Statistical properties of random density matrices

Hans-Jürgen Sommers\textsuperscript{1} and Karol Życzkowski\textsuperscript{2,3}

\textsuperscript{1}Fachbereich Physik, Universität Duisburg-Essen, Campus Essen, 45117 Essen, Germany
\textsuperscript{2}Instytut Fizyki im. Smoluchowskiego, Uniwersytet Jagielloński, ul. Reymonta 4, 30-059 Kraków, Poland
\textsuperscript{3}Centrum Fizyki Teoretycznej, Polska Akademia Nauk, Al. Lotników 32/44, 02-668 Warszawa, Poland

(Dated: July 16, 2004)

Statistical properties of ensembles of random density matrices are investigated. We compute traces and von Neumann entropies averaged over ensembles of random density matrices distributed according to the Bures measure. The eigenvalues of the random density matrices are analyzed: we derive the eigenvalue distribution for the Bures ensemble which is shown to be broader than the quarter–circle distribution characteristic of the Hilbert–Schmidt ensemble. For measures induced by partial tracing over the environment we compute exactly the two-point eigenvalue correlation function.

I. INTRODUCTION

Analyzing the density matrices of a finite size $N$ one is often interested in the properties of typical states. In general the properties depend on the measure, according to which the random density matrices are distributed. Nowadays it is widely accepted, that it is not possible to single out the only, unique measure in the set $\mathcal{M}_N$ of all density matrices of size $N$.

However, several possible measures are distinguished by different mathematical and physical arguments. For instance the Hilbert-Schmidt (HS) measure arises if one constructs random pure states $|\Psi\rangle$ distributed according to the natural (Fubini–Study) measure on the space of pure states on a composite Hilbert space $\mathcal{H}_N \otimes \mathcal{H}_N$ and obtains mixed states by partial tracing, $\rho = \text{Tr}_N(|\Psi\rangle\langle\Psi|)$ \cite{1}. Moreover, the HS measure may be defined by the HS distance,

$$D_{\text{HS}}(\rho, \sigma) = \sqrt{\text{Tr}[(\rho - \sigma)^2]}, \quad (1.1)$$

which induces the flat geometry in $\mathcal{M}_N$. For instance, the set of $N = 2$ mixed states analyzed with respect to the HS distance displays the geometry of the 3–ball (the Bloch ball), with the Bloch sphere, containing the pure states, at its boundary.

Another measure in the space of mixed states, which should be distinguished, is the Bures measure \cite{2}. It is induced by the Bures metric \cite{3,4}

$$D_B(\rho, \sigma) = \sqrt{2[1 - \text{Tr}(\sqrt{\rho}\sqrt{\sigma})^{1/2}]}^{1/2}. \quad (1.2)$$

which is Riemannian and monotone. It is a Fisher–adjusted metric \cite{5}, since in the subspace of diagonal matrices it induces the statistical distance \cite{6}. Moreover, the Bures metric is Fubini–Study adjusted, since at the space of pure states both metrics do agree \cite{7}. These unique features of the Bures distance are used to support the claim that without any prior knowledge on a certain density matrix, the optimal way to mimic it is to generate it at random with respect to the Bures measure.

In this work we analyze statistical properties of ensembles of random states. In section II we provide the definitions of the Hilbert–Schmidt and the Bures ensembles of random density matrices and recall their joint distribution functions for their spectra. The expectation values of the moments $\langle \text{Tr}\rho^j \rangle$ and von Neumann entropies are computed in section III. In section IV we analyze the eigenvalue density of random states: the quarter–circle distribution characteristic of the Hilbert–Schmidt ensemble is rederived and compared with an explicit distribution computed for the Bures ensemble. Section V is devoted to the ensembles of random matrices obtained by partial tracing, for which the average traces and the eigenvalue correlation functions are computed.

II. ENSEMBLES OF RANDOM STATES

We are concerned with ensembles of random states, for which the probability measure has a product form and may be factorized \cite{1,2}.

$$d\mu_x = d\nu_x(\lambda_1, \lambda_2, ..., \lambda_N) \times dh. \quad (2.1)$$
The latter factor, $dh$, determining the distribution of the eigenvectors of the density matrix, is the unique, unitarily invariant, Haar measure on $U(N)$. On the other hand, the first factor describing the distribution of eigenvalues $\lambda_i$ of $\rho$ depends on the measure used (the label $\gamma$ denotes any of the product measures investigated).

The Hilbert-Schmidt measure induces the following joint distribution function in the simplex of eigenvalues

$$P_{HS}(\vec{\lambda}) = \frac{\Gamma(N^2)}{\prod_{j=0}^{N-1} \Gamma(N-j)\Gamma(N-j+1)} \delta\left(1 - \sum_{j=1}^{N} \lambda_j\right) \prod_{i<j}^{N} (\lambda_i - \lambda_j)^2.$$  \hspace{1cm} (2.2)

This distribution may be considered as a special case of the family of measures induced by partial tracing. Consider a pure state of $|\Phi\rangle \in \mathcal{H}_N \otimes \mathcal{H}_K$ of a composite bi–partite system of size $NK$. Tracing over the $K$–dimensional environment one obtains a mixed state of size $N$, namely $\rho = T_K(|\Phi\rangle\langle\Phi|)$. A natural assumption that $|\Phi\rangle$ is a random pure state distributed according to the unique, unitarily invariant measure on the set of pure states leads to the family of measures

$$P_{N,K}(\vec{\lambda}) = \frac{\Gamma(KN)}{\prod_{j=0}^{N-1} \Gamma(K-j)\Gamma(N-j+1)} \delta\left(1 - \sum_{j=1}^{N} \lambda_j\right) \prod_{i<j}^{KN} (\lambda_i - \lambda_j)^2,$$  \hspace{1cm} (2.3)

labeled by the size $K$ of the environment. Such induced measures were discussed many times in the literature \cite{Bures, Hall, Slater}, while the normalisation constant was derived in \cite{Bures}. Note that in the symmetric case $K = N$ the induced measure reduces to the Hilbert-Schmidt measure \cite{Bures}.

It is worth to emphasize a link to known ensembles of random matrices. In order to construct a random density matrix according to the measure $\mu_N$, it is sufficient to generate a rectangular Gaussian matrix $X$ of size $N \times K$ and to compute $\rho = X^{\dagger}X/\text{Tr}(X^{\dagger}X)$ \cite{Bures}. In the special case $K = N$ this fact shows a relation between the Ginibre ensemble of non Hermitian random matrices \cite{Ginibre} and the Hilbert-Schmidt measure.

The Bures measure in the simplex of eigenvalues may be derived from an assumption that any ball in the sense of the Bures distance of a fixed radius belonging to the set $M_N$ has the same volume. The Bures probability distribution in the simplex of eigenvalues was obtained by Hall \cite{Hall}

$$P_B(\vec{\lambda}) = C_N \frac{\delta(\lambda_1 + \lambda_2 + \ldots + \lambda_N - 1)}{\sqrt{\lambda_1 \lambda_2 \ldots \lambda_N}} \prod_{i<j} (\lambda_i - \lambda_j)^2.$$  \hspace{1cm} (2.4)

The normalization constants $C_N$ were found by Slater \cite{Slater} for low values of $N$, while the general formula

$$C_N = 2^{N^2-N} \frac{\Gamma(N^2/2)}{\pi^{N/2} \prod_{j=1}^{N} \Gamma(j + 1)}$$  \hspace{1cm} (2.5)

was derived in \cite{Slater}. The volume of the set of mixed quantum states and the area of its boundary were computed with respect to both measures in \cite{Slater} \cite{Haake}.

### III. EXPECTATION VALUES

To characterize, to what extent a given state $\rho$ is mixed one may use the moments, $\text{Tr}\rho^q$, with any $q > 0$. The simplest to compute is the second trace $r = \text{Tr}\rho^2$, called purity, which is closely related to the linear entropy $1 - r$ and inverse participation ratio, $R = 1/r$. Mean purity averaged over the HS measure is smaller then the average over the Bures measure,

$$\langle \text{Tr}\rho^2 \rangle_{HS} = \frac{2N}{N^2 + 1} < \langle \text{Tr}\rho^2 \rangle_B = \frac{5N^2 + 1}{2N(N^2 + 2)}.$$  \hspace{1cm} (3.1)

This result reflects the fact that the Bures measure is more concentrated on the states of higher purity, than the Hilbert-Schmidt measure \cite{Bures}. It is not so simple to get such results for an arbitrary exponent $q$. However, it is easier to perform averaging in the asymptotic regime, $N >> 1$. Mean traces, averaged over the Hilbert-Schmidt measure are

$$\langle \text{Tr}\rho^q \rangle_{HS} = N^{1-q} \frac{\Gamma(1+2q)}{\Gamma(1+q)\Gamma(2+q)} \left(1 + O\left(\frac{1}{N}\right)\right).$$  \hspace{1cm} (3.2)
The analogous average with respect to the Bures measure reads

$$\langle \text{Tr}\rho^3 \rangle_B = N^{1-2q} \frac{\Gamma[(3q+1)/2]}{\Gamma[(1+q)/2] \Gamma(2+q)} \left( 1 + O\left( \frac{1}{N} \right) \right) .$$

Again we find $\langle \text{Tr}\rho^3 \rangle_{\text{HS}} < \langle \text{Tr}\rho^3 \rangle_B$. As a measure of the degree of mixing one often uses the von Neumann entropy, $S(\rho) = -\text{Tr}\rho \ln \rho$. It varies from $S(\rho) = 0$ for any pure state and $S(\rho) = \ln N$ for the maximally mixed state. Since $S(\rho) = -\lim_{q \to 1} \partial \text{Tr}\rho^q / \partial q$ the mean von Neumann entropy may be obtained by differentiation of (3.2) and (3.3) with respect to the parameter, $\langle S \rangle = -\lim_{q \to 1} (\partial \text{Tr}\rho^q / \partial q) = -\lim_{q \to 1} \partial \langle \text{Tr}\rho^q \rangle / \partial q$. The results are

$$\langle S(\rho) \rangle_{\text{HS}} = \ln N - \frac{1}{2} + O\left( \frac{\ln N}{N} \right) ,$$

for the Hilbert–Schmidt measure and

$$\langle S(\rho) \rangle_B = \ln N - \ln 2 + O\left( \frac{\ln N}{N} \right) ,$$

for the Bures measure. Note that the former result is larger, since the HS measure favors more mixed states. Although the mean value of the traces which respect to the HS measure appeared several times in the literature [8, 9, 10], the results for the Bures measure are new. Their derivation is sketched in appendix A. Using the expansion of the generating functions there, it is possible to give some more moments for the Bures measure. We compare them with the previously known Hilbert–Schmidt averages

$$\langle \text{Tr}\rho^3 \rangle_{\text{HS}} = \frac{5N^2 + 1}{(N^2 + 1)(N^2 + 2)} , \quad \langle \text{Tr}\rho^3 \rangle_B = \frac{8N^2 + 7}{(N^2 + 2)(N^2 + 4)} .$$

$$\langle \text{Tr}\rho^4 \rangle_{\text{HS}} = \frac{14N^3 + 10N}{(N^2 + 1)(N^2 + 2)(N^2 + 3)} , \quad \langle \text{Tr}\rho^4 \rangle_B = \frac{21(11N^4 + 25N^2 + 4)}{8N(N^2 + 2)(N^2 + 4)(N^2 + 6)} .$$

### IV. DISTRIBUTION OF EIGENVALUES

We are going to evaluate the distribution of the rescaled eigenvalue $x := N\lambda_1$ in the limit of large dimension $N$ of density matrices. To derive the probability distribution $P(x)$, we analyze $q$-th moments of these distributions. For the Hilbert–Schmidt measure we obtain

$$f_{\text{HS}}(q) = \int P_{\text{HS}}(x)x^q dx = \frac{1}{\pi} 2^{2q} \frac{\Gamma(q+1/2)\Gamma(1/2)}{\Gamma(q+2)} ,$$

while the moments for the Bures measure are

$$f_B(q) = \int P_B(x)x^q dx = \frac{1}{2\pi} 2^{3q/2} \frac{\Gamma(q/2 + 1/6)\Gamma(q/2 + 5/6)}{\Gamma(q+2)} .$$

The above results follow from eqs. (3.2) and (3.3) by use of the duplication and triplication formula for the Gamma function. They allow us to obtain the explicit form of the level density, exact in the asymptotic limit of large $N$. The distribution obtained for the HS measure

$$P_{\text{HS}}(x) = \frac{1}{2\pi} \sqrt{\frac{4}{x} - 1} \quad \text{for} \quad x \in [0, 4]$$

diverges as $x^{-1/2}$ for $x \to 0$ and becomes a quarter–circle law in the rescaled variable, $y = \sqrt{x} - $ see Fig. 1. It is comforting to verify that this law forms a special case of the distribution obtained by Page for the induced measures [10] and later derived in a different context in [15]. On the other hand, the distribution for the Bures measure

$$P_B(x) = \frac{3}{4a\pi} \left[ \frac{a}{x} + \sqrt{\frac{(a/x)^2 - 1}{x}} \right]^{2/3} - \left( \frac{a}{x} - \sqrt{\frac{(a/x)^2 - 1}{x}} \right)^{2/3} \quad \text{for} \quad x \in (0, a]$$

for $a > 0$. The expression is valid for $x^q$ with $q$ being an integer, while the continuous distribution is obtained for $0 < q < 1$. It satisfies the conditions

$$P_B(x) \rightarrow \delta(x) \quad \text{for} \quad x \rightarrow 0 , \quad \int P_B(x) dx = 1 .$$
is defined on a larger support, $x \leq a = 3\sqrt{3}$, and diverges for $x \to 0$ as $x^{-2/3}$. Level repulsion for the Bures ensemble compared to the HS ensemble will be reduced at $x = 0$ but enhanced at the maximum of the spectrum.

The above distributions may be derived in an alternative way by minimization of the action functional

$$A_{\text{HS}} = -\int dx dx' P(x)P(x') \ln |x-x'|$$

for the HS measure, and

$$A_{\text{B}} = A_{\text{HS}} + \frac{1}{2} \int dx dx' P(x)P(x') \ln (x+x') .$$

for the Bures measure. Both (unknown) solutions of these minimization problems should satisfy the normalization condition, $\int P(x) dx = 1$ and the relation induced by the unit trace constraint, $\int x P(x) dx = 1$. Both conditions can be implemented with the help of Lagrange multipliers. The resulting integral equations for $P(x)$ may be solved by the Green function

$$G(t) = \int dx \frac{P(x)}{x-t}$$

where the cut along the real axis gives the densities (4.5, 4.6). The Hilbert–Schmidt measure leads to a rather simple Green function

$$G_{\text{HS}}(t) = \frac{1}{2} \left( \sqrt{1 - \frac{4}{t}} - 1 \right),$$

for $t < 0$ and otherwise given by analytic continuation, corresponding to (4.8). The Green function corresponding to the Bures measure is more complicated

$$G_{\text{B}}(t) = \frac{1}{6} \left( z + \frac{1}{z} - 1 \right) \quad \text{with} \quad z = \left( -\frac{a}{t} \right)^{2/3} \left( 1 - \sqrt{1 - \left( \frac{t}{a} \right)^2} \right)^{2/3}$$

for $-a < t < 0$ (otherwise given by analytic continuation) and leads to the distribution (4.9). The Green functions fulfill the generalized Pastur equations

$$G_{\text{HS}}(t) = \frac{-1/t}{1+G_{\text{HS}}(t)} \quad \text{and} \quad G_{\text{B}}(t) = \frac{-1/t}{\sqrt{1+2G_{\text{B}}(t)}} .$$

which suggests an interpolation formula between the Hilbert–Schmidt measure, ($\alpha = 1$), and the Bures measure, ($\alpha = 2$) :

$$G_{\alpha}(t) = \frac{-1/t}{1+\alpha G_{\alpha}(t)^{1/\alpha}} .$$

It would be interesting to analyze the family of interpolating measures which lead to the above Green functions.
V. EIGENVALUE DENSITY AND EIGENVALUE CORRELATION FOR INDUCED MEASURES

In this section we are going to investigate statistical properties of the induced measures defined in the space $\mathcal{M}_N$ of density matrices of size $N$. An integer $K \geq N$, represents the size of an environment and may be treated as a parameter labeling the measure.

A. Eigenvalue density

The one-point density $P(\lambda)$ is obtained from the Green function

$$G(\lambda) = \left\langle \frac{1}{N} \sum_i \frac{1}{\lambda_i - \lambda} \right\rangle_{N,K} \quad \text{with} \quad P(\lambda) = \frac{1}{\pi} \text{Im} G(\lambda + i\delta) ,$$

and the Green function will be derived from the generating function

$$Z(\mu) = \left\langle \prod_i \left( \frac{\lambda_i - \mu}{\lambda_i - \lambda} \right) \right\rangle_{N,K} \quad \text{with} \quad G(\lambda) = -\frac{1}{N} \frac{\partial}{\partial \mu} Z|_{\mu=\lambda} .$$

Due to the structure of $P_{N,K}$ with the van der Monde determinant we may write $Z(\mu)$ as inverse Laplace transform of a determinantal function:

$$Z(\mu) \propto \int_{-\infty+\epsilon}^{+\infty+\epsilon} ds \frac{e^s}{2\pi i} \det \left( \int_0^\infty dx e^{-sx} x^{K-N+i+j-2} \left( \frac{x - \mu}{x - \lambda} \right) \right)$$

with $i, j = 1, 2, \ldots, N$. From this we immediately obtain

$$G(\lambda) = \frac{\Gamma(KN)}{N} \int_{-\infty+\epsilon}^{+\infty+\epsilon} ds \frac{e^s}{2\pi i} \sum_{i,j=1}^N W_{j,i}^{-1} \int_0^\infty dx \frac{e^{-sx} x^{K-N+i+j-2}}{(x - \lambda)^s} .$$

Here the matrix $W$ of size $N$ is given by

$$W_{i,j} = \Gamma(K - N + i + j - 1) .$$

An explicit form of the matrix $W^{-1}$ reads

$$(W^{-1})_{i,j} = (-1)^{i+j} \sum_{k=\max(i,j)}^N \left( \begin{array}{c} k-1 \\ i-1 \end{array} \right) \left( \begin{array}{c} k-1 \\ j-1 \end{array} \right) \frac{\Gamma(K - N + k)}{i \Gamma(K - N + i) \Gamma(K - N + j)} .$$

It turns out that the $x$ integral in Eq. (5.4) contributes only for $x < 1$ and we obtain as result

$$G(\lambda) = \frac{\Gamma(KN)}{N} \sum_{i,j=1}^N W_{j,i}^{-1} \int_0^1 dx \frac{x^{K-N+i+j-2}(1-x)^{KN-(K-N+i+j)}}{\Gamma[KN - (K - N + i + j - 1)](x - \lambda)} .$$

The normalization constant in eqs. (5.5) has been restored by the known asymptotic behavior of $G(\lambda)$. Thus we immediately obtain the density

$$P(\lambda) = \frac{\Gamma(KN)}{N \Gamma(KN + q)} \sum_{i,j=1}^N W_{j,i}^{-1} \frac{\lambda^{K-N+i+j-2}(1-\lambda)^{KN-(K-N+i+j)}}{\Gamma(KN - (K - N + i + j - 1))} .$$

The above rather complicated form of obtaining the density already derived by Page [11], allows us to calculate the moments with the help of Euler’s Beta-function

$$\langle \lambda^q \rangle = \frac{\Gamma(KN)}{N \Gamma(KN + q)} \sum_{i,j=1}^N W_{j,i}^{-1} \Gamma(K - N + i + j - 1 + q) .$$
From this follows the mean von Neumann entropy
\begin{equation}
\langle S \rangle = -N\langle \lambda \ln \lambda \rangle = \psi(KN + 1) - \frac{1}{KN} \sum_{i,j} W_{j,i}^{-1} \Gamma(K - N + i + j) \psi(K - N + i + j).
\end{equation}

where $\psi(x)$ is Euler’s Digamma-function $= \Gamma'(x)/\Gamma(x)$, and where we have suppressed the index $N, K$ at the angular brackets. The result for the average entropy has been conjectured by Page [10] and later proved in [17, 18, 19]. It is a rational number. All the formulas are valid for $K \geq N$. Again for $K < N$ one has to interchange $K$ and $N$, obtaining density and moments of the $K$ positive eigenvalues.

One may explicitly give some average traces over the induced measure $\mu \mu, K$
\begin{align}
\langle \text{Tr} \rho^2 \rangle &= \frac{K + N}{KN + 1}, \quad \langle \text{Tr} \rho^3 \rangle = \frac{(K + N)^2 + KN + 1}{(KN + 1)(KN + 2)}, \\
\langle \text{Tr} \rho^4 \rangle &= \frac{(K + N)[(K + N)^2 + 3KN + 5]}{(KN + 1)(KN + 2)(KN + 3)},
\end{align}

the first of which appeared already in the paper of Lubkin [5], the others are consistent with the recent work of Malacarne et al. [20]. Here $\rho$ means again the density matrix and not the eigenvalue density. To find the coefficients of the polynomial in the denominator it is useful to know its order and symmetry as can be found going back to a Gaussian integral writing the density matrix as a matrix of Wishart form $\rho = \psi \psi^\dagger$. In principle they are contained in formula (5.14).

**B. Eigenvalue correlation**

The eigenvalue correlation can be obtained from the Green-function correlation
\begin{align}
\left\langle \frac{1}{N} \sum_i \frac{1}{\lambda_i - \lambda} \frac{1}{\sum_i \lambda_i - \mu} \right\rangle,
\end{align}

which can be derived from the generating function
\begin{align}
\left\langle \prod_i \frac{(\lambda_i - \kappa_1)(\lambda_i - \kappa_2)}{\lambda_i - \lambda} (\lambda_i - \mu) \right\rangle.
\end{align}

The result for the two-eigenvalue density $P(\lambda, \mu)$, which can be obtained along the same lines as in subsection A is then:
\begin{align}
P(\lambda, \mu) = \theta(1 - \lambda - \mu) \frac{\Gamma(KN)}{N(N - 1)} \sum_{\lambda, i, k, l = 1}^N \frac{[W_{j,i}^{-1} W_{l,k}^{-1} W_{i,j}^{-1} W_{l,k}]^{\lambda - N + i + l - 2 \mu - K - N + i + j - l - 1}}{\Gamma(KN - 2K + 2N - i - j - k - l + 2)}. \label{eq:5.15}
\end{align}

We have checked: $P(\lambda) = \int P(\lambda, \mu) d\mu$. In order to prove this, scale $\mu$ with $1 - \lambda$, integrate over $\mu$ with the help of Euler’s Beta–function, and use eq. (5.5). Then the factor $N - 1$ in the denominator of eq. (5.15) cancels and the result is eq. (5.8). The first bracket under the sum in (5.15) ensures level repulsion $\propto (\lambda - \mu)^2$ for $\lambda - \mu \to 0$. Furthermore there is additional repulsion from the boundaries at $\lambda = 0, \mu = 0, 1 - \lambda - \mu = 0$.

It is again easy to calculate the moments with the help of Euler’s Beta–function
\begin{align}
\langle \lambda^L \mu^M \rangle = \frac{\Gamma(KN)}{N(N - 1) \Gamma(KN + L + M)} \sum_{\lambda, i, k, l = 1}^N \frac{[W_{j,i}^{-1} W_{l,k}^{-1} W_{i,j}^{-1} W_{l,k}]^{\lambda - N + i + j - 1 + L}}{\Gamma(KN + L + M)}. \label{eq:5.16}
\end{align}

For the entropy correlation we have
\begin{align}
\langle S S \rangle &= N(N - 1) \langle \lambda \ln \lambda \rangle \langle \mu \ln \mu \rangle + N \langle \lambda^2 \ln \lambda^2 \rangle,
\end{align}

which can be obtained by double differentiation of $\langle \lambda^L \mu^M \rangle$ and $\langle \lambda^{L+M} \rangle$ with respect to $L$ and $M$ at $L = M = 1$. Again one may obtain formulas for $K < N$ by interchange of $K$ and $N$. 

VI. CONCLUDING REMARKS

It is well known that there is no single, naturally distinguished probability measure in the set of mixed quantum states of a fixed size $N$. Guessing a mixed state on random without any additional information whatsoever, it is legitimate to use the Bures measure (2.4), related to the statistical distance and distinguishability. On the other hand, if it is known that the mixed state has arisen by the partial tracing over a $K$ dimensional environment, one uses the induced measure (2.3), which reduces to the Hilbert-Schmidt measure in the special case $K = N$.

In this work we investigated statistical properties of ensembles of density matrices of a fixed size generated according to the Bures or the Hilbert-Schmidt measure. We computed the averages over the set of mixed quantum states with respect to both measures and derived the level density in the asymptotic limit of large matrices. Furthermore, for measures obtained from random pure states of a composite system by partial tracing we computed the one-point eigenvalue density, the exact two-point eigenvalue density, the corresponding moments and average entropies. On one hand, results concerning average traces and average entropy may be useful from the point of view of the theory of quantum information [21]: the von Neumann entropy of a mixed state $\rho$ is equal to the entanglement of the pure state $|\Psi\rangle$ belonging to a composed Hilbert space, which purifies $\rho$. On the other hand, results obtained contribute to the theory of random matrices: the ensembles of random density matrices distributed according to the Bures measure display different properties than the standard Gaussian ensembles of Wigner and Dyson [11].

It is worth to add that the ensembles of random states analyzed in this work do not cover all the cases of a physical interest. For instance it is natural to assume that in a concrete experiment a mixed state $\rho$ is formed by applying a known quantum channel $\Phi$ (completely positive, trace preserving map) on a random pure state

$$\rho = \Phi(|\psi\rangle\langle\psi|).$$  

(6.1)

Without any information concerning the pure state $|\psi\rangle \in \mathcal{H}_N$ one has to assume that it is generated according to the natural, Fubini–Study measure in the set of pure states. In this manner any quantum channel $\Phi$ induces by (6.1) a certain measure in the space of mixed quantum states. Hence it would be interesting to repeat the computation performed in this work for ensembles of mixed states obtained by physically motivated quantum channels. Such a research will be a subject of a forthcoming publication.

It is a pleasure to thank D. Savin for fruitful discussions. Financial support by the Sonderforschungsbereich/Transregio 12 der Deutschen Forschungsgemeinschaft and Komitet Badań Naukowych in Warsaw under the grant 1 P03B 042 26 is gratefully acknowledged.

APPENDIX A: MOMENTS FOR BURES MEASURE

One may derive all moments for the Bures distribution $P_B(\rho)$ from a Laguerre type ensemble

$$P_B^L(\rho) \propto \theta(\rho)e^{-\text{Tr} \rho} \prod_{i,j} (\lambda_i + \lambda_j)^{-1/2},$$  

(A1)

where $\lambda_i$ denote eigenvalues of $\rho$. Moments are related by

$$\int M_\rho(\rho) P_B(\rho) D\rho = \frac{\Gamma(N^2/2)}{\Gamma(N^2/2 + p)} \int M_\rho(\rho) P_B^L(\rho) D\rho,$$  

(A2)

where $M_\rho(\rho)$ is a homogeneous function of $\rho$ of degree $p$ and $D\rho$ is the matrix volume element of a Hermitian matrix. Next we may write

$$P_B^L(\rho) \propto \theta(\rho) \int DA \ e^{-\text{Tr} [\rho (1 + A^2)]}$$  

(A3)

with a Hermitian matrix $A$. Let us denote its eigenvalues by $\{A_i\}$. In the following we use the formula

$$\theta(\rho)e^{-\text{Tr}(\rho \epsilon)} = B_N \det(\delta/\delta \rho + \epsilon)^{-N} \delta(\rho)$$  

(A4)

with a positive definite Hermitian matrix $\epsilon$ and

$$B_N = \pi^{N(N-1)/2} \Gamma(1)\Gamma(2)\ldots\Gamma(N).$$  

(A5)
Note that on the left hand side of Eq. (A4) there is the restriction \( \rho \geq 0 \), while on the right hand side we have no restriction on integration for \( \rho \). With the above formula it is easy to compute the matrix Laplace transform

\[
\int e^{-\text{Tr}(E\rho)} P^L_B(\rho) D\rho \propto \int DA \det(E + 1 + A^2)^{-N}
\]

(A6)

with a nonnegative matrix \( E \) of size \( N \). For \( E = 0 \) this formula leads to the normalization constant for the Bures measure \( \mathbb{B} \). Let \( \{E_1, \ldots, E_N\} \) denote the eigenvalues of \( E \). With the help of the Itzykson-Zuber integral the right hand side of Eq. (A6) is proportional to

\[
\int dA_1 \ldots dA_N \frac{[\Delta(A)]^2}{\Delta(A^2) \Delta(-E)} \det\left[ \frac{1}{1 + A_i^2 + E_j} \right]
\]

(A7)

with the van der Monde determinant \( \Delta(A) = \prod_{i<j}(A_i - A_j) \). Finally one may perform the \( A_i \) integrations in the complex plane, arriving at the generating function

\[
Z^L_B(E) = \int e^{-\text{Tr}(E\rho)} P^L_B(\rho) D\rho = \prod_{i=j}^{1 \ldots N} \frac{2}{\sqrt{1 + E_i} + \sqrt{1 + E_j}}.
\]

(A8)

This expression has a rather simple expansion in powers of \( E \). It starts like

\[
Z^L_B(E) = 1 - \frac{N}{2} \sum E_i + \left( \frac{N^2}{8} + \frac{1}{16} (\sum E_i)^2 + \frac{3N}{16} \sum E_i^2 + O(E^3) \right).
\]

(A9)

Thus it is possible to obtain all moments by matrix derivation, e.g.

\[
\langle \text{Tr} F(\rho) \rangle^L_B = \text{Tr} F(-\delta/\delta E) Z^L_B(E) \big|_{E=0}.
\]

(A10)

The corresponding generating function for the Hilbert-Schmidt measure is even more simple and reads:

\[
Z^L_{HS}(E) = \int e^{-\text{Tr}(E\rho)} P^L_{HS}(\rho) D\rho = \prod_{i=1}^{N} \frac{1}{(1 + E_i)^N}.
\]

(A11)

Its expansion starts like

\[
Z^L_{HS}(E) = 1 - N \sum E_i + \frac{N^2}{2} (\sum E_i)^2 + \frac{N}{2} \sum E_i^2 + O(E^3).
\]

(A12)

To obtain the matrix derivatives \( \delta/\delta E \) is not so easy in general, since everything is expressed in eigenvalues \( E_i \) e.g.

\[
\langle \text{Tr} \rho^\theta \rangle^L_B = \text{Tr} (-\delta/\delta E)^\theta Z^L_B(E) \big|_{E=0} = \frac{1}{\Delta(E)} \sum_{i=1}^{N} (-\partial/\partial E_i)^\theta |\Delta(E)| Z^L_B(E) \big|_{E=0}.
\]

(A13)

One can proof the last equation with the help of the Itzykson–Zuber integral. The above formulae were used to derive the few results for Bures moments in section III.

[1] Życzkowski K and Sommers H-J 2001 J. Phys. A 34 7111
[2] Hall M J W 1998 Phys. Lett. A 242 123
[3] Bures D J C 1969 Trans. Am. Math. Soc. 135 199
[4] Uhlmann A 1992 in Groups and related Topics Gielerak R et. al. (eds.) (Dodrecht: Kluver)
[5] Petz D and Sudár C 1996 J. Math. Phys. 37 2662
[6] Braunstein S L and Caves C M 1994 Phys. Rev. Lett. 72 3439
[7] Uhlmann A 1995 Rep. Math. Phys. 36 461
[8] Lubkin E 1978 J. Math. Phys. 19 1028
[9] Lloyd S and Pagels H 1988 Ann. Phys. (N.Y.) 188 186
[10] Page D 1993 Phys. Rev. Lett. 71 1291
[11] Mehta M L 1991 Random Matrices II ed. (New York: Academic)
[12] Slater P B 1999 J. Phys. A 32 8231
[13] Sommers H-J and Życzkowski K 2003 J. Phys. A 36 10083
[14] Życzkowski K and Sommers H-J 2003 J. Phys. A 36 10115
[15] Sengupta A M and Mitra P P 1999 Phys. Rev. E 60 3389
[16] Pastur L A 1972 Th. Math. Phys. 10 67
[17] Foong S K and Kanno S 1994 Phys. Rev. Lett. 72 1148
[18] Sánchez–Ruiz J 1995 Phys. Rev. E 52 5653
[19] Sen S 1996 Phys. Rev. Lett. 77 1
[20] Malacarne L C and Mandes R S and Lenzi E K 2002 Phys. Rev. E 65 046131
[21] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[22] Itzykson C and Zuber J B 1980 J. Math. Phys. 21 411
[23] Chandra H 1957 Am. J. Math. 79 87