Learning to coordinate in a complex and non-stationary world

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We study analytically and by computer simulations a complex system of adaptive agents with finite memory. Borrowing the framework of the Minority Game and using the replica formalism we show the existence of an equilibrium phase transition as a function of the ratio between the memory \( \lambda \) and the learning rates \( \Gamma \) of the agents. We show that, starting from a random configuration, a dynamic phase transition also exists, which prevents the system from reaching any Nash equilibria. Furthermore, in a non-stationary environment, we show by numerical simulations that agents with infinite memory play worst than others with less memory and that the dynamic transition naturally arises independently from the initial conditions.

Social interactions pose many coordination problems to individuals. Generally social agents face problems of sharing and distributing limited resources in an optimal way. Examples range from the use of public roads and the Internet, to exchanging what we produce with what we consume. A solution to problem of this kind invokes the intervention of a public authority who finds the social optimum and imposes or suggests the optimal behavior to agents. While such a solution may be easy to find, its implementation may be difficult to enforce in practical situations.

Self-enforcing solutions – where agents achieve optimal allocation of resources while pursuing their self-interests, without explicit communication or agreement with others – are of great practical importance. Competitive markets are the prototypical example of such a solution: With everybody maximizing his own profit and no one really caring for global optimality, competitive markets perform the remarkable task of leading to system wide optimality.

Micro-economics and Game Theory have gone quite far in explaining what equilibria can one expect in social interactions. However most of these studies deal with unrealistic cases with either few players or with many, but identical, agents. Secondly the analysis is restricted to the equilibria which deductively rational players would agree upon. Such an approach seems unrealistic in cases involving many individuals with different goals and characteristics. The computational complexity required by deductive rationality may easily go far beyond the capabilities of agents. Inductive thinking, as suggested by Arthur, may be a more suited model of how real people behave. A growing effort has indeed been put in recent years in understanding under what conditions bounded inductively rational agents may reach optimal outcomes. Several learning rules have been found to lead to optimal outcomes when a single agent “plays” against nature. Similar results hold for games with few players, even though non-trivial dynamical effects can also arise.

In this letter we address the problem of how many heterogeneous adaptive agents learn to coordinate in a complex, eventually non-stationary, world. We draw inspiration from recent work on the Minority Game, in order to model a typical situation where a large number of agents pursue different individual goals, using a certain number of distributed resources. Optimal use of resources becomes then a complex coordination problem.

We focus on agents with finite memory and finite learning rates. We find that, when agents need to “learn” collectively a fixed structure of interactions, they can attain a close to optimal coordination, provided that their memory extends far enough into the past. As the memory decreases, the system undergoes a phase transition to a state where agents are unable to learn and play in a random way.

More interestingly we find situations where the agents are unable to coordinate and to converge to a Nash equilibrium. Thus the game ends in a stationary regime with no cooperation. This is a completely dynamical effect which prevents the system from a proper convergence to equilibrium and makes useless the standard analysis based on Nash equilibria. This is a further clear evidence of the relevance of tools and ideas of statistical mechanics in the study of complex socio-economic systems, indeed dynamical transitions are very well known in statistical mechanics.

The model we study is closely related to the Minority Game (MG). The reason for this choice is that this allows us to benefit from the detailed understanding which has been recently uncovered by the statistical mechanics approach. On one hand we can make reference to exact results, on the other we can extend our understanding of this keystone model of complex adaptive systems.

The model is precisely defined as follows: Agents live in a world which can be in one of \( P \) states, labelled by an integer \( \mu = 1, \ldots, P \). Each agent \( i = 1, \ldots, N \) can choose between two personal strategies, labeled by a spin variable \( s_i \), which prescribe an action \( a_{s_i}^{\mu} \) for each state...
\( \mu \). These actions are drawn from a bimodal distribution for all \( i, s \) and \( \mu \), such that there are two possible actions, do something (\( a^\mu_{s,i} = 1 \)) or do the opposite (\( a^\mu_{s,i} = -1 \)).

The payoff received by an agent who plays strategy \( s_i \), while her opponents take strategies \( s_{-i} = \{s_j, \forall j \neq i \} \), is, in the state \( \mu \),

\[
    u^\mu_t(s_i, s_{-i}) = -a^\mu_{s_i,i}A^\mu ,
\]

where \( A^\mu = \sum_j a^\mu_{s,j,j} \). The total payoff to agents is always negative: The majority of agents receives a negative payoff whereas only the minority of them gain.

The game is repeated many times; the state \( \mu \) is drawn from a uniform distribution \( \rho^\mu = 1/P \) at each time and agents try to estimate, on the basis of past observations, which of their strategies is the best one. More precisely, if \( s_i(t) \) is the strategy played by agent \( i \) at time \( t \), we assume as in [7] that

\[
    \text{Prob}[s_i(t) = s] \propto \exp[\Gamma U_{s,i}(t)],
\]

where \( U_{s,i}(t) \) is the score of strategy \( s \) at time \( t \) and \( \Gamma \) is a positive constant [10]. Each agent monitors the scores \( U_{s,i}(t) \) of each of her strategies \( s \) by

\[
    U_{s,i}(t+1) = (1-\lambda/P)U_{s,i}(t) + u^\mu_t[s, s_{-i}(t)]/P,
\]

where the last term is the payoff agent \( i \) would have received if she had played strategy \( s \) at time \( t \) – see Eq. (1) – against the strategies \( s_{-i}(t) = \{s_j(t), \forall j \neq i \} \) played by her opponents at that time.

In words, Eqs. (2) model agents who play more likely strategies which have performed better in the past. Eqs. (2) belong to a class of learning models which has received much attention recently [8].

The relevant parameter [7] is the ratio \( \alpha = P/N \) between the “information complexity” \( P \) and the number of agents, and the key quantity we shall look at is the global efficiency defined as \( \sigma^2 = \langle A^2 \rangle \).

This model differs from the MG [1] for two important aspects: First agents compute correctly the payoff for strategies \( s \neq s_i \) (t) which they did not play. In the MG agents only account for the explicit dependence of \( u^\mu_i \) on \( s \) which arises from \( a^\mu_{s,j,i} \) – see Eq. (1) – whereas they neglect the fact that if they had taken a different decision also \( A^\mu \) would have changed. This seems reasonable at first sight because \( A^\mu \) is an aggregate quantity and its dependence on each individual agent is weak. A more careful analysis [1] however shows that if agents properly account for their impact on \( A^\mu \) as in Eq. (3) a radically different scenario arises: Rather than converging to an unique stationary state as in the MG, the dynamics (with \( \lambda = 0 \)) converges to one of exponentially many (in \( N \)) states – which are Nash equilibria [2] – characterized by an optimal coordination. This change emerges in the statistical mechanics approach with the breakdown of replica symmetry (RS): While the Minority Game is described by a replica symmetric theory, Nash equilibria are described by a full replica symmetry broken (RSB) phase [10]. Our aim is precisely that of studying the coordination of adaptive agents in a complex world with exponentially many optimal states (Nash equilibria).

The second key feature is that previous work has only explored the dynamics of learning with an infinite memory – i.e. with \( \lambda = 0 \) in Eq. (2) – and for a fixed structure of interactions – i.e. with fixed (quenched) disorder \( a^\mu_{s,i} \). Our goal is to clarify the role of different time-scales involved in the learning dynamics. We shall first study the case where the structure of interactions is fixed – which corresponds to \( a^\mu_{s,i} \) being the usual quenched disorder – and then move to the more realistic case where the structure of interactions changes over long time-scales.

Following the lines of reasoning of Refs. [7,13], we introduce a continuum time \( \tau = \Gamma t/P \) and variables \( y_i(\tau) = \Gamma[U_{+,i}(t) - U_{-,i}(t)]/2 \) in terms of which the dynamics reads

\[
    \frac{dy_i}{d\tau} = \frac{\lambda}{\Gamma}y_i - h_i - \sum_{j \neq i} J_{i,j} \tanh(y_j) + \eta_i(\tau),
\]

\[
    h_i = \frac{1}{P} \sum_{\mu=1}^N \left( a^\mu_{s,i} - a^\mu_{-i} \right),
\]

\[
    J_{i,j} = \frac{1}{P} \sum_{\mu=1}^N \left( a^\mu_{s,i} - a^\mu_{-i} \right),
\]

with \( \eta_i(\tau) \) a white noise with zero mean and correlations

\[
    \langle \eta_i(\tau) \eta_j(\tau') \rangle \approx \frac{\Gamma \sigma^2}{\alpha N} \delta_{i,j} \delta(\tau - \tau').
\]

Refs. [7,13] have shown that, for \( \lambda = 0 \), the stationary states of this dynamics are related to the local minima of

\[
    \sigma^2 = H_0 + 2 \sum_i h_im_i + \sum_{j \neq i} J_{i,j}m_im_j,
\]

where \( H_0 \) is a constant and \( m_i = \langle \tanh(y_i) \rangle \). These states are also Nash equilibria [14], which means that agents achieve an optimal coordination. Since \( \sigma^2 \) takes its minima for \( m_i = \pm 1 \) – which correspond to \( y_i \to \pm \infty \) – the stochastic force \( \eta_i(\tau) \) is irrelevant in the late stages of the dynamics, which is dominated by the deterministic drift towards the Nash equilibrium.

For \( \lambda/\Gamma > 0 \) we expect the stochastic force \( \eta_i(\tau) \), whose strength is itself proportional to \( \sigma^2 \), to compete with the deterministic drift. Indeed the distribution of \( y_i \) will be cutoff for \( |y_i| \gg \Gamma/\lambda \); For small \( \lambda \) we expect that \( \langle \tanh(y_i) \rangle \) is close to the values \( m_i^{(\infty)} \) which minimize \( \sigma^2 \), and a spread in the distribution of \( y_i \) around its average which is maintained by the stochastic force. When \( \lambda \) increases we expect a transition to a phase where agents are unable to coordinate because their memory is too short for learning correctly the interaction structure:
The dynamics is dominated by the stochastic force $\eta_i$, which is made even stronger by the fact that $\sigma^2/N \approx 1$ is much larger than in the coordinated state. This transition is captured by the statistical mechanics approach of Ref. [3]. Neglecting stochastic fluctuations induced by $\eta_i$, which is legitimate only for $\Gamma \ll 1$, one can easily prove, following Ref. [7], that $m_i^{(\lambda)} = \langle \tanh y_i \rangle$ are given by the solution of the minimization of the function

$$H = \sigma^2 + \frac{1}{\Gamma} \sum_i \left[ \log(1 - m_i^2) + 2m_i \tanh^{-1}(m_i) \right]. \quad (5)$$

In order to study the ground state properties of $H$ we follow the same steps of Ref. [3]: We introduce an inverse temperature $\beta$, we compute the partition function and the free energy per agent and then we take averages over the disordered variables $a_{\mu,i}$ with the replica method [14]. The free energy, within the RS Ansatz, reads

$$f(q, r, Q, R) = \frac{\alpha}{\beta} \ln \left[ 1 + \frac{\beta(Q - q)}{\alpha} \right] - \frac{\alpha}{2} \frac{1 + q}{\alpha + \beta(Q - q)} + \frac{1 - Q}{2} - \frac{1}{\beta} \left( \int_{-1}^{1} \ln e^{-\beta V_q(m)} + \frac{\alpha \beta}{2} (Q - q) \right), \quad (6)$$

where $Q = \frac{1}{2} \sum_i (m_i)^2$ and $q = \langle m_i^a m_i^b \rangle$ with $a \neq b$ labelling different replicas of the systems; $R$ and $r$ arise as Lagrange multipliers and $V_q(m) = -\sqrt{\alpha \Gamma} m + \frac{\alpha \Gamma}{2} (r - R)m^2 + \frac{\alpha}{2} \log(1 - m^2) + 2m \tanh^{-1}(m)$. The ground state properties of $H$ are obtained solving the saddle point equations (14) in the limit $\beta \to \infty$.

In the inset of Fig. [4] we compare the analytical predictions for $\sigma^2$ and $Q$ with simulations results. We focus on small $\alpha$ (i.e. $\alpha = 0.1$) where the effects we wish to discuss are more evident. Little discrepancies between numerical data and analytical curves are maybe due to RSB effects. Note that a phase transition occurs at $\lambda_c \approx 0.46\Gamma$ where both $\sigma^2$ and $Q$ change their analytical behaviour. We have studied this equilibrium phase transition in the $(\lambda, 1/\Gamma)$ plane, confirming the critical line $\lambda_c = 0.46\Gamma$: Open symbols in Fig. [4] refer to a static experiment where we let the system equilibrate to a Nash equilibrium for $\lambda = 0$ and then we move it slowly along lines $\Gamma = \text{const.}$.

The situation changes when the system starts from scratch $\left[ U_{s,i}(0) = 0 \forall\{s, i\} \right]$ in each run. Depending on $\lambda$ and $\Gamma$, the dynamics may lead the system to a stationary regime (different from the static one) which is characterized by larger fluctuations (i.e. larger $\sigma^2$). These dynamical effects make the phase diagram more complex in the $\lambda < \lambda_c$ region (see Fig. [4]): In I the system always relaxes to the static equilibrium, in II it sometimes converges to equilibrium and sometimes get trapped in the metastable regime with large fluctuations, while in III it never reaches equilibrium. The presence of this dynamical transition implies that the analysis in terms of Nash equilibria is no longer enough to predict the collective behavior of the system in a large part of the phase diagram, i.e. for high learning rates and short memory.

When the external world is non-stationary, i.e. changes with time, the adaptation task becomes still harder. We mimic the external world modification as follows: every $\tau$ time steps a randomly chosen state of the world is removed and a new one replaces it (in order to keep $P$ constant). Actually we randomly choose a $\mu$ index and we re-extract the strategies $a_{\mu,i}$ for all $i$ and $s$.

Here we focus on the results of the simulations done with $\tau = 10^3$, $\Gamma = \infty$, $NP = 10^4$ and many $\lambda$ values. The results do not dependent on the initial conditions.

In the upper panel of Fig. [4] we show the relaxation of $\sigma^2/N$ for $\lambda = 2.5$: As expected, it starts from 1 and converges to its equilibrium value. Note that $\tau = 10^3$ has been chosen in order to allow the system to reach a cooperative behaviour before the world starts changing. For this value of $\lambda$ the system is robust with respect to
changes of the world: Apart from occasional excursions to states with large $\sigma^2$, agents are able to adapt themselves to the evolving interaction structure.

In the lower panel we present the evolution of $\sigma^2/N$ for $\lambda = 3.5$ (i.e. with shorter memory) in 50 different samples. The behaviour is now completely different: After having reached a low value of $\sigma^2/N$ (cooperation) the system undergoes a sharp transition and $\sigma^2/N$ jumps to a high value. The players are no longer able to adapt to the changing world and they start playing in a wrong way. Occasionally agents may achieve a good coordination with small $\sigma^2$, but they eventually always go back to uncoordinated states with large $\sigma^2$.

For large times, the instantaneous values of $\sigma^2/N$ have a roughly bimodal distribution: They are either low ($\sim 10^{-2}$) or high ($\sim 1$). In Fig. 3 we plot the average of the low ($\circ$) and of the high ($\Box$) values (these averages can be defined in an unambiguous way thanks to the gap between low and high $\sigma^2$ values). In the inset we report the fraction of samples that spend the last decade in the high $\sigma^2$ regime. In a whole intermediate range around $\lambda_c \approx 3.3$ we find that coordinated states with small $\sigma^2$ coexist with wildly fluctuating states ($\sigma^2 > 1$).

Is worth noticing some facts in Fig. 3. The minimum of $\sigma^2$, corresponding to the best cooperation, is no longer located in $\lambda = 0$ (i.e. infinite memory). In other words, in a non-stationary environment the agents play better with a finite memory, which allows them to take decision based more on the recent past rather than on the far past. The minimum they can attain is very near to the $\sigma^2/N$ value in an unchanging world (shown with a horizontal line in Fig. 3). The second remarkable fact is that the transition from a coordinated state to a high $\sigma^2$ regime when $\lambda$ increases – which was continuous in a fixed world – shows features of first order transitions such as discontinuities and phase coexistence.

In conclusion, we have extended the replica solution of the Minority Game to the case where agents have finite memory and finite learning rates. We have proven that a phase transition between phases with low and high $\sigma^2$ exists as a function of $\lambda/\Gamma$. We have also shown, by means of computer simulations, that a dynamical phase transition exists for high values of $\lambda$ (short memories), and that this dynamic phase transition is responsible for a non-cooperative behaviour of agents. Furthermore we have shown, by numerical simulation, that when the structure of the interactions is non-stationary, agents with infinite memory behave worse than agents with a finite memory. Under these conditions we recover again a scenario where agents with too short memory display a first order transition from a cooperative to a non-cooperative phase.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3}
\caption{Average low ($\circ$) and high ($\Box$) $\sigma^2/N$ as a function of $\lambda$ ($NP = 10^4$, $\alpha = 0.1$, $\Gamma = \infty$ and $\tau = 10^3$). The arrow indicates a transition from the cooperative to the non-cooperative regime. The horizontal dotted line is the $\sigma^2/N$ value with fixed world ($\tau = \infty$). Inset: Probability of being in a non-cooperative regime as a function of $\lambda$.}
\end{figure}

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