Anomalous Hall Effect in non-commutative mechanics

P. A. Horváthy *
Laboratoire de Mathématiques et de Physique Théorique
Université de Tours
Parc de Grandmont
F-37200 TOURS (France)

January 3, 2022

Abstract

The anomalous velocity term in the semiclassical model of a Bloch electron deviates the trajectory from the conventional one. When the Berry curvature (alias noncommutative parameter) is a monopole in momentum space, as found recently in some ferromagnetic crystals while observing the anomalous Hall effect, we get a transverse shift, similar to that in the optical Hall effect.

cond-mat/0606472

1 Introduction

The Anomalous Hall Effect (AHE), characterized by the absence of a magnetic field, is observed in some ferromagnetic crystals. While this has been well established experimentally, its explanation is still controversial. One, put forward by Karplus and Luttinger [1] fifty years ago, suggests that the effect is due to an anomalous current.

Many years later, it has been argued [2] that the semiclassical dynamics of a Bloch electron in a crystal should involve a Berry curvature term, Θ.

*e-mail: horvathy@lmpt.univ-tours.fr.
In the \( n^{th} \) band the equations of motion read, in an electromagnetic field,

\[
\dot{\mathbf{r}} = \frac{\partial \epsilon_n(\mathbf{k})}{\partial \mathbf{k}} - \dot{\mathbf{k}} \times \Theta, \tag{1}
\]

\[
\dot{\mathbf{k}} = e\mathbf{E} + e\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}), \tag{2}
\]

where \( \mathbf{r} = (x^i) \) and \( \mathbf{k} = (k_j) \) denote the electron’s intracell position and quasimomentum, respectively; \( \epsilon_n(\mathbf{k}) \) is the band energy. The relation (1) exhibits an anomalous velocity term, \( \dot{\mathbf{k}} \times \Theta \), which is the mechanical counterpart of the anomalous current.

The model is distinguished by the non-commutativity of the position coordinates: in the absence of a magnetic field, \( \{x^i, x^j\} = \epsilon_{ijn} \Theta \) \([3, 4]\). In the free noncommutative model in 3 space dimensions, \( \Theta \) can only be momentum-dependent such that \( \partial_{k^i} \Theta^i = 0 \) \([3]\).

A remarkable discovery concerns the AHE in the metallic ferromagnet \( \text{SrRuO}_3 \). Fang et al. \([5]\) found in fact that the experimental data are consistent with \( \Theta \) taking the form of a monopole in momentum space,

\[
\Theta = \theta \frac{\mathbf{k}}{k^3}, \tag{3}
\]

\( k \neq 0 \). \([3]\) is, furthermore, the only possibility consistent with rotational symmetry \([3]\).

Here we propose to study the AHE in the semiclassical framework (as advocated in \([6]\)), with non-commutative parameter \([3]\). For \( \mathbf{B} = 0 \) and a constant electric field, \( \mathbf{E} = \text{const.} \) and assuming a parabolic profile \( \epsilon_n(\mathbf{k}) = k^2/2 \), eqn. (2), \( \mathbf{k} = e\mathbf{E} \), is integrated as \( \mathbf{k}(t) = e\mathbf{E}t + \mathbf{k}_0 \). The velocity relation (1) becomes in turn

\[
\dot{\mathbf{r}} = \mathbf{k}_0 + e\mathbf{E}t + e\theta \frac{Ek_0}{k^3} \hat{n}, \tag{4}
\]

where \( \hat{n} = \hat{k}_0 \times \hat{E} \) [“hats” denote vectors normalized to unit length]. The component of \( \mathbf{k}_0 \) parallel to \( \mathbf{E} \) has no interest; we can assume therefore that \( \mathbf{k}_0 \) is perpendicular to the electric field. Writing \( \mathbf{r}(t) = x(t)\hat{k}_0 + y(t)\hat{E} + z(t)\hat{n} \), eqn. (4) yields that the component parallel to \( \mathbf{k}_0 \) moves uniformly, \( x(t) = k_0 t \), and its component parallel to the electric field is uniformly accelerating, \( y(t) = \frac{1}{2}eEt^2 \). (Our choices correspond to choosing time so that the turning point is at \( t = 0 \).) However, owing to the anomalous term in (1), the particle is also deviated perpendicularly to \( \mathbf{k}_0 \) and \( \mathbf{E} \), namely by

\[
z(t) = \frac{\theta}{k_0} \frac{eEt}{\sqrt{k_0^2 + e^2E^2t^2}}. \tag{5}
\]
Figure 1: The anomalous velocity term deviates the trajectory from the plane.

It follows that the trajectory leaves its initial plane and suffers, between \( t = -\infty \) to \( t = \infty \), a finite transverse shift, namely

\[
\Delta z = \frac{2\theta}{k_0}.
\]  

(6)

Most contribution to the shift comes when the momentum is small, i.e., "near the \( k \)-monopole."

\( \theta \) becomes a half-integer upon quantization, \( \theta = N/2 \), and hence (6) is indeed \( N/k_0 \). The constant \( k_0 \neq 0 \), the minimal possible value of momentum, plays the role of an impact parameter. Let us observe that while (6) does not depend on the field \( E \) or the electric charge \( e \), the limit \( eE \to 0 \) is singular. For \( eE = 0 \), the motion is uniform along a straight line.

The transverse shift is reminiscent of the recently discovered optical Hall effect [7] and can also be derived, just like in the optical case, using the angular momentum. The free expression \( [3] \), \( J = \mathbf{r} \times \mathbf{k} - \theta \hat{\mathbf{k}} \), is plainly broken by the electric field to its component parallel to \( \mathbf{E} \),

\[
J = J_y = z(t)k_0 - \theta \frac{eEt}{\sqrt{k_0^2 + e^2E^2t^2}}.
\]  

(7)
whose conservation yields once again the shift $\left< \mathbf{k} \right>$. How can the same argument work for a Bloch electron and for light? The answer relies, for both problems, on having the same “$k$-monopole” contribution, $-\theta \mathbf{k}$, in the angular momentum.

Our model is plainly not realistic: what we described is, rather, the deviation of a freely falling non-commutative particle from the classical parabola found by Galileo. Particles in a metal are not free, though, and their uniform acceleration in the direction of $\mathbf{E}$ should be damped by some mechanism. It is nevertheless remarkable that we obtain qualitative information from such a toy model.

Note added. I am indebted to Dr. S. Murakami for calling my attention to similar work in the context of the spin Hall effect in semiconductors [8]. I would also like to thank Dr. Y. Kats for informing me about the experimental status of the AHE, see [9].

References

[1] R. Karplus and J. M. Luttinger, Phys. Rev. 95, 1154 (1954).

[2] M. C. Chang and Q. Niu, Phys. Rev. Lett. 75, 1348 (1995).

[3] A. Bérard and H. Mohrbach, Phys. Rev. D 69, 127701 (2004) [hep-th/0310167].

[4] C. Duval et al. Mod. Phys. Lett. B20, 373 (2006) [cond-mat/0506051].

[5] Fang et al. Science 302, 92 (2003) [cond-mat/030232].

[6] T. Jungwirth, Q. Niu, and A. H. MacDonald, Phys. Rev. Lett. 90, 207208 (2002).

[7] M. Onoda, S. Murakami, N. Nagaosa, Phys. Rev. Lett. 93, 083901 (2004); K. Yu. Bliokh and Yu. P. Bliokh, Phys. Lett. A333, 181 (2004) [physics/0402110]; A. Bérard and H. Mohrbach, Phys. Lett. A352, 190; [hep-th/0404165]; C. Duval, Z. Horváth, P. A. Horváthy, [math-ph/0509031], to appear in Journ. Geom. Phys.

[8] S. Murakami, N. Nagaosa, and S.-C. Zhang, Science 301, 1348 (2003) [cond-mat/0308167]

[9] Y. Kats et al., Phys. Rev. B 70, 180407(R) (2004).