Anomalous Scaling of the SO(8) Symmetric Phases in the Two-Leg Ladder

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We carry out a complete analytical stability study for the SO(8) symmetric phases in the weakly-interacting two-leg ladder. It is shown that the SO(8) symmetry is robust under generic perturbations. Since there are no fixed points in the one-loop renormalization group equations, the conventional classification of relevant and irrelevant perturbations fails in this case. A new classification is defined and explained in detail. It leads to the anomalous scaling in the ratios of excitation gaps and an universal exponent 1/3 is extracted. This new method is also applied to the well studied 2D Ising model and similar exponent is calculated.

Introduction. — Ladder materials have attracted lots of attentions in the past decade because of both theoretical and experimental interest [1]. In particular, lots of efforts were focused on the simplest two-leg ladder [2–4]. Without doping, there is one electron per site on average (usually referred as “half-filled”). Because of the Coulomb repulsion, charge excitations acquire a gap which makes the two-leg ladder a Mott’s insulator. Besides, due to the tendency for singlet bond formation across the rungs of the ladder, spin-liquid behavior is expected. Both numerical and analytical approaches support that the ground state at half filling is in the Mott-insulating spin-liquid phase [2–4].

In general, one expects that charge and spin gaps could be rather different. Surprisingly, a complete degeneracy of charge, spin and single particle gaps emerges in the asymptotic weak coupling limit. Based on the one-loop renormalization group (RG) analysis, the effective low-energy theory for the two-leg ladder is described by the exactly soluble Gross-Neveu model with a global SO(8) symmetry. Other than the exact degeneracy of excitation gaps, many interesting results can be drawn from the Bethe Ansatz solution. There have been critics [12–13] about the unexpected restoration of the SO(8) symmetry in weak coupling, concerning about the stability of the symmetry under generic perturbations and the limitation of the RG equations derived in weak coupling. It was argued that the SO(8) symmetry derived by perturbative calculations is fragile because the system eventually flows toward the fixed point in strong coupling.

In this Letter, we address both points by performing a complete an analytical study for the stability in the vicinity of the SO(8) symmetric phases. It should be emphasized that this beautiful symmetry restoration in weak coupling does not implies that the fixed point in strong coupling is SO(8) symmetric at all. On the contrary, the symmetry reflects the local topology of RG flows near the trivial Fermi liquid fixed point which can be safely described by perturbative calculations. Indeed it is rather obvious that, when the interaction $U$ is much larger than the bandwidth $t$, the charge gap is much larger than the spin gap. Thus, the symmetry in the strong coupling is completely destroyed. As to the stability of the symmetry under generic perturbations, the complete stability check shows that the SO(8) symmetry is robust in weak coupling. Even though some couplings are relevant according to the conventional classification, they do not destroy the symmetry but only give rise to anomalous corrections which scale as $(U/t)^{1/3}$. A new classification of “relevant” and “irrelevant” perturbations is necessary here. Finally, we apply the new method to the well-known 2D Ising model and show that the anomalous scaling behavior near the critical point can be very general, as long as more than two relevant couplings are present.

The SO(8) symmetric rays. — For the two-leg ladder at half filling, the number of possible interactions is greatly reduced to nine in weak coupling. Within the one-loop RG calculations, these nine couplings $g_i$ are described by a set of coupled first-order differential equations,

$$\frac{dg_i}{dl} = A_{jk}^i g_j g_k,$$

where $A_{jk}^i$ are nine $9 \times 9$ constant matrices. These matrices can be found by rewriting the RG equations given in Ref. [3]. For a generic interacting Hamiltonian, the bare values for these nine couplings can be straightforwardly determined. However, it is generally very difficult to obtain the solution for these coupled flow equations in analytical form.

Simple analytical solutions emerge if the interactions are chosen in a specific way. These special solutions are later referred as “symmetric rays”. Suppose the bare couplings of the specific interacting Hamiltonian are $g_i(0) = r_i g(0)$, where $g(0) = (U/t) \ll 1$ is small while $r_i$ are order one constants which satisfy the algebraic constraint,

$$r_i = A_{jk}^i r_j r_k.$$  

It is straightforward to show that the ratios between couplings remain the same and the nine complicated equations reduce to single one, $\hat{g} = g^2$ ! The solution is $g(t) = 1/(l_0 - t)$, where the divergent length scale $l_0 = (t/U)$. Of course, one should keep in mind that the solution $g(t)$ is only valid when it does not flow out.
of the weak coupling regime. These special Hamiltonians, whose couplings are described by these symmetric rays, turn out to be SO(8) symmetric. For the two-leg ladder, four different phases (named as D-Mott, S-Mott, CDW and SP in Ref. 3) are of the central concerns.

It was shown previously that the two-leg ladder in weak coupling always scales into one of the four different symmetric phases 3. However, this numerical approach was criticized that the SO(8) symmetric phases might have instabilities which happen not to be tackled by the limited types of interactions considered in the numerical study. To make up the fissure, a complete stability check near the SO(8) symmetric rays is desirable.

**Stability analysis.**—To describe the RG flows in the vicinity of the SO(8) symmetric rays, it is sufficient to consider the linearized version of Eq. (4). For a generic interaction, the couplings are separated into symmetric and asymmetric parts, $g_i(l) = r_i g(l) + \Delta g_i(l)$. In the vicinity of the symmetric rays, the deviations are small, $\Delta g_i(l) \ll g(l)$. Keeping the leading order term, the linearized RG equations are

$$\frac{d(\Delta g_i)}{dt} = \frac{B_{ij}}{(l_d - l)} \Delta g_j,$$

where $B_{ij} = 2A_{ij} r_k r_k$. The matrix $B_{ij}$ can be brought into diagonal form by a linear transformation. As a consequence, the RG equations decouple into nine independent ones,

$$\frac{d(\delta g_i)}{dt} = \frac{\lambda_i}{(l_d - l)} \delta g_i,$$

where $\delta g_i$ are couplings after the linear transformation and $\lambda_i$ are the eigenvalues of the matrix $B_{ij}$. Although the matrix $B_{ij}$ are different for each SO(8) symmetric rays, the eigenvalues are identically the same

$$\lambda_i = 2, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}. $$

This coincidence implies that the results of the stability check only rely on the symmetry group but not on the details of the phases 4.

So far, we have a single equation, $\dot{y} = y^2$, describing the renormalization along the symmetric rays and nine for deviations from the rays as in Eq. (4). Apparently, there must be one redundant equation among them because the original number of equations is only nine. It doesn’t take long to find out that it corresponds to the flow equation with largest eigenvalue $\lambda = 2$. This corresponds to a trivial case that all couplings are shifted along the symmetric ray, i.e. $\delta g_i = r_i \delta g$. Since the flow along the ray is described by $\dot{y} = y^2$, linearization leads to $\dot{\delta g} = 2g(l) \delta g$. Since we only need eight equations to describe the deviations from the symmetric rays, the $\lambda = 2$ equation is only an artifact and should be ignored.

The other eight eigenvalues describe how the flow goes once the bare couplings are off the ray. Starting from the bare values $\delta g_i(0) = \delta r_i g(0)$, where $\delta r_i \ll r_i$. The solutions of Eq. (6) are

$$\delta g_i(l) = \frac{\delta r_i}{(l_d - l)^{\lambda_i}} \left( \frac{U}{t} \right)^{1-\lambda_i}.$$  

According to conventional classification 12, for $\lambda_i < 0$ the deviations diminish under RG transformations as shown in FIG. 1(a) and thus are classified as irrelevant couplings. For $\lambda_i > 0$, any small deviations get enhanced and the flow is pushed away from the symmetric rays as in FIG. 1(b) and 1(c). These are classified as relevant couplings. The conventional wisdom tells us that there are three relevant couplings ($\lambda_i = 2/3$) and five irrelevant ones ($\lambda_i = -1/3$). Since a generic interaction in principle could generate asymmetric deviations in all couplings, one might rush to the incorrect conclusion that the SO(8) symmetry is not stable. However, the first-glance guess is wrong because the conventional classification is based on the perturbative analysis near a fixedpoint while we are dealing with running symmetric rays. A new set of rules to identify relevant perturbations is in order.

![FIG. 1. The topology of RG flows near the symmetric ray with (a) $\lambda < 0$, (b) $0 < \lambda < 1$ and (c) $\lambda > 1$. It is clear that the coupling is irrelevant for $\lambda < 0$ and relevant for $\lambda > 1$. The RG flow is more subtle for $0 < \lambda < 1$. In this case, although the deviation from the symmetric ray is growing, the slope remains the same as the ray.](image-url)

The crucial criterion is whether the deviations $\delta g_i(l)$ grow larger than the symmetric coupling $g(l) = (l_d - l)^{-\lambda}$. Just growing larger than the bare values under RG transformations is not qualified as a relevant perturbation. This subtle but important difference is best illustrated by calculating gap functions using scaling arguments. If the degeneracy of excitation gaps is maintained, the SO(8) symmetry is robust and vice versa.

**Anomalous scaling.**—Since the RG equations are only valid in weak coupling, we cut off the RG procedure when the symmetric coupling $g(l_c) = 1$. At this cutoff length scale $l_c$, the deviations in Eq. (6) are

$$\delta g_i(l_c) \sim \left( \frac{U}{t} \right)^{1-\lambda_i}.$$  

For $\lambda_i > 1$, the deviations are larger than the symmetric coupling and should be classified as “relevant”. As long
as \( \lambda_i < 1 \), the deviations at the cutoff length scale are still vanishingly small and should be viewed as “irrelevant”. This new classification is different from the conventional one for \( 0 < \lambda_i < 1 \). We would see clearly soon that the new classification is appropriate for stability check near a running symmetric ray. The grey regime \( 0 < \lambda_i < 1 \) between the new and conventional rules gives rise to anomalous scaling exponents.

Under RG transformations, the gap functions scale like,
\[
\Delta_i [g(0), \delta g_i(0)] = e^{-\lambda_i} \Delta_i [1, \delta g_i(l_c)],
\]
where \( \delta g_i(l_c) \) are at most of order \((U/t)^{1/3}\). The key point is that, although some perturbations are enhanced to order \((U/t)^{1/3}\), which is larger than the bare order \(U/t\) values, they are still small at the cutoff length scale. The effective Hamiltonian can be separated into two parts \( H = H_0 + \delta H \) – the SO(8) symmetric and asymmetric parts. Since the asymmetric part is of order \((U/t)^{1/3}\), the changes of the gaps would be of the same order by standard perturbation theory. Without the deviations, the SO(8) symmetry guarantees the exact degeneracy of all gaps, i.e. \( \Delta[1,0] = \Delta \). The presence of perturbations modifies the gap functions,
\[
\Delta_i [1, \delta g_i(l_c)] = \Delta \left[ 1 + c_i \left( \frac{U}{T} \right)^{\frac{1}{3}} + \ldots \right].
\]
It is clear that, in the weak coupling limit \( U/t \to 0 \), the degeneracy of all gaps is recovered. It implies that the SO(8) symmetry is indeed robust under generic perturbations.

![FIG. 2](image)

**FIG. 2.** Ratios of charge, spin and single particle gaps plotted versus the interaction strength \( U/t \). The ratios approach unity in the asymptotic \( U/t \to 0 \) limit with anomalous corrections of order \((U/t)^{1/3}\).

The anomalous scaling exponent \( 1/3 \) is clearly seen in the ratios between charge, spin and single particle gaps (see FIG. 2),
\[
\frac{\Delta_i}{\Delta_j} \approx 1 + c_{ij} \left( \frac{U}{t} \right)^{\frac{1}{3}}.
\]
This simple exponent is rather fascinating because the RG equations we started from are very messy. But, despite of which SO(8) symmetric phases the system flows into, the exponent of the gap function corrections is universally equal to \( 1 - \lambda_i = 1/3! \).

\[\text{FIG. 3. The RG flows near the critical point in the 2D Ising model. Notice that the local topology of flows near the critical point is similar to FIG. 1(b).}\]

**2D Ising model.** — The anomalous scaling discussed here is not limited to the particular two-leg ladder at all. As long as more than two relevant couplings (according to the conventional classification) are present, this interesting phenomena shows up somewhere. Here we use the 2D Ising model as another example. The RG flow diagram for the model is shown in FIG. 3. There are two relevant perturbations near the critical point – the magnetic field \( h \) and the temperature deviation from the critical point \( t \). It is well known that the magnetization \( m[h,t] \) scales differently with respect to these two relevant perturbations,
\[
m[u,0] \sim u^{1/\delta},
\]
\[
m[0,u] \sim u^\beta.
\]
Here \( 1/\delta = 1/15 \) and \( \beta = 1/8 \). Although both are relevant, the magnetic field grow faster than the later under RG transformations. We would find out that the scaling of magnetization \( m[h,t] \) in the presence of both perturbations looks similar to the scaling behavior with only non-zero magnetic field. Suppose now we start from a bare coupling in the vicinity of the critical point, \((h,t) = u(\cos \theta, \sin \theta)\) with \( 0 < \theta < \pi/2 \). The scaling function of the magnetization is \( m[h,t] = h^{1/\delta} F(t^\Delta/h) \) with the exponent \( \Delta = \beta \delta = 15/8 > 1 \). Since the argument inside the scaling function \( t^\Delta/h \sim u^{\Delta - 1} \) is small, we can expand the function around zero. After a bit algebra, the magnetization is
\[
\frac{m[u \cos \theta, u \sin \theta]}{m[u,0]} \approx 1 + c(\theta) u^{7/8}.
\]
Notice that the exponent comes from \( \Delta - 1 = 7/8 \). It is clear that, when close to the critical point \( u \to 0 \), the magnetization scales as if the temperature deviation is not present at all. In this sense, the temperature deviation \( t \) should be called “irrelevant”. However, it does lead to a non-trivial correction with anomalous exponent \( 7/8 \).

In summary, we have shown analytically that the SO(8) symmetric phases are stable in weak coupling.
Since we are dealing with symmetric rays but not fixed points, a new classification is defined and explained in detail. Corrections to the SO(8) symmetry at finite but still weak couplings acquire a non-trivial exponent $1/3$ in the ratios of excitation gaps.

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