Discrete phase space - II:  
The second quantization of free relativistic wave fields

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Abstract

The Klein-Gordon equation, the Maxwell equation, and the Dirac equation are presented as partial difference equations in the eight-dimensional covariant discrete phase space. These equations are also furnished as difference-differential equations in the arena of discrete phase space and continuous time. The scalar field and electromagnetic fields are quantized with commutation relations. The spin-1/2 field is quantized with anti-commutation relations. Moreover, the total momentum, energy and charge of these free relativistic quantized fields in the discrete phase space and continuous time are computed exactly. The results agree completely with those computed from the relativistic fields defined on the space-time continuum.

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1. Introduction

In the preceding paper [1], we have presented the Lagrangian formalism for the relativistic wave fields in the covariant discrete phase space, as well as in the discrete phase space and continuous time. In this paper, we shall choose three special examples of relativistic fields [2]. These are the free scalar field, electromagnetic fields, and the Dirac spin-1/2 field. Moreover, we shall quantize these fields with the usual commutation and anti-commutation rules. We generalize variational techniques for the operator-valued second-quantized wave fields. The non-singular Green’s functions for various difference equations are provided [2,3]. Finally, we compute the totally conserved four-momentum and charge for three different fields exactly. These computations are performed for the wave-equations satisfying difference-differential equations in the discrete phase space and continuous time only. (We do not consider wave fields in the covariant discrete phase space to calculate the totally conserved quantities for some physical reasons.) The results of these

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computations are identical to those calculated from the usual relativistic quantum theory of the free fields in the Minkowski space-time. One may wonder about the utility of such a complicated, alternate formulation of the quantum theory of free fields! Actually, the present treatment of the free quantized fields in the discrete phase space and continuous time is just a prelude to the more exciting investigations of the interacting fields and the $S$-matrix in the following paper III.

2. Definitions and notations

We use physical units such that $h = c = \ell = 1$. (Here, $\ell$ is a characteristic length.) All physical quantities are expressed as dimensionless numbers. Greek indices take from $\{1, 2, 3, 4\}$ and roman indices take from $\{1, 2, 3\}$. The summation convention is followed. The Minkowski metric is denoted by $\eta_{\mu\nu}$ and the signature of the metric is $+2$. We denote the set of all real numbers by $\mathbb{R}$ and all non-negative integers by $\mathbb{N}$. A bold roman letter indicates a three-dimensional vector. The equations in the covariant discrete phase space are denoted by $(..A)$, whereas the equations in the discrete phase space and continuous time are labelled by $(..B)$.

An integer $n^\mu$ is associated with a phase space circle of radius $\sqrt{2n^\mu + 1}$ for each $\mu \in \{1, 2, 3, 4\}$. Let a function be defined by $f : \mathbb{N}^4 \to \mathbb{R}$ (or $\mathbb{N}^4 \to \mathbb{C}$). The right-difference, the left-difference, and the weighted mean difference are defined respectively by:

\[
\Delta_\mu f(n) := f(\ldots, n^\mu + 1, \ldots) - f(\ldots, n^\mu, \ldots),
\]

\[
\Delta_\mu' f(n) := f(\ldots, n^\mu, \ldots) - f(\ldots, n^\mu - 1, \ldots),
\]

\[
\Delta_\mu^# f(n) := (1/\sqrt{2}) \left( \sqrt{n^\mu + 1} f(\ldots, n^\mu + 1, \ldots) - \sqrt{n^\mu} f(\ldots, n^\mu - 1, \ldots) \right).
\]

Let us work out an example to elaborate the definition (1iii). Suppose that $k_1$ is a real number and $H_{n^1}(k_1)$ is a Hermite polynomial. (See Appendix I.) In that case

\[
\Delta_1^# \xi_{n^1}(k_1) := \Delta_1^# \left[ \frac{(i)^{n_1} e^{-(k_1)^2/2} H_{n^1}(k_1)}{\pi^{1/4} 2^{n_1/2} \sqrt{(n^1)!}} \right] = (i k_1) \xi_{n^1}(k_1).
\]

In the case of $\phi(n) \equiv \phi(n^1, n^2, n^3, n^4)$ is an operator-valued function over a domain of $\mathbb{N}^4$, we adopt the same definitions as in (1i,ii,iii).
3. The second quantization of a free non-hermitian scalar field

Let \( \rho, \rho_\mu \) be five linear operators over a non-separable Hilbert space \([4]\). Let \( \rho^\dagger, \rho_\mu^\dagger \) denote the corresponding adjoint or hermitian-conjugate operators. The linear operator-valued Lagrangian function \( L \) of ten operators is defined to be \([5]\):

\[
L(\rho, \rho^\dagger; \rho_\mu, \rho_\mu^\dagger) := - (\eta^{\mu\nu} \rho_\mu^\dagger \rho_\nu + \mu^2 \rho^\dagger \rho),
\]

\[
\frac{\partial L}{\partial \rho} = -\mu^2 \rho^\dagger, \quad \frac{\partial L}{\partial \rho^\dagger} = -\mu^2 \rho,
\]

\[
\frac{\partial L}{\partial \rho_\mu} = -\eta^{\mu\nu} \rho_\nu^\dagger, \quad \frac{\partial L}{\partial \rho_\mu^\dagger} = -\eta^{\mu\nu} \rho_\nu,
\]

\[
\frac{\partial^2 L}{\partial \rho \partial \rho^\dagger} = -\mu^2 \mathbb{I}, \quad \frac{\partial^2 L}{\partial \rho_\mu \partial \rho_\mu^\dagger} = -\eta^{\mu\nu} \mathbb{I}, \quad \frac{\partial^2 L}{\partial \rho_\mu \partial \rho_\mu^\dagger} = \frac{\partial^2 L}{\partial \rho_\mu^\dagger \partial \rho_\mu} = 0.
\]

(3)

Here, \( \mu > 0 \) is the mass parameter, \( \mathbb{I} \) is the identity operator, and \( \mathbb{O} \) is the zero operator. The linear operators do not commute necessarily. Therefore, the order in which they appear must be preserved. The \( j \)-th partial derivatives of \( L \) in (3) with respect to \( \rho, \rho^\dagger, \rho_\mu, \rho_\mu^\dagger \) are all zero operators for \( j \geq 3 \).

Assuming that the Euler-Lagrange equations (38A, B) of paper I are valid for the operator-valued functions \( \rho = \phi(n) \) and \( \rho = \phi(n,t) \), we obtain \([2]\) from (3)

\[
\eta^{\mu\nu} \Delta^\#_\mu \Delta^\#_\nu \phi(n) - \mu^2 \phi(n) = 0,
\]

(4A)

\[
\delta^{ab} \Delta^\#_a \Delta^\#_b \phi(n,t) - (\partial_t)^2 \phi(n,t) - \mu^2 \phi(n,t) = 0.
\]

(4B)

(Strictly speaking, \( \phi \) is a section of the fibre bundle \([6]\) of linear operators over the base “manifold” \( \mathbb{N}^4 \) or \( \mathbb{N}^3 \times \mathbb{R} \).

Now we shall derive the operator difference conservation equations. Adapting the equations (A.II.2), (A.II.3), and (A.II.4A) of paper I to the Lagrangian in (3) and noting that operator \( \frac{\partial^2 L}{\partial \rho \partial \rho^\dagger} \) etc. are proportional to the identity, we obtain relativistic conservation equations:

\[
\left\{ \Delta^\#_\mu \left[ \frac{\partial L}{\partial \rho_\mu^\dagger} \right] \cdot \Delta^\#_\mu \phi + \frac{\partial L}{\partial \rho_\mu} \cdot \Delta^\#_\mu \Delta^\#_\mu \phi + (\text{h.c.}) - \Delta^\#_\mu L(\ldots) \right\}
\]

\[
+ \frac{1}{2} \left[ \frac{\partial^2 L}{\partial \rho \partial \rho^\dagger} + \frac{\partial^2 L}{\partial \rho \partial \rho^\dagger} \right]_{\ldots} \cdot \left\{ \Delta^\#_\mu \phi^\dagger(n) \cdot \phi(n) \right\}
\]

\[
- \phi^\dagger(n) \cdot \Delta^\#_\mu \phi - [\Delta^\#_\mu \phi^\dagger] \cdot \phi(n)
\]

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\[
\frac{1}{2} \left[ \frac{\partial^2 L(\ldots)}{\partial \rho_\nu \partial \rho_\nu} + \frac{\partial^2 L(\ldots)}{\partial \rho_\sigma \partial \rho_\nu} \right] \cdot \left\{ \Delta^\mu_{\nu} \left[ \Delta^\mu_{\nu} \phi^\dagger \cdot \Delta^\mu_{\nu} \phi \right] - \Delta^\mu_{\nu} \phi^\dagger \cdot \Delta^\mu_{\nu} \phi - \left[ \Delta^\mu_{\nu} \Delta^\mu_{\nu} \phi^\dagger \right] \cdot \Delta^\mu_{\nu} \phi \right\} \\
+ \left[ \Delta^\mu_{\nu}(1) \right] \left\{ L(\ldots)_\nu \cdot \frac{\partial L(\ldots)}{\partial \rho_\nu} \cdot \phi(n) \right. \\
- \phi^\dagger(n) \cdot \frac{\partial L(\ldots)}{\partial \rho_\nu} \cdot \Delta^\mu_{\nu} \phi - \Delta^\mu_{\nu} \phi^\dagger \cdot \frac{\partial L(\ldots)}{\partial \rho_\nu} \\
+ \frac{1}{2} \left[ \frac{\partial^2 L(\ldots)}{\partial \rho_\nu \partial \rho_\nu} + \frac{\partial^2 L(\ldots)}{\partial \rho_\nu \partial \rho_\nu} \right] \cdot \phi^\dagger(n) \phi(n) \\
+ \frac{1}{2} \left[ \frac{\partial^2 L(\ldots)}{\partial \rho_\nu \partial \rho_\nu} + \frac{\partial^2 L(\ldots)}{\partial \rho_\nu \partial \rho_\nu} \right] \cdot \Delta^\mu_{\nu} \phi^\dagger \cdot \Delta^\mu_{\nu} \phi \right\} = 0. 
\]

Here, \( (h.c.) \) stands for the hermitian-conjugation of the preceding terms.

Since the Lagrangian function \( L \) is a second degree polynomial, the relativistic conservation equation (5A) is \textit{exact} and no additional terms indicated by \ldots are necessary.

Moreover, the last curly bracket in (5A) is exactly the \textit{zero} operator. Using (5v) of paper I and (6i), after a long calculation, the relativistic equation (5A) yields the \textit{exact} conservation equation:

\[
\Delta_\nu \left\{ \sqrt{\frac{n_\nu}{2}} \left[ \frac{\partial L(\ldots)}{\partial \rho_\nu(\ldots,n_\nu-1,\ldots)} \cdot \Delta^\mu_{\nu} \phi + \frac{\partial L(\ldots)}{\partial \rho_\nu} \cdot [\Delta^\mu_{\nu} \phi]_{(\ldots,n_\nu-1,\ldots)} \right] \right. \\
- \delta^\nu_{\mu} \left[ - \eta^{\sigma\tau} \left( \Delta^\mu_{\nu} \phi^\dagger_{(\ldots,n_\nu-1,\ldots)} \cdot \Delta^\mu_{\nu} \phi + \Delta^\mu_{\nu} \phi^\dagger \cdot [\Delta^\mu_{\nu} \phi]_{(\ldots,n_\nu-1,\ldots)} \right) \right] \\
- \mu^2 \left( \phi^\dagger_{(\ldots,n_\nu-1,\ldots)} \cdot \phi(n) + \phi^\dagger(n) \cdot \phi_{(\ldots,n_\nu-1,\ldots)} \right) \right\} = 0. 
\]

In a similar fashion, we can derive \textit{exact} difference-differential equations (which are equivalent to the relativistic equations):

\[
\Delta_\delta T^\delta_{\alpha} + \partial_\delta T^4_{\alpha} = 0, \quad (6Bi) \\
\Delta_\delta T^4_{\alpha} + \partial_\delta T^4_{\alpha} = 0, \quad (6Bii) 
\]
\[ T^b_a(n, t) := \sqrt{\frac{n^a}{2}} \left\{ \frac{\partial L(\cdot)}{\partial \rho_{bl}(..., n^b-1, \cdot)} \right\} \cdot \Delta^a_\phi \]
\[ + \frac{\partial L(\cdot)}{\partial \rho_{bl}(...)} \cdot [\Delta^a_\phi|_{(..., n^b-1, \cdot)} + (\text{h.c.})] \]
\[ - \delta^b_a [\Delta^a_\phi^\dagger|_{(..., n^b-1, \cdot)} \cdot \Delta^a_\phi + \Delta^a_\phi \cdot (\Delta^a_\phi)|_{(..., n^b-1, \cdot)}] \]
\[ + \left[ (\partial_t \phi^\dagger|_{(..., n^b-1, \cdot)} \cdot \partial_t \phi + \partial_t \phi^\dagger \cdot (\partial_t \phi)|_{(..., n^b-1, \cdot)} \right] \]
\[ - \mu^2 \left[ \phi^\dagger(..., n^b - 1, \cdot) \cdot \phi(n) + \phi^\dagger(n) \cdot \phi(..., n^b - 1, \cdot) \right], \]
\[ (6Biii) \]

\[ T^4_a(n, t) := \frac{\partial L(\cdot)}{\partial \rho_{4l}(...)} \Delta^a_\phi + \Delta^a_\phi^\dagger \cdot \frac{\partial L(\cdot)}{\partial \rho^4_{4l}(...)}, \]
\[ (6Biv) \]

\[ T^0_a(n, t) := \sqrt{\frac{n^a}{2}} \left\{ \frac{\partial L(\cdot)}{\partial \rho_{4l}(...)} \cdot \partial_t \phi + (\text{h.c.}) \right\}, \]
\[ (6Bv) \]

\[ T^4_1(n, t) := \left\{ \frac{\partial L(\cdot)}{\partial \rho_{4l}(...)} \cdot \partial_t \phi + (\text{h.c.}) - L(\cdot)|_{(...)} \right\}. \]
\[ (6Bvi) \]

It is instructive to compare and contrast the equations (39Bi,ii,iii,iv,v,vi) of paper I with the equations (6Bi,ii,iii,iv,v,vi). The first set of \( T^\mu_a(n, t) \) satisfy possibly approximate conservation equations whereas the second set satisfy \textit{exact} equations. Neither of these \( T^\mu_a(n, t) \) obey exactly the tensorial transformation rules (34B). However, the \textit{relativistic} total four-momentum components \( P_\mu \)'s and the invariant total charge \( Q \) can be elicited from

\[ P_b = - \sum_{n=0}^{\infty} \left\{ \frac{\partial L(\cdot)}{\partial \rho_{4l}(...)} \cdot \Delta^a_\phi + \Delta^a_\phi^\dagger \cdot \frac{\partial L(\cdot)}{\partial \rho^4_{4l}(...)} \right\} |_{t=0}, \]
\[ (7Bi) \]

\[ H = -P_4 = \sum_{n=0}^{\infty} \left\{ \frac{\partial L(\cdot)}{\partial \rho_{4l}(...)} \cdot \partial_t \phi + \partial_t \phi^\dagger \cdot \frac{\partial L(\cdot)}{\partial \rho^4_{4l}(...)} - \frac{L(\cdot)}{(...)} \right\} |_{t=0}, \]
\[ (7Bii) \]

\[ Q = -ie \sum_{n=0}^{\infty} \left\{ \frac{\partial L(\cdot)}{\partial \rho_{4l}(...)} \cdot \phi(n, t) - \phi^\dagger(n, t) \cdot \frac{\partial L(\cdot)}{\partial \rho^4_{4l}(...)} \right\} |_{t=0}. \]
\[ (7Biii) \]

Note that above \textit{relativistic} equations are \textit{exact} and no additional terms denoted by \ldots are necessary. We furnish a general class of exact solutions [2]
(“the plane wave superposition”) of the (generalized) Klein-Gordon equations (4A,B) in the following:

\[
\phi(n) = \int_{\mathbb{R}^3} [2\omega(k)]^{-1/2} \left\{ a(k) \left[ \prod_{\mu=1}^{4} \xi_{n\mu}(k_\mu) \right] \right. \\
+ \left. b^\dagger(k) \left[ \prod_{\mu=1}^{4} \xi_{n\mu}(k_\mu) \right] \right\} d^3k \\
= : \phi^-(n) + \phi^+(n),
\]

(8A)

\[
\phi(n,t) = \int_{\mathbb{R}^3} [2\omega(k)]^{-1/2} \left\{ a(k) \left[ \prod_{j=1}^{3} \xi_{n\mu}(k_j) \right] e^{-i\omega t} \right. \\
+ \left. b^\dagger(k) \left[ \prod_{j=1}^{3} \xi_{n\mu}(k_j) \right] e^{i\omega t} \right\} d^3k \\
= : \phi^-(n,t) + \phi^+(n,t),
\]

(8B)

\[(k) := (k_1,k_2,k_3), \ d^3k := dk_1,dk_2,dk_3, \quad (9i)\]

\[-k_4 \equiv \omega(k) := +\sqrt{(k_1)^2 + (k_2)^2 + (k_3)^2 + \mu^2} > 0, \quad (9ii)\]

\[\xi_{n\mu}(k_\mu) := (i)^{n\mu} \frac{\exp[-(k_\mu)^2/2] \cdot H_{n\mu}(k_\mu)}{(\pi)^{1/4} 2^{n\mu/2} \sqrt{(n^\mu)!}}, \quad (9iii)\]

Here, the indices \(\mu\) and \(j\) are not summed. Moreover, \(H_{n\mu}(k_\mu)\) stands for a Hermite polynomial. (For the properties of orthonormal complex-polynomials \(\xi_{n\mu}(k_\mu)\), see equation (2) and Appendix I.) The functions “\(a\)” and “\(b\)” are some sections of the fibre-bundle [6] of the linear operators over the base manifold \(\mathbb{R}^3\) (the momentum-space). The operator-valued improper integrals

\[
\int_{\mathbb{R}^3} a^\dagger(k)a(k)d^3k, \int_{\mathbb{R}^3} a(k)a^\dagger(k)d^3k, \int_{\mathbb{R}^3} b^\dagger(k)b(k)d^3k, \int_{\mathbb{R}^3} b(k)b^\dagger(k)d^3k, \quad \text{etc.}
\]

should converge in certain sense for the existence of (8A,B). There are more restrictions on these operators which follow from the quantum theory. These are the following canonical quantum rules to be imposed on the operators \(a, a^\dagger, b, b^\dagger\):

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The eigenvalues of \(N\) are the following (see Appendix II):

\[
[\phi^-(n), \phi^-(\tilde{n})] = [\phi^+(n), \phi^+(\tilde{n})] = [(\phi^-(n))^\dagger, (\phi^-(\tilde{n}))^\dagger]
\]

\[
= [(\phi^+(n))^\dagger, (\phi^+(\tilde{n}))^\dagger] = 0,
\]

\[
[\phi^-(n), (\phi^-)\hat{}(\tilde{n})] = (i/2\pi)\Delta_+(n, \tilde{n}; \mu) I,
\]

\[
[\phi^+(n), (\phi^+)\hat{}(\tilde{n})] = (i/2\pi)\Delta_-(n, \tilde{n}; \mu) I,
\]

\[
[\phi(n), \phi(\tilde{n})] = [(\phi(n))^\dagger, (\phi(\tilde{n}))^\dagger] = 0,
\]

\[
[\phi(n), (\phi(\tilde{n}))^\dagger] = (i/2\pi)\Delta(n, \tilde{n}; \mu) I.
\]

Here, \(\delta(k_j - \tilde{k}_j)\) denotes a Dirac-delta distribution function, \(I(k)\) stands for the identity operator, and "0" denotes the zero operator. The linear operators \(a(k), a^\dagger(k)\) are called the destruction and creation operators for particles (or field quanta). The particle and anti-particle vacuum is denoted by the Hilbert vector \(|\psi_0\rangle\). The particle and anti-particle (occupation) number operators \(N^+(k), N^-(k)\) are defined by and satisfy the following equations:

\[
N^+(k) := a^\dagger(k)a(k), \quad N^-(k) := b^\dagger(k)b(k),
\]

\[
a(k)|\psi_0\rangle = b(k)|\psi_0\rangle = N^+(k)|\psi_0\rangle = N^-(k)|\psi_0\rangle = |0\rangle,
\]

\[
\langle\psi_0|\psi_0\rangle = 1, \quad \langle0|0\rangle = 0,
\]

\[
[N^+(k), a(\tilde{k})] = -\delta^3(k - \tilde{k})a(k),
\]

\[
[N^+(k), a^\dagger(\tilde{k})] = \delta^3(k - \tilde{k})a^\dagger(k),
\]

\[
[N^-(k), b(\tilde{k})] = -\delta^3(k - \tilde{k})b(k),
\]

\[
[N^-(k), b^\dagger(\tilde{k})] = \delta^3(k - \tilde{k})b^\dagger(k),
\]

\[
[N^+(k), b(\tilde{k})] = [N^+(k), b^\dagger(\tilde{k})] \equiv 0,
\]

\[
[N^-(k), a(\tilde{k})] = [N^-(k), a^\dagger(\tilde{k})] \equiv 0,
\]

\[
[N^+(k), N^-(\tilde{k})] \equiv 0.
\]
Moreover, the covariant commutation relations in the difference-differential representation are:

\[
[\phi^-(n, t), \phi^-(\hat{n}, \hat{t})] = [\phi^+(n, t), \phi^+(\hat{n}, \hat{t})] = [(\phi^-(n, t))^\dagger, (\phi^-(\hat{n}, \hat{t}))^\dagger] = [(\phi^+(n, t))^\dagger, (\phi^+(\hat{n}, \hat{t}))^\dagger] \equiv 0,
\]

\[
[\phi^-(n, t), (\phi^-)(\hat{n}, \hat{t})]^\dagger] = i\Delta_+(n, t; \hat{n}, \hat{t}; \mu) I,
\]

\[
[\phi^+(n, t), (\phi^+)(\hat{n}, \hat{t})]^\dagger] = i\Delta_-(n, t; \hat{n}, \hat{t}; \mu) I,
\]

\[
[\phi(n, t), (\phi^+)(\hat{n}, \hat{t})] = [\phi(n, t)^\dagger, (\phi^+(\hat{n}, \hat{t}))^\dagger] \equiv 0,
\]

\[
(12B)
\]

\[
[\phi(n, t), (\phi^+)(\hat{n}, \hat{t})] = i\Delta(n, t; \hat{n}, \hat{t}; \mu) I,
\]

\[
[\phi(n, t), (\phi^+)(\hat{n}, \hat{t})]^\dagger\bigr|_{\hat{t}=t} \equiv 0 \text{ for } n \neq \hat{n},
\]

\[
[\partial_t \phi(n, t), (\phi^+)(\hat{n}, \hat{t})]^\dagger\bigr|_{\hat{t}=t} = -i \left( \prod_{j=1}^{3} \delta_{n_j \hat{n}_j} \right) I = -\delta_{n \hat{n}} I ,
\]

\[
[\partial_t \phi(n, t), (\partial_{\hat{t}} \phi(\hat{n}, \hat{t}))^\dagger\bigr|_{\hat{t}=t} = 0 \text{ for } n \neq \hat{n}.
\]

The last three commutators in (12B) resemble the three fundamental postulates of quantum mechanics, namely

\[
[Q_a, Q_b] \equiv 0, \quad [P_a, Q_b] = -i\delta_{ab} I, \quad [P_a, P_b] \equiv 0.
\]

Now we shall compute the total three-momentum components \(P_j\), the total energy \(H\), and the total charge \(Q\) from the equations (7Bi,ii,iii). (See the comments at the end of Section 5 of paper I for not considering equations labelled A.) Elaborate computations are explicitly performed in Appendix III. We summarize the results in the following equations:

\[
P_j = \int_{\mathbb{R}^3} [N^+(k) + N^-(k)] k_j d^3k , \quad (13i)
\]

\[
H = -P_4 = \int_{\mathbb{R}^3} [N^+(k) + N^-(k) + \delta^3(0) I(k)] \omega(k) d^3k , \quad (13ii)
\]

\[
Q = e \int_{\mathbb{R}^3} [N^+(k) - N^-(k)] d^3k . \quad (13iii)
\]

These results are identical to those derived from the usual relativistic quantum theory of a free non-hermitian scalar field [7] in the (flat) space-time continuum. The equation (13ii) shows that the divergence of the null-point energy cannot be remedied by the discrete phase space approach.

We now use the commutators in the equation (10), (11), the field \(\phi\) in (8B), and the conserved operators in (13i,ii,iii) to derive:

\[
[P_j, \phi(n, t)] = -i\Delta^j\phi(n, t) ,
\]

\[
[H, \phi(n, t)] = -i\partial_t \phi(n, t) ,
\]

\[
[Q, \phi(n, t)] = -e\phi(n, t) .
\]
The above equations prove that the operators $P_j, H, Q$ are the generators for the space translation, the time translation, and the gauge transformation. (We have put here $\delta^{ij}(0)0 = 0$.)

Conservations of the total momentum components $P_j$ and the total charge $Q$ can be proved alternatively by the commutation relations:

$$[H, P_j] \equiv 0,$$

$$[H, Q] \equiv 0.$$  \hfill (15)

4. Quantization of free electro-magnetic field

The Lagrangian function $L$ for the electro-magnetic field is chosen to be the following second degree polynomial function of the twenty \cite{5} linear, self-adjoint operators $y_{\mu}$ and $y_{\mu\nu}$:

$$L(y_{\mu}; y_{\mu\nu}) := -(1/2)\eta^{\mu\nu}\eta^{\alpha\beta}y_{\mu\alpha}y_{\nu\beta}$$

$$= -(1/2)\left[\delta^{ab}\delta^{cd}y_{ac}y_{bd} - \delta^{ab}(y_{a4}y_{b4} + y_{a4}y_{b4}) + (y_{44})^2\right],$$

$$\frac{\partial L(\ldots)}{\partial y_{\mu}} \equiv 0,$$

$$\frac{\partial L(\ldots)}{\partial y_{\rho\tau}} = -\eta^{\rho\mu}\eta^{\tau\nu}y_{\mu\nu},$$

$$\frac{\partial L(\ldots)}{\partial y_{ab}} = y_{4a},$$

$$\frac{\partial L(\ldots)}{\partial y_{a4}} = y_{a4},$$

$$\frac{\partial L(\ldots)}{\partial y_{44}} = -y_{44},$$

$$\frac{\partial^2 L(\ldots)}{(\partial y_{\mu})(\partial y_{\rho\tau})} = -\eta^{\rho\mu}\eta^{\tau\nu}I,$$

$$\frac{\partial^2 L(\ldots)}{(\partial y_{ab})^2} = \frac{\partial^2 L(\ldots)}{(\partial y_{a4})^2} = \frac{\partial^2 L(\ldots)}{(\partial y_{44})^2} = -I,$$

$$\frac{\partial^2 L(\ldots)}{(\partial y_{4a})(\partial y_{cd})} \equiv 0.$$  \hfill (16)

The third and higher order partial derivatives of $L$ are obviously all zero operators.

The Euler-Lagrange equations (36A,B) of paper I extracted from (16) with $y_{\mu} = A_{\mu}(n)$ etc. yield:

$$\eta^{\mu\nu}\Delta^\#_\mu\Delta^\#_\nu A_{\sigma}(n) = 0,$$  \hfill (17A)

$$\delta^{ab}\Delta^\#_a\Delta^\#_b A_{\sigma}(n, t) - (\partial_t)^2 A_{\sigma}(n, t) = 0.$$  \hfill (17B)

These equations are further augmented by the Lorentz-gauge constraint on the allowable state vectors $|\psi\rangle$ (in a “Hilbert space” with indefinite metric):
\[ \langle \psi | \Delta^\mu_\nu A^\mu(n)|\psi \rangle = 0, \quad (18A) \]
\[ \langle \psi | \Delta^\mu_\nu A^\mu(n, t) + \partial_t A^4(n, t)|\psi \rangle = 0. \quad (18B) \]

The Maxwell’s equations (17A,B) and the Lorentz-gauge constraint (18A,B) are preserved by the restricted gauge transformations involving a hermitian operator \( \Omega \):

\[ \hat{A}_\mu(n) = A_\mu(n) - \Delta^\mu_\nu \Omega(n), \quad (19A) \]
\[ \hat{A}_j(n, t) = A_j(n, t) - \Delta^\mu_\nu \Omega(n, t), \quad (19Bi) \]
\[ \hat{A}_4(n, t) = A_4(n, t) - \partial_t \Omega(n, t), \quad (19Bii) \]
\[ \eta^{\mu\nu} \Delta^\mu_\nu \Delta^\mu_\nu \Omega(n) = 0, \quad (20A) \]
\[ \delta^{ab} \Delta^\mu_\nu \Delta^\mu_\nu \Omega(n, t) - (\partial_t)^2 \Omega(n, t) = 0, \quad (20B) \]

By the Lagrangian (16) and the equations (39Biv) and (39Bvi) of paper I, we obtain

\[ P_j = \sum_{n=0}^{\infty(3)} \left[ (\Delta^\mu_\nu A^\mu) \cdot (\partial_t A_\mu) \right]_{t=0}, \quad (21Bi) \]
\[ H = -P_4 = (1/2) \sum_{n=0}^{\infty(3)} \left[ \delta^{ab} (\Delta^\mu_\nu A^\mu) \cdot (\Delta^\mu_\nu A_\mu) + (\partial_t A^\mu) \cdot (\partial_t A_\mu) \right]_{t=0}. \quad (21Bii) \]

The above relativistic equations are exact and no additional terms are necessary. The “plane wave” decomposition of the four-potential operator \( A_\mu(n, t) \) is given by [2]:

\[ A_\mu(n, t) = A^\dagger_\mu(n, t) = \int_{\mathbb{R}^3} [2\nu(k)]^{-1/2} \left\{ a_\mu(k) \left[ \prod_{j=1}^{3} \xi_{nj}(k_j) \right] e^{-i\nu t} \right\} d^3k =: A^-_\mu(n, t) + A^+_\mu(n, t), \]

\[ \nu(k) := +\sqrt{\delta^{ab} k_a k_b} =: -k_4, \]
\[ \Delta^\mu_\nu A_\mu(n, t)|_{t=0} = i \int_{\mathbb{R}^3} [2\nu(k)]^{1/2} k_b \left\{ a_\mu(k) \left[ \prod_{j} \xi_{nj}(k_j) \right] \right\} d^3k, \]
\[ -a^\dagger_\mu(k) \left[ \prod_{j} \bar{\xi}_{nj}(k_j) \right] \}

10
\[ \partial_t A_\mu(n, t)|_{t=0} = i \int_{\mathbb{R}^3} \left[ \nu(k)/2 \right]^{1/2} \left\{ a_{\mu}(k) \left[ \prod_j \xi_{n^j}(k_j) \right] - a^\dagger_{\mu}(k) \left[ \prod_j \xi^{n^j}(k_j) \right] \right\} d^3 k. \] 

\text{(22B)}

The canonical quantization rules are assumed to be:
\[ [a_\mu(k), a_\nu(\hat{k})] = [a^\dagger_\mu(k), a^\dagger_\nu(\hat{k})] = 0, \]
\[ [a_\mu(k), a^\dagger_\nu(\hat{k})] = \eta_{\mu\nu} \delta^3(k - \hat{k}) I(k). \] 

\text{(23B)}

The covariant commutation rules, which follow from (22B), (23B), and (A.II.5B), are summarized below:
\[ [A^+_\mu(n, t), A^\dagger_\nu(\hat{n}, \hat{t})] = [A^-_\mu(n, t), A^-_\nu(\hat{n}, \hat{t})] = 0, \]
\[ [A^+_\mu(n, t), A^-_\nu(\hat{n}, \hat{t})] = \eta_{\mu\nu} D_+(n, t; \hat{n}, \hat{t}) I, \]
\[ [A^+_\mu(n, t), A^-_\nu(\hat{n}, \hat{t})] = \eta_{\mu\nu} D_-(n, t; \hat{n}, \hat{t}) I, \]
\[ [A_\mu(n, t), A_\nu(\hat{n}, \hat{t})] = \eta_{\mu\nu} D(n, t; \hat{n}, \hat{t}) I, \]
\[ [A_\mu(n, t), A_\nu(\hat{n}, \hat{t})]|_{t=\hat{t}} = 0 \quad n \neq \hat{n}, \]
\[ [\partial_t A_\mu(n, t), A_\nu(\hat{n}, \hat{t})]|_{t=\hat{t}} = -i \eta_{\mu\nu} \delta^3 \|_{n=\hat{n}} I, \]
\[ [\partial_t A_\mu(n, t), \partial_t A_\nu(\hat{n}, \hat{t})]|_{t=\hat{t}} = 0 \quad \text{for} \quad n \neq \hat{n}. \]

The computations of the total momentum-energy from (21Bi,ii) and (22B) yield:
\[ P_j = (1/2) \int_{\mathbb{R}^3} \eta^{\mu\nu} [a^+_\mu(k)a_\nu(k) + a_\mu(k)a^\dagger_\nu(k)] k_j d^3 k, \]
\[ H = -P_4 = (1/2) \int_{\mathbb{R}^3} \eta^{\mu\nu} [a^+_\mu(k)a_\nu(k) + a_\mu(k)a^\dagger_\nu(k)] \nu(k) d^3 k. \] 

\text{(25B)}

We can choose a special gauge so that (25B) simplifies considerably. We assume the condition
\[ \langle \psi | k_\mu a^\mu(k) | \psi \rangle = 0. \] 

\text{(26)}

This is a sufficient condition for the satisfaction of the Lorentz-gauge conditions (18A,B). We introduce a special restricted gauge condition (see (22B)) with the help of the hermitian operator-valued function
\[ \Omega(n, t) := -4i \int_{\mathbb{R}^3} |2\nu(k)|^{-5/2} k_b \left\{ a^b(k) \left[ \prod_j \xi_{n^j}(k_j) \right] e^{-ivt} \right. \]
\[ \left. - a^{\dagger b}(k) \left[ \prod_j \xi^{n^j}(k_j) \right] e^{ivt} \right\} d^3 k. \] 

\text{(27B)}
Under this gauge transformation, the new field operators $\hat{a}_\mu(k)$ (corresponding to the field $\hat{A}_\mu(n,t)$) undergo the following transformations:

$$\hat{a}_\mu(k) = a_\mu(k) - [\nu(k)]^{-2} k_\mu [k_\beta a^\beta(k)] , \quad (28i)$$

$$\langle \psi | k_\mu \hat{a}^\mu(k) | \psi \rangle = 0 , \quad (28ii)$$

$$\langle \psi | \hat{a}_4(k) | \psi \rangle = 0 . \quad (28iii)$$

Thus, by (28iii) the temporal component $\hat{a}_4(k)$ drops off. Next, let us consider the “orthonormal” tetrad [8] $e_\mu^{(\lambda)}(k)$ (also see equation (23) of paper I) which in general satisfy:

$$\eta^{(\lambda\sigma)} e_\mu^{(\lambda)}(k) e_\nu^{(\sigma)}(k) = \eta^{\mu\nu} ,$$

$$\eta_{\mu\nu} e_\mu^{(\lambda)}(k) e_\nu^{(\sigma)}(k) = \eta^{\lambda\sigma} ,$$

$$a_\mu(k) := a_\mu(k) e_\mu^{(\lambda)}(k) ,$$

$$a^{(\lambda)}(k) = e_\mu^{(\lambda)}(k) a_\mu(k) . \quad (29)$$

We can choose prudently (for $\nu(k) > 0$) two of the tetrad vectors by the following:

$$(e_1^{(3)}(k), e_2^{(3)}(k), e_3^{(3)}(k), e_4^{(3)}(k)) := [\nu(k)]^{-1}(k_1, k_2, k_3, 0) ,$$

$$e_\mu^{(4)}(k) := \delta_\mu^4 . \quad (30)$$

The choice of the other two vectors $e_\mu^{(1)}(k)$ and $e_\mu^{(2)}(k)$ is arbitrary up to a two-dimensional orthogonal transformation.

The condition (28ii), by (30) yields

$$k_\mu e_\mu^{(\lambda)}(k) \langle \psi | \hat{a}^{(\lambda)}(k) | \psi \rangle = k_\mu e_\mu^{(3)}(k) \langle \psi | \hat{a}^{(3)}(k) | \psi \rangle = [\nu(k)] [\nu(k)]^{-1}(k_1, k_2, k_3, 0) \langle \psi | \hat{a}^{(3)}(k) | \psi \rangle = 0 . \quad (31)$$

Thus, the expectation value of the longitudinal component $\langle \psi | \hat{a}^{(3)}(k) | \psi \rangle$ vanishes. Dropping circumflexes in the sequence, we obtain from (25B), (28iii), (31), and (29), the simplified versions of the expectation values of the total momentum-energy as:

$$\langle \psi | P_3 | \psi \rangle = \sum_{\lambda=1}^{2} \int \langle \psi | N_j(\lambda)(k) | \psi \rangle k_j d^3k , \quad (32B)$$

$$\langle \psi | H | \psi \rangle = \sum_{\lambda=1}^{2} \int \langle \psi | N(\lambda)(k) + (1/2) \delta^3(0) I(\lambda)(k | \psi \rangle \nu(k) d^3k .$$

Only the two degrees of (linear) polarization ($\lambda \in \{1, 2\}$) contribute to the total momentum-energy of the photons. The equations in (32B) are identical to those obtained by the usual relativistic theory [7].
5. Quantization of free spin-1/2 field

The 4 × 4 Dirac matrices \( \gamma^\mu \) satisfy [5]:

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}I, \tag{33}
\]

\[
[\gamma^j]^\dagger = \gamma^j, [\gamma^4]^\dagger = -\gamma^4.
\]

The Dirac bispinor field \( \rho = \psi(n) \) or \( \rho = \psi(n, t) \) is a 4 × 1 column vector of operators in the second quantized theory. We denote

\[
\tilde{\rho} := i\rho^\dagger \gamma^4, \quad [\tilde{\rho}\rho]^\dagger = \tilde{\rho}\rho. \tag{34}
\]

The Lagrangian function for a massive, spin-1/2 field operator \( \rho \) is taken to be:

\[
L(\rho, \tilde{\rho}; \rho_\mu, \tilde{\rho}_\mu) := -(1/2)(\tilde{\rho}\gamma^\mu \rho_\mu - \tilde{\rho}_\mu \gamma^\mu \rho) - m\tilde{\rho}\rho; \quad m > 0,
\]

\[
\frac{\partial L(\rho)}{\partial \rho} = (1/2)\tilde{\rho}\gamma^\mu - m\tilde{\rho}, \quad \frac{\partial L(\rho)}{\partial \tilde{\rho}} = -(1/2)\gamma^\mu \rho_\mu - m\rho,
\]

\[
\frac{\partial L(\rho)}{\partial \rho_\mu} = -(1/2)\tilde{\rho}\gamma^\mu, \quad \frac{\partial L(\rho)}{\partial \tilde{\rho}_\mu} = (1/2)\gamma^\mu \rho,
\]

\[
\frac{\partial^2 L(\rho)}{\partial \tilde{\rho}\partial \rho} = -mI = \frac{\partial^2 L(\rho)}{\partial \rho \partial \tilde{\rho}} = \frac{\partial^2 L(\rho)}{\partial \rho_\mu \partial \tilde{\rho}_\mu} = \frac{\partial^2 L(\rho)}{\partial \rho_\mu \partial \rho_\nu} = \frac{\partial^2 L(\rho)}{\partial \tilde{\rho}_\mu \partial \rho_\nu} = \frac{\partial^2 L(\rho)}{\partial \tilde{\rho}_\mu \partial \tilde{\rho}_\nu} = 0. \tag{35}
\]

The triple and higher partial derivatives of \( L \) are all identically zero operators.

The Euler-Lagrange operator equations from (35) and equations (A.II.5A) and (A.II.6Bi,ii) of paper I are furnished by:

\[
\gamma^\mu \Delta^\#_\mu \psi(n) + m\psi(n) = 0, \tag{36A}
\]

\[
[\Delta^\#_\mu \tilde{\psi}(n)] \gamma^\mu - m\tilde{\psi}(n) = 0, \tag{36A'}
\]

\[
\gamma^j \Delta^\#_j \psi(n, t) + \gamma^4 \partial_t \psi(n, t) + m\psi(n, t) = 0, \tag{36B}
\]

\[
[\Delta^\#_j \tilde{\psi}(n, t)] \gamma^j + [\partial_t \tilde{\psi}(n, t)] \gamma^4 - m\tilde{\psi}(n, t) = 0. \tag{36B'}
\]

By the equation (35), and equations (A.II.5A) and (A.II.6Bi,ii) of paper I, we derive:
\[
\left\{ \Delta^\#_{\mu} \left[ \frac{\partial L(\cdots)}{\partial \rho_{\nu}} \right] \cdot \Delta^\#_{\mu} \psi + \left[ \frac{\partial L(\cdots)}{\partial \rho_{\nu}} \right] \cdot \Delta^\#_{\mu} \Delta^\#_{\mu} \psi + (\text{h.c.}) - \Delta^\#_{\mu} L(\cdots) \right\} \\
+ \left[ \Delta^\#_{\mu} \left( \hat{\psi}(n) \cdot \psi(n) \right) - \hat{\psi}(n) \Delta^\#_{\mu} \psi - (\Delta^\#_{\mu} \hat{\psi}) \cdot \psi(n) \right]
\]

\[
+ \left\{ \Delta^\#_{\mu} \left[ \hat{\psi}(n) \cdot \left( \frac{\partial^2 L(\cdots)}{\partial \rho \partial \rho_{\nu}} \right) \right] \cdot \Delta^\#_{\mu} \psi \\
- \hat{\psi}(n) \cdot \Delta^\#_{\mu} \left[ \left( \frac{\partial^2 L(\cdots)}{\partial \rho \partial \rho_{\nu}} \right) \Delta^\#_{\mu} \psi \right] - \Delta^\#_{\mu} \hat{\psi} \cdot \left[ \frac{\partial^2 L(\cdots)}{\partial \rho \partial \rho_{\nu}} \right] \Delta^\#_{\mu} \psi \right\}
\]

\[
(37A)
\]

\[
+ \left\{ \Delta^\#_{\mu} \left[ \hat{\psi} \cdot \left( \frac{\partial^2 L(\cdots)}{\partial \rho \partial \rho_{\nu}} \right) \right] \psi(n) \\
- \Delta^\#_{\mu} \hat{\psi} \cdot \left[ \frac{\partial^2 L(\cdots)}{\partial \rho \partial \rho_{\nu}} \right] \Delta^\#_{\mu} \psi - (\Delta^\#_{\mu} \Delta^\#_{\mu} \hat{\psi}) \cdot \left[ \frac{\partial^2 L(\cdots)}{\partial \rho \partial \rho_{\nu}} \right] \cdot \psi(n) \right\}
\]

\[
+ 0 =: \Delta_{b} T^a_{b}(n, t) = 0,
\]

\[
\Delta_{b} T^b_{a}(n, t) + \partial_{a} T^4_{b}(n, t) = 0, \quad (37B_i)
\]

\[
\Delta_{b} T^b_{a}(n, t) + \partial_{b} T^4_{a}(n, t) = 0, \quad (37B_{ii})
\]

\[
T^b_{a}(n, t) := \sqrt{\frac{n^b}{2}} \left\{ \left[ \frac{\partial L(\cdots)}{\partial \rho_{b}(\cdots, n^{b-1}, \cdots)} \right] \cdot \Delta^\#_{\mu} \psi \\
+ \frac{\partial L(\cdots)}{\partial \rho_{b}(\cdots, n^{b-1}, \cdots)} \cdot \left( \Delta^\#_{\mu} \psi \right)_{(\cdots, n^{b-1}, \cdots)} + (\text{h.c.}) \right\}
\]

\[
- \delta_{a} \left[ \frac{1}{2} \hat{\psi}(\cdots, n^b - 1, \cdots) \cdot \gamma^c \Delta^\#_{\epsilon} \psi + \hat{\psi}(n, t) \cdot \gamma^c (\Delta^\#_{\epsilon} \psi)_{(\cdots, n^b - 1, \cdots)} \right]
\]

\[
+ \hat{\psi}(\cdots, n^b - 1, \cdots) \gamma^a \partial_{a} \psi + \hat{\psi}(n, t) \gamma^a (\partial_{a} \psi)_{(\cdots, n^b - 1, \cdots)} \quad (37B_{iii})
\]

\[
- (\Delta^\#_{\epsilon} \hat{\psi})_{(\cdots, n^b - 1, \cdots)} \cdot \gamma^c \psi(n, t)
\]

\[
- (\Delta^\#_{\epsilon} \hat{\psi})_{(\cdots, n^b - 1, \cdots)} \cdot (\partial_{a} \hat{\psi})_{(\cdots, n^b - 1, \cdots)} \cdot \gamma^4 \cdot \psi(n, t)
\]

\[
- (\partial_{a} \hat{\psi}) \cdot \gamma^4 \cdot \psi(n, t, \cdots, n^b - 1, \cdots)
\]

\[
+ m(\hat{\psi}(\cdots, n^b - 1, \cdots) \cdot \psi(n, t) + \hat{\psi}(n, t) \cdot \psi(n, t, \cdots, n^b - 1, \cdots))
\]},
\[ T_a^4(n, t) := \frac{\partial L(\ldots)}{\partial \partial a_{4\ldots}} \cdot \Delta a^\#_\psi + (\Delta a^\#_\tilde{\psi}) \cdot \frac{\partial L(\ldots)}{\partial \partial a_{4\ldots}}, \quad (37Biv) \]

\[ T_b^4(n, t) := \sqrt{\frac{n}{2}} \left[ \frac{\partial L(\ldots)}{\partial \partial b_{4\ldots}} \cdot \partial_t \psi + \frac{\partial L(\ldots)}{\partial \partial b_{4\ldots}} \cdot (\partial_t \tilde{\psi})_{\ldots, n^b - 1, \ldots} + (h.c.) \right], \quad (37Bv) \]

\[ T_4^4(n, t) := \left[ \frac{\partial L(\ldots)}{\partial \partial 4_{4\ldots}} \cdot (\partial_t \psi) + (\partial_t \tilde{\psi}) \cdot \frac{\partial L(\ldots)}{\partial \partial 4_{4\ldots}} - L(\ldots)_{\ldots, 0} \right]. \quad (37Bvi) \]

The above equations are exact. Note that the tricky ordering of the operators in the above equations is crucial.

The relativistic total momentum-energy and the charge are given by (see equations (7Bi,ii,iii) and (37Biv,vii)):

\[ P_j = \frac{i}{2} \sum_{n=0}^{\infty} \left[ \Delta^\#_j \psi^\dagger(\ldots, n, t) \cdot \psi(\ldots, n, t) - \psi^\dagger(\ldots, n, t) \cdot \Delta^\#_j \psi \right]_{t=0}, \quad (38Bi) \]

\[ H = -P_4 = \frac{i}{2} \sum_{n=0}^{\infty} \left[ \psi^\dagger(\ldots, n, t) \cdot \partial_t \psi - \partial_t \psi^\dagger(\ldots, n, t) \cdot \psi(\ldots, n, t) \right]_{t=0}, \quad (38Bii) \]

\[ Q = e \sum_{n=0}^{\infty} \psi^\dagger(\ldots, n, 0) \cdot \psi(\ldots, n, 0). \quad (38Biii) \]

A class of “plane wave” solutions of the Dirac equations (36A) and (36B) is provided by (see reference [2] and Appendix I):

\[ \psi(n) = \int_{\mathbb{R}^3} [m/E(p)]^{1/2} \left\{ 2 \sum_{r=1}^{2} \left[ \alpha_r(p) u_r(p) \left( \prod_{\mu} \xi_{\eta^\mu}(p_\mu) \right) \right. \right. \]
\[ + \left. \beta_r^\dagger(p) v_r(p) \left( \prod_{\mu} \xi_{\eta^\mu}(p_\mu) \right) \right] d^3 p \]
\[ =: \psi^-(n) + \psi^+(n), \quad (39Ai) \]

\[ \tilde{\psi}(n) = \int_{\mathbb{R}^3} [m/E(p)]^{1/2} \left\{ 2 \sum_{r=1}^{2} \left[ \alpha_r^\dagger(p) \tilde{u}_r(p) \left( \prod_{\mu} \xi_{\eta^\mu}(p_\mu) \right) \right. \right. \]
\[ + \left. \beta_r(p) \tilde{v}_r(p) \left( \prod_{\mu} \xi_{\eta^\mu}(p_\mu) \right) \right] d^3 p \]
\[ =: \tilde{\psi}^+(n) + \tilde{\psi}^-(n), \quad (39Aii) \]
\[ \psi(n, t) := \int_{\mathbb{R}^3} \left\{ \sum_{r=1}^{2} \left[ \alpha_r(p)u_r(p) \left( \prod_j \xi_{nj}(p_j) \right) e^{-iEt} + \beta_r^\dagger(p)v_r(p) \left( \prod_j \xi_{nj}(p_j) \right) e^{iEt} \right] \right\} d^3p \]

\[ =: \psi^-(n, t) + \psi^+(n, t), \]

\[ \tilde{\psi}(n, t) := \int_{\mathbb{R}^3} \left\{ \sum_{r=1}^{2} \left[ \alpha_r^\dagger(p)\tilde{u}_r(p) \left( \prod_j \xi_{nj}(p_j) \right) e^{iEt} + \beta_r(p)\tilde{v}_r(p) \left( \prod_j \xi_{nj}(p_j) \right) e^{-iEt} \right] \right\} d^3p \]

\[ =: \tilde{\psi}^+(n, t) + \tilde{\psi}^-(n, t), \]

\[ p^4 = -p_4 \equiv E(p) := +\sqrt{\delta^{ab}p_ap_b + m^2} > 0, \]

\[ u_r^\dagger(p)u_r(p) = v_r^\dagger(p)v_r(p) = [E(p)/m] \delta_{rs}, \]

\[ u_r^\dagger(p)v_s(-p) = v_r^\dagger(p)u_s(-p) \equiv 0. \]

The canonical quantization rules for a spin-1/2 field operator are furnished by the anti-commutators:

\[ [A, B]_+ := AB + BA, \]

\[ [\alpha_r(p), \alpha_s(\tilde{p})]_+ = [\beta_r(p), \beta_s(\tilde{p})]_+ \]

\[ = [\alpha_r^\dagger(p), \alpha_s^\dagger(\tilde{p})]_+ = [\beta_r^\dagger(p), \beta_s^\dagger(\tilde{p})]_+ \equiv 0, \]

\[ [\alpha_r(p), \beta_s(\tilde{p})]_+ = [\alpha_r^\dagger(p), \beta_s^\dagger(\tilde{p})]_+ \]

\[ = [\alpha_r(p), \beta_s^\dagger(\tilde{p})]_+ = [\alpha_r^\dagger(p), \beta_s(\tilde{p})]_+ \equiv 0, \]

\[ [\alpha_r(p), \alpha_s^\dagger(\tilde{p})]_+ = [\beta_r(p), \beta_s^\dagger(\tilde{p})]_+ = \delta_{rs}\delta^3(p - \tilde{p})\mathbf{I}. \]

The particle and anti-particle occupation number operators are defined by:

\[ N_r^-(p) := \alpha_r^\dagger(p)\alpha_r(p), \]

\[ N_r^+(p) := \beta_r^\dagger(p)\beta_r(p). \]

Here, the subscript “\( r \)” is not summed. The occupation number operators \( N_r^-(p) \) and \( N_r^+(p) \) take eigenvalues from \( \{0, 1\} \). Therefore, this quantiza-
tion is compatible with the Fermi-Dirac statistics. The particle and anti-
particle vacuum state $|\psi_0\rangle$ is characterized by:

$$\alpha_r(p)|\psi_0\rangle = \beta_r(p)|\psi_0\rangle = |0\rangle,$$
$$\langle\psi_0|\psi_0\rangle = 1, \quad \langle0|0\rangle = 0.$$  \hspace{1cm} (42)

We can derive the covariant quantization rules for a spin-1/2 particle
field by equations (40), (39A), and (39Bi,ii). These are provided by the
following anti-commutation relations:

$$[\psi^-(n), \bar{\psi}^-(\bar{n})]_+ = [\psi^+(n), \bar{\psi}^+(\bar{n})]_+ \equiv 0,$$ \hspace{1cm} (43Ai)

$$[\psi^-(n), \psi^+(\bar{n})]_+ = [\bar{\psi}^-(n), \bar{\psi}^+(\bar{n})]_+ \equiv 0,$$ \hspace{1cm} (43Aii)

$$[\psi(n), \psi(\bar{n})]_+ = [\bar{\psi}(n), \bar{\psi}(\bar{n})]_+ \equiv 0,$$ \hspace{1cm} (43Aiii)

$$[\psi^-(n), \bar{\psi}^+(\bar{n})]_+ = iS_+(n, \bar{n}) I,$$ \hspace{1cm} (43Aiv)

$$[\psi^+(n), \bar{\psi}^-(\bar{n})]_+ = -iS_-(n, \bar{n}) I,$$ \hspace{1cm} (43Av)

$$[\psi(n), \bar{\psi}(\bar{n})]_+ = -iS(n, \bar{n}) I,$$ \hspace{1cm} (43Avi)

$$[\psi^-(n, t), \bar{\psi}^-(\bar{n}, t)]_+ = [\psi^+(n, t), \bar{\psi}^+(\bar{n}, t)]_+ \equiv 0,$$ \hspace{1cm} (43Bi)

$$[\psi^-(n, t), \psi^+(\bar{n}, \bar{t})]_+ = [\bar{\psi}^-(n, t), \bar{\psi}^+(\bar{n}, \bar{t})]_+ \equiv 0,$$ \hspace{1cm} (43Bii)

$$[\psi(n, t), \psi(\bar{n}, \bar{t})]_+ = [\bar{\psi}(n, t), \bar{\psi}(\bar{n}, \bar{t})]_+ \equiv 0,$$ \hspace{1cm} (43Biii)

$$[\psi^-(n, t), \bar{\psi}^+(\bar{n}, \bar{t})]_+ = -iS_+(n, t; \bar{n}, \bar{t}) I,$$ \hspace{1cm} (43Biv)

$$[\psi^+(n, t), \bar{\psi}^-(\bar{n}, \bar{t})]_+ = -iS_-(n, t; \bar{n}, \bar{t}) I,$$ \hspace{1cm} (43Bv)

$$[\psi(n, t), \bar{\psi}(\bar{n}, \bar{t})]_+ = -iS(n, t; \bar{n}, \bar{t}) I,$$ \hspace{1cm} (43Bvi)

$$[\psi(n, 0), \psi^+(\bar{n}, 0)]_+ = \delta_{nn}^3 I.$$ \hspace{1cm} (43Bvii)

Here, the Green’s functions $S(n, \bar{n}) \rangle$ and $S(n, t; \bar{n}, \bar{t})$ are borrowed from
Appendix II.

Now, we shall compute the total momentum, energy, and charge by the
equations (39Bi), (38Bi,ii,iii), (40), and (41). After laborious calculations,
similar to those in Appendix III, we obtain rather neat results:

$$P_j = \sum_{r=1}^{2} \int_{\mathbb{R}^3} [N_r^-(p) + N_r^+(p)] p_j d^3p,$$ \hspace{1cm} (44Bi)
\[ H = -P_4 = \sum_{r=1}^{2} \int_{\mathbb{R}^3} \left[ N_r^- (p) + N_r^+ (p) - \delta^3 (0) \mathbf{i}_r (p) \right] E (p) d^3 p, \quad (44\text{Bii}) \]

\[ Q = e \sum_{r=1}^{2} \int_{\mathbb{R}^3} \left[ N_r^- (p) - N_r^+ (p) + \delta^3 (0) \mathbf{i}_r (p) \right] d^3 p, \quad (44\text{Biii}) \]

In the case of the electron-position field, we choose \( N_r^- (p) \) as the number operator for electrons, and the charge parameter \( e = -\sqrt{4\pi/137} \); in contrast to the charged scalar particles in (13iii). The results (44Bi,ii,iii) coincide exactly with the usual relativistic quantum theory of a free spin-1/2 field in the (flat) space-time continuum [7].

**Appendix I: Hermite and related complex polynomials**

The definitions and the useful formulae for the Hermite polynomials [9] are provided here (\( k \in \mathbb{R}; \ n \in \{0, 1, 2, \ldots\} \)):

\[ H_n (k) := (-1)^n e^{k^2} \frac{d^n}{dk^n} (e^{-k^2}) , \quad (A.I.1) \]

\[ H_n (-k) := (-1)^n H_n (k) , \quad (A.I.2) \]

\[ \frac{d^2 H_n (k)}{dk^2} - 2k \frac{dH_n (k)}{dk} + 2nH_n (k) = 0 , \quad (A.I.3) \]

\[ \frac{dH_n (k)}{dk} = 2nH_{n-1} (k) , \quad n \geq 1 , \quad (A.I.4) \]

\[ H_{n+1} (k) = 2kH_n (k) - 2nH_{n-1} (k) , \quad n \geq 1 . \quad (A.I.5) \]

The complex, orthonormal polynomials \( \xi_n (k) \) satisfy the following definition and equations:

\[ \xi_n (k) := \frac{(i)^n e^{-k^2/2} H_n (k)}{\pi^{1/4} 2^{n/2} \sqrt{n!}} := (i)^n f_n (k) , \quad \xi_{-n-1} (k) := 0 , \quad \text{for} \ n \in \mathbb{N} , \quad (A.I.6) \]

\[ \xi_n (-k) = (-1)^n \xi_n (k) = \overline{\xi_n (k)} , \quad (A.I.7) \]

\[ \frac{d^2 \xi_n (k)}{dk^2} + (-k^2 + 2n + 1) \xi_n (k) = 0 , \quad (A.I.8) \]

\[ \frac{d\xi_n (k)}{dk} + k \xi_n (k) = i \sqrt{2n} \xi_{n-1} (k) , \quad (A.I.9) \]

\[ \Delta^* \xi_n (k) = i k \xi_n (k) , \quad (A.I.10) \]
\[ \xi_{2n}(k) = \frac{(-1)^n e^{-k^2/2}}{\pi^{1/4} 2^n \sqrt{2n!}} \cdot \left[ (2k)^{2n} + \sum_{j=1}^{n} (-1)^j 2^{2n-j} (2j - 1)! \left( \frac{2n}{2j} \right) k^{2(n-j)} \right], \tag{A.I.11a} \]

\[ \xi_{2n+1}(k) = \frac{i(-1)^n e^{-k^2/2}}{\pi^{1/4} 2^{n+1/2} \sqrt{(2n + 1)!}} \cdot \left[ (2k)^{2n+1} + \sum_{j=1}^{n} (-1)^j 2^{2n+1-j} (2j - 1)! \left( \frac{2n + 1}{2j} \right) k^{2n+1-2j} \right], \tag{A.I.11b} \]

\[(2j - 1)!! := (2j - 1)(2j - 3) \ldots 5.3.1 \, )\]

\[ \xi_0(k) = \frac{e^{-k^2/2}}{\pi^{1/4}}, \tag{A.I.12} \]

\[ \xi_1(k) = \frac{i\sqrt{2} k e^{-k^2/2}}{\pi^{1/4}}, \tag{A.I.13} \]

\[ \xi_{2n}(0) = \frac{(2n - 1)!!}{\pi^{1/4} \sqrt{(2n)!}} = \frac{\sqrt{(2n)!}}{\pi^{1/4} 2^n n!}, \tag{A.I.14a} \]

\[ \xi_{2n+1}(0) \equiv 0, \tag{A.I.14b} \]

\[ \exp \left\{ \left[ (t^2 + k^2)/2 \right] + i \sqrt{2} tk \right\} = \sum_{n=0}^{\infty} \frac{\xi_n(k) t^n}{\sqrt{n!}}, \tag{A.I.15} \]

\[ 2^{n/2} \exp \left\{ - \left[ (k - p)^2/2 \right] \xi_n[(k + p)/\sqrt{2}] \right\} = \pi^{1/4} \sum_{j=0}^{n} \sqrt{\binom{n}{j}} \xi_{n-j}(k) \xi_j(p), \tag{A.I.16} \]

\[ (k - \tilde{k}) \sum_{n=0}^{N} \frac{\xi_n(k) \xi_n(\tilde{k})}{\xi_{N+1}(k) \xi_{N+1}(\tilde{k})} = i \sqrt{(N + 1)/2} \, \left[ \xi_{N+1}(k) \xi_N(\tilde{k}) - \xi_{N+1}(\tilde{k}) \xi_N(k) \right], \tag{A.I.17} \]

\[ \int_{-\infty}^{\infty} \xi_m(k) \xi_n(k) \, dk = \int_{-\infty}^{\infty} f_m(k) f_n(k) \, dk = \delta_{mn}, \tag{A.I.18} \]

\[ \sum_{n=0}^{\infty} \xi_n(k) \xi_n(p) = \delta(k - p), \tag{A.I.19} \]
\[
\sum_{n=0}^{\infty} \xi_n(k) \xi_n(p) = \sum_{n=0}^{\infty} \overline{\xi_n(k)} \overline{\xi_n(p)} = \delta(k + p) , \tag{A.I.20}
\]
\[
\xi_n(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_n(x) e^{ikx} dx . \tag{A.I.21}
\]

**Appendix II: Non-singular Green’s functions**

The relativistic invariant Green’s functions [2] for the finite difference and the difference-differential Klein-Gordon equations (4A) and (4B) respectively are given by (see Fig. 1):

\[
\Delta_{(a)}(n; \hat{n}; \mu) := \int_{\mathbb{R}^3} \int_{C_{(a)}} \left\{ (\eta^{\alpha\beta} k_\alpha k_\beta + \mu^2)^{-1} \right. \\
\left[ \prod_{\mu=1}^{4} \xi_{n\mu}(k_\mu) \xi_{\hat{n}\mu}(k_\mu) \right] dk^4 \right\} d^3 k , \tag{A.II.1A}
\]
\[
\Delta_{(a)}(n, t; \hat{n}, \hat{t}; \mu) := (2\pi)^{-1} \int_{\mathbb{R}^3} \left\{ \prod_{j=1}^{3} \xi_{n,j}(k_j) \xi_{\hat{n},j}(k_j) \right. \\
\left[ \int_{C_{(a)}} (\eta^{\alpha\beta} k_\alpha k_\beta + \mu^2)^{-1} \exp[i k^4 (t - \hat{t})] \right] dk^4 \right\} d^3 k . \tag{A.II.1B}
\]

![FIG. 1 The complex $k^4$-plane.](image-url)
The Green’s functions involving the closed contours in Fig. 1 are called homogeneous, whereas the Green’s functions involving the open contours are called inhomogeneous. Assuming appropriate uniform convergences of the improper integrals in (A.II.1A, B) and using the equation (A.I.18), we derive that

\begin{equation}
\eta_{\mu \nu} \Delta_{\mu}^{\#} \Delta_{\nu}^{\#} \Delta_{(a)}(n; \bar{n}; \mu) - \mu^2 \Delta_{(a)}(n; \bar{n}; \mu) = \int_{R^3} \left\{ \int_{C(a)} \prod_{\mu=1}^{4} \xi_{\mu \nu}(k_{\mu}) \xi_{\mu \nu}(k_{\mu}) \right\} dk^4 \right\} d^3 k
\end{equation}

(A.II.2A)

\begin{equation}
\delta^{jl} \Delta_{jl}^{\#} \Delta_{r}^{\#} \Delta_{(a)}(n, t; \bar{n}, \hat{t}; \mu) - \partial_{t}^{2} \Delta_{(a)}(n, t; \bar{n}, \hat{t}; \mu)
\end{equation}

\begin{equation}
= - \left(2\pi\right) \frac{-1}{3} \left\{ \int_{R^3} \left[ \prod_{j=1}^{3} \xi_{n j}(k_{j}) \xi_{\hat{n} j}(k_{j}) \right] \right\} d^3 k
\end{equation}

(A.II.2B)

Consider the particular homogeneous Green’s function \( \Delta(n, t; \bar{n}, \hat{t}; \mu) \) in (A.II.1B) which is associated with the closed contour \( C \) in Fig. 1. Performing the closed contour integration in the complex \( k^4 \)-plane, we obtain

\begin{equation}
\Delta(n, t; \bar{n}, \hat{t}; \mu) = \int_{R^4} \left[ \prod_{j=1}^{3} \xi_{n j}(k_{j}) \xi_{\bar{n} j}(k_{j}) \right] \left[ \omega^{-1} \sin(\omega(t-\hat{t})) \right] d^3 k,
\end{equation}

(A.II.3B)

\[ \Delta(n, t; \bar{n}, \hat{t}; \mu) \equiv 0 \quad \text{for} \quad n \neq \bar{n}, \]

\[ [\partial_{t} \Delta(n, t; \bar{n}, \hat{t}; \mu)]_{|t=t} = -\delta_{n \bar{n}}, \]

\[ [\partial_{t} \partial_{\hat{t}} \Delta(n, t; \bar{n}, \hat{t}; \mu)]_{|t=t} \equiv 0. \]

There exist the following relations among the Green’s functions:
\[ \Delta(n, t; \hat{n}, \hat{t}; \mu) = \Delta_+(n, t; \hat{n}, \hat{t}; \mu) + \Delta_-(n, t; \hat{n}, \hat{t}; \mu), \]
\[ \Delta_-(n, t; \hat{n}, \hat{t}; \mu) = \overline{\Delta_+(n, t; \hat{n}, \hat{t}; \mu)}, \]
\[ \Delta(\hat{n}, \hat{t}; n, t; \mu) = -\Delta(n, t; \hat{n}, \hat{t}; \mu), \]
\[ \Delta_F(n, t; \hat{n}, \hat{t}; \mu) := -\theta(t - \hat{t})\Delta_+(n, t; \hat{n}, \hat{t}; \mu) + \theta(\hat{t} - t)\Delta_-(n, t; \hat{n}, \hat{t}; \mu), \]
\[ \Delta_F(\hat{n}, \hat{t}; n, t; \mu) = \Delta_F(n, t; \hat{n}, \hat{t}; \mu), \]
\[ \theta(t) := \left( \frac{1}{2} \right) (1 + t/|t|) \text{ for } t \neq 0. \]

We define the Green’s functions for the massless case \( \mu = 0 \) by:

\[ D_{(a)}(n; \tilde{n}; 0) := \Delta_{(a)}(n; \tilde{n}; 0), \quad (A.II.5A) \]
\[ D_{(a)}(n, t; \hat{n}, \hat{t}) := \Delta_{(a)}(n, t; \hat{n}, \hat{t}; 0). \quad (A.II.5B) \]

In the second quantization of the spin-1/2 fields, we encounter the 4×4 matrix-valued Green’s functions [2] \( S_{(a)}(n; \tilde{n}; m) \) and \( S_{(a)}(n, t; \hat{n}, \hat{t}; m) \). These are defined by:

\[ S_{(a)}(n; \tilde{n}; m) := (\gamma^\mu \Delta_{(a)} - m I) \Delta_{(a)}(n; \tilde{n}) \]
\[ = \int_{\mathbb{R}^3} \left\{ \int_{C_{(a)}} (\eta^{\alpha\beta} p_\alpha p_\beta + m^2)^{-1}(i\gamma^\mu p_\mu - m I) \right. \]
\[ \left. \left[ \prod_{\nu = 1}^{4} \xi_{n\nu}(p_\nu) \overline{\xi_{\tilde{n}\nu}(p_\nu)} \right] dp^4 \right\} d^3p, \quad (A.II.6A) \]

\[ S_{(a)}(n, t; \hat{n}, \hat{t}; m) := (\gamma^j \Delta_{(a)} + \gamma^4 \partial_t - m I) \Delta_{(a)}(n, t; \hat{n}, \hat{t}; m) \]
\[ = (2\pi)^{-1} \int_{\mathbb{R}^3} \left\{ \int_{C_{(a)}} (\eta^{\alpha\beta} p_\alpha p_\beta + m^2)^{-1}(i\gamma^j p_j + i\gamma^4 p_4 - m I) \right. \]
\[ \left. \left[ \prod_{b = 1}^{3} \xi_{n\nu}(p_b) \overline{\xi_{\tilde{n}\nu}(p_b)} \right] \exp(ip_4(t - \hat{t})) dp^4 \right\} d^3p. \quad (A.II.6B) \]

Here, \( m \geq 0 \) is the mass parameter, \( \gamma^\mu \) are Dirac matrices, and \( C_{(a)} \) are contours in the complex \( p^4 \)-plane (exactly similar to those in Fig.1).

We can prove that

\[ (\gamma^\mu \Delta_{(a)} + m I) S_{(a)}(n; \tilde{n}; m) = \begin{cases} -\delta_{n\tilde{n}}^j I & \text{for the inhomogeneous,} \\ 0 & \text{for the homogeneous.} \end{cases} \quad (A.II.7A) \]
\[
(\gamma^j \Delta_j^\# + \gamma^4 \partial_t + m I) S_{(a)}(n, t; \hat{n}, \hat{t}; m) \\
= \left\{ \begin{array}{ll}
-\delta_{nn}^3 \delta(t - \hat{t}) I & \text{for the inhomogeneous,} \\
0 & \text{for the homogeneous.}
\end{array} \right. \quad (A.II.7B)
\]

There exist linear relationships
\[
S(n; \hat{n}; m) = S_{+}(n; \hat{n}; m) + S_{-}(n; \hat{n}; m), \quad (A.II.8A)
\]
\[
S(n, t; \hat{n}, \hat{t}; m) = S_{+}(n, t; \hat{n}, \hat{t}; m) + S_{-}(n, t; \hat{n}, \hat{t}; m), \quad (A.II.8B)
\]
\[
S_F(n, t; \hat{n}, \hat{t}; m) := -\theta(t - \hat{t}) S_{+}(n, t; \hat{n}, \hat{t}; m) + \theta(t - \hat{t}) S_{-}(n, t; \hat{n}, \hat{t}; m). \quad (A.II.8C)
\]

**Appendix III: Total momentum, energy, and charge of the scalar field**

We shall compute here the total momentum \( P_j \), the total energy \( H = -P_4 \), and the total charge \( Q \) from the non-hermitian scalar field operator \( \phi(n, t) \) in the equations (7Bi,ii,iii). We have from (8B),

\[
\phi(n, t) = \int_{\mathbb{R}^3} [2\omega(k)]^{-1/2} \left\{ a(k) \left[ \prod_{j=1}^{3} \xi_{n,j}(k_j) \right] e^{-i \omega t} \\
+ b^\dagger(k) \left[ \prod_{j=1}^{3} \xi_{n,j}(k_j) \right] e^{i \omega t} \right\} d^3 k,
\]
\[
\phi^\dagger(n, t) = \int_{\mathbb{R}^3} [2\omega(k)]^{-1/2} \left\{ a^\dagger(k) \left[ \prod_{j=1}^{3} \xi_{n,j}(k_j) \right] e^{i \omega t} \\
+ b(k) \left[ \prod_{j=1}^{3} \xi_{n,j}(k_j) \right] e^{-i \omega t} \right\} d^3 k, \quad (A.III.1)
\]
\[
\Delta_j^\# \phi(n, t)_{t=0} = i \int_{\mathbb{R}^3} k_j [2\omega(k)]^{-1/2} \left\{ a(k) \left[ \prod_{j=1}^{3} \xi_{n,j}(k_j) \right] \\
- b^\dagger(k) \left[ \prod_{j=1}^{3} \xi_{n,j}(k_j) \right] \right\} d^3 k,
\]
\[
\partial_t \phi(n, t)_{t=0} = -i \int_{\mathbb{R}^3} \omega(\hat{k}) [2\omega(\hat{k})]^{-1/2} \left\{ a(\hat{k}) \left[ \prod_{\ell=1}^{3} \xi_{n,\ell}(\hat{k}_\ell) \right] \\
- b^\dagger(\hat{k}) \left[ \prod_{\ell=1}^{3} \xi_{n,\ell}(\hat{k}_\ell) \right] \right\} d^3 \hat{k}.
\]
We shall first compute the total charge $Q$ from (7Bi) and (A.III.1) for the sake of simplicity.

$$Q = i e \sum_{n=0}^{\infty(3)} \left[ \phi^\dagger(n,t) \partial_t \phi(n,t) \right]_{t=0} + \text{(h.c.)}$$

$$= (e/2) \sum_{n=0}^{\infty(3)} \int \int (\bar{\omega}/\omega)^{1/2} \left\{ \left[ a^\dagger a \left( \prod_j \prod_\ell \xi_n^{j,\ell} \xi_{n^t}^{j,\ell} \right) - b^\dagger b \left( \prod_j \prod_\ell \xi_n^{j,\ell} \xi_{n^t}^{j,\ell} \right) \right] + \left[ b a \left( \prod_j \prod_\ell \xi_n^{j,\ell} \xi_{n^t}^{j,\ell} \right) - a^\dagger b^\dagger \left( \prod_j \prod_\ell \xi_n^{j,\ell} \xi_{n^t}^{j,\ell} \right) \right] \right\} + \text{(h.c.)}.$$ 

Carrying out the triple sum $\sum_n^{(3)}$ with help of the completeness relations (A.19) and (A.20), we obtain

$$Q = (e/2) \int \frac{d^3 k}{\mathbb{R}^3} \left[ a^\dagger(k) a(k) - b(k) b^\dagger(k) \right] + \left[ b(k) a(-k) - a^\dagger(k) b^\dagger(-k) \right] d^3 k + \text{(h.c.)}$$

$$= e \int \frac{d^3 k}{\mathbb{R}^3} \left[ a^\dagger(k) a(k) - b(k) b^\dagger(k) \right] d^3 k.$$ 

Using the definitions (11) and the commutators (10), we get

$$Q = e \int \frac{d^3 k}{\mathbb{R}^3} \left[ N^+(k) - N^-(k) - \delta^3(0) I(k) \right] d^3 k.$$ 

The last divergent term in the above equation could have been avoided by modifying [10] the Lagrangian (3). Thus we derive the equation (13iii).

Now, we shall compute the total momentum components $P_j$ from (A.III.1), (7Bi), and (10). The result is

$$P_j = - \sum_{n=0}^{\infty(3)} \left[ \partial_t \phi^\dagger(n,t) \cdot \Delta^\#_j \phi(n,t) \right]_{t=0} + \text{(h.c.)}$$

$$= (1/2) \sum_{n=0}^{\infty(3)} \int \int \frac{k_j (\bar{\omega}/\omega)^{1/2}}{\mathbb{R}^3} \prod_j \prod_\ell \xi_n^{j,\ell} \xi_{n^t}^{j,\ell} \right] + \left[ b b^\dagger \left( \prod_j \prod_\ell \xi_n^{j,\ell} \xi_{n^t}^{j,\ell} \right) \right]$$
\[-[\hat{a}\hat{b}\left(\prod\prod_{\ell j} \xi_{\ell j}\xi_{\ell j}'\right) + \hat{b}\hat{a}\left(\prod\prod_{\ell j} \xi_{\ell j}'\xi_{\ell j}\right)]\}\} d^3k d^3\hat{k} + (h.c.)
\]

\[= (1/2) \int_{\mathbb{R}^3} k_j \{[a^\dagger(k)a(k) - b(k)b^\dagger(k)] - [a^\dagger(-k)b^\dagger(k)
+b(-k)a(k)]\} d^3k + (h.c.)
\]

\[= \int_{\mathbb{R}^3} k_j [a^\dagger(k)a(k) + b^\dagger(k)b(k) + \delta^3(0) I(k)] d^3k \quad \text{(A.III.3)}
\]

Here, we put \(\delta^3(0) \int_{\mathbb{R}^3} k_j I(k) d^3k = 0\) in the sense of the Cauchy-Principal-Value and thus derive the equation (13i).

Finally, we calculate the total energy \(H\) by the equations (A.III.1), (7Bii), (10), and (11). We obtain

\[H = -P_4\]

\[= \sum_{n=0}^{\infty} \left[ \delta^{ab}(\Delta^n_\# \phi^\dagger \cdot \Delta^n_\# \phi) + \partial_t \phi^\dagger \cdot \partial_t \phi + \mu^2 \phi^\dagger n(t) \phi(n, t) \right]_{t=0}
\]

\[= (1/2) \sum_{n=0}^{\infty} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\omega\omega)^{-1/2} \left[ (\delta^{ab}k_a\tilde{k}_b + \omega\omega + \mu^2) \right.
\]

\[\left. (a^\dagger\tilde{a} + b\tilde{b}^\dagger) \delta^3(k - \tilde{k})
\]

\[-(\delta^{ab}k_a\tilde{k}_b + \omega\omega - \mu^2)(a^\dagger\tilde{b}^\dagger + \tilde{b}\tilde{a}) \delta^3(k + \tilde{k}) \right] d^3k d^3\tilde{k}
\]

\[= \int_{\mathbb{R}^3} \omega(k) \left[ a^\dagger(k)a(k) + b(k)b^\dagger(k) \right] d^3k \cdot \quad \text{(A.III.4)}
\]

By the above equation, the commutation relation (10), and the number operators in (11), the total energy in (13ii) is obtained.

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FIG. 1 The complex $k^4$-plane.