Casimir Energy of an irregular membrane

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We compute the Casimir energy which arises in a bi-dimensional surface due to the quantum fluctuations of a scalar field. We assume that the boundaries are irregular and the field obeys Dirichlet condition. We re-parametrize the problem to one which has flat boundary conditions and the irregularity is treated as a perturbation in the Laplace-Beltrami operator which appears. Later, to compute the Casimir energy, we use zeta function regularization. It is compared the results coming from perturbation theory with the WKB method.

I. INTRODUCTION

The Casimir effect can be considered among the few macroscopic manifestation of quantum phenomena. As it is well known, the Casimir effect originally appeared as a relative force between conductor or semiconductor surfaces due to the quantum fluctuations of vacuum \cite{1}, \cite{2}. The Casimir effect is an interdisciplinary subject, which plays an important role in Quantum Field Theory, Condensed Matter, Gravitation and Mathematical Physics \cite{3}, \cite{4}. In particular, for the subject of membrane theory, one can refer to \cite{5}. \cite{6}.

We consider a problem in 2+1 dimensions, with a bi-dimensional surface bounded by an irregular border, where the Casimir force arise due to the quantum fluctuations of a free scalar field \cite{7}. We consider this problem as the case of an idealized membrane where the phonon fluctuations are responsible of the Casimir force between borders.

We work in Euclidean space and the spatial coordinates are re-parametrized, in order to convert the irregular borders in two parallel plates, so, the scalar dynamic is given by the resulting Laplace-Beltrami operator, due to the coordinate transformation.

In order to compute the Casimir energy at zero temperature and its resulting force, we use the zeta function regularization as it is done in \cite{8}. Were it is necessary to compute the determinant of the Laplace-Beltrami operator which arises from the integration of the scalar fields. For the sake of simplicity, we consider a rectangular shape of size \( L \times a \) with Dirichlet boundary condition. So, zero modes are avoided. Of course, we consider the length \( L \) much bigger than the width \( a \), as it is usually done.

The free energy in terms of zeta function, is given by

\[
E_{\text{cas}} = \frac{1}{2} \left[ FP\zeta(-1/2) + \text{Res}\zeta(-1/2) \ln(\mu) \right],
\]

as it is shown in \cite{3}. It means that the residues of the zeta function carry an ambiguity in the determination of the free energy, since it appears an arbitrary scale \( \mu \), which is harmless if the residue does not depend on the parameter \( a \), the separation. But, if it appears a dependence on \( a \), it means that the method is not enough to determine the force on the borders. So it is necessary to try another approach.

We first use perturbation theory, and later, the WKB method \cite{9}, \cite{10}. Implying that we can compare both methods. We conclude that the dominant terms are not the same when \( L \rightarrow \infty \). But, if we consider \( L \) finite, there appear differences in the contribution for the energy. We assume that the better method is the WKB, since it is also useful for obtaining the residues of the zeta function, which are related to the geometry of the system \cite{11}.

II. GENERAL SETTING

A. Laplacian of a non regular object

We consider a nearly rectangle membrane,

A free scalar field on the membrane obeys the Laplace equation

\[
- \triangle_s \phi = \lambda \phi.
\]

If we assume an irregular boundary, we can rescale the irregular lenght in order to obtain a rectangular boundary, so the scalar dynamic is given by the resulting Laplace-Beltrami operator, due to the coordinate transformation.

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If we assume an irregular boundary, we can rescale the irregular lenght in order to obtain a rectangular boundary, so the scalar dynamic of the scalar field is given by the Laplace-Beltrami operator

\[
\triangle_s = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ik} \frac{\partial}{\partial x^k} \right).
\]

If we consider two dimensions; \( x \) and \( y \), where \( 0 \leq x \leq L \) and \( 0 \leq y \leq H(x) \). We assume \( a \ll L \), with

\[
H(x) = a + h(x) \quad \text{and} \quad h(x) \ll a.
\]

Rescaling \( x \) and \( y \),

\[
x = uL, \quad y = H(u)v,
\]

where \( u, v \) are fixed coordinates,
whose determinant

\[ g_{ik} = \begin{bmatrix} L^2 + (vh'(u))^2 & (a + h(u))vh'(u) \\ (a + h(u))vh'(u) & (a + h(u))^2 \end{bmatrix}, \]

whose determinant

\[ G(u) = \det(g_{ik}) = (a + h(u))^2L^2, \]

We end up with a nondiagonal spacial metric tensor

\[ g_{ik} = \begin{bmatrix} L^2 + (vh'(u))^2 & (a + h(u))vh'(u) \\ (a + h(u))vh'(u) & (a + h(u))^2 \end{bmatrix}, \]

In order to perform perturbation theory, we identify

\[ \lambda = \frac{1}{L^2} \frac{1}{a^2} \frac{∂^2 φ}{∂u^2} - \frac{2vh'(u)}{(a + h(u))^2} \frac{∂^2 φ}{∂v^2} = λφ. \]

We shall assume \( 1 \gg h(u), h(u) \gg h'(u)/L \) and \( h(u) \gg h''(u)/L \), so, keeping to the quadratic terms in \( h(u) \), we end up with

\[ -\triangle_s φ = -\frac{1}{L^2} \frac{∂^2 φ}{∂u^2} - \frac{1}{a^2} \frac{∂^2 φ}{∂v^2} - \left( \frac{2h(u)}{a^3} - \frac{3h(u)^2}{a^4} \right) \frac{∂^2 φ}{∂v^2}. \]

In order to perform perturbation theory, we identify

\[ V(u, v)φ = \left( \frac{2h(u)}{a^3} + \frac{3h(u)^2}{a^4} \right) \frac{∂^2 φ}{∂v^2} \]

IV. PERTURBATION THEORY

First, we solve the standard Laplace equation

\[ -\triangle_s φ^0 = -\frac{1}{L^2} \frac{∂^2 φ^0}{∂u^2} - \frac{1}{a^2} \frac{∂^2 φ^0}{∂v^2} = λ^0 φ^0, \]

whose solution and eigenvalues are given by

\[ φ_{n,m}^0(u, v) = 2\sin(mπu)\sin(nπv), \]

\[ λ_{n,m}^0 = \left( \frac{nπ}{a} \right)^2 + \left( \frac{mπ}{L} \right)^2, \]

since we are using Dirichlet boundary conditions, \( n \) and \( m \) are positive integers

\[ 1 \leq n < \infty, \quad 1 \leq m < \infty. \]

So, the first order correction term is

\[ δλ_{n,m}^p = \int_0^1 \int_0^1 φ_{n,m}^{0,*} V(u, v) φ_{n,m}^0 dudv \]

\[ = \int_0^1 \int_0^1 φ_{n,m}^{0,*} G(u) \frac{∂^2 φ_{n,m}^0}{∂v^2} dudv, \]

giving the eigenvalue

\[ λ_{n,m} = \left( \frac{nπ}{a} \right)^2 + \left( \frac{mπ}{L} \right)^2 + π^2 n^2 \int_0^1 G(u) du, \]
where \( h(u) \), introduced in [3], shall be considered as a periodic function in the variable \( u \).

The generalized zeta function is given by

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\alpha m^2 t} \sin(\alpha n^2 t) dt,
\]

where the parameters \( \alpha \) and \( \beta \) are

\[
\alpha = \left( \frac{\pi}{L} \right)^2,
\]

\[
\beta = \left( \frac{\pi}{a} \right)^2 + \pi^2 \int_0^1 G(u) \, du.
\]

In order to obtain the thermodynamical potential, it is necessary to compute the generalized zeta function for \( s = -1/2 \)

\[
\zeta(-1/2) = \frac{1}{24} \left( \sqrt{\beta} + \sqrt{\alpha} \right) - \frac{\zeta_R(3)}{16 \sqrt{\alpha \beta \pi^2}} (\beta^{3/2} + \alpha^{3/2}) + \frac{\pi^3}{8 \sqrt{\alpha \beta}} \beta^{3/2} \left( -\alpha \sqrt{\beta} \sqrt{\frac{\beta + \alpha \pi^2}{\alpha}} + \alpha \pi \sqrt{\beta} - \sqrt{\alpha \beta} \right).
\]

For the leading term in the Limit \( \omega = 0 \) and \( L \to \infty \), we have the energy and force per unit length given by

\[
\frac{E_{\text{cas}}}{L} = \frac{E_0}{L} = -\frac{\zeta_R(3)}{16 \pi a^2}.
\]

\[
E_{\text{cas}} = \frac{-a \zeta_R(3)}{16 \pi L^2} + \frac{\pi^4}{8 a^3 L} + \frac{\pi}{48 L} - \frac{\pi^3 \sqrt{L^2 + \pi^2 a^2}}{8 a^4 L} - \frac{\zeta_R(3) L}{16 \pi a^2} + \frac{\pi}{48 a} - \frac{\pi^3}{8 a^4} + \left( \frac{\zeta_R(3) \sin(\omega) \cos(\phi)}{16 \omega \pi L^2} - \frac{3 \pi^4 \sin(\omega) \cos(\phi)}{8 \omega a^4 L} + \cdots \right) \epsilon + \left( \frac{\pi}{64 a^3} - \frac{3 \zeta_R(3) L}{32 \pi a^4} + \cdots \right) \epsilon^2.
\]

In the limit \( L \to \infty \), we have

\[
E_{\text{cas}} = \frac{\zeta_R(3) L}{16 \pi a^2} - \frac{3 \zeta_R(3) L}{32 \pi a^4} \epsilon^2.
\]

For the case where we have two irregular surfaces parametrized by

\[
h(u) = \epsilon_2 \cos(\omega u + \phi) - \epsilon_1 \cos(\omega u),
\]

\[
\zeta(1/2) = \frac{1}{24} \left( \sqrt{\beta} + \sqrt{\alpha} \right) - \frac{\zeta_R(3)}{16 \sqrt{\alpha \beta \pi^2}} (\beta^{3/2} + \alpha^{3/2})
\]

we end with an expression for the Casimir energy

\[
F_0 = \frac{\zeta_R(3)}{8 \pi a^3}.
\]

If we assume a regular behavior for \( h \), in order to parametrize it as a trigonometric function

\[
h(u) = \epsilon \cos(\omega u + \phi), \quad \omega = \delta L,
\]

assuming \( \delta \ll 1 \), \( \epsilon \ll 1 \) and \( \epsilon \ll \delta \). In such case, the energy is given by

\[
E_{\text{cas}} = \frac{-a \zeta_R(3) L}{16 \pi a^2} - \frac{\pi^4}{8 a^3 L} + \frac{\pi}{48 L} - \frac{\pi^3 \sqrt{L^2 + \pi^2 a^2}}{8 a^4 L} - \frac{\zeta_R(3) L}{16 \pi a^2} + \frac{\pi}{48 a} - \frac{\pi^3}{8 a^4} + \left( \frac{\zeta_R(3) \sin(\omega) \cos(\phi)}{16 \omega \pi L^2} - \frac{3 \pi^4 \sin(\omega) \cos(\phi)}{8 \omega a^4 L} + \cdots \right) \epsilon + \left( \frac{\pi}{64 a^3} - \frac{3 \zeta_R(3) L}{32 \pi a^4} + \cdots \right) \epsilon^2.
\]

with \( \omega = \delta L, \delta \ll 1 \) and \( \epsilon_1, \epsilon_2 \ll 1 \). The energy we obtain, is the following

\[
E_{\text{cas}} = \frac{-a \zeta_R(3) L}{16 \pi a^2} - \frac{3 \zeta_R(3) L}{32 \pi a^4} \left( \epsilon_1^2 + \epsilon_2^2 - 2 \epsilon_1 \epsilon_2 \cos(\phi) \right) L.
\]
IV. WKB METHOD

We can rewrite (10)

$$-\Delta s \phi = -\frac{1}{L^2} \frac{\partial^2 \phi}{\partial u^2} - Q(u) \frac{\partial^2 \phi}{\partial u^2} = \lambda \phi. \quad (28)$$

Where, \(Q(u)\) is given by

$$Q(u) = \frac{1}{a^2} - G(u) = \frac{1}{a^2} - \left( \frac{2h(u)}{a^3} + \frac{3h(u)^2}{a^4} \right), \quad (29)$$

with \(G(u)\) defined in (11). If we use the following Ansatz

$$\phi(u,v) = M_{n,\lambda}(u) \sin(n \pi v), \quad n \in \mathbb{N}^+,$$

it leads us to the equation

$$M_{n,\lambda}''(u) + (\lambda - n^2 \pi^2 Q(u)) L^2 M_{n,\lambda}(u) = 0. \quad (30)$$

The eigenvalues of the system are given by the zeros of \(M_{n,\lambda}(1)\). We define the normalized function as it is done in (11)

$$D_n(\lambda) = \frac{M_{n,\lambda}(1)}{M_{n,0}(1)},$$

so, the zeta function can be expressed as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{2 \pi i} \int_{1}^{\infty} \lambda^{-s} \frac{d \ln D_n(\lambda)}{d \lambda} d\lambda. \quad (31)$$

Choosing an appropriate contour integral, where \(\lambda \to iz\), we have

$$\zeta(s) = \frac{\sin(\pi s)}{\pi} \sum_{n=1}^{\infty} (\pi n)^{-2s} \int_{0}^{\infty} z^{-s} \frac{d \ln D_n(i z)}{d z} dz. \quad (32)$$

Since the contour has been rotated, we replace \(\lambda = -n^2 z \pi^2\) in (30),

$$M_{n,z}''(u) - n^2 L^2 \pi^2 (z + Q(u)) M_{n,z}(u) = 0. \quad (33)$$

We can look for a solution of the form

$$M_{n,z}(u) = e^{\int_{0}^{u} S(\sigma, z, n) \, d\sigma}, \quad (34)$$

where \(S(u, z, n)\) obeys

$$S^2(u, z, n) + \frac{\partial S(u, z, n)}{\partial u} = n^2 L^2 \pi^2 (z + Q(u)). \quad (35)$$

We search a solution in powers of \(n^{-1}\)

$$S(u, z, n) = \sum_{i=-1}^{N} a_i(u, z) n^{-i}. \quad (36)$$

So, we can obtain the coefficients \(a_i\) recursively. The first three terms are

$$a_{-1}(u, z) = \pm \pi L \sqrt{z + Q(u)},$$

$$a_0(u, z) = -\frac{1}{2a_{-1}(u, z)} \frac{\partial a_{-1}(u, z)}{\partial u},$$

$$a_1(u, z) = -\frac{1}{2a_{-1}(u, z)} \left( \frac{\partial a_{0}(u, z)}{\partial u} + a_{0}^2(u, z) \right), \quad (37)$$

and so on.

The general solution is given by

$$M_{n,z}(u) = Ae^{\int_{0}^{u} S_+(\sigma, z, n) \, d\sigma} + Be^{\int_{0}^{u} S_-(\sigma, z, n) \, d\sigma}, \quad (38)$$

where \(S_\pm\) is the splitting

$$S_\pm(u, z, k) = \pm S_1 + S_2, \quad (39)$$

with

$$S_1(u, z, n) = a_{-1}(u, z) n + \frac{a_1(u, z)}{n} + \frac{a_3(u, z)}{n^3} + \cdots,$$

$$S_2(u, z, n) = a_0(u, z) n + \frac{a_2(u, z)}{n^2} + \frac{a_4(u, z)}{n^4} + \cdots. \quad (40)$$

Since we are dealing with Dirichlet boundary conditions, we have

$$M_{n,z}(0) = M_{n,z}(1) = 0 \quad \text{and} \quad M_{n,z}'(0) = 1. \quad (41)$$

We end with the solution for \(M_n\)

$$M_{n,z}(u) = \frac{e^{\int_{0}^{u} S_+(\sigma, z, n) \, d\sigma}}{2\sqrt{\int_{1}^{0} S_1(1, z, n) S_1(0, z, n) \, d\sigma}} \times \left( 1 - e^{-2 \int_{0}^{u} S_-(\sigma, z, n) \, d\sigma} \right). \quad (42)$$

The term in (32) reads

$$\ln D_n(-n^2 z) = \ln \left( \frac{M_{n,z}(1)}{M_{n,0}(1)} \right), \quad (43)$$
the derivative of the previous expression assumes the form

\[
\frac{\partial \ln S_1(u, z, n)}{\partial z} = \sum_{i=0}^{\infty} b_{2i}(u, z)n^{-2i}.
\]

Expanding \( S_1 \)

\[
\ln(S_1(u, z, n)) = \ln(a_{-1}n) + \frac{a_1}{a_{-1}n^2} + \frac{a_3}{a_{-1}n^4} - \frac{1}{2} \left( \frac{a_1}{a_{-1}n^2} + \frac{a_3}{a_{-1}n^4} \right)^2 + \cdots
\]

The generalized zeta function can be expressed as

\[
\zeta(s) = \frac{\sin(\pi s)}{\pi} \int_0^1 z^{-s} \sum_{i=0}^{\infty} \pi^{-2s} \zeta_R(2s + 2i - 1) \frac{\partial a_{2i-1}(\sigma, z)}{\partial z} d\sigma dz
\]

\[
- \frac{\sin(\pi s)}{2\pi} \int_0^1 z^{-s} \sum_{i=0}^{\infty} (b_{2i}(1, z) + b_{2i}(0, z)) \pi^{-2s} \zeta_R(2s + 2i) dz
\]

\[
+ \frac{\sin(\pi s)}{\pi} \sum_{n=1}^{\infty} (n\pi)^{-2s} \int_0^{\infty} dz z^{-s} \frac{2e^{-2\int_0^1 S_1(\sigma, z, n) d\sigma}}{1 - e^{-2\int_0^1 S_1(\sigma, z, n) d\sigma}} \frac{\partial}{\partial z} \int_0^1 S_1(\sigma, z, n) d\sigma.
\]

Since the free energy is given by

\[
E_{\text{cas}} = \frac{1}{2} \left[ FP \zeta(-1/2) + \text{Res}\zeta(-1/2) \ln(\mu) \right],
\]

\[
h(u) = \epsilon \cos(\omega u + \phi),
\]

If we choose a regular boundary of the form

\[
\text{with } \omega = \delta L, \delta \ll 1 \text{ and } \epsilon \ll 1. \text{ From (47), we obtain}
\]

\[
FP\zeta(-1/2) = -\frac{\zeta_R(3)L}{8\pi a^2} + \frac{\pi}{24a^2} + \frac{\pi}{24L} + \left( \frac{\zeta_R(3)\sin(\omega)\cos(\phi)L}{4\pi a^3} - \frac{\pi \cos(\phi)}{48a^2} + \frac{\omega \sin(\omega)\cos(\phi)}{a\pi L} + \cdots \right) \epsilon
\]

\[
+ \left( -\frac{3\zeta_R(3)L}{16\pi a^3} + \frac{3\omega \cos(\phi)\sin(\phi)}{16\pi a^2L} + \cdots \right) \epsilon^2,
\]

and the residue

\[
\text{Res}\zeta(-1/2) = \left( \frac{\omega^2 \cos(\omega + \phi)}{64\pi aL^2} + \cdots \right) \epsilon
\]

\[
E_{\text{cas}} = E_0 - \frac{3\zeta_R(3)L}{32\pi a^3} \epsilon^2 + \frac{\delta^2 L}{32\pi a^2} \epsilon^2,
\]

with \( E_0 \), defined in [21]
The Casimir energy when the two surfaces are irregular is given by,

\[ E_0 = -\frac{\zeta_R(3)}{16\pi a^2}. \]

\[ E_{\text{cas}} = -\frac{\zeta_R(3)}{16\pi a^2} \frac{3\zeta_R(3)}{32\pi a^4} (\epsilon_1^2 + \epsilon_2^2 - 2\epsilon_1\epsilon_2 \cos(\phi))L \]
\[ + \frac{\delta^2}{32\pi a^2} (\epsilon_1^2 + \epsilon_2^2 - 2\epsilon_1\epsilon_2 \cos(\phi))L + \cdots, \] (51)

it does not coincide with the term obtained in (27).

V. CONCLUSIONS

In this work, we computed the free energy of a bi-dimensional spatial surface with irregular boundaries, where the physics is played by a scalar field. We assumed a rectangular shape with the width \( a \) much smaller than the length \( L \) of the surface. For the sake of simplicity, we assume Dirichlet boundary condition on the borders. First, we used perturbation theory and after, the WKB method.

In general, we find that perturbation theory does not coincide with the WKB method, for finite \( L \). In the limiting case \( L \to \infty \), there is also a discrepancy with perturbation theory. We conclude that perturbation theory, at least at first order, is not enough to describe the physics of the system.

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