Descent on toric fibrations

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Abstract

We describe descent on families of torsors of a constant torus. A recent result of Browning and Matthiesen then implies that the Brauer–Manin obstruction controls the Hasse principle and weak approximation when the ground field is \( \mathbb{Q} \) and the singular fibres are all defined over \( \mathbb{Q} \).

1 Introduction

Let \( T \) be a torus over a number field \( k \). Let \( X \) be a smooth, proper, geometrically integral variety with a surjective morphism \( f : X \to \mathbb{P}^1_k \) whose generic fibre \( X_{k(t)} \) is geometrically integral and is birationally equivalent to a \( k(t) \)-torsor of \( T \). The main result of this note says that the set \( X(k) \) is dense in \( X(\mathbb{A}_k) \) \( \text{Br} \) if certain auxiliary varieties satisfy the Hasse principle and weak approximation.

These varieties are given by explicit equations. Choose \( \mathbb{A}^1_k \subset \mathbb{P}^1_k \) so that the fibre of \( f \) at \( \infty = \mathbb{P}^1_k \setminus \mathbb{A}^1_k \) is smooth. Let \( P_1, \ldots, P_r \) be closed points of \( \mathbb{P}^1_k \) such that each fibre of the restriction of \( f \) to \( \mathbb{P}^1_k \setminus \{ P_1 \cup \ldots \cup P_r \} \) is split, i.e. contains an irreducible component of multiplicity 1 that is geometrically irreducible. By a well known result (Lemma 3.1) the fibre \( X_{P_i} \), for \( i = 1, \ldots, r \), has an irreducible component of multiplicity 1. We fix such a component in each \( X_{P_i} \) and define \( k_i \) as the algebraic closure of the residue field \( k(P_i) \) in the function field of this component.

Let \( i = 1, \ldots, r \) let \( p_i(t) \in k[t] \) be the monic irreducible polynomial such that \( P_i \) is the zero set of \( p_i(t) \) in \( \mathbb{A}^1_k \), and let \( a_i \) be the image of \( t \) in \( k(P_i) = k[t]/(p_i(t)) \). Let \( u, v \) be independent variables, and let \( z_i \) be a \( k_i \)-variable, for \( i = 1, \ldots, r \). For \( \alpha = \{ \alpha_i \} \), where \( \alpha_i \in k(P_i)^* \), we define the quasi-affine variety \( W_\alpha \subset \mathbb{A}^2_k \times \prod_{i=1}^r R_{k_i/k}(\mathbb{A}^1_{k_i}) \) by

\[
\alpha_i(u - a_i v) = N_{k_i/k(P_i)}(z_i), \quad (u, v) \neq (0, 0), \quad i = 1, \ldots, r,
\]

(1)

The varieties \( W_\alpha \) are smooth and geometrically irreducible. Over an algebraic closure of \( k \) such a variety is given by \( \sum_{i=1}^r [k(P_i): k] \) equations in \( 2 + \sum_{i=1}^r [k_i: k] \) variables. We can now state our main result.

**Theorem 1.1** Suppose that for each \( \alpha \in \prod_{i=1}^r k(P_i)^* \) the variety \( W_\alpha \) satisfies the Hasse principle and weak approximation. Then \( X(k) \) is dense in \( X(\mathbb{A}_k)^{\text{Br}} \).
The results of this kind are obtained by the descent method of Colliot-Thélène and Sansuc, and have a long history. When the relative dimension and the number of singular geometric fibres of \( f \) are small, geometric proofs of the Hasse principle and weak approximation for \( W_\alpha \) were obtained by Colliot-Thélène, Sansuc, Swinnerton-Dyer and others, see [7, 4, 19, 24, 9] and [21, Ch. 7]. The analytic tool in these proofs is Dirichlet’s theorem on primes in an arithmetic progression. Over \( k = \mathbb{Q} \) one can do more if one uses analytic methods: the circle method [16, 8], sieve methods [1, 10] and recent powerful results from additive combinatorics [11, 12, 3, 15, 22]. Note that the circle method can sometimes be applied over arbitrary number fields, see [23, 18].

When \( k = k(P_1) = \ldots = k(P_r) = \mathbb{Q} \), a recent result of Browning and Matthiesen obtained by methods of additive combinatorics [2, Thm 1.3] establishes the Hasse principle and weak approximation for \( W_\alpha \). Hence we deduce the following corollary of Theorem 1.1.

**Corollary 1.2** Let \( X \) be a smooth, proper, geometrically integral variety over \( \mathbb{Q} \), and let \( f : X \to \mathbb{P}_\mathbb{Q}^1 \) be a surjective morphism satisfying the following properties.

(a) There is a torus \( T \) over \( \mathbb{Q} \) such that the generic fibre \( X_{\mathbb{Q}(t)} \) of \( f \) is birationally equivalent to a \( \mathbb{Q}(t) \)-torsor of \( T \times_\mathbb{Q} \mathbb{Q}(t) \).

(b) There exists a finite subset \( E \subset \mathbb{P}^1_\mathbb{Q}(Q) \) such that \( X \setminus f^{-1}(E) \to \mathbb{P}^1_\mathbb{Q} \setminus E \) has split fibres.

Then \( X(\mathbb{Q}) \) is dense in \( X(\mathbb{A}_\mathbb{Q})^{Br} \).

This generalises a recent result due to A. Smeets, namely the unconditional counterpart of [22, Thm. 1.1, Rem. 1.3]. For a higher-dimensional version of this statement see Proposition 3.4.

For a number field \( k \) of degree \( n = [k : \mathbb{Q}] \) we write \( x = (x_1, \ldots, x_n) \) and denote by \( N_{k/\mathbb{Q}}(x) \) the norm form \( \text{Norm}_{k/\mathbb{Q}}(x_1\omega_1 + \ldots + x_n\omega_n) \), where \( \omega_1, \ldots, \omega_n \) is a basis of \( k \) as a vector space over \( \mathbb{Q} \). The following corollary extends [2, Thm. 1.1] to the case of a product of norm forms. It generalises the unconditional version of [22, Cor. 1.6], a number of statements from [15, Section 4] and the main result of [18] in the case of the ground field \( \mathbb{Q} \).

**Corollary 1.3** Let \( k_1, \ldots, k_n \) be number fields and let \( m_1, \ldots, m_n \) be positive integers with \( \gcd(m_1, \ldots, m_n) = 1 \). Let \( L_i \in \mathbb{Q}[t_1, \ldots, t_s] \) be polynomials of degree 1, for \( i = 1, \ldots, r \). Let \( X \) be a smooth and proper variety over \( \mathbb{Q} \) that is birationally equivalent to the affine hypersurface

\[
\prod_{i=1}^{r} L_i(t_1, \ldots, t_s) = \prod_{j=1}^{n} N_{k_j/\mathbb{Q}}(x_j)^{m_j}.
\]

Then \( X(\mathbb{Q}) \) is dense in \( X(\mathbb{A}_\mathbb{Q})^{Br} \).
Note that repetitions among \( L_1, \ldots, L_r \) are allowed here. Corollary 1.3 is a particular case of Proposition 3.5 which deals with several equations like (2) and extends [2, Thm. 1.3].

This note consists of two sections. In §2 we make preliminary remarks, some of which, like Corollary 2.3, are not needed in the proof of our main results but could possibly be of independent interest. In §3 we prove Theorem 1.1, Corollary 1.3 and their generalisations.

Our proof of Theorem 1.1 uses descent like [3], [10] or [18], and not the fibration method like [22] or [15]. It was inspired by the approach of Colliot-Thélène and Sansuc [5] to degeneration of torsors of tori, and by their computation of universal torsors on conic bundles [6, Section 2.6]. We apply open descent based on Harari’s formal lemma as in [9], with an improvement found in [8].

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2 Reduction of varieties defined over a discretely valued field

Let \( R \) be a discrete valuation ring with the field of fractions \( K \), the maximal ideal \( \mathfrak{m} \) and the residue field \( \kappa = R/\mathfrak{m} \). We assume that \( \kappa \) is perfect. Let \( j : \text{Spec}(K) \to \text{Spec}(R) \) and \( i : \text{Spec}(\kappa) \to \text{Spec}(R) \) be the embeddings of the generic and the special points, respectively. We have an exact sequence of étale sheaves on \( \text{Spec}(R) \):

\[
0 \to \mathbb{G}_{m,R} \to j_* \mathbb{G}_{m,K} \to i_* \mathbb{Z}_\kappa \to 0,
\]

see [17, Examples II.3.9]. Let \( T \) be a torus over \( R \). Applying \( \text{Hom}_R(\widehat{T}, \cdot) \) we obtain an exact sequence of abelian groups

\[
H^1(R, T) \to H^1(K, T) \to \text{Ext}^1_\kappa(\widehat{T}, \mathbb{Z}),
\]

see [21, p. 70] for a detailed proof. We note that \( \text{Ext}^1_\kappa(\widehat{T}, \mathbb{Z}) = H^1(\kappa, \widehat{T}^\circ) \), where \( T^\circ \) is the \( \kappa \)-torus dual to \( T \), that is, such that \( \widehat{T}^\circ = \text{Hom}(\widehat{T}, \mathbb{Z}) \) as Galois modules.

For an \( R \)-scheme \( X \) we denote by \( X_K \) and \( X_\kappa \) the generic and special fibres of \( X \), respectively.

**Lemma 2.1** Let \( Y \) be a \( K \)-torsor of \( T_K \). Suppose that there is an integral normal scheme \( X \) and a surjective morphism \( X \to \text{Spec}(R) \) with integral fibres such that the generic fibre \( X_K \) is birationally equivalent to \( Y \).

(i) Let \( \kappa' \) be the algebraic closure of \( \kappa \) in the function field of the special fibre \( \kappa(X_\kappa) \). Then the image of \( [Y] \) in \( H^1(\kappa, \widehat{T}^\circ) \) is in the kernel of the restriction map \( H^1(\kappa, \widehat{T}^\circ) \to H^1(\kappa', \widehat{T}^\circ) \).
(ii) If $X_\kappa$ is geometrically integral, then there is an $R$-torsor $Z$ of $T$ such that $Y \cong Z_K$.

Proof. Let $R'$ be the local ring of the special fibre $X_\kappa$. Since $X$ is normal and $X_\kappa$ is integral, $R'$ is a discrete valuation ring. Since $X$ is integral, the field of fractions of $R'$ is $K(X_K)$. A local parameter of $R$ is also a local parameter of $R'$, because $X_\kappa$ is integral. Thus $\mathfrak{m}' = \mathfrak{m} \otimes_R R'$ is the maximal ideal of $R'$, and the residue field $R'/\mathfrak{m}'$ is the field of functions on the special fibre $X_\kappa$.

The pullback from $R$ to $R'$ gives rise to the commutative diagram

$$
\begin{array}{ccc}
H^1(R', T) & \rightarrow & H^1(K(X_K), T) \\
\uparrow & & \uparrow \\
H^1(R, T) & \rightarrow & H^1(K, T)
\end{array}
$$

The restriction of the diagonal $Y \rightarrow Y \times_K Y$ to the generic fibre of $Y$ is a $K(Y)$-point of $Y$. Hence the torsor $Y$ is split by the field extension $K(X_K) = K(Y)$, so that the class $[Y] \in H^1(K, T)$ goes to zero in $H^1(K(X_K), T)$.

The right vertical map in the diagram factorises as follows:

$$
H^1(\kappa, \hat{T}) \longrightarrow H^1(\kappa', \hat{T}) \longrightarrow H^1(\kappa, \hat{T}).
$$

The second map here is an isomorphism because $\hat{T}$ is a finitely generated free abelian group and $\kappa'$ is algebraically closed in $\kappa(X_\kappa)$. This implies (i).

If the $\kappa$-scheme $X_\kappa$ is geometrically integral, the field $\kappa$ is algebraically closed in $\kappa(X_\kappa)$, that is, $\kappa' = \kappa$. Then $[Y]$ is in the image of the map $H^1(R, T) \rightarrow H^1(K, T)$, and this proves (ii). □

The statement of Lemma 2.1 (i) leaves open the question to what extent $\kappa'$ is determined by the field $K(X_K)$. We treat this as a question about integral, regular, proper schemes over a discretely valued field, see Corollary 2.3 below.

For an integral variety $V$ over a field $k$, we write $k_V$ for the algebraic closure of $k$ in $k(V)$.

**Proposition 2.2** Let $Y$ and $Y'$ be integral regular schemes that are proper over $R$. If there is a dominant rational map from $Y_k$ to $Y'_k$, then for any irreducible component $C \subset Y_\kappa$ of multiplicity 1 there exists an irreducible component $C' \subset Y'_\kappa$ of multiplicity 1 such that $\kappa_{C'} \subset \kappa_C$.

Proof. Write $F = K(Y_K)$ and $F' = K(Y'_K)$. We are given an inclusion $F' \subset F$, or, equivalently, a morphism $\text{Spec}(F) \rightarrow \text{Spec}(F')$. Let $\mathcal{O}_C$ be the local ring of the generic point of $C$. Since $Y$ is regular, $\mathcal{O}_C$ is a discrete valuation ring. The multiplicity of $C$ is 1, that is, the maximal ideal of $\mathcal{O}_C$ is $\mathfrak{m}\mathcal{O}_C = \mathfrak{m} \otimes_R \mathcal{O}_C$. Since $Y$ is integral, the field of fractions of $\mathcal{O}_C$ is $F$. The residue field of $\mathcal{O}_C$ is $\kappa(C)$. 

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Remark 2.2, C have κ of dimension 2. The generic fibre is a conic, and, more generally, a Severi–Brauer variety, or a quadric.

By Proposition 2.2 there exists a birational invariant of the generic fibre extends to a section of the morphism.

Let U ⊂ Y′ be the smooth locus of Y′. Since U contains a κ(C)-point, it is non-empty. By Stein factorisation, the structure morphism U → Spec(κ) factors through the surjective morphism U → Spec(L) with geometrically connected fibres, where L is an étale κ-algebra. Explicitly, L is the direct sum of finite field extensions of κ, each of which is the algebraic closure of κ in the function field of an irreducible component of Y′ of multiplicity 1. The morphism of the composed map Spec(κ(C)) → Spec(L) is connected, hence this is Spec(κ′′) for some irreducible component C′ ⊂ Y′ of multiplicity 1. Thus κ′′ ⊂ κ(C) and hence κ′′ ⊂ κC.

**Corollary 2.3** Let X be an integral regular scheme that is proper over R. Let ΣX be the partially ordered set of irreducible components of multiplicity 1 of X, where C dominates D if κD ⊂ κC. The set of finite field extensions κ ⊂ κC, where C is a minimal element of ΣX, is a birational invariant of the generic fibre X.

Proof. Suppose that proper R-schemes X and Y are integral and regular with birationally equivalent generic fibres, that is, K(X) ∼= K(Y). Let C be a minimal element of ΣX. By Proposition 2.2, there exists C′ ∈ ΣY such that κC′ ⊂ κC. By the same proposition, there is C′′ ∈ ΣX such that κC′′ ⊂ κC′. By minimality of C we have κC′′ = κC, hence κC = κC′′. If C′ is not minimal in ΣY, then, by Proposition 2.2, C is not minimal in ΣX.

This set of finite extensions of the residue field can be explicitly determined when the generic fibre is a conic, and, more generally, a Severi–Brauer variety, or a quadric of dimension 2.

**Remark.** Olivier Wittenberg suggested a somewhat different approach to Proposition 2.2 and Corollary 2.3. Consider a discrete valuation v : K(X) → Z such that the restriction of v to K* is the given discrete valuation of K. Let R′ be the valuation ring of v, and let κ′ be the algebraic closure of κ in the residue field of R′. Let us call ΘX the resulting set of finite field extensions of κ partially ordered by inclusion. It is clear that ΘX is a birational invariant of X. The discrete valuation
associated to an irreducible component of $X_K$ of multiplicity 1 is an example of such a valuation, so we have an inclusion of partially ordered sets $\Sigma_X \subset \Theta_X$. Since $X$ is proper over $R$ and regular, it can be shown that any extension of the given discrete valuation of $K$ to $K(X_K)$ gives rise to a morphism of $R$-schemes $\text{Spec}(R') \to X$ that factors through the smooth locus of $X/R$. Hence $\Sigma_X$ and $\Theta_X$ have the same set of minimal elements, which is thus a birational invariant of $X_K$.

3 Torsors over toric fibrations

Let $k$ be a field of characteristic zero. Let $\bar{k}$ be an algebraic closure of $k$, and let $\Gamma_k = \text{Gal}(\bar{k}/k)$. For a variety $X$ over $k$ we write $X = X \times_k \bar{k}$.

Let $T$ be a $k$-torus. We write $\hat{T}$ for the $\Gamma_k$-module of characters of $T$.

Let $X$ be a smooth, proper, geometrically integral variety with a surjective morphism $f : X \to \mathbb{P}^1_k$ and the geometrically integral generic fibre $X_{k(t)}$ which is birationally equivalent to a $k(t)$-torsor of $T$.

Lemma 3.1 Each fibre of $X \to \mathbb{P}^1_k$ has an irreducible component of multiplicity 1.

Proof. The generic fibre of $\bar{X} \to \mathbb{P}^1_k$ is birationally equivalent to a torsor of $\bar{T} \simeq \mathbb{G}_{m,k}$, where $d = \dim(T)$. By Hilbert’s theorem 90 we have $H^1(\bar{k}(t), T) = H^1(\bar{k}(t), \mathbb{G}_m)^d = 0$, so this torsor has a $\bar{k}(t)$-point. By the lemma of Lang and Nishimura, $X_{k(t)}$ has a $\bar{k}(t)$-point too. By the valuative criterion of properness, this point extends to a section of the proper morphism $\bar{X} \to \mathbb{P}^1_k$. Since $X$ is smooth, by a standard argument (see the proof of [20, Lemma 1.1 (b)], or [25]) any section intersects each fibre of $\bar{X} \to \mathbb{P}^1_k$ in an irreducible component of multiplicity 1. The lemma follows.

Proof of Theorem 1.1. We keep the notation of §1. By Lemma 3.1 there is an irreducible component of multiplicity 1 in each $X_{P_i}$, for $i = 1, \ldots, r$. We mark these components. We also mark a geometrically integral irreducible component of multiplicity 1 in each of the remaining fibres of $f$. Define $Y \subset X$ as the complement to the union of all the unmarked irreducible components of the fibres of $f : X \to \mathbb{P}^1_k$. It is clear that $Y$ is a dense open subset of $X$. The restriction of $f$ to $Y$ is a surjective morphism $f : Y \to \mathbb{P}^1_k$ with integral fibres, and with proper and geometrically integral generic fibre $Y_{k(t)} = X_{k(t)}$. It follows that $\bar{k}[Y]^* = \bar{k}^*$ and $\text{Pic}(\bar{Y})$ is torsion-free.

Let $\text{Pic}(\bar{Y}) \to \text{Pic}(Y_{k(t)})$ be the homomorphism of $\Gamma_k$-modules induced by the inclusion of the generic fibre $Y_{k(t)}$ into $\bar{Y}$. This homomorphism is surjective since $Y$ is smooth. Let $S$ be a $k$-torus defined by the exact sequence of $\Gamma_k$-modules

$$0 \to \hat{S} \to \text{Pic}(\bar{Y}) \to \text{Pic}(Y_{k(t)}) \to 0.$$ (5)
Thus the abelian group $\widehat{S}$ is generated by the geometric irreducible components of the fibres $Y_{P_1}, \ldots, Y_{P_r}$. Recall that a \textit{vertical torsor} $\mathcal{T} \to Y$ is a torsor of $S$ whose type is the injective map $\widehat{S} \to \text{Pic}(Y)$ from $\{$. According to $\{$ [21, Prop. 4.4.1] such torsors exist. For any vertical torsor we have $\bar{k}[\mathcal{T}]^* = \bar{k}^*$ and the abelian group

$$\text{Pic}(\mathcal{T}) \cong \text{Pic}(Y_{\bar{k}(t)}) \cong \text{Pic}(X_{\bar{k}(t)})$$

is torsion-free. It follows that $\text{Br}_1(\mathcal{T})/\text{Br}_0(\mathcal{T})$ is finite.

Recall that for $i = 1, \ldots, r$ we write $k(P_i)$ for the residue field of $P_i$ and $k_i$ for the algebraic closure of $k(P_i)$ in the function field $k(Y_{P_i})$. The variety $W_\alpha$ is defined by $\{$, which is also equation (4.34) of $\{$ [21].

**Lemma 3.2** The variety $W_\alpha$ is smooth and geometrically irreducible, and the morphism $\pi : W_\alpha \to \mathbb{P}^1_k$ given by $(u : v)$ is faithfully flat. We have $\bar{k}[W_\alpha]^* = \bar{k}^*$ and $\text{Pic}(W_\alpha) = 0$.

*Proof.* The first statement is $\{$ [21, Prop. 4.4.5]. The second statement is a straightforward adaptation of $\{$ [6, Lemme 2.6.1]. $\square$

**Proposition 3.3** Any vertical torsor over $Y$ is birationally equivalent to $W_\alpha \times_k Z$, where $\alpha \in \prod_{i=1}^r k(P_i)^*$ and $Z$ is a $k$-torsor of $T$.

*Proof.* The local description of torsors due to Colliot-Thélène and Sansuc, see $\{$ [21, Thm. 4.3.1, Cor. 4.4.6], can be stated as follows. Let

$$U' = \mathbb{A}^1_k \setminus \{ P_1, \ldots, P_r \}, \quad U = Y \cap f^{-1}(U'), \quad V_\alpha = \pi^{-1}(U').$$

Then for any vertical torsor $\mathcal{T}\big|_Y$ there exists an $\alpha \in \prod_{i=1}^r k(P_i)^*$ such that the restriction $\mathcal{T}_U = \mathcal{T} \times_Y U$ is isomorphic to the fibred product $U \times_{U'} V_\alpha$. Let us write $W = W_\alpha, V = V_\alpha$.

Let $j' : \text{Spec}(k(t)) \to U'$ be the embedding of the generic point of $\mathbb{A}^1_k$ into the open set $U'$. Let $i_P : P \to U'$ be the embedding of a closed point $P \subset U'$. Since $\mathbb{A}^1_k$ is smooth, there is the following exact sequence of étale sheaves on $U'$:

$$0 \to \mathbb{G}_{m,U'} \to j'_* \mathbb{G}_{m,k(t)} \to \bigoplus_{P \in U'} i_P^* \mathbb{Z}_{k(P)} \to 0,$$

see $\{$ [17, Examples II.3.9, III.2.22]. Similarly to the proof of Lemma $\{$ [21] an application of $\text{Hom}_{U'}(\widehat{T}, \cdot)$ to this sequence produces an exact sequence

$$H^1(U', T) \to H^1(k(t), T) \to \bigoplus_{P \in U'} H^1(k(P), \widehat{T}^\circ). \quad (6)$$

Let $\xi \in H^1(k(t), T)$ be the class of a $k(t)$-torsor of $T$ birationally equivalent to $X_{\bar{k}(t)}$. The map $H^1(k(t), T) \to H^1(k(P), \widehat{T}^\circ)$ in $\{$ can be computed in the local ring $R = O_P$. The fibres of $f : U \to U'$ are geometrically integral, hence Lemma
\textbf{2.1} (ii) implies that this map is trivial. Thus we see from (6) that \( \xi \) comes from some \( \xi' \in H^1(U', T) \). By Lemma \textbf{2.1} (i) applied to \( f : Y \to \mathbb{P}^1_k \) the image of \( \xi' \) in \( H^1(k(P_i), \hat{T}^0) \) goes to zero in \( H^1(k_i, \hat{T}^0) \).

The fibre \( W_i = \pi^{-1}(P_i) \) is the product of a principal homogeneous space of a \( k(P_i) \)-torus and the affine \( k(P_i) \)-variety defined by \( N_{k_i/k(P_i)}(zi) = 0 \). Thus \( W_i \) is integral over \( k(P_i) \) and the field \( k_i \) is the algebraic closure of \( k(P_i) \) in the function field of \( W_i \). If we set \( P_0 = \infty \), then \( W_0 = \pi^{-1}(P_0) \) is geometrically integral by construction, so the algebraic closure of \( k(P_0) = k \) in \( k(W_0) \) is \( k_0 = k \). We have a commutative diagram similar to the one in the proof of Lemma 2.1:

\[
\begin{array}{ccc}
H^1(W, T) & \to & H^1(V, T) \to \oplus_{i=0}^r H^1(k(W_i), \hat{T}^0) \\
\uparrow & & \uparrow \\
H^1(k, T) & \to & H^1(U'_i, T) \to \oplus_{i=0}^r H^1(k(P_i), \hat{T}^0)
\end{array}
\]

By the structure of degenerate fibres of \( \pi : W \to \mathbb{P}^1_k \) described above, the image of \( \xi' \) in \( H^1(k(W_i), \hat{T}^0) \) is zero for \( i = 0, \ldots, r \). Now the upper sequence in the diagram shows that \( \pi^*(\xi') \in H^1(V, T) \) comes from \( H^1(W, T) \).

By Lemma 3.2 we have \( \bar{k}[W_0]^* = \bar{k}^* \) and \( \text{Pic}(\bar{W}_0) = 0 \), and thus the fundamental exact sequence of Colliot-Thélène and Sansuc (see [6, Thm. 1.5.1] or [21, Cor. 2.3.9]) shows that the natural map \( H^1(k, T) \to H^1(W, T) \) is an isomorphism. It follows that any \( W \)-torsor of \( T \) is the product of \( W \) and a \( k \)-torsor of \( T \).

End of proof of Theorem 1.1

Let \((M_v) \in X_{\Br}(A_k)\). By a theorem of Grothendieck, \( \Br_1(X) \) is naturally a subgroup of \( \Br_1(Y) \). We have \( \bar{k}[Y]^* = \bar{k}^* \), and this implies that \( \Br_1(Y)/\Br_0(Y) \) is a subgroup of \( H^1(k, \text{Pic}(\bar{Y})) \), which is finite because \( \text{Pic}(\bar{Y}) \) is torsion-free. Thus we can use [9, Prop. 1.1] (a consequence of Harari’s formal lemma) which says that the natural injective map of topological spaces \( Y_{\Br_1}(Y) \to X_{\Br_1}(X) \) has a dense image. Thus we can assume without loss of generality that \( (M_v) \in Y_{\Br_1}(Y) \).

The main theorem of the descent theory of Colliot-Thélène and Sansuc states that every point in \( Y_{\Br_1}(Y) \) is in the image of the map \( \bar{T}_0(A_k) \to Y(A_k) \), where \( \bar{T}_0 \to Y \) is a universal torsor (see [6, Section 3] or [21, Thm. 6.1.2(a)]). Thus we can find a point \((N_v) \in \bar{T}_0(A_k)\) such that the image of \( N_v \) in \( Y \) is \( M_v \) for all \( v \).

The structure group of \( \bar{T}_0 \to Y \) is the Néron–Severi torus \( T_0 \) defined by the property \( \bar{T}_0 = \text{Pic}(\bar{Y}) \). The dual of the injective map in \([3]\) is a surjective morphism of tori \( T_0 \to S \). Let \( T_1 \) be its kernel. Then \( T = \bar{T}_0/T_1 \) is a \( Y \)-torsor with the structure group \( S \) whose type is the natural map \( \hat{S} \to \text{Pic}(\bar{Y}) \), so \( T \) is a vertical torsor. Since \( \bar{T}_0 \) is a universal torsor, we have \( \bar{k}[\bar{T}_0]^* = \bar{k}^* \) and \( \text{Pic}(\bar{T}_0) = 0 \), hence \( \Br_1(\bar{T}_0) = \Br_0(\bar{T}_0) \). Let \((P_v) \in T(A_k)\) be the image of \((N_v) \). By the functoriality of the Brauer–Manin pairing we see that \((P_v) \in T_{\Br_1}(T) \).

By Proposition 3.3 the variety \( T \) is birationally equivalent to \( E \times W_\alpha \) for some \( \alpha \in \prod_{i=1}^r k(P_i)^* \). Let \( E^c \) be a smooth compactification of \( E \). Then we have a
rational map \( g \) from the smooth variety \( \mathcal{T} \) to the proper variety \( E^c \), and a rational map \( h \) from \( \mathcal{T} \) to \( W_\alpha \). By the valuative criterion of properness there is an open subset \( \Omega \subset \mathcal{T} \) with complement of codimension at least 2 in \( \mathcal{T} \) such that \( g \) is a morphism \( \Omega \to E^c \). Let \( \Omega' \subset \Omega \) be a dense open subset such that \( (g, h) \) defines an open embedding \( \Omega' \subset E^c \times W_\alpha \).

By Grothendieck’s purity theorem the natural restriction map \( Br(\mathcal{T}) \to Br(\Omega) \) is an isomorphism. Thus \( g^*Br_1(E^c) \subset Br_1(\Omega) \), so that \( g^*Br_1(E^c) \subset Br_1(\mathcal{T}) \). (This argument is borrowed from [9].)

Since \( Br_1(\mathcal{T})/Br_0(\mathcal{T}) \) is finite, by a small deformation we can assume \( P_v \in \Omega(k_v) \) for each \( v \). Thus \((g(P_v))\) is a well defined element of \( E^c(A_k)^{Br_1(E^c)} \). By Sansuc’s theorem, \( E(k) \) is then non-empty and, moreover, is dense in \( E^c(A_k)^{Br_1(E^c)} \).

Let \( \Sigma \) be a finite set of places of \( k \) containing all the places where we need to approximate. By assumption we can find a \( k \)-point in \( \Omega' \) that is arbitrarily close to \((g(P_v), h(P_v))\) for \( v \in \Sigma \). We conclude that there is a \( k \)-point in \( \mathcal{T} \) that is arbitrarily close to \( P_v \) for \( v \in \Sigma \). The image of this point in \( Y \) approximates \((M_v)\). This finishes the proof of Theorem [14]. □

**Proposition 3.4** Let \( X \) be a smooth, proper, geometrically integral variety over \( \mathbb{Q} \), and let \( f : X \to \mathbb{P}_\mathbb{Q}^n \) be a surjective morphism satisfying the following properties.

(a) There is a torus \( T \) over \( \mathbb{Q} \) such that the generic fibre \( X_{\mathbb{Q}(\eta)} \) of \( f \) is birationally equivalent to a \( \mathbb{Q}(\eta) \)-torsor of \( T \times_\mathbb{Q} \mathbb{Q}(\eta) \).

(b) There exist hyperplanes \( H_1, \ldots, H_r \subset \mathbb{P}_\mathbb{Q}^n \) such that \( f \) has split fibres at all points of codimension 1 of \( \mathbb{P}_\mathbb{Q}^n \) other than \( H_1, \ldots, H_r \).

Then \( X(\mathbb{Q}) \) is dense in \( X(A_\mathbb{Q})^{Br} \).

**Proof.** We deduce this from a fibration theorem due to D. Harari [14] Thm. 3.2.1. Choose a point \( M \in \mathbb{P}_\mathbb{Q}^n(\mathbb{Q}) \) such that the fibre \( X_M \) is smooth. The projective lines through \( M \) are in a natural bijection with the points of \( \mathbb{P}_\mathbb{Q}^{n-1} \). Thus an appropriate open subset \( V \subset X \) is a quasi-projective variety equipped with a surjective morphism \( p : V \to \mathbb{A}^{n-1}_\mathbb{Q} \) with split fibres. Any point of \( X_M(\overline{\mathbb{Q}}) \) defines a section of \( p \) over \( \overline{\mathbb{Q}} \). The generic fibre of \( p \) is birationally equivalent to a family of torsors of \( T \) over an open subset of the projective line, hence it is geometrically integral and geometrically rational. In particular, a smooth and proper model of the generic fibre has trivial geometric Brauer group and torsion-free geometric Picard group. The fibres of \( p \) over \( \mathbb{Q} \)-points of a non-empty open subset of \( \mathbb{A}_\mathbb{Q}^{n-1} \) satisfy the assumptions of Corollary [12] hence all the conditions of [14] Thm. 3.2.1] are satisfied. An application of this result proves the proposition. □

**Proposition 3.5** Let \( k_1, \ldots, k_n \) be number fields, and let \( L_i \in \mathbb{Q}[t_1, \ldots, t_s] \) be polynomials of degree 1, for \( i = 1, \ldots, r \). Let \( m_{ij} \geq 0 \), for \( i = 1, \ldots, \ell \) and \( j = 1, \ldots, n \), be integers such that the sublattice of \( \mathbb{Z}^n \) generated by the rows of the matrix \((m_{ij})\)
is primitive. Finally, let $d_{hi} \geq 0$ be integers, where $h = 1, \ldots, \ell$ and $i = 1, \ldots, r$. Let $X$ be a smooth and proper variety over $\mathbb{Q}$ that is birationally equivalent to the affine variety given by the system of equations

$$c_h \prod_{i=1}^{r} L_i (t_1, \ldots, t_s)^{d_{hi}} = \prod_{j=1}^{n} N_{k_j/\mathbb{Q}} (x_j)^{m_{hj}}, \quad h = 1, \ldots, \ell,$$

(7)

where $c_h \in \mathbb{Q}^*$. Then $X(\mathbb{Q})$ is dense in $X(\mathbb{A}_\mathbb{Q})^{Br}$.

**Proof.** The condition on the matrix $(m_{ij})$ implies that the affine variety given by (7) has a morphism $g : Y \to \mathbb{A}^s_\mathbb{Q}$ given by the coordinates $t_1, \ldots, t_s$. Let $H_i$ be the hyperplane $L_i = 0$. The restriction of $g$ to $\mathbb{A}^s_\mathbb{Q} \setminus (H_1 \cup \ldots \cup H_r)$ is a torsor of $T$. We can choose a smooth compactification $X$ of the smooth locus $Y_{\text{sm}}$ in such a way that there is a surjective morphism $f : X \to \mathbb{P}^s_\mathbb{Q}$ extending $g : Y_{\text{sm}} \to \mathbb{A}^s_\mathbb{Q}$. Now we apply Proposition 3.4. □

For $\ell = 1$ this statement is Corollary 1.3.

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