GALAXY-GALAXY FLEXION: WEAK LENSING TO SECOND ORDER

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ABSTRACT

In this paper, we develop a new gravitational lensing inversion technique. While traditional approaches assume that the lensing field varies little across a galaxy image, we note that this variation in the field can give rise to a “flexion,” or bending of a galaxy image, which may then be used to detect a lensing signal with increased signal-to-noise ratio. Since the significance of the flexion signal increases on small scales, this is ideally suited to galaxy-galaxy lensing. We develop an inversion technique based on the “shapelets” formalism of Refregier. We then demonstrate the proof of this concept by measuring a flexion signal in the Deep Lens Survey. Assuming an intrinsically isothermal distribution, we find from the flexion signal alone a velocity width of $v_r = 221 \pm 12 \text{ km s}^{-1}$ for lens galaxies of $r < 21.5$, subject to uncertainties in the intrinsic flexion distribution.

Subject headings: galaxies: halos — galaxies: structure — gravitational lensing

1. INTRODUCTION

The past several years have seen an explosion in the analysis of weakly gravitational lensing images of galaxies by galaxies (e.g., Brainerd et al. 1996; Hoekstra et al. 2004), clusters (e.g., Smail et al. 1997; Wittman 2001; Gray et al. 2002; Taylor et al. 2004), and large-scale structure (e.g., Wittman et al. 2002; Hoekstra et al. 2002; Jarvis et al. 2003; Brown et al. 2003; Pen et al. 2003; Bacon et al. 2003), producing an unprecedented glimpse into the underlying matter distribution of the universe. Galaxy-galaxy lensing, in particular, has proven a fertile test bed of our understanding of structure formation.

Dynamical estimates of galaxy masses are subject to uncertainties about whether a system is relaxed and are limited by the lack of luminous dynamical probes beyond a few tens of kpc from the center of a galaxy. Gravitational lensing techniques, on the other hand, provide an accurate way of computing the surface density of distant objects without recourse to dynamical estimates, since the distortion of images is dependent only on the potential field of a lens and not on either its composition or its dynamics. The success of these endeavors has been such that there are a number of ongoing surveys of weak-lensing fields (e.g., the Deep Lens Survey [DLS; Wittman et al. 2002], the CFHT Legacy Survey1) and instruments (e.g., Advanced Camera for Surveys [Clampin et al. 2000], the Dark Matter Telescope, the Supernova/Acceleration Probe [Rhodes et al. 2004]) largely designed around the acquisition of a large amount of high-quality lensing data. Moreover, there remains a potential bounty of information to be found in existing and ongoing survey data (e.g., SDSS [York et al. 2000], the Great Observatories Origins Deep Survey [GOODS; Giavalisco et al. 2004], the Medium Deep Survey [Ratnatunga et al. 1999]). Given the difficulty and expense in collecting high-quality surveys of lensed galaxies, it is imperative that we extract as much information as possible from them.

One of the great advantages of studying weak-lensing fields is that the physics underlying gravitational lensing is well understood (see, e.g., Blandford & Narayan 1992; Kaiser & Squires 1993; Kaiser et al. 1995; Mellier 1999; Bartelmann & Schneider 2001 for reviews of weak lensing). The Kaiser et al. (1995) approach provides the standard for most current analyses of weak lensing and is based on estimating the ellipticity of a lensed galaxy as a probe of the local shear field. By measuring the ensemble properties of ellipticity and orientation for a number of sources, one can make a determination of the properties of the lens.

While the currently applied approaches have done an excellent job in estimating the matter distribution of gravitational lenses, they ultimately measure only the ellipticity of an image (the second moments) and thus potentially drop significant information from sources with significant substructure. In order to improve on this, we plan to extend the work of Goldberg & Natarajan (2002), who suggested that second-order effects in gravitational lensing fields may give rise to an octopole moment in the light distribution, which expresses itself as “flexion” in the image.

The method of Goldberg & Natarajan ultimately relied on a very complicated form of the second-order shear operator, which made a practical inversion difficult. Instead, here we cast this approach into a “shapelets” formalism (Refregier 2003; Refregier & Bacon 2003), a novel approach to both image and lensing analysis. Rather than analyzing image shapes according to their multipole moments, shapelets methodology decomposes images into combinations of Hermite polynomials. As a reminder to the reader, the reduced Hermite polynomials are the eigenfunctions of the simple harmonic oscillator in quantum mechanics, $B_n(x)$. These functions have a number of useful properties, including orthogonality and a Gaussian factor, which localizes the function.

In § 2 we begin by reviewing basic properties of the “forward” problem in weak gravitational lensing and define our notation within the present work. We then proceed to introduce the second-order term in the lensing operator, and then discuss a particularly useful approach to the second-order problem using the shapelets formalism. In § 3 we discuss a practical technique for inversion of the density gradient signal. In § 4 we provide a

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1 See http://www.cfht.hawaii.edu/Science/CFHTLS-DATA.
proof of concept by measuring the second-order shear signal from galaxy-galaxy lensing in several DLS shear fields. We conclude with a discussion of future prospects.

2. SECOND-ORDER LENSING

2.1. Review of Weak-Lensing Formalism

We begin with a brief review of weak-lensing theory. An excellent discussion of this material can be found in Bartelmann & Schneider (2001), from which we borrow our conventions. Imagine that we are observing a set of extended sources at an angular diameter distance $D_s$ that are gravitationally lensed by a mass distribution at an angular diameter distance $D_l$. We thus define a dimensionless surface density, the convergence, $\kappa$, such that

$$\kappa(\mathbf{x}) = \frac{D_s D_l 4\pi G\Sigma(\theta)}{D_s^2 c^2},$$

where $\Sigma$ is the surface density of the lens, $D_s$ is the distance between lens and source, and $\mathbf{x}$ represents the image coordinates as seen by the observer (neglecting a constant coordinate transform).

The convergence may be thought of as a source term for a potential, $\psi(\theta)$, and related via a Poisson-like equation:

$$\nabla^2 \psi(\mathbf{x}) = 2\kappa(\mathbf{x}),$$

where all gradients and divergences are calculated in two dimensions.

Since lensing conserves surface brightness, a mapping from foreground to background coordinates is sufficient to determine a background brightness map from a foreground one (or vice versa), provided a full knowledge of the geometry of the system (cosmology plus the redshifts of the source and lens) and mass distribution of the lens. Thus, we can expand around the origin to determine a deprojection operator on a foreground light distribution, which yields the amplification matrix,

$$A(x) \equiv \frac{\partial x'}{\partial x} = \delta_{ij} - \frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} \equiv \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix},$$

where $x'$ are the coordinates that the lensed image would have in the absence of lensing. Note that for convenience we set the origins of both the foreground and background coordinate systems to be the centers of light in their respective planes.

Equation (3) is the first term in a Taylor series expansion of the distortion operator. The term $\gamma$ is a complex shear term, representing the anisotropic part of the distortion, with $\gamma = |\gamma|e^{2i\varphi}$ and the real and imaginary parts being denoted with the subscripts “1” and “2,” respectively, as per convention.

If we assume that the convergence and shear field were constant along the scale of the lensed image, the brightness map of a galaxy would transform with the simple relation

$$x'_i = A_{ij} x_j,$$

2.2. Expansion to Second Order

Of course, the field is not a constant. Thus, we can imagine expanding the field to second order such that

$$x'_i \simeq A_{ij} x_j + \frac{1}{2} D_{ijk} x_j x_k,$$

where

$$D_{ijk} = \frac{\partial A_{ij}}{\partial x_k}.$$
However, we can simplify this problem considerably by decomposing the image into basis coefficients.

Following Refregier & Bacon (2003), we expand the image into reduced Hermite polynomials, or “shapelets.” The light distribution \( I(x, y) \) of a galaxy is expanded as a combination of two-dimensional basis functions:

\[
f(x) = \sum_{n,m} f_{nm} B_{nm}(x),
\]

where

\[
B_{nm}(x; \beta) = \beta^{-1} \phi_n(\beta^{-1} x_1) \phi_m(\beta^{-1} x_2),
\]

and where \( \beta \) is a scaling factor and \( \phi_n \) are the reduced Hermite polynomials,\(^{1}\)

\[
\phi_n(x) = \left( \frac{2^n \pi^{1/2} n!}{n} \right)^{-1/2} H_n(x) e^{-x^2/2},
\]

such that

\[
H_n'' - 2xH_n' + 2nH_n = 0.
\]

Since the lowest order polynomials resemble Gaussian light profiles, Refregier (2003) demonstrates that for typical Hubble Space Telescope (HST) images, convergence can rapidly be reached using a few \( \times 10 \) coefficients.

It should be noted that although this expansion is not the same as a multipole expansion, it has many similar properties. For example, an image with \( f_{200} = a^2 \) and \( f_{020} = b^2 < a^2 \) will look like an ellipse with a Gaussian radial profile. Likewise, any combinations for which \( n + m \) is even produces an image that is symmetric with respect to 180° rotations (see, e.g., Fig. 2 in Refregier 2003). The canonical pictures of both spiral and elliptical galaxies have precisely this symmetry, and thus it is possible that typical galaxies can be reconstructed almost exclusively from even moments.

If, taking our example from the simple harmonic oscillator in quantum mechanics, we define

\[
\hat{p}_i = \frac{\partial}{\partial x_i}, \quad \hat{x}_i = x_i,
\]

and expand equation (12) to first order in \( \gamma \), we then get

\[
f(x) \simeq \left[ 1 + (A - I) \right] \hat{x}_i \hat{p}_i + \frac{1}{2} D_{ij} \hat{x}_j \hat{p}_i \hat{p}_j \hat{x}_i f'(x).
\]

Again following Refregier & Bacon (2003), we expand our light function in shapelets and can apply the distortion operator as a combination of raising and lowering operators. Note that we have implicitly defined our coordinates such that the \( \beta \)-parameter used in Refregier (2003) is equal to 1. Recall that

\[
\hat{\xi}_i = \frac{1}{\sqrt{2}} \left( \hat{a}_i + \hat{a}^\dagger_i \right), \quad \hat{\phi}_i = \frac{1}{\sqrt{2}} \left( \hat{a}_i - \hat{a}^\dagger_i \right),
\]

and that these can operate on a shapelet (e.g., with \( i = 1 \)) as

\[
\hat{\xi}_i | \phi_{nm} \rangle = \sqrt{n} | \phi_{n-1,m} \rangle,
\]

\[
\hat{\phi}_i | \phi_{nm} \rangle = \sqrt{n+1} | \phi_{n+1,m} \rangle,
\]

and similarly for the \( y \)-axis.

Now, the first thing to note is that with the addition of the \( \hat{D} \) operator, there are odd combinations of \( \hat{\xi} \) and \( \hat{\phi} \), meaning that there can be coupling between \( \Delta n + \Delta m = \text{odd modes} \). Contrary to ordinary weak lensing, in which an image is lensed symmetrically around its center of light (and thus the centroid remains fixed), second-order effects will cause a shift in the center of light. We must thus compute the shift in the center of light.

Because of the inherent symmetries in the linear lensing operator, the only contribution to the centroid shift comes from the second-order operator:

\[
\langle x_i \rangle = \int \frac{1}{2} D_{ijk} \int d^2x \, x_k \partial_k \partial_j f\left( x' \right). \quad (22)
\]

Expanding this out and integrating, we find that the center of light in the foreground will be shifted by

\[
\langle x_1 \rangle = -\langle x_{1x}^2 \rangle \left( \frac{3}{2} D_{111} + D_{112} \right) - \langle x^2 \rangle \left( \frac{1}{2} D_{122} \right),
\]

\[
\langle x_2 \rangle = -\langle x_{1x}^2 \rangle \left( \frac{1}{2} D_{211} \right) - \langle x^2 \rangle \left( \frac{3}{2} D_{212} + D_{222} \right),
\]

where the second moments of the light distribution can be computed from the foreground field (since changes in the moments from background to foreground will necessarily be second order in \( \gamma \) and thus negligible).

A translational shift can be given by the operator:

\[
\hat{T}_i = \frac{1}{\sqrt{2}} \left( \hat{a}_i^\dagger - \hat{a}_i \right). \quad (24)
\]

Combining all of this yields the following lensing operator:

\[
f(x) \simeq \left( 1 + \kappa \hat{K} + \gamma \hat{S}_1^{(1)} + \gamma_{ij} \hat{S}_2^{(2)} \right) f(x')\quad (25),
\]

where \( \hat{K} \) is the convergence operator,

\[
\hat{K} = 1 + \frac{1}{2} \left( \hat{a}_1^2 + \hat{a}_2^2 - \hat{a}_1^2 + \hat{a}_2^2 \right),
\]

and \( \hat{S}_1^{(1)} \) is the linear shear operator defined by Refregier (2003),

\[
\hat{S}_1^{(1)} = \frac{1}{2} \left( \hat{a}_1^2 - \hat{a}_2^2 + \hat{a}_1^2 + \hat{a}_2^2 \right),
\]

\[
\hat{S}_2^{(2)} = \hat{a}_1^2 \hat{a}_2^2 - \hat{a}_1^2 \hat{a}_2. \quad (28)
\]

It is straightforward, albeit tedious, to derive the explicit form of the second-order operator, \( \hat{S}^{(2)} \), from equations (18) and (22)–(24). We write out the explicit second-order transformation as

\[
\hat{S}^{(2)}_{11} = \frac{1}{4\sqrt{2}} \left[ -2\hat{a}_1^2 + \hat{a}_1 (4 - 2 \hat{N} + 12 \langle xx \rangle) + 8 \langle xy \rangle \hat{a}_2 \right.
\]

\[
-8 \langle xy \rangle \hat{a}_2^2 + \hat{a}_1^2 \left( -6 + 2 \hat{N} - 12 \langle xx \rangle \right) + 2 \hat{a}_1^3 \],
\]

(29)
we can consider only the "odd" (meaning different, and it would be helpful to consider in which regime nearby galaxy pairs, as found in galaxy-galaxy fields—this signal will be particularly well pronounced.

It should also be noted that in general, the observed galaxy image is convolved with a point-spread function (PSF). Refregier & Bacon (2003) discuss a PSF inversion technique. We do not apply the inversion in the present work, restricting ourselves to large galaxies where the impact of the PSF is small. However, we will explicitly perform a PSF inversion in a forthcoming paper. It should further be noted that inversion of the PSF gives rise to a nondiagonal covariance matrix, and we thus use the more general form of the covariance matrix in our parameter estimation below.

3.2. Inversion of the Shapelet Coefficients

In order to compute the lensing coefficients, we need to invert equation (25). In practice, this is somewhat simpler than it might initially appear, since, for example, the convergence, \( \kappa \), cannot be uniquely estimated from a given shear field (a constant \( \kappa \)-value can be arbitrarily added to the reconstruction). This is known as the "mass-sheet degeneracy" (Kaiser & Squires 1993), and, excluding this effect, we can write down a goodness-of-fit relation as

\[
\chi^2 \equiv \left[ \mu_{n,m_1} - f_{n,m_1} + \left( \gamma \hat{S}_i^{(1)} + \gamma_i \hat{S}_j^{(2)} \right) \vec{f}_{n,m_1} \right] \times V^{-1}_{n_1,m_1,n_2,m_2} \left[ \mu_{n_2,m_2} - f_{n_2,m_2} + \left( \gamma \hat{S}_i^{(1)} + \gamma_i \hat{S}_j^{(2)} \right) \vec{f}_{n_2,m_2} \right],
\]

where the lensing operators implicitly bring power from the primed to unprimed indices. We have defined \( \mu_{n,m} \) as the "expected" source signal. For \( n + m = \text{even} \), this is simply the average of the measured signal (since the universe has no preferred direction), and for \( n + m = \text{odd} \), we set this to zero. Likewise, \( f_{n,m} \) is the best-estimate unlensed signal. This is subtly different since, although the odd terms are still set to zero, the even terms are the observed coefficients. Since lensing (especially in galaxy-galaxy fields) is expected to be small, the best estimate of an intrinsic moment is the measured moment itself.

Because we have defined the source terms for the odd moments as zero, we may break the above expression into two separate terms:

\[
\chi^2(\text{even}) = \left[ \mu_{n,m_1} + \gamma \hat{S}_i^{(1)} - 1 \right] f_{n,m_1} \times V^{-1}_{n_1,m_1,n_2,m_2} \left[ \mu_{n_2,m_2} - \left( 1 + \gamma \hat{S}_i^{(1)} \right) f_{n_2,m_2} \right],
\]

\[
\chi^2(\text{odd}) = \left[ f_{n_1,m_1} - \gamma \hat{S}_i^{(2)} f_{n_1,m_1} \right] \times V^{-1}_{n_1,m_1,n_2,m_2} \left[ f_{n_2,m_2} - \gamma_i \hat{S}_j^{(2)} f_{n_2,m_2} \right],
\]
Since the first-order term only lenses even moments to even moments, and the second-order term only lenses even moments to odd moments, the two can be computed independently. In other words, the second-order effects represent an entirely new calculation, while the first-order \( \chi^2 \) can be minimized exactly as described by Refregier & Bacon (2003).

Once the gradient of the complex shear has been estimated via minimization of \( \chi^2 \), we can relate this to the gradient of the convergence using equation (7). We refer to the bending of the observed image as the estimated “flexion”:

\[
\mathcal{F} = (\tilde{\gamma}_{1,1} + \tilde{\gamma}_{2,2})i + (\tilde{\gamma}_{2,1} - \tilde{\gamma}_{1,2})f.
\]

where \( \tilde{\gamma}_{i,j} \) is the estimated inversion of the shear derivatives from the \( \chi^2 \) minimization. Note that in principle, \( \mathcal{F} \) could be used to reconstruct a convergence field. In the present worked example, however, noise plays too much of a role.

3.3. Noise

With no atmospheric or instrumental effects, the gradient estimator can be related to the true signal via the relation

\[
\mathcal{F} = \nabla_{\mathbf{k}} \pm \sigma_{\mathcal{F},p} \pm \sigma_{\mathcal{F},s}.
\]

We have already discussed the Poisson measurement noise (\( \sigma_{\mathcal{F},p} \)) above. However, much like in conventional lensing, we must also take into account the intrinsic scatter in shapes among real galaxies.

To measure the intrinsic scatter, we observed two complementary regimes. First, we measured the shear and flexion variance in the DLS fields themselves. Second, we examined the shear and flexion in two clusters from the \textit{HST} archive, A665 and A2390, and selected elliptical and spiral galaxies. We found 75 ellipticals and 53 spirals in our cluster sample that were large and bright enough to classify by eye. No classification was done on DLS galaxies.

For these samples, we measure the distribution function of ellipticity (shear) and flexion; the results for clusters are shown in Figure 1. We define the flexion in units of the inverse of the semimajor axis of the observed galaxy. In this way, it is a dimensionless and distance-independent measure of the shape. We find that for both elliptical and spiral galaxies, the standard deviation in shear is relatively low, \( \sigma_{\gamma_e} = 0.10 \) and \( \sigma_{\gamma_s} = 0.13 \), respectively.

The scatter in the shear is somewhat lower than that which is normally measured. For example, in the COMBO-17 sample discussed in Brown et al. (2003), the standard deviation of the ellipticity is approximately 0.25, approximately twice as large, and consistent with the estimate from Brainerd et al. (1996). However, taking the same sample and using only those galaxies with \( r < 17 \), \( \sigma_e = 0.16 \). Since traditional approaches use the Kaiser et al. (1995) technique to invert the PSF for small galaxies (an intrinsically noisy inversion), additional scatter is produced in the shear. Since we are interested in the intrinsic scatter for the present work, we use our lower estimate of \( \sigma_e \). We caution, however, that even compared to the bright galaxy sample in COMBO-17, this may be a modest underestimate of the true value.

For the flexion, we find a much larger scatter among spirals than ellipticals, with a standard deviation of \( \sigma_{\mathcal{F},s} = 0.041 \) and \( \sigma_{\mathcal{F},E} = 0.012 \), respectively. This gives the expected result that early-type galaxies are significantly more regular than late type ones.

Since sources with lower resolution will be harder to classify, we would like to take a typical flexion and shear for field galaxies. Making the approximation that 60% of a randomly selected background population will be spirals (e.g., Postman & Geller 1984), we find a typical shear of \( \sigma_e = 0.12 \), and \( \sigma_F = 0.029 \). These results are similar to the result found in the well-resolved (but morphologically unclassified) sample from the DLS, yielding \( \sigma_{\gamma_{DLS}} = 0.14 \) and \( \sigma_{\mathcal{F}_{DLS}} = 0.04 \).

Using the \textit{HST} estimates, we may perform a simple estimate of the signal-to-noise ratio (S/N). For the first-order signal, we find

\[
\left( \frac{S}{N} \right)_1 = \frac{\gamma}{0.12}.
\]

and likewise, for the flexion signal,

\[
\left( \frac{S}{N} \right)_2 = \frac{\gamma}{R_{\text{Lens}}} \frac{1}{0.029/a_{\text{gal}}},
\]

since for an isothermal sphere

\[
\frac{ds}{dr} = -\frac{d\gamma}{dr} = \frac{\gamma}{R_{\text{Lens}}},
\]

since \( \gamma \propto 1/R \).

The signals from first- and second-order effects will be comparable if

\[
R_{\text{Lens}} \simeq 4.1 a_{\text{gal}}.
\]

Thus, at modest separations, the S/N from the second-order signal will be comparable to the traditional shear signal. This
ratio will rise to a factor of 10 if the background sample consists entirely of elliptical galaxies. Moreover, it provides an entirely new source of information, since it is a direct estimator of the underlying surface density of the lens.

3.4. Parameter Estimation with a Nearly Circular PSF

In practice, we do not generally directly estimate either the shear or the flexion directly from averages over the ellipticities. Even a perfectly circular PSF will alter the value, but not the orientation, of the estimated parameters. As a result, we generally wish to consider only the relative orientation of the shear or the flexion with respect to the candidate lens.

In practice, this means measuring the flexion and shear of a large number of background sources that are separated from their respective lenses by a fixed range of distances. The distribution functions of the relative orientation angles of both flexion and shear are then computed. In the absence of lensing, we would assume these distributions would be drawn from a uniform prior. However, lensing will tend to orient the shear perpendicular to the lens, and the flexion toward the lens.

We must thus relate the distribution function of relative orientation angles, $P_\phi$, to the induced shear (in first order) or flexion (in second order) of the lens. Traditional galaxy-galaxy lensing inversion techniques (Brainerd et al. 1996) yield a relation

$$P_\phi = \frac{2}{\pi} (1 - \langle \gamma \rangle \cos 2\phi_1 \langle e^{-1} \rangle)$$

over the domain $\phi_1 = [0, \pi/2]$. The normalization term $\langle e^{-1} \rangle$ is exceedingly noisy and can be unstable for small values of ellipticity. Thus, using the parametric model of Brainerd et al. (1996),

$$P(e) \propto e \exp(-Ae).$$

Using this relation, we find $\langle e^{-1} = 14.4 \rangle$ from the $\sigma_i$ discussed above. Note, again, that this is somewhat larger than the value of 8 found by Brainerd et al. (1996).

A virtually identical derivation will yield a relation for the orientation of the flexion:

$$P_\phi = \frac{1}{\pi} \left[ 1 - \langle A \frac{dF}{dr} \rangle \cos \phi_2 \langle (a\text{gal}F^{-1})^{-1} \rangle \right].$$

Assuming a similar shape to the shear and flexion distribution functions and using the ratio $\sigma_e/\sigma_F = 4.1$ found above, our estimate of the flexion deviation is $\langle (a\text{lens}F^{-1})^{-1} \rangle \simeq 59$ over $\phi_2 = [0, \pi]$.

4. PROOF OF CONCEPT: THE DEEP LENS SURVEY

As a proof of concept of this technique, we apply it to the DLS. The DLS (Wittman et al. 2002) is an ongoing deep optical survey of $7' \times 4'$ fields taken on the NOAO 4 m Blanco and Mayall telescopes. Each field will have an integrated 18 ks exposure in the $R$ band and exposures of 12 ks each in the $B, V$, and $z'$ bands. For the present test, we use only $R$-band photometry. While the DLS is designed for measurements of lensing from large-scale structure, it is also ideally suited for our purposes, since the fields are selected around otherwise empty regions in the sky and thus the primary lensing potential will come from galaxies or large-scale structure only.

We looked at all 17 of the DLS subfields from the first three public data releases, amounting to a total area of approximately 5.4 deg$^2$. We select as potential lensed sources only those galaxies that have $21.5 < r < 23$ and that have semimajor axes $A > 0.9$, since objects significantly smaller than that are not well resolved into high-order shapelets. This is a far more conservative cut on a shallower sample than that used by Brainerd et al. (1996); however, we wish to reemphasize that our goal in this work is to demonstrate the detectability of the flexion signal. Thus, taking as potential lenses those galaxies with $18 < r < 21.5$, in total we found 4833 potential pairs (similar to the number found in the Brainerd et al. 1996 sample), with separations of less than 60$''$ over 17 subfields.

We then decompose all background galaxies into shapelet coefficients as described above and estimate the best-fit flexion and shear by $\chi^2$ minimization, as in equation (37). Since seeing produces a significant reduction in the magnitude of the shear and flexion, we use only the orientation angles.

The distribution of relative orientation angles for pairs with separations of less than 16$''$ and greater than 5$''$ is shown in Figure 2, along with a best-fit curve representing a shear of $\gamma = 0.006 \pm 0.006$ and $F = 0.0027 \pm 0.0012$ arcsec$^{-1}$. We use a K-S test in all cases to determine best-fit parameters.

Since the shear and flexion should be preferentially aligned either perpendicular (shear) or toward (flexion) the source, a rotation of 45$^\circ$ for shear and 90$^\circ$ for flexion (known hereafter as the “$B$-field signal”) should produce an entire random signal. Applying these rotations to each observed source and computing the corresponding shear and flexion produces a means of normalizing the error bars for our sample and checking for consistency.

For example, if we divide a given sample into $N_{\text{bin}}$ bins of lens-source separation, then we expect that the $B$-field terms will yield a signal with $\chi^2 = N_{\text{bin}} - 1$, with the error bars in each bin being proportional to $(r_{\text{bin}})^{1/2}$, the number of galaxies within that bin. Moreover, since the error bars associated with the $B$-field should be the same as for the $E$-field, we can use this test as a means of normalizing the $E$-field error bars. We linearly

![Figure 2](image-url)
correct and extend the $B$ error bars (initially assumed to be Poisson noise) such that the reduced $\chi^2$ is 1. It should be noted that this produces approximately a 10% correction in both the flexion and shear error bars from those based on an analytic estimate of uniformly sampled orientation angles.

This appears to be assuming that there is no systematic error contribution to $B$; however, we are simply applying a useful fiction to obtain error bars for $E$ that contain the systematic error contribution. The method does not explicitly find the level of this systematic error in $B$, but accounts for the systematic and statistical errors when quoting an $E$ error.

We then determine an average radial profile for both terms. We plot the average cumulative shear (measured within a disk from separations of $5''$ to $r$) for both terms in Figure 3.

The radial profiles represent cumulative estimates of the shear and flexion, and thus the error bars are strongly coupled within a given plot. At highest significance (around $10''$ for the shear and $20''$ for the flexion), the shear and flexion result in a 1 and 2 $\sigma$ detection, respectively. The shear estimates are also consistent with estimates from other researchers, notably Brainerd et al. (1996), who find a shear of 0.0055 for separations of $R_{\text{lim}} < 20''$.

In order to fit the data to physical parameters, we select a simple isothermal sphere model for the lens and assume that all lenses are drawn from the same population. We then expect the relations (Bartelmann & Schneider 2001)

$$ |\gamma| = \frac{2\pi}{R_{\text{lim}}} \frac{v_c^2}{c^2} \frac{D_{ls}}{D_s}, $$

$$ \left| \frac{d\kappa}{dr} \right| = \frac{2\pi}{R_{\text{lim}}} \frac{v_c^2}{c^2} \frac{D_{ls}}{D_s}, $$

where $D_{ls}$ is the angular diameter distance from source to the lens. We set this ratio to be 0.5 for our discussion since lensing signals are typically maximized around this ratio (e.g., Bartelmann & Schneider 2001), and we normalize the results to this value. Note that an error in this ratio is systematic, since it effects both our model from the flexion and that from the shear identically.

We fit the isothermal sphere model in Figure 4. Note that we can fit a velocity for either the shear or flexion estimate. For the shear, we find a fit of $v_c = (107^{+24}_{-22} \text{ km s}^{-1})(D_{ls}/D_s/0.5)^{0.5}$, and for the flexion $v_c = (209^{+13}_{-12} \text{ km s}^{-1})(D_{ls}/D_s/0.5)^{0.5}$. These two fits can be combined to provide a best fit of $(201^{+11}_{-11} \text{ km s}^{-1})(D_{ls}/D_s/0.5)^{0.5}$. While here we are quoting only the random error, there may also be a systematic error based on errors in the assumed distribution function of ellipticities and flexions.

The shear estimate produces a significantly lower velocity than that found by Brainerd et al. (1996; 220 km s$^{-1}$), although the two estimates straddle the lower velocity estimate of 135 km s$^{-1}$ found by Hoekstra et al. (2004) in the Red Cluster Survey. Nevertheless, it should be noted that the shear and flexion estimates within our own sample are not statistically consistent. There are several possibilities.

First, an isothermal sphere may not be the best model. Consider a circularly symmetric power-law density field,

$$ \kappa(r) = Ar^{-\eta}, $$

where $\eta = 1$ for an isothermal sphere. The flexion for such a lens will be

$$ |F| = A\eta r^{-\eta-1}. $$

Likewise, symmetry dictates that the shear will be

$$ |\gamma| = \tilde{\kappa}(r) - \kappa(r) = \frac{\eta}{2-\eta} \kappa(r). $$
Thus,

$$\frac{F}{\gamma} = \frac{2 - \eta}{\rho}.$$  \hspace{1cm} (52)

And thus the flexion, which indicates a very high velocity dispersion compared to the shear, may in fact be indicative of an \( \eta < 1 \). In other words, such a discrepancy may be caused by a density distribution dropping off more slowly than isothermal.

Second, and more likely, recall that although the statistical significance of the flexion signal can be determined internally from the data, the normalization of the signal must be estimated from the intrinsic distribution, in the case of a K-S test. However, if we have overestimated the intrinsic variance of the flexion, then we will overestimate the flexion as well. Consider that for a fixed distribution of orientation angles, the normalizations of the curves can be related as

$$\frac{\nu_{2, \text{flexion}}}{\nu_{2, \text{shear}}} \propto \frac{\sigma_F}{\sigma_\gamma}.$$  \hspace{1cm} (53)

Thus, the two estimates can be reconciled if the ratio \( \sigma_F/\sigma_\gamma \) is reduced by a factor of \((107/221)^2 = 0.23\). If this is the sole source of the discrepancy, then the flexion may dominate the signal over shear on larger scales than originally suggested by equation (43), yielding the new relation

$$R_{\text{lens, equality}} \simeq 17.8\alpha_{\text{gal}}$$  \hspace{1cm} (54)

as the crossover scale between flexion dominance and shear dominance.

5. FUTURE PROSPECTS

We believe that we have convincingly laid out and demonstrated the feasibility of interpreting new information from weak-lensing fields by extending that analysis to second order.

However, the present work leads to a number of additional areas of investigation both theoretical and observational.

First, the present work does not incorporate inversion of the PSF. It is therefore clear that we are presently unable to extract a significant signal from galaxies that are on order the same size as the PSF. As a result, we are forced to remove many potentially lensed galaxies from our sample despite the fact that information could be potentially extracted from them. We will address this issue in future work.

Second, the present work has actually thrown away some information with regard to measurements of the shear derivatives. Since we have four derivatives and reduce that data to a 2-vector \((\mathbf{F})\), we have, in essence, thrown away information. In principle, those derivatives could yield, in addition to \( \partial s/d\rho \), a measurement of the variations of, say, the shear as a function of radius. Since for an isothermal sphere these numbers are identical (up to a minus sign), this could potentially be an important test for isothermality.

As a complementary effort, we will apply this technique to space-based data. Of particular use are the Medium Deep Survey (Ratnatunga et al. 1999) and GOODS (Giavalisco et al. 2004), which will provide the opportunity to measure more precisely the intrinsic distribution of flexions in background galaxies. Moreover, since the PSF is much smaller, we will be able to directly measure the component of the shear and flexion parallel to the lens-source displacement.

In summary, we have detected the second-order lensing effect, the “flexion.” This effect will be of great value on its own and in conjunction with first-order lensing, in order to measure the properties of galaxy halos; it affords direct, local information on the gradient of the halo density.

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