AN ISOPERIMETRIC INEQUALITY FOR GAUSS–LIKE PRODUCT MEASURES

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Abstract. This paper deals with various questions related to the isoperimetric problem for smooth positive measure $d\mu = \varphi(x)\,dx$, with $x \in \Omega \subset \mathbb{R}^N$. Firstly we find some necessary conditions on the density of the measure $\varphi(x)$ that render the intersection of half spaces with $\Omega$ a minimum in the isoperimetric problem. We then identify the unique isoperimetric set for a wide class of factorized finite measures. These results are finally used in order to get sharp inequalities in weighted Sobolev spaces and a comparison result for solutions to boundary value problems for degenerate elliptic equations.

Key words: Relative isoperimetric inequalities, Polya-Szegö principle, Degenerate elliptic equations.

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1. Introduction

This paper deals with relative isoperimetric inequalities in the setting of manifold with density. More precisely let $\Omega$ be a Lebesgue measurable set in $\mathbb{R}^N$ and let $\mu$ be a positive finite measure on $\Omega$ given by

\[(1.1) \quad d\mu(x) = \varphi(x)\,dx,\]

where $\varphi$ is a positive function in $C^0(\Omega)$ and $\mu(\Omega) < +\infty$. For any Borel measurable subset $M$ of $\Omega$, the $\mu$–perimeter of $M$ relative to $\Omega$ is given by

\[(1.2) \quad P_\mu(M, \Omega) := \sup \left\{ \int_M \text{div} (\varphi v) \, dx : v \in C^1_0(\Omega, \mathbb{R}^N), |v| \leq 1 \text{ in } \Omega \right\}.\]

As well known, the above distributional definition of weighted perimeter is equivalent to the following

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The function \( \varphi(x) \), which appears both in the volume and in the perimeter, is called the 
\textit{density}. We say that a set is isoperimetric or solves the isoperimetric problem relative to 
\( \Omega \) if it minimizes the weighted perimeter \( P_{\mu}(M, \Omega) \) among all the sets \( M \subset \Omega \) with fixed 
weighted measure \( \mu(M) \). This subject has attracted a growing interest starting from the 
papers by Sudakov-Tsirel’son and Borell (see \cite{30} and \cite{7}) on the isoperimetric problem 
for Gaussian density, where it turned out that the isoperimetric set is a half-space. Since 
then the isopermetric problem has been solved for various class of weights (see, e.g., \cite{19}, 
\cite{6}, \cite{26}, \cite{16}, \cite{9}, \cite{5}, \cite{28}, \cite{10}, \cite{29}, \cite{12}, \cite{4}, \cite{11}, \cite{17}, \cite{20} and \cite{13}). Clearly such a 
bibliography if far from being exhaustive.

We are interested in two types of questions. Firstly we find some necessary conditions on 
smooth positive measures \( \mu \) that render the intersection of half spaces with \( \Omega \) a minimum in 
the relative isoperimetric problem. Among other things, we show that the weight function 
\( \varphi \) must be in separated form

\[
\varphi(x) = \rho(x_1, \ldots, x_{N-1})\sigma(x_N)
\]

for some positive functions \( \rho \) and \( \sigma \) (see Theorem \ref{thm:sep} in the next section). In Theorem \ref{thm:main} 
our main result, we identify the unique isoperimetric set for a wide class of factorized finite 
measures.

In order to state this last result we need some notation. For \( i = 1, \ldots, N-1 \), \( N \geq 2 \), let 
\( -\infty \leq a_i < b_i \leq +\infty \), and let \( A_i \in C^1(a_i, b_i) \) be real functions such that

\[
A_i'(x) \geq 1 \quad \text{on} \quad (a_i, b_i),
\]

and

\[
\lim_{x \to a_i^+} A_i(x) = -\infty \quad \text{and} \quad \lim_{x \to b_i^-} A_i(x) = +\infty.
\]

Further, let

\[
S' := (a_1, b_1) \times \cdots \times (a_{N-1}, b_{N-1}) \quad \text{and} \quad S := S' \times \mathbb{R},
\]

and, finally, let \( \mu \) be the measure on \( S \), given by

\[
d\mu(x) := \varphi(x) \, dx' \, dx_N = \varphi(x) \, dx,
\]

where

\[
\varphi(x) := \exp \left\{ - \sum_{i=1}^{N-1} \frac{A_i(x_i)^2}{2} \, - \frac{x_N^2}{2} \right\} \prod_{i=1}^{N-1} A_i'(x_i), \quad x \in S.
\]
If \( \lambda \in \mathbb{R} \), let \( S_\lambda \) be the intersection of the halfspace \( \{ x_N > \lambda \} \) with \( S \), that is, 
\[
S_\lambda = S' \times (\lambda, \infty).
\]

**Theorem 1.1.** Let \( M \) be a Lebesgue measurable subset of \( S \) and fix \( \lambda \) such that
\[
\mu(M) = \mu(S_\lambda).
\]
Then
\[
P_\mu(M, S) \geq P_\mu(S_\lambda, S).
\]
Moreover equality holds in (1.8) if and only \( M = S_\lambda \).

As we will show in Corollary 4.1, the conclusion of Theorem 1.1 holds for measures \( \mu \) of the type
\[
d\mu(x) := \exp\left\{ -\frac{|x|^2}{2} - \sum_{i=1}^{N-1} B_i(x_i) \right\} \, dx,
\]
where \( B_i \in C^2(a_i, b_i) \) with \( B''_i(x_i) \geq 0 \) on \((a_i, b_i), (i = 1, \ldots, N - 1)\).

Our isoperimetric inequality Theorem 1.1 is proved in Section 4. It generalizes the results contained in [28] in two directions. We consider more general factorized perturbations of the Gaussian measure and we allow these perturbations to affect not just one but \( N - 1 \) variables. Note that in view of Lemma 3.1 and the remark following Corollary 4.1 in Section 4, the weight function in (1.6) is indeed more general than the one of (1.9). The main ingredient in the proof of Theorem 1.1 consists in using a map that coincides with the optimal transport Brenier map and that pushes the measure \( d\mu \) forward to the Gaussian measure. We explicitly remark that the relevant property of the gradient of such a map is proved by means of elementary and self-contained tools. While in [28] the analogous question is faced by using a result by Cafarelli (see [15]). An important example for (1.9) is given by \( a_i = 0, b_i = +\infty, B_i(x_i) = -k_i \log x_i \) with \( k_i \geq 0, (i = 1, \ldots, N - 1) \), that is,
\[
d\mu(x) = \exp\left\{ -\frac{|x|^2}{2} \right\} \prod_{i=1}^{N-1} x_i^{k_i} \, dx.
\]

Finally, in Section 5, using a kind of symmetrization, related to the isoperimetric inequalities that we have proved, we give some sharp apriori bounds to the solutions of a class of elliptic second order Pde’s (see Theorem 5.1).

### 2. Necessary Conditions

Below we will introduce some weighted spaces: for \( p \in [1, +\infty] \), let \( L^p(\Omega, d\mu) \) be the standard weighted \( L^p \)-space (corresponding to the weight \( d\mu = \varphi(x) dx \)). By \( W^{1,2}(\Omega, d\mu) \) we denote the weighted Sobolev space,
\[
W^{1,2}(\Omega, d\mu) := \{ u \in W^{1,1}_{loc}(\Omega) : u, |\nabla u| \in L^2(\Omega, d\mu) \}.
\]
We begin our analysis with some necessary conditions on smooth positive finite measures $\mu$ that render the intersection of half spaces with $\Omega$ a minimum in the relative isoperimetric problem.

Following [29] and [28], we introduce the notion of stationarity and stability of sets. Let $\Omega$ be a smooth set with boundary $\Sigma$ and inward unit normal vector $\nu$. We consider a one-parameter variation $\{\phi_t\}_{|t|<\epsilon} : \mathbb{R}^N \to \mathbb{R}^N$ with associated infinitesimal vector field $X = d\phi_t/dt$ with normal component $u = \langle X, \nu \rangle$. Let $\Omega_t = \phi_t(\Omega)$ and $\Sigma_t = \phi_t(\Sigma)$. The volume and perimeter functions of the variation are $V(t) := \mu(\Omega_t)$ and $P(t) := P_{\mu}(\Omega_t)$, respectively. We say that a given variation $\{\phi_t\}_{t}$ preserves volume if $V(t)$ is constant for any small $|t|$. We say that $\Omega$ is stationary if $P'(0) = 0$ for any volume-preserving variation. Obviously any isoperimetric region is also stationary. Finally, we say that $\Omega$ is stable if it is stationary and if $P''(0) \geq 0$ for any volume-preserving variation of $\Omega$. We note that the first and second variation of the volume and perimeter, $V'(0)$, $P'(0)$, $V''(0)$ and $P''(0)$, respectively, were given in [29].

The following notation for points and the gradient in $\mathbb{R}^N$ will be in force throughout the paper

$$x = (x', x_N), \quad x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}, \quad x_N \in \mathbb{R}$$

and

$$\nabla = (\nabla', \partial/\partial x_N), \quad \nabla' = (\partial/\partial x_1, \ldots, \partial/\partial x_{N-1}).$$

If $\Omega'$ is a domain in $\mathbb{R}^{N-1}$ we set

$$\Omega_\lambda := \{(x', x_N) : x_N > \lambda, x' \in \Omega'\}, \quad \lambda \in \mathbb{R}.$$

Our first result is

**Theorem 2.1.** Let $\Omega := \Omega' \times \mathbb{R}$ where $\Omega'$ is a domain in $\mathbb{R}^{N-1}$ with Lipschitz boundary, and let $\mu$ be a measure on $\Omega$ given by

$$d\mu(x) = \varphi(x) \, dx, \quad x \in \Omega,$$

where $\varphi \in C^1(\Omega)$ and $\varphi(x) > 0$ on $\Omega$.

(i) If $\Omega_\lambda$ is stationary in the relative isoperimetric problem for $\mu$ and $\Omega$, for every $\lambda \in \mathbb{R}$, then

$$\varphi(x) = \rho(x')\sigma(x_N) \quad \forall x \in \Omega,$$

where $\rho \in C^1(\Omega')$ and $\sigma \in C^1(\mathbb{R})$ are positive.

(ii) If $S_\lambda$ is stable in the relative isoperimetric problem for $\mu$ and $\Omega$, for every $\lambda \in \mathbb{R}$, then

$$\kappa_1 \geq \tau,$$

where

$$\tau := \sup \left\{ \left( \frac{\sigma'(t)}{\sigma(t)} \right)^2 - \frac{\sigma''(t)}{\sigma(t)} : t \in \mathbb{R} \right\},$$

(2.5)
\[ (2.6) \quad \kappa_1 := \inf \left\{ \frac{\int_{\Omega'} |\nabla' v|^{2} \rho \, dx'}{\int_{\Omega'} v^{2} \rho \, dx'} : v \in W^{1,2}(\Omega', \rho \, dx'), v \neq 0 \right\}. \]

**Remark 2.1.** (a) Observe that \( \kappa_1 \) is the first nontrivial eigenvalue of the Neumann problem

\[ (2.7) \begin{cases} - \sum_{k=1}^{N-1} \frac{\partial}{\partial x_k} \left( \rho \frac{\partial}{\partial x_k} u \right) = \kappa \rho u & \text{in } \Omega' \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega', \end{cases} \]

where \( u \in W^{1,2}(\Omega', \rho \, dx') \), and \( n \) is the exterior unit normal to \( \partial \Omega' \).

(b) Let \( \Omega = \mathbb{R}^N \). If \( \sigma(t) = e^{-ct^2} \) for some \( c > 0 \), then \( \tau = 2c \), and if \( \rho(x') = e^{-c|x|^2} \), then also \( \kappa_1 = 2c \) (see [32], p.105 ff.), so that condition (2.4) is satisfied for Gauss measures, \( \varphi(x) = e^{-c|x|^2}, (x \in \mathbb{R}^N) \).

**Proof of Theorem 2.1:** Proceeding similarly as in [9], we define volume-preserving perturbations from \( \Omega_\lambda \). Let \( u \in C^2(\Omega') \). Then the Implicit Function Theorem tells us that there exists a number \( \varepsilon_0 > 0 \) and a function \( s \in C^2(-\varepsilon_0, \varepsilon_0) \) with \( s(0) = 0 \), such that

\[ \Omega_\lambda(\varepsilon) := \{ (x', x_N) : x_N > u(x', \varepsilon), x' \in \Omega \}, \]

where

\[ u(x', \varepsilon) := \lambda + \varepsilon u(x') + s(\varepsilon), \]

and \( \Omega_\lambda \) have the same \( \mu \)-measure, that is

\[ \mu(\Omega_\lambda(\varepsilon)) = \int_{\Omega'} \int_{u(x', \varepsilon)}^{+\infty} \varphi(x', t) \, dt \, dx' = \mu(\Omega_\lambda). \]

This implies

\[ \mu(\Omega_\lambda(\varepsilon)) = \int_{\Omega'} \varphi(x', u(x', \varepsilon)) (u(x') + s'(\varepsilon)) \, dx', \]

for \( |\varepsilon| < \varepsilon_0 \). Writing \( s_1 := s'(0) \) and \( s_2 := s''(0) \), we obtain

\[ 0 = \frac{\partial}{\partial \varepsilon} \mu(\Omega_\lambda(\varepsilon)) \bigg|_{\varepsilon=0} = \int_{\Omega'} \varphi(x', \lambda) (u(x') + s_1) \, dx'. \]

Further, we have

\[ P_\mu(\Omega_\lambda(\varepsilon), \Omega) = \int_{\Omega'} \varphi(x', u(x', \varepsilon)) \sqrt{1 + \varepsilon^2 |\nabla' u(x')|^2} \, dx', \]
so that

$$
\frac{\partial}{\partial \varepsilon} P_{\mu}(\Omega_{\lambda}(\varepsilon), \Omega)
= \int_{\Omega'} \left\{ \varphi_{xN}(x', u(x', \varepsilon)) (u(x') + s'(\varepsilon)) \sqrt{1 + \varepsilon^2 |\nabla' u(x')|^2 + \varepsilon \varphi(x', u(x', \varepsilon)) (1 + \varepsilon^2 |\nabla' u(x')|^2)^{-1/2} |\nabla' u(x')|^2} \right\} dx'.
$$

(2.12)

Now assume that $\Omega_{\lambda}$ is stationary for every $\lambda \in \mathbb{R}$. Then (2.12) gives

$$
0 = \frac{\partial}{\partial \varepsilon} P_{\mu}(\Omega_{\lambda}(\varepsilon), \Omega) \bigg|_{\varepsilon=0} = \int_{\Omega'} \varphi_{xN}(x', \lambda)(u(x') + s_1) dx'.
$$

This, together with (2.10) implies that $\int_{\Omega'} \varphi_{xN}(x', \lambda)v(x') dx' = 0$ for all functions $v \in C^1(\Omega')$ satisfying $\int_{\Omega'} \varphi(x', \lambda)v(x') dx' = 0$. Then the Fundamental Lemma in the Calculus of Variations tells us that there is a number $k = k(\lambda) \in \mathbb{R}$ such that

$$
\varphi_{xN}(x', \lambda) = k(\lambda) \varphi(x', \lambda) \quad \forall x' \in \Omega',
$$

which implies (2.3). Hence we have from (2.9)

$$
0 = \int_{\Omega'} \rho(x') \sigma(u(x', \varepsilon))(u(x') + s'(\varepsilon)) dx'.
$$

For $\varepsilon = 0$ this yields

$$
0 = \int_{\Omega'} \rho(x')(u(x') + s_1) dx'.
$$

(2.16)

Differentiating (2.16) we obtain for $\varepsilon = 0$,

$$
0 = \int_{\Omega'} \rho(x') 
\left[ (u(x') + s_1)^2 \sigma'(\lambda) + s_2 \sigma(\lambda) \right] dx'.
$$

(2.17)

(ii) Next assume that $\varphi \in C^2(\Omega)$, and that $\Omega_{\lambda}$ is stable for every $\lambda \in \mathbb{R}$. Then $\rho \in C^2(\Omega')$ and $\sigma \in C^2(\mathbb{R})$. First, by (2.3) and (2.12) we have

$$
\frac{\partial}{\partial \varepsilon} P_{\mu}(\Omega_{\lambda}(\varepsilon), \Omega)
= \int_{\Omega} \rho(x') \left\{ \sigma'(u(x', \varepsilon))(u(x') + s'(\varepsilon)) \sqrt{1 + \varepsilon^2 |\nabla' u(x')|^2 + \varepsilon \sigma(u(x', \varepsilon)) (1 + \varepsilon^2 |\nabla' u(x')|^2)^{-1/2} |\nabla' u(x')|^2} \right\} dx'.
$$

Differentiating this gives

$$
0 \leq \frac{\partial^2}{\partial \varepsilon^2} P_{\mu}(\Omega_{\lambda}(\varepsilon)) \bigg|_{\varepsilon=0}
= \int_{\Omega'} \rho(x') \left\{ \sigma''(\lambda) (u(x') + s_1)^2 + \sigma'(\lambda)s_2 + \sigma(\lambda)|\nabla' u(x')|^2 \right\} dx'.
$$
In view of (2.17) we obtain
\[ \int_{\Omega'} \rho(x') |\nabla' u(x')|^2 \, dx' \geq \left( \frac{\sigma'(\lambda)}{\sigma(\lambda)} \right)^2 \int_{\Omega'} \rho(x') (u(x') + s_1)^2 \, dx'. \]
Hence (2.4) follows by (2.16).

3. A one-dimensional auxiliary result

Let \( I := (a, b) \), where \(-\infty \leq a < b \leq +\infty\) and \( B \in C^2(I) \) with \( B''(x) \geq 0 \) on \( I \). Further, let
\[
E(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} \, dt, \quad y \in \mathbb{R},
\]
\[
c := \frac{1}{\int_{a}^{b} e^{-t^2/2-B(t)} \, dt},
\]
and let \( A \in C^3(I) \) be given by
\[
A(x) := E^{-1}\left( c \int_{a}^{x} e^{-t^2/2-B(t)} \, dt \right), \quad x \in I.
\]
Note that the convexity of \( B \) ensures the convergence of the integrals on the right-hand sides of (3.1) and (3.2), and that
\[
e^{-A(x)^2/2} A'(x) = c \sqrt{2\pi} e^{-x^2/2-B(x)}, \quad x \in I,
\]
\[
\lim_{x \to a^+} A(x) = -\infty, \quad \lim_{x \to b^-} A(x) = +\infty.
\]
We also emphasize that the map \( A \) of (3.2) coincides with the optimal transport Brenier map pushing the measure
\[
d\mu_1(x) := ce^{-x^2/2-B(x)} \, dx,
\]
defined on \((a, b)\), forward to the one-dimensional Gauss measure,
\[
d\gamma_1(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy,
\]
(see [15], Thm. 1 and 2). Hence we can use a result of [15], Thm.11, to obtain the following Lemma 3.1. For the convenience of the reader, we include an elementary proof.

**Lemma 3.1.** Under the assumptions above,
\[
A'(x) \geq 1, \quad x \in I.
\]
Proof: We first claim:

\[(3.6) \text{ If } A' \text{ has a local minimum at } x_0 \in I, \text{ then } A'(x_0) \geq 1.\]

Assume that \(A'\) has a local minimum at \(x_0 \in I\) and \(A'(x_0) < 1\). Identity \((3.3)\) gives

\[
\begin{align*}
A''(x) &= A'(x) (A(x)A'(x) - x - B'(x)) \quad \text{and} \\
A'''(x) &= A''(x) (A(x)A'(x) - x - B'(x)) + \\
& \quad + A'(x) ([A^2 + A(x)A''(x) - 1 - B''(x)])
\end{align*}
\]

on \(I\). Since \(A''(x_0) = 0 \leq A'''(x_0)\), this implies

\[
0 \leq A'(x_0) ([A'(x_0)]^2 - 1 - B''(x_0)) \\
\leq A'(x_0) ([A'(x_0)]^2 - 1) < 0,
\]

a contradiction. Hence \((3.6)\) holds.

This implies that \((3.5)\) holds for points inside \(I\). It remains to show that

\[(3.7) \quad \liminf_{x \to a^+} A'(x) \geq 1\]

and

\[(3.8) \quad \liminf_{x \to b^-} A'(x) \geq 1.\]

We only show \((3.7)\). The proof of \((3.8)\) is similar and is left to the reader.

Assume by absurd that

\[
\liminf_{x \to a^+} A'(x) =: L < 1.
\]

By \((3.6)\) this implies that \(\lim_{x \to a^+} A'(x)\) exists and

\[(3.9) \quad \lim_{x \to a^+} A'(x) = L.\]

By \((3.3)\) and \((3.4)\), this means in particular that \(\lim_{x \to a^+} (x^2/2 + B(x)) = +\infty\). In view of the convexity of \(B\), we deduce

\[(3.10) \quad \lim_{x \to a^+} (x + B'(x)) = -\infty.\]
Then the generalized Mean Value Theorem tells us that for every $x \in (a, b)$ there exists a number $y \in (a, x)$ such that

$$A'(x) = c\sqrt{2\pi} \frac{e^{-x^2/2-B(x)}}{e^{-A(x)^2/2}}$$

$$= c\sqrt{2\pi} \frac{[e^{-y^2/2-B(y)}]'}{[e^{-A(y)^2/2}]'}$$

$$= c\sqrt{2\pi} \frac{(y + B'(y))e^{-y^2/2-B(y)}}{A(y)A'(y)e^{-A(y)^2/2}}$$

$$= \frac{y + B'(y)}{A(y)}.$$

In view of (3.4) and (3.10) and since $L < 1$, we find a strictly decreasing sequence $\{x_n\}$ with $x_n \to a$ such that

$$1 - \frac{A(x_n)}{x_n + B'(x_n)} \geq \frac{L(L + 1)}{2}.$$

Using once more the generalized Mean Value Theorem and (3.12) we find another sequence $\{y_n\}$ with $x_{n+1} < y_n < x_n$ and such that

$$\frac{x_{n+1} + B'(x_n+1)}{A(x_{n+1})} = \frac{x_n + B'(x_n) - x_{n+1} - B'(x_{n+1})}{A(x_n) - A(x_{n+1})} \cdot \frac{1 - \frac{A(x_n)}{x_n + B'(x_n)}}{1 - \frac{A(x_n)}{x_{n+1} + B'(x_{n+1})}}$$

$$= \frac{1 + B''(y_n)}{A'(y_n)} \cdot \frac{1 - \frac{A(x_n)}{x_n + B'(x_n)}}{1 - \frac{x_n + B'(x_n)}{x_{n+1} + B'(x_{n+1})}}$$

$$\geq \frac{1 + B''(y_n)}{2A'(y_n)} \cdot \frac{L(L + 1)}{2}$$

$$\geq \frac{L(L + 1)}{2A'(y_n)} \to \frac{L + 1}{2}, \quad \text{as } n \to \infty.$$

Hence

$$\limsup_{x \to a^+} \frac{x + B'(x)}{A(x)} \geq \frac{L + 1}{2}.$$

By (3.11) this means that also

$$\limsup_{x \to a^+} A'(x) \geq \frac{L + 1}{2} > L,$$

contradicting (3.9). It follows that $L \geq 1$. \qed
4. The isoperimetric inequality

In this section we prove our main result Theorem 1.1. Let $\gamma_N$ denote the $N$-dimensional Gauss measure on $\mathbb{R}^N$, given by

$$d\gamma_N(x) := (2\pi)^{-N/2}e^{-|x|^2/2}dx.$$  

(4.1)

For any $U$ Lebesgue measurable subset of $\mathbb{R}^N$, let $P_{\gamma_N}(U)$ denotes its Gaussian perimeter.

Proof of Theorem 1.1: Define a diffeomorphism $T$ between $S$ and $\mathbb{R}^N$, by

$$T(x', x_N) := (\tilde{T}(x'), x_N),$$

where

$$\tilde{T}(x') := (A_1(x_1), \ldots, A_{N-1}(x_{N-1})), \ (x', x_N) \in S,$$

and let

$$H_\lambda := \{(x', x_N) : x_N > \lambda, x' \in \mathbb{R}^{N-1}\}.$$

Clearly we have

$$T(S_\lambda) = H_\lambda \text{ and } \mu(M) = (2\pi)^{N/2}\gamma_N(T(M)), \ \mu(S_\lambda) = (2\pi)^{N/2}\gamma_N(T(H_\lambda)).$$

Hence (1.7) together with the isoperimetric inequality in Gauss space yields

$$P_{\gamma_N}(T(M)) \geq P_{\gamma_N}(H_\lambda).$$

(4.2)

Since also

$$P_\mu(S_\lambda, S) = \int_{S'} \varphi(x', \lambda) \, dx'$$

$$= \int_{\mathbb{R}^{N-1}} \exp \left\{ -\frac{|x'|^2 + \lambda^2}{2} \right\} \, dx'$$

$$= (2\pi)^{N/2}P_{\gamma_N}(H_\lambda),$$

it remains to show that

$$P_\mu(M, S) \geq (2\pi)^{N/2}P_{\gamma_N}(T(M)).$$

(4.3)

To prove (4.3), we first consider the case that $\Sigma$ is an open subset of $S \cap \partial M$ given in the form

$$\Sigma = \{(x', u(x')) : x' \in \Sigma'\},$$

(4.4)

where $u \in C^1(\Sigma')$ and $\Sigma'$ is an open subset of $S'$. We write $y' \equiv \tilde{T}(x')$, $v(y') := u(x')$, $(x' \in \Sigma')$, and $\tilde{T}(\Sigma') := \{(\tilde{T}(x') : x' \in \Sigma')$, so that

$$T(\Sigma) = \{(y', v(y')) : y' \in \tilde{T}(\Sigma')\}.$$
Since $A_i'(x_i) \geq 1$, $(i = 1, \ldots, N-1, x' \in S')$, we find

$$\int_{\Sigma} \varphi(x) \mathcal{H}_{N-1}(dx)$$

$$= \int_{\Sigma'} \exp \left\{ -\sum_{i=1}^{N-1} \frac{A_i(x_i)^2}{2} - \frac{u(x')^2}{2} \right\} \prod_{i=1}^{N-1} A_i'(x_i) \sqrt{1 + \sum_{i=1}^{N-1} u_{x_i}(x')^2} dx'$$

$$= \int_{\tilde{T}(\Sigma')} \exp \left\{ -\frac{|y'|^2}{2} - \frac{v(y')^2}{2} \right\} \sqrt{1 + \sum_{i=1}^{N-1} v_{y_i}(y')^2} \, dy'$$

$$\geq \int_{\tilde{T}(\Sigma')} \exp \left\{ -\frac{|y'|^2}{2} - \frac{v(y')^2}{2} \right\} \sqrt{1 + \sum_{i=1}^{N-1} v_{y_i}(y')^2} \, dy'$$

(4.5)

$$= \int_{T(\Sigma)} e^{-|x|^2/2} \mathcal{H}_{N-1}(dx).$$

Next assume that $S \cap \partial M$ is a finite, disjoint union of graphs $\Sigma_k$ as in (4.4), and of a compact set $U$ whose projection into the $x'$-hyperplane has $\mathcal{H}_{N-1}$-measure zero,

(4.6)

$$S \cap \partial M = U \cup \bigcup_k \Sigma_k.$$

Clearly we have

(4.7)

$$\int_U \varphi(x) \mathcal{H}_{N-1}(dx) = \int_{T(U)} e^{-|x|^2/2} \mathcal{H}_{N-1}(dx).$$

Using (4.5) and (4.7) we find

$$P_\mu(M, S) = \sum_k \int_{\Sigma_k} \varphi(x) \mathcal{H}_{N-1}(dx) + \int_U \varphi(x) \mathcal{H}_{N-1}(dx)$$

$$\geq \sum_k \int_{T(\Sigma_k)} e^{-|x|^2/2} \mathcal{H}_{N-1}(dx) + \int_{T(U)} e^{-|x|^2/2} \mathcal{H}_{N-1}(dx)$$

$$= (2\pi)^{N/2} P_{\gamma_N} (T(M)).$$

If $M$ is a smooth subset of $S$, we can approximate it by sets satisfying (4.6), so that inequality (4.3) holds in the general case, too.

Finally assume that equality holds in (1.8). Then we have

$$P_\mu(M, S) = (2\pi)^{N/2} P_{\gamma_N} (T(M))$$

and

$$P_{\gamma_N} (T(M)) = P_{\gamma_N} (H_\lambda).$$
Since the Gaussian isoperimetric inequality is achieved only for half-space, modulo a rotation, we deduce that $T(M)$ is a half-space. Hence the conclusion follows by the definition of $T$. □

In view of Lemma 3.1 one has the following

**Corollary 4.1.** The conclusion of Theorem 1.1 holds for measures $\mu$ like (1.9), that is

$$d\mu(x) = \exp \left\{ -\frac{|x|^2}{2} - \sum_{i=1}^{N-1} B_i(x_i) \right\} dx,$$

where $B_i \in C^2(a_i, b_i)$ with $B''_i(x_i) \geq 0$ on $(a_i, b_i)$, $(i = 1, \ldots, N - 1)$.

**Proof:** Define $A_i \in C^3(a_i, b_i)$ by

$$A_i(y) := E^{-1} \left( c_i \int_{a_i}^{y} e^{-t^2/2-B_i(t)} dt \right), \quad a_i < y < b_i,$$

where

$$c_i := \frac{1}{\int_{a_i}^{b_i} e^{-t^2/2-B_i(t)} dt}, \quad i = 1, \ldots, N - 1.$$

Then

$$\prod_{i=1}^{N-1} A_i(x_i) \exp \left\{ -\frac{x_N^2}{2} - \frac{1}{2} \sum_{i=1}^{N-1} A_i(x_i)^2 \right\} = (2\pi)^N \prod_{i=1}^{N-1} c_i \exp \left\{ -\frac{|x|^2}{2} - \sum_{i=1}^{N-1} B_i(x_i) \right\},$$

and by Lemma 3.1 we have

$$A'_i(y) \geq 1 \quad \text{for} \quad a_i < y < b_i, \quad i = 1, \ldots, N - 1.$$

Now the assertion follows from Theorem 1.1. □

**Remark 4.1.** (a) Assume that $A \in C^3(I)$ is given and satisfies (3.4) and (3.5), and define a function $B \in C^2(a, b)$ by (3.3) with $c = 1$. Such assumptions, as the following example shows, do not imply that $B''(x) \geq 0$ on $(a, b)$.

Let $A(x) := x + \alpha x^3$, with $\alpha > 0$, and $a = -\infty$, $b = +\infty$. Then a short computation shows that

$$B(x) = \alpha x^4 + \alpha^2 x^6/2 - \log(1 + 3\alpha x^2) + \log \sqrt{2\pi},$$

that is,

$$B''(x) = 12\alpha x^2 + 15\alpha^2 x^4 + \frac{-6\alpha + 18\alpha^2 x^2}{(1 + 3\alpha x^2)^2}.$$  

Hence $B''(0) = -6\alpha < 0$.

(b) The above example also shows that Theorem 1.1 does not follow from Corollary 4.1.

(c) In the case that $\mu$ is the Gauss measure $\gamma_N$ and that $\omega$ is convex, it has been proved that $\kappa_1 \geq 1$, see [2] and [8]. Together with Theorem 2.1 this suggests the following conjecture:

If $\omega$ is convex, then the sets $S_\lambda$, with $\lambda \in \mathbb{R}$, are minimizers in the isoperimetric inequality for $\Omega$. 


5. Applications

For sake of simplicity we consider the measure \( \mu \) defined by \((1.9)\). We need some notation. Throughout this Section \( G \) will denote a smooth domain in \( S \). We will denote by \( C_\mu \) the constant

\[
C_\mu = \int_{S'} \exp \left( -\frac{\|x'\|^2}{2} - \sum_{i=1}^{N-1} B_i(x_i) \right) \, dx'.
\]

We will use the function \( F : \mathbb{R} \to \mathbb{R}_+ \) defined by

\[
F(t) = \int_t^{+\infty} \exp \left( -\frac{\sigma^2}{2} \right) \, d\sigma,
\]

for any \( t \in \mathbb{R} \). Such a function is strictly decreasing and belongs to \( C^\infty(\mathbb{R}) \); we will denote by \( F^{-1} : \mathbb{R}_+ \to \mathbb{R} \) its inverse function.

If \( \Gamma \) is an open portion of \( \partial \Omega \) with \( \mathcal{H}_{N-1}(\Gamma) > 0 \), let \( W_{\Gamma}(\Omega, d\mu) \) be the weighted Sobolev space consisting in the set of all weakly differentiable functions \( u \) satisfying the following conditions:

\[
\|u\|_{W_{\Gamma}(\Omega, d\mu)}^2 := \int_{\Omega} |Du|^2 \, d\mu + \int_{\Omega} |u|^2 \, d\mu < +\infty;
\]

\[
\begin{cases}
\text{there exists a sequence of functions } u_n \in C^1(\overline{\Omega}) \text{ such that } u_n(x) = 0 \text{ on } \Gamma \text{ and } \\
\lim_{n \to \infty} \left( \int_{\Omega} |D(u_n - u)|^2 \, d\mu + \int_{\Omega} |u_n - u|^2 \, d\mu \right) = 0.
\end{cases}
\]

The space \( W_{\Gamma}(\Omega, d\mu) \) will be endowed with the norm defined by \((5.3)\).

Now we recall a few definitions and properties about weighted rearrangements. For exhaustive treatment on this subject we refer, e.g., to [18], [23] and [27].

Let \( u \) be a Lebesgue measurable function defined in \( G \). Then the distribution function of \( u \) with respect to \( d\mu \) is the function \( m_u : [0, \text{ess sup}|u|] \to [0, \mu(G)] \) defined by

\[
m_u(t) = \mu(\{x \in G : |u(x)| > t\}), \quad \forall t \in \mathbb{R}_+.
\]

The decreasing rearrangement with respect to \( \mu \) of \( u \) is the function

\[
u^* : [0, +\infty[ \to [0, \text{ess sup}|u|[\]

defined by

\[
u^*(s) = \inf \{t \in \mathbb{R} : m_u(t) \leq s\}, \quad s \in [0, \mu(G)].
\]

Let \( G^\star \) be the set defined by

\[
G^\star := \{(x', x_N) : x_N > \lambda, x' \in S'\}, \quad \text{with } \lambda = F^{-1} \left( \frac{\mu(G)}{C_\mu} \right).
\]
The weighted rearrangement of \( u \) (with respect to \( \mu \)) is the function \( u^\star : G^\star \to [0, +\infty] \) defined by

\[
u^\star(x) = u^\star(C_\mu F(x_N)), \quad \forall x \in G^\star,
\]
where \( F \) is the function given by (5.2) and \( C_\mu \) is the constant defined by (5.1). Observe that by definition \( u^\star \) depends just on one variable, it is a decreasing function and moreover the functions \( u \) and \( u^\star \) are equimeasurable. Therefore by Cavalieri's principle, we have

\[
||u||_{L^p(G, d\mu)} = ||u^\star||_{L^p(G^\star, d\mu)}, \quad 1 \leq p \leq +\infty.
\]

Let \( \Gamma := \partial G \cap S \).

By a result contained in [31], we deduce that any nonnegative function belonging to the space \( W_\Gamma(G, d\mu) \) satisfies the following Pólya-Szegő - type inequality.

**Theorem 5.1.** Let \( u \) be a nonnegative function in \( W_\Gamma(G, d\mu) \). Then it holds

\[
\int_\Omega |Du|^2 d\mu \geq \int_{G^\star} |Du^\star|^2 d\mu.
\]

As a consequence of the inequality (5.7) one deduces that \( W_\Gamma(G, d\mu) \) is continuously embedded in \( L^2(G, d\mu) \).

**Corollary 5.1.** For any function \( u \) belonging to \( W_\Gamma(G, d\mu) \), we have

\[
\int_G |u|^2 d\mu \leq C \int_G |Du|^2 d\mu,
\]
where \( C \) is a positive constant depending only on \( \mu(G) \).

**Proof:** By using (5.7), (5.6) and a result contained in [25] (Theorem 1, p. 40), one has that there exist a constant \( K = K(\mu(G)) \in (0, +\infty) \) such that

\[
\frac{\int_G |Du|^2 d\mu}{\int_G |u|^2 d\mu} \geq \int_{G^\star} |Du^\star|^2 d\mu = \int_{\lambda}^{+\infty} \left( \frac{du^\star}{dx_N} \right)^2 \exp \left( -\frac{x_N^2}{2} \right) dx_N \geq K,
\]

for any \( u \in W_\Gamma(G, d\mu) \). \( \square \)

**Remark 5.1.** We explicitly observe that by Corollary 5.1 the norm defined by (5.3) is equivalent to the norm

\[
||u||_{W_\Gamma(G, d\mu)} = \left( \int_G |Du|^2 d\mu \right)^{1/2}.
\]

Henceforth we will endow the space \( W_\Gamma(G, d\mu) \) with the norm (5.8).

Finally, we recall the classical Hardy inequality (see [18], for instance).
Proposition 5.1. Let \( f \) be a function belonging to \( L^1(G,d\mu) \) and \( E \) a measurable subset of \( G \). Then the following inequality holds true
\[
\int_E |f|d\mu \leq \int_0^{\mu(E)} f^*(r)dr.
\]

Now we consider the following class of boundary value problems
\[
\begin{align*}
-\text{div} (A(x)\nabla u) &= \exp \left\{ -\frac{|x|^2}{2} - \sum_{i=1}^{N-1} B_i(x_i) \right\} f(x) \quad \text{in } G, \\
u &= 0 \quad \text{on } \partial G \cap S
\end{align*}
\]
where \( G \) is an open connected subset of \( S \), \( (N \geq 2) \), \( A(x) = (a_{ij}(x)) \) is a symmetric \((N \times N)\)-matrix with measurable coefficients satisfying
\[
\exp \left\{ -\frac{|x|^2}{2} - \sum_{i=1}^{N-1} B_i(x_i) \right\} |\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq C \exp \left\{ -\frac{|x|^2}{2} - \sum_{i=1}^{N-1} B_i(x_i) \right\} |\xi|^2,
\]
for some \( C \geq 1 \), for almost everywhere \( x \in G \) and for all \( \zeta \in \mathbb{R}^N \). Moreover we assume that \( f \in L^2(G,d\mu) \).

Let \( \Gamma := G \cap S \). A solution to problem (5.10) is a function \( u \) belonging to \( W_1^1(G,d\mu) \) such that
\[
\int_{\Omega} A(x)\nabla u \nabla \psi d\mu = \int_{\Omega} f\psi d\mu,
\]
for every \( \psi \in C^1(\overline{G}) \) such that \( \psi = 0 \) on \( \Gamma \). The following Theorem 5.2 gives a-priori estimates for problem (5.10). More precisely, it states that every rearrangement invariant norm (with respect to \( \mu \)) of the solution \( u \) of (5.10) can be estimated with the same norm of the solution \( v = v^\star \) to the problem corresponding to the operator
\[
L^\star = -\text{div} \left( \exp \left\{ -\frac{|x|^2}{2} - \sum_{i=1}^{N-1} B_i(x_i) \right\} \right) \nabla v
\]
and the domain \( G^\star \).

Theorem 5.2. Let \( u \) be the solution to problem (5.10). Denote by \( G^\star \) the subset of \( \mathbb{R}^N \) defined in (5.5) and by \( v \) the function
\[
v(x) = v(x_N) = \int_\lambda^{x_N} \left[ \exp \left( \frac{\rho^2}{2} \right) \int_\rho^\infty \exp \left( -\frac{\xi^2}{2} \right) f^\star(\xi) d\xi \right] d\rho
\]
which is the solution to the problem

\[
\begin{cases}
-\text{div} \left( \exp \left( -\frac{|x|^2}{2} - \sum_{i=1}^{N-1} B_i(x_i) \right) \nabla v \right) = \exp \left( -\frac{|x|^2}{2} - \sum_{i=1}^{N-1} B_i(x_i) \right) f^\star & \text{in } G^\star, \\
v(\lambda) = 0.
\end{cases}
\]

Then

\[
u^\star(x_1) \leq v(x_1) \text{ a.e. in } G^\star,
\]

and

\[
\int_G |Du|^q \, d\mu \leq \int_{G^\star} |Dv|^q \, d\mu, \quad 0 < q \leq 2.
\]

We will omit the proof since it follows closely the lines, for instance, of Theorem 1.1 in [9] (see also [1]).

**Remark 5.2.**

(i) The existence and the uniqueness of the solutions to problems (5.10) and (5.13), respectively, are an easy consequence of Lax-Milgram Theorem and Corollary 5.1.

(ii) Let us assume that the right-hand side \( f \) satisfies the following summability condition

\[
\int_{\lambda}^{+\infty} \left[ \exp \left( \frac{\rho^2}{2} \right) \int_{\rho}^{\infty} \exp \left( -\frac{\xi^2}{2} \right) f^\star(\xi) \, d\xi \right] \, d\rho < +\infty.
\]

Then inequality (5.14) gives an estimate of the norm of \( u \) in \( L^\infty(G, d\mu) \equiv L^\infty(G) \), i.e.

\[
\text{ess sup}|u| = u^\star(0) \leq \text{ess sup}|v| = v^\star(0) = \frac{1}{C_{\mu}} \int_{\lambda}^{\infty} \left[ \exp \left( \frac{\rho^2}{2} \right) \int_{0}^{C_{\mu}F(\rho)} f^\star(\sigma) \, d\sigma \right] \, d\rho.
\]

(iii) Since the solution \( v \) to problem (5.13) depends just on the variable \( x_1 \), it solves the one-dimensional equation

\[
- \frac{d}{dx_1} \left( \exp \left( -\frac{x_1^2}{2} \right) \frac{dv}{dx_1} \right) = \exp \left( -\frac{x_1^2}{2} \right) f^\star \quad \text{in } (\lambda, \infty),
\]

with \( v(\lambda) = 0 \).

**References**

[1] Alvino A., Trombetti G., Sulle migliori costanti di maggiorazione per una classe di equazioni ellittiche degeneri, Ricerche Mat. 27(2) (1978), 413–428.
[2] Andrews B., Ni L., Eigenvalue comparison on Bakry-Emery manifolds, Comm. Partial Differential Equations 37 (2012), no. 11, 2081–2092.
[3] Bandle C. Isoperimetric inequalities and applications, Monographs and Studies in Mathematics 7, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1980.
[4] Barthe F., Bianchini C., Colesanti A., Isoperimetry and stability of hyperplanes for product probability measures, Ann. Mat. Pura Appl. (4) 192 (2013), no. 2, 165–190.
[5] Betta M. F., Brock F., Mercaldo A., Posteraro M. R., Weighted isoperimetric inequalities on Rn and applications to rearrangements, Math. Nachr. 281 (2008), no. 4, 466–498.
[6] Bobkov S. G., An isoperimetric inequality for the discrete cube, and an elementary proof of the isoperimetric inequality in Gauss space, Ann. Probab. 25 (1997), no. 1, 206–214.
[7] Borell C., The Brunn-Minkowski inequality in Gauss space, Invent. Math. 30 (1975) 207–211.
[8] Brandolini B., Chiacchio F., Henrot H., Trombetti C., An optimal Poincare-Wirtinger inequality in Gauss space, Math. Res. Lett. 20 (2013), no. 3, 449–457.
[9] Brock F., Chiacchio F., Mercaldo A., A class of degenerate elliptic equations and a Dido’s problem with respect to a measure, J. Math. Anal. Appl. 348 (2008), no. 1, 356–365.
[10] Brock F., Chiacchio F., Mercaldo A., Weighted isoperimetric inequalities in cones and applications, Nonlinear Anal. 75 (2012), no. 15, 5737–5755.
[11] Brock F., Chiacchio F., Mercaldo A., A weighted isoperimetric inequality in an orthant, Potential Anal. 41 (2014), no. 1, 171–186.
[12] Brock F., Mercaldo A., Posteraro M. R., On isoperimetric inequalities with respect to infinite measures, Rev. Mat. Iberoam. 29 (2013), no. 2, 665–690.
[13] Chambers G. R., Proof of the Log-Convex Density Conjecture, arXiv:1311.4012
[14] Cabr´e X., Ros-Oton X., Sobolev and isoperimetric inequalities with monomial weights, J. Differential Equations 255 (2013), no. 11, 4312–4336.
[15] Caffarelli L. A., Monotonicity properties of optimal transportation and the FKG and related inequalities, Comm. Math. Phys. 214 (2000), no. 3, 547–563.
[16] Carlen E. A., Kerce C., On the cases of equality in Bobkov’s inequality and Gaussian rearrangement, Calc. Var. Partial Differential Equations 13 (2001), no. 1, 1–18.
[17] Cianchi A., Fusco, N., Maggi F., Pratelli A., On the isoperimetric deficit in Gauss space, Amer. J. Math. 133 (2011), no. 1, 131–186.
[18] Chong K. M., Rice N. M., Equimeasurable Rearrangements of Functions, Queen’s Papers in Pure and Applied Mathematics, No. 28, Queen’s University, 1971.
[19] Ehrhard A., Inégalités isopérimétriques et intégrales de Dirichlet gaussiennes, (French) [Isoperimetric inequalities and Gaussian Dirichlet integrals] Ann. Sci. École Norm. Sup. (4) 17 (1984), no. 2, 317–332.
[20] Feo F.; Posteraro M. R., Roberto C., Quantitative isoperimetric inequalities for log-convex probability measures on the line, J. Math. Anal. Appl. 420 (2014), no. 2, 879–907.
[21] Henrot A., Extremum problems for eigenvalues of elliptic operators, Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.
[22] Henrot A., Pierre M., Variation et optimisation de formes. Une analyse géométrique, (Shape variation and optimization. A geometric analysis) Mathématiques & Applications (Berlin) (Mathematics & Applications), vol. 48, Springer-Verlag, Berlin, 2005.
[23] Kawohl B., Rearrangements and Convexity of Level Sets in PDE, Lecture Notes in Mathematics 1150. New York: Springer Verlag, 1985.
[24] Kesavan S., Symmetrization & applications, Series in Analysis, 3. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.
[25] Maz’ja V.G., Sobolev Spaces, Springer-Verlag, 1980.
[26] Maderna C., Salsa S., Sharp estimates of solutions to a certain type of singular elliptic boundary value problems in two dimensions, Applicable Anal. 12 (1981), no. 4, 307–321.
[27] Rakotoson J. M., Simon B., Relative rearrangement on a measure space application to the regularity of weighted monotone rearrangement, I, II, Appl. Math. Lett. 6 (1993), 75–78, 79–82.
[28] Rosales C., *Isoperimetric and stable sets for log-concave perturbations of Gaussian measures*, Anal. Geom. Metr. Spaces 2 (2014), 328–358.

[29] Rosales C., Canete A., Bayle V., Morgan F., *On the isoperimetric problem in Euclidean space with density*, Calc. Var. 31 (2008), 27–46.

[30] Sudakov V.N., Tsirel’son B.S., *Extremal properties of half-spaces for spherically invariant measures*, Zap. Naum. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 41 (1974) 14–24.

[31] Talenti G., *A weighted version of a rearrangement inequality*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 43 (1997) 121–133.

[32] Taylor, M.E., *Partial Differential Equations, Vol. II, Qualitative Studies of Linear Equations*, Appl. Math. Sciences 116, Springer, N.Y. (1996).