Dynamic models using score copula innovations

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Abstract

This paper introduces a new class of observation driven dynamic models. The time evolving parameters are driven by innovations of copula form. The resulting models can be made strictly stationary and the innovation term is typically chosen to be Gaussian. The innovations are formed by applying a copula approach for the conditional score function which has close connections the existing literature on GAS models. This new method provides a unified framework for observation-driven models allowing the likelihood to be explicitly computed using the prediction decomposition. The approach may be used for multiple lag structures and for multivariate models. Strict stationarity can be easily imposed upon the models making the invariant properties simple to ascertain. This property also has advantages for specifying the initial conditions needed for maximum likelihood estimation. One step and multi-period forecasting is straight-forward and the forecasting density is either in closed form or a simple mixture over a univariate component. The approach is very general and the illustrations focus on volatility models and duration models. We illustrate the performance of the modelling approach for both univariate and multivariate volatility models.

Key words: GARCH, heteroskedasticity, GAS models, Copula, Time Series.
1. Introduction

This paper introduces a new class of dynamic models which are driven by innovations of copula form. In particular, the models are imposed to be strictly stationary. The observed time series will be denoted as \( y_t \) for \( t = 1, \ldots, T \) whilst the time varying parameter is denoted as \( \alpha_t \).

A useful distinction, following Cox (1981) and Shephard (1996), is to consider time series as being of two types: observation-driven and parameter-driven models. The first type of model allows \( \alpha_t \) to be a function of lagged values of \( y_t \). For example the autoregressive conditional heteroscedasticity (ARCH) models introduced by Engle (1982) are observation-driven. The advantage of these models is that the likelihood can be explicitly written down as the forecasting density for the observations is available. For parameter driven models a stochastic term is involved in the evolution process for \( \alpha_t \). The stochastic volatility (SV) model (see Shephard (1996)) is an example of a parameter driven model. The properties of these latent variable models are straightforward as the conditions for strict stationarity are easily verified. The marginal distribution for \( y_t \) when the model is stationary is also easily established. However, the likelihood is intractable and one-step and multi-step prediction densities are usually unavailable.

The new class of models introduced in this article involves the construction of an observation-driven model. In particular, the models provide the innovations of the parameter process and use the Generalised Autoregressive Score (GAS), see Creal et al. (2013), as guidance for the appropriate copula function. We call this new model class innovation GAS copula (iGASC) models. The idea is to take the distribution of the generalized score function as the variable of interest within a probability transformation. In the case of multivariate time varying parameters \( \alpha_t \), the vector score function may be transformed into a multivariate distribution with uniform marginals prior to being transformed into a multivariate distribution with specified marginals. A multivariate distribution with uniform marginals is known as a copula with a textbook treatment provided by Joe (1997) and econometric applications given by Patton (2009).

The models we introduce have many of the advantages of parameter-driven processes whilst retaining the computational convenience of observation driven models. Firstly, the likelihood is explicitly computed as the one step ahead forecasts are available. Secondly, Section 2.1.1 shows that is straightforward to obtain the marginal distribution of the observations, whereas this is not the case for most data driven models such as GARCH. Thirdly, the expression of
the parameter process $\alpha_t$ as a Gaussian ARMA($p,q$) model is immediate with the constraints for strict stationarity usually imposed. Fourthly, as a consequence the marginal distribution for any collection of the initial sequence $(\alpha_1, ..., \alpha_p)$ is easily derived and can be used for the initialisation for maximum likelihood (ML) estimation. Finally, Section 2.1.1 shows that one-step and multi-period forecasts are straight-forward.

The structure of this paper is as follows. The univariate approach is described in Section 2 and illustrated for a volatility model in Section 2.1. The marginal and temporal properties of the model is discussed in Section 2.1.1. In Section 2.1.2 forecasting multiple periods ahead is considered together with an ARMA representation for the state transition. The extension to volatility models for which the standardised return are t-distributed in provided in 2.2. The simple approach for duration models, where the observations follow a conditionally Exponential distribution, is given in Section 2.3. The generalization of the duration models to conditional Weibull distributions is explored in Section 2.3.1. In Section 3, the new iGASC models for volatility are applied to Nikkei 225 returns at both the daily and weekly frequency. The multivariate extension of the models is considered in Section 4. The multivariate volatility model is described in Section 4.2 and applied to three stock indices in Section 4.3. Finally, in Section 5 we conclude.

2. Univariate models

This Section considers univariate innovation GAS copula (iGASC) models. Let $y_t$ denote a dependent variate with a time-varying parameter $\alpha_t$. Then the simple data generating process (DGP) may be written as

$$y_t \sim f(y_t; \alpha_t; \nu)$$

$$\alpha_{t+1} = \mu + \phi \alpha_t + \psi \eta_t.$$  

This model may be generalised to be multivariate and to have various lags of $\alpha_t$ and $\eta_t$ on the right hand side for the parameter generating equation. Sections 2.1 and 2.3 discuss the crucial issue of constructing the innovation term $\eta_t$, which is a function of current and past data. The four parameters of (1) will be represented as $\theta := (\mu, \phi, \psi, \nu)'$.

The innovation in the iGASC model is defined by

$$\eta_t = \Phi^{-1} \left[ F_g \{ g(y_t; \alpha_t, \theta) \mid \alpha_t; \theta \} \right];$$
Φ(•) is a standard normal cdf; the univariate function \( g \) is discussed below; \( F_g \) is the cdf of the random variable \( g(Y_t; \alpha_t, \theta) \), where \( Y_t \sim f(y_t | \alpha_t; \theta) \).

If \( y_t \sim f(y_t | \alpha_t; \theta) \), so that the DGP of (1) is correct, then \( F_g \{ g(y_t; \alpha_t, \theta) | \alpha_t; \theta \} \) are independent and identically uniformly distributed on \([0, 1]\) by the probability integral transform (see, e.g. Angus 1994). Hence the innovations \( \eta_t \) will be standard normal and independent.

Even though the choice of \( g \) in (2) can be quite general in principle, its choice in practice depends on two issues. The first is to choose a for which \( F_g \) is straightforward to evaluate. The second is that the function \( g \) makes sense in terms of the relationship between the observations and the innovations. In general the choice of \( g \) is taken to be the score function

\[
g(y_t; \alpha_t, \theta) = \frac{\partial \ln f(y_t | \alpha_t; \theta)}{\partial \alpha_t},
\]

This choice is motivated by the associated GAS literature, see Creal et al. (2011) and Creal et al. (2013). Sections 2.1 and 2.3 show that this choice typically satisfies both criteria.

2.1. Volatility model: conditionally Gaussian

Consider the conditional Gaussian model for the return \( y_t = \sigma_t \varepsilon_t \), where \( \varepsilon_t \stackrel{iid}{\sim} \mathcal{N}(0, 1) \). Let \( \alpha_t = \ln \sigma_t^2 \) be the time-varying log-volatility. Then

\[
y_t | \alpha_t \sim \mathcal{N}(0, e^{\alpha_t}), \quad \text{and} \quad \alpha_{t+1} = \mu + \phi \alpha_t + \psi \eta_t.
\]

This is a special case of the model of (1), where the term \( \alpha_t \) is the log-variance. The model is similar in appearance to both the stochastic volatility model; see Shephard (1996), as well as the EGARCH model of Nelson (1991). For the EGARCH model, the innovations are structured to be a Martingale Difference (MD) sequence.

The innovations \( \eta_t \) are defined in (2), where the function \( g \) is taken to be the score function of (3). For this model,

\[
\ln f(y_t | \alpha_t; \theta) = -\frac{1}{2} \ln(2\pi) - \frac{1}{2} y_t^2 e^{-\alpha_t} - \frac{1}{2} \alpha_t,
\]

\[
g(y_t; \alpha_t, \theta) = \frac{\partial \ln f(y_t | \alpha_t; \theta)}{\partial \alpha_t}
\]

\[
= \frac{1}{2} y_t^2 e^{-\alpha_t} - \frac{1}{2} = \frac{1}{2} (\varepsilon_t^2 - 1).
\]
The distribution function $F_g \{ g(y_t; \alpha_t, \theta) \mid \alpha_t; \theta \}$ is therefore

$$F_g \{ g(y_t; \alpha_t, \theta) \mid \alpha_t; \theta \} = \Pr \left( Z^2 \leq \varepsilon_t^2 \right) = F_{\chi_1^2}(\varepsilon_t^2),$$

where $Z$ is standard normal and $F_{\chi_1^2}$ is the distribution function for the chi-square distribution with one degree of freedom.

In this case the score approach, suggested by the GAS models, for the function $g$ appears appropriate. The innovation term, defined in (2),

$$\eta_t = \Phi^{-1} \left( F_{\chi_1^2}(\varepsilon_t^2) \right)$$

is easily calculated, and $\eta_t$ is a strictly monotonically increasing function of the magnitude of standardised returns $|\varepsilon_t|$. The top panel of Figure 1 plots the relationship between the two; this is very similar to the form of EGARCH processes, Nelson (1991) when an asymmetric term is absent. This is consistent with the belief that larger absolute standardised returns $|\varepsilon_t|$ lead to larger variance at the following time point.

![Figure 1](image_url)

**Figure 1:** The relationship between $\eta_t$ and $\varepsilon_t$. **TOP:** $\eta_t$ against $|\varepsilon_t|$ for the volatility model where $\varepsilon_t$ is standard Gaussian and $t_\nu$. **BOTTOM:** $\eta_t$ against $\varepsilon$ where $\varepsilon_t$ is standard Exponential, Weibull (for $k = 2$ to $4$) for the duration model.

The likelihood is easily calculated by using the prediction decomposition as the forecast distribution is readily available by construction. For a time series $y_{1:T} = (y_1, ..., y_T)'$ the forecast density $f(y_t \mid y_{1:t-1}; \theta) = f(y_t \mid \alpha_t; \theta)$, as $\alpha_t$ is constructed from the previous values $y_{1:t-1}$. The
log-likelihood is calculated via the prediction decomposition as

$$\ell(\theta) = \sum_{t=1}^{T} \log f(y_t|y_{1:t-1}; \theta)$$

$$= -\frac{1}{2} \sum_{t=1}^{T} y_t^2 e^{-\alpha_t} - \frac{1}{2} \sum_{t=1}^{T} \alpha_t - \frac{T}{2} \ln(2\pi),$$

where the log-variance $\alpha_t$ is given by the evolution of (4) and the calculation of $\eta_t$ through (5).

In practice, we add a very small positive constant offset term to $\varepsilon_t^2$ of $10^{-4}$ when returns are expressed as percentages. This avoid any numerical instability in the calculation of (5) for returns which are exactly zero. This is exactly the tiny correction term used to prevent instability when calculating $\log y_t^2$, see for example [Harvey et al., 1994].

### 2.1.1. Properties of iGASc models

The iGASc model has the advantages of the observation driven class of time series models. Specifically, the forecasting density can be evaluated enabling maximum likelihood as seen in (6). The iGASc also shares many attractive properties of the parameter driven class of models. The marginal distribution of both the states $\alpha_t$ and the observations $y_t$ may be easily derived. Consider the model of (4) then the process for $\alpha_t$ is simply a Gaussian autoregression and provided that $|\phi| < 1$,

$$\alpha_t \sim N(\mu_\alpha, \sigma^2_\alpha),$$

$$\mu_\alpha = \frac{\mu}{1 - \phi}, \quad \sigma^2_\alpha = \frac{\psi^2}{1 - \phi^2}.$$

This in turn means that the marginal density for the observations can be computed as a simply univariate mixture

$$f(y_t) = \int N(y_t|0, e^{\alpha_t})N(\alpha_t|\mu_\alpha, \sigma^2_\alpha) d\alpha_t.$$ 

This is very similar to the properties of the parameter driven stochastic volatility (SV) model; see [Shephard, 1996]. This means the marginal properties of the observation process are readily available. For example, the kurtosis of the returns may be calculated as

$$K_y = \frac{E[y_t^4]}{E[y_t^2]^2} = 3e^{\sigma^2_\alpha}.$$
The temporal properties of the model are also similar to SV models. In particular the autocorrelation function may be derived by considering the unobserved components representation of (4) as
\[
\log(y_t^2) = \alpha_t + \log(\varepsilon_t^2),
\]
where, see Shephard (1996), the error \(\log(\varepsilon_t^2)\) is a log chi-squared variable with variance 4.93. We note that \(\varepsilon_t\) is independent of \(\alpha_t\) as the latter is constructed from previous information \(y_{1:t-1} = (y_1, ..., y_{t-1})'\). We therefore obtain that the autocorrelation for the series \(\log(y_t^2)\) at lag \(\tau\) is
\[
\rho(\tau) = \frac{\phi^\tau \sigma_\alpha^2}{\sigma_\alpha^2 + 4.93}.
\]
Whilst the properties of the iGASc models have been illustrated for the simple volatility model of (4), these properties apply more generally, for example to the duration model of Section 2.3.

### 2.1.2. Forecasting and ARMA representation

As discussed in Section 2.1, the state \(\alpha_t\) is formed from past observations \(y_{1:t-1} = (y_1, ..., y_{t-1})'\) and the model parameters \(\theta\). Forecasting one step ahead is therefore explicit as \(f(y_t \mid y_{1:t-1}; \theta) = f(y_t \mid \alpha_t; \theta)\). To illustrate forecasting several steps ahead, the model of (1) is considered. In this case,
\[
\alpha_{t+h} \mid y_{1:t}; \theta \sim \mathcal{N}(\phi^{h-1} \alpha_t + \{1 - \phi^{h-1}\} \mu_\alpha, \{1 - \phi^{2(h-1)}\} \sigma_\alpha^2),
\]
where the quantities \(\mu_\alpha, \sigma_\alpha^2\) are given in (7). The forecast of \(y_{t+h}\) is therefore a simple univariate mixture
\[
f(y_{t+h} \mid y_{1:t}; \theta) = \int f(y_{t+h} \mid \alpha_{t+h}; \theta) f(\alpha_{t+h} \mid y_{1:t}; \theta) d\alpha_{t+h},
\]
where the state forecast density is given by (8). The calculation of moments of \(y_{t+h}\) under \(f(y_{t+h} \mid y_{1:t}; \theta)\) is typically straight-forward.

The model is readily extended to an ARMA\((p,q)\) process in \(\alpha_t\) by writing
\[
\alpha_{t+1} = \mu + \sum_{i=1}^p \phi_i \alpha_{t-i+1} + \sum_{j=1}^q \psi_j \eta_{t-j} + \psi_0 \eta_t;
\]
where the roots of the lag polynomial \(\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - ... - \phi_p z^p\) need to lie outside the unit circle for the process to be stationary. The innovation terms \(\eta_t\) and \(\eta_{t-j}\) are again constructed as in (2), and for the volatility model the innovations are calculated according to (5). Strict stationarity coincides with weak stationarity when the innovations \(\eta_t\) are Gaussian.
2.2. Volatility models: heavy tailed extensions

A natural extension of the volatility model of Section 2.1 is to assume that the standardised returns $\varepsilon_t$ have a heavier tailed distribution than a standard Gaussian variate. For example, $y_t = \sigma_t \varepsilon_t$, where $\varepsilon_t$ is now an independent and identically (IID) $t$-distributed variate with $\nu$ degrees of freedom and unit variance. The density of $\varepsilon_t$ is given as

$$f_\varepsilon(\varepsilon_t) = c \left(1 + \frac{\varepsilon_t^2}{\nu - 2}\right)^{-\frac{\nu+1}{2}}, c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\nu - 2)\pi}}.$$

Similarly to (4) where $\alpha_t = \ln \sigma_t^2$,

$$f(y_t | \alpha_t; \theta) = e^{-\frac{\alpha_t^2}{2}} f_\varepsilon(y_t e^{-\frac{\alpha_t^2}{2}}).$$

The score function is

$$g(y_t; \alpha_t, \theta) = \frac{\partial \ln f(y_t | \alpha_t; \theta)}{\partial \alpha_t} = -\frac{1}{2} + \frac{(\nu + 1)}{2} \frac{1}{(\nu - 2)\varepsilon_t^{-2} + 1}.$$

The corresponding distribution function is

$$F_g \{g(y_t; \alpha_t, \theta) | \alpha_t; \theta\} = \Pr \left(T_\nu^2 \leq \frac{\nu}{\nu - 2} \varepsilon_t^2\right),$$

where $T_\nu$ is a random variable following a $t$-distribution with $\nu$ degrees of freedom. That is,

$$F_g \{g(y_t; \alpha_t, \theta) | \alpha_t; \theta\} = F_{F_1,\nu} \left(\frac{\nu}{\nu - 2} \varepsilon_t^2\right), \quad \eta_t = \Phi^{-1} \left(F_{F,\nu} \left(\frac{\nu}{\nu - 2} \varepsilon_t^2\right)\right).$$

where $F_{F_1,\nu}$ is the distribution function for the $F$-distribution with parameters 1 and $\nu$ degrees of freedom. Similarly to the volatility model with Gaussian innovations $\varepsilon_t$, the innovation term $\eta_t$ is again monotonically increasing as a function of the standardised return $|\varepsilon_t|$. The top panel of Figure 1 plots the relationship between the two, for various values of $\nu$.

The forecasting density and likelihood function can be calculated straightforwardly similarly to (6). The extension to a Gaussian ARMA($p,q$) model of (9) is immediate as the innovations $\eta_t$ are again standard Gaussian.

2.3. Duration models

The iGASC model can be used to model duration data. The conditional model will be used to model the time between successive trades $y_i = \tau_i - \tau_{i-1}$, with $\tau_i$ representing the time of the $i$-th trade for $i = 1, 2, ..., N$. Here, $\tau_0$ and $\tau_N$ represent the time of the first and last trade.
respectively. For this model, $y_i = \lambda_i \varepsilon_i$ where, following the approach of Engle & Russell (1998), $\varepsilon_i \sim \text{Exp}(1)$, a standard Exponential distribution. Transforming to the real line $\alpha_i = \ln \lambda_i$, the conditional density is

$$p(y_i \mid \alpha_i; \theta) = \frac{1}{\lambda_i} e^{-y_i/\lambda_i} = \exp \left(-\alpha_i - y_i e^{-\alpha_i}\right).$$

The parameter evolution is

$$\alpha_{i+1} = \mu + \phi \alpha_i + \psi \eta_i.$$

We now construct the innovation term $\eta_i$. The conditional score is

$$g(y_i; \alpha_i, \theta) = \frac{\partial \ln p(y_i \mid \alpha_i; \theta)}{\partial \alpha_i} = -1 + y_i e^{-\alpha_i} = -1 + \varepsilon_i.$$

Therefore, the distribution function is

$$F_\varepsilon \{g(y_i; \alpha_i, \theta) \mid \alpha_i; \theta\} = F_\varepsilon(\varepsilon_i) = 1 - e^{-\varepsilon_i},$$

where $F_\varepsilon$ is the distribution function for the standard exponential distribution. Therefore, the innovation term is

$$\eta_i = \Phi^{-1}\left\{F_\varepsilon(\varepsilon_i)\right\} = \Phi^{-1}\left(1 - e^{-\varepsilon_i}\right).$$

Clearly, the innovation $\eta_i$ is a monotonically increasing function of the standardised duration $\varepsilon_i$. The bottom panel of Figure 1 plots relationship between the two variables, for various values of $\nu$. This model therefore has a qualitative similarity to that of Engle & Russell (1998) for which the durations themselves are used to drive the parameter process. The advantages of the iGASC model are again that the marginal distribution is simply calculated and the forecasts are of closed distributional form even many steps ahead.

### 2.3.1. Duration models: conditional Weibull variates

The extension to a conditional Weibull, a generalisation of the Exponential distribution. In this case $\varepsilon_i \sim \text{Weib}(\beta, k)$ where $\beta = 1/\Gamma(1 + 1/k)$ to ensure $E[\varepsilon_i] = 1$. This standardised variate is therefore indexed by a single parameter $k$. The density of $\varepsilon_i$ is

$$f_\varepsilon(\varepsilon_i) = \frac{k}{\beta} \left( \frac{\varepsilon_i}{\beta} \right)^{k-1} \exp \left\{ - \left( \frac{\varepsilon_i}{\beta} \right)^k \right\}.$$
When \( k = 1 \), this simply reduces to the conditional Exponential model. As before \( y_t = \lambda_i \varepsilon_i \) following the approach of Engle & Russell (1998). Again, taking \( \alpha_i = \ln \lambda_i \),

\[
p(y_i | \alpha_i; \theta) = e^{-\alpha_i} f_{\varepsilon_i}(y_i e^{-\alpha_i}) \]

\[
\log p(y_i | \alpha_i; \theta) = \text{const} - \alpha_i + (k - 1) \log(y_i e^{-\alpha_i}) - \left( \frac{y_i e^{-\alpha_i}}{\beta} \right)^k
\]

\[
g(y_i; \alpha_i, \theta) = \frac{\partial \ln p(y_i | \alpha_i; \theta)}{\partial \alpha_i} = -k + k \left( \frac{y_i}{\beta} e^{-\alpha_i} \right)^k = k \left( \frac{\varepsilon_i}{\beta} \right)^k - k
\]

Now consider the distribution function \( F_g\{g(y_i)\} \) of \( g(y_i) \) under the conditional density \( p(y_i | \alpha_i; \theta) \), where we denote \( \varepsilon \) as a \( \text{Wei}(\beta, k) \) random variable,

\[
F_g\{g(y_i) \mid \alpha_i; \theta\} = \Pr(\varepsilon \leq \varepsilon_i) = \Phi\left(\frac{\varepsilon_i}{\beta}\right) = \Phi\left(1 - \exp\left(-\frac{(\varepsilon_i/\beta)^k}{k}\right)\right).
\]

The Gaussian innovation \( \eta_i \) as a function of the standardised duration \( \varepsilon_i \) is show in Figure 1 for various values of \( k \). The iGASC model Weibull duration model reduces to the conditionally Exponential model when \( k = 1 \) and so \( \beta = 1/\Gamma(1 + 1/k) = 1 \).

The duration model the extension to a Gaussian ARMA\((p, q)\) model of (9) is straight-forward as the innovations \( \eta_i \) are again standard Gaussian. The likelihood is readily available by using the prediction decomposition.

3. Results: univariate model

3.1. Synthetic data

Simulation studies are carried out for the univariate iGASC-t volatility models of Section 2.2. The parameters are \( \mu = 0.3, \phi = 0.2, \psi = 0.7 \) and \( \nu = 10 \). The replications are over 200 data sets of length \( T = 1000, 5000, 10000 \) and 20000, respectively. Maximum likelihood is used to estimate the parameters of interest and the mean, variance, bias and mean squared errors (MSE) of the resulting estimates are summarised in Table 1. As shown in Table 1 the method produces accurate estimates for the parameter of interests. With the increase of the length of the time period \( T \), the variation of the resulting estimates significantly decreases.
| Parameter | $\mu$ | $\phi$ | $\psi$ | $\nu$ |
|-----------|-------|-------|-------|-------|
| True value | 0.3   | 0.2   | 0.7   | 10    |

| $T = 1000$ | Mean | Variance | Bias | MSE |
|------------|------|----------|------|-----|
|             | 0.29348 | 0.00587  | 0.00651 | 0.00586 |
|             | 0.20431 | 0.00678  | 0.00431 | 0.00673 |
|             | 0.69805 | 0.00286  | 0.00194 | 0.00283 |
|             | 11.70043| 24.01240 | 1.70043 | 26.66371 |

| $T = 5000$ | Mean | Variance | Bias | MSE |
|------------|------|----------|------|-----|
|             | 0.30068 | 0.00105  | 0.00068 | 0.00104 |
|             | 0.19969 | 0.00116  | 0.00030 | 0.00115 |
|             | 0.70081 | 0.00041  | 0.00081 | 0.00041 |
|             | 10.23158| 1.57374  | 0.23158 | 1.61164 |

| $T = 10000$ | Mean | Variance | Bias | MSE |
|------------|------|---------|------|-----|
|             | 0.29759 | 0.00057  | 0.00240 | 0.00057 |
|             | 0.20179 | 0.00055  | 0.00179 | 0.00055 |
|             | 0.69968 | 0.00033  | 0.00031 | 0.00032 |
|             | 10.24054| 1.08449  | 0.24054 | 1.13150 |

| $T = 20000$ | Mean | Variance | Bias | MSE |
|------------|------|---------|------|-----|
|             | 0.29764 | 0.00017  | 0.00235 | 0.00017 |
|             | 0.20149 | 0.00026  | 0.00149 | 0.00026 |
|             | 0.70072 | 0.00011  | 0.00072 | 0.00011 |
|             | 10.03524| 0.38072  | 0.03524 | 0.37815 |

Table 1: Results over 200 replications of the univariate iGASC-t volatility model of Section 2.2. The mean, variance, bias and MSE of the resulting estimates are shown. Different lengths $T$ for the time series are used.

### 3.2. Return data: Nikkei index

The iGASC volatility models of Section 2.1 are applied to the daily closing prices for the Nikkei 225. The two different conditional densities for the returns, standard Gaussian and standard t, are considered. These are compared to the GARCH(1,1) and GARCH(1,1)-t models, see Bollerslev (1987). The returns are continuously compounded and calculated as $y_t = 100(\log S_t - \log S_{t-1})$, where $S_t$ represents the closing price of the stock index. The data is from 2009-04-30 to 2019-04-30 leading to 2476 daily and 520 weekly percentage returns. The parameter estimates for the daily returns are provided in Table 2. It is clear that the persistence for the GARCH class, given by $(\alpha + \beta)$, is a little higher than the persistence, given by $\phi$, for the two iGASC models.

The estimates for the degrees of freedom parameter $\nu$, for the conditional distribution of returns, are very similar for both the GARCH and iGASC model. The corresponding estimates for the 520 weekly returns are provided in Table 3. It is again apparent that the iGASC models result in less persistence when compared to the corresponding GARCH models. However, the degrees of freedom parameter $\nu$ is estimated to be higher for the iGASC-t model than for the GARCH-t.
| Model  | iGASC Gaussian | GARCH(1,1) Gaussian | iGASC Student t | GARCH(1,1) Student t |
|--------|----------------|---------------------|----------------|---------------------|
| \(\mu\) | 0.03568 (0.02334, 0.04802) | 0.01651 (0.00582, 0.02720) | 0.01651 (0.00582, 0.02720) | 0.04235 (0.01301, 0.07169) |
| \(\omega\) | 0.05761 (0.02939, 0.08583) | 0.04235 (0.01301, 0.07169) | 0.04235 (0.01301, 0.07169) | 0.04235 (0.01301, 0.07169) |
| \(\phi\) | 0.95231 (0.93347, 0.97115) | 0.96651 (0.94794, 0.98508) | 0.96651 (0.94794, 0.98508) | 0.11838 (0.07844 0.15832) |
| \(\alpha\) | 0.13411 (0.10221, 0.16601) | 0.14169 (0.10322, 0.18016) | 0.14169 (0.10322, 0.18016) | 0.86439 (0.81875, 0.91003) |
| \(\psi\) | 0.16096 (0.13140, 0.19052) | 0.83982 (0.80158, 0.87806) | 0.83982 (0.80158, 0.87806) | 0.86439 (0.81875, 0.91003) |
| \(\beta\) | 0.16096 (0.13140, 0.19052) | 0.83982 (0.80158, 0.87806) | 0.83982 (0.80158, 0.87806) | 0.86439 (0.81875, 0.91003) |
| \(\nu\) | 6.29119 (4.75189, 7.83049) | 6.27660 (4.72964, 7.82356) |

Table 2: Parameter estimates and 95% confidence intervals for iGASC and GARCH(1,1) models applied to daily Nikkei returns. The distribution of the standardised return \(\varepsilon_t\) is considered as Gaussian or t–distributed.

| Model  | iGASC Gaussian | GARCH(1,1) Gaussian | iGASC Student t | GARCH(1,1) Student t |
|--------|----------------|---------------------|----------------|---------------------|
| \(\mu\) | 0.49686 (0.05370, 0.94002) | 0.33798 (-0.11183 0.78779) | 0.33798 (-0.11183 0.78779) | 1.04019 (-0.63373, 2.71411) |
| \(\omega\) | 1.78208 (-0.74113, 4.30529) | 0.82583 (0.59707, 1.05459) | 0.82583 (0.59707, 1.05459) | 1.04019 (-0.63373, 2.71411) |
| \(\phi\) | 0.74993 (0.52614, 0.97372) | 0.15256 (0.01837, 0.28675) | 0.15256 (0.01837, 0.28675) | 0.10748 (-0.01055, 0.22551) |
| \(\alpha\) | 0.15256 (0.01837, 0.28675) | 0.82583 (0.59707, 1.05459) | 0.82583 (0.59707, 1.05459) | 0.10748 (-0.01055, 0.22551) |
| \(\psi\) | 0.21434 (0.10272, 0.32596) | 0.61593 (0.19244, 1.03942) | 0.61593 (0.19244, 1.03942) | 0.75617 (0.45099, 1.06135) |
| \(\beta\) | 0.21434 (0.10272, 0.32596) | 0.61593 (0.19244, 1.03942) | 0.61593 (0.19244, 1.03942) | 0.75617 (0.45099, 1.06135) |
| \(\nu\) | 9.98325 (2.52799, 17.43851) | 8.62882 (2.93902, 14.31862) |

Table 3: Parameter estimates and 95% confidence intervals for iGASC and GARCH(1,1) models applied to weekly Nikkei returns. The distribution of the standardised return \(\varepsilon_t\) is considered as Gaussian or t–distributed.

Model, although the confidence intervals are wide in both cases. The standard deviations \(\sigma_t\) for the GARCH(1,1)-t model and \(\sigma_t = \exp(\alpha_t/2)\) for the iGASC-t volatility model are displayed for the weekly Nikkei returns in Figure 2. Diagnostics may be performed for the fit of the models by using the probability integral transform (PIT) method. The forecast cumulative distribution function is evaluated as

\[
u_t = F(y_t | y_{1:t-1}; \theta) = \Phi\left(\frac{y_t e^{-\alpha_t/2}}{2}\right).
\]

Under the assumption that the model and parameters are correct, the statistics \(u_t \overset{IID}{\sim} Un(0,1)\) for \(t = 1, \ldots, T\). A Kolmogorov-Smirnov test may be applied to test whether the residuals \(u_t\) are
| Model Standardised return | iGASC Gaussian | GARCH Gaussian | iGASC Student t | GARCH Student t |
|--------------------------|----------------|----------------|----------------|----------------|
| Daily                    | -4000.082      | -4002.294      | -3941.135      | -3944.288      |
| Weekly                   | -1249.165      | -1252.573      | -1244.543      | -1246.397      |

| Kolmogorov-Smirnov test (daily series) |
|----------------------------------------|
| Statistic D                            |
| 0.047565                               |
| p-value                                |
| 2.725e-05                              |

| Kolmogorov-Smirnov test (weekly series) |
|----------------------------------------|
| Statistic D                            |
| 0.036716                               |
| p-value                                |
| 0.4849                                 |

Table 4: Results for 2476 daily and 520 weekly percentage returns of Nikkei 225. The distribution of the standardised return $\varepsilon_t$ is considered as Gaussian or t–distributed. The maximized log-likelihoods and the Kolmogorov-Smirnov residual tests are reported for the four series.

Marginal distributions are summarised in Table 4 for both the daily and weekly Nikkei returns. All four models are acceptable under this test at the weekly frequency. However, only the iGASC-t volatility model is acceptable (at 5% size) for daily data. Both the conditionally Gaussian and conditionally t-distribution models outperform their GARCH counterparts in terms of the log-likelihood at the weekly and daily frequency. These log-likelihood comparisons are given in Table 4.

4. Multivariate models

4.1. The copula distribution

The extension of the model to multivariate states is, in principle, straightforward. The multivariate version of the probability transform approach, of [1] in Section 2, leads to a copula distribution function. A good review of copula distributions, which are simply multivariate distributions with univariate marginals is given by, for example, [Joe (1997)].

A copula $C(u_1, u_2, \ldots, u_N)$ is defined as the joint cumulative distribution function of variables $(U_1, U_2, \ldots, U_N)$ where each $0 \leq U_i \leq 1$ for $i = 1, \ldots, N$. So

\[
C(u_1, u_2, \ldots, u_N) = \Pr(U_1 \leq u_1, U_2 \leq u_2, \ldots, U_N \leq u_N). \tag{10}
\]

Each marginal distribution is uniformly distributed on the interval $[0, 1]$ so $\Pr(U_i \leq u_i) = u_i$ for $i = 1, \ldots, N$.

The use of copulas is widespread as the uniform variables can then be mapped into any distribution of interest, for example to extreme value distributions using the probability integral transform. For the purposes of this paper, the copula will be formed by considering a $N \times 1$
Figure 2: Results for the 520 weekly continuously compounded percentage returns for the Nikkei. The standard deviations $\sigma_i$ for the GARCH(1,1)-t model and $\sigma_i = \exp(\alpha_i/2)$ for the iGASC-t model, of Sec 2.2, are displayed with the return.

continuous random vector $X$, where the marginal of each component of $X$ are known. In the standard copula approach the random vector

$$(U_1, U_2, \ldots, U_N) = (F_1(X_1), F_2(X_2), \ldots, F_N(X_N)),$$

where $F_i(x) = \Pr(X_i \leq x)$. In this case, the copula function which we shall call $C_g(\cdot)$ can be written as

$$C_g(u_1, u_2, \ldots, u_d) = \Pr[X_1 \leq F_1^{-1}(u_1), X_2 \leq F_2^{-1}(u_2), \ldots, X_N \leq F_N^{-1}(u_N)]$$

$$= \Pr[F_1(X_1) \leq u_1, F_2(X_2) \leq u_2, \ldots, F_N(X_N) \leq u_N].$$

Perhaps an easier way of considering this copula is to consider generating random variates from $C_g(\cdot)$. The first step would be to generate the vector $X$ from its distribution function and then simply to evaluate $U_i = F_i(X_i)$ for $i = 1, \ldots, N$. 
4.2. Multivariate stochastic volatility models

We now consider the iGASC approach for the modelling of multivariate stochastic volatility models. The modelling is similar to the parameter driven approach taken by Harvey et al. (1994). The $N \times 1$ vector of continuous compounded returns $y_t$ is modelled as jointly Gaussian,

$$y_t = e^{\alpha_t/2} \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0; \Sigma), \quad (11)$$

where $\Sigma$ is a positive definite correlation matrix with $N(N - 1)/2$ elements and has $1$’s down the diagonal. Equivalently,

$$y_t \mid \alpha_t; \theta \sim \mathcal{N}_N(0; D\Sigma D),$$

where $D$ is a diagonal matrix with elements $D_{ii} = e^{\alpha_{it}/2}$. In a departure from the parameter driven model, the iGASC model has

$$\alpha_{it+1} = \mu_i + \phi_i \alpha_{it} + \psi_i \eta_{it},$$

where the $\eta_{it}$ innovations terms need to be specified.

The simplest approach is to consider the marginal conditional score function. That is, we consider

$$g(y_{it}; \alpha_{it}, \theta) = \frac{\partial \ln f(y_{it} \mid \alpha_{it}; \theta)}{\partial \alpha_{it}}, \quad (12)$$

where $f(y_{it} \mid \alpha_{it}, \mathcal{F}_{t-1}; \theta)$ represents the marginal distribution of $y_{it}$ conditional upon $\alpha_{it}$. In this case,

$$y_{it} \mid \alpha_{it}; \theta \sim \mathcal{N}(0; e^{\alpha_{it}}),$$

and we consider the distribution of $\varepsilon_{it}^2$ under this conditional density for $y_{it}$.

The approach now becomes similar to the univariate method of Section 2.1. The marginal score function is

$$g(y_{it}; \alpha_{it}, \theta) = \frac{1}{2} y_{it}^2 e^{-\alpha_{it}} - \frac{1}{2} = \frac{1}{2} (\varepsilon_{it}^2 - 1).$$

The distribution function for the function $g$ under $f(y_{it} \mid \alpha_{it}, \mathcal{F}_{t-1}; \theta)$ is

$$F_g \{ g(y_{it}; \alpha_{it}, \theta) \mid \alpha_{it}; \theta \} = F_{\chi^2_1}(\varepsilon_{it}^2),$$

where $F_{\chi^2_1}$ is the distribution function for the chi-square distribution with one degree of freedom.

This is similar to Section 2.1. We denote $u_t = (u_{1t}, ..., u_{Nt})'$ where $u_{it} = F_{\chi^2_1}(\varepsilon_{it}^2)$. It is clear
Trivariate iGASC-Gaussian

|            | Nikkei | DAX     | Hang Seng | Correlation |
|------------|--------|---------|-----------|-------------|
| $\mu_1$   | 0.03874| 0.01744 | 0.01402   | 0.29456     |
| $\rho_{N,D}$ | (0.02550, 0.05198) | (0.00996, 0.02492) | (0.00807, 0.01997) | (0.25733, 0.33179) |
| $\phi_1$  | 0.94421| 0.97053 | 0.97193   | 0.35937     |
| $\rho_{D,HS}$ | (0.92430, 0.96412) | (0.95703, 0.98403) | (0.95977, 0.98409) | (0.32367, 0.39507) |
| $\psi_1$  | 0.15444| 0.10583 | 0.08174   | 0.52086     |
| $\rho_{N,HS}$ | (0.00996, 0.02492) | (0.07856, 0.13310) | (0.06166, 0.10182) | (0.49110, 0.55062) |

Table 5: Parameter estimates and 95% confidence intervals for trivariate iGASC model applied to daily returns for the Nikkei, DAX and Hang Seng indices. The distribution of the standardised return $\varepsilon_t$ is considered as multivariate Gaussian.

that $u_t$ is distributed according to a copula, according to the definition given in Section 4.1.

The innovation term $\eta_t$, an $N \times 1$ vector, is derived as

$$
\eta_t = \Phi^{-1}(u_t) = \Phi^{-1}\left[F_{\chi^2_1}(\varepsilon_{it}^2)\right].
$$

(13)

The resulting iGASC models is clearly observation-driven as the multivariate prediction density $f(y_t|y_{1:t-1}; \theta)$ can be calculated. In this case $\theta$ represents all the $N(N-1)/2$ elements of $\Sigma$ and the parameters $(\mu_i, \phi_i, \psi_i)'$ for $i = 1, ..., N$.

There is a caveat to note in the formation of the innovations vector $\eta_t$. Whilst each element $\eta_{it}$ will be marginally standard Gaussian, the vector will $\eta_t$ not be multivariate Gaussian. This is a consequence of the copula construction as the terms $u_{it}$ in (13) are marginally uniform but the vector $u_t$ is not a multivariate uniform random variate. For the multivariate iGASc model the marginal and predictive distributions of $\alpha_{it}$ remain simple to calculate and have a form analogous to those of Sections 2.1.1 and 2.1.2. However, the multivariate forecast distribution for the vector $\alpha_t$, for more than one period ahead, is unavailable in closed form.

4.3. Multivariate results

The multivariate model of (11) is applied to three daily and weekly stock index returns. As in Section 3.2 continuously compounded returns in percentages are analysed. The three series considered are the Nikkei, DAX and Hang Seng indices. The data is again from 2009-04-30 to
Trivariate iGASC-Gaussian

|       | Nikkei       | DAX          | Hang Seng    | Correlation |
|-------|--------------|--------------|--------------|-------------|
| \( \mu_1 \) | 0.27965      | 0.14609      | 0.51559      | \( \rho_{N,D} \) |
| \( \mu_2 \) | (-0.00149, 0.56079) | (0.02917, 0.26301) | (-3.17686, 4.20804) | (0.53818, 0.65040) |
| \( \mu_3 \) | (-0.00149, 0.56079) | (0.02917, 0.26301) | (-3.17686, 4.20804) | (0.53818, 0.65040) |
| \( \phi_1 \) | 0.86064      | 0.92443      | 0.71730      | \( \rho_{D,HS} \) |
| \( \phi_2 \) | (0.71877, 1.00251) | (0.86295, 0.98591) | (-1.29089, 2.72549) | (0.45458, 0.60532) |
| \( \phi_3 \) | 0.15184      | 0.11469      | 0.13833      | \( \rho_{N,HS} \) |
| \( \psi_1 \) | (0.06820, 0.23548) | (0.05573, 0.17365) | (-0.29974, 0.57640) | (0.52842, 0.64846) |

Table 6: Parameter estimates and 95% confidence intervals for trivariate iGASC model applied to weekly Nikkei returns, weekly DAX returns and weekly Hang Seng returns. The distribution of the standardised return \( \varepsilon_t \) is considered as multivariate Gaussian.

2019-04-30. We denote the elements of the correlation matrix as

\[
\Sigma = \begin{pmatrix}
1 & \rho_{N,D} & \rho_{D,HS} \\
\rho_{N,D} & 1 & \rho_{N,HS} \\
\rho_{D,HS} & \rho_{N,HS} & 1
\end{pmatrix},
\]

where \( \rho_{N,D} \) is the conditional correlation between Nikkei and Hang Seng indices, \( \rho_{D,HS} \) the correlation between the DAX and Hang Seng indices and \( \rho_{N,HS} \) the correlation between the Nikkei and Hang Seng indices. Maximum likelihood is employed to estimate the parameters. This is extremely simple to implement and runs very quickly.

The parameter estimates from the maximum likelihood procedure applied to the daily data are provided in Table 5. The estimates for \( (\mu_1, \phi_1, \psi_1)' \) for the Nikkei daily returns (first column) are quite similar to those in Table 2, as would be expected. The estimates of the three elements of \( \Sigma \) (final column) are positive and clearly far from zero, indicating that modelling the correlation is important. The persistence parameters for the DAX and Hang Seng indices, \( \phi_2 \) and \( \phi_3 \), are considerably higher than for the Nikkei index. The results for weekly data are provided in Table 6. The multivariate iGASC model of (11) implies the marginal univariate models of Section 2.1. The parameter estimates will not be the same as the information in the likelihood will be richer for the multivariate models. However, there should be close agreement and this can be seen in Figure 3. The conditional standard deviation for the daily Nikkei index is plotted for both the univariate model and for the multivariate model. It can be seen that there is very close correspondence between the two series. This closeness is also seen for the weekly data in Figure 17.
Figure 3: Results for the daily continuously compounded percentage returns for the Nikkei index. The standard deviations $\sigma_t = \exp(\alpha_t/2)$ for the iGASC-Gaussian univariate and trivariate model.

5. Discussion

The paper shows a new approach for the modelling of dynamic time series. The resulting models inherit many of the attractive properties associated with latent parameter driven models. However, in contrast to parameter driven models, the one step ahead prediction density is available enabling maximum likelihood to be performed straightforwardly. The approach relies on the probability integral transform method, applied to a function of interest, for univariate series. For multivariate evolving parameters, this extends naturally to consideration of a copula method applied to the function of interest. We have deliberately restricted the function of interest to the conditional score and have seen that this results in sensible properties. This ties into the related literature on generalized autoregressive score (GAS) modelling. We have shown how to construct these iGASc models for both dynamic volatility and duration models. The methods appear competitive with existing approaches, for example GARCH models, whilst having tractable marginal and predictive densities. The multivariate extension of the models
Figure 4: Results for the 516 weekly continuously compounded percentage returns for the Nikkei. The standard deviations $\sigma_t = \exp(\alpha_t/2)$ for the iGASC-Gaussian univariate and trivariate model.

has been considered for modelling the volatility of several returns. This leads to a simple model which is applied to three stock indices. This new approach can be easily extended. A very broad class of dynamic models can be considered by adopting the modelling strategy. Additionally, there are many ways in which the copula function of interest can be constructed for multivariate modelling. We have chosen to use the marginal conditional score function. However, in many contexts it may be the full conditional score is more appropriate. An open questions remains when the number of observations, at each time period, is either greater than or less than the number of evolving states at each time period.
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References

Angus, J. E. (1994). The probability integral transform and related results. *SIAM review, 36*, 652–654.

Bollerslev, T. (1987). A conditionally heteroskedastic time series model for speculative prices and rates of return. *The review of economics and statistics, (pp. 542–547).*

Cox, D. R. (1981). Statistical analysis of time series: some recent developments. *Scand J Stat, 8*, 93–115.

Creal, D., Koopman, S. J., & Lucas, A. (2011). A dynamic multivariate heavy-tailed model for time-varying volatilities and correlations. *Journal of Business & Economic Statistics, 29*, 552–563.

Creal, D., Koopman, S. J., & Lucas, A. (2013). Generalized autoregressive score models with applications. *Journal of Applied Econometrics, 28*, 777–795.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica: Journal of the econometric society, (pp. 987–1007).*

Engle, R. F., & Russell, J. R. (1998). Autoregressive conditional duration: a new model for irregularly spaced transaction data. *Econometrica, (pp. 1127–1162).*

Harvey, A., Ruiz, E., & Shephard, N. (1994). Multivariate stochastic variance models. *The Review of Economic Studies, 61*, 247–264.

Joe, H. (1997). *Multivariate models and multivariate dependence concepts.* CRC Press.

Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica: Journal of the Econometric Society, (pp. 347–370).*

Patton, A. J. (2009). Copula–based models for financial time series. In *Handbook of financial time series* (pp. 767–785). Springer.

Shephard, N. (1996). Statistical aspects of arch and stochastic volatility. *Monographs on Statistics and Applied Probability, 65*, 1–68.