COMPUTING A SPECTRAL SEQUENCE OF FINITE HEISENBERG GROUPS OF PRIME POWER ORDER

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Abstract. Let \( p \geq 5 \) be a prime number, let \( n \geq 2 \) be a natural number and let \( \text{Heis}(p^n) \) denote the Heisenberg group modulo \( p^n \). We study the Lyndon-Hochschild-Serre spectral sequence \( E(\text{Heis}(p^n)) \) associated to \( \text{Heis}(p^n) \) considered as a split extension, and show that, \( E(\text{Heis}(p^n)) \) collapses in the third page. Moreover, for a fixed \( p \), the spectral sequences \( E(\text{Heis}(p^n)) \) are isomorphic from the second page on.

1. Introduction

Group cohomology provides a framework to analyse intrinsic algebraic properties of a given group (see [9, Section 2.1], [15] for instance) or to study automorphisms of groups (compare [11], [12] and [20]) and it also has applications in algebra and number theory (see [13] and references therein). It is also interesting to know which type of graded rings can occur as cohomology rings of finite groups and how many of them are distinct (compare [4], [7], [18]). However, computing cohomology is extremely complicated and thus, there are few examples of such rings in the literature. One of the most powerful tools in computing such rings is the Lyndon-Hochschild-Serre spectral sequence (LHSss, for short) and in this paper, we provide one of the first infinite families of groups of prime power order, whose associated LHSss collapse in the same page. More precisely, let \( p \) denote an odd prime number, let \( n \geq 1 \) be an integer and let

\[ G = \text{Heis}(p^n) = C_{p^n} \ltimes (C_{p^n} \times C_{p^n}) \]

be the Heisenberg group modulo \( p^n \). Note that \( G \) is just a finite quotient of the infinite Heisenberg group \( \hat{G} = \mathbb{Z} \ltimes (\mathbb{Z} \times \mathbb{Z}) \). Let moreover \( K \) denote a field of characteristic \( p \) with trivial \( G \)-action and let \( \text{H}^\bullet(G) = \text{H}^\bullet(G; K) \) denote the cohomology ring of \( G \) with coefficients in \( K \). We study the LHSss \( E \) associated to \( G \) as a split extension of \( C_{p^n} \) by \( C_{p^n} \times C_{p^n} \). We show that, for all prime numbers \( p \geq 5 \) and integers \( n \geq 2 \), the spectral sequence \( E \) collapses in the third page, and that for such fixed \( p \), the spectral sequences \( E \) are isomorphic from the second page on; independently of \( n \). To obtain that result, we follow Siegel’s techniques [17], where he computes the spectral sequence associated to \( \text{Heis}(p) \).

We summarise the main results and give an outline of the paper below. We start by setting the notation in Section 2, and in Section 3 we describe the additive and multiplicative structure of the second page \( E_2 \) of the spectral sequence \( E \) (see Proposition 3.1 and Theorem 3.4). In Section 4, we use maps between cohomology rings to detect some of the generators

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in $E_2$ that survive to the infinity page $E_\infty$. In Section 5, we provide a generalization of [17, Corollary 2]; being the key step to deduce the image of the second differential of the remaining generators in $E_2$ (see Theorem 5.1 and Propositions 5.3 and 5.4, respectively). We postpone the statement of Theorem 5.1 to Section 5, as it requires introducing a considerable amount of notation, and its proof can be found in Appendix A. In Sections 6 and 7, we describe the third page $E_3$ of the spectral sequence and we show that all the remaining differentials are trivial. In turn, we attain the main result of this paper.

**Theorem 1.** Let $p \geq 5$ be a prime number and let $n \geq 2$ be an integer. Then, the following statements hold:

(i) The LHS spectral sequence $E(\text{Heis}(p^n))$ collapses in the third page.

(ii) For a fixed prime number $p$, the spectral sequences $E(\text{Heis}(p^n))$ are isomorphic as bigraded $K$-algebras from the second page on.

The description of the infinity page $E_\infty(\text{Heis}(p^n))$ determines the dimension of the $K$-vector space $H^k(\text{Heis}(p^n))$, for every $k \geq 0$. Therefore, we obtain the principal result of Section 8.

**Corollary 2.** Let $p \geq 5$ be a prime number and let $n \geq 2$ be an integer. Then, the Poincaré series of $H^\bullet(\text{Heis}(p^n))$ is given as follows:

$$P(t) = \frac{1 + t^2 - t^3 + t^4 - t^5 + t^{2p+1}}{(1-t)^2(1-t^p)}.$$  

In Section 9, we consider the case where $K$ is a finite field of characteristic $p$ and we obtain the next result (see Corollary 9.1).

**Corollary 3.** Let $p \geq 5$ be a prime number and assume that $K$ is a finite field of characteristic $p$. Then, there are only finitely many isomorphism types of (graded) algebras in the infinite collection $\{H^\bullet(\text{Heis}(p^n))\}^\infty_{n \geq 1}$.

The above result is not surprising as the rank of $G$ is two (see [18]), and it also motivates us to state a conjecture (see Conjecture 1).

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### 2. Background and notation

Throughout, let $p$ denote an odd prime number, let $n \geq 1$ be an integer and let $K$ denote a field of characteristic $p$. We write $G = C_{p^n} \times (C_{p^n} \times C_{p^n})$ for the Heisenberg group modulo $p^n$ and we set $M = C_{p^n} \times C_{p^n} = \langle a, b \rangle$ and $Q = C_{p^n} = \langle \sigma \rangle$. Note that the element $\sigma \in Q$ acts (on the right) on $M$ via $a^\sigma = ab$ and $b^\sigma = b$.

The cohomology ring of $M$ with coefficients in $K$ is

$$H^\bullet(M) = \Lambda(x_1, y_1) \otimes_K K[x_2, y_2] = \Lambda(x_1, y_1) \otimes_K [x_2, y_2],$$

with $|x_i| = |y_i| = i$, for $i = 1, 2$ (see [3, Proposition 4.5.4]). We can take

$$x_1 = a^*, \quad y_1 = b^*, \quad x_2 = \beta_n(x_1), \quad y_2 = \beta_n(y_1),$$

where $\beta_n(x)$ denotes the $n$th Betti number of $x$.
where $(\cdot)^*$ denotes the dual element and $\beta_n: H^1(M) \to H^2(M)$ is the $n$-th Bockstein homomorphism [14, Section 6.2, p.197]. The (left) action of $\sigma$ on $H^\bullet(M)$ can be shown to be given by

$$\begin{align*}
\sigma \cdot x_1 &= x_1, \\
\sigma \cdot y_1 &= x_1 + y_1, \\
\sigma \cdot x_2 &= x_2, \\
\sigma \cdot y_2 &= x_2 + y_2.
\end{align*}$$

For a group $\tilde{G}$ with normal subgroup $\tilde{M}$, there exists a first quadrant spectral sequence $E_r^2(\tilde{G})$ converging to $H^\bullet(\tilde{G})$ (see [9, Section 7.2] and references therein). It is called the Lyndon-Hochschild-Serre spectral sequence (LHSss, for short), and satisfies that

$$E^{r,s}_2(\tilde{G}) = H^r(\tilde{M}; H^s(\tilde{Q})) \implies H^{r+s}(\tilde{G}),$$

with $r, s \geq 0$. In the case under study, $M$ is a normal subgroup of $G$ with quotient $Q$ and, for simplicity, we will denote by $E$ the LHSss associated to the split extension

$$1 \to M \to G \to Q \to 1.$$

3. Description of the second page of the spectral sequence

We follow the notation in the previous section and unless otherwise stated, we additionally assume until the end of the manuscript that $n \geq 2$. We use the minimal $KQ$-resolution ([2, Section 1.6]) to compute the cohomology groups $E^{r,s}_2$. Let $N(\sigma) = \sum_{i=0}^{p-1} \sigma^i \in KM$ and as $n \geq 2$, it can be readily checked that, for all $\varphi \in H^\bullet(M)$, $\sigma^p \cdot \varphi = \varphi$ and $N(\sigma) \cdot \varphi = 0$ hold. The second page of the spectral sequence then takes the following form:

$$E^{r,s}_2 = H^r(Q; H^s(M)) \cong \begin{cases} H^q(M)^Q, & \text{if } r \text{ is even,} \\ \frac{H^q(M)}{(\sigma-1) \cdot H^q(M)}, & \text{if } r \text{ is odd.} \end{cases}$$

Let now

$$z_{2p} = \prod_{i=0}^{p-1} \sigma^i \cdot y_2 = \prod_{i=0}^{p-1} (ix_2 + y_2) \in H^{2p}(M),$$

and observe that the element $z_{2p}$ is invariant under the action of $\sigma$. Furthermore, if we write

$$W = \Lambda^*[x_1, y_1] \otimes \langle x_i^j y_j^i \mid i \geq 0, \ 0 \leq j < p \rangle,$$

$$D_2^{r,*} = \begin{cases} W^Q, & \text{if } r \text{ is even,} \\ \frac{W}{(\sigma-1) \cdot W}, & \text{if } r \text{ is odd,} \end{cases}$$

we have that $H^\bullet(M) = K[z_{2p}] \otimes W$, and so

$$E^{r,*}_2 = K[z_{2p}] \otimes D_2^{r,*} = \begin{cases} K[z_{2p}] \otimes W^Q, & \text{if } r \text{ is even,} \\ K[z_{2p}] \otimes \frac{W}{(\sigma-1) \cdot W}, & \text{if } r \text{ is odd.} \end{cases}$$

Consequently, it suffices to study the structure of $D_2$ so that the structure of $E_2$ is determined.
3.1. Additive structure. The first step will be determining a basis of the $K$-vector space $D_2^{r,s}$ for each $r, s \geq 0$.

Proposition 3.1.

(i) For $s \geq 0$, the basis elements of $(W^s)^Q$ are the following:

| $2i + 1 \geq 1$ | $x_2^i, x_1 y_1 x_2^{i-1}$ |
|------------------|-------------------------------|
| $2i \geq 2$ | $x_1 x_2^i, (x_1 y_2 - y_1 x_2) x_2^{i-1}$ |
| $s$ | $(W^s)^Q$ |

(ii) For $s \geq 1$, the basis elements of $(\sigma - 1) W^s$ are the following:

| $2i + 1 \geq 3$ | $x_1 x_2^j y_2^k$, with $j \geq 1$, $0 \leq k \leq p - 2$, $j + k = i$ |
|----------------|------------------------------------------------------------------|
| $2i \geq 2$ | $x_2^j y_1^k$, with $j \geq 2$, $0 \leq k \leq p - 3$, $j + k = i$ |
| | $x_1 x_2^i x_2^{j+1} y_2^{k-1}$, with $j \geq 0$, $0 \leq k \leq p - 1$, $j + k = i$ |
| $1$ | $x_1$ |
| $s$ | $(\sigma - 1) W^s$ |

(iii) For $s \geq 0$, the basis elements of $W^s/(\sigma - 1) W^s$ are the following:

| $2i + 1 \geq 3$ | $x_1 y_1^{i-\varepsilon} y_2^{\varepsilon}$, with $\varepsilon = 0, 1$, $0 \leq k \leq p - 1$, $k = i$ |
|----------------|--------------------------------------------------------------------------------------------------|
| $2i \geq 2$ | $(x_1 y_1)^{\varepsilon} y_2^i$, with $\varepsilon = 0, 1$, $0 \leq k \leq p - 1$, $\varepsilon + k = i$ |
| | $(x_1 y_1)^{\varepsilon} x_2 y_2^{i-\varepsilon}$, with $\varepsilon = 0, 1$, $j \geq 0$, $\varepsilon + j + k + p - 1 = i$ |
| $1$ | $x_1$ |
| $0$ | $1$ |
| $s$ | $W^s/(\sigma - 1) W^s$ |

Proof. The proof follows verbatim that of [17, Proposition 3].

Using this result, we can write a table with the basis elements of $D_2^{r,s}$:
3.2. Multiplicative structure. Following [17, Section 4] (see also [9, Sections 3.2 and 7.3]) and using the diagonal approximation, we describe the multiplicative structure of $E_2$, that is, the bigraded algebra structure of $E_2$ over $K$. For $r, s, r', s' \geq 0$, let $\varphi \in H^r(M)$ and $\varphi' \in H^{r'}(M)$ represent the elements $\bar{\varphi} \in E_2^{r,s}$ and $\bar{\varphi}' \in E_2^{r',s'}$, respectively. Then, their product in $E_2$ is the element $\bar{\varphi} \bar{\varphi}' \in E_2^{r+r',s+s'}$ with

$(-1)^{r's'} \bar{\varphi} \bar{\varphi}' = \begin{cases} \varphi \circ \varphi', & \text{if } r \text{ or } r' \text{ is even,} \\ \sum_{0 \leq i < j < p^n} \sigma^i \cdot \varphi \circ \sigma^j \cdot \varphi', & \text{if } r \text{ and } r' \text{ are odd.} \end{cases}$

**Lemma 3.2.** Let $\bar{\varphi} \in E_2^{r,s}$ and $\bar{\varphi}' \in E_2^{r',s'}$ be as above with $r$ and $r'$ odd. Then, $\bar{\varphi} \bar{\varphi}' = 0$.

**Proof.** For simplicity, write

$$N_0(\sigma) = 0, \quad \text{and for } k \geq 1, \quad N_k(\sigma) = \sum_{i=0}^{k-1} \sigma^i.$$ 

In particular, we have that $N(\sigma) = N_{p^n}(\sigma)$. Furthermore, note that, for $0 \leq i \leq p-1$ and $k \geq 1$, we have that

$$\sigma^{i+kp} \cdot \varphi = \sigma^i \cdot \varphi \quad \text{and} \quad N_{i+kp}(\sigma) \cdot \varphi = N_i(\sigma) \cdot \varphi + k N_p(\sigma) \cdot \varphi.$$ 

Then, we compute

$$\sum_{0 \leq i < j < p^n} \sigma^i \cdot \varphi \circ \sigma^j \cdot \varphi' = \sum_{j=0}^{p^n-1} N_j(\sigma) \cdot \varphi \circ \sigma^j \cdot \varphi' = \sum_{j=0}^{p^n-1} \left( \sum_{k=0}^{p^{n-1}-1} N_{j+k}(\sigma) \right) \cdot \varphi \circ \sigma^j \cdot \varphi'$$

$$= \sum_{j=0}^{p^n-1} \left( \sum_{k=0}^{p^{n-1}-1} N_j(\sigma) + k N_p(\sigma) \right) \cdot \varphi \circ \sigma^j \cdot \varphi'$$

$$= \sum_{j=0}^{p^n-1} \left( p^{n-1} N_j(\sigma) + p^{n-1} (p^{n-1}-1) N_p(\sigma) \right) \cdot \varphi \circ \sigma^j \cdot \varphi' = 0$$

As a consequence, $\bar{\varphi} \bar{\varphi}' = 0$ in $E_2$.   □
In order to describe the multiplicative structure of $E_2$, we fix the following notation.

$$\lambda_1 = x_1 \in E_2^{0,1}, \quad \lambda_2 = x_2 \in E_2^{0,2},$$

$$\nu_2 = x_1 y_1 \in E_2^{0,2}, \quad \nu_3 = x_1 y_2 - y_1 x_2 \in E_2^{0,3}, \quad \nu_{2p} = z_{2p} \in E_2^{0,2p},$$

$$\gamma_1 = \overline{1} \in E_2^{1,0}, \quad \gamma_2 = \overline{1} \in E_2^{2,0},$$

for $1 \leq i \leq p$, 

$$\mu_{2i} = y_1 y_2^{i-1} \in E_2^{1,2i-1},$$

for $1 \leq i \leq p - 1$, 

$$\mu_{2i+1} = \overline{y_2} \in E_2^{1,2i}.$$

**Proposition 3.3.** Multiplication by the elements $\nu_{2p}, \gamma_2, \lambda_2$ induces vector space homomorphisms as follows:

(i) Multiplication $\cdot \nu_{2p} : E_2^{r,s} \to E_2^{r,s+2p}$ is injective for all $r, s \geq 0$.

(ii) Multiplication $\cdot \gamma_2 : E_2^{r,s} \to E_2^{r+2,s}$ is an isomorphism for all $r, s \geq 0$.

(iii) Multiplication $\cdot \lambda_2 : D_2^{r,s} \to D_2^{r,s+2}$ is an isomorphism for all $s \geq 2p - 1$.

**Proof.** The first claim follows from Equation (3). Using the identifications in Proposition 3.1, note that multiplication by $\gamma_2 = 1$ is simply the identity homomorphism and so, the second item holds. The last statement is clear by the description of the bases in Proposition 3.1 \(\square\)

Using the previous results, we can deduce the multiplicative structure of $E_2$.

**Theorem 3.4.** The structure of the second page can be described as follows:

(i) The graded commutative algebra structure of the zeroth column is given by the following tensor product:

$$E_2^{0,*} = K[\nu_{2p}] \otimes K[\lambda_1, \lambda_2, \nu_2, \nu_3]/(\nu_2^2, \lambda_1 \nu_2, \nu_2 \nu_3, \lambda_1 \nu_3 + \lambda_2 \nu_2).$$

(ii) For $r = 0, 1$ and $s \geq 0$, the basis elements of $D_2^{r,s}$ are the following:

| $2i + 1 \geq 2p + 1$ | $\lambda_1 \lambda_2^i \lambda_3^{i-1} \nu_3$ | $\lambda_2^{i-p+1} \mu_{2p}$ | $\lambda_1 \lambda_2^{i-p+1} \mu_{2p-1}$ |
|----------------------|---------------------------------|----------------------------|--------------------------------|
| $2i \geq 2p$         | $\lambda_1 \lambda_2^i \lambda_3^{i-1} \nu_2$ | $\lambda_1 \lambda_2^{i-p} \mu_{2p}$ | $\lambda_2^{i-p+1} \mu_{2p-1}$ |
| $3 \leq 2i + 1 < 2p$ | $\lambda_1 \lambda_2^i \lambda_3^{i-1} \nu_3$ | $\mu_{s+1}$, $\lambda_1 \mu_s$ |
| $3 \leq 2i < 2p$     | $\lambda_1 \lambda_2^i \lambda_3^{i-1} \nu_2$ | $\mu_{s+1}$, $\lambda_1 \mu_s$ |
| $s$                  | $D_2^{0,s}$                      | $D_2^{1,s}$                |

For $r \geq 2$ and $s \geq 0$, we have that $D_2^{r,s} = D_2^{r-2,s} \gamma_2$.

(iii) We can write $E_2 = K[\nu_{2p}] \otimes D_2$. Furthermore, $E_2$ is generated by the elements $

\lambda_1, \lambda_2, \nu_2, \nu_3, \nu_{2p}, \gamma_1, \gamma_2, \mu_2, \ldots, \mu_{2p}$.

**Proof.** The first statement can be obtained as in [17, Corollary 4 (iii)] and the remaining assertions follow from Propositions 3.1 and 3.3 \(\square\)

We encapsulate the previous result in the following table:
In \([\nu] \text{commutes with differentials, we conclude that} \) 
\[E \text{generators of } \bar{E} \text{differential on} \]
\[\text{multiplicative generators of } E \text{have that} \]
\[\nu \text{in cohomology and, by a slight abuse of notation, we also write res} \]
\[\tilde{\nu} \text{position } H \]

It is clear that Proposition 4.1.

This yields that 
\[\nu \tilde{2} \in E \]
\[\tilde{\nu} \text{in cohomology and, by a slight abuse of notation, we also write res} \]
\[\tilde{\nu} \text{position } H \]

For \(G \rightarrow \tilde{\nu} \in E \), consider the inflation homomorphism inf: \(E_2(\text{Heis}(p)) \longrightarrow E_2\). In particular, for \(\tilde{\nu} \in E_2(\text{Heis}(p)) \text{defined analogously to } \nu \) (see [17, Corollary 4], where Siegel uses \(y_2\)), we have that \(\nu_2 = \text{inf}(\tilde{\nu}_2)\). By [17, Theorem 5], \(\tilde{\nu}_2 \in E_\infty(\text{Heis}(p))\), and since the inflation map commutes with differentials, we conclude that \(\nu_2 \in E_\infty\).

**Remark 3.5.** In [17, Corollary 4], using analogous notation to ours, Siegel obtains that the multiplicative generators of \(E_2(\text{Heis}(p))\) are \(\lambda_1, \lambda_2, \nu_2, \nu_3, \nu_2, \gamma_1, \gamma_2, \mu_2, \ldots, \mu_{2p-3}\).

4. **NON-DIRECT SECOND DIFFERENTIAL COMPUTATIONS**

In this section, we use restriction, inflation and the norm maps to determine some of the generators of \(E_2\) that survive to the infinity page \(E_\infty\). We fix the following notation: the inclusion homomorphism \(M \hookrightarrow G\) induces the restriction map 

\[\text{res}_{G \rightarrow M}: H^*(G) \longrightarrow H^*(M)\]

in cohomology and, by a slight abuse of notation, we also write \(\text{res}_{G \rightarrow M}\) to denote the composition \(H^*(G) \longrightarrow H^*(M) \longrightarrow H^*(M)^Q\).

**Proposition 4.1.** The elements \(\lambda_1, \lambda_2, \nu_2, \nu_3, \nu_2, \gamma_1, \gamma_2, \mu_2, \mu_3\) survive to \(E_\infty\).

**Proof.** It is clear that \(\gamma_1, \gamma_2 \in E_\infty\). Since the extension (2) splits, the image of the second differential on \(E_2^{*,0}\) is trivial. Consequently, \(\lambda_1, \mu_2 \in E_\infty\).

For \(\lambda_2 = \beta_n(\lambda_1) \in E_2^{0,2} = H^2(M)^Q\), consider the map \(\pi: H^1(G;Z/p^nZ) \rightarrow H^1(G)\) and let \(\tilde{\lambda}_1 \in H^1(G;Z/p^nZ)\) be such that \(\pi(\tilde{\lambda}_1) = \lambda_1\). It can be readily checked that \(\lambda_1 = \text{res}_{G \rightarrow M} \circ \pi(\tilde{\lambda}_1)\) and thus,

\[\lambda_2 = (\beta_n \circ \text{res}_{G \rightarrow M} \circ \pi)(\tilde{\lambda}_1) = \text{res}_{G \rightarrow M} \circ \beta_n(\tilde{\lambda}_1)\]

This yields that \(\lambda_2 \in \text{Im}(\text{res}_{G \rightarrow M}) = E_2^{0,2}\).

For \(\nu_2\), consider the inflation homomorphism inf: \(E_2(\text{Heis}(p)) \longrightarrow E_2\). In particular, for \(\tilde{\nu}_2 \in E_2^{0,2}(\text{Heis}(p))\) defined analogously to \(\nu_2\) (see [17, Corollary 4], where Siegel uses \(y_2\)), we have that \(\nu_2 = \text{inf}(\tilde{\nu}_2)\). By [17, Theorem 5], \(\tilde{\nu}_2 \in E_\infty(\text{Heis}(p))\), and since the inflation map commutes with differentials, we conclude that \(\nu_2 \in E_\infty\).
For $\mu_3$, consider the subgroup $H = Q \times (C_{p^n} \times C_{p^n})$ of $G$. The action of $\sigma$ on
$$H^\bullet(C_{p^n} \times C_{p^n}) = \Lambda(w_1, \tilde{y}_1) \otimes K[w_2, \tilde{y}_2]$$
is trivial, and so
$$E_2(H) = H^\bullet(Q) \otimes H^\bullet(C_{p^n} \times C_{p^n}) = E_\infty(H).$$

The restriction homomorphism $\text{res}_{G \to H}: E_2 \to E_2(H)$ then sends $\mu_3 = \tilde{y}_2$ to
$$\text{res}_{G \to H}(\mu_3) = \tilde{y}_2 \gamma_1 \neq 0.$$ Furthermore, $d_2(\mu_3) \in (\mu_2 \gamma_2)$ and $\text{res}_{G \to H}(\mu_2 \gamma_2) = \tilde{y}_2 \gamma_1 \gamma_2 \neq 0$. Nevertheless, we have that $d_2(\tilde{y}_2 \gamma_1) = 0$ and, as a consequence, $d_2(\mu_3) = 0$. Hence, $\mu_3 \in E_\infty$.

Finally, we will study the generator $\nu_2$. The subgroup $L = C_{p^n} \times M$ of $G$ is normal, and so we have that $L \backslash G / M = G / LM = G / L$. Applying the properties in [9, Theorem 6.1.1] of the Evens norm map $N$, we obtain that, for any $\varphi \in H^\bullet(M)$,
$$\text{res}_{G \to M}(N_{L \to G}(\varphi)) = \prod_{g \in G / L} N_{M \to M}(\text{res}_{G \to L}(g \cdot \varphi)) = \prod_{g \in G / L} \text{res}_{L \to M}(g \cdot \varphi)$$
$$= \prod_{g \in G / L} g \cdot \text{res}_{L \to M}(\varphi).$$ Moreover, since the action of $\sigma^p$ on $H^\bullet(M) = \Lambda(\tilde{x}_1, \tilde{y}_1) \otimes K[\tilde{x}_2, \tilde{y}_2]$ is again trivial, we have that
$$E_2(L) = H^\bullet(C_{p^n}) \otimes H^\bullet(M) = E_\infty(L),$$
and we can write $y_2 = \text{res}_{L \to M}(\tilde{y}_2)$. Therefore,
$$\nu_2 = z_2 = \prod_{g \in C_p} g \cdot y_2 = \text{res}_{G \to M}(N_{L \to G}(\tilde{y}_2))$$
and we deduce that $\nu_2 \in E_\infty$. \hfill $\square$

5. Generalisation of Siegel’s result

In this section, we explicitly compute the image of the second differential on the remaining generators of $E_2$. To that aim, we employ a generalization of Siegel’s result [17, Corollary 2], which is derived from a theorem by Charlap and Vasquez [6]. To avoid technicalities in the current section, we collect most of the details and computations of the proof of Theorem 5.1 in Appendix A.

We introduce the necessary notation to state our result. Let $P_\bullet \to K$ be the minimal projective $KM$-resolution and let $V$ be a $KG$-module with trivial $M$ action. Furthermore, for each $g \in Q$, write $P^g_\bullet$ for the $KM$-complex with underlying $K$-complex $P_\bullet$ and $M$-action given as follows: for $h \in M$ and $x \in P_\bullet$, we set $h \cdot x = h^g x^{-1}$. Also, for every $i \in \mathbb{N}$, we write $\text{Hom}_{KM}(P_\bullet, P_\bullet)_i$ to denote $\prod_{k=0}^i \text{Hom}_{KM}(P_k, P_{k+i})$.

**Theorem 5.1.** Let $\alpha: P_\bullet \to P_\bullet^{s-1}$ be a $KM$-chain map commuting with the augmentation, and $\tau \in \text{Hom}_{KM}(P_\bullet, P_\bullet)_1$ such that $\partial \tau + \tau \partial = 1 - \alpha^p$. Suppose that $\zeta \in E_2^{r,s}$ with $r \geq 0, s \geq 1$ is represented by $f \in \text{Hom}_{KM}(P_s, V)$. Then, $d_2(\zeta)$ is represented by $(-1)^r f \circ \tau$.

**Proof.** See Appendix A. \hfill $\square$
5.1. Chain maps $\alpha$ and $\tau$. The problem of computing $d_2$ is reduced to finding appropriate maps $\alpha$ and $\tau$ satisfying the hypotheses in the previous theorem. We start by defining such maps.

Let $P'_* \rightarrow K$ and $P''_* \rightarrow K$ be the minimal projective resolutions of $K$ as a module over $K\langle a \rangle$ and $K\langle b \rangle$, respectively. For each $k \geq 0$, let $e'_k$ and $e''_k$ be the basis elements of $P'_k$ and $P''_k$, respectively. We can then write $P'_k = K\langle a \rangle e'_k$ and $P''_k = K\langle b \rangle e''_k$, and so $P_* = P'_* \otimes P''_* \rightarrow K$ is the minimal projective $KM$-resolution of $K$. If we set

$$
e_i^j = \begin{cases} e'_{i-j} \otimes e''_j, & \text{if } 0 \leq j \leq i, \\ 0, & \text{otherwise,} \end{cases}$$

then, for each $k \geq 0$, the elements $e'_0, \ldots, e'_k$ constitute a basis of $P_k$ as a $KM$-module. Using the duality $H^*(M) \cong H_*(M)^*$ and the fact that $H_* (M) = P_* \otimes_{KM} K$ is a quotient of $P_*$ via the canonical map $P_* \rightarrow P_* \otimes_{KM} K$, with a slight abuse of notation we can identify the elements of $H^*(M)$ as follows:

$$\text{for } i_1, i_2, j_1, j_2 \geq 0, \quad x_{11}^{i_1} y_1^{j_1} x_{2}^{j_2} = (e'_{i_1} \otimes e''_{j_2})^*.$$  

Consider the elements $\rho, \kappa \in KM$ given by

$$\rho = \sum_{0 \leq j \leq i < p^n} a^i b^j, \quad \kappa = \sum_{i=0}^{p^n-1} (i+1) a^i,$$

and define the maps $\alpha \in \text{Hom}_{KM}(P'_*, P''_*^{-1})$ and $\tau \in \text{Hom}_{KM}(P'_*, P'_*)$ as the homomorphisms that for $0 \leq j \leq i < p^n$ satisfy the following equalities:

$$\alpha(\rho^i) = \sum_{j \leq k \leq i} k \binom{k}{j} (e'_{2k} - \kappa e''_{2k+1}), \quad \tau(\rho^i) = -(j+1)\kappa e''_{2j+2},$$

$$\alpha(e''_{2j+1}) = \sum_{j \leq k \leq i} k \binom{k}{j} e'_{2k+1}, \quad \tau(e''_{2j+1}) = -(j+1)e'_{2j+3},$$

$$\alpha(e'^{2i+1}) = \sum_{j \leq k \leq i} k \binom{k}{j} (e''_{2k} + e'^{2i+1}_{2k+1}), \quad \tau(e'^{2i+1}) = -(j+1)e'^{2i+2}_{2j+2},$$

$$\alpha(e'^{2i+1}_{2j+1}) = \sum_{j \leq k \leq i} k \binom{k}{j} e'^{2i+1}_{2k+1}, \quad \tau(e'^{2i+1}_{2j+1}) = -(j+1)\kappa e'^{2i+2}_{2j+3}.$$

**Lemma 5.2.** The maps $\alpha$ and $\tau$ defined as above satisfy the equalities $\partial \alpha = \alpha \partial = 0$ and $\partial \tau - \tau \partial = 1 - a^p\alpha$.

**Proof.** See Appendix A.2. \qed

5.2. Direct second differential computations. Using Theorem 5.1 and the maps in Lemma 5.2, we can now compute the second differential of the remaining generators.

**Proposition 5.3.** The second differential of the elements $\mu_4, \ldots, \mu_{2p}$ is as follows:

(i) For $2 \leq i \leq p$, we have that

$$d_2(\mu_{2i}) = -(i - 1)\lambda_1 \mu_{2i-2} \gamma_2.$$
(ii) For $2 \leq i \leq p - 1$, we have that
\[ d_2(\mu_{2i+1}) = -i\lambda_1\mu_{2i-1}\gamma_2. \]

Proof. Consider $\mu_{2i+2} = y_1y_2 \in E_2^{1,2i+1}$ with $1 \leq i \leq p - 1$, which, by (5), is represented by the map $f: P_{2i+1} \rightarrow K$ with $f = (e_{2i+1}^2)^\ast$. We can easily compute $f \circ \tau$ to obtain that, for $0 \leq j \leq k < p^n$, we have that
\[
(f \circ \tau)(e_{2j}^{2k+1}) = 0, \quad (f \circ \tau)(e_{2j+1}^{2k+1}) = 0, \quad (f \circ \tau)(e_{2j}^{2k}) = 0, \quad (f \circ \tau)(e_{2j+1}^{2k}) = 0,
\]
\[
(f \circ \tau)(e_{2j}^{2k}) = \begin{cases} -i, & \text{if } k = i \text{ and } j = i - 1, \\ 0, & \text{otherwise}. \end{cases}
\]

Hence, $-(f \circ \tau) = i(e_{2i-1}^2)^\ast$, which represents $-i\lambda_1\mu_{2i}\gamma_2 = i\bar{x}_1y_1y_2^{-1}$. Consequently,
\[ d_2(\mu_{2i+2}) = -i\lambda_1\mu_{2i}\gamma_2. \]

Take now $\mu_{2i+1} = \bar{y}_2 \in E_2^{1,2i}$ with $2 \leq i \leq p - 1$, which is represented by the map $f: P_{2i} \rightarrow K$ with $f = (e_{2i}^2)^\ast$. We compute $f \circ \tau$ to obtain that
\[
(f \circ \tau)(e_{2j}^{2k}) = 0, \quad (f \circ \tau)(e_{2j+1}^{2k}) = 0, \quad (f \circ \tau)(e_{2j}^{2k+1}) = 0, \quad (f \circ \tau)(e_{2j+1}^{2k+1}) = 0,
\]
\[
(f \circ \tau)(e_{2j}^{2k}) = \begin{cases} -i, & \text{if } k = i - 1 \text{ and } j = i - 1, \\ 0, & \text{otherwise}. \end{cases}
\]

Hence, $-(f \circ \tau) = i(e_{2i-2}^{2-1})^\ast$, which represents $-i\lambda_1\mu_{2i-1}\gamma_2 = i\bar{x}_1y_2^{-1}$. Hence,
\[ d_2(\mu_{2i+1}) = -i\lambda_1\mu_{2i-1}\gamma_2. \]

The proof of the next result is verbatim to the previous one and we leave it to the reader.

**Proposition 5.4.** The second differential of the element $\nu_3$ is trivial.

6. **Third page of the spectral sequence**

Using the results in Sections 4 and 5.2, we can now determine the structure of the third page $E_3$. First, write $D_3 = E_3/\langle \nu_{2p} \rangle$, and define the elements
for $4 \leq i \leq 2p + 1$, \[ \omega_i = -\lambda_1\mu_{i-1} \in E_2^{1,i-1}, \]
\[ \omega_{2p+2} = \lambda_2\mu_{2p} \in E_2^{1,2p+1}, \]
\[ \xi_{2p+1} = \lambda_2\mu_{2p-1} \in E_2^{1,2p}. \]

One can easily verify that these elements have trivial second differential, and so they are in fact elements of $E_3$.

**Proposition 6.1.** Multiplication by the elements $\nu_{2p}, \gamma_2, \lambda_2$ induces vector space homomorphisms as follows:

(i) Multiplication $\cdot \nu_{2p}: E_3^{r,s} \rightarrow E_3^{r,s+2p}$ is injective for all $r, s \geq 0$. As a consequence, $E_3 = K[\nu_{2p}] \otimes D_3$.

(ii) Multiplication $\cdot \gamma_2: E_3^{r,s} \rightarrow E_3^{r+2,s}$ is surjective for all $r, s \geq 0$, and an isomorphism for all $r \neq 1$, as is $\cdot \gamma_2: D_3^{1,s} \rightarrow D_3^{3,s}$ for $s \geq 2p - 1$.

(iii) Multiplication $\cdot \lambda_2: D_3^{r,s} \rightarrow D_3^{r,s+2}$ is an isomorphism for all $s \geq 2p$. 

Proof. The proofs of (i) and (ii) are based on the proof of [17, Corollary 6].

We start with the first statement. For \( r, s \geq 0 \), let \( \varphi \in E_{2}^{r,s} \) be such that \( d_2(\varphi) = 0 \), and suppose that \( \varphi \nu_{2p} \) is a trivial element in \( E_3 \), i.e. there exists \( \psi \in E_2^{r-2,s+2p+1} \) such that \( \varphi \nu_{2p} = d_2(\psi) \). Then, since \( \langle \nu_{2p} \rangle \cap d_2(E_2 \setminus \langle \nu_{2p} \rangle) = 0 \), there exists \( \psi \in E_2^{r-2,s+1} \) such that \( \psi = \nu \nu_{2p} \). Consequently, \( \varphi \nu_{2p} = d_2(\nu)\nu_{2p} \) and, because \( \nu \nu_{2p} : E_{2}^{r,s} \to E_{2}^{r,s+2p} \) is injective (see Proposition 3.3(i)), we have that \( \varphi = d_2(\nu) \), i.e. \( \varphi = 0 \) in \( E_3 \).

For the next claim, we first show that multiplication by \( \gamma_2 \) is surjective. Take \( \varphi \in E_{2}^{r+2,s} \) with \( r, s \geq 0 \) such that \( d_2(\varphi) = 0 \). By Proposition 3.3(ii), there is some \( \psi \in E_2^{r,s} \) such that \( \varphi = \psi \gamma_2 \) in \( E_2 \). Then, we have that \( d_2(\psi)\gamma_2 = d_2(\varphi) = 0 \) and, because the product \( \cdot \gamma_2 : E_{2}^{r+2,s-1} \to E_{2}^{r+4,s-1} \) is injective, we deduce that \( d_2(\psi) = 0 \), i.e. \( \psi \) survives to \( E_3 \) and \( \varphi = \psi \gamma_2 \) in \( E_3 \).

We will now study the injectivity of the multiplication by \( \gamma_2 \). Let \( \varphi \in E_{2}^{r,s} \) with \( r \neq 1 \) or \( s \geq 2p - 1 \) such that \( d_2(\varphi) = 0 \). Suppose that there exists \( \psi \in E_{2}^{r,s+1} \) such that \( \varphi \gamma_2 = d_2(\psi) \) and we want to deduce that \( \varphi = 0 \). If \( r = 0 \), or if \( \varphi \in D_2^{r,s} \) with \( s \geq 2p - 1 \), then \( d_2(\psi) = 0 \), and by the injectivity of \( \cdot \gamma_2 : E_{2}^{r,s} \to E_{2}^{r+2,s} \) we obtain that \( \varphi = 0 \). Otherwise, if \( r \geq 2 \) we have that \( \psi = \nu \gamma_2 \) with \( \nu \in E_2^{r-2,s+1} \). Hence, \( \varphi \gamma_2 = d_2(\nu)\gamma_2 \) and, because \( \gamma_2 : E_{2}^{r,s} \to E_{2}^{r+2,s} \) is injective, we have that \( \varphi = d_2(\nu) \), i.e. \( \varphi = 0 \) in \( E_3 \).

Now, let us show that multiplication by \( \lambda_2 \) is surjective for \( s \geq 2p \). Take \( \varphi \in E_{2}^{r,s+2} \) with \( s \geq 2p \) such that \( d_2(\varphi) = 0 \). By Proposition 3.3(iii), there is some \( \psi \in E_2^{r,s} \) such that \( \varphi = \psi \lambda_2 \) in \( E_2 \). Then, we have that \( d_2(\psi)\lambda_2 = d_2(\varphi) = 0 \) and, because the product \( \cdot \lambda_2 : E_{2}^{r+2,s-1} \to E_{2}^{r+2,s+1} \) is injective, we deduce that \( d_2(\psi) = 0 \), i.e. \( \psi \) survives to \( E_3 \) and \( \varphi = \psi \lambda_2 \) in \( E_3 \).

Finally, we show that multiplication by \( \lambda_2 \) is injective for \( s \geq 2p \). Let \( \varphi \in E_{2}^{r,s} \) with \( g \geq 2p \) such that \( d_2(\varphi) = 0 \). Suppose that \( \varphi \lambda_2 = d_2(\psi) \) for some \( \psi \in E_2^{r-2,s+3} \). Then, as \( \langle \lambda_2 \rangle \cap d_2(E_2 \setminus \langle \lambda_2 \rangle) = 0 \), there exists \( \psi \in E_2^{r-2,s+1} \) such that \( \psi = \nu \lambda_2 \). Therefore, \( \varphi \lambda_2 = d_2(\nu)\lambda_2 \) and, because \( \cdot \lambda_2 : E_{2}^{r,s} \to E_{2}^{r,s+2} \) is injective, we have that \( \varphi = d_2(\nu) \), i.e. \( \varphi = 0 \) in \( E_3 \).

We can now fully determine the structure of \( E_3 \).

Theorem 6.2. The structure of the third page can be described as follows:

(i) For \( r \geq 0 \) even, we have that \( E_3^r = E_2^r \). For \( r \geq 5 \) odd, we have that \( D_3^{r,s} = D_3^{r-2,s} \cdot \gamma_2 \). For \( r = 1, 3 \), the basis elements of \( D_3^{r,s} \) are the following:
Proof. We can deduce from Theorem 3.4 that $E^3_{s+1} = \langle \omega_{s+1} \rangle$ for $3 \leq s \leq 2p - 1$. Nevertheless, we can easily compute

$$-\omega_4 = \lambda_1 \mu_3, \quad -\omega_5 = \nu_3 \mu_2, \quad \frac{3}{2} \omega_6 = \nu_3 \mu_3.$$ 

Everything else follows from Propositions 5.3 and 6.1. □

7. TO INFINITY AND BEYOND

Our objective in this section is to show that if $p \geq 5$ the spectral sequence $E$ collapses at $E_3$, i.e. $E_3 = E_\infty$. In order to achieve our goal, we will define two group automorphisms that will help us show that all the differentials starting with $d_3$ are trivial. Let $u \in \mathcal{U}(\mathbb{Z}/p^n\mathbb{Z})$ be a generator, i.e. $u^{p_i-1(p-1)} = 1$ but $u^i \neq 1$ for any $1 \leq i < p^{n-1}(p-1)$. For $0 \leq i, j, k \leq p^n - 1$, we define the group automorphisms $\Phi: G \rightarrow G$ and $\Psi: G \rightarrow G$ by

$$\Phi(\sigma^k a^i b^j) = \sigma^k a^i b^uj, \quad \text{and} \quad \Psi(\sigma^k a^i b^j) = \sigma^k a^{ui} b^{uj}.$$ 

Because $\Phi(M), \Psi(M) \leq M$, for every $m \geq 2$, there are induced automorphisms $\Phi^*: E_m \rightarrow E_m$ and $\Psi^*: E_m \rightarrow E_m$. These automorphisms act on the generators of $D_3$ by multiplying each of them by a power of $u$ as described in the following table:

| $2i + 1 \geq 2p + 1$ | $\lambda_2^{i-p+1} \omega_{2p}$ | $\lambda_2^{i-p} \omega_{2p+1}$ | $\lambda_2^{i-p+1} \omega_{2p} \gamma_2$ | $\lambda_2^{i-p} \omega_{2p+1} \gamma_2$ |
|-----------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| $2p - 1$             | $\omega_{2p}$                 | $\omega_{2p} \gamma_2$       |                               |                               |
| $6 \leq s < 2p - 2$  | $\omega_{s+1}$                | $\emptyset$                   |                               |                               |
| 5                     | $\omega_6$                    | $\emptyset$                   |                               |                               |
| 4                     | $\mu_2 \nu_3$                 | $\emptyset$                   |                               |                               |
| 3                     | $\lambda_1 \mu_3$             | $\emptyset$                   |                               |                               |
| 2                     | $\mu_3$                       | $\mu_3 \gamma_2$             |                               |                               |
| 1                     | $\mu_2$                       | $\mu_2 \gamma_2$             |                               |                               |
| 0                     | $\gamma_1$                    | $\gamma_1 \gamma_2$          |                               |                               |
| $s$                   | $D_3^{1,s}$                   | $D_3^{3,s}$                   |                               |                               |

Additionally, if $p \geq 5$ we have that $\omega_6 = 2\nu_3 \mu_3$, and so $D_3^{1,5} = \langle \nu_3 \mu_3 \rangle$. (ii) We can write $E_3 = K[\nu_2 \mu_3] \otimes D_3$. Furthermore, for $p = 3$, the third page $E_3$ is generated by the elements

$$\lambda_1, \lambda_2, \nu_2, \nu_3, \nu_6, \gamma_1, \gamma_2, \mu_2, \mu_3, \omega_6, \omega_7, \omega_8, \xi_7,$$

and for $p \geq 5$, by the elements

$$\lambda_1, \lambda_2, \nu_2, \nu_3, \nu_2 \mu_3, \gamma_1, \gamma_2, \mu_2, \mu_3, \omega_7, \omega_8, \ldots, \omega_{2p+2}, \xi_{2p+1}.$$
**Proposition 7.1.** For \( p \geq 5 \), the element \( \xi_{2p+1} \in E_3 \) survives to \( E_\infty \).

*Proof.* Assume by induction that, for \( m \geq 3 \), \( \xi_{2p+1} \in E_m \), and we will show that \( \xi_{2p+1} \in E_{m+1} \). Consider first the case \( m = 2j + 1 \) with \( j \geq 1 \). We have that

\[
d_{2j+1}(\xi_{2p+1}) = t_1 \lambda_2^{p-j} \gamma_2^{j+1} + t_2 \lambda_2^{p-j-1} \nu_2 \gamma_2^{j+1}
\]

with \( t_1, t_2 \in K \). Applying \( \Psi \), we obtain that

\[
u^p d_{2j+1}(\xi_{2p+1}) = t_1 u^{p-j} \lambda_2^{p-j} \gamma_2^{j+1} + t_2 u^{p-j+1} \lambda_2^{p-j-1} \nu_2 \gamma_2^{j+1}
\]

and, equating coefficients with those in (6), we get the conditions

\[
\begin{aligned}
  t_1(1 - u) &= 0, \\
  t_2(1 - u^{j-1}) &= 0.
\end{aligned}
\]

From these, we deduce that \( t_1 = 0 \) for all \( j \geq 1 \), and \( t_2 = 0 \) for all \( j > 1 \). If \( j = 1 \), applying \( \Phi \) to (6) we deduce that \( t_2(1 - u^{p-3}) = 0 \) and \( t_2 = 0 \) for \( p \geq 5 \). Therefore, \( \xi_{2p+1} \in E_{2j+1} \) survives to \( E_{2j+2} \).

If \( m = 2j \) with \( j \geq 2 \), the only case in which the differential might be non-trivial is \( j = p \). We have that

\[
d_{2p}(\xi_{2p+1}) = t \mu_2 \gamma_2^p
\]

with \( t \in K \). Applying \( \Phi \), we obtain that

\[
u^p d_{2p}(\xi_{2p+1}) = tu^{p+2} \mu_2 \gamma_2^p,
\]

which implies that \( t(1 - u^2) = 0 \), and so \( t = 0 \). Therefore, \( \xi_{2p+1} \in E_{2j} \) survives to \( E_{2j+1} \). \( \square \)

**Proposition 7.2.** For \( p \geq 3 \), the element \( \nu_3 \in E_3 \) survives to \( E_\infty \).

*Proof.* Observe that, for some \( t \in K \), we have \( d_3(\nu_3) = t \mu_2 \gamma_3 \in \langle \mu_2 \gamma_2 \rangle \). Applying \( \Phi \) we obtain that \( \Phi(d_3(\mu_3)) = tu^2 \mu_2 \gamma_2 \). Then, \( t(u^2 - 1) \mu_2 \gamma_2 = 0 \) implies that \( t = 0 \), as desired. \( \square \)

**Proposition 7.3.** For \( p \geq 3 \), the elements \( \omega_6, \omega_7, \ldots, \omega_{2p+2} \in E_3 \) survive to \( E_\infty \).

*Proof.* The proof for the elements \( \omega_7, \ldots, \omega_{2p+2} \in E_3 \) with any \( p \geq 3 \), and for \( \omega_6 \) with \( p = 3 \), is analogous to the proof of Proposition 7.1, and can be done following the proof of [17, Theorem 7]. For \( p \geq 5 \), it is clear that \( \omega_6 = \frac{2}{3} \nu_3 \mu_3 \) also survives to \( E_\infty \). \( \square \)

Therefore, Propositions 7.1, 7.2 and 7.3 prove Theorem 1, which we state below.

**Theorem 7.4.** Let \( n \geq 2 \) and let \( p \geq 5 \). Then, the LHSSS \( E \) associated to \( G \) collapses in the third page, i.e. \( E_3 = E_\infty \).

**Remarks 7.5.**

(i) For \( p = 3 \), following the proof of Proposition 7.1, we are only able to show that \( d_3(\xi_7) = t \lambda_2 \nu_2 \gamma_2^2 \) for some \( t \in K \). If \( t = 0 \), then \( E_3 = E_\infty \). Otherwise, the spectral sequence does not converge until at least the fourth page. This stands in contrast with [17, Theorem 5], where it is shown that \( E_2(\text{Heis}(3)) = E_\infty(\text{Heis}(3)) \).

(ii) For \( p \geq 5 \), combining our result with [17, Theorem 7], we have that \( E_3(\text{Heis}(p^n)) = E_\infty(\text{Heis}(p^n)) \) for all \( n \geq 1 \).
8. Poincaré series

In this section, we will compute the Poincaré series of $H^\bullet(G)$, i.e. the power series

$$P(t) = \sum_{k=0}^{\infty} (\dim H^k(G))t^k = \sum_{k=0}^{\infty} \sum_{r=0}^{k} (\dim E_{\infty}^{r,k-r})t^k.$$ 

Let $D_\infty = E_\infty/(\nu_{2p}) = D_3$, which is the subring of $E_\infty$ generated by all the generators except for $\nu_{2p}$. Given that $E_\infty = K[\nu_{2p}] \otimes D_\infty$, in order to obtain the Poincaré series of $E_\infty$ we only need to compute the Poincaré series of $D_\infty$ and multiply it by the Poincaré series of $K[\nu_{2p}]$.

For $k \geq 0$, write

$$D_k = \bigoplus_{r+s=k} D_{\infty}^{r,s},$$

so that

$$\dim D_k = \sum_{r=0}^{k} \dim D_{\infty}^{r,k-r}.$$ 

Then, the Poincaré series of $D_\infty$ is given by the power series $P_D(t) = \sum_{k=0}^{\infty} (\dim D_k)t^k$, and so we first need to obtain the values $\dim D_k$ for each $k \geq 0$. Note that, for every $r, s \geq 0$, the number $\dim D_{\infty}^{r,s}$ is computed in Theorem 6.2. Indeed, for $i \geq 0$, we have that

$$\dim D_{\infty}^{1,s} = \begin{cases} 1, & 0 \leq s \leq 2p-1, \\ 2, & s \geq 2p, \end{cases}$$

$$\dim D_{\infty}^{2i,s} = \begin{cases} 1, & s = 0, 1, \\ 2, & s \geq 2, \end{cases}$$

$$\dim D_{\infty}^{2i+3,s} = \begin{cases} 1, & s = 0, 1, 2, 2p-1, \\ 0, & 3 \leq s \leq 2p-2, \\ 2, & s \geq 2p. \end{cases}$$

This information can be showcased in the following table:

| $r$ | $s$ | $\dim D_{\infty}^{r,s}$ |
|-----|-----|------------------------|
| $2p + 1$ | $2$ | $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ | $2p$ | $2$ | $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ $2$ | $2p - 1$ | $2$ | $1$ $2$ $1$ $2$ $1$ $2$ $1$ $2$ | $2p - 2$ | $2$ | $1$ $2$ $0$ $2$ $0$ $2$ $0$ | $2$ | $2$ | $1$ $2$ $1$ $2$ $1$ $2$ | $1$ | $1$ | $1$ $1$ $1$ $1$ $1$ $1$ | $0$ | $1$ | $1$ $1$ $1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |

**Figure 3.** Dimension of $D_{\infty}^{r,s}$ for $0 \leq r \leq 6$ and $0 \leq s \leq 2p + 1$. 
Lemma 8.1. For \( k \geq 0 \), we have that

\[
\dim D^k_\infty = \begin{cases} 
  k + 1, & k = 0, 1, \\
  k + 2, & k = 2, 3, \\
  k + 3, & 4 \leq k \leq 2p, \\
  2k - 2p + 3, & k \geq 2p + 1.
\end{cases}
\]

Proof. The values \( \dim D^k_\infty \) for \( 0 \leq k \leq 3 \) can be easily computed from the table in Figure 3. Let \( 4 \leq k \leq 2p \) and write \( k = 2i + \varepsilon \) with \( \varepsilon = 0, 1 \). Then, we can compute

\[
\sum_{r=2}^{k-3} \dim D^{r,k-r}_\infty = 2(i - 2 + \varepsilon) = k - 4 + \varepsilon,
\]

\[
\dim D^{k-2,2}_\infty = 2 - \varepsilon.
\]

Therefore, we obtain that

\[
\dim D^k_\infty = \sum_{r=2}^{k-3} \dim D^{r,k-r}_\infty + \dim D^{k-2,2}_\infty + 5 = (k - 4 + \varepsilon) + (2 - \varepsilon) + 5 = k + 3.
\]

Let now \( k \geq 2p + 1 \) and write \( k = 2i + \varepsilon \) with \( \varepsilon = 0, 1 \). Then, we can compute the following values:

\[
\sum_{r=0}^{k-2p} \dim D^{r,k-r}_\infty = 2(k - 2p + 1) = 2k - 4p + 2,
\]

\[
\dim D^{k-2p+1,2p-1}_\infty = 1 + \varepsilon,
\]

\[
\sum_{r=k-2p+2}^{k-3} \dim D^{r,k-r}_\infty = 2(p - 2) = 2p - 4,
\]

\[
\dim D^{k-2,2}_\infty = 2 - \varepsilon.
\]

Therefore, we obtain that

\[
\dim D^k_\infty = \sum_{r=0}^{k-2p} \dim D^{r,k-r}_\infty + \dim D^{k-2p+1,2p-1}_\infty + \sum_{r=k-2p+2}^{k-3} \dim D^{r,k-r}_\infty + \dim D^{k-2,2}_\infty + 2
\]

\[
= (2k - 4p + 2) + (2p - 4) + (1 + \varepsilon) + (2 - \varepsilon) + 2
\]

\[
= 2k - 2p + 3.
\]

□

As a result, we can compute the Poincaré series of \( H^\bullet(G) \).

Theorem 8.2. The Poincaré series of \( H^\bullet(G) \) is

\[
P(t) = \frac{1 + t^2 - t^3 + t^4 - t^5 + t^{2p+1}}{(1 - t)^2(1 - t^{2p})}.
\]
Proof. Using Lemma 8.1, we can compute the Poincaré series for $D_\infty$ as follows:

$$P_D(t) = \sum_{k=0}^{\infty} (\dim D^k_{\infty})t^k$$

$$= 1 + 2t + 4t^2 + 5t^3 + \sum_{k=4}^{2p} (k + 3)t^k + \sum_{k=2p+1}^{\infty} (2k - 2p + 3)t^k$$

$$= 1 + t^2 - t^3 + t^4 - t^5 + t^{2p+1}$$

Therefore, because $E_\infty = K[\nu_{2p}] \otimes D_\infty$, we have that

$$P(t) = \frac{P_D(t)}{(1 - t^{2p})} = \frac{1 + t^2 - t^3 + t^4 - t^5 + t^{2p+1}}{(1 - t^2)(1 - t^{2p})}.$$ 

$\square$

9. Conclusion and further questions

We follow the notation introduced in Section 2. As a consequence of Theorem 7.4, we obtain that, for a prime number $p \geq 5$, the LHSs $E$ of $G$ are isomorphic from the second page on as bigraded $K$-algebras. We have not however determined the ring structure of $H^\bullet(\text{Heis}(p^n))$ and we encourage the ambitious reader to do so.

Assume now that $K$ is a finite field of characteristic $p$. Then, by [4, Theorem 2.1], there are finitely many liftings of $E_\infty(\text{Heis}(p^n))$ to the cohomology ring $H^\bullet(\text{Heis}(p^n))$. This in particular yields the following result.

**Corollary 9.1.** Let $p \geq 5$ be a prime number. Then, there are only finitely many isomorphism types of $K$-algebras in the infinite collection $\{H^\bullet(\text{Heis}(p^n))\}_{n \geq 1}$.

The above result is in slight analogy with the previously obtained results in the area [4], [7], [8], [10], [18]. Let $G(\cdot)$ denote an affine group scheme over a ring. For example, the Heisenberg group $\tilde{G}$ and the group $G$ are obtained by applying such a functor $G(\cdot)$ to $\mathbb{Z}$ and to $\mathbb{Z}/p^n\mathbb{Z}$, respectively. The presentation of the cohomology rings of such groups is intrinsically hard to obtain. For instance, in [16], Quillen described the cohomology rings of the general linear groups $GL_n(K)$ over a field $K$ of characteristic $p$ with coefficients in a finite field $F$ of characteristic coprime to $p$. However, the case where $K$ and $F$ have the same characteristic is widely open. Based on Corollary 9.1, we ask whether the following conjecture holds or not.

**Conjecture 1.** Let $p$ be a prime number and let $G(\cdot)$ be an affine group scheme over the $p$-adic integers $\mathbb{Z}_p$. Then, there exists a natural number $f = f(p, G)$ that depends only on $p$ and on $G$, such that for each $p$ and for all $n \geq f$, the cohomology rings $H^\bullet(G(\mathbb{Z}_p/p^n\mathbb{Z}_p); K)$ are isomorphic, where $K$ is a field of characteristic $p$ with trivial $G(\mathbb{Z}_p/p^n\mathbb{Z}_p)$-action.

The first reason to support the previous conjecture is that the Quillen categories of the groups $G(\mathbb{Z}_p/p^n\mathbb{Z}_p)$ are isomorphic. That is, the cohomology rings $H^\bullet(G(\mathbb{Z}_p/p^n\mathbb{Z}_p); K)$ are $F$-isomorphic (see [15]). Secondly, observe that for each $n \geq 2$, there is an extension

$$G^1(\mathbb{Z}_p/p^n\mathbb{Z}_p) \to G(\mathbb{Z}_p/p^n\mathbb{Z}_p) \to G(\mathbb{Z}_p/p\mathbb{Z}_p),$$
where $G^1(\mathbb{Z}_p/p^n\mathbb{Z}_p)$ denotes the first congruence subgroup of $G(\mathbb{Z}_p/p^n\mathbb{Z}_p)$. It is known that $G^1(\mathbb{Z}_p/p^n\mathbb{Z}_p)$ is a powerful $p$-central group with the $\Omega$-extension property and thus, for every $n \geq 2$, the cohomology rings $H^*(G^1(\mathbb{Z}_p/p^n\mathbb{Z}_p); K)$ are isomorphic ([19]). Moreover, the actions of $G(\mathbb{Z}_p/p\mathbb{Z}_p)$ on $H^*(G^1(\mathbb{Z}_p/p^n\mathbb{Z}_p); K)$ are isomorphic, in the sense of [7, Definition 5.5]. In turn, the spectral sequences $E_2(G(\mathbb{Z}_p/p^n\mathbb{Z}_p))$ are isomorphic as bigraded $K$-algebras. Therefore, based on [7, Conjecture 6.1], we would expect that the above conjecture holds by taking $f$ to be equal to 2.
In this section, we will state a theorem by Charlap and Vasquez [6] regarding the computation of the second differential of the LHss associated to a split extension of finite groups and then provide a generalization of [17, Corollary 2] for split extensions of cyclic $p$-groups.

We start by introducing the necessary definitions and notation to state the aforementioned result by Charlap and Vasquez. Let $G = Q \rtimes M$ be a split extension of $Q$ by the finite group $M$ and let $V$ be a $KG$-module with trivial $M$-action.

Let $X_\bullet \to K$ be a projective $KG$-resolution, let $Y_\bullet \to K$ be the $KQ$-bar resolution and let $P_\bullet \to K$ be the minimal $KM$-resolution. If $E = E(G)$ is the LHSS associated to the split extension of $Q$ by $M$, the following identifications hold ([9, Section 7.2]):

$$E_0 = \text{Hom}_{KQ}(Y_\bullet, \text{Hom}_{KM}(X_\bullet, V)),$$

$$E_1 = \text{Hom}_{KQ}(Y_\bullet, \text{Hom}_{KM}(P_\bullet, V)).$$

For each $g \in Q$, we write $P^g_\bullet$ for the $KM$-complex with underlying $K$-complex $P_\bullet$ and $M$-action given by

$$\text{for } h \in M \text{ and } x \in P_\bullet, \text{ set } h \cdot x = h^{g^{-1}} \cdot x.$$ 

Also, for every $i \in \mathbb{N}$, we write $\text{Hom}_{KM}(P_\bullet, P^g_\bullet)_i$ to denote $\prod_{k=0}^{i} \text{Hom}_{KM}(P_k, P^g_{k+i})$. Then, for each $g, g' \in Q$ the Comparison Theorem guarantees (see [1, Theorem 2.4.2] and subsequent remark) the existence of maps $A(g) \in \text{Hom}_{KM}(P_\bullet, P^g_\bullet)_0$ and $U(g, g') \in \text{Hom}_{KM}(P_\bullet, P^{gg'}_\bullet)_1$ satisfying the following conditions:

(i) $\partial A(g) - A(g)\partial = 0$ and $\varepsilon A(g) - \varepsilon = 0$,

(ii) $\partial U(g, g') + U(g, g')\partial = A(gg') - A(g)A(g')$.

**Theorem A.1** ([17, Theorem 1]). Let $A$ and $U$ as above. Let $r \geq 0, s \geq 1$ and suppose that $\zeta \in E_2^{r,s}$ is represented by $f \in \text{Hom}_{KM}(P_\bullet, V)$. Then $d_2(\zeta)$ is represented by $(-1)^r D_2(f)$, where

$$D_2(f)[g_1 | \cdots | g_{r+2}] = g_1 g_2 \circ f[g_3 | \cdots | g_{r+2}] \circ U(g_2^{-1}, g_1^{-1}).$$

Although the previous result is for a split extension of a general finite group $Q$, it requires the use of the $KQ$-bar resolution of $K$. In [17], the previous result has been extended for the minimal resolution of a cyclic group $Q$ of size $p$. We generalise Siegel’s result to the case where $Z_\bullet \to K$ is the minimal $KQ$-resolution with $Z_k = KQ e_k$, for $k \geq 0$, and where $Q = C_{p^n}$ is a cyclic $p$-group of size $p^n$, with $n \geq 1$.

**A.1. Proof of Theorem 5.1.** The aim of this section is to finish the proof of Theorem A.3. We follow the notation introduced in the beginning of Appendix A and additionally assume that $Z_\bullet \to K$ is the minimal $KQ$-resolution with $Z_k = KQ e_k$, for $k \geq 0$, and where $Q = C_{p^n}$ is a cyclic $p$-group of size $p^n$, with $n \geq 1$. Under those hypotheses, the first page of the LHSS described in (7) can be identified with

$$E_1 = \text{Hom}_{KQ}(Z_\bullet, \text{Hom}_{KM}(P_\bullet, V)).$$

In order to use Theorem A.1 for the above description of the spectral sequence, we first need explicit chain maps between the bar resolution $Y_\bullet$ and the minimal resolution $Z_\bullet$. For that purpose, we define the following maps:
(i) For $k \geq 1$ and $0 \leq i_1, \ldots, i_{2k+1} \leq p^n - 1$, let $\theta : Y_* \to Z_*$ be a $K$-map that satisfies the next identifications:

$$
\theta[] = e_0, \\
\theta[\sigma^{i_1}] = e_1,
$$

$$
\theta[\sigma^{i_1} \cdots |\sigma^{i_{2k}}] = \begin{cases} 
\sigma^{i_2k}, & \text{if } i_{2j-1} + i_{2j} \geq p^n \text{ for all } 1 \leq j \leq k, \\
0, & \text{otherwise},
\end{cases}
$$

$$
\theta[\sigma^{i_1} \cdots |\sigma^{i_{2k+1}}] = \begin{cases} 
\sum_{i=0}^{i_{2k}+1} \sigma^i e_{k+1} = N_{i_1}(\sigma)e_{2k+1}, & \text{if } i_{2j} + i_{2j+1} \geq p^n \text{ for all } 1 \leq j \leq k, \\
0, & \text{otherwise}.
\end{cases}
$$

(ii) For $k \geq 1$, let $\eta : Z_* \to Y_*$ be a $K$-map that satisfies the following identifications:

$$
\eta(e_0) = [], \\
\eta(e_1) = [\sigma],
$$

$$
\eta(e_{2k}) = \sum_{0 \leq i_1, \ldots, i_k < p^n} [\sigma^{i_1} |\sigma| \cdots |\sigma^{i_k} |\sigma], \\
\eta(e_{2k+1}) = \sum_{0 \leq i_1, \ldots, i_k < p^n} [\sigma|\sigma^{i_1}| \cdots |\sigma|\sigma^{i_k} |\sigma].
$$

**Lemma A.2.** The above maps $\theta$ and $\eta$ are $K$-chain maps.

Proof. We start by showing that $\theta$ is a chain map. To that aim, we need to show that for all $k \geq 1$, the following equalities hold $(\theta \theta) - (\theta \theta)(Y_{2k}) = (\theta \theta - \theta \theta)(Y_{2k+1}) = 0$. We will only show the equality for the even case, $Y_{2k}$, as the odd case follows similarly. Observe that, for every $1 \leq j \leq k - 1$ such that $i_{2j} + i_{2j+1} \geq p^n$ and $i_{2j+1} + i_{2j+2} \geq p^n$, we have that

$$(i_{2j} + i_{2j+1} \mod p^n) + i_{2j+2} = i_{2j} + i_{2j+1} - p^n + i_{2j+2} = i_{2j} + (i_{2j+1} + i_{2j+2} \mod p^n),$$

and thus

$$
\theta[\sigma^{i_1} \cdots |\sigma^{i_{2j}+i_{2j+1}} |\sigma^{i_{2j+2}} \cdots |\sigma^{i_{2k}}] = \theta[\sigma^{i_1} \cdots |\sigma^{i_{2j}} |\sigma^{i_{2j+1}+i_{2j+2}} \cdots |\sigma^{i_{2k}}].
$$

Also note that, if there is some $1 \leq l \leq k - 1$ such that $i_{2l} + i_{2l+1} < p^n$, then

$$
\theta[\sigma^{i_1} \cdots |\sigma^{i_{2l}} \cdots |\sigma^{i_{2l}+i_{2l+1}} \cdots |\sigma^{i_{2k}}] = 0,
$$

for every $2l + 2 \leq j \leq 2k - 1$. Therefore, using (9) we obtain that

$$
\theta \theta[\sigma^{i_1} \cdots |\sigma^{i_{2k}}] = \theta(\sigma^{i_1} |\sigma^{i_2} \cdots |\sigma^{i_{2k}}) + \sum_{j=1}^{2l+1} (-1)^j \theta[\sigma^{i_1} \cdots |\sigma^{i_{2j}+i_{2j+1}} \cdots |\sigma^{i_{2k}}]
$$

$$
+ \sum_{j=2l+2}^{2k-1} (-1)^j \theta[\sigma^{i_1} \cdots |\sigma^{i_{2j}+i_{2j+1}} \cdots |\sigma^{i_{2k}}] + \theta[\sigma^{i_1} \cdots |\sigma^{i_{2k}}],
$$

$$
= \theta(\sigma^{i_1} |\sigma^{i_2} \cdots |\sigma^{i_{2k}}) + \sum_{j=1}^{2l+1} (-1)^j \theta[\sigma^{i_1} \cdots |\sigma^{i_{2j}+i_{2j+1}} \cdots |\sigma^{i_{2k}}].
$$
Analogously, if there is some \(1 \leq m \leq k\) such that \(i_{2m-1} + i_{2m} < p^n\), then

\[
\theta \partial[\sigma^{i_1} \cdots |\sigma^{i_{2k}}] = \sum_{j=2m-2}^{2k-1} (-1)^j \theta[\sigma^{i_1} \cdots |\sigma^{i_{j+i_{j+1}}} \cdots |\sigma^{i_{2k}}] + \theta[\sigma^{i_1} \cdots |\sigma^{i_{2k-1}}].
\]

In order to show that equations (10) and (11) are identical, we need to distinguish four different cases:

(i) There is a smallest \(l\) with \(1 \leq l \leq k - 1\) such that \(i_{2l} + i_{2l+1} < p^n\), and a largest \(m\) with \(1 \leq m \leq k\) such that \(i_{2m-1} + i_{2m} < p^n\).

(ii) There is a largest \(m\) with \(1 \leq m \leq k\) such that \(i_{2m-1} + i_{2m} < p^n\), but \(i_{2j} + i_{2j+1} \geq p^n\) for every \(1 \leq j \leq k - 1\).

(iii) There is a smallest \(l\) with \(1 \leq l \leq k - 1\) such that \(i_{2l} + i_{2l+1} < p^n\), but \(i_{2j} + i_{2j+1} \geq p^n\) for every \(1 \leq j \leq k\).

(iv) For every \(1 \leq j \leq k\), we have that \(i_{2j_1} + i_{2j} \geq p^n\), and for every \(1 \leq j \leq k\), we have that \(i_{2j_0} + i_{2j_0+1} \geq p^n\).

We will study the first case carefully and we omit the rest of the cases as the steps to follow are identical. On the one hand, we can easily see that

\[
\partial[\sigma^{i_1} \cdots |\sigma^{i_{2k}}] = 0.
\]

On the other hand, for \(m > l + 1\), it is clear that

\[
\theta \partial[\sigma^{i_1} \cdots |\sigma^{i_{2k}}] = 0.
\]

Furthermore, the equalities in (10) and (11) yield that, for \(2 \leq m \leq l + 1\),

\[
\theta \partial[\sigma^{i_1} \cdots |\sigma^{i_{2k}}] = \sum_{j=2m-2}^{2l+1} (-1)^j \theta[\sigma^{i_1} \cdots |\sigma^{i_{j+i_{j+1}}} \cdots |\sigma^{i_{2k}}].
\]

If \(2 \leq m \leq l\), using (8), the expression (12) is reduced to

\[
\theta \partial[\sigma^{i_1} \cdots |\sigma^{i_{2k}}] = \theta[\sigma^{i_1} \cdots |\sigma^{i_{2m-2}+i_{2m-1}} \cdots |\sigma^{i_{2k}}] - \theta[\sigma^{i_1} \cdots |\sigma^{i_{2m-1}+i_{2m}} \cdots |\sigma^{i_{2k}}]
\]

\[
+ \theta[\sigma^{i_1} \cdots |\sigma^{i_{2l}+i_{2l+1}} \cdots |\sigma^{i_{2k}}] - \theta[\sigma^{i_1} \cdots |\sigma^{i_{2l+1}+i_{2l+2}} \cdots |\sigma^{i_{2k}}]
\]

\[
= 0 - N_{i_1}(\sigma)e_{2k-1} + N_{i_1}(\sigma)e_{2k-1} - 0
\]

\[
= 0.
\]

Likewise, if \(m = l + 1\) we obtain that

\[
\theta \partial[\sigma^{i_1} \cdots |\sigma^{i_{2k}}] = \theta[\sigma^{i_1} \cdots |\sigma^{i_{2m-2}+i_{2m-1}} \cdots |\sigma^{i_{2k}}] - \theta[\sigma^{i_1} \cdots |\sigma^{i_{2m-1}+i_{2m}} \cdots |\sigma^{i_{2k}}]
\]

\[
= N_{i_1}(\sigma)e_{2k-1} - N_{i_1}(\sigma)e_{2k-1}
\]

\[
= 0.
\]
Finally, if \( m = 1 \) then
\[
\theta(\sigma^{i_1} \cdots | \sigma^{i_{2k}}) = \theta(\sigma^{i_1} [\sigma^{i_2} \cdots | \sigma^{i_{2k}}]) + \sum_{j=1}^{2k+1} (-1)^j \theta(\sigma^{i_1} | \cdots | \sigma^{i_j+1} \cdots | \sigma^{i_{2k}}) ^{2k+1}
\]
\[
= \theta(\sigma^{i_1} [\sigma^{i_2} \cdots | \sigma^{i_{2k}}]) - \theta(\sigma^{i_1+i_2} \cdots | \sigma^{i_{2k}}) + \theta(\sigma^{i_1} \cdots | \sigma^{i_{2}+i_{2k+1}} \cdots | \sigma^{i_{2k}}) - \theta(\sigma^{i_1} \cdots | \sigma^{i_{2k+1}+i_{2k+2}} \cdots | \sigma^{i_{2k}})
\]
\[
= \sigma^{i_1} N_{i_2}(\sigma)e_{2k-1} - N_{i_1+i_2}(\sigma)e_{2k-1} + N_{i_1}(\sigma)e_{2k-1} - 0
\]
\[
= 0.
\]

Let us now show that \( \eta \) is a chain map. Once again, we will focus on the even case and only show that \( (\partial \eta - \eta \partial)(e_{2k}) = 0 \) for \( k \geq 1 \). On the one hand, because the initial sum covers all possible exponents \( 0 \leq i_1, \ldots, i_k < p^n \), it is easy to see that
\[
\sum_{0 \leq i_1, \ldots, i_k < p^n} \sum_{j=1}^{k-1} [\sigma^{i_1} \cdots \sigma^{i_j} | \sigma^{i_{j+1}} \sigma^{i_{k-j}}] = \sum_{0 \leq i_1, \ldots, i_k < p^n} \sum_{j=1}^{k-1} [\sigma^{i_1} \cdots \sigma^{i_j} | \sigma^{i_{j+1}} \sigma^{i_{k-j}}] \sum_{0 \leq i_1, \ldots, i_k < p^n} [\sigma^{i_1} \cdots \sigma^{i_k} | \sigma^{i_{k+1}}],
\]
and so we have that
\[
\partial \eta(e_{2k}) = \partial \left( \sum_{0 \leq i_1, \ldots, i_k < p^n} [\sigma^{i_1} | \sigma^{i_2} | \cdots | \sigma^{i_k} | \sigma] \right)
\]
\[
= \sum_{0 \leq i_1, \ldots, i_k < p^n} \left( [\sigma^{i_1} | \sigma^{i_2} | \cdots | \sigma^{i_k} | \sigma] - \sum_{j=1}^{k} [\sigma^{i_1} | \cdots | \sigma^{i_{j-1}} | \sigma^{i_j} | \sigma^{i_{j+1}} | \sigma | \cdots | \sigma^{i_k} | \sigma] + \sum_{j=1}^{k} [\sigma^{i_1} | \cdots | \sigma^{i_{j-1}} | \sigma^{i_j} | \sigma^{i_{j+1}} | \sigma | \cdots | \sigma^{i_k} | \sigma] \right)
\]
\[
= \sum_{0 \leq i_1, \ldots, i_k < p^n} \sigma^{i_1} | \sigma^{i_2} | \cdots | \sigma^{i_k} | \sigma.
\]

On the other hand,
\[
\eta \partial(e_{2k}) = \eta \left( \sum_{i=0}^{p^n-1} \sigma^i e_{2k-1} \right)
\]
\[
= \sum_{0 \leq i_1, \ldots, i_k < p^n} \sigma^i | \sigma^{i_1} | \cdots | \sigma^{i_{k-1}} | \sigma.
\]

Therefore, \( (\partial \eta - \eta \partial)(e_{2k}) = 0 \).

We will now state and prove Theorem 5.1.

**Theorem A.3.** Let \( \alpha: P_\bullet \rightarrow P_\bullet^{s-1} \) be a KM-chain map commuting with the augmentation, and \( \tau \in \text{Hom}_{KM}(P_\bullet, P_\bullet)^1 \) such that \( \partial \tau + \tau \partial = 1 - \alpha p^n \). Suppose that \( \zeta \in E_{2}^{r,s} \) with \( r \geq 0, s \geq 1 \) is represented by \( f \in \text{Hom}_{KM}(P_\bullet, V) \). Then \( d_2(\zeta) \) is represented by \( (-1)^{s} f \circ \tau \).

**Proof.** The proof of this result can be done by following that of [17, Corollary 2], using the chain maps from Lemma A.2 and writing \( p^n \) instead of \( p \) where appropriate. \( \square \)
A.2. Proof of Lemma 5.2. In this section, we will give the explicit computations required in the proof of Lemma 5.2. To that aim, we display the equalities that will be used during our computations while the proof of such properties is left for the reader.

Lemma A.4. Let $a, b$ denote the generators of $M$ and let $e_j^i$ be as in (4).

(i) The following identities hold:

\[
\begin{align*}
\rho(b - 1) &= bN(ab) - N(a), \\
\rho(a - 1) &= N(b) - N(ab), \\
\rho(ab - 1) &= N(b) - N(a), \\
\kappa(a - 1) &= -N(a).
\end{align*}
\]

(ii) The differential of the elements $e_j^i$ is as follows:

\[
\begin{align*}
\partial(e_{2j}^{2i}) &= N(a)e_{2j-1}^{2i-1} + N(b)e_{2j-1}^{2i-1}, \\
\partial(e_{2j+1}^{2i}) &= (a - 1)e_{2j+1}^{2i-1} - (b - 1)e_{2j}^{2i-1}, \\
\partial(e_{2j+1}^{2i+1}) &= N(a)e_{2j+1}^{2i} + (b - 1)e_{2j}^{2i}.
\end{align*}
\]

Proposition A.5. The map $\alpha$ is a chain map, i.e. $\partial \alpha - \alpha \partial = 0$.

Proof. We will only check that $(\partial \alpha - \alpha \partial)(e_{2j}^{2i}) = 0$ as the other cases follow similarly. We will use Lemma A.4 during the computations. On the one hand,

\[
\begin{align*}
\partial \alpha(e_{2j}^{2i}) &= \sum_{j \leq k \leq i} \binom{k}{j} \partial(e_{2k}^{2i} - \rho e_{2k+1}^{2i}) \\
&= \sum_{j \leq k \leq i} \binom{k}{j} \left( (N(a) + \rho(b - 1))e_{2k-1}^{2i-1} + N(b)e_{2k-1}^{2i-1} - \rho(a - 1)e_{2k+1}^{2i-1} \right) \\
&= \sum_{j \leq k \leq i} \binom{k}{j} \left( bN(ab)e_{2k}^{2i-1} + N(b)e_{2k-1}^{2i-1} + (N(ab) - N(b))e_{2k+1}^{2i-1} \right) \\
&= \sum_{j \leq k \leq i} \binom{k}{j} \left( bN(ab)e_{2k}^{2i-1} + N(ab)e_{2k+1}^{2i-1} \right) + \sum_{j \leq k+1 \leq i} \binom{k+1}{j} N(b)e_{2k+1}^{2i-1} \\
&= \sum_{j \leq k \leq i} \binom{k}{j} \left( bN(ab)e_{2k}^{2i-1} + N(ab)e_{2k+1}^{2i-1} \right) + \sum_{j \leq k+1 \leq i} \binom{k}{j-1} N(b)e_{2k+1}^{2i-1}.
\end{align*}
\]

On the other hand,

\[
\alpha \partial(e_{2j}^{2i}) = \alpha \left( N(a)e_{2j-1}^{2i-1} + N(b)e_{2j-1}^{2i-1} \right)
\]

\[
= \sum_{j \leq k \leq i} \binom{k}{j} N(ab)(be_{2k}^{2i-1} + e_{2k+1}^{2i-1}) + \sum_{j \leq k \leq i} \binom{k}{j-1} N(b)e_{2k+1}^{2i-1}
\]

\[
= \sum_{j \leq k \leq i} \binom{k}{j} N(ab)(be_{2k}^{2i-1} + e_{2k+1}^{2i-1}) + \sum_{j \leq k+1 \leq i} \binom{k}{j-1} N(b)e_{2k+1}^{2i-1}.
\]

Therefore, $(\partial \alpha - \alpha \partial)(e_{2j}^{2i}) = 0$. \qed

We are left to prove that the identity $\partial \tau + \tau \partial = 1 - \alpha p^n$ holds. In order to do that, we first show the identities that will be used throughout the proof.

Lemma A.6.

(i) We have that $\sum_{r=0}^{n-1} \rho^r b^r = \kappa N(b)$. 


(ii) For any \(i, j \geq 0\) and \(m \geq 1\), we have that
\[
\sum_{j \leq k \leq l \leq i} m^{k-j} \binom{l}{k} \binom{k}{j} = \sum_{j \leq l \leq i} (m+1)^{l-j} \binom{l}{j}.
\]

**Proposition A.7.** The maps \(\alpha\) and \(\tau\) satisfy the identity \(\partial \tau + \tau \partial = 1 - \alpha^{p^n}\).

**Proof.** We will only show that \((\partial \tau + \tau \partial)(e^{2i}_{2j}) = (1 - \alpha^{p^n})(e^{2i}_{2j})\) since the other cases can be done in a similar way. First, we compute \((\partial \tau + \tau \partial)(e^{2i}_{2j})\) using Lemma A.4. On the one hand,
\[
\partial \tau(e^{2i}_{2j}) = \partial( - (j+1)\kappa e^{2i+1}_{2j+2}) = -(j+1)\kappa(a-1)e^{2i}_{2j+2} + (j+1)\kappa N(b)e^{2i+1}_{2j+1}
= (j+1)N(a)e^{2i}_{2j+2} + (j+1)\kappa N(b)e^{2i+1}_{2j+1}.
\]

On the other hand,
\[
\tau \partial(e^{2i}_{2j}) = \tau(N(a)e^{2i-1}_{2j} + N(b)e^{2i-1}_{2j-1}) = -(j+1)N(a)e^{2i}_{2j+2} - jN(b)\kappa e^{2i}_{2j+1}.
\]

As a consequence,
\[(\partial \tau + \tau \partial)(e^{2i}_{2j}) = \kappa N(b)e^{2i}_{2j+1}.
\]

Now, we compute \((\partial \tau + \tau \partial)(e^{2i}_{2j})\). Applying \(\alpha\) repeatedly to \(e^{2i}_{2j}\) and using Lemma A.6, we obtain that
\[\alpha^m(e^{2i}_{2j}) = \sum_{j \leq k \leq i} m^{k-j} \binom{k}{j} \left( e^{2i}_{2k} - \sum_{r=0}^{m-1} \rho^r b^r e^{2i}_{2k+1} \right)\]
for any \(1 \leq m \leq p^n\). Therefore,
\[\alpha^{p^n}(e^{2i}_{2j}) = \sum_{j \leq k \leq i} p^{n(k-j)} \binom{k}{j} \left( e^{2i}_{2k} - \kappa N(b)e^{2i}_{2k+1} \right) = e^{2i}_{2j} - \kappa N(b)e^{2i}_{2j+1},\]
and thus \((\partial \tau + \tau \partial)(e^{2i}_{2j}) = (1 - \alpha^{p^n})(e^{2i}_{2j})\). \(\square\)

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