TIME-CONSISTENT STRATEGY FOR A MULTI-PERIOD MEAN-VARIANCE ASSET-LIABILITY MANAGEMENT PROBLEM WITH STOCHASTIC INTEREST RATE

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Abstract. In this paper, we investigate a multi-period mean-variance asset-liability management problem with stochastic interest rate and seek its time-consistent strategy. The financial market is assumed to be composed of one risk-free asset and multiple risky assets, and the stochastic interest rate is characterized by the discrete-time Vasicek model proposed by Yao et al. (2016a)[38]. We regard this problem as a non-cooperative game whose equilibrium strategy is the desired time-consistent strategy. We derive the analytical expressions of the equilibrium strategy, the equilibrium value function and the equilibrium efficient frontier by the extended Bellman equation. Some special cases of our model are discussed, and some properties of our equilibrium strategy, including a two-fund separation theorem, are proposed. Finally, a numerical example with real data is given to illustrate our theoretical results.

1. Introduction. Asset-liability management (ALM), also known as surplus management, is concerned with the difference between the asset value and the liability. This is an important aspect of their concern, both for financial institutions such as banks, funds and insurance companies, and individuals. Thus, the ALM problem,
which is of both theoretical interest and practical importance, has received considerable attention for the last decade in the actuarial and the financial literature, and is the topic of this work.

The mean-variance model, which was proposed by Markowitz (1952)\cite{19}, is an important criterion for modelling ALM problems. In the mean-variance model, the investor’s target is choosing an optimal investment strategy to maximize the expected terminal wealth and to minimize the risk measured by the variance of her terminal wealth at the same time. The pioneering work on mean-variance ALM problems can be traced back to Sharpe and Tint (1990)\cite{26}. Progress of researches on ALM has been made in recent years, especially after the big breakthrough of solving dynamic (multi-period or continuous-time) mean-variance models by Li and Ng (2000)\cite{16} and Zhou and Li (2000)\cite{42}. For example, Leippold et al. (2004)\cite{12} studied a multi-period ALM problem under the mean-variance criterion, in which the liabilities are uncontrollable. Xie (2009)\cite{35} investigated the mean-variance portfolio selection with liability and regime switching in a continuous-time setting. Chen and Yang (2011)\cite{5} studied the mean-variance ALM with regime switching under a multi-period framework. Yao et al. (2016)\cite{36} further studied ALM in a Markov market with stochastic cash flows. For more discussion on this subject, please refer to Chen et al. (2008)\cite{6}, Leippold et al. (2011)\cite{13}, Li and Li (2012)\cite{14}, Yao et al. (2013)\cite{39}, Chiu and Wong (2014)\cite{7} and references therein.

Due to the non-separability of the variance operator, the Bellman optimality principle can not be applied directly to dynamic mean-variance models, and thus the optimal investment strategies of these models do not satisfy the time consistency. Actually, in all literature mentioned above, the optimal portfolio strategies are called pre-commitment strategies. A pre-commitment strategy is the optimal strategy made at the initial time. At any future time $k$, the decision-makers must commit themselves to following it even if it is not optimal anymore at that time. Hence, the pre-commitment strategy is time-inconsistent, which has been criticized for lacking rationality. For example, since the investment psychology and tastes will often change over time, the rational investor at a later time can not commit to following a strategy that is not optimal at that time. For this reason, scholars have studied some methods to obtain time-consistent strategies. The most commonly used one is the game theory approach. Björk and Murgoci (2014)\cite{3} and Björk et al. (2017)\cite{2} made a detailed study of this theory in discrete-time case and continuous-time case. Research on time-consistent strategies for ALM problems is very little. Wei et al. (2013)\cite{32} and Li et al. (2013)\cite{15} obtained the time-consistent strategies for ALM problems with the mean-variance criterion. Wei and Wang (2017)\cite{31} studied the time-consistent solution of an ALM problem with random coefficients. However, all these works are done in the framework of continuous-time. For all we know, time-consistent strategies for ALM problems under the multi-period mean-variance framework have not been studied yet.

Interest rate is the most important determinant of investment according to classical, neo-classical and contemporary economists (see Mushtaq and Siddiqui (2016)\cite{23}). In practice, the interest rate often changes inevitably. Moreover, marketization of interest rate results in more frequent changes in interest rate. Hence, for quite a long time, the consensus is that the interest rate can be modeled by a stochastic process and, naturally, the stochastic interest rate is one of the important uncertain factors in investment. Commonly used stochastic interest rate models are Vasicek model, Cox-Ingersoll-Ross (CIR) model and Ho-Lee model. Some scholars
investigated the portfolio problems with stochastic interest rate. For instance, Korn and Kraft (2002)[11] considered an optimal portfolio selection problem with the Vasicek and Ho-Lee stochastic interest rate models. Ferland and Watier (2010)[8] analyzed a portfolio selection problem with a extended CIR stochastic interest rate model. Yao et al. (2016c) [37] investigated the dynamic mean-variance asset allocation with stochastic interest rate and inflation rate, where the interest rate follows the Vasicek model. For other portfolio selection problems with stochastic interest rate, the readers can refer to Lioui and Poncet (2001)[18], Hainaut (2009)[9], Munk and Sørensen (2010)[22] and Shen and Siu (2012)[27], etc. Meanwhile, some scholars studied ALM problems with stochastic interest rate. For example, Liang and Ma (2015)[17] and Pan and Xiao (2017a)[24] solved the ALM problems under the Vasicek stochastic interest rate model in the expected utility framework. Chang (2015)[4] and Pan and Xiao (2017b)[25] studied the ALM problems with Vasicek stochastic interest rate model in the mean-variance framework. However, all these research works involving stochastic interest rate are limited to continuous-time case and obtained the pre-commitment strategy. Yao et al. (2016a)[38] proposed a discrete-time Vasicek model to investigate a multi-period mean-variance portfolio selection problem, and also obtained a pre-commitment strategy. As far as we know, time-consistent strategies for multi-period mean-variance portfolio selection problems with stochastic interest rate have not been studied in the literature, not to mention a multi-period mean-variance ALM problem with stochastic interest rate. In order to fill up these gaps in the above literature, and considering that time-consistent strategy is more in line with the behavioral characteristics of investors, this paper studies the time-consistent strategy for a multi-period mean-variance ALM problem with stochastic interest rate.

We use the discrete-time Vasicek stochastic interest rate model proposed by Yao et al. (2016a)[38] to describe the stochastic process of interest rate. Similar to Björk and Murgoci (2014)[3], we regard the problem studied by this paper as a non-cooperative game whose equilibrium strategy is the desired time-consistent strategy, and derive the analytical expressions of the equilibrium strategy, the equilibrium value function and the equilibrium efficient frontier by the extended Bellman equation. In addition, we discuss some special cases of our model, and give some properties of our equilibrium strategy, especially a multi-period version of the two-fund separation theorem. Finally, we use real data to analyze the impact of the liability and the stochastic interest rate on the equilibrium strategy and the corresponding efficient frontier, and compare the equilibrium results with the pre-commitment results obtained by Yao et al. (2016a)[38].

The main contributions of this paper are as follows. (i) The time-consistent strategy of a multi-period mean-variance ALM problem is studied for the first time. (ii) We are also the first to study the time-consistent strategy for the multi-period mean-variance investment problem with stochastic interest rate. (iii) We obtain the equilibrium strategy and the corresponding efficient frontier for our problem in closed form. (iv) Compared with Björk and Murgoci (2014)[3], the calculation process is more difficult and complex in this paper. On the one hand, Björk and Murgoci (2014)[3] only consider the case with one risk-free asset and one risky asset, and thus the sign (positive or negative) of the quadratic term’s coefficient of the corresponding strategy is easy to be decided and no further proof is needed. As a result, the optimal strategy can be obtained by the first-order condition. However, since the problem considered in this paper is the case with one risk-free asset and
multiple risky assets, the quadratic coefficient of the strategy is a matrix and thus the method of Björk and Murgoci (2014)[3] can not be used directly. To guarantee that the optimal strategy can be obtained by the first-order condition, we have to prove that this matrix is positive definite, which is quite difficult. To overcome this difficulty, we first use matrix technology to prove \( W_k \geq 0 \) and \( \lambda_k > 0 \) (see Lemma 3.3), and then further prove the positive definiteness of the matrix \( A_k \) (see Lemma 3.4). On the other hand, both the optimal time-consistent strategy and the optimal value function in this paper are quite different from and much more complicated than those in Björk and Murgoci (2014)[3]. For example, there are many exponential function terms in our optimal time-consistent strategy and optimal value function.

The remaining of this paper is organized as follows. We model the multi-period mean-variance ALM problem with stochastic interest rate in Section 2. In Section 3, we obtain the analytical expressions for the equilibrium strategy and the corresponding efficient frontier. We discuss some special cases of our model in Section 4. Section 5 derives some properties of the equilibrium strategy. A numerical example with real data is given in Section 6. Finally, we conclude this paper in Section 7.

2. Problem formulation. Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. Suppose that all the stochastic processes and random variables described below are well-defined on this probability space.

In a time horizon with \( T \) time periods, we consider a financial market consisting of one risk-free asset and \( n \) risky assets. For \( k = 0, 1, \cdots, T - 1 \), let \( R_k \) be the total return of the risk-free asset over period \( k \) (i.e., the time interval \([k, k + 1)\)) and \( S^l_k \) be the excess return of risky asset \( l \) \((l = 1, \cdots, n)\) over the risk-free asset during period \( k \). Then the return of risky asset \( l \) during period \( k \) is \( R_k + S^l_k \). Moreover, for \( k = 0, 1, \cdots, T - 1 \), let \( S_k = (S^1_k, S^2_k, \cdots, S^n_k)' \) be the excess return vector. Throughout this paper, for any matrix \( A \), let \( A' \) be its transpose.

Consider an investor who joins the financial market at time 0 with an initial wealth \( x_0 \). For \( k = 0, 1, \cdots, T - 1 \), let \( \pi_k = (\pi^1_k, \pi^2_k, \cdots, \pi^n_k) \) be the amount invested in the \( n \) risky assets at time \( k \), \( \pi_{k+} = (\pi_k, \pi_{k+1}, \cdots, \pi_{T-1}) \) represent the strategy starting from time \( k \), and shortly denote \( \pi_{0+} \) by \( \pi \). Denote by \( X^\pi_k \) the wealth of the investor under strategy \( \pi \) at time \( k \). Then the amount invested in the risk-free asset is \( X^\pi_k - \sum_{l=1}^n \pi^l_k \).

Hence, the wealth of the investor at time \( k + 1 \) is

\[
X^\pi_{k+1} = (X^\pi_k - \sum_{l=1}^n \pi^l_k) R_k + \sum_{l=1}^n \pi^l_k (R_k + S^l_k) = X^\pi_k R_k + S^l_k \pi_k. \tag{1}
\]

Suppose that the investor also faces a liability. Then the problem that we consider is referred to an ALM problem. Let \( L_k \) be the liability of the investor at the beginning of period \( k \) \((k = 0, 1, \cdots, T - 1)\), where \( L_0 = l_0 \) is a given constant. Similar to Leippold et al. (2004)[12], we assume the liability follows the dynamics

\[
L_{k+1} = \eta_k L_k, \tag{2}
\]

where \( \eta_k \) is an exogenous random variable, which can be understood as the random growth rate of the liability.

In the process of investment, although the current interest rate is deterministic, the future interest rate will change due to some factors such as the market, the inflation and so on. In other words, the future interest rate is stochastic. So it is necessary to incorporate a stochastic interest rate in the ALM problem.
Let \( r \) be the long-run mean of the interest rate. For \( k = 0, 1, \cdots, T - 1 \), let \( \varphi_k \) be a constant with \( 0 < \varphi_k \leq 1 \), \( \theta_k = 1 - \varphi_k \) describe the degree of mean reversion, \( \sigma_k \) be the volatility of interest rate, and \( \varepsilon_k \) be a time-\( k \) measurable random variable with expectation \( \mathbb{E}[\varepsilon_k] = 0 \) and variance \( \text{Var}[\varepsilon_k] = 1 \). We adopt the discrete-time Vasicek stochastic interest rate model proposed by Yao et al. (2016a) [38]:

\[
R_{k+1} = e^{(1-\varphi_k)\tau + \sigma_k \varepsilon_k} R_k^{\sigma_k} = b_k R_k^{\sigma_k},
\]

where \( R_0 = r_0 > 0 \) is a given constant, and

\[
b_k = e^{(1-\varphi_k)\tau + \sigma_k \varepsilon_k}.
\]

Obviously, \( R_k > 0 \) for all \( k = 0, 1, \cdots, T - 1 \). In particular, if we set \( \varphi_k = 1 \) and \( \sigma_k = 0 \), then \( R_k = r_0 \) for all \( k = 0, 1, \cdots, T - 1 \).

Throughout this paper, for a symmetric matrix \( M \), we denote \( M \succ 0 \) (\( M \succeq 0 \)) if \( M \) is positive definite (semidefinite) and denote its determinant by \( |M| \). Similar to most of the existing literature, we make the following assumptions in this paper.

**Assumption 2.1.** Each of the stochastic processes \( \{S_k\}, \{\eta_k\} \) and \( \{\varepsilon_k\} \) is statistically independent, and \( \varepsilon_k \) and \( \eta_k \) are independent for all \( k \in \{0, 1, \cdots, T - 1\} \).

**Assumption 2.2.** \( \text{Var}(S_k) = \mathbb{E}\left[(S_k - \mathbb{E}(S_k))(S_k - \mathbb{E}(S_k))'\right] > 0 \), \( k = 0, 1, \cdots, T - 1 \).

Since \( \mathbb{E}(S_k S_k') = \text{Var}(S_k) + \mathbb{E}(S_k) \mathbb{E}(S_k') \), Assumption 2.2 implies that \( \mathbb{E}(S_k S_k') \succ 0 \) for all \( k = 0, 1, \cdots, T - 1 \).

**Assumption 2.3.** For all \( k = 0, 1, \cdots, T - 1 \), \( \mathbb{E}(S_k) \neq 0_n \), where \( 0_n \) is the \( n \)-dimensional zero vector.

**Assumption 2.4.** Transaction cost and tax are not considered and short-selling is allowed.

In the ALM, we are concerned with the surplus which is the difference between the asset value and the liability. We denote the surplus process of the investor by

\[
U_k^T = X_k^T - L_k - X_k^T R_k + S_k^T \pi_k - \eta_k L_k, \quad k = 0, 1, \cdots, T - 1.
\]

Let \( \varphi_k \) be the family of filters, denoting the information available to the investor up to time \( k \), i.e., \( \varphi_k := \sigma \{ (X_s^T, R_s, L_s) | 0 \leq s \leq k \} \), which is a \( \sigma \)-field. An investment strategy starting from time \( k \), \( \pi_{k+1} \), is called time-\( k \) admissible if \( \pi_j \) is adapted to \( \varphi_j \) for all \( j = k, k + 1, \cdots, T - 1 \). Denote by \( \Theta_k \) the collection of all time-\( k \) admissible investment strategies.

The investor’s aim is to maximize the expected final surplus \( U_T^T \) and to minimize the investment risk measured by the variance of the final surplus. Therefore, we use the following mean-variance model to formulate the investment problem:

\[
\begin{aligned}
\max \{ \omega \mathbb{E}(U_T^T) - \text{Var}(U_T^T) \}, \\
\text{s.t. } X_k^T, \ L_k \text{ and } R_k \text{ satisfy (1), (2) and (3), respectively},
\end{aligned}
\]

where \( \omega \geq 0 \) measures the risk tolerance of the investor. When \( \omega = 0 \), we seek for a strategy with the global minimum risk, which is called the global minimum-variance strategy.

As stated in the introduction, problem (6) is a time-inconsistent problem. Strotz (1955) [28] proposed two main methods to handle this time-inconsistent problem. The first one is that fix an initial point, such as \( (0, x_0, r_0, l_0) \), and then try to find the optimal strategy \( \hat{\pi}_{0+} \) for problem (6), simply disregarding whether the later parts
of strategy $\tilde{\pi}_{0+}$ are optimal or not. In previous literature, this strategy is called the pre-commitment strategy. We should note that, this strategy is a global optimal strategy but is not a time-consistent strategy. Namely, the investment strategy $\pi_{0+}$ is optimal at time 0 but not optimal at some later time $m$ with $0 < m \leq T - 1$. The second one is that take the time inconsistency seriously and use the game theory to obtain an equilibrium strategy. The basic idea of this method is that take the decision-making process as a non-cooperative game, and suppose that there is one decision maker at each time $k$. At time $k$, the decision maker can only choose the current control $\pi_k$, and the controls in future times $k + 1, \ldots, T - 1$ are determined by the future decision makers. This decision-making process guarantees that the strategy starting from any time $k$ is optimal, i.e., the strategy is time-consistent.

In reality, rational investors can’t accept a strategy that is not optimal for themselves in some future periods. They will modify it in the future and seek the time-consistent strategy. So in this paper we use the game theory to deal with problem (6) to obtain an equilibrium strategy. In this case, the investor updates her target at each time $k$ upon the information $(x_k, r_k, l_k)$ at that time with the objective function

$$J_k (x_k, r_k, l_k, \pi_{k+}) = \omega \mathbb{E}_{x_k, r_k, l_k} \left( U_T^r \right) - \text{Var}_{x_k, r_k, l_k} \left( U_T^r \right),$$

and solves a series of mean-variance models

$$\begin{aligned}
\max_{\pi_{k+} \in \Theta_k} \{ J_k (x_k, r_k, l_k, \pi_{k+}) \}, \\
\text{s.t. } X_k^\pi, L_k \text{ and } R_k \text{ satisfy (1), (2) and (3), respectively,}
\end{aligned}$$

where $\mathbb{E}_{x_k, r_k, l_k} (\cdot)$ and $\text{Var}_{x_k, r_k, l_k} (\cdot)$ denote the conditional expectation and conditional variance given the information $X_k^\pi = x_k, R_k = r_k, L_k = l_k$.

As in Björk and Murgoci (2014)[3], we define equilibrium strategy in discrete-time case as follows.

**Definition 2.5.** (Equilibrium Strategy) Let $\pi^*$ be a given time-0 admissible strategy. For an arbitrary point $(k, x_k, r_k, l_k)$ and an arbitrary decision $\pi_k$ adapted to $\mathcal{F}_k$, define the time-$k$ admissible strategy $\pi_{k+} = (\pi_k, \pi_{k+1}, \ldots, \pi_{T-1})$.

Then $\pi^*$ is said to be a subgame perfect Nash equilibrium strategy (shortly, an equilibrium strategy) if for every $k$, it satisfies

$$\max_{\pi_k} \{ J_k (x_k, r_k, l_k, \pi_{k+}) \} = J_k (x_k, r_k, l_k, \pi_{k+}^*),$$

where $\pi_{k+}^* = (\pi_k, \pi_{k+1}^*, \ldots, \pi_{T-1}^*)$. Furthermore, if an equilibrium strategy $\pi^*$ exists, the equilibrium value function is defined as

$$V_k (x_k, r_k, l_k) = J_k (x_k, r_k, l_k, \pi_{k+}^*).$$

From Definition 2.5, in order to find an equilibrium strategy, we only need, at any time $k$, for any given wealth $X_k^\pi = x_k$, interest rate $R_k = r_k$ and liability $L_k = l_k$, to solve the following problem:

$$\begin{aligned}
\max_{\pi_k} \{ J_k (x_k, r_k, l_k, \pi_{k+}) \} = \max_{\pi_k} \{ J_k (x_k, r_k, l_k, \pi_{k+}^*) \}, \\
\text{s.t. } X_k^\pi, L_k \text{ and } R_k \text{ satisfy (1), (2) and (3), respectively.}
\end{aligned}$$
3. **Equilibrium strategy and efficient frontier.** In order to derive the equilibrium strategy and the equilibrium value function, we use the backward induction method to solve the problem (9).

From equation (7) we have

\[
J_k (x_k, r_k, l_k, \pi_{k+}) = \omega \mathbb{E}_{x_k, r_k, l_k} (U_T^k) - \text{Var}_{x_k, r_k, l_k} (U_T^k)
\]

\[
= \omega \mathbb{E}_{x_k, r_k, l_k} (U_T^k) - \mathbb{E}_{x_k, r_k, l_k} \left( (U_T^k)^2 \right) + \mathbb{E}_{x_k, r_k, l_k} (U_T^k)^2
\]

\[
= \omega \mathbb{E}_{x_k, r_k, l_k} \left( \mathbb{E}_{X_{k+1}^k, R_{k+1}, L_{k+1}} (U_T^k) \right) - \mathbb{E}_{x_k, r_k, l_k} \left( \mathbb{E}_{X_{k+1}^k, R_{k+1}, L_{k+1}} (U_T^k)^2 \right)
\]

\[
+ \left( \mathbb{E}_{x_k, r_k, l_k} \left( \mathbb{E}_{X_{k+1}^k, R_{k+1}, L_{k+1}} (U_T^k) \right)^2 \right) - \mathbb{E}_{x_k, r_k, l_k} \left( \left( \mathbb{E}_{X_{k+1}^k, R_{k+1}, L_{k+1}} (U_T^k) \right)^2 \right)
\]

\[
= \mathbb{E}_{x_k, r_k, l_k} \left( \omega \mathbb{E}_{X_{k+1}^k, R_{k+1}, L_{k+1}} (U_T^k) - \mathbb{E}_{X_{k+1}^k, R_{k+1}, L_{k+1}} (U_T^k)^2 \right)
\]

\[
+ \left( \mathbb{E}_{x_k, r_k, l_k} \left( \mathbb{E}_{X_{k+1}^k, R_{k+1}, L_{k+1}} (U_T^k) \right)^2 \right) - \mathbb{E}_{x_k, r_k, l_k} \left( \left( \mathbb{E}_{X_{k+1}^k, R_{k+1}, L_{k+1}} (U_T^k) \right)^2 \right)
\]

\[
= \mathbb{E}_{x_k, r_k, l_k} \left( J_{k+1} \left( X_{k+1}^k, R_{k+1}, L_{k+1}, \pi_{k+1+} \right) \right)
\]

\[= \mathbb{E}_{x_k, r_k, l_k} \left( \pi^* \right).
\]

Let \( \pi^* \) be the equilibrium strategy. Fix an arbitrarily chosen initial point \((x_k, r_k, l_k)\) \((k = 0, 1, \cdots, T)\), and denote

\[
g_k (x_k, r_k, l_k) = \mathbb{E}_{x_k, r_k, l_k} \left( U_T^k \right).
\]

Then, according to Definition 2.5 and equations (10) and (11), the equilibrium value function satisfies the extended Bellman equation

\[
V_k (x_k, r_k, l_k) = \max_{\pi_k} \left\{ J_k (x_k, r_k, l_k, (\pi_k, \pi_{k+1}, \cdots, \pi_{T-1})) \right\}
\]

\[
= \max_{\pi_k} \left\{ \mathbb{E}_{x_k, r_k, l_k} \left( J_{k+1} \left( X_{k+1}^k, R_{k+1}, L_{k+1}, \pi_{k+1+} \right) \right) \right\}
\]

\[
= \max_{\pi_k} \left\{ \mathbb{E}_{x_k, r_k, l_k} \left( \left( \mathbb{E}_{X_{k+1}^k, R_{k+1}, L_{k+1}} (U_T^k) \right)^2 \right) \right\}
\]

\[
= \max_{\pi_k} \left\{ \mathbb{E}_{x_k, r_k, l_k} \left( \left( \mathbb{E}_{X_{k+1}^k, R_{k+1}, L_{k+1}} (U_T^k) \right)^2 \right) \right\}
\]

\[
= \max_{\pi_k} \left\{ \mathbb{E}_{x_k, r_k, l_k} \left( \left( \mathbb{E}_{X_{k+1}^k, R_{k+1}, L_{k+1}} (U_T^k) \right)^2 \right) \right\}
\]

with terminal condition

\[
V_T (x, r, l) = \omega (x - l),
\]
where
\[ g_k(x_k, r_k, l_k) = E_{x_k, r_k, l_k} \left[ g_{k+1} \left( X_{k+1}^T, R_{k+1}, L_{k+1} \right) \right], \quad k = T - 1, \ldots, 0, \]
\[ g_T(x, r, l) = x - l. \]

To obtain the equilibrium strategy and the equilibrium value function in closed-form, we construct the following backward time series. For \( k = T, T - 1, \cdots, 0, \)
\[
\psi_k = 1 + \varphi_k \psi_{k+1}, \quad \psi_T = 0, \tag{14}
\]
\[
W_k = W_{k+1} E \left( b_k^{2\psi_{k+1}} \right) + c_{k+1} Var \left( b_k^{\psi_{k+1}} \right)
- \left( W_{k+1} E \left( b_k^{2\psi_{k+1}} \right) + \lambda_{k+1} Var \left( b_k^{\psi_{k+1}} \right) \right)^2
\]
\[
\times E \left( S'_k \right) \left[ W_{k+1} E \left( b_k^{2\psi_{k+1}} \right) E \left( S_k S'_k \right) + \lambda_{k+1} Var \left( b_k^{\psi_{k+1}} S_k \right) \right]^{-1} E \left( S_k \right)
\]
\[
= B_k - B_k^2 E \left( S'_k \right) A_k^{-1} E \left( S_k \right), \quad W_T = 0, \tag{15}
\]
\[
\lambda_k = \lambda_{k+1} \left[ E \left( \psi_{k+1} \right) - E \left( \psi_{k+1} \right) \left( W_{k+1} E \left( b_k^{2\psi_{k+1}} \right) + \lambda_{k+1} Var \left( b_k^{\psi_{k+1}} \right) \right) \right]
\]
\[
\times E \left( S'_k \right) \left[ W_{k+1} E \left( b_k^{2\psi_{k+1}} \right) E \left( S_k S'_k \right) + \lambda_{k+1} Var \left( b_k^{\psi_{k+1}} S_k \right) \right]^{-1} E \left( S_k \right)
\]
\[
= \lambda_{k+1} \left[ E \left( \psi_{k+1} \right) - E \left( \psi_{k+1} \right) B_k E \left( S'_k \right) A_k^{-1} E \left( S_k \right) \right], \quad \lambda_T = 1, \tag{16}
\]
\[
H_k = H_{k+1} \left[ E \left( \psi_{k+1} \eta_k \right) - \left( W_{k+1} E \left( b_k^{2\psi_{k+1}} \right) + \lambda_{k+1} Var \left( b_k^{\psi_{k+1}} \right) \right) \right]
\times E \left( \eta_k b_k^{\psi_{k+1}} S'_k \right) \left( W_{k+1} E \left( b_k^{2\psi_{k+1}} \right) E \left( S_k S'_k \right) + \lambda_{k+1} Var \left( b_k^{\psi_{k+1}} S_k \right) \right]^{-1} E \left( S_k \right)
\]
\[
+ \lambda_{k+1} Var \left( \psi_{k+1} S_k \right) \right] E \left( S_k \right) - \lambda_{k+1} \beta_{k+1} \left[ \text{cov} \left( \psi_{k+1}, \eta_k \right) \right]
\]
\[
- \left( W_{k+1} E \left( b_k^{2\psi_{k+1}} \right) + \lambda_{k+1} Var \left( b_k^{\psi_{k+1}} \right) \right)
\times D_k \left( W_{k+1} E \left( b_k^{2\psi_{k+1}} \right) E \left( S_k S'_k \right) + \lambda_{k+1} Var \left( b_k^{\psi_{k+1}} S_k \right) \right]^{-1} E \left( S_k \right)
\]
\[
= H_{k+1} E \left( b_k^{\psi_{k+1}} \eta_k \right) - \lambda_{k+1} \beta_{k+1} \text{cov} \left( \psi_{k+1}, \eta_k \right)
\]
\[
- B_k C_k A_k^{-1} E \left( S_k \right), \quad H_T = 0, \tag{17}
\]
\[
\alpha_k = \lambda_{k+1} E^2 \left( b_k^{\psi_{k+1}} \right) E \left( S'_k \right) A_k^{-1} E \left( S_k \right) + \alpha_{k+1}, \quad \alpha_T = 0, \tag{18}
\]
\[
\beta_k = \beta_{k+1} E \left( \eta_k \right) + \lambda_{k+1} E \left( b_k^{\psi_{k+1}} \right) \left( H_{k+1} E \left( \eta_k b_k^{\psi_{k+1}} S'_k \right) - \lambda_{k+1} \beta_{k+1} D'_k \right)
\]
\[
\times \left( W_{k+1} E \left( b_k^{2\psi_{k+1}} \right) E \left( S_k S'_k \right) + \lambda_{k+1} Var \left( b_k^{\psi_{k+1}} S_k \right) \right]^{-1} E \left( S_k \right)
\]
\[
= \beta_{k+1} E \left( \eta_k \right) + \lambda_{k+1} E \left( b_k^{\psi_{k+1}} \right) C_k A_k^{-1} E \left( S_k \right), \quad \beta_T = -1, \tag{19}
\]
\[
\gamma_k = \gamma_{k+1} E \left( \eta_k^2 \right) + \left( H_{k+1} E \left( \eta_k b_k^{\psi_{k+1}} S'_k \right) - \lambda_{k+1} \beta_{k+1} D'_k \right)
\times \left( W_{k+1} E \left( b_k^{2\psi_{k+1}} \right) E \left( S_k S'_k \right) + \lambda_{k+1} Var \left( b_k^{\psi_{k+1}} S_k \right) \right]^{-1}
\times \left( H_{k+1} E \left( \eta_k b_k^{\psi_{k+1}} S_k \right) - \lambda_{k+1} \beta_{k+1} D_k \right) - \beta_{k+1}^2 \text{Var} \left( \eta_k \right)
\]
\[
= \gamma_{k+1} E \left( \eta_k^2 \right) + C_k A_k^{-1} C_k - \beta_{k+1}^2 \text{Var} \left( \eta_k \right), \quad \gamma_T = 0. \tag{20}
\]
where

\[ A_k = W_{k+1} E \left( b_{k+1}^{\psi_{k+1}} \right) E S_k' + \lambda_{k+1}^2 \text{Var} \left( b_{k+1}^{\psi_{k+1}} S_k \right), \]

\[ B_k = W_{k+1} E \left( b_{k+1}^{\psi_{k+1}} \right) + \lambda_{k+1}^2 \text{Var} \left( b_{k+1}^{\psi_{k+1}} \right), \]

\[ C_k = H_{k+1} E \left( \eta_k b_{k+1}^{\psi_{k+1}} S_k \right) - \lambda_{k+1} \beta_{k+1} D_k, \]

\[ D_k = E \left( \eta_k b_{k+1}^{\psi_{k+1}} S_k \right) - E \left( \eta_k \right) E \left( b_{k+1}^{\psi_{k+1}} S_k \right). \]

For convenience, we define \( \prod_{i=k}^{k-1} (\cdot) = 1 \) and \( \sum_{i=m}^{k-1} (\cdot) = 0 \) for \( k = 0, 1, \cdots, T, m \geq k \). After a simple recursive calculation, we can obtain \( \psi_k > 1 \) when \( k = 0, 1, \cdots, T-2 \).

3.1. **Some useful lemmas.** In this subsection, we give some lemmas which will help us to obtain our main results.

**Lemma 3.1.** For all \( k = 0, 1, \cdots, T-1 \), \( \text{Var} \left( b_{k+1}^{\psi_{k+1}} S_k \right) > 0 \).

*Proof.* See Appendix A.

Let \( \Gamma \) be a square matrix partitioned as \( \Gamma = \left( \begin{array}{cc} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{array} \right) \), where \( \Gamma_{11} \) and \( \Gamma_{22} \) are \( m \)- and \( n \)-square matrices, respectively. Then the following lemma holds.

**Lemma 3.2.** (Zhang (2011) [40]):

\[ |\Gamma| = |\Gamma_{11} \Gamma_{22} - \Gamma_{21} \Gamma_{12}|, \quad \text{if } \Gamma_{11} \Gamma_{21} = \Gamma_{21} \Gamma_{11}; \]

\[ |\Gamma| = |\Gamma_{22}| |\Gamma_{11} - \Gamma_{12} \Gamma_{22}^{-1} \Gamma_{21}|, \quad \text{if } \Gamma_{22} \text{ is invertible}. \]

Let

\[ Q_k = b_{k+1}^{\psi_{k+1}} \left( \begin{array}{c} 1 \\ S_k \end{array} \right). \]

**Lemma 3.3.** For arbitrary \( k = 0, 1, \cdots, T \), we have \( W_k \geq 0 \) and \( \lambda_k > 0 \).

*Proof.* See Appendix B.

From Lemmas 3.1 and 3.3, we can easily get the following lemma.

**Lemma 3.4.** For arbitrary \( k = 0, 1, \cdots, T-1 \), \( A_k \) is positive definite.

According to the recursive formula (18), we can get the expression of \( \alpha_k \).

**Lemma 3.5.** For arbitrary \( k = 0, 1, \cdots, T-1 \),

\[ \alpha_k = \sum_{l=k}^{T-1} \lambda_l^2 \text{Var} \left( b_{l+1}^{\psi_{l+1}} \right) E \left( S_l' \right) A^{-1}_l E \left( S_l \right) > 0. \]

*Proof.* See Appendix C.
3.2. Equilibrium strategy and value function. Based on the preliminary results of the subsection 3.1, we can now solve the extended Bellman equation (12) to get the equilibrium strategy and the equilibrium value function which are summarized in the following theorem.

**Theorem 3.6.** For $k = 0, 1, \cdots , T - 1$, $X^*_k = x_k$, $R_k = r_k$ and $L_k = l_k$, the equilibrium strategy is given by

$$
\pi^*_k = A^{-1}_k \left( \frac{\omega \lambda_{k+1} \mathbb{E} \left( b^*_{k+1} \right) \mathbb{E} (S_k)}{2 r^*_{k} \psi_{k+1}} - B_k \mathbb{E} (S_k) r_k x_k + r_k^{-1} \psi_{k+1} C_k l_k \right),
$$

the equilibrium value function is given by

$$
V_k (x_k, r_k, l_k) = -W_k r_k^2 x_k^2 + \omega \lambda_k r_k^2 x_k + \frac{\omega^2}{4} \alpha_k + 2 H_k r_k^2 x_k l_k + \gamma_k l_k^2 + \omega \beta k l_k,
$$

and $g_k$ is given by

$$
g_k (x_k, r_k, l_k) = \lambda_k r_k^2 x_k + \beta_k l_k + \frac{\omega}{2} \alpha_k,
$$

where $\psi_k, W_k, \lambda_k, H_k, \alpha_k, \beta_k, \gamma_k, A_k, B_k$ and $C_k$ are defined by (14)-(23).

**Proof.** See Appendix D.

From equation (27) we find that, (i) for the equilibrium investment strategy, the portfolio at any future time $k$ is independent of the initial state, including the initial wealth, initial liability and initial interest rate. This is different from the pre-commitment investment strategy obtained by Yao et al. (2016a)[38]. The reason is that, at any time $k$, the decision made by the investor with the equilibrium strategy is based on the forthcoming information while the investor with the pre-commitment strategy aims to find the globally optimal strategy from the state of the initial time; (ii) at any time $k$, the portfolio $\pi^*_k$ depends on both the current wealth level $x_k$ and the current liability level $l_k$. Besides, the randomness of the interest rate dictated by $b_k$, the randomness of the risky assets characterized by $S_k$, and the randomness of the liability featured by $\eta_k$, affect the portfolio $\pi^*_k$ together.

**Remark 1.** If $\eta_k, \varepsilon_k$, and $S_k$ are uncorrelated with each other for all $k = 0, 1, \cdots , T - 1$, then $C_k = 0$ and

$$
\pi^*_k = \frac{\omega \lambda_{k+1} \mathbb{E} \left( b^*_{k+1} \right) \mathbb{E} (S_k)}{2 r^*_{k} \psi_{k+1}} - B_k \mathbb{E} (S_k) r_k x_k \bigg| A^{-1}_k \mathbb{E} (S_k).
$$

This means that in this case, the equilibrium investment strategy is independent of the liability. This is quite different from the pre-commitment strategy obtained by Yao et al. (2016a)[38], which depends on the liability, even though $\eta_k, \varepsilon_k$, and $S_k$ are uncorrelated with each other.

3.3. Equilibrium efficient frontier. We consider the efficient frontier starting from any time $k \in \{0, 1, \cdots , T - 1\}$ and arbitrary initial point $(x_k, r_k, l_k)$ at that time with wealth $X^*_k = x_k$, interest rate $R_k = r_k$ and liability $L_k = l_k$. By equations (11), (28) and (29), we have

$$
\mathbb{E}_{x_k, r_k, l_k} \left( U^*_T \right) = g_k (x_k, r_k, l_k) = \lambda_k r_k^2 x_k + \beta_k l_k + \frac{\omega}{2} \alpha_k.
$$
and

\[
V_k(x_k, r_k, l_k) = J_k(x_k, r_k, l_k, \pi^*_k) = \omega \mathbb{E}_{x_k, r_k, l_k} \left( U^*_T \right) - \text{Var}_{x_k, r_k, l_k} \left( U^*_T \right)
\]

\[
= \omega \mathbb{E}_{x_k, r_k, l_k} \left( U^*_T \right) - \text{Var}_{x_k, r_k, l_k} \left( U^*_T \right)
\]

\[
= \omega \lambda_k r^2_k x_k + \omega \beta_k l_k + \frac{\omega^2}{2} \alpha_k - \text{Var}_{x_k, r_k, l_k} \left( U^*_T \right)
\]

\[
= -W_k r^2_k x^2_k + \omega \lambda_k r^2_k x_k + \frac{\omega^2}{4} \alpha_k + 2H_k r^2_k x_k l_k + \gamma_k l^2_k + \omega \beta_k l_k. \quad (32)
\]

Hence

\[
\text{Var}_{x_k, r_k, l_k} \left( U^*_T \right) = W_k r^2_k x^2_k + \frac{\omega^2}{4} \alpha_k - 2H_k r^2_k x_k l_k - \gamma_k l^2_k. \quad (33)
\]

Since \( \alpha_k > 0 \) from Lemma 3.5, equation (31) gives

\[
\frac{\omega}{2} = \frac{\mathbb{E}_{x_k, r_k, l_k} \left( U^*_T \right) - \lambda_k r^2_k x_k - \beta_k l_k}{\alpha_k}.
\]

(34)

Substituting equation (34) into equation (33), we obtain the efficient frontier

\[
\text{Var}_{x_k, r_k, l_k} \left( U^*_T \right) = \left( \frac{\mathbb{E}_{x_k, r_k, l_k} \left( U^*_T \right) - \lambda_k r^2_k x_k - \beta_k l_k}{\alpha_k} \right)^2 + W_k r^2_k x^2_k - 2H_k r^2_k x_k l_k - \gamma_k l^2_k. \quad (35)
\]

4. Special cases. This section discusses some special cases of our model and presents the corresponding simplified results.

**Special case 1**: The case without liability, that is, \( L_k = l_k = 0 \) for all \( k = 0, 1, \cdots, T \). Then the equilibrium strategy and the corresponding efficient frontier become

\[
\hat{\pi}^\text{WL}_k = \left( \frac{\omega \lambda_{k+1} \mathbb{E} \left( b^2_{k+1} \right)}{2 r^2_k \sigma^2_{k+1}} - B_k r_k x_k \right) A^{-1}_k \mathbb{E}(S_k), \quad (36)
\]

\[
\text{Var}_{x_k, r_k} \left( X^\text{WL}_T \right) = \left( \frac{\mathbb{E}_{x_k, r_k} \left( X^\text{WL}_T \right) - \lambda_k r^2_k x_k}{\alpha_k} \right)^2 + W_k r^2_k x^2_k. \quad (37)
\]

Conclusions (36)-(37) have not been given in the previous literature. The equilibrium strategy (36) with stochastic interest rate tells us that the portfolio \( \hat{\pi}^\text{WL}_k \) at any time \( k \) is relevant to the current wealth \( x_k \). This is different from the equilibrium strategy with deterministic interest rate obtained by Wu and Chen (2015)[34].

Equation (37) shows that the equilibrium efficient frontier is a hyperbola in the standard deviation-mean plane, and the variance in the efficient frontier can not reach to zero even if all the wealth is invested in the risk-free asset. This is also different from Wu and Chen (2015)[34]. This phenomenon is reasonable, since the investor has to undertake the stochastic interest rate risk in our model and hence can not fully hedge all risks by investing in the financial market.

**Special case 2**: The case of time variant and deterministic interest rate, i.e., \( \sigma_k = 0 \) for all \( k = 0, 1, \cdots, T - 1 \). In this case, \( b^2_{k+1} = e^{(1-\varphi_k)} r^2_{k+1} \) is time variant but deterministic. Hence,

\[
\text{Var} \left( b^2_{k+1} \right) = 0, \quad \mathbb{E} \left( b^2_{k+1} \right) = e^{(1-\varphi_k)} r^2_{k+1}.
\]
For convenience, we define \( \delta_k = \mathbb{E} (\eta_k S_k) - \mathbb{E} (\eta_k) \mathbb{E} (S_k) \). Then we have the following lemma.

**Lemma 4.1.** Suppose \( \sigma_k = 0 \), \( k = 0, 1, \ldots, T \). Then, for all \( k = 0, 1, \ldots, T \), we have

\[
W_k = 0, \\
\lambda_k = e^{r(T-k-\psi_k)}, \\
\alpha_k = \sum_{j=k}^{T-1} \mathbb{E} (S_j' \Var^{-1} (S_j) \mathbb{E} (S_j)), \\
A_k = e^{2r(T-k-\psi_k)} \Var (S_k), \\
B_k = 0, \\
H_k = 0, \\
\beta_k = - \prod_{l=k}^{T-1} \left[ \mathbb{E} (\eta_l) - \delta_l \Var^{-1} (S_l) \mathbb{E} (S_l) \right], \\
C_k = - e^{r(T-k-\psi_k)} \beta_{k+1} \delta_k, \\
\gamma_k = - \sum_{l=k}^{T-1} \beta_{l+1} \left[ \Var (\eta_l) - \delta_l \Var^{-1} (S_l) \delta_l \right] \prod_{m=k}^{l-1} \mathbb{E} (\psi^2_m).
\]

*Proof.* See Appendix E. \( \square \)

By Lemma 4.1, the equilibrium strategy and the corresponding efficient frontier can be simplified as

\[
\pi_{TVDIR}^k = \Var^{-1} (S_k) \left( \frac{\omega \mathbb{E} (S_k) + 2 \left( \prod_{l=k+1}^{T-1} \left[ \mathbb{E} (\eta_l) - \delta_l \Var^{-1} (S_l) \mathbb{E} (S_l) \right] \delta_l l_k \right)}{2 \rho_k \psi_k^{\psi_k+1} e^{r(T-k-\psi_k)}} \right), \tag{47}
\]

\[
\Var_{x_k, r_k, l_k} \left( U_{TVDIR}^k \right) = \frac{\left( \mathbb{E} (S_k') U_{TVDIR}^{k+1} \mathbb{E} (S_k' \psi_k x_k - \beta_k l_k) \right)^2 - \gamma_k l_k^2}{\sum_{j=k}^{T-1} \mathbb{E} (S_j') \Var^{-1} (S_j) \mathbb{E} (S_j)}. \tag{48}
\]

Clearly, the equilibrium strategy is independent of the current wealth \( x_k \) in this case, which is the same as Wu and Chen (2015)[34], but is different from the equilibrium strategy with stochastic interest rate.

**Special case 3:** The case without liability and with constant interest rate, that is, \( L_k = l_k = 0 \), \( \sigma_k = 0 \) and \( \varphi_k = 1 \) for all \( k = 0, 1, \ldots, T - 1 \). Then \( b_k \equiv 1 \) and \( R_k \equiv r \), where \( r \) is a fixed constant. So we have

\[
\psi_k = T - k, \ B_k = 0, \ W_k = 0, \ A_k = \Var (S_k), \\
\lambda_k = 1, \ \alpha_k = \sum_{j=k}^{T-1} \mathbb{E} (S_j') \Var^{-1} (S_j) \mathbb{E} (S_j).
\]

In this case, the equilibrium strategy and the corresponding efficient frontier become

\[
\pi_{k}^{WL,CHR} = \frac{\omega}{2 \rho^{T-k-1}} \Var^{-1} (S_k) \mathbb{E} (S_k), \tag{49}
\]
The conclusions (49)-(50) are consistent with Wu (2013) [33]. From equation (50) we find that the global minimum standard deviation of the terminal wealth is equal to zero and the efficient frontier is a straight line in the standard deviation-mean plane in this case, which is quite different from Special case 1 with stochastic interest rate.

5. Properties of equilibrium strategy. By equation (27), we immediately obtain the equilibrium global minimum variance strategy by setting \( \omega \) to be zero:

\[
\pi_{\text{MIN}}^* = A_k^{-1} \left( -B_k \mathbb{E}(S_k) r_k x_k + r_k^{-\psi_k} \psi_k r_k C_k l_k \right). 
\]

Hence, we can obtain a decomposition of the equilibrium strategy \( \pi_k^* \):

\[
\pi_k^* = \pi_{\text{MIN}}^* + \omega f_k, 
\]

where \( f_k = \frac{\lambda_{k+1} \mathbb{E}(S_{k+1}) + \lambda_{k-1} \mathbb{E}(S_k)}{2 r_k^{-\psi_k} \psi_k + 1} \).

Moreover, by equation (36), we can also obtain the equilibrium global minimum variance strategy without liability

\[
\pi_{\text{MIN,WL}}^* = -B_k r_k x_k A_k^{-1} \mathbb{E}(S_k). 
\]

Hence,

\[
\pi_k^* = \pi_{\text{MIN,WL}}^* + f_k, 
\]

where \( f_k = r_k^{-\psi_k} \psi_k r_k C_k l_k \).

From equations (52) and (54), we have

\[
\pi_k^* = \pi_{\text{MIN,WL}}^* + f_k + \omega f_k. 
\]

Notice that \( \pi_k^* \) is only the equilibrium portfolio for the risky assets at time \( k \). The equilibrium amount invested in the risk-free asset (indexed as asset 0) is

\[
(\pi_0^*)^* = x_k - \sum_{l=1}^{n} (\pi_l^*)^* = x_k - 1^* \pi_k^* = -1^* \omega f_k - 1^* f_k^* + \left( x_k - 1^* \pi_{\text{MIN,WL}}^* \right) \enspace (56)
\]

where \( 1 = (1, 1, \ldots, 1) \in \mathbb{R}^n \). Therefore, the equilibrium portfolio for all assets is

\[
\Pi_k^* = \left( \begin{array}{c} \pi_0^* \\ \pi_k^* \end{array} \right) = \omega F_k + F_k^{L} + \Pi_{k}^{\text{MIN,WL}}, 
\]

where

\[
F_k = \left( \begin{array}{c} -1^* f_k \\ f_k \end{array} \right), \quad F_k^{L} = \left( \begin{array}{c} -1^* f_k^* \\ f_k^* \end{array} \right), \quad \Pi_{k}^{\text{MIN,WL}} = \left( x_k - 1^* \pi_{\text{MIN,WL}}^* \right). 
\]

Clearly, \( F_k \) and \( F_k^{L} \) are self-financing portfolios—buying some assets equals shorting other assets.

It is easy to see that \( \Pi_k^* := F_k + \Pi_{k}^{\text{MIN,WL}} \) is the equilibrium global minimum variance strategy for all assets. And equation (57) can be rewritten as

\[
\Pi_k^* = \omega F_k + \Pi_{k}^{\text{MIN}}. 
\]
Remark 2. Equations (57) and (58) imply that (i) the equilibrium global minimum variance strategy \( \Pi_{k}^{MIN} \) is composed of the equilibrium global minimum variance strategy without liability \( \Pi_{k}^{MIN, WL} \) and a self-financing portfolio \( F_{k}^{L} \); (ii) starting from the equilibrium global minimum variance strategy \( \Pi_{k}^{MIN} \) and moving along the direction of the self-financing portfolio \( F_{k} \) with a distance \( \omega \), then the equilibrium strategy \( \Pi_{k}^{*} \) is obtained; (iii) since the effects of liabilities are all included in the term \( F_{k}^{L} \), the introduction of liability only leads to a parallel movement of the set of equilibrium strategies for all investors; (iv) \( \Pi_{k}^{*} \) is a linear function of the risk tolerance coefficient \( \omega \). To emphasize the dependence of \( \omega \), we write \( \Pi_{k}^{*} = \Pi_{k}^{*}(\omega) \).

No matter from the theoretical viewpoint or the practical viewpoint, the two-fund separation theorem is very important. It was derived first by Tobin (1958)\(^{30}\) in the case of single-period mean-variance model. It says that an arbitrary efficient portfolio can be linearly represented by any other two efficient portfolios, and conversely any linear combination of any two given efficient portfolios is also an efficient portfolio. Based on equation (58), we can show that the two-fund separation theorem also holds for our equilibrium strategy.

Theorem 5.1. (Two-fund separation theorem) Let \( \Pi_{k}^{*}(\omega_{1}) \) and \( \Pi_{k}^{*}(\omega_{2}) \) be two different equilibrium strategies characterized by \( \omega_{1} \geq 0 \) and \( \omega_{2} \geq 0 \), respectively.

(i) For any equilibrium strategy \( \Pi_{k}^{*}(\omega) \), there must exist a real number \( \alpha \) such that \( \Pi_{k}^{*}(\omega) = \alpha \Pi_{k}^{*}(\omega_{1}) + (1 - \alpha) \Pi_{k}^{*}(\omega_{2}) \);

(ii) For any given real number \( \alpha \) such that \( \alpha \omega_{1} + (1 - \alpha) \omega_{2} \geq 0 \), \( \alpha \Pi_{k}^{*}(\omega_{1}) + (1 - \alpha) \Pi_{k}^{*}(\omega_{2}) \) is also an equilibrium strategy.

Proof. See Appendix F.

Theorem 5.1 implies that investing in any two existing equilibrium strategies with certain weights is equivalent to investing in all \( n + 1 \) assets, no matter what degree of the risk tolerance of the investor is. What she needs to do is to choose the investment proportions \( \alpha \) and \( 1 - \alpha \) in these two equilibrium strategies. It is practically meaningful as it can save the transaction cost.

6. Numerical example. Using real data from the China market, this section gives a numerical example to illustrate our results.

We take the monthly interest rate of the Shanghai Interbank Offered Rate (Shibor) in China as the interest rate, and select three stocks listed in China Stock Market as the risky assets, which are TCL (000100.SZ), Shanghai Pudong Development Bank (600000.SH), ZTE (000063.SZ) and labeled by stocks 1, 2, 3, respectively. Collecting the historical data of the monthly interest rate of Shibor and these stocks from October, 2006 to August, 2017, we get \( n = 131 \) month-return samples. In order to improve the precision of the calculation, we multiply the net returns by 12 to convert them into the annual returns. For the need of calculating market parameters about the interest rate, we also transform historical data of the monthly interest rate of Shibor into their logarithm. In order to better describe the effect of liability on our results, we consider two investors.

Suppose that an investor (indexed as the first investor) who does not face a liability enters the market at time 0 with initial wealth \( x_{0} = 10 \), and exits from the market at time \( T = 10 \). She can invest in the three stocks, and can also deposit or borrow money with a stochastic interest rate governed by equation (3). Assume that the initial interest rate is \( r_{0} = 1.0300 \) and the risk-tolerance degree of this investor
is \( \omega = \frac{1}{2} \). For convenience, suppose that the market parameters are independent of time \( k \) and \( \varepsilon_k \sim N (0, 1) \) (standard normal distribution). Based on the data set above, for \( k = 0, 1, \cdots, 9 \), we obtain the related parameters as follows

\[
\bar{r} = 0.0348, \quad \varphi_k = 0.8388, \quad \sigma_k = 0.0042, \quad \mathbb{E} (S_k) = (0.1141, 0.0772, 0.0597)',
\]

and

\[
\text{Var} (S_k) = \begin{bmatrix}
0.0200 & 0.0061 & 0.0072 \\
0.0061 & 0.0192 & 0.0060 \\
0.0072 & 0.0060 & 0.0220
\end{bmatrix}.
\]

Now suppose that another investor (indexed as the second investor) confronts with a liability and her initial liability is \( t_0 = 1 \). She enters the market at the same time and with the same initial wealth and the same risk-tolerance degree as the first investor. She invests in the same assets as the first investor and can also lend or borrow money with a stochastic interest rate governed by equation (3). We take the month return of SSE (Shanghai Stock Exchange) Treasury Bond Index (000012.SH) in China as the growth rate of the liability. In order to improve the precision of the calculation, we also multiply the growth rate of the liability by 12. Based on the historical data of SSE Treasury Bond Index and the three stocks stated above from October, 2006 to August, 2017, we obtain the parameters related with the liability as follows

\[
\mathbb{E} (\eta_k) = 1.0337, \quad \text{Var} (\eta_k) = 0.0023, \quad \mathbb{E} \left( b_{i_0}^{\psi_{\theta_0}} \right) = 1.0634,
\]

and

\[
\mathbb{E} \left( b_{1}^{\psi_{\eta_1}} \right) = 1.0619, \quad \mathbb{E} \left( b_{2}^{\psi_{\eta_2}} \right) = 1.0602, \quad \mathbb{E} \left( b_{3}^{\psi_{\eta_3}} \right) = 1.0581,
\]

\[
\mathbb{E} \left( b_{4}^{\psi_{\eta_4}} \right) = 1.0556, \quad \mathbb{E} \left( b_{5}^{\psi_{\eta_5}} \right) = 1.0526, \quad \mathbb{E} \left( b_{6}^{\psi_{\eta_6}} \right) = 1.0491,
\]

\[
\mathbb{E} \left( b_{7}^{\psi_{\eta_7}} \right) = 1.0449, \quad \mathbb{E} \left( b_{8}^{\psi_{\eta_8}} \right) = 1.0400, \quad \mathbb{E} \left( b_{9}^{\psi_{\eta_9}} \right) = 1.0342,
\]

\[
\mathbb{E} \left( b_{i_0}^{\psi_{\eta_0} S_0} \right) = \begin{bmatrix}
0.1221 \\
0.0829 \\
0.0643
\end{bmatrix}, \quad \mathbb{E} \left( b_{1}^{\psi_{\eta_1} S_1} \right) = \begin{bmatrix}
0.1219 \\
0.0827 \\
0.0642
\end{bmatrix}, \quad \mathbb{E} \left( b_{2}^{\psi_{\eta_2} S_2} \right) = \begin{bmatrix}
0.1217 \\
0.0826 \\
0.0641
\end{bmatrix},
\]

\[
\mathbb{E} \left( b_{3}^{\psi_{\eta_3} S_3} \right) = \begin{bmatrix}
0.1215 \\
0.0824 \\
0.0639
\end{bmatrix}, \quad \mathbb{E} \left( b_{4}^{\psi_{\eta_4} S_4} \right) = \begin{bmatrix}
0.1212 \\
0.0823 \\
0.0638
\end{bmatrix}, \quad \mathbb{E} \left( b_{5}^{\psi_{\eta_5} S_5} \right) = \begin{bmatrix}
0.1208 \\
0.0820 \\
0.0636
\end{bmatrix},
\]

\[
\mathbb{E} \left( b_{6}^{\psi_{\eta_6} S_6} \right) = \begin{bmatrix}
0.1204 \\
0.0818 \\
0.0634
\end{bmatrix}, \quad \mathbb{E} \left( b_{7}^{\psi_{\eta_7} S_7} \right) = \begin{bmatrix}
0.1200 \\
0.0814 \\
0.0632
\end{bmatrix},
\]

\[
\mathbb{E} \left( b_{8}^{\psi_{\eta_8} S_8} \right) = \begin{bmatrix}
0.1194 \\
0.0811 \\
0.0629
\end{bmatrix}, \quad \mathbb{E} \left( b_{9}^{\psi_{\eta_9} S_9} \right) = \begin{bmatrix}
0.1187 \\
0.0806 \\
0.0625
\end{bmatrix}.
\]
6.1. **Numerical analysis of the equilibrium strategy.** In this subsection, we analyze the effect of liability and stochastic interest rate on the equilibrium strategy. Moreover, we compare the equilibrium strategy with the pre-commitment strategy obtained by Yao et al. (2016a). We mark the first investor as $l_0 = 0$.

![Figure 1](image1.png)  
**Figure 1.** The equilibrium strategies with and without liability

![Figure 2](image2.png)  
**Figure 2.** The surplus process of the equilibrium strategies with and without liability

![Figure 3](image3.png)  
**Figure 3.** The effect of liability on the equilibrium strategy

Figure 1 shows the equilibrium strategies with and without liability. We find that no matter for the first or the second investor, the amount invested in each risky asset is increasing with time. The reason is that the surplus is increasing with time (see Figure 2). In order to investigate the effect of liability on the equilibrium strategy, we compare the amount invested in the risky assets (here and hereafter, we define the sum of the amount invested in the three risky assets as the amount invested in the risky assets) between the first and the second investors. The numerical results are depicted in Figure 3, which shows that the investor with liability invests more in the risky assets than the investor without liability. That is, the investor invests more in the risky assets when the wealth level decreases, which is consistent with the results of He and Liang (2013)\cite{10} and Sun et al (2016)\cite{29}. One possible explanation for this phenomenon is that the decrease of the wealth level forces the
investor to invest more in the risky assets in order to have an opportunity to increase the wealth level.

We show in Figure 4 the effect of stochastic interest rate on the equilibrium strategies with and without liability. We find that no matter for the first investor or the second investor, the amount invested in the risky assets in the case with stochastic interest rate is less than that in the case with deterministic interest rate. The reason may be that when the interest rate is stochastic, the money market provides less buffer (or, hedging) against the risky assets, so that an investor dare not invest as much as in the deterministic interest rate case. Furthermore, the difference between the two cases is decreasing as the time being close to the terminal time $T$. This is because, the uncertainty in stochastic interest rate is decreasing as the time being close to the terminal time $T$. Specially, at time $T-1$, the uncertainty in stochastic interest rate disappears.

![Figure 4. The effect of stochastic interest rate on the equilibrium strategy](image)

![Figure 5. A comparison of the equilibrium strategy and pre-commitment strategy](image)

Under the same conditions, we compare the equilibrium strategy with the pre-commitment strategy (Yao et al. (2016a)[38]). Figure 5 shows the difference between the equilibrium strategy and the pre-commitment strategy. For the pre-commitment strategy, the amount invested in the risky assets shows a decreasing trend, which is consistent with the results of Yao et al. (2016b)[36] and Menoncin
and Vigna (2017)[20], and also in line with real investment practice in financial markets. In general, investors reduce their investment amount in the risky assets as it is near the end of the investment activity, which is called “age effect” in Zhang et al (2018)[41]. However, for the equilibrium strategy, the amount invested in the risky assets is relatively stable in every period. The reason may be that the amount invested in the risky assets in every period is a balance struck between the current self and all future selves. In addition, we find that the investor with the pre-commitment strategy invests more in the risky assets than the investor with the equilibrium strategy does, which is consistent with the results of Bian et al. (2018)[1].

6.2. **Numerical analysis of equilibrium efficient frontier.** This subsection illustrates the effect of liability and stochastic interest rate on the equilibrium efficient frontier. For convenience, we only consider the equilibrium efficient frontier at time 0.

![Figure 6](image1.png)

*Figure 6. The effect of liability on the equilibrium efficient frontier*

![Figure 7](image2.png)

*Figure 7. The effect of stochastic interest rate on the equilibrium efficient frontier*

Figure 6 shows the effect of liability on the equilibrium efficient frontier. We find that the equilibrium efficient frontier with \( l_0 = 1 \) is below the one with \( l_0 = 0 \), that is, to obtain the same expected final surplus, the investor with liability needs to
undertake more risks than the investor without liability. The reason is that liability can generate additional risk during the process of the investment.

The effect of stochastic interest rate on the equilibrium efficient frontier is presented in Figure 7. We find that no matter for the liability case or for the no liability case, the equilibrium efficient frontier with stochastic interest rate is below the one with deterministic interest rate. Moreover, Figure 7 (a) shows that, in the no liability case, the global minimum standard deviation of the final surplus with deterministic interest rate is equal to zero and the efficient frontier is a straight line in the standard deviation-mean plane, while the one with the stochastic interest rate is strictly larger than zero and the efficient frontier is a hyperbola. Figure 7 (b) displays that, for both cases, the global minimum standard deviation of the final surplus is strictly larger than zero and the efficient frontier is a hyperbola in the standard deviation-mean plane. This clearly indicates that the randomness of the interest rate and the liability both increase the risk.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{A comparison of the equilibrium efficient frontier and pre-commitment efficient frontier}
\end{figure}

We compare the difference between the equilibrium efficient frontier and the pre-commitment efficient frontier in Figure 8. We find that the pre-commitment efficient frontier lies above the equilibrium efficient frontier. That is, to obtain the same expected final surplus, the investor with the equilibrium strategy needs to face more risks than the investor with the pre-commitment strategy. The reason is that what the investor with the pre-commitment strategy focuses on is the globally optimal strategy, while what the investor with the equilibrium strategy concerns is to take the non-cooperative game to obtain the time-consistent strategy, which increases the risk.

7. Conclusion. In this paper, we investigate the time-consistent strategy for a multi-period mean-variance ALM problem. We assume that the financial market consists of one risk-free asset and \( n \) risky assets, and the interest rate is stochastic and characterized by the discrete-time Vasicek model proposed by Yao et al. (2016a)[38]. We handle the problem as a non-cooperative game whose equilibrium strategy is the desired time-consistent strategy. Using the extended Bellman equation, which was originally proposed by Björk and Murgoci (2014)[3], we obtain the equilibrium strategy and the equilibrium efficient frontier in closed form. We find that the equilibrium strategy is independent of the initial state (initial wealth, initial liability and initial interest rate), but only depends on the current wealth, current
liability and current interest rate, which is different from the pre-commitment strategy obtained by Yao et al. (2016a)[38]. Simultaneously, we find that the equilibrium strategy is a linear combination of the equilibrium global minimum variance strategy and a self-financing strategy, based on which a multi-period version of two-fund separation theorem is proved. In addition, a numerical example is presented. From which we find that, (i) the investor confronted with liability (stochastic interest rate) will increase (decrease) the investment amount in the risky assets; (ii) compared with the investor with the pre-commitment strategy, the investor with the equilibrium strategy invests less in the risky assets; (iii) to get the same expected final surplus, a greater risk will be undertook if the stochastic interest rate (liability) is taken into consideration, the investor with the equilibrium strategy needs to face more risks than the investor with the pre-commitment strategy. Finally, we point out that our work can be extended in several aspects. For example, we can further add stochastic salary into our model to consider DC pension fund problems, and we can also incorporate other background risks, such as inflation risk, into our model.

Appendix A. The proof of Lemma 3.1.

Proof. Mathematically, if \( \epsilon_k \) is independent of \( S_k \), then \( b_k^{\psi k+1} = e^{(1-\psi_k)\lambda_{k+1}} \epsilon_k \psi_k+1 \) is also independent of \( S_k \). So, by Assumption 2.1, we have

\[
E\left(b_k^{\psi k+1} S_k\right) = E\left(b_k^{\psi k+1}\right)E\left(S_k\right), \quad E\left(b_k^{2\psi k+1} S_k S_k^r\right) = E\left(b_k^{2\psi k+1}\right)E\left(S_k S_k^r\right). \quad (59)
\]

From equation (4), we know

\[
E\left(b_k^{\psi k+1}\right) > 0. \quad (60)
\]

Then from equations (59), (60) and Assumption 2.2, we obtain

\[
\begin{align*}
\text{Var}\left(b_k^{\psi k+1} S_k\right) & = E\left(b_k^{2\psi k+1} S_k S_k^r\right) - E\left(b_k^{\psi k+1} S_k\right) E\left(b_k^{\psi k+1} S_k^r\right) \\
& = E\left(b_k^{2\psi k+1}\right)E\left(S_k S_k^r\right) - E^2\left(b_k^{\psi k+1}\right)E\left(S_k\right)E\left(S_k^r\right) \\
& = \left(\text{Var}\left(b_k^{\psi k+1}\right) + E^2\left(b_k^{\psi k+1}\right)\right)E\left(S_k S_k^r\right) - E^2\left(b_k^{\psi k+1}\right)E\left(S_k\right)E\left(S_k^r\right) \\
& = \text{Var}\left(b_k^{\psi k+1}\right)E\left(S_k S_k^r\right) + E^2\left(b_k^{\psi k+1}\right)\text{Var}\left(S_k\right) \\
& > 0.
\end{align*}
\]

\( \square \)

Appendix B. The proof of Lemma 3.3.

Proof. We prove this lemma by mathematical induction for \( k \). For \( k = T \), because \( W_T = 0 \) and \( \lambda_T = 1 \), this lemma holds.

Suppose that this lemma holds for \( T, T - 1, \ldots, k + 1 \), that is, \( W_{k+1} \geq 0 \) and \( \lambda_{k+1} > 0 \). From Lemma 3.1 we know that \( \text{Var}\left(b_k^{\psi k+1} S_k\right) \) is positive definite. Then

\[
A_k = W_{k+1}E\left(b_k^{2\psi k+1}\right)E\left(S_k S_k^r\right) + \lambda_{k+1}^2 E\left(b_k^{\psi k+1} S_k\right) > 0,
\]

that is, \( A_k \) is also positive definite. Hence, \( |A_k| > 0 \).
Thus, Lemma 3.2, we know $\xi_k$ is semidefinite. Hence, $|\xi_k| \geq 0$. Because $A_k$ is positive definite, from Lemma 3.2, we know

$$|\xi_k| = |A_k| |B_k - B_k^2 E(S_k') A_k^{-1} E(S_k)|.$$  

Thus,

$$W_k = B_k - B_k^2 E(S_k') A_k^{-1} E(S_k) \geq 0.$$  

Set

$$N_k = \begin{pmatrix}
\mathbb{E}(b_k^{\psi_k+1}) & W_{k+1} E\left( b_k^{2\psi_k+1} \right) + \lambda_{k+1}^2 \text{Var}\left( b_k^{\psi_k+1} \right) E(S_k') \\
E\left( b_k^{\psi_k+1} \right) E(S_k) & W_{k+1} E\left( b_k^{2\psi_k+1} \right) E(S_k S'_k) + \lambda_{k+1}^2 \text{Var}\left( b_k^{\psi_k+1} S_k \right)
\end{pmatrix}.$$  

Because $\mathbb{E}(b_k^{\psi_k+1}) E(b_k^{\psi_k+1}) E(S_k) = \mathbb{E}(b_k^{\psi_k+1}) E(S_k) E(b_k^{\psi_k+1})$, from Lemma 3.2 we know

$$|N_k| = | \mathbb{E}(b_k^{\psi_k+1}) (W_{k+1} E\left( b_k^{2\psi_k+1} \right) E(S_k S'_k) + \lambda_{k+1}^2 \text{Var}\left( b_k^{\psi_k+1} S_k \right)) - E\left( b_k^{\psi_k+1} \right) E(S_k) (W_{k+1} E\left( b_k^{2\psi_k+1} \right) + \lambda_{k+1}^2 \text{Var}\left( b_k^{\psi_k+1} \right)) E(S_k') |.$$  

Since $W_{k+1} \geq 0$, $\lambda_{k+1} > 0$, $\mathbb{E}(b_k^{\psi_k+1}) > 0$, $\mathbb{E}(b_k^{2\psi_k+1}) > 0$, and $\text{Var}(S_k)$ is positive definite by Assumption 2.2, we have $|N_k| > 0$. Again from Lemma 3.2, we know

$$|N_k| = |A_k| | \mathbb{E}(b_k^{\psi_k+1}) - E\left( b_k^{\psi_k+1} \right) B_k E(S_k') A_k^{-1} E(S_k) |.$$  

Denote $\xi_k = W_{k+1} E(Q_k Q'_k) + \lambda_{k+1}^2 |E(Q_k Q'_k) - E(Q_k) E(Q'_k)|$. Since $W_{k+1} \geq 0$, $\lambda_{k+1} > 0$, $E(Q_k Q'_k)$ is positive definite by Proposition 1 of Yao et al. (2016a)[38] and $\text{Var}(Q_k) = E(Q_k Q'_k) - E(Q_k) E(Q'_k)$ is semidefinite by Muirhead (1982)[21]. Then $\xi_k$ is semidefinite. Hence, $|\xi_k| \geq 0$. Because $A_k$ is positive definite, from Lemma 3.2, we know

$$E(Q_k Q'_k) = E\left( b_k^{\psi_k+1} \right) E(S_k')$$

Then

$$W_{k+1} E(Q_k Q'_k) + \lambda_{k+1}^2 |E(Q_k Q'_k) - E(Q_k) E(Q'_k)| = \begin{pmatrix}
B_k & B_k E(S_k') \\
B_k E(S_k) & A_k
\end{pmatrix}.$$  

(63)
Then

$$\lambda_k = \lambda_{k+1} \left[ \mathbb{E} \left( b_k^\omega \right) - \mathbb{E} \left( b_k^\omega \right) B_k \mathbb{E} \left( S_k^T \right) A_k^{-1} \mathbb{E} \left( S_k \right) \right] > 0.$$  \hfill (66)

Equations (64) and (66) show that this lemma holds for \(k\). By the principle of mathematical induction, this lemma holds for all \(k = 0, 1, \ldots, T\), and thus this lemma is proved.

**Appendix C. The proof of Lemma 3.5.**

**Proof.** For arbitrary \(k = 0, 1, \ldots, T - 1\), by applying the equation (18) recursively, we can easily obtain

$$\alpha_k = \sum_{l=k}^{T-1} \lambda_l^2 \mathbb{E}^2 \left( b_l^\omega \right) \mathbb{E} \left( S_l^T \right) A_l^{-1} \mathbb{E} \left( S_l \right).$$  \hfill (67)

Since for all \(l = k, k+1, \ldots, T - 1\), \(A_l\) is positive definite by Lemma 3.4, so is \(A_l^{-1}\). Assumption 2.3 shows that \(\mathbb{E} \left( S_l \right) \neq 0_n\). Then we have \(\mathbb{E} \left( S_l^T \right) A_l^{-1} \mathbb{E} \left( S_l \right) > 0\). Moreover, from Lemma 3.3 and equation (4), we know \(\lambda_{l+1} > 0\) and \(\mathbb{E} \left( b_{l+1}^\omega \right) > 0\), respectively. Hence, \(\alpha_k > 0\). 

**Appendix D. The proof of Theorem 3.6.**

**Proof.** We prove this theorem by mathematical induction. For \(k = T - 1\), by equations (1), (2), (3), (12) and (13), we have

$$V_{T-1} (x_{T-1}, r_{T-1}, l_{T-1})$$

$$= \max_{\tau_{T-1}} \left\{ \mathbb{E}_{x_{T-1}, r_{T-1}, l_{T-1}} \left( V_T (X_T^T, R_T, L_T) \right) - \mathbb{E}_{x_{T-1}, r_{T-1}, l_{T-1}} \left( g_T (X_T^T, R_T, L_T) \right) \right\}$$

$$= \max_{\tau_{T-1}} \left\{ \mathbb{E}_{x_{T-1}, r_{T-1}, l_{T-1}} \left( \omega (X_T^T - L_T) \right) - \mathbb{E}_{x_{T-1}, r_{T-1}, l_{T-1}} \left( (X_T^T - L_T)^2 \right) \right\}$$

$$= \max_{\tau_{T-1}} \left\{ \omega \mathbb{E} \left( x_{T-1} r_{T-1} + S_{T-1}^{\omega} \pi_{T-1} - \eta_{T-1} l_{T-1} \right) \right. - \eta_{T-1}^2 \mathbb{E} (\eta_{T-1}) - \mathbb{E}^2 (\eta_{T-1})$$

$$- \pi_{T-1}^2 \left[ \mathbb{E} \left( S_{T-1}^T \right) - \mathbb{E} \left( S_{T-1} \right) \mathbb{E} \left( S_{T-1}^T \right) \right] \pi_{T-1}$$

$$+ 2 l_{T-1} \left[ \mathbb{E} (\eta_{T-1} S_{T-1}) - \mathbb{E} (\eta_{T-1}) \mathbb{E} (S_{T-1}) \right] \pi_{T-1}$$

$$= \omega x_{T-1} r_{T-1} - \omega \mathbb{E} (\eta_{T-1}) l_{T-1} - \eta_{T-1}^2 \mathbb{E} (\eta_{T-1})$$

$$+ \max_{\pi_{T-1}} \left\{ \omega \mathbb{E} (S_{T-1}^{\omega} \pi_{T-1}) - \pi_{T-1} Var (S_{T-1}) \pi_{T-1} + 2 l_{T-1} D_{T-1} \pi_{T-1} \right\}. \hfill (68)$$

Since \(Var (S_{T-1})\) is positive definite by Assumption 2.2, the application of the first order condition to \(\pi_{T-1}\) yields the optimal solution

$$\pi_{T-1} = Var^{-1} (S_{T-1}) \left( \frac{\omega \mathbb{E} (S_{T-1})}{2} + D_{T-1} l_{T-1} \right). \hfill (69)$$

Substituting equation (69) into equations (68) and (13), respectively, we obtain

$$V_{T-1} (x_{T-1}, r_{T-1}, l_{T-1})$$

$$= \omega x_{T-1} r_{T-1} + \omega \left[ D_{T-1} Var^{-1} (S_{T-1}) \mathbb{E} (S_{T-1}) - \mathbb{E} (\eta_{T-1}) \right] l_{T-1}$$
Equations (76)-(78) show that equations (27)-(29) hold for \( k \) on the other hand, by equations (14)-(23), we have

\[
T = \max_{\lambda} \psi \pi T = \max_{\lambda} \psi (S'_{T-1} - \text{Var}(S))
\]

Now suppose that equations (27)-(29) hold for \( \psi = \max_{\lambda} \psi (S'_{T-1} - \text{Var}(S)) \).

From equations (69)-(75), we obtain

\[
\pi_{T-1} = A_{T-1}^{-1} \left( \frac{\omega \lambda_T \mathbb{E} \left( b_{T-1}^{\psi_T} \right) \mathbb{E} (S_{T-1})}{2n^{\psi_T}_{T-1}} - B_{T-1} \mathbb{E} (S_{T-1}) r_{T-1} x_{T-1} \right.
\]

\[
+ r_{T-1}^{\psi_T-1} \psi_C (T-1) l_{T-1} \bigg)
\]

\[
V_{T-1} (x_{T-1}, r_{T-1}, l_{T-1}) = -W_{T-1} \left( x_{T-1}^{2} - x_{T-1} \right) + \omega \lambda_{T-1} r_{T-1} x_{T-1} + \frac{\omega^2}{4} \alpha_{T-1}
\]

\[
+ 2H_{T-1} r_{T-1} x_{T-1} l_{T-1} + \gamma_{T-1} l_{T-1}^2 + \omega \beta_{T-1} l_{T-1},
\]

and

\[
g_{T-1} (x_{T-1}, r_{T-1}, l_{T-1}) = \lambda_{T-1} r_{T-1}^{\psi_T-1} x_{T-1} + \beta_{T-1} l_{T-1} + \frac{\omega}{2} \alpha_{T-1}.
\]

Equations (76)-(78) show that equations (27)-(29) hold for \( k = T - 1 \).

Now suppose that equations (27)-(29) hold for \( T - 1, T - 2, \ldots, k + 1 \). Then for \( k \), by the extended Bellman equation (12), we have

\[
V_k (x_k, r_k, l_k)
\]

\[
= \max_{\pi_k} \left\{ \mathbb{E}_{x_k, r_k, l_k} \left( V_{k+1} (X_{k+1}, R_{k+1}, L_{k+1}) \right) \right.
\]

\[
- \mathbb{E}_{x_k, r_k, l_k} \left( g_k^2 + (X_{k+1}, R_{k+1}, L_{k+1}) \right)
\]

\[
+ \left[ \mathbb{E}_{x_k, r_k, l_k} \left( g_k^2 + (X_{k+1}, R_{k+1}, L_{k+1}) \right) \right]^2 \bigg}\}

\[
= \max_{\pi_k} \left\{ \mathbb{E}_{x_k, r_k, l_k} \left( -W_{k+1} R^{\psi_{k+1}}_{k+1} (X_{k+1})^2 + \omega \lambda_{k+1} R^{\psi_{k+1}}_{k+1} X_{k+1} + \frac{\omega^2}{4} \alpha_{k+1}
\]

\[
+ 2H_{k+1} R^{\psi_{k+1}}_{k+1} X_{k+1} L_{k+1} + \gamma_{k+1} L_{k+1}^2 + \omega \beta_{k+1} L_{k+1} \bigg)
\]

\[
- \mathbb{E}_{x_k, r_k, l_k} \left( \left( \lambda_{k+1} R^{\psi_{k+1}}_{k+1} X_{k+1} + \beta_{k+1} L_{k+1} + \frac{\omega}{2} \alpha_{k+1} \right)^2 \right)
\]

\[
+ \left[ \mathbb{E}_{x_k, r_k, l_k} \left( \lambda_{k+1} R^{\psi_{k+1}}_{k+1} X_{k+1} + \beta_{k+1} L_{k+1} + \frac{\omega}{2} \alpha_{k+1} \right) \right]^2 \bigg}\}
\[
= \max_{\pi_{k+1}} \left\{ \mathbb{E} \left( -W_{k+1} b_k^{2\psi_{k+1}} r_k^{2\psi_{k+1}} (x_k r_k + S_k' \pi_k) \right)^2 + \gamma_{k+1} \eta_k^2 \right\} + \omega_{k+1} \eta_k \right) \\
+ \frac{\omega^2}{4} \alpha_{k+1} + 2H_{k+1} b_k^{2\psi_{k+1}} r_k^{2\psi_{k+1}} \eta_k \pi_k r_k + \frac{\omega^2}{4} \alpha_{k+1} \\
+ \omega \lambda_{k+1} r_k^{2\psi_{k+1}} (x_k r_k + S_k' \pi_k) \\
+ \left\{ \mathbb{E} \left( \left( \lambda_{k+1} r_k^{2\psi_{k+1}} (x_k r_k + S_k' \pi_k) + \beta_{k+1} \eta_k \right) + \frac{\omega}{2} \alpha_{k+1} \right)^2 \right\} \\
+ \left[ \mathbb{E} \left( \lambda_{k+1} r_k^{2\psi_{k+1}} (x_k r_k + S_k' \pi_k) + \beta_{k+1} \eta_k \right) \right] \\
+ \left[ \mathbb{E} \left( \left( \lambda_{k+1} r_k^{2\psi_{k+1}} (x_k r_k + S_k' \pi_k) + \beta_{k+1} \eta_k \right) + \frac{\omega}{2} \alpha_{k+1} \right)^2 \right] \\
= -W_{k+1} \mathbb{E} \left( r_k^{2\psi_{k+1}} \right) r_k^{2\psi_{k+1}} x_k + \omega \lambda_{k+1} \mathbb{E} \left( b_k^{2\psi_{k+1}} \right) r_k^{2\psi_{k+1}} x_k \\
+ \frac{\omega^2}{4} \alpha_{k+1} + 2H_{k+1} \mathbb{E} \left( b_k^{2\psi_{k+1}} \right) \mathbb{E} \left( \left( \lambda_{k+1} r_k^{2\psi_{k+1}} (x_k r_k + S_k' \pi_k) + \beta_{k+1} \eta_k \right) \right) \\
+ \left\{ \mathbb{E} \left( \left( \lambda_{k+1} r_k^{2\psi_{k+1}} (x_k r_k + S_k' \pi_k) + \beta_{k+1} \eta_k \right) + \frac{\omega}{2} \alpha_{k+1} \right)^2 \right\} \\
+ \left[ \mathbb{E} \left( \lambda_{k+1} r_k^{2\psi_{k+1}} (x_k r_k + S_k' \pi_k) + \beta_{k+1} \eta_k \right) \right] \\
+ \left[ \mathbb{E} \left( \left( \lambda_{k+1} r_k^{2\psi_{k+1}} (x_k r_k + S_k' \pi_k) + \beta_{k+1} \eta_k \right) + \frac{\omega}{2} \alpha_{k+1} \right)^2 \right] \\
Since \( r_k = R_k > 0, \) and \( A_k \) is positive definite by Lemma 3.4, the first order condition to \( \pi_k \) yields the optimal solution

\[
\pi_k^* = A_k^{-1} \left( \omega \lambda_{k+1} E \left( b_k^{\psi+1} \right) - \frac{B_k \psi(k)}{2 r_k^{\psi+1}} - B_k \psi(k) r_k x_k + r_k \psi(k) C_k l_k \right). \tag{80}
\]

Substituting equation (80) into equations (79) and (13), respectively, we obtain

\[
V_k(x_k, r_k, l_k) = - \left[ B_k - B_k^2 \psi(k) A_k^{-1} \psi(k) \right] r_k^{\psi+1} x_k^2 \]
\[
+ \omega \lambda_{k+1} \left[ \psi(k) - \frac{B_k \psi(k) A_k^{-1} \psi(k)}{2 r_k^{\psi+1}} \right] r_k^{\psi+1} x_k \]
\[- \omega \left[ \beta_{k+1} E (\eta k) + \lambda_{k+1} E \left( b_k^{\psi+1} \right) C_k \psi(k) A_k^{-1} \psi(k) \right] \]
\[- 2 \left[ H_{k+1} E \left( b_k^{\psi+1} \eta k \right) - \lambda_{k+1} \beta_{k+1}^{\psi+1} \psi(k) \right] \]
\[- B_k C_k A_k^{-1} \psi(k) \]
\[- \gamma_{k+1} E (\eta k^2) + C_k A_k^{-1} C_k - \beta_{k+1} \psi(k) \text{Var} (\eta_k) \right] l_k^2 \]
\[
= - W_k r_k^{\psi+1} x_k^2 + \omega \lambda_k r_k^{\psi+1} x_k + \frac{\omega^2}{4} \alpha_k + 2 H_k r_k^{\psi+1} x_k l_k + \gamma_k l_k^2 + \omega \beta_k l_k, \tag{81}
\]

and

\[
g_k(x_k, r_k, l_k) = E_x, r_k, l_k \left[ g_{k+1} \left( X_{k+1}^x, R_{k+1}, L_{k+1} \right) \right] \]
\[
= E_x, r_k, l_k \left[ \lambda_{k+1} r_{k+1} x_{k+1}^2 + \beta_{k+1} l_{k+1} + \frac{\omega}{2} \alpha_{k+1} \right] \]
\[
= E_{\lambda_{k+1} \beta_{k+1} \gamma_{k+1} \psi(k)} \left( x_k r_k + S_k^x \pi_k \right) \]
\[
= \lambda_{k+1} E \left( \psi(k) \right) r_k^{\psi+1} x_k + \lambda_{k+1} \psi(k) A_k^{-1} \psi(k) \]
\[
+ \beta_{k+1} E (\eta_k) l_k + \frac{\omega}{2} \alpha_{k+1} \]
\[
= \lambda_{k+1} E \left( \psi(k) \right) r_k^{\psi+1} x_k + \beta_{k+1} E \left( \psi(k) \right) B_k \psi(k) A_k^{-1} \psi(k) \]
\[
+ \left[ \beta_{k+1} E (\eta_k) + \lambda_{k+1} E \left( \psi(k) \right) \right] C_k \psi(k) A_k^{-1} \psi(k) \]
\[
+ \frac{\omega}{2} \left[ \beta_{k+1} \psi(k) \right] \]
\[
= \lambda_{k+1} E \left( \psi(k) \right) r_k^{\psi+1} x_k + \beta_{k+1} E (\eta_k) + \lambda_{k+1} E \left( \psi(k) \right) C_k \psi(k) A_k^{-1} \psi(k) + \alpha_{k+1} \]
\[
= \lambda_{k+1} E \left( \psi(k) \right) r_k^{\psi+1} x_k + \beta_{k+1} E (\eta_k) + \lambda_{k+1} E \left( \psi(k) \right) C_k \psi(k) A_k^{-1} \psi(k) + \alpha_{k+1} \]
\[
= \lambda_{k+1} E \left( \psi(k) \right) r_k^{\psi+1} x_k + \beta_{k+1} E (\eta_k) + \lambda_{k+1} E \left( \psi(k) \right) C_k \psi(k) A_k^{-1} \psi(k) + \alpha_{k+1}. \tag{82}
\]

Equations (80)-(82) show that equations (27)-(29) hold for \( k \). Hence, equations (27)-(29) hold for all \( k = 0, 1, \cdots, T - 1 \) and the theorem is thus proved. \( \square \)

**Appendix E. The proof of Lemma 4.1.**

**Proof.** We prove this lemma by mathematical induction. First, we prove equation (38) holds. From equation (15) we know \( W_T = 0 \), which means that equation (38) holds for \( k = T \). Suppose that equation (38) holds for \( T, T - 1, \cdots, k + 1 \). We
consider the case for $k$. Because $B_k = W_{k+1} \mathbb{E} \left( b_k^{2\psi_{k+1}} \right) + \lambda_{k+1}^2 \text{Var} \left( b_k^{\psi_{k+1}} \right) = 0$, we have
\[
W_k = B_k - B_k^2 \mathbb{E} (S'_k) A_k^{-1} \mathbb{E} (S_k) = 0,
\]
which means that equation (38) holds for $k$. Hence equation (38) holds for all $k = 0, 1, \cdots, T$.

Second, we prove equation (39) holds. $\lambda_T = e^{r(T-T-\psi_T)} = 1$ implies that equation (39) holds for $k = T$. Suppose that equation (39) holds for $T, T-1, \cdots, k+1$. We consider the case for $k$. Since $B_k=0$, we have
\[
\lambda_k = \lambda_{k+1} \left[ \mathbb{E} \left( b_k^{\psi_{k+1}} \right) - \mathbb{E} \left( b_k^{\psi_{k+1}} \right) B_k \mathbb{E} (S'_k) A_k^{-1} \mathbb{E} (S_k) \right]
\]
which shows that equation (39) holds for $k$. Hence, equation (39) holds for all $k = 0, 1, \cdots, T$.

Third, we prove equation (40) holds. For $k = T$,
\[
\alpha_T = \sum_{j=1}^{T-1} \mathbb{E} (S'_j) \text{Var}^{-1} (S_j) \mathbb{E} (S_j) = 0,
\]
which shows that equation (40) holds for $k = T$. Suppose that equation (40) holds for $T, T-1, \cdots, k+1$. We consider the case for $k$. Because $W_{k+1} = 0$, we have
\[
A_k = W_{k+1} \mathbb{E} \left( b_k^{2\psi_{k+1}} \right) \mathbb{E} (S_k S'_k) + \lambda_{k+1}^2 \text{Var} \left( b_k^{\psi_{k+1}} S'_k \right) = \lambda_{k+1}^2 b_k^{2\psi_{k+1}} \text{Var} (S_k).
\]
So
\[
\alpha_k = \lambda_{k+1}^2 \mathbb{E}^2 \left( b_k^{\psi_{k+1}} \right) \mathbb{E} (S'_k) A_k^{-1} \mathbb{E} (S_k) + \alpha_{k+1}
\]
\[
= \mathbb{E} (S_k) \text{Var}^{-1} (S_k) \mathbb{E} (S_k) + \sum_{j=k+1}^{T-1} \mathbb{E} (S'_j) \text{Var}^{-1} (S_j) \mathbb{E} (S_j)
\]
\[
= \sum_{j=k}^{T-1} \mathbb{E} (S'_j) \text{Var}^{-1} (S_j) \mathbb{E} (S_j),
\]
which means that equation (40) holds for $k$. Hence, equation (40) holds for all $k = 0, 1, \cdots, T$.

Equations (41) and (42) can be proved easily. The proof of equations (43)-(46) is similar to that of equations (38)-(40), and we omit it here.

**Appendix F. The proof of Theorem 5.1.**

**Proof.** (i) Since $\Pi_k^1(\omega_1)$ and $\Pi_k^2(\omega_2)$ are two equilibrium strategies corresponding to $\omega_1$ and $\omega_2$, respectively, by equation (58) we have
\[
\Pi_k^1(\omega_1) = \omega_1 F_k + \Pi_k^\text{MIN},
\]
\[
\Pi_k^2(\omega_2) = \omega_2 F_k + \Pi_k^\text{MIN}.
\]
Note that $\omega_1 \neq \omega_2$ and set $\alpha = \frac{\omega_2 - \omega_1}{\omega_2 - \omega_1}$. Then $\omega = \alpha \omega_1 + (1-\alpha) \omega_2$. It follows that
\[
\Pi_k^1(\omega) = (\alpha \omega_1 + (1-\alpha) \omega_2) F_k + (\alpha + 1 - \alpha) \Pi_k^\text{MIN} = \alpha (\omega_1 F_k + \Pi_k^\text{MIN}) + (1-\alpha) (\omega_2 F_k + \Pi_k^\text{MIN})
\]
\[
\alpha \Pi^*_k (\omega_1) + (1 - \alpha) \Pi^*_k (\omega_2).
\]

(ii) Since \( \alpha \omega_1 + (1 - \alpha) \omega_2 \geq 0 \), and

\[
\alpha \Pi^*_k (\omega_1) + (1 - \alpha) \Pi^*_k (\omega_2) = \alpha \left( \omega_1 F_k + \Pi^*_k \right) + (1 - \alpha) \left( \omega_2 F_k + \Pi^*_k \right)
\]

\[
= (\alpha \omega_1 + (1 - \alpha) \omega_2) F_k + \Pi^*_k
\]

\[
= \Pi^*_k (\alpha \omega_1 + (1 - \alpha) \omega_2).
\]

Thus, \( \alpha \Pi^*_k (\omega_1) + (1 - \alpha) \Pi^*_k (\omega_2) \) is an equilibrium strategy. \( \square \)

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