Inverse Boundary Spectral Problem for Riemannian Polyhedra

Kirpichnikova A., Kurylev Ya.

February 27, 2022

We consider an admissible Riemannian polyhedron with piece-wise smooth boundary. The associated Laplace defines the boundary spectral data as the set of eigenvalues and restrictions to the boundary of the corresponding eigenfunctions. In this paper we prove that the boundary spectral data prescribed on an open subset of the polyhedron boundary determine the admissible Riemannian polyhedron uniquely.

1 Introduction

Recent years have seen some very significant achievements in the study of inverse boundary-value problems in a single component body. Mathematically such body is described by a PDE or a system of PDE’s with relatively smooth coefficients. Starting from the pioneering works [7] and [12], inverse boundary-value problems were solved, at least on the level of uniqueness and sometimes conditional stability, for a wide range of scalar inverse problems, both isotropic and anisotropic, see e.g. [4], [8], [9], [20], [23], [29], [30], [31], [32], [35], [38], [41] for a far from complete list of references, with further references in monographs [17] or [21]. Moreover, for such media there appeared a number of important results in the study of the inverse boundary-value problems for systems of PDE’s corresponding to physically important models of electromagnetism, elasticity and Dirac equations, see e.g. [18], [27], [28], [33], [34], [36], [37].

Much less is known, however, about the inverse boundary-value problems for a multicomponent medium. Mathematically, such medium is described
by PDE or system of PDE’s with piece-wise smooth coefficients with different subdomains of the regularity of coefficients corresponding to different components of the medium. Clearly, the study of inverse problems for the multicomponent media is of substantial importance for practical applications. Imagine, for example, a human body with bones, muscle tissue, lungs, etc. each of those having distinctive values of material parameters, or an upper crust of the Earth which is a composition of clay, sand, rock, oil, water, etc. A complete answer to the inverse boundary problems in a multicomponent medium, at least when the data are measured on the whole boundary, is obtained only for the two-dimensional case. Namely, it was shown in [2], [3] that the Calderon inverse boundary problem in the 2D case has a unique solution in the class of $L^\infty$-coefficients. Clearly, these results cover also the case of a multicomponent medium. In higher dimensions, the results are restricted mainly to the inverse obstacle problem. In these problems the goal is to find a shape of an inclusion inside a given medium which parameters are known a priori. In the case when parameters of a medium and/or inclusion are unknown they are assumed to be homogeneous throughout each component, see e.g. [1], [14], [15], [16], [26]. Having said so, we should note that there exist powerful methods to find singularities for coefficients of lower order, see e.g. [11].

This paper is devoted to the study of the inverse boundary spectral problem for the Laplace operator in a multicomponent medium. To be more precise, we assume that the domain occupied by the medium consists of a finite number of subdomains with piece-wise smooth boundaries between them. The metric tensor in each subdomain is smooth but does have jump singularity across the interfaces, i.e. the boundaries between adjacent subdomains. Adding proper transmission conditions across the interfaces and boundary conditions on the domain’s boundary, defines a Laplace operator which, from the spectral point of view, has effectively the same properties as the Laplace operator in a single component medium. Mathematically, the considered medium may be described as a Riemannian polyhedron. Leaving exact definitions of an appropriate Riemannian polyhedron to the next section, imagine an $n$—dimensional simplicial complex $\mathcal{M}$ where simplexes can be glued together, pairwise, along their $(n-1)$—dimensional faces which we continue to call interfaces (sometimes $(n-1)$— interfaces). Imagine now that each simplex has its own smooth metric $g$ which, in principle, may have jumps across interfaces between adjacent simplexes. This, together with some additional geometric/combinatoric conditions described in section 2, defines
INTRODUCTION

a Riemannian polyhedron \((\mathcal{M}, g)\). Starting from the corresponding Dirichlet form on \(H^1(\mathcal{M}, g)\)—functions and using standard methods of spectral theory, the Laplace operator with Neumann boundary conditions, \(\Delta\), is then well-defined in \(L^2(\mathcal{M}, g)\). Denote by \(\{\lambda_k, \varphi_k\}_{k=1}^{\infty}\) the set of all eigenvalues, counting multiplicity, and corresponding orthonormal eigenfunctions of \(\Delta\). Let \(\Gamma \subset \partial \mathcal{M}\) be open.

**Definition 1.1** The collection \((\Gamma, \{\lambda_k, \varphi_k|_\Gamma\}_{k=1}^{\infty})\) is called the (local) boundary spectral data (LBSD) of the Riemannian polyhedron \((\mathcal{M}, g)\).

Let now \((\mathcal{M}, g)\) and \((\widetilde{\mathcal{M}}, \widetilde{g})\) be two Riemannian polyhedra with LBSD \((\Gamma, \{\lambda_k, \varphi_k|_\Gamma\}_{k=1}^{\infty})\) and \((\tilde{\Gamma}, \{\tilde{\lambda}_k, \tilde{\varphi}_k|_{\tilde{\Gamma}}\}_{k=1}^{\infty})\), correspondingly.

**Definition 1.2** LBSD for \((\mathcal{M}, g)\) and \((\widetilde{\mathcal{M}}, \widetilde{g})\) are equivalent if

1. \(\Gamma\) and \(\tilde{\Gamma}\) are homeomorphic, \(\varphi: \Gamma \to \tilde{\Gamma}\);
2. \(\lambda_k = \tilde{\lambda}_k\), \(k = 1, 2, \ldots\);
3. If \(\lambda_k\) has multiplicity \(m + 1\), \(m = 0, 1, \ldots\), i.e. \(\lambda_k = \lambda_{k+1} = \lambda_{k+m}\), then there is an \((m + 1) \times (m + 1)\) unitary matrix \(U_k\) such that
   \[
   (\varphi_k|_\Gamma, \ldots, \varphi_{k+m}|_\Gamma) = U_k (\varphi_{\tilde{\lambda}_{k+1}}|_{\tilde{\Gamma}}, \ldots, \varphi_{\tilde{\lambda}_{k+m}}|_{\tilde{\Gamma}}).
   \]

We can now formulate the main result of the paper:

**Theorem 1.3** Let \((\mathcal{M}, g)\) and \((\widetilde{\mathcal{M}}, \widetilde{g})\) be two admissible Riemannian polyhedra. Let, in addition, the metric tensors \(g\) and \(\widetilde{g}\) do have jumps across all \((n-1)\)—interfaces in \(\mathcal{M}\) and \(\widetilde{\mathcal{M}}\), correspondingly. Assume that LBSD \((\Gamma, \{\lambda_k, \varphi_k|_\Gamma\}_{k=1}^{\infty})\) and \((\tilde{\Gamma}, \{\tilde{\lambda}_k, \tilde{\varphi}_k|_{\tilde{\Gamma}}\}_{k=1}^{\infty})\) are equivalent. Then \((\mathcal{M}, g)\) and \((\widetilde{\mathcal{M}}, \widetilde{g})\) are isometric.

Let us make some comments on this theorem:

1. If \(\Omega_i\) is an \(n\)—dimensional simplex of \(\mathcal{M}\) with a smooth metric \(g_i\), then \(g_i\) determines an inner metric on any \(l\) dimensional, \(l < n\), simplex of \(\mathcal{M}\) which lies in \(\Omega_i\) (here and later we assume each simplex to be close). In particular, any \((n-1)\)—interface \(\gamma\) of \(\mathcal{M}\) belongs to two adjacent \(n\)—simplices, which we often denote in such case \(\Omega_-\) and \(\Omega_+\),
and therefore, has two different metric tensors $g_{-|\gamma}$ and $g_{+|\gamma}$. By a metric tensor $g$ having a jump singularity across $\gamma$ we mean that, for any $p \in \gamma$,

$$g_-(p) \neq g_+(p).$$

This assumption is of a technical nature and, in section 6 we will significantly weaken it.

2. As shown in section 2, any Riemannian polyhedron has a natural structure of a metric space. The isometry of $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ is understood with respect to these metric structures.

3. The boundary $\partial \mathcal{M}$ of a Riemannian polyhedron $(\mathcal{M}, g)$ is itself a $(n-1)$-dimensional Riemannian polyhedron, probably disconnected. As $\Gamma, \tilde{\Gamma}$ are open subsets of $\partial \mathcal{M}, \partial \tilde{\mathcal{M}}$, by reducing them if necessary we assume that $\Gamma$ and $\tilde{\Gamma}$ are open subsets of some $(n-1)$-dimensional simplex of $\partial \mathcal{M}, \partial \tilde{\mathcal{M}}$, correspondingly. In the future, we will always assume this condition to be true.

The plan of the paper is as follows: In section 2 we provide some preliminary material on geometry of Riemannian polyhedra and properties of the Laplace operator on them. Section 3 is devoted to the description and some properties of the non-stationary Gaussian beams on a Riemannian polyhedron. We prove Theorem 1.3 in sections 4 and 5. The last section 6 is devoted to some generalizations and open questions.

2 Preliminary constructions

2.1 Admissible Riemannian polyhedron

In this section we will introduce, following mainly [10] and [6], an admissible Riemannian polyhedron which is the main object of the paper. We start with a closed $n$-dimensional finite simplicial complex

$$\mathcal{M} = \bigcup_{i=1}^{l} \Omega_i,$$
2 PRELIMINARY CONSTRUCTIONS

Figure 1: Case (A) is prohibited because its structure is not \((n-1)\)-chainable; Case (B) is prohibited as it is not dimensionally homogeneous; Case (C) is appropriate

where \(\Omega_i\) are closed \(n\)-dimensional simplexes of \(\mathcal{M}\), with \(\Omega_i^{\text{int}}\) standing for the interior of \(\Omega_i\) which is an open subset of \(\mathcal{M}\). We assume that \(\mathcal{M}\) is dimensionally homogeneous, i.e. any \(k\)-simplex, \(0 \leq k \leq n\), of \(\mathcal{M}\) is contained in at least one \(\Omega_i\). We assume also that any \((n-1)\)-dimensional simplex \(\gamma\) belongs either to two different \(n\) simplexes, \(\Omega_i\) and \(\Omega_j\), which in this case we will often denote by \(\Omega_-\) and \(\Omega_+\), or to only one \(n\) simplex \(\Omega_i\). In the former case we call \(\gamma\) an interface (sometimes \((n-1)\)-dimensional interface) between \(\Omega_-\) and \(\Omega_+\), in the latter case we call \(\gamma\) a boundary \((n-1)\)-simplex with \((n-1)\)-simplexes having this property forming the boundary \(\partial \mathcal{M}\). We denote by \(\mathcal{M}^k, 0 \leq k \leq n\) the \(k\)-skeleton of \(\mathcal{M}\) which consists of all \(k\)-simplexes of \(\mathcal{M}\). Clearly, \(\mathcal{M} = \mathcal{M}^n\). We use notations

\[
\mathcal{M}^{\text{int}} = \bigcup \Omega_i^{\text{int}}, \quad \mathcal{M}^{\text{reg}} = \mathcal{M} \setminus \left( \bigcup_{k=0}^{n-2} \mathcal{M}^k \right).
\]

Following [11], we assume that \(\mathcal{M}\) is \((n-1)\)-chainable, i.e. \(\mathcal{M}^{\text{reg}}\) is path connected, see Fig. 1

Assume now that each \(n\)-simplex \(\Omega_i\) is equipped with a smooth (up to \(\partial \Omega_i\)) Riemannian metric \(g_i\), i.e. \((\Omega_i, g_i)\) is a smooth Riemannian manifold with a piecewise smooth boundary. This makes it possible to introduce the arclength for admissible paths \(\eta : [0, a] \to \mathcal{M}\). We call a path \(\eta\) admissible if \(\eta^{-1}(\mathcal{M}^{\text{int}}) \subset [0, a]\) is a (relatively) open subset of \([0, a]\) of full measure and, if \(\eta(\alpha, \beta)\) is in some \(n\)-simplex \(\Omega^{\text{int}}\), then \(\eta : (\alpha, \beta) \to \Omega^{\text{int}}\) is piecewise smooth. Naturally, the arclength \(|\eta(\alpha, \beta)|\) of the path \(\eta\) between \(\eta(\alpha)\) and
\( \eta(\beta) \) is taken as

\[
|\eta(\alpha, \beta)| = \int_{\alpha}^{\beta} \left[ g_{mj}(\eta(t)) \dot{\eta}_m(t) \dot{\eta}_j(t) \right]^{1/2} dt,
\]

where \( \eta_j(t), \alpha < t < \beta \) are, for example, baricentric coordinates in \( \Omega \). As
\( \eta^{-1}(\mathcal{M}^{\text{int}}) \cap (0, a) \) consists of at most a countable number of open intervals
\( (\alpha_i, \beta_i) \) we define

\[
|\eta[0, a]| = \sum_i |\eta(\alpha_i, \beta_i)|.
\]

Next we introduce, for any \( p, q \in \mathcal{M} \), the distance, \( d(p, q) \),

\[
d(p, q) = \inf_{\eta} |\eta|,
\]

where infenum is taken over all admissible paths connecting \( p \) and \( q \). This makes \( (\mathcal{M}, g) \) into a metric space with its metric topology being the same as the topology of a simplicial complex, see [10].

**Definition 2.1** \((\mathcal{M}, g)\) is an admissible Riemannian polyhedron if, for any \( p, q \in \mathcal{M} \),

\[
d(p, q) = \inf_{\tilde{\eta}} |\tilde{\eta}|,
\]

where \( \tilde{\eta} \) run over the subset of admissible paths between \( p \) and \( q \) such that

\[
\tilde{\eta}^{-1} \left( \bigcup_{k=0}^{n-2} \mathcal{M}^k \right) \setminus (\{0\} \cup \{1\}) = \emptyset.
\]

As \( \mathcal{M} \) is finite, the above condition is independent of a particular choice of metric \( g \).

### 2.2 Boundary normal and interface coordinates

In addition to the baricentric coordinates in any \( \Omega^{\text{int}} \), we will often use boundary normal or interface coordinates associated with \( (n-1) \)-subsimplices of \( \Omega \).
Figure 2: A sample of two topologically different Riemannian polyhedra having the same spectral properties. It is stronger than the \((n-1)\)-chainability and is aimed at avoiding topologically different Riemannian polyhedra which, however, can have the same spectral properties.

Let first \(\gamma \in \Omega \cap \partial \mathcal{M}\) be a boundary \((n-1)\)-dimensional simplex with its \((n-1)\)-dimensional interior denoted by \(\gamma^{\text{int}}\). We introduce boundary normal coordinates in an relatively open subset \(U \subset \Omega\) as

\[
p \rightarrow (s(p), \sigma(p)), \quad p \in U.
\]

Here \(\sigma(p) = d(p, \partial \mathcal{M})\), and we assume that there is a unique \(q \in \gamma^{\text{int}}\) with \(d(p, \partial \mathcal{M}) = d(p, q)\), such that \(p\) lies on the normal geodesic to \(q\), \(\varsigma(q) = q\), \(\varsigma(0, d(p, \partial \mathcal{M})) \in \Omega^{\text{int}}\). If \(s(q) = (s^1, \ldots, s^{(n-1)})\) are some (local) coordinates on \(\gamma\), e.g. baricentric coordinates, then \(s(p) = s(q)\).

Let now \(\gamma \subset \Omega_- \cap \Omega_+\) be an \((n-1)\)-interface between \(n\)-simplices \(\Omega_-\) and \(\Omega_+\). Let \(U_{\pm}\) be relatively open subsets of \(\Omega_{\pm}\) with the nearest point on \(\partial \Omega_{\pm}\) lying on \(\gamma\) such that \((U_- \cap \gamma) = (U_+ \cap \gamma)\). Denote by \((s, \sigma_{\pm})\) the boundary normal coordinates in \(U_{\pm}\), where \(s = (s^1, \ldots, s^{(n-1)})\) are some local coordinates on \(\gamma\), e.g. baricentric coordinates with respect to \(\Omega_-\) or \(\Omega_+\). We introduce the interface coordinates \((s, \sigma)\) on \(U_- \cup U_+\):

\[
(s, \sigma) = \begin{cases} 
(s, -\sigma), & \text{in } U_-; \\
(s, \sigma), & \text{in } U_+.
\end{cases}
\]
Then the metric element in these coordinates takes the form,

\[(dl)^2 = (d\sigma)^2 + (g_{\pm})_{\alpha\beta}(s, \sigma) \, ds^\alpha \, ds^\beta.\]  \hspace{1cm} (3)

Throughout the paper we assume that the following condition takes place:

**Condition 2.2** For any interface \( \gamma \) and any point \( q = (s, 0) \) on \( \gamma \), the metric tensor \( g_{\alpha\beta} \) has a jump singularity at \( q \).

### 2.3 Laplace operator

Let \( H^1(\mathcal{M}) \) be the Sobolev space of functions \( u \in L^2(\mathcal{M}) \) such that \( u_i = u|_{\Omega_i} \in H^1(\Omega_i) \) and, for any interface \( \gamma \) between \( \Omega_i \) and \( \Omega_j \),

\[u_i|_{\gamma} = u_j|_{\gamma}.\]

The inner product on \( H^1(\mathcal{M}) \) determines the closed non-negative Dirichlet form,

\[\mathbb{D}[u, v] = \sum_{i=1}^{I} (u_i, v_i)_{H^1(\Omega_i)}, \quad u, v \in H^1(\mathcal{M}).\]

By the standard technique of the theory of quadratic forms, the form \( \mathbb{D} \) determines a self-adjoint operator in \( L^2(\mathcal{M}) \), namely, the Laplace operator with Neumann boundary condition, \( \Delta \). The domain \( \mathcal{D}(\Delta) \) is defined by

\[\mathcal{D}(\Delta) = \{ u \in H^1(\mathcal{M}) : \mathbb{D}[u, v] = (f, v)_{L^2(\mathcal{M})} \text{ for some } f \in L^2(\mathcal{M}) \}, \hspace{1cm} (4)\]

where \( v \in H^1(\mathcal{M}) \) is arbitrary. Analysing condition (4), we see that \( u \in \mathcal{D}(\Delta) \) if \( u \in H^2(\Omega_i), \, i = 1, \ldots, I \) and, on any interface \( \gamma \subset \Omega_- \cap \Omega_+ \),

\[u_-|_{\gamma} = u_+|_{\gamma}, \quad [\sqrt{g_-} \partial_\sigma u_-]|_{\gamma} = [\sqrt{g_+} \partial_\sigma u_+]|_{\gamma}. \hspace{1cm} (5)\]

where \( g_{\pm}(s) = \det[ (g_{\pm})_{\alpha\beta}(s, 0) ] \).

As, due to the finiteness of \( \mathcal{M} \), the embedding of \( H^1(\mathcal{M}) \) into \( L^2(\mathcal{M}) \) is compact, the spectrum of \( \Delta \) is pure discrete,

\[0 = \lambda_1 < \lambda_2 \leq \ldots, \quad \lambda_k \to \infty,\]

with the corresponding basis of orthonormal eigenfunctions to be denoted by \( \{ \varphi_k \}_{k=1}^{\infty} \). Standard considerations, see e.g. \cite{9} or \cite{43} show that \( \{ \varphi_k \}_{k=1}^{\infty} \) distinguish points in \( \mathcal{M}^{\text{int}} \), i.e. for \( p \neq q \in \mathcal{M}^{\text{int}} \), there is \( k \) with \( \varphi_k(p) \neq \varphi_k(q) \).
Proposition 2.3  Let \( \{ \varphi_k \}_{k=1}^\infty \) be an orthonormal basis of eigenfunctions of the Laplace operator \( \Delta \). Then \( \{ \varphi_k \}_{k=1}^\infty \) form local coordinates near any \( p \in \mathcal{M}^{\operatorname{int}} \), i.e. there are \( k_1(p), \ldots, k_n(p) \) such that \( (\varphi_{k_1}, \ldots, \varphi_{k_n}) \) form local coordinates near \( p \).

3 Gaussian Beams near interfaces

3.1 Gaussian beams on smooth manifolds

In this section we briefly recall some results on the non-stationary Gaussian beams on smooth manifolds. Their theory goes back to the pioneering works [5], [19], [40]. In our exposition we follow mainly section 2.4 of [21]. Non-stationary Gaussian beams are some (formal) solutions of the wave equation

\[ U_{tt} - \Delta U = 0, \tag{6} \]

which are concentrated, at each moment of time \( t \), near a point \( x(t) \). The point \( x(t) \) moves with a unit speed along a geodesic on a smooth Riemannian manifold \( (\mathcal{N}, h) \) with \( \Delta \) being the Laplacian corresponding to \( (\mathcal{N}, h) \).

Introducing a moving frame

\[ y(t) = x - x(t), \]

a formal Gaussian beam has a form as a formal series

\[ U_\varepsilon(t, y) \asymp M_\varepsilon \exp \left\{ -(i\varepsilon)^{-1} \Theta(t, y) \right\} \sum_{l \geq 0} u_l(t, y) (i\varepsilon)^l. \tag{7} \]

Here \( M_\varepsilon = (\pi \varepsilon)^{-\frac{n}{4}} \), \( 0 < \varepsilon \ll 1 \); \( \Theta \) and \( u_l \), \( l = 0, 1, \ldots \), are formal series in powers of \( y \). They are usually represented as sums of homogeneous polynomials in \( y \) with coefficients depending on \( t \),

\[ \Theta \asymp \sum_{m \geq 1} \theta_m(t, y), \quad u_l = \sum_{m \geq 1} u_{lm}(t, y), \]

\( \theta_m \) and \( u_{lm} \) being homogeneous polynomials on \( y \). The polynomials \( \theta_m \) and \( u_{lm} \) are chosen so that, considered as formal series with respect to \( y \) and \( (i\varepsilon) \),

\[ \partial_t^2 U_\varepsilon - \Delta U_\varepsilon = 0. \tag{8} \]

Note that "\( \asymp \)" exactly means that the formal series (7) satisfies formally equation (8).

The most important properties of the non-stationary Gaussian beams are:
(a) $\theta_1(t, y) = (\xi(t), y(t)) = \xi_j(t)y^j(t)$, where $\xi_j(t)$ is the unit covector corresponding to the geodesic $x(t)$;

(b) $\theta_2(t, y) = \langle H(t)y, y \rangle$, where $H(t)$ is a symmetric matrix, satisfying $\text{Im} \langle H(t)y, y \rangle \geq C(T)|y|^2$, for $-T < t < T$.

Remark 3.1 From now on throughout this paper we use the following notations $C$ (or, $C_L(t)$) is a generic constant, $C > 0$, independent of $\varepsilon$; $\mu(L)$ is defined for sufficiently large positive integers $L$ such that $\mu(L) \to \infty$ when $L \to \infty$.

Conditions (a) and (b) imply that $U$ decays exponentially outside an $\varepsilon^{1/2}$ - neighborhood of $x(t)$. It is important to note that, starting from a formal Gaussian beam $U_\varepsilon$ we can construct a family of solutions to the wave equation (6), which “looks like” $U_\varepsilon$. To this end, we start with a finite series

$$U^L_\varepsilon = M_\varepsilon \exp \left\{ - (i\varepsilon)^{-1} \Theta^L(t, y) \right\} \sum_{l=0}^{L} u^L_l(t, y) \chi(d^2(x, x(t))\varepsilon^{-5/6}), \quad (9)$$

$$\Theta^L = \sum_{l=0}^{L} \theta_l; \quad u^L_l = \sum_{l=0}^{L} u_{lm},$$

where $\chi(s)$ is a smooth cut-off function equal to 1 near $s = 0$. Then

$$\| \partial^2_t U^L_\varepsilon - \Delta U^L_\varepsilon \|_{C^{\mu(L)}(\mathcal{N} \times [-T, T])} \leq C_L(T) \varepsilon^{-\mu(L)}.$$

By standard hyperbolic estimates there exists a solution $U^L_\varepsilon$ to (6) such that

$$\| (U^L_\varepsilon - U_\varepsilon^L) \|_{C^N(\mathcal{N} \times [-T, T])} \leq C_L(T) \varepsilon^{-\mu(L)}.$$

Moreover, if we generate a wave inside $\mathcal{N}$ by a boundary source

$$U_\varepsilon |_{\partial \mathcal{N} \times [-T, T]} = f_\varepsilon(t, s). \quad (10)$$

Let $f_\varepsilon(t, s), s \in \partial \mathcal{N}, t \in [-T, T]$ be given by a formal expansion

$$f_\varepsilon(t, s) \asymp \exp \left\{ - (i\varepsilon)^{-1} \tilde{\Theta}(t, s) \right\} \sum_{l \geq 0} \tilde{u}_l(t, s)(i\varepsilon)^l,$$

where

$$\tilde{\Theta}(t, s) \asymp -t + \xi_\alpha s^\alpha + < \hat{H}((s, t), (s, t)) > + \sum_{m \geq 2} \tilde{\theta}_m(t, s); \quad g^{\alpha\beta}(0)\xi_\alpha \xi_\beta < 1,$$
\[ \hat{u}(t, s) \simeq \sum_{m \geq 0} \hat{u}_{lm}(t, s), \] (12)

with \( \hat{\theta}_m, \hat{u}_{lm} \) being homogeneous polynomials of degree \( m \) with respect to \((t, s)\). Assume that \( Im \hat{H} > 0 \). Then there is a unique formal Gaussian beam \( U_\varepsilon \) satisfying (8), and the boundary condition (11) for \(-t_0 < t < t_0\) with some \( t_0 > 0 \) depending only on geometry of \((\mathcal{N}, \partial \mathcal{N}, h)\). Moreover, the corresponding geodesic \( x(t) \) starts, at \( t = 0 \), from the point \( s = 0 \), into the (co)divergion \( (\xi_\alpha, \xi_\beta) \) with \( \xi_n = [g^{\alpha\beta} \xi_\alpha \xi_\beta]^{1/2} \) geodesic starting at \( s = 0 \).

We will refer to this result saying that we can guarantee a non-stationary Gaussian beam propagating transversally to \( \partial \mathcal{N} \) by a proper choice of a boundary source (for these and other results on non-stationary Gaussian beams see e.g. [20], [21]).

### 3.2 Gaussian beams at interfaces

In this section we consider reflection and transmission of the non-stationary Gaussian beams from and through an \((n-1)\)-dimensional interface \( \gamma \) between two \( n \)-simplexes \( \Omega_- \) and \( \Omega_+ \). As our constructions will be of a local nature we can, without loss of generality, restrict them to the reflection/transmission of the Gaussian beams from a smooth interface inside a smooth manifold. These questions, for the incidence angle less then critical, were considered in detail in [25] and [39], in the latter restricted to the isotropic media. Assuming that \( x(0) \in \gamma \) and introducing the interface normal coordinates \((s, \sigma)\) with \( s = 0 \) corresponding to \( x(0) \), we have, for the incident Gaussian beam, \( U_{\varepsilon}^{in} \)

\[ U_{\varepsilon}^{in}(t, s)|_{\sigma=0} \simeq M_\varepsilon \exp\{(i\varepsilon)^{-1}\hat{\Theta}^{in}(t, s)\} \sum_{l=0}^{\infty} (i\varepsilon)^l u_{l}^{in}(t, s) \] (13)

\[ \left[ \sqrt{g} - \partial_\sigma U_{\varepsilon}^{in}(t, s) \right]|_{\sigma=0} \simeq M_\varepsilon \exp\{(i\varepsilon)^{-1}\hat{\Theta}^{in}(t, s)\} \sum_{l=-1}^{\infty} (i\varepsilon)^l \hat{u}_{l}^{in}(t, s). \]

Here \( \hat{\Theta}^{in}, u_{l}^{in}, \hat{u}_{l}^{in} \) are sums of homogeneous polynomials with respect to \((t, s)\) with

\[ \hat{\Theta}^{in}(t, s) = -t + \xi_\alpha s^\alpha + \hat{H}((s, t), (s, t)) + \ldots; \]

\[ u_{l=0,0}^{in} = \xi_n u_{0,0}; \quad \xi_n > 0; \quad g^{\alpha\beta} \xi_\alpha \xi_\beta + (\xi_n)^2 = 1, \]
with \((\xi^\text{in}_\alpha, \xi^\text{in}_n)\) being the (co)direction of the Gaussian beam at \(t = 0\). When \(g^{\alpha\beta}_+ (0) \xi^\text{in}_\alpha \xi^\text{in}_\beta < 1\), it is possible to construct two formal non-stationary Gaussian beams \(U^r_\varepsilon\) and \(U^{tr}_\varepsilon\) in \(\Omega_-\) and \(\Omega_+\), correspondingly such that

\[
U_\varepsilon = \begin{cases} 
U^\text{in}_\varepsilon + U^r_\varepsilon, & \text{in } \Omega_-; \\
U^{tr}_\varepsilon & \text{in } \Omega_+ 
\end{cases}
\]  

(14)

satisfies the wave equation (8) and transmission conditions (5). To this end we use the technique briefly described in section 3.1, which reduces the problem to finding boundary conditions of form (11), (12) for \(U^r_\varepsilon\) and \(U^{tr}_\varepsilon\) at \(\sigma = 0\). In turn, this is possible utilizing transmission condition (5) if \(g^{\alpha\beta}_+ (0) \xi^\text{in}_\alpha \xi^\text{in}_\beta < 1\). Summarizing considerations of [25], we obtain the following result

**Lemma 3.2** Let \(U^\text{in}_\varepsilon\) be a formal non-stationary Gaussian beam which hits the interface \(\sigma = 0\) at \(s = 0, t = 0\) with its (co)direction \((\xi^\text{in}_\alpha, \xi^\text{in}_n)\) satisfying \(g^{\alpha\beta}_+ (0) \xi^\text{in}_\alpha \xi^\text{in}_\beta < 1\). Then there are two formal Gaussian beams \(U^r_\varepsilon\) in \(\Omega_-\) and \(U^{tr}_\varepsilon\) in \(\Omega_+\) such that the total wave \(U_\varepsilon\) satisfies the transmission condition (5). The (co)directions \((\xi^r_\alpha, \xi^r_n)\) and \((\xi^{tr}_\alpha, \xi^{tr}_n)\) of the geodesics, corresponding to \(U^r_\varepsilon\) and \(U^{tr}_\varepsilon\) satisfy at \(t = 0, s = 0, \sigma = 0\) the equation (Snell’s Law):

\[
\xi^r_\alpha = \xi^r_\beta = \xi^\text{in}_\alpha, \quad \xi^{tr}_n = -\xi^\text{in}_n, \quad \xi^{tr}_n = (1 - g^{\alpha\beta}_+(0) \xi^\text{in}_\alpha \xi^\text{in}_\beta)^{1/2}.
\]

The main amplitude coefficients \(u^r_{0,0}\) and \(u^{tr}_{0,0}\) of \(U^r_\varepsilon\) and \(U^{tr}_\varepsilon\) are related to \(u^\text{in}_{0,0}\) at \(t = 0, s = 0, \sigma = 0\) by

\[
\begin{align*}
u^{tr}_{0,0} &= -\sqrt{g_- \xi^\text{in}_n + \sqrt{g_+ \xi^\text{in}_n}} u^\text{in}_{0,0}; & u^r_{0,0} &= \sqrt{g_- \xi^\text{in}_n - \sqrt{g_+ \xi^\text{in}_n}} u^\text{in}_{0,0},
\end{align*}
\]

(15)

where \(g_{\pm} = \text{det}^{-1} \left[ g^{\alpha\beta}_+ (0) \right] \).

We note that transmission condition (5) for \(U^\text{in}_\varepsilon + U^r_\varepsilon\) and \(U^{tr}_\varepsilon\) is understood in the formal sense. Namely, \(U^r_\varepsilon\) and \(U^{tr}_\varepsilon\) may be expressed in the form (7) with \(U^r_\varepsilon\) and \(U^{tr}_\varepsilon\) having decomposition of form (13). Then (5) means that

\[
\hat{\Theta}^\text{in} = \hat{\Theta}^r = \hat{\Theta}^{tr}
\]

and

\[
\hat{u}^\text{in}_i + \hat{u}^r_i = \hat{u}^{tr}_i, \quad \hat{\hat{u}}^\text{in}_i + \hat{\hat{u}}^r_i = \hat{\hat{u}}^{tr}_i
\]
as polynomial with respect to \((t, s)\).

Observe that the condition \(\left[ g_{\alpha\beta}(0) \right]_{\alpha,\beta=1}^{n-1} \neq \left[ g^{\alpha\beta}(0) \right]_{\alpha,\beta=1}^{n-1} \) implies that \(u_{0,0}^\beta \neq 0\) for almost all \((\xi^m, \xi^n)\).

When dealing with an incoming non-formal Gaussian beam (9), (10), which we will denote by \(U_{\in,L}^\varepsilon\), similar to the above we find \(U_{\in,L}^\varepsilon\) and \(U_{\tr,L}^\varepsilon\) by formulae (9) with \(\Theta_{\in,L}^\varepsilon, u_{\in,L}^\varepsilon\) and \(\Theta_{\tr,L}^\varepsilon, u_{\tr,L}^\varepsilon\) instead of \(\Theta_{\in,L}^\varepsilon, u_{\in,L}^\varepsilon\), correspondingly. Clearly, they give use to approximate transmission conditions

\[
\| (U_{\in,L}^\varepsilon + U_{\tr,L}^\varepsilon - U_{\in,L}^\varepsilon) \|_{C^p(\gamma \times (-t_0, t_0))} \leq C(T) \varepsilon^{-\mu(L)},
\]

\[
\| (\sqrt{g^-} \partial_\sigma (U_{\in,L}^\varepsilon + U_{\tr,L}^\varepsilon)) - \sqrt{g^+} \partial_\sigma U_{\tr,L}^\varepsilon ) \|_{C^p(\gamma \times (-t_0, t_0))} \leq C(T) \varepsilon^{-\mu(L)}.
\]

Add to \(U_{\tr,L}^\varepsilon\) a function \(\Psi_{\varepsilon}^L(t, s, \sigma)\),

\[
\Psi_{\varepsilon}^L(t, s, \sigma) = \chi(\sigma) \sum_{k=0}^L \sigma^k \Psi_k(s, t),
\]

where \(\Psi_k(s, t)\) are chosen so that

\[
\tilde{U}_{\varepsilon}^\alpha = U_{\varepsilon}^\alpha + \Psi_{\varepsilon}^L
\]

satisfies

\[
\left[ \Delta^p (U_{\varepsilon}^\in + \tilde{U}_{\varepsilon}^\tr) \right]_{\gamma} = \Delta^p U_{\varepsilon}^\tr, \quad \left[ \sqrt{g^-} \partial_\sigma \left( \Delta^p (U_{\varepsilon}^\in + U_{\varepsilon}^\tr) \right) \right]_{\gamma} = \left[ \sqrt{g^+} \partial_\sigma (\Delta^p U_{\varepsilon}^\tr) \right]_{\gamma},
\]

for \(0 \leq p \leq \left[ \frac{L}{2} \right] \). Clearly, (16), (17) imply that

\[
\| \Psi_{\varepsilon}^L(s, t, \sigma) \|_{C^p(\Omega_\varepsilon \times (-t_0, t_0))} \leq C(T) \varepsilon^{-\mu(L)}.
\]

Together with (18), (19), it follows from the wave equation (8) that we can modify \(U_{\varepsilon}^\tr\) and \(\tilde{U}_{\varepsilon}^\tr\).

\[
\tilde{U}_{\varepsilon}^\tr = U_{\varepsilon}^\tr + \Phi_{\varepsilon}^\tr; \quad \tilde{U}_{\varepsilon}^\tr = U_{\varepsilon}^\tr + \Phi_{\varepsilon}^\tr,
\]

with

\[
\| \partial_\mu (\Phi_{\varepsilon}^L \|_{L^2(M)} \leq C_L(t_0) \varepsilon^{-\mu(L)};
\]

\[
\| \Delta^{[\mu(L)/2]} \Phi_{\varepsilon}^L \|_{L^2(M)} \leq C_L(t_0) \varepsilon^{-\mu(L)};
\]
so for $-t_0 \leq t \leq t_0$. Therefore,

$$u_\varepsilon^L = \begin{cases} 
U_\varepsilon^{in,L} + U_\varepsilon^{r,L}, & \sigma \leq 0, \\
U_\varepsilon^{tr,L}, & \sigma \geq 0
\end{cases}$$

satisfies the wave equation (8) and coincide with $U_\varepsilon^{in,L}$ for negative $t$, more precisely, for $t \leq -C_\varepsilon^{-5/12}$. Here $\Phi_\varepsilon^L$ is given by $\Phi_\varepsilon^{r,L}$ for $\sigma < 0$ and $\Phi_\varepsilon^{tr,L}$ for $\sigma > 0$.

**Remark 3.3** When $\gamma$ is $(n-1)-$simplex in $\partial \mathcal{M}$, we can modify the previous construction to find the Gaussian beam reflected from $\gamma$. The part of Lemma 3.2 related to $U_\varepsilon^r$ remains valid with formula for the main term $u_{0,0}^r$ taking the form

$$u_{0,0}^r = -u_{0,0}^{in}. \quad (20)$$

Thus, by the considerations similar to the above it is possible to find solutions to the wave equation (8) which satisfy the Dirichlet boundary condition and look like a Gaussian beam.

## 4 First $n-$simplex

### 4.1

In this section we start proving the uniqueness Theorem 1.3. Recall that we are given diffeomorphic open subsets $\Gamma \subset \gamma$, $\tilde{\Gamma} \subset \tilde{\gamma}$, where $\gamma$ and $\tilde{\gamma}$ are boundary $(n-1)-$simplexes of $n-$simplexes $\Omega \subset \mathcal{M}$, $\tilde{\Omega} \subset \tilde{\mathcal{M}}$. We assume, after proper unitary transformations in finite-dimensional spaces corresponding to eigenvalues of higher multiplicity, that

$$\lambda_k = \tilde{\lambda}_k, \quad \varphi_k |_{\Gamma} = \varphi_k^* |_{\tilde{\Gamma}}, \quad k = 1, 2, \ldots, \quad (21)$$

where $\varphi : \Gamma \rightarrow \tilde{\Gamma}$ is a diffeomorphism. Our goal in this section is to prove the following result

**Lemma 4.1** Let $\varphi : \Gamma_0 \rightarrow \tilde{\Gamma}_0$, $\Gamma_0 \in \gamma$, $\tilde{\Gamma}_0 \in \tilde{\gamma}$ be a diffeomorphism satisfying conditions of Definition 1.2. Then there is an isometry $X : \Omega \rightarrow \tilde{\Omega}$, such that

$$\varphi_k |_{\Omega} = X^* \tilde{\varphi}_k |_{\tilde{\Omega}}; \quad k = 1, 2, \ldots, \quad X |_{\Gamma_0} = \varphi. \quad (22)$$
To prove this lemma, observe first that if $\Gamma_0 \Subset \Gamma$, $\tilde{\Gamma}_0 \Subset \tilde{\Gamma}$ and $\tau > 0$ satisfy:

$$\exp_{\partial \Omega} : \Gamma_0 \times [0, \tau) \to \mathcal{M}, \ \exp_{\partial \tilde{\Omega}} : \tilde{\Gamma}_0 \times [0, \tau) \to \tilde{\mathcal{M}}$$

are regular, i.e. map into $\Omega$ and $\tilde{\Omega}$ correspondingly and provide a diffeomorphism between $\Gamma_0 \times [0, \tau)$ and $\exp_{\partial \Omega}(\Gamma_0 \times [0, \tau)) \subset \Omega$ and $\tilde{\Gamma}_0 \times [0, \tau)$ and $\exp_{\partial \tilde{\Omega}}(\tilde{\Gamma}_0 \times [0, \tau)) \subset \tilde{\Omega}$ and, in addition,

$$d(\Gamma_0, \mathcal{M} \setminus \Omega), \ d(\tilde{\Gamma}_0, \tilde{\mathcal{M}} \setminus \tilde{\Omega}) > \tau,$$

then the diffeomorphism

$$\hat{X} = \pi \times \mathbb{I} : \Gamma_0 \times (0, \tau) \to \tilde{\Gamma}_0 \times (0, \tau)$$

satisfies

$$\hat{X}^* \varphi_k|_{\Gamma_0 \times (0, \tau)} = \varphi_k|_{\Gamma_0 \times (0, \tau)}.$$

Moreover, $\hat{X}$ is actually an isometry between $\Gamma_0 \times (0, \tau)$ and $\tilde{\Gamma}_0 \times (0, \tau)$ considered as domains in $(\Omega^{int}, g)$ and $(\tilde{\Omega}^{int}, \tilde{g})$, correspondingly. The proof of this fact is identical to the smooth case and is given in Section 4.4 of [21].

Assume now that $\omega_1, \omega_2 \subset \Omega^{int}$ and $\tilde{\omega}_1, \tilde{\omega}_2 \subset \tilde{\Omega}^{int}$ are open subsets with

$$\hat{X}_i : \omega_i \to \tilde{\omega}_i$$

being isometries satisfying

$$\hat{X}_i^* \varphi_k|_{\omega_i} = \varphi_k|_{\omega_i}, \quad i = 1, 2, \ k = 1, 2, \ldots$$

(23)

By Proposition 1, $\hat{X}_1|_{\omega_1 \cap \omega_2} = \hat{X}_2|_{\omega_1 \cap \omega_2}$ which makes it possible to extend $\hat{X}_i$ into an isometry $\hat{X} : \omega_1 \cup \omega_2 \to \tilde{\omega}_1 \cup \tilde{\omega}_2$ which also satisfies (23).

Consider the family of all pairs of open subsets $\omega \subset \Omega$ and $\tilde{\omega} \subset \tilde{\Omega}$ which are isometric to each other and satisfy (23). Clearly, this family is partially ordered by induction, by the above we can consider its maximal element which we denote by $(\Omega_m, \tilde{\Omega}_m)$ with the corresponding isometry denoted by $\hat{X}_m$. We want to show that $\Omega_m = \Omega^{int}$, $\tilde{\Omega}_m = \tilde{\Omega}^{int}$.

To proceed, recall the following result from [21], which is proven for smooth manifolds but remains valid for Riemannian polyhedra under the conditions formulated below.

**Theorem 4.2** 1. Let $S \subset \Omega$ (or $\tilde{S} \subset \tilde{\Omega}$) be a smooth subdomain such that

$$\{\varphi_k(p)\}^\infty_k \ (or \ \{\tilde{\varphi}_k(p)\}^\infty_k)$$

are known for $p \in S$ (or for $\tilde{p} \in \tilde{S}$). Assume that for $\tau > 0$

$$\exp_{\partial S} : \partial S \times (0, \tau) \to \Omega \setminus S,\ \exp_{\partial \tilde{S}} : \partial \tilde{S} \times (0, \tau) \to \tilde{\Omega} \setminus \tilde{S}$$
are regular. If, in addition, \( \tau < \{d(S, M \setminus \Omega), d(\tilde{S}, \tilde{M} \setminus \tilde{\Omega})\} \) then these data determine uniquely \( \varphi_k|_{S_r}, \tilde{\varphi}_k|_{\tilde{S}_r} \), where

\[
S_r = S \bigcup \exp_{\partial S} (\partial S \times (0, \tau)), \quad \tilde{S}_r = \tilde{S} \bigcup \exp_{\partial \tilde{S}} (\partial \tilde{S} \times (0, \tau)).
\]

2. Let \( u, \tilde{u} \) be solutions of the initial boundary value problem,

\[
\begin{align*}
  u_{tt} - \Delta u &= F \in C^\infty_0 (S \times \mathbb{R}_+); & u_{tt} - \Delta \tilde{u} &= \tilde{F} \in C^\infty_0 (\tilde{S} \times \mathbb{R}_+), \\
  u|_{t=0} &= f \in C^\infty_0 (S); & \tilde{u}|_{t=0} &= \tilde{f} \in C^\infty_0 (\tilde{S}), \\
  u_t|_{t=0} &= \phi \in C^\infty_0 (S); & \tilde{u}_t|_{t=0} &= \tilde{\phi} \in C^\infty_0 (\tilde{S}).
\end{align*}
\]

Then these data determine \( u, \tilde{u} \) on \( S \times \mathbb{R}_+, \tilde{S} \times \mathbb{R}_+ \), correspondingly. In particular, if \( S \) and \( \tilde{S} \) are isometric, with isometry \( X \) satisfying (23), there is an extended isometry \( X_r \),

\[
X_r : S_r \to \tilde{S}_r
\]

with

\[
\varphi_k|_{S_r} = X_r^* \tilde{\varphi}_k|_{\tilde{S}_r}, \quad u^j|_{S_r \times \mathbb{R}_+} = X_r^* \tilde{u}^j|_{\tilde{S}_r \times \mathbb{R}_+},
\]

when \( f = X^* \tilde{f}, \phi = X^* \tilde{\phi}, F = X^* \tilde{F} \).

### 4.2

Based on Theorem 4.2, we will finish the proof of Lemma 4.1. Assume, that a maximal element (\( \Omega_m, \tilde{\Omega}_m \)) \( \neq (\Omega_{\text{int}}, \tilde{\Omega}_{\text{int}}) \), where without loss of generality we can take \( \Omega_m \neq \Omega_{\text{int}} \). Therefore, there is a point \( p \in \mathcal{C}l(\Omega_m) \cap \Omega_{\text{int}} \). Observe, that as \( \tilde{X}_m : \Omega_m \to \tilde{\Omega}_m \) is an isometry, it may be extended to the mapping \( \tilde{X}_m : \mathcal{C}l(\Omega_m) \to \mathcal{C}l(\tilde{\Omega}_m) \). Consider the following possible scenarios:

1. \( \bar{p} = \tilde{X}_m(p) \in \tilde{\Omega}_{\text{int}} \). Denote by \( \delta = \min \{d(p, \partial \Omega), d(\bar{p}, \partial \tilde{\Omega})\} \) and by \( \rho = \min \{i(\Omega, \rho), i(\tilde{\Omega}, \rho)\} \), where \( i(N, h) \) stands for the injectivity radius of the normal coordinates of Riemannian manifold \( (N, h) \). Let \( \delta_0 = \frac{1}{4} \min(\delta, \rho) \) and \( p_0 \in \Omega_m, \bar{p}_0 = \tilde{X}_m(p_0) \in \tilde{\Omega}_m \) satisfy \( d(\bar{p}_0, \bar{p}) < \delta_0 \). Let \( 0 \leq \sigma < \delta_0 \) satisfies \( B(p_0, \sigma) \subset \Omega_m, B(\bar{p}_0, \sigma) = \tilde{X}_m(B(p_0, \sigma)) \subset \tilde{\Omega}_m \), where \( B(p, r) \) is a closed ball of radius \( r \) centered at \( p \). Taking \( S, \tilde{S} \) in Theorem 4.2 to be \( B(p_0, \sigma), B(\bar{p}_0, \sigma) \), we see that conditions of this
Theorem are satisfied for \( \tau = 2\delta_0 \) with \( S_\sigma = B(p_0, \sigma + 2\delta_0), \tilde{S}_\sigma = B(\tilde{p}_0, \sigma + 2\delta_0) \). Therefore, \( \tilde{X}_m \) can be extended to \( \Omega_m \cup B(p_0, \sigma + 2\delta_0) \) containing \( p \), which contradicts the definition of \( \Omega_m \).

2. \( \tilde{p} = \tilde{X}_m(p) \in \tilde{\gamma}^{int} \), where \( \tilde{\gamma} \) is some \((n - 1)\)-subsimplex of \( \tilde{\Omega} \). Let now \( \delta = \min(d(p, \partial \Omega)), d(\tilde{p}, \partial(\tilde{\Omega} \cup \tilde{\Omega}_1)) \), where \( \tilde{\Omega}_1 \) is another \( n \)-simplex adjacent to \( \tilde{\gamma} \) (if \( \tilde{\gamma} \in \partial(\tilde{M} \cup \tilde{\Omega}) \) we take \( \partial(\tilde{M} \cup \tilde{\Omega}) \)). As earlier, let \( \delta_0 = \frac{1}{4} \min(\delta, \rho, i_\tilde{\gamma}(\tilde{p})) \), where \( i_\tilde{\gamma}(\tilde{p}) \) is the radius of injectivity of the interface normal coordinates related to \( \tilde{p} \), (or to the boundary normal coordinates of \( \tilde{p} \in \partial \tilde{M} \)). Introduce \( p_0, \tilde{p}_0 = \tilde{X}_m(p_0) \) as in the case 1 and take balls \( B(p_0, \sigma), B(\tilde{p}_0, \sigma), 0 < \sigma < \delta_0 \) similar to the case 1. Consider now the non-stationary Gaussian beams \( \tilde{U}_\varepsilon^{in,L} \) on \( \tilde{M} \) which start at \( t = 0 \) at \( \tilde{p}_0 \), in direction close to the normal direction from \( \tilde{p}_0 \) to \( \tilde{\gamma} \), i.e. with the initial co-vector \( \tilde{\xi} \) close to \((0, 0, ..., 1)\). These Gaussian beams reflect from \( \tilde{\gamma} \) and return to \( B(\tilde{p}_0, \sigma) \) approximately at the time \( t = 2 d(\tilde{p}_0, \tilde{\gamma}) - \sigma \). Thus, for \( \tilde{\xi} \) close to \((0, ..., 0, 1)\), the total Gaussian beam \( \tilde{U}_\varepsilon^{L} = \tilde{U}_\varepsilon^{r,L} + \tilde{U}_\varepsilon^{in,L} \) in \( \tilde{\Omega} \) satisfies

\[
\max |\tilde{U}_\varepsilon^{L}(\tilde{q}, t)| > C,
\tag{26}
\]

when \( \tilde{q} \in B(\tilde{p}_0, \sigma), t \in [2d(\tilde{p}_0, \tilde{\gamma}) - \sigma, 2 d(\tilde{p}_0, \tilde{\gamma}) + \sigma] \).

On the other hand, the corresponding Gaussian beam \( U_\varepsilon^{in,L} \) in \( \Omega \) moves from \( B(p_0, \sigma) \) and has no reflected part for \( t \in (0, 2 d(\tilde{p}_0, \tilde{\gamma}) + \sigma) \), so that

\[
\max |U_\varepsilon^{L}(q, t)| \leq C \varepsilon^{-\mu(L)}, \quad \text{for } q \in B(p_0, \sigma).
\tag{27}
\]

When \( \varepsilon \) is sufficiently small and \( L \) is sufficiently large, (26), (27) contradict the second equation (25), of Theorem 4.2. As \( \varphi_k|_{B(p_0, \sigma)} \) and \( \tilde{\varphi}_k|_{B(\tilde{p}_0, \sigma)} \) are known we can evaluate the Fourier coefficients \( u_k(t) = \tilde{u}_k(t) \) of \( U_\varepsilon^{L}(t) \) and \( \tilde{U}_\varepsilon^{L}(t) \) and, next,

\[
U_\varepsilon^{L}(q, t) = \sum u_k(t) \varphi_k(q); \quad \tilde{U}_\varepsilon^{L}(\tilde{q}, t) = \sum u_k(t) \tilde{\varphi}_k(\tilde{q}),
\]

for \( q \in B(p_0, \sigma), \tilde{q}_0 \in B(\tilde{p}_0, \sigma) \).

3. Let now \( \tilde{p} \in \tilde{M}^{n-2} \). Using e.g. baricentric coordinates in \( \tilde{\Omega} \), we see that there are \( C > 1 \) and \( \delta > 0 \) such that if \( d(\tilde{p}_0, \partial \tilde{\Omega}) < \delta \), then there is a curve \( \tilde{x} : [0, 1] \rightarrow \tilde{\Omega}^{int}, \tilde{x}(0) = \tilde{p}_0, \tilde{x}(1) = \tilde{q}_0 \) such that:
\(d(\tilde{x}(t), \partial \tilde{\Omega}) > C^{-1}d(\tilde{p}_0, \partial \tilde{\Omega});\)

(ii) \(|\tilde{x}|[0, 1]| \leq C\,d(p_0, \partial \tilde{\Omega})\), where \(|\tilde{x}|[a, b]|\) is the arclength of \(\tilde{x}(t)\) between \(\tilde{x}(a)\) and \(\tilde{x}(b)\);

(iii) \(d(\tilde{q}_0, \partial \tilde{\Omega}) = d(\tilde{p}_0, \partial \tilde{\Omega})\). However, we can assume that there is a unique nearest point \(\tilde{q} \in \partial \tilde{\Omega}\) to \(\tilde{q}_0\) and, in addition,

(iv) \(\tilde{q}_0 \in \tilde{\gamma}^{int}\), where \(\tilde{\gamma}\) is an \((n-1)\)-interface between \(\tilde{\Omega}\) and some \(\tilde{\Omega}_1\) (or \(\tilde{\gamma} \subset \partial \tilde{M}\));

(v) Interface (boundary) normal coordinates centered at \(\tilde{q}_0\) are regular in \(4\,d(\tilde{q}_0, \tilde{q})\)-vicinity of \(\tilde{q}_0\).

Returning to the consideration of the case \(\tilde{p} \in \tilde{M}^{n-2}\), let \(p_0 \in \Omega_m\) satisfy \(d(p_0, p) \leq \min(\tilde{\delta}, \frac{1}{\infty} \delta)\), where \(\delta\) is defined as in the case 1. With \(\tilde{p}_0 = \tilde{X}_m(p_0)\), let \(\tilde{x}(t)\) be a curve in \(\tilde{\Omega}^{int}\) described earlier. Then the diffeomorphism \(\tilde{X}_m^{-1}\) can be extended onto some open neighborhood of \(\tilde{x}(t)\). Indeed, by the construction of the step 1, we can move recurrently along \(\tilde{x}(t)\), using the balls of radius \(\frac{1}{\infty} d(\tilde{p}_0, \partial \tilde{\Omega})\). By the maximality of \((\Omega_m, \tilde{\Omega}_m), \tilde{q}_0 \in \tilde{\Omega}_m\) with \(q_0 = \tilde{X}_m^{-1}(\tilde{q}_0)\) \(\in \Omega_m\) satisfying

\[
d(q_0, \partial \Omega) \geq 4\,d(\tilde{q}_0, \tilde{q}).
\] (28)

Inequality (28) makes it possible to use the same considerations as in the case 2, proving that the case \(\tilde{p} \in \tilde{M}^{n-2}\) is possible.

We finish the section by the following

**Corollary 4.3** Let \(X : \Omega \to \tilde{\Omega}\), where \(\Omega, \tilde{\Omega}\) are \(n\)-simplexes of \(\mathcal{M}\) and \(\tilde{\mathcal{M}}\), satisfy (22). If \(\gamma, \tilde{\gamma} = X(\gamma)\) are \((n-1)\)-simplexes of \(\Omega\) and \(\tilde{\Omega}\) correspondingly. Then \(\gamma\) is an interface if and only if \(\tilde{\gamma}\) is an interface.

**Proof** Let \(\tilde{f} = f^L_\varepsilon, \tilde{\phi} = \phi^L_\varepsilon\) be the initial data for Gaussian beam \(\tilde{U}^{in,L}_\varepsilon\) starting in \(\tilde{\Omega}\) towards \(\tilde{\gamma}\) and \((f^L_\varepsilon, \phi^L_\varepsilon) = X^* (f^L_\varepsilon, \phi^L_\varepsilon)\) be the initial data of the Gaussian beam \(U^{in,L}_\varepsilon\). Comparing \(\tilde{U}^{r,L}_\varepsilon\) and \(U^{r,L}_\varepsilon = X^* \tilde{U}^{r,L}_\varepsilon\) and using formulae (15) and (20), we obtain the desired result. \(\square\)
5 Global Isometry

5.1

In this section we show that isometry on $X : \Omega \to \tilde{\Omega}$ which satisfies (22) can be extended to a global isometry $X : \mathcal{M} \to \tilde{\mathcal{M}}$ satisfying

$$\varphi_k(p) = X^* \tilde{\varphi}_k(\tilde{p}), \quad k = 1, 2, \ldots, \ p \in \mathcal{M}^\text{int}, \ \tilde{p} = X(p) \in \tilde{\mathcal{M}}^\text{int}, \quad (29)$$

$$d(p, q) = d(\tilde{p}, \tilde{q}), \ \tilde{p} = X(p), \ \tilde{q} = X(q), \ p, q \in \mathcal{M}. \quad (30)$$

We start with the following result which is a partial generalization of Theorem 4.2.

**Lemma 5.1** Let $\gamma$ be the interface between $\Omega_-$ and $\Omega_+$ in $\mathcal{M}$ and $\tilde{\gamma} = X_- (\gamma)$ is the interface between $\tilde{\Omega}_-$ and $\tilde{\Omega}_+$ in $\tilde{\mathcal{M}}$. Assume that $X_- : \Omega_- \to \tilde{\Omega}_-$ is an isometry satisfying (22). Then $X$ can be extended to an isometry $X : \Omega_- \cup \Omega_+ \to \tilde{\Omega}_- \cup \tilde{\Omega}_+$ which satisfies (22) on $\Omega_-^\text{int} \cup \Omega_+^\text{int}$.

**Proof** Let $D, \tilde{D}$ be smooth subdomains in $\Omega_-^\text{int}$ and $\tilde{\Omega}_+^\text{int}$ with the "upper" part of the boundary parallel to $\gamma, \tilde{\gamma}$, i.e. the parts of $\partial D, \partial \tilde{D}$ are given by $x = (s, -\sigma_0), \ \tilde{p} = (\tilde{s}, -\tilde{\sigma}_0), \ s \in \Gamma \in \gamma, \ \tilde{s} \in \tilde{\Gamma}_0 \in \tilde{\gamma}$. Assume, without loss of generality that $d(D, \partial(\Omega_- \cup \Omega_+)), d(\tilde{D}, \partial(\tilde{\Omega}_- \cup \tilde{\Omega}_+)) > 10\sigma_0, \ \sigma_0 < \frac{1}{8}(\min(i_{\Gamma}, i_{\tilde{\Gamma}}))$, where $i_\Gamma, i_{\tilde{\Gamma}}$ are the injectivity radii of interface coordinates related to $\Gamma, \tilde{\Gamma}$, correspondingly. We want to show that equation (22) implies that

$$\varphi_k(s, \sigma) = \tilde{\varphi}_k(\tilde{s}, \tilde{\sigma}), \ \text{for} \ s \in \Gamma, \ \tilde{s} = X_-(\gamma)(s), \ -2\sigma_0 < \sigma < 2\sigma_0. \quad (30)$$

We have the following rather straightforward generalization of Tataru’s approximate controllability:

$$Cl_{L^2(\mathcal{M})}\{u^f(t_0), f \in C_0^\infty(D \times (0, t_0))\} = L^2(U_{t_0}(D)), \ 0 \leq t_0 \leq 4\sigma_0, \quad (31)$$

with a similar identity for $\tilde{D}$. Here $U_{t_0}(D)$ is a $t_0$–neighborhood of $D$ in $\mathcal{M}$ and $u^f$ is a solution to the initial boundary value problem

$$u^f_{tt} - \Delta u^f = f, \ u^f|_{t=0} = 0, \ u^f|_{\partial\mathcal{M} \times \mathbb{R}_+} = 0. \quad (32)$$

As usual this result follows immediately from observability, see e.g. Section 2.5 of [21]. To formulate the desired observability, let

$$\begin{cases}
    v_{tt} - \Delta v = 0, & \text{in} \ \mathcal{M} \times (-t_0, 0), \ v|_{\partial\mathcal{M} \times (-t_0, 0)} = 0, \\
    v|_{D \times (-t_0, 0)} = 0,
\end{cases}$$
where $0 < t_0 < 4\sigma_0$. Then $v = 0$ in double cone

$$K(D, t_0) = \{(p, t) \in \mathcal{M} \times (-t_0, t_0) : d(p, D) < t_0 - |t|\}.$$ 

To use this we first observe that for any $q$ with $d(p, D) < 4\sigma_0$ and any $\delta > 0$ there is a piece-wise smooth curve $x : [0, 1] \to \Omega_- \cup \Omega_+$ with $x(0) = p \in D$, $x(1) = q$ and $|x[0, 1]| < d(q, D) + \delta$. Moreover, $x(t)$ may be chosen to cross $\gamma$ transversally. Then we can continue $v$ by 0 along this curve $x(t)$ so that for $x(s)$

$$v = 0 \quad \text{in} \quad V_\varepsilon \times ((-t_0 - |x[0, s]|, t_0 - |x(0, s)|),$$

where $V_\varepsilon$ is a small vicinity of $x(s)$. Indeed, this is obvious for pieces of the path $x(s)$ lying inside either $\Omega_-^{\text{int}}$ or $\Omega_+^{\text{int}}$. To cross $\gamma$ we just observe that, if $v, \partial_x v = 0$ on $\Sigma \times (-\hat{t}, \hat{t})$, $\Sigma \in \gamma$, when approaching $\gamma$ from $\Omega_+$, then by (8),

$$v, \partial_x v = 0, \quad \text{on} \quad \Sigma \times (-\hat{t}, \hat{t}),$$

when approaching $\gamma$ from $\Omega_-$, and, therefore, $v$ can be continued by 0 further along $x(t)$ into $\Omega_+$ and vice versa from $\Omega_+$ into $\Omega_-$. Clearly, to use (5) we should assume $v$ to be sufficiently regular, however, by smoothing $v$ with respect to time $t$, we can extend it to non-smooth solutions, see e.g. [22].

This implies that $v(q, t) = 0$ for $|t| < t_0 - d(q, D) - \delta$. As $\delta > 0$ is arbitrary, we obtain that $v = 0$ in $K(D, t_0)$.

Identity (31) makes it possible, starting from $\{\lambda_k, \varphi_k|_D\}_{k=1}^{\infty}$, and from $\{\bar{\lambda}_k, \bar{\varphi}_k|_D\}_{k=1}^{\infty}$, such that $\lambda_k = \bar{\lambda}_k$ and $\varphi_k(s, \sigma) = \bar{\varphi}_k(s, \sigma)$, for $-2\sigma_0 < \sigma < -\sigma_0$, $\bar{s} = (X_-^\ast)(s)$, $s \in \Gamma$ to construct $\varphi_k(s, \sigma)$ and $\bar{\varphi}_k(s, \sigma)$ for $2\sigma_0 < \sigma < 2\sigma$, $\bar{s} = (X_-^\ast)(s)$, $s \in \Gamma$, see [20], Chapter 4.4 of [21]. In particular, the construction in [20], [21] imply that $\bar{\varphi}_k(X_-(\gamma)(s), \sigma) = \varphi_k(s, \sigma)$. Observe now that $\Gamma \times [0, 2\sigma_0], \Gamma \times [0, 2\sigma_0]$ from a relatively open subdomains in $\Omega_+, \Omega_+$, respectively with $X_+ : (s, \sigma) \to ((X_+|_\gamma)(s), \sigma), 0 \leq \sigma \leq 2\sigma$, being a diffeomorphism satisfying (23).

Mimicking the proof of Lemma 4.1 in section 4.2 we extend $X_+$ to be a diffeomorphism, $X_+ : \Omega_+ \to \bar{\Omega}_+$, satisfying (22). As, by construction, $X_+|_\gamma = X_-|_\gamma$, $X$ defined as $X_-$ on $\Omega_-$ and $X_+$ on $\Omega_+$ is a desired isometry between $\Omega_- \cup \Omega_+ \cup \bar{\Omega}_+$.

### 5.2 Identification of n-simplices

We have proven that, for any chain of $n$-simplices $\Omega_1 = \Omega_2(1), \ldots, \Omega_2(n)$ in $\mathcal{M}$ which are pairwise adjoint, i.e. there is an $(n - 1)$-interface $\gamma^{(k)}$ be-
5 GLOBAL ISOMETRY

tween \( \Omega_{i(k)} \) and \( \Omega_{i(k+1)} \), \( k = 1, \ldots, m - 1 \), there is a chain of \( n \)-simplexes \( \tilde{\Omega}_1 = \tilde{\Omega}_{i(1)}, \ldots, \tilde{\Omega}_{i(m)} \) in \( \tilde{\mathcal{M}} \), such that \( \Omega_{i(k)} \) and \( \tilde{\Omega}_{i(k)} \) are diffeomorphic with diffeomorphism \( X_{i(k)} \) satisfying (22) (strictly speaking \( X_{i(k)} \) may depend on a chain from \( \Omega_1 \) to \( \Omega_{i(k)} \)), see Fig. 3. Let us show that

(a) For any \( n \)-simplex \( \Omega_j \subset \mathcal{M} \) there is an \( n \)-simplex \( \tilde{\Omega}_j \subset \tilde{\mathcal{M}} \) which is diffeomorphic to \( \Omega_j \) with diffeomorphism \( X_j \) satisfying (22) and, likewise, for any \( \tilde{\Omega}_j \subset \tilde{\mathcal{M}} \) there is \( \Omega_j \subset \mathcal{M} \) with described properties;

(b) If \( \Omega_j \) and \( \tilde{\Omega}_j \) are diffeomorphic with diffeomorphisms \( X_j \) and \( X_j' \) satisfying (22) then \( X_j = X_j' \);

(c) If \( \Omega_j \) is diffeomorphic to \( \tilde{\Omega}_j \) and \( \tilde{\Omega}_j' \) by \( X_j \) and \( X_j' \) satisfying (22), then \( \tilde{\Omega}_j = \tilde{\Omega}_j' \)

Let now \( \Omega_{j(1)}, \tilde{\Omega}_{j(1)} \) be \( \Omega, \tilde{\Omega} \) are the \( n \)-simplexes with \( \gamma, \tilde{\gamma} \) used in Lemma 4.1 being their boundary \( (n-1) \)-subsimplexes. Then, by Lemma 4.1 there is \( X_1 : \Omega_{j(1)} \to \tilde{\Omega}_{j(1)} \) satisfying (22). Denote by \( \gamma_1 \) the \( (n-1) \)-interface between \( \Omega_{j(1)} \) and \( \Omega_{j(2)} \). By Corollary 4.3 there is \( \tilde{\Omega}_{j(2)} \) adjacent to \( \tilde{\Omega}_{j(1)} \) with the \( (n-1) \)-interface \( \tilde{\gamma}_1 = \tilde{\Omega}_{j(1)} \cap \tilde{\Omega}_{j(2)} \) such that \( \tilde{\gamma}_1 = X_1(\gamma_1) \). By Lemma 5.1 there is an isometry \( X_2 : \Omega_{j(2)} \to \tilde{\Omega}_{j(2)} \) with \( X_1|_{\gamma_1} = X_2|_{\gamma_1} \).
Continuing this process, we obtain an isometry \( X_m : \Omega_{j(m)} = \Omega_j, \tilde{\Omega}_{j(m)} := \tilde{\Omega}_j \) which satisfies (22).

To prove (b), let \( p \in \Omega_j^{int} \), and \( \tilde{p} \in X_j(p), \tilde{p}' = X'_j(p) \in \tilde{\Omega}_j^{int} \). As \( X_j, X'_j \) satisfy (22), \( \tilde{\varphi}_k(\tilde{p}) = \tilde{\varphi}_k(\tilde{p}') \), \( k = 1, 2, \ldots \). By Proposition 2.3, \( \tilde{p} = \tilde{p}' \). As \( p \in \Omega_j^{int} \) is arbitrary, \( X_j = X'_j \) on \( \Omega_j \). Using the fact that \( \{\tilde{\varphi}_k\}_{k=1}^\infty \) distinguish points in \( \tilde{M}^{int} \), see Proposition 2.3, we prove property (c).

Based on properties (a)-(c) we show the following result.

**Lemma 5.2** Let LBSD for the Riemannian polyhedron \((\mathcal{M}, g)\) and \((\tilde{\mathcal{M}}, \tilde{g})\) be equivalent. Then

1. For any \( \Omega_i \) in \( \mathcal{M} \) there is a unique diffeomorphism \( X_i : \Omega_i^{int} \to \tilde{\Omega}_i^{int} \), which satisfies (22).

2. \( \gamma \) is an \((n-1)\)-interface between \( \Omega_- \) and \( \Omega_+ \) if and only if the corresponding \( \tilde{\gamma} \) is an \((n-1)\)-interface between \( \tilde{\Omega}_- \) and \( \tilde{\Omega}_+ \). In this case \( X_-|_{\gamma} = X_+|_{\gamma} \),

   where \( X_- \) is the closures of the described above diffeomorphisms \( X_\pm \) on \( \Omega_\pm^{int} \).

3. The diffeomorphisms \( \cup X \) can be uniquely extended to an isometry \( X : \mathcal{M}^{reg} \to \tilde{\mathcal{M}}^{reg} \).

   Here \( X \) is an isometry of \( \mathcal{M}^{reg} \) and \( \tilde{\mathcal{M}}^{reg} \) considered as metric space with the distance function given in Definition 2.1.

**Proof** By (a)-(c) it remains to prove the part of this lemma dealing with \((n-1)\)-simplices. Let \( \gamma \) be an interface between \( \Omega_- \) and \( \Omega_+ \) with \( \tilde{\Omega}_- \) and \( \tilde{\Omega}_+ \) being the corresponding \( n \)-simplices in \( \tilde{\mathcal{M}} \). Crossing \( \gamma \) we move from \( \tilde{\Omega}_- \) to \( \tilde{\Omega}_+ \) in \( \tilde{\mathcal{M}} \) so that, by the previous constructions, \( \tilde{\Omega}_+^{int} \) is a diffeomorphic to \( \Omega_+ \) with diffeomorphism satisfying (22). But also \( \tilde{\Omega}_- \) and \( \tilde{\Omega}_+ \) are diffeomorphic with diffeomorphism satisfying (22). Thus \( \tilde{\Omega}_+ = \tilde{\Omega}_+ \) and the diffeomorphisms \( X_\pm : \Omega_\pm^{int} \to \tilde{\Omega}_\pm^{int} \) are uniquely extendable to an isometry \( X_-+_\pm : \Omega_-^{int} \cup \Omega_+^{int} \cup \tilde{\gamma} \to \tilde{\Omega}_-^{int} \cup \tilde{\Omega}_+^{int} \cup \tilde{\gamma} \), where \( \tilde{\gamma} = X(\gamma) \). As the distance functions in \( \mathcal{M}^{reg} \), \( \tilde{\mathcal{M}}^{reg} \) employ only curves being in \( \mathcal{M}\backslash\mathcal{M}^{n-2}, \tilde{\mathcal{M}}\backslash\tilde{\mathcal{M}}^{n-2} \), the above results concerning the local isometries yield the desired global isometry.
Theorem 1.3 immediately follows from Lemma 5.2 taking into account the Definition 2.1 of an admissible Riemannian polyhedron. Note that this definition implies that the topology of a Riemannian polyhedron considered as a metric space is the same as the topology of the underlying polyhedron.

6 Further generalizations and estimates

6.1

Condition 2.2 that the metric tensor does have a jump singularity at every point of any $(n-1)$-dimensional interface, i.e. Condition 2.2 may be too restrictive. In this section we relax it a bit. namely, we assume that it

- either the metric tensor $g$ does have a jump singularity at any point $p \in \gamma^{\text{int}}$, for an $(n-1)$-interface $\gamma$, between $\Omega_-$ and $\Omega_+$,
- or $g$ is smooth across $\gamma$ at any $p \in \gamma^{\text{int}}$.

The latter condition means, that in the interface coordinates related to $\gamma$, see section 1.2, the metric tensor $g_{\alpha\beta}(s, \sigma)$ is smooth at $\sigma = 0$. In this case we call $\gamma$ an artificial interface and would like to treat $\Omega_- \cup \Omega_+$ together, introducing $\Omega_-^{\text{int}} \cup \gamma^{\text{int}} \cup \Omega_+^{\text{int}}$. Further removing artificial interfaces and taking into account that Riemannian polyhedron consists of a finite number of $n$-simplexes, it is natural to introduce the following object.

**Definition 6.1** A chamber $\Omega$ of a Riemannian polyhedron $(\mathcal{M}, g)$ is the maximal union of open $n$-simplexes together with open artificial interfaces adjacent to them.

It is natural to treat each chamber $\Omega$ as an open Riemannian submanifold of $\mathcal{M}$ with a piece-wise smooth boundary. However, due to [24], there may be topological obstruction to that. Namely, it may happen that, for $p \in \mathcal{M}^{n-2}$, there is no open neighborhood of $p$ (in $\mathcal{M}$) which is homeomorphic to $\mathbb{R}^n$. In this connection we introduce

**Definition 6.2** An admissible Riemannian polyhedron $(\mathcal{M}, g)$ is called weakly admissible if any of its open chamber $\Omega^{\text{int}}$ is an open $n$-dimensional Riemannian manifold with piece-wise smooth boundary and, if $(n-1)$-subsimplex $\gamma$ of $\mathcal{M}$ is not artificial, they either $\gamma \in \partial \mathcal{M}$ or the metric $g$ has a jump singularity across $\gamma$. 
Then, similar to Theorem 1.3 we can prove

**Theorem 6.3** Let \((\mathcal{M}, g)\) and \((\tilde{\mathcal{M}}, \tilde{g})\) be weakly admissible Riemannian polyhedra. Let local boundary spectral data corresponding to \((\mathcal{M}, g)\) and \((\tilde{\mathcal{M}}, \tilde{g})\) are equivalent. Then there is an isometry \(X : \mathcal{M} \to \tilde{\mathcal{M}}\) such that for any open chamber \(\Omega^\text{int}_i \subset \mathcal{M}\), \(X|_{\Omega^\text{int}_i}\) is a diffeomorphism onto an open chamber \(\tilde{\Omega}^\text{int}_i \subset \tilde{\mathcal{M}}\), which satisfy equation (22).

Local boundary spectral data in this case consist of a “smooth” open subset \(\Gamma_0 \subset \partial \mathcal{M}\), such that there is an open neighborhood \(U \subset \mathcal{M}\), of \(\Gamma_0\), which is diffeomorphic to a half-ball \(\{x \in B(0, r) : x_n \geq 0\}\) with \(\Gamma_0\) being an open subset of \(B(0, r) \cap \{x_n = 0\}\), and eigenpairs \(\{\lambda_k, \partial_\nu \varphi_k|_{\Gamma_0}\}_{k=1}^\infty\).

### 6.2

Similar to the smooth case, the uniqueness Theorem 1.3 remains valid when, instead of the boundary spectral data, we have a local non-stationary Dirichlet-to-Neumann maps,

\[ \Lambda_{T_0}^\Gamma : \dot{C}^\infty(\Gamma_0 \times (0, T)) \to \dot{C}^\infty(\Gamma_0 \times (0, T)), \]

where \(\dot{C}^\infty(\Gamma_0 \times (0, T))\) consists of smooth functions which are equal to 0 near \(t = 0\), \(\Lambda_{T_0}^\Gamma\) is then defined as

\[ \Lambda_{T_0}^\Gamma : f \to \partial_\nu u^f|_{\Gamma_0 \times (0,T)}, \]

where \(u^f\) is the solution to the inverse boundary value problem

\[ u^f_{tt} - \Delta u^f = 0, \quad u^f|_{t<0} = 0, \quad u^f|_{\partial \mathcal{M} \times (0,T)} = f. \]

Condition 2.2 may be replaced by a condition that the metric tensor \(g\) is not smooth across any \((n - 1)-\)interface \(\gamma\) at any point \(p \in \gamma^\text{int}\). This condition means that, for any \(p \in \gamma^\text{int}\) and any interface conditions with \(s = 0\) corresponding to \(p\), the metric tensor \(g_{\alpha\beta}(0, \sigma)\) or some its derivatives \(\partial^k_\sigma g_{\alpha\beta}(0, \sigma)\) has a jump at \(\sigma = 0\). Then Theorem 1.3 remains valid, if for regions of \(\gamma\) where \(g_{\alpha\beta}(s, \sigma)\) is continuous across \(\sigma = 0\) together with its first \((k - 1)-\)derivatives with respect to \(\sigma\), but \(\partial^k_\sigma g_{\alpha\beta}(0, \sigma)\), does have a jump across \(\sigma = 0\), the reflected Gaussian beam \(U_{r}^\varepsilon\) is of order \(k\). Namely, in representation (3) for \(U_{r}^\varepsilon(t, y)\) the first non-zero \(u_1^\varepsilon(t, y)\) is \(u_k(t, y)\). This, however, does not alter considerations of section 5.1 with the only difference that in section 5.1 the estimate for the reflected Gaussian beam in (26) is changed to \(|\tilde{U}_{r}^{\varepsilon,L}(\tilde{q}, t)| \geq C\varepsilon^k\).
6.3 Open problems

(i) This paper deals with uniqueness in the inverse boundary spectral problem for Riemannian polyhedron. It does not provide an algorithm to its reconstruction. In particular, trying to apply the technique of Section 4 \[21\], when approaching $M^{n-2}$ the size of a step of the reconstruction procedure tends to zero. Therefore, it is impossible to reconstruct $(M, g)$ by a finite number of steps.

(ii) Even with generalization described in sections 5.1, 5.2, the class of Riemannian Polyhedron considered in this paper does not cover an important case when the metric $g$ is smooth across some part of the interface $\gamma$ but is not smooth across other part of $\gamma$, i.e. we have "holes" in interfaces. We intend to study such cases in the forthcoming paper.

(iii) Another important open question is the one of stability which could relate observation error with, in addition to curvature, injectivity radii, etc., \[4\], size of $n$–simplices, value of jumps, etc.

7 Acknowledgements

The authors would like to express their gratitude to Prof. M. Lassas for numerous stimulating discussions regarding the various aspects of the problem, Prof. Yu. Burago and Dr. N. Kossovski for consultations on the geometric issues and Prof. A. Kachalov for the useful discussions on the asymptotical aspects of non-stationary Gaussian beams.

The research of the first author was financially supported EPSRC grant Ep/D065771/1.

References

[1] Alessandrini, G., Morassi, A., Rosset, E. Detecting cavities by electrostatic boundary measurements, *Inv. Probl.*, 18 (2002), 1333–1353.

[2] K. Astala, M. Lassas, L. Päivärinta, Calderon’s inverse problem for anisotropic conductivity in the plane, Communications in Partial Differential Equations 30(2005), no. 1-3, 207-224.
REFERENCES

[3] Astala, K., Päivärinta, L., Lassas, M. Caldern’s inverse problem for anisotropic conductivity in the plane, Comm. Part. Diff. Eq., 30 (2005), 207–224.

[4] M. Anderson, A. Katsuda, Ya. Kurylev, M. Lassas, M. Taylor, Boundary Regularity for the Riccati equation, Geometry Convergence and Gelfand’s Inverse Boundary Problem, Inventiones Mathematicae 158(2004), 261-321.

[5] V. Babich, V. Ulin, The complex space-time ray method and ”quasiphotoons”, (Russian) Zap. Nauchn. Sem. LOMI, 117(1981), 5-12.

[6] W. Ballman, A volume estimate for piecewise smooth metrics on simplicial complexes, Rendiconti Sem. Mat. Fis. Milano 66(1996), 323-331.

[7] M. Belishev, An approach to multidimensional inverse problems for the wave equation, (Russian) Dokl. Akad. Nauk SSSR 297(1987), no.3, 524-527; translated in Soviet Math. Dokl. 36(1988),no.3 481-484.

[8] M. Belishev, Wave basis in multidimensional inverse problems, (Russian) Mat. Sb. 180(1989), 584-602.

[9] M. Belishev, Ya. Kurylev, To the construction of a Riemannian manifold via its spectral data (BC-method), Communications in Partial Differential Equations 17(1992), no.5-6, 591-594.

[10] J. Eells, B. Fuglede, Harmonic maps between Riemannian polyhedra. With a preface of Gromov M., Cambridge Tracts in Mathematics, 142, Cambridge University Press, Cambridge, 2001.

[11] Greenleaf, A., Uhlmann, G. Recovering singularities of a potential from singularities of scattering data, Comm. Math. Phys., 157 (1993), 549–572.

[12] H. Federer, Geometric Measure Theory, Springer-Verlag, Berlin, Heidelberg, New York, 1996.

[13] B. Fuglede, Finite energy maps from Riemannian polyhedra to metric spaces, Ann.Acad.Sci.Fenn.Math., 28(2003), no.2, 433-458.

[14] Ikehata, M. Reconstruction of inclusion from boundary measurements, J. Inverse Ill-Posed Probl., 10 (2002), 37–65.
REFERENCES

[15] Ikehata, M., Siltanen, S. Electrical impedance tomography and Mittag-Leffler’s function, Inv. Prob., 20 (2004), 1325–1348.

[16] Isakov V., On uniqueness of recovery of a discontinuous conductivity coefficient, Comm.Pure Appl.Math., 41(1988), no.7, 865-877.

[17] Isakov V. Inverse Problems for Partial Differential Equations. Springer, New York, 2006, 344 pp.

[18] Isozaki, H. Inverse scattering theory for Dirac operators. Ann. Inst. H. Poincare Phys. Theor., 66 (1997), 237-270.

[19] A. Kachalov, Gaussian beams, Hamilton-Jacobi equations and Finsler geometry, Zapiski Nauchn. Semin.POMI, 297(2003), 66-92.

[20] Kachalov, A., Kurylev, Y. Multidimensional inverse problem with incomplete boundary spectral data. Comm. PDE, 23 (1998), 55-95.

[21] A. Kachalov, Ya. Kurylev, M. Lassas, Inverse Boundary Spectral Problems, Chapman Hall / CRC 123, 2001.

[22] A. Katchalov, Y. Kurylev, M. Lassas, Energy measurements and equivalence of boundary data for inverse problems on non-compact manifolds. IMA volumes in Mathematics and Applications (Springer Verlag) “Geometric Methods in Inverse Problems and PDE Control” Ed. C. Croke, I. Lasiecka, G. Uhlmann, M. Vogelius, (2004), 183-214.

[23] Kenig C., Sjoestrand J., Uhlmann G. The Calderón problem with partial data, preprint arXiv math.AP/0405486.

[24] M. Kervaire, A manifold which does not admit any differentiable structure, New York (USA), Commentarii mathematici Helvetici, 34(1960), 257-270.

[25] A. Kirpichnikova, Propagation of a Gaussian beam near an interface in an anisotropic medium, Zapiski Nauchn. Sem. POMI, 324(34)(2005), 77-109.

[26] A. Kirsch, L. Päivärinta, On recovering obstacles inside inhomogeneities, Math.Meth.Appl.Sci., 21(1998), 619-651.
REFERENCES

[27] Kurylev y., Lassas M., Somersalo E. Maxwell’s equations with a polarization independent wave velocity: Direct and inverse problems, *J. Mathem. Pures Appl.* 86 (2006), 237-270.

[28] Kurylev Y., Lassas M. Inverse problem for a Dirac-type equation on a vector bundle, arXiv math.AP/0501049.

[29] Lassas, M. Taylor, M., Uhlmann, G. The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary, *Comm. Anal. Geom.* 11 (2003), 207-22.

[30] Lee, J., Uhlmann, G. Determining anisotropic real-analytic conductivities by boundary measurements. *Comm. Pure Appl. Math.* 42 (1989), no. 8, 1097–1112.

[31] Nachman, A. Reconstructions from boundary measurements. *Ann. of Math.* (2) 128 (1988), no. 3, 531–576.

[32] Nachman, A. Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math.* (2) 143 (1996), no. 1, 71–96.

[33] Nakamura, G., Uhlmann, G. Global uniqueness for an inverse boundary value problem arising in elasticity, *Invent. Math.* 118 (1994), 457–474, 152 (2003), 205–207.

[34] Nakamura, G., Tsuchida, T. Uniqueness for an inverse boundary value problem for Dirac operators. *Comm. Part. Diff. Eq.* 25 (2000), 1327–1369.

[35] Novikov R. A multidimensional inverse spectral problem for the equation: \(\Delta \psi + (v(x) - Eu(x))\psi = 0\), Funkt. Anal. i ego Priloz. (in Russian), 22 (1988), 11-22.

[36] Ola, P., Päivärinta, L., Somersalo, E. An inverse boundary value problem in electrodynamics, *Duke Math. J.*, 70 (1993), 617–653.

[37] Ola, P., Somersalo, E. Electromagnetic inverse problems and generalized Sommerfeld potentials, *SIAM J. Appl. Math.*, 56 (1996), 1129–1145.

[38] Pestov, L., Uhlmann, G. Two dimensional compact simple Riemannian manifolds are boundary distance rigid. *Ann. of Math.*, 161 (2005), 1093-1110.
[39] M. Popov, *Ray Theory and Gaussian Beam Method for Geophysicists*, Edufba, Salvator-Bahia, 2002.

[40] J. Ralston, *Gaussian beams and propagation of singularities*, Studies in PDE, MAA Studies in Mathematics, 23, Walter Littman ed, 1983.

[41] Sylvester, J. An anisotropic inverse boundary value problem. *Comm. Pure Appl. Math.* 43 (1990), no. 2, 201–232

[42] Sylvester J., Uhlmann G. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math.*, 125 (1987), 153-169.

[43] M. Taylor, *Tools for PDE*, AMS, Providence, R.I., 2000.