Periodic solutions with long period for the Mackey–Glass equation

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

Tibor Krisztin

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Szeged, H–6720, Hungary

Received 8 November 2020, appeared 21 December 2020
Communicated by Gennaro Infante

Abstract. The limiting version of the Mackey–Glass delay differential equation $x' (t) = -ax(t) + b f (x(t - 1))$ is considered where $a, b$ are positive reals, and $f (\xi) = \xi$ for $\xi \in [0, 1)$, $f (1) = 1/2$, and $f (\xi) = 0$ for $\xi > 1$. For every $a > 0$ we prove the existence of an $\epsilon_0 = \epsilon_0 (a) > 0$ so that for all $b \in (a, a + \epsilon_0)$ there exists a periodic solution $p = p (a, b) : \mathbb{R} \to (0, \infty)$ with minimal period $\omega (a, b)$ such that $\omega (a, b) \to \infty$ as $b \to a +$. A consequence is that, for each $a > 0$, $b \in (a, a + \epsilon_0 (a))$ and sufficiently large $n$, the classical Mackey–Glass equation $y' (t) = -ay(t) + b y(t - 1) / [1 + y^n (t - 1)]$ has an orbitally asymptotically stable periodic orbit, as well, close to the periodic orbit of the limiting equation.

Keywords: Mackey–Glass equation, periodic solution, limiting nonlinearity, discontinuous right-hand side, long period.

2020 Mathematics Subject Classification: 34K13, 34K39, 34K06.

1 Introduction

The Mackey–Glass equation

$$y' (t) = -ay(t) + b \frac{y(t - \tau)}{1 + y^n (t - \tau)}$$

with positive parameters $a, b, \tau, n$ was proposed to model blood production and destruction in the study of oscillation and chaos in physiological control systems by Mackey and Glass [13]. This simple-looking differential equation with a single delay attracted the attention of many mathematicians since its hump-shaped nonlinearity causes entirely different dynamics compared to the case where the nonlinearity is monotone. See [16] for a similar equation. There exist several rigorous mathematical results, numerical and experimental studies on the Mackey–Glass equation showing convergence of the solutions, oscillations with different
characteristics, and the complexity of the dynamics, see e.g. [1,3,6,7,9,15,17–19,22,23]. Despite the intense research, the dynamics is not fully understood yet.

The recent paper [2] studies the classical Mackey–Glass delay differential equation

$$y'(t) = -ay(t) + bf_n(y(t-1))$$  \hspace{1cm} (E_n)

where $a, b, n$ are positive reals, $f_n(\xi) = \xi / [1 + \xi^n]$ for $\xi \geq 0, \tau = 1$ can be assumed by rescaling the time. [2] constructs stable periodic solutions of $(E_n)$ for some $b > a > 0$ and large $n$. The periodic solutions can have complicated shapes, see [2]. A limiting version of $(E_n)$ plays a key role in the construction. The function $f(\xi) = \lim_{n \to \infty} f_n(\xi)$ is given by $f(\xi) = \xi$ for $\xi \in [0,1)$, $f(1) = 1/2$, and $f(\xi) = 0$ for $\xi > 1$. The equation

$$x'(t) = -ax(t) + bf(x(t-1))$$  \hspace{1cm} (E_\infty)

is called the limiting Mackey–Glass equation.

Let $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote the set of real numbers, complex numbers and positive integers, respectively. Let $\mathbb{C}$ be the Banach space $\mathbb{C}([-1,0], \mathbb{R})$ equipped with the norm $\|\varphi\| = \max_{s \in [-1,0]}|\varphi(s)|$. For a continuous function $u : I \to \mathbb{R}$ defined on an interval $I$, and for $t, t-1 \in I$, $u_t \in \mathbb{C}$ is given by $u_t(s) = u(t+s), s \in [-1,0]$. Introduce the subsets

$$C^+ = \{\varphi \in C : \varphi(s) > 0 \text{ for all } s \in [-1,0]\},$$

$$C^+_t = \{\varphi \in C^+ : \varphi^{-1}(c) \text{ is finite for all } c \in (0,1]\}$$

of $\mathbb{C}$ where $\varphi^{-1}(c) = \{s \in [-1,0] : \varphi(s) = c\}$. $C^+$ and $C^+_t$ are metric spaces with the metric $d(\varphi, \psi) = \|\varphi - \psi\|$.

A solution of equation $(E_n)$ on $[-1, \infty)$ with initial function $\varphi \in C^+$ is a continuous function $y : [-1, \infty) \to \mathbb{R}$ so that $y_0 = \varphi$, the restriction $y|_{(0,\infty)}$ is differentiable, and equation $(E_n)$ holds for all $t > 0$. The solutions are easily obtained from the variation-of-constants formula for ordinary differential equations on successive intervals of length one,

$$y(t) = e^{-a(t-k)}y(k) + b \int_k^t e^{-a(t-s)}f_n(y(s-1)) \, ds$$  \hspace{1cm} (1.1)

where $k \in \mathbb{N} \cup \{0\}, k \leq t \leq k + 1$. Hence it is well known that each $\varphi \in C^+$ uniquely determines a solution $y = y^{\varphi} : [-1, \infty) \to \mathbb{R}$ with $y_0^{\varphi} = \varphi$, and $y^{\varphi}(t) > 0$ for all $t \geq 0$.

For equation $(E_\infty)$ with the discontinuous $f$, we use formula (1.1) with $f$ instead of $f_n$ to define solutions. A solution of equation $(E_\infty)$ with initial function $\varphi \in C^+$ is a continuous function $x = x^\varphi : [-1, t_\varphi) \to \mathbb{R}$ with some $0 < t_\varphi \leq \infty$ such that $x_0 = \varphi$, the map $[0, t_\varphi) \ni s \mapsto f(x(s-1)) \in \mathbb{R}$ is locally integrable, and

$$x(t) = e^{-a(t-k)}x(k) + b \int_k^t e^{-a(t-s)}f(x(s-1)) \, ds$$  \hspace{1cm} (1.2)

holds for all $k \in \mathbb{N} \cup \{0\}$ and $t \in [0, t_\varphi)$ with $k \leq t \leq k + 1$.

It is easy to show that, for any $\varphi \in C^+$, there is a unique solution $x^\varphi$ of equation $(E_\infty)$ on $[-1, \infty)$. However, comparing solutions with initial functions $\varphi > 1, \varphi \equiv 1$, one sees that there is no continuous dependence on initial data in $C^+$. Therefore we restrict our attention to the subset $C^+_t$ of $C^+$. The choice of $C^+_t$ as a phase space guarantees not only continuous dependence on initial data, but also allows to compare certain solutions of equations $(E_\infty)$ and $(E_n)$ for large $n$. This is not used here, but it is important in [2]. [2] proves that for each $\varphi \in C^+_t$
there is a unique maximal solution \( x^\omega : [-1, \infty) \rightarrow \mathbb{R} \) of equation \((E_\infty)\). The maximal solution \( x^\omega \) satisfies \( x^\omega_t \in C^+ \) for all \( t \geq 0 \); and if \( t > 0 \) and \( x^\omega(t - 1) \neq 1 \), then \( x^\omega \) is differentiable at \( t \), and equation \((E_\infty)\) holds at \( t \).

One of the main results of [2] is as follows.

**Theorem 1.1.** If the parameters \( b > a > 0 \) are given so that

\( \text{(H) equation } (E_\infty) \text{ has an } \omega \text{-periodic solution } p : \mathbb{R} \rightarrow \mathbb{R} \text{ with the following properties:} \)

\begin{enumerate}
\item \( p(0) = 1, \ p(t) > 1 \text{ for all } t \in [-1,0), \)
\item \( (p(t), p(t-1)) \neq (1,a/b) \text{ for all } t \in [0,\omega] \)
\end{enumerate}

holds then there exists an \( n^* \geq 4 \) such that, for all \( n \geq n^* \), equation \((E_n)\) has a periodic solution \( p^n : \mathbb{R} \rightarrow \mathbb{R} \) with period \( \omega^n > 0 \) so that the periodic orbits

\[ O^n = \{ p^n_t : t \in [0, \omega^n] \} \]

are hyperbolic, orbitally stable, exponentially attractive with asymptotic phase, moreover, \( \omega^n \rightarrow \omega, \) \( \text{dist} \{ O^n, O \} \rightarrow 0 \text{ as } n \rightarrow \infty, \) where \( O = \{ p_t : t \in [0,\omega] \}. \)

[2] shows that in case \( b \) is large comparing to \( a \), namely \( b > \max\{ae^a, e^a - e^{-a}\} \), then \( \text{(H)} \) is satisfied. In addition, by using a rigorous computer-assisted technique, [2] gives parameter values \( a, b \) such that \( \text{(H)} \) is valid, and the obtained stable periodic orbits for the Mackey–Glass equation may have complicated structures.

[2] remarks that \( \text{(H)} \) holds if \( b > a > 0 \) and \( b \) is sufficiently close to \( a \), and refers to this work for the proof. The aim of this paper is to prove this fact, namely the following result.

**Theorem 1.2.** For every \( a > 0 \) there exists an \( \epsilon_0 = \epsilon_0(a) > 0 \) such that for the parameters \( a,b \) with \( b \in (a,a + \epsilon_0) \) condition \( \text{(H)} \) holds.

In particular, for the periodic solution \( p = p(a,b) \) of equation \((E_\infty)\) the minimal period \( \omega = \omega(a,b) \) satisfies \( \omega > 5 \), and there exists a \( \sigma = \sigma(a,b) \in (4,\omega - 1) \) so that

\[ 0 < p(t) < 1 \text{ for all } t \in (0,\sigma); \ p(t) > 1 \text{ for all } t \in (\sigma, \omega). \]

Moreover, if \( a > 0 \) is fixed and \( (b_k)_{k=1}^\infty \) is a sequence in \( (a,a + \epsilon_0(a)) \), \( \lim_{k \rightarrow \infty} b_k = a \) then \( \sigma(a, b_k) \rightarrow \infty, \omega(a, b_k) \rightarrow \infty \) as \( k \rightarrow \infty \).

Theorems 1.1 and 1.2 immediately imply the following result for equation \((E_n)\).

**Theorem 1.3.** For each \( a > 0 \) there exists an \( \epsilon_0 = \epsilon_0(a) > 0 \) such that for every \( b \in (a,a + \epsilon_0) \) there exists an \( n^* = n^*(a,b) \geq 4 \) so that, for all \( n \geq n^* \), equation \((E_n)\) has a periodic solution \( p^n : \mathbb{R} \rightarrow \mathbb{R} \) with minimal period \( \omega^n(a,b) \) so that the periodic orbits

\[ O^n = \{ p^n_t : t \in [0, \omega^n] \} \]

are hyperbolic, orbitally stable, exponentially attractive with asymptotic phase. Moreover, if \( (b_k)_{k=1}^\infty \) is a sequence in \( (a,a + \epsilon_0(a)) \) with \( \lim_{k \rightarrow \infty} b_k = a, n_k > n^*(a, b_k) \) then \( \omega^n(a, b_k) \rightarrow \infty \) as \( k \rightarrow \infty \).

Note that the papers [8] by Karakostas et al. and [5] by Gopalsamy et al. give conditions for the global attractiveness of the unique positive equilibrium of \((E_n)\) for \( b > a > 0 \), and \( n \) is below a certain constant given in terms of \( a,b \). Theorem 1.3 requires \( n \) to be large.

Section 2 contains the proof of Theorem 1.2. The proof requires the study of a special solution of a linear autonomous delay differential equation. If \( \varphi \in C^+_0 \) is any function such
that \( \varphi(s) > 1 \) for \( s \in [-1,0) \) and \( \varphi(0) = 1 \) then the unique solution \( x = x^0 \) of equation \( (E_\infty) \) satisfies \( x(t) = e^{-at} \) for \( t \in [0,1] \). In order to find a periodic solution of \( (E_\infty) \) as stated in Theorem 1.2 we consider the linear autonomous equation

\[
u'(t) = -au(t) + bu(t - 1)
\]

for \( t > 1 \) with \( u(t) = e^{-at}, t \in [0,1] \). If we find a \( T > 0 \) such that \( u(t) < 1 \) for \( t \in (0,T) \), \( u(T) = 1 \), \( u(t) > 1 \) for \( t \in (T,T+1) \), then it is straightforward to see that \( x(t) = u(t) \) for all \( t \in [0,T+1] \). Then, equation \( (E_\infty) \) gives \( x'(t) = -ax(t) \) for all \( t > T + 1 \) as long as \( x(t-1) > 1 \). Hence there exists an \( \omega > T + 1 \) with \( x(\omega) = 1 \) and \( x(t) > 1 \) for all \( t \in (T,\omega) \). By the fact \( f(\xi) = 0 \) for \( \xi > 1 \), the solution \( x \) does not change on \( [0,\infty) \) if \( \varphi \) is replaced by \( x_w \), and consequently \( x(t) = x(t + \omega) \) follows for all \( t \geq -1 \). Therefore the proof of Theorem 1.2 is reduced to the existence of a \( T > 0 \) with \( u(t) < 1 \) for \( t \in (0,T) \), \( u(T) = 1 \), \( u(t) > 1 \) for \( t \in (T,T+1) \). Property (H)(ii) is guaranteed by \( u'(T) > 0 \).

We remark that the use of a limiting equation in order to study nonlinear delay differential equations when the nonlinearity is close to its limiting function is not new. We refer to the papers [10–12,21,24–26] where the limiting step function reduces the search of periodic solutions to a finite dimensional problem. The limiting Mackey–Glass nonlinearity \( f \) is not a step function. The introduction of the limiting Mackey–Glass equation does not reduce the search for periodic solutions to a finite dimensional problem, nevertheless it can simplify it. The paper [14] considered the limiting Mackey–Glass nonlinearity to construct periodic solutions for an equation different from \( (E_n) \). The result of [14] is analogous to the case when \( b \) is large comparing to \( a \), mentioned above for the Mackey–Glass equation.

### 2 The proof of Theorem 1.2

The proof is divided into eight steps. The desired periodic solution of equation \( (E_\infty) \) will be an \( \omega \)-periodic extension of a function \( w : [0,\omega] \to \mathbb{R} \). We construct \( w \) in the remaining part of this section.

**Step 1.** Let \( a > 0 \) be fixed, and consider the characteristic function

\[
h : \mathbb{C} \times \mathbb{R} \ni (z,\varepsilon) \mapsto z + a - (a + \varepsilon)e^{-z} \in \mathbb{C}
\]

of the linear delay differential equation \( \nu'(t) = -av(t) + (a + \varepsilon)v(t - 1) \). By \( h(0,0) = 0 \), \( D_1h(0,0) = 1 + a \), and \( D_2h(0,0) = -1 \), the Implicit Function Theorem can be applied to get that there are \( \varepsilon_1 \in (0,\min\{a,1/4\}) \), \( r_1 \in (0,1) \) and a \( C^1 \)-smooth map \( \lambda_0 : (-\varepsilon_1,\varepsilon_1) \to \mathbb{C} \) such that \( \lambda_0(0) = 0 \), \( h(\lambda_0(\varepsilon),\varepsilon) = 0 \), and \( (\lambda_0(\varepsilon),\varepsilon) \) is the unique solution of \( h(z,\varepsilon) = 0 \) in the set \( \{z \in \mathbb{C} : |z| < r_1 \} \times (-\varepsilon_1,\varepsilon_1) \). Since \( a \) and \( \varepsilon \) are real in the equation \( h(z,\varepsilon) = 0 \), \( (z,\varepsilon) \) is a solution together with \( (z,\varepsilon) \). Then, by uniqueness, it follows that \( \lambda_0(\varepsilon) \in \mathbb{R}, \varepsilon \in (-\varepsilon_1,\varepsilon_1) \).

Chapter XI of [4] applies to get that the zeros of the characteristic function \( h(z,\varepsilon) \) for \( \varepsilon \in (-\varepsilon_1,\varepsilon_1) \) are \( \lambda_0(\varepsilon) \in \mathbb{R} \) and a sequence of pairs \( (\lambda_j(\varepsilon),\overline{\lambda_j(\varepsilon)}) \) with

\[
\lambda_0(\varepsilon) > \text{Re}\lambda_1(\varepsilon) > \text{Re}\lambda_2(\varepsilon) > \cdots > \text{Re}\lambda_j(\varepsilon) \to -\infty \quad \text{as} \quad j \to \infty
\]

and

\[
\text{Im}\lambda_j \in ((2j - 1)\pi, 2j\pi) \quad (j \in \mathbb{N}).
\]

If \( \varepsilon = 0 \) then \( \lambda_0(0) = 0 \), and consequently \( \text{Re}\lambda_1(0) < 0 \). Fix \( c \in (0,a) \) so that

\[
\text{Re}\lambda_1(0) < -2c.
\]
Notice that the choice of \( c \) depends only on \( a \).

Differentiating the equation \( h(\lambda_0(\varepsilon), \varepsilon) = 0 \) with respect to \( \varepsilon \) we obtain \( \lambda_0'(0) = 1/(1 + a) \), and thus
\[
\lambda_0(\varepsilon) = \frac{\varepsilon}{1 + a} + \eta(\varepsilon)
\]
with a function \( \eta : (-\varepsilon_1, \varepsilon_1) \to \mathbb{R} \) satisfying \( \lim_{\varepsilon \to 0} \eta(\varepsilon)/\varepsilon = 0 \). Applying the above representation for \( \lambda_0(\varepsilon) \), we assume (in addition to the above properties of \( \varepsilon_1 \)) that \( \varepsilon_1 \) is so small that
\[
\lambda_0(\varepsilon) < \frac{2\varepsilon}{1 + 2a} \quad \text{for all } \varepsilon \in (0, \varepsilon_1),
\]
where the equality \( 2\varepsilon/(1 + 2a) = \varepsilon/(1 + a) + \varepsilon/[(1 + a)(1 + 2a)] \) shows that this is possible.

By Rouché’s theorem [20] there exists an \( \varepsilon_2 \in (0, \varepsilon_1) \) such that
\[
\Re \lambda_1(\varepsilon) < -2c \quad \text{for all } \varepsilon \in [0, \varepsilon_2].
\]

In particular, \( h(\varepsilon, \varepsilon) \neq 0 \) on the line \( \{ -c + is : s \in \mathbb{R} \} \) for all \( \varepsilon \in [0, \varepsilon_2] \).

**Step 2.** For \( \varepsilon \in (0, \varepsilon_2) \) consider the unique solution \( v : [-1, \infty) \to \mathbb{R} \) of the linear equation
\[
v'(t) = -av(t) + (a + \varepsilon)v(t - 1) \quad (t > 0)
\]
with initial function \( v_0(s) = e^{-a(s+1)}, \quad -1 \leq s \leq 0 \). Remark that \( v \) and \( \lambda_0 \) depend on \( \varepsilon \) as well. Taking the Laplace transform of both sides of (2.2) and expressing the Laplace transform \( \mathcal{L}(v)(z) \) of \( v \),
\[
\mathcal{L}(v)(z) = \frac{1}{h(z, \varepsilon)} \left[ e^{-a} + (a + \varepsilon) \frac{1 - e^{-(z+a)}}{z + a} \right]
\]
is obtained where the right hand side can be written as \( F(z, \varepsilon) = F_1(z) + F_2(z, \varepsilon) \) with
\[
F_1(z) = \frac{e^{-a}}{z + a}, \quad F_2(z, \varepsilon) = \frac{a + \varepsilon}{(z + a)h(z, \varepsilon)}.
\]

According to Chapter I of [4], by taking the inverse Laplace transform, function \( v \) can be written as
\[
v(t) = e^{\lambda_0 t} \text{Res}_{\lambda_0} F(z, \varepsilon) + \frac{1}{2\pi i} e^{-ct} \lim_{T \to \infty} \int_{-T}^{T} e^{ist} \text{Res}_{\lambda_0} F(-c + is, \varepsilon) \, ds \quad (t > 0).
\]

As \( F_1(z) \) is holomorphic in a neighborhood of \( \lambda_0 \), one finds \( \text{Res}_{\lambda_0} F(z, \varepsilon) = \text{Res}_{\lambda_0} F_2(z, \varepsilon) \). By using that \( h(z, \varepsilon) \) has a simple zero at \( \lambda_0 \), and \( \lambda_0 + a = (a + \varepsilon)e^{-\lambda_0} \), we get
\[
\text{Res}_{\lambda_0} F(z, \varepsilon) = \frac{a + \varepsilon}{(\lambda_0 + a) D_t h(\lambda_0, \varepsilon)} = \frac{a + \varepsilon}{(\lambda_0 + a)(1 + (a + \varepsilon)e^{-\lambda_0})} = \frac{e^{\lambda_0}}{1 + a + \lambda_0}.
\]

For \( t \geq 1 \), integration by parts leads to
\[
\int_{-T}^{T} e^{ist} F_1(-c + is) \, ds = \left[ \frac{e^{ist} e^{-a}}{it (a - c + is)} \right]_{s=-T}^{s=T} - \int_{-T}^{T} \frac{ie^{-a}}{it (a - c + is)^2} \, ds.
\]
Thus
\[
\left| \lim_{T \to \infty} \int_{-T}^{T} e^{ist} F_1(-c + is) \, ds \right| \leq \int_{-\infty}^{\infty} \left| \frac{e^{ist}}{it (a - c + is)^2} \right| \, ds \leq K_1
\]
with
\[
K_1 = 2 \int_0^\infty \frac{e^{-a}}{(a-c)^2 + s^2} ds.
\]
Let \( s_0 = 2(a+1)e^c \). The continuous function \((s, \varepsilon) \mapsto h(-c + is, \varepsilon) \in \mathbb{C}\) is nonzero on the set \([-s_0, s_0] \times [0, \varepsilon_2] \). So there exists \( k > 0 \) such that \(|F_2(-c + is, \varepsilon)| \leq k\) on the compact set \([-s_0, s_0] \times [0, \varepsilon_2] \). If \( |s| \geq s_0, \varepsilon \in [0, \varepsilon_2] \) then, by the choice of \( s_0 \),
\[
|h(-c + is, \varepsilon)| \geq |a - c + is| - |(a + \varepsilon)e^{-is}| \geq [(a - c)^2 + s^2]^{1/2} - (a + 1)e^c \\
\geq \frac{1}{2} [(a - c)^2 + s^2]^{1/2}.
\]
Consequently
\[
\lim_{T \to \infty} \int_{-T}^T e^{ist} F_2(-c + is, \varepsilon) \, ds \leq \int_{-\infty}^\infty |F_2(-c + is, \varepsilon)| \, ds \\
\leq 2 \int_0^{s_0} k \, ds + 2 \int_{s_0}^\infty \frac{a + 1}{(1/2)[(a - c)^2 + s^2]} \, ds \\
= K_2
\]
with
\[
K_2 = 2ks_0 + 4 \int_{s_0}^\infty \frac{(a + 1)}{(a-c)^2 + s^2} \, ds.
\]
Notice that both \( K_1 \) and \( K_2 \) are independent of \( \varepsilon \in (0, \varepsilon_2) \).

Summarizing the above estimations we obtain that
\[
v(t) = \frac{e^{\lambda_0(t+1)}}{1 + a + \lambda_0} + \hat{r}(t) \quad (t \geq 1)
\]
for some continuous function \( \hat{r} : [1, \infty) \to \mathbb{R} \) satisfying
\[
|\hat{r}(t)| \leq \hat{K}e^{-ct} \quad (t \geq 1)
\]
with \( \hat{K} = (K_1 + K_2)/(2\pi) \). Note that \( \hat{r} \) depends on \( \varepsilon \), however \( \hat{K} \) and \( c \) are independent of \( \varepsilon \).

**Step 3.** For \( \varepsilon \in (0, \varepsilon_2) \) define the function \( u : [0, \infty) \to \mathbb{R} \) by \( u(t) = v(t - 1), t \geq 0 \). Then \( u(t) = e^{-at} \) for \( t \in [0, 1] \), \( u \) is differentiable on \((1, \infty)\) and satisfies
\[
u'(t) = -au(t) + (a + \varepsilon)u(t - 1) \quad (t > 1).
\]
Moreover, defining \( r(t) = \hat{r}(t - 1) \) for \( t \geq 2, K = \hat{K}e^c \), \( u \) has the representation
\[
u(t) = \frac{e^{\lambda_0 t}}{1 + a + \lambda_0} + r(t) \quad (t \geq 2)
\]
with the continuous function \( r : [2, \infty) \to \mathbb{R} \) satisfying
\[
|r(t)| \leq Ke^{-ct} \quad (t \geq 2).
\]
From equation (2.3)
\[
u(t) = e^{-a(t-1)}u(1) + \int_1^t (a + \varepsilon)e^{-a(s-s)}e^{-a(s-1)} \, ds \\
= e^{-at} [1 + (a + \varepsilon)e^a(t - 1)] \quad (t \in [1, 2])
\]
and
\[ u'(t) = e^{-at} \left[ -a - a(a + \varepsilon)e^a(t - 1) + (a + \varepsilon)e^a \right] \quad (t \in (1, 2]). \]

Define
\[ t_0 = t_0(\varepsilon) = 1 + \frac{1}{a} - \frac{1}{(a + \varepsilon)e^a}. \]

Choose \( \varepsilon_3 \in (0, \varepsilon_2) \) so that
\[ \varepsilon_3 < \frac{a}{1 - a}(e^{-a} - 1 + a) \]
provided \( a \in (0, 1) \), and let \( \varepsilon_3 = \varepsilon_2 \) if \( a \geq 1 \).

Suppose \( \varepsilon \in (0, \varepsilon_3) \). Then \( t_0 = t_0(\varepsilon) \in (1, 2) \) is the unique zero of \( u' \) in \((1, 2)\), and it is easy to see that
\[
\max_{t \in [1, 2]} u(t) = u(t_0) = e^{-a t_0} \left[ 1 + (a + \varepsilon)e^a(t_0 - 1) \right] = \frac{a + \varepsilon}{a} \exp \left[ \frac{ae^{-a}}{a + \varepsilon} - 1 \right]. \tag{2.6}
\]

**Step 4.** In this step we show the following

CLAIM:

(i) For each \( k \in \mathbb{N} \)

\[
\max_{t \in [k+1, k+2]} u(t) \leq \left( 1 + \frac{\varepsilon}{a} \right) \max_{t \in [k, k+1]} u(t),
\]

and

(ii) for each \( N \in \mathbb{N} \)

\[
\max_{t \in [N+1, N+2]} u(t) \leq \left( 1 + \frac{\varepsilon}{a} \right)^N \max_{t \in [1, 2]} u(t).
\]

Let \( k \in \mathbb{N} \) be given. If \( \max_{t \in [k+1, k+2]} \leq \max_{t \in [k, k+1]} u(t) \) then the stated inequality obviously holds for \( k \). If \( \max_{t \in [k+1, k+2]} u(t) > \max_{t \in [k, k+1]} u(t) \), then there exists a \( t_1 \in (k + 1, k + 2] \) such that \( u'(t_1) \geq 0 \) and \( u(t_1) = \max_{t \in [k+1, k+2]} u(t) \). Equation (2.3) at \( t = t_1 \) and \( u'(t_1) \geq 0 \) imply the inequality \(-au(t_1) + (a + \varepsilon)u(t_1 - 1) \geq 0 \). Hence
\[
\max_{t \in [k+1, k+2]} u(t) = u(t_1) \leq \frac{a + \varepsilon}{a} u(t_1 - 1) \leq \left( 1 + \frac{\varepsilon}{a} \right) \max_{t \in [k, k+1]} u(t),
\]
that is, the stated inequality is satisfied. This proves (i).

A repeated application of (i) gives (ii):
\[
\max_{t \in [N+1, N+2]} u(t) \leq \left( 1 + \frac{\varepsilon}{a} \right) \max_{t \in [N, N+1]} u(t) \leq \left( 1 + \frac{\varepsilon}{a} \right)^2 \max_{t \in [N-1, N]} u(t) \leq \cdots \leq \left( 1 + \frac{\varepsilon}{a} \right)^N \max_{t \in [1, 2]} u(t).
\]

**Step 5.** Choose \( \xi_0 \in (\exp(e^{-a} - 1), 1) \). The function
\[
(0, \infty) \ni \varepsilon \mapsto \frac{a + \varepsilon}{a} \exp \left[ \frac{ae^{-a}}{a + \varepsilon} - 1 \right] \in \mathbb{R}
\]
strictly increases and its limit is \( \exp(e^{-a} - 1) \) as \( \varepsilon \to 0^+ \). Therefore there exists an \( \varepsilon_4 \in (0, \varepsilon_3) \) such that
\[
\frac{a + \varepsilon}{a} \exp \left[ \frac{ae^{-a}}{a + \varepsilon} - 1 \right] < \xi_0
\]
for all $\varepsilon \in (0, \varepsilon_4)$.

By the equality (2.6) in Step 3 and the choice of $\varepsilon_4$, for all $\varepsilon \in (0, \varepsilon_4)$, the inequality $\max_{t \in [1,2]} u(t) < \xi_0$ holds. Then by the CLAIM in Step 4

$$\max_{t \in [1,N+2]} u(t) < \left(1 + \frac{\varepsilon}{4}\right)^N \xi_0$$  \hspace{1cm} (2.7)

follows for all $N \in \mathbb{N}$.

For a given $N \in \mathbb{N}$, from (2.7) one gets

$$\max_{t \in [1,N+2]} u(t) < 1$$

provided $\varepsilon \in (0, \varepsilon_4)$ is so small that

$$\varepsilon < a \left(1/\xi_0\right)^{1/N} - 1.$$  \hspace{1cm} (2.8)

**Step 6.** Let $N \in \mathbb{N} \setminus \{1,2\}$ be given. We look for a condition on $\varepsilon \in (0, \varepsilon_4)$ to guarantee

$$u'(t) > 0 \quad \text{for all } t > N.$$  \hspace{1cm} (2.9)

Equation (2.3) gives that

$$au(t) < (a + \varepsilon)u(t-1) \quad \text{for all } t > N$$  \hspace{1cm} (2.10)

is sufficient to yield (2.9). By the representation (2.4) condition (2.10) is equivalent to

$$\frac{a}{1 + a + \lambda_0} e^{\lambda_0 t} \left[\left(1 + \frac{\varepsilon}{a}\right) e^{-\lambda_0} - 1\right] > ar(t) - (a + \varepsilon)r(t-1) \quad (t > N),$$

that is

$$\left(1 + \frac{\varepsilon}{a}\right) e^{-\lambda_0} - 1 > \frac{1 + a + \lambda_0}{a} e^{-\lambda_0 t} [ar(t) - (a + \varepsilon)r(t-1)] \quad (t > N).$$

From $\varepsilon < 1$, $0 < \lambda_0(\varepsilon) < 1$ and (2.5) one obtains

$$\frac{1 + a + \lambda_0}{a} e^{-\lambda_0 t} [ar(t) - (a + \varepsilon)r(t-1)] < \frac{(a + 2)(2a + 1)}{a} Ke^{-c(t-1)}$$

$$< \frac{(a + 2)(2a + 1)}{a} Ke^{-cN} \quad (t > N).$$

Recall that, by the choice of $\varepsilon_1$ in Step 1,

$$\lambda_0(\varepsilon) < \frac{2\varepsilon}{2a + 1}.$$  \hspace{1cm}

Hence

$$e^{-\lambda_0(\varepsilon)} > 1 - \lambda_0(\varepsilon) > 1 - \frac{2\varepsilon}{2a + 1}.$$  \hspace{1cm}

Thus, by using $\varepsilon_1 < 1/4$ as well,

$$\left(1 + \frac{\varepsilon}{a}\right) e^{-\lambda_0(\varepsilon)} - 1 > \left(1 + \frac{\varepsilon}{a}\right) \left(1 - \frac{2\varepsilon}{2a + 1}\right) - 1$$

$$= \frac{\varepsilon - 2\varepsilon^2}{a(2a + 1)} > \frac{\varepsilon}{2a(2a + 1)},$$
Consequently, (2.9) holds if, in addition to $\varepsilon \in (0, \varepsilon_4)$,
\[ \varepsilon > \xi_1 e^{-cN} \]  
(2.11)
with $\xi_1 = 2(a + 2)(2a + 1)^2 K e^\varepsilon$.

**Step 7.** In order to satisfy conditions (2.8) and (2.11) simultaneously consider $a \left[ (1/\xi_0)^{1/N} - 1 \right]$ and $\xi_1 e^{-cN}$. By L'Hospital's rule
\[ \lim_{N \to \infty} \frac{\xi_1 e^{-cN}}{a \left[ (1/\xi_0)^{1/(N+1)} - 1 \right]} = 0. \]
Therefore there exists an integer $N_0 > 2$ such that
\[ \frac{\xi_1 e^{-cN}}{a \left[ (1/\xi_0)^{1/(N+1)} - 1 \right]} < 1 \quad \text{for all integers } N \geq N_0. \]  
(2.12)
Define $\varepsilon_* \in (0, \varepsilon_4)$ so that
\[ \varepsilon_* < a \left[ (1/\xi_0)^{1/N_0} - 1 \right]. \]
Let $\varepsilon \in (0, \varepsilon_*)$ be fixed. By $\varepsilon < \varepsilon_*$ and $\lim_{N \to \infty} a \left[ (1/\xi_0)^{1/N} - 1 \right] = 0$ there exists a maximal integer $N(\varepsilon) \geq N_0$ so that
\[ \varepsilon < a \left[ (1/\xi_0)^{1/N(\varepsilon)} - 1 \right]. \]  
(2.13)
The maximality of $N(\varepsilon) \geq N_0$ and inequality (2.12) imply
\[ \xi_1 e^{-cN(\varepsilon)} < a \left[ (1/\xi_0)^{1/(N(\varepsilon)+1)} - 1 \right] \leq \varepsilon. \]
Therefore, we arrive at the inequality
\[ \xi_1 e^{-cN(\varepsilon)} < \varepsilon < a \left[ (1/\xi_0)^{1/N(\varepsilon)} - 1 \right], \]  
(2.14)
that is, for every $\varepsilon \in (0, \varepsilon_*)$ inequalities (2.11) and (2.8) hold with $N = N(\varepsilon)$.

**Step 8.** By Steps 5–7, for each $\varepsilon \in (0, \varepsilon_*)$ there exists an integer $N = N(\varepsilon) > 2$ such that the unique continuous function $u = u(\varepsilon) : [0, \infty) \to \mathbb{R}$ satisfying $u(t) = e^{-at}$ for $t \in [0, 1]$, and equation (2.3) on $(1, \infty)$ has the properties
\[ 1 = u(0) > u(t) > 0 \quad \text{for all } t \in (0, N + 2), \]
\[ u'(t) > 0 \quad \text{for all } t > N, \]
\[ u(t) \to \infty \quad \text{as } t \to \infty. \]  
(2.15)
The last property is clear from $\lambda_0(\varepsilon) > 0$, (2.4) and (2.5).

From (2.15) it follows that there exits a unique $\sigma(\varepsilon) > N(\varepsilon) + 2 > 4$ so that $u(\sigma(\varepsilon)) = 1$ and $u'(\sigma(\varepsilon)) > 0$. From $u'(\sigma(\varepsilon)) > 0$ it is clear that $u(\sigma(\varepsilon) - 1) \neq a/(a + \varepsilon)$. The maximality of $N(\varepsilon)$ in inequality (2.13) implies that $N(\varepsilon) \to \infty$, $\sigma(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Let $\omega(\varepsilon) = \sigma(\varepsilon) + 1 + (1/a) \log u(\sigma(\varepsilon) + 1) > 5$. Define the function $w : [0, \omega(\varepsilon)] \to \mathbb{R}$ by
\[ w(t) = \begin{cases} 
  u(t) & \text{if } t \in [0, \sigma(\varepsilon) + 1], \\
  u(\sigma(\varepsilon) + 1)e^{-a(t-\sigma(\varepsilon)-1)} & \text{if } t \in [\sigma(\varepsilon) + 1, \omega(\varepsilon)].
\end{cases} \]
Then \( w(t) > 1 \) for all \( t \in (\sigma, \omega) \), and \( w(\omega) = 1 \). Let \( p : \mathbb{R} \to \mathbb{R} \) be the \( \omega(\epsilon) \)-periodic extension of \( w \) to \( \mathbb{R} \).

For the fixed \( a > 0 \) set \( \epsilon_0 = \epsilon_* \). Observe that \( c, K \), and consequently \( \xi_0, \xi_1 \), depend only on \( a \). Then relation (2.12) shows that \( N_0 \) is also a function of \( a \). Therefore, \( \epsilon_0 \) depends only on \( a \).

If \( b \in (a, a + \epsilon_0) \) then the above constructed \( p(\epsilon) \) with \( \epsilon = b - a \in (0, \epsilon_*) \) is clearly an \( \omega(\epsilon) \)-periodic solution of equation \( (E_{\infty}) \) satisfying (H). Setting \( \omega(a,b) = \omega(\epsilon) \) and \( \sigma(a,b) = \sigma(\epsilon) \), we see that all statements of Theorem 1.2 are satisfied, and the proof is complete.

The typical shape of the periodic solutions obtained in this paper for \( (E_{\infty}) \) is shown in Figure 2.1 with \( a = 9, b = 9.7 \).

![Figure 2.1: The periodic solution of \( (E_{\infty}) \) for \( a = 9, b = 9.7 \)](image)

**Acknowledgements**

This research was supported by the grants NKFIH-K-129322, NKFIH-1279-2/2020 of the Ministry for Innovation and Technology, Hungary, and by the EU-funded Hungarian Grant EFOP-3.6.2-16-2017-0015.

The author thanks the reviewer for the relevant comments that contributed to improving the paper.

**References**

[1] P. Amil, C. Cabeza, A. C. Marti, Exact discrete-time implementation of the Mackey–Glass delayed model, IEEE Transactions on Circuits and Systems II: Express Briefs 62 (2015), 681–685. https://doi.org/10.1109/TCSII.2015.2415651

[2] F. Á. Bartha, T. Krisztin, A. Vígh, Stable periodic orbits for the Mackey–Glass equation, preprint.

[3] J. B. van den Berg, C. Groothedde, J.-P. Lessard, A general method for computer-assisted proofs of periodic solutions in delay differential problems, J. Dynam. Differential Equations (2020). https://doi.org/10.1007/s10884-020-09908-6
[4] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, H.-O. Walther, *Delay equations. Functional, complex, and nonlinear analysis*, Applied Mathematical Sciences, Vol. 110, Springer-Verlag, New York, 1995. https://doi.org/10.1007/978-1-4612-4206-2; MR1345150

[5] K. Gopalsamy, S. I. Trofimchuk, N. R. Bantsur, A note on global attractivity in models of hematopoiesis, *Ukrain. Mat. Zh.* **50**(1998), 5–12; reprinted in *Ukrainian Math. J.* **50**(1998), 3–12. https://doi.org/10.1007/BF02514684; MR1669243

[6] U. an der Heiden, H.-O. Walther, Existence of chaos in control systems with delayed feedback, *J. Differential Equations* **47**(1983), 273–295. https://doi.org/10.1016/0022-0396(83)90037-2; MR688106

[7] L. Junges, J. A. C. Gallas, Intricate routes to chaos in the Mackey–Glass delayed feedback system, *Phys. Lett. A* **376**(2012), 2109–2116. https://doi.org/10.1016/j.physleta.2012.05.022

[8] G. Karakostas, Ch. G. Philos, Y. G. Sficas, Stable steady state of some population models, *J. Dynam. Differential Equations* **4**(1992), 161–190. https://doi.org/10.1007/BF01048159; MR1150401

[9] G. Kiss, G. Röst, Controlling Mackey–Glass chaos, *Chaos* **27**(2017), 114321. https://doi.org/10.1063/1.5006922; MR3716183

[10] T. Krisztin, M. Polner, G. Vas, Periodic solutions and hydra effect for delay differential equations with nonincreasing feedback, *Qual. Theory Dyn. Syst.* **16**(2017), 269–292. https://doi.org/10.1007/s12346-016-0191-2; MR3671725

[11] T. Krisztin, G. Vas, Large-amplitude periodic solutions for differential equations with delayed monotone positive feedback, *J. Dynam. Differential Equations* **23**(2011), 727–790. https://doi.org/10.1007/s10884-011-9225-2; MR2859940

[12] T. Krisztin, G. Vas, The unstable set of a periodic orbit for delayed positive feedback, *J. Dynam. Differential Equations* **28**(2016), 805–855. https://doi.org/10.1007/s10884-014-9375-0; MR3537356

[13] M. C. Mackey, L. Glass, Oscillation and chaos in physiological control systems, *Science* **197**(1977), 278–289. https://doi.org/10.1126/science.267326

[14] M. C. Mackey, C. Ou, L. Pujo-Menjouet, J. Wu, Periodic oscillations of blood cell population in chronic myelogenous leukemia, *SIAM J. Math. Anal.* **38**(2006), 166–187. https://doi.org/10.1137/04061578X; MR2217313

[15] B. Lani-Wayda, Erratic solutions of simple delay equations, *Trans. Am. Math. Soc.* **351**(1999), 901–945. https://doi.org/10.1090/S0002-9947-99-02361-X; MR1615995

[16] A. Lasota, Ergodic problems in biology, *Astérisque* **50**(1977), 239–250. MR0490015

[17] E. Liz, G. Röst, Dichotomy results for delay differential equations with negative Schwarzian derivative, *Nonlinear Anal. Real World Appl.* **11**(2010), 1422–1430. https://doi.org/10.1016/j.nonrwa.2009.02.030; MR2646558
[18] E. Liz, E. Trofimchuk, S. Trofimchuk, Mackey–Glass type delay differential equations near the boundary of absolute stability, *J. Math. Anal. Appl.* 275(2002), 747–760. https://doi.org/10.1016/S0022-247X(02)00416-X; MR1943777

[19] G. Röst, J. Wu, Domain-decomposition method for the global dynamics of delay differential equations with unimodal feedback, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 463(2007), 2655–2669. https://doi.org/10.1098/rspa.2007.1890; MR2352875

[20] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York–Toronto, Ont.-London, 1966. MR0210528

[21] A. L. Skubachevskii, H.-O. Walther, On the Floquet multipliers of periodic solutions to non-linear functional differential equations, *J. Dynam. Differential Equations* 18(2006), 257–355. https://doi.org/10.1007/s10884-006-9006-5; MR2229980

[22] R. Szczelina, A computer assisted proof of multiple periodic orbits in some first order non-linear delay differential equations, *Electron. J. Qual. Theory Differ. Equ.* 2016, No. 83, 1–19. https://doi.org/10.14232/ejqtde.2016.1.83; MR3547459

[23] R. Szczelina, P. Zgliczynski, Algorithm for rigorous integration of delay differential equations and the computer-assisted proof of periodic orbits in the Mackey–Glass equation, *Found. Comput. Math.* 18(2018), 1299–1332. https://doi.org/10.1007/s10208-017-9369-5; MR3875841

[24] G. Vas, Configurations of periodic orbits for equations with delayed positive feedback, *J. Differential Equations* 262(2017), 1850–1896. https://doi.org/10.1016/j.jde.2016.10.031; MR3582215

[25] H.-O. Walther, Contracting return maps for some delay differential equations, in: T. Faria, P. Freitas (eds.), *Topics in functional differential and difference equations (Lisbon, 1999)*, Fields Inst. Commun., Vol. 29, Amer. Math. Soc., Providence, RI, 2001, pp. 349–360. MR1821790

[26] H.-O. Walther, Contracting return maps for monotone delayed feedback, *Discrete Contin. Dyn. Syst.* 7(2001), 259–274. https://doi.org/10.3934/dcds.2001.7.259; MR1808399