THE BIOT-SAVART OPERATOR AND ELECTRODYNAMICS ON SUBDOMAINS OF THE THREE-SPHERE

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ABSTRACT. We study the generalization of the Biot-Savart law from electrodynamics in the presence of curvature. We define the integral operator $BS$ acting on all vector fields on subdomains of the three-dimensional sphere. By doing so, we establish a geometric setting for electrodynamics in positive curvature. When applied to a vector field, the Biot-Savart operator behaves like a magnetic field; we demonstrate that Maxwell’s equations hold for this context. In particular, for vector fields that act like currents, the curl operator is a left inverse to $BS$; thus the Biot-Savart operator is important in the study of curl eigenvalue energy-minimization problems in geometry and physics. We show that the Biot-Savart operator is self-adjoint and bounded. In all instances, the formulas we give are geometrically meaningful: they are preserved by orientation-preserving isometries of the three-sphere.

1. INTRODUCTION

The Biot-Savart law in electrodynamics calculates the magnetic field $B$ arising from a current flow $V$ in a smoothly bounded region $\Omega$ of $\mathbb{R}^3$ as

$$B(V)(y) = \frac{\mu_0}{4\pi} \int_{\Omega} V(x) \times \frac{y - x}{|y - x|^3} \, dx.$$  

Taking the curl of $B$ recovers the flow $V$, provided there is no time-dependence for this system. The Biot-Savart law can be extended to an operator $BS$ which acts on all smooth vector fields $V$ defined in $\Omega$. Cantarella, DeTurck, and Gluck studied the properties of this operator and its applications in [6].

The Biot-Savart operator is closely connected to the helicity of a vector field, which measures the average linking of its flowlines. Helicity can be computed as the $L^2$ inner product $\langle V, BS(V) \rangle$. Woltjer [20] first applied helicity to astrophysics in 1958. Moffatt [13] in 1969 gave helicity its name and proved it is an invariant of ideal fluid flow. Arnol’d [11] proved several invariance results about helicity in 1974 (an English translation appeared in 1986). All three needed to invert the curl operator, or equivalently to compute $d^{-1}\alpha$ for a 2-form $\alpha$, and utilized a version of the Biot-Savart operator. For more background on helicity and fluid dynamics, see for instance the book [2, Chapter III] and the recent overview [11].

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Helicity for vector fields is analogous to the linking number of a link and to the writhing number of a knot; many authors have explored these connections; see for instance [1, 2, 3, 5, 12, 14]. For a cohomological view of helicity, involving the Biot-Savart operator for $(k + 1)$-forms on $(2k + 1)$-dimensional subdomains of Euclidean space, see [8].

In [10], DeTurck and Gluck extend the Biot-Savart operator to the three-dimensional sphere $S^3$ and hyperbolic space $H^3$. We continue the investigation of how this story changes in the presence of curvature by looking at perhaps the most natural analog of the Euclidean setting, subdomains $\Omega \subset S^3$.

1.1. **Our aims.** In this work, we develop an approach to electrodynamics on bounded subdomains of the three-sphere via the Biot-Savart operator. We provide integral formulas for Maxwell’s equations and show precisely when $BS$ acts as a right inverse operator to curl. Our formulas are geometrically meaningful, in that their integrands are preserved by orientation-preserving isometries of $S^3$. Though verifying that Maxwell’s equations hold on orientable 3-manifolds is an elementary exercise in differential forms, there exists a paucity of geometric formulas for electrodynamics in the presence of curvature; we aim to remedy this situation.

Electrodynamics on the entire 3-sphere was developed in [10], with applications to geometric knot theory and to the helicity of vector fields. Electrodynamics on compact subdomains (with boundary) of $S^3$ is the natural analog of the Euclidean setting, and it raises a rich and interesting set of issues:

- The Hodge Decomposition Theorem for vector fields on $\Omega$ is more complicated than on the three-sphere, because curl is no longer a self-adjoint operator and divergence is no longer the (negative) adjoint of gradient.
- Current flows on bounded domains can deposit electric charge on boundaries and thereby affect Maxwell’s equations.
- Nonsingular current flows can be restricted to tubular neighborhoods of knots, enabling connections between the writhing number of the core knot and both the helicity and flux of these flows.

For modeling helicity and MHD on open, unbounded domains, such as the exterior of the sun in a Euclidean universe, the methods referenced above have had limited success. We propose an alternative for solar physics: modeling the universe as a giant three-sphere, and viewing the exterior of the sun as $S^3$ minus a ball, which forms a compact domain. The results herein might produce a useful model of solar plasmas.

1.2. **Listing of main results.** The next section surveys the known results for the Biot-Savart operator in $\mathbb{R}^3$ and in $S^3$. Then, we demonstrate in Theorem 3.1 that Maxwell’s equations hold on subdomains of $S^3$ via our setting, where we take a current $V$, its induced magnetic field $BS(V)$ and the appropriate electric field. We then generalize one of Maxwell’s equations:
Theorem 3.3. For $\Omega$ a compact subset of $S^3$ with smooth boundary, and $V$ a smooth vector field on $\Omega$, then

$$\nabla_y \times BS(V)(y) = \begin{cases} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{cases} - \nabla_y \int_\Omega \phi_0 (\nabla_x \cdot V(x)) \, dx + \nabla_y \int_{\partial \Omega} \phi_0 V(x) \cdot \hat{n} \, d(area_x).$$

This theorem implies that for $V$ divergence-free and tangent to the boundary, the curl operator acts as a left inverse to the Biot-Savart operator. These vector fields are the only ones for which $BS$ inverts curl, as we show in Proposition 4.7. Any such $V$ in this space that is also an eigenfield of $BS$ must furthermore be an eigenfield of curl, with reciprocal eigenvalue, i.e., if $BS(V) = \lambda V$, then $\nabla \times V = (1/\lambda) V$.

Also in section 4, we show that $BS$ is a self-adjoint operator. We describe its image and find its kernel.

Theorem 4.2. The kernel of the Biot-Savart operator on $\Omega$ is precisely the subspace of gradients that are always orthogonal to the boundary $\partial \Omega$.

2. Background on the Biot-Savart Operator

On October 30, 1820, Jean-Baptiste Biot (1774-1862) and his junior colleague Felix Savart (1791-1841) [4] announced their landmark experiment which deduced the magnetic field associated to a current flow in a vertical wire and gave rise to the law (1), which bears their names. Soon after, Ampere deduced that for steady currents, $\nabla \times B = \mu_0 J$, a law named for him. (Henceforth, we shall choose units so that both $\mu_0$ and $\epsilon_0$ each equal 1.) For an excellent history of the origins of electrodynamics, see [18].

2.1. Euclidean results from the Biot-Savart operator. The Biot-Savart law (1) holds for currents $J$, which we regard as divergence-free fields that are tangent to the boundary of the containment region; i.e., these are fields with no gradient component, by the Hodge Decomposition Theorem for vector fields (see [7]). The Biot-Savart operator extends the law to act upon all vector fields; for $x \in \Omega$, a compact subdomain of $\mathbb{R}^3$, define it as

$$BS(V)(y) = \frac{1}{4\pi} \int_\Omega V(x) \times \frac{y-x}{|y-x|^3} \, dx = \int_\Omega V(x) \times \nabla \phi(x,y) \, dx,$$

where the function $\phi(x,y) = 1/(4\pi|y-x|)$ is the fundamental solution to the Laplacian.

This operator was introduced in [6], which showed that $BS$ is a bounded, compact, and self-adjoint operator. Furthermore, they showed that $BS$ inverted curl for divergence-free vector fields tangent to the boundary. Our Theorems 3.3 and 4.2 are the direct analogs of their results, suitably adjusted in the presence of positive curvature.
2.2. **Defining the Biot-Savart operator on $S^3$.** In this section, we define the Biot-Savart operator on the three-sphere. We follow the approach of DeTurck and Gluck [10].

Let $VF(\Omega)$ denote the space of smooth vector fields on $\Omega$. For a current $J \in VF(\Omega)$, its magnetic field is smooth except across the boundary $\partial \Omega$, where the field remains continuous; magnetic fields are linear in $J$. Therefore to define $BS$ on the three-sphere we will work in the category of smooth, linear operators.

To motivate the definition, let us first understand what essential properties a magnetic field in $\mathbb{R}^3$ possesses. First, magnetic fields are divergence-free. Second, Ampere’s Law dictates that, for a steady current $J$, the curl of its associated magnetic field must equal $J$. Finally, we must describe how the Biot-Savart operator acts to gradients on $S^3$; for natural reasons, we require that it does so trivially.

Let us explain this last requirement. In the Euclidean subdomain case, the kernel of $BS$ is precisely the complement of the image of curl. On $S^3$, the Hodge Decomposition Theorem for vector fields is straightforward; the image of curl is precisely the space of all gradients. Thus all gradients should lie in the kernel of $BS$.

**Theorem 2.1** (Hodge Decomposition Theorem on $S^3$, cf. [19, Theorem 6.8]). Vector fields on $S^3$ decompose as

$$VF(S^3) = K(S^3) \oplus G(S^3),$$

and we call these spaces knots and gradients, respectively. Furthermore,

$$K(S^3) = \ker \text{div} = \text{image curl}$$

$$G(S^3) = \text{image grad} = \ker \text{curl}.$$

We note that the space of steady currents is precisely the divergence-free fields, $K(S^3)$. A special three-dimensional subspace is the set of left-invariant vector fields. These are all divergence-free, and are eigenfields of curl: $\nabla \times U = -2U$, for left-invariant $U$.

The three properties above depict our definition.

**Definition 2.2.** The Biot-Savart operator $BS : VF(S^3) \rightarrow VF(S^3)$ is defined to be the smooth linear operator satisfying these three properties:

1. BS is divergence free
   $$\nabla \cdot BS(V) = 0$$

2. For $V \in K(S^3)$, curl inverts $BS(V)$
   $$\nabla \times BS(V) = V$$

3. BS vanishes on gradients
   $$BS(\nabla f) = 0 \quad \forall \nabla f \in VF(S^3)$$

A straightforward exercise shows that $BS$ is uniquely determined among smooth linear operators from $VF(S^3)$ to $VF(S^3)$. Suppose not, with $B_1$ and $B_2$ two distinct operators. Then $B_1(V) - B_2(V)$ is both curl-free and divergence-free for all $V$. Hence by the Hodge Theorem, it is zero; thus $B_1 = B_2$.

Another approach defines $BS$ using the Green’s operator on vector fields, which inverts the Laplacian. The Green’s operator commutes with all partial derivatives and hence with div, grad, and curl. See [16] for a description of Green’s operators and Laplacians for vector fields of 3-manifolds.
**Proposition 2.3.** The Biot-Savart operator is the negative of the curl of the Green’s operator:

\[
BS(V) = -\nabla \times Gr(V)
\]

**Proof.** We show that this formula satisfies the three properties in Definition 2.2. The operator \(-\nabla \times Gr(V)\) is divergence-free because it is the curl of a vector field. Taking the curl of this operator yields

\[
\nabla \times (-\nabla \times Gr(V)) = \Delta(Gr(V)) - \nabla (Gr(\nabla \cdot V))
\]

where \(\Delta\) is the Laplacian on vector fields (see [16] for more details about this operator on three-manifolds). So if \(V\) is divergence-free, then the iterated curl above returns \(V\), as desired. n.b., above we must also utilize the scalar Green’s operator on functions.

The last condition states that the operator should vanish on gradients. All gradients are mapped to zero because \(-\nabla \times Gr(\nabla f) = -Gr(\nabla \times \nabla f) = -Gr(0) = 0\). \(\square\)

2.3. **Integral formulas for the Biot-Savart operator.** The integral formula [2] for the Biot-Savart operator in Euclidean space requires the addition of vectors lying in different tangent spaces. To obtain an analogous formula on the three-sphere, we must decide how to move vectors among tangent spaces. Two natural choices exist: parallel transport along a minimal geodesic or left translation (or right translation) using the group structure of \(S^3\), which we may view as the group of unit quaternions or as \(SU(2)\). Each method has its advantages and disadvantages; we utilize both throughout this article.

Let \(P_{yx}V\) denote parallel transport of \(V \in T_x S^3\) to the tangent space \(T_y S^3\); likewise let \((L_{yx}^{-1})_*V\) or \(L_* V\) denote left-translation. Let \(\alpha(x, y)\) be the distance on the three-sphere between \(x\) and \(y\).

Regarding notation, it is important to keep track of at which tangent space a derivative is taken. We affix a lowercase subscript to the nabla operator to accomplish this, e.g., \(\nabla_x f\) is the gradient of \(f\) at \(x\).

**Theorem 2.4** ([10, Theorem 2]). As an integral in which vectors are moved via parallel transport, the Biot-Savart operator is given by the formula:

\[
BS(V)(y) = \int_\Omega P_{yx} V(x) \times \nabla_y \phi(x, y) \, dx.
\]

As an integral in which vectors are moved via left translation, the Biot-Savart operator is given by the formula:

\[
BS(V)(y) = \int_\Omega (L_{yx}^{-1})_* V(x) \times \nabla_y \phi_0 \, dx
\]

\[
- \frac{1}{4\pi^2} \int_\Omega (L_{yx}^{-1})_* V(x) \, dx + 2\nabla_y \int_\Omega (L_{yx}^{-1})_* V(x) \cdot \nabla_y \phi_1 \, dx.
\]
This theorem expresses $BS(V)$ as a convolution of $V$, thought of as an electric current, with (the gradient of) appropriate potential functions. The left-translation version of $BS$ requires two different convolutions, while the parallel transport version requires only one. The potential functions are

\[
\phi_0(\alpha(x, y)) = -\frac{1}{4\pi^2}(\pi - \alpha) \cot(\alpha) \\
\phi_1(\alpha(x, y)) = -\frac{1}{16\pi^2}(\pi - \alpha) \cot(\alpha) \\
\phi_1(\alpha(x, y)) = -\frac{1}{16\pi^2}(\pi - \alpha) \cot(\alpha)
\]

The function $\phi_0$ is the fundamental solution of the Laplacian on $S^3$ and defines $\phi_1$ as follows:

\[
\Delta \phi_0 = \delta(\alpha) - \frac{1}{2\pi^2} \\
\Delta \phi_1 = \phi_0 - [\phi_0].
\]

Here $\delta(\alpha)$ represents the Dirac delta function and $[f]$ denotes the average value of $f$ on $S^3$. Recall that the Laplacian of a function has average value zero on a closed manifold; hence we require the constant $1/2\pi^2$.

Likewise, the function $\phi$ is the fundamental solution of the shifted Laplacian $(\Delta - 1)$; i.e., $\Delta \phi - \phi = \delta(\alpha)$. We note for future reference that $\phi_0(\alpha) = \phi(\alpha) \cos \alpha$.

Using the Biot-Savart operator, DeTurck and Gluck [10] obtain explicit integrals for calculating the linking number of two knots on $S^3$. They also generalize Călugăreanu’s formula $\text{LINK} = \text{TWIST} + \text{WRTHE}$ [9] by finding explicit integrals for the twist and for the writhe of a knot on $S^3$.

3. Electrodynamics on subdomains of $S^3$

This section investigates electric and magnetic fields on subdomains $\Omega$ of the three-sphere via the Biot-Savart operator. We assume $\Omega$ is compact with piecewise smooth boundary. We define $BS$ to be the same operator from Definition 2.2 with integral formulas given by (5) and (6); however, the region of integration is now restricted to $\Omega$. We may then view this operator as $BS : VF(\Omega) \to VF(\Omega)$. For $y \notin \Omega$, these formulas also define the behavior of $BS(V)$ outside of the domain.

3.1. Maxwell’s equations. We begin by considering a current $V(x)$ on $\Omega$ and computing $BS(V)$ as its corresponding magnetic field. If the vector field $V$ either has a nonzero divergence or is not tangent to the boundary $\partial \Omega$, then it no longer represents a steady current contained in $\Omega$. In this case we take a time-varying electric field, so that our system is ‘closed’. If $V$ is not divergence-free, charge will accumulate at a rate of minus the divergence, so let $\rho(x, t) = -(\nabla \cdot V)t$ represent the volume charge density in $\Omega$. If $V$ flows across the boundary, charge will accumulate on the boundary; let $\sigma(x, t) = (V \cdot \hat{n})t$ represent the surface charge density on $\partial \Omega$. Each of these charge distributions determines
an electric field:

\[
E_{\rho}(y, t) = -\left( \nabla_y \int_{\Omega} \phi_0 \left( \nabla_x \cdot V(x) \right) \, dx \right) \quad t = \nabla_y \int_{\Omega} \phi_0 \rho \, dx
\]

\[
E_{\sigma}(y, t) = \left( \nabla_y \int_{\partial \Omega} \phi_0 V(x) \cdot \hat{n} \, d(area_x) \right) \quad t = \nabla_y \int_{\partial \Omega} \phi_0 \sigma \, d(area_x)
\]

We can view \( \partial E_{\rho} / \partial t \) as the time rate of change of the electrodynamic field due to \( \partial \rho / \partial t \), the change in volume charge density; also we can view \( \partial E_{\sigma} / \partial t \) as the time rate of change of the electrodynamic field due to \( \partial \sigma / \partial t \), the change in surface charge density. We also may view \( \partial E_{\rho} / \partial t \) as an electrostatic field itself due to the time-independent volume charge \( \partial \rho / \partial t = \nabla \cdot V \); similarly, we may view \( \partial E_{\sigma} / \partial t \) as an electrostatic field due to the time-independent surface charge \( \partial \rho / \partial t = V \cdot \hat{n} \). In this section we adopt the former viewpoint, while later we will make use of the latter.

We consider the sum of these as the electric field \( E(y, t) = E_{\rho} + E_{\sigma} \). The following theorem demonstrates that this electrodynamic setup obeys Maxwell’s Equations.

**Theorem 3.1.** Let \( V = V(x) \) be a smooth vector field on \( \Omega \); we view \( V \) as a steady current. For the electric field \( E(y, t) \) defined above and the magnetic field \( B(y) = BS(V) \) corresponding to \( V \), Maxwell’s four equations hold at each \( y \in \Omega \):

1. \( \nabla \cdot E = \rho \)
2. \( \nabla \times E = \frac{\partial B}{\partial t} \)
3. \( \nabla \cdot B = 0 \)
4. \( \nabla \times B = V + \frac{\partial E}{\partial t} \)

(n.b., we have chosen units so that both the permittivity of free space \( \epsilon_0 \) and the permeability of free space \( \mu_0 \) are identically 1.)

By the end of this subsection, we will have proven Theorem 3.1. The following proposition restates Maxwell’s third equation.

**Proposition 3.2.** For \( V \) a smooth vector field, \( BS(V) \) is divergence-free, whether defined on \( \Omega \) or on \( S^3 \).

Though Definition 2.2 demands this to be true for \( BS \), we state it here since we provide an independent proof of this fact for its parallel transport formula (5) in section 3.4.

Next, we compute the curl of \( BS \):

**Theorem 3.3.** For \( \Omega \) a compact subset of \( S^3 \) with smooth boundary, and \( V \) a smooth vector field on \( \Omega \), then

\[
\nabla_y \times BS(V)(y) = \begin{cases} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{cases}
\]

\[
- \nabla_y \int_{\Omega} \phi_0 \left( \nabla_x \cdot V(x) \right) \, dx \quad + \quad \nabla_y \int_{\partial \Omega} \phi_0 V(x) \cdot \hat{n} \, d(area_x).
\]
If \( V \) is divergence-free and tangent to \( \partial \Omega \), notice that \( \text{curl} \) acts as a left inverse to \( BS \),

\[
\nabla_y \times BS(V)(y) = \begin{cases} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{cases}.
\]

The above theorem is useful in solving certain energy minimization problems for vector fields on \( S^3 \) (e.g., analogs of the Woltjer and Taylor problems in plasma physics). Solutions are often eigenfields of \( \text{curl} \) and, by (10), also of \( BS \) (assuming they have no gradient component).

We provide two proofs of this theorem in section 3.6, one using parallel transport of vectors and one using left-translation. The first argument relies on a result, the Key Lemma, described in section 3.5.

Recognizing the last two terms of (9) as \( \partial E_\rho/\partial t \) and \( \partial E_\sigma/\partial t \), respectively, we obtain as an immediate corollary that Maxwell’s fourth equation holds.

**Corollary 3.4.** For \( y \in \Omega \),

\[
\nabla_y \times BS(V)(y) = V(y) + \frac{\partial E_\rho}{\partial t} + \frac{\partial E_\sigma}{\partial t}.
\]

**Remark 3.5.** When \( \Omega = S^3 \), Theorem 3.3 above reduces to

\[
\nabla_y \times BS(V)(y) = V(y) - \nabla_y \int_{S^3} \phi_0 (\nabla_x \cdot V(x)) \, dx.
\]

In this case Proposition 2.3 and (4) provide an easy proof:

\[
\nabla \times BS(V) = \nabla \times (-\nabla \times Gr(V)) = V - \nabla Gr(\nabla \cdot V).
\]

The last term is precisely the gradient in (12). Alas, the Green’s operator for \( \Omega \neq S^3 \) can be much more complicated, and we will require a more difficult argument to prove Theorem 3.3 in general.

**Example 3.6.** There exists a three-dimensional subspace of \( V F(S^3) \) consisting of all left-invariant vector fields. Any such field \( U \) is divergence free and satisfies \( \nabla \times U = -2U \). On \( S^3 \), we calculate that \( BS(U) = -(1/2)U \), and so \( \nabla \times BS(U) = U \), as expected by (12).

Consider the solid torus \( \Omega_a \) that consists of all points within spherical distance \( a \) of the circle \( x^2 + y^2 = 1 \), which we denote as \( \gamma \). Let \( \hat{u}_1 \) be the unit, left-invariant field with an orbit along \( \gamma \). Then, \( BS(\hat{u}_1) = -\frac{1}{2} \hat{u}_1 + \frac{1}{2} \cos^2 a \, W \), where \( W \) is the unique curl-free, divergence-free field that has unit length along \( \gamma \); all orbits of \( W \) are longitudes of \( \Omega_a \). (In terms of the Hodge Decomposition Theorem (section 4.1), \( W \) is a harmonic knot.) Then,

\[
\nabla \times BS(\hat{u}_1) = \frac{1}{2} \nabla \times \hat{u}_1 + \frac{\cos^2 a}{2} \nabla \times W = \hat{u}_1,
\]

as expected by Theorem 3.3 since \( \hat{u}_1 \) is divergence-free and tangent to \( \partial \Omega_a \).

We now return to prove Theorem 3.1.
**Proof of Theorem 3.1.** We have already seen through Proposition 3.2 and Corollary 3.4 that the last two of Maxwell’s equations hold in our setting. We must now show the first and second equations do as well.

The divergence of $E$ in $\Omega$ can be calculated by taking the divergence of both sides of Maxwell’s fourth equation:

$$\nabla \cdot \nabla \times BS(V) = \nabla \cdot V + \nabla \cdot \frac{\partial E}{\partial t}$$

$$0 = \nabla \cdot V + \frac{\partial}{\partial t} (\nabla \cdot E)$$

Since the current $V$ does not depend upon time, we integrate to obtain that $\nabla \cdot E = - (\nabla \cdot V) t = \rho$, Maxwell’s first equation.

Maxwell’s second equation holds trivially. The electric field $E$ is defined by two gradients and ergo its curl is zero. The magnetic field $BS$ does not depend upon time, even when the electric field $E$ is time-dependent; hence $\nabla \cdot E = 0 = \partial B / \partial t$. □

The remainder of section 3 is devoted to proving the results claimed above. First, we need to define a computational aid in $\mathbb{R}^4$, the triple product.

### 3.2. Triple products in $\mathbb{R}^4$. Let $A, B, C$ be vectors (or vector fields) on $\mathbb{R}^4$. Let $\alpha \in [0, \pi]$ be the angle between vectors $A$ and $B$.

**Definition 3.7.** The triple product of $A, B, C$ is the vector in $\mathbb{R}^4$

$$[A, B, C] = \text{det} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & \hat{x}_4 \end{bmatrix}.$$

The triple product of three vectors in $\mathbb{R}^4$ is the analog of the cross product of two vectors in $\mathbb{R}^3$. Indeed, the product of $n - 1$ vectors in $\mathbb{R}^n$ is similarly defined as the determinant of an $n \times n$ matrix.

Three useful properties of triple products are that

1. They are multilinear and alternating, i.e., $[A, B, C] = [B, C, A] = -[A, C, B]$.
2. $[A, B, C]$ is orthogonal to $A, B,$ and $C$. If $A, B,$ and $C$ are linearly independent, then $\{A, B, C, [A, B, C]\}$ forms a basis that agrees with the standard orientation on $\mathbb{R}^4$. If $A, B,$ and $C$ are linearly dependent, then $[A, B, C] = 0$.
3. If $A$ is a point in $S^3$ (i.e., $|A| = 1$), and $B$ and $C$ are tangent to the three-sphere at $A$, i.e., $B, C \in T_A S^3$, then $[A, B, C]$ lies in $T_A S^3$ and is equal to the cross product $B \times C$. More generally, for $A \in S^3$, the triple product $[A, B, C] = B^\perp \times C^\perp$, where $B^\perp = B - (A \cdot B) A \in T_A S^3$, is the component of $B$ perpendicular to $A$ (and likewise $C^\perp \in T_A S^3$).

Often, calculations require a formula for an iterated double product.
Lemma 3.8. Let $A, B, C$ be vectors in $\mathbb{R}^4$. Let $C^\perp$ represent the component of $C$ that is orthogonal to the plane spanned by $A$ and $B$.

\begin{equation}
[A, B, [A, B, C]] = -|A|^2 |B|^2 \sin^2(\alpha) C^\perp
\end{equation}

Proof. Let $\alpha$ be the angle between $A$ and $B$. We use Gram-Schmidt to find $B^\perp$ orthogonal to $A$ and to find $C^\perp$ orthogonal to both $A$ and $B^\perp$:

\begin{align}
B^\perp &= B \sin(\alpha) \\
C^\perp &= C - \frac{|B|(A \cdot C) - |A| \cos \alpha (B \cdot C)}{|A|^2 |B|^2 \sin^2(\alpha)} A - \frac{|A|(B \cdot C) - |B| \cos \alpha (A \cdot C)}{|A||B|^2 \sin^2(\alpha)} B
\end{align}

Let $D = [A, B, C]$. Assume $\{A, B, C\}$ are linearly independent, else $D = 0$. Then $D$ is orthogonal to the span of $A, B, C$ and the basis $\{A, B, C, D\}$ has positive orientation in $\mathbb{R}^4$. The length of $D$ is $|D| = |A||B| |C^\perp| \sin \alpha$.

Let $E = [A, B, D] = [A, B, [A, B, C]]$. Then $E$ is orthogonal to $D$, so it is a linear combination of $A, B,$ and $C$. Since $E$ is also orthogonal to $A$ and $B$, it must be a multiple of $C^\perp$. To ensure that the basis $\{A, B, D, E\}$ has positive orientation in $\mathbb{R}^4$, the vector $E$ must point in the direction of $-C^\perp$. The length of $E$ is $|E| = |A||B| |D| \sin \alpha = |A|^2 |B|^2 |C^\perp| \sin^2 \alpha$, which proves the lemma. \hfill $\square$

3.3. The calculus of parallel transport. Let $x, y$ be non-antipodal points on $S^3$, we will view them as unit vectors in $\mathbb{R}^4$. Recall, $\alpha$ is the distance on $S^3$ between them. The component of $y$ perpendicular to $x$ is $y^\perp = (y - x \cos \alpha)$.

Since the three-sphere has unit radius, the gradient of $\alpha$ with respect to $x$ (in $T_x S^3$) must be a unit vector which points away from $y$ along $-y^\perp$. Thus

\begin{align}
\nabla_x \alpha(x, y) &= \frac{-y^\perp}{|y^\perp|} = \frac{x \cos \alpha - y}{\sin \alpha} \\
\nabla_y \alpha(x, y) &= \frac{-x^\perp}{|x^\perp|} = \frac{y \cos \alpha - x}{\sin \alpha}
\end{align}

For any two unit vectors $x, y \in \mathbb{R}^n$ such that $x \neq \pm y$, the unique map $M \in SO(n)$ that maps $x$ to $y$ and fixes all vectors orthogonal to both $x$ and $y$ is

$$M(V) = V - \frac{(V \cdot (x + y))}{1 + (x \cdot y)} x + \frac{(V \cdot x) (1 + 2(x \cdot y)) - (V \cdot y)}{1 + (x \cdot y)} y$$

For $V$ a tangent vector at $x \in S^3$, this map $M$ precisely describes its parallel transport to the tangent space at $y$. The expression above simplifies to

\begin{equation}
P_{yx} V = V - \frac{(V \cdot y)}{1 + (x \cdot y)} (x + y) = M(V)
\end{equation}

Remark 3.9. For a vector $v$ at $x \in S^3$ that points parallel to the geodesic $\gamma$ running through $x$ and $y \in S^3$, left-translation from $x$ to $y$ is exactly the same as parallel transport. The two
methods differ only in how they treat components that are perpendicular to the geodesic $\gamma$. Hence we compute

$$\nabla_y\alpha = -P_{yx} \nabla_x\alpha = -(L_{yx^{-1}})_* \nabla_x\alpha.$$  

The interested reader is invited to show this directly using the group structure of $S^3$ and equations (17) and (18). 

**Remark 3.10.** For any function $f(\alpha)$ that depends only upon the distance $\alpha(x, y)$ from $x$ to $y$, its gradient with respect to $x$ variables is $\nabla_x f(\alpha) = f'(\alpha) \nabla_x\alpha$. Thus the methods of transporting its gradient vector are equivalent,

$$\nabla_y f(\alpha) = -P_{yx} \nabla_x f(\alpha) = -(L_{yx^{-1}})_* \nabla_x f(\alpha).$$

### 3.4. The divergence of BS

Though Definition 2.2 requires the Biot-Savart operator to have zero divergence, here we furnish an independent proof of this fact; we utilize the parallel transport formula (5) for BS.

**Proof of Proposition 3.2** We start by taking the divergence of (5),

$$\nabla_y \cdot BS(V)(y) = \nabla_y \cdot \int_{\Omega} P_{yx} V \times \nabla_y \phi \, dx = \int_{\Omega} \nabla_y \cdot (P_{yx} V \times \nabla_y \phi) \, dx$$

Apply the vector identity $\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - (\nabla \times B) \cdot A$ to the integrand:

$$\nabla_y \cdot BS(V)(y) = \int_{\Omega} \nabla_y \phi \cdot (\nabla_y \times P_{yx} V) \, dx,$$

since the curl of $\nabla_y \phi$ is zero. Next we compute the curl of $P_{yx} V$ using (19):

$$\nabla_y \times P_{yx} V = \nabla_y \times V(x) - \nabla_y \times \frac{V \cdot y}{1 + \cos \alpha} (x + y)$$

Since $x$ is fixed with respect to $y$, $\nabla_y \times x = 0$. Applying the Leibniz rule for curls, we obtain

$$\nabla_y \times P_{yx} V = -\nabla_y \left( \frac{V \cdot y}{1 + \cos \alpha} \right) \times (x + y) - \frac{V \cdot y}{1 + \cos \alpha} \nabla_y \times (x + y)$$

The cross product above is taken in the tangent space at $y$, so a vector crossed with $y$ contributes nothing. Similarly $\nabla_y \times y = 0$.

$$\nabla_y \times P_{yx} V(x) = -\nabla_y \left( \frac{V \cdot y}{1 + \cos \alpha} \right) \times x$$

Now calculate the gradient above:

$$\nabla_y \left( \frac{V \cdot y}{1 + \cos \alpha} \right) = \frac{1}{1 + \cos \alpha} \nabla_y (V \cdot y) + \frac{V \cdot y}{(1 + \cos \alpha)^2} \nabla_y \cos \alpha$$
To proceed, we need to calculate $\nabla_y (V \cdot y)$. It should point along the component of $V$ that lies perpendicular to $y$. Let $\theta$ be the angle between $V(x)$ and $y$ in $\mathbb{R}^4$. Since $(V \cdot y) = |V| \cos \theta$, we calculate

$$\nabla_y (V \cdot y) = \frac{d}{d\theta} (|V| \cos \theta) \nabla_y \theta = -|V| \sin \theta \frac{y \cos \theta - V/|V|}{\sin \theta}$$

(23) $\nabla_y (V \cdot y) = V - (V \cdot y) y = V^\perp$

A similar formula holds for the gradient of the inner product of any two vectors in $\mathbb{R}^n$. As a check, we compute $\nabla_y \cos \alpha = -\sin \alpha \nabla_y \alpha = x^\perp$, by (18), which agrees with (23). Then,

$$\nabla_y \left( \frac{V \cdot y}{1 + \cos \alpha} \right) = \frac{1}{1 + \cos \alpha} V^\perp - \frac{V \cdot y}{(1 + \cos \alpha)^2} x^\perp$$

After substituting this in, we may express equation (22) as a triple product, described in section 3.2.

$$\nabla_y \times P_{yx} V = \left( -\frac{1}{1 + \cos \alpha} V^\perp + \frac{V \cdot y}{(1 + \cos \alpha)^2} x^\perp \right) \times x$$

$$\nabla_y \times P_{yx} V = \left[ y, -\left( -\frac{V^\perp}{1 + \cos \alpha} + \frac{V \cdot y}{(1 + \cos \alpha)^2} x^\perp \right), x \right]$$

A triple product is zero if the three vectors are not linearly independent, so the $x^\perp$ term above vanishes.

$$\nabla_y \times P_{yx} V = \frac{1}{1 + \cos \alpha} [y, V, x]$$

Recall that the triple product is unchanged if we switch to vectors perpendicular to $y$: $[y, V, x] = [y, V^\perp, x^\perp]$. Again, let the triple product represent a cross product.

$$\nabla_y \times P_{yx} V = \frac{1}{1 + \cos \alpha} [y, V^\perp, x^\perp] = \frac{V^\perp \times x^\perp}{1 + \cos \alpha} \in T_y S^3$$

We finally return to (21),

$$\nabla_y \cdot BS(V)(y) = \int_\Omega \nabla_y \phi \cdot (\nabla_y \times P_{yx}) \, dx = \int_\Omega \phi'(\alpha) \nabla_y \alpha \cdot \left( \frac{V^\perp \times x^\perp}{1 + \cos \alpha} \right) \, dx$$

$$= \int_\Omega \frac{\phi'(\alpha)}{1 + \cos \alpha} \frac{-x^\perp}{|x^\perp|} \cdot (V^\perp \times x^\perp) \, dx,$$

where we have substituted (18). The product $x^\perp \cdot (V^\perp \times x^\perp) \equiv 0$, so the integrand vanishes identically at each point. We conclude that the Biot-Savart operator is divergence-free. □
3.5. **Key Lemma.** In this section, we prove an important lemma relating vector fields and functions on $S^3$. DeTurck and Gluck [10] have also proven this lemma; we furnish an independent proof.

**Lemma 3.11.** [Key Lemma] Let $x, y$ be two non-antipodal points in $S^3 \subset \mathbb{R}^4$. Let $\phi = \phi(\alpha)$ be a function, depending only on $\alpha$, which may have a singularity at $\alpha = 0$ but is otherwise smooth. Let $V$ be a tangent vector at $x \in S^3$. Then,

$$\nabla_y \times \{ P_{yx} V \times \nabla_y \phi \} - \nabla_y \{ V \cdot \nabla_x (\phi \cos \alpha) \} = (\Delta \phi - \phi) (V - (V \cdot y) y).$$

Here the Laplacian is taken in terms of functions on $S^3$, where $\alpha$ can be taken as a coordinate. We will use the Key Lemma in the next section to prove Ampere’s Law. Another application of the Key Lemma is in proving an integral formula for the linking number of two knots on $S^3$; refer to [10] for details.

**Proof.** Denote the two terms on the left-hand side of the Key Lemma as terms $C$ and $G$, respectively. We will show that

$$C - G = (\Delta \phi - \phi) (V - (V \cdot y) y).$$

We convert the cross product in $C$ into a triple product and utilize Remark 3.10 to compute the gradient of $\phi$,

$$C = \nabla_y \times \left[ y, P_{yx} V, \frac{\phi'(\alpha)}{\sin \alpha} (y \cos \alpha - x) \right].$$

We calculate the parallel transport of $V(x)$ via (19), and the triple product above becomes

$$[y, P_{yx} V, (y \cos \alpha - x)] = \left[ y, \left( V - \frac{V \cdot y}{1 + \cos \alpha} (x + y) \right), -x \right] = [y, V, -x],$$

since the triple product is linear and the terms with repeated variables vanish. Hence, $C$ becomes

$$C = \nabla_y \times \frac{\phi'(\alpha)}{\sin \alpha} [y, x, V],$$

Applying the Leibniz rule for curls, we obtain

$$C = \nabla_y \left( \frac{\phi'(\alpha)}{\sin \alpha} \right) \times [y, x, V] + \frac{\phi'(\alpha)}{\sin \alpha} \nabla_y \times [y, x, V]$$

(26)

Call these two terms $C_1$ and $C_2$ respectively, so $C = C_1 + C_2$. We analyze them separately. The first one, $C_1$, is converted to an iterated triple product.

$$C_1 = \left[ y, \nabla_y \frac{\phi'(\alpha)}{\sin \alpha}, [y, x, V] \right]$$

$$C_1 = \frac{d}{d\alpha} \left( \frac{\phi'}{\sin \alpha} \right) \left[ y, \frac{y \cos \alpha - x}{\sin \alpha}, [y, x, V] \right]$$

$$C_1 = -\frac{\phi'' \sin \alpha - \phi' \cos \alpha}{\sin^3 \alpha} [y, x, [y, x, V]]$$
Now we utilize Lemma 3.8 to evaluate the iterated triple product. In order to do so, we must first calculate $V^*$, the component of $V$ orthogonal to $x$ and $y$ from equation (16).

$$V^* = V - \frac{V \cdot y}{\sin^2 \alpha} y + \frac{\cos \alpha}{\sin^2 \alpha} (V \cdot y) x$$

Then $C_1$ becomes

$$C_1 = \frac{\phi''}{\sin^3 \alpha} \sin \alpha - \frac{\phi'}{\sin \alpha} \cos \alpha (V^*)$$

(27)

$$C_1 = \frac{\phi''}{\sin^3 \alpha} \sin \alpha - \frac{\phi'}{\sin \alpha} \cos \alpha (\sin^2 \alpha V + (V \cdot y) V - \cos \alpha (V \cdot y) x)$$

Now we consider $C_2$ from (26). Since $[y, x, V] = [x, V, y] = [x^\perp, V^*, y]$, we can rewrite it as

$$C_2 = \frac{\phi'}{\sin \alpha} \nabla y \times [y, x, V] = \frac{\phi'}{\sin \alpha} \nabla y \times [x^\perp, V^*, y].$$

Claim. $\nabla y \times [x^\perp, V^*, y] = 2(x \cdot y) V - 2(V \cdot y) y$.

The claim follows in a bare-hands computation similar to the ones above. Start by expressing the triple product as a curl, and then use vector identities and (23) to compute all necessary derivatives. We leave this to the reader.

Applying the claim to (28), we obtain

$$C_2 = 2\phi \frac{\cos \alpha}{\sin \alpha} V - 2\phi \frac{1}{\sin \alpha} (V \cdot y) x$$

Summing terms $C_1$ and $C_2$, we obtain the following expression for $C$:

$$C = \left( \phi'' + \phi \frac{\cos \alpha}{\sin \alpha} \right) V$$

(29)

$$+ \left( -\phi'' \frac{1}{\sin^2 \alpha} + \phi \frac{\cos \alpha}{\sin^3 \alpha} \right) (V \cdot y) y$$

$$+ \left( \phi'' \frac{\cos \alpha}{\sin^2 \alpha} - \phi \frac{\cos^2 \alpha}{\sin^3 \alpha} - 2\phi \frac{1}{\sin \alpha} \right) (V \cdot y) x$$

Next, we turn our attention to the second term $G = \nabla_y (V \cdot \nabla_x (\phi \cos \alpha))$ in the statement of the Key Lemma.

$$G = -\nabla_y \left( V \cdot \frac{d}{d\alpha} (\phi \cos \alpha) \nabla_\alpha \right) = \nabla_y \left( (\phi' \cos \alpha - \phi \sin \alpha) V \cdot \frac{x \cos \alpha - y}{\sin \alpha} \right)$$
Since $V \in T_x S^3$, the product $V \cdot x = 0$. Thus,

$$G = -\nabla_y \left[ \frac{\phi' \cos \alpha - \phi \sin \alpha}{\sin \alpha} (V \cdot y) \right]$$

$$G = -(V \cdot y) \nabla_y \left( \frac{\phi \cos \alpha}{\sin \alpha} - \phi \right) - \left( \frac{\phi \cos \alpha}{\sin \alpha} - \phi \right) \nabla_y (V \cdot y)$$

(30) $$G = -(V \cdot y) \frac{d}{d\alpha} \left( \frac{\phi \cos \alpha}{\sin \alpha} - \phi \right) \nabla_y \alpha - \left( \frac{\phi \cos \alpha}{\sin \alpha} - \phi \right) \nabla_y (V \cdot y)$$

By (23), $\nabla_y (V \cdot y) = V - (V \cdot y) y$. We substitute via (18) for $\nabla_y \alpha$ to obtain

$$G = -(V \cdot y) \left( \phi'' \frac{\cos \alpha}{\sin^2 \alpha} - \phi' \frac{1}{\sin^3 \alpha} + \phi' \frac{1}{\sin \alpha} \right) (y \cos \alpha - x)$$

$$+ \left( \frac{\phi' \cos \alpha}{\sin \alpha} - \phi \right) (V - (V \cdot y) y)$$

(31)

Finally, $C - G$, which forms the left-hand side of the Key Lemma statement, is merely the difference of equations (29) and (31). All $x$ terms cancel, leaving

$$C - G = \left( \phi'' + 2 \phi' \frac{\cos \alpha}{\sin \alpha} - \phi \right) V + \left( -\phi'' - 2 \phi' \frac{\cos \alpha}{\sin \alpha} + \phi \right) (V \cdot y) y$$

(32) $$C - G = \left( \phi'' + 2 \phi' \frac{\cos \alpha}{\sin \alpha} - \phi \right) (V - (V \cdot y) y)$$

In terms of the coordinate $\alpha$, the Laplacian of $\phi(\alpha)$ on $S^3$ is $\Delta \phi(\alpha) = \phi'' + 2 \phi' \frac{\cos \alpha}{\sin \alpha}$. Thus we have proven the Key Lemma, since

$$C - G = (\Delta \phi - \phi) (V - (V \cdot y) y) \ . \ □$$

3.6. Proving Theorem 3.3

3.6.1. Parallel transport proof of Theorem 3.3 The Key Lemma, as perhaps its most important consequence, directly proves Theorem 3.3. Maxwell’s fourth equation, which states

$$\nabla_y \times BS(V)(y) = \begin{cases} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{cases}$$

(9)

$$-\nabla_y \int_{\Omega} \phi_0 (\nabla_x \cdot V(x)) \ dx + \nabla_y \int_{\partial \Omega} \phi_0 V(x) \cdot \hat{n} \ d(area_x)$$

Proof. We begin by inserting the function $\phi(\alpha) = -\frac{1}{\sin \alpha} (\pi - \alpha) \csc \alpha$, which appears in the parallel transport formula (5) for $BS$, into the Key Lemma. Recall that $\phi_0(\alpha) = \phi(\alpha) \cos \alpha$ and $\Delta \phi - \phi = \delta(\alpha) = \delta(x, y)$. The Key Lemma states

$$\nabla_y \times \{ P_{yx} V \times \nabla_y \phi \} - \nabla_y \{ V \cdot \nabla_x (\phi \cos \alpha) \} = (\Delta \phi - \phi) (V - (V \cdot y) y) \ .$$

(24)

Now, integrate both sides over $\Omega$ with respect to $x$:

$$\int_\Omega \nabla_y \times \{ P_{yx} V \times \nabla_y \phi \} \ dx - \int_\Omega \nabla_y \{ V \cdot \nabla_x \phi_0 \} \ dx = \int_\Omega \delta(x, y) (V(x) - (V \cdot y) y) \ dx$$
We may interchange the integral in $x$ variables with the gradient and curl operators on the left-hand side, since they are in terms of $y$.

$$\nabla_y \times \int_{\Omega} P_{yx} V(x) \times \nabla_y \phi \, dx - \nabla_y \int_{\Omega} V(x) \cdot \nabla_x \phi_0 \, dx = \int_{\Omega} \delta(x, y) (V(x) - (V \cdot y) y) \, dx$$

The first term on the left-hand side is precisely the curl of the Biot-Savart operator. Let us substitute that into the equation above.

$$\nabla_y \times BS(V) = \nabla_y \int_{\Omega} V \cdot \nabla_x \phi_0 \, dx + \int_{\Omega} \delta(x, y) (V(x) - (V \cdot y) y) \, dx \tag{33}$$

We now apply the vector identity $V \cdot \nabla \phi_0 = \nabla \cdot (\phi_0 V) - \phi_0 \nabla \cdot V$ to expand the first RHS integral:

$$\int_{\Omega} V \cdot \nabla_x \phi_0 \, dx = \int_{\Omega} \nabla_x \cdot (\phi_0 V) \, dx - \nabla_y \int_{\Omega} \phi_0 \nabla_x \cdot V \, dx$$

after applying the Divergence Theorem. Substituting into (33), we have found three of our four desired terms.

$$\nabla_y \times BS(V) = \nabla_y \int_{\partial \Omega} \phi_0 V \cdot \hat{n} \, d(\text{area}_x) - \nabla_y \int_{\Omega} \phi_0 \nabla_x \cdot V \, dx + \int_{\Omega} \delta(x, y) (V(x) - (V \cdot y) y) \, dx$$

To evaluate the last integral, we substitute $y$ for $x$ in its integrand to obtain $V(y) - (V(y) \cdot y) y$. Since $V(y)$ lies in the tangent space of $y$, this reduces to just $V(y)$, which is assumed to be zero outside of $\Omega$. We write this fact explicitly,

$$\int_{\Omega} \delta(x, y) (V(x) - (V(x) \cdot y) y) \, dx = \left\{ \begin{array}{ll} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{array} \right.,$$

which completes the proof of the theorem. \hfill \Box

3.6.2. Left-translation proof of Theorem 3.3

Proof. The left-translation formula for $BS$ is comprised of three integrals:

$$BS(V)(y) = \int_{\Omega} (L_{yx^{-1}})_* V(x) \times \nabla_y \phi_0 \, dx$$

$$- \frac{1}{4\pi^2} \int_{\Omega} (L_{yx^{-1}})_* V(x) \, dx + 2\nabla_y \int_{\Omega} (L_{yx^{-1}})_* V(x) \cdot \nabla_y \phi_1 \, dx \tag{6}$$

We take the curl of each term separately; the first integral requires significant analysis while the others do not. The third is a gradient; its curl must be zero. Meanwhile, the second term produces a vector field $-\frac{1}{2}[V]$, where $[V]$ measures the average value of $V$.
on all of $S^3$; such a vector field is left-invariant. Recall that all left-invariant fields are eigenfields of curl, so $\nabla \times -\frac{1}{2}[V] = [V]$.

Now we calculate the curl of the first integral, which for convenience we denote as $B$. We truncate notation to $L_* V$ in lieu of $(L_{yx^{-1}})_* V(x)$.

$$\nabla_y \times B = \int_\Omega \nabla_y \times (L_* V \times \nabla_y \phi_0) \, dx$$

Apply the vector identity for the curl of a cross product:

$$\nabla \times (U \times W) = [W, U] + (\nabla \cdot W) U - (\nabla \cdot U) W,$$

where we set $U = L_* V$ and $W = \nabla_y \phi_0$. We obtain three terms to analyze.

$$\nabla_y \times B = \int_\Omega [W, U] \, dx + \int_\Omega (\Delta_y \phi_0) \, L_* V \, dx - \int_\Omega (\nabla_y \cdot L_* V) \, \nabla_y \phi_0 \, dx$$  

*The third term of (34).* We claim this term is zero. For fixed $x$ and varying $y$, the vector field $L(y) = (L_{yx^{-1}})_* V(x)$ is a left-invariant vector field. Left-invariant vector fields on $S^3$ are divergence-free, so this integrand of this term vanishes pointwise.

*The second term of (34).* The function $\phi_0$ was defined to be the fundamental solution of the Laplacian, i.e., $\Delta \phi_0(\alpha) = \delta(\alpha) - 1/2\pi^2$. Then, the second term can be rewritten as

$$\int_\Omega (\Delta_y \phi_0) \, L_* V \, dvol_x = \int_\Omega \delta(\alpha) \, (L_{yx^{-1}})_* V(x) \, dx - \frac{1}{4\pi^2} \int_\Omega (L_{yx^{-1}})_* V(x) \, dx$$

Since $\delta(\alpha)$ vanishes except when $\alpha = 0$, i.e., when $x = y$, the first RHS integral simply takes on the value of the vector field when $x = y$. The second RHS integral is the formula for $[V]$. Thus the second term of (34) is

$$\int_\Omega (\Delta_y \phi_0) \, L_* V \, dvol_x = \begin{cases} V(y) & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{cases} - [V]$$

*The first term of (34).* We will need this lemma, proven in [17].

**Lemma 3.12.** On $S^3$, let $U$ be a left-invariant vector field, let $W$ be any smooth vector field, and let $G$ be a gradient. Then $^1$

$$\nabla W U = W \times U \quad \text{and} \quad \nabla(U \cdot G) = [U, G].$$

The lemma allows us to rewrite the first term as

$$\int_\Omega [W, U] \, dx = - \int_\Omega \nabla_y(W \cdot U) \, dx = \nabla_y \int_\Omega \nabla_y \phi_0 \cdot L_* V \, dx$$  

^1Regarding notation, we utilize uppercase subscripts on $\nabla$ to denote covariant derivatives, e.g., $\nabla W U$. Lowercase subscripts, e.g. $\nabla_y$, continue to refer to the point at which the nabla operator is applied.
Recall from (20) that the gradient of $\phi_0$ changes sign when switching variables from $y$ to $x$ appropriately, $\nabla_y \phi_0 = -(L_{yx^{-1}})_* \nabla_x \phi_0$. Therefore,

$$\int_{\Omega} [W, U] \, dx = -\nabla_y \int_{\Omega} - (L_{yx^{-1}})_* \nabla_x \phi_0 \cdot (L_{yx^{-1}})_* V(x) \, dx = \nabla_y \int_{\Omega} \nabla_x \phi_0 \cdot V(x) \, dx,$$

since there is no need to left translate before taking the dot product. Now, we apply the Leibniz rule for divergences to obtain

$$\int_{\Omega} [W, U] \, dx = \nabla_y \int_{\Omega} \nabla_x \phi_0 \cdot V(x) \, dx - \nabla_y \int_{\partial \Omega} \phi_0 V \cdot \hat{n} \, (\text{area}_x).$$

Gathering our results from (35) and (37), along with the $[V]$ contribution from the second integral that comprises $BS$, we have proved Theorem 3.3.

$$\nabla_y \times B = \left( \begin{array}{c} V(y) \text{ inside } \Omega \\ 0 \text{ outside } \Omega \end{array} \right) - [V]$$

$$+ \nabla_y \int_{\Omega} \phi_0 (\nabla_x \cdot V) \, dx - \nabla_y \int_{\partial \Omega} \phi_0 V \cdot \hat{n} \, (\text{area}_x)$$

$$\nabla_y \times -\frac{1}{2}[V] = [V]$$

The sum of these two terms is the curl of $BS$, as desired.

4. PROPERTIES OF BIOT-SAVART ON SUBDOMAINS

In this section we calculate the kernel of the Biot-Savart operator and prove a result about its image. As an aid to applications, we determine for which vector fields is curl a left-inverse to $BS$. Finally, $BS$ is self-adjoint, bounded operator. We prove self-adjointness in section 4.4; our upcoming paper [15] proves boundedness.

To discuss the kernel and image, we first need to understand how vector fields decompose on subdomains of the three-sphere.

4.1. Hodge Decomposition Theorem for $\Omega \subset S^3$. In order to discuss the properties of $BS$, we require an understanding of vector fields on subdomains of $S^3$ decompose. The Hodge Decomposition Theorem provides this, in relation to the operators div, grad, and curl. See [7] for a detailed proof for vector fields in $\mathbb{R}^3$; slight modifications to that argument prove the following version on $S^3$.

Theorem 4.1 (Hodge Decomposition Theorem for $S^3$). Let $\Omega$ be a compact, three dimensional submanifold of $S^3$ with $\partial \Omega$ piecewise smooth. Then, there exists a decomposition of $VF(\Omega)$ into five mutually orthogonal subspaces,

$$VF(\Omega) = FK \oplus HK \oplus CG \oplus HG \oplus GG,$$

where,
$FK = \text{fluxless knots} = \{ \nabla \cdot V = 0, V \cdot n = 0, \text{all interior fluxes} = 0 \}$

$HK = \text{harmonic knots} = \{ \nabla \cdot V = 0, V \cdot n = 0, \nabla \times V = 0 \}$

$CG = \text{curly gradients} = \{ V = \nabla \phi, \nabla \cdot V = 0 \}$

$HG = \text{harmonic gradients} = \{ V = \nabla \phi, \text{locally constant on } \partial \Omega \}$

$GG = \text{grounded gradients} = \{ V = \nabla \phi, \phi \big|_{\partial \Omega} = 0 \}$

The subspaces $HK$ and $HG$ are finite dimensional and

\[
HK \cong H_1(\Omega, \mathbb{R}) \cong \mathbb{R}^{\text{genus } \partial \Omega} \\
HG \cong H_2(\Omega, \mathbb{R}) \cong \mathbb{R}^{|\text{components of } \partial \Omega| - |\text{components of } \Omega|}.
\]

Furthermore,

\[
\ker \text{div} = FK \oplus HK \oplus CG \oplus HG \\
\text{image curl} = FK \oplus HK \oplus CG \\
\ker \text{curl} = HK \oplus CG \oplus HG \oplus GG \\
\text{image grad} = CG \oplus HG \oplus GG.
\]

We refer to the subspace $K = FK \oplus HK$ as the fluid knots, or simply as knots; it consists of all vector fields that are both divergence-free and tangent to the boundary.

4.2. Kernel of the Biot-Savart operator. By definition, the Biot-Savart operator on $S^3$ maps the subspace of gradients to zero. No knot on $S^3$ lies in the kernel of $BS$ or else Ampere’s Law (Theorem 3.3) would fail. Hence the kernel of Biot-Savart on the three-sphere is precisely the space of gradients. Gradients on $S^3$ all behave like grounded gradients found on a compact subset $\Omega \subset S^3$. There the Hodge Decomposition Theorem for vector fields is more complicated, so we ponder, how do the other subspaces behave? What is the kernel of $BS$ on $\Omega$?

**Theorem 4.2.** Let $\Omega \subset S^3$ be as described above. The kernel of the Biot-Savart operator on $\Omega$ is precisely those gradients that are orthogonal to the boundary, i.e.,

\[
\ker BS = HG(\Omega) \oplus GG(\Omega).
\]

When discussing the kernel, we must note carefully that as an operator $BS$ maps into $VF(\Omega)$. Though we often extend $BS$ to defining a vector field on $S^3 - \Omega$, that is not its natural target space. Therefore, a vector field $V$ lies in the kernel if and only if $BS(V) = 0$ inside $\Omega$; a priori nothing is known of its behavior on $S^3 - \Omega$. In proving Theorem 4.2 we will show that if $BS(V) = 0$ throughout $\Omega$, then $BS(V)$ must vanish identically on the entire three-sphere.

In order to prove this theorem, we require a few preliminary results.

**Lemma 4.3.** Let $\Omega$ be as above, and let $\hat{n}$ be the outward pointing normal vector on $\partial \Omega$. Consider a vector field $V \in VF(\Omega)$ and let $y \in S^3$. Then,

\[
\int_{\Omega} (L_{y^2-1})_s (\nabla_x \times V(x)) + 2(L_{y^2-1})_s V(x) \, dx = -\int_{\partial \Omega} (L_{y^2-1})_s (V(x) \times \hat{n}) \, d(area_x)
\]
Proof. Let $U(x)$ be any left-invariant vector field on $S^3$. Apply the divergence theorem to $V \times U$:

$$\int_{\Omega} \nabla \cdot (V \times U) \, dx = \int_{\partial \Omega} (V \times U) \cdot \hat{n} \, d(area_x) \tag{38}$$

Now examine the right-hand side of this equation.

$$\int_{\partial \Omega} (V \times U) \cdot \hat{n} \, d(area_x) = - \int_{\partial \Omega} U(x) \cdot (V(x) \times \hat{n}) \, d(area_x)$$

$$= - \int_{\partial \Omega} (L_{yx^{-1}})_* U(x) \cdot (L_{yx^{-1}})_* (V(x) \times \hat{n}) \, d(area_x)$$

$$= - \int_{\partial \Omega} U(y) \cdot (L_{yx^{-1}})_* (V(x) \times \hat{n}) \, d(area_x)$$

$$= - U(y) \cdot \int_{\partial \Omega} (L_{yx^{-1}})_* (V(x) \times \hat{n}) \, d(area_x)$$

Now examine the left-hand side of equation (38).

$$\int_{\Omega} \nabla \cdot (V \times U) \, dx = \int_{\Omega} U(x) \cdot \nabla \times V(x) - V(x) \cdot \nabla \times U(x) \, d(area_x)$$

Since $U$ is left-invariant, $\nabla \times U = -2U$.

$$\int_{\Omega} \nabla \cdot (V \times U) \, dx = \int_{\Omega} U \cdot (\nabla \times V + 2V) \, dx$$

$$= \int_{\Omega} (L_{yx^{-1}})_* U(x) \cdot (L_{yx^{-1}})_* (\nabla \times V(x) + 2V(x)) \, dx$$

$$= \int_{\Omega} U(y) \cdot (L_{yx^{-1}})_* (\nabla \times V(x) + 2V(x)) \, dx$$

$$= U(y) \cdot \int_{\Omega} (L_{yx^{-1}})_* (\nabla \times V(x) + 2V(x)) \, dx$$

The two sides of equation (38) are equal, which implies

$$U(y) \cdot \int_{\Omega} (L_{yx^{-1}})_* (\nabla \times V + 2V) \, dx = - U(y) \cdot \int_{\partial \Omega} (L_{yx^{-1}})_* (V(x) \times \hat{n}) \, d(area_x).$$

Since this holds for any left-invariant field $U$, we may conclude that the projections of the two integrals onto the space of left-invariant vector fields must be equal. Both are integrals of left-translated fields and hence both integrals are left-invariant vector fields depending upon $y$. Therefore, these two integrals must be equal, which proves the lemma. □

We require another lemma which involves a particular energy estimate. In section 3, we defined the electrostatic field $\dot{E}_\sigma$ due to the time-independent surface charge $\sigma = V \cdot \hat{n}$ on $\partial \Omega$. (We also viewed $\dot{E}_\sigma$ as the time derivative of an electrodynamic field (8) due to a
surface charge \((V \cdot \hat{n}) t\) but do not adopt that view in this section.) For convenience, we now drop the derivative notation and hereafter refer to this field as 

\begin{equation}
E_\sigma(y) = \nabla_y \int_{\partial \Omega} \phi_0 V(x) \cdot \hat{n} \, d(area_x).
\end{equation}

Call \(\psi\) the potential function for the electrostatic field, \(E_\sigma = -\nabla \psi\).

**Lemma 4.4.** For \(V = \nabla f \in VF(\Omega)\), let \(E_\sigma\) the electrostatic field that it generates as described above. Then the energy of \(E_\sigma\) is related to its potential \(\psi\) as

\[
\int_{S^3} |E_\sigma|^2 \, dy = \int_{\partial \Omega} \psi(y) \nabla f \cdot \hat{n} \, d(area_y).
\]

**Proof.** Let \(V = \nabla f\), and let \(\sigma = \nabla f \cdot \hat{n}\), which we regard as a surface charge on \(\partial \Omega\). Then we can write

\[
E_\sigma = \nabla_y \int_{\partial \Omega} \phi_0 \sigma \, d(area_x).
\]

Now, smoothly extend the surface charge \(\sigma\) on \(\partial \Omega\) to a volume charge \(\rho\) on a thickened, compact neighborhood \(N(\partial \Omega)\) of the boundary \(\partial \Omega\). We choose \(\rho\) to be \(C^\infty\) smooth with support in \(N(\partial \Omega)\). The electrostatic field \(E_\rho\) resulting from \(\rho\) approximates \(E_\sigma\) and is expressed as

\begin{equation}
E_\rho = \nabla_y \int_{N(\partial \Omega)} \phi_0 \rho \, dx.
\end{equation}

If we regard \(E_\rho = -\nabla \psi_\rho\), then the divergence of \(E_\rho\) is \(\nabla \cdot E_\rho = -\Delta \psi_\rho\). We may extend the domain of integration of \(\psi_\rho\) to be all of \(S^3\), since \(\rho\) vanishes outside of \(N(\partial \Omega)\).

\[
\psi_\rho = -\int_{S^3} \phi_0 \rho \, dx
\]

Recall that \(\phi_0\) is the fundamental solution to the Laplacian on \(S^3\), hence \(\Delta \psi_\rho = -(\rho - \lfloor \rho \rfloor)\). Thus, the divergence \(\nabla \cdot E_\rho = \rho - \lfloor \rho \rfloor\).

We consider now the integral of \(\psi\) times \(\rho\), and again extend the domain of integration to be \(S^3\):

\[
\int_{N(\partial \Omega)} \psi \rho \, dx = \int_{S^3} \psi \rho \, dx = \int_{S^3} \psi \nabla \cdot E_\rho \, dx + \int_{S^3} \psi \lfloor \rho \rfloor \, dx
\]

\[
\int_{N(\partial \Omega)} \psi \rho \, dx = \int_{S^3} \nabla \cdot (\psi E_\rho) - \nabla \psi \cdot E_\rho \, dx + \lfloor \rho \rfloor \int_{S^3} \psi \, dx
\]

\begin{equation}
\int_{N(\partial \Omega)} \psi \rho \, dx = 0 + \int_{S^3} |E_\rho|^2 \, dx + \lfloor \rho \rfloor \lfloor \psi \rfloor
\end{equation}

Now we shrink the neighborhood \(N(\partial \Omega)\) so that it approaches the boundary \(\partial \Omega\). Then \(\rho \to \sigma\) and \(\lfloor \rho \rfloor \to 0\). Thus the LHS of (41) converges to \(\int_{\partial \Omega} \psi \sigma \, d(area_x)\) while the RHS converges to \(\int_{S^3} |E_\sigma|^2 \, dx\). Thus we obtain our desired result, as equation (41) converges to

\[
\int_{\partial \Omega} \psi \sigma \, d(area_x) = \int_{S^3} |E_\sigma|^2 \, dx.
\]
One more result, the following energy estimate, is required before beginning the proof of the kernel of \( BS \).

**Proposition 4.5.** Let \( V \) be a divergence-free vector field on \( \Omega \subset S^3 \), and let \( E_\sigma \) be its associated electrostatic field. Then,

\[
\int_{S^3} |E_\sigma|^2 \, dy \leq \int_{\Omega} |V|^2 \, dy.
\]

**Proof.** When \( V \) is divergence-free and tangent to the boundary, the electrostatic field \( E_\sigma = 0 \). So it suffices to prove the proposition for \( V \) a divergence-free gradient, i.e., \( V \in CG \oplus HG \). We write \( V = \nabla f \), which implies \( f \) is harmonic. The preceding lemma states

\[
\int_{S^3} |E_\sigma|^2 \, dy = \int_{\partial\Omega} \psi(y) \nabla f \cdot \hat{n} \\text{d}(\text{area}_y)
\]

Apply Green’s first identity:

\[
\int_{S^3} |E_\sigma|^2 \, dy = \int_{\Omega} \nabla \psi \cdot \nabla f + \psi \Delta f \, dy = \int_{\Omega} -E_\sigma \cdot \nabla f \, dy \tag{42}
\]

This is the \( L^2 \) inner product of \(-E_\sigma \) and \( \nabla f \); apply the Cauchy-Schwarz inequality:

\[
\int_{\Omega} -E_\sigma \cdot \nabla f \, dy \leq \left( \int_{\Omega} |E_\sigma|^2 \, dy \right)^{1/2} \left( \int_{\Omega} |\nabla f|^2 \, dy \right)^{1/2}
\]

\[
\leq \left( \int_{S^3} |E_\sigma|^2 \, dy \right)^{1/2} \left( \int_{\Omega} |\nabla f|^2 \, dy \right)^{1/2}
\]

Substitute this inequality into equation (42), to conclude

\[
\int_{S^3} |E_\sigma|^2 \, dy \leq \left( \int_{S^3} |E_\sigma|^2 \, dy \right)^{1/2} \left( \int_{\Omega} |\nabla f|^2 \, dy \right)^{1/2},
\]

which proves the proposition. \( \Box \)

Finally we are ready to prove Theorem 4.2 that the kernel of \( BS \) is \( HG(\Omega) \oplus GG(\Omega) \).

**Proof of Theorem 4.2.** First, we show that the subspace \( HG(\Omega) \oplus GG(\Omega) \) is contained in the kernel of \( BS \). Let \( V = \nabla f \) be in this subspace; then \( f \) is locally constant on each boundary component \( \partial\Omega_i \) and \( V \) must lie orthogonal to the boundary.

We compute \( BS(\nabla f) \) using the left-translation formula (6) for \( BS \),

\[
BS(\nabla f)(y) = \int_{\Omega} L_\ast \nabla_x f(x) \times \nabla_y \phi_0 \, dx
\]

\[
-\frac{1}{4\pi^2} \int_{\Omega} L_\ast \nabla_x f(x) \, dx + 2\nabla_y \int_{\Omega} L_\ast \nabla_x f(x) \cdot \nabla_y \phi_1 \, dx
\]

Call these three terms (i), (ii), and (iii), respectively. We compute them individually, beginning with the second term. By Lemma 4.3, term (ii) becomes

\[
(ii) = \frac{1}{8\pi^2} \int_{\Omega} L_\ast (\nabla_x \times \nabla_x f(x)) \, dx + \frac{1}{8\pi^2} \int_{\partial\Omega} L_\ast (\nabla_x f(x) \times \hat{n}) \, dx
\]
Both terms on the right vanish since \( \nabla f \) is orthogonal to the boundary and lies in the kernel of curl. Hence, the second integral (ii) is zero.

By Remark 3.10, we may convert the gradient to \( x \) variables.

\[
(i) = \int_{\Omega} (L_{yx}^{-1})_* \nabla_x f(x) \times - (L_{yx}^{-1})_* \nabla_x \phi_0 \, dx
\]

\[
(i) = - \int_{\Omega} (L_{yx}^{-1})_* (\nabla_x f(x) \times \nabla_x \phi_0) \, dx
\]

The Leibniz rule for curls implies

\[
\nabla f \times \nabla \phi_0 = \nabla \times f \nabla \phi_0 - f(\nabla \times \nabla \phi_0) = \nabla \times f \nabla \phi_0.
\]

After substituting this identity into (i), we apply Lemma 4.3:

\[
(i) = -2 \int_{\Omega} f(x) \nabla_y \phi_0 \, dx + \sum_i f_i \int_{\partial \Omega_i} L_* (\nabla_x \phi_0 \times \hat{n}_i) \, dx.
\]

Now apply Lemma 4.3 again to the summed integrals:

\[
(i) = -2 \int_{\Omega} f(x) \nabla_y \phi_0 \, dx - \sum_i f_i \int_{\Omega_i} L_* (\nabla \times \nabla \phi_0) \, dx - 2 \sum_i f_i \int_{\Omega_i} (L_{yx}^{-1})_* \nabla_x \phi_0 \, dx
\]

We may now change back to gradients in terms of \( y \) variables:

\[
(43) \quad (i) = -2 \int_{\Omega} f(x) \nabla_y \phi_0 \, dx - 0 + 2 \sum_i f_i \int_{\Omega_i} \nabla_y \phi_0 \, dx
\]

We are now ready to compute the third integral, (iii); it will cancel the contribution of (i). Our first maneuver is to change gradients as above (cf. Remark 3.10).

\[
(iii) = 2 \nabla_y \int_{\Omega} (L_{yx}^{-1})_* \nabla_x f(x) \cdot \nabla_y \phi_1 \, dx
\]

\[
(iii) = -2 \nabla_y \int_{\Omega} \nabla_x f(x) \cdot \nabla_x \phi_1 \, dx
\]

\[
(iii) = -2 \nabla_y \int_{\Omega} \nabla_x (f(x) \nabla_x \phi_1) + f \Delta \phi_1 \, dx
\]

Recall that \( \Delta \phi_1 = \phi_0 + 1/8\pi^2 \). Apply the divergence theorem to the first term,

\[
(iii) = -2 \nabla_y \int_{\partial \Omega} f(x) \nabla_x \phi_1 \cdot \hat{n} \, d(area_x) + 2 \nabla_y \int_{\Omega} f \phi_0 \, dx + 2 \nabla_y \int_{\Omega} f(x) \frac{1}{8\pi^2} \, dx
\]
The last integral returns \([f]\), the average value of \(f\) on the three-sphere, which is a constant; its gradient is zero. To analyze the first term, recall that \(f\) is a constant \(f_i\) on each boundary component \(\partial \Omega_i\). Then,

\[
(iii) = -2 \nabla_y \sum_i f_i \int_{\partial \Omega_i} \nabla_x \phi_1 \cdot \hat{n}_i \ d(\text{area}_x) + 2 \nabla_y \int_{\Omega} f \phi_0 \ dx
\]

where we have applied the divergence theorem. Now substitute for the Laplacian of \(\phi_1\):

\[
(iii) = -2 \sum_i f_i \nabla_y \int_{\Omega_i} \nabla_x \phi_0 \ dx + 2 \nabla_y \int_{\Omega} f(x) \phi_0 \ dx
\]

Examining equations (43) and (44), we see that (iii) is the negative of (i). Hence, \(BS(\nabla f) = (i) + (ii) + (iii) = 0\). Therefore, the subspace \(HG \oplus GG\) is indeed contained in the kernel of \(BS\).

Now we prove that \(\ker BS \subset HG \oplus GG\). Let \(V \in FK \oplus HK \oplus CG\) and decompose it as \(V = V_K + V_C\), where \(V_K \in FK \oplus HK\) and \(V \in CG\). Maxwell’s fourth equation (Theorem 3.3) implies that

\[
\nabla \times BS(V) = V_K + V_C - \int_{\partial \Omega} \phi_0 V_C \cdot \hat{n} \ d(\text{area})
\]

If \(V_K \neq 0\), then it cannot possibly be cancelled by the two gradients above; in this case \(\nabla \times BS(V) \neq 0\) and consequently \(BS(V) \neq 0\). So it suffices to show that no curly gradients are in the kernel of \(BS\).

Assume \(V = \nabla f\) is a curly gradient that is in the kernel of \(BS\). We will show then that \(V = 0\). Since curly gradients are divergence-free, Maxwell’s fourth equation states, for \(y \in \Omega\), that \(\nabla \times BS(V)(y) = V - E_\sigma\), where \(E_\sigma\) is the electrostatic field (39). Since we are assuming \(BS(V) = 0\) on \(\Omega\), the curl of \(BS(V)\) must also vanish; thus \(V = E_\sigma\) on \(\Omega\). By Proposition 4.5, the energy of \(V\) on \(\Omega\) is no less than the energy of \(E_\sigma\) throughout the three-sphere. Since \(E_\sigma\) equals \(V\) inside \(\Omega\), it has no available energy left on the complement \(S^3 - \Omega\), and so \(E_\sigma\) must be identically zero there. This implies that \(BS(V) = 0\) on all of the three-sphere, a fact which we could not conclude \textit{a priori}.

Since \(E_\sigma\) equals a gradient \(-\nabla \psi\), the potential function \(\psi\) must be locally constant on \(S^3 - \Omega\). In particular, \(\psi\) is constant on each boundary component, which implies that \(E_\sigma\) lies in \(HG \oplus GG(\Omega)\). But this subspace is orthogonal to the curly gradients, hence \(V\) must be trivial.

Thus we have shown that the kernel of \(BS\) cannot contain any curly gradients or knots. Thus it must be included in \(HG \oplus GG\); by the first half of the proof, the kernel is precisely that subspace. \(\square\)
Indeed, in proving Theorem 4.2, we have shown an even stronger result, namely that no curly gradient can lie in the kernel of $\nabla \times BS$. No knot lies in this kernel due to Ampere’s Law. Thus, the kernel of $\nabla \times BS$ is exactly the kernel of $BS$.

**Theorem 4.6.** The kernel of $\nabla \times BS$ is precisely $HG(\Omega) \oplus GG(\Omega)$.

### 4.3. Curl of the Biot-Savart operator.

An important property of the Biot-Savart operator is that, for certain vector fields, it acts as an inverse operator to curl. As discussed in the introduction, inverting curl is important to problems in plasma physics and fluid dynamics. Here, we precisely state when $BS$ inverts curl.

**Proposition 4.7.** (1) The equation $\nabla \times BS(V) = V$ holds on $\Omega \subset S^3$ if and only if $V$ is a divergence-free field tangent to the boundary $\partial \Omega$, i.e., $V \in FK(\Omega) \oplus HK(\Omega)$.

(2) The equation $\nabla \times BS(V) = 0$ holds on $S^3 - \Omega$ if and only if $V \in FK(\Omega) \oplus HK(\Omega) \oplus HG(\Omega) \oplus GG(\Omega)$.

**Proof.** We begin with the first statement, which is far more complicated to prove than the second. One inclusion is immediate from Theorem 3.3, which guarantees that $\nabla \times BS(V) = V$ holds on $\Omega \subset S^3$ for a fluid knot $V$.

Now for the other inclusion. Assume $\nabla \times BS(V) = V$ holds. Then $V$ lies in the image of curl, namely $FK \oplus HK \oplus CG$. The equation $\nabla \times BS(V) = V$ holds for any fluid knot, and so it suffices to show that it cannot hold for a curly gradient.

Let $V = \nabla f \in CG(\Omega)$. Theorem 3.3 implies that $\nabla \times BS(V) = V + E_\sigma$ on $\Omega$, where $E_\sigma$ is the electrostatic field (39). Suppose on $\Omega$ that $\nabla \times BS(V) = V$, which implies $E_\sigma = 0$ on $\Omega$. We show that this also implies $V$ is trivial.

Our first step is to show that $E_\sigma = 0$ outside $\Omega$. As before, express $E_\sigma = -\nabla \psi$. Since $E_\sigma = 0$ on $\Omega$ by assumption, the potential function $\psi$ is locally constant on $\Omega$; in particular $\psi$ is a constant $\psi_i$ on each boundary component $\partial \Omega_i$.

An exercise in Euclidean electrodynamics shows that any electrostatic field derived from charge on closed surface will have a jump discontinuity across the surface, but will be divergence-free on the interior and exterior of the surface. The same result follows from our definitions on the three-sphere. Applying the Leibniz rule for divergences,

$$\nabla \cdot \psi E_\sigma = \nabla \psi \cdot E_\sigma + \psi (\nabla \cdot E_\sigma) = -|E_\sigma|^2 + \psi \sigma,$$

where $\sigma = V \cdot n$ is defined on $\partial \Omega$. Now we apply the Divergence Theorem,

$$\int_{S^3 - \Omega} -|E_\sigma|^2 \, dx = \int_{S^3 - \Omega} \nabla \cdot \psi E_\sigma - \psi \sigma \, dx$$

$$= \int_{\partial \Omega} \psi (E_\sigma \cdot \hat{n} - V \cdot \hat{n}) \, d(area_x)$$

$$= \sum_i \psi_i \int_{\partial \Omega_i} (E_\sigma \cdot \hat{n}_i - V \cdot \hat{n}_i) \, d(area_x),$$
where \( \hat{n}_i \) is the normal vector on \( \partial \Omega_i \) pointing out of \( S^3 - \Omega \). Since \( V \) is a curly gradient, its flux through any boundary component is zero. Thus,

\[
\int_{S^3 - \Omega} -|E_\sigma|^2 \, dx = \sum_i \psi_i \int_{\partial \Omega_i} E_\sigma \cdot \hat{n}_i \, d(area_x)
\]

(46)

In Euclidean space, Gauss’s law states that the flux of an electrostatic field like \( E_\sigma \) over a closed surface equals the total charge enclosed by this surface. An analogous result holds on \( S^3 \):

\[
\int_{\partial \Omega_i} E_\sigma \cdot \hat{n}_i \, d(area_x) = \text{charge } Q \text{ enclosed by } \partial \Omega_i
\]

Above when we write that charge is enclosed by \( \partial \Omega_i \), we intend that the charge lies in \( S^3 - \partial \Omega_i \). The only possible charge in the region \( S^3 - \partial \Omega_i \) is a surface charge \( \sigma(x) = V(x) \cdot \hat{n}_j \) which lies on other boundary components \( \partial \Omega_j \) that lie outside \( \Omega_i \). Thus,

\[
\int_{\partial \Omega_i} E_\sigma \cdot \hat{n}_i \, d(area_x) = \sum_j (\pm 1) \int_{\partial \Omega_j} \sigma \, d(area_x) = \sum_j (\pm 1) \int_{\partial \Omega_j} V \cdot \hat{n}_j \, d(area_x)
\]

The sign on the right-hand side is determined according to whether the orientations of \( n_i \) and \( n_j \) agree. Since \( V \) is a curly gradient, it has zero flux over any boundary component; thus each integral above vanishes. By equation (46) we conclude that

\[
\int_{S^3 - \Omega} -|E_\sigma|^2 \, dx = 0,
\]

and hence \( E_\sigma = 0 \) on \( S^3 - \Omega \).

We now have that \( E_\sigma = 0 \) on the interiors of \( \Omega \) and \( S^3 - \Omega \). Now we apply a pillbox argument to a point \( x \in \partial \Omega \). Take a neighborhood of \( x \) in \( \partial \Omega \) and extend this into a “pillbox” \( P \) of small height \( \epsilon \). Then by Gauss’s law:

\[
\int_{\partial P} E_\sigma \cdot \hat{n} \, d(area_x) = \text{charge } Q_{\text{enclosed}} = \int_{P \cup \partial \Omega} \sigma \, d(area_x).
\]

But \( E_\sigma = 0 \) on \( \partial P - \partial \Omega \), so the left-hand side above is zero. Thus, for any choice of pillbox, \( \int_{P \cup \partial \Omega} \sigma \, d(area_x) = 0 \). Thus \( \sigma \) is identically zero on \( \partial \Omega \). That implies that \( V \) is tangent to the boundary, so it is orthogonal to the curly gradients \( CG \). Thus \( V = 0 \).

Thus we have shown that the only gradients for which \( \nabla \times BS(V) = V \) holds are trivial; the proof of the first statement is now complete.

The second statement is far easier to prove. For a fluid knot \( V \), Ampere’s Law guarantees that \( \nabla \times BS(V) = 0 \) holds on \( S^3 - \Omega \). The other subspaces \( HG \oplus GG \) lie in the kernel of \( \nabla \times BS \) by Theorem 4.6. Thus any \( V \in FK \oplus HK \oplus HG \oplus GG \) satisfies \( \nabla \times BS(V) = 0 \) outside \( \Omega \).

To prove the reverse implication, it suffices to show, for \( V \) a curly gradient, that \( \nabla \times BS(V) \) is nonzero outside \( \Omega \). If it is zero, then the proof of Theorem 4.2 implies that \( V = 0 \). Thus the second statement holds. \( \square \)
4.4. **Self-adjointness and image of the Biot-Savart operator.** In this section we show that the Biot-Savart operator is self-adjoint, whether it is defined on a subdomain or on all of $S^3$. As a corollary, we learn something about its image.

**Proposition 4.8.** The Biot-Savart operator is self-adjoint.

**Proof.** Let $V, W \in VF(\Omega)$. We use the parallel transport version of Biot-Savart to show

$$\langle BS(V), W \rangle = \langle V, BS(W) \rangle.$$

$$\langle BS(V)(y), W(y) \rangle = \int_{\Omega \times \Omega} P_{yx} V(x) \times \nabla_y \phi \cdot W(y) \, dx \, dy$$

$$\langle BS(V), W \rangle = -\int_{\Omega \times \Omega} W(y) \times \nabla_y \phi \cdot P_{yx} V(x) \, dx \, dy,$$

by merely rearranging vectors. Since $\nabla_y \phi = -P_{yx} \nabla_x \phi$, we have

$$\langle BS(V), W \rangle = \int_{\Omega \times \Omega} W(y) \times P_{yx} \nabla_x \phi \cdot P_{yx} V(x) \, dx \, dy$$

Now parallel translate all vectors from $y$ to $x$. For $y \neq -x$, clearly $P_{xy} P_{yx}$ is the identity map.

$$\langle BS(V), W \rangle = \int_{\Omega \times \Omega} P_{xy} W(y) \times P_{xy} P_{yx} \nabla_x \phi \cdot P_{yx} V(x) \, dx \, dy$$

$$\langle BS(V), W \rangle = \int_{\Omega} \left[ \int_{\Omega} P_{xy} W(y) \times \nabla_y \phi \, dy \right] \cdot V(x) \, dx$$

$$\langle BS(V), W \rangle = \int_{\Omega} BS(W)(x) \cdot V(x) \, dx$$

$$\langle BS(V), W \rangle = \langle BS(W)(x), V(x) \rangle \quad \square$$

**Corollary 4.9.** For $\Omega \subset S^3$, the image of Biot-Savart lies in $FK \oplus HK \oplus CG$.

For comparison, the image of $BS$ on all of $S^3$ is precisely the space of fluid knots $K(S^3)$.

**Proof.** Since $BS$ is self-adjoint, its image must be orthogonal to its kernel. In more detail, let $W$ be the component of $BS(V)$ that lies in $HG \oplus GG$, the kernel of $BS$. Then $BS(W) = 0$ and self-adjointness implies that $\langle BS(V), W \rangle = \langle V, BS(W) \rangle = 0$. But, $\langle BS(V), W \rangle = \langle W, W \rangle$. So $W$ must be trivial, and thus the image of $BS$ never has a nonzero $HG \oplus GG$ component. $\square$

5. **Acknowledgments**

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