Characterizations of Almost Strongly $N_{nc}\theta e$-continuous Functions

A. Vadivel $^1$, V. Sudha $^2$ and S. Tamilselvan$^3$

$^1$ Department of Mathematics, Government Arts College (Autonomous), Karur, Tamil Nadu - 639 005, India
$^2$ Department of Mathematics, Periyar Government Arts College, Cuddalore, Tamil Nadu-607 001, India and
$^3$ Mathematics Section (FEAT), Annamalai University, Annamalainagar, Tamil Nadu-608 002, India

E-mail: $^1$avmaths@gmail.com, $^2$sudhasowjimath@gmail.com and $^3$tamil_au@yahoo.com

Abstract. In this paper several characterizations concerning almost strongly $N_{nc}\theta e$-continuous functions are obtained. Also we investigate the relationships between almost strongly $N_{nc}\theta e$-continuous functions and separation axioms and almost strongly $N_{nc}e$-closedness of graphs of functions.

1. Introduction

Smarandache’s neutrosophic framework have wide scope of constant applications for the fields of Computer Science, Information Systems, Applied Mathematics, Artificial Intelligence, Mechanics, dynamic, Medicine, Electrical & Electronic, and Management Science and so forth [1, 2, 3, 4, 23, 24]. Topology is an classical subject, as a generalization topological spaces numerous kinds of topological spaces presented throughout the year. Smarandache [17] characterized the Neutrosophic set on three segment Neutrosophic sets (T-Truth, F-Falsehood, I-Indeterminacy). Neutrosophic topological spaces (nts’s) presented by Salama and Albawi [14]. Lellies Thivagar et al. [9] was given the geometric existence of $N$ topology, which is a non-empty set equipped with $N$ arbitrary topologies. Lellis Thivagar et al. [10] introduced the notion of $N_{n}$-open (closed) sets and $N_{n}$-continuous in $N$ neutrosophic crisp topological spaces. Al-Hamido et al. [5] investigate the chance of extending the idea of neutrosophic crisp topological spaces into $N$-neutrosophic crisp topological spaces and examine a portion of their essential properties. The concept of continuity is the most important subject in topology. In 2008, the notion of $e$-continuous functions was introduced and studied by Ekici [7] and in 2010, the notion of strongly $\theta e$-continuous function was introduced by Özkoc and Aslim [12]. In 1984, Noiri and Kang introduced the notion of almost strong $\theta$-continuity. Recently, three generalizations of almost strong $\theta$-continuity are obtained by Beceren et al. [6], Park et al. [13] and Noiri and Zorlutuna [11]. In this paper we introduce and investigate some fundamental properties of almost strongly $N_{nc}\theta e$-continuous functions defined via $N_{nc}e$-open sets in $N_{nc}$ topological spaces. It turns out that almost strong $N_{nc}\theta e$-continuity is stronger than $N_{nc}e$-continuity and weaker than strong $N_{nc}\theta e$-continuity, almost strong $N_{nc}e$-semicontinuity and almost strong...
\[ N_{n, \theta}\text{-precontinuity. Moreover, we obtain some results related to separation axioms and graphs properties.} \]

2. Preliminaries
Salama and Smarandache [16] presented the idea of a neutrosophic crisp set in a set \( P \) and defined the inclusion between two neutrosophic crisp sets, the intersection (union) of two neutrosophic crisp sets, the complement of a neutrosophic crisp set, neutrosophic crisp empty (resp., whole) set as more than two types. And they studied some properties related to neutrosophic crisp set operations. However, by selecting only one type, we define the inclusion, the intersection (union), and neutrosophic crisp empty (resp., whole) set again and discover a few properties.

**Definition 2.1** Let \( P \) be a non-empty set. Then \( H \) is called a neutrosophic crisp set (in short, \( ncs \)) in \( P \) if \( H \) has the form \( H = (H_1, H_2, H_3) \), where \( H_1, H_2, \) and \( H_3 \) are subsets of \( P \).

The neutrosophic crisp empty (resp., whole) set, denoted by \( \phi_n \) (resp., \( P_n \)) is an \( ncs \) in \( P \) defined by \( \phi_n = (\phi, \phi, P) \) (resp. \( P_n = (P, P, \phi) \)). We will denote the set of all \( ncses \) in \( P \) as \( ncS(P) \).

In particular, Salama and Smarandache [15] classified a neutrosophic crisp set as the followings.

A neutrosophic crisp set \( H = (H_1, H_2, H_3) \) in \( P \) is called a neutrosophic crisp set of Type 1 (resp. 2 & 3) (in short, \( ncs \)-Type 1 (resp. 2 & 3)), if it satisfies \( H_1 \cap H_3 = H_2 \cap H_3 = H_3 \cap H_1 = \phi \) (resp. \( H_1 \cap H_2 = H_2 \cap H_3 = H_3 \cap H_1 = \phi \) and \( H_1 \cup H_2 \cup H_3 = P \) & \( H_1 \cap H_2 \cap H_3 = \phi \) and \( H_1 \cup H_2 \cup H_3 = P \)). \( ncS_1(P) \) (\( ncS_2(P) \) and \( ncS_3(P) \)) means set of all \( ncs \) Type 1 (resp. 2 and 3).

**Definition 2.2** Let \( H = (H_1, H_2, H_3), M = (M_1, M_2, M_3) \in ncS(P) \). Then \( H \) is said to be contained in (resp. equal to) \( M \), denoted by \( H \subseteq M \) (resp. \( H = M \)), if \( H_1 \subseteq M_1, H_2 \subseteq M_2 \) and \( H_3 \supseteq M_3 \) (resp. \( H \subseteq M \) and \( M \subseteq H \)); \( H^c = (H_3^c, H_2^c, H_1^c) \); \( H \cap M = (H_1 \cap M_1, H_2 \cap M_2, H_3 \cap M_3); \)

\( H \cup M = (H_1 \cup M_1, H_2 \cup M_2, H_3 \cup M_3) \). Let \( (S_j)_{j \in J} \subseteq ncS(P) \), where \( H_j = (H_{j_1}, H_{j_2}, H_{j_3}) \). Then

\[ \bigcap_{j \in J} H_j \] (simply \( \bigcap_{j} H_j \)) = \( (\bigcap_{j_1} H_{j_1}, \bigcap_{j_2} H_{j_2}, \bigcup_{j_3} H_{j_3}); \)

\[ \bigcup_{j \in J} H_j \] (simply \( \bigcup_{j} H_j \)) = \( (\bigcup_{j_1} H_{j_1}, \bigcup_{j_2} H_{j_2}, \bigcap_{j_3} H_{j_3}) \).

The following are the quick consequences of Definition 2.2.

**Proposition 2.1** [8] Let \( L, M, O \in ncS(P) \). Then

(i) \( \phi_n \subseteq L \subseteq P_n \),

(ii) if \( L \subseteq M \) and \( M \subseteq O \), then \( L \subseteq O \),

(iii) \( L \cap M \subseteq L \) and \( L \cap M \subseteq M \),

(iv) \( L \subseteq L \cup M \) and \( M \subseteq L \cup M \),

(v) \( L \subseteq M \) if \( L \cap M = L \),

(vi) \( L \subseteq M \) if \( L \cup M = M \).

Likewise the following are the quick consequences of Definition 2.2.

**Proposition 2.2** [8] Let \( L, M, O \in ncS(P) \). Then

(i) \( L \cup L = L \), \( L \cap L = L \) (Idempotent laws),

(ii) \( L \cup M = M \cup L \), \( L \cap M = M \cap L \) (Commutative laws),

(iii) (Associative laws) : \( L \cap (M \cup O) = (L \cap M) \cup O \), \( L \cap (M \cap O) = (L \cap M) \cap O \),

(iv) (Distributive laws:) \( L \cup (M \cap O) = (L \cup M) \cap (L \cup O) \), \( L \cap (M \cup O) = (L \cap M) \cup (L \cap O) \),

(v) (Absorption laws) : \( L \cup (L \cap M) = L \), \( L \cap (L \cup M) = L \),

(vi) (DeMorgan’s laws) : \( (L \cup M)^c = L^c \cap M^c \), \( (L \cap M)^c = L^c \cup M^c \).
(vii) \((L^c)^c = L\),

(viii) (u) \(L \cup \phi_n = L\), \(L \cap \phi_n = \phi_n\),

(u) \(L \cup P_n = P_n\), \(L \cap P_n = L\),

(u) \(P_n^c = \phi\), \(P_n^c = P_n\),

(u) in general, \(L \cup L^c \neq P_n\), \(L \cap L^c \neq \phi_n\).

Proposition 2.3 [8] Let \(L \in ncS(P)\) and let \((L_j)_{j \in J} \subseteq ncS(P)\). Then

(i) \((\bigcap L_j)^c = \bigcup L_j^c\), \((\bigcup L_j)^c = \bigcap L_j^c\),

(ii) \(L \cap (\bigcup L_j) = \bigcup (L \cap L_j)\), \(L \cup (\bigcap L_j) = \bigcap (L \cup L_j)\).

Definition 2.3 [15] A neutrosophic crisp topology (vriefly, \(ncs\)) on a non-empty set \(P\) is a family \(\tau\) of \(nc\) subsets of \(P\) satisfying the following axioms

(i) \(\phi_n, P_n \in \tau\).

(ii) \(H_1 \cap H_2 \in \tau \forall H_1 \& H_2 \in \tau\).

(iii) \(\bigcup_a H_a \in \tau\), her any \(\{H_a : a \in J\} \subseteq \tau\).

Then \((P, \tau)\) is a neutrosophic crisp topological space (vriefly, \(ncs\)) in \(P\). The \(\tau\) elements are called neutrosophic crisp open sets (vriefly, \(ncos\)) in \(P\). A \(ncs\) \(C\) is closed set (vriefly, \(ncs\)) if its complement \(C^c\) is \(ncos\).

Definition 2.4 [5] Let \(P\) be a non-empty set. Then \(nc\tau_1, nc\tau_2, \ldots, nc\tau_N\) are \(N\)-arbitrary crisp topologies defined on \(P\) and the collection \(Nnc\tau = \{S \subseteq U : S = \bigcup_{j=1}^{N} H_j \cup \bigcap_{j=1}^{N} L_j\}\), \(H_j, L_j \in nc\tau\) is called \(Nnc\)-topology on \(P\) if the axioms are satisfied:

(i) \(\phi_n, P_n \in Nnc\tau\).

(ii) \(\bigcap_{j=1}^{\infty} S_j \in Nnc\tau \forall \{S_j\}_{j=1}^{\infty} \in Nnc\tau\).

(iii) \(\bigcap_{j=1}^{n} S_j \in Nnc\tau \forall \{S_j\}_{j=1}^{n} \in Nnc\tau\).

Then \((P, Nnc\tau)\) is called a \(Nnc\)-topological space (vriefly, \(Nnc\tau\)) on \(P\). The \(Nnc\tau\) elements are called \(Nnc\)-open sets (\(Nncos\)) on \(P\) and its complement is called \(Nnc\)-closed sets (\(Nncs\)) on \(P\). The elements of \(P\) are known as \(Nnc\)-sets (\(Nnc\)) on \(P\).

Definition 2.5 [5] Let \((P, Nnc\tau)\) be \(Nnc\tau\) on \(P\) and \(H\) be an \(Nnc\) on \(P\), then the \(Nnc\) interior of \(H\) (vriefly, \(Nncint(H)\)) and \(Nnc\) closure of \(H\) (vriefly, \(Nnccl(H)\)) are defined as

(i) \(Nncint(H) = \bigcup\{S : S \subseteq H \& S \text{ is a } Nncos \text{ in } U\} \& Nnccl(H) = \bigcap\{C : H \subseteq C \& C \text{ is a } Nncs \text{ in } U\}\).

(ii) \(Nnc\)-regular open [18] set (vriefly, \(Nncros\)) if \(H = Nncint(Nnccl(H))\).

(iii) \(Nnc\)-pre open set (vriefly, \(Nncpos\)) if \(H \subseteq Nncint(Nnccl(H))\).

(iv) \(Nnc\)-semi open set (vriefly, \(Nncsos\)) if \(H \subseteq Nnccl(Nncint(H))\).

(v) \(Nnc\)-alpha open set (vriefly, \(Nncaos\)) if \(H \subseteq Nncint(Nnccl(Nncintg(H)))\).

(vi) \(Nnc\)-beta open set [19] (vriefly, \(Nncbos\)) if \(H \subseteq Nncint(Nnccl(Nncintg(H)))\).

(vii) \(Nnc\)-gamma open set (vriefly, \(Nncgos\)) if \(H \subseteq Nnccl(Nncint(H)) \cup Nncint(Nnccl(H))\).
The complement of an $N_{nc}ros$ (resp. $N_{nc}Sos$, $N_{nc}Pos$, $N_{nc}aos$, $N_{nc}bos$, $N_{nc}aos \& N_{nc}gos$) is called an $N_{nc}$-regular (resp. $N_{nc}$-semi, $N_{nc}$-pre, $N_{nc}$-$\alpha$, $N_{nc}$-$\beta$, $N_{nc}$-$aos \& N_{nc}$-$gos$) closed set (vriefly, $N_{nc}rcs$ (resp. $N_{nc}Scs$, $N_{nc}Pcs$, $N_{nc}acs$, $N_{nc}bes$, $N_{nc}acs \& N_{nc}$-$gos$)) in $P$.

The family of all $N_{nc}ros$ (resp. $N_{nc}rcs$, $N_{nc}Pos$, $N_{nc}Pcs$, $N_{nc}Sos$, $N_{nc}Scs$, $N_{nc}acs$, $N_{nc}bos$, $N_{nc}bes$, $N_{nc}aos$, $N_{nc}acs$, $N_{nc}gos$, $N_{nc}ros$, $N_{nc}$-$os$) of $P$ is denoted by $N_{nc}$-$ROS(P)$ (resp. $N_{nc}$-$RCS(P)$, $N_{nc}$-$POS(P)$, $N_{nc}$-$PCS(P)$, $N_{nc}$-$SOS(P)$, $N_{nc}$-$SCS(P)$, $N_{nc}$-$oS(P)$, $N_{nc}$-$CS(P)$, $N_{nc}$-$OS(P)$, $N_{nc}$-$CS(P)$).

**Definition 2.6** [20] A set $H$ is said to be a

(i) $N_{nc}$$\delta$ interior of $H$ (vriefly, $N_{nc}$-$int(H)$) is defined by $N_{nc}$-$int(H) = \bigcup \{S : S \subseteq H \land S$ is a $N_{nc}ros\}$.

(ii) $N_{nc}$$\delta$ closure of $H$ (vriefly, $N_{nc}$-$cl(H)$) is defined by $N_{nc}$-$cl(H) = \bigcup \{p \in P : N_{nc}$-$int(N_{nc}$-$cl(H)) \cap H \neq \emptyset, p \in H \land H$ is a $N_{nc}os\}$.

**Definition 2.7** [20] A set $H$ is said to be a

(i) $N_{nc}$$\delta$-open set (vriefly, $N_{nc}dos$) if $H = N_{nc}$-$int(H)$.

(ii) $N_{nc}$$\delta$-pre open set (vriefly, $N_{nc}$$\delta$-Pos) if $H \subseteq N_{nc}$-$int(N_{nc}$-$cl(H))$.

(iii) $N_{nc}$$\delta$-semi open set (vriefly, $N_{nc}$$\delta$-Sos) if $H \subseteq N_{nc}$-$cl(N_{nc}$-$int(H))$.

(iv) $N_{nc}$$\delta$- open set [21] (vriefly, $N_{nc}$-$os$) if $H \subseteq N_{nc}$-$cl(N_{nc}$-$int(H)) \cup N_{nc}$-$int(N_{nc}$-$cl(H))$.

The complement of an $N_{nc}$$\delta$os (resp. $N_{nc}$$\delta$Pos, $N_{nc}$$\delta$Sos & $N_{nc}$$\delta$o) is called an $N_{nc}$$\delta$ (resp. $N_{nc}$$\delta$-pre, $N_{nc}$$\delta$-semi & $N_{nc}$$\delta$) closed set (vriefly, $N_{nc}$$\delta$cs (resp. $N_{nc}$$\delta$Pcs, $N_{nc}$$\delta$Scs & $N_{nc}$$\delta$cs)) in $P$.

The family of all $N_{nc}$$\delta$os (resp. $N_{nc}$$\delta$Pcs, $N_{nc}$$\delta$Scs, $N_{nc}$$\delta$Sos, $N_{nc}$$\delta$aos, $N_{nc}$$\delta$acs, $N_{nc}$$\delta$os & $N_{nc}$$\delta$ecs) of $P$ is denoted by $N_{nc}$-$ROS(P)$ (resp. $N_{nc}$-$RCS(P)$, $N_{nc}$-$POS(P)$, $N_{nc}$-$PCS(P)$, $N_{nc}$$\delta$POS(P), $N_{nc}$$\delta$PCS(P), $N_{nc}$$\alpha$OS(P), $N_{nc}$$\alpha$CS(P), $N_{nc}$-$oS(P)$ & $N_{nc}$-$CS(P)$).

The $N_{nc}$$\xi$-interior of $H$ (vriefly, $N_{nc}$$\xi$-$int(H)$) and $N_{nc}$$\xi$-closure of $H$ (vriefly, $N_{nc}$$\xi$-$cl(H)$) are defined as $N_{nc}$$\xi$-$int(H) = \bigcup \{G : G \subseteq H \land G$ is a $N_{nc}$$\xi$ set in $U\}$ & $N_{nc}$$\xi$-$cl(H) = \bigcap \{F : H \subseteq F \land F$ is a $N_{nc}$$\xi$ set in $U\}$.

The $N_{nc}$ semiclosure, $N_{nc}$ preclosure, $N_{nc}$-$b$-closure and $N_{nc}$$\alpha$-closure are similarly defined and are denoted by $N_{nc}$-$ScS(S)$, $N_{nc}$-$PcS(S)$, $N_{nc}$-$bcl(S)$ and $N_{nc}$$\alpha$-$cl(S)$ respectively.

**Definition 2.8** [22] Let $S$ be a $N_{nc}$ set on a $N_{nc}$ts $P$ is said to be $N_{nc}$$\delta$-regular (vriefly, $N_{nc}$$\delta$) if it is $N_{nc}$$\delta$o and $N_{nc}$$\delta$.$

A point $p$ of $P$ is called an $N_{nc}$$\delta$-$\theta$-cluster point of $S$ if $N_{nc}$-$cl(L) \cap S \neq \emptyset$ for every $N_{nc}$$\delta$o set $L$ containing $p$. The set of all $N_{nc}$$\delta$-$\theta$-cluster points of $S$ is called the $N_{nc}$$\delta$-$\theta$-closure of $S$ and is denoted by $N_{nc}$-$cl\theta(S)$. A subset $S$ is said to be $N_{nc}$$\delta$-$\theta$-closed (vriefly, $N_{nc}$$\delta$$\theta$c) if $S = N_{nc}$-$cl\theta(S)$. The complement of an $N_{nc}$$\delta$$\theta$c set is called an $N_{nc}$$\delta$$\theta$-open (vriefly, $N_{nc}$$\delta$$\theta$o) set. Also it is noted in that

$$N_{nc}$$\delta$$\theta \Rightarrow N_{nc}$$\delta$$\theta$$o \Rightarrow N_{nc}$$\delta$$o$$

The family of all $N_{nc}$$\delta$o (resp. $N_{nc}$$\delta$c, $N_{nc}$$\delta$er, $N_{nc}$$\delta$$\theta$o, $N_{nc}$$\delta$$\theta$c) subsets of $P$ is denoted by $N_{nc}$$\delta$OS($P$) (resp. $N_{nc}$$\delta$CS($P$), $N_{nc}$$\delta$RS($P$), $N_{nc}$$\delta$$\theta$OS($P$), $N_{nc}$$\delta$$\theta$CS($P$)). The family of all $N_{nc}$$\delta$c (resp. $N_{nc}$$\delta$c, $N_{nc}$$\delta$er, $N_{nc}$$\delta$$\theta)o, $N_{nc}$$\delta$$\theta$c) sets of $P$ containing a point $p$ of $P$ is denoted by $e$OS($P$, $p$) (resp. $N_{nc}$$\delta$CS($P$, $p$), $N_{nc}$$\delta$RS($P$, $p$), $N_{nc}$$\delta$$\theta$OS($P$, $p$), $N_{nc}$$\delta$$\theta$CS($P$, $p$)).

**Definition 2.9** [22] The family of $N_{nc}$$\delta$ro sets of a space $(P, N_{nc}$$\tau$)$\phi$"""ns a base for a smaller topology $N_{nc}$$\tau$"""n on $P$ called $N_{nc}$$\tau$ semi-regularization of $N_{nc}$$\tau$. The space $(P, N_{nc}$$\tau$)$\phi$"""ns to be $N_{nc}$ semi-regular if $N_{nc}$$\tau$"""n = $N_{nc}$$\tau$.\""\"
A space \((P, N_{nc}\tau)\) is called almost \(N_{nc}\) regular (vriefly, \(aN_{nc}Reg\)) if for any \(N_{nc}o\) set \(L \subseteq P\) and each paint \(p \in L\), there is a \(N_{nc}o\) set \(M\) of \(P\) such that \(p \in V \subseteq N_{nc}cl(M) \subseteq L\).

**Lemma 2.1** [22] Let \(P\) be a \(N_{nc}ts\). If \(S\) is a \(N_{nc}Po\) set in \(P\), then \(N_{nc}Scl(S) = N_{nc}Int(N_{nc}cl(S))\).

**Lemma 2.2** [22] Let \(P\) be a \(N_{nc}ts\) and \(S \subseteq P\) and \(\{S_\alpha | \alpha \in \Lambda\} \subseteq \mathcal{P}(P)\). Then the following statements hold:

1. \(S \in N_{nc}eO(P)\) iff \(N_{nc}ecl(S) \in N_{nc}eR(P)\).
2. \(S\) is \(N_{nc}e\) in \(P\) iff for each \(p \in S\), there exixts \(O \in N_{nc}eRS(P, p)\) such that \(O \subseteq S\).
3. If \(S_\alpha\) is \(N_{nc}e\) in \(P\) hor each \(\alpha \in \Lambda\), then \(\bigcup \alpha S_\alpha\) is \(N_{nc}e\) in \(P\).
4. \(S \in N_{nc}eRS(P)\) iff \(S\) is \(N_{nc}e\) and \(N_{nc}e\).

**Lemma 2.3** [22] In a \(N_{nc}\) space \(P\), the intersection of an \(N_{nc}a\)-open set and an \(N_{nc}eo\) set is an \(N_{nc}eo\) set.

**Definition 3.1** A function \(h : P \to Q\) is said to be almost strongly \(N_{nc}\theta e\)-continuous (vriefly, \(astN_{nc}\theta eCts\)) if for each \(p \in P\) and each \(N_{nc}o\) set \(M\) containing \(h(p)\), there exists an \(N_{nc}eo\) set \(L\) in \(P\) containing \(p\) such that \(h(N_{nc}ecl(L)) \subseteq N_{nc}int(N_{nc}cl(M))\).

3. Comparisons and Some Properties

**Definition 3.2** A function \(h : P \to Q\) is called \(astN_{nc}\theta eCts\) (resp. \(astN_{nc}o\)) if for each \(p \in P\) and each \(N_{nc}o\) set \(M\) containing \(h(p)\), there is an \(N_{nc}eo\) set \(L\) containing \(p\) such that \(h(N_{nc}ecl(L)) \subseteq M\) (resp. \(h(L) \subseteq M\)).

**Definition 3.3** A function \(h : P \to Q\) is called \(N_{nc}\theta e\)-continuous (vriefly \(N_{nc}\theta eCts\)) if for each \(p \in P\) and each \(N_{nc}o\) set \(M\) containing \(h(p)\), there is an \(N_{nc}eo\) set \(L\) containing \(p\) such that \(h(N_{nc}ecl(L)) \subseteq N_{nc}cl(M)\).

**Remark 3.1** From Definitions 3.1, 3.2 and 3.3, we have the following diagram.

![Diagram](image)

However, none of these implications is reversible as shown by the following examples.

**Example 3.1** Let \(P = \{u, v, w, x, y\}\), \(N_{nc}\tau_1 = \{\phi, P, S, S_2, S_3\}\), \(N_{nc}\tau_2 = \{\phi, P\}\). \(S_1 = \{\{w\}, \{x, y\}\}, S_2 = \{\{u, v\}, \{\phi\}\}, S_3 = \{\{u, v, w\}, \{\phi\}\}, \theta = \{\{u, v, w, x, y\}\}\). Then we have \(2_{nc}\tau = \{\phi, P, S, S_2, S_3\}\). Define \(h : (P, 2_{nc}\tau) \to (P, 2_{nc}\tau)\) be an identity function. Then \(h\) is a...
(i) $ast_{nc}thetaSCts$ but not $ast_{nc}thetaCts$.
(ii) $ast_{nc}thetaCts$ but not $ast_{nc}thetaPCts$.
(iii) $ast_{nc}thetaCts$ but not $ast_{nc}thetaPCts$.
(iv) $2_{nc}eCts$ but not $2_{nc}eCts$.

Example 3.2 Let $P = \{u, v, w, x, y\} = Y$, $\{\phi, P_N, S_1, S_2, S_3\}$, $\{\phi, P_N\}$, $S_1 = \{\{w\}, \{\phi\}, \{u, v, x, y\}\}$, $S_2 = \{\{u, v\}, \{\phi\}, \{w, x, y\}\}$, $S_3 = \{\{u, v, w\}, \{\phi\}, \{x, y\}\}$, then we have $2_{nc}^\tau = \{\phi, P_N, S_1, S_2, S_3\}$. $\{\phi, Q_N, D, E\}$, $\{\phi, Q_N\}$, $D = \{\{v, x\}, \{\phi\}, \{u, w, y\}\}$, $E = \{\{u, w, y\}, \{\phi\}, \{v, x\}\}$, then we have $2_{nc}^\sigma = \{\phi, Q_N, D, E\}$. Define $h : (P, 2_{nc}^\tau) \to (Q, 2_{nc}^\sigma)$ be an identity function. Then $h$ is a

(i) $ast_{nc}thetaPCts$ but not $ast_{nc}thetaCts$.
(ii) $ast_{nc}thetaCts$ but not $ast_{nc}thetaSCts$.
(iii) $ast_{nc}thetaCts$ but not $ast_{nc}thetaSCts$.

Example 3.3 Let $P = \{u, v, w, x, y\} = Y$, $\{\phi, P_N, S_1, S_2, S_3\}$, $\{\phi, P_N\}$, $S_1 = \{\{w\}, \{\phi\}, \{u, v, x, y\}\}$, $S_2 = \{\{u, v\}, \{\phi\}, \{w, x, y\}\}$, $S_3 = \{\{u, v, w\}, \{\phi\}, \{x, y\}\}$, then we have $2_{nc}^\tau = \{\phi, P_N, S_1, S_2, S_3\}$. $\{\phi, Q_N, D, E\}$, $\{\phi, Q_N\}$, $D = \{\{u, w\}, \{\phi\}, \{v, x, y\}\}$, then we have $2_{nc}^\sigma = \{\phi, Q_N, D\}$. Define $h : (P, 2_{nc}^\tau) \to (Q, 2_{nc}^\sigma)$ be an identity function. Then $h$ is an $ast_{nc}thetaCts$ but not $ast_{nc}thetaSCts$.

Example 3.4 Let $P = \{u, v, w, x\} = Y$, $\{\phi, P_N, S_1, S_2, S_3, D\}$, $\{\phi, P_N\}$, $S_1 = \{\{u\}, \{\phi\}, \{v, w, x\}\}$, $S_2 = \{\{w\}, \{\phi\}, \{u, v, x\}\}$, $S_3 = \{\{u, w\}, \{\phi\}, \{v, x\}\}$, $D = \{\{u, w, x\}, \{\phi\}, \{v\}\}$, $E = \{\{w, x\}, \{\phi\}, \{u, v\}\}$, then we have $2_{nc}^\tau = \{\phi, P_N, S_1, S_2, S_3, D\}$. $\{\phi, Q_N, F, G, H\}$, $\{\phi, Q_N\}$, $D = \{\{w, x\}, \{\phi\}, \{u, v\}\}$, $H = \{\{u, w, x\}, \{\phi\}, \{v\}\}$, $I = \{\{x\}, \{\phi\}, \{u, v, w\}\}$, $J = \{\{v, w, x\}, \{\phi\}, \{u\}\}$, then we have $2_{nc}^\sigma = \{\phi, Q_N, F, G, H, I, J\}$. Define $h : (P, 2_{nc}^\tau) \to (Q, 2_{nc}^\sigma)$ be $h(u) = h(w) = h(p) = u, h(v) = w$, then $h$ is an $ast_{nc}thetaCts$ but not $ast_{nc}thetaCts$.

Theorem 3.1 Let $h : P \to Q$ be a function. Then the following statements hold:

(i) If $h : P \to Q$ is $a_{nc}Reg$ and $Q$ is $a_{nc}Reg$, then $h$ is an $ast_{nc}thetaCts$.
(ii) If $h : P \to Q$ is an $ast_{nc}thetaCts$ and $Q$ is $N_{nc}$ semi-regular, then $h$ is $st_{nc}thetaCts$.

Proof. (i) Let $h$ be $N_{nc}Cts$ and $Q$ an $a_{nc}Reg$. We have $(p \in P)(M \in N_{nc}ROS(Q, h(p)))$, $Q$ is an $a_{nc}Reg$

$$\Rightarrow (\exists O \in N_{nc}ROS(Q, h(p)))(O \subseteq N_{nc}cld(O) \subseteq M); h \text{ is } N_{nc}Cts$$

$$\left\{\begin{array}{l}
\Rightarrow (\exists L \in N_{nc}OS(P, p))(h(L) \subseteq O \Rightarrow L \subseteq h^{-1}(O)) \\
q \notin N_{nc}cld(O) \Rightarrow (\exists \underline{G} \in U(q))(\underline{G} \cap h(\phi) \Rightarrow h^{-1}(G) \cap h^{-1}(O) = \phi)
\end{array}\right\} \Rightarrow h^{-1}(G) \cap L = \phi.
$$

(1)

$$G \in \underline{U}(q); h \text{ is } N_{nc}Cts \Rightarrow h^{-1}(G) \in N_{nc}OS(P)
$$

(2)
equation (1), (2) $\Rightarrow h^{-1}(G) \cap N_{nc}cld(L) = \phi \Rightarrow G \cap h(N_{nc}cld(L)) = \phi \Rightarrow q \notin h(N_{nc}cld(L))$.

(ii) Let $h$ be an $ast_{nc}thetaCts$ and $Q$ $N_{nc}$ semi-regular. We have $(p \in P)(M \in U(Q, h(p)))$, $Q$ is

$N_{nc}$ semi-regular

$$\Rightarrow (\exists O \in N_{nc}ROS(P, p))(O \subseteq M); h \text{ is } ast_{nc}thetaCts. \Rightarrow (\exists O \in N_{nc}OS(P, p))(h(N_{nc}cld(L)) \subseteq O \subseteq M).$$
Theorem 3.2 Let $Q$ be a $N_{nc}$ semi-regular space. Then $h : P \to Q$ is $astN_{nc}\theta\thetaCts$ iff $h : P \to Q$ is $stN_{nc}\theta\thetaCts$.

Proof. It follows clearly from Theorem 3.1.

Corollary 3.1 Let $Q$ be a $N_{nc}Reg$ space. Then the following statements are equivalent for a function $h : P \to Q$ :

(i) $h$ is $stN_{nc}\theta\thetaCts$,
(ii) $h$ is $astN_{nc}\theta\thetaCts$,
(iii) $h$ is $N_{nc}\theta\thetaCts$,
(iv) $h$ is $N_{nc}e\thetaCts$.

Definition 3.4 A space $P$ is called (i) $N_{nc}$ submaximal if each $N_{nc}$ dense subset of $P$ is $N_{nc}o$ in $P$, (ii) $N_{nc}$ extremally disconnected (briefly, $N_{nc}ED$) if the $N_{nc}$ closure of each $N_{nc}o$ set of $P$ is $N_{nc}o$ in $P$.

In an $N_{nc}ED$ submaximal regular space, $N_{nc}o$, $N_{nc}P\theta o$, $N_{nc}S\theta o$, $N_{nc}b\theta o$ and $N_{nc}eo$ sets are equivalent. Then we have the following corollary:

Corollary 3.2 Let $P$ be an $N_{nc}ED$ submaximal regular space and let $Q$ be a $N_{nc}$ regular space. Then the following statements are equivalent for a function $h : P \to Q$ :

(i) $h$ is $astN_{nc}\theta\thetaCts$,
(ii) $h$ is $astN_{nc}\theta\thetaPCts$,
(iii) $h$ is $astN_{nc}\theta\thetaSCts$,
(iv) $h$ is $astN_{nc}\theta\thetaCts$,
(v) $h$ is $astN_{nc}\theta\thetaCts$,
(vi) $h$ is $stN_{nc}\theta\thetaCts$,
(vii) $h$ is $astN_{nc}\theta\thetaCts$,
(viii) $h$ is $N_{nc}b\theta Cts$,
(ix) $h$ is $N_{nc}e\theta Cts$.

4. Fundamental Properties

Lemma 4.1 Let $P$ be a $N_{nc}ts$ and $P_0$ an $N_{nc}ao$ set in $P$. Then :

(i) $P_0 \cap N_{nc}eOS(P) := \{ P_0 \cap E \mid E \in N_{nc}eOS(P) \} = N_{nc}eOS(P_0)$.
(ii) If $S \subseteq P_0$ and $S \in N_{nc}eOS(P_0)$, then $S \in N_{nc}eOS(P)$.
(iii) If $F \subseteq P_0$ and $F \in N_{nc}eCS(P_0)$, then $F \in N_{nc}eCS(P)$.

Proof. (i) Let $S \in N_{nc}eOS(P_0)$. Then

$$S \in N_{nc}eOS(P_0) \implies S \in P_0 \cap N_{nc}eOS(P) \implies (\exists E \in N_{nc}eOS(P))(S = P_0 \cap E) \implies S \in N_{nc}eOS(P).$$

(ii) Let $F \in N_{nc}eCS(P_0)$. Then

$$F \in N_{nc}eCS(P_0) \implies X \setminus F \in N_{nc}eOS(P_0) \implies X \setminus F \in N_{nc}eOS(P) \implies F \in N_{nc}eCS(P).$$
**Lemma 4.2** If $S \subseteq P_0 \subseteq P$ and $P_0$ is an $N_{nc}ao$ set in $P$, then $N_{nc}ecl(S) \cap P_0 = N_{nc}ecl_{P_0}(S)$, where $N_{nc}ecl_{P_0}(S)$ denotes the $N_{nc}$-closure of $S$ in the subspace $P_0$.

**Proof.** Let $p \in N_{nc}ecl(S) \cap P_0$ and $L \in N_{nc}eOS(P_0, p)$. We have

$$
(p \in N_{nc}ecl(S) \cap P_0)(L \in N_{nc}eOS(P_0, p)) \iff (\exists M \in N_{nc}eOS(P, p))(L = M \cap P_0),
$$

$$
p \in N_{nc}ecl(S)
$$

$\Rightarrow \phi \neq M \cap S = L \cap P_0 \Rightarrow p \in N_{nc}ecl_{P_0}(S)$. Then we have $N_{nc}ecl(S) \cap P_0 \subseteq N_{nc}ecl_{P_0}(S)$.

$$
(p \in N_{nc}ecl_{P_0}(S))(L \in N_{nc}eOS(P, p)) \iff (L \cap P_0 \in N_{nc}eOS(P, p))(\phi \neq S \cap (L \cap P_0) = S \cap L)
$$

$\Rightarrow p \in N_{nc}ecl(S)$

(3)

$$
p \in N_{nc}ecl_{P_0}(S) \subseteq P_0 \Rightarrow p \in P_0
$$

(4)

(3), (4) $\Rightarrow p \in N_{nc}ecl(S) \cap X$. Then we have $N_{nc}ecl_{P_0}(S) \subseteq N_{nc}ecl(S) \cap P_0$.

**Lemma 4.3** Let $G \subseteq P_0 \subseteq P$ and $P_0$ be an $N_{nc}ao$ set in $P$. If $G$ is an $N_{nc}e\theta o$ set in $P_0$, then $G$ is an $N_{nc}e\theta o$ set in $P$.

**Proof.** Let $G \in N_{nc}e\theta OS(P_0, p)$. Then

$$
G \in N_{nc}e\theta OS(P_0, p) \iff (\exists L \in N_{nc}eOS(P_0, p))(L \subseteq N_{nc}ecl(L) \subseteq G)
$$

$\iff N_{nc}ecl_{P_0}(L) \in N_{nc}eCS(P_0)$

$\iff \forall L \subseteq N_{nc}ecl(L) \subseteq N_{nc}ecl(N_{nc}ecl_{P_0}(L)) = N_{nc}ecl_{P_0}(L) \subseteq G
$$

$\Rightarrow p \in N_{nc}ecl_{P_0}(L) \subseteq G
$$

**Lemma 4.4** If $P_0$ is an $N_{nc}ao$ set and $L$ is an $N_{nc}e\theta o$ set in $P$, then $L \cap P_0$ is an $N_{nc}e\theta o$ set in the relative topology of $P_0$.

**Proof.** Let $P_0$ be an $N_{nc}ao$ set in $P$ and $L \in N_{nc}e\theta OS(P)$. Then

$$
p \in L \cap P_0 \Rightarrow (p \in L)(p \in P_0); L \in N_{nc}e\theta OS(P) \iff (\exists T \in N_{nc}eOS(P, p))(N_{nc}ecl(T) \subseteq L)
$$

$\iff (T \cap P_0 \in N_{nc}eOS(P_0, p))(T \cap P_0 \subseteq N_{nc}ecl(T \cap P_0) \subseteq L \cap P_0)
$$

$\iff N_{nc}ecl(T \cap P_0) \subseteq N_{nc}ecl(T) \cap P_0 \subseteq L \cap P_0)
$$

$\Rightarrow p \in N_{nc}ecl_{P_0}(L \cap P_0).
$$

**Corollary 4.1** If $P_0$ is an $N_{nc}ao$ set and $L$ is an $N_{nc}e\theta o$ set in $P$, then $L \cap P_0$ is an $N_{nc}e\theta o$ set in $P$. 


Theorem 4.1 Let \( \{L_\alpha | \alpha \in \Lambda \} \) be an \( N_{nc,ao} \) cover of a \( N_{nc} \)ts \( P \). A function \( h: (P, N_{nc}) \rightarrow (Q, N_{nc}) \) is \( astN_{nc}e\theta Cts \) iff the restriction \( h|_{L_\alpha}: (L_\alpha, N_{nc} \cap L_\alpha) \rightarrow (Q, N_{nc}) \) is \( astN_{nc}e\theta Cts \) for each \( \alpha \in \Lambda \).

**Proof.** Necessity. Let \( h \) be \( astN_{nc}e\theta Cts \) and \( \alpha_0 \in \Lambda \) and \( p \in L_{\alpha_0} \). Then
\[
(h(p) \in M \in N_{nc}) (\text{hast} N_{nc}e\theta Cts) \Rightarrow (\exists G \in N_{nc}eOS(P,p))(h(N_{nc}ecl(G)) \subseteq N_{nc}int(N_{nc}cl(M))) \nonumber
\]
\[
O := G \cap L_{\alpha_0}
\]
\[
\text{Lemma 4.1, 4.2} \quad (p \in O \in N_{nc}eOS(L_{\alpha_0}))(N_{nc}ecl_{L_{\alpha_0}}(O) \subseteq N_{nc}ecl(O)) \nonumber
\]
\[
\Rightarrow (O \in N_{nc}eOS(L_{\alpha_0},p))(h|_{L_{\alpha_0}}(N_{nc}ecl_{L_{\alpha_0}}(O)) = h(N_{nc}ecl_{L_{\alpha_0}}(O)) \subseteq h(N_{nc}ecl(O)) \subseteq N_{nc}int(N_{nc}cl(M))). \nonumber
\]

Sufficiency. Let \( h|_{L_\alpha} \) be a \( astN_{nc}e\theta Cts \) for all \( \alpha \in \Lambda \) and \( M \in N_{nc}eROS(Q) \). Then

\[
M \in N_{nc}eROS(Q), h|_{L_\alpha} \text{ is } astN_{nc}e\theta Cts \Rightarrow \text{Theorem 3.1 in [22]} \quad (\forall \alpha \in \Lambda)((h|_{L_\alpha})^{-1}(M) \in N_{nc}e\theta OS(L_\alpha)) \nonumber
\]
\[
\text{Lemma 4.3 sub} \quad (\forall \alpha \in \Lambda)((h|_{L_\alpha})^{-1}(M) \in N_{nc}e\theta OS(P)) \nonumber
\]
\[
\Rightarrow h^{-1}(M) = h^{-1}(M) \cap P = h^{-1}(M) \cap \left( \bigcup_{\alpha \in \Lambda} L_\alpha \right) = \bigcup \{h^{-1}(M) \cap L_\alpha | \alpha \in \Lambda \} \nonumber
\]
\[
\Rightarrow h^{-1}(M) = \bigcup \{(h|_{L_\alpha})^{-1}(M) | \alpha \in \Lambda \} \quad (5)
\]

(5), (6) \Rightarrow h^{-1}(M) \in N_{nc}e\theta OS(P).

**Definition 4.1** A function \( h: P \rightarrow Q \) is called an \( N_{nc}R \)-map if the preimage of every \( N_{nc}ro \) set of \( Q \) is \( N_{nc}ro \) in \( P \).

**Definition 4.2** A function \( h: P \rightarrow Q \) is called \( N_{nc}\delta \)-continuous [20] (briefly, \( N_{nc}\delta Cts \)) if for each \( p \in P \) and each \( N_{nc,o} \) set \( M \) containing \( h(p) \), there is an \( N_{nc,o} \) set \( L \) containing \( p \) such that \( h(N_{nc}int(N_{nc}cl(L))) \subseteq N_{nc}int(N_{nc}cl(M)) \).

**Theorem 4.2** Let \( h: P \rightarrow Q \) and \( g: Q \rightarrow Z \) be two functions. Then:

(i) If \( h \) is \( astN_{nc}e\theta Cts \) and \( g \) is an \( N_{nc}R \)-map, then \( g \circ h \) is \( astN_{nc}e\theta Cts \).

(ii) If \( h \) is \( astN_{nc}e\theta Cts \) and \( g \) is \( N_{nc}\delta Cts \), then \( g \circ h \) is \( astN_{nc}e\theta Cts \).

**Proof.** Clear.

**Theorem 4.3** Let \( h: P \rightarrow Q \) be a function and \( g: Q \rightarrow Z \) an injective \( N_{nc}R \)-map which preserves \( N_{nc}ro \) sets. Then \( h \) is \( astN_{nc}e\theta Cts \) iff \( g \circ h \) is \( astN_{nc}e\theta Cts \).

**Proof.** Necessity. It follows from Theorem 4.2.

Sufficiency. Let \( g \circ h \) be \( astN_{nc}e\theta Cts \) and let \( g \) be an injective \( N_{nc}R \)-map which preserves \( N_{nc}ro \) sets.

\[
v \in N_{nc}ROS(Q) \Rightarrow v(M) \in N_{nc}ROS(Z) \Rightarrow g(M) \in N_{nc}ROS(Q) \Rightarrow g \text{ is } N_{nc}R \text{-map and injective} \nonumber
\]
\[
\Rightarrow M = g^{-1}(g(M)) \in N_{nc}ROS(Q) \Rightarrow h^{-1}(M) = h^{-1}(g^{-1}(g(M))) = (g \circ h)^{-1}(g(M)), g \circ h \text{ is } astN_{nc}e\theta Cts \nonumber
\]
\[
\Rightarrow h^{-1}(M) \in N_{nc}e\theta OS(P). \nonumber
\]
Theorem 4.4 Let \( \{Q_\alpha | \alpha \in \Lambda \} \) be a family of spaces. If a function \( h : P \rightarrow \Pi Q_\alpha \) is ast\( N_{nc}e\theta Cts \) then \( P_\alpha \circ h : P \rightarrow Q_\alpha \) is ast\( N_{nc}e\theta Cts \) for each \( \alpha \in \Lambda \), where \( P_\alpha \) is the projection of \( \Pi Q_\alpha \) onto \( Q_\alpha \).

Proof. This is obvious from theorem 4.2 because every \( N_{nc} \) open continuous surjection \( P_\alpha \) is an \( N_{nc}R \)-map.

5. Separation Axioms

Definition 5.1 A space \( P \) is called almost \( N_{nc} \)-regular (vriefly, \( aN_{nc}e\)Reg) if for any \( N_{nc}c\) set \( F \subseteq P \) and any point \( p \in P \backslash F \), there exist disjoint \( N_{nc}eo \) sets \( L \) and \( M \) such that \( p \in L \) and \( F \subseteq M \).

Theorem 5.1 The following statements are equivalent for a space \( P \):

(i) \( P \) is \( aN_{nc}e \) regular,

(ii) for each \( p \in P \) and for each \( N_{nc}c \) set \( L \) of \( P \) containing \( p \), there exists \( M \in N_{nc}e\)OS\( (P) \) such that \( p \in V \subseteq N_{nc}ecl(M) \subseteq L \),

(iii) for each \( N_{nc}c \) set \( F \) of \( P \), \( F = \cap \{N_{nc}ecl(M)|F \subseteq M \) and \( M \in N_{nc}e\)OS\( (P) \} \),

(iv) for each subset \( S \subseteq P \) and each \( N_{nc}c \) set \( F \) such that \( S \cap F = \emptyset \), there exist disjoint \( L, M \in N_{nc}e\)OS\( (P) \) such that \( S \cap L \neq \emptyset \) and \( F \subseteq M \),

(v) for each subset \( S \subseteq P \) and each \( N_{nc}c \) set \( L \) such that \( S \cap L \neq \emptyset \), there exists \( O \in N_{nc}e\)OS\( (P) \) such that \( S \cap W \neq \emptyset \) and \( N_{nc}ecl(O) \subseteq L \).

Proof. It can be proved directly.

Theorem 5.2 If a \( N_{nc}Cts \) function \( h : P \rightarrow P \) is ast\( N_{nc}e\theta Cts \) then \( P \) is \( aN_{nc}e \) regular.

Proof. Let \( h \) be the identity function. Then \( h \) is \( N_{nc}Cts \) and ast\( N_{nc}e\theta Cts \) so, \( p \in L \in N_{nc}e\)OS\( (P) \), \( h \) is identity and ast\( N_{nc}e\theta Cts \)

\[ \Rightarrow p \in h^{-1}(L) = L \in N_{nc}e\)OS\( (P) \]

\[ \text{Lemma 5.2} \ (\exists M \in N_{nc}e\)OS\( (P, p))(M \subseteq N_{nc}ecl(M) \subseteq L)). \]

Theorem 5.3 An \( N_{nc}R \)-map \( h : P \rightarrow P \) is ast\( N_{nc}e\theta Cts \) iff \( P \) is \( aN_{nc}e \) regular.

Proof. Necessity. Obvious.

Sufficiency. Let \( h \) be an \( N_{nc}R \)-map and \( P \) be \( aN_{nc}e \) regular.

\( (p \in P)(M \in N_{nc}e\)OS\( (Q, h(p)) \)), \( h \) is \( N_{nc}R \)-map

\[ \Rightarrow (p \in h^{-1}(M) \in N_{nc}e\)OS\( (P)): P \) is almost \( N_{nc}e \) regular

\[ \text{Theorem 5.1} \ (\exists L \in N_{nc}e\)OS\( (P, p))(N_{nc}ecl(L) \subseteq h^{-1}(M)) \]

\[ \Rightarrow (\exists UN_{nc}e\)OS\( (P, p))(h(N_{nc}ecl(L)) \subseteq M)). \]

Definition 5.2 A space is called \( N_{nc}e \) regular if for any \( N_{nc}c \) set \( F \subseteq P \) and any point \( p \in P \backslash F \), there exist disjoint \( N_{nc}eo \) sets \( L \) and \( M \) such that \( p \in L \) and \( F \subseteq M \).

Definition 5.3 A function \( h : P \rightarrow Q \) is called almost \( N_{nc} \) continuous (vriefly, \( aN_{nc}Cts \)) if the preimage of every \( N_{nc}c \) set of \( Q \) is \( N_{nc}eo \) in \( P \).

Theorem 5.4 If \( h : P \rightarrow Q \) is \( aN_{nc}Cts \) and \( P \) is \( N_{nc}e \) regular, then \( h \) is ast\( N_{nc}e\theta Cts \).
**Proof.** Let \( p \in P \) and let \( M \in N_{nc}ROS(Q, h(p)) \). Then

\[
(p \in P)(M \in N_{nc}ROS(Q, h(p))), h \text{ is } aN_{nc}Cts \Rightarrow p \in h^{-1}(M) \in N_{nc}r, X \text{ is } N_{nc}r\text{regular}
\]

\[
\Rightarrow (\exists L \in N_{nc}eOS(P, p))(N_{nccel}(L) \subseteq h^{-1}(M))
\]

\[
\Rightarrow (\exists L \in N_{nc}eOS(P, p))(h(N_{nccel}(L)) \subseteq M).
\]

**Theorem 5.5** Let \( h : P \to Q \) be a function and let \( g : P \to P \times Q \), given by \( g(p) = (p, h(p)) \) for each \( p \in P \) be graph function. Then \( g \) is ast\( N_{nc}e\thetaCts \) if \( h \) is ast\( N_{nc}e\thetaCts \) and \( P \) is a\( N_{nc}e \) regular.

**Proof.** Necessity. Let \( p \in P \) and let \( M \in N_{nc}ROS(Q, h(p)) \). Then

\[
(p \in P)(M \in N_{nc}ROS(Q, h(p))) \Rightarrow g(p) = (p, h(p)) \in P \times M; P \times M \in N_{nc}ROS(P \times Q),
\]

\[
g \text{ is ast}N_{nc}e\thetaCts \Rightarrow (\exists L \in N_{nc}eRS(P, p))(g(L) \subseteq P \times M)
\]

\[
\Rightarrow (\exists L \in N_{nc}eRS(P, p))(h(L) \subseteq M). \text{ Then } h \text{ is ast}N_{nc}e\thetaCts
\]

\[
L \in N_{nc}ROS(P, p) \Rightarrow g(p) \in L \times Q \in N_{nc}ROS(P \times Q); g \text{ is } astN_{nc}e\thetaCts
\]

\[
(\exists O \in N_{nc}eOS(P, p))(g(N_{nc}cel(O)) \subseteq L \times Q)
\]

\[
(\exists O \in N_{nc}eOS(P, p))(O \subseteq N_{nc}cel(O) \subseteq L).
\]

Then \( P \) is a\( N_{nc}e \) regular.

**Sufficiency.** Let \( p \in P \) and let \( M \in N_{nc}ROS(P \times Q, g(p)) \). Then

\[
(p \in P)(M \in N_{nc}ROS(P \times Q, g(p))) \Rightarrow (\exists M_1 \in N_{nc}ROS(P))(\exists M_2 \in N_{nc}ROS(Q))g(p) =
\]

\[
(p, h(p)) \in M_1 \times M_2 \subseteq M, h \text{ is ast}N_{nc}e\thetaCts
\]

\[
\Rightarrow (\exists L_0 \in N_{nc}eRS(P, p))(h(L_0) \subseteq M_2)
\]

(7)

\[
L := L_0 \cap M_1 \overset{\text{Lemma 4.4}}{\Rightarrow} L \in N_{nc}e\thetaOS(M_1) \overset{\text{Lemma 4.3}}{\Rightarrow} L \in N_{nc}e\thetaOS(P)
\]

(8)

(7), (8) \Rightarrow (\exists L \in N_{nc}e\thetaOS(P))(g(L) \subseteq L \times h(L) \subseteq L \times h(L_0) \subseteq M_1 \times M_2 \subseteq M).

**Definition 5.4** A space \( P \) is said to be:

(i) \( N_{nc}rT_0 \) if for each pair of distinct points \( p \) and \( q \) in \( P \), there exists a \( N_{nc}r \) set \( L \in N_{nc}ROS(P) \) such that either \( p \in L \) and \( q \notin L \) or \( q \in L \) and \( p \notin L \).

(ii) \( N_{nc}rT_2 \) if for each pair of distinct points \( p \) and \( q \) in \( P \), there exist \( N_{nc}ro \) sets \( L \) and \( M \) of \( P \) containing \( p \) and \( q \), respectively, such that \( L \cap M = \phi \).

**Theorem 5.6** If \( h : P \to Q \) is an ast\( N_{nc}e\thetaCts \) injection and \( Q \) is \( N_{nc}rT_0 \), then \( P \) is \( N_{nc}rT_2 \).

**Proof.** Let \( p_1, p_2 \in P \) and \( p_1 \neq x_2 \). Then

\[
(p_1, p_2) \in P(p_1 \neq x_2) \text{ (} h \text{ is injective) } \Rightarrow h(p_1) \neq h(p_2), Q \text{ is } N_{nc}rT_0
\]

\[
\Rightarrow (\exists M \in N_{nc}ROS(Q, h(p_1)))(\exists O \in N_{nc}ROS(Q, h(p_2)))(h(p_1) \notin W \cup h(p_2) \notin M).
\]

**Case I.** Let \( M \in N_{nc}ROS(Q, h(p_1)) \) and \( h(p_2) \notin M \).

\[
M \in N_{nc}ROS(Q, h(p_1)), h \text{ is ast}N_{nc}e\thetaCts \Rightarrow (\exists L \in N_{nc}eOS(P, p_1))(h(N_{nc}cel(L)) \subseteq M),
\]

\[
h(p_2) \notin M \Rightarrow h(p_2) \notin h(N_{nc}cel(L))
\]

\[
\Rightarrow (p_2) \notin N_{nc}cel(L) \Rightarrow x_2 \in P \backslash N_{nc}cel(L).
\]

**Case II.** It can be proved similarly.
Corollary 5.1 If $h: P \to Q$ is an $ast N_{nc}e\theta Cts$ injection and $Q$ is $N_{nc}$ Hausdorff, then $P$ is $N_{nc}\text{-}T_2$.

Proof. it is obvious since every $N_{nc}$ Hausdorff space is $N_{nc}rT_0$.

Theorem 5.7 Let $h, g: P \to Q$ be functions and $Q$ a $N_{nc}$ Hausdorff space. If $h$ is $ast N_{nc}e\theta Cts$ and $g$ is an $N_{nc}R$-map, then the set $S = \{p \in P| h(p) = g(p)\}$ is $N_{nc}ec$ in $P$.

Proof. Let $p \notin S$. Then

$$p \notin A \Rightarrow h(p) \neq g(p), \text{ } Q \text{ is } N_{nc} \text{ Hausdorff} \Rightarrow (\exists M_1 \in \mathcal{U}(h(p)))(\exists M_2 \in \mathcal{U}(x(p)))(M_1 \cap M_2 = \phi)$$

$$\Rightarrow (\exists M_1 \in \mathcal{U}(h(p)))(\exists M_2 \in \mathcal{U}(g(p)))(N_{nc}\text{int}(N_{nccl}(M))) \cap N_{nc}\text{int}(N_{nccl}(M_2) = \phi) \tag{9}$$

$$N_{nc}\text{int}(N_{nccl}(M_1)) \in N_{nc}\text{ROS}(Q, h(p_1)); h \text{ is } ast N_{nc}e\theta Cts \Rightarrow (\exists G \in N_{nc}\text{ROS}(P, p))(h(N_{nccl}(G)) \subseteq N_{nc}\text{int}(N_{nccl}(M_1))) \tag{10}$$

$$N_{nc}\text{int}(N_{nccl}(M_2)) \in N_{nc}\text{ROS}(Q, h(p_2)); g \text{ is } N_{nc} R\text{-map} \Rightarrow g^{-1}(N_{nc}\text{int}(N_{nccl}(M_2))) \subseteq N_{nc}\text{ROS}(P, p) \tag{11}$$

$$L := G \cap g^{-1}(N_{nc}\text{int}(N_{nccl}(M))) \xrightarrow{Lemma \text{-} 2.3} L \in N_{nc}\text{ROS}(P, p) \tag{12}$$

(9), (10), (11), (12) $\Rightarrow (L \in N_{nc}\text{ROS}(P, p))(L \cap S = \phi) \Rightarrow x \notin N_{nc}\text{ec}(S)$.

6. Preservation Properties

Definition 6.1 A space $P$ is called:

(i) nearly $N_{nc}$ compact (resp. nearly $N_{nc}$ countable compact) if every $N_{nc}ro$ cover (resp. countable $N_{nc}ro$ cover) of $P$ has a finite subcover.

(ii) $N_{nc}ec$ (resp. countable $N_{nc}ec$) if every $N_{nc}$ cover (resp. countable $N_{nc}$ cover) of $P$ by $N_{nc}eo$ sets has a finite subcover whose $N_{nc}c$-closures cover $P$.

A subset $S$ of a $N_{nc}clts$ $P$ is said to be $N_{nc}ec$ (resp. $N$-$N_{nc}ec$) relative to $P$ if for every $N_{nc}$ cover $\{M_{\alpha}| \alpha \in I\}$ of $S$ by $N_{nc}eo$ (resp. $N_{nc}ro$) sets of $P$, there exists a finite subset $I_0$ of $I$ such that $S \subseteq \bigcup\{N_{nccl}(M_{\alpha})| \alpha \in I_0\}$ (resp. $S \subseteq \bigcup\{M_{\alpha}| \alpha \in I_0\}$)

Theorem 6.1 If $h: P \to Q$ is an $ast N_{nc}e\theta Cts$ function and $S$ is $N_{nc}ec$ relative to $P$, then $h(S)$ is $N$-$N_{nc}ec$ relative to $Q$.

Proof. It can be proved directly.

Corollary 6.1 Let $h: P \to Q$ be an $ast N_{nc}e\theta Cts$ surjection. Then the following statements hold:

(i) If $P$ is $N_{nc}ec$, then $Q$ is nearly $N_{nc}$ compact.

(ii) If $P$ is countable $N_{nc}ec$, then $Q$ is $N_{nc}$ nearly countable compact.

Definition 6.2 The graph $G(h)$ of a function $h: P \to Q$ is said to be $N_{nc}e\theta$ closed if for each $(p, q) \in (P \times Q) \setminus G(h)$, there exist $L \in N_{nc}ecos(P, p)$ and an $N_{nc}o$ set $M$ containing $q$ such that $(N_{nc}cl(L) \times stN_{nccl}(M)) \cap G(h) = \phi$.

Definition 6.3 The graph $G(h)$ of a function $h: P \to Q$ is said to be $ast N_{nc}ec$ if for each $(p, q) \in (P \times Q) \setminus G(h)$, there exist $L \in N_{nc}ecos(P, p)$ and a $N_{nc}ro$ set $M$ containing $q$ such that $(N_{nc}cl(L) \times M) \cap G(h) = \phi$.  

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Corollary 6.2 If the graph $G(h)$ of a function $h : P \rightarrow Q$ is $N_{nc}e\theta$ closed, then it is $astN_{nc}e$ closed.

Lemma 6.1 The graph $G(h)$ of a function $h : P \rightarrow Q$ is $astN_{nc}e$-closed in $P \times Q$ iff for each $(p, q) \in (P \times Q) \setminus G(h)$, there exist $L \in N_{nc}eOS(P, p)$ and a $N_{nc}ro$ set $M$ containing $q$ such that $h(N_{nc}ecl(L)) \cap M = \phi$.

Proof. It follows immediately from the definition.

Definition 6.4 The graph $G(h)$ of a function $h : P \rightarrow Q$ is said to be $stN_{nc}e$-closed if for each $(p, q) \in (P \times Q) \setminus G(h)$, there exist $L \in N_{nc}eOS(P, p)$ and an $N_{nc}o$ set $M$ containing $q$ such that $(N_{nc}ecl(L) \times M) \cap G(h) = \phi$.

It is obvious that if the graph of a function is $astN_{nc}e$-closed, then it is $stN_{nc}e$-closed.

Theorem 6.2 If $h : P \rightarrow Q$ is $astN_{nc}e\thetaCts$ and $Q$ is $N_{nc}$ Hausdorff, then the graph $G(h)$ of $h$ is $astN_{nc}e$-closed in $P \times Q$.

Proof. Let $(p, q) \notin G(h)$. Then $(p, q) \notin G(h) \Rightarrow y \neq h(p), Q$ is $N_{nc}$ Hausdorff

$$\Rightarrow (\exists M_1 \in U(Q, h(p)))(\exists M_2 \in U(Q, q))(M_1 \cap M_2 = \phi)$$

$$\Rightarrow (\exists M_1 \in U(h(p)))(\exists M_2 \in U(g(p))(N_{nc}int(N_{nc}cl(M_1))) \cap (N_{nc}int(N_{nc}cl(M_2)) = \phi),$$

$h$ is $astN_{nc}e\thetaCts$

$$\Rightarrow (\exists L \in N_{nc}eOS(P, p))(h(N_{nc}ecl(L)) \cap N_{nc}int(N_{nc}cl(M_2)) = \phi).$$

Then $G(h)$ is $astN_{nc}e$-closed in $P \times Q$ by Lemma 6.1.

Theorem 6.3 If a function $h : P \rightarrow Q$ has an $astN_{nc}e$-closed graph, then $h(K)$ is $N_{nc}e\delta$-closed in $Q$ for each subset $K$ which is $N_{nc}e\delta$ relative to $P$.

Proof. Let $h$ be $astN_{nc}e\thetaCts$ and $q \notin h(K)$. Then

$q \notin h(K) \Rightarrow (\forall p \in K)((p, q) \notin G(h)), G(h)$ is $astN_{nc}e$-closed

$$\Rightarrow (\exists L_p \in N_{nc}eOS(P, p))(\exists M_p \in N_{nc}ROS(Q, q))(h(N_{nc}ecl(L_p)) \cap M_p = \phi)$$

$$\Rightarrow ((\{L_p\} \subset K) \subseteq N_{nc}eOS(P))(K \subseteq \bigcup \{L_p\} \subset K); K is N_{nc}e\delta relative to X$$

$$\Rightarrow (\exists K^* \subset K)(|K^*| < N_0)(K \subseteq \bigcup \{N_{nc}ecl(L_p)|p \in K^*\}), V := \bigcap_{p \in K^*} M_p \in N_{nc}ROS(Q, q)$$

$$\Rightarrow (M \in N_{nc}ROS(Q, q))(h(K) \cap M \subseteq \bigcup_{p \in K^*} h(N_{nc}ecl(L_p))) \cap M = \phi$$

$$\Rightarrow (M \in N_{nc}ROS(Q, q))(h(K) \cap M = \phi) \Rightarrow x \notin N_{nc}e\delta cl(h(K)).$$

Corollary 6.3 If $h : P \rightarrow Q$ is an $astN_{nc}e\thetaCts$ function and $Q$ is $N_{nc}$ Hausdorff, then $h(K)$ is $N_{nc}e\delta c$ in $Q$ for each subset $K$ which is $N_{nc}e\delta c$ relative to $P$.

Theorem 6.4 Let $P$ be a $N_{nc}$ submaximal extremally disconnected regular space and $Q$ be a $N_{nc}$ compact Hausdorff space. Then the following statements are equivalent:

(i) $h$ is $stN_{nc}e\thetaCts$,

(ii) $G(h)$ is $stN_{nc}e$-closed in $P \times Q$,

(iii) $h$ is $stN_{nc}\thetaCts$,
(iv) $h$ is $N_{nc}Cts$,
(v) $h$ is $N_{nc}eCts$.

**Corollary 6.4** Let $P$ be a $N_{nc}$ submaximal extremally disconnected regular space and $Q$ a $N_{nc}$ compact Hausdorff space. Then the following properties are equivalent:

(i) $h$ is $stN_{nc}e\theta Cts$,
(ii) $h$ is $ast\theta eCts$,
(iii) $G(h)$ is $astN_{nc}e$-closed in $P \times Q$,
(iv) $G(h)$ is $stN_{nc}e$-closed in $P \times Q$,
(v) $h$ is $stN_{nc}\theta Cts$,
(vi) $h$ is $N_{nc}Cts$,
(vii) $h$ is $N_{nc}eCts$.

**Proof.** $(2) \Rightarrow (3)$: It follows from Theorem 6.2. Other implications follow from Theorem 6.4.

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