Free Brownian motion and evolution towards □-infinite divisibility for k-tuples

Serban T. Belinschi Alexandru Nica *

Abstract

Let \( D_c(k) \) be the space of (non-commutative) distributions of \( k \)-tuples of selfadjoint elements in a \( C^* \)-probability space. For every \( t \geq 0 \) we consider the transformation \( B_t : D_c(k) \to D_c(k) \) defined by

\[
B_t(\mu) = \left( \mu^{\boxplus(1+t)} \right)^{(1)/(1+t)}, \quad \mu \in D_c(k),
\]

where \( \boxplus \) and \( \boxdot \) are the operations of free additive convolution and respectively of Boolean convolution on \( D_c(k) \). We prove that \( B_s \circ B_t = B_{s+t} \), \( \forall s, t \geq 0 \). For \( t = 1 \) we prove that \( B_1(D_c(k)) \) is precisely the set \( D_c^{\text{inf-div}}(k) \) of distributions in \( D_c(k) \) which are infinitely divisible with respect to \( \boxplus \), and that the map \( D_c(\mu) \ni \mu \mapsto B_1(\mu) \in D_c(\mu)^{\text{inf-div}} \) coincides with the multi-variable Boolean Bercovici-Pata bijection put into evidence in our previous paper \([1]\). Thus for a fixed \( \mu \in D_c(k) \), the process \{\( B_t(\mu) \mid t \geq 0 \} \) can be viewed as some kind of “evolution towards □-infinite divisibility”.

On the other hand we put into evidence a relation between the transformations \( B_t \) and free Brownian motion. More precisely, we introduce a map \( \Phi : D_c(\mu) \to D_c(\mu) \) which transforms the free Brownian motion started at an arbitrary \( \nu \in D_c(k) \) into the process \{\( B_t(\mu) \mid t \geq 0 \} \) for \( \mu = \Phi(\nu) \).

1. Introduction

1.1 Review of past work. The study of noncommutative forms of independence for random variables has led to several “convolution operations” that can be defined on the space \( \mathcal{M} \) of probability distributions on \( \mathbb{R} \). Two such operations are the free (additive) convolution \( \boxplus \) and the Boolean convolution \( \boxdot \), which reflect the operations of addition of freely independent and respectively Boolean independent random variables.

In the paper \([2]\) we introduced a family \{\( B_t \mid t \geq 0 \} \) of transformations of \( \mathcal{M} \), defined by the formula

\[
B_t(\mu) = \left( \mu^{\boxplus(1+t)} \right)^{(1)/(1+t)} , \quad \forall t \geq 0 , \; \forall \mu \in \mathcal{M}.
\]

The transformations \( B_t \) turn out to form a semigroup: \( B_s \circ B_t = B_{s+t} \), \( \forall s, t \geq 0 \). On the other hand for \( t = 1 \) it turns out that the range set \( B_1(\mathcal{M}) \) is precisely the set \( \mathcal{M}^{\text{inf-div}} \) of distributions in \( \mathcal{M} \) which are infinitely divisible with respect to \( \boxplus \); and moreover, the map \( \mathcal{M} \ni \mu \mapsto B_1(\mu) \in \mathcal{M}^{\text{inf-div}} \) coincides with a remarkable bijection discovered by Bercovici

*Research supported by a Discovery Grant of NSERC, Canada.
and Pata [3] in their study of relations between infinite divisibility with respect to $\boxplus$ and to $\oplus$.

Due to the above properties of the transformations $\mathbb{B}_t$, for a fixed $\mu \in \mathcal{M}$ the process $t \mapsto \mathbb{B}_t(\mu)$ can be viewed as a kind of “evolution towards $\boxplus$-infinite divisibility” (where infinite divisibility is always reached by the time $t = 1$). In [2] it was observed that this process is related to free Brownian motion. Recall that the free Brownian motion started at $\nu \in \mathcal{M}$ is the process $\{\nu \boxplus \gamma_t \mid t \geq 0\}$, where $\gamma_t \in \mathcal{M}$ is the centered semicircular distribution of variance $t$. The connection between this and the transformations $\mathbb{B}_t$ is described as follows.

For a distribution $\mu \in \mathcal{M}$ let $G_\mu$ and $F_\mu$ denote the Cauchy transform and respectively the reciprocal Cauchy transform of $\mu$; that is, we have

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(s)}{z - s}, \quad \text{and} \quad F_\mu(z) = 1/G_\mu(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$  

By using basic facts from the theory of the Cauchy transform, one easily sees that for every distribution $\nu \in \mathcal{M}$ there exists a unique $\mu \in \mathcal{M}$ such that

$$F_\mu(z) = z - G_\nu(z), \quad \forall \ z \in \mathbb{C} \setminus \mathbb{R}. \quad (1.2)$$

One can thus define a map $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ by putting $\Phi(\nu) := \mu$ with $\mu$ and $\nu$ related as in (1.2). The map $\Phi$ turns out to be one-to-one, with image $\Phi(\mathcal{M})$ consisting precisely of those distributions $\mu \in \mathcal{M}$ which have $\int_{-\infty}^{\infty} t^2 \, d\mu(t) = 1$ and $\int_{-\infty}^{\infty} t \, d\mu(t) = 0$. (A detailed presentation of these facts appears in Section 2 of the paper [4].) The relation between the transformations $\mathbb{B}_t$ and free Brownian motion can be expressed by using the map $\Phi$ and the following formula:

$$\Phi(\nu \boxplus \gamma_t) = \mathbb{B}_t(\Phi(\nu)), \quad \forall \nu \in \mathcal{M}, \ \forall \ t > 0. \quad (1.3)$$

In other words the free Brownian motion started at $\nu$ corresponds exactly, via $\Phi$, to the process $\{\mathbb{B}_t(\mu) \mid t \geq 0\}$ started at $\mu = \Phi(\nu)$.

### 1.2 Description of results of this paper

In this paper we find multi-variable analogues for the results described above. Let $k$ be a positive integer, and let $\mathcal{D}_c(k)$ denote the space of non-commutative distributions of $k$-tuples of selfadjoint elements in a $C^*$-probability space. The convolution operations $\boxplus$ and $\oplus$ can be defined on $\mathcal{D}_c(k)$, and for every $\mu \in \mathcal{D}_c(k)$ it makes sense to define convolution powers $\mu^{\boxplus p}, \forall p \geq 1$ and $\mu^{\oplus q}, \forall q > 0$. One can thus define a family $\{\mathbb{B}_t(\mu) \mid t \geq 0\}$ of transformations of $\mathcal{D}_c(k)$ by exactly the same formula as in (1.1):

$$\mathbb{B}_t(\mu) = \left(\mu^{\boxplus(1+t)}\right)^{\omega(1/(1+t))}, \quad \forall t \geq 0, \ \forall \mu \in \mathcal{D}_c(k). \quad (1.4)$$

We prove that $\mathbb{B}_s \circ \mathbb{B}_t = \mathbb{B}_{s+t}, \forall s, t \geq 0$. For $t = 1$ we prove that $\mathbb{B}_1(\mathcal{D}_c(k))$ is precisely the set $\mathcal{D}_c^{\text{inf-div}}(k)$ of distributions in $\mathcal{D}_c(k)$ which are infinitely divisible with respect to $\boxplus$, and that the map $\mathcal{D}_c(k) \ni \mu \mapsto \mathbb{B}_1(\mu) \in \mathcal{D}_c(k)^{\text{inf-div}}$ coincides with the multi-variable Boolean Bercovici-Pata bijection put into evidence in our previous paper [1]. Thus for a fixed $\mu \in \mathcal{D}_c(k)$, the process $\{\mathbb{B}_t(\mu) \mid t \geq 0\}$ can still be viewed as a kind of evolution towards $\boxplus$-infinite divisibility, which is now taking place in the framework of $\mathcal{D}_c(k)$.

Moreover, we prove that the transformations $\mathbb{B}_t$ relate to the multi-variable free Brownian motion in a similar way to the one presented above in the 1-dimensional case. The free
Brownian motion started at $\nu \in D_c(k)$ is the process $\{\nu \boxplus \gamma_t \mid t \geq 0\}$ where $\gamma_t \in D_c(k)$ now stands for the joint distribution of a free family $x_1, \ldots, x_k$ of selfadjoint elements in a $C^*$-probability space, such that every $x_i$ has a centered semicircular distribution of variance $t$.

In order to connect this to the transformations $B_t$, we use a multi-variable analogue for the map $\Phi$ which had been defined via Equation (1.2). The multi-variable version of Equation (1.2) involves formal power series in $k$ non-commuting indeterminates $z_1, \ldots, z_k$ (instead of complex analytic functions of one variable $z$). For $\mu \in D_c(k)$ let $M_\mu$ be its moment series, $M_\mu(z_1, \ldots, z_k) := \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1}^{k} \mu(X_{i_1} \cdots X_{i_n}) z_{i_1} \cdots z_{i_n}$; and let us moreover denote $\eta_\mu := M_\mu ((1 + M_\mu)^{-1}, \mu \in D_c(k)$, where $(1 + M_\mu)^{-1}$ is the inverse of $1 + M_\mu$ under multiplication in the ring of power series $\mathbb{C}((z_1, \ldots, z_k))$. With these notations, the multi-variable analogue for (1.2) is

$$\eta_\mu(z_1, \ldots, z_k) = \sum_{i=1}^{k} z_i \left(1 + M_\nu(z_1, \ldots, z_k)\right) z_i$$

(equality of formal power series). We prove that for every $\nu \in D_c(k)$ there exists a unique $\mu \in D_c(k)$ such that (1.3) holds. One can thus define a map $\Phi : D_c(k) \to D_c(k)$ by putting $\Phi(\nu) := \mu$ with $\mu, \nu$ as in (1.3), and it turns out that we then have the analogue of (1.3):

$$\Phi(\nu \boxplus \gamma_t) = B_t(\Phi(\nu)), \quad \forall \nu \in D_c(k), \forall t > 0. \quad (1.6)$$

Concerning the seemingly different appearance of Equations (1.2) and (1.5), we make the following comment. Suppose that $k = 1$ and that $\mu, \nu \in D_c(1)$ are identified as compactly supported distributions on $\mathbb{R}$. Then $\eta_\mu$ and $M_\nu$ can be viewed as analytic functions on a neighbourhood of 0, and (for $z$ running in a suitable domain of $\mathbb{C}$) we have

$$F_\mu(z) = z \left(1 - \eta_\mu(1/z)\right), \quad G_\nu(z) = \frac{1}{z} \left(1 + M_\nu(1/z)\right).$$

By substituting these formulas into Equation (1.2), and by replacing $z$ with $1/z$, we bring (1.2) to the form

$$\eta_\mu(z) = z^2 \left(1 + M_\nu(z)\right),$$

which is exactly the 1-dimensional version of Equation (1.5).

1.3 Further remarks. The results in [2] were proved by using complex analytic functions. The methods used in this paper are completely different, they rely on the combinatorics of non-crossing partitions. For most of the paper we use the larger algebraic framework of $D_{\text{alg}}(k)$, the space of all possible joint distributions of $k$-tuples in a (sheer algebraic) non-commutative probability space. It makes sense to define $B_t$ as a bijective transformation of $D_{\text{alg}}(k)$, then prove the algebraic statements made about $\{B_t \mid t \geq 0\}$ in
this larger framework; these properties of the $B_t$ are summarized in Theorem 4.11 of the paper. In the same theorem we also point out an additional property of $B_t$, that

$$B_t(\mu \boxtimes \nu) = B_t(\mu) \boxtimes B_t(\nu), \quad \forall t \geq 0, \forall \mu, \nu \in D_{\text{alg}}(k),$$

(1.7)

where $\boxtimes$ (the free multiplicative convolution) is the operation on $D_{\text{alg}}(k)$ which corresponds to the multiplication of free $k$-tuples of random variables in a non-commutative probability space.

The map $\Phi$ defined via Equation (1.5) and the relation between the transformations $B_t$ and free Brownian motion given in (1.6) can be considered on $D_{\text{alg}}(k)$ as well. The proof of formula (1.6) is in fact done in this algebraic framework, in Theorem 6.2 of the paper.

After the algebraic results are established, what remains to be done is make sure that the transformations $B_t$ do indeed the job they are supposed to, when they are restricted to the smaller space $D_c(k)$. Some of the verifications needed here are direct consequences of things proved in our preceding paper [1]. But there is one verification that we are left with, namely that the map $\Phi : D_{\text{alg}}(k) \to D_{\text{alg}}(k)$ carries $D_c(k)$ into itself. We prove this fact by providing an operator model for how $\Phi$ works on $D_c(k)$; this operator model is discussed in Remark 7.4 and Theorem 7.5 of the paper.

We conclude the introduction with an outline of how the paper is organized. Throughout the whole paper $k$ is a fixed positive integer – the “number of indeterminates” we are working with. In Section 2 we review the algebraic framework of $D_{\text{alg}}(k)$, and the $R$ and $\eta$ transforms for distributions in $D_{\text{alg}}(k)$. Section 3 is a review section as well, devoted to an important bijection on power series introduced in [1], the bijection “Reta” sending $R_\mu \mapsto \eta_\mu$ for every $\mu \in D_{\text{alg}}(k)$; this bijection is the workhorse for many of the computations with power series done in the present paper. In Section 4 we introduce $B_t$ as a bijective transformation of $D_{\text{alg}}(k)$ and we prove some of its basic properties; the results of the section are summarized in Theorem 4.11. In Section 5 we establish a formula for moments of the free Brownian motion, which is needed in the proof of the connection between free Brownian motion and the $B_t$. The proof of this connection is then done in Section 6, Theorem 6.2. The final Section 7 of the paper deals with the framework of $D_c(k)$, the main point of the section being the operator model for $\Phi$.

2. Non-commutative convolutions and transforms on $D_{\text{alg}}(k)$

Definition 2.1. (Non-commutative distributions.)

1° We denote by $\mathbb{C}\langle X_1, \ldots, X_k \rangle$ the algebra of non-commutative polynomials in $X_1, \ldots, X_k$. Thus $\mathbb{C}\langle X_1, \ldots, X_k \rangle$ has a linear basis

$$\{1\} \cup \{X_{i_1} \cdots X_{i_n} \mid n \geq 1, 1 \leq i_1, \ldots, i_n \leq k\},$$

(2.1)

where the monomials in the basis are multiplied by concatenation. When needed, $\mathbb{C}\langle X_1, \ldots, X_k \rangle$ will be viewed as a $*$-algebra, with $*$-operation determined uniquely by the fact that each of $X_1, \ldots, X_k$ is selfadjoint.

2° Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space; that is, $\mathcal{A}$ is a unital algebra over $\mathbb{C}$, and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional, normalized by the condition that $\varphi(1_\mathcal{A}) = 1$. 

4
For \(x_1, \ldots, x_k \in A\), the joint distribution of \(x_1, \ldots, x_k\) is the linear functional \(\mu_{x_1, \ldots, x_k} : \mathbb{C}(X_1, \ldots, X_k) \to \mathbb{C}\) which acts on the linear basis (2.1) by the formula

\[
\begin{align*}
\mu_{x_1, \ldots, x_k}(1) &= 1 \\
\mu_{x_1, \ldots, x_k}(X_{i_1} \cdots X_{i_n}) &= \varphi(x_{i_1} \cdots x_{i_n}),
\end{align*}
\]

(2.2)

3° We will denote

\[D_{\text{alg}}(k) := \{\mu : \mathbb{C}(X_1, \ldots, X_k) \to \mathbb{C} \mid \mu\text{ linear}, \mu(1) = 1\}.\]

(2.3)

It is immediate that \(D_{\text{alg}}(k)\) is precisely the set of linear functionals on \(\mathbb{C}(X_1, \ldots, X_k)\) that can arise as joint distribution for some \(k\)-tuple \(x_1, \ldots, x_k\) in a non-commutative probability space.

**Remark 2.2.** (The operations \(\boxplus\) and \(\boxplus\) on \(D_{\text{alg}}(k)\).) These operations are defined via the general principle that if one has a form of independence for non-commutative random variables, then the addition of independent \(k\)-tuples of random variables will induce a “convolution” operation on \(D_{\text{alg}}(k)\).

The operation \(\boxplus\) arises in this way, in connection to the concept of free independence. Given \(\mu, \nu \in D_{\text{alg}}(k)\), one can always find random variables \(x_1, \ldots, x_k, y_1, \ldots, y_k\) in a non-commutative probability space \((A, \varphi)\) such that the joint distribution of the \(k\)-tuple \(x_1, \ldots, x_k\) is equal to \(\mu\), the joint distribution of \(y_1, \ldots, y_k\) is equal to \(\nu\), and such that \(\{x_1, \ldots, x_k\}\) is freely independent from \(\{y_1, \ldots, y_k\}\) in \((A, \varphi)\). The joint distribution of the \(k\)-tuple \(x_1 + y_1, \ldots, x_k + y_k\) turns out to depend only on \(\mu\) and \(\nu\); and the free additive convolution \(\mu \boxplus \nu\) is equal, by definition, to the joint distribution of \(x_1 + y_1, \ldots, x_k + y_k\).

The operation \(\boxplus\) is defined in the same way, but where we use the concept of Boolean independence: given \(\mu, \nu \in D_{\text{alg}}(k)\), the Boolean convolution \(\mu \boxplus \nu\) is the (uniquely determined) distribution of \(x_1 + y_1, \ldots, x_k + y_k\), where the joint distribution of \(x_1, \ldots, x_k\) is equal to \(\mu\), the joint distribution of \(y_1, \ldots, y_k\) is equal to \(\nu\), and \(\{x_1, \ldots, x_k\}\) is Boolean independent from \(\{y_1, \ldots, y_k\}\).

A commonly used method for studying the operations \(\boxplus\) and \(\boxplus\) goes by considering cumulants for distributions in \(D_{\text{alg}}(k)\): in relation to \(\boxplus\) one considers the free cumulants introduced in [7], while for \(\boxplus\) one uses the Boolean cumulants which go back all the way to [8]. In this paper we will work with the concepts, equivalent to cumulants, of linearizing transforms for \(\boxplus\) and \(\boxplus\). Specifically, for a distribution \(\mu \in D_{\text{alg}}(k)\) we will work with the R-transform \(R_\mu\) (a formal power series which records the free cumulants of \(\mu\)) and with the \(\eta\)-series \(\eta_\mu\) (which does the same job in connection to the Boolean cumulants of \(\mu\)).

The precise definitions of \(R_\mu\) and \(\eta_\mu\) will be reviewed in the next notation and remark. The meaning of the statement that \(R\) and \(\eta\) are “linearizing transforms” for \(\boxplus\) and respectively for \(\boxplus\) is that we have:

\[R_{\mu \boxplus \nu} = R_\mu + R_\nu, \quad \forall \mu, \nu \in D_{\text{alg}}(k),\]

(2.4)

and

\[\eta_{\mu \boxplus \nu} = \eta_\mu + \eta_\nu, \quad \forall \mu, \nu \in D_{\text{alg}}(k).\]

(2.5)
Remark 2.6. Coming out of (2.9) – see Lectures 11 and 16 of [6].

When one writes explicitly the relations between the coefficients of $D$ it follows that $R$ is not necessarily integer. More precisely, for every $\mu \in D$ and its coefficients are called the free cumulants $\eta$.

Remark 2.4. (Review of the series $M_\mu$, $R_\mu$, $\eta_\mu$). Let $\mu$ be a distribution in $D_{alg}(k)$.

1° We will denote by $M_\mu$ the series in $C_0(\langle z_1, \ldots, z_k \rangle)$ given by

$$M_\mu(z_1, \ldots, z_k) := \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1}^{k} \mu(X_{i_1} \cdots X_{i_n}) z_{i_1} \cdots z_{i_n}. \tag{2.7}$$

$M_\mu$ is called the moment series of $\mu$, and its coefficients $(\mu(X_{i_1} \cdots X_{i_n}))$, with $n \geq 1$ and $1 \leq i_1, \ldots, i_n \leq k$ are called the moments of $\mu$.

2° The $\eta$-series of $\mu$ is

$$\eta_\mu := M_\mu(1 + M_\mu)^{-1} \in C_0(\langle z_1, \ldots, z_k \rangle), \tag{2.8}$$

where $(1 + M_\mu)^{-1}$ is the inverse of $1 + M_\mu$ under multiplication, in the ring of series $C(\langle z_1, \ldots, z_k \rangle)$. The coefficients of $\eta_\mu$ are called the Boolean cumulants of $\mu$.

3° There exists a unique series $R_\mu \in C_0(\langle z_1, \ldots, z_k \rangle)$ which satisfies the functional equation

$$R_\mu\left(z_1(1 + M_\mu), \ldots, z_k(1 + M_\mu)\right) = M_\mu. \tag{2.9}$$

Indeed, it is easily seen that Equation (2.9) amounts to a recursion which determines uniquely the coefficients of $R_\mu$ in terms of those of $M_\mu$. The series $R_\mu$ is called the $R$-transform of $\mu$, and its coefficients are called the free cumulants of $\mu$. (See the discussion in Lecture 16 of [6], and specifically Theorem 16.15 and Corollary 16.16 of that lecture.)

Remark 2.5. It is immediate that for every $f \in C_0(\langle z_1, \ldots, z_k \rangle)$ there exists a unique distribution $\mu \in D_{alg}(k)$ such that $\eta_\mu = f$. (This is because, as one immediately checks, the equation $M_\mu(1 + M_\mu)^{-1} = f$ is equivalent to $M_\mu = f(1 - f)^{-1}$.) Thus the map $\mu \mapsto \eta_\mu$ is a bijection from $D_{alg}(k)$ onto $C_0(\langle z_1, \ldots, z_k \rangle)$.

Likewise, the map $D_{alg}(k) \ni \mu \mapsto R_\mu \in C_0(\langle z_1, \ldots, z_k \rangle)$ is bijective. The fact that for every $g \in C_0(\langle z_1, \ldots, z_k \rangle)$ there exists a unique $\mu \in D_{alg}(k)$ such that $R_\mu = g$ is easily seen when one writes explicitly the relations between the coefficients of $R_\mu$ and $M_\mu$ that are coming out of (2.9) – see Lectures 11 and 16 of [6].

Remark 2.6. (Convolution powers.) For $\mu \in D_{alg}(k)$ and a positive integer $n$ one denotes the $n$-fold convolution $\mu \boxtimes \cdots \boxtimes \mu$ by $\mu^{\boxtimes n}$. From the additivity (2.4) of the $R$-transform it follows that $R_\mu^{\boxtimes n} = n \cdot R_\mu$, and the latter formula can be extended to the case when $n$ is not necessarily integer. More precisely, for every $\mu \in D_{alg}(k)$ and $t \in (0, \infty)$ one defines
the convolution power $µ^{⊞t}$ to be the unique distribution in $D_{alg}(k)$ which has $R$-transform equal to

$$R_{µ^{⊞t}} = t \cdot R_µ.$$  \hfill (2.10)

It is immediate that the $⊞$-convolution powers defined in this way satisfy the usual rules for operating with exponents:

$$µ^{⊞s} ⊞ µ^{⊞t} = µ^{⊞(s+t)} \text{ and } (µ^{⊞s})^{⊞t} = µ^{⊞st}, \quad ∀ µ ∈ D_{alg}(k), \forall s, t > 0.$$  \hfill (2.11)

Note that, as a consequence, one has that for every fixed $t ∈ (0, ∞)$ the map $µ ↦ µ^{⊞t}$ is a bijection from $D_{alg}(k)$ onto itself.

A similar discussion can be made in connection to the convolution powers with respect to $⊎$: for every $µ ∈ D_{alg}(k)$ and $t ∈ (0, ∞)$ one defines the convolution power $µ^{⊎t}$ to be the unique distribution in $D_{alg}(k)$ which has $η$-series equal to

$$η_{µ^{⊎t}} = t \cdot η_µ.$$  \hfill (2.12)

Then the $⊎$-convolution powers satisfy the usual rules of operating with exponents, and for every fixed $t ∈ (0, ∞)$ the map $µ ↦ µ^{⊎t}$ is a bijection from $D_{alg}(k)$ onto itself.

### 3. The bijection “Reta”, and its combinatorial properties

**Definition 3.1.** We will denote

$$\text{Reta} := η \circ R^{-1} : C_0(⟨⟨z_1, \ldots, z_k⟩⟩) → C_0(⟨⟨z_1, \ldots, z_k⟩⟩),$$  \hfill (3.1)

where $R, η : D_{alg}(k) → C_0(⟨⟨z_1, \ldots, z_k⟩⟩)$ are the bijections $µ ↦ R_µ$ and respectively $µ ↦ η_µ$ that were discussed in Remark 2.5. In other words, Reta is the bijection from $C_0(⟨⟨z_1, \ldots, z_k⟩⟩)$ onto itself which is uniquely determined by the requirement that

$$\text{Reta}(R_µ) = η_µ, \quad ∀ µ ∈ D_{alg}(k).$$  \hfill (3.2)

**Remark 3.2.** The bijection Reta was introduced in our previous paper [1]. Its name was chosen by looking at Equation (3.2) (the transformation of $C_0(⟨⟨z_1, \ldots, z_k⟩⟩)$ that “converts $R$ into $η$”). It is very useful that one can alternatively describe Reta via an explicit formula which gives directly the coefficients of the series Reta$(f)$ in terms of those of $f$, for $f ∈ C_0(⟨⟨z_1, \ldots, z_k⟩⟩)$. This formula is reviewed (following [1]) in Proposition 3.5 below. It involves summations indexed by non-crossing partitions, and in order to present it we will start with a very concise review (intended mostly for setting notations) of the lattice $NC(n)$ of non-crossing partitions. For a more detailed introduction to $NC(n)$ and to how it is used in free probability, we refer to [6], Lectures 9 and 10.
Remark 3.3. (Review of NC(n).) Let \( n \) be a positive integer.

1° Let \( \pi = \{B_1, \ldots, B_p\} \) be a partition of \( \{1, \ldots, n\} \) – i.e. \( B_1, \ldots, B_p \) are pairwise disjoint non-void sets (called the blocks of \( \pi \)), and \( B_1 \cup \cdots \cup B_p = \{1, \ldots, n\} \). We say that \( \pi \) is non-crossing if for every \( 1 \leq i < j < k < l \leq n \) such that \( i \) is in the same block with \( k \) and \( j \) is in the same block with \( l \), it necessarily follows that all of \( i, j, k, l \) are in the same block of \( \pi \). The set of all non-crossing partitions of \( \{1, \ldots, n\} \) will be denoted by \( NC(n) \).

2° For \( \pi \in NC(n) \), the number of blocks of \( \pi \) will be denoted by \( |\pi| \).

3° On \( NC(n) \) we consider the partial order given by reversed refinement: for \( \pi, \rho \in NC(n) \), we write “\( \pi \leq \rho \)” to mean that every block of \( \rho \) is a union of blocks of \( \pi \). The minimal and maximal element of \( (NC(n), \leq) \) are denoted by \( 0_n \) (the partition of \( \{1, \ldots, n\} \) into \( n \) singleton blocks) and respectively \( 1_n \) (the partition of \( \{1, \ldots, n\} \) into only one block).

4° In the considerations about Reta, an important role is played by another partial order relation on \( NC(n) \), which was introduced in [1] and is denoted by “\( \ll \)”. For \( \pi, \rho \in NC(n) \) we will write “\( \pi \ll \rho \)” to mean that \( \pi \leq \rho \) and that, in addition, the following condition is fulfilled:

\[
\left\{ \begin{array}{l}
\text{For every block } C \text{ of } \rho \text{ there exists a block } B \text{ of } \pi \text{ such that } \min(C), \max(C) \in B.
\end{array} \right. \tag{3.3}
\]

It is immediately verified that “\( \ll \)” is indeed a partial order relation on \( NC(n) \). It is much coarser than the reversed refinement order. For instance, the inequality \( \pi \ll 1_n \) is not holding for all \( \pi \in NC(n) \), but it rather amounts to the condition that the numbers 1 and \( n \) belong to the same block of \( \pi \). At the other end of \( NC(n) \), the inequality \( \pi \gg 0_n \) can only take place when \( \pi = 0_n \).

Definition 3.4. (coefficients for series in \( \mathbb{C}_0(\langle z_1, \ldots, z_k \rangle) \)).

1° For \( n \geq 1 \) and \( 1 \leq i_1, \ldots, i_n \leq k \) we will denote by

\[
\text{Cf}_{(i_1, \ldots, i_n)} : \mathbb{C}_0(\langle z_1, \ldots, z_k \rangle) \to \mathbb{C}
\]

the linear functional which extracts the coefficient of \( z_{i_1} \cdots z_{i_n} \) in a series \( f \in \mathbb{C}_0(\langle z_1, \ldots, z_k \rangle) \). Thus for \( f \) written as in Equation (2.6) we have \( \text{Cf}_{(i_1, \ldots, i_n)}(f) = a_{(i_1, \ldots, i_n)} \).

2° Suppose we are given a positive integer \( n \), some indices \( i_1, \ldots, i_n \in \{1, \ldots, k\} \), and a partition \( \pi \in NC(n) \). We define a (generally non-linear) functional

\[
\text{Cf}_{(i_1, \ldots, i_n)}(\pi) : \mathbb{C}_0(\langle z_1, \ldots, z_k \rangle) \to \mathbb{C},
\]

as follows. For every block \( B = \{b_1, \ldots, b_m\} \) of \( \pi \), with \( 1 \leq b_1 < \cdots < b_m \leq n \), let us use the notation

\[
(i_1, \ldots, i_n)|B := (i_{b_1}, \ldots, i_{b_m}) \in \{1, \ldots, k\}^m.
\]

Then we define

\[
\text{Cf}_{(i_1, \ldots, i_n)}(\pi)(f) := \prod_{B \text{ block of } \pi} \text{Cf}_{(i_1, \ldots, i_n)}(f), \quad \forall f \in \mathbb{C}_0(\langle z_1, \ldots, z_k \rangle). \tag{3.6}
\]

(For example if we had \( n = 5 \) and \( \pi = \{\{1, 4, 5\}, \{2, 3\}\} \), and if \( i_1, \ldots, i_5 \) would be some fixed indices from \( \{1, \ldots, k\} \), then the above formula would become

\[
\text{Cf}_{(i_1, i_2, i_4, i_5)}(f) = \text{Cf}_{(i_1, i_4, i_5)}(f) \cdot \text{Cf}_{(i_2, i_3)}(f),
\]

\( f \in \mathbb{C}_0(\langle z_1, \ldots, z_k \rangle) \).) The quantities \( \text{Cf}_{(i_1, \ldots, i_n)}(\pi)(f) \) will be referred to as generalized coefficients of the series \( f \).
Proposition 3.5. Let $f, g$ be series in $C_0\langle\langle z_1, \ldots, z_k\rangle\rangle$ such that $\Reta(f) = g$. Then for every $n \geq 1$ and $1 \leq i_1, \ldots, i_n \leq k$ we have

$$Cf_{(i_1, \ldots, i_n)}(g) = \sum_{\pi \in NC(n), \pi \ll 1_n} Cf_{(i_1, \ldots, i_n);\pi}(f). \quad (3.7)$$

More generally, we have the following formula for a generalized coefficient $Cf_{(i_1, \ldots, i_n);\rho}(g)$, where $\rho$ is an arbitrary partition in $NC(n)$:

$$Cf_{(i_1, \ldots, i_n);\rho}(g) = \sum_{\pi \in NC(n), \pi \ll \rho} Cf_{(i_1, \ldots, i_n);\pi}(f). \quad (3.8)$$

Remark 3.6. 1° For the proof of the above formulas (3.7) and (3.8) we refer to Proposition 3.9 of [1]. Let us mention here that the same Proposition 3.9 of [1] also gives an explicit formula for how Equation (3.7) can be inverted in order to write the coefficients of $f$ in terms of those of $g$. This latter formula says that for every $n \geq 1$ and $1 \leq i_1, \ldots, i_n \leq k$ we have:

$$Cf_{(i_1, \ldots, i_n)}(f) = \sum_{\pi \in NC(n), \pi \ll 1_n} (-1)^{1+|\pi|}Cf_{(i_1, \ldots, i_n);\pi}(g). \quad (3.9)$$

Note that, since $(-1)^{1+|\pi|}Cf_{(i_1, \ldots, i_n);\pi}(g)$ can also be written as “$-Cf_{(i_1, \ldots, i_n);\pi}(-g)$”, an equivalent way of recording the formula (3.9) is by stating that $\Reta^{-1}(g) = -\Reta(-g), \forall g \in C_0\langle\langle z_1, \ldots, z_k\rangle\rangle$. \hfill□

2° Let $f, g$ be two series in $C_0\langle\langle z_1, \ldots, z_k\rangle\rangle$ such that $\Reta(f) = g$. An immediate consequence of Equation (3.7) is that the linear and quadratic coefficients of $g$ are identical with the corresponding coefficients of $f$:

$$Cf_{(i_1, \ldots, i_n)}(f) = Cf_{(i_1, \ldots, i_n)}(g), \forall 1 \leq i, i_1, i_2 \leq k.$$ 

(This is because for $n \leq 2$ the only partition $\pi \in NC(n)$ which satisfies $\pi \ll 1_n$ is $1_n$ itself.)

The first time when we see a difference between $f$ and $g$ is when we look at coefficients of order 3:

$$Cf_{(i_1, i_2, i_3)}(g) = Cf_{(i_1, i_2, i_3)}(f) + Cf_{(i_1, i_2)}(f) \cdot Cf_{(i_2)}(f), \quad \text{for } 1 \leq i_1, i_2, i_3 \leq k.$$ 

3° In Section 4 of the paper we will need a formula for the iterations of $\Reta$, which we derive in Proposition 3.8 below. The proof of this formula is based on a property of the partial order $\ll$ which was proved in Proposition 2.13 of [1], and goes as follows.
Lemma 3.7. Let $\pi$ be a partition in $NC(n)$ such that $\pi \ll 1_n$. For every integer $p$ satisfying $1 \leq p \leq |\pi|$, we have that:

$$\text{card}\left\{ \rho \in NC(n) \mid \rho \gg \pi \text{ and } |\rho| = p \right\} = \left( \frac{|\pi| - 1}{p - 1} \right).$$

\square

Proposition 3.8. Let $f$ be a series in $C_0(\langle\langle z_1, \ldots, z_k \rangle\rangle)$ and let $s$ be in $\mathbb{R}$, $s \neq -1$. We have

$$\text{Ret}(s \text{ Ret}(f)) = \frac{s}{1 + s} \text{ Ret}(1 + s f).$$

(3.11)

Proof. Fix $n \geq 1$ and $1 \leq i_1, \ldots, i_n \leq k$ for which we verify the equality of the coefficients of $z_{i_1} \cdots z_{i_n}$ for the series on the two sides of Equation (3.11). We start from the left-hand side of this equation, and compute:

$$\text{Cf}_{(i_1, \ldots, i_n)} \left( \text{Ret}(s \text{ Ret}(f)) \right) = \sum_{\rho \in NC(n)} \text{Cf}_{(i_1, \ldots, i_n); \rho} \left( s \text{ Ret}(f) \right) \text{ (by (3.7))}
= \sum_{\rho \in NC(n)} s^{|\rho|} \text{Cf}_{(i_1, \ldots, i_n); \rho} \left( \text{Ret}(f) \right)
= \sum_{\rho \in NC(n)} \left( s^{|\rho|} \sum_{\pi \in NC(n) \mid \pi \ll \rho} \text{Cf}_{(i_1, \ldots, i_n); \pi} (f) \right) \text{ (by (3.8)).}
$$

By reversing the order of summation in the double sum that has appeared, we continue our sequence of equalities with:

$$= \sum_{\pi \in NC(n) \mid \pi \ll 1_n} \text{Cf}_{(i_1, \ldots, i_n); \pi} (f) \cdot \left( \sum_{\rho \in NC(n) \mid \pi \ll \rho \ll 1_n} s^{|\rho|} \right)
= \sum_{\pi \in NC(n) \mid \pi \ll 1_n} \text{Cf}_{(i_1, \ldots, i_n); \pi} (f) \cdot \left( \sum_{p=1}^{|\pi|} \left( \frac{|\pi| - 1}{p - 1} \right) s^p \right) \text{ (by Lemma 3.7)}
= \sum_{\pi \in NC(n) \mid \pi \ll 1_n} s (1 + s)^{|\pi| - 1} \text{Cf}_{(i_1, \ldots, i_n); \pi} (f)
= \frac{s}{1 + s} \cdot \sum_{\pi \in NC(n) \mid \pi \ll 1_n} \text{Cf}_{(i_1, \ldots, i_n); \pi} \left( (1 + s) f \right)
= \text{Cf}_{(i_1, \ldots, i_n)} \left( \frac{s}{1 + s} \text{ Ret}(1 + s f) \right) \text{ (by (3.8)).} \quad \square
Remark 3.9. In the case when $s = -1$, the expression “$\text{Reta} \left( s \text{Reta}(f) \right)$” is not treated by the preceding proposition, but rather by using Equation (3.10) of Remark 3.6.1, which gives us that

$$\text{Reta} \left( -\text{Reta}(f) \right) = -f, \ \forall f \in \mathcal{D}_{\text{alg}}(k). \quad (3.12)$$

4. The transformations $\mathbb{B}_t$ on $\mathcal{D}_{\text{alg}}(k)$

Definition 4.1. For every $t \geq 0$ define a transformation $\mathbb{B}_t : \mathcal{D}_{\text{alg}}(k) \to \mathcal{D}_{\text{alg}}(k)$ by the formula

$$\mathbb{B}_t(\mu) = \left( \mu^{\Xi(1+t)} \right)^{(1/(1+t))}, \ \forall \mu \in \mathcal{D}_{\text{alg}}(k). \quad (4.1)$$

Every $\mathbb{B}_t$ is a bijection from $\mathcal{D}_{\text{alg}}(k)$ onto itself (which happens because, as noticed in Remark 2.6, both the maps $\mathcal{D}_{\text{alg}}(k) \ni \mu \mapsto \mu^{\Xi(1+t)} \in \mathcal{D}_{\text{alg}}(k)$ and $\mathcal{D}_{\text{alg}}(k) \ni \nu \mapsto \nu^{\eta(1/(1+t))} \in \mathcal{D}_{\text{alg}}(k)$ are bijective). The transformations $\{\mathbb{B}_t | t \geq 0\}$ form in fact a semigroup under composition; this will follow from a “commutation relation”, stated in the next proposition, satisfied by the convolution powers with respect to $\oplus$ and to $\sqcup$.

Proposition 4.2. Let $p, q$ be two real numbers such that $p \geq 1$ and $q > (p - 1)/p$. We have

$$\left( \mu^{\Xi p} \right)^{\sqcup q} = \left( \mu^{\eta q'} \right)^{\oplus p'}, \ \forall \mu \in \mathcal{M}, \quad (4.2)$$

where the new exponents $p', q' > 0$ are defined by

$$p' := pq/(1-p+pq), \quad q' := 1-p+pq. \quad (4.3)$$

Proof. If $q = 1$ then it follows that $q' = 1$ and $p' = p$, and both sides of Equation (4.2) are equal to $\mu^{\Xi p}$. For the rest of the proof we will assume that $q \neq 1$, which implies that $q' \neq 1$ as well. Our strategy is to prove that the distributions on the two sides of Equation (4.2) have equal $R$-transforms. We prove this by calculating explicitly the $R$-transforms in question, where we take advantage of the fact that the convolution powers with respect to $\oplus$ and with respect to $\sqcup$ are scaled by the $R$-transform and respectively by the $\eta$-series (Equations (2.10) and (2.12) in Remark 2.6). The calculations may occasionally come to the point where we deal with the $R$-transform of a $\sqcup$-power, or with the $\eta$-series of a $\oplus$-power; in such a situation we apply $\text{Reta}$ (or $\text{Reta}^{-1}$) and go on, remaining that the compositions of $\text{Reta}$’s that arise in this way are dealt with by using Proposition 3.8. To be specific, on the left-hand side of (4.2) we calculate:

$$R_{(\mu^{\Xi p})^{\sqcup q}} = \text{Reta}^{-1} \left( \eta_{(\mu^{\Xi p})^{\sqcup q}} \right) = \text{Reta}^{-1} \left( q \cdot \eta_{\mu^{\Xi p}} \right) \quad (\text{by (2.12)})$$

$$= \text{Reta}^{-1} \left( q \cdot \text{Reta}(R_{\mu^{\Xi p}}) \right) = \text{Reta}^{-1} \left( q \cdot \text{Reta}(p R_{\mu}) \right) \quad (\text{by (2.10)})$$

$$= -\text{Reta} \left( -q \cdot \text{Reta}(p R_{\mu}) \right) \quad (\text{by (3.10)})$$

$$= \frac{q}{1-q} \text{Reta} \left( (1-q)p R_{\mu} \right) \quad (\text{by (3.11)}).$$

11
On the right-hand side of (4.2) we calculate:

\[ R_{(\mu \oplus q') \oplus p'} = p' R_{\mu \oplus q'} \quad \text{(by (2.10))} \]
\[ = p' \text{Reta}^{-1}(\eta_{\mu \oplus q'}) = p' \text{Reta}^{-1}(q' \eta_\mu) \quad \text{(by (2.12))} \]
\[ = p' \text{Reta}^{-1}(q' \text{Reta}(R_\mu)) = -p' \text{Reta}(-q' \text{Reta}(R_\mu)) \quad \text{(by (3.10))} \]
\[ = (-p') \frac{-q'}{1-q'} \cdot \text{Reta} \left( (1-q') R_\mu \right) \quad \text{(by (3.11)).} \]

It only remains to observe that the definition of \( p' \) and \( q' \) ensures that \( 1-q' = (1-q)p \) and \( p'q'/(1-q') = q/(1-q) \), hence the two \( R \)-transforms calculated above are indeed equal to each other. \( \square \)

**Corollary 4.3.** We have that \( B_s \circ B_t = B_{s+t}, \ \forall s, t \geq 0 \).

**Proof.** For every \( s, t \geq 0 \) and \( \mu \in D_{\text{alg}}(k) \) we have

\[ B_s(B_t(\mu)) = B_s \left( \left( \mu^{\frac{\text{B}t+1}{t+1}} \right)^{\frac{1}{s+t+1}} \right) \]
\[ = \left[ \left( \mu^{\frac{\text{B}t+1}{t+1}} \right)^{\frac{1}{s+t+1}} \right]^{\frac{1}{s+t+1}} \]
\[ = \left( \mu^{\frac{s+t+1}{s+t+1}} \right)^{\frac{1}{s+t+1}} \]
\[ = B_{s+t}(\mu), \]

where at the third equality sign we used Proposition 4.2 with \( p = (s+t+1)/(t+1) \) and \( q = (s+1)/(s+t+1) \). \( \square \)

**Remark 4.4.** If in the calculation for the \( R \)-transform of \( \left( \mu^{\frac{\text{B}t}{t}} \right)^{\frac{1}{t+1}} \) that was shown in the proof of Proposition 4.2 we make \( p = 1+t \) and \( q = 1/(1+t) \) (for some \( t > 0 \)) we obtain

\[ R_{\text{B}_t(\mu)} = \frac{1}{t} \text{Reta}(tR_\mu), \ \forall \mu \in D_{\text{alg}}(k), \ \forall t > 0. \quad (4.4) \]

We leave it as an exercise to the reader to check that the similar calculation done with \( \eta \)-series instead of \( R \)-transforms leads to the analogous formula

\[ \eta_{\text{B}_t(\mu)} = \frac{1}{t} \text{Reta}(t\eta_\mu), \ \forall \mu \in D_{\text{alg}}(k), \ \forall t > 0. \quad (4.5) \]

**Remark 4.5.** (Relation to the Boolean Bercovici-Pata bijection \( \mathbb{B} \) from [1].) In [1] we studied a bijection \( \mathbb{B} : D_{\text{alg}}(k) \to D_{\text{alg}}(k) \) defined via the requirement that

\[ R_{\mathbb{B}(\mu)} = \eta_\mu, \ \forall \mu \in D_{\text{alg}}(k). \quad (4.6) \]
It is immediate that $\mathcal{B}$ coincides with the transformation $\mathcal{B}_1$ obtained by putting $t = 1$ in Definition 4.1. Indeed, for every $\mu \in \mathcal{D}_{\text{alg}}(k)$ we have

$$R_{\mathcal{B}_1}(\mu) = \text{Ret}(R_{\mu}) \quad \text{(by making $t = 1$ in Equation (4.4))}$$

$$= \eta_{\mu} \quad \text{(by definition of Ret);}$$

this implies that $\mathcal{B}_1(\mu) = \mathcal{B}(\mu)$, since $\mathcal{B}_1(\mu)$ and $\mathcal{B}(\mu)$ have the same $R$-transform.

**Remark 4.6.** An intriguing property of the map $\mathcal{B}$ which was observed in [1] is that it is a homomorphism with respect to the operation of free multiplicative convolution $\boxtimes$ on $\mathcal{D}_{\text{alg}}(k)$. This operation is defined as follows. Given $\mu, \nu \in \mathcal{D}_{\text{alg}}(k)$, one can always find random variables $x_1, \ldots, x_k, y_1, \ldots, y_k$ in a non-commutative probability space $(\mathcal{A}, \varphi)$ such that the joint distribution of the $k$-tuple $x_1, \ldots, x_k$ is equal to $\mu$, the joint distribution of the $k$-tuple $y_1, \ldots, y_k$ is equal to $\nu$, and such that $\{x_1, \ldots, x_k\}$ is freely independent from $\{y_1, \ldots, y_k\}$ in $(\mathcal{A}, \varphi)$. The joint distribution of the $k$-tuple $x_1 y_1, \ldots, x_k y_k$ turns out to depend only on $\mu$ and $\nu$; and the free multiplicative convolution $\mu \boxtimes \nu$ is equal, by definition, to the joint distribution of $x_1 y_1, \ldots, x_k y_k$.

In the remaining part of this section we will show that every $\mathcal{B}_t$ is a homomorphism with respect to $\boxtimes$. The argument is short, because it takes advantage of what had already been proved in [1] – the essential point is to use Theorem 7.3 of that paper. We mention that in the 1-dimensional case another derivation of the $\boxtimes$-homomorphism property of $\mathcal{B}_t$ can be obtained by using the concept of S-transform (see Section 3 of [2]).

In the proof that $\mathcal{B}_t$ is a $\boxtimes$-homomorphism we will also use a binary operation denoted by $\boxdot$ on $\mathcal{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$, which was introduced in [5], and is uniquely determined by the fact that

$$R_{\mu} \boxdot R_{\nu} = R_{\mu \boxtimes \nu}, \quad \forall \mu, \nu \in \mathcal{D}_{\text{alg}}(k). \quad (4.7)$$

In other words, $\boxdot$ is the operation with formal power series which reflects the multiplication of two free $k$-tuples in terms of their $R$-transforms.

A remarkable fact proved in Theorem 7.3 of [1] is that we also have

$$\eta_{\mu} \boxdot \eta_{\nu} = \eta_{\mu \boxtimes \nu}, \quad \forall \mu, \nu \in \mathcal{D}_{\text{alg}}(k). \quad (4.8)$$

That is, $\boxdot$ is at the same time the operation with formal power series which reflects the multiplication of two free $k$-tuples in terms of their $\eta$-series. It is immediate that formula (4.8) is actually just another form of stating the $\boxtimes$-multiplicativity of $\mathcal{B}$. We prefer this formula which makes explicit use of $\boxdot$ because we want to combine it with other properties that $\boxdot$ has, in connection to dilations and scalar multiplication of power series (as reviewed in Remark 4.8 below).

**Definition 4.7.** For $\mu \in \mathcal{D}_{\text{alg}}(k)$ and $r > 0$ we denote by $\mu \circ D_r$ the distribution in $\mathcal{D}_{\text{alg}}(k)$ determined by the condition that

$$(\mu \circ D_r)(X_{i_1} \cdots X_{i_n}) = r^n \cdot \mu(X_{i_1} \cdots X_{i_n}), \quad \forall n \geq 1, \forall 1 \leq i_1, \ldots, i_n \leq k. \quad (4.9)$$

$\mu \circ D_r$ is called the dilation of $\mu$ by $r$. 

13
For $f \in C_0(\langle z_1, \ldots, z_k \rangle)$ and $r > 0$ we denote by $f \circ D_r$ the series in $C_0(\langle z_1, \ldots, z_k \rangle)$ determined by the condition that

$$\text{Cf}(i_1, \ldots, i_n)(f \circ D_r) = r^n \cdot \text{Cf}(i_1, \ldots, i_n)(f), \ \forall n \geq 1, \ \forall 1 \leq i_1, \ldots, i_n \leq k. \quad (4.10)$$

$f \circ D_r$ is called the dilation of $f$ by $r$.

**Remark 4.8.** It is easy to see, directly from the definitions, that all three series $M_\mu$, $R_\mu$, $\eta_\mu$ associated to a distribution $\mu \in D_{\text{alg}}(k)$ behave well with respect to dilations; that is, we have

$$M_{\mu \circ D_r} = M_\mu \circ D_r, \quad R_{\mu \circ D_r} = R_\mu \circ D_r, \quad \eta_{\mu \circ D_r} = \eta_\mu \circ D_r, \quad \forall \mu \in D_{\text{alg}}(k), \ \forall r > 0. \quad (4.11)$$

Let us also record here two formulas from [5] which involve dilations and the operation $\boxtimes$. The first formula simply says that $\boxtimes$ behaves well with respect to dilations:

$$(f \circ D_r) \boxtimes g = f \boxtimes (g \circ D_r) = (f \boxtimes g) \circ D_r, \ \forall f, g \in C_0(\langle z_1, \ldots, z_k \rangle), \ \forall r > 0. \quad (4.12)$$

The second formula puts into evidence a special connection with scalar multiplication of series. While $\boxtimes$ is highly non-linear (and doesn’t generally behave well with respect to scalar multiplication), it is remarkable that we have

$$(rf) \boxtimes (rg) = r \left( (f \boxtimes g) \circ D_r \right), \ \forall f, g \in C_0(\langle z_1, \ldots, z_k \rangle), \ \forall r > 0. \quad (4.13)$$

For the proof of (4.12) and (4.13) we refer to Notation 4.1 and Lemma 4.4 of [5].

In order to prove that $B_t$ is a homomorphism with respect to $\boxtimes$ we will show in the next proposition that, in fact, each of the two kinds of convolution powers involved in the definition of $B_t$ is “only a dilation away” from being itself a $\boxtimes$-homomorphism.

**Proposition 4.9.** For every $t > 0$ and every $\mu, \nu \in D_{\text{alg}}(k)$ we have

$$(\mu^{\boxtimes t}) \boxtimes (\nu^{\boxtimes t}) = (\mu \boxtimes \nu)^{\boxtimes t} \circ D_t \quad (4.14)$$

and

$$(\mu^{\ast t}) \boxtimes (\nu^{\ast t}) = (\mu \boxtimes \nu)^{\ast t} \circ D_t. \quad (4.15)$$

**Proof.** In order to establish the formula (4.14) we check that the distributions appearing on the two sides of this formula have the same $R$-transform:

$$R_{(\mu^{\boxtimes t}) \boxtimes (\nu^{\boxtimes t})} = R_{\mu^{\boxtimes t}} \boxtimes R_{\nu^{\boxtimes t}} \quad \text{(by (4.7))}$$

$$= tR_{\mu^{\boxtimes t}} \boxtimes tR_{\nu^{\boxtimes t}} \quad \text{(by (2.10))}$$

$$= t \left( R_\mu \boxtimes R_\nu \right) \circ D_t \quad \text{(by (4.13))}$$

$$= \left( tR_\mu \boxtimes tR_\nu \right) \circ D_t \quad \text{(by (4.13))}$$

$$= \left( R_{(\mu \boxtimes \nu)^{\boxtimes t}} \right) \circ D_t \quad \text{(by (2.10))}$$

$$= R_{(\mu \boxtimes \nu)^{\boxtimes t} \circ D_t} \quad \text{(by (4.11)).}$$
The verification of (4.15) is done in a similar way, where now we check that the distributions on the two sides of the formula have identical \( \eta \)-series. The calculation is virtually identical to the one shown in the verification of (4.14), only that we have to replace everywhere \( R \)-transforms by \( \eta \)-series, and \( \boxplus \)-powers by \( \boxminus \)-powers. (An important point included in this “mutatis mutandis” argument is that, right at the beginning of the calculation, we can invoke the formula (4.8) relating \( \eta \)-series to the operation \( \boxtimes \).)

\[ \square \]

**Corollary 4.10.** For every \( t \geq 0 \), the transformation \( B_t \) of \( \mathcal{D}_{\text{alg}}(k) \) is a homomorphism for \( \boxtimes \). That is, we have

\[ B_t(\mu \boxtimes \nu) = B_t(\mu) \boxtimes B_t(\nu), \quad \forall \mu, \nu \in \mathcal{D}_{\text{alg}}(k). \tag{4.16} \]

**Proof.** This is a straightforward consequence of Proposition 4.9: the dilation factors which appear when we take successively the powers “\( \boxplus (t+1) \)” and “\( \boxslash 1/(t+1) \)” cancel each other, and we are left with the plain \( \boxtimes \)-multiplicativity stated in Equation (4.16). \( \square \)

The results of this section are thus summarized in the following theorem, which puts together Corollary 4.3, Remark 4.5, and Corollary 4.10.

**Theorem 4.11.** The bijections \( B_t : \mathcal{D}_{\text{alg}}(k) \to \mathcal{D}_{\text{alg}}(k) \) introduced in Definition 4.1 have the following properties:

1. \( B_{s+t} \circ B_t = B_{s+t} \), for every \( s, t \geq 0 \).
2. \( B_1 = B \), the multi-variable Boolean Bercovici-Pata bijection introduced in [[1]].
3. Every \( B_t \) is a homomorphism for the free multiplicative convolution \( \boxtimes \) on \( \mathcal{D}_{\text{alg}}(k) \).

\[ \square \]

5. A formula for the moments of the free Brownian motion

Our goal in this section is to prove an explicit formula via summations over non-crossing partitions for moments \( (\nu \boxplus \gamma_t)(X_{i_1} \cdots X_{i_n}) \), where \( \nu \) is an arbitrary distribution in \( \mathcal{D}_{\text{alg}}(k) \) and \( \gamma_t \) is defined as follows.

**Notation 5.1.** For \( t > 0 \) we will denote by \( \gamma_t \in \mathcal{D}_{\text{alg}}(k) \) the joint distribution of a \( k \)-tuple \( (x_1, \ldots, x_k) \) where \( x_1, \ldots, x_k \) form a free family, and every \( x_i \) has a centered semicircular distribution of variance \( t \).

The formula for moments which is the main result of the section will be stated in Proposition 5.3. We start by introducing a few natural conventions of notations for non-crossing partitions that will be useful in Proposition 5.4.
Remark 5.2. 1° It will be convenient that instead of sticking strictly to “NC(n)”, we use the more general notation “NC(M)” for an arbitrary totally ordered finite set M. Of course, \( NC(M) \) can always be identified canonically to \( NC(|M|) \), where one uses the unique increasing bijection from M onto \( \{1, \ldots, |M| \} \) in order to identify partitions of M with partitions of \( \{1, \ldots, |M| \} \).

2° Let M be a totally ordered finite set, and let L be a non-empty subset of M. For \( \pi \in NC(M) \) we can consider the restricted partition \( \pi | L \) of L into blocks of the form \( A \cap L \), with A block of \( \pi \) such that \( A \cap L \neq \emptyset \). It is immediately verified that \( \pi | L \in NC(L) \) (where L is endowed with the total order inherited from M).

3° Let M be a totally ordered finite set, and suppose that \( M = L_1 \cup L_2 \), disjoint union. If \( \pi_1 \) is a partition of \( L_1 \) and \( \pi_2 \) is a partition of \( L_2 \), then there is an obvious way of putting \( \pi_1 \) and \( \pi_2 \) together to form a partition of M; we will denote this partition by \( \pi_1 \sqcup \pi_2 \).

It is clear that in order to have \( \pi_1 \sqcup \pi_2 \in NC(M) \) it is necessary but not sufficient that \( \pi_1 \in NC(L_1) \) and \( \pi_2 \in NC(L_2) \).

4° Let \( M, L_1, L_2 \) be as above and let \( \pi_1 \) be a fixed partition in \( NC(L_1) \). It is easy to see that among the partitions \( \pi_2 \in NC(L_2) \) with the property that \( \pi_1 \sqcup \pi_2 \in NC(M) \) there is one, \( \widehat{\pi} \), which is larger than all the others with respect to reversed refinement order on \( NC(L_2) \). So \( \widehat{\pi} \in NC(L_2) \) is characterized by the fact that for a partition \( \pi_2 \in NC(L_2) \) we have the equivalence

\[
\pi_1 \sqcup \pi_2 \in NC(A) \iff \pi_2 \leq \widehat{\pi}.
\] (5.1)

The formula for moments that will be proved in Proposition [5.4] uses the class of non-crossing partitions discussed in the following notation.

Notation 5.3. Let \( n \) be a positive integer.

1° We will denote by \( NC_{\leq 2}(n) \) the set of partitions \( \rho \in NC(n) \) such that every block of \( \rho \) has either 1 or 2 elements.

2° For a partition \( \rho \) in \( NC_{\leq 2}(n) \) we will denote by \( D(\rho) \) the union of all doubletons (2-element blocks) of \( \rho \), and by \( S(\rho) \) the union of all singletons (1-element blocks) of \( \rho \). Thus \( D(\rho) \cup S(\rho) = \{1, \ldots, n\} \) (disjoint union).

3° Let \( \rho \) be in \( NC_{\leq 2}(n) \), and let us consider the partition \( \rho | D(\rho) \in NC(D(\rho)) \). We will denote by \( \widehat{\rho} \) the maximal partition in \( NC(S(\rho)) \) that can be combined with \( \rho | D(\rho) \) into a non-crossing partition of \( \{1, \ldots, n\} \), in the sense discussed in part 4° of the preceding remark. Thus \( \widehat{\rho} \) is characterized by the fact that for a partition \( \sigma \in NC(S(\rho)) \) we have the equivalence

\[
\left( \rho | D(\rho) \right) \sqcup \sigma \in NC(n) \iff \sigma \leq \widehat{\rho}.
\] (5.2)

[A concrete example illustrating the parts 2° and 3° of this notation: say that \( n = 9 \) and that

\[
\rho = \left\{ \{1\}, \{2,8\}, \{3\}, \{4,5\}, \{6\}, \{7\}, \{9\} \right\} \in NC_{\leq 2}(9).
\] (5.3)

Then \( D(\rho) = \{2,4,5,8\}, \ S(\rho) = \{1,3,6,7,9\} \), and we have \( \rho | D(\rho) = \{2,8\}, \{4,5\} \in NC(D(\rho)) \) and \( \widehat{\rho} = \{1,9\}, \{3,6,7\} \in NC(S(\rho)) \).]
Proposition 5.4. Let \( \nu \) be a distribution in \( \mathcal{D}_{\text{alg}}(k) \), and let \( \gamma_t \) be as described in Notation 5.1. For every \( n \geq 1 \) and \( 1 \leq i_1, \ldots, i_n \leq k \) we have

\[
(\nu \boxplus \gamma_t)(X_{i_1} \cdots X_{i_n}) = \sum_{\rho \in \mathcal{N}C_{\leq 2}(n)} \left( \prod_{B \text{ 2-element block of } \rho} t_{\delta_{ip,iq}} \right) \cdot \text{Cf}_{(i_1, \ldots, i_n) \pi(S(\rho))}(M_{\nu}) .
\] (5.4)

Remark 5.5. Let us comment a bit on what is achieved by the formula (5.4). An important point is, of course, that we explicitly identify a combinatorial structure – namely \( \mathcal{N}C_{\leq 2}(n) \) – which indexes the sum leading to \( (\nu \boxplus \gamma_t)(X_{i_1} \cdots X_{i_n}) \). Let us moreover fix a partition \( \rho \in \mathcal{N}C_{\leq 2}(n) \) and let us examine the term indexed by \( \rho \) on the right-hand side of (5.4).

First there is an issue of compatibility. Let us say that “\( \rho \) is compatible with \((i_1, \ldots, i_n)\)” when the following happens: whenever \( B = \{p, q\} \) is a 2-element block of \( \rho \), it follows that \( i_p = i_q \). If \( \rho \) is not compatible with \((i_1, \ldots, i_n)\), then the term indexed by \( \rho \) on the right-hand side of (5.4) vanishes.

Suppose then that \( \rho \) is compatible with \((i_1, \ldots, i_n)\). Let \( S(\rho) = \{b_1 < b_2 < \cdots < b_m\} \) be the set of singletons of \( \rho \), and let \( \hat{\rho} \) be the non-crossing partition of \( S(\rho) \) that was put into evidence in Notation 5.3. The term indexed by \( \rho \) on the right-hand side of (5.4) is then equal to

\[
t^d \text{Cf}_{(i_{b_1},i_{b_2},\ldots,i_{b_m})\hat{\rho}}(M_{\nu}),
\] (5.5)

where \( d = (n-m)/2 \) is the number of doubletons of \( \rho \), and where the generalized coefficient \( \text{Cf}_{(i_{b_1},i_{b_2},\ldots,i_{b_m})\hat{\rho}}(M_{\nu}) \) is as in the above Definition 3.4. (Note the detail that in (5.5) the partition \( \hat{\rho} \) is viewed, in the canonical way, as a partition from \( \mathcal{N}C(m) \).

A concrete example: look again at the example of \( \rho \in \mathcal{N}C(9) \) given for illustration at the end of Notation 5.3. There we had \( S(\rho) = \{1, 3, 6, 7, 9\} \), and \( \hat{\rho} = \{1, 9\}, \{3, 6, 7\} \in \mathcal{N}C(S(\rho)) \). Thus the generalized coefficient of \( M_{\nu} \) we have to look at is \( \text{Cf}_{(i_1,i_3,i_6,i_7,i_9)}(M_{\nu}) \), which is just \( \nu(X_{i_1}X_{i_3})\nu(X_{i_6}X_{i_7}X_{i_9}) \). Hence the term indexed by \( \rho \) in the sum on the right-hand side of (5.4) is in this concrete example equal to

\[
\begin{cases} 
  t^2 \nu(X_{i_1}X_{i_3})\nu(X_{i_6}X_{i_7}X_{i_9}) & \text{if } i_2 = i_8 \text{ and } i_4 = i_5 \\
  0 & \text{otherwise.}
\end{cases}
\]

Remark 5.6. We now move towards proving the formula stated in Proposition 5.4. In preparation of the proof, let us review the basic “moments vs. free cumulants” formula which expresses the moments of a distribution \( \mu \in \mathcal{D}_{\text{alg}}(k) \) in terms of its free cumulants – that is, in terms of the coefficients of the \( R \)-transform \( R_{\mu} \). This formula says that

\[
\text{Cf}_{(i_1, \ldots, i_n)}(M_{\mu}) = \sum_{\pi \in \mathcal{N}C(n)} \text{Cf}_{(i_1, \ldots, i_n)\pi}(R_{\mu}), \quad \forall n \geq 1, \forall 1 \leq i_1, \ldots, i_n \leq k;
\] (5.6)

and more generally, that for any \( \rho \in \mathcal{N}C(n) \) we have

\[
\text{Cf}_{(i_1, \ldots, i_n)\rho}(M_{\mu}) = \sum_{\pi \in \mathcal{N}C(n), \pi \leq \rho} \text{Cf}_{(i_1, \ldots, i_n)\pi}(R_{\mu})
\] (5.7)
(where Equation (5.6) corresponds to the case when $\rho = 1_n$). For more details on this, see Lectures 11 and 16 of [6].

Also in preparation of the proof of Proposition 5.4 it is convenient to introduce the following elements of notation.

**Notation 5.7.** Let $n$ be a positive integer.

1° For $\rho \in NC_{\leq 2}(n)$ and $\pi \in NC(n)$ we will write “$\rho \prec \pi$” to mean that every 2-element block $B$ of $\rho$ also is a block of $\pi$. (That is: if the 2-element blocks of $\rho$ are $B_1, \ldots, B_p$, then $\pi$ must be of the form $\pi = \{B_1, \ldots, B_p, C_1, \ldots, C_q\}$, with $q \geq 0$ and $C_1 \cup \cdots \cup C_q = S(\rho)$.)

2° Let $i_1, \ldots, i_n$ be some indices in $\{1, \ldots, k\}$. We will denote by $S_{\rho \prec \pi}$ the set of partitions $\rho \in NC_{\leq 2}(n; i_1, \ldots, i_n)$ with the property that whenever $B = \{p, q\}$ is a 2-element block of $\rho$, it follows that $i_p = i_q$.

**Proof of Proposition 5.4.** We fix for the whole proof a positive integer $n$ and some indices $1 \leq i_1, \ldots, i_n \leq k$ for which we will prove that Equation (5.4) holds.

We start from the left-hand side of the equation. From the moment-cumulant formula (5.6) and the fact that $R_{\nu \boxplus \gamma_t} = R_{\nu} + R_{\gamma_t}$, we have:

$$
(\nu \boxplus \gamma_t)(X_{i_1} \cdots X_{i_n}) = \sum_{\pi \in NC(n)} \text{Cf}_{(i_1, \ldots, i_n); \pi}(R_{\nu} + R_{\gamma_t}). 
$$

(5.8)

Now let us fix for the moment a partition $\pi \in NC(n)$, and let us look at the term indexed by $\pi$ on the right-hand side of (5.8). We write this term explicitly:

$$
\text{Cf}_{(i_1, \ldots, i_n); \pi}(R_{\nu} + R_{\gamma_t}) = \prod_{\text{A block of } \pi} \left( \text{Cf}_{(i_1, \ldots, i_n); A}(R_{\nu}) + \text{Cf}_{(i_1, \ldots, i_n); A}(R_{\gamma_t}) \right)
$$

(5.9)

and we expand the product on the right-hand side of (5.9) into a sum of $2^{\vert \pi \vert}$ terms. The general term of the sum is obtained by splitting the set of blocks of $\pi$ into two sets of blocks $S_1$ and $S_2$, and by forming the product

$$
\left( \prod_{A \in S_1} \text{Cf}_{(i_1, \ldots, i_n); A}(R_{\nu}) \right) \cdot \left( \prod_{B \in S_2} \text{Cf}_{(i_1, \ldots, i_n); B}(R_{\gamma_t}) \right).
$$

(5.10)

But a fundamental fact about free semicircular systems is that the $R$-transform of $\gamma_t$ is just

$$
R_{\gamma_t}(z_1, \ldots, z_k) = t(z_1^2 + \cdots + z_k^2)
$$

(see [6], Lectures 11 and 16). Thus the second product in (5.10) is non-zero if and only if every block $B \in S_2$ is of the form $B = \{p, q\}$ with $1 \leq p < q \leq n$ such that $i_p = i_q$. When this requirement is satisfied, the set $S_2$ of blocks of $\pi$ corresponds naturally to a partial pairing $\rho \in NC_{\leq 2}(n; i_1, \ldots, i_n)$ such that $\rho \prec \pi$ (where Notation 5.7 is used). For our fixed $\pi \in NC(n)$ we thus arrive to an equation of the form

$$
\text{Cf}_{(i_1, \ldots, i_n); \pi}(R_{\nu} + R_{\gamma_t}) = \sum_{\rho \in NC_{\leq 2}(n; i_1, \ldots, i_n) \text{ such that } \rho \prec \pi} \text{term}_\rho
$$

(5.11)
where the quantities “term_ρ” are further discussed in the next paragraph.

So let π ∈ NC(n) be as in the preceding paragraph, and let ρ ∈ NC≤2(n;i_1,...,i_n) such that ρ < π. In connection to this ρ we will use the notations D(ρ), S(ρ) and ̃ρ ∈ NC(S(ρ)) that were introduced in Notation 5.3. The contribution “term_ρ” to the inner sum over π as (ρ ⊳ π) that every partition π appeared in (5.13). By taking into account the discussion from Notation 5.3, it is immediate that we have

\[ \prod_{A \in S_1} \text{Cf}_{(i_1,\ldots,i_n)} |A(R_\nu) = \text{Cf}_{(i_1,\ldots,i_n)} |S(\rho);(\pi|S(\rho)) (R_\nu). \] (5.12)

The conclusion of the preceding two paragraphs of the proof is that for every π ∈ NC(n) we have

\[ \text{Cf}_{(i_1,\ldots,i_n);\pi}(R_\nu + R_\pi) = \sum_{\rho \in NC\leq2(n;i_1,\ldots,i_n)} t |D(\rho)|/2 \cdot \text{Cf}_{(i_1,\ldots,i_n)} |S(\rho);(\pi|S(\rho)) (R_\nu). \] (5.12)

We now sum over π in Equation (5.12). On the left-hand side the sum over π gives us (ν ⊢ γ_π)(X_i_1 · · · X_i_n), as we knew since (5.8). On the right-hand side of (5.12) we get a double sum, over π and ρ; we interchange the order of summation in this double sum, to obtain:

\[ \sum_{\rho \in NC\leq2(n;i_1,\ldots,i_n)} t |D(\rho)|/2 \left( \sum_{\pi \in NC(n)} \text{Cf}_{(i_1,\ldots,i_n)} |S(\rho);(\pi|S(\rho)) (R_\nu) \right). \] (5.13)

It is now the turn of ρ to be fixed, while we examine the summation over π that has appeared in (5.13). By taking into account the discussion from Notation 5.3 it is immediate that every partition π ∈ NC(n) with the property that ρ < π is obtained in a unique way as (ρ | D(ρ)) ⊳ σ, where σ ∈ NC(S(ρ)) is such that σ ≤ ̃ρ (see the equivalence (5.2) in Notation 5.3). It follows that the inside sum over π in (5.13) is equal to

\[ \sum_{\sigma \in NC(S(\rho))} \text{Cf}_{(i_1,\ldots,i_n)} |S(\rho);\sigma (R_\nu). \] (5.13)

But the latter quantity is in turn equal to \text{Cf}_{(i_1,\ldots,i_n)} |S(\rho);\sigma (M_\nu), due to the moments vs. free cumulant formula (used now in the more general form that was reviewed in (5.7)). Replacing this in (5.13) takes us precisely to the right-hand side of Equation (5.14), and this concludes the proof.

\[ \square \]

6. Relation between \( \mathbb{B}_t \) and the free Brownian motion

Recall from Remark 2.3 that the map \( \mu \to \eta_\mu \) is a bijection from \( D_{\text{alg}}(k) \) onto the space of series \( \mathbb{C}_0(\langle z_1,\ldots,z_k \rangle) \). It thus makes sense to define a map \( \Phi : D_{\text{alg}}(k) \to D_{\text{alg}}(k) \) via the \( \eta \)-series prescription described as follows.
**Definition 6.1.** For every $\nu \in \mathcal{D}_{\text{alg}}(k)$, we let $\Phi(\nu)$ be the unique distribution $\mu \in \mathcal{D}_{\text{alg}}(k)$ which has $\eta$-series given by:

$$
\eta_\mu(z_1, \ldots, z_k) = \sum_{i=1}^{k} z_i \left(1 + M_\nu(z_1, \ldots, z_k)\right) z_i.
$$

(6.1)

Our goal in the present section is to prove the following result.

**Theorem 6.2.** Let $\nu$ be a distribution in $\mathcal{D}_{\text{alg}}(k)$. We have that

$$
\Phi(\nu \boxplus \gamma_t) = B_t(\Phi(\nu)), \quad \forall \ t > 0,
$$

(6.2)

where $\gamma_t \in \mathcal{D}_{\text{alg}}(k)$ is the distribution of the scaled free semicircular system from Notation 5.1.

A key point in the proof of Theorem 6.2 will be to use a natural combinatorial construction of “assigning singletons to doubletons” in a partial pairing, which is described next.

**Remark 6.3.** (“Assigning singletons to doubletons for $\rho \in NC_{\leq 2}(n)$.”) Let a partition $\rho \in NC_{\leq 2}(n)$ be given. We will denote by $\alpha(\rho)$ the non-crossing partition of $\{0, 1, \ldots, n, n+1\}$ which is obtained as follows. Start with the partial pairing of $\{0, 1, \ldots, n, n+1\}$ that is obtained by adding to $\rho$ the 2-element block $\{0, n+1\}$. Consider the picture of this new partial pairing (drawn in the usual way – with the points $0, 1, \ldots, n, n+1$ on a horizontal line, and with a family of non-intersecting “hooks” drawn under that horizontal line, to represent the 2-element blocks of the partial pairing). In this picture we draw some additional vertical line segments, starting at every singleton of $\rho$, and going down until they meet a hook representing a doubleton. When these new vertical segments are added to the picture, we now have the picture of a non-crossing partition of $\{0, 1, \ldots, n, n+1\}$, which will be denoted by $\alpha(\rho)$.

A concrete example: if $n = 9$ and $\rho \in NC_{\leq 2}(n)$ is as in (5.3) from Notation 5.3 then $\alpha(\rho) = \{ \{0, 1, 9, 10\}, \{2, 3, 6, 7, 8\}, \{4, 5\} \}$, and the pictures of $\rho$ and of $\alpha(\rho)$ look as follows:

$$
\rho = \begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\
\end{array} \quad \Rightarrow \quad \alpha(\rho) = \begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
$$

Clearly, the definition of $\alpha(\rho)$ could also be stated without referring to pictures. That is, the rule for assigning the singletons of $\rho$ to doubletons (in order to create $\alpha(\rho)$) can be expressed in plain algebraic terms. Indeed, for every 1-element block $\{i\}$ of $\rho$, exactly one of the following two possibilities (1) and (2) applies:

(1) Either there is no 2-element block $B = \{p, q\}$ of $\rho$ such that $p < i < q$. In this case $i$ is assigned to the doubleton $\{0, n+1\}$ that was added to $\rho$.

(2) Or there exist 2-element blocks $B = \{p, q\}$ of $\rho$ such that $p < i < q$. Due to the fact that $\rho$ is non-crossing, among these blocks there has to exist one, $B_o = \{p_o, q_o\}$, which is
nested inside all the others (we have \( p < q \) and \( q > q \), for every block \( B = \{p, q\} \), \( B \neq B_o \), such that \( p < i < q \). In this case the singleton \( i \) is assigned to the doubleton \( B_o \).

The construction of \( \alpha(p) \) described above defines a map

\[
\alpha : NC_{\leq 2}(n) \to NC(\{0, 1, \ldots, n + 1\}),
\]

the “assign-singletons-to-doubletons” map. It is easily checked that the image of \( \alpha \) is

\[
\left\{ \pi \in NC(\{0, 1, \ldots, n + 1\}) \mid 0 \sim_{\pi} n + 1 \text{ and } \pi \text{ has no 1-element blocks} \right\},
\]

where the notation “\( 0 \sim_{\pi} n + 1 \)” in (6.4) is a shorthand for “\( 0 \) and \( n + 1 \) belong to the same block of \( \pi \)”. It is also immediate that the map \( \alpha \) from (6.3) is one-to-one. The map

\[
\beta : \left\{ \pi \in NC(\{0, 1, \ldots, n + 1\}) \mid 0 \sim_{\pi} n + 1 \text{ and } \pi \text{ has no 1-element blocks} \right\} \to NC_{\leq 2}(n)
\]

which is inverse to \( \alpha \) is described as follows. Let \( \pi = \{A_1, \ldots, A_p\} \) be a partition from the set (6.4), and say that \( A_1 \) is the block of \( \pi \) that contains 0 and \( n + 1 \). Then

\[
\beta(\pi) = \{B_2, \ldots, B_p\} \cup \{\{i\} \mid i \in \{1, \ldots, n\} \setminus (B_2 \cup \cdots \cup B_p)\},
\]

where for every \( 2 \leq i \leq p \) we denoted \( B_i := \{\min(A_i), \max(A_i)\} \subseteq \{1, \ldots, n\} \).

**Proof of Theorem 6.2.** Fix \( t > 0 \) for which we will prove that (6.2) holds. We will prove this equality by showing that the distributions on its two sides have the same \( \eta \)-series:

\[
\eta_{\Phi(\nu \boxplus \gamma_t)} = \eta_{\beta_t(\Phi(\nu))}.
\]

We first observe that on both sides of (6.7) we have series in \( C_0(\langle z_1, \ldots, z_k \rangle) \) that are of the form

\[
\left( \sum_{i=1}^{k} z_i^2 \right) + (\text{terms of order } \geq 3).
\]

Indeed, from the definition of \( \Phi \) in Equation (6.1) it is clear that \( \eta_{\Phi(\sigma)} \) is of the form (6.8) for every \( \sigma \in D_{\text{alg}}(k) \), and this applies in particular to the left-hand side of (6.7). On the right-hand side of (6.7) we first invoke Remark 4.4 and write

\[
\eta_{\beta_t(\Phi(\nu))} = \frac{1}{t} \text{Reta} \left( t \eta_{\Phi(\nu)} \right);
\]

then we use the fact that \( \eta_{\Phi(\nu)} \) is of the form (6.8), combined with the observation (see Remark 3.6) that applying \( \text{Reta} \) does not change the linear and quadratic terms of a series in \( C_0(\langle z_1, \ldots, z_k \rangle) \).

In order to prove (6.7), we should thus fix a monomial of length \( \geq 3 \) in \( z_1, \ldots, z_k \), and prove that the coefficients for this monomial in \( \eta_{\Phi(\nu \boxplus \gamma_t)} \) and in \( \eta_{\beta_t(\Phi(\nu))} \) are equal to each other. It will be convenient to denote our fixed monomial in \( z_1, \ldots, z_k \) as \( z_{i_0} z_{i_1} \cdots z_{i_n} z_{i_{n+1}} \) for some \( n \geq 1 \) and \( i_0, i_1, \ldots, i_{n+1} \leq k \). Our job for the remaining of the proof is to verify that

\[
\text{Cl}_{(i_0, i_1, \ldots, i_{n+1})} \left( \eta_{\Phi(\nu \boxplus \gamma_t)} \right) = \text{Cl}_{(i_0, i_1, \ldots, i_{n+1})} \left( \eta_{\beta_t(\Phi(\nu))} \right),
\]

(6.9)
for this fixed $n$ and $i_0, i_1, \ldots, i_{n+1}$.

On the left-hand side of (6.9) we have

$$\text{Cf}_{(i_0, i_1, \ldots, i_{n+1})}(\eta \Phi(\nu \gamma_\ell)) = \delta_{i_0, i_{n+1}} \cdot \text{Cf}_{(i_1, \ldots, i_n)}(M_{\nu \gamma_\ell}) \quad \text{(by Equation (6.1))}$$

$$= \delta_{i_0, i_{n+1}} \cdot (\nu \gamma_\ell)(X_{i_1} \cdots X_{i_n}).$$

The latter moment is exactly of the kind studied in Section 5 of the paper, and can be expressed (by Proposition 5.4) as a summation indexed by $NC_{\leq 2}(n)$. Thus for the left-hand side of (6.9) we have

$$\text{Cf}_{(i_0, i_1, \ldots, i_{n+1})}(\eta \Phi(\nu \gamma_\ell)) = \delta_{i_0, i_{n+1}} \cdot \sum_{\rho \in NC_{\leq 2}(n)} \text{term}'_\rho,$$  \hspace{1cm} (6.10)

where for every $\rho \in NC_{\leq 2}(n)$ the contribution $\text{term}'_\rho$ of $\rho$ is as on the right-hand side of Equation (5.4) in Proposition 5.4.

On the right-hand side of Equation (6.9) we go as follows:

$$\text{Cf}_{(i_0, i_1, \ldots, i_{n+1})}(\eta \Phi(\nu \gamma_\ell)) = \frac{1}{l} \sum_{\pi \in NC\{0,1,\ldots,n+1\}} \text{Cf}_{(i_0, i_1, \ldots, i_{n+1});\pi}(t \eta \Phi(\nu)) \quad \text{(by Proposition 3.5)}$$

$$= \sum_{\pi \in NC\{0,1,\ldots,n+1\}} t^{\|\pi|-1} \text{Cf}_{(i_0, i_1, \ldots, i_{n+1});\pi}(\eta \Phi(\nu)).$$  \hspace{1cm} (6.11)

Observe that the summation in (6.11) may in fact be restricted to those partitions in $\pi \in NC\{0,1,\ldots,n+1\}$ which (in addition to the condition that $0 \ncong n + 1$) are required to have no singleton blocks; this is because $\eta \Phi(\nu)$ has no linear terms (see the discussion around (6.8) above), thus $\text{Cf}_{(i_0, i_1, \ldots, i_{n+1});\pi}(\eta \Phi(\nu)) = 0$ whenever $\pi$ has singleton blocks. So for the right-hand side of (6.9) we arrive to the formula

$$\text{Cf}_{(i_0, i_1, \ldots, i_{n+1})}(\eta \Phi(t \nu \gamma_\ell)) = \sum_\pi \text{term}''_\pi,$$  \hspace{1cm} (6.12)

where $\pi$ runs precisely in the set described in (6.4) of Remark 6.3, and where for such $\pi$ we put

$$\text{term}''_\pi := t^{|\pi|-1} \cdot \text{Cf}_{(i_0, i_1, \ldots, i_{n+1});\pi}(\eta \Phi(\nu))$$

$$= t^{|\pi|-1} \cdot \prod_{A \text{ a block of } \pi} \delta_{i_{m_1}, i_{m_p}} \nu(X_{i_{m_2}} \cdots X_{i_{m_{p-1}}}).$$  \hspace{1cm} (6.13)

When writing (6.13) we also took into account how $\eta \Phi(\nu)$ is defined by Equation (6.1).

Let us next observe that if the indices $i_0, i_1, \ldots, i_{n+1}$ fixed since (6.9) do not satisfy the condition $i_0 = i_{n+1}$, then the right-hand sides of both (6.10) and (6.12) vanish. This is clear
for (6.10), while for (6.12) we argue as follows: if \( i_0 \neq i_{n+1} \) then the product in (6.13) is guaranteed to vanish (since one of the blocks of \( \pi \) contains 0 and \( n+1 \)), hence every term \( \text{term}_\pi'' \) on the right-hand side of (6.12) is equal to 0.

So let us then assume that \( i_0 = i_{n+1} \). The equality (6.9) that we have to prove is reduced (by virtue of (6.10) and (6.12)) to

\[
\sum_{\rho \in NC_{\leq 2}(n)} \text{term}_\rho' = \sum_{\pi \text{ in the set from (6.4)}} \text{term}_\pi'',
\]

where the quantities \( \text{term}_\rho' \) and \( \text{term}_\pi'' \) are described in Equations (5.4) and (6.11), respectively. But the equality (6.14) is immediately verified by using the “assign-singletons-to-doubletons” construction from Remark 6.3. Indeed, in Remark 6.3 we pointed out a natural bijection \( \beta \) from the set in (6.4) onto \( NC_{\leq 2}(n) \), and by using the explicit description provided there for \( \beta \) it is immediately seen that \( \text{term}_\pi'' = \text{term}_\beta''(\pi) \), for every \( \pi \) in the set from (6.4). Thus \( \beta \) provides a term-by-term identification of the sums on the two sides of (6.14), and this completes the proof. \( \square \)

7. Restricting to the framework of \( D_c(k) \)

In this section we show that the results from the Sections 4–6 of the paper continue to hold when we work in \( C^* \)-framework.

**Definition 7.1.** We denote

\[
D_c(k) = \left\{ \mu \in D_{\text{alg}}(k) \mid \exists \text{ } C^*\text{-probability space } (A, \varphi) \text{ and selfadjoint elements } x_1, \ldots, x_k \in A \text{ such that } \mu \circ x_1,\ldots,x_k = \mu \right\}
\]

(7.1)

(where the joint distribution \( \mu \circ x_1,\ldots,x_k \) is defined as in Equation (2.2) from Definition 2.1). The fact that \( (A, \varphi) \) is a \( C^* \)-probability space means here that \( A \) is a unital \( C^* \)-algebra and that \( \varphi : A \to \mathbb{C} \) is a positive linear functional such that \( \varphi(1_A) = 1 \).

The notation “\( D_c(k) \)” is chosen to remind of “distributions with compact support” – indeed, in the case when \( k = 1 \) we have a natural identification between \( D_c(1) \) and the set of probability distributions with compact support on \( \mathbb{R} \).

**Remark 7.2.** In the preceding sections, the operations \( \boxplus \) and \( \boxdot \) and the convolution powers with respect to them were considered in the larger framework of the space \( D_{\text{alg}}(k) \). But by considering sums of freely independent and respectively Boolean independent \( k \)-tuples of selfadjoint elements in a \( C^* \)-probability space, one sees that if \( \mu, \nu \in D_c(k) \) then \( \mu \boxplus \nu \) and \( \mu \boxdot \nu \) belong to \( D_c(k) \) as well. Hence \( \boxplus \) and \( \boxdot \) make sense as binary operations on \( D_c(k) \). Moreover, concerning convolution powers we have that

\[
\mu \in D_c(k) \Rightarrow \left\{ \begin{array}{ll}
\text{(a)} & \mu^\boxplus t \in D_c(k) \quad \forall t \geq 1, \text{ and } \\
\text{(b)} & \mu^\boxdot t \in D_c(k) \quad \forall t > 0.
\end{array} \right.
\]

(7.2)
The fact stated in (7.2(a)) was proved in [5], by using compressions with free projections. The proof of (7.2(b)) is done by constructing an operator model for $\mu^{\text{out}}$ – see Remark 4.7 and Proposition 4.8 of [1].

The following result is then an immediate consequence of (7.2) and of what was proved in algebraic framework in Theorem 4.11.

Corollary 7.3. 1° For every $t \geq 0$ it makes sense to define $B_t: D_c(k) \rightarrow D_c(k)$ by the formula

$$B_t(\mu) = \left(\mu \boxplus (1+t)\right) \boxplus (1/1+t)), \forall \mu \in D_c(k).$$ (7.3)

2° The transformations of $D_c(k)$ defined by (7.3) form a semigroup: $B_s \circ B_t = B_{s+t}$, $\forall s, t \geq 0$.

3° For $t = 1$ we have $B_1(\mu) = B(\mu), \forall \mu \in D_c(k)$, where $B$ is the multi-variable Boolean Bercovici-Pata bijection from Theorems 1 and 1’ of the paper [1]. □

In the remaining part of this section we will show that the above Theorem 6.2 also carries through to the $C^*$-framework. The main point that needs to be addressed is that the map $\Phi : D_{\text{alg}}(k) \rightarrow D_{\text{alg}}(k)$ introduced in Definition 6.1 sends $D_c(k)$ into itself. We will prove this via an “operator model” for $\Phi$, described in the next remark and theorem.

Remark 7.4. (The operator model for $\Phi$.) The input for this operator model is a system

$$(\mathcal{H}; a_1, \ldots, a_k; \xi_0)$$

where $\mathcal{H}$ is a Hilbert space, $a_1, \ldots, a_k \in B(\mathcal{H})$ are selfadjoint operators, and $\xi_0 \in \mathcal{H}$ is a unit vector. Starting from this data, we proceed as follows:

(i) We consider the Hilbert space $\mathcal{K} := \mathbb{C} \oplus \left(\bigoplus_{j=1}^k \mathcal{H}\right)$, and the unit vector $\Omega_0 := 1 \oplus 0 \oplus 0 \oplus \cdots \oplus 0 \in \mathcal{K}$. For $1 \leq j \leq k$ we let $v_j: \mathcal{H} \rightarrow \mathcal{K}$ be the embedding defined by

$$v_j(\xi) = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus \xi \oplus 0 \oplus \cdots \oplus 0 \in \mathcal{K}, \quad \xi \in \mathcal{H}.$$ The direct sum defining $\mathcal{K}$ can thus also be written as $\mathcal{K} = \mathbb{C}\Omega_0 \oplus v_1(\mathcal{H}) \oplus \cdots \oplus v_k(\mathcal{H})$.

(ii) For $1 \leq j \leq k$ we denote $v_j\xi_0 =: \Omega_j \in \mathcal{K}$, and we consider the rank-one partial isometry $w_j \in B(\mathcal{K})$ which carries $\Omega_0$ to $\Omega_j$. The operator $w_j$ and its adjoint are thus described by the formulas:

$$w_j \eta = \langle \eta, \Omega_0 \rangle \Omega_j, \quad \text{and} \quad w_j^* \eta = \langle \eta, \Omega_j \rangle \Omega_0, \quad \forall \eta \in \mathcal{K}.$$ (iii) For $1 \leq j \leq k$ we consider the selfadjoint operators $x_j, y_j \in B(\mathcal{K})$ defined by

$$x_j := 0 \oplus a_1 \oplus \cdots \oplus a_j \quad \text{and} \quad y_j := w_j + x_j + w_j^*.$$ The system $(\mathcal{K}; y_1, \ldots, y_k; \Omega_0)$ will be called the output of the operator model for $\Phi$. The terms “input” and “output” used in this construction are justified by the following theorem.
**Theorem 7.5.** Let \((\mathcal{H}; a_1, \ldots, a_k; \xi_o)\) and \((\mathcal{K}; y_1, \ldots, y_k; \Omega_0)\) be as in Remark 7.4. Let \(\nu\) be the joint distribution of \(a_1, \ldots, a_k\) with respect to the vector-state \(\langle \cdot, \xi_o, \xi_o \rangle\) on \(B(\mathcal{H})\), and let \(\mu\) be the joint distribution of \(y_1, \ldots, y_k\) with respect to the vector-state \(\langle \cdot, \Omega_0, \Omega_0 \rangle\) on \(B(\mathcal{K})\). Then \(\Phi(\nu) = \mu\).

**Remark 7.6.** In preparation of the proof of Theorem 7.5 we review here the “moments vs. Boolean cumulants” formula, which expresses the moments of a distribution \(\mu \in \mathcal{D}_{\text{alg}}(k)\) in terms of its Boolean cumulants – that is, in terms of the coefficients of the \(\eta\)-series \(\eta_{\mu}\). This is very similar to the moment-cumulant formula reviewed in Remark 5.6 in connection to free cumulants, with the difference that we now only consider summations over the subposet of \(\text{NC}(n)\) consisting of interval partitions.

A partition \(\pi\) of \(\{1, \ldots, n\}\) is said to be an interval partition when every block of \(\pi\) is of the form \([a, b] \cap \mathbb{Z}\) for some \(a \leq b\) in \(\{1, \ldots, n\}\). The set of all interval partitions of \(\{1, \ldots, n\}\) will be denoted by \(\text{Int}(n)\). It is clear that \(\text{Int}(n) \subseteq \text{NC}(n)\). The “moments vs. Boolean cumulants” formula says that for a distribution \(\mu \in \mathcal{D}_{\text{alg}}(k)\) we have

\[
\text{Cf}_{(i_1, \ldots, i_n)}(M_{\mu}) = \sum_{\pi \in \text{Int}(n)} \text{Cf}_{(i_1, \ldots, i_n); \pi}(\eta_{\mu}), \quad \forall n \geq 1, \quad \forall 1 \leq i_1, \ldots, i_n \leq k.
\]

Equation (7.4) is easily seen to be equivalent to the formula “\(\eta_{\mu} = \mu(1 + M_{\mu})^{-1}\)” used in the above Remark 7.4 as definition for the \(\eta\)-series of \(\mu\) (for a proof of this equivalence, see for instance Proposition 3.5 in [1]).

**Remark 7.7.** We now return to the notations from Remark 7.4 and record how the operators \(w_i, x_i, w_i^* (1 \leq i \leq k)\) behave with respect to the direct sum decomposition \(\mathcal{K} = \mathbb{C}\Omega_0 \oplus v_1(\mathcal{H}) \oplus \cdots \oplus v_k(\mathcal{H})\): we have that

\[
\begin{align*}
\{ & w_i \text{ sends } \mathbb{C}\Omega_0 \text{ to } v_i(\mathcal{H}) \text{ and sends } v_1(\mathcal{H}), \ldots, v_k(\mathcal{H}) \text{ to } 0; \\
& x_i \text{ sends } \mathbb{C}\Omega_0 \text{ to } 0 \text{ and sends every } v_i(\mathcal{H}) \text{ into itself, } 1 \leq i \leq k; \\
& w_i^* \text{ sends } v_i(\mathcal{H}) \text{ into } \mathbb{C}\Omega_0 \text{ and sends } \mathbb{C}\Omega_0 \text{ and every } v_j(\mathcal{H}) \text{ with } j \neq i \text{ to } 0.
\end{align*}
\]

The verification of (7.5) is immediate from the explicit formulas describing \(w_i, x_i, w_i^*\) in Remark 7.4.

**Lemma 7.8.** Consider the notations from Remark 7.4 and let \(\nu\) denote the joint distribution of \(a_1, \ldots, a_k\) with respect to the vector-state \(\langle \cdot, \xi_o, \xi_o \rangle\) on \(B(\mathcal{H})\). Let \(j_1, \ldots, j_m\) and \(i', i''\) be some indices in \(\{1, \ldots, k\}\). Then we have

\[
w_{i'}x_{j_1} \cdots x_{j_m}w_{i''}\Omega_0 = \lambda\Omega_0,
\]

where \(\lambda = \text{Cf}_{(i', j_1, \ldots, j_m, i'')} \left( \eta_{\Phi(\nu)} \right)\).
Proof. If \( i' \neq i'' \) then both sides of Equation (7.6) are equal to 0: the right-hand side vanishes because of how \( \eta_{\Phi(\nu)} \) is defined (see Definition 6.1), while the vanishing on the left-hand side follows immediately from the operating rules described in (7.5). So we will assume that \( i' = i'' =: i \), when the relation that has to be proved becomes:

\[
W_i^* x_{j_1} \cdots x_{j_m} w_i \Omega_0 = \nu(X_{j_1} \cdots X_{j_m}) \Omega_0.
\]

We have \( w_i(\Omega_0) = \Omega_i \), and directly from the definition of \( x_1, \ldots, x_k \) we observe that \( x_{j_1} \cdots x_{j_m} \Omega_i = v_i(a_{j_1} \cdots a_{j_m} \xi_0) \). But then:

\[
W_i^* x_{j_1} \cdots x_{j_m} w_i \Omega_0 = W_i^* v_i(a_{j_1} \cdots a_{j_m} \xi_0)
\]

\[
= \langle v_i(a_{j_1} \cdots a_{j_m} \xi_0), \Omega_i \rangle \Omega_0
\]

\[
= \langle a_{j_1} \cdots a_{j_m} \xi_0, \xi_0 \rangle \Omega_0 \quad \text{(since } v_i^* \Omega_i = \xi_0 \text{)}
\]

\[
= \nu(X_{j_1} \cdots X_{j_m}) \Omega_0,
\]

as required. \( \square \)

Lemma 7.9. Consider the notations from Remark 7.4, and let \( \nu \) denote the joint distribution of \( a_1, \ldots, a_k \) with respect to the vector-state \( \langle \cdot, \xi_0, \Omega \rangle \) on \( B(H) \). Let \( i_1, \ldots, i_n \) be some indices in \( \{1, \ldots, k\} \). Let \( \pi \) be a partition in \( \text{Int}(n) \) which has no 1-element blocks, and which is written explicitly as \( \pi = \{ \{a_1, \ldots, b_1\}, \ldots, \{a_p, \ldots, b_p\} \} \), with \( 1 = a_1 < b_1 < \cdots < a_p < b_p = n \) (and where \( a_0 = b_1 + 1, \ldots, a_p = b_p - 1 \)). Consider the operators \( u_1, \ldots, u_n \in B(K) \) defined as follows:

\[
\begin{cases}
  u_{a_1} = W_{i_{a_1}}^*, \ldots, u_{a_p} = W_{i_{a_p}}^*, \\
  u_{b_1} = W_{i_{b_1}}^*, \ldots, u_{b_p} = W_{i_{b_p}}^*, \\
  u_c = x_{i_c} \text{ for every } c \in \{1, \ldots, n\} \setminus \{a_1, b_1, \ldots, a_p, b_p\}.
\end{cases}
\]

Then we have

\[
\langle u_1 \cdots u_n \Omega_0, \Omega_0 \rangle = Cf_{(i_1, \ldots, i_n); \pi}(\eta_{\Phi(\nu)}).
\]  

(7.8)

Proof. By picking out the last \( b_p - a_p + 1 \) factors in the product \( u_1 \cdots u_n \) applied to the vector \( \Omega_0 \) we get:

\[
u_{a_p} u_{a_p+1} \cdots u_{b_p-1} u_{b_p} \Omega_0 = W_{i_{a_p}}^* \cdot \prod_{a_p < c < b_p} x_{i_c} \cdot W_{i_{b_p}} \Omega_0 = Cf_{(i_{a_p}, i_{a_p+1}, \ldots, i_{b_p})}(\eta_{\Phi(\nu)}) \Omega_0,
\]

(7.9)

where at the second equality sign we invoked Lemma 7.8. Thus

\[
u_{a_p} \cdots u_{b_p-1} \Omega_0 = Cf_{(i_{a_p}, i_{a_p+1}, \ldots, i_{b_p})}(\eta_{\Phi(\nu)}) (u_{a_p} \cdots u_{b_p} \Omega_0)
\]

\[
u_{a_p} \cdots u_{b_p-1} \Omega_0 = Cf_{(i_{a_p}, i_{a_p+1}, \ldots, i_{b_p})}(\eta_{\Phi(\nu)}) (u_{a_p} \cdots u_{b_p} \Omega_0)
\]

\[
u_{a_p} \cdots u_{b_p-1} \Omega_0 = Cf_{(i_{a_p}, i_{a_p+1}, \ldots, i_{b_p})}(\eta_{\Phi(\nu)}) (u_{a_p} \cdots u_{b_p} \Omega_0)
\]
Now the same trick as in (7.9) can be applied to the right-most piece $u_{a_{p-1}} \cdots u_{b_{p-1}} \Omega_0$ of $u_1 \cdots u_{b_{p-1}} \Omega_0$. By iterating this trick we arrive to required the conclusion that

$$\langle u_1 \cdots u_n \Omega_0, \Omega_0 \rangle = \text{Cf}(i_{a_1}, \ldots, i_{b_n})(\eta \Phi(\nu)) \cdots \text{Cf}(i_{a_p}, \ldots, i_{b_p})(\eta \Phi(\nu)) = \text{Cf}(i_{1}, \ldots, i_{n}; \pi)(\eta \Phi(\nu)).$$

□

Proof of Theorem 7.5. We fix for the whole proof a positive integer $n$ and some indices $1 \leq i_1, \ldots, i_n \leq k$, for which we verify that

$$\mu(X_{i_1} \cdots X_{i_n}) = (\Phi(\nu))(X_{i_1} \cdots X_{i_n}). \quad (7.10)$$

By the definition of $\mu$, the left-hand side of (7.10) is

$$\mu(X_{i_1} \cdots X_{i_n}) = \langle y_{i_1} \cdots y_{i_n} \Omega_0, \Omega_0 \rangle = \langle (w_{i_1} + x_{i_1} + w^*_{i_1}) \cdots (w_{i_1} + x_{i_1} + w^*_{i_1}) \Omega_0, \Omega_0 \rangle,$n

and the latter quantity expands as a sum of $3^n$ terms of the form $\langle u_1 \cdots u_n \Omega_0, \Omega_0 \rangle$, with $u_1 \in \{w_{i_1}, x_{i_1}, w^*_{i_1}\}, \ldots, u_n \in \{w_{i_n}, x_{i_n}, w^*_{i_n}\}$. (7.11)

But from the rules (7.5) for how the operators $w_i, x_i, w^*_i$ act on the decomposition $C \Omega_0 \oplus v_1(\mathcal{H}) \oplus \cdots \oplus v_k(\mathcal{H})$ of $\mathcal{K}$ it follows that many of these $3^n$ terms vanish. We leave it as an easy exercise to the reader to verify that we have in fact $\langle u_1 \cdots u_n \Omega_0, \Omega_0 \rangle = 0$ whenever $u_1, \ldots, u_n$ from (7.11) are not chosen according to the recipe (7.7) from Lemma 7.9. By taking Lemma 7.9 into account, we thus find that

$$\mu(X_{i_1} \cdots X_{i_n}) = \sum_{\pi \in \text{Int}(n), \text{with no singletons}} \text{Cf}(i_{1}, \ldots, i_{n}; \pi)(\eta \Phi(\nu)). \quad (7.12)$$

It remains to note that on the right-hand side of (7.12) it does not cost anything to add the terms $\text{Cf}(i_{1}, \ldots, i_{n}; \pi)(\eta \Phi(\nu))$ where $\pi \in \text{Int}(n)$ has some singleton blocks; indeed, each of these added terms is in fact equal to 0, because the linear terms of the series $\eta \Phi(\nu)$ vanish. So from (7.12) we can write:

$$\mu(X_{i_1} \cdots X_{i_n}) = \sum_{\pi \in \text{Int}(n)} \text{Cf}(i_{1}, \ldots, i_{n}; \pi)(\eta \Phi(\nu)) = \text{Cf}(i_{1}, \ldots, i_{n})(M \Phi(\nu)) \quad \text{(by Remark 7.6)}$$

$$= (\Phi(\nu))(X_{i_1} \cdots X_{i_n}),$$

which is what we wanted to obtain. □

Corollary 7.10. The map $\Phi : D_{\text{alg}}(k) \rightarrow D_{\text{alg}}(k)$ introduced in Definition 6.1 carries $D_c(k)$ into itself.
Proof. Let $\nu$ be in $\mathcal{D}_c(k)$. By using the GNS construction one can realize $\nu$ as the joint distribution of a $k$-tuple $a_1, \ldots, a_k$ of selfadjoint operators on a Hilbert space $\mathcal{H}$, with respect to a vector-state $(\cdot, \xi_0, \xi_0)$ on $B(\mathcal{H})$. Then Theorem 7.5 gives $\Phi(\nu)$ as the joint distribution of $y_1, \ldots, y_k \in B(K)$ with respect to $(\cdot, \Omega_0, \Omega_0)$, where $(K; y_1, \ldots, y_k; \Omega_0)$ are constructed as in Remark 7.4. This implies that $\Phi(\nu) \in \mathcal{D}_c(k)$. □

It thus follows that the statement of Theorem 6.2 also holds in $C^*$-framework:

**Corollary 7.11.** Let $\nu$ be a distribution in $\mathcal{D}_c(k)$. Then for every $t > 0$ we have

$$
\Phi(\nu \boxplus \gamma_t) = \mathbb{B}_t(\Phi(\nu)) \in \mathcal{D}_c(k),
$$

(7.13)

where $\gamma_t$ is the distribution of the scaled free semicircular system from Notation 5.1. □

**References**

[1] S. T. Belinschi, A. Nica. $\eta$-series and a Boolean Bercovici–Pata bijection for bounded $k$-tuples, to appear in *Advances in Mathematics*. Available at arXiv:math.OA/0608622.

[2] S.T. Belinschi, A. Nica. On a remarkable semigroup of homomorphisms with respect to free multiplicative convolution. Preprint 2007. Available at arXiv:math.OA/0703295.

[3] H. Bercovici, V. Pata. Stable laws and domains of attraction in free probability theory. With an Appendix by P. Biane: The density of free stable distributions, *Annals of Mathematics* 149 (1999), 1023-1060.

[4] H. Maassen. Addition of freely independent random variables, *Journal of Functional Analysis* 106 (1992), 409–438.

[5] A. Nica, R. Speicher. On the multiplication of free $n$-tuples of noncommutative random variables. With an appendix by D. Voiculescu: Alternative proofs for the type II free Poisson variables and for the free compression results, *American Journal of Mathematics* 118 (1996), 799-837.

[6] A. Nica, R. Speicher. *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series 335, Cambridge University Press, 2006.

[7] R. Speicher. Multiplicative functions on the lattice of noncrossing partitions and free convolution, *Mathematische Annalen* 298 (1994), 611-628.

[8] W. von Waldenfels. An approach to the theory of pressure broadening of spectral lines, in *Probability and Information Theory II* (M. Behara, K. Krickeberg, J. Wolfowitz, Editors), Springer Lecture Notes in Mathematics 296 (1973), 19-69.
Serban T. Belinschi: University of Waterloo and IMAR.  
Address: Department of Pure Mathematics, University of Waterloo,  
Waterloo, Ontario N2L 3G1, Canada.  
Email: sbelinsc@math.uwaterloo.ca

Alexandru Nica: University of Waterloo.  
Address: Department of Pure Mathematics, University of Waterloo,  
Waterloo, Ontario N2L 3G1, Canada.  
Email: anica@math.uwaterloo.ca