Research Article
Subdirect Sums of Doubly Strictly Diagonally Dominant Matrices

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1. Introduction

In 1999, the concept of subdirect sums of square matrices was introduced by Fallat and Johnson, which is a generalization of the usual sum of matrices [1], and arises in several contexts, such as matrix completion problems, overlapping subdomains in domain decomposition problems, and global stiffness matrices in finite element methods [2].

For a given class matrix, an important problem is that whether the $k$-subdirect sums of matrices belong to the same class or not, which has been widely considered for different classes of matrices, such as nonsingular $M$-matrices [3], $S$-strictly diagonally dominant matrices [4], $\Sigma$-strictly diagonally dominant matrices [5], doubly diagonally dominant matrices [6], Nekrasov matrices [7, 8], and SDD1 matrices [9].

In this paper, we focus on the subdirect sum of doubly strictly diagonally dominant (shortly as DSDD) matrices, which is a subclass of $H$-matrices [10], and some sufficient conditions such that the $k$-subdirect sums of DSDD matrices belong to DSDD matrices are given, and these sufficient conditions only depend on the elements of the given matrices. Moreover, examples are presented to illustrate the corresponding results.

Now, some notations and definitions are listed, which can also be found in [1, 11–13].

Let $n$ be an integer number. $\mathbb{C}^{n \times n}$ is the set of complex matrices.

Definition 1 (see [1]). Let $A$ and $B$ be square matrices of orders $n_1$ and $n_2$, respectively, and $k$ be an integer number such that $1 \leq k \leq \min\{n_1, n_2\}$. Let $A$ and $B$ be partitioned in a $2 \times 2$ block as follows:

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix},
\]

where $A_{22}$ and $B_{11}$ are the square matrices of order $k$. We call the following square matrix of order $n = n_1 + n_2 - k$:

\[
C = \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{21} & A_{22} + B_{11} & B_{12} \\
0 & B_{21} & B_{22}
\end{bmatrix},
\]

the $k$-subdirect sum of $A$ and $B$, and we denote it by $C = A \oplus_k B$.

In order to more explicitly express each element of $C$ in terms of the ones of $A$ and $B$, we can define the following set of indices:

\[
S_1 = \{1, 2, \ldots, n_1 - k\},
\]

\[
S_2 = \{n_1 - k + 1, n_1 - k + 2, \ldots, n_1\},
\]

\[
S_3 = \{n_1 + 1, n_1 + 2, \ldots, n\}.
\]
Then, $C$ can be expressed as follows:

\[
\begin{bmatrix}
  a_{11} & \cdots & a_{1,t} & a_{1,p} & \cdots & a_{1,n_1} & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
  a_{t,1} & \cdots & a_{t,t} & a_{t,p} & \cdots & a_{t,n_t} & 0 & \cdots & 0 \\
  a_{p,1} & \cdots & a_{p,t} & a_{p,p} + b_{11} & \cdots & a_{p,n_1} + b_{1,n_1-t} & b_{1,n_1-t+1} & \cdots & b_{1,n-t} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
  a_{n_t,1} & \cdots & a_{n_t,t} & a_{n_t,n_t} + b_{n_t,n_t-t} & \cdots & b_{n_t,n_t-t+1} & b_{n_t,n_t-t+1} & \cdots & b_{n_t,n-t} \\
  0 & \cdots & 0 & b_{n_t-t+1,n_t-t} & \cdots & b_{n_t-t+1,n_t-t+1} & b_{n_t-t+1,n_t-t+1} & \cdots & b_{n_t-t,n-t} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & b_{n-t,n-t} & \cdots & b_{n-t,n-t+1} & b_{n-t,n-t+1} & \cdots & b_{n-t,n-t} \\
\end{bmatrix}
\]  

(4)

where $t = n_1 - k$ and $p = t + 1$.

**Definition 2** (see [12]). Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $A$ is called (row) diagonally dominant (DD) if

\[
|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \quad i = 1, 2, \ldots, n, \tag{5}
\]

where

\[
R_i(A) = \sum_{j=1}^{n} |a_{ij}|, \quad i = 1, 2, \ldots, n. \tag{6}
\]

If the inequality in (5) holds strictly for all $i$, we say that $A$ is strictly diagonally dominant (SDD).

**Definition 3** (see [13]). The matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ is a doubly strictly diagonally dominant (DSDD) matrix if

\[
|a_{ii}||a_{jj}| > R_i(A)R_j(A), \quad i, j = 1, 2, \ldots, n, i \neq j. \tag{7}
\]

### 2. Subdirect Sums of DSDD Matrices

In general, the subdirect sum of two DSDD matrices is not always a DSDD matrix. We show this in the following example.

**Example 1.** Let

\[
A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{bmatrix}, \tag{8}
\]

\[
B = \begin{bmatrix} 3 & -2 & -1 \\ -1 & 2 & 0 \\ 1 & 1 & 4 \end{bmatrix}
\]

be two DSDD matrices, but

\[
C = A \oplus_1 B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 7 & -2 & -1 \\ -1 & -1 & 6 & 0 \\ 0 & 1 & 1 & 4 \end{bmatrix} \tag{9}
\]

is not a DSDD matrix, since $|c_{11}||c_{22}| = 7 < R_1(C)R_2(C) = 8$.

Example 1 shows that the subdirect sum of DSDD matrices is not a DSDD matrix; then, a meaningful discussion is concerned: under what conditions, the subdirect sum of DSDD matrices is in the class of DSDD matrices?

In order to obtain the main results, we need the following lemma.
Lemma 1. Let \( A = [a_{ij}] \) and \( B = [b_{ij}] \) be DSDD square matrices of orders \( n_1 \) and \( n_2 \) partitioned as in (1), respectively. Let \( k \) be an integer number such that \( 1 \leq k \leq \min\{n_1, n_2\} \), \( S_1, S_2, S_3 \) be defined as in (4), and all diagonal entries of \( A_{22} \) and \( B_{11} \) are positive (or all negative), with \( C = AB \cdot B \), then

(a) For \( i \in S_1 \), we have \( R_i (C) = R_i (A) \).

(b) For \( k = 1 \), \( i \in S_2 \), we have \( R_i (C) = R_{n_1} (C) = R_{n_1} (A) + R_1 (B) \).

(c) For \( k \geq 2 \), \( i \in S_2 \), we have \( R_i (C) = R_{n_1} (A) + R_{n_1} (B) \).

(d) For \( i \in S_3 \), we have \( R_i (C) = R_{n_1} (B) \).

Proof. For \( i \in S_1 \), we can write

\[
R_i (C) = \sum_{j \in S_1, j \neq i} |a_{ij}| + \sum_{j \in S_2} |a_{ij}| + 0 = R_i (A). \tag{10}
\]

For \( k = 1 \), \( i \in S_2 \), we obtain

\[
R_i (C) = R_{n_1} (C) = \sum_{j \in S_1} |a_{ij}| + \sum_{j \in S_2} |b_{1j-t-t'}| = R_{n_1} (A) + R_1 (B). \tag{11}
\]

For \( k \geq 2 \), \( i \in S_2 \), we obtain

\[
R_i (C) = \sum_{j \in S_1} |a_{ij}| + \sum_{j \in S_2, j \neq i} |a_{ij}| + \sum_{j \in S_2} |b_{1j-t-t'}| \leq \sum_{j \in S_1} |a_{ij}| + \sum_{j \in S_2, j \neq i} |a_{ij}| + \sum_{j \in S_2} |b_{1j-t-t'}| + \sum_{j \in S_2} |b_{t-t'}| = R_{n_1} (A) + R_{n_1} (B). \tag{12}
\]

For the rest case of \( i \in S_3 \), the proof is similar to the proof of \( i \in S_1 \).

Firstly, we study the 1-subdirect sum of DSDD matrices.

Theorem 1. Let \( A \) and \( B \) be DSDD matrices of orders \( n_1 \) and \( n_2 \) partitioned as in (1), respectively, and \( k = 1 \). Then, \( C = AB \cdot B \) is a DSDD matrix if all diagonal entries of \( A_{22} \) and \( B_{11} \) are positive (or all negative) and for \( i \in S_1 \),

\[
|a_{ii}| > |a_{n_1,n_1}|, \tag{13}
\]

\[
R_{n_1} (A) \geq R_i (A), \tag{14}
\]

\[
\max_{j \in S_2 \cup S_3} \left| b_{j-t-t'} \right| \leq R_{n_1} (A) \leq |a_{n_1,n_1}|, \quad (t = n_1 - 1). \tag{15}
\]

Proof. Since \( k = 1 \) and \( A \) and \( B \) are the DSDD matrices of orders \( n_1 \) and \( n_2 \) respectively, it is obvious that \( S_1 = \{1, 2, \ldots, n_1 - 1\}, S_2 = \{n_1\}, S_3 = \{n_1 + 1, n_1 + 2, \ldots, n\} \).

Case 1: for \( i, j \in S_1 \), from (a) of Lemma 1, we have

\[
|c_{ij}| = |a_{ii}|, \quad |c_{jj}| = |a_{jj}|, \tag{16}
\]

\[
R_i (C) = R_i (A), \quad R_j (C) = R_j (A). \tag{17}
\]

Since \( A \) is DSDD, we obtain that for \( i, j \in S_1 \),

\[
|c_{ij}| = |a_{ij}| > |a_{i1}|R_{n_1} (A) + R_{n_1} (B). \tag{18}
\]

Case 2: for \( i \in S_1 \) and \( j \in S_2 = \{n_1\} \), from (a) of Lemma 1, it is easy to obtain

\[
|c_{ij}| = |a_{ij}|, \quad R_i (C) = R_i (A). \tag{19}
\]

Since all diagonal entries of \( A_{22} \) and \( B_{11} \) are positive (or all negative), we obtain

\[
|c_{jj}| = |b_{n_1,n_1}| + |b_{11}|. \tag{20}
\]

Since \( A \) is DSDD, and from inequalities (13)–(15), we have

\[
|c_{ij}| = |a_{ii}|, \quad R_i (C) = R_i (A), \quad |c_{jj}| = |b_{11}|, \quad R_j (C) = R_j (B). \tag{21}
\]

From (b) of Lemma 1, it is easy to obtain that for \( i \in S_1 \), \( j \in S_2 = \{n_1\} \),

\[
|c_{ij}| > R_i (C)R_j (C). \tag{22}
\]

Case 3: for \( i \in S_1 \) and \( j \in S_3 \), from (a) and (d) of Lemma 1, we have

\[
|c_{ij}| > R_i (C)R_j (C). \tag{23}
\]
Then, from inequalities (13)–(15), we have that for $i \in S_1$ and $j \in S_3$,

$$
\left| c_{ij} \right| = \left| a_{ii} \right| \left| b_{j-t,j-t} \right| + \left| a_{i,j} \right| \left| b_{j-t,j-t} \right| \geq \left| b_{j-t,j-t} \right| \max_{j \in S_3} \frac{R_{j-t}(B)}{\left| b_{j-t,j-t} \right|} R_n(A) 
$$

$$
\geq R_n(A)R_{j-t}(B) \geq R_1(A)R_{j-t}(B) = R_1(C)R_j(C). \tag{23}
$$

Case 4: for $i \in S_2 = \{n_1\}$ and $j \in S_3$, from (b) and (d) of Lemma 1, we obtain

$$
\left| c_{ij} \right| = \left| a_{n,n} \right| + \left| b_{11} \right|, \quad \left| c_{jj} \right| = \left| b_{j-t,j-t} \right|. \tag{24}
$$

Case 5: for $i, j \in S_3$, from (d) of Lemma 1, we obtain

$$
\left| c_{ij} \right| = \left| a_{n,n} \right| \left| b_{j-t,j-t} \right| \left| b_{j-t,j-t} \right| 
$$

$$
= \left| a_{n,n} \right| \left| b_{j-t,j-t} \right| + \left| b_{11} \right| \left| b_{j-t,j-t} \right| 
$$

$$
> \left| b_{j-t,j-t} \right| \max_{j \in S_3} \frac{R_{j-t}(B)}{\left| b_{j-t,j-t} \right|} R_n(A) + R_1(B)R_{j-t}(B) 
$$

$$
\geq R_n(A)R_{j-t}(B) + R_1(B)R_{j-t}(B) 
$$

$$
= (R_n(A) + R_1(B))R_{j-t}(B) = R_1(C)R_j(C). \tag{25}
$$

Since $B$ is DSDD, and from inequality (15), for $i \in S_2 = \{n_1\}$ and $j \in S_3$, we have

$$
\left| c_{ij} \right| = \left| a_{n,n} \right| \left| b_{11} \right| + \left| b_{11} \right| \left| b_{11} \right| 
$$

$$
= \left| a_{n,n} \right| \left| b_{j-t,j-t} \right| \left| b_{j-t,j-t} \right| 
$$

$$
\left| c_{ij} \right| = \left| a_{n,n} \right| \left| b_{j-t,j-t} \right| \left| b_{j-t,j-t} \right| \geq \left| b_{j-t,j-t} \right| \max_{j \in S_3} \frac{R_{j-t}(B)}{\left| b_{j-t,j-t} \right|} R_n(A) 
$$

$$
\geq R_n(A)R_{j-t}(B) \geq R_1(A)R_{j-t}(B) = R_1(C)R_j(C). \tag{26}
$$

Since $B$ is DSDD, we have

$$
\left| c_{ij} \right| = \left| b_{11} \right| \left| b_{11} \right| \geq \left| b_{j-t,j-t} \right| \left| b_{j-t,j-t} \right| \geq R_n(A)R_{j-t}(B) = R_1(C)R_j(C). \tag{27}
$$

Therefore, we can draw a conclusion that for any $i, j \in \{1, 2, \ldots, n\}, [c_{ij}] > R_1(C)R_j(C)$, that is, $C = AB_1B$ is a DSDD matrix.

**Example 2.** The matrices

$$
A = \begin{bmatrix}
3 & 0 & -1 \\
-1 & 4 & 0 \\
-2 & 0 & 2
\end{bmatrix}, \tag{28}
$$

$$
B = \begin{bmatrix}
3 & 1 & -1 \\
-1 & -2 & 0 \\
1 & 0 & 2
\end{bmatrix}
$$

are two DSDD matrices, and from Theorem 1, it is easy to verify that

$$
C = AB_1B = \begin{bmatrix}
3 & 0 & -1 & 0 \\
-1 & 4 & 0 & 0 \\
-2 & 0 & 2 & 0
\end{bmatrix}
$$

is a DSDD matrix since $3 = |a_{11}| > |a_{13}| = 2.4 = |a_{22}| > |a_{33}| = 2.2 = R_1(A) > R_1(A) = 1.2 = R_3(A) > R_2(A) = 1$, and $\max_{j \in S_2, j \neq i} \left( R_{j-t}(B) / \left| b_{j-t,j-t} \right| \right) \left| R_n(A) \right| = \left( R_3(B) / |b_{33}| \right) R_3(A) = (4/7) \times 3 < |a_{33}| = 2$.

However,

$$
C = AB_2B = \begin{bmatrix}
3 & 0 & -1 & 0 \\
-1 & 7 & 1 & -1 \\
-2 & -1 & 0 & 0 \\
0 & 1 & 0 & 2
\end{bmatrix}
$$

is not DSDD since $|c_{11}| > |c_{33}| = 0 < 3 = R_1(C)R_3(C)$.

Examples 2 motivates the search for other conditions such that $C = AB_1B (k \geq 2)$ is also a DSDD matrix, where $A$ is a DSDD matrix and $B$ is a DSDD matrix.
Next, some sufficient conditions ensuring that the $k$-subdirect sum of DSDD matrices is a DSDD matrix are given.

**Theorem 2.** Let $A$ and $B$ be matrices of orders $n_1$ and $n_2$ partitioned as in (1), respectively, and $k$ is an integer number such that $2 \leq k \leq \min\{n_1, n_2\}$. Let $A$ and $B$ be DSDD matrices, if all diagonal entries of $A_{22}$ and $B_{11}$ are positive (or all negative) and for any $i \in S_1 \cup S_2$,

$$\max_{j \in S_j \cup S_i} R_{j-i}(B) B_{j-i-i} R_j(A) < |a_{ii}|, \quad (t = n_1 - k), \quad (31)$$

and then the $k$-subdirect sum $C = A \oplus_k B$ is DSDD.

**Proof.** Let $A$ and $B$ be DSDD matrices of orders $n_1$ and $n_2$ respectively; thus, it is obvious that $S_1 = \{1, 2, \ldots, n_1 - k\}, S_2 = \{n_1 - k + 1, n_1 - k + 2, \ldots, n_1\}$, and $S_3 = [n_1 + 1, n_1 + 2, \ldots, n]$.

Case 1: for $i, j \in S_1$, from (a) of Lemma 1, we have

$$|c_{ii}| = |a_{ii}|,$$

$$R_i(C) = R_i(A).$$

Since $A$ is DSDD, we obtain

$$|c_{ii}| = |a_{ii}| R_i(C) = R_i(A) R_i(C). \quad (32)$$

Case 2: for $i \in S_1$ and $j \in S_2$, from (a) and (c) of Lemma 1, we obtain

$$|c_{jj}| = |a_{jj}|,$$

$$R_j(C) = R_j(A),$$

$$R_j(C) \leq R_j(A) + R_{j-i}(B).$$

Since all diagonal entries of $A_{22}$ and $B_{11}$ are positive (or all negative), we obtain

$$|c_{jj}| = |a_{jj}| + |b_{j-i,j-i}|. \quad (33)$$

Case 3: for $i \in S_1$ and $j \in S_2$, from (a) and (d) of Lemma 1, we conclude

$$|c_{ii}| = |a_{ii}|,$$

$$R_i(C) = R_i(A),$$

$$|c_{jj}| = |b_{j-i,j-i}|,$$

$$R_j(C) = R_{j-i}(B).$$

Then, from inequality (31), we have

$$|c_{ii}| = |a_{ii}| R_i(C) = R_i(A) R_i(C). \quad (34)$$

Case 4: for $i, j \in S_2$, from (c) of Lemma 1, we obtain

$$|c_{ii}| = |a_{ii}|,$$

$$R_i(C) = R_i(A),$$

$$|c_{jj}| = |b_{j-i,j-i}|,$$

$$R_j(C) \leq R_j(A) + R_{j-i}(B).$$

Since $A$ and $B$ are DSDD, and from inequality (31), we conclude
Case 5: for \( i, j \in S_3 \), from (c) and (d) of Lemma 1, we obtain
\[
|c_{ij}| = |a_{ii} + b_{i-t,j-t}|, \\
R_i(C) = R_{j-t}(B), \quad R_j(C) = R_{j-t}(B).
\]
Since \( B \) is DSDD, we obtain
\[
|c_{ij}| = |a_{ii} + b_{i-t,j-t}| > R_{j-t}(B)R_{j-t}(B) = R_i(C)R_j(C). 
\]
(45)

Example 3. Let
\[
A = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 5 & -2 \\ 0 & -1 & 2 \end{bmatrix}, \\
B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 8 & 0 \\ -1 & 2 & 5 \end{bmatrix}
\]
(46)
From Definitions 2 and 3, it is easy to show that SDD matrices are contained into DSDD matrices. Therefore, from Theorem 2, we obtain the following corollaries, which present sufficient conditions such that $k$-subdirect sum $C = A \oplus_k B$ is DSDD.

**Corollary 1.** Let $A$ and $B$ be square matrices of orders $n_1$ and $n_2$ partitioned as in (1), respectively, and $k$ is an integer number such that $1 \leq k \leq \min\{n_1, n_2\}$. We assume that $A$ is a DSDD matrix and $B$ is a SDD matrix. If there exists an $i_0 \in S_1 \cup S_2$ such that

$$\max_{j \in S_1 \cup S_2} \frac{R_{j,t}(B)}{|b_{j-t,j-t}|} R_{i_0}(A) < |a_{i_0,j}| < R_{i_0}(A),$$

(49)

then $a_{i_0,j} \geq R_i(A)$, $i \in S_1 \cup S_2 \setminus \{i_0\}$.

And, all diagonal entries of $A_{22}$ and $B_{11}$ are positive (or all negative), then the $k$-subdirect sum $C = A \oplus_k B$ is a DSDD matrix.

**Proof.** Without loss of generality, we can assume $i_0 = 1$ such that $\max_{j \in S_1 \cup S_2} (R_{j,t}(B)/|b_{j-t,j-t}|)R_1(A) < |a_{11}| < R_1(A)$ and $|a_{11}| \geq R_1(A)$, $i = 2, 3, \ldots, n_1$.

Case 1: for $i, j \in S_1$, from (a) of Lemma 1, we have

$$|c_{11}| = |a_{11}|(|a_{j_1} + |b_{j-t,j-t}|) = |a_{11}||a_{j_1}| + |a_{11}||b_{j-t,j-t}|$$

$$> R_1(A)R_1(A) + |b_{j-t,j-t}| \max_{j \in S_1 \cup S_2} \frac{R_{j,t}(B)}{|b_{j-t,j-t}|} R_1(A) \geq R_1(A)R_j(A) + |b_{j-t,j-t}| \frac{R_{j,t}(B)}{|b_{j-t,j-t}|} R_1(A)$$

(55)

$$= R_1(A)R_j(A) + R_{j-t}(B)R_1(A) = R_1(A)(R_j(A) + R_{j-t}(B)) \geq R_1(C)R_j(C).$$

If $i = 2, 3, \ldots, n_1 - k$, since $B$ is SDD, and from inequality (50), we can write

$$|c_{11}| = |a_{11}||a_{j_1}| + |b_{j-t,j-t}| > R_1(A)(R_j(A) + R_{j-t}(B)) \geq R_1(C)R_j(C).$$

(56)

Case 3: for $i \in S_1$ and $j \in S_3$, from (a) and (d) of Lemma 1, we have

$$|c_{i1}| = |a_{i1}| = |a_{i1}||a_{j_1}| + |b_{j-t,j-t}|$$

$$R_i(C) = R_i(A),$$

$$|c_{j1}| = |b_{j-t,j-t}|,$n

(57)

$$R_j(C) = R_{j-t}(B).$$

If $i = 1$, from inequality (49), we have

$$|c_{11}| = |a_{11}|,$n

$$|c_{j1}| = |b_{j-t,j-t}|,$

$$R_i(C) = R_i(A),$$

$$R_j(C) = R_{j-t}(B).$$

(58)

Since $A$ is DSDD, we obtain that for $i, j \in S_1$,

$$|c_{i1}| = |a_{i1}| + |b_{j-t,j-t}| > R_i(A)R_j(A) = R_i(C)R_j(C).$$

(52)

Case 2: for $i \in S_1$ and $j \in S_2$, from (a) of Lemma 1, it is easy to obtain

$$|c_{i1}| = |a_{i1}|,$n

$$R_i(C) = R_i(A).$$

(53)

Since all diagonal entries of $A_{22}$ and $B_{11}$ are positive (or all negative), we obtain

$$|c_{j1}| = |a_{j1}| + |b_{j-t,j-t}|.$$

(54)

If $i = 1$, since $A$ is DSDD, and from inequality (49), we can obtain

$$|c_{11}| = |a_{11}| + |b_{j-t,j-t}|.$$
\[ |c_{ii}||c_{jj}| = |a_{ii}||b_{j-i,j-i}| > R_i(A)R_{j-t}(B) = R_{i}(C)R_{j}(C). \]  
(59)

Case 4: for \( i, j \in S_2 \), from (b) and (c) of Lemma 1, we obtain
\[ |c_{ii}| = |a_{ii}| + |b_{j-i,j-i}|, \]
\[ R_i(C) \leq R_i(A) + R_{j-t}(B). \]
(60)

Since \( B \) is SDD, and from inequality (50), we can obtain
\[ |c_{ij}| = |a_{ij}| + |b_{j-i,j-i}| \]
\[ \geq R_i(A) + R_{j-t}(B) \]
(61)

Since \( B \) is SDD, and from inequality (50), we can obtain
\[ |c_{ij}| \geq R_i(A) + R_{j-t}(B) \geq R_i(C)R_j(C). \]
(63)

Case 6: for \( i, j \in S_3 \), from (d) of Lemma 1, we obtain
\[ |c_{ii}| = |b_{i-t,j-t}|, \]
\[ R_i(C) = R_{j-t}(B). \]
(64)

Since \( B \) is SDD, we can obtain
\[ |c_{ij}| = |b_{j-i,j-i}| \]
\[ > R_{j-t}(B)R_{j-t}(B) = R_i(C)R_j(C). \]
(65)

Therefore, we can draw a conclusion that for any \( i, j \in \{1, 2, \ldots, n\} \), \( |c_{ij}| > R_i(C)R_j(C) \); that is, \( C = A \mathbb{R}_k B \) is a DSDD matrix.

\[ \square \]

**Corollary 2.** Let \( A \) and \( B \) be square matrices of orders \( n_1 \) and \( n_2 \) partitioned as in (1), respectively, and \( k \) is an integer number such that \( 1 \leq k \leq \min\{n_1, n_2\} \). We assume that \( A \) is a SDD matrix and \( B \) is a DSDD matrix. If there exists a \( j_0 \in S_2 \cup S_3 \) such that
\[ \max_{i \in S_2 \cup S_3} \frac{R_i(A)}{|a_{ii}|} R_{j_0}(B) \leq |b_{j_0-t,j_0-t}| < R_{j_0-t}(B), \]
(66)
\[ |a_{ii}| > R_{j-t}(B), \quad j \in S_2 \cup S_3 \setminus \{j_0\} \]  
(67)

And, all diagonal entries of \( A_{22} \) and \( B_{11} \) are positive (or all negative), then the \( k \)-subdirect sum \( C = A \mathbb{R}_k B \) is a DSDD matrix.

Case 5: for \( i \in S_2 \) and \( j \in S_3 \), from (c) and (d) of Lemma 1, we obtain
\[ |c_{ii}| = |a_{ii}| + |b_{j-i,j-i}|, \]
\[ R_i(C) \leq R_i(A) + R_{j-t}(B), \]
\[ |c_{jj}| = |b_{j-i,j-i}|, \]
(62)
\[ R_j(C) = R_{j-t}(B). \]

Since \( B \) is SDD, and from inequality (50), we obtain
\[ |c_{ij}| = |a_{ij}| + |b_{j-i,j-i}| \]
\[ \geq R_i(A) + R_{j-t}(B) \geq R_i(C)R_j(C). \]
(63)

**Proof.** Without loss of generality, we can assume \( j_0 = n \) such that \( \max_{i} R_i(A)R_{n-t}(B) < |b_{n-t,n-t}| \leq R_{n-t}(B) \) and \( |c_{ij}| \geq R_{n-t}(B) \), \( j = n_1 - k + 1, n_1 - k + 2, \ldots, n - 1 \).

Case 1: for \( i, j \in S_1 \), from (a) of Lemma 1, we have
\[ |c_{ii}| = |a_{ii}|, \]
\[ |c_{jj}| = |a_{jj}|, \]
(68)
\[ R_i(C) = R_i(A), \]
\[ R_j(C) = R_j(A). \]

Since \( A \) is SDD, we obtain that for \( i, j \in S_1 \),
\[ |c_{ij}| = |a_{ij}| > R_i(A)R_{j}(A) = R_i(C)R_j(C). \]  
(69)

Case 2: for \( i \in S_1 \) and \( j \in S_2 \), from (a) of Lemma 1, it is easy to obtain
\[ |c_{ii}| = |a_{ii}|, \]
(70)
\[ R_i(C) = R_i(A). \]

Since all diagonal entries of \( A_{22} \) and \( B_{11} \) are positive (or all negative), we obtain
\[ |c_{ij}| = |a_{ij}| + |b_{j-i,j-i}| \]
(71)
Since \( A \) is SDD and from inequality (67), we have
\[ |c_{ii}\parallel c_{jj}| = |a_{ii}| \left( |a_{jj}| + |b_{j-t,j-t}| \right) > R_{i}(A) \left( R_{j}(A) + R_{j-t}(B) \right) \geq R_{i}(C)R_{j}(C). \] (72)

Case 3: for \( i \in S_{1} \) and \( j \in S_{2} \), from (a) and (d) of Lemma 1, we have
\[ |c_{ii}| = |a_{ii}|, \]
\[ R_{i}(C) = R_{i}(A), \]
\[ |c_{jj}| = |b_{j-t,j-t}|, \]
\[ R_{j}(C) = R_{j-t}(B). \] (73)

If \( j = n \), from inequality (66), we have
\[ |c_{ii}\parallel c_{mn}| = |a_{ii}| |b_{n-t,n-t}| > |a_{ii}| \max_{i \in S_{1} \cup S_{2}} \frac{R_{i}(A)}{|a_{ii}|} R_{n-t}(B) \]
\[ \geq |a_{ii}| \frac{R_{i}(A)R_{n-t}(B)}{|a_{ii}|} = R_{i}(A)R_{n-t}(B) = R_{i}(C)R_{n}(C). \] (74)

If \( j = n_{1} - k + 1, n_{1} - k + 2, \ldots, n - 1 \), since \( A \) is SDD, and from inequality (67), we can write
\[ |c_{ii}\parallel c_{jj}| = |a_{ii}| |b_{j-t,j-t}| > R_{i}(A)R_{j-t}(B) = R_{i}(C)R_{j}(C). \] (75)

\[ |c_{ii}\parallel c_{mn}| = (|a_{ii}| + |b_{j-t,j-t}|)|b_{n-t,n-t}| \]
\[ = |a_{ii}| |b_{n-t,n-t}| + |b_{j-t,j-t}| |b_{n-t,n-t}| \]
\[ > |a_{ii}| \max_{i \in S_{1} \cup S_{2}} \frac{R_{i}(A)}{|a_{ii}|} R_{n-t}(B) + R_{i-t}(B)R_{n-t}(B) \]
\[ \geq |a_{ii}| \frac{R_{i}(A)}{|a_{ii}|} R_{n-t}(B) + R_{i-t}(B)R_{n-t}(B) \]
\[ = R_{i}(A)R_{n-t}(B) + R_{i-t}(B)R_{n-t}(B) \]
\[ = (R_{i}(A) + R_{i-t}(B))R_{n-t}(B) \geq R_{i}(C)R_{n}(C). \] (79)

If \( j = n_{1} - k + 1, n_{1} - k + 2, \ldots, n - 1 \), since \( A \) is SDD, and from inequality (67), we can write
\[ |c_{ii}\parallel c_{jj}| = (|a_{ii}| + |b_{j-t,j-t}|)|b_{j-t,j-t}| > (R_{i}(A) + R_{i-t}(B))R_{j-t}(B) \geq R_{i}(C)R_{j}(C). \] (80)

Case 4: for \( i, j \in S_{2} \), from (c) of Lemma 1, we obtain
\[ |c_{ii}| = |a_{ii}| + |b_{i-t,j-t}|, \]
\[ R_{i}(C) \leq R_{i}(A) + R_{i-t}(B). \] (76)

Since \( A \) is SDD, and from inequality (67), we can obtain
\[ |c_{ii}\parallel c_{jj}| = (|a_{ii}| + |b_{i-t,j-t}|)(|a_{jj}| + |b_{j-t,j-t}|) \]
\[ > (R_{i}(A) + R_{i-t}(B))(R_{j}(A) + R_{j-t}(B)) \geq R_{i}(C)R_{j}(C). \] (77)

Case 5: for \( i \in S_{2}, j \in S_{3} \), from (c) and (d) of Lemma 1, we obtain
\[ |c_{ii}| = |a_{ii}| + |b_{i-t,j-t}|, \]
\[ R_{i}(C) \leq R_{i}(A) + R_{i-t}(B), \]
\[ |c_{jj}| = |b_{j-t,j-t}|, \]
\[ R_{j}(C) = R_{j-t}(B). \] (78)

If \( j = n \), since \( B \) is DSDD, and from inequality (66), we have
\[ |c_{ii}\parallel c_{jj}| = (|a_{ii}| + |b_{i-t,j-t}|)|b_{j-t,j-t}| \]
\[ = |a_{ii}| |b_{j-t,j-t}| + |b_{i-t,j-t}| |b_{j-t,j-t}| \]
\[ > |a_{ii}| \max_{i \in S_{1}} \frac{R_{i}(A)}{|a_{ii}|} R_{n-t}(B) + R_{i-t}(B)R_{n-t}(B) \]
\[ \geq |a_{ii}| \frac{R_{i}(A)}{|a_{ii}|} R_{n-t}(B) + R_{i-t}(B)R_{n-t}(B) \]
\[ = R_{i}(A)R_{n-t}(B) + R_{i-t}(B)R_{n-t}(B) \]
\[ = (R_{i}(A) + R_{i-t}(B))R_{n-t}(B) \geq R_{i}(C)R_{n}(C). \] (81)
Case 6: for $i, j \in S_3$, from (d) of Lemma 1, we obtain

$$|c_{ii}| = |b_{i-t,i-t}|, \quad R_i(C) = R_i(B).$$

Since $B$ is DSDD, we can obtain

$$|c_{ii} c_{jj}| = |b_{i-t,i-t} b_{j-t,j-t}| > R_i(B) R_j(B) = R_i(C) R_j(C).$$

(82)

In conclusion, for any $i, j \in S_1 \cup S_2 \cup S_3$, $|c_{ii} c_{jj}| > R_i(C) R_j(C)$. Therefore, $C = A \oplus_B B$ is a DSDD matrix. □

3. Conclusions

In this paper, some sufficient conditions such that the subdirect sum of DSDD matrices is in the class of DSDD matrices are given. Moreover, numerical examples are also presented to illustrate the corresponding results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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References

[1] S. M. Fallat and C. R. Johnson, "Sub-direct sums and positivity classes of matrices," Linear Algebra and Its Applications, vol. 288, pp. 149–173, 1999.
[2] R. Bru, F. Pedroche, and D. B. Szyld, "Additive schwarz iterations for Markov chains," SIAM Journal on Matrix Analysis and Applications, vol. 27, no. 2, pp. 445–458, 2005.
[3] R. Bru, F. Pedroche, and D. B. Szyld, "Subdirect sums of nonsingular $M$-matrices and of their inverse," The Electronic Journal of Linear Algebra, vol. 13, no. 1, pp. 162–174, 2005.
[4] R. Bru, F. Pedroche, and D. B. Szyld, "Subdirect sums of $S$-strictly diagonally dominant matrices," The Electronic Journal of Linear Algebra, vol. 15, no. 1, pp. 201–209, 2006.
[5] R. Bru, L. Cvetković, V. Kostić, and F. Pedroche, "Sums of $\Sigma$-strictly diagonally dominant matrices," Linear and Multilinear Algebra, vol. 58, no. 1, pp. 75–78, 2010.
[6] Y. Zhu and T. Z. Huang, "Subdirect Sum of doubly diagonally dominant matrices," The Electronic Journal of Linear Algebra, vol. 16, no. 1, pp. 171–182, 2007.
[7] J. Zhao, D. Liu, and R. Y. Hu, "Subdirect sums of Nekrasov matrices and Nekrasov matrices," Advances Applied Mathematics, vol. 64, no. 2, pp. 1–11, 2015.
[8] L. Gao, H. Huang, and C. Q. Li, "Subdirect sums of Nekrasov matrices," Linear and Multilinear Algebra, vol. 64, no. 2, pp. 208–218, 2016.
[9] X. Chen and Y. Wang, "Subdirect sums of SDD$_1$ matrices," Journal of Mathematics, vol. 2020, Article ID 3810423, 20 pages, 2020.
[10] P.-F. Dai, "A note on diagonal dominance, Schur complements and some classes of $H$-matrices and $P$-matrices," Advances in Computational Mathematics, vol. 42, no. 1, pp. 1–4, 2016.
[11] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, MA, USA, 1985.
[12] A. Berman and R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, NY, USA, 1979.
[13] B. Li and M. J. Tsatsomeros, "Doubly diagonally dominant matrices," Linear Algebra and Its Applications, vol. 261, no. 1–3, pp. 221–235, 1997.