The cubic matrix model
and a duality
between strings and loops

Lee Smolin

Center for Gravitational Physics and Geometry
Department of Physics
The Pennsylvania State University
University Park, PA, USA 16802
and
The Blackett Laboratory
Imperial College of Science, Technology and Medicine
South Kensington, London SW7 2BZ, UK
smolin@phys.psu.edu

June 6, 2000

ABSTRACT

We find evidence for a duality between the standard matrix formulations of $\mathcal{M}$ theory
and a background independent theory which extends loop quantum gravity by replacing
$SU(2)$ with a supersymmetric and quantum group extension of $SU(16)$. This is deduced
from the recently proposed cubic matrix model for $\mathcal{M}$ theory which has been argued to have
compactifications which reduce to the IKKT and dWHN-BFSS matrix models. Here we
find new compactifications of this theory whose Hilbert spaces consist of $SU(16)$ conformal
blocks on compact two-surfaces. These compactifications break the $SU(N)$ symmetry of the
standard $\mathcal{M}$ theory compactifications, while preserving $SU(16)$, while the BFSS model pre-
serve the $SU(N)$ but break $SU(16)$ to the $SO(9)$ symmetry of the 11 dimensional light cone
coordinates. These results suggest that the supersymmetric and quantum deformed $SU(16)$
extension of loop quantum gravity provides a dual, background independent description of
the degrees of freedom and dynamics of the $\mathcal{M}$ theory matrix models.
1 Introduction

One of the oldest and deepest ideas in gauge theories is the conjectured duality between a loop description, based on the Wilson loops, or holonomies, of a gauge theory and a string description, in which the position of a string of quantized electric flux are taken as the fundamental coordinates\cite{1}. Taken into the gravitational context, this suggests that there should be a duality between string theory and loop quantum gravity, as the latter is based on the quantum dynamics of the Wilson loops of the spacetime connection\cite{2}-\cite{5}. In this paper we propose a specific form for such a string/loop duality, by finding evidence that a particular extension of loop quantum gravity is dual to the standard dWHN-BFSS\cite{6, 7, 8} and IKKT\cite{9} matrix models. We do this by arguing that both arise from different compactifications of a single matrix model, called the cubic matrix model\cite{10}.

In a recent paper the cubic matrix models were proposed, and we presented some evidence that one of them has compactifications which reproduce the IKKT and dWHN-BFSS matrix models\cite{10}. In this paper we study a new class of compactifications of these models, which lead directly to a background independent description of the theory. This turns out to be an extension of loop quantum gravity in which the usual $SU(2)$ algebra has been replaced by a supersymmetric and quantum extension of $SU(16)$. The $SU(16)$ means that the model extends the symmetries of the 9 dimensional Clifford algebra, while the quantum group extension means that it is also in a class of theories previously proposed as background independent membrane field theories\cite{11, 12}. In \cite{13} we argued that when based on an $SU(16)$ symmetry this may provide a background independent formulation of $M$ theory. Here we provide independent evidence for this claim by showing how it can be obtained by compactification of a model that we argue has other compactifications that lead to the standard $M$ theory matrix models. This suggests that the full duality group of $M$ theory relates the standard background dependent descriptions of $M$ theory to this new class of background independent theories.

The model we defined in \cite{10} is based on two simple ideas. First the degrees of freedom are an $N \times N$ matrix, $M$, whose elements themselves are matrices, valued in $Osp(1\mid 32)$. Thus, the matrix elements refer to elements of an algebra rather than to positions in a flat background geometry. The different background geometries must then arise from expansions around classical solutions that break the algebra to one that generates the symmetry group of a background spacetime. The algebra $Osp(1\mid 32)$ is chosen because there is evidence that it may be the full symmetry group of $M$ theory\cite{14, 15}. Second, the action has the simplest non-trivial form possible, which is the trace of the cube of $M$.

In \cite{10} we argued that this theory has a compactification which break the symmetries of $M$ down to the Super-Poincare group in $9+1$ dimensions. The one loop effective action describing fluctuations around the classical solution representing the compactification was argued to reproduce the $IKKT$ matrix model. Another compactification was described, based on a classical solution that breaks the symmetry to the Super-Euclidean group of the light cone gauge of $10+1$ dimensional Minkowski spacetime. We argued in \cite{10} that the one
loop effective action includes the dWHN-BFFS matrix model.1

The model has, however, many classical solutions in which the $Osp(1|32)$ is broken to a group which is not the symmetry group of any spacetime. The small fluctuations around one of these solutions describes a quantum mechanical system which cannot be interpreted in terms of a background spacetime. We may call these non-geometrical compactifications. As these incorporate more of the $Osp(1|32)$ group which has been conjectured to be the full symmetry group of $\mathcal{M}$ theory, it is necessary to understand these solutions to understand the full physical content of the theory and the full range of duality transformations which operate on its solutions.

Among the nongeometrical compactifications of the cubic matrix model are a special set whose continuum limits are Chern-Simons field theories. As these are topological field theories, they are independent of any metric. Further, quantum Chern-Simons theory is exactly solvable and its state space is understood in terms of the representation theory of quantum groups. This gives rise to a completely algebraic description of the physics of these non-geometrical compactifications. This can be understood in several different ways, one of which involves spaces of conformal blocks, or intertwiners of quantum groups, on 2-surfaces. As was pointed out first by Crane and Kauffman, the $k \to \infty$ limit of this structure (where $k$ is the level, or the coupling constant of the Chern-Simons theory) reproduces the spin networks which label a normalizable bases of diffeomorphism invariant states in loop quantum gravity descriptions of quantum gravity and supergravity. We will see here that this can be used to derive an extension of loop quantum gravity from the Chern-Simons compactifications of the cubic matrix model.

To make these claims precise we must distinguish two meanings of background independence. We can have a quantum field theory that depends on the topology and differential structure of a manifold of some given dimension, but is independent of any classical fields. Such theories must then have active diffeomorphisms as part of their gauge group. As a result all fields are represented by quantum operators which are subject to evolution by dynamical laws. We may call these field-background independent theories. Loop quantum gravity has provided examples of such theories, in several different dimensions, with and without supersymmetry.

However, one can require that the theory not depend on even the dimension, topology or differential structure of a manifold. We may call such a theory manifold-background independent. A theory with both properties may be called totally-background independent.

If string theory has a background independent formulation it must be totally background independent, because its different solutions are defined on manifolds of different dimension and topology. Loop quantum gravity, in the form originally given in, is not such a theory. This is because its derivation through a rigorous quantization of general relativity and supergravity required that the manifold, dimension and differential structure be fixed. However, it is possible to extend the structure of loop quantum gravity to make it totally background independent. As explained in this requires thickening the spin

1These claims are supported by one loop calculations that will be described in detail in.
networks to 2-surfaces labeled by conformal blocks, or intertwiners of a quantum group. This extension is also motivated by the fact that it solves a key problem in loop quantum gravity, because it introduces certain terms in the Hamiltonian or action which are required both for the existence of a classical limit\[25, 26\] and the recovery of spacetime diffeomorphism invariance\[24, 27\].

It was noted in \[11, 12\] that the resulting theories may be understood as background independent membrane field theories, and that the $SL(2,\mathbb{Z})$ duality group can be easily represented in a background independent fashion. This recalls the old arguments that the fundamental objects in \(\mathcal{M}\) theory should be membranes[14]. To realize this idea in the present context we must invent a theory in which the embeddings of the membrane in different background can be extracted from the intertwiners of a quantum group, by symmetry breaking. A detailed conjecture for how to do this was proposed in \[13\]. It was found there that to match the dWHN-BFSS matrix model the quantum group must be a quantum deformation of a super-Lie algebra that extends $SU(16)$. Here we will derive a closely related theory from a non-geometrical compactification of the cubic matrix model, but with a simpler dynamics, which is also very reminiscent of the dynamics generated by the Hamiltonian constraint in quantum gravity and supergravity[28].

The version of the cubic matrix model we study here was introduced in [10], and is characterized by the fact that it uses a complexification of the $Osp(1|32)$ degrees of freedom which is represented in terms of $SU(16,16|1)$ matrices. We posit here the simplest possible action, given by

\[ I = \frac{k}{4\pi} Tr M^3 \]  

where $M_{iA}^B$ is a double matrix with $i, j = 1, \ldots, N$ and $A, B = 1, \ldots, 33$. For fixed $i, j$, $M_{iA}^B$ is an element in the 33 dimensional adjoint representation of $SU(16,16|1)$. This was called the gauged action in [10].

It is not difficult to carry out the same steps as described in [10] to show that this model has compactifications that reduce at the one loop level to the IKKT and BFSS models. This will be discussed very briefly in section 8 below and described in full detail elsewhere[16]. We may note that these compactifications involve breaking the $SU(16,16|1)$ symmetry to $SO(9,1)$ and $SO(9)$, respectively. The non-geometric compactifications we will study break the symmetry in a different way.

\[ SU(16,16) \to Sp(2) \oplus SU(16) \]

In the next section we introduce the model and some of its properties. In section 3 we describe a Chern-Simons compactification leading to an $SU(16)$ Chern-Simons theory. In section 4 we show how related compactifications give rise to a set of interacting Chern-Simons theories. In sections 5 to 7 we describe the quantization of these compactifications and show how they reproduce the extension of loop quantum gravity described in [11, 12, 13]. An argument for how the dWHN-BFSS theory may be recovered from the effective action in a different limit of the theory is sketched in section 8. This allows us, in section 9 to describe
the explicit duality that holds between the standard background dependent matrix models of $\mathcal{M}$ theory and the background independent description derived here. This allows us to answer questions such as what corresponds to $D0$ branes in the background independent language. The paper closes with a brief mention of some of the important open problems raised by the results reported here.

2 The $SU(16,16|1)$ matrix theory

We begin by describing the model and its basic properties. We recall first the definition of $Osp(1|32)$, which consists of supermatrices of 32 even dimensions and one odd dimension, $M^B_A$ that preserve the graded antisymmetric metric

$$G^B_A = \begin{bmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3)$$

(where the first two rows and columns are $16 \times 16$ bosonic blocks, while the third row and column is in the one fermionic coordinate) so that

$$M \cdot G = -G \cdot M^T \quad (4)$$

where $T$ stands for transpose. We complexify this by considering complex matrices of the same form, which satisfy

$$M \cdot G = -G \cdot M^\dagger \quad (5)$$

where $\dagger$ means hermitian conjugate. These generate a supergroup by $R = \exp M$, which satisfies $R \cdot G \cdot R^\dagger = G$. This group is $SU(16,16|1)$, as can be seen from the fact that $iG^B_A$ is an hermitian metric of signature $(16,16)$. We may decompose $M^B_A$ as follows,

$$M^B_A = \begin{pmatrix} A_2 + Y & A_- & \Psi \\ A_+ & -A_2 + Y & \Phi \\ \Phi^\dagger & -\Psi^\dagger & 0 \end{pmatrix} \quad (6)$$

where the $A_2, A_\pm$ are three $16 \times 16$ hermitian matrices, $Y$ is a tracefree $16 \times 16$ antihermitian matrix and $\Psi^A$ and $\Phi^{A'}$ are 16 component spinors. We will also find it convenient to use $A_0 = A_+ - A_-$ and $A_1 = A_+ + A_-$. The components of $Y$ and $A_a$ for $a = 0, 1, 2$ are even Grassman variables, while the components of $\Phi$ and $\Psi$ are odd Grassman. We will decompose the 33 dimensional indices $A, B, \ldots$ as $A = (P, P', \cdot)$ where $P = 1, \ldots, 16$, $P' = 1', \ldots, 16'$, and $\cdot$ is the lone fermionic index. We will find it very useful to consider the decomposition

$$SU(16,16) \rightarrow Sp(2) \oplus SU(16) \quad (7)$$

where the $SU(16)$ is generated by

$$W^B_A = \begin{pmatrix} Y & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (8)$$
and the generators of $Sp(2)$ are given by

$$
\tau^0 = \begin{pmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau^1 = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

These satisfy

$$
\frac{1}{32} Tr \tau^a \tau^b = \eta^{ab}
$$

where $\eta^{ab}$ is the 2 + 1 dimensional metric

$$
\eta^{ab} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

thus showing $Sp(2) = SO(2,1)$.

The $SU(16)$ transformations can be extended to the superalgebra $SU(16|1)$, leading to a reduction of superalgebras,

$$SU(16, 16|1) \rightarrow Sp(2) \oplus SU(16|1).
$$

We now extend the matrix by considering each entry to be an $N \times N$ matrix parameterized by $i, j = 1, \ldots, N$. We thus have a double matrix $M_{Ai}^{Bj}$. We define the action

$$I_{gauged} = \frac{k}{4\pi} Tr M^3
$$

where the multiplication and trace is on both sets of indices. $k$ is the coupling constant of the theory. This action is somewhat simpler than that studied in [10] and has more gauge symmetry, as we can see by writing it more explicitly

$$I_{gauged} = \frac{k}{4\pi} \sum_{ijk} Tr_{SU(16,16|1)} M_{i}^{j} \cdot M_{j}^{k} \cdot M_{k}^{i}
$$

where for each $i, j, k$ the multiplications and trace are over the $SU(16,16|1)$ indices. The action is then invariant under transformations

$$M_{i}^{j} \rightarrow U(i) \cdot M_{i}^{j} \cdot U(j)^\dagger
$$

where, for each $i$, $U(i) \in SU(16,16|1)$. Similarly, for each $A$ and $B$ we have the gauge symmetry, suppressing now the $SU(N)$ indices

$$M_{A}^{B} \rightarrow V(A) \cdot M_{A}^{B} \cdot V(B)^\dagger
$$

where $V(A) \in SU(N)$. For this reason we call it a gauged matrix action.
It is useful to decompose the action in terms of the variables defined in (6). We have
\[
I_{\text{gauged}} = \frac{k}{4\pi} \text{Tr}_{ij} \text{Tr}_{PQ} \left\{ \epsilon^{abc}(A_a A_b A_c) + Y A_a A_b \eta^{ab} + Y^3 \right\} \\
+ 3 \text{Tr}_{ij} \left\{ \Phi^P A^Q_{-P} \Phi_Q - \Psi^P A^Q_{+P} \Psi_Q - A_{2PQ} \{ \Psi^Q, \Phi^P \} + Y_{PQ} \{ \Psi^Q, \Phi^P \} \right\}
\]  
(17)

We see explicitly here the decomposition into \(Sp(2) \oplus SU(16|1)\).

It is also interesting to note that when the matrix elements are restricted to be real, so that the symmetry is reduced from \(SU(16, 16|1)\) to \(Osp(1|32)\) the action has a translation invariance given by
\[
A^Q_{aP} \rightarrow A^Q_{aP} = A^Q_{aP} + \delta_i^Q V_{aP} 
\]
(18)
with \(V_{aP} = V^\dagger_{aP}\). This is reduced from the symmetry of the model studied in [10] but the remaining translation symmetry includes that of the dWHN-BFSS and IKKT models, when we identify the fields of those models with components of \(M^R_B\) as described in [10].

It is also of interest to consider the theory without the \(Y\) degrees of freedom. This theory is invariant under the \(Sp(2) \oplus SU(16|1)\) subalgebra of \(SU(16, 16|1)\). Its action is given by
\[
I^{\text{MCS}} = I_{Y=0}^{\text{gauged}} = \frac{k}{4\pi} \text{Tr}_{ij} \left\{ \epsilon^{abc} \text{Tr}_{PQ}(A_a A_b A_c) + \chi^{ijP} \cdot \tau^a \cdot A^Q_{aP} \chi_Q \right\}
\]  
(19)

We will call this \textit{matrix Chern-Simons theory} in the following.

### 3 A simple Chern-Simons compactification

The Chern-Simons compactification comes about because for each \(a\), \(A_a = (A^Q_{aP})^j_i\) is an \(N \times N\) matrix of 16 dimensional hermitian matrices. There are then compactifications of the model in which the \(\epsilon^{abc}(A_a A_b A_c)\) term in the action becomes an \(SU(16)\) Chern-Simons theory. To see this we compactify the theory on a three-torus, making use of a modification of the route studied in [10]. We consider the classical solutions given by
\[
A^Q_{aiP} = \delta^Q_P (\partial_a)^i_j
\]
(20)
with the fermionic fields vanishing. We divide the indices so as to make three derivative operators. We choose \(i = i_0, i_1, i_2\) where \(i_a = -M_a, \ldots, 0, 1, \ldots, M_a\) such that \(N = \Pi_{a=0,1,2}(2M_a + 1)\). We then choose
\[
(\partial_0)^j_i = (\partial_0)^{i_0j_i} = i_0 \delta_{i_0}^j \delta_{i_1}^i \delta_{i_2}^j 
\]
(22)
and similarly for the other two derivative operators. Clearly we have \([\partial_a, \partial_b] = 0\). We then expand around this classical solution, using the usual matrix compactification trick [30, 8], defining variables
\[
A^Q_{aiP} = \delta^Q_P (\partial_a)^i_j + a^Q_{aiP}
\]
(23)
Following the usual translation into continuum fields\[30, 8\], we find in the limit \(M_a \to \infty\)

\[
I^{\text{gauged}} = \frac{k}{4\pi} \int_{\mathcal{T}^3} Tr_{SU(16)} \left\{ a \wedge da + \frac{2}{3} a^3 + \chi^\dagger \tau^a D_a \chi \right\}
+ (Y_P^Q)^3 + Y_{PQ} \eta^{ab} (D_a \bar{a}_b)^{QP} + Y_{PQ} [\Psi_P, \Phi_Q]
\] (24)

where \(\bar{\eta}_{PQ}\) is tracefree.

If we neglect the coupling to the \(Y\) field this is a Chern-Simons theory on \(T^3\).

Note that had we begun with the bosonic part matrix Chern-Simons theory \([13]\), whose action is just \(e^{abc} Tr(A_a A_b A_c)\), the result would just be the first two terms of (24). In this case something remarkable has happened, which is that the dependence on a background metric on the torus, which is implicit in the definition of the compactification, has gone away in the limit \(M_a \to \infty\), leaving a topological field theory. We may note that with the ultraviolet cutoff \(l_{\text{Planck}}\) held fixed, this is equivalent to the limit in which all three compactification radii are taken to infinity. The fact that a topological field theory emerges from the limit is a consequence of the large amount of gauge symmetry in the original action, together with the taking of a limit in which the length scales set by the compactification radii are removed.

The coupling to the spinor variables \(\chi\) depends on the background \(\tau^a\) which define the compactification directions. These result in a supersymmetrization of the state space of Chern-Simons theory as we will discuss in \([33]\).

The theory with the \(Y\) terms present is more subtle, but it is possible that they will not change the topological character of the theory in the limit of infinite compactification radii, up to renormalizations of the coupling constant of Chern-Simons theory. The reason is that we can use the gauge symmetry \([13]\) to set \(Y^i_j = 0\) locally in \(i\) and \(j\). Given any sequence of values for \(i\), with no repeats, given by \(i_\mu = i_1, \ldots, i_L\), we can set \(Y^{i_{\mu+1}}_{i_\mu} = 0\). To see this note that by \([13]\)

\[
\delta Y^i_j = \delta M^a_{i\alpha P} = U(j)^{\gamma R}_{\alpha P} M_i^{\alpha Q}_\gamma R = M^i_{\alpha P} U(j)^{\alpha Q}_{\gamma R}
\] (25)

Thus, \(Y^i_j\) transforms like a gauge potential, it can be set to zero along any open curve in the space labeled by \(i, j\) but its effect on closed curves in those indices must be taken into account. In a matrix compactification the indices \(i\) and \(j\) refer to Fourier modes, in particular the local field is constructed from sums such as \(Y(x) = \sum_k Y^i_{i_0 + k} e^{-ik \cdot x}\). This means we have a kind of gauge invariance in momentum space. As a result, we can eliminate the \(Y\)’s from any scattering matrix involving other degrees of freedom. But we cannot eliminate closed loops of \(Y\)’s in any Feynman diagram, as they will correspond to a closed loop of matrix entries. The effect on the classical equations of motion will only be through the \(Y\) equation itself. The bosonic equations of motion which follow from (24) have the form

\[
e^{abc} F_{bc} - \chi^\dagger \tau^c \chi - \eta^{cd} D_d Y = 0
\] (26)

\[
Y^2_{PQ} + (D_a a_b)_{PQ} \eta^{ab} + [\Psi_P, \Phi_Q] = 0
\] (27)

If we set \(Y = 0\) at a point we have the equations of Chern-Simons theory plus the condition

\[
(D_a a_b)_{PQ} \eta^{ab} + [\Psi_P, \Phi_Q] = 0
\] (28)
But this differs only by the fermion term from the standard gauge fixing term which is used to define Chern-Simons theory perturbatively. This means that at least up to the effects of closed loops in the $Y$ variables, the theory given by (24) is a supersymmetric extension of the $SU(16)$ Chern-Simons theory.

This does not completely resolve the question of the influence of the $Y$ degrees of freedom. However, it suggests that the effect of the $Y$’s on the Chern-Simons compactification can only be to renormalize the coupling of the Chern-Simons theory. For this reason we will neglect the effects of the $Y$’s in the next sections where we study the physics of the Chern-Simons compactifications.

4 The (Chern-Simons)\(^P\) compactification

We now introduce a different set of compactifications that reduce the theory to a coupled set of $P$ Chern-Simons theories, for any $P$. The idea is to blow up each of the $i_a - j_a$ entries of the previous compactification into $P \times P$ blocks. We then have a symmetry, for each $i_a, j_a$, which is $Sp(2) \oplus SU(16) \oplus U(P)$. When we make a compactification on the three torus, as just described, we find a 3 dimensional quantum field theory with symmetry $SU(16) \oplus U(P)$.

There are however, several different ways of taking the limit that defines the field theory, which result in a different set of fluctuating fields with different symmetry. One way, which we will sketch in section 8, preserves the $U(P)$ symmetry and breaks the $SU(16)$ down to the $SO(9)$ symmetry of the lightcone gauge of 10 + 1 dimensional spacetime.

Here we will consider a way to preserve the full $SU(16)$, but break the $U(P)$ symmetry completely. This leads not to one 3d quantum field theory with $U(P)$ symmetry, but to $P$ 3d field theories, each of which defines a Chern-Simons theory. These interact via bi-local fields that create and annihilate pairs of punctures that join the tori on which these Chern-Simons theories are defined. As we will show in the next several sections, this gives a background independent theory. This theory is a version of the background independent membrane field theory\([11, 12]\) which was proposed in\([13]\) as a background independent version of $M$ theory.

We begin the demonstration by choosing integer factors $M_a$ and $P$ such that $\prod_a (2M_a + 1)P = N$. We will write

$$i = i_0, i_1, i_2, I$$

with $I = 1, \ldots, P$ and $i_a = -M_a, \ldots, M_a$.

We then decompose $M^J_I$ according to (with all the other indices suppressed)

$$M^J_I = \begin{bmatrix} A_1^1 & B_1^2 & B_1^3 & \ldots \\ B_2^1 & A_2^2 & B_2^3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

That is, we define

$$A_{\alpha I_0 i_1 i_2 P}^{J=I_0 j_1 j_2 Q} = A_{\alpha a i_1 i_2 P}^{I_0 j_1 j_2 Q}$$

and for the off diagonal terms,

$$A_{\alpha I_0 i_1 i_2 P}^{J=I_0 j_1 j_2 Q} \gamma_\alpha = B_{\alpha a i_1 i_2 P}^{\beta j_1 j_2 Q}$$
We continue to neglect the role of the $Y$ field. The dynamics is given by (with the $i_a$ indices suppressed)

\[
I^{\text{gauged}} = \frac{k}{4\pi} T_{I_a} \left\{ \sum_I \epsilon^{abc} (A^{I Q}_{a P} A^{I R}_{b Q} A^{I P}_{c R}) + \chi_a I_{\alpha \beta} A^{I Q}_{a P} \chi^{\beta I} \\
+ \sum_{J \neq I} \left[ 3B^{I Q}_{a P} b^{J \alpha}_{a Q} B^{I P}_{J R} + \chi_{\alpha I}^{J \beta} A^{I Q}_{a P} \chi^{\beta I}_Q + \chi_{\alpha I}^{P J} B^{I Q}_{P J} \chi^{\beta I}_Q \\
+ \sum_{J \neq I \neq K} \left[ B^{I Q}_{a P} B^{K R}_{J Q} B^{I P}_{K R} + \chi_{\alpha I}^{P J} B^{K Q}_{P J} \chi^{\beta I}_Q \right] \right\}
\]

(33)

We see we get $P$ theories described by an $A^I$ coupled by terms involving the $B^J_I$ and $\chi^{\beta I}_{Q K}$ variables.

The theory has a $U(P)$ symmetry mixing up the $I, J$ indices. We now define a class of compactifications which can lead to a quantum theory that breaks this symmetry. To do this we will define a Chern-Simons compactification to an $SU(16)$ Chern-Simons theory separately for each $I$. For each of the $P$ diagonal $A^I$'s we define the compactification to an $SU(16)$ gauge theory described in the last section. This leads, for each $I$, to a Chern-Simons theory on a three torus based on the fusion algebra of $SU(16)$.

This multiple-Chern-Simons compactification also induces a transformation on the $B$ variables. We find (for $J \neq I$) that when $M_a \to \infty$ the $B^{I j \beta j}_{I a j}$ become bilocal fields whose domain are pairs of the $T^3$ on which the Chern-Simons theories are defined.

\[
B^{I j \beta j}_{I a j} \to B^{I j \beta j}_{I a j}(x_I, x_J)
\]

(34)

We also have bilocal fermionic fields given by

\[
\chi^{a j}_{Q I a j I} \to \chi^{a j}_{Q I}(x_I, x_J)
\]

(35)

Thus after the $P$ simultaneous three-torus compactifications and the $M_a \to \infty$ limits the action of the theory becomes

\[
I^{\text{gauged}} = \frac{k}{4\pi} \sum_I S_{CS}^I(A^I, \chi^I) + \sum_{J \neq I} \frac{k}{4\pi} \int_{T^3_I} d^3 x_I \int_{T^3_J} d^3 x_J \left\{ T_{R G} \left( B^{I \beta}_{I a}(x_I, x_J) \gamma^\alpha_{\beta \gamma} D^{\beta}_{a b} B^{I a}_{J b}(x_J, x_I) \right) \\
+ \chi_{\alpha I}^{J P}(x_I, x_J) \gamma^\alpha_{\beta \gamma} D^{\beta}_{a b} \chi^{I \beta\gamma}_{Q I}(x_I, x_J) + \chi_{\alpha I}^{P J}(x_J, x_I) \gamma^\alpha_{\beta \gamma} D^{\beta}_{a b} \chi^{I \beta\gamma}_{Q I}(x_I, x_J) \\
+ \sum_{I \neq J \neq K} \frac{k}{4\pi} \int_{T^3_I} d^3 x_I \int_{T^3_J} d^3 x_J \int_{T^3_K} d^3 x_K T_{R G} \left( B^{I \beta}_{I a}(x_I, x_J) B^{K \gamma}_{J a}(x_J, x_K) B^{I a}_{K a}(x_K, x_I) \\
+ \chi_{\alpha I}^{J P}(x_I, x_J) B^{K Q}_{J a}(x_J, x_K) \chi^{I \beta\gamma}_{Q K}(x_K, x_I) \right) \right\}
\]

(36)
where
\[ S^I_{CS}(A^I, \chi^I) = \int_{T^3} Tr_{SU(16)} \left\{ a \wedge da + \frac{2}{3}a^3 + \chi^I \sigma^a D_a \chi \right\} \]

Thus, we have \( P \) Chern-Simons theories, each defined on a distinct 3-torus, which interact with each other via the bi-local fields \( B^I_{\alpha}(x_I, x_J) \).

### 5 Hamiltonian dynamics of the \((\text{Chern-Simons})^P\) compactification

We now study the dynamics of the theory defined by the multiple Chern-Simons compactification. To simplify the discussion we will ignore the fermion fields and consider only the bosonic parts of (36). The fermion terms give a supersymmetric completion of the structure we will describe here; details of this will be described elsewhere [33].

To uncover the dynamics of the theory in this compactification we make a \( 2+1 \) splitting in each of the \( P \) 3-tori. This gives coordinates \( x^3_I = (t_I, x^m_I) \), where we define spatial indices \( l, m, n = 1, 2 \). We also let the compactification radius for each time coordinate go to infinity, so that the Hamiltonian theory becomes defined on a domain which is \( P \) copies of \( R \times T^2 \). The action then takes the form,

\[ I^{\text{gauged}} = \sum_I \frac{k}{4\pi} \int dt I \int d^2 x I Tr_{SU(16)} \left[ \epsilon^{mn} A^I_m A^I_n - A^I_0 G^I \right] \]

\[ + \sum_{I>l} \frac{k}{4\pi} \int dt I d^2 x I dt J d^2 x J Tr \left\{ (B^I_{\alpha}(x_I, t_I; x_J, t_J))_{\gamma}^\beta \tau_{\gamma}^0 \left( \frac{\partial}{\partial t_I} - \frac{\partial}{\partial t_J} \right) B^I_{\alpha}(x_I, t_I, x_J, t_J) \right\} \]

\[ + \sum_{I\neq J\neq K} \frac{k}{4\pi} \int d^3 x I \int_{T^3} d^3 x J \int_{T^3} d^3 x K Tr \left( B^I_{\alpha}(x_I, x_J) B^J_{\beta}(x_J, x_K) B^K_{\gamma}(x_K, x_I) \right) \]

where the Gauss's law constraint is

\[ G^I(x_I, t) = \frac{1}{2} \epsilon^{mn} F_{mn}^I(x_I, t) + \sum_{J>I} \int dt J \int d^2 x_J (B^I_{\alpha}(x_I, t_I; x_J, t_J))_{\gamma}^\beta \tau_{\gamma}^0 \left( B^I_{\alpha}(x_I, t_I, x_J, t_J) \right) \]

\[ - \sum_{K<J} \int dt_K \int d^2 x_J (B^K_{\gamma}(x_I, t_I; x_J, t_K))_{\alpha}^\beta \tau_{\gamma}^0 \left( B^K_{\gamma}(x_K, t_K, x_I, t_I) \right) \]

We see that the theory appears to be non-local in time. We will shortly see that this is an expression of a many fingered time gauge invariance, in that the time can be evolved independently on each torus.

The \( A^I_m \) have conventional momenta,

\[ \pi^{im}(x_I, t) = \frac{\delta I^{\text{cubic}}}{A^I_m(x_I, t)} = \frac{k}{2\pi} \epsilon^{mn} A^I_n(x_I, t) \]
The $B_I^J$'s depend on two time coordinates, one in each of the two tori they live in. The momenta then similarly depend on two time and two space variables. As we see from the momenta of the $B_I^J$'s depend on the difference of its two time coordinates,

$$\Pi^J(x_I, t_I; x_J, t_J)_{\alpha \beta}^{\gamma} \equiv \frac{\delta^{\text{cubic}}}{(\partial \tau - \partial \tau_J)B_I^J} \equiv \frac{k}{4\pi}(B_I^J)_{\alpha \beta}^{\gamma}(x_I, t_I, x_J, t_J)_{\tau^\gamma}$$

To define a polarization we pick an arbitrary ordering of the $P$ tori. This breaks the $U(P)$ symmetry. We then take the variables $B_I^J(x_I, t, x_J, s)$ for $J > I$ to be configuration variables, while the momenta are coded in the $B_I^J$ for $J < I$.

Note also that there is some freedom of the density transformation properties of the fields defined by the compactification. This must be defined so that all the integrands are densities. A sensible choice seems to be to define the compactification in such a way that the $B_I^J(x^a_I, x^b_J)$ are densities on the second spacetime variables $x^a_J$ and ordinary functions on the first. The momenta will then have the opposite density weights, i.e. weight one in the first slot and zero in the second.

Finally, we write the action in Hamiltonian form

$$I^{\text{gauged}} = \sum_I \int dt_I \int d^2 x_I T_{\text{SU}(16)} \left[ \pi^{I m} A_m^I - A_0^I G_I^I \right]$$

$$+ \sum_{J > I} \int dt_I d^2 x_I dt_J d^2 x_J T_{\text{SU}(16)} \left[ \Pi^J_I(x_I, t_I; x_J, t_J)_{\alpha \beta}^{\gamma} \left( \frac{\partial}{\partial t_I} - \frac{\partial}{\partial t_J} \right) B_I^J(x_I, t_I, x_J, t_J) \right]$$

$$+ \sum_{J > I} \int dt_I dt_J H_2^{IJ}(t_I, t_J) + \sum_{I < J < K} \int dt_I dt_J dt_K H_3^{IKJ}(t_I, t_J, t_K)$$

where the Gauss’s law constraint is now

$$G_I(x_I, t) = \frac{1}{2} \epsilon_{mn} F_{mn}^I(x_I, t) - j_I^I(x_I, t) = 0$$

where the $SU(16)$ current for each Chern-Simons theory is given by

$$j_Q^{KP}(x_K, t_K) = \sum_{I < K} \int d^3 x_J T_{\text{Sp}(2)} \Pi^K_{IR} \cdot B_{IQ}^{KR} - \sum_{K < J} \int d^3 x_J T_{\text{Sp}(2)} \Pi^K_{IR} \cdot B_{IQ}^{KR}$$

and the two and three time Hamiltonians are given by

$$H_2^{IJ}(t_I, t_J) = \int d^2 x_I d^2 x_J T_{\text{SU}(16)} \Phi_I^J(x_I, t; x_J, s)_{\alpha \beta}^{\gamma} \left( D^I_m - D^J_m \right) B_{IJ}^\alpha(x_I, t_I, x_J, s)$$

$$H_3^{JK}(t_I, t_J, t_K) = 3 \int d^2 x_I d^2 x_J d^2 x_K T_{\text{tr}} \left( B_{IK}^\gamma \Phi_I^\gamma \Phi_I^\gamma + B_{IJ}^\gamma \Phi_I^\gamma \Phi_I^\gamma \right)$$

Here $\Phi_I^J = I_{J}^{\gamma} \gamma_J^{I \gamma}$.
6 Quantization of the \((CS)^P\) compactification

What we have is a multi-time theory in which a set of \(P\) Chern-Simons theories, each with a time coordinate, are coupled at all pairs and triples of times. This can be treated to some extent like a many fingered time, as we will now see.

The canonical commutation relations for the \(B\)’s have the structure, for \(I < J\) and \(K < L\),

\[
[B^J_{I\alpha}(x_I, t, x_J, s)_{AB}, \Pi^K_L(x_K, t; x_L, u)_{\gamma\delta}] = \delta^K_I \delta^J_L \delta(s, u) \delta^2(x_K, x_I) \delta^2(x_J, x_L) \delta^K_I \delta^J_L \delta_{AB} \delta_{\gamma\delta} \delta_{CD} (48)
\]

To realize these quantum mechanically we are going to have to have a Hilbert space for every set of \(P\) times \(t_I\). We will call this space \(H(t_I)\). For each set of \(t_I\) we will have operators that act on this space that satisfy (48). This kind of structure is common to certain histories formulations of quantum theory, such as those studied in [34, 35, 36]. In those papers it is shown that this kind of multitime quantum theory is natural for histories formulations of systems with spacetime diffeomorphism invariance.

The form of the Poisson brackets suggests that we quantize in a generalization of the connection representation in which the states are of the form

\[
\Psi[\{t_I\}] = \Psi[A_I(t_I), B^J_I(t_I, t_J)] (49)
\]

for \(J > I\).

The first step in the quantization will be to find solutions to the Gauss’ law constraint (44). To define this we note that the states are defined to be functions only of the configuration variables at a set of \(P\) fixed times \(t_I\). Thus, when acting on a state \(\Psi(t_I)\) the integral in 45 will for each \(J\) only pick up times in that list. The \(\delta(s, t)\) in (48) absorbs the integral over \(s\); thus, the action of \(\hat{j}^I(x_I)\) on states has the form

\[
\hat{j}^I(x_I, t_I) \Psi[A_I(t_I), B^J_I(t_I, t_J)] \equiv \left\{ \sum_{J > I} \int d^3x_J B^J_I(x_I, t_I, x_J, t_J) \delta \frac{\delta B_I^J(x_I, t_I, x_J, t_J)}{\delta B_I^J(x_I, t_I, x_J, t_J)} \right\} \Psi[A_I(t_I), B^J_I(t_I, t_J)] (50)
\]

A natural set of solutions to these equations may be obtained by making an ansatz

\[
\Psi[A_I(t_I), B^J_I(t_I, t_J)] = \chi[B_I^J] \prod_K \Phi_K[A^K] (51)
\]

To solve this we want to find the action of the operator representing the current (45), which acts as

\[
\hat{j}^K_Q(x_K) \chi[B_I^J] = \left\{ \sum_{I < K} \int d^3x_I Tr_{Sp(2)} B^K_I \delta B^Q_K \delta B^R_K \right\} \chi[B_I^J]. (52)
\]

13
It is clear that there are solutions to the Gauss law which involve finite products of $B^I_J$’s. In such a state, for each pair of tori $T^2_I$ and $T^2_J$, there will be a finite number of points $(p^a_I, p^b_J) \in T^2_I \times T^2_J$ for $a = 1, \ldots, n_{IJ}$ on which there are $B^I_J$’s in the product \((51)\). We will call these punctures. The current will then have the form

$$J^K_P(x_K) = \sum_{I<K}^{n_{IK}} \sum_{a=1}^{n_{K}} \delta^2(p^a_K, x_K) J^a[B^K_I] + \sum_{K<J}^{n_{KJ}} \sum_{b=1}^{n_{J}} \delta^2(p^b_K, x_K) J^b[B^K_J]$$

where the currents $J^a[B^K_I]$ depend on the $B$’s indicated. As a consequence $\Phi_K(A^K)$ satisfies the condition

$$\frac{k}{2\pi} \tilde{F}_{12}^K(x_K) \Phi_K(A^K) = \left( \sum_{I<K}^{n_{IK}} \sum_{a=1}^{n_{K}} \delta^2(p^a_K, x_K) J^a[B^K_I] - \sum_{K<J}^{n_{KJ}} \sum_{b=1}^{n_{J}} \delta^2(p^b_K, x_K) J^b[B^K_J] \right) \Phi_K(A^K)$$

But this is a familiar equation from quantum Chern-Simons theory \([17, 18, 20, 19]\). The connection on each tori is flat except for a finite set of punctures at which there is a delta function contribution, which depends on the $B$’s. We can solve this in the standard way, expressing the solutions in terms of conformal blocks or intertwiners of $G$ on the punctured two torus. The dependence of the states on the $B$’s will be expressed in terms of the representations labeling the punctures. As a consequence the punctures will satisfy braid statistics. This means care must be taken if we create two punctures on top of each other. A simple solution to this problem which we will employ is to construct the states with all punctures distinct and use the recoupling relations of the quantum group associated to $SU(16)$ to extend their values to the cases of multiple punctures at the same point.

But once the punctures are distinct we can use the gauge freedom in \((15)\) to reduce the number of independent components of the $B^I_J$’s. Taking $U(i)$ in \((13)\) to be valued in $Sp(2)$ we have transformations,

$$A^a_I(x_I) \tau^a \rightarrow U_I^{-1}(x_I) \cdot A^a_I(x_I) \tau^a \cdot U_I(x_I)$$

$$B^I_J(x_I, x_J) \rightarrow U_I^{-1}(x_I) \cdot B^I_J(x_I, x_J) \cdot U_J(x_J)$$

Since the punctures are distinct we can use this gauge freedom to diagonalize the $B^I_J$. Given the definition of the momenta, this gives us, for $I < J$

$$B^I_J = \begin{bmatrix} b^I_J & 0 \\ 0 & b^J_I \end{bmatrix}$$

while for $J > I$ we define momenta\(^2\)

$$B^I_J = \begin{bmatrix} p^I_J & 0 \\ 0 & p^J_I \end{bmatrix}$$

\(^2\)It is convenient here to take for the time variable the coefficient of $\tau^2$ in the parameterization we used previously.

14
The commutation relations are then,

\[ [b_{\pm I}^J(x_I, t, x_J, s)_{PQ}, p_{\pm K}^L(x_K, t; x_L, u)]^{RS} = \pm i\delta^K_I\delta^J_{\pm I} \delta(s, u)\delta^2(x_K, x_I) \delta^2(x_J, x_L) \delta^{RS}_{PQ} \] (59)

The two degrees of freedom of the \( B_I^J \) correspond to the two cases in which the current flows from \( I \) to \( J \) (with \( I < J \)) in a positive or a negative sense. The current is now,

\[ J^K_P(x_K, t_K) = \sum_{\pm} (\pm 1) \left\{ \sum_{I < K} \int d^3x d^J_{P\pm KR} \delta^K_I \pm \sum_{K < J} \int d^3x J^P_{JKR} \delta^K_I \right\} \] (60)

We then work with the reduced Hilbert space

\[ \Psi[A_I^J(x_I, t_I), b_{\pm I}^J(x_I, t_I; x_J, t_J)] \] (61)

for all ordered pairs \( J > I \). On this space we will have the kinematical operators,

\[ \hat{A}_I^J(x_I, t_I) = A_I^J(x_I, t_I) \] (62)

\[ \hat{A}_I^J(x_I, t_I) = i\hbar \frac{\delta}{\delta A_I^J(x_I, t_I)} \] (63)

\[ \hat{b}_{\pm I}^J(x_I, t_I; x_J, t_J) = b_{\pm I}^J(x_I, t_I; x_J, t_J) \] (64)

\[ \hat{p}_{\pm I}^J(x_I, t_I; x_J, t_J) = \pm i\hbar \frac{\delta}{\delta b_{\pm I}^J(x_I, t_I; x_J, t_J)} \] (65)

We can then describe the solutions to the Gauss’s law constraint as follows.

Let us then pick \( R \) pairs of punctures, each member of a pair is a point on a distinct 2-torus. The pairs are labeled by \( w = 1, \ldots, R \) and each pair is given by a point on the 2-torus’s \( I_w \) and \( J_w \) labeled by,

\[ p_w = (p_w^+, p_w^-) \in T_{I_w}^2 \times T_{J_w}^2 \] (66)

Each puncture has a polarity \( \epsilon(w) = \pm \). We then consider an ansatz for the wavefunction,

\[ \Psi[A^I, b_{\pm I}^J] = \prod_w b_{\epsilon(w) I}^J(p_w^+, p_w^-) \prod_I \Phi_I(A^I) \] (67)

These satisfy

\[ \hat{j}_I^J(x_I) \Psi[A^I, b_{\pm I}^J] = \sum_w \epsilon(w) \left( J_w \delta I_w^2(x_I, p_w^+) - J_w \delta I_w^2(x_I, p_w^-) \right) \Psi[A^I, b_{\pm I}^J] \] (68)

where \( J_w \in SU(16) \). This means that \( \Phi_I(A^I) \) satisfies the condition

\[ \frac{k}{2\pi} \hat{F}^{I}_{12}(x_I) \Phi_I(A^I) = \sum_w \epsilon(w) \left( J_w \delta I_w^2(x_I, p_w^+) - J_w \delta I_w^2(x_I, p_w^-) \right) \Phi_I(A^I) \] (69)
Figure 1: A typical solution to the Gauss’s law constraint contains $P$ punctured tori joined in pairs. On each torus the state lives in the space of conformal blocks or intertwiners defined by putting each puncture in the fundamental representation of $SU(16)$ quantum deformed to level $k$.

We can solve this by using the usual methods from Chern-Simons theory \[17, 18, 20, 19\]. We may picture a typical state in terms of punctured tori whose punctures are connected in pairs as in Figure 1.

Finally, we may note that once we have fixed the $B_I^J$ to the diagonal gauge the quadratic terms in the Hamiltonian, $H_{IJ}^2$, all vanish. This suggests that the role of these terms is in implementing spatial diffeomorphism invariance on each of the 2-tori. Since the choice of representation of the $Sp(2)$ matrices defined which components of the matrix variables corresponded to the spatial coordinates of the tori it is not surprising that there is in this way a coupling between the $Sp(2)$ gauge freedom and spatial diffeomorphisms.

We may then finally describe the space of gauge invariant and diffeomorphism invariant states as follows. We can first describe an auxiliary linear space $H_{aux}^P$ constructed from all ways to join $P$ tori along pairs of punctures. Given a set of $R$ punctures labeled by $\sigma = (I_w, p^+_w, J_w, p^-_w)$ we join our $P$ tori into a set of compact 2-surfaces by joining them along each pair of punctures in the set. For each $w$ we draw a circle around the puncture $p^+_w$ on the $I_w$'th torus and another circle around the puncture $p^-_w$ on the $J_w$'th torus. We then remove the interiors of each circle and join the two boundaries together. The resulting single circle $c_w$ is oriented and labeled with an $s$, corresponding to a projection operator which restricts the current flowing across $c_w$ to be in the spinor representation.

Once all these operations are completed we have a (generally disjoint) 2-surface we may call $S_\sigma$. The result of our construction is a Hilbert space $V_\sigma^P$ which is the space of intertwiners on $S_\sigma$ subject to the condition that there are projection operators for the spinor representation on the $R$ oriented circles $c_w$.

For each $P$ and $R$ the space of states lives in the auxiliary Hilbert space given by

$$
H_{aux}^{P,R} = \sum_{\sigma} V_\sigma^P
$$

where the sum is over all sets $\sigma$ of $R$ pairs of punctures. For each $P$ we may let $R \to \infty$ to
define
\[ \mathcal{H}_{aux}^P = \lim_{R \to \infty} \mathcal{H}_{aux}^{P,R}. \] (71)

The auxiliary space (71) is larger than the physical state space in that a given conformal block may be created many different ways by joining states on punctured tori in this way. As a result there is a non-trivial action of the group, \( \mathcal{R} \) of modular transformations\([18, 20, 19]\), or recoupling relations\([21]\) of the conformal blocks on \( \mathcal{H}_{aux}^P \). (There are also linear relations between states in different \( \mathcal{H}_{aux}^P \) for different \( P \).) We then define the physical state space to be,
\[ \mathcal{H}_{phys}^P = \mathcal{H}_{aux}^P / \mathcal{R}. \] (72)

It is not difficult to argue that any conformal block on any compact surface of genus higher than 1 can be built up in this way. As a result we have
\[ \mathcal{H}_{phys}^P = \sum_{g \geq 1} \mathcal{V}_g \] (73)

where \( \mathcal{V}_g \) is the space of intertwiners on the compact surface of genus \( g \). For each \( P \) this is the physical Hilbert space of the theory. We note that because there is no limit on the numbers of punctures all the Hilbert spaces are identical for \( P \geq 2 \).

The translation from states on punctured tori to states on compact surfaces is illustrated in Figure (2). We note that each \( \mathcal{S}_\sigma \) can be decomposed non-uniquely into a set of \( P \) 2-tori joined on pairs of circles. Given a basis for each of the spaces of intertwiners on the punctured two tori then each such decomposition gives a basis for \( \mathcal{H}_{aux}^P \). We arbitrarily fix one such basis, which we will call the reference basis. This is necessary because the construction of the Hilbert space from a set of solutions to the quantum Gauss law constraint produces such a basis. As we will see Hamiltonian comes expressed in that basis.

Finally we note that the state space (described in 72) is almost the same as that discussed in \([1, 12, 13]\) The main difference is that the zero genus state is excluded from the present theory, and in the proposal of \([1, 12, 13]\) we considered only states arising from
connected two-surfaces. This means that the state space we have arrived at is a natural background independent extension of the spin network states of quantum general relativity and supergravity, with $SU(16)$ in place of $SU(2)$ or $SU(2|1)$.

7 Action of the Hamiltonian and evolution rules

We now construct the action of the remaining cubic term (47) in the Hamiltonian on the Hilbert space (72). The Hamiltonian is constructed in terms of the operators $b_{±I}^I$ and $p_{±I}^I$ which means their action is defined on the auxiliary space (71). The action on the physical space (72) is defined by imposing equivalence under modular transformations. We have

$$
\sum_{I<J<K} \int dt_It_Jt_K H^{IJK}(t_I, t_J, t_K)\Psi(\{t_I\}) =
$$

$$
3 \sum_{±} \int d^2x_Id^2x_Jd^2x_K \text{Tr}_G \left\{ \sum_{I<J<K} b_{±I}^Jb_{±J}^Kp_{±I}^K + \frac{4\pi}{k} \sum_{I<K<J} b_{±I}^Jp_{±J}^KP_{±I}^K \right\} \quad (74)
$$

The action of $H_3$ on states is given in Figure (3). $H_3$ has two kinds of actions. Two of the terms eliminate two links labeled by the $b$’s and creates a new one as shown in Fig. (3). In this case the action of $H$ eliminates a positive and negative puncture on a single torus. It then induces a map from $\mathcal{V}_s \otimes \bar{s} \rightarrow \mathcal{V}_{r_2 \ldots}$, where $s$ is the fundamental representation, which is defined by taking the intertwiner $s \otimes \bar{s} \rightarrow \text{Id}$. The other terms act in the opposite way to eliminate a $b^K_I$ joining tori $I$ and $K$ and create, for every distinct torus $I \neq J \neq K$ a link from $I$ to $J$ and a link from $J$ to $K$. The action on the space of intertwiners on $J$ then acts in the opposite way, through the channel $\text{Id} \rightarrow s \otimes \bar{s}$.

Translated to an action on states on compact two surfaces rather than punctured tori the action of $H_3$ is illustrated in Figure (4).

We may note that the action has exactly the form of the Hamiltonian constraint for quantum general relativity, found in [28] with the replacement of $SU(2)$ by the quantum

---

Footnote: different variants are discussed in [29].
deformation of $SU(16)$. This can be seen in Figure (5) where we indicate schematically the action of the Hamiltonian constraint of general relativity found in [28, 29].

This means that if we restrict $SU(16)$ to an $SU(2)$ subgroup, the dynamics predicted by the cubic matrix model will be of the same form as that found from canonical quantization of quantum general relativity. In terms of the dual picture of [24] this gives the $1 \rightarrow 3$ and $3 \rightarrow 1$ Pachner moves. Both the $1 \rightarrow 3$ and $3 \rightarrow 1$ moves arise in the theory with the same amplitude, showing that the Hamiltonian is hermitian.

The fact that the present theory is at finite $k$ means, moreover, that the difficulty found for quantum general relativity, in which there is no long range propagation of information, and hence no chance of a classical limit [25, 26] need not be present. The reason, as pointed out in [11] is that at finite $k$ the missing moves which are required to have propagation of information are the $2 \rightarrow 2$ Pachner moves, shown in Figure (6). However, these moves are present for all theories with finite $k$ as they are just changes of basis in the space of intertwiners as illustrated in Figure (6). Thus, if we consider a sequence of moves for finite $k$ induced by $H_3$ we may insert at any point the change of basis shown in Fig. 5. For finite $k$ this is just a change of basis and the history in terms of finite $k$ states is unchanged whether we do it or not. But this operation does not commute with the limit $k \rightarrow \infty$. This means that the histories that must be included in the sum over histories at $k \rightarrow \infty$ on the spin network states includes the insertion of all possible finite $k$ change of basis moves. Since these are no longer changes of bases for $k = \infty$ this is equivalent to the insertion in the $k = \infty$ hamiltonian constraint of the change of basis move, with the amplitudes given by the $k \rightarrow \infty$ limit of the $6j$ symbol. This is equivalent to saying that the theory has these $2 \rightarrow 2$ moves with the amplitudes given by the BF or Crane-Yetter-Ooguri theory, while the $1 \rightarrow 3$ and $3 \rightarrow 1$ moves have the non-topological form given by $H_3$. However, with the inclusion of these $2 \rightarrow 2$ moves the problem with lack of long range correlations found in [25, 26] is solved, as pointed out in [27, 24].

Finally, we note that the fact that the $2 \rightarrow 2$ moves have a different amplitude than the $1 \leftrightarrow 3$ move means that there is a dependence of the amplitudes on the causal structure of
Figure 5: The action of the Hamiltonian constraint on spin network states in quantum general relativity indicated schematically. The dependence on representations and the coefficients of the different terms are not indicated but can be found in [28, 29]. The action of the constraint induces Wilson lines as shown, inserted into the edges as shown by the dots. The summation indicates a summation over certain representations, which are suppressed. The $k \to \infty$ limit has been taken so that the compact two surfaces dressed with intertwiners become graphs dressed by representations.

Figure 6: The change of basis formula for finite $k$ induces new $2 \to 2$ moves in the limit $k \to \infty$. 
the quantum history[24, 11]. The theory does not have the crossing symmetry required by a Euclidean quantum gravity theory[27, 38], it is then intrinsically a causal theory in the sense of [24, 11].

8 Compactification to the dWHN-BFSS model

In this section we describe very briefly a different compactifications which appears to lead to the dWHN-BFSS model at the one loop level. This will be discussed in detail in [16]. The required compactification involves a simultaneous compactification on three directions, but it is different than the one described in section 4. There we considered a compactification that broke the $U(P)$ symmetry involving the indices $I, J, . . .$ while preserving the whole $SU(16)$ symmetry. Now we consider a compactification that does the reverse: we preserve the $SU(16)$ symmetry, but break the $SU(16)$ symmetry.

To do this we break the fields in terms of $SU(16)$ traces and trace-free parts:

$$A_{aiI_0i_1i_2}^{J_{j_0j_1j_2}Q} = \delta_P^Q A_{aiI_0i_1i_2}^{J_{j_0j_1j_2}Q} \tilde{A}_{aiI_0i_1i_2}^{J_{j_0j_1j_2}Q} \quad (75)$$

where $\tilde{A}_{aiI_0i_1i_2}^{J_{j_0j_1j_2}P} = 0$.

We begin as before by expanding around the classical solution

$$A_{aiI_0i_1i_2}^{J_{j_0j_1j_2}Q} = \delta_P^Q \left( \delta_P^Q a_{aiI_0i_1i_2}^{J_{j_0j_1j_2}} + \tilde{A}_{aiI_0i_1i_2}^{J_{j_0j_1j_2}Q} \right) \quad (76)$$

However, what is different in this case is that we define the compactification in such a way that all of the $SU(16)$ components of the fields becoming $SU(P)$ matrices. To do this we employ a parameterization that preserves the $U(P)$ gauge symmetry which is

$$A_{aiI_0i_1i_2}^{J_{j_0j_1j_2}Q} = \delta_P^Q \left( \delta_P^Q a_{aiI_0i_1i_2}^{J_{j_0j_1j_2}} + \tilde{A}_{aiI_0i_1i_2}^{J_{j_0j_1j_2}Q} \right) \quad (77)$$

We will also restrict the fields to real values, so that the symmetry is reduced from $SU(16, 16|1)$ to $Osp(1|32)$.

We now compactify on the three directions associated with $x^+, x^−$ and $x^2$. This is similar to the compactification we used up till this point, but we will treat it in a rather different manner. Rather than considering the limit in which all three compactification radii go to infinity, which leads us to a Chern-Simons type theory, we will keep the compactification radii $L_{2−} = (2M_{2−} + 1)L_{Pl}$ small, while taking only the third radius $L_+ = (2M_+ + 1)L_{Pl}$ large. This breaks the $SO(2, 1) = Sp(2)$ symmetry. The gauged action can be written as

$$I^{gauged} = \frac{k}{4\pi} \int_{T^3} dx^+ dx_− dx_2 Tr_{U(P)} \left\{ Y^{CS}(a) + \epsilon^{abc} \left( 3\tilde{A}_{aP}^Q D_b \tilde{A}_{cQ}^R + \tilde{A}_{aP}^Q \tilde{A}_{bQ}^R \tilde{A}_{cR}^P \right) + Y^3 + \eta^{ab} \tilde{Y}_P^Q \left( \{D_a, \tilde{A}_{bQ}^P \} + \tilde{A}_{aQ}^R \tilde{A}_{bR}^P \right) + \chi P \tau^a_{PQ} D_a \chi^Q \right\}$$

21
although we must remember that as we have not taken the compactification radii to infinity what is really meant is the cutoff version of this theory with $2M_a + 1$ modes in each direction. An important consequence of not taking the limit of infinite compactification radii is that the theory knows about the $2 + 1$ dimensional metric $\eta^{ab}$.

The next step is to compute the effective action. This will be described in [16], here we only discuss how the symmetries of the theory constrain its form.

The form of the effective potential will be governed by the symmetries of the action which are left unbroken by the compactification. The local bosonic symmetries includes $U(P)$. In the case of compactifications with independent compactification radii $L_+ \neq L_- \neq L_2$, the $SU(16, 16)$ symmetry has been broken down into a subgroup, which is $SO(9)$. This can be seen from the fact that in a standard parameterization of the components of the matrix $M$ in terms of 32 component $\Gamma$ matrices for $Spin(10, 1)$, the $+$, $-$ and 2 components that defined the three $\tau^a$ matrices that generate the $Sp(2)$ symmetry are in the 0, 10 and 0 $\wedge$ 10 boost directions [10].

There are in addition the remaining translation symmetries involving the tracefree parts, $\tilde{A}_{aP}$. These are symmetries of the action in the case that we restrict all fields to the real slice. This is one reason we have restricted the fields to the real case. Finally, one can check that 16 of the 32 supersymmetries have been broken, so that there remain 16 unbroken supersymmetry generators.

The lowest dimensional terms that can appear in the effective action, consistent with these symmetries are,

$$I^{1\ \mathrm{loop}} = \int_{T^3} dx_+ dx_- dx dT U(P) \left\{ f_{ab} f_{cd} \eta^{ac} \eta^{bd} + [D_a, \tilde{A}_b][D_c, \tilde{A}_d] \eta^{ac} \eta^{bd} + (D_a \tilde{A}_b \eta^{ab})^2 + \text{fermions + interactions} \right\}$$

(79)

We now consider the limit

$$L_-, L_2 \to l_{Pl}, \quad L_+ \to \infty.$$  

(78)

At the same time we boost to the infinite momentum frame in the + direction. All fields decouple except those that have the maximal number of $\partial_+$ derivatives. These turn out to be the $\tilde{A}_-$ and $\Psi$ fields.

In this limit the only terms that survive in the kinetic energy are

$$I^{1\ \mathrm{loop}} = \int dx_+ Tr U(P) \left\{ (\partial_+ \tilde{A}_-)^2 + \Psi^P \partial_+ \Psi_P \right\}$$

(79)

We see that the fields which remain consist of one 16 component fermion together with the symmetric tensor $A_{-PQ}$. This decomposes as

$$A_{-PQ} = \Gamma_{PQ}^\mu X_\mu + \Gamma_{PQ}^{\mu\nu\rho\sigma} X_{\mu\nu\rho\sigma}$$

(80)

The $X_\mu$ are the nine transverse matrices of the dWHN-BFSS theory, which correspond to the $D0$ brane coordinates as well as to the light cone gauge coordinates of the embedding
of a membrane. The $X_{\mu\nu\rho\sigma}$ are additional degrees of freedom that do not appear in that model.

The leading terms in a derivative expansion of the effective action will be completed by interaction terms amongst these degrees of freedom. These will be determined by the unbroken symmetries, which are precisely the symmetries of the dWHN-BFSS theory. The action involving $\Psi^P$ and $X^\mu$ with these symmetries is exactly the dWHN-BFSS theory. One can check that this is extended by terms involving the four form field $X_{\mu\nu\rho\sigma}$. The full action invariant under the translations (18), $SO(9)$ rotations and 16 supersymmetry generators is

$$I^{1\, \text{loop}} = \int dx^+ Tr_U(P) \left\{ \left( \partial_+ \tilde{A}_- \right)^2 + \Psi^P \partial_+ \Psi_P + \Psi^P [\tilde{A}_{-PQ}, \Psi^Q] + \tilde{A}_{-Q} \tilde{A}^P_{-R} \tilde{A}^R_S \right\}$$

One can check that the $X_{\alpha\beta\gamma\delta}$ terms do not break the supersymmetry of the dWHN-BFSS theory. Instead, the symmetry algebra is extended by central charges that exist for any $P$. In the standard dWHN-BFSS model these central charges appear only in the limit $P \to \infty$. In that case the central charges are interpreted to describe certain components the 5-brane. This suggests that the new degrees of freedom $X_{\alpha\beta\gamma\delta}$ provide an additional description of the 5-brane, wrapped on the $X^+$ and $X^-$ directions. This will be discussed in more detail elsewhere.

9 The duality between strings and loops

From the results of the last several sections it is possible to describe precisely how the BFSS matrix model and the $SU(16)_q$ extension of loop quantum gravity give equivalent descriptions of the same degrees of freedom. We can do this by tracing how each description is derived from a compactification of the cubic matrix model.

In both cases we make use of the decomposition $SU(16,16|1) \to Sp(2) \oplus SU(16|1)$ and decompose the fields in terms of the basic degrees of freedom, $A_{aIa,P}^{JbQ}$

The $a$ is the $Sp(2)$ index and the $P$ and $Q$ are $SU(16)$ indices. We have then decomposed the $N \times N$ components of the matrices in terms of two sets of indices so that

$$i = (i_a, I)$$
where \( i_a = 1, \ldots, M_a \) are the indices that will give rise to compactification on a three torus, and \( I = 1, \ldots, P \) defines a remaining \( U(P) \) symmetry.

In both cases we begin by defining a compactification on a 3-torus, leaving aside a \( U(P) \) symmetry. Each compactification is then defined by an expansion around the classical solution

\[
(A^0)_{IaP} = \delta^I_I \delta^Q_Q \partial_a a^Q_{ia} 
\]  

Each then defines a fluctuating field

\[
A^{JQ}_{iaP} = \delta^J_I \delta^Q_Q \partial_a a^Q_{ia} + A^{JQ}_{iaP} 
\]  

The stringy and loopy descriptions then part company. In each case one of the two remaining symmetries, which are \( U(P) \) and \( SU(16) \), are broken and the other is treated as a gauge symmetry in the background created by the compactification. The two compactifications differ as to which components of the fluctuating field \( A^{JQ}_{iaP} \) are treated as a gauge field and which are taken as a matter-like field.

There is freedom to chose which symmetry appears as an ordinary gauge symmetry because before compactification the fields have three index types: \( A_{iaIP} \). The \( i_a \) describe the fourier modes of the spatial dependence on the 3 torus, while the \( I \) and \( P \) parameterize respectively \( SU(P) \) and \( SU(16) \). The gauge symmetry as it appears as a local symmetry in the three dimensional field theory on the \( T^3 \)'s comes from a symmetry of the form of either \((\mathbb{4})\) or \((\mathbb{15})\). But there are now three sets of indices, which must be grouped into two sets. The first set determines which indices the gauge parameters depend on, while the second determines the gauge group. For example if we group the \( i_a \) with the \( I \) then we have gauge transformations of the form

\[
A^{jQ}_{iaP} \rightarrow U(i_aI) \cdot A^{jQ}_{iaP} \cdot U(j_aJ)^\dagger 
\]  

where \( U(i_aI) \in SU(16) \). This gives an \( SU(16) \) gauge symmetry that acts locally in each of the tori labeled by \( I \). This is the symmetry of the background independent multi-tori compactifications. In this representation of the theory the \( U(P) \) symmetry is hidden. It is possible that it is expressed by the recoupling relations of the intertwiners of \( SU(16) \) (equivalently the modular group of the 2-surfaces acting on the conformal blocks).

In this case we defined the components of the fluctuating fields as

\[
A^{jQ}_{iaP} = \delta^Q_I A^{IQ}_{iaP} + B^{jQ}_{iaP} 
\]  

We treated the diagonal components \( A^{IQ}_{iaP} \) as gauge fields for the \( SU(16) \) symmetry. There are \( P \) of them, corresponding to the diagonal components of \( U(P) \) matrices, so we have a separate gauge invariance on each of \( P \) tori. The off-diagonal (in \( U(P) \)) components \( B^{jQ}_{iaP} \) defined only for \( I \neq J \) define interactions between the \( P \) Chern-Simons theories defined by the diagonal components \( a^{IQ}_{iaP} \). The \( U(P) \) symmetry is broken by the choice of polarization, which requires an arbitrary ordering of the \( P \) Chern-Simons theories.
The other choice is to make manifest the gauge symmetry that comes from grouping the $i_a$ indices with the $SU(16)$ indices. In this case we write,

$$A_{i_a P}^{j_b} \rightarrow W(i_a P) \cdot A_{i_a P}^{j_b} \cdot W(j_a Q)$$

where $W(i_a P) \in SU(P)$. In this case we get a local $SU(P)$ gauge symmetry on a single torus. This leads to the case of the stringy compactification, which is expressed in terms of a local $SU(P)$ gauge theory, which we argued leads to the dWHN-BFSS matrix model.

In this case we we defined the components of the fluctuating field as

$$A_{aI_i}^{Jj_b} = \delta^{Q}_{aI_i} a_{aI_i}^{Jj_b} + \tilde{A}_{aI_i}^{Jj_b}$$

where $\tilde{A}_{aI_i}^{Jj_b} = 0$. The scalar (in $SU(16)$ terms) components, $a_{aI_i}^{Jj_b}$ define the fluctuations in the compactification radii, and then must become the string theory moduli. The theory is treated from that point on as an $U(P)$ gauge theory, where the scalars $a_{aI_i}^{Jj_b}$ play the role of gauge fields in the compactified directions and the $SU(16|1)$ symmetry is broken down to the superpoincare algebra in 10 + 1 dimensions or the super-euclidean algebra in 9 dimensions.

The different theories are treated differently in other ways, particularly in the fact that to get the standard matrix models two or three of the compactified radii must be taken to the Planck scale, leading to a low energy theory defined in either 0 + 1 or 0 + 0 spacetime dimensions (in the dWHN-BFSS and IKKT cases, respectively.) But the essential difference between them is defined by these two different coordinatizations of the fluctuating fields around the classical solution (85).

As a consequence of the identifications defined here we have a genuine translation between the loopy and stringy descriptions of the kinematics and dynamics of the cubic matrix model. This may be used to translate problems from one description to the other. For example, the $P \rightarrow \infty$ limit which is problematic for the conventional matrix models is clearly related to the limit of loop quantum gravity in which the universe grows infinitely large. On the other side we can say that the continuum limit of the loopy description may involve a restoration of the $U(P)$ symmetry which is broken by the quantization described here. The correspondence may also be used to translate the $D$-brane description of black hole horizons into the loop quantum gravity description, which is based on describing the state space on the horizon in terms of conformal blocks. This may make possible a description of black holes which is not restricted to the near extremal case of positive specific heat. Similarly, we may try to use the correspondence to extend the description of boundaries with non-zero cosmological constant given in to arrive at a detailed description of the AdS/CFT correspondence in 3 + 1 dimensions.

The duality can also be expressed by considering how the fundamental excitations are described in the two pictures. In the multi-Chern-Simons compactifications, the fundamental degrees of freedom are the pairs of punctures which are created or destroyed by the dynamical terms in the Hamiltonian. It is then tempting to see them as the background independent analogue of $D_0$ branes. There is, in fact, some more direct evidence that these excitations are related to $D_0$ branes, which is described in. We then have two different representations of
these degrees of freedom, either in terms of the light cone gauge components of the matrices $A_{-I}$, which lead to the standard matrix description of $D0$ branes, or as the operators which create and annihilate punctures which join the $P$ tori. By tracing through the correspondence we have just discussed we can see explicitly how the two descriptions of the fundamental degrees of freedom may be translated into each other. It is then perhaps fitting to call this the string/loop duality.

10 Conclusions

In this paper we have studied a new kind of compactification for $M$ theory, which is defined, not in terms of a background geometry, but in terms of an algebra, that is the fusion algebra of conformal blocks for the quantum deformation of $SU(16)$ on arbitrary 2-surfaces. We may then call this an algebraic compactification. It is clear from the construction that many other algebraic compactifications can be defined corresponding to the reduction of the representation theory of quantum deformed $SU(16)$ to the representation theory of its subalgebras.

There are many issues raised by this formulation. Some are technical. The most pressing of these include the need to study in detail the quantum group extension of $SU(16)$ defined
by a Chern-Simons theory with a connection valued in that algebra and the need to develop the details of its representation theory. Another set of issues to be discussed in [33] involve the details of the supersymmetric extension of the results described here. We want to understand also the way in which the modular group acting on compact two surfaces is related to the $U(P)$ symmetry that was broken in the multi-Chern-Simons compactification. The computation of the effective action which we sketched above will be discussed in detail in [19]. The relationship between the real and complex forms of the theory needs more investigation. Finally, there remain subtle issues associated with the role of the $Y$ degrees of freedom.

On the conceptual side, the role of the many fingered time of the form found here needs further study. The action appears to be non-local in time, but the multi-time quantization discussed here seems to lead to a sensible notion of a history. One may begin with an initial state at a time $t_I = 0$ for all $t_I$ and then act repeatedly by the cubic Hamiltonian to evolve, on each action, triples of Hilbert spaces forward in time. In this way one can generate a history, which seems to be of a form which is closely related to that described in [11]. In that case acting repeatedly with the moves generates a causal history, of the general form studied in [35, 36]. The causal structure results from the ordering dependence of the action of the different terms in the Hamiltonian, the time labels themselves seem irrelevant apart from ordering. This structure is very suggestive but deserves further study. The relationship to the histories projection operator formulation developed in [34] is also very suggestive.

Another interesting issue to discuss is the relationship of this formulation with the holographic principle and the related problem of the interpretation of quantum theories of cosmology. The construction in [11, 12, 13] was motivated in part by the holographic principle, which we showed in [32, 42] appears naturally in loop quantum gravity when boundaries are considered. It is interesting to consider each one of the Hilbert spaces $H(t_I)$ to be a screen, in a background independent formulation of the holographic principle of the kind described in [37]. We may then try to follow the argument there and construct the quantum geometry by defining the area of each screen to be the log of the dimension of its space of intertwiners, as in [32].

When we take the limit $k \to \infty$ we reproduce, as argued in [11] exactly the spin network states of quantum general relativity or supergravity. Moreover, the cubic action gives rise to evolution rules which are exactly of the same form as follow from a first principles, canonical quantization of general relativity [28]. The main difference is that because the theory is defined at finite $k$, new evolution rules must appear in the $k \to \infty$ limit, exactly of the form required to cure a major problem of loop quantum gravity, which is the absence of long ranged correlations at zero cosmological constant [25, 26].

It is very interesting to note that there are other derivations of path integral formulations for loop quantum gravity from matrix models [38, 27, 39]. It is possible that there is a direct derivation from the cubic matrix model to matrix models of the form used in those derivations, which is induced by quantum corrections to the present model. This is presently under investigation. Also of interest is the question of whether the topological field theory parameterization of the degrees of freedom of 11 dimensional supergravity introduced in [40]
can be derived directly from the cubic matrix model studied here.

Finally, any approach to a background independent formulation of $\mathcal{M}$ theory must answer the question of how the particular structures which seem required by string theory for perturbative consistency of a quantum theory of gravity are picked out at the more fundamental, background independent level. It is quite possible that the theory presented here has a more fundamental formulation in which the choice of algebra is not arbitrary. The possibility of a reformulation in terms of Jordan algebras and octonions comes naturally to mind. The $SU(16,16)$ structure reminds one of an octonionic extension of twistor theory. There is also an intriguing similarity to Chern-Simons inspired formulations of string field theory[43] that deserves further investigation.

ACKNOWLEDGEMENTS

I am grateful to Louis Crane, Bartomeu Fiol, Willy Fischler, Mike Green, Chris Hull, Chris Isham, Yi Ling, Fotini Markopoulou, George Minic, Mike Reisenberger, Kelle Stelle and Dennis Sullivan for discussions, suggestions and encouragement during the course of this work. In addition, Richard Levine contributed a number of extremely helpful observations and corrections. I am grateful also for hospitality at the Institute in Theoretical Physics in Santa Barbara, where this work was begun, and to the theory group at Rutgers University, where it was completed. This work was supported by the NSF through grant PHY95-14240 and by an SPG grant. I would also like to thank the Jesse Phillips Foundation for support and encouragement.

References

[1] A. M. Polyakov, GAUGE FIELDS AND STRINGS. Harwood Academic, 1987. 301p. (Contemporary Concepts in Physics, Vol. 3); G. ’t Hooft, UNDER THE SPELL OF THE GAUGE PRINCIPLE. World Scientific, 1994. 683p. (Advanced Series in Mathematical Physics, v. 19)

[2] T. Jacobson and L. Smolin, Nonperturbative quantum geometries Nuclear Physics B299 (1988) 295-345.

[3] C. Rovelli and L. Smolin, Phys Rev Lett 61 (1988) 1155; Nucl Phys B133, 80 (1990); Discreteness of area and volume in quantum gravity Nuclear Physics B 442 (1995) 593. Erratum: Nucl. Phys. B 456 (1995) 734; “Spin networks and quantum gravity” gr-qc/9505006. Physical Review D 52 (1995) 5743-5759.

[4] R. Gambini and J. Pullin, Loops, knots, gauge theories and quantum gravity Cambridge University Press, 1996.
[5] For reviews see, L. Smolin: in *Quantum Gravity and Cosmology*, eds J. Pérez-Mercader *et al.*, World Scientific, Singapore 1992; The future of spin networks gr-qc/9702030 in the Penrose Feshschrift; C. Rovelli, gr-qc/9710008.

[6] M. Claudson and M. Halpern, Nucl. Phys. B250 (1985) 689.

[7] B. DeWitt, J. Hoppe, H. Nicolai, Nuclear Physics B305 (1988) 545.

[8] T. Banks, W. Fishler, S. H. Shenker, L. Susskind, *M theory as a matrix model: a conjecture* hep-th/9610043. Phys. Rev. D55 (1997) 5112.

[9] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large N reduced model as superstring* hep-th/9612115. Nucl. Phys. B498 (1997) 467; M. Fukuma, H. Kawai, Y. Kitazawa and A. Tsuchiya, *String field theory from IIB matrix model* hep-th/9705128. Nucl. Phys. B (Proc. Suppl) 68 (1998) 153.

[10] L. Smolin, *M theory as a matrix extension of Chern-Simons theory*, hep-th/0002009.

[11] F. Markopoulou and L. Smolin *Quantum geometry with intrinsic local causality* preprint, Dec. 1997, gr-qc/9712067. Phys. Rev. D58 (1998) 084032.

[12] F. Markopoulou and L. Smolin *Non-perturbative dynamics of algebraic (p,q) string networks* preprint, December 1997, hep-th/9712148. Phys. Rev. D58 (1998) 084033.

[13] L. Smolin *A candidate for the background independent formulation of M theory* hep-th/9903166.

[14] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. Phys. B348 (1995) 109; E. Witten, *String theory in various dimensions*, Nucl. Phys. B443 (1995) 85; P. Townsend, *(M)embrane theory on T⁹*, Nucl. Phys. (Proc. Suppl) 68 (1998) 11-16;[22]; *M-theory from its superalgebra*, in ‘Strings, branes and dualities’, Cargèse 1997, ed. L. Baulieu et al., Kluwer Academic Publ., 1999, p. 141, hep-th/9712004; hep-th/9507048, in Particles, Strings and Cosmology, ed. J. Bagger et al (World Scientific, 1996); hep/9612121; I. Bars, hep-th/9608061, hep-th/9607122. Petr Horava, *M-Theory as a Holographic Field Theory*, hep-th/9712130. Phys. Rev. D59 (1999) 046004.

[15] Eric Bergshoeff and Antoine Van Proeyen, *The many faces of OSp(1 j 32)*, hep-th/0003261.

[16] R. Levine and L. Smolin, in preparation.

[17] E. Witten, *Quantum field theory and the Jones Polynomial* Commun. Math. Phys. 121 (1989) 351

[18] G. Moore and N. Seiberg, *Classical and quantum conformal field theories* Comun. Math. Phys. 123 (1988) 177.
[19] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. B 300 (1988) 360; G. Moore and N. Yu. Reshetikhin, *A comment on quantum group symmetry in conformal field theory*, Nucl. Phys. B 328 (1989) 557; N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. 127 (1990) 1; *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. 103 (1991) 547; V Turaev O Viro: *Topology*, 31 (1992) 865.

[20] L. Crane, Commun. Math. Phys. 135 (1991) 615; Phys. Lett. B 259 (1991) 243.

[21] L.Kauffman,S.L.Lins, *Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds*, Princeton U Press, 1994.

[22] D. Armand-Ugon, R. Gambini, O. Obregon, J. Pullin, *Towards a loop representation for quantum canonical supergravity*, hep-th/9508030, Nucl.Phys. B460 (1996) 615; H. Kunitomo and T. Sano *The Ashtekar formulation for canonical N=2 supergravity*, Prog. Theor. Phys. suppl. (1993) 31; Takashi Sano and J. Shiraiishi, *The Non-perturbative Canonical Quantization of the N=1 Supergravity*, Nucl. Phys. B410 (1993) 423; [hep-th/9211104] *The Ashtekar Formalism and WKB Wave Functions of N=1,2 Supergravities*, [hep-th/9211103] K. Ezawa, *ASHTEKAR’S FORMULATION FOR N=1, N=2 SUPERGRAVITIES AS CONSTRAINED BF THEORIES*, Prog.Theor.Phys.95:863-882,1996. [hep-th/9511047] T. Kadotomi and S. Nojiri, *N=3 AND N=4 TWO FORM SUPERGRAVITIES*, Mod.Phys.Lett.A12:1165-1174,1997, [hep-th/9703149] L. F. Urrutia *Towards a loop representation of connection theories defined over a super-lie algebra* hep-th/9609010.

[23] Yi Li and L. Smolin, *Supersymmetric Spin Networks and Quantum Supergravity*, hep-th/9904010, to appear in Physical Review D

[24] F. Markopoulou, *Dual formulation of spin network evolution preprint*, March 1997, gr-qc/9704013.

[25] L. Smolin, *The classical limit and the form of the hamiltonian constraint in nonperturbative quantum gravity* preprint CGPG-96/9-4, gr-qc/9609034.

[26] R. Loll, gr-qc/9708025, Class.Quant.Grav. 15 (1998) 799-809; R. Gambini, J. Lewandowski, D. Marolf, J. Pullin, *On the consistency of the constraint algebra in spin network quantum gravity*, gr-qc/9710018, Int.J.Mod.Phys. D7 (1998) 97-109; J. Lewandowski and D. Marolf, Int.J.Mod.Phys. D7 (1998) 299-330, gr-qc/9710016; D. Neville, Phys.Rev. D59 (1999) 044032, gr-qc/9803066.

[27] M. Reisenberger and C. Rovelli, *“Sum over Surfaces” form of Loop Quantum Gravity*, gr-qc/9612035, Phys.Rev. D56 (1997) 3490-3508.
[28] C. Rovelli and L. Smolin, *The physical hamiltonian in nonperturbative quantum gravity*, Phys. Rev. Lett. 72 (1994) 446.

[29] T. Thiemann, *Quantum Spin Dynamics I-V*, Class. Quant. Grav. 15 (1998) 839-905, 1207-1512; gr-qc/9606092, gr-qc/9606083, gr-qc/9606090, gr-qc/9705020, gr-qc/9705021, gr-qc/9705019, gr-qc/9705018, gr-qc/9705017.

[30] W. Taylor, *D brane field theory on compact spaces* hep-th/9611042, Phys. Lett. B394 (1997) 283; O.J. Ganor, S. Ramgoolam and W. Taylor *Branes, Fluxes and Duality in (M)atrix theory*, hep-th/961202.

[31] K. Krasnov, gr-qc/9603023, Phys. Rev. D55 (1997) 3505-3513; gr-qc/9605047, Gen. Rel. Grav. 30 (1998) 53-68; A. Ashtekar, J. Baez, A. Corichi, K. Krasnov, Phys. Rev. Lett. 80 (1998) 904-907, gr-qc/9710007, gr-qc/9902015.

[32] L. Smolin, *Linking topological quantum field theory and nonperturbative quantum gravity* gr-qc/9505028, J. Math. Phys. 36 (1995) 6417.

[33] R. Levine and L. Smolin, in preparation.

[34] C.J. Isham, N. Linden, K. Savvidou, S. Schreckenberg, quant-ph/9711031, J. Math. Phys. 39 (1998) 1818-1834; Ntina Savvidou, Charis Anastopoulos, gr-qc/9912077; Konstantina Savvidou, gr-qc/9912076, gr-qc/9811078, J. Math. Phys. 40 (1999) 5657-5674.

[35] F. Markopoulou, *The internal description of a causal set: What the universe looks like from the inside*, gr-qc/9811053, Quantum causal histories, hep-th/9904009.

[36] F. Markopoulou, *An insiders guide to quantum causal histories*, hep-th/9912137, to appear in the proceedings of Quantum Gravity 99 meeting, held in Sardinia, Italy, on Sept. 1999.

[37] F. Markopoulou and L. Smolin, *Holography in a quantum spacetime*, hep-th/9910146.

[38] C. Rovelli, *The projector on physical states in loop quantum gravity*, gr-qc/9806121, Phys. Rev. D59 (1999) 104015.

[39] R. De Pietri, L. Freidel, K. Krasnov, C. Rovelli, *Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space*, hep-th/9907154.

[40] Y. Ling and L. Smolin, 11 *dimensional supergravity as a constrained topological field theory*, hep-th/0003283.

[41] T. Banks, N. Seiberg and S. Shenker, hep-th/9612157, T. Banks, hep-th/99110682.

[42] L. Smolin, Phys Rev D15 DW6333 hep-th/9808191.
[43] E. Witten, Nucl. Phys. B276, 291 (1986); G. T. Horowitz, J. Lykken, R. Rohm and A. Strominger, Phys. Rev. Lett. 57, 283 (1986); H. Hata, K. Itoh, T. Kugo, H. Kunitomo and K. Ogawa, Phys. Rev. D34, 2360 (1986). A. Strominger, Nucl. Phys. B294, 93 (1987).