Bernoulli Fractional Differential Equation Solution Using Adomian Decomposition Method

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Abstract. Fractional calculus relates with derivatives, integrals, and differential equations of order not integers. The Bernoulli Differential Equation is a form of the first-order ordinary differential equation. This paper aims to solve the Bernoulli Differential Equation with α fractional-order using the Adomian Decomposition Method, where 0 < α ≤ 1. The fractional derivative used in this paper is the fractional derivative of Caputo. Based on several numerical examples presented in this paper, the results show that the Adomian Decomposition Method is easy and very effective to use for solving Bernoulli Differential Equations with fractional order α.

1. Introduction

Fractional calculus is the area of calculus that deals with derivatives, integrals, and differential equations of fractional order. In calculus, the derivatives are usually of the order of a non-negative integer denoted by $D^n_x y(x)$ or $\frac{d^n y}{dx^n}$, where $n = 1, 2, 3, ...$. This notation can be generalized to the derivative of the fractional-order α denoted by $D^\alpha_x y(x)$ or $\frac{d^\alpha y}{dx^\alpha}$, where $\alpha \in \mathbb{R}$. Most people assume that fractional calculus is an abstract area of mathematics that is of little use and almost no application. Nowadays, various researches on the application of fractional calculus, particularly fractional differential equations, are starting to emerge in various fields, such as physics, engineering, chemistry, biology, environment, economics, and finance [1]-[4].

The Bernoulli Differential Equation is a form of the first-order ordinary differential equation [5]. This differential equation has a general form, viz

$$\frac{dy}{dx} + P(x)y = Q(x)y^m,$$

where $P$ and $Q$ are functions of $x$ or a constant, and $m$ is a real number. Bernoulli Differential Equation is a first-order differential equation which can be linear or nonlinear. For $m = 0$ or $m = 1$, equation (1) becomes the Linear Bernoulli Differential Equation. For $m ≥ 2$, equation (1) becomes the Nonlinear Bernoulli Differential Equation. Equation (1) can be generalized to Bernoulli Differential Equation with a fractional-order which is written as follows

$$D^\alpha_x y(x) + P(x)y = Q(x)y^m,$$

where $\alpha \in \mathbb{R}$, with $0 < \alpha ≤ 1$. 
This paper aims to solve the Bernoulli Differential Equation with $\alpha$ fractional-order using the Adomian Decomposition Method, where the fractional derivative used is the fractional derivative of Caputo. Previous research has been conducted by Agom and Ogunfiditimi (2014) [6] who used the Adomian Decomposition Method to solve the classical Bernoulli Differential Equation, in other words, it has no fractional order.

George Adomian (1980) [7] first introduced the Adomian Decomposition Method for solving systems of stochastic equations. This Adomian Decomposition Method can be an effective and easy procedure for solving differential or integral equations without linearization, perturbation, discretization, or transformation [8]. This method can solve differential equations with natural or fractional order, ordinary or partial, with initial or boundary value problems, with constant or variable coefficients, linear or nonlinear, homogeneous or non-homogeneous [9]-[11].

Other research on Fractional Bernoulli Differential Equation can be seen in [12]-[16].

2. Fractional Calculus
This section presents the basic theories and properties that are related to and support this research, such as derivatives and fractional integrals.

**Definition 1.** (Podlubny, 1999) [17] Caputo fractional derivative of the function $f$ to $t$ of order $\alpha$, where $\alpha > 0$ and $n - 1 < \alpha \leq n$, is defined as

$$\frac{\text{c}D^\alpha_t f(t)}{\text{a}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-u)^{n-\alpha-1} f^n(u) du.$$

**Theorem 2.** (Mathai and Haubold, 2017) [18] Caputo fractional derivative of order $\alpha > 0$, with $n - 1 < \alpha \leq n$, from the function $f(t) = t^\beta$, where $\beta \geq 0$, is

$$\frac{\text{c}D^\alpha_t t^\beta}{\text{a}} = \begin{cases} \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha} & ; \beta > n - 1, \\ 0 & ; \beta \leq n - 1. \end{cases}$$

Based on Theorem 2, Caputo fractional integral of order $\alpha > 0$, with $n - 1 < \alpha \leq n$, from the function $f(t) = t^\beta$, where $\beta \geq 0$, is

$$I^\alpha_t b^\beta = \frac{\text{c}D_{t}^{-\alpha} t^\beta}{\text{a}} = \begin{cases} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} t^{\beta + \alpha} & ; \beta > n - 1, \\ 0 & ; \beta \leq n - 1. \end{cases}$$

**Theorem 3.** (Mathai dan Haubold, 2017) [18] Let $f$ be continuous in $[a, b]$ and $n - 1 < \alpha \leq n$, then

$$I^\alpha f(t) = D_{t}^{-\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0).$$

3. Adomian Decomposition Method
The Adomian Decomposition Method assumes that the solution is decomposed into an infinite series, the nonlinear form (if any) is decomposed into an Adomian polynomial, and an iterative algorithm is constructed to recursively determine the solution. This method can solve differential equations with natural or fractional order, ordinary or partial, with initial or boundary value problems, with constant or variable coefficients, linear or nonlinear, homogeneous or non-homogeneous. The solution of the Fractional Ordinary Differential Equation using the Adomian Decomposition Method can refer to [19]-[22].

4. Numeric Examples
This section presents two examples of numerical solutions to Bernoulli Fractional Differential Equation using the Adomian Decomposition Method.

**Example 1.** Given the Linear Bernoulli Fractional Differential Equation as follows

$$D^\alpha_x y - 2y = 0,$$  \hspace{1cm} (3)
with \(0 < \alpha \leq 1\) and initial condition \(y(0) = 1\). The exact solution of equation (3) is
\[
y = \exp(2x).
\]
Meanwhile, the approximate solution of equation (3) using the Adomian Decomposition Method based on the recursive formula in equation (2) is
\[
y_0 = 1, \quad y_{n+1} = I^\alpha [2y_n], n = 0,1,2, ... \tag{5}
\]
Figure 1 depicts a numerical simulation for the comparison of the approximate solutions of the Linear Bernoulli Fractional Differential Equation using the Adomian Decomposition Method (5), where \(\alpha = 1\) and \(n = 5\), with the exact solution (4). The results show that in the initial iteration the approach solution for \(\alpha = 1\) is close to the exact solution.

**Figure 1.** Comparison of the approximate solutions of the Linear Bernoulli Fractional Differential Equation using the Adomian Decomposition Method with its exact solution

Furthermore, Figure 2 illustrates a numerical simulation for the approximate solution of the Linear Bernoulli Fractional Differential Equation using the Adomian Decomposition Method (5), where \(n = 5\), with different \(\alpha\) values. The results show that if the \(\alpha\) value increases, then the \(y\) graph decreases.

**Example 2.** Given the Nonlinear Bernoulli Fractional Differential Equation as follows
\[
6D_x^{\alpha} y - 2y = xy^4, \tag{6}
\]
with \(0 < \alpha \leq 1\) and initial condition \(y(0) = -2\). The exact solution of equation (6) is
\[
y = -\frac{2}{(4x - 4 + 5 \exp(-x))^{\frac{1}{5}}}. \tag{7}
\]
Meanwhile, the approximate solution of equation (6) using the Adomian Decomposition Method based on the recursive formula in equation (2) is
\[
y_0 = -2, \quad y_{n+1} = I^\alpha \left[ \frac{1}{6} xA_n \right] - I^\alpha \left[ -\frac{1}{3} \frac{y_n}{y} \right], n = 0,1,2, ... \tag{8}
\]
where \(A_n\) is an Adomian polynomial, with the nonlinear operator \(Ny = y^4\).
Figure 2. The solution to the approach of the Linear Bernoulli Fractional Differential Equation using the Adomian Decomposition Method with different $\alpha$ values.

Figure 3 illustrates the numerical simulation for comparison of the approximate solutions of the Nonlinear Bernoulli Fractional Differential Equation using the Adomian Decomposition Method (5), where $\alpha = 1$ and $n = 5$, with the exact solution (7). The results show that in the initial iteration the approach solution for $\alpha = 1$ is close to the exact solution.

Figure 3. Comparison of the approximate solution of the Nonlinear Bernoulli Fractional Differential Equation using the Adomian Decomposition Method with its exact solution.
Next, Figure 4 depicts a numerical simulation for the approximate solution of the Nonlinear Bernoulli Fractional Differential Equation using the Adomian Decomposition Method (6), where \( n = 5 \), with different \( \alpha \) values. The results show that if the \( \alpha \) value increases, then the \( y \) graph decreases.

![Graph showing the approach solution of the Nonlinear Bernoulli Fractional Differential Equation using the Adomian Decomposition Method with different \( \alpha \) values](image)

**Figure 4.** The approach solution of the Nonlinear Bernoulli Fractional Differential Equation using the Adomian Decomposition Method with different \( \alpha \) values

5. Conclusion

The Bernoulli Differential Equation is a form of the first-order ordinary differential equation. This differential equation can be of integer or fractional order, it can also be linear or nonlinear. The Adomian Decomposition Method is an approach method that can solve ordinary and partial differential equations, linear or nonlinear, and can be of integer or fractional order. Two numerical examples presented in this paper show that the Adomian Decomposition Method is reliable and easy to use to solve the Linear and Nonlinear Fractional Bernoulli Differential Equations.

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