Interacting many-body systems as non-cooperative games

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Abstract

We explore the possibility that physical phenomena arising from interacting multi-particle systems, can be usefully interpreted in terms of multi-player games. We show how non-cooperative phenomena can emerge from Ising Hamiltonians, even though the individual spins behave cooperatively. Our findings establish a mapping between two fundamental models from condensed matter physics and game theory.
The spatial prisoner’s dilemma game of Nowak and May [1] showed that complex macroscopic patterns can develop and evolve in a system even when the microscopic interactions are local in space and time [1, 2, 3]. This is reminiscent of the observation in multi-particle interacting systems in condensed matter physics (e.g. a spatial array of spins) whereby different macroscopic ‘phases’ can emerge. In a separate development, multi-spin models have been shown to offer new tools and viewpoints to help in understanding problems in information processing and optimization problems [4]. Meanwhile multi-agent games have recently caught physicists’ attention via Econophysics [5]; just as in more conventional condensed matter systems, it has been shown that a renormalization of inter-agent interactions to form non-interacting ‘crowd-anticrowd’ clusters provides a good description of the macroscopic fluctuations [6]. Game-theoretic language has also emerged in the quantum domain via the new field of quantum games [7]. Of relevance to biologically-motivated physics is the finding that mesoscopic biological systems such as viruses can play games, in particular the fundamental prisoner’s dilemma game [8]. Furthermore, it is known that such games can exhibit a variety of non-linear dynamical phenomena when evolved in time [9].

Given all this circumstantial evidence, is there possibly a deeper connection between games and physical phenomena? At first sight, games and physics do appear to have some common elements: both consider collections of interacting objects following certain local rules which can give rise to a number of macroscopic configurations. However there is a fundamental problem with the analogy. Physical systems are always driven to minimize the free energy hence the global optimal solution is preferred: particles’ behavior is therefore cooperative. On the other hand, the games which are arguably most interesting and important are non-cooperative: as we explain below, this means that their Nash equilibria are not Pareto optimal.

This Letter shows that non-cooperative behavior can indeed emerge in physical systems, even with the individual particles behaving cooperatively. We specifically consider Ising spin models because they serve as basic models in condensed matter physics, and focus on the classical regime to avoid the complication of quantum entanglement [7]. By treating each site or subset of sites in the Ising-model as a game-playing agent, we show that non-cooperative behaviour is indeed possible. In particular, we show the emergence of a prisoner’s dilemma game being ‘played’ within a multi-spin system.

We start with some necessary background material. The two-player prisoner’s dilemma game [10] is characterized by the following
payoff matrix:

\[
M := \begin{pmatrix}
(2, 2) & (0, 3) \\
(3, 0) & (1, 1)
\end{pmatrix}
\]

The rows designate Rose’s choice of strategy (A or B) while the columns designate Colin’s choice of strategy (A or B). The entry \((a, b)\) indicates that Rose receives payoff \(a\) while Colin receives \(b\). A Nash equilibrium is the set of strategy choices that provides the optimal payoff for each player individually: no individual player can improve his payoff by changing strategy. There is a Nash equilibrium \((1, 1)\) with payoffs 1 to each player. However the global or ‘Pareto’ optimum is \((2, 2)\) with payoffs 2 to each player. Since the players have no way of ‘planning’ coordinated moves, the system will end up choosing a payoff which is macroscopically inferior yet microscopically superior for the individual players: hence the game is non-cooperative. Next we motivate the introduction of game-theoretic language to a physical system, by means of the following simple example involving two spins with a ferromagnetic spin-spin interaction:

\[
H = -2|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + 2|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + 2|\downarrow\uparrow\rangle\langle\downarrow\uparrow| - 2|\downarrow\downarrow\rangle\langle\downarrow\downarrow|.
\]

At zero temperature and at equilibrium, the system will adopt the state with the lowest energy, i.e. \(|\uparrow\uparrow\rangle\) or \(|\downarrow\downarrow\rangle\). (Recall we are ignoring quantum effects such as entanglement for the present discussion). The energy of the system will now be \(-2\). In comparison with the other two possible states (\(|\uparrow\downarrow\rangle\) and \(|\downarrow\uparrow\rangle\)), the two particles have gained in stability by losing energy. The negative form of the Hamiltonian can be used to quantify this gain (and hence loss in energy) at zero temperature. Hence the system has gained a payoff of 2 as compared to the value with the spins separated to infinity. Because of the system’s symmetry, there is no change if we exchange particles 1 and 2 – hence we quantify the gain for each particle by dividing the total gain by two. Hence particles 1 and 2 have gained a payoff of 1. With these definitions, we may use the following game payoff matrix to represent the physical system:

\[
H := \begin{pmatrix}
(1, 1) & (-1, -1) \\
(-1, -1) & (1, 1)
\end{pmatrix}
\]

which has two equivalent Nash equilibria \((1, 1)\).

We now consider the case of two more general subsystems each having two possible choices of states. These subsystems may be individual spins, blocks of spins, or more general subsets of the total many-particle system. For simplicity, we assume the two subsystems are identical in their composition. We will prove that the global optimal solutions of the total system will always be at a Nash equilibrium and hence the interesting dilemma associated with the prisoner’s game cannot unfortunately arise. We suppose that the states of subsystems
1 and 2 are completely described by the vectors $\vec{x}$ and $\vec{y}$ respectively. A general Hamiltonian for the combined system will have the form: $H_1 + H_2 + H_{12}$ where $H_1$ and $H_2$ are the internal energies of the subsystems when separated (or equivalently, when the interaction is turned off) while $H_{12}$ is the interaction term. The symmetry within the system means that we can denote $H_1 = H_2 \equiv H$ and $H_{12} = H_{21} \equiv I$. The condition $H(\vec{x}) = H(\vec{y})$ implies that no one particular state is preferred and so the two subsystems will have equal incentive to start in any of the two states before the interaction is turned on. Since the two subsystems share the payoff in the interaction term, they each have the payoff $[-H(.) - I(.)]/2$. It follows that whichever strategy they each adopt, the two subsystems will always have the same payoffs.

To complete the proof, we first define the set of best replies with respect to $\vec{y}$ for system 1 to be $B_1(\vec{y}) := \{\vec{x}^* : P_1(\vec{x}^*, \vec{y}) \geq P_1(\vec{x}, \vec{y}), \forall \vec{x}\}$. In words, given that system 2 has adopted state-vector $\vec{y}$ then the choice of state-vector $\vec{x}^*$ by system 1 yields the maximum payoff $P_1$. The set of best replies is defined similarly for system 2. Using this notation, a Nash equilibrium represents an operator profile $(\vec{x}, \vec{y})$ for which $\vec{x} \in B_1(\vec{y})$ and $\vec{y} \in B_2(\vec{x})$. If $(\vec{x}^*, \vec{y}^*)$ is a global optimal state, then by definition $\vec{x}^* \in B_1(\vec{y}^*)$ and $\vec{y}^* \in B_2(\vec{x}^*)$ and so $(\vec{x}^*, \vec{y}^*)$ is a solution at Nash equilibrium. Therefore, in this ‘two-player-two-move’ symmetric scenario, the global optimal solutions are always at Nash equilibrium. We will refer to this situation as cooperative, noting that this word has a somewhat more general meaning in game theory. The above analysis indicates that any effective two-body (i.e. two-subsystem) Hamiltonian system containing just two strategies/states, is cooperative.

So can macroscopic non-cooperative behavior ever arise? We now show that it can, if the subsystem under consideration has several possible configurations and hence more states from which to choose. In particular, we will search for non-cooperative phenomena in two subsystems each containing many spins. We concentrate on the specific example of non-cooperation offered by the prisoner’s dilemma, since this is the only symmetric two-player two-strategy game with a unique Nash equilibrium which never coincides with the global optimum. We start by specifying more precisely the criterion for the existence of non-cooperative behaviour. A system composed of two-body Ising Hamiltonians such as

$$H = a \begin{pmatrix} |\uparrow\uparrow\rangle\langle\uparrow\uparrow| & |\uparrow\downarrow\rangle\langle\uparrow\downarrow| & |\downarrow\uparrow\rangle\langle\downarrow\uparrow| & |\downarrow\downarrow\rangle\langle\downarrow\downarrow| \\ |\uparrow\downarrow\rangle\langle\uparrow\downarrow| & |\uparrow\uparrow\rangle\langle\uparrow\uparrow| & |\downarrow\uparrow\rangle\langle\downarrow\uparrow| & |\downarrow\downarrow\rangle\langle\downarrow\downarrow| \\ |\downarrow\uparrow\rangle\langle\downarrow\uparrow| & |\downarrow\uparrow\rangle\langle\downarrow\uparrow| & |\uparrow\uparrow\rangle\langle\uparrow\uparrow| & |\downarrow\downarrow\rangle\langle\downarrow\downarrow| \\ |\downarrow\downarrow\rangle\langle\downarrow\downarrow| & |\downarrow\downarrow\rangle\langle\downarrow\downarrow| & |\downarrow\downarrow\rangle\langle\downarrow\downarrow| & |\uparrow\uparrow\rangle\langle\uparrow\uparrow| \end{pmatrix},$$

may easily be represented as a graph. For example, consider the following graphical representation of the total Hamiltonian for a particu-
lar five-particle spin system comprising various two-body interactions:

\[
\begin{array}{cccccccc}
\uparrow_1 & \cdots & (-2) & \cdots & \uparrow_2 \\
\vdots & & \vdots & & \vdots \\
(3) & \cdots & (3) & \cdots & \equiv H_5 \\
\vdots & & \vdots & & \vdots \\
\downarrow_3 & \cdots & (-2) & \cdots & \downarrow_4 & \cdots & (-1) & \cdots & \downarrow_5
\end{array}
\]

where entries in brackets denotes the \( a \)-parameters in the five-body Hamiltonian. The total Hamiltonian may then be represented by the following matrix:

\[
H_5 \equiv \begin{bmatrix}
0 & -2 & 3 & 0 & 0 \\
-2 & 0 & 0 & 3 & 0 \\
3 & 0 & 0 & -2 & 0 \\
0 & 3 & -2 & 0 & -1 \\
0 & 0 & 0 & -1 & 0
\end{bmatrix}.
\]

We now consider the coexistence of two subsystems, each containing \( n \) spins. In order that the total system can be considered as exhibiting the prisoner’s dilemma scenario, the following requirements must be met: (1) there are two locally stable states that each subsystem can be in when they are separated, i.e. in the absence of interaction terms between the subsystems; (2) when the two subsystems are put together, i.e. when the interactions between them are turned on, the final state of the two subsystems depends on the initial states of these two subsystems; (3) the final payoff for each subsystem (as defined above) must reproduce the same form as the prisoner’s dilemma payoff matrix. By symmetry, we require that the two subsystems are identical and that all interactions are invariant under the exchange of the two subsystems. Therefore, it must be possible to write the overall Hamiltonian in the following form:

\[
M := \begin{bmatrix}
X & Y \\
Y & X
\end{bmatrix},
\]

where \([X]\) denotes the Hamiltonian for each system when isolated, and \([Y]\) denotes the interaction between the two subsystems. We note that \( X \) and \( Y \) must be symmetric matrices in order to render the resulting game symmetric. We will represent \( \uparrow \) by 1 and \( \downarrow \) by \(-1\) and can hence represent the state of the whole system by a string \( \delta_1 \cdots \delta_{2n} \) with \( \delta_k = 1 \) or \(-1\). The total energy of the system will be \( \frac{1}{2} \sum_{i,j} \delta_i \delta_j M_{ij} \). The payoff for subsystem 1 will be the negative of the sum corresponding to the upper half of the matrix \( M \), while the payoff
for subsystem 2 will be the negative of the sum corresponding to the lower half of the matrix $M$. From now on, we will use $A$ and $B$ to denote the two states corresponding to the two strategies.

With all the criteria for non-cooperativity laid out, we are now ready to discuss under what conditions non-cooperative behavior is possible. We will concentrate on the infinite-range Ising model and will look for the minimal number of restrictions for two interacting subsystems which each contain $n$ spins. By condition (1), we see that we need to construct two wells with the same energy at the bottom. In general, this would mean one equality and $2n$ inequalities. The equality comes from the equal energies for the two ground states of the subsystems before interacting, and the $2n$ inequalities come from the fact that the whole system is equivalent to a $2n$-dimensional vector space where each element has $2n$ neighbors in terms of Hamming distance. But since we are looking for the minimum number of restrictions, we may choose the two states corresponding to the bottoms of the two wells to be adjacent: for example, the state $|00\ldots00\rangle$ and $|00\ldots01\rangle$ (where for simplicity, we are now representing $\uparrow$ by 0 and $\downarrow$ by 1). In this case only one equality and $2n - 2$ inequalities need to be satisfied. However the two initial states cannot be adjacent since we do not want $(A, A) = |00\ldots00\rangle \otimes |00\ldots00\rangle$ to be adjacent to $(A, B) = |00\ldots00\rangle \otimes |00\ldots01\rangle$; if it were adjacent, then there is an incentive for the whole system to go from one configuration to another, thereby implying that the resulting game may not have well-defined payoffs. Fortunately, due to the symmetry in the Ising Hamiltonians, we may replace $00\ldots01$ by $11\ldots10$. Therefore, we designate $00\ldots00$ and $11\ldots10$ to be the two initial states (i.e. strategies) $A$ and $B$ respectively. We note that since the interactions have infinite range, it does not matter where the 1s and 0s are located. In terms of the entries in matrix (I), we have the following equalities and inequalities:

$$\sum_{1 \leq j \leq n} x_{nj} = 0 \quad (2)$$

$$\sum_{1 \leq j \leq n} x_{kj} > 0, \ 1 \leq k \leq n - 1 \quad (3)$$

$$\sum_{1 \leq j \leq n-1} x_{kj} - x_{kn} > 0, \ 1 \leq k \leq n - 1. \quad (4)$$

We now come to condition 2. In order to obtain the least number of restrictions, we again require that the state $(A, A)$ is at a local minimum of the whole system, which amounts to $n$ inequalities:

$$\sum_{1 \leq j \leq n} (x_{kj} + y_{kj}) > 0, \ 1 \leq k \leq n. \quad (5)$$
Similarly, requiring the state \((B, B)\) to be at local minimum means:

\[
\sum_{1 \leq j \leq n-1} (x_{kj} - y_{kj}) - x_{kn} - y_{kn} > 0, \quad 1 \leq k \leq n-1 
\]

\[
\sum_{1 \leq j \leq n-1} (-x_{nj} - y_{nj}) - x_{nn} + y_{nn} > 0 
\]

Using Eq. 4, we can write Eq. 7 as:

\[
\sum_{1 \leq j \leq n-1} (-y_{nj}) + y_{nn} > 0 
\]

If the initial state of the whole system is at state \((A, B)\), we then require that the first system, which is at state \(|00\ldots00\rangle\), transform itself to state \(|10\ldots00\rangle\). Therefore, the whole system will then be at state \(|10\ldots00\rangle \otimes |11\ldots10\rangle\). We further require that it is at a local minimum. Hence the following inequalities apply:

\[
\sum_{1 \leq j \leq n-1} (y_{1j} - x_{1j}) - y_{1n} > 0 
\]

\[
\sum_{1 \leq j \leq n-1} (x_{kj} - y_{kj}) + x_{kn} + y_{kn} > 0, \quad 2 \leq k \leq n 
\]

\[
\sum_{1 \leq j \leq n-1} (x_{kj} - y_{kj}) - x_{kn} - y_{kn} > 0, \quad 1 \leq k \leq n-1 
\]

\[
\sum_{1 \leq j \leq n} y_{nj} > 0. 
\]

Finally, since the prisoner’s payoff table is desired, we need the following three inequalities concerning the payoffs in different scenarios:

\[
\sum_{1 \leq j \leq n} y_{nj} - y_{nn} > 0 
\]

\[
-y_{n1} - \sum_{1 \leq k \leq n-1} y_{kj} > 0 
\]

\[
2 \sum_{1 \leq j \leq n-1} x_{1j} + \sum_{2 \leq k \leq n-1} \sum_{1 \leq j \leq n-1} y_{kj} - \sum_{2 \leq j \leq n-1} y_{nj} > 0. 
\]

Indeed, there are \(6n\) inequalities and 1 equality in total. We know that if there are \(6n + 1\) arbitrary parameters, then the above system of inequalities has a solution \(11\). However by exploiting many of the redundancies inherent in the system, we will now show that only \(n + 2\) non-zero parameters are actually needed. Table 1 summarizes the ranges of these inequalities. From Table 1, we see that for \(k = 1\)
we may set every parameter to zero except $b_1, d_1$ and $f_1$, which satisfy the following inequalities:

\begin{align*}
  b_1 &> 0 \\
  d_1 &> 0 \\
  f_1 &< 0 \\
  d_1 &> b_1 + f_1.
\end{align*}

For $k = n$, since we have set $f_1$ to be non-zero and $Y$ is symmetric, we have $d_n = f_1$. Hence we may now set every parameter to zero except $d_n, e_n$ and $f_n$. The following inequalities then follow:

\begin{align*}
  e_n &> -f_1 \\
  f_n &> 0 \\
  f_n &> e_n + f_1.
\end{align*}

For $2 \leq k \leq n - 1$, we may set all parameters to zero except $b_k$. Recall that we have already set $e_n > 0$ hence we may now also set $b_k > e_n$.

Finally, we come back to Eqs 14 and 15, which yields two further inequalities:

\begin{align*}
  -2f_1 &> e_n \\
  2b_1 &> f_1.
\end{align*}

The following form corresponds to two $n = 3$ spin subsystems, and is one of many possible scenarios which satisfies the above requirements:

\begin{align*}
  X := \begin{bmatrix}
    0 & 2 & 0 \\
    2 & 0 & 0 \\
    0 & 0 & 0
  \end{bmatrix}; & \\
  Y := \begin{bmatrix}
    2 & 0 & -1 \\
    0 & 0 & 1.5 \\
    -1 & 1.5 & 2
  \end{bmatrix},
\end{align*}

The corresponding payoff matrix is:

\begin{align*}
  M := \begin{bmatrix}
    (4.5, 4.5) & (1.5) \\
    (5.1) & (3.5, 3.5)
  \end{bmatrix}.
\end{align*}

By considering the number of non-zero parameters required, one can deduce that an $n = 3$-spin subsystem ‘playing’ against another $n = 3$-spin subsystem is the smallest total system which exhibits non-cooperative behaviour within the Ising framework: the phrase ‘two’s company but three’s a crowd’ therefore comes to mind. The above discussion has been mostly concerned with infinite-range interactions, however we note that one may also analyse the positions
of the non-zero parameters in order to discuss various other models such as nearest-neighbor interaction models, etc.

We now return to discuss whether the Ising-based games considered here exhibit similar behavior to that observed in spatial games. One interesting phenomenon observed in the spatial prisoner’s game is the coexistence of different strategies up to some critical point [1]. It turns out that this same phenomenon also occurs for our Ising games. To show this, we consider a large system with many n-spin subsystems such that each subsystem is playing a prisoner’s game with its immediate neighbors. In order that the subsystems evolve their strategies, we add in fluctuations which we assume to be thermal. At low enough temperature, only very few subsystems (which are statistically far apart) will have the chance to change their states at a given timestep. These subsystems will, with high probability, change to a state which minimizes their free energy – we assume this to be close to the energy that we use to construct the payoff matrix table. Regardless of what their neighbors’ states actually are, these subsystems will always choose state $B$. (If their neighbors are in state $A$, they can exploit their neighbors by adopting state $B$, whereas if their neighbors are in state $B$, then adopting state $B$ is the best they can do). We note that this is in marked contrast to previous assumptions used in spatial games [1, 3], where agents were usually assumed to imitate the strategy of the best player in their neighborhood; since a subsystem cannot ‘know’ the surrounding subsystems’ payoffs, it can only maximize its own payoff given any particular situation. At low enough temperature, the whole system will evolve to the situation where every subsystem is in state $B$ given any initial condition. This resembles the phenomenon of condensation. By contrast, if the temperature is high enough that many subsystems are allowed to change their states simultaneously but still low enough that the free energies exhibit a payoff matrix similar to the prisoner’s dilemma (see Eq. (25)), then with high probability neighboring sites or blocks of neighboring sites may be able to change their states together. Then with high probability, the subsystems in the block will all end up in state $A$ because it maximizes the payoff for the whole block. Therefore at high temperature, the majority of sites will adopt state $A$. Just as for the spatial games considered previously in the literature, a whole spectrum of scenarios is possible for the Ising games even though the detailed dynamics of these two models may have different origins. Beside thermal fluctuations, we may also consider a fluctuation in the payoffs. In Ref. [1], Nowak and May used the following payoff matrix:

$$M := \begin{pmatrix}
(1, 1) & (0, b) \\
(b, 0) & (0, 0)
\end{pmatrix}$$

(26)
where \( b > 1 \). They found that a region where the coexistence of both strategies is encouraged, appears for \( 1.8 < b < 2 \). In our Ising games, this is also what one expects to happen: when \( b \) is close to the prescribed range, the configurations \((A,A)\), \((A,B)\) and \((B,A)\) will all have similar global energies, hence all three configurations are preferred at high enough temperature such that neighboring subsystems can evolve together. This similarity in the region of interest further strengthens the tie between spatial games and condensed matter systems.

Finally we note that the kind of fluctuation considered here is different from the conventional models considered in the fluctuating systems literature associated with spin glasses and other condensed matter systems. Instead it is usually assumed that given enough time, a system will achieve equilibrium: the fluctuation of the whole system is taken to mean that the subsystems fluctuate as well, hence the whole system has an incentive to move to equilibrium. Here we encounter something different: given the initial conditions, some subsystems can actually benefit from the fluctuation and so have no incentive to increase the global payoff. For more general models, such as the Heisenberg model, one would expect quantum fluctuations to play an essential role, thereby suggesting a possible future role for quantum game theory \[\text{[7]}.\]
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Table 1: Table of inequalities Eq. (3)–(13) where $a_k x_k + b_k \sum_{2 \leq j \leq n-1} x_{kj} + c_k x_{kn} + d_k y_{k1} + e_k \sum_{2 \leq j \leq n-1} y_{kj} + f_k y_{kn} > 0$.

|     | $a_k$ | $b_k$ | $c_k$ | $d_k$ | $e_k$ | $f_k$ |
|-----|-------|-------|-------|-------|-------|-------|
| $k = 1$ | 1     | 1     | 1     | 1     | 1     | 1     |
|       | 1     | 1     | -1    | -1    | -1    | 1     |
|       | 1     | 1     | 1     | 0     | 0     | 0     |
|       | 1     | 1     | -1    | 0     | 0     | 0     |
|       | -1    | -1    | 0     | 1     | 1     | -1    |
| $k = n$ | 0     | 0     | 0     | -1    | -1    | 1     |
|       | 0     | 0     | 0     | 1     | 1     | 1     |
|       | 0     | 0     | 0     | 1     | 1     | 0     |
| $2 \leq k \leq n-1$ | 1     | 1     | 1     | 1     | 1     | 1     |
|       | 1     | 1     | -1    | -1    | -1    | -1    |
|       | 1     | 1     | 1     | 0     | 0     | 0     |
|       | 1     | 1     | -1    | 0     | 0     | 0     |
|       | 1     | 1     | 1     | -1    | -1    | 1     |