = 0. \]  
(2.6)

Recall that this equation implies that the space-time has constant scalar curvature proportional to \( \Lambda \):

\[ g_{\mu\nu}(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \Lambda g^{\mu\nu}) = -\frac{1}{2} R + 3\Lambda = 0. \]  
(2.7)

Therefore, \( R = 6\Lambda \). Note that this result is independent of the signature of the space-time and therefore, of the gauge group we are considering.

At this point it is convenient to use a fact from 3 dimensions: the dimensions of the “Lorentz group” and the manifold coincide. That is, the Lie algebra \( so(2,1)(so(3)) \) and the internal space \( W \) can be identified. Furthermore, the Killing-Cartan metric on the Lie algebra \( k_{IJ} \) is proportional to the internal metric \( g_{IJ} \). Note that the connection is a “Lie-Algebra valued” 1–form, and its components are labeled by two vectorial indices (the natural labels in the vectorial or defining representation). However, in 3 dimensions, the adjoint representation and the defining representation of \( so(2,1)(so(3)) \) coincide. It is simpler to work with 1–index labels (the natural ones in the adjoint representation). We therefore define

\[ \omega_I^J := \epsilon^J_{IK} \omega^K. \]  
(2.8)

It is in this step that the Euclidean and Lorentzian theories have different expressions. This can be understood from the fact that we are using the internal metric \( g_I^J \) to raise the indices of the \( \epsilon_{IJK} \) in equation (2.8): \( \epsilon_I^{JM} \epsilon_{MIK} \). We can now rewrite the generalized covariant derivative in terms of \( \omega_I^J \) as,

\[ D_{\mu} v^I = \partial_{\mu} v^I + [\omega_{\mu}, v]^I, \]  
(2.9)

where \( [\omega_{\mu}, v]^I = \epsilon^I_{JK} \omega_{\mu}^J v^K \). From (2.8) it also follows that the generalized curvature tensor two–form \( F^I \) is,

\[ F^I = d\omega^I + \frac{1}{2} \epsilon^I_{JK} \omega^J \wedge \omega^K. \]  
(2.10)

The Palatini action can be rewritten as,

\[ S_L[e, \omega] = \frac{1}{2\pi} \int_M \left( 2F(\omega)_I \wedge e^I - \epsilon_{IJK} \frac{\Lambda_L}{3} e^I \wedge e^J \wedge e^K \right) \]  
Lorentzian, \hspace{1cm} (2.11)

\[ S_E[e, \omega] = -\frac{1}{2\pi} \int_M \left( 2F(\omega)_I \wedge e^I + \epsilon_{IJK} \frac{\Lambda_E}{3} e^I \wedge e^J \wedge e^K \right) \]  
Euclidean. \hspace{1cm} (2.12)

The actions (2.11) and (2.12), in spite of the fact that are defined for different Lie groups and have different signs, are in a precise sense, equivalent. This can be seen at different levels (for details see [18]). The easiest way of showing their equivalence is to generalize the observation of Witten [4] that one can define a new connection \( A^a \) from the old connection \( \omega \) and the co-triad \( e \), living on a larger Lie algebra. This generalization, originally due to Romano [20], can be applied to any action that has the structure of actions (2.11) and (2.12). It is defined, in particular, for any semi-simple Lie group \( G \) and depends on a free
The result is that any such action is equal to a Chern-Simons action based on the $\kappa$-extension, $\kappa G$, of the original group $G$. For the cases of interest, that is, for the groups $SO(3)$ and $SO(2,1)$, their $\kappa$-extension coincide, being the group $SO(3,1)$, when $\kappa = -1$. The gravity actions can be put in the BF form if we construct the $B$ field from the drei-bein $e$ as follows: $B^I_\mu := \sqrt{|\Lambda|} e^I_\mu$. Then, the gravity actions (2.11), (2.12) give rise to the same Chern-Simons action when $\Lambda_L > 0$ and $\Lambda_E = -\Lambda_L$. For details of this equivalence, manifested also at the phase space level see [18]. The other cases give rise to the groups $\kappa G = SO(2,2)$ when $G = SO(2,1)$ and $\Lambda_L < 0$, and $\kappa G = SO(4)$ when $G = SO(3)$ and $\Lambda_E > 0$.

In this note we shall restrict our attention to the cases in which the $\kappa$-extension corresponds to the De-Sitter group $SO(3,1)$. Solutions to the equation of the motion coming from the Chern-Simons action are flat connections, and two solutions related by a gauge transformation are regarded as physically indistinguishable. Thus, the physical phase space $\hat{\Gamma}$ is the moduli space of flat $SO(3,1)$ connections or, in other words, the homomorphisms from $\pi_1(M)$ to $SO(3,1)$ (modulo the adjoint action of the group).

There are several remarks. First, a point in $\hat{\Gamma}$ represents both (an equivalence class of) Lorentzian space-times (3-geometries) with a positive cosmological constant and Euclidean space-times with a negative cosmological constant! Thus, there exists, at the fundamental level, a mathematical equivalence between the two theories. It is precisely this duality that we shall explore in this note. Second, it is no coincidence that the group $SO(3,1)$ appears as the Chern-Simons gauge group, since it is also the global symmetry group of the De-Sitter (dS) space in three dimensions. In fact, one knows that a space-time of the desired topology can be found by appropriate identifications and/or quotients of the dS space by a subgroup of the $SO(3,1)$ group [14].

The relation between the elements of the group $G$, associated to every non-contractible loop, and the quotient construction is what allows us to reconstruct a spacetime given a set of group elements. Furthermore, one is able to reconstruct two space-times: one Lorentzian ($\Lambda > 0$) and one Euclidean ($\Lambda < 0$). For, the global De-Sitter solutions can be viewed as embedded in four dimensional Minkowski space-time; the Lorentzian De-Sitter (dSL) is the hyperboloid defined by $x_0^2 + x_1^2 + x_2^2 - x_3^2 = l^2$, where $l^2 = 1/|\Lambda|$, and the Euclidean De-Sitter (dSE) can be taken as one of the hyperboloids defined by $x_0^2 + x_1^2 + x_2^2 - x_3^2 = -l^2$. It is easy to see that the scalar curvature of these space-times is positive for dSL and negative for dSE, so they are indeed solutions to the Einstein equations (2.6).

It is important to note that this “holonomy” duality is, in a sense, complementary to the well known Wick rotation (analytic continuation). In this later case, one in changing the signature of the embedding space so the symmetry group of the global, maximally extended, geometry is not preserved, whereas the sign of the cosmological constant, and therefore the

\[ S_{BF} = \int \text{tr}(B \wedge (2F + \frac{\kappa}{3} B \wedge B)). \]

\[ \text{Our use of the term De Sitter is sometimes different from what has been used in the literature (see for instance [18]). For us, a De-Sitter spacetime, Euclidean or Lorentzian is one for which the (local) isometry group is the De-Sitter group SO(3,1).} \]
scalar curvature, is preserved. For instance, the Wick rotation of Anti De Sitter (AdS), a Lorentzian space-time with \( \Lambda < 0 \) and group \( G = SO(2, 2) \) is Euclidean De-Sitter \( dS_E \) for which \( \Lambda < 0 \) and \( G = SO(3, 1) \).

III. DUAL SPACE-TIMES: SOME EXAMPLES

In this section, we construct explicitly the duality mentioned in previous sections for a particular family of (very well known) solutions: the BTZ family. In the Euclidean side this includes the BTZ black hole (of positive mass), going through the zero mass solution and the so called “mass gap” finishing with the global Euclidean De-Sitter. We shall study their Lorentzian duals and comment on their thermodynamic properties.

A. De Sitter space-times

1. Lorentzian De Sitter

The Lorentzian De Sitter Spacetime \( dS_L \) can be defined as the surface

\[
x_0^2 + x_1^2 + x_2^2 - x_3^2 = l^2 ,
\]

embedded in the four dimensional Minkowski spacetime with line element,

\[
ds^2 = dx_0^2 + dx_1^2 + dx_2^2 - dx_3^2 .
\]

The topology of the space-time is \( S^2 \times \mathbb{R} \). This spacetime is maximally symmetric, i.e., it has six Killing vectors (three rotations and three boosts), which form a \( SO(3, 1) \) isometry group.

A standard choice of coordinates \( (t, \theta, \phi) \) that cover the entire space-time are \[21\],

\[
\begin{align*}
x_0 &= l \cosh t \sin \theta \sin \phi ; \\
x_1 &= l \cosh t \sin \theta \cos \phi \\
x_2 &= l \cosh t \cos \theta \\
x_3 &= l \sinh t
\end{align*}
\]

The induced metric then takes the form,

\[
ds^2 = l^2 \left[ -dt^2 + \cosh^2 t (d\theta^2 + \sin^2 \theta d\phi^2) \right]
\]

where \( t \in (-\infty, \infty) \), \( \phi \in [0, 2\pi) \), \( \theta \in [0, \pi) \). As can be easily seen, the spacetime is a two sphere that contracts to its minimum area \( 4\pi l^2 \) at \( t = 0 \) and then expands again.

In spite of the convenience of working on a globally defined coordinate patch, we shall introduce a popular choice of coordinates in which the metric looks “static”. First let us introduce some other embedding coordinates that will cover the region \( x_0^2 + x_1^2 \leq l^2 \) and \( x_2 \geq 0 \),

\[
\begin{align*}
x_0 &= l \sin \chi \sin \phi ; \\
x_1 &= l \sin \chi \cos \phi \\
x_2 &= l \cos \chi \cosh \tau \\
x_3 &= l \cos \chi \sinh \tau
\end{align*}
\]
Here $\phi \in [0, 2\pi)$, $\chi \in [0, \pi/2]$ and $\tau \in (-\infty, \infty)$. The metric induced in this region is,

$$ds_I^2 = -l^2 \cos^2 \chi \, d\tau^2 + l^2 d\chi^2 + l^2 \sin^2 \chi \, d\phi^2 \, .$$  \hspace{1cm} (3.6)

Defining the coordinates $r = l \sin \chi$ and $t = l \tau$ we get,

$$ds_I^2 = -\left(1 - \frac{r^2}{l^2}\right) dt^2 + \left(1 - \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2 \, .$$  \hspace{1cm} (3.7)

where $t \in (-\infty, \infty)$, $\phi \in [0, 2\pi)$, $r \in [0, l]$. In this “Schwarzschild” coordinates, $r = 0$, the origin, is the location of an observer and $r = l$ is its cosmological (event) horizon [22]. Using standard arguments in which one analytically continues the solution to a Euclidean (in this case, locally isometric to the three sphere $S^3$) spacetime, and requires regularity in the $(r, \tau)$ plane, one can assign a temperature to the horizon to be [23],

$$T_h = \frac{\hbar}{2\pi l} \, .$$  \hspace{1cm} (3.8)

This temperature is just $(1/2\pi$ times) the surface gravity of the (properly normalized) killing vector generating the horizon [22]. In the global coordinate system (3.4) the observer is located at the “north pole” of the two-sphere ($\theta = 0$). The antipodal point would be represented, in a Penrose-Carter diagram, as living in another “asymptotic region” [22], and is not covered by the coordinate patch (3.6).

2. Euclidean De Sitter

The Euclidean De Sitter space-time $dS_E$ can be defined as the surface

$$x_0^2 + x_1^2 + x_2^2 - x_3^2 = -l^2 \, ,$$  \hspace{1cm} (3.9)

embedded in the four dimensional Minkowski spacetime with line element,

$$ds^2 = dx_0^2 + dx_1^2 + dx_2^2 - dx_3^2 \, .$$  \hspace{1cm} (3.10)

This space-time is disconnected, each connected component being a hyperboloid. It is standard to choose one of them, say, the upper one as the space-time. The topology of the space-time is $R^3$. This spacetime is maximally symmetric, i.e., it has six Killing vectors (three rotations and three boosts), which form a $SO(3, 1)$ isometry group.

A standard choice of coordinates $(\tau, \theta, \phi)$ that cover all the spacetime ($x_3 \geq 0$) are, [21],

$$x_0 = l \sinh \tau \sin \theta \sin \phi \, ; \, \quad x_1 = l \sinh \tau \sin \theta \cos \phi$$

$$x_2 = l \sinh \tau \cos \theta \, ; \, \quad x_3 = l \cosh \tau \, .$$  \hspace{1cm} (3.11)

The induced metric then takes the form,

$$ds^2 = l^2 \left[d\tau^2 + \sinh^2 \tau (d\theta^2 + \sin^2 \theta d\phi^2)\right] \, .$$  \hspace{1cm} (3.12)
where $\tau \in [0, \infty)$, $\phi \in [0, 2\pi)$, $\theta \in [0, \pi)$. The point $\tau = 0$ is a regular point and is just a singular point of the coordinate system.

This space-time can also be put in a static form. For that we need to introduce some new embedding (covering the same region of spacetime) as follows,

$$
\begin{align*}
    x_0 &= l \sinh \rho \sin \phi \\
    x_1 &= l \sinh \rho \cos \phi \\
    x_2 &= l \cosh \rho \sinh t' \\
    x_3 &= l \cosh \rho \cosh t'
\end{align*}
$$

The induced metric then takes the form,

$$
\text{ds}^2 = l^2 \left[ \cosh^2 \rho \text{d}t'^2 + \text{d}\rho^2 + \sinh^2 \rho \text{d}\phi^2 \right]
$$

where $\rho \in [0, \infty)$, $\phi \in [0, 2\pi)$, $t' \in (-\infty, \infty)$.

Defining the coordinates $r = l \sinh \rho$ and $t = lt'$ we get,

$$
\text{ds}^2 = \left(1 + \frac{r^2}{l^2}\right) \text{d}t^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} \text{d}r^2 + r^2 \text{d}\phi^2
$$

where $t \in (-\infty, \infty)$, $\phi \in [0, 2\pi)$, $r \in [0, \infty)$. This is precisely the “Wick transform” of (a patch of) the global Anti–de Sitter (AdS) spacetime through the transformation $t \mapsto \text{i}t$.

**B. The Boost Sector: BTZ Black Hole vs. Cosmological $T^2$**

1. **Euclidean Signature**

   In this part we construct, in an explicit fashion, a Euclidean BTZ black hole of mass $M$. We do this in order to illustrate the procedure that will lead to the (one parameter family of) dual Lorentzian cosmological space-times. For simplicity we shall consider the non-rotating black hole. The rotating case can be found in the Appendix. The starting point is Euclidean De Sitter $\text{dS}_E$ (3.9). Next, we consider the Killing vector field,

$$
\xi = x_2 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_2},
$$

which generates boosts in the plane $(x_2, x_3)$ and it is tangent to the De Sitter space. Note that $|\xi|^2 = x_3^2 - x_2^2 = l^2 + x_0^2 + x_1^2$. Let us now define the embedding (3.14), changing the names of the coordinates as follows,

$$
\begin{align*}
    x_0 &= l \sinh \rho \sin \chi \\
    x_1 &= l \sinh \rho \cos \chi \\
    x_2 &= l \cosh \rho \sinh \varphi \\
    x_3 &= l \cosh \rho \cosh \varphi
\end{align*}
$$

we have the line element,

$$
\text{ds}^2 = l^2 \left[ \sinh^2 \rho \text{d}\chi^2 + \text{d}\rho^2 + \cosh^2 \rho \text{d}\varphi^2 \right]
$$

where $\rho \in [0, \infty)$, $\chi \in [0, 2\pi)$, $\varphi \in (-\infty, \infty)$. Note that $\xi = \frac{\partial}{\partial \varphi}$, and $\varphi$ is an affine parameter of the finite “boost” along the integral curves of $\xi$. The metric (3.18) induces naturally the
topology $R^3$ over the region covered by the embedding $\dot{3}$[17]. Note however, that from a canonical perspective, where we require the space-time to be of the form $M = \Sigma \times R$, we are forced to take out the point $\rho = 0$ from the space-time, and as a consequence of that, the topology will change to $S^1 \times R^2$. We shall construct the solution and see that in the BTZ case, one can add the point $\rho = 0$ to the space ‘generated’ via the Hamiltonian $2 + 1$ evolution and still have a regular space-time with a different topology.

The BTZ solution is constructed by periodically identifying $\varphi$ with $\varphi + \varphi_0$, where $\varphi_0 = \frac{2\pi r_+}{l}$. Here $r_+$ is simply a parameter labeling the space-time. What one does is to compactify along $\chi$. If we consider the point $\rho = 0$ to be part of space-time we have a $S^1 \times R^2$ topology, and if we remove it, we end up with a space-time of topology $T^2 \times R$ (needed from the canonical perspective). Note that even when we pick this later choice, we are respecting the $2\pi$ periodicity of the $\chi$ coordinate. This will translate into a trivial holonomy along its (artificially constructed, topological nontrivial) orbits. Let us see that we indeed get the BTZ solution. Writing, $\chi = (r_+ / l^2) \tau$ and $\varphi = (r_+ / l) \phi$, we have

\[ ds^2 = \frac{r_+^2}{l^2} \sinh^2 \rho \, d\tau^2 + l^2 d\rho^2 + r_+^2 \cosh^2 \rho \, d\phi^2 \]  

(3.19)

where $\rho \in [0, \infty)$, $\tau \in [0, \frac{2\pi l^2}{r_+})$, $\phi \in [0, 2\pi)$. Now, changing coordinates $r = r_+ \cosh \rho$ we get,

\[ ds^2 = \left( \frac{r^2}{l^2} - \frac{r_+^2}{l^2} \right) d\tau^2 + \left( \frac{r^2}{l^2} - \frac{r_+^2}{l^2} \right)^{-1} dr^2 + r^2 d\phi^2 \]  

(3.20)

We see that the horizon is at $r = r_+ \ (\rho = 0)$ and the $\tau$ coordinate, the parameter along the Killing field $\frac{\partial}{\partial \tau}$ is naturally periodic with period $(2\pi l^2) / r_+$. From here we see that the temperature that we should assign is,

\[ T_{bh} = \frac{h r_+}{2\pi l^2} \]  

(3.21)

Finally, the mass of the space-time is $M := \frac{r_+^2}{l^2}$, where the value $M = 0$ is assigned, as usual, to the solution with $r_+ = 0$.

The following questions come to mind: what is the relation between this construction and the reduced phase space in terms of $\pi_1(M) \to SO(3,1)$? In what sense can we say that the holonomies along the orbits of $\xi$ are related to the parameter $r_+ / l$? The relation between the appearance of this non-trivial holonomies and the construction described above deserves more attention. The procedure involves the construction of a 3-dimensional space-time $M$, out of a point of the reduced phase space. This procedure naturally implies some sort of “gauge fixing” in order to select a representative form the equivalence class of flat $SO(3,1)$ connections, that will have a corresponding space-time locally isometric to $M^4$. To see that the space-time constructed via identifications has associated to it a flat $SO(3,1)$ connection whose holonomies parameterize the reduced phase space we need to use the structure available in $M^4$. The $SO(3,1)$ 3-d connection is constructed out of the structure defined on the embedding space-time, namely, the flat $SO(3,1)$ connection compatible with
the Minkowski metric\(^4\) and the action \(\phi\) of the group \(SO(3, 1)\) acting on \(M^4\) (i.e., \(\phi : SO(3, 1) \to \text{Diff}(M^4), \ g \mapsto \phi(g) : M^4 \to M^4\)). The hyperboloids corresponding to De-Sitter space-times are invariant subspaces of the \(SO(3, 1)\) action \(\phi\), and the group acts transitively. That is, given any two points \(p\) and \(q\) on \(M\), we can always find an element \(g \in SO(3, 1)\) such that \((\phi(g))(p) = q\). Let us now define a canonical (auxiliary) 3-d flat connection on \(M\) as follows: Given an open curve \(\gamma\) on \(M\), with \(p\) as its starting point and \(q\) as its end point, and an initial vector \(\vec{v}(p)\), we parallel transport the vector using the flat \(SO(3, 1)\) connection of the embedding space-time. Thus, the connection so defined is a true \(SO(3, 1)\) connection on \(M\). Furthermore, given that the action \(\phi\) of \(SO(3, 1)\) on \(M^4\) is linear, the derivative \(\phi^*\) coincides with it, and is also given by an \(SO(3, 1)\) action on tangent vectors. Thus, if we parallel transport a 3-dimensional vector (using the natural connection) along the orbits of the KVF \(\xi\) from the origin \(\varphi = 0\) to say, the point \(q\) defined by \(\varphi = \varphi_0\), the original vector and the parallel transported one will be the “same” (for instance in a Cartesian coordinate system). How can we say that the parallel transported vector will be different from the original one when we are using the flat constructed above? Since we are identifying \(q\) with the origin, we are at that step closing the curve \(\gamma\) into a loop, and we have to “bring back” the vector to the starting point. That is, we “push” forward the vector from \(q\) to \(p\) using the mapping \((\phi_*(g))^{-1}\). Thus, the element of the group \(SO(3, 1)\) that provides the parallel transport (in this case, a boost), becomes the holonomy along the non-contractible loop \(\gamma\). Therefore the “parallel transported” vector has as its holonomy precisely the group element in \(SO(3, 1)\) corresponding to a boost along \(\xi\) with parameter \(\varphi_0\). This is the relation between holonomies and space-time identifications. To see more about the relation between geometry and holonomies (known as geometric structures) see \[24\].

Now, the topology of the spacetime is \(T^2 \times \mathbb{R}\) and therefore we have two non-contractible generators of \(\pi_1(M) = \mathbb{Z}_2\). The gauge invariant information, i.e., the coordinates of \(\hat{\Gamma}\) are four parameters \((a_1, s_1, a_2, s_2)\) \[23\], where \(a_i\) are rotation parameters corresponding to a holonomy along the \([\gamma_i]\) generator of \(\pi_1(M)\), and \(s_i\) correspond to boost parameters along the dual plane in \(M^4\). Concretely, representatives of the \([\gamma_1]\) equivalence class are given by the closed orbits of the \(\partial/\partial \tau\) vector field and representatives of \([\gamma_2]\) are given by the orbits of \(\partial/\partial \varphi\). In this language, a (Euclidean) BTZ black hole of mass \(M\) and \(J = 0\) has coordinates in \(\hat{\Gamma}\) equal to \((2\pi, 0, 0, (2\pi r_+)/l)\).

2. Lorentzian Signature

In order to construct the holonomy-duals to the BTZ black hole, we will proceed analogously. First, we take the same killing vector

\[
\xi = x_2 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_2},
\]

\[3.22\]

\(^4\)This \(SO(3, 1)\) connection parallel transports vectors with internal indices. In Minkowski space-time there is a canonical, globally defined soldering form that translates internal vectors into space-time vectors. We shall use this structure to refer to internal vectors as simply ‘vectors’
and identify points in $dS_L$ space along this killing vector. This procedure will induce closed time-like curves in the region where $|\xi|^2 < 0$, therefore we will take out this region from the spacetime and take the surface $|\xi| = 0$ as a “chronological singularity”.

The region of interest is covered by the following patch of coordinates

$$
x_0 = l \cosh t \sin \chi \quad ; \quad x_1 = l \cosh t \cos \chi \quad (3.23)
$$

$$
x_2 = l \sinh t \sinh \varphi \quad ; \quad x_3 = l \sinh t \cosh \varphi
$$

with $-\infty < t, \varphi < \infty$ and $0 \leq \chi < 2\pi$. In these coordinates the metric takes the form,

$$
ds^2 = -l^2 dt^2 + l^2 \cosh^2 t \, d\chi^2 + l^2 \sinh^2 t \, d\varphi^2 \quad (3.24)
$$

We can now safely identify $\varphi$ with $\varphi + 2\pi b$, which makes the surfaces of constant time compact $T^2$ manifolds. Notice that in order to relate it to the BTZ black hole we have the correspondence $b \leftrightarrow r_+/l$. The chronological singularity $|\xi|^2 = 0$ is now located at $t = 0$, which is a naked “initial” singularity. For big values of $t$, the spatial slices have the topology of a 2–torus whose radius are expanding exponentially in time. The parameter $b$, labeling the particular space-time, gives the ratio between the two radii. For small $t$ one of the radius goes to $l$ while the other shrinks to zero (see Figure 1).

In order to put this solution in a familiar form set $\chi = (b/l)\chi', \phi = (1/b)\phi$ and $T = lb \sinh t$. Thus, we have,

$$
ds^2 = -\left( b^2 + \frac{T^2}{l^2} \right)^{-1} dT^2 + \left( b^2 + \frac{T^2}{l^2} \right) d\chi'^2 + T^2 d\varphi^2 \quad (3.25)
$$

where $T \geq 0, \chi' \in [0, (2\pi l)/b)$, and $\phi \in [0, 2\pi)$.

These solutions are not new. They are a subclass of the general solution found by Ezawa \cite{25} (see also \cite{3} and \cite{26}). It is important to note that in terms of the reduced phase space language, these solutions have coordinates $(2\pi, 0, 0, 2\pi b)$, and that the holonomy-duality is relating space-times $(3.20)$ and $(3.25)$ when we set $b = r_+/l$. 


FIGURE 1: The “Big Bang” spacetime (a) vs. The BTZ spacetime (b). In (b) we represent the plane \((t, \rho)\) in the coordinates given by the metric (3.19). Each point of this plane has a circle on it, which completes the spacetime (with its \(R^2 \times S^1\) topology. \(r_+\) represents the radius of the circle at the origin. In (a), the parameter \(b\) defines the geometry of the \(t = \text{constant}\) torus. The duality identifies \(bl\) with \(r_+\).

C. The Rotation Sector: Particle Solutions

In this part we shall consider another class of solutions. Just as the \(T^2\)-BTZ-cosmological solutions were found by means of an identification along a boost, the solutions in this section will be constructed using a rotation.

1. Lorentzian Signature

Point particle solutions for De-Sitter spacetime were studied by Deser and Jackiw [2]. They are a generalization of the point particle solutions for flat 3-dimensional gravity studied in [1]. Recall from Sec. III A that the global De Sitter has a \(S^2 \times R\) topology. Thus, as argued by Deser and Jackiw, one cannot construct a one-particle space-time by removing a “wedge” from the \(S^2\) spatial slice since there will be, at least, two conical singularities. In our coordinate system \((t, \theta, \phi)\) (3.4) the two particles will be located at antipodal points on the sphere \((\theta = 0\) and \(\theta = \pi)\). However, the Schwarzschild patch (3.6) only sees one of the observers/particles and the other one is hidden behind the cosmological horizon. The
‘remove a wedge’ construction of Deser and Jackiw can be interpreted in our language as, again, an identification along the orbits of a Killing field. The main difference with the BTZ construction is that now we take a rotation KVZ $\zeta = \frac{\partial}{\partial \phi}$ instead of a boost. When one uses a rotation field to construct a new spacetime, one needs due care. For, one cannot simply identify, say, a point with angle coordinate $\phi$ with the point $\phi + \phi_0$, for $\phi_0 < 2\pi$. This naive quotient procedure leads (for almost all values of $\phi_0$) to a space that is not even a manifold. Instead, the construction involves the arbitrary choice of, say $\phi = 0$, such that the points $\phi = 0$ and $\phi = \phi_0$ are to be identified and the wedge between $\phi_0$ and $2\pi$ is removed from the manifold. Even though the construction procedure is different from the one used in the BTZ case, the picture of having an element of $SO(3,1)$, the holonomy, for each non-contractible loop remains unchanged. This is so since by cutting the wedge, one has two curvature-singular points at the poles of the sphere that have to be removed from the manifold. Thus, the resulting topology of $M$ is $S^1 \times \mathbb{R}^2$ and we have one non-trivial loop around the $S^1$. As an end result, we have that the holonomy along this loop, the (new) orbit of the vector $\partial_\phi$, is a rotation (in $SO(3,1)$) by $\phi_0$.

In this case we take the region of $dS_L$ space covered by the patch of coordinates described below.

$$
\begin{align*}
  x_0 &= l \sin \chi \sin \phi; & x_1 &= l \sin \chi \cos \phi \\
  x_2 &= l \cos \chi \cosh \tau; & x_3 &= l \cos \chi \sinh \tau
\end{align*}
$$

(3.26)

Here $\phi \in [0, 2\pi)$, $\chi \in [0, \pi/2]$ and $\tau \in (-\infty, \infty)$. The metric induced in this region is,

$$
ds^2 = -l^2 \cos^2 \chi \, d\tau^2 + l^2 d\chi^2 + l^2 \sin^2 \chi \, d\phi^2.
$$

(3.27)

Cutting the wedge means restricting the range of $\phi$ to $[0, 2\pi\alpha)$ and identifying $\phi = 0$ with $\phi = 2\pi\alpha$, where $\alpha \in (0, 1]$. Thus, the space-time has a deficit angle of $\Omega = 2\pi (1 - \alpha)$. When $\alpha = 1$, there is no deficit and we recover global De Sitter. If we now change coordinates to $\varphi = \phi/\alpha$, $\tau = (\alpha/l)t$ and $r = \alpha \sin \chi$ we get,

$$
ds^2 = -\left(\alpha^2 - \frac{r^2}{l^2}\right) dt^2 + \left(\alpha^2 - \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\varphi^2
$$

(3.28)

where $\varphi \in [0, 2\pi)$, $r \in [0, \alpha l]$ and $t \in (-\infty, \infty)$. The cosmological horizon is now located at $r_c = \alpha l$ and, following the argument given in the Introduction, the temperature of the horizon is $T_h = \alpha \bar{h}/(2\pi l)$. The deficit angle is manifested by the fact that the ‘area’ of the horizon is now $A_h = 2\pi \alpha l$.

2. Euclidean Signature

Let us now consider the Euclidean case. The starting point is Euclidean De Sitter spacetime (3.9) with topology $R^3$. Now it is possible to have a single point particle located at the ‘origin’ $\rho = 0$. If we remove this world-line, we are left with a space-time with topology $S^1 \times R^2$, with a single non-contractible loop. We shall use the same rotational KVZ $\zeta$ as in the case of Lorentzian signature to identify along its orbits. The resulting space-time, with deficit angle $\Omega = 2\pi (1 - \alpha)$ is given by,
which corresponds to the Wick transform of the ‘point particle’ Anti-De Sitter spacetime of mass $M = -\alpha^2$. Just as we did for the $T^2$-cosmological space-times, we could, formally, assign a temperature to this point particle solution to be equal to $T_p = \alpha \hbar/(2\pi l)$.

D. The Vacuum

This spacetime is the limit $r_+ \to 0$ in both Lorentzian and Euclidean Signatures. For the Euclidean signature, we can simply take the limit of (3.20) when $r_+ \to 0$,

$$d\tau^2 = \frac{r^2}{L^2} d\tau^2 + \frac{l^2}{r^2} dr^2 + r^2 d\phi^2$$

(3.30)

now, $\tau$ is no longer a periodic coordinate (recall that its range, for the case $r_+ > 0$ was $\tau \in [0, \frac{2\pi l}{r_+})$), so we have a space-time with topology $S^1 \times R$ and zero temperature. Note that we would have arrived at the same space-time in the limit $\alpha \to 0$ in (3.28). In this case, $\alpha$ had the interpretation of the range of $\phi$ after one has removed the wedge from the spacetime, so the limit $\alpha \to 0$ might be interpreted as the limit in which the conical singularity ‘opens up’ to form a cylinder, thus recovering the $S^1 \times R$ topology. This space-time is referred as the vacuum since it is the Wick rotation of the BTZ black hole of zero mass and this is considered to be the zero energy solution [9].

The $r_+ \to 0$ limit of our Lorentzian solutions has also some similar features. That is, starting from the $T^2$-cosmological solutions (3.25), and taking the limit $b \to 0$ we get,

$$d\tau^2 = \frac{l^2}{T^2} dT^2 + \frac{T^2}{l^2} d\chi^2 + T^2 d\phi^2$$

(3.31)

where $\chi$ is not periodic and it takes values in $(-\infty, \infty), T \geq 0$ and $\phi \in [0, 2\pi)$. It is again a space-time with topology $S^1 \times R^2$. Starting from the point-particle family (3.28), it corresponds to taking the limit $\alpha \to 0$, where the cosmological horizon shrinks to zero, and it can be interpreted as the 2-sphere opening up to a cylinder. Again, the temperature we would assign to this vacuum state would be zero.

IV. DISCUSSION AND OUTLOOK

The main results of this note can be summarized in Fig.2. Here, we have the four possibilities of signatures and signs of the cosmological constant. In the left column we have Lorentzian space-times and in the right their corresponding analytic continuations (Wick transforms), as shown by the ‘W’ arrows (for Wick). We have listed the space-times considered in this note. The ‘Holonomy’ transform (H-duality) corresponds to the diagonal ‘H’ arrows going from the lower-left corner to the upper right corner and relating the space-times studied in this note.
There are several remarks that need to be made.

1. **Relation to Wick Transform.** As already discussed, the standard Wick transform and the Holonomy duality are in a sense, complementary. First, the Wick-duality for space-times is restricted to complex space-times that admit both an Euclidean and a Lorentzian real section. Normally this is possible only in the presence of a time-like KVF. Furthermore, the mapping in not strictly between globally defined space-times, since, for instance, it is only the exterior region of the Lorentzian BTZ black hole that gets “mapped” to the Euclidean sector. On the other hand, the Holonomy transform, being a duality at the phase space level, relates globally defined space-times. Furthermore, the existence of a KVF is not essential for the definition of the duality. The Wick rotation has been extensively used in gravitational physics. Not only is it used for QFT calculations, but also in connection with gravitational thermodynamics. Standard arguments of Hawking and Gibbons assign a temperature to the Lorentzian space-time by looking at the period of the Wick-rotated time in the dual space-time. This physical equivalence between space-times is used regularly in the study of thermodynamics of black holes. One may wonder if the Holonomy dual space-times can be regarded as having more than just the phase-space equivalence. Is it justified to assign thermodynamical properties to the dual space-time to define, say, a temperature and entropy for space-times that do not even have a horizon? Can we perform a calculation in one domain and conclude something about the other domain just as we do with the Wick transform?

Affirmative answers to this intriguing questions would indeed be of some interest since we could, for instance, pass from the physics in AdS to dS, composing a W and a H-transform. What would the relevance be for an hypothetical dS/CFT conjecture remains an open question.

2. **Thermodynamics.** Let us elaborate on the possible assignment of a temperature to the space-times studied in previous section. Let us first recall the standard argument
to define a physical temperature for a Euclidean space-time. First, we start we a Lorentzian stationary black-hole space-time, with an $N - 1$-dimensional (null) horizon (with $N$ being the dimension of space-time). In the Euclidean sector, the horizon is now $N - 2$ dimensional and ‘time’ is a periodic coordinate. One asks for regularity of the space-time at the horizon in order to fix the periodicity of the Euclidean time. Even if we start with a Euclidean solution with non-trivial topology, the fact that it is regular is not enough to define a temperature. We need, as in the Schwarzschild black hole case, the Lorentzian time that measures proper time at infinity in order to have the ‘right’ temperature. The same is true for the de-Sitter space-time where one needs to choose the time-parameter as the proper time of the observer whose cosmological horizon we want to consider. The periodicity of the Wick-transformed time is then related to the temperature.

Let us now consider the space-times of interest for this note. The (non-rotating) Euclidean BTZ solutions have a naturally defined temperature $T_{bh} = \frac{\hbar r_+}{(2\pi l^2)}$. On the other hand, the $H$-dual space-time is a spatially closed space-time, with no time-like KVF and no horizon. One may wonder what, if any, would be the meaning of formally defining thermodynamic parameters and, in particular, a constant temperature to the whole space-time. We shall return to this question shortly.

The Lorentzian De-Sitter space-time together with the point particle solutions (3.28) have horizons with a temperature (assigned via the Wick rotation) given by $T_h = \frac{\hbar \alpha}{(2\pi l)}$. The dual space-times (3.29) do not have a periodic ‘time’ coordinate $\tau$ and therefore, no natural temperature. However, if we declare the temperature of this space-times to be $T_p := \frac{\hbar \alpha}{(2\pi l)}$, (i.e., the temperature of its $H$-dual) then the natural period for $\tau$ is $2\pi l/\alpha$. Now, the space-times (3.29) are the Wick transform of the family of point particle $AdS$ space-time (with $M \in [-1,0]$). Therefore, using the standard Wick transform argument, we can assign a temperature to the family of point-particle-$AdS$ space-times to be,

$$T_\alpha := \frac{\hbar \alpha}{2\pi l}$$  \hspace{1cm} (4.1)

Note that with this definition, the $AdS$ space-time (corresponding to $M = -1$ and $\alpha = 1$) is a ‘thermal state’ with temperature $T_{AdS} = \frac{\hbar}{(2\pi l)}$, recovering the suggestion of Strominger and others [13,27].

In recent years there have been several attempts to calculate the entropy of the BTZ black hole. One of those calculations, originally due to Carlip [15] (see also [28]), starts by considering the $SL(2,C)$ Chern-Simons phase space for a $T^2 \times R$ space-time. Then, one imposes boundary conditions, motivated by the Lorentz black hole horizon, and computes the Bekenstein-Hawking entropy using the WZW theory induced on the horizon. One intriguing fact about this calculation is that, in the Euclidean domain, one does not have the geometrical interpretation of the Lorentzian horizon, and without the knowledge of its origin, we would only be faced with some boundary conditions on phase space inducing a WZW theory on the boundary.

Since the $H$-duality is precisely based on the fact that the two gravity theories share the same phase space, then we could perfectly well interpret the result of the calculation as
giving us the entropy of the Lorentzian cosmological space-time. The correspondence is given as follows. The horizon $\rho = 0$ of the Euclidean black hole corresponds in the Lorentzian side to the initial $T = 0$ singularity (this is a striking property, because $T = 0$ is a *chronological* singularity and the horizon $\rho = 0$, is just a coordinate singularity). This singularity is a mild one, since it is manifested by the fact that the spatial slice is a degenerate torus. If would be of interest to geometrically interpret the boundary conditions in the Lorentzian theory and to identify the calculated entropy with some geometrical invariant, the analogous of the horizon area in the BH case.

Another interesting calculation refers to the entropy associated to the cosmological horizon in De-Sitter. The first approach, very much in the style of Carlip is given in [14]. Maldacena and Strominger again impose some boundary conditions in the $SL(2, C)$ Chern-Simons phase space and compute the Bekenstein-Hawking entropy using the CFT on the boundary of the region, in this case, the horizon. The computed entropy is, as expected, one fourth of the cosmological horizon radius, $S = 2\pi l/(4G\hbar)$. The other calculation [16] computes the Gibbons-Hawking entropy by considering the partition function in the Wick-rotated theory (corresponding to a Chern-Simons theory on $so(3) \times so(3)$). The Maldacena-Strominger results can be re-interpreted in the same way that Carlip’s results were (see also [27]). That is, one can argue that the calculation very well applies to the $H$-dual theory, namely, to a Euclidean De-Sitter space-time with topology $S^1 \times R$ (3.29), and assigns to it an entropy $S := 2\pi l\alpha/(4G\hbar)$. Now, recall that these space-times are the Wick-duals of the point particle $AdS$ space-times, and therefore one is naturally lead to assign to these space-times an entropy as well. For the global $AdS$ the associated entropy would be $S_{AdS} := 2\pi l/(4G\hbar)$. Just as for the cosmological space-times, we would like to interpret this entropy in geometrical terms.

3. Path Integrals, Wick Rotations, etc..

The fact that three dimensional gravity has a dual description, both as a theory of triads and connections ($\omega, e$) and as a theory of space-time metrics, poses a puzzle. On the one hand, there is the not so unpopular wisdom that there is a deep relation between the Lorentzian and Euclidean path integral formulation of gravity. Independently of the viewpoint one adopts, be it that the Euclidean path integral is useful for calculating Lorentzian transition amplitudes, or that the Euclidean regime is fundamental, there is some belief that Euclidean methods are relevant for quantum gravity and BH-thermodynamics. In fact, this is the standard justification for the extended use of the Wick-rotation. On the other hand, when the theory is formulated as a diffeomorphism invariant theory of connections, one looses even the notion of a space-time metric; it now becomes a *derived* notion and, therefore, the role of the Wick rotation turns out to be not so obvious (see [29]). For, in the completely solvable models like $BF$ theories and $2 + 1$ gravity with no cosmological constant, the rigorously defined measures on the space of histories have heuristic analogs involving a factor of the form $\exp(iS)$ for both gauge groups ($SO(3)$ and $SO(2, 1)$), so the signature of the space-time metric seems to play no role. If one takes the viewpoint that the heuristic path integrals based on the actions (2.11, 2.12) fully represent quantum gravity in three
dimensions, then one is naturally lead to conclude that the $H$-duality is, in a sense, more fundamental and natural than the Wick transform. For, the $H$-dual theories not only share the space of histories, in their Chern-Simons version, but also have the property that the actions \((2.11, 2.12)\), when evaluated on the shared history, differ only by a sign [18]. Thus, the path integral for these two actions must be equivalent (since the factor involved is of the form $\exp(iS)$ and the actions are real).

Implications of the previous discussion are intriguing. First, one can justify the formal assignments of thermodynamical properties like entropy and temperature to cases where, classically, there is no time-like KVF and no horizon. This can be done by applying the standard arguments of Gibbons-Hawking together with the $H$-duality [30]. Second, there seems to be some tension between the relevance of the Wick transform for quantum gravity in its metrical/geometrodynamical formulation and the holonomy duality for its connection-dynamics formulation. It would be of interest to fully understand the role of these dualities in the context of path integrals for diffeomorphism invariant theories. Finally, we could speculate about some implications for the $H$-dual of the $AdS/CFT$ correspondence and what it could tell us about quantum gravity in higher dimensions.

ACKNOWLEDGMENTS

The authors would like to thank the participants of the 15th Pacific Coast Gravity Meeting, February 1999, and in particular Abhay Ashtekar, Máximo Bañados and Steven Carlip for discussions. This research was supported in part by the National Science Foundation under Grant No. PHY94-07194. AC would like to thank the ITP for its hospitality during which part of this work was completed. AC was also supported by DGAPA-UNAM Proy. No. IN121298 and by CONACyT (México) Ref. No. I25655-E. The work of AG was also supported in part by National Science Foundation grant PHY97-22362, and by funds from Syracuse University.

APPENDIX: ROTATING SOLUTIONS: BOOST WITH A TWIST

1. BTZ-$T^2$ Family

Let us start by considering the Euclidean BTZ black hole with angular momentum,

$$
\text{ds}^2 = \left( -M + \frac{r^2}{l^2} - \frac{J^2}{4r^2} \right) \text{d}r^2 + \left( -M + \frac{r^2}{l^2} - \frac{J^2}{4r^2} \right)^{-1} \text{d}r^2 + r^2 \left( \text{d}\phi - \frac{J}{4r^2} \text{d}r \right)^2 \tag{A1}
$$

where

$$
M = \frac{r_+^2 + r_-^2}{l^2} ; \quad J = \pm \frac{2r_+r_-}{l} \tag{A3}
$$

\[r_\pm = \left\{ \frac{Ml^2}{2} \left[ 1 \pm \left( 1 + \left( \frac{J^2}{M^2l^2} \right) \right)^{1/2} \right] \right\}^{1/2} \tag{A2} \]
(recall that $8G = 1$). In order to construct this space-time, we can again start with the metric (3.18),

$$ds^2 = l^2 \left[ \sinh^2 \rho \, d\chi^2 + d\rho^2 + \cosh^2 \rho \, d\varphi^2 \right]$$

where $\rho \in [0, \infty)$, $\chi \in [0, 2\pi)$, $\varphi \in (-\infty, \infty)$. Note that $\xi = \frac{\partial}{\partial \varphi}$; and $\varphi$ is an affine parameter of the finite “boost” along the integral curves of $\xi$. The region that the embedding (3.17) covers has the topology $S^1 \times R^2$.

The rotating BTZ solution is constructed by specifying the points that need to be identified in order to have a two torus. Just as in the non-rotating case, one of the generators is given by the (closed) orbits of the $\partial/\partial \chi$ KVF, with period $2\pi$. Thus, one only needs to specify the other generator. This is done by periodically identifying the point $(\varphi = 0, \chi = 0)$ with the point given by a boost along $\xi$ with parameter $\frac{2\pi r_-}{l}$, together with a rotation along the KVF $\frac{\partial}{\partial \chi}$ equal to $2\pi \lvert r_- \rvert$. Here $(r_+, \lvert r_- \rvert)$ are the parameters labeling the space-time. What one does is to compactify along the line, in the plane $(\varphi, \chi)$, connecting, say $(0, 0)$ and $(\frac{2\pi r_+}{l}, \frac{2\pi \lvert r_- \rvert}{l})$ and one ends up with a space-time of topology $T^2 \times R$.

We can define the new periodic coordinates $(\phi', \chi')$ along the generators of the torus,

$$\varphi = \frac{r_+}{l} \phi' \quad ; \quad \chi = \chi' + \frac{\lvert r_- \rvert}{l} \phi'$$

where $\chi' \in [0, 2\pi)$ and $\phi' \in [0, 2\pi)$. Then, the line element has the form,

$$ds^2 = l^2 \left[ \sinh^2 \rho (d\chi' + \frac{\lvert r_- \rvert}{l} d\phi')^2 + d\rho^2 + \cosh^2 \rho (\frac{r_+}{l} d\phi')^2 \right]$$

In order to put it in the standard form, let us define new coordinates

$$\chi' = \frac{(r_+^2 + \lvert r_- \rvert^2)}{2r_+} \tau \quad ; \quad \phi' = \phi + \frac{\lvert r_- \rvert}{r_+} \tau$$

where $\phi \in [0, 2\pi)$ and $\tau \in [0, \frac{2\pi r_+}{r_+ + \lvert r_- \rvert}]$. Then, the metric takes the “proper radial coordinates” form [28],

$$ds^2 = \sinh^2 \rho \left( \frac{r_+}{l} d\tau + \lvert r_- \rvert d\phi \right)^2 + l^2 d\rho^2 + \cosh^2 \rho \left( \frac{r_+}{l} d\tau - r_+ d\phi \right)^2$$

Finally, if we define $r^2 := r_+^2 \cosh^2 \rho + \lvert r_- \rvert^2 \sinh^2 \rho$, we recover the standard form (A11).

In terms of the gauge invariant information, i.e., the coordinates of $\hat{\Gamma}$ given by the four parameters $(a_1, s_1, a_2, s_2)$ [28][185], a (Euclidean) BTZ black hole of mass $M$ and angular momentum $J$ has coordinates in $\hat{\Gamma}$ equal to $(2\pi, 0, (2\pi \lvert r_- \rvert)/l, (2\pi r_+)/l)$.

The corresponding Lorentzian dual space-times are constructed using the same holonomy parameters. Starting from the metric (3.24),

$$ds^2 = -l^2 dt^2 + l^2 \cosh^2 t \, d\chi^2 + l^2 \sinh^2 t \, d\varphi^2$$

Along the $\chi$ direction we keep the $2\pi$ periodicity, and the other generator is constructed via a boost in the $\partial/\partial \varphi$ direction with parameter $2\pi b$ together with a rotation along $\partial/\partial \chi$ with parameter $2\pi a$. Thus, the spacetime has has coordinates in $\hat{\Gamma}$ equal to $(2\pi, 0, 2\pi a, 2\pi b)$. It
is the holonomy dual to the BTZ black hole when we set $a = \frac{|r_+|}{r}$ and $b = \frac{r_-}{r}$. We can define the new periodic coordinates $(\phi', \chi')$ along the generators of the torus,

$$\varphi = b \phi' \quad ; \quad \chi = \chi' + a \phi'$$  \hspace{1cm} (A10)

where $\chi' \in [0, 2\pi)$ and $\phi' \in [0, 2\pi)$. Then, the line element has the form,

$$ds^2 = l^2 \left[ -dt^2 + \cosh^2 t (d\chi' + a d\phi')^2 + \sinh^2 t (b d\phi')^2 \right]$$  \hspace{1cm} (A11)

$$= -l^2 dt^2 + l^2 \cosh^2 t |d\chi + z(t) d\phi|^2,$$  \hspace{1cm} (A12)

where $z(t) = a + ib \tanh t$ is the so called modulus of the torus, which completely determines the conformal geometry of it. For $t \to \infty$, this parameter goes to the constant $a + ib$, and the torus will evolve blowing up with the conformal factor $\exp(-2t)$. For $t = 0$ the torus degenerates to a circle of radius $2\pi(1 + a)$.

Finally, the line element can be put in the form,

$$ds^2 = -\frac{-dT^2}{\left((b^2 - a^2) + \frac{T^2}{r^2} - \frac{a^2 b^2}{T^2}\right)} + \left((b^2 - a^2) + \frac{T^2}{r^2} - \frac{a^2 b^2}{T^2}\right) dR^2 + T^2 (d\phi + \frac{\wp}{\rho} dR)^2$$  \hspace{1cm} (A13)

that can be heuristically obtained from (A1) by the substitution $l \mapsto il$, $\tau \mapsto it$, $J \mapsto iJ$ and by the relabeling $t \mapsto R$ and $r \mapsto T$.

The reduced phase space for a $T^2 \times R$ topology space-time is four dimensional and parameterized by $(s_i, a_i)$. However, the BTZ black hole family spans only a two dimensional surface (given by $r_+$ and $|r_-|$, with $0 \leq |r_-| \leq r_+$), while the Lorentzian cosmological solutions are well behaved for any point on $\hat{\Gamma}$. What are then, the Euclidean duals to the cosmological solutions that are off the BTZ torus? These solutions have been studied by Carlip and Teitelboim [31] who refer to them as “off-Shell” black holes. They are (Euclidean) $T^2 \times R$ solutions, with a conical singularity and a twist at the $\rho = 0$ horizon. Thus, one can not include this point in the (maximally evolved) space-time and obtain a regular space-time with the $S^1 \times R^2$ topology of the black holes. If one did not ask for regularity at the horizon and only asked for a space-time defined in the region $\rho > 0$ (as in the cosmological case for $T > 0$), the “off-shell” black holes would be perfectly valid, as solutions to the Einstein equations.

## 2. Rotating Particles

Let us first construct the Euclidean space-times corresponding to a ‘rotating particle’ at the origin. These spaces are the Wick transforms of the (naked singularity) Lorentzian space-times with a rotating particle at the origin. The topology of these space-times, both Lorentzian and Euclidean is $S^1 \times R$. Thus, we only need to specify the holonomy along the unique non-trivial loop. The explicit procedure for the construction of the spacetime includes some cutting and gluing together, starting from the Euclidean De-Sitter space. Recall that in the case of zero angular momentum, the construction involved cutting a wedge, removing the points inside the wedge, and gluing together the lines that defined the original wedge (for a constant time). Now, when angular momentum is present, we still cut a wedge, but now
we shall identify points that are not only related by a rotation but also a boost. Note that
the particular rotation and boost that we have chosen in the construction of this Appendix
are such that they leave invariant orthogonal planes, which implies that, as elements of the
group $SO(3, 1)$, they commute. The starting point is again the region of $M^4$ parameterized
by (3.14) and with line element,

$$ds^2 = l^2 \left[ \cosh^2 \rho \, dt'^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2 \right] \quad (A14)$$

where $\rho \in [0, \infty)$, $\phi \in [0, 2\pi)$, $t' \in (-\infty, \infty)$. Here, the KVF along which the identification
will be made are $\zeta := \frac{d}{d\phi}$ and $\xi := \frac{d}{dt'}$. We again cut a wedge with angle $2\pi \alpha$ and identify
points that differ by a rotation by $2\pi \alpha$ along $\zeta$ and a boost with parameter $2\pi a$ along $\xi$. In
order to implement these identifications, let us define the new coordinates

$$t := t' - \frac{a}{\alpha} \phi \quad ; \quad \phi := \frac{\phi}{\alpha} \quad (A15)$$

then,

$$ds^2 = l^2 \left[ \cosh^2 \rho \left( dt + ad\phi \right)^2 + d\rho^2 + \sinh^2 \rho \left( ad\phi \right)^2 \right] \quad (A16)$$

where $t \in (-\infty, \infty)$, $\phi \in [0, 2\pi)$ and $\rho \in (0, \infty)$. Now, we define the new coordinates
$r := l\alpha \sinh \rho$ and $\tau := (l/\alpha)t$, we get,

$$ds^2 = \left( \alpha^2 + \frac{r^2}{l^2} \right) \left( d\tau + \frac{\alpha}{l} d\phi \right) + \left( \alpha^2 + \frac{r^2}{l^2} \right)^{-1} dr^2 + r^2 d\phi^2 \quad (A17)$$

note that this is the Wick transform of the AdS-point particle solution given in [32].

The construction of the Lorentzian solution follows the same steps. The starting point
are the embeddings (3.26) and the induced metric in De-Sitter,

$$ds^2 = -l^2 \cos^2 \chi \, d\tau^2 + l^2 d\chi^2 + l^2 \sin^2 \chi \, d\phi^2 \quad . \quad (A18)$$

Again, we cut a wedge of size $2\pi \alpha$ and identify using a rotation by $2\pi \alpha$ along $\zeta$ together
with a boost along $\xi$ of parameter $2\pi a$. With the new coordinates $\bar{\tau} := \tau - (a/\alpha)\phi$ and
$\bar{\phi} := (l/\alpha)\phi$ we get,

$$ds^2 = -l^2 \cos^2 \chi (d\bar{\tau} + a d\phi)^2 + l^2 d\chi^2 + l^2 \sin^2 \chi (a d\phi)^2 \quad (A19)$$

which becomes, after defining $r := l\alpha \sin \chi$, $t := (l/\alpha)\bar{\tau}$,

$$ds^2 = - \left( \alpha^2 - \frac{r^2}{l^2} \right) \left( dt + \frac{\alpha}{l} d\phi \right)^2 + \left( \alpha^2 - \frac{r^2}{l^2} \right)^{-1} dr^2 + r^2 d\phi^2 \quad (A20)$$
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