Abstract

A family of harmonic superspaces associated with four-dimensional Minkowski spacetime is described. Applications are given to free massless supermultiplets, invariant integrals and super Yang-Mills. The generalisation to curved spacetimes is given with emphasis on conformal supergravities.
1 Introduction

Harmonic superspace was first introduced by Gikos in 1984 [3] and has proved to be a useful tool in the analysis of four-dimensional supersymmetric field theories, in particular $N = 2$ [3, 4] super Yang-Mills, the $N = 2$ non-linear sigma model [7, 8] and $N = 2$ supergravity [9, 6]. The harmonic formalism is closely related to the projective superspace formalism [18, 19] and to twistor theory [22].

In this article we develop the theory further by studying a family of superspaces in four dimensions which includes those previously studied as special cases. We call this family of superspaces $(N, p, q)$ harmonic superspaces (where $p + q \leq N$), and the subset mentioned above corresponds to $(p, q) = (1, 1)$, although $(3, 2, 1)$ superspace was also discussed in [5]. We shall introduce these superspaces firstly in the complex setting where they can be viewed as coset spaces of the complex superconformal group $SL(4|N)$; in fact the complex superspaces are examples of flag supermanifolds, studied in particular by Manin [20]. As in twistor theory [26], one can fit together sets of three such superspaces into double fibrations and it is in this way that they are exploited in, for example, Yang-Mills theory. In the real setting, the double fibration can be replaced by a single fibration over super Minkowski space and real $(N, p, q)$ harmonic superspaces have the property that they are CR supermanifolds, a CR structure being a generalisation of a complex structure, a property observed in the $(1, 1)$ case in [23]. Viewed in this light, harmonic analyticity is related to CR-analyticity. In a recent paper, an analogous family of superspaces for spacetimes of dimension less than four was introduced [14]. In section 2 we construct the $(N, p, q)$ superspaces and study their properties.

From the point of view of applications to field theory, it seems that the $(p, q) = (1, 1)$ superspaces remain the most useful, but we shall show that there are field theoretic uses of the more general superspaces. In particular, in section 8 we show that the field strength superfields of some massless multiplets can be viewed as CR-analytic superfields on $(N, p, N - p)$ superspaces; in section 4, we show that harmonic integration is equivalent to the superaction formulae introduced in ref. [16] and rewrite the 3-loop counter-term for linearised $N = 8$ supergravity as a harmonic superspace integral. This counterterm was first constructed by Kallosh [17] in a non-manifestly symmetric way and subsequently recast as a superaction in [16]. The version
given here has the feature of being manifestly supersymmetric. In section 5 we study super Yang-Mills theory in \((N, p, q)\) harmonic superspaces. For \((p, q) > (1, 1)\), we cannot impose flatness as can be done for \((p, q) = (1, 1)\), but we show that the non-Abelian field strength superfield is itself covariantly CR-analytic for appropriate choices of \((p, q)\). Finally, in section 6 we study the conformal constraints of \(N\)-extended supergravity and show that \((N, p, q)\) superspaces also have a rôle in this context.

### 2 Harmonic Superspaces

We begin in the complex setting. The complex superconformal group is \(SL(4|N)\), the supergroup of \((4 + N) \times (4 + N)\) complex supermatrices with unit superdeterminant, (for \(N = 4\) this group is not simple). All the superspaces which have been used in supersymmetric field theory can be obtained as coset spaces of the form \(P\backslash SL(4|N)\) where \(P\) is a parabolic subsupergroup of \(SL(4|N)\). These spaces are called flag supermanifolds and this point of view has been developed at length in an article in preparation by the authors [13]. In the present context, the parabolic subgroups have block lower triangular form. For example, complex \(N\)-extended super Minkowski space \(\widetilde{\mathbb{M}}_N\) corresponds to the subgroup

\[
P_N = \left\{ \begin{array}{c|c}
\times & \times \\
\times & \\
\times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\end{array} \right\}
\]

(1)

The top left corner corresponds to Minkowski space, while the odd directions of the coset correspond to the blank entries in the off-diagonal blocks. We shall use blackboardbold letters to denote complex superspaces, and tildes to indicate spaces with compact bodies.

The harmonic superspaces we are interested in have the property that their bodies, which correspond to the diagonal part of the subgroup, are schematically of the form \(\mathbb{M}_0\) (complex Minkowski space) times an internal flag manifold, while they have the same number, \(4N\), of odd coordinates.
as super Minkowski space. We then define complex \((N,p,q)\) harmonic superspace \(\tilde{M}_N(p,q)\), \((p + q \leq N)\) as the coset space of \(SL(4|N)\) with parabolic subgroup

\[
P_N(p,q) = \begin{pmatrix}
\times & \times \\
\times & & \times \\
\times & & & \times \\
\times & & & & \times \\
\times & & & & & \times \\
& & & & & & \times \\
& & & & & & & \times \\
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& & & & & & & & & & & & & & & & & & & & & & & \times \\
& & & & & & & & & & & & & & & & & & & & & & & & \times \\
\end{pmatrix}
\]

\((2)\)

The internal space, represented by the bottom right-hand corner, is the (ordinary) flag manifold \(F_{p,N-q}(N)\), the space of flags of type \((p,N-q)\) in \(\mathbb{C}^N\). (We recall that a flag of type \((k_1 \ldots k_l)\) in \(\mathbb{C}^N\) is a set of subspaces \(V_{k_1} \subset V_{k_2} \subset \ldots V_{k_l} \subset \mathbb{C}^N\) where \(k_1 < k_2 < \ldots < k_l < N\).) For each such superspace we define an associated \((N,p,q)\) analytic superspace \(\tilde{M}_{N,A}(p,q)\)
as the coset space of $SL(4|N)$ with parabolic subgroup

\[
P_{NA}(p,q) = \left\{ \begin{array}{c|c}
\times & \times \\
\times & \times \\
\times & \times \\
\times & \times \\
\times & \times \\
\times & \times \\
\times & \times \\
\times & \times \\
\end{array} \right\}
\]

(3)

Given any simple complex Lie (super)group $G$ and parabolic sub(super) groups $P_1, P_2, P_{12} = P_1 \cap P_2$, there is a double fibration

\[
P_{12} \backslash G \\
\pi_L \downarrow \\
P_2 \backslash G \quad \iff \quad P_1 \backslash G \\
\pi_R
\]

(4)

A point $p_1 \in P_1 \backslash G$ corresponds to a subset $\pi_L \circ \pi_R^{-1}(p_1)$ of $P_2 \backslash G$, and a point $p_2 \in P_2 \backslash G$ to a subset $\pi_R \circ \pi_L^{-1}(p_2)$ of $P_1 \backslash G$. In the twistor theory context the double fibration allows one to code information about theories in the space of interest, $P_1 \backslash G$, into information on the twistor space $P_2 \backslash G$ via the correspondence space $P_{12} \backslash G$. 
For each choice of \((N, p, q)\), we have the following double fibration

\[
\begin{array}{ccc}
\tilde{M}_N(p, q) & \xrightarrow{\pi_L} & \tilde{M}_{NA}(p, q) \\
\pi_R & & \xleftarrow{\pi_L} \tilde{M}_N
\end{array}
\]  

(5)

In the case \((p, q) = (1, 1)\) it is this mathematical fact which underlies the interpretation of the constraints of super Yang Mills theory in terms of analytic superspace. In the applications, we are not interested in the compactified superspaces, and we define an open set in \(\tilde{M}_N\) to be complex super Minkowski space, \(\mathbb{M}_N\). We can do this as follows: let \(z \in \mathbb{M}_N\) (= \(\mathbb{C}^{4|4N}\)) and consider the element \(s(z)\) in \(SL(4|N)\) defined by

\[
s(z) = \begin{pmatrix}
1 & -iX^{\alpha\dot{\beta}} & -i\theta^{\alpha\dot{j}} \\
0 & 1 & 0 \\
0 & -i\varphi_i^{\beta} & 1
\end{pmatrix}
\]

where \(X^{\alpha\dot{\beta}} = x^{\alpha\dot{\beta}} - \frac{i}{2}{\theta^{\alpha\dot{i}}\varphi_i^{\dot{\beta}}}\). We can regard \(s(z)\) as a section of \(SL(4|N)\) considered as a bundle over \(P_N \setminus SL(4|N)\). Using standard homogeneous space techniques one can then find how infinitesimal \(SL(4|N)\) transformations act on \(z\) and thereby confirm that \(SL(4|N)\) is indeed the complex superconformal group. The Maurer-Cartan form on \(SL(4|N)\) pulled back to \(\mathbb{M}_N\) is

\[
ds(z)s^{-1}(z) = \begin{pmatrix}
0 & -iE^{\alpha\dot{\beta}} & -iE^{\alpha\dot{j}} \\
0 & 0 & 0 \\
0 & -iE_i^{\beta} & 0
\end{pmatrix}
\]

(7)

from which we identify the usual basis \(\{E^{\alpha\dot{\alpha}}, E^{\alpha\dot{i}}, E_i^{\dot{\alpha}}\}\) of super Minkowski space, namely:

\[
E^{\alpha\dot{\alpha}} = dx^{\alpha\dot{\alpha}} + \frac{i}{2}d\theta^{\alpha\dot{i}}\varphi_i^{\dot{\alpha}} + \frac{i}{2}d\varphi_i^{\dot{\alpha}}\theta^{\alpha\dot{i}}
\]

(8)

\[
E^{\alpha\dot{i}} = d\theta^{\alpha\dot{i}}
\]

(9)

\[
E_i^{\dot{\alpha}} = d\varphi_i^{\dot{\alpha}}
\]

(10)
We define non-compact harmonic superspace \( M_{N}(p, q) \) as \( \pi_{R}^{-1}(M_{N}) \) and the corresponding analytic superspace \( M_{NA}(p, q) \) as \( \pi_{L}(M_{N}(p, q)) \). \( M_{N}(p, q) \) is in fact simply \( M_{N} \times \mathbb{F}_{p,N-q}(N) \), so that the non-compact version of the double fibration (5) is

\[
\begin{align*}
M_{N}(p, q) &= M_{N} \times \mathbb{F}_{p,N-q}(N) \\
\pi_{L} &\quad \pi_{R} \\
M_{NA}(p, q) &\iff M_{N}
\end{align*}
\]

(11)

In the following we shall denote the internal flag manifold as simply \( \mathbb{F} \). Finally, we need to consider real super Minkowski space, \( M_{N} \), which can be defined to be the subspace of \( M_{N} \) such that \( x \) is real and \( \varphi^{\alpha}_{i} = \bar{\theta}^{\alpha}_{i} \). Correspondingly, real \( (N, p, q) \) harmonic superspace, \( M_{N}(p, q) \) is \( \pi_{R}^{-1}(M_{N}) = M_{N} \times \mathbb{F} \). However, we now find that \( \pi_{L}(M_{N}(p, q)) \) embeds \( M_{N}(p, q) \) as a real subsupermanifold of \( M_{NA}(p, q) \), the latter being essentially complex. Indeed, the compactified versions of \( M_{N} \) and \( M_{N}(p, q) \) are coset spaces of the real superconformal group \( SU(2,2|N) \), whereas there is no corresponding analogue of \( M_{NA}(p, q) \). For this reason, it is sometimes simpler, in the real context, to consider the single fibration \( M_{N}(p, q) \rightarrow M_{N} \). The rôle of the analytic space is then replaced by CR-analyticity on \( M_{N}(p, q) \) which we now explain.

Let \( M \) be a real (super) manifold of dimension \( 2n + m \), where \( n, m \in \mathbb{Z} \) for manifolds, and \( n, m \in \mathbb{Z}^{2} \) for supermanifolds. A CR structure on \( M \) is a subbundle \( K \) of the complexified tangent bundle, \( T_{c} \) of rank \( n \) such that

\[
\begin{align*}
K \cap \bar{K} &= \emptyset \\
[K, K] &\subset K
\end{align*}
\]

where the latter statement means that the commutator of any two vector fields belonging to \( K \) also belongs to \( K \), i.e. \( K \) is involutive. This notion generalises that of a complex structure which is recovered in the case \( m = 0 \). Given a function \( f \) on \( M \) we can define a CR operator \( \bar{\partial}_{K} \) by

\[
\bar{\partial}_{K} f = \pi \circ df
\]

(12)
where \( \pi \) is the projection: 1-forms on \( M \to \) sections of \( \bar{K}^* \), the dual space of \( \bar{K} \). A function \( f \) such that \( \bar{\partial}_K f = 0 \) is called CR-analytic, and the involutivity of \( K \) ensures that this is consistent. The holomorphic bundle of vectors tangent to the fibre, \( T_{\bar{F}} \), is a subbundle of \( K \), and it is convenient to write

\[
K = T_{\bar{F}} \oplus T_G
\]  

(13)

where \( T_G \) is the odd part of \( K \). The derivative \( \bar{\partial}_K \) can correspondingly be written as

\[
\bar{\partial}_K = \bar{\partial} + \bar{D}
\]  

(14)

where \( \bar{\partial} \) is essentially the usual \( \bar{\partial} \) operator, on \( F \) and \( \bar{D} \) is the odd part of \( \bar{\partial}_K \). Following Gikos we define a field \( f \) to be Grassmann analytic, or simply G-analytic, if

\[
\bar{D}f = 0
\]  

(15)

A G-analytic superfield on \( M_N(p,q) \) can be thought of as the pull-back of a function defined on \( \bar{M}_{NA}(p,q) \) to \( M_N(p,q) \).

We shall now describe the above in local coordinates. It is convenient to work on the space \( M_N \times SU(N) \), as advocated in [3], which we call \( \hat{M}_N \), instead of on \( M_N(p,q) = M_N \times \mathbb{F} \) directly. Since \( \mathbb{F} \) is a coset space of \( SU(N) \):

\[
\mathbb{F} = S(U(p) \times U(p) \times U(N - p - q)) \backslash SU(N)
\]  

(16)

a field on \( \mathbb{F} \) is the same as a field on \( SU(N) \) invariant under the subgroup \( P_N(p,q) = S(U(p) \times U(p) \times U(N - p - q)) \). More generally, we can consider fields on \( SU(N) \) which take their values in a vector space, \( V \), which is a representation space of \( P_N(p,q) \) and which are equivariant under the action of \( P_N(p,q) \), i.e. fields \( \phi(u), u \in SU(N) \), such that

\[
\phi(hu) = T(h)\phi(u)
\]  

(17)

where \( T(h) \) is the representation of \( P_N(p,q) \) on the vector space \( V \). Such fields correspond to sections of various bundles over \( \mathbb{F} \). In practice, they are fields with various \( H_N \) indices. The extension to fields which depend also on the coordinates of superspace is straightforward.

Explicitly, we let \( u_I^i \) be an element of \( SU(N) \) where \( SU(N) \) acts on \( i \) to the right and \( P_N(p,q) \) on \( I \) to the left. We denote the inverse of \( u \) by \( u_i^I \). The right invariant vector fields on \( SU(N) \) are

\[
D_I = u_I^i \frac{\partial}{\partial u_i} - \frac{1}{N} \delta_I^j u_K^j \frac{\partial}{\partial u_K^i}
\]  

(18)
so that
\[ D_I^J u_K^i = \delta_K^J u_I^i - \frac{1}{N} \delta_I^J u_K^i \] (19)

Clearly,
\[ [D_I^J, D_K^L] = \delta_K^J D_I^L - \delta_I^L D_K^J \] (20)

The coordinates of \( \hat{M}_N \) are then \((x^{\alpha\dot{\alpha}}, \theta^{\alpha}, \bar{\theta}_{\dot{\alpha}}, u_I^i)\). The supercovariant derivatives dual to \( E^{\alpha i} \) and \( E_i^{\dot{\alpha}} \) are
\[
D_{\alpha i} = \frac{\partial}{\partial \theta^{\alpha i}} - \frac{i}{2} \bar{\theta}_{\dot{\alpha}} \frac{\partial}{\partial \theta^{\alpha \dot{\alpha}}} \] (21)
\[
\bar{D}_{\dot{\alpha}}^i = -\frac{\partial}{\partial \theta^{\alpha i}} + \frac{i}{2} \theta^{\alpha i} \frac{\partial}{\partial \theta^{\alpha \dot{\alpha}}} \] (22)

and we define
\[
D_{\alpha i} = u_I^i D_{\alpha i} \quad \bar{D}_{\dot{\alpha}}^i = u_I^i \bar{D}_{\dot{\alpha}}^i \\
\theta^{\alpha i} = u_I^i \theta^{\alpha i} \quad \bar{\theta}_{\dot{\alpha}} = u_I^i \bar{\theta}_{\dot{\alpha}} \] (23)

For \((N, p, q)\) superspace, we split the index \( I \) up as follows:
\[
I = (R, R'', R')
\]
\[ R = 1, \ldots, p; \quad R'' = 1, \ldots, N - (p + q); \quad R' = 1, \ldots, q \] (24)

The derivatives corresponding to the isotropy group \( P_N(p, q) \) are
\[
\{ D_R^S, D_{R''}^{S''}, D_{R'}^{S'} \} \] (25)

and the components of the CR operator \( \bar{\partial}_K \) are
\[
\{ D_{\alpha R}, \bar{D}^{R'}_{\dot{\alpha}}, D_R^{S''}, D_{R'}^{S'}, D_R^{S'} \} \] (26)

while the components of \( \partial \) and \( \bar{\partial} \) are
\[
\{ D_R^{S''}, D_R^{S'}, D_{R''}^{S''} \} \] (27)

and
\[
\{ D_{\alpha R}, \bar{D}^{R'}_{\dot{\alpha}} \} \] (28)

respectively. A CR-analytic field on \( M_N(p, q) \) is defined as a field \( f \) on \( \hat{M}_N \) which satisfies \( \bar{\partial}_K f = 0 \) and which is equivariant with respect to \( P_N(p, q) \).
Since \( \mathbb{F} \) is a compact, complex manifold, such a field is severely constrained as a function of the complex coordinates of \( \mathbb{F} \), because it is a holomorphic section of a complex vector bundle over \( \mathbb{F} \). However, G-analytic fields are not so constrained. In index notation, a G-analytic field satisfies

\[
D_{\alpha R} f = \bar{D}_{\bar{\alpha}}^{R'} f = 0
\]

(29)
as well as being equivariant under \( P_N(p, q) \).

In the case that \( p = q \), i.e. \((N, p, p)\) superspace, we can define a transformation on \( SU(N) \)

\[
u \to ku = u'
\]

(30)
where

\[
k = \begin{pmatrix}
0 & 0 & 1_p \\
0 & 1_{N-2p} & 0 \\
-1_p & 0 & 0 \\
\end{pmatrix}
\]

(31)
Although \( k \) acts to the left as \( SU(N) \) it is easy to see that it induces a transformation of the flag manifold \( \mathbb{F} \) because \( k\h k^{-1} \in H \) for any \( h \in H \).

To define a real structure, we combine the above transformation with complex conjugation, i.e.

\[
u \to \bar{ku} = \bar{u}
\]

(32)
Explicitly

\[
\begin{align*}
u_R^i & \to u_i^{R'} \\
u_{R'}^i & \to -u_i^R \\
u_{R''}^i & \to u_i^{R''}
\end{align*}
\]

(33)
For a field \( F(x, \theta, u) \) we define

\[
\bar{F}(x, \theta, u) = \overline{F(x, \theta, ku)}
\]

(34)
Note that this transformation preserves G-analyticity, since

\[
\begin{align*}
\left( D_{\alpha R} F \right)(x, \theta, u) &= \bar{D}_{\bar{\alpha}}^{R'} \left( \bar{F}(x, \theta, u) \right) \\
\left( \bar{D}_{\alpha R}^{R'} F \right)(x, \theta, u) &= -D_{\alpha R} \left( \bar{F}(x, \theta, u) \right)
\end{align*}
\]

(35)
(36)
It also preserves CR-analyticity, as can easily be checked. A general field \( F \) will transform under a representation of \( H_N \). Depending on the representation it may be possible to have real fields, i.e. fields for which \( \bar{F}(x, \theta, u) = F(x, \theta, u) \).
We conclude this section with a brief discussion of the superconformal
group. Since (compactified) \( M_N(p, q) \) is a coset space of \( SU(2, 2|N) \) we can easily find how this group acts. However, we are interested in representations of the Lie superalgebra \( su(2, 2|N) \) on \( G \)-analytic fields, and it turns out that the straightforward representation is not the required one. Instead, we can show that the following vector fields, \( V \), give a representation of \( su(2, 2|N) \).

\[
\begin{align*}
V = f^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} + f^{\alpha R'} D_{\alpha R'} + f^{\alpha R''} D_{\alpha R''} - \bar{f}_{\dot{\alpha}} D_{\dot{\alpha}} - \bar{f}_{\dot{\alpha}} D_{\dot{\alpha}}' + f_{R}^{S'} D_{S'} R + f_{R}^{S''} D_{S''} R + f_{R}^{S'} D_{S'} R'' 
\end{align*}
\]  

(37)

where 

\[
\begin{align*}
f^{\alpha I} = f^{\alpha i} u^I_i, \quad \bar{f}_{\dot{\alpha}} = u^i_I \bar{f}_{\dot{\alpha}}, \text{ and where} \\
D_{\alpha I} F^{\beta \dot{\beta}} + i \delta_{\alpha}^{\beta} \bar{f}^{\dot{\beta}} = 0 
\end{align*}
\]  

(38)

and

\[
\begin{align*}
f_{I}^{J} = \frac{1}{2}(D_{\alpha I} f^{\alpha I} - \frac{1}{N} \delta_{I}^{J} D_{\alpha K} f^{\alpha K}) 
\end{align*}
\]  

(39)

A superconformal vector field on super Minkowski space is given by \( F^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} + f^{\alpha i} D_{\alpha i} - \bar{f}_{\dot{i}} \bar{D}_{\dot{i}} \) (for a discussion of the \( N = 1 \) case see, for example, [2]) with the components constrained as above, and the rest of the conditions show that the components of \( V \) are also given in terms of \( F \). The vector field \( V \) preserves the CR-structure (up to isotropy group terms) and \( G \)-analyticity. Thus for a scalar \( G \)-analytic superfield, \( f \), we can define a superconformal transformation by

\[
\delta f = V f
\]  

(40)

Note that this is a representation of \( su(2, 2|N) \) on fields; it is not the standard representation on fields on \( M_N(p, q) \) that one constructs using homogeneous space techniques.

3 Massless Field Strength Superfields

It is well-known that a free, massless supermultiplet of highest spin \( s \) can be described by a field-strength superfield, \( W \), which is subject to certain constraints. For \( N \)-extended supersymmetry, when the greatest spin, \( s < \frac{N}{2} \),
$W$ has 2s internal indices, is totally antisymmetric and satisfies
\[ D_\alpha^i W_{j_1 \ldots j_{2s}} = \frac{2(-1)^{2s-1}s}{N-2s+1} \delta_{[i_1}^i D_\alpha^k W_{j_2 \ldots j_{2s}] k} \]
\[ D_{\alpha i} W_{j_1 \ldots j_{2s}} = D_{\alpha[i} W_{j_1 \ldots j_{2s}]} \]  
(41)

(See, for instance [13].) When $s = \frac{1}{4} N$, there is an additional self-duality condition:
\[ \bar{W}^{i_1 \ldots i_{2s}} = \frac{1}{(2s)!} \varepsilon^{i_1 \ldots i_{2s} j_1 \ldots j_{2s}} W_{j_1 \ldots j_{2s}} \]  
(42)

These superfields can be described naturally in $(N, p, N-p)$ harmonic superspace, with $p = 2s$, as we shall now demonstrate. It is convenient to modify our notation a little by noticing that the isotropy subalgebra of the flag manifold $F$ is $\mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathfrak{su}(r) \oplus \mathfrak{u}(1)$, where $r = N - (p + q)$. We can therefore write
\[ D^R_S = \tilde{D}^R_S + \frac{1}{N} \delta^R_S D_o \]  
(43)
\[ D^{R'}_{S'} = \tilde{D}^{R'}_{S'} + \frac{1}{N} \delta^{R'}_{S'} D'_o \]  
(44)
\[ D^{R''}_{S''} = \tilde{D}^{R''}_{S''} + \frac{1}{N} \delta^{R''}_{S''} D''_o \]  
(45)
where
\[ D''_o = -\frac{p D_o + q D'_o}{r} \]  
(46)
and where the tilded derivatives correspond to the basis elements of the $\mathfrak{su}$ algebras, so that, e.g. $\tilde{D}^R_S = 0$. The $U(1)$ charges of the $u$'s, with respect to $(D_o, D'_o)$ are given by
\[ u_R^i : \left( \frac{(N-p)}{p}, -1 \right), \quad u_R^i : \left( -1, \frac{(N-q)}{q} \right), \quad u_R^{i'} : \left( -1, -1 \right) \]  
(47)
with the inverse matrix elements having the opposite charges. For $(p, q) = (p, N-p)$, the flag manifold $F$ is simply the Grassmannian of $p$-planes in $\mathbb{C}^N$, and the isotropy algebra is $\mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathfrak{u}(1)$ (with $q = N - p$). In this case we have
\[ D^R_S = \tilde{D}^R_S + \frac{1}{N} \delta^R_S D_o \]  
(48)
\[ D^{R'}_{S'} = \tilde{D}^{R'}_{S'} + \frac{1}{N} \delta^{R'}_{S'} D'_o \]  
(49)
with $D'_o = -(p/q)D_o$. The $U(1)$ charges (with respect to $D_o$) of $u_R^i$ and $u_R'^i$ are $q/p$ and $-1$ respectively.

We claim that the on-shell superfield $W_{i_1...i_p}$ as defined above is equivalent to a CR-analytic superfield, $W$, on $M_N$ which is invariant under $\mathfrak{su}(p) \oplus \mathfrak{su}(q)$ and has $U(1)$ charge $N - p = q$. First, suppose we are given $W_{i_1...i_p}$. We can define

$$W_{R_1...R_p} = u_{R_1}^{i_1} ... u_{R_p}^{i_p} W_{i_1...i_p}$$

(50)

Because of the antisymmetry of $W_{i_1...i_p}$ we can set

$$W_{R_1...R_p} = \varepsilon_{R_1...R_p} W$$

(51)

where

$$W = \frac{1}{p!} \varepsilon_{R_1...R_p} u_{R_1}^{i_1} ... u_{R_p}^{i_p} W_{i_1...i_p}$$

(52)

To prove that $W$ is G-analytic, first apply $D_{\alpha R}$,

$$D_{\alpha R} W = \frac{1}{p!} \varepsilon_{R_1...R_p} u_{R_1}^{j} u_{R_1}^{i_1} ... u_{R_p}^{i_p} D_{\alpha j} W_{i_1...i_p}$$

(53)

Since $D_{\alpha j} W_{i_1...i_p}$ is totally antisymmetric, it follows that the product of $u$’s must be also and hence this product must vanish as $R$ runs only from 1 to $p$. We also have

$$\bar{D}_{\dot{\alpha}}^{R'} W = \frac{1}{p!} \varepsilon_{R_1...R_p} u_{R_1}^{i_1} ... u_{R_p}^{i_p} u_{j}^{R'} \bar{D}_{\dot{\alpha}}^{j} W_{i_1...i_p}$$

(54)

Since $\bar{D}_{\dot{\alpha}}^{j} W_{i_1...i_p}$ consists of a sum of terms with $\delta_{ij}$, and $u^{i_1} u^{S'}_{i} = 0$, it follows that $\bar{D}_{\dot{\alpha}}^{R'} W = 0$. Hence $W$ is G-analytic. To prove that it is CR-analytic it is enough to observe that $D_{R}^{S'} u_{T}^{i} = 0$ which immediately implies that $D_{R}^{S'} W = 0$.

We can also prove the converse, namely, that a CR-analytic field $W$ on $M_N(p, q)$ with charge $N - p$ defines a superfield $W_{i_1...i_p}$ on super Minkowski space which satisfies the constraints defining an on-shell superfield with highest helicity $s = p/2$. To prove this one expands $W$ is harmonics on $\mathbb{F}$. Since $W$ is invariant under $\mathfrak{su}(p) \oplus \mathfrak{su}(q)$ it can only depend on the $u$’s via the $\mathfrak{su}(p) \oplus \mathfrak{su}(q)$-invariants

$$a^{i_1...i_p} = \frac{1}{p!} \varepsilon_{R_1...R_p} u_{R_1}^{i_1} ... u_{R_p}^{i_p}$$

(55)
and
\[ b_{i_1\ldots i_q} = \frac{1}{ql!} \varepsilon^{R_1'\ldots R_q'} u_{R_1'i_1} \ldots u_{R_q'i_q} \] (56)

which is actually the dual of the complex conjugate of \( a \). Since \( a \) has \( U(1) \) charge \( q \) and \( b \) has \( U(1) \) charge \(-q\), it follows that \( W \) must be of the form
\[ W \sim \sum_{k=1} \left(a^k \right)^k (b)^{k-1} W(k) \] (57)

However, \( D_{RS'} a = 0 \) whereas
\[ D_{RS'} b_{i_1\ldots i_q} = \frac{1}{(q-1)!} \varepsilon^{S'R_1'\ldots R_q'} u_{R_1'i_1} u_{R_2'i_2} \ldots u_{R_q'i_q} \] (58)
from which it follows that only the \( k = 1 \) term survives in the sum when we impose \( D_{RS'} W = 0 \). Hence we find
\[ W = a^{i_1\ldots i_p} W_{i_1\ldots i_p} \] (59)
where \( W_{i_1\ldots i_p} \) depends only on \( x, \theta \) and \( \bar{\theta} \). Imposing G-analyticity we easily recover the on-shell constraints described above as may easily be checked.

Finally, the self-duality condition on \( W_{i_1\ldots i_p} \) when \( s = N/4 \), i.e. \( N = 2p \), is equivalent to requiring \( W \) to be real with respect to the real structure introduced in the previous section. We have
\[ W = \frac{1}{p!} \varepsilon^{R_1\ldots R_p} u_{R_1'i_1} \ldots u_{R_p'i_p} W_{i_1\ldots i_p} \] (60)
so
\[ \tilde{W} = \frac{1}{p!} \varepsilon^{R_1'\ldots R_p'} u_{i_1} R_1' \ldots u_{i_p} R_p' \tilde{W}_{i_1\ldots i_p} \] (61)
Hence, if \( \tilde{W}_{i_1\ldots i_p} = \frac{1}{p!} \varepsilon^{i_1\ldots i_p j_1\ldots j_p} W_{j_1\ldots j_p} \), \( W \) is real and vice versa.

4 Harmonic Integration and Superactions

Chiral integration in \( N = 1 \) supersymmetry generalises to harmonic superspace in a straightforward fashion, as shown by Gikos for the case \((p, q) = (1, 1)\). In this section we extend this to arbitrary \((p, q)\) and show that the
resulting integrals are equivalent to a class of superactions of the type discussed in [16]. As an application, we rewrite the linearised $N = 8$ supergravity three-loop counterterm as an $(N, p, q) = (8, 4, 4)$ harmonic superspace integral.

The $G$-analytic measure on $(N, p, q)$ superspace, $d\mu$, is defined as follows

$$d\mu = d^4x\, du\, d^2\theta\, d^2\bar{\theta} \, d^2q\, d^2p\, d^2\theta'$$

(62)

where $\theta \sim \theta^{\alpha R}$, $\theta' \sim \theta^{\alpha R'}$, $\theta'' \sim \theta^{\alpha R''}$ and $du$ is the Haar measure on the coset space $F$. Since Grassmann integration is actually differentiation, one has e.g.

$$d^2q\, \theta' \sim (D^{\alpha R'} D^S_{\sigma} f^S)^q$$

(63)

Moreover, since one is integrating over all $\theta^{\alpha R}$'s, etc., the measure is invariant under $SU(p) \times SU(q) \times SU(r)$, $(r = N - (p + q))$, but it has $U(1)$ charges given by $2(-(N-p+q), (N+p-q))$. We can therefore use it to integrate $G$-analytic superfields $L$ on $\hat{M}_N$ which are invariant under $SU(p) \times SU(q) \times SU(r)$ and which have charges $2(N-p+q, -(N+p-q))$, to get integrals

$$I = \int d\mu \, L$$

(64)

Such integrals are invariant under super Poincaré transformations (with internal symmetry $SU(N)$) and invariant under superconformal symmetry if

$$\delta L = \mathcal{V} + \Delta L$$

(65)

where $\mathcal{V}$ is given in (37) and

$$\Delta = \partial_{\alpha\dot{\alpha}} F^{\alpha\dot{\alpha}} - D_{\alpha R'} f^{\alpha R'} - D_{\alpha R''} f^{\alpha R''} + \bar{D}_{\dot{\alpha} R} \bar{f}_{\dot{\alpha} R} + \bar{D}_{\dot{\alpha} R'} \bar{f}_{\dot{\alpha} R'} + D_R^S f_{S R} + D_R^{S''} f_{S'' R} + D_{R''} S f_{S'' R'} + D_{R''} S f_{S'' R'}$$

(66)

The simplest example is the case $N = 2$, $p = 1$, $q = 1$, the original Gikos example. In this case we set $u_1^i = (u_1^i, u_2^i)$, where $u_1^i(u_2^i)$ has $U(1)$ charge $+1(-1)$ respectively. The measure can be written in the form

$$d\mu = d^4x\, du_1^i u_1^i u_1^k u_1^l D_{ij} D_{kl}$$

(67)

where

$$D_{ij} \equiv D_{\alpha\dot{\alpha}} D^S_{ij}, \quad \bar{D}^{ij} \equiv \bar{D}^i_{\dot{\alpha}} \bar{D}^{\dot{\alpha}j}$$

(68)
and where internal indices are raised and lowered with the $\varepsilon$-tensor for $N = 2$. The integrand $\mathcal{L}$, which has charge 4, can be expanded in harmonics on $\mathbb{F} = \mathbb{C}P^1 = U(1) \backslash SU(2)$ in the form

$$
\mathcal{L} = u_1^i u_2^j u_4^k u_1^l L_{ijkl} + u_1^i u_1^j u_1^k u_1^l u_2^m u_2^n L_{ijklmn} + \ldots
$$

(69)

The products of $u$'s in irreducible $SU(2)$ representations are the spherical harmonics and are orthogonal with respect to the Haar measure $d\mu$, so that if we carry out the integration over $\mathbb{C}P^1$ only the first term in the expansion of $\mathcal{L}$ contributes. Hence

$$
I = \int d\mu \mathcal{L} = \int d^4 x D^{ij} \bar{D}^{kl} L_{ijkl}
$$

(70)

The last expression is an example of what was called a superaction in [16]; there, it was supposed that $D^{\alpha(i} L_{jklm)} = 0$ $D_{\dot{\alpha}(i} L_{jklm)} = 0$ (71)

and this is certainly sufficient for invariance of $I$ (under Poincaré supersymmetry). However, if $L_{ijkl}$ is the first component of a $G$-analytic field $\mathcal{L}$ one finds that

$$
D_{\dot{\alpha}(i} L_{jklm)} \sim D_{\alpha}^{\dot{\alpha}(i} L_{jklm)}
$$

(72)

so that the superaction is a particular case of a harmonic integral for an $\mathcal{L}$ which is CR-analytic, instead of just $G$-analytic. Alternatively, one can say that the constraint on $L_{ijkl}$ necessary for the superaction to be invariant may be weakened from (71) to the set of conditions which follow from $G$-analyticity of $\mathcal{L}$.

The $(N, p, q)$ case is the natural generalisation of the above. If we let $u_R^i = (u_{R_1}^i, u_{R_2}^i)$ and $u_{\bar{R}}^i = (u_{\bar{R}_1}^i, u_{\bar{R}_2}^i)$ we can write the odd part of the measure in the form

$$
A^{i_1 \ldots i_n} A^{j_1 \ldots j_n} \bar{B}^{k_1 \ldots k_m} \bar{B}^{l_1 \ldots l_m} D^{k_{i_1} \ldots k_{i_n} l_{j_1} \ldots l_{j_n}}
$$

(73)

where

$$
A^{i_1 \ldots i_n} = \frac{1}{n!} \varepsilon_{R_1 \ldots R_n} u_{R_1}^{i_1} \ldots u_{R_n}^{i_n}
$$

$$
B^{i_1 \ldots i_m} = \frac{1}{m!} \varepsilon_{\bar{R}_1 \ldots \bar{R}_m} u_{\bar{R}_1}^{i_1} \ldots u_{\bar{R}_m}^{i_m}
$$

$$
D^{k_{i_1} \ldots k_{i_n} l_{j_1} \ldots l_{j_n}} = D_{i_1 j_1} \ldots D_{i_n j_n} D^{k_{i_1} \ldots k_{i_n} l_{j_1} \ldots l_{j_n}}
$$

(74)
with \( n = N - p, \ m = N - q \). We have
\[
A^{i_1...i_n} = \varepsilon^{i_1...i_n j_1...j_p} \tilde{a}_{j_1...j_p} \\
B^{i_1...i_m} = \varepsilon^{i_1...i_m j_1...j_q} \tilde{b}_{j_1...j_q}
\]
so that, using the \( \varepsilon \)-tensor to dualise the indices on the \( D \)'s, we can rewrite the measure as
\[
d^4 x \, du \, Y_{i_1 j_1,...,i_p j_p} \tilde{D}^{i_1 j_1,...,i_p j_p}_{k_1 l_1,...,k_q l_q}
\]
where the harmonic function \( Y \) is given by
\[
Y_{i_1 j_1,...,i_p j_p} = \tilde{a}_{i_1...i_p} \tilde{b}_{j_1...j_p} b^{k_1...k_q} b^{l_1...l_q} - \text{traces}
\]
It belongs to the following irreducible representation \((p \geq q)\) of \( SU(N) \):

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\]

Integrating over \( u \) we again pick out the first component in the harmonic expansion of \( \mathcal{L} \) so that
\[
I = \int d\mu \mathcal{L} = \int d^4 x \tilde{D}^{i_1 j_1,...,i_p j_p}_{k_1 l_1,...,k_q l_q} L^{k_1 l_1,...,k_q l_q}_{i_1 j_1,...,i_p j_p}
\]
where
\[
\mathcal{L} = Y^{i_1 j_1,...,i_p j_p}_{k_1 l_1,...,k_q l_q} L^{k_1 l_1,...,k_q l_q}_{i_1 j_1,...,i_p j_p} + \ldots
\]
Again, if \( \mathcal{L} \) is CR-analytic and not just \( G \)-analytic, its harmonic expansion stops at the first term and the constraints imposed on \( \mathcal{L} \) by analyticity are precisely those given in [16]. The set of all harmonic measures therefore corresponds to the set of superactions which have the total number of \( D \)'s and \( \tilde{D} \)'s greater than or equal to \( 2N \).

As an example, consider \((N, p, q) = (8, 4, 4)\). In this case, \( \mathbb{F} = Gr_4(8) \) so the isotropy group is \( SU(4) \times U(4) \). The measure has \( U(1) \) charge \(-16\), while the supergravity field strength superfield \( W \) has charge \( 4 \). Therefore, the harmonic integral
\[
I = \int d\mu \mathcal{L}
\]
with
\[ \mathcal{L} = W^4 \]  
(81)
is manifestly Poincaré supersymmetric. It is in fact the linearised \( N = 8 \) supergravity three-loop counterterm. If one integrates over the coset space one recovers the superaction form given in [16] with the Lagrangian \( L \) being \( W^4 \) in the \( 4 \times 4 \) square tableau representation of \( SU(8) \).

5 Super Yang Mills

The constraints on the field strength two-form in super Yang-Mills are (with \( N \leq 4 \) and a real gauge group) [25],

\[
\begin{align*}
F_{\alpha i\beta j} &= \varepsilon_{\alpha\beta} W_{ij} \\
F^{i\bar{j}}_{\dot{a}\bar{\beta}} &= \varepsilon_{\dot{a}\bar{\beta}} W^{ij} \\
F_{\alpha i\dot{j}} &= 0
\end{align*}
\]  
(82)

where the field strengths are defined as usual. In \( N = 4 \) the field strength \( W \) is self-dual:

\[ \bar{W}^{ij} = \frac{1}{2} \varepsilon^{ijkl} W_{kl} \]  
(83)
The above constraints are off-shell for \( N = 1, 2 \) and on-shell for \( N = 3, 4 \).

If we consider the same theories on \( \hat{M}_N, N \geq 2 \), the constraints can be restated in the form

\[
\begin{align*}
F_{\alpha R\beta S} &= \varepsilon_{\alpha\beta} W_{RS} \\
F_{\dot{a}R'S'}_{\dot{\alpha}} &= \varepsilon_{\dot{a}\bar{\alpha}} \bar{W}^{R'S'} \\
F_{\alpha R\beta} &= 0
\end{align*}
\]  
(84)

where \( F_{\alpha R\beta S} = u_{R}^i u_{S}^j F_{\alpha i\beta j} \), etc. and \( R, R' \) run from 1 to \( p \) and 1 to \( q \) as usual. For \( (p, q) = (1, 1) \) the right-hand sides vanish and so the constraints of \( N = 2, 3 \)

Yang-Mills can be understood as vanishing curvatures in (1,1) harmonic superspace. For \( N = 4 \) these constraints

still need to be supplemented by self-duality. Hence in \( N = 2, 3 \) the constraints can be solved in terms of pure gauges; for \( N = 2 \) this gives the
harmonic form of the off-shell theory \[3\], whereas in \(N = 3\) it gives the field equations, although one can then go off-shell by allowing the field strength to be non-zero in the \(F\)-directions \[4\].

We shall consider here two other cases, namely, \((4, 2, 2)\) and \((3, 2, 1)\) harmonic superspaces. In both cases we have

\[
F_{\alpha R \beta S} = \varepsilon_{\alpha \beta} \varepsilon_{RS} W
\]

\[
F_{\alpha R' \beta} = 0
\]  

(85)

For \(N = 3\) the third equation is

\[
F_{\bar{\alpha} R' \bar{\beta}} = 0
\]  

(86)

while for \(N = 4\) we have

\[
F_{\bar{\alpha} R' \bar{\beta}} = \varepsilon_{\bar{\alpha} \bar{\beta}} \varepsilon^{R'S'} W
\]  

(87)

The Bianchi identities then imply that \(W\) is covariantly G-analytic

\[
\nabla_{\alpha R} W = \nabla_{\bar{\alpha} R'} W = 0
\]  

(88)

and \(F\)-analytic

\[
D_{R} S' W = 0
\]  

(89)

where \(\nabla\) is the Yang-Mills covariant derivative. In \(N = 4\) the self-duality condition is implemented by demanding that \(W\) be real, as in the linearised case, \(W = \bar{W}\).

It therefore seems to be the case that the \((1, 1)\) superspace formalism is more powerful since it allows one to solve the constraints. However, it is at least interesting that the non-linear on-shell \(N = 3\) or \(4\) theory has such a simple description in terms of a one-component superfield obeying simple constraints. A small application of this formalism is the construction of the \(N = 4\) supercurrent superfield, \(J\), for the fully interacting Yang-Mills theory. It is given by

\[
J = \text{Tr} (W^2)
\]  

(90)

and is clearly CR-analytic and real. It has \(U(1)\) charge 4 and couples to a linearised potential for \(N = 4\) conformal supergravity, \(V\), by

\[
\int d\mu J V
\]  

(91)
where \(d\mu\) is the \((4,2,2)\) measure discussed in the previous section.

The potential \(V\) is \(G\)-analytic, real and has charge 4. The gauge transformations which leave the above interaction invariant are

\[
\delta V = D_{R'}^{S'} X_S^R
\]

where the gauge parameter \(X\) is \(G\)-analytic and real.

6 Conformal Supergravity

In this section we shall apply the harmonic superspace formalism to supergravity theories, in particular, conformal supergravity theories which exist for \(N = 1, 2, 3, 4\). The superspace constraints for these theories were written down in [12]. For \(N = 2\) our results give a geometrical interpretation of the work of [9, 6] on the subject, while for \(N = 3, 4\) we show how this geometry generalises to higher \(N\), although we shall not go into any detail about solving the constraints here. In fact the same formalism can also be applied to \(N = 5, 6, 7, 8\), and in these cases we find that the constraints imposed by harmonic superspace considerations lead to the conformal constraints proposed in [11]; these constraints are not fully off-shell, as was noted in [10], but they are compatible with the field equations of on-shell Poincaré supergravity [11]. Thus the geometry proposed below summarises a large part of the known results on superspace supergravity from the harmonic point of view.

In order to derive these results we shall have to make some assumptions about the basic geometrical structures that are to be imposed. We start with a \((4|4N)\)-dimensional real supermanifold \(M\) which is equipped with a choice of odd tangent bundle, \(F\) (rank \((0|4N)\)), \(F \subset T\), where \(T\) is the tangent bundle. In addition we shall suppose that \(F\) is maximally non-integrable in the sense that

\[
[F,F] \mod F = B := T/F
\]

that is, the even tangent bundle, \(B\), defined as the quotient of \(T\) by \(F\) is spanned by the commutators of odd vector fields. We shall further suppose that the structure group of the bundle of odd frames \(LF\) can be reduced from \(GL(4N,\mathbb{R})\) to \(SL(2,\mathbb{C}) \cdot U(N) \times \mathbb{R}^+\) where \(\mathbb{R}^+\) corresponds to Weyl rescalings. This implies (at least locally) that
the complexification of \( F, F_c \) can be written as \( \mathcal{F} \oplus \bar{\mathcal{F}} \) where \( \mathcal{F} = S \otimes V \), with \( S \) having rank \((0|2)\) and \( V \) having rank \((N|0)\). The above structure defines a tensor \( T \) which we shall call the structure tensor of the theory and which is a section of \( \wedge^2 F^* \otimes B \). If we introduce local basis vector fields \( \{ E_{\alpha i}, \bar{E}_i^\beta, E^a \} \) for \( \mathcal{F}, \bar{\mathcal{F}} \) and \( B^* \) respectively, the components of the structure tensor can be expressed as

\[
T_{\alpha i\beta j}^c = - \langle [E_{\alpha i}, E_{\beta j}], E^c \rangle \\
T_{\tilde{\alpha} \tilde{\beta}}^i j = - \langle [E_{\tilde{\alpha} i}, E_{\tilde{\beta} j}], E^c \rangle \\
T_{\alpha i} j^c = - \langle [E_{\alpha i}, E^j], E^c \rangle
\]

(94)

the second equation being the complex conjugate of the first.

The conformal constraints are simply that the components of \( T \) are the same as in the flat case, i.e.

\[
T_{\alpha i\beta j}^c = T_{\tilde{\alpha} \tilde{\beta}}^i j = 0
\]

(95)

and

\[
T_{\alpha i} j^c = -i(\sigma^c)_{\alpha \beta} \delta_i^j
\]

(96)

More precisely, if \( T \) is flat, one can choose a basis for \( B \) as a subbundle of \( T \) and an \( \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{u}(N) \) connection such that the conformal constraints are recovered.

The result which we shall demonstrate here is that the above constraints can be interpreted geometrically as the statement that \((1,1)\) harmonic superspace is a CR supermanifold with CR bundle \( K \) of rank \((2N - 3|4)\). In the gravitational context harmonic superspace is the \( U(N) \) bundle with fibre \( \mathbb{F}(=\mathbb{F}_{p,N-q}(N) \text{ in the } (p,q) \text{ case}) \) associated with the principal \( U(N) \) bundle (assuming it exists) which corresponds to the \( U(N) \) part of the structure group. It is often convenient to work with \( U(N) \) instead of \( SU(N) \) in supergravity and \( \mathbb{F} \) can be viewed as a coset space of \( U(N) \) with isotropy group \( U(p) \times U(q) \times U(N - (p + q)) \). If we denote the principal \( U(N) \) bundle by \( \hat{M}_N \) we can study harmonic superspace \( \hat{M}_N \) by working with fields on \( \hat{M}_N \) that are equivariant with respect to the isotropy subgroup.

To prove the result we first split \( T = F \oplus B \) and introduce an \( \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{u}(N) \) connection \( \Gamma \). If \( E_A = \{ E_{\alpha i}, \bar{E}_i^\alpha, E^a \} \) denotes a set of local basis vectors for \( T \), the horizontal lifts, \( \hat{E}_A \), of these vector fields in \( \hat{M}_N \)
are defined by
\[ \hat{E}_A = E_A - \Gamma_{AI}^JD_J^I \]  
(97)
where \( D_I^J \) are now the right-invariant vector fields on \( U(N) \) defined by
\[ D_I^J = u^i_I \frac{\partial}{\partial u^J_i} \]  
(98)
and
\[ \Gamma_{AI}^J = u^i_I \Gamma_{AI}^J u^J_i \]  
(99)
is the \( u(N) \) part of the connection. A local basis for \( \hat{M}_N \) is given by \{\( \hat{E}_A, D_I^J \)}
The claim is that the basis vectors of the CR subbundle, \( K \), of the tangent bundle of \( M_N(1, 1) \) are the following:
\[ D_1^r, D_r^N, D_1^N; \hat{E}_\alpha^1, \hat{\bar{E}}_{\dot{\alpha}}^N \]  
(100)
where \( r = 2, \ldots (N - 1) \) and
\[ \hat{E}_{\alpha I} = u^i_I \hat{E}_{\alpha i}; \quad \hat{\bar{E}}^I_{\dot{\alpha}} = u^i_I \hat{\bar{E}}^i_{\dot{\alpha}} \]  
(101)
We have
\[ [\hat{E}_{\alpha I}, \hat{E}_{\beta J}] = -T_{\alpha I\beta J}^C \hat{E}_C + \Gamma_{\alpha I,\beta}^\gamma \hat{E}_{\gamma J} + \Gamma_{\beta J,\alpha}^\gamma \hat{E}_{\gamma I} - R_{\alpha I,\beta J,KL}^L D_L^K \]  
(102)
and similarly for dotted indices where \( \Gamma_{A,\beta}^\gamma \) is the \( \mathfrak{sl}(2, \mathbb{C}) \) part of the connection, the little \( U(N) \) indices are converted to capital ones by using \( u \) as usual, and \( R \) is the \( U(N) \) curvature tensor. Since we are interested in establishing the involutivity of \( K \) in \( M_N(1, 1) \) we can take the above vector fields to be act on functions which are invariant under the isotropy group \( U(1) \times U(N - 2) \times U(1) \).
The conditions for involutivity of \( K \) therefore include, at dimension zero,
\[ T_{\alpha I\beta 1}^c = T_{\alpha 1}^{N c} = 0 \]  
\( T_{\alpha 1}^{N c} = 0 \)  
(103)
at dimension one-half,
\[ T_{\alpha 1\beta 1,1}^{\gamma} = T_{\alpha 1\beta 1,r}^{\gamma} = 0 \]  
\[ T_{\dot{\alpha} \dot{\beta}}^{NN\gamma N} = T_{\dot{\alpha} \dot{\beta}}^{N\gamma r} = 0 \]  
(104)
and at dimension one,

\[ R_{\ldots,1} = R_{\ldots,\alpha} = R_{\ldots,1} = 0 \quad (105) \]

where the missing indices can be any of the pairs \( \{ (\alpha_1, \beta_1), (\alpha_1, \beta), (\alpha, \beta_1) \} \), and where \( T \) now denotes the torsion tensor. The dimension zero constraints imply flatness of the structure tensor (which becomes the dimension zero part of the torsion tensor) when supplemented by algebraic constraints, and the rest can be shown to be consistent after using a little algebra.

The use of the group \( U(N) \) is particularly useful in conformal supergravity since the corresponding spacetime gauge field belongs to the conformal supermultiplet. However, for \( N = 4 \), there are only gauge fields for the \( SU(4) \) subgroup, and so an additional constraint is needed in this case [12].

The computation carried out here can be repeated for any choice of \( (p, q) \). It is found that, for \( N = 3 \), \( (3, 2, 1) \) harmonic superspace is a CR supermanifold, and for \( N = 4 \) \( (4, 2, 2) \) superspace is a CR supermanifold. However, if one lifts the structure tensor up to \( M_N(p, q) \) by replacing the \( E \)'s by the \( \hat{E} \)'s in (94), one sees that the connections drop out. Hence the constraints can also be generated by demanding that the lifted structure tensor vanish along \( K \) for any choice of \( (p, q) \) even though \( K \) may not be involutive.

The geometry of \( N = 1 \) conformal supergravity does not fit into the above scheme due to the fact that the internal symmetry group is in this case only \( U(1) \). However, the Ogievetsky-Sokatchev formalism [21, 24] for complexified supergravity can be described by the double fibration \( M_L \leftarrow M \rightarrow M_R \), where \( M_L(M_R) \) are respectively the left and right chiral superspaces [24]. In the real case the intrinsic geometrical formulation of \( N = 1 \) conformal supergravity may be stated as follows. \( M \) is a real \( (4|4) \) CR supermanifold with CR bundle \( \mathcal{F} \) of rank \( (0|2) \) such that

\[ [\mathcal{F}, \overline{\mathcal{F}}] \mod F_c = B_c \quad (106) \]

where \( F_c = \mathcal{F} \oplus \overline{\mathcal{F}} \) and \( B_c = T_c / F_c \). Given this structure one can reconstruct the OS formalism straightforwardly.

In conclusion, we have seen that the harmonic superspace formalism, for various values of \( (p, q) \), can be used to describe the constraints of \( N = 2, 3, 4 \) conformal supergravity and the conformal constraints of \( N \geq 5 \) supergravity. In the conformal case one would like to solve these constraints to obtain new potentials for the \( N = 3, 4 \) theories. However, this is not as simple
as in $N = 2$, since the complex dimension of the internal flag manifold is greater than one. In addition, one can see on dimensional grounds that there will be complications compared to $N = 2$; for example, the linearised harmonic potential $V$ for $N = 4$ introduced in the previous section has dimension $-2$. In the case of Poincaré supergravity one needs to find the geometrical formulation of the additional constraints that arise; it is possible that extended supergravity theories with sufficiently large $N$ may turn out to be in some sense integrable, that is, the constraints defining the field equations are given by vanishing curvatures on some appropriate superspace.

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