Lower Bound for the Discrete Norm of a Polynomial on the Circle

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1. INTRODUCTION AND STATEMENT OF THE RESULT

For the approximation of functions, a uniform grid of values of the arguments is often chosen. In this connection, it is natural to pose the question of how the discrete norm on a given grid relates to the uniform norm of the corresponding function on a given set. In the comparatively recent paper [1], Sheil-Small showed that, for algebraic polynomials $P$ of degree $n$ and natural $N > n$, the following estimate holds:

$$\max_{\omega^{N}=1} |P(\omega)| \geq \sqrt{\frac{N - n}{n}} \max_{|z|=1} |P(z)|. \quad (1)$$

Earlier Rakhmanov and Shekhtman [2] proved the inequality

$$\max_{\omega^{N}=1} |P(\omega)| \geq \left(1 + C \log \frac{N}{N - n}\right)^{-1} \max_{|z|=1} |P(z)|, \quad (2)$$

where the absolute constant $C$ can be estimated by the number 16 (see [2, p. 3, 5]). This result generalizes an estimate due to Marcinkiewicz [3], who obtained (2) for the case $N = n + 1$. Inequality (2) is better than inequality (1) for $n/N$ close to 1, but worse than the Sheil-Small estimate for small values of $n/N$. In the present paper, we prove the following statement.

**Theorem.** Let $P$ be a polynomial of degree $n$, and let $N$ be a natural number, $N \geq 2n$. Then, for the discrete norm of the polynomial $P$, the following inequality holds:

$$\max_{\omega^{N}=1} |P(\omega)| \geq \cos \frac{\pi n}{2N} \max_{|z|=1} |P(z)|. \quad (3)$$

The equality in (3) is attained in the case $P(z) = (z \exp(i\pi/N))^n + 1$ and for any $N$ which is a multiple of $n$.

Estimate (3) holds for all $N > n$. However, for $n < N < 2n$, it is worse than estimate (1). In the case $N = 2n$, estimates (1) and (3) coincide and, for $N > 2n$, inequality (3)
strengthens inequalities (1) and (2). Moreover, for numbers $N$ which are multiples of $n$, inequality (3) is sharp, and we obtain the equality
\[ \sup \left\{ \left( \max_{|z|=1} |P(z)| \right) / \left( \max_{\omega=1} |P(\omega)| \right) \right\} = \left( \cos \left( \frac{\pi n}{2N} \right) \right)^{-1}, \]
where $N$ is a multiple of $n$, and the upper bound is taken over all polynomials $P$ of degree $n$ (see [2, Theorem 1]).

Note that, as far back as 1931, Bernstein [4] obtained inequalities for trigonometric sums close to inequalities (1)–(3) (see also [5, pp. 147, 149, 154]). In particular, the corollary on p. 154 of [5] implies inequality (3) in the case of even degrees $n$. Our proof is different from the proofs in [1]–[5]. It is based, essentially, only on the maximum principle of the modulus of the suitable analytic function. Following [6], we can strengthen inequality (3) by taking the constant term and the leading coefficient of the polynomial $P$ into account.

2. AUXILIARY RESULT

We introduce the notation
\[ m(P) = \min_{|z|=1} |P(z)|, \quad M(P) = \max_{|z|=1} |P(z)|. \]
We shall need the following analog of the Schwartz lemma in one its particular cases.

**Lemma.** Let $P$ be a polynomial of degree $n$ for which $P(0) \neq 0$ and $m(P) \neq M(P)$, and let the function $\zeta = \Phi(w)$ conformally and univalently map the exterior of the closed interval $\gamma = [m^2(P), M^2(P)]$ onto the disk $|\zeta| < 1$ so that $\Phi(\infty) = 0$ and $\Phi(m^2(P)) = -1$. Then the function
\[ f(z) = \Phi \left( \frac{P(z)}{P(1/z)} \right) \]
is regular on the set
\[ G = \left\{ z : |z| < 1, \, \frac{P(z)}{P(1/z)} \notin \gamma \right\}, \]
analytically continuable to the set
\[ E = \left\{ z : |z| = 1, \, |P(z)| \neq m(P), \, |P(z)| \neq M(P) \right\}, \]
and, at the points of the set $E$, the following inequality holds:
\[ |f'(z)| \leq n. \]
Proof. The smoothness of the function $f$ on the sets $G$ and $E$ can easily be verified. Further, in a neighborhood of the origin, the following expansion is valid:

$$f(z) = \frac{M^2(P) - m^2(P)}{4\pi c_n}z^n + \ldots,$$

where $c_0$ is the constant term and $c_n$ is the leading coefficient of the polynomial $P$. In addition, $f(z) \neq 0$ in $G \setminus \{0\}$. Therefore, the function $z^n/f(z)$ is regular on the open set $G$. At the points of the boundary of this set, the modulus of this function does not exceed 1. By the maximum principle for the modulus, we find that the inequality

$$|f(z)| \geq |z^n|$$

holds everywhere on the set $G$. Now, let $z$ be an arbitrary fixed point of the set $E$. Taking inequality (5) into account, we obtain

$$|f'(z)| = \left| \frac{\partial |f(z)|}{\partial |z|} \right| = \lim_{r \to 1} \frac{|f(z)| - |f(rz)|}{1 - r} \leq \lim_{r \to 1} \frac{1 - r^n}{1 - r} = n.$$

The lemma is proved.

3. PROOF OF THE THEOREM

We can assume that $M(P) = 1$ and $P(0) \neq 0$. Under these conditions, $m(P) < M(P)$. Indeed, otherwise, the polynomial $P$ maps the circle $|z| = 1$ into itself and, in view of the equality $P(\infty) = \infty$, the symmetry principle leads to a contradiction: $P(0) = 0$. Let us show that, for any point $z = e^{i\varphi}$ on the circle $|z| = 1$, the following inequality holds:

$$\left| \left| P(z) \right|^2 \varphi' \right| \leq n\sqrt{|P(z)|^2(1 - |P(z)|^2)}$$

(see [6, Theorem 2]). In view of the continuity, it suffices to verify this inequality for all points $z$ such that $\pi$ belongs to the set $E$ from the lemma. Suppose that $z \in E$. Then, for the function $f$ from the lemma, we have

$$|f'(z)| = \Phi' \left( \frac{P(\pi)P(\frac{1}{z})}{z} \right) \left| \frac{P'(\pi)P(\frac{1}{z})}{P(\pi)} - \frac{1}{z^2} \frac{P'(\pi)}{P(\frac{1}{z})} \right| =$$

$$= \Phi' \left( |P(\pi)|^2 \right) \left| \frac{P^2(\pi)}{P(\pi)} - \frac{1}{z^2} P'(\frac{1}{z}) \right| =$$

$$= \Phi' \left( |P(\pi)|^2 \right) \left| 2\text{Im} \frac{PP'(\pi)}{P(\pi)} \right| = \Phi' \left( |P(\pi)|^2 \right) \left| \left( |P(\pi)|^2 \right)' \varphi' \right|.$$
Before calculating the derivative $\Phi'$, note that the inverse function $\Phi^{-1}(\zeta)$ is of the form

$$
\Phi^{-1}(\zeta) = \frac{1}{4} \left( \zeta + \frac{1}{\zeta} \right) (M^2(P) - m^2(P)) + \frac{1}{2} (M^2(P) + m^2(P)).
$$

Hence

$$
|\Phi'(|P(\varpi)|^2)|^{-1} = \frac{1}{4} \left( 1 - e^{-2i\theta} \right) (M^2(P) - m^2(P)) = \frac{1}{2} |\sin \theta|(M^2(P) - m^2(P)),
$$

where $\Phi^{-1}(e^{i\theta}) = |P(\varpi)|^2$ and, therefore,

$$
\cos \theta = \frac{2|P(\varpi)|^2 - (M^2(P) + m^2(P))}{M^2(P) - m^2(P)}.
$$

Finally,

$$
|f'(z)| = \frac{\left| \left( |P(\varpi)|^2 \right)' \right|}{\sqrt{|(|P(\varpi)|^2)'_\omega|}}.
$$

Using inequality (4), we obtain inequality (6) for $\varpi$.

Let us now pass to the proof of inequality (3). Let $z_0 = e^{i\varphi_0}$ denote one of the points at which the maximum $M(P) = |P(z_0)| = 1$ is attained, and let $\omega_k$ be the $N$th root of 1 for which the arc of the circle

$$
\left\{ z : |z| = 1, \ |\arg z - \arg \omega_k| \leq \frac{\pi}{N} \right\}
$$

contains the point $z_0$. Suppose that, for some branch of the argument, the following inequality holds:

$$
\arg \omega_k - \frac{\pi}{N} \leq \varphi_0 \leq \arg \omega_k.
$$

Dividing both sides of inequality (6) by the quadratic root on the right and integrating the resulting relation on the interval $(\varphi_0, \arg \omega_k)$ with the replacement $|P(z)|^2 = u(\varphi)$, we obtain

$$
n(\arg \omega_k - \varphi_0) \geq \int_{\varphi_0}^{\arg \omega_k} \frac{-u'_\varphi d\varphi}{\sqrt{u(1-u)}} = -\int_1^u \frac{du}{\sqrt{u(1-u)}} =
$$

$$
= -2 \int_1^{u_k} \frac{dt}{\sqrt{1-t^2}} = -2 \arcsin u_k + \pi,
$$

where $u_k = u(\arg \omega_k)$. Hence

$$
2 \arcsin |P(\omega_k)| \geq \pi - n \frac{\pi}{N} > 0,
$$
and
\[ |P(\omega_k)| \geq \sin \left( \frac{\pi}{2} - \frac{n\pi}{2N} \right) = \cos \frac{n\pi}{2N}. \]

Passing to the maximum, we obtain inequality (3). In the case \( \arg \omega_k \leq \varphi_0 \), similar arguments yield the same inequality.

If \( P(z) = (z \exp(i\pi/N))^n + 1 \) and \( N = nl \), where \( l \geq 2 \) is a natural number, then
\[
\max_{|z|=1} |P(z)| = 2.
\]

On the other hand, direct calculations yield
\[
\max_{\omega^n = 1} |P(\omega)| = \max_{0 \leq k \leq l-1} |P(\omega_k)| = \max_{0 \leq k \leq l-1} 2 \left| \cos \left( \frac{\pi}{l}k + \frac{\pi}{2l} \right) \right| = 2 \cos \frac{\pi}{2l}.
\]

Thus, for the given polynomial \( P \), we have the equality sign in (3). The theorem is proved.

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