SHARP UNIVERSAL RATE FOR STABLE BLOW-UP OF COROTATIONAL WAVE MAPS

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Abstract. We consider the energy-critical (corotational) $1$-equivariant wave maps into the two-sphere. By the seminal work [53] of Raphaël and Rodnianski, there is an open set of initial data whose forward-in-time development blows up in finite time with the blow-up rate $\lambda(t) = (T-t)e^{-\sqrt{|\log(T-t)|} + O(1)}$. In this paper, we show that this $e^{O(1)}$-factor in fact converges to the universal constant $2e^{-1}$, and hence these solutions contract at the universal rate $\lambda(t) = 2e^{-1}(T-t)e^{-\sqrt{|\log(T-t)|}(1 + o_{t\to T}(1))}$. Our proof is inspired by recent works on type-II blow-up dynamics for parabolic equations. The key improvement is in the construction of an explicit invariant subspace decomposition for the linearized operator perturbed by the scaling generator in the dispersive case, from which we obtain a more precise ODE system determining $\lambda(t)$.

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1. Introduction

1.1. Equivariant wave maps. We consider the energy-critical wave maps on $\mathbb{R}^{1+2}$ with the $S^2$-target. These are defined by formal critical points to the action

$$\mathcal{L}(\phi) = \frac{1}{2} \int_{\mathbb{R}^{1+2}} (-|\partial_t \phi|^2 + |\nabla \phi|^2) dx dt.$$  

The Euler–Lagrange equation associated with the action (1.1), which is

$$\partial_{tt} \phi - \Delta \phi = (-|\partial_t \phi|^2 + |\nabla \phi|^2) \phi$$

in our case, is called the wave maps equation from $\mathbb{R}^{1+2}$ into $S^2$. The model under consideration is also known as the $O(3)$ sigma model on the plane in the physics literature. It is one of the simplest nonlinear scalar field models that admit topological solitons given by harmonic maps. A particular interest is drawn from the fact that harmonic maps come from solutions to the first-order Bogomol’nyi equation [4], manifesting the self-dual structure of the equation. Mathematically, the wave maps equation is a natural geometric generalization of the free wave equation whose nonlinear structure is tied to the geometry of the target manifold.

We refer to [46, 25, 57] for a nice introduction to the subject.

The action (1.1) is invariant under the symmetries of both the Minkowski space and the target space $S^2$. In particular, from the time-translation symmetry, the energy functional is conserved under the flow (1.2):

$$E(\phi, \partial_t \phi) = \frac{1}{2} \int_{\mathbb{R}^2} (|\partial_t \phi|^2 + |\nabla \phi|^2) dx$$

Moreover, (1.2) is scaling invariant: if $\phi(t, x)$ is a solution to (1.2), then so is $\phi_\lambda(t, x) = \phi(\frac{t}{\lambda}, \frac{x}{\lambda})$. (1.2) is called energy-critical because this scaling preserves the energy.

In this paper, we consider (1.2) within 1-equivariance. This means that we restrict to solutions of the form

$$\phi(t, r, \theta) = (\sin u(t, r) \cos \theta, \sin u(t, r) \sin \theta, \cos u(t, r)) \in S^2,$$

where $(r, \theta)$ denote polar coordinates on $\mathbb{R}^2$ and $u = u(t, r)$ is the new unknown function taking values in $\mathbb{R}$. The wave maps equation (1.2) then reads

$$\partial_{tt} u - \partial_{rr} u - \frac{1}{r} \partial_r u + \frac{\sin(2u)}{2r^2} = 0.$$  

We note that one can also consider $k$-equivariant maps, where $k \in \mathbb{N}$ and the map $\phi$ takes the form (1.3) with the two $\theta$s in the right hand side being replaced by $k \theta$. The equation (1.4) is then changed by multiplying $k^2$ to the last term.

We will use the first-order formulation of (1.4). In other words, we consider the $\mathbb{R}^2$-valued function $u = u(t, r)$ defined by

$$u := \begin{bmatrix} u \\ \dot{u} \end{bmatrix} := \begin{bmatrix} u \\ \partial_t u \end{bmatrix}$$

and rewrite the equation (1.4) as

$$\partial_t u = \partial_t \begin{bmatrix} u \\ \dot{u} \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{u} + \frac{1}{r} \partial_r u - \frac{\sin(2u)}{2r^2} \end{bmatrix}.$$  

The energy functional reads

$$E(u) = 2\pi \int_0^\infty \frac{1}{2} \left( |\dot{u}|^2 + |\partial_r u|^2 + \frac{\sin^2 u}{r^2} \right) dr.$$  

The equation is known to be well-posed in the energy space $\mathcal{E}$, defined by the set of 1-equivariant maps with finite energy. For any $u \in \mathcal{E}$ both limits of $u(r)$ as $r \to +\infty$ and $r \to 0$ exist and take values in $\pi \mathbb{Z}$. Thus the space $\mathcal{E}$ is a disjoint
union of its connected components $\mathcal{E}_{\ell,m}$ for $\ell, m \in \mathbb{Z}$, whose elements $u$ additionally satisfy $\lim_{r \to 0} u(r) = \ell \pi$ and $\lim_{r \to +\infty} u(r) = m \pi$. The dynamics of (1.4) for finite energy solutions are then considered separately on each component $\mathcal{E}_{\ell,m}$. Note that we may always assume $\ell = 0$ because (1.4) is invariant under the transforms $u \mapsto u + \pi$ and $u \mapsto -u$.

In the energy space $\mathcal{E}$, there exists a nontrivial static solution (i.e., nonconstant harmonic map)

$$Q(r) := 2 \arctan(r),$$

which is unique up to sign and addition of a multiple of $\pi$. $Q := (Q, 0)$ belongs to $\mathcal{E}_{0,1}$, has energy

$$E(Q) = 4 \pi,$$

and is characterized (up to scaling symmetry) as the energy minimizer in the class $\mathcal{E}_{0,1}$. Moreover, $Q$ is the ground state without symmetry (the corresponding map $\phi$ being the stereographic projection) in the sense that $Q$ has the least energy among nontrivial harmonic maps. Within $k$-equivariance for general $k \in \mathbb{N}$, the ground state is given by

$$Q^{(k)}(r) := 2 \arctan(r^k) \quad \text{with} \quad E(Q^{(k)}) = 4 \pi k.$$

The local-in-time Cauchy problem of (1.2) is by now well-understood. The local well-posedness for regular solutions, being a semilinear problem, is classical. Through the efforts of many authors, the regularity threshold for local well-posedness has been pushed down to the scaling critical regularity; see for example [37, 38, 63, 39, 65]. We refer to [25] and references therein for more discussions on the local theory of wave maps.

Moreover, there have been spectacular developments in the description of the long-term dynamics for wave maps. Earlier works include the global regularity results by Christodoulou–Tahvildar-Zadeh [5], Shatah–Tahvildar-Zadeh [58, 59], and the bubbling result of Struwe [62], under symmetry conditions. These works also say that, if a (symmetric) wave map develops a singularity in finite time then (i) the energy does not concentrate at the backward lightcone and (ii) the solution bubbles off a nontrivial harmonic map as approaching to the blow-up time. The latter fact suggests the following threshold conjecture: if a wave map has energy less than the ground state energy, then it exists globally (and scatters). Note that the threshold is taken to be $+\infty$ if there are no nontrivial harmonic maps, and the threshold for the $S^2$-target is $E(Q) = 4 \pi$. Even without symmetry, Krieger–Schlag [12], Sterbenz–Tataru [60, 61], and Tao [64] established the threshold conjecture for reasonable targets. (See also [14] for equivariant data.) For the $S^2$-target, $4 \pi$ is the real threshold as demonstrated by the finite-time blow-up constructions with energy arbitrarily close to $4 \pi$, due to Krieger–Schlag–Tataru [43] and Raphaël–Rodnianski [53]. We will come back to these blow-up solutions in more details.

A further interesting observation in the threshold conjecture is that, if one restricts to degree zero wave maps (this corresponds to $u \in \mathcal{E}_{0,0}$ in our equivariance reduction) then the real threshold is the twice of the energy of the ground state ($2E(Q) = 8 \pi$ for the $S^2$-target) due to topological reasons; see [12, 45]. For the $S^2$-target within $k$-equivariance, the complete classification of degree zero wave maps with the threshold energy $8 \pi k$ is provided by Jendrej, Lawrie, and Rodríguez [28, 29, 30, 51, 59]. These threshold wave maps can develop a singularity only by forming a pure two-bubble solution, which exists globally in time for $k \geq 2$ and blows up in finite time for $k = 1$.\footnote{Within $k$-equivariance, the threshold energy is $8 \pi k$ because $Q^{(k)}$ is the ground state with energy $4 \pi k$.}
Remarkably, recent works by Jendrej–Lawrie \[32\] (for all \(k \geq 1\)) and Duyckaerts–Kenig–Martel–Merle \[19\] (for \(k = 1\) and 4D radial critical wave equation (NLW)) established soliton resolution for equivariant wave maps into the \(S^2\)-target: any finite energy equivariant wave maps asymptotically decompose into the sum of decoupled (in scales) harmonic maps and a radiation. In the context of equivariant wave maps, some prior works to the resolution are \[11\] \[12\] \[13\] \[51\] \[52\] (where deep insights from the works \[20\] \[21\] \[22\] on the critical NLW play a crucial role. See also \[17\] \[18\] \[24\] for nonradial results in this direction. We refer to \[22\] \[23\] \[19\] \[6\] for soliton resolution for the radial critical wave equations in various dimensions.

In this paper, we focus on the singularity formation via one bubble. More precisely, we consider finite energy blow-up solutions \(u(t, r)\) which decompose as

\[
u(t) - Q_{\lambda(t)}^{(k)} \to z^* \quad \text{as} \ t \to T,
\]

where \(Q_{\lambda(t)}^{(k)} = Q^{(k)}(\cdot/\lambda(t))\), \(k \geq 1\), \(T \in (0, +\infty]\), and \(z^* = z^*(r)\) is the asymptotic profile (or radiation). The convergence in (1.6) indeed holds in the energy topology, but it suffices to consider much weaker convergences at this moment.

The first rigorous constructions of one bubble blow-up solutions are due to Krieger–Schlag–Tataru \[13\] (for \(k = 1\)), Rodnianski–Sterbenz \[55\] (for \(k \geq 4\)) and Raphaël–Rodnianski \[53\] (for all \(k \geq 1\)). However, the blow-up solutions in (1.3) have quite different characters from those of \[53\] \[55\]. In \[43\] (and \[24\]), the authors construct 1-equivariant finite-time blow-up solutions with the blow-up rates

\[
\lambda(t) = (T - t)^{1+\nu}
\]

for any \(\nu > 0\) via the method of backward construction. These solutions have very limited regularity at the backward light cone \(r = |T - t|\) especially for small values of \(\nu\). As the method heavily uses the fact that the linearized operator (see (1.18) below) when \(k = 1\) has the zero resonance

\[
\Lambda Q(r) = \frac{2r}{1 + r^2},
\]

it seems difficult to extend the construction for higher \(k\) (see however \[44\]). Recently, for small values of \(\nu\), Krieger, Miao, and Schlag \[40\] \[41\] proved the stability of these solutions under smooth non-equivariant(!) perturbations supported inside the light cone, so that the perturbation still preserves the shock at the light cone. Moreover, Pillai \[51\] \[52\] extended the approach to global-in-time solutions and constructed infinite-time blow-up (and also oscillating or relaxing) solutions for \(k = 1\).

On the other hand, the authors in \[55\] \[53\] describe a stable finite-time blow-up regime via the method of forward construction. More precisely, there is an open set (in \(H^2 \times H^1\) topology) of initial data within \(k\)-equivariance for all \(k \geq 1\), containing smooth finite energy initial data, such that forward-in-time maximal solutions starting from this set blow up in finite time in a universal regime:

\[
\lambda(t) = \begin{cases} 
 c_k (T - t)|\log(T - t)|^{-\frac{2}{\nu - 2}}(1 + o_{t \to T}(1)) & \text{if } k \geq 2, \\
 (T - t)e^{-\sqrt{|\log(T - t)|} + O(1)} & \text{if } k = 1,
\end{cases}
\]

where \(c_k\) is some universal constant depending only on \(k\). Key features of the proof are approximation of the blow-up dynamics by a finite-dimensional dynamics of well-prepared blow-up profiles and the forward-in-time control of remainders using monotonicity (more precisely, repulsivity).

Let us finally mention a recent work \[33\] of Jendrej, Lawrie, and Rodriguez, who investigated the relation between the blow-up speed \(\lambda(t)\) and the asymptotic profile \(z^*(r)\).
1.2. Main results. The goal of this paper is to refine the description of the finite-time blow-up solutions constructed in [53] for $k = 1$.

Before we recall this result, we first give the definition of the function space $\mathcal{H}_Q^2$, where the initial data of these smooth blow-up solutions belong to. Roughly speaking, $\mathcal{H}_Q^2$ will look like $(\dot{H}^2 \times \dot{H}^1) \cap \mathcal{E}_{0,1}$. We define the function spaces $\mathcal{H}_1^1$ and $\hat{H}_1^1$ with the norms:

$$
\| f \|_{\mathcal{H}_1^1}^2 := \| \partial_y y f \|_{L^2}^2 + \left\| \frac{1}{y(\log y)} |f| \right\|_{L^2}^2,
$$

$$
\| g \|_{\hat{H}_1^1}^2 := \| \partial_y g \|_{L^2}^2 + \frac{1}{y^2} \| g \|_{L^2}^2.
$$

The affine space $\mathcal{H}_Q^2$ is then defined by

$$
(1.10) \quad \mathcal{H}_Q^2 = Q + \mathcal{H}^2, \quad \text{where} \quad \mathcal{H}^2 = (\dot{H}_1^1 \cap \dot{H}_1^2) \times H_1^1.
$$

We are now ready to recall the result of [53] for $k = 1$.

**Theorem 1.1** (Raphael–Rodnianski blow-up solutions for $k = 1$ [53]). There exists an open set $\mathcal{O}$ in $\mathcal{H}_Q^2$ such that for any $u_0 = (\bar{u}_0, \bar{\bar{u}}_0) \in \mathcal{O}$ its forward-in-time solution $u = (u, \bar{u})$ to (1.3) satisfies the following:

- (Finite-time blow-up) $u$ blows up in finite time $T = T(u_0) \in (0, \infty)$;
- (Description of the blow-up) There exist $\lambda(t) \in C^1([0, T), \mathbb{R}_+)$ and $u^* = (u^*, \bar{u}^*) \in \dot{H}_1^1 \times L^2$ such that

$$
\begin{align*}
\lim_{t \to T^-} u(t) - Q_{\lambda(t)} & \to u^* \quad \text{in} \quad \dot{H}_1^1 \times L^2 \\
\lambda(t) & = (T - t) e^{-\sqrt{\log(T-t) + O(1)}}.
\end{align*}
$$

- (Additional properties) $u$ also satisfies the properties in Proposition 4.7.

Note that the blow-up rate (1.11) has a room of $O(1)$-freedom. The question of whether this $O(1)$-freedom is present or not has been left open, though the numerical simulations in Ovchinnikov–Sigal [50] suggest the non-existence of such $O(1)$-freedom and the convergence of $e^{O(1)}$ to some universal constant. Our main result provides a rigorous proof of this universality and computes this value precisely.

**Theorem 1.2** (Sharp universal blow-up rates). Let $u$ be a finite-time blow-up solution as in Theorem 1.1. Then, we have

$$
(1.12) \quad \lambda(t) = 2 e^{-1}(T - t) e^{-\sqrt{\log(T-t) + O(1)}} (1 + o_{t \to T}(1)).
$$

**Remark 1.3** (Sharp universal blow-up rate). Our result is inspired by recent works [17, 33] by Collot–Ghouli–Masmoudi–Nguyen on the 2D Keller–Segel system, who applied the eigenfunction expansion method introduced by Hadžić and Raphaël [27] to refine the blow-up construction in Raphaël–Schweyer [54]. They in particular obtained the sharp asymptotics of the blow-up rate, proved the stability of the blow-up under nonradial perturbations, and also constructed other blow-up regimes that are conditionally stable (of finite codimension).

Our universal constant for the blow-up rate, $2 e^{-1} \approx 0.736$, does not match with the value $0.416 \approx 0.382$ obtained numerically in [50]. $2 e^{-1}$ is close to the twice of this numerical value.

---

$^1$We note that the space $\mathcal{H}_Q^2$ defined here and the space $\mathcal{H}^2$ used in [53] (1.19) are the same. Indeed, $\| \partial_y y f \|_{L^2}^2 + \|1_{(0,1)} \frac{1}{y(\log y)} |f| \|_{L^2}^2$ (plus some subcoercive term $\|1_{\sim 1} f \|_{L^2}^2$ of $\mathcal{H}^2$) can control $\| \partial_y y f \|_{L^2}^2$ of $\mathcal{H}_Q^2$. Conversely, for $f \in \mathcal{H}_Q^2$ (note that $f \equiv 1$ is not allowed), $\| \partial_y y f \|_{L^2}^2$ can control $\|1_{(0,1)} \frac{1}{y(\log y)} |f| \|_{L^2}^2$ of $\mathcal{H}^2$. See for example the proof of (1.1) of this paper.
Remark 1.4 (Comments on the proof). The key ingredient of the proof of Theorem 1.2 is the extension of (some parts of) the approach of \cite{27, 9, 7, 8} on type-II blow-up problems to the dispersive setting. More precisely, we will obtain an explicit formal invariant subspace decomposition (or, partial diagonalization) for the linearized operator $M_\nu$ (see (1.17) below) arising in the self-similar coordinates. As a consequence, we deduce Theorem 1.2 from Theorem 1.1. The part that could not be extended in this paper is the dissipativity of $M_\nu$ with respect to our invariant subspace decomposition. See Remark 3.4 for more discussions. This is the reason why our blow-up analysis is built upon the result of \cite{53}. However, to put it differently, our work demonstrates that the approach of \cite{53} combined with the invariant subspace decomposition for $M_\nu$ yields (i) the existence of stable blow-up regime for (1.4) and (ii) the sharp universality of the blow-up rate for such blow-up solutions.

Remark 1.5 (On Krieger–Schlag–Tataru blow-up solutions). As mentioned above, the finite-time blow-up solutions considered here arise from smooth initial data. This is also reflected in our analysis of the linear operator $M_\nu$, as our blow-up regime is derived from determining some smooth eigenpairs $(\lambda_j, \varphi_j)$ to $M_\nu$ and studying the dynamics of the eigenfunctions $\varphi_j$ under (1.4) (more precisely, the dynamics of $P$; see (1.28)). The same operator also appears in the blow-up construction of \cite{43}. However, the latter construction involves approximate formal inversions of the operator $M_\nu - \lambda$ with various choices of $\lambda < 0$ depending on the prescribed blow-up rate (1.7) and hence results in non-smooth solution ansatz at the light cone $r = |T - t|$ (See for example the behavior of the fundamental solutions $h_1$ and $h_2$ at $z = 0$ in Section 2.2 of this paper).

Remark 1.6 (Extension to weaker equivariance and rotational instability). The equivariant symmetry used in this paper is a strong restriction; $\phi(t, r, \theta)$ with fixed $\theta$ is confined in one great circle of $S^2$. Thus one may also consider the dynamics under a weaker version of equivariant symmetry, which only requires the property $\phi(t, r, \theta) = R(\theta)\phi(t, r, 0)$ where $R(\theta)$ is the rotation on $S^2$ by the angle $\theta$ around the $z$-axis. It is believed that the blow-up regime considered here is no longer stable under this symmetry. The blow-up is expected to be stable under codimension-1 perturbations and rather exhibit the rotational instability: solutions stop concentrating at small but nonzero scale, take a quick rotation by the angle $\pi$, and then start to spread out. We refer to the discussions in \cite{47, 66} for Schrödinger maps and the rigorous construction of solution families exhibiting the rotational instability \cite{35} for the self-dual Chern–Simons–Schrödinger equation (which is also a self-dual model). It is an interesting open problem to show this rotational instability rigorously.

### 1.3. Strategy of the proof.

We use modulation analysis to prove Theorem 1.2. We proceed similarly as in recent works \cite{27, 9, 7, 8} on type-II blow-up dynamics for parabolic equations. We extend some parts of this approach (namely, the invariant subspace decomposition) to type-II blow-up problems in the dispersive setting, and show by its consequence that our main Theorem 1.2 can be deduced from Theorem 1.1.

Since we will assume the result of Theorem 1.1 we already know the blow-up time $T < +\infty$ of $u$ and its refined description near the blow-up time. As the blow-up speed is almost self-similar, it is natural to study the dynamics of $u$ in the self-similar coordinates $(\tau, \rho)$ defined by

\[
\tau := -\log(T - t) \quad \text{and} \quad \rho := \frac{r}{T - t}.
\]
Denoting by $v$ the renormalized solution of $u$

\begin{equation}
\label{eq:1.14}
v(\tau, \rho) = \frac{v(\tau, \rho)}{v(\tau, \rho)} = \left[ \frac{u(t, r)}{(T - t)\dot{u}(t, r)} \right],
\end{equation}

the equation \((1.15)\) reads

\begin{equation}
\label{eq:1.15}
\partial_\tau v = \left[ \begin{array}{c}
v \\
\dot{\varphi}
\end{array} \right] = \left[ \begin{array}{c}
-\Lambda v + \dot{v} \\
\frac{1}{\rho} \partial_\rho v - \frac{\sin(2\nu)}{\nu} - \Lambda_0 \dot{v}
\end{array} \right],
\end{equation}

Note that $v$ can be written as

$$v(\tau, \rho) = Q_v(\tau)(\rho) + \tilde{v}(\tau, \rho),$$

where $\nu(\tau) = \lambda(t)/(T - t)$ is a slowly varying scale going to 0 as $\tau \to +\infty$ and $\tilde{v}$ is the remainder term. Linearizing \((1.15)\) around $Q_v$ in the self-similar coordinates, we have

\begin{equation}
\label{eq:1.16}
\partial_\tau \tilde{v} = -(\partial_\tau Q_v + \Lambda Q_v) + M_v \tilde{v} = \left[ \begin{array}{c}
0 \\
R_{NL}(\tilde{v})
\end{array} \right],
\end{equation}

where $M_v$ is the linear operator defined by

\begin{equation}
\label{eq:1.17}
M_v := \left[ \begin{array}{cc}
-\Lambda & 1 \\
-H_v & -\Lambda_0
\end{array} \right],
\end{equation}

$H_v$ is the usual linearized operator around $Q_v$

\begin{equation}
\label{eq:1.18}
H_v := -\partial_\rho \rho - \frac{1}{\rho} \partial_\rho V_v := -\partial_\rho \rho - \frac{1}{\rho} \partial_\rho + \frac{\cos(2\nu_v)}{\nu_v^2},
\end{equation}

and $R_{NL}(\tilde{v})$ is a nonlinear term in $\tilde{v}$. As seen in \([53]\) for $k = 1$, the nonlinear term $R_{NL}(\tilde{v})$ does not affect the modulation equation (i.e., the evolution law) for $\nu$. Therefore, \textit{any sharper information on the modulation dynamics lies in the analysis of the linear operator $M_v$ in the regime $0 < \nu \ll 1$}.

1. \textit{Construction of the first two eigenpairs for $M_v$.} We construct the first two eigenpairs $(\lambda_0, \varphi_0)$ and $(\lambda_1, \varphi_1)$ of $M_v$ with $\lambda_0 \approx 1$ and $\lambda_1 \approx 0$. We refer to Remark \(2.5\) for more discussions on these eigenvalues, and let us focus on the construction of these eigenpairs here. We find that the matching argument of \([27, 9, 8]\) in the parabolic case can be extended to our operator $M_v$.

To explain the argument, we first notice that if $(\lambda, \varphi)$ is an eigenpair of $M_v$, then $\varphi$ has the structure

$$\varphi = \left[ \begin{array}{c}
\varphi \\
(\lambda + \Lambda) \varphi
\end{array} \right]$$

and $\varphi$ solves the second-order differential equation

\begin{equation}
\label{eq:1.19}
[H_v + (\lambda + \Lambda_0)(\lambda + \Lambda)] \varphi = 0.
\end{equation}

We need to find eigenvalues near 0 and 1 such that $\varphi$ is a smooth solution to ~\((1.19)\), in particular on $[0, 1]$.

The matching argument consists of the following three steps. First, one observes that for any $\lambda$ (with $\lambda \approx 0$ or $\lambda \approx 1$), one can construct a solution $\varphi_{in}$ in the region $\rho \leq 2\delta_0 \ll 1$ (which we call the \textit{inner eigenfunction}) to ~\((1.19)\) which is smooth at $\rho = 0$. When $\lambda$ and $\nu$ are fixed, such a solution is unique up to multiplication by scalars. Next, one observes that for any $\lambda$, one can construct a solution $\varphi_{out}$ in the region $\rho \geq \frac{1}{2}\delta_0$ (which we call the \textit{outer eigenfunction}) to ~\((1.19)\) which is smooth at the light cone $\rho = 1$. Again, such a solution is unique up to multiplication by scalars. The final step is to glue the functions $\varphi_{in}$ and $\varphi_{out}$ at $\rho = \delta_0$. Since we have a freedom of multiplying $\varphi_{in}$ or $\varphi_{out}$ by any scalars, one can have $\varphi_{in}(\rho = \delta_0) = \varphi_{out}(\rho = \delta_0)$ for any $\lambda$. However, the first derivatives of $\varphi_{in}$ and $\varphi_{out}$ at $\rho = \delta_0$ are in general different, and matching them is possible only for \textit{non-generic} values of $\lambda$. These
non-generic values are the eigenvalues, because the glued function becomes a smooth solution to (1.19).

For the construction of the inner eigenfunctions, since \( \rho \ll 1 \), we can treat \((\lambda + \Lambda_0)(\lambda + \Lambda)\varphi\) as a perturbative term. Starting from \( \frac{1}{\rho} \mathcal{L}Q_{\nu} \), which is a smooth kernel element of \( H_{\nu} \), we perform a fixed-point argument to construct \( \varphi_{\text{in}} \). However, due to the critical nature of our spectral problem similarly as in [5], we will need to study more refined structure of \( \varphi_{\text{in}} \); see Section 2.3. For the construction of the outer eigenfunctions, the term \((\lambda + \Lambda_0)(\lambda + \Lambda)\varphi\) should be taken into account, but we can approximate \( H_{\nu} \) by the 1-equivariant Laplacian \(-\Delta_{1}\), thanks to \( \nu \ll 1 \). The operator \(-\Delta_{1} + (\lambda + \Lambda_0)(\lambda + \Lambda)\) can be transformed into the hypergeometric differential operator and we can construct the outer eigenfunction \( \varphi_{\text{out}} \) by starting from a hypergeometric function that is smooth at \( \rho = 1 \). Finally, we can match \( \varphi_{\text{in}} \) and \( \varphi_{\text{out}} \) for non-generic values of \( \lambda \) in view of the implicit function theorem.

2. Formal invariant subspace decomposition of \( \mathbf{M}_{\nu} \). Having constructed the eigenpairs \((\lambda_0, \varphi_0) \) and \((\lambda_1, \varphi_1) \), we construct two linear functionals \( \ell_0 \) and \( \ell_1 \) such that their kernels are invariant under \( \mathbf{M}_{\nu} \) and transversal to the eigenfunctions \( \varphi_0 \) and \( \varphi_1 \). These linear functionals are crucial ingredients of the proof of Theorem 1.2 because they will serve as test functions to detect the refined modulation equations in our later blow-up analysis.

In contrast to the parabolic case, a new input is required to find such linear functionals because \( \mathbf{M}_{\nu} \) is highly non-self-adjoint. In the present work, we find a very simple explicit form of these linear functionals \( \ell_0 \) and \( \ell_1 \):

\[
\ell_j(\epsilon) \coloneqq \langle (\lambda_j + \Lambda_0)\epsilon + \dot{\epsilon}, g_j \varphi_j \rangle, \tag{1.20}
\]

\[
g_j(\nu; \rho) \coloneqq 1_{(0, 1)}(\rho) \cdot (1 - \rho^2)^{\lambda_j - \frac{1}{2}}. \tag{1.21}
\]

The formula is not limited to our situation, and it should be extended to other wave equations in the self-similar coordinates (see Remark 3.3). The appearance of a weight function of the form \( (1 - \rho^2)^p \) in self-similar coordinates is in fact classical (see e.g., [2]), but the formula of the invariant linear functional (1.20) itself does not seem to appear explicitly in the literature. Thus we take this opportunity to state this explicitly and use it to derive refined modulation equations in Section 4.

3. Review of the Raphaël–Rodnianski blow-up solutions. As mentioned above, we work with finite-time blow-up solutions \( u \) constructed in [53]. In particular, we already know that \( u \) blows up in finite time \( T \), and it moreover admits the decomposition

\[
u(t, r) = [P_{\text{RR}}(\tilde{b}(t); \cdot) + \epsilon_{\text{RR}}(t, \cdot)](\frac{r}{\hat{\lambda}(t)})
\]

with some modified profile \( P_{\text{RR}} \) for the blow-up part and \( \tilde{b}(t) \coloneqq -\hat{\lambda}(t) > 0 \), where \( \hat{\lambda} \) and \( \tilde{b} \) satisfy (see Appendix A)

\[
\left| \frac{\hat{\lambda}}{T - t} - \tilde{b} \right| \lesssim \frac{\tilde{b}}{\log \tilde{b}}, \tag{1.22}
\]

and \( \epsilon_{\text{RR}} \) satisfies the smallness estimate inside the lightcone: (see Section 4.4 for more precise statements and notation)

\[
\|\epsilon_{\text{RR}}\|_{\dot{H}^1(y \leq \tilde{b}^{-1})} + \|\epsilon_{\text{RR}}\|_{\dot{H}^1(y \geq \tilde{b}^{-1})} \lesssim \frac{\tilde{b}^2}{\log \tilde{b}}. \tag{1.23}
\]

The logarithmic gain in the above display is crucial. This gain was one of the key observations in [53] and is also indispensable for our error analysis. To put the
above information into our setting, we translate it into the first-order formulation and in terms of the self-similar coordinates, say
\[ \mathbf{v}(\tau, \rho) = \mathbf{P}_{\nu, \mathbf{b}}^\text{RR} (\hat{\nu}(\tau); \rho) + \mathbf{e}_{\nu, \mathbf{b}}^\text{RR} (\tau, \rho). \]

4. Decomposition of solutions. Next, we introduce a new decomposition of \( v \):
\[ \mathbf{v}(\tau, \rho) = \mathbf{P}(\nu(\tau), \mathbf{b}(\tau); \rho) + \mathbf{e}(\tau, \rho), \]
where \( \mathbf{P} = \mathbf{P}(\nu, \mathbf{b}; \rho) \) is a new modified profile
\[ \mathbf{P} = Q + \frac{b}{2} \left\{ (\varphi_0 - \varphi_1) + \left[ \frac{0}{|\log \nu|^2} \right] \right\}. \]

\( \mathbf{P} \) is chosen to achieve the following two goals: (i) \( \mathbf{P} \) is described in terms of \( Q \) and \( \varphi_j \) so that it is fitted for applying the previous linear analysis (and computations) of \( \mathbf{M}_\nu \), and (ii) \( \mathbf{P} \) is sufficiently close to \( \mathbf{P}^\text{RR} \) (after correcting parameters) so that the smallness estimates (1.23) for \( \mathbf{e}^\text{RR} \) can be transferred to \( \mathbf{e} \). Note that \( \mathbf{e} \) does not satisfy orthogonality conditions anymore, but its smallness will suffice.

5. Refined modulation equations. To obtain sharp evolution laws of the modulation parameters \( \nu \) and \( \mathbf{b} \), we simply test the evolution equation of \( v \) against each \( \ell_j \).

This improves the precision of the modulation equations in (53).

First, as our definition of \( \ell_j \) is sharply localized to (the inside of) the light cone \( \rho \leq 1 \) with the correct weight function \( g_j \), the computations do not encounter any cutoff errors even at the self-similar scale \( \rho \approx 1 \). This allows us to obtain the modulation equations
\[
\begin{aligned}
\nu \tau - \left( \frac{b}{\nu} - 1 \right) + \frac{4}{|\log \nu|^2} &= (\epsilon\text{-term}) + O\left( \frac{1}{|\log \nu|^2} \right), \\
b \tau - \left( \frac{b}{\nu} - 1 \right) \frac{1}{|\log \nu|} + \frac{1}{2} \frac{1}{|\log \nu|^2} + \frac{1}{2} \frac{\log^2 2}{|\log \nu|^2} &= (\epsilon\text{-term}) + O\left( \frac{1}{|\log \nu|^2} \right).
\end{aligned}
\]

The first improvement is the determination of the precise \( \frac{1}{|\log \nu|^2} \)-order term in (1.24) for \( \mathbf{b} \). See also Remark 4.3.

The second improvement is in the structure of \( (\epsilon\text{-term}) \), thanks to the invariance of \( \ell_j \). We refer to Section 4.3 for details. We remark that \( (\epsilon\text{-term}) \) itself cannot be considered as a perturbative term. However, the additional structure allows us to integrate the refined modulation equations (1.24) by absorbing \( (\epsilon\text{-term}) \) as a correction to the modulation parameters. This part crucially uses the logarithmic gain in the smallness estimate (1.23).

6. Integration of the modulation equations. To integrate the modulation equations (1.24), assuming \( \epsilon = 0 \) temporarily, we reduce (1.24) into a single equation of \( \mathbf{b} \).

This requires a more refined relation between \( \nu \) and \( \mathbf{b} \) than (1.22). For this purpose, we observe that the quantity
\[ \frac{b}{\nu} - 1, \]
which is a priori of size \( O\left( \frac{1}{|\log \nu|} \right) \) by (1.22), is exponentially unstable under the evolution (1.24). Thanks to this exponential instability, a backward-in-time integration of the evolution equation for \( \frac{b}{\nu} - 1 \) with the boundary condition \( \frac{b}{\nu} - 1 \to 0 \) as \( \tau \to +\infty \) gives a refined relation
\[ \frac{b}{\nu} - 1 \approx \frac{1}{|\log \nu|} + (\epsilon\text{-term}) \]
with a certain structure on \((\epsilon\text{-term})\). Substituting this into the \(b_r\)-equation of (1.24) gives
\[
\frac{b_r}{b} + \frac{\frac{1}{2}}{|\log b|} + \frac{\frac{1}{2} - \frac{\log 2}{2}}{|\log b|^2} = (\epsilon\text{-term}) + O\left(\frac{1}{|\log b|^2}\right).
\]
Integrating this equation by absorbing \((\epsilon\text{-term})\) as a correction to \(b\) and using \(\nu = b(1 + o_r \to \infty(1))\) again, the sharp asymptotics of the scaling parameter \(\nu(r)\) follows.
Substituting this into \(\lambda(t) = (T - t)\nu(t)\) completes the proof of Theorem 1.2.

1.4. Organization of the paper. Sections 2–3 are devoted to the linear analysis of the operator \(\mathbf{M}_\nu\). In Section 2, we construct and study the refined properties of the first two eigenpairs \((\lambda_0, \varphi_0)\) and \((\lambda_1, \varphi_0)\). In Section 3, we use these eigenpairs to obtain a formal invariant subspace decomposition for \(\mathbf{M}_\nu\). In particular, we construct key linear functionals \(\ell_0\) and \(\ell_1\) that are invariant under \(\mathbf{M}_\nu\). Section 4 is devoted to the blow-up analysis and the proof of Theorem 1.2.

1.5. Notation. For \(A \in \mathbb{R}\) and \(B > 0\), we use the standard asymptotic notation \(A \lesssim B\) or \(A = O(B)\) to denote the relation \(|A| \leq CB\) for some positive constant \(C\). The dependencies of \(C\) are specified by subscripts, e.g., \(A \lesssim E B \iff A = O_E(B) \iff |A| \leq C(E)B\). We also introduce the shorthands \(\langle x \rangle = (1 + x^2)^{\frac{1}{2}}, \quad \log_+ x = \max\{0, \log x\}, \quad \log_- x = \max\{0, -\log x\}\).

We let \(\chi\) be a smooth radial cutoff function such that \(\chi(r) = 1\) for \(r \leq 1\) and \(\chi(r) = 0\) for \(r \geq 2\). For any \(R > 0\), we define \(\chi_R(r) := \chi(r/R)\) and \(\chi_{\geq R} := 1 - \chi_R\).
We also use the sharp cutoff function on a set \(A\), denoted by \(\Lambda_A\).

For \(f : (0, \infty) \to \mathbb{R}\), we use the shorthand for integrals:
\[
\int f := \int f(r)dr = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(|x|)dx.
\]

For functions \(f, g : (0, \infty) \to \mathbb{R}\), their \(L^2\) inner product is defined by
\[
\langle f, g \rangle := \int f \, g.
\]

All the \(L^p\) norms are equipped with the \(rdr\)-measure unless otherwise stated. For \(\lambda > 0\), we define
\[
f_\lambda(r) := f\left(\frac{r}{\lambda}\right).
\]
For \(s \in \mathbb{R}\), let \(\Lambda_s\) be the infinitesimal generator of the \(\dot{H}^s\)-invariant scaling:
\[
\Lambda_s f := \left. \frac{d}{d\lambda} \right|_{\lambda=1} \chi^{1-s} f(\lambda r) = (r \partial_r + 1 - s)f,
\]
\[
\Lambda f := \Lambda_1 f.
\]
For a vector \(f = (f, \hat{f})^t\), we define
\[
\Lambda_s (r) := \begin{bmatrix} f(\frac{r}{\lambda}) \\ \frac{1}{\lambda} \hat{f}(\frac{r}{\lambda}) \end{bmatrix}.
\]

We also use the \(1\)-equivariant Laplacian \(\Delta_1 := \partial_r + \frac{1}{r} \partial_r - \frac{1}{r^2}\). For a function \(f\) and a norm \(\| \cdot \|_X\), we write \(f = O_X(B)\) to denote \(\|f\|_X \lesssim B\). For \(k \in \mathbb{N}\), we define
\[
|f|_k := \sup_{0 \leq \ell \leq k} |r^\ell \partial^\ell_r f| \quad \text{and} \quad |f|_{-k} := \sup_{0 \leq \ell \leq k} |r^{-\ell} \partial^\ell_r f|.
\]

We also need notation related to hypergeometric functions. We denote by \(\Gamma\) the usual gamma function. We also use Pochhammer’s symbol
\[
(z)_n = \frac{\Gamma(z + n)}{\Gamma(z)} = \begin{cases} 1 & \text{if } n = 0, \\ z(z + 1) \cdots (z + n - 1) & \text{if } n = 1, 2, \ldots. \end{cases}
\]
We will also need the digamma function
\[
\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)},
\]
i.e., the logarithmic derivative of the gamma function $\Gamma$. Let us record several known properties of $\psi$ (see [1] Section 6.3):
\[
\begin{align*}
\psi(1) &= -\gamma, & \psi\left(\frac{1}{2}\right) &= -\gamma - 2\log 2, \\
\psi(2) &= -\gamma + 1, & \psi\left(\frac{3}{2}\right) &= -\gamma - 2\log 2 + 2,
\end{align*}
\]
where $\gamma$ is the Euler–Mascheroni constant, and
\[
\psi(z + 1) = \psi(z) + \frac{1}{z}, \quad \psi(z) - \psi(w) = \sum_{n=0}^{\infty} \left(\frac{1}{n + w} - \frac{1}{n + z}\right).
\]

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## 2. First two eigenpairs for $M_{\nu}$

In this section, we need two small parameters $\nu^*$ and $\delta_0$ to be chosen in the course of the proof, satisfying the parameter dependence $0 < \nu^* < \delta_0 < 1$. Next, $\nu$ will vary in the range $(0, \nu^*)$. Also, $\lambda$ in this section always means a spectral parameter which ranges in either $|\lambda| \lesssim \frac{1}{|\log \nu|}$ or $|\nu - 1| \lesssim \frac{1}{|\log \nu|}$.

The goal (Proposition 2.1) of this section is to construct and study the refined properties of the first two smooth eigenpairs $(\lambda_0, \varphi_0)$ and $(\lambda_1, \varphi_1)$ to the linear operator
\[
M_{\nu} = \begin{bmatrix}
-\Lambda & 1 \\
-\Lambda_0 & -\Lambda_0
\end{bmatrix}
\]
in the regime $0 < \nu < 1$ such that $\lambda_0 \approx 1$, $\lambda_1 \approx 0$, and each $\varphi_j$ is smooth (on $[0, 1]$, in particular). Note that $\lambda_j$ and $\varphi_j$ depend on $\nu$. Recall that $M_{\nu}$ naturally appears in the linearization of (1.13) in the self-similar coordinates.

The two eigenpairs $(\lambda_0, \varphi_0)$ and $(\lambda_1, \varphi_1)$ are crucial ingredients for our blow-up analysis. They will directly appear in our blow-up profile ansatz (1.28) and in the explicit construction of the linear functionals $\ell_j$ (1.20) that are invariant under $M_{\nu}$. The latter functionals will be the key testing functions that enable the refined modulation estimates as well as the sharp universal blow-up rate (1.12).

To be explained in Section 2.3, the first two eigenvalues $\lambda_0$ and $\lambda_1$ turn out to be approximate solutions to $p(\nu; \lambda) = 0$, where
\[
(2.1) \quad p(\nu; \lambda) := \lambda(\lambda - 1)(|\log \nu| - 1 - \frac{d_{\nu}(\lambda)}{2}) + \lambda - \frac{5}{6},
\]
\[
(2.2) \quad d_{\nu}(\lambda) := -\psi(1) - \psi(2) + \psi\left(\frac{\lambda}{2} + 1\right) + \psi\left(\frac{\lambda + 1}{2}\right).
\]
For each $j \in \{0, 1\}$, it is easy to see by the implicit function theorem that
\[
(2.3) \quad \exists! \lambda_j = \hat{\lambda}_j(\nu) \text{ satisfying } p(\nu; \hat{\lambda}_j) = 0 \text{ in the class } |\lambda_j - (1 - j)| \lesssim \frac{1}{|\log \nu|}.
\]

\footnote{In the blow-up analysis, $\lambda$ or $\lambda(t)$ in general denotes the scale of the soliton $Q_{\lambda(t)}$.}
\footnote{Recall the digamma function $\psi = \Gamma'/\Gamma$.}
Moreover, since (due to (1.27))
\[ d_0(1 - j) = 1 - 2 \log 2 - 2j, \]
each \( \tilde{\lambda}_j \) satisfies the following \( \frac{1}{|\log \nu|} \)-expansion:
\[
\begin{align*}
\tilde{\lambda}_0 &= 1 + \frac{\lambda_j}{|\log \nu|} + \frac{\hat{\lambda}_j}{|\log \nu|^2} + O\left( \frac{1}{|\log \nu|^3} \right), \\
\tilde{\lambda}_1 &= \frac{\lambda_j}{|\log \nu|} + \frac{\hat{\lambda}_j}{|\log \nu|^2} + O\left( \frac{1}{|\log \nu|^3} \right).
\end{align*}
\]

The above expansion also holds after taking \( \nu \partial \nu \) to the both sides. We can now state the main result of this section.

**Proposition 2.1** (First two eigenpairs). There exists \( \nu^* > 0 \) such that for any \( \nu \in (0, \nu^*) \) and \( j \in \{0, 1\} \), there exist unique eigenvalue \( \lambda_j = \lambda_j(\nu) \) and corresponding eigenfunction \( \varphi_j = \varphi_j(\nu; \rho) \) to the linear operator \( M_\nu \) with the following properties.

- **(Smooth eigenfunction \( \varphi_j \))** The eigenfunctions \( \varphi_j(\nu; \cdot) \) are globally defined and analytic. \( \varphi_j \) has the structure
\[
\varphi_j = \left[ (\lambda_j + \Lambda) \varphi_j \right]
\]
and \( \varphi_j \) solves the differential equation
\[
[H_\nu + (\lambda_j + \Lambda_0)(\lambda_j + \Lambda)]\varphi_j = 0.
\]
- **(Estimates for the eigenvalues)** The eigenvalues \( \lambda_j = \lambda_j(\nu) \) satisfy
\[
|\lambda_j - \lambda_j| \lesssim \nu^2 |\log \nu|, \\
|\nu \partial \nu \lambda_j| \lesssim \frac{1}{|\log \nu|^2}.
\]
In particular, the expansion (2.4) holds for \( \lambda_j \).
- **(Sharp estimates of \( \varphi_j \) inside the light cone)** For \( \rho \in (0, 1] \), we have
\[
\varphi_j(\nu; \rho) = \frac{1}{\nu^2} [\Lambda Q + \nu T_1 + (2\lambda_j - 1)\nu S_1 + \nu(\lambda_j - 1)\nu U_1] \left( \frac{2}{\nu^3} \right) + \chi_{\nu, \rho} \lambda_j(\lambda_j - 1)U_\infty(\lambda_j; \rho) + \tilde{\varphi}_j(\nu; \rho),
\]
where the profiles \( T_1(y), S_1(y), U_1(y) \), and \( U_\infty(\lambda_j; \rho) \) are defined in (2.21) and 2.22, and the remainder \( \tilde{\varphi}_j \) satisfies for any \( k \in \mathbb{N} \)
\[
1_{(0,1]}(\tilde{\varphi}_j[k] + |\nu \partial \nu \tilde{\varphi}_j[k]|) \lesssim k \nu^2 \left( 1_{(0,1]}(\nu^2) \right)^4 + 1_{[0,1]}(\rho^2 |\log(\frac{2}{\nu^3})|)^2.
\]
- **(Rough estimates of \( \varphi_j \))** For any \( k \in \mathbb{N} \), we have
\[
1_{(0,1]}|\tilde{\varphi}_j[k] | \lesssim k 1_{(0,1]} \rho \left( \frac{2}{\nu^3} \right)^4 + 1_{[0,1]}(\rho^2 |\log(\frac{2}{\nu^3})|)^2,
\]
and
\[
1_{(0,1]}|\nu \partial \nu \tilde{\varphi}_j[k] | \lesssim k 1_{(0,1]} \rho^2 \left( \frac{2}{\nu^3} \right)^4 + 1_{[0,1]}(\rho^2 |\log(\frac{2}{\nu^3})|)^2.
\]
Remark 2.2 ($\frac{1}{|\log \nu|}$-expansion). Our spectral problem has a critical nature as in [3]: 
\[ \hat{\lambda}_j \] (and hence \( \lambda_j \)) does not have a polynomial expansion, but rather has an \( \frac{1}{|\log \nu|} \)-expansion. In our derivation of the refined modulation equations (see Section 4.3), the precise \( \frac{1}{|\log \nu|} \)-order term of \( \lambda_1 \) is necessary. However, we only need the \( \frac{1}{|\log \nu|} \)-order term for \( \lambda_0 \).

Remark 2.3 (Structure of \( \varphi_j \)). The profiles in the first line of RHS (2.9) will be derived in the inner scale analysis, and this ansatz is sufficiently accurate in the region \( \rho \ll 1 \). However, the ansatz is no more accurate in the region \( \rho \sim 1 \), in which region the eigenfunction \( \varphi_j(\nu, \rho) \) in fact behaves like a hypergeometric function that is smooth at the light cone \( \rho = 1 \). The \( U_\infty \)-correction term of RHS (2.9) is introduced to incorporate this mismatch.

Remark 2.4 (Technical bounds on eigenfunctions). Roughly speaking, by substituting \( \rho \sim 1 \), the bounds (2.10), (2.12), and (2.11) might be considered as \( O(\nu^2 |\log \nu|^2) \), \( O(\frac{1}{|\log \nu|}) \), and \( O(1) \) bounds, respectively. Thus (2.10) is a very sharp bound. However, the rough bounds (2.11)-(2.14) will suffice in many places of our blow-up analysis.

Remark 2.5 (Comparison with the spectral property for exact self-similar solutions). For higher-dimensional wave maps, which are energy-supercritical, there are exact self-similar (type-I) blow-up solutions [3]. There is always an unstable eigenvalue 1, corresponding to the time-translation symmetry. Except this eigenvalue, the mode stability result of [10] shows that there is no other eigenvalue \( \lambda \) with \( \text{Re}(\lambda) \geq 0 \) for some exact self-similar blow-up solutions. In particular, 0 is not an eigenvalue for that problem. For our spectral problem, although our blow-up solution is not exactly self-similar, there is an eigenvalue \( \lambda_1 \) near 0. The eigenvalue \( \lambda_1 \) near 0 (in fact \( \lambda_1(\nu) \to 0 \) as \( \nu \to 0 \)) is special to our situation and responsible for our almost self-similar blow-up.

The rest of this section is devoted to the proof of Proposition 2.1. We use the matching argument as explained in Section 1.3

2.1. Rough inner eigenfunctions. The goal of this subsection is to construct inner eigenfunctions to the problem (1.19) in the inner region \( \rho \leq 2\delta_0 \). In this region, in the sense of constructing the ansatz, the operator \((\lambda + \Lambda_0)(\lambda + \Lambda)\) of (1.19) is perturbative with respect to \( H_\nu \); we rewrite the eigenproblem as

\[ H_\nu \phi = -(\Lambda_0 + \lambda)(\lambda + \Lambda)\phi, \]

Since \( H_\nu \phi \) is the main linear term, it is convenient to zoom in to the soliton scale. Thus we introduce the inner variables

\[ \phi_{in}(\lambda, \nu; \rho) = \frac{1}{\nu} \phi_{in}(\lambda, \nu; y), \quad y = \frac{\rho}{\nu}, \]

and rewrite the eigenproblem (2.15) as

\[ H \phi_{in} = -\nu^2(\Lambda_0 + \lambda)(\lambda + \Lambda)\phi_{in}. \]

We then construct the inner eigenfunction by considering the RHS (2.17) as a perturbative term.

If RHS (2.17) were zero, \( \phi \) must be a kernel element of \( H \). As we want \( \phi \) to be smooth at the origin \( y = 0 \), we may choose \( \phi = \Lambda Q \). Starting from this, it is natural to consider an expansion of the form

\[ \phi_{in} = \Lambda Q + \nu^2(T_1 + (2\lambda - 1)S_1 + \lambda(\lambda - 1)U_1) + O(\nu^4). \]
Substituting this expansion into (2.17) and taking the $\nu^2$-order terms, we obtain the equations for the profiles $T_1, S_1, U_1$:

$$\begin{align*}
HT_1 + \Lambda_0 \Lambda_0 \Lambda Q &= 0, \\
HS_1 + \Lambda_0 \Lambda Q &= 0, \\
HU_1 + \Lambda Q &= 0.
\end{align*}$$

(2.18)

To construct these profiles, we need to invert the operator $H$. A fundamental system associated with $H$ is $\{J_1, J_2\}$ with

$$J_1 := \Lambda Q \quad \text{and} \quad J_2 := \Lambda Q \int_1^y \frac{dy'}{y' (\Lambda Q)^y}.$$  

(2.19)

Note that $J_2$ is singular at the origin $y = 0$. As mentioned above, we require $\phi$ to be smooth at $y = 0$. Thus we determine the higher order expansions using the outgoing Green’s function of $H$; we choose a formal right inverse of $H$ by the formula

$$H^{-1} f := J_1 \int_0^y f J_2 y' dy' - J_2 \int_0^y f J_1 y' dy'.$$

(2.20)

so that repeated applications of $H^{-1}$ improve regularity (or, degeneracy) at the origin $y = 0$. Using this $H^{-1}$, we define the smooth profiles $T_1, S_1, U_1$ by

$$\begin{align*}
&T_1 := -H^{-1} \Lambda_0 \Lambda_0 \Lambda Q, \\
&S_1 := -H^{-1} \Lambda_0 \Lambda Q, \\
&U_1 := -H^{-1} \Lambda Q.
\end{align*}$$

(2.21)

The following pointwise estimates are immediate consequences of (2.18) and (2.20), whose proof is omitted.

**Lemma 2.6** (Pointwise estimates for inner profiles). For any $k \in \mathbb{N}$, the following estimates hold.

- **(Pointwise bounds)** We have

$$|\Lambda Q|_k = |J_1|_k \lesssim_k 1_{(0,1]} y + 1_{[1,\infty)} y^{-1},$$

(2.22)

and

$$|J_2|_k \lesssim_k 1_{(0,1]} y^{-1} + 1_{[1,\infty)} y,$$

(2.23)

- **(Asymptotics and degeneracy estimates as $y \to \infty$)** For $y \geq 1$, we have

$$|\Lambda Q - \frac{2}{y} - \Lambda_0 \Lambda Q|_k \lesssim_k \frac{1}{y^3}.$$  

(2.25)

and

$$\begin{align*}
&T_1 + \frac{1}{3} y k + |S_1 - \frac{1}{2} y|_k + |U_1 - (y \log y - y)|_k \lesssim_k \frac{(\log y)}{y}, \\
&|\Lambda_2 T_1|_k + |\Lambda_2 S_1|_k + |\Lambda_2 U_1 - y|_k \lesssim_k \frac{(\log y)}{y}. \\
\end{align*}$$

(2.26)

- **(Mapping property of $H^{-1}$)** We have

$$|H^{-1} f|_2 \lesssim |J_1|_2 \int_0^y |J_2 y' dy' + |J_2|_2 \int_0^y |J_1 y' dy'$$

$$+ y^2 \left(|(y \partial_y J_1) J_2| + |(y \partial_y J_2) J_1|\right) |f|.$$  

(2.27)

By a simple fixed-point argument, we can construct inner eigenfunctions:
Lemma 2.7 (Rough inner eigenfunctions). For any \( \lambda \) with either \( |\lambda| \lesssim 1 \) or \( |\lambda - 1| \lesssim \frac{1}{\log\nu} \) or \( |\lambda - 1| \lesssim \frac{1}{\log\nu} \), there exists unique smooth solution \( \tilde{\phi}_n(\lambda, \nu; y) \) to (2.17) in the region \( y \in (0, \frac{2\nu}{\lambda}) \), which admits the decomposition
\[
\phi_n(\lambda, \nu; y) = \Lambda_0(y) + \nu^2(T_1(y) + (2\nu - 1)S_1(y) + \lambda(\nu - 1)U_1(y)) + \tilde{\phi}_n(\lambda, \nu; y)
\]
with the pointwise estimates
\[
|\tilde{\phi}_n|_k \lesssim k \nu^4(1_{[0,1]}y^5 + 1_{[1,2\nu/\lambda]}y^3), \quad \forall k \in \mathbb{N}.
\]

Proof. By (2.18), \( \tilde{\phi}_n \) should solve the equation
\[
(2.28) \quad H\tilde{\phi}_n(y) = -\Psi_n - \nu^2(\Lambda_0 + \lambda)(\Lambda + \lambda)\tilde{\phi}_n(y),
\]
where \( \Psi_n \) is the inhomogeneous error term from our ansatz:
\[
(2.29) \quad \Psi_n := \nu^4(\Lambda_0 + \lambda)(\Lambda + \lambda)(T_1(y) + (2\nu - 1)S_1(y) + \lambda(\nu - 1)U_1(y)).
\]

Thus we may construct \( \tilde{\phi}_n \) via the integral equation
\[
(2.30) \quad \tilde{\phi}_n(y) := -H^{-1}\{\Psi_n + \nu^2(\Lambda_0 + \lambda)(\Lambda + \lambda)\tilde{\phi}_n(y)\}.
\]

Next, we set up a fixed-point argument for (2.30). For a function \( f \) defined on \( (0, \frac{2\nu}{\lambda}) \), define the norm \( \|f\|_X \) by the least number satisfying
\[
(2.31) \quad \|f\|_X(y) \leq \|f\|_X(1_{(0,1]}y^5 + 1_{[1,2\nu/\lambda]}y^3).
\]

With this norm, we claim that
\[
(2.32) \quad \|H^{-1}\tilde{\phi}_n\|_X \lesssim \nu^4,
\]
\[
(2.33) \quad \|H^{-1}\{\nu^2(\Lambda_0 + \lambda)(\Lambda + \lambda)f\}\|_X \lesssim (\delta_0)^2\|f\|_X.
\]

Proof of (2.32). By \( |\lambda(\lambda - 1)| \lesssim \frac{1}{\log\nu} \) and (2.24), we have
\[
(2.34) \quad |\tilde{\phi}_n|_k \lesssim k \nu^4(1_{[0,1]}y^5 + 1_{[1,2\nu/\lambda]}y^3)
\]
for any \( k \in \mathbb{N} \). Applying (2.27), we have
\[
|H^{-1}\tilde{\phi}_n|_2 \lesssim \nu^4(1_{(0,1]}y^5 + 1_{[1,2\nu/\lambda]}y^3).
\]

This completes the proof of (2.32).

Proof of (2.33). By (2.27),
\[
|H^{-1}\{\nu^2(\Lambda_0 + \lambda)(\Lambda + \lambda)f\}|_2 \leq \nu^2 \left\{ |J_1|_2 \int_0^y |f|_2 |J_2|y'dy' + |J_2|_2 \int_0^y |f|_2 |J_1|y'dy' + y^2 \left( |(y\partial_y J_1)J_2| + |(y\partial_y J_2)J_1| \right) |f|_2 \right\}.
\]

Substituting the estimates (2.22), (2.23) and (2.31) into the above, we have
\[
(2.35) \quad \|H^{-1}\{\nu^2(\Lambda_0 + \lambda)(\Lambda + \lambda)f\}\|_2 \lesssim \nu^2\|f\|_X(1_{(0,1]}y^5 + 1_{[1,2\nu/\lambda]}y^3) \lesssim (\delta_0)^2\|f\|_X(1_{(0,1]}y^5 + 1_{[1,2\nu/\lambda]}y^3).
\]

This completes the proof of (2.33).

Having settled the proofs of the claims (2.32), (2.33), provided that \( \delta_0 \ll 1 \), an application of the contraction principle yields the unique existence of \( \tilde{\phi}_n \) in the space \( \|\tilde{\phi}_n\|_X \lesssim \nu^4 \).
It remains to show the smoothness of $\phi_{\text{in}}$ and higher derivative estimates of $\tilde{\phi}_{\text{rough}}$ in the region $y \in (0, \frac{2\delta_0}{\nu}]$. Substituting the identity

$$H + \nu^2 (\Lambda_0 + \lambda)(\Lambda + \lambda) = -(1 - (\nu y)^2) \partial_y - \left(\frac{1}{y} - \nu^2 (2\lambda + 2)y\right) \partial_y + \left(\frac{V}{y^2} + \nu^2 \lambda (\lambda + 1)\right)$$

into (2.28), we obtain

$$\partial_y \tilde{\phi}_{\text{rough}} = \frac{1}{(1 - (\nu y)^2)} \left\{ \Psi_{\text{rough}} - \left(\frac{1}{y} - \nu^2 (2\lambda + 2)y\right) \partial_y \phi_{\text{rough}} + \left(\frac{V}{y^2} + \nu^2 \lambda (\lambda + 1)\right) \phi_{\text{rough}} \right\}.$$  

Repeated applications of the above identity with (2.34) completes the proof.  

The rough ansatz in Lemma 2.7 is not quite accurate in the region $\nu y \sim 1$ and it is insufficient to derive several delicate bounds in Proposition 2.1. In Section 2.3, we will find a higher order correction to our rough ansatz to understand the refined structure of the inner eigenfunctions.

2.2. Outer eigenfunctions. The goal of this subsection is to construct outer eigenfunctions to the problem (1.19) in the outer region $\rho \geq \frac{1}{2} \delta_0$. In this region, the operator $(\lambda + \Lambda_0)(\lambda + \Lambda)$ of (1.19) is no longer perturbative and it must be included into the main linear operator. On the other hand, we may approximate the potential $\frac{V}{\rho^2}$ by $\frac{1}{\rho^2}$:

$$\frac{V}{\rho^2} = \frac{1}{\rho^2} - \frac{8\nu^2}{(\nu^2 + \rho^2)^2} = \frac{1}{\rho^2} + O(\delta_0 \nu^2).$$

This motivates us to rewrite the eigenproblem (1.19) as

$$(-\Delta_1 + (\Lambda_0 + \lambda)(\Lambda + \lambda)) \phi = \frac{8\nu^2}{(\nu^2 + \rho^2)^2} \phi.$$  

We will construct outer eigenfunctions by treating the RHS as a perturbative term.

The homogeneous linear differential equation associated with (2.35) is rewritten as

$$K_\lambda \phi_{\text{out}} = \frac{2\nu^2}{(\nu^2 + 1 - z)^2} \phi_{\text{out}},$$

then (2.35) is rewritten as

$$K_\lambda \phi_{\text{out}} = \frac{2\nu^2}{(\nu^2 + 1 - z)^2} \phi_{\text{out}},$$

$$K_\lambda := z(1 - z) \partial_z + \left(\lambda + \frac{1}{2}\right)(1 - z) \partial_z - \frac{\lambda(\lambda - 1)}{4}.$$  

In order to construct $\phi_{\text{out}}$ that is smooth at the light cone $\rho = 1$, we construct $\phi_{\text{out}}$ that is smooth at $z = 0$.

The homogeneous linear equation associated with (2.36), i.e.,

$$K_\lambda h = 0,$$

5Recall the 1-equivariant Laplacian $\Delta_1 = \partial_{\rho} \rho + \frac{1}{\rho} \partial_{\rho} - \frac{1}{\rho^2}$.
has a unique solution that is smooth at $z = 0$ and has value 1 at $z = 0 \, ^6$ We denote it by

$$h_1(\lambda; z) := F\left(\frac{\lambda}{2}, \frac{\lambda - 1}{2}; \lambda + \frac{1}{2}; z\right),$$

where $F$ is the (Gaussian) hypergeometric function.\(^7\)

$$F(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n.$$\(^8\)

Note that $\lambda + \frac{1}{2}$ is never a negative integer for $\lambda \approx 1$ or $\lambda \approx 0$. Thus the series converges absolutely when $|z| < 1$. Moreover, as (2.38) has no singularities in $(-\infty, 0)$, the function $h_1(\lambda; \cdot)$ analytically extends over the region $(-\infty, 0)$. Thus $h_1(\lambda; \cdot)$ is defined and analytic on $(-\infty, 1)$. Note that there is another solution to (2.38) that is linearly independent of $h_1$:

$$h_2(\lambda; z) := z^{\frac{1}{2} - \lambda} F\left(\frac{1}{2} - \lambda, \frac{3}{2} - \lambda; -\lambda; z\right).$$

Notice that $h_2$ is defined for $z \in (0, 1)$ and is not smooth at $z = 0$ due to the asymptotics $h_2(\lambda; z) \approx z^{\frac{1}{2} - \lambda}$ as $z \to 0^+$.

In Lemma 2.10 below, we construct the outer eigenfunction $\phi_{\text{out}}$ using the Frobenius method. For this purpose, we need to ensure certain growth bounds (in fact, the polynomial growth) on the Taylor coefficients of $h_1$ and find it convenient to introduce the following definition. For a function $f(\lambda, \nu; z)$ defined for $\lambda$ with either $|\lambda| \lesssim \frac{1}{\log \nu}$ or $|\lambda - 1| \lesssim \frac{1}{\log \nu}$, $\nu \in (0, \nu^*)$, $|z| < 1$, and $K \in \mathbb{R}$, we say

$$f \in \mathcal{P}_K \iff f(\lambda, \nu; z) = \sum_{n=0}^{\infty} f^{(n)}(\lambda, \nu) z^n, \quad \text{and} \quad |f^{(n)}(\lambda, \nu)| \leq C(n + 1)^K \quad \text{for some } C > 0.$$\(^9\)

We also say

$$f \in \mathcal{P} \quad \iff \quad f \in \mathcal{P}_K \quad \text{for some } K \in \mathbb{R}.$$\(^9\)

We record some useful asymptotic formulas and pointwise estimates of $h_1$:

**Lemma 2.8 (Estimates for $h_1$).** For any $\lambda$ with either $|\lambda| \lesssim \frac{1}{\log \nu}$ or $|\lambda - 1| \lesssim \frac{1}{\log \nu}$, the following estimates hold.

- (An expansion for $\frac{1}{\rho} h_1(\lambda; 1 - \rho^2)$) We have the expansion

$$\frac{1}{\rho} h_1(\lambda; 1 - \rho^2) = \frac{\Gamma(\lambda + \frac{1}{2})}{2^{\lambda} \Gamma(\frac{1}{2}) \Gamma(\lambda + 1)} \left\{ \sum_{n=0}^{\infty} c_n \rho^{2n} [\log \rho + \frac{d_n}{2}] \right\}. \quad (2.43)$$

Here, the constants $c_n = c_n(\lambda)$ and $d_n = d_n(\lambda)$ are defined by\(^{10}\)

$$c_n(\lambda) := \frac{(\lambda + 1)_n (\lambda + 2)_n}{n!(n + 1)!}, \quad d_n(\lambda) := -\psi(n + 1) - \psi(n + 2) + \psi\left(\lambda + 1 + n\right). \quad (2.44)$$

- (Polynomial growth of Taylor coefficients) We have

$$h_1, \partial_\lambda h_1 \in \mathcal{P}. \quad (2.45)$$

\(^6\)See $h_2$ below for a singular solution.

\(^7\)Recall Pochhammer’s symbols \(\Gamma(n + 1)\).

\(^8\)Recall the digamma function $\psi = \Gamma'/\Gamma$.\(^{11}\)
\begin{itemize}
  \item (Rough pointwise bounds) For any \( k \in \mathbb{N} \) and \( z \in (0, 1 - \frac{1}{\delta}\delta) \), we have
  \begin{equation}
  |\partial^k_z h_1| + |\partial^k_z (\partial_z h_1)| \lesssim_{k, \delta} 1,
  \end{equation}
  \end{itemize}

Remark 2.9. It is possible to derive sharper pointwise estimates for \( \partial^k_z h_1 \) and \( \partial^k_z (\partial_z h_1) \) in the region \( z \in (0, 1) \) using the expansion (2.47) below. However, we chose to state the rough estimate (2.46) for simplicity.

Proof. We first show (2.43). We use the connection formula for the hypergeometric function \([11] \text{p.559 (15.3.11)}\) to have
\begin{equation}
  h_1(\lambda; z) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma\left(\frac{1}{2} + 1\right)\Gamma\left(\frac{1}{2} + \lambda\right)} \times \left\{ 1 + \frac{\lambda(\lambda - 1)}{4}(1 - z) \sum_{n=0}^{\infty} c_n(1 - z)^n[\log(1 - z) + d_n] \right\}.
\end{equation}

Substituting \( z = 1 - \rho^2 \) and multiplying the above by \( \rho^{-1} \), (2.43) follows.

The proof of (2.46) is immediate from the following two easy facts: (i) For any \( a, b \notin \{-1, -2, \ldots\} \), the sequence of ratios \( ((a)_n/(b)_n)_{n \in \mathbb{N}} \) is of polynomial growth. (ii) The digamma function \( \psi(z) \) diverges logarithmically as \( z \to +\infty \) and the trigamma function \( \psi''(z) \) behaves like \( \frac{1}{z} \) as \( z \to +\infty \)(in particular they are of polynomial growth). Finally, (2.46) follows directly from (2.45). This completes the proof.

Using the estimates in Lemma 2.8 we are now ready to construct the solutions \( \phi_{\text{out}} \) to the problem (2.36).

\textbf{Lemma 2.10 (Outer eigenfunctions in the \( z \)-variable).} For any \( \lambda \) with either \( |\lambda| \lesssim \frac{1}{|\log \nu|} \) or \( |\lambda - 1| \lesssim \frac{1}{|\log \nu|} \), there exists unique analytic solution \( \phi_{\text{out}}(\lambda, \nu; z) \) to (2.36) in the region \( z \in (-\infty, 1) \) such that it admits the decomposition
\begin{equation}
  \phi_{\text{out}}(\lambda, \nu; z) = h_1(\lambda; z) + \tilde{\phi}_{\text{out}}(\lambda, \nu; z)
\end{equation}

with the following pointwise estimates in the region \( z \in [0, 1 - \frac{\delta^2}{4}] \):
\begin{equation}
  |\tilde{\phi}_{\text{out}}| + |\nu \partial_\nu \tilde{\phi}_{\text{out}}| + |\partial_\lambda \tilde{\phi}_{\text{out}}| \lesssim_{\delta} \nu^2 z,
\end{equation}
\begin{equation}
  |\partial^k_z \tilde{\phi}_{\text{out}}| + |\partial^k_z (\nu \partial_\nu \tilde{\phi}_{\text{out}})| + |\partial^k_z (\partial_\lambda \tilde{\phi}_{\text{out}})| \lesssim_{k, \delta} \nu^2, \quad \forall k \in \mathbb{N}.
\end{equation}

Proof. The equation for \( \tilde{\phi}_{\text{out}} \) is
\begin{equation}
  K_\lambda \tilde{\phi}_{\text{out}} = -\frac{2\nu^2}{(\nu^2 + 1 - z)^2}(h_1 + \tilde{\phi}_{\text{out}}).
\end{equation}

In order to construct \( \tilde{\phi}_{\text{out}} \) (as well as \( \nu \partial_\nu \tilde{\phi}_{\text{out}} \) and \( \partial_\lambda \tilde{\phi}_{\text{out}} \)) using the Frobenius method, it is more convenient to introduce the operator
\begin{equation}
  K_\lambda := z \partial_z + \left( \lambda + \frac{1}{2} \right) \partial_z
\end{equation}

and further rewrite (2.49) as
\begin{equation}
  K_\lambda \tilde{\phi}_{\text{out}} = \frac{1}{1 - z} \left\{ -\frac{2\nu^2}{(\nu^2 + 1 - z)^2}(h_1 + \tilde{\phi}_{\text{out}}) + \frac{\lambda(\lambda - 1)}{4} \tilde{\phi}_{\text{out}} \right\}.
\end{equation}

This motivates us to introduce
\begin{equation}
  \Psi_{\text{out}}(\lambda, \nu; z) := -\frac{2\nu^2 h_1(\lambda; z)}{(1 - z)(\nu^2 + 1 - z)^2},
\end{equation}
\begin{equation}
  V_{\text{out}}(\lambda, \nu; z) := -\frac{1}{1 - z} \left( \frac{2\nu^2}{(\nu^2 + 1 - z)^2} - \frac{\lambda(\lambda - 1)}{4} \right).
\end{equation}
and rewrite (2.50) as the following system:

\[
\begin{align*}
K_\lambda \phi_{\text{out}} &= \Psi_{\text{out}} + V_{\text{out}} \phi_{\text{out}}, \\
K_\lambda (\nu \partial_\nu \phi_{\text{out}}) &= (\nu \partial_\nu \Psi_{\text{out}}) + V_{\text{out}} (\nu \partial_\nu \phi_{\text{out}}) + (\nu \partial_\nu V_{\text{out}}) \phi_{\text{out}}, \\
K_\lambda (\partial_\lambda \phi_{\text{out}}) &= (\partial_\lambda \Psi_{\text{out}}) + V_{\text{out}} (\partial_\lambda \phi_{\text{out}}) + (-\partial_\lambda K_\lambda + \partial_\lambda V_{\text{out}}) \phi_{\text{out}}.
\end{align*}
\]

(2.51)

Note that the second and third rows of (2.51) are obtained by taking $\nu \partial_\nu$ and $\partial_\lambda$ to the first row of (2.51), respectively. Note also that $\partial_\lambda K_\lambda$ is simply $\partial_\lambda$. These equations are necessary to estimate $\nu \partial_\nu \phi_{\text{out}}$ and $\partial_\lambda \phi_{\text{out}}$.

Next, we claim that

\[
\begin{align*}
\Psi_{\text{out}}, \nu \partial_\nu \Psi_{\text{out}}, \partial_\lambda \Psi_{\text{out}} &\in \nu^2 \mathcal{P}, \\
V_{\text{out}}, \nu \partial_\nu V_{\text{out}}, \partial_\lambda V_{\text{out}} &\in \mathcal{P}_1.
\end{align*}
\]

(2.52) (2.53)

The claim (2.52) easily follows from (2.45) and $(\nu^2 + 1 - z)^{-2} \in \mathcal{P}$ (and also for its $\nu \partial_\nu$-derivative), and the algebra property of $\mathcal{P}$ (i.e., $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$). For the proof of (2.53), since $(1 - z)^{-1} \in \mathcal{P}_0$ and $\mathcal{P}_0 \times \mathcal{P}_0 \to \mathcal{P}_1$, it suffices to show

\[
\frac{\nu^2}{(\nu^2 + 1 - z)^2}, \frac{\nu^4}{(\nu^2 + 1 - z)^3} \in \mathcal{P}_0,
\]

which follow from the expansions

\[
\frac{\nu^2}{(\nu^2 + 1 - z)^2} = \sum_{n=0}^{\infty} \frac{(n + 1) \nu^2}{(1 + \nu^2)^{n+2}} z^n,
\]

\[
\frac{\nu^4}{(\nu^2 + 1 - z)^3} = \sum_{n=0}^{\infty} \frac{\nu^4}{(1 + \nu^2)^{n+3}} z^n,
\]

and the inequalities $(1 + \nu^2)^{n+2} \geq (n + 2) \nu^2$ and $(1 + \nu^2)^{n+3} \geq \frac{1}{2}(n + 2)(n + 3) \nu^4$.

By the claims (2.52)-(2.53), if we write $(\phi_{\text{out}}, \nu \partial_\nu \phi_{\text{out}}, \partial_\lambda \phi_{\text{out}}) = (f_1, f_2, f_3)$ and $f_k = \sum_{n=0}^{\infty} f_k^{(n)} z^n$, then (2.51) takes the form

\[
\begin{align*}
K_\lambda \sum_{n=0}^{\infty} f_1^{(n)} z^n &= \nu^2 \sum_{n=0}^{\infty} \Psi_1^{(n)} z^n + \sum_{n=0}^{\infty} \left( \sum_{\ell + m = n} V_1^{(\ell)} f_1^{(m)} \right) z^n, \\
K_\lambda \sum_{n=0}^{\infty} f_2^{(n)} z^n &= \nu^2 \sum_{n=0}^{\infty} \Psi_2^{(n)} z^n + \sum_{n=0}^{\infty} \left( \sum_{\ell + m = n} (V_2^{(\ell)} f_1^{(m)} + V_1^{(\ell)} f_2^{(m)}) \right) z^n, \\
K_\lambda \sum_{n=0}^{\infty} f_3^{(n)} z^n &= \nu^2 \sum_{n=0}^{\infty} \Psi_3^{(n)} z^n + \sum_{n=0}^{\infty} \left( - (n + 1) f_1^{(n+1)} \\
&\quad + \sum_{\ell + m = n} (V_3^{(\ell)} f_1^{(m)} + V_1^{(\ell)} f_3^{(m)}) \right) z^n,
\end{align*}
\]

(2.54)

where $\Psi_i^{(n)}$ and $V_i^{(n)}$ are polynomially growing sequences and $V_i^{(n)}$ satisfy the linear growth bounds $|V_i^{(n)}| \lesssim n + 1$. In view of

\[
K_\lambda \sum_{n=0}^{\infty} f_k^{(n)} z^n = \sum_{n=0}^{\infty} (n + 1)(n + \lambda + \frac{1}{2}) f_k^{(n+1)} z^n,
\]

the system (2.54) gives recurrence relations for the coefficients $f_k^{(n+1)}$.

Now, we can construct a solution to the system (2.54) starting from the initial condition

\[
f_k^{(0)} = 0, \quad \forall k \in \{1, 2, 3\}.
\]
The recurrence relations uniquely define formal power series solutions $f_k$. We show that $f_k \in \nu^2 \mathcal{P}$. To show $f_1 \in \nu^2 \mathcal{P}$, we show the bound $f_1^{(n)} \le C_1 \nu^2 (n+1)^{K}$ for some constants $C_1, K_1 \gg 1$ (to be chosen later) by induction. This inductive bound combined with the recurrence relations give the following inequality (where we fixed some $K$ such that $|\Psi_1^{(n)}| \lesssim (n+1)^K$):

$$
|f_k^{(n+1)}| = \left| \nu^2 \Psi_1^{(n)} + \sum_{\ell+m=n} V_1^{(\ell)} f_1^{(m)} \right|
\lesssim \nu^2 (n+1)^K + C_1 \sum_{\ell=1}^n (\ell+1)(n+1-\ell) K
\lesssim \nu^2 (n+1)^K + C_1 K (n+2)^{K_1}.
$$

The right hand side can be bounded by $C_1 \nu^2 (n+2)^{K_1}$ if $C_1$ and $K_1$ are chosen sufficiently large. We remark that the property $V_{\text{out}} \in \mathcal{P}_1$ (or, $|V_1^{(n)}| \lesssim n+1$) is crucially used. By induction, we obtain $f_1 \in \nu^2 \mathcal{P}$. We omit the proof of $f_2, f_3 \in \mathcal{P}$, which can be proved in a similar fashion.

If we set $\tilde{\phi}_{\text{out}} = f_1 \in \nu^2 \mathcal{P}$, then $\tilde{\phi}_{\text{out}}$ exists (as an analytic function of $z$) in the region $z \in (-1, 1)$, and $\nu \partial \nu \tilde{\phi}_{\text{out}} = f_2$ and $\partial_{\lambda} \tilde{\phi}_{\text{out}} = f_3$ by uniqueness. The estimate \ref{2.35} follows from the fact that $\tilde{\phi}_{\text{out}}, \nu \partial_{\nu} \tilde{\phi}_{\text{out}},$ and $\partial_{\lambda} \tilde{\phi}_{\text{out}} \in \nu^2 \mathcal{P}$ and the zero initial condition. Finally, the solution $\phi_{\text{out}} = h_1 + \tilde{\phi}_{\text{out}}$ analytically extends over the region $z \in (-\infty, 1)$ because the linear differential equation \ref{2.36} does not have singular points in $(-\infty, 0)$. This completes the proof.

In terms of the self-similar variable $\rho$, we have the following.

**Corollary 2.11 (Outer eigenfunctions).** For any $\lambda$ with either $|\lambda| \lesssim \frac{1}{\log \nu}$ or $|\lambda - 1| \lesssim \frac{1}{\log \nu}$, define

(2.55) \hspace{1cm} \varphi_{\text{out}}(\lambda, \nu; \rho) := \frac{1}{\rho} \phi_{\text{out}}(\lambda, \nu; 1 - \rho^2)

for all $\rho \in (0, \infty)$. Then, $\varphi_{\text{out}}$ solves \ref{2.35}. Moreover, in the region $\rho \in \left[\frac{1}{2} h_0, 1\right]$, it admits the decomposition

(2.56) \hspace{1cm} \varphi_{\text{out}}(\lambda, \nu; \rho) = \frac{1}{\rho} h_1(\lambda; 1 - \rho^2) + \tilde{\varphi}_{\text{out}}(\lambda, \nu; \rho)

and satisfies the pointwise estimates

(2.57) \hspace{1cm} |\tilde{\varphi}_{\text{out}}|_k + |\nu \partial_{\nu} \tilde{\varphi}_{\text{out}}|_k + |\partial_{\lambda} \tilde{\varphi}_{\text{out}}|_k \lesssim_{k, \delta_0} \nu^2, \quad \forall k \in \mathbb{N}.

**Proof.** This follows from the definition \ref{2.55} and Lemma 2.11. Note that it is convenient to use

$$
\rho \partial_{\nu} \left( \frac{1}{\rho} f(\lambda, \nu; 1 - \rho^2) \right) = \frac{1}{\rho} [(2z - 2) \partial_z f - f(\lambda, \nu; 1 - \rho^2)],
$$

where $z = 1 - \rho^2$. We omit the details. \hfill \Box

---

9In fact, one can prove uniqueness in a much larger function space; by showing that the linear differential equation \ref{2.36} has a fundamental system $\{h_1, h_2\}$ with $h_1 = 1 + O(z)$ and $h_2 = z^{\frac{1}{2} - \lambda}(1 + O(z))$, the uniqueness holds in any function space that removes the freedom of adding $h_1$ or $h_2$. 

SHARP BLOW-UP RATE FOR STABLE BLOW-UP 20
2.3. Refined inner eigenfunctions. The goal of this subsection is to obtain refined description of the inner eigenfunctions. As mentioned in Section 2.1, the ansatz for the inner eigenfunction used in Lemma 2.7 is not quite accurate in the region \( \nu y \sim 1 \) (i.e., \( \rho \sim 1 \)) and its higher order expansion is still missing. In this subsection, we will look more carefully at the structure of this higher order expansion. As a byproduct, we also motivate the definitions of \( p(\nu; \lambda) \) and \( \tilde{\lambda}_j \) introduced at the beginning of this subsection.

In view of Corollary 2.11 the outer eigenfunction \( \varphi_\text{out} \) in the region \( \rho \sim \delta_0 \) is well-approximated by \( \tilde{\rho} \frac{h_1(\lambda; 1 - \rho^2)}{\rho} \), which has the following \( \rho \)-expansion (2.14):

\[
\varphi_\text{out} \approx \frac{\Gamma(\frac{\lambda}{2} + 1)}{2\Gamma(\frac{\lambda}{2} + 1)\Gamma(\frac{\lambda}{2} + 1)} \left\{ \frac{2}{\rho} + \lambda(\lambda - 1)\rho \sum_{n=0}^\infty c_n\rho^{2n}[\log \rho + \frac{d_n}{2}] \right\}.
\]

On the other hand, using the rough ansatz from Lemma 2.7 and (2.26), the inner eigenfunction \( \varphi_\text{in} \) (see (2.10)) in the region \( \rho \sim \delta_0 \) has rough asymptotics

\[
(2.59) \quad \varphi_\text{in}(\lambda, \nu; \rho) = \frac{1}{\nu} \phi_\text{in}(\lambda, \nu; \rho) \approx \frac{2}{\rho} - \frac{1}{3}\rho + (2\lambda - 1)\frac{1}{2}\rho + \lambda(\lambda - 1)\rho(\log(\frac{\rho}{\nu}) - 1).
\]

If the matching happens at \( \rho \equiv \delta_0 \ll 1 \) for some \( \lambda \), then the inner eigenfunction (which has a rough asymptotics (2.59) and the outer eigenfunction (which has the asymptotics (2.58)) must be parallel in the region \( \rho \sim \delta_0 \). First, looking at the \( \frac{1}{\rho} \)-order terms of (2.58) and (2.59), we may define our connection coefficient

\[
(2.60) \quad c_{\text{conn}}(\lambda) := \frac{2\Gamma(\frac{\lambda}{2} + 1)\Gamma(\frac{\lambda}{2} + 1)}{\Gamma(\frac{\lambda}{2})},
\]

so that (2.58) multiplied by \( c_{\text{conn}} \) matches (2.59) at the \( \frac{1}{\rho} \)-order. Next, we look at the \( \rho \)-order terms. We rewrite RHS (2.60)

\[
(2.61) \quad \varphi_\text{in}(\lambda, \nu; \rho) \approx \left\{ \frac{2}{\rho} + \lambda(\lambda - 1)\rho \cdot (\log \rho + \frac{d_0}{2}) \right\} + p(\nu; \lambda)\rho
\]

with

\[
(2.61) \quad p(\nu; \lambda) = \lambda(\lambda - 1)\left( |\log \nu| - 1 - \frac{d_0(\lambda)}{2} \right) + \lambda - \frac{5}{6}
\]

and observe that the terms in the curly bracket of (2.61) match the \( \frac{1}{\rho} \)- and \( \rho \)-order terms in the curly bracket of (2.58). Therefore, it is natural to expect that the inner eigenfunction would have large \( y \) asymptotics of the form

\[
\left\{ \frac{2}{\rho} + \lambda(\lambda - 1)\rho \sum_{n=0}^\infty c_n\rho^{2n}[\log \rho + \frac{d_n}{2}] \right\} + p(\nu; \lambda)(\rho + \cdots).
\]

Therefore, (i) the missing part in our rough ansatz for the inner eigenfunction is \( \lambda(\lambda - 1)U_\infty \), where \( U_\infty = U_\infty(\lambda; \rho) \) is defined by\[10\]

\[
(2.62) \quad U_\infty(\lambda; \rho) := \rho \sum_{n=1}^\infty c_n\rho^{2n}[\log \rho + \frac{d_n}{2}],
\]

and (ii) the eigenvalues \( \lambda \) enabling the matching procedure is an approximate solution to the equation

\[
p(\nu; \lambda) = 0.
\]

This motivates the definitions of \( p(\nu; \lambda) \) and \( \tilde{\lambda}_j \).

In the following lemma, we make the above discussion into rigorous estimates.

**Lemma 2.12** (**\( U_\infty \)-correction**). Let \( \lambda \) satisfy either \( |\lambda| \lesssim \frac{1}{|\log \nu|} \) or \( |\lambda - 1| \lesssim \frac{1}{|\log \nu|} \).

\[10\]Note that this series converges absolutely for all \( \rho \in (0, 1) \). See also Lemma 2.12 below.
• (Connection coefficient) We have
\begin{align*}
(2.63) \quad c_{\text{conn}} &= 2 + O\left(\frac{1}{\log \rho}\right) \quad \text{and} \quad |\partial \lambda c_{\text{conn}}| + |\partial \lambda c_{\text{conn}}| \lesssim 1.
\end{align*}

• (Analytic extension and pointwise estimates of $U_\infty$) The function $U_\infty(\lambda; \rho)$ extends analytically over the region $\rho \in (0, +\infty)$. Moreover, for $\rho \in (0, 1]$ and $k \in \mathbb{N}$, we have the pointwise estimates
\begin{align*}
|U_\infty|_k + |\partial \lambda U_\infty|_k \lesssim_k \rho^3 (\log \rho).
\end{align*}

• (Connection to the outer eigenfunction) Let
\begin{align*}
\tilde{\Psi}_{\text{conn}}(\lambda, \nu; \rho) := \left[\frac{1}{\nu} \Lambda Q_\nu + \nu \Theta_1 + (2 \lambda - 1) \nu S_1 + \lambda(\lambda - 1) \nu U_1\right] \left(\frac{\rho}{\nu}\right)
+ \lambda(\lambda - 1) U_\infty(\lambda; \rho) - c_{\text{conn}}(\lambda) \frac{1}{\rho} h_1(\lambda; 1 - \rho^2) - p(\nu; \lambda) \rho.
\end{align*}

Then, we have the following pointwise estimates for $\rho \in [\nu, 1]$ and $k \in \mathbb{N}$:
\begin{align*}
|\tilde{\Psi}_{\text{conn}}|_k + |\rho \partial_{\nu} \tilde{\Psi}_{\text{conn}}|_k + |\partial \lambda \tilde{\Psi}_{\text{conn}}|_k \lesssim_k \rho^2 \left(\frac{1}{\rho^2} + \frac{\log(\frac{\rho}{\nu})}{\rho}\right).
\end{align*}

Proof. (1) (2.63) is clear from the definition of (2.66); we omit the proof.

(2) Since the power series (2.62) converges absolutely when $\rho < 1$, $U_\infty(\lambda; \rho)$ is well-defined for $\rho \in (0, 1)$. In particular, the pointwise estimate (2.64) in the region $\rho \leq \frac{1}{2}$ directly follows from the series expansion (2.62).

For larger values of $\rho$, we recall that the definition of $U_\infty$ is related to $h_1$; by (2.43), (2.60), and (2.62), we have
\begin{align*}
(2.66) \quad c_{\text{conn}}(\lambda) \frac{1}{\rho} h_1(\lambda; 1 - \rho^2)
= \frac{2}{\rho} + \lambda(\lambda - 1) \rho \log \rho + \frac{d_0(\lambda)}{2} + \lambda(\lambda - 1) U_\infty(\lambda; \rho).
\end{align*}

The above identity holds a priori for $\rho \in (0, 1)$ but suggests the following alternative definition of $U_\infty$:
\begin{align*}
(2.67) \quad U_\infty(\lambda; \rho) = \frac{1}{\lambda(\lambda - 1)} \cdot \frac{1}{\rho} \left(c_{\text{conn}}(\lambda) h_1(\lambda; 1 - \rho^2) - \rho \log \rho + \frac{d_0(\lambda)}{2}\right).
\end{align*}

In this alternative definition, $\lambda = 0$ and $\lambda = 1$ are removable singularities due to
\begin{align*}
(2.68) \quad c_{\text{conn}} h_1 - 2 = c_{\text{conn}}(h_1 - 1) + (c_{\text{conn}} - 2),
\end{align*}
(2.63), and (letting $z = 1 - \rho^2$)
\begin{align*}
h_1(\lambda; z) = \frac{\lambda(\lambda - 1)}{4 \lambda + 2} F\left(\frac{\lambda}{2} + 1, \frac{\lambda + 1}{2}; \lambda + \frac{3}{2}; z\right) \cdot z.
\end{align*}

Note that the function $F\left(\frac{\lambda}{2} + 1, \frac{\lambda + 1}{2}; \lambda + \frac{3}{2}; z\right)$ extends analytically over the region $z \in (-\infty, 1)$ by the same reason as $h_1$. Therefore, the alternative definition (2.68) defines an analytic function on the region $z \in (-\infty, 1)$, i.e., $\rho \in (0, +\infty)$. In particular, we have the pointwise estimate (2.64) also for $\rho \in [\frac{1}{2}, 1]$.

(3) Let $\rho \in [\nu, 1]$. Using the identity (2.67), we have
\begin{align*}
\tilde{\Psi}_{\text{conn}} &= \left[\frac{1}{\nu} \Lambda Q_\nu + \nu \Theta_1 + (2 \lambda - 1) \nu S_1 + \lambda(\lambda - 1) \nu U_1\right] \left(\frac{\rho}{\nu}\right)
- \left(\frac{2}{\rho} + \lambda(\lambda - 1) \rho \log \rho + \frac{d_0(\lambda)}{2}\right) - p(\nu; \lambda) \rho.
\end{align*}
We rearrange the above using the definition (2.1) of \( p(\nu; \lambda) \) as

\[
\tilde{\Psi}_{\text{conn}} = \left( \frac{1}{\nu} \Lambda Q \left( \frac{\nu}{\nu} \right) - \frac{2}{\rho} \right) + (\nu T_1 \left( \frac{2}{\nu} \right) + \frac{1}{3} \rho) + (2\lambda - 1) \left( \nu S_1 \left( \frac{2}{\nu} \right) - \frac{1}{2} \rho \right) + \lambda (\lambda - 1) \left( \nu U_1 \left( \frac{2}{\nu} \right) - \rho \left( \log \left( \frac{\nu}{\nu} \right) - 1 \right) \right).
\]

Applying (2.26), we have

\[
|\tilde{\Psi}_{\text{conn}}|_k + |\nu \partial_\nu \tilde{\Psi}_{\text{conn}}|_k + |\partial_\lambda \tilde{\Psi}_{\text{conn}}|_k \lesssim_k \frac{1}{\nu} \frac{1}{\nu y^2} + \nu \frac{\langle \log y \rangle}{y} \lesssim_k \nu^2 \left( \frac{1}{\rho^3} + \frac{\langle \log (\frac{\nu}{\nu}) \rangle}{\rho} \right).
\]

This completes the proof of (2.60). \( \square \)

We are now ready to obtain a refined description of the inner eigenfunctions.

**Lemma 2.13** (Refined inner eigenfunctions in the \( y \)-variable). For any \( \lambda \) with either \( |\lambda| \lesssim \frac{1}{\log \nu} \) or \( |\lambda - 1| \lesssim \frac{1}{\log \nu} \), there exists unique eigenfunction \( \phi_{\text{in}}(\lambda, \nu; y) \) in the region \( y \in (0, \frac{2\kappa}{\nu}) \) with the following properties.

- **(Decomposition)** The inner eigenfunction \( \phi_{\text{in}}(\lambda, \nu; y) \) admits the decomposition

  \[
  \phi_{\text{in}}(\lambda, \nu; y) = \Lambda Q(y) + \nu^2 T_1(y) + (2\lambda - 1)\nu^2 S_1(y) + \lambda (\lambda - 1)\nu^2 U_1(y) + \lambda (\lambda - 1)\nu^2 U_1(y) + \lambda (\lambda - 1)\nu^2 U_1(y) + \cdots \]

  with the following structure of the remainder

  \[
  \phi_{\text{in}}(\lambda, \nu; y) = \phi_{\text{in},1}(\lambda, \nu; y) + p(\nu; \lambda)\phi_{\text{in},2}(\lambda, \nu; y).
  \]

- **(Estimates for the remainder)** For any \( k \in \mathbb{N} \), we have

  \[
  |\phi_{\text{in},1}|_k + |\nu \partial_\nu \phi_{\text{in},1}|_k \lesssim_k \nu^4 \left( \frac{1}{\nu} \right) + 1 + \frac{1}{2\kappa} \nu^2(1 + \log y)^2
  \]

  \[
  |\partial_\lambda \phi_{\text{in},1}|_k \lesssim_k \nu^4 \left( \frac{1}{\nu} \right) + 1 + \frac{1}{2\kappa} \nu^2(\log y)^2
  \]

  and

  \[
  |\phi_{\text{in},2}|_k + |\nu \partial_\nu \phi_{\text{in},2}|_k + |\partial_\lambda \phi_{\text{in},2}|_k \lesssim_k \nu^4 \left( \frac{1}{\nu} \right) + 1 + \frac{1}{2\kappa} \nu^2(\log y)^2.
  \]

**Proof.** In the proof, we always assume \( y \in (0, \frac{2\kappa}{\nu}) \) and write \( \rho = \nu y \). Let us write

\[
\phi_{\text{rough}}(\lambda, \nu; y) := \Lambda Q(y) + \nu^2 T_1(y) + (2\lambda - 1)\nu^2 S_1(y) + \lambda (\lambda - 1)\nu^2 U_1(y),
\]

\[
\phi_{\text{in}}(\lambda, \nu; y) := \phi_{\text{rough}}(\lambda, \nu; y) + (\lambda - 1)\nu^2 U_1(y),
\]

**Step 1.** Computation of the inhomogeneous error term.

In this step, we compute the total inhomogeneous error term

\[
\Psi_{\text{in}} := \left[ H + \nu^2 (\Lambda_0 + \lambda)(\Lambda + \lambda) \right] \phi_{\text{in}}.
\]

More precisely, our aim is to derive (2.70) below.

We begin by recalling the inhomogeneous error from the rough ansatz (2.29):

\[
(2.69) \quad \left[ H + \nu^2 (\Lambda_0 + \lambda)(\Lambda + \lambda) \right] \phi_{\text{rough}} = \nu^2 (\Lambda_0 + \lambda)(\lambda + \lambda) (\phi_{\text{rough}} - \Lambda Q).
\]

Next, in order to incorporate the \( U_\infty \)-correction, we note that

\[
(2.70) \quad \left[ H + \nu^2 (\Lambda_0 + \lambda)(\Lambda + \lambda) \right] \phi_{\text{in}} = \lambda (\lambda - 1)\nu^3 \chi_{\geq 1}(y) \left[ -\Delta_1 + (\Lambda_0 + \lambda)(\Lambda + \lambda) \right] U_\infty(\rho) + \Psi_1,
\]

where

\[
\Psi_1 := -\lambda (\lambda - 1)\nu \left( \frac{8}{(1 + \nu^2)^2} \right) \chi_{\geq 1}(y) \left[ -\Delta_1 + (\Lambda_0 + \lambda)(\Lambda + \lambda), \chi \right] \left( U_\infty(\lambda; \nu y) \right).
\]
Finally, we rewrite $\Psi(2.75)$:

\[
(- \Delta_1 + (\Lambda_0 + \lambda)(\Lambda + \lambda)) U_{\infty}(\lambda; \rho) = -\Delta_1 \left(c_1 \rho^\lambda (\log \rho + \frac{d_1}{2}) \right) = -(\Lambda_0 + \lambda)(\Lambda + \lambda) \left\{ \rho (\log \rho + \frac{d_0}{2}) \right\}.
\]

Substituting this into (2.70) yields

\[
[H + \nu^2 (\Lambda_0 + \lambda)(\Lambda + \lambda)] \left( \lambda (\lambda - 1) \nu \chi_{\geq 1}(y) U_{\infty}(\nu y) \right) = -\lambda(\lambda - 1) \nu^2 \chi_{\geq 1}(y) \cdot (\Lambda_0 + \lambda)(\Lambda + \lambda) \left\{ \rho (\log \rho + \frac{d_0}{2}) \right\} + \Psi_1.
\]

Summing up (2.69) and (2.72), we obtain the total inhomogeneous error:

\[
(2.73) \quad \Psi_{in} = \Psi_1 + \nu^4 (\Lambda_0 + \lambda)(\Lambda + \lambda) \Psi_2,
\]

where (recall that $\Psi_1$ was defined in (2.71))

\[
(2.74) \quad \Psi_2 := \nu^{-2} (\tilde{\phi}_{\text{rough}} - \Lambda Q) - \lambda(\lambda - 1) \nu \chi_{\geq 1}(y) \cdot \left( y \log(\nu y) + \frac{d_0}{2} y \right).
\]

Finally, we rewrite $\Psi_2$ as (c.f. the proof of (2.66))

\[
(2.75) \quad \Psi_2 = \left( T_1 + \chi_{\geq 1} \frac{1}{3} y \right) + (2\lambda - 1) \left( S_1 - \chi_{\geq 1} \frac{1}{3} y \right) + \lambda(\lambda - 1) \left( U_1 - \chi_{\geq 1} (y \log y - y) \right) + \chi_{\geq 1} p(\nu; \lambda) y =: \Psi_{2,1} + \chi_{\geq 1} p(\nu; \lambda) y.
\]

Substituting (2.75) into (2.73), we finally obtain

\[
(2.76) \quad \Psi_{in} = \tilde{\Psi}_1 + p(\nu; \lambda) \tilde{\Psi}_2,
\]

where

\[
(2.77) \quad \tilde{\Psi}_1 := \Psi_1 + \nu^4 (\Lambda_0 + \lambda)(\Lambda + \lambda) \Psi_{2,1},
\]

\[
(2.78) \quad \tilde{\Psi}_2 := \nu^4 (\Lambda_0 + \lambda)(\Lambda + \lambda) (\chi_{\geq 1} y),
\]

and $\Psi_1$ and $\Psi_{2,1}$ were defined in (2.71) and (2.75).

**Step 2. Estimates for $\Psi_{in}$**

We first estimate $\tilde{\Psi}_1$. Recall that

\[
\tilde{\Psi}_1 = \Psi_1 + \nu^4 (\Lambda_0 + \lambda)(\Lambda + \lambda) \Psi_{2,1}.
\]

To estimate $\Psi_1$, we use (2.71), $|\lambda(\lambda - 1)| \lesssim \frac{1}{\log \nu}$, and the pointwise estimate (2.64) for $U_{\infty}$, we have

\[
| \log \nu | (|\Psi_1|_k + |\nu \partial_\nu \Psi_1|_k) + | \partial_\lambda \Psi_1|_k \\
\lesssim_k \nu \left( 1_{[1,2\delta_0/\nu]} \frac{1}{\nu} |U_{\infty}(\lambda; \nu y)|_{k+1} + 1_{[1,2]} |U_{\infty}(\lambda; \nu y)|_{k+2} \right) \\
+ \frac{\nu}{| \log \nu |} \left( 1_{[1,2\delta_0/\nu]} \frac{1}{\nu} |\partial_\lambda U_{\infty}(\lambda; \nu y)|_k + 1_{[1,2]} |\partial_\lambda U_{\infty}(\lambda; \nu y)|_{k+1} \right) \\
\lesssim_k \nu^4 1_{[1,2\delta_0/\nu]} \frac{\log(\nu y)}{y}.
\]

Next, using the definition (2.75) of $\Psi_{2,1}$ and the asymptotics (2.26) for the profiles $T_1, S_1, U_1$, we obtain

\[
|\Psi_{2,1}|_k + |\nu \partial_\nu \Psi_{2,1}|_k + | \partial_\lambda \Psi_{2,1}|_k \lesssim_k 1_{(0,1]} y^3 + 1_{[1,2\delta_0/\nu]} \frac{\log y}{y}.
\]
Substituting the above estimates into (2.77), we have
\[
(2.79) \quad \begin{cases}
|\hat{\Psi}_1|_k + |\nu \partial_\nu \hat{\Psi}_1|_k \lesssim_k \nu^4 \left(1_{(0,1]} y^3 + 1_{[1,2\delta_0/v]} \log y \right), \\
|\partial_\lambda \hat{\Psi}_1|_k \lesssim_k \nu^4 \left(1_{(0,1]} y^3 + 1_{[1,2\delta_0/v]} |\log \nu|^\frac{1}{2} \right).
\end{cases}
\]

We turn to estimate \(\hat{\Psi}_2\). We recall from (2.78) that
\[
\hat{\Psi}_2 = \nu^4 (\lambda_0 + \lambda) (\Lambda + \lambda) (\chi_{\geq 1} \cdot y).
\]

Thus we easily have
\[
(2.80) \quad |\hat{\Psi}_2|_k + |\nu \partial_\nu \hat{\Psi}_2|_k + |\partial_\lambda \hat{\Psi}_2|_k \lesssim_k \nu^4 1_{[1,2\delta_0/v]} y.
\]

**Step 3. Completion of the proof.**
In view of the structure (2.76) of \(\Psi_{in}\), we can construct \(\tilde{\phi}_{in}\) by
\[
\phi_{in} = \tilde{\phi}_{in,1} + p(\nu; \lambda) \tilde{\phi}_{in,2},
\]
where \(\tilde{\phi}_{in,1}\) and \(\tilde{\phi}_{in,2}\) solve the differential equations
\[
H \tilde{\phi}_{in,1} = -[\hat{\Psi}_1 + \nu^2 (\Lambda_0 + \lambda) (\Lambda + \lambda) \tilde{\phi}_{in,1}],
\]
\[
H \tilde{\phi}_{in,2} = -[\hat{\Psi}_2 + \nu^2 (\Lambda_0 + \lambda) (\Lambda + \lambda) \tilde{\phi}_{in,2}],
\]
respectively. Introducing the short-hand notation
\[
H_{rem} := \nu^2 (\Lambda_0 + \lambda) (\Lambda + \lambda)
\]
and taking \(\nu \partial_\nu\) and \(\partial_\lambda\) to the above differential equations, we arrive at the system
\[
\begin{cases}
\tilde{\phi}_{in,t} = -H^{-1} \left[ \hat{\Psi}_t + H_{rem} \tilde{\phi}_{in,t} \right], \\
(\nu \partial_\nu \tilde{\phi}_{in,t}) = -H^{-1} [(\nu \partial_\nu \hat{\Psi}_t) + (\nu \partial_\nu H_{rem}) \tilde{\phi}_{in,t} + H_{rem} (\nu \partial_\nu \tilde{\phi}_{in,t})], \\
(\partial_\lambda \tilde{\phi}_{in,t}) = -H^{-1} [(\partial_\lambda \hat{\Psi}_t) + (\partial_\lambda H_{rem}) \tilde{\phi}_{in,t} + H_{rem} (\partial_\lambda \tilde{\phi}_{in,t})].
\end{cases}
\]
for each \(t \in \{1, 2\}\). As in the proof of Lemma 2.7, we set up the function space \(X_t\) whose norm is defined by the smallest number satisfying
\[
|f_1|_2 + |f_2|_2 \leq \|f\|_{X_t} \left(1_{(0,1]} y^5 + 1_{[1,2\delta_0/v]} |\log y|^2 \right),
\]
\[
|f_3|_2 \leq \|f\|_{X_t} \left(1_{(0,1]} y^5 + 1_{[1,2\delta_0/v]} |\log \nu|^2 |\log y| \right),
\]
and
\[
|f_1|_2 + |f_2|_2 + |f_3|_2 \leq \|f\|_{X_t} \cdot 1_{[1,2\delta_0/v]} y^3,
\]
respectively. Here, \(f_1, f_2, f_3\) correspond to \(\tilde{\phi}_{in,t}, \nu \partial_\nu \tilde{\phi}_{in,t}, \nu \partial_\nu \tilde{\phi}_{in,t}\). Thanks to the estimates \((2.70-2.80)\) for \(\hat{\Psi}_t, (2.27)\) for \(H^{-1}\), and the smallness \(\delta_0 \ll 1\), one can apply the contraction principle. As a result, one obtains the unique existence of \(\phi_{in,t}\) in \(X_t\) as well as the desired pointwise estimates. Finally, smoothness and higher derivative estimates follow in a similar fashion as in the proof of Lemma 2.7 which we will not repeat. This completes the proof. \(\square\)

Translating the above results in terms of the self-similar variable \(\rho\), we get:

**Corollary 2.14 (Refined inner eigenfunctions).** For any \(\lambda\) with either \(|\lambda| \lesssim \frac{1}{\log v}\) or \(|\lambda - 1| \lesssim \frac{1}{\log v}\), the inner eigenfunction defined by
\[
(2.81) \quad \tilde{\phi}_{in}(\lambda; \nu; \rho) := \frac{1}{\nu} \phi_{in}(\lambda, \nu; \rho)
\]
in the region \((0, 2\delta_0]\) satisfies the following properties.
• (Decomposition) The inner eigenfunction \( \varphi_{in}(\lambda, \nu; y) \) admits the decomposition

\[
\varphi_{in}(\lambda, \nu; \rho) = \left[ \frac{1}{\nu} \Lambda Q + \nu T_1 + (2\lambda - 1)\nu S_1 + \lambda(\lambda - 1)\nu U_1 \right] \left( \frac{\rho}{\nu} \right) + \chi_{\geq \nu}(\rho) \lambda(\lambda - 1)U_\infty(\lambda; \rho) + \tilde{\varphi}_{in}(\lambda, \nu; \rho)
\]

with the following structure of the remainder

\[
\tilde{\varphi}_{in}(\lambda, \nu; \rho) = \tilde{\varphi}_{in.1}(\lambda, \nu; \rho) + p(\nu; \lambda)\tilde{\varphi}_{in.2}(\lambda, \nu; \rho).
\]

• (Estimates for the remainder) For any \( k \in \mathbb{N} \), we have

\[
|\tilde{\varphi}_{in.1}|_k + |\nu \partial \tilde{\varphi}_{in.1}|_k \lesssim_k \nu^2 \left( 1_{(0, \nu)}(\rho) \left( \frac{\rho}{\nu} \right)^4 + 1_{[\nu, 2\delta_0]}(\rho)(\log\left( \frac{\rho}{\nu} \right))^2 \right),
\]

\[
|\partial^2 \tilde{\varphi}_{in.1}|_k \lesssim_k \nu^2 \left( 1_{(0, \nu)}(\rho) \left( \frac{\rho}{\nu} \right)^4 + 1_{[\nu, 2\delta_0]}(\rho) \left( \log\left( \frac{\rho}{\nu} \right) \right)^2 \right),
\]

and

\[
|\tilde{\varphi}_{in.2}|_k + |\nu \partial \tilde{\varphi}_{in.2}|_k + |\partial^2 \tilde{\varphi}_{in.2}|_k \lesssim_k 1_{[\nu, 2\delta_0]}(\rho)^3.
\]

Proof. This follows from the definition \ref{eq:decomposition} and Lemma \ref{lem:gradient}. Note that

\[
\nu \partial \left[ \frac{1}{\nu} f(\lambda, \nu; \frac{\rho}{\nu}) \right] = \frac{1}{\nu} [\nu \partial \nu - \lambda_0] f(\lambda, \nu; \frac{\rho}{\nu}).
\]

We omit the details.

\[\square\]

2.4. Matching. In the previous subsections, we have constructed inner and outer eigenfunctions for any \( \lambda \) with \( |\lambda| \lesssim \frac{1}{\log \nu} \) or \( |\lambda - 1| \lesssim \frac{1}{\log \nu} \). For any \( \lambda \), we may consider the function

\[
\varphi(\lambda, \nu; \rho) = \begin{cases} 
\varphi_{in}(\lambda, \nu; \rho) & \text{if } \rho \leq \delta_0, \\
[\varphi_{in}/\varphi_{out}](\lambda, \nu; \delta_0) \cdot \varphi_{out}(\lambda, \nu; \rho) & \text{if } \rho \geq \delta_0,
\end{cases}
\]

where \( \varphi \) is well-defined at \( \rho = \delta_0 \). In general, this function has different left and right derivatives at \( \rho = \delta_0 \), and hence it is not a solution to \ref{eq:ODE}. However, for non-generic values of \( \lambda \), it is possible that \( \varphi \) has the same left and right derivatives, which we call matching. When the matching happens, \( \varphi_{in}(\rho) = [\varphi_{in}/\varphi_{out}](\delta_0) \cdot \varphi_{out}(\rho) \) for all \( \rho \in (\frac{1}{2}\delta_0, 2\delta_0) \) (by the uniqueness of solutions to second-order ODEs) and hence \( \varphi \) becomes a smooth solution to \ref{eq:ODE}.

The first goal of this subsection is to show that the matching happens for some unique \( \lambda = \lambda_j \) near 1 - \( j \) and in fact \( \lambda_j \approx \lambda_j \) (see \ref{eq:matching}). Using this, we also finish the proof of Proposition \ref{prop:matching}.

Lemma 2.15 (Some preliminaries for matching). For any \( \lambda \) with either \( |\lambda| \lesssim \frac{1}{\log \nu} \) or \( |\lambda - 1| \lesssim \frac{1}{\log \nu} \), the following hold for \( \rho \in [\frac{1}{\delta_0}, 2\delta_0] \).

• (Relation between inner and outer eigenfunctions) We have

\[
\varphi_{in}(\lambda, \nu; \rho) = c_{\text{conn}}(\lambda) \varphi_{out}(\lambda, \nu; \rho) + p(\nu; \lambda)(\rho + \varphi_{in.2}(\lambda, \nu; \rho)) + \Psi_{\text{conn}}(\lambda, \nu; \rho),
\]

where we recall \( c_{\text{conn}}(\lambda) \) from \ref{eq:connexion} and \( \Psi_{\text{conn}} \) satisfies

\[
|\Psi_{\text{conn}}|_k + |\nu \partial \Psi_{\text{conn}}|_k + |\partial^2 \Psi_{\text{conn}}|_k \lesssim_k \nu^2 \left( \log \nu \right)^2 \delta_0, \quad \forall k \in \mathbb{N}.
\]
\( (\text{Decay rates of outer eigenfunctions}) \) We have

\[
\left| \frac{\varphi'_\text{out}(\lambda, \nu; \rho)}{\varphi_\text{out}(\lambda, \nu; \rho)} + \frac{1}{\rho} \right| \lesssim \frac{1}{|\log \nu|} \delta_0 |\log \delta_0|,
\]

\[
|\nu \partial_\nu \left( \frac{\varphi'_\text{out}(\lambda, \nu; \rho)}{\varphi_\text{out}(\lambda, \nu; \rho)} \right) | \lesssim \delta_0 \nu^2,
\]

\[
|\partial_\lambda \left( \frac{\varphi'_\text{out}(\lambda, \nu; \rho)}{\varphi_\text{out}(\lambda, \nu; \rho)} \right) | \lesssim \delta_0 |\log \delta_0|.
\]

\( (\text{Properties of } p(\nu; \lambda)) \) We have

\[
|\log \nu| |\nu \partial_\nu p| + (|p| + |\nu \partial_\nu \partial_\lambda p|) + \frac{1}{|\log \nu|} (|\partial_\lambda p| + |\partial_\lambda \lambda p|) \lesssim 1,
\]

\[
\delta_\lambda \rho \gtrsim |\log \nu|.
\]

**Proof.** In the proof, we always assume \( \rho \in [\frac{1}{2} \delta_0, 2 \delta_0] \) or equivalently \( y \in [\frac{d_0}{2 \delta_0}, \frac{2 d_0}{\delta_0}] \). In particular, we can ignore the cutoff \( \chi_{\geq \nu}(\rho) \) in the refined ansatz for the inner eigenfunction.

First, we show (2.87). By (2.86), (2.56), (2.82), and (2.63), we have

\[
\Psi_{\text{conn}} = \varphi_\text{in} - c_{\text{conn}} \varphi_\text{out} - p(\nu; \lambda)(\rho + \varphi_{\text{in}, 2}) = \hat{\Psi}_{\text{conn}} + \hat{\varphi}_{\text{in}, 1} - c_{\text{conn}} \hat{\varphi}_{\text{out}}.
\]

Applying (2.66), (2.84), and (2.54) to the above (with the parameter dependence \( \nu^2 \ll \delta_0 \)) completes the proof of (2.87).

Next, we show (2.88)-(2.90). Recall that

\[
\varphi_\text{out}(\lambda, \nu; \rho) = \frac{1}{\rho} h_1(\lambda; 1 - \rho^2) + \tilde{\varphi}_\text{out}(\lambda, \nu; \rho).
\]

Using the expansion (2.43) of \( h_1(\lambda; 1 - \rho^2) \), we have

\[
\frac{1}{\rho} h_1(\lambda; 1 - \rho^2) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2} + 1) \Gamma(\frac{1}{2})} \cdot \frac{1}{\rho} \left( 1 + \tilde{h}_1(\lambda; \rho) \right),
\]

where

\[
\tilde{h}_1(\lambda; \rho) = \frac{\lambda(\lambda - 1)}{2} \rho^2 \sum_{n=0}^{\infty} c_n \rho^{2n} [\log \rho + \frac{d_n}{2}].
\]

Thus

\[
\frac{\varphi'_\text{out}}{\varphi_\text{out}} = \frac{\partial_\rho \left\{ \frac{1}{\rho} (1 + \tilde{h}_1) + \frac{1}{2} c_{\text{conn}} \tilde{\varphi}_\text{out} \right\}}{\frac{1}{\rho} (1 + \tilde{h}_1) + \frac{1}{2} c_{\text{conn}} \tilde{\varphi}_\text{out}}.
\]

Now (2.88)-(2.90) follow from applying the pointwise estimates

\[
|\log \nu| |\tilde{h}_1|_k + |\partial_\lambda \tilde{h}_1|_k \lesssim k \delta_0^2 |\log \delta_0|, \quad \forall k \in \mathbb{N},
\]

\( \nu \partial_\nu \tilde{h}_1 \equiv 0 \), (2.57), and (2.63).

Finally, (2.91) and (2.92) are immediate consequences of the definition (2.4) of \( p(\nu; \lambda) \) and \( |\lambda(\lambda - 1)| \lesssim \frac{1}{|\log \nu|} \). This completes the proof. \( \square \)

With Lemma 2.15 at hand, we are now ready to achieve the matching.

**Lemma 2.16 (Matching).** For each \( j \in \{0, 1\} \), there exists unique \( \lambda_j = \lambda_j(\nu) \) in the class \( |\lambda_j - (1 - j)| \lesssim \frac{1}{|\log \nu|} \) such that the matching condition

\[
\varphi'_\text{in}(\lambda_j, \nu; \delta_0) = \frac{\varphi'_\text{out}(\lambda_j, \nu; \delta_0)}{\varphi_\text{out}(\lambda_j, \nu; \delta_0)} \varphi_\text{in}(\lambda_j, \nu; \delta_0)
\]

holds. Moreover, we have the following.
(Refined estimates for the eigenvalues) We have
\[ |\lambda_j - \widehat{\lambda}_j| \lesssim \nu^2 |\log \nu|, \]
\[ |\nu \partial_\nu (\lambda_j - \widehat{\lambda}_j)| \lesssim \nu^2 |\log \nu|, \]
where we recall \( \widehat{\lambda}_j \) from (2.3). In particular, we have
\[ |\nu \partial_\nu \lambda_j| \lesssim \frac{1}{|\log \nu|^2}. \]

(Estimates for \( p(\nu; \lambda_j) \)) We have
\[ |p(\nu; \lambda_j)| \lesssim \nu^2 |\log \nu|^2, \]
\[ |\nu \partial_\nu p(\nu; \lambda_j)| \lesssim \nu^2 |\log \nu|^2. \]

(Estimates for the matching constants) The matching constants defined by
\[ c_{j, \text{match}}(\nu) := \frac{\varphi_{\text{in}}(\lambda_j(\nu), \nu \cdot \delta_0)}{\varphi_{\text{out}}(\lambda_j(\nu), \nu \cdot \delta_0)}, \quad j \in \{0, 1\} \]
satisfy
\[ |c_{j, \text{match}} - c_{\text{conn}}(\widehat{\lambda}_j)| + |\nu \partial_\nu (c_{j, \text{match}} - c_{\text{conn}}(\widehat{\lambda}_j))| \lesssim \nu^2 |\log \nu|^2. \]

Proof. Define
\[ \Phi(\nu, \delta_0; \lambda) := \varphi'_{\text{in}}(\lambda, \nu \cdot \delta_0) - \varphi'_{\text{out}}(\lambda, \nu \cdot \delta_0) \cdot \frac{\varphi_{\text{in}}(\lambda, \nu \cdot \delta_0)}{\varphi_{\text{out}}(\lambda, \nu \cdot \delta_0)}. \]
We rewrite \( \Phi \) as
\[ \Phi(\nu, \delta_0; \lambda) = \left[ (\partial_\rho - \frac{\varphi'_{\text{out}}}{\varphi_{\text{out}}}) \varphi_{\text{in}} \right](\lambda, \nu \cdot \delta_0) \]
\[ = \left[ (\partial_\rho - \frac{\varphi'_{\text{out}}}{\varphi_{\text{out}}}) (\varphi_{\text{in}} - c_{\text{conn}} \varphi_{\text{out}}) \right](\lambda, \nu \cdot \delta_0) \]
\[ = \left[ (\partial_\rho - \frac{\varphi'_{\text{out}}}{\varphi_{\text{out}}}) (p \cdot (\rho + \tilde{\varphi}_{\text{in}, 2}) + \Psi_{\text{conn}}) \right](\lambda, \nu \cdot \delta_0), \]
where in the last equality we used (2.86). Note that the matching condition (2.94) is equivalent to
\[ \Phi(\nu, \delta_0; \lambda) = 0. \]

Step 1. Preliminary estimates for \( \Phi \).
In this step, we claim the following estimates for \( \Phi \):
\[ \Phi = p \cdot \left( \partial_\rho - \frac{\varphi'_{\text{out}}}{\varphi_{\text{out}}} \right) (\rho + \tilde{\varphi}_{\text{in}, 2}) + O(\nu^2 |\log \nu|^2), \]
\[ \nu \partial_\nu \Phi = \nu \partial_\nu p \cdot \left( \partial_\rho - \frac{\varphi'_{\text{out}}}{\varphi_{\text{out}}} \right) (\rho + \tilde{\varphi}_{\text{in}, 2}) + p \cdot O(\delta_0^2 |\log \delta_0|) + O(\nu^2 |\log \nu|^2), \]
\[ \partial_\lambda \Phi = \partial_\lambda p \cdot \left( \partial_\rho - \frac{\varphi'_{\text{out}}}{\varphi_{\text{out}}} \right) (\rho + \tilde{\varphi}_{\text{in}, 2}) + p \cdot O(\delta_0^2 |\log \delta_0|) + O(\nu^2 |\log \nu|^2). \]
Indeed, (2.104) follows from (2.102) and (2.87). As the proofs of (2.105) and (2.106) are very similar, so we only show (2.106). We start from writing
\[ \partial_\lambda \Phi = \partial_\lambda p \cdot \left( \partial_\rho - \frac{\varphi'_{\text{out}}}{\varphi_{\text{out}}} \right) (\rho + \tilde{\varphi}_{\text{in}, 2}) \]
\[ + \left( \partial_\rho - \frac{\varphi'_{\text{out}}}{\varphi_{\text{out}}} \right) (p \cdot \partial_\lambda \tilde{\varphi}_{\text{in}, 2} + \partial_\lambda \Psi_{\text{conn}}) \]
\[ - \partial_\lambda \left( \frac{\varphi'_{\text{out}}}{\varphi_{\text{out}}} \right) (p \cdot (\rho + \tilde{\varphi}_{\text{in}, 2}) + \Psi_{\text{conn}}). \]
We keep the first line of RHS (2.107). For the second line of RHS (2.107), we use (2.88), (2.85), and (2.91) to have
\[
\left( \partial_p - \frac{\varphi_{\text{out}}}{\varphi_{\text{out}}} \right) (p \cdot \partial_\lambda \tilde{\varphi}_{\text{in},2} + \partial_\lambda \Psi_{\text{conn}}) = p \cdot O(\delta_0^2) + O(\nu^2 |\log \nu|^2).
\]
For the third line of RHS (2.107), we use (2.92), (2.88), (2.85), and (2.91) to have
\[
-\partial_\lambda \left( \frac{\varphi_{\text{out}}}{\varphi_{\text{out}}} \right) (p \cdot (p + \tilde{\varphi}_{\text{in},2}) + \Psi_{\text{conn}}) = p \cdot O(\delta_0^2 |\log \delta_0|) + O(\nu^2 |\log \nu|^2).
\]
Substituting the above two displays into (2.107) completes the proof of (2.106).

**Step 2.** Unique existence of \( \lambda_j \), and proofs of (2.93) and (2.98).

We first claim that
\[
(2.108) \quad |\Phi(\nu, \delta_0; \lambda_j)| \lesssim \nu^2 |\log \nu|^2,
\]
\[
(2.109) \quad \partial_\lambda \Phi(\nu, \delta_0; \lambda) \gtrsim |\log \nu|.
\]
Indeed, (2.108) follows from (2.102) and \( p(\nu; \tilde{\lambda}_j) = 0 \). The proof of (2.109) follows from applying (2.89), (2.85), (2.82), and (2.91) to (2.106):
\[
\partial_\lambda \Phi = \partial_\lambda p \cdot \left( \partial_\lambda - \frac{\varphi_{\text{out}}}{\varphi_{\text{out}}} \right)(p \cdot (p + \tilde{\varphi}_{\text{in},2}) + p \cdot O(\delta_0^2 |\log \delta_0|) + O(\nu^2 |\log \nu|^2)
\]
\[
= \partial_\lambda p \cdot (2 + O(\delta_0^2)) + O(\delta_0^2 |\log \delta_0|)
\]
\[
\gtrsim |\log \nu|.
\]
Having settled the proofs of the claims (2.108)-(2.109), the unique existence of \( \lambda_j \) satisfying \( \Phi(\nu, \delta_0; \lambda_j) = 0 \) in the class \( |\lambda - j| \lesssim \frac{1}{|\log \nu|} \) and the bound (2.93) follow from the (quantitative) implicit function theorem. (2.98) now follows from (2.91) and (2.95):
\[
|p(\nu; \lambda_j)| = |p(\nu; \lambda_j) - p(\nu; \lambda)| \lesssim |\log \nu| \cdot \nu^2 |\log \nu| \lesssim \nu^2 |\log \nu|^2.
\]

**Step 3.** Proofs of the derivative bounds (2.96), (2.97), and (2.99).

We first show (2.96). We start from the identities
\[
\nu \partial_\nu \lambda_j = -\left[ \frac{\nu \partial_\nu \Phi}{\partial_\lambda \Phi} \right] (\nu, \delta_0; \lambda_j) \quad \text{and} \quad \nu \partial_\nu \tilde{\lambda}_j = -\left[ \frac{\nu \partial_\nu p}{\partial_\lambda p} \right] (\nu, \lambda_j).
\]
This implies
\[
(2.110) \quad \nu \partial_\nu (\lambda_j - \tilde{\lambda}_j) = \left[ \nu \partial_\nu p \cdot \partial_\lambda \Phi - \partial_\lambda p \cdot \nu \partial_\nu \Phi \right] (\nu, \delta_0; \lambda_j) + \left[ \nu \partial_\nu p \right] \tilde{\lambda}_j.
\]
The second term of RHS (2.110) is of size \( O(\nu^2) \) due to (2.91), (2.92), and (2.95). For the first term of RHS (2.110), we notice that its denominator has size \( \gtrsim |\log \nu|^2 \) due to (2.92) and (2.109). Therefore, we have proved that
\[
(2.111) \quad |\nu \partial_\nu (\lambda_j - \tilde{\lambda}_j)| \lesssim \frac{|\nu \partial_\nu p \cdot \partial_\lambda \Phi - \partial_\lambda p \cdot \nu \partial_\nu \Phi|(\nu, \delta_0; \lambda_j)}{|\log \nu|^2} + \nu^2.
\]
We then recall from (2.105) and (2.106) (after applying (2.98)) that
\[
\nu \partial_\nu \Phi = \nu \partial_\nu p \cdot \left( \partial_\lambda - \frac{\varphi_{\text{out}}}{\varphi_{\text{out}}} \right)(p + \tilde{\varphi}_{\text{in},2}) + O(\nu^2 |\log \nu|^2),
\]
\[
\partial_\lambda \Phi = \partial_\lambda p \cdot \left( \partial_\lambda - \frac{\varphi_{\text{out}}}{\varphi_{\text{out}}} \right)(p + \tilde{\varphi}_{\text{in},2}) + O(\nu^2 |\log \nu|^2).
\]
Therefore, we have
\[
|\nu \partial_\nu p \cdot \partial_\lambda \Phi - \partial_\lambda p \cdot \nu \partial_\nu \Phi|(\nu, \delta_0; \lambda_j) \lesssim \nu^2 |\log \nu|^3.
\]
Substituting this into (2.111) completes the proof of (2.96).
Note that (2.97) is immediate from (2.96) and $|ν∂ν\tilde{λ}_j| \lesssim \frac{1}{|\log ν|^2}$. For the proof of (2.99), note that

$$ν∂ν[p(ν; λ_j)] = ν∂ν[p(ν; λ_j) − p(ν; \tilde{λ}_j)]$$

$$= [ν∂ν,p(ν; λ_j) − [ν∂ν,p(ν; \tilde{λ}_j)]$$

$$+ ν∂ν,λ_j · [∂λp(ν; λ_j) − ν∂ν,λ_j · [∂λp(ν; \tilde{λ}_j)].$$

Thus (2.99) follows from (2.91), (2.95), and (2.96):

$$|ν∂ν[p(ν; λ_j)]| \lesssim (\sup |ν∂ν,λ_j| + \frac{1}{|\log ν|^2}) \sup |∂λp(ν)||λ_j − \tilde{λ}_j| + (\sup |∂λp(ν)|) |ν∂ν(λ_j − \tilde{λ}_j)|$$

$$\lesssim ν^2|\log ν|^2,$$

where the supremum is taken over $λ$ with $|λ − (1 − j)| \lesssim \frac{1}{|\log ν|}$.

**Step 4. Proof of (2.101).**

In this last step, we show the estimate (2.101) for the matching constants. We start from writing

$$c_j,\text{match}(ν) − c_\text{conn}(\tilde{λ}_j)$$

$$= \frac{1}{\varphi_\text{out}(0, ν; ν_0)} \varphi_\text{in}(c_\text{conn}(λ_j), ν_0) (c_\text{conn}(λ_j) − c_\text{conn}(\tilde{λ}_j))$$

(2.112)

$$= \frac{1}{\varphi_\text{out}(0, ν; ν_0)} (p \cdot (ρ + \varphi_\text{in}(2, ν; ν_0)) (c_\text{conn}(λ_j) − c_\text{conn}(\tilde{λ}_j)),$$

where in the last equality we used (2.86). Using $\frac{1}{\varphi_\text{out}(0, ν; ν_0)} \sim ρ \sim δ_0$. Using (2.98), and (2.87), the first term of RHS (2.112) is of size $O(δ_0^4 ν^2 |\log ν|^2)$. The same bound also holds for $ν∂ν$ taken on the first term of RHS (2.112). On the other hand, using (2.94), the second term of RHS (2.112) is of size $O(ν^2 |\log ν|)$. The same bound also holds after taking $ν∂ν$. This completes the proof.

We finish the proof of Proposition 2.1.

**Proof of Proposition 2.7.** We fix small parameters $0 < δ_0 \ll 1$ and $0 < ν^* = ν^*(δ_0) \ll 1$ so that all the previous lemmas hold. Now we can safely ignore any dependence on $δ_0$ in the following analysis.

(1) Define

$$φ_j(ν; ρ) := \begin{cases} φ_\text{in}(λ_j(ν), ν; ρ) & \text{if } ρ < δ_0, \\ c_j,\text{match}(ν)φ_\text{out}(λ_j(ν), ν; ρ) & \text{if } ρ ≥ δ_0, \end{cases}$$

and define $\tilde{φ}_j$ via the formula (2.9). By the definitions (2.100) and (2.94), the values and the first derivatives of $φ_\text{in}(λ_j(ν), ν; ρ)$ and $c_j,\text{match}(ν)φ_\text{out}(λ_j(ν), ν; ρ)$ at $ρ = δ_0$ are the same. Since both are solutions to the second-order differential equation (2.6b), whose singular points do not belong to $(0, 1)$, they are in fact equal for all $ρ \in (0, 2δ_0)$. Therefore, $φ_j$ is a globally-defined analytic solution to (2.6).

(2) The estimates (2.8) and (2.8) are already proved in (2.94).

(3) We show the sharp eigenfunction estimate (2.11). In the inner region $ρ \in (0, δ_0)$, we have

$$\tilde{φ}_j(ν; ρ) = \tilde{φ}_\text{in}(λ_j(ν), ν; ρ).$$

Substituting the structure (2.83) of $\tilde{φ}_\text{in}$ into the above, we have

(2.113) $$\tilde{φ}_j(ν; ρ) = \tilde{φ}_\text{in,1}(λ_j, ν; ρ) + p(ν; λ_j) \tilde{φ}_\text{in,2}(λ_j, ν; ρ).$$
Taking $\nu \partial_\nu$, we also have
\[
[\nu \partial_\nu \tilde{\varphi}_j](\nu; \rho) = [\nu \partial_\nu \tilde{\varphi}_{1m,1} + (\nu \partial_\nu \lambda_j \partial_\lambda \tilde{\varphi}_{1m,1})(\lambda_j, \nu; \rho) + \nu \partial_\nu [p(\nu; \lambda_j)] \cdot \tilde{\varphi}_{1n,2}(\lambda_j, \nu; \rho) + p(\nu; \lambda_j) \cdot \left( [\nu \partial_\nu \tilde{\varphi}_{1n,2} + (\nu \partial_\nu \lambda_j \partial_\lambda \tilde{\varphi}_{1n,2})(\lambda_j, \nu; \rho) \right).
\] (2.114)

Applying the estimates (2.83)-(2.85) of $\tilde{\varphi}_{1m}$ and (2.98)-(2.99) of $p(\nu; \lambda_j)$ to the equations (2.113)-(2.114), we obtain (2.10) for $\rho \in \rho_0, 1]$. Henceforth, we show (2.12) and (2.14).

Next, we use more precise asymptotics (2.115)-(2.116) above, we obtain (2.10) for $\rho \in \rho_0, 1]$. From (2.24), substituting the above display into (2.117) completes the proof of (2.12).

We rearrange the above as
\[
\tilde{\varphi}_j(\nu; \rho) = (c_{j, \text{match}}(\nu) - c_{\text{conn}}(\lambda_j))\rho^{-1}h_1(\lambda_j; 1 - \rho^2) + c_{j, \text{match}}(\nu)\tilde{\varphi}_{\text{out}}(\lambda_j, \nu; \rho) - \tilde{\Psi}_{\text{conn}}(\lambda_j, \nu; \rho) - p(\nu; \lambda_j).\] (2.115)

Taking $\nu \partial_\nu$, we also have
\[
[\nu \partial_\nu \tilde{\varphi}_j](\nu; \rho) = \nu \partial_\nu (c_{j, \text{match}}(\nu) - c_{\text{conn}}(\lambda_j)) \cdot \rho^{-1}h_1(\lambda_j; 1 - \rho^2) + \left( c_{j, \text{match}}(\nu) - c_{\text{conn}}(\lambda_j) \right) \cdot \rho^{-1}[\partial_\nu h_1](\lambda_j; 1 - \rho^2) + c_{j, \text{match}}(\nu)\nu \partial_\nu \tilde{\varphi}_{\text{out}}(\lambda_j, \nu; \rho) + c_{j, \text{match}}(\nu)[\partial_\nu \tilde{\varphi}_{\text{out}} + (\nu \partial_\nu \lambda_j \partial_\lambda \tilde{\varphi}_{\text{out}})(\lambda_j, \nu; \rho)] - [\nu \partial_\nu \tilde{\varphi}_{\text{conn}} + (\nu \partial_\nu \lambda_j \partial_\lambda \tilde{\varphi}_{\text{conn}})(\lambda_j, \nu; \rho)]\rho.
\] (2.116)

We note as a consequence of (2.10) and $|\nu \partial_\nu \lambda_j| \lesssim |\log \nu|^{-2}$ that
\[
|c_{j, \text{match}}| \lesssim 1 \quad \text{and} \quad |\nu \partial_\nu c_{j, \text{match}}| \lesssim |\log \nu|^{-2}.
\]

Applying the estimates (2.10), (2.37), (2.60), (2.95), (2.99), and (2.8) to the displays (2.115)-(2.116) above, we obtain (2.10) for $\rho \in \rho_0, 1]$. (4) Finally, we show (2.11)-(2.14). We note that (2.11) and (2.13) are immediate from (2.12) and (2.14), respectively. Henceforth, we show (2.12) and (2.14).

Let $\rho \in (0, 1]$. For the proof of (2.12), we apply $|\lambda_j(\lambda_j - 1)| \lesssim \frac{1}{|\log \nu|}$, (2.64), and (2.10) to (2.19) to have
\[
(2.117) \quad \left| \left( \tilde{\varphi}_j - \frac{1}{\rho^2} \Lambda Q_\nu \right) - \nu \left[ T_1 + (2\lambda_j - 1)S_1 + \lambda_j(\lambda_j - 1)U_1 \right] \right| \lesssim_k \frac{3^3|\log \rho|}{|\log \nu|}.
\]

Next, we use more precise asymptotics
\[
(2.118) \quad \lambda_j(\lambda_j - 1) = -\frac{1}{|\log \nu|} + O\left( \frac{1}{|\log \nu|^2} \right)
\]
and (2.26) to have
\[
(2.119) \quad 1_{(0, \nu^{-1})}|T_1 + (2\lambda_j - 1)S_1 + \lambda_j(\lambda_j - 1)U_1| \lesssim_k 1_{(0, 1)}y^2 + 1_{(1, \nu^{-1})}y|\log(\nu y)|.
\]

Notice that for large $y \sim \nu^{-1}$ this bound is logarithmically better than a rough bound $O(y)$ from (2.24). Substituting the above display into (2.117) completes the proof of (2.12).
For the proof of (2.14), we take $\nu \partial_\nu$ to (2.9) to have

$$
\nu \partial_\nu \left( \varphi_j - \frac{1}{\nu} \Lambda Q_\nu \right) = -\nu \Lambda_2 \{ T_1 + (2\lambda_j - 1) S_1 \}_\nu + (\nu \partial_\nu \lambda_j) 2 \nu [ S_1 ]_\nu + \nu \left[ -\lambda_j (\lambda_j - 1) \Lambda_2 U_1 + (\nu \partial_\nu \lambda_j) (2\lambda_j - 1) U_1 \right]_\nu + [ \Lambda \lambda ]_\nu (\rho) \lambda_j (\lambda_j - 1) U_\infty (\lambda_j; \rho) + [ \nu \partial_\nu \varphi_j ](\nu; \rho) + (\nu \partial_\nu \lambda_j) \chi_{\geq \nu} (\rho) \partial_\nu \{ \lambda (\lambda - 1) U_\infty \}(\lambda_j; \rho).
$$

(2.119)

For the first line of RHS (2.119), we can apply (2.24), (2.26), and (2.8). For the third and fourth lines of RHS (2.119), we can apply $|\lambda_j (\lambda_j - 1)| \lesssim \frac{1}{|\log \nu|}$, (2.64), and (2.10). As a result, we have

$$
(\nu \partial_\nu \lambda_j) (2\lambda_j - 1) = \frac{-1}{|\log \nu|^2} + O \left( \frac{1}{|\log \nu|^3} \right),
$$

and (2.26) to have

$$
1_{[0, \nu^{-1}]} - \lambda_j (\lambda_j - 1) \Lambda_2 U_1 + (\nu \partial_\nu \lambda_j) (2\lambda_j - 1) U_1 \right|_k \lesssim k \ 1_{[0, \nu]} \frac{\rho^2}{|\log \nu|^2} + 1_{[1, \nu^-1]} \frac{\nu^2 (\log (\nu y))}{|\log \nu|^2}.
$$

Substituting this into (2.120) completes the proof of (2.14). The proof of Proposition 2.1 is complete.

□

3. Formal invariant subspace decomposition for $M_\nu$

In this section, we exhibit a formal invariant subspace decomposition for the operator $M_\nu$. The eigenfunctions $\varphi_j$ constructed in the previous section form a two-dimensional space span{ $\varphi_0, \varphi_1$ } that is invariant under $M_\nu$. From now on, we aim to find two linear functionals $\ell_0, \ell_1$ such that the kernel of $\ell_0, \ell_1$ complements the space span{ $\varphi_0, \varphi_1$ } (i.e., it is of codimension two and transversal to span { $\varphi_0, \varphi_1$ }) and is invariant under $M_\nu$.

Note that we did not specify the total space, say $X$, for the invariant subspace decomposition. However, whatever the total space $X$ is, if $X$ contains the eigenfunctions $\varphi_j$ and the linear functionals $\ell_j$ are well-defined (and $X$ has a dense subspace where $M_\nu$ is defined), then one should have the invariant subspace decomposition

$$
X = \text{span} \{ \varphi_0, \varphi_1 \} \oplus \ker X \{ \ell_0, \ell_1 \},
$$

where $\ker X \{ \ell_0, \ell_1 \}$ is the kernel of $\ell_0$ and $\ell_1$ in $X$. In this paper, the total space $H^2 \times H^1$ restricted in the region $\rho \leq 1$ will work.

For the computational purposes in our later blow-up analysis, we need to find explicit formula of each $\ell_j$. This requires a new input because the operator $M_\nu$ is in general non-self-adjoint on usual (weighted-)Sobolev spaces. To compare the situation with the parabolic case (the harmonic map heat flow), the analogue of $M_\nu$ takes the form

$$
\partial_{\rho \rho} + \frac{1}{\rho} \partial_\rho - \frac{1}{2} \rho \partial_\rho - \frac{V_\nu}{\rho^2},
$$

(3.1)
and it is already a classical fact that this operator is self-adjoint in the weighted 
$L^2$-space $L^2(w \rho d \rho)$ with the Gaussian weight

$$w(\rho) = e^{-\frac{1}{2} \rho^2}.$$  

Therefore, one can simply choose $\ell_j = \langle \epsilon, \varphi \rangle_{L^2(w \rho d \rho)}$ with the eigenfunctions $\varphi_j$ for the operator (3.1), in the parabolic case. In fact, the self-adjointness says more; one can also obtain a spectral gap estimate.

Let us come back to our operator $M_\nu$. There seems to be no natural Hilbert space that makes $M_\nu$ self-adjoint. However, we can find an explicit formula for $\ell_j$:

$$\ell_j(\epsilon) = \langle (\lambda_j + \Lambda_0) \epsilon + \epsilon_j \varphi_j \rangle,$$

$$g_j(\nu; \rho) = 1_{(0,1)}(\rho) \cdot (1 - \rho^2)^{\frac{3}{2}}.$$  

Notice that $g_j$ is sharply localized in the interior of the light cone $\rho \leq 1$. In Proposition 3.1 below, we will check the invariance and transversality properties of $\ell_j$.

We formally derive $\ell_j$ as follows. We notice that finding a linear functional $\ell$ whose kernel is invariant under $M_\nu$ is equivalent to finding an eigenfunction of the transpose $M^*_\nu$ of $M_\nu$. If $M_\nu$ has a real eigenvalue $\lambda$, then the transpose $M^*_\nu$ might also have the eigenvalue $\lambda$. Thus we want to find $\ell$ such that

$$(M^*_\nu - \lambda) \ell = 0.$$  

Now, by considering some weighted $L^2$-space $L^2(g \rho d \rho)$ ($g$ will be chosen soon), we identify $M^*_\nu$ by $M^*_\nu$, where $A^*$ denotes the formal adjoint of $A$ with respect to the $L^2(g \rho d \rho)$-inner product. We then compute $M^*_\nu - \lambda$. We start from writing $M_\nu - \lambda$ into a triangular form:

$$M_\nu - \lambda = \begin{bmatrix} 1 & 0 \\ \Lambda + \lambda & 1 \end{bmatrix} \begin{bmatrix} -H_\nu - (\Lambda_0 + \lambda)(\Lambda + \lambda) & 1 \\ -\Lambda_0 - \Lambda - 2\lambda & -\Lambda - \lambda - 1 \end{bmatrix}.$$  

Next, we take the $L^2(g \rho d \rho)$-adjoint. If we choose (c.f. (1.21))

$$g(\rho) = (1 - \rho^2)^{\frac{3}{2}},$$

then the operator $-H_\nu - (\Lambda_0 + \lambda)(\Lambda + \lambda)$ becomes formally self-adjoint (c.f. (3.6) below) and hence we have

$$M^*_\nu - \lambda = \begin{bmatrix} 1 & -\Lambda^* + \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -H_\nu - (\Lambda_0 + \lambda)(\Lambda + \lambda) & 1 \\ -\Lambda^*_0 - \Lambda^* - 2\lambda & -\Lambda^* + \lambda - 1 \end{bmatrix},$$

where $\Lambda^*$ and $\Lambda^*_0$ denote the formal $L^2(g \rho d \rho)$-adjoints of $\Lambda$ and $\Lambda_0$. From this, we see that

$$\begin{bmatrix} 1 & \Lambda^* + \lambda \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} (\Lambda^*_0 + \Lambda^* + 2\lambda) \varphi_j \\ \varphi_j \end{bmatrix} = \begin{bmatrix} (\Lambda^*_0 + \lambda) \varphi_j \\ \varphi_j \end{bmatrix}$$

is a kernel element of $M^*_\nu - \lambda$. This means that the linear functional $\ell$ defined by

$$\ell(f) = \langle f, (\Lambda^*_0 + \lambda) \varphi \rangle_{L^2(g \rho d \rho)} + \langle \dot{f}, \varphi \rangle_{L^2(g \rho d \rho)}$$

$$= \langle (\Lambda_0 + \lambda) f + \dot{f}, \varphi \rangle_{L^2(g \rho d \rho)} = \langle (\Lambda_0 + \lambda) f + \dot{f}, g \varphi \rangle_{L^2(g \rho d \rho)}$$

is a kernel element of $M^*_\nu - \lambda$. This gives the expression (1.20).

**Proposition 3.1** (Formal invariant subspace decomposition).

- **(Invariance)** For smooth $\epsilon$, we have

$$\ell_j(M_\nu \epsilon) = \lambda_j \ell_j(\epsilon).$$

- **(Transversality)** We have

$$\begin{bmatrix} \ell_0(\varphi_0) & \ell_1(\varphi_0) \\ \ell_0(\varphi_1) & \ell_1(\varphi_1) \end{bmatrix} = \begin{bmatrix} 4 \log \nu + O(1) & 0 \\ 0 & -4 \log \nu + O(1) \end{bmatrix}.$$
Remark 3.2 (Algebraic multiplicity of $\lambda_j$). Each eigenvalue $\lambda_j$ has algebraic multiplicity 1. Note that the geometric multiplicity is 1 because of the unique existence in Proposition 3.2. To show that the algebraic multiplicity is 1, suppose that $(M_\nu - \lambda_j)\zeta_j = \varphi_j$ for some $\zeta_j$. By the invariance (3.2) one has $0 = \ell_j((M_\nu - \lambda_j)\zeta_j) = \ell_j(\varphi_j)$. This is absurd due to (3.3).

Remark 3.3 (Extension to exact self-similar solutions). The above formal derivation of $\ell_j$ can be extended to the linearized operators around exact self-similar solutions to wave equations. For example, for $(1+d)$-dimensional wave maps, one may replace $\Lambda_0$ by $\rho d\rho + 1$ and $g_j\rho d\rho$ by $1_{(0,1)}(\rho) \cdot (1 - \rho^2)^{\frac{d+1}{2}}\rho^{d-1}d\rho$ in the formula of $\ell_j$. Note however that there are some subtleties to use this formula for high dimensions (e.g., $d \geq 5$ for $\lambda = 1$) because the weight $g$ is not integrable near $\rho = 1$. We believe that one may perform some renormalization (e.g., integrating by parts the weight $(1 - \rho^2)^{\frac{d+1}{2}}$ to soften the singularity) to obtain a correct form of $\ell$. Using this, one may also obtain an explicit formula of the Riesz projection onto its discrete spectrum, as well as the analogue of the previous remark.

Remark 3.4 (On dissipativity). This remark requires the notation in Section 4.1. For $f$, consider the orthogonality condition

$$\langle f, [\chi_M \Lambda Q]_{\nu} \rangle = 0 = \langle f, [\chi_M \Lambda Q]_{\nu} \rangle,$$

where $M \gg 1$ is some large fixed constant (see Remark 4.2). We expect that the modified energy functional from [53, Lemma 6.5] suggests the existence of some inner product $\langle f, g \rangle_{X_{\nu}}$ defined on the space of functions satisfying (3.1) at the level $H^2 \times H^1$ with the dissipativity property

$$\langle M_\nu f, f \rangle_{X_{\nu}} \leq -(1 + o_{\nu \to 0}(1)) \langle f, f \rangle_{X_{\nu}}.$$

This is why we called $(\lambda_0, \varphi_0)$ and $(\lambda_1, \varphi_1)$ the first two eigenpairs. However, the orthogonality condition (3.4) is not preserved under the flow of $\partial_\tau - M_\nu$. We do not know how to obtain analogous dissipativity under the orthogonality condition $\ell_\nu(f) = 0 = \ell_1(f)$ which are invariant under the flow of $\partial_\nu - M_\nu$. More technically, the implicit constants in the coercivity estimates (4.21)-(4.22) start to depend on $\nu$ under this orthogonality condition and could be dangerous.

Note that in the parabolic case the dissipativity easily manifests after identifying the eigenvalues, thanks to the self-adjoint structure (under the weighted Gaussian measure). In the dispersive case, proving the dissipativity is in fact a difficulty. There are some results on such dissipativity or decay estimates for dispersive equations in the self-similar coordinates (in various topologies), with applications to the (conditional) stability problem of self-similar blow-up. See for example [10, 15, 19]. Note that we circumvent this problem in our type-II blow-up analysis by assuming the result of [53].

Proof of Proposition 3.2.

**Step 1.** Proof of the invariance (3.2).

This follows from a direct computation. We start from writing

$$\ell_j(M_\nu e) = \langle (\lambda_j + \Lambda_0)(-\Lambda e + \epsilon) + (-H_\nu e - \Lambda_0 \epsilon), g_j \varphi_j \rangle$$

$$-\langle [H_\nu + (\lambda_j + \Lambda_0)(\lambda_j + \Lambda)] e, g_j \varphi_j \rangle + \lambda_j \ell_j(\epsilon).$$

Thus the proof of (3.2) is complete if we show

$$\langle [H_\nu + (\lambda_j + \Lambda_0)(\lambda_j + \Lambda)] e, g_j \varphi_j \rangle = 0.$$
Now, we expand
\[
H_\nu + (\lambda_j + \Lambda_0)(\lambda_j + \Lambda) \\
= -\partial_{\rho\rho} - \frac{1}{\rho}\partial_\rho + \frac{V_\nu}{\rho^2} + (\rho\partial_\rho + \lambda_j + 1)(\rho\partial_\rho + \lambda_j) \\
= -(1 - \rho^2)\partial_{\rho\rho} - \left\{ \frac{1}{\rho} - (2\lambda_j + 2)\rho \right\}\partial_\rho + \frac{V_\nu}{\rho^2} + \lambda_j(\lambda_j + 1).
\]
Using the operator identity (for \( \rho < 1 \))
\[
(1 - \rho^2)\partial_{\rho\rho} + \left\{ \frac{1}{\rho} - (2\lambda_j + 2)\rho \right\}\partial_\rho = \frac{1}{\rho g_j}\partial_\rho \rho (1 - \rho^2) g_j \partial_\rho,
\]
we obtain the following operator identity (for \( \rho < 1 \)):
\[
(3.6) \quad H_\nu + (\lambda_j + \Lambda_0)(\lambda_j + \Lambda) = -\frac{1}{\rho g_j}\partial_\rho \rho (1 - \rho^2) g_j \partial_\rho + \frac{V_\nu}{\rho^2} + \lambda_j(\lambda_j + 1).
\]
Although the function \( g_j \) is possibly singular at \( \rho = 1 \), the factor \( (1 - \rho^2) \) in the principal symbol is enough to compensate this singularity. Thus the above identity says that \( H_\nu + (\lambda_j + \Lambda_0)(\lambda_j + \Lambda) \) is formally self-adjoint with respect to the measure \( g_j \rho \). As a result, we have
\[
\langle (H_\nu + (\lambda_j + \Lambda_0)(\lambda_j + \Lambda))\epsilon, g_j \varphi_j \rangle \\
= \langle \epsilon, g_j [H_\nu + (\lambda_j + \Lambda_0)(\lambda_j + \Lambda)]\varphi_j \rangle = 0,
\]
where in the last equality we used (2.6). This completes the proof of (3.2).

**Step 2.** Proof of transversality (3.3).

The proof of (3.3) for the off-diagonal entries, i.e., \( \ell_j(\varphi_k) = 0 \) for \( j \neq k \), is an easy consequence of (3.2). Indeed, since \( \varphi_k \) is a smooth eigenfunction of \( M_\nu \) with the eigenvalue \( \lambda_k \), we have
\[
\lambda_k \ell_j(\varphi_k) = \ell_j(M_\nu \varphi_k) = \lambda_j \ell_j(\varphi_k),
\]
where in the second equality we used (3.2). Since \( \lambda_0 \neq \lambda_1 \), we have \( \ell_j(\varphi_k) = 0 \) for \( j \neq k \).

Next, we show (3.3) for the diagonal entries. Let \( j \in \{0, 1\} \). We first claim that
\[
(3.7) \quad \ell_j(\varphi_j) = \left\langle [2\lambda_j - 1 + 2\Lambda_0]^{1/2} \Lambda Q_\nu, g_j^{1/2} \Lambda Q_\nu \right\rangle + O\left( \frac{1}{|\log \nu|} \right).
\]
To show this, we note by (2.20) and (2.5) that
\[
\ell_j(\varphi_j) = \left\langle [2\lambda_j - 1 + 2\Lambda_0] \varphi_j, g_j \varphi_j \right\rangle.
\]
We rewrite this as
\[
\ell_j(\varphi_j) = \left\langle [2\lambda_j - 1 + 2\Lambda_0]^{1/2} \Lambda Q_\nu, g_j^{1/2} \Lambda Q_\nu \right\rangle \\
+ \left\langle [2\lambda_j - 1 + 2\Lambda_0]^{1/2} \Lambda Q_\nu, g_j \left( \varphi_j - \frac{1}{\nu} \Lambda Q_\nu \right) \right\rangle \\
+ \left\langle [2\lambda_j - 1 + 2\Lambda_0] \left( \varphi_j - \frac{1}{\nu} \Lambda Q_\nu \right), g_j \varphi_j \right\rangle.
\]
In the above display, the last two terms are errors; we use (2.12), \( |\frac{1}{2} \Lambda Q_\nu|_1 + |\varphi_j|_1 \lesssim \frac{1}{\nu} \), and \( g_j \lesssim \frac{1}{|\log \nu|} \cdot (1 - \rho)^{-1/2} \) to see that these terms are of size \( O\left( \frac{1}{|\log \nu|} \right) \). Therefore, (3.7) follows.
Using the formula (1.5) for $\Lambda Q$ and the improved decay (2.25) for $\Lambda_0\Lambda Q$, we have

$$
\left\langle \frac{1}{\nu} \Lambda Q_{\nu}, g_j \frac{1}{\nu} \Lambda Q_{\nu} \right\rangle = 4|\log\nu| + O(1),
$$

$$
\left\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_{\nu}, g_j \frac{1}{\nu} \Lambda Q_{\nu} \right\rangle = O(1).
$$

Substituting the above display and $|\lambda_j - (1 - j)| \lesssim \frac{1}{|\log\nu|}$ into (3.7) completes the proof of (3.3). □

Computations with $O(1)$ errors in (3.3) will not be sufficient in the later refined modulation estimates, and we will need the following more precise computations of relevant quantities.

Lemma 3.5 (Computations adapted to modulation equations). For each $j \in \{0, 1\}$, define

$$
b_j := \ell_j \left( \frac{1}{\nu} \Lambda Q_{\nu} \right),
$$

$$
c_j := \frac{1}{2} \left\{ \ell_j (\varphi_0 - \varphi_1) + \left\langle \frac{1}{\nu} \Lambda Q_{\nu}, g_j \varphi_j \right\rangle \right\}.
$$

Then, we have the following.

- (Expansions of $b_j$ and $c_j$) We have

$$
\begin{cases}
  b_0 = 4|\log\nu| + \left( -\frac{14}{3} + 4 \log 2 \right) + O\left( \frac{1}{|\log\nu|} \right), \\
  b_1 = -\frac{4}{3} + \frac{8}{|\log\nu|} + O\left( \frac{1}{|\log\nu|^2} \right),
\end{cases}
$$

and

$$
\begin{cases}
  c_0 = 4|\log\nu| + \left( -\frac{14}{3} + 4 \log 2 \right) + O\left( \frac{1}{|\log\nu|} \right), \\
  c_1 = 4|\log\nu| + \left( -\frac{2}{3} + 4 \log 2 \right) + O\left( \frac{1}{|\log\nu|} \right).
\end{cases}
$$

- (Bounds for $\nu \partial_\nu b_j$ and $\nu \partial_\nu c_j$) For each $j \in \{0, 1\}$, we have

$$
|\nu \partial_\nu b_j| + |\nu \partial_\nu c_j| \lesssim 1.
$$

- (One more computation) We have

$$
\ell_j (\nu \partial_\nu (\varphi_0 - \varphi_1)) - \left\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_{\nu}, g_j \varphi_j \right\rangle = -4 + O\left( \frac{1}{|\log\nu|} \right).
$$

Proof. **Step 1.** Computations of $\left\langle \frac{1}{\nu} \Lambda Q_{\nu}, g_j \frac{1}{\nu} \Lambda Q_{\nu} \right\rangle$ and $\left\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_{\nu}, g_j \frac{1}{\nu} \Lambda Q_{\nu} \right\rangle$. As seen in the proof of (3.3), the key quantities are $\left\langle \frac{1}{\nu} \Lambda Q_{\nu}, g_j \frac{1}{\nu} \Lambda Q_{\nu} \right\rangle$ and $\left\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_{\nu}, g_j \frac{1}{\nu} \Lambda Q_{\nu} \right\rangle$. In this step, we claim

$$
\left\langle \frac{1}{\nu} \Lambda Q_{\nu}, g_j \frac{1}{\nu} \Lambda Q_{\nu} \right\rangle = 4|\log\nu| - 2 + 2(\psi(1) - \psi(\lambda_j + \frac{1}{2})) + O(\nu^2 |\log\nu|),
$$

$$
\left\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_{\nu}, g_j \frac{1}{\nu} \Lambda Q_{\nu} \right\rangle = 2 + O(\nu^2 |\log\nu|).
$$

We first show (3.14). We write

$$
\left\langle \frac{1}{\nu} \Lambda Q_{\nu}, g_j \frac{1}{\nu} \Lambda Q_{\nu} \right\rangle = \left\langle \frac{1}{\nu} \Lambda Q_{\nu}, 1_{(0,1)} \frac{1}{\nu} \Lambda Q_{\nu} \right\rangle + \left\langle \frac{1}{\nu} \Lambda Q_{\nu}, (g_j - 1_{(0,1)}) \frac{1}{\nu} \Lambda Q_{\nu} \right\rangle
$$

$$
= (\Lambda Q, 1_{(0,1)} \Lambda Q) + \left\langle 2, (g_j - 1_{(0,1)}) \frac{2}{\rho} \right\rangle + O(\nu^2 |\log\nu|),
$$

11In Lemma 5.3 below, we will compute these quantities more precisely.
where in the last equality we used $|g_j - 1_{(0,1)}| \lesssim \rho^2 (1 - \rho^2) - \hat{\chi}$. We then compute
\[
\langle \Lambda Q, 1_{(0,\nu^{-1})} \Lambda Q \rangle = \int_0^1 \frac{4y^3}{(1+y^2)^2} dy = 2 \log \left( 1 + \frac{1}{\nu^2} \right) - \frac{2}{1 + \nu^2} = 4|\log \nu| - 2 + O(\nu^2)
\]
and
\[
\left\langle \frac{2}{\rho}, (g_j - 1_{(0,1)}) \frac{2}{\rho} \right\rangle = 4 \int_0^1 \left( (1 - \rho^2)^{\lambda_j - \frac{3}{2}} - 1 \right) \frac{d\rho}{\rho}
\]
\[
= 2 \int_0^1 \left( \frac{1}{z^{\lambda_j - \frac{3}{2}} - 1} \right) \frac{dz}{1 - z} = 2(\psi(1) - \psi(\lambda_j + \frac{1}{2}))
\]
where in the last equality we used the series expansion $\frac{1}{z} = 1 + z + z^2 + \cdots$ and (1.23). This completes the proof of (3.14).

For the proof of (3.15), we can use the improved spatial decay $|\Lambda_0 Q| \lesssim \langle y \rangle^{-3}$ to have
\[
\left\langle \frac{1}{\nu}, \Lambda_0 \Lambda Q_v, g_j \frac{1}{\nu} \Lambda Q_v \right\rangle = \left\langle \frac{1}{\nu}, \Lambda_0 \Lambda Q_v, 1_{(0,1)} \frac{1}{\nu} \Lambda Q_v \right\rangle + \left\langle \frac{1}{\nu}, \Lambda_0 \Lambda Q_v, (g_j - 1_{(0,1)}) \frac{1}{\nu} \Lambda Q_v \right\rangle
\]
\[
= \langle \Lambda_0 \Lambda Q, 1_{(0,\nu^{-1})} \Lambda Q \rangle + O(\nu^2|\log \nu|)
\]
\[
= 2 + O(\nu^2|\log \nu|).
\]
This completes the proof of (3.15).

**Step 2.** Expansions for $\langle \frac{1}{\nu} \Lambda Q_v, g_j \varphi_j \rangle$ and $\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_v, g_j \varphi_j \rangle$.

In this step, we claim
\[
\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_v, g_j \varphi_j \rangle = 4|\log \nu| + (4j - 6 + 4 \log 2) + O\left( \frac{1}{|\log \nu|} \right)
\]
\[
\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_v, g_j \varphi_j \rangle = 2 + O(\nu^2|\log \nu|).
\]
To show (3.16), we use (3.12) and $\frac{1}{\nu} \Lambda Q_v \lesssim \frac{1}{\rho}$ to have
\[
\langle \frac{1}{\nu} \Lambda Q_v, g_j \varphi_j \rangle = \langle \frac{1}{\nu} \Lambda Q_v, 1_{(0,1)} \frac{1}{\nu} \Lambda Q_v \rangle + O\left( \frac{1}{|\log \nu|} \right)
\]
Applying (3.14) and (1.27) to the above yields (3.16). To show (3.17), we use (3.12) and $1_{(0,\nu)} |\frac{1}{\nu} \Lambda_0 \Lambda Q_v| \lesssim 1_{(0,1)} \frac{1}{\nu} \Lambda Q_v + 1_{[\nu^{-1}, \nu]} \frac{1}{\nu} \Lambda Q_v$ to have
\[
\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_v, g_j \varphi_j \rangle = \langle \frac{1}{\nu} \Lambda_0 \Lambda Q_v, 1_{(0,1)} \frac{1}{\nu} \Lambda Q_v \rangle + O(\nu^2|\log |\nu|)|
\]
Applying (3.15) to the above yields (3.17).

**Step 3.** Proof of (3.10) and (3.11).

For (3.10), we note that
\[
b_j = \ell_j \left( \frac{1}{\nu} \Lambda Q_v \right) = \lambda_j \left( \frac{1}{\nu} \Lambda Q_v, g_j \varphi_j \right) + \left\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_v, g_j \varphi_j \right\rangle
\]
Substituting (2.3) and (3.10–3.11) into the above completes the proof of (3.10) (Notice that $|\lambda_1| \lesssim \frac{1}{|\log \nu|}$ is used to have the improved $O\left( \frac{1}{|\log \nu|^2} \right)$ error for $b_1$).

For (3.11), we recall that
\[
c_j = \frac{1}{2} \left\{ \ell_j (\varphi_0 - \varphi_1) + \left( \frac{1}{\nu} \Lambda Q_v, g_j \varphi_j \right) \right\}.
\]
Using \( \ell_j(\varphi_k) = 0 \) for \( j \neq k \), we have
\[
\epsilon_j = \frac{1}{2} \left\{ (-1)^j \ell_j(\varphi_j) + \left\langle \frac{1}{\nu} \Lambda Q_{\nu}, g_j \varphi_j \right\rangle \right\}
\]
\[
(3.19)
\]
We rewrite this as
\[
\epsilon_j = \frac{1}{2} \left\{ (-1)^j (2 \lambda_j - 1 + 2 \Lambda_0) \varphi_j + \frac{1}{\nu} \Lambda Q_{\nu}, g_j \varphi_j \right\}.
\]
By (3.21) and (2.11), the first line of RHS (3.22) is of size \( O(\frac{1}{|\log \nu|}) \), thanks to (2.12) and
\[
|g_j \varphi_j| \lesssim 1_{(0, \frac{1}{2})} \rho^{-1} + 1_{[\frac{1}{2}, 1]} \cdot (1 - \rho)^{-\frac{1}{2}}.
\]
Therefore, we have proved that
\[
\epsilon_0 = \lambda_0 \left\langle \frac{1}{\nu} \Lambda Q_{\nu}, g_0 \varphi_0 \right\rangle + \left\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_{\nu}, g_0 \varphi_0 \right\rangle + O\left( \frac{1}{|\log \nu|} \right),
\]
\[
\epsilon_1 = (1 - \lambda_1) \left\langle \frac{1}{\nu} \Lambda Q_{\nu}, g_1 \varphi_1 \right\rangle - \left\langle \frac{1}{\nu} \Lambda_0 \Lambda Q_{\nu}, g_1 \varphi_1 \right\rangle + O\left( \frac{1}{|\log \nu|} \right).
\]
Substituting (2.8) and (3.16)-(3.17) into the above completes the proof of (3.11).

**Step 4. Proof of (3.12).**
Let us first note that
\[
\nu \partial_{\nu} g_j = (\nu \partial_{\nu} \lambda_j) 1_{(0, 1]} \cdot (1 - \rho^2)^{\lambda_j - \frac{1}{2}} \log(1 - \rho^2).
\]
Using (2.8), we have the following estimate:
\[
\left| (\nu \partial_{\nu} \lambda_j) g_j \right| + |\nu \partial_{\nu} g_j| \lesssim \frac{1}{|\log \nu|^2} 1_{(0, 1]} \cdot (1 - \rho)^{-\frac{1}{2}}.
\]
We show (3.12) for \( \mathbf{b}_j \). Taking \( \nu \partial_{\nu} \) to (3.18), we have
\[
(3.22)
\]
By (3.21) and (2.11), the first line of RHS (3.22) is of size \( O(\frac{1}{|\log \nu|}) \). By (2.13), the second line of RHS (3.22) is of size \( O(1) \). This shows (3.12) for \( \mathbf{b}_j \).

We turn to show (3.12) for \( \epsilon_j \). Taking \( \nu \partial_{\nu} \) to (3.19), we have
\[
(3.23)
\]
By (3.21) and (2.11), the first two lines of RHS (3.23) are of size \( O(\frac{1}{|\log \nu|}) \). By (2.13), the last two lines of RHS (3.23) are of size \( O(1) \). This shows (3.12) for \( \epsilon_j \).

**Step 5. Proof of (3.13).**
To prove (3.13), note that
\[
LHS(3.13) = \left\langle (\lambda_j + \Lambda_0) \nu \partial_{\nu} (\varphi_0 - \varphi_1) + \nu \partial_{\nu} ((\lambda_0 + \Lambda) \varphi_0 - (\lambda_1 + \Lambda) \varphi_1) - \frac{1}{\nu} \Lambda_0 \Lambda Q_{\nu}, g_j \varphi_j \right\rangle.
\]
We rearrange this as

\[(3.24) \quad \text{LHS} (3.13) = \left\langle - (\lambda_0 - \lambda_1 + 1) \frac{1}{\nu} \lambda_0 \Lambda Q_\nu, g_j \varphi_j \right\rangle + \left\langle \lambda_j + \Lambda_0 + \Lambda \nu \partial_\nu (\varphi_0 - \varphi_1), g_j \varphi_j \right\rangle + \left\langle \lambda_0 \nu \partial_\nu (\varphi_0 - \frac{1}{\nu} \Lambda Q_\nu) - \lambda_1 \nu \partial_\nu (\varphi_1 - \frac{1}{\nu} \Lambda Q_\nu), g_j \varphi_j \right\rangle + \left\langle (\nu \partial_\nu \lambda_0) \varphi_0 - (\nu \partial_\nu \lambda_1) \varphi_1, g_j \varphi_j \right\rangle. \]

By (3.17), the first line of RHS (3.24) becomes

\[= - (\lambda_0 - \lambda_1 + 1) \left\{ 2 + O \left( \frac{1}{\log \nu} \right) \right\} = -4 + O \left( \frac{1}{\log \nu} \right). \]

The remaining terms are considered as errors. Indeed, using (3.20) and (2.14), the second and third lines of RHS (3.24) are of size \( O \left( \frac{1}{\log \nu}^2 \right) \). Using (2.8) and (2.11), the last line of RHS (3.24) is of size \( O \left( \frac{1}{\log \nu} \right) \). This completes the proof of (3.13).

The proof is complete. □

4. Blow-up analysis

In this section, we prove Theorem 1.2. We will perform modulation analysis upon the result of Raphaël–Rodnianski [53]. In particular, we directly work with a finite-time blow-up solution \( u \) as in Theorem 1.1, whose construction is established in [53]. The readers familiar with it may assume Corollary 4.3 below and jump to Section 4.2.

Raphaël–Rodnianski blow-up solutions can be roughly described as follows. One decomposes a solution \( u \) to (1.4) as

\[ u(t,r) = P_{RR}(b(t); \frac{r}{\lambda(t)}) + w_{RR}(t,r), \]

where \( v \) is the renormalized solution in the self-similar coordinates and \( P(\nu; b; \rho) \) is a new modified profile, which is carefully chosen to resemble the modified profiles in [53]. In Section 4.3, we test the evolution equation for \( \epsilon \) against \( \ell_j \) to obtain refined modulation equations for \( \nu \) and \( b \). These two subsections are the points where all the properties of \( \lambda_j, \varphi_j, \ell_j \) (Propositions 2.1 and 3.1) as well as the explicit computations (Lemma 3.5) gather. Finally, in Section 4.4, we integrate the resulting modulation equations to finish the proof of Theorem 1.2.

4.1. Raphaël–Rodnianski blow-up solutions. In this subsection, we recall refined properties of Raphaël–Rodnianski blow-up solutions constructed in [53]. The readers familiar with it may assume Corollary 4.3 below and jump to Section 4.2.

Raphaël–Rodnianski blow-up solutions can be roughly described as follows. One decomposes a solution \( u \) to (1.4) as

\[ u(t,r) = P_{RR}(b(t); \frac{r}{\lambda(t)}) + w_{RR}(t,r), \]

where \( b = -\lambda_t, P_{RR}(b; \cdot) \) is some modified profile satisfying \( P_{RR}(0; \cdot) = Q \), and \( w_{RR} \) is an error (or, radiative) part of \( u \). At the initial time \( t = 0 \), one assumes that \( w_{RR}(0) \) is sufficiently small and \( u_t(0) \approx \frac{b}{\lambda} \Lambda Q_\lambda \) so that \( u \) contracts forwards in time.

The modified profile \( P_{RR} \) almost mimics the profile of formal self-similar solutions to (1.4) (i.e., the solution to (1.4) with \( b(t) \equiv b \) constant), but it is different
at the self-similar scale $\frac{t}{b} \sim b^{-1}$; the profile $P^{RR}$ as well as the formal $b_t$-equation

\begin{equation}
\frac{b_t}{b} + \frac{b^2}{2|\log b|} \approx 0
\end{equation}

defined in (4.4) is nonnegative and repulsive [55]:

\begin{equation}
\tilde V_{\lambda} \geq 0, \quad \partial_r \tilde V_{\lambda} \leq 0, \quad \partial_{rr} \tilde V_{\lambda} \leq 0,
\end{equation}

where the last one is guaranteed in the blow-up regime $b = -\lambda_t > 0$. The conjugated linearized operator $\tilde H_{\lambda}$ naturally arises after conjugating $A_{\lambda}$ to the equation for $w^{RR}$. The repulsivity enables certain Morawetz-type monotonicity for $A_{\lambda}w^{RR}$ and $\partial_t(A_{\lambda}w^{RR})$ and hence the radiative part $w^{RR}$ can be controlled forwards in time; see [55] [53] for more details. After a careful bootstrap argument, one in particular justifies the formal dynamics $\lambda_t = -b$ and (4.6), which lead to the blow-up rate (1.1).

Let us simply write $A = A_1$, $\tilde H = \tilde H_1$, and so on. From now on, we recall the precise definition of the modified profile in [53]. For $0 < b \ll 1$, this modified profile is defined in the following form

\begin{equation}
P^{RR}(b; y) := (1 - \chi_{B_1}(y))\pi + \chi_{B_1}(y)(Q(y) + b^2T^{RR}(b; y)),
\end{equation}

where $B_1 := |\log b|/b$ is a localization radius and $T^{RR}(b; y)$ is some corrector to be defined in (4.7) below. Define

\begin{equation}
\tilde T^{RR}(b; y) := J_1(y) \int_1^y g(b; y')J_2(y'y')dy' - \int_0^y g(b; y')J_1(y'y')dy',
\end{equation}

where $J_1$ and $J_2$ are as in (2.19), and

\begin{equation}
g := -\Lambda_0\Lambda Q + c_0\Lambda Q \chi_{B_0/4},
\end{equation}

\begin{equation}
c_0 := \frac{\langle \Lambda_0\Lambda Q, \Lambda Q \rangle}{\langle \chi \Lambda Q, \Lambda Q \rangle} = \frac{1}{2|\log b|} + O\left(\frac{1}{|\log b|^2}\right),
\end{equation}

\begin{equation}
B_0 := 1/(b\sqrt{\int y\chi(y)dy}),
\end{equation}

so that $H\tilde T^{RR} = g$. We note that $\langle g, J_1 \rangle = 0$ so the $\int_0^y$-integral in (4.3) can be replaced by the $-\int_0^\infty$-integral if necessary. Next, we define $T^{RR} = T^{RR}(b; y)$ by

\begin{equation}
T^{RR} := \tilde T^{RR} - \frac{\langle \tilde T^{RR}, \chi M \Lambda Q \rangle}{\langle \Lambda Q, \chi M \Lambda Q \rangle} \Lambda Q,
\end{equation}
where $M > 1$ is some fixed large constant (whose role is explained in Remark 4.2 below) so that $T^{RR}$ additionally satisfies the orthogonality condition $(T^{RR}, \chi_M \Lambda Q) = 0$. Using our notation (2.20), we may express $T^{RR}$ in an alternative form:

$$T^{RR} = H^{-1} g - c_{M,b} \Lambda Q, \quad \text{where} \quad c_{M,b} := \frac{\langle H^{-1} g, \chi_M \Lambda Q \rangle}{\langle \Lambda Q, \chi_M \Lambda Q \rangle}.$$  

(4.8)

With this $T^{RR}$, the modified profile $P^{RR}$ is defined by (4.2). We are now ready to state refined properties of the Raphaël–Rodnianski blow-up solutions of Theorem 1.1.

Proposition 4.1 (Raphaël–Rodnianski stable blow-up [53]). There exists an open (initial data) set $\mathcal{O}$ in $\mathcal{H}_0^2$ (see (1.11)) with the following properties. For any $u_0 = (u_0, \dot{u}_0) \in \mathcal{O}$, its forward-in-time solution $u = (u, \dot{u})$ to (1.4) blows up in finite time $T = T(u) \in (0, +\infty)$ with the following descriptions:

- (Decomposition of the solution) There exist $0 < b^* \ll 1$ and $C^1$ functions $(\hat{\lambda}(t), \hat{b}(t)) : [0, T) \to (0, \infty) \times (0, b^*)$ such that the solution $u$ admits the decomposition

$$u(t, r) = P^{RR}(\hat{b}(t), \frac{r}{\hat{\lambda}(t)}) + w^{RR}(t, r)$$

with

$$\langle w^{RR}, (\chi_M \Lambda Q)\hat{\lambda} \rangle = 0,$$

$$\hat{\lambda} + \hat{b} = 0.$$  

(4.9) \hspace{1cm} (4.10) \hspace{1cm} (4.11)

- (Estimates for the modulation parameters $\hat{\lambda}$ and $\hat{b}$) We have

$$\hat{\lambda} |\hat{b}| \lesssim \frac{\hat{b}^2}{|\log \hat{b}|}.$$  

Moreover, as $t \to T^-$, we have

$$\frac{\hat{\lambda}(t)}{T - t} - \hat{b}(t) \lesssim \frac{\hat{b}(t)}{|\log \hat{b}(t)|},$$  

(4.13)

$$\hat{\lambda}(t) = (T - t) e^{-\sqrt{|\log(T - t)|} + O(1)}.$$  

(4.14)

- (Smallness of $w^{RR}$) Let $W^{RR}(t) := A(\hat{\lambda}(t)) w^{RR}(t)$. We have

$$\|w^{RR}\|_{H^1} + \|\partial_t w^{RR}\|_{L^2} = o_{b^* \to 0}(1),$$

(4.15)

$$\hat{\lambda}^2 \int \left( |\partial_t W^{RR}|^2 + |\partial_x W^{RR}|^2 + \frac{\hat{V}}{r^2} |W^{RR}|^2 \right) \lesssim \hat{b}^4,$$

(4.16)

$$\hat{\lambda}^2 \int 1_{(0, 4\lambda \hat{b}]} \left( |\partial_t W^{RR}|^2 + |\partial_x W^{RR}|^2 + \frac{\hat{V}}{r^2} |W^{RR}|^2 \right) \lesssim \frac{\hat{b}^4}{|\log \hat{b}|^2}.$$  

(4.17)

Remark 4.2. We are implicitly assuming that a large constant $M$ is fixed, and then there exists small $b^* = b^*(M) > 0$ satisfying the above statements. The large constant $M$ is fixed in order to have coercivity estimates (see (1.19) and (1.20) below) and to gain a smallness factor to close the bootstrap argument in [53].

Proof. All the statements are essentially proved in [53]. For the decomposition part (4.9)-(4.11), see Section 5.2 of [53]. For (4.12) and (4.13)-(4.17), see (5.34)-(5.36) and Lemma 6.1 of [53]. Next, (4.14) is the blow-up rate proved in [53]. Finally, the estimate (4.13) is not mentioned in [53], but it can be proved in a very similar manner; see Appendix A of this paper. \qed
For our blow-up analysis, (i) we rewrite the above proposition in the first-order formulation and in terms of the self-similar variables, and (ii) derive Hardy-type controls on the error part from the bounds \((4.16)\) and \((4.17)\).

For the former purpose, we also define the profile \(\hat{P}^{RR}(b;y)\) by
\[
P^{RR}(b;y) = (1 - \chi_{B_1}(y))\pi + \chi_{B_1}(y)(Q(y) + b^2T^{RR}(b;y)),
\]
and set \(P^{RR}\) as a vector \((\hat{P}^{RR}, \dot{\hat{P}}^{RR})\).

For the latter purpose, we recall the adapted function spaces \(\tilde{H}^2_t\) and \(\tilde{H}^1_t\) introduced in Section 1.2.

\[
\|f\|_{\tilde{H}^2_t}^2 := \|\partial_{yy}f\|_{L^2}^2 + \left\| \frac{1}{y(\log y)} |f| - 1 \right\|_{L^2}^2,
\]
\[
\|g\|_{\tilde{H}^1_t}^2 := \|\partial_y g\|_{L^2}^2 + \left\| \frac{1}{y} g \right\|_{L^2}^2.
\]

These function spaces were introduced to satisfy the following coercivity estimates (see [53, Appendix B] and also Appendix B of this paper):
\[
(4.19) \quad \|A^*A f\|_{L^2} \sim_M \|f\|_{\tilde{H}^2_t}, \quad \forall f \in \tilde{H}^2_t \cap \{\chi_M A Q\}^\perp,
\]
\[
(4.20) \quad \|A g\|_{L^2} \sim_M \|g\|_{\tilde{H}^1_t}, \quad \forall g \in \tilde{H}^1_t \cap \{\chi_M A Q\}^\perp,
\]
provided that \(M \gg 1\).

Our blow-up analysis will be done in the backward lightcone \(\rho \leq 1\). Thus we also define localized versions of the aforementioned norms
\[
\|f\|_{(\tilde{H}^2_t)_R}^2 := \|1_{(0,R]}\partial_{yy}f\|_{L^2}^2 + \left\| 1_{(0,R]} \frac{1}{y(\log y)} |f| - 1 \right\|_{L^2}^2,
\]
\[
\|g\|_{(\tilde{H}^1_t)_R}^2 := \|1_{(0,R]}\partial_y g\|_{L^2}^2 + \left\| 1_{(0,R]} \frac{1}{y} g \right\|_{L^2}^2,
\]
as well as the scaled version of the local \(\tilde{H}^2_t\)-norm (restricted to the region \(\rho \leq 1\))
\[
\|f\|_{(\tilde{H}^2_t)_1}^2 := \|1_{(0,1]}\partial_{yy}f\|_{L^2}^2 + \left\| 1_{(0,1]} \frac{1}{\rho(\log(\frac{\rho}{y}))} |f| - 1 \right\|_{L^2}^2.
\]

With these localized norms we still have analogous coercivity estimates for \(R \gg M\): (see Appendix B for the proof)
\[
(4.21) \quad \|1_{(0,R]} A^* A f\|_{L^2} \sim_M \|f\|_{(\tilde{H}^2_t)_R}, \quad \forall f \in \tilde{H}^2_t \cap \{\chi_M A Q\}^\perp,
\]
\[
(4.22) \quad \|1_{(0,R]} A g\|_{L^2} \sim_M \|g\|_{(\tilde{H}^1_t)_R}, \quad \forall g \in \tilde{H}^1_t \cap \{\chi_M A Q\}^\perp,
\]
where the implicit constants can be chosen uniformly in \(R\). The above coercivity estimates allow us to transfer the controls \((4.15)-(4.17)\) to the \(\tilde{H}^2\)-norm of the remainder. As a result, we have the following corollary.

**Corollary 4.3** (Raphaël–Rodnianski blow-up solutions in self-similar variables). Let \(u = (u, \hat{u})\) be a finite-time blow-up solution as in Proposition \((4.1)\) and \(T \in (0, \infty)\) be its blow-up time. Let \((\tau, \rho)\) be the self-similar coordinates \((4.13)\) with this \(T\). Define \(v = (v, \hat{v})\) by
\[
u(t,r) = v(\tau, \rho), \quad \dot{u}(t,r) = \frac{1}{T-t} \dot{v}(\tau, \rho),
\]
and the modulation parameters \(\tilde{v}\) and \(\tilde{b}\) by
\[
\tilde{v}(\tau) := \frac{\dot{v}(t)}{T-t} \quad \text{and} \quad \tilde{b}(\tau) := \tilde{b}(t).
\]

\(12\)Abuse of notation for \(\tilde{b}\)
Then, the following hold for all large $\tau$:

- (Decomposition) $v$ admits the decomposition
  \[ v(\tau, \rho) = [P^{RR}(\hat{b}(\tau); \cdot) + \epsilon^{RR}(\tau, \cdot)] \hat{v}(\tau)(\rho). \]

- (Estimates for the modulation parameters $\hat{v}$ and $\hat{b}$) As $\tau \to \infty$, we have
  \[
  \frac{\hat{v}}{\nu} + \left| \frac{\hat{b}}{\nu} \right| + \left| \frac{\hat{b}}{\nu} - 1 \right| \lesssim \frac{1}{\log \nu},
  \]
  \[
  \hat{v}(\tau) \sim \hat{b}(\tau) \sim e^{-\sqrt{\tau}}.
  \]

- (Smallness of $\epsilon^{RR}$) We have
  \[
  \|\epsilon^{RR}\|_{H_1^2} + \|\epsilon^{RR}\|_{L^2} = o_{b^* \to 0}(1),
  \]
  \[
  \|\epsilon^{RR}\|_{H_1^2} + \|\epsilon^{RR}\|_{H_1^2} \lesssim \hat{b}^2,
  \]
  \[
  \|\epsilon^{RR}\|_{(H_1^2)_{2/\nu}} + \|\epsilon^{RR}\|_{(H_1^2)_{2/\nu}} \lesssim \frac{\hat{b}^2}{\log \hat{b}}.
  \]

The proof is given in Appendix C.

### 4.2 Decomposition of solutions

From now on, we pick a solution $v$ as in Corollary $4.3$ and work with that solution. Moreover, by considering only large $\tau$, we may assume $\hat{b}(\tau) \ll \min\{b^*, \nu^*\}$ as well.

The goal of this subsection (Proposition $4.4$) is to write $v$ in a slightly different way:

\[
\nu(\tau, \rho) = P(\nu(\tau), b(\tau) \cdot \rho) + \epsilon(\tau, \rho),
\]

where $P = P(\nu, b; \rho)$ is a new modified profile that satisfies $P(\nu, 0; \cdot) = Q_\nu$ and is described in terms of $Q_\nu$ and $\varphi_j$. Using such a profile will enable more refined modulation estimates for $\nu$ and $b$ in view of the computations in Lemma $4.3$. At the same time, we also want to use a priori information on $v$ in Corollary $4.4$. Thus our modified profile $P$ will be chosen sufficiently close to $P^{RR}$ (Lemma $4.5$) so that the bounds of $\epsilon^{RR}$ can be transferred to those of $\epsilon$ of our new decomposition. Our $\epsilon$ does not satisfy any orthogonality conditions, but its smallness (c.f. (4.27)) will be sufficient.

From the above discussion, we define the modified profile $P = P(\nu, b; \rho)$ by

\[
P = Q_\nu + \frac{b}{2}(\varphi_0 - \varphi_1 + \left[ \frac{\lambda}{\nu_0} A Q_\nu \right]).
\]

Denoting $\hat{P} = (P, \hat{P})$, this is equivalent to saying that

\[
P := Q_\nu + \frac{b}{2}(\varphi_0 - \varphi_1),
\]

\[
\hat{P} := \frac{b}{2}\left((\lambda_0 + \Lambda)\varphi_0 - (\lambda_1 + \Lambda)\varphi_1 + \frac{1}{\nu} A Q_\nu \right).
\]

The main proposition of this subsection is the following.

**Proposition 4.4** (Decomposition). There exist $C^1$ functions $\nu(\tau)$ and $b(\tau)$ defined for all large $\tau$ such that

\[
v(\tau, \rho) = P(\nu(\tau), b(\tau) \cdot \rho) + \epsilon(\tau, \rho)
\]

with the following properties:
Then, the following hold.

(4.30) \[ \left| \frac{\nu_r}{\nu} \right| + \left| \frac{b_r}{b} \right| + \left| \frac{b}{\nu} - 1 \right| \lesssim \frac{1}{\log \nu}, \]
(4.31) \[ \nu(\tau) \sim b(\tau) \sim e^{-\sqrt{\tau}}. \]

• (Estimates for the modulation parameters) We have

(4.32) \[ \|\epsilon\|_{H^2} + \|\epsilon\|_{H^1} \lesssim \frac{\nu}{\log \nu}, \]
(4.33) \[ 1_{(0,1)} \left( \frac{1}{\log(\nu)} - 1 \right) \lesssim \frac{\nu}{\log \nu}, \]
and

(4.34) \[ |\epsilon_j(\nu)| \lesssim \frac{\nu}{|\log \nu|^2}. \]
(4.35) \[ ||\nu \partial_\nu \epsilon_j(\nu)| \lesssim \frac{\nu}{|\log \nu|^2}. \]

First, let us show that the profiles \( P^{RR} = (P^{RR}, \dot{P}^{RR}) \) and \( P \) are close enough in the self-similar region \( \rho \leq 1 \), after changing the parameters \( \nu \) and \( b \) slightly.

Lemma 4.5 (Difference estimate for profiles). For \( \hat{b} \) and \( \hat{\nu} \) satisfying

\[ 0 < \hat{b}, \hat{\nu} \ll \min\{b^*, \nu^*\} \quad \text{and} \quad \left| \frac{b}{\nu} - 1 \right| \lesssim \frac{1}{|\log \nu|}, \]
define \( \nu \) and \( b \) by the relations (see (4.3) for the definition of \( \epsilon_{M,b} \))

(4.36) \[ \log \left( \frac{\nu}{\hat{\nu}} \right) = -c_{M,b} \hat{b}^2 \quad \text{and} \quad \hat{b} = b \cdot \frac{1}{2} (\lambda_0(\nu) - \lambda_1(\nu) + 1). \]

Then, the following hold.

• (Parameter changes are small) We have

(4.37) \[ \left| \frac{\nu}{\hat{\nu}} - 1 \right| + \left| \frac{b}{\nu} - 1 \right| \lesssim \frac{\nu}{|\log \nu|}, \]
(4.38) \[ \left| \frac{\nu}{\hat{\nu}} - 1 \right| \lesssim \left| c_{M,b} \right| b^2 \lesssim \nu^2. \]

• (Derivative estimates) We have

(4.39) \[ \left( \partial_{b} \nu \quad \partial_{b} \nu \right) = \left( \frac{1}{\nu} \quad 0 \right) + O\left( \frac{1}{|\log \nu|} \right). \]

• (Main difference estimate) We have

(4.40) \[ \left\| P^{RR} (\hat{b}; \frac{\rho}{\hat{\nu}}) - P(\nu, b; \rho) \right\|_{H^2} + \left\| \frac{1}{\nu} P^{RR} (\hat{b}; \frac{\rho}{\hat{\nu}}) - \dot{P}(\nu, b; \rho) \right\|_{H^1} \lesssim \frac{\nu}{|\log \nu|}. \]

Proof. Step 1. Bounds of \( H^{-1} g \) and \( \hat{b} \partial_b (H^{-1} g) \) and structure of \( H^{-1} g \).

Recall the definitions (4.4)-(4.9), where \( b \) is replaced by \( \hat{b} \). In other words, we will use \( g = g(\hat{b}; y), c_0 \), and \( B_0 = B_0(\hat{b}) \). In this step, we claim the following pointwise bounds on \( H^{-1} g \) and \( \hat{b} \partial_b (H^{-1} g) \):

(4.41) \[ \|H^{-1} g\|_2 \lesssim 1_{(0,1)} g^3 + 1_{[1, +\infty)} \frac{y (\log (\hat{b} y))}{|\log b|}, \]
(4.42) \[ \|\hat{b} \partial_b (H^{-1} g)\|_2 \lesssim 1_{(0,1)} \frac{y^3}{|\log b|^2} + 1_{[1, B_0]} \frac{1}{|\log b|} + 1_{[B_0 / 4, +\infty)} \frac{1}{b^2 |\log b| y}. \]
and the structure of $H^{-1}g$:

\[ |H^{-1}g - (S_1 - c_bU_1)|_2 \lesssim \begin{cases} 1_{[B_0/4, +\infty]} & \frac{y(\log(by))}{|\log b|}, \\ 1_{[1, B_0/4]} & \frac{1}{|\log b|^2}, \\ 1_{[1, 1/2]} & \frac{1}{|\log b|}. \end{cases} \]

Let us first record the following pointwise bounds on $g$ and $\hat{b}\partial_b g$, which are immediate consequences of (4.41), (4.5), and $\hat{b}\partial_b c_b \lesssim \frac{1}{|\log b|^2}$:

\[ |g| \lesssim 1_{[0, 1]} y + 1_{[1, +\infty]} \frac{y}{|\log b|^2} + 1_{[1, B_0/4]} \frac{1}{|\log b|}, \]

\[ |\hat{b}\partial_b g| \lesssim 1_{[0, 1]} \frac{y}{|\log b|^2} + 1_{[1, B_0/4]} \frac{1}{|\log b|^2} + 1_{[B_0/4, B_0/2]} \frac{1}{|\log b|}. \]

Now we show (4.41) and (4.42). Note that $\hat{b}\partial_b (H^{-1}g) = H^{-1}(\hat{b}\partial_b g)$. In the region $y \leq 1$, taking $H^{-1}$ is essentially the multiplication by $y^2$ and hence (4.41) and (4.42) follow. Henceforth, let us focus on the region $y \geq 1$. Since $g$ and $\hat{b}\partial_b g$ satisfy the orthogonality conditions $\langle g, J_1 \rangle = \langle \hat{b}\partial_b g, J_1 \rangle = 0$ (recall that $J_1 = \Lambda Q$), we have

\[ H^{-1}F = J_1 \int_0^y FJ_2ydy' + J_2 \int_y^{\infty} FJ_1ydy', \quad \text{for } F \in \{g, \hat{b}\partial_b g\}. \]

The first integral is estimated by (for $y \geq 1$)

\[ \left| J_1 \int_0^y gJ_2ydy' \right|_2 \lesssim \frac{\langle \log y \rangle}{y} + \frac{y}{|\log b|}, \]

\[ \left| J_1 \int_0^y (\hat{b}\partial_b g)J_2ydy' \right|_2 \lesssim \frac{y}{|\log b|^2} + 1_{[1, B_0/4]} \frac{1}{|\log b|^2}. \]

The second integral is estimated by (for $y \geq 1$)

\[ \left| J_2 \int_y^{\infty} gJ_1ydy' \right|_2 \lesssim \frac{1}{y} + 1_{[1, B_0/2]} \frac{y(\log(by))}{|\log b|}, \]

\[ \left| J_2 \int_y^{\infty} (\hat{b}\partial_b g)J_1ydy' \right|_2 \lesssim 1_{[1, B_0/2]} \frac{y}{|\log b|}. \]

Summing up the above estimates yields (4.41) and (4.42) for $y \geq 1$. The proof of (4.41) and (4.42) is complete.

We turn to show (4.43). By the definition (2.21), we have

\[ H^{-1}g - (S_1 - c_bU_1) = H^{-1}(g + (\Lambda_0\Lambda Q - c_b\Lambda Q)) = -c_b H^{-1}(\chi_{\geq B_0/4}Q). \]

Applying (4.45) and (2.27), we have

\[ |H^{-1}g - (S_1 - c_bU_1)|_2 \lesssim 1_{[B_0/4, +\infty]} \frac{y(\log(by))}{|\log b|}, \]

completing the proof of (4.43).

**Step 2.** Estimates for $c_M\hat{b}$ and $\hat{b}\partial_b c_M\hat{b}$.

Recall the definition (4.3) of $c_M\hat{b}$. By (4.41) and (4.42), we in particular have

\[ 1_{[0, 2M]} |H^{-1}g| \lesssim 1_{[0, 1]} y^3 + 1_{[1, 2M]} y \]

\[ 1_{[0, 2M]} |\hat{b}\partial_b (H^{-1}g)| \lesssim 1_{[0, 1]} \frac{y^3}{|\log b|^2} + 1_{[1, 2M]} \frac{y}{|\log b|}. \]
Therefore, we have the following estimates for $c_{M,\tilde{b}}$ and $\tilde{b}\partial_\nu c_{M,\tilde{b}}$.

\begin{align}
|c_{M,\tilde{b}}| &\lesssim \frac{M^2}{\log M} = O_M(1), \\
|\tilde{b}\partial_\nu c_{M,\tilde{b}}| &\lesssim \frac{1}{|\log \tilde{b}|} \cdot \frac{M^2}{\log M} = O_M\left(\frac{1}{|\log \tilde{b}|}\right).
\end{align}

**Step 3.** Proof of (4.37)-(4.39).

We note that (4.37) and (4.38) are immediate from (4.44) and $\lambda_0 - \lambda_1 + 1 = 2 + O\left(\frac{1}{\log \nu}\right)$. Henceforth, we show (4.39). First, we differentiate the first equation of (4.36), and use (4.43) in the form
\[
\partial_\nu = \frac{\nu}{\nu - 1} = 1 + \left(\frac{1}{\log \nu}\right),
\]
\[
\partial_\nu = -\nu((\partial_\nu c_{M,\tilde{b}})\tilde{b}^2 + 2c_{M,\tilde{b}}) = O_M(\nu^2) = O(\nu^2 - \nu) .
\]

Next, we differentiate the second equation of (4.36), and use the above display, (4.37), and (2.8) to have
\[
\partial_\nu = \frac{1}{\nu^2} + \frac{1}{\log \nu} = O_M\left(\frac{1}{\log \nu}\right) .
\]

This completes the proof of (4.39).

**Step 4.** Difference of $P^{RR}$ and $P$.

In this step, we show (4.40) for $P$. We are only interested in the estimates inside the light cone, i.e., $\rho \leq 1$. Thus the cutoff $\chi_{B_1}$ in the definition of $P^{RR}$ does not play any role. For $\rho \leq 1$, we note by the definitions (4.2), (4.8), and (4.28) that

\begin{align}
P^{RR}(\tilde{b}, \frac{\rho}{\nu}) - P(\nu, b; \rho) \\
= \left(Q_\nu + \tilde{b}^2 (H^{-1}g - c_{M,\tilde{b}}\Lambda Q)\right) - \left(Q_\nu + \frac{b}{\nu} [\varphi_0 - \varphi_1][\nu; \rho]\right).
\end{align}

For the first term of RHS (4.46), we use (4.44) in the form
\[
\|H^{-1}g - (S_1 - c_\nu U_1)\|_{H_{\nu}^2 L_2} \lesssim \left\|1_{\{b_{\nu}/4, 1/\nu\}} \frac{1}{|\log b_{\nu}|} \right\|_{L^2} \lesssim \frac{1}{|\log \nu|}
\]
to have
\begin{align}
\tilde{b}^2 [H^{-1}g]_{\nu} = \tilde{b}^2 \left(S_1 - c_\nu U_1\right)_{\nu} + O_M\left(\frac{1}{|\log \nu|}\right).
\end{align}

For the second term of RHS (4.46), we use (2.117) to have
\begin{align}
\frac{1}{2} [\varphi_0 - \varphi_1] = \nu [aS_1 + uU_1]_{\nu} + O_M\left(\frac{1}{|\log \nu|}\right),
\end{align}

where
\[
s(\nu) := \lambda_0 - \lambda_1 = 1 + O\left(\frac{1}{|\log \nu|}\right),
\]
\[
u u(\nu) := \frac{1}{2}(\lambda_0(\lambda_0 - 1) - \lambda_1(\lambda_1 - 1)) = -\frac{1}{2|\log \nu|} + O\left(\frac{1}{|\log \nu|^{2}}\right).
\]
Substituting (4.47) and (4.48) into (4.46), we have

\[
P^{RR}(\hat{b}, \frac{\nu}{\bar{\nu}}) - P(\nu, b; \rho) = (Q_{\bar{\nu}} - Q_{\nu} - c_{M, \bar{b}} \bar{b}^2 \Lambda Q_{\bar{\nu}}) + (\hat{b}^2[S_1]_{\bar{\nu}} - \hat{b}^2[S_1]_{\nu})
\]

\[
- (\hat{b}^2 c_b[U_1]_{\bar{\nu}} + \hat{b} \nu U_1)_{\nu}) + O_{(H_1^\infty)} \left( \frac{\nu}{\log \nu} \right).
\]

Therefore, it suffices to show that each term of RHS (4.49) is of order \(O_{(H_1^\infty)} \left( \frac{\nu}{\log \nu} \right)\).

For the first term of RHS (4.49), we note that

\[
Q_{\bar{\nu}} - Q_{\nu} - c_{M, \bar{b}} \bar{b}^2 \Lambda Q_{\bar{\nu}} = (Q - Q_{\nu/\bar{\nu}} - c_{M, \bar{b}} \bar{b}^2 \Lambda Q_{\bar{\nu}}).
\]

Thanks to the relation (4.36), we can write

\[
Q - Q_{\nu/\bar{\nu}} - c_{M, \bar{b}} \bar{b}^2 \Lambda Q = \int_{\nu/\bar{\nu}}^{1} (\Lambda Q_a - \Lambda Q) \frac{da}{a} = \int_{\nu/\bar{\nu}}^{1} \int_{a}^{1} \Lambda Q_{\nu} \frac{da}{a} - \frac{da}{a}.
\]

Hence by (4.38) we have

\[
\left\| (Q - Q_{\nu/\bar{\nu}} - c_{M, \bar{b}} \bar{b}^2 \Lambda Q)_{\bar{\nu}} \right\|_{(H_1^\infty)} \lesssim \frac{1}{\nu} \log \left( \frac{\nu}{\bar{\nu}} \right) \lesssim \nu^{3/2}.
\]

For the second term of RHS (4.49), we write

\[
\hat{b}^2[S_1]_{\bar{\nu}} - \hat{b}^2[S_1]_{\nu} = \hat{b}^2 - (\hat{b} - \hat{b})[\hat{b} S_1]_{\bar{\nu}} + \hat{b}([\hat{b} \nu S_1]_{\bar{\nu}} - [\nu S_1]_{\nu}).
\]

We estimate each term using (4.37) - (4.38) and (2.24) (in particular improved spatial decay of \(\partial_y S_1\) and \(\Lambda_2 S_1\)):

\[
\left\| \hat{b}^2[S_1]_{\bar{\nu}} - \hat{b}^2[S_1]_{\nu} \right\|_{(H_1^\infty)} \lesssim \frac{\nu}{\log \nu},
\]

\[
\left\| \hat{b}([\hat{b} \nu S_1]_{\bar{\nu}} - [\nu S_1]_{\nu}) \right\|_{(H_1^\infty)} \lesssim \left\| \hat{b} \right\| \left\| \int_{1}^{\bar{\nu}} \nu \Lambda_2 S_1 \frac{da}{\nu} \right\|_{(H_1^\infty)} \lesssim \nu \log \left( \frac{\nu}{\bar{\nu}} \right) \lesssim \nu^{3/2}.
\]

Thus we have proved that

\[
\left\| \hat{b}^2[S_1]_{\bar{\nu}} - \hat{b}^2[S_1]_{\nu} \right\|_{(H_1^\infty)} \lesssim \frac{\nu}{\log \nu}.
\]

Finally, for the third term of RHS (4.49), we write

\[
\hat{b}^2 c_b[U_1]_{\bar{\nu}} + \hat{b} \nu U_1_{\nu} = \left( \frac{\hat{b}^2}{\bar{\nu}} c_b + \nu \right) [\hat{b} U_1]_{\nu} + \hat{b}([\hat{b} U_1]_{\bar{\nu}} - [\nu U_1]_{\nu}).
\]

As in the proof of (4.31), we use (4.37) - (4.38) and the logarithmically improved spatial decay for \(y \geq 1\)

\[
\left\| \partial_y U_1 \right\| + \frac{1}{y^2} \left\| \Lambda_2 U_1 \right\| \lesssim \frac{1}{y}
\]

to have (note that \(\left\| 1_{[1, \nu - \nu \frac{1}{\nu}] \frac{1}{\nu} \right\|_{L^2} \sim \log \nu \frac{\nu}{\bar{\nu}}\))

\[
\left\| \left( \frac{\hat{b}^2}{\bar{\nu}} c_b + \nu \right) [\hat{b} U_1]_{\nu} \right\|_{(H_1^\infty)} \lesssim \frac{\nu}{\log \nu} \cdot \left\| \log \nu \right\|_{\bar{\nu}} \lesssim \frac{\nu}{\log \nu} \cdot \frac{\nu}{\log \nu} \lesssim \nu^{3/2},
\]

\[
\left\| \hat{b}([\hat{b} U_1]_{\bar{\nu}} - [\nu U_1]_{\nu}) \right\|_{(H_1^\infty)} \lesssim \frac{\nu}{\log \nu} \cdot \left\| \log \nu \frac{\nu}{\bar{\nu}} \right\| \lesssim \nu^{3/2}.
\]
Thus we have proved that
\begin{equation}
\left\| \partial^2 \tilde{v} [U_1]_{\tilde{\nu}} + ub[U_1]_{\tilde{\nu}} \right\|_{(H^1)_{\tilde{\nu}}} \lesssim \frac{\nu}{| \log \nu |^2}.
\end{equation}
Substituting (4.50)-(4.52) into (4.49) completes the proof of (4.40) for $P$.

\textbf{Step 5. Difference of $\dot{\rho}^{RR}$ and $\dot{P}$}.

In this step, we show (4.40) for $\dot{P}$. As in the previous step, we will only consider the region $\rho \leq 1$ and the cutoff $\chi_{B_1}$ in the definition of $\dot{P}$ plays no role. For $\rho \leq 1$, we note by the definitions (4.18) and (4.28) that
\begin{equation}
\frac{1}{\nu} \dot{\rho}^{RR}(\tilde{\nu}, b; \rho) - \dot{P}(\nu, b; \rho) = \frac{\tilde{b}}{\nu} \left( \lambda Q_{\tilde{\nu}} + \tilde{\nu} \lambda (H^{-1}g - c_{M, \tilde{\nu}} Q)_{\tilde{\nu}} \right)
\end{equation}
where we abbreviated $\lambda_j = \lambda_j(\nu)$. By (4.41) and (4.44), we have
\begin{align*}
\left\| \frac{\tilde{b}}{\nu} \lambda (H^{-1}g)_{\tilde{\nu}} \right\|_{(H^1)_{\tilde{\nu}}} &\lesssim \frac{\nu}{| \log \nu |}, \\
\left\| \frac{\tilde{b}}{\nu} \lambda Q_{\tilde{\nu}} \right\|_{(H^1)_{\tilde{\nu}}} &\lesssim \nu^{-1}.
\end{align*}
By (2.12), we have
\begin{equation}
\left\| \varphi_j - \frac{1}{\nu} \lambda Q_{\tilde{\nu}} \right\|_{(H^1)_{\tilde{\nu}}} \lesssim \frac{1}{| \log \nu |}.
\end{equation}
Substituting the above two displays into (4.53) yields
\begin{equation}
\frac{1}{\nu} \dot{\rho}^{RR}(\tilde{\nu}, b; \rho) - \dot{P}(\nu, b; \rho) = \left( \frac{\tilde{b}}{\nu} \lambda Q_{\tilde{\nu}} - \frac{b}{2} (\lambda_0 - \lambda_1 + 1) \frac{1}{\nu} \lambda Q_{\nu} \right) + O_{(H^1)_{\tilde{\nu}}} \left( \frac{\nu}{| \log \nu |} \right).
\end{equation}
Next, we write
\begin{align*}
\frac{\tilde{b}}{\nu} \lambda Q_{\tilde{\nu}} &- \frac{b}{2} (\lambda_0 - \lambda_1 + 1) \frac{1}{\nu} \lambda Q_{\nu} \\
&= \frac{\tilde{b}}{\nu} \left( \lambda Q_{\tilde{\nu}} - \frac{1}{\nu} \lambda Q_{\nu} \right) + \frac{\tilde{b}}{\nu} \left( \frac{b}{2} (\lambda_0 - \lambda_1 + 1) \frac{1}{\nu} \lambda Q_{\nu} \right).
\end{align*}
Notice that the second term vanishes thanks to the relation (3.36). The first term is small due to (4.38):
\begin{equation}
\left\| \frac{\tilde{b}}{\nu} \left( \lambda Q_{\tilde{\nu}} - \frac{1}{\nu} \lambda Q_{\nu} \right) \right\|_{(H^1)_{\tilde{\nu}}} \lesssim \frac{\tilde{b}}{\nu} \int \left| \left( \frac{\nu}{\tilde{\nu}} \right)^{\nu} \right| \left\| \lambda_0 Q_{\tilde{\nu}} \right\|_{(H^1)_{\tilde{\nu}}} \frac{da}{\nu} \lesssim \left| \log \left( \frac{\tilde{\nu}}{\nu} \right) \right| \lesssim \nu^{-2}.
\end{equation}
Substituting this into (4.54) completes the proof of (4.40) for $\dot{P}$. \hfill \square

To show the $\ell_j(\epsilon)$ bounds (4.34)-(4.35) in Proposition 4.3 we also need the following mapping properties of $\ell_j$

\textbf{Lemma 4.6 (Mapping properties of $\ell_j$).} We have
\begin{align}
\| \ell_j(f) \| &\lesssim | \log \nu |^{\frac{1}{2}} \| f \|_{(H^1)_{\tilde{\nu}}} + \| \tilde{f} \|_{(H^1)_{\tilde{\nu}}}, \\
\| \nu \partial_{\nu} \ell_j(f) \| &\lesssim \frac{1}{| \log \nu |^2} \| \log \nu |^{\frac{1}{2}} \| f \|_{(H^1)_{\tilde{\nu}}} + \| \tilde{f} \|_{(H^1)_{\tilde{\nu}}},
\end{align}
Proof. In the proof, we use the weighted $L^\infty$-bounds in Lemma 4.53, which by scaling reads
\begin{align}
(4.57) \quad & 1_{(0,1]}|f|_1 \lesssim \|f\|_{(\mathcal{H}_T^2)^*} \cdot \rho \log(\frac{R}{\nu})^{\frac{1}{2}}, \\
(4.58) \quad & 1_{(0,1]}|\dot{f}| \lesssim \|\dot{f}\|_{(\mathcal{H}_T^1)^*}.
\end{align}
One important point here is that we only have power $\frac{1}{2}$ in the log factor of RHS (4.57).
(Compare this with the factor $|\log y|^{-1}$ in the $\mathcal{H}_T^2$-norm.)
We first show (4.55). Recall that
\begin{align}
(4.20) \quad & \ell_j(f) = \langle (\lambda_j + \Lambda) f + \dot{f}, g_j \varphi_j \rangle,  \\
(4.21) \quad & |g_j \varphi_j| \lesssim 1_{(0, \frac{1}{2}]\rho^{-1}} + 1_{[\frac{1}{2}, 1]} (1 - \rho)^{-\frac{1}{2}}.
\end{align}
Thus we have (by (4.57))
\begin{align*}
\langle (\lambda_j + \Lambda) f, g_j \varphi_j \rangle & \lesssim \|f\|_{1} \cdot \|g_j \varphi_j\|_{L^\infty} \\
& \lesssim \left( \int_0^\frac{1}{2} \rho \log(\frac{R}{\nu})^{\frac{1}{2}} d\rho + \int_\frac{1}{2}^1 |\log \nu|^{\frac{1}{2}} (1 - \rho)^{-\frac{1}{2}} d\rho \right) \|f\|_{(\mathcal{H}_T^2)^*} \\
& \lesssim \|\dot{f}\|_{(\mathcal{H}_T^2)^*}.
\end{align*}
and (by (4.58))
\begin{align*}
\langle f, g_j \varphi_j \rangle & \lesssim \left( \int_0^\frac{1}{2} d\rho + \int_\frac{1}{2}^1 (1 - \rho)^{-\frac{3}{2}} d\rho \right) \|f\|_{(\mathcal{H}_T^1)^*} \lesssim \|\dot{f}\|_{(\mathcal{H}_T^1)^*}.
\end{align*}
This completes the proof of (4.55).
We turn to show (4.56). Note that
\begin{align}
(4.59) \quad & [\nu \partial_\nu \ell_j](f) = (\nu \partial_\nu \lambda_j) \langle f, g_j \varphi_j \rangle + \langle (\lambda_j + \Lambda_0) f + \dot{f}, (\nu \partial_\nu g_j) \varphi_j \rangle \\
& \quad + \langle (\lambda_j + \Lambda_0) f + \dot{f}, g_j (\nu \partial_\nu \varphi_j) \rangle.
\end{align}
We also recall that
\begin{align}
(4.51) \quad & |(\nu \partial_\nu \lambda_j) g_j| + |\nu \partial_\nu g_j| \lesssim \frac{1}{|\log \nu|^2} 1_{(0, 1]} \cdot (1 - \rho)^{-\frac{1}{2}},
\end{align}
which is just $\frac{1}{|\log \nu|^2}$ times (4.20). Thus the first line of RHS (4.59) is estimated by
\[
\frac{1}{|\log \nu|^2} \langle (\lambda_j + \Lambda_0) f, g_j \varphi_j \rangle + \|\dot{f}\|_{(\mathcal{H}_T^1)^*}
\]
by the proof of (4.55). Using (2.13), the second line of RHS (4.59) can be estimated by
\begin{align*}
\langle ((\lambda_j + \Lambda_0) f, g_j (\nu \partial_\nu \varphi_j) \rangle & \lesssim \left\{ \int_0^\nu \frac{\rho^3 (\log(\frac{R}{\nu}))^{\frac{1}{2}}}{\nu^2} d\rho + \int_\nu^\frac{1}{2} \frac{\rho^2 (\log(\frac{R}{\nu}))^{\frac{1}{2}}}{\log \nu} (\log(\frac{R}{\nu}))^{\frac{1}{2}} d\rho \\
& \quad + \int_\frac{1}{2}^1 (1 - \rho)^{-\frac{3}{2}} d\rho \right\} \|f\|_{(\mathcal{H}_T^1)^*} \lesssim \frac{1}{|\log \nu|^2} \|f\|_{(\mathcal{H}_T^1)^*}
\end{align*}
and
\begin{align*}
\langle \dot{f}, g_j (\nu \partial_\nu \varphi_j) \rangle & \lesssim \left\{ \int_0^\nu \frac{\rho^2}{\nu^2} d\rho + \int_\nu^\frac{1}{2} \frac{\rho^2 (\log(\frac{R}{\nu}))^{\frac{1}{2}}}{\log \nu} d\rho + \int_\frac{1}{2}^1 (1 - \rho)^{-\frac{3}{2}} d\rho \right\} \|\dot{f}\|_{(\mathcal{H}_T^1)^*} \\
& \lesssim \frac{1}{|\log \nu|^2} \|\dot{f}\|_{(\mathcal{H}_T^1)^*}.
\end{align*}
This completes the proof of (4.56). \hfill \Box

\textbf{Proof of Proposition 4.4.} For all large \( \tau \), define \( \nu(\tau) \) and \( b(\tau) \) using Lemma 4.5. The estimates (4.50) and (4.51) follow from combining the corresponding statements (1.23) and (1.24) for \( \hat{v}(\tau) \) and \( \hat{b}(\tau) \) with (1.37)-(1.39). The bound (4.32) follows from (1.27) and (1.29). Now (4.32) implies (1.38), (4.34), and (4.35) thanks to (4.52)-(4.58) and Lemma 4.6. This completes the proof of Proposition 4.4. \hfill \Box

4.3. Modulation estimates. So far, we started with a blow-up solution \( u = (u, \dot{u}) \) described in Proposition 4.1, applied the self-similar transform to \( u \) to get \( v = (v, \dot{v}) \) as in Corollary 4.3, and finally decomposed \( v \) according to Proposition 4.4. The main goal (Proposition 4.4) of this subsection is to obtain refined modulation equations for the parameters \( \nu \) and \( b \) by testing the evolution equation of \( \epsilon \) against \( \ell_j \), where we recall

\[ \ell_j(\epsilon) = \langle (\lambda_j + \Lambda_0)\epsilon + \epsilon, g_j\varphi_j \rangle, \]
\[ g_j(\nu; \rho) = 1_{(0,1)}(\rho) \cdot (1 - \rho^2)^{\lambda_j + \frac{1}{2}}. \]

It is instructive to see how testing against \( \ell_j \) improves the modulation estimates in [53]. The modulation estimates in [53, Section 7] are obtained by testing the equation for \( \epsilon^{RR} \) against \( \Lambda P^{RR}_{B_0} \), where \( P^{RR}_{B_0} \) is defined similarly with (1.2) but the cutoff \( \chi_{B_0} \) is replaced by \( \chi_{B_0} \). This motivates some function \( G(b) \approx 4b|\log b| \) with the property

\[ \lambda\{G(b) + O(b)\}_t = -2b^2 + O\left(\frac{b^2}{|\log b|}\right). \]

Note that (4.60) is not sufficient to deduce the sharp blow-up rate (1.12) because (1.12) requires more refined information on both the \( O(b) \)-term of LHS (4.60) and the \( O\left(\frac{b^2}{|\log b|}\right) \)-order term of RHS (4.60).

The problem is that there are several error terms in (4.60) which we do not know whether they can be improved or not. One type of such terms arises from the cutoff errors; even in the definition of \( G(b) \) there is a cutoff error of size \( O(b) \). Here the spatial decays of the involved profiles are critical in the sense that changing the cutoff radius does not improve the error. Another type of such error terms in RHS (4.60) would be the linear error term

\[ \langle H_{B_0} \epsilon^{RR}, \Lambda P^{RR}_{B_0} \rangle, \]

where \( H_{B_0} \epsilon = -\partial_{yy}\epsilon - \frac{1}{v} \partial_y \epsilon + b^2 \Lambda_0 \Delta \epsilon + \frac{1}{v^2} \cos(P^{RR}_{B_0}) \epsilon \). From the construction of \( \Lambda P^{RR}_{B_0} \), Lemma 4.6 essentially becomes \( \langle \epsilon^{RR}, \Lambda \Psi_{B_0} + 2\Psi_{B_0} \rangle \), where \( \Psi_{B_0} \) is the equation error for the modified profile \( \Lambda P^{RR}_{B_0} \). This error is of critical size \( O\left(\frac{b^2}{|\log b|}\right) \), and improving this error requires either the improvements on the size of \( \epsilon^{RR} \) or the modified profile ansatz, which seem to be difficult.

Our key observation is that, without changing the modified profiles or improving the bounds for \( \epsilon \), one can still derive refined modulation equations if one changes the test function. We use \( \ell_j \) in the self-similar coordinates, instead of \( \langle \cdot, \Lambda P^{RR}_{B_0} \rangle \) in the inner coordinates for \( \epsilon^{RR} \). The linear functional \( \ell_j \) is defined explicitly (1.20) and the sharp cutoff \( 1_{(0,1]} \) in its definition does not cause any problem in the computations, thanks to the choice of the weight \( g_j \) (c.f. the proof of (3.2)). Moreover, due to the invariance (3.2), the linear error term (the analogue of (4.61) in this case) has additional structure, which is simply

\[ \ell_j(M_\nu, \epsilon) = \lambda_j \ell_j(\epsilon). \]

In particular, this error can be improved by a logarithmic factor when \( j = 1 \), thanks to \( |\lambda_1| \lesssim \frac{1}{|\log v|} \).
We now derive the evolution equation for $\epsilon$. Recall (1.15):

$$\partial_t v = \partial_t \left[ \frac{v}{\nu} \right] = \left[ -\Delta v + \frac{\nu}{\nu} - \frac{\sin(2\nu)}{2\nu} - \lambda_0 \bar{\nu} \right].$$

Substituting the decomposition (4.29) of $v$ into the above, we have

$$\partial_t (P + \epsilon) + \Lambda Q_\nu = M_{\nu} \bar{v} - \begin{bmatrix} 0 \\ R_{\text{NL}}(\bar{v}) \end{bmatrix},$$

where

$$\bar{v} := v - Q_\nu = (P - Q_\nu) + \epsilon,$$

$$R_{\text{NL}}(\bar{v}) := \frac{1}{2\rho^2} \{ \sin(2\nu) - \sin(2Q_\nu) - 2\cos(2Q_\nu)\bar{v} \}.$$

We remark that the nonlinear error term $R_{\text{NL}}(\bar{v})$ will not affect the modulation equations.

**Proposition 4.7** (Modulation estimates). We have a rough estimate

(4.65) $$|\partial_{\nu}[\ell_j(\epsilon)]| \lesssim \nu, \quad \forall j \in \{0, 1\}.$$  

Moreover, we have the following refined modulation estimates

(4.66) $$\left| \frac{\nu_t}{\nu} + \left( \frac{b}{\nu} - 1 \right) + \frac{\nu_1}{\log \nu} + \sum_{j=0}^{1} L^{(\nu)}_j (\partial_{\nu} - \lambda_j)[\ell_j(\epsilon)] \right| \lesssim \frac{1}{|\log \nu|^2},$$

and

(4.67) $$\left| \frac{b_t}{b} + \left( \frac{b}{\nu} - 1 \right) \frac{b_{-1}}{\log \nu} + \frac{b_1}{\log \nu} + \frac{b_2}{\log \nu} \right. + \left. \sum_{j=0}^{1} L^{(b)}_j (\partial_{\nu} - \lambda_j)[\ell_j(\epsilon)] \right| \lesssim \frac{1}{|\log \nu|^2},$$

where

(4.68) $$\nu_1 = \frac{1}{3}, \quad b_{-1} = \frac{1}{2}, \quad b_1 = \frac{1}{2}, \quad b_2 = \frac{5}{12} - \frac{\log 2}{2},$$

and $L^{(\nu)}_j$ and $L^{(b)}_j$ are functions satisfying

(4.69) $$|L^{(\nu)}_j| + |L^{(b)}_j| \lesssim \frac{1}{\nu |\log \nu|},$$

(4.70) $$|\partial_{\nu} L^{(\nu)}_j| + |\partial_{\nu} L^{(b)}_j| \lesssim \frac{1}{\nu |\log \nu|^2},$$

(4.71) $$|L^{(b)}_0| + |L^{(\nu)}_1 - L^{(b)}_1| \lesssim \frac{1}{\nu |\log \nu|^2}.$$  

**Remark 4.8.** In order to prove the universality and to determine the precise constant of the blow-up rate (see the proof of Theorem 1.2 in Section 1.3), it is necessary to know precise $\frac{1}{|\log \nu|^2}$-order terms (i.e., the values of $b_{-1}$ and $b_2$) of the modulation equation (1.31) for $b_r$. In contrast, the $\nu_r$-equation (4.66) does not need to be as precise as the $b_r$-equation. This fact allows us to simplify some of the computations of $\lambda_j$, $b_j$, $\epsilon_j$ (see Lemma 3.5); only $\lambda_1$ and $b_1$ require precise $\frac{1}{\log \nu}$-order terms (see (1.31)).

In principle, it is possible to compute precise $\frac{1}{\log \nu^2}$-order terms of (4.66). However, one needs to use the delicate information (2.9) on the eigenfunctions, and the computations of definite integrals are much more involved.
For the second term of LHS (4.73), by (3.8) we have
\begin{equation}
\nu_{\tau}\ell_{j}(\partial_{\nu}P) + b_{\tau}\ell_{j}(\partial_{b}P) + (\partial_{\tau} - \lambda_{j})[\ell_{j}(\epsilon)] + \nu b_{j} - b\lambda_{j}c_{j}
= \nu_{\tau}[\partial_{\nu}\ell_{j}](\epsilon) - \langle R_{\text{NL}}(\tilde{v}), g_{j}\varphi_{j} \rangle.\tag{4.72}
\end{equation}

To see this, we start from taking \( \ell_{j} \) to the equation (4.62):
\begin{equation}
\ell_{j}(\partial_{\tau}P) + \ell_{j}(\partial_{\nu}P) + \ell_{j}(\Lambda Q_{\nu}) = \ell_{j}(M_{\nu}\tilde{v}) - \langle R_{\text{NL}}(\tilde{v}), g_{j}\varphi_{j} \rangle.\tag{4.73}
\end{equation}

For the first term of LHS (4.73), we write
\( \ell_{j}(\partial_{\nu}P) = \nu_{\tau}\ell_{j}(\partial_{\nu}P) + b_{\tau}\ell_{j}(\partial_{b}P) \).

For the second term of LHS (4.73), we write
\( \ell_{j}(\partial_{\nu}P) = \partial_{\nu}[\ell_{j}(\epsilon)] - \nu_{\tau}[\partial_{\nu}\ell_{j}](\epsilon) \).

For the third term of LHS (4.73), by (3.8) we have
\( \ell_{j}(\Lambda Q_{\nu}) = \nu b_{j} \).

For the first term of RHS (4.73), we use the invariance (3.2) of \( M_{\nu} \) and the definitions of \( \tilde{v}, P, \) and \( c_{j} \) (see (4.69), (4.28), and (3.9)) to write
\( \ell_{j}(M_{\nu}\tilde{v}) = \lambda_{j}\ell_{j}(\tilde{v}) = \lambda_{j}\ell_{j}(P - Q_{\nu}) + \lambda_{j}\ell_{j}(\epsilon) = b\lambda_{j}c_{j} + \lambda_{j}\ell_{j}(\epsilon) \).

Substituting the previous displays into (4.73) completes the proof of (4.72).

**Step 1.** Algebraic computations.

In this step, we claim the following algebraic identity:
\begin{equation}
\nu_{\tau}\ell_{j}(\partial_{\nu}P) + b_{\tau}\ell_{j}(\partial_{b}P) + (\partial_{\tau} - \lambda_{j})[\ell_{j}(\epsilon)] + \nu b_{j} - b\lambda_{j}c_{j} = \nu_{\tau}[\partial_{\nu}\ell_{j}](\epsilon) - \langle R_{\text{NL}}(\tilde{v}), g_{j}\varphi_{j} \rangle.\tag{4.72}
\end{equation}

**Remark 4.9.** On the other hand, because of the necessary accuracies for the modulation equations, we cannot simply regard \((\partial_{\tau} - \lambda_{j})\ell_{j}(\epsilon)\)-type terms in (4.66)-(4.67) as perturbative terms (mostly due to the rough estimate (4.65)). When integrating the modulation equations in Section 4.3, we will always need to incorporate these \( \epsilon \)-dependent terms as corrections to our modulation parameters \( \nu \) and \( b \), and also utilize the structure (4.69)-(4.71) of \( L_{j} \)’s.

**Proof of Proposition 4.7.** Recall \( b_{j} \) and \( c_{j} \) defined from Lemma 3.3. These quantities will be important to compute the universal constants shown in (4.68).

**Step 2.** Computation of \( \ell_{j}(\partial_{\nu}P) \) and \( \ell_{j}(\partial_{b}P) \).

In this step, we claim
\begin{align*}
(4.74) & \quad \ell_{j}(\partial_{\nu}P) = -(b_{j} + 2) + O\left(\frac{1}{\log \nu}\right), \\
(4.75) & \quad \ell_{j}(\partial_{b}P) = c_{j}.
\end{align*}

Assuming these claims and using (4.30), (4.72) becomes
\begin{equation}
-\nu_{\tau}(b_{j} + 2) + b_{\tau}c_{j} + (\partial_{\tau} - \lambda_{j})[\ell_{j}(\epsilon)] + \nu b_{j} - b\lambda_{j}c_{j}
= \nu_{\tau}[\partial_{\nu}\ell_{j}](\epsilon) - \langle R_{\text{NL}}(\tilde{v}), g_{j}\varphi_{j} \rangle + O\left(\frac{\nu}{\log \nu}\right).\tag{4.76}
\end{equation}

From now on, we show (4.74) and (4.75).

**Proof of (4.74).** Observe that
\begin{equation}
\ell_{j}(\partial_{\nu}P) = -\ell_{j}\left(\frac{1}{\nu}\Lambda Q_{\nu}\right) + \frac{b}{2\nu}\left\{ \ell_{j}(\nu\partial_{b}(\varphi_{0} - \varphi_{1})) - \frac{1}{\nu}\Lambda_{\varphi_{0}}Q_{\nu}, g_{j}\varphi_{j} \right\}.\tag{4.77}
\end{equation}

Applying (3.8), (3.13), and \( \frac{b}{\nu} - 1 \leq \frac{1}{\log \nu} \) to the above, we get (4.74).

**Proof of (4.75).** This directly follows from the definition of \( P \) and \( c_{j} \) (see (4.28) and (3.9)).

**Step 3.** Treatment of the error terms.
In this step, we claim that \( \text{RHS}(4.76) \) is treated as an error:

\[
(4.77) \quad \nu \tau_j|\partial_x \ell_j(\epsilon)| \lesssim \frac{\nu}{| \log \nu|^2},
\]

\[
(4.78) \quad |\langle R_{NL}(\tilde{v}), g_j \varphi_j \rangle| \lesssim \nu^3 |\log \nu|.
\]

Assuming these claims, (4.79) becomes

\[
(4.79) \quad | - \nu \tau_j(b_j + 2) + b_x \epsilon_j + (\partial_x - \lambda_j)[\ell_j(\epsilon)] + \nu b_j - b \lambda_j \epsilon_j | \lesssim \frac{\nu}{| \log \nu|^2}.
\]

Dividing the both hand sides by \(-\nu\), we arrive at

\[
(4.80) \quad \left| \frac{\nu}{\nu} (b_j + 2) - \frac{b_x}{\nu} \left( \frac{b}{\nu} \epsilon_j \right) + \frac{b}{\nu} \lambda_j \epsilon_j - b_j - \frac{1}{\nu} (\partial_x - \lambda_j)[\ell_j(\epsilon)] \right| \lesssim \frac{1}{\nu^2}.
\]

From now on, we show (4.77) and (4.78).

**Proof of (4.77).** This is indeed immediate from (4.30) and (4.35).

**Proof of (4.78).** Recall the definition (4.64) of \( R_{NL}(\tilde{v}) \). We apply the trigonometric identity to have

\[
R_{NL}(\tilde{v}) = \frac{1}{2\rho^2} \{ \sin(2Q_{\nu} + 2\tilde{v}) - \sin(2Q_{\nu}) - 2 \cos(2Q_{\nu}) \tilde{v} \}
\]

\[
= \frac{1}{2\rho^2} \{ \sin(2Q_{\nu}) (\cos(2\tilde{v}) - 1) + \cos(2Q_{\nu}) (\sin(2\tilde{v}) - 2\tilde{v}) \}.
\]

Applying \( \sin(2Q_{\nu}) = 2 \cos(Q_{\nu}) \Lambda Q_{\nu} \), we have

\[
|R_{NL}(\tilde{v})| \lesssim \frac{1}{\rho^2} \{ |\Lambda Q_{\nu}| \cos(2\tilde{v}) - 1 | + | \sin(2\tilde{v}) - 2\tilde{v} | \}
\]

\[
\lesssim \frac{1}{\rho^2} \{ |\Lambda Q_{\nu}| |\tilde{v}|^2 + |\tilde{v}|^3 \}.
\]

Note that \( |\tilde{v}| \lesssim b|\varphi_0 - \varphi_1| + |\epsilon| \) by the definition (4.63) of \( \tilde{v} \). Using \( b \sim \nu \), (2.12), and (4.33), we have

\[
1_{(0,1)}|\tilde{v}| \lesssim 1_{(0,1)} \{ b|\varphi_0 - \varphi_1| + |\epsilon| \}
\]

\[
\lesssim \nu \cdot \left( 1_{(0,\nu)} \rho \left( \frac{\rho}{\nu} \right)^2 \right) + 1_{[\nu,1)} \rho \langle \log \rho \rangle + 1_{(0,1)} \rho \langle \log \left( \frac{\rho}{\nu} \right) \rangle \lesssim 1_{(0,1)} \nu^3 
\]

Using \( |\Lambda Q_{\nu}| \lesssim \frac{\nu}{\rho} \), we have a rough pointwise bound of \( R_{NL}(\tilde{v}) \):

\[
1_{(0,1)} |R_{NL}(\tilde{v})| \lesssim 1_{(0,1)} \nu^3 \frac{\nu}{\rho}.
\]

Combining this with the pointwise bound

\[
|g_j \varphi_j| \lesssim g_j \frac{\nu}{\rho} \Lambda Q_{\nu} \lesssim 1_{(0,\nu)} \nu \frac{\rho}{\nu^2} + 1_{[\nu,1)} \nu \frac{\rho}{\nu} (1 - \nu)^{-\frac{3}{2}}
\]

gives

\[
|\langle R_{NL}(\tilde{v}), g_j \varphi_j \rangle| \lesssim \int_0^1 \left( 1_{(0,\nu)} \nu + 1_{[\nu,1)} \nu \frac{\nu^3}{\rho^2} (1 - \nu)^{-\frac{3}{2}} \right) \rho d\rho \lesssim \nu^3 |\log \nu|,
\]

completing the proof of (4.78).

**Step 4.** Proof of the rough estimate (4.65).

The rough estimate (4.65) directly follows from estimating each term of (4.79) except \( \partial_x [\ell_j(\epsilon)] \) using (4.30), (3.10)-(3.11), and (4.34). Note that although each of \( \nu b_0 \) and \( b \lambda_0 \) is of size \( O(\nu |\log \nu|) \), \( \nu b_0 - b \lambda_0 \) is of size \( O(\nu) \).

**Step 5.** Proofs of the refined modulation estimates (4.66) and (4.67).

In this step, we show (4.66) and (4.67). Our starting point is (4.80).
We first show (4.66). We subtract (4.80) for $j = 1$ from (4.80) for $j = 0$ and then divide it by $b_0 - b_1$ to have
\[
\left| \frac{\nu_\tau}{\nu} - \frac{b_\tau}{b} \left( \frac{b_0 \cdot (c_0 - c_1)}{b_0 - b_1} \right) + \frac{b \lambda_0 c_0 - \lambda_1 c_1}{\nu (b_0 - b_1)} - 1 \right| + \sum_{j=0}^{\nu} L_{\nu}^{(j)} (\partial_\tau - \lambda_j) [\ell_j (\epsilon)] \lesssim \frac{1}{|\log \nu|^3},
\]
where
\[
L_{\nu}^{(0)} := -\frac{1}{\nu (b_0 - b_1)} \quad \text{and} \quad L_{\nu}^{(1)} := \frac{1}{\nu (b_0 - b_1)}.
\]
We look at each term at (4.81). First, we use (4.80) and (3.10)-(3.11) to have
\[
\left| - \frac{b_\tau}{b} \left( \frac{b_0 \cdot (c_0 - c_1)}{b_0 - b_1} \right) \right| \lesssim \frac{1}{|\log \nu|} \cdot \frac{1}{|\log \nu|} \lesssim \frac{1}{|\log \nu|^2}.
\]
Next, we use (4.30), (2.4), and (3.10)-(3.11) to have
\[
\frac{b \lambda_0 c_0 - \lambda_1 c_1}{\nu (b_0 - b_1)} - 1 = \frac{b}{\nu} \left( 1 + \frac{1}{|\log \nu|} + O \left( \frac{1}{|\log \nu|^2} \right) \right) - 1
\]
\[
= \left( \frac{b}{\nu} - 1 \right) + \frac{b}{\nu} + O \left( \frac{1}{|\log \nu|^2} \right).
\]
Substituting the previous two displays into (4.81), we have proved
\[
\left| \frac{\nu_\tau}{\nu} + \left( \frac{b}{\nu} - 1 \right) + \frac{1}{|\log \nu|} + \sum_{j=0}^{\nu} L_{\nu}^{(j)} (\partial_\tau - \lambda_j) [\ell_j (\epsilon)] \right| \lesssim \frac{1}{|\log \nu|^3},
\]
which is (4.66).

We turn to the proof of (4.67). We divide (4.80) for $j = 1$ by $-\frac{b}{\nu} c_1$ to obtain
\[
\left| \frac{b_\tau}{b} - \frac{\nu_\tau}{\nu} \left( \frac{b_0 (b_1 + 2)}{c_1} \right) + \frac{\nu b_1}{b} \frac{b_1}{c_1} - \lambda_1 - \tilde{L}_1^{(b)} (\partial_\tau - \lambda_1) [\ell_1 (\epsilon)] \right| \lesssim \frac{1}{|\log \nu|^3},
\]
where
\[
\tilde{L}_1^{(0)} := 0, \quad \tilde{L}_1^{(1)} := \frac{1}{b c_1}.
\]
Let us look at each term of (4.83). First, we use (4.30) and (3.10)-(3.11) to have
\[
\frac{\nu_\tau}{\nu} \left( \frac{b_0 (b_1 + 2)}{c_1} \right) = \frac{\nu_\tau}{\nu} \left( -\frac{b}{\nu} + O \left( \frac{1}{|\log \nu|^2} \right) \right) = \frac{\nu_\tau}{\nu} - \frac{b}{\nu} \frac{1}{|\log \nu|} + O \left( \frac{1}{|\log \nu|^3} \right).
\]
Next, we use $\frac{b}{\nu} = 1 - (\frac{b}{\nu} - 1) + O \left( |\log \nu - 1| \right) = 1 - (\frac{b}{\nu} - 1) + O \left( \frac{1}{|\log \nu|^2} \right)$, (3.10)-(3.11), and (2.4) to have
\[
\frac{b_\tau}{b} - \lambda_1 = \left( \frac{b}{\nu} - 1 \right) - \frac{b_1}{c_1} - \lambda_1 + O \left( \frac{1}{|\log \nu|^3} \right)
\]
\[
= \left( \frac{b}{\nu} - 1 \right) - \frac{b_1}{c_1} - \lambda_1 + O \left( \frac{1}{|\log \nu|^3} \right) + O \left( \frac{b_1}{c_1} - \lambda_1 \right) + O \left( \frac{1}{|\log \nu|^2} \right).
\]
Substituting the previous two displays into (4.83), we have
\[
\left| \frac{b_\tau}{b} + \frac{\nu_\tau}{\nu} \frac{b}{|\log \nu|} + \left( \frac{b}{\nu} - 1 \right) \frac{1}{|\log \nu|} + \left( \frac{b}{\nu} - 1 \right) \frac{1}{|\log \nu|^2} \right| \lesssim \frac{1}{|\log \nu|^3}.
\]
Finally, substituting (4.82) into (4.85) (in other words, we add \( \frac{1}{6|\log \nu|} \)) into (4.85) times (4.82)), we obtain
\[
|\frac{b_x}{b} + \left( \frac{b}{\nu} - 1 \right) \frac{1}{\log \nu}| + \frac{\nu}{\log \nu} + \frac{b}{\log \nu} + \sum_{j=0}^{1} L_j^{(b)} (\partial_{\tau} - \lambda_j) [f_j(\epsilon)] | \lesssim \frac{1}{\log \nu^2},
\]
where
\[
L_j^{(b)} := \tilde{L}_j^{(b)} + \frac{f_j^{(b)}}{6|\log \nu|}.
\]
This completes the proof of (4.67).

**Step 6.** Proofs of (4.69)-(4.71).

We finish the proof by showing (4.69)-(4.71). Recall that
\[
L_0^{(\nu)} = -\frac{1}{\nu(b_0 - b_1)}, \quad L_1^{(\nu)} = \frac{1}{\nu(b_0 - b_1)},
\]
\[
L_0^{(b)} = \frac{f_0^{(b)}}{6|\log \nu|}, \quad L_1^{(b)} = \frac{1}{b\epsilon_1} + \frac{L_1^{(\nu)}}{6|\log \nu|}.
\]
Note that (4.69) is immediate from (3.10)-(3.11). The proof of (4.71) follows from (4.69), (3.10)-(3.11), and |\( \frac{b}{\nu} - 1 \)| \( \lesssim \frac{1}{\log \nu} \):
\[
|L_0^{(b)}| \lesssim \frac{|L_0^{(\nu)}|}{|\log \nu|} \lesssim \frac{1}{\nu |\log \nu|^2},
\]
\[
|L_1^{(\nu)} - L_1^{(b)}| \leq \left| \frac{\nu(b_0 - b_1)}{b\epsilon_1} \right| + \frac{|L_1^{(\nu)}|}{6|\log \nu|} \lesssim \frac{1}{6|\log \nu|}.
\]
It remains to show (4.70). From the formulas of \( L_0^{(\nu)} \) and \( L_1^{(b)} \) and (4.30), we have
\[
|\partial_{\tau} L_0^{(\nu)}| + |\partial_{\tau} L_1^{(b)}| \lesssim \left( \left| \frac{\nu c_1}{\nu} \right| + \left| \frac{b_x}{b} \right| \right) \frac{1}{\nu |\log \nu|} + \frac{|\partial_{\tau}(b_0 - b_1)|}{\nu |\log \nu|} \lesssim \frac{1}{\nu |\log \nu|^2}.
\]
Applying (4.14.2) to the above yields (4.70). This completes the proof. □

### 4.4. Proof of Theorem 1.2

In this final subsection, we finish the proof of Theorem 1.2. In the following lemma, we show that (4.69) and (4.67) improve the rough estimate (4.30) to a more refined relation between \( b \) and \( \nu \).

**Lemma 4.10** (Compatibility of \( \nu \) and \( b \) for blow-up). We have
\[
(4.86) \quad \frac{b}{\nu} = 1 + \frac{b_1 - \nu_1}{|\log \nu|} + L_0^{(\nu)} \ell_0(\epsilon) + O\left( \frac{1}{|\log \nu|^2} \right),
\]

**Proof.** Let
\[
(4.87) \quad \beta := \frac{b}{\nu} - 1 - \frac{b_1}{|\log \nu|} - L_0^{(\nu)} \ell_0(\epsilon),
\]
where \( \beta_1 := b_1 - \nu_1 \). It suffices to show that \( |\beta| \lesssim \frac{1}{|\log \nu|^2} \).

**Step 1.** Computation of \( \beta_\tau \).

Thanks to (4.30), we have
\[
\beta_\tau = \left( \frac{b_\tau}{\nu} - \frac{\nu c_1}{\nu} \right) - \left( L_0^{(\nu)} \partial_{\tau} \ell_0(\epsilon) \right) + O\left( \frac{1}{|\log \nu|^2} \right).
\]
Thanks to (4.30), (4.70), and (4.34), we have
\[
\beta_\tau = \left( \frac{b_\tau}{\nu} - \frac{\nu c_1}{\nu} \right) - L_0^{(\nu)} \partial_{\tau} \ell_0(\epsilon) + O\left( \frac{1}{|\log \nu|^2} \right).
\]
Next, we apply (4.66), (4.67), and (4.30) to have
\[
\beta_\tau = \left(\frac{b}{\nu} - 1\right) + \frac{\nu - b_1}{|\log \nu|} - L_0^{(\nu)}\lambda_0 \ell_0(\epsilon) - \frac{1}{|\log \nu|^2}.
\]
Using (4.71), (4.65), and (4.34), the terms in the second line of the above display are of size \( O \left( \frac{1}{|\log \nu|} \right) \). Thus we have
\[
\beta_\tau = \left(\frac{b}{\nu} - 1\right) + \frac{\nu - b_1}{|\log \nu|} - L_0^{(\nu)}\lambda_0 \ell_0(\epsilon) + O \left( \frac{1}{|\log \nu|^2} \right).
\]
In view of (4.87), we replace \( \frac{b}{\nu} - 1 \) by \( \frac{b_1}{\log \nu} + L_0^{(\nu)}\ell_0(\epsilon) + \beta \) to arrive at
\[
\beta_\tau = \frac{\beta_1 + \nu - b_1}{|\log \nu|} + (1 - \lambda_0)L_0^{(\nu)}\ell_0(\epsilon) + \beta + O \left( \frac{1}{|\log \nu|^2} \right).
\]
Let us look at each term of (4.88). The first term vanishes thanks to the choice \( \beta_1 = b_1 - \nu \). For the second term, we use (4.69) and (4.34) to have
\[
|(1 - \lambda_0)L_0^{(\nu)}\ell_0(\epsilon)| \lesssim \frac{1}{|\log \nu|} \cdot \frac{1}{\nu |\log \nu|} \cdot |\nu| \lesssim \frac{1}{|\log \nu|^{3/2}}.
\]
Finally by (4.31) we have \( \frac{1}{|\log \nu|^2} \sim \tau^{-1} \). As a result, (4.88) becomes
\[
\beta_\tau = \beta + O(\tau^{-1}).
\]

Step 2. Backward integration of the \( \beta_\tau \)-equation.

To conclude (4.89), we integrate the \( \beta_\tau \)-equation (4.89) backwards in time from \( \tau = +\infty \). We rewrite (4.89) as
\[
|e^{-\tau \beta(\tau)}| \lesssim e^{-\tau^{-1}}.
\]
Integrating this from \( +\infty \) to \( \tau \) yields
\[
|\beta(\tau)| \lesssim e^\tau \int_\tau^{+\infty} e^{-\tau'} \left( \tau' \right)^{-1} d\tau' \lesssim \tau^{-1} \sim \frac{1}{|\log \nu|^{3/2}}.
\]
This completes the proof of (4.89).

Substituting the refined relation (4.86) into the the modulation equation for \( b \), we obtain the following sharp modulation equation only in terms of \( b \) and \( \epsilon \).

**Corollary 4.11** (Sharp modulation equation for \( b \)). We have
\[
\begin{align*}
\left| \frac{b_\tau}{b} + \frac{1}{|\log b|} + \frac{1}{|\log b|^2} + \sum_{j=0}^{1} L_j^{(b)}[\ell_j(\epsilon)] \right| \lesssim \frac{1}{|\log b|^2}.
\end{align*}
\]

**Proof.** Substituting (4.86) into (4.67) yields
\[
\begin{align*}
\left| \frac{b_\tau}{b} + \frac{b_1}{|\log \nu|} + \frac{b_2 + (b_1 - \nu_1)b_1}{|\log \nu|^2} + \frac{b_1 L_0^{(\nu)}}{|\log \nu|}\ell_0(\epsilon) \right| + \sum_{j=0}^{1} L_j^{(b)}[\ell_j(\epsilon)] \lesssim \frac{1}{|\log \nu|^3}.
\end{align*}
\]
In the above display, the following terms are considered as errors. We use (4.69), \( |\lambda_1| \lesssim \frac{1}{|\log \nu|} \) and (4.34) to have
\[
\begin{align*}
\left| \frac{b_1 L_0^{(\nu)}}{|\log \nu|}\ell_0(\epsilon) \right| + \left| - L_1^{(b)}\lambda_1 \ell_1(\epsilon) \right| \lesssim \frac{1}{|\log \nu|^{3/2}}.
\end{align*}
\]
Next, we use (4.71) and (4.34) to have
\[ | -L_0^{(b)} \lambda_0 \ell_0(\epsilon) | \lesssim \frac{1}{| \log \nu |^{5/2}}. \]
Substituting the above error estimates into (4.91), we have
\[ | b_\tau + \frac{b_1}{| \log \nu |} + \frac{b_2 + (b_1 - b_1) b_1}{| \log |^2} \lambda_0 \ell_0(\epsilon) | \lesssim \frac{1}{| \log \nu |^{5/2}}. \]
Replacing \( \nu_1, b_1, b_2, b_2 \) by their values (4.68) and applying \( \log \nu = \log b + O\left( \frac{1}{| \log b |} \right) \) (due to (4.30)) completes the proof.

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** The proof follows from integrating (4.90).

**Step 1.** A correction \( \tilde{b} \).

Let us introduce a correction \( \tilde{b} \) to \( b \):
\[
\tilde{b} := b \left( 1 + \sum_{j=0} L_j^{(b)} \ell_j(\epsilon) \right).
\]
Using (4.69) and (4.34), we see that \( \tilde{b} \) is a correction of \( b \) in the sense that
\[
b = b \left( 1 + O\left( \frac{1}{| \log \nu |^{5/2}} \right) \right).
\]
With this \( \tilde{b} \), we have
\[
\frac{b_\tau}{\tilde{b}} = b_\tau + \frac{\sum_{j=0}^1 L_j^{(b)} \partial_{\tau} [\ell_j(\epsilon)]}{1 + \sum_{j=0}^1 L_j^{(b)} \ell_j(\epsilon)} + \frac{\sum_{j=0}^1 \partial_{\tau} L_j^{(b)} \ell_j(\epsilon)}{1 + \sum_{j=0}^1 L_j^{(b)} \ell_j(\epsilon)}.
\]
Using (4.69), (4.70), (4.34), and (4.65), we have
\[
\frac{b_\tau}{\tilde{b}} = b_\tau + \left( \sum_{j=0}^1 L_j^{(b)} \partial_{\tau} [\ell_j(\epsilon)] \right) \left( 1 + O\left( \frac{1}{| \log \nu |^{5/2}} \right) \right) + O\left( \frac{1}{| \log \nu |^{5/2}} \right).
\]
Substituting (4.90) into the above, we get
\[
\left| \frac{b_\tau}{\tilde{b}} + \frac{\frac{1}{2}}{| \log b |} + \frac{\frac{1}{2} \frac{\log 2}{2} | \log b |^2} \right| \lesssim \frac{1}{| \log b |^{5/2}}.
\]
Finally using (4.93), we have
\[
\frac{b_\tau}{\tilde{b}} + \frac{\frac{1}{2}}{| \log b |} + \frac{\frac{1}{2} \frac{\log 2}{2} | \log b |^2} \lesssim \frac{1}{| \log b |^{5/2}}.
\]

**Step 2.** Integration of (4.94).

It is more convenient to introduce the variable
\[
\mu := | \log \tilde{b} | = - \log b \sim \sqrt{\tau}
\]
so that (4.94) becomes
\[
| \mu_\tau - \frac{\frac{1}{2}}{\mu} + \frac{\frac{1}{2} \frac{\log 2}{2}}{\mu^2} | \lesssim \frac{1}{| \tau |^{5/4}}.
\]
Multiplying the above by $2\mu$ and using $\mu = \sqrt{\tau} + O(1)$ from (4.31), we have

$$ \left| (\mu^2)_{\tau} - 1 - \frac{1 - \log 2}{\sqrt{\tau}} \right| \lesssim \frac{1}{\tau^{3/4}}. $$

Integrating this, we have

$$ \mu^2 = \tau + (1 - \log 2) \cdot 2\sqrt{\tau} + O(\tau^{1/4}) $$

and hence

$$ \mu = \sqrt{\tau} + (1 - \log 2) + O(\tau^{-1/4}). $$

Going back to the variable $\hat{b}$, we have proved that

$$ (4.95) \quad \hat{b} = 2e^{-1}e^{-\sqrt{\tau}(1 + o_{\tau \to \infty}(1))}. $$

**Step 3. Conclusion.**

By (4.30) and (4.93), we have

$$ \nu = \hat{b}(1 + o_{\tau \to \infty}(1)). $$

Thus by (4.95), we have

$$ \nu(\tau) = 2e^{-1}e^{-\sqrt{\tau}(1 + o_{\tau \to \infty}(1))}. $$

In terms of the original spacetime coordinates $(t, r)$, this reads

$$ \lambda(t) = 2e^{-1}(T - t)e^{-\sqrt{\log(T - t)}(1 + o_{t \to T}(1))}. $$

This completes the proof of Theorem 4.2. $\square$

**APPENDIX A. PROOF OF (4.13)**

Proof of (4.13). In this proof, we denote $\hat{\lambda}$ and $\hat{b}$ by $\lambda$ and $b$, respectively, and abbreviate the integral $\int_t^T dt'$ as $\int_t^T$. We will also rely on the following facts:

- [53, p.108] Almost monotonicity of $b$: for $t' \in [t, T)$,
  \[
  \frac{b^2(t')}{|\log b(t')|} \leq 2 \frac{b^2(t)}{|\log b(t)|}, \tag{A.1}
  \]

- [53, (5.34)] Control of $b_{t}$:
  \[
  b_t |b| \lesssim \frac{b^2}{|\log b|}, \tag{A.2}
  \]

The overall scheme of the proof is to follow the proof in [53] pp.107–108 Step 2, but we also need to keep $\frac{1}{|\log b|}$-smallness. We study the quantity

$$ \int_t^T b^2(t')dt' $$

in two ways.

On one hand, using $\lambda_t + b = 0$ and (A.2), we observe

$$ \int_t^T b^2 = \int_t^T -b\lambda_t = b(t)\lambda(t) + \int_t^T \lambda b_t = b(t)\lambda(t) + \int_t^T O\left( \frac{b^2}{|\log b|} \right). $$

Thus we have

$$ b(t)\lambda(t) = \int_t^T \left( 1 + O\left( \frac{1}{|\log b|} \right) \right)b^2 = \left( 1 + O\left( \frac{1}{|\log b(t)|} \right) \right) \int_t^T b^2, \tag{A.3} $$

where in the last equality we used the almost monotonicity (A.1) of $b$. 

On the other hand, we integrate by parts and use (4.12) to have
\[
\frac{1}{(T-t)b^2(t)} \int_t^T b^2 - 1 = \frac{1}{(T-t)b^2(t)} \int_t^T b \cdot (T-t') \leq \frac{1}{(T-t)|\log b(t)|} \int_t^T \frac{b}{\lambda} (T-t').
\]
Using \( \log \lambda(t) = \log(T-t) - \sqrt{|\log(T-t)|} + O(1) \) of (4.14), we have
\[
\int_t^T \frac{b}{\lambda} (T-t') = \int_t^T \frac{\lambda}{\lambda}(T-t') = (T-t) \log \lambda(t) - \int_t^T \log \lambda = O(T-t).
\]
Combining the above two displays yields
\[
(A.4) \quad \frac{1}{(T-t)b^2(t)} \int_t^T b^2 = 1 + O\left(\frac{1}{|\log b(t)|}\right).
\]
The estimates (A.3) and (A.4) yield
\[
\frac{\lambda(t)}{(T-t)b(t)} = \frac{1}{(T-t)b^2(t)} \cdot b(t)\lambda(t) = 1 + O\left(\frac{1}{|\log b(t)|}\right),
\]
completing the proof of (4.13).

\[\square\]

**Appendix B. Localized Hardy Inequalities**

In this appendix, we show the localized coercivity estimates (4.21)-(4.22) as well as the weighted \( L^\infty \)-estimates (Lemma B.3) used in Sections 4.1-4.2.

**Lemma B.1** (Local subcoercivity estimates and kernel characterization). For \( R \gg 1 \), we have
\[
(B.1) \quad \|1_{(0,R]}A^*Af\|_{L^2} + \|1_{y<R}f\|_{L^2} \sim \|f\|_{(H^1_\lambda)_\infty}, \quad \forall f \in \mathcal{H}_1,
\]
\[
(B.2) \quad \|1_{(0,R]}Ag\|_{L^2} + \|1_{y<R}g\|_{L^2} \sim \|g\|_{(H^1_\lambda)_\infty}, \quad \forall g \in \mathcal{H}_1,
\]
where the implicit constants are uniform in \( R \). Moreover, the kernels of \( A^*A : (\mathcal{H}^1_\lambda)_R \to L^2(r<R) \) and \( A : (\mathcal{H}^1_\lambda)_R \to L^2(r<R) \) are equal to the span of \( \Delta Q \).

**Proof.** Without localization \( 1_{(0,R]} \), these subcoercivity estimates are already proved in [33 Lemma B.2]. The key point here is to keep track of the localization \( 1_{(0,R]} \) while we reproduce the proof.

**Step 1. Prooff of (B.3).**

Let us first show (B.2). As the \( \lesssim \)-inequality is clear, we focus on the proof of the \( \gtrsim \)-inequality. An integration by parts yields the identity
\[
\int_0^R |Ag|^2 ydy = \int_0^R \left(|\partial_y g|^2 + \frac{V}{y^2} |g|^2\right) ydy + \frac{1 + R}{1 + R^2} |g|^2(R).
\]
For \( R \gg 1 \), the last term is nonnegative (which is a good sign). Thus we have
\[
\int_0^R |Ag|^2 ydy \geq \int_0^R \left(|\partial_y g|^2 + \frac{1}{y^2} |g|^2 + \frac{V}{y^2} |g|^2\right) ydy
\]
\[
= \|g\|^2_{(H^1_\lambda)_\infty} - O\left(\int_0^R \frac{1}{(1 + y^2)^2} |g|^2 ydy\right).
\]
Since
\[
\int_0^R \frac{1}{(1 + y^2)^2} |g|^2 ydy \lesssim \|1_{(r_0^-, r_0)}g\|_{L^2}^2 + r_0^{-2} \|g\|_{(H^1_\lambda)_\infty}^2,
\]
taking \( r_0 > 1 \) large yields
\[
\int_0^R |Ag|^2 ydy + \|1_{(r_0^-, r_0)}g\|_{L^2}^2 \gtrsim \|g\|_{(H^1_\lambda)_\infty}^2.
\]
This completes the proof of (B.2).

**Step 2.** Localized Hardy controls from $\Delta_1$ and $\partial_{yy}$.
From now on, we turn to the more delicate proof of (B.1). We follow the proof of [36, Lemma A.7], but we also need to keep track of the localization $1_{(0,R]}$. In this step, we claim the following Hardy controls from $-\Delta_1$ and $\partial_{yy}$:

$$
(B.3) \quad \|1_{(0,R]} \Delta_1 f\|_{L^2} \sim \|1_{(0,R]} \left(\partial_y - \frac{1}{y}\right)f\|_{L^2} \sim \|1_{(0,R]} \partial_{yy} f\|_{L^2}.
$$

To see this, let us note the operator identity

$$
\Delta_1 = (\partial_y + \frac{1}{y} - \frac{1}{y})\left(\partial_y - \frac{1}{y}\right),
$$

$$
\partial_{yy} = (\partial_y + \frac{1}{y} - \frac{1}{y}).
$$

We turn to the proof of the Step 3. Proof of (B.1).

In this step, we use the Hardy controls (B.3) to prove (B.1). We use

$$
A^* A = -\partial_{yy} - \frac{1}{y} \partial_y + \frac{V}{y^2} = -\Delta_1 + O\left(\frac{1}{(1 + y^2)^2}\right),
$$

to have

$$
(B.4) \quad \|1_{(0,R]} A^* A f\|_{L^2} - \|1_{(0,R]} \Delta_1 f\|_{L^2} \lesssim \|1_{(0,R]} \partial_{yy} f\|_{L^2} + \|\tilde{f}\|_{(\tilde{H}_1^*) n}.
$$

Now the $\lesssim$-inequality of (B.1) easily follows from (B.3) and (B.4):

$$
\|1_{(0,R]} A^* A f\|_{L^2} \lesssim \|1_{(0,R]} \partial_{yy} f\|_{L^2} + \|\tilde{f}\|_{(\tilde{H}_1^*) n} \lesssim \|1_{(0,R]} \partial_{yy} f\|_{L^2} + \|\tilde{f}\|_{(\tilde{H}_1^*) n}.
$$

We turn to the proof of the $\gtrsim$-inequality of (B.1). First, by (B.3) and (B.4), we have

$$
(B.5) \quad \|1_{(0,R]} \left(\partial_y - \frac{1}{y}\right)f\|_{L^2} \gtrsim \|1_{(0,R]} \partial_{yy} f\|_{L^2} \lesssim \|1_{(0,R]} A^* A f\|_{L^2} + \|1_{(0,R]} \partial_{yy} f\|_{L^2} + \|\tilde{f}\|_{(\tilde{H}_1^*) n}.
$$

Next, we use

$$
\|1_{(0,R]} \frac{1}{y (\log y)} |f|_{-1}\|_{L^2} \lesssim \|1_{(0,R]} \frac{1}{y (\log y)} \left(\partial_y - \frac{1}{y}\right)f\|_{L^2} + \|1_{(0,R]} \frac{1}{y^2 (\log y)} f\|_{L^2},
$$

the following logarithmic Hardy’s inequality

$$
\|1_{(0,R]} \frac{1}{y^2 (\log y)} f\|_{L^2} \lesssim \|1_{(0,R]} \frac{1}{y} \left(\partial_y - \frac{1}{y}\right)f\|_{L^2} + \|1_{(1,2]} f\|_{L^2}.
$$
Combining (B.5) and (B.6), we have shown that

\[ f \sim 0 \]

and (B.5) to have

\[ \|f\|_{\hat{H}^2_1 R} \lesssim \|1_{(0,R)}(\partial_y - \frac{1}{y})f\|_{L^2} + \|1_{(1,2)}f\|_{L^2} \]

\[ \lesssim \|1_{(0,R)} A^* A f\|_{L^2} + \|1_{(r_0^{-1},r_0)}f\|_{L^2} + r_0^{-2} \|f\|_{(\hat{H}^2_1 R)^n}. \]

Combining (B.5) and (B.6), we have shown that

\[ \|f\|_{(\hat{H}^2_1 R)^n} \lesssim \|1_{(0,R)}(\partial_y - \frac{1}{y})f\|_{L^2} + \|1_{(0,R)} \frac{1}{y(y \log y)}[f]_{-1}\|_{L^2} \]

\[ \lesssim \|1_{(0,R)} A^* A f\|_{L^2} + \|1_{(r_0^{-1},r_0)}f\|_{L^2} + r_0^{-2} \|f\|_{(\hat{H}^2_1 R)^n}. \]

Taking \( r_0 > 1 \) large, the \( \gg \)-inequality of (B.7) follows.

For the proof of (B.8), the fact that \( (\partial_y - \frac{1}{y})f \) enjoys better Hardy control than \( \partial_y f \) will play a crucial role. More precisely, we will use the following control (as a consequence of (B.3)):

\[ \|1_{(0,R)}(\partial_y - \frac{1}{y})f\|_{L^2} \lesssim \|f\|_{(\hat{H}^2_1 R)^n}. \]
Noting that
\[ \partial_y \left( \frac{|f|^2}{y^2 \log y} \right) = \frac{1}{\log y} \partial_y \left( \frac{|f|^2}{y^2} \right) - \frac{\log y}{y^3 (\log y)^3} |f|^2, \]
we have
\[ \partial_y \left( \frac{|f|^2}{y^2 \log y} \right) \lesssim \frac{1}{y} \left( \partial_y - \frac{1}{y} f \right) \cdot \frac{|f|}{y \log y} + \frac{|f|^2}{y^3 (\log y)^2}. \]

Similarly, we have
\[ \partial_y \left( \frac{\partial_y f^2}{\log y} \right) \lesssim |\partial_{yy} f| \cdot \frac{|\partial_y f|}{\log y} + \frac{|\partial_y f|^2}{y \log y^2}. \]

Integrating (B.10) and (B.11) over \((0, y]\), exploiting \(|f|_{-1} \leq C(f)\) as \(y \to 0\) if \(f \in \mathcal{H}_1^2\) is smooth at the origin (due to the density argument), and using \(|\partial_{yy} f| \lesssim |(\partial_y - \frac{1}{y}) f|_{-1}\), we have
\[ 1_{(0, R)} \frac{|f|^2}{y^2 \log y} \lesssim 1_{(0, R)} \int_0^y \left\{ \left| (\partial_y - \frac{1}{y}) f \right|^2 + \left| \frac{|f|_{-1}}{y \log y^3} \right|^2 \right\} y' dy' \lesssim \|f\|_{(\mathcal{H}_1^2)R}^2. \]

This completes the proof of (B.8). The proof of (B.9) is much easier and we omit the proof. □

**Appendix C. Proof of Corollary 4.3**

**Proof of Corollary 4.3.** We only show the estimates (4.25)-(4.27) as all the other statements are immediate from Proposition 4.1. For the sake of simplicity we write \(\lambda = \lambda, \hat{b} = b,\) and \(R = 4/b\) in the proof. We keep \(M\)-dependences in the following estimates, but one can simply ignore them because \(M\) is already fixed in Proposition 4.1 (see also Remark 4.2). Notice that we stated (4.25)-(4.27) without \(M\)-dependence.

**Step 1.** Proof of the estimates (4.25)-(4.27) for \(\epsilon^{RR}\).

By definition, we have
\[
\left\{ \begin{array}{l}
\epsilon^{RR}(\tau, y) = w^{RR}(t, \lambda(t)y), \\
A^{RR}(\tau, y) = \lambda(t)W^{RR}(t, \lambda(t)y),
\end{array} \right.
\]
and this \(\epsilon^{RR}\) satisfies the orthogonality
\[ \langle \epsilon^{RR}, \chi_M A_Q \rangle = 0. \]

Note that (4.25) for \(\epsilon^{RR}\) easily follows from (4.13) and scaling. To show the estimates (4.26)-(4.27) for \(\epsilon^{RR}\), we first apply the coercivity estimates (4.19) and (4.21) to have
\[ \|\epsilon^{RR}\|_{\mathcal{H}_1^2}^2 \sim_M \|A^* A^{RR}\|_{L^2}^2, \]
\[ \|\epsilon^{RR}\|_{(\mathcal{H}_1^2)R}^2 \sim_M \|1_{(0, R)} A^* A^{RR}\|_{L^2}^2. \]

An integration by parts shows that the RHS of the above display can be written as
\[ \|A^* A^{RR}\|_{L^2}^2 = \int \left( |\partial_y A^{RR}|^2 + \frac{\tilde{V}}{y^2} |A^{RR}|^2 \right), \]
\[ \|1_{(0, R)} A^* A^{RR}\|_{L^2}^2 = \int 1_{(0, R)} \left( |\partial_y A^{RR}|^2 + \frac{\tilde{V}}{y^2} |A^{RR}|^2 \right) + \frac{2}{1 + R^2} |A^{RR}|^2(R). \]
By (C.1) and (4.16)-(4.17), we have
\[
\int \left( |\partial_y (A e^{\text{RR}})|^2 + \frac{\bar{v}}{y^2} |A e^{\text{RR}}|^2 \right) \lesssim b^4,
\]
\[
\int 1_{(0,R)} \left( |\partial_y (A e^{\text{RR}})|^2 + \frac{\bar{v}}{y^2} |A e^{\text{RR}}|^2 \right) \lesssim \frac{b^4}{|\log b|}.
\]
By the localized $L^\infty$-control (5.8), the boundary term is controlled by
\[
\frac{2}{1+R^2} |A e^{\text{RR}}|^2 (R) \lesssim \frac{\log R}{R^2}, \quad \frac{|e^{\text{RR}}|_{-1}^2 (R)}{\log R} \lesssim o_b \omega_b (1) \cdot \| e^{\text{RR}} \|_{(\tilde{H}^1_2)_N}^2.
\]
Therefore, we have shown that
\[
\| e^{\text{RR}} \|_{\tilde{H}_2}^2 \lesssim_M b^4,
\]
\[
\| e^{\text{RR}} \|_{(\tilde{H}^1_2)_N} \lesssim_M \frac{b^4}{|\log b|} + o_b \omega_b (1) \cdot \| e^{\text{RR}} \|_{(\tilde{H}^1_2)_N}^2.
\]
This completes the proof of (4.20)-(4.27) for $e^{\text{RR}}$.

**Step 2.** Proof of the estimates (4.20)-(4.27) for $e^{\text{RR}}$.

By definition, we have
\[
e^{\text{RR}}(\tau, y) = \lambda(t) [\partial_t w^{\text{RR}}] (t, \lambda(t) y) + \lambda(t) b(t) [\partial_y w^{\text{RR}}] (b(t); y).
\]
First, we show that (4.20)-(4.27) holds for $\lambda b [\partial_y w^{\text{RR}}] (b; y)$ in place of $e^{\text{RR}}$. Note that we know $|\lambda b| \lesssim \frac{b^3}{|\log b|}$ by (4.12). For $\partial_y w^{\text{RR}}$, one has the following pointwise estimate (5.3) (3.61):
\[
|\partial_y w^{\text{RR}}| \lesssim 1_{(0,R)} \frac{b}{|\log b|} y \log (1+y) + 1_{(0,R)} \frac{1}{b|\log b|} \frac{1}{1+y}.
\]
Therefore, we have (recall that $B_0 \sim b^{-1}$ and $B_1 \sim b^{-1} |\log b|$)
\[
\| \partial_y w^{\text{RR}} \|_{L^2} \lesssim \frac{1}{b} + C(M) b |\log b| \lesssim \frac{1}{b},
\]
\[
\| \partial_y w^{\text{RR}} \|_{H^1_2} \lesssim \frac{1}{|\log b|} + C(M) b \lesssim \frac{1}{|\log b|}.
\]
Multiplying the above estimates by $|\lambda b| \lesssim \frac{b^3}{|\log b|}$ completes the proof of (4.20)-(4.27) for $\lambda b [\partial_y w^{\text{RR}}] (b; y)$.

It remains to show (4.20)-(4.27) for $\lambda(t) [\partial_t w^{\text{RR}}] (t, \lambda(t) y) = \tilde{\epsilon} (\tau, y)$ in place of $e^{\text{RR}}$. Note that (4.25) for $\tilde{\epsilon}$ is immediate from (4.15). For the proof of (4.20)-(4.27) for $\tilde{\epsilon}$, we would like to use the coercivity estimates (4.20) and (4.22) as in the case of $e^{\text{RR}}$, but the orthogonality condition (4.2) for $\tilde{\epsilon}$ does not hold. However, the orthogonality condition is almost satisfied in the sense that
\[
\langle \tilde{\epsilon}, \chi_M \Lambda Q \rangle = \langle \partial_t w^{\text{RR}}, \frac{1}{\chi_M \Lambda Q} \rangle \lesssim_M \frac{b^3}{|\log b|},
\]
where in the last inequality we used (4.27) for $e^{\text{RR}}$. Thus (4.20) and (4.22) become
\[
\| \tilde{\epsilon} \|_{H^1_2} \lesssim_M \| A \tilde{\epsilon} \|_{L^2} + |\tilde{\epsilon}, \chi_M \Lambda Q| \lesssim_M \| A \tilde{\epsilon} \|_{L^2} + \frac{b^3}{|\log b|},
\]
\[
\| \tilde{\epsilon} \|_{(\tilde{H}^1_2)_N} \lesssim_M 1_{(0,R)} \| A \tilde{\epsilon} \|_{L^2} + |\tilde{\epsilon}, \chi_M \Lambda Q| \lesssim_M 1_{(0,R)} \| A \tilde{\epsilon} \|_{L^2} + \frac{b^3}{|\log b|}.
\]
Next, we use the identity
\[ A_\lambda \partial_t w = \partial_t W + [A_\lambda, \partial_t] w = \partial_t W - \frac{b}{\lambda^2} \left( \frac{4y}{(1 + y^2)^2} \right) w \]
to have
\[ ||A\bar{c}||_{L^2} = \lambda ||A_\lambda \partial_t w||_{L^2} \lesssim \lambda ||\partial_t W||_{L^2} + \frac{4y}{(1 + y^2)^2} \epsilon^{RR} \]
\[ ||1_{(0, R)} A\bar{c}||_{L^2} = \lambda ||1_{(0, \lambda R)} A_\lambda \partial_t w||_{L^2} \lesssim \lambda ||1_{(0, \lambda R)} \partial_t W||_{L^2} + \frac{4y}{(1 + y^2)^2} \epsilon^{RR} \]
Applying (4.16)–(4.17) for \( \partial_t W \) and (4.20)–(4.27) for \( \epsilon^{RR} \), we obtain
\[ ||A\bar{c}||_{L^2} \lesssim b^2 \text{ and } ||1_{(0, R)} A\bar{c}||_{L^2} \lesssim \frac{b^2}{|\log b|} \]
Therefore, we conclude that
\[ ||\bar{c}||_{H^1_t} \lesssim_M ||A\bar{c}||_{L^2} + \frac{b^3}{|\log b|} \lesssim_M b^2, \]
\[ ||\bar{c}||_{H^1_t} \lesssim ||1_{(0, R)} A\bar{c}||_{L^2} + \frac{b^3}{|\log b|} \lesssim_M b^2 \]
This completes the proof of (4.20)–(4.27) for \( \bar{c} \). The proof is complete. \( \square \)

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