Boundary motive, relative motives and extensions of motives

by

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May 5, 2011

Abstract

We explain the role of the boundary motive in the construction of certain Chow motives, and of extensions of Chow motives. Our two main examples concern proper, singular surfaces and fibre products of a universal elliptic curve.

Keywords: weight structures, boundary motive, relative motives, intersection motive, interior motive, extensions of motives.

Math. Subj. Class. (2010) numbers: 19E15 (14F25, 14F42, 14G35).

*Partially supported by the Agence Nationale de la Recherche, project no. ANR-07-BLAN-0142 “Méthodes à la Voevodsky, motifs mixtes et Géométrie d’Arakelov”.
0 Introduction

This article contains largely extended notes of a short series of lectures delivered during the Ecole d'été franco-asiatique "Autour des motifs", which took place at the IHES in July 2006. The task which I was assigned was to explain the role of the boundary motive, and I hope that the present article will make a modest contribution to this effect.

By definition [W1], the boundary motive $\partial M(X)$ of a variety $X$ over a perfect field $k$ fits into a canonical exact triangle

\[
\begin{align*}
\partial M(X) & \longrightarrow M(X) \longrightarrow M^c(X) \longrightarrow \partial M(X) \Rightarrow 1
\end{align*}
\]

in the category $DM^{eff}_{gm}(k)$ of effective geometrical motives. This triangle establishes the relation of the boundary motive to $M(X)$ and $M^c(X)$, the motive of $X$ and its motive with compact support, respectively [V1].

One way to explain its interest is to start with the notion of extensions. Indeed, most of the existing attempts to prove the Beilinson or Bloch–Kato conjectures on special values of $L$-functions necessitate the construction of extensions of (Chow) motives, and the explicit control of their realizations (Betti, de Rham, étale...). Often, the source of these extensions is localization, which expresses the motive with compact support of a non-compact variety $X$ as an extension of the motive of a compactification $X^*$ by the motive of the complement $X^* - X$. The realizations of these extensions then correspond to cohomology with compact support of $X$. This approach is clearly present e.g. in Harder’s work on special values [H].

Thus, given two Chow motives, one may try to use localization to construct an extension of one by the other. Here, we base ourselves on the principle that the given Chow motives are “basic”, and that the extension
is “difficult” to obtain. But one may also invert the logic: given a “mixed” motive, try to use localization to construct the Chow motives used to build it up; let us refer to this problem as “resolution of extensions”.

The purpose of this article is to establish that the boundary motive plays a role both for the construction and for the resolution of extensions via localization. In Section 1, we start by making precise the relation between localization and the boundary motive. In fact, the triangle \((\ast)\) turns out to be obtained by “splicing” the localization triangle and its dual. We chose to discuss this relation first in the Hodge theoretic realization, and in the special case of a complement \(X\) of two points in an elliptic curve over \(\mathbb{C}\) (Examples 1.1, 1.3 and 1.5), and deduce from that discussion the general picture in Hodge theory (Theorems 1.6 and 1.7), concerning compactifications of a fixed variety \(X\) over \(\mathbb{C}\). We observe in particular (Corollary 1.8) that when \(X\) is smooth, then any smooth compactification induces a weight filtration on the boundary cohomology of \(X\), i.e., on the Hodge realization of the boundary motive.

In order to formulate the motivic analogues of these results, we need the right notion of weights for motives. It turns out that this notion is given by weight structures, as recently introduced and studied by Bondarko [Bo2]. We review the definition, and the basic properties of weight structures, including their application to motives (Theorem 1.11): according to Bondarko, there is a canonical such structure on the triangulated category \(DM_{gm}^{eff}(k)\), and its heart equals the category \(CHM_{eff}(k)\) of effective Chow motives. The motivic analogue of Corollary 1.8 holds: according to Corollary 1.16, any smooth compactification of a fixed variety \(X\) which is smooth over \(k\) induces a weight filtration on \(\partial M(X)\).

Then we try to invert this process (hoping for this inversion to allow us to resolve extensions). The precise statement is given in Theorem 1.18 which states that for fixed \(X\), there is a canonical bijective correspondence (discussed at length in Construction 1.17) between isomorphism classes of two types of objects: (1) weight filtrations on \(\partial M(X)\), and (2) certain effective Chow motives \(M_0\) through which the morphism \(M(X) \to M^c(X)\) factors. An analogous statement (Variant 1.23) holds for direct factors of \(\partial M(X)\), \(M(X)\), and \(M^c(X)\), provided that they are images of an idempotent endomorphism of the whole exact triangle \((\ast)\). In this correspondence, the passage to isomorphism classes cannot be avoided because of the necessity to choose cones of certain morphisms in the triangulated category \(DM_{gm}^{eff}(k)\). This causes (at least) one important problem, namely the lack of functoriality of the representatives of the isomorphism classes. In order to obtain functoriality, Construction 1.17 thus needs to be rigidified.

In the rest of Section 1, we describe the approach from [W3] to rigidifi-
fication, hence functoriality. It is based on the notion of motives avoiding certain weights. If a direct factor \( \partial M(X)^e \) of \( \partial M(X) \) is without weights \(-1\) and \(0\), then an effective Chow motive \( M_0 \) is canonically and functorially defined (Complement 1.24). Given the nature of the realizations of \( M_0 \), it is natural to call it the \( e \)-part of the interior motive of \( X \). Its main properties are established in [W3, Sect. 4]. Note however (Problem 1.22) that the above condition on absence of weights is never satisfied for the whole of \( \partial M(X) \) unless \( \partial M(X) = 0 \). In order to make this approach work, we thus need an idempotent endomorphism \( e \) of the exact triangle (\*), giving rise to a direct factor

\[
\partial M(X)^e \rightarrow M(X)^e \rightarrow M^c(X)^e \rightarrow \partial M(X)^e[1].
\]

Section 2 shows how the theory of smooth relative Chow motives can be employed to construct endomorphisms of the exact triangle (\*). Fix a base scheme \( S \), which is smooth over \( k \). Theorem 2.2 establishes the existence of a functor from the category of smooth relative Chow motives over \( S \) to the category of exact triangles in \( DM^{\text{eff}}_{gm}(k) \). On objects, it is given by mapping a proper, smooth \( S \)-scheme \( X \) to the exact triangle

\[
\partial M(X) \rightarrow M(X) \rightarrow M^c(X) \rightarrow \partial M(X)[1].
\]

We should mention that as far as the \( M(X) \)-component is concerned, the functoriality statement from Theorem 2.2 is just a special feature of results by Dégilse [Dég2], Cisinski–Dégilse [CiDég] and Levine [L] (see Remarks 2.3 and 2.13 for details). However, the application of the results from [loc. cit.] to the functor \( \partial M \) is not obvious. This is one of the reasons why we follow an alternative approach. It is based on a relative version of moving cycles [W1, Thm. 6.14]. This also explains why we are forced to suppose the base field \( k \) to admit a strict version of resolution of singularities. Theorem 2.5 and Corollary 2.15 then analyze the behaviour of the functor from Theorem 2.2 under change of the base \( S \). Another reason for us to choose a cycle theoretic approach was that it becomes then easier to keep track of the correspondences on \( X \times_k X \) commuting with our constructions. Our main application (Example 2.16) thus concerns correspondences “of Hecke type” yielding endomorphisms of the exact triangle (\*).

In Section 3 we apply these principles to Abelian schemes. More precisely, the main result of [DeMu] on the Chow–Künneth decomposition of the relative motive of an Abelian scheme \( A \) over \( S \) (recalled in Theorem 3.1) yields canonical projectors in the relative Chow group. Given our analysis from Section 2 it follows that they act idempotently on the exact triangle

\[
\partial M(A) \rightarrow M(A) \rightarrow M^c(A) \rightarrow \partial M(A)[1].
\]

In Sections 4 and 5 we discuss two examples. Section 4 concerns normal, proper surfaces \( X^* \). We first recall the basic construction of the intersection motive \( M^*(X^*) \) of \( X^* \), following previous work of Cataldo and
Migliorini \[\text{CatMi}\], and review some of the material from \[\text{W5}\]. In particular (Proposition 4.3), we recall that \(M^\times(X^*)\) is co- and contravariantly functorial under finite morphisms of proper surfaces. We then analyze the precise relation to the weight filtration of the boundary motive of a dense, open subscheme \(X \subset X^*\), which is smooth over \(k\) (Theorem 4.4), following the lines of Construction 1.17. We finish the section with a discussion of the case of Baily–Borel compactifications of Hilbert surfaces. We recall, following \[\text{W5, Sect. 6 and 7}\], that localization allows to construct non-trivial extensions of a certain Artin motive by a direct factor of \(M^\times(X^*)\). Using Proposition 4.3, we then establish stability of \(M^\times(X^*)\) under the correspondences “of Hecke type” constructed in Example 2.16.

In Section 5, we discuss fibre products of the universal elliptic curve over the modular curve of level \(n \geq 3\). We review some of the material from \[\text{Sch}\] and \[\text{W3, Sect. 3 and 4}\]. Notably (Proposition 5.3), we recall that in this geometrical setting, the condition from Complement 1.24 on the absence of weights \(-1\) and \(0\) in the boundary motive is satisfied. Thus, the interior motive can be defined. The new ingredient is Example 5.4 where we use rigidity of our construction to give a proof “avoiding compactifications” of equivariance of the interior motive under the correspondences “of Hecke type”.

As mentioned above, this article is primarily intended to be a general introduction to the construction and to the applications of boundary motives. For many details of the proofs, we shall refer to our earlier articles \[\text{W1}\] and \[\text{W3}\]. Let us however indicate that various parts of this paper discuss original constructions. This is true in particular for Section 2 (on relative motives and functoriality), including our study of Hecke equivariance. We expect these constructions to be of interest in other contexts than those discussed in Sections 4 and 5.

For further developments of the theory of boundary motives and their applications to special classes of algebraic varieties and to their associated motives, in particular to the motives of Shimura varieties, we refer also to \[\text{W2}\], \[\text{W4}\].

Part of this work was done while I was enjoying a *modulation de service pour les porteurs de projets de recherche*, granted by the *Université Paris 13*, and during a stay at the *Universität Zürich*. I am grateful to both institutions. I wish to thank the organizers of *Autour des motifs* for the invitation to Bures-sur-Yvette, and J. Ayoub, F. Déglise, D. Hébert, B. Kahn, F. Lecomte and M. Levine for useful discussions and comments. Special thanks go to J.-B. Bost for insisting on this article to be written, and for his helpful suggestions to improve an earlier version.
Note that the notation and conventions are as follows: $k$ denotes a fixed perfect base field, $\text{Sch}/k$ the category of separated schemes of finite type over $k$, and $\text{Sm}/k \subset \text{Sch}/k$ the full sub-category of objects which are smooth over $k$. When we assume $k$ to admit resolution of singularities, then it will be in the sense of [FV, Def. 3.4]: (i) for any $X \in \text{Sch}/k$, there exists an abstract blow-up $Y \to X$ [FV, Def. 3.1] whose source $Y$ is in $\text{Sm}/k$, (ii) for any $X,Y \in \text{Sm}/k$, and any abstract blow-up $q : Y \to X$, there exists a sequence of blow-ups $p : X_n \to \ldots \to X_1 = X$ with smooth centers, such that $p$ factors through $q$. We say that $k$ admits strict resolution of singularities, if in (i), for any given dense open subset $U$ of the smooth locus of $X$, the blow-up $q : Y \to X$ can be chosen to be an isomorphism above $U$, and such that arbitrary intersections of the irreducible components of the complement $Z$ of $U$ in $Y$ are smooth (e.g., $Z \subset Y$ a normal crossing divisor with smooth irreducible components).

As far as motives are concerned, the notation of this paper follows that of [V1]. We refer to Levine’s lecture notes (this volume) for a review of this notation, and in particular, of the definition of the categories $\text{DM}^{\text{eff}}(k)$ and $\text{DM}^{\text{gm}}(k)$ of (effective) geometrical motives over $k$, and of the motive $M(X)$ and the motive with compact support $M_c(X)$ of $X \in \text{Sch}/k$. Let $F$ be a commutative flat $\mathbb{Z}$-algebra, i.e., a commutative unitary ring whose additive group is without torsion. The notation $\text{DM}^{\text{eff}}_{gm}(k)_F$ and $\text{DM}^{\text{gm}}_{gm}(k)_F$ stands for the $F$-linear analogues of $\text{DM}^{\text{eff}}_{gm}(k)$ and $\text{DM}^{\text{gm}}_{gm}(k)$ defined in [A, Sect. 16.2.4 and Sect. 17.1.3]. Similarly, let us denote by $\text{CHM}^{\text{eff}}(k)$ and $\text{CHM}(k)$ the categories opposite to the categories of (effective) Chow motives, and by $\text{CHM}^{\text{eff}}(k)_F$ and $\text{CHM}(k)_F$ the pseudo-Abelian completion of the category $\text{CHM}^{\text{eff}}(k) \otimes_{\mathbb{Z}} F$ and $\text{CHM}(k) \otimes_{\mathbb{Z}} F$, respectively. Using [V2, Cor. 2] ([V1, Cor. 4.2.6] if $k$ admits resolution of singularities), we canonically identify $\text{CHM}^{\text{eff}}(k)_{F}$ and $\text{CHM}(k)_{F}$ with a full additive sub-category of $\text{DM}^{\text{eff}}_{gm}(k)_{F}$ and $\text{DM}^{\text{gm}}_{gm}(k)_{F}$, respectively. Note in particular that with these conventions, $\text{CHM}(k)_{\mathbb{Q}}$ is actually opposite to the category denoted by the same symbol in [W5].

1 Motivation

Let us start by recalling the geometrical interpretation of (cup product with) the Chern class in a very special context.

Example 1.1. Let $E$ be an elliptic curve over the field $\mathbb{C}$ of complex numbers, and $P \in E(\mathbb{C})$ a point unequal to zero. Put $X := E - \{0, P\}$, and consider the complementary inclusions

$$X \xrightarrow{i} E \xleftarrow{j} \{0, P\}.$$
Let us prepare the reader that the following aspect of this situation will be generalized in the sequel: $E$ is a smooth compactification of $X$. The associated long exact localization sequence for (singular) cohomology with coefficients in $\mathbb{Q}$ reads as follows.

$$
0 \rightarrow H^0_c(X(\mathbb{C}), \mathbb{Q}) \rightarrow H^0(E(\mathbb{C}), \mathbb{Q}) \rightarrow H^0(\{0\}, \mathbb{Q}) \oplus H^0(\{P\}, \mathbb{Q}) \rightarrow H^1_c(X(\mathbb{C}), \mathbb{Q}) \rightarrow H^1(E(\mathbb{C}), \mathbb{Q}) \rightarrow H^2_c(X(\mathbb{C}), \mathbb{Q}) \rightarrow H^2(E(\mathbb{C}), \mathbb{Q}) \rightarrow 0
$$

It shows that $H^0_c(X(\mathbb{C}), \mathbb{Q}) = 0$, that $H^2_c(X(\mathbb{C}), \mathbb{Q}) \cong H^2(E(\mathbb{C}), \mathbb{Q})$, and, most interestingly, that $H^1_c(X(\mathbb{C}), \mathbb{Q})$ is a Yoneda one-extension of $H^1(E(\mathbb{C}), \mathbb{Q})$ by the cokernel of

$$i^*: H^0(E(\mathbb{C}), \mathbb{Q}) \rightarrow H^0(\{0\}, \mathbb{Q}) \oplus H^0(\{P\}, \mathbb{Q}) .$$

The Hodge structures on the three groups $H^0(E(\mathbb{C}), \mathbb{Q})$, $H^0(\{0\}, \mathbb{Q})$ and $H^0(\{P\}, \mathbb{Q})$ all equal $\mathbb{Q}(0)$, and under these identifications, $i^*$ corresponds to the diagonal embedding

$$\Delta: \mathbb{Q}(0) \hookrightarrow \mathbb{Q}(0) \oplus \mathbb{Q}(0) .$$

We identify its cokernel with $\mathbb{Q}(0)$ by choosing the class of $(-1, 1) \in H^0(\{0\}, \mathbb{Q}) \oplus H^0(\{P\}, \mathbb{Q})$ as its generator. The one-extension then takes the form

$$0 \rightarrow \mathbb{Q}(0) \rightarrow H^1_c(X(\mathbb{C}), \mathbb{Q}) \rightarrow H^1(E(\mathbb{C}), \mathbb{Q}) \rightarrow 0 .$$

Let us denote it by $\text{Ext}_P$. Defining $\text{Ext}_0$ to be the trivial extension, we thus get a map

$$\text{Ext}: E(\mathbb{C}) \rightarrow \text{Ext}^1(H^1(E(\mathbb{C}), \mathbb{Q}), \mathbb{Q}(0)) , \ P \mapsto \text{Ext}_P$$

$(\text{Ext}^1 :=$ the group of one-extensions in the category of mixed $\mathbb{Q}$-Hodge structures), which can be checked to be a morphism of groups.

An analogous construction is possible for $\ell$-adic cohomology and elliptic curves over a field of characteristic unequal to $\ell$. The one-extensions then take place in the category of modules over the absolute Galois group. Note that in the context considered in Example 1.1, the morphism $\text{Ext}$ induces an isomorphism

$$E(\mathbb{C}) \otimes_\mathbb{Z} \mathbb{Q} \cong \text{Ext}^1(H^1(E(\mathbb{C}), \mathbb{Q}), \mathbb{Q}(0)) .$$

Here is what we would like the reader to recall from the above.
**Principle 1.2.** *Localization potentially leads to interesting extensions of Hodge structures or Galois modules.*

Actually, Principle 1.2 admits a more general version, where we replace “localization” by “the formalism of six operations”. In the sequel of this article, we shall however concentrate on localization and its dual.

**Example 1.3.** We continue to consider the situation from Example 1.1. (a) The long exact sequence dual to the localization sequence associated to

\[ X \xrightarrow{\ i \ } E \xrightarrow{\ j \ } \{0, P\} \]

\( (X = E - \{0, P\} \) as before) will be referred to as the *co-localization sequence*. It shows that

\[ H^0(E(\mathbb{C}), \mathbb{Q}) \xrightarrow{\sim} H^0(X(\mathbb{C}), \mathbb{Q}) , \]

that \( H^2(X(\mathbb{C}), \mathbb{Q}) = 0 \), and that \( H^1(X(\mathbb{C}), \mathbb{Q}) \) is a Yoneda one-extension

\[ 0 \longrightarrow H^1(E(\mathbb{C}), \mathbb{Q}) \longrightarrow H^1(X(\mathbb{C}), \mathbb{Q}) \longrightarrow \mathbb{Q}(-1) \longrightarrow 0 \]

(with the identifications dual to the one used in Example 1.1).

(b) Let us now compare cohomology and cohomology with compact support of \( X \). This comparison is expressed by a third long exact sequence, which we shall refer to as the *boundary sequence*.

\[ H^0_c(X(\mathbb{C}), \mathbb{Q}) \longrightarrow H^0(X(\mathbb{C}), \mathbb{Q}) \longrightarrow \partial H^0(X(\mathbb{C}), \mathbb{Q}) \]

\[ \longrightarrow H^1_c(X(\mathbb{C}), \mathbb{Q}) \longrightarrow H^1(X(\mathbb{C}), \mathbb{Q}) \longrightarrow \partial H^1(X(\mathbb{C}), \mathbb{Q}) \]

\[ \longrightarrow H^2_c(X(\mathbb{C}), \mathbb{Q}) \longrightarrow H^2(X(\mathbb{C}), \mathbb{Q}) \]

The third term in this sequence is *boundary cohomology* of \( X \), defined as cohomology of \( E(\mathbb{C}) \) with coefficients in the complex \( i^* j_* \mathbb{Q}_X \). Here, the symbol \( j_* \mathbb{Q}_X \) denotes the total direct image of \( \mathbb{Q}_X \) under \( j \); following the convention used in [BBD], we drop the letter “\( R \)” from our notation. The morphism from \( H^1_c(X(\mathbb{C}), \mathbb{Q}) \) to \( H^1(X(\mathbb{C}), \mathbb{Q}) \) factors over \( H^1(E(\mathbb{C}), \mathbb{Q}) \). Therefore, by what was said before, we see that the boundary sequence is obtained by splicing the two sequences

\[ 0 \longrightarrow H^0_c(X(\mathbb{C}), \mathbb{Q}) \longrightarrow H^0(X(\mathbb{C}), \mathbb{Q}) \longrightarrow \partial H^0(X(\mathbb{C}), \mathbb{Q}) \]

\[ \longrightarrow H^1_c(X(\mathbb{C}), \mathbb{Q}) \longrightarrow H^1(E(\mathbb{C}), \mathbb{Q}) \longrightarrow \partial H^1(X(\mathbb{C}), \mathbb{Q}) \]

and

\[ 0 \longrightarrow H^1(E(\mathbb{C}), \mathbb{Q}) \longrightarrow H^1(X(\mathbb{C}), \mathbb{Q}) \longrightarrow \partial H^1(X(\mathbb{C}), \mathbb{Q}) \]

\[ \longrightarrow H^2_c(X(\mathbb{C}), \mathbb{Q}) \longrightarrow H^2(X(\mathbb{C}), \mathbb{Q}) \longrightarrow 0 . \]
This allows us in particular to identify boundary cohomology:

$$\partial H^0(X(\mathbb{C}), \mathbb{Q}) \cong \mathbb{Q}(0)^2, \quad \partial H^1(X(\mathbb{C}), \mathbb{Q}) \cong \mathbb{Q}(-1)^2.$$  

(c) We claim that the above “first half” of the boundary sequence

\[ \begin{array}{ccccccccc}
0 & \longrightarrow & H^0_\epsilon(X(\mathbb{C}), \mathbb{Q}) & \longrightarrow & H^0(X(\mathbb{C}), \mathbb{Q}) & \longrightarrow & \partial H^0(X(\mathbb{C}), \mathbb{Q}) & \longrightarrow & 0 \\
& & H^1_\epsilon(X(\mathbb{C}), \mathbb{Q}) & \longrightarrow & H^1(E(\mathbb{C}), \mathbb{Q}) & \longrightarrow & 0
\end{array} \]

equals the localization sequence

\[ \begin{array}{ccccccccc}
0 & \longrightarrow & H^0_\epsilon(X(\mathbb{C}), \mathbb{Q}) & \longrightarrow & H^0(E(\mathbb{C}), \mathbb{Q}) & \longrightarrow & H^0(\{0\}, \mathbb{Q}) \oplus H^0(\{P\}, \mathbb{Q}) & \longrightarrow & 0 \\
& & H^1_\epsilon(X(\mathbb{C}), \mathbb{Q}) & \longrightarrow & H^1(E(\mathbb{C}), \mathbb{Q}) & \longrightarrow & 0
\end{array} \]

from Example 1.1. Indeed, the identification is achieved by the canonical isomorphism

$$H^0(\{0\}, \mathbb{Q}) \oplus H^0(\{P\}, \mathbb{Q}) \cong \partial H^0(X(\mathbb{C}), \mathbb{Q}),$$

induced by the adjunction $i^*\mathbb{Q}_E \rightarrow i^*j_*\mathbb{Q}_X$. A dual statement relates the co-localization sequence from (a) to the “second half” of the boundary sequence.

(d) Altogether, we see that the long exact boundary sequence allows us to recover cohomology of $E$, together with the localization and co-localization sequences. One “half” of boundary cohomology (namely $\partial H^0$) contributes to the localization sequence, the other “half” (namely $\partial H^1$) to the co-localization sequence.

Here is what we would like the reader to recall from the above.

**Principle 1.4.** The boundary sequence allows to recover cohomology of a smooth compactification of $X$, together with the localization and co-localization sequences.

A few precisions are necessary. First, given that $X$ is a curve, there is only one possible choice of smooth compactification (namely $E$). But this changes of course in higher dimensions. Second, the “recovery” of the localization and co-localization sequences from the boundary sequence seems to require a choice of additional data, namely a division of boundary cohomology into two “halves”. In order to address both points in a satisfactory manner (see Theorem 1.6 below), we need to formalize the problem.

Since we wish the discussion to apply to the triangulated category of motives, for which no $t$-structure is available at present, it is best placed in the context of triangulated categories. In the context we chose to discuss, namely that of Hodge theory, the appropriate triangulated category is the category of **algebraic** $\mathbb{Q}$-Hodge modules $\mathbb{Q}$. We should immediately reassure readers...
not familiar with this theory: for our purposes, only its formal properties (localization, purity, proper base change,...) will be needed. Therefore, in order to motivate what is to follow, we might just as well have placed ourselves in the context of ℓ-adic sheaves, which would allow to argue in a completely analogous fashion. Readers wishing nonetheless to have a survey on Hodge theory at their disposal might find it useful to consult [St].

**Example 1.5.** The relation to the geometric situation

\[ X \leftarrow \overset{j}{\rightarrow} E \leftarrow \overset{i}{\rightarrow} \{0, P\} \]

studied before is as follows. The localization sequence from Example 1.1 is the result of application of the cohomological functor \( H^*(E(\mathbb{C}), \bullet) \) to the exact localization triangle

\[
\begin{align*}
    j_! Q_X(0) & \longrightarrow Q_E(0) \longrightarrow i_* i^! Q_E(0) \longrightarrow j_! Q_X(0)[1]
\end{align*}
\]

of algebraic \( \mathbb{Q} \)-Hodge modules on \( E \) [Sa] (4.4.1). In the same way, the co-localization sequence from Example 1.3 (a) is induced by the exact co-localization triangle

\[
\begin{align*}
    i_* i^! Q_E(0) & \longrightarrow Q_E(0) \longrightarrow j_* Q_X(0) \longrightarrow i_* i^! Q_E(0)[1]
\end{align*}
\]

of Hodge modules on \( E \) [Sa] (4.4.1), using in addition that thanks to purity, we have a canonical identification

\[
i_* i^! Q_E(0) \cong Q_E(-1)[-2].
\]

Applying localization to the Hodge module \( j_* Q_X(0) \) (or equivalently, co-localization to \( j_! Q_X(0) \)), we obtain the exact boundary triangle

\[
\begin{align*}
    j_! Q_X(0) & \longrightarrow j_* Q_X(0) \longrightarrow i_* i^* j_* Q_X(0) \longrightarrow j_! Q_X(0)[1],
\end{align*}
\]

which induces the boundary sequence from Example 1.3 (b).

Note that the three triangles (localization, co-localization and boundary) exist for any pair of complementary immersions. The following results from Saito’s formalism of six operations on algebraic Hodge modules [Sa].

**Theorem 1.6.** Let

\[ X \leftarrow \overset{j}{\rightarrow} \overline{X} \leftarrow \overset{i}{\rightarrow} D \]

be complementary immersions (\( j \) open, \( i \) closed) of separated schemes of finite type over \( \mathbb{C} \).

(a) If \( \overline{X} \) is proper, then the result of applying \( H^*(\overline{X}(\mathbb{C}), \bullet) \) to the boundary triangle

\[
\begin{align*}
    j_! Q_X(0) & \longrightarrow j_* Q_X(0) \longrightarrow i_* i^* j_* Q_X(0) \longrightarrow j_! Q_X(0)[1],
\end{align*}
\]

does only depend on \( X \).
(b) The morphisms

\[ i_\ast i^! Q_X(0) \to i_\ast i^* Q_X(0) \]

and

\[ i_\ast i^* Q_X(0) \to i_\ast i^* j_\ast Q_X(0) \]

induced by the respective adjunctions fit into a canonical exact triangle

\[ i_\ast i^\! Q_X(0) \to i_\ast i^* Q_X(0) \to i_\ast i^* j_\ast Q_X(0) \to i_\ast j_\ast Q_X(0)[1] \].

It is the third column of a diagram of exact triangles

\[
\begin{array}{cccccc}
0 & \to & i_\ast i^! Q_X(0) & \to & i_\ast i^* Q_X(0) & \to & 0 \\
| & | & | & | & | & | \\
| & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & | \\
| & j_\ast Q_X(0) & \to & Q_X(0) & \to & i_\ast i^* Q_X(0) & j_\ast Q_X(0)[1] \\
| & | & | & | & | & | \\
| & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & | \\
| & j_\ast Q_X(0) & j_\ast Q_X(0) & \to & i_\ast i^* j_\ast Q_X(0) & j_\ast Q_X(0)[1] \\
| & | & | & | & | & | \\
| & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & | \\
| & 0 & \to & i_\ast i^! Q_X(0)[1] & \to & i_\ast i^* Q_X(0)[1] & \to & 0 \\
\end{array}
\]

whose second and third rows are the localization and boundary triangles, and whose second column is the co-localization triangle.

(c) If \( \overline{X} \) is smooth of constant dimension \( d \), then there is a canonical isomorphism

\[ i_\ast i^! Q_X(0) \cong \mathbb{D}_X(i_\ast i^* Q_X(d))[2d] \]

(\( \mathbb{D}_X := \text{duality for Hodge modules on } \overline{X} \) \([\text{Sa}, (4.1.5)]\)).

Proof. (a) is a consequence of proper base change \([\text{Sa}, (4.4.3)]\). As for (b), let us define the morphism

\[ i_\ast i^* j_\ast Q_X(0) \to i_\ast i^! Q_X(0)[1] \]

as \( i_\ast i^* \) of the morphism

\[ j_\ast Q_X(0) \to i_\ast i^! Q_X(0)[1] \]

occurring in the co-localization triangle. Together with the morphisms defined before, it yields the diagram of the statement. Exactness of its third column is then a consequence of exactness of the first and second column. Finally, part (c) results from duality \([\text{Sa}, (4.3.5)]\). \( \text{q.e.d.} \)

Fix \( X \in \text{Sch}/\mathbb{C} \). In the sequel, the boundary triangle

\[ j_\ast Q_X(0) \to j_\ast Q_X(0) \to i_\ast i^* j_\ast Q_X(0) \to j_\ast Q_X(0)[1] \]

will always be assumed to be formed using a compactification \( j : X \hookrightarrow \overline{X} \) of \( X \), with complement \( i : D \hookrightarrow \overline{X} \). Theorem \([\text{Lb}]\) gives the precise relation between the boundary triangle on the one hand, and the localization and co-localization triangles on the other hand. While boundary cohomology, i.e.,
cohomology of $i_* i^* j_* \mathbb{Q}_X(0)$ does not depend on $\overline{X}$, cohomology of the two other terms of the triangle

$$i_* i^! \mathbb{Q}_{\overline{X}}(0) \rightarrow i_* i^* \mathbb{Q}_{\overline{X}}(0) \rightarrow i_* i^* j_* \mathbb{Q}_X(0) \rightarrow i_* i^! \mathbb{Q}_{\overline{X}}(0)[1].$$

from Theorem [1.6] (b) in general does (unless $X$ is itself proper). Saito’s formalism allows to put restrictions on the Hodge structures potentially occurring as such cohomology groups.

**Theorem 1.7.** Let $n$ be an integer. Assume $\overline{X}$ to be proper and smooth (hence $X$ is smooth).

(a) The Hodge structure $H^n(\overline{X}, i_* i^* \mathbb{Q}_{\overline{X}}(0))$ is of weights at most $n$.

(b) The Hodge structure $H^n(\overline{X}, i_* i^! \mathbb{Q}_{\overline{X}}(0)[1])$ is of weights at least $n + 1$.

**Proof.** The scheme $\overline{X}$ being proper, $H^n$ maps complexes of Hodge modules of weights $\leq 0$ to Hodge structures of weights $\leq n$, and complexes of Hodge modules of weights $\geq 1$ to Hodge structures of weights $\geq n + 1$. We thus need to show that $i_* i^* \mathbb{Q}_{\overline{X}}(0)$ is of weights $\leq 0$, and $i_* i^! \mathbb{Q}_{\overline{X}}(0)$ of weights $\geq 0$.

The scheme $\overline{X}$ being smooth, $\mathbb{Q}_{\overline{X}}(0)$ is pure of weight 0. Therefore [Sa, (4.5.2)], $i^* \mathbb{Q}_{\overline{X}}(0)$ is of weights $\leq 0$, and $i^! \mathbb{Q}_{\overline{X}}(0)$ of weights $\geq 0$, and the same remains true after application of the functor $i_*$.

q.e.d.

In “triangulated” language, Theorem [1.7] says that the objects

$$\overline{\pi}_*(i_* i^* \mathbb{Q}_{\overline{X}}(0)) \quad \text{and} \quad \overline{\pi}_*(i_* i^! \mathbb{Q}_{\overline{X}}(0)[1])$$

($\overline{\pi} :=$ the structure morphism of $\overline{X}$) of the derived category of Hodge structures are of weights $\leq 0$ and $\geq 1$, respectively, when the compactification $\overline{X}$ is smooth.

**Corollary 1.8.** Let $X \in Sm/\mathbb{C}$, and

$$\ldots \rightarrow A^n \rightarrow \partial H^n(X(\mathbb{C}), \mathbb{Q}) \rightarrow B^n \rightarrow A^{n+1} \rightarrow \ldots$$

a long exact sequence of mixed $\mathbb{Q}$-Hodge structures. This sequence is the result of applying $H^*(\overline{X}(\mathbb{C}), \bullet)$ to the triangle

$$i_* i^! \mathbb{Q}_{\overline{X}}(0) \rightarrow i_* i^* \mathbb{Q}_{\overline{X}}(0) \rightarrow i_* i^* j_* \mathbb{Q}_X(0) \rightarrow i_* i^! \mathbb{Q}_{\overline{X}}(0)[1],$$

for a suitable smooth compactification $j : X \hookrightarrow \overline{X}$, only if $A^n$ is of weights at most $n$, and $B^n$ is of weights at least $n + 1$, for all $n \in \mathbb{Z}$.

Theorems [1.6] and [1.7] and Corollary [1.8] admit motivic analogues (Theorems [1.13] and [1.15], and Corollary [1.16] below), which we shall develop now. To do so, it is necessary to use the right notion of weights on triangulated categories. Let us recall the following definitions and results of Bondarko [Bo2].
Definition 1.9. Let $\mathcal{C}$ be a triangulated category. A weight structure on $\mathcal{C}$ is a pair $w = (\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0})$ of full sub-categories of $\mathcal{C}$, such that, putting
\[ \mathcal{C}_{w_{\leq n}} := \mathcal{C}_{w_{\leq 0}}[n] \quad \text{and} \quad \mathcal{C}_{w_{\geq n}} := \mathcal{C}_{w_{\geq 0}}[n] \quad \forall \ n \in \mathbb{Z}, \]
the following conditions are satisfied.

1. The categories $\mathcal{C}_{w_{\leq 0}}$ and $\mathcal{C}_{w_{\geq 0}}$ are Karoubi-closed: for any object $M$ of $\mathcal{C}_{w_{\leq 0}}$ or $\mathcal{C}_{w_{\geq 0}}$, any direct summand of $M$ formed in $\mathcal{C}$ is an object of $\mathcal{C}_{w_{\leq 0}}$ or $\mathcal{C}_{w_{\geq 0}}$, respectively.

2. (Semi-invariance with respect to shifts.) We have the inclusions
\[ \mathcal{C}_{w_{\leq 0}} \subset \mathcal{C}_{w_{\leq 1}} \quad \text{and} \quad \mathcal{C}_{w_{\geq 0}} \supset \mathcal{C}_{w_{\geq 1}} \]
of full sub-categories of $\mathcal{C}$.

3. (Orthogonality.) For any pair of objects $M \in \mathcal{C}_{w_{\leq 0}}$ and $N \in \mathcal{C}_{w_{\geq 1}}$, we have
\[ \text{Hom}_{\mathcal{C}}(M, N) = 0. \]

4. (Weight filtration.) For any object $M \in \mathcal{C}$, there exists an exact triangle
\[ A \rightarrow M \rightarrow B \rightarrow A[1] \]
in $\mathcal{C}$, such that $A \in \mathcal{C}_{w_{\leq 0}}$ and $B \in \mathcal{C}_{w_{\geq 1}}$.

By condition 1.9 (2),
\[ \mathcal{C}_{w_{\leq n}} \subset \mathcal{C}_{w_{\leq 0}} \]
for negative $n$, and
\[ \mathcal{C}_{w_{\geq n}} \subset \mathcal{C}_{w_{\geq 0}} \]
for positive $n$. There are obvious analogues of the other conditions for all the categories $\mathcal{C}_{w_{\leq n}}$ and $\mathcal{C}_{w_{\geq n}}$. In particular, they are all Karoubi-closed, and any object $M \in \mathcal{C}$ is part of an exact triangle
\[ A \rightarrow M \rightarrow B \rightarrow A[1] \]
in $\mathcal{C}$, such that $A \in \mathcal{C}_{w_{\leq n}}$ and $B \in \mathcal{C}_{w_{\geq n+1}}$. By a slight generalization of the terminology introduced in condition 1.9 (4), we shall refer to any such exact triangle as a weight filtration of $M$.

Remark 1.10. (a) Our convention concerning the sign of the weight is actually opposite to the one from [Bo2, Def. 1.1.1], i.e., we exchanged the roles of $\mathcal{C}_{w_{\leq 0}}$ and $\mathcal{C}_{w_{\geq 0}}$.

(b) Note that in condition 1.9 (4), “the” weight filtration is not assumed to be unique.

As observed by Bondarko, weight structures are relevant to motives thanks to the following result.
Theorem 1.11. Let $F$ be a commutative flat $\mathbb{Z}$-algebra, and assume $k$ to admit resolution of singularities.

(a) There is a canonical weight structure on the category $DM_{gm}^{\text{eff}}(k_F)$. It is uniquely characterized by the requirement that its heart equal $\text{CHM}_{gm}^{\text{eff}}(k_F)$.

(b) There is a canonical weight structure on the category $DM_{gm}(k_F)$ extending the weight structure from (a). It is uniquely characterized by the requirement that its heart equal $\text{CHM}(k_F)$.

(c) Statements (a) and (b) hold without assuming resolution of singularities provided $F$ is a $\mathbb{Q}$-algebra.

Proof. For $F = \mathbb{Z}$ and $k$ of characteristic zero, this is the content of [Bo2, Sect. 6.5 and 6.6]. For the modifications of the proof in the remaining cases, see [W3, Thm. 1.13]. q.e.d.

The following result is formally implied by Theorem 1.11 and the fundamental properties of the category $DM_{gm}(k_F)$, notably localization and duality [VI] Prop. 4.1.5 and Thm. 4.3.7]. For details of the proof, we refer to [W3, Cor. 1.14] ([Bo1, Thm. 6.2.1 (1) and (2)] if $k$ is of characteristic zero).

Corollary 1.12. Let $X \in \text{Sch}/k$ be of (Krull) dimension $d$. Assume $k$ to admit resolution of singularities.

(a) The motive with compact support $M^c(X)$ lies in $DM_{gm}^{\text{eff}}(k)_{w \geq 0} \cap DM_{gm}^{\text{eff}}(k)_{w \leq d}$.

(b) If $X \in \text{Sm}/k$, then the motive $M(X)$ lies in $DM_{gm}^{\text{eff}}(k)_{w \geq -d} \cap DM_{gm}^{\text{eff}}(k)_{w \leq 0}$.

Fix $X \in \text{Sch}/k$. The motivic analogue of (the complex computing) boundary cohomology (for $k = \mathbb{C}$) is given by the boundary motive $\partial M(X)$ of $X$ [VI] Def. 2.1]. The analogue of Theorem 1.11 reads as follows: there as in the sequel, we shall denote by $M^*$ the dual of a geometrical motive $M$ [VI] Thm. 4.3.7].

Theorem 1.13. (a) There is a canonical exact boundary triangle $\partial M(X) \rightarrow M(X) \rightarrow M^c(X) \rightarrow \partial M(X)[1]$ in $DM_{gm}^{\text{eff}}(k)$.

(b) Assume $k$ to admit resolution of singularities. Let $X \xleftarrow{j} \overline{X} \xrightarrow{i} D$ be complementary immersions ($j$ open, $i$ closed) of schemes in $\text{Sch}/k$. Assume $\overline{X}$ to be proper. There is a canonical morphism $\alpha : \partial M(X) \rightarrow M(D)$. Define $M(\overline{X}/X)$ as the relative motive of $\overline{X}$ modulo $X$ [VI] Conv. 1.2], and
let $\beta : M(D) \to M(\overline{X}/X)$ be induced by the morphism $i_* : M(D) \to M(\overline{X})$. Then the morphisms $\alpha$ and $\beta$ fit into a canonical exact triangle

$$M(\overline{X}/X)[-1] \to \partial M(X) \xrightarrow{\alpha} M(D) \xrightarrow{\beta} M(\overline{X}/X).$$

It is the third column of a canonical diagram of exact triangles

$\xymatrix@C=10pt{ & M(\overline{X}/X) \ar[dl] \ar[dr] & M(\overline{X}/X) \ar[dl] \ar[dr] & 0 \ar[d] \cr 0 \ar[r] & M^c(\overline{X}) \ar[r] & M(\overline{X}) \ar[r] & M(D) \ar[r] & M^c(X)[-1] \ar[u] \cr M^c(X) \ar[r] & M(\overline{X}) \ar[r] & M(D) \ar[r] & M^c(X)[-1] \ar[u] \cr 0 \ar[r] & M(\overline{X}/X)[-1] \ar[r] & M(\overline{X}/X)[-1] \ar[r] & 0 \ar[u] }$

whose second and third row are the localization [VI Prop. 4.1.5] and boundary triangles, and whose second column is canonically associated to the relative motive $M(\overline{X}/X)$.

(c) In the situation of (b), assume $\overline{X}$ to be proper and smooth of constant dimension $d$ (hence $X$ is smooth). There is a canonical morphism $\alpha^* : M(D)^*(d)[2d-1] \to \partial M(X)$. Let $\gamma := i^* i_* : M(D) \to M(D)^*(d)[2d]$ be the composition of $i_* : M(D) \to M(\overline{X})$ and its dual. Then the morphisms $\alpha^*, \alpha$ (from (b)) and $\gamma$ form a canonical exact triangle

$$M(D)^*(d)[2d-1] \xrightarrow{\alpha^*} \partial M(X) \xrightarrow{\alpha} M(D) \xrightarrow{\gamma} M(D)^*(d)[2d].$$

It is the third column of a second canonical diagram of exact triangles

$\xymatrix@C=10pt{ & M(D)^*(d)[2d] \ar[dl] \ar[dr] & M(D)^*(d)[2d] \ar[dl] \ar[dr] & 0 \ar[d] \cr 0 \ar[r] & M^c(D) \ar[r] & M(D) \ar[r] & M^c(X)[-1] \ar[u] \cr M^c(X) \ar[r] & M(\overline{X}) \ar[r] & M(D) \ar[r] & M^c(X)[-1] \ar[u] \cr 0 \ar[r] & M(D)^*(d)[2d-1] \ar[r] & M(D)^*(d)[2d-1] \ar[r] & 0 \ar[u] }$

whose second and third row are the localization and boundary triangles, and whose second column is dual, up to a twist by $(d)$ and a shift by $[2d]$, to the localization triangle (it will be referred to as the co-localization triangle). This diagram is isomorphic to the diagram from (b).

**Proof.** For (a), let us briefly recall the definition of $\partial M(X)$. First, according to [VI pp. 223, 224], a monomorphism of Nisnevich sheaves with transfers $i:X \hookrightarrow L(X)$ is associated to $X$: the sheaf $L(X)$ is formed
using finite correspondences, and $L^c(X)$ is formed using quasi-finite correspondences. Next [VI] pp. 207, 208, there is a functor $RC$ associating to a Nisnevich sheaf with transfers its singular simplicial complex. Voevodsky goes on to define the motive $M(X)$ as $RC(L(X))$, and the motive with compact support $M^c(X)$ as $RC(L^c(X))$. Set
\[ \partial M(X) := RC(coker \iota_X)[-1] \]
[VI, Def. 2.1]. Claim (a) is then a direct consequence of this definition. As for (b), we refer to [VI, Prop. 2.4]. It remains to show part (c). The morphism $\alpha^*$ is defined as the dual of $\alpha$, twisted by $(d)$ and shifted by $[2d-1]
M(D)^*(d)[2d-1] \rightarrow \partial M(X)^*(d)[2d-1],$
followed by the auto-duality isomorphism
\[ \partial M(X)^*(d)[2d-1] \isom \partial M(X) \]
[VI, Thm. 6.1]. Note that by duality [VI, Thm. 4.3.7 3],
\[ M(X)^*(d)[2d] \isom M(X) \quad M(X)^*(d)[2d] \isom M^c(X) \]
canonically, and under these identifications, the dual of the canonical morphism $M(X) \rightarrow M^c(X)$ occurring in the localization triangle equals the canonical morphism $M(X) \rightarrow M(X)$ occurring in the co-localization triangle. It remains to show that the composition
\[ M(D)^*(d)[2d-1] \xrightarrow{\alpha^*} \partial M(X) \rightarrow M(X) \]
equals the morphism
\[ M(D)^*(d)[2d-1] \rightarrow M(X) \]
in the co-localization sequence. But this identity can be checked after applying duality. Note that the boundary triangle is auto-dual [VI, Thm. 6.1]. Therefore, the dual of the above composition equals
\[ M^c(X) \rightarrow \partial M(X) \xrightarrow{\alpha} M(D), \]
which in turn equals the morphism
\[ M^c(X) \rightarrow M(D) \]
in the localization sequence.

Recall that motives à la Voevodsky behave homologically; this is why the sense of the arrows is inversed when compared to cohomology.

**Remark 1.14.** If $\overline{X}$ is proper and smooth of constant dimension, there should be a canonical choice of isomorphism between the two canonical diagrams from Theorem 1.13 (b) and (c). If $D$ is (proper and) smooth, then such a choice is induced by purity [VI, Prop. 3.5.4].

Here is the motivic analogue of Theorem 1.7; it follows directly from Corollary 1.12.
Theorem 1.15. Assume $k$ to admit resolution of singularities. Let 

$$X \xrightarrow{j} \overline{X} \xrightarrow{i} D$$

be complementary immersions ($j$ open, $i$ closed) of schemes in $\text{Sch}/k$. Assume $\overline{X}$ to be proper and smooth (hence $X$ is smooth).

(a) The motive $M(D)$ lies in $\text{DM}_{\text{eff}}^{\text{gm}}(k)_{w \geq 0}$.

(b) The motive $M(D)^*(d)[2d - 1]$ lies in $\text{DM}_{\text{eff}}^{\text{gm}}(k)_{w \leq -1}$.

In particular, the exact triangle

$$M(D)^*(d)[2d - 1] \xrightarrow{\alpha^*} \partial M(X) \xrightarrow{\alpha} M(D) \xrightarrow{\gamma} M(D)^*(d)[2d] .$$

from Theorem 1.13 (c) is then a weight filtration of $\partial M(X)$.

Corollary 1.16. Assume $k$ to admit resolution of singularities. Let

$$A \longrightarrow \partial M(X) \longrightarrow B \longrightarrow A[1]$$

be an exact triangle in $\text{DM}_{\text{eff}}^{\text{gm}}(k)$, for $X \in \text{Sm}/k$. This triangle is isomorphic to the triangle

$$M(D)^*(d)[2d - 1] \xrightarrow{\alpha^*} \partial M(X) \xrightarrow{\alpha} M(D) \xrightarrow{\gamma} M(D)^*(d)[2d] .$$

for a suitable smooth compactification $j : X \hookrightarrow \overline{X}$, only if it is a weight filtration of $\partial M(X)$: $A \in \text{DM}_{\text{eff}}^{\text{gm}}(k)_{w \leq -1}$ and $B \in \text{DM}_{\text{eff}}^{\text{gm}}(k)_{w \geq 0}$.

Altogether, for fixed $X \in \text{Sm}/k$, we get a functor from the category of smooth compactifications of $X$ to the category of weight filtrations of $\partial M(X)$. It turns out to be very instructive to see what one gets when trying to invert this functor.

Construction 1.17. Assume $k$ to admit resolution of singularities. Fix a weight filtration

$$\partial M(X)_{\leq -1} \xrightarrow{e^c} \partial M(X) \xrightarrow{e^c} \partial M(X)_{\geq 0} \xrightarrow{\delta} \partial M(X)_{\leq -1}[1]$$

of $\partial M(X)$, for $X \in \text{Sm}/k$:

$$\partial M(X)_{\leq -1} \in \text{DM}_{\text{eff}}^{\text{gm}}(k)_{w \leq -1} \quad \text{and} \quad \partial M(X)_{\geq 0} \in \text{DM}_{\text{eff}}^{\text{gm}}(k)_{w \geq 0} .$$

Consider the boundary triangle

$$\partial M(X) \xrightarrow{v} M(X) \xrightarrow{u} M^c(X) \xrightarrow{v^c} \partial M(X)[1] .$$

According to axiom TR4' of triangulated categories (see [BB]) Sect. 1.1.6]
for an equivalent formulation), the diagram of exact triangles

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \partial M(X)_{\leq -1} & \rightarrow & \partial M(X)_{\leq -1} & \rightarrow & 0 \\
\downarrow & & \downarrow \delta & & \downarrow \delta & & \downarrow \\
M^c(X) & \rightarrow & \partial M(X)_{\geq 0} & \rightarrow & \partial M(X)_{\geq 0} & \rightarrow & M^c(X)[-1] \\
\uparrow v_- & & \uparrow c_+ & & \uparrow c_+ & & \uparrow \partial M(X)_{\leq -1} \\
M^c(X) & \rightarrow & M(X) & \rightarrow & M(X) & \rightarrow & M^c(X)[-1] \\
\uparrow v_- & & \uparrow \partial M(X)_{\leq -1} & & \uparrow \partial M(X)_{\leq -1} & & \uparrow \partial M(X)_{\leq -1} \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

can be completed to give

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \partial M(X)_{\leq -1} & \rightarrow & \partial M(X)_{\leq -1} & \rightarrow & 0 \\
\downarrow & & \downarrow \delta & & \downarrow \delta & & \downarrow \\
M^c(X) & \rightarrow & i_0 & \rightarrow & M_0 & \rightarrow & \partial M(X)_{\geq 0} & \rightarrow & \partial M(X)_{\geq 0} & \rightarrow & M^c(X)[-1] \\
\uparrow \delta_+ & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \uparrow \partial M(X)_{\leq -1} \\
M^c(X) & \rightarrow & M(X) & \rightarrow & M(X) & \rightarrow & M^c(X)[-1] \\
\uparrow v_- & & \uparrow \partial M(X)_{\leq -1} & & \uparrow \partial M(X)_{\leq -1} & & \uparrow \partial M(X)_{\leq -1} & & \uparrow \partial M(X)_{\leq -1} \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

Note that this completion necessitates choices of \( M_0 \) and of factorizations \( u = i_0 \pi_0 \) and \( \delta = \delta_- \delta_+ \). In general, the object \( M_0 \) is unique up to possibly non-unique isomorphism; it is this problem that will be addressed in the last part of this section.

For the moment, note that whatever choice we make, \( M_0 \) will be in the heart of our weight structure: indeed, the second row of the diagram, together with Corollary 1.12 (a) shows that \( M_0 \in \text{DM}^{eff}_{gm}(k)_{w \geq 0} \), and the second column, together with Corollary 1.12 (b) shows that \( M_0 \in \text{DM}^{eff}_{gm}(k)_{w \leq 0} \). According to Theorem 1.11 (a), it is therefore an effective Chow motive. Note that it comes equipped with a factorization

\[
M(X) \xrightarrow{\pi_0} M_0 \xrightarrow{i_0} M^c(X)
\]

of the canonical morphism \( u : M(X) \rightarrow M^c(X) \), and that the triangles

\[
\begin{array}{c}
\partial M(X)_{\geq 0} \xrightarrow{\delta_+} M_0 \xrightarrow{i_0} M^c(X) \xrightarrow{(c_+)[1]} \partial M(X)_{\geq 0}[1] \\
\partial M(X)_{\leq -1} \xrightarrow{v_- c_-} M(X) \xrightarrow{\pi_0} M_0 \xrightarrow{\delta} \partial M(X)_{\leq -1}[1]
\end{array}
\]

are weight filtrations of \( M^c(X) \) and of \( M(X) \), respectively.

Let us summarize the discussion.
Theorem 1.18. Assume $k$ to admit resolution of singularities, and fix $X \in \text{Sm}_k$. The map

$$\left\{ (\partial M(X)_{\leq -1}, \partial M(X)_{\geq 0}) \right\}/\cong \rightarrow \left\{ (M_0, \pi_0, i_0) \right\}/\cong$$

from the preceding construction is a bijection between

1. the isomorphism classes of weight filtrations of the boundary motive $\partial M(X)$,
2. the isomorphism classes of effective Chow motives $M_0$, together with a factorization

$$M(X) \xrightarrow{\pi_0} M_0 \xrightarrow{i_0} M^c(X)$$

of the canonical morphism $u : M(X) \rightarrow M^c(X)$, such that both $i_0$ and $\pi_0$ can be completed to give weight filtrations of $M^c(X)$ and of $M(X)$, respectively.

There are obvious $F$-linear versions of Theorem 1.18 for any commutative flat $\mathbb{Z}$-algebra $F$. Recall that we started off with special choices of Chow motives factorizing $u$, namely the motives of smooth compactifications of $X$. But Theorem 1.18 should yield more general Chow motives $M_0$. For example, one might hope for the motivic version of intersection cohomology of a singular compactification of $X$ to occur. For surfaces, this will be spelled out in Section 4.

Note that we are forced to pass to the level of isomorphism classes because of the choices made in Construction 1.17. One important problem caused by this is the lack of functoriality. Thus, an endomorphism of a given weight filtration

$$\partial M(X)_{\leq -1} \rightarrow \partial M(X) \rightarrow \partial M(X)_{\geq 0} \rightarrow \partial M(X)_{\leq -1}[1]$$

will in general not yield an endomorphism of any of the Chow motives $M_0$ representing the associated isomorphism class.

Principle 1.19. In order to obtain functoriality, Construction 1.17 needs to be rigidified.

It turns out that an ad hoc geometrical method suffices to achieve rigidification in the setting of surfaces (see Section 4). Let us finish this section by describing another method (namely, that of [W3]), based again on the formalism of weights. It will be illustrated in the setting of self-products of the universal elliptic curve over a modular curve (see Section 5).

Remark 1.20. Getting back to Construction 1.17 and starting again with a weight filtration

$$\partial M(X)_{\leq -1} \xrightarrow{c} \partial M(X) \xrightarrow{c^+} \partial M(X)_{\geq 0} \xrightarrow{\delta} \partial M(X)_{\leq -1}[1],$$
let us see what obstacles there are for the triple \((M_0, \pi_0, i_0)\) to be unique up to unique isomorphism. Note that \((M_0, \pi_0)\) is a cone of

\[
\varphi : \partial M(X) \leq -1 \to \partial M(X) \to M(X).
\]

Any other choice of cone would map isomorphically to \((M_0, \pi_0)\), the isomorphism in question being unique up to the image of an element in

\[
\text{Hom}_{\text{DM}^\text{eff}_{\text{gm}}(k)}(\partial M(X) \leq -1[1], M_0).
\]

In general, the object \(\partial M(X) \leq -1\) belonging to \(\text{DM}^\text{eff}_{\text{gm}}(k)_{w \leq -1}\), hence

\[
\partial M(X) \leq -1[1] \in \text{DM}^\text{eff}_{\text{gm}}(k)_{w \leq 0},
\]

there is no way of preventing such elements from being non-zero. However, if

\[
\partial M(X) \leq -1 \in \text{DM}^\text{eff}_{\text{gm}}(k)_{w \leq -2} \subset \text{DM}^\text{eff}_{\text{gm}}(k)_{w \leq -1},
\]

then

\[
\text{Hom}_{\text{DM}^\text{eff}_{\text{gm}}(k)}(\partial M(X) \leq -1[1], M_0) = 0
\]

by orthogonality [L.9] (3) (recall that \(M_0\) is of weight zero). Thus, under this hypothesis, the pair \((M_0, \pi_0)\) is rigid. As for \(i_0\), the same type of reasoning shows unicity provided that

\[
\partial M(X) \geq 0 \in \text{DM}^\text{eff}_{\text{gm}}(k)_{w \geq 1} \subset \text{DM}^\text{eff}_{\text{gm}}(k)_{w \geq 0}.
\]

We are thus led naturally to make the following definition [W3 Def. 1.6 and 1.10].

**Definition 1.21.** Let \(M \in \text{DM}^\text{eff}_{\text{gm}}(k)\), and \(m \leq n\) two integers (which may be identical). A **weight filtration of \(M\) avoiding weights \(m, m+1, \ldots, n-1, n\)** is an exact triangle

\[
M_{\leq m-1} \to M \to M_{\geq n+1} \to M_{\leq m-1}[1],
\]

with \(M_{\leq m-1} \in \text{DM}^\text{eff}_{\text{gm}}(k)_{w \leq m-1}\) and \(M_{\geq n+1} \in \text{DM}^\text{eff}_{\text{gm}}(k)_{w \geq n+1}\). If such a weight filtration exists, then we say that \(M\) is **without weights \(m, \ldots, n\)**.

Weight filtrations avoiding weights \(m, \ldots, n\) behave functorially [W3 Prop. 1.7]. In particular, if \(M \in \text{DM}^\text{eff}_{\text{gm}}(k)\) is without weights \(m, \ldots, n\), then its weight filtration avoiding weights \(m, \ldots, n\) is unique up to unique isomorphism. Remark [L.20] therefore shows that we can rigidify Construction [L.17] provided that the boundary motive \(\partial M(X)\) is without weights \(-1\) and 0.

**Problem 1.22.** The boundary motive \(\partial M(X)\) of \(Sm/k\) is never without weights \(-1\) and 0 — unless it is altogether trivial.
Here is a heuristic reason, using the weights occurring in boundary cohomology over $k = \mathbb{C}$: to say that $\partial M(X)$ is not trivial implies that $X$ is not proper. On the one hand, the cokernel of
\[ H^0(X(\mathbb{C}), \mathbb{Q}) \to H^0(X(\mathbb{C}), \mathbb{Q}) \cong \mathbb{Q}(0)^r \]
is then non-trivial. On the other hand, it injects into $\partial H^0(X(\mathbb{C}), \mathbb{Q})$.

Therefore, the approach of “avoiding weights” cannot work on the whole of the boundary motive. We need to restrict to direct factors. Fix a commutative flat $\mathbb{Z}$-algebra $F$, and let
\[ (\ast) \quad \partial M(X) \to M(X) \to M^e(X) \to \partial M(X)[1] \]
be the boundary triangle associated to a fixed object $X$ of $Sm/k$, viewed as a triangle in $DM^{eff}_{gm}(k)_F$. Fix an idempotent endomorphism $e$ of the triangle (\ast), that is, fix idempotent endomorphisms of each of $M(X)$, $M^e(X)$ and $\partial M(X)$, viewed as objects of $DM^{eff}_{gm}(k)_F$, which yield an endomorphism of the triangle. Denote by $M(X)^e$, $M^e(X)^e$ and $\partial M(X)^e$ the images of $e$ on $M(X)$, $M^e(X)$ and $\partial M(X)$, respectively, and consider the canonical morphism $u : M(X)^e \to M^e(X)^e$. By Corollary 1.12 and condition 1.9 (1), the object $M(X)^e$ belongs to $DM^{eff}_{gm}(k)_F$, and $M^e(X)^e$ to $DM^{eff}_{gm}(k)_F$. In this situation, Construction 1.17 yields the following.

**Variant 1.23.** The map
\[ \{ (\partial M(X)^e_{\leq -1}, \partial M(X)^e_{\geq 0}) \} / \cong \to \{ (M_0, \pi_0, i_0) \} / \cong \]
is a bijection between
1. the isomorphism classes of weight filtrations of $\partial M(X)^e$,
2. the isomorphism classes of effective Chow motives $M_0 \in CHM^{eff}(k)_F$, together with a factorization
\[ M(X)^e \xrightarrow{\pi_0} M_0 \xrightarrow{i_0} M^e(X)^e \]
of the canonical morphism $u : M(X)^e \to M^e(X)^e$, such that both $i_0$ and $\pi_0$ can be completed to give weight filtrations of $M^e(X)$ and of $M(X)^e$, respectively.

**Complement 1.24.** Let $e$ denote an idempotent endomorphism of the boundary triangle (\ast) as above, and assume that $\partial M(X)^e$ is without weights $-1$ and $0$. Then the isomorphism class $(M_0, \pi_0, i_0)$ associated to the weight filtration of $\partial M(X)^e$ avoiding weights $-1$ and $0$ essentially contains one single object, which is unique up to unique isomorphism.

This is the principle exploited in [W3, Sect. 4]. Due to the behaviour of its realizations, the object $M_0$ is referred to as the $e$-part of the interior motive of $X$. It has very strong functoriality properties. They will be illustrated in our Section 5, where we shall establish equivariance under the Hecke algebra of $M_0$ in a special geometrical context. Before that, we need to address two
very concrete questions:

(I) How does one get endomorphisms of the boundary triangle

$\partial M(X) \to M(X) \to M^e(X) \to \partial M(X)[1]$ ?

(II) How can one show that a given such endomorphism is idempotent?

The following two sections attempt to answer these questions, at least partially.

2 Relative motives and functoriality of the boundary motive

In this and the next section, the base field $k$ is assumed to admit strict resolution of singularities. For $X \in \text{Sm}/k$, the algebra of finite correspondences $c(X, X)$ acts on $M(X)$ [VI, p. 190]. In order to apply the constructions from Section II, we need to construct endomorphisms of the whole boundary triangle

$\partial M(X) \to M(X) \to M^e(X) \to \partial M(X)[1]$.

One of the aims of this section is to show that the theory of relative motives provides a source of such endomorphisms. This result is a special feature of an analysis of the functorial behavior of the exact triangle (*) under morphisms of relative motives (Theorems 2.2 and 2.5, Corollary 2.15). The main application (Example 2.16) concerns endomorphisms of (*) “of Hecke type”.

Let us fix a base scheme $S \in \text{Sm}/k$. Recall that by definition, objects of $\text{Sm}/k$ are separated over $k$. Thus, for any two schemes $X$ and $Y$ over $S$, the natural morphism

$X \times_S Y \to X \times_k Y$

is a closed immersion. Therefore, cycles on $X \times_S Y$ can and will be considered as cycles on $X \times_k Y$. Denote by $\text{Sm}/S$ the category of separated smooth schemes of finite type over $S$, by $\text{PropSm}/S \subset \text{Sm}/S$ the full sub-category of objects which are proper and smooth over $S$, and by $\text{ProjSm}/S \subset \text{Sm}/S$ the full sub-category of projective, smooth $S$-schemes.

**Definition 2.1.** Let $X, Y \in \text{Sm}/S$. Denote by $c_S(X, Y)$ the subgroup of $c(X, Y)$ of correspondences whose support is contained in $X \times_S Y$.

The group $c_S(X, Y)$ is at the base of the theory of (effective) geometrical motives over $S$, as defined and developed (for arbitrary regular Noetherian bases $S$) in Dé1, Dé2. Note that any cycle $\mathfrak{z}$ in $c_S(X, Y)$ gives rise to a morphism from $M(X)$ to $M(Y)$, which we shall denote by $M(\mathfrak{z})$. Recall from [DeMu, Sect. 1.3, 1.6] the definition of the categories of smooth
(effective) Chow motives over $S$; note that the approach of [loc. cit.] does not necessitate passage to $\mathbb{Q}$-coefficients, and that one may choose to perform the construction using schemes in $\text{PropSm}/S$ instead of just schemes in $\text{ProjSm}/S$. Denote by $\text{CHM}^{s,eff}(S)$ and $\text{CHM}^*(S)$ the respective opposites of these categories. Note that for $X, Y \in \text{PropSm}/S$ and $3 \in c_S(X, Y)$, the class of $3$ in the Chow group $\text{CH}^*(X \times_S Y)$ of cycles modulo rational equivalence lies in the right degree, and therefore defines a morphism from the relative Chow motive $h(X/S)$ of $X$ to the relative Chow motive $h(Y/S)$. Our aim is to prove the following.

**Theorem 2.2.** (a) There is a canonical additive covariant functor, denoted $(\partial M, M, M^c) = (\partial M, M, M^c)_S$, from $\text{CHM}^*(S)$ to the category of exact triangles in $\text{DM}_{\text{gm}}(k)$. On objects, it is characterized by the following properties:

(a1) for $X \in \text{PropSm}/S$, the functor $(\partial M, M, M^c)$ maps $h(X/S)$ to the boundary triangle

\[(*) = \partial M(X) \rightarrow M(X) \rightarrow M^c(X) \rightarrow \partial M(X)[1],\]

(a2) the functor $(\partial M, M, M^c)$ is compatible with Tate twists.

On morphisms, the functor $(\partial M, M, M^c)$ maps the class of a cycle $3 \in c_S(X, Y)$ in $\text{CH}^*(X \times_S Y)$, for $X, Y \in \text{PropSm}/S$, to a morphism $(*)_X \rightarrow (*)&_Y$ whose $M$-component $M(X) \rightarrow M(Y)$ coincides with $M(3)$.

(b) There is a canonical additive contravariant functor $(\partial M, M, M^c)^* = (\partial M, M, M^c)^*_{S}$ from $\text{CHM}^*(S)$ to the category of exact triangles in $\text{DM}_{\text{gm}}(k)$. On objects, it is characterized by the following properties:

(b1) for an object $X \in \text{PropSm}/S$ which is of pure absolute dimension $d_X$, the functor $(\partial M, M, M^c)^*$ maps $h(X/S)$ to the triangle

\[(*)^*_X := (*)_X(-d_X)[-2d_X],\]

(b2) the functor $(\partial M, M, M^c)^*$ is anti-compatible with Tate twists.

On morphisms, the functor $(\partial M, M, M^c)^*$ maps the class of a cycle $3 \in c_S(X, Y)$ in $\text{CH}^*(X \times_S Y)$, for $X, Y \in \text{PropSm}/S$ of pure absolute dimensions $d_X$ and $d_Y$, respectively, to a morphism $(*)^*_Y \rightarrow (*)_X^*$ whose $M^c$-component coincides with the dual of $M(3)$.

(c) The functor $(\partial M, M, M^c)^*$ is canonically identified with the composition of $(\partial M, M, M^c)$ and duality in $\text{DM}_{\text{gm}}(k)$.

Recall that by [VI] Thm. 4.3.7 3], the object $M^c(X)$ is indeed dual to $M(X)(-d_X)[-2d_X]$. Note also [VI] Cor. 4.1.6] that the functor from Theorem 2.2(a) maps the full sub-category $\text{CHM}^{s,eff}(S)$ to the full sub-category [VI] Thm. 4.3.1] of exact triangles in $\text{DM}^*_{\text{gm}}(k)$. Also recall that by convention, the Tate twist $(n)$ in $\text{CHM}^*(S)$ corresponds to the (componentwise) operation $M \mapsto M(n)[2n]$ in $\text{DM}_{\text{gm}}(k)$. Thus, anti-compatibility of
the functor \((\partial M, M, M^c)^*\) with Tate twists means that for any object \(X\) of \(CHM^s(S)\), there is a functorial isomorphism
\[
(\partial M, M, M^c)^*(X(n)) \xrightarrow{\sim} ((\partial M, M, M^c)^*(X))(-n)[-2n]
\]

**Remark 2.3.** As far as the \(M\)- and \(M^c\)-components are concerned, Theorem 2.2, or at least its restriction to the full sub-category \(CHM^s(S)_{\text{proj}}\) of \(CHM^s(S)\) generated by the motives of projective smooth \(S\)-schemes, is a consequence of the main results of [De2], especially [De2, Thm. 5.23] together with the existence of an adjoint pair \((\La_a, a^\#)^s\) of exact functors [CId, Ex. 4.12, Ex. 7.15] linking the category \(DM_{gm}(S)\) of geometrical motives over \(S\) to \(DM_{gm}(k)\) (here we let \(a_S : S \to \text{Spec} \ k\) denote the structure morphism of \(S\)). We should also mention that this approach would allow to avoid the hypothesis on strict resolution of singularities. However, the application of the results of [loc. cit.] to the functor \(\partial M\) is not obvious. We are therefore forced to follow an alternative approach.

**Remark 2.4.** The following sheaf-theoretical phenomenon explains why one should expect a statement like Theorem 2.2. Writing \(a = a_X\) for the structure morphism \(X \to \text{Spec} \ k\), for \(X \in \text{Sch}/k\), there is an exact triangle of exact functors
\[
(+) \to a_! \to a_* \to a^!/a_![-1]
\]
from the derived category \(D^+(X)\) of complexes of étale sheaves on \(X\) (say), bounded from below, to \(D^+(\text{Spec} \ k)\). Here, \(a_*\) denotes the derived functor of the direct image, \(a_!\) is its analogue “with compact support”, and \(a_*/a_!\) is a canonical choice of cone (which exists since the category of compactifications of \(X\) is filtered). The triangle \((+)\) is contravariantly functorial with respect to proper morphisms. Up to a twist and a shift, it is covariantly functorial with respect to proper smooth morphisms. This shows that a suitable version of Theorem 2.2 (a) is likely to extend to the sub-category of \(DM_{gm}(S)\) generated by the relative motives of schemes which are (only) proper over \(S\).

For any proper smooth morphism \(f : T \to S\) in the category \(Sm/k\), denote by \(f_* : CHM^s(T) \to CHM^s(S)\) the canonical functor induced by \(h(X/T) \to h(X/S)\), for any proper smooth scheme \(X\) over \(T\) (hence, over \(S\)). For any morphism \(g : U \to S\) in \(Sm/k\), denote by \(g^* : CHM^*(S) \to CHM^*(U)\) the canonical tensor functor induced by \(h(Y/S) \to h(Y \times_S U/U)\), for any proper smooth scheme \(Y\) over \(S\). When \(g\) is proper and smooth, the functor \(g_*\) is left adjoint to \(g^*\). The following summarizes the behaviour of \((\partial M, M, M^c)^s\) and \((\partial M, M, M^c)^s\) under change of the base \(S\).

**Theorem 2.5.** (a) Let \(f : T \to S\) be a proper smooth morphism in \(Sm/k\). There are canonical isomorphisms of additive functors
\[
\alpha_{f_*} : (\partial M, M, M^c)_S \circ f_* \xrightarrow{\sim} (\partial M, M, M^c)_T
\]
\[ \alpha^*_{f_T} : (\partial M, M, M^c)^*_T \to (\partial M, M, M^c)^*_S \circ f_T \]
on CHM^*(T). The formation of both \( \alpha^*_{f_T} \) and \( \alpha^*_{f_T} \) is compatible with composition of proper smooth morphisms in \( Sm/k \). Under the identification of Theorem \[\text{(c)}\], the equivalence \( \alpha^*_{f_T} \) corresponds to the dual of the equivalence \( \alpha_{f_T} \).

(b) Let \( g : U \to S \) be a proper smooth morphism in \( Sm/k \). Then there exists a canonical transformation of additive functors

\[ \beta_{g^*, \text{id}_S} : (\partial M, M, M^c)_U \circ g^* \to (\partial M, M, M^c)_S. \]
The formation of \( \beta_{g^*, \text{id}_S} \) is compatible with composition of proper smooth morphisms in \( Sm/k \).

(c) The transformations \( \alpha_{f_T} \) and \( \beta_{g^*, \text{id}_S} \) commute in the following sense: let \( f : T \to S \) and \( g : U \to S \) be proper smooth morphisms in \( Sm/k \). Consider the cartesian diagram

\[
\begin{array}{ccc}
V = T \times_S U & \xrightarrow{f} & U \\
\downarrow{g'} & & \downarrow{g} \\
T & \xrightarrow{f} & S
\end{array}
\]

and the canonical identification of natural transformations

\[ f_T \circ g'^* = g^* \circ f_T \]
of functors from \( \text{CHM}^*(T) \) to \( \text{CHM}^*(U) \). Then the transformations

\[ \beta_{g^*, \text{id}_T} \circ (\alpha_{f_T} \circ g'^*) , \quad \alpha_{f_T} \circ (\beta_{g^*, \text{id}_S} \circ f_T) \]
of functors on \( \text{CHM}^*(T) \)

\[ (\partial M, M, M^c)_U \circ g^* \circ f_T \to (\partial M, M, M^c)_T \]
coincide.

(d) Let \( g : U \to S \) be a proper smooth morphism in \( Sm/k \). Then there exists a canonical transformation of additive functors

\[ \gamma_{\text{id}_S, g^*} : (\partial M, M, M^c)_S \to (\partial M, M, M^c)_U \circ g^* . \]
The formation of \( \gamma_{\text{id}_S, g^*} \) is compatible with composition of proper smooth morphisms in \( Sm/k \). Under the identification of Theorem \[\text{(c)}\], the transformation \( \gamma_{\text{id}_S, g^*} \) corresponds to the dual of the transformation \( \beta_{g^*, \text{id}_S} \).

(e) The transformations \( \alpha^*_{f_T} \) and \( \gamma_{\text{id}_S, g^*} \) commute in the following sense: let \( f : T \to S \) and \( g : U \to S \) be proper smooth morphisms in \( Sm/k \). Consider
the cartesian diagram

\[
\begin{array}{ccc}
V = T \times_S U & \xrightarrow{f'} & U \\
\downarrow{g'} & & \downarrow{g} \\
T & \xrightarrow{f} & S
\end{array}
\]

Then the transformations

\[
(\gamma_{id_T, g^*} \circ f'_z) \circ \alpha_{f'_z}^* \quad , \quad (\alpha_{f'_z}^* \circ g'^*) \circ \gamma_{id_T, g'^*}
\]

of functors on \(CHM^*(T)\)

\[
(\partial M, M, M^c)^*_T \rightarrow (\partial M, M, M^c)^*_U \circ g^* \circ f'_z
\]

coincide.

**Remark 2.6.** Sheaf-theoretical considerations show that parts (b)–(e) of Theorem 2.5 should hold more generally for morphisms \(g\) which are (only) proper. While this could be shown to be indeed the case, we chose to prove the statements only under the more restrictive assumption on \(g\): the proof then simplifies considerably since it is possible to make use of the functor \(g'_z\), which only exists when \(g\) is (proper and) smooth.

Let us prepare the proofs of Theorems 2.2 and 2.5. They are based on the following result.

**Theorem 2.7** ([W1, Thm. 6.14, Rem. 6.15]). Let \(W \in Sm/k\) be of pure dimension \(m\), and \(Z \subset W\) a closed sub-scheme such that arbitrary intersections of the irreducible components of \(Z\) are smooth. Fix \(n \in \mathbb{Z}\).

(a) There is a canonical morphism

\[
cyc : h^0 (z_{equi}(W, m - n)Z)(Spec k) \rightarrow \text{Hom}_{DM_{gm}^eff(k)}(M(W/Z), Z(n)[2n])
\]

(b) The morphism \(cyc\) is compatible with passage from the pair \(Z \subset W\) to \(Z' \subset U\), for open sub-schemes \(U\) of \(W\), and closed sub-schemes \(Z'\) of \(Z \cap U\) such that arbitrary intersections of the irreducible components of \(Z'\) are smooth.

(c) When \(Z\) is empty, then

\[
cyc : h^0 (z_{equi}(W, m - n))(Spec k) \rightarrow \text{Hom}_{DM_{gm}^eff(k)}(M(W), Z(n)[2n])
\]

coincides with the morphism from [W1, Cor. 4.2.5]. In particular, it is then an isomorphism.

Some explanations are necessary. First, by definition [W1, Def. 6.13], the Nisnevich sheaf with transfers \(z_{equi}(W, m - n)Z\) associates to \(T \in Sm/k\) the group of those cycles in \(z_{equi}(W, m - n)(T)\) [W1, p. 228] having empty intersection with \(T \times_k Z\). In particular, the group \(z_{equi}(W, m - n)Z(Spec k)\)
equals the group of cycles on $W$ of dimension $m - n$, whose support is disjoint from $Z$. Recall then [W1, p. 207] that the group 

$$h^0\left(\zeta_{equi}(W, m - n)_Z\right)(\text{Spec } k)$$

is the quotient of $\zeta_{equi}(W, m - n)_Z(\text{Spec } k)$ by the image under the differential “pull-back via 1 minus pull-back via 0” of $\zeta_{equi}(W, m - n)_Z(\mathbb{A}^1_k)$. Finally the object $M(W/Z)$ denotes the relative motive associated to the immersion of $Z$ into $W$ [W1, Def. 6.4].

**Remark 2.8.** One may speculate about the validity of Theorem 2.7 for arbitrary closed sub-schemes $Z$ of $W \in \text{Sm}/S$. While the author is optimistic about this possibility, he notes that the tools developed in [W1] to prove Theorem 2.7 require $Z$ to satisfy our more restrictive hypotheses. It is for that reason that we are forced to suppose $k$ to admit strict resolution of singularities.

Now note the following.

**Proposition 2.9.** In the above situation, let in addition $V \subset W$ be a closed sub-scheme in $\text{Sm}/S$, which is disjoint from $Z$. Then the natural map 

$$\zeta_{equi}(V, m - n) \to \zeta_{equi}(W, m - n)_Z$$

induces a morphism 

$$\text{CH}_{m-n}(V) \to h^0\left(\zeta_{equi}(W, m - n)_Z\right)(\text{Spec } k).$$

**Corollary 2.10.** Let $W \in \text{Sm}/k$ be of pure dimension $m$, $V, Z \subset W$ closed sub-schemes, and $n \in \mathbb{Z}$. Suppose that arbitrary intersections of the irreducible components of $Z$ are smooth, and that $V \cap Z = \emptyset$. Then there is a canonical morphism 

$$cyc : \text{CH}_{m-n}(V) \to \text{Hom}_{\text{DM}_{gm}^{eff}}(k)(M(W/Z), Z(n)[2n]).$$

Given an open immersion $j : U \hookrightarrow W$ and a closed sub-scheme $Z'$ of the intersection $Z \cap U$ such that arbitrary intersections of the irreducible components of $Z'$ are smooth, the diagram

$$\begin{array}{ccc}
\text{CH}_{m-n}(V) & \xrightarrow{cyc} & \text{Hom}_{\text{DM}_{gm}^{eff}}(k)(M(W/Z), Z(n)[2n]) \\
\downarrow j^* & & \downarrow j^* \\
\text{CH}_{m-n}(V \cap U) & \xrightarrow{cyc} & \text{Hom}_{\text{DM}_{gm}^{eff}}(k)(M(U/Z'), Z(n)[2n])
\end{array}$$

commutes.

Now fix $X, Y \in \text{PropSm}/S$. Choose a compactification (over $k$) $\overline{S}$ of $S$, and compactifications $\overline{X}$ of $X$, and $\overline{Y}$ of $Y$ together with cartesian diagrams

$$\begin{array}{ccc}
X & \rightarrow & \overline{X} \\
\downarrow & \downarrow & \downarrow \\
S & \rightarrow & \overline{S}
\end{array}$$
and

\[
\begin{array}{c}
X \\ \downarrow \\
S
\end{array}
\quad \begin{array}{c}
Y \\ \downarrow \\
S
\end{array}
\quad \begin{array}{c}
Y \\ \downarrow \\
S
\end{array}
\]

(this is possible since \(X\) and \(Y\) are proper over \(S\)). The hypothesis on \(k\) ensures that arbitrary intersections of the irreducible components of the complements \(\partial X\) of \(X\) in \(X\) and \(\partial Y\) of \(Y\) in \(Y\) can be supposed to be smooth. Each of the three constituents \(M, M^c, \partial M\) of the exact triangle (*) will correspond to an application of Corollary 2.10 with different choices of \((W, Z)\).

1. For \(M\), we define \(W := X \times_k Y\);
2. For \(M^c\), we define \(W := X \times_k Y\);
3. For \(\partial M\), we define \(W := X \times_k Y - \partial X \times_k \partial Y\).

In all three cases, we put \(Z := W - X \times_k Y\). That is,

1. \(Z = X \times_k \partial Y\),
2. \(Z = \partial X \times_k Y\),
3. \(Z = X \times_k \partial Y \cup \partial X \times_k Y\).

We also let \(V := X \times_S Y \subset X \times_k Y\) in all three cases. These choices satisfy the hypotheses of Corollary 2.10 thanks to the following.

**Lemma 2.11.** The scheme \(X \times_S Y\) is closed in \(X \times_k Y - \partial X \times_k \partial Y\).

**Proof.** Indeed, the diagram

\[
\begin{array}{c}
X \times_S Y \\
\downarrow \\
S
\end{array}
\quad \begin{array}{c}
X \times_k Y - \partial X \times_k \partial Y \\
\downarrow \\
S \times_k S
\end{array}
\]

is cartesian.

**q.e.d.**

**Proof of Theorem 2.2.** We may clearly assume \(S, X\) and \(Y\) to be of pure absolute dimension \(d_S, d_X\) and \(d_Y\), respectively.

Let us treat \(M\) first. Note that by [V1, Thm. 4.3.7 3], the group of morphisms in \(DM_{gm}(k)\) from \(M(X)\) to \(M(Y)\) is canonically isomorphic to

\[
\text{Hom}_{DM_{gm}(k)}(M(X) \otimes M^c(Y), \mathbb{Z}(d_Y)[2d_Y]).
\]

Localization for the motive with compact support [V1, Prop. 4.1.5] shows that \(M^c(Y) = M(Y/\partial Y)\). Given the definition of the tensor structure on \(DM_{gm}(k)\), the above therefore equals

\[
\text{Hom}_{DM_{gm}(k)}(M(X \times_k Y/X \times_k \partial Y), \mathbb{Z}(d_Y)[2d_Y]).
\]

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As in [W1, pp. 650–651], one shows that \(\partial M\) cyclically to the relative motive of morphisms in \(DM\) to \(\mathbb{Z}\) the target of the morphism \(cyc\). By Corollary 2.10, applied to the setting (1), this group is the target of the arrow, say \(\alpha\). Hence the group of morphisms \(\text{Hom}_{DM}(M^{c}(X) \otimes M(Y), \mathbb{Z}[dY][2dY])\). By localization, this group then equals

\[
\text{Hom}_{DMgm(k)}(M(X \times_k Y)/\partial X \times_k Y, \mathbb{Z}[dY][2dY])
\]

By Corollary 2.10 applied to the setting (2), this group is the target of the morphism \(cyc_{2}\) on \(CH^{dY-dS}(X \times_k Y)\). In order to show that for a cycle class \(z\) in \(CH^{dY-dS}(X \times_k Y)\), the diagram

\[
\begin{array}{ccc}
\partial M & \xrightarrow{\text{cyc}_{1}(z)} & \partial M(X) \\
\downarrow{\text{cyc}_{2}(z)} & & \downarrow{\text{cyc}_{2}(z)} \\
M(Y) & \xrightarrow{\text{cyc}_{3}(z)} & M^{c}(Y)
\end{array}
\]

commutes, we need to study the group of morphisms in \(DMgm(k)\) from \(M(X)\) to \(M^{c}(Y)\). Again by duality, it is canonically isomorphic to

\[
\text{Hom}_{DMgm(k)}(M(X \times_k Y), \mathbb{Z}[dY][2dY])
\]

The above commutativity then follows from the compatibility of \(cyc\) under restriction from \(X \times_k Y\), resp. \(X \times_k Y\), to \(X \times_k Y\) (Corollary 2.10). Now let us treat \(\partial M\). Note that by [W1] Thm. 6.1, the group of morphisms in \(DMgm(k)\) from \(\partial M(X)\) to \(\partial M(Y)\) is canonically isomorphic to

\[
\text{Hom}_{DMgm(k)}(\partial M(X) \otimes \partial M(Y)[1], \mathbb{Z}[dY][2dY])
\]

As in [W1] pp. 650–651, one shows that \(\partial M(X) \otimes \partial M(Y)[1]\) maps canonically to the relative motive

\[
M\left((X \times_k Y - \partial X \times_k \partial Y)/(X \times_k Y - \partial X \times_k \partial Y - X \times_k Y)\right)
\]

Hence the group of morphisms \(\text{Hom}_{DMgm(k)}(\partial M(X), \partial M(Y))\) receives an arrow, say \(\alpha\), from the group of morphisms from

\[
M\left((X \times_k Y - \partial X \times_k \partial Y)/(X \times_k Y - \partial X \times_k \partial Y - X \times_k Y)\right)
\]

to \(\mathbb{Z}[dY][2dY]\). By Corollary 2.10 applied to the setting (3), this group is the target of the morphism \(cyc_{3}\) on \(CH^{dY-dS}(X \times_k Y)\). In order to show that for a cycle class \(z\) in \(CH^{dY-dS}(X \times_k Y)\), the diagram

\[
\begin{array}{ccc}
M^{c}(X) & \xrightarrow{\text{cyc}_{2}(z)} & \partial M(X)[1] \\
\downarrow{\text{cyc}_{3}(z)} & & \downarrow{\text{cyc}_{3}(z)}[1] \\
M^{c}(Y) & \xrightarrow{\text{cyc}_{3}(z)} & \partial M(Y)[1]
\end{array}
\]

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commutes, we need to study the group of morphisms in $DM_{gm}(k)$ from $M^c(X)$ to $\partial M(Y)[1]$. Again by [W1 Thm. 6.1], it is canonically isomorphic to

$$\text{Hom}_{DM_{gm}(k)}(M^c(X) \otimes \partial M(Y), \mathbb{Z}(d_Y)[2d_Y]).$$

But $M^c(X) \otimes \partial M(Y)$ maps canonically to $M^c(X) \otimes M(Y)$, which was already identified with the relative motive $M(\overline{X} \times_k Y/\partial \overline{X} \times_k Y)$. Hence the group of morphisms $\text{Hom}_{DM_{gm}(k)}(M^c(X), \partial M(Y)[1])$ receives an arrow, say $\beta$, from

$$\text{Hom}_{DM_{gm}(k)}(M(\overline{X} \times_k Y/\partial \overline{X} \times_k Y), \mathbb{Z}(d_Y)[2d_Y]).$$

The desired commutativity then follows from the compatibility of $cyc$ under restriction from $\overline{X} \times_k \overline{Y} - \partial \overline{X} \times_k \partial \overline{Y}$ to $\overline{X} \times_k Y$ (Corollary 2.10), and from the compatibility of $\beta$ and the map $\alpha$ from above. The latter is a consequence of the compatibility of the isomorphism

$$\partial M(Y)[1] \xrightarrow{\sim} \partial M(Y)^*(d_Y)[2d_Y]$$

with duality $M(Y) \cong M^c(Y)^*(d_Y)[2d_Y]$ [W1 Thm. 6.1].

The proof of the commutativity of

$$\begin{array}{ccc}
\partial M(X) & \longrightarrow & M(X) \\
\downarrow_{cyc(z)} & & \downarrow_{cyc(z)} \\
\partial M(Y) & \longrightarrow & M(Y)
\end{array}$$

is similar.

Altogether, this proves part (a) of the statement. As for parts (b) and (c), simply compose the functor from (a) with duality in $DM_{gm}(k)$, using [V1 Thm. 4.3.7 3] and [W1 Thm. 6.1].

By [W1 Rem. 6.15], our construction is independent of the compactifications $\overline{S}$, $\overline{X}$, $\overline{Y}$.

**Proof of Theorem 2.5.** We keep the notations of the previous proof. Choose compactifications $\overline{T}$ of $T$, and $\overline{U}$ of $U$ together with cartesian diagrams

$$\begin{array}{ccc}
T' & \longrightarrow & \overline{T} \\
\downarrow f & & \downarrow \\
S' & \longrightarrow & \overline{S}
\end{array}$$

and

$$\begin{array}{ccc}
U' & \longrightarrow & \overline{U} \\
\downarrow s & & \downarrow \\
S' & \longrightarrow & \overline{S}
\end{array}$$
(\(f\) and \(g\) are proper).

(a) Checking the definitions, the transformation \(\alpha\) is in fact given by the identity. Indeed, both \((\partial M, M, M^c)_S \circ f^*_S\) and \((\partial M, M, M^c)_T\) map the object \(h(X/T)\), for \(X \in \text{PropSm}/T\), to the exact triangle

\[
(*)_X \quad \partial M(X) \longrightarrow M(X) \longrightarrow M^c(X) \longrightarrow \partial M(X)[1].
\]

Note that on morphisms, the functor \(f^*_S\) corresponds to the push-forward

\[
\text{CH}_*(X \times_T Y) \longrightarrow \text{CH}_*(X \times_S Y)
\]

along the closed immersion \(X \times_T Y \hookrightarrow X \times_S Y\). The latter factors the closed immersion

\[
X \times_T Y \longrightarrow \overline{X \times_Y Y} - \partial \overline{X \times_Y Y}.
\]

The construction (see the preceding proof) shows then that the effects of the functors \((\partial M, M, M^c)_S \circ f^*_S\) and of \((\partial M, M, M^c)_T\) coincide also on \(\text{CH}_*(X \times_T Y)\). This shows the first half of statement (a). The second is implied formally by Theorem 2.2 (c).

(b) We first consider an auxiliary functor. The morphism \(g\) being proper and smooth, we may consider the composition \(g^*_T \circ g^*\) on \(\text{CHM}_*(S)\), which on objects is given by \(h(Y/S) \mapsto h(Y \times_S U/S)\), for any proper smooth scheme \(Y\) over \(S\). Projection onto the first component then yields a transformation of functors, namely the adjunction

\[
\beta g^*, \id_S : g^*_T \circ g^* \longrightarrow \id_{\text{CHM}_*(S)}.
\]

Then define \(\beta g^*, \id_S\) to be the composition of transformations

\[
\beta g^*, \id_S := (\partial M, M, M^c)_S \circ \beta g^*, \id_S \circ (\alpha g^* \circ g^*)^{-1},
\]

observing the equivalence

\[
\alpha g^* \circ g^* : (\partial M, M, M^c)_S \circ g^*_T \circ g^* \longrightarrow (\partial M, M, M^c)_U \circ g^*.
\]

from part (a). We leave it to the reader to check the compatibility of this construction with composition of proper smooth morphisms in \(\text{Sm}/k\).

(c) Similarly, this commutativity statement is left as an exercice.

(d), (e) Given Theorem 2.2 (c), these statements follow formally from (b) and (c), respectively. q.e.d.

For \(X, Y \in \text{PropSm}/S\), denote by \(\bar{c}_S(X, Y)\) the quotient of \(c_S(X, Y)\) by the group of cycles \(3\) satisfying

\[
M(3) = 0 \quad , \quad M^c(3) = 0 \quad , \quad \partial M(3) = 0.
\]

Note that composition of correspondences induces a well-defined composition on \(\bar{c}_S\). In particular, for any \(X \in \text{PropSm}/S\), the group \(\bar{c}_S(X, X)\) carries the structure of an algebra.

**Corollary 2.12.** Let \(X\) and \(Y\) be in \(\text{PropSm}/S\). Then the projection

\[
c_S(X, Y) \longrightarrow \bar{c}_S(X, Y)
\]

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factors through the image of $c_S(X, Y)$ in $\text{CH}^*(X \times_S Y)$. In other words, two cycles $\mathfrak{z}_1, \mathfrak{z}_2 \in c_S(X, Y)$ induce the same morphisms $M(\mathfrak{z}_1)$, resp. $M^c(\mathfrak{z}_1)$, resp. $\partial M(\mathfrak{z}_1)$, if they are rationally equivalent on $X \times_S Y$.

Remark 2.13. (a) Let $X, Y \in \text{Sm}/S$. As shown in [L, Lemma 5.18], the map

$$c_S(X, Y) \to \text{CH}_{d_X}(X \times_S Y)$$

is surjective, whenever $Y$ is projective, and $X$ of pure absolute dimension $d_X$. Therefore, by Corollary [2.12], the group $\tilde{c}_S(X, Y)$ is canonically a quotient of $\text{CH}_{d_X}(X \times_S Y)$ if $X \in \text{PropSm}/S$ and $Y \in \text{ProjSm}/S$.

(b) The observation from (a) fits in the functorial picture sketched in Remark 2.3. Indeed, [Dé2, Thm. 5.23] implies that the restriction of the functor $M, M : \text{CHM}^*(S)_{\text{proj}} \to \text{DM}_{gm}(k)$ factors canonically through a fully faithful embedding

$$\text{CHM}^*(S)_{\text{proj}} \to \text{DM}_{gm}(S).$$

(c) In [L, Prop. 5.19], an embedding result analogous to (b) is proven for a dg-version of $\text{DM}_{gm}(S)$, denoted $\text{SmMot}(S)$ in [loc. cit.], from which the embedding (b) can be deduced [L, Cor. 7.13].

(d) Using [Dé2, Thm. 5.23], one can show that

$$\text{Hom}_{\text{DM}_{gm}(S)}(M_1, M_2[i]) = 0$$

for any two smooth relative Chow motives $M_1, M_2 \in \text{CHM}^*(S)_{\text{proj}}$, and any integer $i > 0$.

(e) When $S = \text{Spec} k$, results (b) and (d) are contained in [V1, Cor. 4.2.6].

Remark 2.14. Fix a non-negative integer $d$, and consider the full subcategory $\text{CHM}^*(S)_d$ of $\text{CHM}^*(S)$ of smooth relative Chow motives generated by the Tate twists of $h(X/S)$, for $X \in \text{PropSm}/S$ of pure absolute dimension $d$. The construction of the duality isomorphisms [V1, Thm. 4.3.7 3], [W1, Thm. 6.1] shows that the identification

$$(\partial M, M, M^c)^* = (\partial M, M, M^c)(-d)[-2d]$$

of the restriction of the functors from Theorem 2.2 to $\text{CHM}^*(S)_d$ admits an alternative description, when $S$ is of pure absolute dimension, say $s$: on $\text{CHM}^*(S)_d$, the functor $(\partial M, M, M^c)^*$ equals then composition of duality in the category $\text{CHM}^*(S)_d$ with $(\partial M, M, M^c)$, followed by the functor $M \mapsto M(-s)[-2s]$. Note that on morphisms, duality in $\text{CHM}^*(S)_d$ corresponds to the transposition $\text{CH}_*(X \times_S Y) \to \text{CH}_*(Y \times_S X)$.

This observation allows to deduce the following statements from Theorem 2.5.
Corollary 2.15. (a) Let \( g : U \to S \) be a proper smooth morphism in \( Sm/k \) of pure relative dimension \( d_g \). Then there exists a canonical transformation of additive functors
\[
\delta_{\text{id}_S,g^*} : (\partial M, M, M^c)_S \to (\partial M, M, M^c)_{U}(-d_g)[-2d_g] \circ g^* .
\]
The formation of \( \delta_{\text{id}_S,g^*} \) is compatible with composition of proper smooth morphisms in \( Sm/k \) of pure relative dimension.
(b) Let \( g : U \to S \) be a finite étale morphism in \( Sm/k \) of constant (fibrewise) degree \( u \). Then the endomorphism \( \beta_{g^*,\text{id}_S} \circ \delta_{\text{id}_S,g^*} \) of the functor \( (\partial M, M, M^c)_S \) equals multiplication by \( u \).

Proof. (a) We may assume \( S \) to be of pure absolute dimension, say \( s \).
Consider the transformation \( \gamma_{\text{id}_S,g^*} : (\partial M, M, M^c)^*_S \to (\partial M, M, M^c)^*_U \circ g^* \) from Theorem 2.5 (d). Composition with duality \( D_S \) in \( CHM^*(S) \) gives
\[
\gamma_{\text{id}_S,g^*} \circ D_S : (\partial M, M, M^c)^*_S \circ D_S \to (\partial M, M, M^c)^*_U \circ g^* \circ D_S .
\]
Now observe the formula
\[
D_U \circ g^* = g^* \circ D_U
\]
(\( D_U := \text{duality in } CHM^*(U) \)). Define \( \delta_{\text{id}_S,g^*} \) as the composition of \( \gamma_{\text{id}_S,g^*} \circ D_S \) and \( M \mapsto M(s)[2s] \), observing that source and target of \( \delta_{\text{id}_S,g^*} \) are identified with \( (\partial M, M, M^c)_S \) and \( (\partial M, M, M^c)_U(-d_g)[-2d_g] \circ g^* \), respectively.
(b) The morphism \( g \) being finite and étale, we have
\[
D_S \circ g_2 = g_1 \circ D_U .
\]
This shows that \( g_2 \) is also right adjoint to \( g^* \). Checking the definitions, the composition \( \beta_{g^*,\text{id}_S} \circ \delta_{\text{id}_S,g^*} \) equals up to twist and shift the composition of the two adjunctions
\[
\xi : \text{id}_{CHM^*(S)} \to g_2 \circ g^* \to \text{id}_{CHM^*(S)} ,
\]
preceded by duality, and followed by \( (\partial M, M, M^c)^*_S \). These functors being additive, it suffices to show that \( \xi \) equals multiplication bu \( u \). But this identity on morphisms of smooth relative Chow motives is classical. \( \text{q.e.d.} \)

The main results of this section have obvious \( F \)-linear versions, for any commutative \( \mathbb{Q} \)-algebra \( F \). Let us now describe how our analysis of the functor \( (\partial M, M, M^c) \) will be used in the sequel.

Example 2.16. Let \( g_1, g_2 : U \to S \) be two finite étale morphisms in \( Sm/k \). Fix an object \( X \in \text{PropSm}/S \), an idempotent \( e \) on \( h(X/S) \) (possibly belonging to \( \text{CH}^*(X \times_S X) \otimes_{\mathbb{Z}} F \), for some commutative \( \mathbb{Q} \)-algebra \( F \)), and a morphism
\[
\varphi : g_1^*(h(X/S)^c) \to g_2^*(h(X/S)^c)
\]
in $CHM^e(U)$ (or $CHM^e(U)_F$).
(a) Let us define an endomorphism of $(\partial M, M, M^e)(h(X/S)^e)$ “of Hecke type”, denoted $\varphi(g_1, g_2)$, by composing

$$\delta_{\text{id}_S, g_1^1} : (\partial M, M, M^e)(h(X/S)^e) \rightarrow (\partial M, M, M^e)(g_1^1(h(X/S)^e))$$

first with $(\partial M, M, M^e) \circ \varphi$, and then with

$$\beta_{g_2^2, \text{id}_S} : (\partial M, M, M^e)(g_2^2(h(X/S)^e)) \rightarrow (\partial M, M, M^e)(h(X/S)^e).$$

(b) Note that unless $g_1 = g_2$, the endomorphism $\varphi(g_1, g_2)$ is in general not the image of an endomorphism on the smooth relative Chow motive $h(X/S)^e$ under the functor $(\partial M, M, M^e)$.
(c) If $\varphi$ is an isomorphism, with inverse $\psi$, then using the construction from (a), the endomorphism $\psi(g_2, g_1)$ on $(\partial M, M, M^e)(h(X/S)^e)$ can be defined. If $X$ is of pure absolute dimension $d_X$, then $\psi(g_2, g_1)$ equals the dual of $\varphi(g_1, g_2)$, twisted by $d_X$ and shifted by $2d_X$, under the identification

$$(\partial M, M, M^e)^*(h(X/S)) = (\partial M, M, M^e)(h(X/S))(-d_X)[-2d_X]$$

from Theorem 2.2 (b1). We leave the details of the verification to the reader.
(d) In practice, the morphism $\varphi : g_1^1(h(X/S)^e) \rightarrow g_2^2(h(X/S)^e)$ will be obtained from a morphism of smooth relative Chow motives over $U$

$$\varphi : h(X \times_{s,g_1} U/U) = g_1^1(h(X/S)) \rightarrow g_2^2(h(X/S)) = h(X \times_{s,g_2} U/U)$$

satisfying the equation

$$\varphi \circ g_1^1(e) = g_2^2(e) \circ \varphi$$

in $CH^*((X \times_{s,g_1} U) \times_U(X \times_{s,g_2} U))$ (or $CH^*((X \times_{s,g_1} U) \times_U(X \times_{s,g_2} U)) \otimes \mathbb{Z}F$).
In that case, $\varphi(g_1, g_2)$ can be seen as an endomorphism of the whole of $(\partial M, M, M^e)(h(X/S))$ commuting with $e$.
(e) In the setting of (d), assume that the morphism

$$\varphi : h(X \times_{s,g_1} U/U) \rightarrow h(X \times_{s,g_2} U/U)$$

is represented by the cycle $3$ in

$$c_U(X \times_{s,g_1} U, X \times_{s,g_2} U)$$

(or in $c_U(X \times_{s,g_1} U, X \times_{s,g_2} U) \otimes \mathbb{Z}F$). Checking the definitions, one sees that the $M$-component of $\varphi(g_1, g_2)$ is then represented by the image of $3$ under the direct image

$$(g_1 \times_k g_2)_* : c_U(X \times_{s,g_1} U, X \times_{s,g_2} U) \rightarrow c(X, X)$$

(or under $(g_1 \times_k g_2)_* \otimes F$).
3 Motives associated to Abelian schemes

Fix a field $k$ admitting strict resolution of singularities, and a base $S \in Sm/S$. In this section, we combine the main result from [DeMu] with the theory developed in Section 2. Recall the following.

**Theorem 3.1** ([DeMu Thm. 3.1, Prop. 3.3]). (a) Let $A/S$ be an Abelian scheme of relative dimension $g$. Then there is a unique decomposition of the class of the diagonal $\Delta \in CH^g(A \times_S A) \otimes \mathbb{Q}$,

$$\Delta = \sum_{i=0}^{2g} p_{A,i}$$

such that $p_{A,i} \circ (\Gamma_{[n]_A}) = n^i \cdot p_{A,i}$ for all $i$, and all integers $n$. The $p_{A,i}$ are mutually orthogonal idempotents, and $(\Gamma_{[n]_A}) \circ p_{A,i} = n^i \cdot p_{A,i}$ for all $i$.

(b) For any morphism $f : A \to B$ of Abelian schemes over $S$, and any $i$,

$$p_{B,i} \circ (\Gamma_f) = (\Gamma_f) \circ p_{A,i} \in CH^*(A \times_S B) \otimes \mathbb{Q}.$$  

In other words, the decomposition in (a) is covariantly functorial in $A$.

(c) For any isogeny $g : B \to A$ of Abelian schemes over $S$, and any $i$,

$$p_{B,i} \circ (\Gamma g) = (\Gamma g) \circ p_{A,i} \in CH^*(A \times_S B) \otimes \mathbb{Q}.$$  

In other words, the decomposition in (a) is contravariantly functorial under isogenies.

We use the notation $\Gamma_h$ for the graph of a morphism $h$ of $S$-schemes, $[n]_A$ for the multiplication by $n$ on the Abelian scheme $A$, $\mathfrak{3}$ for the class of a cycle $3$, and $\mathfrak{3}'$ for its transposition. Let

$$h(A/S) = \bigoplus_i h_i(A/S)$$

be the decomposition of the relative motive of $A$ corresponding to the decomposition $\Delta = \sum_i p_{A,i}$. Thus, on the term $h_i(A/S)$, the cycle class $(\Gamma_{[n]_A})$ acts via multiplication by $n^i$.

Now recall the exact triangle

$$(*)_A \quad \partial M(A) \to M(A) \to M^c(A) \to \partial M(A)[1].$$

By Theorem 2.2 (a), the cycle classes $p_{A,i}$ induce endomorphisms of $(*)_A$, when considered as an exact triangle in $DM_{gm}^{eff}(k)_\mathbb{Q}$.

**Theorem 3.2.** (a) Let $A/S$ be an Abelian scheme of relative dimension $g$. For $0 \leq i \leq 2g$, denote by $M(A)_i$, $M^c(A)_i$ and $\partial M(A)_i$ the images of the idempotent $p_{A,i}$ on $M(A)$, $M^c(A)$ and $\partial M(A)$, respectively, considered as objects of the category $DM_{gm}^{eff}(k)_\mathbb{Q}$. Then for any $i$, the triangle

$$(*)_A \quad \partial M(A)_i \to M(A)_i \to M^c(A)_i \to \partial M(A)_i[1]$$
in $\text{DM}_{gm}^{\text{eff}}(k)_{Q}$ is exact.

(b) The direct sum of the triangles $(\ast)_{A,i}$ yields a decomposition

$$(\ast)_A = \bigoplus_{i=0}^{2g} (\ast)_{A,i}.$$ 

It has the following properties:

(b1) for any integer $n$, the decomposition is respected by $[n]_A$.

(b2) for each $i$ and $n$, the induced morphisms $[n]_{A,i}$ on the three terms of $(\ast)_{A,i}$ equal multiplication by $n^i$.

(c) As a decomposition of $(\ast)_A$ into some finite direct sum of exact triangles in $\text{DM}_{gm}^{\text{eff}}(k)_{Q}$,

$$(\ast)_A = \bigoplus_i (\ast)_{A,i}$$ 

is uniquely determined by properties (b1) and (b2). More precisely, it is uniquely determined by the following properties:

(c1) for some integer $n \neq -1, 0, 1$, the decomposition is respected by $[n]_A$.

(c2) for the choice of $n$ made in (c1) and each $i$, the induced morphism $[n]_{A,i}$ on the three terms of $(\ast)_{A,i}$ equals multiplication by $n^i$.

(d) The decomposition

$$(\ast)_A = \bigoplus_i (\ast)_{A,i}$$

is covariantly functorial under morphisms, and contravariantly functorial under isogenies of Abelian schemes over $S$.

Proof. Part (a) is a formal consequence of the fact that the $p_{A,i}$ are idempotent.

Parts (b) and (d) follow from Theorem 3.1 and the functoriality statement from Theorem 2.2 (a).

Part (c) is left to the reader. q.e.d.

The following seems worthwhile to note explicitly.

Corollary 3.3. Let $A/S$ be an Abelian scheme of relative dimension $g$. Then the boundary motive $\partial M(A)$ decomposes functorially into a direct sum

$$\partial M(A) = \bigoplus_{i=0}^{2g} \partial M(A)_i.$$ 

On $\partial M(A)_i$, the endomorphism $[n]_A$ acts via multiplication by $n^i$, for any integer $n$, and any $0 \leq i \leq 2g$. 

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Here is an illustration of the surjectivity proved in [L, Lemma 5.18].

**Proposition 3.4.** Let $A/S$ be an Abelian scheme. The elements $p_{A,i}$ of $\text{CH}^*(A \times_S A) \otimes_{\mathbb{Z}} \mathbb{Q}$ lie in the image of

$$c_S(A, A) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \text{CH}^*(A \times_S A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$  

More precisely, for any integer $n \neq -1, 0, 1$,

$$\pi_{A,i,n} := \prod_{j \neq i} \frac{\Gamma[|n|_A] - n^j}{n^i - n^j}$$

is a pre-image of $p_{A,i}$ in $c_S(A, A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. On each of the direct factors $h_j(A/S) \subset h(A/S)$, the projector $p_{A,i}$ acts via multiplication by the Kronecker symbol $\delta_{ij}$, while $(\Gamma[|n|_A])$ acts via multiplication by $n^j$. Therefore,

$$p_{A,i} = \prod_{j \neq i} \frac{(\Gamma[|n|_A]) - n^j}{n^i - n^j} \in \text{CH}^*(A \times_S A) \otimes_{\mathbb{Z}} \mathbb{Q},$$

for any integer $n \neq -1, 0, 1$. Therefore, the element $\pi_{A,i,n} \in c_S(A, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is indeed a pre-image of $p_{A,i}$.

q.e.d.

4 The intersection motive of a surface

Fix a normal, proper surface $X^*$ over $k$. Let us first recall, following [CatMi], the construction and the basic properties of the *intersection motive* of $X^*$. Choose

$$X^* \longrightarrow X^* \hookrightarrow Z$$

where $Z$ is a closed sub-scheme of $X^*$ which is finite over $k$, and whose complement $X$ is smooth. Choose a resolution of singularities. More precisely, consider in addition the following diagram, assumed to be cartesian:

$$X \xrightarrow{\pi} \tilde{X} \xleftarrow{i} D$$

$$X \xrightarrow{\pi} X^* \xrightarrow{i} Z$$

where $\pi$ is proper (and birational), $\tilde{X}$ is smooth (and proper), and $D$ is a divisor with normal crossings, whose irreducible components $D_m$ are smooth (and proper).

Recall [Sch, Sect. 1.13] that the “degree 2 parts” $M_2(D_m)$ are canonically defined as sub-objects of the motives $M(D_m)$ (we remind the reader
that throughout the article, we use homological notation). Hence there is a canonical morphism
\[ i_{*2} : M_2(D) := \bigoplus_m M_2(D_m) \hookrightarrow \bigoplus M(D_m) \rightarrow M(\tilde{X}) \]
of Chow motives. Similarly \cite{Sch}, Sect. 1.11, there is a canonical morphism
\[ i_0^* : M(\tilde{X}) \rightarrow \bigoplus M(D_m)(1) \hookrightarrow \bigoplus M_0(D_m)(1)[2] , \]
where \( M_0(D_m) \) denote the “degree 0 parts”, canonically defined as quotients of \( M(D_m) \). The following is a special case of \cite{CatMi}, Sect. 2.5 (see also \cite{W5}, Thm. 2.2).

**Theorem 4.1.** (i) The composition \( \alpha := i_0^* i_{*2} \) is an isomorphism in the \( \mathbb{Q} \)-linear category \( DM^{eff}_{gm}(k)_\mathbb{Q} \).
(ii) The composition \( p := i_{*2} \alpha^{-1} i_0^* \) is an idempotent on \( M(\tilde{X}) \in DM^ {eff}_{gm}(k)_\mathbb{Q} \). Hence so is the difference \( \text{id}_X - p \).
(iii) The image \( \text{im} p \in DM^{eff}_{gm}(k)_\mathbb{Q} \) is canonically isomorphic to \( M_2(D) \).

The proof relies on the non-degeneracy of the intersection pairing on the components of \( D \).

**Definition 4.2** \cite{CatMi} p. 158, \cite{W5} Def. 2.3). The intersection motive of \( X^* \) is defined as
\[ M^{l*}(X^*) := \text{im}(\text{id}_X - p) \in DM^{eff}_{gm}(k)_\mathbb{Q} . \]
The name is motivated by the behaviour of the realizations of the intersection motive. Its functoriality properties are given in \cite{W5} Prop. 2.5. It will be useful to recall in particular the behaviour under finite morphisms \( f : Y^* \rightarrow X^* \) between normal, proper surfaces over \( k \). Assume that \( Z \) is such that the pre-image under \( f \) of \( X = X^* - Z \) is dense, and smooth (this can be achieved by enlarging \( Z \), if necessary). The closed sub-scheme \( f^{-1}(Z) \) of \( Y \) contains the singularities of \( Y^* \). We thus can find a cartesian diagram of desingularizations of \( X^* \) and \( Y^* \) of the type considered before:

\[ \begin{array}{c c c}
\tilde{Y} & \rightarrow & C \\
F \downarrow & & F \\
\tilde{X} & \rightarrow & \tilde{D}
\end{array} \]
The following is the content of \cite{W5} Prop. 2.5 (iii) and (iv), Prop. 2.4.

**Proposition 4.3.** (a) Both \( F^* \) and \( F_* \) respect the decompositions
\[ M(\tilde{X}) = M^{l*}(X^*) \oplus M_2(D) \]
and
\[ M(\tilde{Y}) = M^{l*}(Y^*) \oplus M_2(C) \]
of $M(\tilde{X})$ and of $M(\tilde{Y})$, respectively. The composition $F_* F^*$ equals multiplication with the degree of $f$.

(b) The definition of $M^!(X^*)$ is independent of the choices of the finite sub-scheme $Z$ containing the singularities, and of the desingularization $\tilde{X}$ of $X^*$.

Next, let us establish the connection to the boundary motive of $X$, and to the constructions of Section 1. To do so, assume $k$ to admit resolution of singularities, fix a dense open sub-scheme $X \subset X^*$ which is smooth, and choose

\[
\begin{array}{ccc}
X & \xrightarrow{i} & \tilde{X} \\
\downarrow \pi & & \downarrow \pi \\
X & \xrightarrow{} & X^* \xleftarrow{} Z
\end{array}
\]

as above. Recall the diagram of exact triangles

\[
\begin{array}{cccc}
0 & \xrightarrow{} & M(D)^*(2)[4] & \xrightarrow{} M(D)^*(2)[4] & \xrightarrow{} 0 \\
\downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\
M^c(X) & \xrightarrow{i^*} & M(\tilde{X}) & \xrightarrow{i^*} M(D) & \xrightarrow{} M^c(X)[-1] \\
\downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\
M^c(X) & \xrightarrow{} M(X) & \xrightarrow{} \partial M(X) & \xrightarrow{} M^c(X)[-1] \\
\downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\
0 & \xrightarrow{} M(D)^*(2)[3] & \xrightarrow{} M(D)^*(2)[3] & \xrightarrow{} 0
\end{array}
\]

from Theorem 1.13 (c), and let us refer to it using the symbol $(A)$. It turns out that the three components of the idempotent $p = i_* \alpha^{-1} i^*_0$ on $M(\tilde{X})$ all extend to give morphisms of diagrams of exact triangles: the first, denoted $i^*_0$, maps $(A)$ to

\[
\begin{array}{cccc}
0 & \xrightarrow{} & \bigoplus_m M_0(D_m)(1)[2] & \xrightarrow{} \bigoplus_m M_0(D_m)(1)[2] & \xrightarrow{} 0 \\
\downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\
0 & \xrightarrow{} & \bigoplus_m M_0(D_m)(1)[2] & \xrightarrow{} \bigoplus_m M_0(D_m)(1)[2] & \xrightarrow{} 0 \\
\downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\
0 & \xrightarrow{} & 0 & \xrightarrow{} 0 & \xrightarrow{} 0 \\
\downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\
0 & \xrightarrow{} & \bigoplus_m M_0(D_m)(1)[1] & \xrightarrow{} \bigoplus_m M_0(D_m)(1)[1] & \xrightarrow{} 0
\end{array}
\]

The second component $\alpha^{-1}$ maps this diagram isomorphically to the follow-
ing, which we shall denote by \((B)\).

$$
\begin{array}{c}
\uparrow & \uparrow & \uparrow \\
0 & M_2(D) & \text{by} \quad M_2(D) & 0 \\
\uparrow & \uparrow & \uparrow \\
0 & M_2(D) & M_2(D) & 0 \\
\uparrow & \uparrow & \uparrow \\
0 & 0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
0 & M_2(D)[-1] & M_2(D)[-1] & 0
\end{array}
$$

Finally the third component \(i_2\) maps \((B)\) back to \((A)\). The composition of the three morphisms, denoted by \(p: (A) \rightarrow (A)\), is idempotent. Its image is diagram \((B)\). Denote the image of \(id - p\) on \(M(D)\) by \(M_{\leq 1}(D)\). Then the image of \(id - p\) on the whole diagram equals

$$
\begin{array}{c}
\uparrow & \uparrow & \uparrow \\
0 & M_{\leq 1}(D)^*(2)[3] & M_{\leq 1}(D)^*(2)[4] & 0 \\
\uparrow & \uparrow & \uparrow \\
M^c(X) & M^c(X^*) & i^* & M_{\leq 1}(D) & M^c(X)[-1] \\
\uparrow & \uparrow & \uparrow \\
M^c(X) & M(X) & \partial M(X) & M^c(X)[-1] \\
\uparrow & \uparrow & \uparrow \\
0 & M_{\leq 1}(D)^*(2)[3] & M_{\leq 1}(D)^*(2)[3] & 0
\end{array}
$$

**Theorem 4.4.** Assume \(k\) to admit resolution of singularities.

(a) For fixed \(X^*\) and smooth \(X \subset X^*\), the above diagram of exact triangles is independent of the choice of \(\tilde{X}\).

(b) The diagram is covariantly and contravariantly functorial under finite morphisms \(f: Y^* \rightarrow X^*\) of normal, proper surfaces, which are compatible with the choices of smooth sub-schemes \(X \subset X^*\) and \(Y \subset Y^*: f^{-1}(X) = Y\).

(c) The third column of the diagram

\[
M_{\leq 1}(D)^*(2)[3] \rightarrow \partial M(X) \rightarrow M_{\leq 1}(D) \rightarrow M_{\leq 1}(D)^*(2)[4]
\]

is a weight filtration of \(\partial M(X)\):

\[
M_{\leq 1}(D)^*(2)[3] \in DM^{eff}_{gm}(k)_{Q,w \leq -1} \quad \text{and} \quad M_{\leq 1}(D) \in DM^{eff}_{gm}(k)_{Q,w \geq 0}.
\]

(d) The isomorphism classes of the weight filtration from (c) and of the Chow motive \(M^*(X^*)\) correspond under the bijection from Theorem 1.18 (\(Q\)-linear version).

**Proof.** Parts (a) and (b) follow from Proposition 4.3. Part (c) follows from stability under passage to direct factors 1.9 (1), and the fact that

\[
M(D)^*(2)[3] \rightarrow \partial M(X) \rightarrow M(D) \rightarrow M(D)^*(2)[4]
\]
Remark 4.5. Let us discuss Construction 1.17 in the present geometrical setting. The weight filtration of $\partial M(X)$ is

$$M_{\leq 1}(D)^*(2)[3] \xrightarrow{c_+} \partial M(X) \xrightarrow{c_-} M_{\leq 1}(D) \xrightarrow{\delta} M_{\leq 1}(D)^*(2)[4];$$

according to Theorem 4.4 (a), it is independent of the choice of $\tilde{X}$ (hence of $D$). It fits into the diagram of exact triangles

$$
\begin{array}{cccccc}
0 & \longrightarrow & M_{\leq 1}(D)^*(2)[4] & \longrightarrow & M_{\leq 1}(D)^*(2)[4] & \longrightarrow 0 \\
\uparrow & & \uparrow & & \uparrow & \\
M^c(X) & \xrightarrow{\delta} & M_{\leq 1}(D) & \xrightarrow{c_+(v_+[-1])} & M^c(X)[-1] & \\
\uparrow & & \downarrow & & \downarrow & \\
M^c(X) & \xrightarrow{v_-} & M(X) & \xrightarrow{v_+[-1]} & M^c(X)[-1] & \\
\uparrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & M_{\leq 1}(D)^*(2)[3] & \longrightarrow & M_{\leq 1}(D)^*(2)[3] & \longrightarrow 0
\end{array}
$$

In the general situation of Construction 1.17, what we did next was to choose some object $M_0$ completing the diagram. In the specific situation we are considering at present, there is only one choice, up to unique isomorphism, which in addition is compatible with any of the diagrams of type (A) associated to desingularizations $\tilde{X}$ of $X^*$. This choice is $M^{i*}(X^*)$. We thus obtain rigidification of the intersection motive, while the condition from Complement 1.24 on the absence of weights $-1$ and $0$ in the boundary motive is clearly not satisfied — unless $X^* = X$ is itself (proper and) smooth (cmp. Problem 1.22).

Let us finish this section by an example, which will allow us to illustrate both Principle 1.2 (on extensions) and Principle 1.19 (on functoriality).

Example 4.6. Our base field $k$ equals the field of rational numbers $\mathbb{Q}$. Fix a real quadratic number field $L$, and let $X$ be a Hilbert modular surface associated to $L$ and some level $K$. We view $K$ as an open compact subgroup of the group $G(\mathbb{A}_f)$ of (finite) adelic points of the group scheme $G$ from [R, Sect. 1.27]. The subgroup $K \subset G(\mathbb{A}_f)$ is assumed to be sufficiently small, a condition which ensures that $X$ is smooth over $\mathbb{Q}$. Denote by $X^*$ its Baily–Borel compactification; it is normal and projective over $\mathbb{Q}$.

(a) The morphism

$$i_* : M_{\leq 1}(D) \longrightarrow M^{i*}(X^*)$$

occurring in the weight filtration

$$M_{\leq 1}(D) \xrightarrow{i_*} M^{i*}(X^*) \longrightarrow M^c(X) \longrightarrow M_{\leq 1}(D)[1]$$
of $M^c(X)$ can be used to construct a morphism of certain sub-quotients of its source and target,

$$M_1(D) \rightarrow M_2^h(X^*)$$

[W5, Thm. 6.6]. This morphism can be interpreted as an element of

$$\text{Ext}^1_{DM^\text{eff}(\mathbb{Q})} (M_1(D)[-1], M_2^h(X^*)[-2]),$$

i.e., a one-extension in the triangulated category $DM^\text{eff}(\mathbb{Q})$ (note that according to [W5, Prop. 6.5], $M_1(D)[-1]$ is an Artin motive). Following [W5, Ex. 7.4], it can be related to the Kummer–Chern–Eisenstein extensions considered in [Cas]. In particular [W5, Ex. 7.4 (6)], the extension is non-trivial.

(b) The intersection motive $M^h(X^*)$ carries a natural action of the Hecke algebra $R(K, G(\mathbb{A}_f))$ associated to $K \subset G(\mathbb{A}_f)$. More precisely, let $x \in G(\mathbb{A}_f)$. The Hilbert surface $X$ is the target of two finite étale morphisms $g_1, g_2 : Y \rightarrow X$, where $Y$ denotes the Hilbert surface of level $K' := K \cap x^{-1}Kx$. In standard notation from the theory of Shimura varieties, the morphism $g_1$ corresponds to $[\cdot]$, and the morphism $g_2$ to $[\cdot x^{-1}]$. Both morphisms can be extended to finite morphisms between the Baily–Borel compactifications $g_i : Y^* \rightarrow X^*$, satisfying the formulae $g_i^{-1}(X) = Y$, $i = 1, 2$. According to Theorem 4.4 (b), the diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & M_1(D)^*(2)[4] & \rightarrow & M_1(D)^*(2)[4] & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
M^c(X) & \rightarrow & M^h(X^*) & \rightarrow & M_1(D) & \rightarrow & M^c(X)[-1] \\
\uparrow & & \uparrow & & \uparrow & & \\
M^c(X) & \rightarrow & M(X) & \rightarrow & \partial M(X) & \rightarrow & M^c(X)[-1] \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & M_1(D)^*(2)[3] & \rightarrow & M_1(D)^*(2)[3] & \rightarrow & 0
\end{array}
$$

is therefore stable under the composition $g_2 \circ g_1^*$. By definition, this composition equals the action of the class $K x K \in R(K, G(\mathbb{A}_f))$. In particular, the Hecke algebra acts on the whole of the above diagram. It is useful to note that its effect on the third row, i.e., on the boundary triangle, is the one described in Example 2.16 (d), for $X = S$, $\phi = \text{id}_Y$, and $e = \text{id}_X$.

5 The interior motive of a product of universal elliptic curves

In this section, our base field $k$ equals the field of rational numbers $\mathbb{Q}$. Fix integers $n \geq 3$ and $r \geq 0$, and let $S \in Sm/\mathbb{Q}$ denote the modular curve
parametrizing elliptic curves with level $n$ structure. Write $X \to S$ for the universal elliptic curve, and

$$X^r := X \times_S \ldots \times_S X$$

for the $r$-fold fibre product of $X$ over $S$. Recall the decomposition

$$h(X/S) = \bigoplus_{i=0}^2 h_i(X/S)$$

of the relative motive of $X$ from Theorem 3.1.

**Definition 5.1.** Define $rV \in \text{CH}^r(S) \mathbb{Q}$ as

$$rV := \text{Sym}^r h_1(X/S).$$

The tensor product is in $\text{CH}^r(S) \mathbb{Q}$, and the symmetric powers are formed with the usual convention concerning the (twist of) the natural action of the symmetric group on a power of $X$ over $S$ (see e.g. [Sch, Sect. 1.1.2]). Thus, $rV$ is a direct factor of $h(X^r/S) \in \text{CH}^r(S) \mathbb{Q}$. That is, it is associated to an idempotent

$$e \in \text{CH}^r(X^r \times_S X^r) \otimes \mathbb{Q}.$$  

From Theorem 2.2 (a), we get a natural action of $e$ on the boundary triangle of $X^r$. In particular, we have the following result.

**Proposition 5.2.** The triangle

$$(*)_{X^r} \quad \partial M(X^r)^e \to M(X^r)^e \to M^c(X^r)^e \to \partial M(X^r)^e[1]$$

in $DM^{\text{eff}}_{gm}(k) \mathbb{Q}$ is exact.

This triangle equals the image $(\partial M, M, M^c)(h(X^r/S)^c)$ of the smooth relative Chow motive $rV = h(X^r/S)^c$ under the functor $(\partial M, M, M^c)_S$ from Theorem 2.2. It is not very difficult to check that the idempotent $e$ coincides with the one used in [Sch, Sect. 1] and [W3, Sect. 3 and 4].

**Proposition 5.3** ([W3, Ex. 4.16 (d)]). The direct factor $\partial M(X^r)^e$ of the boundary motive of $X^r$ is without weights $-1$ and $0$ whenever $r \geq 1$.

By Complement 1.24, the $e$-part of the interior motive of $X^r$ can be constructed, and is unique up to unique isomorphism. It is shown in [W3, Thm. 3.3 (b) and Cor. 3.4 (b)] that it is canonically isomorphic to the Chow motive $r\mathcal{W}$ constructed in [Sch] out of a compactification of $X^r$. In that article, the action of the Hecke algebra on that compactification, hence on $r\mathcal{W}$ is then used to construct the Grothendieck motive $M(f)$ for elliptic normalized newforms $f$ of level $n \geq 3$ and weight $w = r + 2 \geq 3$. Let us finish this section by giving an alternative description of the action of the Hecke algebra on $r\mathcal{W}$, which avoids compactifications.
Example 5.4. Assume that \( r \geq 1 \), and consider the diagram

\[
\begin{array}{ccl}
0 & \longrightarrow & \partial M(X^r)_{\leq -2}[1] \\
\downarrow & & \downarrow \\
M^c(X^r)^{\leq -2} & \longrightarrow & \partial M(X^r)_{\leq -2}[1] \\
\downarrow & & \downarrow \\
M^c(X^r)^{\leq -2} & \longrightarrow & \partial M(X^r)_{\leq -2}[1] \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \partial M(X^r)_{\leq -2}
\end{array}
\]

from Complement 1.24 associated to the weight filtration of \( \partial M(X^r) \) avoiding weights \(-1\) and \(0\). We shall show that this diagram, and hence \( \nabla \mathcal{W} \) in particular, carries a natural action of the Hecke algebra \( R(K_n, GL_2(K_f)) \) associated to the principal subgroup \( K_n \subset GL_2(K_f) \) of level \( n \). Let \( x \in GL_2(K_f) \).

The curve \( S \) is the target of two finite étale morphisms \( g_1, g_2 : U \to S \), where \( U \) denotes the modular curve of level \( K' := K_n \cap x^{-1}K_nx \). In standard notation from the theory of Shimura varieties, the morphism \( g_1 \) corresponds to \([\cdot - 1]\), and the morphism \( g_2 \) to \([\cdot x^{-1}]\). Denote by \( X_1, X_2 \) the base changes of the universal elliptic curve \( X \to U \) via \( g_1 \) and \( g_2 \), respectively. To the data \( K_n \) and \( x \), the following are canonically associated: a third elliptic curve \( Y \) over \( U \), and isogenies \( f_1 : Y \to X_1 \) and \( f_2 : Y \to X_2 \). Now note that \( \varphi := \Gamma f_2 \circ \Gamma f_1 \) defines a morphism of smooth relative Chow motives over \( U \),

\[
\varphi : h(X^r_1 / U) = g_1^*(h(X^r / S)) \longrightarrow g_2^*(h(X^r / S)) = h(X^r_2 / U).
\]

Since both \( f_1 \) and \( f_2 \) are isogenies, this morphism is compatible with the external products of the idempotents \( p_{X_i,1} \) projecting onto the \( h_1 \) (Theorem 3.1 (b) and (c)). The morphism \( \varphi \) is also compatible with the action of the symmetric group; hence it is compatible with the cycle classes \( g_1^*(e_r) \in CH^r(X^r_1 \times_U X^r_1) \otimes Z \). This means that we have the relation

\[
\varphi \circ g_1^*(e_r) = g_2^*(e_r) \circ \varphi
\]

of morphisms of smooth relative Chow motives over \( U \). We are thus in the situation of Example 2.16 (d), and may therefore define the endomorphism \( \varphi(g_1, g_2) \) of the boundary triangle \((\ast)^{\lambda}_{X^r} \), i.e., of the third row of the above diagram. The weight filtration of \( \partial M(X^r)^{\leq -2} \) being functorial, \( \varphi(g_1, g_2) \) induces an endomorphism of the third column. Finally, thanks to Complement 1.24 the endomorphism extends uniquely to \( M_0 \). Altogether, \( \varphi(g_1, g_2) \) extends to the whole of the above diagram of exact triangles. By definition, this is the action of the class \( K_nxK_n \) we aimed at.
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