Single-Field Model of Gravitational-Scalar Instability.  
I. Evolution of Perturbations  
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Abstract—On the basis of the previously formulated mathematical model of a statistical system with a scalar interaction of fermions and the theory of gravitational-scalar instability of a cosmological model based on a two-component statistical system of scalarly charged degenerate fermions, a numerical model of the cosmological evolution of gravitational-scalar perturbations for a one-component cosmological system with a canonical scalar interaction is constructed and studied. The influence of the magnitude of the scalar charge of fermions on differential and integral parameters of the instability is revealed. It is shown that the gravitational-scalar instability at early stages of expansion in the model under study arises at sufficiently small scalar charges. Four fundamentally different types of perturbations are identified, as well as four types of gravitational–scalar instability, determined by the fundamental parameters of the model. Examples of numerical models are given that provide large values of the increments of the increase in the perturbation magnitudes.

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1. INTRODUCTION

Currently, there is an unsolved problem in astrophysics and cosmology related to the lack of sufficiently convincing models for the formation of supermassive BHs (BHs) with a mass of the order of \(10^9-10^{10} M_\odot\), which are central objects of luminous quasars in the early Universe at \(z > 6\) and even \(z > 7\), which, according to modern concepts, corresponds to the age of the Universe of \(0.65-1\) billion years (see, e.g., [1–3]). The astrophysical origin of such supermassive BHs in the early Universe is still insufficiently understood, since observational data raise the question of the mechanism of formation and rapid growth of such objects in the early Universe. The results of a numerical simulation [4] impose a number of restrictions on the formation parameters of supermassive BHs. For example, it is shown that light BH nuclei with a mass \(M \leq 10^3 M_\odot\) cannot grow to masses of the order of \(10^8 M_\odot\) by \(z = 6\) even with supercritical accretion. For the formation of supermassive BHs with masses \(10^8-10^9 M_\odot\), heavier nuclei are needed by this time, 

\[ M_{nc} \sim 10^4-10^6 M_\odot, \]  

(1)

and gas-rich galaxies containing quasars. However, at present there are no sufficiently convincing models for the appearance of such heavy nuclei in the early Universe.

The interest in formation mechanisms of supermassive BHs, taking into account the fact of their dominant presence in the composition of quasars, is caused, in particular, by the fact that such BHs are formed at fairly early stages of the Universe evolution, before the formation of stars. This circumstance opens up the possibility of supermassive BH formation under the conditions where scalar fields and dark matter can significantly influence this process. Note that the numerical simulation in [4] was carried out within the framework of the standard gas accretion model, which does not take into account the possible influence of scalar fields on the process of BH formation. In this connection, we note the papers [5–7], in which the possibility of the existence of scalar halos and scalar hair in the vicinity of supermassive BHs is considered.

In this regard, it is necessary to construct theoretical models for the formation of supermassive BHs in the early Universe based on mechanisms different from the standard gas accretion mechanism. In the papers [8, 9], the theory of short-wavelength longitudinal perturbations was formulated in a cosmological model based on a degenerate one-component system of scalarly charged fermions. In this case, the fermions were assumed to bear a single canonical or phantom scalar charge, the potential energy of the corresponding scalar fields was assumed to be Higgs, and the perturbations of the cosmological model were...
studied in the stiff WKB approximation:
\[ n \eta \gg 1, \]
\[ n^2 \gg a^2 m^2, \]  
(2)
(3)
where \( n \) is the wave number of perturbations in the spatially flat Friedmann metric
\[
ds_0^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \\
\equiv a^2(t)(dt^2 - dx^2 - dy^2 - dz^2),
\]  
(4)
where \( t = \int ad\eta \), and \( m \) is the mass of scalar field quanta. In [9], the following result was obtained: in the cosmological model based on the canonical scalar field, there always occurs a moment of time at which all short-wavelength longitudinal\(^1\) perturbations become unstable, and all perturbations of scalar fields over a finite time interval reach infinitely large values. In this case, the metric perturbations remain small. The cosmological model based on phantom interaction turns out to be stable. In [10], this instability was called gravitational-scalar instability.\(^2\)

The stiffness condition of the WKB approximation (3) used in [9] did not allow us to take into account the specifics of the Higgs potential and to study the evolution of perturbations at sufficiently long times. In this regard, in [10], based on the complete theory of a two-component system of degenerate scalarly charged fermions with Higgs scalar fields [11], and the results of numerical modelling of the corresponding cosmological model [12], to be referred to as the model \( \mathcal{M}_1 \), a complete theory of the evolution of longitudinal perturbations was constructed with the rigid WKB condition removed—the (3) approximation. First, in [10], closed formulas with respect to a given cosmological background were obtained for the perturbation eikonal functions in the WKB approximation (2), and the assumption of smallness of the fermion scalar charges \( e_{(a)} \)
\[ n \eta \gg 1, \quad n^2 \gg e^4_{(a)}, \quad a^2 m^2_{(a)} \gg e^4_{(a)}, \]  
(5)
where \( m_{(a)} \) is the mass of quanta of the scalar field \( \Phi_{(a)} \).\(^3\) Secondly, the possibility of a short-wave scalar-gravitational instability of the cosmological model in the case of a canonical scalar singlet (a canonical field \( \Phi \)) and an asymmetric scalar doublet (a canonical field \( \Phi \) and a phantom field \( \varphi \)) was strictly substantiated, as well as the short-wavelength stability of the cosmological system based on a phantom scalar singlet.

Further, in [13], a numerical model of the evolution of scalar-gravitational perturbations was constructed for the case of an asymmetric scalar doublet \( \mathcal{M}_1 \), examples of the development of instability in a cosmological system were given, and some features of this process were revealed.

As we noted above, it is the canonical scalar field \( \Phi \) that is directly responsible for the emergence of an instability in the short-wavelength sector of oscillations of a degenerate system of scalarly charged fermions. Therefore, the purpose of this paper is, firstly, to construct a numerical model for the evolution of short-wave perturbations of the \( \mathcal{M}_1 \) cosmological model for the case of a canonical scalar singlet and a one-component system of degenerate fermions when the condition of stiff WKB approximation (3) is removed. Secondly, the purpose of this work is to study the influence of the model parameters on the process of the emergence and development of disturbance instability and to identify the possible features of this process.

The second part of the work will be devoted to a study of the possibility of BH formation in the early Universe using the mechanism of gravitational-scalar instability.

2. MATHEMATICAL MODEL OF THE COSMOLOGICAL SYSTEM OF DEGENERATE SCALARLY CHARGED FERMIONS

Bearing in mind the application of a number of results of this article in its second part, we first consider the general mathematical model \( \mathcal{M}_1 \) for the case of an asymmetric scalar doublet represented by the canonical scalar field \( \Phi \) and a phantom scalar field \( \varphi \). The cosmological model for the canonical scalar singlet \( \Phi \) will be considered as a special case of the general \( \mathcal{M}_1 \) model emerging in the limit of a number of its parameters.

2.1. Self-Consistent Set of Equations for Degenerate Scalarly Charged Fermions

Consider a cosmological model based on a self-gravitating two-component system of singly scalarly charged degenerate fermions interacting through a pair of scalar Higgs fields, a canonical one (\( \Phi \)), and a phantom one (\( \varphi \)). This model is described, firstly, by the Einstein equations
\[ R_{ik}^i - \frac{1}{2} \delta^i_k \dot{R} = 8\pi T^i_k + \delta^i_k \Lambda_0, \]  
(6)
where \( \delta^i_k \) is the Kronecker delta and \( \Lambda_0 \) is the cosmological constant.
where \( T^i_k = T^{i(s)}_{k} + T^{i(p)}_{k} \), and \( T^{i(s)}_{k} \) is the energy-momentum tensor (EMT) of the scalar fields,

\[
T^{i(s)}_{k} = \frac{1}{16\pi} \left(2\Phi^4 \Phi_k - \delta^i_k \Phi_j \Phi, j + 2V_z(\Phi)\delta^i_k\right) - \frac{1}{16\pi} (2\phi^i \phi_k - \delta^i_k \phi_j \phi, j - 2V_\zeta (\phi) \delta^i_k),
\]

and

\[
V_z (\Phi) = -\frac{\alpha}{4} \left( \phi^2 - \frac{m^2}{\alpha} \right)^2,
\]

\[
V_\zeta (\phi) = -\frac{\beta}{4} \left( \phi^2 - \frac{m^2}{\beta} \right)^2
\]

are the potential energies of the scalar fields, \( \alpha \) and \( \beta \) are their self-action constants, \( m \) and \( m \) are their quantum masses. As a carrier of scalar charges, we consider a two-component degenerate system of fermions, in which the carriers of a canonical charge \( z \)-fermions have the canonical charge \( e_z \) and the Fermi momentum \( \pi \), and the carriers of a phantom charge are \( \zeta \)-fermions have the phantom charge \( e_\zeta \) and Fermi momentum \( \pi \). The dynamic masses of these fermions in the case of zero bare masses are \( 11 \)

\[
m_z = e_z \Phi; \quad m_\zeta = e_\zeta \Phi.
\]

The seed cosmological constant \( \Lambda_0 \), which appears in the right-hand side of the Einstein equations (6), is related to its observed value \( \Lambda \) by

\[
\Lambda = \Lambda_0 - 1/4 \sum_r \frac{m^4}{\alpha_r}.
\]

Further, the EMT of an equilibrium statistical system is

\[
T^{i(p)}_{k} = (\varepsilon_p + p_p) u^i u_k - \delta^i_k p_p,
\]

where \( u^i \) is the macroscopic velocity vector of the statistical system, \( \varepsilon_p \) and \( p_p \) are its energy density and pressure. These macroscopic scalars, as well as other scalar functions that determine the macroscopic characteristics of the statistical system, are equal for a two-component statistical system of degenerate fermions (see, e.g., \( 11 \)):

\[
n^{(a)} = \frac{1}{\pi^2} \pi^{3(a)},
\]

\[
\varepsilon_p = \frac{e^4_1 \Phi^4}{8\pi^2} F_2 (\psi_z) + \frac{e^4_\zeta \phi^4}{8\pi^2} F_2 (\psi_\zeta),
\]

\[
p_p = \frac{e^4_1 \Phi^4}{24\pi^2} (F_2 (\psi_z) - 4F_1 (\psi_z)) + \frac{e^4_\zeta \phi^4}{24\pi^2} (F_2 (\psi_\zeta) - 4F_1 (\psi_\zeta)),
\]

\[
\sigma^z = \frac{e^4_1 \Phi^3}{2\pi^2} F_1 (\psi_z), \quad \sigma^\zeta = \frac{e^4_\zeta \phi^3}{2\pi^2} F_1 (\psi_\zeta),
\]

where the macroscopic scalars are \( n^{(a)} \)—scalar particle number density, and \( \sigma^z \) and \( \sigma^\zeta \)—scalar charge densities of \( e_z \) and \( e_\zeta \):

\[
\psi_z = \frac{\pi (z)}{|e_z \Phi|}, \quad \psi_\zeta = \frac{\pi (\zeta)}{|e_\zeta \phi|}
\]

and also to shorten the notations, the functions \( F_1 (\psi) \) and \( F_2 (\psi) \) are introduced:

\[
F_1 (\psi) = \psi \sqrt{1 + \psi^2} - \ln (\psi + \sqrt{1 + \psi^2}),
\]

\[
F_2 (\psi) = \psi \sqrt{1 + \psi^2} (1 + 2\psi^2) - \ln (\psi + \sqrt{1 + \psi^2}).
\]

In addition, we write down the expression we need below for the densities of scalar charges, \( \rho^{(a)} \), which are defined using the charge number density \( n^{(a)} \) and do not coincide, in general, with the scalar charge densities \( \sigma^z \) and \( \sigma^\zeta \) introduced above:

\[
\rho^{(a)} = e^{(a)} n^{(a)} = \frac{e^{(a)} \pi^{3(a)}}{\pi^2}.
\]

Finally, the scalar field equations for the system under study take the form

\[
\Box \Phi + m^2 \Phi - \alpha \Phi^3 = \frac{4}{\pi^2} e^4_1 \Phi^4 F_1 (\psi_z),
\]

\[
\Box \phi + m^2 \phi - \beta \phi^3 = \frac{4}{\pi^2} e^4_\zeta \phi^4 F_1 (\psi_\zeta).
\]

Note that the scalar charge densities \( \sigma^z \) and \( \sigma^\zeta \) are sources of the corresponding scalar fields \( \Phi \) and \( \phi \), while the scalar charge densities \( \rho_z \) and \( \rho_\zeta \) are defined by a prime number of charges and, unlike \( 14 \), are related to the corresponding conserved scalar charges

\[
Q^{(a)} = \int \rho^{(a)} dV.
\]

2.2. Background State of the Cosmological Model \( M_1 \)

Let us further consider the spatially flat model of the Friedman universe (4). A strict consequence of the general-relativistic kinetic theory for statistical systems of completely degenerate fermions is the Fermi momentum conservation law \( \pi^{(a)} \) for each component,

\[
a(t) \pi^{(a)} (t) = \text{const}.
\]

Assuming in what follows, for definiteness, \( a(0) = 1 \) and

\[
\xi = \ln a, \quad \xi \in (-\infty, +\infty), \quad \xi (0) = 0,
\]

\[
\pi^{(z)} = \pi^z e^{-\xi}, \quad \pi^{(\zeta)} = \pi^\zeta e^{-\xi},
\]

\[
(\pi^z = \pi^{(z)} (0), \quad \pi^\zeta = \pi^{(\zeta)} (0)),
\]
let us write down the complete normal set of Einstein-
scalar equations for this two-component system of
scalarly charged degenerate fermions [11] in a clearly
nonsingular form:
\[ \dot{h} = - \frac{Z^2}{2} + \frac{z^2}{2} + e^{-3\xi} \left( \frac{\pi}{3} e^{2\xi} + e^{4\Phi^2} \right) \]
\[ + \pi f \sqrt{\pi e^{-2\xi} + e^{2\varphi^2}}, \] (25)
and the invariant curvature of 4D space \( K \),
\[ K = \sqrt{R_{ijkl}R^{ijkl}} = H^2 \sqrt{6(1 + \Omega^2)} \]
\[ \equiv \sqrt{6H^4 + (H^2 + \dot{H})^2} \geq 0. \] (31)

3. GRAVITATIONAL-SCALAR INSTABILITY
OF THE \( \mathcal{M}_1 \) MODEL IN THE
SHORT-WAVELENGTH LIMIT
FOR A ONE-COMPONENT SYSTEM
3.1. WKB Approximation of Instability Theory
For the case of a classical scalar Higgs field and a
one-component system of singly scalarly charged
degenerate fermions, we reformulate the main results
of [10] (see also [13]), in which the evolution
of gravitational-scalar perturbations of the (4) metric
and scalar fields in the \( \mathcal{M}_1 \) model, for the case of
purely longitudinal perturbations of the metric (4) in
the form [15] (for details, see [16])
\[ ds^2 = ds_0^2 - a^2(t)h_{a\beta}dx^a dx^\beta, \] (32)
where \( ds_0 \) is the unperturbed spatially flat Friedmann
metric (4) in the conformally flat form, and, for
definiteness, the wave vector is directed along the \( Oz \)
axis:
\[ h_{11} = h_{22} = \frac{1}{3} \left( \lambda(t) + \frac{1}{3} \mu(t) \right) e^{inz}, \]
\[ h = \mu(t)e^{inz}, \]
\[ h_{12} = h_{13} = h_{23} = 0, \]
\[ h_{33} = \frac{1}{3} [-2\lambda(t) + \mu(t)] e^{inz}. \] (33)
At the same time, the matter in the \( \mathcal{M}_1 \) model in
the case of a classical scalar Higgs singlet and a
one-component system of degenerate scalarly charged
fermions is completely determined by two scalar func-
tions, \( \Phi(z, \eta) \) and \( \pi(z, \eta, \eta\) eta), as well as the velocity
t vector \( u^i(z, \eta) \). Let us expand these functions in
a series in terms of smallness of perturbations with
respect to the corresponding functions against the
background of the Friedmann metric (4).4
\[ \Phi(z, \eta) = \Phi(\eta) + \delta \Phi(\eta)e^{inz}, \]
\[ \pi(z, t) = \pi(\eta)(1 + \delta(\eta)e^{inz}), \]
\[ \sigma^2(z, \eta) = \sigma^2(\eta) + \delta \sigma^2(\eta)e^{inz}, \]
\[ u^i = \frac{1}{a} \delta^i_4 + \delta_3^i \nu(\eta)e^{inz}, \] (34)
where \( \delta \Phi(\eta), \delta(\eta), s_\eta(\eta), \) and \( \nu(\eta) \) are functions of
the first order of smallness as compared to their unperturbed
values.

4 For scalar singlets, see [8]. To avoid cumbersome notations,
we have retained the notation for the perturbed values of the
functions, distinguishing them only by arguments.
In [10] (see also [13]), the evolution of longitudinal gravitational scalar perturbations of the \( \mathcal{M}_1 \) model was studied in the short-wavelength and low-charge approximations (5). At the same time, in contrast to [8, 9], the condition of the rigid WKB approximation was not imposed, which makes it possible to consider sufficiently large wavelengths:

\[
n^2 \gtrsim a^2 \{ m^2 \Phi, \alpha \Phi^3, m^2 \varphi, \beta \varphi^3 \}.
\]

In accordance with the WKB method, we present the perturbation functions \( f(\eta) \) in the form

\[
 f = f(\eta) = e^{i \int u(\eta) d\eta} \quad (|u\eta| \sim |n\eta| \gg 1),
\]

where \( f(\eta) \) and \( u(\eta) \) are functions of the perturbation amplitude and the eikonal that vary slightly along with the scale factor.

In this paper, in contrast to the case of a scalar doublet, we will not impose an additional smallness condition of the scalar charge used in [10] to simplify the dispersion equation

\[
n^2 \gg e^4_{(a)}, \quad a^2 \{ m^2, m^2 \} \gg e^4_{(a)}.
\]

The equations for perturbation amplitudes in the zero WKB approximation (2) take the form of a linear homogeneous system of algebraic equations (\( \nu = \lambda + \mu \))

\[
\begin{bmatrix}
 n^2 - u^2 + \gamma_{11} & 0 & n^2 \gamma_{13} \\
 0 & u^2 & 0 \\
 0 & n^2 \gamma_{33} - u^2 & 0
\end{bmatrix}
\begin{bmatrix}
 \delta \Phi \\
 \lambda \\
 \nu
\end{bmatrix}
= 0,
\]

where the coefficients \( \gamma_{\alpha \beta} \) for the case under study are

\[
\begin{align*}
\gamma_{11} & \equiv a^2 (m^2 - 3 \alpha \Phi^2 + 8 \pi \delta S_{\Phi}^2), \\
\gamma_{13} & \equiv \frac{e^4 \Phi^3 \psi^2}{6 \pi^2 \varepsilon_p^2 \sqrt{1 + \psi^2}}, \\
\gamma_{33} & \equiv 1 + \frac{p^2}{e^2}, \\
\gamma_{31} & \equiv -3 a^2 [\Phi (m^2 - \alpha \Phi^2) - 8 \pi P \Phi].
\end{align*}
\]

The coefficients of the theory of gravitational-scalar instability [10] included in Eqs. (36) and expressed in terms of the basic functions of the unperturbed model \( \mathcal{M}_1 \) \( a(t) \) and \( \Phi(t) \) as well as the kinetic coefficients \( \psi_\alpha(t) \) (15), are

\[
\begin{align*}
\varepsilon_p^\delta & = \frac{1}{2} e^4 \Phi^4 \psi^2 \sqrt{1 + \psi^2} > 0, \\
\varepsilon_p^\Phi & = \frac{e^4 \Phi^3}{2 \pi^2} F_1 (\psi_z), \\
p_p^\delta & = \frac{1}{2} \sqrt{1 + \psi^2} > 0, \\
P^\Phi & = \frac{e^4 \Phi^3}{2 \pi^2} F_1 (\psi_z) - p_p^\delta \Delta \Phi,
\end{align*}
\]

The dispersion equation (40) is a biquadratic equation with respect to the eikonal function \( u(t) \), whose solution gives

\[
\begin{align*}
\delta \sigma^2 &= \frac{\varepsilon_p^\Phi^2}{1 + \psi_z^2} n^2 \nu + S_{\Phi}^2, \\
\delta \sigma^2 &= \frac{\varepsilon_p^\delta}{2 \lambda^2} n^2 \nu + P \Phi \delta \Phi.
\end{align*}
\]

### 3.2. Dispersion Equation, Modes, and Types of Perturbations

A necessary and sufficient condition for nontrivial solvability of the set of equations (36) is that the determinant of the matrix of this system is equal to zero, which gives the necessary dispersion equation on the eikonal function \( u(t) \) of the perturbations. In this case, two zero modes \( u^0_{(0)} = 0 \) are immediately distinguished, corresponding to perturbations of the \( \lambda \) metric (see [13]), which are eliminated by admissible transformations (for details, see [16]). The four oscillation modes \( u^0_{(\pm)} \) corresponding to perturbations of the classical scalar field \( \delta \Phi \) and those of the \( \nu \) metric are determined by the dispersion equation

\[
\text{Det}(\bar{A}) = \begin{vmatrix} n^2 - u^2 + \gamma_{11} & n^2 \gamma_{13} \\ \gamma_{31} & n^2 \gamma_{33} - u^2 \end{vmatrix} = 0.
\]

The dispersion equation (40) is a biquadratic equation with respect to the eikonal function \( u(t) \), whose solution gives

\[
\begin{align*}
\nu^2 \pm \varepsilon_p^\delta (n^2 \nu + 1 + \gamma_{33}) + \gamma_{11} &= \frac{1}{2} \sqrt{\left[ n^2 \nu + 1 + \gamma_{33} \right]^2 + 4 \gamma_{11} \gamma_{33}}, \\

\Rightarrow \quad u^\pm &= \pm \sqrt{\frac{c \pm \sqrt{b}}{2}}.
\end{align*}
\]

where the upper signs correspond to the signs before the external radical, the lower ones to the sign before the internal radius, and we denote

\[
c = n^2 + \gamma_{11} + n^2 \gamma_{33},
\]
With the relation
\[
\frac{1}{4}(c^2 - b) \equiv n^2\gamma_{33}(n^2 + \gamma_{11}) - \gamma_{13}\gamma_{31},
\]
with the relation
\[
b = [n^2(1 - \gamma_{33}) + \gamma_{11}]^2 + 4\gamma_{13}\gamma_{31},
\]
(43)

The solutions (42) satisfy the relations
\[
u_u = -\nu_u,
\quad \nu_{\bar{u}} = \nu_{\bar{u}},
\quad \nu_{\bar{u}u} = \nu_{uu} = \sqrt{c^2 - b}.
\]
(45)

According to (42), there are only four types of perturbations, depending on the ratio between the quantities \(a\) and \(b\):

1. \(b > c^2 \Rightarrow \Im(\nu_u) = \Im(\nu_{\bar{u}}) = 0, \Re(\nu_u) = -\Re(\nu_{\bar{u}}), \Re(\nu_{\bar{u}}) = \Re(\nu_u) = 0, \Im(\nu_{\bar{u}}) = -\Im(\nu_u).\) (46)

2. \(c > 0, 0 < b < c^2 \Rightarrow \Re(\nu_u) = 0, \Re(\nu_{\bar{u}}) = 0, \Im(\nu_u) = \Im(\nu_{\bar{u}}) = 0, \Re(\nu_{\bar{u}}) = -\Re(\nu_u).\) (47)

3. \(b < 0 \Rightarrow (\nu_u)^* = \nu_u, (\nu_{\bar{u}})^* = \nu_{\bar{u}}.\) (48)

4. \(c < 0, 0 < b < c^2 \Rightarrow \Re(\nu_u) = 0, \Im(\nu_u) = 0, \Re(\nu_{\bar{u}}) = \Re(\nu_u) = 0, \Im(\nu_{\bar{u}}) = -\Im(\nu_u).\) (49)

Of the listed four types of perturbations (46)–(49), the first one, (46), represents a superposition of a pair of standing growing and damped modes and a pair of undamped modes\(^5\) of retarded and advanced waves; the second type (47) represents two pairs of undamped waves (leading and retarded) with different frequencies, type 3 (48) two pairs of traveling waves (leading and retarded) with different frequencies, having damped and growing modes, type 4 (49) represents pairs of damped and growing modes with different decrements/increments of standing waves. Since, according to (43), the coefficients \(c\) and \(b\) are functions of time \((c(t), b(t))\), the listed types of perturbations can transform one into another at times \(t_k:\)

\[
b(t) > 0 \iff b(t_k) = 0 \iff b(t) < 0,
\]
\[
c(t) > 0 \iff c(t_k) = 0 \iff c(t) < 0,
\]
\[
e^2(t) - b(t) > 0 \iff c(t_k)^2 - b(t_k) = 0 \iff c(t)^2 - b(t) < 0.
\]
(50)

As a result, in the course of the cosmological evolution, rather a complex picture of alternating stages of wave oscillations and instability stages, determined by the presence of the imaginary part of the eikonal, can be obtained.

\(^5\) A standard weak amplitude decay occurs only due to the geometric factor \(a(t)\).

### 3.3. Oscillation Frequencies, Decrements, and Increments

Further, due to linearity of the perturbations, the final expressions for them according to Eq. (35) and the found eikonal functions (42) can be written in the form
\[
f = e^{in\eta} \sum_{+} \sum_{-} \tilde{f}_{\pm}(\eta) e^{i\int a_{\pm}(\eta) d\eta} + \text{CC},
\]
(51)

where \(\tilde{f}_{\pm}(\eta)\) are slowly varying perturbation amplitudes, \(\nu(\eta, z)\) and \(\delta\Phi(\eta, z)\), corresponding to the above four oscillation modes \(u_{\pm}\), and \(\text{CC}\) means a complex conjugate quantity. Thus unstable modes can only correspond to the eikonal functions \(u_{\pm}(\eta)\) in the region of their complex values.

Passing to the cosmological time using Eq. (4) in the expressions (51) and separating the real and imaginary parts in the eikonal functions, we obtain
\[
i \int_{\eta_0}^{\eta} u_{\pm} d\eta = i \int_0^t \frac{\nu_{\pm}(t)}{a(t)} dt = \int_0^t \omega_{\pm}(t) dt - \int_0^t \gamma_{\pm}(t) dt,
\]
(52)

where \(\omega(t)\) and \(\gamma(t)\) are the local frequency, decay decrements or growth increments of oscillations in the cosmological time scale \(t\):

\[
\omega_{\pm} = e^{-\xi(t)} \Re(\nu_{\pm}) \equiv e^{-\xi(t)} \nu_{\pm},
\]
\[
\gamma_{\pm} = -e^{-\xi(t)} \Im(\nu_{\pm}) \equiv e^{-\xi(t)} \nu_{\pm},
\]
(53)

\(\tilde{\omega}_{\pm}\) and \(\tilde{\gamma}_{\pm}\) are the local frequency, decay decrements or growth increments of fluctuations in the scale of the time variable \(\eta\).

The parity of the eikonal (45) implies the parity property of the oscillation frequency and the increment (if there are corresponding real and imaginary parts of the eikonal function)

\[
\omega_{\pm} = -\omega_{\mp} \equiv \omega_{\mp}, \quad \gamma_{\pm} = -\gamma_{\mp} \equiv \gamma_{\mp}.
\]
(54)

Therefore, the terms under the double sum (51) corresponding to each pair of modes can be written in the form
\[
\tilde{f}_{\pm} e^{i(nz + \int \omega_{\pm}(t) dt)} e^{-\int \gamma_{\mp}(t) dt} + \tilde{f}_{\mp} e^{i(nz - \int \omega_{\pm}(t) dt)} e^{\int \gamma_{\mp}(t) dt}.
\]
Now, adding to this expression its complex conjugate, we obtain, according to (51), the final expression for perturbations

\[
    f = \left( f^+ e^{i(nz+f \omega_+ \check{t})} \right) + (f^\pm)^* e^{-i(nz+f \omega_+ \check{t})} - f^{-} e^{-f \gamma_+ \check{t}} + (f^\pm)^* e^{i(nz-f \omega_+ \check{t})} + (f^\pm)^* e^{-i(nz-f \omega_+ \check{t})} \right) e^{f \gamma_+ \check{t}}.
\]

In the general case, the disturbances represent two groups of retarded and advanced waves propagating with a phase velocity

\[
v_f = \frac{\omega_\mp}{n} \equiv a \frac{\omega_\mp}{n},
\]

with exponentially decaying or growing amplitudes

\[
    \tilde{f}^-(\eta)e^{-f \tilde{\gamma}(n,n) \eta d\eta}, \quad \tilde{f}^+(\eta)e^{f \tilde{\gamma}(n,n) \eta d\eta}.
\]

The growing oscillation modes correspond to instability of the homogeneous unperturbed state of the cosmological model. As we noted above, this mode is associated with \(\delta \Phi, \psi\) disturbances, so the instability, if it exists, is essentially \textit{gravitational–scalar} in nature. Further, according to (57), the amplitude of the growing disturbance mode at time \(t\), \textit{the growth factor of the disturbance amplitude}, is determined by the expression

\[
    \chi(t) = \int_{t_1}^{t} \gamma(t) dt,
\]

where \(t_1\) is the initial moment of instability occurrence. Let \(t_2\) be the end time of the unstable phase, so that at \(t > t_2\), \(\gamma(t) = 0\). Thus, during the development of instability on the interval \(\Delta t = t_2 - t_1\), the perturbation amplitude is fixed at \(\tilde{f}^+(t) \exp(\chi_\infty)\), where

\[
    \chi_\infty = \int_{t_1}^{t_2} \gamma(t) dt.
\]

4. PRELIMINARY REMARKS ON NUMERICAL MODELLING

4.1. Parameters and Initial Conditions

The general background model \(\mathfrak{M}_1\) is defined by an ordered set of nine parameters [12],

\[
    \mathbf{P} = [\alpha, m, e, \pi_c, \beta, \mu, \epsilon, \pi_f, \Lambda]
\]

and initial conditions

\[
    \mathbf{I} = [\Phi_0, Z_0, \varphi_0, z_0, \kappa],
\]

where \(\kappa = \pm 1\), and the value \(\kappa = +1\) corresponds to a nonnegative initial value of the Hubble parameter, \(H_0 = H_+ \geq 0\), while \(\kappa = -1\) corresponds to a negative initial value of the Hubble parameter, \(H_0 = H_- < 0\). In this paper, we consider a particular case of the \(\mathfrak{M}_1\) cosmological model based on a one-component degenerate system of fermions charged with a canonical scalar charge for a one-field model of a scalar field. This special case is obtained from the \(\mathfrak{M}_1\) model with the following parameter values (60) and initial conditions (61):

\[
    \begin{align*}
        \mathbf{P} &= [\alpha, m, e, \pi_c, \Lambda], \\
        \mathbf{I} &= [\Phi_0, Z_0, \kappa].
    \end{align*}
\]

In what follows, we denote such a model by the symbol \(\mathfrak{M}_{1c}\).

Note, firstly, that according to Eq. (26), in the absence of a phantom field in the \(\mathfrak{M}_1\) model, always

\[
    (26) \Rightarrow \dot{H} \leq 0, \quad \varphi = 0
\]

and the zero value \(\dot{H} = 0\) can be reached only for \(\Phi = \Phi_0\) and \(\xi \to +\infty \Rightarrow a(t) \to +\infty\), i.e., in the infinite future. In this case, (27) implies that \(\Phi_0\) is determined by a stationary singular point of a dynamical system with a vacuum scalar field (see [14]). However, in the general case, the existence of this stationary point may contradict the Einstein equation (29), which imposes a rigid connection between the possible fundamental parameters of the cosmological model [11].

In what follows we will need the coordinates of singular points of the dynamical system of the unperturbed cosmological model with a vacuum canonical Higgs field in the phase plane \([\Phi, H]\), \(Z = 0\). There can be eight such points under the condition \(\alpha > 0, \lambda \geq 0\) (see [14]) making four symmetrical pairs:

\[
    M_0^\pm = \left[0, \pm \sqrt{\frac{\Lambda}{3}} \right],
\]

\[
    M_{\pm 1}^\pm = \left[\pm m \sqrt{\alpha}, \pm \sqrt{\frac{\Lambda}{3} + \frac{m^4}{12\alpha}} \right].
\]

According to the results of [14] adapted to the case of a one-field model, \textit{under the condition that singular points exist}, (65), they have the following nature: the points \(M_0^+\) are attracting, the remaining points: \(M_0^-\); \(M_{\pm 1}^\pm\) are saddle. As can be seen from (65), the attracting points \(M_0^+\) exist only if the cosmological constant \(\Lambda\) is non-negative. As a result, the behavior of the basic functions of the \(\mathfrak{M}_1\) model critically depends on the sign of \(\Lambda\). In this case, the coordinates of the singular points for the vacuum Higgs scalar field
Below, using the autonomy of the dynamical system, we set \( \xi(0) = 0 \) everywhere. Thus the \( \mathfrak{M}_1 \) model is determined by five fundamental parameters and three initial conditions. Further, keeping in mind the still too large number of parameters of the model, in this paper we will fix some of them, assuming in the future

\[
P = [[1, 1, e, 0.1], \Lambda], \quad I = [1, 0, 1].
\]  

(66)

Thus the model under study here has only two parameters \( e \) and \( \Lambda \). In this case, the coordinates of the singular points of the dynamical system of the unperturbed cosmological model with the vacuum canonical Higgs field (65) take the following values, depending only on \( \Lambda \):

\[
M_0^\pm = \left[0, \pm \sqrt{\frac{\Lambda}{3}}\right],
\]

\[
M_{\pm1} = \left[\pm 1, \pm \sqrt{\frac{1}{12} + \frac{\Lambda}{3}}\right].
\]  

(67)

4.2. Remarks on the Instability Mechanism

In [13] and [10], as well as earlier in [9], it is noted that the gravitational-scalar instability of short-wavelength perturbations in a system of scalarly charged particles develops due to the canonical scalar field, as long as the true strict WKB condition holds. Below are plots of the dependence of the squared eikonal function \( u^2_\pm \) (41) on the wavenumber \( n \) and the scale function \( \xi \) in the case of a scalar singlet with the potential \( \Phi = 1 \). The plots below in Figs. 1 and 2 are constructed for the following values of the fundamental parameters:\footnote{Note that the eikonal function \( u(t) \) does not explicitly depend on the value of \( \Lambda \), its dependence is determined indirectly in terms of the basic functions \( e(t) \) and \( \Phi(t) \), whose evolution essentially depends on the value of \( \Lambda \).}

\[
P_0 = [[1, 1, 1, 0.1], \Lambda].
\]  

(68)

A necessary and sufficient condition for the onset of instability is

\[
\text{Im}(u) \neq 0.
\]  

(69)

In particular, instabilities arise in the region of negative values of the squared eikonal function,

\[
u^2 < 0.
\]  

(70)

We emphasize, firstly, that the condition (70) is not necessary but only sufficient since the squared eikonal function (41) can also be a complex quantity provided that the expression under the square root in (41)

\[
b \equiv (n^2 + \gamma_{11} - n^2\gamma_{33})^2 + 4\gamma_{13}\gamma_{31} < 0,
\]  

(71)

which can be satisfied for sufficiently large values of the scalar charge \( e \). In this case, all functions of the eikonal \( u^\pm_\pm(t) \) automatically become complex, which ensures the occurrence of instability (3rd type

Fig. 1. Dependence of the squared eikonal function \( u^2_\pm \) on the wave number at \( \xi = 1 \) for (68). The dashed line is \( u^2_+ \), the solid line is \( u^2_- \).

Fig. 2. Dependence of the real and imaginary parts of the squared eikonal function \( u^2_\pm \) on the scaling function \( \xi \) for \( n = 1 \), for the parameters (68). Solid line—\( \text{Re}(u^2_\pm) \), dashed-dotted line—\( \text{Re}(u^2_-) \), dashed line—\( \text{Im}(u^2_-) \), dotted line—\( \text{Im}(u^2_+) \).
of perturbations in (48)). This case just corresponds to Fig. 2.

Next, Figs. 3 and 4 show the dependence of the oscillation frequency \( \omega = \text{Re}(u_-) \) and the increment or decrement of growth/damping of the perturbation amplitude \( \gamma = \text{Im}(u_-) \) on the value of the scale function \( \xi \).

The above examples show the fundamental possibility of the existence of a gravitational-scalar instability in a system of scalarly charged fermions with a classical scalar Higgs interaction.

However, the question of the emergence of an instability in the course of cosmological evolution remains open. Indeed, the condition for the occurrence of instability (69) is, in essence, an algebraic condition. But for the emergence of a gravitational-scalar instability, it is necessary that in the process of cosmological evolution, on a certain time interval, the values of the basic functions of the model \( a(t), \Phi(t) \) turn out to be such that the condition (69) holds,

\[
\text{Im}(u(a(t), \Phi(t), \varphi(t), n)) = \text{Im}(u(t, n)) \neq 0. 
\]  

(72)

Further on we will study the model based on the analysis of the behavior of its basic functions: the scale factor \( a(t) \) (or \( \xi(t) \)), the Hubble parameter \( H(t) \), the invariant cosmological acceleration \( \Omega(t) \) (30), and the invariant curvature \( K(t) \) (31), the local perturbation growth rate \( \gamma(t) \) (53), the perturbation amplitude growth factor \( \chi(t) \) (58), and its final value \( \chi_\infty \) (59).

Let us note here, not to return to this later, that a constant value of the Hubble parameter \( H(t) = \text{const} \) corresponds to the inflationary mode \( \Omega = 1 \): inflationary expansion for \( H > 0 \) or inflationary contraction for \( H < 0 \). These two modes are mutually invertible under time inversion, \( t \leftrightarrow -t \).

Let us make the following remark regarding the parameters of the models under consideration: below, we will mainly consider models with small values of the scalar charge \( e \lesssim 10^{-4} \), which in ordinary units corresponds to the values

\[
e \lesssim 10^{-4} m_{pl}^{1/2} \sim \sqrt{10^{15}} \text{ Gev},
\]

since even these values already lie at the level of the values of the parameters of the SU(5) field-theoretic models. Values of scalar charges of the order of 1 would mean that scalar charges must have a gravitational nature, which would require a reformulation of the theory of gravity. For such values of the scalar charges, according to the WKB applicability condition (5), even in the case of small values of the wave number \( n \lesssim 1 \), we can use the results of the WKB instability theory for \( n\eta \gg 1 \), i.e., for fulfillment of the condition

\[
nR(t) \gg 1 \left( R(t) \equiv \int_0^t e^{-\xi} dt \right). 
\]  

(73)
5. NUMERICAL MODELING

5.1. Negative Values of the Cosmological Constant, \( \Lambda < 0 \)

The studies show (see [12]) that negative values of the cosmological constant correspond to cosmological models with finite lifetime. At the same time, in the case of sufficiently large values of the scalar charges and the cosmological constant, the model remains stable. This corresponds, for example, to the case \( \mathbf{P} = [[1, 1, 0.001, 1], -0.001] \). With large negative values of \( \Lambda \), among other things, the Universe has a too short lifetime. Let us therefore consider the case of sufficiently small values of the scalar charges and the cosmological constant:

\[
\mathbf{P}_1 = [[1, 1, 10^{-5}, 0.1], -10^{-5}]. \tag{74}
\]

Figures 5 and 6 show the evolution of the functions \( \xi(t) \) and \( H(t) \) for these parameter values. As can be seen from these figures, the lifetime of the cosmological model in the case of parameters (74) is less than 1200 Planck times.

In this case, the cosmological model is mostly in a state with an almost zero value of the Hubble parameter, \( H \approx 0 \) (Fig. 6).

Note that the initial singularity at the parameters (74) in our model corresponds to the time \( t_0 \approx -20.12 \). Figure 7 shows the function \( R(t) \) (73) for the present model parameters. Thus, the WKB approximation (73) for \( t \geq 0 \) in the case under study takes the form:

\[
1200n \gg 1,
\]

which allows us to study perturbations with wave numbers \( n \gtrsim 10^{-2} \) in the WKB approximation. We will not return to this issue in the future.

Figures 8 and 9 show the evolution of the perturbation frequency \( \omega(t) \) and the oscillation growth increment \( \gamma(t) \). As can be seen from these figures, firstly, in the case under consideration, there are four oscillation modes ((42) and (53)). Secondly, the perturbation growth rate for modes \( \left( \mp \right) \) is strictly equal to
SINGLE-FIELD MODEL OF GRAVITATIONAL-SCALAR INSTABILITY

Fig. 8. Evolution of the oscillation frequency $\omega_+^\pm$ (dotted lines and $\omega_-^+=$dashed-dotted lines) and the oscillation growth increment $\gamma_+^\pm$ (dashed and $\gamma_-^+=$solid lines) in the case of the parameters (74) and $n = 5$.

Fig. 9. Evolution of the oscillation frequency $\omega_+^\pm$ (dotted lines and $\omega_-^+=$dashed-dotted lines) and the growth rate of oscillations $\gamma_+^\pm$ (dashed and $\gamma_-^+=$solid lines) in the case of the parameters (74) and $n = 5$.

Fig. 10. Evolution of the scalar potential $\Phi(t)$ in the case of parameters (74) (solid line) and (75) (dashed line).

In this case, for a long time, the background solution coincides with the inflationary $H = H_0 \approx 0.2887$ with high accuracy. Figure 10 shows the evolution of the scalar field potential $\Phi(t)$ for this case and the parameters (74).

This background solution corresponds to the singular point $M_{1/1}^+$ with $H = H_0, \Phi = 1$.

zero, $\gamma_+^\pm = 0$ (in Fig. 7 the lines $\gamma_+^\pm(t)$ merge). This means that the perturbation modes ($\pm$) in this case represent two pairs of undamped waves with a phase velocity (56). This case corresponds to the 2nd type of disturbances in (47).

Thirdly, the oscillation mode $(-)$ on the interval $[t_1, t_2] \approx [5, 25]$ is unstable, while the mode $(\pm)$ is damped on this interval. Meanwhile, the oscillation frequencies $\omega_\pm^\pm$ vanish. Thus, on the interval $t \in [5, 25]$, the perturbation modes $(-)$ represent a superposition of growing and damping standing waves, i.e., the perturbations on this interval belong to the 1st type of perturbations (46). Outside this interval, the perturbations are pairs of nondamped waves, which corresponds to the second type of perturbations (47). In what follows, for convenience, we will refer to the type of instability corresponding to Fig. 9 narrow-band instability.

To remove a possible confusion in connection with the isolation of the negative frequency part of perturbations, we recall that we must add complex-conjugate quantities to the final expressions for perturbations according to Eq. (55), and therefore isolation of the negative frequency part of perturbations turns out to be fictitious.

Let us give an example with even smaller values of the charges and the cosmological constant

$$P_1 = [1, 1, 10^{-6}, 0.1, -10^{-7}].$$

In this case, for a long time, the background solution coincides with the inflationary $H = H_0 \approx 0.2887$ with high accuracy. Figure 10 shows the evolution of the scalar field potential $\Phi(t)$ for this case and the parameters (74).

This background solution corresponds to the singular point $M_{1/1}^+$ with $H = H_0, \Phi = 1$. 
Figure 11 shows the evolution of the perturbation frequency \( \omega^\pm_t \) (dashed lines) and the increment \( \gamma^\pm_t \) (dotted lines) almost over the entire time interval \( t \in (0, 600) \) representing a pair of growing and damping standing waves with the increment \( \gamma^\pm \approx \pm 1.43 \), and the perturbation modes \( \pm \) almost over the entire time interval \( t \in (0, 600) \) represent a pair of undamped delayed and advanced waves with oscillation frequencies decreasing with time \( \omega^\pm_t \rightarrow 0 (t \rightarrow \infty) \). This means that, almost on the entire interval \( t \in (0, 600) \), perturbations are of the first type (46). In what follows, for convenience, we will call the one shown in Fig. 11 broad-band instability. Its distinguishing feature is the constancy of \( \gamma(t) \) over a wide time interval. This instability arises for the perturbation mode \( \pm \) in the case of a sufficiently small scalar charge, \( e \leq 10^{-5} \).

5.2. Zero Value of the Cosmological Constant, \( \Lambda = 0 \)

As shown by the studies [12], in the case of zero cosmological constant, the cosmological model \( \mathcal{M}_1^2 \) has an infinite lifetime, the scaling function \( \xi(t) \), like the scaling factor \( a(t) \), is monotonically increasing, the Hubble parameter is monotonically decreasing with zero asymptote, \( H(t) \rightarrow 0 \) as \( t \rightarrow +\infty \). Figure 13 shows the evolution of \( H(t) \) for the parameters

\[
P_2 = [1, 1, 10^{-6}, 0.1, 0].
\]

Figure 14 shows the evolution of the perturbation frequencies \( \omega^\pm_t \) and the oscillation growth incre-
\[ \frac{d^2 \gamma(t)}{dt^2} + \omega^2 \gamma(t) = 0. \]

As can be seen from the figures, this case is similar to that of a negative cosmological constant considered in Section 5.1. In this case, \( \gamma \approx 1.41 \).

Thus this case also corresponds to the 2nd type of disturbances (47).

### 5.3. Positive Values of the Cosmological Constant

Figure 16 shows the evolution of the function \( H(t) \) for the parameters

\[ P_3 = [1, 1, 10^{-5}, 0.1, 10^{-5}] \]  \( (77) \)

In the \( M_1^3 \) model with a positive cosmological constant, a singularity appears at some time \( t_0 \), which should be considered as the beginning of the Universe \[ 12 \] in the chosen time scale. In the case of the parameters (77) and the initial conditions \( I = [1, 0, 1] \), this time corresponds to \( t_0 \approx -20.1 \). The function \( \xi(t) \) is monotonically increasing, and the Hubble parameter \( H(t) \) is monotonically decreasing with the asymptote \( H(t) \rightarrow H_0 \approx 0.2886809080 \) as \( t \rightarrow +\infty \), corresponding to the stationary point of the dynamical system for the Higgs vacuum scalar field (65). The values of the Hubble parameter \( H \) corresponding to singular points (65) are shown in Fig. 16 with dashed lines. The asymptote \( H(t) \rightarrow H_0 \approx 0.18 \) as \( t \rightarrow +\infty \) corresponds to an inflationary expansion of the model.

Next, Figure 17 shows the evolution of the oscillation frequency \( \omega \) and the increment/decrement of oscillation growth \( \gamma \) for the oscillation modes \( \pm \), and Fig. 18 for the oscillation modes \( \pm \) in the case of the parameters (77) and the wave number \( n = 5 \).

Comparing the corresponding plots in Sections 5.1 (\( \Lambda < 0 \)), 5.2 (\( \Lambda = 0 \)) and 5.3 (\( \Lambda > 0 \)), we see almost the same behavior of the increments or decrements of the perturbation amplitude. However, this is only a superficial similarity: firstly, the frequencies of \( \omega^\pm \) in these cases differ significantly: for \( \Lambda < 0 \), \( |\omega^\pm| \lesssim 0.3 \), for \( \Lambda = 0 \), \( |\omega^\pm| \lesssim 0.5 \), for \( \Lambda > 0 \), \( |\omega^\pm| \lesssim 300 \) (Figs. 11–18). Secondly, the behavior of the function \( \gamma(t) \) coincides in the case of completely different values of the parameters \( \epsilon, \Lambda \) in the case of
\( \Lambda > 0 \), the behavior of \( \gamma(t) \) is similar to that with the parameters (77) but is achieved with the parameters (75) for which the values of \( e, |\Lambda| \) are an order of magnitude and two orders of magnitude smaller than the corresponding parameters (77). In the case of the same absolute value of the parameters (74) and (77), the behavior of the functions \( \omega(t) \) and \( \gamma(t) \) in these cases is fundamentally different (cf. Figs. 8, 9 and 17, 18).

5.4. Influence of the Value of the Wave Number on the Growth Rate of Oscillations

Figure 19 shows the wave number dependence of the oscillation growth rate evolution for \( n = 1—10000 \), in the (±) mode.

Thus, as the wave number \( n \) increases, the beginning of the instability phase slowly shifts towards later
times, while the absolute value of the growth rate $\gamma$ does not change.

6. LARGE SCALAR CHARGES: A TYPE OF QUASIPERIODIC INSTABILITY

The studies have shown that for sufficiently small values of the scalar charge and the cosmological constant, $e \leq 10^{-5}$ and $\Lambda \leq 10^{-5}$, the functions $\gamma(t)$ are practically independent from the charge and cosmological constant values. However, as the scalar charge increases in the $e \geq 10^{-5}$ range, the situation radically changes.

Figure 20 shows the dependence of the evolution of the oscillation growth increment on the value of the scalar charge, $e = 10^{-5} - 7 \times 10^{-5}$, for the $(\pm)$ mode at $n = 3$. This figure clearly shows how, with an increase in the scalar charge, the broad-band instability transforms into a narrow-band one, which, in turn, transforms into a quasiperiodic instability. This last type of instability is characterized by alternating pauses of slow growth or decay, $[t_i, t_i + \Delta t/2]$ of trapezoidal bursts of the growth rate of the disturbance amplitude $\gamma(t)$ and pauses $[t_i + \Delta t/2, t_i + \Delta t]$ with zero increment $\gamma(t) = 0$. So, in Fig. 20, $\Delta t \approx 6$.

Figure 21 demonstrates the dependence of the evolution of the oscillation growth rate on the wave number, $n = 3$–300, for the $(\pm)$ mode. Here, as in the previous cases, the wave number value affects only the initial stages of the development of instability— with an increase in $n$, the beginning of the instability phase shifts to later times. Next, Figure 22 shows the evolution of the oscillation frequency $\omega$ and the increment or decrement of oscillations $\gamma(t)$ for the modes $\pm$ in the case of the parameters (77) and the wave number $n = 10$.

In the transition region of sufficiently large values of the scalar charge $e \approx 10^{-5}$ and very large values of the cosmological constant $\Lambda \approx 0.1$, the behavior of perturbations becomes much more complicated: while maintaining the oscillatory nature of the function $\gamma(t)$, its quasi-periodicity disappears (Fig. 23). This transient type of instability can be called aperiodic instability.
7. CONCLUSION

Let us briefly list the main results of this part of the paper.

1. A closed mathematical model is constructed for a self-consistent description of the cosmological evolution of longitudinal short-wavelength perturbations in a one-component system of degenerate scalarly charged fermions and a canonical Higgs scalar field. This model contains a system of ordinary nonlinear differential equations describing the background functions $a(t)$ and $\Phi(t)$—the scale factor and the scalar potential, as well as a set of equations describing the evolution of short-wave disturbances relative to this background.

2. On the basis of the formulated mathematical model, expressions are obtained for the eikonal functions in the WKB approximation, which describe four nontrivial perturbation modes $\left(\frac{\pi}{4}\right)$ (Eqs. (41), (42)). At the same time, the limitations of the author's previous works, which prevent the study of perturbations at large times and sufficiently large values of scalar charges, are removed. An analysis of the eikonal functions made it possible to identify four different types of perturbations, whose existence is determined by the fundamental parameters of the model (46)–(49).

3. Using the found expressions for the eikonal, the values of the frequency $\omega_{\pm}$ and the growth rate of the oscillation amplitude $\gamma_{\pm}$ are determined as functions of the fundamental parameters of the model for given background values $\{a, \Phi\}$. With the help of numerical simulation, the fundamental possibility of the existence of regions with a nonzero growth rate of the disturbance amplitude, i.e., the possibility of a gravitational-scalar instability (Figs. 1–4) has been demonstrated.

4. Four types of evolution of the $\gamma(t)$ increment are distinguished: 1.—narrow-band instability, characterized by a short burst at early stages of the expansion (Fig. 9, $e > 10^{-5}, \Lambda \gtrsim 10^{-5}$); 2.—broad-band instability characterized by a long phase with an approximately constant value of $\gamma_{\pm} \approx \text{const}$ (Figs. 12, 15, 18—$e < 10^{-5}, \Lambda < 10^{-5}$); 3.—quasiperiodic instability, characterized by periodic alternation of pauses with a trapezoidal increment function $\gamma_{\pm}(t)$, with decreasing value over time and pauses with $\gamma(t) = 0$ (Figs. 20, 22—$e > 5 \times 10^{-5}, \Lambda \gtrsim 10^{-5}$); 4.—aperiodic instability characterized by aperiodic instability phase alternation with a constant maximum value $\gamma_{\pm}(t)$ (Fig. 23—$e > 10^{-5}, \Lambda \gg 10^{-5}$). It is shown that the value of the wave number $n$ affects the development of instability only at its earliest stages: as $n$ increases, the beginning of the unstable phase slightly shifts to later times.

5. Since, according to (58) and (59), the disturbance amplitude growth factor is determined by the area under the plot of the $\gamma(t)$ function, we can conclude that in the case of a narrow-band instability (Fig. 9) $\chi_{\infty} \approx 28$, in the case of broad-band instability $\chi(t) \approx 1.4 \cdot t$ (moreover, $t$ can reach at least values of the order of 600), in the case of quasi-periodic instability (Fig. 20), taking into account that the duration of the “empty” phases is equal to half the period, we can estimate $\chi(t)$ as a sum of the area of the first burst and half the area of the curvilinear trapezium described by individual bursts, $\gamma_{\pm}(t) > 0$:

$$\gamma_{\pm}(t) \approx 42 + 0.5 \cdot (t - 35).$$

Since even in the case of narrow-band instability, the amplitude of perturbations $\gamma(t)$ can increase very strongly ($e^{28} \sim 10^{12}$), there is a real possibility of explaining the formation mechanism of supermassive BHs in the early Universe using the gravitational-scalar instability in the \(\mathbb{M}_F\) model, built on a one-component system of degenerate scalarly charged fermions and a classical Higgs scalar field. This issue will be explored in detail in the next part of the article.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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