F-RATIONALITY OF REES ALGEBRAS

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To Prof. Craig Huneke

1. RESULTS

Let $R$ be a noetherian ring and $I$ an $R$-ideal. The Rees algebra of $R$ with respect to $I$ is $R(I) := \oplus_{n \geq 0} I^n$; the extended Rees algebra of $R$ with respect to $I$ is $R'(I) := \oplus_{n \in \mathbb{Z}} I^n$, where, for $n \leq 0$ $I^n := R$.

Several authors have studied the singularities of Rees algebras: E. Hyry [Hyr99] for rational singularities in characteristic zero; A. K. Singh [Sin00] in prime characteristic for strong $F$-regularity and $F$-purity; and N. Hara, K.-i. Watanabe and K.-i. Yoshida [HWY02a] and [HWY02b] in prime characteristic for $F$-rationality and $F$-regularity respectively.

In this article, we study the $F$-rationality of $R(I)$ and $R'(I)$. Our primary aim is to understand some questions of N. Hara, K.-i. Watanabe and K.-i. Yoshida [HWY02a, Section 1], which ask for necessary and sufficient conditions for Rees algebras to be $F$-rational. We list our results below, postponing definitions to Section 2.

Theorem 1.1. Let $(R, \mathfrak{m})$ be an excellent local domain of prime characteristic and $I$ an $\mathfrak{m}$-primary $R$-ideal. Then the following are equivalent:

(a) $R'(I)$ is $F$-rational;
(b) $R$ and $R(I)$ are $F$-rational.

This settles [HWY02a, Conjecture 4.1] which asserted the conclusion of the above theorem. That the $F$-rationality of $R(I)$ implies the $F$-rationality of $R'(I)$ is [HWY02a, Theorem 4.2], but we give a different proof, which follows directly from some observations on the tight closure of zero in the local cohomology modules of $R(I)$ and of $R'(I)$ and on the $F$-rationality of $\text{Proj } R(I)$ that we discuss in Section 3. As applications of the results of Section 3, we get a sufficient (but not necessary) condition for the $F$-rationality of $R$ given the $F$-rationality of $R(I)$ (Proposition 4.3) and recover [HWY02a, Theorem 3.1] about the $F$-rationality of Rees algebras of integrally closed ideals in two-dimensional $F$-rational rings (Theorem 4.5).

Our next result partially answers [HWY02a, Question 3.7], which asked whether the result holds (for $R(I)$), without any restriction on the dimension. Since our proof uses the principalization result of V. Cossart and O. Piltant [CP08, Proposition 4.2], we put some conditions on $R$.

Theorem 1.2. Let $R$ be a three-dimensional finite-type domain over a field of prime characteristic and $\mathfrak{m}$ a maximal ideal. Assume that $R$ is a rational singularity. Let $I$ be an $\mathfrak{m}$-primary ideal. Let $S$ be a graded $R$-algebra with $R(I) \subseteq S \subseteq \overline{R(I)}$. Suppose that $\text{Proj } S$ is $F$-rational. Then the ring $\oplus_{n \geq 0} S_{Nn}$ is $F$-rational for every integer $N \gg 0$.

This paper arose from trying to understand whether the results of Hyry [Hyr99] (who, in characteristic zero, relate the rationality of a Rees algebra to that of the corresponding blow-up) have counterparts for $F$-rationality.

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In Section 2, we give the definitions, some known results needed in our proofs and some preliminary lemmas. The subsequent sections contain the proofs of the above theorems.

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2. Preliminaries

For the remainder of this paper, unless otherwise indicated, all rings considered are excellent, and of prime characteristic $p$. For a ring $S$, the Frobenius endomorphism, denoted $F_S$ (or, merely $F$, if no confusion is likely to arise), is the map $s \mapsto s^p$. When used in the context of Frobenius endomorphisms, tight closure, $F$-rationality etc, $q$ denotes a power of $p$, and the expression “$q \gg 0$" is synonymous with “$q = p^e$ for every integer $e \gg 0.”$ By $S^0$, we mean the complement in $S$ of the union of minimal primes of $S$. For a ring $S$, $\overline{S}$ denotes its normalization.

Throughout this paper, $(R, m)$ is a $d$-dimensional ring with $d \geq 2$, $I$ an $m$-primary ideal, admitting a reduction $I = (f_1, \ldots, f_d)$. We write $\mathcal{R} := R(I)$, $\mathcal{R}' := R'(I)$, $J = (f_1, \ldots, f_d)R$. $\mathfrak{M} := m\mathcal{R} + \mathcal{R}_+$. $R := \sqrt{\mathcal{R} + (t^{-1})\mathcal{R}'}$ and $X = \text{Proj} R$. Write $E$ for the exceptional divisor of $X$, i.e., the effective Cartier divisor defined by $\mathcal{O}_X$. Write $G = \text{gr}_I(R)$ and $G_+$ for the $G$-ideal $\oplus_{n \geq 0} G_n$. Note that $E = \text{Proj} G$.

Local cohomology. We discuss some properties of local cohomology and the Frobenius action on it.

Discussion 2.1. Using the two exact sequences

$$0 \to \mathcal{R}_+(1) \to \mathcal{R} \to G \to 0 \quad \text{and} \quad 0 \to \mathcal{R}_+ \to \mathcal{R} \to R \to 0,$$

and the fact that $\mathfrak{M} G = \sqrt{G_+}$ and $\mathfrak{M} R = m$, one can conclude that $H^{d+1}_{\mathfrak{M} R}(\mathcal{R})_n = 0$ for every $n \geq 0$ and that $H^{d+1}_{\mathfrak{M} G}(\mathcal{R})_{-1} \neq 0$; see, e.g., [GN94, Lemma 3.3, p. 87]. A similar conclusion holds also for a graded $R$-algebra with $\mathcal{R} \subseteq S \subseteq \overline{R}$.

Note that $H^i_{\mathcal{R}_+}(R) = 0$ for every $i > 0$. Hence the two exact sequences above give an exact sequence

$$\cdots \to H^2_{\mathcal{R}_+}(\mathcal{R})(1) \to H^2_{\mathcal{R}_+}(\mathcal{R}) \to H^2_{G_+}(G) \to H^3_{\mathcal{R}_+}(\mathcal{R})(1) \to \cdots \to H^d_{\mathcal{R}_+}(\mathcal{R})(1) \to H^d_{\mathcal{R}_+}(\mathcal{R}) \to H^d_{G_+}(G) \to 0.$$

For $i \geq 2$, $H^i_{\mathcal{R}_+}(\mathcal{R})_j = H^{-1}(X, (1\mathcal{O}_X)^j)$, so $H^i_{\mathcal{R}_+}(\mathcal{R})_j = 0$ for every $j \gg 0.$ \hfill \Box

Remark 2.2. Suppose that $R$ is Cohen-Macaulay. Then $\mathcal{R}'$ is Cohen-Macaulay if and only if $G$ is Cohen-Macaulay. Further, $\mathcal{R}$ is Cohen-Macaulay if and only if $G$ is Cohen-Macaulay and $(H^d_{\mathcal{R}_+}(G))_n = 0$ for every $n \geq 0$. See [GS82, Theorem 1.1], along with Remark 3.10 on p. 218 and the discussion surrounding (*) and (***) on pp. 202–203 in [GS82]. Observe from Discussion 2.1 that $(H^d_{G_+}(G))_n = 0$ for every $n \geq 0$ if and only if $H^i_{\mathcal{R}_+}(\mathcal{R})_n = 0$ for every $n \geq 0$. \hfill \Box

Remark 2.3. Let $(S, n)$ be an $n$-dimensional Cohen-Macaulay ring and $x_1, \ldots, x_n$ a system of parameters. Write $x = x_1 \cdots x_n$. Let $a \in S$. Then the Čech cycle

$$\frac{a}{\prod x_i}$$

gives the zero element of $H^n(S)$ if and only if $a \in (x_1^j, x_2^j, \ldots, x_n^j)$; see [LT81, Proof of Theorem 2.1, pp. 104–105]. \hfill \Box
Observation 2.4. Let \( S := \bigoplus_{n \in \mathbb{N}} S_n \) be a \( d \)-dimensional graded ring. Suppose that there exists an \( S \)-regular element \( x \in S_1 \). Then we have an exact sequence
\[
\cdots \to H^{d-1}_{S_{-x}}(S/(x)) \to H^d_{S_x}(S)(-1) \to H^d_{S_x}(S) \to 0.
\]
From this it follows that for every \( j \in \mathbb{Z} \), if \( H^d_{S_x}(S)_j = 0 \) then \( H^d_{S_x}(S)_{j+1} = 0 \).

The Frobenius map on a ring \( S \) commutes with the localization map \( S \to S_a \) for each \( a \in S \). Therefore, for every sequence \( a_1, \ldots, a_n \in S \), we get a map of the Čech complex \( \check{C}^\bullet(a_1, \ldots, a_n; S) \) and hence of the local cohomology modules \( H^*_a(a_1, \ldots, a_n; S) \). We now discuss some properties of this action.

Discussion 2.5. Let \( S \) be a ring, \( x \in S \) a non-zero-divisor on \( S \), and \( a = (a_1, \ldots, a_n) \). We have the following commutative diagram of abelian groups, in which the rows are exact sequences of \( S \)-modules:

\[
\begin{array}{cccccc}
0 & \to & S & \xrightarrow{x} & S & \to S/(x) & \to 0 \\
\parallel & \parallel & \parallel & F & \parallel & \parallel & \\
0 & \to & S & \xrightarrow{x_{p^{-1}F}} & S & \to S/(x) & \to 0 \\
\end{array}
\]

These maps commute with localization, thus giving us the following commutative diagram (of abelian groups, in which the rows are exact sequences of \( S \)-modules):

\[
\begin{array}{cccccc}
0 & \to & \check{C}^\bullet(a_1, \ldots, a_n; S) & \xrightarrow{\check{C}^\bullet(a_1, \ldots, a_n; S)} & \check{C}^\bullet(a_1, \ldots, a_n; S) & \to 0 \\
\parallel & \parallel & \parallel & \parallel & \parallel & \\
0 & \to & \check{C}^\bullet(a_1, \ldots, a_n; S) & \xrightarrow{\check{C}^\bullet(a_1, \ldots, a_n; S)} & \check{C}^\bullet(a_1, \ldots, a_n; S) & \to 0 \\
\end{array}
\]

This gives a corresponding commutative diagram of local cohomology modules \( H^*_a(-) \). We observe that even though the vertical maps are maps of abelian groups, their kernels are \( S \)-modules. Now assume that \( S \) is graded and \( x \) is a homogeneous element of degree \( m \). This gives the following commutative diagram of local cohomology modules

\[
\cdots \to H^{d-1}_{a}((S/\langle x \rangle)_{-m+j}) \xrightarrow{x} H^d_{a}(S)_{-m+j} \xrightarrow{F} H^d_{a}(S)_{j} \xrightarrow{F} H^d_{a}(S)_{j} \xrightarrow{F} H^d_{a}(S)_{j} \to H^d_{a}(S)_{j} \to \cdots
\]

\[
(2.6)
\]

F-rationality. Let \( S \) be a ring and \( M \) an \( S \)-module. For a positive integer \( c \), write \( \epsilon S \) for the \( S \)-module \( S \) considered through the \( \epsilon \)th iteration of the Frobenius ring endomorphism \( S \to S \). Write \( \epsilon M \) for \( M \otimes S \epsilon S \). For \( z \in M \), \( z^\epsilon \) is the image of \( z \) under the map \( M \to \epsilon M \). For a submodule \( N \) of \( M \), \( N^{\epsilon \langle q \rangle} \) denotes the \( S \)-submodule of \( \epsilon M \) generated by the image of \( \epsilon N \). The tight closure of \( N \) in \( M \) is

\[
N^{\epsilon \langle q \rangle}_M := \left\{ z \in M \mid \text{there exists } c \in S^0 \text{ such that for all } q \gg 0, cz^\epsilon \in N^{\epsilon \langle q \rangle}_M \right\}.
\]

We say that \( N \) is tightly closed in \( M \) if \( N^{\epsilon \langle q \rangle}_M = N \). For an \( S \)-ideal \( I \), we write \( I^{\epsilon} = I_S^{\epsilon} \) and say that \( I \) is tightly closed if it is tightly closed in \( S \).

Of particular interest to us is \( 0^{\epsilon}_{H_d(S)} \) for a \( d \)-dimensional Cohen-Macaulay local ring \((S,n)\). Let \( a_1, \ldots, a_d \) be a system of parameters for \( S \). Then for all \( b \in S \), the class of the Čech cycle
2.13 Let \( S \) be the homomorphic image of a Cohen-Macaulay ring. If \( S \) is F-rational, then all the local rings are F-rational. A ring \( A \) is said to be F-rational if the local ring at every closed point is F-rational. A ring \( A \) is said to be F-rational if the ideal generated some system of parameters is tightly closed. A ring \( A \) is said to be F-rational if every maximal S-ideal \( n \). A scheme is F-rational if the local ring at every closed point is F-rational.

For the rings and the schemes that we consider in this paper, if the local ring at every maximal ideal / closed point is F-rational, then all the local rings are F-rational, by [HH94, (4.2f)].

**Proposition 2.8.** Let \( S \) be the homomorphic image of a Cohen-Macaulay ring. If \( S \) is F-rational, then \( S[T] \) and \( S[T, T^{-1}] \) are F-rational.

**Proof.** The first statement is [Vél95, Proposition 3.2]. Since \( S[T, T^{-1}] \) is the homomorphic image of a Cohen-Macaulay ring, it suffices to show that \( S[T, T^{-1}]I_n \) is F-rational for every maximal ideal \( n \) of \( S[T, T^{-1}] \) [HH94, Theorem 4.2]. This holds since \( S[T] \) is F-rational.

**Definition 2.9.** Let \( S \) be an equi-dimensional local ring. The parameter test ideal \( \tau_p(S) \) of \( S \) is the set of all the elements \( c \in S \) such that \( cI^N \subseteq I \) for every ideal generated by part of a system of parameters. A parameter test element is an element \( c \in \tau_p(S) \cap S_0 \).

**Proposition 2.10.** [Smi95, Proposition 4.4]. Let \( (S, n) \) be a Cohen-Macaulay local ring. Then \( \tau_p(S) = \text{Ann}_S (\text{Ann}_S^0 (\text{Hdim}_n^S(S))) \).

**Observation 2.11.** Let \((R, \mathfrak{m})\) be an excellent local domain and \( S \) a domain that is finitely generated as an \( R \)-algebra. Let \( n \) be a maximal ideal of \( S \). Suppose that Spec \( S \setminus \{n\} \) is F-rational. Then, by [Vél95, Theorem 3.9], for every \( c \in n \), there exists a positive integer \( N \) such that \( c^N \) is a parameter test element. Hence \( \sqrt{\tau_p(S_n)} = n \) or \( \tau_p(S_n) = S_n \). We see from the proof of [Smi95, Proposition 4.4(ii)] that \( \tau_p(S_n) \subseteq \text{Ann}_R (\text{Ann}_R^0 (\text{Hdim}_n^S(S_n))) \); so \( 0^*_n = 0^*_n \).

**Rational singularities.** In order to prove Theorem 1.2, we need to discuss desingularization and rational singularities. We restrict our attention to integral schemes. A scheme \( X \) is said to be regular if the local ring \( \mathcal{O}_{X,x} \) is a regular local ring for every \( x \in X \).

**Definition 2.12.** Let \( X \) be an integral scheme. A desingularization of \( X \) is a proper birational morphism \( Y \rightarrow X \) such that \( Y \) is regular.

**Definition 2.13.** Let \( X \) be a scheme. We say that \( X \) is a rational singularity if there exists a desingularization \( g : Y \rightarrow X \) such that \( \mathcal{O}_X \rightarrow \mathcal{R}g_* \mathcal{O}_Y \) is a quasi-isomorphism, i.e., \( g_* \mathcal{O}_Y = \mathcal{O}_X \) and \( \mathcal{R}^ig_* \mathcal{O}_Y = 0 \) for every \( i > 0 \).

Suppose that \( X \) is Cohen-Macaulay with a dualizing sheaf \( \omega_X \). Write \( \omega_Y \) for the left-most non-zero cohomology for the complex \( g^!\omega_X \). Then \( \omega_Y \) is a dualizing sheaf for \( Y \). Moreover, \( \mathcal{O}_X \rightarrow \mathcal{R}g_* \mathcal{O}_Y \) is a quasi-isomorphism if and only if \( \mathcal{R}g_* \omega_Y \rightarrow \omega_X \) is a quasi-isomorphism [Lip94, Lemma (4.2)].

In characteristic zero, it is known that if \( X \) is a rational singularity, then every desingularization \( Y \rightarrow X \) satisfies the conditions of Definition 2.13. This can be proved using ‘principalization of ideal sheaves in regular schemes’ and vanishing of cohomology for finite sequence of blow-ups along regular centres (Lemma 2.15). Principalization of ideal sheaves is not known in general in positive characteristic, in dimensions greater than three. We use the following result about principalization in our proof of Theorem 1.2.
**Proposition 2.14** ([CP08, Proposition 4.2]). Let $Y$ be a three-dimensional quasi-projective regular variety over a field and $I \subseteq \mathcal{O}_Y$ an ideal sheaf. Then there exists a finite sequence of morphisms

$$Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 := Y$$

such that $Y_{i+1} \longrightarrow Y_i$ is the blow-up along a regular subscheme of $Y_i$, for each $i$, and $I\mathcal{O}_{Y_n}$ is an invertible sheaf.

The following lemma is perhaps well-known, but we include a proof here for the sake of completeness.

**Lemma 2.15.** Let $\mu$ the composite of a sequence of morphisms

$$Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 := Y$$

such that $Y_i$ is regular for each $i$ and the morphism $Y_{i+1} \longrightarrow Y_i$ is the blow-up along a regular subscheme of $Y_i$, for each $i$. Then the natural maps $\mathcal{O}_Y \longrightarrow R\mu_*\mathcal{O}_{Y_n}$ and $R\mu_*\mu'^*\omega_Y \longrightarrow \omega_Y$ are quasi-isomorphisms.

**Proof.** We first note that the two assertions are equivalent to each other [Lip94, Lemma (4.2)], so we will prove that $\mathcal{O}_Y \longrightarrow R\mu_*\mathcal{O}_{Y_n}$ is a quasi-isomorphism. Since $Y$ is normal, the map $\mathcal{O}_Y \longrightarrow \mu_*\mathcal{O}_{Y_n}$ is an isomorphism. Hence we need to show that for every $i > 0$, $R^i\mu_*\mathcal{O}_{Y_n} = 0$. We do this by induction on $n$. Assume that $n = 1$. The question is local on $Y$, so we may assume that $Y = \text{Spec} \, S$ for some regular local ring $(S, n)$ and that $Y_1$ is obtained by the blow-up along an ideal generated by a regular sequence $x_1, \ldots, x_c \in n \setminus n^2$. The Rees algebra $\mathcal{R}_S((x_1, \ldots, x_c))$ is Cohen-Macaulay, so $H^f(Y_1, O_{Y_1}) = 0$ for every $i > 0$.

Now assume that $n > 1$. Write $\nu$ for the composite morphism $Y_n \longrightarrow Y_1$ and $\nu'$ for the morphism $Y_1 \longrightarrow Y$. Then $\nu_*\mathcal{O}_{Y_n} = \mathcal{O}_{Y_1}$ and $R^i\nu_*\mathcal{O}_{Y_n} = 0$ for every $i > 0$. Hence the spectral sequence (see, e.g., [Wei94, Theorem 5.8.3])

$$E_2^{ij} = R^i\nu'_*R^j\nu_*\mathcal{O}_{Y_n} \implies R^{i+j}\mu_*\mathcal{O}_{Y_n}$$

proves the lemma. $\square$

**Pseudo-rational rings.** Lipman [Lip78, Section 1a, p. 156] in dimension two and Lipman and Teissier [LT81, Section 2, p. 102] more generally defined pseudo-rational rings. Again, as we did with the definition of $F$-rationality, we present the definition relevant for excellent rings.

**Definition 2.16.** A $d$-dimensional local ring $(S, n)$ is said to be **pseudo-rational** if it is Cohen-Macaulay and normal and for every proper birational morphism $f : Y \longrightarrow \text{Spec} \, S$, the map $H^0_d(S) \longrightarrow H^d_{f^{-1}(\{n\})}(\mathcal{O}_Y)$ (an edge map in the Leray spectral sequence $H^0_d(R^if_*\mathcal{O}_Y) \implies H^d_{f^{-1}(\{n\})}(\mathcal{O}_Y)$) is injective.

**Theorem 2.17** ([Sm97, Theorem 3.1]). Every excellent $F$-rational local ring is pseudo-rational.

**Remark 2.18.** Let $(S, n)$ be two-dimensional pseudo-rational singularity and $Y \longrightarrow \text{Spec} \, S$ a proper birational morphism with $Y$ normal. Write $F$ for the closed fiber in $Y$, i.e., inverse image of $\{n\}$. Then $H^1_F(Y, O_Y) = 0$ [Lip78, Theorem (2.4), p. 177]. Using duality with supports [Lip78, Theorem, p. 188], we see that $H^1(Y, \omega_Y) = 0$. Additionally, $H^1(Y, \omega_Y) = 0$ [LT81, Examples (a), p. 103]. Suppose that $Y = \text{Proj} \, \mathcal{R}_S(J)$ for some integrally closed $n$-primary ideal $J$. We see from the proof of [LT81, Corollary 5.4] that $H^1(Y, J^n\omega_Y) = 0$ for every $n \geq 0$. $\square$

3. **Rees algebras**

In this section, we prove some properties of the local cohomology modules of Rees algebras and extended Rees algebras, which will be used in the proofs of the theorems.
Discussion 3.1. Assume that $R$ and $\mathcal{R}$ are Cohen-Macaulay. We now argue that there exists a commutative diagram

$$
\begin{array}{cccc}
0 & \rightarrow & H_{\mathcal{R}}^d(\mathcal{R}) & \phi \rightarrow \bigoplus_{n<0} H_n^d(R)t^n & \rightarrow & H^{d+1}_{\mathcal{R}}(\mathcal{R}) & \rightarrow & 0 \\
& & \downarrow & \downarrow & & \downarrow & & \\
0 & \rightarrow & H_{\mathcal{R}}^d(\mathcal{R})' & \phi \rightarrow \bigoplus_{n\in\mathbb{Z}} H_n^d(R)t^n & \rightarrow & H^{d+1}_{\mathcal{R}}(\mathcal{R})' & \rightarrow & 0 \\
\end{array}
$$

with exact rows. We prove this as follows. The top row is from [HWY02a, Lemma 2.7]. (Note that $H_{\mathcal{R}}^d(\mathcal{R}) = 0$ for every $j \geq 0$, by Remark 2.2.) The lower row is from the exact sequence

$$
\cdots \rightarrow H_{\mathcal{R}}^{d-1}(\mathcal{R}_{j-1}) \rightarrow H_n^d(\mathcal{R}) \rightarrow H_{\mathcal{R}}^d(\mathcal{R})' \rightarrow H_{\mathcal{R}}^d(\mathcal{R}_{j-1}) \rightarrow H^{d+1}_{\mathcal{R}}(\mathcal{R})' \rightarrow \cdots.
$$

Since $\mathcal{R}$ and $R$ are Cohen-Macaulay, $\mathcal{R}'$ is Cohen-Macaulay. We now describe the maps. The left-most vertical map is identity, since $\mathcal{R}' / \mathcal{R}$ is a $\mathcal{R}$-torsion module and $d \geq 2$. The middle vertical map is the natural inclusion map; the right-most vertical map $\gamma$ is induced by the commutativity of the left square.

To prove the commutativity of the diagram, we need to show that the left square is commutative. For this, we need to describe the two maps $\phi$ and $\psi$. Write $f = f_1 \cdots f_d$. Every homogeneous element of $H_{\mathcal{R}}^d(\mathcal{R})$ is the class in cohomology of a Čech cycle

$$
\frac{a t^n}{f_1 \cdots f_d}
$$

with $a \in \mathcal{R}_n$ and some $l$. Then

$$
\phi\left(\frac{a t^n}{f_1 \cdots f_d}\right) = \left[\frac{a}{f}\right] \cdot t^{n-d_l}.
$$

(See [HWY02a, Remark, p. 166]). On the other hand, the map $\psi$ is the map

$$
H_{\mathcal{R}}^d(\mathcal{R})' \rightarrow H_{\mathcal{R}}^d(\mathcal{R}_{j-1})
$$

which maps

$$
\left[\frac{a t^n}{f_1 \cdots f_d}\right], \quad a \in \mathcal{R}_n'
$$

to itself, with $a$ now thought of as an element of $(\mathcal{R}_{j-1})_n$. Hence the left square commutes. Also note that the right-most vertical map is injective. Moreover, for each $j < 0$, the map $H_{\mathcal{R}}^{d+1}(\mathcal{R})_j \rightarrow H_{\mathcal{R}}^{d+1}(\mathcal{R})'_j$ induced by $\gamma$ (i.e., $j$th graded piece of $\gamma$) is an isomorphism.

We now prove some results relating the $F$-rationality of $\mathcal{R}$, $\mathcal{R}'$ and $X$. To make sense of the following observation, recall that the map $\gamma$ is bijective in negative degrees.

**Lemma 3.2.** Adopt the notation of Discussion 3.1. Let $0 \neq c \in R$, $e$ a positive integer and $\xi \in H_{2n}^{d+1}(\mathcal{R})$ be such that $cF^e(\xi) = 0$. Then $cF^e(\gamma(\xi)) = 0$. Conversely, let $0 \neq c \in R$, $e$ a positive integer, $j$ a negative integer, and $\xi \in H_{2n}^{d+1}(\mathcal{R})'_j$ be such that $cF^e(\xi) = 0$. Then $cF^e(\gamma^{-1}(\xi)) = 0$.

**Proof.** Without loss of generality, $\xi$ is homogeneous. Lift it to $\xi'$ in $\bigoplus_{n<0} H_n^d(R)t^n$. Then $cF^e(\xi') \in \text{Im}(\phi)$. Reading this in the lower row of the diagram of Discussion 3.1 we see that $cF^e(\gamma(\xi)) = 0$. A similar argument proves the other assertion. \qed

**Remark 3.3.** From [HWY02a, Corollary 1.10], we know that there exists $c \in R$ that is a parameter test element for $R$ and $\mathcal{R}$. The same argument shows that there exists $c \in R$ that is a parameter test element for $R$, $\mathcal{R}$ and $\mathcal{R}'$. 
As an immediate corollary, we get the following:

**Proposition 3.4.** Adopt the notation of Discussion 3.1. Then

$$\gamma(0^*_{H^d_{\mathfrak{m}}(\mathcal{R})}) = \bigoplus_{j < 0} \left(0^*_{H^d_{\mathfrak{m}}(\mathcal{R})}\right)_j.$$  

In particular, if \( \mathcal{R}'_{\mathfrak{m}} \) is F-rational, then so is \( \mathcal{R}_{\mathfrak{m}}. \)

**Lemma 3.5.** Suppose that \( R \) is normal. Let \( S \) be a graded R-algebra with \( \mathcal{R} \subseteq S \subseteq \mathcal{R}. \) Then \( \sqrt{\mathfrak{m}_S} \) is the homogeneous maximal ideal of \( S. \) Further, \( \text{Proj} S \) is F-rational if and only if \( \text{Spec} S \setminus \{ \sqrt{\mathfrak{m}_S} \} \) is F-rational.

**Proof.** Note that \( S_0 = R. \) For every \( j > 0, \) \( S_j \subseteq \overline{t^j}, \) so there exists \( N \) such that \( S_j^N \subseteq \overline{t^j}N \subseteq \mathfrak{m}_S \). Hence \( \sqrt{\mathfrak{m}_S} \) is the homogeneous maximal ideal of \( S. \)

Write \( X = \text{Proj} S. \) Note that \( \sqrt{\mathfrak{m}_S} = S_+ \), so \( mS + S_+ = \sqrt{\mathfrak{m}_S} \) and the affine schemes \( \text{Spec} S_{ij}, 1 \leq i \leq d, \) form an open covering for \( \text{Spec} S \setminus V(S_+) \) and, by [GD67, Chapitre II, Corollaire (2.3.14)], \( \text{Spec} S_{(f_t)}, 1 \leq i \leq d, \) form an open covering for \( X. \) Since \( S_{(f_t)} \simeq S_{(f_t)[T, T^{-1}]} \) [GD67, Chapitre II, (2.2.1)], we see that \( X \) is F-rational if and only if \( \text{Spec} S \setminus V(S_+) \) is F-rational. (Use Proposition 2.8.)

Now assume that \( \text{Spec} S \setminus \{ \sqrt{\mathfrak{m}_S} \} \) is F-rational. Then \( \text{Spec} S \setminus V(S_+) \) is F-rational, so \( X \) is F-rational. Conversely assume that \( X \) is F-rational. Let \( \Omega \in \text{Spec} S \setminus \{ \sqrt{\mathfrak{m}_S} \}. \) If \( \Omega \cap R = m, \) then \( \Omega \in \text{Spec} S \setminus V(S_+) \), so \( \Omega \) is F-rational. On the other hand, if \( \Omega \cap R \subseteq m, \) then there exists \( 1 \leq i \leq d \) such that \( f_i \notin \Omega, \) so \( S_\Omega \simeq (S_{(f_t)})_{\Omega f_t}. \) Now observe that \( \text{Spec} R \setminus \{ m \} \simeq \text{Proj} S \setminus V(IS) \) is F-rational, so \( S_{(f_t)} \simeq R_{(f_t)}[T] \) is F-rational. \( \square \)

**Lemma 3.6.** \( \text{Spec} \mathcal{R}' \setminus \{ \mathfrak{m} \} \) is F-rational if and only if \( R \) and \( X \) are F-rational.

**Proof.** Since \( \mathfrak{m} = \sqrt{(f_1t, \ldots, f_dt)\mathcal{R}' + (t^{-1})\mathcal{R}'}, \) we see that \( \text{Spec} \mathcal{R}' \setminus \{ \mathfrak{m} \} \) is F-rational if and only if \( \mathcal{R}'_{i-1} \) and \( \mathcal{R}'_{i}, 1 \leq i \leq d \) are F-rational.

Now \( \mathcal{R}'_{i-1} \simeq R[t, t^{-1}], \) so \( \mathcal{R}'_{i-1} \) is F-rational if and only if \( R \) is F-rational. Moreover, every element of the \( \mathcal{R} \)-module \( \mathcal{R}' / \mathcal{R} \) is annihilated by a power of \( \mathcal{R}_+, \) so the inclusion \( \mathcal{R}_{(f_t)} \rightarrow \mathcal{R}'_{(f_t)} \) is an isomorphism for every \( 1 \leq i \leq d. \) Hence, arguing as in the proof of Lemma 3.5, we conclude that \( X \) if F-rational if and only if \( \mathcal{R}'_{(f_t)} \) is F-rational for every \( 1 \leq i \leq d. \) \( \square \)

4. **Theorem 1.1**

In this section we prove Theorem 1.1 and some corollaries.

**Lemma 4.1.** If \( R \) is F-rational and \( \mathcal{R}' \) is Cohen-Macaulay, \( \mathcal{R} \) is Cohen-Macaulay.

**Proof.** Since \( \mathcal{R}' \) is Cohen-Macaulay, \( G \simeq \mathcal{R}'/ (t^{-1}) \mathcal{R}' \) is Cohen-Macaulay. Note that \( R \) is pseudo-rational. Hence the lemma follows from [Lip94, Theorem (5)]. \( \square \)

**Lemma 4.2.** Suppose that \( R \) is Cohen-Macaulay and that \( \mathcal{R} \) is F-rational. Then \( R \) is F-rational if and only if \( 0^*_{H^d_{\mathfrak{m}}(\mathcal{R})} = 0. \)

**Proof.** We see from the lower row of the diagram in Discussion 3.1 that \( H^d_{\mathfrak{m}}(R) \subseteq H^d_{\mathfrak{m}}(\mathcal{R}) \). Hence, if \( 0^*_{H^d_{\mathfrak{m}}(\mathcal{R})} = 0, \) then \( 0^*_{H^d_{\mathfrak{m}}(R)} \subseteq \left(0^*_{H^d_{\mathfrak{m}}(\mathcal{R})}\right)_0 = 0, \) and, therefore, \( R \) is F-rational.

Conversely, assume that \( R \) is F-rational. Every homogeneous element of \( H^l_{\mathfrak{m}}(\mathcal{R}) \) is the class of a \( \check{\text{C}} \)ech cycle of the form

$$\frac{a t^k}{f_1^{l(d-1)}}.$$
with \( at^k \in I_1^q \) and \( f = f_1 \cdots f_d \). (We use the \( \Omega \)-primary ideal \( (f_1 t, \ldots, f_d t, t^{-1}) \) for describing \( \mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R}) \).) Suppose that
\[
\left[ \frac{at^k}{f^t(t-1)^j} \right] \in 0^{*}_{\mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R})}.
\]
Then from the lower row of the commutative diagram in Discussion 3.1, we see that \( k \geq (d-1)l \), so this element of \( \mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R}) \) is the image of
\[
\left[ \frac{a}{f^t} \right] \cdot t^{k-(d-1)l} \in \oplus_{n \geq 0} \mathcal{H}_m^d(R) t^n.
\]
Since there exists \( c \in R \) such that
\[
\left[ \frac{ca^q t^q}{f^t(t-1)^j} \right] = 0
\]
we conclude that
\[
\left[ \frac{ca^q}{f^t} \right] \cdot t^{k-(d-1)l} = 0 \in \oplus_{n \geq 0} \mathcal{H}_m^d(R) t^n.
\]
so
\[
\left[ \frac{ca^q}{f^t} \right] = 0 \in \mathcal{H}_m^d(R).
\]
Since \( R \) is \( F \)-rational, \( 0^{*}_{\mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R})} = 0 \). \( \square \)

**Proof of Theorem 1.1.** (a) \( \implies \) (b): \( \text{Spec } \mathcal{R} \setminus \{ \Omega \} \) is \( F \)-rational by Lemmas 3.6 and 3.5. On the other hand, \( \mathcal{R}_\Omega \) is \( F \)-rational by Lemma 4.1 and Proposition 3.4.

(b) \( \implies \) (a): \( \text{Spec } \mathcal{R}' \setminus \{ \Omega \} \) is \( F \)-rational by Lemmas 3.6 and 3.5. Lemma 4.2 implies that \( \mathcal{R}'_\Omega \) is \( F \)-rational. \( \square \)

In general, even if \( \mathcal{R} \) is \( F \)-rational, \( R \) need not be \( F \)-rational. Hara, Watanabe and Yoshida show that if \( \mathcal{R} \) is \( F \)-rational and the \( a \)-invariant of the associated graded ring \( G \) is at most \(-2\) then \( R \) is \( F \)-rational [HWY02a, Corollary 2.13]. In [HWY02a, Example 3.9], they exhibit an example where \( R \) is not \( F \)-rational but \( \mathcal{R} \) is. As an application of our earlier results, we get a sufficient condition for the \( F \)-rationality of \( R \), given the \( F \)-rationality of \( \mathcal{R}' \), which recovers [HWY02a, Corollary 2.13].

**Proposition 4.3.** Let \( (R, m) \) be an excellent Cohen-Macaulay local domain of prime characteristic and \( I \) an \( m \)-primary \( R \)-ideal. Suppose that \( \mathcal{R}_K(1) \) is \( F \)-rational. If the Frobenius map \( \mathcal{H}_m^d(1) \to \mathcal{H}_m^d(1) \) is injective, then \( R \) and \( \mathcal{R}_K(1) \) are \( F \)-rational.

**Proof.** We first show that \( 0^{*}_{\mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R})} = 0 \). By way of contradiction, assume that \( 0^{*}_{\mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R})} \neq 0 \). By Proposition 3.4 we may pick a non-zero \( \xi \in 0^{*}_{\mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R})} \) of smallest degree. Write \( j + 1 \) for this degree. Then \( j \geq -1 \). Note that \( t^{-1} \xi = \xi \), so \( \xi \) is the image of a non-zero element \( \zeta \in \mathcal{H}_m^d(1) \). Since \( \mathcal{R} \) is Cohen-Macaulay, \( j < 0 \). Hence \( j = -1 \).

Now consider the diagram (2.6) from Discussion 2.5, with \( S = \mathcal{R}', x = t^{-1} \) and \( m = -1 \). Rewriting it with only the relevant cohomology groups we get the following:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{H}_m^d(1) & \longrightarrow & \mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R})_0 & \overset{t^{-1}}{\longrightarrow} & \mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R})_{-1} & \longrightarrow & 0 \\
& & \uparrow F & & \uparrow t^{-1} & & \uparrow F & & \\
0 & \longrightarrow & \mathcal{H}_m^d(1)_{-p} & \longrightarrow & \mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R})_{-1-p} & \overset{t^{-1}}{\longrightarrow} & \mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R})_{-p} & \longrightarrow & 0.
\end{array}
\]

Note that \( F(\xi) \in \left( 0^{*}_{\mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R})} \right)_0 \), so \( t^{-1} F(\xi) = 0 \). On the other hand, by hypothesis, \( F(\xi) \neq 0 \), a contradiction. Hence \( 0^{*}_{\mathcal{H}_{\Omega 1}^{d+1}(\mathcal{R})} = 0 \).
Hence $\mathcal{R}_R$ and $R$ are $F$-rational (Lemma 4.2). Since $\text{Proj } \mathcal{R}$ is $F$-rational, $\text{Spec } \mathcal{R}' \setminus \{\mathfrak{N}\}$ is $F$-rational (Lemma 3.6). Therefore $\mathcal{R}'$ is $F$-rational.

The next example shows that the hypothesis on $H^2_{G_z}(G)$ in Proposition 4.3 is only sufficient, but not necessary.

**Example 4.4.** Let $R = k[[X,Y,Z]]/(X^2 + Y^3 + Z^5)$, with $k$ a field of characteristic $p \geq 7$. It is a two-dimensional $F$-rational ring. Since $m = (X,Y,Z)$ is integrally closed, the Rees algebra $R_R(m)$ is $F$-rational by [HWY02a, Theorem 3.1] (Theorem 4.5 below). Note that $G = k[x,y,z]/(x^2)$, where $x,y,z$ correspond to $X,Y,Z$. Compute $H^2_{G_z}(G)$ from the Čech complex $\check{C}^*(y,z;G)$. Use Remark 2.3 to conclude that the class

$$\left[ \frac{x}{yz} \right] \in H^2_{G_z}(G)_{-1}$$

of the Čech cycle in non-zero, but

$$F \left( \left[ \frac{x}{yz} \right] \right) = \left[ \frac{x^p}{y^p z^p} \right] = 0 \in H^2_{G_z}(G)_{-p}.$$ 

Hence the map $H^2_{G_z}(G)_{-1} \to H^2_{G_z}(G)_{-p}$ is not injective. □

As another application of our results we recover the following result of Hara, Watanabe and Yoshida [HWY02a, Theorem 3.1]. (They mention that Huneke and Smith also had obtained this result.)

**Theorem 4.5.** Let $(R, m)$ be a two-dimensional excellent $F$-rational domain of prime characteristic. Let $I$ be an $m$-primary integrally closed ideal. Then $\mathcal{R}_R(I)$ is $F$-rational.

**Remark 4.6.** By [Lip94, Example (3)], we know that $\mathcal{R}$ is Cohen-Macaulay. Hence $G$ and $\mathcal{R}'$ are Cohen-Macaulay.

**Proof of Theorem 4.5.** We claim that the ideal $(f_1, f_2, t^{-1})$ of $\mathcal{R}'$ is tightly closed. Assume the claim. Then, by [HH90, Proposition 4.14], $(f_1, f_2, t^{-1})\mathcal{R}_R$ is tightly closed. Since $\mathcal{R}_R$ is Cohen-Macaulay, it is $F$-rational [HH94, Theorem 4.2]. Discussion 3.1 and Proposition 3.4 apply to $\mathcal{R}$, so $\mathcal{R}_R$ is $F$-rational. By [HWY02a, Lemma 1.12], $\mathcal{R}$ is $F$-rational.

Now to prove the claim, consider an element $at^k \in (f_1, f_2, t^{-1})^{*}$, with $a \in I^k$. (It suffices to consider homogeneous elements.) If $k < 0$, then $at^k \in (t^{-1})$, so we may assume that $k \geq 0$. If $k \geq 2$, then $I^k = I^{k-1}$ [LT81, Corollary 5.4], so $at^k \in (f_1, f_2)$. Therefore we need to consider $k = 0$ and $k = 1$. For every $0 \neq c \in I$, $\mathcal{R}_R \simeq R_c[t, t^{-1}]$ is $F$-rational, so we may pick a parameter test element $c$ from $I$. If $k = 0$, then for all sufficiently large $q = p^r$, there exist $a_q, \beta_q \in I$ and $\gamma_q \in I^q$ such that $ca^q = a_q t^{-1} f_1^q t^q + \beta_q t^{-1} f_2^q t^q + \gamma_q t^q t^{-q} \in \langle f_1^q, f_2^q \rangle + I^q \subseteq I^q$. Since $I$ is integrally closed, $a \in I$. Since $I = (t^{-1})$, $a \in (t^{-1})$. If $k = 1$, then for all sufficiently large $q = p^r$, there exist $a_q, \beta_q \in I$ and $\gamma_q \in I^{q^2}$ such that $ca^q t^q = a_q f_1^q t^q + \beta_q f_2^q t^q + \gamma_q t^{2q} t^{-q}$, so $ca^q \in \langle f_1^q, f_2^q \rangle + I^{q^2} \subseteq \langle f_1^q, f_2^q \rangle$, where the last inclusion follows from the pseudo-rationality of $R$ [LT81, Corollary 5.4]. Since $R$ is $F$-rational, $a \in (f_1, f_2)$, so $at \in (f_1, f_2)$. □

5. **THEOREM 1.2**

**Notation 5.1.** Let $S = \bigoplus_{n \in \mathbb{N}} S_n$ be a graded ring with $S_0$ a local ring with maximal ideal $n_0$. Write $n$ for the homogeneous maximal ideal $n_0 S + S_+$. For a positive integer $N$, we write $S^{(N)} := \bigoplus_{n \geq 0} S_{Nn}$ and $n^{(N)}$ for the homogeneous maximal ideal of $S^{(N)}$.

**Observation 5.2.** Note that for integers $N > 0$ and $i \geq 0$,

$$H^i_{n^{(N)}}(S^{(N)}) = \bigoplus_{n \in \mathbb{Z}} \left( H^i_{n}(S) \right)_{Nn}.$$
Write $W = 0^i_{H^0_i(S)}$. It is a graded $S$-module. Let $m \in \mathbb{Z}$ and $\xi \in W_m$. Then we may assume that there exists $c \in S^{(m)}$ such that $c^q \xi = 0$ for every sufficiently large power $q$ of $p$. Hence

$$
\bigoplus_{n \in \mathbb{Z}} W_{Nn} \subseteq 0^*_n H^i_{H'(0)}(S^{(N)}).
$$

On the other hand, we can check directly that

$$
0^*_H H^i_{H'(0)}(S^{(N)}) \subseteq \bigoplus_{n \in \mathbb{Z}} W_{Nn},
$$

so

$$
0^*_H H^i_{H'(0)}(S^{(N)}) = \bigoplus_{n \in \mathbb{Z}} W_{Nn}. \quad \square
$$

One of the ingredients of the proof of Theorem 1.2 is the following proposition, which is analogous to the results of E. Hyry [Hyr99] relating the rationality $\text{Proj } R(I)$ to that of $R(I)$ in characteristic zero.

**Proposition 5.3.** Let $(R, \mathfrak{m})$ be an excellent noetherian local normal Cohen-Macaulay domain of prime characteristic and $I$ an $m$-primary $R$-ideal. Let $S$ be a graded $R$-algebra with $R(I) \subseteq S \subseteq R(I)$. Write $X = \text{Proj } S$. Suppose that $X$ is $F$-rational and that $H^i(X, \delta_X) = 0$ for every $i \geq 1$. Then for every integer $N \gg 0$, the subring $\bigoplus_{n \geq 0} S_{Nn}$ is $F$-rational.

**Lemma 5.4.** With the hypothesis in Proposition 5.3, $0^*_H H^i_{H'(0)}(S)$ is a finite-length $S$-module. In particular, for every integer $N \gg 0$, $0^*_H H^i_{H'(0)}(S) = 0$.

**Proof.** From Lemma 3.5 we see that $\text{Spec } S \setminus \{ \sqrt{\mathfrak{m}S} \}$ is $F$-rational. Now apply Observation 2.11 to prove the first assertion. The second assertion follows from this and Observation 5.2, after noting that $H^i_{H'(0)}(S)_n = 0$ for every $n \geq 0$. \hfill $\square$

**Proof of Proposition 5.3.** Write $n = \sqrt{\mathfrak{m}S}$. For every integer $N \gg 0$, $S^{(N)}$ is Cohen-Macaulay by [Lip94, Theorem (4.1)] and $0^*_H H^i_{H'(0)}(S^{(N)}) = 0$ by Lemma 5.4. Hence $(S^{(N)})_{\mu^{(N)}}$ is $F$-rational by [Smi95, Proposition 4.4(ii)]. On the other hand, by Lemma 3.5, $\text{Spec } S^{(N)} \setminus n^{(N)}$ is $F$-rational since $\text{Proj } S^{(N)} = \text{Proj } S$ is $F$-rational. \hfill $\square$

We now prove some lemmas needed in the proof of Theorem 1.2. For the remainder of this section we assume the hypothesis of the theorem and adopt the following notation: Write $X = \text{Proj } S$ and $f$ for the structure morphism $X \to \text{Spec } R$. Since $R$ is a rational singularity, it is Cohen-Macaulay. Write $\omega_R$ for a dualizing module for $R$, which exists since $R$ is of finite type over a field and a Cohen-Macaulay ring. Write $\omega_X$ for the left-most non-zero cohomology sheaf of the complex $f^! \omega_R$; since $X$ is $F$-rational (and, a fortiori, Cohen-Macaulay), it is a dualizing sheaf for $X$.

**Lemma 5.5.** There exists a desingularization $g : Y \to X$ such that $H^0(Y, \omega_Y) = \omega_R$ and $H^i(Y, \omega_Y) = H^i(Y, \delta_Y) = 0$ for every $i > 0$.

**Proof.** Since $R$ is normal, $H^i(Y, \delta_Y) = R$, so by [Lip94, Lemma (4.2)] the assertions about $\omega_Y$ would follow from showing that $H^i(Y, \delta_Y) = 0$ for every $i > 0$. Since $R$ is a rational singularity, pick a desingularization $\mu : Y' \to \text{Spec } R$ such that the natural maps $R \to R_{\mu\ast}\delta_Y$ and $R_{\mu\ast}\mu\ast\omega_R \to \omega_R$ are quasi-isomorphisms. By Proposition 2.14, we can find a proper birational morphism $\mu' : Y'' \to Y$ such that the ideal sheaf $I\delta_{Y'}$ is invertible. Now use Lemma 2.15 to conclude that the map $\delta_{Y'} \to R_{\mu\ast}\mu\ast\omega_R$ is a quasi-isomorphism. Therefore the spectral sequence

$$
E^j_2 = R_{\mu\ast}R_{\mu'}^j\omega_{Y'}, \quad \Rightarrow \quad R^{i+j}(\mu\mu')_\ast\delta_{Y'},
$$

converges to $0$.
helps us conclude that $H^i(Y', \mathcal{O}_{Y'}) = 0$ for every $i > 0$.

Since $Y'$ is normal and $I\mathcal{O}_{Y'}$ is invertible the map $(\mu\mu') : Y' \to \operatorname{Spec} R$ factors as $fg$ for some $g : Y' \to X$. Relabel $Y'$ as $Y$.

Lemma 5.6. Let $g : Y \to X$ be as in Lemma 5.5. Then
(a) The maps $\mathcal{O}_X \to g_*\mathcal{O}_Y$ and $g_*\omega_Y \to \omega_X$ are isomorphisms. In particular, $H^0(X, \mathcal{O}_X) = R$ and $H^0(X, \omega_X) = \omega_R$.
(b) For every $i \geq 1$, $R^ig_*\mathcal{O}_Y$ and $R^ig_*\omega_Y$ have zero-dimensional support, if they are non-zero.

Proof. (a): The map $\mathcal{O}_X \to g_*\mathcal{O}_Y$ is an isomorphism since $X$ is normal. That the map $g_*\omega_Y \to \omega_X$ is an isomorphism follows from the characterization of pseudo-rational rings in [LT81, Corollary (of (iii)), p. 107], after noting that $X$ is pseudo-rational [Smi97, Theorem 3.1]. The remaining assertions follow from the above isomorphisms and Lemma 5.5.

(b): Let $x \in X$ be a point with $\mathcal{O}_{X,x} \simeq R$ and consider the cartesian square

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{g_1} & Y \\
\downarrow g & & \downarrow g \\
\operatorname{Spec} \mathcal{O}_{X,x} & \to & X
\end{array}
$$

Then $g_1 : Y_1 \to \operatorname{Spec} \mathcal{O}_{X,x}$ is a desingularization of the two-dimensional pseudo-rational ring $\mathcal{O}_{X,x}$; by Remark 2.15, $R^1g_1_*\mathcal{O}_{Y_1} = R^1g_1_*\omega_{Y_1} = 0$. Since $\mathcal{O}_{X,x} = \mathcal{O}_X$, we need to use [Ver69, Theorem 2, p. 394] for the second equality. Hence these sheaves have zero-dimensional support, if they are non-zero.

Proof of Theorem 1.2. In view of Proposition 5.3, it suffices to show that $H^i(X, \mathcal{O}_X) = 0$ for every $i > 0$. Let $g : Y \to X$ be as in Lemma 5.5. Write $h = fg$. Let $\mathcal{F}$ be $\mathcal{O}_Y$ or $\omega_Y$. Then in the spectral sequence

$$E_2^{ij} = H^i(X, R^jg_*\mathcal{F}) = R^jf_*R^ig_*\mathcal{F} \Rightarrow R^{i+j}h_*\mathcal{F} = H^{i+j}(Y, \mathcal{F}),$$

$E_2 = 0$ if $i > 0$ and $j > 0$ by Lemma 5.6(b). This gives an exact sequence

$$0 \to H^1(X, \mathcal{F}) \to H^1(Y, \mathcal{F}) \to H^0(X, \mathcal{O}_X) \to H^2(X, \mathcal{F}) \to H^2(Y, \mathcal{F}) \to H^0(X, \omega_X) \to 0.
$$

Use Lemma 5.5 to conclude that $H^1(X, \mathcal{F}) = 0$, that the map $H^0(X, \mathcal{O}_X) \to H^2(X, \omega_X)$ is an isomorphism, and (since $\mathcal{F}$ has zero-dimensional support) that $H^2(X, \omega_X) = 0$.

We now have the following situation: $H^0(X, \mathcal{O}_X) = R$, $H^1(X, \mathcal{O}_X) = 0$, $H^0(X, \omega_X) = \omega_R$ and $H^1(X, \omega_X) = 0$. Note, also, that $H^2(X, \mathcal{O}_X)$ is a finite-length $R$-module. We need to show that $H^2(X, \mathcal{O}_X) = 0$.

We apply duality for proper morphisms [Har66, Chapter VII, Theorem 3.3, p. 379] to $f$ to see that

$$Rf_*\omega_X \simeq R\operatorname{Hom}_R(Rf_*\mathcal{O}_X, \omega_R).$$

The right-side can be computed using a second-quadrant spectral sequence with

$$E_2^{ij} = \operatorname{Ext}_R^j(H^i(X, \mathcal{O}_X), \omega_R) \Rightarrow H^{i+j}(X, \omega_X).$$

Since the only possibly non-zero terms are $E_2^{0,0} = \omega_R$ and $E_2^{2,3} = \operatorname{Ext}_R^3(H^2(X, \mathcal{O}_X), \omega_R)$, we get an exact sequence (with $a$ coming from the $E_3$-page)

$$0 \to H^0(X, \omega_X) \to \omega_R \to \operatorname{Ext}_R^3(H^2(X, \mathcal{O}_X), \omega_R) \to H^1(X, \omega_X) \to 0.$$
The above proof also gives the following proposition:

**Proposition 5.7.** $X$ has rational singularities.

**Proof.** For $\mathcal{F} = \mathcal{O}_Y$ or $\mathcal{F} = \omega_Y$, we established the following: (a) $R^2g_*\mathcal{F} = 0$; (b) $R^1g_*\mathcal{F}$ has zero-dimensional support, if it is non-zero; (c) $H^0(X, R^1g_*\mathcal{F}) = H^2(X, \mathcal{O}_X) = 0$. Hence $R^1g_*\mathcal{F} = 0$. □

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