Discrete symmetries, roots of unity, and lepton mixing

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Abstract

We investigate the possibility that the first column of the lepton mixing matrix $U$ is given by $u_1 = (2, -1, -1)^T / \sqrt{6}$. In a purely group-theoretical approach, based on residual symmetries in the charged-lepton and neutrino sectors and on a theorem on vanishing sums of roots of unity, we discuss the finite groups which can enforce this. Assuming that there is only one residual symmetry in the Majorana neutrino mass matrix, we find the almost unique solution $\mathbb{Z}_q \times S_4$ where the cyclic factor $\mathbb{Z}_q$ with $q = 1, 2, 3, \ldots$ is irrelevant for obtaining $u_1$ in $U$. Our discussion also provides a natural mechanism for achieving this goal. Finally, barring vacuum alignment, we realize this mechanism in a class of renormalizable models.

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1 Introduction

The recent measurements of a rather large reactor mixing angle $\theta_{13}$ [1, 2] disfavour tri-bimaximal mixing [3] and, therefore, also such models—see [4, 5] for reviews on models for neutrino masses and mixing. While the third column of the tri-bimaximal mixing matrix $U_{\text{TBM}}$ is now definitively in disagreement with the data, the first or the second column of $U_{\text{TBM}}$ could still occur in the mixing matrix $U$. These cases are denoted by TM$_1$ and TM$_2$, respectively, in [6]. However, in the case of TM$_2$ the solar mixing angle $\theta_{12}$ is related to the reactor mixing angle via $\sin^2 \theta_{12}(1 - \sin^2 \theta_{13}) = 1/3$, which creates a tension with the data but is still compatible at the $3\sigma$ level. Therefore, it is more interesting to consider TM$_1$ [7, 8, 9, 10] where the first column in $U = (u_1, u_2, u_3)$ is given by

$$u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$ (1)

Recently, a purely group-theoretical approach has been developed for the investigation of the effect of finite family symmetry groups on the mixing matrix [11, 12, 13, 14, 15, 16, 17]. Apart from assuming that the left-handed neutrino fields and the left-handed charged lepton fields are in the same gauge doublet of the Standard Model gauge group, no other assumption concerning the interactions in the lepton sector is made. On the one hand, taking into account extant data on lepton mixing, such a general approach allows a systematic investigation of the possible symmetry groups, see for instance the scan of groups performed in [18, 19]. On the other hand, this approach has its limitations since it is not entirely clear how its results relate to concrete models [20]; we will address this point later in this paper.

The goal of the present investigation is to find all possible finite family symmetry groups underlying TM$_1$, without restricting the other two columns $u_2$ and $u_3$ beyond orthonormality. In order to accomplish this task, we will use a theorem on vanishing sums of roots of unity. In the course of this investigation we will also come across a mechanism for the implementation of TM$_1$. In the following we will use this mechanism to write down a class of models based on $S_4$ and the type II seesaw mechanism where TM$_1$ is realized.

The paper is organized as follows. In section 2 the group-theoretical method of [15] is reviewed using our notation. In order to be as clear as possible, some arguments are emphasized by formulating them as propositions. This also applies to section 3 where the symmetry group for TM$_1$ is determined; some technical points needed in the course of our argumentation are deferred to appendix A. In the same section we also find a mechanism for the implementation of TM$_1$, which is then used in section 4 for the construction of a class of models. The summary of our findings is presented in section 5. As a supporting material, we provide a set of generators of $S_4$ and the three-dimensional irreducible representations of this group in appendix B.
2 Residual symmetries in the mass matrices

The class of models we have in mind as an application of the following discussion are extensions of the Standard Model in the scalar and fermion sectors. Typical examples would be several Higgs doublets and right-handed neutrino singlets which facilitate the seesaw mechanism \cite{21}, or Higgs triplet extensions with the type II seesaw mechanism \cite{22}. We further assume that before spontaneous symmetry breaking (SSB) the theory is invariant under a finite family symmetry group $G$ and that there are three lepton families. We want to investigate the case that there are residual symmetries in the charged-lepton and Majorana neutrino mass matrices, left over from SSB of $G$, and study their effect on the lepton mixing matrix $U$ \cite{11,15}.

Such residual symmetries will occur whenever the vacuum expectation values (VEVs) of the neutral components of scalar gauge multiplets are invariant under some transformations of $G$. However, we are not interested in the full symmetry of the vacuum, which might very well be trivial, but in the symmetries of the vacua in the respective sectors whose VEVs lead to charged-lepton and neutrino masses. It is well known that the mismatch of these symmetries is responsible for predictions in the mixing matrix—see for instance \cite{4,23}. Therefore, we must distinguish between the symmetries in $M_\ell M_\ell^\dagger$ where $M_\ell$ the charged-lepton mass matrix and those in the Majorana neutrino mass matrix $\mathcal{M}_\nu$. These symmetries are supposed to generate the subgroups $G_\ell$ and $G_\nu$ of $G$, pertaining to $M_\ell M_\ell^\dagger$ and $\mathcal{M}_\nu$, respectively. Obviously, we have the relation

$$G_\ell \subseteq U(1) \times U(1) \times U(1), \quad G_\nu \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

(2)
due to the Dirac and Majorana natures of charged and neutral leptons, respectively. The groups $G_\ell$ and $G_\nu$ will be quite small and very often be generated by just one symmetry. In the following we will assume that this is the case for $G_\nu$, but $G_\ell$ will in principle be allowed to contain several non-trivial elements though the analysis will be phrased in terms of a single matrix $T \in G_\ell$.

The mass Lagrangian—obtained through SSB—has the form

$$\mathcal{L}_{\text{mass}} = -\bar{\ell}_LM_\ell\ell_R + \frac{1}{2}\nu_L^T S^T \mathcal{M}_\nu \nu_L + \text{H.c.}$$

(3)

with the residual symmetries

$$T^\dagger M_\ell M_\ell^\dagger T = M_\ell M_\ell^\dagger, \quad S^T \mathcal{M}_\nu S = \mathcal{M}_\nu,$$

(4)

where $T$ and $S$ are unitary matrices and $\mathcal{M}_\nu$ is symmetric but complex in general. Furthermore, the diagonalizing matrices $U_\ell$ and $U_\nu$, and the lepton mixing matrix are given

$$U_\ell^\dagger M_\ell M_\ell^\dagger U_\ell = \text{diag} \left( m_e^2, m_\mu^2, m_\tau^2 \right), \quad U_\nu^T \mathcal{M}_\nu U_\nu = \text{diag} \left( m_1, m_2, m_3 \right), \quad U = U_\ell^\dagger U_\nu,$$

(5)

respectively. We denote the weak basis of $M_\ell$ and $\mathcal{M}_\nu$ by basis 1. Since we are interested in the effect of $T$ and $S$ on the lepton mixing matrix, the action of the residual symmetry on right-handed lepton fields is irrelevant for us, which is the reason to consider $M_\ell M_\ell^\dagger$ instead of $M_\ell$.  

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It is useful to distinguish basis 1 from the weak basis, where the charged-lepton mass matrix is diagonal, in which case we use the phrase basis 2. All matrices in this basis are indicated by a tilde. Usually, basis 1 is the weak basis where the matrices of the representation $D(G)$ on the left-handed lepton gauge doublets have a “nice” form. Depending on the representations of $G$ used in the Lagrangian it can happen that basis 1 coincides with basis 2, but in general the two bases will be different. Because all charged-lepton masses are different, we conclude that $\tilde{T}$ is a diagonal matrix.

In the following analysis we will assume that there are no accidental symmetries in $M_\ell M_\ell^\dagger$ and $M_\nu$. Then, since $S$ and $T$ are given in a weak basis, they must both belong to $D(G)$, the group of representation matrices of $G$ acting on the left-handed lepton doublets. The matrices $S$ and $T$ generate a group denoted by $\bar{G}$. In this case it is clear that $\bar{G}$ is a subgroup of the group $D(G)$ and both are subgroups of $U(3)$, i.e. $\bar{G} \subseteq D(G) \subset U(3)$. In the simplest case $\bar{G}$ is identical with $D(G)$.

Due to equation (2) and assuming $\det S = 1$ without loss of generality, the form of $S$ is given by

$$S = 2uu^\dagger - 1$$

(6)

with a unit vector $u$ and $S^2 = 1$.

**Proposition 1** If $S^T M_\nu S = M_\nu$ with $S = 2uu^\dagger - 1$, then $M_\nu u \propto u^*$.

Proof: By construction, the matrix $S$ fulfills $Su = u$, with a unique eigenvalue 1. Therefore, $S^T M_\nu u = M_\nu u$, and due to the hermiticity of $S$ and $S^2 = 1$ we find $S^*(M_\nu u) = M_\nu u$. Since $S^*$ has a unique eigenvalue 1 with corresponding eigenvector $u^*$, we conclude that $M_\nu u$ is proportional to $u^*$. \[\square\]

In basis 2 the mixing matrix $U$ diagonalizes the neutrino mass matrix:

$$U^T \tilde{M}_\nu U = \text{diag} (m_1, m_2, m_3).$$

(7)

With column vectors $u_j$ and $U = (u_1, u_2, u_3)$, equation (7) is reformulated as

$$\tilde{M}_\nu u_j = m_j u_j^*.$$ 

(8)

Comparing with proposition[1] and denoting the unit vector associated with $\tilde{S}$ by $\bar{u}$, we immediately come to the following conclusion.

**Proposition 2** If $\tilde{S}^T \tilde{M}_\nu \tilde{S} = \tilde{M}_\nu$, then, apart from irrelevant phases, $\bar{u}$ is one of column vectors of the lepton mixing matrix $U$.

This proposition is the basis of all discussions concerning residual symmetries in the mass matrices.

Since we assume that $G$ is a finite group, it is a necessary condition that the matrices $T$ and $ST$ have finite orders:

$$T^m = (ST)^n = 1$$

(9)

for some natural numbers $m$ and $n$. Note that $S$ has order two by construction[1].

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[1] Groups generated by $S$ and $T$ such that the orders of $S$, $T$ and $ST$ are finite are called von Dyck groups [15]; such groups are not necessarily finite because because finite orders of group generators does in general not imply that the group is finite.
The interesting observation is that equation (9) connects group properties with information on the mixing matrix [11][15]. In effect, one uses the following proposition.

**Proposition 3** Suppose we have a $3 \times 3$ matrix $M$ which is a function of a set of parameters $x = (x_1, \ldots, x_r)$ such that for all values of $x$ $M$ is unitary. Then the values of $x$ where $M$ has the eigenvalues $\lambda_k$ ($k = 1, 2, 3$ and $|\lambda_k| = 1$) are determined by the two equations

$$\text{Tr } M(x) = \lambda_1 + \lambda_2 + \lambda_3 \quad \text{and} \quad \det M(x) = \lambda_1 \lambda_2 \lambda_3.$$  

(10)

Proof: The characteristic polynomial of $M$ is

$$P_M(\lambda) = \lambda^3 - M_2 \lambda^2 + M_1 \lambda - M_0.$$  

In terms of the eigenvalues of $M$ the coefficients $M_i$ are given by

$$M_1 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1, \quad M_2 = \lambda_1 + \lambda_2 + \lambda_3, \quad M_0 = \lambda_1 \lambda_2 \lambda_3.$$  

Unitarity of $M$ means that the eigenvalues are located on the unit circle. Therefore, we find $M_1 = M_2^* M_0$. This means that, as long as equation (10) is satisfied, then automatically $M$ gives the correct coefficient $M_2$ in $P_M(\lambda)$ leading to the eigenvalues $\lambda_k$. \hfill \Box

We make the identification $M = \tilde{S} \tilde{T}$. With

$$\tilde{T} = \text{diag}(e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau}) \quad \text{and} \quad \tilde{u} = \begin{pmatrix} U_{ei} \\ U_{\mu i} \\ U_{\tau i} \end{pmatrix}$$  

equation (10) reads [15]

$$\sum_{\alpha = e, \mu, \tau} (2 |U_{\alpha i}|^2 - 1) e^{i\phi_\alpha} = \lambda_1 + \lambda_2 + \lambda_3 \quad \text{and} \quad \prod_{\alpha} e^{i\phi_\alpha} = \lambda_1 \lambda_2 \lambda_3,$$  

(12)

where the $e^{i\phi_\alpha}$ are $m$-th roots and the $\lambda_k$ are $n$-th roots of unity. The parameters in $M$ are the mixing angles, the CP phase and the $\phi_\alpha$. Equation (12) provides the necessary and sufficient conditions that $ST$ is of order $n$.

Equation (12) can be used both ways: Given the group $\bar{G}$ generated by $S$ and $T$, relations among the mixing parameters are obtained [15]; vice versa, assuming a specific column vector in $U$, we can infer the group $\bar{G}$.

### 3 A symmetry for TM$_1$

Now we discuss the possible symmetry groups $\bar{G}$ leading to TM$_1$. From the discussion in the previous section we have learned that in this case we have to take $\tilde{u} \equiv u_1$ with $u_1$ given by equation (11), which leads to

$$\bar{S} = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix}.$$  

(13)
Specifying equation (12) to this case, we obtain
\[-e^{i\phi_e} + 2e^{i\phi_\mu} + 2e^{i\phi_r} + 3\lambda_1 + 3\lambda_2 + 3\lambda_3 = 0 \quad \text{and} \quad e^{i\phi_e}e^{i\phi_\mu}e^{i\phi_r} = \lambda_1\lambda_2\lambda_3. \quad (14)\]

Our aim is to find all cases of roots of unity \(e^{i\phi_\alpha} (\alpha = e, \mu, \tau)\) and \(\lambda_k (k = 1, 2, 3)\) which satisfy these two relations. For this purpose we use the following theorem [24].

**Theorem 1** For any vanishing sum \(\alpha_1 + \alpha_2 + \cdots + \alpha_k = 0\) of \(k\) roots of unity there is a decomposition of \(k\) into a sum of \(r\) prime numbers, \(k = p_1 + p_2 + \cdots + p_r\), and a set of \(r\) roots of unity, \(\epsilon_i\), such that the \(k\) roots can be written as
\[
\epsilon_1, \epsilon_1\xi_1, \ldots, \epsilon_1\xi_1^{p_1-1}, \epsilon_2, \epsilon_2\xi_2, \ldots, \epsilon_2\xi_2^{p_2-1}, \ldots, \epsilon_r, \epsilon_r\xi_r, \ldots, \epsilon_r\xi_r^{p_r-1}
\]
with \(\xi_i = \exp\left(2\pi i/p_i\right)\).

In other words, a vanishing sum of roots of unity consists of subsets which are each similar to a complete set of roots of prime order and which, therefore, have themselves a vanishing sum:
\[
\epsilon_i + \epsilon_i\xi_i + \cdots + \epsilon_i\xi_i^{p_i-1} = \epsilon_i \left(1 + \xi_i + \cdots + \xi_i^{p_i-1}\right) = 0.
\]

The first relation of equation (14) constitutes a vanishing sum of 14 roots of unity in which there are three sets of three equal roots and two sets of two equal roots. According to the theorem we first have to find all decompositions of 14 into a sum of prime numbers. This is done in appendix A where it is also shown that out of a total of ten such decompositions only one physically viable case remains, namely 14 = 3 + 3 + 2 + 2 + 2 + 2, and the 14 roots are written as
\[
\epsilon_1, -\epsilon_1, \epsilon_2, -\epsilon_2, \epsilon_3, -\epsilon_3, \epsilon_4, -\epsilon_4, \epsilon_5(1, \omega, \omega^2), \epsilon_6(1, \omega, \omega^2) \quad \text{with} \quad \omega = e^{2\pi i/3}. \quad (15)
\]

Let us first assume that there are no three equal roots among the \(\pm\epsilon_i\) with \(i = 1, 2, 3, 4\). This means that in order to achieve three sets of three equal roots we have to take, without loss of generality, \(\epsilon_5 = \epsilon_6\) and choose, for instance, \(\epsilon_1 = \epsilon_5, \epsilon_2 = \epsilon_5\omega\) and \(\epsilon_3 = \epsilon_5\omega^2\). Then only \(\pm\epsilon_4\) remains at our disposal, but together with \(-\epsilon_i\ (i = 1, 2, 3)\) we cannot obtain two sets of two equal roots. Having ruled out this possibility, we find that, without loss of generality, we must require \(\epsilon_1 = \epsilon_2 = \epsilon_3\), which automatically leads to two sets of three equal roots. Then, in order to obtain a third set of three equal roots, we choose for instance \(\epsilon_4 = \epsilon_5 = \epsilon_6\). Thus we end up with the desired set of 14 roots of unity which in a suitable notation reads
\[
\epsilon, -\epsilon \quad \text{(threelfold degenerate)}, \\
\eta, -\eta, \\
\eta(1, \omega, \omega^2) \quad \text{(twofold degenerate)}. \quad (16)
\]

By comparison with equation (14) we find
\[
e^{i\phi_e} = \eta, \quad e^{i\phi_\mu} = \eta\omega, \quad e^{i\phi_r} = \eta\omega^2, \quad \lambda_1 = \epsilon, \quad \lambda_2 = -\epsilon, \quad \lambda_3 = \eta. \quad (17)
\]

The second condition of equation (14) yields
\[
\eta^3 = -\epsilon^2\eta \quad \text{or} \quad \epsilon = \pm i\eta. \quad (18)
\]
However, this equation will be of no use in the following. As a result of this lengthy consideration we have found
\[ \tilde{T} = \eta \text{diag} \left( 1, \omega, \omega^2 \right). \] (19)
Thus \( \tilde{T} \) is completely determined up to an unknown root of unity \( \eta \). We remind the reader that the matrices \( \tilde{S} \) of equation (13) and \( \tilde{T} \) of equation (19) are given in basis 2.

A suitable basis 1—see appendix B—is given by a basis transformation with
\[ U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \] (20)
The resulting matrices are
\[ S = U_\omega \tilde{S} U_\omega^\dagger = S_1 B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad T = U_\omega \tilde{T} U_\omega^\dagger = \eta E = \eta \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \] (21)
where \( S_1 \) is defined in equation (B1) and \( E \) and \( B \) in equation (B2).

Let us assume that \( \eta \) is a primitive root of order \( q \). The following proposition determines the group \( \bar{G} \).

**Proposition 4** The matrix \( \tilde{S} \) of equation (13) and \( \tilde{T} \) of equation (19) generate the group \( \bar{G} = \mathbb{Z}_q \times S_4 \).

Proof: For simplicity of notation we use basis 1. From \((ST)^2 = \eta^2 S_3 \) and \( T^\dagger \eta^2 S_3 T = \eta^2 S_1 \) we find that \( \eta^2 S_1 S = \eta^2 B \in \bar{G} \) and, therefore,
\[ (\eta^2 B)^\dagger T(\eta^2 B) = \eta \text{diag} \left( 1, \omega^2, \omega \right) \in \bar{G}. \]
Eventually, with
\[ T(\eta^2 B)^\dagger T(\eta^2 B) = \eta^2 \mathbb{1} \in \bar{G} \quad \text{and} \quad T^3 = \eta^3 \mathbb{1} \in \bar{G} \]
we conclude \( \eta \mathbb{1} \in \bar{G} \). Therefore, \( \bar{G} \) contains \( E \), \( S_1 \) and \( B \), which is a set of generators of \( S_4 \)—see appendix B. It is then almost trivial to show that every element of \( g \in \bar{G} \) can uniquely be decomposed into \( g = \eta^k h \) with \( k \in \{0, 1, \ldots, q-1\} \) and \( h \in S_4 \). \( \square \)

In summary, we have found all finite groups which enforce \( \text{TM}_1 \) in the mixing matrix because we have seen that any \( \tilde{T} \) of finite order satisfying equation (14) leads to \( S_4 \times \mathbb{Z}_2 \) times a cyclic factor. In [18] it was stated that \( S_4 \) is the smallest such group. Here we have shown that actually \( S_4 \) is unique up to a trivial factor with a cyclic group. We formulate our result as a theorem.

**Theorem 2** Under the premises that \( G_\nu \) is a \( \mathbb{Z}_2 \) generated by \( \tilde{S} \) of equation (13) and that \( G_\ell \) contains at least one matrix \( \tilde{T} \) which is not proportional to the unit matrix, the only symmetry groups \( \bar{G} \) generated by the residual symmetries of the mass matrices which are able to enforce \( \text{TM}_1 \) in the lepton mixing matrix \( U \) are \( \mathbb{Z}_q \times S_4 \).
With our derivation of theorem \[2\] we have also demonstrated that

\[ E^\dagger \left( M_\ell M_\ell^\dagger \right) E = M_\ell M_\ell^\dagger. \]  \hspace{1cm} (22)

This has the following consequence.

**Proposition 5** For every charged-lepton mass matrix which fulfills equation \[22\] it follows that \[U_\omega^\dagger \left( M_\ell M_\ell^\dagger \right) U_\omega\] is diagonal.

**Proof:** The eigenvectors of \( E \) are identical with the column vectors of \( U_\omega \), i.e. \( U_\omega = (x_1, x_2, x_3) \) and \( Ex_k = \omega^{k-1}x_k \). Therefore, we have

\[ \left( M_\ell M_\ell^\dagger \right) E x_k = \omega^{k-1} \left( M_\ell M_\ell^\dagger \right) x_k = E \left( M_\ell M_\ell^\dagger \right) x_k, \]

whence we conclude that \( \left( M_\ell M_\ell^\dagger \right) x_k \) is an eigenvector of \( E \) to the eigenvalue \( \omega^{k-1} \). Since eigenvectors to non-degenerate eigenvalues are unique up to a multiplicative factor, we arrive at \( \left( M_\ell M_\ell^\dagger \right) x_k = \nu_k x_k \) where the quantities \( |\nu_k| \) are identical with the charged-lepton masses. \( \square \)

Turning to the neutrino mass matrix \( M_\nu \), we remember that we have started with the requirement that it is invariant under \( S \), i.e. \( S^T M_\nu S = M_\nu \). In the present discussion \( S \) is given by equation \[21\]. By construction, this matrix is an involution—see also equation \[6\]—with a unique eigenvalue 1:

\[ Su = u = U_\omega u_1 \quad \text{with} \quad u = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \]  \hspace{1cm} (23)

Then we know from proposition \[4\] that \( u \) is also an eigenvector of \( M_\nu \). Therefore, in our basis 1 the mechanism for achieving \( TM_1 \) boils down to

\[ U_\omega^\dagger u = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}. \]  \hspace{1cm} (24)

This mechanism has recently been used in \[10\] for the construction of a model which exhibits \( TM_1 \).

4 Realizing \( TM_1 \) in a concrete \( S_4 \) scheme

Though the mechanism for obtaining \( TM_1 \) from the mass matrices is unique, it is not unique how to embed it into an \( S_4 \) model. Below we introduce a scheme with the type II seesaw mechanism \[22\].

Our starting point is the tensor product (see for instance \[25\])

\[ 3 \otimes 3 = 1 \oplus 2 \oplus 3 \oplus 3', \]  \hspace{1cm} (25)
where the 1 is the trivial one-dimensional representation and the 3 and 3’ are the two inequivalent irreducible three-dimensional representations of $S_4$—see appendix B for the generators and the three-dimensional representations. The 3’ and 3 correspond to the off-diagonal symmetric and antisymmetric parts, respectively, in the tensor product. The 1 and the 2 comprise the diagonal part. If we assign to both the left-handed lepton gauge doublets and the right-handed lepton gauge singlets a 3 of $S_4$, then the right-hand side of equation (25) shows the possible irreducible $S_4$ representations of Higgs doublets.

Since we require the validity of equation (22), we need the vacuum to be invariant under $s = (123)$, which is mapped in the 3 and 3’ into $E$—see appendix B. However, for the two-dimensional irreducible representation we have

$$s = (123) \mapsto \left( \begin{array}{cc} \omega & 0 \\ 0 & \omega^2 \end{array} \right),$$

which means that no non-trivial VEV is invariant under this representation of $s$ and the 2 cannot contribute to $M_\ell$. For 3 and 3’, invariance of the VEVs under $s$ means that the VEVs of the three Higgs doublets have to be equal. Eventually, we arrive at the most general $M_\ell$, invariant under $E$ and compatible with equation (25):

$$M_\ell = \begin{pmatrix} a & b + c & b - c \\ b - c & a & b + c \\ b + c & b - c & a \end{pmatrix}.$$  \hspace{2cm} (27)

Indeed, here $U_\omega$ diagonalizes not only $M_\ell M_\ell^\dagger$ but also $M_\ell$:

$$U_\omega^\dagger M_\ell U_\omega = \text{diag} \left( a + 2b, a - b + \sqrt{3}ic, a - b - \sqrt{3}ic \right).$$  \hspace{2cm} (28)

This result shows that $M_\ell$ of equation (27) is rich enough to accommodate three different charged-lepton masses, albeit with finetuning.

Using equation (25) in the neutrino sector, we note that the antisymmetric part on the right-hand side, the 3, is not allowed according to the assumed Majorana nature of the neutrinos. Dropping also the 2, we have four scalar gauge triplets $\Delta_k$ ($k = 0, 1, 2, 3$) in $1 \oplus 3'$. From equation (21) we know that $M_\nu$ has to be invariant under $S_1B$. Therefore, the triplet VEVs $w_k$ ($k = 0, 1, 2, 3$) have to be invariant under the $S_1B$ transformation corresponding to $S_1B$ in the representation of the scalar triplets. Leaving out the trivial case of $w_0$ and using that $S_1B$ acts as $-S_1B$ on the 3’, the VEVs $w_k$ ($k = 1, 2, 3$) are determined by

$$-S_1B \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \Rightarrow w_3 = -w_2,$$  \hspace{2cm} (29)

and, therefore, the neutrino mass matrix has the form

$$M_\nu = \begin{pmatrix} A & B & -B \\ B & A & C \\ -B & C & A \end{pmatrix}.$$  \hspace{2cm} (30)

The mass matrices $M_\ell$ and $M_\nu$ of this section have recently been obtained in [10] in a different model.
5 Summary

Before we summarize our findings, we want to point out the caveats and limitations attached to the group-theoretical method reviewed in section 2. It is useful to distinguish between three groups: \( G \) is the family symmetry group of the Lagrangian, the group \( D(G) \) is the \( U(3) \) subgroup given by the representation matrices of \( G \) on the three left-handed leptonic gauge doublets, and \( \bar{G} \) is the \( U(3) \) subgroup generated by the residual symmetries of the mass matrices \( M_\ell M_\ell^\dagger \) and \( M_\nu \). The method of section 2 is a prescription for the determination of \( \bar{G} \). How \( \bar{G} \) is related to \( G \) in a specific model and what \( \bar{G} \) tells us about model building, is another matter. This always has to be kept in mind when assessing results obtained by the group-theoretical method of section 2. Below, whenever we use the phrase “mass matrices,” we mean \( M_\ell M_\ell^\dagger \) and \( M_\nu \). Our list of caveats is the following:

- The method of section 2 explicitly assumes that the family symmetry group \( G \) of the Lagrangian is finite and that neutrinos have Majorana nature.
- Since this method is purely group-theoretical and uses only information contained in the mass matrices, it can yield at most \( D(G) \).
- It is well known that accidental symmetries can occur in the mass matrices, which contribute, therefore, to \( \bar{G} \). An accidental symmetry cannot be elevated to a symmetry of the Lagrangian. In this case, \( \bar{G} \) is not even a subgroup of \( D(G) \)—see for instance [20].
- The method does not apply to models where VEVs break \( G \) totally.

Note that it is possible that a model is predictive because of an accidental symmetry\(^2\). Even if \( G \) is totally broken and there are no accidental symmetries, the model can be predictive because of the restrictions imposed by \( G \) on the Yukawa couplings or because the VEVs have an alignment but this does not correspond to a subgroup of \( G \). For instance, the typical neutrino mass matrix resulting from \( \Delta(27) \) is a case where the group is completely broken, but has a predictive neutrino mass matrix in specific models [26].

The methods and results of the paper can be summarized as follows:

i) We have used the group-theoretical method of [15], together with theorem 1 on vanishing sums of roots of unity, to determine all possible groups \( \bar{G} \) which result from the requirement that in the lepton mixing matrix \( U \) the first column is given by equation (1), i.e. identical with the first column of the tri-bimaximal mixing matrix. This is called TM\(_1\) in [6].

ii) The result is amazingly simple. Only groups \( \bar{G} \) of the form \( \mathbb{Z}_q \times S_4 \) with \( q = 1, 2, 3, \ldots \) are capable to enforce TM\(_1\) without fixing the columns \( u_2 \) and \( u_3 \) in \( U \). Note that we have not only shown that all such groups contain \( S_4 \) [18], from our investigation.

\(^2\)The typical \( A_4 \) models [4] have a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry in \( M_\nu \), but only one \( \mathbb{Z}_2 \) is the residual symmetry of \( A_4 \).
it follows that any such group larger than \( S_4 \) is obtained from \( S_4 \) by multiplication with a cyclic factor\(^3\)

iii) Furthermore, in the basis where the Klein four-group, which is a subgroup of \( S_4 \)—see appendix\(^3\) is represented by diagonal matrices, we have found a unique mechanism for achieving \( TM_1 \): the first column in \( U_\nu \) must be the vector \( u \) of equation (23), while \( U_\ell = U_\omega \)—see equation (20). This mechanism was recently used in [10].

iv) Finally, we have pointed out how to straightforwardly implement the mechanism of the previous item in a class of renormalizable \( S_4 \) models with type II seesaw mechanism. It is fair to mention that we have not solved the VEV alignment problem in this context.

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### A Roots of unity and the eigenvalues of \( T \) and \( ST \)

Here we discuss the general solution of equation (14). We conceive the first relation as a vanishing sum of 14 roots of unity in which there are three sets of three equal roots and two sets of two equal roots. Thus, among the 14 roots, at most six of them are different. According to theorem 1 we must first find all possibilities of decomposing 14 into a sum of prime numbers \( p_i \). There are 10 such possibilities:

\[ 14 = 2 + 2 + 2 + 2 + 2 + 2 + 2 \]
\[ = 3 + 3 + 2 + 2 + 2 + 2 \]
\[ = 3 + 3 + 3 + 3 \]
\[ = 5 + 5 + 2 \]
\[ = 5 + 3 + 3 \]
\[ = 5 + 3 + 2 + 2 \]
\[ = 7 + 5 \]
\[ = 7 + 3 + 2 \]
\[ = 11 + 3. \]

For the sake of simplicity we call the subsets of \( p_i \) roots of unity whose sums vanish by themselves \( p_i \)-sets. There are three simple rules. All roots within a \( p_i \)-set are different. Furthermore, two \( p_i \)-sets are either identical or have no element in common. Finally, a \( p_i \)-set and a \( p_j \)-set with \( p_i \neq p_j \) have at most one element in common. With the first rule we can exclude all cases which contain 7 or 11, since—as noted above—at most six roots are different. Next we consider the cases which contain 5 + 3. The third rule tells us that

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\(^3\)In [11] \( S_4 \) was identified as the minimal group \( \tilde{G} \) for tri-bimaximal mixing, but it is also the group for \( TM_1 \) alone [18, 20]; this follows for instance from proposition 4. The reason is that \( S_4 \) contains not only \( \tilde{S}_1 \) of equation (18) but also \( \tilde{S}_i = 2u_2u_i^\dagger - 1 \) \( (i = 2, 3) \) where \( u_2 \) and \( u_3 \) are the second and third column of \( U_{TBM} \), respectively. When \( S_4 \) is the group of \( TM_1 \), then \( \tilde{S}_i \) with \( i = 2, 3 \) is broken and not part of \( G_\nu \).
all cases with $5 + 3$ contain at least seven different roots and are, therefore, excluded. Next we consider the case $5 + 5 + 2 + 2$. Here both 5-sets have to be identical, otherwise there are too many different roots. But then it is impossible to accommodate three sets of three identical roots.

The case $3 + 3 + 3 + 3 + 2$ is more complicated. Since we must not have more than six different roots, there cannot be three different 3-sets. Thus we assume now that among the four 3-sets there are two different ones: either three 3-sets are identical and the fourth one is different, or two identical 3-sets differ from the other two identical 3-sets. In either case, we have already six different roots. Thus, if we denote the roots of the 2-set by $\epsilon$ and $-\epsilon$, then $\epsilon$ must be identical with a root of one 3-set and $-\epsilon$ must be identical with a root of the other 3-set. It is then easy to check that in this way it is impossible to accommodate at the same time three sets of three identical roots and two sets of two identical roots. The same applies to the remaining case where all four 3-sets are identical.

The remaining two cases need a separate treatment. The second case is the only physically viable one; it is discussed in section \[4\]. The first one will be studied below.

The case $2 + 2 + 2 + 2 + 2 + 2 + 2$ allows to accommodate the three sets of three equal roots and the two sets of two equal roots. It turns out that there is a unique way to achieve this, given by the following grouping:

$$\epsilon_1, \epsilon_1, -\epsilon_1, -\epsilon_1, \epsilon_2, \epsilon_2, -\epsilon_2, -\epsilon_2, \epsilon_2.$$ 

Comparing with equation (13), we write

$$-(-\epsilon_2) + 2(-\epsilon_2) + 2(-\epsilon_2) + 3(\epsilon_1 - \epsilon_1 + \epsilon_2) = 0,$$

whence we deduce

$$e^{i\phi_\epsilon} = e^{i\phi_\mu} = e^{i\phi_\tau} = -\epsilon_2,$$

which means that $\tilde{T}$ is proportional to the unit matrix. Therefore, the symmetry transformation pertaining to this case cannot achieve $T_{M1}$.

\section*{B Generators of $S_4$}

The generators of a finite group are not unique. For our purpose it is useful to consider the Klein four-group, which is an Abelian subgroup of $S_4$, given by $k_1 = (12)(34)$, $k_2 = (14)(23)$, $k_3 = (13)(24)$ and the unit element. We further need one three-cycle, say $s = (123)$, and one transposition, say $t = (12)$. All elements of $S_4$ can be obtained as products of these permutations—see for instance \[25\]. However, what we are really interested in is a faithful three-dimensional irreducible representation of $S_4$. Here we display the 3 as derived in \[25\]:

$$k_1 \mapsto S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$k_2 \mapsto S_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

(B1)
\[ k_3 \mapsto S_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

and

\[ s \mapsto E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad t \mapsto B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \]  \tag{B2}

In the spirit of section 2 we call this basis of the 3 of \( S_4 \) basis 1. The second three-dimensional irreducible representation, \( 3' \), is obtained is obtained from 3 by a sign change in equation \( \text{(B2)} \), namely \( t \mapsto -B \).

It is also useful to have the above generators in the basis where \( E \) is diagonal. With the similarity transformation \( U^\dagger A U = \tilde{A} \) where \( U_\omega \) is given by equation \( (20) \) we obtain (see for instance \([5, 27]\))

\[ \tilde{S}_1 = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \tilde{S}_2 = \frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & -1 & 2\omega \\ 2\omega^2 & 2\omega & -1 \end{pmatrix}, \quad \tilde{S}_3 = \frac{1}{3} \begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & -1 & 2\omega^2 \\ 2\omega^2 & 2\omega & -1 \end{pmatrix} \]  \tag{B3}

and

\[ \tilde{E} = \text{diag} (1, \omega, \omega^2), \quad \tilde{B} = B. \]  \tag{B4}

In the spirit of section 2 this is the 3 given in basis 2.

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