UNBOUNDED INDUCED REPRESENTATIONS OF Δ*-ALGEBRAS.

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Dedicated to the memory of A.U. Klimyk (14.04.1939–22.07.2008)

Abstract. Induced representations of Δ*-algebras by unbounded operators in Hilbert space are investigated. Conditional expectations of a Δ*-algebra \( \mathcal{A} \) onto a unital Δ*-subalgebra \( \mathcal{B} \) are introduced and used to define inner products on the corresponding induced modules. The main part of the paper is concerned with group graded Δ*-algebras \( \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \) for which the Δ*-subalgebra \( \mathcal{B} := \mathcal{A}_e \) is commutative. Then the canonical projection \( p : \mathcal{A} \to \mathcal{B} \) is a conditional expectation and there is a partial action of the group \( G \) on the set \( \mathcal{B}^+ \) of all characters of \( \mathcal{B} \) which are nonnegative on the cone \( \sum \mathcal{A}_g^2 \cap \mathcal{B} \). The complete Mackey theory is developed for Δ*-representations of \( \mathcal{A} \) which are induced from characters of \( \mathcal{B}^+ \). Systems of imprimitivity are defined and two versions of the Imprimitivity Theorem are proved in this context. A concept of well-behaved Δ*-representations of such Δ*-algebras \( \mathcal{A} \) is introduced and studied. It is shown that well-behaved representations are direct sums of cyclic well-behaved representations and that induced representations of well-behaved representations are again well-behaved. The theory applies to a large variety of examples. For important examples such as the Weyl algebra, enveloping algebras of the Lie algebras \( su(2), su(1,1) \), and of the Virasoro algebra, and Δ*-algebras generated by dynamical systems our theory is carried out in great detail.

1. Introduction

Induced representations are a fundamental tool in representation theory of groups and algebras. They were first defined in 1898 for finite groups by G. Frobenius and in 1955 for arbitrary algebras by D.G. Higman. If \( \mathcal{B} \) is a subalgebra of an algebra \( \mathcal{A} \) and \( V \) is a left \( \mathcal{B} \)-module, then the left \( \mathcal{A} \)-module \( \mathcal{A} \otimes_V \mathcal{B} \) \( \mathcal{V} \) with action defined by \( a_0(a \otimes v) := a_0a \otimes v \) is called induced module of \( V \).

In his seminal paper [R] M. Rieffel introduced induced representations for \( C^* \)-algebras and developed a major part of Mackey’s theory in the context of \( C^* \)-algebras. Another pioneering paper is due to J.M.G. Fell [F2]. Detailed treatments of this theory are given in the monographs [FD] by J.M.G. Fell and R.S. Doran and [L] by N.P. Landsman. An essential step in Rieffel’s inducing process is the definition of an inner product on the algebraic tensor product \( \mathcal{A} \otimes \mathcal{B} \mathcal{V} \) by means of a Hilbert \( C^* \)-module or by a conditional expectation. More precisely, if there exists a conditional expectation \( p \) from a \( C^* \)-algebra \( \mathcal{A} \) onto its \( C^* \)-subalgebra \( \mathcal{B} \) and if a Hilbert space \( (V, \langle \cdot , \cdot \rangle) \) is a Hermitian \( \mathcal{B} \)-module (that is, \( \langle bx, y \rangle = \langle x, b^* y \rangle \) for \( x, y \in V \) and \( b \in \mathcal{B} \)), then there exists a pre-inner product \( \langle \cdot , \cdot \rangle_0 \) on \( \mathcal{A} \otimes \mathcal{B} \mathcal{V} \) such that

\[
\langle a_1 \otimes v_1, a_2 \otimes v_2 \rangle_0 := p(a_2^* a_1) v_1, v_2
\]

and the quotient space of \( \mathcal{A} \otimes \mathcal{B} \mathcal{V} \) by the null space of the form \( \langle \cdot , \cdot \rangle_0 \) is a Hermitian \( \mathcal{A} \)-module.

The aim of the present paper is to develop the basics of a theory of unbounded induced \( \ast \)-representations for complex unital \( \ast \)-algebras. In contrast to the case of \( C^* \)-algebras there are various notions of positivity for general \( \ast \)-algebras that lead to different possible definitions of conditional expectations. We shall define (see Definition 4 below) a conditional expectation from a unital \( \ast \)-algebra \( \mathcal{A} \) to a unital \( \ast \)-subalgebra \( \mathcal{B} \) to be a \( \mathcal{B} \)-linear projection \( p \) of \( \mathcal{A} \) onto \( \mathcal{B} \)

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which preserves involution and units and satisfies the following positivity condition:

\[ pl(\sum A^2) \subseteq B \cap \sum A^2. \]

Then a cyclic Hermitian \( B \)-module \( V \) is "inducible" to \( A \) via \( p \) if and only if every element of \( B \cap \sum A^2 \) is represented by a positive symmetric operator on \( V \).

In this paper we shall show that large classes of (unbounded and bounded) \( * \)-representations of important \( * \)-algebras \( A \) are induced from one-dimensional representations (characters) of some appropriate commutative \( * \)-subalgebras \( B \). Before we turn to the content of the paper, let us briefly explain this for the first Weyl algebra. We shall not carry out all details of proofs. Note that the Weyl algebra is a special case of the \( * \)-algebra treated in Section 10 below.

**Example 1.** Let \( A \) be the Weyl algebra \( \mathbb{C}\langle a, a^* | aa^* - a^*a = 1 \rangle \) and let \( B \) be the unital \( * \)-subalgebra \( \mathbb{C}[N] \) of polynomials in \( N := a^*a \). Each element \( x \in A \) can be written as

\[ x = \sum_{r=0}^{n} a^* f_r(N) + \sum_{s=1}^{l} a^{**} f_{-s}(N) \]

with polynomials \( f_j \in \mathbb{C}[N] \) uniquely determined by \( x \). Defining \( p(x) = f_0(N) \), we obtain a conditional expectation \( p \) from \( A \) to \( B \). It can be proved (see [FS] or formula (14) below) that an element \( f(N) \in \mathbb{C}[N] \) belongs to \( B \cap \sum A^2 \) if and only if there are polynomials \( g_0, \ldots, g_k \in \mathbb{C}[N] \) such that

\[ f(N) = g_0(N)^* g_0(N) + N g_1(N)^* g_1(N) + \cdots + N(N - 1) \cdots (N - k + 1) g_k(N)^* g_k(N). \]

For \( \lambda \in \mathbb{R} \), let \( V_\lambda = \mathbb{C} \) be the one-dimensional \( B \)-module given by \( N = \lambda \). It is not difficult to show that \( f(N) = f(\lambda) \geq 0 \) for each polynomial \( f(N) \) of the form (2) if and only if \( \lambda \in \mathbb{N}_0 \).

Now suppose that \( \lambda \in \mathbb{N}_0 \). Let \( H_\lambda \) denote the Hilbert space obtained from the pre-inner product (1) on \( A \otimes_B V_\lambda \). Clearly, the vectors \( a^r \otimes 1, a^{r+1} \otimes 1 \), where \( r \in \mathbb{N}_0 \), form a base of the vector space \( A \otimes_B V_\lambda \). From the relation \( aa^* - a^*a = 1 \) it follows that

\[ a^r a^{*r} = (N + 1) \cdots (N + r), \quad a^{*r} a^r = N(N - 1) \cdots (N - r + 1) \]

for \( r \in \mathbb{N}_0 \). If \( r > \lambda \), then \( p(a^r a^{*r}) = 0 \), so \( a^r \otimes 1 \) belongs to the kernel of the form (1). Set

\[ e_k := \sqrt{k! \lambda!^{-1}} a^{*k} \otimes 1 \quad \text{for} \quad k = 0, \ldots, \lambda \quad \text{and} \quad e_{k+\lambda} := \sqrt{\lambda!(\lambda + k)!^{-1}} a^k \otimes 1 \quad \text{for} \quad k \in \mathbb{N}. \]

From (1) and (3) we easily compute that \( \langle e_k, e_n \rangle_0 = \delta_{kn} \) for \( k, n \in \mathbb{N}_0 \). Hence \( \{e_k : k \in \mathbb{N}_0\} \) is an orthonormal base of \( H_\lambda \). From the definition of \( e_k \) we immediately obtain that

\[ a^{*k} e_k = \sqrt{k!} \sum_{j=0}^{k} \nabla^j e_{k-j+1} \quad \text{and} \quad a e_k = \sqrt{k!} \sum_{j=0}^{k} \nabla^j e_{k-j-1} \quad \text{for} \quad k \in \mathbb{N}_0, \quad \text{where} \quad e_{-1} := 0. \]

This shows that for each \( \lambda \in \mathbb{N}_0 \), the Hermitian \( A \)-module induced from the \( B \)-module \( V_\lambda \) via \( p \) is nothing but the Bargman-Fock representation of the Weyl algebra.

If \( \lambda \notin \mathbb{N}_0 \), the form (1) is not positive semi-definite. Indeed, by (3) we have \( \langle a \otimes 1, a \otimes 1 \rangle_0 = \lambda < 0 \) if \( \lambda < 0 \) and \( \langle a^{k+1} \otimes 1, a^{k+1} \otimes 1 \rangle_0 = \lambda \cdots (\lambda - k + 1)(\lambda - k) < 0 \) if \( k - 1 < \lambda < k \) for \( k \in \mathbb{N} \).

Summarizing, we have shown that the \( B \)-module \( V_\lambda \) is inducible to a Hermitian \( A \)-module if and only if \( f(N) = f(\lambda) \geq 0 \) for all \( f \in B \cap \sum A^2 \) or equivalently if \( \lambda \in \mathbb{N}_0 \).

Our paper is organized in the following way. In Section 2 we study induced \( * \)-representations defined by rigged modules. We follow mainly the approach given in Chapter XI of [FD] with some necessary modifications needed for unbounded representations. As an application we show that the well-behaved representations of \( * \)-algebras defined in [S2] by means of compatible pairs are induced representations coming from certain rigged modules. Section 3 is concerned with conditional expectations of general \( * \)-algebras. We give various definitions depending on the corresponding positivity conditions and develop a number of examples for these notions. Section 4 is devoted to \( G \)-graded \( * \)-algebras \( A = \bigoplus_{g \in G} A_g \) for a discrete group \( G \). If \( H \) is a subgroup of \( G \), then there exists a canonical conditional expectation of \( A \) on the \( * \)-subalgebra \( A_H = \bigoplus_{h \in H} A_h \). Hence \( * \)-representations of \( A_H \) can be induced to \( * \)-representations of \( A \). From Section 6 we are dealing with \( G \)-graded \( * \)-algebras \( A = \bigoplus_{g \in G} A_g \) for which the \( * \)-subalgebra \( B := A_e \) commutative.

There is a large variety of \( G \)-graded \( * \)-algebras (Weyl
algebra, enveloping algebras of $su(2)$ and $su(1,1)$, quotients of the enveloping algebra of the Virasoro algebra, $*$-algebras associated with dynamical systems, quantum disc algebras, Podles’ quantum spheres, quantum algebras, and many others) that have this property. In Section 5 we study systems of imprimitivity and prove our first Imprimitivity Theorem. In Section 6 we show that there is a partial action of the group $G$ on the set $\hat{B}^+$ of all characters of the commutative $*$-algebra $B$ which are nonnegative on the cone $B \cap \sum \mathcal{A}^2$. This partial action is used for a detailed study of the inducing process from characters of the set $\hat{B}^+$. In particular, we characterize irreducible representations and equivalent representations in terms of stabilizer groups of characters. A fundamental problem in unbounded representation theory is how to define and characterize well-behaved representations of a general $*$-algebra. In Section 7 we develop a concept of well-behaved representations for $*$-subalgebras $\mathcal{A}$ of $\mathcal{B}$ with commutative $*$-subalgebra $\mathcal{A}_c$. Among others it is shown that well-behaved representations decompose into direct sums of cyclic well-behaved representations. In Section 8 we define well-behaved systems of imprimitivity and prove an Imprimitivity Theorem for well-behaved representations. The next two sections of the paper are devoted to detailed treatments of some important examples. In Section 9 we study enveloping algebras of three Lie algebras. For the real Lie algebras $su(2)$ and $su(1,1)$ we prove that the induced representations from characters of $\hat{B}^+$ are precisely the representations $dU$, where $U$ is an irreducible unitary representation of the Lie group $SU(2)$ resp. of the universal covering group of $SU(1,1)$. For the enveloping algebra of the Virasoro algebra we characterize irreducible $*$-representations with finite dimensional weight spaces as induced representations from characters of $\hat{B}^+$. In Section 10 we investigate the $*$-algebra with a single generator $a$ and defining relation $aa^* = f(a^*a)$, where $f$ is a polynomial. The special case $f(t) = t + 1$ of this $*$-algebra is just the Weyl algebra. It turns out that for these examples all well-behaved representations according to our definition in Section 7 coincide with distinguished ”nice” representations of the corresponding $*$-algebras thereby showing the usefulness of our concept. In Section 11 we mention a number of further examples for which our theory applies.

We close this introduction by collecting some definitions and notations.

By a $*$-algebra we mean a complex associative algebra $\mathcal{A}$ equipped with a mapping $a \mapsto a^*$ of $\mathcal{A}$ into itself, called the involution of $\mathcal{A}$, such that $(\lambda a + \mu b)^* = \lambda a^* + \mu b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$. The unit of $\mathcal{A}$ (if it exists) will be denoted by $1_\mathcal{A}$ and the group of all $*$-automorphisms of $\mathcal{A}$ by $\text{Aut}\mathcal{A}$. We shall say that a group $G$ acts as automorphism group on $\mathcal{A}$ if there is a group homomorphism $g \mapsto \alpha_g$ of $G$ into $\text{Aut}\mathcal{A}$. A subset $\mathcal{C}$ of $\mathcal{A}_0 := \{a \in \mathcal{A} : a = a^*\}$ is called a pre-quadratic module if $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$, $\mathbb{R}_+\mathcal{C} \subseteq \mathcal{C}$, and $a^*\mathcal{C} \subseteq \mathcal{C}$ for all $a \in \mathcal{A}$. A quadratic module of $\mathcal{A}$ is a pre-quadratic module $\mathcal{C}$ such that $1_\mathcal{A} \in \mathcal{C}$ (see e.g. [5]). The wedge

$$\sum \mathcal{A}^2 := \left\{ \sum_{j=1}^n a_j^2 a_j ; \ a_1, \ldots, a_n \in \mathcal{A}, n \in \mathbb{N} \right\}$$

of all finite sums of squares is obviously the smallest quadratic module of $\mathcal{A}$.

Throughout this paper we use some terminology and results from unbounded representation theory in Hilbert space (see e.g. in [11]). In particular, we shall speak about $*$-representations rather than Hermitian modules. Let us repeat some basic notions and facts.

Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$. A $*$-representation of a $*$-algebra $\mathcal{A}$ on $\mathcal{D}$ is an algebra homomorphism $\pi$ of $\mathcal{A}$ into the algebra $L(\mathcal{H})$ of linear operators on $\mathcal{D}$ such that $\langle \pi(a) \varphi, \psi \rangle = \langle \varphi, \pi(a^*) \psi \rangle$ for all $\varphi, \psi \in \mathcal{D}$ and $a \in \mathcal{A}$. We call $\mathcal{D}(\pi) := \mathcal{D}$ the domain of $\pi$ and write $\mathcal{H}_\pi := \mathcal{H}$. Two $*$-representation $\pi_1$ and $\pi_2$ of $\mathcal{A}$ are (unitarily) equivalent if there exists an isometric linear mapping $U$ of $\mathcal{D}(\pi_1)$ onto $\mathcal{D}(\pi_2)$ such that $\pi_2(a) = U \pi_1(a) U^{-1}$ for $a \in \mathcal{A}$. The direct sum representation $\pi_1 \oplus \pi_2$ acts on the domain $\mathcal{D}(\pi_1) \oplus \mathcal{D}(\pi_2)$ by $(\pi_1 \oplus \pi_2)(a) = \pi_1(a) \oplus \pi_2(a)$, $a \in \mathcal{A}$. A $*$-representation $\pi$ is called irreducible if a direct sum decomposition $\pi = \pi_1 \oplus \pi_2$ is only possible when $\mathcal{D}(\pi_1) = \{0\}$ or $\mathcal{D}(\pi_2) = \{0\}$. If $T$ is a Hilbert space operator, $\mathcal{D}(T), \text{Ran}T, \mathcal{T}$ and $T^*$ denote its domain, its range, its closure and its adjoint, respectively.
Suppose that $\pi$ is a $\ast$-representation of $\mathcal{A}$. If $\mathcal{C}$ is a pre-quadratic module of $\mathcal{A}$, $\pi$ is called $\mathcal{C}$-positive if $(\pi(c)\varphi, \varphi) \geq 0$ for all $c \in \mathcal{C}$ and $\varphi \in \mathcal{D}(\pi)$. We denote by $\operatorname{Res}_B \pi$ the restriction of $\pi$ to a $\ast$-subalgebra $\mathcal{B}$. The graph topology of $\pi$ is the locally convex topology on the vector space $\mathcal{D}(\pi)$ defined by the norms $\varphi \mapsto \|\varphi\| + \|\pi(a)\varphi\|$, where $a \in \mathcal{A}$.

If $D(\pi)$ denotes the completion the $\mathcal{D}(\pi)$ in the graph topology of $\pi$, then $\pi(a) \dagger D(\pi)$, $a \in \mathcal{A}$, defines a $\ast$-representation of $\mathcal{A}$ with domain $D(\pi)$, called the closure of $\pi$. In particular, $\pi$ is closed if and only if $D(\pi)$ is complete in the graph topology of $\pi$. By a core for $\pi$ we mean a dense linear subspace $\mathcal{D}_0$ of $D(\pi)$ with respect to the graph topology of $\pi$. A $\ast$-representation $\pi$ is called non-degenerate if $\pi(A)\mathcal{D}(\pi) := \operatorname{Lin} \{\pi(a)\varphi; a \in \mathcal{A}, \varphi \in \mathcal{D}(\pi)\}$ is dense in $D(\pi)$ in the graph topology of $\pi$. Further, $\pi$ is called self-adjoint if $D(\pi)$ is the intersection of all domains $D(\pi(a)^\ast)$, where $a \in \mathcal{A}$. The (strong) commutant $\pi(\mathcal{A})'$ consists of all bounded operators $T$ on $H_\mathcal{A}$ such that $T\mathcal{D}(T) \subseteq \mathcal{D}(T)$ and $\pi(a)T = T\pi(a)$ for $a \in \mathcal{A}$. If $\pi$ is self-adjoint, $\pi(\mathcal{A})'$ is a von Neumann algebra. A closed $\ast$-representation $\pi$ of a commutative $\ast$-algebra $\mathcal{B}$ is called integrable if $\pi(b^\ast) = \pi(b)^\ast$ for all $b \in \mathcal{B}$.

2. Rigged Modules and Induced Representations

2.1. Let $\mathcal{B}$ be a $\ast$-algebra. From [PD], p. 1078, we repeat the following

**Definition 1.** A right $\mathcal{B}$-rigged module is a right $\mathcal{B}$-module $\mathfrak{X}$ equipped with a map $\langle \cdot, \cdot \rangle_B : \mathfrak{X} \times \mathfrak{X} \to \mathcal{B}$ which is $\mathbb{C}$-linear in the first variable and $\mathbb{C}$-anti-linear in the second variable and satisfies the following conditions:

(i) $\langle x, y \rangle_B = \langle y, x \rangle_B^\ast$ for $x, y \in \mathfrak{X}$,

(ii) $\langle xb, y \rangle_B = \langle x, y \rangle_B b$ for $x, y \in \mathfrak{X}$ and $b \in \mathcal{B}$.

Clearly, (i) and (ii) are equivalent to the conditions (i) and (ii), where

(ii) $\langle xy, y \rangle_B = b^\ast \langle x, y \rangle_B$ for $x, y \in \mathfrak{X}$ and $b \in \mathcal{B}$.

Suppose that $(\mathfrak{X}, \langle \cdot, \cdot \rangle)$ is a right $\mathcal{B}$-rigged module. By (ii) we have

(i) $\langle xb_1, yb_2 \rangle_B = b_2^\ast \langle x, y \rangle_B b_1$ for $x, y \in \mathfrak{X}$ and $b_1, b_2 \in \mathcal{B}$.

Suppose that $\rho$ is a $\ast$-representation of $\mathcal{B}$ on $(\mathcal{D}(\rho), \langle \cdot, \cdot \rangle)$. Let $\mathfrak{X} \otimes B \mathcal{D}(\rho)$ denote the quotient of the tensor product $\mathfrak{X} \otimes \mathcal{B} \mathcal{D}(\rho)$ over $\mathbb{C}$ by the subspace

$$\mathcal{N}_\rho = \left\{ \sum_{k=1}^r x_k b_k \otimes \varphi_k - \sum_{k=1}^r x_k \otimes \rho(b_k)\varphi_k; \ x_k \in \mathfrak{X}, \ b_k \in \mathcal{B}, \ \varphi_k \in \mathcal{D}(\rho), \ r \in \mathbb{N} \right\}.$$

**Lemma 1.**

(4) $\sum_{k \in \mathbb{N}} x_k \otimes \varphi_k, \sum_{l \in \mathbb{N}} y_l \otimes \psi_l)$ := $\sum_{k,l} (\rho((x_k, y_l)_B) \varphi_k, \psi_l)$,

where $x_k, y_l \in \mathfrak{X}$ and $\varphi_k, \psi_l \in \mathcal{D}(\rho)$, is a well-defined Hermitian sesquilinear form $\langle \cdot, \cdot \rangle_0$ on the tensor products $\mathfrak{X} \otimes \mathcal{D}(\rho)$ and $\mathfrak{X} \otimes B \mathcal{D}(\rho)$.

**Proof.** Obviously, $\langle \cdot, \cdot \rangle_0$ is well-defined on the tensor product $\mathfrak{X} \otimes \mathcal{D}(\rho)$ over $\mathbb{C}$. To prove that $\langle \cdot, \cdot \rangle_0$ is also well-defined on the tensor product $\mathfrak{X} \otimes B \mathcal{D}(\rho)$ it suffices to show that $\langle \zeta, \eta \rangle_0 = 0$ and $\langle \eta, \zeta \rangle_0 = 0$ for arbitrary vectors $\eta = \sum y_j \otimes y_j \in \mathfrak{X} \otimes \mathcal{D}(\rho)$ and $\zeta = \sum x_k b_k \otimes \varphi_k - \sum x_k \otimes \rho(b_k)\varphi_k \in \mathcal{N}_\rho$. From (ii) we obtain

$$\sum_{k,l} (\rho((x_k b_k, y_l)_B) \varphi_k, \psi_l) = \sum_{k,l} (\rho((x_k, y_l)_B) \rho(b_k) \varphi_k, \psi_l).$$

Using condition (i) it follows from the latter that $\langle \zeta, \eta \rangle_0 = 0$. Similarly, (i) and (ii) yield $\langle \zeta, \eta \rangle_0 = 0$. Condition (i) implies that $\langle \cdot, \cdot \rangle_0$ is Hermitian (that is $\langle \zeta, \eta \rangle_0 = \langle \eta, \zeta \rangle_0$ for all $\zeta, \eta \in \mathfrak{X} \otimes \mathcal{D}(\rho)$ resp. $\zeta, \eta \in \mathfrak{X} \otimes B \mathcal{D}(\rho)$).

Let $\mathcal{C}$ be the set of finite sums of elements $\langle x, x \rangle_B$, where $x \in \mathfrak{X}$. Then $\mathcal{C}$ is a pre-quadratic module of the $\ast$-algebra $\mathcal{B}$. Indeed, condition (ii) implies that $b^\ast cb \in \mathcal{C}$ for $b \in \mathcal{B}$ and $c \in \mathcal{C}$.\\

\[\square\]
Let $\text{Rep}_c \mathcal{B}$ denote the family of all direct sums of cyclic $*$-representations of $\mathcal{B}$. Note that each cyclic $*$-representation is obviously non-degenerate.

**Lemma 2.** If $\rho \in \text{Rep}_c \mathcal{B}$ and $\rho$ is $\mathcal{C}$-positive, then $\langle \cdot, \cdot \rangle_0$ is a nonnegative sesquilinear form on $\mathfrak{X} \otimes_\mathcal{B} \mathcal{D}(\rho)$.

**Proof.** Assume first that $\rho$ is a cyclic representation with a cyclic vector $\xi \in \mathcal{D}(\rho)$. Take $\eta = \sum_{k=1}^n x_k \otimes \psi_k \in \mathfrak{X} \otimes_\mathcal{B} \mathcal{D}(\rho)$ and fix $\varepsilon > 0$. Since $\xi$ is cyclic, there exist $b_1, \ldots, b_n \in \mathcal{B}$ such that $\| \rho(b_k)\xi - \psi_k \| < \varepsilon$ and $\| \rho(x_k x_i) \mathcal{B}(\rho(b_k)\xi - \psi_k) \| < \varepsilon$ for all $k, l = 1, \ldots, n$. Then for $k, l = 1, \ldots, n$ we get

$$\langle (\rho(x_k x_i) \mathcal{B}(\rho(b_k))\xi, \rho(b_l)\xi) \rangle \leq \langle \rho(x_k x_i) \mathcal{B}(\rho(b_k)\xi) - \rho(x_k x_i) \mathcal{B}(\rho(b_l)\xi), \rho(b_l)\xi \rangle \leq \langle \rho(x_k x_i) \mathcal{B}(\rho(b_l))\xi \rangle + \| \rho(x_k x_i) \mathcal{B}(\rho(b_l))\xi \| \varepsilon \leq ||\rho(x_k x_i)\mathcal{B}(\rho(b_l))\xi|| \varepsilon + \| \rho(b_l)\xi \| \varepsilon + \varepsilon^2.$$

Therefore $\langle \eta, \eta \rangle_0 = \sum_{k,l=1}^n \langle \rho(x_k x_i) \mathcal{B}(\rho(b_k))\xi, \rho(b_l)\xi \rangle$ can be approximated as small as we want by

$$\sum_{k,l=1}^n \langle \rho(x_k x_i) \mathcal{B}(\rho(b_k))\xi, \rho(b_l)\xi \rangle = \sum_{k,l=1}^n \langle \rho(x_k b_k x_l b_l) \mathcal{B}(\rho(b_k))\xi, \xi \rangle = \langle \sum_{k=1}^n x_k b_k, \sum_{k=1}^n x_k b_k \rangle \mathcal{B}(\rho),$$

which is nonnegative. This implies that $\langle \eta, \eta \rangle_0$ is also nonnegative.

In the case when $\rho$ is a direct sum of cyclic representations $\rho_i$ use the equality $\mathfrak{X} \otimes_\mathcal{B} \mathcal{D}(\rho) = \sum_i \mathfrak{X} \otimes_\mathcal{B} \mathcal{D}(\rho_i)$. \(\Box\)

**Remark.** There is a counter-part of Lemma 2 for $*$-representations $\rho$ of $\mathcal{B}$ which are not necessarily direct sums of cyclic $*$-representations. If $\rho$ is non-degenerate and completely positive with respect to the corresponding matrix ordering (see [11], 11.1 and 11.2, for this concept), then the sesquilinear form $\langle \cdot, \cdot \rangle_0$ is nonnegative on $\mathfrak{X} \otimes \mathcal{D}(\rho)$ resp. $\mathfrak{X} \otimes_\mathcal{B} \mathcal{D}(\rho)$.

2.2. Now let $\mathcal{A}$ be another $*$-algebra.

**Definition 2.** A right $\mathcal{B}$-rigged left $\mathcal{A}$-module is a right $\mathcal{B}$-rigged module $(\mathfrak{X}, \langle \cdot, \cdot \rangle_\mathcal{B})$ which is a left $\mathcal{A}$-module such that

(iii) $\langle ax, y \rangle_\mathcal{B} = \langle x, a^* y \rangle_\mathcal{B}$ for $a \in \mathcal{A}, x, y \in \mathfrak{X}$.

A right $\mathcal{B}$-rigged $\mathcal{A}$–$\mathcal{B}$-bimodule is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module satisfying

(iv) $\langle ax b \rangle = a \langle xb \rangle$ for $a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathfrak{X}$.

**Lemma 3.** Suppose $(\mathfrak{X}, \langle \cdot, \cdot \rangle_\mathcal{B})$ is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module (resp. $\mathcal{A}$–$\mathcal{B}$-bimodule). Then

$$\langle \pi_0(a) \sum_k x_k \otimes \varphi_k \rangle = \sum_k ax_k \otimes \varphi_k, a \in \mathcal{A},$$

where $x_k \in \mathfrak{X}, \varphi_k \in \mathcal{D}(\rho)$, is a well-defined homomorphism of $\mathfrak{A}$ into the linear mappings of the vector space $\mathfrak{X} \otimes \mathcal{D}(\rho)$ (resp. $\mathfrak{X} \otimes_\mathcal{B} \mathcal{D}(\rho)$) such that

$$\langle \pi_0(a) \zeta, \eta \rangle_0 = \langle \zeta, \pi_0(a^*) \eta \rangle_0 \text{ for } a \in \mathcal{A}, \zeta, \eta \in \mathfrak{X} \otimes \mathcal{D}(\rho) \text{ resp. } \zeta, \eta \in \mathfrak{X} \otimes_\mathcal{B} \mathcal{D}(\rho).$$

**Proof.** Since $\mathfrak{X}$ is a left $\mathcal{A}$-module, $\pi_0$ is an algebra homomorphism into $L(\mathfrak{X} \otimes \mathcal{D}(\rho))$. Equation (6) follows then immediately by combining (4), (5) and Definition 2 (iv).

If $\mathfrak{X}$ is an $\mathcal{A}$–$\mathcal{B}$-bimodule, $\pi_0$ is well-defined on $\mathfrak{X} \otimes_\mathcal{B} \mathcal{D}(\rho)$, since by (iv) we have

$$\pi_0(a) \sum_k x_k b_k \otimes \varphi_k = \sum_k ax_k b_k \otimes \varphi_k = \sum_k (ax_k) b_k \otimes \varphi_k$$

and

$$= \sum_k ax_k \otimes \rho(b_k) \varphi_k = \pi_0(a) \sum_k x_k \otimes \rho(b_k) \varphi_k.$$

\(\Box\)
Lemma 4. Suppose $\mathcal{X}$ is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module and $\rho$ is a $*$-representation of $\mathcal{B}$ such that the sesquilinear form $\langle \cdot, \cdot \rangle_0$ on $\mathcal{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$ is nonnegative. Let $\langle \cdot, \cdot \rangle$ be the scalar product on the quotient space $\mathcal{D}(\pi_0) := (\mathcal{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho))/\mathcal{K}_\rho$ defined by $\langle [\eta], [\zeta] \rangle = \langle \eta, \zeta \rangle_0$, where $\mathcal{K}_\rho := \{ \eta : \langle \eta, \eta \rangle_0 = 0 \}$ and $[\eta] := \eta + \mathcal{K}_\rho$. Then

$$
\pi_0(a)[\eta] = [\pi_0(a)\eta], \quad a \in \mathcal{A}, \quad \eta \in \mathcal{X} \otimes \mathcal{D}(\rho),
$$

defines a $*$-representation $\pi_0$ of $\mathcal{A}$ on the pre-Hilbert space $(\mathcal{D}(\pi_0), \langle \cdot, \cdot \rangle)$.

**Proof.** Because of Lemma 3 it suffices to check that $\pi(a)$ is well-defined on $\mathcal{D}(\pi_0)$, that is, $\pi_0(a)\mathcal{K}_\rho \subseteq \mathcal{K}_\rho$. Let $\eta \in \mathcal{K}_\rho$. Using (4) and the Cauchy-Schwarz inequality for the nonnegative sesquilinear form $\langle \cdot, \cdot \rangle_0$ we obtain

$$
\langle \pi_0(a)\eta, \pi_0(a)\eta \rangle_0 = \langle \eta, \pi_0(a)^*\pi_0(a)\eta \rangle_0 = \langle \eta, \pi_0(a^*a)\eta \rangle_0 \leq \langle \eta, \eta \rangle_0^{1/2} \langle \pi_0(a^*a)\eta, \pi_0(a^*a)\eta \rangle_0^{1/2} = 0.
$$

That is, $\pi_0(a)\eta \in \mathcal{K}_\rho$. \qed

Let $\pi$ denote the closure of the $*$-representation $\pi_0$ from Lemma 3.

**Definition 3.** We say the $*$-representation $\pi$ of $\mathcal{A}$ is induced from the $*$-representation $\rho$ of $\mathcal{B}$ via the right $\mathcal{B}$-rigged left $\mathcal{A}$-module $\mathcal{X}$ or simply $\pi$ is induced from $\rho$. A $*$-representation $\rho$ of $\mathcal{B}$ is called inducible (from $\mathcal{B}$ to $\mathcal{A}$) if the sesquilinear form (4) is nonnegative.

We denote $\pi$ by $\text{Ind}_{\mathcal{B}^*\mathcal{A}}\rho$ or simply by $\text{Ind}\rho$ if no confusion can arise. The main assertions of the preceding lemmas are summarized by the following proposition.

**Proposition 1.** Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras and $\mathcal{X}$ is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module. If $\rho$ is a $*$-representation of $\mathcal{B}$ such that the sesquilinear form $\langle \cdot, \cdot \rangle_0$ on $\mathcal{X} \otimes \mathcal{D}(\rho)$ given by (4) is nonnegative, then $\text{Ind}\rho$ is a closed $*$-representation of $\mathcal{A}$ defined on the core $(\mathcal{X} \otimes \mathcal{D}(\rho))/\mathcal{K}_\rho$ by

$$
\text{Ind}\rho(a) \left[ \sum_k x_k \otimes \varphi_k \right] = \left[ \sum_k ax_k \otimes \varphi_k \right], \quad a \in \mathcal{A}, x_k \in \mathcal{X}, \varphi_k \in \mathcal{D}(\rho).
$$

If $\rho$ is a $\mathcal{C}$-positive $*$-representation from $\text{Rep}_\mathcal{B}\mathcal{B}$, then the form $\langle \cdot, \cdot \rangle_0$ is nonnegative and hence the induced representation $\text{Ind}\rho$ exists. If $\mathcal{X}$ is a right $\mathcal{B}$-rigged $\mathcal{A} - \mathcal{B}$-bimodule, then the core $(\mathcal{X} \otimes \mathcal{D}(\rho))/\mathcal{K}_\rho$ is a quotient of the tensor product $\mathcal{X} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$.

For applications the following proposition is convenient.

**Proposition 2.** Let $\mathcal{A}$ and $\mathcal{B}$ $*$-algebras and let $\mathcal{X}$ be a right $\mathcal{B}$-rigged left $\mathcal{A}$-module. Let $\rho$ be a $*$-representation of $\mathcal{B}$. Assume that there exists a Hilbert space $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_1)$ and a (well-defined) linear mapping $\Phi : \mathcal{X} \otimes \mathcal{D}(\rho) \to \mathcal{H}_1$ such that $\mathcal{D}_1 := \Phi(\mathcal{X} \otimes \mathcal{D}(\rho))$ is dense in $\mathcal{H}_1$ and

$$
\langle \Phi(x \otimes \varphi), \Phi(y \otimes \psi) \rangle_1 = \langle \rho((x,y)_B) \varphi, \psi \rangle, \quad x,y \in \mathcal{X}, \varphi, \psi \in \mathcal{D}(\rho).
$$

Then $\rho$ is inducible and $\text{Ind}\rho$ is unitarily equivalent to the closure of the $*$-representation $\pi_1$ on $\mathcal{D}_1$ defined by $\pi_1(a) \Phi(x \otimes \varphi) = \Phi(ax \otimes \varphi)$, where $a \in \mathcal{A}, x \in \mathcal{X}, \varphi \in \mathcal{D}(\rho)$.

**Proof.** Define a the linear mapping $U$ of $\mathcal{X} \otimes \mathcal{D}(\rho)$ onto $\mathcal{D}_1$ by $U(\eta) = \Phi(\sum_k x_k \otimes \varphi_k)$ for $\eta = \sum_k x_k \otimes \varphi_k$. Comparing (4) and (7) we see that the form (4) is nonnegative, so $\rho$ is inducible. Further it follows that $\eta \in \mathcal{K}_\rho$ if and only if $\Phi(\sum_k x_k \otimes \varphi_k) = 0$. Hence $U$ yields an isometric linear mapping, denoted again by $U$, of the unitary space $((\mathcal{X} \otimes \mathcal{D}(\rho))/\mathcal{K}_\rho, \langle \cdot, \cdot \rangle)$ onto the unitary space $(\mathcal{D}_1, \langle \cdot, \cdot \rangle_1)$ such that $\pi_1(a) = U \text{Ind}\rho(a)U^{-1}$, $a \in \mathcal{A}$. \qed

**Remark.** Above we have defined induced representations for a right $\mathcal{B}$-rigged left $\mathcal{A}$-module $\mathcal{X}$. However, except for Example 2 in all applications below $\mathcal{X}$ is even a right $\mathcal{B}$-rigged $\mathcal{A} - \mathcal{B}$-bimodule. Moreover, if $\mathcal{X}$ is a right $\mathcal{B}$-rigged left $\mathcal{A}$-module, then using the axioms (ii) and (iii) we compute

$$
\langle (ax)b - a(xb), y \rangle_B = \langle ax, y \rangle_B - \langle xb, a^*y \rangle_B = \langle x, a^*y \rangle_B - \langle x, a^*y \rangle_B = 0.
$$
for arbitrary \( a \in \mathcal{A} \), \( b \in \mathcal{B} \) and \( x, y \in \mathcal{X} \). That is, all elements \((ax)b - a(xb)\) are annihilated by \( \mathcal{X} \) with respect to the \( \mathcal{B} \)-valued form \( \langle \cdot, \cdot \rangle_\mathcal{B} \). In particular, if this form is nondegenerate, then the right \( \mathcal{B} \)-rigged left \( \mathcal{A} \)-module \( \mathcal{X} \) is a right \( \mathcal{B} \)-rigged \( \mathcal{A}-\mathcal{B} \)-bimodule.

The following lemma is needed in Section 7 below.

**Lemma 5.** Suppose \( \mathcal{X} \) is a right \( \mathcal{B} \)-rigged left \( \mathcal{A} \)-module (resp. \( \mathcal{A} - \mathcal{B} \)-bimodule) and \( \rho \) is an inducible cyclic \( * \)-representation of \( \mathcal{B} \) with cyclic vector \( v \in \mathcal{D}(\rho) \). Then the linear subspace of vectors \( [x \otimes v] \), where \( x \in \mathcal{X} \), is a core of \( \pi = \text{Ind} \rho \).

**Proof.** It suffices to show that for arbitrary \( \varepsilon > 0 \), \( a \in \mathcal{A} \), \( x \in \mathcal{X} \), and \( w \in \mathcal{D}(\rho) \) there exists \( b \in \mathcal{B} \) such that \( \| \pi(a)([x \otimes w] - [x \otimes \rho(b)v]) \| < \varepsilon \). Since \( v \) is cyclic, there is a \( b \in \mathcal{B} \) such that \( \| \rho(ax, ax)_{\mathcal{B}}(\rho(b)v - w) \| < \varepsilon \) and \( \| \rho(b)v - w \| < \varepsilon \). Using the Cauchy-Schwarz inequality we get

\[
\| \pi(a)([x \otimes w] - [x \otimes \rho(b)v]) \|^2 = \| [ax \otimes (w - \rho(b)v)] \|^2 \\
= \langle \rho((ax, ax)_{\mathcal{B}})(w - \rho(b)v), (w - \rho(b)v) \rangle_0 < \varepsilon^2.
\]

\[\square\]

The next lemma is a standard fact about induced representations. We omit its simple proof.

**Lemma 6.** Suppose \( \mathcal{X} \) is a right \( \mathcal{B} \)-rigged left \( \mathcal{A} \)-module (resp. \( \mathcal{A} - \mathcal{B} \)-bimodule) and \( \rho \) is a \( * \)-representation of \( \mathcal{B} \). Assume that \( \rho \) is a direct sum of representations \( \rho_i, i \in I \). Then \( \rho \) is inducible if and only if each \( \rho_i \) is inducible. Moreover, \( \text{Ind} \rho = \bigoplus_{i \in I} \text{Ind} \rho_i \).

We close this section by showing that the considerations of \[S2\] fit nicely into the theory of induced representations.

**Example 2.** **Compatible pairs in the sense of \[S2\]**

Let \( \mathcal{A} \) and \( \mathcal{B} \) be two \( * \)-algebras. Following \[S2\], we call \( (\mathcal{A}, \mathcal{B}) \) a compatible pair if \( \mathcal{B} \) is a left \( \mathcal{A} \)-module, with a left action denoted by \( \rhd \), such that

\[
(a \rhd b)^* c = b^*(a^* \rhd c) \quad \text{for} \quad a \in \mathcal{A} \quad \text{and} \quad b \in \mathcal{B}.
\]

(8)

Now let \( (\mathcal{A}, \mathcal{B}) \) be such a compatible pair. We equip \( \mathcal{X} = \mathcal{B} \) with the \( \mathcal{B} \)-valued sesquilinear form \( \langle b, c \rangle_{\mathcal{B}} := c^*b, \quad b, c \in \mathcal{B} \), and with the right \( \mathcal{B} \)-action given by the multiplication. Then \( (\mathcal{X}, \langle \cdot, \cdot \rangle_{\mathcal{B}}) \) is a right \( \mathcal{B} \)-rigged left \( \mathcal{A} \)-module. Indeed, axioms (i) and (ii) are obvious. Axiom (iii) follows from \[S3\], since for arbitrary \( a \in \mathcal{A} \) and \( b, c \in \mathcal{B} \) we have

\[
\langle a \rhd b, c \rangle_{\mathcal{B}} = c^*(a \rhd b) = (a^* \rhd c)^*b = \langle b, a^* \rhd c \rangle_{\mathcal{B}}.
\]

Suppose that \( \rho \in \text{Rep}_c \mathcal{B} \). Since bounded \( * \)-representations acting on the whole Hilbert space are obviously in \( \text{Rep}_c \mathcal{B} \), this covers all representations of \( \mathcal{B} \) considered in \[S2\]. Since the pre-quadratic module \( \mathcal{C} \) for the form \( \langle \cdot, \cdot \rangle_{\mathcal{B}} \) is \( \sum \mathcal{B}^2 \), \( \rho \) is \( \mathcal{C} \)-positive. Therefore, by Proposition 1, \( \rho \) induces a \( * \)-representation \( \pi = \text{Ind} \rho \) of \( \mathcal{A} \). We shall give a more explicit description of this representation \( \pi \) expressed by formula \(9\) below.

Clearly, an element \( \zeta = \sum b_k \otimes \varphi_k \in \mathcal{X} \otimes \mathcal{D}(\rho) \) belongs to the kernel \( \mathcal{K}_\rho \) of the sesquilinear form \( \langle \cdot, \cdot \rangle_0 \) if and only if

\[
\langle \zeta, \zeta \rangle_0 = \sum_{k,l} \langle \rho((b_k, b_l)_{\mathcal{B}})\varphi_k, \varphi_l \rangle = \sum_k \rho(b_k)\varphi_k, \sum_l \rho(b_l)\varphi_l = 0
\]

or equivalently if \( \sum_k \rho(b_k)\varphi_k = 0 \). Hence \( \mathcal{K}_\rho \) is the kernel of the mapping

\[
\mathcal{B} \otimes \mathcal{D}(\rho) \ni \sum_k b_k \otimes \varphi_k \mapsto \sum_k \rho(b_k)\varphi_k \in \rho(\mathcal{B})\mathcal{D}(\rho),
\]

so we have an isomorphism of vector spaces \( \mathcal{D}(\pi_0) = (\mathcal{B} \otimes \mathcal{D}(\rho))/\mathcal{K}_\rho \) and \( \rho(\mathcal{B})\mathcal{D}(\rho) \). If we identify \( \mathcal{D}(\pi_0) \) and \( \rho(\mathcal{B})\mathcal{D}(\rho) \) by identifying \( b \otimes \varphi \) and \( \rho(b)\varphi \), then we have

\[
\pi(a)(\sum_k \rho(b_k)\varphi_k) = \pi_0(a)(\sum_k \rho(b_k)\varphi_k) = \sum_k \rho(a \rhd b_k)\varphi_k
\]

(9)
for \( a \in \mathcal{A} \). This formula shows that the \( * \)-representation \( \pi_0 \) and its closure \( \pi = \text{Ind}\rho \) as defined above are precisely the \( * \)-representations \( \rho \) and \( \rho' \) as defined in [S2], Proposition 1.1. That is, all well-behaved \( * \)-representations \( \rho' \) of \( \mathcal{A} \) associated with the compatible pair \( (\mathcal{A}, \mathcal{B}) \) in the sense of [S2] are induced \( * \)-representations \( \text{Ind} \rho \). Note that the well-behaved \( * \)-representations in the sense of [S2] are closely related to representations constructed from unbounded \( C^* \)-seminorms (see [APT], Chapter 8, for details).

In [S2] a number of examples of compatible pairs are developed. A typical example of a compatible pair \( (\mathcal{A}, \mathcal{B}) \) is obtained as follows: \( \mathcal{B} \) is the \( * \)-algebra \( C_0^* (G) \) of a Lie group \( G \) with convolution multiplication, \( \mathcal{A} \) is the enveloping algebra \( U(\mathfrak{g}) \) of the Lie algebra \( \mathfrak{g} \) of \( G \) and \( x \triangleright f \) is the action of \( x \in U(\mathfrak{g}) \) as a right-invariant differential operator on \( f \in C_0^* (G) \). Note that as in all other examples of compatible pairs treated in [S2] the \( * \)-algebra \( \mathcal{B} \) has no unit.

Moreover, all examples described in [S2] are of the following form: \( \mathcal{A} \) and \( \mathcal{B} \) are \( * \)-subalgebras of a common unital \( * \)-algebra \( \mathfrak{A} \) and the left action of \( a \in \mathcal{A} \) on \( b \in \mathcal{B} \) is just the multiplication in the larger algebra \( \mathfrak{A} \). In this case it follows at once from the \( * \)-algebra axioms that condition (\[8\]) is valid and that \( (\mathfrak{X}, \langle \cdot, \cdot \rangle_\mathfrak{X}) \) is a right \( \mathcal{B} \)-rigged \( \mathcal{A} - \mathcal{B} \)-bimodule.

3. Conditional expectations

In the rest of this paper we assume that \( \mathcal{B} \) is a unital \( * \)-subalgebra of a unital \( * \)-algebra \( \mathcal{A} \).

Most examples of rigged modules are derived from conditional expectations. This is a fundamental concept for this paper. Since positivity will play a crucial role in what follows, we require various versions of this notion.

**Definition 4.** A linear map \( p : \mathcal{A} \to \mathcal{B} \) is called a conditional expectation of \( \mathcal{A} \) onto \( \mathcal{B} \) if

\[ p(a^*) = p(a^*), \quad p(b_1 a b_2) = b_1 p(a) b_2 \quad \text{for all } a \in \mathcal{A}, \ b_1, b_2 \in \mathcal{B}, \ p(1_\mathcal{A}) = 1_\mathcal{B}, \]

and \( p \) is positive in the sense that

\[ p(\sum A^2) \subseteq \sum A^2 \cap \mathcal{B}. \]

A linear map \( p \) satisfying only condition (i) is called a \( \mathcal{B} \)-bimodule projection of \( \mathcal{A} \) onto \( \mathcal{B} \).

A conditional expectation \( p \) will be called a strong conditional expectation if

\[ p(\sum A^2) \subseteq \sum B^2. \]

Let \( \mathcal{C}_A \) and \( \mathcal{C}_B \) be pre-quadratic modules of \( \mathcal{A} \) resp. \( \mathcal{B} \). A \( \mathcal{B} \)-bimodule projection \( p \) will be called \( (\mathcal{C}_A, \mathcal{C}_B) \)-conditional expectation of \( \mathcal{A} \) onto \( \mathcal{B} \) if

\[ p(\mathcal{C}_A) \subseteq \mathcal{C}_B. \]

Note that axiom (i) implies that any \( \mathcal{B} \)-bimodule projection of \( \mathcal{A} \) onto \( \mathcal{B} \) is indeed a projection of \( \mathcal{A} \) onto \( \mathcal{B} \).

The bridge of these notions to rigged modules is given by the following simple lemma.

**Lemma 7.** Suppose that \( p : \mathcal{A} \to \mathcal{B} \) is a \( \mathcal{B} \)-bimodule projection of \( \mathcal{A} \) onto \( \mathcal{B} \) and define \( \langle b, c \rangle_\mathcal{B} := p(c^* b) \) for \( b, c \in \mathcal{B} \) and \( \mathfrak{X} := \mathcal{A}. \) Then \( (\mathfrak{X}, \langle \cdot, \cdot \rangle_\mathfrak{X}) \) is a right \( \mathcal{B} \)-rigged \( \mathcal{A} - \mathcal{B} \)-bimodule with left and right actions given by the multiplications in \( \mathcal{A} \).

**Proof.** Conditions (i), (ii)$_1$, (iii) and (iv) in Definitions 1 and 2 follow at once from (i) in Definition 4 and the \( * \)-algebra axioms. For instance, we verify (ii)$_1$. If \( x, y \in \mathfrak{X}(= \mathcal{A}) \) and \( b \in \mathcal{B} \), then using axiom (i) in Definition 4 we have \( \langle xb, y \rangle_\mathcal{B} = p(y^* xb) = p(y^* x)b = \langle x, y \rangle_\mathcal{B}b. \)

**Definition 5.** A \( \mathcal{B} \)-bimodule projection \( p \) of \( \mathcal{A} \) onto \( \mathcal{B} \) is called faithful if \( p(a^* x) = 0 \) for some \( x \in \mathcal{A} \) implies that \( x = 0 \).

The next lemma illustrates the importance of this notion.

**Lemma 8.** Suppose that \( p \) is a faithful \( \mathcal{B} \)-bimodule projection of \( \mathcal{A} \) onto \( \mathcal{B} \). Let \( \pi_i, \ i \in I \), be a family of inducible \( * \)-representations of \( \mathcal{B} \) which separates the elements of \( \mathcal{B} \). Then the family \( \text{Ind}\pi_i, \ i \in I \), separates the elements of \( \mathcal{A} \).

**Proof.** Let \( a \in \mathcal{A}, \ a \neq 0. \) Since \( p \) is faithful, \( p(a^* a) \neq 0. \) Since the family \( \pi_i, i \in I \), separate the elements of \( \mathcal{B} \), there exist a representation \( \pi_{i_0}, i_0 \in I \), and a vector \( \varphi \in \mathcal{D}(\pi_{i_0}) \) such that \( \pi_{i_0}(p(a^* a))\varphi \neq 0. \) Then we have \( \|\text{Ind}\pi_{i_0}(a)[1 \otimes \varphi]\| = \|\pi_{i_0}(p(a^* a))\varphi\| \neq 0. \)
The following simple proposition is taken from \cite{Y}. It characterizes a $\mathcal{B}$-bimodule projection in terms of its kernel.

**Proposition 3.** There exists a $\mathcal{B}$-bimodule projection from $\mathcal{A}$ onto $\mathcal{B}$ if and only if there exists a $\ast$-invariant subspace $\mathcal{T} \subseteq \mathcal{A}$ such that $\mathcal{A} = \mathcal{B} \oplus \mathcal{T}$ and

\[
(10) \quad BTB \subseteq \mathcal{T}.
\]

If this is true, the $\mathcal{B}$-bimodule projection $p$ is uniquely defined by the requirement $\ker p = \mathcal{T}$ and we have $p(\sum A^2) = \sum B^2 + p(\sum T^2)$.

**Proof.** Let $p$ be a $\mathcal{B}$-bimodule projection from $\mathcal{A}$ onto $\mathcal{B}$ and put $\mathcal{T} = \ker p$. For $t \in \mathcal{T}$ and $b_1, b_2 \in \mathcal{B}$ we have $p(b_1tb_2) = b_1p(t)b_2 = 0$ and $p(t^\ast) = p(t)^\ast = 0$, so that $\mathcal{T}$ satisfies (10) and is $\ast$-invariant. For arbitrary $a \in \mathcal{A}$ we have $p(a) \in \mathcal{B}$ and $a - p(a) \in \mathcal{T}$, so that $\mathcal{A} = \mathcal{B} \oplus \mathcal{T}$.

Conversely, if $\mathcal{T}$ is given, one easily checks that the linear map $p$ defined by $p(b) = b$, $b \in \mathcal{B}$, and $p(t) = 0$, $t \in \mathcal{T}$, is indeed a $\mathcal{B}$-bimodule projection. \qed

In the remaining part of this section we develop a number of examples. In the first example we use Proposition 3 to show that there is no $\mathcal{B}$-bimodule projection.

**Example 3.** Let $\mathcal{A}$ be the Weyl algebra from Example 1. As it is well-known, the Hermitian elements $\mathcal{P} = \frac{1}{\sqrt{2}}i(a^\ast - a)$ and $\mathcal{Q} = \frac{1}{\sqrt{2}}(a^\ast + a)$ satisfy the commutation relation $\mathcal{P}\mathcal{Q} - \mathcal{Q}\mathcal{P} = -i$.

We show that there is no $\mathcal{B}$-bimodule projection of $\mathcal{A}$ onto $\mathcal{B} := \mathbb{C}[\mathcal{P}]$. Assume to the contrary there is such a projection $p$ and let $\mathcal{T}$ be its kernel. Then, since $\mathcal{A} = \mathcal{B} \oplus \mathcal{T}$, there exists a polynomial $f \in \mathbb{C}[t]$ such that $\mathcal{Q} + f(\mathcal{P}) \in \mathcal{T}$. By (10) we have $\mathcal{P}\mathcal{Q} + f(\mathcal{P})\mathcal{Q} - \mathcal{Q}\mathcal{P} = -i \in \mathcal{T}$. Hence $\mathbf{1}_\mathcal{A} \in \mathcal{T}$ and so $p = 0$ which is a contradiction.

Using Proposition 3 one can check that the map $p$ defined in Example 1 is the unique $\mathcal{B}$-bimodule projection from $\mathcal{A}$ onto $\mathcal{B} := \mathbb{C}[\mathcal{N}]$.

**Example 4.** Let $q_1, \ldots, q_n \in \mathcal{A}$ be a decomposition of unit of the unital $\ast$-algebra $\mathcal{A}$, that is, $q_1 + \cdots + q_n = 1$ and $q_i^2 = q_i$ for $i = 1, \ldots, n$. It is not difficult to show that $q_iq_j = 0$ for all $i \neq j$ and that the map

$$p : a \mapsto q_1aq_1 + \cdots + q_naq_n$$

is a conditional expectation of $\mathcal{A}$ onto the $\ast$-subalgebra $\mathcal{B} = \{b \in \mathcal{A} : b = p(b)\}$. If $\mathcal{A}$ is an $\mathcal{O}^\ast$-algebra, then $p$ is faithful.

**Example 5.** Suppose that $G$ is a discrete group and $H$ is a subgroup of $G$. Let $\mathcal{A} = \mathbb{C}[G]$ and $\mathcal{B} = \mathbb{C}[H]$ be the group algebras of $G$ and $H$, respectively. Recall that the group algebra $\mathbb{C}[G]$ of a discrete group $G$ is a unital $\ast$-algebra with multiplication given by the convolution and involution determined by the inversion of group elements. More precisely, $\mathbb{C}[G]$ is a complex vector space with basis given by the group elements of $G$ and the product of two base element $g$ and $h$ is just the group product $gh$ and $g^\ast$ is the inverse $g^{-1}$. Let $p$ be the canonical projection of $\mathbb{C}[G]$ onto $\mathbb{C}[H]$ defined by $p(g) = g$ if $g \in H$ and $p(g) = 0$ if $g \notin H$.

**Proposition 4.** $p$ is a faithful strong conditional expectation of $\mathbb{C}[G]$ onto $\mathbb{C}[H]$.

**Proof.** It is clear from its definition that $p$ satisfies condition (i) of the Definition 3 so $p$ is a $\mathbb{C}[H]$-bimodule projection.

We shall prove that $p(\sum \mathbb{C}[G]^2) \subseteq \sum \mathbb{C}[H]^2$. Let us fix precisely one element $k_i \in G$ in each left coset $t \in G/H$. Take an arbitrary element $a = \sum_{g \in G} \theta_g g$ of the group algebra $\mathbb{C}[G]$. Then there exist elements $a_i \in \mathbb{C}[H]$, $i \in G/H$, such that $a = \sum_{g \in G} \theta_g g = \sum_{i \in G/H} k_i a_i$. If $i, j \in G/H$ and $i \neq j$, then $k_i^{-1}k_j \notin H$ and hence $p(k_i^{-1}k_j) = 0$. Using this fact we obtain

\[
p(a^\ast a) = p \left( \left( \sum_{i \in G/H} k_i a_i \right)^\ast \left( \sum_{j \in G/H} k_j a_j \right) \right) = \sum_{i, j \in G/H} p(a_i^\ast k_i^{-1}k_j a_j) = \sum_{i, j \in G/H} a_i^\ast p(k_i^{-1}k_j) a_j = \sum_{i \in G/H} a_i^\ast a_i,
\]
so $p(a^*a) \in \sum \mathbb{C}[H]^2$. That is, $p$ is a strong conditional expectation.

From the preceding equality it follows also that $p$ is faithful. Indeed, if $p(a^*a) = 0$, then $\sum_i a_i^*a_i = 0$ which implies that $a_i = 0$ for all $i \in G/H$ and hence $a = 0$. \hfill \Box$

A large source of conditional expectations is obtained from groups of $\ast$-automorphisms. The idea is taken from the following standard construction of conditional expectations of $C^\ast$-algebras reproduced from [R], Example 1.5.

**Example 6.** Suppose that $A$ is a $C^\ast$-algebra and $G$ is a compact group such that there is a continuous action $g \mapsto \alpha_g$ of $G$ as automorphism group of $A$. Let $dg$ denote the normalized Haar measure of $G$. Then the map

$$a \mapsto \int_G \alpha_g(a)dg, \ a \in A,$$

is a strong conditional expectation of $A$ onto the $C^\ast$-subalgebra $B$ of stable elements. \hfill \Box

We now generalize this example to the case of general $\ast$-algebras.

**Example 7.** Suppose that $G$ is a compact group which acts by $\ast$-automorphisms $\alpha_g$, $g \in G$, on a $\ast$-algebra $A$. Assume in addition that the action is locally finite-dimensional, that is, for every $a \in A$ there exists a finite-dimensional linear subspace $V \subset A$ such that $a \in V$, $\alpha_g(V) \subseteq V$ for all $g \in G$, and the map $g \to \alpha_g(a)$ of $G$ into $V$ is continuous. Then the mapping $p$ given by

$$p(a) = \int_G \alpha_g(a)dg, \ a \in A,$$

is well-defined. One easily verifies that $p$ is a $B$-bimodule projection from $A$ onto the $\ast$-subalgebra $B := \{a \in A : \alpha_g(a) = a \text{ for all } g \in G\}$ of stable elements.

Every $G$-invariant finite-dimensional subspace $V \subseteq A$ is a unitarizable $G$-module. Since $G$ is compact, $A$ is a direct sum of submodules $A_t$, $t \in \hat{G}$, where $A_t$ denotes the direct sum of submodules in $A$ isomorphic to $t \in \hat{G}$. In the case when $A$ is a $C^\ast$-algebra, the subspaces $A_t$, $t \in \hat{G}$, are called spectral subspaces, see e.g. [HLS] and [ES]. The mapping $p$ is nothing but the projection of the direct sum $A = \bigoplus_{t \in \hat{G}} A_t$ onto the spectral subspace $A_0$ corresponding to the trivial representation.

An analogue of the map $p$ was considered in [CKS]. Suppose $R$ is a real closed field, $R[V]$ is the coordinate ring of an affine variety $V$ and $G$ is a linear algebraic group over $R$ acting on $R[V]$. If $G$ is reductive, there is a canonical projection $\rho$ from $R[V]$ onto the subring $R[V]^G$ of $G$-invariants called Reynolds operator (see [CKS] for details). In the case when $G(R)$ semi-algebraically compact, Corollary 3.6 in [CKS] states that $\rho(\sum R[V]^2) \subseteq \sum R[V]^2$.

**Proposition 5.** The map $p$ defined by (11) is a conditional expectation of $A$ onto $B$.

**Proof.** It remains to show that $p(\sum A^2) \subseteq \sum A^2$. Let $a \in A$. Then there is a finite-dimensional $G$-invariant subspace $V$ of $A$ containing $a$. Then $V$ is a finite direct sum of submodules $V^{(t)}$, where $V^{(t)}$ is multiple of $t \in \hat{G}$. Fix $t \in \hat{G}$ and let $V^{(t)} = \bigoplus_i V^{(t)}_i$ be a decomposition of $V^{(t)}$ into a direct sum of irreducible $G$-modules. We can choose an orthonormal base $a^{(t)}_{ij}$ in each space $V^{(t)}_i$ such that the matrices corresponding to $\alpha_g$ are unitary and equal for all $i$, i.e. we have

$$\alpha_g(a^{(t)}_{ij}) = \sum_k u^{(t)}_{kj}(g)a^{(t)}_{ik}, \ g \in G, \ t \in \hat{G}.$$
Let us fix elements \( a_{i,j_1}^{(t)}, a_{i_2,j_2}^{(s)} \in V \subseteq A \). Using the orthogonality relations of matrix elements \( u_{k,j_1}^{(t)} \) and \( u_{m,j_2}^{(s)} \) on the compact group \( G \) we compute

\[
p((a_{i,j_1}^{(t)})^*a_{i_2,j_2}^{(s)}) = \int \left( \sum_k u_{k,j_1}^{(t)}(g)(a_{i,j_1}^{(t)})^* \right) \cdot \left( \sum_m u_{m,j_2}^{(s)}(g)a_{i_2,m}^{(s)} \right) \, dg =
\]

\[
= \sum_{k,m} \int u_{k,j_1}^{(t)}(g)u_{m,j_2}^{(s)}(g) \, dg \cdot (a_{i,j_1}^{(t)})^* a_{i_2,m}^{(s)} =
\]

\[
= \delta_{i,i_1} \delta_{j_1,j_2} \dim t \sum_k (a_{i_1,k}^{(t)})^* a_{i_2,k}^{(s)}.
\]

Since \( a \in V \), we can write \( a \) as a finite sum \( a = \sum_{i,j,t} \lambda_{ij}^{(t)} a_{ij}^{(t)} \), where \( \lambda_{ij}^{(t)} \in \mathbb{C} \). Applying the preceding equality we obtain

\[
p(a^*a) = p \left( \sum_{i,j,t} \lambda_{ij}^{(t)} (a_{ij}^{(t)})^* \cdot \sum_{k,l,s} \lambda_{kl}^{(s)} u_{kl}^{(s)} \right) = \sum_{j,t} p \left( \sum_i \lambda_{ij}^{(t)} (a_{ij}^{(t)})^* \cdot \sum_k \lambda_{kj}^{(t)} a_{kj}^{(t)} \right) =
\]

\[
= \sum_{j,t} \frac{1}{\dim t} \left( \sum_i \lambda_{ij}^{(t)} a_{ij}^{(t)} \right)^* \cdot \left( \sum_k \lambda_{kj}^{(t)} a_{kj}^{(t)} \right) \in \sum A^2.
\]

In general this conditional expectation \( p \) is not strong, i.e. \( p(\sum A^2) \) is not contained in \( \sum B^2 \).

4. Group graded \(*\)-algebras

The algebraic representation theory of group graded algebras has been extensively studied, see e.g. the books [NO] and [M]. The monograph [FD] deals with \(*\)-algebraic bundles which can be considered as generalizations of \( G \)-graded \(*\)-algebras to the case when \( G \) is a topological group. However, in [FD] only bounded Hilbert space representations are treated. As we shall see below, there are a plenty of important \( G \)-graded \(*\)-algebras (Weyl algebra, enveloping algebras etc.) for which most \(*\)-representations are unbounded.

Definition 6. Let \( G \) be a (discrete) group. A \( G \)-graded \(*\)-algebra is a \(*\)-algebra \( A \) which is a direct sum \( A = \bigoplus_{g \in G} A_g \) of vector spaces \( A_g \), \( g \in G \), such that

\[
(12) \quad A_g \cdot A_h \subseteq A_{g\cdot h} \quad \text{and} \quad (A_g)^* \subseteq A_{g^{-1}} \quad \text{for} \quad g, h \in G.
\]

From the two conditions in (12) it follows that a \( G \)-grading of a \(*\)-algebra \( A \) is completely determined if we know the corresponding components for a set of generators of the algebra \( A \). In what follows we shall describe most of our \( G \)-gradings of \(*\)-algebras in this manner.

Example 8. In this example we use some basics from the theory of semi-simple Lie algebras. All facts we need can be found in the monograph [D2], 7.0 and 7.4.1. Suppose that \( g \) is a semi-simple complex Lie algebra. We denote by \( h \) a Cartan subalgebra, by \( Q \) the root lattice and by \( H_1, \ldots, H_l, X_{-\alpha_1}, \ldots, X_{-\alpha_n}, X_{\alpha_1}, \ldots, X_{\alpha_n} \) a Cartan-Weyl basis of the Lie algebra \( g \). If we consider the complex universal enveloping algebra \( U(g) \) of \( g \) as a \( h \)-module and so as an \( h \)-module by the adjoint representation, we obtain a direct sum decomposition \( U(g) = \sum_{\lambda \in Q} U(g)_\lambda \). This means that \( U(g) \) is a \( G \)-graded algebra, where \( G \) is the abelian group \( Q \). If \( U(g) \) is equipped with an involution such that \( (X_{\alpha})^* = \varepsilon_{\alpha} X_{-\alpha} \) and \( (H_k)^* = H_k \) for all \( j, k \), then we have \( (U(g))_\lambda^* = U(g)_{-\lambda} \) and hence \( U(g) \) is a \( Q \)-graded \(*\)-algebra. The algebra \( U(g)_0 \) is just the commutant of the Cartan algebra \( h \) in \( U(g) \). Its structure is described in [D2], 7.4.2.

Example 9. Let \( F = \mathbb{C}(z_1, \ldots, z_d, w_1, \ldots, w_d) \) be the free polynomial algebra with generators \( z_1, \ldots, z_d, w_1, \ldots, w_d \) and involution determined by \( (z_j)^* = w_j \), \( j = 1, \ldots, d \). Then \( F \) is a \( \mathbb{Z} \)-graded \(*\)-algebra with \( \mathbb{Z} \)-grading given by \( z_j \in F_1 \).

To derive further examples we shall use the following lemma. We omit its simple proof.
Lemma 9. If $F = \bigoplus_{g \in G} F_g$ is a $G$-graded $\ast$-algebra and $J$ is a two-sided $\ast$-ideal of $F$ generated by subsets of $F_g$, $g \in G$, then the quotient $\ast$-algebra $F/J$ is also $G$-graded.

The proofs of the existence of gradings for all examples occurring in this paper follow by the same pattern: We first define the corresponding grading on the free $\ast$-algebra (Example 9). If the polynomials of the defining relations belong to single components of this grading, Lemma 9 applies and gives the grading of the $\ast$-algebra. We illustrate this by a number of examples in the last section.

Throughout the rest of this section $G$ is a discrete group with unit element $e$, $H$ denotes a subgroup of $G$ and $A = \bigoplus_{g \in G} A_g$ is a unital $G$-graded $\ast$-algebra. The subspace $A_e$ is a $\ast$-subalgebra of $A$ which will be denoted by $B$. Clearly, $1_A \in B$, so that $1_A = 1_B$.

For a subset $X \subseteq G$ we denote by $A_X$ the linear subspace $\bigoplus_{g \in X} A_g$ of $A$. From (12) we conclude that $A_H$ is a $\ast$-subalgebra of $A$ for the subgroup $H$ of $G$.

Proposition 6. Let $p_H$ be the canonical projection of $A$ onto $A_H$, that is, $p_H(a) = \sum_{g \in H} a_g$ for $a = \sum_{g \in G} a_g$, where $a_g \in A_g$. Then $p_H$ is a conditional expectation of $A$ onto $A_H$.

Proof. Condition (i) of Definition 4 follows at once from (12). Our proof is complete once we have shown that $p_H(\sum A^2) \subseteq \sum A^2$.

We choose one element $k_i \in G$, $i \in G/H$, in each left coset of $H$ in $G$. Let $a = \sum_{i \in G/H} b_i$, where $b_i \in A_{k_iH}$. If $i, j \in G/H$, then $b_i b_j \in A_{Hk_j^{-1}k_iH}$, hence we have $p_H(b_i b_j) = b_i b_j$ and $p_H(b_i b_j) = 0$ if $i \neq j$. Using the latter facts we obtain

$$p_H(a^*a) = p_H(\sum_{i \in G/H} b_i^* b_i) = \sum_{i \in G/H} b_i^* b_i \subseteq \sum A^2.$$  

The map $p_H$ from Proposition 6 is called the canonical conditional expectation of the $G$-graded $\ast$-algebra $A$ onto the $\ast$-subalgebra $A_H$.

Equation (13) shows that $p_H$ is faithful when $\sum_{k=1}^n a_k^* a_k = 0$ for arbitrary $a_1, \ldots, a_n \in A$ implies that $a_1 = \cdots = a_n = 0$. In particular, $p_H$ is faithful when $A$ is an $O^\ast$-algebra.

Another immediate consequence of (13) is stated as

Corollary 1. An element $a \in A$ belongs to the cone $\sum A^2 \cap A_H$ if and only if it can be presented as a finite sum of squares $\sum b_i^* b_i$, where each $b_i$ belongs to some $A_{gH}$, $gH \in G/H$.

Example 10. Let $A = \langle a, a^\ast | aa^\ast - a^\ast a = 1 \rangle$ be the Weyl algebra (see Example 1). Then $A$ is a $\mathbb{Z}$-graded $\ast$-algebra with $\mathbb{Z}$-grading defined by $a \in A_1$, $a^\ast \in A_{-1}$ and we have $B = \mathbb{C}[N]$, where $N = a^\ast a$. We now use Corollary 1 to describe the cone $\sum A^2 \cap B$.

Suppose $k \in \mathbb{N}$. Let $a_k \in A_k$. Then $a_k$ is of the form $a_k = a_k^\ast p_k$, where $p_k \in \mathbb{C}[N]$, and

$$a_k^\ast a_k = p_k^\ast a_k^\ast a_k p_k = N(N-1) \cdots (N-k+1)p_k^\ast p_k.$$  

For $a_{-k} \in A_{-k}$ we have $a_{-k} = a_{-k}^\ast p_{-k}$, where $p_{-k} \in \mathbb{C}[N]$, and

$$a_{-k}^\ast a_{-k} = p_{-k}^\ast a_{-k}^\ast a_{-k} p_{-k} = (N+1)(N+2) \cdots (N+k)p_{-k}^\ast p_{-k}.$$  

One easily verifies that $a_{-k}^\ast a_{-k}$ belongs to $\sum B^2 + N \sum B^2$. Hence from Corollary 1 we obtain

$$\sum A^2 \cap B = \sum B^2 + N \sum B^2 + N(N-1) \sum B^2 + \cdots.$$  

This result was derived in [FS] by other methods. Among others it shows that $\sum A^2 \cap B \neq \sum B^2$ and that the canonical conditional expectations $p : A \to B$ is not strong.

Example 11. Let $G$ be a discrete group and $H$ a normal subgroup of $G$. Then the group algebra $\mathbb{C}[G]$ becomes a $G/H$-graded $\ast$-algebra in canonical manner. The canonical conditional expectation coincides with the one from the Example 5, so by Proposition 4 it is strong. In particular, we have $\sum \mathbb{C}[G]^2 \cap \mathbb{C}[H] = \sum \mathbb{C}[H]^2$.  

Example 12. Let $A$ be a unital $*$-algebra. Let $G$ be a (discrete) group which acts as $*$-automorphism group $g \mapsto \alpha_g$ on $A$. Recall that the crossed product $*$-algebra $\mathcal{A} = A \times_\alpha G$ is defined as follows. As a linear space $\mathcal{A}$ is the tensor product $A \otimes \mathbb{C}[G]$ or equivalently the vector space of $A$-valued functions on $G$ with finite support. Product and involution on $\mathcal{A}$ are determined by $(a \otimes g)(b \otimes h) = ab\alpha_a(b) \otimes gh$ and $(a \otimes g)^* = \alpha_{g^{-1}}(a^*) \otimes g^{-1}$, respectively. If we identify $b$ with $b \otimes e$ and $g$ with $1 \otimes g$, then the $*$-algebra $A \times G$ can be considered as the universal $*$-algebra generated by the two $*$-subalgebras $A$ and $\mathbb{C}[G]$ with cross commutation relations $gb = \alpha_g(b)g$ for $b \in A$ and $g \in G$. Set $\mathcal{A}_g := A \otimes g$ for $g \in G$. Then $\mathcal{A}$ becomes a $G$-graded $*$-algebra with canonical conditional expectation $p$ onto $B = A_e$ given by $p(a \otimes g) = \delta_{g,e} a \otimes e$.  

Proposition 7. The canonical conditional expectation $p : A \times G \to B$ is strong.  

Proof. Let $x = \sum_{g \in G} a_g \otimes g$, $a_g \in A$, be an element of the $A \times G$. Then  

$$p(xx^*) = p \left( \sum_{g \in G} \sum_{h \in G} (a_g \otimes g)(a_h \otimes h)^* \right) = p \left( \sum_{g \in G} \sum_{h \in G} a_g \alpha_{g^{-1}}(a_h^*) \otimes gh^{-1} \right) = \sum_{g \in G} a_g a^*_g \otimes e = \sum_{g \in G} (a_g \otimes e)(a_g \otimes e)^* \in \sum B^2. \quad \square$$ 

Example 13. Let $G$ be a compact abelian group. Then the dual group $\hat{G}$ is a discrete abelian group. We now establish a duality between actions of $G$ and $\hat{G}$-gradings on a $*$-algebra $A$ (cf. Example 7).  

Suppose that an action $\alpha : G \to Aut(A)$ is given. Assume, in addition, that the action is locally finite-dimensional (see Example 7). For $\psi \in \hat{G}$, $\psi : G \to \mathbb{T}$ put  

$$\mathcal{A}_\psi = \{a \in A | \alpha_g(a) = \psi(g)a, \text{ for all } g \in G\}. \quad (15)$$  

If $\mathcal{A}$ is a $\hat{G}$-graded $*$-algebra, we define an action of $\hat{G} = G$ on $A$ as follows. For $a = \sum_{\psi \in \hat{G}} a_\psi$, $a_\psi \in \mathcal{A}_\psi$ and $g \in G$, define a $*$-automorphism $\alpha_g$ by putting  

$$\alpha_g(a) := \sum_{\psi \in \hat{G}} \psi(g)a_\psi. \quad (16)$$  

Proposition 8. Equations (13) and (16) give a one-to-one correspondence between locally finite-dimensional actions of $G$ on $A$ and $\hat{G}$-gradings of $A$.  

Proof. Let $\alpha : G \to Aut(A)$ be locally finite-dimensional action and let $\mathcal{A}_\psi$ be defined by (15). We consider $A$ as $G$-module and $\mathcal{A}_\psi$ as unitary $G$-submodule. Take a finite-dimensional $\alpha$-invariant linear subspace $V$ of $A$. Since $G$ is compact, $V$ is unitarizable and hence spanned by its subspaces $\mathcal{A}_\psi$. Since the action of $G$ is locally finite-dimensional, $A$ is spanned by such subspaces $V$ and so by $\mathcal{A}_\psi$, $\psi \in \hat{G}$. It is easily checked that $A = \bigoplus_{\psi \in \hat{G}} \mathcal{A}_\psi$ is a $\hat{G}$-grading of $A$.  

Conversely, suppose $\mathcal{A}$ is a $\hat{G}$-graded $*$-algebra. It is clear that (16) defines an action of $G$ on $A$. Each element $a \in A$ is of the form $a = \sum_{i=1}^k a_{\psi_i}$, where $a_{\psi_i} \in \mathcal{A}_{\psi_i}$ and the elements $\psi_i \in \hat{G}$ are pairwise distinct. The elements $a_{\psi_i}$ span a finite-dimensional subspace of $A$ which is obviously invariant under the action (16). Hence the action (13) is locally finite-dimensional.  

Remark. For the study of modules over a $G$-graded ring $A = \bigoplus_{g \in G} A_g$, it is usually assumed that for all $g,h \in G$ the linear span of $A_g A_h$ is equal to $A_{gh}$, see [NO], [M]. Likewise in [FD] it is supposed that this linear span is dense in $A_{gh}$. We have not made such an assumption, because it is not satisfied in most of our standard examples. For instance, if $A$ is the Weyl algebra (Example 10), then we have $B = \mathbb{C}[N]$, $A_1 = aB$ and $A_{-1} = a^*B = Ba^*$. Therefore, the linear span of $A_{-1} \cdot A_1$ is equal to $N \cdot \mathbb{C}[N]$ which is different from $B$.  

UNBOUNDED INDUCED REPRESENTATIONS OF $\ast$-ALGEBRAS.
5. Systems of imprimitivity

Let $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be a $G$-graded $*$-algebra. We retain the notation of the previous section. Recall that for a subgroup $H \subseteq G$, the left $G$-space of left $H$-cosets is denoted by $G/H$.

**Definition 7.** Let $\pi$ be a $*$-representation of the $*$-algebra $\mathcal{A}$ and let $E$ be a mapping from the set $G/H$ to the set of projections of the underlying Hilbert space $\mathcal{H}_\pi$ such that

(i) $E(t_1)E(t_2) = 0$ for all $t_1, t_2 \in G/H$, $t_1 \neq t_2$, and $\sum_{t \in G/H} E(t) = I$,

(ii) $E(gt)\pi(a_g) \subseteq \pi(a_g)E(t)$ for all $g \in G$, $t \in G/H$, $a_g \in \mathcal{A}_g$.

We call the pair $(\pi, E)$ a system of imprimitivity for the algebra $\mathcal{A}$ over $G/H$.

Let $(\pi, E)$ be a system of imprimitivity. Let $t \in G/H$ and set $D_t(\pi) := \text{Ran} E(t) \cap D(\pi)$. The conditions in Definition 7 imply that

$$E(t)D(\pi) \subseteq D(\pi), \quad \pi(\mathcal{A}_g)D_t(\pi) \subseteq D_{gt}(\pi) \text{ for } g \in G, \quad \text{and } D(\pi) \subseteq \widetilde{\bigoplus}_{t \in G/H} D_t(\pi),$$

where $\widetilde{\oplus}$ denotes the direct Hilbert sum.

A system of imprimitivity $(\pi, E)$ is called non-degenerate if for all $t \in G/H$ the subspace $\pi(\mathcal{A}_t)D_H(\pi)$ is dense in $D_t(\pi)$ with respect to the graph topology of $\pi$. Otherwise, we say that $(\pi, E)$ is degenerate.

**Lemma 10.** Let $H$ be a subgroup of $G$ and let $(\pi, E)$ be a system of imprimitivity for the algebra $\mathcal{A}$ over $G/H$. Then the pair $(\pi_t, E_t)$ is again a system of imprimitivity for $\mathcal{A}$ over $G/H$. Moreover, if $(\pi, E)$ is non-degenerate, then $(\pi_t, E_t)$ is also non-degenerate.

**Proof.** From condition (ii) we obtain $\|\pi(a_g)E(t)\phi\| \leq \|\pi(a_g)\phi\|$ for $a_g \in \mathcal{A}_g$ and $\phi \in D(\pi)$. This shows that $E(t)$ is a continuous mapping of $D(\pi)$ with respect to the graph topology of $\pi$. Hence condition (2) extends by continuity to the closure $\pi_t$ of $\pi$. Obviously, $(\pi_t, E_t)$ is non-degenerate if $(\pi, E)$ is. \qed

Systems of imprimitivity arise from induced representations in the following way (see e.g. [FD], p.1248, for the case of finite groups). Let $\rho$ be a non-zero inducible representation of the algebra $\mathcal{A}_H$ on a dense domain $D(\rho)$ of the Hilbert space $\mathcal{H}_\rho$ and let $\pi = \text{Ind}_{\mathcal{A}_H} \mathcal{A}_\rho$.

Since $\mathcal{A} = \bigoplus_{t \in G/H} \mathcal{A}_t$, we get

$$\mathcal{A} \otimes \mathcal{A}_H D(\rho) = \bigoplus_{t \in G/H} \mathcal{A}_t \otimes \mathcal{A}_H D(\rho).$$

Recall that the representation space $\mathcal{H}_\pi$ of $\pi$ is the completion of the quotient space of the tensor product $\mathcal{A} \otimes \mathcal{A}_H D(\rho)$ by the kernel $\mathcal{K}_\rho$ of the sesquilinear form $\langle \cdot, \cdot \rangle_0$ defined by $\mathcal{H}_\rho$. Let $\mathcal{H}_{t,0}$ denote the subspace of vectors $\xi_t \in \mathcal{A}_t \otimes \mathcal{A}_H D(\rho)$, $t \in G/H$, such that $\langle \xi_t, \xi_t \rangle_0 = 0$. Take $\eta = \sum_{t \in G/H} \eta_t \in \mathcal{H}_0$, where $\eta_t \in \mathcal{A}_t \otimes \mathcal{A}_H D(\rho)$. Since $\langle \eta_t, \eta_s \rangle_0 = 0$ for $t \neq s$ we get

$$0 = \langle \eta, \eta \rangle_0 = \sum_{s, t \in G/H} \langle \eta_s, \eta_t \rangle_0 = \sum_{t \in G/H} \langle \eta_t, \eta_t \rangle_0,$$

that is, every $\eta_t$ belongs to $\mathcal{H}_{t,0}$. This implies that $\mathcal{H}_0 = \bigoplus_{t \in G/H} \mathcal{H}_{t,0}$ and hence

$$\left(\mathcal{A} \otimes \mathcal{A}_H D(\rho)\right)/\mathcal{H}_0 = \bigoplus_{t \in G/H} \left(\mathcal{A}_t \otimes \mathcal{A}_H D(\rho)\right)/\mathcal{H}_{t,0}.$$

Note that for different left cosets $t \in G/H$ the subspaces $\mathcal{A}_t \otimes \mathcal{A}_H D(\rho)/\mathcal{H}_{t,0}$ are pairwise orthogonal. For $t \in G/H$, we denote by $E(t)$ the orthogonal projection from $\mathcal{H}_\pi$ onto the completion of the subspace $\mathcal{A}_t \otimes \mathcal{A}_H D(\rho)/\mathcal{H}_{t,0}$.

**Proposition 9.** The pair $(\pi, E)$ constructed above is a non-degenerate system of imprimitivity for the algebra $\mathcal{A}$ over $G/H$. 
Proof. Because of Lemma 10 it suffices to check the conditions in Definition 7 for the restriction of \( \pi \) to its core \((A \otimes_{A_H} D(\rho))/\mathcal{H}_0\). One easily verifies condition (i). We now show that condition (ii) is satisfied. Since the vectors \([a_t \otimes v], a_t \in A_t, t \in G/H, v \in D(\rho), \) span a core for \( \pi \), it is enough to check (ii) for vectors of this form. Let us fix elements \( g \in G, \ a_g \in A_g, \ s, t \in G/H \) and \( v \in D(\rho) \). Then we have

\[
\pi(a_g)E(s)[a_t \otimes v] = \begin{cases} [a_ga_t \otimes v], & \text{if } s = t; \\ 0, & \text{otherwise.} \end{cases}
\]

Since the same result is obtained for \( E(gs)\pi(a_g)[a_t \otimes v] = E(gs)[a_ga_t \otimes v] \), (ii) holds.

The equality \( \pi(a_t)[1_A \otimes v] = [a_t \otimes v] \) implies that the span of \( \pi(A_t)D_H(\pi) \) is equal to \( D(\pi) \), so \( (\pi, E) \) is non-degenerate.

We call the pair \((\pi, E)\) from Proposition 9 the system of imprimitivity induced by \( \rho \).

**Theorem 1. (First Imprimitivity Theorem)** Let \( A = \bigoplus_{g \in G} A_g \) be a \( G \)-graded \(*\)-algebra and \( H \) a subgroup of \( G \). Suppose that \( \pi \) is a closed \(*\)-representation of \( A \) and \((\pi, E)\) is a non-degenerate system of imprimitivity for \( A \) over \( G/H \). Then there exists a unique, up to unitary equivalence, closed \(*\)-representation \( \rho \) of \( A_H \) such that

(i) \( \rho \) is inducible,

(ii) \((\pi, E)\) is unitarily equivalent to the system of imprimitivity induced by \( \rho \).

Proof. By condition (ii) in Definition 7 the projection \( E(H) \) commutes with the operators \( \pi(a_H), a_H \in A_H \). Hence the restriction of the representation \( \text{Res}_{A_H}\pi \) to the subspace \( \text{Ran}E(H) \) is a well-defined \(*\)-representation of the \(*\)-algebra \( A_H \) denoted by \( \rho \). The domain \( D(\rho) \) is equal to \( \text{Ran}E(H) \cap D(\pi) \) and the representation space \( \mathcal{H}_0 \) is \( \text{Ran}E(H) \).

First we prove that \( \rho \) is inducible. We have to show that the form \( \langle \cdot, \cdot \rangle_0 \) is nonnegative. Take a vector \( \xi = \sum_r a_r \otimes v_r \in A \otimes A_H D(\rho) \), where \( v_r \in D(\rho), a_r \in A \). Each \( a_r \) can be presented as a finite sum \( a_r = \sum_{t \in G/H} a_{r,t} \), where \( a_{r,t} \in A_t, t \in G/H \). Then we have

\[
\langle \xi, \xi \rangle_0 = \sum_r \sum_s \langle a_r \otimes v_r, a_s \otimes v_s \rangle_0 = \sum_{r,s} \langle \rho(p_H(a_s^*a_r))v_r, v_s \rangle = \sum_{r,s} \langle \rho(\sum_{t \in G/H} a_{s,t}^*a_{r,t})v_r, v_s \rangle = \sum_{t \in G/H} \sum_{r,s} \langle \pi(a_{s,t})v_r, \pi(a_{r,t})v_s \rangle = \sum_{t \in G/H} \sum_r \sum_s \pi(a_{s,t})v_r, \pi(a_{r,t})v_s \geq 0.
\]

This shows that \( \rho \) is inducible.

Let \((\pi_1, E_1)\) denote the system of imprimitivity on the space \( \mathcal{H}_{\pi_1} \), induced by \( \rho \). We have to prove that \((\pi_1, E_1)\) is unitarily equivalent to \((\pi, E)\). Define a linear mapping \( F_0 : A \otimes D(\rho) \to D(\pi) \) by putting \( F_0(a \otimes v) = \pi(a)v, \) where \( v \in D(\rho) \subseteq D(\pi) \), \( a \in A \). It is clear that \( F_0 \) maps \( A \otimes A_H D(\rho) \) into \( D(\pi) \). Recall that \( \mathcal{K}_p \) denotes the kernel of the sesquilinear form \( \langle \cdot, \cdot \rangle_0 \). Reasoning in the same manner as in (17) it follows that for any \( \xi \in A \otimes A_H D(\rho) \) we have \( \langle \xi, \xi \rangle_0 = \langle F_0(\xi), F_0(\xi) \rangle \). Therefore, the quotient mapping from \( A \otimes A_H D(\rho)/\mathcal{K}_p \) to \( \mathcal{H}_\pi \) is a well-defined isometric linear mapping. We extend this mapping by continuity to an isometry \( F : \mathcal{H}_{\pi_1} \to \mathcal{H}_\pi \).

We claim that \( F \) intertwines the systems \((\pi, E)\) and \((\pi_1, E_1)\). Take \( k \in G, \ a_k \in A_k, \ a_t \in A_t, \ t \in G/H, \ v \in D(\rho) \). Then we obtain

\[
F(\pi_1(a_k)([a_t \otimes v])) = F(a_ka_t \otimes v) = \pi(a_k)a_tv = \pi(a_k)\pi(a_t)v = \pi(a_k)v = F([a_t \otimes v])
\]

which means that \( F \) intertwines \( \pi \) and \( \pi_1 \).

For \( v \in D(\rho), a_t \in A_t, \ t \in G/H \) condition (ii) in Definition 7 implies that \( \pi(a_t)v \in D(\pi) \). The subspace \( D(\pi_1) \), is spanned by the vectors \([a_t \otimes v], a_t \in A_t, v \in D(\rho), \) and we have \( F([a_t \otimes v]) = \pi(a_t)v \in D(\pi) \). Thus, \( F(D(\pi_1)) \subseteq D(\pi) \) and \( F \) intertwines \( E \) and \( E_1 \).
Since \((\pi, E)\) is non-degenerate, the vectors \(F([a_t \otimes v]) = \pi(a_t)v, a_t \in A_t, v \in \mathcal{D}(\rho)\), span a dense linear subspace \(\mathcal{D}_t(\pi_1)\) of \(\mathcal{D}_t(\pi)\) in the graph topology of \(\pi\). In particular, we have \(F(\text{Ran} E_2(t)) = \text{Ran} E(t)\), so that \(F\) is a unitary operator. Since the graph topology on \(F(\mathcal{D}_t(\pi_1))\) is the same as that of \(\pi\) and \(\pi_1\) is closed by definition, we have \(F(\mathcal{D}_t(\pi_1)) = \mathcal{D}_t(\pi)\) for each \(t \in G/H\), which implies that \(F(\mathcal{D}(\pi_1)) = \mathcal{D}(\pi)\). That is, \(\pi\) and \(\pi_1\) are unitarily equivalent.

Let \(\rho_1\) be an inducible closed \(*\)-representation of \(A_t\) on the Hilbert space \(\mathcal{H}_{\rho_1}\) and let \((\pi_2, E_2)\) be the system of imprimitivity for \(A\) over \(G/H\) induced by \(\rho_1\). It follows from the previous considerations that \(\rho_2 := \text{Res}_{A_t} \pi_2 \uparrow \text{Ran} E_2(H)\) is well-defined \(*\)-representation of \(A_H\). One immediately verifies that the canonical isomorphism \(v \leftrightarrow [1_A \otimes v]\) of \(\mathcal{H}_{\rho_1}\) and \(\text{Ran} E_2(H)\) defines a unitary equivalence of \(\rho_1\) and \(\rho_2\).

Summarizing, we have shown that there is a one-to-one correspondence between unitary equivalence classes of inducible representations of \(A_H\) and unitary equivalence classes of non-degenerate closed systems of imprimitivity for \(A\) over \(G/H\). In particular, the inducing representation \(\rho\) is determined uniquely up to unitary equivalence by the system of imprimitivity.

The following example shows that the non-degeneracy assumption of the system of imprimitivity is crucial in Theorem \(\Box\)

**Example 14.** Let \(A_q\) be the \(*\)-algebra \(\mathbb{C}\langle a, a^\ast | aa^\ast - qa^\ast a = 1 \rangle\), where \(q > -1\). Put \(\lambda_0 = 0\) and \(\lambda_k = \sqrt{1 + q + q^2 + \cdots + q^{k-1}}, k \in \mathbb{N}\). Let \(\mathcal{H}\) be a Hilbert space with orthonormal base \(\{e_k, k \in \mathbb{N}_0\}\). There is a \(*\)-representations \(\pi\) of \(A_q\) on \(\mathcal{D}(\pi) = \text{Lin} \{e_k, k \in \mathbb{N}_0\}\) such that
\[
\pi(a)e_k = \lambda_k e_{k-1}, \pi(a^\ast )e_k = \lambda_{k+1} e_{k+1}, \quad \text{for } k \in \mathbb{N}_0,
\]
where \(e_{-1} := 0\). The representation \(\pi\) is bounded if and only if \(-1 < q \leq 0\). Note that in the case \(q = 1\) the algebra \(A_q\) is just the Weyl algebra and \(\pi\) is the Fock-Bargmann representation.

Let \(E(n), n \in \mathbb{N}\), be the orthogonal projection onto \(\mathbb{C}e_{n-1}\) and put \(E(n) := 0\) for \(n \leq 0\). Then the pair \((\pi, E)\) is a system of imprimitivity for \(A\) over \(G = \mathbb{Z}\). Since \(E(0) = 0\), it follows immediately from the construction of the induced system of imprimitivity that \((\pi, E)\) is not induced by a \(*\)-representation of \(B\).

We now define another construction of systems of imprimitivity. It will also include the system of imprimitivity in the latter example. Fix a system of imprimitivity \((\pi, E)\) for \(A\) over \(G/H\) and an element \(f \in G\). Define a mapping \(E^f\) from the set \(G/\text{fHf}^{-1}\) into the set of projections on the space \(\mathcal{H}_\pi\) by \(E^f(k(\text{fHf}^{-1})) := E(k\text{fH})\), \(k \in G\).

**Proposition 10.** The pair \((\pi, E^f)\) constructed above is a well-defined system of imprimitivity for \(A\) over \(G/\text{fHf}^{-1}\).

**Proof.** Take \(k_1(\text{fHf}^{-1}), k_2(\text{fHf}^{-1}) \in G/\text{fHf}^{-1}\), where \(k_1, k_2 \in G\). The cosets \(k_1(\text{fHf}^{-1})\) and \(k_2(\text{fHf}^{-1})\) are equal if and only if \(k_2^{-1}k_1 \in \text{fHf}^{-1}\) which is equivalent to \(k_1 \text{fH} = k_2 \text{fH}\). This implies that \(E^f\) is well-defined. It is straightforward to verify that \((\pi, E^f)\) satisfies the two conditions in Definition \(\Box\).

**Definition 8.** If \((\pi, E), f \in G\), \((\pi, E^f)\) are as above, we say that the system \((\pi, E^f)\) is conjugated to the system \((\pi, E)\) by the element \(f \in G\).

Our second Imprimitivity Theorem describes systems of imprimitivity which are not necessarily non-degenerate. We prove it now for bounded representations (cf. also the Imprimitivity Theorem in [FD], p.1192). In Section 8 we formulate its analogue for well-behaved systems of imprimitivity (Theorem 11).

The following definition and the subsequent lemma are used in the proof of Theorem 11 below.

**Definition 9.** Let \((\pi, E)\) be a system of imprimitivity for \(A\) over \(G/H\) and let \(fH \in G/H\). We say that \((\pi, E)\) is generated by the projection \(E(fH)\) if for every \(gH \in G/H\) the linear subspace \(\pi(\mathcal{A}_{gHf^{-1}})(\mathcal{D}_{fH}(\pi))\) is dense in \(\mathcal{D}_{gH}(\pi)\) with respect to the graph topology of \(\pi\).

**Lemma 11.** A system of imprimitivity \((\pi, E)\) is generated by the projection \(E(fH), fH \in G/H, f \in G\), if and only if the conjugated system of imprimitivity \((\pi, E^f)\) over \(G/\text{fHf}^{-1}\) is non-degenerate.
The simple proof of Lemma 11 will be omitted. The next theorem says that for bounded representations each system of imprimitivity over \( G/H \) can be obtained as a direct sum of conjugated systems by elements of \( G \).

**Theorem 2.** (Second Imprimitivity Theorem) Let \( \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \) a \( G \)-graded \( \ast \)-algebra, \( H \) a subgroup of \( G \) and \((\pi, E)\) a system of imprimitivity for \( \mathcal{A} \) over \( G/H \). Suppose the \( \ast \)-representation \( \pi \) acts by bounded operators on \( D(\pi) = \mathcal{H}_\pi \). We fix one element \( k_i \in G \), \( t \in G/H \), in each left coset from \( G/H \). Then for every \( t \in G/H \) there exists a bounded \( \ast \)-representation \( \rho_t \) of \( \mathcal{A}_{k_i H k_i^{-1}} \) on a Hilbert space \( \mathcal{H}_t \) such that:

(i) \( \rho_t \) is inducible,

(ii) \((\pi, E)\) is the direct sum of systems of imprimitivity \((\pi_t, E_t)\), \( t \in G/H \), where \((\pi_t, E_t)\) is conjugated by the element \( k_i \) to the system of imprimitivity induced by \( \rho_t \), \( t \in G/H \).

**Proof.** Let \((\pi_1, E_1)\) be a subsystem of imprimitivity of \((\pi, E)\) over \( G/H \), that is, \( \pi_1 \subseteq \pi \) is a subrepresentation of \( \pi \) on a Hilbert subspace \( \mathcal{H}_1 \subseteq \mathcal{H}_\pi \) and for all \( gH \in G/H \) we have \( \text{Ran} E_1(gH) \subseteq \text{Ran} E(gH) \). Since \( \pi_1 \) is a bounded \( \ast \)-representation, there is a \( \ast \)-representation \( \pi_2 \) on \( \mathcal{H}_2 := \mathcal{H}_\pi \ominus \mathcal{H}_1 \) such that \( \pi = \pi_1 \oplus \pi_2 \). Put \( E_2(gH) := E(gH) \ominus \text{Ran} E_1(gH) \) for \( gH \in G/H \). Then \((\pi_2, E_2)\) is again a system of imprimitivity for \( \mathcal{A} \) over \( G/H \). Indeed, condition (i) in Definition 7 is obvious and condition (ii) follows immediately by subtracting the equation \( \pi_1(a_g)E_1(fH) = E_1(gfH)\pi_1(a_g) \) from \( \pi(a_g)E(fH) = E(gfH)\pi(a_g) \), where \( g \in G, a_g \in \mathcal{A}_g \), \( fH \in G/H \). That is, we have shown that every subsystem of imprimitivity has a complement.

Now we fix \( fH \in G/H \). Let \( E_1(gH) \) denote the orthogonal projection onto the closure of \( \text{Ran} \pi(\mathcal{A}_{gHf^{-1}})E(fH) \) and set \( \mathcal{H}_1 := \bigoplus_{t \in G/H} \text{Ran} E_1(t) \). It is easily checked that the family of projections \( E_1(t), \ t \in G/H, \) satisfies condition (i) of Definition 7. Let \( g \in G, a_g \in \mathcal{A}_g \) and \( kH \in G/H \). Then we have

\[
\pi(a_g)\text{Ran} E_1(kH) = \pi(a_g)\text{Ran} \pi(\mathcal{A}_{kHf^{-1}})E(fH) \subseteq \text{Ran} \pi(\mathcal{A}_{gHf^{-1}})E(fH) = E_1(gkH),
\]

which shows that the subspace \( \mathcal{H}_1 \) is invariant under all operators \( \pi(a_g), \ a \in \mathcal{A} \). If we denote by \( \pi_1 \) the restriction of \( \pi \) to \( \mathcal{H}_1 \), then condition (ii) in Definition 7 holds for the pair \((\pi_1, E_1)\). Therefore, \((\pi_1, E_1)\) is an subsystem of imprimitivity for \( \mathcal{A} \) over \( G/H \). The system \((\pi_1, E_1)\) is generated by \( E_1(fH) = E(fH) \).

Combining the considerations of the preceding paragraphs with Zorn’s lemma we conclude that there exist system of imprimitivity \((\pi_t, E_t)\), \( t \in G/H \), for \( \mathcal{A} \) over \( G/H \) such that every \((\pi_t, E_t)\) is generated by the projection \( E_t(k_i H) \), \( t \in G/H \), and \((\pi, E)\) is equal to the orthogonal direct sum of \((\pi_t, E_t)\), \( t \in G/H \).

Lemma 11 together with Theorem 2 imply that each conjugated system \((\pi_t, E_t^k)\), \( t \in G/H \), is induced by some representation \( \rho_t \) of the \( \ast \)-algebra \( \mathcal{A}_{k_i H k_i^{-1}} \). By the construction of \( \rho_t \) (see the proof of the Theorem 2), \( \rho_t \) it is a bounded \( \ast \)-representation.

**Remark.** We do not know a generalization of Theorem 2 for general unbounded representations. The main difficulty lies in the fact that for a closed subrepresentation \( \pi_1 \) of closed \( \ast \)-representation \( \pi \) in general there is no representation \( \pi_2 \) such that \( \pi = \pi_1 \oplus \pi_2 \).

6. A PARTIAL GROUP ACTION DEFINED BY THE GRADING

Throughout this section we assume that \( \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \) is a \( G \)-graded unital \( \ast \)-algebra and that the \( \ast \)-subalgebra \( \mathcal{B} := \mathcal{A}_e \) is commutative. The canonical conditional expectation of \( \mathcal{A} \) onto \( \mathcal{B} \) is denoted by \( \rho \).

Let \( \widehat{\mathcal{B}} \) be the set of all characters of \( \mathcal{B} \), that is, \( \widehat{\mathcal{B}} \) is the set of nontrivial \( \ast \)-homomorphisms \( \chi : \mathcal{B} \to \mathbb{C} \). The set of characters from \( \widehat{\mathcal{B}} \) which are nonnegative on the cone \( \sum \mathcal{A}_g^2 \cap \mathcal{B} \) is denoted by \( \widehat{\mathcal{B}}^+ \).

In addition we assume in this section that all characters \( \chi \in \widehat{\mathcal{B}}^+ \) satisfy the following condition:

\[
\chi(c^*d)\chi(d^*c) = \chi(c^*c)\chi(d^*d) \text{ for all } \chi \in \widehat{\mathcal{B}}^+, \ g \in G, \text{ and } c, d \in \mathcal{A}_g.
\]

\[\text{(18)} \]
Note that for \( c, d \in \mathcal{A}_g \) we have \( c^*d, d^*c, \, d^*d \in \mathcal{A}_{g^{-1}} \cdot \mathcal{A}_g \subseteq \mathcal{A}_g = \mathcal{B} \). Hence all expressions in the equation (18) are well-defined.

**Proposition 11.** Let \( \mathcal{A} \) denote the crossed product algebra \( A \times_\alpha G \) from Example 12. Assume that \( \mathcal{A} \) is commutative, so that \( \mathcal{B} = A \otimes e \) is commutative. Then condition (18) is satisfied.

Proposition 11 follows at once from the more general

**Proposition 12.** Assume that for every \( g \in G \) there exists an element \( a_g \in \mathcal{A}_g \) such that \( \mathcal{A}_g = a_g \mathcal{B} \) or \( \mathcal{A}_g = \mathcal{B} a_g \). Then condition (18) is satisfied.

**Proof.** Fix a \( g \in G \). Assume that there exist an element \( a_g \in \mathcal{A}_g \) such that \( \mathcal{A}_g = a_g \mathcal{B} \). Take \( \chi \in \hat{\mathcal{B}}^+ \) and \( c,d \in \mathcal{A}_g \).

Then there exist \( c_1, d_1 \in \mathcal{B} \) such that \( c = a_g c_1 \) and \( d = a_g d_1 \). We now compute

\[
\chi(c^*d)\chi(d^*c) = \chi(c_1^*)\chi((a_g^*)a_g)\chi(d_1^*)\chi(d_1)\chi((a_g^*)a_g)\chi(c_1) = \chi(c^*)\chi(d^*d).
\]

In the same way one proves (18) in the case when \( \mathcal{A}_g = \mathcal{B} a_g \), \( a_g \in \mathcal{A}_g \).

The main content of this section is the following partial action of \( G \) on the set \( \hat{\mathcal{B}}^+ \).

**Definition 10.** Let \( \chi \in \hat{\mathcal{B}}^+ \) and \( g \in G \). We say that \( \chi^g \) is defined if there exists an element \( a_g \in \mathcal{A}_g \) such that \( \chi(a_g^* a_g) \neq 0 \). In this case we set

\[
\chi^g(b) := \frac{\chi(a_g^* b a_g)}{\chi(a_g^* a_g)} \text{ for } b \in \mathcal{B}.
\]

For \( g \in G \) we denote by \( \mathcal{D}_g \) the set of all characters \( \chi \in \hat{\mathcal{B}}^+ \) such that \( \chi^g \) is defined.

**Remarks.** 1. One could also define \( \chi^g \) as it was done in [FD]. As noted in [FD], the space \( \mathcal{A}_g, \, g \in G \), has a natural structure of a \( \mathcal{B} \)-rigged \( \mathcal{B} \)-\( \mathcal{B} \)-bimodule, where \( \mathcal{B} \) acts by the multiplication and the \( \mathcal{B} \)-valued product is

\[
[c, d] := d^*c, \, c, d \in \mathcal{A}_g.
\]

Then \( \chi^g \) is defined as the representation of \( \mathcal{B} \) induced from \( \chi \) via \( \mathcal{A}_g \). Condition (18) ensures that \( \chi^g \) is again a character.

2. Crossed-products defined by partial group actions on \( C^* \)-algebras appeared in [Ex]. Our \( G \)-graded \( * \)-algebra \( \mathcal{A} \) can be considered as another generalization of crossed-product algebras. We shall not elaborate the details here.

**Proposition 13.** The map \( \chi \mapsto \chi^g \) is a well-defined partial action of \( G \) on the set \( \hat{\mathcal{B}}^+ \), that is:

(i) \( \chi^g(b) \) in (19) does not depend on the choice of \( a_g \) and we have \( \chi^g \in \hat{\mathcal{B}}^+ \),

(ii) if \( \chi^g \) and \( (\chi^g)^h \) are defined, then \( (\chi^g)^h \) is defined and equal to \( (\chi^g)^h \),

(iii) if \( \chi^g \) is defined, then \( (\chi^g)^{g^{-1}} \) is defined and equal to \( \chi \),

(iv) \( \chi^e \) is defined and equal to \( \chi \).

**Proof.** (i): Let \( \chi \in \hat{\mathcal{B}}^+ \), \( g \in G \), and \( c,d \in \mathcal{A}_g \) such that \( \chi(d^*d) \neq 0 \) and \( \chi(c^*c) \neq 0 \). Since \( \mathcal{B} \) is commutative, we have \( bcd^* = cd^*b \) for \( b \in \mathcal{B} \). Therefore we obtain

\[
\chi(c^*bc)\chi(d^*d) = \chi(c^*bcd^*d) = \chi(c^*cd^*bd) = \chi(c^*c)\chi(d^*bd),
\]

so that

\[
\frac{\chi(c^*bc)}{\chi(c^*c)} = \frac{\chi(d^*bd)}{\chi(d^*d)}.
\]

We show that \( \chi^g \) is again a character belonging to \( \hat{\mathcal{B}}^+ \). Let \( b_1, b_2 \in \mathcal{B} \). Since \( \mathcal{B} \) is commutative, we have \( a_g a_g^* b_1 = b_1 a_g a_g^* \). Hence we get

\[
\chi^g(b_1 b_2) = \frac{\chi(a_g^* b_1 a_g^* a_g b_2)}{\chi(a_g^* a_g)} = \frac{\chi(a_g^* a_g b_1 b_2 a_g)}{\chi(a_g^* a_g)} = \frac{\chi(a_g^* b_1 a_g^* a_g b_2 a_g)}{\chi(a_g^* a_g)} = \chi^g(b_1) \chi^g(b_2).
\]

Next we prove the positivity of \( \chi^g \). For take \( b \in \sum \mathcal{A}_2 \). Since \( \chi(\sum \mathcal{A}_2) \geq 0 \) and \( a_g^* b a_g \in \sum \mathcal{A}_2 \) we have \( \chi^g(b) > 0 \).
(ii): Let $\chi \in \hat{B}^+$ and $g, h \in G$ such that $(\chi^g)^h$ is defined. Then there exists $a_g \in \mathcal{A}_g$ such that $\chi(a_g^*a_g) \neq 0$. Since $(\chi^g)^h$ is defined, there exists $a_h \in \mathcal{A}_h$ such that

$$
\chi^g(a_h^*a_h) = \frac{\chi(a_h^*a_h a_g)}{\chi(a_g^*)} \neq 0,
$$

that is, $\chi((a_h a_g)^*a_h a_g) \neq 0$. Since $a_h a_g \in \mathcal{A}_{hg}$, $\chi^{hg}$ is well-defined. It is straightforward to check that $(\chi^g)^h = \chi^{hg}$.

(iii): Assume that $\chi^g$ is defined. Then there exists $a_g \in \mathcal{A}_g$ such that $\chi(a_g^*a_g) \neq 0$. We have $a_g^* \in \mathcal{A}_{g^{-1}}$ and

$$
\chi^g(a_g a_g^*) = \frac{\chi(a_g^*a_g a_g^*)}{\chi(a_g^*)} = \chi(a_g^*a_g) \neq 0.
$$

Hence $(\chi^g)^{g^{-1}}$ is defined. One easily verifies that $(\chi^g)^{g^{-1}} = \chi$.

(iv) is trivial. \qed

**Remark.** It follows from Proposition 13 that for each $g \in G$ the mapping $\chi \mapsto \chi^g$ defines a bijection $\alpha_g : \mathcal{D}_g \to \mathcal{D}_{g^{-1}}$ such that:

(i) $\mathcal{D}_e = \hat{B}^+$ and $\alpha_e$ is the identity mapping of $\hat{B}^+$,

(ii) $\alpha_g(D_g \cap D_h) = D_{g^{-1}} \cap D_{hg}$,

(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$, for $x \in D_g \cap D_{gh}$.

In what follows, we shall use both notations $\alpha_g(\chi)$ and $\chi^g$ for the partial action of $g \in G$ on $\chi \in \hat{B}^+$ and we freely use the properties (i) – (iii).

It should be emphasized that up to now condition (18) has not been used for the partial action. For the next proposition assumption (18) is needed.

**Proposition 14.** Let $a_g, c_g \in \mathcal{A}_g, g \in G$, and $\chi \in \hat{B}^+$ be such that $\chi(a_g^*c_g) \neq 0$. Then we have $\chi \in \mathcal{D}_g$ and

$$
\chi^g(b) = \frac{\chi(a_g^*bc_g)}{\chi(a_g^*c_g)} \text{ for all } b \in \mathcal{B}.
$$

**Proof.** Since $\chi(a_g^*c_g) \neq 0$, we have $\chi(c_g a_g) = \chi(a_g^*c_g) \neq 0$, so that (18) implies $\chi(a_g^*a_g) \neq 0$, i.e. $\chi \in \mathcal{D}_g$. Now (20) follows from the equality

$$
\chi(a_g^*ba_g)\chi(a_g^*c_g) = \chi(a_g^*a_g a_g^*bc_g) = \chi(a_g^*a_g)\chi(a_g^*bc_g).
$$

\qed

Examples developed below show that in general $\chi^g$ is not always defined, so that in general $\chi \mapsto \chi^g$ is not a group action.

We introduce some more notation which will be kept till the end of the paper. For a fixed $\chi \in \hat{B}^+$ let

$$
G_\chi = \{g \in G | \chi^g \text{ is defined} \}.
$$

We denote by $\text{Orb}_\chi \subseteq \hat{B}^+$ the orbit of the $\chi$, that is,

$$
\text{Orb}_\chi = \{\chi^g | \chi^g \text{ is defined} \}.
$$

Further, let $\text{St}_\chi \subseteq G_\chi$ denote the stabilizer of the element $\chi$, that is,

$$
\text{St}_\chi = \{g \in G | \chi^g \text{ is defined and equal to } \chi \}.
$$

A number of elementary properties of the partial action of $G$ are collected in the following

**Proposition 15.** Let $\chi \in \hat{B}^+$. Then we have:

(i) $\text{St}_\chi$ is a subgroup of $G$,

(ii) The union of sets $G_\psi, \psi \in \text{Orb}_\chi$ equipped with the multiplication derived from $G$ is a groupoid with identity element,

(iii) if $\psi \in \hat{B}^+$, then $\psi \in \text{Orb}_\chi$ if and only if $\text{Orb}_\psi = \text{Orb}_\chi$,

(iv) if $\psi \in \text{Orb}_\chi$, then $\text{St}_\chi$ and $\text{St}_\psi$ are conjugate subgroups of $G$. 

Now we illustrate these concepts by a few examples.

**Example 15.** Let $A$ be a commutative $*$-algebra and $\mathcal{A} = A \times_\alpha G$ be the crossed-product algebra from Example 12. It was shown therein that $\sum \mathcal{A}^2 \cap \mathcal{B} = \sum \mathcal{B}^2$. This implies that $\mathcal{B}^+ = \widehat{\mathcal{B}} = \widehat{A}$ and the partial action defined by (19) coincides with the usual group action of $G$ on $\widehat{A}$ induced by the action of $G$ on $A$.

**Example 16.** Let $\mathcal{A}$ be the Weyl algebra. We retain the notation from Examples 11 and 10. It follows from (14) that a character $\chi \in \widehat{\mathcal{B}}$ is non-negative on the cone $\sum \mathcal{A}^2 \cap \mathcal{B}$ if and only if $\chi(N) \in \mathbb{N}_0$. For $k \in \mathbb{N}_0$, let $\chi_k$ denote the character of $\mathcal{B}^+$ defined by $\chi_k(N) = k$.

Suppose that $n \in \mathbb{N}_0$. Clearly, any element of the $\mathcal{A}_n$ has the form $a^n p(N)$, where $p \in \mathbb{C}[N]$, and $\chi_k((a^n p(N))^* a^n p(N)) \neq 0$ implies $\chi_k(a^n a^n) \neq 0$. So we obtain that

$$(\alpha_n(\chi_k))(N) = \frac{\chi_k(a^n N a^n)}{\chi_k(a^n a^n)} = \frac{\chi_k(N(N-1) \ldots (N-n+1)(N-n))}{\chi_k(N(N-1) \ldots (N-n+1))}$$

is defined if and only if $k \geq n$ and $(\alpha_n(\chi_k))(N) = \chi_{k-n}(N)$.

Analogously we conclude that

$$(\alpha_{-n}(\chi_k))(N) = \frac{\chi_k(a^n N a^n)}{\chi_k(a^n a^n)} = \frac{\chi_k((N+1)(N+2) \ldots (N+n))}{\chi_k((N+1)(N+2) \ldots (N+n))}$$

is defined for all $n \in \mathbb{N}$ and $(\alpha_{-n}(\chi_k))(N) = \chi_{k+n}(N)$, i.e. $\alpha_{-n}(\chi_k) = \chi_{k+n}$.

The partial action is transitive, so $\mathcal{B}^+$ consists of a single orbit. The stabilizer $\text{St}_{\chi_k}$ of each character $\chi_k$ is trivial, the set $G_{\chi_k}$ is equal to $\{ n \in \mathbb{Z} | n \leq k \}$.

The next proposition gives explicit formulas for representations induced from characters. Recall that a character $\chi \in \widehat{\mathcal{B}}^+$ is a one-dimensional $*$-representation of $\mathcal{B}$ on the space $\mathbb{C}$ and the representation space $\mathcal{H}_\pi$ of $\pi = \text{Ind}_\chi$ is spanned by the vectors $[a \otimes 1]$, $a \in \mathcal{A}$ (see Section 2).

**Proposition 16.** Let $\chi \in \widehat{\mathcal{B}}^+$ and $\pi = \text{Ind}_\chi$. Fix elements $a_g \in \mathcal{A}_g$, $g \in G_\chi$, such that $\chi(a_g^* a_g) \neq 0$, $g \in G_\chi$. Then we have:

(i) The vectors

$$e_g = \frac{[a_g \otimes 1]}{\sqrt{\chi(a_g^* a_g)}} , \text{ } g \in G_\chi,$$

form an orthonormal base of the representation space $\mathcal{H}_\pi$ of $\text{Ind}_\chi$.

(ii) For $b_h \in \mathcal{A}_h$ and $h \in G$ we have

$$\pi(b_h) e_g = \frac{\chi(a_{hg}^* b_h a_g)}{\sqrt{\chi(a_{hg}^* a_{hg})}} e_h, \text{ if } h g \in G_\chi$$

and $\pi(b_h) e_g = 0$ otherwise. In particular, if $b \in \mathcal{B}$, then we have

$$\pi(b) e_g = \frac{\chi(a_g^* b a_g)}{\sqrt{\chi(a_g^* a_g)}} e_g = \chi^g(b) e_g.$$

**Proof.** First suppose that $b_g \in \mathcal{A}_g$ and $g \notin G_\chi$. Then $\|[b_g \otimes 1]\|^2 = \chi(b_g^* b_g) = 0$, so $\mathcal{H}_\pi$ is spanned by the vectors $[b_g \otimes 1]$, where $b_g \in \mathcal{A}_g$ and $g \in G_\chi$.

For $b_g \in \mathcal{A}_g$ and $g \in G$ the equality (18) applied to $a_g$ and $b_g$ is equivalent to the equation

$$||[a_g \otimes 1], [b_g \otimes 1]||^2 = ||a_g \otimes 1||^2 ||b_g \otimes 1||^2,$$

that is, we have equality in the Cauchy-Schwartz inequality. This implies that $[a_g \otimes 1] = \lambda [b_g \otimes 1]$ for some complex number $\lambda$. Hence it follows that the elements $[a_g \otimes 1]$, $g \in G_\chi$, span the space $\mathcal{H}_\pi$. Since $\langle [a_g \otimes 1], [a_h \otimes 1] \rangle = \chi(p(a_g^* a_g)) = \chi(0) = 0$ for $g \neq h$, the elements $[a_g \otimes 1]$ are pairwise orthogonal. The square of the norm of $[a_g \otimes 1]$ is equal to $\langle [a_g \otimes 1], [a_g \otimes 1] \rangle = \chi(a_g^* a_g)$. Thus we have shown that the elements $e_g$, $g \in G_\chi$, form an orthonormal base of $\mathcal{H}_\pi$. 


Now let \( b_h \in \mathcal{A}_h,\ h \in H \). If \( hg \in G_\chi \) we have
\[
\pi(b_h)e_g = \frac{[b_ha_g \otimes 1]}{\sqrt{\chi(a_g^*a_g)}} = \frac{[a_hg \otimes 1]}{\sqrt{\chi(a_g^*a_g)}} = \frac{\lambda}{\sqrt{\chi(a_g^*a_g)}} e_{hg},
\]
where \( \lambda \) is equal to
\[
\frac{\langle [b_ha_g \otimes 1], [a_hg \otimes 1] \rangle}{\langle [a_hg \otimes 1], [a_hg \otimes 1] \rangle} = \frac{\chi(a_h^*b_g)}{\chi(a_g^*a_hg)}.
\]
This yields the second statement of the theorem. \( \square \)

In Section 8 we will derive a simple criterion of the irreducibility of the induced representation by showing that \( \text{Ind}_\chi, \chi \in \hat{B}^+ \), is irreducible if and only if the stabilizer group \( \text{St}_\chi \) is trivial.

7. WELL-BEHAVED REPRESENTATIONS

There is an essential difference between unbounded and bounded representation theory of \(*\)-algebras in Hilbert space. The problem of classifying all or even all self-adjoint unbounded \(*\)-representations is not well-posed for arbitrary \(*\)-algebras. Let us explain this for the \(*\)-algebra \( \mathbb{C}[x_1, x_2] \) of polynomials in two variables. In [S3] it was proved that for any properly infinite von Neumann algebra \( \mathcal{N} \) on a separable Hilbert space there exists a self-adjoint \(*\)-representation \( \pi \) of \( \mathbb{C}[x_1, x_2] \) such that the operators \( \pi(x_1) \) and \( \pi(x_2) \) are self-adjoint and their spectral projections generate \( \mathcal{N} \). This result has been used in [ST] to show the representation theory of \( \mathbb{C}[x_1, x_2] \) is wild. Such a pathological behavior can be overcome if we restrict to integrable representations. For the \(*\)-algebra \( \mathbb{C}[x_1, x_2] \) a self-adjoint representation \( \pi \) is integrable if and only the operators \( \pi(x_1) \) and \( \pi(x_2) \) are self-adjoint and their spectral projections commute. However, for arbitrary \(*\)-algebras no method is known to single out such a class of well-behaved representations. To define and classify well-behaved representations of general \(*\)-algebras is a fundamental problem in unbounded representation theory. One possible proposal was given in [S2]. In this section we develop a concept of well-behaved representations for \( G \)-graded \(*\)-algebras \( \mathcal{A} \) with commutative \(*\)-subalgebras \( \mathcal{A}_e \). We begin with some simple technical facts.

**Lemma 12.** Let \( \pi \) be a \(*\)-representation of a \( G \)-graded \(*\)-algebra \( \mathcal{A} \) and \( \mathcal{B} = \mathcal{A}_e \). Then the graph topologies of \( \pi \) and of \( \text{Res}_B \pi \) coincide. In particular, \( \pi \) is closed if and only if \( \text{Res}_B \pi \) is closed.

**Proof.** Since \( \mathcal{B} \) is a \(*\)-subalgebra of \( \mathcal{A} \), the graph topology of \( \text{Res}_B \pi \) is obviously weaker than that of \( \pi \). For \( a_g \in \mathcal{A}_g \) and \( \varphi \in \mathcal{D}(\pi) \), we have
\[
\|\pi(a_g)\varphi\| = (\pi(a_g^*a_g)\varphi, \varphi)^{1/2} \leq \|\pi(a_g^*a_g)\varphi\| + \|\varphi\|.
\]
Since \( a_g^*a_g \in \mathcal{B} \), the graph topology of \( \pi \) is weaker than the graph topology of \( \text{Res}_B \pi \). Hence both topologies coincide. Since closedness of a \(*\)-representation is equivalent to the completeness in the graph topology (see [S1], 8.1), it follows that \( \pi \) is closed if and only if \( \text{Res}_B \pi \) is closed. \( \square \)

Throughout the rest of this section we assume that \( \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \) is a \( G \)-graded \(*\)-algebra such that \( \mathcal{A}_e = \mathcal{B} \) is commutative and condition (13) is satisfied.

We begin with some preliminaries. An element \( b \in \mathcal{B} \) can be viewed as a function \( f_b \) on the set \( \hat{B}^+ \), that is, \( f_b(\chi) = \chi(b) \) for \( b \in \mathcal{B} \) and \( \chi \in \hat{B}^+ \). Let \( \tau \) denote the weakest topology on the set \( \hat{B}^+ \) for which all functions \( f_b, b \in \mathcal{B} \), are continuous. This topology is generated by the sets \( f_b^{-1}((c, d]), -\infty \leq c \leq d \leq \infty \). Clearly, the topology \( \tau \) on \( \hat{B}^+ \) is Hausdorff. We assume in addition that the topology \( \tau \) on \( \hat{B}^+ \) is locally compact.

The topology \( \tau \) on \( \hat{B}^+ \) defines a Borel structure which is generated by all open sets. Since the domain \( \mathcal{D}_g \) of the mapping \( \alpha_g \) is the union of open sets \( f_{a_g^*a_g}^{-1}((0, +\infty)), a_g \in \mathcal{A}_g \), the set \( \mathcal{D}_g \) is open and hence Borel.

**Lemma 13.** Let \( \tau_g, g \in G \), be the weakest topology on \( \mathcal{D}_g \) for which all functions \( f_{a^*_ga_g}, a_g \in \mathcal{A}_g \), are continuous. Then \( \tau_g \) is induced from the topology \( \tau \) on \( \hat{B}^+ \).
**Proof.** Let $\chi \in \mathcal{D}_g$. Since the topology $\tau$ on $\hat{B}^+$ is locally compact, there is a compact neighborhood $\Omega$ of $\chi$. Since $\mathcal{D}_g$ is open, $\Omega_1 = \Omega \cap \mathcal{D}_g$ is again a neighborhood of $\chi$. The elements of $\mathcal{B}$ separate the points of $\hat{B}^+$. The set $\{b^2|b = b^*, b \in \mathcal{B}\}$ generates $\mathcal{B}$, so it also separates the points of $\hat{B}^+$. It follows that the set $\{a^*_g a_g, a_g \in A_g\}$ separates the elements of $\mathcal{D}_g$. Since $\Omega$ is compact, $\overline{\Omega_1}$ is also compact. Since the functions $f a^*_g a_g$ are continuous on $\Omega_1$ and vanish on the set $\overline{\Omega_1}\setminus \Omega_1$, they belong to the $C^*$-algebra $C_0(\Omega_1)$ of continuous functions vanishing at infinity. By the Stone-Weierstraß theorem, the functions $f a^*_g a_g$, where $a_g \in A_g$, generate a $*$-algebra which is dense in $C_0(\Omega_1)$. Hence the induced topology of $\tau_g$ on $\Omega_1$ coincides with the induced topology of $\tau$. Since $\chi \in \mathcal{D}_g$ is arbitrary, $\tau_g$ is induced from the topology $\tau$ on $\hat{B}^+$.

For $\Delta \subseteq \hat{B}^+$ and $g \in G$, we define $\Delta^g$ by
\[ \Delta^g = \{ \chi^g | \chi \in D_g \cap \Delta \}. \]
By definition, $\emptyset^g$ is $\emptyset$. In particular, if $\Delta \cap \mathcal{D}_g = \emptyset$, then $\Delta^g = \emptyset$. We also write $\alpha_g(\Delta)$ for $\Delta^g$.

**Lemma 14.**

(i) For any $g \in G$, the mapping $\alpha_g$ is a homeomorphism of $D_g$ onto $D_g^{-1}$.
(ii) If $\Delta \subseteq \mathcal{D}_g$ is open (resp. Borel), then $\Delta^g$ is open (resp. Borel).

**Proof.** (i): By Proposition 13, $\alpha_g$ is a bijection. The equality $f a^*_g a_g(\chi) = f a^*_g a_g(\chi^g)$, $a_g \in A_g$, implies that for every open subset $X$ of $\mathbb{R}$ the set $(f a^*_g a_g(X))^g = f a^*_g a_g(X)$ is open. Therefore, by Lemma 13 $\alpha_g^{-1}$ is continuous. Replacing $g$ by $g^{-1}$ we conclude that $\alpha_g$ is continuous. Since $\alpha_g$ and $\alpha_g^{-1}$ are inverse to each other, $\alpha_g$ is a homeomorphism.

(ii) As noted above, $\mathcal{D}_g$ is open. Therefore, if $\Delta$ is open (resp. Borel), then $\Delta \cap \mathcal{D}_g$ is open (resp. Borel). Since $\alpha_g$ is a homeomorphism, $\Delta^g = (\Delta \cap \mathcal{D}_g)^g$ is also open (resp. Borel).

After these preliminaries we are ready to give the main definition of this section.

**Definition 11.** A $*$-representation $\pi$ of $A$ is well-behaved if the following two conditions are satisfied:

(i) The restriction $\text{Res}_{B^+} \pi$ of $\pi$ to $B$ is integrable and there exists a spectral measure $E_\pi$ on the locally compact space $\hat{B}^+[\tau]$ such that
\[ \overline{\pi(b)} = \int_{\hat{B}^+} f_\sigma(\chi) dE_\pi(\chi) \text{ for } b \in B. \]

(ii) For all $a_g \in A_g$, $g \in G$, and all Borel subsets $\Delta \subseteq \hat{B}^+$, we have
\[ E_\pi(\Delta^g) \pi(a_g) \subseteq \pi(a_g) E_\pi(\Delta). \]

If (i) is fulfilled, we shall say that the spectral measure $E_\pi$ is associated with $\pi$.

We give some equivalent forms of the conditions in Definition 11. From Theorem 5 in the Appendix it follows that condition (i) is already fulfilled if $\text{Res}_{B^+} \pi$ is integrable and $B$ is countably generated. The next proposition contains a number of reformulations of condition (ii).

**Proposition 17.** Let $\pi$ be a $*$-representation of $A$ satisfying condition (i) of Definition 11. Let $\mathcal{F}_\pi$ denote the set of Borel functions $f$ on $\hat{B}^+$ such that the operator $\int f dE_\pi$ maps the domain $D(\pi)$ into itself. For $a_g \in A_g$, $g \in G$, let $U_g C_g$ be the polar decomposition of $\pi(a_g)$. Then the following statements are equivalent:

(i) Condition (ii) of Definition 11 is fulfilled.
(ii) For all $a_g \in A_g$, $g \in G$, and all Borel sets $\Delta \subseteq \hat{B}^+$ we have $U_g E_\pi(\Delta) = E_\pi(\Delta^g) U_g$.
(iii) For any $E$-measurable function $f$ on $\hat{B}^+$ and $a_g \in A_g$, $g \in G$, we have
\[ U_g \int f(\chi) dE_\pi(\chi) \subseteq \int_{D_g^{-1}} f(\alpha_g^{-1}(\chi)) dE_\pi(\chi) U_g. \]
(iv) For any \( f \in \mathcal{F}_\pi \), \( a_g \in \mathcal{A}_g \), \( g \in G \), and \( \varphi \in \mathcal{D}(\pi) \), we have

\[
\pi(a_g) \int f(\chi) d\mathcal{E}_\pi(\chi) \varphi = \int_{\mathcal{D}_\pi^{-1}} f(\alpha_{g^{-1}}(\chi)) d\mathcal{E}_\pi(\chi) \pi(a_g) \varphi.
\]

**Proof.** (i) \( \Rightarrow \) (ii): Fix \( \Delta \subseteq \hat{B}^+ \). Since \( \text{Res}_B \pi \) is integrable, \( \overline{\pi(a_g^*a_g)} \) is self-adjoint. But \( \overline{\pi(a_g^*a_g)} \) is self-adjoint extension of \( \pi(a_g^*a_g) \), so that \( C_g^2 = \pi(a_g^*a_g) = \overline{\pi(a_g^*a_g)} \). Since \( \pi(a_g^*a_g) \) commutes with the projections \( \mathcal{E}_\pi(\cdot) \), \( C_g^2 \) and hence \( C_g \) commute with \( \mathcal{E}_\pi(\cdot) \). Thus we get \( U_{\tilde{\pi}}(\Delta) \subseteq U_g C_g E(\Delta) = \pi(a_g) E(\Delta) \). From Definition 11, (i) it follows that the kernel of \( C_g^2 = \pi(a_g^*a_g) \) is equal to \( \text{Ran} \mathcal{E}_\pi(f_{a_g}^{-1}(0)) \). By the properties of the polar decomposition, this kernel equals to \( \ker U_g = \ker C_g \). If \( v \in \ker C_g \), then \( E(\Delta) U_g v = 0 \) and, since \( P_0 := \mathcal{E}_\pi(f_{a_g}^{-1}(0)) \) commutes with \( \mathcal{E}_\pi(\cdot) \), we get \( U_{\tilde{\pi}}(\Delta) v = U_g E(\Delta) P_0 v = U_g P_0 E(\Delta) v = 0 \). Thus the bounded operators \( U_g E(\Delta) \) and \( E(\Delta) U_g \) coincide on the dense set \( \text{Ran} C_g + \ker C_g \), so they coincide everywhere.

(ii) \( \Rightarrow \) (iii): From (ii) we get (iii) for characteristic functions, then for simple functions and by a limit procedure for arbitrary measurable functions \( f \in \mathcal{F}_\pi \).

(iii) \( \Rightarrow \) (iv): This follows from the relation \( \pi(a_g) \varphi = U_g C_g \varphi \) combined with the fact that \( C_g \) and the first integral commute on vectors \( \varphi \in \mathcal{D}(\pi) \).

(iv) \( \Rightarrow \) (i): Since \( \pi \) is integrable, \( \pi \) is closed and so is \( \text{Res}_B \pi \) by Lemma 12. Therefore, \( \mathcal{D}(\pi) = \cap_{b \in B} \mathcal{D}(\pi(b)) \). By 21 the latter implies that \( E(\Delta) \) leaves the domain \( \mathcal{D}(\pi) \) invariant. Hence the characteristic function of \( \Delta \) belongs to \( \mathcal{F}_\pi \) and (i) follows from (iv) applied to this characteristic function.

Many notions on unbounded operators are derived from appropriate reformulations of the corresponding notions on bounded operators. The next proposition says that bounded \(*\)-representations satisfy the two conditions in Definition 11. This observation was in fact the starting point for our definition of well-behaved representations.

**Proposition 18.** If \( \pi \) is a bounded \(*\)-representation of the \(*\)-algebra \( \mathcal{A} \) such that \( \mathcal{D}(\pi) = \mathcal{H}_\pi \), then \( \pi \) is well-behaved.

**Proof.** Since the representation \( \pi \) is bounded, the closure of \( \pi(B) \) in the operator norm is a commutative \( C^* \)-algebra. Hence condition (i) follows from Theorem 12.22 in [Ru].

Fix \( g \in G \), \( a_g, b_g \in \mathcal{A}_g \). From assumption 18 we obtain that \( f_{a_g b_g} \in \mathcal{D}(\pi) \) on \( \hat{B}^+ \). Therefore, by condition (i) we have \( \pi(a_g^* b_g) = \pi(a_g^* b_g) \) which can be rewritten in the form

\[
\pi(a_g^* b_g) = \pi(a_g^* b_g) (a_g^* b_g).
\]

Since \( \pi(b_g^*) \) commutes with \( \pi(a_g^*) \), it also commutes with the projection onto the range of \( \pi(a_g) \). This implies that \( \pi(b_g^*) (\text{Ran} \pi(a_g)) \) is contained in \( \overline{\text{Ran} \pi(a_g)} \), so the range of the operator \( \pi(b_g^* a_g) \) is contained in \( \text{Ran} \pi(a_g) \). The range of the operator \( \pi(b_g^* a_g) \) is evidently contained in \( \text{Ran} \pi(a_g) \). From the relation \( \text{Ran} \pi(a_g) = \ker(\pi(a_g))^\perp \) it follows that \( \pi(a_g^*) \) restricted to \( \overline{\text{Ran} \pi(a_g)} \) is injective. Therefore, from 22 we get \( \pi(a_g b_g^*) = \pi(b_g^* a_g) \) and so

\[
\pi(a_g) \pi(b_g^*) = \pi(b_g^*) \pi(a_g),
\]

for all \( b_g \in \mathcal{A}_g \). Now we use a standard approximation procedure. The preceding relation yields

\[
\pi(a_g) p_n (\pi(b_g^*) ) = p_n (\pi(b_g^*) ) \pi(a_g),
\]

for all polynomials \( p_n \in \mathbb{C}[t] \) which implies that

\[
\pi(a_g) E_{\pi(b_g^*)}(X) = E_{\pi(b_g^*)}(X) \pi(a_g),
\]

for all \( b_g \in \mathcal{A}_g \).
where $E_{\pi(\cdot)}$ denotes the spectral measure of the self-adjoint operator $\pi(\cdot)$ and $X$ is a Borel subset of $\mathbb{R}$. The spectral measure $E_\pi$ on the space $\hat{\mathcal{B}}^+$ associated with $\pi$ is related to the spectral measure of the operator $\pi(b_h^*b_h)$, $b_h \in \mathcal{A}_h$, $h \in G$, by the equation

$$E_\pi(b_h^*b_h)(X) = E_\pi(f_{b_h^*b_h}^{-1}(X)),$$

where $f_{b_h^*b_h}$ is the function on $\hat{\mathcal{B}}^+$ defined by the element $b_h^*b_h \in \mathcal{B}$. From the equality

$$\alpha_h(f_{b_h^*b_h}^{-1}(X)) = f_{b_h^*b_h}^{-1}(X)$$

we obtain

$$(23) \quad \pi(a_g)E_\pi(\Delta) = E_\pi(\Delta^\circ)\pi(a_g),$$

where $g \in G$, $a_g \in \mathcal{A}_g$, $\Delta = f_{c_ga_g}^{-1}(X)$, and $X$ is a Borel subset $\mathbb{R}$. Since $(23)$ is valid for such sets $\Delta$, it holds for the for all sets from the $\sigma$-algebra generated by the sets $\Delta$ as well. From Lemma [13] we conclude that $(23)$ holds for all Borel sets $\Delta \subseteq \mathcal{D}_g$.

In particular, equation $(23)$ is true for $\Delta = \mathcal{D}_g$, so also for $\Delta = \hat{\mathcal{B}}^+\setminus \mathcal{D}_g$. Therefore we have

$$\pi(a_g)E_\pi(\hat{\mathcal{B}}^+\setminus \mathcal{D}_g) = 0$$

which implies that $\pi(a_g)E_\pi(D_0) = 0$ for all Borel subsets $D_0 \subseteq \hat{\mathcal{B}}^+\setminus \mathcal{D}_g$.

Since $E_\pi(\alpha_g(D_0)) = E_\pi(\emptyset) = 0$, $(23)$ is valid for all Borel sets $D_0$ of $\hat{\mathcal{B}}^+\setminus \mathcal{D}_g$. Hence condition $(ii)$ of Definition [11] is satisfied.

In the rest of this section we derive some basic properties of well-behaved representations.

**Proposition 19.** Let $\pi$ be a well-behaved representation of $\mathcal{A}$. Then any self-adjoint subrepresentation $\pi_0 \subseteq \pi$ is well-behaved.

**Proof.** Since $\pi$ is well-behaved, it is self-adjoint. By Corollary 8.3.13 in [S1], there exists a representation $\pi_1$ of $\mathcal{A}$ such that $\pi = \pi_0 \oplus \pi_1$. Since $\text{Res}_B\pi$ is integrable, $\text{Res}_B\pi_0$ is integrable by Proposition 9.1.17 (i) in [S1]. Let $P \in \pi(\mathcal{A})'$ denotes the projection on the representation space $\mathcal{H}_{\pi_0}$ of $\pi_0$. Then $P\pi(\cdot) \upharpoonright \mathcal{H}_{\pi_0}$ is a spectral measure $E_{\pi_0}(\cdot)$ associated with $\pi_0$. Let $a_g \in \mathcal{A}_g$, $g \in G$, and let $\Delta$ be a Borel subset in $\hat{\mathcal{B}}^+$ such that $\Delta^\circ$ is defined. Suppose that $\varphi \in \mathcal{D}(\pi_0)$. Using Definition [11] $(ii)$ for $\pi$ we obtain

$$E_{\pi_0}(\Delta^\circ)\pi_0(a_g)\varphi = P\pi(a_g)E_{\pi_0}(\Delta)\varphi = \pi_0(a_g)E_{\pi_0}(\Delta)\varphi,$$

that is, $E_{\pi_0}(\Delta^\circ)\pi_0(a_g) \subseteq \pi_0(a_g)E_{\pi_0}(\Delta)$, so condition $(ii)$ of Definition [11] holds for $\pi_0$. Hence $\pi_0$ is well-behaved. □

**Lemma 15.** As above, $\mathcal{H}$ denotes a subgroup of $G$. Let $\rho$ be a well-behaved inducible representation of $\mathcal{A}_H$, $E_\rho$ a spectral measure on $\hat{\mathcal{B}}^+$ associated with $\rho$ and $\pi$ the induced representation $\text{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}}\rho$. Suppose that $b \in \mathcal{B}$ and $g \in G$. Then the domain of the operator $\int_{\mathcal{D}_g} f_b(\alpha_g(\chi))dE_\rho(\chi)$ contains $\mathcal{D}(\rho)$ and for arbitrary $a_g \in \mathcal{A}_g$ and $v \in \mathcal{D}(\rho)$ we have

$$(24) \quad \pi(b)[a_g \otimes v] = [ba_g \otimes v] = [a_g \otimes \left( \int_{\mathcal{D}_g} f_b(\alpha_g(\chi))dE_\rho(\chi) \right) v].$$

**Proof.** Let $[c_g \otimes w] \in \mathcal{H}_\pi$, where $c_g \in \mathcal{A}_g$, $w \in \mathcal{D}(\rho)$. Then we have

$$\langle \pi(b)[a_g \otimes w], [c_g \otimes w] \rangle = \langle [ba_g \otimes v], [c_g \otimes w] \rangle = \langle \rho(c_g^*ba_g)v, w \rangle = \int_{\hat{\mathcal{B}}^+} f_{c_g^*ba_g}(\chi)d(E_\rho(\chi)v, w).$$

From Proposition [14] we obtain the equalities $f_{c_g^*ba_g}(\chi) = f_b(\alpha_g(\chi))f_{c_ga_g}(\chi)$ for $\chi \in \mathcal{D}_g$ and $f_{c_g^*ba_g}(\chi) = 0$ for $\chi \in \hat{\mathcal{B}}^+\setminus \mathcal{D}_g$, so the preceding is equal to

$$\int_{\mathcal{D}_g} f_b(\alpha_g(\chi))f_{c_ga_g}(\chi)d(E_\rho(\chi)v, w) = \left( \int_{\mathcal{D}_g} f_b(\alpha_g(\chi))f_{c_g^*a_g}(\chi)dE_\rho(\chi) \right) v, w.$$

Since \( v \) belongs to the domains of \( \int_{D_g} f_b(\alpha_g(\chi))c_{z_a g}(\chi)dE_{\rho}(\chi) \) and \( \int_{D_g} f_c(\alpha_g(\chi))dE_{\rho}(\chi) \), the multiplicativity property of the spectral integral (see e.g. \([R]\), 13.24) implies that \( v \) belongs to the domain of \( \int_{D_g} f_b(\alpha_g(\chi))dE_{\rho}(\chi) \) and we can proceed

\[
\langle \pi(b)[a_g \otimes v], [c_g \otimes w] \rangle = \langle \left( \int_{D_g} f_c(\alpha_g(\chi))dE_{\rho}(\chi) \right) \left( \int_{D_g} f_b(\alpha_g(\chi))dE_{\rho}(\chi) \right) v, w \rangle
\]

\[
= \langle \rho(c^*a_g) \left( \int_{D_g} f_b(\alpha_g(\chi))dE_{\rho}(\chi) \right) v, \rangle = \langle [a_g \otimes \left( \int_{D_g} f_b(\alpha_g(\chi))dE_{\rho}(\chi) \right) v], [c_g \otimes w] \rangle.
\]

Since the linear span of vectors \( [c_g \otimes w] \), where \( c_g \in A_g \) and \( w \in D(\rho) \), is dense in the closed subspace to which \( [b_g \otimes v] \) and \( [a_g \otimes (\int f_b(\alpha_g(\chi))dE_{\rho}(\chi))v] \) belong, the assertion follows. \( \square \)

**Proposition 20.** Assume that \( \mathcal{B} \) is countably generated. If \( \rho \) is a well-behaved inducible cyclic representation of the \(*\)-algebra \( A_H \), then the induced representation \( \pi = \text{Ind}_{A_H \uparrow A}(\rho) \) is a well-behaved representation of the \(*\)-algebra \( A \).

**Proof.** Let \( E_{\rho} \) be a spectral measure on \( \hat{\mathcal{B}}^+ \) associated with \( \rho \). It follows from the Theorem \([7]\) \((i)\) that \( E_{\rho} \) is supported on \( \hat{\mathcal{B}}^+ \). We first show that \( \text{Res}_{\mathcal{B}}\pi \) is defined by a spectral measure, i.e. condition \((i)\) in Definition \([11]\) holds for some spectral measure \( E_\pi \) on \( \hat{\mathcal{B}}^+ \).

Let \( a_g \in A_g \), \( g \in G \), \( w \in D(\rho) \), and let \( \Delta \) be a Borel subset of \( \hat{\mathcal{B}}^+ \). We define a linear operator \( E_\pi(\Delta) \) on the tensor product \( A \otimes D(\rho) \) by putting \( E_\pi(\Delta)(a_g \otimes w) := a_g \otimes E_\rho(\Delta g^{-1})w \). Note that the vector \( E_\rho(\Delta g^{-1})w \) belongs to \( D(\rho) \). Let \( h \in H \) and \( a_h \in A_h \). Using Proposition \([17]\) \((i)\) we get that

\[
E_\pi(\Delta)(a_g a_h \otimes w - a_g \otimes \rho(h)w) = a_g a_h \otimes E_\rho(\Delta h^{-1}g^{-1})w - a_g \otimes E_\rho(\Delta g^{-1})\rho(h)w = a_g a_h \otimes E_\rho(\Delta h^{-1}g^{-1})w - a_g \otimes \rho(h)E_\rho(\Delta h^{-1}g^{-1})w,
\]

belongs to the kernel of the quotient mapping \( A \otimes D(\rho) \to A \otimes A_H \otimes D(\rho) \), so \( E_\pi(\Delta) \) defines a linear operator on \( A \otimes D(\rho) \) which we denote again by \( E_\pi(\Delta) \).

Let \( v \in D(\rho) \) be a cyclic vector for \( \rho \). Take \( a \otimes v \in A \otimes A_H \otimes D(\rho) \). We write \( a \) as a finite sum \( \sum_{i,k} a_{ik}, a_{ik} \in A_{g_{ik}} \), where \( g_{ik} \in G \) are pairwise distinct and \( g_{ik}^{-1}g_{jm} \in H \) if and only if \( k = m \). Then we have \( \langle a_{ik} \otimes v, a_{jm} \otimes v \rangle_0 = 0 \) for \( k \neq m \) and remembering that \( \rho \) is well-behaved we get

\[
\langle E_\pi(\Delta)(a \otimes v), E_\pi(\Delta)(a \otimes v) \rangle_0 = \sum_{i,k} a_{ik} \otimes E_\rho(\Delta g_{ik}^{-1})v, \sum_{i,k} a_{ik} \otimes E_\rho(\Delta g_{ik}^{-1})v \rangle_0 = \sum_{i,k} \langle E_\rho(\Delta g_{ik}^{-1})v, E_\rho(\Delta g_{ik}^{-1})v \rangle_0 = \sum_{i,j} \rho(a_{kj} a_{ik})E_\rho(\Delta g_{ik}^{-1})v, \rho(\Delta g_{ik}^{-1})v \rangle = \sum_{i,j} \rho(a_{kj} a_{ik})v, E_\rho(\Delta g_{ik}^{-1})v \rangle = \sum_{i,j} \rho(a_{kj} a_{ik})v, E_\rho(\Delta g_{ik}^{-1})v \rangle = \langle a \otimes v, E_\pi(\Delta)(a \otimes v) \rangle_0
\]

Assume that \( a \otimes v \in K_\rho \), that is, \( \langle a \otimes v, a \otimes v \rangle_0 = 0 \). The preceding calculation implies that \( E_\pi(\Delta)(a \otimes v) \in K_\rho \), so \( E_\pi(\Delta) \) is a well-defined linear operator on the linear span of vectors \( a \otimes v \in D(\pi) \) defined by

\[
E_\pi(\Delta)[a_g \otimes v] := [a_g \otimes E_\rho(\Delta g^{-1})v].
\]

Since \( v \) is cyclic, the set of vectors \( [a \otimes v] \) is dense in \( H_\pi \) by Lemma \([5]\). It follows from \((25)\) that \( E_\pi(\Delta) \) is bounded and can be extended by continuity to \( H_\pi \). From now on we consider \( E_\pi(\Delta) \) on the subspace \( H_\pi \).

It can be easily seen that \( E_\pi(\Delta)^2 = E_\pi(\Delta) \). We prove that \( E_\pi(\Delta) \) is self-adjoint. For this it suffices to show that

\[
\langle E_\pi(\Delta)[a_{g_1} \otimes v], [a_{g_2} \otimes v] \rangle = \langle [a_{g_1} \otimes v], E_\pi(\Delta)[a_{g_2} \otimes v] \rangle
\]
for \(a_{g_1} \in A_{g_1}, \ a_{g_2} \in A_{g_2}, \ g_1, g_2 \in G\). First we consider the case when \(g_1 H \neq g_2 H\). Then we get

\[
\langle E_\pi(\Delta)[a_{g_1} \otimes v], [a_{g_2} \otimes v]\rangle = \langle [a_{g_1} \otimes E_\rho(\Delta^g v^{-1})v], [a_{g_2} \otimes v]\rangle =
\]

\[
= \langle \rho(p_H(a_{g_2}^* a_{g_1})) E_\rho(\Delta^g v^{-1})v, v \rangle = 0,
\]

since \(p_H(a_{g_2}^* a_{g_1}) = 0\). Analogously, \(\langle [a_{g_1} \otimes v], E_\pi(\Delta)[a_{g_2} \otimes v]\rangle = 0\), so that \((27)\) holds in this case. Now suppose that \(g_1 H = g_2 H\). Then we have

\[
\langle E_\pi(\Delta)[a_{g_1} \otimes v], [a_{g_2} \otimes v]\rangle = \langle [a_{g_1} \otimes E_\rho(\Delta^g v^{-1})v], [a_{g_2} \otimes v]\rangle = \langle \rho(a_{g_2}^* a_{g_1}) E_\rho(\Delta^g v^{-1})v, v \rangle.
\]

Since \(\rho\) is well-behaved and \(a_{g_2}^* a_{g_1} \in A_{g_2^{-1} g_1}\), the preceding equals to

\[
\langle \rho(\Delta^g v^{-1}) \rho(a_{g_2}^* a_{g_1}) v, v \rangle = \langle \rho(a_{g_2}^* a_{g_1}) v, E_\rho(\Delta^g v^{-1}) v \rangle = \langle [a_{g_1} \otimes v], E_\pi(\Delta)[a_{g_2} \otimes v] \rangle.
\]

Thus, \(E_\pi(\Delta)\) is self-adjoint.

Take \(a_g \in A_g\), a Borel set \(\Delta \subseteq \hat{B}^+\) and \(a_k \in A_k\). Then we get

\[
(28) \quad \pi(a_g) E_\pi(\Delta)[a_k \otimes v] = \pi(a_g)[a_k \otimes E_\rho(\Delta^g v^{-1})v] = [a_g a_k \otimes E_\rho(\Delta^g v^{-1})v] = [a_g a_k \otimes \rho((\Delta^g (g_k)^{-1})v)] = E_\pi(\Delta^g)[a_g a_k \otimes v] = E_\pi(\Delta^g) a_g a_k \otimes v.
\]

Next we prove that \(E_\pi(\Delta) D(\pi) \subseteq D(\pi)\). Take \(d_g \in A_g\), \(g \in G\). Using \((28)\) we obtain

\[
\|E_\pi(\Delta)[a \otimes v]\|^2_{d_g} = \|\pi(d_g) E_\pi(\Delta)[a \otimes v]\|^2 = \langle \pi(d_g) E_\pi(\Delta)[a \otimes v], \pi(d_g) E_\pi(\Delta)[a \otimes v] \rangle =
\]

\[
= \langle E_\pi(\Delta^g) \pi(d_g)[a \otimes v], E_\pi(\Delta^g) \pi(d_g)[a \otimes v] \rangle = \langle \pi(d_g)[a \otimes v], E_\pi(\Delta^g) \pi(d_g)[a \otimes v] \rangle =
\]

\[
= \langle \pi(d_g), \pi(d_g) E_\pi(\Delta)[a \otimes v] \rangle \leq \|a \otimes v\|_{d_g} \cdot \|E_\pi(\Delta)[a \otimes v]\|_{d_g},
\]

and hence \(\|E_\pi(\Delta)[a \otimes v]\|_{d_g} \leq \|a \otimes v\|_{d_g}\). By Lemma \((\ref{24})\) the set of vectors \([a \otimes v]\) is a core for \(\pi\). Therefore, the preceding shows that \(E_\pi(\Delta)\) is continuous in the graph topology of \(\pi\). This in turn implies that \(E_\pi(\Delta) D(\pi) \subseteq D(\pi)\).

Now we prove that \(E_\pi(\cdot)\) defines a spectral measure on \(\hat{B}^+\). For \(a_g \in A_g\) we have

\[
\langle E_\pi(\hat{B}^+)[a_g \otimes v], [a_g \otimes v] \rangle = \langle [a_g \otimes E_\rho(\mathcal{D}g)v], [a_g \otimes v] \rangle =
\]

\[
= \langle \rho(a_g^* a_g) E_\rho(\mathcal{D}g)v, v \rangle = \langle \rho(a_g^* a_g) v, v \rangle = \langle [a_g \otimes v], [a_g \otimes v] \rangle
\]

which shows that \(E_\pi(\hat{B}^+) = I\). The countable additivity \(E_\pi(\cdot)\) follows at once from the countable additivity of \(E_\rho(\cdot)\).

Next we show that \(\text{Res}_g \pi\) is an integrable representation associated with spectral measure \(E_\pi\). It suffices to prove that

\[
(29) \quad \langle \pi(b)[a_{g_1} \otimes v], [a_{g_2} \otimes v] \rangle = \int f_b(\chi) d\langle E_\pi(\chi)[a_{g_1} \otimes v], [a_{g_2} \otimes v] \rangle.
\]

for all \([a_{g_1} \otimes v], [a_{g_2} \otimes v] \in H_\pi\). In the case \(g_1 H \neq g_2 H\) one easily checks that the both sides of \((29)\) are equal to zero. In the case \(g_1 H = g_2 H\) we use \((24)\) and compute

\[
\langle \pi(b)[a_{g_1} \otimes v], [a_{g_2} \otimes v] \rangle = \langle [a_{g_1} \otimes \int \mathcal{D}_{g_1} f_b(\alpha_{g_1}(\chi)) dE_\rho(\chi) \otimes v], [a_{g_2} \otimes v] \rangle
\]

\[
= \langle \rho(a_{g_2}^* a_{g_1}) \int \mathcal{D}_{g_1} f_b(\alpha_{g_1}(\chi)) dE_\rho(\chi) v, v \rangle.
\]
Applying Proposition 17\textsuperscript{(iv)} we continue
\[
\int_{D_{g_2}} f_b(\alpha_{g_2}(\chi))dE_\rho(\chi)\rho(\alpha_{g_2}^{-1}a_{g_1})v, v) = \int_{D_{g_2}} f_b(\alpha_{g_2}(\chi))dE_\rho(\chi)\rho(a_{g_2}^{-1}a_{g_1})v, v)
\]
\[
= \int_{D_{g_2}} f_b(\alpha_{g_2}(\chi))d(\rho(a_{g_2}^{-1}a_{g_1})E_\rho(\alpha_{g_2}^{-1}a_{g_1}(\chi)))v, v) = \int_{D_{g_2}^{-1}} f_b(\chi)d(\rho(a_{g_2}^{-1}a_{g_1})E_\rho(\alpha_{g_2}^{-1}(\chi)))v, v)
\]
\[
= \int_{D_{g_2}^{-1}} f_b(\chi)d([a_{g_1} \otimes E_\rho(\alpha_{g_2}^{-1}(\chi))]v, [a_{g_2} \otimes v]) = \int_{D_{g_2}^{-1}} f_b(\chi)d(E_\rho(\chi)[a_{g_1} \otimes v], [a_{g_2} \otimes v])
\]
\[
= \int_{D_{g_2}^{-1}} f_b(\chi)d([a_{g_1} \otimes v], [a_{g_2} \otimes E_\rho(\alpha_{g_2}^{-1}(\chi))v]) = \int_{D_{g_2}^{-1}} f_b(\chi)d(E_\rho(\chi)[a_{g_1} \otimes v], [a_{g_2} \otimes v]).
\]

It follows from (28) that the equality in the Definition 11\textsuperscript{(ii)} holds on the span of vectors \([a \otimes v] \in D(\pi)\) which is a core of \(\pi\) by Lemma 15. Since \(\pi(a_g)\) and \(E_\rho(\Delta)\) are continuous in the graph topology of \(\pi\), condition (ii) in Definition 11 holds for \(\pi\). This completes the proof.

In what follows, we want to induce from arbitrary well-behaved representations of subalgebras \(\mathcal{A}_H\). For this reason we shall need the decomposition of well-behaved representations into direct sums of cyclic well-behaved representations. This aim will be achieved by Proposition 22 below. First we develop some more preliminaries.

**Lemma 16.** Suppose that \(\pi\) is a well-behaved representation of \(\mathcal{A}\). Let \(a_g \in \mathcal{A}_g\) and let \(UC\) be the polar decomposition of \(\pi(a_g)\). Then \(U\) belongs to \(\pi(\mathcal{A})''\).

**Proof.** Let \(T \in \pi(\mathcal{A})'\). As noted already in the proof of Proposition 17 we have \(C^2 = \pi(a_g^*a_g)\). Since \(T\) commutes with \(\pi(a_g^*a_g)\), it commutes with \(C^2\) and therefore with \(C\).

Take \(\varphi \in D(C)\). Then we obtain \(TU(UC)\varphi = T\pi(a_g)\varphi = \pi(a_g)T\varphi = UCT\varphi = UT(C)\varphi\). Now let \(\psi \in \ker C = \ker U = \ker \pi(a_g)\). Then we have \(\pi(a_g)T\psi = T\pi(a_g)\psi = 0\), i.e. \(T\ker U \subseteq \ker U\), so that \(TU\psi = 0 = TU\psi\). Therefore, \(T\) and \(U\) commute on the linear dense subspace \(\ker C + \text{Ran} C\). Since \(T\) and \(U\) are bounded, they commute on \(\mathcal{H}_\pi\). This shows that \(U \in \pi(\mathcal{A})''\).

**Lemma 17.** If \(\pi\) is a well-behaved representation of \(\mathcal{A}\), then we have:

1. \(\pi(a_g^*) = \pi(a_g)^*\) for \(a_g \in \mathcal{A}_g\).
2. \(\pi(a_ga_k) = \pi(a_g) \cdot \pi(a_k)\) for \(a_g \in \mathcal{A}_g\) and \(a_k \in \mathcal{A}_k\).

**Proof.** (i) It is clear that \(\pi(a_ga_g^*) \subseteq (\pi(a_g^*)^*)\pi(a_g^*)\). Since \(\pi\) is well-behaved, \(\text{Res}_B\pi\) is integrable, so \(\pi(a_ga_g^*)\) is essentially self-adjoint (SLI, 9.1.2). Hence it follows that \(\pi(a_ga_g^*) = \pi(a_g^*)^*\pi(a_g^*) = (\pi(a_g^*))^2\). By the same reasoning we obtain \(\pi(a_g^*) = \pi(a_g)\pi(a_g)^* = (\pi(a_g))^2\). Combining these relations with the fact that \(D(T) = D(|T|)\) for a closed operator \(T\) we get

\[
D(\pi(a_g^*)) = D(|\pi(a_g^*)|) = D((\pi(a_g^*))^{1/2}) = D(|\pi(a_g)|) = D(\pi(a_g)^*).
\]

Since \(\pi(a_g^*) \subseteq \pi(a_g)^*\), the preceding implies that \(\pi(a_g^*) = \pi(a_g)^*\).

(ii) Clearly, \(\pi(a_k^*a_ga_ga_k) \subseteq (\pi(a_g) \cdot \pi(a_k))^* \pi(a_g^*) \pi(a_k)\). Since \(a_k^*a_ga_ga_k \in \mathcal{B}\), the operator \(\pi(a_k^*a_ga_ga_k)\) is self-adjoint, so we have the equality \(\pi(a_k^*a_ga_ga_k) = (\pi(a_g) \cdot \pi(a_k))^* \pi(a_g^*) \pi(a_k)\) which yields \(D((\pi(a_k^*a_ga_ga_k))^{1/2}) = D(\pi(a_g) \cdot \pi(a_k))\). As shown in the proof of (i) we also have that \(D(\pi(a_ga_k)) = D((\pi(a_k^*a_ga_ga_k))^{1/2})\). Combining these two equalities with the obvious inclusion \(\pi(a_ga_k) \subseteq \pi(a_g) \cdot \pi(a_k)\), the assertion follows.

\(\square\)
Lemma 18. Let $\pi$ be a well-behaved $\ast$-representation of $\mathcal{A}$. We denote by $\mathcal{U}_\pi$ the set of all partial isometries in the polar decompositions of elements $\pi(a_g)$, where $a_g \in \mathcal{A}_g$, $g \in G$. Then

$$\mathfrak{A}_0 = \left\{ \sum_{i=1}^{n} \lambda_i U_i E_{\pi}(\Delta_i) : \lambda_i \in \mathbb{C}, U_i \in \mathcal{U}_\pi, \Delta_i \subseteq \widehat{B}^+, \Delta_i \text{ is a Borel set} \right\}$$

is a dense $\ast$-subalgebra of $\pi(\mathcal{A})''$ in the strong operator topology.

**Proof.** Since $\mathcal{U}_\pi \subseteq \pi(\mathcal{A})''$ by Lemma 16 and the spectral projections $E_{\pi}(\cdot)$ belong to $\pi(\mathcal{B})'' \subseteq \pi(\mathcal{A})''$, we conclude that $\mathfrak{A}_0 \subseteq \pi(\mathcal{A})''$.

We prove that $\mathfrak{A}_0$ is a $\ast$-algebra. Take $a_g \in \mathcal{A}_g$ and let $U_g[\pi(a_g)]$ be the polar decomposition of the closed operator $\pi(a_g)$. By Lemma 17 (i) we have $\pi(a_g^*) = \pi(a_g)^*$. It is well-known (see e.g. [K], p. 421), that $U_g^*[\pi(a_g)]$ is the polar decomposition of the adjoint operator $\pi(a_g)^* = \pi(a_g)^*$ of $\pi(a_g)$. Therefore, $U_g^* \in \mathfrak{A}_0$ which proves that $\mathfrak{A}_0$ is $\ast$-invariant.

Take another element $a_k \in \mathcal{A}_k$, $k \in G$ and let $U_k C_k$ be the polar decomposition of $\pi(a_k)$. Then using Lemma 17 and Proposition 17 (iii) we get

$$\pi(a_g a_k) \supseteq U_g C_g U_k C_k \supseteq U_g U_k \int_{\mathcal{D}_k} f_{a_g a_k}^* (\alpha_k(\chi)) dE_\pi(\chi) \cdot C_k.$$  \hfill (30)

From the properties of the polar decomposition and the equality $\pi(a_g a_k) = \int f_{a_g a_k}^* dE_\pi$ we conclude that $U_g^* U_k^* = E_\pi(f_{a_g a_k}^{-1}(0, +\infty))$. Similarly, $U_k^* U_k = E_\pi(f_{a_g a_k}^{-1}(0, +\infty))$. Using Proposition 17 (ii) it follows that

$$U_g^* U_k^* U_g U_k = U_k^* E_\pi(f_{a_g a_k}^{-1}(0, +\infty)) U_k = U_k^* U_k E_\pi(\alpha_{k-1}(\mathcal{D}_k \cap f_{a_g a_k}^{-1}(0, +\infty))) =$$

$$= E_\pi(f_{a_g a_k}^{-1}(0, +\infty)) E_\pi(\alpha_{k-1}(\mathcal{D}_k \cap f_{a_g a_k}^{-1}(0, +\infty)))$$

is a projection. Hence $U_g U_k$ is a partial isometry. We denote by $S_{g k}$ the closure of the operator $\int_{\mathcal{D}_k} f_{a_g a_k}^* (\alpha_k(\chi)) dE_\pi(\chi) \cdot C_k$. From (31) and the properties of the partial action we conclude that the kernels of $U_g U_k$ and $S_{g k}$ are equal. Since $S_{g k}$ is positive and its domain $\mathcal{D}(S_{g k})$ contains $\mathcal{D}(\pi)$, it follows from (30) that the polar decomposition of $\pi(a_g a_k)$ is $U_g U_k S_{g k}$. Hence $U_g U_k$ belongs to $\mathcal{U}_\pi$. By Proposition 17 (ii), $\mathfrak{A}_0$ is closed under multiplication. That is, $\mathfrak{A}_0$ is a unital $\ast$-algebra.

Since any $T \in \mathfrak{A}_0$ commutes with $U_\pi$ and with the spectral projections $E_{\pi}(\cdot)$, we have $T \in \pi(\mathcal{A})'$. That is, $\mathfrak{A}_0' \subseteq \pi(\mathcal{A})'$ and so $\mathfrak{A}_0'' \supseteq \pi(\mathcal{A})''$ which implies that $\mathfrak{A}_0'' = \pi(\mathcal{A})''$. Hence $\mathfrak{A}_0$ is dense in $\pi(\mathcal{A})''$ in the strong operator topology.

**Proposition 21.** Suppose that $\pi$ is a well-behaved representation of algebra $\mathcal{A}$ such that the graph topology of $\pi$ is metrizable. Then $\pi$ is cyclic if and only if the von Neumann algebra $\pi(\mathcal{A})''$ is cyclic.

**Proof.** Suppose that $\varphi_0 \in \mathcal{H}_\pi$ is a cyclic vector for $\pi$. Let $\psi \in \mathcal{D}(\pi)$ and $\varepsilon > 0$. Then there exists an element $a \in \mathcal{A}$ such that $\|\pi(a)\varphi_0 - \psi\| < \varepsilon$. Clearly, $a$ is a finite sum $a_1 + a_2 + \cdots + a_k$, where each $a_i$ belong to some vector space $\mathcal{A}_g$, $g \in G$. Let $\pi(a_i) = U_i C_i$ be the polar decomposition of $\pi(a_i)$. Since the operators $U_i$ (by Lemma 18) and the spectral projections $E_{C_i}(\cdot)$ of $C_i$ belong to $\pi(\mathcal{A})''$, the operators

$$A_{i,r} := U_i \int_{-r}^{r} \lambda dE_{C_i}(\lambda), \ r \in \mathbb{N},$$

are in the von Neumann algebra $\pi(\mathcal{A})''$. We choose $r \in \mathbb{N}$ such that $\|(A_{i,r} - \pi(a_i))\varphi_0\| < \varepsilon/k$, $i = 1, \ldots, k$, and put $A_r := A_{1,r} + \cdots + A_{k,r}$. Then we have

$$\|A_r \varphi_0 - \psi\| \leq \|(A_r - \pi(a))\varphi_0\| + \|\pi(a)\varphi_0 - \psi\| \leq \sum_{i=1}^{k} \|(A_{i,r} - \pi(a_i))\varphi_0\| + \|\pi(a)\varphi_0 - \psi\| < 2\varepsilon.$$  

Since $A_r \in \pi(\mathcal{A})''$, this shows that $\varphi_0$ is cyclic for $\pi(\mathcal{A})''$. 

□
Conversely, suppose that \( \varphi_0 \) is a cyclic vector for the von Neumann algebra \( \pi(\mathcal{A})'' \). Let \( P_0 \) be the orthogonal projection onto the closure of \( \pi(\mathcal{B})''\varphi_0 \). Obviously, \( P_0 \in \pi(\mathcal{B})' \). Since \( \text{Res}_{\pi} \) is self-adjoint by Definition \([11]\), \( P_0 \) reduces \( \text{Res}_{\pi} \) to a self-adjoint subrepresentation \( \rho \) \((\text{[SI]}, \text{8.3.11}) \) which is also integrable \((\text{[SI]}, \text{9.1.17}) \). The graph topology of \( \pi \) is metrizable by assumption, so are the graph topologies of \( \text{Res}_{\pi} \) and \( \rho \) by Lemma \([12]\) \((i) \). Therefore, a theorem of R.T. Powers \((\text{[PW]} \), see \([\text{SI}], \text{9.2.1}) \) applies and states that \( \rho \) is cyclic, that is, there exists a vector \( \psi_0 \in \mathcal{D}(\rho) \) such that \( \rho(\mathcal{B})\psi_0 \) is dense in \( \mathcal{D}(\rho) \) in the graph topology. In particular \( \rho(\mathcal{B})\psi_0 = P_0 \mathcal{H}_{\pi} = \pi(\mathcal{B})''\varphi_0 \). Hence \( \psi_0 \) is also cyclic for the commutative von Neumann algebra \( \rho(\mathcal{B})'' = P_0 \pi(\mathcal{B})''P_0 \). Our aim is to show that \( \psi_0 \) is cyclic for \( \pi \), that is, \( \pi(\mathcal{A})\psi_0 \) is dense in \( \mathcal{D}(\pi) \) in the graph topology of \( \pi \).

We first show that the subspace \( \mathcal{H}_0 := \pi(\mathcal{A})\psi_0 \) is dense in \( \mathcal{H}_{\pi} \). Let \( \mathfrak{A}_0 \) be as in Lemma \([18]\). Since \( \mathfrak{A}_0 \) is dense in \( \pi(\mathcal{A})'' \) in the strong operator topology, the vector \( \varphi_0 \) is also cyclic for \( \mathfrak{A}_0 \). Let \( U_g \in \pi_{\mathcal{A}} \) and \( a_g \in \mathfrak{A}_g \), \( g \in \mathcal{G} \), be such that the polar decomposition of \( \overline{\pi}(a_g) \) is \( U_g C_g \). It suffices to show, that for any Borel \( \delta_0 \subseteq \mathbb{B}^+ \) and \( \varepsilon > 0 \) there exists \( b_1 \in \mathcal{B} \) such that

\[
\|U_g E_{\pi}(\delta_0)\varphi_0 - \pi(a_g b_1)\psi_0\| < \varepsilon.
\]

Let \( b_0 \) be such that \( \|\rho(b_0)\psi_0 - E_{\pi}(\delta_0)\varphi_0\| < \varepsilon/3 \). Denote by \( E_{C_g} \) the spectral measure on \( \mathbb{R}_+ \) associated with \( C_g \). Since \( U_g E_{C_g}([0, +\infty)) = U_g E_{C_g}((0, +\infty)) \), we can choose \( n \) such that

\[
\|U_g (E_{C_g}([0, 1/n]) + E_{C_g}([n, +\infty]))\rho(b_0)\psi_0\| < \varepsilon/3.
\]

Further, let \( f \) be the function on \( \mathbb{R} \) defined by \( f(x) = 1/x \) if \( x \in (1/n, n) \) and \( f(x) = 0 \) otherwise. Then the bounded operator \( f(C_g) \) is quasi-inverse to \( C_g \), that is, we have

\[
\text{Id}_{\mathcal{H}_0} = C_g f(C_g) + E_{C_g}([0, 1/n]) + E_{C_g}([n, +\infty]).
\]

Since \( \psi_0 \) is strongly cyclic and \( \pi(a_g b_1) = C_g^2 \), there exists \( b_1 \in \mathcal{B} \) such that

\[
\| (1 + C_g^2)(f(C_g)\rho(b_0) - \rho(b_1))\psi_0 \| < \varepsilon/3.
\]

Using \((33)\) and \((34)\) we derive

\[
\|U_g E_{\pi}(\delta_0)\varphi_0 - \pi(a_g b_1)\psi_0\| \leq \|U_g E_{\pi}(\delta_0)\varphi_0 - \rho(b_0)\psi_0\| + \|U_g (\rho(b_0) - C_g\rho(b_1))\psi_0\| \\
\leq \|U_g\| \varepsilon/3 + \|U_g (E_{C_g}([0, 1/n]) + E_{C_g}([n, +\infty]))\rho(b_0)\psi_0\| + \|U_g (C_g f(C_g)\rho(b_0) - C_g\rho(b_1))\psi_0\| \\
\leq \varepsilon/3 + \varepsilon/3 + \|U_g C_g (1 + C_g^2)^{-1}\| \cdot \| (1 + C_g^2)(f(C_g)\rho(b_0) - \rho(b_1))\psi_0 \| < \varepsilon.
\]

Thus we have shown that \( \mathcal{H}_0 \) is dense in \( \mathcal{H}_{\pi} \).

Let \( \mathcal{D}_0 \) denote the closure of \( \pi(\mathcal{A})\psi_0 \) in the graph topology of \( \pi \). We show that the representation \( \pi_0 := \pi \restriction \mathcal{D}_0 \) of \( \mathcal{A} \) is self-adjoint. Since \( \rho \) is a restriction of \( \text{Res}_{\pi} \), it is inducible. Let \( \mathcal{H}_1 \) denote the representation space of \( \text{Ind}\rho \). Define a linear operator \( T : \mathcal{A} \otimes \mathcal{D}(\rho) \to \mathcal{D}_0 \subseteq \mathcal{D}(\pi) \) by \( T(a \otimes \psi_0) := \pi(a)\psi_0 \). One easily checks that \( T \) gives rise to a unitary operator \( \tilde{T} \) of \( \mathcal{H}_1 \) onto \( \mathcal{H}_0 \) such that \( \tilde{T}[a \otimes \psi_0] = \pi(a)\psi_0 \) and that \( \tilde{T} \) defines a unitary equivalence of representations \( \text{Ind}\rho \) and \( \pi_0 \). Since \( \rho \) is cyclic and well-behaved, \( \text{Ind}\rho \) is well-behaved by Proposition \([20]\) and hence self-adjoint by Lemma \([12]\). Therefore, \( \pi_0 \) is self-adjoint. Since \( \mathcal{D}(\pi_0) = \mathcal{D}_0 \) is dense in \( \mathcal{H}_{\pi} \) as shown in the preceding paragraph, the \( * \)-representation \( \pi \) of \( \mathcal{A} \) is an extension of the self-adjoint representation \( \pi_0 \) acting on the same Hilbert space \( \mathcal{H}_0 \). By Corollary 8.3.12 in \([\text{SI}]\) this implies that \( \mathcal{D}_0 = \mathcal{D}(\pi) \), that is, \( \psi_0 \) is a cyclic vector for \( \pi \).

\[
\text{Proposition 22. Let } \pi \text{ be a well-behaved representation of } \mathcal{A} \text{ on the Hilbert space } \mathcal{H}_{\pi} \text{ such that the graph topology of } \pi \text{ is metrizable. Then } \pi \text{ can be decomposed into a direct orthogonal sum of cyclic well-behaved representations.}
\]

\[
\text{Proof. The identity representation of the von Neumann algebra } \pi(\mathcal{A})'' \text{ can be decomposed into a direct sum of cyclic representations, i.e. there exists a decomposition } \mathcal{H}_{\pi} = \bigoplus_{i \in I} \mathcal{H}_i \text{ such that the orthogonal projections } P_i \text{ onto } \mathcal{H}_i \text{ belong to } \pi(\mathcal{A})' \text{ and each von Neumann algebra } P_i \pi(\mathcal{A})'' \text{ is cyclic on } \mathcal{H}_i. \text{ By Proposition 8.3.11 in } \text{[SI]} \text{ each representation } \pi_i := \pi \restriction P_i \mathcal{D}(\pi) \text{ is self-adjoint. It is straightforward to check that } \pi = \bigoplus_{i \in I} \pi_i. \text{ Since } \pi \text{ is well-behaved, it follows}
\]
from Proposition [19] that \(\pi_i, i \in I\), is well-behaved. By Proposition [21] each representation \(\pi_i\) is cyclic.

\[\square\]

Proposition [22] combined with Lemmas [2] and [12] implies the following

**Proposition 23.** Let \(H\) be a subgroup of \(G\) and let \(\rho\) be a well-behaved representation of \(A_H\) with metrizable graph topology. Then \(\rho\) is inducible to a \(*\)-representation of \(A\) if and only \(\rho\) is \(C\)-positive, where \(C := \sum A^2 \cap A_H\).

8. Well-behaved systems of imprimitivity

In this section we shall prove an analogue of the Imprimitivity Theorem for well-behaved representations. A crucial step for this is to show that representations induced from well-behaved ones are again well-behaved. In the view of Proposition [20] we assume for this section that \(B\) is countably generated. We retain the notation from the previous section. Throughout \(H\) denotes a subgroup of the group \(G\).

**Definition 12.** A system of imprimitivity \((\pi, E)\) for \(A\) over \(G/H\) is called well-behaved if

(i) \(\pi\) is a well-behaved representation of \(A\),
(ii) the projections \(E\) and \(E_\pi\) commute, that is, \(E(t)E_\pi(\Delta) = E_\pi(\Delta)E(t)\) for all \(t \in G/H\) and all Borel subsets \(\Delta\) of \(\hat{B}^+\).

From Propositions [20] and [22] we obtain the following result.

**Proposition 24.** If \(\rho\) is a well-behaved inducible representation of the \(*\)-algebra \(A_H\) with metrizable graph topology, then the induced representation \(\pi = \text{Ind}_{A_H \uparrow A}(\rho)\) is a well-behaved representation of the \(*\)-algebra \(A\).

The next proposition is an analogue of Proposition [9]

**Proposition 25.** If \(\rho\) is a well-behaved inducible \(*\)-representation of \(A_H\), then the system of imprimitivity induced by \(\rho\) is non-degenerate and well-behaved.

**Proof.** Let \((\pi, E)\) be the system of imprimitivity induced by \(\rho\) and let \(E_\pi(\cdot)\) be a spectral measure associated with \(\pi\). It follows from Proposition [9] that \((\pi, E)\) is non-degenerate. By Proposition [24] the representation \(\pi\) is well-behaved. From the construction of \(E(\cdot)\) (see Section [11] and relation [26]) it follows easily that \(E(\cdot)\) and \(E_\pi(\cdot)\) commute. \(\square\)

**Theorem 3.** (Imprimitivity Theorem for well-behaved representations) Let \(H\) be a subgroup of \(G\) and let \((\pi, E)\) be a non-degenerate well-behaved system of imprimitivity for \(A\) over \(G/H\). Then there exists a unique, up to unitary equivalence, inducible well-behaved representation \(\rho\) of \(A_H\) such that \((\pi, E)\) is unitarily equivalent to the system of imprimitivity induced by \(\rho\).

**Proof.** Define \(\rho\) as in the proof of the Theorem [11]. By Theorem [11] we only need to prove that \(\rho\) is well-behaved. Recall that the representation space \(\mathcal{H}_\rho\) is defined as \(\text{Ran}E(H)\) and the domain \(\mathcal{D}(\rho)\) of \(\rho\) is \(\mathcal{D}(\pi) \cap \text{Ran}E(H)\). For a Borel set \(\Delta \subseteq \hat{B}^+\) put \(E_\rho(\Delta) := E_\pi(\Delta)E(H)\). Since \(E_\pi(\cdot)\) commutes with \(E(\cdot)\), \(E_\rho(\cdot)\) is a well-defined spectral measure on \(\hat{B}^+\) whose values are projections in the Hilbert space \(\text{Ran}E(H) = \mathcal{H}_\rho\). One easily checks that \(\text{Res}_{\mathcal{B}}\rho\) is integrable and defined by \(E_\rho(\cdot)\).

Let \(a_h \in A_h, h \in H, v \in \mathcal{D}(\rho)\), and let \(\Delta \subseteq \hat{B}^+\) be a Borel set. Since \(\pi(a_h)v = E(H)\pi(a_h)v\), we compute

\[\rho(a_h)E_\rho(\Delta)v = \pi(a_h)E_\pi(\Delta)v = E_\pi(\Delta^h)\pi(a_h)v = E_\rho(\Delta^h)\rho(a_h)v.\]

Hence \(\rho\) is well-behaved. \(\square\)

For the sake of completeness we formulate an analogue of Theorem [2] for well-behaved representations. Using the fact that well-behaved subrepresentations have complements, the proof is similar to that of Theorem [2].
Theorem 4. Let $H$ be a subgroup of $G$ and let $(\pi, E)$ be a well-behaved system of imprimitivity for $A$ over $G/H$. Fix one element $k_t \in G$, $t \in G/H$, in each left coset from $G/H$. Then for every $t \in G/H$ there exists a well-behaved $*$-representation $\rho_t$ of $A_{k_tHk_t^{-1}}$ on a Hilbert space $\mathcal{H}_t$ such that:

(i) $\rho_t$ is irreducible,
(ii) $(\pi, E)$ is the direct sum of systems of imprimitivity $(\pi_t, E_t)$, $t \in G/H$, where $(\pi_t, E_t)$ is conjugated by the element $k_t$ to the system of imprimitivity induced by $\rho_t$, $t \in G/H$.

Definition 13. Let $\pi$ be a well-behaved representation of $A$. We say that $\pi$ is associated with an orbit $\text{Orb}\chi$, where $\chi \in \hat{\mathcal{B}}^+$, if the spectral measure $E_\pi$ associated with $\pi$ is supported on the set $\text{Orb}\chi$.

The next theorem is a central result of the Mackey analysis (cf. [FD], p. 1251 and p. 1284).

Theorem 5. Assume that the group $G$ is countable. Let $\chi \in \hat{\mathcal{B}}^+$ be a character and let $H = \text{St}\chi$ be its stabilizer group. Then the map

$\rho \mapsto \text{Ind}_{A_H \uparrow A}(\rho) = \pi$

is a bijection from the set of unitary equivalence classes of inducible representations $\rho$ of $A_H$ for which $\text{Res}_B\rho$ corresponds to a multiple of the character $\chi$ onto the set of unitary classes of well-behaved representations $\pi$ of $A$ associated with $\text{Orb}\chi$. A $*$-representation $\rho$ satisfying (35) is bounded and irreducible. Moreover, the von Neumann algebras $\rho(A_H)'$ and $\pi(A)'$ are isomorphic. In particular, $\pi$ is irreducible if and only if $\rho$ is irreducible.

Proof. Let $\pi$ be a well-behaved representation of $A$ associated with $\text{Orb}\chi$, $\chi \in \hat{\mathcal{B}}^+$. Since $G$ is countable, the orbit $\text{Orb}\chi$ is also countable. Therefore the spectral measure $E_\pi$ is discrete. From the definition of $E_\pi$ it follows that $E_\pi(\{\psi\})$, $\psi \in \text{Orb}\chi$, is the eigenspace of each operator $\pi(b)$, $b \in \mathcal{B}$, corresponding to the eigenvalue $\psi(b)$. Hence for all $\psi \in \text{Orb}\chi$ the range $\text{Ran}E_\pi(\{\psi\})$ is contained in the domain of $\text{Res}_B\pi$ which is equal to $\mathcal{D}(\pi)$.

Since $H$ is the stabilizer of $\chi$, the projections $E_\pi(\{\chi\}^a)$ and $E_\pi(\{\chi\}^{g_a})$ are equal if $g_1H = g_2H$ and for all $v \in \mathcal{D}(\pi)$ we have

$\pi(a_g)E_\pi(\{\chi\}^k)v = E_\pi(\{\chi\}^{g_k})\pi(a_g)v$.

(Note that if $\chi \in D_g$, then $E_\pi(\{\chi\}^g)$ is equal to $E_\pi(\{a_g(\chi)\})$, otherwise it is zero projection.) Therefore, we can define a system of imprimitivity $E$ for $A$ over $G/H$ by putting $E(gH) := E_\pi(\{\chi\}^g)$.

We show that $(\pi, E)$ is non-degenerate. Let $g \in G$ be such that $\chi \in D_g$ and let $e_{\chi^g} \in \text{Ran}E(gH)$ be a non-zero vector. Since $\chi^g \in D_{g^{-1}}$, there exists $a_{g^{-1}} \in A_{g^{-1}}$ such that $\chi^g = (a_{g^{-1}})^{-1}\pi(a_{g^{-1}})e_{\chi^g}$. Then, $a_{g^{-1}} \in A_g$ and $\pi(a_{g^{-1}})e_{\chi^g} \in \text{Ran}E_\pi(\{\chi\}^g)$, we obtain

$\pi(a_{g^{-1}})e_{\chi} = (\chi^g)^{-1}(a_{g^{-1}})^{-1}\pi(a_{g^{-1}})e_{\chi^g} = e_{\chi^g}$.

Thus, we have shown that the set $\{\pi(a_g)e_{\chi} | a_g \in A_g, e_{\chi} \in \text{Ran}E(H)\}$ is equal to $\text{Ran}E(gH)$, that is, $(\pi, E)$ is non-degenerate. Since $E(H)$ is equal to $E_\pi(\{\chi\})$, condition (36) is satisfied.

Conversely, let $\rho$ be a $*$-representation of $A_H$ satisfying condition (36). Since $\rho(a_h^*a_h)$, $a_h \in A_h$, $h \in H$, is a multiple of the identity, $\rho(a_h)$ is bounded. Therefore each $\rho(a)$, $a \in A$, is bounded, in particular $\mathcal{D}(\rho) = \mathcal{H}_\rho$. We will show later (see Proposition 28) that every representation $\rho$ satisfying (36) is positive on the cone $A^2$. Since $\rho$ is bounded, it is a direct sum of cyclic representations and hence inducible by Lemma 2. Proposition 20 together with Lemma 6 imply that $\pi = \text{Ind}_{A_H \uparrow A}\rho$ is well-behaved. Let $E_\pi$ be the spectral measure associated with $\pi$. The equality (20) implies that $E_\pi$ is supported on $\text{Orb}\chi$ which means that $\pi$ is associated with $\text{Orb}\chi$. 
It was shown in the proof of the Theorem \[\Box\] that the map
\[\pi \mapsto \text{Res}_{\mathcal{A}_H} \pi \upharpoonright \text{Ran}(E)\]
is the inverse of the map \[\Box\]. Thus, we have proved that the mapping \[\Box\] is indeed a bijection.

Now we prove that \(\rho(\mathcal{A}_H)' = \pi(\mathcal{A})'\). Let \(T \in \rho(\mathcal{A}_H)'\). Define linear operator \(\widetilde{T}\) on \(\mathcal{A} \otimes \mathcal{H}_\rho\) by putting
\[(37) \quad \widetilde{T}(a \otimes v) = a \otimes Tv, \ a \in \mathcal{A}, v \in \mathcal{H}_\rho.\]
Let \(c_H \in \mathcal{A}_H\). Then for arbitrary \(a \in \mathcal{A}\) and \(v \in \mathcal{H}_\rho\) we have
\[
\widetilde{T}(ac_H \otimes v - a \otimes c_H v) = ac_H \otimes Tv - a \otimes Tc_H v = ac_H \otimes Tv - a \otimes c_H Tv,
\]
so \(\widetilde{T}\) defines a linear operator on \(\mathcal{A} \otimes \mathcal{A}_H \mathcal{H}_\rho\) which is also denoted by \(\widetilde{T}\).

Let \(a \in \mathcal{A}\), \(v \in \mathcal{H}_\rho\). We denote by \(\|\cdot\|_0\) the seminorm \(\langle \cdot, \cdot \rangle_{0}^{1/2}\). Since \(\rho\) is inducible, \(S := \rho(p_H(a^*a))\) is a positive operator on \(\mathcal{H}_\rho\) commuting with \(T\). Hence \(T\) commutes with \(S^{1/2}\) and we get
\[
\|\widetilde{T}(a \otimes v)\|^2_0 = \langle \widetilde{T}(a \otimes v), \widetilde{T}(a \otimes v) \rangle_0 = \langle \rho(p_H(a^*a))Tv, Tv \rangle = \langle S^{1/2}Tv, S^{1/2}Tv \rangle
\]
\[
= \langle TS^{1/2}v, TS^{1/2}v \rangle \leq \|T\|^2 \langle S^{1/2}v, S^{1/2}v \rangle = \|T\|^2 \langle \rho(p_H(a^*a))v, v \rangle = \|T\|^2 \|a \otimes v\|^2_0.
\]

Let \(\rho\) be a direct sum of cyclic representations \(\rho_i\) with cyclic vectors \(v_i, \ i \in I\). Take \(\xi = \sum a_k \otimes v_k \in \mathcal{A} \otimes \mathcal{A}_H \mathcal{H}_\rho\), where \(a_k \in \mathcal{A}\) and \(v_k\) are distinct, hence pairwise orthogonal, cyclic vectors. Then the vectors \(a_k \otimes v_k\) are pairwise orthogonal with respect to \(\langle \cdot, \cdot \rangle_0\). Using the preceding inequality and the latter fact we derive
\[
\|\widetilde{T}\xi\|^2 = \left\|\sum_k a_k \otimes v_k \right\|^2_0 \leq \left(\sum_k \|T\| \|a_k \otimes v_k\|_0\right)^2 = \|T\|^2 \sum_k \langle a_k \otimes v_k, a_k \otimes v_k \rangle_0
\]
\[
= \|T\|^2 \left(\sum_k a_k \otimes v_k, \sum_k a_k \otimes v_k \right)_0 = \|T\|^2 \left\|\sum_k a_k \otimes v_k \right\|^2_0 = \|T\|^2 \|\xi\|^2.
\]
This shows that \(\widetilde{T}\) gives rise to a bounded operator on \(\mathcal{H}_\rho\), which we denoted again by \(\widetilde{T}\). It is straightforward to check that \(\widetilde{T}\) commutes with all operators \(\pi(a), a \in \mathcal{A}\), and that the map \(\beta : T \mapsto \widetilde{T}\) is a \(*\)-homomorphism from \(\rho(\mathcal{A}_H)'\) into \(\pi(\mathcal{A})'\).

If \(\widetilde{T} = 0\), then in particular \(\langle Tv, Tv \rangle = \left\|\widetilde{T}(1 \otimes v)\right\|^2 = 0\) for all \(v \in \mathcal{D}(\rho)\) which implies that \(T = 0\). That is, \(\beta\) is injective.

We prove that \(\beta\) is surjective. Let \(S\) be an operator from \(\pi(\mathcal{A})'\). Then \(S \in \pi(\mathcal{B})'\). Since the restrictions of \(\text{Res}_{\mathcal{B}} \pi\) to \(\text{Ran}(E(gH)) = \text{Ran}E_{\pi}(\{\chi\}_g)\) are disjoint representations for distinct cosets \(gH \in G/H\), \(S\) commutes with all operators \(E(gH)\). In particular, \(S_1 := S \upharpoonright \text{Ran}(E)\) is a bounded operator on the Hilbert space \(\text{Ran}(E)\) which commutes with all operators \(\pi(a) \upharpoonright \text{Ran}(E)\), where \(a \in \mathcal{A}_H\). By the canonical isomorphism of \(\mathcal{H}_\rho\) and \(\text{Ran}(E)\), \(S_1\) is a bounded operator on \(\mathcal{H}_\rho\). By construction we have \(S_1 \in \rho(\mathcal{A}_H)'\). One easily verifies that \(\beta(S_1)\) is equal to \(S\). This shows that \(\beta\) is surjective. Summarizing the preceding, we have proved that the mapping \(\beta\) is an isomorphism of von Neumann algebras \(\rho(\mathcal{A}_H)'\) and \(\pi(\mathcal{A})'\).

Remark. Suppose that \(\rho\) is an inducible well-behaved representation of \(\mathcal{A}_H\). If condition \[\Box\] does not hold, then the mapping \(\beta : T \mapsto \widetilde{T}\) of \(\rho(\mathcal{A}_H)'\) into \(\pi(\mathcal{A})'\) is not surjective in general.

We now derive an important corollary from the previous theorem.

**Proposition 26.** Let \(\chi \in \tilde{\mathcal{B}}^+\). Then the induced representation \(\pi = \text{Ind}_\chi\) is irreducible if and only if its stabilizer group \(\text{St}_\chi\) is trivial.

**Proof.** If the stabilizer \(\text{St}_\chi\) is trivial, then \(\pi\) is irreducible by Theorem \[\Box\].
Assume that the stabilizer group is not trivial. Then there exists \( h \in H = \text{St}_\chi \) such that \( h \neq e \). We choose an element \( a_h \in \mathcal{A}_h \) such that \( \chi(a_h^*a_h) = 1 \). Using similar arguments as in the
proof of the Theorem, one shows that there is a linear operator \( T_h \) on the \( \mathcal{H}_\pi \) defined by
\[
T_h([a_g \otimes 1]) = [a_g a_h \otimes 1], \quad a_g \in \mathcal{A}_g, \ g \in G.
\]
For vectors \([a_1 \otimes 1], [a_2 \otimes 1] \in \mathcal{H}_\pi \), where \( a_i \in \mathcal{A}_g, \ g \in G, i = 1, 2 \), we have
\[
\langle T_h[a_1 \otimes 1], T_h[a_2 \otimes 1] \rangle = \langle [a_1 a_h \otimes 1], [a_2 a_h \otimes 1] \rangle = \chi(p(a_h^*a_h^*a_h^*a_h)).
\]
If \( g_1 \neq g_2 \), the latter is equal to \( 0 = \langle [a_1 \otimes 1], [a_2 \otimes 1] \rangle \). If \( g_1 = g_2 \), then \( a_2 a_1 \in \mathcal{B} \) and hence
\[
\chi(p(a_h^*a_h^*a_h^*a_h)) = \chi(a_h^*a_h^*a_h^*a_h) = \chi(a_h^*a_h^*a_h^*a_h) = \langle [a_1 \otimes 1], [a_2 \otimes 1] \rangle.
\]
This shows that \( T_h \) is unitary. Since \( T_h \) acts as a weighted shift (see Proposition 16), it is not a scalar multiple of the identity. One easily verifies that \( T_h \) commutes with all representation operators. Since the commutant of \( \pi \) contains a non-trivial unitary, \( \pi \) is not irreducible. \( \square \)

We now classify all representations of \( \mathcal{A}_H \) satisfying condition (36). The result is the same as in the case when \( \mathcal{A} \) is the group algebra \( \mathbb{C}[G] \) and \( \mathcal{B} \) is the group algebra \( \mathbb{C}[N] \) of a commutative normal subgroup (see [K] and [FD], pp. 1252-1258). That is, we establish a correspondence between \( * \)-representations \( \rho \) of \( \mathcal{A}_H \) satisfying (36) and unitary projective representations of \( H \).

Let \( \chi \in \hat{\mathcal{B}}^+ \) and let \( H \) be the stabilizer group of \( \chi \). Take a representation \( \rho \) satisfying (36). Since \( \chi^h \) is defined for all \( h \in H \), we can find elements \( a_h \) in each \( \mathcal{A}_h \), \( h \in H \), such that \( \chi(a_h^*a_h^*) = \chi^h(a_h^*a_h^*) = \chi(a_h^*a_h^*) \neq 0 \). From (36) it follows that for \( h \in H \) the operator
\[
(38) \quad \zeta(h) := \chi(a_h^*a_h^*)^{-1/2}\rho(a_h)
\]
is unitary and for any \( b_h \in \mathcal{A}_h \) the operator \( \rho(b_h^*a_h) \) is a scalar multiple of the identity, so \( \rho(a_h) \) differs from \( \rho(b_h) \) by a scalar. Thus, the operators \( \zeta(h) \) define a unitary projective representation of \( H \). Hence (see [K]) there exists a 2-cocycle \( \tau : H \times H \rightarrow \mathbb{T} \) such that
\[
(39) \quad \zeta(hk) = \tau(h,k)\zeta(h)\zeta(k), \ h, k \in H.
\]
For \( k \in H \) we have the equality \( \rho(a_k)^{-1} = \chi(a_k^*a_k)^{-1}\rho(a_k) \), in particular, \( \chi(a_k^*a_k) = \chi(a_k^*a_k) \). Using this we calculate
\[
\zeta(hk) = \chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\rho(a_hk) = \chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\rho(a_h)\rho(a_h) = \\
= \chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\zeta(h)\chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\zeta(k)\chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\zeta(k)\chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\zeta(k)\chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\zeta(k).
\]
Thus we have
\[
(40) \quad \tau(h,k) = \chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\chi(a_h^*a_h^*a_h^*a_h)^{-1/2}\chi(a_h^*a_h^*a_h^*a_h), \ h, k \in H.
\]
The mapping \( \zeta \) satisfying (39) will be called \( \tau \)-representation. Let \( t \) be the element of the cohomology group \( Z^2(H, \mathbb{T}) \) of \( H \) with values in \( \mathbb{T} \) defined by the cocycle \( \tau \). Analogously to the group case we call \( t \) the Mackey obstruction of \( \chi \).

Conversely, having a cocycle \( \tau \) of the form (40) and a \( \tau \)-representation \( \zeta \) of \( H \) it is straightforward to verify that (36) defines a \( * \)-representation \( \rho \) of \( \mathcal{A}_H \) satisfying (36). Moreover, \( \rho \) is irreducible if and only if \( \zeta \) is irreducible.

We now show that condition (36) implies \( \sum \mathcal{A}^2 \)-positivity.

**Proposition 27.** The Mackey obstruction \( t \) of \( \chi \) is trivial if and only if \( \chi \) can be extended to a character \( \tilde{\chi} \) of the algebra \( \mathcal{A}_H \). Equation (38) defines a one-to-one correspondence between unitary equivalence classes of \( \tau \)-representations \( \zeta \) of \( H \) and unitary equivalence classes of \( * \)-representations \( \rho \) of \( \mathcal{A}_H \) satisfying (36). Moreover, \( \rho \) is irreducible if and only if \( \zeta \) is irreducible.

**Proposition 28.** Let \( \chi \in \hat{\mathcal{B}}^+ \) and let \( H \) be its stabilizer. If \( \rho \) is a \( * \)-representation of \( \mathcal{A}_H \) satisfying condition (36), then \( \rho \) is nonnegative on the cone \( \sum \mathcal{A}^2 \cap \mathcal{A}_H \).
Proof. It suffices to show that for any \( a \in \mathcal{A} \), \( \rho(p_H(a^*a)) \) is a positive operator. It is enough to consider the case when \( a \) belongs to \( \mathcal{A}_{gh} \) for some \( gh \in G/H \), i.e. \( a = \sum_{h \in H} a_{gh}, \ a_{gh} \in \mathcal{A}_{gh} \).

Using that \( H \) is the stabilizer group of \( \chi \), we get

\[
\chi(a_{gh}^*a_{gh}) = \chi^{gh}(a_{gh}^*a_{gh}) = \chi^{ghk}(a_{gh}^*a_{gh}) = \chi(a_{gh}^*a_{gh})\chi(a_{gh}^*a_{gh}).
\]

Using (38) and the latter equality we calculate

\[
\rho(p_H(a^*a)) = \rho(a^*a) = \sum_{k,h \in H} \rho(a_{gh}^*a_{gh}) = \sum_{k,h \in H} \chi(a_{gh}^*a_{gh})^{1/2}(k^{-1}h) = \sum_{k,h \in H} \chi(a_{gh}^*a_{gh})^{1/2}\zeta(k)^*\zeta(h) = \left( \sum_{h \in H} \chi(a_{gh}^*a_{gh})^{1/2}\zeta(h) \right)^* \sum_{h \in H} \chi(a_{gh}^*a_{gh})^{1/2}\zeta(h),
\]

which implies that \( \rho(p_H(a^*a)) \) is positive. \( \square \)

Next we want to associate well-behaved irreducible representations with orbits. Under some technical assumption this aim will be achieved by Proposition 29 below. For this some preparations are necessary.

Definition 14. A Borel subset \( \Delta \) of \( \hat{\mathcal{B}}^+ \) is called invariant under the partial action of \( G \) if \( \Delta^g \subseteq \Delta \) for every \( g \in G \). A spectral measure \( E \) on \( \hat{\mathcal{B}}^+ \) is called ergodic under the partial action of \( G \) on \( \hat{\mathcal{B}}^+ \) if for every invariant Borel subset \( \Delta \) of \( \hat{\mathcal{B}}^+ \) either \( E(\Delta) \) or \( E(\hat{\mathcal{B}}^+ \setminus \Delta) \) is zero.

Lemma 19. Let \( \pi \) be a well-behaved irreducible representation of the \( * \)-algebra \( \mathcal{A} \) and let \( E_\pi \) be an associated spectral measure. Then \( E_\pi \) is ergodic.

Proof. Let \( \Delta \) be a Borel subset of \( \hat{\mathcal{B}}^+ \) which invariant under the partial action of \( G \). From Proposition 17(i), it follows that \( E_\pi(\Delta) \) is a projection commuting with \( \pi(A_g) \) for all \( g \in G \) and hence with \( \pi(A) \). Since \( \pi \) is irreducible, \( E_\pi(\Delta) \) is trivial, i.e. \( E_\pi(\Delta) = 0 \) or \( E_\pi(\Delta) = I \). \( \square \)

The following concepts are taken from the paper [Ef].

We shall say that a measurable space \((Y, \mathfrak{B})\) is countably separated if there exists a countable subfamily \( \mathfrak{B}_0 \) of \( \mathfrak{B} \) such that for any two points in \( Y \) there exists a member of \( \mathfrak{B}_0 \) containing one point but not the other. A measurable subset \( \Gamma \subseteq Y \) is said to be countably separated if \((\Gamma, \mathfrak{B}_\Gamma)\) is countably separated, where \( \mathfrak{B}_\Gamma \) is the induced Borel structure.

A subset \( \Gamma \subseteq \hat{\mathcal{B}}^+ \) is called a section of the partial action of \( G \) on \( \hat{\mathcal{B}}^+ \) if it contains precisely one point from each orbit. Recall that a (spectral) measure is called an atom if it attains only two values. An atom is called trivial if it is supported at a single point.

The proof of the following simple lemma is borrowed from the proof of Theorem 2.6 in [Ef].

Lemma 20. Let \( E \) be a spectral measure on a countably separated measurable space \((X, \mathfrak{B})\). If \( E \) is an atom, then it is trivial.

Proof. Let \( \{B_k; k \in \mathbb{N}\} \) be a countable family of Borel subsets of \( X \) which separates the points of \( X \) and is closed under taking complements. Let \( B_{k_n}, n \in \mathbb{N}, \) be those sets with \( E(B_{k_n}) = I \) and put \( B = \cap_{n \in \mathbb{N}} B_{k_n} \). Then we have \( E(B_{k_1}) \cap \cdots \cap B_{k_n} = E(B_{k_1}) \cdots E(B_{k_n}) = I \) which implies that \( E(B) = I \) and \( B \neq \emptyset \).

Assume to the contrary that there exist distinct points \( p \) and \( q \) in \( B \). Then there exists \( j \in \mathbb{N} \) such that \( p \in B_j \) and \( q \notin B_j \). Due to the latter relation, we have \( B_j \notin \{B_{k_n}\} \) and \( X \setminus B_j \notin \{B_{k_n}\} \) which implies that \( E(B_j) \) and \( E(X \setminus B_j) \) are zero. Hence \( E(X) = 0 \) which is a contradiction. \( \square \)

Proposition 29. Let \( G \) be a countable group. Suppose that the partial action of \( G \) on \( \hat{\mathcal{B}}^+ \) possesses a measurable countably separated section \( \Gamma \). Then every ergodic spectral measure \( E \) on \( \hat{\mathcal{B}}^+ \) is supported on a single orbit. In particular, each irreducible well-behaved representation of \( \mathcal{A} \) is associated with an orbit.

Proof. We first show that the spectral measure \( E \) restricted to \( \Gamma \) is either zero or an atom. Suppose that \( E \) restricted to \( \Gamma \) is non-zero. Assume to the contrary that \( E \) restricted to \( \Gamma \) is not an atom. Then \( \Gamma \) is a disjoint union of two Borel sets \( \Gamma_1 \) and \( \Gamma_2 \) such that \( E(\Gamma_1) \neq 0 \) and \( E(\Gamma_2) \neq 0 \) and...
most countable set \( E \) that \( U^\cap \) which is a contradiction. Thus, \( E \) restricted to \( \Gamma \) is an atom.

Since \( \Gamma \) is countably separated, Proposition 13 implies that all \( \Gamma^g \), \( g \in G \), are countably separated. Since \( \hat{\mathcal{B}}^+ \) is the union of sets \( \Gamma^g \), it follows from Lemma 20 that there exist points \( \chi_k \in \Gamma^k \), \( k \in I \subseteq G \), such that \( E_\chi(k) \neq 0 \) for all \( k \in I \) and \( E \) is supported on the (at most countable) set \( \{ \chi_k \}_{k \in I} \). Since the set \( \text{Orb}_\chi \) is invariant and \( E(\text{Orb}_\chi) \neq 0 \) for all \( k \), the ergodicity of \( \hat{E} \) implies that all \( \chi_k \) belong to a single orbit. \( \square \)

9. Example: Enveloping algebras of some complex Lie algebras

In this section we illustrate the concepts of the previous sections on three examples: enveloping algebras \( U(su(2)) \), \( U(su(1,1)) \) and \( U(Vir) \), where \( Vir \) denotes the Virasoro algebra [CP], [FQS]. Is is easily checked that in these cases condition (18) is satisfied and the space \( \hat{\mathcal{B}}^+ \) is locally compact, so the theory developed in the preceding sections applies.

First let \( g \) be one of the real Lie algebras \( su(2) \) or \( su(1,1) \) and let \( g_\mathbb{C} \) be its complexification. Then \( g_\mathbb{C} = sl(2, \mathbb{C}) \) has a vector space basis \( \{ E, F, H \} \) with commutation relations

\[
[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.
\]

From (41) it follows that in the complex universal enveloping algebra \( U(g) \) we have

\[
Eq(H) = q(H - 2)E, \quad Fq(H) = q(H + 2)F
\]

(42)

\[
HE^n = E^n(H + 2n), \quad FE^n = E^{n-1}(EF - n(H + n - 1)), \quad n \in \mathbb{N},
\]

(43)

\[
HF^n = F^n(H - 2n), \quad EF^n = F^{n-1}(FE + n(H - n + 1)), \quad n \in \mathbb{N}.
\]

(44)

for each polynomial \( q \in \mathbb{C}[x] \) and that the Casimir element

\[ C := 2(EF + FE) + H^2 = 4FE + H(H + 2) = 4EF + H(H - 2) \]

belongs to the center of \( U(g) \).

The complex unital algebra \( U(g) \) becomes a \(*\)-algebra with involution determined by \( x^* = -x \) for \( x \in g \). In terms of the generators \( \{ E, F, H \} \) of the algebra \( U(g) \) this means that

\[
E^* = F, \quad H^* = H \quad \text{for} \quad g = su(2),
\]

(45)

\[
E^* = -F, \quad H^* = H \quad \text{for} \quad g = su(1,1).
\]

(46)

Using the commutation relation (41) it follows by induction that

\[ U(g)_0 := \text{Lin} \left\{ E^i F^j H^k ; \; k, l \in \mathbb{N}_0 \right\} = \text{Lin} \left\{ (EF)^i H^k ; \; k, l \in \mathbb{N}_0 \right\} = \text{Lin} \left\{ C^i H^k ; \; k, l \in \mathbb{N}_0 \right\}. \]

In particular, \( B := U(g)_0 \) is a commutative unital \(*\)-subalgebra of \( A = U(g) \). For \( n \in \mathbb{N}_0 \), let

\[ A_n = E^n B = \text{Lin} \left\{ E^{n+i} F^j H^k ; \; k, l \in \mathbb{N}_0 \right\}, \quad A_{-n} = F^n B = \text{Lin} \left\{ E^i F^{n+i} H^k ; \; k, l \in \mathbb{N}_0 \right\}. \]

By the Poincare-Birkhoff-Witt theorem, \( \{ E^i F^j H^l ; \; i, j, l \in \mathbb{N}_0 \} \) is a vector space basis of \( U(g) \).

From this fact and the definitions (45) and (46) of the involution we derive that

\[
A = \bigoplus_{n \in \mathbb{Z}} A_n
\]

(47)

is a \( \mathbb{Z} \)-graded \(*\)-algebra. Let \( p : A \to B \) be the canonical conditional expectation (see Proposition 9). In both cases \( g = su(2) \) and \( g = su(1,1) \) the conditional expectation \( p \) is not strong, because we have \( E^* E \notin \sum A^2 \cap B \), but \( E^* E \notin \sum B^2 \).

Remarks 1. The \( \mathbb{Z} \)-graded \(*\)-algebra (47) is the special case \( g = sl(2, \mathbb{C}) \) of Example 8. In this case, \( Q = \mathbb{Z} \) and \( B = U(g)_0 \) is just the commutant of the element \( H \) in the algebra \( U(g) \). Note that \( sl(2, \mathbb{C}) \) is the only simple Lie algebra \( g \) for which \( B = U(g)_0 \) is commutative.

2. For the real Lie algebra \( g = sl(2, \mathbb{R}) \) the involution of the enveloping algebra \( U(g) \) is given by \( E^* = E, \quad F^* = F, \quad H^* = -H \). In this case the decomposition (47) remains valid and shows that \( U(g) \) is a \( \mathbb{Z} \)-graded algebra. But since \( (U(g)_n)^* = U(g)_n \) for \( n \in \mathbb{Z}, \quad U(g) = \bigoplus_n U(g)_n \) is not a \( \mathbb{Z} \)-graded \(*\)-algebra.
We derive three simple lemmas which will be needed below.

**Lemma 21.** Let $\mathfrak{g}$ be one of the real Lie algebras $su(2)$ or $su(1,1)$. A character $\chi \in \hat{B}$ belongs to $\hat{B}^+$ if and only $\chi(F^kF^k) \geq 0$ and $\chi(E^kE^k) \geq 0$ for all $k \in \mathbb{N}$.

**Proof.** Recall that $\chi \in \hat{B}^+$ if and only if $\chi(b) \geq 0$ for all $b \in \mathcal{A}^2 \cap \mathcal{B}$. Hence the necessity of the condition is obvious. We prove that it is also sufficient. By Corollary 1, it suffices to show $\chi(a_n^*a_n) \geq 0$ for all homogeneous elements $a_n \in \mathcal{A}_n$, $n \in \mathbb{Z}$.

Let $n \in \mathbb{N}_0$ and take $a_n \in \mathcal{A}_n$. By the definition of $\mathcal{A}_n$ we have $a_n = E^n b$ for some $b \in \mathcal{B}$. Since $\chi(E^nE^n) \geq 0$ by assumption, $\chi(a_n^*a_n) = \chi(b^*E^nE^n b) = \chi(E^nE^n) \chi(b^*b) \geq 0$. Similarly, for $n < 0$ the inequality $\chi(F^nF^n) \geq 0$ implies that $\chi(a_n^*a_n) \geq 0$ for all $a_n \in \mathcal{A}_n$. \hfill $\square$

**Lemma 22.** For $n \in \mathbb{N}$ we have

\begin{align}
E^nF^n &= EF(EF + H - 2)(EF + H - 2 + H - 4) \cdots (EF + H - 2 + \cdots + H - (n - 1)), \\
F^nE^n &= (EF - H - (H+2) - \cdots - (H+2(n-1))) \cdots (EF - H - (H+2))(EF - H) \\
&= FE(FE - (H+2)) \cdots (FE - (H+2) - \cdots - (H+2(n-1)))
\end{align}

**Proof.** We prove the first equality (48) by induction on $n$. The two equalities concerning $F^nE^n$ are verified in a similar manner. Using the commutation relation (41) we compute

\begin{align*}
E^{n+1}F^{n+1} &= E^n(FE + H)F^n = E^nFEF^n + (H - 2n)E^nF^n = \\
&= E^{n-1}(FE + H)EF^n + (H - 2n)E^nF^n = \\
&= E^{n-1}FE^2F^n + (H - 2(n - 1))E^nF^n + (H - 2n)E^nF^n = \\
&\quad \cdots = (EF + H - 2 + \cdots + (H-2n))E^nF^n.
\end{align*}

Inserting the induction hypothesis (48) for $n$ and remembering that all elements $E^kF^k$ and $H^l$ mutually commute, we obtain (48) for $n + 1$. \hfill $\square$

**Lemma 23.** $\mathcal{B} \equiv \mathcal{U}(\mathfrak{g})_0 = \mathbb{C}[EF, H] = \mathbb{C}[C, H]$.

**Proof.** Since the elements $EF$ and $H$ of $\mathcal{U}(\mathfrak{g})$ commute, there is an algebra homomorphism $\sigma : \mathbb{C}[x_1, x_2] \rightarrow \mathcal{U}(\mathfrak{g})$ given by $\sigma(x_1) = EF$ and $\sigma(x_2) = H$. From the Poincare-Birkhoff-Witt theorem we derive easily that $\sigma$ is injective which gives $\mathcal{U}(\mathfrak{g})_0 = \mathbb{C}[EF, H]$. Clearly, we have also $\mathbb{C}[EF, H] = \mathbb{C}[C, H]$. \hfill $\square$

Lemma 23 implies that the map $\hat{B} \ni \chi \mapsto (\chi(C), \chi(H)) \in \mathbb{R}^2$ is bijective. Denote by $\chi_{st} \in \hat{B}$; $s, t \in \mathbb{R}$ a character such that

$$
\chi_{st}(C) = s, \ \chi_{st}(H) = t.
$$

Propositions 31 and 33 below describe the set of parameters $s, t \in \mathbb{R}$ for which $\chi_{st} \in \hat{B}^+$ in the cases $\mathfrak{g} = su(2)$ and $\mathfrak{g} = su(1,1)$, respectively.

**Proposition 30.** Let $\mathfrak{g}$ be one of the real Lie algebras $su(2)$ or $su(1,1)$. If a character $\chi_{st}$ belongs to $\hat{B}^+$ and if $\chi_{st}^n$ is defined for $n \in \mathbb{Z}$, then we have

$$
\chi_{st}^n = \chi_{s,t+2n}.
$$

**Proof.** For $n = 0$ the proof is trivial. Assume that $n > 0$. In the case $n < 0$ the proof is similar. Since $\chi_{st}^n$ is defined, $\chi_{st}(E^nE^n) > 0$. We compute

$$
\chi_{st}^n(H) = \frac{\chi_{st}(F^nH^nE^n)}{\chi_{st}(F^nE^n)} = \frac{\chi_{st}(F^nE^n(H+2n))}{\chi_{st}(F^nE^n)} = \chi_{st}(H+2n) = t + 2n = \chi_{s,t+2n}(H).
$$

Since $C$ belongs to the center of $\mathcal{A}$, we have $\chi_{st}^n(C) = \chi_{s,t+2n}(C)$. By the definition of $\chi_{st}$ we obtain (51). \hfill $\square$
9.1. The case $\mathfrak{g} = su(2)$. In this subsection we let $\mathcal{A} = \mathcal{U}(su(2))$ and $\mathcal{B} = \mathcal{A}_0 = \mathbb{C}[EF, H] = \mathbb{C}[C, H]$. The next proposition describes the set $\hat{\mathcal{B}}^+$.

**Proposition 31.** A character $\chi_{st}$ defined by (54) belongs to $\hat{\mathcal{B}}^+$ if and only if $t \in \mathbb{Z}$ and $s = (t + 2n)(t + 2n + 2)$ for some $n \in \mathbb{N}_0$ such that $n + t \geq 0$.

**Proof.** Since $E^{*n} = F^n$, Lemmas 21 and 22 imply that $\chi$ belongs to $\hat{\mathcal{B}}^+$ if and only if the following inequalities are fulfilled for arbitrary $k \in \mathbb{N}$:

\[(52) \quad \chi(E^kF^k) \equiv \chi(EF)\chi(EF + H - 2) \cdots \chi(EF + H - 2 \cdots + H - 2k) \geq 0,\]

\[(53) \quad \chi(F^kE^k) \equiv \chi(EF - H)\chi(EF - H - (H + 2)) \cdots \chi(EF - H - \cdots - (H + 2k)) \geq 0.\]

We claim that for every $\chi \in \hat{\mathcal{B}}^+$ there exist $m, n \in \mathbb{N}_0$ such that

\[(54) \quad \chi(EF + m(H - (m + 1))) = 0, \quad \chi(EF - (n + 1)(H + n)) = 0.\]

Assume to the contrary that $\chi(EF + k(H - (k + 1))) \neq 0$ for all $k \in \mathbb{N}_0$. It follows from (52) that $\chi$ is positive on all factors in (52), that is,

\[\chi(EF + H - 2 + \cdots + H - 2k) = \chi(EF + k(H - (k + 1))) = \chi(EF) + k(\chi(H) - (k + 1)) > 0\]

for all $k \in \mathbb{N}_0$ which is a contradiction. Hence $\chi(EF + m(H - (m + 1))) = 0$ for some $m \in \mathbb{N}$. In the same way one proves the second equality in (54).

The solution of the system of equations (54) is

\[(55) \quad \chi(EF) = m(n + 1), \quad \chi(H) = m - n.\]

It is easy to verify that for all $m, n \in \mathbb{N}_0$ the characters $\chi$ defined by (55) satisfy both inequalities (52) and (53).

Putting $t = m - n$ in (55) we get

\[\chi(C) = 4\chi(EF) + \chi(H^2 - 2H) = 4m(n + 1) + (m - n)^2 - 2m + 2n = (m + n)(m + n + 2) = (t + 2n)(t + 2n + 2),\]

i.e. $\chi = \chi_{st}$ where $t = m - n \in \mathbb{Z}$ and $s = (t + 2n)(t + 2n + 2)$. Clearly, we have $m, n \in \mathbb{N}_0$ if and only if $t \in \mathbb{Z}$, $n + t \geq 0$. \qed

We denote by $\psi_n$, $n \in \mathbb{N}_0$, the character $\chi_{n(n+2),-n} \in \hat{\mathcal{B}}^+$ and by $\Gamma$ the subset $\{\psi_n, n \in \mathbb{N}_0\}$ of $\hat{\mathcal{B}}^+$. By Propositions 30 and 31 each orbit under the partial action of $\mathbb{Z}$ on $\hat{\mathcal{B}}^+$ contains precisely one of the characters from $\Gamma$, i.e. $\Gamma$ is a section of the partial action of $\mathbb{Z}$ on $\hat{\mathcal{B}}^+$.

**Proposition 32.** The representations $\text{Ind} \chi$, $\chi \in \Gamma$, are pairwise non-equivalent and irreducible. Each irreducible well-behaved representation of $\mathcal{A}$ is unitarily equivalent to $\text{Ind} \chi$ for some $\chi \in \Gamma$. An $\ast$-representation $\pi$ of $\mathcal{A} = \mathcal{U}(su(2))$ is well-behaved (in the sense of Definition 17) if and only if $\pi$ is integrable (that is, $\pi = dU$ for some unitary representation $U$ of the Lie group $SU(2)$).

**Proof.** Clearly, the bijection $\chi_{st} \mapsto (s, t)$ of the space $\hat{\mathcal{B}}$ onto $\mathbb{R}^2$ (by Lemma 23) is a homeomorphism. Hence Proposition 31 implies that $\hat{\mathcal{B}}^+$ is a discrete space. It follows from the formulas for the partial action of $\mathbb{Z}$ that $\Gamma$ is a Borel section. By Proposition 29 all irreducible well-behaved representations are associated with orbits. Therefore, by Theorem 5 we have that $\text{Ind} \chi$, $\chi \in \Gamma$, are up to unitary equivalence all irreducible well-behaved representations. It follows from Proposition 31 that $\text{Orb} \psi_n$, $n \in \mathbb{N}_0$ consists of $n + 1$ elements, and Proposition 16 implies that $\text{Ind} \psi_n$, $n \in \mathbb{N}_0$ has dimension $n + 1$. The latter implies in particular that each representation $\text{Ind} \chi$, $\chi \in \Gamma$ is integrable.

Let $\pi$ be a well-behaved representation of $\mathcal{A}$ and let $E_\pi$ be the associated spectral measure on $\hat{\mathcal{B}}^+$. Denote by $\rho$ the restriction of $\text{Res}_{\hat{\mathcal{B}}} \pi$ to $\text{Ran}(E_\pi(\Gamma))$. It is easily checked that $\pi$ is unitarily equivalent to $\text{Ind} \rho$. Since $\hat{\mathcal{B}}^+$ is discrete, $\rho$ is equivalent to a direct sum of characters $\chi \in \Gamma$ (taken with multiplicities), so that $\pi$ is equivalent to a direct sum of representations $\text{Ind} \chi$, $\chi \in \Gamma$. Because $\text{Ind} \chi$ is integrable as shown in the preceding paragraph, $\pi$ is integrable.
Conversely, if $\pi$ is an integrable representation, $\pi$ is a direct sum of integrable irreducible representations $\pi_i$. Since each representation $\pi_i$ is finite dimensional and hence well-behaved by Proposition 18, $\pi$ is well-behaved. □

It is well-known that for each $n \in \mathbb{N}_0$ the spin $\frac{n}{2}$ representation is the unique (up to unitary equivalence) irreducible $(n+1)$-dimensional $\ast$-representation of $\mathcal{A} = \mathcal{U}(su(2))$. Since the $\ast$-representation $\text{Ind}\psi_n$ of $\mathcal{A}$ is irreducible and of dimension $n+1$, $\text{Ind}\psi_n$ is equivalent to the spin $\frac{n}{2}$ representation. We want to establish this equivalence by explicit formulas.

Recall from Proposition 16 (i) that the vectors

$$\left\{ \frac{[E^k \otimes 1]}{\|E^k \otimes 1\|}, \ k = 0, 1 \ldots n \right\}$$

form an orthonormal base of the representation space of $\text{Ind}\psi_n$. By definition of $\psi_n$ we have $\psi_n(H) = -n$ and $\psi_n(EF) = \frac{1}{2}\psi_n(C - H^2 + 2H) = 0$. Using Lemma 22 we compute

$$\|E^k \otimes 1\|^2 = \psi_n(F^kE^k) = \psi_n((EF - H)(EF - 2(H + 1)) \ldots (EF - k(H + k - 1))) = n(2(n-1)) \ldots (k(n-k+1)) = \frac{k! \cdot n!}{(n-k)!}, \ k = 0, 1 \ldots, n.$$ 

Putting $l = \frac{n}{2}$, $\pi_l := \text{Ind}\psi_n$ and

$$e_m := \frac{[E^{l+m} \otimes 1]}{\|E^{l+m} \otimes 1\|} = \sqrt{\frac{(l-m)!}{(2l)!(l+m)!}} [E^{l+m} \otimes 1], \ m = -l, l+1, \ldots, l,$$

we calculate

$$\pi_l(E)e_m = \frac{[E^{l+m+1} \otimes 1]}{\|E^{l+m+1} \otimes 1\|} = \frac{\|E^{l+m+1} \otimes 1\|}{\|E^{l+m} \otimes 1\|} e_{m+1} = \sqrt{\frac{(2l)!(l+m+1)!}{(l-m)!}} \frac{(l-m)!}{(2l)!(l+m)!} e_{m+1} =$$

$$= \sqrt{(l-m)(l+m+1)}e_{m+1}, \ m = -l, l+1, \ldots, l.$$ 

In the same manner we derive

$$\pi_l(F)e_m = \sqrt{(l-m+1)(l+m)}e_{m-1}, \ \pi_l(H)e_m = 2me_m, \ m = -l, l+1, \ldots, l.$$ 

These are the formulas for the actions of $E, F, H$ in the spin $l$ representation of $\mathcal{U}(su(2))$.

We now show that the representations $\pi_l$ can be also induced from the $\ast$-subalgebra $\mathcal{C} = \mathbb{C}[H]$. Let $p_3 = p_2 \circ p_1$, where $p_1$ is the canonical conditional expectation $p_1 : \mathcal{A} \to \mathcal{B}$ and $p_2 : \mathcal{B} \to \mathcal{C}$ is conditional expectation defined by $p_2((EF)^k) = 0, p_2(H^k) = H^k, k \in \mathbb{N}$. Using Lemma 22 we obtain

$$p_3(\sum A^2) = \sum C^2 - H \sum C^2 + H(H + (H + 2)) \sum C^2 - H(H + (H + 2)) \sum C^2 + \cdots =$$

$$= \sum C^2 - H \sum C^2 + H(H + 1) \sum C^2 - H(H + 1)(H + 2) \sum C^2 + \cdots + (-1)^k H(H + 1)(H + 2) \cdots (H + k - 1) \sum C^2 + \cdots.$$ 

Obviously, $p_3$ is a $(\sum A^2, p_3(\sum A^2))$-conditional expectation. It is easy to check that $\sum A^2 \cap \mathbb{C}[H] = \sum C^2$. Since $p_3(\sum A^2)$ is strictly larger than $\sum C^2$, $p_3$ is not a conditional expectation according to Definition 4. In particular we have seen that the composition of two conditional expectations is not a conditional expectation in general.

It is clear from the preceding formulas that the set of characters on $\mathbb{C}[H]$ which are non-negative on the cone $p_3(\sum A^2)$ and hence inducible via $p_3$ is the set $\{ \chi_k, k \in \mathbb{N}_0 \}$. Note that $\chi_k(H) = -k$. It is not difficult to compute that the corresponding induced representation $\text{Ind}\chi_{2l}, l \in \frac{1}{2}\mathbb{N}_0$, is unitarily equivalent to $\pi_l$. 
9.2. **The case** $\mathfrak{g} = su(1, 1)$. In this subsection let $\mathcal{A} = \mathcal{U}(su(1, 1))$ and $\mathcal{B} = \mathcal{A}_0 = \mathbb{C}[EF, H] = \mathbb{C}[C, H]$.

We denote by $\chi_{st} \in \hat{B}$ the characters determined by (50). It is convenient to introduce the following subsets of $\hat{B}$:

$$X_{00} = \{\chi_{00}\},$$

$$X_{1k} = \{\chi_{st}|2k \leq t < 2k + 2, -\infty < s < (t - 2k)(t - 2(k + 1))\}, \ k \in \mathbb{Z},$$

$$X_{2k} = \{\chi_{st}|2k < t < 2k + 2, s = (t - 2k)(t - 2(k + 1))\}, \ k \in \mathbb{Z},$$

$$X_{3k} = \{\chi_{st}|t \geq 2k + 2, s = (t - 2k)(t - 2(k + 1))\}, \ k \in \mathbb{Z}_0,$$

$$X_{4k} = \{\chi_{st}|t \leq 2k, s = (t - 2k)(t - 2(k + 1))\}, \ k \in \mathbb{Z}\backslash\mathbb{N}_0.$$

The following two propositions describe the set $\hat{B}^+$ and the partial action of $\mathbb{Z}$ on it.

**Proposition 33.** The set $\hat{B}^+$ is equal to the disjoint union

$$X_{00} \cup \bigcup_{k \in \mathbb{Z}} X_{1k} \cup \bigcup_{k \in \mathbb{Z}} X_{2k} \cup \bigcup_{k \in \mathbb{N}_0} X_{3k} \cup \bigcup_{k \in \mathbb{Z}\backslash\mathbb{N}_0} X_{4k}.$$

**Proof.** The equality $E^{*n} = (-1)^n F^n$ and Lemmas 21 and 22 imply that a character $\chi \in \hat{B}$ belongs to $\hat{B}^+$ if and only if the following inequalities hold:

$$(56) \quad (-1)^k \chi((EF - H)(EF - (H + 2)) \cdot \cdot \cdot (EF - H - (H + 2) - \cdot \cdot \cdot - (H + 2(k - 1))) \geq 0, \ k \in \mathbb{N},$$

$$(57) \quad (-1)^k \chi((EF - H)(EF - (H + 2)) \cdot \cdot \cdot (EF - H - (H + 2) - \cdot \cdot \cdot - (H + 2(k - 1))) \geq 0, \ k \in \mathbb{N}.$$

Straightforward calculations show that the solutions of the latter system of inequalities are precisely the characters belonging to one of the above sets $X_{ij}$. One easily verifies that the sets $X_{ij}$ are pairwise disjoint for different $(i, j)$. \qed

**Proposition 34.**

(i) $\chi_{00}^n$ is defined only for $n = 0$.

(ii) For $\chi_{st} \in X_{1k} \cup X_{2k}$, $k \in \mathbb{Z}$, the $\chi_{st}^n$ is defined for all $n \in \mathbb{Z}$.

(iii) For $\chi_{st} \in X_{3k}$, $k \in \mathbb{N}_0$, the $\chi_{st}^n$ is defined for $n \geq -k$.

(iv) For $\chi_{st} \in X_{4k}$, $k \in \mathbb{Z}$, the $\chi_{st}^n$ is defined for $n \leq k - 1$.

**Proof.** Follows directly from Propositions 30 and 33. \qed

Set

$$\Gamma := X_{00} \cup X_{10} \cup X_{20} \cup X_{30} \cup X_{4,-1} \subseteq \hat{B}^+.$$

It follows from the previous propositions that each orbit under the partial action of $\mathbb{Z}$ on $\hat{B}^+$ intersects $\Gamma$ exactly in one point, i.e. $\Gamma$ is a section of the partial action. As in the case of $su(2)$, the topology on $\hat{B}^+$ is induced from the standard topology on $\mathbb{R}^2$. Hence $\Gamma$ is a countably separated Borel section of the partial action of $\mathbb{Z}$ on $\hat{B}^+$.

Explicit formulas for the representations $\text{Ind}\chi$, $\chi \in \Gamma$, can be derived in a similar manner as in case of $su(2)$. We omit the details. In the standard terminology of representation theory of Lie algebras we have:

- the representation $\text{Ind}\chi$, $\chi \in X_{00}$, is the trivial representation,

- the representations $\text{Ind}\chi$, $\chi \in X_{10}$, form the principal unitary series,

- the representations $\text{Ind}\chi$, $\chi \in X_{20}$, form the supplementary unitary series,

- the representations $\text{Ind}\chi$, $\chi \in X_{30} \cup X_{40}$, form the discrete unitary series.

Using this description we obtain the following

**Proposition 35.** The representations $\text{Ind}\chi$, $\chi \in \Gamma$, are pairwise non-equivalent and irreducible. Each irreducible well-behaved representation of $\mathcal{A}$ is unitarily equivalent to $\text{Ind}\chi$ for precisely one $\chi \in \Gamma$. A $\ast$-representation of $\mathcal{A} = \mathcal{U}(su(1, 1))$ is well-behaved (in the sense of Definition
if and only it is of the form $dU$ for some unitary representation $U$ of the universal covering group of the Lie group $SU(1, 1)$.

We close this subsection with the following

**Remark.** For a character $\chi \in \mathcal{B}^+$ the following three statements are equivalent:

(i) $\chi$ belongs to one of the series $X_{1k}$ or $X_{2k}$, $k \in \mathbb{Z}$, corresponding to the principal or supplementary unitary series,

(ii) $\chi^k$ is defined for all $k \in \mathbb{Z}$,

(iii) $\chi(C) < 0$, where $C$ is the Casimir element defined above.

9.3. Enveloping algebra of the Virasoro algebra. Recall that the Virasoro algebra is the complex Lie algebra $Vir$ with generators $L_n$, $n \in \mathbb{Z}$, and $C$ and defining relations

$$[L_n, L_m] = (m - n)L_{n+m} + \delta_{n,-m}(n^3 - n)/12C \quad \text{and} \quad [L_n, C] = 0 \text{ for } n, m \in \mathbb{Z}. \quad (58)$$

In this subsection we show that the unitary representations with finite-dimensional weight spaces of the Virasoro algebra can be identified with the well-behaved representations with respect to a canonical grading of a quotient algebra of its enveloping algebra. For results on unitary representations of $Vir$ we refer to [CP] and references therein.

Let $\mathcal{W}$ denote the enveloping algebra of $Vir$, that is, $\mathcal{W}$ is the unital $\ast$-algebra with generators $L_n$, $n \in \mathbb{Z}$, and $C$ and the same defining relations (58). It is a $\ast$-algebra with involution determined by $L_n^\ast = L_{-n}$ for $n \in \mathbb{Z}$ and $C^\ast = C$. Lemma 9 implies that $\mathcal{W}$ is $\mathbb{Z}$-graded such that $L_n \in \mathcal{W}_n$ and $C \in \mathcal{W}_0$.

The main result in [CP] states that there are precisely two families of irreducible unitary representations of $\mathcal{W}$ with finite-dimensional weight spaces. The first series consists of highest weight representations, i.e. representations generated by a vector $a \in \mathcal{W}$.

These representations are uniquely defined by the pair $(a, z) \in \mathbb{C}^2$. The possible values of $(a, z)$ for the representation to be unitary (that is, a $\ast$-representation in our terminology) are the following ones (see [FQS]):

$$a \geq 0, \ z \geq 1, \ or \ z_n = 1 - \frac{6}{n(n+1)} \quad \text{or} \quad a(p,q) = \frac{(np+q)^2 - 1}{4n(n+1)}, \quad (59)$$

where the integers $n, p, q$ satisfy $n \geq 2$ and $0 \leq p < q < n$.

The other series of unitary representations are defined on spaces of $\lambda$-densities (see [CP]). They can be described as follows. Let $\{w_k\}_{k \in \mathbb{Z}}$ be an orthonormal base of $l^2(\mathbb{Z})$. Then the action of $\mathcal{W}$ on $l^2(\mathbb{Z})$ is given by

$$L_k w_n = (n + a + k\lambda)w_{n+k}, \ Cw_n = 0, \ k, n \in \mathbb{Z}, \ \lambda \in \frac{1}{2} + i\mathbb{R}, \ a \in \mathbb{R}. \quad (60)$$

Let $\mathcal{I}$ denote the two-sided $\ast$-ideal of $\mathcal{W}$ generated by elements

$$bd - db, \ b, d \in \mathcal{W}_0 \ \text{and} \ a_k^i c_k c_k^i a_k - a_k^i a_k c_k^i c_k, \ a_k, c_k \in \mathcal{W}_k, \ k \in \mathbb{Z}.$$ 

**Lemma 24.** $\mathcal{I}$ is contained in the intersection of all kernels of representations described above.

**Proof.** We prove the assertion for $\ast$-representations defined by (60). For highest and lowest weight representations the proof is similar.

We fix a $\ast$-representation $\pi$ given by (60), $k \in \mathbb{Z}$ and $a_k, c_k \in \mathcal{W}_k$. It follows from (60) that $\pi(a_k)w_m = \mu_m w_{m+k}, \ \pi(c_k)w_m = \nu_m w_{m+k}, \ m \in \mathbb{Z}$, for some $\mu_m, \ \nu_m \in \mathbb{C}$. This implies that

$$\pi(a_k^i c_k c_k^i a_k)w_m = \lambda_m \nu_m \nu_{m+1} \lambda_m \cdot w_m = \pi(a_k^i a_k c_k^i c_k)w_m, \quad (\forall m \in \mathbb{Z}).$$

Therefore $\mathcal{I}$ is contained in $\ker \pi$. 

In view of Lemma 24 we introduce the $\ast$-algebra $\mathcal{A} = \mathcal{W}/\mathcal{I}$. Let $i : \mathcal{W} \to \mathcal{A}$ be the quotient mapping and put $l_k := i(L_k)$ for $k \in \mathbb{Z}$ and $c = i(C)$. Since the generators of $\mathcal{I}$ are homogeneous,
Lemma [9] implies that \( A \) is again a \( \mathbb{Z} \)-graded \(*\)-algebra such that \( l_k \in A_k, \ k \in \mathbb{Z} \), and \( c \in A_0 \). As usual we denote by \( B \) the subalgebra \( A_0 \).

Because of the PBW-theorem there are two "natural" bases of the vector space \( \mathcal{W} \):

\[
B_1 = \left\{ C^k L_{n_1} L_{n_2} \ldots L_{n_r} \mid n_1 \leq n_2 \leq \cdots \leq n_r, \ k, r \in \mathbb{N}_0, n_i \in \mathbb{Z} \right\},
\]

\[
B_2 = \left\{ C^k L_{n_1} L_{n_2} \ldots L_{n_r} \mid n_1 \geq n_2 \geq \cdots \geq n_r, \ k, r \in \mathbb{N}_0, n_i \in \mathbb{Z} \right\}.
\]

Fix \( i = 1, 2 \). Since all elements in \( B_i \) are homogeneous, the elements \( C^k L_{n_1} L_{n_2} \ldots L_{n_r} \in B_i, \sum_j n_j = 0 \), form a vector space base of the algebra \( \mathcal{W}_0 \). To define a character of \( \mathcal{W}_0 \), it is therefore sufficient to define it on these elements \( C^k L_{n_1} L_{n_2} \ldots L_{n_r} \in B_i \).

Let \( \pi \) be an irreducible unitary highest weight representation of \( \text{Vir} \) with weight vector \( v \). It defines a \(*\)-representation of \( \mathcal{W} \) denoted also by \( \pi \). One easily checks that the subspace \( \mathbb{C} \cdot v \) is invariant under all operators \( \pi(b), b \in \mathcal{W}_0 \). Therefore it defines a character \( \chi \) on \( \mathcal{W}_0 \) given by \( \chi(L_{n_1} \ldots L_{n_k}) = 0, \chi(L_0) = a, \chi(C) = z \), where \( n_1 \leq \cdots \leq n_k \), \( \sum_r n_r > 0 \), and \( (a, z) \) is one of the pairs defined by (61), (62), (63) and (64). By Lemma [24] \( \chi \) annihilates the ideal \( \mathcal{I} \), so it gives a character on the quotient algebra \( \mathcal{B} = \mathcal{I}(\mathcal{W}_0) \) which we denote again by \( \chi \). It is defined by

\[
\chi(l_{n_1} \ldots l_{n_k}) = 0, \ \chi(l_0) = a, \ \chi(c) = z, \ \text{where} \ n_1 \leq \cdots \leq n_k \neq 0, \ \sum_r n_r = 0,
\]

where \( (a, z) \) is given by (64). The character \( \chi \) obviously belongs to \( \hat{\mathcal{B}}^+ \).

From the lowest weight representations we get another series of characters \( \chi \in \hat{\mathcal{B}}^+ \) determined by

\[
\chi(l_{n_1} \ldots l_{n_k}) = 0, \ \chi(l_0) = a, \ \chi(c) = z, \ \text{where} \ n_1 \geq \cdots \geq n_k \neq 0, \ \sum_r n_r = 0,
\]

where \( (a, z) \) is as in (64).

Let \( \pi \) be a representation given by (60). Considering the restriction of \( \pi \) to the subspace \( \mathbb{C} \cdot w_0 \) we obtain a series of characters \( \chi \in \hat{\mathcal{B}}^+ \) defined by

\[
\chi(l_{n_1} \ldots l_{n_k}) = \prod_{r=1}^{k} (a - \sum_{s=1}^{r} n_s + n_r \lambda), \ \chi(c) = 0,
\]

where \( a \in \mathbb{R}, \ \lambda \in \frac{1}{2} + i\mathbb{R} \).

Let \( \Gamma \subseteq \hat{\mathcal{B}}^+ \) denotes the union of all characters defined by the equations (61), (62) and (63).

**Proposition 36.** Each orbit under the partial action of \( \mathbb{Z} \) on \( \hat{\mathcal{B}}^+ \) contains precisely one character from \( \Gamma \). The stabilizer of each character in \( \hat{\mathcal{B}}^+ \) is trivial. For every \( \chi \in \hat{\mathcal{B}}^+ \), \( \iota \circ \text{Ind} \chi \) is a \(*\)-representation of \( \mathcal{W} \) with finite-dimensional weight spaces. Every irreducible \(*\)-representation of \( \mathcal{W} \) with finite-dimensional weight spaces is unitarily equivalent to \( \iota \circ \text{Ind} \chi \) for precisely one \( \chi \in \Gamma \).

**Proof.** A straightforward computation shows that

\[
[l_0, l_{n_1} l_{n_2} \ldots l_{n_r}] = (n_1 + n_2 + \cdots + n_r)l_{n_1} l_{n_2} \ldots l_{n_r}, \ n_i \in \mathbb{Z}, \ r \geq 1.
\]

Since every \( a_n \in A_n \) is a linear combination of the elements \( l_{n_1} l_{n_2} \ldots l_{n_r}, \ n_1 + n_2 + \cdots + n_r = n \), it follows that

\[
[l_0, a_n] = na_n, \ \text{for all} \ a_n \in A_n, \ n \in \mathbb{Z}.
\]

Let \( \chi \in \hat{\mathcal{B}}^+ \) and \( n \in \mathbb{Z} \). Assume that \( \chi^n \) is defined. Then there exists an \( a_n \in A_n \) such that \( \chi(a_n^* a_n) > 0 \). Using (64) we get

\[
\chi^n(l_0) = \frac{\chi(a_n^* l_0 a_n)}{\chi(a_n^* a_n)} = \frac{\chi(a_n^* a_n l_0 + na_n^* a_n)}{\chi(a_n^* a_n)} = \chi(l_0) + n.
\]

Let \( \pi := \text{Ind} \chi \). Since \( \chi \) satisfies condition (18), we can choose an orthonormal base of vectors \( e_k \) of the representation space \( \mathcal{H}_B \) such that \( \pi(l_0) e_k = \lambda_k e_k \), where \( \lambda_k = \chi(l_0) + k \). This implies that \( \pi(l_0) \) acts as a semisimple operator and that all eigenspaces of \( \pi(l_0) \) are
finite dimensional. It is also clear that the stabilizer of \( \chi \) is trivial, so the representation \( \pi \) is irreducible by Proposition [26]. Therefore, by Theorem 0.5 in [CP] the representation \( \iota \circ \pi \) is unitarily equivalent either to a highest or lowest weight representation or to a representation defined by (60).

On the other hand, one easily verifies that Ind\( \chi \) gives rise via \( \iota \) either to a highest or lowest weight representation or to a representation defined by (60). This implies that \( \mathcal{B}^+ \) is equal to the union of all orbits \( \text{Orb} \chi \), where \( \chi \in \Gamma \).

10. EXAMPLE: Representations of dynamical systems

Let \( f \in \mathbb{R}[x] \) be a fixed polynomial. In this section we consider the *-algebra
\[
\mathcal{A} = \mathbb{C} \langle a, a^* | aa^* = f(a^*a) \rangle.
\]
Representations of the relation \( aa^* = f(a^*a) \) for a measurable real-valued function \( f \) have been studied in detail in [OS] by other means. From the very beginning this important example gave us intuition for developing our theory.

By Lemma [3] the *-algebra \( \mathcal{A} \) is \( \mathbb{Z} \)-graded with grading determined by \( a \in \mathcal{A}_1 \) and \( a^* \in \mathcal{A}_{-1} \). From the definition of \( \mathcal{A} \) it follows that every element of \( \mathcal{A} \) is a linear combination of elements
\[
a^m, \ m \geq 0; \ a^{*k}, \ k > 0; \ a^{*k_1}a^{m_1} \cdots a^{*k_r}a^{m_r}, \ r \geq 1, \ k_1 > 0, \ m_r > 0.
\]
This implies that \( \mathcal{A}_n \) is the linear span of elements
\[
a^{*k_1}a^{m_1} \cdots a^{*k_r}a^{m_r}, \ r \geq 1, \ k_1 \geq 0, \ m_r \geq 0, \sum m_j - \sum k_i = n.
\]
From the defining relation \( aa^* = f(a^*a) \) we easily derive that
\[
(66) \quad ap(a^*a) = p(f(a^*a))a, \ p(a^*a)a^* = a^*p(f(a^*a)) \quad \text{for} \quad p \in \mathbb{C}[t].
\]

Lemma 25. The *-algebra \( \mathcal{B} \) is commutative and spanned by the Hermitian elements
\[
a^{*k_1}a^{m_1} \cdots a^{*k_r}a^{m_r}, \ r \geq 1, \ k_1 > 0, \ m_r > 0, \sum k_i = \sum m_j.
\]
Proof. For \( k \in \mathbb{N} \), let \( \mathcal{B}_k \) be the subalgebra of \( \mathcal{B} \) generated by words \( w \) in \( a^* \) and \( a \) satisfying equation (67) and of length \( |w| \) less or equal to \( 2k \).

We first prove by induction on \( k \) that the algebra \( \mathcal{B}_k \) is generated by words \( w, \ |w| \leq 2k \), of the form \( a^*Q \) for some word \( Q \). For \( k = 1 \) the assertion holds, since \( \mathcal{B}_1 \) is generated by the element \( a^*a \). Suppose that the assertion is valid for \( k > 1 \). Let \( w \in \mathcal{B}, \ |w| \leq 2k + 2, k > 1 \). If \( w = a^*Q \) for some word \( Q \), then the induction proof is complete. Let \( w = a^*a^*P, \ r > 0, \) for some word \( P \). Using (66) we get
\[
w = a^*a^*P = a^{r-1}f(a^*a)P = a^{r-2}f(f(a^*a))aP = \cdots = f^r(a^*a)a^{r-1}P.
\]
The word \( a^{r-1}P \) belongs to the algebra \( \mathcal{B}_{k-1} \) and the element \( f^r(a^*a) \) belongs to \( \mathcal{B}_1 \). It follows that \( w \in \mathcal{B}_{k-1} \) and the induction hypothesis applies. This completes our first induction proof.

A second similar induction proof shows that \( \mathcal{B}_k, \ k \geq 1, \) is generated by words \( w, \ |w| \leq 2k \), of the form \( a^*Qa \) for some word \( Q \).

We now prove by induction on \( k \) that \( \mathcal{B} \) is commutative. The algebra \( \mathcal{B}_1 \) is generated by the single element \( a^*a \), so it is commutative. Suppose that \( \mathcal{B}_k, \ k \geq 1, \) is commutative. Let \( w_1 \) and \( w_2 \) be words of length between \( 2k \) and \( 2k + 2 \). Then, it is enough to consider the case when the words \( w_i \) have the form \( a^*P_i^a, \ i = 1, 2 \), for some words \( P_i \). Remembering that \( aa^* \in \mathcal{B}_1 \subseteq \mathcal{B}_k \) and using the induction hypothesis we compute
\[
w_1w_2 = a^*P_1aa^*P_2a = a^*aa^*P_1P_2a = a^*aP_2aa^*P_1a = a^*P_2aa^*P_1a = w_2w_1.
\]
Thus, \( \mathcal{B}_{k+1} \) is commutative.

Remark. The algebra \( \mathcal{B} \) is in general rather "large" when the polynomial \( f \) is not linear. We shall see this from the description of the set \( \widehat{\mathcal{B}^+} \subseteq \widehat{\mathcal{B}} \) given below.

The following Proposition allows us to use the theory developed in the Section [6]

Proposition 37. The \( \mathbb{Z} \)-grading of the algebra \( \mathcal{A} \) introduced above satisfies condition (18).
We now describe the set $\hat{B}^+$, the partial action of $\mathbb{Z}$ on it and the representations associated with orbits of this partial action.

Let $\chi \in \hat{B}^+$ be fixed and let $\pi$ be the induced representation $\text{Ind}\chi$. Let $h_k$ denote the vector $[a^k \otimes 1] \in \mathcal{H}_\pi$ for all $k \in \mathbb{Z}$. We always put $a^{-k} := a^k$ for $k \in \mathbb{N}$ and $a^0 := 1_A$.

If $h_k = 0$ for some $k > 0$, then for any $c_k \in \mathcal{A}_k$ we have $[c_k \otimes 1] = 0$. Indeed, by (68) there exists $b \in \mathcal{B}$ such that $c_k = b a^k$ which implies $[c_k \otimes 1] = [b a^k \otimes 1] = \pi(b)[a^k \otimes 1] = 0$. Moreover, for all $m > 0$ we have $h_{k+m} = \pi(a^m) h_k = 0$.

Analogously, if $h_{-k} = 0$ for some $k > 0$, then for any $c_{-k} \in \mathcal{A}_{-k}$ we have $[c_{-k} \otimes 1] = 0$. Indeed, by (68) there exists $b \in \mathcal{B}$ such that $c_{-k} = a^k b$. It implies $[c_{-k} \otimes 1] = [a^k b \otimes 1] = [a^k \otimes \chi(b)] = \chi(b)[a^k \otimes 1] = 0$. For all $m > 0$ we have $h_{-k-m} = \pi(a^m) h_{-k} = 0$.

Summarizing the above considerations we conclude that there exist $K, M \in \mathbb{N} \cup \{\pm \infty\}$, $K < 0 < M$ such that $h_k \neq 0$ if and only if $K < k < M$. All $h_k$ are pairwise orthogonal and Proposition 16 implies that the vectors $h_k$ span $\mathcal{H}_\pi$. Using Proposition 16 we also conclude that $\pi(a) h_k = \mu_k h_{k+1}$ for some $\mu_k \in \mathbb{C}$. We choose numbers $\nu_k \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{Z}$, $\nu_0 = 1$, such that the vectors $e_k := \nu_k h_k$, $k \in \mathbb{Z}$ are of the norm 1 if $h_k \neq 0$ and

$$
\pi(a) e_k = \lambda_k e_{k+1}, \quad \pi(a^* e_k = \lambda_{k-1} e_{k-1}
$$

for some $\lambda_k \geq 0$, $k \in \mathbb{Z}$.

Thus the vectors $e_k$, $K < k < M$, form an orthonormal base of $\mathcal{H}_\pi$. Furthermore, $\lambda_k > 0$ for $K < k < M - 1$ and relation (69) together with the defining relation $aa^* = f(a^* a)$ imply $\lambda_k^2 = f(\lambda_k^2)$ for all $K < k < M$. In the case when $K$ resp. $M$ is finite we have also $f(\lambda_{K+1}^2) = \lambda_K^2 = 0$, resp. $\lambda_{M-1} = 0$, $f(0) = \lambda_{M-2}^2$.

For the fixed character $\chi \in \hat{B}^+$ we consider the possible cases depending on $K$ and $M$.

1. Let $K < 0$ and $M > 0$ be finite, so that $\lambda_{k-1}^2 = f(\lambda_k^2)$ for $K < k < M$, $f(\lambda_{K+1}^2) = 0$, $f(0) = \lambda_{M-2}^2$. Since $\chi(c_k^* c_k) = ||[c_k \otimes 1]||^2 = 0$ for all $c_k \in \mathcal{A}_k$, $k \leq K$, $k \geq M$, the character $\chi^k$ is defined only for $K < k < M$. It implies that the stabilizer of $\chi$ is trivial. Thus $\pi$ is an irreducible finite-dimensional representation. Using (69) we get

$$
\pi(a) e_k = \lambda_k e_{k+1}, \quad \text{for } K < k < M - 1, \quad \pi(a) e_{M-1} = 0,
$$

$$
\pi(a^* e_k = \lambda_{k-1} e_{k-1} \quad \text{for } K + 1 < k < M, \quad \pi(a^* e_{K+1} = 0.
$$

According to the terminology of [OS], $\pi$ is the Fock representation.

2. Let only $M > 0$ be finite, so that $\lambda_{k-1}^2 = f(\lambda_k^2)$ for all $k < M$ and $f(0) = \lambda_{M-2}^2$. As in the previous case we have that the stabilizer of $\chi$ is trivial. Thus $\pi$ is an irreducible infinite-dimensional representation. By (69) we have

$$
\pi(a) e_k = \lambda_k e_{k+1}, \quad \text{for } k < M - 1, \quad \pi(a) e_{M-1} = 0,
$$

$$
\pi(a^* e_k = \lambda_{k-1} e_{k-1} \quad \text{for } k < M.
$$

According to the terminology of [OS], $\pi$ is called anti-Fock representation.

3. Let only $K < 0$ be finite, so that $\lambda_{k-1}^2 = f(\lambda_k^2)$ for $K < k$, $f(\lambda_{K+1}^2) = 0$. As in the case 1, the stabilizer of $\chi$ is trivial. Thus $\pi$ is an irreducible infinite-dimensional representation. From (69) we obtain

$$
\pi(a) e_k = \lambda_k e_{k+1}, \quad \text{for } K < k, \quad \pi(a^* e_k = \lambda_{k-1} e_{k-1} \quad \text{for } K + 1 < k, \quad \pi(a^* e_{K+1} = 0.
$$

In the terminology of [OS], $\pi$ is called anti-Fock representation.

4. Let both $K$ and $M$ be infinite, so that $\lambda_{k-1}^2 = f(\lambda_k^2)$ for $k \in \mathbb{Z}$. Recall that a sequence $\{\lambda_k\}_{k \in \mathbb{Z}}$ is called periodic if there exists $m \in \mathbb{N}$, such that $\lambda_k = \lambda_{k+m}$ for all $k \in \mathbb{Z}$. The smallest such $m$ is called period of the sequence $\{\lambda_k\}_{k \in \mathbb{Z}}$. We consider two subcases.

4.1. Let $\{\lambda_k^2\}_{k \in \mathbb{Z}}$ be not periodic. Then, in particular all numbers $\lambda_k$, $k \in \mathbb{Z}$, are pairwise different. From (69) we have $\pi(a^* a) e_k = \lambda_k^2 e_k$ and Proposition 16 (ii) implies that $\chi^k(a^* a) = \lambda_k^2$. 

Since \( \{\lambda_k^2\}_{k \in \mathbb{Z}} \) is not periodic, all characters \( \chi^k \), \( k \in \mathbb{Z} \), are different. Thus, the stabilizer of \( \chi \) is trivial and representation \( \pi \) defined by (69) is irreducible.

4.2. Let \( \{\lambda_k^2\}_{k \in \mathbb{Z}} \) be periodic with a period \( m \in \mathbb{N} \). Repeating the arguments from the previous case it follows that the stabilizer \( H \) of \( \chi \) is equal to \( m\mathbb{Z} \subset \mathbb{Z} \). Let \( \mathcal{H}_{\pi,m} \) be the Hilbert subspace spanned by the vectors \( e_{rm} \), \( r \in \mathbb{Z} \). Let \( p \in \mathbb{N} \) and \( c_{pm} \in \mathcal{A}_{pm} \). Then (68) implies that \( c_{pm} = b_1 a^{pm} \) for some \( b_1 \in \mathcal{B} \). Using (69) and Proposition 16 (ii) we get

\[
\pi(c_{pm}) e_{rm} = \chi^{pm}(b_1)(\lambda_0 \lambda_1 \ldots \lambda_{m-1})^p e_{(r+p)m} = \chi(b_1)(\lambda_0 \lambda_1 \ldots \lambda_{m-1})^p e_{(r+p)m}.
\]

Thus \( \pi(c_{pm}) \) acts as a scalar multiple of the bilinear shift on \( \mathcal{H}_{\pi,m} \). This implies that

\[
(70) \quad \widetilde{\chi}(b_1 a^{pm}) := \chi(b_1)(\lambda_0 \lambda_1 \ldots \lambda_{m-1})^p, \; p \in \mathbb{N},
\]
defines a character on the algebra \( \mathcal{A}_H \). The restriction of \( \widetilde{\chi} \) to \( \mathcal{B} \) coincides with \( \chi \). Therefore, by Proposition 27 the Mackey obstruction of \( \chi \) is trivial. We denote by \( \zeta_z \), \( z \in \mathbb{T} \), the character of the group \( H = m\mathbb{Z} \) defined by \( \zeta_z(m) = z \). Then, using (68) and (70), we see that all representations \( \rho_z \), \( z \in \mathbb{T} \), of \( \mathcal{A}_H \) satisfy condition (38). These representations are one-dimensional, that is, they are characters. For \( c_{pm} = ba^{pm} \), \( p \in \mathbb{N} \), \( b \in \mathcal{B} \), we have

\[
\rho_z(c_{pm}) = \chi(c_{pm})^{1/2} \zeta_z(pm) = \widetilde{\chi}(c_{pm})^{1/2} \chi(c_{pm})^{1/2} z^p = \chi(b^* b)^{1/2} (\lambda_0 \lambda_1 \ldots \lambda_{m-1})^p,
\]

where \( z \in \mathbb{T} \).

We now compute the representations induced from \( \rho_z \), \( z \in \mathbb{T} \). Let \( \pi_z \) denotes the induced representation \( \text{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}} \rho_z \) on the space \( \mathcal{H}_z \). One easily verifies that the vectors

\[
f_k = \chi(a^{k} a^k)^{-1/2}[a^k \otimes 1], \quad k = 0, \ldots, m - 1,
\]
form an orthogonal base of the space \( \mathcal{H}_z \). We calculate the action of \( \pi(a) \) on the base vectors \( f_k \). Using Proposition 16 (ii) and formulas (69) we find that \( \chi(a^{k} a^k) = \lambda_0^2 \lambda_1^2 \ldots \lambda_{k-1}^2 \), \( k \in \mathbb{N} \). Take \( r = 0, \ldots, m - 2 \). Then we have

\[
\pi_z(a) f_r = \frac{\chi(a^{(r+1)k} a^{r+1})^{1/2}}{\chi(a^{rk} a^r)^{1/2}} f_{r+1} = \lambda_r f_{r+1}.
\]

For \( f_{m-1} \) we get

\[
\pi_z(a) f_{m-1} = \chi(a^{(m-1)k} a^{m-1})^{-1/2}[a^m \otimes 1] = \chi(a^{(m-1)k} a^{m-1})^{-1/2}[1_A \otimes \rho_z(a^m)] = \chi(a^{(m-1)k} a^{m-1})^{-1/2} \widetilde{\chi}(a^m) [1_A \otimes 1] = z \lambda_{m-1} f_0.
\]

Now suppose we are given a sequence \( \lambda_k > 0 \), \( K < k < M - 1 \), where \( -\infty \leq K < 0 < M \leq \infty \). Suppose also that \( f(\lambda_{K+1}^2) = 0 \) resp. \( f(0) = \lambda_{M-2}^2 \) in the case when \( K \) resp. \( M \) is finite. We call such a sequence a nonnegative orbit of the dynamical system \((f, [0, +\infty))\). Then (69) defines a *-representation \( \pi \) of \( \mathcal{A} \) and the restriction of \( \text{Res}_{\mathcal{E}_0} \pi \) to \( \mathbb{C} \cdot e_0 \) gives a character \( \chi \in \hat{\mathcal{B}}^+ \). Let us describe these characters \( \chi \) in the case 4. explicitly. Take an element \( a^{k_1} a^{m_1} \ldots a^{k_r} a^{m_r} \in \mathcal{B}, \; r \geq 1, \; k_1 > 0, \; m_r > 0, \; \sum k_i = \sum m_j \). Using formulas (69) we obtain

\[
\chi(a^{k_1} a^{m_1} \ldots a^{k_r} a^{m_r}) = \prod_{i=0}^{m_r-1} \lambda_i \prod_{i=1}^{k_1} \lambda_{m_r-i} \ldots \prod_{i=1}^{k_1} \lambda_{m_r-k_r+m_r-1-\ldots+m_1-i}.
\]

We summarize the above discussion in the following

**Proposition 38.** The equations (69) give a one-to-one correspondence between nonnegative orbits of the dynamical system \((f, [0, +\infty))\) and orbits of the partial action of \( \mathbb{Z} \) on \( \hat{\mathcal{B}}^+ \). A representation \( \pi \) defined by (69) is reducible if and only if the sequence \( \lambda_k \) is periodic and \( \lambda_k > 0 \) for all \( k \in \mathbb{Z} \).

Finally, we consider the problem of associating irreducible well-behaved representations of \( \mathcal{A} \) with orbits in \( \hat{\mathcal{B}}^+ \) (cf. also [OS]).
Proposition 39. Assume that the function \( f \) is one-to-one and there exists a measurable set \( \Gamma \subseteq [0, +\infty) \) containing precisely one point from each nonnegative orbit of the dynamical system \( (f, [0, +\infty)) \). Then every irreducible well-behaved representation of \( \mathcal{A} \) is associated with an orbit in \( \hat{\mathcal{B}}^+ \).

**Sketch of proof.** Let \( \pi \) be an irreducible well-behaved representation of \( \mathcal{A} \). Then \( \pi(a^*a) \) is essentially self-adjoint. Using Proposition 33 in [OS] we conclude that the spectral measure of \( \pi(a^*a) \) is ergodic with respect to \( f \). Applying Proposition 34 in [OS] it follows that the spectral measure of \( \pi(a^*a) \) is concentrated on a single orbit of the dynamical system \( (f, [0, +\infty)) \). \( \square \)

For the case, when \( f \) is not bijective, we refer to Theorem 15 in [OS].

11. FURTHER EXAMPLES

In this section we mention and briefly discuss some other classes of examples, where the theory developed in the previous sections can be applied.

**Example 17.** *(Quantum disk algebra.)* Suppose that \( 0 \leq \mu \leq 1, 0 \leq q \leq 1 \), and \( (\mu, q) \neq (0, 1) \). The two-parameter unit quantum disk \(*\)-algebra \( \mathcal{A} \) has generators \( a \) and \( a^* \) and the defining relation
\[
qaa^* - a^*a = q - 1 + \mu(1 - aa^*)(1 - a^*a).
\]
Then \( \mathcal{A} \) is \( \mathbb{Z} \)-graded such that \( a \in \mathcal{A}_1 \) and \( a^* \in \mathcal{A}_{-1} \). As in the case of the dynamical systems in the previous section one shows that \( \mathcal{B} = \mathcal{A}_0 \) is commutative and condition \( (\text{IS}) \) is satisfied. There is a one-to-one correspondence between orbits in \( \hat{\mathcal{B}}^+ \) and orbits of the dynamical system \( (f, [0, +\infty)) \) where
\[
f(\lambda) = \frac{(q + \mu)\lambda + 1 - q - \mu}{\mu\lambda + 1 - \mu}.
\]

For a more detailed analysis of this \(*\)-algebra see [KL] and [OS], p.101.

**Example 18.** *(Podles’ quantum spheres.)* Let \( q \in (0, \infty) \). For \( r \in [0, \infty) \), \( \mathcal{O}(S^2_{qr}) \) is the unital \(*\)-algebra with generators \( A = A^*, B, B^* \) and defining relations (see [Pd] or [KS], 4.5)
\[
AB = q^{-2}BA, \quad AB^* = q^2B^*A, \quad B^*B = A - A^2 + r, \quad BB^* = q^2A - q^4A^2 + r.
\]
For \( r = \infty \), the defining relations of \( \mathcal{O}(S^2_{q,\infty}) \) are
\[
AB = q^{-2}BA, \quad AB^* = q^2B^*A, \quad B^*B = -A^2 + 1, \quad BB^* = -q^4A^2 + 1.
\]
In both cases \( \mathcal{A} = \mathcal{O}(S^2_{qr}) \) is \( \mathbb{Z} \)-graded such that \( B \in \mathcal{A}_1, B^* \in \mathcal{A}_{-1} \) and \( A \in \mathcal{A}_0 \). One can check that \( \mathcal{B} = \mathcal{A}_0 \) is commutative and condition \( (\text{IS}) \) is fulfilled. It follows immediately from the defining relations that all \(*\)-representations of \( \mathcal{A} \) are bounded.

**Example 19.** *(Compact quantum group algebras)* The simplest example is the quantum group \( SU_q(2), q \in \mathbb{R} \). The corresponding \(*\)-algebra has two generators \( a \) and \( c \) and defining relations
\[
ac = qca, \quad c^*c = cc^*, \quad aa^* + q^2cc^* = 1, \quad a^*a + c^*c = 1.
\]
Then \( \mathcal{A} \) is \( \mathbb{Z}^2 \)-graded such that \( a \in \mathcal{A}_{g_1}, c \in \mathcal{A}_{g_2} \) where \( g_1, g_2 \) are generators of the group \( \mathbb{Z}^2 \).

**Example 20.** *(Deformations of CAR algebra)* Let \( q \in (0, 1) \) be fixed. The twisted canonical anti-commutation relations (briefly, TCAR) \(*\)-algebra \( \mathcal{A} = \mathcal{A}_q \) is generated by elements \( a_i, a_i^*, i = 1, \ldots, d \), with defining relations (see [P])
\[
a_i^*a_i = 1 - a_i a_i^* - (1 - q^2) \sum_{j<i} a_j a_j^*, \quad i = 1, \ldots, d,
\]
\[
a_i a_j = -qa_j a_i^*, \quad a_i a_j = -qa_j a_i, \quad i < j, a_i^2 = 0, \quad i = 1, \ldots, d.
\]
For \( q = 1 \) we get the ”usual” CAR algebra. For all \( q \in (0, 1) \), \( \mathcal{A} \) is \( (\mathbb{Z}/2\mathbb{Z})^d \)-graded such that \( a_k, a_k^* \in \mathcal{A}_{g_k} \) where \( g_1, \ldots, g_d \) are generators of \( (\mathbb{Z}/2\mathbb{Z})^d \), the subalgebra \( \mathcal{B} = \mathcal{A}_0 \) is commutative and condition \( (\text{IS}) \) is satisfied.
The Wick analogue of TCAR (denoted as WTCAR) was studied in [JŚW, PT, PST]. The WTCAR $\ast$-algebra $\mathcal{A}$ is obtained from TCAR by omitting the relations between $a_i$ and $a_j$. Hence $\mathcal{A}$ is $\mathbb{Z}^d$-graded such that $a_k \in \mathcal{A}_{g_k}$ where $g_1, \ldots, g_d$ are generators of $\mathbb{Z}^d$. In this case the $\ast$-subalgebra $\mathcal{B} = \mathcal{A}_0$ is not commutative. However, it was shown in [JŚW, PT] that in any irreducible representation of WTCAR the relations
\[
a_j a_i = -qa_i a_j, \quad i < j, \quad a_i^2 = 0, \quad i = 1, \ldots, d-1,
\]
hold automatically. Then our theory applies to the quotient of WTCAR $\ast$-algebra by the latter relations.

Example 21. (Quantum algebras $\mathcal{U}_q(su(2))$ and $\mathcal{U}_q(su(1,1)))$ For $q \in \mathbb{R}$, $q^2 \neq 1$, the $q$-deformed enveloping algebra $\mathcal{U}_q(sl(2))$ is the complex unital (associative) algebra with generators $E, F, K, K^{-1}$ and defining relations
\[
KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.
\]
The involutions defining the $\ast$-algebras $\mathcal{U}_q(su(2))$ and $\mathcal{U}_q(su(1,1))$ are given by the formulas
\[
E^* = F, \quad F^* = E, \quad K^* = K, \quad K^{-1*} = K^{-1},\]
\[
E^* = -F, \quad F^* = -E, \quad K^* = K, \quad K^{-1*} = K^{-1},
\]
respectively. Let $\mathcal{A}$ be one of the $\ast$-algebras $\mathcal{U}_q(su(2))$ or $\mathcal{U}_q(su(1,1))$. Then $\mathcal{A}$ is $\mathbb{Z}$-graded with grading determined by $E \in \mathcal{A}_1$, $F \in \mathcal{A}_{-1}$, and $K, K^{-1} \in \mathcal{A}_0$, the $\ast$-subalgebra $\mathcal{B} = \mathcal{A}_0$ is commutative, and condition (18) is valid. The Mackey analysis for $\mathcal{A}$ is similar to that of $\mathcal{U}(su(2))$ and $\mathcal{U}(su(1,1))$.

The algebra $\mathcal{U}_q(sl(2))$ was introduced in [KR], see e.g. [KS], 3.1. Representations of $\mathcal{U}_q(su(2))$ and $\mathcal{U}_q(su(1,1))$ have been investigated in [VS] and [BK], respectively.

Example 22. (CAR algebras). Let $\mathcal{A}$ be the direct limit of matrix $\ast$-algebras $M_{2k} (\mathbb{C})$, $k \in \mathbb{N}$, where the embedding $M_{2k} (\mathbb{C}) \hookrightarrow M_{2k+1} (\mathbb{C})$ is given by the canonical injection $M_{2k} (\mathbb{C}) \otimes I_2 \hookrightarrow M_{2k+1} (\mathbb{C})$. Here $I_2 \in M_2 (\mathbb{C})$ is the identity matrix. The representation theory of $\mathcal{A}$ was studied in [GW], see also [Sa] and [KR].

Each matrix algebra $M_n (\mathbb{C})$ has a natural $\mathbb{Z}$-grading such that each matrix unit $e_{ij}$ belongs to the $(i-j)$-component. Since the embeddings $M_{2k} (\mathbb{C}) \hookrightarrow M_{2k+1} (\mathbb{C})$ respect this grading, $\mathcal{A}$ is also $\mathbb{Z}$-graded. One checks that condition (18) is valid for $M_{2k} (\mathbb{C})$ which implies that the $\mathbb{Z}$-grading on $\mathcal{A}$ also satisfies (18). The $\ast$-subalgebra $\mathcal{B} = \mathcal{A}_0$ is the direct limit of commutative algebras $\mathbb{C}^{2k}$. It can be considered as a (dense) $\ast$-subalgebra of the $\ast$-algebra of all continuous functions on the Cantor set. The conditional expectation defined by the $\mathbb{Z}$-grading is strong, so $\mathcal{B}^+$ coincides with $\mathcal{B}$ which is equal to the Cantor set. All representations of $\mathcal{A}$ are bounded. The partial action of $\mathbb{Z}$ on $\mathcal{B}^+$ has trivial stabilizers. All irreducible representations associated with orbits in $\mathcal{B}^+$ are direct limits of representations. In this case the assumptions of Proposition 29 are not satisfied and there exist irreducible representations of $\mathcal{A}$ arising from ergodic measures under the partial action of $\mathbb{Z}$ on $\mathcal{B}^+$ which are not supported on single orbits.

**APPENDIX**

The main result of this Appendix (Theorem 7) is related to condition (i) of Definition 11 but it is also of interest in itself. Its proof is based on the spectral theorem for countable families of commuting self-adjoint operators, see [Sa], Theorem 1. We equip $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \ldots$ with the product topology and denote by $\mathcal{B}(\mathbb{R}^\infty)$ the Borel structure on $\mathbb{R}^\infty$ induced by this topology.

**Theorem 6.** For each family $A_k$, $k \in \mathbb{N}$, of strongly commuting self-adjoint operators there exists a unique resolution of the identity $E$ on the Borel space $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ such that
\[
A_k = \int \lambda_k dE(\lambda_1, \lambda_2, \ldots) \text{ for all } k \in \mathbb{N}.
\]
In the notation of Theorem 6 the joint spectrum of the family \( A_k, \ k \in \mathbb{N} \), is the intersection of all closed subsets \( X \) of \( \mathbb{R}^\infty \) such that \( E(X) = E(\mathbb{R}^\infty) \).

Let \( B \) be a commutative unital *-algebra. As in Section 7 we equip the set \( \hat{B} \) of all characters of \( B \) with the weakest topology for which all functions \( f_b, \ b \in B, \) are continuous, where \( f_b \) is defined by \( f_b(\chi) = \chi(b) \) for \( \chi \in \hat{B} \). Clearly, if \( B \) is generated by elements \( b_n, \ n \in \mathbb{N}, \) then this topology coincides with the weakest topology for which all functions \( f_{b_n}, \ n \in \mathbb{N}, \) are continuous.

**Theorem 7.** Suppose that \( B \) is a countably generated commutative unital *-algebra. We equip \( \hat{B} \) with the Borel structure induced by the weak topology. Let \( C \) be a quadratic module of \( B \) and let \( \hat{B}^+ \) denote the set of all characters \( \chi \in \hat{B} \) which are nonnegative on \( C \). If \( \pi \) is an integrable representation of \( B \), then:

(i) There exists a unique spectral measure \( E_\pi \) on \( \hat{B} \) such that

\[
\pi(b) = \int f_b(\lambda) \, dE_\pi(\lambda) \text{ for all } b \in B.
\]

(ii) Assume in addition that \( \langle \pi(c)\varphi, \varphi \rangle \geq 0 \) for all \( c \in C \) and \( \varphi \in \mathcal{D}(\pi) \). Then the spectral measure \( E_\pi \) is supported on \( \hat{B}^+ \) which is a closed subset of \( \hat{B} \).

**Proof.** (i) First we fix a sequence of self-adjoint generators \( b_k, \ k \in \mathbb{N}, \) of the *-algebra \( B \) and consider \( \hat{B} \) as a subset of \( \mathbb{R}^\infty \) by identifying

\[
\hat{B} \ni \chi \longleftrightarrow (\chi(b_1), \chi(b_2), \chi(b_3), \ldots) \in \mathbb{R}^\infty.
\]

We prove that \( \hat{B} \) is closed in \( \mathbb{R}^\infty \), hence Borel. Let \( \chi_n = (\chi_n(b_1), \chi_n(b_2), \ldots) \in \hat{B}, \ n \in \mathbb{N} \) be a sequence of characters converging to \( \chi \in \mathbb{R}^\infty \) in the product topology. We claim that there is a character \( \chi \) on \( B \) such that \( \chi(b_k) := \lim_{n \to \infty} \chi_n(b_k) \). Indeed, let \( m \in \mathbb{N} \) and \( p \in \mathbb{C}[t_1, \ldots, t_m] \) be a polynomial such that \( p(b_1, \ldots, b_m) = 0 \). Since

\[
p(\chi_n(b_1), \ldots, \chi_n(b_m)) = \chi_n(p(b_1, \ldots, b_m)) = 0,
\]

we conclude that

\[
p(\chi(b_1), \ldots, \chi(b_m)) = p(\lim_{n \to \infty} \chi_n(b_1), \ldots, \lim_{n \to \infty} \chi_n(b_m)) = 0
\]

for all \( n \in \mathbb{N} \). Therefore \( \chi \in \mathbb{R}^\infty \) defines a character on \( B \), i.e. \( \chi \in \hat{B} \).

A sequence \( \chi_n \in \hat{B} \) converges to \( \chi \in \hat{B} \) if and only if \( \chi_n(b_k) = f_{b_k}(\chi_n) \) converges to \( \chi(b_k) = f_{b_k}(\chi) \) for every fixed \( k \) as \( n \to \infty \). Since the elements \( b_k, \ k \in \mathbb{N}, \) generate \( B \), it follows that the topology on \( \hat{B} \) induced from \( \mathbb{R}^\infty \) coincides with the weak topology. In particular, the Borel structure on \( \hat{B} \) coincides with the one induced from \( \mathbb{R}^\infty \).

Since \( \pi \) is integrable, the operators \( \pi(b_k), \ k \in \mathbb{N}, \) are self-adjoint and pairwise strongly commuting ([SI], Corollary 9.1.14). Therefore, by Theorem 6 there exist a spectral measure \( E_\pi \) on the set \( \mathbb{R}^\infty \) such that

\[
\pi(b_k) = \int \lambda_k dE_\pi(\lambda_1, \lambda_2, \ldots).
\]

for all \( k \in \mathbb{N} \). For every polynomial \( p \in \mathbb{R}[t_1, \ldots, t_m] \) the operator \( p(\pi(b_1), \ldots, \pi(b_m)) \) is essentially self-adjoint and from basic properties of spectral integrals we obtain

\[
(71) \quad p(\pi(b_1), \ldots, \pi(b_m)) = \int p(\lambda_1, \ldots, \lambda_m) dE_\pi(\lambda_1, \lambda_2, \ldots).
\]

Next we show that the spectral measure \( E_\pi \) is supported on \( \hat{B} \subseteq \mathbb{R}^\infty \), or equivalently, that the joint spectrum \( \sigma(\pi(b_1), \pi(b_2), \ldots) \) of the family \( \pi(b_k), \ k \in \mathbb{N} \), is contained in \( \hat{B} \). Let \( x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty \) be a point in \( \sigma(\pi(b_1), \pi(b_2), \ldots) \). Again, let \( m \in \mathbb{N} \) and \( p_0 \in \mathbb{R}[t_1, \ldots, t_m] \) be such that \( p_0(b_1, \ldots, b_m) = 0 \). Then we obtain

\[
\pi(p_0(b_1, \ldots, b_m)) = 0.
\]
Assume to the contrary that \( p_0(x_1, x_2, \ldots, x_m) \neq 0 \). Then for every open neighborhood \( O(x) \) we have \( E_\pi(O(x)) \neq 0 \). Using (71) we get
\[
0 = \pi(p_0(b_1, \ldots, b_m)) = p_0(\pi(b_1), \ldots, \pi(b_m)) = \int p_0(\lambda_1, \ldots, \lambda_m) dE_\pi(\lambda_1, \lambda_2, \ldots) \neq 0,
\]
which is a contradiction. That is, we have \( p_0(x_1, x_2, \ldots, x_m) = 0 \). Thus we have shown that \( \chi(b_k) := x_k \) defines a character and \( E_\pi \) is supported on \( \hat{\mathcal{B}} \). The uniqueness of the spectral measure \( E_\pi \) follows at once from the corresponding assertion in Theorem 6.

(ii) Since \( \hat{\mathcal{B}} \) is a closed subset of the separable space \( \mathbb{R}^\infty \), \( \hat{\mathcal{B}} \) is also separable. Similar arguments as used in the proof of (i), show that \( \hat{\mathcal{B}}^+ \) is closed in \( \hat{\mathcal{B}} \).

Assume to the contrary that \( E_\pi(\hat{\mathcal{B}} \setminus \hat{\mathcal{B}}^+) \neq 0 \). Since \( \hat{\mathcal{B}} \) is separable and \( \hat{\mathcal{B}}^+ \) is a closed subset of \( \hat{\mathcal{B}} \), there exists a countable dense subset \( \{ \chi_i \}_{i \in \mathbb{N}} \) of \( \hat{\mathcal{B}} \setminus \hat{\mathcal{B}}^+ \). For every \( \chi_i \) there exists an element \( c_i \) of \( \mathcal{C} \) such that \( \chi_i(c_i) < 0 \). Since \( \{ \chi_i \}_{i \in \mathbb{N}} \) is dense in \( \hat{\mathcal{B}} \setminus \hat{\mathcal{B}}^+ \), the open sets \( f_{c_i}^{-1}((-\infty, 0)) \) cover \( \hat{\mathcal{B}} \setminus \hat{\mathcal{B}}^+ \). From the latter it follows that there exists a \( k \in \mathbb{N} \) such that \( E_\pi(f_{c_k}^{-1}((-\infty, 0))) \neq 0 \). Hence there exists a vector \( \varphi \in \text{Ran}E_\pi(f_{c_k}^{-1}((-\infty, 0))) \cap \mathcal{D}(\pi) \) such that \( \langle \pi(c_k)\varphi, \varphi \rangle < 0 \) which contradicts our assumption. \[ \square \]

**Definition 15.** If \( \mathcal{B}, \pi \) and \( E_\pi \) are as in the previous theorem, we shall say that the integrable representation \( \pi \) and the spectral measure \( E_\pi \) are **associated** with each other.

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