Dissipative Josephson effect in coupled nanolasers

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Abstract

Josephson effects are commonly studied in quantum systems in which dissipation or noise can be neglected or do not play a crucial role. In contrast, here we discuss a setup where dissipative interactions do amplify a photonic Josephson current, opening a doorway to dissipation-enhanced sensitivity of quantum-optical interferometry devices. In particular, we study two coupled nanolasers subjected to phase coherent drivings and coupled by a coherent photon tunneling process. We describe this system by means of a Fokker–Planck equation and show that it exhibits an interesting non-equilibrium phase diagram as a function of the coherent coupling between nanolasers. As we increase that coupling, we find a non-equilibrium phase transition between a phase-locked (PL) and a non-phase-locked (NPL) steady-state, in which phase coherence is destroyed by the photon tunneling process. We show this system by means of a Fokker–Planck equation and show that it exhibits an interesting non-equilibrium phase diagram as a function of the coherent coupling between nanolasers. As we increase that coupling, we find a non-equilibrium phase transition between a phase-locked (PL) and a non-phase-locked (NPL) steady-state, in which phase coherence is destroyed by the photon tunneling process. In the coherent, PL regime, an imbalanced photon number population appears if there is a phase difference between the nanolasers, which appears in the steady-state as a result of the competition between competing local dissipative dynamics and the Josephson photo-current. The latter is amplified for large incoherent pumping rates and it is also enhanced close to the lasing phase transition. We show that the Josephson photocurrent can be used to measure optical phase differences. In the quantum limit, the accuracy of the two nanolaser interferometer grows with the square of the photon number and, thus, it can be enhanced by increasing the rate of incoherent pumping of photons into the nanolasers.

1. Introduction

Among the striking effects of quantum coherence, the Josephson effect is one of widest used in nowadays technologies [1]. To name a few examples, the superconducting quantum interference device (SQUID) is one of the oldest and most sensitive magnetic sensors [2, 3], and Josephson junctions are integral building blocks to construct artificial two-level systems in quantum information processing [4, 5]. In precision metrology, the Josephson effect has been used as a practical standard of voltage and the elementary charge, $e$ [6, 7].

The Josephson effect occurs when two quantum systems having both well-defined quantum phases, $\phi_{1,2}$, are weakly coupled so that quantum tunneling is enabled between them. It is manifested as a net current,

$$I = I_c \sin(\Delta \phi),$$

between the two subsystems depending on the phase difference, $\Delta \phi = \phi_1 - \phi_2$. This effect was discovered by Josephson, who predicted a macroscopic electric current along a superconducting tunnel junction [8, 9]. Since then, extensions of these ideas have been proposed and tested in several platforms, such as Bose–Einstein condensates [10–15], superfluid $^3$He [16], photonic [17–19] and optomechanical systems [20] and polaritons [21, 22]. While the effect of dissipation has been studied in some cases [17], it has generally been considered as detrimental for the observation of the Josephson current and its applications.
In this work we unveil a dissipative Josephson effect in a fully photonic setup consisting of two coherently coupled nanolasers. We show that this system presents interesting non-equilibrium phases and it also has an exciting outlook for applications in quantum metrology and sensing. Nanolasers are probably the most fundamental example of active dissipative systems with a non-trivial non-equilibrium phase diagram. They can be implemented in several platforms such as photonic [23–27], plasmonic [28], or nano-mechanical systems [29]. One interesting aspect of nanolasers is that they can work in a mesoscopic regime in which photon numbers are large enough to allow for a semiclassical description in terms of Fokker–Planck equations, however, finite size effects and fluctuations are still important. A natural extension from the single nanolaser model into the many-body regime arises when we consider networks of local nanolasers coupled by means of photon tunneling terms of the form $H_{tc} = t_c (a_1^+ a_2 + a_1 a_2^+)$. [30, 31]. Jaynes–Cummings resonators coupled by photon tunneling terms have been investigated in the realm of circuit QED, both in theory [32] and experiment [33], showing non-equilibrium and dissipation-induced phase transitions. In the present work, interesting phenomena may appear as a result of the interplay between coherent tunneling and on-site non-linear dissipative terms. The latter are responsible for sustaining the quantum coherence, whereas coherent photon tunneling leads to a number of intriguing effects, such as the appearance of a photonic Josephson current when the two nanolasers are phase-locked (PL) to external drives (see figure 1).

Our theoretical description relies on a reduced Fokker–Planck equation for the phase dynamics, since we will be in a regime where photon number fluctuations can be neglected. In the past, phase synchronization between lasers has been studied both in theory and experiments. However, in most cases a dissipative coupling has been considered, since this is the kind of coupling found in typical experiments with macroscopic lasers. This is the case, for example, of coupled semiconductor lasers [34] and laser networks [35, 36]. Dissipative couplings have been shown to lead to phase synchronization effects that can be described in terms of Kuramoto models or stochastic Kuramoto models if phase fluctuations are included [37]. Our model, on the contrary, includes what is known in the literature as coherent, dispersive or reactive coupling [34]. Systems of coherently coupled Van der Pol oscillators have been theoretically studied, for example in [38], with a focus on a limit of small photon numbers where quantum effects are more relevant. In [39], a system of two coupled micromasers has been numerically studied and, interestingly, the authors found very different behaviours between the cases with dispersive and coherent couplings. To the best of our knowledge, a phase Fokker–Planck equation for coherently coupled nanolasers has never been derived and studied before. In this work we derive such equation, which can be considered as an extension of the stochastic Kuramoto model [37].

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**Figure 1.** General scheme: (a) each nanolaser is a single-qubit laser subjected to incoherent qubit pumping with rate $\gamma$, photon loss with rate $\kappa$, and a coherent driving field with amplitude $\epsilon = |\epsilon|e^{i\phi}$. In the lasing phase the photon field is well described by a semiclassical probability distribution for a complex variable, $a \rightarrow \alpha$. In the presence of a external coherent drive, the cavity mode gets phase locked to the external drive with the photonic phase shifted by $\pi/2$ with respect to the coherent drive’s phase, $\theta = \phi - \pi/2$. (b) We show that coupling the two nanolasers with a small photon tunneling process leads to a photocurrent $\propto \sin(\theta_1 - \theta_2)$, where $\theta_j$ are the phases at which each nanolaser is locked. The photocurrent leads to a photon imbalance that can also be related to the original phases in the coherent drive, $\propto \sin(\phi_1 - \phi_2)$. (c) The situation is, thus, analogous to the Josephson effect in a superconductor–insulator-superconductor tunnel junction, since the optical phases in the cavity modes, $\theta_j$, play the same role as the superconductor phases. (d) For very large values of the photon tunneling term, $t_c$, the effect of the coupling is to destroy the phase coherence of the system, leading to a cancellation of the photonic Josephson current. The transition between regimes (b) and (d) is actually a dissipative phase transition, as shown numerically in section 3.
This article presents the following results: (i) we derive a semi-classical description of a dissipative interferometer consisting of two nanolasers coupled through a coherent photon tunneling process and subjected to coherent drivings with phase difference, $\phi_1 - \phi_2$. (ii) We identify two limiting regimes of the steady-state of this system. If the coherent coupling is small, the lasers are phase locked to each individual coherent driving. As we increase the coherent coupling between nanolasers, the system goes through a non-equilibrium phase transition into a steady-state in which phase coherence is lost. (iii) We present analytical and numerical evidence of the existence of a photonic Josephson current between the two nanolasers. This current causes a photon number imbalance in the photonic steady-state proportional to $\sin(\phi_1 - \phi_2)$. (iv) We analyse the performance of this system as an interferometer and we show that there is an amplification effect by which the accuracy grows as $\sqrt{n}$ in the shot noise limit, with $n$ the number of photons in each of the nanolasers. Furthermore, the accuracy is optimal close to the critical point of the lasing phase transition.

2. Theoretical description of coherently coupled nanolasers

2.1. Dimer of single-qubit lasers

Our system consists of two single-qubit lasers coupled by coherent photon tunneling. This scheme can be implemented in several setups including circuit QED (for example using the ideas proposed in [40]) and trapped ion phonon lasers [29]. The discussion below is focused on the case of optical nanolasers, however, our results are independent of any particular realization in photonics, vibronics or optomechanics.

Each nanolaser consists of a two-level system (qubit) with levels $|g\rangle$ and $|e\rangle$, coupled on resonance with the local cavity mode through a Jaynes–Cummings type interaction. The two-level system is incoherently pumped with a rate $\gamma$ and the photonic mode has a decay rate $\kappa$. In addition, the cavity interacts with a weak coherent driving on resonance with the photonic mode. The two nanolasers are connected by a photon coherent tunneling term that couples the photonic cavities. We assume that the two cavity modes and qubits have the same transition frequency. In an interaction picture rotating at the mode frequency, the following master equation for the system density matrix, $\rho$, captures the complete quantum dynamics of the system (we use units such that $\hbar = 1$),

$$\dot{\rho} = -i [H, \rho] + \sum_{j=1,2}\left(\mathcal{L}_{(\sigma_j^+ \sigma_j)}(\rho) + \mathcal{L}_{(\sigma_j \sigma_j^+)}(\rho)\right).$$

We use the notation $\mathcal{L}_{(O_1, O_1)}(\rho) = \Gamma(2O_1\rho O_1^\dagger - O_1^\dagger O_1\rho - \rho O_1^\dagger O_1)$ for Lindblad super-operators. The Hamiltonian in equation (1) is

$$H = g \sum_{j=1,2} (\sigma_j^+ a_j + a_j^\dagger \sigma_j^-) + \sum_{j=1,2} (\epsilon_j a_j + \epsilon_j^* a_j^\dagger) + H_c.$$

The first term represents the qubit-field coupling, with strength $g$. The second one describes external driving terms acting on the nanolasers, with driving amplitudes $\epsilon_j = |\epsilon| e^{i\phi_j}$. In this work we will consider that both driving amplitudes have the same strength, $|\epsilon|$, but there may be a phase difference, $\Delta \phi = \phi_2 - \phi_1$. Finally, the last term represents the coherent photon tunneling term, with amplitude $t_c$,

$$H_c = -t_c \left( a_1^\dagger a_2 + a_2^\dagger a_1 \right).$$

This coherent coupling occurs in systems of single-mode nano-cavity arrays [31, 41], superconducting circuits [30] or nano-mechanical systems such as trapped ions (where it takes the form of a phonon-tunneling term, see [42, 43]). The minus sign is added to the coupling so that for positive $t_c$, the lowest energy mode is the symmetric or center of mass mode. At first sight, we would expect a term like equation (2) to induce some synchronization of the phase between nanolasers, however, we will prove later on that this intuition is wrong and, actually, strong tunneling constants, $t_c$, tend to destroy the quantum coherence in the system.

For low enough values of the amplitude of the coherent driving field, $|\epsilon|$, the steady-state of each nanolaser is governed by the parameter

$$C_p = \frac{\sum \epsilon_j^2}{\kappa \gamma},$$

such that, for $C_p < 1$, nanolasers are in a non-lasing steady-state, whereas $C_p > 1$ is the lasing phase, with a number of photons that roughly scales like $\gamma_0 / \kappa$, as we show below. In this paper, we will assume that the local nanolaser dynamics is much faster than the photon tunneling term, and in particular, $t_c \ll \gamma$, such...
that we can safely assume that the lasing transition stays at \( C_p = 1 \). The mean-field prediction for the number of photons is [44],

\[
\begin{align*}
\text{n}_{\text{mf}} &= 0, \quad C_p < 1 \\
\text{n}_{\text{mf}} &= \frac{1}{2} \frac{C_p - 1}{C_p} \frac{\gamma}{\kappa}, \quad C_p \geq 1
\end{align*}
\]  

(4)

From equation (4), we learn that the parameter that determines the effective size of our nanolaser model is actually the ratio \( \gamma/\kappa \), which determines any finite-size scaling effects and plays a role that is analogous to the number of sites in a quantum lattice model (see appendix A). Equation (4) is obtained under the assumption of small coherent driving amplitudes, which amounts to neglecting the number of photons that is generated by the coherent drive, compared to the mean photon number, \( \langle n \rangle \). This conditions can be quantified as \( |\epsilon|/\sqrt{\langle n \rangle} \ll \kappa \), either by using a Fokker–Planck calculation, as shown for example in our reference [45], or by directly using the Heineberg equations of motion to evaluate the number of photons, see equation (35) below.

In the lasing regime, the coherent drive will act as a symmetry breaking term, and the optical phase will get PL to the phase of the coherent drive, with a \( \pi/2 \)-shift [46],

\[
\langle a_j \rangle \approx \sqrt{\langle n \rangle} e^{i(\phi_j - \pi/2)}. 
\]  

(5)

The latter result is strictly valid in the large photon number limit only, since finite size effects can decrease the optical coherence (see [45, 46]). The phase locking of a laser to an external driving field, as well as the role played by finite-size effects can be quantitatively addressed with the Fokker–Planck equation, see for example [47], or the more recent discussion in the context of quantum sensing in our own work in [45]).

### 2.2. Effective non-linear photonic master equation

In this work, we will focus on the physics occurring in a lasing regime of large photon numbers in which each nanolaser can be approximated by a self-sustained quantum oscillator. Rather than considering the full thermodynamic limit, we are interested in a mesoscopic regime, such that the system is amenable of a description in terms of a Fokker–Planck equation, but yet finite size effects and classical fluctuations are relevant. This regime may be attained for a strong pumping of the qubits such that

\[
\gamma \gg g, \kappa, |\epsilon|, t_c.
\]  

(6)

All along this paper we will be working in this regime, in which we can simplify our model by adiabatically eliminating the qubits dynamics [47]. This step will ultimately allow us to obtain a semiclassical description in terms of a Fokker–Planck equation. Note that, equation (6) implies that \( \gamma \gg g \), and thus, in order to enter into the lasing regime, the condition of strong coupling \( g \gg \kappa \) is required, such that \( C_p > 1 \).

After the adiabatic elimination of the qubits we get the following master equation (see appendix A for details),

\[
\dot{\rho}_t = i t_c \left[ a_1 a_2 + a_1 a_1, \rho_t \right] - i \sum_{j=1,2} \left[ \epsilon_j a_j + \epsilon_j a_j^\dagger, \rho_t \right] + \sum_{j=1,2} \mathcal{L}_{\{a_j a_j^\dagger\}} (\rho_t)
\]

\[
+ \sum_{j=1,2} \left( \mathcal{L}_{\{a_j^\dagger a_j\}} (\rho_t) + \frac{1}{2} \mathcal{L}_{\{a_j a_j\}} (\rho_t) - \frac{1}{2} \mathcal{L}_{\{a_j^\dagger a_j, a_j^\dagger a_j\}} (\rho_t) + \frac{1}{2} \mathcal{L}_{\{a_j^\dagger a_j, a_j\}} (\rho_t) \right),
\]  

(7)

with the coefficients:

\[
A = \frac{g^2}{\gamma}, \quad B = 2 \frac{g^4}{\gamma^3}, \quad C = \kappa.
\]  

(8)

\( \rho_t = \text{Tr}_{\text{qubit}} (\rho) \) is the reduced density matrix of the photonic subsystem, obtained after tracing out the qubits. The last two terms in equation (7), proportional to the coefficient \( B \), account for the non-linear matter–light interaction and they are ultimately responsible for the non-equilibrium phase transition into the lasing phase.

Equation (7) is strictly valid only below the critical point, \( C_p < 1 \), and slightly above it, \( C_p \geq 1 \), since it has been obtained under the assumption of total qubit population inversion (see appendix A). To quantify better the regime of validity of this approximation we can calculate the ground-state population of the qubits in the steady-state, \( \langle \sigma_j^+ \sigma_j \rangle_{ss} \), and check whether it can really be neglected. Actually, by using the
equations for the single-qubit case, we find [45],

\[ \langle \sigma_+^j \sigma_-^j \rangle_u = \left( \frac{g}{\gamma} \right)^2 \left( 1 + \langle a_i^d a_i^d \rangle_u \right) \approx \frac{1}{2} \frac{C_p - 1}{C_p}, \quad C_p > 1 \]

\[ \approx 0, \quad C_p < 1. \]  

(9)

We confirm that condition \( \langle \sigma_+^j \sigma_-^j \rangle_u \ll 1 \) is met close to the phase transition at \( C_p \gtrsim 1 \), and in the non-lasing phase \( (C_p < 1) \). It is, however, highly desirable to extend equation (7) well into the lasing phase \( (C_p > 1) \), to fully understand the system’s behavior. This can be done with a proper renormalization of the coefficient that takes into account the neglected terms in the adiabatic elimination. As we show in appendix A, this procedure amounts to replacing

\[ B \rightarrow B/C_p \equiv B_t, \]

(10)

in equation (7). The new parameter \( B_t \) includes the effect of processes of higher order in \( g \), and it ensures the right prediction on the photon number in the steady-state.

Finally, we address the effect of a dephasing term in the Liouvillian describing each of the nanolasers. This is an effect that may be relevant in solid-state platforms, see for example [48]. Pure dephasing with rate \( \gamma_{\text{dep}} \) would require the addition of a superoperator term,

\[ \mathcal{L}_{\text{dec}}(\rho) = \frac{\gamma_{\text{dec}}}{2} (\sigma_+^j \rho \sigma_-^j - \rho) \]

(11)

to the Liouvillian in equation (1). It can be easily checked (see appendix A... that the effect of pure dephasing is just a renormalization of the cooperativity,

\[ C_p \rightarrow \frac{g^2}{\kappa(\gamma + \gamma_{\text{dec}})} \]

(12)

after which all expressions above still hold. Additionally, parameters governing the reduced photonic master equation also have to be modified, \( A \rightarrow g^2/(\gamma + \gamma_{\text{dec}}) \), \( B \rightarrow g^4/(\gamma(\gamma + \gamma_{\text{dec}})^2) \). Apart from those changes, our main conclusions and discussion below remain unchanged.

2.3. Semiclassical Fokker–Planck equation

Although we are dealing here with only two nanolasers, the solution of equation (7) can become computationally demanding for an exact diagonalization. For example, if we want to describe a lasing phase with an average number, \( \langle n \rangle = 200 \) photons, we would need to describe each cavity mode with a number of \( n_{\text{cut}} = 1000 \) of Fock states, to stay within the limit \( n_{\text{cut}} \gg \langle n \rangle \). Thus, the total Hilbert space dimension would be \( 2.5 \times 10^5 \) states. Calculating the steady-state requires finding the zero-eigenstate of a Liouvillian \( \mathcal{L}(\rho) = 0 \), which is a super-operator that can be written as a non-Hermitian matrix of dimension \( n_{\text{cut}}^2 = 10^{12} \), equivalent to a system of 40 qubits. This is very challenging and, in practice, we have been unable to get exact diagonalization results for regimes with \( \langle n \rangle > 10 \), way below the large photon limit.

In the limit of high-photon numbers an analytical approach based on phase-space methods can be used to further simplify equation (7). Concretely, we shall employ the Glauber–Sudarshan \( P \) representation [47] of the effective master equation. We will see that this method is not only amenable to be solved numerically, but it will also allow us to get useful analytical insights. This representation is defined as the pseudo-probability distribution satisfying

\[ \rho(t) = \int d^2 \alpha P(\alpha, \alpha^*, t) |\alpha\rangle \langle \alpha|, \]

(13)

where \( |\alpha\rangle \) is the coherent state \( |\alpha\rangle = \exp(\alpha a^d - \alpha^* a)|0\rangle \). The function \( P(\alpha, \alpha^*) \) plays the role of a classical probability distribution over \( |\alpha\rangle \langle \alpha| \), with the normalization condition \( \int d^2 \alpha P(\alpha, \alpha^*) = 1 \). Expectation values of normal ordered operators can be evaluated with the identity,

\[ \langle (a_i^d)^p a_i^q \rangle = \int d^2 \alpha (\alpha^*)^p \alpha^q P(\alpha, \alpha^*). \]

(14)

The substitution of equation (13) into equation (7) transforms the master equation into a Fokker–Planck equation for \( P(\alpha, \alpha^*, t) \) [49], see appendix C. We can achieve further simplification by working with polar coordinates,

\[ \alpha_j = r e^{i \theta_j} \quad (j = 1, 2). \]

(15)
The radial components, \( r_j \), are related to the photon number observable in each cavity,

\[
\langle r_j^2 \rangle = \int d^2\alpha |\alpha_j|^2 P(\alpha, \alpha^*, t) = \langle n_j \rangle \equiv \langle a_j a_j^\dagger \rangle.
\] (16)

In the last equation and along the rest of this work, we will understand \( \langle O \rangle \) as referring to both quantum average, or average with respect to the semiclassical distribution, \( P \), depending on whether \( O \) is an operator or a Fokker–Planck variable. The key to simplifying the Fokker–Planck equation is the observation that the fluctuations in the radial components, \( r_j \), may be neglected as long as we are deep enough in the lasing regime (\( C_\rho > 1 \) and \( \langle n_j \rangle \gg 1 \)). Thereby we can assume that radial variables remain close to their steady-state values \( r_j \approx r_j^0 \). In this case we can work out the radial variables and consider a reduced description in terms of phase variables, \( \theta_j \). Formally, this is accomplished by assuming a factorized \( P(\alpha_1, \alpha_2) \approx R(r_1)R(r_2)P(\theta_1, \theta_2) \). Each \( R(r_1) \) is a Gaussian distribution properly normalized around \( r_j^0 \) (average value of \( r_j \)), which corresponds to the radial probability distribution of each nanolaser in the lasing regime. This procedure is discussed in details in appendix C, and leads to an equation that depends on the phases of the nanolasers only,

\[
\frac{\partial P_\theta}{\partial t} = \frac{A}{2} \sum_{j=1,2} \frac{\partial^2 P_\theta}{\partial \theta_j^2} + \sum_{j=1,2} \frac{\partial}{\partial \theta_j} \left( -\epsilon \frac{r_{j+1}^0}{r_j^0} \cos(\theta_{j+1} - \theta_j) + \frac{|\epsilon|}{r_j^0} \cos(\theta_j - \phi_j) \right) P_\theta,
\] (17)

in which \( n_j^0 = \langle r_j^0 \rangle^2 \) stands for the steady-state average number of bosons at site \( j \). In the lasing regime and in the absence of tunneling, this quantity is independent of the site and is given by \( n_j^0 = n_0 \), with

\[
n_0 \equiv \frac{A - C}{B_\gamma} = \frac{1}{2} \frac{C_p}{C_p} - 1 \frac{\gamma}{\hbar}.
\] (18)

In the homogeneous case we have \( r_j^0 = r_0 \), which leads to a homogeneous Fokker–Planck equation for the phases,

\[
\frac{\partial P_\theta}{\partial t} = \sum_{j=1,2} \left( D_\theta \frac{\partial^2 P_\theta}{\partial \theta_j^2} + \frac{\partial}{\partial \theta_j} \left( -\epsilon \cos(\theta_{j+1} - \theta_j) + \frac{|\epsilon|}{r_0} \cos(\theta_j - \phi_j) \right) \right) P_\theta,
\] (19)

representing a uniform photon density together with a single phase diffusion rate,

\[
D_\theta \equiv \frac{A}{2n_0}.
\] (20)

We will see later that this picture has to be corrected to account for photon imbalance induced by the Josephson current between nanolasers.

Equation (19) is one of the most important results for our work. We remark that the novel element in this equation is the coupling between phases induced by \( \epsilon_c \). Actually, this equation is closely related to the dissipative Kuramoto model, however, in our phase model there is a coherent coupling term which differs from the usual dissipative couplings considered in coupled laser or synchronization models. This will have severe consequences in the non-equilibrium phase diagram of the model, as we see below.

3. Dissipative phase transition induced by coherent photon tunneling

We investigate now the effect of the coherent photon coupling in the steady-state of our Fokker–Planck equation (19) and show that, surprisingly, it does not lead to any synchronization effect between the nanolasers. On the contrary a coherent photon tunneling process leads to the loss of quantum coherence in the system.

3.1. Phase-locked and non-phase-locked steady-states

To simplify the discussion we consider first an homogeneous driving with \( \phi_1 = \phi_2 = \phi \) and \( r_1^0 = r_2^0 = r_0 \). In equation (19) there are two limiting cases:

(a) \( \epsilon/r_0 \gg \epsilon_c \). In this limit we expect that the system is well approximated by two independent single qubit lasers PL to the driving fields, as explored in reference [45]. In this case the Fokker–Planck equation can be exactly solved and we get

\[
P_\theta(\theta_1, \theta_2) \propto e^{-\frac{|\epsilon| n_0}{\hbar} \sum_j \sin(\theta_j - \phi)}.
\] (21)
Figure 2. Non-equilibrium phases of the nanolaser dimer in the lasing regime. In the PL phase (left), nanolasers are phase locked to the external coherent fields. The semiclassical probability distribution is strongly concentrated around $\theta_j = \phi_j - \pi/2$. In the NPL phase the photon tunneling between nanolasers is the dominant process, and it induces a total loss of phase coherence in the system.

Hence, here we find non-zero coherences $|\langle a_i \rangle| \neq 0$. We will refer to this steady-state as a PL phase.

(b) $t_c \gg |\epsilon|/r_0$, where we expect that the coherent tunneling dominates the system’s dynamics. To understand this limit it is useful to study equation (19) with $|\epsilon| = 0$,

$$
\frac{\partial P_\theta}{\partial t} = D_p \left( \frac{\partial^2 P_\theta}{\partial \theta_1^2} + \frac{\partial^2 P_\theta}{\partial \theta_2^2} \right) - t_c \left( \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} \right) \cos(\theta_2 - \theta_1) P_\theta.
$$

Equation (22) can be solved by a change of variables to collective coordinates, $\theta_+ = (\theta_1 + \theta_2)/\sqrt{2}$, $\theta_- = (\theta_2 - \theta_1)/\sqrt{2}$. In the new variables we find the steady-state solution,

$$
P_\theta(\theta_1, \theta_2) = \frac{1}{(2\pi)^2},
$$

with vanishing coherences, $\langle a_i \rangle = 0$. Furthermore, in this limiting case, the two-point correlation $G = \langle a_1^{\dagger} a_2 \rangle$ also becomes zero, which implies the absence of any synchronization between the modes [50]. We will refer to a steady-state where $t_c$ is dominant, and the system does not retain any coherence, as non-phase-locked (NPL) phase. We note that equation (23) is obtained under the assumption that no periodic driving is present, and thus, the absence of phase coherence cannot be understood as resulting from the energy shift induced by $t_c$ in the collective mode energies of the two cavity system.

We arrive to the somehow counter-intuitive conclusion that the presence of a coherent photon tunneling term does not induce any correlations between nanolasers, see figure 2 for a summary of our discussion. This situation strongly differs from the case of a dissipative coupling between nanolasers, as induced, e.g. by incoherent tunneling through evanescent modes or by an intermediate lossy cavity, in which coupling does induce a phase correlation between nanolasers (see [51]). This result is in agreement with a theoretical investigation of coupled micromasers, see for example the work by Davis-Tilley and Armour [39], which studies a very similar system to the coupled single-qubit lasers considered here. Numerical calculations with moderate occupation numbers showed in that work that a coherent coupling induces correlations between micromasers, although they are very weak as compared to dissipative couplings. This situation seems to differ from the case of coupled quantum Van der Pol oscillators studied in [38], where it was found that coherent coupling does induce phase correlations between oscillators.
3.2. Phase transition between PL and NPL steady-states: driving with homogeneous phases ($\phi_1 = \phi_2$)

What happens in the intermediate regime between the PL and NPL steady-states identified above? To address this question, we assume that we have a constant driving term, $\epsilon$, and we increase the tunneling from $t_c \ll \epsilon/\rho_0$ to $t_c \gg \epsilon/\rho_0$. In particular, we are interested to know whether a dissipative phase transition separates the two phases. The coherences $|\langle a|\rangle|$ can be used as the order parameter to distinguish between the PL and NPL steady-states. We also need to define a valid thermodynamic limit to establish the existence of critical properties. Even though this is a two-site system, a thermodynamic limit is obtained by letting the number of photons in the steady state, $n_0$, play the role of the system size [45, 52]. The number of photons is essentially regulated by the ratio $\gamma/\kappa$ as shown in equation (4), so that the thermodynamic limit will be reached in a limit of strong pumping, $\gamma \gg \kappa$. We have solved numerically equation (19) by discretizing the angular variables, $\theta_1, \theta_2$, in a number of values, $n_0$, running from 0 to $2\pi$. This procedure allows us to express $P_0$ as a vector and $\partial P_0/\partial t$ as a non-Hermitian matrix, and to calculate numerically the steady-state solution, $\partial P_0/\partial t = 0$. We expect that this numerical method is accurate as long as

$$\frac{n_0}{2\pi} \gg \frac{1}{P_0} \frac{\partial P_0}{\partial \theta}_i,$$

or, equivalently, that the angular probability distribution function does not change much within a phase interval $\Delta \theta = 2\pi/n_0$. We have checked that in all the calculations shown in this work the numerical results have converged for the values of $n_0$ used. Numerical calculations allow us to find the reduced probability distributions,

$$P_1(\alpha_1) = \int d\alpha_2 P(\alpha_1, \alpha_2) \approx R(\alpha_1) \int d\theta_2 P_0(\theta_1, \theta_2),$$

where in the last terms we have re-expressed the reduced probability for $\alpha_1$ in polar coordinates. An analogous definition holds for $P_2(\alpha_1)$. In figure 3 we show our numerical results for reduced probability distributions for increasing coherent coupling $t_c$. We observe two main qualitative effects. First, the probability distribution spreads in phase space as we increase $t_c$, in agreement with the expected transition from the PL to the NPL phase. Second, we observe a rotation of the probability distribution: at values $t_c \approx 0$ the two nanolasers are PL to $\theta = 0$, however as we increase $t_c$, the distribution rotates to an increasing angle around $\theta = \pi/2$. This effect can be qualitatively understood with the equations of motion for the coherences of the bosonic modes,

$$\frac{d\langle a_1\rangle}{dt} = \frac{d\langle a_1\rangle}{dt}|_{at} + |\epsilon| e^{i(\phi_1 - \pi/2)} + t_c \langle a_2\rangle e^{ix/2},$$

$$\frac{d\langle a_2\rangle}{dt} = \frac{d\langle a_2\rangle}{dt}|_{at} + |\epsilon| e^{i(\phi_2 - \pi/2)} + t_c \langle a_1\rangle e^{ix/2}. \quad (26)$$

All non-linear and dissipative effects are included in the single nanolaser contribution to the time-evolution of the coherences, $d\langle a\rangle/dt|_{at}$. The second and third terms in the rhs of equations (26) and (27) are the external drives and coherent coupling terms, respectively. Note that the external drives—proportional to $|\epsilon|$—contribute with a term that is out of phase by an angle $\pi/2$. The coherent coupling can be understood as an additional external drive with a phase and amplitude determined by the coherence in the nearby cavity mode. If we assume that $t_c$ is small, then cavities are phase locked to an angle $\theta_j = \phi_j - \pi/2$. However, as we increase $t_c$, the photon tunneling process induce an effective external driving, with a phase rotated by an angle $\pi/2$. This explains qualitatively the rotation of the probability distribution in figure 3.

To gain a quantitative understanding, we have explored numerically the PL--NPL phase transition as a function of $t_c$, see figure 4. The coherence in each of the nanolasers is calculated with the distribution $P_\theta$ by using the expression,

$$|\langle a_i\rangle|^2 = r_0^2 \left(\cos(\theta_i)\right)^2 + \left(\sin(\theta_i)\right)^2. \quad (28)$$

We find in figure 4(a) that an abrupt transition between a PL and a UL phase happens at $t_c = 1$, at which the coherences seem to become a non-analytic function of $t_c$. The same behaviour is qualitatively observed in the correlation function between cavity modes in figure 4(c), which can be obtained from the Fokker–Planck equation by using the equivalence,

$$\langle a_1|a_2\rangle = r_0^2 \left(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\right). \quad (29)$$

We see in figure 4(c) that the relative phase also tends to zero while increasing $t_c$, although with a slower decay. In order to evaluate the finite-size scaling at the critical point we consider a constant coherent coupling $t_c$. We then scale $\epsilon$ such that we stay at the critical point $\epsilon/r_0 = \epsilon/\sqrt{r_0} = \epsilon$. Increasing the number of
Figure 3. Reduced probability distribution $P_j(\alpha)$ (with $j = 1$ or $2$), calculated with the Fokker–Planck equation, equation (19) with parameters $D_{p} = 10^{-2}$, $\epsilon = 2$, $n_{0} = 100$, $C_{p} = 2$ and $\phi_{1,2} = \pi/2$ and different values of $t_{c}$. The transition between PL and NPL regimes is predicted to occur at values $t_{c} = \epsilon/r_{0} = \epsilon/\sqrt{n_{0}} = 0.2$. These set of parameters can be obtained by setting from values $\kappa = 1$, $\gamma = 10^{3}$, $g = 44.7$. These rates can be obtained with the values $\kappa = 20$, $g = 565.7$, $\gamma = 8 \times 10^{3}$ ($D_{p} = 0.2$) and $\kappa = 0.5$, $g = 14.1$, $\gamma = 200$ ($D_{p} = 0.005$) of the original parameters in equation (1). Since $D_{p} \propto 1/n_{0}$, smaller values of $D_{p}$ are closer to the thermodynamic limit of the problem.

Figure 4. Numerical investigation of the PL–NPL phase transition. All plots are obtained for two nanolasers subjected to an external driving, $\epsilon/r_{0} = 1$. In (a)–(c) we plot the evolution of the coherences with values $D_{p} = 0.2$ (dashed lines) and $D_{p} = 0.005$ (continuous lines). Those rates can be obtained with the values $\kappa = 20$, $g = 565.7$, $\gamma = 8 \times 10^{3}$ ($D_{p} = 0.2$) and $\kappa = 0.5$, $g = 14.1$, $\gamma = 200$ ($D_{p} = 0.005$) of the original parameters in equation (1). Since $D_{p} \propto 1/n_{0}$, smaller values of $D_{p}$ are closer to the thermodynamic limit of the problem. (a) Evolution of the coherences in each nanolaser as a function of $t_{c}$. (b) Dependence of the cos (blue lines) and sin (black lines) components of the photon coherence as a function of $t_{c}$. (c) Evolution of the two-point photon correlation function as a function of $t_{c}$. (d) Dots are calculated by numerically solving equation (19) at the critical point $t_{c} = |\epsilon|/r_{0}$, whereas the red line is a linear fit.

photons has thus the effect of decreasing the phase decoherence rate which scales as $D_{p} \propto 1/n_{0}$. Our numerical results show that the coherence at the critical point follows a power-law dependence,

$$1 - \frac{|\langle a_{j} \rangle|^{2}}{n_{0}} \propto n_{0}^{-\beta},$$

and we find the critical exponent $\beta \approx 0.63$ from our numerical calculations (see figure 4(d)). The critical point is thus a non-analytical point in the thermodynamical limit. By considering the different contributions to the bosonic coherences we have checked the self-rotation of the phase-locking angle induced by the coherent coupling, which is apparent in an increase in the average $\langle \sin(\theta) \rangle$ relative to $\langle \cos(\theta) \rangle$ as we approach the critical point in figure 4(b).

Finally, we study the asymptotic dependence of the field correlation functions and coherences for $t_{c} \gg |\epsilon|/r_{0}$. Figures 4(a)–(c) seem to indicate that both of those quantities vanish for large values of $t_{c}$, however the decay seems to slow down, specially in the case of the correlation function, $\langle a_{1}^{\dagger}a_{2} \rangle$. In figure 5 we present numerical results obtained with equation (19) with values of $t_{c}$ varying several orders of
magnitude, which show that both $\langle a_1^\dagger a_2 \rangle$ and $\langle a_2 \rangle$ do indeed vanish, following a power-law dependence $\propto t_c^{-1}$.

3.3. PL–NPL phase transition with inhomogeneous phases ($\phi_1 \neq \phi_2$)

So far we have considered the case of homogeneous driving. An interesting behaviour is found if we study the case of different driving phases $\phi_1 \neq \phi_2$. Consider for simplicity that we fix $\phi_1 = 0$ and change $\phi_2$. We plot numerical results in figure 6. We compare three cases by keeping the same scaling criteria, namely, we fix $\epsilon/t_0 = 1$ and calculate the coherence in the nanolaser system as a function of both $t_c$ and the angle $\phi_2$. We observe the PL/UL phase transition at $t_c = 1$ for values $\phi_2 = 0, \pi$, in agreement with our results in figure 4. However, as we approach the value $\phi_2 = \pi/2$, we observe that the critical $t_c$ required to enter into the UL phase increases to values $t_c \gg 1$. In other words, close to $\phi_2 - \phi_1 = \pi/2$ the PL regime is more robust to a coherent coupling between nanolasers. This effect is more pronounced for large photon numbers and thus lower values of the phase decoherence rate (see in figures 6(b) and (c)). We can understand this effect by, again, looking at the equations (26) and (27). If we assume small coherent couplings, then the condition $\phi_1 = 0$ implies, according to equation (26), that the coherent driving of the first cavity mode induced phase locking into $\theta_1 = -\pi/2$. Equation (27) then becomes

$$\langle \Delta a_2 \rangle = \langle \Delta a_1 \rangle + |\epsilon|e^{i\phi_2 - \pi/2} + t_c |\langle a_1 \rangle|.$$  

We find that the coherent driving on the second nanolaser and the photon tunneling term have the same phase, only if $\phi_2 = \pi/2$. This qualitative argument explains the trend observed in the numerical calculation that the PL regime is more resilient to the coherent coupling term, $t_c$, when $\Delta \phi = \pi/2$. 

![Figure 5](image1.png)

**Figure 5.** Asymptotic dependence of the correlation functions and coherences as a function of the photon tunneling constant, $t_c$. We have considered values $D_p = 0.2$, and other parameters are the same as in figure 4. In (a) we plot the normalized correlation function and in (b) the cavity field coherences. Orange circles are results from the numerical solution of the Fokker–Planck equation in equation (19). Blue lines are a fit to a power-law decay, $F(t_c) = A t_c^{-\alpha}$. Since we aim to capture the asymptotic dependence we have selected values with $10^3 < t_c < 10^4$ for the fit.

![Figure 6](image2.png)

**Figure 6.** We plot the value of the average optical coherence in the nanolaser dimer, $C = \left( \langle |a_1|^2 / n_0 + |a_2|^2 / n_0 \rangle \right) / 2$. This quantity is calculated by using the Fokker–Planck equation, equation (19), and the equivalence $C = (1/2) \sum_{j=1} \left( \langle \cos(\theta_j)^2 \rangle + \langle \sin(\theta_j)^2 \rangle \right)$, $\phi_1 = 0$ and $|\epsilon|/t_0 = 1$ in all the plots. (a) $D_p = 0.5$, $n_0 = 20 (K = 10$, $g = 1.27 \times 10^2$, $\gamma = 8 \times 10^2$ in the original parameters of equation (1)). (b) $D_p = 0.05$, $n_0 = 100 (K = 5, g = 1.41 \times 10^2$, $\gamma = 2 \times 10^2$). (c) $D_p = 0.005$, $n_0 = 500 (K = 2.5, g = 1.58 \times 10^2$, $\gamma = 5 \times 10^2$).
4. Photonic Josephson current

4.1. Josephson photo-current

By restricting ourselves to the previously mentioned PL steady-states, interesting phenomena emerge as a result of the interplay between local nanolaser dynamics and the coherent tunneling term ($t_c$). In particular, we will show that a photonic analog of the Josephson current is generated between the two nanolasers in the case of a finite difference between the optical phases into which they are locked.

To get a qualitative understanding of this effect, let us examine the evolution of the average photon number, $\langle n_j \rangle$, in each nanolaser ($j = 1, 2$), which can be calculated by means of the Heisenberg equations obtained with the master equation, equation (7). We distinguish two contributions. First, we find a term that is given by the non-linear local dynamics of the nanolasers laser, which, in the limit of large photon numbers, reads

$$\langle n_j \rangle_{nl} = 2(A - C)\langle n_j \rangle - 2B\epsilon\langle n_j \rangle^2 - 2\Im(\epsilon\langle a_j \rangle).$$

Note that, apart from corrections arising from the coherent drive, proportional to $\epsilon$, the steady-state solution in the lasing phase is $\langle n_j \rangle = n_0$, given by equation (4).

Second, we have a contribution arising from the coherent hopping of photons between sites,

$$\langle n_1 \rangle_c = 2t_c\Im\left(\langle a_1 a_2 \rangle\right),$$

(32)

(analogously for $\langle n_2 \rangle_c$). For weak coherent coupling, $t_c$, and small values of $\epsilon$, the lowest order approximation of equations (31) and (32) can be found by assuming that the effect of the tunneling and coherent drive is negligible in the calculation of the two-point correlation function, $\langle a_1 a_2 \rangle$. In that case, we find an approximate expression,

$$\langle n_1 \rangle_c = -2t_c n_0 \sin(\phi_1 - \phi_2),$$

which fundamentally indicates a Josephson effect by which a phase difference, $\Delta \phi$, of the coherent drivings induces a photo-current between sites. This current arises from the coherent nature of the coupling, and it does not appear in the case of a dissipative-mediated coupling [51].

Finally, in the limit of perfect phase-locking, there will be a fixed phase relation between the nanolaser and the coherent drive, such that

$$\Im(\epsilon^* a_j) \approx |\epsilon|\sqrt{n_0}.\quad (34)$$

We can obtain now an approximate expression for the steady average photon number by adding up (31) and (32), and imposing the conditions $\langle \dot{n}_j \rangle_{nl} + \langle \dot{n}_j \rangle_c = 0$, leading to

$$\langle n_j \rangle_{ss} = n_0 + \frac{\sqrt{n_0}|\epsilon|}{\kappa(C_p - 1)} - \frac{t_c}{2\kappa^2 C_p} \gamma \sin(\phi_1 - \phi_2).$$

Hence, we find an imbalanced average photon number given by,

$$\Delta n_{ss} = \langle n_1 \rangle_{ss} - \langle n_2 \rangle_{ss} = n_1 \sin(\phi_2 - \phi_1), \quad n_1 = \frac{t_c \gamma}{\kappa C_p \kappa}.\quad (36)$$

The result in equation (36) arises from the balance between the individual nanolaser dynamics and the photon tunneling between nanolasers. Note that we have written the solution so as to make explicit the dependence $n_1 \propto \gamma / \kappa$, which implies that the photon number imbalance can be increased by increasing the incoherent pumping rate. This indicates that the system can act as a good sensor to estimate the phase difference, $\Delta \phi$, by simply measuring the imbalance of the average photon number in the steady state of the system.

4.2. Numerical investigation of the Josephson current between nanolasers

So far, our results apply in a limit of strict phase-locking and small coherent couplings. However, we have seen in previous sections that the effect of $t_c$ is to destroy the coherence in the coupled nanolasers system. We have to expect that it is not possible to simply increase the Josephson photocurrent by means of increasing the coupling $t_c$ since at some point the system will enter into the NPL, incoherent, steady-state.

To investigate the interplay between Josephson photocurrent and the PL–NPL phase transition we need to carry out numerical calculations that bring us beyond the approximations considered in the previous subsection, in particular beyond the approximation of small $t_c$ values. Unfortunately, exact diagonalization of equation (1) is a numerically challenging task, since we need to deal with a very large Hilbert space. We resort to two approximate methods:
The last two steps are repeated until convergence is reached.

The Gutzwiller ansatz approximates the steady state of Liouvillian (1) by assuming a separable state of the form $\rho \approx \rho_1 \otimes \rho_2$, where each $\rho_j$ follows a local Liouvillian,

$$\dot{\rho}_j = -i\mathcal{H}_j\rho_j + i t_c [a_j a_{j+1}^\dagger + a_j^\dagger a_{j+1}, \rho_j],$$

where $a_j = \text{Tr} (\rho_j a)$.

To summarize the two methods: the self-consistent Fokker–Planck method allows us to include correlations between phases, but it relies on the validity of the semiclassical approximation. The Gutzwiller ansatz, on the other hand, allows us to describe exactly the quantum dynamics at the single nanolaser level, but it does not allow to include correlations. In the lasing regime and with small $t_c$, the two methods must yield the same results.
Our numerical calculations are presented in figure 7, where we plot the photon number imbalance caused by the Josephson photocurrent. There is a reasonable agreement between the Gutzwiller ansatz and the self-consistent Fokker–Planck approaches. At large values of the coherent coupling $t_c$, the Gutzwiller ansatz does not converge numerically to a steady-state value, and we must assume that neglecting correlations between cavities is not a valid approximation. The self-consistent Fokker–Planck equation, on the other hand, is more robust and converges up to higher values of the coupling term, $t_c$. We see in figure 7 that the Josephson effect decreases for large values of $t_c$, as expected from our discussion on the PL–NPL transition in the previous section.

Figures 7(a) and (b) shows that the photocurrent imbalance grows with $t_c$ up to a certain value at which increasing the coupling is detrimental to the coherence between nanolasers, since the system enters into the NPL steady-state.

5. The photonic Josephson current as a metrological resource

The result in equation (36) indicates that the nanolaser dimer considered in this work is very sensitive to a phase difference $\phi = \phi_2 - \phi_1$. If we calculate the derivative,

$$\frac{\partial \Delta n_{as}}{\partial \phi}|_{\phi=0} = n_j = \frac{t_c \gamma}{\kappa C_p \kappa},$$

we find that the sensitivity of the system grows linearly with $\gamma/\kappa$, which physically can be interpreted in terms of the bosonic amplification of the Josephson photocurrent. Furthermore, we also find that the sensitivity is increased as we approach the critical point of the lasing phase transition at $C_p = 1$.

We have compared the prediction of the sensitivity around the critical point by using the self-consistent numerical methods introduced in the previous section. We have confirmed that the critical point is indeed the optimal operating point of view for detecting a phase difference, as shown in figure 8. This result can be qualitatively explained from the dependence of the photon number dynamics on the typical rate scale $A = C_p \kappa$, see equation (31): close to $C_p$, the local photon number dynamics slows down, thus leading two a stronger effect from the Josephson photocurrent.

Finally, we can also estimate the uncertainty in the value of a phase measurement carried out by the coupled nanolaser system in the quantum limit. The latter assumes that experimental error in photon number measurements is entirely due to the quantum fluctuations of the photon number observable. The latter can be estimated to be

$$\Delta(n_1 - n_2)_{as} \approx 2 \left(\frac{1}{2} \frac{\gamma}{\kappa} \right)^{1/2},$$

where we have used the calculation of photon number fluctuations obtained with the single nanolaser radial Fokker–Planck equation, equation (C.11). Equation (41), together with equation (40) allows us to estimate the uncertainty in the estimation of a small phase difference, $\delta \phi$, from a measurement of the difference in photon numbers in the coupled nanolasers,

$$\delta \phi = \frac{\Delta(n_1 - n_2)_{as}}{\Delta(n_{as})_{\phi=0}} = \sqrt{2} \frac{C_p}{t_c} \frac{\kappa}{\kappa} \frac{\gamma}{\kappa}^{-1/2}.$$

The last expression shows that the error grows with $C_p$, and it scales like $\delta \phi \propto 1/\sqrt{\gamma/\kappa} \approx 1/\sqrt{\gamma}$. This result shows that the amplification effect survives in the quantum limit, in which the accuracy can be enhanced by increasing the number of photons, for example by increasing the excitation rate of the qubits, $\gamma$.

Equation (42) may lead to the wrong conclusion that $\delta \phi$ can be indefinitely decreased just by increasing the photon tunneling rate, $t_c$. However, there is a limitation in the maximum value of $t_c$ that we can consider, since, according to our results from section 3.2, phase coherence is destroyed by values $t_c \gg \epsilon/\sqrt{n_0}$. According to that limitation, and using the expression for $n_0$ given by equation (18), we find the ultimate precision limit,

$$\delta \phi > \sqrt{C_p(C_p - 1)} \frac{\kappa}{\epsilon}.$$

This result shows that the accuracy of the nanolaser dimer for measuring a phase gradient improves as the system gets closer to the lasing phase transition, where $C_p = 1$.

Let us comment now on the realization of our ideas in practice. Photon and phonon nanolasers can be fabricated in highly controllable quantum setups, like trapped ions and superconducting circuits. The latter is a specially relevant platform, since a photonic Josephson system like the one described here would allow to measure properties of incoming microwave fields. In appendix D we propose a scheme where lasing is
achieved with the aid of a periodic driving of the qubit’s frequency, following ideas first introduced in [40]. In circuit QED, typical qubit–cavity and cavity–cavity couplings are in the range of MHz, whereas $\kappa$ can be tuned to significantly smaller values of tens of KHz [53]. Incoherent pumping can be achieved by using Floquet engineering together with the coupling of the cavity mode to a highly dissipative auxiliary cavity. A realistic choice of parameters (see appendix D) leads to values $\kappa/2\pi = 10$ kHz, $\gamma/2\pi = 10$ MHz, $C_p = 2$. $t_c$ is typically of the order of MHz [33], but it can be made arbitrarily smaller by fabrication, since it decreases with the cavity–cavity separation. Working at the limiting value of the PL regime at $t_c = \epsilon/\sqrt{n_0}$, and assuming $n_0 = 100$ photons and coherent drivings such that $\epsilon \approx \kappa$, would lead to a required value of $t_c = 1/2\pi$ kHz. Since $C_p$ can be tuned to reach values close to the lasing phase transition, this implies that the maximum accuracy limit in equation (43) can be achieved with $\delta \phi \ll 1$. Even though this is a rough estimation, we can see that the energy scales of typical superconducting setups are very promising for the implementation of nanolaser Josephson interferometers.

6. Conclusions & outlook

We have presented a theoretical study of two nanolasers coupled by a photon tunneling term. We have arrived to two main conclusions. The first is that a photon tunneling term is a source of decoherence which can ultimately destroy the phase locking of each individual nanolaser to an external driving field. The second conclusion is that, in a limit of small photon tunneling rates, a photonic Josephson effect is induced that can be used to measure the phase difference between external fields.

The model of interferometer proposed in this article may be implemented in circuit QED systems, as shown in appendix D. In that experimental platform, single-qubit nanolasers have already been demonstrated [24], and schemes for controlling the properties of single-qubit laser light, including the generation of entangled states of light, have been proposed [40]. Photon tunneling is actually the main mechanism that couples microwave cavities in circuit QED [30], thus making this system an ideal experimental platform for implementing lattices of coupled nanolasers. Our ideas can also be translated to the realm of phonons, for example in trapped ion setups. Here, the coupling between internal electronic states acting as quibts and the vibrational degrees of freedom can be controlled with optical lasers. Single ion phonon lasing has actually been experimentally demonstrated [29]. Vibrational modes of coupled trapped ions can often be described in terms of local phononic modes, with the Coulomb interaction between ions inducing a phonon hopping process [42], an effect that has been experimentally observed [43, 54]. Trapped ions actually offer the possibility to study few-mode coupled lasers or include highly controllable qubit–phonon interactions [55].

Our work could be extended to larger lattice sizes, something that could potentially allow high sensitive estimation of phase gradients. Additionally, the simultaneous implementation of coherent couplings and dissipative-mediated couplings [51] is expected to exhibit interesting dissipative phase transitions of photonic phases, also interesting for studying quantum synchronization. An intriguing possibility here is the investigation of non-reciprocal couplings and topological effects, as well as topological amplification.
Finally, these results invite to study further Josephson effects or configurations, like the ac Josephson effect or the SQUID [57].

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Adiabatic elimination

In this appendix we show how the adiabatic elimination of the qubits leads to equation (7). For this we employ a generalization of the procedure discussed in references [45, 51]. Since we assume that photon tunneling rates smaller than local energy scales, we consider the single nanolaser case for this derivation.

Firstly, we trace over the qubits from the master equation. Since the effect of the photon decay, not affected by the adiabatic elimination procedure, we set $\kappa = 0$, and reinsert it later in the calculation,

$$\dot{\rho}_f = -ig(\rho^{ee} a d - \rho^{ee} a^\dagger),$$  \hspace{1cm} (A.1)

where $\rho_f$ is the reduced density matrix of the field after tracing over the qubit states, $\rho_f = \text{Tr}_q \{ \rho \} = (g|\rho|g^\dagger + (e|\rho|e^\dagger),$ and we employ the notation $\rho^{ee} = (g|\rho|e^\dagger).$ To obtain a closed equation for $\rho_f$ we need to eliminate the operators $\rho^{ee}, \rho^{ee}$ from equation (A.1). The corresponding equations of motion for these operators are,

$$\dot{\rho}^{ee} = -ig(a^\dagger \rho^{ee} - \rho^{ee} a^\dagger) - \gamma \rho^{ee},$$  \hspace{1cm} (A.2)

The operators $\rho^{ee}$ and $\rho^{ee}$ may now be adiabatically eliminated (in the limit $\gamma \gg \kappa, g, |\epsilon|$) from equation (A.1) by taking $\dot{\rho}_f \approx 0$ in equation (A.2) and substituting their steady-state solutions,

$$\rho^{ee} = -i\frac{g^2}{\gamma} (a^\dagger \rho^{ee} - \rho^{ee} a^\dagger).$$  \hspace{1cm} (A.3)

Now we can use equation (A.3) and insert it into equation (A.1) and get,

$$\dot{\rho}_f = -\frac{g^2}{\gamma} a^\dagger a^\dagger \rho^{ee} + \frac{g^2}{\gamma} a d \rho^{ee} a^\dagger + \frac{g^2}{\gamma} a^\dagger \rho^{ee} a - \frac{g^2}{\gamma} a^\dagger a \rho^{ee} + \text{H.c.}$$  \hspace{1cm} (A.4)

We need now equations of motion for $\rho^{ee}$ and $\rho^{ee}$, which can be derived again from equation (1), leading to

$$\dot{\rho}^{ee} = -ig(a^\dagger \rho^{ee} - \rho^{ee} a^\dagger) + 2\gamma \rho^{ee}$$  \hspace{1cm} (A.5)

$$\dot{\rho}^{ee} = -ig(a^\dagger \rho^{ee} - \rho^{ee} a^\dagger) - 2\gamma \rho^{ee}$$  \hspace{1cm} (A.6)

We may now reach a perturbative solution to the steady-states of equations (A.5) and (A.6) in terms of the field density matrix $\rho_f$ by adiabatically eliminating $\rho^{ee}$ and $\rho^{ee}$. This is done by taking $\dot{\rho}^{ee} \approx 0$ in equation (A.6),

$$\rho^{ee} = -\frac{ig}{2\gamma} (a^\dagger \rho^{ee} - \rho^{ee} a^\dagger) = \frac{g^2}{2\gamma^2} (2a^\dagger \rho^{ee} a - a^\dagger a \rho^{ee} - \rho^{ee} a^\dagger a).$$  \hspace{1cm} (A.7)

The ground state population of each qubit is expected to be negligible as a result of the fast pumping of the qubits. Thus, in first order we can assume $\rho^{ee} \approx 0$ and $\rho^{ee} = \rho_f - \rho^{ee} \approx \rho_f$. A second order approximation is achieved by inserting this first order approximation into equation (A.7), hence

$$\rho^{ee} = \rho_f - \frac{g^2}{\gamma^2} a^\dagger \rho_f a$$  \hspace{1cm} (A.8)

$$\rho^{ee} = \rho_f - \rho^{ee} = \rho_f - \rho^{ee}$$  \hspace{1cm} (A.9)
The desired closed equation for $\rho_t$ is accomplished by plugging equations (A.8) and (A.9) into equation (A.4), leading to

$$\dot{\rho}_t = \frac{g^2}{\gamma} (2a^\dagger \rho a - aa^\dagger \rho_t - \rho_t aa^\dagger) + \frac{2g^4}{\gamma^3} (aa^\dagger \rho aa^\dagger - a^2 \rho a^2) + \frac{2g^4}{\gamma^3} (a^\dagger \rho a)$$  \hspace{1cm} (A.10)

We can check, by using the commutation relations of $a$ that the equation can be written in Lindblad form,

$$\dot{\rho}_t = A (2a^\dagger \rho a - aa^\dagger \rho_t - \rho_t aa^\dagger) + B \left(aa^\dagger \rho aa^\dagger - \frac{1}{2}aa^\dagger a a^\dagger \rho_t - \frac{1}{2} \rho_t a a^\dagger a^\dagger\right)$$

$$- B \left(a^2 \rho a^2 - \frac{1}{2} a^2 a^\dagger a^\dagger \rho_t - \frac{1}{2} \rho a^2 a^\dagger\right) + B \left(a^\dagger \rho a - \frac{1}{2} a^\dagger a^\dagger \rho a - \frac{1}{2} \rho a a^\dagger a^\dagger\right).$$  \hspace{1cm} (A.11)

with definitions

$$A = \frac{g^2}{\gamma},$$

$$B = \frac{2g^4}{\gamma^3}.$$  \hspace{1cm} (A.12)

$A = \frac{g^2}{\gamma}$, and $B = \frac{2g^4}{\gamma^3}$. If we include now the photon decay term, we finally get to equation (7).

We discuss now how the addition of a pure dephasing term, like the one in equation (11), would change this scenario. Pure dephasing would contribute to the decoherence of density matrix non-diagonal elements,

$$\rho^{ge} = -i \frac{g}{\gamma + \gamma_{dec}} (a^\dagger \rho^{se} - \rho^{se} a^\dagger),$$  \hspace{1cm} (A.13)

however, the equations of motion of diagonal terms, equations (A.5) and (A.6), remain unchanged. Using the above equation we arrive to the same equation (A.11) with new parameters

$$A = \frac{g^2}{\gamma + \gamma_{dec}},$$

$$B = \frac{2g^4}{\gamma (\gamma + \gamma_{dec})^2}.$$  \hspace{1cm} (A.14)

We can see that the effect of pure qubit dephasing is just a shift in the parameters values.

**Appendix B. Renormalization of the adiabatic equation**

Equation (A.10) is a perturbative description in the limit of adiabatic elimination of the fast qubit dynamics. This equation is strictly valid below the critical point, $C_p < 1$, and slightly above it, $C_p > 1$ [45], which implies a severe limitation in the range of applicability of our results.

For the single qubit laser the average number of bosons in the steady state predicted by equation (7) is,

$$n_{ph} \approx \frac{A - C}{B} = \frac{1}{2} \frac{C_p - 1}{C_p} \frac{\gamma}{\kappa},$$  \hspace{1cm} (B.1)

which tends to zero for large values of the pumping parameter. Equation (B.1) can be obtained by finding the steady-state value of the photon number operator and neglecting photon number fluctuations. This prediction differs from the mean field result [45],

$$n_{mf} = \frac{1}{2} \frac{C_p - 1}{C_p} \frac{\gamma}{\kappa}.$$  \hspace{1cm} (B.2)

The two expressions agree only close to $C_p = 1$. Still, one may think of a renormalization procedure that allows us to perform a summation of the neglected terms in the perturbative series with respect to $(g/\gamma)^2$.

The root of the perturbative nature of equation (A.10) is given by the truncation performed in equation (A.8). We search for a new adequate parameter $\beta$ that takes into account the remaining terms of the series such that,

$$\rho^{se} = \beta \frac{g^2}{\gamma^2 a^\dagger a^\dagger} \rho a.$$  \hspace{1cm} (B.3)

We discuss now a procedure to compute $\beta$ by using an exact relation that holds in the steady state and which is easily inferred from the equation (1) of the letter. Let us calculate the Heisenberg equation of motion for the observable $N = a^\dagger a + \sigma^2/2$, which reads, for the single site case,

$$\frac{d\langle N \rangle}{dt} = -2\kappa \langle a^\dagger a \rangle - \gamma \left(\langle \sigma^2 \rangle - 1 \right).$$  \hspace{1cm} (B.4)
In the steady state, $\frac{d\langle N \rangle}{dt} = 0$, which implies

$$1 - \langle \sigma^z \rangle = \frac{2\kappa}{\gamma}\langle \sigma^z \rangle,$$

(B.5)

where $n = a^\dagger a$. Notice that this is an exact relation. On the other hand, by taking traces in both sides of (B.3), we find the following equation

$$2 \text{Tr}(\rho\gamma) = 1 - \langle \sigma^z \rangle = 2\beta\gamma_n^2((\langle n \rangle + 1)).$$

(B.6)

We equate now equations (B.5) and (B.6), and take the limit $\langle n \rangle \gg 1$, and get,

$$\beta = \frac{1}{C_p}.$$

(B.7)

This eventually leads to a renormalization of the parameter $B$ that amounts to $B \rightarrow B/C_p \equiv B_r$. The latter leads to a correction of equation (B.1), such that

$$n_{ph} \rightarrow A - \frac{B_r}{C} \equiv n_{mf},$$

(B.8)

so that the number of photons finally actually agrees with the mean result. Furthermore, in figure 9(a) we present numerical results that show that the photon number calculated with equation (7) agrees very well with an exact single qubit laser calculation using equation (1). In figure 9(b) we show calculations carried out with the renormalized $B_r$ parameter, that show that the number of photons curves converge to the mean-field result for large values of $\gamma/\kappa$.

Note that for smaller values of $\langle n \rangle$ it is possible to make an $n$-dependent definition of $\beta$. One could then carry out an iteration procedure (e.g. estimate $\langle n \rangle$ with $\beta = 1/C_p$, then use this value to re-calculate $\beta$, calculate $\langle n \rangle$ again, and so on until reaching convergence). The added benefit of this is, however, not very high, since our renormalization procedure works very well in practice.

Finally, we observe that the whole discussion remains valid even in the presence of the qubit pure dephasing term in equation (11). In particular, we can see that renormalizing parameters $A$ and $B$ to the updated values in equation (A.14), amounts to the substitution of $C_p$ by its renormalized value in equation (12).
Appendix C. Fokker–Planck equation

C.1. Derivation of the Fokker–Planck equation

Let us sum up the derivation of the Fokker–Planck equations used in this work. Recall that we employ the Glauber–Sudarshan $P$ representation of the density matrix [47], defined by

$$\rho(t) = \int d^2 \alpha P(\alpha, \alpha^*, t)|\alpha\rangle\langle\alpha|,$$

where $|\alpha\rangle$ is the coherent state $|\alpha\rangle = \exp(\alpha \alpha^* - \alpha^* \alpha)|0\rangle$. We transform the master equation (A.10) with the help of the following relations for coherent states,

$$a|\alpha\rangle = \alpha|\alpha\rangle,$$
$$|\alpha\rangle a^\dagger = \alpha^*|\alpha\rangle,$$
$$a^\dagger|\alpha\rangle = \left(\frac{\partial}{\partial \alpha} + \alpha^*\right)|\alpha\rangle,$$
$$|\alpha\rangle a = \left(\frac{\partial}{\partial \alpha^*} + \alpha\right)|\alpha\rangle.$$

With equations (C.2) we can write the master equation as an equation of motion for $P(\alpha, \alpha^*, t)$, after an integration by parts with the assumption of zero boundary conditions at infinity. This procedure is simplified in the limit $|\alpha|^2 \gg 1$, and $g \ll \gamma$ in which we drop any contribution smaller than $B|\alpha|^2\alpha$, as $B$ happens to be a very small coefficient compared to $A$, $B/A \propto (g/\gamma)^2 \ll 1$. In doing so, we arrive at the Fokker–Planck equation,

$$\frac{\partial P}{\partial t} = +2\sum_{j=1,2} \frac{\partial^2 P}{\partial \alpha_j \partial \alpha_j^*} - \sum_{j=1,2} \frac{\partial}{\partial \alpha_j} [(A - C - B_j|\alpha|^2)\alpha_j - it\epsilon_{j+1} - i\epsilon_j]P + C.c.$$  (C.3)

Our equation of motion can be written in polar coordinates with the aid of the equivalences,

$$\alpha_j = r_j e^{i\theta_j}, \quad \alpha_j^* = r_j e^{-i\theta_j},$$
$$\frac{\partial}{\partial \alpha_j} = \frac{1}{2} e^{-i\theta_j} \left( \frac{\partial}{\partial r_j} - i \frac{\partial}{\partial \theta_j} \right),$$
$$\frac{\partial}{\partial \alpha_j^*} = \frac{1}{2} e^{i\theta_j} \left( \frac{\partial}{\partial r_j} + i \frac{\partial}{\partial \theta_j} \right).$$  (C.4)

Equation (C.3) then reads,

$$\frac{\partial P}{\partial t} = A \sum_j \frac{\partial^2 P}{\partial r_j^2} - \sum_j \left\{ \frac{1}{r_j \partial r_j} \left[ r_j^2 (A - C - Br_j^2) P \right] + \frac{A}{2} \left[ \frac{\partial^2}{\partial r_j^2} + \frac{1}{r_j} \frac{\partial}{\partial r_j} \right] P \right\}$$
$$+ |\epsilon| \sum_j \sin(\theta_j - \theta) \frac{\partial P}{\partial r_j} + |\epsilon| \sum_j \cos(\theta_j - \theta) \frac{\partial P}{\partial \theta_j}$$
$$+ t \sum_j r_{j+1} \sin(\theta_{j+1} - \theta_j) \frac{\partial P}{\partial r_j} - i \sum_j \frac{r_{j+1}}{r_j} \cos(\theta_j - \theta_{j+1}) \frac{\partial P}{\partial \theta_j}.$$  (C.5)

This is a very complicated equation, whose analytical or even numerical solution is very challenging. We are going to see below how a simplification can be justified in the lasing regime.

C.2. Single nanolaser Fokker–Planck equation

Before addressing the case of two coupled nanolasers, let us consider the solution of a single nanolaser Fokker–Planck equation without any external drive ($\epsilon = 0$). In this case, $P(r, \theta) = R(r)P_0(\theta)$, with $P_0 = 1/(2\pi)$. The radial function, $R(r)$, satisfies the differential equation,

$$\frac{dR}{dr} = \frac{1}{r} \frac{\partial}{\partial r} \left[ r^2 (A - C - Br^2) R \right] + \frac{A}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) R.$$  (C.6)
The steady-state solution of equation (C.6) with \( \frac{dR}{dr} = 0 \) is,

\[
R(r) = \frac{1}{N} e^{-\frac{r_0^2}{2\sigma^2}} = \frac{1}{N} e^{-\frac{r^2}{2\sigma^2}},
\]

(C.7)

where \( N = \int_0^\infty dr r R(r) \), is a normalization constant.

In the lasing regime \( (C_p > 1) \) and in the limit of large photon numbers \( (\gamma/\kappa > 1) \), the steady-state radial probability function takes the form,

\[
R(r) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-r_0)^2}{2\sigma^2}},
\]

(C.8)

with the constants

\[
(r_0)^2 = \frac{A - C}{B_0} = \frac{1}{2} \frac{C_p - 1}{C_p} \frac{\gamma}{\kappa},
\]

\[
\sigma^2 = \frac{1}{4A - C} = \frac{1}{4} \frac{C_p}{C_p - 1}.
\]

(C.9)

The radial distribution can be used to calculate the mean and variance of the photon number,

\[
\langle n \rangle = (r_0)^2,
\]

(C.10)

\[
\Delta^2 n = \langle n^2 \rangle - \langle n \rangle^2 \approx \frac{1}{2} \frac{\gamma}{\kappa}.
\]

(C.11)

Note that equation (C.10) agrees with the mean-field calculation \( (n_{mf} = (r_0)^2) \). Also, equation (C.11) agrees with the calculation of the variance from a coherent state in the limit of \( C_p \gg 1 \), for which we recover the well-known relation \( \Delta^2 n = \langle n \rangle [58] \).

### C.3. Reduced Fokker–Planck equation for two nanolasers

In the lasing regime, we can simplify equation (C.3) by assuming that the radial variables are settled around their steady-state mean values \( \tau_j \approx r_0^j \), and \( P(\tau, \alpha) \) can be well approximated by a factorized form,

\[
P(\tau, \theta) = R(\tau_1) R(\tau_2) P_\theta(\theta_1, \theta_2),
\]

(C.12)

where each \( R(\tau_j) \) is a properly normalized Gaussian function around \( r_0^j \),

\[
R(\tau_j) = \frac{1}{N_j} \exp \left( -\frac{(\tau_j - r_0^j)^2}{2\sigma^2} \right), \quad N_j = \sqrt{2\pi\sigma^2 r_0^j}
\]

(C.13)

Equation (C.12) is valid as long as the phase variable evolves on much slower time scales as the radial variables. The typical times scales for phase \( \tau_\phi \) and radial \( \tau_r \) variables can be estimated, in the lasing regime and assuming values of \( \tau_r \) close to \( r_0^r \), from equation (C.5),

\[
1/\tau_\phi = A/2\langle n \rangle = \kappa C_p,
\]

\[
1/\tau_r = A/2.
\]

(C.14)

Additionally, the relaxation time associated to the photon number variable can also be estimated by means of the Heisenberg equations, see equation (31). We find, thus, that \( \tau_r \approx n \tau_\phi \), such that in the large photon number limit, the nanolaser radial coordinate can be considered to relax very fast compared with the phase variable. This behaviour is consistent with other theoretical descriptions of the laser’s dynamics [58].

If we now integrate both sides of equation (C.5) in the radial variables \( \int_0^\infty dr \tau_r \partial R \), the result is an equation of motion for the angular part. In this derivation it is important to notice that the terms \( \partial R/\partial t \) can be cancelled together with other terms by using the identity equation (C.6). Also, the integration of the first derivative \( \partial \tau_r \) is eliminated through the relation,

\[
\int_0^\infty \tau_j d\tau_r \partial R = -\frac{1}{N_j} \int_0^\infty \tau_j d\tau_j \frac{(\tau_j - r_0^j)^2}{\sigma^2} \exp \left( -\frac{(\tau_j - r_0^j)^2}{2\sigma^2} \right)
\]

\[
= -\frac{1}{N} \int_0^\infty (r' + r_0) d\tau' \frac{(r')^2}{\sigma^2} \exp \left( -\frac{(r')^2}{2\sigma^2} \right)
\]
Our scheme may be experimentally implemented in a variety of setups, like trapped ion phonon lasers [29] or artificial superconducting atoms [24], where both single-qubit lasing and photon/phonon cavity–cavity tunneling terms have been demonstrated. In the following we sketch an implementation in a superconducting circuit setup by following the approach presented by one of us in [40], where the theoretical framework and range of possible experimental parameters is provided with more detail.

We start first by considering how to build a single nanolaser. The main idea is to achieve the incoherent pumping of the qubit with the aim of a periodic driving of the qubit frequency. We consider first the following single superconducting qubit–cavity Hamiltonian,

\[ H = H_0 + H_g + H_d(t), \]
\[ H_0 = \omega a a^\dagger + (\epsilon_q/2)\sigma^z, \quad H_g = g \left( a + a^\dagger \right) \sigma^x, \quad H_d(t) = \eta \Omega \cos(\Omega t) \sigma^z. \]  

(D.1)

In this model, \( H_0 \) represents the free Hamiltonian of a superconducting microwave cavity and a superconducting qubit, \( H_g \) is the electromagnetic qubit–cavity coupling, and \( H_d(t) \) is a time-dependent periodic driving with frequency \( \Omega \) and amplitude \( \eta \Omega \) of the qubit’s frequency. These are quite basic elements in circuit QED steps (see for example [30, 31]), see [40] for a more detailed description of the circuit architecture.

We consider periodic drivings such that \( \Omega = \epsilon_q + \omega \). If, additionally, \( g \ll \epsilon_q, \omega \), then we can move to the interaction picture with respect to \( H_0 + H_d(t) \) and neglect non-resonant terms, such that, in the new interaction picture [40],

\[ H_g \approx g' \left( a^\dagger \sigma^+ + a \sigma^- \right), \]  

(D.2)

with \( g' = f_0(2\eta)J_1(2\eta), \) where \( f_n(x) \) are \( n \)’th Bessel functions [40]. Other schemes in which it is the coupling \( g \) that is periodically driven are also possible, see for example [59]. The big advantage of having obtained an effectively counter-rotating qubit–cavity coupling is that qubit decay is now the equivalent of incoherent pumping. Namely, imagine that we consider the following Liouvillian, which includes now dissipative terms,

\[ L(\rho) = L_{[\sigma^-;\gamma]}(\rho) + L_{[\alpha;\kappa]}(\rho) - i[H_d,\rho]. \]  

(D.3)

We find that after the trivial transformation \( \sigma^\dagger \sigma \rightarrow \sigma^+ \sigma^- \), we recover the single qubit nanolaser contributions to equation (1). The advantage of this scheme is that it can allows to transform the qubit decay, which can be controlled by placing a dissipative cavity nearby, into incoherent excitation of the qubit in the new, inverted basis. The other components of equation (1) appear quite naturally: the incoming coherent driving fields, \( \epsilon_i \), are classical microwave signals, whereas the coherent tunneling term \( t_c \) is the usual coupling between microwave cavities [30].

In practice, we could consider typical parameters in circuit QED like \( \epsilon_q/2\pi = 10 \text{ GHz} \), \( \omega/2\pi = 4.5 \text{ GHz} \), \( g/2\pi = 2.38 \text{ MHz} \), and damping rate \( \kappa/2\pi = 10 \text{ kHz} \) [53]. With a modulation amplitude parameter \( \eta = 0.2 \), this would lead to an effective couplings \( g'/2\pi = 0.48 \text{ MHz} \). Now, \( \gamma \) can be tuned by fabrication, for example, by approximating a very dissipative cavity to the qubit. A typical decay rate \( \gamma/2\pi = 10 \text{ MHz} \), would finally lead to \( C_0 = 2, \gamma/\kappa = 10^3 \), and \( n_0 = 475 \text{ photons} \), just to given numbers that could be achieved in this setup. On the other hand, typical values for cavity–cavity couplings, \( t_c \), are in the regime of MHz, but they can be made arbitrarily small simply by widening the separating barrier by fabrication.

Appendix D. Physical implementation in a circuit QED setup

\[ \approx -\frac{1}{Nj} \int_{-\infty}^{\infty} dr' \left( \frac{r'^2}{\sigma^2} \right) \exp \left( -\frac{(r')^2}{2\sigma^2} \right) \]
\[ = -\frac{1}{N} \sqrt{2\pi} \sigma = -\frac{1}{r_j^2}. \]  

(C.15)

After grouping terms, the resulting equation adopts the form claimed in equation (17),

\[ \frac{\partial P_{\rho}}{\partial t} = \frac{A}{2} \sum_{j=1,2} \frac{1}{n_j^2} \frac{\partial^2 P_{\rho}}{\partial \theta_j^2} + \sum_{j=1,2} \frac{\partial}{\partial \theta_j} \left( \frac{\sqrt{\gamma}}{\kappa} \cos(\theta_{j+1} - \theta_j) + |\epsilon| \cos(\theta_j - \phi_j) P_{\rho} \right), \]  

(C.16)

Notice that this a sort of quantum version of the stochastic Kuramoto model, that is, it represents two Fokker–Planck equation coupled by a term that was originated by coherent photon tunneling.
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