Quantum no-singularity theorem from geometric flows

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In this paper, we analyze the classical geometric flow as a dynamical system. We obtain an action for this system, such that its equation of motion is the Raychaudhuri equation. This action will be used to quantize this system. As the Raychaudhuri equation is the basis for deriving the singularity theorems, we will be able to understand the effects such a quantization will have on the classical singularity theorems. Thus, quantizing the geometric flow, we can demonstrate that a quantum space-time is complete (non-singular). This is because the existence of a conjugate point is a necessary condition for the occurrence of singularities, and we will be able to demonstrate that such conjugate points cannot occur due to such quantum effects.

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I. INTRODUCTION

Even though general relativity (GR) is one of the most well tested theories, it predicts its own breakdown due to the occurrence of singularities. Furthermore, the Penrose-Hawking singularity theorems demonstrate that the occurrence of these singularities is an intrinsic property built into the structure of classical GR, and not a mathematical artifact. At the singularities the space-time is no longer a smooth manifold, and the laws of physics cannot be meaningful when studied at these points. Thus, the removal of such singularities is very important physically, and it is generally argued that singularities should be removed due to quantum gravitational effects.

It has been argued that the problem would be resolved due to string-theoretical effects like the propagation of strings across singularities, colliding Branes in heterotic M-theory, string theoretical effect in a Kasner background, string gas cosmology, and bouncing branes with negative-tension. All these cosmological models are motivated from string theory, however, the string theory also predicts the existence of higher dimensions and supersymmetry, both of which have not been experimentally observed. Similarly, it has been argued that loop quantum gravity (LQG) can also lead to resolution of singularities, but it has not been possible to recover the Einstein equation in LQG. Hence, the physical validity of these models that are based on string theory and LQG can be questioned. Even though the singularities have also been removed using other phenomenological approaches to quantum gravity, such as the existence of a minimum measurable length scale, a modified Wheeler-DeWitt equation, quantum gravity condensates, and information-theoretic networks, all of these approaches have problems associated with them. Furthermore, all the work on removal of singularities by quantum effects has been done using different proposals, and all of which depend on the specifics of a particular approach to quantum gravity. However, rarely, there were attempts to show quantum completeness of space-time without referring to a particular model of gravity.

A quantum mechanical counterpart of the singularity theorems would be needed to rigorously prove the absence of singularities and completion of quantum space-time, in a model-independent approach. It may be noted that the classical Penrose-Hawking singularity theorems were derived using the Raychaudhuri equation (RE), thus it is expected that a quantum mechanical generalization of the RE could be used to analyze the quantum mechanical version of the classical Penrose-Hawking singularity theorems. In fact, semiclassical corrections to the RE from Bohmian trajectories has been constructed, and it has been argued that such an equation can resolve singularities in cosmology, and in black holes. These trajectories get corrected because of the quantum corrections to the flow of geometries. Furthermore, to derive the quantum version of the classical Penrose-Hawking singularity theorems, we would need to understand the full quantum mechanical behavior of the geometric flows.

In this paper, we shall study the dynamics of these geometric flows. We will construct a classical action for these...
We start by studying a congruence of test particle motion in quantum space-time using these quantum geometric flows. We will demonstrate that quantum gravitational effects will indeed remove the singularities, and we will obtain quantum no-singularity theorems. The classical Penrose-Hawking singularity theorems will be recovered in the Ehrenfest limit of these quantum geometric flows.

II. GEOMETRIC FLOWS

We proceed further by computing the variation \( \delta L \). We define the dynamical evolution of the transverse metric \( h_{\alpha\beta} \) by the equation

\[
\frac{\partial h_{\alpha\beta}}{\partial \lambda} = \theta_{\alpha\beta} = 2\sigma_{\alpha\beta} + \frac{2}{n} h_{\alpha\beta}\theta. \tag{2}
\]

With \( \sigma_{\alpha\beta} \) being the traceless shear tensor, and \( \theta \) is the expansion parameter. This equation is a form of a geometric flow equation for the equivalent class of manifolds \( \mathcal{F} \), or more accurately for \( \sigma_\lambda \). We may take the trace of (2) to get,

\[
h_{\alpha\beta} \partial_{\lambda} h_{\alpha\beta} = \frac{2}{n} \theta. \tag{3}
\]

Then, we multiply by \( \sqrt{\det h} \), obtaining:

\[
\sqrt{\det h} h_{\alpha\beta} \partial_{\lambda} h_{\alpha\beta} = \frac{2}{n} \sqrt{\det h} \theta. \tag{4}
\]

Using the identity \( \delta (\det h) = hh_{\alpha\beta} \delta h_{\alpha\beta} \) we identify the LHS of above equation with \( \dot{\rho} \). Thus we have,

\[
\dot{\rho} = \frac{2}{n} \rho \theta. \tag{5}
\]

We now make the following ansatz about the action for the geometric flow

\[
S[\rho, \dot{\rho}] = \int d\lambda \frac{n}{4} \frac{1}{\rho^2} - \mathcal{R} \rho - V_\sigma[\rho], \tag{6}
\]

with \( \mathcal{R} \), being the Raychaudhuri scalar \( \mathcal{R} := R_{\mu\nu}\xi^\mu\xi^\nu \), and \( V_\sigma[\rho] \) is the shear potential, satisfying:

\[
\frac{\delta V_\sigma[\rho]}{\delta \rho} = 2\sigma^2 \tag{7}
\]

We may identify the canonical conjugate momentum to \( \rho \) using the Lagrangian,

\[
\Pi = \frac{\delta L}{\delta \dot{\rho}} = \frac{n}{2} \rho^{-1} \dot{\rho}, \quad \frac{n}{2} \rho^{-1}(\frac{2}{n} \rho \theta) = \theta. \tag{8}
\]

Thus, as expected, the expansion parameter is the conjugate momentum to the dynamical degree of freedom. We proceed further by computing the variation \( \delta L/\delta \rho \)

\[
\frac{\delta L}{\delta \rho} = -\frac{n}{4} \rho^{-2} \dot{\rho}^2 - 2\sigma^2 - \mathcal{R}
\]

\[
= \left[ \frac{n}{4} \rho^{-2} \left( \frac{4}{n^2} \rho^2 \theta^2 \right) \right] - 2\sigma^2 - \mathcal{R}
\]

\[
= -\frac{1}{n} \theta^2 - 2\sigma^2 - \mathcal{R}. \tag{9}
\]
Now, we write the Euler-Lagrange equations in the ρ configuration space,
\[
\frac{d}{d\lambda} \theta = \frac{\delta L}{\delta \rho}
\]
(10)

Thus, we obtain the Raychaudhuri equation,
\[
\dot{\theta} = -\frac{1}{n} \theta^2 - 2\sigma^2 - \mathcal{R}
\]
(11)

This is the expected result, since the dynamics of the geometric flow should generate the dynamics of the congruence ‘above’ it. Indicating that the action (6) is the correct ansatz about the dynamical description of the geometric flows.

We can moreover write the effective Hamiltonian for geometric flows by preforming a Legendre transformation on the Lagrangian \( L \), The effective Hamiltonian can be written as:
\[
H = \frac{1}{n} \rho \theta^2 + \mathcal{R}\rho + V_\sigma[\rho]
\]
(12)

Raychaudhuri equation can also be recovered from the Poisson brackets
\[
\dot{\theta} = -\{\theta, H\} = -\frac{1}{n} \theta^2 - 2\sigma^2 - \mathcal{R},
\]
(13)

We shall use the canonical formalism for the geometric flows dynamics in order to canonically quantize it, as in the next section.

### III. CANONICAL QUANTIZATION

We define the operators \( \hat{\rho} \) and \( \hat{\theta} \) acting on the Hilbert space of geometric flows \( \mathcal{H} \), and the geometric flow state \( \Psi \). We may use the \( \rho \)-representation for these operators, such that the Hilbert space is identified to be \( \mathcal{H} := L^2(\mathbb{R}^+; d\mu[\rho]) \), since the configuration space for \( \rho \) consists only of non-negative values \( \rho \). This Hilbert space is equipped with the measure \( d\mu[\rho] := \rho d\rho \). The states become wave functions of \( \rho \) and time \( \Psi[\rho, \lambda] \). In fact, they are wave functionals of the coordinates on \( \sigma_\lambda \) as they are defined by,
\[
\Psi[\rho, \lambda] := \int_{\sigma_\lambda} \psi(\rho(x^\alpha, \lambda))\sqrt{h}d^nx.
\]
(14)

The pair \( \hat{\rho} \) and \( \hat{\theta} \) are self-adjoint operators, that satisfy the canonical commutation relations (CCR),
\[
[\hat{\rho}, \hat{\theta}] = i\hbar \mathcal{I}
\]
(15)

1. The determinant of a Riemannian manifold is non-negative. Hence, there is no need to assume any boundary conditions for \( \rho < 0 \).

In the \( \rho \)-representation, they are identified with \( \mathbb{R} \)
\[
\hat{\rho} = \rho : \mathcal{I}(\mathbb{R}^+) \rightarrow \mathcal{I}(\mathbb{R}^+),
\]
(16a)
\[
\hat{\theta} = -\frac{i\hbar}{\sqrt{\rho}} \delta \rho : \mathcal{I}(\mathbb{R}^+) \rightarrow \mathcal{I}(\mathbb{R}^+).
\]
(16b)

Here, \( \mathcal{I}(\mathbb{R}^+) \) is a subset of \( L^2(\mathbb{R}^+) \) and \( \Psi[\rho, \lambda] \) belongs to it. Now, for this wavefunction to be valid for solving Schrödinger’s equation \( \mathcal{I}(\mathbb{R}^+) \), it should be at least \( C^2(\mathbb{C}) \). We therefore have a well-defined Hilbert space and operators as endomorphisms acting on it. We can write the effective Hamiltonian operator,
\[
\hat{H} = -\frac{\hbar^2}{n} \left[ \frac{1}{\sqrt{\rho}} \frac{\delta}{\delta \rho} \left( \rho^{-1} \frac{\delta}{\delta \rho} \sqrt{\rho} \right) \right] \Psi[\rho, \lambda] + (\mathcal{R}\rho + V_\sigma[\rho]) \Psi[\rho, \lambda] = i\hbar \frac{d}{d\lambda} \Psi[\rho, \lambda].
\]
(18)

This equation is very similar to Wheeler-DeWitt’s [31], in terms that the wave function \( \Psi[\rho, \lambda] \) is a wave functional of an extended object of a background geometry. Nevertheless, as this equation is for the quantum flow of geometries, that is a subsystem of the universe not the universe as a whole. Here, we are studying the space-time locally, as an ensemble of ‘atoms’ of geometry. In addition, this equation contains a real effective Hamiltonian, not a Hamiltonian constraint. Hence, it contains an evolution parameter \( \lambda \), which acts as a real time for this system. Thus, this equation does not have the problem of time that is associated with the usual Wheeler-Dewitt equation. Furthermore, here the foliation of space-time is based on the flows of geodesics to a longitudinal and transverse directions, relative to the congruence. Unlike the standard ADM-foliation that foliates the whole space-time, arbitrarily [32].
IV. CONJUGATE POINTS AND SINGULARITIES

Since the RE is the equation of motion of the geometric flow, we shall discuss the singularity theorems using the language of geometric flows. Therefore, we will discuss singularities in the dynamical language of geometric flows. Using the dynamical foliation adopted at the beginning of section II we have Global hyperbolicity for $\mathcal{M}$ [1] implied from using the dynamical foliation. Thus, the strong causal conditions hold for $\mathcal{M}$. Now, we can recall the following preposition [6, 7],

Preposition 1 If the strong causal conditions hold on $\mathcal{M}$, it implies that the strong energy conditions hold as well,

$$R > 0$$

Which leads us to the focusing theorem [27]

Theorem 1 If the strong energy conditions hold on $\mathcal{M}$, then a congruence of time-like geodesics will encounter a conjugate point/caustic at a time not greater than $\frac{\rho}{\theta_0}$. For a given initial value of the expansion parameter $\theta_0 < 0$. That is, if congruence enters an isolated horizon.

The implication of this theorem to the geometric flows is that, if the initial canonical momentum was less than zero, the geometric flow will evolve in time such that the cross sectional volume decreases quickly and reaches zero $\rho \rightarrow 0$, when $\theta \rightarrow -\infty$, and $\lambda = n/\theta_0$.

The existence of a conjugate point does not imply that the geometric flow hits a singularity, because its volume might expand again after sometime, if $\theta$ is defined at that point. However, the existence of a conjugate point is an essential step for showing the existence of a singularity [1].

We shall not discuss the classical singularity theorems in terms of geometric flows any further, as they are kept for later extended study. However, we shall demonstrate the absence of conjugate points, due to quantum effects. As the existence of conjugate points is a necessary condition for the occurrence of singularities, we will be able to demonstrate that singularities will not form in quantum space-time.

V. COMPLETENESS OF QUANTUM SPACE-TIME

Now, we turn to the existence of conjugate points in the context of quantum geometric flows. The focusing theorem stated above will only hold in the Ehrenfest limit for the operator $\hat{\theta}$. We need to investigate this theorem for the operator $\hat{\theta}$ without referring to the classical argument.

The CCR relation between $\hat{\rho}$ and $\hat{\theta}$ leads to the uncertainty relation,

$$\Delta \rho \Delta \theta \geq \frac{\hbar}{2}.$$  \hspace{1cm} (19)

Indicating that quantum geometric flows do not tend to focus forming conjugate points, due to the uncertainty in measuring the volume, without causing them to expand rapidly, indicating that gravity at the small scale $\sim \ell_p$ possess a quantum repulsive force.

In order to show that conjugate points do not form for quantum geometric flows, we need to show that the spectrum of the operator $\zeta(\hat{\theta})$ is bounded below. This can be done by showing that the operator itself is bounded from below [4], i.e.

$$\left| \left( \Psi, \hat{\theta} \Psi \right) \right| \geq c,$$  \hspace{1cm} (20)

for some constant $c$.

Such that the expansion will not blow down to minus infinity, as opposed to the classical case. We know that the spectrum of $\hat{\rho}$ is bounded below since it takes only non-negative values $\zeta(\hat{\rho}) \in \mathbb{R}^+$. Thus, the Hilbert space is, as mentioned earlier, the space $\mathcal{H} := L^2(\mathbb{R}^+)$. We define the inner product on the Hilbert space $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ by [2]

$$(\Phi, \Psi) := \int_0^\infty \Phi[\rho; \lambda]^* \Psi[\rho, \lambda] \rho d\rho$$  \hspace{1cm} (21)

and the norm from this inner product:

$$\|\Psi\| = |(\Phi, \Psi)|$$  \hspace{1cm} (22)

Therefore, we define the norm of the operator $\hat{\theta}$:

$$\|\hat{\theta} \Psi\| = |i \hbar \int_0^\infty d\rho \rho \Psi[\rho, \lambda]^* \Psi[\rho, \lambda]|$$  \hspace{1cm} (23)

with the comma denoting the differentiation. However, the norm of $\Psi$ satisfies the inequality [3]

$$\|\Psi\|^2 = \int_0^\infty d\rho \rho \Psi[\rho, \lambda]^* \Psi[\rho, \lambda] \leq \int_0^\infty d\rho |\Psi|^2 \leq \sup_{\rho \in \mathbb{R}^+} |\Psi|^2$$  \hspace{1cm} (24)

We can use the uniform norm to obtain:

$$\|\hat{\theta} \Psi\| \geq \|\Psi\| \Rightarrow \|\hat{\theta} \Psi\| \geq \sup_{\rho \in \mathbb{R}^+} |\Psi|$$  \hspace{1cm} (25)

That proves the operator $\hat{\theta}$, is bounded below, hence its spectrum $\zeta(\hat{\theta}) > -\infty$, proving that conjugate points do not exist in space-time with quantum geometric flows. This can be understood easily if we recalled that the probability measure density $\rho|\Psi|^2$ should vanish at the end points 0 and $+\infty$ in order to satisfy the Born conditions. Hence, the wave function of the geometric flow will give vanishing probability at the conjugate point $\rho = 0$, implying that singularities do not form in quantum space-time. The previous analysis was made for congruence of time-like geodesics, but it can be repeated for the null geodesics with letting $n \rightarrow n - 1$, and considering the optical RE [1].
VI. NO-SINGULARITY THEOREM

It is possible to state a ‘no-singularity’ theorem for a quantum space-time, from the argument in the previous section.

Theorem 2 There is a zero probability for the quantum space-time to have incomplete geodesics due to singularities. Hence, the quantum space-time is complete.

a. Proof Although the condition for the operator $\hat{\theta}$ to be bounded-below is a sufficient to prove the above theorem. We can use the Schrödinger-like equation (18) to have a more detailed proof.

We observe that (18) has an effective potential term $V_{\text{eff}} \propto \frac{1}{\rho}$, that diverges at conjugate points i.e., $V_{\text{eff}} \to \infty$ as $\rho \to 0$.

This will affect the behavior of the wavefunction near $\rho \to 0$, acting like a boundary condition, from a diverging effective potential. Therefore, there will be a vanishing probability to measure the geometry forming a conjugate point, and conversely a singularity, see Figure VI 0 a.

Due to the effective mass diverging at singularities, we may conjecture that the curvature $\mathcal{R}$ is regular even close to the singularity. Even if this is not the case and $\mathcal{R} \to \infty$ at the ‘classical singularity’. This could be translated as an infinite potential barrier at $\rho$, forming yet another boundary condition for the wavefunction, such that $\Psi[0] = 0$, and the above theorem still remains valid.

VII. CONCLUSION

In this paper, we have introduced a new dynamical system for the space-time, the geometric flow associated with the motion of a congruence of time-like or null geodesics along an $n + 1$ dimensional globally hyperbolic space-time. We have identified the dynamical degree of freedom for such system being the cross-sectional volume of the congruence at a given time $\rho(\lambda)$. Then, we wrote the action for such degree of freedom. Variation of the action with respect to $\rho$ yielded the Raychaudhuri equation, as the dynamics of the geometric flows should coincide with the dynamics of the congruence described by Raychaudhuri equation. Then, we identified the conjugate momentum to $\rho$ being the expansion parameter $\theta$, and wrote the effective Hamiltonian from the Legendre transformation of the Lagrangian. The system was canonically quantized using the standard techniques. We were able to show from studying the quantum geometric flows that quantum mechanically the space-time is free from singularities, and these singularities occur only in the Ehrenfest limit. Thus, we were able to obtain quantum no-singularity theorem, such that the classical Penrose-Hawking singularity theorems where the Ehrenfest limits of such no-singularity theorems. The proof of completeness of quantum space-time is made by showing that the expansion operator $\hat{\theta}$ is bounded from below. Moreover, analysis of the wavefunction $\Psi[\rho, \lambda]$ as a solution to (18) with a diverging effective mass at $\rho = 0$ explicitly shows a null probability for the congruence to form a conjugate point, and thereby showing that quantum space-time is complete. Although there is a real singularity in the space-time manifold $\mathcal{M}$, there is a different behavior in the $\rho$ configuration space. Due to the diverging effective mass, there is an essential singularity at $\rho = 0$. However, the solution to Schrödinger-like equation (18) has a non-essential (removable) singularity at $\rho = 0$. Therefore, we expect that analytic extension to the dynamical variable would indicated the presence of pre-singular geometry, without loosing regularity. The absence of singularity means the absence of inconsistency in the laws of nature describing our universe, that shows a particular importance in studying black holes and cosmology.

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2 We have not attempted to quantize the curvature directly here, this is left for future work.
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