Reparametrization-Invariant Path Integral in GR and ”Big Bang” of Quantum Universe

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Abstract

The reparametrization-invariant generating functional for the unitary and causal perturbation theory in general relativity in a finite space-time is obtained. The region of validity of the Faddeev-Popov-DeWitt functional is studied. It is shown that the invariant content of general relativity as a constrained system can be covered by two ”equivalent” unconstrained systems: the ”dynamic” (with ”dynamic” evolution parameter as the metric scale factor) and ”geometric” (given by the Levi-Civita type canonical transformation to the action-angle variables where the energy constraint converts into a new momentum).

”Big Bang”, the Hubble evolution, and creation of matter fields by the ”geometric” vacuum are described by the inverted Levi-Civita transformation of the geometric system into the dynamic one. The particular case of the Levi-Civita transformations are the Bogoliubov ones of the particle variables (diagonalizing the dynamic Hamiltonian) to the quasiparticles (diagonalizing the equations of motion). The choice of initial conditions for the ”Big Bang” in the form of the Bogoliubov (squeezed) vacuum reproduces all stages of the evolution of the Friedmann-Robertson-Walker universe in their conformal (Hoyle-Narlikar) versions.

1. Introduction

”Big Bang” as the beginning of the Hubble evolution of a universe is described as a pure classical phenomenon on the basic of particular solutions of the Einstein equations in general relativity in the homogeneous approximation. A strange situation consists in that the highest level of the theory, i.e., the Faddeev-Popov-DeWitt generating functional for unitary S-matrix \[1, 2\], neglects the questions about the evolution of a universe which are in the competence of the simplest classical approximation. There is an opinion that the solution of the ”Big Bang” problem in quantum theory goes beyond the scope of the unitary perturbation theory and even of general relativity. To answer these questions, we need a more general theory of the type of superstring \[3\].

According to another point of view, the reason of theoretical difficulties in understanding the ”Big Bang” phenomenon is not the Einstein theory, but the non-invariant method of its quantization. In particular, for the Faddeev-Popov-DeWitt unitary S-matrix the non-invariant coordinate time is considered as the time of evolution, whereas an observer in a universe can observe and measure only...
invariants of group of diffeomorphisms of the Hamiltonian dynamics, which includes reparametrizations of the coordinate time [4, 5, 6, 7].

In the present paper we try to construct the unitary $S$-matrix for general relativity in a finite space-time in terms of the reparametrization-invariant evolution parameter, and to answer the questions: What do Quantum Universe and Quantum Gravity mean? What is the status of the ”Big Bang” evolution in quantum theory? What does creation of Quantum Universe mean? on the level of perturbation theory, using the scheme of the time-reparametrization-invariant Hamiltonian reduction [3].

In the context of the Dirac generalized Hamiltonian theory for constrained systems [8, 9, 10, 11], this scheme means the explicit resolving of the first class constraints to determine the constraint-shell action directly in terms of invariants. In other words, we use the invariant reduction of the action (to obtain an equivalent unconstrained system) instead of the generally accepted non-invariant reduction of the phase space by fixing gauges [1, 2, 12] (see Fig. 1).

The example of the application of such an invariant reduction of the action is the Dirac formulation of QED [13] directly in terms of the gauge-invariant (dressed) fields as the proof of the adequateness of the Coulomb gauge with the invariant content of classical equations. Recall that the invariant reduction of the action is the way to obtain the unconstrained Feynman integral [14] for the foundation of the intuitive Faddeev-Popov functional integral in gauges theories [12] and to reveal collective excitations of the gauge fields in the form of zero-modes of the first class constraints [15] (see Fig. 2).

A constructive idea of the considered invariant Hamiltonian reduction of general relativity is to introduce the dynamic evolution parameter as the zero-mode collective excitation of metric [4, 5, 6, 7, 16, 17, 18, 19]. This dynamic evolution parameter can be identified with the zero-Fourier harmonic of the space-metric determinant [5, 6] (treated in cosmology as the cosmic scale factor), whereas its conjugate momentum, i.e, the second (external) form, plays the role of the localizable Hamiltonian of evolution.

The separation of this zero-mode evolution parameter on the level of the action allows us to determine also the invariant geometric time formed by averaging the time-like component of a metric over the space coordinates [5, 6, 7].

An observer reveals the evolution of the universe (with ”Big Bang”) as the dependence of the dynamic evolution parameter (i.e. cosmic scale factor) on the geometric time.

The evolution of both a classical and a quantum universes in terms of the geometric time has the form of the canonical transformations [20, 21, 22] of the initial dynamic variables to a new set of variables for which the total energy constraint becomes a new momentum, and its conjugate variable (i.e., a new dynamic evolution parameter) coincides with the geometric time onto equations of motion.

The content of the paper is the following. Section 2 is devoted to the reparametrization-invariant Hamiltonian reduction of general relativity. In section 3 we construct the generating functional for the unitary perturbation theory which includes ”Big Bang” and Hubble evolution. In Section 4, ”Big Bang” and Hubble evolution are reproduced in lowest order of perturbation theory. In Section 5, we research the conditions of validity of the conventional quantum field theory in the infinite space-time limit.

2. Reparametrization-invariant Hamiltonian dynamics of GR

2.1. Action and variables

General relativity (GR) is given by the singular Einstein-Hilbert action with the matter fields

$$ W(g|\mu) = \int d^4x \sqrt{-g} \left[ -\frac{\mu^2}{6} R(g) + L_f \right] \quad \left( \mu^2 = M^2_{\text{Planck}} \frac{3}{8\pi} \right) $$

(1)
and by a measurable interval
\[(ds)_c^2 = g_{\alpha\beta}dx^\alpha dx^\beta.\] (2)

They are invariant with respect to general coordinate transformations
\[x_\mu \to x'_\mu = x'_\mu(x_0, x_1, x_2, x_3).\] (3)

The generalized Hamiltonian approach to GR was formulated by Dirac and Arnovit, Deser and Misner [23] as a theory of system with constraints in 3 + 1 foliated space-time
\[(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu = N^2dt^2 - (3)g_{ij}\ddot{x}^i\ddot{x}^j \quad (\ddot{x}^i = dx^i + N^i dt)\] (4)

with the lapse function \(N(t, \vec{x})\), three shift vectors \(N^i(t, \vec{x})\), and six space components \(g_{ij}(t, \vec{x})\) depending on the coordinate time \(t\) and the space coordinates \(\vec{x}\). The Dirac-ADM parametrization of metric (4) characterizes a family of hypersurfaces \(t = \text{const}\) with the unit normal vector \(\nu = (1/N, -N^k/N)\) to a hypersurface and with the second (external) form
\[
\frac{1}{N}(\tilde{g}_{ij}) - \Delta_i N_j - \Delta_j N_i)
\] (5)

that shows how this hypersurface is embedded into the four-dimensional space-time.

Coordinate transformations conserving the family of hypersurfaces \(t = \text{const}\).
\[t \to \tilde{t} = \tilde{t}(t); \quad x_i \to \tilde{x}_i = \tilde{x}_i(t, x_1, x_2, x_3),\] (6)

\[\tilde{N} = N \frac{dt}{dt}; \quad \tilde{N}^k = N^i \frac{\partial \tilde{x}^k}{\partial x_i} dt - \frac{\partial \tilde{x}^k}{\partial x_i} \frac{\partial x^i}{\partial t}\] (7)

are called a kinemetric subgroup of the group of general coordinate transformations [3, 4, 24]. The group of kinemetric transformations is the group of diffeomorphisms of the generalized Hamiltonian dynamics. It includes reparametrizations of the nonobservable time coordinate \(\tilde{t}(t)\) that play the principal role in the procedure of the reparametrization-invariant reduction discussed in the previous Sections. The main assertion of the invariant reduction is the following: the dynamic evolution parameter is not the coordinate but the variable with a negative contribution to the energy constraint. (Recall that this reduction is based on the explicit resolving of the global energy constraint with respect to the conjugate momentum of the dynamic evolution parameter to convert this momentum into the Hamiltonian of evolution of the reduced system.)

A negative contribution to the energy constraint is given by the space-metric-determinant logarithm. Therefore, following papers [17, 5, 18, 1, 16, 19] we introduce an invariant evolution parameter \(\varphi_0(t)\) as the zero Fourier harmonic component of this logarithm (treated, in cosmology, as the cosmic scale factor). This variable is distinguished in general relativity by the Lichnerowicz conformal-type transformation of field variables \(f\) with the conformal weight \(n\)
\[\langle n \rangle \tilde{f} = \langle n \rangle f \left(\frac{\varphi_0(t)}{\mu}\right)^{-n},\] (8)

where \(n = 2, 0, -3/2, -1\) for the tensor, vector, spinor, and scalar fields, respectively, \(\tilde{f}\) is so-called conformal-invariant variable used in GR for the analysis of initial data [16, 25]. In particular, for metric we get
\[g_{\mu\nu}(t, \vec{x}) = \left(\frac{\varphi_0(t)}{\mu}\right)^2 \bar{g}_{\mu\nu}(t, \vec{x}) \Rightarrow (ds)^2 = \left(\frac{\varphi_0(t)}{\mu}\right)^2 [\tilde{N}^2 dt^2 - (3)\tilde{g}_{ij}\ddot{x}^i\ddot{x}^j].\] (9)

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As the zero Fourier harmonic is extracted from the space metric determinant logarithm, the space metric \( \tilde{g}_{ij}(t, \vec{x}) \) should be defined in a class of nonzero harmonics

\[
\int d^3x \log ||\tilde{g}_{ij}(t, \vec{x})|| = 0.
\]  

(10)

The transformational properties of the curvature \( R(g) \) with respect to the transformations (9) lead to the action (1) in the form [5]

\[
W(g|\mu) = W(\tilde{g}|\varphi_0) - \frac{t_2}{t_1} \int d^3x \varphi_0 \frac{d}{dt}(\frac{\varphi_0 \sqrt{\tilde{g}}}{N}).
\]  

(11)

This form define the global lapse function \( N_0 \) as the average of the lapse function \( \tilde{N} \) in the metric \( \tilde{g} \) over the kinemetric invariant space volume

\[
N_0(t) = \frac{V_0}{\int V_0 d^3x \sqrt{\tilde{g}(t, \vec{x})}}, \quad \tilde{g} = \text{det}(\tilde{g})^3, \quad V_0 = \int V_0 d^3x,
\]  

(12)

where \( V_0 \) is a free parameter which in the perturbation theory has the meaning of a finite volume of the free coordinate space. The lapse function \( \tilde{N}(t, \vec{x}) \) can be factorized into the global component \( N_0(t) \) and the local one \( \mathcal{N}(t, \vec{x}) \)

\[
\tilde{N}(t, \vec{x})\tilde{g}^{-1/2} := N_0(t)\mathcal{N}(t, \vec{x}) := N_q,
\]  

(13)

where \( \mathcal{N} \) fulfills normalization condition:

\[
I[\mathcal{N}] := \frac{1}{V_0} \int \frac{d^3x}{\mathcal{N}} = 1
\]  

(14)

that is imposed after the procedure of variation of action, to reproduce equations of motion of the initial theory. In the Dirac harmonical variables [8] chosen as

\[
q^{jk} = \tilde{g}^{jk},
\]  

(15)

the metric (4) takes the form

\[
(ds)^2 = \frac{\varphi_0(t)^2}{2q^{1/2}} q^{ij}(N^2_q dt^2 - q_{ij} dx^i dx^j), \quad (q = \text{det}(q^{ij})).
\]  

(16)

The Dirac-Bergmann version of action (11) in terms of the introduced above variables reads [8, 9]

\[
W = \int_{t_1}^{t_2} dt \left\{ L + \frac{1}{2} \partial_t (P_0\varphi_0) \right\},
\]  

(17)

\[
L = \left[ \int_{V_0} d^3x \left( \sum_F P_F \dot{F} - N^i P_i \right) \right] - P_0 \varphi_0 - N_0 \left[ -\frac{P_0^2}{4V_0} + I^{-1}H(\varphi_0) \right],
\]  

(18)

where

\[
\sum_F P_F \dot{F} = \sum_F p_f \dot{f} - \pi_{ij} \dot{q}^{ij};
\]  

(19)
is the total Hamiltonian of the local degrees of freedom,
\[ H(\varphi_0) = \int d^3 x \mathcal{N} \mathcal{H}(\varphi_0) \] (20)

is the total Hamiltonian of the local degrees of freedom,
\[ \mathcal{H}(\varphi_0) = \frac{6}{\varphi_0^2} q^{ij} q^{kl} [\pi_{ik}\pi_{jl} - \pi_{ij}\pi_{kl}] + \frac{\varphi_0^2 q^{1/2} (3) R(\bar{g}) + \mathcal{H}_f ,} \] (21)

\[ \mathcal{P}_i = 2[\nabla_k (q^{kl} \pi_{il}) - \nabla_i (q^{kl} \pi_{kl})] + \mathcal{P}_f \] (22)

are the densities of energy and momentum and \( \mathcal{H}_f, \mathcal{P}_f \) are contributions of the matter fields. In the following, we call the set of the field variables \( F, \varphi_0 \) with the dynamic evolution parameter \( \varphi_0 \) the field world space. The local part of the momentum of the space metric determinant
\[ \pi(t, x) := q^{ij}\pi_{ij} \] (23)

is given in the class of functions with the non-zero Fourier harmonics, so that
\[ \int d^3 x \pi(t, x) = 0 . \] (24)

The geometric foundation of introducing the global variable \( \bar{g} \) in GR was given in [19] as the assertion about the nonzero value of the second form in the whole space. This assertion (which contradicts the Dirac gauge \( \pi = 0 \)) follows from the global energy constraint, as, in the lowest order of the Dirac perturbation theory, positive contributions of particle-like excitations to the zero Fourier harmonic of the energy constraint can be compensated only by the nonzero value of the second form.

The aim of this paper is to obtain the dynamic "equivalent" unconstrained system in the field world space \( (F, \varphi_0) \) by explicit resolving the global energy constraint and to study the possibility of the Hamiltonian description of the "equivalent" unconstrained system in terms of the reparametrization-invariant evolution parameter \( T \) defined by equation \( N_0 dt = dT \).

2.2. Local constraints and equations of motion

Following Dirac [8] we formulate generalized Hamiltonian dynamics for the considered system \( (\mathcal{L}) \). It means the inclusion of momenta for \( \mathcal{N} \) and \( N_i \) and appropriate terms with Lagrange multipliers
\[ W^D = \int_{t_1}^{t_2} dt \left\{ L^D + \frac{1}{2} \partial_t (P_0 \varphi_0) \right\}, \quad L^D = L + \int d^3 x (P_0 \dot{\mathcal{N}} + P_{N} \dot{\mathcal{N}}) - \lambda^0 P_N - \lambda_i P_{N^i} . \] (25)

We can define extended Dirac Hamiltonian as
\[ H^D = N_0 \left[ -\frac{P_0^2}{4V_0} + I^{-1} H(\varphi_0) \right] + \int d^3 x (\lambda^0 P_N + \lambda_i P_{N^i}) . \] (26)

The equations obtained from variation of \( W^D \) with respect to Lagrange multipliers are called first class primary constraints
\[ P_N = 0, \quad P_{N^i} = 0 . \] (27)

The condition of conservation of these constraints in time leads to the first class secondary constraints
\[ \{ H^D, P_N \} = \mathcal{H} - \frac{\int d^3 x N \mathcal{H}}{V_0 N^2} = 0, \quad \{ H^D, P_{N^i} \} = \mathcal{P}_i = 0 . \] (28)
For completeness of the system we have to include set of secondary constraints. According Dirac we choose them in the form
\[ N(t, \vec{x}) = 1; \quad N^j(t, \vec{x}) = 0; \] (29)
\[ \pi(t, \vec{x}) = 0; \quad \chi^j := \partial_j(q^{-1/3}q^{ij}) = 0. \] (30)

The equations of motion obtained for our system are
\[ \frac{dF}{dT} = \partial H(\varphi_0) \partial P_F, \quad - \frac{dP_F}{dT} = \partial H(\varphi_0) \partial F, \] (31)
where \( H(\varphi_0) \) is given by the equation (20), and we introduced reparametrization-invariant geometric time \( T \)
\[ N_0 dt \overset{\text{def}}{=} dT. \] (32)

2.3. Global constraints and equations of motion.

The physical meaning of the geometric time \( T \), the dynamic variable \( \varphi_0 \) and its momentum is given by the explicit resolving of the zero-Fourier harmonic of the energy constraint
\[ \frac{\delta W^E}{\delta N_0(t)} = - \frac{P_0^2}{4V_0} + H(\varphi_0) = 0. \] (33)

This constraint has two solutions for the global momentum \( P_0 \):
\[ (P_0)_\pm = \pm 2\sqrt{V_0H(\varphi_0)} \equiv H^*_{\pm}. \] (34)

The equation of motion for this global momentum \( P_0 \) in gauge [29] takes the form
\[ \frac{\delta W^E}{\delta P_0} = 0 \Rightarrow \left( \frac{d\varphi}{dT} \right)_\pm = \frac{(P_0)_\pm}{2V} = \pm \sqrt{\rho(\varphi_0)}; \quad \rho(\varphi_0) = \frac{\int d^3x H}{V_0} = \frac{H(\varphi_0)}{V_0}. \] (35)

The integral form of the last equation is
\[ T(\varphi_1, \varphi_2) = \int_{\varphi_1}^{\varphi_2} d\varphi \rho^{-1/2}(\varphi). \] (36)

Equation obtained by varying the action with respect to \( \varphi_0 \) follows independently from the set of other constraints and equations of motion.

Equations (35), (36) in the homogeneous approximation of GR are the basis of observational cosmology where the geometric time is the conformal time connected with the world time \( T_f \) of the Friedmann cosmology by the relation
\[ dT_f = \frac{\varphi_0(T)}{\mu} dT, \] (37)
and the dependence of scale factor (dynamic evolution parameter \( \varphi_0 \)) on the geometric time \( T \) is treated as the evolution of the universe. In particular, equation (34) gives the relation between the present-day value of the dynamic evolution parameter \( \varphi_0(T_0) \) and cosmological observations, i.e., the density of matter \( \rho \) and the Hubble parameter
\[ H^*_{\text{hub}} = \frac{\mu_0}{\varphi_0} = \frac{\mu_0}{\varphi_0^2} \Rightarrow \varphi_0(T_0) = \left( \frac{\mu_0}{H^*_{\text{hub}}} \right)^{1/2} := \mu \Omega_{0}^{1/4} \quad (0.6 < (\Omega_0^{1/4})_{\text{exp}} < 1.2). \] (38)
The dynamic evolution parameter as the cosmic scale factor and a value of its conjugate momentum (i.e., a value of the dynamic Hamiltonian) as the density of matter (see equations (35), (36)) are objects of measurement in observational astrophysics and cosmology and numerous discussions about the Hubble parameter, dark matter, and hidden mass.

Our aim is to find the equivalent unconstrained Hamiltonian system that describes evolution of the field world space \((F, \varphi_0)\) in terms of the geometric time \(T\).

### 2.4. Equivalent Unconstrained Systems

Assume that we can solve the constraint equations and pass to the reduced space of independent variables \((F^*, P^*_F)\). The explicit solution of the local and global constraints has two analytic branches with positive and negative values for scale factor momentum \(P^*_0 (34)\). Therefore, inserting solutions of all constraints into the action we get two branches of the equivalent Dynamic Unconstrained System (DUS)

\[
W_{\pm}^{DUS}[F|\varphi_0] = \int_{\varphi_1}^{\varphi_2} d\varphi_0 \left\{ \left[ \int d^3x \sum_{F^*} P^*_F \frac{\partial F^*}{\partial \varphi_0} \right] - H^\pm_F + \frac{1}{2} \partial_{\varphi_0}(\varphi_0 H^\pm_F) \right\},
\]

where \(\varphi_0\) plays the role of evolution parameter and \(H^\pm_F\) (defined by equation (34) plays the role of the evolution Hamiltonian in the reduced phase space of independent physical variables \((F^*, P^*_F)\) with equations of motion

\[
\frac{dF^*_G}{d\varphi_0} = \frac{\partial H^*_G}{\partial P^*_F}, \quad -\frac{dP^*_G}{d\varphi_0} = \frac{\partial H^*_G}{\partial F^*}.
\]

The evolution of the field world space variables \((F^*, \varphi_0)\) with respect to the geometric time \(T\) is not contained in DUS (39). This geometric time evolution is described by supplementary equation for nonphysical momentum \(P^*_0 (35)\) that follows from the initial extended system.

To get an equivalent unconstrained system in terms of the geometric time (we call it the Geometric Unconstrained System (GUS)), we need the Levi-Civita canonical transformation (LC) \([20, 21, 22]\) of the field world phase space

\[
(F^*, P^*_F|\varphi_0, P_0) \Rightarrow (F^*_G, P^*_G|Q_0, \Pi_0)
\]

which converts the energy constraint (33) into the new momentum \(\Pi_0\) (see the similar transformations for a relativistic particle in Appendix A).

In terms of geometrical variables the action takes the form

\[
W^G = \int_{t_1}^{t_2} dt \left\{ \left[ \int d^3x \sum_{F^*_G} P^*_G \dot{F}^*_G \right] - \Pi_0 \dot{Q}_0 + N_0 \Pi_0 + \frac{dt}{dt} S^{LC} \right\}
\]

where \(S^{LC}\) is generating function of LC transformations. Then the energy constraint and the supplementary equation for the new momentum take trivial form

\[
\Pi_0 = 0 ; \quad \frac{\delta W}{\delta \Pi_0} = 0 \quad \Rightarrow \quad \frac{dQ_0}{dt} = N_0 \quad \Rightarrow \quad dQ_0 = dT.
\]

Equations of motion are also trivial

\[
\frac{dP^*_G}{dT} = 0, \quad \frac{dF^*_G}{dT} = 0,
\]

and their solutions are given by the initial data

\[
P^*_G = P^*_G^0, \quad F^*_G = F^*_G^0.
\]
Substituting solutions of (43) and (44) into the inverted Levi-Civita transformations
\[ F^* = F^*(Q_0, \Pi_0|F_G^*, P_G^*), \quad \varphi_0 = \varphi_0(Q_0, \Pi_0|F_G^*, P_G^*) \] (46)
and similar for momenta, we get formal solutions of (40) and (36)
\[ F^* = F^*(T, 0|F_G^0, P_G^0), \quad P_F^* = P_F^*(T, 0|F_G^0, P_G^0), \quad \varphi_0 = \varphi_0(T, 0|F_G^0, P_G^0). \] (47)
We see that the geometric time evolution of the dynamic variables is absent in DUS. The geometric
time evolution of the dynamic variables can be described in the form of the LC (inverted) canonical
transformation of GUS into DUS (46), (47).

To obtain the geometric time Hamiltonian evolution, it is sufficient to use the weak form of Levi-
Civita-type transformations to GUS \( (F^*, P_F^*) \Rightarrow (\tilde{F}, \tilde{P}) \) with a new constraint
\[ \tilde{\Pi}_0 - \tilde{H}(\tilde{Q}_0, \tilde{F}, \tilde{P}) = 0. \] (48)
We get the constraint shell action
\[ W^{GUS} = \int dT \left\{ \int d^3x \sum_{\tilde{F}} \tilde{P} \frac{d\tilde{F}}{dT} - \tilde{H}(T, \tilde{F}, \tilde{P}) \right\}, \] (49)
that allows us to choose the initial cosmological data with respect to the geometric time.

The considered invariant reduction reveals the difference of reparametrization-invariant theory
from the gauge-invariant theory: in gauge-invariant theory the superfluous (longitudinal) variables
are completely excluded from the reduced system; whereas, in reparametrization-invariant theory the
superfluous (longitudinal) variables leave the sector of the Dirac observables (i.e., the phase space
\( (F^*, P_F^*) \)) but not the sector of measurable quantities: superfluous (longitudinal) variables become the
dynamic evolution parameter and dynamic Hamiltonian of the reduced theory.

2.5. Quantization and the arrow of the time

In quantum theory of GR (like in quantum theories of a particle considered in Appendix A), we get
two Schrödinger equations
\[ i \frac{d}{d\varphi_0} \Psi^\pm(F|\varphi_0, \varphi_1) = H^\pm_\pm(\varphi_0)\Psi^\pm(F|\varphi_0, \varphi_1) \] (50)
with positive and negative eigenvalues of \( P_0 \) and normalizable wave functions with the spectral series
over quantum numbers \( Q \)
\[ \Psi^+(F|\varphi_0, \varphi_1) = \sum_Q A_Q^+ < F|Q > < Q|\varphi_0, \varphi_1 > \theta(\varphi_0 - \varphi_1) \] (51)
\[ \Psi^-(F|\varphi_0, \varphi_1) = \sum_Q A_Q^- < F|Q > < Q|\varphi_0, \varphi_1 > \theta(\varphi_1 - \varphi_0), \] (52)
where \( < F|Q > \) is the eigenfunction of the reduced energy \( \frac{34}{34} \)
\[ H^\pm_\pm(\varphi_0) < F|Q > = \pm E(Q, \varphi_0) < F|Q > \]
\[ < Q|\varphi_0, \varphi_1 > = \exp[-i \int \varphi_0 \varphi_1 \frac{dE(Q, \varphi)}{\varphi_0 \varphi_1}], \quad < Q|\varphi_0, \varphi_1 > = \exp[i \int \varphi_0 \varphi_1 \frac{dE(Q, \varphi)}{\varphi_0 \varphi_1}]. \] (54)
The coefficient $A_Q^+$ in "secondary" quantization, can be treated as the operator of creation of a universe with positive energy; and the coefficient $A_Q^-$, as the operator of annihilation of a universe also with positive energy. The "secondary" quantization means $[A_Q^+, A_Q^-] = \delta_{Q, Q'}$. The physical states of a quantum universe are formed by the action of these operators on the vacuum $<0|$, $|0>$ in the form of out-state ($|Q > = A_Q^{|0 >}$) with positive "frequencies" and in-state ($< Q| = < 0|A_Q^-$) with negative "frequencies". This treatment means that positive frequencies propagate forward ($\varphi_0 > \varphi_1$); and negative frequencies, backward ($\varphi_1 > \varphi_0$), so that the negative values of energy are excluded from the spectrum to provide the stability of the quantum system in quantum theory of GR. In other words, instead of changing the sign of energy, we change that of the dynamic evolution parameter, which leads to the causal Green function

$$G_c(F_1, \varphi_1|F_2, \varphi_2) = G_+ (F_1, \varphi_1|F_2, \varphi_2) \theta(\varphi_2 - \varphi_1) + G_-(F_1, \varphi_1|F_2, \varphi_2) \theta(\varphi_1 - \varphi_2)$$  \hspace{1cm} (55)

where $G_+(F_1, \varphi_1|F_2, \varphi_2) = G_-(F_2, \varphi_2|F_1, \varphi_1)$ is the "commutative" Green function

$$G_+(F_2, \varphi_2|F_1, \varphi_1) = <0|\Psi^-(F_2|\varphi_2, \varphi_1)\Psi^+(F_1|\varphi_1, \varphi_1)|0>$$  \hspace{1cm} (56)

For this causal convention, the geometric time (54) is always positive in accordance with the equations of motion (33):

$$\left(\frac{dT}{d\varphi_0}\right)_\pm = \pm \sqrt{\rho} \Rightarrow T_\pm(\varphi_1, \varphi_0) = \pm \int_{\varphi_1}^{\varphi_0} d\varphi \rho^{-1/2}(\varphi) \geq 0.$$  \hspace{1cm} (57)

Thus, the causal structure of the field world space immediately leads to the arrow of the geometric time (57) and the beginning of evolution of a universe with respect to the geometric time $T = 0$.

As it was shown in [3], the way to obtain conserved integrals of motion in classical theory and quantum numbers $Q$ in quantum theory is the Levi-Civita-type canonical transformation of the field world space $(F, \varphi_0)$ to a geometric set of variables $(V, Q_0)$ with the condition that the geometric evolution parameter $Q_0$ coincides with the geometric time $dT = dQ_0$ (see Fig. 3).

3. Reparametrization invariant path integral

Following Faddeev-Popov procedure we can write down the path integral for local fields of our theory using constraints and gauge conditions (27-30):

$$Z_{local}(F_1, F_2|P_0, \varphi_0, N_0) = \int_{F_1}^{F_2} D(F, P_f) \Delta_s \Delta_t \exp \{i\hat{W}\},$$

where

$$D(F, P_f) = \prod_{t,x} \left( \prod_{i<k} \frac{d\eta_{ik} d\pi_{ik}}{2\pi} \prod_{f} \frac{df dp_f}{2\pi} \right)$$  \hspace{1cm} (59)

are functional differentials for the metric fields $(\pi, q)$ and the matter fields $(p_f, f)$,

$$\Delta_s = \prod_{t,x,i} \delta(P_i)) \delta(\chi^j) det\{P_i, \chi^j\},$$

$$\Delta_t = \prod_{t,x} \delta(\mathcal{H}(\mu)) \delta(\pi) det\{\mathcal{H}(\varphi_0) - \rho, \pi\}, \quad \left( \rho = \frac{\int d^3 x \mathcal{H}(\varphi_0)}{V_0} \right)$$  \hspace{1cm} (61)
are the F-P determinants, and
\[
\tilde{W} = \int_{t_1}^{t_2} dt \left\{ \int d^3x \left( \sum_F P_F \dot{F} - P_0 \varphi_0 - N_0 \left[ -\frac{P_0^2}{4V_0} + H(\varphi_0) \right] + \frac{1}{2} \partial_i (P_0 \varphi_0) \right) \right\}
\]
(62)
is extended action of considered theory.

By analogy with SR considered in Appendix A we define a commutative Green function as an integral over global fields \((P_0, \varphi_0)\) and the average over reparametrization group parameter \(N_0\)

\[
G_+(F_1, \varphi_1|F_2, \varphi_2) = \prod_{\varphi_1} \left( \frac{d\varphi_0 dP_0 d\tilde{N}_0}{2\pi} \right) Z_{\text{local}}(F_1, F_2|P_0, \varphi_0, N_0),
\]
(63)
where
\[
\tilde{N} = N/2\pi \delta(0), \quad \delta(0) = \int dN_0.
\]
The causal Green function in the world field space \((F, \varphi_0)\) is defined as the sum

\[
G_+(F_1, \varphi_1|F_2, \varphi_2) = G_+(F_1, \varphi_1|F_2, \varphi_2)\theta(\varphi_1 - \varphi_2) + G_+(F_2, \varphi_1|F_2, \varphi_1)\theta(\varphi_2 - \varphi_1).
\]
(65)
This function will be considered as generating functional for the unitary S-matrix elements [26]

\[
S[\varphi_1, \varphi_2] = <\text{out} (\varphi_2)|T_\varphi \exp \left\{ -i \int d\varphi (H^*_T) \right\} |(\varphi_1) \text{in} >,
\]
(66)
where \(T_\varphi\) is a symbol of ordering with respect to parameter \(\varphi_0\), and \(<\text{out} (\varphi_2)|\text{in} >\) are states of quantum Univers in the lowest order of the Dirac perturbation theory \((N = 1; \ N^k = 0; \ q^{ij} = \delta_{ij} + h^T_{ij})\), \(H^*_T\) is the interaction Hamiltonian

\[
H^*_T = H^* - H^*_0, \quad H^* = 2\sqrt{V_0H(\varphi)}, \quad H^*_0 = 2\sqrt{V_0H_0(\varphi)},
\]
(67)
\(H_0\) is a sum of the Hamiltonians of "free" fields (gravitons, photons, massive vectors, and spinors) where all masses (including the Planck mass) are replaced by the dynamic evolution parameter \(\varphi_0\) [7]. For example for gravitons the "free" hamiltonian takes the form:

\[
H_0(\varphi_0) = \int d^3x \left( \frac{6(\pi_{(0)})^2}{\varphi_0^2} + \frac{\varphi_0^2}{24} (\partial_i h^T)^2 \right); \quad (h^T_{ii} = 0; \ \partial_i h^T_{ij} = 0).
\]
(68)
In order to reproduce Faddeev-Popov integral for general relativity in infinite space-time [4], one should fix the dynamic evolution parameter at its present-day value \(\varphi_0 = \mu\) [8], remove all the zero-mode dynamics \(P_0 = \varphi_0 = 0, N_0 = 1\), and neglect the surface Newton term in the Hamiltonian. We get

\[
Z^{FP}(F_1, F_2) = Z_{\text{local}}(F_1, F_2|P_0 = 0, \varphi_{0\exp} = \mu, N_0 = 1),
\]
(69)
or

\[
Z^{FP}(F_1, F_2) = \int_{F_1}^{F_2} D(F, P_f) \Delta_s \Delta_t \exp \{ iW_{fp} \},
\]
(70)
where

\[
W_{fp} = \int_{-\infty}^{+\infty} dt \int d^3x \left( \sum_F P_F \dot{\Phi} - \mathcal{H}_{fp}(\mu) \right), \quad \mathcal{H}_{fp}(\mu) = \mathcal{H}(\mu) - \frac{\mu^2}{6} \partial_i \partial_j q^{ij},
\]
(71)
and
\[ \Delta_t = \prod_{t,x} \delta(H(\mu))\delta(\pi)\det\{H(\mu), \pi\}. \tag{72} \]

The F-P integral (70) is considered as the generating functional for unitary perturbation theory in terms of S-matrix elements
\[ S[-\infty|+\infty] = \langle \text{out}\,|T\exp \left\{ -i \int_{-\infty}^{+\infty} dt H_I(\mu) \right\}|\text{in} \rangle. \tag{73} \]

Strictly speaking, the approximation (69) is not a correct procedure, as it breaks the reparametrization-invariance. The region of validity of FP integral (70) is discussed in next sections.

4. Evolution of ”Free” Quantum Universe

4.1. Dynamic unconstrained system

Possible states of a free quantum universe in S-matrix (66) are determined by the lowest order of the Dirac perturbation theory given by the well-known system of ”free” conformal fields (8), (68) in a finite space-time volume \[ W_E = \int_{t_1}^{t_2} dt \left( \int d^3x \sum_F P_F \dot{F} - P_0 \dot{\varphi}_0 - N_0 [-\frac{P_0^2}{4V_0} + H_0(\varphi_0)] + \frac{1}{2} \partial_0(P_0 \dot{\varphi}_0) \right), \tag{74} \]

where \( H_0 \) is a sum of the Hamiltonians of ”free” fields (gravitons \[ (8), \] photons, massive vectors, and spinors) where all masses (including the Planck mass) are replaced by the dynamic evolution parameter \( \varphi_0 \). The classical equations for the action (74)
\[ \frac{dF}{dT} = \frac{\partial H_0}{\partial P_F}, \quad - \frac{dP_F}{dT} = \frac{\partial H_0}{\partial F}, \quad P_0 = \pm 2\sqrt{V_0 H_0} := H_0^\pm \tag{75} \]

contain two invariant times: the geometric \( T \) and the dynamic \( \varphi_0^\pm \) connected by the geometro-dynamic (back-reaction) equation
\[ \frac{d\varphi_0^\pm}{dT} = \pm \sqrt{\rho_0(\varphi_0^\pm)}, \quad \left( \rho_0 = \frac{H_0}{V_0} \right). \tag{76} \]

Solving the energy constraint we get the action for dynamic system
\[ W_0^E(\text{constraint}) = W_0^D = \int_{\varphi(t_1)}^{\varphi(t_2)} d\varphi \left( \int d^3x \sum_F P_F \dot{\varphi} F - H_0^\pm + \frac{1}{2} \partial_\varphi(\varphi H_0^\pm) \right), \tag{77} \]

that has two branches for a universe with a positive energy \( (P_0 > 0) \), and a universe with a negative energy \( (P_0 < 0) \). We interpret the branch with negative energy as an ”anti-universe” which propagates backward \( (\varphi < 0) \) with positive energy to provide the stability of a quantum system.

The content of matter in the universe is described by the number of particles \( N_{F,k} \) and their energy \( \omega_F(\varphi_0, k) \) (which depends on the dynamic evolution parameter \( \varphi_0 \) and quantum numbers \( k \), momenta, spins, etc.). Detected particles are defined as the field variables \( F = f \)
\[ f(x) = \sum_k \frac{C_f(\varphi_0) \exp(ik_1 x_1)}{\sqrt{V_0^{3/2}/2\omega_f(\varphi_0, k)}} \left( a_f^+(\gamma) + a_f^-(\gamma) \right) \tag{78} \]
which diagonalize the operator of the density of matter

\[ \rho_0 = \sum_{f,k} \frac{\omega_f(\varphi_0, k)}{V_0} \hat{N}_{f,k}, \quad \hat{N}_f(a) = \frac{1}{2}(a_f^+ a_f^- + a_f^- a_f^+) . \]  

(79)

We restrict ourselves to gravitons (f=h) \( C_h(\varphi_0) = \varphi_0 \sqrt{12} \), \( \omega_h(\varphi_0, k) = \sqrt{k^2} \), and massive vector particles (f=v) \( C_v(\varphi_0) = 1, \omega_v(\varphi_0, k) = \sqrt{k^2 + y^2 \varphi_0^2} \), where y is the mass in terms of the Planck constant.

### 4.2. Geometric unconstrained system

The equations of motion (76) in terms of \( a^+, a^- \) are not diagonal

\[ \frac{d}{dT} \chi := i \chi' = -\hat{H}_{a_f} \chi_{a_f}, \quad \chi_{a_f} = \begin{pmatrix} a_f^+ \\ a_f^- \end{pmatrix}, \quad \hat{H}_{a_f} = \begin{pmatrix} \omega_{a_f} & -i \Delta_f \\ -i \Delta_f & -\omega_{a_f} \end{pmatrix}, \]  

(80)

where nondiagonal terms \( \Delta_{f=h,v} \) are proportional to the Hubble parameter \( [28] \)

\[ \Delta_{f=h} = \frac{\varphi'_0}{\varphi_0}, \quad \Delta_{f=v} = -\frac{\omega'_v}{2\omega_v}, \quad \varphi'_0 = \sqrt{\rho_0} . \]  

(81)

The "geometric system" \( (b^+, b^-) \) is determined by the transformation to the set of variables which diagonalize equations of motion (73) and determine a set of integrals of motion of equations (75) (as conserved numbers \{Q\}).

To obtain integrals of motion and to choose initial conditions for the universe evolution we use the Bogoliubov transformations [28] and define "quasi-particles"

\[ b^+ = \cosh(r)e^{-i\theta}a^+ - i \sinh(r)e^{i\theta}a, \quad b = \cosh(r)e^{i\theta}a + i \sinh(r)e^{-i\theta}a^- , \]  

(82)

or

\[ \chi_b = \begin{pmatrix} b^+ \\ b^- \end{pmatrix} = \hat{O} \chi_a , \]

which diagonalize the classical equations expressed in terms of "particles" \( (a^+, a^-) \), so that the "number of quasiparticles" is conserved

\[ \frac{d(b^+ b^-)}{dt} = 0, \quad b = \exp(-i \int_0^T d\bar{T} \bar{\omega}_b(\bar{T}))b_0 . \]  

(83)

Functions \( r \) and \( \theta \) in (83), and the quasiparticle energy \( \bar{\omega}_b \) in (83) are determined by the equation of diagonalization

\[ i \frac{d}{dT} \chi_b = \begin{pmatrix} \bar{\omega}_b & 0 \\ 0 & -\bar{\omega}_b \end{pmatrix} \chi_b \equiv \begin{pmatrix} \omega_f - \theta_f & (\Delta_f \cos 2\theta_f) \sinh(2r_f) \\ (\Delta_f \cos 2\theta_f) \cosh(2r_f) & \omega_f + \theta_f \end{pmatrix} \]

in the form obtained in (85)

\[ \omega_{fb} = (\omega_f - \theta_f') \cosh(2r_f) - (\Delta_f \cos 2\theta_f) \sinh(2r_f) , \]

(85)

\[ 0 = (\omega_f - \theta'_f) \sinh(2r_f) - (\Delta_f \cos 2\theta_f) \cosh(2r_f), \quad \theta' = -\Delta_f \sin 2\theta_f . \]  

(84)
Equations (81)–(85) are closed by the definition of "observable particles" in terms of quasiparticles

$$\rho(\varphi) = \frac{H_0}{V} = \sum f \frac{\omega_f(\varphi)\{a_f^+ a_f\}}{V_0}; \quad \{a^+ a\} = \{b_0^+ b_0\} \cosh 2r - \frac{i}{2}(b^+ b - b b^+) \sinh 2r$$

with

$$\tilde{\omega}_{fb} = \sqrt{(\omega_f - \theta_f)^2 + (r'')^2 - \Delta_f^2}, \quad \theta_f = -\frac{1}{2} \left( \frac{r''}{\Delta_f} \right)^0 \left[ 1 - \frac{(r'')^2}{\Delta_f^2} \right]^{-1/2}, \quad \cosh(2r_f) = \frac{\omega_f - \theta_f}{\tilde{\omega}_{fb}}.$$  

The constrained system in terms of geometric variables is described by the action

$$\tilde{W}^G = \int dt \left\{ \sum f \frac{i}{2} (b\partial_t b^+ - b^+ \partial_t b)_f - \bar{\Pi}_0 \dot{Q}_0 - N_0 \left[ -\bar{\Pi}_0 + \sum_f \omega_f^2(Q_0) N_f(b) \right] \right\},$$

where the new dynamic evolution parameter $Q_0$ coincides with geometric time $T$ on the equations of motion

$$\frac{\delta \tilde{W}^E}{\delta \bar{\Pi}_0} = 0 \Rightarrow \frac{dQ_0}{dt} \Rightarrow dQ_0 = dT.$$  

Reduction of this system leads to the weak version of Geometric Unconstrained System (49)

$$\tilde{W}^{GUS} = \int dT \left\{ \sum f \frac{i}{2} (b\partial_T b^+ - b^+ \partial_T b)_f - \sum_f \omega_f^2(T) N_f(b) \right\}.$$  

We choose the initial data appropriate for the dynamics described by GUS (80).

4.3. Quantization

The initial data $b_0, b_0^+$ of quasiparticle variables (83) form the set of quantum numbers in quantum theory.

Let us suppose that we manage to solve equations (83)–(87) with respect to the geometric time $T$ in terms of conserved numbers $b_0^+, b_0$. This means that the wave function of a quantum universe can be represented in the form of a series over the conserved quantum numbers $Q = n_{f,k} = <Q | b_f^+ b_f | Q >$ of the Bogoliubov states

$$\Phi_Q(T) = \prod_{f,n_f} \exp \left\{ -i \int_0^T dT n_f \tilde{\omega}_f(T) \right\} \frac{(b_f^+)^n_f}{\sqrt{n_f!}} | 0 >_b.$$  

In this geometric system, we have an arrow of the geometric time $T$ for the universe

$$T^+(\varphi_2, \varphi_1) = \int_{\varphi_1}^{\varphi_2} d\varphi \rho(\varphi)^{-1/2} > 0, \quad \varphi_2 > \varphi_1,$$  

and for anti-universe

$$T^-(\varphi_2, \varphi_1) = -\int_{\varphi_2}^{\varphi_1} d\varphi \rho(\varphi)^{-1/2} = \int_{\varphi_2}^{\varphi_1} d\varphi \rho(\varphi)^{-1/2} > 0, \quad \varphi_1 > \varphi_2.$$
The dynamic system of particle variables $a^+, a$ is connected with the geometric one by the Bogoliubov transformations. Using these transformations we can find wave functions of a universe, for $\varphi_2 > \varphi_1$ and an anti-Universe, for $\varphi_1 > \varphi_2$

$$\Psi_Q(T) = A_Q^+ \Phi_Q(T_+(\varphi_2, \varphi_1))\theta(\varphi_2 - \varphi_1) + A_Q^- \Phi_Q^*(T_-(\varphi_2, \varphi_1))\theta(\varphi_1 - \varphi_2),$$

(94)

where the first term and the second one are positive ($P_0 > 0$) and negative ($P_0 < 0$) frequency parts of the solutions with the spectrum of quasiparticles $\tilde{\omega}_b$. $A_Q^+$ is the operator of creation of a universe with a positive "frequency" (which propagates in the positive direction of the dynamic evolution parameter) and $A_Q^-$ is the operator of annihilation of a universe (or creation of an anti-universe) with a negative "frequency" (which propagates in the negative direction of the dynamic evolution parameter).

We can see that the creation of the universe in the field world space and the creation of dynamic particles by the geometric vacuum ($b^+|0> = 0$) are two different effects.

The second effect disappears if we neglect gravitons and massive fields. In this case, $d\rho/d\varphi = 0$, and one can represent a wave function of Universe in the form of the spectral series over eigenvalues $\rho_Q$ of the density $\rho$

$$\Psi(f|\varphi_2, \varphi_1) = \sum_Q \left[ A_Q^+ \sqrt{2\rho_Q} \exp \left\{ -i(\varphi_2 - \varphi_1) \sum \tilde{\omega}_n \sqrt{\rho_Q} \right\} < f|Q > + A_Q^- \sqrt{2\rho_Q} \exp \left\{ i(\varphi_2 - \varphi_1) \sum \tilde{\omega}_n \sqrt{\rho_Q} \right\} < f|Q >^\ast \right],$$

(95)

where $< f|Q >$ is a product of normalizable Hermite polynomials.

4.4. Evolution of quantum universe

The equations of diagonalization (84) for the Bogoliubov coefficients (82) and the quasiparticle energy $\tilde{\omega}_b$ (85) play the role of the equations of state of the field matter in the universe. We can show that the choice of initial conditions for the "Big Bang" in the form of the Bogoliubov (squeezed) vacuum $b|0>_b = 0$ reproduces all stages of the evolution of the Friedmann-Robertson-Walker universe in their conformal versions: anisotropic, inflation, radiation, and dust.

The squeezed vacuum (i.e., the vacuum of quasiparticles) is the state of "nothing". For small $\varphi$ and a large Hubble parameter, at the beginning of the universe, the state of vacuum of quasiparticles leads to the density of matter

$$b < \rho(a^+, a) >_b = \rho_0 \frac{1}{2} \left( \frac{\varphi^2(0)}{\varphi^2(T)} + \frac{\varphi^2(T)}{\varphi^2(0)} \right), \quad \theta = \frac{\pi}{4}$$

(96)

where $\varphi(0)$ is the initial value, and $\rho_0$ is the density of the Casimir energy of vacuum of "quasiparticles". The first term corresponds to the conformal version of the rigid state equation (in accordance with the classification of the standard cosmology) which describes the Kasner anisotropic stage $T_{\pm}(\varphi) \sim \pm \varphi^2$ (considered on the quantum level by Misner [29]). The second term of the squeezed vacuum density (96) (for an admissible positive branch) leads to the stage with inflation of the dynamic evolution parameter $\varphi$ with respect to the geometric time $T$

$$\varphi(T)_{(+) \simeq \varphi(0) \exp[T \sqrt{2\rho_0}/\varphi(0)].}$$

It is the stage of intensive creation of "measurable particles". After the inflation, the Hubble parameter goes to zero, and gravitons convert into photon-like oscillator excitations with the conserved number of particles.
At the present-day stage, the Bogoliubov quasiparticles coincide with particles, so that the measurable density of energy of matter in the universe is a sum of relativistic energies of all particles

$$\rho_0(\varphi) = \frac{E}{V_0} = \sum_{n_f} n_f \frac{1}{V_0} \sqrt{k^2 + y_f^2 \varphi^2(T)},$$

(97)

where \(y_f\) is the mass of a particle in units of the Planck mass. The case of massless particles \((y = 0, \rho_0(\varphi) = \text{constant})\) correspond to the conformal version of radiation stage of the standard FRW-cosmology. And the massive particles at rest \((k = 0, \rho_0(\varphi) = \rho_{\text{barions}} \varphi / \mu)\) corresponds to the conformal version of the dust universe of the standard cosmology with the Hubble law

$$\varphi' = \pm \sqrt{\rho_0} \Rightarrow \varphi(T) = \frac{\rho_{\text{barions}}}{4\mu} T^2; \quad q = \frac{\varphi''}{\varphi'^2} = \frac{1}{2}. \quad (98)$$

The dynamic evolution parameter is expressed through the geometric time of a quantum asymptotic state of the universe \(|\text{out}>\) and conserved quantum numbers of this state: energy \(E_{\text{out}}\) and density \(\rho_0 = E_{\text{out}} / V_0\).

It is well-known that \(E_{\text{out}}\) is a tremendous energy \((10^{79} \text{GeV})\) in comparison with possible real and virtual deviations of the free Hamiltonian in the laboratory processes:

$$\hat{H}_0 = E_{\text{out}} + \delta H_0, \quad <\text{out}|\delta H_0|\text{in}> \ll E_{\text{out}}. \quad (99)$$

We have seen that the dependence of the scale factor \(\varphi_0\) on the geometric time \(T\) (or the "relation" of two classical unconstrained systems: dynamic and geometric) describes the "Big Bang" and evolution of a universe.

Therefore, from the point of view of unconstrained system "Big Bang" is effect of evolution of the geometric interval with respect to the dynamic evolution parameter which goes beyond the scope of Hamiltonian description of a single classical unconstrained system.

Reparametrization-invariant dynamic of General Relativity is covered by Geometric and Dynamic Unconstrained Systems connected by the Levi-Civita transformation of fields of matter into the vacuum fields of initial data with respect to geometric time (see Fig. 3).

5. QFT limit of Quantum Gravity

The simplest way to determine the QFT limit of Quantum Gravity and to find the region of validity of the FP-integral\(^7\) is to use the quantum field version of the reparametrization-invariant integral\(^6\) in the form of S-matrix elements \(^2\) (see \(^3\), \(^7\)). We consider the infinite volume limit of the S-matrix element \(^5\) in terms of the geometric time \(T\) for the present-day stage \(T = T_0, \varphi(T_0) = \mu, \) and \(T(\varphi_1) = T_0 - \Delta T, T(\varphi_2) = T_0 + \Delta T = T_{\text{out}}.\) One can express this matrix element in terms of the time measured by an observer of an out-state with a tremendous number of particles in the Universe using equation \(^1\) \(d\varphi = dT_{\text{out}} \sqrt{\rho_{\text{out}}}\) and approximation \(^9\) to neglect "back-reaction". In the infinite volume limit, we get from \(^1\)

$$d\varphi_0[H^1_I] = 2d\varphi_0 \left( \sqrt{V_0(H_0 + H_I)} - \sqrt{V_0H_0} \right) = dT_{\text{out}}[\hat{F}H_I + O(1/E_{\text{out}})]$$

(100)

where \(H_I\) is the interaction Hamiltonian in GR, and

$$\hat{F} = \sqrt{\frac{E_{\text{out}}}{H_0}} = \sqrt{\frac{E_{\text{out}}}{E_{\text{out}} + \delta H_0}}$$

(101)
is a multiplier which plays the role of a form factor for physical processes observed in the ”laboratory” conditions when the cosmic energy $E_{out}$ is much greater than the deviation of the free energy

$$\delta H_0 = H_0 - E_{out};$$

(102)
due to creation and annihilation of real and virtual particles in the laboratory experiments.

The measurable time of the laboratory experiments $T_2 - T_1$ is much smaller than the age of the universe $T_0$, but it is much greater than the reverse ”laboratory” energy $\delta$, so that the limit

$$\int_{T(\varphi_1)}^{T(\varphi_2)} dT_{out} \Rightarrow \int_{-\infty}^{+\infty} dT_{out}$$

is valid. If we neglect the form factor (101) that removes a set of ultraviolet divergences, we get the matrix element (73) that corresponds to the standard FP functional integral (70) and $S$-matrix element (73) where the coordinate time is replaced by the geometric (conformal) time $t \rightarrow T_{out}$:

$$S[-\infty|+\infty] = \langle \text{out}|T \exp \left\{-i \int_{-\infty}^{+\infty} dT_{out} \hat{F} H_f(\mu) \right\}|\text{in} > \quad \left( \hat{F} = 1 \right).$$

(103)

Thus, the standard FP-integral (i.e., the Hamiltonian description of the evolution of fields with respect to the geometric time) appears as the nonrelativistic approximation of tremendous mass of a universe and its very large life-time (see Fig. 4). Now, it is evident that the conventional FP unitary S-matrix are not valid for the description of the early universe given in the finite spatial volume and the finite positive interval of geometrical time when the early universe only begins to create matter.

On the other hand, we revealed that standard quantum field theory (103) is expressed in terms of the conformal-invariant Lichnerowicz variables and coordinates including the conformal time ($T_{out}$) as the time of evolution of these variables. The reparametrization-invariant description of the ”Big Bang” evolution distinguishes conformal variables (8) and coordinates. The conformal invariance of the variables can testify to the conformal invariance of the initial theory of gravity. Such the theory can be a scalar version of the Weyl conformal invariant theory (3) (that dynamically coincides with the Einstein General Relativity) where these conformal variables, coordinates, geometric time $T$, and the conformal Hubble parameter $H_{hub} = \varphi'/\varphi$ can be considered as measurable quantities. In particular, to get ”accelerating” universe with $q > 0$ in the dust stage (see (98)), is enough to count that we measure the relative Weyl time ($T_{out}$) of quantum field theory (103).

6. Conclusions

The main result of the paper is the reparametrization-invariant generating functional for the unitary and causal perturbation theory in general relativity in a finite space-time. We show that the classical cosmology of a universe and the Faddeev-Popov-DeWitt functional correspond to different orders of decomposition of this functional over the inverse ”mass” of a universe.

This result is based on the assertion that the measurable time in any reparametrization-invariant system is not the coordinate time, but the time-like dynamic variable of an extended phase space of this system (of the type of the conformal scale factor). Accordingly, the measurable Hamiltonian is a solution of the energy constraint with respect to the conjugate momentum of this dynamic evolution parameter $\hat{F}$. This definition of the dynamic evolution parameter and Hamiltonian supposes the reduction of an action for constructing an ”equivalent” unconstrained system.
The second assertion is that such an unconstrained dynamic system cannot cover the physical content of a relativistic reparametrization-invariant system. This content can be covered by two "equivalent" unconstrained systems – the mentioned above dynamic system and the geometric system constructed by the Levi-Civita type canonical transformation so that a new dynamic evolution parameter coincides with the geometric time.

The "geometric" variables are the Bogoliubov quasiparticles which diagonalize equations of motion and give cosmological initial conditions. The coefficients of the Bogoliubov transformations, in the conventional QFT perturbation theory, determine the evolution density of matter. For the vacuum initial data this evolution reproduces all stages of the standard FRW evolution of the universe in their conformal versions.

Consistent QFT limits of the generating functional in classical gravitation and cosmology and the conventional quantum field theory (in the form of the Faddeev-Popov generating functional for an infinite space-time) can be fulfilled for a tremendous value of the universe mass and the universe life-time (see Fig. 4). The quantum field theory limit distinguish the conformal treatment of general relativity developed in [5, 6, 30].

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APPENDIX A. Special Relativity
To answer the question: Why is the reparametrization-invariant Hamiltonian reduction needed?, let us consider relativistic mechanics [5] in the Hamiltonian form

\[
W[P, X|N|\tau_1, \tau_2] = \int_{\tau_1}^{\tau_2} d\tau [-P_\mu \dot{X}^\mu - \frac{N}{2m} (P_\mu^2 - m^2)] .
\] (A.1)

This action is invariant with respect to reparametrizations of the coordinate evolution parameter

\[
\tau \to \tau' = \tau'(\tau), \quad N' d\tau' = N d\tau
\] (A.2)
given in the one-dimensional space with the invariant interval

\[
dT := N d\tau, \quad T = \int_0^\tau d\tau N(\tau) .
\] (A.3)

We called this invariant interval the geometric time [3] whereas the dynamic variable \(X_0\) (with a negative contribution in the constraint) we called dynamic evolution parameter.

In terms of the geometric time [A.3] the classical equations of the generalized Hamiltonian system (A.1) takes the form

\[
\frac{dX_\mu}{dT} = \frac{P_\mu}{m}, \quad \frac{dP_\mu}{dT} = 0, \quad P_\mu^2 - m^2 = 0.
\] (A.4)
The problem is to obtain the equivalent unconstrained theories directly in terms of the invariant times $X_0$ or $T$ with the invariant Hamiltonians of evolution with respect to these times. The solution of this problem is called the dynamic (for $X_0$), or geometric (for $T$) reparametrization-invariant Hamiltonian reductions.

The dynamic reduction of the extended system (A.1) means the substitution, into it, of the explicit resolving of the energy constraint ($-P_\mu^2 + m^2 = 0$) with respect to the momentum $P_0$ with a negative contribution

$$\frac{\delta W}{\delta N} = 0 \Rightarrow P_0 = \pm \sqrt{m^2 + P_i^2}. \quad (A.5)$$

In accordance with the two signs of the solution (A.5), after the substitution of (A.5) into (A.1), we have two branches of the dynamic unconstrained system

$$W(\text{constraint}) = W^D_\pm[\bar{P}_i, X_i|X_0(1), X_0(2)] = \int_{X_0(\tau_1)=X_0(1)}^{X_0(\tau_2)=X_0(2)} dX_0 \left[ P^i_i \frac{dX_i}{dX_0} \mp \sqrt{m^2 + P_i^2} \right]. \quad (A.6)$$

The role of the time of evolution, in this action, is played by the variable $X_0$ which abandons the Dirac sector of "observables" $P_i, X_i$, but not the sector of "measurable" quantities. At the same time, its conjugate momentum $P_0$ converts into the corresponding Hamiltonian of evolution, values of which are the energy of a particle.

This invariant reduction of the action gives the "equivalent" unconstrained system together with definition of the invariant evolution parameter $X_0$ corresponding to a non-zero Hamiltonian $P_0$.

Thus, we need the reparametrization-invariant Hamiltonian reduction to determine the invariant evolution parameter and its invariant Hamiltonian for reparametrization-invariant systems.

In quantum relativistic theory, we get two Schrödinger equations

$$i \frac{d}{dX_0} \Psi(\pm)(X|P) = \pm \sqrt{m^2 + P_i^2} \Psi(\pm)(X|P), \quad (A.7)$$

with positive and negative values of $P_0$ and normalized wave functions

$$\Psi(\pm)(X|P) = \frac{A^\pm_0 \theta(\pm P_0)}{\sqrt{(2\pi)^{3/2} \sqrt{2P_0}}} \exp(-iP_\mu X^\mu), \quad \left[ A^-_P, A^+_P \right] = \delta^3(P_i - P'_i). \quad (A.8)$$

The coefficient $A^+_P$, in the secondary quantization, is treated as the operator of creation of a particle with positive energy; and the coefficient $A^-_P$, as the operator of annihilation of a particle also with positive energy. The physical states are formed by action of these operators on the vacuum $<0|0>$ in the form of out-state ($|P> = A^+_P|0>$) with positive frequencies and in-state ($<P| = <0|A^-_P$) with negative frequencies. This treatment means that positive frequencies propagate forward ($X_{02} > X_{01})$; and negative frequencies, backward ($X_{01} > X_{02}$), so that the negative values of energy are excluded from the spectrum to provide the stability of the quantum system in QFT [26]. For this causal convention the geometric time (A.3) is always positive in accordance with the equations of motion (A.4)

$$(\frac{dT}{dX_0})_\pm = \pm \frac{m}{\sqrt{P_i^2 + m^2}} \Rightarrow T(X_{02}, X_{01}) = \pm \frac{m}{\sqrt{P_i^2 + m^2}} (X_{02} - X_{01}) \geq 0 \quad (A.9)$$

In other words, instead of changing the sign of energy, we change that of the dynamic evolution parameter, which leads to the arrow of the geometric time (A.3) and to the causal Green function

$$G^s(X) = G_+(X)\theta(X_0) + G_-(X)\theta(-X_0) = i \int \frac{d^4P}{(2\pi)^4} \exp(-iPX) \frac{1}{P^2 - m^2 - i\epsilon}, \quad (A.10)$$

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where \( G_+(X) = G_-(-X) \) is the "commutative" Green function \[26\]

\[ G_+(X) = \int \frac{d^3 P}{(2\pi)^3} \exp(-iPX)\delta(P^2 - m^2)\theta(P_0) = \frac{1}{2\pi} \int d^3 P d^3 P' <0|\Psi-(X|P)\Psi+(0|P')|0> . \tag{A.11} \]

To obtain the reparametrization-invariant form of the functional integral adequate to the considered gauge-less reduction \[A.4\], and the causal Green function \[A.10\], we use the version of composition law for the commutative Green function with the integration over the whole measurable sector \(X_{i\mu}\)

\[ G_+(X - X_0) = \int G_+(X - X_1)\tilde{G}_+(X_1 - X_0)dX_1 \quad (\tilde{G}_+ = \frac{G_+}{2\pi\delta(0)} ) , \tag{A.12} \]

where \(\delta(0) = \int dN\) is the infinite volume of the group of reparametrizations of the coordinate \(\tau\). Using the composition law \(n\)-times, we got the multiple integral

\[ G_+(X - X_0) = \int G_+(X - X_1) \prod_{k=1}^n \tilde{G}_+(X_k - X_{k+1})dX_k , \quad (X_{n+1} = X_0) . \tag{A.13} \]

The continual limit of the multiple integral with the integral representation for \(\delta\)-function

\[ \delta(P^2 - m^2) = \frac{1}{2\pi} \int dN \exp[iN(P^2 - m^2)] \]

can be defined as the path integral in the form of the average over the group of reparametrizations

\[ G_+(X) = \int_{X(\tau_1)=0}^{X(\tau_2)=X} \frac{dN(\tau_2)d^4 P(\tau_2)}{(2\pi)^3} \prod_{\tau_1 \leq \tau \leq \tau_2} \left\{ d\bar{N}(\tau) \prod_{\mu} \left( \frac{dP_\mu(\tau)dX_\mu(\tau)}{2\pi} \right) \right\} \exp(iW[P,X|N_{\tau_1},\tau_2]) , \tag{A.14} \]

where \(\bar{N} = N/2\pi\delta(0)\), and \(W\) is the initial extended action \[A.1\].

This functional integral has the form of the average over the group of reparametrization of the integral over the sector of "measurable" variables \(P_\mu, X_\mu\).

The Hamiltonian unconstrained system in terms of the geometric time \(T\) can be obtained by the canonical Levi-Civita - type transformation \[28, 29, 22\]

\[ (P_\mu, X_\mu) \Rightarrow (\Pi_\mu, Q_\mu) \tag{A.15} \]

to the variables \((\Pi_\mu, Q_\mu)\) for which one of equations identifies \(Q_0\) with the geometric time \(T\). This transformation \[20\] converts the constraint into a new momentum

\[ \Pi_0 = \frac{1}{2m}[P_0^2 - P_i^2] , \quad \Pi_i = P_i , \quad Q_0 = X_0 \frac{m}{P_0} , \quad Q_i = X_i - X_0 \frac{P_i}{P_0} \tag{A.16} \]

and has the inverted form

\[ P_0 = \pm\sqrt{2m\Pi_0 + \Pi_i^2} , \quad P_i = \Pi_i , \quad X_0 = \pm Q_0 \sqrt{\frac{2m\Pi_0 + \Pi_i^2}{m}} , \quad X_i = Q_i + Q_0 \frac{\Pi_i}{m} \tag{A.17} \]

After transformation \[A.16\] the action \[A.1\] takes the form

\[ W = \int d\tau \left[ -\Pi_\mu Q_\mu - N(-\Pi_0 + \frac{m}{2}) - \frac{d}{d\tau} S^lc \right] , \quad S^lc = (Q_0\Pi_0) . \tag{A.18} \]
The invariant reduction is the resolving of the constraint \( \Pi_0 = m/2 \) which determines a new Hamiltonian of evolution with respect to the new dynamic evolution parameter \( Q_0 \), whereas the equation of motion for this momentum \( \Pi_0 \) identifies the dynamic evolution parameter \( Q_0 \) with the geometric time \( T \) \((Q_0 = T)\). The substitution of these solutions into the action \((A.18)\) leads to the reduced action of a geometric unconstrained system

\[
W(\text{constraint}) = W^G[\Pi, Q_i|T_1, T_2] = \int_{T_1}^{T_2} dT \left( \Pi_i \frac{dQ_i}{dT} - \frac{m}{2} - \frac{d}{dT}(S^{lc}) \right) \quad (S^{lc} = Q_0 \frac{m}{2}), \quad (A.19)
\]

where variables \( \Pi_i, Q_i \) are cyclic ones and have the meaning of initial conditions in the comoving frame

\[
\delta W/\delta \Pi_i = dQ_i/d\tau = 0 \Rightarrow Q_i = Q_i^{(0)}, \quad \delta W/\delta Q_i = d\Pi_i/d\tau = 0 \Rightarrow \Pi_i = \Pi_i^{(0)}. \quad (A.20)
\]

The substitution of all geometric solutions

\[
Q_0 = T, \quad \Pi_0 = \frac{m}{2}, \quad \Pi_i = \Pi_i^{(0)} = P_i, \quad Q_i = Q_i^{(0)} \quad (A.21)
\]

into the inverted Levi-Civita transformation \((A.17)\) leads to the conventional relativistic solution for the dynamical system

\[
P_0 = \pm \sqrt{m^2 + P_i^2}, \quad P_i = \Pi_i^{(0)}, \quad X_0(T) = T \frac{P_0}{m}, \quad X_i(T) = X_i^{(0)} + T \frac{P_i}{m}. \quad (A.22)
\]

The Schrödinger equation for the wave function

\[
\frac{d}{dT} \Psi^{lc}(T, Q_i|\Pi_i) = \frac{m}{2} \Psi^{lc}(T, Q_i|\Pi_i), \quad \Psi^{lc}(T, Q_i|\Pi_i) = \exp(-iT \frac{m}{2}) \exp(i \Pi_i^{(0)} Q_i) \quad (A.23)
\]

contains only one eigenvalue \( m/2 \) degenerated with respect to the cyclic momentum \( \Pi_i \). We see that there are differences between the dynamic and geometric descriptions. The dynamic evolution parameter is given in the whole region \(-\infty < X_0 < +\infty\), whereas the geometric one is only positive \(0 < T < +\infty\), as it follows from the properties of the causal Green function \((A.10)\) after the Levi-Civita transformation \((A.16)\)

\[
G^c(Q_\mu) = \int_{-\infty}^{+\infty} d^4\mu \frac{\exp(i Q_\mu \Pi_\mu)}{2m(\Pi_0 - m/2 - i\epsilon/2m)} = \frac{\delta^3(Q)}{2m} \theta(T), \quad T = Q_0.
\]

Two solutions of the constraint (a particle and antiparticle) in the dynamic system correspond to a single solution in the geometric system.

Thus, the reparametrization-invariant content of the equations of motion of a relativistic particle in terms of the geometric time is covered by two “equivalent” unconstrained systems: the dynamic and geometric. In both the systems, the invariant times are not the coordinate evolution parameter, but variables with the negative contribution into the energy constraint. The Hamiltonian description of a relativistic particle in terms of the geometric time can be achieved by the Levi-Civita-type canonical transformation, so that the energy constraint converts into a new momentum. Whereas, the dynamic unconstrained system is suit for the secondary quantization and the derivation of the causal Green function that determine the arrow of the geometric time.
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FIGURE CAPTIONS

Fig. 1. The tree of modern theoretical physics grew from two different roots ("particle" and "field") which gave the VARIATIONAL method and SYMMETRY principles for formulating modern physical theories as constrained systems. To obtain unambiguous physical results, one should construct Equivalent Unconstrained Systems compatible with the symplest variational method. It is just the problem discussed in the present paper.

Fig. 2. An equivalent unconstrained system $W^*(p^*, q^*)$ can be obtained in the case when the operations of varying and constraining commute with each other. The next problem is to establish the range of validity of the standard Faddeev-Popov (FP) integral.

Fig. 3. Reparametrization-invariant dynamics of General Relativity is covered by the Dynamic Unconstrained Systems (DUS) and the Geometric Unconstrained Systems (GUS) connected by the Levi-Civita transformation of fields of MATTER into the vacuum fields of initial data with respect to geometric TIME.

Fig. 4. Reparametrisation-invariant dynamics of General Relativity. The Big Bang of Quantum Universe from point of view of Geometric and Dynamic Unconstrained Systems connected by Levi-Civita canonical transformations.
Fig. 1
\[ W^{\text{sing}} \]

\[ Z_F = \int dp^* dq^* e^{iW^*} \]

\[ Z_{FP} = \int dp dq \Delta_{FP} \delta_{\text{gauge}} \delta_{\text{const}} e^{iW^{\text{sing}}} \]

UNITARY PERTURBATION THEORY

Fig. 2
General Relativity

**GEOMETRY:**
\[
\left(\frac{d\varphi}{dT}\right)_\pm = \sqrt{\frac{H(\varphi)}{V_0}}
\]

**DYNAMICS:**
\[
\left(\frac{dF}{dT}\right) = \{H(\varphi), F\}
\]

**LC transformations**
\[(F, \varphi) \rightarrow (V, Q_0)\]
\[
\frac{dQ_0}{dT} = 1
\]

**GUS**
\[
\frac{dV}{dT} = \{H^v(T), V\}
\]

**DUS**
\[
\frac{dF}{d\varphi} = \{V_0 H(\varphi), F\}
\]

Fig. 3
Cosmic evolution and matter creating

INITIAL COSMIC DATA

TIME VACUUM

GUS=(V|T)

Levi-Civita transformation

M, T → ∞

DUS=(F|φ)

CREATION OF UNIVERSE

MATTER

QFT = (F | T)

LABORATORY EXPERIMENT

Fig. 4