A POSITIVE MASS THEOREM FOR TWO SPATIAL DIMENSIONS

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Abstract. We observe that an analogue of the Positive Mass Theorem in the time-symmetric case for three-space-time-dimensional general relativity follows trivially from the Gauss-Bonnet theorem. In this case we also have that the spatial slice is diffeomorphic to \( \mathbb{R}^2 \).

In this short note we consider Einstein’s equation without cosmological constant, that is
\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu},
\]
on \( 1 + 2 \) dimensional space-times. This theory has long been considered as a toy model with possible applications to cosmic strings and domain walls, or to quantum gravity. For a survey please refer to [Bro88, Car98] and references within. This low dimensional theory is generally considered as un-interesting [Col77] due to the fact that Weyl curvature vanishes identically in \( (1 + 2) \) dimensions, a fact often interpreted in the physics literature as the theory lacking gravitational degrees of freedom. Furthermore, the theory does not reduce in a Newtonian limit [BBL86]: the exterior space-time to compact sources is necessarily locally flat and is typically asymptotically conical [Car98, DJtH84, Des85].

For these space-times, by considering point-sources, it is revealed [DJtH84] that the mass should be identified with the angle defects near spatial infinity. For static space-times with spatial sections diffeomorphic to \( \mathbb{R}^3 \), it is also known that the asymptotic mass can be related to the integral of scalar curvature on the spatial slice, and hence under a dominant energy assumption must be positive.

The purpose of this note is to remark that the topological assumption is unnecessary.

Throughout we shall assume that \((\Sigma, g)\) is a complete two-dimensional Riemannian manifold which represents a time-symmetric spatial slice in a three-dimensional Lorentzian space-time \((M, \bar{g})\) (that is, trace of the second fundamental form of \( \Sigma \leftrightarrow M \) vanishes identically; in other words, the slice is maximal). We assume that the dominant energy condition holds for \( \bar{g} \), and in particular the ambient Einstein tensor satisfies \( \bar{G}_{\mu\nu} \xi^\mu \xi^\nu \geq 0 \) for any time-like \( \xi^\mu \). The Gauss equation then immediately implies that \( g \) has non-negative scalar curvature.

**Definition 1.** A complete two-dimensional Riemannian manifold \((\Sigma, g)\) is said to be asymptotically conical if there exists a compact subset \( K \subseteq \Sigma \) where \( \Sigma \setminus K \) has finitely many connected components, and such that if \( E \) is a connected component of \( \Sigma \setminus K \), there exists a diffeomorphism \( \phi : E \to (\mathbb{R}^2 \setminus \bar{B}(0,1)) \) where in the Cartesian

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coordinates on \( \mathbb{R}^2 \) the line element satisfies
\[
\mathrm{d}s^2 = \left[ \mathrm{d}x^2 + \mathrm{d}y^2 - \frac{1 - P^2}{x^2 + y^2} (x \, \mathrm{d}y - y \, \mathrm{d}x)^2 \right] \in O_2 \left( (x^2 + y^2)^{-\epsilon} \right)
\]
for some \( \epsilon > 0 \) and \( P > 0 \). The notation \( f \in O_2(r^{-2\epsilon}) \) is a shorthand for
\[
|f| + r \left| \partial f \right| + r^2 \left| \partial^2 f \right| \leq Cr^{-2\epsilon}.
\]

**Remark 2.** The decay condition is sufficient to imply that the scalar curvature \( S \) of \( g \) is integrable on \( \Sigma \). Note that in polar coordinates \( x = r \cos \theta \) and \( y = r \sin \theta \) the conical metric in the square brackets can be written as the conical
\[
\mathrm{d}r^2 + P^2 r^2 \mathrm{d}\theta^2
\]
where we see that \( m = 2\pi (1 - P) \) is the angle defect for parallel transport around the tip of the cone.

**Theorem 3.** If \((\Sigma, g)\) is a complete asymptotically conical two-dimensional orientable Riemannian manifold with pointwise non-negative scalar curvature, then \( \Sigma \) is diffeomorphic to \( \mathbb{R}^2 \) and \( m = 2\pi (1 - P) \) is non-negative. If furthermore \( m = 0 \) then \((\Sigma, g)\) is isometric to the Euclidean plane.

**Proof.** Enumerate from \( 1 \ldots N \) the asymptotic ends \((E_i, g_i)\) with diffeomorphisms \( \phi_i \) and constant \( P_i \). By the asymptotic structure, for sufficiently large \( R \), the curve \( \gamma_i = \{ \phi_i^1(x, y) \mid x^2 + y^2 = R_i^2 \} \) in the end \( \Sigma_i \) has positive geodesic curvature, if we choose the orientation so that the inward normal is toward the compact set \( K \).

Let \( \Sigma_0 \supseteq K \) denote the compact manifold with boundary in \( \Sigma \) that is bounded by the \( \gamma_i \). Applying Gauss-Bonnet theorem, using the fact that the geodesic curvatures are all signed and the scalar curvature is non-negative, we have that \( \Sigma_0 \) has positive Euler characteristic. As \( \Sigma_0 \) is orientable and connected, and has nonempty boundary, it must be diffeomorphic to a disc. Hence \( \Sigma \) has only one asymptotic end and is diffeomorphic to \( \mathbb{R}^2 \).

For sufficiently large \( R \) we let \( \Sigma_R \) denote the compact region bounded by \( x^2 + y^2 = R^2 \). Using the Gauss-Bonnet theorem again, along with the decay properties of the metric, we see that \( m \), the angle defect, is in fact given by
\[
m = \lim_{R \to \infty} \frac{1}{2} \int_{\Sigma_R} S \, \mathrm{dvol}_g = \frac{1}{2} \int_{\Sigma} S \, \mathrm{dvol}_g.
\]
Hence \( m \) is necessarily nonnegative, with equality to \( 0 \) only in the case \( S \equiv 0 \). \( \Box \)

**Remark 4.** One can analogously define the “quasilocal mass” \( m_\gamma \) associated to \( \gamma \subseteq \Sigma \) a simple closed curve by letting \( m_\gamma \) be the angle defect for parallel transport around \( \gamma \). Then it is easy to see the this quantity has a monotonicity property: if \( \gamma_1 \) is to the “outside” of \( \gamma_2 \), let \( \Sigma_{1,2} \) be the annular region bounded by the two curves, we must have
\[
m_{\gamma_1} - m_{\gamma_2} = \frac{1}{2} \int_{\Sigma_{1,2}} S \, \mathrm{dvol}_g.
\]

**Remark 5.** It was pointed out to the author by Julien Cortier that some similar considerations in the asymptotically hyperbolic case was mentioned by Chruściel and Herzlich; see Remark 3.1 in \[CH03\].

Indeed, Theorem 3 follows also from some more powerful classical theorems in differential geometry. The topological classification can be deduced from, e.g. Proposition 1.1 in \[L T91\]. One can also deduce the theorem (with some work) from Shiohama’s Theorem A \[Shi85\]. As shown above, however, in the very restricted
case considered in this note the desired result can be obtained with much less machinery.

**Remark 6.** The author would also like to thank Gary Gibbons for pointing out that a similar argument to the proof of Theorem 3 was already used by Comtet and Gibbons (see end of section 2 of [CG88]) to establish a positive mass condition on cylindrical space-times about a cosmic string; the main difference is that in the above theorem we contemplate, and rule out, the possibility of multiple asymptotic ends, as well as non-trivial topologies inside a compact region.

**Remark 7.** One can also ask about asymptotically cylindrical spaces, which can be formally viewed as a limit of cones. Indeed, if in Definition 1 we replace the asymptotic condition
\[ ds^2 \to dr^2 + P^2 r^2 d\theta^2 \]
with
\[ ds^2 \to dr^2 + (P^2 r^2 + p^2) d\theta^2 , \]
then the limit \( P = 0 \) is no longer degenerate, and in fact corresponds to a spatial slice that is cylindrical at the end. In this case, however, the topological statement in Theorem 3 is no longer true: the standard cylinder \( S^1 \times \mathbb{R} \) is flat, is asymptotically cylindrical, and has two asymptotic ends. However, it is easily checked using the same method of proof as Theorem 3 that this is the only multiple-ended asymptotically cylindrical surface to support a non-negative scalar curvature.

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