THE REALIZATION PROBLEM FOR VON NEUMANN
REGULAR RINGS

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Abstract. We survey recent progress on the realization problem for von Neumann regular rings, which asks whether every countable conical refinement monoid can be realized as the monoid of isoclasses of finitely generated projective right $R$-modules over a von Neumann regular ring $R$.

This survey consists of four sections. Section 1 introduces the realization problem for von Neumann regular rings, and points out its relationship with the separativity problem of [7]. Section 2 surveys positive realization results for countable conical refinement monoids, including the recent constructions in [5] and [4]. We analyze in Section 3 the relationship with the realization problem of algebraic distributive lattices as lattices of two-sided ideals over von Neumann regular rings. Finally we observe in Section 4 that there are countable conical monoids which can be realized by a von Neumann regular $K$-algebra for some countable field $K$, but they cannot be realized by a von Neumann regular $F$-algebra for any uncountable field $F$.

1. The problem

All rings considered in this paper will be associative, and all the monoids will be commutative.

For a unital ring $R$, let $\mathcal{V}(R)$ denote the monoid of isomorphism classes of finitely generated projective right $R$-modules, where the operation is defined by

$$[P] + [Q] = [P \oplus Q].$$

This monoid describes faithfully the decomposition structure of finitely generated projective modules. The monoid $\mathcal{V}(R)$ is always a conical monoid, that is, whenever $x + y = 0$, we have $x = y = 0$. Recall that an order-unit in a monoid $M$ is an element $u$ in $M$ such that for every $x \in M$ there is $y \in M$ and $n \geq 1$ such that $x + y = nu$. Observe that $[R]$ is a canonical order-unit in $\mathcal{V}(R)$. By results of Bergman [11] Theorems 6.2 and 6.4] and Bergman and Dicks [12, page 315], any conical monoid with an order-unit appears as $\mathcal{V}(R)$ for some unital hereditary ring $R$.

Date: February 13, 2008.

2000 Mathematics Subject Classification. Primary 16D70; Secondary 06A12, 06F05, 46L80.

Key words and phrases. von Neumann regular ring, Leavitt path algebra, refinement monoid.

Partially supported by the DGI and European Regional Development Fund, jointly, through Project MTM2005-00934, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.
A monoid $M$ is said to be a refinement monoid in case any equality $x_1 + x_2 = y_1 + y_2$ admits a refinement, that is, there are $z_{ij}$, $1 \leq i, j \leq 2$ such that $x_i = z_{i1} + z_{i2}$ and $y_j = z_{1j} + z_{2j}$ for all $i, j$, see e.g. [8]. If $R$ is a von Neumann regular ring, then the monoid $\mathcal{V}(R)$ is a refinement monoid by [20, Theorem 2.8].

The following is still an open problem:

**R1. Realization Problem for von Neumann Regular Rings** Is every countable conical refinement monoid realizable by a von Neumann regular ring?

A related problem was posed by K.R. Goodearl in [22]:

**FUNDAMENTAL OPEN PROBLEM** Which monoids arise as $\mathcal{V}(R)$’s for a von Neumann regular ring $R$?

It was shown by Wehrung in [35] that there are conical refinement monoids of size $\aleph_2$ which cannot be realized. If the size of the monoid is $\aleph_1$ the question is open. Wehrung’s approach is related to Dilworth’s Congruence Lattice Problem (CLP), see Section 3. A solution to the latter problem has recently appeared in [12].

Problem R1 is related to the separativity problem. A class $C$ of modules is called separative if for all $A, B \in C$ we have

$$A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B.$$ 

A ring $R$ is separative if the class $\text{FP}(R)$ of all finitely generated projective right $R$-modules is a separative class. Separativity is an old concept in semigroup theory, see [16]. A commutative semigroup $S$ is called separative if for all $a, b \in S$ we have $a + a = a + b = b + b \implies a = b$. An alternative characterization is that a commutative semigroup is separative if and only if it can be embedded in a product of monoids of the form $G \sqcup \{\infty\}$, where $G$ is an abelian group. Clearly a ring $R$ is separative if and only if $\mathcal{V}(R)$ is a separative semigroup. Separativity provides a key to a number of outstanding cancellation problems for finitely generated projective modules over exchange rings, see [7].

Outside the class of exchange rings, separativity can easily fail. In fact it is easy to see that a commutative ring $R$ is separative if and only if $\mathcal{V}(R)$ is cancellative. Among exchange rings, however, separativity seems to be the norm. It is not known whether there are non-separative exchange rings. This is one of the major open problems in this area. See [3] for some classes of exchange rings which are known to be separative. We single out the problem for von Neumann regular rings. (Recall that every von Neumann regular ring is an exchange ring.)

**SP. Is every von Neumann regular ring separative?**

We have ($R1$ has positive answer $) \implies (SP$ has a negative answer $)$. To explain why we have to recall results of Bergman and Wehrung concerning existence of countable non-separative conical refinement monoids.
Recall that every monoid $M$ is endowed with a natural pre-order, the so-called algebraic pre-order, by $x \leq y$ iff there is $z \in M$ such that $y = x + z$. This is the only order on monoids that we will consider in this paper. A monoid homomorphism $f : M \to M'$ is an order-embedding in case $f$ is one-to-one and, for $x, y \in M$, we have $x \leq y$ if and only if $f(x) \leq f(y)$.

**Proposition 1.1.** (cf. [36]) Let $M$ be a countable conical monoid. Then there is an order-embedding of $M$ into a countable conical refinement monoid.

Let us apply the above Proposition to the conical monoid $M$ generated by $a$ with the only relation $2a = 3a$. Then

$$a + a = a + (2a) = (2a) + (2a)$$

but $a \neq 2a$ in $M$. By Proposition 1.1 there exists an order-embedding $M \to M'$, where $M'$ is a countable conical refinement monoid, and $M'$ cannot be separative.

Thus if R1 is true we can represent $M'$ as $\mathcal{V}(R)$ for some von Neumann regular ring and $R$ will be non-separable.

2. Known cases

It turns out that only a few cases of R1 are known. In this section I will describe the positive realization results of which I am aware.

The first realization result is by now a classical one. Recall that a monoid $M$ is said to be unperforated if, for $x, y \in M$ and $n \geq 1$, the relation $nx \leq ny$ implies that $x \leq y$. A dimension monoid is a cancellative, refinement, unperforated conical monoid. These are the positive cones of the dimension groups [20, Chapter 15]. Recall that, by definition, an ultramatricial $K$-algebra $R$ is a direct limit of a sequence of finite direct products of matrix algebras over $K$. Clearly every ultramatricial algebra is von Neumann regular.

**Theorem 2.1.** ([18], [19, Theorem 3.17], [20, Theorem 15.24(b)]) If $M$ is a countable dimension monoid and $K$ is any field, then there exists an ultramatricial $K$-algebra $R$ such that $\mathcal{V}(R) \cong M$.

A $K$-algebra is said to be locally matricial in case it is a direct limit of a directed system of finite direct products of matrix algebras over $K$, see [23, Section 1]. It was proved in [23, Theorem 1.5] that if $M$ is a dimension monoid of size $\leq \aleph_1$, then it can be realized as $\mathcal{V}(R)$ for a locally matricial $K$-algebra $R$. Wehrung constructed in [35] dimension monoids of size $\aleph_2$ which cannot be realized by regular rings. Indeed the monoids constructed in [35] are the positive cones of dimension groups which are vector spaces over $\mathbb{Q}$. A refinement of the method used in [35] gave a dimension monoid counterexample of size $\aleph_2$ with an order unit of index two [41], thus answering a question posed by Goodearl in [21].

Another realization result was obtained by Goodearl, Pardo and the author in [6].

**Theorem 2.2.** ([6, Theorem 8.4]). Let $G$ be a countable abelian group and $K$ any field. Then there is a purely infinite simple regular $K$-algebra $R$ such that $K_0(R) \cong G$. 
Recall that a simple ring $R$ is purely infinite in case it is not a division ring and, for every nonzero element $a \in R$ there are $x, y \in R$ such that $xay = 1$ (see [6, Section 1], especially Theorem 1.6). Since $\mathcal{V}(R) = K_0(R) \sqcup \{0\}$ for a purely infinite simple regular ring [6, Corollary 2.2], we get that all monoids of the form $G \sqcup \{0\}$, where $G$ is a countable abelian group, can be realized.

As Fred Wehrung has kindly pointed out to me, another class of conical refinement monoids which can be realized by von Neumann regular rings is the class of continuous dimension scales, see [28, Chapter 3]. All the monoids in this class satisfy the property that every bounded subset has a supremum, as well as some additional axioms, see [28, Definition 3-1.1]. Indeed, if $M$ is a commutative monoid, then $M \cong \mathcal{V}(R)$ for some regular, right self-injective ring $R$ if and only if $M$ is a continuous dimension scale with order-unit [28, Corollary 5-3.15]. These monoids have unrestricted cardinality, indeed they are usually quite large.

A recent realization result has been obtained by Brustenga and the author in [5]. As we will note later, the two results mentioned above (Theorem 2.1 and Theorem 2.2) can be seen as particular cases of the main result in [5]. Before we proceed with the statement of this result, and in order to put it in the right setting, we need to recall a few monoid theoretic concepts.

**Definition 2.3.** Let $M$ be a monoid. An element $p \in M$ is prime if for all $a_1, a_2 \in M$, $p \leq a_1 + a_2$ implies $p \leq a_1$ or $p \leq a_2$. A monoid is primely generated if each of its elements is a sum of primes.

**Proposition 2.4.** [14, Corollary 6.8] Any finitely generated refinement monoid is primely generated.

We have the following particular case of question R1.

**R2.** Realization Problem for finitely generated refinement monoids: Is every finitely generated conical refinement monoid realizable by a von Neumann regular ring?

We conjecture that R2 has a positive answer. Our main tool to realize a large class of finitely generated refinement monoids is the consideration of some regular algebras associated with quivers.

Recall that a quiver (= directed graph) consists of a ‘vertex set’ $E^0$, an ‘edge set’ $E^1$, together with maps $r$ and $s$ from $E^1$ to $E^0$ describing, respectively, the range and source of edges. A quiver $E = (E^0, E^1, r, s)$ is said to be row-finite in case, for each vertex $v$, the set $s^{-1}(v)$ of arrows with source $v$ is finite. For a row-finite quiver $E$, the graph monoid $M(E)$ of $E$ is defined as the quotient monoid of $F = F_E$, the free abelian monoid with basis $E^0$ modulo the congruence generated by the relations

$$v = \sum_{\{e \in E^1 \mid s(e) = v\}} r(e)$$
for every vertex \( v \in E^0 \) which emits arrows (that is \( s^{-1}(v) \neq \emptyset \)).

It follows from Proposition \([2.4]\) and \([8, \text{Proposition 4.4}]\) that, for a finite quiver \( E \), the monoid \( M(E) \) is primely generated. Note that this is not always the case for a general row-finite graph \( E \). An example is provided by the graph:

\[
\begin{array}{c}
p_0 \\
\downarrow \ \\
p_1 \\
\downarrow \ \\
p_2 \\
\downarrow \ \\
p_3 \\
\vdots
\end{array}
\]

The corresponding monoid \( M \) has generators \( a, p_0, p_1, \ldots \) and relations given by \( p_i = p_{i+1} + a \) for all \( i \geq 0 \). One can easily see that the only prime element in \( M \) is \( a \), so that \( M \) is not primely generated.

Now we have the following result of Brookfield:

**Theorem 2.5.** \([14, \text{Theorem 4.5 and Corollary 5.11}(5)]\) Let \( M \) be a primely generated refinement monoid. Then \( M \) is separative.

In fact, primely generated refinement monoids enjoy many other nice properties, see \([14]\) and also \([37]\).

It follows from Proposition \([2.4]\) and Theorem \([2.5]\) that a finitely generated refinement monoid is separative. In particular all the monoids associated to finite quivers are separative. For a row-finite quiver \( E \) the result follows by using the fact that the monoid \( M(E) \) is the direct limit of monoids associated to certain finite subgraphs of \( E \), see \([8, \text{Lemma 2.4}]\).

**Theorem 2.6.** \(([5, \text{Theorem 4.2, Theorem 4.4}])\) Let \( M(E) \) be the monoid corresponding to a finite quiver \( E \) and let \( K \) be any field. Then there exists a unital von Neumann regular hereditary \( K \)-algebra \( Q_K(E) \) such that \( \mathcal{V}(Q_K(E)) \cong M(E) \). Furthermore, if \( E \) is a row-finite quiver, then there exists a (not necessarily unital) von Neumann regular \( K \)-algebra \( Q_K(E) \) such that \( \mathcal{V}(Q_K(E)) \cong M_E \).

Note that, due to unfortunate lack of convention in this area, the Cuntz-Krieger relations used in \([5]\) are the opposite to the ones used in \([8]\), which are the ones we are following in this survey, so that the result in \([5, \text{Theorem 4.4}]\) is stated for column-finite quivers instead of row-finite ones. The regular algebras \( Q_K(E) \) are related to the Leavitt path algebras \( L_K(E) \) of \([1], [2], [8]\).

We now observe that Theorem \([2.1]\) and Theorem \([2.2]\) are particular cases of Theorem \([2.6]\). This follows from the fact that the monoids considered in these theorems are known to be graph monoids \( M(E) \) for suitable quivers \( E \). Indeed, taking into account \([8, \text{Theorem 3.5 and Theorem 7.1}]\), we see that the case of dimension monoids follows from \([31, \text{Proposition 2.12}]\) and the case of monoids of the form \( \{0\} \sqcup G \), with \( G \) a countable abelian group, follows from \([34, \text{Theorem 1.2}]\). As Pardo pointed out to me, the quiver \( E \) can be chosen to be finite in case \( M = \{0\} \sqcup G \) for a finitely generated
abelian group $G$. To see this, note that such a monoid admits a presentation given by a finite number of generators $a_1, \ldots, a_n$, and relations of the form $a_i = \sum_{j=1}^{n} \gamma_{ji}a_j$, where all $\gamma_{ji}$ are strictly positive integers and $\gamma_{ii} \geq 2$ for all $i$. The corresponding finite quiver will have $\gamma_{ji}$ arrows from the vertex $i$ to the vertex $j$.

Now we would like to describe how this construction sheds light on problem R2. The answer is completely known for the class of antisymmetric finitely generated refinement monoids. The monoid $M$ is said to be **antisymmetric** in case the algebraic pre-order is a partial order, that is, in case $x \leq y$ and $y \leq x$ imply that $x = y$. Note that every antisymmetric monoid is conical.

We say that a monoid $M$ is **primitive** if it is an antisymmetric primely generated refinement monoid \cite[Section 3.4]{[30]}. A primitive monoid $M$ is completely determined by its set of primes $\mathbb{P}(M)$ together with a transitive and antisymmetric relation $\prec$ on it, given by $q \prec p$ iff $p + q = p$. Indeed given such a pair $(\mathbb{P}, \prec)$, the abelian monoid $M(\mathbb{P}, \prec)$ defined by taking as a set of generators $\mathbb{P}$ and with relations given by $p = p + q$ whenever $q \prec p$, is a primitive monoid, and the correspondences $M \mapsto (\mathbb{P}(M), \prec)$ and $(\mathbb{P}, \prec) \mapsto M(\mathbb{P}, \prec)$ give a bijection between isomorphism types of primitive monoids and isomorphism types of pairs $(\mathbb{P}, \prec)$, where $\mathbb{P}$ is a set and $\prec$ a transitive antisymmetric relation on $\mathbb{P}$, see \cite[Proposition 3.5.2]{[30]}.

Let $M$ be a primitive monoid and $p \in \mathbb{P}(M)$. Then $p$ is said to be **free** in case $p \nLeftarrow p$. Otherwise $p$ is **regular**, see \cite[Section 2]{[10]}. So giving a primitive monoid is equivalent to giving a poset $(\mathbb{P}, \leq)$ which is a disjoint union of two subsets: $\mathbb{P} = \mathbb{P}_{\text{free}} \cup \mathbb{P}_{\text{reg}}$. If $M$ is a finitely generated primitive monoid then $\mathbb{P}(M)$ is a finite set, indeed $\mathbb{P}(M)$ is the minimal generating set of $M$.

We can now describe the finitely generated primitive monoids which are graph monoids. Recall that a **lower cover** of an element $p$ of a poset $\mathbb{P}$ is an element $q$ in $\mathbb{P}$ such that $q < p$ and $[q, p] = \{q, p\}$. The set of lower covers of $p$ in $\mathbb{P}(M)$ is denoted by $L(M, p)$, and $L_{\text{free}}(M, p)$ and $L_{\text{reg}}(M, p)$ denote the sets of free and regular elements in $L(M, p)$ respectively.

**Theorem 2.7.** (cf. \cite[Theorem 5.1]{[10]}) Let $M$ be a finitely generated primitive monoid. Then the following statements are equivalent:

1. $M$ is a graph monoid.
2. $M$ is a direct limit of graph monoids.
3. $|L_{\text{free}}(M, p)| \leq 1$ for each $p \in \mathbb{P}_{\text{free}}(M)$.

For the monoids as in the statement there is a hereditary, von Neumann regular ring $Q(E)$ such that $\mathcal{V}(Q(E)) = M_E = M$ (Theorem \cite[2.6]{[2.6]}). It is worth mentioning that, in some cases, an infinite quiver is required in Theorem \cite[2.7]{[2.7]}. A slightly more general result is indeed presented in \cite[Theorem 5.1]{[10]}. Namely the same characterization holds when $M$ is a primitive monoid such that $L(M, p)$ is finite for every $p$ in $\mathbb{P}(M)$.

In view of Theorem \cite[2.7]{[2.7]} the simplest primitive monoid which is not a graph monoid is the monoid

$$M = \langle p, a, b \mid p = p + a = p + b \rangle.$$
In this example $\mathbb{P}(M) = \mathbb{P}_{\text{free}}(M) = \{p, a, b\}$, and $p$ has two free lower covers $a, b$. Thus, by Theorem 2.7, the monoid $M$ is not even a direct limit of graph monoids (with monoid homomorphisms as connecting maps). However $M$ can be realized as the monoid of a suitable von Neumann regular ring, as follows. Fix a field $K$ and consider two indeterminates $t_1, t_2$ over $K$. We consider the regular algebra $Q_{K(t_2)}(S_1)$ over the quiver $S_1$ with two vertices $v_{0,1}, v_{1,1}$ and two arrows $e_1, f_1$ such that $r(e_1) = s(e_1) = v_{1,1} = s(f_1)$ and $r(f_1) = v_{0,1}$. The picture of $S_1$ is as shown in Figure 2.

The algebra $Q_1 := Q_{K(t_2)}(S_1)$ has a unique non-trivial (two-sided) ideal $M_1$, which coincides with its socle, so that we get an extension of rings:

$0 \rightarrow M_1 \rightarrow Q_1 \xrightarrow{\pi_1} K(t_2)(t_1) = K(t_1, t_2) \rightarrow 0$

However the element $t_1$ does not lift to a unit in $Q_1$, rather there are $z_1, \overline{z}_1$ in $Q_1$ such that $\overline{z}_1 z_1 = 1$, but $z_1 \overline{z}_1 \neq 1$, and $\pi_1(z_1) = t_1$. Some additional information on the algebra $Q_F(S_1)$, where $F$ denotes an arbitrary field, can be found in [5, Examples 4.3].

Let $S_2$ be a copy of $S_1$, now with vertices labelled as $v_{0,2}, v_{1,2}$ and arrows labelled as $e_2, f_2$, and set $Q_2 = Q_{K(t_1)}(S_2)$. There is a corresponding diagram

$0 \rightarrow M_2 \rightarrow Q_2 \xrightarrow{\pi_2} K(t_1)(t_2) = K(t_1, t_2) \rightarrow 0$

Let $P$ be the pullback of the maps $\pi_1$ and $\pi_2$, so that $P$ fits in the following commutative square:

$$
\begin{array}{ccc}
P & \rightarrow & Q_1 \\
\rho_2 \downarrow & & \downarrow \pi_1 \\
Q_2 & \rightarrow & K(t_1, t_2)
\end{array}
$$

(2.1)

Then $P$ is a von Neumann regular ring and $\mathcal{V}(P) = M$, see [4]. Indeed a wide generalization of this method gives the following realization result:
**Theorem 2.8.** ([4, Theorem 2.2]) Let \( M \) be a finitely generated primitive monoid such that all primes of \( M \) are free and let \( K \) be a field. Then there is a unital regular \( K \)-algebra \( Q_K(M) \) such that \( \mathcal{V}(Q_K(M)) \cong M \).

Moreover both the regular algebras associated with quivers [5] and the regular algebras constructed in [4] are given explicitly in terms of generators and relations (including universal localization [33]).

Recall that a monoid \( M \) is strongly separative in case \( a + a = a + b \) implies \( a = b \) for \( a, b \in M \). A ring \( R \) is said to be strongly separative in case \( \mathcal{V}(R) \) is a strongly separative monoid, see [7] for background and various equivalent conditions. As we mentioned above, every primely generated refinement monoid is separative [14, Theorem 4.5]. In particular every primitive monoid is separative. Moreover, a primitive monoid \( M \) is strongly separative if and only if all the primes in \( M \) are free, see [14, Theorem 4.5, Corollary 5.9]. Thus, the class of monoids covered by Theorem 2.8 coincides exactly with the strongly separative finitely generated primitive monoids. The case of a general finitely generated primitive monoid remains open, although it seems amenable to analysis in the light of [5] and [4].

### 3. Realizing distributive lattices

Let \( R \) be a regular ring. Then the lattice \( \text{Id}(R) \) of all (two-sided) ideals of \( R \) is an algebraic distributive lattice. Here an algebraic lattice means a complete lattice such that each element is the supremum of all the compact elements below it. The set of compact elements in \( \text{Id}(R) \) is the set \( \text{Id}_c(R) \) of all finitely generated ideals of \( R \), and it is a distributive semilattice, see for example [27]. Here a semilattice means a \( \lor \)-semilattice with least element 0. A semilattice is the same as a monoid \( M \) such that \( x + x = x \) for every \( x \) in \( M \), and a distributive semilattice is just a semilattice satisfying the refinement axiom [27 Lemma 2.3]. Observe that, if \( M \) is a semilattice then \( x \lor y = x + y \) gives the supremum of \( x, y \) in \( M \).

The famous Congruence Lattice Problem (CLP) asks whether an algebraic distributive lattice is the congruence lattice of some lattice; equivalently, whether every distributive semilattice is the semilattice of all the compact congruences of a lattice. This problem has been recently solved in the negative by Fred Wehrung [12], who constructed for each \( \aleph \geq \aleph_{\omega+1} \) an algebraic distributive lattice with \( \aleph \) compact elements such that it cannot be represented as the congruence lattice of any lattice, see also [25]. His methods have been refined by Růžička [32] to cover the case \( \aleph \geq \aleph_2 \). It is worth to remark that Wehrung had previously shown in [39] that every algebraic distributive lattice with \( \leq \aleph_1 \) compact elements can be realized as the ideal lattice for some von Neumann regular ring \( R \), and thus is isomorphic to the congruence lattice of the lattice \( L(R_R) \) of principal right ideals of \( R \) [38 Corollary 4.4]. So the formulation of Růžička is best possible (concerning the number of compact elements). The “hard core” of Wehrung’s proof in [39] is a ring-theoretic amalgamation result proved by P. M. Cohn in [17 Theorem 4.7].
One can ask: What is the relationship between the CLP, or more concretely, the representation problem of algebraic distributive lattices as lattices of ideals of regular rings, and our problem R1? The answer is that, for a regular ring \( R \), the lattice \( \text{Id}(R) \) is only a small piece of information compared with the information contained in the monoid \( \mathcal{V}(R) \), in the sense that \( \mathcal{V}(R) \) determines \( \text{Id}(R) \), but generally the structure of \( \mathcal{V}(R) \) can be much more complicated than the structure of \( \text{Id}(R) \), e.g. for simple rings. Indeed we have a lattice isomorphism \( \text{Id}(R) \cong \text{Id}(\mathcal{V}(R)) \), where for a conical monoid \( M \), \( \text{Id}(M) \) is the lattice of all order-ideals of \( M \), cf. [27, Proposition 7.3]. Recall that an order-ideal of \( M \) is a submonoid \( I \) of \( M \) with the property that whenever \( x \leq y \) in \( M \) and \( y \in I \), we have \( x \in I \). If \( L \) is an algebraic distributive lattice which is not the congruence lattice of any lattice and \( M \) is any conical refinement monoid such that \( \text{Id}(M) \cong L \), then \( M \) cannot be realized as \( \mathcal{V}(R) \) for a regular ring \( R \). For every algebraic distributive lattice \( L \) there is at least one such conical refinement monoid, namely the semilattice \( L_c \) of compact elements of \( L \), but we should expect a myriad of such monoids to exist. Wehrung proved in [39] that if \( |L_c| \leq \aleph_1 \) then the semilattice \( L_c \) can be realized as \( \mathcal{V}(R) \) where \( R \) is a von Neumann regular ring, and he showed in [40] that there is a distributive semilattice \( S_{\omega_1} \) of size \( \aleph_1 \) which is not the semilattice of finitely generated, idempotent-generated ideals of any exchange ring of finite stable rank. In particular there is no locally matricial \( K \)-algebra \( A \) over a field \( K \) such that \( \text{Id}_c(A) \cong S_{\omega_1} \); see [40] for details. This contrasts with Bergman’s result [13] stating that every distributive semilattice of size \( \leq \aleph_0 \) is the semilattice of finitely generated ideals of an ultramatricial \( K \)-algebra, for every field \( K \).

Say that a subset \( A \) of a poset \( \mathbb{P} \) is a lower subset in case \( q \leq p \) and \( p \in A \) imply \( q \in A \). The set \( \mathcal{L}(\mathbb{P}) \) of all lower subsets of \( \mathbb{P} \) forms an algebraic distributive lattice, which is a sublattice of the Boolean lattice \( 2^\mathbb{P} \). Now if \( L \) is a finite distributive lattice, then by a result of Birkhoff ( [24, Theorem II.1.9]) there is a finite poset \( \mathbb{P} \) such that \( L \) is the lattice of all lower subsets of \( \mathbb{P} \). In the case of a finite Boolean algebra \( 2^n \) with \( n \) atoms, the poset \( \mathbb{P} \) is just an antichain with \( n \) points, and \( 2^n = \mathcal{P}(\mathbb{P}) = \mathcal{L}(\mathbb{P}) \). Our construction in [4] gives a realization of \( L \) as the ideal lattice of a regular \( K \)-algebra \( Q_K(\mathbb{P}) \), where \( K \) is an arbitrary fixed field, such that the monoid \( \mathcal{V}(Q_K(\mathbb{P})) \) is the monoid \( M(\mathbb{P}) \) associated with \( (\mathbb{P}, \prec) \), with all elements in \( M(\mathbb{P}) \) being free, that is, \( M(\mathbb{P}) \) is the abelian monoid with generators \( \mathbb{P} \) and relations given by \( p = q + p \) whenever \( q < p \) in \( \mathbb{P} \). Moreover \( Q_K(\mathbb{P}) \) satisfies the following properties ([4, Proposition 2.12, Remark 2.13, Theorem 2.2]):

(a) There is a canonical family of commuting idempotents \( \{ e(A) : A \in L \} \) such that
   
   (i) \( e(A) e(B) = e(A \cap B) \)
   
   (ii) \( e(A) + e(B) - e(A) e(B) = e(A \cup B) \)

   (iii) \( e(\emptyset) = 0 \) and \( e(\mathbb{P}) = 1 \).

   (iv) \( e(A) Q_K(\mathbb{P}) e(A) \cong Q_K(A) \).

(b) Let \( I(A) \) be the ideal of \( Q_K(\mathbb{P}) \) generated by \( e(A) \). Then the assignment

\[
A \mapsto I(A)
\]
defines a lattice isomorphism from $L = \mathcal{L}(\mathbb{P})$ onto $\text{Id}(Q_K(M))$.

(c) The map $M(\mathbb{P}) \to \mathcal{V}(Q_K(\mathbb{P}))$ given by $p \mapsto [e(\mathbb{P} \downarrow p)]$, for $p \in \mathbb{P}$, is a monoid isomorphism. Here $\mathbb{P} \downarrow p = \{q \in \mathbb{P} : q \leq p\}$ is the lower subset of $\mathbb{P}$ generated by $p$.

The set $\text{Idem}(R)$ of idempotents of a ring is a poset in a natural way, by using the order $e \leq f$ iff $e = fe = ef$. This poset is a partial lattice, in the following sense: every two commuting idempotents $e$ and $f$ have an infimum $ef$ and a supremum $e + f - ef$ in $\text{Idem}(R)$. Say that a map $\phi: L \to \text{Idem}(R)$ from a lattice $L$ to $\text{Idem}(R)$ is a lattice homomorphism in case $\phi(x)$ and $\phi(y)$ commute and $\phi(x \lor y) = \phi(x) \lor \phi(y)$ and $\phi(x \land y) = \phi(x) \land \phi(y)$, for every $x, y \in L$.

The above results can be paraphrased as follows: The canonical mapping $\text{Idem}(Q_K(\mathbb{P})) \to \text{Id}(Q_K(\mathbb{P})) = \mathcal{L}(\mathbb{P})$ sending each element $e$ in $\text{Idem}(Q_K(\mathbb{P}))$ to the ideal generated by $e$ has a distinguished section $e: \text{Id}(Q_K(\mathbb{P})) \to \text{Idem}(Q_K(\mathbb{P}))$, $A \mapsto e(A)$, which is a lattice homomorphism.

Write $Q = Q_K(\mathbb{P})$. Observe that we have lattice isomorphisms

$$L = \mathcal{L}(\mathbb{P}) \cong \text{Id}(Q) \cong \text{Id}(Q) \cong \mathcal{V}(Q)/\simeq.$$

Here $\mathcal{V}(Q)/\simeq$ is the maximal semilattice quotient of the monoid $\mathcal{V}(Q)$, see [27, Section 2]. The finite distributive lattice $L$ can be represented in many other ways as an ideal lattice of a regular ring, for instance using ultramatricial algebras [13], but the monoids corresponding to these ultramatricial algebras have little to do with $M(\mathbb{P})$. Indeed as soon as $\mathbb{P}$ is not an antichain we will have that $\mathcal{V}(R)$ is non-finitely generated for every ultramatricial algebra $R$ such that $\text{Id}(R) \cong \mathcal{L}(\mathbb{P})$.

4. THE DEPENDENCE ON THE FIELD

We like to work with von Neumann regular rings which are algebras over a field $K$. A natural question is whether the field $K$ plays any role concerning the realization problem. So we ask the following variant of R1.

**R(K). Realization Problem for von Neumann Regular $K$-algebras** Let $K$ be a fixed field. Is every countable conical refinement monoid realizable by a von Neumann regular $K$-algebra?

The answer to this question is known for uncountable fields, thanks to an argument due to Wehrung. Indeed the basic counter-example comes from a construction due to Chuang and Lee [15]. Their remarkable example gave a negative answer to five open questions in Goodearl’s book [20] on von Neumann regular rings.

We take this opportunity to present the complete argument, including Wehrung’s result and a version of the Chuang-Lee construction. We thank Fred Wehrung for allowing us to include his unpublished result here.

Let $I$ be an order-ideal of a monoid $M$. The equivalence relation $\equiv_I$ defined on $M$ by the rule

$$x \equiv_I y \iff (\exists u, v \in I)(x + u = y + v), \text{ for all } x, y \in M$$

defines a lattice isomorphism from $L = \mathcal{L}(\mathbb{P})$ onto $\text{Id}(Q_K(M))$. 

\[ (c) \text{ The map } M(\mathbb{P}) \rightarrow \mathcal{V}(Q_K(\mathbb{P})) \text{ given by } p \mapsto [e(\mathbb{P} \downarrow p)], \text{ for } p \in \mathbb{P}, \text{ is a monoid isomorphism. Here } \mathbb{P} \downarrow p = \{q \in \mathbb{P} : q \leq p\} \text{ is the lower subset of } \mathbb{P} \text{ generated by } p. \]
We claim that $\Lambda = \bigcup \{P \in \mathcal{V}(R) \mid P = PJ\}$ is an order-ideal of $\mathcal{V}(R)$ and $\mathcal{V}(R/J) \cong \mathcal{V}(R)/\mathcal{V}(J)$, see [7, Proposition 1.4].

A simplicial monoid is a monoid $M$ isomorphic to $(\mathbb{Z}^+)^k$ for some positive integer $k$.

**Proposition 4.1.** (Wehrung) Let $(M, u)$ be a conical refinement monoid with order-unit such that there is a sequence of order-ideals $(I_n)$ in $M$ such that $\bigcap_n I_n = \{0\}$ and $M/I_n$ is simplicial for all $n$. Assume that $M$ is not cancellative. Then there is no regular algebra $R$ over an uncountable field $F$ such that $\mathcal{V}(R) \cong M$.

**Proof.** Assume that $R$ is a regular $F$-algebra over an uncountable field $F$ with $\mathcal{V}(R) \cong M$, and consider the ideals $J_n = \{x \in R \mid [xR] \in I_n\}$ associated with $I_n$. Clearly we can assume that $R$ is unital and that $[1]$ corresponds to $u$ under the isomorphism $\mathcal{V}(R) \cong M$.

By [26, Theorem 2.2], it suffices to prove that there is no uncountable independent family of nonzero right or left ideals of $R$. Since $\mathcal{V}(R) \cong \mathcal{V}(R^{op})$, where $R^{op}$ is the opposite ring of $R$, we see that it suffices to show this fact for right ideals. Indeed, once this is established, we get from [26, Theorem 2.2] and [7, Proposition 4.12] that $R$ is unit-regular, and thus $\mathcal{V}(R)$ must be cancellative by [7, Theorem 4.5], a contradiction with our hypothesis.

There is a surjective lattice homomorphism from the lattice $L(R_R)$ of principal right ideals of $R$ onto the corresponding lattice $L((R/J_n)_{R/J_n})$ of principal right ideals of $R/J_n$, sending a principal right ideal $I$ over $R$ to the principal right ideal $(I + J_n)/J_n$. (To see that this map preserves intersections, suppose that $I$ and $J$ are right ideals of a regular ring $R$ with $I \cap J = 0$, and $K$ a two-sided ideal. Denote by $\mathcal{T}$ and $\mathcal{J}$ the images of $I$ and $J$ under the canonical projection $\pi : R \to R/K$. Then addition gives us a monomorphism of right $R$-modules $I + J \to R$. Since $R$ is regular, $R/K$ is a flat left $R$-module, so the induced map $(I + J) \otimes_R (R/K) \to R \otimes_R (R/K)$ is monic. But this boils down to the obvious map $I/IK \oplus J/JK \to R/K$, so the image, which is just $\mathcal{T} + \mathcal{J}$, must be the direct sum of $\mathcal{T}$ and $\mathcal{J}$. Therefore $\mathcal{T} \cap \mathcal{J} = 0$.)

Observe that $\mathcal{V}(R/J_n)$ is simplicial. It follows indeed that $R/J_n$ is semisimple artinian. For any independent family $(K_\lambda)_{\lambda \in \Lambda}$ of nonzero right ideals $K_\lambda$ of $R$, the family $(K_\lambda + J_n/J_n)_{\lambda \in \Lambda}$ is also independent (because being independent is a lattice theoretic property). So there is a finite subset $X_n$ of $\Lambda$ such that for $\lambda \in \Lambda \setminus X_n$, we have $K_\lambda \subseteq J_n$. We claim that $\Lambda = \bigcup_{n=1}^{\infty} X_n$. Indeed if $\lambda_0 \in \Lambda \setminus \bigcup_{n=1}^{\infty} X_n$, then $K_{\lambda_0} \subseteq \bigcap_{n=0}^{\infty} J_n = \{0\}$. Thus $K_{\lambda_0} = \{0\}$, a contradiction. This shows that $\Lambda = \bigcup_{n=1}^{\infty} X_n$. Since all subsets $X_n$ are finite we get that $\Lambda$ is countable. Thus there is no uncountable independent family of nonzero right ideals of $R$, as desired.

Now we are going to recall the example of Chuang and Lee [15]. We will give a presentation which is a little bit more general. Let $R$ be a $\sigma$-unital regular ring, that
is, a regular ring having an increasing sequence \((e_n)\) of idempotents in \(R\) such that \(R = \bigcup_{n=1}^\infty e_nRe_n\). Put \(R_n = e_nRe_n\). Recall that the multiplier ring \(\mathcal{M}(R)\) is the completion of \(R\) with respect to the strict topology; see [9]. Write

\[\mathcal{R} = \{(x_n) \mid (x_n) \text{ is a Cauchy sequence in the strict topology } \} \subseteq \prod_{n=1}^\infty R_n.\]

By the continuity of operations, \(\mathcal{R}\) is a unital subring of \(\prod_{n=1}^\infty R_n\). There is an obvious canonical surjective homomorphism \(\Phi : \mathcal{R} \to \mathcal{M}(R)\) whose kernel is \(I = \{(x_n) \mid x_n \to 0\}\), where the convergence is with respect to the strict topology.

**Lemma 4.2.** \(I\) is always a (non-unital) regular ring. If each \(e_nRe_n\) is unit-regular, then \(I\) is unit-regular, meaning that \(eIe\) is unit-regular for every idempotent \(e\) in \(I\).

**Proof.** Let \(x = (x_n) \in I\). Choose a sequence of integers \(m_1 < m_2 < \cdots\) such that for all \(m \geq m_i\) we have \(x_me_i = e_ix_m = 0\). Now for \(m_i \leq m < m_{i+1}\), choose a quasi-inverse \(y_m\) of \(x_m\) in \((e_m - e_i)R(e_m - e_i)\). (Note that \(x_m \in (e_m - e_i)R(e_m - e_i)\) for \(m_i \leq m < m_{i+1}\).) We get a quasi-inverse \(y = (y_n)\) of \(x\) such that \(y_n \to 0\) strictly, so \(y \in I\) and \(I\) is regular. The last part is easy, and is left to the reader. \(\square\)

Observe that if \(Q\) is any regular ring such that \(Q \subseteq \mathcal{M}(R)\), then \(\Phi^{-1}(Q)\) is a regular ring ([20, Lemma 1.3]) which is a subdirect product of the regular rings \((R_n)\). In particular \(\Phi^{-1}(Q)\) is stably finite if each \(R_n\) is so.

Now we see that when \(K\) is a countable field the regular algebra \(Q_K(E)\) of the quiver \(E\) with \(E^0 = \{v_0, v_1\}\) and \(E^1 = \{e, f\}\), with \(r(e) = s(e) = s(f) = v_1\) and \(r(f) = v_0\) gives an example that fits in the above picture. (Note that the quiver \(E\) is the same as the quiver \(S_1\) of Figure 2) Since the field \(K\) is countable, the algebra \(Q_K(E)\) is also countable.

Write \(Q = Q_K(E)\), and let \(I\) be the ideal of \(Q\) generated by \(v_0\). Then \(I = \text{Soc}(Q)\) is a simple (non-unital) ring, and \(I\) is countable, so we have \(I \cong M_\infty(K)\) (because \(v_0Qv_0\) is isomorphic to \(K\)), see [9, Remark 2.9]. Here \(M_\infty(K)\) denotes the \(K\)-algebra of countably infinite matrices with only a finite number of nonzero entries. Since this is a crucial argument here, let us recall the details. The ring \(I\) is countable and simple with a minimal idempotent \(v_0\), so by general theory there is a dual pair \(V, W\) of \(K\)-vector spaces such that \(I \cong \mathcal{F}_W(V)\), the algebra of all adjointable operators on \(V\) of finite rank. Since \(I\) is countable, both \(V\) and \(W\) are countably dimensional \(K\)-vector spaces. By an old result of G. W. Mackey [29, Lemma 2], there are dual bases \((v_i)\) and \((w_j)\) for \(V\) and \(W\) respectively, that is, we have \(\langle v_i, w_j \rangle = \delta_{ij}\) for all \(i, j\), which shows that \(\mathcal{F}_W(V) \cong M_\infty(K)\). Since \(I\) is essential in \(Q\) we get an embedding of \(Q\) into the multiplier algebra \(\mathcal{M}(I)\). Observe that \(\mathcal{M}(I) \cong RCFM(K)\), the algebra of row-and-column-finite matrices with coefficients in \(K\), and that \(Q/I \cong K(t)\). So the algebra \(Q\) has the same essential properties as the Chuang and Lee algebra, see [15]. Now \(M_\infty(K)\) is clearly \(\sigma\)-unital and unit-regular. Indeed there is a \(\sigma\)-unit \((e_n)\) for \(I\) consisting of idempotents such that \(e_nIe_n \cong M_n(K)\). Now Lemma 4.2 together with
Lemma 1.3] gives that $S := \Phi^{-1}(Q)$ is regular, and it is residually artinian. The ring $S$ is not countable but it can be easily modified to get a countable algebra with similar properties. Indeed consider the $K$-subalgebra $S_0$ of $S$ generated by $\bigoplus_{n=1}^{\infty} M_n(K)$ and $a, b$ where $a, b$ are elements in $S$ such that $\Phi(a)\Phi(b) = 1$ and $\Phi(b)\Phi(a) \neq 1$. Observe that $S_0$ is countable. We can build a sequence of countable $K$-subalgebras of $S$: $$S_0 \subset S_1 \subset S_2 \subset \cdots \subset S$$ such that each element in $S_i$ is regular in $S_{i+1}$ for all $i$. It follows that $S_\infty = \bigcup S_i$ is a countable, regular $K$-algebra, which is embedded in $\prod_{n=1}^{\infty} M_n(K)$. Moreover $S_\infty$ cannot be unit-regular because it has a quotient ring which is not directly finite. It follows that $M = \mathcal{V}(S_\infty)$ is not cancellative and it is a countable monoid satisfying the hypothesis of Proposition 4.1, so $M$ gives a counterexample to $R(F)$ for uncountable fields $F$, although by definition it can be realized over some countable field $K$.

ACKNOWLEDGMENTS

It is a pleasure to thank Gene Abrams, Ken Goodearl, Kevin O’Meara and Enrique Pardo for their helpful comments. I am specially grateful to Fred Wehrung for his many valuable comments and suggestions.

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