Abstract: It is known that a connected simple graph $G$ associates a simple polytope $P_G$ called a graph associahedron in Euclidean space. In this paper we show that the set of facet vectors of $P_G$ forms a root system if and only if $G$ is a cycle graph and that the root system is of type A.

Key words: graph associahedron; facet vector; root system.

1. Introduction. Let $G$ be a connected simple graph with $n+1$ nodes and its node set $V(G)$ be $[n+1] = \{1, 2, \ldots, n+1\}$. We can construct the graph associahedron $P_G$ in $\mathbb{R}^n$ from $G$ ([8]). We call a primitive (inward) normal vector to a facet of $P_G$ a facet vector and denote by $F(G)$ the set of facet vectors of $P_G$. One can observe that when $G$ is a complete graph, $F(G)$ agrees with the primitive edge vectors of the fan formed by the Weyl chambers of a root system of type A ([1]), in other words, $F(G)$ is dual to a root system of type A when $G$ is a complete graph. Motivated by this observation, we ask whether $F(G)$ itself forms a root system for a connected simple graph $G$. It turns out that $F(G)$ forms a root system if and only if $G$ is a cycle graph (Theorem [2]). On the way to prove it, we show that $F(G)$ is centrally symmetric (this is the case when $F(G)$ forms a root system) if and only if $G$ is a cycle graph or a complete graph.

2. Construction of graph associahedra. We set

$$B(G) := \{ I \subset V(G) \mid G[I] \text{ is connected} \},$$

where $G[I]$ is a maximal subgraph of $G$ with the node set $I$ (i.e. the induced subgraph). The empty set $\emptyset$ is not in $B(G)$. We call $B(G)$ a graphical building set of $G$. We take an $n$-simplex in $\mathbb{R}^n$ such that its facet vectors are $e_1, \ldots, e_n$, and $-e_1 - \cdots - e_n$, where $e_1, \ldots, e_n$ are the standard basis of $\mathbb{R}^n$. Each facet vector $e_i$ ($1 \leq i \leq n$) corresponds to an element $\{i\}$ in $B(G)$, and the facet vector $-e_1 - \cdots - e_n$ corresponds to an element $\{n+1\}$ in $B(G)$. We truncate the $n$-simplex along faces in increasing order of dimension. Let $F_i$ denote the facet of the simplex corresponding to $\{i\}$ in $B(G)$. For every element $I = \{i_1, \ldots, i_k\}$ in $B(G) \setminus \{n+1\}$ we truncate the simplex along the face $F_{i_1} \cap \cdots \cap F_{i_k}$ in such a way that the facet vector of the new facet, denoted $F_I$, is the sum of the facet vectors of the facets $F_{i_1}, \ldots, F_{i_k}$. Then the resulting polytope, denoted $P_G$, is called a graph associahedron. We denote by $F(G)$ the set of facet vectors of $P_G$.

3. Facet vectors associated to complete graphs. As mentioned in the Introduction, $F(G)$ is dual to a root system of type A when $G$ is a complete graph. We shall explain what this means. If $G$ is a complete graph $K_{n+1}$ with $n+1$ nodes, then the graphical building set $B(K_{n+1})$ consists of all subsets of $[n+1]$ except for $\emptyset$ so that the graph associahedron $P_{K_{n+1}}$ is a permutohedron obtained by cutting all faces of the $n$-simplex with facet vectors $e_1, \ldots, e_n, -(e_1 + \cdots + e_n)$. It follows that

$$F(K_{n+1}) = \left\{ \pm \sum_{i \in I} e_i \mid \emptyset \neq I \subset [n] \right\}.$$  

On the other hand, consider the standard root system $\Delta(A_n)$ of type $A_n$ given by

$$\Delta(A_n) = \left\{ \pm (e_i - e_j) \mid 1 \leq i < j \leq n + 1 \right\}$$

which lies on the hyperplane $H$ of $\mathbb{R}^{n+1}$ with $e_1 + \cdots + e_{n+1}$ as a normal vector. Take $e_1 - e_2, e_2 - e_3, \ldots, e_n - e_{n+1}$ as a base of $\Delta(A_n)$ as usual. Then their dual base with respect to the standard inner product on $\mathbb{R}^{n+1}$ is what is called the fundamental dominant weights given by

$$\lambda_i = (e_1 + \cdots + e_i) - \frac{1}{n+1}(e_1 + \cdots + e_{n+1})$$

($i = 1, 2, \ldots, n$)

which also lie on the hyperplane $H$. The Weyl group action permutes $e_1, \ldots, e_{n+1}$ so that it preserves $H$. We identify $H$ with the quotient vector space $H^*$ of $\mathbb{R}^{n+1}$ by the line spanned by $e_1 + \cdots + e_{n+1}$ using the inner product, namely put the condition $e_1 + \cdots + e_{n+1} = 0$. Then the set of elements obtained from the orbits of $\lambda_1, \ldots, \lambda_n$ by the Weyl group action is

$$\left\{ \sum_{j \in J} e_j \mid \emptyset \neq J \subset [n+1] \right\} \text{ in } H^*.$$
This set agrees with $F(K_{n+1})$ in (1) because $e_{n+1} = -(e_1 + \cdots + e_n)$. In this sense $F(K_{n+1})$ is dual to $\Delta(A_n)$.

4. Main theorem. We note that $F(K_{n+1})$ itself forms a root system (of type $A_n$) when $n = 1$ or 2. However the following holds.

Lemma 1. If $n \geq 3$, then $F(K_{n+1})$ does not form a root system.

Proof. Suppose that $F(K_{n+1})$ forms a root system for $n \geq 3$. Then $F(K_{n+1})$ is of rank $n$ and the number of positive roots is $2^n - 1$ by (1). On the other hand, no irreducible root system of rank $n(\geq 3)$ has $2^n - 1$ positive roots (see Table 1 in p.66). Therefore, it suffices to show that $F(K_{n+1})$ is irreducible if it forms a root system.

Let $V$ be an $m$-dimensional linear subspace of $\mathbb{R}^n$ such that $E = F(K_{n+1}) \cap V$ is a root subsystem of $F(K_{n+1})$. We consider the mod 2 reduction map

$$\varphi: Z^n \cap V \to (Z^n \cap V) \otimes Z/2$$

where $Z/2 = \{0, 1\}$. Since $(Z^n \cap V) \otimes Z/2$ is a vector space over $Z/2$ of dimension $\leq m$, it contains at most $2^m - 1$ nonzero elements. On the other hand, since the coordinates of an element in $F(K_{n+1})$ are either in $\{0, 1\}$ or $\{0, -1\}$ by (1), the number of elements in $\varphi(E)$ is exactly equal to the number of positive roots in $E$.

Now suppose that the root system $F(K_{n+1})$ decomposes into the union of two nontrivial components $E_i$ for $i = 1, 2$. Then there are $m_i$-dimensional linear subspaces $V_i$ of $\mathbb{R}^n$ such that $E_i = F(K_{n+1}) \cap V_i$ and $m_1 + m_2 = n$, where $m_2 \geq 1$. Since the number of positive roots in $E_i$, denoted by $p_i$, is at most $2^{m_i} - 1$ by the observation above, we have

$$p_1 + p_2 \leq (2^{m_1} - 1) + (2^{m_2} - 1) < 2^n - 1.$$

However, since $F(K_{n+1}) = E_1 \cup E_2$ and the number of positive roots in $F(K_{n+1})$ is $2^n - 1$ as remarked before, we must have $2^n - 1 = p_1 + p_2$. This is a contradiction. Therefore, $F(K_{n+1})$ must be irreducible if it forms a root system.

The following is our main theorem.

Theorem 2. Let $G$ be a connected finite simple graph with more than two nodes. Then the set $F(G)$ of facet vectors of the graph associahedron associated to $G$ forms a root system if and only if $G$ is a cycle graph. Moreover, the root system associated to the cycle graph with $n+1$ nodes is of type $A_n$.

The rest of this paper is devoted to the proof of Theorem 2. We begin with the following lemma.

Lemma 3. Let $C_{n+1}$ be the cycle graph with $n+1$ nodes. Then $F(C_{n+1})$ forms a root system of type $A_n$.

Proof. An element $I$ in the graphical building set $B(C_{n+1})$ different from the entire set $[n+1]$ is one of the following:

(I) $\{i, i+1, \ldots, j\}$ where $1 \leq i \leq j \leq n$,

(II) $\{i, i+1, \ldots, n+1\}$ where $2 \leq i \leq n + 1$,

(III) $\{i, i+1, \ldots, n+1, 1, \ldots, j\}$ where $1 \leq j \leq i \leq n+1$ and $i - j \geq 2$.

Therefore the facet vector of the facet corresponding to $I$ is respectively given by

$$\sum_{k=i}^j e_k, \quad -\sum_{k=1}^{i-1} e_k, \quad -\sum_{k=j+1}^{i-1} e_k$$

according to the cases (I), (II), (III) above. Hence

$$F(C_{n+1}) = \left\{ \pm \sum_{k=i}^j e_k \mid 1 \leq i < j \leq n \right\}.$$

This set forms a root system of type $A_n$. Indeed, an isomorphism from $Z^n$ to the lattice

$$\{(x_1, \ldots, x_{n+1}) \in Z^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\}$$

sending $e_i$ to $e_i - e_{i+1}$ for $i = 1, 2, \ldots, n$ maps $F(C_{n+1})$ to the standard root system $\Delta(A_n)$ of type $A_n$ in $A_2$.

The following lemma is a key observation.

Lemma 4. Let $G$ be a connected simple graph. Then $F(G)$ is centrally symmetric, which means that $\alpha \in F(G)$ if and only if $-\alpha \in F(G)$ (note that $F(G)$ is centrally symmetric if $F(G)$ forms a root system) if and only if the following holds:

$$I \in B(G) \iff V(G) \setminus I \in B(G).$$

Proof. Let $V(G) = [n+1]$ as before and let $I$ be an element in $B(G)$ and $\alpha_I$ be the facet vector of the facet of $F(G)$ corresponding to $I$. If we set $e_{n+1} := -(e_1 + \cdots + e_n)$, then $\alpha_I = \sum_{i \in I} e_i$. Since $\alpha_I + \sum_{i \in [n+1] \setminus I} e_i = \sum_{i \in [n+1]} e_i = 0$, we obtain $-\alpha_I = \sum_{i \in [n+1] \setminus I} e_i$ and this implies the lemma.

Using Lemma 4 we prove the following.

Lemma 5. Let $G$ be a connected finite simple graph. Then $B(G)$ satisfies (4) if and only if $G$ is a cycle graph or a complete graph.
Proof. If \( G \) is a cycle or complete graph, then \( F(G) \) is centrally symmetric by (1) or (3) and hence \( B(G) \) satisfies (4) by Lemma 4. So the “if” part is proven.

We shall prove the “only if” part, so we assume that \( B(G) \) satisfies (4). Suppose that \( G \) is not a complete graph. Then there are \( i \neq j \in V(G) \) such that \( \{i, j\} \) is not contained in \( B(G) \). By (3), \( V(G) \setminus \{i, j\} \) is not contained in \( B(G) \), which means that the induced subgraph \( G[(V(G) \setminus \{i, j\})] \) is not connected.

On the other hand, since \( B(G) \) contains \( \{i\} \) and \( \{j\} \), \( B(G) \) contains \( V(G) \setminus \{i\} \) and \( V(G) \setminus \{j\} \) by (4). Hence

\[
G[(V(G) \setminus \{i\})] \quad \text{and} \quad G[(V(G) \setminus \{j\})]
\]

are connected.

Let \( k \) be the number of connected components of \( G[(V(G) \setminus \{i, j\})] \) and we denote its \( k \) components by \( G_1, \ldots, G_k \) (Figure 1). By (5), the nodes \( i \) and \( j \) are respectively joined to every connected component by at least one edge. Since \( G[(V(G_1) \cup \{i, j\})] \) is connected, \( G[(V(G_2) \cup \cdots \cup V(G_k))] \) is also connected by (4). However, \( G[(V(G_2) \cup \cdots \cup V(G_k))] \) is the disjoint union of the connected subgraphs \( G_2, \ldots, G_k \). Therefore we have \( k = 2 \).

\begin{figure}[h!]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure}

If \( G_1 \) and \( G_2 \) are both path graphs and the node \( i \) is joined to one end node of \( G_1, G_2 \) respectively and the node \( j \) is joined to the other end node of \( G_1, G_2 \), then \( G \) is a cycle graph (Figure 2).

\begin{figure}[h!]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Figure 2. the case of cycle graph}
\end{figure}

We consider the other case, that is, either

(I) \( G_1 \) or \( G_2 \) is not a path graph, or

(II) both \( G_1 \) and \( G_2 \) are path graphs but the nodes \( i \) and \( j \) are not joined to the end points of \( G_1 \) and \( G_2 \) (see Figure 3 left).

Then there exist nodes \( i_1, j_1 \in V(G_1) \) and \( i_2, j_2 \in V(G_2) \) such that

- \( i_1 \) and \( i_2 \) are joined to \( i \),
- \( j_1 \) and \( j_2 \) are joined to \( j \), and
- either the shortest path \( P_1 \) from \( i_1 \) to \( j_1 \) in \( G_1 \) is not the entire \( G_1 \) or the shortest path \( P_2 \) from \( i_2 \) to \( j_2 \) in \( G_2 \) is not the entire \( G_2 \).

\begin{figure}[h!]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Figure 3. the other case}
\end{figure}

Without loss of generality we may assume that \( P_1 \neq G_1 \). Since \( G[(V(P_1) \cup \{i_1, j_1, i_2, j_2\})] \) is connected, so is \( G[(V(G) \setminus (V(P_1) \cup \{i_1, j_1, i_2, j_2\}))] \) by (4). This means that there is at least one edge joining \( G_1 \) and \( G_2 \) (Figure 3 right), and hence \( G[(V(G) \setminus \{i, j\})] \) is connected. This contradicts that \( G[(V(G) \setminus \{i, j\})] \) consists of two connected components.

\( \square \)

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REFERENCES

[1] H. Abe, Young diagrams and intersection numbers for toric manifolds associated with Weyl chambers, Electron. J. Combin. 22(2) #P2.4, 2015.
[2] J. E. Humphreys, Introduction to Lie algebras and representation theory, Springer-Verlag, Grad. Texts in Math. vol. 9, 1972.
[3] A. Postnikov, Permutohedra, associahedra, and beyond, Int. Math. Res. Not. IMRN 2009, no. 6, 1026–1106.