Vibration Analysis of a Simply Supported Beam with Multiple Breathing Cracks

K. Vigneshwaran* and R.K. Behera

Department of Mechanical Engineering, National Institute of Technology, Rourkela-769008, India

*E-mail ID: vikkydesigns@gmail.com

Abstract

The dynamic characteristics of a beam with multiple breathing cracks are studied in this paper. A systematic approach has been adopted in the present investigation to develop theoretical expressions for evaluation of natural frequencies and mode shapes. A simple elastic simply supported beam with two breathing cracks is considered for the dynamic analysis. The stiffness of the cracked beam is found out by using influence coefficients. The influence coefficients are calculated by using castigiliano’s theorem and strain energy release rate (SERR). The equation of motion of the beam was derived by using Hamilton’s principle. The stiffness and natural frequencies for the multiple cracked beam calculated using eigen value approach. It is seen that due to presence of cracks, the stiffness and natural frequency changes. The mode shapes and the FRF for the uncracked, cracked and breathing cracked cantilever beam also obtained and compared. The mode shape changes considerably due to the presence of crack.

Keywords: Breathing crack, Mode shape, Frequency response (FRF), FEM, Vibration analysis.

Introduction

Beams are one of the most commonly used structural elements in numerous engineering applications (ex. aerodynamic structures, tankers and rotors, etc.,) and experience a wide variety of static and dynamic loads. The crack modelling is an important step in studying the behaviour of damaged structures. The majority of published studies assume that the crack in a structural member always remains open during vibration. The cracks are always open in vibration is not realistic because due to repetitive loads the crack may open or close. The relation between the applied load and the strain energy concentration around the tip of crack is explained by Irwin (1957). Qian et al. (1990) developed a finite element model of an edge cracked beam. In his paper, an element stiffness matrix of a beam with crack is first derived from an integration of stress intensity factors. The finite element model is applied to a cantilever beam with an edge crack, and the eigen frequencies are determined for different crack lengths and
locations. Douka et al. (2005) investigated time–frequency analysis of the free vibration response of a beam with a breathing crack using the empirical mode decomposition and Hilbert transform for single degree of freedom system (SDOF) of a beam. The influence of a crack in a welded joint on the dynamic behaviour of a structural member is discussed by Chondros and Dimaragonas (1980). The vibration analysis of a cracked beam and cracked shaft have been studied by Dimarogonas et.al (1983).They derived the flexibility matrix of the cross section containing the crack. The vibrational response of the harmonic force of a cantilever beam with cracks of different size and locations has been analysed by Pugno et al. (2000) using harmonic balance method. They analysed and compared the results for the free end dynamic response of beam with numerical integration. The effects of closure of cracks on the dynamics of a cracked cantilever beam have been analysed by Kisa et al. (2000) using the finite element method, component mode synthesis and time stepping scheme. Modelling of a beam with a breathing edge crack subjected to harmonic loading has been investigated by Nundi et al. (2002). They considered the frictionless contact model with single degree of freedom. Andreaus et al. (2007) studied the nonlinear dynamics of a cracked cantilever beam under harmonic excitation using two dimensional finite elements. They considered the frictionless contact model and single degree of freedom to solve the problem and outlined the response with respect to the linear one. Dynamic characteristics of a cantilever beam with transverse cracks are analytically investigated by Behera et al. (2006) using the contour plots for finding out the various effect of cracks. In this thesis finite element method is proposed to model the beams with a breathing crack using fracture mechanics methods. In modelling, the effect of the crack on the deformation of a beam has often been considered similar to that of an elastic hinge (or) plastic hinge. The behaviour of beam with crack and without crack is studied. An investigation of crack depth and its location are provided here directly by using the mode shape analysis.

2. Finite Element Formulation

2.1. Derivation of Shape Function

For finite element formulation, the beam element of fig. (2) is taken here with the length of \( l \) and axial load co-ordinate \( x \) and transverse local co-ordinate \( y \). The local transverse nodal displacements are assumed as \( u_1 \) and \( u_2 \). The rotations are given by \( \phi_1 \) and \( \phi_2 \). The local nodal forces are given by \( F_1 \) and \( F_2 \). The bending moments are given by \( m_1 \) and \( m_2 \). For simplicity, the beam is divided into 21 numbers of discretized finite elements as shown in fig. (1).

Figure 1a. Schematic diagram of a beam.  
Figure 1b. Beam element with positive
nodal displacements, rotations, forces, and moments.

From fig. 1(b), the transverse displacement variation through the element length to be approximated as a polynomial function of quadratic equation as,

\[ v(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0. \]  

(1)

The above equation can be rewritten in matrix form of \( v(x) = [N]{\delta}^e \) as,

\[ v(x) = [N_1\quad N_2\quad N_3\quad N_4]\begin{bmatrix} u_1 \\ u_2 \\ \phi_1 \\ \phi_2 \end{bmatrix}. \]  

(2)

Where, 

\( N_1, N_2, N_3 \), and \( N_4 \) are shape functions for beam element.

\begin{align*}
N_1 &= \frac{1}{l^3} \left( 2x^3 - 3x^2l + l^3 \right), \quad N_2 = \frac{1}{l^2} \left( x^2l - 2x^2l^2 + x^3l \right), \quad N_3 = \frac{1}{l^3} \left( -2x^3 + 3x^2l \right), \quad N_4 = N_2 = \frac{1}{l^3} \left( x^3l - x^2l^2 \right).
\end{align*}

### 2.2. The Stiffness Matrix for Beam Element

The element stiffness matrix of the beam element can be derived as,

\[ [K_e] = \begin{bmatrix}
12 & 6l & -12 & 6l \\
6l & 4l^2 & -6l & 2l^2 \\
-12 & -6l & 12 & -6l \\
6l & 2l^2 & -6l & 4l^2
\end{bmatrix}. \]  

(3)

### 2.3. Mass Matrix for the Beam Element

The shape function is used as the weight functions in Galerkin’s method. Substituting the shape functions in the weak form equation which is given as below,

\[ \int W(x)EI \frac{d^2v}{dx^2} \, dx - \int \frac{dW}{dx} EI \frac{d^2v}{dx^2} \, dx + \int EI \frac{d^2v}{dx^2} \frac{d^2W}{dx^2} \, dx - \int \rho A \sigma^2 W(x)v(x) \, dx = 0. \]  

(4)

and evaluating the integrals with respect to each of the weighing functions one can obtain the governing equations at a time and rewrite those terms in matrix, one can obtain the element mass matrix of an Euler Bernoulli beam as,

\[ [M_e] = \frac{\rho A l}{420} \begin{bmatrix}
156 & 22l & 54 & -13l \\
22l & 4l^2 & 13l & -3l^2 \\
54 & 13l & 156 & -22l \\
-13l & -3l^2 & -22l & 4l^2
\end{bmatrix}. \]  

(5)

Where, \( l \) = Length of the beam element in m, \( \rho \) = Density of the beam material in kg/m^3 and \( A \) = Area of cross section in m^2. Only a transverse crack under bending has been used to establish the element stiffness matrix.

### 3. Modelling of Cracks

A cantilever beam of length \( L \), of uniform rectangular cross section \( b \times h \) with a crack located at positions \( l \), is considered (Fig. 1). The crack is assumed to be open and close periodically. The depth of crack is \( a \). The beam is subjected to transverse & bending load, \( P \) & \( M \) respectively.
3.1. Local Flexibility of a Cracked Beam under Bending

The presence of a transverse crack of depth \( a \) introduces a local flexibility matrix. The dimension of the local flexibility matrix is \( (2 \times 2) \), as each side has two degrees of freedom. The off-diagonal elements of the flexibility matrix are considered as coupling elements. According to the principle of Saint Venant, the stress field only affects the region which is adjacent to the crack. The flexibility co-efficient is expressed by means of stress intensity factor which can be easily derived by using Castigliano’s theorem in the linear elastic range. With shearing action neglected, the strain energy of an element without a crack can be written as,

\[
W^{(0)} = \int_0^l \frac{M^2}{2EI} dx. \tag{6}
\]

Where,
- \( I \) = Moment of inertia of the uniform beam,
- \( E, E' \) = Elastic modulus and
- \( M \) = Bending moment due to the applied force.

Total bending moment due to the transverse force can be written as (see fig. (1))

\[
M = P_x + M. \tag{7}
\]

Where
- \( P_x \) = Transverse Load and \( M \) = Total Bending moment.

Therefore eq. (6) can be rewritten as,

\[
W^{(0)} = \int_0^l \frac{(P_x + M)^2}{2EI} dx. \tag{8}
\]

After simplification the strain energy of an element without a crack can be obtained as,

\[
W^{(0)} = (M^2 l + MPl^2 + P^2 l^3/12)/2EI. \tag{9}
\]

Where, \( l \) is the Length of the beam element. The additional strain energy due to the crack can be written as (Tada H (1973), Qian et al. (1990)),

\[
W^{(1)} = \int_0^l \left[ \left( \frac{K_1^2 + K_2^2}{E} \right) + (1 + \nu) \frac{K_{mn}^2}{E} \right] da. \tag{10}
\]

Where \( E = E / (1 - \nu^2) \) for plain strain and \( E' = E \) for plane stress problem condition. \( E \) is the Young’s modulus of elasticity and \( \nu \) is the Poisson’s ratio. \( K_1, K_2, K_{mn} \) are stress intensity factors for opening, sliding and tearing mode cracks. The amplitude of vibration is assumed to be well below the crack opening due to preloading.

With the action of axial force neglected, the above eq. (10) becomes,

\[
W^{(1)} = \int_0^l \left[ \left( \frac{(K_{1m} + K_{1p})^2 + K_{2m}^2}{E} \right) \right] da. \tag{11}
\]

The stress intensity factors from elementary fracture mechanics are given as,

\[
K_{1m} = (6M/bh^2)\sqrt{\pi a} F_1(s), \quad K_{1p} = (3Pl/bh^2)\sqrt{\pi a} F_1(s) \quad \text{and} \quad K_{1m} = (P/bh)\sqrt{\pi a} F_1(s). \tag{12}
\]

Where \( b \) and \( h \) are the cross-section dimensions and \( a \) is the crack depth as shown in fig. (1). The relative crack depth, the crack location from fixed end and the relative crack location are denoted as \( s = (a/h) \), \( l \), and \( l' = (l/L) \) respectively. The functions \( F_1(s) \) and \( F_p(s) \) are dependent on the crack depth \( a \) and are approximated (Stephen H C 1978) by

\[
F_1(s) = \sqrt{2/\pi s} \tan (\pi s/2) \frac{0.923 + 0.199[1 - \sin(\pi s/2)]^4}{\cos(\pi s/2)} \quad \text{and} \quad F_2(s) = (3s - 2s^3) \frac{1.122 - 0.561s + 0.08s^2 + 0.18s^3}{\sqrt{1 - s}}.
\]

Here we consider the \( F_1(s) \) for calculating the stress intensity factors for open mode crack. In addition, the crack produces a local additional displacement \( u_i \) between the right and left sections of the crack, in a similar way as an
equivalent spring. These displacements \( U_i \) in the \( i \) direction, under the action of force \( P_i \) are given, according to Castigiliano’s theorem. The flexibility co-efficient for an element without a crack is,

\[
c_{ij}^{(0)} = \frac{\partial^2 W^{(0)}}{\partial P_i \partial P_j}, \quad (13)
\]

Substituting \( P_1 = P, P_2 = M \); \( i, j = 1, 2 \) in eq. (13) one can obtain the resulting equation as,

\[
c_{11}^{(0)} = \frac{I^2}{3EI}, \quad c_{12}^{(0)} = c_{21}^{(0)} = \frac{I^2}{2EI}, \quad \left( \frac{\partial^2 W^{(0)}}{\partial P \partial M} = \frac{\partial^2 W^{(0)}}{\partial M \partial P} \right) \text{ and } \quad c_{22}^{(0)} = \frac{I}{EI}.
\]

Similarly the additional flexibility co-efficient for the cracked element can be obtained as,

\[
c_{ij}^{(1)} = \frac{\partial^2 W^{(1)}}{\partial P \partial P_j}, \quad (14)
\]

Therefore the total flexibility co-efficient for the cracked beam is,

\[
C = c_{ij}^{(0)} + c_{ij}^{(1)}. \quad (15)
\]

The order of the flexibility matrix is \( 2 \times 2 \). From the equilibrium condition of the cracked beam element as shown in fig. 1(b) one can have

\[
(P_i, M_i, P_{i+1}, M_{i+1}) = [T](P_{i+1}, M_{i+1}). \quad (16)
\]

Where,

\[
[T] = \text{Transformation matrix and is given by}
\]

\[
[T] = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}. \quad (17)
\]

Using the principle of virtual work, the stiffness matrix of the cracked element can be obtained as,

\[
[K_c] = [T]^T C^{-1} [T]. \quad (18)
\]

3.2. Mathematical Modelling of Breathing Crack

The stiffness of a structure with breathing crack changes with respect to the time and it can be expressed as a continuous function of time in the form,

\[
K(t) = K_c + K_{nc}(1 + \cos \omega t).
\]

4. Equation of Motion of an Euler Bernoulli Beam

Hamilton’s equation can be written as below,

\[
\delta \{ T (U) + \delta W_{nc} \} = 0 \quad (19)
\]

Substituting the values of \( T, U \) and \( \delta W_{nc} = f \delta u - c \delta \dot{u} \) as in eq. (19) one can obtain the equation of motion of a beam as

\[
M \ddot{u} + \left( K_c + K_{nc}(1 + \cos \omega t) \right) u = 0. \quad (20)
\]

The non-conservative forces are not considered here.

4.2. Modal Analysis

The global stiffness matrices \([K]\) and the global mass matrices \([M]\) for the beam is obtained by means of applying the boundary conditions for the beam as, \( U = 0 \) and \( \theta \neq 0 \) for simply supported beam. The natural frequencies (eigen value) and the mode shapes (eigen vector) are obtained by means of using Eigen value analysis.
using matlab (using the command eig (K, M)) using finite element methodology. The crack only affects the stiffness of the structure. There is a minor difference \( \Delta K \) between the general stiffness matrixes of the cracked and uncracked beam.

5. Numerical Analysis

For numerical simulations, mild steel beam of total length 0.8m and rectangular cross section 0.05×0.006m is considered. A crack of varying depth is introduced at \( l_1 = 0.05, 0.1, 0.2, 0.3, 0.4 \) and 0.60 from the fixed end. A Young’s modulus of \( 2.1 \times 10^{11} \text{ N/m}^2 \), a density of \( 7850 \text{ kg/m}^3 \), a Poisson ratio of 0.3 and a damping factor of 0.01 were used. The dynamic response of the beam was obtained. The initial conditions were taken as \( u'(0) = 0 \text{ m/s} \) and \( u(0) = 0 \text{ m} \). The numerical results are presented in the tabular & graphical form.

**Table 1. Natural frequencies of an open cracked and breathing cracked beam.**

| Sl no | First natural frequency \( \omega_1 \) (rad/sec) | Second natural frequency \( \omega_2 \) (rad/sec) | Third natural frequency \( \omega_3 \) (rad/sec) |
|-------|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| 1     | 138.1514                                     | 552.6086                                    | 1243.397                                    |

**Table 2. Natural frequencies of an uncracked beam.**

| Sl no | Crack location | Crack depth | First natural frequency \( \omega_1 \) (rad/sec) | Second natural frequency \( \omega_2 \) (rad/sec) | Third natural frequency \( \omega_3 \) (rad/sec) |
|-------|----------------|-------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| 0.001 | 0.076          | 0.001       | 138.149                                    | 138.150                                    | 552.5748                                    | 552.5917                                    | 1243.258                                    | 1243.328                                    |
|       |                | 0.002       | 138.1468                                   | 138.1491                                   | 552.5443                                    | 552.5765                                    | 1243.139                                    | 1243.268                                    |
|       |                | 0.003       | 138.1296                                   | 138.1407                                   | 552.3075                                    | 552.4597                                    | 1242.214                                    | 1242.812                                    |
| 0.002 |                | 0.001       | 138.1469                                   | 138.1492                                   | 552.5431                                    | 552.576                                    | 1243.113                                    | 1243.255                                    |
|       |                | 0.002       | 138.1447                                   | 138.1481                                   | 552.5126                                    | 552.5607                                    | 1242.993                                    | 1243.196                                    |
|       |                | 0.003       | 138.1275                                   | 138.1396                                   | 552.2758                                    | 552.444                                    | 1242.069                                    | 1242.74                                     |
| 0.003 | 0.001         | 0.001       | 138.1418                                   | 138.1467                                   | 552.466                                     | 552.538                                    | 1242.758                                    | 1243.081                                    |
|       |                | 0.002       | 138.1396                                   | 138.1456                                   | 552.4354                                    | 552.5228                                    | 1242.638                                    | 1243.021                                    |
|       |                | 0.003       | 138.1224                                   | 138.1371                                   | 552.1987                                    | 552.406                                    | 1241.714                                    | 1242.565                                    |
| 0.011 | 0.001         | 0.001       | 138.1457                                   | 138.1486                                   | 552.5495                                    | 552.5791                                    | 1243.258                                    | 1243.328                                    |
|       |                | 0.002       | 138.1393                                   | 138.1454                                   | 552.4868                                    | 552.5478                                    | 1243.139                                    | 1243.268                                    |
|       |                | 0.003       | 138.0892                                   | 138.1207                                   | 552.0012                                    | 552.3081                                    | 1242.218                                    | 1242.813                                    |
| 0.012 | 0.001         | 0.001       | 138.1436                                   | 138.1475                                   | 552.5179                                    | 552.5633                                    | 1243.113                                    | 1243.255                                    |
|       |                | 0.002       | 138.1372                                   | 138.1443                                   | 552.4552                                    | 552.5321                                    | 1242.994                                    | 1243.196                                    |
|       |                | 0.003       | 138.0871                                   | 138.1196                                   | 551.9697                                    | 552.2924                                    | 1242.073                                    | 1242.741                                    |
| 0.013 | 0.001         | 0.001       | 138.1385                                   | 138.145                                    | 552.4407                                    | 552.5254                                    | 1242.758                                    | 1243.081                                    |
|       |                | 0.002       | 138.1321                                   | 138.1418                                   | 552.378                                     | 552.4941                                    | 1242.639                                    | 1243.021                                    |
|       |                | 0.003       | 138.0832                                   | 138.1171                                   | 551.8927                                    | 552.2545                                    | 1241.72                                     | 1242.566                                    |
Figure 2. Schematic diagram of an element.

Figure 3(a). First mode of transverse vibration, $s=0.5$ and $l_c=0.5$.

Figure 3(b). Magnified view of first mode of transverse, $s=0.5$ and $l_c=0.5$.

Figure 4. Second mode of transverse vibration, $s=0.5$ and $l_c=0.05$.

Figure 5(a). Third mode of transverse vibration, $s=0.5$ and $l_c=0.05$.

Figure 5(b). Magnified view of third mode of transverse vibration, $s=0.5$ and $l_c=0.05$.

Figure 6. FRF for uncracked, open cracked and breathing cracked beam for $s=0.5$ and $l_c=0.05$ within the range of 0 Hz to 1000 Hz.
6. Results and Discussion

The transverse natural frequencies for the open and breathing cracked beam of mild steel are calculated and presented in Table 1. It is noticed that as the relative crack depth increases the natural frequency decrease and the same for uncracked is shown in Table (2). When, the crack position shifts from the supported end to the centre, the natural frequency decreases. For deep crack (s=0.5), the difference between mode shape for breathing crack and open crack are clearly noticed, as shown in Figure 3(a) and Figure 3(b). However the effect of small crack can be obtained by magnified view (Fig. 3(b)) of the first mode shape at the crack location. Comparisons and magnified view of the transverse mode shapes of the uncracked, open cracked and breathing cracked beam are shown in Figures 3(a), 4 & 5(a) and Figure 3(b) & 5(b) respectively. The Frequency response of the open cracked, uncracked and breathing cracked beam are shown in Figure 6 within the frequency range of 0 Hz to 1000 Hz.

7. Conclusions

The stiffness and natural frequency decreases due to the presence of crack. The position of the crack can be identified from the deviation of mode shape between the cracked & uncracked one. For minor crack small change in mode shape takes place. The mode shape changes considerably due to the presence of crack. The FRF also changes due to the change in stiffness of the structure. When, the crack position shifts from the supported end to the centre, the natural frequency decreases. The most significant conclusion is that the increase in relative crack depth decreases the natural frequencies.

References

[1] Andreaus U, Casini P and Vestroni F, Nonlinear dynamics of a cracked cantilever beam under harmonic excitation, International Journal of Nonlinear Mechanics 42 (2007) 566–575.
[2] Behera R K, Parhi D R K and Sahu S K, Dynamic characteristics of a cantilever beam with transverse cracks, International Journal of Acoustics and Vibration 11 (2006) 3-18.
[3] Chondros T G and Dimaragonas A D, Identification of cracks in welded joints of complex structures, Journal of Sound and Vibration 69 (1980) 531-538.
[4] Dimaragonas A D and Paipetis S A, Analytical methods in rotor dynamics, London: Applied Science Publishers (1983) chapter 6.
[5] Douka E and Hadjileontiadis L J, Time frequency analysis of the free vibration response of a beam with a breathing crack. NDT&E International 38 (2005) 3-10.
[6] Ferreira A J M, Matlab codes for finite element analysis. Portugal, Springer publication.
[7] Irwin G R, Analysis of stresses and strains near the end of a crack traversing a plate. Journal of Applied Mechanics 24 (1957) 361-364.
[8] Kapoor K K and Hartz B J, Stability of plates using the finite element method, Journal of Engineering Mechanics Div. Proceedings of the ASCE EM2 (1966) 177-195.
[9] Kassimallli A, Structural Analysis, Pws - Kent Publish Company (1993).
[10] Kisa M and Brandon J, The effects of closure of cracks on the dynamics of a cracked cantilever beam, Journal of Sound and Vibration 238 (2000) 1-18.
[11] Lee W K, Lee K S and Pak C H, Stability analysis for non-planer free vibrations of a cantilever beam by using nonlinear normal modes, Nonlinear Dynamics 52 (2008) 217-225.
[12] Nandi A and Neogy S, Modelling of a beam with a breathing edge crack and some observations for crack detection, Journal of Vibration and Control 8 (2002) 673–693.
[13] Pugno N, Surace C and Ruotolo R, Evaluation of the non-linear dynamic response to harmonic excitation of a beam with several breathing cracks, Journal of Sound and Vibration 235 (2000) 749-762.
[14] Qian G L, Gu S N and Jiang J S, The Dynamic behaviour and crack detection of a beam with a crack, Journal of Sound and Vibration 138 (1980) 233-243.
[15] Seshu P A, Textbook of finite element analysis, New Delhi, PHI learning private limited, (2009).
[16] Stoykov S and Ribeiro P, Vibration analysis of rotating 3D beams by the p-version finite element method, Finite Elements in Analysis and Design 65 (2013) 76–88.
[17] Tada H, Paris P C and Irwin G R, The stress analysis of cracks handbook, Hellertown Pennsylvania, Del Research Corporation (1973).