Locally convex spaces with the strong Gelfand–Phillips property

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Abstract
We introduce the strong Gelfand–Phillips property for locally convex spaces and give several characterizations of this property. We characterize the strong Gelfand–Phillips property among locally convex spaces admitting a stronger Banach space topology. If \( C_T(X) \) is a space of continuous functions on a Tychonoff space \( X \), endowed with a locally convex topology \( T \) between the pointwise topology and the compact-open topology, then: (a) the space \( C_T(X) \) has the strong Gelfand–Phillips property iff \( X \) contains a compact countable subspace \( K \subseteq X \) of finite scattered height such that for every functionally bounded set \( B \subseteq X \) the complement \( B \setminus K \) is finite, (b) the subspace \( C^b_T(X) \) of \( C_T(X) \) consisting of all bounded functions on \( X \) has the strong Gelfand–Phillips property iff \( X \) is a compact countable space of finite scattered height.

Keywords The strong Gelfand–Phillips property · Locally convex space · Banach space · function space

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1 Introduction

All topological spaces are assumed to be Tychonoff, all locally convex spaces are infinite-dimensional over the field \( \mathbb{F} \) of real or complex numbers, and all operators...
between locally convex spaces are linear and continuous. For a locally convex space $E$ (lcs for short), we denote by $E'$ the dual space of $E$. For a bounded subset $B \subseteq E$ and a functional $\chi \in E'$, we put $\|\chi\|_B := \sup \{ |\chi(x)| : x \in B \cup \{0\} \}$.

Let $E$ be a Banach space. A bounded subset $B$ of $E$ is called limited if every weak* null sequence $(\chi_n)_{n \in \omega}$ in $E'$ converges uniformly on $B$, that is $\lim_n \|\chi_n\|_B = 0$. A Banach space $E$ is called Gelfand–Phillips if every limited set in $E$ is precompact, i.e., has compact closure in $E$.

Gelfand–Phillips Banach spaces were intensively studied by many authors, see for example [12, 13, 24, 25] and more recent articles [9, 17, 19]. Since every Banach space is (isometrically) embedded in a $C(K)$-space, it is important to recognize Gelfand–Phillips spaces among Banach spaces of continuous functions in terms of the compact space $K$. Some sufficient conditions on compact spaces $K$ to have Gelfand–Phillips space $C(K)$ were obtained by Drewnowski [12], Drewnowski and Emmanuele [13], and by Schlumprecht in [24, 25].

In our recent article [5], we obtained several new characterizations of Banach spaces $E$ with the Gelfand–Phillips property using different idea: instead of limited sets in $E$, we consider bounded non-precompact subsets of $E$. This approach leads us to the following characterization of the Gelfand–Phillips property:

**Theorem 1.1** A Banach space $E$ is Gelfand–Phillips if and only if for every bounded non-precompact set $B \subseteq E$, there is a weak* null sequence $(\chi_n)_{n \in \omega}$ in $E'$ such that $\|\chi_n\|_B \not\to 0$.

The importance of this characterization of Gelfand–Phillips spaces is that it gives not only a new characterization of the Gelfand–Phillips property based on a much more handle class of bounded non-precompact sets than the class of limited sets, but it also allows to introduce and study the following strong version of that property.

**Definition 1.2** ([5]) A Banach space $E$ is defined to have the strong Gelfand–Phillips property if $E$ admits a weak* null-sequence $(\chi_n)_{n \in \omega}$ in $E'$ such that $\|\chi_n\|_B \not\to 0$ for every bounded non-precompact set $B \subseteq E$. In this case we shall say that $E$ is strongly Gelfand–Phillips.

It turns out that the class of strongly Gelfand–Phillips Banach spaces is rather narrow.

**Theorem 1.3** [5] A Banach space $E$ is strongly Gelfand–Phillips if and only if it embeds into $c_0$.

In [5], we also obtained the following complete characterization of $C(K)$-Banach spaces with the strong Gelfand–Phillips property (for the definitions of scattered height and pseudocompactness, see Sect. 3). We note that for the case when $K$ is a compact space, the equivalences (ii)$\iff$(iii)$\iff$(iv) are the classical result of Bessaga and Pelczyński [7] (see also Theorem 4.5.2 in [1]).

**Theorem 1.4** [5] For an infinite pseudocompact space $K$ the following conditions are equivalent:

(i) the Banach space $C(K)$ is strongly Gelfand–Phillips;
(ii) the Banach space $C(K)$ is isomorphic to a subspace of $c_0$.
(iii) the Banach space $C(K)$ is isomorphic to $c_0$;
(ii) the space $K$ is countable, compact and has finite scattered height.

The aforementioned results and discussion naturally lead us to the problem of extending the notion of a (strongly) Gelfand–Phillips space to the class of all locally convex space. It turns out that such an extension cannot be done directly and needs some preliminary notions defined below.

Let $E$ be a locally convex space. We say that a subset $B \subseteq E$ is barrel-bounded if for any barrel $U \subseteq E$ there is an $n \in \omega$ such that $B \subseteq nU$. A subset $A$ of $E$ is called barrel-precompact if for any barrel $B \subseteq E$ there exists a finite set $F \subseteq E$ such that $A \subseteq F + B$. Clearly, each barrel-precompact set is barrel-bounded. It is easy to see that if $E$ is a Banach space, then a subset $B$ of $E$ is barrel-bounded but not barrel-precompact if and only if $B$ is a bounded non-precompact subset of $E$.

Analogously to the case of Banach spaces, we defined in [3] the notion of limited sets and Gelfand–Phillips spaces as follows. A bounded subset $A$ of a locally convex space $E$ is called limited if every weak* null sequence $\{\chi_n\}_{n \in \omega}$ in $E'$ converges uniformly on $A$, that is, $\lim_n \|\chi_n\|_A = 0$. A locally convex space $E$ is called a Gelfand–Phillips space or has the Gelfand–Phillips property if every limited subset of $E$ is barrel-precompact. In [3], we obtain the following characterization of Gelfand–Phillips spaces which is similar to Theorem 1.1.

**Theorem 1.5** [3] A locally convex space $E$ is Gelfand–Phillips if and only if for every barrel-bounded non-barrel-precompact set $B \subseteq E$, there is a weak* null sequence $\{\chi_n\}_{n \in \omega}$ in $E'$ such that $\|\chi_n\|_B \not\to 0$.

The main goal of this article is to define and study the strong Gelfand–Phillips property in the class of all locally convex spaces, in particular, in various classes of function spaces.

The discussion above shows that a Banach space $E$ has the strong Gelfand–Phillips property if and only if it satisfies the following definition.

**Definition 1.6** A locally convex space $E$ is said to have the strong Gelfand–Phillips property (strong $(GP)$ property) if $E$ admits a weak* null sequence $\{\chi_n\}_{n \in \omega}$ in $E'$ such that $\|\chi_n\|_B \not\to 0$ for every barrel-bounded set $B \subseteq E$ which is not barrel-precompact. Locally convex spaces with the strong Gelfand–Phillips property are called strongly Gelfand–Phillips.

Our motivation for studying of strongly Gelfand–Phillips spaces is explained also by the following. In [4], we introduce and study the class of Josefson–Nissenzweig locally convex spaces. For a locally convex space $E$, we denote by $\beta^*(E', E)$ the topology on $E'$ whose neighborhood base at zero consists of the polars of barrel-bounded subsets of $E$.

**Definition 1.7** [4] A locally convex space $E$ is said to have the Josefson–Nissenzweig property (briefly, the JNP) if the identity map $(E', \sigma(E', E)) \to (E', \beta^*(E', E))$ is not sequentially continuous.

By the classical Josefson–Nissenzweig theorem, every Banach space has the JNP, however, by [8] (see also [4]), a Fréchet space has the JNP if and only if it is not Montel.
It is clear that for a locally convex space containing a barrel-bounded but not barrel-precompact set, we have the implications

$$\text{strong } (GP) \implies (GP) \implies \text{JNP}$$

and, by [5], none of these implications is reversible even in the class of Banach spaces.

Our study of the strongly Gelfand–Phillips spaces is also motivated by the following very natural relationship between the notions introduced above in the simplest case of Banach spaces. Let $E$ be a Banach space. Then the Josefson–Nissenzweig theorem states that there exists a weak* null sequence $\{\chi_n\}_{n \in \omega}$ in $E'$ such that $\|\chi_n\|_{BE} \not\to 0$. Taking into account that $BE$ is trivially a barrel-bounded, non-barrel-precompact subset of $E$ and keeping in mind the general case of locally convex spaces, one can naturally ask: When the sequence $\{\chi_n\}_{n \in \omega}$ is universal by modulo the Josefson–Nissenzweig theorem in the sense that $\|\chi_n\|_B \not\to 0$ for every barrel-bounded, non-precompact subset $B$ of $E$? Therefore, we can consider the strong $(GP)$ property as a universal version of the Josefson–Nissenzweig property. On the other hand, we can also ask: When for a chosen barrel-bounded, non-barrel-precompact subset of $E$ there exists a weak* null sequence $\{\chi_n\}_{n \in \omega}$ in $E'$ (now it depends also on $B$) such that $\|\chi_n\|_B \not\to 0$? Therefore, the $(GP)$ property can be considered as a version of the Josefson–Nissenzweig property which holds everywhere.

We can now describe the content of the paper. In Sect. 2 we characterize strongly Gelfand–Phillips spaces, see Theorem 2.3. The study of Banach spaces endowed with the weak topology is one of the most important directions in Banach Space Theory, see for example [10, 14, 16]. So locally convex space $E$, which admits a stronger Banach space topology, is of independent interest. We shall say that a locally convex space $E$ is $\beta$-Banach if the space $E$ endowed with the topology whose neighborhood base at zero consists of barrels is a Banach space. In Theorem 2.6 we characterize $\beta$-Banach spaces which have the strong Gelfand–Phillips property. As a corollary, we obtain that for no Banach space $E$, the dual space $E'_{w^*}$ endowed with the weak* topology is strongly Gelfand–Phillips. We show that the class of strongly Gelfand–Phillips spaces is closed under finite products, but even the countable power of a strongly Gelfand–Phillips Banach space can fail to have the strong Gelfand–Phillips property, see Proposition 2.14 and Example 4.1. In Example 4.3 we show that the completion of a strongly Gelfand–Phillips normed space can fail to be strongly Gelfand–Phillips, as well as a complete strongly Gelfand–Phillips space may contain a dense subspace without the strong Gelfand–Phillips property, see Example 4.4. As an application, we present a characterization of Josefson–Nissenzweig spaces in the terms of operators to spaces with the (strong) Gelfand–Phillips property, see Theorem 2.17.

For a Tychonoff space $X$, let $C(X)$ be the space of all continuous functions on $X$. Denote by $C^b(X)$ the subspace of $C(X)$ consisting of all bounded functions. Locally convex properties of the spaces $C(X)$ and $C^b(X)$ endowed with the topologies of uniform convergence on various families of sets in $X$ are widely studied in Functional Analysis, we refer the reader to the books [11, 20, 26]. In Sect. 3 using general Theorem 3.7, we obtain complete characterizations of spaces $X$ for which the spaces $C(X)$ and $C^b(X)$ endowed with one of those topologies have the strong Gelfand–Phillips property, see Theorems 3.15 and 3.19, respectively.
In Sect. 4, we provide numerous examples to clarify relationships between different notions considered in the article, and in Theorem 4.7, we show that any locally convex space $E$ in the weak topology is isomorphic to a closed subspace of a locally convex space with the strong Gelfand–Phillips property. As a corollary, we obtain that every Tychonoff space $X$ is homeomorphic to a closed subspace of a barrelled strongly Gelfand–Phillips $C_p$-space, see Corollary 4.8. The condition that $E$ is considered with the weak topology is essential because, by Example 4.9, the Banach space $\ell_\infty$ cannot be embedded into any Gelfand–Phillips space.

2 Locally convex spaces with the strong Gelfand–Phillips property

In this section, we characterize locally convex spaces with the strong Gelfand–Phillips property. First, we recall some definitions. Let $E$ be a locally convex space. We denote by $E_w$ the space $E$ endowed with the weak topology and by $E_w^\ast$ the dual space $E'$ endowed with the weak* topology. A subset $A$ of $E$ is called precompact if for any neighborhood $V$ of zero in $E$ there is a finite subset $F \subseteq E$ such that $A \subseteq F + V$. Let $U$ be a neighborhood of zero in $E$. A subset $A$ of $E$ is called $U$-separated if $a - a' \notin U$ for any distinct $a, a' \in A$. A subset of $E$ is uniformly discrete if it is $U$-separated for some neighborhood $U$ of zero in $E$. We shall use the following result which follows from Theorem 5 of [6].

**Proposition 2.1** A subset $A$ of a locally convex space $E$ is precompact if and only if every uniformly discrete subset of $A$ is finite.

Let us recall also the following well-known description of precompact subsets of the Banach space $c_0$, where $e_n'$ is the $n$th coordinate functional of $c_0$.

**Proposition 2.2** A subset $A$ of $c_0$ is precompact if and only if $\lim_{n \to \infty} \|e'_n\|_A = 0$.

For a locally compact space $X$, let

$$C_0(X) := \{ f \in C(X) : \text{the set} \{ x \in X : |f(x)| \geq \varepsilon \} \text{is compact for each } \varepsilon > 0 \}$$

be the space of $\mathbb{F}$-valued continuous functions on $X$ tending to zero at the infinity. The space $C_0(X)$ endowed with the sup-norm $\|f\|_X := \sup\{|f(x)| : x \in X\}$ is a Banach space. We denote by $C_0^0(X)$ the space $C_0(X)$ endowed with the subspace topology inherited from $C_p(X)$, where $C_p(X)$ denotes the space of $\mathbb{F}$-valued continuous functions on $X$ endowed with the pointwise topology.

Let $E$ be a locally convex space. Denote by $E_\beta$ the space $E$ endowed with the locally convex topology $\beta(E, E')$ whose neighborhood base at zero consists of barrels. Observe that a subset $A$ of $E$ is barrel-precompact if and only if $A$ is precompact in $E_\beta$, and $E$ is barrelled if and only if $E = E_\beta$. Recall that a subset $A$ of $E$ is defined to be barrel-precompact if for any barrel $B \subseteq E$ there exists a finite set $F \subseteq E$ such that $A \subseteq F + B$. In other words, $A$ is barrel-precompact if and only if $A$ is precompact in the topology $\beta(E, E')$.

In the next theorem, we characterize strongly Gelfand–Phillips spaces.

**Theorem 2.3** For a locally convex space $E$ the following assertions are equivalent:
(i) \(E\) has the strong Gelfand–Phillips property.

(ii) There exists a null sequence \(\{\chi_n\}_{n \in \omega}\) in \(E'_{w^*}\) such that for any infinite barrel-bounded, barrel-separated subset \(D\) of \(E\), there are an infinite subset \(D_0 \subseteq D\) and \(\delta > 0\) such that

\[
\sup_{n \in \omega} |\chi_n(x - x')| > \delta \quad \text{for every distinct } x, x' \in D_0.
\]

(iii) There is a continuous operator \(T : E \to C^0_p(\omega)\) such that for any infinite barrel-bounded barrel-separated set \(D\) in \(E\), the set \(T(D)\) is not precompact in the Banach space \(c_0\).

**Proof**

(i)⇒(ii) Let \(S = \{\chi_n\}_{n \in \omega}\) be a null sequence in \(E'_{w^*}\) witnessing the strong Gelfand–Phillips property. We show that it satisfies (ii). Indeed, let \(D\) be an infinite barrel-bounded, barrel-separated subset of \(E\). Then, by Proposition 2.1, \(D\) is not barrel-precompact. By the choice of \(S\), we have \(\|\chi_n\|_D \neq 0\). Choose \(\delta > 0\) and a strictly increasing sequence \(\{n_k\}_{k \in \omega} \in \omega\) such that \(\|\chi_{n_k}\|_D > 2\delta\) for every \(k \in \omega\). Now, choose arbitrarily \(x_0 \in D\) such that \(||\chi_{n_0}(x_0)| > 2\delta\) and set \(k_0 := 0\). Since \(S\) is a null sequence in \(E'_{w^*}\), by induction on \(i > 0\), we can find \(k_i > k_{i-1}\) and \(x_i \in D\) such that

\[\chi_{n_k}(x_i) < \delta \quad \text{and} \quad |\chi_{n_k}(x_j)| > 2\delta \quad \text{for every } 0 \leq i < j.\]

Set \(D_0 := \{x_i\}_{i \in \omega}\). Then, by construction, we have

\[
\sup_{n \in \omega} |\chi_n(x_i - x_j)| \geq |\chi_{n_{k_j}}(x_i - x_j)| \geq |\chi_{n_{k_j}}(x_j) - \chi_{n_{k_j}}(x_i)| > \delta
\]

for every distinct \(i < j\) in \(\omega\), as desired.

(ii)⇒(iii) Let \(S = \{\chi_n\}_{n \in \omega}\) be a null sequence in \(E'_{w^*}\) satisfying (ii). Then the operator

\[T : E \to C^0_p(\omega), \quad T(x) := (\chi_n(x))_{n \in \omega},\]

is well-defined and continuous. Now, let \(D\) be an infinite barrel-bounded barrel-separated set in \(E\). The choice of the sequence \(S\) implies that there is a sequence \(D_0 = \{x_n\}_{n \in \omega} \subseteq D\) and \(\delta > 0\) such that

\[
\sup_{n \in \omega} |\chi_n(x_i - x_j)| > \delta \quad \text{for every distinct } i, j \in \omega. \quad (2.1)
\]

We claim that \(\|\chi_n\|_{D_0} \not\to 0\). Indeed, assume for a contradiction that \(\|\chi_n\|_{D_0} \to 0\). Take \(N \in \omega\) such that \(\|\chi_n\|_{D_0} < \frac{\delta}{4}\) for every \(n > N\). From this and (2.1), it follows that

\[
\sup_{n \leq N} |\chi_n(x_i - x_j)| > \delta \quad \text{for every distinct } i, j \in \omega. \quad (2.2)
\]

Since \(D\) is barrel-bounded, it is bounded and hence \(\{\chi_n(x_i)\}_{i \in \omega}\) is a bounded subset of \(\mathbb{F}\) for every \(n \leq N\). Therefore, there exist a sequence \(\{x_{i_k}\}_{n \in \omega}\) and numbers \(c_0, \ldots, c_N \in \mathbb{R}\) such that...
such that \( \chi_n(x_i) \rightarrow c_n \) for every \( n \leq N \). But then there is \( M \in \omega \) such that

\[
\sup_{n \leq N} |\chi_n(x_i - x_j)| < \delta \quad \text{for every distinct } i, j > M,
\]

which contradicts (2.2). The claim is proved.

Since, by the claim, \( \|\chi_n\|_{D_0} \not\rightarrow 0 \), Proposition 2.2 implies that \( T(D_0) \) is not pre-compact in \( c_0 \). Thus, also \( T(D) \) is not precompact in \( c_0 \), as desired.

\((iii)\Rightarrow(i)\) Let \( T : E \rightarrow C^0_p(\omega) \) be a continuous operator such that for any infinite barrel-bounded barrel-separated set \( D \) in \( E \), the set \( T(D) \) is not precompact in the Banach space \( c_0 \). For every \( n \in \omega \), set \( \chi_n := e_n' \circ T \), where \( e_n' \) is the \( n \)-th coordinate functional on \( C^0_p(\omega) \). Clearly, the sequence \( S = \{\chi_n\}_{n \in \omega} \) is null in \( E_w^* \). We show that \( S \) witnesses the strong Gelfand–Phillips property of \( E \). To this end, fix a barrel-bounded subset \( B \) of \( E \) which is not barrel-precompact. Then, by Proposition 2.1 applied to the topology \( \beta(E, E') \), \( B \) contains an infinite barrel-separated subset \( D \) of \( E \). As a subset of \( B \), the set \( D \) is also barrel-bounded. Then the choice of the operator \( T \) implies that \( T(D) \) is not precompact in \( c_0 \). Finally, by Proposition 2.2, we obtain that \( \|\chi_n\|_B = \|e_n' \circ T\|_B \not\rightarrow 0 \). Thus \( E \) has the strong Gelfand–Phillips property. \( \square \)

Let \( E \) be a Banach space. A subset \( A \) of the dual space \( E' \) of \( E \) is called norming if there is a real constant \( \lambda \geq 1 \) such that

\[
\sup\{ |\chi(x)| : \chi \in A \cap B_{E'} \} \geq \frac{1}{\lambda} \|x\|
\]

for every \( x \in E \), where \( B_{E'} \) is the closed unit ball of \( E' \).

We shall use repeatedly the following assertion.

**Lemma 2.4** Let \( E \) be a Banach space, \( L \) be a norming subspace in \( E' \), and let \( T \) be a weaker locally convex topology on \( E \) such that \( \sigma(E, L) \subseteq T \). Then \( (E, T) \beta = E \). In particular:

(i) \( \left(C^0_p(\omega)\right)_\beta = c_0 \);

(ii) if \( X \) is a pseudocompact space and \( T \) is a locally convex vector topology on \( C(X) \) such that \( T_p \subseteq T \subseteq T_n \), where \( T_n \) is the norm topology on \( C(X) \), then \( (C_T(X))_\beta = C(X) \).

(iii) if \( E = L' \) for some Banach space \( L \), then \( (E_w^*)_\beta = E \).

**Proof** Choose \( \lambda \geq 1 \) such that for \( A = L \), the inequality (2.3) holds for every \( x \in E \). Define \( S_L := \{\chi \in L : \|\chi\| = 1\} \) and set \( H := (E, T) \).

Now, let \( U \) be a barrel in \( H \). Then \( U \) is a barrel in \( E \), and hence \( E \) is a neighborhood of zero in the Banach space \( E \). Thus, \( \beta(H, H') \) is contained in the topology \( T_n \) generated by the norm of the Banach space \( E \).

To prove that \( \beta(H, H') \supseteq T_n \), it suffices to show that the \( T \)-closure \( \bar{B} \) of the closed unit ball \( B = \{x \in E : \|x\| \leq 1\} \) of \( E \) is contained in the ball \( 2\lambda B \) (because \( \bar{B} \) is a barrel in \( H \), so \( \frac{1}{2\lambda} \bar{B} \subseteq B \) is a \( \beta(H, H') \)-neighborhood of zero). To this end, for any \( x \in E \setminus 2\lambda B \), choose \( \chi_x \in S_L \) such that

\[
|\chi_x(x)| > \frac{1}{2\lambda} \|x\| > 1.
\]
On the other hand, if \( y \in B \), we have \( |\chi_x(y)| \leq \|x\| \leq 1 \). Since \( \sigma(E, L) \leq T \), each \( \chi_x \) is \( T \)-continuous and hence

\[
\overline{B} \subseteq \bigcap_{x \in E \setminus 2\lambda B} \{ y \in E : |\chi_x(y)| \leq 1 \} \subseteq \bigcap_{x \in E \setminus 2\lambda B} E \setminus \{ x \} = 2\lambda B.
\]

as desired.

(i) To prove the equality \( \left( C^0_p(\omega) \right)_\beta = c_0 \), let \( L := \text{span} \{ e'_n : n \in \omega \} \), where \( e'_n \) is the projection of \( C^0_p(\omega) \) onto the \( n \)th coordinate. Then \( L \) is a norming subspace of \( c_0 \) and \( C^0_p(\omega) = (c_0, \sigma(c_0, L)) \). As we proved above \( \left( C^0_p(\omega) \right)_\beta = c_0 \).

(ii) It is clear that the subspace \( L = C_p(X)' \) of \( C(X)' \) is norming for the Banach space \( C(X) \), and \( \sigma(C(X), L) = T_p \subseteq T \subseteq T_n \). Thus, \( (C_T(X))_\beta = C(X) \).

(iii) follows from the fact that \( L \) is a norming subspace of \( E' \).

We define a locally convex space \( E \) to be \( \beta \)-Banach if the space \( E_\beta \) is topologically isomorphic to a Banach space. In particular, each Banach space is \( \beta \)-Banach. Since each linear continuous functional on the space \( E \) remains continuous in the topology of the space \( E_\beta \), we can identify the dual space \( E' \) of \( E \) with a subspace of \( (E_\beta)' \).

To prove Theorem 2.6 we shall use the following simple lemma.

**Lemma 2.5** Each continuous operator \( T : L \to C^0_p(\omega) \) from a barrelled space \( L \) remains continuous as an operator from \( L \) to \( c_0 \).

**Proof** Let \( \{ e'_n \}_{n \in \omega} \) be the sequence of coordinate functionals on the Banach space \( c_0 \). The definition of the topology of the space \( C^0_p(\omega) \) ensures that each functional \( e'_n \) remains continuous on the locally convex space \( C^0_p(\omega) \). Observe that the intersection

\[
B := \bigcap_{n \in \omega} \{ x \in C^0_p(\omega) : |e'_n(x)| \leq 1 \}
\]

coincides with the closed unit ball of the Banach space \( c_0 \). Since \( B \) is a barrel also in \( C^0_p(\omega) \), the continuity of the operator \( T \) implies that the set \( T^{-1}(B) \) is a barrel in \( L \). Since \( L \) is barrelled, \( T^{-1}(B) \) is a neighborhood of zero, which means that the operator \( T : L \to c_0 \) is continuous.

Below we characterize \( \beta \)-Banach locally convex spaces with the strong Gelfand–Phillips property. This result essentially generalizes Theorem 1.3.

**Theorem 2.6** A \( \beta \)-Banach locally convex space \( E \) has the strong Gelfand–Phillips property if and only if the space \( E_\beta \) embeds into the Banach space \( c_0 \) and \( E' \) is dense in the Banach space \( (E_\beta)' \).

**Proof** Assume that \( E \) has the strong Gelfand–Phillips property. Then the dual space \( E'_{w^*} \) contains a null sequence \( \{ \chi_n \}_{n \in \omega} \) such that \( \|\chi_n\|_B \not\to 0 \) for any barrel-bounded subset \( B \subseteq E \), which is not barrel-precompact. This implies that each barrel-bounded subset \( B \subseteq Z \) in the closed linear subspace \( Z = \bigcap_{n \in \omega} \chi_n^{-1}(0) \) of \( E \) is barrel-precompact. Then each bounded subset in the subspace \( Z \) of the Banach space \( E_\beta \) is precompact, which implies that the space \( Z \) is finite-dimensional. Unifying the
null sequence \( \{\chi_n\}_{n \in \omega} \) with a finite set of functionals separating points of the finite-dimensional subspace \( Z \), we can assume that \( Z = \{0\} \). In this case, the operator

\[
T : E \to C^0_p(\omega), \quad T : x \mapsto (\chi_n(x))_{n \in \omega},
\]

is injective and continuous.

We claim that the operator \( T : E_\beta \to c_0 \) is an isomorphic topological embedding. Indeed, since the space \( E_\beta \) is Banach and hence barrelled, Lemma 2.5 ensures that the operator \( T : E_\beta \to c_0 \) is continuous. By our assumption, the topology \( \beta(E, E') \) of the space \( E_\beta \) is generated by a (complete) norm \( \| \cdot \| \).

Assuming that the operator \( T : E_\beta \to c_0 \) is not an embedding, we can find a sequence \( \{x_n\}_{n \in \omega} \subseteq E \) of elements of norm \( \|x_n\| = 1 \) such that \( \sup_{m \in \omega} |\chi_m(x_n)| = \|T(x_n)\| \leq \frac{1}{2^m} \) for every \( n \in \omega \). We claim that the set \( B = \{x_n\}_{n \in \omega} \) is not precompact in the Banach space \( E_\beta \). Indeed, in the opposite case, by the completeness of \( E_\beta \), the sequence \( \{x_n\}_{n \in \omega} \) would contain a subsequence \( \{x_{n_k}\}_{k \in \omega} \) that converges in \( E_\beta \) to some element \( x_\infty \in E \) of norm \( \|x_\infty\| = 1 \). The continuity of the operator \( T \) ensures that \( T(x_\infty) = \lim_{n \to \infty} T(x_n) = 0 \), which contradicts the injectivity of \( T \). This contradiction shows that the set \( B \) is not precompact in \( E_\beta \) and hence is not barrel-precompact in \( E \). Now the choice of the sequence \( \{\chi_n\}_{n \in \omega} \) ensures that the sequence \( \{\|\chi_n\|_B\}_{n \in \omega} \) does not converge to zero. On the other hand, for every \( \varepsilon > 0 \), we can find an \( n \in \omega \) such that \( \frac{1}{2^n} < \varepsilon \). Since the sequence \( \{\chi_k\}_{k \in \omega} \) converges to zero in \( E'_{\text{w*,}} \), there exists a natural number \( m \) such that \( |\chi_k(x_i)| < \varepsilon \) for all \( i \leq n \) and \( k \geq m \). Then for every \( k \geq m \), we have

\[
\|\chi_k\|_B = \sup_{i \in \omega} |\chi_k(x_i)| = \max \left\{ \max_{i \leq n} |\chi_k(x_i)|, \sup_{i > n} |\chi_k(x_i)| \right\} \\
\leq \max \left\{ \varepsilon, \sup_{i > n} \frac{1}{2^n} \right\} \leq \max \left\{ \varepsilon, \frac{1}{2^m} \right\} = \varepsilon,
\]

which means that \( \|\chi_k\|_B \to 0 \). This contradiction shows that the operator \( T : E_\beta \to c_0 \) is a topological embedding.

To show that \( E' \) is dense in the Banach space \( (E_\beta)' \), we observe first that the adjoint map \( T^* : (c_0)' = \ell_1 \to (E_\beta)' \) is surjective because \( T \) is an embedding. Therefore, it suffices to show that \( E' \) contains all vectors \( T^*(e'_n) \), where \( \{e'_n\}_{n \in \omega} \) is the canonical basis in \( \ell_1 \). But this follows from the construction of the operator \( T \) since \( T^*(e'_n) = \chi_n \) for every \( n \in \omega \).

Conversely, assume now that the space \( E_\beta \) embeds into the Banach space \( c_0 \) and \( E' \) is dense in the Banach space \( (E_\beta)' \). We shall identify the space \( E_\beta \) with a subspace of \( c_0 \). Let \( B = \{x \in E_\beta : \|x\| \leq 1\} \) be the closed unit ball in the subspace \( E_\beta \) of \( c_0 \).

For every \( n \in \omega \), let \( \chi_n = e'_n | E \) be the restriction of the coordinate functional \( e'_n \in c_0' = \ell_1 \) to the subspace \( E_\beta \subseteq c_0 \). By our assumption, \( \chi_n \in \overline{E'} \subseteq (E_\beta)' \), so there exists a functional \( \mu_n \in E' \) such that

\[
\|\mu_n - \chi_n\| = \sup_{x \in B} |\mu_n(x) - \chi_n(x)| < \frac{1}{2^n}. \tag{2.4}
\]
Since the sequence \( \{e_n\}_{n \in \omega} \) is null in \((c_0)_{\omega^*}\), the sequence \( \{\chi_n\}_{n \in \omega} \) is \( \sigma((E_\beta)', E_\beta)\)-null in the dual space \((E_\beta)'\), and (2.4) implies that \( \{\mu_n\}_{n \in \omega} \) is \( \sigma(E', E)\)-null in \( E' \).

Now take any barrel-bounded subset \( P \subseteq E \) which is not barrel-precopact. Then there exists a barrel \( D \subseteq E \) such that \( P \not\subseteq F + D \) for any finite set \( F \subseteq E \). Since \( D \) is a barrel in the Banach space \( E_\beta \), there exists \( \varepsilon > 0 \) such that \( \varepsilon B \subseteq D \). Then \( P \not\subseteq F + \varepsilon B \) for any finite set \( F \subseteq E \). This allows us to select inductively a sequence \( \{x_n\}_{n \in \omega} \subseteq P \) such that \( \|x_m - x_n\| > \varepsilon \) for any \( n < m \). Since the set \( P \subseteq E_\beta \subseteq c_0 \) is barrel-bounded, it is bounded in \( \mathbb{R}^{\omega} \), and hence has compact closure in \( \mathbb{R}^{\omega} \). Replacing the sequence \( \{x_n\}_{n \in \omega} \) by a suitable subsequence, we can assume that it converges to some element of \( \mathbb{R}^{\omega} \).

We claim that there are two strictly increasing sequences \( \{n_k\}_{k \in \omega} \) and \( \{m_k\}_{k \in \omega} \) in \( \omega \) such that
\[
|e'_{m_k}(x_{n_k})| > \frac{1}{3}\varepsilon. \tag{2.5}
\]

We proceed by induction on \( k \). Since \( \inf_{i \neq j} \|x_i - x_j\| \geq \varepsilon \), there exists a number \( n_0 \in \omega \) such that \( \|x_{n_0}\| > \frac{\varepsilon}{3} \). By the definition of the norm \( \|x_{n_0}\| \), there exists a number \( m_0 \in \omega \) such that \( |e'_{m_0}(x_{n_0})| > \frac{1}{3}\varepsilon \). Now, assume that for some \( k \in \omega \), we constructed strictly increasing sequences \( \{n_i\}_{i < k} \) and \( \{m_i\}_{i < k} \). Since the sequence \( \{x_n\}_{n \in \omega} \) converges in \( \mathbb{R}^{\omega} \), there exists a number \( l_k > n_{k-1} \) such that
\[
|e'_{p}(x_i - x_j)| < \frac{\varepsilon}{3} \quad \text{for all } i, j \geq l_k \text{ and } p \leq m_{k-1}. \tag{2.6}
\]

As \( \|x_{l_k} - x_{l_k+1}\| \geq \varepsilon \), (2.6) implies that there exists a natural number \( m_k > m_{k-1} \) such that \( |e'_{m_k}(x_{l_k} - x_{l_k+1})| > \frac{\varepsilon}{3} \). Then, for some number \( n_k \in \{l_k, l_k + 1\} \), we get
\[
|e'_{m_k}(x_{n_k})| > \frac{1}{3}\varepsilon. \quad \text{Noting that } n_k \geq l_k > n_{k-1}, \text{ we complete the inductive step.}
\]

Now, for every \( k \in \omega \), (2.5) implies
\[
\|\chi_{m_k}\|_P \geq |e'_{m_k}(x_{n_k})| > \frac{1}{3}\varepsilon.
\]

Observe also that, by (2.4), we have
\[
|\mu_m(x_{n_k}) - \chi_{m_k}(x_{n_k})| \leq \|\mu_m - \chi_{m_k}\| \cdot \|P\| < \frac{1}{2m_k} \|P\|.
\]

where \( \|P\| = \sup_{x \in P} \|x\| \) is finite as \( P \) is bounded in \( E_\beta \subseteq c_0 \). Then (2.5) ensures that
\[
\|\mu_m\|_P \geq \|\chi_{m_k}(x_{n_k})\| - |\mu_{m_k}(x_{n_k}) - \chi_{m_k}(x_{n_k})| > \frac{1}{3}\varepsilon - \frac{1}{2m_k} \cdot \|P\|,
\]

which implies that \( \|\mu_n\|_P \not\to 0 \), witness that the space \( E \) is strongly Gelfand–Phillips. \( \square \)

**Corollary 2.7** The space \( C^0_p(\omega) \) is strongly Gelfand–Phillips.
Proof By Lemma 2.4, we have \((C^0_p(\omega))_\beta = c_0\). To apply Theorem 2.6 it remains to note that the dual space \(C^0_p(\omega)'\) coincides with the linear hull of the standard basis of the Banach space \(\ell_1 = (c_0)' = (C^0_p(\omega))_\beta'\).

A locally convex topology \(T\) on a Banach space \(E\) is called compatible if \((E, T)' = E'\).

Corollary 2.8 For a Banach space \(E\) the following assertions are equivalent:

(i) for every compatible locally convex topology \(T\) on \(E\), the space \((E, T)\) is strongly Gelfand–Phillips;

(ii) \(E_w\) is a strongly Gelfand–Phillips space;

(iii) \(E\) embeds into the Banach space \(c_0\).

Proof The implication (i)⇒(ii) is trivial.

(ii)⇒(iii) By Lemma 2.4, we have \((E_w)_\beta = E\). Since \((E_w)' = E'\) the space \(E_w\) is strongly Gelfand–Phillips, by Theorem 2.6.

(iii)⇒(i) By Lemma 2.4, we have \((E, T)_\beta = E\). Since \((E, T)' = E'\) the space \((E, T)\) is a strongly Gelfand–Phillips space by Theorem 2.6.

It follows from Corollary 2.8 that there are Banach spaces which in the weak topology are strongly Gelfand–Phillips, for example \(E = (c_0)_w\). It is natural to ask whether there exist a Banach space \(E\) such that the weak* dual \(E'_{w^*}\) is strongly Gelfand–Phillips. Below we answer this question in the negative. We recall that a Banach space \(E\) is called reflexive if the canonical map \(I : E \to E''\), \(I(x)(\chi) := \chi(x)\), is surjective. Observe that since \(I\) is an embedding and \(E\) is complete, \(E\) is reflexive if and only if \(I(E)\) is dense in the bidual Banach space \(E''\).

Corollary 2.9 For every Banach space \(E\), the weak* dual \(E'_{w^*}\) of \(E\) does not have the strong Gelfand–Phillips property.

Proof Suppose for a contradiction that \(E'_{w^*}\) has the strong Gelfand–Phillips property. By Lemma 2.4, we have \((E'_{w^*})_\beta\) is the dual Banach space \(E'\). Therefore \(E'_{w^*}\) is \(\beta\)-Banach. By Theorem 2.6, \(E'\) embeds into \(c_0\) and \(E = (E'_{w^*})'\) is a dense subspace of the Banach bidual space \(E''\). Hence, the Banach space \(E\) is reflexive. Therefore, also \(E'\) is reflexive. On the other hand, by Proposition 2.a.2 in [21], each infinite-dimensional subspace of \(c_0\) contains a copy of \(c_0\), and hence the space \(E' \subseteq c_0\) is not reflexive. This contradiction shows that \(E'_{w^*}\) is not strongly Gelfand–Phillips.

Example 2.10 There exists a separable locally convex space \(E\) such that the space \(E_\beta\) is isomorphic to the Banach space \(c_0\) but \(E\) is not strongly Gelfand–Phillips.

Proof Let \(Z\) be a closed, proper, and norming subspace of \(\ell_1\), and let \(E := (c_0, \sigma(c_0, Z))\). Then, by Lemma 2.4, \(E_\beta\) is isomorphic to the space \(c_0\). Since \(Z\) is a closed, proper subspace of \(\ell_1\), Theorem 2.6 implies that \(E\) is not strongly Gelfand–Phillips. Mikhail Ostrovskii suggested to take for \(Z\) the linear subspace of \(\ell_1\) consisting of all sequences \(z = (z_n)_{n \in \omega} \in \ell_1\) such that \(\sum_{n \in \omega} z_n = 0\).  

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Another example with the same properties can be found among function spaces \( C_p(K) \).

**Example 2.11** The locally convex space \( c_p = \{ x \in \mathbb{F}^\omega : \exists \lim_{n \to \infty} x(n) \} \) is not strongly Gelfand–Phillips but \((c_p)_\beta\) is isomorphic to the Banach space \( c_0 \).

**Proof** Observe that the topology of \( c_p \) is the topology on the Banach space \( c \) defined by the norming subspace \( L = \text{span}(e_n')_{n \in \omega} \) of \( c' \), where \( e_n' \) is the \( n \)th coordinate functional. Therefore, by Lemma 2.4, \((c_p)_\beta = c\) and hence \( c_p \) is \( \beta \)-Banach. It is well known that \( c \) is isomorphic to \( c_0 \). Since \((c_p)' = L\) is not dense in \((c_p)_\beta' = c'\), Theorem 2.6 implies that the space \( c_p \) is not strongly Gelfand–Phillips. \( \square \)

Now we consider some hereditary properties of the class of strongly Gelfand–Phillips spaces. We define a linear subspace \( X \) of a locally convex space \( E \) to be \( \beta \)-embedded if the identity inclusion \( X_\beta \to E_\beta \) is a topological embedding. It is easy to see that \( X \) is \( \beta \)-embedded in \( E \) if and only if for any barrel \( B \subseteq X \) there exists a barrel \( D \subseteq E \) such that \( D \cap X = B \).

**Proposition 2.12** A subspace \( X \) of a locally convex space \( E \) is \( \beta \)-embedded if one of the following conditions is satisfied:

(i) \( X \) is complemented in \( E \);

(ii) \( X \) is barrelled;

(iii) \( X \) and \( E \) are \( \beta \)-Banach and \( X_\beta \) is closed in \( E_\beta \).

**Proof** Given a barrel \( B \subseteq X \), we should find a barrel \( D \subseteq E \) such that \( D \cap X = B \).

(i) If \( X \) is complemented in \( E \), then there exists a linear continuous operator \( R : E \to X \) such that \( R(x) = x \) for all \( x \in X \). In this case the set \( D = R^{-1}(B) \) is a barrel in \( E \) with \( D \cap X = B \).

(ii) If \( X \) is barrelled, then the barrel \( B \) is a neighborhood of zero. Since \( X \) is a subspace of \( E \), there exists a barrel neighborhood \( D \subseteq E \) of zero such that \( D \cap X \subseteq B \).

(iii) Assume that the spaces \( X \) and \( E \) are \( \beta \)-Banach and \( X_\beta \) is closed in \( E_\beta \). Then the identity inclusion \( I : X_\beta \to E_\beta \) is a continuous injective operator between Banach spaces such that the image \( I(X_\beta) \) is closed in \( E_\beta \). By the Banach Open Mapping Principle, the operator \( I : X_\beta \to E_\beta \) is a topological embedding. \( \square \)

In the next proposition, we give some sufficient conditions on a subspace of a strong Gelfand–Phillips space to have the strong Gelfand–Phillips property.

**Proposition 2.13** Assume that a locally convex space \( E \) is strongly Gelfand–Phillips. Then:

(i) Every \( \beta \)-embedded subspace of \( E \) is strongly Gelfand–Phillips.

(ii) Every barrelled subspace of \( E \) is strongly Gelfand–Phillips.

(iii) If \( E_\beta \) is barrelled (for example, \( E \) is \( \beta \)-Banach), then \( E_\beta \) is strongly Gelfand–Phillips.

**Proof** (i) Fix a weak* null sequence \( \{ x_n \}_{n \in \omega} \) in \( E' \) such that \( \| x_n \|_A \not\to 0 \) for every barrel-bounded set \( A \subseteq E \) which is not barrel-precompact. Let \( X \) be a \( \beta \)-embedded subspace of \( E \). Since the identity operator \( I : X \to E \) is continuous, the sequence

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(\chi_n \circ I)_{n \in \omega} is null in the weak* topology on \(X'\). We claim that the weak* null sequence \((\chi_n \circ I)_{n \in \omega}\) witnesses the strong Gelfand–Phillips property of \(X\). To this end, let \(A\) be an infinite barrel-bounded but not barrel-precompact subset of \(X\). Since \(I\) is continuous the set \(I(A)\) is barrel-bounded in \(E\). To show that \(I(A)\) is not barrel-precompact in \(E\), we observe that, by Proposition 2.1, the set \(A\) contains an infinite subset \(D\) which is \(B\)-separated for some barrel \(B \subseteq X\). Since \(X\) is \(\beta\)-embedded, there exists a barrel \(\hat{B} \subseteq E\) such that \(\hat{B} \cap X \subseteq B\). Observe that \(I(D)\) is also \(\hat{B}\)-separated in \(E\). Therefore, by Proposition 2.1, \(I(A)\) is not barrel-precompact in \(E\). Now the choice of \((\chi_n)_{n \in \omega}\) implies \(\|\chi_n \circ I\|_A = \|\chi_n\|_{I(A)} \not\to 0\). Thus \(X\) is strongly Gelfand–Phillips.

(ii) follows from (i) and Proposition 2.12(ii).

(iii) Assume that the space \(E_{\hat{\beta}}\) is barrelled. By Theorem 2.3, there is a continuous operator \(T : E \to C^0_\beta(\omega)\) such that for any infinite barrel-bounded barrel-separated set \(D \subseteq E\), the set \(T(D)\) is not precompact in the Banach space \(c_0\).

Set \(\hat{T} := T \circ i : E_{\hat{\beta}} \to C^0_\beta(\omega)\), where \(i : E_{\hat{\beta}} \to E\) is the identity inclusion. Now take any infinite barrel-bounded barrel-separated subset \(D \subseteq E_{\hat{\beta}}\). Find a barrel \(B \subseteq E_{\hat{\beta}}\) such that \(D\) is \(B\)-separated. Since \(E_{\hat{\beta}}\) is barrelled, \(B\) is a neighborhood of zero in \(E_{\hat{\beta}}\). By the definition of \(E_{\hat{\beta}}\), we can assume that \(B\) is a barrel also in \(E\). Hence, \(D\) is an infinite barrel-bounded barrel-separated subset of \(E\). Then the image \(T \circ i(D) = T(D)\) is not precompact in \(c_0\). Thus, by Theorem 2.3, \(E_{\hat{\beta}}\) is strongly Gelfand–Phillips.

\[\square\]

**Proposition 2.14** If \(E\) and \(L\) are locally convex spaces, then \(E \times L\) is strongly Gelfand–Phillips if and only if so are \(E\) and \(L\).

**Proof** Assume that \(E \times L\) is a strongly Gelfand–Phillips space. By Proposition 2.12(i), \(E\) and \(L\) are \(\beta\)-embedded in \(E \times L\) and therefore, by Proposition 2.13(i), \(E\) and \(L\) are strongly Gelfand–Phillips.

Conversely, let \(E\) and \(L\) be strongly Gelfand–Phillips spaces. Choose weak* null sequences \((\chi_n)_{n \in \omega} \subseteq E'\) and \((\eta_n)_{n \in \omega} \subseteq L'\) witnessing the strong Gelfand–Phillips property. Identify \(E \times L\) with the direct sum \(E \oplus L\) of the spaces \(E\) and \(L\). Consider the null sequence \((\xi_n)_{n \in \omega} \subseteq (E \oplus L)'_{w*}\) defined by \(\xi_{2n} := \chi_n\) and \(\xi_{2n+1} = \eta_n\) for every \(n \in \omega\). We claim that \(\|\xi_n\|_B \not\to 0\) for any barrel-bounded set \(B \subseteq E \times L\) which is not barrel-precompact. Given such a set \(B\), observe that the projections \(B_E\) and \(B_L\) of \(B\) onto \(E\) and \(L\), respectively, are barrel-bounded. Indeed, if \(U\) is a barrel in \(E\), then \(U \times L\) is a barrel in \(E \times L\). Therefore, there is \(a > 0\) such that \(B \subseteq a(U \times L)\). So \(B_E \subseteq aU\) and hence \(B_E\) is barrel-bounded. Analogously one can prove that \(B_L\) is barrel-bounded. We claim that at least one of the sets \(B_E\) or \(B_L\) is not barrel-precompact. To derive a contradiction, assume that \(B_E\) and \(B_L\) are barrel-precompact. Observe that for every barrel \(D\) in \(E \oplus L\), the intersection \(D \cap E\) and \(D \cap L\) are barrels in \(E\) and \(L\), respectively. By the barrel-precompactness of the sets \(B_E\) and \(B_L\), there are finite sets \(F_E \subseteq E\) and \(F_L \subseteq L\) such that \(B_E \subseteq F_E + \frac{1}{2}D\) and \(B_L \subseteq F_L + \frac{1}{2}D\). Then \(B \subseteq B_E + B_F \subseteq (F_E + F_L) + D\) and hence \(B\) is barrel-precompact in \(E \oplus L\), which contradicts the choice of \(B\). This contradiction shows that one of the sets \(B_E\) and \(B_L\) is not barrel-precompact. Without loss of generality we assume that \(B_E\) is not barrel-precompact. Then the choice of \((\chi_n)_{n \in \omega}\) ensures that \(\|\xi_{2n}\|_B = \|\chi_n\|_{B_E} \not\to 0\).
Thus, the sequence \((\xi_n)_{n \in \omega}\) witnesses the strong Gelfand–Phillips property of \(E \times L\).

\[\square\]

**Remark 2.15** In Example 4.1 we shall construct a strongly Gelfand–Phillips Banach space whose countable power is not strongly Gelfand–Phillips, and in Example 4.6 we construct a strongly Gelfand–Phillips locally convex space containing a closed subspace which is not strongly Gelfand–Phillips.

\[\square\]

**Problem 2.16** Let \(X\) be a closed subspace of a locally convex space \(Y\).

(i) Is the quotient space \(Y/X\) strongly Gelfand–Phillips if so is the space \(Y\)?

(ii) Is \(Y\) strongly Gelfand–Phillips if \(X\) and \(Y/X\) are strongly Gelfand–Phillips?

We finish this section with the following characterization of Josefson–Nissenzweig spaces in the terms of operators to spaces with the (strong) Gelfand–Phillips property.

We recall (see [3]) that an operator \(T : X \to Y\) between locally convex spaces is \(\beta\)-to-\(\beta\) precompact if for any barrel-bounded set \(B \subseteq X\) the image \(T(B)\) is barrel-precompact in \(Y\).

**Theorem 2.17** For a locally convex space \(E\) the following conditions are equivalent:

(i) \(E\) has the Josefson-Nissenzweig property;

(ii) there exists a continuous operator \(T : E \to C^0_\omega(\omega)\), which is not \(\beta\)-to-\(\beta\) precompact;

(iii) for some locally convex space \(Y\) with the strong Gelfand–Phillips property, there exists a continuous operator \(T : E \to Y\), which is not \(\beta\)-to-\(\beta\) precompact;

(iv) for some locally convex space \(Y\) with the Gelfand–Phillips property, there exists a continuous operator \(T : E \to Y\), which is not \(\beta\)-to-\(\beta\) precompact.

**Proof** The equivalence (i) \(\iff\) (ii) is proved in [4].

The implication (ii) \(\Rightarrow\) (i), assume that \(E\) admits a continuous operator \(T : E \to Y\) into a locally convex space \(Y\) with the Gelfand–Phillips property such that \(T\) is not \(\beta\)-to-\(\beta\) precompact. Then for some barrel-bounded set \(B \subseteq X\), the image \(T(B)\) is not barrel-precompact in \(Y\). Since \(Y\) has the \((GP)\) property, there exists a null sequence \((\mu_n)_{n \in \omega} \subseteq Y'_{w^*}\) such that \(\|\mu_n\|_{T(B)} \not\to 0\). For every \(n \in \omega\), consider the functional \(\eta_n = \mu_n \circ T \in E'\) and observe that the sequence \((\eta_n)_{n \in \omega}\) converges to zero in the topology \(\sigma(E', E)\). However, since

\[\|\eta_n\|_{B} = \|\mu_n\|_{T(B)} \not\to 0 \quad \text{as} \quad n \to \infty,\]

the sequence \((\eta_n)_{n \in \omega}\) does not converge to zero in the topology \(\beta^*(E', E)\). This means that \(E\) has the JNP.

\[\square\]

### 3 Function spaces with the strong Gelfand–Phillips property

Let \(X\) be a set, and let \(f : X \to \mathbb{F}\) be a function to the field \(\mathbb{F}\) of real or complex numbers. For a subset \(A \subseteq X\) and \(\varepsilon > 0\), let

\[\|f\|_A := \sup(|f(x)| : x \in A \cup \{0\}) \in [0, \infty],\]
and if $\mathcal{F}$ is a subfamily of $\mathbb{P}^X$, we set

$$[A; \varepsilon]_{\mathcal{F}} := \{ f \in \mathcal{F} : \|f\|_A \leq \varepsilon \}.$$ 

If the family $\mathcal{F}$ is clear from the context, then we shall omit the subscript $\mathcal{F}$ and write $[A; \varepsilon]$ instead of $[A; \varepsilon]_{\mathcal{F}}$. A family $S$ of subsets of $X$ is directed if for any sets $A, B \in S$ the union $A \cup B$ is contained in some set $C \in S$.

For a Tychonoff space $X$, we denote by $C(X)$ the space of all continuous functions $f : X \to \mathbb{F}$ on $X$. A subset $A \subseteq X$ is called functionally bounded if $\|f\|_A < \infty$ for any continuous function $f \in C(X)$. A Tychonoff space $X$ is pseudocompact if $X$ is functionally bounded in $X$.

A Tychonoff space $X$ is defined to be a $\mu$-space if every functionally bounded subset of $X$ has compact closure in $X$. We denote by $\nu X$, $\mu X$ and $\beta X$ the Hewitt completion (=realcompactification), the Dieudonné completion and the Stone-Čech compactification of $X$, respectively. It is known ([15, 8.5.8]) that $X \subseteq \mu X \subseteq \nu X \subseteq \beta X$. Also it is known that all paracompact spaces and all realcompact spaces are Dieudonné complete and each Dieudonné complete space is a $\mu$-space, see [15, 8.5.13]. On the other hand, each pseudocompact $\mu$-space is compact.

A topological space $X$ is scattered if each nonempty subspace of $X$ has an isolated point. For a topological space $X$, let $X^{(0)} := X$ and let $X^{(1)}$ be the space of non-isolated points of $X$. For a non-zero ordinal $\alpha$, let $X^{(\alpha)} := \bigcap_{\beta<\alpha} (X^{(\beta)})^{(1)}$. It is well known that a topological space $X$ is scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal $\alpha$. The smallest ordinal $\alpha$ with $X^{(\alpha)} = \emptyset$ is called the scattered height of $X$.

For a Tychonoff space $X$, the space $C(X)$ carries many important locally convex topologies, i.e., topologies turning $C(X)$ into a locally convex space. For a locally convex topology $\mathcal{T}$ on $C(X)$, we denote by $C_\mathcal{T}(X)$ the space $C(X)$ endowed with the topology $\mathcal{T}$.

Each directed family $S$ of functionally bounded sets in a Tychonoff space $X$ induces a locally convex topology $\mathcal{T}_S$ on $C(X)$ whose neighborhood base at zero consists of the sets $[S; \varepsilon]$ where $S \in S$ and $\varepsilon > 0$. The topology $\mathcal{T}_S$ is called the topology of uniform convergence on sets of the family $S$. The topology $\mathcal{T}_S$ is Hausdorff if and only if the union $\bigcup S$ is dense in $X$.

If $S$ is the family of all finite (resp. compact or functionally bounded) subsets of $X$, then the topology $\mathcal{T}_S$ will be denoted by $\mathcal{T}_p$ (resp. $\mathcal{T}_k$ and $\mathcal{T}_b$), and the function space $C_{\mathcal{T}_S}(X)$ will be denoted by $C_p(X)$ (resp. $C_k(X)$ or $C_b(X)$).

**Lemma 3.1** A Tychonoff space $X$ is a $\mu$-space if and only if $C_k(X) = C_b(X)$.

**Proof** If $X$ is a $\mu$-space, then each closed functionally bounded set in $X$ is compact, which implies the equality $C_b(X) = C_k(X)$. Now assume conversely that $C_b(X) = C_k(X)$. Given any closed functionally bounded set $B \subseteq X$, consider the neighborhood of zero $[B; 1]$ in $C_b(X)$. Since $C_b(X) = C_k(X)$, there exists a compact set $K \subseteq X$ and $\varepsilon > 0$ such that $[K; \varepsilon] \subseteq [B; 1]$. We claim that $B \subseteq K$. In the opposite case, by the Tychonoff property of $X$, we could find a continuous function $f : X \to \mathbb{R}$ such that $f(K) \subseteq [0]$ and $f(x) > 1$ for some point $x \in B \setminus K$. Then $f \in [K; \varepsilon] \subseteq [B; 1]$ and hence $|f(x)| \leq 1$, which contradicts the choice of $f$. This contradiction shows that $B \subseteq K$ and hence $B$ is compact. □
Let $X$ be a dense subspace of a Tychonoff space $M$ (for example, $M = \mu X$, $\nu X$ or $\beta X$). Then the union $\bigcup S$ of the directed family of all finite (resp. compact) subsets of $X$ is dense in $M$. Therefore, $S$ naturally defines a Hausdorff locally convex vector topology on the space $C(M)$, which will be denoted by $T_S|X$. For simplicity of notations we shall denote the function spaces $C_{T_p|X}(M)$ and $C_{T_k|X}(M)$ by $C_p|X(M)$ and $C_k|X(M)$, respectively.

For a Tychonoff space $X$ and a linear functional $\mu \in C(X)'$, the support $\text{supp}(\mu)$ of $\mu$ is the set of all points $x \in X$ such that for every neighborhood $O_x \subseteq X$ of $x$ there exists a function $f \in C(X)$ such that $\mu(f) \neq 0$ and $\text{supp}(f) \subseteq O_x$, where $\text{supp}(f) = \{x \in X : f(x) \neq 0\}$. This definition implies that the support $\text{supp}(\mu)$ is a closed subset of $X$. Although the next lemma is proved in [4, Lemma 3.2], we add its short proof to make the paper self-contained.

**Lemma 3.2** Let $X$ be a Tychonoff space, and let $S$ be a directed set of functionally bounded sets in $X$. If a functional $\mu \in C(X)'$ is continuous in the topology $T_S$, then $\text{supp}(\mu) \subseteq \overline{S}$ for some set $S \in S$ such that $[S; 0] \subseteq \mu^{-1}(0)$.

**Proof** By the continuity of $\mu$ in the topology $T_S$, there exist a set $S \subseteq S$ and $\varepsilon > 0$ such that $\mu([S; \varepsilon]) \subseteq (-1, 1)$. Then

$$[S; 0] = \bigcap_{n \in \mathbb{N}} [S; \frac{\varepsilon}{n}] \subseteq \bigcap_{n \in \omega} \mu^{-1}((-\frac{1}{n}, \frac{1}{n})) = \mu^{-1}(0).$$

It remains to prove that $\text{supp}(\mu) \subseteq \overline{S}$. In the opposite case, we can find a function $f \in C(X)$ such that $\mu(f) \neq 0$ and $\text{supp}(f) \cap \overline{S} = \emptyset$. On the other hand, $f \in [S; 0] \subseteq \mu^{-1}(0)$ and hence $\mu(f) = 0$. This contradiction shows that $\text{supp}(\mu) \subseteq \overline{S}$. \hfill $\Box$

**Lemma 3.3** Let $X$ be a Tychonoff space, and let $D$ be a dense subset of $X$. If a linear functional $\mu \in C(X)'$ is continuous in the topology $T_{k|D}$, then $\text{supp}(\mu)$ is a compact subset of $D$ and $[\text{supp}(\mu); 0] \subseteq \mu^{-1}(0)$.

**Proof** By the continuity of $\mu$ in the topology $T_{k|D}$, there exist a compact subset $K \subseteq D$ and $\varepsilon > 0$ such that $\mu([K; \varepsilon]) \subseteq (-1, 1)$. By (the proof of) Lemma 3.2, $\text{supp}(\mu) \subseteq K$ and $[K; 0] \subseteq \mu^{-1}(0)$. Since $\text{supp}(\mu)$ is a closed subset of $X$, $\text{supp}(\mu)$ is closed in $K$ and hence $\text{supp}(\mu)$ is a compact subset of $D$.

It remains to prove that $[\text{supp}(\mu); 0] \subseteq \mu^{-1}(0)$. This can be done repeating the arguments of the proof of Lemma 3.3 in [4]. Nevertheless, we provide a detailed proof for the reader convenience. To derive a contradiction, assume that $[\text{supp}(\mu); 0] \nsubseteq \mu^{-1}(0)$ and hence there exists a continuous function $f \in C(X)$ such that $\mu(f) \neq 0$ but $f|_{\text{supp}(\mu)} = 0$. Multiplying $f$ by a suitable constant, we can assume that $\mu(f) = 2$. Embed the space $X$ into its Stone–Čech compactification $\beta X$. By the Tietze–Urysohn Theorem, there exists a continuous function $\tilde{f} \in C(\beta X)$ such that $\tilde{f}|_K = f|_K$. It follows from $[K; 0] \subseteq \mu^{-1}(0)$ that $\mu(f) = \mu(\tilde{f}|_X)$.

Consider the open neighborhood $U = \{x \in \beta X : |\tilde{f}(x)| < \varepsilon\}$ of $\text{supp}(\mu)$ in $\beta X$. By the definition of support $\text{supp}(\mu)$, every point $x \in K \setminus U$ has an open neighborhood $O_x \subseteq \beta X$ such that $\mu(g) = 0$ for any function $g \in C(X)$ with $\text{supp}(g) \subseteq O_x \cap X$. Observe that $U \cup \bigcup_{x \in K \setminus U} O_x$ is an open neighborhood of the compact set $K$ in $\beta X$. 

\vspace{0.5cm}

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So there is a finite family \( F \subseteq K \setminus U \) such that \( K \subseteq U \cup \bigcup_{x \in F} O_x \). Let \( 1_{\beta X} \) denote the constant function \( \beta X \to \{1\} \). By the paracompactness of the compact space \( \beta X \), there is a finite family \( \{\lambda_0, \ldots, \lambda_n\} \) of continuous functions \( \lambda_i : \beta X \to [0, 1] \) such that \( \sum_{i=0}^n \lambda_i = 1_{\beta X} \) and for every \( i \in \{0, \ldots, n\} \), the support \( \text{supp}(\lambda_i) \) is contained in some set \( V \in \{\beta X \setminus K, U\} \cup \{O_x : x \in F\} \). We lose no generality assuming that

\[
\bigcup_{i=0}^j \text{supp}(\lambda_i) \subseteq \beta X \setminus K, \quad \bigcup_{i=j+1}^s \text{supp}(\lambda_i) \subseteq U,
\]

and for every \( i \in \{s+1, \ldots, n\} \) there exists \( x_i \in F \) such that \( \text{supp}(\lambda_i) \subseteq O_{x_i} \).

Replacing the functions, \( \lambda_0, \ldots, \lambda_j \) by the single function \( \sum_{i=0}^j \lambda_i \) and the functions, \( \lambda_{j+1}, \ldots, \lambda_s \) by the single function \( \sum_{i=j+1}^s \lambda_i \), we can assume that \( j = 0 \) and \( s = 1 \). In this simplified case we have supp(\( \lambda_0 \)) \( \subseteq \beta X \setminus K \), supp(\( \lambda_1 \)) \( \subseteq U \) and supp(\( \lambda_i \)) \( \subseteq O_{x_i} \) for all \( i \in \{2, \ldots, n\} \).

For every \( i \in n \), consider the function \( f_i \in C(\beta X) \), defined by \( f_i(x):= \lambda_i(x) \cdot \tilde{f}(x) \) for \( x \in \beta X \). Then \( \tilde{f} = \sum_{i \in n} f_i \).

It follows from \( \text{supp}(f_0) \subseteq \text{supp}(\lambda_0) \subseteq \beta X \setminus K \) and \( [K; 0] \subseteq \mu^{-1}(0) \) that \( f_0\mid_K = 0 \) and \( \mu(f_i\mid_X) = 0 \).

Since \( K \subseteq U \), \( \tilde{f}(U) \subseteq (-\epsilon, \epsilon) \) and \( \text{supp}(f_1) \subseteq \text{supp}(\lambda_1) \subseteq U \), the function \( f_1\mid_X = (\tilde{f} \cdot \lambda_1)\mid_X \) belongs to the set \([K; \epsilon] \subseteq \mu^{-1}((-1, 1))\) and hence

\[
|\mu(f_1\mid_X)| \leq 1. \tag{3.1}
\]

For every \( i \in \{2, \ldots, n\} \), we have \( \text{supp}(f_i) \subseteq \text{supp}(\lambda_i) \subseteq O_{x_i} \) and hence \( \mu(f_i\mid_X) = 0 \) by the choice of \( O_{x_i} \).

Now we see that

\[
2 = \mu(f) = \mu(\tilde{f}\mid_X) = \mu(f_0\mid_X) + \mu(f_1\mid_X) + \mu\left(\sum_{i=2}^n f_i\mid_X\right) = \mu(f_1\mid_X),
\]

which contradicts (3.1).

The following example shows that Lemma 3.3 is not true for linear functionals which are continuous only in the topology \( T_b \).

**Example 3.4** Let \( \omega_1 \) be the space of countable ordinals, endowed with the order topology. It is well known that the space \( \omega_1 \) is pseudocompact and every continuous function \( f : \omega_1 \to \mathbb{F} \) is eventually constant and hence has \( \lim_{\omega \to \omega_1} f(\alpha) = \text{lim}_{\alpha \to \omega_1} f(\alpha) \). Then \( \lim_{\omega \to \omega_1} C(\omega_1) \to \mathbb{F}, \lim_{\omega \to \omega_1} f \mapsto \lim_{\alpha \to \omega_1} f(\alpha) \), is a well-defined continuous linear functional on the Banach space \( C_b(\omega_1) \). It is easy to show that \( \text{supp}(\lim_{\omega \to \omega_1}) = \emptyset \) and hence \( \text{supp}(\lim_{\omega \to \omega_1}; 0) = \emptyset \).

The next lemma follows from Lemma 3.4 from [4] because \( T_S \subseteq T_b \).

**Lemma 3.5** ([4, Lemma 3.4]) Let \( X \) be a Tychonoff space, and let \( S \) be a directed family of functionally bounded sets in \( X \). For any bounded subset \( M \subseteq (C_{T_S}(X))'_* \), the set \( \text{supp}(M) = \bigcup_{\mu \in M} \text{supp}(\mu) \) is functionally bounded in \( X \).
Lemma 3.6 Let $X$ be a dense subspace a Tychonoff space $K$, and let $Z$ be a finite-dimensional linear subspace of $C(K)$. Then there exists a finite set $F \subseteq X$ such that the restriction operator $R : C(K) \to \mathbb{F}^F$, $R : f \mapsto (f(x))_{x \in F}$, is injective on the set $Z$.

Proof The density of $X$ in $K$ implies that the restriction operator $R : C(K) \to \mathbb{F}^X$ is injective. Let $e_1, \ldots, e_n$ be a basis of the finite-dimensional linear space $Z$. Then the vectors $e_1, \ldots, e_n$ are linearly independent in $C(K)$ and, by the injectivity of the operator $R$, the vectors $R(e_1), \ldots, R(e_n)$ are linearly independent in $\mathbb{F}^X$. Consider the compact set $S = \{(x_1, \ldots, x_n) \in \mathbb{F}^n : \sum_{i=1}^{n} |x_i| = 1\}$ in $\mathbb{F}^n$ and the continuous map

$$\Lambda_X : S \times (\mathbb{F}^X)^n \to \mathbb{F}^X, \quad \Lambda_X : ((x_i)_{i=1}^{n}, (f_i)_{i=1}^{n}) \mapsto \sum_{i=1}^{n} x_i f_i.$$ 

The linear independence of the vectors $R(e_1), \ldots, R(e_n)$ implies that $0_X \notin \Lambda_X \left( S \times \{(R(e_i))_{i=1}^{n}\} \right)$, where $0_X : X \to \{0\} \subseteq \mathbb{F}$ is the zero of the linear space $\mathbb{F}^X$. By the compactness of $S$, there exists a neighborhood $U$ of zero in $\mathbb{F}^X$ such that $U \cap \Lambda \left( S \times \prod_{i=1}^{n} (R(e_i) + U) \right) = \emptyset$. By the definition of the Tychonoff product topology on $\mathbb{F}^X$, there exists a finite set $F \subseteq X$ such that $\operatorname{pr}_F^{-1}(0_F) \subseteq U$, where $\operatorname{pr}_F : \mathbb{F}^X \to \mathbb{F}^F$, $\operatorname{pr}_F : f \mapsto f|_F$, is the projection map and $0_F$ is the zero of the linear space $\mathbb{F}^F$. The choice of $F$ and $U$ ensure that the vectors $e_1|_F, \ldots, e_n|_F$ are linearly independent in $\mathbb{F}^F$ and hence the restriction operator $R : C(K) \to \mathbb{F}^F$ is injective on the set $Z$. \qed

A subset $X$ of a topological space $M$ is called

- $k$-dense in $M$ if any compact set $K \subseteq M$ is contained in a compact set $S \subseteq M$ such that $S \cap X$ is dense in $S$;
- sequentially closed in $M$ if for any sequence $(x_n)_{n \in \omega} \subseteq X$ that converges in $M$ we have $\lim_{n \to \infty} x_n \in X$.

The following theorem is the main result of this section.

Theorem 3.7 For a $\mu$-space $M$ and a dense subset $X$ in $M$, the following assertions are equivalent:

(i) for every locally convex topology $T$ on $C(M)$ satisfying $T_{p|X} \subseteq T \subseteq T_k|X$, the space $C_T(M)$ is strongly Gelfand–Phillips;
(ii) there is a locally convex topology $T$ on $C(M)$ such that $T_{p|X} \subseteq T \subseteq T_k|X$ and the space $C_T(M)$ is strongly Gelfand–Phillips;
(iii) $X$ contains a compact countable subspace $K$ of finite scattered height such that for every compact set $F \subseteq M$ the complement $F \cap X \setminus K$ is finite.

If $X$ is $k$-dense and sequentially closed in $M$, then the conditions (i)–(iii) are equivalent to

(iv) for every locally convex topology $T$ on $C(M)$ satisfying $T_{p|X} \subseteq T \subseteq T_k$, the space $C_T(M)$ is strongly Gelfand–Phillips;
(v) there is a locally convex topology $T$ on $C(M)$ such that $T_p|X \subseteq T \subseteq T_k$ and the space $C_T(M)$ is strongly Gelfand–Phillips;

(vi) $X = M$ and $X$ contains a compact countable subspace $K$ of finite scattered height such that for every compact set $F \subseteq X$ the complement $F \setminus K$ is finite.

If $X$ is $k$-dense in $M$, then the condition (v) implies the condition

(vii) $M$ contains a compact countable subspace $K$ of finite scattered height such that $M = X \cup K$ and for every compact set $F \subseteq M$ the complement $F \setminus K$ is finite.

If (iii) holds, then a sequence $\{\mu_n\}_{n \in \omega} \subseteq C_T(X)'$ witnessing the strong Gelfand–Phillips property of $C_T(X)$ in (i) and (iv) can be chosen such that all $\mu_n$ have finite support supp$(\mu_n)$ contained in $K$.

**Proof** The implications (i)$\Rightarrow$(ii), (iv)$\Rightarrow$(v), and (vi)$\Rightarrow$(iii) are trivial.

(ii)$\Rightarrow$(iii) and (v)$\Rightarrow$(vi,vii). Assume that for some locally convex topology $T$ on $C(M)$ with $T_p|X \subseteq T \subseteq T_k$, the locally convex space $C_T(X)$ is strongly Gelfand–Phillips. We also assume that either $X$ is $k$-dense in $M$ or $T \subseteq T_k|X$.

By the strong Gelfand–Phillips property of $C_T(M)$, there exists a null sequence $(\mu_n)_{n \in \omega}$ in $(C_T(M))'_{w^*}$ such that $\|\mu_n\|_B \neq 0$ for any barrel-bounded set $B \subseteq C_T(M)$, which is not barrel-precompact. The continuity of the identity operator $C_k(M) \to C_T(M)$ implies that the sequence $(\mu_n)_{n \in \omega}$ is null also in $C_k(M)'_{w^*,*}$.

By Lemma 3.5, the set $S := \bigcup_{n \in \omega} \text{supp}(\mu_n)$ is functionally bounded and hence has compact closure $\bar{S}$ in the $\mu$-space $M$.

**Claim 3.8** For every compact set $F \subseteq M$, the complement $F \cap X \setminus \bar{S}$ is finite.

**Proof** Replacing $F$ by the closure of $F \cap X$ in $M$, we can assume that $X \cap F$ is dense in $F$. Consider the bounded subset $B = \{f \in C(M) : f|_{\bar{S}} = 0 \text{ and } \|f\|_M \leq 1\}$ of the locally convex space $C_k(M)$. Since $M$ is a $\mu$-space, the locally convex space $C_k(M)$ is barrelled by the Nachbin–Sirota Theorem [18, 11.7.5]. Then $B$ is barrel-bounded in $C_k(M)$ and, by the continuity of the identity operator $C_k(M) \to C_T(M)$, the set $B$ is barrel-bounded in $C_T(M)$. Applying Lemma 3.3 and taking into account that every function $f \in B$ is zero on the compact set $\bar{S} \supseteq \bigcup_{n \in \omega} \text{supp}(\mu_n)$, we conclude that $\|\mu_n\|_B = 0$ for every $n \in \omega$. Now the choice of the sequence $(\mu_n)_{n \in \omega}$ ensures that the set $B$ is barrel-precompact in $C_T(M)$. Observe that the set $[F; 1] = [F \cap X; 1]$ is a barrel in $C_p|X(M)$ and hence a barrel in $C_T(M)$. Since the set $B$ is barrel-precompact, for every $\epsilon > 0$ there exists a finite set $A_\epsilon \subseteq B$ such that $B \subseteq A_\epsilon + \epsilon \cdot [F; 1]$. Then for the restriction operator $T : C(M) \to C(F)$, $T : f \mapsto f|_F$, we have $T(B) \subseteq T(A_\epsilon) + \epsilon \cdot T([F; 1])$. The Tietze–Urysohn Extension Theorem implies that $T([F; 1])$ coincides with the closed unit ball of the Banach space $C(F)$ and $T(B)$ coincides with the closed unit ball of the Banach subspace $L = \{f \in C(F) : f|_{F \cap \bar{S}} = 0\}$ of $C(F)$. Now we see that the closed unit ball $T(B)$ of the Banach space $L$ is precompact in the Banach space $C(F)$ and hence the Banach space $L$ is finite-dimensional, which implies that the set $F \setminus \bar{S}$ is finite and so is the set $F \cap X \setminus \bar{S} \subseteq F \setminus \bar{S}$.

**Claim 3.9** If $X$ is $k$-dense in $M$, then $M = X \cup \bar{S}$.
Proof By the $k$-density of $X$ in $M$, every point $y \in M$ is contained in a compact set $F \subseteq M$ such that the set $F \cap X$ is dense in $F$. By Claim 3.8, the set $F \cap X \setminus \bar{S}$ is finite and hence the set $(F \cap X) \cup \bar{S}$ is compact. Since the latter set is dense in $F \cup \bar{S}$, we obtain that $y \in F \cup \bar{S} = (F \cap X) \cup \bar{S} \subseteq X \cup \bar{S}$.

If $T \subseteq T_{k\cap X}$, then Lemma 3.3 implies that $S = \bigcup_{n \in \omega} \text{supp}(\mu_n) \subseteq X$ and hence $\bar{S} \cap X \subseteq S$ is dense in $\bar{S}$. In this case we put $K = \bar{S}$. If $T \nsubseteq T_{k\cap X}$, then $X$ is $k$-dense in $M$. In this case, we can find a compact set $K \subseteq M$ such that $\bar{S} \subseteq K$ and $K \cap X$ is dense in $K$. In both cases we have a compact set $K \subseteq M$ such that $S \subseteq K$ and $K \cap X$ is dense in $K$. Since $\bar{S} \subseteq K$, Claim 3.8 ensures that $F \cap X \setminus K$ is finite for any compact set $F \subseteq M$. Moreover, if $X$ is $k$-dense in $M$, then $M = X \cup K$ by Claim 3.9.

Let us prove that $K$ is countable and has finite scattered height. We shall derive this from Theorem 1.4 by showing that the Banach space $C(K) = C_k(K)$ is isomorphic to a subspace of the Banach space $c_0$.

Let $R : C_k(M) \to C(K)$, $R : f \mapsto f|_K$, be the restriction operator. The Tietze–Urysohn Extension Theorem implies that this operator is surjective. For every $n \in \omega$, the inclusion $\text{supp}(\mu_n) \subseteq K$ and Lemma 3.3 imply the existence of a linear continuous functional $\lambda_n \in C(K)'$ such that $\mu_n = \lambda_n \circ R$. The surjectivity of the operator $R$ ensures that the sequence $(\lambda_n)_{n \in \omega}$ is null in $C(K)_{\omega^*}$. Observe that the operator

$$\Lambda : C(K) \to c_0, \quad \Lambda(f) := (\lambda_n(f))_{n \in \omega},$$

which is continuous by the Uniform Boundedness Principle.

Claim 3.10 The kernel of the operator $\Lambda : C(K) \to c_0$ is finite-dimensional.

Proof Let $Z = \{f \in C(K) : \Lambda(f) = 0\}$ be the kernel of $\Lambda$. Since the locally convex space $C_k(M)$ is barrelled, the bounded subset $P = \{f \in C_k(M) : \|f\|_X \leq 1$ and $f|_K \in Z\}$ of $C_k(M)$ is barrel-bounded in $C_k(M)$ and hence barrel-bounded in $C_T(M)$ (as $T \subseteq T_k$). Observe that for every $f \in P$, we have $\mu_n(f) = \lambda_n(f|_K) = 0$ for all $n \in \omega$. Consequently, $\|\mu_n\|_P = 0$ for all $n \in \omega$. Now the choice of the sequence $(\mu_n)_{n \in \omega}$ ensures that the barrel-bounded set $P$ is barrel-precompact in $C_T(M)$. Observe that the set $[K; 1] = [K \cap X; 1]$ is a barrel in $C_p(M)$ and hence in $C_T(M)$. Since $P$ is barrel-precompact in $C_T(M)$, for every $\varepsilon > 0$ there exists a finite set $A_\varepsilon \subseteq P$ such that $P \subseteq A_\varepsilon + \varepsilon \cdot [K; 1]$ and hence $R(P) \subseteq R(A_\varepsilon) + \varepsilon \cdot R([K; 1])$. By the Tietze-Urysohn Extension Theorem, the set $R([K; 1])$ coincides with the closed unit ball of the Banach space $C(K)$ and the set $R(P)$ coincides with the closed unit ball of the Banach subspace $Z$ of $C(K)$. Now we see that the closed unit ball $R(P)$ of the Banach space $Z$ is precompact in the Banach space $C(K)$, which implies that the space $Z$ is finite-dimensional.

Applying Lemma 3.6, we can append to the sequence $(\mu_n)_{n \in \omega}$ finitely many Dirac functionals supported at points of the set $K \cap X$, which separate points of the finite-dimensional space $Z$, and assume that the operator $\Lambda : C(K) \to c_0$ is injective.

Claim 3.11 $\Lambda$ is an isomorphic embedding.
Then we can choose a sequence of functions \( F = \{ f_n \}_{n \in \omega} \subseteq C(K) \) such that \( \| f_n \|_K = 1 \) for all \( n \in \omega \) and \( \Lambda(f_n) \to 0 \) in the Banach space \( c_0 \). Let \( (e'_n)_{n \in \omega} \) be the sequence of coordinate functionals in \( c_0 \). Since the set \( F_0 = \{ 0 \} \cup \Lambda(F) \) is compact in \( c_0 \), Proposition 2.2 implies \( \| e'_n \|_{F_0} \to 0 \).

By the Tietze–Urysohn Theorem, for every \( n \in \omega \), there exists a function \( \hat{f}_n \in C(M) \) such that \( \hat{f}_n |_K = f_n \) and \( \| \hat{f}_n \|_M = 1 \). It follows that the set \( \hat{F} = \{ \hat{f}_n : n \in \omega \} \) is bounded in \( C_k(M) \). Since \( M \) is a \( \mu \)-space, the locally convex space \( C_k(M) \) is barreled. Then the bounded set \( \hat{F} \) is barrel-bounded in \( C_k(M) \) and also in \( C_T(M) \). For every \( n, m \in \omega \), we have \( \mu_n(\hat{f}_m) = \lambda_n(f_m) \) and hence

\[
\| \mu_n \|_{\hat{F}} = \sup_{m \in \omega} |\mu_n(\hat{f}_m)| = \sup_{m \in \omega} |\lambda_n(f_m)| = \| e'_n \|_{F_0} \to 0.
\]

The choice of the sequence \( (\mu_n)_{n \in \omega} \) implies that the barrel-bounded set \( \hat{F} \) is barrel-precompact in \( C_T(M) \). Observe that the set \( [K; 1] = [K \cap X; 1] \) is a barrel in \( C_p|X \cap K(M) \) and hence in \( C_T(M) \). For every \( \varepsilon > 0 \), by the barrel-precompactness of \( \hat{F} \), there exists a finite set \( A_\varepsilon \subseteq \hat{F} \) such that \( \hat{F} \subseteq A_\varepsilon + \varepsilon \cdot [K; 1] \) and hence \( F = R(\hat{F}) \subseteq R(A_\varepsilon) + \varepsilon \cdot R([K; 1]) \). The Tietze–Urysohn Theorem ensures that \( R([K; 1]) \) coincides with the closed unit ball of the Banach space \( C(K) \). Now we see that the set \( F = R(\hat{F}) \) is precompact in the Banach space \( C(K) \) and hence the sequence \( (f_n)_{n \in \omega} \) contains a subsequence \( (f_{n_k})_{k \in \omega} \) that converges to some function \( f_\infty \in C(K) \) with \( \| f_\infty \|_K = 1 \). On the other hand, \( \Lambda(f_\infty) = \lim_{n \to \infty} \Lambda(f_{n_k}) = 0 \), which contradicts the injectivity of \( \Lambda \). This contradiction shows that the Banach space \( C(K) \) is isomorphic to a subspace of \( c_0 \).

By Theorem 1.4 and Claim 3.11, the compact space \( K \) is countable and has finite scattered height. This completes the proof of the implication \( (v) \Rightarrow (vii) \).

The following claim completes the proof of the implication \( (ii) \Rightarrow (iii) \).

**Claim 3.12** If \( T \subseteq T_k|X \), then the function space \( E = C_k|X \cap K(K) \) is strongly Gelfand–Phillips and \( K \subseteq X \).

**Proof** The continuity of the functionals \( \mu_n \) in the topology \( T \subseteq T_k|X \) implies the continuity of the functionals \( \lambda_n \) in the topology \( T_k|X \cap K \) on the function space \( C(K) \). Then \( (\lambda_n)_{n \in \omega} \) is a null sequence in the dual \( E'_u \). We claim that \( \| \lambda_n \|_B \not\to 0 \) for any barrel-bounded set \( B \subseteq E \), which is not barrel-precompact. The continuity of the identity operator \( C_k|X \cap K(K) \to C_p|X \cap K(K) \) implies that \( B \) is barrel-bounded in the space \( C_p|X \cap K(K) \). The density of \( K \subseteq X \cap K \) implies that the barrel-bounded set \( B \) is bounded in the Banach space \( C_k(K) \). Since \( B \) is not barrel-precompact in \( E \), there exists a barrel \( D \) in \( E \) such that \( B \nsubseteq A + D \) for every finite set \( A \subseteq B \). Then we can choose a sequence of functions \( \{ f_n \}_{n \in \omega} \subseteq B \) such that \( f_n \not\in f_k + D \) for every numbers \( k < n \). By the Tietze–Urysohn Theorem, for every \( n \in \omega \) there exists a function \( \hat{f}_n \in C(M) \) such that \( \hat{f}_n |_K = f_n \) and \( \| \hat{f}_n \|_M = \| f_n \|_K \). Since \( \sup_{n \in \omega} \| \hat{f}_n \|_M = \sup_{n \in \omega} \| f_n \|_K < \infty \), the set \( \{ f_n \}_{n \in \omega} \) is barrel-bounded in \( C_k(M) \) and hence is barrel-bounded in \( C_T(M) \).

Since \( D \) is a barrel in the Banach space \( C_k(K) \), there exists \( \varepsilon > 0 \) such that \( [K; \varepsilon] \subseteq D \). It follows from \( f_n \not\in f_k + [K; \varepsilon] \) that \( \hat{f}_n \not\in \hat{f}_k + [K; \varepsilon] \) for every \( k < n \).
Since the set $K \cap X$ is dense in $K$, the set $[K]; 1]_{C(M)} = [K \cap X; 1]_{C(M)}$ is a barrel in $C_p[X](M)$ and hence a barrel in $C_T(M)$. Now we see that the set $\hat{B} = \{ \hat{f}_n : n \in \omega \}$ is barrel-bounded and not barrel-precompact in $C_T(M)$. The choice of the sequence $(\mu_n)_{n \in \omega}$ ensures that $\|\mu_n\|_{\hat{B}} \neq 0$. By Lemma 3.3, the inclusion $\text{supp}(\mu_n) \subseteq K$ and equality $\hat{f}_m|K = f_m$ imply that $\lambda_n(f_m) = \mu_n(\hat{f}_m)$ for any $n, m \in \omega$. Then $\|\lambda_n\|_{\hat{B}} \geq \|\mu_n\|_{\hat{B}}$ and hence $\|\lambda_n\| \neq 0$, witnessing that the space $E$ is strongly Gelfand–Phillips.

Since $K \cap X$ is dense in $K$, the span $L$ of $K \cap X$ in $E'$ is norming, and hence, by Lemma 2.4, the space $E$ is $\beta$-Banach with $E_\beta = C(\hat{K})$. By Theorem 2.6, the dual space $E'$ is dense in the dual Banach space $C_k(K)'$, which is isometric to the Banach space $\ell_1(K)$ because $K$ is countable. On the other hand, by Lemma 3.3, each functional $\mu \in E'$ has compact support in the set $X \cap K$, which implies that $E' \subseteq \ell_1(K \cap X)$. The density of $E'$ in $C_k(K)'$ ensures that $K \cap X = K$ and hence $K \subseteq X$. □

If the space $X$ is sequentially closed in $M$, then the dense subset $K \cap X$ of the metrizable compact space $K$ is closed in $K$ and hence $K = \overline{K \cap X} = K \cap X \subseteq X$. This completes the proof of the implication (v)⇒(vi).

(iii)⇒(i, iv) Assume that $X$ contains a compact countable subspace $K$ of finite scattered height such that for every compact set $S \subseteq X$ the complement $S \cap X \setminus K$ is finite. Let $T$ be any locally convex topology on $C(M)$ such that $T|_S \subseteq T \subseteq T_k$. If $X$ is not $\kappa$-dense in $M$, then we shall also assume that $T \subseteq T_k|_X$.

By Theorem 1.4, the Banach space $C(K)$ is isomorphic to $c_0$. Observe that $C_p(K)'$ consists of finitely supported functionals and is dense in the dual Banach space $C(K)'$, which is isometric to $\ell_1(K)$. By Lemma 2.4 and Theorem 2.6, the function space $C_p(K)$ is strongly Gelfand–Phillips. Consequently, there exists a null sequence $(\mu_n)_{n \in \omega} \in C_p(K)'_{w^*}$ such that $\|\mu_n\|_{B} \neq 0$ for any barrel-bounded subset $B \subseteq C_p(K)$ which is not barrel-precompact. The measures $\mu_n$ have finite support in $K \subseteq X \subseteq M$, and hence can be identified with finitely supported linear functionals on $C_p[X](M)$. The continuity of the restriction map $C_p[X](M) \rightarrow C_p(K)$ implies that $(\mu_n)_{n \in \omega}$ is a null sequence in $C_p[X](M)'_{w^*}$.

Since the identity operator $C_T(M) \rightarrow C_p[X](M)$ is continuous, the sequence $(\mu_n)_{n \in \omega}$ remains null in $C_T(M)'_{w^*}$. We claim that $\|\mu_n\|_{B} \neq 0$ for any barrel-bounded subset $B \subseteq C_T(M)$, which is not barrel-precompact. To derive a contradiction, assume that $\|\mu_n\|_{B} \rightarrow 0$. Since $B$ is not barrel-precompact, there exists a barrel $D \subseteq C_T(M)$ such that $B \subseteq A + D$ for any finite set $A \subseteq B$. In this case we can inductively construct a sequence of functions $\{f_n\}_{n \in \omega} \subseteq B$ such that $f_n \notin f_k + D$ for any numbers $k < n$.

By the continuity of the identity operator $C_k(M) \rightarrow C_T(M)$, the set $D$ is also a barrel in $C_k(M)$. Since $M$ is a $\mu$-space, the space $C_k(M)$ is barrelled according to the Nachbin–Shirota Theorem [18, 11.7.5]. Then $D$ is a neighborhood of zero in $C_k(M)$ and hence $[S; \varepsilon] \subseteq D$ for some compact set $S \subseteq M$ and some $\varepsilon > 0$. Replacing $S$ by $S \cup K$, we can assume that $K \subseteq S$.

**Claim 3.13** $[S \cap X; \varepsilon] \subseteq D$.

**Proof** If $X$ is $\kappa$-dense in $M$, then $S$ is contained in a compact set $\hat{S} \subseteq M$ such that $\hat{S} \cap X$ is dense in $\hat{S}$. On the other hand, the set $\hat{S} \cap X \setminus K$ is finite and hence the set $\hat{S} \cap X$ is compact and $S \subseteq \hat{S} = \hat{S} \cap X \subseteq \hat{S} \cap X \subseteq X$. Then $[S \cap X; \varepsilon] = [S; \varepsilon] \subseteq D$.  

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So, assume that $X$ is not $k$-dense in $M$. In this case $T \subseteq T_{k\mid X}$. Suppose for a contradiction that $[S \cap X; \varepsilon] \setminus D$ contains some function $f$. Since the barrel $D$ is closed in the topology $T \subseteq T_{k\mid X}$, there exist $\delta > 0$ and a compact set $C \subseteq X$ such that $(f + [C; \delta]) \cap D = \emptyset$. If follows from $f \in [S \cap X; \varepsilon]$ that $f(C \cap S) \subseteq f(X \cap S) \subseteq \varepsilon B$, where $B$ is the closed unit ball of the field $F$. By the Tietze–Urysohn Theorem, there exists a continuous function $g : M \to \varepsilon B$ such that $g(x) = f(x)$ for every $x \in C \cap S$. Define the function $\varphi : C \cup S \to F$ by the formula

$$
\varphi(x) = \begin{cases} 
  f(x) & \text{if } x \in C; \\
  g(x) & \text{if } x \in S;
\end{cases}
$$

and observe that it is well defined and continuous. By the Tietze–Urysohn Theorem, the function $\varphi$ can be extended to a bounded continuous function $\psi : M \to F$. Then $\psi \in (f + [C; \delta]) \cap [S; \varepsilon] \subseteq (f + [C; \delta]) \cap D$, which contradicts the choice of $C$ and $\delta$. This contradiction shows that $[S \cap X; \varepsilon] \subseteq D$. □

Since $[S \cap X; \varepsilon] \subseteq D$, we can replace the set $S$ by $S \cap X$ and assume that $S \cap X$ is dense in $S$. Since the identity operator $C_T(M) \to C_{p\mid X}(M)$ is continuous, the set $B$ is barrel-bounded in $C_{p\mid X}(M)$. Consider the restriction operator $R : C_{p\mid X}(M) \to C_p(K)$ and observe that $R(B)$ is barrel-bounded in $C_p(K)$. By the barrelledness of the Banach space $C(K)$, the barrel-bounded subset $R(B)$ of $C_p(K)$ is bounded in $C(K)$. Since $\bigcup_{n \in \omega} \text{supp}(\mu_n) \subseteq K$, we obtain $\|\mu_n\|_{R(B)} = \|\mu_n\|_B = 0$. The choice of the sequence $\{\mu_n\}_{n \in \omega}$ ensures that the set $R(B)$ is barrel-precompact and hence $R(B) \subseteq R(P) + [K; \varepsilon] \subseteq C(K)$ for some finite set $P \subseteq B$. It follows that $B \subseteq P + [K; \varepsilon] \subseteq C(M)$, and hence for every $n \in \omega$ there exists a function $g_n \in P$ such that $f_n - g_n \in [K; \varepsilon]$. Since the set $P$ is finite, for some $g \in P$ the set $\Omega = \{n \in \omega : g_n = g\}$ is infinite. For any two numbers $n < m$ in $\Omega$, we have

$$
f_m - f_n = (f_m - g) - (f_n - g) \in [K; \varepsilon] + [K; \varepsilon] \subseteq [K; \varepsilon].
$$

It follows from $f_m - f_n \notin [S; \varepsilon]$ and $K \subseteq S$ that $f_m - f_n \notin [S \setminus K; \varepsilon] = [S \cap X \setminus K; \varepsilon]$. This means that the bounded subset $\{f_n\}_{n \in \Omega}$ of the finite-dimensional space $C(S \cap X \setminus K)$ is not precompact, which is not possible. This contradiction shows that $\|\mu_n\|_B \not\to 0$. Therefore, the sequence $(\mu_n)$ witnesses that the space $C_T(M)$ is strongly Gelfand–Phillips. This completes the proof of the implications (iii) $\Rightarrow$ (i,iv). □

**Example 3.14** The compact space $M = \omega + 1 = \omega \cup \{\omega\}$ endowed with the order topology and its $k$-dense subspace $X = \omega$ satisfy the condition (vii) of Theorem 3.7 but not the conditions (iii) and (vi). This example shows that (vii) is not equivalent to any of the conditions (i)–(vi). Theorem 3.7 implies that the locally convex space $C_{p\mid \omega}((\omega + 1)$ is not strongly Gelfand–Phillips in spite of the fact that the Banach space $C(\omega + 1) = (C_{p\mid \omega}((\omega + 1))_p$ is isomorphic to $c_0$. □

Theorem 3.7 implies the following results.

**Theorem 3.15** For a Tychonoff space $X$ the following assertions are equivalent:

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(i) for every locally convex topology $T$ on $C(X)$ satisfying $T_p \subseteq T \subseteq T_k$, the space $C_T(X)$ is strongly Gelfand–Phillips;
(ii) there is a locally convex vector topology $T$ on $C(X)$ such that $T_p \subseteq T \subseteq T_k$ and the space $C_T(X)$ is strongly Gelfand–Phillips;
(iii) $X$ is a $\mu$-space containing a compact countable subspace $K \subseteq X$ of finite scattered height such that for every compact set $S \subseteq X$ the complement $S \setminus K$ is finite.

If (i)–(iii) hold, a sequence $(\mu_n)_{n \in \omega} \subseteq C_T(X)'$ witnessing the strong Gelfand–Phillips property of $C_T(X)$ can be chosen such that all $\mu_n$ have finite support contained in $K$.

**Proof** The implication (i)$\Rightarrow$(ii) is trivial.

To prove that (ii)$\Rightarrow$(iii), assume that $T$ is a locally convex topology on $C(X)$ such that $T_p \subseteq T \subseteq T_k$ and the lcs $C_T(X)$ is strongly Gelfand–Phillips. By the classical result of Hewitt [15, 3.11.16], $X$ is a dense subspace of some $\mu$-space $M$ such that the restriction operator $C(M) \to C(X), f \mapsto f \upharpoonright_X$, is bijective. If $X$ is a $\mu$-space, then we can (and will) assume that $M = X$. Let $\tau$ be a unique locally convex topology on the function space $C(M)$ such that the restriction operator $C_\tau(M) \to C_\tau(X)$ is a topological isomorphism. Then the spaces $C_p|_X(M), C_\tau(M)$ and $C_k|_X(M)$ are topologically isomorphic to the spaces $C_p(X), C_\tau(X)$ and $C_k(X)$, respectively. By Theorem 3.7, $X$ contains a compact countable set $K$ of finite scattered height such that for any compact subset $F \subseteq M$ the set $F \cap X \setminus K$ is finite. In particular, for any compact subset $S \subseteq X$, the complement $S \setminus K = S \cap X \setminus K$ is finite. It remains to prove that $X$ is a $\mu$-space. As the restriction operator $C(M) \to C(X)$ is bijective, every functionally bounded set $F$ in $X$ is functionally bounded in $M$. Since $M$ is a $\mu$-space, $F$ has compact closure $\bar{F}$ in $M$. Since the set $\bar{F} \cap X \setminus K$ is finite, the set $(\bar{F} \cap X) \cup K \subseteq X$ is compact. Now we see that the functionally bounded set $F$ is contained in the compact subset $(\bar{F} \cap X) \cup K$ of $X$, which means that $X$ is a $\mu$-space.

The implication (iii)$\Rightarrow$(i) follows from Theorem 3.7 applied to the $\mu$-space $M = X$ and its subset $X$.

By Lemma 3.1, if $X$ is a $\mu$-space, then $T_k = T_b$. This fact and Theorem 3.15 immediately imply the next result.

**Corollary 3.16** For a $\mu$-space $X$, the following assertions are equivalent:

(i) for every locally convex topology $T$ on $C(X)$ satisfying $T_p \subseteq T \subseteq T_b$, the space $C_T(X)$ is strongly Gelfand–Phillips;
(ii) there is a locally convex vector topology $T$ on $C(X)$ such that $T_p \subseteq T \subseteq T_b$ and the space $C_T(X)$ is strongly Gelfand–Phillips;
(iii) $X$ contains a compact countable subspace $K \subseteq X$ of finite scattered height such that for every compact set $S \subseteq X$ the complement $S \setminus K$ is finite.

If (i)–(iii) hold, a sequence $(\mu_n)_{n \in \omega} \subseteq C_T(X)'$ witnessing the strong Gelfand–Phillips property of $C_T(X)$ can be chosen such that all $\mu_n$ have finite support contained in $K$.

Since the weak topology of $C_T(X)$ is stronger than the pointwise topology, Corollary 3.16 implies

**Corollary 3.17** Let $X$ be a $\mu$-space, and let $\mathcal{S}$ be a directed family of functionally bounded subsets of $X$ containing $\mathcal{F}(X)$. Then the following assertions are equivalent:
(i) the space $C_{T_S}(X)$ is strongly Gelfand–Phillips;
(ii) the space $C_{T_S}(X)_w$ is strongly Gelfand–Phillips;
(iii) $X$ contains a compact countable subspace $K \subseteq X$ of finite scattered height such that for every compact set $S \subseteq X$ the complement $S \setminus K$ is finite.

A topological space $X$ is defined to be bounded-finite if every functionally bounded set in $X$ is finite. It is clear that each bounded-finite space is a $\mu$-space. Corollary 3.16 implies the following helpful assertion.

**Corollary 3.18** For any bounded-finite Tychonoff space $X$, the function space $C_p(X) = C_k(X) = C_b(X)$ is strongly Gelfand–Phillips.

In the next corollary of Theorem 3.7, we characterize Tychonoff spaces $X$ for which the space of bounded functions $C^b(X)$ endowed with a locally convex vector topology $T$ between $T_p$ and $T_n$ has the strong Gelfand–Phillips property, where $T_n$ denotes the topology on $C^b(X)$ generated by the norm $\| \cdot \|_X$.

**Theorem 3.19** For any Tychonoff space $X$, the following assertions are equivalent:

(i) for every locally convex topology $T$ on $C^b(X)$ satisfying $T_p \subseteq T \subseteq T_n$, the space $C^b_T(X)$ is strongly Gelfand–Phillips;
(ii) there is a locally convex topology $T$ on $C^b(X)$ such that $T_p \subseteq T \subseteq T_n$ and the space $C^b_T(X)$ is strongly Gelfand–Phillips property;
(iii) $X$ is a compact countable space of finite scattered height.

**Proof** The implication (i)⇒(ii) is trivial.

(ii)⇒(iii) Let $T$ be a locally convex topology on $C^b(X)$ such that $T_p \subseteq T \subseteq T_n$ and the space $C^b_T(X)$ is strongly Gelfand–Phillips. Let $M = \beta X$ be the Stone-Čech compactification of $X$. It is clear that the restriction map $R : C(M) \to C^b(X)$, $R : f \mapsto f|_X$, is bijective. Let $\tau$ be the unique locally convex topology on $C(M)$ such that the restriction operator $R : C_\tau(M) \to C^b_T(X)$ is a topological isomorphism. The inclusions $T_p \subseteq T \subseteq T_n$ imply that $T_p|_X \subseteq \tau \subseteq T_k$. Since the space $M = \beta X$ is compact, the dense subspace $X$ of $M$ is $k$-dense in $M$. Then the implication (v)⇒(vii) in Theorem 3.7 yields the existence of a compact countable subspace $K \subseteq M$ of finite scattered height such that $M = X \cup K$ and for every compact set $S \subseteq M$ the complement $S \setminus K \subseteq X$ is finite. In particular, for $S = M$ we obtain that the set $M \setminus K = X \setminus K$ is finite and hence the compact space $M = (M \setminus K) \cup K$ is countable and metrizable. We claim that $X$ is closed in $M$. In the opposite case, we can find a point $x_\infty \in M \setminus X$ and a one-to-one sequence $(x_n)_{n \in \omega} \subseteq X$ converging to $x_\infty$. Therefore, the set $D = \{x_n\}_{n \in \omega}$ is closed and discrete in $X$. Since $X$ is metrizable, it is normal and hence, by Corollary 3.6.8 of [15], the closure $\overline{D}$ of $D$ in $M = \beta X$ is homeomorphic to $\beta D = \beta \omega$ which is impossible since $\beta \omega$ is not metrizable. This contradiction shows that $X$ is closed in $M$ and hence $M = X$ by the density of $X$ in $M$. Since $M \setminus K$ is finite, the space $M = X$ is compact, countable and has finite scattered height.

(iii)⇒(i) If $X$ is a countable compact space of finite scattered height, then $X$ is a $\mu$-space, $C^b_T(X) = C_T(X)$ and the assertion follows from Theorem 3.7. $\square$

As an immediate corollary of Theorem 3.19, we obtain the following complement of Theorem 1.4.
**Corollary 3.20** For a pseudocompact space $X$, the following assertions are equivalent:

(i) for every locally convex topology $T$ on $C(X)$ satisfying $T_p \subseteq T \subseteq T_b = T_n$, the space $C_T(X)$ has the strong Gelfand–Phillips property;

(ii) there is a locally convex topology $T$ on $C(X)$ such that $T_p \subseteq T \subseteq T_b = T_n$ and the space $C_T(X)$ has the strong Gelfand–Phillips property;

(iii) $X$ is a countable compact space of finite scattered height.

Since the weak topology of $C_T(X)$ is finer than $T_p$, Corollary 3.20 implies

**Corollary 3.21** Let $X$ be a pseudocompact space, and let $S$ be a directed family of functionally bounded subsets of $X$ containing $F(X)$. Then the following assertions are equivalent:

(i) the space $C_{TS}(X)$ is strongly Gelfand–Phillips;

(ii) the space $C_{TS}(X)_w$ is strongly Gelfand–Phillips;

(iii) $X$ is a countable compact space of finite scattered height.

\[ \square \]

4 Examples

In this section, we show that the class of all locally convex spaces with the strong Gelfand–Phillips property behaves badly with respect to standard operations as taking completions, infinite products or (closed) subspaces. We also construct a Banach space that cannot be embedded into a strongly Gelfand–Phillips locally convex space.

We start from the operation of taking infinite power of a locally convex space. By Proposition 2.14 the finite product of strongly Gelfand–Phillips spaces is strongly Gelfand–Phillips. But for infinite products, this is not true in general as the next example shows.

**Example 4.1** The Banach space $c_0$ is strongly Gelfand–Phillips, but its countable power $(c_0)^\omega$ is not strongly Gelfand–Phillips.

**Proof** By Theorem 1.4, the Banach space $c_0$ is strongly Gelfand–Phillips. To show that $(c_0)^\omega$ is not strongly Gelfand–Phillips, we recall first that the Banach space $c_0$ is isomorphic to the Banach space $C(K)$ for any countable compact space $K$ with a unique non-isolated point. Then the countable power $(c_0)^\omega$ is isomorphic to the function space $C_k(K \times \omega)$ where $\omega$ is endowed with the discrete topology. By Theorem 3.15, the function space $C_k(K \times \omega)$ is not strongly Gelfand–Phillips. \[ \square \]

Let us recall that a locally convex space $E$ is feral if every bounded subset of $E$ is finite-dimensional. Observe that if a normed space $E$ is feral, then the closed unit ball of $E$ is finite-dimensional and hence $E$ is finite-dimensional as well. We shall consider a wider class of locally convex spaces introduced in [2]: a locally convex space $E$ is called $b$-feral if every barrel-bounded subset of $E$ is finite-dimensional. Clearly, every feral spaces is $b$-feral, but the converse is not true in general, see Example 4.3 below. The next simple assertion is proved in [2], we give its proof to make the paper more self-contained.

**Proposition 4.2** ([2]) Every $b$-feral space $E$ is strongly Gelfand–Phillips.
**Proof** By definition every barrel-bounded subset of $E$ is finite-dimensional and hence is barrel-precompact. Then the constant sequence of zero functionals on $E$ witnesses that $E$ is strongly Gelfand–Phillips.

Although the $b$-feralness of $E$ in the next example follows from a general assertion proved in [4], we provide a simple direct proof for the reader convenience and to make the article self-contained.

**Example 4.3** Let $E$ be the subspace of the Banach space $\ell_1$ consisting of all eventually zero sequences. The normed space $E$ is $b$-feral (but not feral) and hence it has the strong Gelfand–Phillips property. However, the completion $\ell_1$ of $E$ is not strongly Gelfand–Phillips.

**Proof** Suppose for a contradiction that there is an infinite-dimensional barrel-bounded subset $A$ of $E$. So there are a strictly increasing sequence $(n_k)_{k \in \omega}$ of natural numbers and a sequence $\{x_k\}_{k \in \omega} \subseteq A$ such that $\text{supp}(x_k) \subseteq \{0, \ldots, n_k+1 - 1\}$ and $a_{n_k} := |x_k(n_k)| \neq 0$. For every $n \in \omega \setminus \{n_k\}_{k \in \omega}$ we put $a_n := 1$. Consider the set $U := E \cap \prod_{n \in \omega} \left[ -\frac{a_n}{n+1}, \frac{a_n}{n+1} \right]$. It is clear that $U$ is a barrel in $E$. However, for every $\lambda \in \omega$ there is $k \in \omega$ such that $\lambda < n_k$ and hence $x_k \notin \lambda U$. Thus $A$ is not barrel-bounded, a contradiction. Thus $E$ is $b$-feral. Being an infinite-dimensional normed space, $E$ cannot be feral.

By Proposition 4.2, the space $E$ has the strong Gelfand–Phillips property. However, the completion $\ell_1$ of $E$ is not strongly Gelfand–Phillips by Theorem 1.3 (or Theorem 2.6).

Strongly Gelfand–Phillips spaces can contain dense subspaces without the strong Gelfand–Phillips property as the following example shows. In this example the ordinal $\omega$ is endowed with the discrete topology.

**Example 4.4** The space $C^b_p(\omega) = \mathbb{F}^{\omega}$ is strongly Gelfand–Phillips by Corollary 3.18, but its dense subspace $C^b_p(\omega)$ is not strongly Gelfand–Phillips, by Theorem 3.19.

A topological space $X$ is called *compact-finite* if every compact subspace of $X$ is finite. A map $f : Y \to Z$ is called *hereditarily quotient* if for any subspace $B \subseteq Z$ the map $f|_B : B \to f(B) \subseteq Z$ is quotient. By [15, 2.4.F(a)], a map $f : Y \to Z$ is hereditarily quotient if and only if for any $z \in Z$ and any open set $U \subseteq Y$ containing $f^{-1}(z)$ the image $f(U)$ is a neighborhood of $z$.

Let $X$ be a $T_1$-space. The *compact-finite resolution* of $X$ at a point $x \in X$ is the space

$$\mathcal{R}_x(X) = \{x\} \cup (X \setminus \{x\}) \times \omega$$

endowed with the topology consisting of the sets $U \subseteq \mathcal{R}_x(X)$ such that if $x \in U$, then the set

$$\{x\} \cup \{y \in X : (\{y\} \times \omega) \setminus U \text{ is finite}\}$$

is a neighborhood of $x$ in $X$. It follows that $\mathcal{R}_x(X) \setminus \{x\}$ is an open discrete subspace of $\mathcal{R}_x(X)$. The space $\mathcal{R}_x(X)$ is discrete if and only if $x$ is an isolated point of $X$. If
the point \( x \) is non-isolated in \( X \), then \( x \) is a unique non-isolated points of \( \mathcal{R}_x(X) \). The definition of the topology on the space \( \mathcal{R}_x(X) \) implies that the map \( q_x : \mathcal{R}_x(X) \to X \) defined by

\[
q_x(z) = \begin{cases} 
  x & \text{if } z = x; \\
  y & \text{if } z = (y, n) \in (X \setminus \{x\}) \times \omega;
\end{cases}
\]

is continuous and open at the point \( x \in \mathcal{R}_x(X) \) (recall that a function \( f : Y \to Z \) between topological spaces is called open at a point \( y \in Y \) if for any neighborhood \( O_y \) of the point \( y \) in \( Y \) the image \( f(O_y) \) is a neighborhood of the point \( f(y) \) in \( Z \)). The compact-finite resolution of the space \( X \) is the topological sum

\[
\mathcal{R}(X) = \bigoplus_{x \in X} \mathcal{R}_x(X) = \bigcup_{x \in X} \mathcal{R}_x(X) \times \{x\}.
\]

Example 4.6 below shows that a strongly Gelfand–Phillips space may contain a closed subspace which does not have the strong Gelfand–Phillips property. To construct this example we need the following theorem which is of independent interest.

**Theorem 4.5** Let \( X \) be a \( T_1 \)-space. Then

(i) \( \mathcal{R}(X) \) is a zero-dimensional, paracompact and compact-finite \( \mu \)-space;

(ii) the map \( q : \mathcal{R}(X) \to X, q : (z, x) \mapsto q_x(z) \), is hereditarily quotient;

(iii) \( C_p(X) \) is a closed subspace of the space \( C_p(\mathcal{R}(X)) \).

**Proof** (i) Since for every \( x \in X \), the space \( \mathcal{R}_x(X) \) has only one non-isolated point, it is zero-dimensional and paracompact. Therefore \( \mathcal{R}(X) \) is zero-dimensional and paracompact as well. Being paracompact, \( \mathcal{R}(X) \) is a \( \mu \)-space.

To show that \( \mathcal{R}(X) \) is compact-finite, it suffices to check that the space \( \mathcal{R}_x(X) \) is compact-finite for every \( x \in X \). Fix a compact subset \( K \subseteq \mathcal{R}_x(X) \). Replacing \( K \) by \( K \cup \{x\} \), we can assume that \( x \in K \). Observe that for every \( y \in X \setminus \{x\} \) the subspace \( \{y\} \times \omega \) is closed and discrete in \( \mathcal{R}_x(X) \) and hence \( (\{y\} \times \omega) \cap K \) is finite, being a closed discrete subspace of the compact space \( K \). Now the definition of the topology on \( \mathcal{R}_x(X) \) ensures that the set \( U = \{x\} \cup (\mathcal{R}_x(X) \setminus K) \) is a neighborhood of \( x \). So, \( x \) is an isolated point of \( K \) and hence all points of the compact space \( K \) are isolated, which implies that \( K \) is finite.

(ii) To prove that the map \( q : \mathcal{R}(X) \to X, q : (z, x) \mapsto q_x(z) \), is hereditarily quotient, fix an arbitrary \( x \in X \) and take any open set \( U \subseteq \mathcal{R}(X) \) containing the preimage \( q^{-1}(x) \). We should prove that \( q(U) \) is a neighborhood of \( x \) in \( X \). Since \( (x, x) \in (\mathcal{R}_x(X) \times \{x\}) \cap q^{-1}(x) \subseteq U \), there exists an open neighborhood \( V \subseteq \mathcal{R}_x(X) \) of \( x \) such that \( V \times \{x\} \subseteq U \). Since the map \( q_x : \mathcal{R}_x(X) \to X \) is open at \( x \), the image \( q(U) \supseteq q_x(V) \) is a neighborhood of \( x \) in \( X \).

(iii) It is well known that if \( q : Y \to Z \) is a quotient map, then the adjoint map \( q^* : C_p(Z) \to C_p(Y), q^* : f \mapsto f \circ q \), is an embedding onto a closed subspace. This result and (ii) imply that \( C_p(X) \) is a closed subspace of the space \( C_p(\mathcal{R}(X)) \). \( \square \)
**Example 4.6** Let $K$ be a countable compact space of infinite scattered height. Then:

(i) $C_p(K)$ is not strongly Gelfand–Phillips;
(ii) $C_p(R(K))$ is a separable metrizable strongly Gelfand–Phillips space;
(iii) $C_p(K)$ is a closed subspace of the space $C_p(R(K))$.

**Proof** The assertion (i) follows from Corollary 3.20, and (iii) follows from Theorem 4.5(iii).

(ii) Since $K$ is countable, the space $R(K)$ is countable as well. Therefore $C_p(R(K))$ is a separable metrizable space. It is strongly Gelfand–Phillips by (i) of Theorem 4.5 and Corollary 3.18.

**Theorem 4.7** For every locally convex space $E$ there exists a bounded-finite, zero-dimensional, paracompact Tychonoff space $X$ such that $E_w$ is topologically isomorphic to a closed subspace of the barrelled strongly Gelfand–Phillips space $C_p(X)$.

**Proof** Let $E'$ be the dual space of $E$, endowed with the weak$^*$ topology. Observe that the map $\delta : E_w \to C_p(E')$ assigning to every $x \in X$ the function $\delta_x : E' \to \mathbb{F}$, $\delta_x : f \mapsto f(x)$, is an isomorphic topological embedding. By Theorem 4.5, the compact-finite resolution $R(E')$ of $E'$ is a bounded-finite, zero-dimensional, paracompact Tychonoff space and the space $C_p(E')$ is isomorphic to a closed linear subspace of $C_p(R(E'))$. By Corollary 3.18, the function space $C_p(R(E'))$ is strongly Gelfand–Phillips. Since $R(E')$ is a bounded-finite space, its function space $C_k(R(E')) = C_p(R(E'))$ is barrelled by the Nachbin–Sirota Theorem. Therefore, the space $E_w$ is topologically isomorphic to a subspace of the strongly Gelfand–Phillips barrelled space $C_k(R(E')) = C_p(R(E'))$.

**Corollary 4.8** Every Tychonoff space $X$ is homeomorphic to a closed subspace of a barrelled strongly Gelfand–Phillips $C_p$-space.

**Proof** It is well known that $X$ is homeomorphic to a closed subspace of $C_p(C_p(X))$. Since $C_p(C_p(X))$ carries its weak topology Theorem 4.7 applies.

Example 4.6 and Theorem 4.7 motivate the question of whether every locally convex space embeds into a locally convex space with the strong Gelfand–Phillips property. This question has a negative answer as the following example shows. Recall that a locally convex space $E$ is injective in the class of locally convex spaces if $E$ is complemented in every locally convex space that contains $E$ as a subspace. It is well known that the Banach space $\ell_\infty$ is injective, see Theorems 10.1.2 and 10.1.3 in [22].

**Example 4.9** The Banach space $\ell_\infty$ does not embed into a Gelfand–Phillips space.

**Proof** Suppose for a contradiction that $\ell_\infty$ embeds into a strongly Gelfand–Phillips space $E$. Since $\ell_\infty$ is injective, $\ell_\infty$ is complemented in $E$. Then, by Proposition 2.5 and Corollary 2.6 of [3], $\ell_\infty = C(\beta\omega)$ is Gelfand–Phillips. But this contradicts the main result of [23] which states that $\ell_\infty$ is not a Gelfand–Phillips space.

It is worth mentioning that, by Proposition 2.12 of [4], every locally convex space is topologically isomorphic to a closed subspace of a locally convex space with the Josefson–Nissenzweig property.
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