A Simple Explanation for the Reconstruction of Graphs

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Abstract

The graph reconstruction conjecture states that all graphs on at least three vertices are determined up to isomorphism by their deck.

Any non-regular graph has a proper induced subgraph which is unique due to its structure or the way of its connection to the rest of the graph. The former subgraph is called an anchor and the latter is called a connective anchor, if it is distinguishable in the deck.

Graph anchor can be regarded as a useful tool to prove that an arbitrary non-regular graph is reconstructible. Also, it provides a general framework for the reconstruction conjecture. We show that if a graph has an orbit with at least three vertices whose removal leaves an anchor, or it has two vertices whose removal leaves an anchor with the mentioned condition in the paper, then it is reconstructible. This simple statement, easily, explains the reconstruction of a graph from its deck. For instance, this statement is sufficient to prove that almost every graph, trees and small graphs are reconstructible.

Finally, we have discussed about the sufficiency of the above simple statement for every non-regular graph.

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1 Introduction

The Reconstruction Conjecture is an interesting problem which has remained open for more than 70 years. It states that all graphs on at least three vertices are determined up to isomorphism by their deck [13, 31]. To know more about this problem, see [4, 5]. The deck of a graph $G$ is the multiset of graphs that is obtained from deleting one vertex in every possible way from the graph $G$. The members of a deck are referred to as cards. A graph is reconstructible, if it is determined up to isomorphism by its deck. A class of graphs is reconstructible, if every member of the class is reconstructible. Any property of a graph, i.e. graph invariant, which is determined by the deck of graph, is said to be reconstructible. In this paper, the asymmetry of a graph, i.e. a part of a graph which does not repeat, is employed to reconstruct it form its deck.

First time, the graph reconstruction conjecture was proposed by Ulam [31]. Kelly [14], in his Ph.D thesis, showed that regular graphs, Eulerian graphs, disconnected graphs and trees are reconstructible. Bondy and Hemminger [5] have proposed another proof for trees by employing the counting theorem. Graphs in which no two cycle have a common edge [21], graphs in which all cycles pass through a common vertex [22], outer planner graphs [10], separable graphs without end vertices [3], maximal planar graphs [19], critical blocks and graphs with some specific degrees sequence [25] are some well known classes of reconstructible graphs. Another approach to this problem are attempts to find reconstructible graph invariants. The fundamental result in this area is Kelly’ Lemma [14]. It states that the number of occurrence of any proper subgraph of a graph is reconstructible. Many results in graph reconstruction are based on this lemma. The generalization of this lemma for spanning subgraphs is Kocay’s Lemma [17] which asserts that the number of any disconnected spanning subgraph is reconstructible. Another reconstructible graph invariants are characteristic polynomial [30], chromatic polynomial [30] and planarity [1]. Oliveira and Thatte [26, 29] had a new approach to this problem by studying the matrix of covering numbers of graphs by sequences of subgraphs and proposing a bound for the rank of this matrix.
In this paper, the concept of unique subgraph is employed to explain how a graph is reconstructible from its deck and, consequently, new families of reconstructible graphs are proposed. Unique subgraph is a known key concept for the graph reconstruction problem. Bollobás [2] has employed graphs in which all \((n - 2)\) and \((n - 3)\)-vertex subgraphs are unique to show that almost every graph is reconstructible by three cards. Muller [24] has shown almost every graph is reconstructible using graphs whose all \(n/2\)-vertex subgraphs are unique. Unique subgraphs also have been used by Chinn [6] and Zhu [32] to introduce some families of reconstructible graphs. Ramachandran [28], also, has employed the idea of unique subgraph for the digraph \(N\)-reconstruction.

In this paper, a general framework for the graph reconstruction problem is proposed. Any graph which is not vertex-transitive has a proper induced subgraph which is unique due to its structure or the way of its connection to the rest of the graph. The former subgraph is called an anchor and the latter is called a connective anchor, if it is distinguishable in the deck.

First, we show that how graph anchor helps us in graph reconstruction and introduces significance families of reconstructible graphs. Then, we show that even graph anchor enables us to have a general framework for the graph reconstruction problem.

In this paper, it is shown that a graph \(G\) is reconstructible, if it has either an orbit \(O\) (with at least three vertices) which makes \(G\setminus O\) an anchor or two vertices \(\{v, w\}\) which make \(G\setminus \{v, w\}\) an anchor with the conditions which will be mentioned. Also, we discuss about the sufficiency of this result for any non-regular graph with at least three vertices.

The above simple result is sufficient to prove that many significant families of graphs are reconstructible, such as almost every graph, trees or small graphs. It may be wondering how this simple statement may be sufficient for every non-regular graph. What makes the above result, nearly, comprehensive, is the concept of anchor extension which is defined, in this paper. Accordingly, the graph reconstruction conjecture is reduced to verifying the following simple statement: Any graph has either a \((n - 2)\)-vertex anchor with the mentioned condition or an orbit with at least three vertices whose removal leaves an anchor.

After the definition of anchor in the next section, some results about anchor and graph reconstruction are given in the third section. Then, the concept of anchor extension is employed to draw a general framework for the graph reconstruction problem in Section 4. The reconstruction of graphs
with \((n-2)\)-vertex anchor is discussed in Section 5. Finally, as an application of the suggested framework, it is shown that trees are reconstructible using the anchor extension technique.

# 2 Definitions and Notations

In this section, graph anchor and shadows set are defined. In this paper, any subgraph is a vertex induced subgraph, otherwise, it is mentioned. The neighbors of any vertex \(v\) in a graph is denoted by \(N(v)\). A graph \(G\) is called asymmetric, if \(\text{Aut}(G) = I\). Two vertices \(u\) and \(v\) of a graph \(G\) are called similar, if there is an automorphism of \(G\) which maps \(u\) into \(v\). Dissimilar vertices whose removal leaves isomorphic subgraphs are called pseudo-similar [20, 15]. Similarity is, obviously, an equivalence relation. Thus, the similar vertices are in classes which are called orbits. We call the subset \(S \subseteq V(G)\) invariant under \(\text{Aut}(G)\), if \(\{\theta(S) | \theta \in \text{Aut}(G)\} = S\). If \(H\) is a subgraph of a graph \(G\), \(G \setminus V(H)\) is the induced subgraph on \(V(G) - V(H)\).

**Definition 1.** A proper induced subgraph \(H\) of a graph \(G\) is an anchor, if it either
a) is unique, i.e. occurs exactly once in \(G\), or
b) is not necessarily unique due to its structure, but, it is unique due to the way of its connection to \(G \setminus V(H)\) and this distinct connection is distinguishable from the deck.

We call the former an structural anchor and the latter a connective anchor.

**Example 1:** The anchors of some graphs are shown in Fig. 1. There is just one subgraph isomorphic to \(K_3\) in graph (a) and clearly it is an anchor. In graph (b), there are four copies of \(K_3\). But, just one of them has 6 edges in connection to the rest of the graph and this fact inferable from the deck. In any card that such \(K_3\) exists, at most one of these 6 edges is omitted and 5 edges are remained, while other \(K_3\) have only 2 edges in connection to the rest of the graph. Thus, the specified \(K_3\) is distinguished in any card that it exists, and is a connective anchor.

**Example 2:** Let \(T\) be an arbitrary tree and \(H\) be the induced subgraph on all its vertices which are not leaves. We show \(H\) is a connective anchor. We know that it is unique due to the number of edges between \(H\) and \(T \setminus V(H)\) which is maximum possible value. Thus, it is sufficient to show that it is distinguishable in any card containing it. Clearly, \(H\) just exits in any card
that a leaf is deleted from. Assume $T$ has $k$ leaves. Thus, there is $k - 1$ or $k$
leaves in any card that a leaf is deleted from. If there are $k - 1$ leaves, $H$
can be obtained by deleting all $k - 1$ leaves and if there are $k$ leaves, the deletion
of any subset of $k - 1$ leaves which leads to a subgraph isomorphic to $H$, gives
$H$. Because, if the deletion of two leaves of a tree results in two isomorphic
subgraph, then those two leaves are similar. Therefore, $H$ is distinguishable
in any card containing it.

Unique subgraph, also, has been defined by Entringer and Erdős [7] and
used by Harary and Schwenk [12]. They have used the concept of unique
subgraph as the spanning subgraphs which are unique. Here, in contrast, we
deal with proper vertex induced subgraph.

An anchor is a unique subgraph and, therefore, is distinguished in any
card containing it. Therefore, an anchor in a graph, like a real anchor which
fixes a boat, fixes a part of some cards and makes it possible to compare
them.

**Definition 2.** Let $H$ be an anchor of a graph $G$ and $v$ a vertex out of the
anchor. We call $N(v) \cap V(H)$ as the shadow of $v$ on $H$ and denote it by
$s_v$. Let $S$ be the set of shadows that vertices out of the anchor $H$ make on
it. Let $A, B \subseteq S$. We define $A$ and $B$ are isomorphic shadows, if there is
$\theta \in Aut(H)$ such that $A = \theta(B)$. We call $A \subseteq S$ is an anchor of $S$, if there
is not any $A' \subset S$ which is isomorphic to $A$.

For illustration, Fig. 2 shows a sample set of five shadows $S = \{s_1, s_2, s_3, s_4, s_5\}$
on a sample graph. $\{s_1, s_4\}$ is a subset of this shadows set which is isomorphic
to $\{s_2, s_5\}$. $\{s_2, s_3, s_5\}$ is an anchor for this shadows set.
3 Graph Anchor as a tool for reconstruction

How an anchor of a graph $G$ helps us in the reconstruction of $G$ from its deck? Let $H$ be an anchor of $G$. The subgraph $H$ is unique and, consequently, is the same in any card containing a copy of $H$. Thus, there is a one to one mapping between the vertices of $H$ in any card containing $H$. An anchor provides a partial correspondence between the vertices of cards including $H$. We use such partial correspondence to reconstruct a graph from its deck. Using an anchor of a graph the reconstruction of that graph reduces to a smaller form of the reconstruction. In [8], it is shown that how using an anchor of a graph $G$, the reconstruction of $G$ reduces to the reconstruction of a shadow graph with smaller order. For simplicity, the concept of shadow graph is omitted, here.

Lemma 1. If a graph $G$ has an anchor, then it is distinguishable in any card containing it.

Proof: According to Kelly’s Lemma, the number of occurrence of any induced subgraph $H$ in $G$ on at most $(n-1)$ vertices is reconstructible from the deck of $G$. Therefore, the anchors of a graph can be obtained from the deck. Since any anchor is a unique subgraph, it is distinguishable and the same in any card containing it. A connective anchor is distinguishable in the deck, according to its definition.

Theorem 1.A. If the induced subgraph on the neighbors (non-neighbors) of a vertex $v$ in a graph $G$ is an anchor, then graph $G$ is reconstructible.

Proof: Let $H$ be the induced subgraph on the neighbors of a vertex $v$ which is unique (anchor). If $\deg(v) = n-1$, then $v$ is an isolated vertex of $\overline{G}$. Thus, $\overline{G}$ and $G$ are reconstructible due to being disconnected. Thus, suppose
that \( \text{deg}(v) \leq (n - 2) \). Therefore, \(|V(H)| \leq (n - 2)\) and \( H \) is present in at least two cards. The anchor \( H \) is distinguishable in any card containing it. We gather all such cards. If we omit \( H \) from these cards, we have the deck of \( G \setminus V(H) \). Using them, we find out that there is at least one isolated vertex, say \( x \), in \( G \setminus V(H) \). In a card that \( x \) is deleted from, we add \( x \) without any neighbor in \( G \setminus V(H) \). But, the number of edges of the graph imposes \( x \) to be adjacent to all vertices of \( H \). Thus, \( G \) is reconstructible. \( \diamond \)

**Theorem 1.B.** Let \( G \) be an \( n \)-vertex graph with an asymmetric anchor \( H \) such that \( V(H) < (n - 2) \). If no pairs of vertices in \( V(G) \setminus V(H) \) have the same neighbors in \( H \), then \( G \) is reconstructible.

*Proof:* The anchor \( H \) is distinguishable in any card containing \( H \). The subgraph \( H \) is asymmetric and no pairs of vertices in \( V(G) \setminus V(H) \) have the same neighbors in \( H \). Thus, any vertex out of the anchor \( H \) finds a specific label by its shadow on \( H \). Having specific label, the adjacency of any pair of vertices which are out of the anchor can be recognized from the cards containing the anchor \( H \). \( \diamond \)

In [9], it is shown that almost every graph satisfies in the condition of Theorem 1.B. In this section, we saw that how graph anchor helps us for reconstruction of some graphs. In the next section, we try to reconstruct any non-regular graph from its deck using graph anchor. Even though, just cards containing an anchor of a graph are used for the reconstruction, but all cards are needed to recognize an anchor.

## 4 Graph anchor and a general framework

We know that an anchor provides a partial correspondence between the vertices of cards and helps us in reconstruction from the deck. Therefore, it is favorite to have a larger anchor and, consequently, more information about the correspondence of cards vertices. Here, the concept of anchor extension is introduced to have a larger anchor.

**Definition 3.** Let \( H_0 \) be an anchor of \( G \). If we can find an anchor \( H_1 \) such that \( V(H_0) \subset V(H_1) \), we call anchor \( H_0 \) is extended to \( H_1 \).

Next Lemma shows how we can extend an anchor of a graph.

**Lemma 2.** Let \( H \) be an arbitrary anchor of a graph \( G \).
a) If $H'$ is an anchor of $G \setminus V(H)$, then the induced subgraph on $V(H) \cup V(H')$ is an anchor of $G$.

b) Let $S$ be the shadows set of $V(G) - V(H)$ on the anchor $H$ and $A \subset V(G) - V(H)$. If the shadows of $A$ on $H$ make an anchor for $S$, then the induced subgraph on $V(H) \cup A$ is an anchor of $G$.

Proof: Let $A$ be a subset of $V(G) - V(H)$ where the induced subgraph on $A$ is an anchor of $G \setminus V(H)$ or the shadows of $A$ is an anchor of $S$. Suppose that the induced subgraph on $V(H) \cup A$ is not an anchor of $G$. Thus, there is $B \subset V(G) - V(H)$ that the induced subgraph on $V(H) \cup A$ and $V(H) \cup B$ are isomorphic by mapping $\phi : V(H) \cup A \to V(H) \cup B$. Since, $H$ is an anchor and unique, $\phi(H) = H$ and $\phi(A) = B$. Thus, the restriction of $\phi$ to $V(H)$ is an automorphism of $H$. Therefore, the induced subgraph on $A$ and its shadow is isomorphic to the induced subgraph on $B$ and its shadow, contradicting to $A$ be an anchor of $G \setminus V(H)$ or its shadow be an anchor for the shadows of $G \setminus V(H)$ on $H$. \hfill \Box

Lemma 3. A unique subgraph either includes or excludes both of two similar vertices.

Proof: If some vertices of one orbit are in an anchor while other vertices are not, the action of the automorphism group makes another copy of the anchor, which contradicts the anchor definition. Therefore, an anchor has or has not one orbit, entirely. \hfill \Box

Corollary 1. An anchor of a graph can be extended at most to reach an orbit out of the anchor. In other words, if the vertices which are out of the anchor belong to the same orbit, then the anchor can not be extended.

Please note that we do not know determinedly whether any anchor extension leads to an orbit out of the anchor or not. But, we know that if there is just one orbit out of the anchor, the anchor can not be extended. In the following, we show that if the anchor extension of a graph terminates with an orbit out of the anchor with more than two vertices, then that graph is reconstructible.

Theorem 2. Let $O$ be an orbit of a graph $G$ with at least three vertices. If $G \setminus O$ is an anchor, then $G$ is reconstructible.
Proof: We gather all cards containing the anchor $G \backslash O$ and, then, distinguish the anchor $G \backslash O$ in all of them. Since all vertices which are out of the anchor are similar, all of these cards are the same. Let $v$ be a vertex in $O$. Since the induced subgraph on $O$ is vertex-transitive, the neighbors of $v$ in $O$ is recognizable by regularity. It is sufficient to find the shadow of $v$ on the anchor $G \backslash O$, i.e. $N(v) \cap V(G \backslash O)$. Since $|O| > 2$, there are at least two different vertices, say $w, w'$, out of the anchor in a card. Since they belong to the same orbit, there exists $\alpha \in Aut(G \backslash O)$ such that $\alpha(s_w) = s_{w'}$. We assign a digraph to the shadows of $O$ on $G \backslash O$. For any $s_w, s_{w'}$ of shadows on $G \backslash O$ such that $s_{w'} = \alpha(s_w)$, we draw an arc from $s_w$ to $s_{w'}$. The resulted digraph is vertex transitive. Because, all cards are the same. Thus, it is reconstructible due to regularity. Therefore, in card $G \backslash \{v\}$, $s_v$ is $\alpha(s_w)$ for a shadow $s_w$ on $G \backslash O$. ∘

![Graph G with orbit O and anchor G\O is reconstructible.](image)

Figure 3: Graph $G$ with orbit $O$ and anchor $G \backslash O$ is reconstructible.

Lemma 4.

a) Any graph, which is not vertex-transitive, has a proper induced subgraph which is unique due to either its structure or the way of its connection to the rest of the graph.

b) Any non-regular graph has a proper induced subgraph which is unique due to either its structure or the way of its connection to the rest of the graph.

Proof: a) Let $G$ be a graph which is not vertex-transitive and $O$ be an arbitrary orbit of $G$. We show that $G \backslash O$ is unique due to either its structure or the way of its connection to $O$. Let $H$ be $G \backslash O$. If $H$ is a unique subgraph, then it is finished. Thus, suppose that there is another copy of $H$ in graph $G$, say $H'$. If $G \backslash V(H)$ and $G \backslash V(H')$ are non-isomorphic, then clearly the connection of $H$ and $H'$ to the rest of the graph are different. If $G \backslash V(H)$
and $G\backslash V(H')$ are isomorphic and the connection of $H$ to $G\backslash V(H)$ and $H'$ to $G\backslash V(H')$ are the same, then the vertices in $G\backslash V(H)$, i.e. $O$, are similar to vertices of $G\backslash V(H')$, while it contradicts to the assumption that $O$ is an orbit. Because, an orbit includes all similar vertices of a graph.

b) Let $H$ be the induced subgraph on the vertices with the maximum degree (or the minimum degree). If $H$ is a unique subgraph, then it is finished. Otherwise, since $2E(H) + E(H, G\backslash V(H))$, i.e. $\Sigma_{v \in V(H)}deg(v)$, is maximum possible value, $H$ is unique due to the way of its connection to $G\backslash V(H)$.

Remark 3. Please note that a subgraph which is unique due to its connection to the rest of the graph, is not a connective anchor, necessarily. This subgraph is a connective anchor, if it is distinguishable in the deck. In contrast, a subgraph which is unique due to its structure is, always, distinguishable in the deck.

By anchor extension, we, usually, reach to either an anchor whose removal leaves an orbit or an anchor with $(n-2)$ vertices. In this section, the former was studied, i.e. it is proved that if the anchor extension of a graph $G$ leads to an orbit with at least three vertices, then $G$ is reconstructible. Next section, we study the reconstruction of graphs with an $(n-2)$-vertex anchor.

5 Graphs with $(n-2)$-vertex anchor

The reconstruction of graphs with an $(n-2)$-vertex anchor is important. Because, almost every graph has a $(n-2)$-vertex anchor. In addition, the anchor extension usually results in either an orbit or two vertices out of the anchor. We saw that if we reach to an orbit with more than two vertices, then the graph is reconstructible. Here, we study the cases in which we reach to just two vertices out of the anchor. Thus, we deal with the reconstruction of $n$-vertex graphs with an $(n-2)$-vertex anchor, here.

Lemma 5. Almost every graph has an asymmetric anchor with $(n-2)$ vertices.

Proof: According to [18, 24, 2], we know that in almost every $n$-vertex graph all $(n-3)$-vertex subgraphs are mutually non-isomorphic. In such graphs, any $(n-2)$-vertex subgraph is unique and asymmetric.

Chinn [6] has shown if there exists a vertex $v$ such that all $(n-2)$-vertex subgraphs of $G\backslash \{v\}$ are unique, graph $G$ is reconstructible. Zhu [32] has...
improved this result by showing that at most three of \((n-2)\)-vertex subgraphs of \(G\setminus\{v\}\) can be non-unique. Here, we show that one unique \((n-2)\)-vertex subgraph which is asymmetric, is enough for \(G\) to be reconstructible.

**Theorem 4.** Let \(G\) be a graph with anchor \(H = G\setminus\{v,u\}\). If the neighbors of \(v\) in \(H\), that is \(s_v\), is invariant under \(\text{Aut}(H)\), then \(G\) is reconstructible.

**Proof:** The anchor \(H\) is distinguished in the cards \(G\setminus\{v\}\) and \(G\setminus\{u\}\). Thus, we find the neighbors of \(v\) in \(H\), i.e. \(s_v\), using the card \(G\setminus\{u\}\). Since, \(s_v\) is invariant under \(\text{Aut}(H)\), there is just one way to add \(v\) to \(H\) in the card \(G\setminus\{v\}\). The existence of edge between \(v\) and \(u\) can be inferred from the number of edges which is reconstructible.

**Corollary 2.** Any \(n\)-vertex graph with an asymmetric anchor of order \((n-2)\) is reconstructible.

**Proof:** In the proof of Theorem 4, if we assume that \(H\) is asymmetric, then \(\text{Aut}(H)=I\) and, consequently, any subset of \(H\) is invariant under \(\text{Aut}(H)\).

**Corollary 3.** Almost every graph is reconstructible.

**Proof:** Using Lemma 5 and Corollary 2.

**Lemma 6.** Let \(O\) be an orbit of a graph \(G\) with two vertices such that \(G\setminus\{v\}\) is an anchor and the order of any element of \(\text{Aut}(G\setminus\{v\})\) is at most three, then graph \(G\) is reconstructible.

**Proof:** Let \(O = \{v_1, v_2\}\) and \(\{s_{v_1}, s_{v_2}\}\) be the shadows of \(\{v_1, v_2\}\) on \(G\setminus\{v\}\). Since, \(v_1\) and \(v_2\) are similar in \(G\), there is \(\theta \in \text{Aut}(G\setminus\{v\})\) where \(s_{v_1} = \theta(s_{v_2})\). Since \(\text{ord}(\theta) = 1, 2\) or \(3\), there are, exactly, two possibilities for \(s_{v_1}\) and \(s_{v_2}\): they are the same, or different. The degree sequence of the graph vertices which is reconstructible by the deck, separates these two cases and chooses one of them.

In a graph \(G\) with anchor \(H = G\setminus\{v,w\}\), let \(s_v\) and \(s_w\) be the neighbors of \(v\) and \(w\) in \(H\), respectively. \(s_v\) and \(s_w\) are recognizable from cards \(G\setminus\{w\}\) and \(G\setminus\{v\}\), respectively. But, \(s_v\) and \(s_w\) can arbitrary move on \(H\) by the action of \(\text{Aut}(H)\). Thus, it may make different possibilities for the state of placing both of them together and, consequently, two cards containing the anchor are not sufficient to reconstruct the graph.
For example, in Fig. 4, two non-isomorphic graphs (a) and (b) are shown. The subgraph $C_5$ is an anchor for both of them. The set of cards containing the anchor are the same and is shown in the right side. The set of cards containing the anchor is not sufficient to discriminate these two non-isomorphic graphs. Therefore, it is necessary to use other cards to find the relative position of two shadows on the anchor. We do not know whether the other $(n - 2)$ cards are, always, sufficient to determine the position of two shadows on the anchor, together. The reconstruction conjecture claims that they are sufficient.

![Figure 4: $s_u$ is isomorphic to $s_{u'}$ and $s_v$ is isomorphic to $s_{v'}$. But, the way of putting these two shadows together on the anchor $C_5$ makes two non-isomorphic graphs. The set of cards containing the anchor $C_5$ is not sufficient to separate these two non-isomorphic graphs.](image)

The following theorem uses the distance of vertices $v$ and $w$ to identify the state of placing both of their neighbors together, when the automorphism group of the anchor is not trivial. For example, in Fig. 4 the distance between $v$ and $w$ in two graphs discriminates these two graphs. Thus, these two graphs are reconstructible due to the next lemma.

**Lemma 7.** Let $G$ be a graph with anchor $H = G\{v,w\}$. If the distance of $v$ and $w$ within $H$ specifies the state of placing both of the neighbors of $v$ and $w$ together in $H$, then graph $G$ is reconstructible.

**Proof:** The neighbors of $w$ and $v$ in $H$ are distinguishable in cards $G\backslash v$ and $G\backslash w$, respectively, up to isomorphism. It is sufficient to know the position of them when they come together. According to the hypothesis, the distance of $v$ and $w$ within $H$ clarifies the relative position of the neighbors of $v$ and $w$ in $H$. Thus, it is sufficient to show the distance of $v$ and $w$ within $H$ is reconstructible.

The number of subgraphs containing $v$ and the number of subgraphs containing $w$ can be obtained from the cards $G\backslash w$ and $G\backslash v$, respectively. In addition,
the subgraphs which include none of them are, exactly, the subgraphs of \( H \)
and, thus, their numbers is reconstructible. Therefore, the number of sub-
graphs which include both \( v \) and \( w \) are reconstructible. The smallest path
which includes \( v \) and \( w \) indicates the distance of \( v \) and \( w \) in \( H \) when \( v \) and
\( w \) are not adjacent in \( G \). If \( v \) and \( w \) are adjacent in \( G \), we consider the
smallest cycle including \( v \) and \( w \). Thus, the distance of \( v \) and \( w \) within \( H \) is
reconstructible. \( \diamond \)

6 An application

As an application of the suggested framework, we will use it to show that
trees are reconstructible. First time, Kelly [13], in his PhD thesis, proved that
trees are reconstructible. His proof is relatively long. Bondy and Hemminger
[5] have proposed another proof for this fact by employing counting theorem.
Here, a new proof is given for trees to be reconstructible.
Before, we show that an arbitrary \( n \)-vertex tree with \((n - 2)\) vertex anchor
is reconstructible, provided that the vertices which are out of the anchor are
tree leaves.

Lemma 8. Let \( T \) be a tree with anchor \( H = T \setminus \{v, w\} \) where \( v \) and \( w \) are
two leaves of \( T \), then \( T \) is reconstructible.

Proof: Let \( s_v \) and \( s_w \) be the shadows of \( v \) and \( w \) on \( H \). We show that
the distance between \( s_v \) and \( s_w \) determine the relative position of \( s_v \) and \( s_w \).
Therefore, \( T \) is reconstructible due to Lemma 7.
We show that if \( d(s_v, s_w) = d(s_v, s_w') \), then there is a mapping in \( Aut(H) \)
which preserves \( s_v \) and maps \( s_w \) to \( s_w' \).
Consider \( H \) and let \( P : s_v = v_0, v_1, ..., v_k = s_w \) be the \( s_v \)-\( s_w \) path, \( Q \) the
\( s_v \)-\( s_w' \) path, and \( t \) the largest integer \((0 < t < k - 1)\) for which \( v_t \in V(P) \cap V(Q) \).
Necessarily, \( d(s_w, v_t) = d(s_w', v_t) \). Let \( T_w \) and \( T_{w'} \) be the
components containing \( s_w \) and \( s_{w'} \), respectively. Since \( s_w \) and \( s_{w'} \) belong to
the same orbit of \( H \) and \( v_t \) is in the middle vertex of path \( s_w \)-\( s_{w'} \), due to [27]
there is an automorphism of \( H \) that interchanges the components \( T_w \) and
\( T_{w'} \), while other vertices are fixed. Since the vertex \( s_v \) does not belong to
neither \( T_w \) nor \( T_{w'} \), it is fixed under such mapping. \( \diamond \)

Theorem 5. Trees are reconstructible. [13]
Proof: Let $T$ be an arbitrary tree with $n$ vertices and $H$ be the induced subgraph on all vertices which are not tree leaves. It was shown that $H$ is an anchor. Any tree has at least two leaves. Thus, there are at least two vertices out of the anchor $H$. Now, we enlarge the anchor as far as possible. According to the number of leaves which will be remained out of the anchor, there are three cases:

Case 1: If the anchor extension leads to just one leaf out of the anchor, say $v$, then the orbit, that $v$ belongs to, has just one element. Also, the neighbor of $v$ in $T$, say $n_v$, does not have any similar vertex in $T$. Additionally, we know that the leaves and end-cutvertices of a tree do not have pseudo-similar mate [11, 16]. Therefore, $T\{v\}$ and $T\{n_v\}$ are unique cards in the deck. $T\{n_v\}$ has two components: an isolated vertex and a tree $T'$. In fact, $T'$ is $T\{v\}$ that a leaf, i.e $n_v$, is deleted from. This specification highlights this card in the deck. Since, a leaf of a tree does not admit pseudo-similar mate, there is just one way to add a leaf to $T'$ to reach $T\{v\}$ up to isomorphism. After $n_v$ is added to $T'$ and $T\{v\}$ is obtained, it is sufficient to add $v$ to its neighbor, i.e. $n_v$.

Case 2: If there are two vertices out of the anchor, then $T$ is reconstructible due to Lemma 8.

Case 3: If there are more than two vertices out of the anchor, but it is not possible to extend the anchor, we show that vertices out of the anchor make an orbit of the tree.

Suppose that $H'$ is an anchor which can not be extended while there are more than two vertices out of the anchor. According to Lemma 2, since $H'$ can not be extended all shadows on $H'$ are isomorphic shadows. In addition, since all vertices which are out of $H'$ are tree leaves, the shadow of any vertex out of $H'$ is a vertex of $H'$. Therefore, the shadows are similar vertices in $H'$. Now, we show that if the vertices out of $H'$ are added to $H'$, then they are similar in $T$.

Let $C$ be the central vertex or edge of $H'$. We assign to each vertex $u \in V(T)$, the number of vertices which are out of $H'$ and their path to $C$ pass through $u$. In any depth, all non-zero values should be the same. Otherwise, if there are vertices $\{v_i\}_I$ in a depth of tree whose values are larger than other non-zero values, then it is possible to enlarge the anchor by addition of vertices out of the anchor whose paths to $C$ pass through $\{v_i\}_I$. Therefore, in any depth the non-zero values are the same and the vacant place of a leaf, in a card that a leaf is omitted from, can be inferred from the disequilibrium of the assigned values. In fact, in this case the vertices which are out of the
anchor make an orbit of $T$ and it is, also, reconstructible due to Theorem 2.\end{flushleft}

The application of the obtained results for small graphs is given in Appendix I. Mckay [23] has shown that graphs with up to 10 vertices are reconstructible by providing the deck of graphs and, then, comparison with the deck of graphs which are probably hypomorphic. But, now using the obtained results without establishing the graphs deck and without any comparison, just by determining a suitable anchor, we can prove that a graph is reconstructible from its deck. The application of anchor for the reconstruction of graphs up to 7 vertices are given in appendix I.

7 Conclusion

In this paper, it is shown that:
If a graph $G$ has either an orbit with at least three vertices whose removal leaves an anchor or a $(n - 2)$-vertex anchor in which the relative position of shadows is reconstructible, then $G$ is reconstructible. Therefore, the graph reconstruction conjecture is reduced to verifying the following simple statement:
Any graph has either a $(n - 2)$-vertex anchor in which the relative position of shadows is reconstructible or an orbit with at least three vertices whose removal leaves an anchor.

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Appendix I: Why small graphs are reconstructible?

All graphs $G$ with at most 6 vertices such that $G$ and $G^c$ are not disconnected or regular, are shown in Fig. 5. For any graph, the anchor that makes it reconstructible, is shown by a gray closed curve.
Figure 5: All graphs with at most 6 vertices which are not disconnected or regular (also their complement) with the anchors that prove they are reconstructible.