New Interpretation of Skyrme Theory

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ABSTRACT: Based on the proposal that the Skyrme theory is a theory of monopole we provide a new interpretation of Skyrme theory, that the theory can also be viewed as an effective theory of strong interaction which is dual to QCD, where the monopoles (not the quarks) are confined through the Meissner effect. This dual picture leads us to predict the existence of a topological glueball in QCD, a chromoelectric knot which is dual to the chromomagnetic Faddeev-Niemi knot in Skyrme theory, whose mass and decay width are estimated to be around 60 GeV and 8 GeV. As importantly, the existence of the magnetic vortex and the magnetic vortex ring in Skyrme theory strongly indicates that the theory could also be interpreted to describe a very interesting low energy condensed matter physics in a completely different environment. These new interpretations of Skyrme theory puts the theory in a totally new perspective.

KEYWORDS: Meissner effect in Skyrme theory, chromoelectric knot in QCD, Faddeev-Niemi knot in condensed matter physics.
1. Introduction

The Skyrme theory has played an important role in physics, in particular in nuclear physics as a successful effective field theory of strong interaction \cite{1, 2, 3, 4}. This is based on the fact that the theory can be viewed as a non-linear sigma model which describes the pion physics. In this view the baryon is identified as a topological soliton made of pions. The purpose of this paper is to argue that the theory allows totally different interpretations which put it in a new perspective. Based on the proposal that the Skyrme theory is a theory of monopole \cite{5, 6}, we argue that the theory can also be viewed as an effective theory of strong interaction which is dual to Quantum Chromodynamics (QCD). More precisely we argue that the theory is an effective theory of confinement in which the monopoles, not the quarks, in QCD are confined. This provides a new interpretation of Skyrme theory which is orthogonal but complementary to the popular view. Furthermore, based on the fact that the Skyrme theory has a built-in Meissner effect which confines the magnetic flux of monopoles, we also argue that the theory can describe a very interesting low energy condensed matter physics in a completely different environment.
A remarkable feature of Skyrme theory is its rich topological structure \cite{5,6}. It has been well-known that the theory allows the skyrmion and the baby skyrmion \cite{1,7}. But recently it has been shown that it also allows the helical baby skyrmion and the Faddeev-Niemi knot \cite{5,6,8}. More importantly, it contains (singular) monopoles which play a fundamental role. In fact all the finite energy topological objects in the theory could be viewed either as dressed monopoles or as confined magnetic flux of the monopole-antimonopole pair. For example, the skyrmion can be viewed as a dressed monopole, and the baby skyrmion can be viewed as a magnetic vortex created by a monopole-antimonopole pair infinitely separated apart. These observations have led us to propose that the theory can be interpreted as a theory of monopole, in which the monopole-antimonopole pairs are confined to make finite energy bound states \cite{5,6}.

In fact the Skyrme theory is a theory of confinement with a built-in Meissner effect, where the confinement is manifest already at the classical level. It is this Meissner effect which allows us to have the baby skyrmion, which is nothing but the confined magnetic flux of a monopole-antimonopole pair infinitely separated apart. This confinement mechanism allows us to construct more interesting topological objects, the helical baby skyrmion and the Faddeev-Niemi knot. The helical baby skyrmion is a twisted chromomagnetic vortex which is periodic in $z$-coordinate. The importance of the helical baby skyrmion is that it assures the existence of the Faddeev-Niemi knot. This is because the Faddeev-Niemi knot can be viewed as a vortex ring made of the helical baby skyrmion, with periodic ends connected together \cite{5,8}. This allows us to interpret the knot as two magnetic flux rings linked together, the first one winding the second $m$ times and the second one winding the first $n$ times.

In this paper we present a consistent knot ansatz and construct a knot solution numerically. Our solution confirms that the knot can indeed be interpreted as two magnetic flux rings linked together, whose quantum number is given by the product of two magnetic flux quanta $mn$, the linking number of two flux rings. This tells that the knot manifests its topology even at the dynamical level, as the linking of two magnetic flux rings. This dynamical manifestation of knot topology assures a supercurrent and an angular momentum which prevent the collapse of the knot, and thus guarantees the dynamical stability of the knot \cite{6}.

The Skyrme theory has always been interpreted as an effective theory of strong interaction, with the skyrmion identified as the baryon. This interpretation is based on the fact that the Skyrme theory can be viewed as a non-linear sigma model which describes the pion physics (in general the flavor dynamics). In this view the baryon is identified as a topological soliton made of mesons. If so, one may ask how the knot can be interpreted in this picture. Since the skyrmion and the knot have different topology and since the knot has a vanishing baryon number, one may naturally interpret the knot as a new type of topological meson. In this view one can estimate the energy of the knot, which turns out to be around $5 \text{ GeV}$.

In this paper we provide an alternative view. The alternative view follows from the
observation made by Faddeev and Niemi that the Skyrme theory is closely related to QCD \cite{9}. Based on this we argue that the Skyrme theory can be viewed as an effective theory of strong interaction which is dual to QCD. This is because QCD is a theory of confinement in which the chromoelectric flux of the quarks and gluons are confined by the dual Meissner effect, but the Skyrme theory can be viewed as a theory of monopole in which the magnetic flux of monopole-antimonopole pairs are confined by the Meissner effect \cite{5, 6}. Obviously this dual picture is completely orthogonal to the popular view that the Skyrme theory describes the flavor dynamics, because this view advocates that the Skyrme theory describes the chromomagnetic dynamics. Nevertheless this alternative interpretation is worth a serious consideration, and has an interesting prediction. Based on this dual picture between Skyrme theory and QCD, we predict the existence of a chromoelectric knot in QCD which is dual to Faddeev-Niemi knot, a topological glueball of the twisted chromoelectric flux ring. We estimate the mass and decay width of such exotic knot glueball to be around 60 \text{GeV} and 8 \text{GeV} \cite{10}.

The interpretation of the Faddeev-Niemi knot as a twisted magnetic vortex ring suggests that it could also be viewed as a topological object in condensed matter. In fact, recently similar knots have been asserted to exist in condensed matter physics, in particular in multi-component Bose-Einstein condensates \cite{11, 12} and in multi-gap superconductors \cite{13, 14}. In this paper we point out a remarkable similarity which exists between these knots in condensed matters and the Faddeev-Niemi knot in Skyrme theory. In particular we show that the Skyrme-Faddeev theory can also be viewed as a theory of self-interacting two-component superfluid, in which the non-linear Skyrme interaction describes the vorticity interaction of the superfluid. This provides yet another interpretation of the Faddeev-Niemi knot, two quantized vorticity flux rings linked together in two-component superfluid. This strongly indicates that the Skyrme theory, with the built-in Meissner effect, could also describe a very interesting low energy condensed matter physics.

The paper is organized as follows. In Section II we review the the various topological objects and their relationships with singular monopoles in Skyrme theory for later purpose. In particular we show that the skyrmion can be viewed as a dressed monopole whose energy is made finite by the dressing of the massless scalar field of Skyrme theory. In Section III we discuss the helical baby skyrmion, a twisted chromomagnetic vortex which is periodic in $z$-coordinate, in Skyrme theory. As importantly we establish the Meissner effect which confines the magnetic flux of the baby skyrmion in Skyrme theory. In Section IV we present a numerical solution for an axially symmetric Faddeev-Niemi knot, and show that the knot is nothing but a twisted magnetic vortex ring made of a helical baby skyrmion. In Section V we estimate the mass of Faddeev-Niemi knot, and discuss the physical significance of the knot in Skyrme theory. In Section VI we discuss the deep connection between Skyrme theory and (massive) $SU(2)$ gauge theory, and show that the Skyrme theory can describe the chromomagnetic dynamics of $SU(2)$ QCD. In Section VII we propose the existence of a chromoelectric knot in QCD, based on the dual relationship between Skyrme theory and QCD. We estimate the mass and decay width of the lightest chromoelectric knot. In Section VIII we argue that, with the built-in Meissner effect, the Skyrme theory could also describe
an interesting condensed matter physics. In particular, we point out an apparent similarity between the Skyrme theory and the $U(1)$ gauge theory of two-component Bose-Einstein condensates and superfluids. Finally in Section IX we discuss the physical implications of our results.

2. Skyrme Theory: A Review

The Skyrme theory has long been interpreted as an effective field theory of flavor dynamics in strong interaction with a remarkable success [2, 3, 4]. However, it can also be interpreted as a theory of monopole in which the monopole-antimonopole pairs are confined through the Meissner effect [5, 6]. It has singular non-Abelian monopoles very similar to the Wu-Yang monopole in $SU(2)$ QCD which play the key role in the theory. All finite energy topological objects in the theory appear as dressed monopol es or confined magnetic flux of monopole-antimonopole pair. To see this, we review the topological objects in Skyrme theory and their relations first.

Let $\omega$ and $\hat{n}$ ($\hat{n}^2 = 1$) be the massless scalar field and the non-linear sigma field in Skyrme theory, and let

$$L_\mu = U \partial_\mu U^\dagger,$$

$$U = \exp\left(\frac{\omega}{2i} \hat{\sigma} \cdot \hat{n}\right) = \cos \frac{\omega}{2} - i(\hat{\sigma} \cdot \hat{n}) \sin \frac{\omega}{2}.$$  \hspace{1cm} (2.1)

With this one can write the Skyrme Lagrangian as [1]

$$\mathcal{L} = \frac{\mu^2}{4} \mathrm{tr} \left( L_\mu^2 + \frac{\alpha}{32} \mathrm{tr} \left( [L_\mu, L_\nu]^2 \right) \right)$$

$$= -\frac{\mu^2}{4} \left[ \frac{1}{2} (\partial_\mu \omega)^2 + 2 \sin^2 \frac{\omega}{2} (\partial_\mu \hat{n})^2 \right]$$

$$- \frac{\alpha}{16} \left[ \sin^2 \frac{\omega}{2} (\partial_\mu \omega \partial_\nu \hat{n} - \partial_\nu \omega \partial_\mu \hat{n})^2 + 4 \sin^4 \frac{\omega}{2} (\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2 \right],$$  \hspace{1cm} (2.2)

where $\mu$ and $\alpha$ are the coupling constants. The Lagrangian has a hidden $U(1)$ gauge symmetry as well as a global $SU(2)$ symmetry. From the Lagrangian one has the following equations of motion

$$\partial^2 \omega - \sin \omega (\partial_\mu \hat{n})^2 + \frac{\alpha}{8\mu^2} \sin \omega (\partial_\mu \omega \partial_\nu \hat{n} - \partial_\nu \omega \partial_\mu \hat{n})^2$$

$$+ \frac{\alpha}{\mu^2} \sin^2 \frac{\omega}{2} \partial_\mu [(\partial_\mu \omega \partial_\nu \hat{n} - \partial_\nu \omega \partial_\mu \hat{n}) \cdot \partial_\nu \hat{n}] - \frac{\alpha}{\mu^2} \sin^2 \frac{\omega}{2} \sin \omega (\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2 = 0,$$

$$\partial_\mu \left[ \sin^2 \frac{\omega}{2} \hat{n} \times \partial_\mu \hat{n} + \frac{\alpha}{4\mu^2} \sin^2 \frac{\omega}{2} [(\partial_\nu \omega)^2 \hat{n} \times \partial_\mu \hat{n} - (\partial_\nu \omega \partial_\mu \omega) \hat{n} \times \partial_\nu \hat{n}]$$

$$+ \frac{\alpha}{\mu^2} \sin^4 \frac{\omega}{2} (\hat{n} \cdot \partial_\mu \hat{n} \times \partial_\nu \hat{n}) \partial_\nu \hat{n} \right] = 0.$$  \hspace{1cm} (2.3)

Notice that the second equation can be interpreted as the conservation of $SU(2)$ current originating from the global $SU(2)$ symmetry of the theory. With the spherically symmetric ansatz

$$\omega = \omega(r), \quad \hat{n} = \pm \hat{r},$$  \hspace{1cm} (2.4)
(3) is reduced to
\[
\frac{d^2 \omega}{dr^2} + \frac{2 d \omega}{r \, dr} - \frac{2 \sin \omega}{r^2} + 2 \alpha \left( \frac{\sin^2(\omega/2) \, d^2 \omega}{r^2} + \frac{\sin \omega \, d \omega}{4 r^2 \, dr} - \frac{\sin \omega \sin^2(\omega/2)}{r^4} \right) = 0. \quad (2.5)
\]

Notice that the energy of the spherically symmetric solutions is given by
\[
E = \frac{\pi}{2} \mu^2 \int_{0}^{\infty} \left\{ \left( r^2 + 2 \alpha \frac{\sin^2 \omega}{2} \right) \left( \frac{d \omega}{dr} \right)^2 + 8 \left( 1 + \frac{\alpha}{2 \mu^2 \, r^2} \sin^2 \frac{\omega}{2} \right) \sin \frac{\omega}{2} \right\} \, dr
= \pi \sqrt{\alpha \mu} \int_{0}^{\infty} \left[ x^2 \left( \frac{d \omega}{dx} \right)^2 + 8 \sin^2 \frac{\omega}{2} \right] \, dx, \quad (x = \frac{\mu}{\sqrt{\alpha}} \, r) \quad (2.6)
\]

where \( x \) is a dimensionless variable. Notice that the last equality follows from the virial theorem. Imposing the boundary condition
\[
\omega(0) = 2\pi, \quad \omega(\infty) = 0, \quad (2.7)
\]
one has the well-known skyrmion which has a finite energy \( \[1, 15\] \)
\[
E \simeq 73 \sqrt{\alpha \mu}. \quad (2.8)
\]

It carries the baryon number
\[
Q_s = \frac{1}{24\pi^2} \int \epsilon_{ijk} \, \text{tr} \, (L_i L_j L_k) \, d^3 r
= \pm \frac{1}{8\pi^2} \int \epsilon_{ijk} \partial_j \omega [\hat{r} \cdot (\partial_j \hat{r} \times \partial_k \hat{r})] \, \sin^2 \frac{\omega}{2} \, d^3 r
= \pm 1, \quad (2.9)
\]
which represents the non-trivial homotopy \( \pi_3(S^3) \) defined by \( U \) in (2.1).

A remarkable point of (2.3) is that
\[
\omega = \pi, \quad (2.10)
\]
becomes a classical solution, independent of \( \hat{n} \). So restricting \( \omega \) to \( \pi \), one can reduce
the Skyrme Lagrangian (2.2) to the Skyrme-Faddeev Lagrangian
\[
\mathcal{L} \rightarrow -\frac{\mu^2}{2} (\partial_\mu \hat{n})^2 - \frac{\alpha}{4} (\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2, \quad (2.11)
\]
whose equation of motion is given by
\[
\hat{n} \times \partial^2 \hat{n} + \frac{\alpha}{\mu^2} (\partial_\mu H_{\mu\nu}) \partial_\nu \hat{n} = 0,
H_{\mu\nu} = \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu C_\nu - \partial_\nu C_\mu. \quad (2.12)
\]
Notice that \( H_{\mu\nu} \) allows a potential \( C_\mu \) because it forms a closed two-form. Again the equation can be viewed as a conservation of \( SU(2) \) current,
\[
\partial_\mu (\hat{n} \times \partial_\mu \hat{n} + \frac{\alpha}{\mu^2} H_{\mu\nu} \partial_\nu \hat{n}) = 0.
\]
It is this equation that allows not only the baby skyrmion and the Faddeev-Niemi knot but also the non-Abelian monopole.

Just like the \(SU(2)\) QCD the Lagrangian (2.11) has singular Wu-Yang type monopole solutions \([5, 6]\)

\[ \hat{n} = \pm \hat{r}. \quad (2.13) \]

They become solutions of (2.12) except at the origin, because

\[ \partial^2 \hat{r} = -\frac{2}{r^2} \hat{r}, \]
\[ \partial_{\mu} [\hat{r} \cdot (\partial_{\mu} \hat{r} \times \partial_{\nu} \hat{r})] = 0. \quad (2.14) \]

Moreover we have

\[ H_{\hat{r}} = \pm \frac{\hat{r}}{r^2}, \quad H_{\theta} = 0, \quad H_{\phi} = 0, \]

so that it carries the magnetic charge \([5, 11]\)

\[ Q_{m} = \frac{\pm 1}{8\pi} \int \epsilon_{ijk} [\hat{r} \cdot (\partial_i \hat{r} \times \partial_j \hat{r})] d\sigma_k \]
\[ = \pm 1, \quad (2.15) \]

which represents the homotopy \(\pi_2(S^2)\) defined by \(\hat{n}\). This is precisely the magnetic field of a singular monopole located at the origin, which is very similar to the Wu-Yang monopole. In \(SU(2)\) QCD we have the well known Wu-Yang monopole \([17, 16]\)

\[ \vec{A}_{\mu} = -\frac{1}{g} \hat{r} \times \partial_{\mu} \hat{r}, \quad \vec{F}_{\mu\nu} = -\frac{1}{g} \partial_{\mu} \hat{r} \times \partial_{\nu} \hat{r}, \quad (2.16) \]

whose magnetic field is defined by

\[ \vec{H}_{\mu\nu} = -\frac{1}{g} \hat{r} \cdot (\partial_{\mu} \hat{r} \times \partial_{\nu} \hat{r}). \quad (2.17) \]

This is almost identical to the monopole solution (2.13), which justifies us to interpret \(C_{\mu}\) and \(H_{\mu\nu}\) as the magnetic potential and magnetic field generated by the monopole.

However, there are also significant differences between the two monopoles. First, in QCD \(\hat{n} = +\hat{r}\) and \(\hat{n} = -\hat{r}\) are gauge equivalent, whereas here they are physically different because the Skyrme theory has no local \(SU(2)\) symmetry. So, unlike in QCD, the monopole and the anti-monopole in Skyrme theory are not equivalent. Secondly, our monopole here has the quadratic interaction in the Lagrangian (2.11). Because of this, the energy of the monopoles has divergent contribution from both the origin and the infinity. In contrast, the energy of Wu-Yang monopole in QCD is divergent only at the origin. Finally, the above solution becomes a solution even without the quartic interaction (i.e., with \(\alpha = 0\)). This justifies the interpretation that the Skyrme theory is indeed a theory of monopole (interacting with the massless scalar field \(\omega\)). But one has to keep in mind that this monopole is not an electromagnetic monopole, but rather a non-Abelian chromomagnetic one.
Actually the Skyrme theory has more complicated monopole solutions. To see this consider (2.3) with the spherical symmetry (2.4) again, and impose the following boundary condition

\[ \omega(0) = \pi, \quad \omega(\infty) = 0, \quad (2.18) \]

With this one can find a monopole solution which reduces to the singular solution (2.13) near the origin. The only difference between this and the singular solution is the non-trivial dressing of the scalar field \( \omega \), so that it could be interpreted as a dressed monopole. This dressing, however, is only partial because this makes the energy finite at the infinity, but not at the origin. One can also impose the following boundary condition

\[ \omega(0) = 0, \quad \omega(\infty) = -\pi. \quad (2.19) \]

and obtain another monopole solution which approaches the singular solution near the infinity. Here again the partial dressing makes the energy finite at the origin, but not at the infinity. So the partially dressed monopoles still carry an infinite energy.

Obviously the dressed monopoles have a unit monopole charge \( Q_m \), but carry a half baryon number due to the boundary conditions (2.18) and (2.19). This must be clear from the definition of the baryon number (2.9). In this sense they could be called half-skyrmions.

The physical significance of the partially dressed monopoles is not that they are physical, but that together they can form a finite energy soliton, the fully dressed skyrmion. This must be clear because putting the boundary conditions (2.18) and (2.19) together one recovers the boundary condition (2.7) of the skyrmion (modulo 2\( \pi \)) whose energy is finite. From this one can conclude that the skyrmion is a finite energy monopole in which the cloud of the massless scalar field \( \omega \) regularizes the energy of the singular monopole both at the origin and the infinity.

### 3. Helical Baby Skyrmion

It has been well-known that the Skyrme theory has a vortex solution known as the baby skyrmion \([7]\). But the theory also has a twisted vortex solution, the helical baby skyrmion \([6]\).

To construct the desired helical vortex we choose the cylindrical coordinates \((\theta, \varphi, z)\), and adopt the ansatz

\[
\hat{n} = \begin{pmatrix}
\sin f(\theta) \cos (n\varphi + mkz) \\
\sin f(\theta) \sin (n\varphi + mkz) \\
\cos f(\theta)
\end{pmatrix},
\]

\[
C_\mu = (\cos f(\theta) + 1)(n\partial_\mu \varphi + mk\partial_\mu z). \quad (3.1)
\]

With this the equation (2.12) is reduced to

\[
\left( 1 + \frac{\alpha}{\mu^2} \left( \frac{n^2}{\theta^2} + m^2 k^2 \right) \sin^2 f \right) \ddot{f}
\]
Figure 1: The baby skyrmion (dashed line) with $m = 0, n = 1$ and the helical baby skyrmion (solid line) with $m = n = 1$ in Skyrme theory. Here $\varrho$ is in the unit $\sqrt{\alpha/\mu}$ and $k = 0.8 \mu/\sqrt{\alpha}$.

\begin{equation}
+ \left( \frac{1}{\varrho} + \frac{\alpha}{\mu^2} \left( \frac{n^2}{\varrho^2} + m^2 k^2 \right) \dot{f} \sin f \cos f \right)
- \frac{\alpha}{\mu^2} \frac{1}{\varrho^2} \left( \frac{n^2}{\varrho^2} - m^2 k^2 \right) \dot{f} \sin^2 f
- \left( \frac{n^2}{\varrho^2} + m^2 k^2 \right) \sin f \cos f = 0.
\end{equation}

So with the boundary condition

\begin{equation}
f(0) = \pi, \quad f(\infty) = 0,
\end{equation}

we obtain the non-Abelian vortex solutions shown in Fig. 1. When $m = 0$, the solution describes the well-known baby skyrmion. But when $m$ is not zero, it describes a helical vortex which is periodic in $z$-coordinate. Our result shows that the size (radius) of vortex can drastically be reduced by twisting.

Notice that the helical vortex has a non-vanishing magnetic potential $C_\mu$ (not only around the vortex but also) along the $z$-axis, so that it has two helical magnetic fields

\begin{equation}
H_\varphi = \frac{1}{\varrho} H_{\varrho \varphi} = \frac{n}{\varrho} \dot{f} \sin f, \quad H_{\varphi} = H_{\varphi \varphi} = mk \dot{f} \sin f,
\end{equation}

which gives two quantized chromomagnetic fluxes. It has a quantized magnetic flux along the $z$-axis

\begin{equation}
\Phi_\varrho = \int H_{\varrho \varphi} d\varrho d\varphi = 4\pi n,
\end{equation}

and a quantized magnetic flux around the $z$-axis (in one period section from $z = 0$ to $z = 2\pi/k$)

\begin{equation}
\Phi_\varphi = \int H_{z \varphi} d\varrho dz = -4\pi m.
\end{equation}
This confirms that the magnetic fluxes are quantized in the unit of $4\pi$, the unit of the monopole.

The origin of this quantization of the magnetic flux, of course, is topological. To see this consider the baby skyrmion with $m = 0$ (the straight vortex) first. In this case $\hat{n}$ defines a mapping $\pi_2(S^2)$ from the compactified $xy$-plane $S^2$ to the target space $S^2$. And the quantized magnetic flux $\Phi_{\hat{z}}$ describes the winding number of this mapping. Similarly $\Phi_{\hat{\varphi}}$ describes the winding number of another mapping $\pi_2(S^2)$ from the compactified half $xz$-plane to the target space $S^2$.

The vortex solutions implies the existence of Meissner effect which confines the magnetic flux of the vortex [5, 6]. To see how the Meissner effect comes about, notice that due to the $U(1)$ gauge symmetry the Skyrme theory has a conserved current, 

$$j_\mu = \partial_\nu H_{\mu\nu}, \quad \partial_\mu j_\mu = 0.$$  \hspace{1cm} (3.7)

So the magnetic flux of the vortex can be thought to come from the helical chromoelectric supercurrent density

$$j_\mu = -(\partial^2 C_\mu - \partial_\mu \partial_\nu C_\nu)$$

$$= n \epsilon_{\hat{\eta} \hat{\xi}} \frac{d}{d\hat{\eta}} \left( \sin f \hat{f} \right) \partial_\mu \varphi + mk \frac{d}{d\hat{\eta}} \left( \hat{g} \sin f \right) \partial_\mu \hat{z}$$

$$= \sin f \left[ n \left( \hat{f} \right) + \frac{\cos f}{\sin f} \hat{f}^2 - \frac{1}{\hat{g}} \right] \partial_\mu \varphi$$

$$+ mk \left( \frac{\hat{f}}{\sin f} \hat{f}^2 + \frac{1}{\hat{g}} \right) \partial_\mu \hat{z}.$$  \hspace{1cm} (3.8)

This produces the supercurrents $i_{\hat{\varphi}}$ (per one period section from $z = 0$ to $z = 2\pi/k$) around the $z$-axis

$$i_{\hat{\varphi}} = n \int_{\varphi = 0}^{\varphi = \infty} \int_{z = 0}^{z = 2\pi/k} \sin f \left( \hat{f} + \frac{\cos f}{\sin f} \hat{f}^2 - \frac{1}{\hat{g}} \right) d\varphi dz$$

$$= \frac{2\pi n}{k} \hat{f}^2 (0),$$  \hspace{1cm} (3.9)

and $i_\hat{z}$ along the $z$-axis

$$i_\hat{z} = mk \int_{\varphi = 0}^{\varphi = \infty} \sin f \left( \hat{f} + \frac{\cos f}{\sin f} \hat{f}^2 + \frac{1}{\hat{g}} \right) d\varphi d\varphi$$

$$= \frac{2\pi mk \hat{f} \sin f}{1} \bigg|_{\varphi = 0} = 0.$$  \hspace{1cm} (3.10)

Notice that, even though $i_\hat{z} = 0$, it has a non-trivial current density.

The helical magnetic fields and supercurrents are shown in Fig. 2 and Fig. 3. Clearly the helical magnetic fields are confined along the $z$-axis, confined by the helical supercurrent.
Figure 2: The supercurrent $i_\hat{\phi}$ (in one period section in $z$-coordinate) and corresponding magnetic field $H_\hat{z}$ circulating around the cylinder of radius $\varrho$ of the helical vortex with $m = n = 1$, where $\varrho$ is in the unit $\sqrt{\alpha}/\mu$ and $k = 0.8 \mu/\sqrt{\alpha}$. The current density $j_\hat{\varphi}$ is represented by the dotted line.

Figure 3: The supercurrent $i_\hat{z}$ and corresponding magnetic field $H_\hat{\varphi}$ flowing through the disk of radius $\varrho$ of the helical vortex with $m = n = 1$, where $\varrho$ is in the unit $\sqrt{\alpha}/\mu$ and $k = 0.8 \mu/\sqrt{\alpha}$. The current density $j_\hat{z}$ is represented by the dotted line.

This is nothing but the Meissner effect, which assures that the Skyrme theory has a built-in confinement mechanism which confines the magnetic flux. Notice that for the monopole solutions (2.13) this supercurrent becomes identically zero, which is why the Meissner effect does not work for the monopoles.

With the ansatz (3.1) the energy (in one periodic section) of the helical vortex is given by

$$E = \frac{2\pi^2 \mu^2}{k} \int_0^{\infty} \left\{ \left( 1 + \frac{\alpha}{\mu^2} \left( \frac{n^2}{\varrho^2} + k^2 m^2 \right) \sin^2 f \right) \left( \frac{df}{dx} \right)^2 + \left( \frac{n^2}{\varrho^2} + k^2 m^2 \right) \sin^2 f \right\} d\varrho \varrho$$

$$= \frac{2\pi^2 \mu^2}{k} \int_0^{\infty} \left\{ \left( 1 + \left( \frac{n^2}{x^2} + \frac{\alpha}{\mu^2} k^2 m^2 \right) \sin^2 f \right) \left( \frac{df}{dx} \right)^2 \right\}$$
\[
+ \left( \frac{n^2}{a^2} + \frac{\alpha}{\mu^2} k^2 m^2 \right) \sin^2 f \right) x dx,
\]

where

\[ x = \frac{\mu}{\sqrt{\alpha}} \theta. \]

One could calculate the energy of the helical baby skyrmion numerically. With \( m = n = 1 \) and \( k = 0.8 \mu/\sqrt{\alpha} \) we find

\[
E \simeq 224.5 \sqrt{\alpha}\mu. \tag{3.12}
\]

In comparison for the baby skyrmion (i.e., for \( m = 0 \) and \( n = 1 \)) of the same length we have

\[
E \simeq 100.9 \sqrt{\alpha}\mu. \tag{3.13}
\]

This shows that twisting requires a considerable amount of energy.

4. Axially Symmetric Faddeev-Niemi Knot: A Numerical Solution

The helical vortex in Skyrme theory is unphysical in the sense that it becomes unstable and decays to the untwisted baby skyrmion unless the periodicity condition is enforced by hand. But it allows us to construct a knot \([5, 6]\). This is because by smoothly connecting two periodic ends of the helical vortex we can naturally enforce the periodicity condition and make it a stable vortex ring. By construction this vortex ring carries two magnetic fluxes, \( m \) unit of flux passing through the disk of the ring and \( n \) unit of flux passing along the ring. Moreover the two fluxes can be thought of two unit flux rings linked together winding each other \( m \) and \( n \) times, whose linking number becomes \( mn \). This implies that the twisted vortex ring becomes a knot.

To confirm this we now construct a knot explicitly with a consistent ansatz. To do this we first introduce the toroidal coordinates \((\eta, \gamma, \varphi)\) defined by

\[
x = \frac{a}{D} \sinh \eta \cos \varphi, \quad y = \frac{a}{D} \sinh \eta \sin \varphi, \quad z = \frac{a}{D} \sin \gamma,
\]

\[ D = \cosh \eta - \cos \gamma, \]

\[
ds^2 = \frac{a^2}{D^2} \left( d\eta^2 + d\gamma^2 + \sinh^2 \eta d\varphi^2 \right),
\]

\[
d^3x = \frac{a^3}{D^3} \sinh \eta d\eta d\gamma d\varphi, \tag{4.1}
\]

where \( a \) is the radius of the knot defined by \( \eta = \infty \). Now we adopt the following axially symmetric knot ansatz \([8]\),

\[
\hat{n} = \begin{pmatrix} \sin f \cos(n\beta + m\varphi) \\ \sin f \sin(n\beta + m\varphi) \\ \cos f \end{pmatrix}, \tag{4.2}
\]
where \( f \) and \( \beta \) are functions of \( \eta \) and \( \gamma \). With the ansatz we have
\[
C_{\mu} = n(\cos f - 1)\partial_\mu \beta + m(\cos f + 1)\partial_\mu \varphi,
\]
\[
H_{\eta \gamma} = -nK \sin f, \quad H_{\gamma \varphi} = -m \sin f \partial_\gamma f,
\]
\[
H_{\varphi \eta} = m \sin f \partial_\eta f,
\]
\[
K = \partial_\eta f \partial_\gamma \beta - \partial_\gamma f \partial_\eta \beta.
\]

(4.3)

Notice that, in the orthonormal frame \((\hat{\eta}, \hat{\gamma}, \hat{\varphi})\), we have
\[
C_{\hat{\eta}} = \frac{nD}{a} (\cos f - 1) \partial_\eta \beta, \quad C_{\hat{\gamma}} = \frac{nD}{a} (\cos f - 1) \partial_\gamma \beta,
\]
\[
C_{\hat{\varphi}} = \frac{mD}{\sinh \eta} (\cos f + 1),
\]
\[
H_{\hat{\eta} \hat{\gamma}} = -\frac{nD^2}{2a^2} K \sin f, \quad H_{\hat{\gamma} \hat{\varphi}} = -\frac{mD^2}{2a^2 \sinh \eta} \sin f \partial_\gamma f,
\]
\[
H_{\hat{\varphi} \hat{\eta}} = \frac{mD^2}{2a^2 \sinh \eta} \sin f \partial_\eta f.
\]

(4.4)

With this the knot equation (2.12) is written as
\[
\left[ \partial_\eta^2 + \partial_\gamma^2 + \left( \cosh \eta \sinh \eta - \frac{\sinh \eta}{D} \right) \partial_\eta - \sin \gamma \frac{D}{\sinh \eta} \partial_\gamma \right] f = 0,
\]
\[
-\left( n^2 (\partial_\eta f)^2 + (\partial_\gamma f)^2 + \frac{m^2}{\sinh^2 \eta} \right) \sin f \cos f
\]
\[
+ \frac{\alpha}{\mu^2} \frac{D^2}{a^2} \left( A \cos f + B \sin f \right) \sin f = 0,
\]
\[
\left[ \partial_\eta^2 + \partial_\gamma^2 + \left( \cosh \eta \sinh \eta - \frac{\sinh \eta}{D} \right) \partial_\eta - \sin \gamma \frac{D}{\sinh \eta} \partial_\gamma \right] \beta
\]
\[
+ 2 \left( \partial_\eta f \partial_\eta \beta + \partial_\gamma f \partial_\gamma \beta \right) \frac{\cos f \sin f}{\sin f} - \frac{\alpha}{\mu^2} \frac{D^2}{a^2} C = 0.
\]

(4.5)

where
\[
A = \left[ n^2 K^2 + \frac{m^2}{\sinh^2 \eta} \left( (\partial_\eta f)^2 + (\partial_\gamma f)^2 \right) \right],
\]
\[
B = \left\{ n^2 \partial_\eta K \partial_\gamma \beta - n^2 \partial_\gamma K \partial_\eta \beta + n^2 K \left[ \left( \cosh \eta \sinh \eta + \sinh \eta \frac{D}{\sinh \eta} \right) \partial_\gamma \beta - \sin \gamma \frac{D}{\sinh \eta} \partial_\gamma \beta \right] \right. \right.
\]
\[
+ \frac{m^2}{\sinh^2 \eta} \left. \left[ \partial_\eta^2 + \partial_\gamma^2 - \left( \cosh \eta \sinh \eta - \frac{\sinh \eta}{D} \right) \partial_\eta + \sin \gamma \frac{D}{\sinh \eta} \partial_\gamma \right] f \right\},
\]
\[
C = \left\{ \partial_\eta K \partial_\gamma f - \partial_\gamma f \partial_\eta K + K \left[ \left( \cosh \eta \sinh \eta + \sinh \eta \frac{D}{\sinh \eta} \right) \partial_\gamma - \sin \gamma \frac{D}{\sinh \eta} \partial_\eta \right] f \right\}.
\]

From the ansatz (4.2) we have the following Hamiltonian
\[
\mathcal{H} = \frac{\mu^2}{2} \frac{D^2}{a^2} \left[ (\partial_\eta f)^2 + (\partial_\gamma f)^2 + n^2 (\partial_\eta \beta)^2 + (\partial_\gamma \beta)^2 + \frac{m^2}{\sinh^2 \eta} \right] \sin^2 f
\]
\[
+ \frac{\alpha}{4} \frac{D^4}{a^4} A \sin^2 f.
\]

(4.6)
and the energy of the knot

\[ E = \int \frac{\mathcal{H} a^3}{D^4} \sinh \eta d\eta d\gamma d\varphi \]

\[ = \sqrt{\alpha \mu} \int \left\{ \frac{\mu}{a} \frac{D}{2D} \left[ (\partial_\eta f)^2 + (\partial_\gamma f)^2 + \left( \frac{m^2}{\sinh^2 \eta} \right)^2 \right] \sin^2 f \right\} \sinh \eta d\eta d\gamma d\varphi. \]  

(4.7)

Minimizing the energy we reproduce the knot equation (4.5), which tells that our ansatz (4.2) is indeed consistent.

In toroidal coordinates, \( \eta = \gamma = 0 \) represents spatial infinity and \( \eta = \infty \) describes the torus center. So we can impose the following boundary condition

\[ f(0, \gamma) = 0, \quad f(\infty, \gamma) = \pi, \]

\[ \beta(\eta, 0) = 0, \quad \beta(\eta, 2\pi) = 2\pi, \]  

(4.8)

to obtain the desired knot. Of course, an exact solution of (4.5) is extremely difficult to obtain, even numerically. In fact many known “knot solutions” are actually the energy profile of knots which minimizes the Hamiltonian [8, 21]. But here we can find an actual profile of the knot because we have the explicit knot ansatz (4.2). For \( m = n = 1 \) we find that the knot radius \( a \) which minimizes the energy is given by

\[ a \simeq 1.21 \frac{\sqrt{\alpha}}{\mu}. \]  

(4.9)

With this we obtain the knot profile for \( f \) and \( \omega \) of the lightest axially symmetric knot shown in Fig. 4 and Fig. 5. The numerical result indicates that the radius of the vortex (the thickness of the knot) \( r_0 \) is roughly about

\[ r_0 \simeq a \text{csch} 2 \simeq \frac{1}{3} \frac{\sqrt{\alpha}}{\mu}, \]  

(4.10)

which means that the radius of the vortex ring is about 3.6 times the radius of the vortex. From this we can construct a three dimensional energy profile of the knot shown in Fig. 4.

We emphasize that the knot ansatz (4.2) has played a crucial role for us to obtain the numerical solution. With the ansatz we were able to obtain an actual knot profile, and estimate the radius of the vortex and the radius of the vortex ring. Notice that, just for the sake of the knot topology, one might have assumed \( \beta = \gamma \) [13, 19]. But one can check that this is inconsistent with the equation of motion, even though this describes the correct knot topology.

With the numerical solution we can check the topology of the knot. From the ansatz (4.2) we have the knot quantum number

\[ Q_k = \frac{1}{32\pi^2} \int \epsilon_{ijk} C_i H_{jk} d^3 x \]

\[ = \frac{mn}{8\pi^2} \int K \sin f d\eta d\gamma d\varphi = \frac{mn}{4\pi} \int \sin f d\eta \frac{d\varphi}{m}, \]  

(4.11)

\[ = mn, \]
where the last equality comes from the boundary condition (4.8). This assures that our ansatz describes the correct knot topology. To understand the meaning of (4.11) we now calculate the magnetic flux of the knot. Since the magnetic field is helical, we have two magnetic fluxes, $\Phi_\gamma$ passing through the knot disk of radius $a$ in the $xy$-plane and $\Phi_\phi$ passing along the knot ring of radius $a$. From (4.4) and (4.8) we have

$$\Phi_\gamma = \int_{\gamma=\pi} H_\gamma \frac{a^2 \sinh \eta}{D^2} d\eta d\varphi = m \int_{\gamma=\pi} \sin f \partial_\eta f d\eta d\varphi = 4\pi m,$$

$$\Phi_\phi = \int H_\phi \frac{a^2}{D^2} d\eta d\gamma = n \int K \sin f d\eta d\gamma = 4\pi n. \quad (4.12)$$

This confirms that the flux is quantized in the unit of $4\pi$. As importantly this tells that the two fluxes are linked, whose linking number is given by $mn$. This is precisely the knot quantum number (4.11). This proves that the knot quantum number is given by the linking number of two magnetic fluxes $\Phi_\gamma$ and $\Phi_\phi$. Notice that all topological quantum numbers are completely fixed by the boundary condition (4.8).

The supercurrent which generates the twisted magnetic flux of the knot is given by the conserved current density

$$j_\mu = \frac{nD^2}{a^2} \left( \partial_\eta \frac{cosh \eta}{\sinh \eta} + \frac{\sinh \eta}{D} \right) K \partial_\mu \gamma$$

$$+ \frac{mD^2}{a^2} \left( \left( \partial_\eta - \frac{cosh \eta}{\sinh \eta} + \frac{\sinh \eta}{D} \right) \sin f \partial_\eta f \right.$$
Figure 5: (Color online). The β configuration of the knot with \( m = n = 1 \) in Skyrme theory. Notice that here the actual configuration shown is \( β - γ \).

\[
+\left( \partial_γ + \frac{\sin γ}{D} \right) \sin f \partial_γ f \right) \partial_μ ϕ, \\
g^{μν} \nabla_μ j_ν = 0. \tag{4.13}
\]

From this we have two quantized supercurrents, \( i_γ \) which flows through the knot disk and \( i_ϕ \) which flows along the knot,

\[
i_γ = \int_{γ=π} j_γ(η, γ) \frac{a^2 \sinh η}{D^2} dη dϕ \\
= \frac{n}{a} \int_{γ=π} D \sinh η \left( \partial_η + \frac{\cosh η}{\sinh η} + \frac{\sinh η}{D} \right) K dη dϕ,
\]

\[
i_ϕ = \int j_ϕ \frac{a^2}{D^2} dη dγ \\
= \frac{m}{a} \int \frac{D}{\sinh η} \left[ \left( \partial_γ - \frac{\cosh γ}{\sinh γ} + \frac{\sinh γ}{D} \right) \sin f \partial_γ f \\
+ \left( \partial_γ + \frac{\sin γ}{D} \right) \sin f \partial_γ f \right] dη dγ. \tag{4.14}
\]

For \( m = n = 1 \) we find numerically

\[
i_γ \approx \frac{23.9}{a}, \quad i_ϕ \approx 0. \tag{4.15}
\]

Notice that \( i_ϕ \) is vanishing, which is consistent with our picture that the knot is a twisted vortex ring made of the helical vortex. Notice, however, that the current density \( j_ϕ \) is non-trivial. This tells that the knot has a net angular momentum around the symmetric axis which stabilizes the knot.

Our result confirms that the knot solution has a dynamical manifestation of knot topology [6]. In mathematics the knot topology has always been described by the linking number
Figure 6: (Color online). The 3-dimensional energy profile of the lightest axially symmetric knot with \( m = n = 1 \) in Skyrme theory. Here the scale is in the unit of \( \sqrt{\alpha/\mu} \).

of the preimage of the Hopf mapping. Indeed the knot topology of the Faddeev-Niemi knot is described by the non-linear sigma field \( \hat{n} \) in (2.12), which defines the Hopf mapping from the compactified space \( S^3 \) to the target space \( S^2 \). When the preimages of two points of the target space are linked, the mapping defines a knot. In this case the knot quantum number of \( \pi_3(S^2) \) is given by the linking number of two preimages fixed by the Chern-Simon index (4.11) of the potential \( C_\mu \) \cite{8,21}.

Our interpretation of the Faddeev-Niemi knot as a helical vortex ring, however, provides an alternative picture of knot. It tells that the knot is made of two real (physical) magnetic flux rings linked together, whose knot quantum number is given by the linking number of two flux rings. This certainly is different from the above mathematical description of knot based on the Hopf mapping. This is a dynamical manifestation of knot, the linking of two physical flux rings which comes from dynamics \cite{3}.

Obviously two flux rings linked together can not be unlinked by any continuous deformation of the field configuration. This guarantees the topological stability of the knot. Furthermore the topological stability is backed up by the dynamical stability which comes from the dynamical manifestation of the knot. This is because the supercurrent which generates the quantized chromomagnetic flux of the knot has two components, the component moving along the knot, and the one moving around the knot tube. And the supercurrent moving along the knot generates an angular momentum around the \( z \)-axis, which provides the centrifugal force preventing the vortex ring to collapse. Another way to understand this is to notice that the supercurrent generates the \( m \) unit of the magnetic flux trapped in the knot disk which can not be squeezed out. Clearly, this flux provides a stabilizing repulsive force which prevent the collapse of the knot. This is how the knot acquires the
dynamical stability [8].

Our analysis tells that the Faddeev-Niemi knot could also be viewed as two flux rings made of non-helical baby skyrmion linked together. Of course, a vortex ring made of straight baby skyrmion is unstable, because such a ring has no knot topology. But if we make two such rings and link them together, it becomes a Faddeev-Niemi knot. Now, two such rings could reconnect and make one ring. However, in this process the knot topology survives, and the reconnected ring becomes a helical vortex ring. This tells that there are actually two complementary ways to view the Faddeev-Niemi knot, as a vortex ring made of the helical baby skyrmion or two vortex rings made of the untwisted baby skyrmion linked together.

The energy of the Faddeev-Niemi knot has been calculated before. Theoretically it has been known that the energy has the following bound [20],

$$c Q^{3/4} \leq E \leq C Q^{3/4}.$$  \hfill (4.16)

Numerically one finds up to $Q_k = 8$ [21]

$$E_{Q_k} \simeq 234 \sqrt{\alpha \mu} Q^{3/4}. \quad (4.17)$$

With our ansatz we can estimate the energy of the axially symmetric knot numerically. For the lightest knot (with $m = n = 1$) we find

$$E_1 \simeq 274.0 \sqrt{\alpha \mu}. \quad (4.18)$$

In general the energy of the axially symmetric knots depends on (not just $Q$ but) $m$ and $n$, because the axially symmetric knot ansatz (4.2) explicitly depends on them. We have calculated the energy up to $Q = 6$ and obtain the result shown in Fig. 7, where we have included the earlier estimate (4.17) for comparison (Notice that there are other estimates of energy based on an inconsistent knot ansatz, which nevertheless turn out to be similar to our result numerically [19]). Our result shows that the energy of the axially symmetric knots is systematically larger than the earlier estimate. We do not know the precise reason for this discrepancy, but this is probably because of the axial symmetry of the knots. Notice that in reality there is no reason that the minimum energy knots should have the axial symmetry, especially for those with $Q_k$ larger than one [21]. This implies that the minimum energy knots in general could have smaller energy than the axially symmetric knots.

5. Physical Interpretation of Knot in Skyrme Theory

It is well-known that the Skyrme theory can be viewed as a non-linear sigma model which describes the pion physics [2, 3]. Indeed with

$$U = \exp(\frac{\omega}{2i} \vec{n}) = f - i\vec{\sigma} \cdot \vec{\pi},$$

$$f = \cos \frac{\omega}{2}, \quad \vec{\pi} = \hat{n} \sin \frac{\omega}{2},$$

$$f^2 + \vec{\pi}^2 = 1,$$ \hfill (5.1)
Figure 7: (Color online). The $Q$-dependence of the energy of the axially symmetric knots. Our estimate is denoted by dots and $(m, n)$, with $Q = mn$. For comparison we included the earlier estimate denoted by crosses, which depends only on $Q$. The solid and dashed lines represent $E = E_1 Q^{3/4}$ curves.

the Skyrme Lagrangian (2.2) can be put into the form

$$
\mathcal{L} = -\frac{\mu^2}{2}((\partial_\mu f)^2 + (\partial_\mu \bar{\pi})^2) - \frac{\alpha}{4}((\partial_\mu f \partial_\nu \bar{\pi} - \partial_\nu f \partial_\mu \bar{\pi})^2 + (\partial_\mu \bar{\pi} \times \partial_\nu \bar{\pi})^2) + \frac{\lambda}{4}(f^2 + \bar{\pi}^2 - 1),
$$

where $\lambda$ is a Lagrange multiplier. In this form $f$ and $\bar{\pi}$ represent the sigma field and the pion field, so that the Skyrme theory describes the pion physics. In this picture the skyrmion becomes a topological soliton made of mesons, which can be identified as the baryon.

In this picture one may choose the parameters $\mu$ and $\alpha$ to be $[2, 3]$

$$
\mu \simeq 93 \, MeV, \quad \alpha \simeq 0.0442, \quad \sqrt{\alpha \mu} \simeq 19.55 \, MeV,
$$

(5.3)

so that from (2.6) one has the baryon mass

$$
m_b \simeq 73 \sqrt{\alpha \mu} \simeq 1.427 \, GeV.
$$

(5.4)

In a slightly different fitting one may choose $[2, 4]$

$$
\mu \simeq 65 \, MeV, \quad \alpha \simeq 0.0336, \quad \sqrt{\alpha \mu} \simeq 11.91 \, MeV,
$$

(5.5)

to have the baryon mass

$$
m_b \simeq 0.869 \, GeV.
$$

(5.6)
In this view one might hope to have composite “mesons” which can be viewed as baryon and anti-baryon bound states. But so far such bound states have not been found in Skyrme theory, probably because such states may not exist as topologically stable solitonic objects. Instead we have the Faddeev-Niemi knot.

Clearly the Faddeev-Niemi knot should describe a non-baryonic state. Indeed, if the skyrmion is identified as the baryon, the Faddeev-Niemi knot should be interpreted as a topological meson (a “glueball”) made of the twisted flux of pion pair. In this view one can easily estimate the mass of the knot from (4.17), because we know $\mu$ and $\alpha$. With (5.3) we find the mass of the lightest knot to be

$$m_k \simeq 305.8 \sqrt{\alpha \mu} \simeq 5.98 \text{ GeV},$$

but with (5.5) we obtain

$$m_k \simeq 3.64 \text{ GeV}. \quad (5.8)$$

Notice that the above estimate suggests that the knot mass is a little too big to be viewed as a baryon-antibaryon bound state. As we have remarked, the knot is not an ordinary bound state, but a topological meson made of pion pair.

As we have argued, the knot must be stable within the framework of Skyrme-Faddeev theory. However, one has to be cautious about the knot stability in Skyrme theory, because here one has an extra massless scalar field $\omega$ which could (in principle) destabilize the knot. To see this remember that the knot is made of the twisted magnetic flux of baryon-antibaryon pair. In Skyrme-Faddeev theory this flux ring can not be cut, because there are no finite energy baryon and anti-baryon in the theory which can terminate the flux at end points. But in Skyrme theory there are. This suggests that in Skyrme theory, there is a possibility that the knot could decay to a baryon-antibaryon pair. This indicates that in Skyrme theory the proof of the stability of the knot is a non-trivial matter. One needs a careful analysis to establish the stability of the knot.

It is remarkable that the Skyrme theory allows such diverse topological objects. In fact it allows all topological objects known in physics. But what is really remarkable is that the Skyrme theory itself can be put in many different forms, which put the theory in a completely different perspective. In the followings we discuss new interpretations of Skyrme theory.

6. Skyrme theory and QCD

The Skyrme theory is known to be closely related to QCD. In this section we expand this fact and show that the Skyrme theory can be viewed as an effective theory of strong interaction which is dual to SU(2) QCD. To show this notice that the Skyrme-Faddeev Lagrangian (2.11) can be put into a very suggestive form,

$$\mathcal{L} = -\frac{\alpha}{4} g^2 \hat{H}_{\mu\nu}^2 - \frac{\mu^2}{2} g^2 \hat{C}_\mu^2,$$

$$\hat{H}_{\mu\nu} = \partial_\mu \hat{C}_\nu - \partial_\nu \hat{C}_\mu + g \hat{C}_\mu \times \hat{C}_\nu,$$

(6.1)
where \( \hat{C}_\mu \) is the well-known magnetic potential of \( SU(2) \) QCD [17, 23, 8, 25]

\[
\hat{C}_\mu = -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n}.
\] (6.2)

This shows that formally the Skyrme-Faddeev Lagrangian (2.11) can be viewed as a massive \( SU(2) \) QCD. In general, the Skyrme Lagrangian (2.2) itself can be expressed as

\[
L = -\frac{\mu^2}{4} \left[ \frac{1}{2} (\partial_\mu \omega)^2 + 2g^2 \sin^2 \frac{\omega}{2} \hat{C}_\mu^2 \right]
- \frac{\alpha}{16} g^2 \left[ \sin^2 \frac{\omega}{2} (\partial_\mu \omega \hat{C}_\nu - \partial_\nu \omega \hat{C}_\mu)^2 + 4 \sin^4 \frac{\omega}{2} \hat{H}_{\mu\nu}^2 \right]
- \frac{\mu^2 (\partial_\mu \sigma)^2}{2 (1 - \sigma^2)} - \frac{\alpha}{4} g^2 (\partial_\mu \sigma \hat{C}_\nu - \partial_\nu \sigma \hat{C}_\mu)^2,
\] (6.3)

where

\( \sigma = \cos \frac{\omega}{2} \).

So the Skyrme theory could be expressed (with the scalar field \( \omega \)) in terms of the magnetic potential (6.2) of \( SU(2) \) QCD.

Notice that for small \( \sigma \) we can approximate the Lagrangian (6.3) near \( \sigma \approx 0 \) with a linear approximation. Neglecting the higher order interactions we have the linearized Skyrme Lagrangian

\[
L \approx -\frac{\alpha}{4} g^2 \hat{H}_{\mu\nu}^2 - \frac{\mu^2}{2} g^2 \hat{C}_\mu^2
- \frac{\mu^2 (\partial_\mu \sigma)^2}{2 (1 - \sigma^2)} - \frac{\alpha}{4} g^2 (\partial_\mu \sigma \hat{C}_\nu - \partial_\nu \sigma \hat{C}_\mu)^2.
\] (6.4)

At this point one cannot miss the remarkable contrast between the Lagrangians (5.2) and (6.4). They describe the same theory, but obviously have totally different implications. The Lagrangian (5.2) describes a non-linear sigma model which has no trace of gauge interaction. But the Lagrangian (6.4) clearly has the structure of gauge interaction which allows us to relate the Skyrme theory to QCD.

Indeed we can derive the linearized Skyrme Lagrangian from QCD [5]. Consider \( SU(2) \) QCD again. Introducing an isotriplet unit vector field \( \hat{n} \) which selects the “Abelian” direction (i.e., the color charge direction) at each space-time point, we can decompose the gauge potential into the restricted potential \( \hat{B}_\mu \) and the valence potential \( \vec{X}_\mu \) [17, 23],

\[
\vec{A}_\mu = A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n} + \vec{X}_\mu = \hat{B}_\mu + \vec{X}_\mu,
(\hat{n}^2 = 1, \; \hat{n} \cdot \vec{X}_\mu = 0),
\] (6.5)

where \( A_\mu = \hat{n} \cdot \vec{A}_\mu \) is the “electric” potential. Notice that the restricted potential is precisely the connection which leaves \( \hat{n} \) invariant under the parallel transport,

\[
\hat{D}_\mu \hat{n} = \partial_\mu \hat{n} + g \hat{B}_\mu \times \hat{n} = 0.
\] (6.6)

\[
\hat{D}_\mu \hat{n} = \partial_\mu \hat{n} + g \hat{B}_\mu \times \hat{n} = 0.
\] (6.6)
Under the infinitesimal gauge transformation
\[ \delta \hat{n} = -\vec{\alpha} \times \hat{n}, \quad \delta \hat{A}_\mu = \frac{1}{g} D_\mu \vec{\alpha}, \] (6.7)
one has
\[ \delta A_\mu = \frac{1}{g} \hat{n} \cdot \partial_\mu \vec{\alpha}, \quad \delta \hat{B}_\mu = \frac{1}{g} \hat{D}_\mu \vec{\alpha}, \]
\[ \delta \hat{X}_\mu = -\vec{\alpha} \times \hat{X}_\mu. \] (6.8)

This shows that \( \hat{B}_\mu \) by itself describes an \( SU(2) \) connection which enjoys the full \( SU(2) \) gauge degrees of freedom. Furthermore \( \hat{X}_\mu \) transforms covariantly under the gauge transformation. Most importantly, the decomposition (6.5) is gauge-independent. Once the color direction \( \hat{n} \) is selected the decomposition follows automatically, independent of the choice of a gauge.

Notice that the unit isotriplet \( \hat{n} \) describes all topological features of the original non-Abelian gauge potential. Clearly the isolated singularities of \( \hat{n} \) defines \( \pi_2(S^2) \) which describes the Wu-Yang monopole \([17, 23]\). Besides, with the \( S^3 \) compactification of \( \mathbb{R}^3 \), \( \hat{n} \) characterizes the Hopf invariant \( \pi_3(S^2) \simeq \pi_3(S^3) \) which classifies the topologically distinct vacua and the instantons \([22, 24]\). The importance of the decomposition (6.5) has recently been apprecciated by many authors in studying various aspects of QCD \([9, 25]\). Furthermore in mathematics the decomposition has been shown to play a crucial role in studying the geometry, in particular the Deligne cohomology, of non-Abelian gauge theory \([27, 28]\).

To understand the physical meaning of the decomposition (6.5) notice that the restricted potential \( \hat{B}_\mu \) actually has a dual structure. Indeed the field strength made of the restricted potential is decomposed as
\[ \hat{B}_{\mu\nu} = \hat{F}_{\mu\nu} + \hat{H}_{\mu\nu} = (F_{\mu\nu} + H_{\mu\nu})\hat{n}, \]
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]
\[ H_{\mu\nu} = -\frac{1}{g} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = -\frac{1}{g} H_{\mu\nu}, \]
\[ = \partial_\mu \hat{C}_\nu - \partial_\nu \hat{C}_\mu, \] (6.9)
where \( \hat{C}_\mu \) is the “magnetic” potential \([14, 23]\). Notice that this magnetic potential in QCD is identical (up to the normalization factor \(-1/g\)) to the magnetic potential \( C_\mu \) we have in Skyrme theory. This confirms that \( \hat{C}_\mu \) which appears in the Skyrme-Faddeev theory in (6.2) is precisely the magnetic potential which provides the dual structure of QCD. This is an indication that the Skyrme theory and QCD are dual to each other.

With (5.5) we have
\[ \hat{F}_{\mu\nu} = \hat{B}_{\mu\nu} + \hat{D}_\mu \hat{X}_\nu - \hat{D}_\nu \hat{X}_\mu + g \hat{X}_\mu \times \hat{X}_\nu, \] (6.10)
so that the Yang-Mills Lagrangian is expressed as
\[ \mathcal{L} = \frac{1}{4} \hat{F}_{\mu\nu}^2. \]
\[ = -\frac{1}{4} \hat{B}_{\mu\nu}^2 - \frac{1}{4} (\hat{D}_\mu \hat{X}_\nu - \hat{D}_\nu \hat{X}_\mu)^2 \]
\[ - \frac{g}{2} \hat{B}_{\mu\nu} \cdot (\hat{X}_\mu \times \hat{X}_\nu) - \frac{g^2}{4} (\hat{X}_\mu \times \hat{X}_\nu)^2. \]  
(6.11)

This tells that QCD can be viewed as a restricted gauge theory made of the binding gluon \( \hat{B}_\mu \) described by the restricted potential, which has an additional valence gluon \( \hat{X}_\mu \) described by the valence potential \[ \text{[17, 23].} \]

Now we can show that one can actually derive the linearized Skyrme theory from a massive QCD. Suppose that the confinement mechanism generates a mass term for the binding gluon. In this case the QCD Lagrangian can be modified to a massive QCD
\[ \mathcal{L} \simeq -\frac{1}{4} \hat{B}_{\mu\nu}^2 - \frac{\mu^2}{2} \hat{B}_\mu^2 - \frac{1}{4} (\hat{D}_\mu \hat{X}_\nu - \hat{D}_\nu \hat{X}_\mu)^2 \]
\[ - \frac{g}{2} \hat{B}_{\mu\nu} \cdot (\hat{X}_\mu \times \hat{X}_\nu) - \frac{g^2}{4} (\hat{X}_\mu \times \hat{X}_\nu)^2. \]  
(6.12)

Of course, this Lagrangian is too simple to describe the real dynamical symmetry breaking in QCD. But notice that, once the confinement sets in, the insertion of the mass term can be justified. Now the above Lagrangian, in the absence of \( A_\mu \) and \( \hat{X}_\mu \), reduces exactly to the Skyrme-Faddeev Lagrangian \[ \text{(2.11).} \]
Furthermore, with
\[ \hat{X}_\mu = f_1 \partial_\mu \hat{n} + f_2 \hat{n} \times \partial_\mu \hat{n}, \]  
(6.13)
the Lagrangian is expressed as
\[ \mathcal{L} \simeq -\frac{1}{4} [F_{\mu\nu} + (1 - g \phi^* \phi) \tilde{H}_{\mu\nu}]^2 \]
\[ + \frac{ig}{2} (D_\mu \phi)^* (D_\nu \phi) \tilde{H}_{\mu\nu} - \frac{g^2}{4} |D_\mu \phi \hat{C}_\nu - D_\nu \phi \hat{C}_\mu|^2 \]
\[ - \frac{\mu^2}{2} (A_\mu^2 + \hat{C}_\mu^2), \]  
(6.14)

where
\[ \phi = f_1 + if_2, \quad D_\mu \phi = (\partial_\mu + ig A_\mu) \phi. \]

So, with
\[ \partial_\mu \phi = 0, \quad A_\mu = \partial_\mu \sigma, \]  
(6.15)
we have
\[ \mathcal{L} \simeq -\frac{(1 - g \phi^* \phi)^2}{4} g^2 \tilde{H}_{\mu\nu}^2 - \frac{\mu^2}{2} g^2 \hat{C}_\mu^2 \]
\[ - \frac{\mu^2}{2} (\partial_\mu \sigma)^2 - \frac{\phi^* \phi}{4} g^2 (\partial_\mu \sigma \hat{C}_\mu - \partial_\nu \sigma \hat{C}_\nu)^2. \]  
(6.16)

Now, with
\[ \alpha = \phi^* \phi = (1 - g \phi^* \phi)^2, \]  
(6.17)
the Lagrangian (6.16) becomes nothing but the linearized Skyrme Lagrangian (6.4). So, if one wishes, one can actually derive the linearized Skyrme theory from QCD [5, 6]. This confirms the deep connection between the Skyrme theory and QCD. More importantly this demonstrates that the Skyrme theory describes the chromomagnetic dynamics, not the chromoelectric dynamics, of QCD. This tells that the two theories are dual to each other.

The new interpretation is completely orthogonal to the popular view. Nevertheless two views are not necessarily contradictory to each other, in the sense that both agrees that the Skyrme theory is an effective theory of strong interaction in which the confinement is manifest already at the classical level. Indeed we believe that the new interpretation should be viewed complementary to the popular view, which can allow us a deeper understanding of the Skyrme theory.

7. Chromoelectric Knot in QCD

Based on the fact that the Skyrme theory is closely related to QCD, Faddeev and Niemi predicted a topological knot in QCD [9]. According to the dual picture, the Faddeev-Niemi knot should be interpreted as a chromomagnetic knot made of twisted magnetic flux, which can not exist in QCD because QCD confines chromoelectric (not chromomagnetic) flux. But the dual picture implies the existence of a chromoelectric knot in QCD which is dual to the chromomagnetic Faddeev-Niemi knot [10]. In other words it could have a knot made of twisted color electric flux ring. This, of course, is a bold conjecture. Proving this conjecture within the framework of QCD will require a detailed knowledge of the confinement mechanism. Nevertheless from the physical point of view there is no reason why such an object can not exist, because one can easily construct such object simply by twisting a $g\bar{g}$ flux and smoothly bending and connecting both ends. Assuming the existence one may estimate the mass of the lightest electric knot. In this case one may identify $\Lambda_{QCD}$ as $(\sqrt{\alpha\mu})_{QCD}$, because this is the only scale one has in QCD. Now, with [28]

$$\Lambda_{QCD} \simeq (\sqrt{\alpha\mu})_{QCD} \simeq 200 \text{ MeV};$$

one can easily estimate the mass of the lightest electric knot. From (4.18) we expect [10]

$$M_K \simeq 305.8 \Lambda_{QCD} \simeq 61.2 \text{ GeV}.$$  (7.2)

It would certainly be very interesting to search for such exotic glueball experimentally.

At this point one might wonder how the small $\Lambda_{QCD}$ can produce such a large mass. To understand this notice that in Skyrme theory we have a large knot energy in spite of the fact that the theory has only one small mass scale $\sqrt{\alpha\mu}$. The reason is because of the large volume of the knot. Here we have exactly the same situation. To see this notice that the radius of the vortex ring of the lightest Faddeev-Niemi knot is about 3.6 times larger than the radius of vortex. So assuming that the radius of vortex of the chromoelectric knot in QCD is about $1/\Lambda_{QCD}$, we expect the volume the knot to be about $7.2 \pi^2/\Lambda_{QCD}^3$. Now assuming that the energy density $\rho$ of the knot is comparable to that of typical low-lying
hadrons, we expect $\rho$ roughly to be of the order of $\Lambda_{QCD}^3 M_p$ (where $M_p$ is the proton mass). From this we expect the energy of the knot to be about $66.7 \text{ GeV}$, which tells that our estimate (7.2) is quite reasonable. This explains how the small $\Lambda_{QCD}$ can produce such a large knot mass. As importantly this implies that, as far as the energy density is concerned, our knot is not much different from ordinary hadrons, in spite of its large mass.

Classically the chromoelectric knot must be stable, because it has both topological and dynamical stability. But this does not guarantee the physical stability of the knot, because in QCD we have other fields, the quarks and gluons, which could destabilize the knot. For example, the knot can be cut and decay to $g\bar{g}$ pairs and thus to lowlying hadrons. Indeed, just as the electric background in QED produces electron-positron pairs [31], the chromoelectric background in QCD becomes unstable and decays to gluon pairs [32]. This tells that the knot must have the quantum instability.

We could estimate the decay width of the knot from the one-loop effective action of QCD, because the effective action tells us what is the pair production probability of the gluons in chromoelectric background. In the presence of pure electric or pure magnetic background the one-loop effective action of $SU(2)$ QCD is given by

$$\mathcal{L}_{eff} = \begin{cases} -\frac{a^2}{2g^2} - \frac{11a^2}{48\pi^2}(\ln \frac{a}{\mu^2} - c), & b = 0 \\ \frac{b^2}{2g^2} + \frac{11b^2}{48\pi^2}(\ln \frac{b}{\mu^2} - c) & a = 0 \\ -\frac{11b^2}{96\pi} & \end{cases}$$

(7.3)

where $c$ is a (subtraction-dependent) constant and

$$a = \frac{g}{2}\sqrt{\sqrt{G^4 + (G\tilde{G})^2} + G^2},$$

$$b = \frac{g}{2}\sqrt{\sqrt{G^4 + (G\tilde{G})^2} - G^2},$$

$$G_{\mu\nu} = F_{\mu\nu} + H_{\mu\nu}.$$  

Notice that $a = gH$ and $b = 0$ represent a pure magnetic background, and $a = 0$ and $b = gE$ represent a pure electric background. According to the effective action the chromoelectric background is unstable and decays to $g\bar{g}$, with the probability $11g^2E^2/96\pi$ per unit volume per unit time [22, 32]. So assuming that the knot is made of $g\bar{g}$ flux ring of thickness $1/\Lambda_{QCD}$ we can estimate the average electric field of the knot to be

$$\bar{E} \simeq \frac{g\Lambda_{QCD}^2}{\pi}.$$ 

(7.4)

Also the lightest knot whose radius is about 3.6 times the thickness of the knot has the volume $7.2 \pi^2/\Lambda_{QCD}^3$. From this we can estimate the decay width $\Gamma$ of the knot [10]

$$\Gamma \simeq \frac{11g^2}{96\pi} \left(\frac{9\Lambda_{QCD}^2}{\pi}\right)^2 \times \frac{7.2 \pi^2}{\Lambda_{QCD}^3} \simeq 41 \alpha_s^2 \Lambda_{QCD}.$$ 

(7.5)
Obviously the decay width depends on the volume of the knot which is large. But more importantly it depends on \( \alpha_s \) which is running. This is because the decay process is a quantum process. So we have to decide what value of \( \alpha_s \) we have to use. There are two extremes, \( \alpha_s(M_p) \approx 1 \) and \( \alpha_s(M_k) \approx 0.13 \). We prefer \( \alpha_s \approx 1 \). This is because, as far as the energy density is concerned, the knot is like ordinary hadrons in spite of the large mass. With \( \alpha_s \approx 1 \) we have \( \Gamma \approx 8.2 \) GeV. So we expect a knot glueball of mass about 60 GeV which has a decay width about 8 GeV. Of course this is a rough estimate, but this implies that the chromoelectric knot can have a typical hadronic decay. In this connection we remark that, had we used \( \alpha_s(M_k) \approx 0.13 \), the decay width would have been around 0.14 GeV which is absurd. In the presence of quarks, a similar knot made of a twisted \( q\bar{q} \) flux could also exist in QCD.

The above exercise teaches us an important lesson. It tells that the classical stability of a topological soliton does not guarantee the physical stability. It can decay through a quantum process. The decay of the chromoelectric knot demonstrates this fact.

At this point one may ask what is the difference between the Faddeev-Niemi knot in Skyrme theory and our knot in QCD. Clearly both theories advocates the existence of a hadronic knot soliton in strong interaction. If one believes in the popular view of Skyrme theory as an effective theory of strong interaction, one must expect a hadronic knot around 5 GeV. But if one believes in the dual picture, one should expect a hadronic knot around 60 GeV. Experiment can tell which view is more realistic. A nice feature of the dual picture is that one can identify not just the decay mechanism, but actually calculate the decay width of the knot within the framework of QCD. This is impossible for the Faddeev-Niemi knot in Skyrme theory.

8. Skyrme Theory and Condensed Matter Physics

The Skyrme theory allows another unexpected interpretation. With the built-in Meissner effect, the (helical) baby skyrmion looks very much like the magnetic vortex in superconductors. This strongly suggests that the Skyrme theory could be related to a condensed matter physics at low energy \[11, 13\]. Now we argue that the Faddeev-Niemi Lagrangian \[(2.11)\] can actually be viewed to describe a \( CP^1 \) model which describes a two-component superfluid.

To see this let \( \phi \) be a complex doublet which describes a two-component Bose-Einstein condensate (BEC)

\[
\phi = \frac{1}{\sqrt{2}} \rho \xi, \quad (\xi^\dagger \xi = 1)
\]

and consider the following “gauged” Gross-Pitaevskii type Lagrangian \[11\]

\[
\mathcal{L} = -|D_\mu \phi|^2 - \frac{\lambda}{2} (\phi^\dagger \phi - \frac{\mu^2}{\lambda})^2 - \frac{1}{4} F_{\mu \nu}^2,
\]
where $D_\mu = \partial_\mu + igA_\mu$, $\mu^2$ and $\lambda$ are the coupling constants. Of course, since we are interested in a neutral condensate, we identify the potential $A_\mu$ with the velocity field of $\xi$.

\[ gA_\mu = -i\xi^\dagger \partial_\mu \xi. \]  

With this the Lagrangian (8.2) is reduced to

\[
\mathcal{L} = -\frac{1}{2}(\partial_\mu \rho)^2 - \frac{\rho^2}{2} \left( |\partial_\mu \xi|^2 - |\xi^\dagger \partial_\mu \xi|^2 \right) \\
- \frac{\lambda}{8} \left( \rho^2 - \rho_0^2 \right)^2 + \frac{1}{4g^2} (\partial_\mu \xi \partial_\nu \xi - \partial_\nu \xi \partial_\mu \xi)^2,
\]

where

\[ \rho_0^2 = \frac{2\mu^2}{\lambda}. \]

Notice that here the gauge field strength $F_{\mu \nu}$ is replaced by the non-vanishing vorticity of the velocity field (8.3), but the Lagrangian still retains the $U(1)$ gauge symmetry of (8.2).

From the Lagrangian we have the following equation of motion

\[
\partial^2 \rho - \left( |\partial_\mu \xi|^2 - |\xi^\dagger \partial_\mu \xi|^2 \right) \rho = \frac{\lambda}{2} \left( \rho^2 - \rho_0^2 \right) \rho, \\
\left\{ (\partial^2 - \xi^\dagger \partial^2 \xi) + 2 \left( \frac{\partial_\mu \rho}{\rho} - \xi^\dagger \partial_\mu \xi \right) \\
+ \frac{1}{g^2 \rho^2} \partial_\alpha (\partial_\mu \xi^\dagger \partial_\alpha \xi - \partial_\alpha \xi^\dagger \partial_\mu \xi) (\partial_\mu - \xi^\dagger \partial_\mu \xi) \right\} \xi \\
= 0.
\]

But remarkably, with

\[ \hat{n} = \xi^\dagger \sigma \xi, \]

we have

\[
(\partial_\mu \hat{n})^2 = 4 \left( |\partial_\mu \xi|^2 - |\xi^\dagger \partial_\mu \xi|^2 \right) \\
H_{\mu \nu} = \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = -2i(\partial_\mu \xi^\dagger \partial_\nu \xi - \partial_\nu \xi^\dagger \partial_\mu \xi) \\
= \partial_\mu C_\nu - \partial_\nu C_\mu.
\]

This tells that the velocity potential (8.3) plays the role of the magnetic potential $C_\mu$ in (2.12) of Skyrme theory.

With (8.7) the Lagrangian (8.4) is reduced to the following $CP^1$ Lagrangian

\[
\mathcal{L} = -\frac{1}{2}(\partial_\mu \rho)^2 - \frac{\rho^2}{2} (\partial_\mu \hat{n})^2 - \frac{\lambda}{8} \left( \rho^2 - \rho_0^2 \right)^2 \\
- \frac{1}{16g^2} (\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2.
\]
Furthermore the equation (8.5) can be put into the form
\[
\partial^2 \rho - \frac{1}{4} (\partial_i \hat{n})^2 \rho = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho,
\]
\[
\hat{n} \times \partial^2 \hat{n} + 2 \frac{\partial_i \rho}{\rho} \hat{n} \times \partial_i \hat{n} + \frac{1}{g^2 \rho^2} \partial_i H_{ij} \partial_j \hat{n} = 0.
\] (8.9)

The reason why we can express (8.5) completely in terms of \( \hat{n} \) (and \( \rho \)) is that the Abelian gauge invariance of (8.2) effectively reduces the target space of \( \xi \) to the gauge orbit space \( S^2 = S^3 / S^1 \), which is identical to the target space of \( \hat{n} \).

This analysis clearly shows that the above theory of two-component BEC is closely related to the Skyrme theory. In fact, in the vacuum
\[
\rho^2 = \rho_0^2,
\] (8.10)
the Lagrangian is reduced to the Skyrme-Faddeev Lagrangian
\[
\mathcal{L} = -\frac{\rho_0^2}{2} \left( |\partial_\mu \xi|^2 - |\xi^\dagger \partial_\mu \xi|^2 \right)
- \frac{1}{4g^2} (\partial_\mu \xi^\dagger \partial_\nu \xi - \partial_\nu \xi^\dagger \partial_\mu \xi)^2
= -\frac{\rho_0^2}{2} (\partial_\mu \hat{n})^2 - \frac{1}{16g^2} (\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2.
\] (8.11)

This tells that three Lagrangians (2.11), (6.1), and (8.11) are all identical to each other, which confirms that the Skyrme theory and the theory of two-component superfluid indeed have an important overlap. This implies that the Skyrme-Faddeev theory could also be regarded as a theory of two-component superfluid.

The above analysis also reveals two important facts. First, it shows that the Skyrme theory has a \( U(1) \) gauge symmetry. This is evident from the fact that the two-form \( H_{\mu \nu} \) admits the gauge potential given by the velocity field (8.3) of \( \xi \). Actually the gauge symmetry really originates from the little group of \( \hat{n} \), the arbitrary \( U(1) \) rotation which leaves \( \hat{n} \) (and thus the Skyrme Lagrangian) invariant. As we have seen, it is this gauge symmetry which have allowed us to establish the existence of the Meissner effect in Skyrme theory. Secondly, it provides a new meaning to \( H_{\mu \nu} \). The two-form now describes the vorticity of the velocity field of the superfluid \( \xi \). In other words, the non-linear Skyrme interaction can be interpreted as the vorticity interaction in superfluids. It has been well-known that the vorticity plays an important role in superfluids [33]. But creating a vorticity in superfluid costs energy. So it makes a perfect sense to include the vorticity interaction in the theory of superfluids. And the Skyrme-Faddeev Lagrangian naturally contains this interaction.

Furthermore the above analysis makes it clear that the above gauge theory of two-component BEC has a knot solution very similar to the Faddeev-Niemi knot [11, 13]. Indeed the following knot ansatz for two-component BEC
\[
\phi = \frac{1}{\sqrt{2}} \rho (\eta, \gamma) \xi,
\]
\[
\xi = \left( \frac{\cos f(\eta, \gamma)}{2} \exp(-im\varphi), \frac{\sin f(\eta, \gamma)}{2} \exp(in\beta(\eta, \gamma)) \right), \quad (8.12)
\]
gives us
\[
\hat{n} = \xi^\dagger \partial \xi = \left( \sin f \cos(n\beta + m\varphi), \sin f \sin(n\beta + m\varphi), \cos f \right),
\]
\[
C_\mu = -2i\xi^\dagger \partial_\mu \xi = n(cos f - 1)\partial_\mu \beta + m(cos f + 1)\partial_\mu \varphi.
\]
This is identical to the knot ansatz \([4,2]\) for the Faddeev-Niemi knot. Indeed with the ansatz \((8.12)\) we can obtain a knot solution very similar to the Faddeev-Niemi knot \([11, 13]\).
The only difference between the two knots is that the one in BEC has a dressing of an extra scalar field \(\rho\) which represents the degree of the condensation. This implies that under a proper circumstance, the condensed matter physics can allow a knot similar to the Faddeev-Niemi knot.

Clearly the knot in two component BEC describes a vorticity knot. But here the complex doublet \(\xi\) provides the knot topology, because it defines the mapping \(\pi_3(S^3)\) from the compactified space \(S^3\) to the target space \(S^3\) of the unit doublet. But with the Hopf fibering of \(S^3\) to \(S^2 \times S^1\), we have \(\pi_3(S^3) \simeq \pi_3(S^2)\). So two mappings \(\pi_3(S^2)\) defined by \(\hat{n}\) and \(\pi_3(S^3)\) defined by \(\xi\) describe an identical knot topology. Indeed in terms of the complex doublet \(\xi\) the knot quantum number is given by
\[
Q_k = \frac{1}{4\pi^2} \int \epsilon_{ijk} \xi^\dagger \partial_j \xi (\partial_i \xi^\dagger \partial_k \xi) d^3x
\]
\[
= \frac{1}{32\pi^2} \int \epsilon_{ijk} C_i H_{jk} d^3x, \quad (8.13)
\]
which is identical to the knot quantum number of Skyrme theory.

The fact that the knot quantum number of Two-component BEC can be described by \(\pi_3(S^3)\) has led to a confusing statement in the literature that the knot can be identified as a skyrmion, because the baryon quantum number in Skyrme theory is also described by \(\pi_3(S^3)\) \([2]\). But we emphasize that this is a misleading statement, because \(\pi_3(S^3)\) of \((8.13)\) in BEC is different from the one which defines the baryon number \((2.9)\) in Skyrme theory. In both cases the target space \(S^3 \simeq S^2 \times S^1\) has the Hopf fiber \(S^1\). But for the \(\pi_3(S^3)\) of the knot in BEC the hidden \(U(1)\) gauge group constitutes the fiber \(S^3\), so that \(\pi_3(S^3)\) defined by \(\xi\) actually reduces to \(\pi_3(S^2)\). On the other hand, for the \(\pi_3(S^3)\) of the skyrmion in \((2.9)\) the massless scalar field \(\omega\) describes the fiber \(S^1\). And the non-trivial \(\omega\) forbids us to reduce this \(\pi_3(S^3)\) to \(\pi_3(S^2)\). So the two \(\pi_3(S^3)\) actually describe different topology. This tells that it is misleading to call the knot in BEC a skyrmion.

The above analysis also makes it clear that alternatively the Faddeev-Niemi knot can also be viewed as a two quantized vorticity rings linked together in a two-component superfluid, whose linking number becomes the knot quantum number \([11]\).
9. Discussion

The Skyrme theory has rich topological structures. It has the Wu-Yang type monopole which has infinite energy, the skyrmion which can be viewed as a finite energy dressed monopole, the baby skyrmion which can be viewed as an infinitely long magnetic flux of a monopole-antimonopole pair, and the Faddeev-niemi knot which can be viewed as a twisted magnetic vortex ring made of a helical baby skyrmion. Both the monopole and the baby skyrmion have the topology $\pi_2(S^2)$. But the same topology describes physically different mapping, and thus physically different objects. Similarly the skyrmion and the knot have different topology. But the topologies of these objects are closely related. With the Hopf fibering $S^3 = S^2 \times S^1$ the monopole topology $\pi_2(S^2)$ can naturally be extended to $\pi_3(S^3)$ of the skyrmion topology. Similarly $\pi_2(S^2)$ of the baby skyrmion is extended to $\pi_3(S^2)$ of the helical baby skyrmion (and the knot) topology.

In this paper we have presented the numerical solution of the knot in Skyrme theory. This was made possible with a consistent knot ansatz. The numerical result confirms that the knot is indeed a twisted magnetic vortex ring made of a helical baby skyrmion. In fact we have shown that the knot can be viewed as two magnetic flux rings linked together, whose linking number is fixed by the knot quantum number. This confirms that the knot has a dynamical manifestation of knot topology which assures the dynamical stability.

But what is really remarkable is that the Skyrme theory itself has many different faces. The Skyrme theory has always been associated to nuclear and/or high energy physics at GeV scale. This traditional view is based on the fact that the theory can be put into the form of a non-linear sigma model which describes the flavor dynamics. As we have shown, however, our analysis tells that theory can also be interpreted as an effective theory of chromomagnetic dynamics which is dual to QCD. More precisely it can be viewed as an effective theory of confinement with a built-in Meissner effect. This view is completely orthogonal to, but not inconsistent with, the traditional view. Both shows that the Skyrme theory can be interpreted as an effective theory of strong interaction. In this sense the new interpretation is complementary to the traditional interpretation.

Both views predict the existence of a non-baryonic topological knot in QCD, but they predict different knot. In the traditional view the mass of the lightest knot is supposed to be around 5 GeV, but in the dual picture the mass of such knot should be around 60 GeV. Experimentally one could tell which is a better view simply by measuring (assuming the existence) the mass of such exotic knot.

The Skyrme theory has another surprising face. The Meissner effect in Skyrme theory suggests that the theory could be interpreted as a theory of condensed matter. In this paper we have argued that the Skyrme-Faddeev Lagrangian (2.11) could be understood to describe a theory of two-component superfluid or two-component Bose-Einstein condensate. In particular, the Faddeev-Niemi knot could be viewed as a vorticity vortex ring in these condensed matters [1]. This implies that the theory could actually describe an interesting low energy physics in a completely different environment at eV scale, in
two-component condensed matters.

If this view is correct, one could actually construct the Faddeev-Niemi knot (or a similar one) in laboratories, in particular in two-component superfluids and/or two-gap superconductors [11, 13]. If so, the challenge now is to confirm the existence of the topological knot experimentally in these condensed matters. Constructing such a knot may be a tricky task at present moment, but we hope that such a knot could be constructed in laboratories in the near future.

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