Abstract. The aim of this paper is to introduce the sublinear Higson corona and show that the sublinear Higson corona of Euclidean cone of $P$ and $X$ is decomposed into the product of $P$ and that of $X$. Here $P$ is a compact metric space and $X$ is unbounded proper metric space. For example, the sublinear Higson corona of $n$-dimensional Euclidean space is homeomorphic to the product of $(n-1)$-dimensional sphere and that of natural numbers.

1. INTRODUCTION

In the coarse geometry, it is very important to study boundaries of metric spaces. If the metric space $X$ is hyperbolic in the sense of Gromov, we can define the ideal boundary and construct a natural metric on it. There are a lot of studies of relations between the coarse geometry of $X$ and topology of the ideal boundary of $X$.

For general proper metric spaces, we can define the boundary, which is called Higson corona. It is a contravariant functor from the category of coarse spaces into that of compact Hausdorff spaces. Since Higson coronae are never metrizable, their topology seem extremely complicated. However, there are several studies of the topology of Higson coronae. For example, if the asymptotic dimension of the metric space is finite, then it is equal to the covering dimension of its Higson corona [1]. Dranishnikov and Ferry showed that $n$-dimensional Euclidean space and Hyperbolic space of the same dimension have different $n$-th Čech cohomologies [2]. The author studied the group action and fixed point theorem on Higson coronae [4].

In this paper, we introduce the sublinear Higson corona $\nu_L X$ for coarse space $X$ and show that sublinear Higson corona of Euclidean cone $P \times_{cone} X$ is decomposed into the product of $P$ and $\nu_L X$ (Theorem 4.3). Here $P$ is a compact metric space and $P \times_{cone} X$ is defined in Section 4. For example, $\nu_L \mathbb{R}^n$ is homeomorphic to $S^{n-1} \times \nu_L \mathbb{N}$. As an application, we consider the linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ of a positive determinant. We show that the induced map on $\nu_L \mathbb{R}^n$ is homotopic to the identity. We remark that this statement does not hold on the Higson corona $\nu \mathbb{R}^n$.

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The organization of this paper is as follows: In section 2, we introduce the coarse category. Our definition is slightly different from usual one. In section 3, we define the sublinear Higson compactification and study its functorial properties. In section 4, we define the Euclidean cone and study the decomposition of the sublinear Higson corona of it. In section 5, we give an application. In Appendix, we give a proof of Lemma 3.5.

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2. COARSE CATEGORY

A metric space $X$ is *proper* if every closed bounded set in $X$ is compact. For a positive number $C$, we say that $X$ is $C$-quasi-geodesic space if for any $x, x' \in X$, there exists a map $f: [0, d(x, y)] \to X$ such that $(1/C) |a - b| - C \leq d(f(a), f(b)) \leq C |a - b| + C$ for any $a, b \in [0, d(x, y)]$. We choose a base point $e \in X$ and define $|x| := d(e, x)$ for $x \in X$. We say that $X$ is *coarse space* if $X$ is proper and $C$-quasi-geodesic space for some $C$ with a base point $e$.

**Definition 2.1.** Let $X$ and $Y$ be coarse spaces and let $f: X \to Y$ be a map (not necessarily continuous). We say that the map $f$ is *coarse* if there exists a positive constant $A$ such that the following two conditions are satisfied.

1. $|f(x)| \geq |x|/A - A$ for every $x \in X$.
2. $d(f(x), f(x')) \leq A d(x, x') + A$ for every $x, x' \in X$.

Let $f, g: X \to Y$ be maps. We define that $f$ is *sublinearly close* to $g$ if for any $\epsilon > 0$, there exists a positive constant $C_\epsilon$ such that $d(f(x), g(x)) \leq \epsilon |x| + C_\epsilon$ for all $x \in X$.

We define that metric spaces $X$ and $Y$ are *coarsely equivalent* if there exist coarse maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f$ and $f \circ g$ are sublinearly close to the identity maps of $X$ and $Y$, respectively. The category of coarse spaces consists of coarse spaces and coarse maps.

**Remark 2.2.** Our definition of the coarse equivalence is different from that of for metric spaces with usual bounded coarse structure. Compare with Definition 1.8 and Example 2.5 of [6]. Our definition is related to the sublinear coarse structure defined in [3].

3. SUBLINEAR HIGSON CORONA

All arguments in this section is based on that in Section 2.3 of [6]. Let $X$ be a coarse space and $\varphi: X \to \mathbb{C}$ be a bounded continuous map. We say that a $\varphi$ is sublinear Higson function on $X$ if there exists a constant $C_\varphi$ such that for all $R > 0$ and $x, x' \in X \setminus B(R)$, we have

$$|\varphi(x) - \varphi(x')| < \frac{C_\varphi d(x, x')}{R}.$$
Here $B(R)$ denotes the ball of radius $R$ centered at the base point $e$. We let $C_{h_L}(X)$ denote the space of all bounded continuous sublinear Higson functions on $X$. Then $C_{h_L}(X)$ is a sub $*$-algebra of $C_b(X)$ and its closure, denoted by $\overline{C_{h_L}(X)}$, is a unital $C^*$-algebra.

By the Gelfand-Naimark theorem, $\overline{C_{h_L}(X)}$ is the algebra of continuous functions on some compactification of $X$.

**Definition 3.1.** The compactification $h_L X$ of $X$ characterized by the property $C(h_L X) = \overline{C_{h_L}(X)}$ is called the **sublinear Higson compactification**. Its boundary $h_L X \setminus X$ is denoted $\nu_L X$, and is called the **sublinear Higson corona** of $X$.

We let $C_0(X)$ denote the algebra of continuous functions on $X$ which banish at infinity. Then we have $C(\nu_L X) \cong \overline{C_{h_L}(X)}/C_0(X)$.

**Example 3.2.** We define an embedding $i : \mathbb{R}^2 \to \mathbb{C}$ by sending $z$ to $z/(1 + |z|)$. For any $\varphi \in C(D^2)$, the composite $\varphi \circ i$ belongs to $\overline{C_{h_L}(\mathbb{R}^2)}$. Thus $i$ induces the injection $i^* : C(D^2) \hookrightarrow \overline{C_{h_L}(R^2)}$, it follows that $i$ extends to the surjection

$$h_L i : h_L \mathbb{R}^2 \to D^2.$$  

**Proposition 3.3.** As is Higson corona, the sublinear Higson corona of an unbounded coarse space is never second countable and its cardinal number is greater than or equal to $2^{2^{\aleph_0}}$.

**Proof.** For second countability, it is enough to show that $\overline{C_{h_L}(X)}$ is not separable. We can choose a sequence $\{x_n\}$ such that $|x_n| > 2|x_{n-1}|$ for all $n \geq 0$. We define a continuous map $\varphi_n : X \to \mathbb{C}$ as follows:

$$\varphi_n(x) = \begin{cases} 1 - \frac{4d(x,x_n)}{|x_n|} & \text{if } d(x,x_n) \leq \frac{|x_n|}{4}, \\ 0 & \text{otherwise.} \end{cases}$$

For any map $P : \mathbb{N} \to \{0,1\}$, we define the continuous map $\psi : X \to \mathbb{C}$ by $\psi_P(x) = \sum_{n \in \mathbb{N}} P(n) \varphi_n(x)$. Thus we obtain an uncountably many family of sublinear Higson functions such that $\|\psi_P - \psi_{P'}\|_1 = 1$ for any pair $(\psi_P, \psi_{P'})$ of distinct $P, P' \in \{0,1\}^\mathbb{N}$. This shows that $\overline{C_{h_L}(X)}$ is not separable. For any $\tilde{\psi} \in C_b(\{x_n\}) = l^\infty(\{x_n\})$, an extension $\tilde{\psi} \in \overline{C_{h_L}(X)}$ is given by $\tilde{\psi}(x) = \sum_{n \in \mathbb{N}} \tilde{\psi}(x) \varphi_n(x)$. This means the restriction $\overline{C_{h_L}(X)} \to C_b(\{x_n\})$ is surjective and thus the inclusion $\{x_n\} \hookrightarrow X$ extends to the embedding $\beta\{x_n\} \to h_L X$. Since $\beta\{x_n\}$ is homeomorphic to $\beta\mathbb{N}$, the cardinal number of $h_L X$ is greater than or equal to that of $\beta\mathbb{N}$, that is, $2^{2^{\aleph_0}}$. \hfill \Box

Let $f : X \to Y$ be a continuous coarse map. Then $f$ extends to the continuous map $h_L f : h_L X \to h_L Y$. The restriction to $\nu_L X$ is denoted $\nu_L f$ or $h_L f |_{\nu_L X}$. However, even if $f$ is coarse but not continuous, we can define the map $\nu_L f : \nu_L X \to \nu_L Y$ as follows.
Let $X$ be a coarse space. Let $B_{h_L}(X)$ denote the algebra of all bounded (not necessary continuous) functions $\varphi : X \to \mathbb{C}$ which satisfy the following condition: There exists a constant $C_\varphi$ such that for all $R > 0$ and $x, x' \in X \setminus B(R)$, we have

$$|\varphi(x) - \varphi(x')| < \frac{C_\varphi d(x, x') + C_\varphi}{R}.$$ 

Let $\overline{B_{h_L}}(X)$ be a closure of $B_{h_L}(X)$ in the algebra $B(X)$ of all bounded functions $X \to \mathbb{C}$ and $B_0(X)$ be the ideal of all bounded functions that tend to 0 at infinity.

**Definition 3.4.** A coarse space $X$ is said to be of bounded geometry if there exists a uniformly bounded cover $U = \{U_\alpha\}$ of $X$ with positive Lebesgue number and of finite degree. That is, there exist constants $L, d$ and $N$ such that Lebesgue number of $U$ is $L$, the diameter of all $U_\alpha \in U$ is bounded by $d$ and no more than $N$ member of $U$ has non-empty intersection.

**Lemma 3.5.** Let $X$ be a coarse space of bounded geometry. Then

(a) $C_0(X) = \overline{C_{h_L}}(X) \cap B_0(X)$.

(b) $\overline{B_{h_L}}(X) = \overline{C_{h_L}}(X) + B_0(X)$.

The part (a) is obvious. For the proof of the part (b), see Appendix.

It follows from the second Isomorphism theorem that

$$C(\nu_L X) = \frac{\overline{C_{h_L}}(X)}{C_0(X)} = \frac{\overline{C_{h_L}}(X) \cap B_0(X)}{\overline{C_{h_L}}(X) \cap B_0(X)} = \frac{B_0(X) + \overline{C_{h_L}}(X)}{B_0(X)} = \frac{\overline{B_{h_L}}(X)}{B_0(X)}.$$

**Proposition 3.6.** Let $X$ and $Y$ be coarse spaces of bounded geometry. A coarse map $f: X \to Y$ extends to a continuous map $\nu_L f: \nu_L X \to \nu_L Y$. If $f, g: X \to Y$ are sublinearly close, then $\nu_L f = \nu_L g$.

**Proof.** A coarse map $f: X \to Y$ induces $*$-homomorphism $f^*: \overline{B_{h_L}}(Y) \to \overline{B_{h_L}}(X)$ and $f^*: B_0(Y) \to B_0(X)$. If $f$ is sublinearly close to $g$ then $f^* - g^*$ maps $\overline{B_{h_L}}(Y)$ to $B_0(X)$.

**Corollary 3.7.** Coarsely equivalent spaces of bounded geometry have homeomorphic sublinear Higson corona.

**Proposition 3.8.** Sublinear Higson corona is a faithful functor from the category of coarse spaces of bounded geometry to that of compact Hausdorff spaces. That is, for any pair of coarse maps $f, g$, the equality $\nu_L f = \nu_L g$ implies that $f$ is sublinearly close to $g$.

**Proof.** Let $f, g: X \to Y$ be coarse maps. We suppose that $f$ is not sublinearly close to $g$. There exist a constant $C$ and a sequence $\{x_n\}$ such that $d(f(x_n), g(x_n)) \geq C \cdot |x|$. We can construct a sublinear Higson function $\varphi$ on $Y$ such that $\varphi(f(x_n)) = 1$ and $\varphi(g(x_n)) = 0$ for all $n \geq 0$, therefore $\nu_L f \neq \nu_L g$. □
Thus we have \( \| \rho \| \leq C \) and showed that sublinear Higson corona of proper metric space is homeomorphic to its Higson corona with respect to the sublinear coarse structure (Theorem 2.11 of [3]).

4. Euclidean cone

Let \( X \) be a coarse space and \( P \) be a compact path metric space. We define the Euclidean (punctured) cone metric on \( P \times X \) as follows. A path \( \gamma \) is a continuous map from an interval \( I \subset [0, \infty) \to P \times X \). We define the length of \( \gamma \) to be

\[
\sup \left\{ \sum_{j=0}^{n-1} d(x_j, x_{j+1}) + \max\{1, |x_j|, |x_{j+1}|\}d(p_j, p_{j+1}) \right\},
\]

where the supremum is taken over all finite sequences \((p_j, x_j)_{j=0}^n\) of points on the path \( \gamma \) with \((p_0, x_0)\) and \((p_n, x_n)\) being the two endpoints. We make \( P \times X \) into a path metric space by defining the distance \( d \) between two points to be the infimum of the length of all possible paths joining them. The Euclidean cone of \( P \) and \( X \), denoted by \( P \times \text{cone} \ X \), is the metric space \( P \times X \) equipped with this metric.

Example 4.1. The Euclidean cone \( S^{n-1} \times \text{cone} \ N \) is coarsely equivalent to \( \mathbb{R}^n \).

We define the map \( \Lambda : C_{h_L}(P \times \text{cone} \ X) \to C(P \times \overline{C_{h_L}(X)}) \) by sending \( \varphi \in C_{h_L}(Y) \) to the map which sends \( p \in P \) to \( \varphi_p \):

\[
\Lambda : C_{h_L}(P \times \text{cone} \ X) \ni \varphi \mapsto (p \mapsto \varphi_p) \in C(P, \overline{C_{h_L}(X)}).
\]

Here \( \varphi_p \) denotes the map \( x \mapsto \varphi(p, x) \) and \( C(P, \overline{C_{h_L}(X)}) \) denotes the C*-algebra of continuous maps from \( P \) to \( \overline{C_{h_L}(X)} \).

Proposition 4.2. \( \Lambda \) is well-defined and extends to a *-monomorphism of C*-algebras:

\[
\Lambda : \overline{C_{h_L}(P \times \text{cone} \ X)} \to C(P, \overline{C_{h_L}(X)}).
\]

Proof. Let \( \varphi \in C_{h_L}(P \times \text{cone} \ X) \). It is clear that \( \varphi_p \) belongs to \( \overline{C_{h_L}(X)} \) for all \( p \in P \). We show that the map \( \Lambda(\varphi) : P \to \overline{C_{h_L}(X)} \) is continuous. Let \( p, p' \in P \). For all \( x \in X \),

\[
|\Lambda(\varphi)(p)(x) - \Lambda(\varphi)(p')(x)| = |\varphi(p, x) - \varphi(p', x)|
\leq \frac{C_{\varphi}}{|x|}d((p, x), (p', x))
\leq C_{\varphi}d(p, p').
\]

Thus we have \( \|\Lambda(\varphi)(p) - \Lambda(\varphi)(p')\| \leq C_{\varphi}d(p, p') \) and therefore \( \Lambda(\varphi) \in C(P, \overline{C_{h_L}(X)}) \). Since \( \Lambda \) is a Lipschitz map with Lipschitz constant less than or equal to 1, we see that \( \Lambda \) extends to \( \overline{C_{h_L}(Y)} \). It is clear that \( \Lambda \) is a *-monomorphism. □

Proposition 4.3. \( \Omega : C(P) \otimes \overline{C_{h_L}(X)} \to \overline{C_{h_L}(P \times \text{cone} \ X)} : \varphi \otimes \psi \mapsto \varphi \cdot \psi \) is well-defined.
Proof. Let $\varphi \otimes \psi \in C(P) \otimes C_{h_L}(X)$. We assume that $\varphi$ is a Lipschitz map with a Lipschitz constant $C_\varphi$. Let $R > 0$. For any $(p, x), (p', x') \in P \timescone X$ with $|x|, |x'| > R$, we have

$$d(x, x') + |x| d(p, p') \leq d((p, x), (p', x')).$$

Thus

$$|\varphi(p)\psi(x) - \varphi(p')\psi(x')| \leq |\varphi(p)\psi(x) - \varphi(p')\psi(x)| + |\varphi(p')\psi(x) - \varphi(p')\psi(x')| \leq C_\varphi \|\psi\| d(p, p') + \frac{C_\varphi \|\varphi\| |d(x, x')|}{R} \leq \frac{C_\varphi \|\psi\| + C_\varphi \|\varphi\|}{R} d((p, x), (p', x')).$$

It follows that $\Omega(\varphi \otimes \psi)$ belongs to $\overline{C_{h_L}(P \timescone X)}$. Since the set of Lipschitz maps is dense in $C(P)$, we have the desired consequence. \qed

To show that $\Omega$ gives the inverse of $\Lambda$, we need the following well-known fact.

Lemma 4.4. Let $P$ be a compact metric space and $A$ be a commutative $C^*$-algebra. Then $C(P) \otimes A \cong C(P, A)$.

Proof. We can construct a family $\{U^n\}_{n \in \mathbb{N}}$ of finite covers of $P$ such that, the diameter of each member $U^n_i$ of $U^n$ is less than $1/n$. We choose points $p^n_i \in U^n_i$ for each $n \in \mathbb{N}$. Let $\{h^n_i\}$, be a partition of unity subordinate to $U^n$. We define $\Psi: C(P) \otimes A \to C(P, A)$ by $\Psi(\varphi \otimes a)(p) = \varphi(p)a$ for $\varphi \in C(P), a \in A$ and $p \in P$. Clearly $\Psi$ is injective. We show that $\Psi$ is surjective. Let $\psi \in C(P, A)$. Set $\psi_n = \sum_i h^n_i \otimes \psi(p^n_i) \in C(P) \otimes A$. Since $\psi$ is uniformly continuous, $\|\psi - \Psi(\psi_n)\|$ tends to 0 as $n$ goes to infinity. Thus $\Psi$ is surjective. \qed

Theorem 4.5. The sublinear Higson compactification of the Euclidean cone $P \timescone X$ is homeomorphic to the product $P \times h_LX$. Especially $\nu_{L}(P \timescone X) = P \times \nu_LX$.

Proof. By Proposition 4.2, 4.3 and Lemma 4.4, $\overline{C_{h_L}(P \timescone X)} \equiv C(P) \otimes \overline{C_{h_L}(X)}$. \qed

Example 4.6. $\nu_{L}(\mathbb{R}^n) = S^{n-1} \times \nu_L\mathbb{N}$.

5. Applications

Definition 5.1. Let $f, g: X \to Y$ be coarse maps. We say that $f$ is cone-homotopic to $g$ if there exists a coarse map $H: [0, 1] \timescone X \to Y$ such that $f = H_0$ and $g = H_1$. Such an $H$ is called cone homotopy between $f$ and $g$.

Theorem 5.2. If $f$ is cone-homotopic to $g$, then the induced map $\nu_Lf$ is homotopic to $\nu_Lg$.

Proof. $H$ induces a continuous map $\nu_LH: [0, 1] \times \nu_LX \to \nu_LY$ such that $\nu_LH(0, x) = \nu_Lf(x)$ and $\nu_LH(1, x) = \nu_Lg(x)$ for all $x \in X$. \qed
EXAMPLE 5.3. Let $T$ be an $n$ by $n$ integer matrix with a positive determinant. Then linear map $T: \mathbb{Z}^n \to \mathbb{Z}^n$ is a coarse map. The induced map $\nu_L T: \nu_L \mathbb{Z}^n \to \nu_L \mathbb{Z}^n$ is homotopic to the identity map $\text{id}_{\nu_L \mathbb{Z}^n}$.

We remark that since $T$ is not close to the identity $I_n$, the induced map $\nu_L T$ is different from the identity $\text{id}_{\nu_L \mathbb{Z}^n}$.

**Proof.** Since $\nu_L \mathbb{Z}^n$ is homeomorphic to $\nu_L \mathbb{R}^n$, it is enough to show that the map $T: \mathbb{R}^n \to \mathbb{R}^n$ is cone-homotopic to the identity map. Since $T$ has a positive determinant, we can choose a continuous path $\Theta: [0,1] \to GL_n(\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det A > 0\}$ such that $\Theta(0) = T$ and $\Theta(1) = I_n$. A map $H(x,t) = \Theta(t)x$ is a cone homotopy between $T$ and the identity $I_n$.

The same statement for the Higson corona of $\mathbb{Z}^n$ does not hold. This is pointed out by Makoto Yamashita. Here we recall the definition of the Higson corona. Let $X$ be a coarse space. We define a $C^*$-algebra by

$$C_h(X) = \{f \in C_b(X) : \lim_{|x| \to \infty} \text{diam}(f(B(x,r))) = 0 \quad \forall r > 0\}.$$ 

Here $B(x,r)$ denotes the $r$-ball centered at $x \in X$. The Higson compactification of $X$, denoted $hX$, is the compactification characterized by $C_h(X) = C(hX)$. The Higson corona, denoted $\nu X$, is defined by $hX \setminus X$. A continuous coarse map $f: X \to Y$ extends to $hf : hX \to hY$. The restriction to $\nu X$ is denoted $\nu f := hf|_{\nu X}$. The following argument is based on that in Section 3 of [5].

**Lemma 5.4.** For $s > 0$, we define a map $\phi_s : \mathbb{R}_+ \to \mathbb{R}$ by $x \mapsto \sqrt{sx}$. Let $e : \mathbb{R} \to S^1$ be a covering $e(x) = \exp(\sqrt{-1}x)$. The composite $e \circ \phi_s$ extends $h(e \circ \phi_s) : h\mathbb{R}_+ \to S^1$. If $h(e \circ \phi_s)$ is homotopic to $h(e \circ \phi_t)$ for some $t > 0$, then $s = t$.

**Proof.** Since $\text{diam}(\phi_s(B(x,r))) \to 0$ as $|x| \to \infty$ for any $r > 0$, we have a $*$-homomorphism $(e \circ \phi_s)^* : C(S^1) \to C_h(\mathbb{R}_+)$. Thus we have the extension $h(e \circ \phi_s) : h\mathbb{R}_+ \to S^1$. We suppose that $h(e \circ \phi_s)$ is homotopic to $h(e \circ \phi_t)$. We define $\psi_{s,t} : h\mathbb{R}_+ \to S^1$ by

$$\psi_{s,t}(x) = \frac{h(e \circ \phi_s)(x)}{h(e \circ \phi_t)(x)}.$$ 

Since $\psi_{s,t}$ is null-homotopic, there exists a lift $\psi_{s,t}^- : hX \to \mathbb{R}$ such that $e \circ \psi_{s,t}^- = \psi_{s,t}$. The restriction $\psi_{s,t}^-|_X$ and $\phi_s - \phi_t$ are both lifts of $\psi_{s,t}|_X$. The image of $\psi_{s,t}^-$ must be bounded since $hX$ is compact. It follows that the image of $\phi_s - \phi_t$ is also bounded. Thus we have $s = t$.

**Proposition 5.5.** For $s > 0$, we define a map $f_s : \mathbb{R}_+ \to \mathbb{R}_+$ by $f_s(x) = sx$. If the induced map $\nu f_s : \nu \mathbb{R}_+ \to \nu \mathbb{R}_+$ is homotopic to $\nu f_t$ for some $t > 0$. Then $s = t$. 
Proof. We suppose the induced map \( \nu f_s : \nu \mathbb{R}^+ \to \nu \mathbb{R}^+ \) is homotopic to \( \nu f_t \) for some \( t > 0 \). Let \( H : [0, 1] \times \nu \mathbb{R}^+ \to \nu S^1 \) be a homotopy between \( h(e \circ \phi_1) \circ \nu f_s \) and \( h(e \circ \phi_1) \circ \nu f_t \). Set \( A = [0, 1] \times \nu \mathbb{R}^n \cup \{0, 1\} \times h \mathbb{R}^n \subset \{0, 1\} \times \mathbb{R}^n \). We define a map \( H' : A \to S^1 \) by

\[
H'(u, x) = \begin{cases} 
H(u, x) & \text{if } 0 < u < 1, \\
h(e \circ \phi_s)(x) & \text{if } u = 0, \\
h(e \circ \phi_t)(x) & \text{if } u = 1.
\end{cases}
\]

Here we remark \( h(e \circ \phi_s) = h(e \circ \phi_1) \circ h f_s \). Since \( S^1 \) is ANR, there exists a neighborhood \( U \) of \( \nu X \) in \( h X \) and a extension \( H'' : U \times [0, 1] \to S^1 \) such that \( H''|_{A \cap U} = H'|_{A \cap U} \). Then we have \( h(e \circ \phi_s)|_U \) is homotopic to \( h(e \circ \phi_t)|_U \). By Lemma 5.4 we have \( s = t \). \( \square \)

Corollary 5.6. Let \( T : \mathbb{Z}^n \to \mathbb{Z}^n \) be a linear map with an eigenvalue \( \lambda \neq 1 \). Then the induced map \( \nu T : \nu \mathbb{Z}^n \to \nu \mathbb{Z}^n \) is not homotopic to the identity \( id_{\nu \mathbb{Z}^n} \).

6. Appendix—Proof of Lemma 3.5

The proof is based on the proof of Lemma 2.40 of [6]. By the assumption of \( X \), there exists a cover \( \mathcal{U} = \{U_\alpha\} \) of \( X \) and constants \( L, d, N \) such that Lebesgue number of \( \mathcal{U} \) is \( L \), the diameter of any \( U_\alpha \) is less than \( d \) and no more than \( N \) members of \( \mathcal{U} \) has non-empty intersection. Then we can construct a partition of unity \( \pi_\alpha \) subordinate to \( \mathcal{U} \) all of whose constituent functions are \( D \)-Lipschitz for some constant \( D = D(L, d, N) \). (See the proof of Theorem 9.9 of [6].)

We choose a point \( x_\alpha \in U_\alpha \) for each \( \alpha \). Now we let \( f \in B_{hL}(X) \) and \( C_f \) be a constant which appears in the definition of \( B_{hL}(X) \). We define

\[
g(x) := \sum_\alpha \pi_\alpha(x)f(x_\alpha).
\]

The function \( g \) is continuous and bounded. For all \( x \in X \), we have

\[
f(x) - g(x) = \sum_\alpha \pi_\alpha(x)(f(x) - f(x_\alpha))
\]

and \( d(x, x_\alpha) < d \) whenever \( \pi_\alpha(x) \neq 0 \). Thus we have \( f - g \in B_0(X) \). Next we will show that \( g \) satisfies (3.1). We assume that \( X \) is a \( C_X \)-quasi-geodesic for some positive constant \( C_X \). Let \( R > 2d \) and \( x, x' \in X \setminus B(R) \) such that \( d(x, x') \leq C_X \). Set \( I_+ := \{\alpha : \pi_\alpha(x) - \pi_\alpha(x') > 0\} \) and \( I_- := \{\alpha : \pi_\alpha(x) - \pi_\alpha(x') < 0\} \). Set \( t := \sum_{\alpha \in I_+} (\pi_\alpha(x) - \pi_\alpha(x')) = -\sum_{\alpha \in I_-} (\pi_\alpha(x) - \pi_\alpha(x')) \). Since each \( \pi_\alpha \) is \( D \)-Lipschitz, we have \( t \leq 2NDd(x, x') \). Set \( f_{\max} := \max\{f(x_\alpha) : \alpha \in I_+\} \) and \( f_{\min} := \min\{f(x_\alpha) : \alpha \in I_-\} \). For any \( \alpha, \alpha' \in I_+ \cup I_- \), we have \( x_\alpha, x_{\alpha'} \in X \setminus B(R - d) \) and \( d(x_\alpha, x_{\alpha'}) < C_X + 2d \). It follows that \( f_{\max} - f_{\min} \leq C_f(C_X + 2d)/(R - d) < 2C_f(C_X + 2d)/R \). We can assume that \( g(x) \geq g(x') \). Then we
have
\[
\begin{align*}
|g(x) - g(x')| &= g(x) - g(x') \\
&= \sum_{\alpha \in I_+} (\pi_\alpha(x) - \pi_\alpha(x')) f(x_\alpha) + \sum_{\alpha \in I_-} (\pi_\alpha(x) - \pi_\alpha(x')) f(x_\alpha) \\
&\leq \sum_{\alpha \in I_+} (\pi_\alpha(x) - \pi_\alpha(x')) f_{\max} + \sum_{\alpha \in I_-} (\pi_\alpha(x) - \pi_\alpha(x')) f_{\min} \\
&= t(f_{\max} - f_{\min}) \\
&\leq \frac{4NDf(C_X + 2d)}{R} d(x, x').
\end{align*}
\]
This shows that \( g \in \overline{C_{hL}}(X) \) and completes the proof of Lemma 3.5.

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