Fixed Point Theorems for Nonexpansive Mappings under Binary Relations

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Abstract: In the present article, we establish relation-theoretic fixed point theorems in a Banach space, satisfying the Opial condition, using the ℛ-Krasnoselskii sequence. We observe that graphical versions (Fixed Point Theory Appl. 2015:49 (2015) 6 pp.) and order-theoretic versions (Fixed Point Theory Appl. 2015:110 (2015) 7 pp.) of such results can be extended to a transitive binary relation.

Keywords: binary relation; ℛ-nonexpansive maps; uniformly convex Banach space; ℛ-Krasnoselskii sequence

1. Introduction

An analogue of the classical Banach contraction principle employing a partial ordering on underlying complete metric space was initiated by Ran and Reurings [1], which was further refined by Nieto and Rodríguez-López [2]. The fixed point theorems of Nieto and Rodríguez-López [2] are generalized and extended by various authors (for instance, see [3–15]). On the other hand, Jachymski [16] obtained a more general approach by using a graph instead of a partial ordering. In 2015, Alam and Imdad [17] established a novel version of the Banach contraction principle, employing an amorphous binary relation. In recent years, various metrical fixed theorems were proved under different types of contractivity conditions, employing certain binary relations (e.g., [18–36]). In such results, the involved contraction conditions remain relatively weaker than the usual contraction conditions, as these are required to hold merely for those elements which are related in the underlying binary relation.

On the other hand, Alfuraidan [37] initiated the idea of monotone nonexpansive mappings on a Banach space (X, ∥·∥) equipped with a directed graph ℜ such that V(ℜ) = K and E(ℜ) ⊇ Δ, where K is a nonempty, convex and bounded subset of X not reduced a single point, and V(ℜ) denotes the set of vertices of ℜ, while E(ℜ) denotes the edges of ℜ. We say that the directed graph ℜ = (V(ℜ), E(ℜ)) is transitive if for all x, y, z ∈ V(ℜ) with (x, y) ∈ E(ℜ) and (y, z) ∈ E(ℜ), we have (x, z) ∈ E(ℜ). Recall that the sets {a, b} = {x ∈ K : (a, x) ∈ E(ℜ)} and {a, b} = {x ∈ K : (x, b) ∈ E(ℜ)}, for any a, b ∈ K, term as ℜ-intervals. Following Alfuraidan [37], a self-mapping T on K is termed as ℜ-monotone if for all x, y ∈ K with (x, y) ∈ E(ℜ), we have (Tx, Ty) ∈ E(ℜ). In addition, if ∥Tx − Ty∥ ≤ ∥x − y∥, for all x, y ∈ K with (x, y) ∈ E(ℜ), then T is said to be ℜ-monotone nonexpansive. Given a Hausdorff topology T on X, the triplet (K, ∥·∥, ℜ) is said to enjoy property (P) if for any sequence {xₙ} ⊂ K satisfies (xₙ, xₙ₊₁) ∈ E(ℜ) for all n ∈ N \ {0}; if a subsequence {xₙₖ} of {xₙ} 3-converges to x, we have (xₙₖ, x) ∈ E(ℜ) for all k ∈ N.

Before mentioning the result of Alfuraidan [37], we summarize the notion of the “Opial condition” in order to make our discussion self-contained.
Definition 1 ([38]). A Banach space \((X, \| \cdot \|)\) is said to enjoy the \(\mathfrak{I}\)-Opial condition (wherein \(\mathfrak{I}\) remains a Hausdorff topology on \(X\)) if for any sequence, \(\{x_n\} \subset X\) satisfying \(x_n \xrightarrow{n \to +\infty} x\), we have the following:

\[
\limsup_{n \to +\infty} \|x_n - x\| < \limsup_{n \to +\infty} \|x_n - x\|
\]

for all \(x \in X\) such that \(x \neq x\).

In particular, under the weak topology \(\mathfrak{I}\), the notion “\(\mathfrak{I}\)-Opial condition” refers to “Opial condition”. While if \(\mathfrak{I}\) is assumed to be the norm topology, the notion “\(\mathfrak{I}\)-Opial condition” is transformed to “Strongly Opial condition”.

Theorem 1 ([37]). Suppose that \((X, \| \cdot \|)\) is a Banach space enjoying the \(\mathfrak{I}\)-Opial condition, and \(K\) is a nonempty, convex, bounded and \(\mathfrak{I}\)-compact subset of \(X\), not reducible to a single point. Let \(G\) be a directed graph on \(X\) such that the triplet \((K, \| \cdot \|, G)\) enjoys property (P). Additionally, assume that the \(G\)-intervals are convex subsets of \(X\) and \(T : K \to K\) is a \(G\)-monotone nonexpansive mapping. If there exists \(x_0 \in K\) such that \((x_0, Tx_0) \in E(G)\), then \(T\) admits a fixed point.

Here, it is worth mentioning that Alfuraidan [37] used the transitivity of graph \(G\) to construct the Krasnosel’skii sequence but failed to mention it.

In this continuation, Bachar and Khamsi [39] obtained a natural version of Theorem 1 by assuming a partial order instead of a graph and utilized the same to obtain nonnegative and nonpositive solutions of an integral equation. Given a partially ordered Banach space \((X, \| \cdot \|, \preceq)\), we say that the mapping \(T : K \subseteq X \to K\) is monotone if for all \(x, y \in K\) with \(x \preceq y\), we have \(T(x) \preceq T(y)\). In addition if for all \(x, y \in K\) with \(x \preceq y\), we have \(\|Tx - Ty\| \leq \|x - y\|\), then \(T\) is called a monotone nonexpansive mapping. For any \(a, b \in X\), the subsets \([a, \rightarrow] = \{x \in X : a \preceq x\}\) and \((+b, b] = \{x \in X : x \preceq b\}\) refer to the order intervals in partial ordered set \((X, \preceq)\) with initial point \(a\) and with end point \(b\), respectively.

Theorem 2 ([39]). Suppose that \((X, \| \cdot \|)\) is a Banach space enjoying the \(\mathfrak{I}\)-Opial condition and \(K\) is a nonempty convex, bounded and \(\mathfrak{I}\)-compact subset of \(X\), not reducible to a single point. Let \(\preceq\) be a partial order on \(X\) such that order intervals are convex and \(\mathfrak{I}\)-closed. Additionally, assume that \(T : K \to K\) is a monotone nonexpansive mapping. If there exists \(x_0 \in K\) such that \(x_0\) and \(T(x_0)\) are comparable, then \(T\) admits a fixed point.

The idea of a monotone nonexpansive mapping in the context of graph as well as partial ordering is generalized and extended by several authors, such as [40–46]. Very recently, Alam et al. [47] initiated the concept of \(R\)-nonexpansive mappings and utilized the same to extend the results of Bin Dehaish and Khamsi [42] up to the transitive binary relation, obtaining a relation-theoretic analogue of the classical Browder-G’ohde fixed point theorem.

In this paper, we shall obtain a sharpened version of Theorems 1 and 2 with respect to the following observations.

- The partial ordering or transitive directed graph is not necessary; it is enough to consider a transitive binary relation.
- The transitive binary relation is not necessarily considered on the whole space \(X\); it suffices the same only on the subset \(K\).
- There is no need to use the assumption that all order intervals in \(X\) are convex and closed. It suffices that the certain relational intervals in \(K\) are convex and closed. Furthermore, the convexity and closedness of whole set \(K\) are also relaxed.
- The boundedness of whole set \(K\) must be replaced by the relatively weaker assumption.

In this paper, we shall obtain a sharpened version of Theorems 1 and 2 with respect to the following observations.
2. Relation-Theoretic Notions

In this section, we give some definitions regarding binary relations, which are utilized to prove our main results. In what follows, \( \mathbb{N} \) stands for the set of natural numbers, while \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Remember that a binary relation on a nonempty set \( K \) is a subset \( R \subseteq K^2 \) (i.e., \( R \subseteq K^2 \)). Naturally, \( \emptyset \) and \( K^2 \) remain trivial binary relations on \( K \). We say that \( \emptyset \) is the empty relation, while \( K^2 \) is the universal relation on \( K \).

Definition 2 ([48]). Given a binary relation \( R \) on a nonempty set \( K \), the relation \( R^{-1} \) on \( K \), defined by
\[
R^{-1} := \{(y, x) \in K^2 : (x, y) \in R\}
\]
is called the inverse or transpose or dual relation of \( R \).

Remark 1. \( R^{-1} \) is transitive iff \( R \) is transitive.

Definition 3 ([48]). Given a binary relation \( R \) on a nonempty set \( K \), the relation \( R^s \) on \( K \), defined by
\[
R^s := R \cup R^{-1}
\]
is called the symmetric closure of \( R \).

Remark 2. \( R^s \) is the smallest symmetric relation on \( K \) containing \( R \).

Definition 4 ([17]). Given a binary relation \( R \) defined on a nonempty set \( K \), two elements \( x \) and \( y \) in \( K \) are said to be \( R \)-comparative if \( (x, y) \in R \) or \( (y, x) \in R \). We denote it by \( [x, y] \in R \).

Proposition 1 ([17]). Given a binary relation \( R \) defined on a nonempty set \( K \), we have the following:
\[
(x, y) \in R^s \iff [x, y] \in R.
\]

Definition 5 ([17]). Given a binary relation \( R \) on a nonempty set \( K \) and a self-mapping \( T \) on \( K \), we say that \( R \) is \( T \)-closed if for all \( x, y \in K \), we have the following:
\[
(x, y) \in R \Rightarrow (Tx, Ty) \in R.
\]

Proposition 2 ([49]). Let \( T \) be self-mapping on a nonempty set \( K \). If a binary relation \( R \) on \( K \) is \( T \)-closed, then it is also \( T^n \)-closed for each \( n \in \mathbb{N}_0 \).

Definition 6 ([47]). Given a subset \( K \) of a Banach space \( (X, \| \cdot \|) \) and a binary relation \( R \) defined on \( K \), we say that the mapping \( T : K \rightarrow K \) is \( R \)-nonexpansive if the following holds:
\[
(x, y) \in R \Rightarrow \|Tx - Ty\| \leq \|x - y\|.
\]

Remark 3 ([47]). The following conclusions are straightforward:
(i) \( T \) is \( R \)-nonexpansive \iff \( T \) is \( R^{-1} \)-nonexpansive.
(ii) $T$ is $\mathcal{R}$-nonexpansive $\iff$ $T$ is $\mathcal{R}^k$-nonexpansive.

(iii) Under universal relation

$$
\underbrace{\text{universal relation}}_{\text{the relation-theoretic analogue of the Krasnoselskii sequence}}
$$

Thus, our result is true for

$$
\underbrace{\text{following}}_{\text{constructed sequence, order intervals}}
$$

Similarly, the preimage of a $\in \mathcal{K}$ or $\mathcal{R}$-interval with end point $a \in \mathcal{K}$ is a subset of $\mathcal{K}$ defined by the following:

$$
\text{PreIm}(a, \mathcal{R}) = \{ x \in \mathcal{K} : (x, a) \in \mathcal{R} \text{ or } x = a \}.
$$

Remark 4 ([47]). The following conclusions are immediate:

$$
\begin{align*}
\text{Im}(a, \mathcal{R}) &= \text{PreIm}(a, \mathcal{R}^{-1}), \\
\text{PreIm}(a, \mathcal{R}) &= \text{Im}(a, \mathcal{R}^{-1}), \\
\text{Im}(a, \mathcal{R}^k) &= \text{PreIm}(a, \mathcal{R}^k).
\end{align*}
$$

Remark 5 ([47]). Under $\mathcal{R} := \preceq$, partial ordering, $\text{Im}(a, \mathcal{R})$ and $\text{PreIm}(a, \mathcal{R})$ coincide with order intervals $[a, \to)$ and $(\leftarrow, a]$ respectively.

Notice that the $\mathcal{R}$-intervals in a Banach lattice are both closed as well as convex under $\mathcal{R} := \preceq$ (cf. [50]).

3. Main Results

Before proving the main results, we present some auxiliary results, which will be needed in our main results. Firstly, we present the way of construction of a sequence, which is the relation-theoretic analogue of the Krasnoselskii sequence [51].

Lemma 1. Let $X$ be a real or complex vector space and $\mathcal{K} \subseteq \mathcal{X}$. Suppose that $T$ is a self-mapping on $\mathcal{K}$ and $\mathcal{R}$ is a $T$-closed transitive binary relation on $\mathcal{K}$. If $x_0 \in \mathcal{K}$ is an element such that $(x_0, Tx_0) \in \mathcal{R}$, then there exists an $\mathcal{R}$-preserving Krasnoselskii sequence $\{x_n\}$ with initial point $x_0$ given by the following:

$$
x_{n+1} = (1 - \lambda)x_n + \lambda T(x_n), \quad 0 \leq \lambda \leq 1
$$

provided $\text{Im}(x_0, \mathcal{R})$ and $\text{PreIm}(Tx_0, \mathcal{R})$ are convex.

Proof. Fix $\lambda \in (0, 1)$. Given that $(x_0, Tx_0) \in \mathcal{R}$. If $x_0 = T(x_0)$, then $(x_0, x_0) \in \mathcal{R}$. In this case, we have $x_n = x_0$ for all $n \geq 1$. Thus, the conclusion is immediate. Now, we have to prove our result in the case that $x_0 \neq T(x_0)$. Due to the availability of $(x_0, Tx_0) \in \mathcal{R}$, we have $x_0, Tx_0 \in \text{Im}(x_0, \mathcal{R}) \cap \text{PreIm}(Tx_0, \mathcal{R})$. Using the convexity of $\text{Im}(x_0, \mathcal{R}) \cap \text{PreIm}(Tx_0, \mathcal{R})$, we obtain the following:

$$
x_1 = (1 - \lambda)x_0 + \lambda Tx_0 \in \text{Im}(x_0, \mathcal{R}) \cap \text{PreIm}(Tx_0, \mathcal{R}).
$$

Thus, our result is true for $n = 0$.

Since $\mathcal{R}$ is $T$-closed, we obtain $(Tx_0, T x_1) \in \mathcal{R}$; hence, by the transitivity of $\mathcal{R}$, we obtain $(x_1, Tx_1) \in \mathcal{R}$. If $x_1 = T(x_1)$, then $(x_1, x_1) \in \mathcal{R}$ and $x_2 = x_1$, thereby yielding $x_n = x_1$ for all $n \geq 2$, hence, in this case, the conclusion is again immediate. Otherwise in
case of \( x_1 \neq T(x_1) \), due to \((x_1, Tx_1) \in \mathcal{R} \), we have \( x_1, T(x_1) \in \text{Im}(x_1, \mathcal{R}) \cap \text{PreIm}(Tx_1, \mathcal{R}) \).

Therefore, by using the convexity of \( \text{Im}(x_1, \mathcal{R}) \cap \text{PreIm}(Tx_1, \mathcal{R}) \), we obtain the following:

\[
x_2 = (1 - \lambda)x_1 + \lambda Tx_1 \in \text{Im}(x_1, \mathcal{R}) \cap \text{PreIm}(Tx_1, \mathcal{R}) \quad \text{(as } x_1 \neq T(x_1))
\]

\[
\implies (x_1, x_2) \in \mathcal{R} \text{ and } (x_2, Tx_1) \in \mathcal{R}.
\]

Thus, our result is true for \( n = 1 \).

Continuing the same procedure, we can construct a Krasnoselskii as well as \( \mathcal{R} \)-preserving sequence \( \{x_n\} \).

A sequence \( \{x_n\} \) constructed as above is called a \( \mathcal{R} \)-Krasnoselskii sequence with initial point \( x_0 \).

**Remark 6.** If we take \((Tx_0, x_0) \in \mathcal{R} \) instead of \((x_0, Tx_0) \in \mathcal{R} \), then the Krasnoselskii sequence \( \{x_n\} \), constructed similarly in Lemma 1, remains \( \mathcal{R} \)-reversing with initial point \( x_0 \). Such a sequence \( \{x_n\} \) is called an inverse \( \mathcal{R} \)-Krasnoselskii sequence with initial point \( x_0 \).

The following result is also a relation-theoretic version of a well-known result proved in [52,53].

**Proposition 3.** In addition to the hypothesis of Lemma 1, if \( T \) is an \( \mathcal{R} \)-nonexpansive mapping, then the \( \mathcal{R} \)-Krasnoselskii sequence \( \{x_n\} \) remains an approximating fixed point sequence of \( T \), i.e., the following:

\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0.
\]

**Proof.** By the induction argument on the index \( i \), we can easily shown the following:

\[
(1 + n\lambda)\|Tx_i - x_i\| \leq \|Tx_{i+n} - x_i\| + (1 - \lambda)^n\left(\|Tx_i - x_i\| - \|Tx_{i+n} - x_i\|\right) \quad \forall i, n \in \mathbb{N}.
\]

(1)

For each \( n \in \mathbb{N}_0 \), we have the following:

\[
x_{n+1} - x_n = \lambda(Tx_n - x_n)
\]

\[
\Rightarrow \|x_{n+1} - x_n\| = \lambda\|Tx_n - x_n\|
\]

which concludes that \( \{\|x_n - Tx_n\|\} \) is decreasing if and only if \( \{\|x_{n+1} - x_n\|\} \) is decreasing.

As \( T \) is an \( \mathcal{R} \)-nonexpansive mapping, for all \( n \in \mathbb{N}_0 \), we have the following:

\[
\|x_{n+2} - x_{n+1}\| = \|(1 - \lambda)x_{n+1} + \lambda Tx_{n+1} - (1 - \lambda)x_n - \lambda Tx_n\|
\]

\[
\leq (1 - \lambda)\|x_{n+1} - x_n\| + \lambda\|Tx_{n+1} - Tx_n\|
\]

\[
\leq \|x_{n+1} - x_n\|
\]

which implies that the sequence \( \{\|x_{n+1} - x_n\|\} \) is decreasing. Consequently, the sequence \( \{\|x_n - Tx_n\|\} \) (of nonnegative real numbers) remains bounded and monotonically decreasing (whereas 0 is its lower bound). Hence \( \lim_{n \to \infty} \|x_n - Tx_n\| = R \geq 0 \).

We claim that \( R = 0 \). To substantiate our claim, on the contrary, suppose that \( R > 0 \). Then, letting \( i \to \infty \) in the inequality (1), for all \( n \in \mathbb{N} \), we obtain the following:

\[
(1 + n\lambda)R \leq \delta(K)
\]
where $\delta(K)$ denotes the diameter of $K$. Hence, we have the following:

$$R \leq \frac{\delta(K)}{1 + n\lambda} \quad \forall n \in \mathbb{N}$$

which contradicts the fact that $\lambda$ and $R$ both are not equal to 0. Therefore, we have the following:

$$R = 0.$$

This completes the proof. \(\Box\)

Now, we are equipped to prove the main result.

**Theorem 3.** Let $(X, \| \cdot \|)$ be a Banach space satisfying the $\mathfrak{J}$-Opial condition for a given Hausdorff topology $\mathfrak{J}$ on $X$. Suppose that $K$ is a nonempty $\mathfrak{J}$-compact subset of $X$, not reducible to a single point; $\mathcal{R}$ remains a transitive binary relation on $K$, and $T : K \to K$ is an $\mathcal{R}$-nonexpansive mapping. Additionally, assume that there exists $x_0 \in K$ such that the following holds:

(i) $(x_0, Tx_0) \in \mathcal{R}$;

(ii) $\{x_n\}$ is bounded;

(iii) $\text{Im}(x_n, \mathcal{R})$ and $\text{PreIm}(Tx_n, \mathcal{R})$ are convex, for each $n \in \mathbb{N}_0$;

(iv) $\text{Im}(x_n, \mathcal{R})$ is $\mathfrak{J}$-closed, for each $n \in \mathbb{N}_0$.

wherein $\{x_n\}$ is the $\mathcal{R}$-Krasnoselskii sequence based on initial point $x_0$. Then, $T$ admits a fixed point.

**Proof.** In view of Lemma 1, assumptions (i) and (iii) guarantee the existence of the $\mathcal{R}$-Krasnoselskii sequence $\{x_n\}$ based on $x_0$ as an initial point. As $K$ is $\mathfrak{J}$-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is $\mathfrak{J}$-convergent to a point $z \in K$. Suppose that $\tau : K \to [0, \infty)$ is the type function generated by $\{x_{n_k}\}$, i.e., the following:

$$\tau(x) = \limsup_{n \to \infty} \|x_{n_k} - x\|, \quad x \in K.$$  

Using Lemma 3, we obtain the following:

$$\limsup_{n \to \infty} \left(\|x_{n_k} - x\| - \|Tx_{n_k} - x\|\right) \leq \limsup_{n \to \infty} \|x_{n_k} - Tx_{n_k}\| = 0$$

which yields the following:

$$\limsup_{n \to \infty} \|Tx_{n_k} - x\| = \limsup_{n \to \infty} \|x_{n_k} - x\| = \tau(x). \quad (2)$$

Now, we use assumption (iv). Denote $M_n := \text{Im}(x_n, \mathcal{R})$. Take $u \in M_{n+1}$, then $(x_{n+1}, u) \in \mathcal{R}$. Further, as $\{x_n\}$ is $\mathcal{R}$-preserving, we have $(x_n, x_{n+1}) \in \mathcal{R}$. Hence, by the transitivity of $\mathcal{R}$, we obtain $(x_n, u) \in \mathcal{R}$, thereby yielding $M_{n+1} \subseteq M_n$. It follows that $\{M_n\}$ is a decreasing family of sets under the inclusion “$\subseteq$”. Consequently, if $\mathcal{U}_z \in \mathfrak{J}$ is a neighborhood of $z$, the $\mathfrak{J}$-limit of subsequence $\{x_{n_k}\}$; then, $\mathcal{U}_z \cap M_n \neq \emptyset$, for all $n \in \mathbb{N}_0$. As $M_n$ is $\mathfrak{J}$-closed, we have $z \in M_n$, i.e., $(x_n, z) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$. In particular, we have $(x_{n_k}, z) \in \mathcal{R}$, for all $k \in \mathbb{N}$. Now, as $T$ is $\mathcal{R}$-nonexpansive mapping, we obtain the following:

$$\|Tx_{n_k} - Tz\| \leq \|x_{n_k} - z\|.$$  

Taking limit superior on both sides and using (2) in LHS, we obtain the following:

$$\tau(Tz) \leq \tau(z). \quad (3)$$
Since $X$ satisfies the $\mathfrak{J}$-Opial condition, we have the following:
\[
\limsup_{k \to \infty} \|x_{n_k} - z\| < \limsup_{k \to \infty} \|x_{n_k} - x\|, \quad z \neq x \in K
\]
\[
\implies \tau(z) < \tau(x).
\]

It follows that $z$ is the minimum point of $\tau$. Finally, using (3) and the uniqueness of the minimum point of $\tau$, we have $T(z) = z$, i.e., $z$ is a fixed point of $T$. \qed

If we consider $\mathfrak{J}$ to be the norm topology on the Banach space $(X, \| \cdot \|)$, then the following consequence of Theorem 3 is immediate.

**Corollary 1.** Let $(X, \| \cdot \|)$ be a Banach space, satisfying the strongly Opial condition, and $K$ a nonempty compact subset of $X$ not reducible to a single point. Suppose that $\mathcal{R}$ is a transitive binary relation on $K$, and $T : K \to K$ is an $\mathcal{R}$-nonexpansive mapping. Additionally, assume that there exists $x_0 \in K$ such that the following holds:

(i) $(x_0, Tx_0) \in \mathcal{R}$;

(ii) $\{x_n\}$ is bounded;

(iii) $\text{Im}(x_n, \mathcal{R})$ and $\text{PreIm}(Tx_n, \mathcal{R})$ are convex, for each $n \in \mathbb{N}_0$;

(iv) $\text{Im}(x_n, \mathcal{R})$ is closed, for each $n \in \mathbb{N}_0$.

where $\{x_n\}$ is the $\mathcal{R}$-Krasnoselskii sequence based on initial point $x_0$. Then, $T$ admits a fixed point.

In addition, if we take $\mathfrak{J}$ (the weak topology on the Banach space $(X, \| \cdot \|)$) in Theorem 3, then the following consequence is immediate.

**Corollary 2.** Let $(X, \| \cdot \|)$ be a Banach space, satisfying the Opial condition, and $K$ a nonempty weakly compact subset of $X$ not reducible to a single point. Suppose that $\mathcal{R}$ is a transitive binary relation on $K$, and $T : K \to K$ is an $\mathcal{R}$-nonexpansive mapping. Additionally, assume that there exists $x_0 \in K$ such that the following holds:

(i) $(x_0, Tx_0) \in \mathcal{R}$;

(ii) $\{x_n\}$ is bounded;

(iii) $\text{Im}(x_n, \mathcal{R})$ and $\text{PreIm}(Tx_n, \mathcal{R})$ are convex, for each $n \in \mathbb{N}_0$;

(iv) $\text{Im}(x_n, \mathcal{R})$ is weakly closed, for each $n \in \mathbb{N}_0$.

where $\{x_n\}$ is the $\mathcal{R}$-Krasnoselskii sequence based on initial point $x_0$. Then, $T$ admits a fixed point.

Taking $\mathcal{R}^{-1}$ instead of $\mathcal{R}$ in Theorem 3 and using Remarks 1, 3 and 4, we obtain the following dual result.

**Theorem 4.** Let $(X, \| \cdot \|)$ be a Banach space satisfying the $\mathfrak{J}$-Opial condition for a given Hausdorff topology $\mathfrak{J}$ on $X$. Suppose that $K$ is a nonempty $\mathfrak{J}$-compact subset of $X$ not reducible to a single point, $\mathcal{R}$ remains a transitive binary relation on $K$, and $T : K \to K$ is an $\mathcal{R}$-nonexpansive mapping. Additionally, assume that there exists $x_0 \in K$ such that the following holds:

(i) $(Tx_0, x_0) \in \mathcal{R}$;

(ii) $\{x_n\}$ is bounded;

(iii) $\text{Im}(x_n, \mathcal{R})$ and $\text{PreIm}(Tx_n, \mathcal{R})$ are convex, for each $n \in \mathbb{N}_0$;

(iv) $\text{PreIm}(x_n, \mathcal{R})$ is $\mathfrak{J}$-closed, for each $n \in \mathbb{N}_0$.

where $\{x_n\}$ is the inverse $\mathcal{R}$-Krasnoselskii sequence based on initial point $x_0$. Then, $T$ admits a fixed point.

Similar to Corollaries 1 and 2 of Theorem 3, we can deduce the corollaries corresponding to Theorem 4 by taking $\mathfrak{J}$ as the norm and weak topology, respectively.

Now, we define a notion, which is an analogue of “Property (P)”, utilized in Theorem 1.
Definition 9. Given a Banach space \((X, \|\cdot\|)\) and a Hausdorff topology \(\mathcal{J}\) on \(X\), we say that a binary relation \(\mathcal{R}\) on \(X\) is \(\mathcal{J}\)-self-closed if whenever \(\{x_n\} \subset X\) is an \(\mathcal{R}\)-preserving sequence and \(\{x_{n_k}\}\) is a subsequence of \(\{x_n\}\) such that \(x_{n_k} \xrightarrow{\mathcal{J}} x\), then \([x_{n_k}, x] \in \mathcal{R}\), for all \(K \in \mathbb{N}_0\).

It can be pointed out that in the hypotheses of Theorem 2, it is assumed that “\(\mathcal{R}\)-intervals are \(\mathcal{J}\)-closed” under the restriction \(\mathcal{R} := \leq\), a partial order. Assumption (iv) utilized in Theorem 3 as well as “\(\mathcal{J}\)-self closedness of \(\mathcal{R}\)” both are weaker as compared to the condition that “\(\mathcal{R}\)-intervals are \(\mathcal{J}\)-closed” but independent to each other. Henceforth, we present yet another fixed point theorem.

Theorem 5. Theorem 3 is true if assumption (iv) is replaced by the assumption that “\(\mathcal{R}\) is \(\mathcal{J}\)-self closed”.

Proof. Similar to Theorem 3, we obtain Equation (2). Since \(\{x_n\}\) is \(\mathcal{R}\)-preserving, the \(\mathcal{J}\)-self closed property of \(\mathcal{R}\) provides \([x_{n_k}, z] \in \mathcal{R}\), for all \(k \in \mathbb{N}\). Hence, using \(\mathcal{R}\)-nonexpansive property of \(T\), we obtain the following:

\[
\|Tx_{n_k} - Tz\| \leq \|x_{n_k} - z\|
\]

from which we can obtain the relation (3). The remaining part of the proof can be completed on the lines of the proof of Theorem 3. \(\Box\)

Remark 7. Under the universal relation \(\mathcal{R} = X^2\), Theorem 3 (similarly, Theorems 4 and 5) reduce to a well-known fixed point theorem, which states that “Every nonexpansive mapping on a nonempty \(\mathcal{J}\)-compact closed and bounded subset of a Banach space \((X, \|\cdot\|)\) satisfying the \(\mathcal{J}\)-Opial condition admits a fixed point”. In this case, since the whole set \(K\) is convex and bounded, any arbitrary \(x_0 \in K\) satisfies the assumptions (i), (ii), (iii) and (iv).

4. Conclusions

In the short course of the last five years, the idea of monotone nonexpansive mappings was considered and utilized to prove several core recent results existing in the literature. While doing so, the involved researchers adopted two structures: transitive directed graph and partial ordering. From a mathematical point of view, the set of all edges of a graph on a set remains a reflexive binary relation. Consequently, a transitive graph turns out to be a quasi-order. In this work, we have concluded that all such existing results can be extended up to a transitive binary relation instead of a graph and partial ordering.

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