Instantons and the information metric

David Groisser∗
Department of Mathematics
University of Florida
Gainesville FL 32611–8105
USA
groisser@math.ufl.edu

Michael K. Murray
Department of Pure Mathematics
The University of Adelaide
Adelaide, SA 5005
Australia
mmurray@maths.adelaide.edu.au

12 November 1996

Abstract

The information metric arises in statistics as a natural inner product on a space of probability distributions. In general this inner product is positive semi-definite but is potentially degenerate. By associating to an instanton its energy density, we can examine the information metric $g$ on the moduli spaces $\mathcal{M}$ of self-dual connections over Riemannian 4-manifolds. Compared with the more widely known $L^2$ metric, the information metric better reflects the conformal invariance of the self-dual Yang-Mills equations, and seems to have better completeness properties. In the case of $SU(2)$ instantons on $S^4$ of charge one, $g$ is known to be the hyperbolic metric on the five-ball. We show more generally that for charge-one $SU(2)$ instantons over 1-connected, positive-definite manifolds, $g$ is nondegenerate and complete in the collar region of $\mathcal{M}$, and is ‘asymptotically hyperbolic’ there; $g$ vanishes at the cone points of $\mathcal{M}$. We give explicit formulae for the metric on the space of instantons of charge one on $\mathbb{C}P_2$.

1 Introduction

The information metric arises in statistics as a metric on a manifold of probability distributions $\mathbb{R}$. Its construction is very simple and can be applied, in principle, to any manifold which parametrises a set of probability distributions or measures. However in this general setting the information metric may be degenerate (though it is always positive semidefinite).

∗Supported in part by National Science Foundation grant DMS-9307648
One class of such manifolds are the minimum sets of variational problems. Here each point of the minimum set has associated to it an energy density—a measure of finite total integral—and we can apply the construction of the information metric on the space of energy densities. The metric on the minimum set is then actually the pull-back, under the map that assigns to any point its energy density, of the information metric on the space of energy densities. To show that the resulting pull-back metric is non-degenerate we have to prove that the map sending a point to its energy density is an immersion. Often in these types of problems there is a symmetry group which acts and preserves the energy density. The information metric will clearly be degenerate in directions parallel to the group action so we have to factor these out and construct a metric on the minimum set modulo the symmetry group.

For example consider the space of harmonic maps from two-sphere to itself. Rotations of the target sphere leave the energy density unchanged so that we consider the quotient space of harmonic maps modulo rotations. This is a manifold and it is possible to show that the energy density is an immersion and that therefore the information metric is non-degenerate [M].

The space of harmonic maps from $S^2$ to $S^2$ has often been used as a model for the space of instantons. In this paper we consider the problem of showing that the information metric on the instanton moduli space is non-degenerate and studying its behaviour. One such example is already known [H]. For instantons of charge 1 on the four-sphere the moduli space is a five ball and the information metric, on general grounds, is conformally invariant and hence the hyperbolic metric. Another example is the moduli space of charge one instantons on $CP_2$ which is a cone. We give explicit formulae for the information metric in this case. Recall that Donaldson [D] showed that for a large class of four manifolds $M$ the moduli space of charge one instantons ‘interpolates’ between these two models. That is it is a five-dimensional space having some singular points which are cones over $CP_2$, and having an ideal boundary, a neighbourhood of which looks like $(0,1) \times M$. We show that in such a situation the information metric vanishes at the cone points, and near the ideal boundary is asymptotic to an “asymptotically hyperbolic” metric of the form const $\cdot (dt^2 + g_M)/t^2$ where $g_M$ is the metric on $M$. We do not know in this generality if the information metric is nondegenerate between these two extremes.

The best-known metric on instanton moduli spaces is the $L^2$ metric ([GP1],[GP2],[G1]). In those cases where nondegeneracy of the information metric is known, there are several features distinguishing the the $L^2$ and information metrics on these spaces. First, unlike the $L^2$ metric, the information metric is conformally invariant, reflecting the conformal invariance of the self-duality equations (see §2). Second, in the five-dimensional examples above, and presumably in greater generality, the information metric tends to be complete near the ideal boundary (§3), whereas the $L^2$ metric tends to be incomplete (indeed the $L^2$ completion is in many cases known to be the Donaldson/Uhlenbeck compactification; see [F]). Third, the information metric is truly a quotient metric living on the unbased moduli space; it is degenerate on the based
moduli space (§2), and unlike the $L^2$ metric it cannot be induced by a Riemannian submersion from the based moduli space (§5). Fourth, the asymptotics of the metrics near cone singularities are quite different (§§4,5). And fifth, while for the $L^2$ metric one can easily write down a general formula for the Riemannian connection and curvature in terms of Green operators, for the information metric there is no obvious way to write down such general formulas (§2).

It was Hitchin [H] who first suggested the information metric as an alternative to the $L^2$ metric on these moduli spaces. The completeness of the information metric in the case of 1–instantons on $S^4$, and its better conformal properties in general (as compared with the $L^2$ metric), led Hitchin to speculate that for purpose of differential geometry on instanton moduli spaces the information metric might be the more suitable of the two.

The remainder of this paper is organized as follows. In §2, we review the construction of the information metric and see what form it takes on general instanton moduli spaces. In §3 we restrict attention to the five-dimensional moduli spaces mentioned above. We show that the information metric is nondegenerate in the collar (the region near the ideal boundary) and establish the asymptotics of the metric there. In §4 we show that for general $SU(2)$ moduli spaces the metric vanishes at reducible self-dual connections. In §5 we derive a concrete formula for the information metric on the space of charge-one instantons on $CP_2$. Along the way we discuss the differences noted above between the $L^2$ and information metrics.

Readers interested in the role played by the information metric and differential geometry in statistics more generally should look at Amari [A] and Murray and Rice [MR] and references therein.

## 2 The information metric on instanton moduli space

We start by reviewing the definition and some general properties of the information metric.

Let $M$ be a compact oriented $n$-dimensional manifold, let $\Omega^n(M)$ be the space of all (smooth) $n$-forms, and consider the subspace $\Omega^+_+(M)$ of nonnegative $n$-forms that vanish on no open set. The open set $\Omega_{++}$ of strictly positive $n$-forms can be be viewed as an infinite-dimensional manifold whose (formal) tangent space at any point $\tau$ is naturally isomorphic to $\Omega^n(M)$. For any $\tau \in \Omega_{++}$ and any $\alpha \in \Omega^n(M)$, the ratio $\alpha/\tau$ is well-defined, so we obtain a Riemannian metric on $\Omega_{++}$ by setting

\[
g_{\text{info}}(\alpha, \beta)_{\tau} = \int_M (\alpha/\tau)(\beta/\tau)\tau, \quad \tau \in \Omega_{++}, \quad \alpha, \beta \in T_\tau \Omega_{++} \cong \Omega^n(M).
\]  

(2.1)

This construction can be generalized by dropping some positivity and smoothness conditions. At any $\tau \in \Omega^+_+(M)$ consider the subspace $H_\tau \subset \Omega^n(M)$ of all $n$-forms that are of the form $f\tau$ for $f$ some function which is square integrable with respect
to \( \tau \). There is an inner product on \( H_\tau \) defined by

\[
\mathbf{g}_{\text{info}}(f_\tau, g_\tau) = \int_M fg \tau. \tag{2.2}
\]

(More generality is possible but unnecessary for our purposes.)

We refer to the Riemannian metric (2.1) and its generalization (2.2) as the information metric on the space of “energy densities”. Note that this metric is invariant under the group of orientation-preserving diffeomorphisms of \( M \).

Given another manifold \( S \) and a smooth map \( e : S \to \Omega^*_+(M) \) we define the information metric on \( S \) by

\[
\mathbf{g} = e^*\mathbf{g}_{\text{info}}.
\]

The term “metric” is used loosely here, since at each \( A \in S \) the quadratic form \( \mathbf{g}_A \) is positive semidefinite but is potentially degenerate. For example, if a Lie group \( G \) acts on \( S \) and leaves \( e \) invariant, then for any vector \( \eta \in T_\tau S \) tangent to a \( G \)-orbit we have \( \mathbf{g}(\eta, \cdot) \equiv 0 \). When the quotient space \( \mathcal{M} := S/G \) is a manifold it is more appropriate to consider the induced metric (which we will still denote \( \mathbf{g} \)) on \( \mathcal{M} \)—although here too \( \mathbf{g} \) is potentially degenerate.

To specialize to gauge theory, let \((M, g_M)\) be a compact, oriented Riemannian four-manifold and let \( P \) be a principal bundle over \( M \) with compact semisimple structure group \( G \). Let \( \mathcal{A} \) be the space of smooth connections on \( P \), \( \mathcal{G} \) the gauge group, \( SD \subset \mathcal{A} \) the subspace of self-dual connections, and \( \mathcal{M} = \mathcal{M}(P) = SD/G \) be the moduli space of \( P \)-instantons. (\( \mathcal{M}(P) \) depends on \( g_M \) but we suppress this from the notation.) For any connection \( A \) we can define the energy density

\[
e(A) = (F_A \wedge \star F_A) = (F_A, F_A)Vol_M \tag{2.3}
\]

where \(( , )\) denotes both the invariant inner product on the Lie algebra and the inner product constructed out of a metric on \( M \). The volume form \( Vol_M \) is defined using the metric \( g_M \).

Notice that \( e \) is invariant under the action of the gauge group, so its restriction to \( SD \) induces a smooth map

\[
e : \mathcal{M}(P) \to \Omega^4(M) \tag{2.4}
\]

from the moduli space to the space of all four-forms on \( M \).

Next recall the definition of the tangent space to \( \mathcal{M}(P) \) at \([A]\). Let \( Ad P \) be the adjoint bundle. For each self-dual connection \( A \) there is an elliptic complex

\[
0 \to \Omega^0(Ad P) \xrightarrow{d_A} \Omega^1(Ad P) \xrightarrow{p_-d_A} \Omega^2(Ad P) \to 0 \tag{2.5}
\]

where \( d_A \) is covariant exterior derivative and \( p_- \) projects a two-form onto its anti-self dual part. In this complex, \( \Omega^0(Ad P) \) represents the Lie algebra of \( \mathcal{G} \), \( \Omega^1(Ad P) \) the tangent space \( T_A \mathcal{A} \), and \( \Omega^2(Ad P) \) the tangent space \( T_A SD \) (at least...
formally). The sequence (2.5) is left-exact at every irreducible connection, and for generic metrics $g_M$ (and for some very non-generic metrics) is right-exact at every self-dual connection. We will always assume that $g_M$ is such a metric. In this case $\mathcal{M}^*(P)$, the subspace of irreducible instantons, is a manifold, and the tangent space $T_{[A]} \mathcal{M}^*(P)$ is formally $H^1$ of the complex (2.5). We call an object defined on $\mathcal{M}(P)$ smooth (e.g. the map $e$ in (2.4)) if its restriction to $\mathcal{M}^*(P)$ is smooth.

The curvature of an instanton cannot vanish on an open set (see the proof of Theorem 3.4 in [FU]), and hence the energy density of an instanton is contained in the subspace $\Omega^4(M)$. If the image of the tangent space $T_{[A]} \mathcal{M}$ at the energy density $e(A)$ is in the space $H^1_e(A)$ then we can pull the information metric back to $SD$. To see that it is consider the derivative of the energy density function in the direction of a tangent vector $\eta$:

$$\frac{\partial e}{\partial \eta}(A) = 2(d_A \eta \wedge *F_A) = 2(d_A \eta, F_A) \text{Vol}_M$$

Then

$$(d_A \eta \wedge *F_A) = 2 \frac{(d_A \eta, F_A)}{(F_A, F_A)} e(A)$$

and by Cauchy’s inequality

$$\frac{(d_A \eta, F_A)}{(F_A, F_A)} \leq \frac{|d_A \eta|}{|F_A|}$$

which is certainly square integrable with respect to $e(A)$.

The information metric on $SD$ (actually on all of $A$) is therefore

$$g(\eta_1, \eta_2) = \int_M \left( \frac{\partial e}{\partial \eta_1}(A)/e(A) \right) \left( \frac{\partial e}{\partial \eta_2}(A)/e(A) \right) e(A)$$

$$= 4 \int_M \frac{(d_A \eta_1, F_A)(d_A \eta_2, F_A)}{(F_A, F_A)} \text{Vol}_M$$

$$(2.6)$$

$$= 4 \int_M (d_A \eta_1, \hat{F}_A)(d_A \eta_2, \hat{F}_A) \text{Vol}_M,$$

where $\hat{F}_A = F_A/|F_A|$. The degeneracy of the information metric tangent to orbits of the gauge group can be seen explicitly in (2.6), since for $v \in \Omega^0(Ad P)$ we have

$$(F_A, d_A(\eta + d_A v)) = (F_A, d_A \eta) + (F_A, [F_A, v]) = (F_A, d_A \eta).$$

(2.7)

Because of this degeneracy the same formula (2.6) serves as the definition of the quotient information metric on $\mathcal{M}^*(P)$ at $[A]$.

To have any hope of obtaining a nondegenerate metric on a moduli space, note that it is important that we divide by the action of the full gauge group; the information metric on the based moduli space is automatically degenerate. For if we divide only by the action of the based gauge group, there is a residual action of $G/(\text{center}(G))$ on the quotient—which as we saw above is a recipe for degeneracy along the orbits.
The information metric is still potentially degenerate on $\mathcal{M}^*(P)$; conceivably there exist $\eta \notin \text{im}(d_A)$ for which $(d_A\eta, F_A) \equiv 0$. Since this amounts to an infinite number of conditions on an element of the finite-dimensional space $T_{[A]}\mathcal{M} \cong \ker(d_A^*)/\text{im}(d_A)$ (where $d_A = p \cdot d_A$), it is plausible that for generic metrics $g_M$ the information metric on $\mathcal{M}^*(P)$ is nondegenerate. However, any such theorem will need to make use of irreducibility, since as we show in §4 the information metric vanishes at reducible connections.

Notice that the definition of the information metric does not rely on choosing a particular base metric $g_M$ except in as much as that is needed for the definition of self-duality; a conformal change of $g_M$ leaves $g$ unchanged. Moreover it is well known that the space of instantons is invariant under conformal diffeomorphisms of $M$ homotopic to the identity, and it follows that the information metric is invariant under transformations of $\mathcal{M}(P)$ induced by conformal transformations of $M$. This is in marked contrast to the $L^2$ metric on $\mathcal{M}(P)$ which is only invariant under isometries of $M$. When $M$ is the four-sphere and $P$ is the $SU(2)$-bundle of instanton number 1, $\mathcal{M}(P)$ is well-known to be the five-ball, and (as first noted by Hitchin [H]) the conformal invariance of the information metric implies that it is the hyperbolic metric on the five ball, up to scale.

With a formula for the general information metric in hand, it is natural to try to write down a formula for the Riemannian connection, and from this compute curvature. Indeed, if $\nabla$ is the putative Levi-Civita connection, it is straightforward to write down a formula for $g(\nabla_\alpha \beta, \gamma)$, where $\alpha, \beta, \gamma$ are vector fields on $SD$ obtained by applying the $L^2$ projection $T_A A \to \ker((d_A^*)^*) \cong T_A SD$ to “constant” vector fields. We find

$$g(\nabla_\alpha \beta, \gamma)|_A = \int_M Q(\alpha, \beta)(d_A \gamma, \hat{F}_A)\text{Vol}_M,$$

where

$$Q(\alpha, \beta) = ([\alpha, \beta] - d_A(d_A^*)^*(d_A^*)^{-1}p_-[\alpha, \beta], \hat{F}_A) + |F_A|^{-1}((d_A \alpha, d_A \beta) - (d_A \alpha, \hat{F}_A)(d_A \beta, \hat{F}_A));$$

here $[\alpha, \beta] \in \Omega^2(Ad P)$ is the wedge-bracket of $\alpha, \beta \in \Omega^1(Ad P)$. Were $Q(\alpha, \beta)$ of the form $(d_A(\text{something}), \hat{F}_A)$ we would obtain a formula for the Levi-Civita connection—as one is able to do for the $L^2$ metric—but unfortunately this is not the case here. For this reason we are at present unable to analyze curvature invariants of the information metric, except in those two cases in which the metric is explicitly computable—the spaces of 1-instantons over $S^4$ and $CP_2$.

### 3 The collared moduli spaces

The best understood moduli spaces are those in which the base space and principal bundle satisfy the following topological conditions.
Topological Conditions 3.0 (i) $M$ is a closed, simply connected oriented 4-manifold whose intersection form on $H^2(M)$ is positive-definite; and (ii) $P$ is the principal $SU(2)$-bundle over $M$ with instanton number (Pontryagin index) 1.

Throughout this section we will assume $M$ and $P$ satisfy these conditions, which ensure that at smooth points the dimension of $\mathcal{M}(P)$ is five. We write $\mathcal{M}(P) = \mathcal{M}_1(M) = \mathcal{M}_1$, and assume $g_M$ has been chosen so that the subspace $\mathcal{M}_1^*$ of irreducible instantons is a manifold. A small neighborhood of each reducible point in $\mathcal{M}_1$ is then topologically a cone on $\mathbb{C}P_2$; see [D1] and [FU]. Furthermore there is a compact set in $\mathcal{M}_1$ whose complement—the “collar”—consists of instantons with curvature concentrated near a point, and is diffeomorphic to (open interval) $\times M$ via the map assigning to each sufficiently concentrated connection $A$ a scale $\lambda(A)$ and center point $\text{ctr}(A)$.

In the explicitly computable examples ($M = S^4$ or $\mathbb{C}P_2$) one usually defines $\lambda(A)$ to be the radius of the smallest ball containing half the total energy $\|F_A\|_2^2$, and $\text{ctr}(A)$ to be the center of this ball. More generally this definition is modified non-canonically to ensure differentiability of the scale and center-point functions (see [D1]); this definition coincides asymptotically with the preceding one. In this section, since we deal with general 4-manifolds obeying the topological conditions 3.0(i), we use such a non-canonical definition of $\lambda$ and $\text{ctr}$. This gives us a diffeomorphism $\Psi$ from a region $\mathcal{M}_1^{\lambda_0}$ to $(0, \lambda_0] \times M$ for $\lambda_0$ sufficiently small.

In what follows, we use $\lambda$ to denote both the scale function on $\mathcal{M}$ and the corresponding real variable in the interval $(0, \lambda_0]$. Also we let $g_M$ denote both the metric on $M$ and its pullback to $(0, \lambda_0] \times M$. The letter $c$ is used for a continually updated constant whose value can depend on $g_M$ but is independent of all other parameters of interest.

We will prove the following theorem.

**Theorem 3.1** Let $M, P$ satisfy the topological conditions 3.0. Then as $\lambda \to 0$,

$$
(\Psi^{-1})^* g \sim \frac{128\pi^2}{5}(d\lambda^2 + g_M) := g_{\text{hyp}},
$$

(Here $\sim$ means asymptotic in a $C^0$ sense only: for any $\epsilon > 0$, there exists $\lambda_0 > 0$ such that for any tangent vector $X \in T((0, \lambda_0] \times M)$ we have $|(\Psi^{-1})^* g(X,X) - g_{\text{hyp}}(X,X)| \leq \epsilon g_{\text{hyp}}(X,X)$.)

In particular, for $\lambda_0$ sufficiently small $g$ is nondegenerate and $\mathcal{M}_1^{\lambda_0}$ is complete with respect to the distance function defined by $g$.  

Remark. The metric $(d\lambda^2 + g_M)/\lambda^2$ on $\mathbb{R}^+ \times M$ is “asymptotically hyperbolic” in the sense that as $\lambda \to 0$, all sectional curvatures approach 1, regardless of the metric $g_M$. Also, sitting above every geodesic $\gamma$ in $M$ is an immersed, totally geodesic copy of the hyperbolic plane; if $t$ is an arclength parameter along $\gamma$, the induced metric on $\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \text{image}(\gamma) \subset \mathbb{R}^+ \times M$ is $(d\lambda^2 + dt^2)/\lambda^2$. Every geodesic in $\mathbb{R}^+ \times M$ is contained in such a 2-strip, so $\mathbb{R}^+ \times M$ is geodesically complete in this metric.
It follows that for all \( a \in \mathbb{R}^+ \), \((0,a] \times M\) is complete as a metric space with the induced distance function. Since (3.1) implies that the distance functions induced by \((d\lambda^2 + g_M)/\lambda^2\) and \(g\) are equivalent on \((0,\lambda_0] \times M\), the completeness assertion in Theorem 3.1 is an immediate consequence.

The proof of Theorem 3.1 requires some analysis, to which we devote the rest of this section. In several places we use estimates derived in [GP2], [G2], and [G3].

Given \([A] \in \mathcal{M}_1^*\), \(T_{[A]}\mathcal{M}^*_1\) can be naturally identified with the harmonic space \(H_A = \ker(d^*_{\lambda}) \cap \ker(d_{\lambda}) \subset \Omega^1(AdP)\). More precisely, the harmonic spaces piece together into a gauge-invariant subbundle of \(AdP\), and \(T_{[A]}\mathcal{M}^*_1\) is naturally isomorphic to the space of gauge-invariant sections along the gauge orbit \([A]\). Since the information metric on \(AdP\) is gauge-invariant, to compute the metric at \([A]\) from (2.6) it suffices to choose any \(A \in [A]\) and take \(\eta\) to be \(A\)-harmonic.

To make the inner product effectively computable one needs some idea of what such \(\eta\) look like. In [GP2] an “approximate tangent space” \(\tilde{H}_A\) was introduced for this purpose, approximating elements of \(H_A\) by purely local objects. The accuracy of this approximation in various norms was measured in [GP1] and later sharpened in [G2-G3], and we will use these estimates to determine the error introduced by replacing true tangent vectors in (2.6) by their \(L^2\) projections to \(\tilde{H}_A\). First we recall the definition of the approximate tangent space. For this purpose we fix a smooth cutoff function \(b \in C_0^\infty(\mathbb{R})\) with \(b(t) = 1\) for \(0 \leq t \leq 1\), \(b(t) = 0\) for \(t \geq 2\) and \(0 \leq b(t) \leq 1\) everywhere. Given \(p \in M\), let \(r_p\) denote distance to \(p\) and define \(\beta_p(\cdot) = b(r_p(\cdot)/r_0)\), where \(4r_0\) is less than the injectivity radius of \((M,g_M)\).

**Definition.** Let \([A] \in \mathcal{M}_{\lambda_0}\) have center point \(p = \text{ctr}(A)\), and let \(\{x^i\}_1^4\) be normal coordinates based at \(p\). Given \(a \in T_pM\) and \(a_0 \in \mathbb{R}\), define functions \(\phi = \frac{1}{2}\beta_p r_p^2\), \(\phi_a = \beta_p a_i x^i\) (note that \(\phi_a\) is independent of the choice of normal coordinate system), and \(\phi(a_0,a) = \lambda^{-1}a_0\hat{\phi} + \phi_a\); also define a vector field \(Z(a_0,a) = \nabla_a\phi(a_0,a)\). For any vector field \(Z\) on \(M\) define \(\tilde{Z}^A = t_Z F_A\). The approximate tangent space at \(A\) is the space

\[
\tilde{H}_A := \{\tilde{Z}(a_0,a) \mid (a_0,a) \in \mathbb{R} \times T_p(M)\}.
\]

For \(\lambda_0\) sufficiently small, the \(L^2\)-orthogonal projection \(\pi_A : H_A \rightarrow H_A\) is an isomorphism ([GP2] §5), and thus so is the map

\[
\alpha_{\Psi(A)} : \mathbb{R} \times T_{\text{ctr}(A)}M \rightarrow H_A
\]

\[
(a_0, a) \mapsto -\pi_A \tilde{Z}(a_0,a).
\]

In fact, \(\alpha\) approximately inverts the differential of \(\Psi:\)

\[
|((\Psi_* \circ \alpha_{\Psi(A)}) - \text{Id})(a_0, a)| \leq c(|a_0| + |a|)\lambda.
\]

(The bound \(c(|a_0| + |a|)\lambda^{1-\delta}\) appearing in Proposition 5.2 of [GP2] was strengthened to \(c(|a_0|\lambda^{1+\delta} + |a|\lambda^2)\) in Proposition 1.1 of [G3], but actually any positive power of \(\lambda\) is sufficient for our purposes.)
We will prove Theorem 3.1 by showing that both \((\Psi^{-1})^*g\) and \(g_{\text{hyp}}\) are asymptotic to \(\alpha^*g\).

**Part I: \(\alpha^*g \sim g_{\text{hyp}}\)**

Let \(\epsilon > 0\) be arbitrary. We subdivide our argument further into two steps: computing (2.6) when \(\eta_i\) are replaced by their approximate counterparts \(\tilde{Z}_{(a_0,a)}\), and bounding the error introduced by the approximation \(\tilde{Z}_{(a_0,a)} \approx \pi_A \tilde{Z}_{(a_0,a)} = -\alpha_{\Psi(A)}(a_0,a)\).

Below, we write simply \(\alpha\) for \(\alpha_{\Psi(A)}\).

For the first step, it suffices to consider
\[
g(\tilde{Z}_{(a_0,a)}, \tilde{Z}_{(a_0,a)}) = 4 \int_M \frac{(F_A, d_A \tilde{Z}_{(a_0,a)})^2}{|F_A|^2} \text{Vol}_M
\]
(3.5) since we can determine \(g(\tilde{Z}_{(a_0,a)}, \tilde{Z}_{(b_0,b)})\) from this by polarization.

**Lemma 3.2** For any vector field \(Z\) on \(M\) and any self-dual connection \(A\) we have
\[
(F_A, d_A (\iota_X F_A)) = \frac{1}{2} (\text{div}(X)|F_A|^2 + X(|F_A|^2)).
\]
(3.6)
Here \(\text{div}(X) = -d^*(X^\text{dual})\), where \(X^\text{dual}\) is the metric dual of \(X\).

**Proof:** Let \(F = F_A\) let \(\{e_i\}\) be a local orthonormal basis of \(TM\), and let \(\theta^i\) be the dual coframe. Write \(\nabla\) for the Levi-Civita connection on \(M\), \(\nabla^A\) for the tensor product connection on \(AdP \otimes \Lambda^k T^* M\) and \(\nabla_i = \nabla e_i\). Using the Bianchi identity as in the proof of Lemma 3.1 of [GP2] one finds \(d_A (\iota_X F) = \theta^i \wedge (\iota_{\nabla_i} X F) + \nabla^A_X F\). Furthermore since \(F\) is self-dual,
\[
(F, \theta^i \wedge (\iota_{\nabla_i} X F)) = (\iota_{e_i} F, \iota_{\nabla_i} X F) = \frac{1}{2} (e_i, \nabla_i X) (F, F)
\]
(see [GP2], Lemma 3.4). But \((e_i, \nabla_i X) = \text{div}(X)\) and \((F, \nabla^A_X F) = \frac{1}{2} X(|F|^2)\), so (3.6) follows.

We will apply this with \(X = Z_{(a_0,a)}\) and \(A \in \mathcal{M}_\lambda\). In view of (3.3) we have
\[
g(\tilde{Z}_{(a_0,a)}, \tilde{Z}_{(a_0,a)}) = \int_M (\text{div}(Z_{(a_0,a)}) |F_A| + 2Z_{(a_0,a)}(|F_A|))^2 \text{Vol}_M
\]
(3.7) We break this up into an integral over \(B_{N\lambda}(p)\) (the ball of radius \(N\lambda\) centered at \(p = \text{ctr}(A)\)) and its complement, where \(N\) is to be determined later. (The smaller the \(\epsilon\) in Theorem 3.1, the larger \(N\) must be taken; to simplify estimates, we will always take \(N \geq 1\).)
Interior estimates.

Recall that for any \( k \geq 0 \) and any compact set \( K \subset \mathbb{R}^4 \), for \( \lambda \) sufficiently small, after a suitable gauge choice and rescaling of coordinates \( F_A \) is \( C^k(K) \)-close to the curvature of standard instanton of scale 1 on \( \mathbb{R}^4 \). After undoing the rescalings, this implies that given \( N, \delta > 0 \), there exists \( \lambda_0 > 0 \) such that for all \( [A] \in \mathcal{M}_{\lambda_0} \) and \( x \in B_{N\lambda}(p) \) we have

\[
\lambda^2 | |F_A| - |F_{0,\lambda}| | (x) + \lambda^2 | \nabla|F_A| - \nabla|F_{0,\lambda}| | (x) \leq \delta
\]  
(3.8)

(see Theorem 16 of [D1]). Here, after a normal-coordinate identification of a small ball centered at \( p \in M \) with a small ball centered at \( 0 \in \mathbb{R}^4 \), \( |F_{0,\lambda}| = \sqrt{48\lambda^2/(\lambda^2 + r^2)^2} \) is the norm of the standard instanton on \( \mathbb{R}^4 \) of scale \( \lambda \) centered at \( p \). It is immaterial in (3.8) whether the norms are computed with respect to the metric \( g_M \) or the flat metric in normal coordinates.

For the divergence term in (3.7), letting \( r = r_p \) (with \( p = \text{ctr}(A) \)) we have

\[
\text{div}(Z_{(a_0,a)}) = -\Delta(\phi_{(a_0,a)}) = 4a_0\lambda^{-1} + O(|a| r^2 + |a_0|^2 + r^2) \]  
(3.9)

Choose \( \lambda_0 \) small enough that \( N\lambda_0 \ll r_0 \); thus \( N \leq \text{const} \cdot c\lambda^{-1} \) and on \( B_{N\lambda}(p) \) we have \( r \leq N\lambda \leq \text{const} \), a fact we will use frequently below without further mention. Then the derivative of the cutoff \( \beta \) in the definition of \( Z_{(a_0,a)} \) vanishes on \( B_{N\lambda}(p) \), so, writing \( c_1 = \sqrt{48} \) and noting that \( |Z_{(a_0,a)}| \leq c(|a| + |a_0|\lambda^{-1}r) \) we have

\[
Z_{(a_0,a)}(|F_{0,\lambda}|) = -4c_1\lambda^2(\lambda^2 + r^2)^{-3}(a_i x^i + a_0\lambda^{-1}r + O(|a| r^3))
= -4c_1\lambda^2(\lambda^2 + r^2)^{-3}(a_i x^i + a_0\lambda^{-1}r^2) + O(|a|\lambda^{-1}).
\]

Using (3.8) and letting \( Z = Z_{(a_0,a)} \), on \( B_{N\lambda}(p) \) we then have

\[
\text{div}(Z)|F_A| + 2Z(|F_A|) = \left(4a_0\lambda^{-1}|F_{0,\lambda}| + 2Z(|F_{0,\lambda}|)\right)
+ O(|a| r + |a_0|\lambda^{-1}r^2)|F_A| + 4a_0\lambda^{-1}O(\delta\lambda^{-2}) + O(|Z|\delta\lambda^{-3})
\]

\[
= \frac{4c_1\lambda^2}{(\lambda^2 + r^2)^3} (a_0\lambda^{-1}(\lambda^2 - r^2) - 2a_i x^i)
+ O (|a| + |a_0|)\lambda^{-1}(1 + N\delta\lambda^{-2}) .
\]  
(3.10)

Because the metric on \( B_{N\lambda}(p) \) is Euclidean (in normal coordinates) up to \( O(r^2) \), we can estimate the main term in (3.5) arising from (3.10) by the corresponding Euclidean integral, and similarly we can bound the integrated error terms. Writing \( d^4x \) for the Euclidean volume form, we find

\[
\int_{B_{N\lambda}(p)} \left[ \frac{4c_1\lambda^2}{(\lambda^2 + r^2)^3} (a_0\lambda^{-1}(\lambda^2 - r^2) - 2a_i x^i) \right]^2 d^4x
\]

\[
= 16c_1^2 \text{Vol}(S^3) \int_0^{N\lambda} \left\{ \frac{\lambda^4}{(\lambda^2 + r^2)^6} \left[ a_0^2\lambda^2(\lambda^2 - r^2)^2 + 4 \cdot \frac{1}{4} |a|^2 r^2 + (\text{odd function}) \right] \right\} r^3 dr
\]

\[
= 16 \cdot 48 \cdot 2\pi^2\lambda^{-2} \int_0^{N\lambda} \rho^3 \left[ a_0^2(1 - \rho^2)^2 + |a|^2 \rho^2 \right] d\rho.
\]
As \( \int_0^\infty \frac{\rho^2}{(1 + \rho^2)^4} (1 - \rho^2)^2d\rho \) and \( \int_0^\infty \frac{\rho^2}{(1 + \rho^2)^4} d\rho \) converge (both to 1/60), we can choose \( N \) large enough that the integrals from 0 to \( N \) above differ from their limiting values by less than \( \epsilon/(16 \cdot 48 \cdot 2\pi^2) \). Hence
\[
\left| \int_{B_{N\lambda}(p)} \frac{4c_1 \lambda^2}{(\lambda^2 + r^2)^3} \left(a_0 \lambda^{-1}(\lambda^2 - r^2) - 2a_i x^i\right) \right|^2 d^4x - \| (a_0, a) \|_{\text{hyp}}^2 \leq \epsilon \| (a_0, a) \|_{\text{hyp}}^2
\]
(3.11)
where \( \| \cdot \|_{\text{hyp}} \) is the norm associated with \( g_{\text{hyp}} \).

Now we turn to the integrated error terms (still on \( B_{N\lambda}(p) \)). These arise from two sources: the error term in (3.10), and the \( O(r^2) \) difference between \( d^4x \) and the Riemannian volume form on \( B_{N\lambda}(p) \). Noting that \( \lambda^2 \frac{\lambda}{(\lambda^2 + r^2)^2} |a_0 \lambda^{-1}(\lambda^2 - r^2) - 2a_i x^i| \leq c \frac{\lambda}{(\lambda^2 + r^2)^2} (|a_0| + |a|) \) one finds that the error introduced by the difference in volume forms is bounded by the error term arising from (3.10), and hence by
\[
c(|a_0| + |a|)^2 \int_0^{N\lambda} \lambda^{-1}(1 + N\delta \lambda^{-2}) \lambda \frac{\lambda}{(\lambda^2 + r^2)^2} \lambda^{-1}(1 + N\delta \lambda^{-2}) r^3 dr \leq c \lambda^{-2} (|a_0|^2 + |a|^2) (\lambda^2 + N\delta) (\log N + \lambda^2 N^4 + \delta N^5).
\]
(3.12)
Since \( \log N \leq |\log \lambda| + \text{const} \), by taking \( \delta = \delta(N) \) small enough (and reducing \( \lambda_0 \), if necessary), we can arrange for this last bound to be less than \( \epsilon \| (a_0, a) \|_{\text{hyp}}^2 \). Combining this with (3.10), we arrive at
\[
\left| \int_{B_{N\lambda}(p)} \left( \text{div}(Z_{(a_0, a)}) |F_A| + 2Z_{(a_0, a)}(|F_A|) \right)^2 \text{Vol}_M - \| (a_0, a) \|_{\text{hyp}}^2 \right| \leq 2\epsilon \| (a_0, a) \|_{\text{hyp}}^2.
\]
(3.13)

Exterior estimates.

Let \( \Omega = \Omega(p) = B_{2r_0}(p) - B_{N\lambda}(p) \). Since
\[
|\text{div}(Z) F_A + 2Z(|F_A|)| \leq c \left( |a| (r|F_A| + |\nabla_A F_A|) + |a_0| \lambda^{-1}(|F_A| + r|\nabla_A F_A|) \right)
\]
(3.14)
and the vector fields \( Z_{(a_0, a)} \) are supported in \( B_{2r_0}(p) \), to bound the contribution to (3.7) from the complement of \( B_{N\lambda}(p) \) it suffices to bound \( \| r^k F_A \|_{L^2(\Omega)} \) and \( \| r^k \nabla_A F_A \|_{L^2(\Omega)} \)
for \( k = 0, 1 \). The non-derivative norms are easy to evaluate since for \( [A] \in \mathcal{M}_{\lambda_0} (\lambda_0 \) sufficiently small) we have the pointwise bound
\[
|F_A| \leq \text{const} \frac{\lambda^2}{(\lambda^2 + r^2)^2}
\]
(3.15)
everywhere on \( M \) (see [GP3], §5). Because the ratio of \( \text{Vol}_M \) to \( d^4x \) is bounded on \( B_{2r_0}(p) \), for \( -2 < k < 2 \) we therefore have
\[
\| r^k F_A \|_{L^2(\Omega)} \leq c \left[ \int_{N\lambda}^{r_0} \left\{ \frac{\lambda^4}{(\lambda^2 + r^2)^4} \right\} r^{2k+3} dr \right]^{1/2} \leq c(k) \lambda^k N^{k-2}.
\]
(3.16)
Bounding the norms that involve $\nabla_A F_A$ is less direct because one does not have an analog of (3.15) available. Instead, we introduce a cutoff function $\gamma$ that is identically 1 on $\Omega$:

$$
\gamma(\cdot) = \gamma_p(\cdot) = b\left(\frac{r_p(\cdot)}{2r_0}\right) \left(1 - b\left(\frac{r_p(\cdot)}{2N\lambda}\right)\right).
$$

Thus $\|r^k \nabla_A F_A\|_{L^2(\Omega)}^2 \leq \|\gamma r^k \nabla_A F_A\|_{L^2(\Omega)}^2$, where $\|\cdot\|_2 = \|\cdot\|_{L^2(M)}$. If we integrate the $L^2(M)$-inner product by parts and use the Weitzenböck identity for self-dual 2-forms (which implies that $|\nabla_A^* \nabla_A F_A| \leq c(|F_A| + |F_A|^2)$), we find

$$
\|\gamma r^k \nabla_A F_A\|_2 \leq c\left(\|\gamma kr^{k-1} F_A\|_2 + \|d\gamma r^k F_A\|_2 + \left[\int_M \gamma^2 r^{2k} |F_A|^3 Vol_M\right]^{1/2}\right).
$$

The first term on the right is $O(\lambda^{-1}N^{-k-3})$ as in (3.16), and the third term is similarly seen to be $O(\lambda^{-1}N^{-k-4})$ (if $-2 < k < 4$). For the middle term, note that $|d\gamma| \leq c((N\lambda)^{-1} \chi_{in} + \chi_{out})$, where $\chi_{in}$ and $\chi_{out}$ are the characteristic functions of the annuli $B_{N/\lambda}(p) - B_{N/(2\lambda)}(p)$ and $B_{4r_0}(p) - B_{2r_0}(p)$ respectively. Integrating as in (3.16) one then finds that $\|d\gamma r^k F_A\|_2 \leq c(\lambda^{k-1}N^{-k-3} + \lambda^2) \leq c\lambda^{k-1}N^{-k-2}$.

$$
\|r^k \nabla_A F_A\|_{L^2(\Omega)} \leq \|\gamma r^k \nabla_A F_A\|_2 \leq c(k)\lambda^{k-1}N^{-k-3}
$$

for $-1 < k < 3$. We conclude that

$$
\|\text{div}(Z) F_A + 2Z(|F_A|)\|_{L^2(\Omega)}^2 \leq c\left(\|a\|^2 (\|r F_A\|_{L^2(\Omega)}^2 + \|\nabla_A F_A\|_{L^2(\Omega)}^2) + |a_0|^2 \lambda^{-2} (\|F_A\|_{L^2(\Omega)}^2 + \|r \nabla_A F_A\|_{L^2(\Omega)}^2)\right)
$$

$$
\leq c(|a|^2 + |a_0|^2)\lambda^{-2}N^{-4}.
$$

(3.17)

Increasing $N$, if necessary (and correspondingly decreasing $\delta(N)$ and $\lambda_0$), we can therefore ensure that the contribution to (3.7) from the complement of $B_{N\lambda}(p)$ is less than $\epsilon\|(a_0, a)\|_{\text{hyp}}^2$. Hence

$$
\left| g(Z(a_0, a), Z(\hat{a}, a)) - \|(a_0, a)\|_{\text{hyp}}^2 \right| \leq 3\epsilon \|(a_0, a)\|_{\text{hyp}}^2.
$$

(3.18)

This completes the first step of Part I. For the second step, define $\xi(a_0, a) = \pi_A Z(a_0, a) - Z(a_0, a) = -\alpha(a_0, a) - Z(a_0, a)$. Then from (2.7) we have

$$
|g(\alpha(a_0, a), \alpha(a_0, a)) - g(Z(a_0, a), \hat{Z}(a_0, a))| \leq 4(2\|d_A \xi(a_0, a)\|_2^2 d_A \hat{Z}(a_0, a)\|_2 + \|d_A \xi(a_0, a)\|_2^2).
$$

Pointwise, $d_A \hat{Z}(a_0, a)$ is bounded by the right-hand side of (3.14), and a simpler version of the analysis above shows that $\|d_A \hat{Z}(a_0, a)\|_2 \leq c\lambda^{-1}(|a_0| + |a|)$. From Proposition 5.1 of [G2] we have $\|d_A \xi(a_0, a)\|_2 \leq c(|a_0| + |a|)$. Thus $|g(\alpha_A(a_0, a), \alpha_A(a_0, a)) -$
Theorem 4.1 is not merely degenerate, but actually zero. Let \( P \) be a principal \( SU(2) \)-bundle and assume the base metric \( g_M \) is one for which \( SD \) is a manifold. Let \( g \) denote the information metric on \( SD \). Then at every reducible self-dual connection, \( g = 0 \).

Proof: Let \( A \in SD \) be reducible and let \( \eta \in T_A SD = \Omega^1(Ad P) \); thus \( d_A \eta = *d_A \eta \). Since \( A \) is a reducible \( SU(2) \) connection, there exists a nonzero covariantly
constant section $\Phi \in \Gamma(Ad\, P)$, and moreover $F_A = \Phi \otimes \omega$ for some real-valued self-dual 2-form $\omega$ (see [FU], Theorem 3.1). Momentarily let $(\cdot, \cdot)$ denote the pairing $\Omega^k(Ad\, P) \otimes Ad\, P \to \Omega^k(M)$ given by taking inner product only on the $Ad\, P$ factors, and let $\alpha = (\eta, \Phi) \in \Omega^1(M)$. Since $\Phi$ is covariantly constant, we obtain $d\alpha = (dA\eta, \Phi) = (\ast dA\eta, \Phi) = \ast d\alpha$. Therefore $d^*d\alpha = -\ast dd\alpha = 0$, implying $\langle d\alpha, d\alpha \rangle_{L^2} = \langle \alpha, d^*d\alpha \rangle_{L^2} = 0$, and hence $d\alpha = 0 = (dA\eta, \Phi)$. Thus $dA\eta$ is pointwise perpendicular to $\Phi$, hence to $F_A$, so from (2.6) we have $g(\eta, \cdot) = 0$.

**Remarks.** (1) Even without the assumption that $SD$ is a manifold, our argument shows that the information metric vanishes on the formal tangent space to $SD$. In contrast, the $L^2$ metric is positive definite on every subspace of $\Omega^1(Ad \, P)$.

(2) Because of the quadratic nature of the integrand in the definition of $g$, the analysis above shows that $g$ vanishes to order two at a reducible connection.

## 5 The information metric on $M_1(CP_2)$

Let $g_{FS}$ be the Fubini-Study metric on $M = CP_2$, with sectional curvatures between 1 and 4. The moduli space $M_1 = M_1(CP_2)$ of $SU(2)$ 1-instantons over $M$ is a cone on $CP_2$. Let $\lambda$ be the usual “scale function” on these instantons: $\lambda(A)$ is the radius of the smallest ball containing half the total energy $\|F_A\|_2^2$. The vertex $[A_0]$ of this cone is a reducible (and homogeneous) connection, and $\lambda(A_0) = 1$, while $\lambda(A) \to 0$ as $[A]$ approaches the “ideal boundary” of $M_1$. For $[A] \in M_1$ other than $[A_0]$, the center of the ball defining $\lambda(A)$ is unique, and the map sending $[A]$ to its scale and center point is a diffeomorphism from the punctured cone $M_1^* = M_1 - \{[A_0]\}$ to $(0, 1) \times M$. We will implicitly use this identification of $M_1^* \cong (0, 1) \times M$ below.

**Theorem 5.1** On the punctured moduli space $M_1(CP_2) - \{[A_0]\}$, the information metric $g$ is given by

$$g = \frac{128}{5} \frac{\pi^2}{\lambda^2} \left( f(\lambda)d\lambda^2 + h(\lambda)g_{FS} \right),$$

where

$$f(\lambda) = 1 - \frac{7}{3}\lambda^2 + \frac{14}{9}\lambda^4 - \frac{2}{3}\lambda^6 + \frac{2}{27}\lambda^8 - \frac{30}{81}\lambda^8 - \frac{20}{81}\lambda^{10} = \frac{30}{81} \frac{\lambda^8}{(1 - \lambda^2)^2} \log \frac{\lambda^2}{3 - 2\lambda^2},$$

$$h(\lambda) = 1 - \frac{7}{3}\lambda^2 + \frac{23}{18}\lambda^4 + \frac{93}{108}\lambda^6 - \frac{77}{108}\lambda^8 + \frac{5}{18}\lambda^6 - \frac{10}{27}\lambda^8 + \frac{1}{81}\lambda^{10} = \frac{5}{18} \frac{\lambda^6}{(1 - \lambda^2)^2} \frac{\lambda^2}{3 - 2\lambda^2}. \quad (5.3)$$

Before giving the proof, we review the parametrization of $M_1(CP_2)$ discussed in [G1], where the $L^2$ metric $g_2$ was computed. The $C^2$-bundle over $CP^2$ of instanton
number 1 is \( L \oplus L^{-1} \), where \( L \) is the hyperplane bundle. These are homogeneous bundles under the action of of \( SU(3) \) on \( \mathbb{C}P^2 \) induced by the standard linear action on \( \mathbb{C}^3 \), and \( SU(3) \) preserves the space of self-dual connections. The canonical connection \( A_0 \) on the holomorphic hermitian vector bundle \( L \oplus L^{-1} \) is a fixed point of this action. It was shown in [G1] that an identification of \( \mathbb{C}^3 \) with the formal tangent space \( T_{[A_0]} \mathcal{M}_1 \) induces an identification of \( \mathcal{M}_1 / SU(3) \) with \( \mathbb{C}^3 / SU(3) \cong [0, \infty) \), and of any given orbit in \( \mathcal{M}_1' := \mathcal{M}_1 - \{[A_0]\} \) with \( SU(3)/S(U(1) \times U(2)) \cong \mathbb{C}P_2 \). It is more convenient to replace \([0, \infty)\) with \([0, 1)\). Also given in [G1] is a 1-parameter family of self-dual connections \( \{A_t\} \), \( 0 \leq t = \sqrt{1 - \lambda^2} < 1 \), centered at \( p_0 = [1, 0, 0] \in \mathbb{C}P_2 \). The image of this family in \( \mathcal{M}_1 \) is transverse to the \( SU(3) \)-orbits (i.e. is a section of the fibration \( \mathcal{M}_1 \to \mathcal{M}_1 / SU(3) \)). With these identifications we can speak of “tangential” and “radial” directions in \( T \mathcal{M}_1' \), i.e. those tangent to the \( SU(3) \)-action and those tangent to this 1-parameter family (or to its translates under the \( SU(3) \)-action). For \( t > 0 \), we will write \( \mathbb{C}P^t_2 \) for the \( SU(3) \)-orbit through \([A_t] \).

**Proof of Theorem 5.1.** The explicit formulas in §4 of [G1] for elements of \( T_{[A_t]} \mathcal{M}_1 \), which we do not repeat here, give all the data needed to compute inner products; we only state the relevant consequences below. The computation for the information metric is actually simpler than that for the \( L^2 \) metric \( g_2 \), since by (2.7) there is no need to compute the \( L^2 \)-orthogonal projection of elements of \( T_A \mathcal{S} \mathcal{D} \) to \( \ker(d^*_A) \) (the difference between \( \eta \in \Omega^1(Ad P) \) and its projection is in the image of \( d_A \)). The action of \( SU(3) \) on \( \mathcal{M}_1 \) is preserves \( g \) (as well as \( g_2 \)), so it suffices to determine the metric at each \([A_t] \); the general form (5.1) then follows from symmetry.

If \( \eta_{rad} \) is the “radial” tangent vector \( dA_t/ dt \) (where \([A_t] \) is the family above), one finds from [G1] that

\[
(F_{A_t}, dA_t, \eta_{rad}) = \frac{32t(1 - t^2)D^3}{(D - t^2)^5} \left( -D^2 + D(3 - 4t^2) + 3t^2 - t^4 \right),
\]

where \( D = 1 + |z_1|^2 + |z_2|^2 \), and where \((z_1, z_2)\) are the usual coordinates on a standard \( \mathbb{C}^2 \subset \mathbb{C}P_2 \). The action of \( SU(3) \) induces a complex-linear isometric identification of the dual space \( (\mathbb{C}^2)^* \) (equipped with its standard hermitian metric) with \( T_{p_0} \mathbb{C}P_2 \) (equipped with the metric \( g_{FS} \)), and therefore with \( T_{[A_t]} \mathbb{C}P^t_2 \). If \( \eta_{\mu,t} \in T_{[A_t]} \mathbb{C}P^t_2 \) denotes the “tangential” tangent vector corresponding to \( \mu \in (\mathbb{C}^2)^* \), then from [G1] one finds that

\[
(F_{A_t}, dA_t, \eta_{\mu,t}) = \frac{-96t^2(1 - t^2)^2D^3(D + t^2)Re(\mu(z_1, z_2))}{(D - t^2)^5}.
\]

Finally, one has \(|F_{A_t}|^2 = 16D^3(1 - t^2)^2(D + t^2)(D - t^2)^{-4} \), and the Riemannian volume form on \( \mathbb{C}P^2 \) is \( D^{-3}d^4x \), where \( d^4x \) is the standard volume form on \( \mathbb{R}^4 \cong \mathbb{C}^2 \). The proof of Theorem 5.1 is now a matter of computing integrals and substituting \( t = \sqrt{1 - \lambda^2} \).
The $L^2$ metric $g_2$ also has the form $f(\lambda)d\lambda^2 + h(\lambda)g_{FS}$ (with different coefficient functions $f, h$), and it is interesting to compare the behavior of the two metrics near the vertex $[A_0]$ of the cone. For simplicity, we rescale the metrics as indicated in the table below. Near $[A_0]$ it is more natural to express $g$ in terms of $r$, distance to the vertex, rather than in terms of $\lambda$. For each metric, let $N$ denote a unit vector in $T_{A_0}M_1$ normal to $CP_2^{(r)}$, and let $T, T'$ denote unit vectors tangent to such a $CP_2$, with $T'$ orthogonal both to $T$ and to $JT$, where $J$ is the complex structure on $CP_2$.

All sectional curvatures of $M_1$ (with respect to either metric) can be expressed in terms of the three “primary” sectional curvatures $\sigma_{TN} := \sigma(T, N)$ (the curvature of the two-plane spanned by $T$ and $N$), $\sigma_{TT} := \sigma(T, JT)$, and $\sigma_{TT'} := \sigma(T, T')$. (In the Fubini-Study metric $g_{FS}$ on $CP_2$, these last two sectional curvatures are 4 and 1, respectively.) As $r \to 0$, one has the following asymptotics.

| metric                  | $\sigma_{TN}$           | $\sigma_{TT1}$          | $\sigma_{TT4}$          |
|-------------------------|--------------------------|--------------------------|--------------------------|
| $(\frac{128\pi^2}{5})^{-1}g_{info} = dr^2 + r^2(3 + O(r^2))g_{FS}$ | $- \frac{8}{125} + O(r)$ | $-\frac{2}{3}r^{-2} + O(1)$ | $\frac{1}{3}r^{-2} + O(1)$ |
| $(4\pi^2)^{-1}g_2 = dr^2 + r^2(1 + O(r^2))g_{FS}$                      | $-\frac{3}{2} + O(r^2)$ | $-\frac{3}{2} + O(r^2)$ | $3r^{-2} + O(1)$ |

Table 1: Comparison of the information metric $g = g_{info}$ and the $L^2$ metric $g_2$ on $M_1(CP_2)$.

There are several interesting observations that can be drawn from this table. Recall that $M_1$ is the quotient by $SO(3)$ of a smooth 8-dimensional manifold $\tilde{M}_1$, $SO(3)$ acts freely on the irreducible connections in $\tilde{M}_1$, and the stabilizer of each reducible connection is a circle. In [GP2] it was shown that there is a metric on $\tilde{M}$ for which the quotient map, restricted to irreducible connections, is a Riemannian submersion. It was also shown that given any principal Riemannian submersion of this general type—i.e. with $SO(3)$ acting smoothly on a Riemannian manifold, freely except at isolated points stabilized by circles, so that the singularities in the quotient are cones on $CP_2$ topologically—the asymptotics of the base metric near the singular points is always of the form $dr^2 + r^2g_{FS} + O(r^4)$. The factor of 3 in the asymptotics of the information metric therefore shows that there is no smooth metric on $\tilde{M}_1$ for which the map $\tilde{M}_1^* \to M_1^*$ is a Riemannian submersion. It would be nice to have an interpretation of this “3” in in terms of a model geometry, or family of model geometries, that universally gives the behavior of the information metric near cone singularities.

While the authors do not know at this writing whether there is such a universal model, a candidate is the following. On $\mathbb{R}^6$, let $\rho$ denote distance to the origin in the standard flat metric, let $g_{S^5}$ be the standard metric on $S^5$, and using polar coordinates define $g_{sing} = \rho^2(d\rho^2 + \frac{3}{4}\rho^2g_{S^5})$. This is a smooth field of quadratic forms on $\mathbb{R}^6$, but
vanishes to order two at the origin, just as the information metric on \(SD\) vanishes to order two at a reducible connection. We can still use \(g_{\text{sing}}\) to define distance to the origin; if this distance is \(r\), then \(r = \rho^2/2\). On the complement of the origin, \(g_{\text{sing}} = dr^2 + 3g_{S^2}\). If we now make the identification \(\mathbb{R}^6 \cong \mathbb{C}^3\) and let \(U(1)\) act the usual way, the quotient is a cone on \(\mathbb{CP}_2\). The circle action preserves \(g_{\text{sing}}\), and on the complement of the vertex we obtain a Riemannian submersion metric \(dr^2 + 3g_{FS}\).

While this model works well for \(\mathcal{M}_1(\mathbb{CP}_2)\), an examination of the formula for \(g\) near more general cone singularities gives no reason to believe that such a symmetric model is valid more generally.

Finally, we mention that the table also shows qualitative differences in the sectional curvatures \(g_{\text{info}}\) and \(g_2\): for \(g\), the sectional curvatures are unbounded both positively and negatively, while for \(g_2\) they are only unbounded positively. Note also that the difference between \(\sigma_{TN}\) and its limiting value is \(O(r^2)\) for \(g_2\), but \(O(r)\) for \(g\). Again, the generality of these features is unclear.

Acknowledgements: The financial support of the National Science Foundation (USA), the Australian Research Council and the University of Adelaide are gratefully acknowledged. MKM would like to thank Nick Buchdahl and Alan Carey for many useful conversations. DG would like to thank the Pure Maths department of University of Adelaide for its hospitality during the summer of 1993, when much of this work was completed.

References

[A] Amari, S., Differential Geometrical Methods in Statistics. Lecture Notes in Statistics 28, Springer, Heidelberg (1985).

[D] S. Donaldson, An application of gauge theory to four-dimensional topology, J. Diff. Geom. 18 (1983) 279-315

[F] P. Feehan: Geometry of the ends of the moduli space of anti-self-dual connections, J. Diff. Geom. 42 (1995), 465-553.

[FU] D.S. Freed and K.K. Uhlenbeck, Instantons and Four-Manifolds, Springer-Verlag, New York (1984).

[G1] D. Groisser: The geometry of the moduli space of \(\mathbb{CP}^2\) instantons. Inventiones Mathematicae, 99 (1990) 393-409.

[G2] ________, Curvature of Yang-Mills moduli spaces near the boundary, I, Commun. in Anal. and Geom. 1 (1993), 139-216.

[G3] ________, Curvature of Yang-Mills moduli spaces near the boundary, II: totally geodesic boundary. Preprint, 1995.
[GP1] __________ and T.H. Parker, The Riemannian Geometry of the Yang-Mills Moduli Space, *Commun. Math. Phys.* **112**, (1987) 663–689.

[GP2] ________________, The Geometry of the Yang-Mills Moduli Space for Definite Manifolds, J. Differential Geometry, vol. 29, (1989) 499-544.

[GP3] ________________, Sharp decay estimates for Yang-Mills Fields, *Commun. in Anal. and Geom.*, to appear.

[H] N.J. Hitchin, The Geometry and Topology of Moduli Spaces, in *Global Geometry and Mathematical Physics*, Lecture Notes in Mathematics **1451**, Springer, Heidelberg (1988).

[M] M.K. Murray, *The information metric on rational maps*, Experimental Mathematics **2**, 271–279 (1994).

[MR] __________ and J.W. Rice, *Statistics and Differential Geometry*, Monographs on Statistics and Applied Probability **48**, Chapman and Hall, London (1993).

[R] C.R. Rao, Information and the accuracy attainable in the estimation of statistical parameters, *Bull. Calcutta Math. Soc.* **37**, (1945), 81–91.