Classification of Solutions to Reflection Equation of Two-Component Systems

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Abstract

The symmetries, especially those related to the $R$-transformation, of the reflection equation(RE) for two-component systems are analyzed. The classification of solutions to the RE for eight-, six- and seven-vertex type $R$-matrices is given. All solutions can be obtained from those corresponding to the standard $R$-matrices by $K$-transformation. For the free-Fermion models, the boundary matrices have property $trK_+(0) = 0$, and the free-Fermion type $R$-matrix with the same symmetry as that of Baxter type corresponds to the same form of $K_-$-matrix for the Baxter type. We present the Hamiltonians for the open spin systems connected with our solutions. In particular, the boundary Hamiltonian of seven-vertex models was obtained with a generalization to the Sklyanin’s formalism.

1 Introduction

In the framework of quantum inverse scattering method(QISM)[1, 2, 3, 4], Yang-Baxter equation(YBE)

$$R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u)$$

(1.1)

is a sufficient condition for the integrability of systems with periodic boundary condition(BC). Given a solution $R$-matrix to YBE (1.1), we can construct the Lax operator of certain models at suitable representation of $R$, and hence transfer matrix $t(u)$. The YBE ensures that $t(u)$ commutes with each other for different spectrum parameters. So, if we expand $t(u)$ with respect to spectrum parameter $u$, the coefficients are a set of conserved quantities which satisfy Liouville’s criterion of integrability [3, 3].

However, when considering systems on a finite interval with independent boundary conditions at each end, we have to introduce reflection matrices $K_\pm(u)$ to describe such boundary conditions. Sklyanin assumed that $R$-matrix has the following symmetries [8],

Regularity: $R(0) \propto P$;

P-symmetry: $P_{12}R_{12}(u)P_{12} = R_{21}(u) = R_{12}(u)$;

T-symmetry: $R_{12}^{t_2,t_1} = R_{12}(u)$;

Unitarity: $R_{12}(u)R_{21}(-u) \propto id$;

Crossing Unitarity: $R_{12}^{t_1}(-u - 2\eta) \propto id$,

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where \( \eta \) is crossing parameter and \( R_{12} \) is the permutation matrix. \( t_1, t_2 \) denote transposition in space \( V_1 \) and \( V_2 \) respectively. In order that the BC are compatible with integrability, the reflection matrices should obey so-called reflection equations(RE), or boundary Yang-Baxter equations(BYBE)\[^{[9, 8, 10]}\]. The solutions are divided into three cases, each of corresponds to \( K \)-matrices according to the classification of eight- and six-vertex type solutions of both the YBE and the coloured YBE \[^{[19, 20, 21]}\]. It is tedious to solve the reflection equation for every \( R \)-matrix. Fortunately, all those \( R \)-matrices of two-component systems can be obtained by applying particular solution transformation to standard(or called gauge) ones \[^{[22]}\] which satisfy certain initial conditions(let us call solution transformation of \( R \)-matrix as R-transformation for brevity). The word "two-component" means that there exist two states in the systems: particles and antiparticles in field theory, spin up and down in spin system, arrow up and down or right and left in lattice model (see Ref.\[^{[22]}\]). After a detailed study of reflection equation, we can show that there exists a corresponding transformation to the \( K \)-matrix (we call it \( K \)-transformation) to keep RE invariant under \( R \)-transformation. Therefore, we only need to concentrate on \( K \)-matrices for the standard \( R \)-matrices.

In this paper we shall focus our attention on solutions to reflection equations of two-component systems up to \( K \)-transformation. The solutions are divided into three cases, each of corresponds to eight-vertex(8V), six-vertex(6V) and seven-vertex(7V) models and will be discussed in section 2, 3, 4, respectively. For each case, we first analyze the symmetries of the RE, especially those related to the \( R \)-transformations, and then find solutions to the RE for the Baxter type and free-Fermion type standard \( R \)-matrices, respectively. We put emphasis on new solutions, but for completeness, we also give the solutions obtained by others.

In section 5, for those solutions given in previous sections, we shall construct the corresponding local Hamiltonian of the open spin-chain. The local Hamiltonian means that it only consists of nearest-neighbor interaction terms. A system with such Hamiltonian can be viewed as having coupling with...
magnetic field on its ends. Finally we shall argue that for all boundary conditions to the free-fermion models, the reflection matrices $K_+(u)$ have property $trK_+(0) = 0$. This property requires us to derive the Hamiltonian from the second derivative of the transfer matrix \[16\]. In section 6, we make some remarks and discussions.

2 $K_-$-Matrices to Eight-Vertex Model

In this section, we shall first study the symmetries of RE and give the $K$-transformation corresponding to $R$-transformation in \[20\]. With these discussions, we can concentrate our attention on the RE for the standard 8V $R$-matrices, which are divided into three types: Baxter-type (or XYZ spin-chain[3]), Free-Fermion type I (or XY model[23, 25]), and Free-Fermion type II. All $K_-$-matrices associated with these $R$-matrices are given.

2.1 Symmetries of Reflection Equation

The general eight-vertex $R$-matrix and the corresponding $K_-$-transformation are expressed in the following forms respectively,

\[
R(u) = \begin{pmatrix}
\omega_1(u) & 0 & 0 & \omega_7(u) \\
0 & \omega_2(u) & \omega_5(u) & 0 \\
0 & \omega_6(u) & \omega_3(u) & 0 \\
\omega_8(u) & 0 & 0 & \omega_4(u)
\end{pmatrix},
\]

(2.1)

\[
K_-(u) = \begin{pmatrix}
a_1(u) & a_2(u) \\
a_3(u) & a_4(u)
\end{pmatrix}.
\]

(2.2)

Assuming that $R(u)$ is a solution to YBE \[1.1\], then as studied in \[20\], there are four symmetries for eight-vertex type $R$-matrix \[2.1\].

(R.A) Symmetry of interchanging indices. If we exchange the elements of $R(u)$ as $\omega_1(u) \leftrightarrow \omega_4(u)$, $\omega_5(u) \leftrightarrow \omega_6(u)$ or $\omega_2(u) \leftrightarrow \omega_3(u)$, $\omega_7(u) \leftrightarrow \omega_8(u)$, the new matrix also satisfies YBE \[1.1\].

(R.B) The scaling symmetry. Multiplication of $R(u)$ by an arbitrary function $f(u)$ is still a solution to YBE \[1.1\].

(R.C) Symmetry of spectral parameter. If we take a new spectral parameter $\bar{u} = \lambda u$, where $\lambda$ is an constant complex number, the new matrix $R(\bar{u})$ is still a solution to YBE \[1.1\].

(R.D) Symmetry of weight functions. If we replace weight functions $\omega_7(u), \omega_8(u)$ by the new ones

\[
\tilde{\omega}_7(u) = s^{-1} w_7(u), \quad \tilde{\omega}_8(u) = sw_8(u),
\]

(2.3)

where $s$ is a non-zero complex constant, the new matrix is still a solution to YBE \[1.1\].

The symmetries (R.A)-(R.D) are called solution transformations (or $R$-transformation) of 8V type solution of YBE \[1.1\].

It is convenient for later discussion to use such notation as follows

\[
\omega_i(u - v) = u_i, \quad \omega_i(u + v) = v_i,
\]

\[
a_i(u) = x_i, \quad a_i(v) = y_i.
\]

Substituting the matrices $R$ and $K_-$ into the reflection equation \[1.2\], we get sixteen component equations, which are divided into groups according to symmetries of the indices:

\[
\omega_1(u - v) = u_1, \quad a_1(u) = x_1.
\]

\[
\omega_2(u - v) = u_2, \quad a_2(u) = x_2.
\]

\[
\omega_3(u - v) = u_3, \quad a_3(u) = x_3.
\]

\[
\omega_4(u - v) = u_4, \quad a_4(u) = x_4.
\]

\[
\omega_5(u - v) = u_5, \quad a_5(u) = x_5.
\]

\[
\omega_6(u - v) = u_6, \quad a_6(u) = x_6.
\]

\[
\omega_7(u - v) = u_7, \quad a_7(u) = x_7.
\]

\[
\omega_8(u - v) = u_8, \quad a_8(u) = x_8.
\]

\[
\omega_{11}(u - v) = u_{11}, \quad a_{11}(u) = x_{11}.
\]

\[
\omega_{12}(u - v) = u_{12}, \quad a_{12}(u) = x_{12}.
\]

\[
\omega_{13}(u - v) = u_{13}, \quad a_{13}(u) = x_{13}.
\]

\[
\omega_{14}(u - v) = u_{14}, \quad a_{14}(u) = x_{14}.
\]

\[
\omega_{22}(u - v) = u_{22}, \quad a_{22}(u) = x_{22}.
\]

\[
\omega_{23}(u - v) = u_{23}, \quad a_{23}(u) = x_{23}.
\]

\[
\omega_{24}(u - v) = u_{24}, \quad a_{24}(u) = x_{24}.
\]

\[
\omega_{33}(u - v) = u_{33}, \quad a_{33}(u) = x_{33}.
\]

\[
\omega_{34}(u - v) = u_{34}, \quad a_{34}(u) = x_{34}.
\]

\[
\omega_{44}(u - v) = u_{44}, \quad a_{44}(u) = x_{44}.
\]

\[
\omega_{55}(u - v) = u_{55}, \quad a_{55}(u) = x_{55}.
\]

\[
\omega_{56}(u - v) = u_{56}, \quad a_{56}(u) = x_{56}.
\]

\[
\omega_{57}(u - v) = u_{57}, \quad a_{57}(u) = x_{57}.
\]

\[
\omega_{58}(u - v) = u_{58}, \quad a_{58}(u) = x_{58}.
\]

\[
\omega_{66}(u - v) = u_{66}, \quad a_{66}(u) = x_{66}.
\]

\[
\omega_{67}(u - v) = u_{67}, \quad a_{67}(u) = x_{67}.
\]

\[
\omega_{68}(u - v) = u_{68}, \quad a_{68}(u) = x_{68}.
\]

\[
\omega_{77}(u - v) = u_{77}, \quad a_{77}(u) = x_{77}.
\]

\[
\omega_{78}(u - v) = u_{78}, \quad a_{78}(u) = x_{78}.
\]

\[
\omega_{88}(u - v) = u_{88}, \quad a_{88}(u) = x_{88}.
\]

(2.4)
\begin{align}
\begin{cases}
(u_7v_8 - u_6v_7)x_1y_1 - u_8v_8x_2y_2 + u_7v_3x_3y_3 + u_4v_6(x_2y_3 - x_3y_2) = 0, \\
(u_7v_8 - u_8v_7)x_1y_1 - u_8v_2x_2y_2 + u_7v_3x_3y_3 + u_1v_5(x_2y_3 - x_3y_2) = 0,
\end{cases}
\end{align} \tag{A.1}

\begin{align}
\begin{cases}
(u_2v_1 - u_3v_2)x_1y_1 + u_2v_8x_2y_2 - u_3v_7x_3y_3 + u_5v_4(x_2y_3 - x_3y_2) = 0, \\
(u_2v_1 - u_3v_2)x_1y_1 + u_2v_2x_2y_2 - u_3v_7x_3y_3 + u_6v_4(x_2y_3 - x_3y_2) = 0,
\end{cases}
\end{align} \tag{A.2}

\begin{align}
\begin{cases}
u_1v_1x_1y_1 - u_7v_4x_4y_4 - u_1v_7x_1y_1 + u_4v_7x_4y_4 + u_7(v_5 - v_6)x_3y_2 = 0, \\
u_1v_2x_1y_1 - u_8v_4x_4y_4 + u_8v_1x_1y_1 + u_4v_8x_4y_4 + u_8(v_5 - v_6)x_3y_3 = 0,
\end{cases}
\end{align} \tag{A.3}

\begin{align}
\begin{cases}
u_3v_5x_1y_1 - u_3v_5x_4y_4 - u_5v_3x_1y_1 + u_5u_6v_8x_2y_2 + u_3(v_4 - v_1)x_3y_2 = 0, \\
u_2v_5x_1y_1 - u_2v_5x_4y_4 - u_6v_2x_1y_4 + u_5v_2x_3y_3 + u_2(v_4 - v_1)x_3y_3 = 0,
\end{cases}
\end{align} \tag{A.4}

\begin{align}
\begin{cases}
u_1v_1 - u_3v_2)x_1y_2 + (u_7v_8 - u_5v_5)x_4y_2 - u_5v_1x_2y_1 + u_1v_5x_3y_1 + u_7v_3x_3y_1 + u_7v_3x_3y_1 = 0, \\
u_1v_2 - u_3v_2)x_1y_2 + (u_8v_7 - u_5v_5)x_4y_2 - u_5v_1x_3y_1 + u_1v_5x_3y_1 + u_8v_2x_2y_1 + u_8v_2x_2y_1 = 0,
\end{cases}
\end{align} \tag{A.5}

\begin{align}
\begin{cases}
u_6v_2x_1y_2 - u_1v_7x_1y_2 + u_2v_5x_1y_2 - u_7v_4x_4y_3 + (u_2v_1 - u_1v_2)x_2y_1 + (u_6v_2 - u_7v_6)x_3y_1 = 0, \\
u_6v_3x_1y_3 - u_1v_8x_1y_2 + u_2v_5x_3y_3 - u_8v_4x_4y_3 + (u_2v_1 - u_1v_3)x_3y_1 + (u_6v_6 - u_8v_6)x_2y_3 = 0,
\end{cases}
\end{align} \tag{A.6}

After a careful study of the above equations, we find that if one apply the following transformations to $K_-(u)$ under the transformations (R.A)-(R.D), the system of equations (A) keeps invariant:

(K.A) **The symmetry of interchanging indices.** This symmetry will be discussed for each type of $R$-matrix later.

(K.B) **The scalar symmetry.** If we multiply $K_-(u)$ by an arbitrary function $g(u)$, the new matrix $g(u)K_-(u)$ is still a solution to RE. On the other hand, all the $R$-matrices up to an arbitrary scalar function have the same reflection matrix.

(K.C) **The symmetry of spectral parameters.** If we take a new spectral parameter $\bar{u} = \lambda u$, where $\lambda$ is any constant, the new matrix $K(\bar{u})$ also satisfies RE for $R(\bar{u})$.

(K.D) **The symmetry of weight function.** If applying the transformation (R.D) to $R(u)$, we can make a corresponding $K$-transformation on $K_-(u)$:

$$\bar{a}_3(u) = \sqrt{\bar{s}}a_3(u), \bar{a}_2(u) = \sqrt{s}^{-1}a_2(u)$$ \tag{2.4}

keeping $a_1(u), a_4(u)$ unchanged. The new $K_-(u)$ matrix is a also solution to RE for new $R$-matrix.

Considering the above symmetries for $R$-matrix and $K_-$-matrix, we can concentrate on the standard $R$-matrix with the restrictions \[20\]

$$\omega_5(u) = \omega_6(u) = 1, \; \omega_7(u) = \omega_8(u),$$ \tag{2.5}

and initial condition

$$R_{12}(0) = P_{12}.$$ \tag{2.6}

Note that from condition \[2.3\], we only need consider $R$-transformation (R.A) of interchanging indices $\omega_1 \leftrightarrow \omega_2$ and $\omega_3 \leftrightarrow \omega_4$ hereafter. All $R$-matrices are classified into two classes: Baxter type and free-fermion type, according to whether or not the elements of $R$-matrix satisfy the free-fermion condition \[23\; 24\; 19\]

$$\omega_1(u)\omega_4(u) + \omega_2(u)\omega_3(u) - \omega_5(u)\omega_6(u) - \omega_7(u)\omega_8(u) = 0.$$ \tag{2.7}

The RE corresponding to these two kinds of $R$-matrices has very different properties. We shall discuss solutions to RE for these gauge $R$-matrices respectively.
2.2 Baxter Type

The gauge R-matrix of Baxter-type was first derived by Baxter [5], it has the following parametrization,

\[
\begin{align*}
\omega_1(u) &= \omega_4(u) = \text{sn}(u + h)/\text{sn}h, \\
\omega_2(u) &= \omega_3(u) = \text{sn}u/\text{sn}h, \\
\omega_5(u) &= \omega_6(u) = 1, \\
\omega_7(u) &= \omega_8(u) = k \text{ sn}u(\text{sn}u + h),
\end{align*}
\]

where snu, cnu, dnu are Jacobi elliptic functions of modulus k. It is a high-symmetric one with \(\omega_1(u) = \omega_4(u), w_2(u) = w_3(u)\), so the transformation of interchanging indices (R.A) has no effect in this case.

The \(K_\pm\)-matrices in this case have been widely discussed in [13, 14, 15, 18]. For completeness, we list here main results. The most general one was given in [15, 18] as follows

\[
K_-(u) = \begin{pmatrix} \nu \text{ sn}(\alpha - u) & \mu \text{ sn}(2u) \\ \mu \text{ sn}(2u) & \nu \text{ sn}(\alpha + u) \end{pmatrix} \frac{\lambda(1-k^2\text{ sn}^2u)+1+k\text{ sn}^2u}{1-k^2\text{ sn}^2u} \frac{\lambda(1-k^2\text{ sn}^2u)+1+k\text{ sn}^2u}{1-k^2\text{ sn}^2u},
\]

where \(\mu, \nu, \lambda, \alpha\) are free parameters, and the other special solutions can be obtained by setting these parameters to take special values.

2.3 Free-Fermion Type I

The R-matrix of free fermion type-I is less symmetric than that of Baxter type. In this case, \(w_2(u) = w_3(u), \text{ but } w_1(u) \neq w_4(u)\). The reflection equation is equivalent to five systems of equations:

\[
\begin{align*}
\{ & u_2v_7(x_2y_2 - x_3y_3) + u_5v_1(x_3y_2 - x_2y_3) = 0, \\
& u_2v_7(x_2y_2 - x_3y_3) + u_5v_1(x_3y_2 - x_2y_3) = 0, \\
& u_7v_2(x_2y_2 - x_3y_3) + u_1v_5(x_3y_2 - x_2y_3) = 0, \\
& u_7v_2(x_2y_2 - x_3y_3) + u_4v_5(x_3y_2 - x_2y_3) = 0,
\end{align*}
\]

\[
\begin{align*}
& u_7v_1x_1y_1 - u_1v_7x_1y_1 + u_4v_7x_4y_1 - u_7v_4x_4y_1 + (u_4 - u_1)v_2x_2y_2 = 0, \\
& u_7v_1x_1y_1 - u_1v_7x_1y_1 + u_4v_7x_4y_1 - u_7v_4x_4y_1 + (u_4 - u_1)v_2x_3y_3 = 0, \\
& u_2v_5(x_1y_1 - x_4y_4) + u_5v_2(x_4y_1 - x_1y_4) + u_2v_4 - v_1x_2y_2 = 0, \\
& u_2v_5(x_1y_1 - x_4y_4) + u_5v_2(x_4y_1 - x_1y_4) + u_2v_4 - v_1x_3y_3 = 0,
\end{align*}
\]

\[
\begin{align*}
& u_5v_2x_1y_2 - u_1v_7x_1y_3 + u_2v_5x_4y_2 - u_7v_4x_4y_3 + (u_2v_1 - u_1v_2)x_2y_1 + (u_5v_7 - u_7v_5)x_3y_1 = 0, \\
& u_5v_2x_1y_2 - u_1v_7x_1y_3 + u_2v_5x_4y_2 - u_7v_4x_4y_3 + (u_2v_1 - u_1v_2)x_3y_1 + (u_5v_7 - u_7v_5)x_2y_1 = 0, \\
& u_5v_2x_4y_2 - u_4v_7x_4y_3 + u_2v_5x_1y_2 - u_7v_1x_1y_3 + (u_2v_4 - u_1v_2)x_2y_1 + (u_5v_7 - u_7v_5)x_3y_4 = 0, \\
& u_5v_2x_4y_2 - u_4v_7x_4y_3 + u_2v_5x_1y_2 - u_7v_1x_1y_3 + (u_2v_4 - u_1v_2)x_2y_1 + (u_5v_7 - u_7v_5)x_3y_4 = 0,
\end{align*}
\]

\[
\begin{align*}
& (u_1v_1 - u_2v_2)x_1y_2 + (u_7v_7 - u_5v_5)x_2y_2 + u_5v_1x_3y_1 + u_1v_5x_3y_4 - u_2v_7x_3y_1 - u_7v_2x_3y_4 = 0, \\
& (u_1v_1 - u_2v_2)x_1y_2 + (u_7v_7 - u_5v_5)x_2y_2 + u_5v_1x_3y_1 + u_1v_5x_3y_4 - u_2v_7x_3y_1 - u_7v_2x_3y_4 = 0, \\
& (u_1v_1 - u_2v_2)x_2y_2 + (u_7v_7 - u_5v_5)x_1y_2 + u_5v_1x_3y_1 + u_1v_5x_3y_4 - u_2v_7x_3y_1 - u_7v_2x_3y_4 = 0, \\
& (u_1v_1 - u_2v_2)x_2y_2 + (u_7v_7 - u_5v_5)x_1y_2 + u_5v_1x_3y_1 + u_1v_5x_3y_4 - u_2v_7x_3y_1 - u_7v_2x_3y_4 = 0.
\end{align*}
\]

There also exist symmetries of interchanging indices. The system of equations (B) is invariant under exchange of \(a_1(u) \leftrightarrow a_4(u)\) and \(\omega_1(u) \leftrightarrow \omega_4(u)\) or \(a_2(u) \leftrightarrow a_3(u)\). The gauge R-matrix is [20],

\[
\begin{align*}
\omega_1(u) &= \text{cn}u + H \text{ sn}u \text{ dnu}, \\
\omega_4(u) &= \text{cn}u - H \text{ sn}u \text{ dnu}, \\
\omega_2(u) &= \omega_3(u) = G \text{ sn}u \text{ dnu}, \\
\omega_5(u) &= \omega_6(u) = \text{dnu}, \\
\omega_7(u) &= \omega_8(u) = k \text{ sn}u \text{ dnu},
\end{align*}
\]

(2.10)
where $G, H$ are arbitrary parameters with relation $G^2 - H^2 = 1$. Note that in [2.10] we do not take $\omega_5(u) = \omega_6(u) = 1$ in order to compare our following discussion with other’s work. We will consider the general $R$-matrix which has $w_1(u) \neq \omega_4(u)$, i.e. $H \neq 0$. The case of $H = 0$ is remarked at the end of this subsection. Now we solve the RE (B) case by case.

**Case 2.3.1: Diagonal solution.** From (B.1), $a_2(u) \equiv 0 \Leftrightarrow a_3(u) \equiv 0$. There are only two equations to be considered,

\[
\begin{align*}
\{ u_2v_5(x_1y_1 - x_4y_4) + u_5v_2(x_4y_1 - x_1y_4) = 0, \\
u_7v_1x_1y_1 - u_1v_7x_1y_4 + u_4v_7x_4y_1 - u_7v_4x_4y_4 = 0. \\
\end{align*}
\]

Introducing new variable $g(u) = a_1(u)/a_4(u)$ and solving $g(u)$ from the above equations, we get a solution

\[
K_{-}(u) = \begin{pmatrix} cnu dnu & \pm ik' \text{sn}u & 0 \\ 0 & cnu dnu & \mp ik' \text{sn}u \end{pmatrix},
\]

where $k'$ is the complementary modulus of elliptic function. Note that the diagonal solution of 8V free-fermion type I has no free parameter, which is different from that of Baxter type.

**Case 2.3.2: Skew-diagonal solution.** If $a_2(u) \neq 0$, we conclude from (B.1) that

\[
a_2(u) = \epsilon a_3(u), \quad \epsilon = \pm 1.
\]

Taking $a_1(u) \equiv 0$, we get from (B.4)

\[
x_4(u_2v_5y_2 - u_7v_4y_3) = 0.
\]

With the help of (2.13) and (2.11), the above equation calls for $a_4(u) \equiv 0$. However, this is contradictory to $a_2(u) \neq 0$ as seen from (B.2). So the skew-diagonal solution does not exist due to less symmetry of $R$-matrix.

**Case 2.3.3: General solution.** Because $(u_1 - u_4)v_2 = (v_1 - v_4)u_2$, the following equation is obtained from (B.2), (B.3) and (2.13)

\[
\begin{align*}
u_7v_1x_1y_1 - u_1v_7x_1y_4 + u_4v_7x_4y_1 - u_7v_4x_4y_4 \\
- \epsilon \{ u_2v_5(x_1y_1 - x_4y_4) + u_5v_2(x_4y_1 - x_1y_4) \} = 0.
\end{align*}
\]

Differentiating the above equation with respect to $v$ and setting $v = 0$, we can express $a_1(u), a_4(u)$ as

\[
\begin{align*}
a_1(u) &= (F(u) \text{cnu dnu} - G(u) \text{snu} p(u)/2, \\
a_4(u) &= (F(u) \text{cnu dnu} + G(u) \text{snu} p(u)/2,
\end{align*}
\]

where

\[
F(u) = c_1 + \frac{k(1 - \epsilon kG)c_1 + Hc_2}{\epsilon G - k} \text{sn}^2 u, \tag{2.15}
\]

\[
E(u) = c_2 + \frac{k(1 - \epsilon kG)c_2 - k^2 Hc_1}{\epsilon G - k} \text{sn}^2 u. \tag{2.16}
\]

and $p(u)$ is a meromorphic function to be determined. Substituting the above expressions into (B.5), we get

\[
a_2(u) \equiv 0, \quad p(u) = \mu \text{sn} \text{cnu} \text{dnu},
\]

and an additional restriction between $c_1$ and $c_2$ from (B.2)

\[
k^2 c_1^2 + c_2^2 = 2\mu^2 (G - \epsilon k)/k.
\]

Therefore, the most general solution is

\[
K_{-}(u) = \begin{pmatrix} F(u) \text{cn}(u) \text{dn}(u) + E(u) \text{sn}(u) & 2\mu \text{sn}(u) \text{cn}(u) \text{dn}(u) \\
2\epsilon \mu \text{sn}(u) \text{cn}(u) \text{dn}(u) & F(u) \text{cn}(u) \text{dn}(u) - E(u) \text{sn}(u) \end{pmatrix}, \tag{2.17}
\]

6
The result in [17] is a specific case of \( \mu = 1 \). It is also easy to find that the diagonal solution (2.13) can be obtained by setting \( \mu = 0 \).

**Remark 2.3.1.** There are in fact various parameterizations of free fermionic 8V \( R \)-matrix, one of which is given in [24]

\[
\begin{aligned}
\omega_1(u) &= 1 - e(u) e(h_1) e(h_2), \\
\omega_2(u) &= e(u) - e(h_1) e(h_2), \\
\omega_3(u) &= e(h_1) - e(u) e(h_2), \\
\omega_4(u) &= e(h_2) - e(u) e(h_1), \\
\omega_5(u) &= \omega_6(u) = \sqrt{e(h_1) e(h_2) \sin(h_2)/(1 - e(u))/\sin(\frac{\omega}{2})}, \\
\omega_7(u) &= \omega_8(u) = -i \sqrt{e(h_1) e(h_2) \sin(h_2)(1 + e(u))/\sin(\frac{\omega}{2})},
\end{aligned}
\]

where \( h_1 \) and \( h_2 \) are colour parameters, and \( e(u) \) is the elliptic exponential:

\[
e(u) = cn(u) + i \text{sn}(u).
\]

If we make transformation (R.B) to (2.18) with factor function

\[
\sqrt{e(h_1) e(h_2) \sin(h_1) \sin(h_2)} \frac{1 - e(u)}{\sin(u)/2}
\]

and set

\[h_1 = h_2 = h, \ u \rightarrow u/2, \ G = \frac{1}{\sin h}, \ H = \frac{\cosh h}{\sin h},\]

the new \( R \)-matrix coincides with (2.18), so our solution includes the diagonal solution given in [16].

**Remark 2.3.2.** Let us consider the special case of \( H = 0 \) and \( G = 1 \). In this case, the \( R \)-matrix is

\[
\begin{aligned}
\omega_1(u) &= \omega_4(u) = cn(u), \\
\omega_2(u) &= \omega_3(u) = sn(u) dn(u), \\
\omega_5(u) &= \omega_6(u) = dn(u), \\
\omega_7(u) &= \omega_8(u) = k \sin(u) \cos(u).
\end{aligned}
\]

It has the same symmetry as Baxter type. The calculation shows that this feature is responsible for the fact that both two \( R \)-matrices share the same \( K_-(u) \) as given in (2.9).

### 2.4 Free-Fermion Type II

This kind of \( R \)-matrix takes the form,

\[
\begin{aligned}
\omega_1(u) &= \omega_4(u) = \frac{\cosh(\lambda u)}{\cos(\mu u)}, \\
\omega_2(u) &= -\omega_3(u) = -\frac{\sinh(\lambda u)}{\cos(\mu u)}, \\
\omega_5(u) &= \omega_6(u) = 1, \\
\omega_7(u) &= \omega_8(u) = \tan(\mu u),
\end{aligned}
\]

where \( \lambda, \mu \) are parameters. The RE in component forms is equivalent to the following twelve equations:

\[
\begin{aligned}
&u_2 v_7 (x_2 y_2 + x_3 y_3) + u_5 v_1 (x_3 y_2 - x_2 y_3) = 0, \\
&u_7 v_2 (x_2 y_2 + x_3 y_3) + u_1 v_5 (x_3 y_2 - x_2 y_3) = 0, \\
&u_1 v_7 (x_1 y_1 - x_4 y_4) + u_4 v_1 (x_4 y_1 - x_1 y_4) = 0, \\
&u_2 v_5 (x_1 y_1 - x_4 y_4) + u_5 v_2 (x_4 y_1 - x_1 y_4) = 0,
\end{aligned}
\]

(C.1)

\[
\begin{aligned}
&u_5 v_2 x_1 y_2 - u_1 v_7 x_1 y_3 + u_2 v_5 x_1 y_2 - u_7 v_1 x_4 y_3 + (u_2 v_1 - u_1 v_2) x_2 y_1 + (u_5 v_7 - u_7 v_5) x_3 y_1 = 0, \\
&u_5 v_2 x_1 y_3 - u_1 v_7 x_1 y_2 + u_2 v_5 x_1 y_3 - u_7 v_1 x_4 y_2 + (u_2 v_1 - u_1 v_2) x_3 y_1 + (u_5 v_7 - u_7 v_5) x_2 y_1 = 0, \\
&u_5 v_2 x_4 y_2 - u_1 v_7 x_4 y_3 + u_2 v_5 x_4 y_2 - u_7 v_1 x_1 y_3 + (u_2 v_1 - u_1 v_2) x_3 y_1 + (u_5 v_7 - u_7 v_5) x_3 y_4 = 0, \\
&u_5 v_2 x_4 y_3 - u_1 v_7 x_4 y_2 + u_2 v_5 x_4 y_3 - u_7 v_1 x_1 y_2 + (u_2 v_1 - u_1 v_2) x_3 y_1 + (u_5 v_7 - u_7 v_5) x_2 y_2 = 0,
\end{aligned}
\]

(C.3)
\[
\begin{align*}
\begin{cases}
(u_1v_1 + u_2v_2)x_1y_2 + (u_7v_7 - u_5v_5)x_4y_2 - u_5v_1x_3y_1 + u_1v_5x_2y_4 + u_2v_7x_3y_1 - u_7v_2x_3y_4 = 0, \\
(u_1v_1 + u_2v_2)x_1y_3 + (u_7v_7 - u_5v_5)x_4y_3 - u_5v_1x_3y_1 + u_1v_5x_2y_4 + u_2v_7x_3y_1 - u_7v_2x_2y_4 = 0, \\
(u_1v_1 + u_2v_2)x_3y_3 + (u_7v_7 - u_5v_5)x_1y_3 - u_5v_1x_3y_1 + u_1v_5x_2y_4 + u_2v_7x_3y_1 - u_7v_2x_2y_4 = 0.
\end{cases}
\end{align*}
\]

We find that under \(R\)-transformation of interchanging \(\omega_2(u)\) and \(\omega_3(u)\) in (2.20), one can perform a \(K\)-transformation as follows

\[
\bar{a}_2(u) = -a_2(u), \quad \text{or} \quad \bar{a}_3(u) = -a_3(u),
\]

(2.21)
to keep the system of equations (C) invariant.

The existence of nontrivial solution implies that there exists relation

\[
\frac{u_7v_2}{u_2v_7} = \frac{u_1v_5}{u_5v_1},
\]

which requires \(\lambda = \pm i\mu\). Thus we should consider two different \(R\)-matrices,

\[
\begin{align*}
\begin{cases}
\omega_1(u) = \omega_4(u) = 1, \\
\omega_2(u) = -\omega_3(u) = i \tan u, \\
\omega_5(u) = \omega_6(u) = 1, \\
\omega_7(u) = \omega_8(u) = \tan u,
\end{cases}
\end{align*}
\]

(2.22)

and

\[
\begin{align*}
\begin{cases}
\omega_1(u) = \omega_4(u) = 1, \\
\omega_2(u) = -\omega_3(u) = -i \tan u, \\
\omega_5(u) = \omega_6(u) = 1, \\
\omega_7(u) = \omega_8(u) = \tan u.
\end{cases}
\end{align*}
\]

(2.23)

They are in fact related each other by an exchange \(\omega_2(u) \leftrightarrow \omega_3(u)\). Let us give solution \(K_-(u)\) directly because the calculation procedure has nothing new. For \(R(u)\) in (2.22), we have

\[
K_-(u) = \begin{pmatrix}
\mu_1(1 + \nu_1 \sin 2u) & i\mu_2(1 + \nu_2 \cos 2u) \sin 2u \\
\mu_2(1 - \nu_2 \cos 2u) \sin 2u & \mu_1(1 - \nu_1 \sin 2u)
\end{pmatrix},
\]

(2.24)

while for \(R(u)\) in (2.23), using \(K\)-transformation (2.21), we have

\[
K_-(u) = \begin{pmatrix}
-\mu_1(1 + \nu_1 \sin 2u) & i\mu_2(1 + \nu_2 \cos 2u) \sin 2u \\
-\mu_2(1 - \nu_2 \cos 2u) \sin 2u & \mu_1(1 - \nu_1 \sin 2u)
\end{pmatrix},
\]

(2.25)

where \(\mu_1, \mu_2, \nu_1, \nu_2\) are free parameters.

3 \(K_\text{-Matrix for Six-Vertex Model}\)

The general six-vertex \(R\)-matrix takes the form

\[
R(u) = \begin{pmatrix}
\omega_1(u) & 0 & 0 & 0 \\
0 & \omega_2(u) & \omega_5(u) & 0 \\
0 & \omega_6(u) & \omega_3(u) & 0 \\
0 & 0 & 0 & \omega_4(u)
\end{pmatrix}.
\]

(3.1)

By setting \(u_7, 8 = 0\) and \(v_7, 8 = 0\) in Eqs.(A), we write down the reflection equations for 6V type \(R\)-matrix in component forms:

\[
\begin{align*}
\begin{cases}
(u_4 - u_1)v_2x_2y_2 = 0, \\
(u_4 - u_1)v_3x_3y_4 = 0,
\end{cases}
\end{align*}
\]

(D.1)
\[ \begin{align*}
&\begin{cases}
  u_1 v_5 (x_2 y_3 - x_3 y_2) = 0, \\
u_4 v_6 (x_2 y_3 - x_3 y_2) = 0, 
\end{cases} \\
&\begin{cases}
  (u_2 v_3 - u_3 v_2) x_1 y_4 + u_5 v_1 (x_3 y_2 - x_2 y_3) = 0, \\
u_2 v_3 - u_3 v_2) x_4 y_1 + u_6 v_4 (x_3 y_2 - x_2 y_3) = 0, 
\end{cases} \\
&\begin{cases}
  u_3 v_6 x_1 y_1 - u_3 v_5 x_1 y_4 + u_6 v_3 x_1 y_4 + u_3 v_4 - v_1 = 0, \\
u_2 v_6 x_1 y_1 - u_2 v_5 x_1 y_4 + u_6 v_2 x_1 y_4 + u_5 v_2 x_4 y_1 + u_2 (v_4 - v_1) x_2 y_3 = 0, 
\end{cases} \\
&\begin{cases}
  u_6 v_2 x_1 y_2 + u_2 v_5 x_4 y_2 + (u_2 v_1 - u_1 v_2) x_2 y_1 = 0, \\
u_6 v_3 x_1 y_3 + u_3 v_5 x_4 y_3 + (u_3 v_1 - u_1 v_3) x_3 y_1 = 0, \\
u_5 v_2 x_4 y_2 + u_2 v_6 x_1 y_2 + (u_2 v_4 - u_4 v_2) x_2 y_4 = 0, \\
u_5 v_3 x_4 y_3 + u_3 v_6 x_1 y_3 + (u_3 v_4 - u_4 v_3) x_3 y_4 = 0, 
\end{cases} \\
&\begin{cases}
  (u_1 v_1 - u_3 v_2) x_1 y_2 - u_5 v_5 x_4 y_2 - u_5 v_1 x_2 y_1 + u_1 v_5 x_2 y_4 = 0, \\
u_1 v_1 - u_3 v_3) x_1 y_3 - u_5 v_5 x_4 y_3 - u_5 v_1 x_3 y_3 + u_1 v_5 x_3 y_4 = 0, \\
u_4 v_4 - u_2 v_2) x_4 y_2 - u_6 v_6 x_1 y_2 = u_6 v_4 x_2 y_4 + u_4 v_6 x_2 y_1 = 0, \\
u_4 v_4 - u_2 v_3) x_4 y_3 - u_6 v_6 x_1 y_3 - u_6 v_4 x_3 y_4 + u_4 v_6 x_3 y_1 = 0. 
\end{cases}
\end{align*} \]

(D.2) (D.3) (D.4) (D.5) (D.6)

From Ref. [23], we know that the 6V type solutions of YBE have the same solution-transformation as that for 8V type solutions except for the symmetries of weight functions and of the interchanging indices related to \( \omega_7(u) \) and \( \omega_8(u) \). Now the two symmetries of weight functions are

\[ \tilde{\omega}_2(u) = s \omega_2(u), \quad \tilde{\omega}_3(u) = s^{-1} \omega_3(u) \]

(3.2)

and

\[ \tilde{\omega}_5(u) = e^{c u} \omega_5(u), \quad \tilde{\omega}_6(u) = e^{-c u} \omega_6(u) \]

(3.3)

where \( s, c \) are two nonzero constants. In fact, we find that transformation (3.2) has no effect on the system of equations (D), and if making \( K \)-transformation

\[ \tilde{a}_1(u) = e^{c u} a_1(u), \quad \tilde{a}_4(u) = e^{-c u} a_4(u) \]

(3.4)

the new \( K_- (u) \) is still a solution to RE for the new \( R \)-matrix obtained from \( R \)-transformation (3.3). Due to these symmetries, we will consider the gauge \( R \)-matrices as in 8V model. They are also classified into two classes, the Baxter type

\[ \begin{align*}
\begin{cases}
  \omega_1(u) = \omega_4(u) = \frac{\sin(u + h)}{\sin h}, \\
  \omega_2(u) = \omega_3(u) = \frac{\sin u}{\sin h}, \\
  \omega_5(u) = \omega_6(u) = 1,
\end{cases}
\end{align*} \]

(3.5)

and the free-Fermion type

\[ \begin{align*}
\begin{cases}
  \omega_1(u) = \frac{\sin(u + h)}{\sin h}, \\
  \omega_4(u) = \frac{\sin(-u + h)}{\sin h}, \\
  \omega_2(u) = \omega_3(u) = \frac{\sin u}{\sin h}, \\
  \omega_5(u) = \omega_6(u) = 1.
\end{cases}
\end{align*} \]

(3.6)

For Baxter-type, the general solution to RE was given in [3]

\[ K_- (u) = \begin{pmatrix}
  \lambda \sin(\alpha - u) & \mu \sin (2u) \\
  \nu \sin (2u) & \lambda \sin (\alpha + u)
\end{pmatrix}, \]

(3.7)

which has four free parameters \( \lambda, \alpha, \mu, \) and \( \nu \).

While for free-fermion type, since \( \omega_1(u) \neq \omega_4(u) \), one can immediately see that \( a_2(u) \equiv 0 \) and \( a_3(u) \equiv 0 \) from (D.1). In other words, the RE for the free-Fermion type 6V models only has diagonal solution,

\[ K_- (u) = \begin{pmatrix}
  \sin (\alpha - u) & 0 \\
  0 & \sin (\alpha + u)
\end{pmatrix}. \]

(3.8)
In addition, if setting \( \cosh = 0 \) in (3.6), we have symmetric \( R \) matrix of free-Fermion type as follows,

\[
R(u) = \begin{pmatrix}
\cos u & 0 & 0 & 0 \\
0 & \sin u & 1 & 0 \\
0 & 1 & \sin u & 0 \\
0 & 0 & 0 & \cos u
\end{pmatrix}.
\] (3.9)

Just as discussed in Remark 2.3.2, this \( R \)-matrix shares the same \( K_- \)-matrix in (3.7) with 6V Baxter-type.

So, up to \( K \)-transformation (3.4), we obtain all general solutions (3.7) and (3.8) to reflection equation in six-vertex case.

4 \( K_- \)-Matrices to Seven-Vertex Model

If setting \( \omega_8(u) \equiv 0 \) in eight-vertex \( R \)-matrix (2.1), we get seven-vertex one

\[
R(u) = \begin{pmatrix}
\omega_1(u) & 0 & 0 & \omega_7(u) \\
0 & \omega_2(u) & \omega_5(u) & 0 \\
0 & \omega_6(u) & \omega_3(u) & 0 \\
0 & 0 & 0 & \omega_4(u)
\end{pmatrix}.
\] (4.1)

The classification of solutions to the coloured 7V-type YBE is given recently in [26]. Due to less symmetries, the \( R \)-matrices show a much more different properties from that of both eight-vertex and six-vertex models. At this point, we expect that the corresponding reflection equation reveals new features as well.

First of all, let us study symmetries of reflection equation as do previously for other cases. After removing the terms containing \( u_8 \) and \( v_8 \) in the system of equations (A), we find that there still exists \( K_- \)-transformation (2.4) under \( R \)-transformation (2.3)(note that \( \omega_8 \) is absent!).

In [26], an additional relation \( \omega_5(u)/\omega_6(u) = e^{cu} \) is given, where \( c \) is a constant. When \( c \neq 0 \), there have only trivial \( K_- \)-matrices. The case of \( c = 0 \) or \( \omega_5(u) = \omega_6(u) \) is further classified into three different types: Baxter type, free-fermion type I, II, which will be discussed in the following sections.

4.1 Baxter Type

The parametrization of the \( R \)-matrix is as follows

\[
\begin{align*}
\omega_1(u) &= \omega_4(u) = \frac{\sin(u + h)}{\sin h}, \\
\omega_2(u) &= \omega_3(u) = \frac{\sin u}{\sin h}, \\
\omega_5(u) &= \omega_6(u) = 1, \\
\omega_7(u) &= \sin(u + h) \sin u.
\end{align*}
\] (4.2)

Substituting (4.2) into (A), we solve these equations case by case.

Case 4.1.1: Diagonal Solution. It can be seen from (A.1) that if \( a_2(u) \equiv 0 \) then \( a_3(u) \equiv 0 \)(Note that \( u_8 = 0 = v_8 \) in (A)). We obtain the diagonal solution:

\[
K_-(u) = \begin{pmatrix}
\sin(\alpha - u) & 0 \\
0 & \sin(\alpha + u)
\end{pmatrix}.
\] (4.3)

Case 4.1.2: Skew-diagonal solution. Let \( a_1(u) \equiv 0 \), it requires \( a_4(u) \equiv 0 \) from (A.4). We only need to consider two equations:

\[
\begin{align*}
u_7 v_3 x_3 y_3 + u_1 v_5 (x_2 y_3 - x_3 y_2) &= 0, \\
u_3 v_7 x_3 y_3 + u_5 v_1 (x_2 y_3 - x_3 y_2) &= 0.
\end{align*}
\] (4.4)
Solving Eqs. (4.4), we have two $K_-$-matrices,

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & \rho(u) \\
1 & 0
\end{pmatrix},
\]

where $\rho(u) = (\lambda + \cos 2u)/2$ and $\lambda$ is a free parameter.

**Case 4.1.3:** $a_3(u) \equiv 0$. One can get from case 4.1.1 that

\[a_1(u) = p(u) \sin(\alpha - u), \quad a_4(u) = p(u) \sin(\alpha + u)\]

and find

\[a_2(u)/p(u) = \mu \sin(2u),\]

so the $K_-$-matrix is

\[
K_-(u) = \begin{pmatrix}
\nu \sin(\alpha - u) & \mu \sin(2u) \\
0 & \nu \sin(\alpha + u)
\end{pmatrix},
\]

where $\mu, \nu$ are parameters.

**Case 4.1.4.** Combining the results obtained above, one can easily write down the general $K$-matrix as follows

\[
K_-(u) = \begin{pmatrix}
\nu \sin(\alpha - u) & \mu p(u) \sin(2u) \\
\mu \sin(2u) & \nu \sin(\alpha + u)
\end{pmatrix}.
\]

In summary, we can regard (4.3) and (4.5) as the most general reflection matrices, because others can be obtained by assigning special values to free parameters. Furthermore, comparing (4.3) and (4.5), we see that in the case of 7V type, $a_2(u) = 0$ implies $a_3(u) = 0$, but the reverse does not hold, this is different from the case of 8V type.

### 4.2 Free-Fermion Type I

This kind of $R$-matrix reads

\[
\begin{align*}
\omega_1(u) &= \frac{\sin(u + h)}{\sin h}, \\
\omega_4(u) &= \frac{\sin(-u + h)}{\sin h}, \\
\omega_2(u) &= \omega_3(u) = \frac{\sin u}{\sin h}, \\
\omega_5(u) &= \omega_6(u) = 1, \\
\omega_7(u) &= \frac{\sin 2u}{\sin h}.
\end{align*}
\]

If $\cosh = 0$, or $\omega_1(u) = \omega_4(u)$, we can make the similar calculation as do for the Baxter-type, for both have the same symmetries. The result is almost the same as that given in (4.6) and (4.7) but with $\rho(u) = (\lambda + \cos 2u)$.

When $\omega_1(u) \neq \omega_4(u)$, it force $a_3(u) \equiv 0$ from the second equation of (A.3). The RE reduces to the following equations

\[
\begin{cases}
 u_2 v_5 (x_1 y_1 - x_4 y_4) + u_5 v_2 (x_4 y_1 - x_1 y_4) = 0, \\
u_7 v_1 x_1 y_1 - u_7 v_1 x_4 y_4 + u_4 v_7 x_4 y_1 - u_1 v_7 x_1 y_4 + (u_4 - u_1) v_2 x_2 y_2 = 0,
\end{cases}
\]

\[
\begin{cases}
u_5 v_2 x_1 y_2 + u_2 v_5 x_4 y_2 + (u_2 v_1 - u_1 v_2) x_2 y_1 = 0, \\
u_5 v_2 x_4 y_2 + u_2 v_5 x_1 y_2 + (u_2 v_1 - u_4 v_2) x_2 y_4 = 0 \\
(u_1 v_1 - u_3 v_2) x_1 y_2 - u_5 v_5 x_4 y_2 - u_5 v_1 x_2 y_1 + u_1 v_5 x_2 y_4 = 0, \\
(u_4 v_4 - u_3 v_2) x_1 y_2 - u_5 v_5 x_1 y_2 - u_5 v_4 x_2 y_4 + u_4 v_5 x_2 y_1 = 0.
\end{cases}
\]

**Case 4.2.1:** $a_2(u) \equiv 0$. The two equations of (E.1) are not compatible with each other due to the symmetry between $\omega_1(u)$ and $\omega_4(u)$ being broken. Therefore there exists no diagonal solution for this type of $R$-matrix.
Case 4.2.2: $a_1(u) \equiv 0$. One can deduce from (E.1) that $a_4(u) \equiv 0$ and $a_2(u) \equiv 0$. This is a trivial case.

Case 4.2.3: General solution. From the second equation of (E.1) and (E.2), one can get the same result as that in Baxter type

$$a_1(u) = \sin(\alpha - u), a_4(u) = \sin(\alpha + u), a_2(u) = \mu \sin 2u.$$ 

Substituting them into the second equation of (E.1), one find

$$K_-(u) = \begin{pmatrix} \sin(\alpha - u) & \pm \sin(2u) \\ 0 & \sin(\alpha + u) \end{pmatrix},$$

which also shows that $a_3(u) = 0$ does not imply $a_2(u) = 0$.

4.3 Free-Fermion Type II

In this case, the elements of the $R$-matrix take the following forms

$$\begin{align*}
\omega_1(u) &= \omega_4(u) = \cosh u,
\omega_2(u) &= -\omega_3(u) = \sinh u,
\omega_5(u) &= \omega_6(u) = 1,
\omega_7(u) &= u.
\end{align*}$$

(4.9)

For the sake of brevity, we simply give the result. Note that the nontriviality requires $a_1(u) = \pm a_4(u)$. If $a_1(u) = a_4(u)$, we get

$$K_-(u) = \begin{pmatrix} \alpha & \mu \sinh u \\ 0 & \alpha \end{pmatrix},$$

(4.10)

while if $a_1(u) = -a_4(u)$, we have

$$K_-(u) = \begin{pmatrix} \alpha & \mu \cosh u \\ 0 & \alpha \end{pmatrix}.$$ 

(4.11)

In addition, according to the discussion in section 2.4, the solutions keep invariant under exchange of $\omega_2 \leftrightarrow \omega_3$ since $a_3(u) \equiv 0$.

5 Construction of Boundary Hamiltonian

In this section, we will discuss the Hamiltonians for the systems described by the $R$-matrices and $K$-matrices obtained in the previous sections. The 6V (Baxter type and free-fermion type) and 8V (Baxter type and free-fermion type I) are included in Sklyanin’s formalism. While for 7V models, both Baxter type and free-fermion type I $R$-matrices has only regularity, $P$-symmetry, unitarity and crossing-unitarity symmetries, their $K_+(u)$-matrices are obtained by (1.5). However, all of these cases has the same definition of transfer matrix [8, 12], and we can construct their Hamiltonians in a unified way.

If $K_-(0) \propto id, trK_+(0) \neq 0$, the Hamiltonian for the open systems is defined as

$$H \equiv \frac{1}{2} \sum_{j=1}^{N-1} H_{j,j+1} + \frac{1}{2} K_-(0) K_-(0) + \frac{tr_0 K_+(0) H_{N0}}{trK_+(0)},$$

(5.1)

where two-site Hamiltonian $H_{j,j+1}$ is given by

$$H_{j,j+1} = P_{j,j+1} \frac{d}{du} R_{j,j+1}(u) \big|_{u=0} = \frac{d}{du} R_{j,j+1}(u) \big|_{u=0} P_{j,j+1}.$$ 

(5.2)

All the Baxter type models in two-component systems belong to this case. The boundary Hamiltonian of 6V and 8V Baxter type can be found in [13, 15, 18].
Case 5.1: Baxter type $7V$ with crossing parameter $\eta = h$. From $K_-(u)$ in (4.6) and relation (1.8), we find that

$$K_+(u) = K_-(u - h; -\alpha_+, \mu_+, \nu_+, \lambda_+).$$

(5.3)

According to Eqn.(5.1), the Hamiltonian is

$$H = \frac{1}{4 \sinh h} \sum_{j=1}^{N-1} \mathcal{H}_{j,j+1} - A_- \sigma^+_1 + B_- \sigma^+_1 + C_- \sigma^-_1 - A_+ \sigma^+_N + B_+ \sigma^+_N + C_+ \sigma^-_N,$$

where

$$\mathcal{H}_{j,j+1} = (2 + \sin^2 h) \sigma^+_j \sigma^+_j + (2 - \sin^2 h) \sigma^-_j \sigma^-_j + i \sin^2 h (\sigma^+_j \sigma^-_j \sigma^+_j \sigma^-_j + \sigma^-_j \sigma^+_j \sigma^-_j \sigma^+_j) + 2 \cosh \sigma^+_j \sigma^-_j, \quad (5.5)$$

$$A_\pm = \frac{1}{2} \cot \alpha, \quad B_\pm = \frac{(1 + \lambda_\pm) \mu_\pm}{2 \sin \alpha}, \quad C_\pm = \frac{\mu_\pm}{\nu_\pm \sin \alpha}. \quad (5.6)$$

However, if $\text{tr} K_+(0) = 0$, just as pointed out in Refs.[13, 16], there will have no well-defined Hamiltonian from the first derivative of the transfer matrix as done in (5.1). But if

$$\text{tr}_0 K_+(0) H_{N0} = A \cdot \text{id}, \quad (5.7)$$

where $A$ is a constant, we can still derive the well-defined local Hamiltonian from the second derivative of transfer matrix as follows

$$H = \frac{t''(0)}{4(C + 2A)} = \sum_{j=1}^{N-1} \mathcal{H}_{j,j+1} + \frac{1}{2} K^{-1}_-(0) K'_-(0) + \frac{1}{2(C + 2A)} \{\text{tr}_0 (K_+(0) G_{N0}) + 2 \text{tr}_0 (K'_+(0) H_{N0}) + \text{tr}_0 (K_+(0) H^2_{N0})\}, \quad (5.8)$$

where

$$C = \text{tr} K'_+(0), \quad (5.9)$$

$$G_{j,j+1} \equiv P_{j,j+1} \left. \frac{d^2 R_{j,j+1}(u)}{du^2} \right|_{u=0}. \quad (5.10)$$

The following discussions show that all the boundary conditions corresponding to the free-Fermion type $R$-matrix belong to this case. We argue that it is a common property for all free-Fermion models.

Case 5.2: Free-Fermion type-I $8V$ with crossing parameter $\eta = I$. Here $I$ is the complete elliptic integral of the first kind of modulus $k$. For general boundary condition described by $K_-(u)$ in (2.17), we have

$$K_+(u) = K_+^\perp (-u - I) = \begin{pmatrix}
    k^2 F_+(u) \sin u + E_+(u) \cos u \cos u \\
    2k^2 \mu_+ \sin u + E_+(u) \cos u
\end{pmatrix} \begin{pmatrix}
    2 \mu_+ k^2 \sin u + E_+(u) \cos u \\
    k^2 F_+(u) \sin u + E_+(u) \cos u
\end{pmatrix}, \quad (5.11)$$

where

$$F_+(u) = c_+^2 \sin^2 u + \frac{k((1 - ekG) c_+^2 + Hc_+^2)}{\epsilon G - k} \csc^2 u, \quad (5.12)$$

$$E_+(u) = c_+^2 \sin^2 u + \frac{k((1 - ekG) c_+^2 - k^2 Hc_+^2)}{\epsilon G - k} \csc^2 u. \quad (5.13)$$

From Eq.(5.8), the Hamiltonian reads

$$H = \sum_{j=1}^{N-1} \mathcal{H}_{j,j+1} + A_- \sigma^-_1 + B_- (\sigma^+_1 + \epsilon \sigma^-_1) + A_+ \sigma^-_N + B_+ (\sigma^+_N + \epsilon \sigma^-_N), \quad (5.14)$$
where
\[ \mathcal{H}_{j,j+1} = \frac{H}{2}(\sigma_j^x + \sigma_{j+1}^x) + \frac{G+k}{2}\sigma_j^x\sigma_{j+1}^x + \frac{G-k}{2}\sigma_j^y\sigma_{j+1}^y, \]  
(5.15)

and
\[ A_- = c_2^\dagger/c_1^\dagger, \quad B_- = 2\mu^-/c_1^\dagger, \]
\[ A_+ = k^2(HF_+(0) - E_+(0))/2(k^2F_+(0) + HE_+(0)), \]
\[ B_+ = k^2(G + \epsilon k)\mu^+/(k^2F_+(0) + HE_+(0)). \]

For the diagonal \( K \)-matrix (2.12), we have
\[ \mathcal{H} = \sum_{j=1}^{N-1} H_{j,j+1} + \frac{ik}{2}(\sigma_1^x - \sigma_N^x), \]  
(5.16)

which has been discussed in \cite{14} and is a special case of (5.14).

**Case 5.3: Symmetric free-Fermion type-I 8V.** If considering \( R \)-matrix (2.19) and the corresponding \( K \)-matrix (2.9), we get the following Hamiltonian
\[ \mathcal{H} = \sum_{j=1}^{N-1} \left( \frac{1+k}{2}\sigma_j^x\sigma_{j+1}^x + \frac{1-k}{2}\sigma_j^y\sigma_{j+1}^y \right) - A_-\sigma_1^x + B_-\sigma_1^+ + C_-\sigma_1^- - A_+\sigma_N^x + B_+\sigma_N^+ + C_+\sigma_N^- \]  
(5.17)

where
\[ A_- = \frac{\cna_\pm\dna_\pm}{2\sna_\pm}, \quad A_+ = \frac{\cna_\pm}{2\sna_\pm\dna_\pm}(1-k^2\sna_\pm), \]
\[ B_\pm = \frac{\mu_\pm(\lambda_\pm + 1)}{\sna_\pm}, \quad C_\pm = \frac{\mu_\pm(\lambda_\pm - 1)}{\sna_\pm}. \]

**Case 5.4: Free-Fermion type 6V with \( \eta = \pi/2 \).** For \( R \)-matrix in (3.6) and the general \( K_- \)-matrix (3.8), if setting
\[ K_+(u) = K_\ell^t(-u - \pi/2; \pi/2 - \alpha_+ - h) \]  
(5.18)

we have
\[ \mathcal{H} = \frac{1}{\sinh} \sum_{j=1}^{N-1} (\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \cosh(\sigma_j^x + \sigma_{j+1}^x)) - \cot\alpha_-\sigma_1^x - \cot\alpha_+\sigma_N^x. \]  
(5.19)

**Case 5.5: Symmetric free-Fermion type 6V.** If considering \( R(u) \) in (3.3) together with the general \( K_- \)-matrix (3.7), we can set
\[ K_+(u) = K_\ell^t(-u - \pi/2; -\alpha_+, \mu_+, \nu_+), \]  
(5.20)

thus the Hamiltonian is
\[ \mathcal{H} = \sum_{j=1}^{N-1} (\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) - A_-\sigma_1^x + B_-\sigma_1^+ + C_-\sigma_1^- - A_+\sigma_N^x + B_+\sigma_N^+ + C_+\sigma_N^- \]  
(5.21)

where
\[ A_\pm = \cot\alpha_\pm, \quad B_\pm = \frac{2\mu_\pm}{\sna_\pm}, \quad C_\pm = \frac{2\nu_\pm}{\sna_\pm}. \]

**Case 5.6: Free-Fermion type I 7V with crossing parameter \( \eta = \pi/2 \).** From \( K_- \)-matrix in (1.8) and relation (1.6), we get
\[ K_+(u) = K_-(u - \pi/2; \alpha_+ - h + \pi/2, \mu_+). \]  
(5.22)
and the Hamiltonian is
\[
\mathcal{H} = \frac{1}{2 \sinh} \sum_{j=1}^{N-1} \left\{ \cosh(\sigma_j^x + \sigma_{j+1}^x) + 2\sigma_j^y \sigma_{j+1}^y + i(\sigma_j^x \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^x) \right\} \\
- A_- \sigma_1^x + B_- \sigma_1^+ - A_+ \sigma_N^- + B_+ \sigma_N^+,
\]
where
\[
\mu_-, \mu_+ = \pm 1, \quad A_\pm = \cot \alpha_\pm/2, \quad B_\pm = \frac{\mu_\pm}{\sin \alpha_\pm}.
\]

**Case 5.7: Symmetric free-Fermion type I 7V.** From \( K_- \)-matrix in (4.6) with \( \rho(u) = \lambda + \cos 2u \) and relation (1.0), we have
\[
K_+(u) = K_-(-u - \pi/2; -\alpha_+, \mu_+, \nu_+, \lambda_+),
\]
and
\[
\mathcal{H} = \sum_{j=1}^{N-1} \left\{ \sigma_j^x \sigma_{j+1}^x + i(\sigma_j^y \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^x) \right\} \\
- A_- \sigma_1^x + B_- \sigma_1^+ + C_- \sigma_N^- - A_+ \sigma_N^- + B_+ \sigma_N^+ + C_+ \sigma_N^-
\]
where
\[
A_\pm = \frac{\cot \alpha_\pm}{2}, \quad B_\pm = \frac{(1 + \lambda_\pm) \mu_\pm}{\nu_\pm \sin \alpha_\pm}, \quad C_\pm = \frac{\mu_\pm}{\nu_\pm \sin \alpha_\pm}.
\]

It should be pointed out that for the free-Fermion type II of both 7V and 8V models which have no crossing-unitarity symmetry, how to prove their integrability and to obtain the corresponding Hamiltonians in the case of open boundary condition is an open problem.

### 6 Remarks and Discussions

In this paper we find that symmetries play an important role in solving the reflection equation. For any non-standard \( R \)-matrix which is obtained by applying \( R \)-transformation to the standard one, then the corresponding reflection matrix can be obtained by making \( K \)-transformation to that for standard \( R \)-matrix.

Moreover, all solutions given above indicate that the number of free parameters appeared in \( K_- \)-matrix is determined by symmetries of \( R \)-matrix. The \( R \)-matrices with different forms but the same symmetries share the same \( K_- \)(\( u \)) matrix. The free-Fermion type \( R \)-matrix with \( \omega_1(\( u \)) = \omega_3(\( u \)) \) is just in this case. It has the same form \( K_- \)(\( u \)) as in Baxter type. Also we note that, different from that for six- and eight-vertex cases, the elements \( a_2(\( u \)), a_3(\( u \)) \) of \( K_- \)(\( u \)) in seven-vertex case have no interchanging-symmetry resulting from the symmetry between \( \omega_7(\( u \)) \) and \( \omega_8(\( u \)) \) of \( R \)-matrix being broken.

It is also interesting to note that while constructing Hamiltonian, all reflection matrices for free-Fermion \( R \)-matrices have property of \( \text{tr} K_+(0) = 0 \). We argue that it is a typical property for all free-Fermion models. So the local Hamiltonian for such system are obtained from second derivative of the transfer matrix.

We are sure that our procedure to find solutions of the reflection equation can be applied to high-spin models, though the calculation may be much more involved in this case. With the solutions given in this paper, we can use the Bethe ansatz method to study the physical properties of open spin chains.

Furthermore, recently much attention has been directed to the Yang-Baxter equation with dynamical parameters (24, 25). How to construct the corresponding reflection equation and to seek its solution is an open problem. We wish to discuss some related problems using the method and procedure given in this paper.
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