Abstract

This article studies the dynamics of a nonlinear dissipative reaction-diffusion equation with well-separated stable states which is perturbed by infinite-dimensional multiplicative Lévy noise with a regularly varying component at intensity $\epsilon > 0$. The main results establish the precise asymptotics of the first exit times and locus of the solution $X^\epsilon$ from the domain of attraction of a deterministic stable state, in the limit as $\epsilon \to 0$. In contrast to the exponential growth for respective Gaussian perturbations the exit times grow essentially as a power function of the noise intensity as $\epsilon \to 0$ with the exponent given as the tail index $-\alpha$, $\alpha > 0$, of the Lévy measure, analogously to the case of additive noise in Debussche et al [18]. In this article we substantially improve their quadratic estimate of the small jump dynamics and derive a new exponential estimate of the stochastic convolution for stochastic Lévy integrals with bounded jumps based on the recent pathwise Burkholder-Davis-Gundy inequality by Siorpaes [51]. This allows to cover perturbations with general tail index $\alpha > 0$, multiplicative noise and perturbations of the linear heat equation. In addition, our convergence results are probabilistically strongest possible. Finally, we infer the metastable convergence of the process on the common time scale $t/\epsilon^\alpha$ to a Markov chain switching between the stable states of the deterministic dynamical system.

Keywords: first exit times; first exit locus; metastability; nonlinear reaction-diffusion equation; Morse-Smale property; small noise asymptotics; $\alpha$-stable Lévy process in Hilbert space; multiplicative Lévy noise; regularly varying noise; stochastic heat equation with additive and multiplicative $\alpha$-stable noise; stochastic Chafee-Infante equation with multiplicative noise;

2010 Mathematical Subject Classification: 60H15; 60G51; 60G52; 60G55; 35K05; 35K91; 35K57; 35K55; 37D15; 37L55.

1 Introduction

This article solves the asymptotic first exit problem from the domain of attraction of a stable state in a generic class of scalar dissipative reaction-diffusion equations subject to small multiplicative regularly varying Lévy noise, such as small multiplicative $\alpha$-stable noise. More precisely, the asymptotic first exit time and locus, as the noise intensity $\epsilon$ tends to 0, are determined completely.
The first exit problem of a randomly perturbed dynamical system from the domain of attraction of a stable fixed point in the limit of small noise intensity has a long history in finite dimensions for Gaussian perturbations going back to the works of Cramér and Lundberg and giving rise to the edifice of large deviations theory and the associated Freidlin-Wentzel theory. We refer the reader to the classical works [2, 5, 15, 16, 21, 24, 25, 40] and the references therein. In infinite dimensions this problem was studied for the infinite dimensional Wiener processes for instance in [1, 7, 8, 22, 23]. It is a characteristic feature of small Gaussian perturbations that the first exit times grow exponentially as a function of the inverse of the noise intensity, with the prefactor in the exponent given as the solution of an optimization problem. The convergence of the suitably renormalized process to a Markov chain [26, 37] and its connection between the metastability and the spectrum of the diffusion generator are treated in [3, 5, 6, 38, 39]. In the context of regularly varying Lévy jump noise perturbations, however, the perturbed process exhibits heavy tails and therefore lacks the necessary exponential moments for a large deviations principle (cf. [1, 50]).

The first exit problem for dynamical systems perturbed by small α-stable or more generally regularly varying Lévy perturbations was addressed in different settings in a series of works. After the early work of [27] on a large deviations principle in Skorohod space the first exit times problem is solved in one dimension for additive α-stable noise in [35]. The authors introduced the following purely probabilistic proof technique also used and extended in this article, which we sketch briefly:

Given an α-stable noise perturbation $\varepsilon dL$ the first step is the choice of an $\varepsilon$-dependent jump size threshold $\rho_\varepsilon$, which decomposes the driving noise into the sum $\varepsilon d\xi + \varepsilon d\eta$. $\xi_\varepsilon$ being an infinite intensity process with jumps bounded from above by the threshold $\rho_\varepsilon$ and thus exhibiting exponential moments and $\eta_\varepsilon$ the compound Poisson process of jumps bounded from below by the threshold. Assuming that $\varepsilon \rho_\varepsilon$ tends to 0 as $\varepsilon$ tends to 0 and taking into account that $\xi_\varepsilon$ has exponential moments, it does not come as a surprise that up to the first large compound Poisson jump of $\eta_\varepsilon$, the process $\varepsilon \xi_\varepsilon$ is very small. Hence the resulting flow decomposition of the strong Markov solution $X_\varepsilon$ yields that up to the first large jump of $\varepsilon \eta_\varepsilon$, the process $X_\varepsilon$ remains close to the deterministic solution with overwhelming probability. Therefore in the vast majority of cases it cannot cause the exit from the domain of attraction. Due to the Morse-Smale property and the choice of the noise decomposition the convergence of the deterministic solution to a small ball centered in the stable state is faster than the first large jump time. As a consequence, the first large compound Poisson jump starts from close to the stable state and yields an exit probability of the first large jump in terms of the tail of the Lévy measure. The strong Markov property propagates this exit scenario to all independent waiting time intervals between the large jumps. The exit is hence caused with very high probability by the first successful attempt of a large jump to exit. The resulting geometric exit structure of the exit times happens at a rate given by the tail decay of the Lévy measure governing the large jumps which in the case of regular variation is of polynomial order. In [44] the author shows this result for gradient systems in any finite dimension and multiplicative noise; in particular, he derives an exponential estimate for small deviations from the deterministic system. He obtains exponential estimates for the small noise components, however his treatment of the small jump component depends on the dimension of the driving noise and is not suitable in infinite dimensions. In [51] the results are generalized to the non-gradient case in finite dimensions, in addition the convergence in law of the first exit locus is proved. The well-posedness of reaction-diffusion equations in infinite dimensions in a generic setting is established in [9].
and precisely for this reason only allows for tail indices $-\alpha$ for $0 < \alpha < 2$ there.

This article provides a substantial extension of these results in several directions. We extend the scope of the deterministic forcing of $[18]$ to a general class of weakly dissipative non-linear reaction terms over an interval with Dirichlet boundary conditions for which the system retains the Morse-Smale property of the deterministic system. The most important cases covered here are dissipative polynomials of odd order, such as for the Chafee-Infante equation, and the linear heat equation. Our results are stated for the Laplace operator with Dirichlet conditions on the Sobolev space $H^1_0$ over the standard interval $[0,1]$. We expect the results to hold true for any unbounded operator with negative point spectrum $A$ which generates a generalized contracting analytic semigroup. However, we use the Morse-Smale property of the deterministic dynamical system in $H^1_0$ as well as the smoothness of the separating manifold of the domains of attractions, and to our knowledge these results are not readily available in the literature for general spaces $D(A^\frac{1}{2})$.

The generalizations of the type of stochastic perturbations are twofold. In the first place we study multiplicative noise coefficients as opposed to $[18]$. They are the original motivation of this article and make it necessary to consider the first exit problem localized on large balls. Consequently we get rid of the rather strong point dissipativity of the deterministic dynamical system, and also treat the important new example of the linear heat equation subject to additive and multiplicative $\alpha$-stable noise.

Secondly, we lift the rather strong restriction of a tail index $0 < \alpha < 2$ in $[18]$ with the help of an exponential estimate of the stochastic convolution for multiplicative Poisson random integrals with bounded jumps. It combines the recent pathwise estimate of the stochastic convolution in $[52]$ and the pathwise Burkholder-Davis-Gundy inequality in $[51]$. Since multiplicative noise necessarily leads to working with stopped processes, the non-pathwise estimates of the stochastic convolution available until then were rather difficult to implement. With these new powerful tools at hand an almost sure estimate in the exponent of an exponential moment yields a lift of the right side of the Burkholder-Davis-Gundy inequality to the exponent, which is estimated with the help of a Campbell type formula for the Laplace-transform of Poisson random integrals. Our estimates are rather direct and avoid the adaption of the technically charged large deviation theory introduced by Budhiraja and collaborators; see for instance $[11, 13, 14]$. In comparison to those works we construct explicitly (on the same probability space as the driving noise) a completely understood model of the first exit times and locus respectively to which the original objects converge. Our convergence results are optimal in a probabilistic sense, in that we obtain exponential convergence up to all exponents strictly less than 1, while the limiting object does not have exponential moments of order 1. The same applies to the convergence of the first exit locus, which is essentially a geometric mixing of deformed large jump increments of the noise. Those increments with tail index $-\alpha$, $\alpha > 0$, have moments of order $0 < p < \alpha$ and we show convergence in any such $L^p$-sense towards the limiting object. Finally we infer metastability in the sense of $[32]$ and $[34]$ as a corollary.

The article is organized as follows. In Section 2 we present the general setup, the specific hypotheses, the main results and examples. The proof relies on the mentioned $\varepsilon$-dependent distinction of large and small jump perturbations. In Section 3 we prove an exponential error probability estimate on the smallness of the stochastic convolution between large jumps and its pushforward to the nonlinear equation. In Section 4 we use the preceding result which yields an asymptotic compound Poisson noise structure that essentially contains only large jumps. With the help of the strong Markov property and tailor-made event estimates we identify the asymptotic first exit mechanism of the solution of the fully perturbed nonlinear equation.
2 The object of study and the main result

Notation: For $J = (0, 1)$ we consider the Sobolev space $H := H^1_0(J)$ equipped with the inner product $\langle x, y \rangle = \langle \nabla x, \nabla y \rangle$ for $x, y \in H$ and the norm $\|x\| = \langle \langle x, x \rangle \rangle^{1/2}$, where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(J)$ with $|x| = \langle x, x \rangle^{1/2}$. Let $C_0(J)$ be the space of continuous functions $x : J \to \mathbb{R}$ with $x(0) = x(1) = 0$ equipped with the supremum norm $\| \cdot \|_\infty$. Since $|x| \leq |x|_\infty \leq \|x\|$ for $x \in H$ we have the embeddings $H \hookrightarrow C_0(J) \hookrightarrow L^2(J)$, in particular, $|x| \leq \Lambda_0 \|x\|$ for all $x \in H$ and the Sobolev constant $\Lambda_0 > 0$.

2.1 The underlying deterministic dynamics

The unperturbed PDE: The object of study is the effect of random perturbations of the deterministic dynamical system given for any $t \geq 0$ as the solution map $x \mapsto u(t; x)$ of the following nonlinear reaction-diffusion equation over the interval $J$ with Dirichlet boundary conditions. We consider

$$\frac{\partial}{\partial t} u(t, \zeta) = \Delta u(t, \zeta) + f(u(t, \zeta)) \quad \text{with} \quad u(t, 0) = u(t, 1) = 0 \quad \text{and} \quad u(0, \zeta; x) = x(\zeta), \quad (2.1)$$

for $t \geq 0$, $x \in H$ and $\zeta \in J$, where the non-linearity $f \in C^2(\mathbb{R}, \mathbb{R})$ satisfies the growth condition

$$\limsup_{|r| \to \infty} f'(r) < \Lambda_0. \quad (2.2)$$

This equation has unique and well-posed weak and mild solutions in $L^2(J)$ and $H$ (cf. [18, 23]). The solutions are most regular for any $t > 0$ and $x \in L^2(J)$, that is, $u(t; x) \in C^{\infty}(J) \cap C_0(J)$.

Remark 2.1. In case of $f(r) = \sum_{j=1}^{2n-1} b_j r^j$ with $b_{2n-1} < 0$ for $n \in \mathbb{N}$ it is well-known in the literature [23, 29, 40] that for a generic choice of $(b_0, \ldots, b_n) \in \mathbb{R}^{n+1}$, that is, up to a nowhere dense set, the dynamical system generated by (2.1) is of Morse-Smale type. In other words, there is a finite number of fixed points, all of which are hyperbolic and whose stable and unstable manifold intersect transversally. The paradigmatic example of the Chafee-Infante equation is studied in [12, 23, 29] where $f(r) = -a(r^3 - r)$, $r \in \mathbb{R}$, whenever $a > \pi^2$ and $a \neq (\pi n)^2$, $n \in \mathbb{N}$. Since all finitely many equilibria are elements of $H \subseteq L^\infty(J)$, the Morse-Smale property only involves $f$ on a bounded set. On bounded sets a general function $f \in C^2(\mathbb{R}, \mathbb{R})$ is approximated in $C^2$-norm by polynomials and the generic Morse-Smale property is inherited by $f$.

The deterministic dynamics: It is well-known that the solution of equation (2.1) has the nonnegative potential function $\mathcal{V}(x) = \int_J \big((\nabla x(\zeta))^2 + F(x(\zeta))\big) d\zeta$ on $H$ where $F(r) = \int_0^r f(s) ds$ for some $r_0$. Therefore, equation (2.1) reads as the gradient system

$$\frac{\partial}{\partial t} u(t, \zeta) = -(DV)(u(t, \zeta)) \quad \text{with} \quad u(0, \zeta; x) = x(\zeta) \quad \text{for} \quad x \in H.$$

The level sets of $\mathcal{V}$ are bounded in $H$ and positive invariant under the system (2.1). For $r > 0$ set

$$\mathcal{U}_r := \{ x \in H \mid \mathcal{V}(x) \leq d_*(r) \}, \quad d_*(r) := \inf \{ s > 0 \mid B_r(0) \subseteq \mathcal{V}^{-1}[0, s] \}, \quad d(r) := \sup_{x \in \mathcal{U}_r} \|x\|. \quad (2.3)$$

As a consequence, $\mathcal{V}$ serves as a Lyapunov function and yields the following result (cf. [23, 30]).
Proposition 2.2. Denote by $\mathcal{P} \subseteq H$ the set of fixed points of (2.1). Then $0 < |\mathcal{P}| < \infty$ and for any $x \in H$ there exists a stationary state $\phi \in \mathcal{P}$ of the system (2.1) such that $\lim_{t \to \infty} u(t; x) = \phi$.

For $\phi \in \mathcal{P}$ we define the domain of attraction $D(\phi) := \{x \in H \mid \lim_{t \to \infty} u(t; x) = \phi\}$. The set of stable states is the subset $\mathcal{P}^-$ of all $\phi \in \mathcal{P}$ such that $D(\phi)$ contains an open ball in $H$. For $\phi^* \in \mathcal{P}^-$, $1 \leq t \leq |\mathcal{P}|$, we denote its domain of attraction $D^{t} = D(\phi^*)$ and the separating manifold between them by $\mathcal{S} := H \setminus \bigcup D^t$. For a generic choice of coefficients the Morse-Smale property implies that $\mathcal{S}$ is a closed $C^1$-manifold without boundary in $H$ of codimension 1 separating all elements of $(D^{t})_{\phi^* \in \mathcal{P}^-}$ and containing all unstable fixed points $\mathcal{P} \setminus \mathcal{P}^-$ (cf. [48]).

Reduced domains of attraction: Note that $f : H \to H$ is locally Lipschitz continuous. For any subset $D^t \subseteq D^t$ with $C^1$ boundary, such that $\Delta + f$ on $\partial D^t$ is uniformly inward pointing we have

$$\kappa_1 := \inf_{v \in \partial D^t \cap C^3(I)} \frac{\langle \nabla v, \Delta v + f(v) \rangle}{\|\Delta v + f(v)\|} > 0,$$

where $\nabla v$ is the normalized inner normal at the foot point $v \in \partial D^t$ and $C^3_0(I) = C^3(I) \cap C_0(I)$. In the sequel we define the following nested reduced domains of attraction of $D^t$ in order to formulate the nondegeneracy of the noise perturbations in Hypothesis (S.4) below. Fix a radius $R_0 > 0$ such that $\mathcal{P} \subseteq B_{R_0}/2(0)$ and $U^{R_0} \cap \partial D^t \neq \emptyset$ for all $\phi^* \in \mathcal{P}^-$. We define for $\delta \geq 0$, $i = 1, \ldots, 3$, $\mathcal{R} \geq R_0$ and $G$ the function appearing in (2.9) below the following reductions of $D^t$:

$$D^t_1(\mathcal{R}) := D^t \cap U^\mathcal{R},$$
$$D^t_2(\delta_1, \mathcal{R}) := \{x \in D^t_1(\mathcal{R}) \mid B_{\delta_1}(x) \subseteq D^t_1(\mathcal{R})\},$$
$$D^t_3(\delta_1, \delta_2, \mathcal{R}) := \{x \in D^t_2(\delta_1, \mathcal{R}) \mid \bigcap_{v \in B_{\delta_2}(x)} \{v + G(v, z)\} \subseteq D^t_2(\delta_1, \mathcal{R})\}.$$  

(2.5)

For convenience we set $D^t_3(\delta_1, \mathcal{R}) := D^t_3(\delta_1, \delta_1, \mathcal{R})$. The reduced domains of attraction are nested by construction and $D^t = \bigcup_{\mathcal{R} \geq R_0, \delta \in (0, 1)} D^t_3(\delta, \mathcal{R})$ (cf. [17]). For any $\mathcal{R} > 0$ and $\delta \in (0, \delta_0]$, $\delta_0 \in (0, 1]$ sufficiently small, the reduced domains of attraction $D^t_3(\delta, \mathcal{R})$ (and $D^t_3(\delta, \mathcal{R})$) are positive invariant under the dynamical system (2.1) due to the uniformly inward pointing property of $f$ on $\partial D^t$.

Proposition 2.3. For any choice of $f$ such that (2.1) is Morse-Smale, $D^t \subseteq D^t$ with $\partial D^t \in C^1$ satisfying (2.4) and $\mathcal{R} \geq R_0$ there exists a constant $\kappa_0 > 0$ which satisfies the following. For any function $\gamma_c : (0, 1) \to (0, 1)$ with $\lim_{t \to 0} \gamma_c = 0$ there is a constant $\varepsilon_0 \in (0, 1]$ such that for each $\varepsilon \in (0, \varepsilon_0]$ the conditions $t \geq \kappa_0 |\ln(\gamma_c)|$ and $x \in D^t_1(\mathcal{R})$ imply $\|u(t; x) - \phi^*\| < \frac{1}{5} \gamma_c$. In addition, (2.1) is Morse-Smale if and only if the equilibrium points are hyperbolic.

This result is based on the existence of a Lyapunov function, the uniform inward pointing property of $f$ on $\partial D^t$, and the hyperbolicity of the fixed points. In [17] it is shown for a stronger form of approximation for the Chafee-Infante equation. Its generalization is straightforward. The second part of the statement is given by Theorem 2.2.1 in [28] and the references therein.

2.2 The stochastic reaction-diffusion equation

The Lévy driver: Given a filtered probability space $\Omega = (\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ satisfying the usual conditions in the sense of Protter [10], let $L = (L(t))_{t \geq 0}$ be a càdlàg version of a Lévy process in $(H, \mathcal{B}(H))$. We denote by $\mathcal{M}_0(H)$ the class of Radon measures $\nu$ on $\mathcal{B}(H)$ satisfying $\nu(A) < \infty$ for
A ∈ ℬ(H) with 0 ∉ Ā. The Lévy-Chinchine representation establishes a unique Lévy triplet (h, Q, ν) with h ∈ H, a positive trace-class operator Q ∈ L1+(H) and ν ∈ ℬ0(H) satisfying ν({0}) = 0 and 
\[\int_H (1 + \|y\|^2)ν(dy) < \infty\] such that the characteristic function \(\phi_{L(t)}(u) := \mathbb{E}\left[\exp(i\langle u, L(t)\rangle)\right]\) has the exponent
\[t\left(i\langle h, u\rangle - \frac{1}{2}\langle Qu, u\rangle + \int_H \left(e^{i\langle u, z\rangle} - 1 - i\langle z, u\rangle\mathbb{1}\{\|z\| \leq 1\}\right)ν(dz)\right), \quad u \in H, \ t \geq 0.\]

By the Lévy-Itô representation of L there exist a Q-Wiener process \((B_Q(t))_{t \geq 0}\) and a Poisson random measure N on \(\mathfrak{Ω}\) with intensity measure \(dt \otimes ν(dz)\) on \([0, \infty) \times H\) such that \(\mathbb{P}\)-a.s.
\[L(t) = ht + B_Q(t) + \int_0^t \int_{\|z\| \leq 1} z\tilde{N}(d sdz) + \int_0^t \int_{\|z\| > 1} zN(dsdz), \quad \text{for all} \ t \geq 0, \ (2.6)\]
where \(\tilde{N}((a, b] \times A) := N((a, b] \times A) - (b - a)ν(A)\) for \(a \leq b, A \in ℬ(H), 0 \notin Ā\) is the compensated Poisson random measure of N. For a comprehensive account of Lévy processes in Hilbert spaces with refer to [47]. In this study we set \(h = 0\) and \(B_Q = 0\), since their exit contributions are asymptotically insignificant compared to the pure jump part, if ν ≠ 0.

The multiplicative nonlinearity is given as a map \(G : H \times H \to H\) which satisfies the following standard boundedness and Lipschitz conditions. There are constants \(K_1, K_2 > 0\) such that
\[\int_{B_1(0)} \|G(x, z)\|^2ν(dz) \leq K_1(1 + \|x\|^2) \quad \text{for all} \ x \in H, \ (2.7)\]
\[\|G(x_1, z) - G(x_2, z)\| \leq K_2\|x_1 - x_2\| \quad \text{for all} \ x_1, x_2, z \in H. \ (2.8)\]

The perturbed equation: We consider the formal stochastic reaction diffusion equation for \(t > 0, x \in H, ζ \in J\) and \(ε \in (0, 1)\)
\[dX^ε(t, ζ) = (ΔX^ε(t, ζ) + f(X^ε(t, ζ)))dt + G(X^ε(t−, ζ), εdL(t, ζ)) \quad \text{with} \ X^ε(t, 0) = X^ε(t, 1) = 0 \quad \text{and} \ X^ε(0, ζ) = x(ζ), \ (2.9)\]
where
\[G(X^ε(t, ζ), εdL(t, ζ)) = \int_{|z| \leq 1} G(X^ε(t−, ζ), εz(ζ))\tilde{N}(dtdz) + \int_{|z| > 1} G(X^ε(t−, ζ), εz(ζ))N(dtdz).\]

**Proposition 2.4.** Assume the setting of Subsection 2.1, in particular, the growth rate (2.2) for \(f \in C^2(\mathbb{R}, \mathbb{R})\) and the conditions (2.7) and (2.8) of G. Then for any mean zero càdlàg \(L^2(\mathbb{P}; H)\)-martingale \(ξ = (ξ(t))_{t \geq 0}\) on \(\mathfrak{Ω}, T > 0\), and initial value \(x \in H\), equation (2.9) driven by εdξ instead of εdL has a unique càdlàg mild solution \((X^ε(t, x))_{t \in [0, T]}\). The transition kernels of the solution process \(X^ε\) induce a homogeneous Markov family satisfying the Feller property and hence the strong Markov property.

The proof relies on the local Lipschitz continuity and the dissipativity of \(f : H \to H\). A proof for dissipative polynomials \(f\) is given in [47], Chapter 10, and for the Chafee-Infante equation in [17] and can be extended to our situation straightforwardly. By interlacing of large jumps, this notion of solution is extended to the heavy-tailed process \(L\), as carried out in [47], Subsec. 9.7, pp.170.
Corollary 2.5. For \( x \in H \) equation (2.4) has a global càdlàg mild solution \( (X^x(t; x))_{t \geq 0} \), which satisfies the strong Markov property.

2.3 The specific hypotheses and the main results

Under the standing assumptions of Subsection 2.1 and 2.2 we impose the following additional hypotheses.

Hypotheses:

(D.1) The function \( f \) is generic in the sense that the solution flow of (2.1) defines a Morse-Smale system. In addition, we assume \( 2 \leq |P^-| < \infty \).

(D.2) The function \( f \) satisfies the eventual monotonicity condition (2.2).

(D.3) Consider a subset \( D^x \subseteq D^\varepsilon \) with \( C^1 \)-boundary such that the operator \( \Delta + f \) is uniformly inward pointing on \( \partial D^x \) in the sense of (2.4).

(S.1) There is a globally Lipschitz continuous function \( G_1 : H \rightarrow [0, \infty) \) such that

\[
\|G(y, z)\| \leq G_1(y)\|z\|, \quad y, z \in H.
\]

(S.2) The Lévy measure \( \nu \in \mathcal{M}_0(H) \) of \( L \) is regularly varying with index \( -\alpha \), \( \alpha > 0 \), and limit measure \( \mu \in \mathcal{M}_0(H) \).

See for instance Def 3.44, p. 67, in [18]. By [10] and [33] Hypothesis (S.2) is equivalent to the existence of measurable functions \( h, \ell : (0, \infty) \rightarrow (0, \infty) \) such that

\[
\lim_{r \rightarrow \infty} \frac{\nu(rU)}{h(r)\mu(U)} = 1 \quad \text{for any } U \in \mathcal{B}(H) \text{ with } 0 \notin U,
\]

where \( h(r) = r^{-\alpha}\ell(r) \) and \( \ell \) is a slowly varying function. In other words, \( \lim_{r \rightarrow \infty} \frac{\ell(a r)}{\ell(r)} = 1 \) for any \( a > 0 \). We define the set of increment vectors \( z \in H \) sending \( x \in H \) to the set \( U \in \mathcal{B}(H) \) as

\[
J^U(x) := \{z \in H \mid x + G(x, z) \in U\}, \quad x \in H.
\]

For any \( \phi^\varepsilon \in \mathcal{P}^- \) we denote the measure \( m^\varepsilon(U) := \mu(J^U(\phi^\varepsilon)) \), \( U \in \mathcal{B}(H) \) with \( 0 \notin U \), and the scale \( h_\varepsilon := h(\frac{1}{\varepsilon}) \) for \( \varepsilon \in (0, 1] \).

(S.3) For all \( \phi^\varepsilon \in \mathcal{P}^- \) and \( \mathcal{R} \geq \mathcal{R}_0 \) we have \( m^\varepsilon((D^\varepsilon \cap \mathcal{U}(\mathcal{R}))^c) > 0 \).

(S.4) For \( D^\varepsilon \) in (D.3) and all \( \eta > 0 \) there are \( \delta > 0 \) and \( \mathcal{R} \geq \mathcal{R}_0 \) such that \( m^\varepsilon(D^\varepsilon \setminus D^\varepsilon_3(\delta, \mathcal{R})) < \eta \).

Hypothesis (S.3) states that asymptotically there is some non-vanishing mass for large jumps to exit, while (S.4) codes the nondegeneracy of the limiting Lévy measure on the boundary \( \partial D^\varepsilon \), in order to allow for the approximations of \( D^\varepsilon \) by \( D^\varepsilon_3(\delta, \mathcal{R}) \) in terms of \( m^\varepsilon \).

(S.5) For any \( \eta > 0 \) and \( \iota \) there are sets \( D^\varepsilon \subseteq D^\varepsilon \) satisfying (D.3) as well as \( \delta > 0 \) and \( \mathcal{R} \geq \mathcal{R}_0 > 0 \) such that

\[
m^\varepsilon(H \setminus \bigcup_{\iota} D^\varepsilon_3(\delta, \mathcal{R})) < \eta.
\]

Hypothesis (S.5) is a sort of uniform version of (S.4) for all domains of attraction \( D^\varepsilon \).
The first exit time result: For \( \gamma, \varepsilon \in (0, 1], \mathcal{R} \supseteq \mathcal{R}_0, x \in D_2^\varepsilon(\varepsilon^{\gamma}, \mathcal{R}) \) and the càdlàg mild solution \((X^\varepsilon(t); x)_{t \geq 0}\) of (2.9) we define the first exit time from the reduced domain of \( D^\varepsilon \)

\[
\tau_0^\varepsilon(\varepsilon, \mathcal{R}) := \inf\{t > 0 \mid X^\varepsilon(t; x) \notin D_2^\varepsilon(\varepsilon^{\gamma}, \mathcal{R})\}.
\]

We define the characteristic exit rate \( \lambda_\varepsilon^\gamma \) of system (2.9) from \( D^\varepsilon \) by

\[
\lambda_\varepsilon^\gamma := \nu\left( \frac{1}{\varepsilon} \mathcal{J}^{\gamma}(\phi') \right), \quad \varepsilon \in (0, 1].
\]

(2.10)

Then (S.2) implies \( \frac{\lambda_\varepsilon^\gamma}{h_\varepsilon} \xrightarrow{\varepsilon \to 0^+} m'(\varepsilon D^\varepsilon) \). Our asymptotic exit time result reads as follows.

**Theorem 2.6.** Let Hypotheses (D.1)-(D.3) and (S.1)-(S.4) be satisfied for some \( \varepsilon \). Then there is an \( \text{EXP}(1) \)-distributed family of random variables \( (\varsigma'(\varepsilon))_{\varepsilon \in (0, 1]} \) on \( \Omega \) satisfying the following. For any \( c > 0 \) and \( \theta \in (0, 1) \) there are \( \mathcal{R} \supseteq \mathcal{R}_0 \) and \( \varepsilon_0, \gamma, \varepsilon \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) implies

\[
\sup_{x \in D_2^\varepsilon(\varepsilon^{\gamma}, \mathcal{R})} E\left[ e^{\theta |\lambda_\varepsilon^\gamma(\varepsilon, \mathcal{R}) - \varsigma'(\varepsilon)|} \right] \leq 1 + c.
\]

Therefore, we have the convergence of all moments \( \lim_{\varepsilon \to 0} E[|\lambda_\varepsilon^\gamma(\varepsilon, \mathcal{R})|^n] \in [n! - c, n! + c] \) uniformly in \( D_2^\varepsilon(\varepsilon^{\gamma}, \mathcal{R}) \) and the following polynomial behavior

\[
\sup_{x \in D_2^\varepsilon(\varepsilon^{\gamma}, \mathcal{R})} E\left[ \tau_0^\varepsilon(\varepsilon, \mathcal{R}) \right] \in \left[ \frac{1 - c}{\lambda_\varepsilon^\gamma}, \frac{1 + c}{\lambda_\varepsilon^\gamma} \right] \subseteq \left[ \frac{1}{\varepsilon^\alpha \ell(\varepsilon)} m'(\varepsilon D^\varepsilon), \frac{1 + 2c}{\varepsilon^\alpha \ell(\varepsilon)} m'(\varepsilon D^\varepsilon) \right],
\]

where the supremum can be changed to the infimum.

In terms of [5] the memorylessness of \( \varsigma'(\epsilon) \) describes the “unpredictability” of the exit times, however, with a “polynomial” loss of memory as opposed to an “exponential” loss of memory in the case of Gaussian perturbations.

The first exit locus result: For the statement of the main result about the exit locus we write \( \Delta_t L := L(t) - L(t^-), t \geq 0 \) for \( L \) given by (2.6). For some \( \rho \in (0, 1) \) and \( \varepsilon \in (0, 1] \) we define large jump arrival times of \( L \) by

\[
T_0(\varepsilon) := 0, \quad T_k(\varepsilon) := \inf\{t > T_{k-1}(\varepsilon) \mid \|\Delta_t L\| > \varepsilon^{-\rho}\}, \quad k \geq 1,
\]

(2.11)

and large jump increments by \( W_k(\varepsilon) := \Delta T_k(\varepsilon) L, k \in \mathbb{N} \). The family \( (W_k(\varepsilon))_{k \in \mathbb{N}} \) is i.i.d. with

\[
\mathbb{P}(\varepsilon W_k(\varepsilon) \in U) = \frac{\nu(U \cap B_{\varepsilon^{-\rho}}(0))}{\beta_{\varepsilon}}, \quad \text{where } \beta_{\varepsilon} := \nu(B_{\varepsilon^{-\rho}}(0)) \text{ satisfies } \frac{\beta_{\varepsilon}}{\varepsilon^{\alpha} \ell'(\varepsilon^{-\rho})} \xrightarrow{\varepsilon \to 0^+} \mu(B_1^\varepsilon(0)).
\]

We drop the \( \varepsilon \)-argument of \( T_k \) and \( W_k \). The asymptotic exit locus result theorem reads as follows.

**Theorem 2.7.** Let Hypotheses (D.1)-(D.3) and (S.1)-(S.4) be satisfied for some \( \varepsilon \). Then there is a family of random variables \( (\mathcal{R}(\varepsilon))_{\varepsilon \in (0, 1]} \) on \( \Omega \) with \( \mathcal{R}(\varepsilon) \) being GEO(\( \lambda_\varepsilon^\gamma/\beta_{\varepsilon} \)) distributed and satisfying the following. For any \( c > 0 \) and \( 0 < p < \alpha \) there are \( \mathcal{R} \supseteq \mathcal{R}_0 \) and \( \gamma, \mu, \varepsilon_0 \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) implies

\[
\sup_{x \in D_2^\varepsilon(\varepsilon^{\gamma}, \mathcal{R})} E\left[ \left\| X^\varepsilon(\tau_0^\varepsilon(\varepsilon, \mathcal{R}); x) - (\phi' + G(\phi', \varepsilon \mathcal{R}(\varepsilon))) \right\|^p \right] \leq c.
\]
For any \( c > 0 \) there are \( R \geq R_0 \) and \( \gamma, \rho \in (0, 1] \) such that for any \( U \in B(H) \) with \( m^t(U) > 0 \) and \( m^t(\partial U) = 0 \) there is \( \varepsilon_0 \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) yields
\[
\sup_{x \in D^c_\varepsilon} |\mathbb{P}(X^\varepsilon(\tau^\varepsilon_1(\varepsilon, R); x) \in U) - \frac{m^t(U \cap (D^\varepsilon)^c)}{m^t(D^\varepsilon)^c}| \leq c.
\]

**The metastability result:** Under Hypothesis (S.5) we have a good approximation of \( D^\varepsilon \) by sets \( D^c_\varepsilon(\delta, \mathcal{R}) \) with inward pointing \( \Delta + f \) at its boundary in the sense of (2.4). Hence the exit rate \( \varepsilon \to \lambda^\varepsilon \approx \varepsilon^\alpha f(\frac{1}{\varepsilon})m^t((D^\varepsilon)^c) \) asymptotically depends on \( \varepsilon \) only by a prefactor and the process \( X^\varepsilon \) converges on the common (polynomial) time scale \( t/\varepsilon^\alpha \) to a continuous time Markov chain whose transition probabilities from \( D^\varepsilon \) to \( D^\varepsilon \) only depend on the values \( m^t(D^\varepsilon) \), \( \kappa \in \{1, \ldots, |P^-| \} \setminus \{\ell\} \).

This behavior is typical for regularly varying Lévy noises such as \( \alpha \)-stable noise \([18, 32, 34]\) and differs strongly from the Gaussian case (see in particular \([26]\)). In the introduction of \([32]\) it is explained in detail how this behavior corresponds to the degenerate Gaussian case where the time scales are comparable which occurs if and only if the potential barriers are all of exactly the same height. The asymptotic metastability result reads as follows.

**Corollary 2.8.** Let Hypotheses (D.1)-(D.3), (S.1)-(S.3) and (S.5) be satisfied. Then there exists a continuous time Markov chain \( M = (M_t)_{t \geq 0} \) with values in \( P^- \) and infinitesimal generator \( \mathcal{G} \) for \( p = |P^-| \) given as the matrix
\[
\mathcal{G} = \begin{pmatrix}
-m_1((D^1)^c) & m_1(D^2) & \cdots & m_1(D^p) \\
\vdots & \vdots & \ddots & \vdots \\
m_p(D^1) & \cdots & m_p(D^{p-1}) & -m_p((D^p)^c)
\end{pmatrix}, \tag{2.12}
\]
and a constant \( \gamma > 0 \) such that for any \( R \geq R_0 \), \( T > 0 \) and \( x \in D^c_\varepsilon(\gamma, \mathcal{R}) \), \( 1 \leq \ell \leq |P^-| \), we have \( (X^\varepsilon(\frac{1}{\varepsilon}, x))_{t \in [0, T]} \overset{d}{\to} (M_t)_{t \in [0, T]} \) in the sense of convergence of finite-dimensional distributions.

The proof of this result is an analogous construction as in \([32]\) and does not depend on the fact that the system is infinite-dimensional and follows in a fairly straightforward manner.

### 2.4 Examples

#### 2.4.1 The Chafee-Infante equation with multiplicative stable noise

We consider equation \([24]\) for \( f(r) = -a(r^3 - r) \), \( r \in \mathbb{R} \), where \( a > \pi^2 \) and \( a \neq (\pi n)^2 \) for all \( n \in \mathbb{N} \). The respective deterministic dynamical system is a well understood Morse-Smale system and has two stable states \( \phi^\pm \), \( \ell \in \{+,-\} \) with domains of attraction \( D^\pm \), see \([28, 30]\).

The stochastic perturbation of interest is an \( H \)-valued symmetric \( \alpha \)-stable Lévy process \( L(t) = \int_0^t \int_{\|z\| \leq 1} zN(ds, dz) + \int_0^t \int_{\|z\| > 1} zN(ds, dz) \) with characteristic triplet \( (0, 0, \nu) \), where \( \nu(ds, dz) = \sigma(dz)/\pi^{1/2} \), \( \alpha \in (0, 2) \), \( r = \|z\|, \bar{z} = z/\|z\| \) and \( \sigma \) is a symmetric Radon measure on \( \partial B_1(0) \subseteq H \). The characteristic function has the special shape \( \phi_{L(t)}(u) = \exp(-tc_\alpha \|u\|^\alpha) \) for some \( c_\alpha > 0 \). This Lévy measure is selfsimilar in the sense that \( \nu(rA) = r^{-\alpha} \nu(A) \) for \( r > 0 \) and \( A \in B(H) \) with \( 0 \notin \bar{A} \).

In other words, it is regularly varying with index \( -\alpha \) and has the limiting measure \( \mu = \nu \). We take \( G(x, z) := \|x\|z \).
Then the $H$-valued mild solution of the Chafee-Infante equation with multiplicative $\alpha$-stable noise

$$dX^\varepsilon(t) = (\Delta X^\varepsilon(t) - a(X^\varepsilon(t))^{1\alpha}(t))dt + \varepsilon\|X^\varepsilon(t)\|dL(t)$$

has the following first exit times and locus behavior from a set $D = D^i \subseteq D^v$ with inward pointing vector field $\Delta + f$ in the sense of (2.7). For any $c > 0$ there are $\mathcal{R} > 1$ and $\varepsilon_0, \gamma \in (0, 1)$ such that $\varepsilon \in (0, \varepsilon_0]$ implies

$$\sup_{x \in D_2(\varepsilon^\gamma, \mathcal{R})} \mathbb{E}[\tau^\varepsilon_+ (\varepsilon, \mathcal{R})] \leq \frac{1}{\varepsilon^{\alpha} \nu(\frac{1}{\|\varepsilon\|} D^\varepsilon - \varphi)}[1 - c, 1 + c]$$

and for any $U \in \mathcal{B}(H)$ with $\nu(\frac{1}{\|\varepsilon\|} U - \varphi) > 0$ and $\nu(\frac{1}{\|\varepsilon\|} \partial U - \varphi) = 0$ we have

$$\sup_{x \in D_2(\varepsilon^\gamma, \mathcal{R})} \mathbb{P}(X^\varepsilon(\tau; x) \in U) \leq \frac{\nu(\frac{1}{\|\varepsilon\|} (U \cap D)^\varepsilon - \varphi)}{\nu(\frac{1}{\|\varepsilon\|} D^\varepsilon - \varphi)}[1 - c, 1 + c], \text{ where } \tau = \tau^\varepsilon_+ (\varepsilon, \mathcal{R}).$$

Our results also cover the exit times result of the additive case given in [18]. In addition, the continuous time Markov chain $M$ constructed in Corollary 2.8 is switching trivially between the states $\{\phi^+ \cup \phi^-\}$ and $(X^\varepsilon(\frac{\varepsilon}{2}, x))_{t \in [0, T]} \xrightarrow{d} (M_t)_{t \in [0, T]}$ in the sense of Corollary 2.8.

### 2.4.2 The linear heat equation perturbed by additive and multiplicative stable noise

We consider the linear heat equation on $H$ with Dirichlet conditions, that is, $f = 0$, since the eigenvalues of the Dirichlet-Laplacian are strictly negative with upper bound $-\Lambda_0$. The unit ball $D := B_1(0) \subseteq H$ is obviously positive invariant. As in the previous example we treat perturbations by a symmetric $\alpha$-stable process $(L(t))_{t \geq 0}$. We consider the multiplicative coefficient $G(x, z) = \langle x - v, z \rangle_v$ for some $0 \neq v \in H$, with $\|v\| = 1$ fixed. For $\varepsilon > 0, t \geq 0$ and $x \in D$ the equation

$$dX^\varepsilon(t) = \Delta X^\varepsilon(t)dt + \varepsilon\|X^\varepsilon(t) - v, dL(t)\|v$$

with $X^\varepsilon(0) = x,$ has the following first exit times and locus behavior as $\varepsilon \to 0$. We calculate the exit increments using the half space in $v$ direction, $\mathcal{H}(v) = \{z \in H | \langle z, v \rangle > 0\}$, as follows

$$J_{D^v}(0) = \{z \in H | \|z, v\| > 1\} = (\mathcal{H}(v) + v) \cup (-\mathcal{H}(v) - v).$$

Theorem 2.6 states that for any $c > 0$ we find $\varepsilon_0, \gamma \in (0, 1)$ such that $\varepsilon \in (0, \varepsilon_0]$ implies

$$\sup_{x \in B_{1-\varepsilon^\gamma}(0)} \mathbb{E}[\tau_x(\varepsilon)] \leq \frac{1}{\varepsilon^{\alpha} 2

\nu(\mathcal{H}(v) + v)}[1 - c, 1 + c]$$

and Theorem 2.7 yields for $U \in \mathcal{B}(H)$ with $\nu(J_{\mathcal{U} \cap D^v}(0)) > 0$ and $\nu(J_{\partial \mathcal{U} \cap D^v}(0)) = 0$

$$\sup_{x \in B_{1-\varepsilon^\gamma}(0)} \mathbb{P}(X^\varepsilon(\tau; x) \in U) \leq \frac{\nu(U \cap (\mathcal{H}(v) + v) \cup (-\mathcal{H}(v) - v))}{2 \nu(\mathcal{H}(v) + v)}[1 - c, 1 + c],$$

10
where $\tau = \tau_x(\varepsilon)$. Note that our first exit results also cover the additive case, which to our knowledge is also new in the literature for $G(x, z) = z$, with

$$
\sup_{x \in B_{1 - \varepsilon \gamma}(0)} \mathbb{E}[\tau_x(\varepsilon)] \in \frac{1}{\varepsilon \nu(D^\varepsilon)}[1 - c, 1 + c], \quad \text{and}
$$

$$
\sup_{x \in B_{1 - \varepsilon \gamma}(0)} \mathbb{P}(X^\varepsilon(\tau; x) \in U) \in \frac{\nu(U \cap D^\varepsilon)}{\nu(D^\varepsilon)}[1 - c, 1 + c], \quad \text{where } \tau = \tau_x(\varepsilon).
$$

### 3 Exponentially small deviations of the small noise solution

This section is devoted to a large deviations type estimate for the stochastic convolution between consecutive large jumps. It quantifies the fact that in the time interval strictly between two adjacent large jumps the solution of (2.9) is perturbed by only the small noise component and deviates from the solution of the deterministic equation by only a small $\varepsilon$-dependent quantity, with a probability converging to 1 exponentially fast as a function of the inverse noise intensity, $1/\varepsilon$, as $\varepsilon \to 0^+$. 

**Preliminaries and notation:** In this section we fix the domain of attraction $D = D^\phi$ of $\phi = \phi^\iota$ and the invariant subset $D = D^\phi$ with $\partial D^\varepsilon \subset \mathcal{C}^1$ such that $f$ is inward pointing on $\partial D$. We drop all further dependencies on the index $\iota$. For a better understanding of the role of the different scales we formulate and prove our results for abstract scale functions

$$
\rho^\phi : (0, 1] \to [1, \infty), \quad \lim_{\varepsilon \to 0^+} \rho^\varepsilon = \infty, \quad \lim_{\varepsilon \to 0^+} \varepsilon \rho^\varepsilon = 0,
$$

$$
\gamma : (0, 1] \to (0, 1], \quad \lim_{\varepsilon \to 0^+} \gamma = 0,
$$

$$
T : (0, 1] \to [1, \infty), \quad \lim_{\varepsilon \to 0^+} T^\varepsilon = \infty, \quad (3.1)
$$

before choosing them numerically in (3.31)-(3.34) below the statement of Proposition 3.3. We fix the following notation for abstract scales

$$
T_0 := 0, \quad T_k := \inf \{ t > T_{k-1} \mid \| \Delta t L \| > \rho^\varepsilon \} \quad \text{and} \quad W_k := \Delta T_k L, \quad k \geq 1. \quad (3.2)
$$

We define the compound Poisson process $\eta^\varepsilon$ which consists only of large jumps of $L$ with intensity $\beta^\varepsilon := \nu(B^\varepsilon_{\rho^\varepsilon}(0))$ and the jump probability distribution by $\mathbb{P}(W_k \in U) = \nu(U \cap B^\varepsilon_{\rho^\varepsilon}(0))/\beta^\varepsilon$. Then

$$
\eta^\varepsilon_t := \int_0^t \int_{\| z \| > \rho^\varepsilon} zN(ds dz) + \sum_{k=1}^{\infty} W_k \mathbf{1}_{(T_k < t)}, \quad t \geq 0.
$$

The complementary small jumps process $\xi^\varepsilon_t := L_t - \eta^\varepsilon_t$ has the following shape

$$
\xi^\varepsilon_t = \int_0^t \int_{\| z \| \leq 1} z\tilde{N}(ds dz) + \int_1^t \int_{\| z \| \leq \rho^\varepsilon} zN(ds dz) = \int_0^t \int_{\| z \| \leq \rho^\varepsilon} z\tilde{N}(ds dz) + \int_0^t \int_{1 < \| z \| \leq \rho^\varepsilon} z\nu(dz)ds.
$$
The process $\xi^\varepsilon$ has for any $\varepsilon \in (0,1]$ exponential moments of any order due to the uniformly bounded jump size and $\xi^\varepsilon - t \int_{\|z\| \leq \rho^\varepsilon} \nu(dz)$ is a mean zero $(\mathcal{F}_t)_{t \geq 0}$-martingale in $H$.

Denote by $(S(t))_{t \geq 0}$ the semigroup $(e^{t\Delta})_{t \geq 0}$ on $H$. It is a generalized contraction $C_0$-semigroup satisfying several regularization properties. We refer for our special setup to [18] p.13-14 and for general cases to [19].

The i.i.d. family of $\text{EXP}(\beta^\varepsilon)$-distributed waiting times between successive large jumps of $\eta^\varepsilon$ is given by $t_0 = 0$ and $t_k := T_k - T_{k-1}$ for $k \geq 1$. We denote the process $L$ between the waiting times by $\xi^\varepsilon,t_k(t) := L_{t + T_{k-1}} - L_{T_{k-1}}$ for $t \in [0, t_k)$. Then the i.i.d. families $(t_k)_{k \in \mathbb{N}}$, $(W_k)_{k \in \mathbb{N}}$, $(\xi^\varepsilon,t_k(t))_{t \in [0,t_k), k \in \mathbb{N}}$ are independent. We call $Y^\varepsilon(t, \xi^\varepsilon) = \Phi^\varepsilon_{t} = \Phi^\varepsilon_{t} \in \mathbb{R}$ in terms of $\Phi^\varepsilon_{t}$ of order $\varepsilon$ is very small, i.e. of order $\varepsilon_0^2$, for $\varepsilon \in (0, 1]$ with a probability which tends to 1 in terms of $\gamma_\varepsilon$ exponential fast as long as $Y^\varepsilon$ and the stochastic convolution remain bounded by $\mathcal{R}$. For the solution $Y^\varepsilon(t; x)$ of (3.3) with $\mathcal{R} \geq \mathcal{R}_0$, $\varepsilon \in (0,1]$ and $x \in D_2(\gamma_\varepsilon, \mathcal{R})$ we consider the multiplicative stochastic convolution process

\[
\Psi^\varepsilon_{t} := \int_{0}^{t} S(t - s) G(Y^\varepsilon(s^-; x), \varepsilon \xi^\varepsilon(s)) \, ds \in \mathbb{R}^{q+x}
\]

\[
= \int_{0}^{t} \int_{0 < \|z\| \leq \rho^\varepsilon} S(t - s) G(Y^\varepsilon(s^-; x), \varepsilon z) \tilde{N}(dsdz) + \int_{0}^{t} \int_{1 < \|z\| \leq \rho^\varepsilon} S(t - s) G(Y^\varepsilon(s; x), \varepsilon z) \nu(dz) \, ds
\]

\[
=: \Phi^\varepsilon_{t} + b^\varepsilon_{t-x}.
\]

The process $(\Psi^\varepsilon_{t})_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$-adapted càdlàg process. For $\mathcal{R} \geq \mathcal{R}_0$, $\varepsilon \in (0,1]$ and $x \in D_2(\gamma_\varepsilon, \mathcal{R})$ we set

\[
\sigma^2 := \sigma^2_{\mathcal{R}, x}(\varepsilon) := \inf \{ t > 0 \mid \Psi^\varepsilon_{t} \notin \mathcal{U}^{\mathcal{R}} \} \quad \text{and} \quad \sigma := \sigma^1 \wedge \sigma^2.
\]

3.1 Exponential estimate of stochastic convolutions with bounded jumps

In this subsection we show that the stochastic convolution with respect to $G(Y^\varepsilon(s^-; x), \varepsilon \xi^\varepsilon(s))$ for small $\varepsilon > 0$ is very small, i.e. of order $\leq \gamma_\varepsilon^{2}$ for some $q \geq 1$, with a probability which tends to 1 in terms of $\gamma_\varepsilon$ exponentially fast as long as $Y^\varepsilon$ and the stochastic convolution remain bounded by $\mathcal{R}$. For the solution $Y^\varepsilon(t; x)$ of (3.3) with $\mathcal{R} \geq \mathcal{R}_0$, $\varepsilon \in (0,1]$ and $x \in D_2(\gamma_\varepsilon, \mathcal{R})$ we consider the multiplicative stochastic convolution process

\[
\Psi^\varepsilon_{t} := \int_{0}^{t} S(t - s) G(Y^\varepsilon(s^-; x), \varepsilon \xi^\varepsilon(s)) \, ds \in \mathbb{R}^{q+x}
\]

\[
= \int_{0}^{t} \int_{0 < \|z\| \leq \rho^\varepsilon} S(t - s) G(Y^\varepsilon(s^-; x), \varepsilon z) \tilde{N}(dsdz) + \int_{0}^{t} \int_{1 < \|z\| \leq \rho^\varepsilon} S(t - s) G(Y^\varepsilon(s; x), \varepsilon z) \nu(dz) \, ds
\]

\[
=: \Phi^\varepsilon_{t} + b^\varepsilon_{t-x}.
\]

The process $(\Psi^\varepsilon_{t})_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$-adapted càdlàg process. For $\mathcal{R} \geq \mathcal{R}_0$, $\varepsilon \in (0,1]$ and $x \in D_2(\gamma_\varepsilon, \mathcal{R})$ we set

\[
\sigma^2 := \sigma^2_{\mathcal{R}, x}(\varepsilon) := \inf \{ t > 0 \mid \Psi^\varepsilon_{t} \notin \mathcal{U}^{\mathcal{R}} \} \quad \text{and} \quad \sigma := \sigma^1 \wedge \sigma^2.
\]
Proposition 3.1. Let the Hypotheses (D.1)-(D.3), (S.1) and (S.2) be satisfied and the functions $\rho, \gamma$ be given by (7.1) and fulfill for some $q \geq 1$ the limit relation
\[
\lim_{\varepsilon \to 0} \Gamma(\varepsilon) = 0 \quad \text{where} \quad \Gamma(\varepsilon) := \frac{(\varepsilon \rho)^{2T\varepsilon}}{\gamma^{4q+6}}.
\]

Then for any $R \geq R_0$ there is $\varepsilon_0 \in (0, 1]$ such that $\varepsilon \in (0, \varepsilon_0]$ implies
\[
\sup_{x \in D_2(\gamma, \varepsilon), \varepsilon} \mathbb{P}\left( \sup_{s \in [0, \sigma \wedge T^\varepsilon]} \|\Phi^\varepsilon_{s,x}\| > \gamma^q\right) \leq 2 \exp(-2(\gamma^q)^{-1}).
\]

Proof. The plan of the proof is as follows. First we get rid of the drift $b^\varepsilon_{x}$ (Step 0). In order to control $\Phi^\varepsilon_{x}$ we start with the exponential Kolmogorov inequality where we introduce the free parameters $\lambda$ and $\chi$. We estimate the stochastic convolution with the help of a result of Salavati and Zangeneh [52] and derive an exponential version of the Burkholder-Davis-Gundy inequality using Siorpaes [51] (Step 1). Then we optimize over the free parameters and use the Campbell representation of the Laplace transform of the quadratic variation of Poisson random integrals and a Campbell type estimate shown in Lemma [92]. This allows for a comparison principle for the characteristic exponent of $\Psi^\varepsilon_{x}$ (Step 2) and allows to conclude (Step 3).

Step 0: Drift estimate. We show that for any $R \geq R_0$ there is $\varepsilon_0 \in (0, 1]$ such that $\varepsilon \in (0, \varepsilon_0]$ implies
\[
\sup_{t \in [0, \sigma \wedge T^\varepsilon]} \sup_{x \in D_2(\gamma, \varepsilon)} \|b^\varepsilon_{t,x}\| < \frac{1}{2} \gamma^q.
\]

We write $Y(t) = Y^\varepsilon(t, x)$ and recall Hypothesis (S.1). For $g_1 := \sup_{x \in U^\gamma} G_1(x)$ the triangular inequality and the norm estimate of the heat semigroup $S$ yield for $x \in D_2(\gamma, \varepsilon)$
\[
\begin{align*}
\sup_{t \in [0, \sigma]} \|b^\varepsilon_{t,x}\| & \leq \sup_{t \in [0, \sigma]} \left\| \int_0^t S(t-s) G(Y(s^-), \varepsilon z) \nu(dz) ds \right\| \\
& \leq \sup_{t \in [0, \sigma]} \int_0^t \left\| S(t-s) G(Y(s^-), \varepsilon z) \| \nu(dz) ds \right\|
\end{align*}
\]
\[
\begin{align*}
& \leq \left( g_1 \sup_{t \in [0, \sigma]} \int_0^t e^{-\Lambda_0(t-s)} \| \varepsilon z \| \nu(dz) ds \right) \\
& \leq \varepsilon g_1 \left( g_1 \frac{\nu(B_1^\gamma(0))}{\Lambda_0} \right) e^{t/2} \leq C_0 \varepsilon \rho^\varepsilon.
\end{align*}
\]

Note that the right side is independent $\sigma$ and $x$. The limit (3.6) yields the existence of a constant $\varepsilon_0 \in (0, 1]$ such that for $\varepsilon \in (0, \varepsilon_0]$ we have $\varepsilon \rho^\varepsilon \leq \gamma^q/2C_0$ and hence satisfies the claim.
Step 1: Exponential estimate of the stochastic convolution: Note that for \( \varepsilon_0 \) of Step 0 and \( \varepsilon \in (0, \varepsilon_0) \) we get

\[
\mathbb{P}(\sup_{t \in [0, \sigma \wedge T^*]} ||\Phi_{t,x}^\varepsilon|| > \frac{\gamma^q}{2}) \leq \mathbb{P}(\sup_{t \in [0, \sigma \wedge T^*]} ||\Phi_{t,x}^\varepsilon|| > \frac{\gamma^q}{2} + \mathbb{P}(\sup_{t \in [0, \sigma \wedge T^*]} ||b_{t,x}^\varepsilon|| > \frac{\gamma^q}{2})
\]

Kolmogorov’s exponential inequality yields with a free parameter \( \vartheta > 0 \) reads

\[
\mathbb{P}(\sup_{t \in [0, \sigma \wedge T^*]} ||\Phi_{t,x}^\varepsilon|| > \frac{\gamma^q}{2}) \leq \exp(-\vartheta \frac{\gamma^q}{2}) \mathbb{E}\left[ \exp(\vartheta \sup_{t \in [0, \sigma \wedge T^*]} ||\Phi_{t,x}^\varepsilon||) \right]. \tag{3.8}
\]

For \( M^{(1)}_t := \int_0^t \int_{0<\|z\|\leq \rho^*} G(Y(s-x), z) \tilde{N}(dsdz) \) the process \( (M^{(1)}_{t\wedge \sigma})_{t \geq 0} \) is an \( (\mathcal{F}_t)_{t \geq 0} \)-martingale. The pathwise estimate of the stochastic convolution in Salavati and Zangeneh \[52\] (Theorem 6) yields for any \( t \geq 0 \) the following \( \mathbb{P} \)-a.s. inequality \((\Phi_{0,x}^\varepsilon = 0)\)

\[
||\Phi_{t,x}^\varepsilon||^2 \leq 2 \int_0^t e^{-2\Lambda_0(s-t)} \langle \Phi_{s-}^\varepsilon, dM^{(1)}_s \rangle + \sum_{0<s\leq t} e^{-2\Lambda_0(s-t)}(||\Phi_{s-}^\varepsilon||^2 - ||\Delta_s\Phi_{s-}^\varepsilon||^2) - 2\langle \Phi_{s-}^\varepsilon, \Delta_s M^{(1)}_s \rangle
\]

\[
= e^{-2\Lambda_0(t)} \left( 2 \int_0^t \int_{\|z\|\leq \rho^*} e^{2\Lambda_0(s)} \langle \Phi_{s-}^\varepsilon, G(Y(s-), z) \rangle \tilde{N}(dsdz) + \int_0^t \int_{\|z\|\leq \rho^*} e^{2\Lambda_0(s)} ||G(Y(s-), z)||^2 N(dsdz) \right)
\]

\[
\leq e^{-2\Lambda_0(t)} (|M^{(2)}_t| + M^{(3)}_t),
\]

where

\[
M^{(2)}_t := 2 \int_0^t \int_{\|z\|\leq \rho^*} e^{2\Lambda_0(s)} \langle \Phi_{s-}^\varepsilon, G(Y(s-), z) \rangle \tilde{N}(dsdz),
\]

\[
M^{(3)}_t := \int_0^t \int_{\|z\|\leq \rho^*} e^{2\Lambda_0(s)} ||G(Y(s-), z)||^2 N(dsdz).
\]

Therefore

\[
\sup_{t \in [0, \sigma \wedge T^*]} ||\Phi_{t,x}^\varepsilon||^2 \leq \sup_{t \in [0, \sigma \wedge T^*]} e^{-2\Lambda_0(t)} |M^{(2)}_t| + \sup_{t \in [0, \sigma \wedge T^*]} e^{-2\Lambda_0(t)} M^{(3)}_t. \tag{3.9}
\]

**Step 1a:** We start with the first term in \((3.7)\). Itô’s formula applied to \( e^{-2\Lambda_0(t)} M^{(2)}_t \) gives

\[
e^{-2\Lambda_0(t)} M^{(2)}_t = \int_0^t e^{-2\Lambda_0(s)} dM^{(2)}_s - 2\Lambda_0 \int_0^t (e^{-2\Lambda_0(s)} M^{(2)}_s) ds. \tag{3.10}
\]
The preceding identity (3.10) defines the following recursion. We replace the expression $e^{-2\Lambda_0 s}M_s^{(2)}$ under the integral by the respective right-hand side

$$e^{-2\Lambda_0 t}M_t^{(2)} = \int_0^t e^{-2\Lambda_0 s_1}dM_{s_1}^{(2)} - 2\Lambda_0 \int_0^t (e^{-2\Lambda_0 s_1}M_{s_1}^{(2)})ds_1$$

$$= \int_0^t e^{-2\Lambda_0 s_1}dM_{s_1}^{(2)} + (2\Lambda_0) \int_0^t \left( \int_0^{s_1} e^{-2\Lambda_0 s_2}dM_{s_2}^{(2)} - 2\Lambda_0 \int_0^{s_1} e^{-2\Lambda_0 s_2}M_{s_2}^{(2)}ds_2 \right)ds_1$$

$$= \int_0^t e^{-2\Lambda_0 s_1}dM_{s_1}^{(2)} + (2\Lambda_0) \int_0^t e^{-2\Lambda_0 s_2}dM_{s_2}^{(2)}ds_1 + (2\Lambda_0)^2 \int_0^t e^{-2\Lambda_0 s_2}M_{s_2}^{(2)}ds_2ds_1.$$  

Repeating this procedure $k + 1$ times an easy induction shows that for all $k \in \mathbb{N}$

$$e^{-2\Lambda_0 t}M_t^{(2)} = \sum_{\ell=0}^{k} \int_0^t \int_0^{s_1} \ldots \int_0^{s_\ell} \left( \int_0^{s_\ell} e^{-2\Lambda_0 s_{\ell+1}}dM_{s_{\ell+1}}^{(2)} \right)ds_{\ell+1} \ldots ds_1$$

$$+ (2\Lambda_0)^{k+1} \int_0^t \int_0^{s_1} \ldots \int_0^{s_k} e^{-2\Lambda_0 s_{k+1}}M_{s_{k+1}}^{(2)}ds_{k+1} \ldots ds_1.$$  

Cauchy’s formula for repeated integrals applied to the second term provides

$$\left| (-2\Lambda_0)^{k+1} \int_0^t \int_0^{s_1} \ldots \int_0^{s_k} e^{-2\Lambda_0 s_{k+1}}M_{s_{k+1}}^{(2)}ds_{k+1} \ldots ds_1 \right| = \left| (-2\Lambda_0) \int_0^t \frac{((-2\Lambda_0)(t-s))^{k}}{k!} e^{-2\Lambda s}M_s^{(2)}ds \right|$$

$$\leq 2\Lambda_0 \int_0^t \frac{2\Lambda_0(t-s)^{k}}{k!} ds \sup_{s \in [0,t]} e^{-2\Lambda s}|M_s^{(2)}|.$$  

Since $\sup_{s \in [0,t]} e^{-2\Lambda s}|M_s^{(2)}| < \infty \text{ P-a.s.}$ we may pass to the limit as $k \to \infty$ and obtain with the help of the monotone convergence theorem that

$$\limsup_{k \to \infty} \left| (-2\Lambda_0)^{k+1} \int_0^t \int_0^{s_1} \ldots \int_0^{s_k} e^{-2\Lambda_0 s_{k+1}}M_{s_{k+1}}^{(2)}ds_{k+1} \ldots ds_1 \right|$$

$$\leq 2\Lambda_0 \limsup_{k \to \infty} \int_0^t \frac{(2\Lambda_0(t-s))^{k-1}}{(k-1)!} ds \sup_{s \in [0,t]} e^{-2\Lambda s}|M_s^{(2)}| = 0, \text{ P-a.s.} \quad (3.11)$$

Hence we have proved the representation

$$e^{-2\Lambda_0 t}M_t^{(2)} = \sum_{\ell=1}^{\infty} \int_0^t \int_0^{s_1} \ldots \int_0^{s_{\ell-1}} \int_0^{s_\ell} (-2\Lambda_0)^{\ell} e^{-2\Lambda_0 s_{\ell+1}}dM_{s_{\ell+1}}^{(2)}ds_{\ell+1} \ldots ds_1, \text{ P-a.s.}$$
For each of the summands we apply Cauchy’s formula for repeated integrals

\[
\int_0^t \cdots \int_0^{s_{t-1}} \int_0^{s_{t-2}} \cdots \int_0^{s_1} (-2\Lambda_0)^\ell e^{-2\Lambda_0s_{\ell+1}} dM^{(2)}_{s_{\ell+1}} ds_\ell \cdots ds_1
\]

\[
= (-2\Lambda_0) \int_0^t \frac{(-2\Lambda_0(t - s))^{\ell-1}}{\ell!} \int_0^s e^{-2\Lambda_0r} dM^{(2)}_r ds.
\]

Fubini’s theorem yields \(\mathbb{P}\)-a.s.

\[
\sum_{\ell=1}^\infty \int_0^t (-2\Lambda_0(t - s))^{\ell-1} (\ell - 1)! \int_0^s e^{-2\Lambda_0r} dM^{(2)}_r ds = \int_0^t \sum_{\ell=1}^\infty (-2\Lambda_0(t - s))^{\ell-1} (\ell - 1)! \int_0^s e^{-2\Lambda_0r} dM^{(2)}_r ds = \int_0^t e^{-2\Lambda_0(t - s)} \int_0^s e^{-2\Lambda_0r} dM^{(2)}_r ds.
\]

We estimate

\[
e^{-2\Lambda_0t} |M^{(2)}_t| \leq |(-2\Lambda_0) \int_0^t e^{-2\Lambda_0(t - s)} \int_0^s e^{-2\Lambda_0r} dM^{(2)}_r ds| \\
\leq |(-2\Lambda_0) \int_0^t e^{-2\Lambda_0(t - s)} ds| \sup_{s \in [0,t]} |M^{(4)}_s| \leq \sup_{s \in [0,t]} |M^{(4)}_s|,
\]

(3.12)

where

\[
\int_0^s e^{-2\Lambda_0r} dM^{(2)}_r = 2 \int_0^t \int_0^\|z\| \leq \rho \langle \Phi_{\epsilon,x}^{s-}, G(Y(s-), z) \rangle \tilde{N}(dsdz) =: M^{(4)}_t.
\]

Since the right-hand side of (3.12) is nondecreasing the first term in (3.9) has the upper bound

\[
\sup_{t \in [0,\sigma \wedge T\epsilon]} e^{-2\Lambda_0t} |M^{(2)}_t| \leq \sup_{t \in [0,\sigma \wedge T\epsilon]} |M^{(4)}_t|.
\]

(3.13)

We continue with the pathwise Burkholder-Davis-Gundy inequality by Siorpaes [51]

\[
\sup_{t \in [0,\sigma \wedge T\epsilon]} |M^{(4)}_t| \leq 6 \sqrt{|M^{(4)}_\sigma |_{\sigma \wedge T\epsilon} + 2 \int_0^{\sigma \wedge T\epsilon} H_s dM^{(4)}_s},
\]

(3.14)

where

\[
H_s := M^{(4)}_s / \sqrt{|M^{(4)}_s| + \sup_{r \in [0,s]} |M^{(4)}_r|^2}.
\]

(3.15)

Note that \(\sup_{s \geq 0}|H_s| \leq 1 \mathbb{P}\)-a.s. by construction.
Step 1b: The second term in (3.9) is easier since the nonnegative integrands allow for a \( \mathbb{P} \)-a.s. monotonicity estimate

\[
\sup_{t \in [0, \sigma \land T^*]} e^{-2\Lambda_0 t} M_t^{(3)} \leq \sup_{t \in [0, \sigma \land T^*]} \int_0^t \int_0^{\sigma \land T^*} \|G(Y(s-), z)\|^2 N(dsdz) \\
= \int_0^{\sigma \land T^*} \|G(Y(s-), z)\|^2 N(dsdz) =: M_{\sigma \land T^*}^{(5)}. \quad (3.16)
\]

Step 1c: We combine (3.9), \( (3.13), (3.14) \) and \( (3.16) \). The subadditivity of the square root and the estimate \( \sqrt{r} \leq r + \frac{1}{4} \) for \( r \geq 0 \) yield

\[
\mathbb{E}\left[ \exp(\theta \sup_{t \in [0, \sigma \land T^*]} \|\Phi_t^{\epsilon, z}\|) \right] \\
\leq \mathbb{E}\left[ \exp \left( \theta \sqrt{6[M^{(4)}]_{\sigma \land T^*} + 2 \int_0^{\sigma \land T^*} H_s dM_s^{(4)} + M_{\sigma \land T^*}^{(5)}} \right) \right] \\
\leq e^{\frac{\theta}{4}} \mathbb{E}\left[ \exp \left( \frac{3\theta^2}{\chi^2} [M^{(4)}]_{\sigma \land T^*} + \frac{3\theta^2 \chi^2}{2} + 2\theta^2 \left( \int_0^{\sigma \land T^*} H_s dM_s^{(4)} \right) + \theta^2 M_{\sigma \land T^*}^{(5)} \right) \right].
\]

Young’s inequality with the additional free parameter \( \chi > 0 \) reads

\[
\sqrt{[M^{(4)}]_{\sigma \land T^*}} \leq \frac{1}{2\chi^2} [M^{(4)}]_{\sigma \land T^*} + \frac{\chi^2}{2}.
\]

Then the estimate \( abcd \leq (a^4 + b^4 + c^4 + d^4)/4 \), \( a, b, c, d > 0 \), provides for (3.17) the upper bound

\[
e^{\frac{\theta}{4}} \mathbb{E}\left[ \exp \left( \frac{3\theta^2}{\chi^2} [M^{(4)}]_{\sigma \land T^*} + \frac{3\theta^2 \chi^2}{2} + 2\theta^2 \left( \int_0^{\sigma \land T^*} H_s dM_s^{(4)} \right) + \theta^2 M_{\sigma \land T^*}^{(5)} \right) \right] \\
\leq e^{\frac{\theta}{4}} \mathbb{E}\left[ \exp \left( \frac{12\theta^2}{\chi^2} [M^{(4)}]_{\sigma \land T^*} \right) \right] + e^{\frac{\theta}{4}} \mathbb{E}\left[ \exp \left( 6\theta^2 \chi^2 \right) \right] \\
+ e^{\frac{\theta}{4}} \mathbb{E}\left[ \exp \left( 8\theta^2 \left( \int_0^{\sigma \land T^*} H_s dM_s^{(4)} \right) \right) \right] + e^{\frac{\theta}{4}} \mathbb{E}\left[ \exp \left( 4\theta^2 M_{\sigma \land T^*}^{(5)} \right) \right] \\
=: J_1(\theta, \chi) + J_2(\theta, \chi) + J_3(\theta) + J_4(\theta). \quad (3.18)
\]

Step 2: Campbell’s formula and Optimization over the free parameters. We now choose the free parameters \( \theta \) and \( \chi \) as \( \varepsilon \)-dependent functions \( \theta_\varepsilon := \frac{1}{\gamma_\varepsilon^2} \) and \( \chi_\varepsilon = \gamma_\varepsilon^{g+2} \) such that the upper bound (3.18) of the right side of (3.8) reads

\[
\exp(-\theta_\varepsilon \frac{1}{2} \gamma_\varepsilon^2) \left( J_1(\theta_\varepsilon, \chi_\varepsilon) + J_2(\theta_\varepsilon, \chi_\varepsilon) + J_3(\theta_\varepsilon) + J_4(\theta_\varepsilon) \right).
\]

In the sequel we estimate the respective terms \( J_1, \ldots, J_4 \) one by one.
\[ J_1(\theta_\varepsilon, \chi_\varepsilon): \quad \text{By } \mathbb{P}\text{-a.s. monotonicity we have} \]

\[
[M^{(4)}]_{\sigma \wedge T^\varepsilon} = \left[ 4 \int_0^T \int \langle \Phi_{s-}^{\chi_\varepsilon}, G(Y(s-), \varepsilon) \rangle \tilde{N}(dsdz) \right]_{\sigma \wedge T^\varepsilon}
\]

\[
= 16 \int_0^T \int \langle \Phi_{s-}^{\chi_\varepsilon}, G(Y(s-), \varepsilon) \rangle^2 \tilde{N}(dsdz)
\]

\[
\leq C_1 \int_0^T \int \| \Phi_{s-}^{\chi_\varepsilon} \|^2 \| \varepsilon \|^2 \tilde{N}(dsdz)
\]

\[
\leq C_2 \int_0^T \int \| \varepsilon \|^2 \tilde{N}(dsdz) \leq C_2 \int_0^T \int \| \varepsilon \|^2 \tilde{N}(dsdz).
\]

The classical Campbell formula for the Poisson random measure \( \tilde{N} \) has the shape

\[
\mathbb{E} \left[ \exp \left( 12 \frac{\partial_\varepsilon}{\lambda_\varepsilon} [M^{(4)}]_{\sigma \wedge T^\varepsilon} \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{C_2}{\gamma_\varepsilon^{4q+6}} \int_0^T \int \| \varepsilon \|^2 \tilde{N}(dsdz) \right) \right]
\]

\[
= \mathbb{E} \left[ \exp \left( T^\varepsilon \int_0^T \left( \exp \left( \frac{C_3}{\gamma_\varepsilon^{4q+6}} \right) - 1 \right) \nu(z) \right) \right]. \quad (3.19)
\]

The limit (3.6) implies for the exponent

\[
\sup_{\| z \| \leq \rho^\varepsilon} \frac{\| \varepsilon \|^2}{\gamma_\varepsilon^{4q+6}} \leq \left( \frac{\varepsilon \rho^\varepsilon}{\gamma_\varepsilon^{4q+6}} \right)^2, \quad \text{as } \varepsilon \to 0.
\]

In the sequel we use \((e^r - 1) \leq (e - 1)r \) for all \( r \in [0, 1] \) under the integral in the exponent of equation (3.19) and choose \( \varepsilon_0 \in (0, 1) \) small enough such that \( \varepsilon \in (0, \varepsilon_0) \) implies \( \frac{C_2}{\gamma_\varepsilon^{4q+6}} \leq 1 \). If, in addition, \( \rho^\varepsilon \geq 1 \) for any \( \varepsilon \in (0, \varepsilon_0) \) we obtain the following upper bound of (3.19)

\[
\mathbb{E} \left[ \exp \left( C_3(e - 1) \frac{e^2 T^\varepsilon}{\gamma_\varepsilon^{4q+6}} \int_{\| z \| \leq \rho^\varepsilon} \| z \|^2 \nu(z) \right) \right]
\]

\[
\leq \mathbb{E} \left[ \exp \left( C_3(e - 1) \frac{e^2 T^\varepsilon}{\gamma_\varepsilon^{4q+6}} \left( \int_{\| z \| \leq 1} \| z \|^2 \nu(z) + (\rho^\varepsilon)^2 \nu(B_\varepsilon(0)) \right) \right) \right] \leq \exp \left( C_4 \frac{(e \rho^\varepsilon)^2 T^\varepsilon}{\gamma_\varepsilon^{4q+6}} \right),
\]

where \( C_4 = C_3(e - 1) \left( \int_{\| z \| \leq 1} \| z \|^2 \nu(z) + \nu(B_\varepsilon(0)) \right) \). Thus \( \varepsilon \in (0, \varepsilon_0) \) yields

\[
\exp(-\frac{\partial_\varepsilon}{2\gamma_\varepsilon^q}) J_1(\theta_\varepsilon, \chi_\varepsilon) = \exp(-\frac{1}{2\gamma_\varepsilon}) J_1(\theta_\varepsilon, \chi_\varepsilon) \leq \frac{C_4^4}{4} \exp(-\frac{1}{2\gamma_\varepsilon}) \exp(C_4 \Gamma(\varepsilon)) \leq \frac{1}{2} \exp(-\frac{1}{2\gamma_\varepsilon}). \quad (3.20)
\]
\( J_2(\vartheta, \chi) \): There is \( \varepsilon_0 \in (0, 1] \) such that for \( \varepsilon \in (0, \varepsilon_0) \) we have the estimate
\[
\exp(-\frac{\vartheta}{2} \gamma_{\varepsilon}^2) J_2(\vartheta, \chi) \leq \frac{e^{\frac{1}{4}}}{4} \exp(-\frac{1}{2} \gamma_{\varepsilon} + 6(\vartheta \chi_{\varepsilon})^2) = \frac{e^{\frac{1}{4}}}{4} \exp(-\frac{1}{2} \gamma_{\varepsilon} + 6(\vartheta \chi_{\varepsilon})^2) \leq \frac{1}{2} \exp(-\frac{1}{2} \gamma_{\varepsilon}).
\] (3.21)

\( J_3(\vartheta) \): Recall that
\[ M^{(4)}_t = 2 \int_0^t \int \langle \Phi^{x,x}_{\varepsilon}, G(Y(s-), \varepsilon z) \rangle \tilde{N}(dsdz) \]
such that for the function \( h_y(s-, \varepsilon z) := 2H_s\langle \langle \Phi^{x,x}_{\varepsilon}, G(Y(s-; x), \varepsilon z) \rangle \rangle \) we have the representation
\[ Z^{x,x}_t := \int_0^t \int_{\|z\| \leq \rho} h_y(s-, \varepsilon z) \tilde{N}(dsdz) \]
leading to
\[ \mathbb{E} \left[ \exp \left( 8 \theta_{\varepsilon}^2 |Z^{x,x}_{\sigma^\wedge T^\varepsilon}| \right) \right] \leq 2. \]

This implies for \( \varepsilon \in (0, \varepsilon_0] \) the estimate
\[ \exp(-\vartheta \frac{1}{4} \gamma_{\varepsilon}^2) J_3(\vartheta) \leq \frac{2}{4e^{\frac{1}{4}}} \exp(-\frac{1}{2} \gamma_{\varepsilon}) \leq \frac{1}{2} \exp(-\frac{1}{2} \gamma_{\varepsilon}). \] (3.23)

\( J_4(\vartheta) \): This case resembles the one of \( J_1(\vartheta, \chi) \). Since we have only positive jumps we estimate \( \mathbb{P}\)-a.s.
\[ M^{(3)}_{\sigma^\wedge T^\varepsilon} = \int_0^{\sigma^\wedge T^\varepsilon} \|G(Y(s-), \varepsilon z)\|^2 N(dsdz) \leq \int_0^{\sigma^\wedge T^\varepsilon} g_1^2 \|\varepsilon z\|^2 N(dsdz), \]
leading to
\[ \mathbb{E} \left[ \exp \left( 2 \theta_{\varepsilon}^2 M^{(3)}_{\sigma^\wedge T^\varepsilon} \right) \right] \leq \mathbb{E} \left[ \exp \left( T^\varepsilon \int_{\|z\| \leq \rho^\varepsilon} \left( \exp(2g_1 \theta_{\varepsilon}^2 \|\varepsilon z\|^2) - 1 \right) \nu(dz) \right) \right]. \]

Analogously to \( J_1(\vartheta, \chi) \) we obtain with the help of the limit (3.6) that for \( \varepsilon \to 0^+ \)
\[ \sup_{\|z\| \leq \rho^\varepsilon} 2g_1 \theta_{\varepsilon}^2 \|\varepsilon z\|^2 \leq 2g_1 \theta_{\varepsilon}^2 (\varepsilon \rho^\varepsilon)^2 \leq 2g_1 \frac{(\varepsilon \rho^\varepsilon)^2}{\gamma_{\varepsilon}} \to 0. \]
The additional restriction of \( \varepsilon_0 \in (0, 1) \) such that \( \varepsilon \in (0, \varepsilon_0] \) simultaneously implies \( 2g_1 \frac{(\rho^r)^2}{\varepsilon^2} \leq 1 \) and \( \rho^r \geq \int_{\|z\| \leq 1} \|z\|^2 \nu(dz)/\nu(B_1(0)) \) yields the estimate

\[
J_4(\vartheta_\varepsilon) \leq \frac{e^{\frac{1}{4}}}{4} \mathbb{E}\left[ \exp \left( \int_0^{\sigma \wedge T^r} \left( \exp \left( 2g_1 C_0 T^r \vartheta_\varepsilon^2 (\varepsilon \rho^r)^2 \right) - 1 \right) \nu(dz)ds \right) \right]
\]

\[
\leq \frac{e^{\frac{1}{4}}}{4} \mathbb{E}\left[ \exp \left( (e - 1) 2g_1 C_0 T^r \vartheta_\varepsilon^2 (\varepsilon \rho^r)^2 \right) \right]
\]

\[
\leq \frac{1}{2} \exp \left( C_7 \frac{(\varepsilon \rho^r)^2 T^r}{\sigma^2} \right).
\]

This implies for \( \varepsilon_0 \in (0, 1] \) sufficiently small and any \( \varepsilon \in (0, \varepsilon_0] \) the estimate

\[
\exp\left( -\vartheta_\varepsilon \frac{1}{2} \gamma_0^2 \right) J_4(\lambda_\varepsilon) \leq \frac{1}{2} \exp\left( -\frac{1}{2} \gamma_0 \right).
\]  
(3.24)

**Step 3:** For \( \varepsilon_0 \in (0, 1] \) sufficiently small such that (3.20) - (3.24) are satisfied we conclude for \( \varepsilon \in (0, \varepsilon_0] \) that

\[
P\left( \sup_{t \in [0, \sigma \wedge T^r]} \|\Psi^\varepsilon_{t,x}\| > \gamma_0^2 \right) \leq \exp\left( -\vartheta_\varepsilon \frac{1}{2} \gamma_0^2 \right) \mathbb{E}\left[ \exp\left( \vartheta_\varepsilon \sup_{t \in \sigma \wedge T^r} \|\Phi^\varepsilon_{t,x}\| \right) \right] \leq 2 \exp\left( -\frac{1}{2} \gamma_0 \right).
\]

Note that our estimates are uniformly over all \( x \in D_2(\gamma_\varepsilon, \mathbb{R}) \). This finishes the proof. \( \square \)

**Lemma 3.2** (An asymptotic Campbell type estimate). Under the hypothesis of Proposition and the notation of Step 2 of the proof of Proposition 3.1 the process \( Z^\varepsilon_{t,x} = (Z^\varepsilon_{t,x})_{t \geq 0} \) given in (3.24) satisfies the following. There is \( \varepsilon_0 \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) implies

\[
\sup_{x \in D_2(\gamma_\varepsilon, \mathbb{R})} \mathbb{E}\left[ \exp\left( 8\vartheta_\varepsilon \|Z^\varepsilon_{\sigma \wedge T^r}\| \right) \right] \leq 2.
\]

The proof is found in Appendix B.5.2

### 3.2 Exponential estimates of the deviations of the small jump equation

For \( \varepsilon, \gamma \in (0, 1], x \in H, T \geq 0 \) we define the events

\[
\mathcal{E}_{x,T}(\gamma, \varepsilon) := \{ \sup_{s \in [0, T]} \|\Psi^\varepsilon_{s,x}\| \leq \gamma \}, \quad \mathcal{E}_{x,T}^\gamma(\gamma, \varepsilon) := \{ \sup_{s \in [0, T]} \|\Psi^\varepsilon_{s,x}\| \leq \gamma \},
\]  
(3.25)

\[
\mathcal{G}_{x,T}(\gamma, \varepsilon) := \{ \sup_{s \in [0, T]} \|Y^\varepsilon_{s,x} - u(s;x)\| \leq \gamma \}, \quad \mathcal{G}_{x,T}^\gamma(\gamma, \varepsilon) := \{ \sup_{s \in [0, T]} \|Y^\varepsilon_{s,x} - u(s;x)\| \leq \gamma \},
\]  
(3.26)

\[
\mathcal{G}_{E,T}(\gamma, \varepsilon) := \{ \sup_{s \in [0, T]} \|Y^\varepsilon_{s,x} - u(s;x)\| \leq \gamma \}, \quad \mathcal{G}_{E,T}^\gamma(\gamma, \varepsilon) := \{ \sup_{s \in [0, T]} \|Y^\varepsilon_{s,x} - u(s;x)\| \leq \gamma \}.
\]  
(3.27)

We suppress the overall dependence on \( \varepsilon \in (0, 1] \). This subsection is dedicated to the proof of the following estimate used in the proof of the main result.

\[
20
\]
Proposition 3.3. Let the Hypotheses (D.1)-(D.3), (S.1) and (S.2) be satisfied (for fixed $\iota$). Furthermore let the functions $\gamma, \rho, T$ be given by (3.1) and $\lambda = \lambda^\iota$ be defined in (2.10). Then there exists a constant $q \geq 1$ such that if $\gamma, \rho, T$ satisfy condition (3.6) for this value of $q$ and additionally
\[
\lim_{\varepsilon \to 0^+} \beta_\varepsilon T^\varepsilon = \infty, \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \lambda_\varepsilon / \beta_\varepsilon = 0 \quad (3.29)
\]
we have the following. For any $R \geq R_0$ and $\theta \in (0, 1)$ there is a constant $\varepsilon_0 \in (0, 1]$ such that $\varepsilon \in (0, \varepsilon_0]$ implies
\[
\sup_{x \in D_2(\gamma_\varepsilon, R)} \mathbb{E} \left[ e^{\beta_\varepsilon T_\varepsilon} \mathbf{1} \left( G_x^\varepsilon \left( \frac{1}{2} \gamma_\varepsilon \right) \right) \right] \leq 2 \exp \left( -\frac{1}{2 \gamma_\varepsilon} \right) + 2 \exp \left( -\frac{\beta_\varepsilon T_\varepsilon}{2} \right). \quad (3.30)
\]

(C) Choice of the scales:

1. For any $\alpha > 0$ and $q \geq 1$ fixed the scales satisfying (3.1) and (3.29) are chosen as follows
\[
\gamma_\varepsilon := \varepsilon \gamma^\star, \quad \rho_\varepsilon := \varepsilon^{-\rho^\star}, \quad \beta_\varepsilon = \nu(\rho^\star B_1^\nu(0)) = O(\varepsilon^{\rho^\star} \ell(\frac{1}{\varepsilon})) \varepsilon \to 0, \quad T_\varepsilon := \varepsilon^{-\theta^\star}, \quad (3.31)
\]
where $\gamma^\star, \rho^\star \in (0, 1)$ satisfy
\[
(2q + 3) \gamma^\star + (1 + \alpha) \rho^\star < 1 \quad (3.32)
\]
and $\theta^\star := 2 \alpha \rho^\star$. Since both $q + 2 > 0$ and $\alpha > 0$ condition (3.32) is easily satisfied for sufficiently small positive values of $\gamma^\star, \rho^\star$. Condition (3.32) directly implies the limits (3.6) and (3.29).

2. For further use in Section 4 we additionally impose the conditions
\[
\gamma^\star < \rho^\star, \quad (3.33)
\]
\[
\frac{\gamma^\star}{\alpha} + 3 \rho^\star < 1, \quad (3.34)
\]
on $\gamma^\star$ and $\rho^\star$, which do not contradict (3.32) since (3.34) is of the same type and (3.33) can be satisfied independently. Then condition (3.33) yields $\lim_{\varepsilon \to 0^+} |\ln(\gamma_\varepsilon)| \varepsilon^{\gamma^\star / \alpha} = 0$ and inequality (3.34) implies the limit $\lim_{\varepsilon \to 0^+} \gamma_\varepsilon \beta_\varepsilon / \rho_\varepsilon = 0$. The latter two are used in the estimates (4.14) and (4.17) respectively of Step 3 in the proof of Proposition 4.3 in Section 4.

The proof of Proposition 3.3: Our strategy consists of two steps. First we show in Proposition 3.4 that for some $q \geq 1$ and small $\varepsilon$ the stopped small perturbation event $\mathcal{E}_{y, T^\varepsilon}(\gamma_\varepsilon^q)$ implies the stopped small deviation event $G_{y, T^\varepsilon}(\frac{1}{2} \gamma_\varepsilon^q)$. In Corollary The second step relates $\mathcal{E}_{y, T^\varepsilon}(\gamma_\varepsilon^q)$ to $\mathcal{E}_{y, T^\varepsilon}(\gamma_\varepsilon^q)$ before finally using the estimate of $\mathcal{E}_{y, T^\varepsilon}(\gamma_\varepsilon^q)$ in Proposition 3.1.

Proposition 3.4. Let the Hypotheses (D.1)-(D.3), (S.1) and (S.2) be satisfied and for some fixed $q \geq 1$ the scales $\gamma, \rho, T$ be chosen as in (C). Then for any $R \geq R_0$ there exists a constant $\varepsilon_0 \in (0, 1]$ such that $\varepsilon \in (0, \varepsilon_0]$ and $x \in D_2(\gamma_\varepsilon, R)$ imply
\[
\mathcal{E}_{x, T^\varepsilon}(\gamma_\varepsilon^q) \subseteq G_{x, T^\varepsilon}(\frac{1}{2} \gamma_\varepsilon). \quad (3.35)
\]
Corollary 3.5. Let the hypotheses of Proposition [3.4] be satisfied. Then for any \( R \geq R_0 \) there exists a constant \( \varepsilon_0 \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) and \( x \in D_2(\gamma_\varepsilon, R) \) imply

\[
E^\sigma_{x,T^\varepsilon}(\gamma^q_\varepsilon) \subseteq \{ \sigma > T^\varepsilon \}.
\]

(3.36)

The proof of Corollary 3.5 is given below the proof of Proposition 3.4 at the end of this subsection.

Corollary 3.6. Let the hypotheses of Proposition 3.4 be satisfied. Then for any \( R \geq R_0 \) there exists a constant \( \varepsilon_0 \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) and \( x \in D_2(\gamma_\varepsilon, R) \) imply

\[
E_{x,T^\varepsilon}(\gamma^q_\varepsilon) \subseteq G_{x,T^\varepsilon}\left(\frac{1}{2} \gamma_\varepsilon\right).
\]

(3.37)

The proof of Corollary 3.6 is found at the end of the current subsection.

Remark 3.7. The proof of Proposition 3.4 given below is based on Gronwall estimates in Lemma 3.8 and 3.9. They yield estimates with right-hand sides which are monotonically growing as a function of an (\( \varepsilon \)-independent) time argument \( T \) and imply the inclusion (3.35) for any fixed \( T \) instead of \( T^\varepsilon \) when \( \varepsilon \) is sufficiently small. In Lemma 3.9 we show the stronger statement that (3.35) is valid for the \( \varepsilon \)-dependent argument \( T = T^\varepsilon \) which grows monotonically \( T^\varepsilon \to \infty \) as \( \varepsilon \to 0+ \). We stress that by the mentioned monotonicity in \( T \) that (3.35) is also valid for \( T^\varepsilon \) being replaced by any \( s \in [0, T^\varepsilon] \) and can be verified below line by line without difficulty. The stopping procedure with \( \sigma \) does not affect this reasoning.

Proof of Proposition 3.3. By Corollary 3.5 there is \( q \geq 1 \) such that for any \( R \geq R_0 \) there is some \( \varepsilon_0 \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) implies

\[
E^\sigma_{x,T^\varepsilon}(\gamma^q_\varepsilon) \subseteq \{ \sigma > T^\varepsilon \}.
\]

(3.38)

This result yields

\[
\begin{align*}
E^\sigma_{x,T^\varepsilon} = E^\sigma_{x,T^\varepsilon} \cap \{ \{ \sigma > T^\varepsilon \} \cup \{ \sigma \leq T^\varepsilon \} \} = (E^\sigma_{x,T^\varepsilon} \cap \{ \sigma > T^\varepsilon \}) \cup (E^\sigma_{x,T^\varepsilon} \cap \{ \sigma \leq T^\varepsilon \}) \\
= (E_{x,T^\varepsilon} \cap \{ \sigma > T^\varepsilon \}) \cup (E_{x,T^\varepsilon} \cap \{ \sigma \leq T^\varepsilon \}) = E_{x,T^\varepsilon} \cap \{ \sigma > T^\varepsilon \}.
\end{align*}
\]

Hence \( \varepsilon \in (0, \varepsilon_0] \) yields

\[
(E^\sigma_{x,T^\varepsilon})^c = (E_{x,T^\varepsilon} \cap \{ \sigma > T^\varepsilon \})^c = E_{x,T^\varepsilon} \cup \{ \sigma > T^\varepsilon \}^c.
\]

(3.39)

The identity (3.39) puts us in the position to prove inequality (3.30).

Due to the independence of \( Y^\varepsilon \) and \( T_1 \) and the statement of Proposition 3.4 there is a constant \( \varepsilon_0 \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) implies (3.35) and additionally due to (3.29) the inequality

\[
\ln\left(\frac{\beta_\varepsilon}{\beta_\varepsilon - \theta \lambda_\varepsilon}\right) + \theta \lambda_\varepsilon T^\varepsilon \leq 2\theta \frac{\lambda_\varepsilon}{\beta_\varepsilon} + \theta \frac{\lambda_\varepsilon}{\beta_\varepsilon} \beta_\varepsilon T^\varepsilon \leq \frac{1}{2} (1 + \beta_\varepsilon T^\varepsilon) \leq \ln(2) + \frac{1}{2} \beta_\varepsilon T^\varepsilon.
\]

(3.40)
With the help of (3.40) and Remark 3.7 \( \varepsilon \in (0, \varepsilon_0] \) yields

\[
\sup_{x \in D_2(\gamma_\varepsilon, R)} \mathbb{E}\left[ e^{\theta_\lambda T_\varepsilon} \mathbf{1}(G^x_{\varepsilon,r}(\frac{1}{2} \gamma_\varepsilon)) \right] = \sup_{x \in D_2(\gamma_\varepsilon, R)} \int_0^{T_\varepsilon} \mathbb{P}(G^x_{\varepsilon,r}(\frac{1}{2} \gamma_\varepsilon)) \beta_\varepsilon e^{-(\beta_\varepsilon - \theta_\lambda) s} ds \leq \int_0^{T_\varepsilon} \sup_{x \in D_2(\gamma_\varepsilon, R)} \mathbb{P}(G^x_{\varepsilon,r}(\gamma_\varepsilon)) \beta_\varepsilon e^{-(\beta_\varepsilon - \theta_\lambda) s} ds + \frac{\beta_\varepsilon e^{-(\beta_\varepsilon - \theta_\lambda) T_\varepsilon}}{\beta_\varepsilon - \theta_\lambda} \leq \sup_{x \in D_2(\gamma_\varepsilon, R)} \mathbb{P}(\mathcal{E}^\varepsilon_{x,T_\varepsilon}(\gamma_\varepsilon)) + 2e^{-\frac{1}{2}\beta_\varepsilon T_\varepsilon}.
\]

(3.41)

Using (3.39) we apply Proposition 3.1 and obtain for \( \varepsilon \in (0, \varepsilon_0] \) the inequality

\[
\sup_{x \in D_2(\gamma_\varepsilon, R)} \mathbb{P}(\mathcal{E}^\varepsilon_{x,T_\varepsilon}(\gamma_\varepsilon)) \leq \sup_{x \in D_2(\gamma_\varepsilon, R)} \mathbb{P}(\mathcal{E}^\varepsilon_{x,T_\varepsilon}(\gamma_\varepsilon) \cup \{\sigma > T_\varepsilon\}^c) \leq \sup_{x \in D_2(\gamma_\varepsilon, R)} \mathbb{P}(\mathcal{E}^\varepsilon_{x,T_\varepsilon}(\gamma_\varepsilon) \cap \{\sigma > T_\varepsilon\}^c) \leq 2e^{-\frac{1}{2}\pi_\varepsilon}.
\]

(3.42)

Combining (3.41) and (3.42) we obtain the desired result.

\[\Box\]

**The proof of Proposition 3.4:** We introduce the nonlinear residuum \( R^\varepsilon \) of the randomness in \( Y^\varepsilon \)

\[
R^\varepsilon_{t,x} := Y^\varepsilon(t; x) - u(t; x) - \Psi^\varepsilon_{t,x}, \quad t \geq 0, \ x \in D_2(\gamma_\varepsilon, R), \ \varepsilon \in (0, 1].
\]

(3.43)

The quantity we have to control in \( G_{x,T} \) has the shape \( Y^\varepsilon - u = \Psi^\varepsilon + R^\varepsilon \). By Proposition 3.1 we have a good estimate of \( \Psi^\varepsilon \). It is therefore natural to control the remainder term \( R^\varepsilon \) in terms of \( \Psi^\varepsilon \), which is done first for large initial values \( x \) (of \( Y^\varepsilon \) and \( u \)) on small time scales in Lemma 3.8 and then for initial values \( x \) (of \( Y^\varepsilon \) and \( u \)) close to the stable state and large time scales in Lemma 3.9. Lemma 3.10 combines the previous two lemmas before concluding the statement of Proposition 3.4.

**Lemma 3.8.** Let the Hypotheses (D.1)-(D.3), (S.1) and (S.2) be satisfied and the scales \( \gamma_, \rho, T \) be chosen as in (3.37). We set \( s^\varepsilon := \kappa_0 \ln(\gamma_\varepsilon), \ \varepsilon \in (0, 1], \) where \( \kappa_0 > 0 \) be given by Proposition 2.3.

Then for all \( R \geq R_0 \) and \( K > 0 \) there is a constant \( q \geq 1 \) such that in case the scales \( \gamma_, \rho, T \) satisfy (3.32), (3.33) and (3.34) with respect to \( q \) we have the following. There is \( \varepsilon_0 \in (0, 1] \) such that for \( \varepsilon \in (0, \varepsilon_0] \), \( x \in D_1(R) \) and \( \omega \in \mathcal{E}^\varepsilon_{x,T_\varepsilon}(\gamma_\varepsilon) \) we have

\[
\sup_{t \in [0, s^\varepsilon \wedge T_\varepsilon \wedge \sigma(\omega)]} \| R^\varepsilon_{t,x}(\omega) \| \leq K\gamma_\varepsilon.
\]

(3.44)

**Proof.** Fix \( R \geq R_0 \) and \( \varepsilon \in (0, 1] \) and \( x \in D_1(R) \). Recall that \( f : H \to H \) are locally Lipschitz continuous, that is, for \( y, u \in H \)

\[
\| f(y) - f(u) \| \leq \ell^\varepsilon(y, u)\|y - u\|,
\]

(3.45)
for some \( \ell^* : H \times H \rightarrow (0, \infty), (y,u) \mapsto \ell^*(y,u) \) jointly continuous and bounded on bounded sets. Consequently, it is globally Lipschitz continuous on any of the bounded level sets \( \mathcal{U}^R \). The process \( R^\varepsilon_t = 0 \). The Gronwall-Bellmann inequality applied to \( x \in D_1(\mathcal{R}) \subseteq \mathcal{U}^R \) and the positive invariance of \( \mathcal{U}^R \) under the deterministic system \( u \) yield

\[
\sup_{x \in D_2(\gamma^*_t, R)} \sup_{t \geq 0} \|u(t;x)\| \leq d(\mathcal{R}) < \infty.
\]

We define the \((F_t)_{t \geq 0}\) stopping time \( \sigma_\varepsilon(t) := \sigma \land \inf\{t > 0 \mid \|R^\varepsilon_t\|^* > 1\} \). Then we obtain for \( t \in [0, s^\varepsilon \land T^\varepsilon \land \sigma_\varepsilon] \) on the event \( \mathcal{E}_{x,s^\varepsilon \land T^\varepsilon}(\gamma^*_t) \) for any arbitrary fixed \( q \geq 1 \)

\[
\|Y^\varepsilon(t;x)\| \leq \|u(t;x)\| + \|\Psi_t^\varepsilon\| + \|R_t^\varepsilon\| \leq d(\mathcal{R}) + 2.
\]

As a consequence, \( \ell_\mathcal{R} := \sup_{(y,u) \in B_2(\mathcal{R})^2(0)} \ell^*(y,u) < \infty \) implies

\[
e^{\Lambda_0 t} \|R_t^\varepsilon\| \leq \ell_\mathcal{R} \left( \int_0^t e^{\Lambda_0 s} \|R_s^\varepsilon\| ds + \int_0^t e^{\Lambda_0 s} \|\Psi_s^\varepsilon\| ds \right).
\]

The Gronwall-Bellmann inequality applied to \( e^{\Lambda_0 t} \|R_t^\varepsilon\| \) with \( e^{\Lambda_0 t} \|R_0^\varepsilon\| = 0 \) yields

\[
e^{\Lambda_0 t} \|R_t^\varepsilon\| \leq \int_0^t \int_0^s e^{\ell_\mathcal{R}(t-s)} e^{\Lambda_0 t} \|\Psi_r^\varepsilon\| drds \leq \sup_{r \in [0,t]} \|\Psi_r^\varepsilon\| \int_0^t \int_0^s e^{\ell_\mathcal{R}(t-s)} e^{\Lambda_0 r} drds.
\]

The elementary calculation of the factor

\[
\int_0^t \int_0^s e^{\ell_\mathcal{R}(t-s)} e^{\Lambda_0 r} drds = \frac{e^{\ell_\mathcal{R} t}}{\ell_\mathcal{R}(\ell_\mathcal{R} - \Lambda_0)} + \frac{1}{\Lambda_0 \ell_\mathcal{R}} - \frac{e^{\Lambda_0 t}}{\Lambda_0 (\ell_\mathcal{R} - \Lambda_0)}
\]

shows for \( \kappa := \ell_\mathcal{R} - \Lambda_0 > 0 \) on the event \( \mathcal{E}_{x,s^\varepsilon \land T^\varepsilon}(\gamma^*_t) \) the estimate

\[
\|R_t^\varepsilon\| \leq \frac{e^{\kappa t}}{\kappa^2} \sup_{r \in [0,t]} \|\Psi_r^\varepsilon\|,
\]

where \( t \in [0, s^\varepsilon \land T^\varepsilon \land \sigma_\varepsilon] \). We set \( q := \kappa_0 \kappa + 3 \) and obtain for any \( K > 0 \) a value \( \varepsilon_0 \in (0,1] \) sufficiently small such that \( \varepsilon \in (0, \varepsilon_0) \) implies on the event \( \mathcal{E}_{x,s^\varepsilon \land T^\varepsilon}(\gamma^*_t) \) the desired estimate

\[
\sup_{t \in [0,s^\varepsilon \land T^\varepsilon \land \sigma_\varepsilon]} \|R_t^\varepsilon\| \leq \frac{e^{\kappa s^\varepsilon}}{\kappa^2} \gamma^*_t \leq K \gamma^*_t.
\]
If \( \varepsilon_0 \in (0, 1] \) is additionally small enough such that \( K \gamma_\varepsilon < 1 \) for \( \varepsilon \in (0, \varepsilon_0] \) we have on the event \( \mathcal{E}_{\varepsilon, \varepsilon^* \wedge T^\varepsilon}(\gamma_\varepsilon^2) \)

\[
\inf \{ t > 0 \mid \| R_t^{\varepsilon, x} \| > 1 \} > \varepsilon^* \wedge T^\varepsilon \wedge \sigma,
\]

which proves (3.44).

\[ \square \]

**Lemma 3.9.** Let the Hypotheses (D.1)-(D.3), (S.1) and (S.2) be satisfied and the scales \( \gamma, \rho, T^\varepsilon \) given by (C) for some \( q \geq 1 \). Then for all \( \mathcal{R} \supseteq \mathcal{R}_0 \) there exist constants \( \delta_0, \delta_1, K_0 > 0 \) such that for all \( x \in B_{\delta_0}(\phi) \), \( \varepsilon \in (0, 1] \) and \( \omega \in \mathcal{E}_{\varepsilon, \varepsilon^* \wedge T^\varepsilon}(\delta_1) \), we have

\[
\sup_{t \in [0, T^\varepsilon \wedge \sigma(\omega)]} \| R_t^{\varepsilon, x}(\omega) \| \leq K_0 \sup_{t \in [0, T^\varepsilon \wedge \sigma(\omega)]} \| \Psi_t^{\varepsilon, x}(\omega) \|. \tag{3.48}
\]

**Proof.** The stability of \( \phi \) implies that the linearization \( \Delta v + f'(\phi)v \) of \( \Delta u + f(u) \) centered in \( \phi \) has strictly negative maximal eigenvalues with strictly negative upper bound, \( -\Lambda_1 < 0 \), say, in that \( (\Delta v + f'(\phi)v, v) \leq -\Lambda_1|v|^2 \) for \( v \in H \). We fix \( \delta_0 \in (0, 1) \) such that we have additionally

\[
\sup_{v \in B_{\delta_0}(\phi)} \| f'(v) \| \leq 2 \| f'(\phi) \| =: C_0, \tag{3.49}
\]

\[
\sup_{v, w \in B_{\delta_0}(\phi)} \| f'(v) - f'(w) \| \leq \frac{\Lambda_1}{4}. \tag{3.50}
\]

The stability also implies the existence of \( \delta_1 \in (0, 1) \) such that for \( x \in B_{\delta_1}(\phi) \)

\[
u(t; x) \in B_{\frac{\Lambda_1}{4}}(\phi) \quad t \geq 0. \tag{3.51}\]

Denote for \( x \in B_{\delta_1}(\phi) \) the \((\mathcal{F}_t)_{t \geq 0}\)-stopping time \( \sigma_* := \sigma \wedge \inf\{ t > 0 \mid \| R_t^{\varepsilon, x} \| > \frac{K_0}{4} \} \). The decomposition (3.43) and the mean value theorem applied to (3.46) read

\[
\frac{d R_t^{\varepsilon, x}}{dt} = \Delta R_t^{\varepsilon, x} + \int_0^1 f'(u(t; x) + \theta(R_t^{\varepsilon, x} + \Psi_t^{\varepsilon, x})) d \theta(R_t^{\varepsilon, x} + \Psi_t^{\varepsilon, x})
\]

\[
= \Delta R_t^{\varepsilon, x} + f'(\phi)R_t^{\varepsilon, x} + \int_0^1 (f'(u(t; x) + \theta(R_t^{\varepsilon, x} + \Psi_t^{\varepsilon, x})) - f'(\phi)) d \theta R_t^{\varepsilon, x}
\]

\[
+ \int_0^1 f'(u(t; x) + \theta(R_t^{\varepsilon, x} + \Psi_t^{\varepsilon, x})) d \theta \Psi_t^{\varepsilon, x}.
\]

We multiply with \( R_t^{\varepsilon, x} \) in \( L^2(J) \) and integrate by parts. Then for any \( \delta_1 < \frac{\Lambda_1}{4} \) the event \( \mathcal{E}_{\varepsilon, T^\varepsilon}(\delta_1) \) together with the embedding \( \| \Psi_t^{\varepsilon, x} \|_\infty \leq \| \Psi_t^{\varepsilon, x} \| \), the deterministic stability \( \| u(t; x) + \theta(R_t^{\varepsilon, x} + \Psi_t^{\varepsilon, x}) \| \leq \delta_0 \) and the definition of the stopping time \( \sigma_* \) imply for \( t \in [0, T^\varepsilon] \) and \( \theta \in [0, 1] \) the estimate

\[
\| u(t; x) + \theta(R_t^{\varepsilon, x} + \Psi_t^{\varepsilon, x}) \| \leq \delta_0.
\]

Hence the inequalities (3.49) and (3.50) and the embedding \( H \subseteq L^2(J) \) give on \( \mathcal{E}_{\varepsilon, T^\varepsilon}(\delta_1) \) for any \( t \in [0, T^\varepsilon \wedge \sigma_*] \) the estimate

\[
\frac{1}{2} \frac{d}{dt} \| R_t^{\varepsilon, x} \|^2 + \Lambda_1 \| R_t^{\varepsilon, x} \|^2 \leq \frac{\Lambda_1}{4} \| R_t^{\varepsilon, x} \|^2 + C_0 \| R_t^{\varepsilon, x} \| \| \Psi_t^{\varepsilon, x} \| \leq \frac{\Lambda_1}{2} \| R_t^{\varepsilon, x} \|^2 + \frac{(C_0)^2}{\Lambda_0} \| \Psi_t^{\varepsilon, x} \|^2,
\]

25
such that we have after rearrangement
\[
\frac{d}{dt} |R_t^{\varepsilon,x}|^2 + \Lambda_1 |R_t^{\varepsilon,x}|^2 \leq 2\frac{(C_0)^2}{\Lambda_1} |\Psi_t^{\varepsilon,x}|^2.
\]
Gronwall’s lemma applied to \(|R_t^{\varepsilon,x}|^2\) with initial condition \(|R_0^{\varepsilon,x}|^2 = 0\) yields on the event \(E_{x,T^\varepsilon}(\delta_1)\) for \(t \in [0, T^\varepsilon \wedge \sigma]\) the estimate
\[
|R_t^{\varepsilon,x}|^2 \leq 2\frac{(C_0)^2}{\Lambda_1} |\Psi_t^{\varepsilon,x}|^2. \tag{3.52}
\]
In order to obtain an estimate in \(H\) we use the smoothing property of the heat semigroup \(S\) and the mean value theorem as well as \(\tag{3.52}\) on \(E_{x,T^\varepsilon}(\delta_1)\) for \(t \in [0, T^\varepsilon \wedge \sigma]\) as follows
\[
\|R_t^{\varepsilon,x}\| \leq C_1 \int_0^t \frac{e^{-\Lambda_0(t-s)}}{\sqrt{t-s}} |f(Y_s^{\varepsilon,x}) - f(u(s;x))| ds
\leq C_1(C_0 + \frac{\Lambda_0}{4}) \int_0^t \frac{e^{-\Lambda_0(t-s)}}{\sqrt{t-s}} (|R_s^{\varepsilon,x}| + |\Psi_s^{\varepsilon,x}|) ds
\leq C_1(C_0 + \frac{\Lambda_0}{4}) \left(2\frac{(C_0)^2}{\Lambda_0} + 1\right) \int_0^t \frac{e^{-\Lambda_0(t-s)}}{\sqrt{t-s}} ds \sup_{r \in [0,t]} |\Psi_r^{\varepsilon,x}| \leq C_2 \sup_{r \in [0,t]} \|\Psi_r^{\varepsilon,x}\|,
\]
where \(C_2 = C_1(C_0 + \frac{\Lambda_0}{4}) \left(2\frac{(C_0)^2}{\Lambda_0} + 1\right) \int_0^\infty \frac{e^{-\Lambda_0r}}{\sqrt{r}} dr < \infty\). If, in addition, \(\delta_1 < \frac{1}{\Lambda_1}\) we obtain on \(E_{x,T^\varepsilon}(\delta_1)\)
\[
\inf\{t > 0 \mid \|R_t^{\varepsilon,x}\| > \frac{\delta_0}{4}\} > T^\varepsilon \wedge \sigma.
\]
Note that we have not used any specific property of \(T^\varepsilon\). This finishes the proof. \(\Box\)

We combine Lemma \(3.8\) and Lemma \(3.9\). For this purpose we assume without loss of generality that the limit
\[
\lim_{\varepsilon \to 0} \frac{s^\varepsilon}{T^\varepsilon} \in \{0, \infty\}. \tag{3.53}
\]
This is justified by the choice of scales in \(3.31\) and Lemma \(3.8\).

**Lemma 3.10.** Let the Hypotheses (D.1) - (D.3), (S.1) and (S.2) be satisfied and the scales \(\gamma, \rho, T\) given by \(3.31\). For the constant \(q \geq 1\) obtained in Lemma \(3.8\) let \(\gamma, \rho, T\) additionally satisfy conditions \(3.32\), \(3.33\) and \(3.34\). Furthermore, we assume \(3.35\).

Then for any \(R \geq R_0\) there is a constant \(\varepsilon_0 \in (0, 1]\) such that for any \(\varepsilon \in (0, \varepsilon_0]\), \(x \in D_2(\gamma_\varepsilon, R)\) and \(\omega \in E_{x,T^\varepsilon}(\gamma_\varepsilon)\) we have
\[
\sup_{t \in [0,T^\varepsilon \wedge \sigma(\omega)]} \|R_t^{\varepsilon,x}(\omega)\| \leq \frac{1}{4} \gamma_\varepsilon. \tag{3.54}
\]
**Proof.** Recall the notation \(s^\varepsilon := \kappa_0 |\ln(\gamma_\varepsilon)|\) with \(\kappa_0\) from the statement of Proposition \(2.8\). Assume \(\varepsilon_0 \in (0, 1]\) is sufficiently small such that \(\gamma_\varepsilon \leq \delta_0\) and \(\gamma_\varepsilon < \delta_1\) given in Lemma \(3.9\). In the first case \(\lim_{\varepsilon \to 0_+} \frac{\varepsilon}{s^\varepsilon} = 0\) the result follows immediately by Lemma \(3.8\) for \(K = \frac{1}{4}\).
In the second case \( \lim_{\varepsilon \to 0^+} \frac{s^2}{\varepsilon^2} = 0 \) there is \( \varepsilon_0 \in (0, 1) \) such that \( T^\varepsilon > s^\varepsilon \) for all \( \varepsilon \in (0, \varepsilon_0] \). Note that if \( \sigma < s^\varepsilon \) then \( s^\varepsilon \wedge \sigma \leq s^\varepsilon \) and we are back in the first case. Thus we only have to consider the case \( \sigma \geq s^\varepsilon \). Using Lemma 3.38 and the stability of \( \phi \) we fix additionally \( \varepsilon_0 \in (0, 1) \) small enough such that \( \varepsilon \in (0, \varepsilon_0] \) implies for \( x \in D_2(\gamma_\varepsilon, \mathcal{R}) \) on \( \mathcal{E}_{\varepsilon,T^\varepsilon}(\gamma_\varepsilon^2) \) and \( \{ \sigma > s^\varepsilon \} \) the estimates

\[
\sup_{t \in [0,s^\varepsilon \wedge \sigma]} \| R^\varepsilon_{t,x} \| \leq \frac{1}{9} \gamma_\varepsilon, \tag{3.55}
\]

\[
\| u(t; x) - \phi \| \leq \frac{1}{4} \gamma_\varepsilon \quad \text{for } t \geq s^\varepsilon. \tag{3.56}
\]

Then (3.55) and (3.56) give for \( x \in D_2(\gamma_\varepsilon, \mathcal{R}) \) on \( \mathcal{E}_{\varepsilon,T^\varepsilon}(\gamma_\varepsilon^2) \) and \( \{ \sigma > s^\varepsilon \} \)

\[
\| Y_\varepsilon(s^\varepsilon; x) - \phi \| \leq \| u(s^\varepsilon; x) - \phi \| + \| R^\varepsilon_{s^\varepsilon} \| + \| \Psi^\varepsilon_{s^\varepsilon} \| \leq \left( 1 + \frac{1}{4} \right) \gamma_\varepsilon + \gamma_\varepsilon^2 \leq \frac{1}{2} \gamma_\varepsilon. \tag{3.57}
\]

As in the proof of Lemma 3.39 the stability of \( \phi \) implies for all \( x \in B_{\delta_0}(\phi) \) that \( u(t; x) \in B_{\delta_1}(\phi) \) for all \( t \geq 0 \). In addition, the linear stability of \( \phi \) gives a constant \( \ell_0 \in (0, 1) \) such that

\[
\| u(t; x) - u(t; y) \| \leq \ell_0 \| x - y \|, \quad \text{for all } x, y \in B_{\delta_0}(\phi), t \geq 0.
\]

Hence we have for \( \varepsilon \in (0, \varepsilon_0] \) and \( x \in D_2(\gamma_\varepsilon, \mathcal{R}) \) on the event \( \mathcal{E}_{\varepsilon,T^\varepsilon}(\gamma_\varepsilon^2) \cap \{ \sigma > T^\varepsilon \} \)

\[
\| u(t; x) - u(t - s^\varepsilon; Y_\varepsilon(s^\varepsilon; x)) \| \leq \ell_0 \| u(s^\varepsilon; x) - Y_\varepsilon(s^\varepsilon; x) \| \leq \| R^\varepsilon_{s^\varepsilon} \| + \| \Psi^\varepsilon_{s^\varepsilon} \| \leq \frac{1}{9} \gamma_\varepsilon + \gamma_\varepsilon^2. \tag{3.58}
\]

Estimate (3.55) provides for \( x \in D_2(\gamma_\varepsilon, \mathcal{R}) \) on \( \mathcal{E}_{\varepsilon,T^\varepsilon}(\gamma_\varepsilon^2) \cap \{ \sigma > s^\varepsilon \} \) and additionally \( s^\varepsilon < t \leq T^\varepsilon \wedge s^\varepsilon \) the inequality

\[
\| R^\varepsilon_{t,x} \| = \| Y_\varepsilon(t - s^\varepsilon, s^\varepsilon, Y_\varepsilon(s^\varepsilon; x)) - u(t - s^\varepsilon; u(s^\varepsilon; x)) - \Psi^\varepsilon_{t,x} \|
\leq \| Y_\varepsilon(t - s^\varepsilon, s^\varepsilon, Y_\varepsilon(s^\varepsilon; x)) - u(t - s^\varepsilon; Y_\varepsilon(s^\varepsilon; x)) - \Psi^\varepsilon_{t,s^\varepsilon,x} \|
\quad + \| u(t - s^\varepsilon; u(s^\varepsilon; x)) - u(t - s^\varepsilon; Y_\varepsilon(s^\varepsilon; x)) \| + \| \Psi^\varepsilon_{t,s^\varepsilon,x} \| + \| \Psi^\varepsilon_{t,s^\varepsilon,x} \|
\leq \sup_{s \in [s^\varepsilon,T^\varepsilon \wedge \sigma]} \left( \sup_{s \in [s^\varepsilon,T^\varepsilon \wedge \sigma]} \| \Psi^\varepsilon_{s^\varepsilon,x} \| + \frac{1}{9} \gamma_\varepsilon + \gamma_\varepsilon^2 \right).
\]

On the other hand estimate (3.57), the Markov property of \( Y_\varepsilon \) for time \( s^\varepsilon \) and Lemma 3.39 guarantee for \( x \in D_2(\gamma_\varepsilon, \mathcal{R}) \) on \( \mathcal{E}_{\varepsilon,T^\varepsilon}(\gamma_\varepsilon^2) \), \( \{ \sigma > s^\varepsilon \} \) and \( s^\varepsilon < t \leq T^\varepsilon \wedge s^\varepsilon \) the inequality

\[
\| R^\varepsilon_{t,x} \| \leq \sup_{\gamma \in B_{\frac{1}{2}\gamma_\varepsilon}(\phi)} \sup_{s \in [0,T^\varepsilon \wedge \sigma - s^\varepsilon]} \| R^\varepsilon_{s,y} \| + \frac{1}{9} \gamma_\varepsilon + \gamma_\varepsilon^2 \quad \text{for } \gamma \in [0,T^\varepsilon \wedge \sigma] \quad \| \Psi^\varepsilon_{s,x} \|
\leq \sup_{\gamma \in B_{\frac{1}{2}\gamma_\varepsilon}(\phi)} \sup_{s \in [0,T^\varepsilon \wedge \sigma]} \| R^\varepsilon_{s,y} \| + \frac{1}{9} \gamma_\varepsilon + \gamma_\varepsilon^2 \quad \text{for } \gamma \in [0,T^\varepsilon \wedge \sigma] \quad \| \Psi^\varepsilon_{s,x} \|
\leq K_0 \sup_{\gamma \in B_{\frac{1}{2}\gamma_\varepsilon}(\phi)} \sup_{s \in [0,T^\varepsilon \wedge \sigma]} \| \Psi^\varepsilon_{s,y} \| + \frac{1}{9} \gamma_\varepsilon + \gamma_\varepsilon^2 \quad \text{for } \gamma \in [0,T^\varepsilon \wedge \sigma] \quad \| \Psi^\varepsilon_{s,x} \|
\leq (K_0 + 2) \gamma_\varepsilon + \frac{1}{9} \gamma_\varepsilon.
\]
We note that the preceding expression is less than \( \frac{1}{4} \gamma \) for all \( \varepsilon \in (0, \varepsilon_0] \) if \( \varepsilon_0 \in (0, 1) \) is chosen sufficiently small. This finishes the proof.

**Proof of Proposition 3.4:** Without loss of generality we assume in the sequel \( T^\varepsilon \geq s^\varepsilon \) for all \( \varepsilon \in (0, \varepsilon_0] \) for some \( \varepsilon_0 \in (0, 1) \). Let the assumptions of Lemma 3.10 be satisfied for some \( R \geq R_0 \) and \( q \geq 1 \) be given by Lemma 3.8. By Lemma 3.10 there exists \( \varepsilon_0 \in (0, 1) \) such that for all \( \varepsilon \in (0, \varepsilon_0] \) and \( x \in D_2(\gamma, R) \) we have \( P \)-a.s.

\[
(\mathcal{G}_{x,T^\varepsilon} \left( \frac{1}{T^\varepsilon} \mathcal{G} \varepsilon \right))^\varepsilon = \left\{ \sup_{t \in [0,T^\varepsilon]} \| Y^\varepsilon (t) - u(t) \| \geq \frac{\gamma \varepsilon}{2} \right\}
\[
= \left\{ \sup_{t \in [0,T^\varepsilon]} \| R^\varepsilon_t \psi + \Psi_t \| \geq \frac{\gamma \varepsilon}{2} \right\}
\[
\subseteq \left\{ \sup_{t \in [0,T^\varepsilon]} \| R^\varepsilon_t \psi \| \geq \frac{\gamma \varepsilon}{4} \right\} \cup \left\{ \sup_{t \in [0,T^\varepsilon]} \| \Psi_t \| \geq \frac{\gamma \varepsilon}{4} \right\}
\[
\subseteq \left\{ \sup_{t \in [0,T^\varepsilon]} \| R^\varepsilon_t \psi \| \geq \frac{\gamma \varepsilon}{4} \right\} \cup \left( \mathcal{E}_{x,T^\varepsilon} (\gamma)^\varepsilon \right) \subseteq (\mathcal{E}_{x,T^\varepsilon} (\gamma)^\varepsilon)^\varepsilon. \quad (3.59)
\]

This finishes the proof of Proposition 3.4.

**Proof of Corollary 3.5:** Proposition 3.4 states the existence of \( q \geq 1 \) such that for all \( R \geq R_0 \) there is \( \varepsilon_0 \in (0, 1) \) such that \( \varepsilon \in (0, \varepsilon_0] \) implies for \( x \in D_2(\gamma, R) \)

\[
\mathcal{E}_{x,T^\varepsilon} (\gamma)^\varepsilon \cap \{ \sigma < T^\varepsilon \} = \mathcal{E}_{x,T^\varepsilon} (\gamma)^\varepsilon \cap \{ \sigma < T^\varepsilon \} \cap \left\{ \sup_{t \in [0,T^\varepsilon]} \| Y^\varepsilon (t) - u(t) \| \leq \frac{1}{2} \gamma \right\}
\[
\subseteq \left\{ \sup_{t \in [0,T^\varepsilon]} \| Y^\varepsilon (t) - u(t) \| \leq \frac{1}{2} \gamma \right\}
\[
= \{ Y^\varepsilon (t, x) = B_{2\gamma}(u(t)) \} \text{ for all } t \in [0, \sigma] \}
\[
\subseteq \{ Y^\varepsilon (t, x) = \bigcup_{t \geq 0} B_{2\gamma}(u(s)) \} \text{ for all } t \in [0, \sigma] \}
\]

By construction, we have that

\[
\bigcup_{x \in D_2(\gamma, R)} \bigcup_{t \geq 0} B_{2\gamma}(u(t)) \subseteq D_1(R) \subseteq U^{R} \setminus \bigcup_{v \in \partial U^{R}} B_{\gamma}(v).
\]

In particular, we obtain

\[
\mathcal{E}_{x,T^\varepsilon} (\gamma)^\varepsilon \cap \{ \sigma < T^\varepsilon \} \subseteq \{ Y^\varepsilon (\sigma, x) = \bigcup_{v \in \partial U^{R}} B_{\gamma}(v) \}.
\]

However, by definition of \( \sigma \) it is clear that \( Y^\varepsilon (\sigma, x) \in (U^{R})^\varepsilon \). Therefore for \( \varepsilon_0 \in (0, 1) \) sufficiently small, \( \varepsilon \in (0, \varepsilon_0] \) implies the desired result

\[
\mathcal{E}_{x,T^\varepsilon} (\gamma)^\varepsilon \cap \{ \sigma < T^\varepsilon \} = \emptyset. \quad (3.60)
\]
Proof of Corollary 3.5. Combining (3.54) and (3.60) ensures a constant \( q \geq 1 \) such that for any \( R \geq R_0 \) there is \( \varepsilon_0 \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) yields

\[
E_{x,T^*}(\gamma_{2}^{q}) \subseteq \mathcal{G}_{x,T^*}(\frac{1}{2} \gamma_{\varepsilon}).
\]

\[\square\]

4 The geometric structure of the large jumps dynamics

4.1 The models of the exit times and exit locus

We now construct on \( \Omega := (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0}) \) the random variables \( (\mathcal{G}(\varepsilon))_{\varepsilon \in (0, 1]} \) of Theorem 2.6 and \( (\mathcal{R}(\varepsilon))_{\varepsilon \in (0, 1]} \) of Theorem 2.7.

Definition 4.1. For given scales \( \rho \) and \( \gamma \) in \( (C) \), \( B_\varepsilon^\rho(\varepsilon) := \{\varepsilon W_j \in \mathcal{J}^{(D)}(\rho') \} \), and the arrival times \( T_k \) of \( W_k \) given in (3.29), we define for \( \varepsilon \in (0, 1] \)

\[
\mathcal{G}(\varepsilon) := \sum_{k=1}^{\infty} T_k \prod_{j=1}^{k-1} (1 - 1(B_j^\rho)) \mathbf{1}(B_k^\rho),
\]

\[
\mathcal{R}(\varepsilon) := \sum_{k=1}^{\infty} k \prod_{j=1}^{k-1} (1 - 1(B_j^\rho)) \mathbf{1}(B_k^\rho).
\]

Lemma 4.2. For given scale \( \rho \) in (4.1) and any \( \varepsilon \in (0, 1] \) the random variable \( \mathcal{G}(\varepsilon) \) is exponentially distributed with rate \( \lambda_1(\varepsilon) \) and the random variable \( \mathcal{R}(\varepsilon) \) is geometrically distributed with rate \( \mathbb{P}(B^\rho) = \lambda_1(\varepsilon) / \beta_\varepsilon \). In particular \( \mathcal{G}_{\varepsilon}(\varepsilon) := \lambda_1(\varepsilon) \mathcal{G}(\varepsilon) \) is exponentially distributed with rate 1.

The proof is elementary and provided in Appendix 5.1.

4.2 Exit events and their estimates

Recall the arrival times \( T_k = t_1 + \cdots + t_k \) of \( W_k \) from (3.29). The following events are the building blocks of the first exit events. For \( x \in H, R \geq R_0 \) and a given rate \( \gamma : (0, 1) \to (0, 1) \) with \( \gamma_\varepsilon \to 0 \) as \( \varepsilon \to 0 \) we define for \( j \in \mathbb{N} \)

\[
A_x^{(j)} := \{Y^\varepsilon(t; x) \in D_2^j(\gamma_\varepsilon, L) \text{ for } t \in [0, t_j] \text{ and } Y^\varepsilon(t_j; x) + G(Y^\varepsilon(t_j; x), \varepsilon \Delta t_j L) \notin D_2^j(\gamma_\varepsilon, L) \},
\]

\[
B_x^{(j)} := \{Y^\varepsilon(t; x) \in D_2^j(\gamma_\varepsilon, L) \text{ for } t \in [0, t_j] \text{ and } Y^\varepsilon(t_j; x) + G(Y^\varepsilon(t_j; x), \varepsilon \Delta t_j L) \notin D_2^j(\gamma_\varepsilon, L) \},
\]

\[
C_x^{(j)} := \{Y^\varepsilon(t; x) \notin D_2^j(\gamma_\varepsilon, L) \text{ for some } t \in [0, t_j] \}.
\]

In order to use the (strong) Markov property in Subsection 4.3 we identify \( \Omega \) with the canonical probability space given as the path space of the driving noise \( \mathbb{D}([0, \infty), H) \). The shift operator
In Proposition 4.3 we show the statement of Theorem 2.6 and additionally the convergence in probability of the first exit locus of Theorem 2.7. We apply this strategy for the sake of efficiency in order to avoid the repetition of arguments. Proposition 4.4 sharpens this result to the convergence in $L^p$ for $p \in (0, \alpha)$ of Theorem 2.7 by showing the respective uniform integrability.

**Proposition 4.3.** Let the assumptions of Theorem 2.7 be satisfied. Then for any $\theta \in (0, 1)$ and $c > 0$ there are $\varepsilon_0, \gamma \in (0, 1]$ and $\mathcal{R} \geq \mathcal{R}_0$ such that $\varepsilon \in (0, \varepsilon_0]$ implies for any $U \in \mathcal{B}(H)$ with $m'(\partial U) = 0$ that

$$\sup_{x \in \mathcal{D}_1(\varepsilon, \mathcal{R})} \mathbb{E} \left[ e^{\theta |\lambda|^s_{\varepsilon}(x, \mathcal{R}) - s'(\varepsilon)} \left( 1 + |\{\mathcal{X}^x(t; x) \in U \} - 1\{W_{\mathcal{R}'}(\varepsilon) \in 1 e^{\mathcal{U} \cup (\mathcal{D}')^c(\phi')} \} | \right) \right] \leq 1 + c.$$ 

The statement of Proposition 4.3 directly implies the statement of Theorem 2.6.

**Proof.** The proof is organized in four consecutive steps. First, the strong Markov property reduces the main expression to four geometric sums, whose limit consists of event involving certain events, which are estimated in Step 2. In Step 3 we estimate the resulting event probabilities using all the previous results available and apply these results in Step 4 to the four sums mentioned above and conclude.
**Step 0: Conventions and assumptions.** We choose the scales \( \gamma, \rho, T \) according to (C) for \( q \geq 1 \) given in Lemma 3.8. Without loss of generality we set \( \theta \in \left( \frac{1}{2}, 1 \right) \). We use Hypothesis (S.4) and fix \( c \in (0, \frac{1}{2}(1 - \theta)), R \geq R_0 \) large enough and \( \delta \in (0, 1] \) sufficiently small such that

\[
\frac{m'(D^s \setminus D^s_3(\delta, R))}{\mu'(D^s)} < c.
\]

In addition, we assume \( \varepsilon_0 \in (0, 1] \) is sufficiently small such that \( \gamma_c \leq \delta \). Due to the ubiquitous dependence of all quantities of \( \varepsilon, R \) and \( \iota \) we drop these dependencies. For convenience we write \( D_i = D_i^c(\gamma_c, R), i = 2, 3 \).

**Step 1: Reduction to expressions based on events on \((0, T_1)\).** We start with the estimate

\[
\sup_{x \in D_2} \mathbb{E} \left[ e^{\theta|\lambda_x|\tau_x - i} \left( 1 + |1\{X^x(\tau_x); x \in U\} - 1\{W_{R^*} \in \mathcal{F}_{U \cap D^s}(\phi)\} \right) \right] \leq S_{11} + S_{12} + S_2 + S_3,
\]

where

\[
S_{11} := \sum_{k=1}^{\infty} \sup_{y \in D_2} \mathbb{E} \left[ e^{\theta|\lambda_x|\tau_x - T_k} \right. \\
\left. \quad \cdot 1\{\tau_y = T_k\} \cap \{s = T_k\} \left( 1 + |1\{X^x(\tau_y; y) \in U\} - 1\{W_{R^*} \in \mathcal{F}_{U \cap D^s}(\phi)\} \right) \right],
\]

\[
S_{12} := \sum_{k=1}^{\infty} \sup_{y \in D_2} \mathbb{E} \left[ e^{\theta|\lambda_x|\tau_y - T_k} |1\{y \in (T_{k-1}, T_k)\} \cap \{s = T_k\} \right],
\]

\[
S_2 := \sum_{k=1}^{\infty} \sum_{\ell=1}^{k-1} \sup_{y \in D_2} \mathbb{E} \left[ e^{\theta|\lambda_x|\tau_y - T_k} |1\{y \in (T_{\ell-1}, T_{\ell})\} \cap \{s = T_k\} \right],
\]

\[
S_3 := \sum_{k=1}^{\infty} \sum_{\ell=k+1}^{\infty} \sup_{y \in D_2} \mathbb{E} \left[ e^{\theta|\lambda_x|\tau_y - T_k} |1\{y \in (T_{\ell-1}, T_{\ell})\} \cap \{s = T_k\} \right].
\]

In the sequel we estimate the preceding expressions using the representations in (4.2) and the strong Markov property with respect to the \((\mathcal{F}_t)_{t \geq 0}\)-stopping times \( T_k \).

**S11:** The term \( S_{11} \) is treated first since it is the only one of order \( O(1)_{\varepsilon \to 0} \), while all other expressions are \( o(1)_{\varepsilon \to 0} \). We denote the symmetric difference \( E_1 \triangle E_2 := (E_1 \setminus E_2) \cup (E_2 \setminus E_1) \) for
events $E_1, E_2$. In the sequel we repeatedly use strong Markov estimates of the following type

$$E \left[ \mathbf{1}\left\{ \tau_y = T_k \right\} \cap \bigcap_{j=1}^{k-1} A^\circ_y \cap B^\circ_k \left( 1 + \mathbf{1}\{X^\varepsilon(T_k; y) \in U \} - \mathbf{1}\{W_k \in \frac{1}{\varepsilon} \mathcal{J}^U \cap \mathcal{D}^\varepsilon(\phi) \} \right) \right]$$

$$\leq E \left[ \left( \bigcap_{j=1}^{k-1} A^\circ_y \cap A^\circ_j \right) \mathbf{1}\left( B^\circ_k \cap B^\circ_k \right) \left( 1 + \mathbf{1}\{X^\varepsilon(T_k; y) \in U \} \triangle \left\{ W_k \in \frac{1}{\varepsilon} \mathcal{J}^U \cap \mathcal{D}^\varepsilon(\phi) \right\} \right) \right]$$

$$= E \left[ \left( \bigcap_{j=1}^{k-1} A^\circ_y \cap A^\circ_j \right) \mathbf{E} \left[ \left( B^\circ_k \cap B^\circ_k \right) \left( 1 + \mathbf{1}\{X^\varepsilon(T_k; y) \in U \} \triangle \left\{ W_k \in \frac{1}{\varepsilon} \mathcal{J}^U \cap \mathcal{D}^\varepsilon(\phi) \right\} \right) \right] \right]$$

$$= E \left[ \left( \bigcap_{j=1}^{k-1} A^\circ_y \cap A^\circ_j \right) \mathbf{E}_{X^\varepsilon(T_k; y)} \left[ \left( B^\circ_k \cap B^\circ_k \right) \left( 1 + \mathbf{1}\{X^\varepsilon(T_k; y) \in U \} \triangle \left\{ W_k \in \frac{1}{\varepsilon} \mathcal{J}^U \cap \mathcal{D}^\varepsilon(\phi) \right\} \right) \right] \right]$$

$$\leq E \left[ \left( \bigcap_{j=1}^{k-1} A^\circ_y \cap A^\circ_j \right) \sup_{y \in D_2} \mathbf{E} \left[ \left( B^\circ_k \cap B^\circ_k \right) \left( 1 + \mathbf{1}\{X^\varepsilon(T_k; y) \in U \} \triangle \left\{ W_k \in \frac{1}{\varepsilon} \mathcal{J}^U \cap \mathcal{D}^\varepsilon(\phi) \right\} \right) \right] \right].$$

The $(k - 1)$-fold iteration of this argument yields

$S_{11} \leq \sum_{k=1}^{\infty} \sup_{y \in D_2} \mathbb{P}(A_y \cap A^\circ)^{k-1} \sup_{y \in D_2} \mathbf{E} \left[ \left( B^\circ_k \cap B^\circ_k \right) \left( 1 + \mathbf{1}\{X^\varepsilon(T_k; y) \in U \} \triangle \left\{ W_k \in \frac{1}{\varepsilon} \mathcal{J}^U \cap \mathcal{D}^\varepsilon(\phi) \right\} \right) \right]$

$$= \sup_{y \in D_2} \mathbf{E} \left[ \left( B^\circ_k \cap B^\circ_k \right) \left( 1 + \mathbf{1}\{X^\varepsilon(T_k; y) \in U \} \triangle \left\{ W_k \in \frac{1}{\varepsilon} \mathcal{J}^U \cap \mathcal{D}^\varepsilon(\phi) \right\} \right) \right] \cdot \left( 1 - \sup_{y \in D_2} \mathbb{P}(A_y \cap A^\circ) \right).$$

(4.4)

$S_{12}$: The remaining diagonal term is estimated as follows

$S_{12} \leq 2 \sum_{k=1}^{\infty} \sup_{y \in D_2} \mathbf{E} \left[ e^{\theta_k \lambda_k T_k} \mathbf{1}\left( \bigcap_{j=1}^{k-1} (A^\circ_y \cap A^\circ_j) \cap (C^k_y \cap B^\circ_k) \right) \right].$

For $k \geq 1$ we obtain by the analogous strong Markov arguments as for the term $S_{11}$

$$\sup_{y \in D_2} \mathbf{E} \left[ e^{\theta_k \lambda_k T_k} \mathbf{1}\left( \bigcap_{j=1}^{k-1} (A^\circ_y \cap A^\circ_j) \cap (C^k_y \cap B^\circ_k) \right) \right] \leq \sup_{y \in D_2} \mathbb{P}(A_y \cap A^\circ)^{k-1} \sup_{y \in D_2} \mathbf{E} \left[ e^{\theta_k \lambda_k T_k} \mathbf{1}\left( C^k_y \cap B^\circ_k \right) \right],$$

such that

$S_{12} \leq 2 \sup_{D_2} \mathbf{E} \left[ e^{\theta_k \lambda_k T_k} \mathbf{1}\left( C^k_y \cap B^\circ_k \right) \right] \sum_{k=1}^{\infty} \sup_{D_2} \mathbb{P}(A_y \cap A^\circ)^{k-1} \leq \frac{2 \sup_{y \in D_2} \mathbf{E} \left[ e^{\theta_k \lambda_k T_k} \mathbf{1}\left( C^k_y \right) \right]}{1 - \sup_{y \in D_2} \mathbb{P}(A_y \cap A^\circ)}.$

(4.5)

$S_2$: The estimate of $\{ \tau_y \in (T_{k-1}, T_k) \}$ and the representation of $\{ s = T_k \}$ yield

$S_2 \leq 2 \sum_{k=1}^{\infty} \sum_{\ell=1}^{k-1} \sup_{y \in D_2} \mathbf{E} \left[ e^{\theta_k \lambda_k (T_{\ell-1} + \cdots + T_k)} \mathbf{1}\left( \bigcap_{j=\ell}^{k-1} (A^\circ_y \cap B^\circ_k) \mathbf{1}\left( \bigcap_{j=1}^{\ell-1} (A^\circ_y \cap A^\circ_j) \cap ((B^\circ_k \cup C^k_y) \cap A^\circ_j) \right) \right) \right].$
For each of the summands \( k \in \mathbb{N} \) and \( k - 1 \geq \ell \geq 1 \) we combine the mutual independence of the families \( (T_k)_{k \in \mathbb{N}} \) and \( (W_k)_{k \in \mathbb{N}} \) with the analogous strong Markov estimate and obtain

\[
sup_{y \in D_2} E \left[ e^{\theta \lambda_c (t_{\ell+1} + \cdots + t_k)} 1 \left( \bigcap_{j=\ell+1}^{k-1} A^y_j \cap B^y_k \right) 1 \left( \bigcap_{j=1}^{\ell-1} (A^y_j \cap A^y_j) \cap (B^y_j \cup C^y_j) \cap A^y_j \right) \right] \\
\leq (1 - P(A^\diamond)) E \left[ e^{\theta \lambda_c T_1} \right] sup_{y \in D_2} P(B^y \cup C^y) E \left[ e^{\theta \lambda_c T_1} \right]^{k-1} P(A^\diamond)^{k-1} \\
\sup_{y \in D_2} P(A^y \cap A^\diamond)^{\ell-1} \left( E \left[ e^{\theta \lambda_c T_1} \right] P(A^\diamond) \right)^{-(\ell-1)}.
\]

Obviously we have

\[
sup_{y \in D_2} P(A^y \cap A^\diamond) \leq E \left[ e^{\theta \lambda_c T_1} \right] P(A^\diamond),
\]

such that for any \( k \geq 1 \)

\[
\sum_{\ell=1}^{k-1} sup_{y \in D_2} P(A^y \cap A^\diamond)^{\ell-1} \left( E \left[ e^{\theta \lambda_c T_1} \right] P(A^\diamond) \right)^{-(\ell-1)} \leq k - 1,
\]

and hence

\[
S_2 \leq 2 \sum_{y \in D_2} P(B^y) sup_{y \in D_2} \sum_{k=1}^\infty (k-1) E \left[ e^{\theta \lambda_c T_1} \right]^{k-1} P(A^\diamond)^{k-1} \\
= 2 P(B^\diamond) sup_{y \in D_2} \frac{P(B^y \cup C^y) \cap A^\diamond}{1 - E \left[ e^{\theta \lambda_c T_1} \right] P(A^\diamond)}.
\]

\( S_3 \): Due to the doubly infinite summation \( S_3 \) turns out to be the cumbersome case here. We rewrite \( S_3 \) in terms of the events

\[
S_3 = 2 \sum_{k=1}^\infty \sum_{\ell=k+1}^\infty sup_{y \in D_2} E \left[ e^{\theta \lambda_c \ell k + \cdots + t_k} 1 \left( \bigcap_{j=\ell+1}^{k-1} (A^y_j \cap A^y_j) \right) 1 \left( \bigcap_{j=1}^{\ell-1} (A^y_j \cap B^y_k) \right) 1 \left( \bigcap_{j=k+1}^{\ell-1} A^y_j \cap (B^y_{j-1} \cup C^y_j) \right) \right] \\
\leq sup_{y \in D_2} P(A^y \cap A^\diamond)^{k-1} sup_{y \in D_2} P(A^y \cap B^\diamond) sup_{y \in D_2} E \left[ e^{\theta \lambda_c T_1} 1(A^y) \right]^{\ell-k} sup_{y \in D_2} E \left[ e^{\theta \lambda_c T_1} 1(B^y \cup C^y) \right].
\]

![Image](image-url)
Assuming that \( \sup_{y \in D_2} \mathbb{E}[e^{	heta \lambda r T_1} \mathbf{1}(A_y)] < 1 \) for \( \varepsilon \in (0, \varepsilon_0) \) for \( \varepsilon_0 \in (0, 1] \) sufficiently small, which we verify in estimate \((4.15)\) of Step 3, we obtain

\[
S_{3/2} \leq \sum_{k=1}^{\infty} \sup_{D_2} \mathbb{P}(A_y \cap A^c)^{k-1} \left( \sup_{D_2} \mathbb{P}(A_y \cap B^c) \mathbb{E} \left[ e^{\theta \lambda r T_1} \mathbf{1}(B_y \cup C_y) \right] \right)
\]

\[
= \sup_{D_2} \mathbb{P}(A_y \cap B^c) \left( \frac{\sup_{D_2} \mathbb{E}[e^{\theta \lambda r T_1} \mathbf{1}(A_y)]}{1 - \sup_{D_2} \mathbb{E}[e^{\theta \lambda r T_1} \mathbf{1}(A_y)]} \right)
\]

\[
\leq \mathbb{E}[e^{\theta \lambda r T_1}] \left( \frac{\sup_{D_2} \mathbb{P}(A_y \cap B^c) \sup_{y \in D_2} \mathbb{E}[e^{\theta \lambda r T_1} \mathbf{1}(B_y \cup C_y)]}{(1 - \sup_{D_2} \mathbb{E}[e^{\theta \lambda r T_1} \mathbf{1}(A_y)])^2} \right).
\]

(4.7)

**Step 2: Precise estimates of the events on \((0, T_1)\).**

**Claim 1:** For \( y \in D_2 \) it follows that

\[
1(A_y) \leq 1 \{ \varepsilon W_1 \in \mathcal{J}^{D_1}(\phi) \} + 1\{\|\varepsilon W_1\| > \frac{\gamma_0}{a}\} 1\{T_1 < \kappa_0|\ln(\gamma_0)|\} + 1(G_y^c), \tag{4.8}
\]

\[
1(B_y) \leq 1 \{ \varepsilon W_1 \in \mathcal{J}^{D_2}(\phi) \} + 1\{\|\varepsilon W_1\| > \frac{\gamma_0}{a}\} 1\{T_1 < \kappa_0|\ln(\gamma_0)|\} + 1\{\|\varepsilon W_1\| \leq \frac{\gamma_0}{a}\} 1\{T_1 < \kappa_0|\ln(\gamma_0)|\} + 1(G_y^c), \tag{4.9}
\]

\[
1(C_y) \leq 1 \{ T_1 < \kappa_1 \gamma_0 \} + 1(G_y^c). \tag{4.10}
\]

**Proof of Claim 1:** We prove \((4.8)\): For \( y \in D_2 \) we have by construction for \( a = 5 \vee g_1(\mathcal{R}) \)

\[
1(A_y) \leq 1(A_y) 1(G_y^c) 1\{\|\varepsilon W_1\| > \frac{\gamma_0}{a}\} + 1\{\|\varepsilon W_1\| \leq \gamma_0\} + 1(G_y^c)
\]

\[
\leq 1(A_y) 1(G_y^c) 1\{\|\varepsilon W_1\| > \frac{\gamma_0}{a}\} \{T_1 \geq \kappa_0|\ln(\gamma_0)|\}
\]

\[
+ 1(A_y) 1(G_y^c) 1\{\|\varepsilon W_1\| > \frac{\gamma_0}{a}\} \{T_1 < \kappa_0|\ln(\gamma_0)|\} + 1\{\|\varepsilon W_1\| \leq \frac{\gamma_0}{a}\} + 1(G_y^c)
\]

\[
\leq 1\{\|\varepsilon W_1\| > \frac{\gamma_0}{a}\} 1\{T_1 < \kappa_0|\ln(\gamma_0)|\} + 1\{\|\varepsilon W_1\| \leq \frac{\gamma_0}{a}\} + 1(G_y^c)
\]

\[
\leq 1\{\|\varepsilon W_1\| > \frac{\gamma_0}{a}\} + 1\{\|\varepsilon W_1\| \leq \frac{\gamma_0}{a}\} 1\{T_1 < \kappa_0|\ln(\gamma_0)|\} + 1(G_y^c)
\]

We use that by definition

\[
\bigcap_{y \in B_{\frac{3}{4}\gamma_0}(\phi)} \{\varepsilon W_1 \in \mathcal{J}^{D_2}(\phi)\} = \bigcap_{y \in B_{\frac{3}{4}\gamma_0}(\phi)} \{y + G(y, \varepsilon W_1) \in D_2\}.
\]

Then for \( y \in B_{\frac{3}{4}\gamma_0}(\phi) \) on \( \{\|\varepsilon W_1\| \leq \frac{\gamma_0}{a}\} \) we obtain for \( \varepsilon \in (0, \varepsilon_0) \) with \( \varepsilon_0 \in (0, 1] \) sufficiently small the estimate

\[
\|y + G(y, \varepsilon W_1) - \phi\| \leq \frac{3}{4}\gamma_0 + \frac{\gamma_0}{a} < \gamma_0.
\]

34
The obvious inclusion \( B_{\gamma_\varepsilon}(\phi) \subseteq D_2 \) for \( \varepsilon \) sufficiently small yields

\[
1\{\|\varepsilon W_\varepsilon\| \leq \frac{\gamma_\varepsilon}{a}\} = 1\{\|\varepsilon W_\varepsilon\| \leq \frac{\gamma_\varepsilon}{a}\} 1\left( \bigcap_{y \in B_{\frac{\gamma_\varepsilon}{a}}(\phi)} \{\varepsilon W_1 \in \mathcal{J}^{D_2}(y)\} \right).
\]

Hence the inclusion \( \mathcal{J}^{D_2}(B_{\gamma_\varepsilon}(\phi)) \subseteq \mathcal{J}^D(\phi) \) provides the desired result (4.8).

\[
1(A_y) \leq 1\{\varepsilon W_1 \in \mathcal{J}^D(\phi)\} + 1\{\|\varepsilon W_\varepsilon\| > \frac{\gamma_\varepsilon}{a}\} 1\{T_1 < \kappa_0|\ln(\gamma_\varepsilon)|\} + 1(G_y^\varepsilon).
\]

We recall the Lipschitz constant \( B \) of \( G \) given in (2.8).

We prove (4.9): Hypothesis (D.3) implies for \( y \in D_2 \) and \( t \geq \kappa_1 \gamma_\varepsilon \) that \( u(t; y) \subseteq D_3 \). Hence on \( G_y \cap \{\|\varepsilon W_1\| \leq \frac{\gamma_\varepsilon}{a}\} \) it follows \( Y^\varepsilon(T_1; x) + G(Y^\varepsilon(T_1; x), \varepsilon W_1) \subseteq D_2 \), which implies \( B_y \cap G_y \cap \{\|\varepsilon W_1\| \leq \frac{\gamma_\varepsilon}{a}\} \cap \{T_1 > \kappa_1 \gamma_\varepsilon\} = \emptyset \). Therefore, we obtain the estimate

\[
1(B_y) \leq 1(B_y) 1(G_y) + 1(G_y^\varepsilon)
\]

We conclude (4.10) by the obvious inclusions

\[
\mathcal{J}^{D_2}(B_{\gamma_\varepsilon}(\phi)) \subseteq \mathcal{J}^{D_3}(\phi), \quad \text{and} \quad D_3 \subseteq D^\varepsilon \cup (D \setminus D_3).
\]

We prove (4.11): By Hypothesis (D.3) \( y \in D_2 \) and \( t \geq \kappa_1 \gamma_\varepsilon \) imply \( u(t; y) \subseteq D_3 \). Hence the event \( G_y \cap \{T_1 \geq \kappa_1 \gamma_\varepsilon\} \) implies that \( Y^\varepsilon(t; y) \subseteq B_{\frac{\gamma_\varepsilon}{a}}(u(t; y)) \subseteq D_2(\gamma_\varepsilon, R) \) for all \( t \in [\kappa_1 \gamma_\varepsilon, T_1] \) and \( C_y \cap G_y \cap \{T_1 \geq \kappa_1 \gamma_\varepsilon\} = \emptyset \). This implies the desired result

\[
1(C_y) \leq 1(C_y) 1(G_y) 1\{T_1 \geq \kappa_1 \gamma_\varepsilon\} + 1\{T_1 < \kappa_1 \gamma_\varepsilon\} + 1(G_y^\varepsilon) = 1\{T_1 < \kappa_1 \gamma_\varepsilon\} + 1(G_y^\varepsilon),
\]

and finishes the proof of Claim 1.

We recall the Lipschitz constant \( K_2 \) of \( G \) given in (2.8).

Claim 2: For \( y \in D_2 \) and \( U \in B(H) \) it follows that

\[
1(A_y \cap B^\circ) \leq 1\{\|\varepsilon W_1\| > \frac{\gamma_\varepsilon}{a}\} 1\{T_1 < \kappa_0|\ln(\gamma_\varepsilon)|\} + 1(G_y^\varepsilon), \quad \text{(4.11)}
\]

\[
1(B_y \cap A^\circ) \leq 1\{\varepsilon W_1 \in \mathcal{J}^{D \setminus D_3(\gamma_\varepsilon)}(\phi)\} + 1\{\|\varepsilon W_\varepsilon\| > \frac{\gamma_\varepsilon}{a}\} 1\{T_1 < \kappa_0|\ln(\gamma_\varepsilon)|\}
\]

\[
+ 1\{T_1 < \kappa_1 \gamma_\varepsilon\} + 1(G_y^\varepsilon), \quad \text{(4.12)}
\]

\[
1(B_y \cap B^\circ) 1\{(X(T_1; y) \in U) \triangle \{\varepsilon W_1 \in \mathcal{J}^U(\phi)\}\} \leq 1\{\varepsilon W_1 \in \mathcal{J}^{B_{(K_2+1)\gamma_\varepsilon}(\beta U) \cap D^\varepsilon}(\phi)\}
\]

\[
+ 1\{\|\varepsilon W_\varepsilon\| > \frac{\gamma_\varepsilon}{a}\} 1\{T_1 < \kappa_0|\ln(\gamma_\varepsilon)|\} + 1\{T_1 < \kappa_1 \gamma_\varepsilon\} + 1(G_y^\varepsilon). \quad \text{(4.13)}
\]
Proof of Claim 2: Estimate (4.11) is a direct consequence of (4.8) in Claim 1. With the help of (4.9) the proof of (4.12) is straightforward. For the proof of (4.13) we use the inclusion $\mathcal{J}^U(B_{y_\varepsilon}(\phi)) \subseteq \mathcal{J}^U(\phi)$ and the global Lipschitz continuity of $y \mapsto y + G(y, z)$ with Lipschitz constant $1 + K_2$ as follows

$$1(B_y \cap B_\varepsilon)1\{X(T_1; y) \in U\} \Delta \{\varepsilon W_1 \in \mathcal{J}^U\} \leq 1\{\varepsilon W_1 \in \mathcal{J}^{D'}(\phi) \cap (\mathcal{J}^U(B_{y_\varepsilon}(\phi)) \Delta \mathcal{J}^U(\phi))\} + 1\{\|\varepsilon W_1\| > \frac{\gamma_\varepsilon}{a}\}1\{T_1 < \kappa_0|\ln(\gamma_\varepsilon)|\} + 1\{T_1 < \kappa_1 \gamma_\varepsilon\} + 1(G^\varepsilon_x).$$

Finally we see for the first term the inclusions

$$\{\varepsilon W_1 \in \mathcal{J}^{D'}(\phi) \cap (\mathcal{J}^U(B_{y_\varepsilon}(\phi)) \Delta \mathcal{J}^U(\phi))\} \subseteq \{\varepsilon W_1 \in \mathcal{J}^{D'}(\phi) \cap (\mathcal{J}^U(\phi) \setminus \mathcal{J}^U(B_{y_\varepsilon}(\phi)))\} \subseteq \{\varepsilon W_1 \in \mathcal{J}^{B_{(k_{2}+1)\gamma_c}(\partial U)\cap D'}(\phi)\}.$$ 

This finishes the proof of (4.13) and of Claim 2.

Step 3: Estimates of the resulting expressions: Step 2 provides the estimates to dominate respectively the term $S_{11}$ by (4.4), $S_{12}$ by (4.5), $S_2$ by (4.6) and $S_3$ by (4.7). In the sequel we estimate the probabilities of the events contained in these expressions.

Event $A_y$: Note that due to Hypothesis (S.2) and the choice $\gamma^* < \rho^*$ in (3.32) we have

$$\lim_{\varepsilon \to 0^+} \mathbb{P}(\|\varepsilon W_1\| > \frac{\gamma_\varepsilon}{a}(\frac{\varepsilon^\alpha}{a \gamma_\varepsilon})^{\beta_\varepsilon})^{-1} = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0^+} |\ln(\gamma_\varepsilon)|\left(\frac{e^{\alpha}}{(a \gamma_\varepsilon)^\alpha}\right)^{\beta_\varepsilon} = 0. \quad (4.14)$$

Together with the estimate (4.8) there is a constant $\varepsilon_0 \in (0, 1]$ such that $\varepsilon \in (0, \varepsilon_0]$ implies

$$\sup_{y \in D_2} \mathbb{P}(A_y \cap A^c) \leq \sup_{y \in D_2} \mathbb{E}[e^{\theta \lambda_\varepsilon T_1}1(A_y)] \leq 1 - \left(\frac{1 - \theta}{1 - \theta / \beta_\varepsilon} - 2c\right) \frac{\lambda_\varepsilon}{\beta_\varepsilon} \leq 1 - \frac{1 - \theta}{1 - \theta / \beta_\varepsilon} \leq 1 - (1 - c) \frac{\lambda_\varepsilon}{\beta_\varepsilon} < 1. \quad (4.15)$$

Event $B_y$: Using that $\nu$ is regularly varying and the initial choice of $\mathcal{R} \geq \mathcal{R}_0$ in (4.3) we obtain

$$\lim_{\varepsilon \to 0} \frac{\mu(\frac{1}{\nu} J^{D \setminus D_3(\gamma_\varepsilon, \mathcal{R})(\phi))}}{\beta_\varepsilon} \left(\frac{\lambda_\varepsilon}{\beta_\varepsilon}\right)^{-1} \leq \lim_{\varepsilon \to 0} \frac{\mu(\frac{1}{\nu} J^{D \setminus D_3(\gamma_\varepsilon, \mathcal{R})(\phi))}}{\nu(\frac{1}{\nu} J^D(\phi))} = \frac{\mu(\mathcal{J}^D(\phi))}{\mu(\mathcal{J}^D(\phi))} \leq c. \quad (4.16)$$

In addition, by the choice of scales (3.34) there is $\varepsilon_0 \in (0, 1]$ such that for $\varepsilon \in (0, \varepsilon_0]$ we have

$$\mathbb{P}(T_1 < \kappa_1 \gamma_\varepsilon) = 1 - e^{-\kappa_1 \gamma_\varepsilon \beta_\varepsilon} \leq 2 c \kappa_1 \gamma_\varepsilon \beta_\varepsilon \leq c \frac{\lambda_\varepsilon}{\beta_\varepsilon}. \quad (4.17)$$

36
Hypothesis (S.2) and (4.3) we have $\varepsilon \in (0, \varepsilon_0]$ implies

$$\sup_{y \in D_2} \mathbb{P}(B_y \cap B^c) \leq \sup_{y \in D_2} \mathbb{E}[e^{\theta \lambda_y T_1} 1(B_y)]$$

$$\leq \mathbb{E}[e^{\theta \lambda_y T_1}] \mathbb{P}(W_1 = \frac{1}{e} \mathcal{F}^D(\phi)) + \mathbb{E}[e^{\theta \lambda_y T_1} 1(T_1 < \kappa_0 \ln(\gamma_\varepsilon))]|\mathbb{P}(\|\varepsilon W_1\| > \gamma_\varepsilon) + \mathbb{E}[e^{\theta \lambda_y T_1} 1(\mathcal{G}_y^c)]$$

$$\leq \frac{\lambda_x}{\beta_\varepsilon - \theta \lambda_x} + (1 + c)\beta_\varepsilon \left(\frac{c\alpha(1 - e^{-\beta_\varepsilon \theta \lambda_x} \kappa_0 \ln(\gamma_\varepsilon))}{\beta_\varepsilon}ight) + \frac{\beta_\varepsilon}{\beta_\varepsilon - \theta \lambda_x} (e^{-\frac{1}{\beta_\varepsilon}} + e^{-\frac{\kappa_0 T}{\beta_\varepsilon}}) \leq (1 + 5c)\frac{\lambda_x}{\beta_\varepsilon}. \quad (4.18)$$

**Event $C_y$:** By estimate (4.10) we have a constant $\varepsilon_0 \in (0, 1]$ such that for $\varepsilon \in (0, \varepsilon_0]$ it holds

$$\sup_{y \in D_2} \mathbb{P}(C_y) \leq \sup_{y \in D_2} \mathbb{E}[e^{\theta \lambda_y T_1} 1(C_y)] \leq \sup_{y \in D_2} \mathbb{E}[e^{\theta \lambda_y T_1} (1(T_1 < \kappa_1 \gamma_\varepsilon) + 1(\mathcal{G}_y^c))] \leq 3c \frac{\lambda_x}{\beta_\varepsilon}. \quad (4.19)$$

**Events $A_y \cap B^c$ and $B_y \cap A^c$:** By (4.11) there is $\varepsilon_0 \in (0, 1]$ such that for $\varepsilon \in (0, \varepsilon_0]$ we obtain with the analogous calculations

$$\sup_{y \in D_2} \mathbb{P}(A_y \cap B^c) \leq \sup_{y \in D_2} \mathbb{E}[e^{\theta \lambda_y T_1} 1(A_y \cap B^c)] \leq \frac{\lambda_x}{\beta_\varepsilon}. \quad (4.20)$$

With the help of (4.12), the regular variation of $\nu$ and (4.3) there is a constant $\varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_0]$ it follows that

$$\sup_{y \in D_2} \mathbb{P}(B_y \cap A^c) \leq \sup_{y \in D_2} \mathbb{E}[e^{\theta \lambda_y T_1} 1(B_y \cap A^c)] \leq \frac{\lambda_x}{\beta_\varepsilon}. \quad (4.21)$$

**Step 4: Concluding estimates of the sums of (4.4):**

**Estimate $S_1$:** Since $m^c(\partial U) = \mu(\mathcal{J}^D(\phi^c)) = 0$ by assumption, the regular variation of $\nu$ by Hypothesis (S.2) and (4.3) we have $\varepsilon_0 \in (0, 1]$ such that $\varepsilon \in (0, \varepsilon_0)$ yields

$$\lim_{\varepsilon \to 0} \mathbb{P}((\varepsilon \times W_1) \in \mathcal{J}^D(\kappa_2 + 1) \cap (\partial U \cap \mathcal{D}^D(\phi))) \left(\frac{\lambda_x}{\beta_\varepsilon}\right)^{-1} = \lim_{\varepsilon \to 0} \frac{\nu((\frac{1}{\varepsilon} \mathcal{J}^D(\kappa_2 + 1) \cap \partial U \cap \mathcal{D}^D(\phi)))}{\nu(\rho^\varepsilon B_1^c(0))} \frac{\nu(\rho^\varepsilon B_1^c(0))}{\nu(\frac{1}{\varepsilon} \mathcal{J}^D(\phi))} \leq \frac{\mu(\mathcal{J}^D(\phi))}{\mu(\mathcal{J}^D(\phi))} \varepsilon. \quad (4.22)$$

37
Hence (4.13), (4.18) and (4.22) combined yield
\[
\sup_{y \in D_2} \mathbb{E} \left[ 1 \left( B_y \cap B^c \right) \left( 1 + 1 \{ X^\varepsilon(T_1; y) \in U \} \triangle \{ \varepsilon W_1 \in \mathcal{J}^{U \cap D^c} \} \right) \right] \\
\leq \mathbb{P}(\varepsilon W_1 \in \mathcal{J}^{D^c}(\phi)) + \mathbb{P}(\varepsilon W_1 \in \mathcal{J}^{B(K_2 + 1) \cap D^c}(\phi)) + \mathbb{P}(T_1 < \kappa_0|\ln(\gamma_\varepsilon)|)\mathbb{P}(\|\varepsilon W_1\| > \frac{\gamma_\varepsilon}{a}) \\
+ \sup_{y \in D_2} \mathbb{P}(\mathcal{G}_y^c) \\
\leq (1 + 3c) \frac{\lambda_\varepsilon}{\beta_\varepsilon}.
\]

Finally, for \( \varepsilon \in (0, \varepsilon_0] \) the sum \( S_{11} \) given in (4.4) satisfies due to \( c \leq \frac{1}{4} \)
\[
S_{11} \leq \frac{1 + 3c}{1 - c} \leq 1 + 6c.
\] (4.23)

\( S_{12} \) given by (4.5): By (4.15) and (4.19) the sum \( S_{12} \) given in (4.5) satisfies for \( \varepsilon \in (0, \varepsilon_0] \)
\[
S_{12} \leq \frac{6c}{1 - c} \leq 8c.
\] (4.24)

\( S_2 \) given by (4.6): Using the estimates (4.19), (4.21) and the choice \( c \in (0, \frac{1 - \theta}{2}) \) the sum \( S_2 \) given in (4.6) satisfies for \( \varepsilon \in (0, \varepsilon_0] \) the estimate
\[
S_2 \leq \frac{8c(1 + 5c)}{((1 - c)\frac{\lambda_\varepsilon}{\beta_\varepsilon})^2} \left( \frac{\lambda_\varepsilon}{\beta_\varepsilon} \right)^2 \leq 48c.
\] (4.25)

\( S_3 \) given by (4.7): Using (4.15) and (4.20) we obtain \( \varepsilon_0 \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) implies
\[
S_3 \leq 4 \frac{c^2}{((1 - c)\frac{\lambda_\varepsilon}{\beta_\varepsilon})^2} \leq \frac{16c^2}{(1 - c)^2} \leq 4c.
\] (4.26)

We finally collect (4.23) - (4.26) and infer the existence of \( \varepsilon_0 \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) yields
\[
\sup_{x \in D_2} \mathbb{E} \left[ e^{\delta x} \| X^\varepsilon(t; x) \| \left( 1 + 1 \{ X^\varepsilon(T_1; x) \in U \} \triangle \{ \| \varepsilon W_1 \| \in \mathcal{J}^{U \cap D^c}(\phi) \} \right) \right] \leq 1 + 66c.
\]

Since \( c \in (0, \frac{1 - \theta}{2}) \) was chosen arbitrary this finishes the proof. \( \square \)

Having established the convergence in probability of the exit locus it is sufficient to establish the uniform integrability. We keep all the notation and the scales of the proof of Proposition 4.3.

**Proposition 4.4.** Under the assumptions of Proposition 4.3 for any \( 0 < p < \alpha \) and \( R \geq R_0 \) there are \( \varepsilon_0, \gamma \in (0, 1] \) and such that
\[
\sup_{\varepsilon \in (0, \varepsilon_0]} \sup_{x \in D_2(\varepsilon^\gamma, R)} \mathbb{E} \left[ \| X^\varepsilon(t; x) - \left( \phi + G(\phi, \varepsilon W_1(\varepsilon)) \right) \|_p \right] < \infty.
\] (4.27)
The proof of Proposition 4.4 is given in Subsection 5.3 of the appendix.

Proof of Theorem 2.7. The convergence $\|X^\varepsilon(\tau;y) - \phi - G(\phi, \varepsilon W_{R^\varepsilon(\varepsilon)})\| \to 0$ in probability as $\varepsilon \to 0$ is established in Proposition 4.3. In addition it holds true uniformly for all $y \in D_2(\varepsilon; R)$. The uniform boundedness of Proposition 4.4 implies the uniform integrability of the family of random variables $(\|X^\varepsilon(\tau;y) - \phi - G(\phi, \varepsilon W_{R^\varepsilon(\varepsilon)})\|^p)_{\varepsilon \in (0, \varepsilon_0)}$ and hence its convergence as $\varepsilon \to 0$ in $L^p$.

The last statement of Theorem 2.7 follows from $\lim_{\varepsilon \to 0} \mathbb{P}(\varepsilon W_{R^\varepsilon(\varepsilon)}(\varepsilon) \in U) = \frac{\mu(U \cap (D^\varepsilon)^Y)}{\mu((D^\varepsilon)^Y)}$ for all $U \in \mathcal{B}(H)$ with $\mu(\partial U) = 0$.

\[ \square \]

5 Appendix

5.1 Proof of Lemma 4.2: the law of the models.

Since the family $(W_k)_{k \in \mathbb{N}}$ is i.i.d. and $B_\varepsilon^k = \{\varepsilon W_k \in (D^\varepsilon)\}$ we have that by construction $R^\varepsilon(\varepsilon)$ is geometrically distributed with rate $\mathbb{P}(B^\varepsilon) = \frac{\lambda_\varepsilon}{\mu_\varepsilon}$. Let $\theta > 0$. We calculate the Laplace transform of $\tilde{g}^\varepsilon(\varepsilon)$

\[ \mathbb{E}\left[e^{-\theta \tilde{g}^\varepsilon(\varepsilon)}\right] = \mathbb{E}\left[e^{-\theta \sum_{k=1}^{\infty} T_k \prod_{j=1}^{k-1}(1-1(B_{j}^\varepsilon))1(B_{j}^\varepsilon)}\right] = \mathbb{E}\left[\prod_{k=1}^{\infty} e^{-\theta T_k \prod_{j=1}^{k-1}(1-1(B_{j}^\varepsilon))1(B_{j}^\varepsilon)}\right] \]

\[ = \sum_{k=1}^{\infty} \mathbb{E}\left[\prod_{j=1}^{k-1}(1-1(B_{j}^\varepsilon))1(B_{j}^\varepsilon)\right] = \sum_{k=1}^{\infty} \mathbb{E}\left[\prod_{j=1}^{k-1} e^{-\theta t_j(1-1(B_{j}^\varepsilon))e^{-\theta t_k 1(B_{k}^\varepsilon)}}\right]. \]

The independence of $(W_k)_{k \in \mathbb{N}}$ and $(T_k)_{k \in \mathbb{N}}$ as well as the stationarity of $(W_k)_{k \in \mathbb{N}}$ yield that each summand takes the form

\[ \mathbb{E}\left[\prod_{j=1}^{k-1} e^{-\theta t_j(1-1(B_{j}^\varepsilon))e^{-\theta t_k 1(B_{k}^\varepsilon)}}\right] = \prod_{j=1}^{k-1} \mathbb{E}\left[e^{-\theta t_j(1-1(B_{j}^\varepsilon))}\right] \mathbb{E}\left[e^{-\theta t_k 1(B_{k}^\varepsilon)}\right] \]

\[ = (\mathbb{E}\left[e^{-\theta t_1(1-1(B_{1}^\varepsilon))}\right])^{k-1} \mathbb{E}\left[e^{-\theta t_1}\mathbb{P}(B_{1}^\varepsilon)\right] \]

\[ = \left(\frac{\beta_\varepsilon}{\theta + \beta_\varepsilon}(1-\frac{\lambda_\varepsilon}{\beta_\varepsilon})\right)^{k-1} \frac{\beta_\varepsilon}{\theta + \beta_\varepsilon} \frac{\lambda_\varepsilon}{\beta_\varepsilon}. \]

Finally we conclude

\[ \mathbb{E}\left[e^{-\theta \tilde{g}^\varepsilon(\varepsilon)}\right] = \sum_{k=1}^{\infty} \left(\frac{\beta_\varepsilon}{\theta + \beta_\varepsilon}(1-\frac{\lambda_\varepsilon}{\beta_\varepsilon})\right)^{k-1} \frac{\beta_\varepsilon}{\theta + \beta_\varepsilon} \frac{\lambda_\varepsilon}{\beta_\varepsilon} \]

\[ = \frac{\beta_\varepsilon}{\theta + \beta_\varepsilon} \frac{\lambda_\varepsilon}{1 - \frac{\mu_\varepsilon}{\beta_\varepsilon}(1 - \frac{\lambda_\varepsilon}{\beta_\varepsilon})} = \frac{\lambda_\varepsilon}{\theta + \frac{\mu_\varepsilon}{\beta_\varepsilon}(1 - \frac{\lambda_\varepsilon}{\beta_\varepsilon})} = \frac{\lambda_\varepsilon}{\theta + \lambda_\varepsilon} = \mathbb{E}\left[\tilde{g}^\varepsilon(\varepsilon)\right]. \]
5.2 Proof of Lemma 3.2: a Campbell type estimate.

Recall the notation from Step 2 of Proposition 3.1. For the \((F_t)_{t\geq 0}\)-predictable process \((H_t)_{t\geq 0}\) given in (3.15) and \(x \in D_2\) we define \(h_x(s-,\varepsilon z) := 2H_s(\langle \Phi_{s,x}^{\varepsilon,z}, G(Y(s-x),\varepsilon z) \rangle)\). Consider the process

\[ Z_t := Z_t^{\varepsilon,x} = \int_0^t \int_{\|z\| \leq \rho} h_x(s-,\varepsilon z) \tilde{N}(dsdz). \]

We define the smooth function \(I_c(r) := \sqrt{r^2 + c^2}\), \(c \in (0, 1]\), with \(I_0(r) = |r|\), which satisfies the following useful properties

\[ |r| \leq I_c(r) \leq |r| + c, \quad r \in \mathbb{R}, \quad (5.1) \]

\[ \sup_{r \in \mathbb{R}} \frac{r}{I_c(r)} = 1 \quad (5.2) \]

\[ I_c(r + h) \leq I_c(r) + I_c(h), \quad r, h \in \mathbb{R}, \quad (5.3) \]

\[ I_c(r)' = \frac{r}{I_c(r)}, \quad r \in \mathbb{R}, \quad I_c(r)'' = \frac{c^2}{I_c^3(r)}, \quad r \in \mathbb{R}. \]

For \(F(r) := \exp(\kappa I_c(r))\) for some parameter \(\kappa > 0\) we first obtain for all \(r \in \mathbb{R}\)

\[ F'(r) = F(r) \frac{\kappa r}{I_c(r)} \quad (5.4) \]

\[ F''(r) = F(r) \left( \frac{\kappa^2 r^2 I_c(r) + \kappa c^2}{I_c(r)^3} \right). \quad (5.5) \]

Applying twice the mean value theorem, and \(5.2\) - \(5.5\) we obtain for all \(r, h \in \mathbb{R}\) the estimate

\[ |F(r + h) - F(r) - F'(r)h| \leq \int_0^1 \int_0^1 |F''(r + \theta' \theta h)|d\theta'd\theta |h^2| \]

\[ \leq \int_0^1 \int_0^1 |F(r + \theta' \theta h)\left( \frac{\kappa^2 r^2 I_c(r + \theta' \theta h) + \kappa c^2}{I_c(r + \theta' \theta h)^3} \right)|d\theta'd\theta |h^2| \]

\[ \leq F(r)F(|h|)\left( \frac{\kappa}{c} \right) |h^2|. \quad (5.6) \]

Itô’s formula for Poisson random measures then yields \(P\)-a.s. for all \(t \geq 0\)

\[ F(Z_t) = 1 + \int_0^t \int_{\|z\| \leq \rho} F(Z_{s-} + h(s-,\varepsilon z)) - F(Z_{s-}) \tilde{N}(dsdz) \]

\[ + \int_0^t \int_{\|z\| \leq \rho} F(Z_{s-} + h(s-,\varepsilon z)) - F(Z_{s-}) - F(Z_{s-}) \frac{\kappa Z_{s-} h(s-,\varepsilon z)}{I_c(Z_{s-})} \nu(dz)ds. \]
Let \( \sigma \) be the \((\mathcal{F}_t)_{t \geq 0}\)-stopping time defined in (3.5) and (3.5). Then for \( \kappa = \kappa^\varepsilon = 8\overline{\theta}_{\varepsilon}^2 = 8\gamma_{\varepsilon}^{2q-2} \) we have
\[
\sup_{s \in [0, \sigma \wedge T^\varepsilon]} \sup_{\|z\| \leq \rho^\varepsilon} (\kappa^\varepsilon)^2 |h(s, \varepsilon z)|^2 \leq \frac{2(\varepsilon \rho^\varepsilon)^2}{\gamma_{\varepsilon}^{4p+4}} \sup_{s \in [0, \sigma \wedge T^\varepsilon]} |H_s| d(\mathcal{R})g_1(\mathcal{R})
\]
\[
\leq 2d(\mathcal{R})g_1(\mathcal{R}) \left( \frac{(\varepsilon \rho^\varepsilon)^2}{\gamma_{\varepsilon}^{4p+4}} \right) \leq 2d(\mathcal{R})g_1(\mathcal{R}) \Gamma(\varepsilon) \to 0, \quad \varepsilon \to 0,
\]
where \( \Gamma(\varepsilon) = \Gamma(\varepsilon)/T^\varepsilon \) in (3.6). Hence there is \( \varepsilon_0 \in (0, 1) \) with \( \sup_{s \in [0, \sigma \wedge T^\varepsilon]} \sup_{\|z\| \leq \rho^\varepsilon} |h(s, \varepsilon z)| \leq 1 \) for all \( \varepsilon \in (0, \varepsilon_0] \). Then due to the optional stopping theorem the second term vanishes. Using \( \rho^\varepsilon \geq 1 \), the constant \( C_1 = \int_{\|z\| \leq 1} \|z\|^2 \nu(dz) + \nu(B_1^0) \) and the parametrization \( c = c_{\varepsilon} = \gamma_{\varepsilon} \) we have
\[
E\left[F(Z_{t \wedge \sigma})\right] \leq 1 + E\left[\int_0^{t \wedge \sigma} \int_{\|z\| \leq \rho^\varepsilon} (F(Z_{s^-} + h(s, \varepsilon z)) - F(Z_{s^-}) - F(Z_{s^-}) \frac{\kappa Z_{s^-} h(s, \varepsilon z)}{\Gamma(Z_{s^-})}) \nu(dz) ds\right]
\]
\[
\leq 1 + E\left[\int_0^{t \wedge \sigma} \int_{\|z\| \leq \rho^\varepsilon} F(Z_{s^-})F(|h(s, \varepsilon z)|(\kappa_{\varepsilon})^2 + \frac{\kappa^\varepsilon}{c}) |h(s, \varepsilon z)|^2 \nu(dz) ds\right]
\]
\[
\leq 1 + E\left[\int_0^{t \wedge \sigma} \int_{\|z\| \leq \rho^\varepsilon} F(Z_{s^- \wedge \sigma})F(|h(s \wedge \sigma, \varepsilon z)|(\kappa_{\varepsilon})^2 + \frac{\kappa^\varepsilon}{c}) |h(s \wedge \sigma, \varepsilon z)|^2 \nu(dz) ds\right]
\]
\[
\leq 1 + C_2 \int_0^{t \wedge \sigma} E\left[F(Z_{s^- \wedge \sigma})\right] F(\sqrt{2d(\mathcal{R})}g_1(\mathcal{R}) \Gamma(\varepsilon))(\kappa^\varepsilon)^2 + \frac{\kappa^\varepsilon}{\gamma_{\varepsilon}^{4p+4}} \|z\|^2 \nu(dz) ds
\]
\[
\leq 1 + C_3 \int_0^{t \wedge \sigma} E\left[F(Z_{s^- \wedge \sigma})\right] ds,
\]
where \( C_2 = 2d(\mathcal{R})g_1(\mathcal{R}) \) and \( C_3 = C_1C_2\Gamma(\varepsilon)F(1) \). Setting
\[
\phi_{\varepsilon}(t) := E\left[F(Z_{t \wedge \sigma})\right], \quad t \geq 0,
\]
we have
\[
\phi_{\varepsilon}(t) \leq 1 + C_3 \Gamma(\varepsilon) \int_0^t \phi_{\varepsilon}(s) ds, \quad t \geq 0.
\]
The Gronwall-Bellman inequality yields \( \phi_{\varepsilon}(t) \leq \exp(C_3 \Gamma(\varepsilon)) \) for all \( t \geq 0 \), and in particular, \( \phi_{\varepsilon}(T^\varepsilon) \leq \exp(C_3 \Gamma(\varepsilon)) \). For \( \varepsilon_0 \in (0, 1] \) sufficiently small, \( \varepsilon \in (0, \varepsilon_0] \) yields that the right-hand side is less than 2. We conclude by (3.1) the existence of \( \varepsilon_0 \in (0, 1] \) such that \( \varepsilon \in (0, \varepsilon_0] \) implies
\[
E\left[\exp\left(\kappa^\varepsilon |Z_{\sigma \wedge T^\varepsilon}|\right)\right] \leq \phi_{\varepsilon}(T^\varepsilon) \leq 2.
\]
Note that our estimates are uniformly for all \( x \in D_2 \). This finishes the proof of Lemma 3.2.
5.3 Proof of Proposition 4.4: uniform integrability.

Fix $p \in (0, \alpha)$. We use the conventions in Step 0 of the proof of Proposition 4.3. Then for $x \in D_2$

$$
\mathbb{E}\left[\|X^\varepsilon(\tau; x) - (\phi + G(\phi, \varepsilon W_{R^\varepsilon(\varepsilon)})\|^p\right] \leq 3^p \left( \mathbb{E}\left[\|X^\varepsilon(\tau; x)\|^p\right] + \|\phi\|^p + G_1(\phi)\mathbb{E}\left[\|\varepsilon W_{R^\varepsilon(\varepsilon)}\|^p\right]\right).
$$

(5.7)

For the last term on the right-hand side of (5.7) we obtain

$$
\mathbb{E}\left[\|\varepsilon W_{R^\varepsilon(\varepsilon)}\|^p\right] = \sum_{k=1}^{\infty} \mathbb{E}\left[\|\varepsilon W_k\|^p\right] \mathbb{P}(R^\varepsilon(\varepsilon) = k) = \varepsilon^p \mathbb{E}\left[\|\varepsilon W_1\|^p\right],
$$

and hence for $\varepsilon_0 \in (0, 1]$ sufficiently small the regular variation of $\nu$ implies for $\varepsilon \in (0, \varepsilon_0]$ that

$$
\mathbb{E}\left[\|\varepsilon W_1\|^p\right] = \int_{\rho^\varepsilon}^{\infty} r^{p-1} \mathbb{P}(\|W_1\| > r) dr = \int_{\rho^\varepsilon}^{\infty} r^{p-1} \frac{\nu(r^\varepsilon B_{0}(0))}{\nu(\rho^\varepsilon B_{0}(0))} dr \leq 2 \int_{\rho^\varepsilon}^{\infty} (\varepsilon \rho^\varepsilon)^{\alpha} dr = 2(\varepsilon \rho^\varepsilon)^{\alpha} \int_{\rho^\varepsilon}^{\infty} r^{p-\alpha-1} dr = 2(\varepsilon \rho^\varepsilon)^{\alpha} \frac{2}{\alpha - p} (\rho^\varepsilon)^{p-\alpha} \leq \frac{2(\varepsilon_0 \rho^\varepsilon)^{\alpha}}{\alpha - p} (\rho^\varepsilon)^{p-\alpha} < \infty.
$$

We calculate the first term on the right side in (5.7)

$$
\mathbb{E}\left[\|X^\varepsilon(\tau; x)\|^p\right] = \int_{0}^{\infty} r^{p-1} \mathbb{P}(\|X^\varepsilon(\tau; x)\| > r) dr = \int_{\mathcal{R}} r^{p-1} \mathbb{P}(\|X^\varepsilon(\tau; x)\| > r) dr.
$$

Using (4.2) and the same strong Markov argument as in Claim 1 we obtain for $x \in D_3$

$$
\mathbb{P}(\|X^\varepsilon(\tau; x)\| > r)
= \sum_{k=1}^{\infty} \mathbb{P}(\\{\|X^\varepsilon(\tau; x)\| > r\} \cap \{\tau = T_k\}) + \sum_{k=1}^{\infty} \mathbb{P}(\\{\|X^\varepsilon(\tau; x)\| > r\} \cap \{\tau \in (T_{k-1}, T_k)\})
\leq \sum_{k=1}^{\infty} \mathbb{P}(\bigcap_{j=1}^{k-1} A_j^k \cap B_j^k \cap (\{\|X^\varepsilon(T_k; x)\| > r\}) + \sum_{k=1}^{\infty} \mathbb{P}(\bigcap_{j=1}^{k-1} A_j^k \cap C_j^k \cap (\{\|X^\varepsilon(\tau_x; x)\| > r\})
\leq \sum_{k=1}^{\infty} \sup_{y \in D_2} \mathbb{P}(A_y)^{k-1} \sup_{y \in D_2} \mathbb{P}(B_y \cap (\{\|X^\varepsilon(T_1; y)\| > r\})
+ \sum_{k=1}^{\infty} \sup_{y \in D_2} \mathbb{P}(A_y)^{k-2} \sup_{y \in D_2} \mathbb{P}(A_y \cap C_y^2 \cap (\{\|Y^\varepsilon(\tau_y; y)\| > r\})
\leq \sup_{y \in D_2} \mathbb{P}(B_y \cap (\{\|X^\varepsilon(T_1; y)\| > r\}) + \left( \sup_{y \in D_2} \mathbb{P}(\|Y^\varepsilon(\tau_y; y)\| > r) \right) \frac{1 - \sup_{y \in D_2} \mathbb{P}(A_y)}{\sup_{y \in D_2} \mathbb{P}(A_y)}.
$$

42
For the first sum we have for \( r > d(R) + 2 \)
\[
\sup_{y \in D_2} \mathbb{P}(B_y \cap \{ \| X^\varepsilon(T_1; y) \| > r \}) = \sup_{y \in D_2} \mathbb{P}(B_y \cap \{ ||Y^\varepsilon(T_1; y) + G(Y^\varepsilon(T_1; y), \varepsilon W_1) || > r \}) \\
\leq \sup_{y \in D_2} \mathbb{P}(\| y + G(y, \varepsilon W_1) \| > r) \\
\leq \sup_{y \in D_2} \mathbb{P}(G_1(y)||\varepsilon W_1\| > r - d(R) - 1) \\
\leq \mathbb{P}(g_1(R)||\varepsilon W_1\| > r - d(R) - 1) \\
\leq \mathbb{P}(\| W_1 \| > \frac{1}{\varepsilon} \frac{r - d(R) - 1}{g_1(R)}).
\]
Without loss of generality we fix \( p' \) by \( \alpha = p' > p > (1 - \rho)\alpha \) and the estimate \(4.15\) of \( A_x \) yields
\[
\sup_{y \in D_2} \frac{\mathbb{P}(B_y \cap \{ \| X^\varepsilon(T_1; y) \| > r \})}{1 - \sup_{D_2} \mathbb{P}(A_x)} \\
\leq \mathbb{E}\left[ \mathbb{E}\left[ \frac{\| W_1 \|^p}{(1 - c)\frac{\lambda_{\varepsilon, R}}{\mu_{\varepsilon, R}}} \right] \right] \\
\leq \varepsilon_0^{p' - \alpha(1 - \rho)} \mathbb{E}\left[ \frac{\| W_1 \|^p}{(1 - c)^{2\mu(B_1^\varepsilon(0))}} \left( r - d(R) - 1 \right)^{p'} < \infty \right.
\]
for any \( \varepsilon \in (0, \varepsilon_0) \) for \( \varepsilon_0 \) sufficiently small. For the last term we have for \( r > d(R) + K_2 + 1 \)
\[
\sup_{y \in D_2} \mathbb{P}(\| Y^\varepsilon(\tau_y(\varepsilon, R); y) \| > r) = 0,
\]
since \( \| Y^\varepsilon(\tau_y(\varepsilon, R); x) \| = d(R) + (K_2 + 1)\varepsilon \rho^\varepsilon < d(R) + K_2 + 1 \) for \( \varepsilon_0 \rho^\varepsilon < 1 \). Therefore for \( \varepsilon_0 \in (0, 1) \) and \( \varepsilon \in (0, \varepsilon_0) \) we have
\[
\mathbb{E}\left[ \frac{\| X^\varepsilon(\tau_y(\varepsilon, R); y) \|^p}{\| \tau_y(\varepsilon, R) \|^{p\varepsilon}} \right] \\
\leq \int_{d(R) + 2}^{\infty} \mathbb{P}(\| X^\varepsilon(\tau_y(\varepsilon, R); y) \| > r)dr \\
\leq \int_{d(R) + 2}^{\infty} \frac{1}{r^{1-p}(r - d(R) - 1)^{p'}} dr < \infty.
\]
This establishes the uniform integrability result \(4.27\).

**Acknowledgments:** This work was supported by the FAPA grant “Stochastic dynamics of Lévy driven systems” of Universidad de los Andes, Bogotá, Colombia, which is greatly acknowledged. The author also thanks the Escuela Venezolana en Matemáticas 2017, namely Prof. Dr. Stella Brassesco, IVIC, Caracas, Venezuela, for the invitation to hold a virtual summer course on the subject. Furthermore, the author is grateful to CIMAT, Guanajuato, México, for the invitation to the conference Mexico-Poland, 1st Meeting in Probability, during which final parts of the work were completed.
References

[1] D. Applebaum. Lévy processes and stochastic calculus. Volume 116 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, second edition, 2009.

[2] N. Berglund, B. Gentz. On the noise-induced passage through an unstable periodic orbit I: Two-level model. J. Stat. Phys., 114(5–6):1577–1618, 2004.

[3] N. Berglund, B. Gentz. The Eyring–Kramers law for potentials with nonquadratic saddles. Markov Processes Relat. Fields 16, 549–598, 2010.

[4] N. Berglund, B. Gentz. Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers’ law and beyond. EJP, 18 (2013), pp. no. 24, 58.

[5] A. Bovier, M. Eckhoff, V. Gayrard, M. Klein. Metastability in reversible diffusion processes I: Sharp asymptotics for capacities and exit times. J. of the EMS, 6 (2004), pp. 399–424.

[6] A. Bovier, V. Gayrard, M. Klein. Metastability in reversible diffusion processes II: Precise asymptotics for small eigenvalues. Journal of the European Mathematical Society, 7 (2005), pp. 69–99.

[7] S. Brassesco. Some results on small random perturbations of an infinite dimensional dynamical system. Stochastic Processes and their Applications, 38:33–53, 1991.

[8] S. Brassesco. Unpredictability of an exit time. Stochastic Processes and their Applications, 63:55–65, 1996.

[9] Z. Brzeźniak, E. Hausenblas, P.-A. Razafimandimby. Stochastic Reaction-diffusion Equations Driven by Jump Processes. Potential Analysis, 49:131–201, 2018.

[10] N. H. Bingham, C. M. Goldie, J. L. Teugels. Regular variation. Cambridge University Press, Cambridge, 1987.

[11] A. Budhiraja, P. Nyquist. Large deviations for multidimensional state-dependent shot-noise processes. J. Appl. Probab., Vol. 52, Number 4 (2015), 1097-1114.

[12] N. Chafee, E. F. Infante. A bifurcation problem for a nonlinear partial differential equation of parabolic type. Applicable Anal. (4), 17-37, 1974/75.

[13] A. Budhiraja, J. Chen, P. Dupuis. Large Deviations for Stochastic Partial Differential Equations Driven by a Poisson Random Measure. Stoch. Proc. App. 123 (2013), no 2, 523-560.

[14] A. Budhiraja, P. Dupuis, A. Ganguly. Moderate Deviation Principles for Stochastic Differential Equations with Jumps. The Annals of Probability, 2016, Vol. 44, No 3., 1723–1775.

[15] M. V. Day. On the exponential exit law in the small parameter exit problem. Stochastics, 8:297–323, 1983.

[16] M. V. Day. Exit cycling for the Van der Pol oscillator and quasipotential calculations. Journal of Dynamics and Differential Equations, 8(4):573–601, 1996.
[17] A. Debussche, M. H"ogele, P. Imkeller. Metastability for the Chafee-Infante equation with small heavy-tailed Lévy noise. Electronic Communications in Probability, 16, 213–225, 2011.

[18] A. Debussche, M. H"ogele, P. Imkeller. The dynamics of nonlinear reaction-diffusion equations with small Lévy noise. Springer Lecture notes in Mathematics, Vol. 2085, 163p. (2013).

[19] J. D. Deuschel, D. W. Stroock. Large Deviations. American Mathematical Society, 2001.

[20] G. DaPrato, J. Zabczyk. Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and Its Applications, vol. 44, (Cambridge University Press 1992).

[21] A. Dembo, O. Zeitouni. Large deviation techniques and applications, second edition. Applications of Mathematics, vol. 38, 1998.

[22] G. W. Faris, G. Jona-Lasinio. Large fluctuations for a nonlinear heat equation with noise. J. Phys. A 15 (1982), 3025-3055.

[23] M. I. Freidlin. Random perturbations of reaction-diffusion equations: the quasideterministic approximation. Transactions of the American Mathematical Society, 305(2):665–697, 1988.

[24] A. D. Ventsel’, M. I. Freidlin. On small random perturbations of dynamical systems. Russian Mathematical Surveys, 25(1):1–55, 1970.

[25] M. I. Freidlin, A. D. Wentzell. Random perturbations of dynamical systems. Volume 260 of Grundlehren der Mathematischen Wissenschaften. Springer, second edition, 1998.

[26] A. Galves, E. Olivieri, M. E. Vares. Metastability for a class of dynamical systems subject to small random perturbations. The Annals of Probability, 15 (1987), pp. 1288–1305.

[27] V. V. Godovanchuk. Asymptotic probabilities of large deviations due to large jumps of a Markov process. Theory of Probability and its Applications, 26:314–327, 1982.

[28] J. Hale. Dynamics of a Scalar Parabolic Equation. CWI Quarterly, 12 (1999), no 3&4, 239-314.

[29] D. Henry. Some infinite-dimensional Morse-Smale systems defined by parabolic partial differential equations. J. Differential Equations, 59 (1985), no 2, 165-205.

[30] D. Henry. Geometric theory of semilinear parabolic equations. Lecture notes in Mathematics, 840. Springer-Verlag, Berlin-New York, 1981.

[31] M. H"ogele, I. Pavlyukevich. The exit problem from the neighborhood of a global attractor for heavy-tailed Lévy diffusions. Stochastic Analysis and Applications, 32 (1), 163–190 (2013).

[32] M. H"ogele, I. Pavlyukevich. Metastability in a class of hyperbolic dynamical systems perturbed by heavy-tailed Lévy type noise. Stoch. and Dyn., 15 (3), 1550019-1 – 1550019-26 (2015).

[33] H. Hulk, F. Lindskog. Regular variation for measures on metric spaces. Publ. Inst. Math. (Beograd) (N.S.) 80 (2006), no 94, 121-140.

[34] P. Imkeller, I. Pavlyukevich. Metastable behaviour of small noise Lévy-driven diffusions. ESAIM: Prob. Stat., 12 (2008), 412-437.
[35] P. Imkeller, I. Pavlyukevich. First exit times of SDEs driven by stable Lévy processes. Stoch. Proc. Appl., 116 (2006), no 4, 611-642.

[36] P. Imkeller, I. Pavlyukevich, M. Stauch. First exit times of non-linear dynamical systems in $\mathbb{R}^d$ perturbed by multifractal Lévy noise. J. Stat. Phys., 141(1):94–119, 2010.

[37] C. Kipnis, C. M. Newman. The metastable behavior of infrequently observed, weakly random, one-dimensional diffusion processes. SIAM Journal on Applied Mathematics, 45 (1985), pp. 972–982.

[38] V. N. Kolokoltsov. Semiclassical analysis for diffusions and stochastic processes. vol. 1724 of Lecture Notes in Mathematics, Springer, 2000.

[39] V. N. Kolokol’tsov, K. A. Makarov. Asymptotic spectral analysis of a small diffusion operator and the life times of the corresponding diffusion process. Russian Journal of Mathematical Physics, 4 (1996), pp. 341–360.

[40] H. A. Kramers. Brownian motion in a field of force and the diffusion model of chemical reactions. Physica, 7 (1940), pp. 284–304.

[41] C. Marinelli, C. Prévôt, M. Röckner. Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise. J. Funct. Anal. 258 (2010), no. 2, 616-649.

[42] C. Marinelli, M. Röckner. Well-posedness and asymptotic behavior for stochastic reaction-diffusion equations with multiplicative Poisson noise. EJP, Vol. 15 (2010), pa. no. 49, p. 1528-1555.

[43] C. Marinelli, M. Röckner. On uniqueness of mild solutions for dissipative stochastic evolution equations. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 13, 363 (2010).

[44] I. Pavlyukevich. First exit times of solutions of stochastic differential equations driven by multiplicative Lévy noise with heavy tails. Stoch. and Dyn., 11(2&3), 2011.

[45] A. Pazy. Semigroups of Linear Operators and Applications to Partial Differential Equations. Berlin-Heidelberg-New York-Tokyo, Springer-Verlag 1983. VIII.

[46] P. E. Protter. Stochastic integration and differential equations. volume 21 of Applications of Mathematics, Springer, second edition, 2004.

[47] S. Peszat, J. Zabczyk. Stochastic partial differential equations with Lévy noise (an evolution equation approach). Cambridge University Press, Cambridge, 2007.

[48] G. Raugel. Global attractors in partial differential equations. in Fiedler, Bernold (ed.), Handbook of dynamical systems. Vol 2, 885-982, North-Holland, Amsterdam, 2002.

[49] C. Rocha. Examples of Attractors in Reaction-Diffusion equations. J. Diff. Equ. 73, 178-195, 1988.

[50] K. Sato. Lévy processes and infinitely divisible distributions. volume 68 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999.
[51] P. Siorpaes. Applications of pathwise Burkholder-Davis-Gundy inequalities. Bernoulli 24,(4B), 3222–3245, 2018.

[52] E. Salavati, B. Z. Zangeneh. A maximal inequality for the $p$-th power of stochastic convolution integrals. volume 68 of Journal of Inequalities and Applications, 155: 1–16, 2016.

[53] R. Temam. Infinite-dimensional dynamical systems in Mechanics and Physics. Springer, New York, Applied Mathematical Sciences Series, vol. 68, 1988. Second augmented edition, 1997.