KNOT FLOER HOMOLOGY AND SEIFERT SURFACES

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Abstract. Let \( K \) be a knot in \( S^3 \) of genus \( g \) and let \( n > 0 \). We show that if \( \text{rk} \hat{HF}(K, g) < 2^{n+1} \) (where \( \hat{HF} \) denotes knot Floer homology), in particular if \( K \) is an alternating knot such that the leading coefficient \( a_g \) of its Alexander polynomial satisfies \( |a_g| < 2^{n+1} \), then \( K \) has at most \( n \) pairwise disjoint non-isotopic genus \( g \) Seifert surfaces. For \( n = 1 \) this implies that \( K \) has a unique minimal genus Seifert surface up to isotopy.

1. Introduction and preliminaries

If \( S_1 \) and \( S_2 \) are Seifert surfaces of a knot \( K \subset S^3 \) then \( S_1 \) and \( S_2 \) are said to be equivalent if \( S_1 \cap X(K) \) and \( S_2 \cap X(K) \) are ambient isotopic in the knot exterior \( X(K) = S^3 \setminus N(K) \), where \( N(K) \) is a regular neighborhood of \( K \). In [4] Kakimizu assigned a simplicial complex \( MS(K) \) to every knot \( K \) in \( S^3 \) as follows.

Definition 1.1. \( MS(K) \) is a simplicial complex whose vertices are the equivalence classes of the minimal genus Seifert surfaces of \( K \). The equivalence classes \( \sigma_0, \ldots, \sigma_n \) span an \( n \)-simplex if and only if for each \( 0 \leq i \leq n \) there is a representative \( S_i \) of \( \sigma_i \) such that the surfaces \( S_0, \ldots, S_n \) are pairwise disjoint.

In [10] it is shown that the complex \( MS(K) \) is always connected. I.e., if \( S \) and \( T \) are minimal genus Seifert surfaces for a knot \( K \) then there is a sequence \( S = S_1, S_2, \ldots, S_k = T \) of minimal genus Seifert surfaces such that \( S_i \cap S_{i+1} = \emptyset \) for \( 0 \leq i \leq k - 1 \).

The main goal of this short note is to show that for a genus \( g \) knot \( K \) and for \( n > 0 \) the condition \( \text{rk} \hat{HF}(K, g) < 2^{n+1} \) implies \( \dim MS(K) < n \), consequently for \( n = 1 \) the knot \( K \) has a unique Seifert surface up to equivalence. This condition involves the use of knot Floer homology introduced by Ozsváth and Szabó in [8] and independently by Rasmussen in [9]. However, when \( K \) is alternating then this condition is equivalent to \( |a_g| < 2^{n+1} \), where \( a_g \) is the leading coefficient of the Alexander polynomial of \( K \). The alternating case is already a new result whose statement doesn’t involve knot Floer homology. On the other hand, the proof of this particular case seems to need sutured Floer homology techniques, which is a generalization of knot Floer homology that was introduced by the author in [3]. At the time of writing this paper there are very few results in knot theory which can only be proved using Floer homology methods.

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The above statement does not hold for \( n = 0 \) since every knot has at least one minimal genus Seifert surface. However, it was shown in [6] and [2] that \( \text{rk} \mathcal{HFK}(K, g) < 2 \) implies that the knot \( K \) is fibred.

To a knot \( K \) in \( S^3 \) and every \( j \in \mathbb{Z} \) knot Floer homology assigns a graded abelian group \( \mathcal{HFK}(K, j) \) whose Euler characteristic is the coefficient \( a_j \) of the Alexander polynomial \( \Delta_K(t) \). In [7] it is shown that if \( K \) is alternating then \( \mathcal{HFK}(K, j) \) is non-zero in at most one grading, thus \( \text{rk} \mathcal{HFK}(K, j) = |a_j| \).

Next we are going to review some necessary definitions and results from the theory of sutured manifolds and sutured Floer homology. Sutured manifolds were introduced by Gabai in [1].

**Definition 1.2.** A sutured manifold \((M, \gamma)\) is a compact oriented 3-manifold \(M\) with boundary together with a set \(\gamma \subset \partial M\) of pairwise disjoint annuli \(A(\gamma)\) and tori \(T(\gamma)\). Furthermore, the interior of each component of \(A(\gamma)\) contains a suture, i.e., a homologically nontrivial oriented simple closed curve. We denote the union of the sutures by \(s(\gamma)\).

Finally every component of \(R(\gamma) = \partial M \setminus \text{Int}(\gamma)\) is oriented. Define \(R_+ (\gamma)\) (or \(R_-(\gamma)\)) to be those components of \(\partial M \setminus \text{Int}(\gamma)\) whose normal vectors point out of (into) \(M\). The orientation on \(R(\gamma)\) must be coherent with respect to \(s(\gamma)\), i.e., if \(\delta\) is a component of \(\partial R(\gamma)\) and is given the boundary orientation, then \(\delta\) must represent the same homology class in \(H_1(\gamma)\) as some suture.

A sutured manifold is called taut if \(R(\gamma)\) is incompressible and Thurston norm minimizing in \(H_2(M, \gamma)\).

The following definition was introduced in [3].

**Definition 1.3.** A sutured manifold \((M, \gamma)\) is called balanced if \(M\) has no closed components, \(\chi(R_+ (\gamma)) = \chi(R_-(\gamma))\), and the map \(\pi_0(A(\gamma)) \to \pi_0(\partial M)\) is surjective.

**Example 1.4.** If \(R\) is a Seifert surface of a knot \(K\) in \(S^3\) then we can associate to it a balanced sutured manifold \(S^3(R) = (M, \gamma)\) such that \(M = S^3 \setminus (R \times I)\) and \(\gamma = K \times I\). Observe that \(R_- (\gamma) = R \times \{0\}\) and \(R_+ (\gamma) = R \times \{1\}\). Furthermore, \(S^3(R)\) is taut if and only if \(R\) is of minimal genus.

Sutured Floer homology is an invariant of balanced sutured manifolds defined by the author in [3], and is a common generalization of the invariants \(\widehat{HF}\) and \(\mathcal{HFK}\). It assigns an abelian group \(SFH(M, \gamma)\) to each balanced sutured manifold \((M, \gamma)\). The following theorem is a special case of [2, Theorem 1.5].

**Theorem 1.5.** Let \(K\) be a genus \(g\) knot in \(S^3\) and suppose that \(R\) is a minimal genus Seifert surface for \(K\). Then

\[
SFH(S^3(R)) \approx \mathcal{HFK}(K, g).
\]

A sutured manifold \((M, \gamma)\) is called a product if it is homeomorphic to \((\Sigma \times I, \partial \Sigma \times I)\), where \(\Sigma\) is an oriented surface with boundary. If \((M, \gamma)\) is a product then \(SFH(M, \gamma) \approx \mathbb{Z}\). Let us recall [2, Theorem 1.4] and [2, Theorem 9.3].

**Theorem 1.6.** If \((M, \gamma)\) is a taut balanced sutured manifold then \(SFH(M, \gamma) \geq \mathbb{Z}\). Furthermore, if \((M, \gamma)\) is not a product then \(SFH(M, \gamma) \geq \mathbb{Z}^2\).

**Definition 1.7.** Let \((M, \gamma)\) be a balanced sutured manifold. An oriented surface \(S \subset M\) is called a horizontal surface if \(S\) is open, \(\partial S = s(\gamma)\) in an oriented sense; moreover, \(|S| = |R_+(\gamma)|\) in \(H_2(M, \gamma)\), and \(\chi(S) = \chi(R_+(\gamma))\).
A horizontal surface $S$ defines a horizontal decomposition

$$(M, \gamma) \rightsquigarrow (M_-, \gamma_-) \coprod (M_+, \gamma_+)$$

as follows. Let $M_{\pm}$ be the union of the components of $M \setminus \text{Int}(N(S))$ that intersect $R_\pm(\gamma)$. Similarly, let $\gamma_{\pm}$ be the union of the components of $\gamma \setminus \text{Int}(N(S))$ that intersect $R_\pm(\gamma)$.

The following proposition is a special case of [2, Proposition 8.6].

**Proposition 1.8.** Suppose that $(M, \gamma)$ is a taut balanced sutured manifold and let $S$ be a horizontal surface in it. Then

$$\text{rkSFH}(M, \gamma) = \text{rkSFH}(M_-, \gamma_-) \cdot \text{rkSFH}(M_+, \gamma_+).$$

The following definition can be found for example in [6].

**Definition 1.9.** A balanced sutured manifold $(M, \gamma)$ is called **horizontally prime** if every horizontal surface $S$ in $(M, \gamma)$ is isotopic to either $R_+(\gamma)$ or $R_-\gamma(\gamma)$ rel $\gamma$.

## 2. The results

**Theorem 2.1.** Let $(M, \gamma)$ be a taut balanced sutured manifold such that both $R_+(\gamma)$ and $R_-(\gamma)$ are connected. Suppose that there is a sequence of pairwise disjoint non-isotopic connected horizontal surfaces $R_-(\gamma) = S_0, S_1, \ldots, S_n = R_+(\gamma)$. Then

$$\text{rkSFH}(M, \gamma) \geq 2^n.$$

**Proof.** We prove the theorem using induction on $n$. If $n = 1$ then $(M, \gamma)$ is not a product since $R_-(\gamma)$ and $R_+(\gamma)$ are non-isotopic. Thus Theorem 1.9 implies that $\text{rkSFH}(M, \gamma) \geq 2$.

Now suppose that the theorem is true for $n - 1$. Since each $S_i$ is connected we can suppose without loss of generality that $S_1$ separates $S_i$ and $S_0$ for every $i \geq 2$. Let $(M_-, \gamma_-)$ and $(M_+, \gamma_+)$ be the sutured manifolds obtained after horizontally decomposing $(M, \gamma)$ along $S_1$. Note that both $(M_-, \gamma_-)$ and $(M_+, \gamma_+)$ are taut. Since $S_0$ and $S_1$ are non-isotopic $(M_-, \gamma_-)$ is not a product so as before $\text{rkSFH}(M_-, \gamma_-) \geq 2$. Applying the induction hypothesis to $(M_+, \gamma_+)$ and to the surfaces $R_-(\gamma_+), S_2, \ldots, S_n = R_+(\gamma_+)$ we get that $\text{rkSFH}(M_+, \gamma_+) \geq 2^{n-1}$. So using Proposition 1.8 we see that $\text{rkSFH}(M, \gamma) \geq 2^n$. \qed

**Corollary 2.2.** If $(M, \gamma)$ is a taut balanced sutured manifold and $\text{rkSFH}(M, \gamma) < 4$ then $(M, \gamma)$ is horizontally prime. More generally, if $n > 0$ and $\text{rkSFH}(M, \gamma) < 2^{n+1}$ then $(M, \gamma)$ can be cut into horizontally prime pieces by less than $n$ horizontal decompositions.

**Proof.** Suppose that $\text{rkSFH}(M, \gamma) < 2^{n+1}$. If $(M, \gamma)$ is not horizontally prime then there is a surface $S_1$ in $(M, \gamma)$ which is not isotopic to $R_\pm(\gamma)$. Decomposing $(M, \gamma)$ along $S_1$ we get two sutured manifolds $(M_-, \gamma_-)$ and $(M_+, \gamma_+)$. If they are not both horizontally prime then repeat the the above process with a non-prime piece and obtain a horizontal surface $S_2$, etc. This process has to end in less than $n$ steps according to Theorem 2.1. \qed

**Theorem 2.3.** Let $K$ be a knot in $S^3$ of genus $g$ and let $n > 0$. If $\text{rkHF}(K, g) < 2^{n+1}$ then $K$ has at most $n$ pairwise disjoint non-isotopic genus $g$ Seifert surfaces, in other words, $\dim MS(K) < n$. If $n = 1$ then $K$ has a unique Seifert surface up to equivalence.
Proof. Suppose that $R, S_1, \ldots, S_n$ are pairwise disjoint non-isotopic Seifert surfaces for $K$. According to Theorem 1.5 we have $\widehat{HF}(K, g) \approx SFH(S^3(R))$. Let $S^3(R) = (M, \gamma)$. If $R_+\gamma$ and $R_-\gamma$ were isotopic then $(M, \gamma)$ would be a product and $R$ would be equivalent. So the surfaces $R_-\gamma = S_0, S_1, \ldots, S_n, S_{n+1} = R_+\gamma$ satisfy the conditions of Theorem 2.1, thus $\text{rk} SFH(S^3(R)) \geq 2^{n+1}$, a contradiction.

In particular, if $n = 1$ then $\dim MS(K) = 0$. But according to [10] the complex $MS(K)$ is connected, so it consists of a single point. □

Corollary 2.4. Suppose that $K$ is an alternating knot in $S^3$ of genus $g$ and let $n > 0$. If the leading coefficient $a_g$ of its Alexander polynomial satisfies $|a_g| < 2^{n+1}$ then $\dim MS(K) < n$. If $|a_g| < 4$ then $K$ has a unique Seifert surface up to equivalence.

Proof. This follows from Theorem 2.3 and the fact that for alternating knots $\text{rk} \widehat{HF}(K, g) = |a_g|$. □

Remark 2.5. In [5] Kakimizu classified the minimal genus Seifert surfaces of all the prime knots with at most 10 crossings. The $n = 1$ case of Corollary 2.4 is sharp since the knot 74 is alternating, the leading coefficient of its Alexander polynomial is 4, and has 2 inequivalent minimal genus Seifert surfaces. On the other hand, the Alexander polynomial of the alternating knot 92 is also 4, but has a unique minimal genus Seifert surface up to equivalence.

Also note that [2, Theorem 1.7] implies that if the leading coefficient $a_g$ of the Alexander polynomial of an alternating knot $K$ satisfies $|a_g| < 4$ then the knot exterior $X(K)$ admits a depth $\leq 1$ taut foliation transversal to $\partial X(K)$. Indeed, for alternating knots $g = g(K)$ and $|a_g| = \text{rk} \widehat{HF}(K, g) \neq 0$, so the conditions of [2, Theorem 1.7] are satisfied.

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