On the exact constant in Jackson-Stechkin inequality for the uniform metric

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Abstract

The classical Jackson-Stechkin inequality estimates the value of the best uniform approximation of a periodic function \( f \) by trigonometric polynomials of degree \( \leq n - 1 \) in terms of its \( r \)-th modulus of smoothness \( \omega_r(f, \delta) \). It reads

\[
E_{n-1}(f) \leq c_r \omega_r\left(f, \frac{2\pi}{n}\right),
\]

where \( c_r \) is some constant that depends only on \( r \). It was known that \( c_r \) admits the estimate \( c_r < r^{ar} \) and, basically, nothing else could be said about it.

The main result of this paper is in establishing that

\[
(1 - \frac{1}{r+1}) \gamma_r^* \leq c_r < 5 \gamma_r^*, \quad \gamma_r^* = \frac{1}{\left(\frac{r}{2}\right)} \times \frac{r^{1/2}}{2},
\]

i.e., that the Stechkin constant \( c_r \), far from increasing with \( r \), does in fact decay exponentially fast. We also show that the same upper bound is valid for the constant \( c_{r,p} \) in the Stechkin inequality for \( L_p \)-metrics with \( p \in [1, \infty) \), and for small \( r \) we present upper estimates which are sufficiently close to \( 1 \cdot \gamma_r^* \).

1 Introduction

The classical Jackson-Stechkin inequality estimates the value of the best uniform approximation of a periodic function \( f \) by trigonometric polynomials of degree \( \leq n - 1 \) in terms of its \( r \)-th modulus of smoothness \( \omega_r(f, \delta) \). It reads

\[
E_{n-1}(f) \leq c_r \omega_r\left(f, \frac{2\pi}{n}\right),
\]

where \( c_r \) is some constant which depends only on \( r \) (see [10] or [3, p.205]).

Besides the case \( r = 1 \), hardly any attempts have been made to find the best value of this constant \( c_r \), or even to determine its dependence on \( r \). Stechkin’s original proof [10] (as well as alternative ones) allows to obtain the estimate \( c_r < r^{ar} \), and, basically, nothing else could be said about it.

The main result of this paper is in establishing that

\[
(1 - \frac{1}{r+1}) \gamma_r^* \leq c_r < 5 \gamma_r^*, \quad \gamma_r^* = \frac{1}{\left(\frac{r}{2}\right)} \times \frac{r^{1/2}}{2},
\]

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i.e., that the Stechkin constant $c_r$, far from increasing with $r$, does in fact decay exponentially fast.

We also show that the same upper bound is valid for the constant $c_{r,p}$ in the Stechkin inequality for $L_p$-metrics with $p \in [1, \infty)$, and for small $r$ we present upper estimates which are sufficiently close to $1 \cdot \gamma_r^r$.

In retrospect, such a result could have been anticipated, since for trigonometric approximation in $L_2$-metric, already in 1967, Chernykh [2] established that

$$E_{n-1}(f)_2 \leq c_{r,2} \omega_r \left( f, \frac{2\pi}{n} \right)_2, \quad c_{r,2} = \frac{1}{\sqrt{\left( \frac{2^r}{r} \right)}} \approx \frac{1}{2r}, \quad (1.3)$$

proving also that such a $c_{r,2}$ is best possible (for the argument $\delta = 2\pi n$ in $\omega_r(f, \delta)$). However, this result was based on Fourier technique for $L_2$-approximation and that does not work in other $L_p$-metrics.

Our method of proving (1.2) is based on deriving first the intermediate inequality

$$\|f\| \leq c_{n,r}(\delta) \omega_r(f, \delta), \quad f \in T_{n-1}^r, \quad (1.4)$$

which is valid for the functions $f$ which are orthogonal to the trigonometric polynomials of degree $\leq n - 1$. This may be viewed as a difference analogue of the classical Bohr-Favard inequality for differentiable functions

$$\|f\| \leq \frac{E_r}{n^r} \|f^{(r)}\|, \quad f \in T_{n-1}^r,$$

and is of independent interest.

We make a pass from the Bohr-Favard-type inequality (1.4) to the Stechkin one (1.1) by approximating $f$ with the de la Valleé Poussin sums $v_{m,n}(f)$ and using the fact that

$$f - v_{m,n}(f) \in T_m^r, \quad \omega_r(f - v_{m,n}(f), \delta) \leq (1 + \|v_{m,n}\|) \omega_r(f, \delta).$$

With that we arrive at the inequality

$$E_{n-1}(f) \leq \|f - v_{m,n}(f)\| \leq c_{m,n,r}(\delta) \omega_r(f, \delta),$$

where we finally minimize the resulting constant over $m$, for given $r, n$ and $\delta$.

### 2 Results

For a continuous $2\pi$-periodic function $f$, we denote by $E_{n-1}(f)$ the value of best approximation of $f$ by trigonometric polynomials of degree $\leq n - 1$ in the uniform norm,

$$E_{n-1}(f) := \inf_{\tau \in T_{n-1}} \|f - \tau\|,$$

and by $\omega_r(f, \delta)$ its $r$-th modulus of smoothness with the step $\delta$,

$$\omega_r(f, \delta) := \sup_{0 < h \leq \delta} \|\Delta_h^r(f, \cdot)\|, \quad \Delta_h^r(f, x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(x + ih);$$

and
where $\Delta_h^r(f, x)$ is the forward difference of order $r$ of $f$ at $x$ with the step $h$.

We will study the best constant $K_{n,r}(\delta)$ in the Stechkin inequality

$$E_{n-1}(f) \leq K_{n,r}(\delta) \omega_r(f, \delta),$$

i.e., the quantity

$$K_{n,r}(\delta) := \sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r(f, \delta)},$$

which depends on the given parameters $n, r \in \mathbb{N}$ and $\delta \in [0, 2\pi]$.

In such a setting (which goes back to Korneichuk and Chernykh) we may safely consider $\delta = \frac{2\pi}{n}$ with some $\alpha$ not necessarily 1 or 2. The choice of particular $\delta$'s can be motivated by two reasons:

1) “nice” look and/or tradition: $\delta = \frac{\pi}{n}$, or $\delta = \frac{2\pi}{n}$, or (why not) $\delta = \frac{1}{n}$, and alike;

2) “nice” result:

$$\sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r(f, \delta)} \asymp c_{n,r}(\delta).$$

Ideally, both approaches should be combined to provide nice results for nice $\delta$'s, but that happens not very often.

In this paper we obtain the following results.

1) First of all, we show that the exact order of the Stechkin constant $K_{n,r}(\delta)$ at $\delta = \frac{2\pi}{n}$ (and in fact at any $\delta \in \left[\frac{2\pi}{n}, \frac{\pi}{r}\right]$) is $r^{1/2} 2^{-r}$, namely

$$K_{n,r}\left(\frac{2\pi}{n}\right) \asymp \gamma^*_r \asymp \frac{r^{1/2}}{2^r},$$

where

$$\gamma^*_r = \frac{1}{\left(\frac{r}{2}\right)} = \begin{cases} \frac{1}{(2k)}, & r = 2k; \\ \frac{1}{(2k - 1)}, & r = 2k - 1. \end{cases}$$

Moreover, we locate the exact value of this constant within quite a narrow interval.

**Theorem 1.** We have

$$c'_r\left(\frac{2\pi}{n}\right) \gamma^*_r \leq \sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r(f, \frac{2\pi}{n})} \leq c_r\left(\frac{2\pi}{n}\right) \gamma^*_r,$$

where

$$c'_r\left(\frac{2\pi}{n}\right) = \begin{cases} 1 - \frac{1}{r+1}, & r = 2k - 1; \\ 1, & r = 2k; \end{cases} \quad n \geq 2r,$$

and

$$c_r\left(\frac{2\pi}{n}\right) = 5, \quad n \geq 1.$$
2) Next, we show that the value of the constant $c_r(\delta)$ remains bounded uniformly in $r$ and $n$ also for $\frac{\pi}{n} < \delta < \frac{2\pi}{n}$ (but it grows to infinity as $\delta$ approaches $\frac{\pi}{n}$).

**Theorem 2.** For any $\alpha > 1$, there exists a constant $c_\alpha$ which depends only on $\alpha$ such that

$$E_{n-1}(f) \leq c_\alpha \gamma_r^* \omega_r \left( f, \frac{\alpha \pi}{n} \right), \quad n \geq 1.$$ 

3) Thirdly, although we did not succeed to reach the argument $\delta = \frac{\pi}{n}$ with an absolute constant in front of $\gamma_r^* \omega_r (f, \delta)$, we prove that this constant grows like $O(\sqrt{r} \ln r)$ at most.

**Theorem 3.** For $\delta = \frac{\pi}{n}$, we have the estimate

$$E_{n-1}(f) \leq c_r \left( \frac{\pi}{n} \right) \gamma_r^* \omega_r \left( f, \frac{\pi}{n} \right), \quad c_r \left( \frac{\pi}{n} \right) = O(\sqrt{r} \ln r), \quad n \geq 1.$$ 

4) Fourthly, for small $r$, the general upper bound $c_r \left( \frac{2\pi}{n} \right) = 5$ can be decreased to the values that are quite close to the lower bound $c_r' \approx 1$, thus giving support to the (upcoming) conjecture that $K_{n,r} (\delta) \leq 1 \cdot \gamma_r^*$ for $\delta \geq \frac{\pi}{n}$.

**Theorem 4.** For $\delta = \frac{\pi}{n}$ and $\delta = \frac{2\pi}{n}$, we have

$$E_{n-1}(f) \leq c_r (\delta) \gamma_r^* \omega_r (f, \delta),$$

where $c_{2k-1}(\delta) = c_{2k}(\delta)$, and the values of $c_{2k}(\delta)$ are given below

$$
\begin{array}{c|c|c|c|c}
 & c_2 (\pi/n) & c_4 (\pi/n) & c_2 (2\pi/n) & c_4 (2\pi/n) & c_6 (2\pi/n) \\
\hline
1/4 & 2/10 & 1/10 & 2/17 & 1/17 & 1/12
\end{array}
$$

5) Finally, all upper estimates in Theorems 1-4 remain valid for any $p \in [1, \infty]$. (There is no need to give a separate proof of this statement, since all the inequalities we used in the text still hold for the $L_p$-metrics, $1 \leq p < \infty$, in particular the Bohr-Favard inequality (4.1) and the inequalities of §5 involving the norms of the de la Vallée Poussin operator.)

**Theorem 5.** For any $p \in [1, \infty]$, we have

$$E_{n-1}(f)_p \leq c_r (\delta) \gamma_r^* \omega_r (f, \delta)_p,$$

with the same constants $c_r (\delta)$ and the same $\delta$’s as in Theorems 1–4. In particular,

$$E_{n-1}(f)_p \leq 5 \gamma_r^* \omega_r \left( f, \frac{2\pi}{n} \right)_p.$$ 

The latter $L_p$-estimate is hardly of the right order for $1 < p < \infty$ because $\gamma_r^* \propto r^{1/2} 2^{-r}$, while, for $p = 2$, Chernykh’s result (1.3) says that $K_{n,r} (\frac{2\pi}{n})_2 \propto r^{1/4} 2^{-r}$, so one may guess that

$$K_{n,r} (\frac{2\pi}{n})_p \propto r^{\max(1/2p,1/2p')} 2^{-r}.$$ 

This guess is partially based on the results of Ivanov [7] who obtained such an upper bound for the values $K_{n,r}(\delta)_p$ with relatively large $\delta = \frac{\pi r^{1/3}}{n}$, and proved that, for $p \in [2, \infty]$, the order of the lower bounds is the same.
6) The value $\delta = \frac{\pi}{n}$ is critical in the sense that the Stechkin constant $K_{n,r}(\delta)$ and the constant $\gamma_r^*$ are no longer of the same (exponential) order for $\delta = \frac{\alpha \pi}{n}$ with $\alpha < 1$. Indeed, in this case, with $f_0(x) := \cos nx$, we have $\omega_r(f_0, \delta) = 2^r \sin^r \frac{\alpha \pi}{2}$ (see (7.2)) and $E_{n-1}(f_0) = 1$, so that, for $\alpha < 1$, we have

$$\frac{K_{n,r}(\frac{\alpha \pi}{n})}{\gamma_r^*} > \frac{c r^{1/2}}{\sin^r \frac{\alpha \pi}{2}} > c_0 \lambda^r_\alpha, \quad \lambda > 1.$$ 

This being said, a natural question arises from the two estimates

$$K_{n,r}(\frac{2\pi}{n}) \asymp \gamma_r^*, \quad K_{n,r}(\frac{\pi}{n}) \leq c \sqrt{r} \ln r \cdot \gamma_r^*$$

whether an extra factor at $\delta = \frac{\pi}{n}$ is essential. We believe it is not, and we are making the following brave conjecture.

**Conjecture 2.1** For all $r \in \mathbb{N}$, we have

$$\sup_{n \in \mathbb{N}} K_{n,r}(\frac{\pi}{n}) := \sup_{n \in \mathbb{N}} \sup_{f \in C} E_{n-1}(f) = 1 \cdot \gamma_r^*, \quad \gamma_r^* = \frac{1}{\left\lfloor \frac{r}{\left\lfloor \frac{\pi}{2} \right\rfloor} \right\rfloor}.$$ 

(Our point is mainly about the upper bound, namely that $K_{n,r}(\delta) \leq 1 \cdot \gamma_r^*$, for any $\delta \geq \frac{\pi}{n}$. The lower bound for even $r = 2k$ is established in this paper, while for odd $r$ we guess that $K_{n,r}(\delta)$ tends to $\gamma_r^*$ at $\delta = \frac{\pi}{n}$ for large $n$, but for $\delta > \frac{\pi}{n}$ it takes smaller values.)

This conjecture is true for $r = 1$, for in this case we have Korneichuk’s result [9]:

$$1 - \frac{1}{2n} \leq K_{n,1}(\frac{\pi}{n}) < 1.$$ 

For $r = 2$, the conjecture gives the estimate $K_{n,2}(\frac{\pi}{n}) = \frac{1}{2}$ which is (to a certain extent) stronger than Korneichuk’s one (because $\omega_2(f, \delta) \leq 2 \omega_1(f, \delta)$), so it would be interesting to prove (or to disprove) it in this particular case. Meanwhile, according to Theorems 1 and 4, we have

$$\frac{1}{2} \leq K_{n,2}(\frac{\pi}{n}) \leq \frac{5}{8}, \quad \frac{1}{2} \leq K_{n,2}(\frac{2\pi}{n}) \leq \frac{17}{32}.$$ 

For arbitrary $r$, it seems unlikely that the value of the Stechkin constant will ever be precisely determined, but it would be a good achievement to narrow the interval for $K_{n,r}(\frac{2\pi}{n})$, say, to $[\gamma_r^*, 2\gamma_r^*]$, and to settle down the correct order of $K_{n,r}(\frac{\pi}{n})$ with respect to $r$.

7) We finish this section with the remark that if, with some constant $c(\delta)$, the inequality

$$E_{n-1}(f) \leq c(\delta) \gamma_r^* \omega_r(f, \delta)$$

is true for an even $r = 2k$, then it is true for the odd $r = 2k - 1$ too, with the same constant $c(\delta)$. Indeed, since $\gamma_{2k-1}^* = 2 \gamma_{2k}^*$ and $\omega_{2k}(f, \delta) \leq 2 \omega_{2k-1}(f, \delta)$, we have

$$E_{n-1}(f) \leq c(\delta) \gamma_{2k}^* \omega_{2k}(f, \delta) \leq c(\delta) \gamma_{2k}^* \cdot 2 \omega_{2k-1}(f, \delta) = c(\delta) \gamma_{2k-1}^* \omega_{2k-1}(f, \delta).$$

Therefore, it is sufficient to prove upper estimates only for even $r = 2k$.  

5
3 Smoothing operators

Here, we present the general idea of our method.

1) For a fixed $k$, with

\[ \tilde{\Delta}^{2k}_k(f, x) := \sum_{i=-k}^{k} (-1)^i \left( \frac{2k}{k+i} \right) f(x + it) \]

being the central difference of order $2k$ with the step $t$, and with $\phi_h$ being an integrable function which satisfies conditions

\begin{align*}
    a) & \quad \phi_h(t) = \phi_h(-t), \\
    b) & \quad \text{supp } \phi_h = [-h, h], \\
    c) & \quad \int_{\mathbb{R}} \phi_h(t) \, dt = 1, \quad (3.1)
\end{align*}

consider the following operator

\[ W_h(f, x) := \frac{1}{(2k)_{k}} \int_{\mathbb{R}} \tilde{\Delta}^{2k}_k(f, x) \phi_h(t) \, dt. \quad (3.2) \]

If a given subspace $S$ is invariant under the operator $W_h$, and if the restriction $W_h$ to $S$ has a bounded inverse, then, for any $f \in S$, we have a trivial estimate

\[ \|f\| \leq \|W_h^{-1}\|_S \|W_h(f)\|, \quad f \in S. \quad (3.3) \]

It follows immediately from the definition that

\[ \|W_h(f)\| \leq \|\phi_h\|_1 \gamma^*_{2k} \omega_{2k}(f, h), \quad \gamma^*_{2k} = \frac{1}{(2k)_{k}}, \quad (3.4) \]

and we arrive at the following inequality:

\[ \|f\| \leq c_{2k}(h) \gamma^*_{2k} \omega_{2k}(f, h), \quad c_{2k}(h) = \|\phi_h\|_1 \|W_h^{-1}\|_S, \quad (3.5) \]

valid for all functions $f$ from a given subspace $S$.

2) Next, we present $W_h$ as $W_h = I - U_h$ what allows us to get some bounds for $\|W_h^{-1}\|$ in (3.5) in terms of $U_h$.

To this end, for integer $i$ (and, in fact, for any $i$), define the dilations $\phi_{ih}$ and the convolution operators $I_{ih}$ by the rule

\[ \phi_{ih}(t) := \frac{1}{(2k)_{k}} \phi_h\left( \frac{t}{2k} \right), \quad I_{ih}(f) := f * \phi_{ih} := \int_{\mathbb{R}} f(x - t) \phi_{ih}(t) \, dt. \]

Then, taking into account that

\[ \int_{\mathbb{R}} f(x - it) \phi_h(t) \, dt = \int_{\mathbb{R}} f(x - \tau) \frac{1}{i} \phi_h\left( \frac{\tau}{i} \right) \, d\tau = I_{ih}(f), \]

and that also $I_{ih} = I_{-ih}$ (because $\phi_{ih}$ is even), we may put $W_h$ in the following form:

\[ W_h = \frac{1}{(2k)_{k}} \sum_{i=-k}^{k} (-1)^i \left( \frac{2k}{k+i} \right) I_{ih} = I - 2 \sum_{i=1}^{k} (-1)^{i+1} a_i I_{ih}, \quad a_i := \frac{(2k)_{k+i}}{(2k)_{k}}. \]
So, with the further notations
\[ U_h := 2 \sum_{i=1}^{k} (-1)^{i+1} a_i I_{ih}, \quad \psi_{kh} := 2 \sum_{i=1}^{k} (-1)^{i+1} a_i \phi_{ih}, \]
we obtain
\[ W_h = I - U_h, \quad U_h(f) = f \ast \psi_{kh}. \]

Respectively, we may rewrite the inequality (3.5) in the following way.

**Lemma 3.1** If the operator \((I - U_h)^{-1}\) is bounded on a given subspace \(S\), then, for all \(f \in S\), we have
\[ \|f\| \leq c_{2k}(h) \gamma_{2k}^* \omega_{2k}(f, h), \quad c_{2k}(h) = \|\phi_h\|_1 \|(I - U_h)^{-1}\|_S. \]

3) Now, we call upon elementary properties of Banach algebras (see, e.g., Kantorovich, Akilov [8, Chapter 5, \S 4]) for the claim that if an operator \(U : S \to S\) satisfies \(\sum_{m=0}^{\infty} \|U^m\| < \infty\), then the operator \(I - U\) is invertible, and the norm of its inverse admits the estimate
\[ \|(I - U)^{-1}\|_S \leq \sum_{m=0}^{\infty} \|U^m\|_S. \]  \hfill (3.6)

**Proposition 3.2** If \(\phi_h\) is such that \(\sum_{m=0}^{\infty} \|U_h^m\|_S = A_h < \infty\), then, for any \(f \in S\), we have
\[ \|f\| \leq c_{2k}(h) \gamma_{2k}^* \omega_{2k}(f, h), \quad c_{2k}(h) = A_h \|\phi_h\|_1. \]

4) Finally, let us make a short remark about the structure of the subspaces \(S\) that may go into consideration. It is clear that, if \(S\) is shift-invariant, i.e., together with \(f\) it contains also \(f(\cdot + t)\) for any \(t\), then \(S\) is invariant under the action of \(W_h\) for any \(h\). A typical example is a subspace \(S\) that contains (or does not contain) certain monomials \((\cos kx, \sin kx)\).

We will consider \(S = T_{n-1}^\perp\), the subspace of functions which are orthogonal to trigonometric polynomials of degree \(\leq n - 1\).

4 **A difference analogue of the Bohr-Favard inequality**

Denote by \(T_{n-1}^\perp\) the set of functions \(f\) which are orthogonal to \(T_{n-1}\), i.e., such that
\[ \int_{-\pi}^{\pi} f(x) \tau(x) \, dx = 0, \quad \forall \tau \in T_{n-1}. \]

The Bohr-Favard inequality for such functions reads
\[ \|f\| \leq \frac{F_r}{n} \|f^{(r)}\|, \quad f \in T_{n-1}^\perp, \]  \hfill (4.1)
where \(F_r\) are the Favard constants, which are usually defined by the formula
\[ F_r := \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i (r+1)^{r+1}}{(2i+1)^{r+1}}. \]
and which satisfy the following relations:

\[ F_0 = 1 < F_2 = \frac{\pi^2}{8} < \cdots < \frac{4}{\pi} < \cdots < F_3 = \frac{\pi^3}{24} < F_1 = \frac{\pi}{2}. \]

In this section we obtain a difference analogue of the Bohr-Favard inequality in the form

\[ \|f\| \leq c_{n,2k}(h) \gamma_k^* \omega_{2k}(f,h), \quad f \in T_{n-1}^+, \]

using the approach from the previous section (Proposition 3.2). Namely, we consider the operator

\[ U_h = 2 \sum_{i=1}^{k} (-1)^{i+1} a_i I_{ih}, \quad a_i = \left( \frac{2k}{k+i} \right) \]

with the following specific choice of \( I_{ih} \) (and respectively of \( \phi_h \)):

\[ I_{ih}(f,x) := \frac{1}{h^2} \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} f(x - t_1 - t_2) \, dt_1 \, dt_2, \]

i.e., taking \( I_{ih}(f) \) as the Steklov function of order 2. It is known that

\[ I_{ih}(f,x) = f \ast \phi_h, \]

where

\[ \phi_h(t) = \begin{cases} \frac{1}{h} (1 - \frac{|t|}{h}), & t \in [-h,h], \\ 0, & \text{otherwise}, \end{cases} \]

i.e., \( \phi_h \) is the \( L_1 \)-normalized B-spline of order 2 (the hat-function) with the step-size \( h \) supported on \([-h,h]\). We also have

\[ I_{ih}''(f,x) = -\frac{1}{(ih)^2} \sum_{i=1}^{k} (-1)^{i+1} a_i f(x) = \frac{1}{(ih)^2} [f(x - ih) - 2f(x) + f(x + ih)]. \]

We denote by \( \|U_h\|_{T_{n-1}^+} \) the norm of the operator \( U_h \) on the space \( T_{n-1}^+ \).

**Lemma 4.1** We have

\[ \|U_h''\| \leq \frac{\pi^2 \mu^2}{h^2}, \quad \mu^2 := \mu_{2k}^2 := \frac{8}{\pi^2} \sum_{\text{odd } i}^{k} \frac{a_i}{i^2} < 1. \]  

**Proof.** 1) We have

\[
U_h''(f,x) = 2 \sum_{i=1}^{k} (-1)^{i+1} a_i I_{ih}''(f,x)
\]

\[
= 2 \sum_{i=1}^{k} (-1)^{i+1} \frac{a_i}{(ih)^2} \left[ f(x - ih) - 2f(x) + f(x + ih) \right]
\]

\[
= \frac{2}{h^2} \sum_{i=1}^{k} (-1)^{i+1} \frac{a_i}{i^2} \left[ f(x - ih) - 2f(x) + f(x + ih) \right]
\]

\[
= \frac{2}{h^2} \sum_{i=1}^{k} (-1)^{i+1} a_i \left[ -2f(x) \right] + \frac{2}{h^2} \sum_{i=1}^{k} (-1)^{i+1} a_i \left[ f(x - ih) + f(x + ih) \right],
\]

\[ 8 \]
where in the last line we put \( a_i' = \frac{a_i}{\sqrt{n}} \). Hence,
\[
\frac{h^2}{4} \| U_h''(f) \| \leq \left| \sum_{i=1}^{k} (-1)^{i+1} a_i' \right| + \left| \sum_{i=1}^{k} a_i' \right| = \sum_{i=1}^{k} (-1)^{i+1} a_i' + \sum_{i=1}^{k} a_i' = 2 \sum_{i \text{ odd}} \frac{a_i}{\sqrt{n}}^2 = \frac{\pi^2}{4} \mu^2
\]
i.e.,
\[
\| U_h'' \| \leq \frac{\pi^2 \mu^2}{h^2}.
\]

2) The estimate for \( \mu^2 \) follows from the fact that \( a_i = \frac{2k}{2k+1} \) < 1, and that
\[
\sum_{i=1}^\infty \frac{1}{i^2} = \frac{\pi^2}{6}:
\]
\[
\frac{\pi^2}{8} \mu^2 = \sum_{i \text{ odd}} \frac{a_i}{i^2} < \sum_{i \text{ odd}} \frac{1}{i^2} = \sum_{i=1}^\infty \frac{1}{i^2} - \sum_{i \text{ even}} \frac{1}{i^2} = \left( 1 - \frac{1}{4} \right) \sum_{i=1}^\infty \frac{1}{i^2} = \frac{\pi^2}{8}.
\]

We will prove in §6 that \( 1 - \mu^2 \approx \frac{1}{\sqrt{2k}} \). \(\Box\)

**Lemma 4.2** We have
\[
\| U_h^m \|_{T_{n-1}^\perp} \leq F_{2m} \left( \frac{\pi^2 \mu^2}{n^2 h^2} \right)^m.
\] (4.4)

**Proof.** 1) If \( f \) is orthogonal to \( T_{n-1} \), then so are its Steklov functions \( I_{ih}(f) \), hence \( U_h(f) \) and the iterates \( U_h^m(f) \) as well. Also, the operators \( D^2 \) (of double differentiation) and \( U_h \) commute (since \( D^2 \) and \( I_{ih} \) clearly commute). Therefore, using the Bohr-Favard inequality with the \((2m)\)-th derivative, we obtain
\[
\| U_h^m(f) \|_{T_{n-1}^\perp} \leq F_{2m} \| D^{2m} U_h^m(f) \| = F_{2m} \| D^{2m} U_h \| \| f \| \leq F_{2m} \| D^2 U_h \| \| f \|. \quad (4.5)
\]

By (4.2), we get \( \| D^2 U_h \| \leq \frac{\pi^2 \mu^2}{h^2} \), hence the conclusion. \(\Box\)

**Remark 4.3** For \( h = \frac{\pi}{n} \), we have equality in (4.4), i.e.,
\[
\| U_h^m \|_{T_{n-1}^\perp} = F_{2m} \mu^2, \quad h = \frac{\pi}{n},
\]
which is attained on the Favard function \( \varphi_n(x) = \sgn \sin nx \). Indeed, for \( h = \frac{\pi}{n} \), we have
\[
\varphi_n(x - ih) - 2 \varphi_n(x) + \varphi_n(x + ih) = \begin{cases} 
-4 \varphi_n(x), & \text{odd } i, \\
0, & \text{even } i,
\end{cases}
\]
and it follows from (4.3) that \( U_h''(\varphi_n) = -\frac{\pi^2 \mu^2}{h^2} \varphi_n \), and respectively
\[
D^{2m} U_h^m(\varphi_n) = (-1)^m \left( \frac{\pi^2 \mu^2}{h^2} \right)^m \varphi_n.
\]

On the other hand, the Bohr-Favard inequality turns into equality on the functions \( f \in T_{n-1}^\perp \) such that \( f^{(2m)}(x) = a \varphi_n(x - b) \), hence on \( U_h^m(\varphi_n) \). Therefore, in (4.5), we have equalities all the way through.
Proposition 4.4 Let $f \in T_{n-1}^\perp$, and let $h > \frac{\pi}{n}\mu$. Then
\[ \|f\| \leq c_{n,2k}(h) \gamma_{2k}^* \omega_{2k}(f,h), \]
where
\[ c_{n,2k}(h) = \left( \cos \frac{\pi}{2} \rho \right)^{-1}, \quad \rho = \frac{\pi \mu}{nh} < 1. \]

Proof. From Proposition 3.2, using the estimate (4.4), we obtain
\[ c_{n,2k}(h) = \sum_{m=0}^{\infty} \|U_h^n\|_{T_{n-1}^\perp} \leq \sum_{m=0}^{\infty} F_{2m} \rho^{2m} = \left( \cos \frac{\pi}{2} \rho \right)^{-1}, \]
the last equality (provided $\rho < 1$) being the Taylor expansion of $\sec \frac{x}{2} = 1/\cos \frac{x}{2}x$. (The latter is usually given in terms of the Euler numbers $E_{2m}$ as $\sec x = \sum_{m=0}^{\infty} \frac{|E_{2m}|}{(2m)!} x^{2m}$, see, e.g., Gradshteyn, Ryzhik [6, § 1.411.9], so we have $\sec \frac{x}{2} = \sum_{m=0}^{\infty} \frac{|E_{2m}|}{(2m)!} x^{2m}$, and we use the fact that $F_{2m} = \frac{|E_{2m}|}{2^{2m}(2m)!}$, see [6, § 0.233.6].) \[ \square \]

Theorem 4.5 If $f \in T_{n-1}^\perp$, then, for any $\alpha > 1$, we have
\[ \|f\| \leq c_{\alpha} \gamma_{2k}^* \omega_{2k}(f, \frac{\alpha n}{\pi}), \quad c_{\alpha} = \left( \cos \frac{\pi}{2\alpha} \right)^{-1}. \]  

Proof. Just put $h = \frac{\pi n}{\alpha}$ in (4.6), and use the fact that $\mu < 1$. \[ \square \]

Let us give some particular cases of Theorem 4.5.

1) $\alpha = 2$, $\quad c_{\alpha} = \left( \cos \frac{\pi}{4} \right)^{-1} = \sqrt{2}$, $\quad \|f\| \leq \frac{1}{2} \gamma_{2k}^* \omega_{2k}(f, \frac{2\pi}{n})$;
2) $\alpha = \frac{3}{2}$, $\quad c_{\alpha} = \left( \cos \frac{\pi}{3} \right)^{-1} = 2$, $\quad \|f\| \leq 2 \gamma_{2k}^* \omega_{2k}(f, \frac{3\pi}{2n})$;
3) $\alpha = \frac{4}{3}$, $\quad c_{\alpha} = \left( \cos \frac{2\pi}{3} \right)^{-1} = 2.61$, $\quad \|f\| \leq 2 \gamma_{2k}^* \omega_{2k}(f, \frac{4\pi}{3n})$;
4) $\alpha = \frac{5}{4}$, $\quad c_{\alpha} = \left( \cos \frac{5\pi}{4} \right)^{-1} = 3.23$, $\quad \|f\| \leq 3 \gamma_{2k}^* \omega_{2k}(f, \frac{5\pi}{4n})$.

From the relations $\cos \frac{x}{2} = \sin \frac{\pi}{2}(1 - x) \geq \frac{\pi}{2}(1 - x^2)$, it follows that, in (4.8),
\[ c_{\alpha} < \frac{4}{\pi} \left( 1 - \frac{1}{\alpha^2} \right)^{-1}, \]
i.e., $c_{\alpha}$ behaves like $\frac{2}{\pi} \frac{1}{\alpha^2}$ as $\alpha \searrow 1$.

Theorem 4.6 If $f \in T_{n-1}^\perp$, then, for $\delta = \frac{\pi}{n}$, we have
\[ \|f\| \leq c_{2k} \gamma_{2k}^* \omega_{2k}(f, \frac{\pi}{n}), \quad c_{2k} = O(\sqrt{2k}). \]

Proof. Putting $h = \frac{\pi n}{\alpha}$ into (4.6), we obtain the inequality (4.10) with the constant
\[ c_{2k} = \left( \cos \frac{\pi}{2} \mu_{2k} \right)^{-1} \leq \frac{4}{\pi} \left( 1 - \mu_{2k}^2 \right)^{-1}, \]
and we are proving in §6 that $1 - \mu_{2k}^2 \approx \frac{1}{\sqrt{2k}}$. \[ \square \]
5 Stechkin inequality for $\frac{\pi}{n} < \delta \leq \frac{2\pi}{n}$

1) Consider the de la Vallée Poussin sum (operator)

$$v_{m,n} = \frac{1}{n-m} \sum_{i=m}^{n-1} s_i,$$  \hspace{1cm} (5.1)

which is an average of $(n-m)$ Fourier sums $s_i$ of degree $i$. For $m = n-1$ and for $m = 0$, it becomes the Fourier sum $s_{n-1}$ and the Fejer sum $\sigma_n = \frac{1}{n} \sum_{i=0}^{n-1} s_i$, respectively.

Since $v_{m,n}(f)$ is the convolution of $f$ with the de la Vallée Poussin kernel $V_{m,n}$, we clearly have

$$\omega_k(v_{m,n}(f), \delta) \leq \|v_{m,n}\| \omega_k(f, \delta),$$

where $\|v_{m,n}\|$ is the norm, or the Lebesgue constant, of the operator $v_{m,n}$.

Stechkin [11] made a detailed study of behaviour of the value $\|v_{m,n}\|$ as a function of $m$ and $n$. We will need just two facts from his work, one of them combined with a later result of Galkin [5].

a) The norm $\|v_{m,n}\|$ depends only on ratio $m/n$, and in a monotone way. Precisely, with

$$\ell(x) := \frac{2}{\pi} \int_0^\infty \left| \frac{\sin xt \cdot \sin t}{t^2} \right| dt,$$

which is (non-trivially) a monotonely increasing function of $x$, we have

$$\|v_{m,n}\| = \ell(x_{m/n}), \quad x_{m/n} := \frac{1 + m/n}{1 - m/n}.$$

b) The values of $\ell$ at integer points can be related to the so-called Watson constants $L_{M/2}$ (for $M = 2N$, they turn into the Lebesgue constants $L_N := \|s_N\|$ of the Fourier operator $s_N$). Namely,

$$\ell(M+1) = L_{M/2},$$

and from the result of Galkin [5] that $L_{M/2} < \frac{4}{\pi^2} \ln(M+1) + 1$, we conclude that

$$\ell(p) < \frac{4}{\pi^2} \ln p + 1 \quad \text{for integer } p,$$

therefore (rather roughly)

$$\ell(x) < \frac{4}{\pi^2} \ln(x+1) + 1 \quad \text{for all } x.$$  \hspace{1cm} (5.3)

2) Now, from definition (5.1), we see firstly that $v_{m,n}(f)$ is a trigonometric polynomial of degree $\leq n-1$, hence

$$E_{n-1}(f) \leq \|f - v_{m,n}(f)\|,$$

and secondly that $v_{m,n}$ acts as identity on $T_m$, therefore

$$f - v_{m,n}(f) \perp T_m.$$
So, we may apply Proposition 4.4 to the difference \( f - v_{m,n}(f) \) to obtain
\[
E_{n-1}(f) \leq \| f - v_{m,n}(f) \| \\
\leq c_{m+1,2k}(h) \gamma_{2k}^* \omega_{2k}(f - v_{m,n}(f), h) \\
\leq c_{m+1,2k}(h) (1 + \| v_{m,n} \|) \gamma_{2k}^* \omega_{2k}(f, h) \\
= \left[ \cos \left( \frac{\pi \mu}{2 (m+1) h} \right) \right]^{-1} \left[ 1 + \ell \left( \frac{1 + m/n}{1 - m/n} \right) \right] \gamma_{2k}^* \omega_{2k}(f, h).
\]

Now, with some parameter \( s \in [0,1) \) which may well depend on \( n \) and \( h \), we put in the last line
\[
m = \lfloor sn \rfloor.
\]
With such an \( m \), we have \( m + 1 > sn \) and \( m/n \leq s \), therefore
\[
E_{n-1}(f) \leq \left[ \cos \left( \frac{\pi \mu}{2 snh} \right) \right]^{-1} \left[ 1 + \ell \left( \frac{1 + s}{1 - s} \right) \right] \gamma_{2k}^* \omega_{2k}(f, h).
\] (5.4)

Finally, taking \( h = \frac{\alpha \pi}{n} \), and evaluating the factor \( 1 + \ell(x_s) \) with the help of (5.3), we obtain
\[
E_{n-1}(f) \leq \left( \cos \frac{\pi \mu}{2 \alpha s} \right)^{-1} \left[ 2 + \frac{4}{\pi^2} \ln \left( \frac{2}{1 - s} \right) \right] \gamma_{2k}^* \omega_{2k}(f, \frac{\alpha \pi}{n}),
\] (5.5)
where we can minimize the right-hand side with respect to \( s \in (\frac{\mu}{\alpha}, 1) \).

3) Now, using the last estimate, we establish Stechkin inequalities for particular \( \alpha \)'s.

**Theorem 5.1** For all \( n \geq 1 \), we have
\[
E_{n-1}(f) \leq c \gamma_{2k}^* \omega_{2k} \left( f, \frac{2\pi}{n} \right), \quad c = 5.
\]

**Proof.** In (5.5), take \( \alpha = 2 \) and majorize \( \mu \) by 1. Then the constant for \( \delta = \frac{2\pi}{n} \) takes the form
\[
c = \left( \cos \frac{\pi}{4s} \right)^{-1} \left[ 2 + \frac{4}{\pi^2} \ln \left( \frac{2}{1 - s} \right) \right].
\]
It turns out that the value \( s = 8/9 \) is almost optimal, and we obtain Stechkin inequality with the constant
\[
c = \left( \cos \frac{9\pi}{32} \right)^{-1} \left[ 2 + \frac{4}{\pi^2} \ln 18 \right] = 4.999144 < 5. \quad (5.6)
\]
To make sure that our step away from 5 is free from a round-off error, we notice that, for \( s = \frac{8}{9} \), we have in (5.4)
\[
\ell \left( \frac{1 + s}{1 - s} \right) = \ell(17) = L_8.
\]
Therefore, in the pass from (5.4) to (5.5), we can use the estimate (5.2) instead of (5.3), thus changing in (5.6) the value \( \ln 18 \) to \( \ln 17 \), and that will give the constant \( c = 4.962628 \). We can make another bit down by computing directly the Lebesgue constant \( L_8 = 2.137730 \), hence getting
\[
c = \left( \cos \frac{9\pi}{32} \right)^{-1} \left[ 1 + L_8 \right] = 4.946034,
\]
so that \( c < 5 \) is secured. \( \square \)
Remark 5.2 Surprising is the fact that, in this theorem, the upper estimate is provided by one and the same linear method of approximation that works for all \( r \) simultaneously. Namely, for any \( r \), the de la Vallée Poussin operator \( v_{m,n} \) with \( m = \lceil \frac{8}{9} n \rceil \) provides

\[
\| f - v_{m,n}(f) \| \leq 5 \gamma^*_r \omega_r \left( f, \frac{2\pi}{n} \right), \quad \forall r \in \mathbb{N}.
\]

Perhaps it makes sense to try to derive such an estimate directly from the properties of \( v_{m,n} \).

Theorem 5.3 For any \( \alpha > 1 \), there exists a constant \( c_\alpha \) that depends only on \( \alpha \) such that

\[
E_{n-1}(f) \leq c_\alpha \gamma^*_{2k} \omega_{2k} \left( f, \frac{\alpha\pi}{n} \right), \quad n \geq 1.
\]

Proof. Putting (a non-optimal) \( s = \frac{1}{\sqrt{\alpha}} \) in (5.5), and again majorizing \( \mu \) by 1, we obtain (5.7) with

\[
c_\alpha = \left( \cos \frac{\pi}{2\sqrt{\alpha}} \right)^{-1} \left( \frac{4}{\pi^2} \ln \left( \frac{2\sqrt{\alpha}}{\sqrt{\alpha} - 1} \right) + 2 \right)
\leq \frac{4}{\pi^2} \left( \frac{4}{\pi^2} \ln \left( \frac{2\sqrt{\alpha}}{\sqrt{\alpha} - 1} \right) + 2 \right),
\]

where we have used the inequality \( \cos \frac{\pi}{2} x \geq \frac{\pi}{4}(1 - x^2) \) for \( |x| \leq 1 \). \( \square \)

6 Stechkin inequality for \( \delta = \frac{\pi}{n} \)

Theorem 6.1 For \( \delta = \frac{\pi}{n} \), and \( r = 2k \), we have

\[
E_{n-1}(f) \leq c_r \left( \frac{\pi}{n} \right) \gamma^*_r \omega_r \left( f, \frac{\pi}{n} \right), \quad n \geq 1,
\]

where

\[
c_r \left( \frac{\pi}{n} \right) = O(\sqrt{n} \ln r).
\]

Proof. From the estimate (5.5), with \( h = \frac{\pi}{n} \) and \( s = \sqrt{\mu} \), we obtain the inequality (6.1) with the constant

\[
c_{2k} \left( \frac{\pi}{n} \right) = \left( \cos \frac{\pi}{2\sqrt{\mu}} \right)^{-1} \left( \frac{4}{\pi^2} \ln \left( \frac{2}{1 - \sqrt{\mu}} \right) + 2 \right)
\leq \frac{4}{\pi^2} \left( \frac{4}{\pi^2} \ln \left( \frac{2}{1 - \sqrt{\mu}} \right) + 2 \right).
\]

The estimate (6.2) follows now from the fact that

\[
1 - \mu_{2k}^2 > \frac{c_1}{\sqrt{2k}}, \quad c_1 = \frac{2}{3},
\]

which we are proving in the next lemma. With the value \( c_1 = \frac{2}{3} \) at hands, we can give the explicit estimate \( c_r \left( \frac{\pi}{n} \right) < 2\sqrt{n} \ln r + 12\sqrt{r} \). \( \square \)
Lemma 6.2 For $\mu_{2k}^2 := \frac{8}{\pi^2} \sum_{\text{odd } i}^k \frac{a_i}{i^2}$, where $a_i := (\frac{2k}{k+i})/\binom{2k}{k}$, we have

$$\frac{c_1}{\sqrt{2k}} < 1 - \mu_{2k}^2 < \frac{c_2}{\sqrt{2k}}, \quad c_1 = \frac{2}{3}, \quad c_2 = \frac{5}{4}. \quad (6.3)$$

Proof. Let us compute the value $\hat{\Delta}_{t}^{2k}(f_0, x)$ for $f_0(x) = \cos x$ at $x = 0$. Since

$$\hat{\Delta}_{t}^{2}(\cos, x) = -\cos(x-t) + 2 \cos x - \cos(x+t) = 2 \cos x(1 - \cos t) = 4 \sin^2 \frac{t}{2} \cos x,$$

we have

$$\hat{\Delta}_{t}^{2k}(f_0, x) \big|_{x=0} = 4k \sin^2 \frac{t}{2}.$$

On the other hand, by the definition,

$$\hat{\Delta}_{t}^{2k}(f_0, x) \big|_{x=0} = \sum_{i=-k}^{k} (-1)^i \binom{2k}{k+i} \cos(x+it) \big|_{x=0} = \binom{2k}{k} \left[ 1 - 2 \sum_{i=1}^{k} (-1)^{i+1} a_i \cos it \right].$$

So, we have

$$1 - 2 \sum_{i=1}^{k} (-1)^{i+1} a_i \cos it = \lambda_k \sin^2 \frac{t}{2}, \quad \lambda_k := \frac{4k}{\binom{2k}{k}}.$$

Integrating both parts twice, first time between 0 and $u$, and then between 0 and $\pi$, we obtain: for the left-hand side

$$\left[ \frac{u^2}{2} + 2 \sum_{i=1}^{k} (-1)^{i+1} \frac{a_i}{i^2} \cos it \right]_{0}^{\pi} = \frac{\pi^2}{2} - 4 \sum_{\text{odd } i}^{k} \frac{a_i}{i^2} = \frac{\pi^2}{2} (1 - \mu_{2k}^2),$$

and for the right-hand side

$$\lambda_k \int_{0}^{\pi} \int_{0}^{u} \sin^{2k} \left( \frac{\tau}{2} \right) du \, dt = \lambda_k \int_{0}^{\pi} (\pi - t) \sin^{2k} \left( \frac{\tau}{2} \right) dt = 4 \lambda_k \int_{0}^{\pi/2} \tau \cos^{2k} (\tau) d\tau$$

(we firstly changed the order of integration and then put $\tau = \frac{\pi}{2} - \frac{t}{2}$). So, equating the rightmost values in the last two lines, we obtain

$$1 - \mu_{2k}^2 = \frac{8}{\pi^2} \frac{4k}{\binom{2k}{k}} \int_{0}^{\pi/2} t \cos^{2k} t \, dt. \quad (6.4)$$

Now, by Wallis inequality, we have

$$\sqrt{\frac{\pi}{2}} \sqrt{\frac{2k}{\pi}} \leq \frac{4k}{\binom{2k}{k}} \leq \sqrt{\frac{\pi}{2}} \sqrt{\frac{2k+1}{\pi}},$$

while the integral admits the two-sided estimate

$$\frac{1}{2k+1} \leq \int_{0}^{\pi/2} t \cos^{2k} (t) \, dt \leq \frac{1}{2k},$$

because $\sin t \leq \frac{\sin t}{t}$ on $[0, \frac{\pi}{2}]$ and $\int_{0}^{\pi/2} \sin(t) \cos^n(t) \, dt = \frac{1}{m+1}$. Hence

$$\frac{8}{\pi^2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{2k}{2k+1}} \leq 1 - \mu_{2k}^2 \leq \frac{8}{\pi^2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{2k+1}{2k}},$$

and (6.3) follows with $c_1 = \frac{8}{\pi^2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{2k}{2k+1}} > \frac{2}{3}$ and $c_2 = \frac{8}{\pi^2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{2k+1}{2k}} < \frac{5}{4}$. \qed
7 On the factor $\sqrt{r}$ at $\delta = \frac{\pi}{n}$

For $\delta = \frac{\pi}{n}$, our estimates for the Stechkin constant (with the lower bound yet to be proved) look as follows:

$$c' \gamma^*_r \leq K_{n,r}(\frac{\pi}{n}) \leq c \sqrt{r} \ln r \gamma^*_r,$$

i.e., the upper and lower bounds do not match. In §2 we already expressed our belief that additional factors on the right are redundant. However, as we show in this section, appearance of the factor $\sqrt{r}$ within our method is unavoidable. (The factor $\ln r$ originates from the use of the de la Vallée Poussin sums, and perhaps can be removed by some more sophisticated technique.)

From our initial steps (3.2)-(3.4), it is easy to see that our upper estimates in all Stechkin inequalities are valid not only for the standard modulus of smoothness $\omega^2_k(f,h)$, but also for the modulus

$$\omega^*_{2k}(f,h) := \left\| \int_{\mathbb{R}} \Delta^k_2(f,\cdot) \phi_h(t) \, dt \right\|,$$

which has a smaller value at every $h$. It is clear that the Stechkin constant defined with respect to a smaller modulus takes larger values, and now we show that, for the modulus $\omega^*_{2k}(f,h)$, the increase at $h = \frac{\pi}{n}$ is exactly by the factor $\sqrt{2k}$.

**Theorem 7.1** For $r = 2k$, we have

$$\frac{\gamma^*_r}{1 - \mu^*_r} \leq \sup_{f \in T^r_{t-1}} \frac{\|f\|_{\omega^*_{2k}(f,\frac{\pi}{n})}}{\omega^*_{2k}(f,\frac{\pi}{n})} \leq \frac{4}{\pi} \frac{\gamma^*_r}{1 - \mu^*_r},$$

where

$$\frac{\gamma^*_r}{1 - \mu^*_r} \asymp \sqrt{r} \frac{\gamma^*_r}{1 - \mu^*_r} \asymp \frac{r}{2^r}.$$

**Proof.** The upper bound was established in (4.10)-(4.11). For the lower bound, take $f_0(x) = \cos nx$. Then

$$\widehat{\Delta^k_2}(f_0, x) = 4^k \sin^2 \left(\frac{nt}{2}\right) \cos nx, \quad \phi_{\pi/n}(t) = \frac{n}{\pi} \left(1 - \frac{n}{\pi} |t|\right), \quad |t| \leq \frac{\pi}{n},$$

hence

$$\omega^*_{2k}(f_0, \frac{\pi}{n}) = \left\| \int_{-\pi/n}^{\pi/n} \Delta^k_2(f_0, \cdot) \phi_{\pi/n}(t) \, dt \right\|$$

$$= 2 \cdot 4^k \int_0^{\pi/n} \sin^2 \left(\frac{nt}{2}\right) \frac{n}{\pi} \left(1 - \frac{n}{\pi} t\right) dt$$

$$= \frac{8}{\pi^2} 4^k \int_0^{\pi/2} \tau \cos^{2k}(\tau) \, d\tau \quad \left(\tau = \frac{\pi}{2} - \frac{nt}{2}\right)$$

$$= \frac{1 - \mu^2_{2k}}{\gamma^*_{2k}},$$

(6.4)

while $\|f_0\| = 1$. \qed
Since also $E_{n-1}(f_0) = 1$, we have the same estimate for the ratio $E_{n-1}(f_0)/\omega^*_r(f_0, \frac{\pi}{n})$, therefore, for the Stechkin constant $K^*_n, (\delta)$ defined with respect to the modulus $\omega^*_r(f, \delta)$, we obtain at $\delta = \frac{\pi}{n}$

$$c' \sqrt{r} \gamma^*_r \leq K^*_n, r \left( \omega_{r}^* (f, \frac{\pi}{n}) \right) := \sup_{f \in C} \frac{E_{n-1}(f)}{\omega^*_r(f, \frac{\pi}{n})} \leq c \sqrt{r} \ln r \gamma^*_r.$$  

8 Lower estimate

Lemma 8.1 For any $n$, $r$ and $\epsilon$, and for any $\delta < \frac{\pi}{r}$, there exists an $f \in C$ such that,

$$E_{n-1}(f) \geq \frac{1}{2} \frac{\gamma^*_r}{\gamma^*_{r-1}} \omega_r(f, \delta) - \epsilon.$$  

Proof. Take the step periodic function 

$$f_0(x) = \begin{cases} 1, & x \in (-\pi, 0]; \\ 0, & x \in (0, \pi]. \end{cases}$$  

For any $x \in [-\pi, \pi]$, and for any $h < \frac{\pi}{r}$, consider the values of this function at the points $x_i = x + ih$, where $0 \leq i \leq r$. It is clear that, for some $m \leq r$, we have either

$$f_0(x_i) = 1, \quad 0 \leq i \leq m, \quad f_0(x_i) = 0, \quad m < i \leq r,$$

or the other way round. Therefore, for the modulus of smoothness $\omega_r(f_0, \delta)$, we have the following relations:

$$\omega_r(f_0, \delta) = \max_{0 < h \leq \delta} x |\Delta^r_h f_0(x)| = \max_{0 < h \leq \delta} x \left| \sum_{i=0}^{r} (-1)^i \binom{r}{i} f_0(x + ih) \right|$$

$$= \max_{0 \leq m \leq r} \left| \sum_{i=0}^{m} (-1)^i \binom{r}{i} \right| = \max_{0 \leq m \leq r} \left| (-1)^m \binom{r-1}{m} \right| = \left( \frac{r-1}{\left\lfloor \frac{r-1}{2} \right\rfloor} \right) = 1/\gamma^*_r,$$

i.e.,

$$\omega_r(f_0, \delta) = 1/\gamma^*_r.$$  

It is also clear that, for the best $L_\infty$-approximation of $f_0$, we have

$$E_{n-1}(f_0) = \frac{1}{2},$$

therefore the result for such an $f_0$ (without $\epsilon$ subtracted).

This is almost what we need except that $f_0$ is not continuous. But we can get a continuous $f$ by smoothing $f_0$ at the points of discontinuity, say, by linearization. For a given $\epsilon$, set

$$f(x) = \frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} f_0(x + t) \, dt.$$
i.e.,

\[ f(x) = \begin{cases} 
1, & x \in [-\pi + \epsilon, -\epsilon]; \\
0, & x \in [\epsilon, \pi - \epsilon]; \\
is linear on \, [-\epsilon, \epsilon] \text{ and } [\pi - \epsilon, \pi + \epsilon].
\]

Then, from the definition (or, more generally, because \( f \) is the convolution of \( f_0 \) with a positive kernel), it follows that

\[ \omega_r(f, \delta) \leq \omega_r(f_0, \delta) = \frac{1}{\gamma^*_r}. \]

As for the best approximation of \( f \), we have

\[ E_{n-1}(f) \geq \frac{1}{2} - \epsilon'. \]

Indeed, since \( E_{n-1}(f) = \| f - t_{n-1} \| \leq \| f \| = 1 \), the polynomial \( t_{n-1} \) of best approximation satisfies \( \| t_{n-1} \| \leq 2 \), therefore, by Bernstein inequality, we have \( \| t'_{n-1} \| \leq 2(n - 1) \), hence, on the interval \([ -\epsilon, \epsilon] \) of the length \( 2\epsilon \), the range of \( t_{n-1} \) is not more than \( 4(n - 1)\epsilon =: 2\epsilon' \), while the function \( f \) on the same interval takes the values 0 and 1.

**Theorem 8.2** For any \( r \), and any \( \delta \leq \frac{\pi}{r} \), we have

\[ K_{n,r}(\delta) := \sup_{f \in C} E_{n-1}(f) \omega_r(f, \delta) \geq c'_r \gamma^*_r \]

where

\[ c'_r = \begin{cases} 
\frac{r}{r + 1}, & r = 2k - 1; \\
1, & r = 2k.
\end{cases} \]

In particular, for any \( r \) and any \( n \geq 2r \) (i.e., when \( \frac{2\pi}{n} \leq \frac{\pi}{r} \)),

\[ K_{n,r}(\frac{2\pi}{n}) := \sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r(f, \frac{2\pi}{n})} \geq c'_r \gamma^*_r, \quad n \geq 2r. \]

**Proof.** The first lower bound is just a reformulation of the previous lemma, because, for \( \gamma^*_r := \left( \frac{r}{\lfloor \frac{r}{2} \rfloor} \right)^{-1} \), we have \( \frac{1}{2} \gamma^*_r = c'_r \gamma^*_r. \)

**Remark 8.3** The order \( r^{1/2} - r^{-r} \) of the lower bound for the Stechkin constant was established earlier by Ivanov [7], but he did not pay attention to the constant (and his extremal function was different from ours).

### 9 Stechkin constants for small \( r \)

For small \( r = 2k \), when \( \mu_r \) is noticeably smaller than 1, our method in §5 will give for the Stechkin constant the upper estimates which are better than \( 5\gamma^*_r \), but they will never be smaller than \( 2\gamma^*_r \) because of the factor \( 1 + \| v_{m,n} \|. \)
Surprisingly, better values (for small \( r \)) which stand quite close to the lower bound \( 1 \cdot \gamma^*_r \) could be obtained through technique of intermediate approximation with Steklov-type functions. (For general \( r \), this technique provides the same overblown estimate \( c_r < r^\alpha r \) as Stechkin’s original proof, therefore a surprise.)

Such a technique is of course well-known (it was introduced probably by Brudnyi [1] and Freud–Popov [4]), and it was exploited repeatedly for proving Stechkin inequalities of various types (e.g., for spline and one-sided approximations). Our only innovation (if any) is the use of the central differences instead of the forward ones, which reduces the constants by the factor \( \left( \frac{2k}{k} \right)^k \), and the will to take a closer look at their actual values.

**Lemma 9.1** We have

\[
E_{n-1}(f) \leq c_{2k} \left( \frac{\alpha \pi}{n} \right) \gamma^*_{2k} \omega_{2k} \left( f, \frac{\alpha \pi}{n} \right),
\]

where

\[
c_{2k} \left( \frac{\alpha \pi}{n} \right) = 1 + F_{2k} \frac{k^{2k}}{(\alpha \pi)^{2k}} \sum_{i=1}^{k} \frac{2b_i}{i^{2k}}, \quad b_i = \left( \frac{2k}{k+i} \right),
\]

and \( F_{2k} \) are the Favard constants.

**Proof.** Given \( f \), with any \( 2k \) times differentiable function \( f_h \), we have

\[
E_{n-1}(f) \leq E_{n-1}(f - f_h) + E_{n-1}(f_h) \leq \| f - f_h \| + \frac{F_{2k}}{n^{2k}} \| f^{(2k)} \| \tag{9.2}
\]

where we used the Favard inequality for the best approximations of \( f_h \). A typical choice of \( f_h \) is via the Steklov functions of order \( 2k \):

\[
I_{i\hbar}(f, x) := \frac{1}{(h/k)^{2k}} \int_{-h/2k}^{h/2k} \cdots \int_{-h/2k}^{h/2k} f(x - i(t_1 + \cdots + t_{2k})) \, dt_1 \cdots dt_{2k},
\]

\[
I_{i\hbar}^{(2k)}(f, x) = \frac{(-1)^k}{(ih/k)^{2k}} \sum_{i=1}^{\frac{2k}{h/k}} I_{i\hbar/k} f(x),
\]

namely

\[
f_h := \frac{1}{(2k)} \sum_{i=0}^{k} (-1)^{i+1} \left( \frac{2k}{k+i} \right) I_{i\hbar}(f) = \gamma^*_{2k} \sum_{i=1}^{k} (-1)^{i+1} 2b_i I_{i\hbar}(f).
\]

Then

\[
\| f - f_h \| \leq \gamma^*_{2k} \omega_{2k}(f, h),
\]

\[
\| f^{(2k)}_h \| \leq \gamma^*_{2k} \sum_{i=1}^{k} \frac{2b_i}{(ih/k)^{2k}} \omega_{2k}(f, ih/k) \leq \gamma^*_{2k} \omega_{2k}(f, h) \frac{k^{2k}}{h^{2k}} \sum_{i=1}^{k} \frac{2b_i}{i^{2k}},
\]

whence applying (9.2)

\[
E_{n-1}(f) \leq c_{2k}(h) \gamma^*_{2k} \omega_{2k}(f, h), \quad c_{2k}(h) = 1 + F_{2k} \frac{k^{2k}}{(ih)^{2k}} \sum_{i=1}^{k} \frac{2b_i}{i^{2k}},
\]
and we take $h = \frac{\alpha \pi}{n}$.

In (9.1), we can obtain a small value only if $\frac{k}{\alpha \pi} < 1$, i.e., we may try $k = (1, 2, 3)$ for $\alpha = 1$, and $k = (1, 2, 3, 4, 5)$ for $\alpha = 2$. So we did (dropping those values for which the resulting constants in (9.1) were not close to 1).

**Theorem 9.2** For $\delta = \frac{\pi}{n}$ and $\delta = \frac{2\pi}{n}$, we have

$$E_{n-1}(f) \leq c_r(\delta) \gamma_r^* \omega_r(f, \delta),$$

where $c_{2k-1}(\delta) = c_{2k}(\delta)$, and the values of $c_{2k}(\delta)$ are given below

| $c_2\left(\frac{\pi}{n}\right)$ | $c_4\left(\frac{\pi}{n}\right)$ | $c_2\left(\frac{2\pi}{n}\right)$ | $c_4\left(\frac{2\pi}{n}\right)$ | $c_6\left(\frac{2\pi}{n}\right)$ |
|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\frac{1}{4}$                  | $\frac{2}{7}$                   | $\frac{1}{16}$                  | $\frac{1}{9}$                   | $\frac{1}{11}$ |

**Proof.** We will use the following values: $F_2 = \frac{\pi^2}{8}$, $F_4 = \frac{5\pi^4}{384}$, $F_6 = \frac{61\pi^6}{46080}$.

1) For $2k = 2$, we have

$$c_2\left(\frac{\alpha \pi}{n}\right) = 1 + \frac{\pi^2}{8} \left(\frac{n}{\alpha \pi}\right)^2 = 1 + \frac{1}{4\alpha^2}.$$  

With $\alpha = 1$ and $\alpha = 2$, we obtain $c_2\left(\frac{\pi}{n}\right) = \frac{5}{4}$ and $c_2\left(\frac{2\pi}{n}\right) = \frac{17}{16}$. Also, with $\alpha = \frac{1}{2}$, we obtain the remarkable inequality

$$E_{n-1}(f) \leq 1 \cdot \omega_2\left(f, \frac{\pi}{2n}\right).$$

2) For $2k = 4$,

$$c_4\left(\frac{\alpha \pi}{n}\right) = 1 + \frac{\pi^4}{384} \left(\frac{n}{\alpha \pi}\right)^4 \cdot 2 \left[\frac{4}{14} + \frac{1}{24}\right] = 1 + \frac{325}{192\alpha^4}.$$  

With $\alpha = 1$ and $\alpha = 2$, we obtain $c_4\left(\frac{\pi}{n}\right) = \frac{517}{192} = 2.6927$, and $c_4\left(\frac{2\pi}{n}\right) = \frac{3397}{3072} = 1.1058$.

3) For $2k = 6$, with $\alpha = 2$, we have

$$c_6\left(\frac{2\pi}{n}\right) = 1 + \frac{61\pi^6}{46080} \left(\frac{2\pi}{n}\right)^6 \cdot 2 \left[\frac{15}{14} + \frac{6}{26} + \frac{1}{36}\right] = 1.4552 < \frac{1}{2}.$$  

Theorem 9.2 provides a certain support to our Conjecture 2.1, which says, in particular, that, for even $r = 2k$, and for $\delta \geq \frac{\pi}{n}$, the best constant in the Stechkin inequality has the value $K_{n,r}(\delta) = 1 \cdot \gamma_r^*$.

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**References**

[1] Yu. A. Brudnyi, On a theorem of local best approximations, Kazan. Gos. Univ. Uchen. Zap. 124 (1964), vol. 6, 43–49 (in Russian).
[2] N. I. Chernykh, The best approximation of periodic functions by trigonometric polynomials in $L_2$, Mat. Zametki 2 (1967), 513–522 = Math. Notes 2 (1967), no. 5-6, 818–821.

[3] R. A. DeVore, G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, 1993.

[4] G. Freud, V. A. Popov, Certain questions connected with approximation by spline-functions and polynomials, Studia Sci. Math. Hungar. 5 (1970), 161–171 (in Russian).

[5] P. V. Galkin, Estimate for Lebesgue constants, Trudy Mat. Inst. Steklov 109 (1971), 3–5 = Proc. Steklov Inst. Math. 109 (1971), 1–4.

[6] I. S. Gradshteyn, I. M. Ryzhik, *Table of integrals, series, and products*, 6-th ed, San Diego, Academic Press, 2000.

[7] V. I. Ivanov, On the approximation of functions in $L_p$ spaces, Mat. Zametki 56 (1994), no. 2, 15–40 = Math. Notes 56 (1994), no. 1-2, 770–789.

[8] L. V. Kantorovich, G. P. Akilov, *Functional Analysis*, 2-oe izd, Moskva, Nauka, 1977 = Oxford, Pergamon, 1982.

[9] N. P. Korneichuk, The exact constant in D. Jackson’s theorem on uniform approximation of continuous periodic functions, Doklady AN SSSR 145(1962), 514-515 = Sov. Math. Dokl. 3 (1962), 1040-1041.

[10] S. B. Stechkin, On the order of the best approximations of continuous functions, Izv. Akad. Nauk SSSR, Ser. Mat. 15 (1951), 219–242 (in Russian).

[11] S. B. Stechkin, On de la Vallée Poussin sums, Doklady Akad. Nauk SSSR, 80 (1951), no. 4, 545–548 (in Russian).

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