Elliptical Symmetry Tests in $\mathbb{R}$

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Abstract

The assumption of elliptical symmetry has an important role in many theoretical developments and applications, hence it is of primary importance to be able to test whether that assumption actually holds true or not. Various tests have been proposed in the literature for this problem. To the best of our knowledge, none of them has been implemented in $\mathbb{R}$. The focus of this paper is the implementation of several well-known tests for elliptical symmetry together with some novel tests. We demonstrate the testing procedures with a real data example, and we conduct a Monte Carlo simulation study of the finite-sample performances of the tests.

Key words: Elliptical symmetry, Hypothesis testing, Skew distributions, Skewness

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1 Introduction

Let $X_1, \ldots, X_n$ denote a sample of $n$ i.i.d. $d$-dimensional observations. A $d$-dimensional random vector $X$ is said to be elliptically symmetric about some location parameter $\theta \in \mathbb{R}^d$ if its density $f$ is of the form

$$x \mapsto f(x; \theta, \Sigma, f) = c_{d,f}|\Sigma|^{-1/2}f\left(\|\Sigma^{-1/2}(x - \theta)\|\right), \quad x \in \mathbb{R}^d,$$

(1)

where $\Sigma \in S_d$ (the class of symmetric positive definite real $d \times d$ matrices) is a scatter parameter, $f : \mathbb{R}^+_d \to \mathbb{R}^+$ is an a.e. strictly positive function called radial density, and $c_{d,f}$ is a normalizing constant depending on $f$ and the dimension $d$. Many well-known and widely used multivariate distributions are elliptical. The multivariate normal, multivariate Student $t$, multivariate power-exponential, symmetric multivariate stable, symmetric multivariate Laplace, multivariate logistic, multivariate Cauchy, and multivariate symmetric general hyperbolic distribution are all examples of elliptical distributions. The family of elliptical distributions has several appealing properties. For instance, it has a simple stochastic representation, clear parameter interpretation, it is closed under affine transformations, and its marginal and conditional distributions are also elliptically symmetric: see Paindaveine (2014) for details. Thanks to its mathematical
tractability and nice properties, it became a fundamental assumption in multivariate analysis and many applications. Numerous statistical procedures therefore rest on the assumption of elliptical symmetry: one- and $K$-sample location and shape problems (Um and Randles, 1998; Hallin and Paindaveine, 2002, 2006; Hallin et al., 2006), serial dependence and time series (Hallin and Paindaveine, 2004), one- and $K$-sample principal component problems (Hallin et al., 2010, 2014), multivariate tail estimation (Dominicy et al., 2017), to cite but a few. Elliptical densities are also considered in portfolio theory (Owen and Rabinovitch, 1983), capital asset pricing models (Hodgson et al., 2002), semiparametric density estimation (Liebscher, 2005), graphical models (Vogel and Fried, 2011), and many other areas.

Given the omnipresence of the assumption of elliptical symmetry, it is essential to be able to test whether that assumption actually holds true or not for the data at hand. Numerous tests have been proposed in the literature, including Beran (1979), Baringhaus (1991), Koltchinskii and Sakhanenko (2000), Manzotti et al. (2002), Schott (2002), Huffer and Park (2007), Cassart (2007) and Babić et al. (2019). Tests for elliptical symmetry based on Monte Carlo simulations can be found in Diks and Tong (1999) and Zhu and Neuhaus (2000); Li et al. (1997) recur to graphical methods and Zhu and Neuhaus (2004) build conditional tests. We refer the reader to Serfling (2006) and Sakhanenko (2008) for extensive reviews and performance comparisons. To the best of our knowledge, none of these tests is available in the open software R. The focus of this paper is to close this gap by implementing several well-known tests for elliptical symmetry together with some novel tests. The test of Beran (1979) is neither distribution-free nor affine-invariant; moreover, there are no practical guidelines to the choice of the basis functions involved in the test statistic. Therefore, we opt not to include it in the package. Baringhaus (1991) proposes a Cramér-von Mises type test for spherical symmetry based on the independence between norm and direction. Dyckerhoff et al. (2015) have shown by simulations that this test can be used as a test for elliptical symmetry in dimension 2. This test assumes the location parameter to be known and its asymptotic distribution is not simple to use (plus no proven validity in dimensions higher than 2), hence we decided not to include it in the package. Thus, the tests suggested by Koltchinskii and Sakhanenko (2000), Manzotti et al. (2002), Schott (2002), Huffer and Park (2007), Cassart (2007) and Babić et al. (2019) are implemented in the package ellipticalsymmetry.

The paper is organized as follows. In Section 2, we describe the tests for elliptical symmetry that have been implemented in the ellipticalsymmetry package, together with a detailed description of the functions that are available in the package. In Section 3, we conduct a Monte Carlo simulation study of the finite-sample performances of the tests. The use of the implemented functions is illustrated using the data set AIS from the package sn of the R environment in Section 4 and conclusions are provided in Section 5.

2 Testing for elliptical symmetry

In this section, we focus on tests for elliptical symmetry that have been implemented in our new ellipticalsymmetry package. Besides formal definitions of test statistics and limiting distributions, we also explain the details on computing.
2.1 Test by Koltchinskii and Sakhanenko

Koltchinskii and Sakhanenko (2000) develop a class of omnibus bootstrap tests for unspecified location that are affine invariant and consistent against any fixed alternative. The estimators of the unknown parameters are as follows: \( \hat{\theta} = n^{-1} \sum_{i=1}^{n} X_i \) and \( \hat{\Sigma} = n^{-1} \sum_{i=1}^{n} (X_i - \hat{\theta})(X_i - \hat{\theta})' \). Define \( Y_i = \Sigma^{-1/2}(X_i - \hat{\theta}) \) and let \( F_B \) be a class of Borel functions from \( \mathbb{R}^d \) to \( \mathbb{R} \). Their test statistics are functionals (for example, sup-norms) of the stochastic process

\[
n^{-1/2} \sum_{i=1}^{n} (f(Y_i) - m_f(||Y_i||)) ,
\]

where \( f \in F_B \) and \( m_f(\rho) \) is the average value of \( f \) on the sphere with radius \( \rho > 0 \). Several examples of classes \( F_B \) and test statistics based on the sup-norm of the above process are considered in Koltchinskii and Sakhanenko (2000). Here, we restrict our attention to \( F_B := \{ I_{0 < ||x|| \leq \psi} \left( \frac{1}{||x||^{d-1}} \right) : \psi \in G_l, ||\psi||_2 \leq 1, t > 0 \} \) where \( I_A \) stands for the indicator function of \( A \), \( G_l \) for the linear space of spherical harmonics of degree less than or equal to \( l \) in \( \mathbb{R}^d \), and \( || \cdot ||_2 \) is the \( L^2 \)-norm on the unit sphere \( S^{d-1} \) in \( \mathbb{R}^d \). With these quantities in hand, the test statistic becomes

\[
Q_{KS}^{(n)} := n^{-1/2} \max_{1 \leq j \leq n} \left( \frac{\dim(G_l)}{j} \left( \sum_{k=1}^{j} \psi_s \left( \frac{Y_{[k]}}{||Y_{[k]}||} \right) - \delta_{s1} \right)^2 \right)^{1/2} ,
\]

where \( Y_{[i]} \) denotes the \( i \)th order statistic from the sample \( Y_1, \ldots, Y_n \) ordered according to their \( L^2 \)-norm, \( \{ \psi_s, s = 1, \ldots, \dim(G_l) \} \) denotes an orthonormal basis of \( G_l \), \( \psi_1 = 1 \), and \( \delta_{ij} = 1 \) for \( i = j \) and 0 otherwise. The test statistic is relatively simple to construct if we have formulas for spherical harmonics. A spherical harmonic of degree \( l \) is the restriction to the unit sphere of a homogeneous polynomial \( p(x) \) on \( \mathbb{R}^d \), of degree \( l \) and such that \( \Delta(p) \equiv 0 \) on \( \mathbb{R}^d \), where \( \Delta \) denotes the Laplace operator \( \Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} \). The number of linearly independent spherical harmonics of degree \( l \), in dimension \( d \), is given by \( N(d, l) = \binom{d+l-1}{d-l+1} - \binom{d-l-1}{d-1} \). In dimension 2 spherical harmonics coincide with sines and cosines on the unit circle. The detailed construction of the test statistic \( Q_{KS}^{(n)} \) for dimensions 2 and 3 can be found in Sakhanenko (2008). In order to be able to use \( Q_{KS}^{(n)} \) in higher dimensions, we need corresponding formulas for spherical harmonics. Using recursive formulas from Müller (1966) and equations given in Manzotti and Quiroz (2001) we obtained spherical harmonics of degree one to four in arbitrary dimension. The reader should bare in mind that larger degree leads to better power performance of this test. A drawback of this test is that it requires bootstrap procedures to obtain critical values.

In our \( \text{R} \) package, this test can be run using a function called \texttt{KoltchinskiiSakhanenko()} . The syntax for this function is very simple:

\[
\text{KoltchinskiiSakhanenko}(X, R=1000),
\]

where \( X \) is an input to this function consisting of a data set which must be a matrix and \( R \) stands for the number of bootstrap replicates. The default number of replicates is set to 1000.
2.2 The MPQ test

Manzotti et al. (2002) develop a test based on spherical harmonics. The estimators of the unknown parameters are the sample mean denoted as $\hat{\theta}$ and the unbiased sample covariance matrix given by $\hat{\Sigma} = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \hat{\theta})(X_i - \hat{\theta})'$. Define again $Y_i = \hat{\Sigma}^{-1/2}(X_i - \hat{\theta})$. When the $X_i$'s are elliptically symmetric, then $Y_i/||Y_i||$ should be uniformly distributed on the unit sphere, and Manzotti et al. (2002) chose this property as the basis of their test. The uniformity of the standardized vectors $Y_i/||Y_i||$ can be checked in different ways. Manzotti et al. (2002) opt to verify this uniformity using spherical harmonics. For a fixed $\varepsilon > 0$, let $\rho_n$ be the $\varepsilon$ sample quantile of $||Y_1||, \ldots, ||Y_n||$. Then, the test statistic is

$$Q_{MPQ}^{(n)} = n \sum_{h \in S_{jl}} \left( \frac{1}{n} \sum_{i=1}^{n} h_i \left( \frac{Y_i}{||Y_i||} \right) I(||Y_i|| > \rho_n) \right)^2$$

for $l \geq j \geq 3$, where $S_{jl} = \bigcup_{j \leq i \leq l} H_i$ and $H_i$ is the set of spherical harmonics of degree $i$ in an orthonormal basis with respect to the uniform probability measure on the unit sphere. In the implementation of this test we used spherical harmonics of degree 3 and 4. By including only spherical harmonics of degree greater than 2, the asymptotic distribution of the test statistic $Q_{MPQ}^{(n)}$ is $(1 - \varepsilon)\chi$, where $\chi$ is a variable with a chi-square distribution with $\nu_{jl}$ degrees of freedom, where $\nu_{jl}$ denotes the total number of functions in $S_{jl}$. The asymptotic distribution does not depend on the parameters $\theta, \Sigma$ and the radial function $f$. However, $Q_{MPQ}^{(n)}$ is only a necessary condition statistic for the null hypothesis of elliptical symmetry and therefore this test does not have asymptotic power against all alternatives. In the *ellipticalsymmetry* package, this test is implemented as the MPQ() function with the following syntax

```r
MPQ(X, epsilon = 0.05)
```

As before, $X$ is a numeric matrix that represents the data while *epsilon* is an option that allows the user to indicate the percentage of points $Y_i$ close to the origin which will not be used in the calculation. By doing this, extra assumptions on the radial density in (1) are avoided. The default value of *epsilon* is set to 0.05.

2.3 Schott’s test

Schott (2002) develops a Wald-type test for elliptical symmetry based on the analysis of covariance matrices. The test compares the sample fourth moments with the expected theoretical ones under ellipticity. Given that the test statistic involves consistent estimates of the covariance matrix of the sample fourth moments, the existence of eight-order moments is required. Furthermore, the test has very low power against several alternatives. The final test statistic is of a simple form, even though it requires lengthy notations.

For an elliptical distribution with mean $\theta$ and covariance matrix $\Sigma$, the fourth moment defined as $M_4 = E[(X - \theta)(X - \theta)' \otimes (X - \theta)(X - \theta)']$, with $\otimes$ the Kronecker product, has the form

$$M_4 = (1 + \kappa)(I_d \otimes K_{dd}) (\Sigma \otimes \Sigma) + vec(\Sigma)vec(\Sigma)'$$

where $K_{dd}$ is a commutation matrix (Magnus, 1988), $I_d$ is the $d \times d$ identity matrix, and $\kappa$ is a scalar which can be expressed using the characteristic function of the elliptical distribution. Here
the vec operator stacks all components of a $d \times d$ matrix $\mathbf{M}$ on top of each other to yield the $d^2$ vector $\text{vec}(\mathbf{M})$. Let $\hat{\Sigma}$ denote the usual unbiased sample covariance matrix and $\tilde{\theta}$ the sample mean. A simple estimator of $\mathbf{M}$ is given by $
abla_4 = \hat{\Sigma}^{-1/2} \hat{\Sigma}_{4}^{-1/2} \hat{\Sigma}_{4}^{-1/2}$ and its standardized version is given by

$$
\hat{\nabla}_4 = \left( \hat{\Sigma}^{-1/2} \otimes \hat{\Sigma}^{-1/2} \right) \hat{\nabla}_4 \left( \hat{\Sigma}^{-1/2} \otimes \hat{\Sigma}^{-1/2} \right).
$$

Then, an estimator of $\text{vec}(\mathbf{M}_4)$ is constructed as $\mathbf{G} = \text{vec}(\mathbf{N}_4)\text{vec}(\mathbf{N}_4')\text{vec}(\hat{\nabla}_4)/(3d(d+2))$, and it is consistent if and only if $\mathbf{M}_4$ is of the form (2); here $\mathbf{N}_4$ represents the value of $\mathbf{M}_4$ under the multivariate standard normal distribution. Note that the asymptotic mean of $\mathbf{v} = n^{1/2}(\text{vec}(\hat{\nabla}_4) - \mathbf{G})$ is 0 if and only if (2) holds and this expression is used to construct the test statistic. Denote the estimate of the asymptotic covariance matrix of $n^{1/2}\mathbf{v}$ as $\hat{\Phi}$. The Wald test statistic is then formalized as $T = \mathbf{v}'\hat{\Phi}^{-1}\mathbf{v}$, where $\hat{\Phi}^{-1}$ is a generalized inverse of $\hat{\Phi}$.

For more technical details we refer the reader to Section 2 in Schott (2002). In order to define Schott’s test statistic, we further have to define the following quantities:

$$
(1 + \kappa) = \frac{1}{nd(d+2)} \sum_{i=1}^{d} \left( (X_i - \tilde{\theta})' \Sigma^{-1} (X_i - \tilde{\theta}) \right)^2,
$$

$$
(1 + \eta) = \frac{1}{nd(d+2)(d+4)} \sum_{i=1}^{d} \left( (X_i - \tilde{\theta})' \Sigma^{-1} (X_i - \tilde{\theta}) \right)^3,
$$

$$
(1 + \omega) = \frac{1}{nd(d+2)(d+4)(d+6)} \sum_{i=1}^{d} \left( (X_i - \tilde{\theta})' \Sigma^{-1} (X_i - \tilde{\theta}) \right)^4.
$$

Moreover, let $\hat{\beta}_1 = (1 + \omega)^{-1}/24$, $\hat{\beta}_2 = -3a\{24(1 + \omega)^2 + 12(d+4)a(1 + \omega)\}^{-1}$, $a = (1 + \omega) + (1 + \kappa)^3 - 2(1 + \kappa)(1 + \eta)$. Finally, the test statistic becomes

$$
T = n \left[ \hat{\beta}_1 \text{tr}(\hat{\nabla}_4^2) + \hat{\beta}_2 \text{vec}(\mathbf{I}_d)' \hat{\nabla}_4^2 \text{vec}(\mathbf{I}_d) - \{3\hat{\beta}_1 + (d+2)\hat{\beta}_2\}d(d+2)(1 + \kappa)^2 \right].
$$

It has an asymptotic chi-squared distribution with degrees of freedom $\nu_d = d^2 + d(d-1)(d^2 + 7d - 6) - 1$.

The Schott test can be performed in our package by using the function `Schott()` with the very simple syntax `Schott(X)`, where X is a numeric matrix of data values.

### 2.4 Test by Huffer and Park

Huffer and Park (2007) propose a Pearson chi-square type test with multi-dimensional cells. Under the null hypothesis of ellipticity the cells have asymptotically equal expected cell counts and after determining the observed cell counts, the test statistic is easily computed. Let $\tilde{\theta}$ be the sample mean and $\hat{\Sigma} = n^{-1} \sum_{i=1}^{n} (X_i - \tilde{\theta}) (X_i - \tilde{\theta})'$ the sample covariance matrix. Define $\mathbf{Y}_i = \mathbf{R}(X_i - \tilde{\theta})$, where the matrix $\mathbf{R} = \mathbf{R}(\hat{\Sigma})$ is a function of $\hat{\Sigma}$ such that $\mathbf{R} \hat{\Sigma} \mathbf{R} = \mathbf{I}_d$. Typically $\mathbf{R} = \hat{\Sigma}^{-1/2}$ as for the previous tests. However, Huffer and Park suggest to use the Gram-Schmidt transformation because that will lead to standardized data whose joint distribution does not depend on $\theta$ or $\Sigma$. In order to compute the test statistic, the space $\mathbb{R}^d$ should be divided into
c spherical shells centered at the origin such that each shell contains an equal number of the scaled residuals $Y_i$. The next step is to divide $\mathbb{R}^d$ into $g$ sectors such that for any pair of sectors there is an orthogonal transformation mapping one onto the other. Therefore, the $c$ shells and $g$ sectors divide $\mathbb{R}^d$ into $gc$ cells which, under elliptical symmetry, should contain $n/(gc)$ of the vectors $Y_i$. The test statistic then has the simple form

$$HP_n = \sum_{\pi} (U_{\pi} - np)^2 / (np),$$

where $U_{\pi}$ are cell counts for $\pi = (i,j)$ with $1 \leq i \leq g$ and $1 \leq j \leq c$ and $p = 1/(gc)$. The asymptotic distribution exists only under normality. It is a linear combination of chi-squared random variables and it depends on eigenvalues of congruent sectors used to divide the space $\mathbb{R}^d$. Some simulation results show that this asymptotic distribution is a good approximation to the true distribution of the test statistic as long as the distribution we are sampling from is close to the normal distribution. Otherwise, bootstrap techniques are required.

In the R package we are considering three particular ways to partition the space: using (i) the $2^d$ orthants, (ii) permutation matrices and (iii) a cyclic group consisting of rotations by angles which are multiples of $2\pi/g$. The first two options can be used for any dimension, while the angles are supported only for dimension 2. Huffer and Park’s test can be run using a function called `HufferPark()`. The syntax, including all options, for the function `HufferPark()` is for instance

```r
HufferPark(X, c, R = NaN, sector = "orthants", g = NaN).
```

We will now provide a detailed description of its arguments. $X$ is an input to this function consisting of a data set. `sectors` is an option that allows the user to specify the type of sectors used to divide the space. Currently supported options are "orthants", "permutation" and "bivariates_angles", the last one being available only in dimension 2. The $g$ argument indicates the number of sectors. The user has to choose $g$ only if `sectors` = "bivariates_angles" and it denotes the number of regions used to divide the plane. In this case, regions consist of points whose angle in polar coordinates is between $2(m-1)\pi/g$ and $2m\pi/g$ for $m \in \{1 \ldots g\}$. If `sectors` is set to "orthants", then $g$ is fixed and equal to $2^d$, while for `sectors` = "permutation" $g$ is $d!$. No matter what type of sectors is chosen, the user has to specify the number of spherical shells that are used to divide the space, which is $c$. The value of $c$ should be such that the average cell counts $n/(gc)$ are not too small. Several settings with different sample size, and different values of $g$ and $c$ can be found in the simulation studies presented in Sections 4 and 5 of Huffer and Park (2007). The asymptotic distribution is available only under `sectors` = "orthants" when the underlying distribution is close to normal. Otherwise, bootstrap procedures are required and the user can freely choose the number of bootstrap replicates, denoted as $R$. Note that by default `sectors` is set to "orthants" and $R = NaN$.

### 2.5 Pseudo-Gaussian test

Cassart (2007) and Cassart et al. (2008) construct Pseudo-Gaussian tests for specified and unspecified location that are most efficient against a multivariate form of Fechner-type asymmetry. This type of asymmetry goes back to univariate families proposed in Fechner (1897), and their
multivariate extension involves densities of the form

\[ x \mapsto c_{d,f_1} \frac{1}{\Sigma^{1/2} f_1 \left( \left( x - \theta \right)^T \Sigma^{-1/2} B \Sigma^{-1/2} (x - \theta) \right)^{1/2}}, x \in \mathbb{R}^d, \]

where \( \theta \in \mathbb{R}^d \) is a location parameter, \( \Sigma \in \mathbb{S}_d \) is a \( d \times d \) scatter matrix, \( c_{d,f_1} \) is a normalization constant, the matrix \( B \) is diagonal with \( (B)_{ij} := (1 - \text{sign}(Z_j) \xi_j) \) for vector \( Z = (Z_1, \ldots, Z_d)' \) and \( \xi = (\xi_1, \ldots, \xi_d)' \in (-1, 1)^d \) is a multivariate skewness parameter. The infinite-dimensional parameter \( f_1 : \mathbb{R}_0^+ \to \mathbb{R}^+ \) is an a.e. strictly positive function.

Pseudo-Gaussian tests are based on Le Cam’s theory of asymptotic experiments. We start by describing the specified-location Pseudo-Gaussian test. The unknown parameter \( \Sigma \) is estimated by using Tyler (1987)'s estimator of scatter which we simply denote by \( \hat{\Sigma} \). Let \( m_k(\theta, \Sigma) := n^{-1} \sum_{i=1}^n \left( \left( \Sigma^{1/2}(X_i - \theta) \right) \right)^k \), \( U_j(\theta, \Sigma) := \frac{\Sigma^{1/2}(X_j - \theta)}{\| \Sigma^{1/2}(X_j - \theta) \|} \) and

\[ S^U_j(\theta, \Sigma) := ((U_{i1}(\theta, \Sigma))^2 \text{sign}(U_{i1}(\theta, \Sigma)), \ldots, (U_{id}(\theta, \Sigma))^2 \text{sign}(U_{id}(\theta, \Sigma)))' \]

The test statistic then has the simple form

\[ Q_{p,\theta}^{(n)} = \frac{d(d+2)}{3nm_4(\theta, \Sigma)} \sum_{i,j=1}^n \left( \| \Sigma^{1/2}(X_i - \theta) \| \right)^2 \| \Sigma^{1/2}(X_j - \theta) \| )^2 S_i^U(\theta, \Sigma) S_j^U(\theta, \Sigma) \]

and follows asymptotically a chi-squared distribution \( \chi^2_d \) with \( d \) degrees of freedom. Finite moments of order four are required.

In most cases the assumption of a specified center is however unrealistic. Cassart (2007) therefore proposes also a test for the scenario when location is not specified. The estimator of the unknown \( \theta \) is the sample mean denoted by \( \hat{\theta} \). Let \( Y_i = \Sigma^{-1/2}(X_i - \hat{\theta}) \). The test statistic takes on the guise

\[ Q_{p,\theta}^{(n)} := (\Delta g(\hat{\theta}, \Sigma))' (\Gamma g(\hat{\theta}, \Sigma))^{-1} \Delta g(\hat{\theta}, \Sigma), \]

where

\[ \Delta g(\hat{\theta}, \Sigma) = n^{-1/2} \sum_{i=1}^n \| Y_i \| \left( c_d(d+1)m_1(\hat{\theta}, \Sigma) U_i(\hat{\theta}, \Sigma) - \| Y_i \| S_i^U(\hat{\theta}, \Sigma) \right) \]

and

\[ \Gamma g(\hat{\theta}, \Sigma) := \left( \frac{3}{d(d+2)} m_4(\hat{\theta}, \Sigma) - 2c_d^3(d+1)m_1(\hat{\theta}, \Sigma)m_3(\hat{\theta}, \Sigma) + c_d^3(d+1)^2m_1(\hat{\theta}, \Sigma)^2 m_2(\hat{\theta}, \Sigma) \right) I_d \]

with \( c_d = 4\Gamma(d/2)/((d^2 - 1)\sqrt{\pi\Gamma(d/2)}) \), \( \Gamma(\cdot) \) being the Gamma function. The test rejects the null hypothesis of elliptical symmetry at asymptotic level \( \alpha \) whenever the test statistic \( Q_{p,\theta}^{(n)} \) exceeds \( \chi^2_{d,1-\alpha} \), the upper \( \alpha \)-quantile of a \( \chi^2_d \) distribution. We refer to Chapter 3 of Cassart (2007) for formal details.

This test can be run in our package by calling the function \texttt{pseudoGaussian()} with the simple syntax

\[ \texttt{pseudoGaussian(X, location = NaN)}. \]

Besides \( X \) which is a numeric matrix of data values, now we have an extra argument location
which allows the user to specify the known location. The default is set to NaN which means that the unspecified location test will be performed unless the user specifies location.

2.6 SkewOptimal test

Recently, Babić et al. (2019) proposed a new test for elliptical symmetry both for specified and unspecified location. These tests are based on Le Cam’s theory of asymptotic experiments and are optimal against generalized skew-elliptical alternatives, but they remain quite powerful under a much broader class of non-elliptical distributions. Generalized skew-elliptical distributions defined by Genton and Loperfido (2005) have densities of the form

\[ x \mapsto 2 c_{d,f}|\Sigma|^{-1/2}f(\|\Sigma^{-1/2}(x - \theta)\|)\Pi(\delta^T\Sigma^{-1/2}(x - \theta)), \ x \in \mathbb{R}^d, \tag{3} \]

where \( \theta, \Sigma, c_{d,f}, \) and \( f \) are defined as in (1), the skewing function \( \Pi \) has values in \([0, 1]\) and satisfies \( \Pi(-r) = 1 - \Pi(r) \) for \( r \in \mathbb{R} \), and \( \delta \in \mathbb{R}^d \) plays the role of a skewness parameter. The density (3) is obtained by perturbing or modulating symmetry in elliptical distributions via multiplication with a skewing function \( \Pi(\cdot) \). Typical choices for \( \Pi \) are univariate distribution functions with symmetric densities, such as the normal or Student ones; see the monograph Genton (2004) for a detailed exposition.

The test statistic for the specified location scenario has a very simple form and an asymptotic chi-squared distribution. For specified location, the test rejects the null hypothesis whenever \( Q_{\theta}^{(n)} = n(\hat{X} - \theta)^T\Sigma^{-1}(\hat{X} - \theta) \) exceeds the \( \alpha \)-upper quantile \( \chi^2_{d,1-\alpha} \). Here, \( \Sigma \) is Tyler (1987)’s estimator of scatter and \( \hat{X} \) is the sample mean.

When the location is not specified, Babić et al. (2019) propose tests that have a simple asymptotic chi-squared distribution under the null hypothesis of ellipticity, are affine-invariant, computationally fast, have a simple and intuitive form, only require finite moments of order 2, and offer much flexibility in the choice of the radial density \( f \) at which optimality (in the maximin sense) is achieved. Note that the Gaussian \( f \) is excluded, though, due to a singular information matrix; see Babić et al. (2019).

We implemented in our package the test statistic based on the radial density \( f \) of the multivariate \( t \) distribution, multivariate power-exponential and multivariate logistic, though in principle any non-Gaussian choice for \( f \) is possible. The test requires lengthy notations, but its implementation is straightforward. For the sake of generality, we will derive the test statistic for a general (but fixed) \( f \), and later on provide the expressions of \( f \) for the three special cases implemented in our package. Let \( \varphi_f(x) = \frac{-f(x)}{f(x)^2} \) and \( Y_i = \Sigma^{-1/2}(X_i - \hat{\theta}) \) where \( \hat{\theta} \) is the sample mean. In order to construct the test statistic, we first have to define the quantities

\[ \Delta_f(\hat{\theta}, \hat{\Sigma}) = 2n^{-1/2}\Pi(0)\sum_{i=1}^n \left[ \frac{d}{\mathcal{K}_{d,f}(\hat{\theta}, \hat{\Sigma})} \varphi_f(\|Y_i\|) \right] \frac{Y_i}{\|Y_i\|} \]

and

\[ \tilde{\Gamma}_f(\hat{\theta}, \hat{\Sigma}) := \frac{4(\Pi(0))^2}{nd} \sum_{i=1}^n \left[ \frac{d}{\mathcal{K}_{d,f}(\hat{\theta}, \hat{\Sigma})} \varphi_f(\|Y_i\|) \right]^2 \mathbf{I}_d \]

where \( \mathcal{K}_{d,f}(\hat{\theta}, \hat{\Sigma}) := \frac{1}{n} \sum_{i=1}^n \left[ \varphi_f'(\|Y_i\|) + \frac{d-1}{\|Y_i\|^2} \varphi_f(\|Y_i\|) \right] \) and \( \Pi \) is the cdf of the standard
normal distribution (we use \( \dot{\Pi}(\cdot) \) for the derivative). Finally, the test statistic is of the form 
\[
Q_f^{(n)} := (\Delta_f(\dot{\theta}, \dot{\Sigma}))'(\dot{\Gamma}_f(\dot{\theta}, \dot{\Sigma}))^{-1}\Delta_f(\dot{\theta}, \dot{\Sigma})
\]
and it has a chi-squared distribution with \( d \) degrees of freedom. The test is valid under the entire semiparametric hypothesis of elliptical symmetry with unspecified center and uniformly optimal against any type of generalized skew-\( f \) alternative.

From this general expression, one can readily derive the test statistics for specific choices of \( f \). In our case, the radial density of the multivariate Student \( t \) distribution corresponds to 
\[
f(x) = (1 + \frac{1}{\nu} x^2)^{-(\nu+d)/2}, \quad \text{where} \quad \nu \in (0, \infty)\]
represents the degrees of freedom, while that of the multivariate logistic distribution is given by 
\[
f(x) = \exp\left(-x^2\right) \left[1 + \exp(-x^2)\right]^{-2}, \quad \text{where} \quad \beta \in (0, \infty)\]
is a parameter related to kurtosis. These tests can be run in R using a function called \texttt{SkewOptimal()} with the syntax

\[
\text{SkewOptimal}(X, \text{location} = \text{NaN}, f = "t", \text{param} = \text{NaN})
\]

Depending on the type of the test some of the input arguments are not required. \( X \) and \textbf{location} are the only input arguments for the specified location test, and have the same role as for the Pseudo-Gaussian test. As before, the default value for \textbf{location} is set to \texttt{NaN} which implies that the unspecified location test will be performed unless the user specifies location. For the unspecified location test, besides the data matrix \( X \), the input arguments are \( f \) and \textbf{param}. The \textbf{f} argument is a string that specifies the type of the radial density based on which the test is built. Currently supported options are "\( t \)", "logistic" and "\text{powerExp}". Note that the default is set to \( t \). The role of the \textbf{param} argument is as follows. If \( f = "t" \) then \textbf{param} denotes the degrees of freedom of the multivariate \( t \) distribution. Given that the default radial density is "\( t \)", it follows that the default value of \textbf{param} represents the degrees of freedom of the multivariate \( t \) distribution and it is set to \( 4 \). Note also that the degrees of freedom have to be greater than \( 2 \). If \( f = "\text{powerExp}" \) then \textbf{param} denotes the kurtosis parameter \( \beta \), in which case the value of \textbf{param} has to be different from \( 1 \) because \( \beta = 1 \) corresponds to the multivariate normal distribution. The default value is set to \( 0.5 \).

### 2.7 Time complexity

We conclude the description of tests for elliptical symmetry by comparing their time complexity in terms of the big O notation (Cormen et al., 2009). More concretely, we are comparing the number of simple operations that are required to evaluate the test statistics and the p-values. Table 1 summarizes the time complexity of the implemented tests.

The test of Koltchinskii and Sakhanenko is computationally more demanding than the bootstrap version of the test of Huffer and Park. Among unspecified location tests that do not require bootstrap procedures, the most computationally expensive test is the MPQ test under the realistic assumption that \( n > d \). Regarding the specified location tests we can conclude that the Pseudo-Gaussian test is more computationally demanding than the SkewOptimal test. Note that both the test of Koltchinskii and Sakhanenko and the MPQ test are based on spherical harmonics up to degree 4. In case we would use spherical harmonics of higher degrees, the tests would of course become even more computationally demanding.

We have seen that several tests require bootstrap procedures and therefore are by default computationally demanding. Such tests require the calculation of the statistic on the resampled
data \( R \) times in order to get the p-value, where \( R \) is the number of bootstrap replicates. Consequently, the time required to obtain the p-value in such cases is \( R \) times the time to calculate the test statistic. For the tests that do not involve bootstrap procedures, the p-value is calculated using the inverse of the cdf of the asymptotic distribution under the null hypothesis, which is considered as one simple operation. The exception here is the test of Huffer and Park whose asymptotic distribution is more complicated and includes \( O(c) \) operations where \( c \) is an integer and represents an input parameter for this test.

| Table 1: Time complexity of the various tests for elliptical symmetry |
|---------------------------------------------------------------|
| statistics | p-value |
| KoltchinskiiSakhanenko | \( O(n \log n + nd^5) \) | \( O(Rn \log n + Rnd^5) \) |
| MPQ | \( O(n \log n + nd^5) \) | \( O(1) \) |
| Schott | \( O(nd^2 + d^5) \) | \( O(1) \) |
| HufferPark | \( O(nd^2 + d^5) \) | \( O(c) \) |
| HufferPark (bootstrap) | \( O(nd^2 + d^5) \) | \( O(Rnd^2 + Rd^3) \) |
| PseudoGaussian (specified location) | \( O(n^2d + nd^2 + d^5) \) | \( O(1) \) |
| PseudoGaussian | \( O(nd^2 + d^5) \) | \( O(1) \) |
| SkewOptimal (specified location) | \( O(nd + d^3) \) | \( O(1) \) |
| SkewOptimal | \( O(nd^2 + d^3) \) | \( O(1) \) |

3 Comparative finite-sample study

In this section, we present simulation results describing the behavior of the described tests for elliptical symmetry in finite samples. In order to test the accuracy of the limiting distributions and bootstrap procedures we consider several elliptically symmetric distributions as null hypotheses: the multivariate normal (MVN), the multivariate \( t \) distribution with \( \nu = 5 \) and 9 degrees of freedom (\( t_\nu \)) and the multivariate generalized Laplace distribution (MVL(\( \lambda \))) suggested by Ernst (1998). Its density function is given by

\[
x \mapsto \frac{\lambda^{d/2}}{2\pi^{d/4}(d/\lambda)^{1/2}} \exp\left\{-\left[(x - \mu)'\Sigma^{-1}(x - \mu)\right]^{\lambda/2}\right\},
\]

where \( \mu \in \mathbb{R}^d \) is a location parameter, \( \Sigma \in S_d \) is a \( d \times d \) scatter matrix, and \( \lambda \in \mathbb{R}^d \) is a shape parameter. By varying \( \lambda \) we get distributions with different tail behaviour. For instance, \( \lambda = 2 \) corresponds to the multivariate normal distribution. For our simulation study we choose \( \lambda = 1 \) and 5. We throughout consider the three-dimensional case, and for all distributions location is \( \mu = 0 \) and scatter is \( \Sigma = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 5 \end{bmatrix} \). The power of these tests is investigated by considering a variety of non-elliptical distributions. We opted to include not just distributions that deviate significantly from elliptical symmetry, but also distributions that are still symmetric though not elliptically. Our alternatives are:

- normal and \( t \) skew-elliptical with increasing skewness values \( \delta \).
- multivariate sinh-arcsinh-normal distributions of Jones and Pewsey (2009) with increasing skewness values and kurtosis parameters fixed to 1.
• mixtures of Gaussian distributions of the form $\frac{1}{2}N_3(\mu_1, \Sigma_1) + \frac{1}{2}N_3(\mu_2, \Sigma_2)$, with various locations $\mu_1$ and $\mu_2$ and scatter matrices $\Sigma_1 = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 5 \end{bmatrix}$ and $\Sigma_2 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix}$, respectively.

• an alternative studied in Sakhanenko (2008). It is a three-dimensional distribution whose first coordinate $X_1$ has the Gamma distribution with shape and scale parameters 2 and 3, respectively, and the remaining coordinates are independent of $X_1$ and have a bivariate standard normal distribution. We denote it as $\Gamma + N_2$.

• an alternative studied in Huffer and Park (2007) which we denote as $I \times N_3$. It is a three-dimensional distribution with density $f(x_1, x_2, x_3) = \frac{2}{(2\pi)^{3/2}} \exp\left(-\frac{r^2}{2}\right)I(x_1 x_2 x_3(r^2 - m) < 0)$, where $r^2 = x_1^2 + x_2^2 + x_3^2$ and $m$ is the median of the $\chi^2_3$ distribution. Note that this alternative is centrally symmetric.

For each of these distributions, we carried out simulations for dimension $d = 3$ and sample size $n = 100$. In each simulation, we generated $N = 3000$ samples. In cases when the bootstrap procedures were required, the number of bootstrap replicates was $R = 1000$. Location and scatter parameters were fixed. When using Huffer and Park’s test we set $c = 5$ for two versions of the test: orthants and permutation. When computing SkewOptimal tests we consider all three options: the test based on the radial density of the multivariate $t$ distribution with 4 degrees of freedom, the test based on the radial density of the power-exponential distribution with kurtosis parameter set to 0.5 and the test based on the multivariate logistic distribution.

Table 2: The levels of the tests under various three-dimensional elliptical distributions

| Test                        | MVN | $t_5$ | $t_9$ | MVL(1) | MVL(5) |
|-----------------------------|-----|-------|-------|--------|--------|
| KoltchinskiiSakhanenko      | 0.042 | 0.075 | 0.056 | 0.054  | 0.059  |
| MPQ                         | 0.054 | 0.060 | 0.050 | 0.052  | 0.051  |
| Schott                      | 0.038 | 0.039 | 0.047 | 0.042  | 0.055  |
| HufferPark-orthants         | 0.038 | 0.063 | 0.045 | 0.047  | 0.112  |
| HufferPark-permutation      | 0.045 | 0.049 | 0.049 | 0.052  | 0.109  |
| Pseudo-Gaussian             | 0.052 | 0.031 | 0.043 | 0.044  | 0.057  |
| SkewOptimal-t              | 0.049 | 0.035 | 0.043 | 0.040  | 0.053  |
| SkewOptimal-powerExp       | 0.048 | 0.042 | 0.040 | 0.044  | 0.050  |
| SkewOptimal-logistic       | 0.044 | 0.044 | 0.045 | 0.035  | 0.048  |

The results under the null hypothesis are summarized in Table 2. The test by Koltchinskii and Sakhanenko is close to its limiting distribution except for the multivariate $t$ distribution. MPQ respects the nominal level constraint very well. We see that Schott’s test can be rather conservative for some distributions that we study. Huffer and Park’s test based on permutations preserves the level slightly better than Huffer and Park’s test based on orthants. They both have problems with the MVL(5). The Pseudo-Gaussian and all SkewOptimal tests overall exhibit a similarly good performance, but probably due to heavy tails the Pseudo-Gaussian and SkewOptimal based on the radial density of the multivariate $t$ distribution also have small finite-sample difficulties.
with the multivariate $t_5$. The SkewOptimal test based on the radial density of the multivariate logistic distribution has difficulties with the MVL(1).

Table 3: The power of the tests under various three-dimensional skew-elliptical distributions

| Test                   | $\delta = (2, 0, 0)$ | $\delta = (1, -0.5, 1.5)$ | $\delta = (2, -1, 3)$ | $\delta = (3, -1.5, 4.5)$ |
|------------------------|-----------------------|---------------------------|-----------------------|---------------------------|
|                        | skew-normal           |                           |                       |                           |
| Koltchinskii Sakanenko | 0.066                 | 0.070                     | 0.137                 | 0.169                     |
| MPQ                    | 0.062                 | 0.077                     | 0.127                 | 0.153                     |
| Schott                 | 0.038                 | 0.032                     | 0.041                 | 0.042                     |
| HufferPark-orthants    | 0.057                 | 0.070                     | 0.143                 | 0.186                     |
| HufferPark-permutation | 0.065                 | 0.072                     | 0.125                 | 0.162                     |
| Pseudo-Gaussian        | 0.158                 | 0.096                     | 0.195                 | 0.235                     |
| SkewOptimal-t          | 0.140                 | 0.181                     | 0.450                 | 0.558                     |
| SkewOptimal-powerExp   | 0.111                 | 0.142                     | 0.377                 | 0.460                     |
| SkewOptimal-logistic   | 0.098                 | 0.127                     | 0.331                 | 0.424                     |
|                        | skew-$t_5$             |                           |                       |                           |
| Koltchinskii Sakanenko | 0.158                 | 0.184                     | 0.299                 | 0.355                     |
| MPQ                    | 0.079                 | 0.083                     | 0.118                 | 0.128                     |
| Schott                 | 0.042                 | 0.044                     | 0.047                 | 0.049                     |
| HufferPark-orthants    | 0.207                 | 0.229                     | 0.416                 | 0.498                     |
| HufferPark-permutation | 0.192                 | 0.207                     | 0.360                 | 0.402                     |
| Pseudo-Gaussian        | 0.267                 | 0.237                     | 0.346                 | 0.369                     |
| SkewOptimal-t          | 0.527                 | 0.581                     | 0.823                 | 0.864                     |
| SkewOptimal-powerExp   | 0.528                 | 0.578                     | 0.823                 | 0.862                     |
| SkewOptimal-logistic   | 0.551                 | 0.622                     | 0.874                 | 0.915                     |
|                        | skew-$t_9$             |                           |                       |                           |
| Koltchinskii Sakanenko | 0.097                 | 0.092                     | 0.186                 | 0.225                     |
| MPQ                    | 0.063                 | 0.058                     | 0.106                 | 0.119                     |
| Schott                 | 0.034                 | 0.033                     | 0.038                 | 0.045                     |
| HufferPark-orthants    | 0.120                 | 0.126                     | 0.271                 | 0.322                     |
| HufferPark-permutation | 0.109                 | 0.118                     | 0.221                 | 0.267                     |
| Pseudo-Gaussian        | 0.243                 | 0.198                     | 0.324                 | 0.354                     |
| SkewOptimal-t          | 0.395                 | 0.444                     | 0.726                 | 0.799                     |
| SkewOptimal-powerExp   | 0.347                 | 0.392                     | 0.671                 | 0.744                     |
| SkewOptimal-logistic   | 0.326                 | 0.373                     | 0.685                 | 0.763                     |
Table 4: The power of the tests under skewed SAS-normal, Gaussian Mixtures, $\Gamma + N_2$ and $I \times N_3$. The last two alternatives are suggested and studied in Sakhanenko (2008) and Huffer and Park (2007).

| Test                        | $\delta = (0.3, 0, 0)$ | $\delta = (0.15, -0.1, 0.2)$ | $\delta = (0.3, -0.2, 0.4)$ | $\delta = (0.45, -0.3, 0.6)$ |
|-----------------------------|------------------------|-------------------------------|-------------------------------|-------------------------------|
| Koltchinskii-Sakhanenko     | 0.098                  | 0.080                         | 0.193                         | 0.423                         |
| MPQ                         | 0.098                  | 0.075                         | 0.185                         | 0.382                         |
| Schott                      | 0.043                  | 0.040                         | 0.048                         | 0.065                         |
| HufferPark-orthants         | 0.082                  | 0.075                         | 0.180                         | 0.395                         |
| HufferPark-permutation      | 0.077                  | 0.077                         | 0.165                         | 0.326                         |
| Pseudo-Gaussian             | 0.416                  | 0.297                         | 0.842                         | 0.991                         |
| SkewOptimal-t               | 0.195                  | 0.164                         | 0.558                         | 0.876                         |
| SkewOptimal-powerExp        | 0.159                  | 0.137                         | 0.468                         | 0.801                         |
| SkewOptimal-logistic        | 0.149                  | 0.131                         | 0.436                         | 0.768                         |

| Gaussian mixtures           | $\mu'_1$               | $\mu'_2$                      | $\nu'_1$                      | $\nu'_2$                      |
|-----------------------------|------------------------|-------------------------------|-------------------------------|-------------------------------|
| Koltchinskii-Sakhanenko     | 0.134                  | 0.304                         | 0.699                         | 0.956                         |
| MPQ                         | 0.061                  | 0.168                         | 0.391                         | 0.604                         |
| Schott                      | 0.124                  | 0.124                         | 0.192                         | 0.342                         |
| HufferPark-orthants         | 0.084                  | 0.225                         | 0.626                         | 0.872                         |
| HufferPark-permutation      | 0.081                  | 0.177                         | 0.488                         | 0.676                         |
| Pseudo-Gaussian             | 0.043                  | 0.130                         | 0.192                         | 0.153                         |
| SkewOptimal-t               | 0.052                  | 0.394                         | 0.879                         | 0.956                         |
| SkewOptimal-powerExp        | 0.054                  | 0.372                         | 0.870                         | 0.968                         |
| SkewOptimal-logistic        | 0.049                  | 0.413                         | 0.919                         | 0.986                         |

The power results for alternatives are summarized in Tables 3 and 4. The SkewOptimal tests outperform the other tests for almost all settings. The Pseudo-Gaussian test performs also very well and outperforms the SkewOptimal tests only for the SAS-normal distribution for all values of $\delta$. The Pseudo-Gaussian and SkewOptimal tests have no power when it comes to the alternative studied in Huffer and Park (2007), the $I \times N_3$ distribution. In general Schott’s test has
very poor power. The Koltchinskii-Sakhanenko, MPQ and HufferPark tests possess poor power against the skew-normal, while their power increases for the skew-$t_9$ and especially for the skew-$t_5$ (except for MPQ). They have moderate power against SAS-normal alternatives compared to the power of the Pseudo-Gaussian and SkewOptimal tests. All tests except the pseudo-Gaussian and Schott’s have very good power against Gaussian mixtures. The SkewOptimal and Pseudo-Gaussian tests have excellent power against $\Gamma + N_2$, the Koltchinskii-Sakhanenko, MPQ and HufferPark tests moderate power, while Schott’s test has rather low power. Finally, only the Koltchinskii-Sakhanenko and HufferPark tests exhibit power against the $I \times N_3$ distribution. The other tests have no power against this centrally symmetric alternative.

4 Illustrations using the AIS data set

In this section we illustrate the usage of our R package ellipticalsymmetry by analyzing a data set containing measurements from high-performance athletes from the Australian Institute of Sport (ais). The data consist of 202 athletes (102 males, 100 females) with, each, 13 variables. Two of them are categorical (sex and sport) while height, weight, lean body mass, red cell count, white cell count, hematocrit, hemoglobin, plasma ferritin concentration, body mass index, sum of skin folds and percent body fat are numeric. The data set is available within the package sn of the R environment. Telford and Cunningham (1991) provide more information on how the data were collected.

We want to test for elliptical symmetry without specifying the center of symmetry. We opt to focus on a 2-dimensional set which allows us to make some visual inspection of our data. We decide to investigate body fat percentage (Bfat) and weight (Wt). First we use all measurements for these two variables and then we split the data set in two according to sex. We perform a 2D kernel density estimation using the function kde2d() from the MASS package and display the results with contours in Figure 1, together with the scatter plot of Body fat percentage and Weight.

The user can obtain the data via

```R
> library(ellipticalsymmetry)
> library(sn)
> data(ais)
> X = ais

We define a new data set Y as the columns that contain Bfat and Wt observations from the data set X.

> Y = cbind(X$Bfat,X$Wt)```
Panels (a) and (b) show the scatter plot and contour lines for female and male athletes, respectively, while panel (c) shows the scatter plot and contour lines for all athletes.

We now make use of the functions available in the `ellipticalsymmetry` package. We start with the test by Koltchinskii and Sakhanenko.

```r
> KoltchinskiiSakhanenko(Y)
Test for elliptical symmetry by Koltchinskii and Sakhanenko
data: Y
statistic = 5.6481, p-value < 2.2e-16
alternative hypothesis: the distribution is not elliptically symmetric
```

The `KoltchinskiiSakhanenko()` output is simple and clear. It reports the value of test statistic
and p-value. For this particular data set the test statistic is equal to 5.6481 and the p-value is very close to zero, indicating a rejection of the null hypothesis of elliptical symmetry at virtually any level of significance. Note that here we did not specify a value of $R$, so the default value for bootstrap replicates was used.

The MPQ test and Schott’s test can be performed by running very simple commands:

```r
> MPQ(Y)

Test for elliptical symmetry by Manzotti et al.

data: Y
statistic = 17.187, p-value = 0.001185
alternative hypothesis: the distribution is not elliptically symmetric

> Schott(Y)

Schott test for elliptical symmetry

data: Y
statistic = 12.246, p-value = 0.01561
alternative hypothesis: the distribution is not elliptically symmetric
```

Given the number of the input arguments, the function for the test by Huffer and Park deserves some further comments. The non-bootstrap version of the test can be performed by running the command

```r
> HufferPark(Y, c = 4)

Test for elliptical symmetry by Huffer and Park

data: Y
statistic = 32.139, p-value = 0.000242
alternative hypothesis: the distribution is not elliptically symmetric
```

By specifying $R$ the bootstrap will be applied:

```r
> HufferPark(Y, c = 4, R = 1000)

The p-value for the bootstrap version of the test is equal to 0.008. Note that in both cases we used the default value for sector, that is "orthants".

Another very easy-to-use test is the Pseudo-Gaussian test:

```r
> PseudoGaussian(Y)

Pseudo-Gaussian test for elliptical symmetry

data: Y
statistic = 12.809, p-value = 0.001654
alternative hypothesis: the distribution is not elliptically symmetric
```
By running the following simple command the SkewOptimal test based on the radial density of the multivariate $t$ distribution with 4 degrees of freedom will be performed (note that the degrees of freedom could be readily changed by specifying the param argument).

```r
> SkewOptimal(Y)
```

SkewOptimal test for elliptical symmetry

data: Y
statistic = 6.2742, p-value = 0.04341
alternative hypothesis: the distribution is not elliptically symmetric

The test based on the radial density of the multivariate logistic distribution can be performed by simply adding `f = "logistic"`:

```r
> SkewOptimal(Y, f = "logistic")
```

This version of the SkewOptimal test yields a p-value equal to 0.03362. Finally, if we want to run the test based on the radial density of the multivariate power-exponential distribution, we have to set `f` to "powerExp". The kurtosis parameter equal to 0.5 will be used unless specified otherwise.

```r
> SkewOptimal(Y, f = "powerExp")
```

The resulting p-value equals 0.03768.

We can conclude that when we use all observations, all tests reject the null hypothesis of elliptical symmetry at the 5% level. We do not report p-values for the male and female observations, but the conclusions are as follows:

(i) When we use measurements of male athletes the null hypothesis is rejected at the 5% level by all tests except by the bootstrap version of the test by Huffer and Park.

(ii) When we use measurements of female athletes the ellipticity is rejected at the 5% level only by the test of Koltchinskii and Sakhanenko.

This clearly shows a difference between male and female athletes, difference that is somehow visible from the plots in Figure 1 but of course needs formal determination as done with the various tests here. Luckily they mostly agree. In general, in situations of discordance between two (or more) tests, a practitioner may compare the essence of the tests as described in this paper and check if, perhaps, one test is more suitable for the data at hand than the other (e.g., if assumptions are not met). The freedom of choice among several tests for elliptical symmetry is an additional feature of our new package.

5 Conclusion

In this paper, we have described several existing tests for elliptical symmetry and explained in details their R implementation in our new package ellipticalsymmetry. The implemented functions are simple to use, and we illustrate this via a real data analysis. In order to give the user an idea about which tests are performing better in which situation, we additionally performed a small simulation study that we presented in this paper. Together with the properties of each test, this should give a guidance as to which test to use when. The availability of several tests for elliptical symmetry is clearly an appealing strength of our new package.
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