Scattering of string-waves on black hole background

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Abstract

We consider the propagation of perturbations along an infinitely long stationary open string in the background of a Schwarzschild black hole. The equations of motion for the perturbations in the 2 transverse physical directions are solved to second order in a weak field expansion. We then set up a scattering formalism where an ingoing wave is partly transmitted and partly reflected due to the interaction with the gravitational field of the black hole. We finally calculate the reflection coefficients to third order in our weak field expansion.
1 Introduction and conclusions

In a previous publication [1] we developed a covariant formalism for physical perturbations propagating along a string in an arbitrary curved background. As applications we considered stationary strings in quasi-Newtonian, Rindler and de Sitter spacetimes, and we presented the wave-equations determining the evolution of the perturbations when propagating along a string in the background of different kinds of black holes. In the latter cases, however, solving these wave-equations explicitly was not attempted, and therefore the physical quantities (scattering amplitudes,...) describing the propagation of the waves were not extracted from the general formalism.

In the present paper we return to these wave-equations in the special case of perturbations propagating along an infinitely long stationary string in the equatorial plane of a Schwarzschild black hole. We make an expansion in the dimensionless parameter \( \lambda \equiv r_g/b \) [2], where \( r_g \) is the gravitational radius of the black hole and \( b \) is the ”impact parameter” measuring the minimal distance between the black hole and the stationary string. We then solve the wave-equations up to second order in \( \lambda \), thus obtaining explicit expressions for the physical perturbations \( \delta x_{\parallel}^{(0-2)}(\sigma_c, \tau) \) and \( \delta x_{\perp}^{(0-2)}(\sigma_c, \tau) \) parallel and perpendicular to the equatorial plane of the black hole, respectively (\( \tau \) and \( \sigma_c \) are the 2 world-sheet coordinates). Far away from the black hole the geometry of spacetime is flat and the string perturbations are simple plane waves. We can then set up a scattering formalism with a localized regime near the black hole where the scattering takes place and 2 ”free” asymptotic regions. We derive the transformation matrix relating the ”out”-amplitudes to the ”in”-amplitudes and we calculate the reflection coefficients for the 2 physical polarizations up to third order in \( \lambda \). The reflection coefficient is largest for perturbations in the equatorial plane, but in both cases it is only appreciably different from zero when the wavelength at infinity \( \lambda_{\infty} \) is comparable to the impact parameter \( b \). Although the physical situation is considerably different it turns out that our analysis is somewhat similar to the analysis of the quantum scattering of fundamental strings by black holes carried out by Vega and Sanchez [2-3].

The paper is organized as follows: In section 2 we give a very short review of the general formalism developed in Ref.1., and in section 3 we use this general formalism to derive the wave-equations for perturbations propagating along a stationary string in the equatorial plane of a Schwarzschild
black hole. In section 4 we introduce the weak field approximation [2] where the perturbations and the potentials are expanded in powers of the small parameter $\lambda$, and in sections 5-6 we solve the wave-equations up to second order. Finally in section 7 we consider a scattering process and the reflection coefficients are calculated up to third order in $\lambda$.

We use sign-conventions of Misner-Thorn-Wheeler [4] and units where $G = 1$, $c = 1$ and the string tension $(2\pi\alpha')^{-1} = 1$. 
2 General approach

In this section we give a short review of the derivation of the scalar wave-equations determining the propagation of perturbations in the transverse directions of a stationary string in a static background. Starting with the Polyakov action [5]:

$$S = \int d\tau d\sigma \sqrt{-h} h^{AB} G_{AB},$$

(2.1)

and the corresponding equations of motion ($\Box \equiv (-h)^{-1/2} \partial_A (\sqrt{-h} h^{AB} \partial_B)$):

$$G_{AB} = \frac{1}{2} h_{AB} G^C_C,$$

(2.2)

$$\Box x^\mu + h^{AB} \Gamma^\mu_{\rho\sigma} x^\rho_A x^\sigma_B = 0,$$

(2.3)

we consider variations/perturbations:

$$\delta h_{AB}, \quad \delta x^\mu = \delta x^R n_R^\mu$$

(2.4)

around an exact solution. $n_R^\mu (R = 2, 3)$ are 2 vectors normal to the surface of the world-sheet of the exact solution, so we only consider perturbations $\delta x^\mu$ in the physical directions.

Variation of Eq. (2.2) gives $\delta h_{AB}$ in terms of $\delta x^R$, which, when used in the variation of Eq. (2.3), leads to a complicated matrix wave-equation for the 2 physical perturbations $\delta x^R$. The details can be found in Ref.1., here we just recall the result:

$$\Box \delta x^R + 2 \mu_{RS} A (\delta x^S)_A + (\nabla_A \mu_{RS} A) \delta x^S - \mu_{RT} A \mu_S T A \delta x^S + \frac{2}{G^C_C} \Omega_R^{AB} \Omega_{S,AB} \delta x^S - h^{AB} x_A^\mu x_B^\nu R_{\mu\rho\sigma\nu} n_R^\mu n_S^\nu \delta x^S = 0,$$

(2.5)

where $\nabla_A$ is the world-sheet covariant derivative, $R^{\mu\rho\sigma\nu}$ is the spacetime Riemann-tensor and $\Omega_{R,AB}$ and $\mu_{RS,A}$ are the second fundamental form and the normal fundamental form, respectively [6]:

$$\Omega_{R,AB} = g_{\mu\nu} n_R^\mu x_A^\nu A \nabla_\rho x_B^\nu,$$

(2.6)

$$\mu_{RS,A} = g_{\mu\nu} n_R^\mu x_A^\nu \nabla_\rho n_S^\nu,$$

(2.7)

where $\nabla_\rho$ is the spacetime covariant derivative.
Equation (2.5) holds for an arbitrary string configuration in an arbitrary curved spacetime. In the case of a stationary string in a static background some simplifications arise. The metric of a static spacetime is conveniently written:

\[ g_{\mu \nu} = \begin{pmatrix} -F & 0 \\ 0 & H_{ij}/F \end{pmatrix}, \] (2.8)

where \( \partial_t F = 0, \partial_t H_{ij} = 0; \ i, j = 1, 2, 3. \) A stationary string can be parametrized by:

\[ t = x^0 = \tau, \quad x^i = x^i(\sigma); i = 1, 2, 3. \] (2.9)

Using Eqs. (2.8)-(2.9) it can be shown that Eq. (2.5) reduces to (in the conformal gauge):

\[ (\partial^2_{\sigma_c} - \partial^2_{\tau}) \delta x_R = U_{RS} \delta x^S, \] (2.10)

where the matrix potential \( U_{RS} \) is given by:

\[ U_{RS} = V \delta_{RS} + F^{-1}V_{RS}; \] (2.11)

\[ V = \frac{3}{4F^2} \left( \frac{dF}{d\sigma_c} \right)^2 - \frac{1}{2F} \frac{d^2F}{d\sigma_c^2}, \] (2.12)

\[ V_{RS} = \frac{dx^i dx^j}{d\sigma_c d\sigma_c} \tilde{R}_{iklj} n_R^k n_S^l. \] (2.13)

Here \( \tilde{R}_{iklj} \) is the Riemann tensor for the metric \( H_{ij} \) and \( n_R^k \) represents the space components of the normal vectors introduced in Eq. (2.4). For the details we refer the reader to Ref.1.

### 3 Schwarzschild black hole

The results of Eqs. (2.10)-(2.13) in the special case of perturbations propagating along a stationary string in the equatorial plane of a Schwarzschild black hole were stated in Ref.1. without any details of the derivation. In this section we show the detailed calculations and in the following sections we then come to the solutions of the wave-equations (2.10).

The Schwarzschild black hole is given by the line element:

\[ ds^2 = -(1 - 2m/r)dt^2 + (1 - 2m/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \] (3.1)
where $m$ is the mass of the black hole. Comparing with Eq. (2.8) we find:

$$F = 1 - 2m/r, \quad (3.2)$$

$$H_{ij} = \text{diag}(1, \Delta, \Delta \sin^2 \theta); \quad \Delta \equiv r^2 - 2mr. \quad (3.3)$$

The Christoffel symbols $\tilde{\Gamma}_{jk}^i$ for the metric $H_{ij}$ are given by:

$$\tilde{\Gamma}_{\phi\phi}^r = \sin^2 \theta \tilde{\Gamma}_{\theta\theta}^r = -(r - m) \sin^2 \theta,$$

$$\tilde{\Gamma}_{\theta\theta}^r = \tilde{\Gamma}_{\phi\phi}^r = \frac{r - m}{\Delta},$$

$$\tilde{\Gamma}_{\phi\theta}^\theta = -\cos \theta \sin \theta, \quad \tilde{\Gamma}_{\phi\phi}^\theta = \cot \theta, \quad (3.4)$$

and the non-vanishing components of the Riemann tensor $\tilde{R}_{iklj}$ are:

$$\tilde{R}_{\phi\phi\phi\phi} = \sin^2 \theta \tilde{R}_{\phi\phi\theta\theta} = \sin^2 \theta \frac{m^2}{\Delta},$$

$$\tilde{R}_{\theta\theta\phi\phi} = -m^2 \sin^2 \theta. \quad (3.5)$$

Let us now consider the stationary string in the background of the black hole given by the line element (3.1). Taking the string in the equatorial plane and still working in the conformal gauge, the configuration is determined by [7]:

$$t = \tau, \quad \theta = \pi/2, \quad (3.6)$$

$$\frac{d\phi}{d\sigma_c} = \frac{b}{r^2}, \quad (3.7)$$

$$\frac{dr}{d\sigma_c} = \pm \sqrt{(1 - \frac{2m}{r})^2 - \frac{b^2}{r^2} \frac{1 - \frac{2m}{r}}{1 - \frac{2m}{r}}}, \quad (3.8)$$

where $b$ is an integration constant. The 2 normalvectors introduced in Eq. (2.4) are then chosen in the following way:

$$n_0^0 \equiv n_0^0 = 0, \quad n_3^0 \equiv n_0^0 = 0, \quad (3.9)$$

$$n_2^k \equiv n_2^k = \frac{1}{r}(0, 1, 0), \quad (3.10)$$

$$n_3^k \equiv n_3^k = \frac{1}{r}(-b, 0, \frac{1}{1 - \frac{2m}{r}} \frac{dr}{d\sigma_c}). \quad (3.11)$$
From Eqs. (2.12)-(2.13), (3.2)-(3.5) and (3.7)-(3.11) we find:

\[ V = \frac{m}{\Delta r^4} [\Delta(2r - 3m) + (4m - 3r)b^2], \]
\[ V_{\perp \parallel} = V_{\parallel \perp} = 0, \]
\[ V_{\perp \perp} = \frac{m^2}{r^4}(2b^2 - \Delta), \quad V_{\parallel \parallel} = -\frac{m^2}{r^4}\Delta. \]  
(3.12)

Eq. (2.10) then leads to the 2 decoupled equations [1]:

\[ \left( \partial_{\sigma_c}^2 - \partial_t^2 \right) \delta x_\perp = \frac{m}{r^5}(2\Delta - 3b^2)\delta x_\perp \equiv U_{\perp \perp} \delta x_\perp, \]  
(3.13)

\[ \left( \partial_{\sigma_c}^2 - \partial_t^2 \right) \delta x_\parallel = \left[ \frac{m}{r^5}(2\Delta - 3b^2) - \frac{2m^2b^2}{\Delta r^4} \right] \delta x_\parallel \equiv U_{\parallel \parallel} \delta x_\parallel, \]  
(3.14)

determining the evolution of the 2 physical polarizations of perturbations (perpendicular and parallel to the equatorial plane). The way to proceed now is to solve Eq. (3.8) for \( r(\sigma_c) \) and then to write \( U_{\perp \perp} \) and \( U_{\parallel \parallel} \) as explicit functions of \( \sigma_c \). Finally we then have to solve Eqs. (3.13)-(3.14) for \( \delta x_\parallel(\sigma_c, \tau) \) and \( \delta x_\parallel(\sigma_c, \tau) \).

### 4 Weak field expansion

Actually solving Eqs. (3.13)-(3.14) turns out to be a very difficult task. Equation (3.8) can of course be solved for \( r(\sigma_c) \) in terms of elliptic (or hyper-elliptic) functions, and the potentials \( U_{\perp \perp} \) and \( U_{\parallel \parallel} \) are then some complicated rational expressions in these elliptic functions. The shape of the potentials are sketched in Fig.1. Far away from the black hole the potentials are obviously flat but there is a well surrounded by potential barriers where the string is closest to the black hole. The potential for perturbations in the equatorial plane is a little deeper than for perturbations perpendicular to the equatorial plane, but otherwise they have the same shape.

We have not been able to find the general analytic solution to Eqs. (3.13)-(3.14). They can of course easily be solved numerically, but we will instead consider approximate analytic solutions based on a weak field expansion [2]. First note that the integration constant \( b \) introduced in Eqs. (3.7)-(3.8) is related to the minimal distance between the infinitely long open string and the black hole by [7]:

\[ r_{\text{min}} = m + \sqrt{m^2 + b^2}. \]  
(4.1)
Thus plays the role of an "impact parameter". For $b = 0$ the string touches the horizon of the black hole while for $b \neq 0$ it is always strictly outside. We then define our dimensionless expansion parameter $\lambda$ by:

$$\lambda \equiv \frac{r_g}{b},$$

(4.2)

where $r_g = 2m$ is the gravitational radius of the black hole. If $\lambda$ is small the string is far away from the black hole, corresponding to a weak field approximation [2]. In the following we will consider $b$ to be a finite positive constant so that the zeroth order in $\lambda$ is obtained by $r_g = 0$ (and not $b = \infty$).

We now expand Eqs. (3.13)-(3.14) in powers of $\lambda$. First consider the coordinates $(r, \phi)$ of the stationary string:

$$r = r(0) + r(1) + r(2) + \ldots,$$

(4.3)

$$\phi = \phi(0) + \phi(1) + \phi(2) + \ldots$$

(4.4)

From Eqs. (3.7)-(3.8) we find to zeroth order in $\lambda$:

$$\frac{d\phi(0)}{d\sigma_c} = \frac{b}{r(0)^2},$$

(4.5)

$$\frac{dr(0)}{d\sigma_c} = \pm \sqrt{1 - \frac{b^2}{r(0)^2}}.$$  

(4.6)

These 2 equations are easily solved by:

$$r(0) = \sqrt{\sigma_c^2 + b^2},$$

(4.7)

$$\phi(0) = \arctan \frac{\sigma_c}{b},$$

(4.8)

i.e. it is just the straight string. This is of course not surprising since at the zeroth order there is no black hole at all, and the only stationary string in flat spacetime is the straight one. At first order in $\lambda$ we find:

$$\frac{dr(1)}{d\sigma_c} = \pm \left( \frac{b^2 - 2r(0)^2}{2r(0)^2 \sqrt{r(0)^2 - b^2}} r_g + \frac{b^2}{r(0)^2 \sqrt{r(0)^2 - b^2}} r(1) \right),$$

(4.9)
\[
\frac{d\phi_{(1)}}{d\sigma_c} = -\frac{2b}{r_0^3} r_{(1)}. \tag{4.10}
\]

Using Eq. (4.7) these 2 equations are solved by:

\[
\begin{align*}
  r_{(1)} &= r_g \left( \frac{1}{2} - \frac{\sigma_c}{\sqrt{\sigma_c^2 + b^2}} \sinh^{-1} \frac{\sigma_c}{b} \right), \tag{4.11} \\
  \phi_{(1)} &= -\frac{br_g}{\sigma_c^2 + b^2} \sinh^{-1} \frac{\sigma_c}{b}, \tag{4.12}
\end{align*}
\]

where the integration constants have been chosen such that:

\[\phi(0) = 0, \quad \frac{dr}{d\sigma_c}(0) = 0. \tag{4.13}\]

We can then continue to solve Eqs. (3.7)-(3.8) order by order, but for our purposes we will only need \(r\) and \(\phi\) up to first order in \(\lambda\).

Now consider the 2 potentials \(U_{\perp\perp}\) and \(U_{\parallel\parallel}\). Obviously the potentials have no zeroth order contributions (flat spacetime), are identical at first order but in general different at second and higher orders. We find:

\[
\begin{align*}
  \frac{m}{r^5}(2\Delta - 3b^2) &= \frac{2r_{(0)}^2 - 3b^2}{2r_{(0)}^4} r_g - \frac{r_g^2}{r_{(0)}^4} + 3 \frac{5b^2 - 2r_{(0)}^2}{2r_{(0)}^6} r_g r_{(1)} + \mathcal{O}(\lambda^3), \tag{4.14} \\
  - \frac{2m^2b^2}{\Delta r^4} &= -\frac{b^2 r_g^2}{2r_{(0)}^6} + \mathcal{O}(\lambda^3). \tag{4.15}
\end{align*}
\]

Using Eqs. (4.7) and (4.11) these 2 expressions give the potentials \(U_{\perp\perp}\) and \(U_{\parallel\parallel}\) as explicit functions of \(\sigma_c\) up to second order in \(\lambda\).

Finally we also expand the perturbations propagating along the stationary string:

\[
\begin{align*}
  \delta x_{\perp} &= \delta x_{\perp}^{(0)} + \delta x_{\perp}^{(1)} + \delta x_{\perp}^{(2)} + \ldots, \tag{4.16} \\
  \delta x_{\parallel} &= \delta x_{\parallel}^{(0)} + \delta x_{\parallel}^{(1)} + \delta x_{\parallel}^{(2)} + \ldots. \tag{4.17}
\end{align*}
\]

Collecting everything we can then write down the wave-equations (3.13)-(3.14) up to the second order in \(\lambda\).
5 Zeroth and first order equations and their solutions

We now come to the solutions of the wave-equations (2.10) up to first order in the expansion described in the previous section. To zeroth order we have:

\[(\partial^2_{\sigma_c} - \partial^2_{\tau})\delta x^{(0)}_\perp = (\partial^2_{\sigma_c} - \partial^2_{\tau})\delta x^{(0)}_\parallel = 0,\]  
\[\text{(5.1)}\]

i.e. it is just the ordinary flat spacetime wave-equations (as it should be!). The solutions are generally written as integrals over the continuous frequency plane waves:

\[\delta x^{(0)}_\perp(\sigma_c, \tau) = \int d\omega \left(a^\perp_\omega e^{-i\omega(\tau-\sigma_c)} + b^\perp_\omega e^{-i\omega(\tau+\sigma_c)}\right),\]  
\[\text{(5.2)}\]

\[\delta x^{(0)}_\parallel(\sigma_c, \tau) = \int d\omega \left(a^\parallel_\omega e^{-i\omega(\tau-\sigma_c)} + b^\parallel_\omega e^{-i\omega(\tau+\sigma_c)}\right),\]  
\[\text{(5.3)}\]

where:

\[(a^\perp_\omega)^* = a^\perp_{-\omega}, \quad (a^\parallel_\omega)^* = a^\parallel_{-\omega},\]
\[(b^\perp_\omega)^* = b^\perp_{-\omega}, \quad (b^\parallel_\omega)^* = b^\parallel_{-\omega}.\]

The first order wave-equations are:

\[(\partial^2_{\sigma_c} - \partial^2_{\tau})\delta x^{(1)}_\perp = U^{(1)}_{\perp\perp}\delta x^{(0)}_\perp,\]  
\[\text{(5.4)}\]

\[(\partial^2_{\sigma_c} - \partial^2_{\tau})\delta x^{(1)}_\parallel = U^{(1)}_{\parallel\parallel}\delta x^{(0)}_\parallel,\]  
\[\text{(5.5)}\]

where we used the fact that the potentials have no zeroth order terms. From Eqs. (3.13)-(3.14) and (4.14)-(4.15) we find:

\[U^{(1)}_{\perp\perp} = U^{(1)}_{\parallel\parallel} = \frac{2r^2_{(0)} - 3b^2}{2r^2_{(0)} - r_g} = m\frac{2\sigma_c^2 - b^2}{(\sigma_c^2 + b^2)^{5/2}}.\]  
\[\text{(5.6)}\]

To proceed it is convenient to Fourier-expand the perturbations \(\delta x^{(1)}_\perp\) and \(\delta x^{(1)}_\parallel\):

\[\delta x^{(1)}_\perp(\sigma_c, \tau) = \int d\omega D^\perp_\omega(\sigma_c)e^{-i\omega\tau},\]  
\[\text{(5.7)}\]
Now equations (5.4)-(5.5) lead to:

\[
\left( \frac{d^2}{d\sigma_c^2} + \omega^2 \right) D^{R}_\omega(\sigma_c) = m \frac{2\sigma_c^2 - b^2}{(\sigma_c^2 + b^2)^{3/2}} \left( a^{R}_\omega e^{i\omega\sigma_c} + b^{R}_\omega e^{-i\omega\sigma_c} \right),
\]

where \( R \) is either "\( \perp \)" or "\( \parallel \)". This equation is solved by:

\[
D^{R}_\omega(\sigma_c) = e^{i\omega\sigma_c} \left( a^{R}_\omega A_\omega(\sigma_c) + b^{R}_\omega B^*_\omega(\sigma_c) \right) + e^{-i\omega\sigma_c} \left( a^{R}_\omega B_\omega(\sigma_c) + b^{R}_\omega A^*_\omega(\sigma_c) \right),
\]

where:

\[
A_\omega(\sigma_c) = \frac{-m}{2i\omega} \frac{\sigma_c}{(\sigma_c^2 + b^2)^{3/2}},
\]

\[
B_\omega(\sigma_c) = \frac{-m}{2i\omega} \int_{-\infty}^{\sigma_c} \frac{2x^2 - b^2}{(x^2 + b^2)^{5/2}} e^{2i\omega x} dx,
\]

and we have imposed the boundary conditions:

\[
D^{\perp}_\omega(-\infty) = D^{\parallel}_\omega(-\infty) = 0.
\]

The first order perturbations are finally given by:

\[
\delta x^{(1)}_{\parallel}(\sigma_c, \tau) = \int d\omega \left[ e^{-i\omega(\tau - \sigma_c)} (a^{R}_\omega A_\omega(\sigma_c) + b^{R}_\omega B^*_\omega(\sigma_c)) + e^{-i\omega(\tau + \sigma_c)} (a^{R}_\omega B_\omega(\sigma_c) + b^{R}_\omega A^*_\omega(\sigma_c)) \right],
\]

Note that \( A_\omega(\sigma_c) \) and \( B_\omega(\sigma_c) \) are independent of the physical polarization, so that up to this order in \( \lambda \) there is no difference in the way the perturbations propagate in the 2 physical directions.

### 6 Second order equation and its solution

At second order in \( \lambda \) things become quite complicated. It is however important to calculate the second order corrections to the plane wave perturbations given by Eqs. (5.2)-(5.3), since this is the order where the perturbations can really begin to reflect the geometry in the sense that this is the lowest order where we can possibly find a difference between the perturbations in the 2 physical directions. For waves on a string in the equatorial plane of a black
hole we obviously expect some differences for the propagation of perturbations in these 2 directions. The second order wave-equations are:

\[(\partial_{\sigma_c}^2 - \partial_{\tau}^2)\delta x_{\perp}^{(2)} = U_{\perp\perp}^{(2)} \delta x_{\perp}^{(0)} + U_{\perp\perp}^{(1)} \delta x_{\perp}^{(1)}, \quad (6.1)\]
\[(\partial_{\sigma_c}^2 - \partial_{\tau}^2)\delta x_{\parallel}^{(2)} = U_{\parallel\parallel}^{(2)} \delta x_{\parallel}^{(0)} + U_{\parallel\parallel}^{(1)} \delta x_{\parallel}^{(1)}. \quad (6.2)\]

So we need explicit expressions for \(U_{\perp\perp}^{(2)}\) and \(U_{\parallel\parallel}^{(2)}\). From Eqs. (3.13)-(3.14) and (4.14)-(4.15) we find:

\[U_{\perp\perp}^{(2)} = 3\frac{5b^2 - 2r^2_{(0)}}{2r^6_{(0)}}r_g r_{(1)} - \frac{r_g}{r_{(0)}^2}, \quad (6.3)\]
as well as:

\[U_{\parallel\parallel}^{(2)} = U_{\perp\perp}^{(2)} - \frac{b^2 r_g^2}{2r^6_{(0)}} = U_{\perp\perp}^{(2)} - \frac{2m^2 b^2}{(\sigma_c^2 + b^2)^3}, \quad (6.4)\]

where also Eq. (4.11) was used. To solve Eqs. (6.1)-(6.2) we Fourier-expand the perturbations \(\delta x_{\perp}^{(2)}\) and \(\delta x_{\parallel}^{(2)}\):

\[\delta x_{\perp}^{(2)}(\sigma_c, \tau) = \int d\omega E_\omega^{\perp}(\sigma_c)e^{-i\omega \tau}, \quad (6.5)\]
\[\delta x_{\parallel}^{(2)}(\sigma_c, \tau) = \int d\omega E_\omega^{\parallel}(\sigma_c)e^{-i\omega \tau}. \quad (6.6)\]

Using the results of section 5, Eqs. (6.1)-(6.2) reduce to (no summation over \(R\)):

\[\left(\frac{d^2}{d\sigma_c^2} + \omega^2\right)E_\omega^R(\sigma_c) = (a_\omega^R f_\omega^R(\sigma_c) + b_\omega^R (g_\omega^R(\sigma_c))^*)e^{i\omega \sigma_c} + (a_\omega^R g_\omega^R(\sigma_c) + b_\omega^R (f_\omega^R(\sigma_c))^*)e^{-i\omega \sigma_c}, \quad (6.7)\]

where \(R\) is either ” \(\perp\) ” or ” \(\parallel\) ”, and the functions \(f_\omega^R\) and \(g_\omega^R\) are given by:

\[f_\omega^R = U_{RR}^{(2)} + A_\omega U_{RR}^{(1)}, \quad (6.8)\]
\[g_\omega^R = B_\omega U_{RR}^{(1)}. \quad (6.9)\]
The solution to equation (6.7) is:

\[ E_\omega^R(\sigma_c) = e^{i\omega\sigma_c} \left( a_\omega^R X_\omega^R(\sigma_c) + b_\omega^R (Y_\omega^R(\sigma_c))^* \right) + e^{-i\omega\sigma_c} \left( a_\omega^R Y_\omega^R(\sigma_c) + b_\omega^R (X_\omega^R(\sigma_c))^* \right) \]

(6.10)

where:

\[ X_\omega^R(\sigma_c) = \frac{1}{2i\omega} \int_{-\infty}^{\sigma_c} (f_\omega^R(x) + e^{-2i\omega x} g_\omega^R(x)) dx, \]

(6.11)

\[ Y_\omega^R(\sigma_c) = -\frac{1}{2i\omega} \int_{-\infty}^{\sigma_c} (g_\omega^R(x) + e^{2i\omega x} f_\omega^R(x)) dx, \]

(6.12)

and we have imposed the boundary conditions:

\[ E_\omega^\perp(-\infty) = E_\omega^\parallel(-\infty) = 0. \]

(6.13)

The second order perturbations are finally given by:

\[ \delta x^{(2)}_R(\sigma_c, \tau) = \int d\omega \left[ e^{-i\omega(\tau-\sigma_c)} (a_\omega^R X_\omega^R(\sigma_c) + b_\omega^R (Y_\omega^R(\sigma_c))^*) + e^{-i\omega(\tau+\sigma_c)} (a_\omega^R Y_\omega^R(\sigma_c) + b_\omega^R (X_\omega^R(\sigma_c))^*) \right] \]

(6.14)

The coefficient \( X_\omega^R \) and \( Y_\omega^R \) now depend on \( R \) through the potentials given by Eqs. (6.3)-(6.4), so that the perturbations propagate differently in the 2 transverse directions. The explicit expressions \( X_\omega^R(\sigma_c) \) and \( Y_\omega^R(\sigma_c) \) in terms of \( \sigma_c \), obtained by combination of Eqs. (6.11)-(6.12), (6.8)-(6.9), (6.3)-(6.4) and (5.11)-(5.12), are not very enlightening. For convenience they are listed in the appendix. In the next section we shall see, however, that it is possible to extract some simple quantitative and qualitative physical consequences from these 2 functions.
7 Scattering formalism

In this section we will consider a scattering process where a plane wave from the asymptotic region $\sigma_c = -\infty$ is travelling along the string towards the black hole. When the wave interacts with the gravitational field of the black hole it will split into a reflected part returning to $\sigma_c = -\infty$ and a transmitted part continuing towards the other asymptotic region $\sigma_c = +\infty$ (see Fig. 2.). The boundary conditions Eqs. (5.13) and (6.13) were chosen such that up to second order we can identify the zeroth order solution at $\sigma_c = -\infty$ with the solution in the asymptotic region $\sigma_c = -\infty$:

$$\delta x^{(0-2)}_{R}(\sigma_c \to -\infty, \tau \rightarrow) = \int d\omega \left(a_{\omega}^{R} e^{-i\omega(\tau-\sigma_c)} + b_{\omega}^{R} e^{-i\omega(\tau+\sigma_c)}\right).$$  \hspace{1cm} (7.1)

In the other asymptotic region $\sigma_c = +\infty$ we find from Eqs. (5.2)-(5.3), (5.14) and (6.14) up to second order in $\lambda$:

$$\delta x^{(0-2)}_{R}(\sigma_c \to +\infty, \tau \rightarrow) = \int d\omega [e^{-i\omega(\tau-\sigma_c)}(a_{\omega}^{R}(1 + \bar{A}_{\omega} + \bar{X}_{\omega}^{R}) + b_{\omega}^{R}(\bar{B}_{\omega} + \bar{Y}_{\omega}^{R})^*) + e^{-i\omega(\tau+\sigma_c)}(a_{\omega}^{R}(\bar{B}_{\omega} + \bar{Y}_{\omega}^{R}) + b_{\omega}^{R}(1 + \bar{A}_{\omega} + \bar{X}_{\omega}^{R})^*)],$$  \hspace{1cm} (7.2)

where the bar indicates evaluation at $\sigma_c = +\infty$ in the functions (5.11)-(5.12) and (6.11)-(6.12). Equation (7.2) can be written more compactly as:

$$\delta x^{(0-2)}_{R}(\sigma_c \to +\infty, \tau \rightarrow) = \int d\omega [c_{\omega}^{R} e^{-i\omega(\tau-\sigma_c)} + d_{\omega}^{R} e^{-i\omega(\tau+\sigma_c)}],$$  \hspace{1cm} (7.3)

where:

$$c_{\omega}^{R} = a_{\omega}^{R}(1 + \bar{A}_{\omega} + \bar{X}_{\omega}^{R}) + b_{\omega}^{R}(\bar{B}_{\omega} + \bar{Y}_{\omega}^{R})^*,$$  \hspace{1cm} (7.4)

$$d_{\omega}^{R} = a_{\omega}^{R}(\bar{B}_{\omega} + \bar{Y}_{\omega}^{R}) + b_{\omega}^{R}(1 + \bar{A}_{\omega} + \bar{X}_{\omega}^{R})^*.$$  \hspace{1cm} (7.5)

Eqs. (7.4)-(7.5) constitute the transformation giving the $(\sigma_c = +\infty)$- amplitudes as a linear superposition of the $(\sigma_c = -\infty)$-amplitudes. Energy conservation is then expressed as:

$$|a_{\omega}^{R}|^2 - |b_{\omega}^{R}|^2 = |c_{\omega}^{R}|^2 - |d_{\omega}^{R}|^2,$$  \hspace{1cm} (7.6)

which up to second order in $\lambda$ reads:

$$1 = 1 + |\bar{A}_{\omega}|^2 - |\bar{B}_{\omega}|^2 + 2Re(\bar{A}_{\omega} + \bar{X}_{\omega}^{R}).$$  \hspace{1cm} (7.7)
From Eqs. (5.11)-(5.12) and (6.11) we find (taking \( \omega > 0 \)):

\[
\bar{A}_\omega \equiv A_\omega (\sigma_c = +\infty) = 0,
\]

(7.8)

\[
\bar{B}_\omega \equiv B_\omega (\sigma_c = +\infty) = -i2\omega bK_o(2\omega b)\frac{r_g}{b},
\]

(7.9)

\[
\text{Re}(\bar{X}_\omega^R) \equiv \text{Re}(X_\omega^R(\sigma_c = +\infty)) = 2\omega^2 b^2 K_o^2(2\omega b)\left(\frac{r_g}{b}\right)^2,
\]

(7.10)

so that Eq. (7.7) is indeed fulfilled. Here \( K_o(x) \) is the Modified Bessel function with the integral representation [8]:

\[
K_o(x) = \int_0^{\infty} \frac{\cos xt}{\sqrt{1 + t^2}} dt ; \quad x > 0.
\]

(7.11)

Let us now consider a scattering process where an ingoing wave is partly reflected and partly transmitted, i.e. we look for solutions up to second order in \( \lambda \) in the form (see Fig.2.):

\[
\delta x^{(0-2)}_R(\sigma_c, \tau) = \left\{ \begin{array}{ll}
a_\omega^R e^{-i\omega(\tau - \sigma_c)} + b_\omega^R e^{-i\omega(\tau + \sigma_c)} & \text{for } \sigma_c \to -\infty \\
c_\omega^R e^{-i\omega(\tau - \sigma_c)} & \text{for } \sigma_c \to +\infty
\end{array} \right.
\]

(7.12)

\( a_\omega^R \) is then the amplitude of the ingoing wave, \( b_\omega^R \) is the amplitude of the reflected wave and \( c_\omega^R \) is the amplitude of the transmitted wave. The reflection coefficient and transmission coefficient are given by:

\[
R_\omega^R = \left| \frac{b_\omega^R}{a_\omega^R} \right|^2 , \quad T_\omega^R = 1 - R_\omega^R = \left| \frac{c_\omega^R}{a_\omega^R} \right|^2 .
\]

(7.13)

Using Eqs. (7.4)-(7.5) with \( d_\omega^R = 0 \) we get:

\[
R_\omega^R = |\bar{B}_\omega^R|^2 + 2\text{Re}[\bar{B}_\omega(\bar{Y}_\omega^R)^*] + \mathcal{O}(\lambda^4).
\]

(7.14)

The leading order of the reflection coefficient is therefore a second order term independent of the polarization and given by (from Eq. (7.9)):

\[
R_\omega^R = (2\omega b)^2 K_o^2(2\omega b)\left(\frac{r_g}{b}\right)^2 + \mathcal{O}(\lambda^3).
\]

(7.15)

The asymptotic behaviour of the Bessel function tells us [8]:

\[
R_\omega^R \sim \left\{ \begin{array}{ll}
(\omega b)^2 e^{-2\omega b} & \text{for } \omega b \gg 1 \\
(\omega b)^2 \ln^2 \omega b & \text{for } \omega b \ll 1
\end{array} \right.
\]

(7.16)
It follows that if the wavelength $\lambda = 2\pi/\omega$ of the free wave in the asymptotic region $\sigma = -\infty$ is considerably different from the "impact parameter" $b$, then the reflection coefficient effectively vanishes and the wave is almost completely transmitted.

To calculate the third order correction to $R^R_\omega$ we need the imaginary part of $Y^R_\omega$ in Eq. (7.14) since $\bar{B}_\omega$ is purely imaginary. The evaluation of $Y^R_\omega$ is quite involved (see the appendix). After some tedious calculations we find:

\begin{equation}
Im(Y_{\omega}^\perp) = -\pi e^{-2\omega b} \left( \frac{7\omega b}{128} - \frac{1}{16} - \frac{1}{32\omega b} \right) \left( \frac{r_g}{b} \right)^2 + 2\pi i \omega^2 b^2 K_0(2\omega b) \left( \frac{r_g}{b} \right)^2, \quad (7.17)
\end{equation}

\begin{equation}
Im(Y_{\omega}^\parallel) = Im(Y_{\omega}^\perp) - \frac{i\pi}{32} e^{-2\omega b} \left( \frac{3}{\omega b} + 6 + 4\omega b \right) \left( \frac{r_g}{b} \right)^2, \quad (7.18)
\end{equation}

from which we can write down explicit expressions for the third order corrections to $R^R_\omega$ (using also Eq. (7.9)). It is more interesting, however, to consider the difference in reflection coefficient in the 2 physical directions:

\begin{equation}
\Delta R_{\omega} \equiv R^\parallel_{\omega} - R^\perp_{\omega} = 2Re[B_{\omega}(Y^\parallel_{\omega} - Y^\perp_{\omega})^*]
\end{equation}

\begin{equation}
= \frac{\pi}{8} K_0(2\omega b) e^{-2\omega b} (3 + 6\omega b + 4\omega^2 b^2) \left( \frac{r_g}{b} \right)^3. \quad (7.19)
\end{equation}

This quantity is always positive so, not surprisingly, the reflection coefficient for waves in the equatorial plane is larger than the reflection coefficient for waves perpendicular to the equatorial plane, i.e. the waves feel the black hole strongest when they are propagating along the string in the equatorial plane. Finally we should mention that independently of the relative size of the wavelength $2\pi/\omega$ and the "impact parameter" $b$ (cf. the comment after equation (7.16)) our reflection coefficients are always quite small when evaluated as pure numbers. This is of course an artifact of our expansion scheme which assumes $r_g/b \ll 1$, i.e. the gravitational field of the black hole, that is responsible for a non-vanishing reflection coefficient, is assumed to be weak.

This concludes our investigations of the propagation of perturbations along a string in the equatorial plane of a Schwarzschild black hole.

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8 Appendix

In this appendix we give the explicit expressions for the functions $X^R_\omega(\sigma_c)$ and $Y^R_\omega(\sigma_c)$ introduced in Eqs. (6.11)-(6.12), and we present the calculations of $\text{Re}(\bar{X}^R_\omega)$ and $\text{Re}(\bar{Y}^R_\omega)$ used in Eqs. (7.10) and (7.17)-(7.18).

The explicit expressions for $X^R_\omega(\sigma_c)$ and $Y^R_\omega(\sigma_c)$ are (section 6):

\[
X^R_\omega(\sigma_c) = \frac{r_g^2}{2i\omega} \int_{-\infty}^{\sigma_c} \frac{5(b^2 - 2x^2)}{4(x^2 + b^2)^3} - \frac{3x(3b^2 - 2x^2)}{2(x^2 + b^2)^{7/2}} \sinh^{-1} \frac{x}{b} \cdot \\
- \frac{b^2}{2(x^2 + b^2)^3} \delta^R || - \frac{1}{8i\omega} \frac{x(2x^2 - b^2)}{(x^2 + b^2)^4} \cdot \\
- \frac{e^{-2i\omega x}}{8i\omega} \frac{2x^2 - b^2}{(x^2 + b^2)^{5/2}} \int_{-\infty}^{x} \frac{2t^2 - b^2}{(t^2 + b^2)^{5/2}} e^{2i\omega t} dt \cdot dx.
\]

\[(8.1)\]

and:

\[
Y^R_\omega(\sigma_c) = -\frac{r_g^2}{2i\omega} \int_{-\infty}^{\sigma_c} [-e^{2i\omega x} \frac{5(b^2 - 2x^2)}{4(x^2 + b^2)^3} - e^{2i\omega x} \frac{3x(3b^2 - 2x^2)}{2(x^2 + b^2)^{7/2}} \sinh^{-1} \frac{x}{b} \cdot \\
- \frac{b^2 e^{2i\omega x}}{2(x^2 + b^2)^3} \delta^R || - \frac{e^{2i\omega x}}{8i\omega} \frac{x(2x^2 - b^2)}{(x^2 + b^2)^4} \cdot \\
- \frac{1}{8i\omega} \frac{2x^2 - b^2}{(x^2 + b^2)^{5/2}} \int_{-\infty}^{x} \frac{2t^2 - b^2}{(t^2 + b^2)^{5/2}} e^{2i\omega t} dt] dx.
\]

\[(8.2)\]

with the only polarization dependence through the $\delta^R ||$-terms. For the calculation of $X^R_\omega$ and $Y^R_\omega$ we will use the following integrals [9]:

\[
\int_{0}^{\infty} \frac{\cos px}{x^2 + a^2} dx = \frac{\pi}{2b} e^{-pa},
\]

\[(8.3)\]

\[
\int_{0}^{\infty} \frac{\cos px}{\sqrt{x^2 + a^2}} dx = K_0(pa),
\]

\[(8.4)\]

\[
\int_{-\infty}^{\infty} \frac{x \cos px}{(x^2 + a^2)^{3/2}} \sinh^{-1} \frac{x}{a} dx = \frac{\pi}{a} (e^{-pa} - paK_0(pa)),
\]

\[(8.5)\]

as well as integrals obtained from them by differentiation with respect to the constants $a$ and $p$. We will also use the equation for the Modified Bessel function [8]:

\[
x^2K''_0(x) + xK'_0(x) - x^2K_0(x) = 0.
\]

\[(8.6)\]
The real part of $\bar{X}_R^\omega$ used in Eq. (7.10) is now easily calculated:

$$
Re(\bar{X}_R^\omega) = \frac{r_g^2}{16\omega^2} \int_{-\infty}^{\infty} \frac{2x^2 - b^2}{(x^2 + b^2)^{5/2}} \cos 2\omega x \int_{-\infty}^{x} \frac{2t^2 - b^2}{(t^2 + b^2)^{5/2}} \cos 2\omega dt \, dx
$$

$$
= \frac{r_g^2}{32\omega^3} \left[ \int_{-\infty}^{\infty} \frac{2x^2 - b^2}{(x^2 + b^2)^{5/2}} \cos 2\omega x \, dx \right]^2
$$

$$
= 2\omega^2 b^2 K_0^2 (2\omega b) \left( \frac{r_g}{b} \right)^2.
$$

(8.7)

The imaginary part of $\bar{Y}_R^\omega$ used in Eqs. (2.17)-(2.18) leads to:

$$
Im(\bar{Y}_R^\omega) = -\frac{r_g^2}{2i\omega} \int_{-\infty}^{\infty} \frac{5(b^2 - 2x^2)}{4(x^2 + b^2)^2} \, dx - \frac{3x(3b^2 - 2x^2)}{2(x^2 + b^2)^{7/2}} \sinh^{-1} \frac{x}{b} \cos 2\omega x \, dx
$$

$$
+ \frac{r_g^2 b^2}{4i\omega} \delta_{\parallel} \int_{-\infty}^{\infty} \frac{\cos 2\omega x}{(x^2 + b^2)^2} \, dx + \frac{r_g^2}{8i\omega^2} \int_{-\infty}^{\infty} \frac{x \sin 2\omega x}{(x^2 + b^2)^4} (2x^2 - b^2) \, dx
$$

$$
\equiv \alpha_\omega + \beta_\omega \delta_{\parallel} + \gamma_\omega,
$$

(8.8)

and from Eqs. (8.3)-(8.6):

$$
\alpha_\omega = -\frac{\pi}{2t} \left[ 4\omega^2 b^2 K_0(2\omega b) + \frac{e^{-2\omega b}}{8\omega b} \left( \frac{3}{4} + \frac{3}{2} \frac{\omega b}{2} - \frac{15}{8} \omega^2 b^2 \right) \left( \frac{r_g}{b} \right)^2 \right],
$$

(8.9)

$$
\beta_\omega = \frac{\pi}{32t} \left[ \frac{3}{\omega b} + 6 + 4\omega b \right] \left( \frac{r_g}{b} \right)^2,
$$

(8.10)

$$
\gamma_\omega = \frac{\pi}{64t} \left[ \frac{1}{\omega b} + 2 - 4\omega b \right] \left( \frac{r_g}{b} \right)^2,
$$

(8.11)

in agreement with Eqs. (7.17)-(7.18).
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Figure captions

Fig.1. The potential $U_{\perp\perp}$ of Eq. (3.13) as a function of the string parameter $\sigma_c$: $\sigma_c = 0$ corresponds to the minimal distance $r = m + \sqrt{m^2 + b^2}$ between the stationary string and the black hole, the 2 zeroes of $U_{\perp\perp}$ correspond to $r = m + \sqrt{m^2 + 3b^2}/2$ while the 2 maxima correspond to $\pm r = 4m/3 + \sqrt{16m^2/9 + 5b^2}/2$. The potential $U_{\parallel\parallel}$ has the same shape, but is a little deeper near the black hole.

Fig.2. Schematic representation of the scattering processes described in section 7: An ingoing wave from the asymptotic region $\sigma_c = -\infty$ is partly reflected and partly transmitted because of the interaction with the gravitational field of the black hole. The circle represents the horizon of the black hole in the equatorial plane.