SOME STATISTICS FOR MEASURING LARGE-SCALE STRUCTURE

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ABSTRACT

Good statistics for measuring large-scale structure in the Universe must be able to distinguish between different models of structure formation. In this paper, two and three dimensional “counts in cell” statistics and a new “discrete genus statistic” are applied to toy versions of several popular theories of structure formation: random phase cold dark matter model, cosmic string models, and global texture scenario. All three statistics appear quite promising in terms of differentiating between the models.

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1. Introduction

Recent redshift surveys extending to greater than $100\,h^{-1}\,\text{Mpc}$ seem, upon visual inspection, to be dominated by voids, sheets and filaments\textsuperscript{1,2}. Are these structures real and are they significant? Are they consistent with currently popular models of structure formation? In order to answer these questions, we need statistical methods. Since most of the currently popular theories of structure formation predict a similar (namely scale invariant) spectrum of density perturbations\textsuperscript{3}, we require statistical measures which can pick out the phase information which distinguishes between the models.

More specifically, we are interested in statistics which can clearly distinguish between the two most popular classes of theories: models based on random phase fluctuations produced during inflation (\textit{e.g.}, the cold dark matter (CDM) model) on one hand and topological defect models on the other. A good statistic must also be able to differentiate between the various topological defect models – as representative examples we pick the global texture scenario, a model based on cosmic string wakes, and a filament model.

In this paper we discuss three promising statistics: two and three dimensional counts in cell (CIC) statistics\textsuperscript{4}, and a discrete genus statistic. We apply these statistics to our toy models of structure formation and conclude that for sufficiently small observational error bars the statistics will be able to clearly differentiate between the models.

In Section 2 we define our toy models. In Section 3 we define the “discrete genus statistic” and apply it to our toy models. In Section 4, we investigate the two dimensional and three dimensional CIC statistics. We compare the results of the two dimensional CIC statistic with the results for two slices of the CFA2 survey\textsuperscript{1}. The final section contains a discussion of the results and ideas for future work.

This paper is based on senior theses by D.K.\textsuperscript{5} and S.R.\textsuperscript{6}.
2. Toy Models

The purpose of this paper is primarily to study the effectiveness of the statistics considered here at distinguishing different models of structure formation. At this stage we are not yet attempting to confirm or rule out concrete models. Hence, we will apply the statistics to toy models of structure formation. These models are designed to mimic key features of specific theories of structure formation, in particular the distinctive non-gaussian and topological aspects.

We consider five models: a model in which galaxies are randomly distributed throughout the sample volume (Poisson model), a CDM model (without non-linearities taken into account), a cosmic string wake model, a cosmic string filament model, and a global texture model.

In order to study the dependence of the statistical measures on topology (rather than number density), we chose all topological defect models to contain the same number of structures, one per Hubble volume at $t_{eq}$, the time of equal matter and radiation. All structures in a given model have the same mass, with the total mass chosen to give a spatially flat Universe.

Taking the structures to have the same size corresponds to a severe truncation of the power spectrum of the actual topological defect models. The justification for this truncation comes from the fact that structures produced at $t_{eq}$ are dominant in both the global texture models and in cosmic string models in which the dark matter is hot. We will come back to this point below.

The numerical simulations produce cubes of data whose side length is 200 Mpc and which contains 222,400 galaxies, chosen such that the number of galaxies per unit volume agrees roughly with the number density of the CFA2 survey\(^1\).

In the texture toy model, spherical balls of galaxies with Gaussian radial density function were placed randomly in the sample volume. The standard deviation of the Gaussian was taken to be 9 Mpc.

This toy model should provide a rough approximation for what happens in
In this theory, density perturbations are caused by contracting topologically nontrivial scalar field configurations. There is a fixed probability $p$ per Hubble volume that at any time $t$ a nontrivial configuration will become smaller than the Hubble radius and start to contract at relativistic speeds, leading to a roughly spherical density perturbation. Hence, a fixed number $p$ of textures per Hubble volume per expansion time are created. Those produced before $t_{eq}$ are washed out by pressure, those produced after $t_{eq}$ have less time to grow by gravitational instability. Hence, the most prominent texture induced perturbations are those laid down at $t_{eq}$.

The cosmic string model of galaxy formation still has many uncertain aspects. It is known that the network of cosmic strings approaches a scale invariant distribution, i.e., the distribution of strings looks statistically the same at all times provided all lengths are scaled by the Hubble radius. The cosmic string ensemble consists of a network of infinite strings with curvature radius comparable to time $t$, and a distribution of loops with radius smaller than $t$. Recent numerical simulations agree that the loops are subdominant. However, there is no agreement on the small scale structure on long strings.

If long strings are straight on small scales, they will form planar density perturbations called wakes (see e.g., Ref. 13 for a recent review of structure formation in the cosmic string model) of planar dimensions $t \times vt$, where $v$ is the velocity of the string in its normal plane. However, if there is small scale structure on the long strings, these strings will move slowly and will exert a local gravitational force on the surrounding matter, leading to the formation of filaments. Because of this uncertainty in the string model we consider two cosmic string toy models, a “wake model” and a “filament model.” They correspond to the two extreme cosmic string scenarios.

In the wake model, rectangular prisms of length and width 40 Mpc and thickness 2 Mpc were placed randomly in the sample volume (subject to the constraint that they lie entirely in the sample volume). The thickness corresponds to the
thickness of the nonlinear region around the wake for a cosmic string model with
hot dark matter and a mass per unit length $\mu$ given by $G\mu = 10^{-6}$, $G$ being
Newton’s constant\textsuperscript{15). This value of $\mu$ is the preferred value based on large-scale
structure analyses\textsuperscript{16} and on the COBE cosmic microwave anisotropy results\textsuperscript{17).}
Note that for $h = 0.5$, the planar dimensions of the wake correspond to the Hubble
radius at $t_{eq}$.

In the filament model, cylinders of length 60 Mpc and radius 4.1 Mpc were
placed randomly in the sample volume. Galaxies were placed at random in the
cylinders, as they were in the wake model.

The linear CDM model was constructed by starting from the power spectrum\textsuperscript{18)

$$|\delta(k)|^2 = \frac{A_k}{(1 + \beta k + \omega k^{1.5} + \gamma k^2)^2}$$

(1)

with

$$\beta = 1.7(\Omega_0 h^2)^{-1}\text{Mpc}$$

$$\omega = 9(\Omega_0 h^2)^{-1.5}\text{Mpc}^{1.5}$$

$$\gamma = 1(\Omega_0 h^2)^{-2}\text{Mpc}^{2}$$

(2)

Fourier transforming to position space, and laying down galaxies according to the
position space density distribution. The transition from Fourier space to position
space was done by taking the lowest $50^3$ Fourier modes (corresponding to the
sample volume) in the first octant of Fourier space, choosing random phases for
all of these modes, by evaluating the Fourier transform at $50^3$ cell centers $x_{ijk}$
in position space, by calculating the number of galaxies in cell $(ijk)$ according to
$\rho(x_{ijk})$ and by laying down the galaxies at random in the cell (for details see Ref.
6).

3. Discrete Genus Statistic

The first statistic we investigate is a variant (developed in Ref. 5) of the genus
statistic which was proposed in 1986 by Gott et al.\textsuperscript{19} as a method of gaining direct
information about the topology of the galaxy distribution.
For a compact surface $S$ in $\mathbb{R}^3$, the genus is defined as

$$g = (\# \text{ of holes}) - (\# \text{ of disconnected components}) + 1. \quad (3)$$

By the Gauss-Bonnet theorem the genus can be computed as a surface integral of the Gaussian curvature $k$:

$$g = -\frac{1}{4\pi} \int_S k \, dA. \quad (4)$$

Given a smooth density distribution $\rho(\vec{x})$, the genus statistic is defined as the curve $g(\rho)$, where $g(\rho)$ is the genus of the surface $\rho(\vec{x}) = \rho$. For a random phase density field, the genus curve can be calculated analytically\(^\text{19}\):

$$g(\nu) = N(1 - \nu^2) \exp(-\nu^2/2), \quad (5)$$

where $\nu$ is the number of standard deviations from the mean density, and $N$ is a constant which depends on the power spectrum. Note that $g(\nu)$ is peaked at $\nu = 0$ and is symmetric. For non-Gaussian models we expect a shift in the peak position and a deviation from symmetry about $\nu = 0$.

The usual method\(^\text{19}\) of applying the genus statistic to a distribution of galaxies is to construct a smooth density field by smearing each galaxy with a Gaussian distribution

$$W(r) = \frac{1}{\pi^{3/2} \lambda^3} \exp(-r^2/\lambda^2), \quad (6)$$

where $r$ is the smoothing length.

The choice of $\lambda$ is critical. $\lambda$ must be large enough such that the density distribution inside structures connect, but small enough such that the topology of the dominant structures is not lost. The results depend crucially on $\lambda$, and this is a big disadvantage of the statistic.
To avoid the above problem, we use a “discrete genus statistic”\(^5\). Given a volume limited redshift survey (or a simulated galaxy distribution), we divide the volume into cells of size smaller than that of the structures we are interested in probing but large enough such that the counts in cell are not dominated by shot noise. In our simulations we chose a cell size of 8 Mpc.

Consider the polygonal surface \(S(n)\) which is the boundary of the complex of cells each of which contains greater than or equal to \(n\) galaxies. The genus \(g(n)\) of this surface is

\[
g(n) = 1 - \frac{1}{2}(V - E + F)
\]

where \(V, E\) and \(F\) are the number of vertices, edges and faces respectively.

The curve \(g(n)\) is the discrete genus curve. The surface \(S(n)\) can be regarded as the surface with galaxy density \(n/(\text{cell volume})\). Hence, we can plot \(g\) as a function of the galaxy number density.

The results of the numerical simulations are shown in Fig. 1. With exception of the CDM model, the data is the average of 20 independent simulations of the model. The statistical error bars are smaller than the symbol sizes. The CDM model results come from a single realization.

The most important conclusion we can draw from this investigation is that the discrete genus statistic is a very powerful discriminant between different models of structure formation. The genus curves for all topological defect models are highly asymmetrical about the mean number density, whereas the Poisson and CDM models are symmetrical. The width of the genus curve for the CDM model is larger than that for the Poisson model which reflects the degree of clustering in the simulation. The difference in the peak density is due to a slightly different normalization of the models.

For the wake the genus curve is positive. This is due to the many holes between the interconnected network of wakes. In contrast, the genus curve for the texture model is overwhelmingly negative since the distribution of galaxies is clumpy (no
holes and many disconnected components). The curve for the filament model lies between the two extreme cases.

4. Counts in Cell Statistics

The counts in cell statistics\(^4\) are very simple. The sample volume is divided into cells of equal volume. For each integer \(n\), the number \(f(n)\) of occurrences of cells with \(n\) galaxies is determined. The graph of \(f(n)\) as a function of \(n\) is the counts in cell statistic (CIC). Counts in cell statistics have been studied extensively by Saslaw and collaborators\(^{20}\), and more recently by Coles and Plionis\(^{21}\) for the Lick galaxy catalog, by Coles et al.\(^{22}\) for CDM models, by Kaiser et al.\(^{23}\) for IRAS galaxies, by Weinberg and Cole\(^{24}\) and by de Lapparent et al.\(^{25}\) in the context of defining a percolation statistic.

For our three dimensional simulations, it is straightforward to evaluate the CIC. We divide the simulation box into 50\(^3\) cells, each on the average containing about two galaxies. Simulations with a smaller number of cells showed more noise whereas the range of \(n\) values with \(f(n) \neq 0\) was too small for a greater number of cells.

The results of the simulations are shown in Figs. 2 and 3. As in Section 3, the results are averages over 20 simulations, except for the CDM model for which only a single realization was considered. In the region of \(n\) values plotted in Fig. 2, the one sigma statistical error bars are of the size of the symbols, as is seen from the individual plots of Fig. 3.

The most obvious conclusion is that the three dimensional CIC statistic can well discriminate between our toy models. The CIC curve for the texture model has the longest tail, a reflection of the dense clusters of galaxies it contains. The length of the tail of the CIC curve decreases as the dimension of the structures of the model increases. This allows a clear distinction between the filament and wake models. All topological defect toy models considered here are more strongly clustered than the CDM model and hence have longer tails of the CIC statistic.
We can also consider two dimensional CIC statistics. They are constructed such that a comparison with data from the CFA2\textsuperscript{1}) redshift survey is possible. Slices of data designed to resemble the CFA2 slices were extracted from cubes of simulated data by generating random orientations for the slices and selecting all galaxies in the cube within the angular (120° × 6°) and radial (100h\textsuperscript{-1}Mpc) bounds of the slice. The slices were divided into 35\textsuperscript{2} cells of equal volume. Less cells per side reduced the resolution and generated a lot of noise, whereas more cells per side caused a significant shortening of the CIC curves. Considering cells of equal area instead of volume would weight nearby and far away galaxies differently. This explains our choices.

In order to calculate the CIC statistic for the CFA data we must correct for the apparent magnitude limitation of the data set. This was done in two steps. First, a volume-limited subsample of the data with radial extent 100h\textsuperscript{-1}Mpc was used. The volume thus selected includes most of the interesting structure in each of the CFA slices, but is small enough such that selection effects can be reliably corrected.

Second, the number of galaxies in each cell was multiplied by a selection function \( f(r) \), \( r \) being the distance of the cell from us, which corrects for the deficiency of galaxies. This function \( f(r) \) can be determined from the Schechter luminosity function\textsuperscript{26})

\[
\varphi(L)dL = \varphi^\ast \left(\frac{L}{L^\ast}\right)^\alpha \exp(-L/L^\ast)d\left(\frac{L}{L^\ast}\right),
\]

where \( \varphi(L)dL \) is the number density of galaxies with luminosity in the interval \([L, L + dL]\), and \( \alpha, \varphi^\ast \) and \( L^\ast \) are parameters determined from the data. The CFA data gives\textsuperscript{25})

\[\alpha = -1.1,\]

\[\varphi^\ast = 0.020h^3 Mpc^{-3},\]

\[M^\ast = -19.2,\]

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where \( M^* \) is the absolute magnitude corresponding to \( L^* \).

The number density of galaxies which can be seen at a distance \( r \) given the apparent magnitude cutoff in the data is

\[
\phi(r) = \int_{L(r)}^{\infty} \varphi(L) dL,
\]

(10)

where \( L(r) \) is the absolute magnitude which at distance \( r \) corresponds to the apparent magnitude cutoff. The selection function \( f(r) \) is

\[
f(r) = \frac{\phi(r_0)}{\phi(r)},
\]

(11)

where \( r_0 \) is a suitably chosen reference distance (20\( h^{-1}\)Mpc in our case).

The results of our simulations are shown in Figs. 4 and 5. In Fig. 4 the results are compared to the average of two CFA2 slices. The error bars of the individual CFA2 data were determined from the uncertainty in the positions due to peculiar velocities. The statistical error bars of the numerical simulations are shown in Fig. 5.

The tendency of the two dimensional CIC curves is the same as for three dimensions: the texture curve has the longest tail, followed by the wake and filamentary models. All three defect models give rise to CIC curves with longer tails than the CDM model. However, the observational error bars are sufficiently large such that only the Poisson model is convincingly ruled out. A \( \chi^2 \) analysis shows that the string filament model fits the data best, significantly better than the CDM model\(^5\).

At this stage, however, it is premature to draw conclusions about the validity of the various models of structure formation. The toy models are too naive to allow any such conclusion. The main lesson is that both two and three dimensional CIC statistics are good ways to analyze large-scale structure data and confront theory with observations.
5. Discussion

We have studied the applicability of a discrete genus statistic and two and three dimensional counts in cell statistics to distinguish the predictions of different models of structure formation. Most theories predict a similar power spectrum of density perturbations, and hence a good statistic must be able to pick out the non-random phases which differentiate between the models.

We conclude that our three statistics give large differences when applied to toy models of structure formation. Topological defect models give rise to long tails in counts in cell statistics, the tail length increasing as the dimension of the prominent structure decreases. The discrete genus statistic is very sensitive to the topology of large-scale structure and shows a large difference between the texture and cosmic string wake toy models.

We have applied the statistics to five toy models of structure formation: Poisson, CDM, global texture, cosmic string wakes and cosmic string filaments. The models are constructed to capture the important topological and statistical properties of the “real” models on scales larger than the horizon at $t_{eq}$. On smaller scales, the topological defect toy models are too rough to give a good approximation to the actual models.

In this paper we have only compared one set of data, namely two slices of the CFA2 redshift survey, with the toy models. In future work we plan to analyze more data. We also plan to construct more realistic toy models for topological defect models which have the correct power spectrum on all scales and are normalized to agree with the CMB anisotropies measured on large angular scales by the COBE-DMR experiment$^{27}$. It will then be realistic to perform a detailed statistical comparison between toy models and observations.

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Figure Captions

**Figure 1:** The discrete genus statistic evaluated for the four models of structure formation considered in the text, and compared to the results for a Poisson distribution of galaxies.

**Figure 2:** 3-d counts in cell statistic evaluated for the four toy models and for a Poisson distribution of galaxies.

**Figure 3:** 3-d counts in cell statistic for the four toy models. One sigma statistical error bars are shown for the wake, filament and texture models.

**Figure 4:** 2-d counts in cell for the filament, texture, Poisson and inflation-based CDM models, compared to the mean $f(n)$ for two CFA slices.

**Figure 5:** 2-d counts in cell (including statistical error bars) for wake, filament, texture and CDM models.