Ghost Kinetic Operator of Vacuum String Field Theory

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ABSTRACT: Using the data of eigenvalues and eigenvectors of Neumann matrices in the 3-string vertex, we prove analytically that the ghost kinetic operator of vacuum string field theory obtained by Hata and Kawano is equal to the ghost operator inserted at the open string midpoint. We also comment on the values of determinants appearing in the norm of sliver state.

KEYWORDS: Bosonic Strings, D-branes, Anomalies in Field and String Theories
1. Introduction

Vacuum string field theory (VSFT) [1] is proposed as the theory around the closed string vacuum after the open string tachyon condensation. (See [2]–[20] for related papers.) Since all the information of tachyon condensation is contained in the kinetic operator $Q$ of this theory, it is important to determine its form and study its properties. One possible form of $Q$ is

$$Q = c_0 + \sum_{n=1}^{\infty} (c_n + (-1)^n c_{-n}) f_n,$$

where $f_n$'s are numerical coefficients. This satisfies the requirement that $Q$ has a trivial cohomology and it is a derivation of the star product.

In [9], Hata and Kawano found that the coefficients $f_n$ are determined uniquely if we demand that the equation of motion has a nontrivial solution in the Siegel gauge. The coefficients $f_{HK}^n$ they found are written in terms of the Neumann coefficients in the 3-string vertex of ghost sector [9, 10]:

$$f_{HK}^n = 1 \frac{1}{1 - \tilde{M}}.$$  

(1.2)

See section 2 for the definition of $\tilde{M}$ and $\tilde{v}$.

Recently, Gaiotto, Rastelli, Sen and Zwiebach [17] discussed that there is a canonical choice of kinetic operator: the ghost insertion at the open string midpoint

$$Q = c \left( \frac{\pi}{2} \right) = \sum_{n=-\infty}^{\infty} c_n \cos \frac{\pi n}{2} = c_0 + \sum_{n=1}^{\infty} (c_n + (-1)^n c_{-n}) \cos \frac{\pi n}{2}.$$

(1.3)
This operator has a form (1.1) with the coefficients \( f_n \) given by
\[
f_{\text{mid}} = \cos \frac{\pi n}{2} = \frac{1}{2} \left[ i^n + (-i)^n \right].
\]
(1.4)

They conjectured that the operator found in [9] is nothing but this canonical kinetic operator, i.e.,
\[
f^{\text{HK}} = f_{\text{mid}},
\]
(1.5)

and analyzed this equality numerically.

In this paper, we will prove (1.5) analytically. To do that, detailed information of the spectrum of matrix \( f_M \) is needed. Since \( f_M \) can be written in terms of the Neumann matrix \( M \) in the matter sector, we can use the spectrum of \( M \) recently obtained in [19].

This paper is organized as follows: In section 2, we review the spectrum of Neumann matrices obtained in [19]. In section 3, we show that the eigenvector of Neumann matrix is \( \delta \)-function normalizable. In section 4, we prove the equation \( f^{\text{HK}} = f_{\text{mid}} \) by showing that their generating functions coincide. In section 5, we define the index of Neumann matrix and compute it for some examples. In section 6, we estimate the values of determinants which appear in the norm of silver state. It is found that the norm of the matter part is vanishing and the norm of the ghost part is divergent.

2. Eigenvalues and eigenvectors of Neumann matrices

In this section, we review the result of [19]. The 3-string vertex in the zero-momentum sector is given by [21]
\[
|V_3\rangle = \exp \left( \sum_{r,s=1}^3 \frac{1}{2} \epsilon^{(r)(s)} V^{rs} a^{(s)\dagger} + c^{(r)\dagger} \tilde{V}^{rs} b^{(s)} + c^{(r)\dagger} \tilde{v}^{rs} b^{(s)} \right) \otimes e^{(r)} e^{(r)} |0\rangle_r.
\]
(2.1)

In this paper, we use the following notation for Neumann matrices:
\[
M = CV^{11}, \quad \tilde{M} = C\tilde{V}^{11}, \quad \tilde{v} = \tilde{v}^{11},
\]
(2.2)

where \( C_{nm} = (-1)^n \delta_{nm} \) is the twist matrix. Note that \( M, \tilde{M} \) and \( \tilde{v} \) are twist-even:
\[
[M, C] = [\tilde{M}, C] = 0, \quad C\tilde{v} = \tilde{v}.
\]
(2.3)

Since \( |V_3\rangle \) is invariant under the action of \( L_1 + L_{-1} \), the matter Neumann matrix \( M \) and the matrix \( K_1 \) can be simultaneously diagonalized, where \( K_1 \) is defined as the representation matrix of \( L_1 + L_{-1} \) on matter oscillators:
\[
[L_1 + L_{-1}, v \cdot a] = (K_1 v) \cdot a - \sqrt{2} v_1 p.
\]
(2.4)

Here \( v \cdot a = \sum_{n=1}^{\infty} v_n a_n \). The explicit form of \( K_1 \) is given by
\[
(K_1)_{nm} = -\sqrt{n(n-1)} \delta_{n-1,m} - \sqrt{n(n+1)} \delta_{n+1,m}.
\]
(2.5)
Since $K_1$ is a real and symmetric matrix, its eigenvalue is a real number. The spectrum of $K_1$ is continuous on the real axis. The eigenvector $v^{(k)}$ of $K_1$ with eigenvalue $k$ is implicitly given by the generating function

$$f_k(z) = \sum_{n=1}^{\infty} \frac{v^{(k)}_n}{\sqrt{n}} z^n = \frac{1}{k} (1 - e^{-k \tan^{-1} z}).$$

(2.6)

This function has a symmetry

$$f_k(-z) = -f_{-k}(z),$$

(2.7)

which reflects the fact that $K_1$ is twist-odd

$$\{K_1, C\} = 0.$$

(2.8)

From $f_{k=0}(z) = \tan^{-1} z$, the eigenvector $v^{(0)}$ with eigenvalue $k = 0$ becomes

$$v^{(0)}_{2l} = 0, \quad v^{(0)}_{2l-1} = \frac{(-1)^{l-1}}{\sqrt{2l-1}}.$$

(2.9)

It is convenient to introduce a bracket notation for the infinite summation in (2.6):

$$f_k(z) = \langle z | E^{-1} | k \rangle = \langle k | E^{-1} | z \rangle,$$

(2.10)

where $E$ is a diagonal matrix defined by

$$E_{nm} = \sqrt{n} \delta_{n,m},$$

(2.11)

and $|z\rangle$ and $|k\rangle$ denote the infinite-dimensional vectors

$$|z\rangle = (z, z^2, z^3, \ldots)^T, \quad |k\rangle = (v^{(k)}_1, v^{(k)}_2, v^{(k)}_3, \ldots)^T.$$

(2.12)

$|z\rangle$ is the transpose of $|z\rangle$, not the hermitean conjugate of $|z\rangle$. Under the twist, $|z\rangle$ and $|k\rangle$ transform as

$$C |z\rangle = | - z \rangle, \quad C |k\rangle = - | - k \rangle.$$

(2.13)

A useful relation satisfied by $|z\rangle$ is

$$z \partial_z |z\rangle = |z\rangle E^2.$$

(2.14)

For a matrix $X$ which commutes with $K_1$, let $X(k)$ denote the eigenvalue for the eigenvector $|k\rangle$

$$X |k\rangle = X(k) |k\rangle.$$

(2.15)

The eigenvalue of $M$ is given by

$$M(k) = - \frac{1}{1 + 2 \cosh \frac{k}{2}}.$$

(2.16)

The width-matrix $T_N$ of the wedge state $|2\rangle$, which is defined by

$$|N\rangle_w = (|0\rangle)^N = \exp \left( - \frac{1}{2} a^\dagger CT_N a^\dagger \right) |0\rangle,$$

(2.17)
also commutes with $K_1$. $T_N$ can be written as 

$$T_N = \frac{T + (-T)^{N-1}}{1 - (-T)^N},$$

(2.18)

where $T$ is the width-matrix of the sliver $[2,3,4]$

$$T = \frac{1}{2M} \left( 1 + M - \sqrt{(1-M)(1+3M)} \right).$$

(2.19)

Note that $T_3 = M$ and $T_\infty = T$. The eigenvalues of $T$ and $T_N$ are given by

$$T(k) = -e^{-\frac{\pi}{2}|k|}, \quad T_N(k) = \frac{\sinh \left( \frac{2-N}{4} \pi k \right)}{\sinh \left( \frac{N}{4} \pi k \right)}.$$  

(2.20)

3. Inner product $\langle k|p \rangle$

In this section, we study the inner product between two eigenvectors $|k\rangle, |p\rangle$ of $K_1$. The inner product of two vectors is defined by

$$\langle v|v' \rangle = \sum_{n=1}^{\infty} v_n v'_n = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{2\pi} \langle v|e^{i\theta} \rangle \langle e^{-i\theta} |v' \rangle.  

(3.1)

Since $K_1$ is symmetric, we can expect that two eigenvectors with different eigenvalues are orthogonal to each other with respect to this inner product. But the norm of eigenvector is divergent, which can be seen from the example (2.9). Therefore, the eigenvector of $K_1$ is non-normalizable in this sense. However, as we will show below, the eigenvector of $K_1$ is $\delta$-function normalizable as usual for the continuous spectrum. To see this, let us calculate the inner product $\langle k|p \rangle$ of two eigenvectors:

$$\langle k|p \rangle = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{2\pi} \langle k|E^{-1}|e^{i\theta} \rangle \langle e^{-i\theta} |E|p \rangle$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{2\pi} \langle k|E^{-1}|e^{i\theta} \rangle \left( z \partial_z \langle z|E^{-1}|p \rangle \big|_{z=e^{-i\theta}} \right)$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{2\pi} \frac{1}{k} \left( 1 - e^{-k \tan^{-1} e^{i\theta}} \right) \left( \frac{1}{2 \cos \theta} e^{-p \tan^{-1} e^{-i\theta}} \right).$$

(3.2)

Here we used (2.14), (2.10) and (2.6).

In order to evaluate this integral, we should specify the branch of the function

$$\tan^{-1} z = \frac{1}{2i} \log \frac{1 + iz}{1 - iz}.$$  

(3.3)

For the consistency of the relation $f_k(0) = 0$ (2.3), we should take a branch such that $\tan^{-1}(0) = 0$. In other words, $\tan^{-1} z$ is given by the Taylor series

$$\tan^{-1} z = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{2l-1} z^{2l-1}.$$  

(3.4)
Figure 1: Image of the unit circle under the map \( f(z) = \tan^{-1} z \).

On this branch of the map \( w = \tan^{-1} z \), the unit disk \( |z| < 1 \) is mapped to the strip \( |\text{Re } w| < \pi/4 \), the right half of the unit circle \((C_R)\) is mapped to the line \( \frac{x}{2} + i \mathbb{R} \) on the \( w \)-plane, and the left half \((C_L)\) is mapped to \( -\frac{x}{2} + i \mathbb{R} \) (see figure 1):

\[
\tan^{-1} e^{i\theta} = \frac{\pi}{4} + \frac{i}{2} \log \tan \left( \frac{\theta}{2} + \frac{\pi}{4} \right) \quad \text{on } C_R = \left\{ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right\},
\]

\[
= -\frac{\pi}{4} - \frac{i}{2} \log \tan \left( \frac{\theta}{2} - \frac{\pi}{4} \right) \quad \text{on } C_L = \left\{ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \right\}. \tag{3.5}
\]

Let us first consider the contribution from \( C_R \). We make a change of integration variable from \( \theta \) to \( x \) defined by

\[
\tan^{-1} e^{i\theta} = \frac{\pi}{4} + ix, \quad \tan^{-1} e^{-i\theta} = \frac{\pi}{4} - ix, \quad \tan \frac{\theta}{2} = \tanh x, \tag{3.6}
\]

\[
\frac{d\theta}{2\cos \theta} = dx, \quad d\theta = \frac{2dx}{\cosh 2x}. \tag{3.7}
\]

Note that \( x \) runs from \(-\infty\) to \( \infty \). Then, the contribution from \( C_R \) to the integral (3.2) becomes

\[
\int_{C_R} \frac{d\theta}{2\pi k} \left( 1 - e^{-k\tan^{-1} e^{i\theta}} \right) \left( \frac{1}{2\cos \theta} e^{-p\tan^{-1} e^{-i\theta}} \right) = \int_{-\infty}^{\infty} \frac{dx}{2\pi k} \frac{1}{k} (1 - e^{-\frac{\pi}{k} - ikx}) e^{-\frac{\pi}{k} + ipx}
\]

\[
= \frac{1}{k} \delta(p) - \frac{1}{k} e^{-\frac{\pi}{k}} \delta(k - p). \tag{3.8}
\]

Next we consider the integral over \( C_L \). Here we also change the integration variable as

\[
\tan^{-1} e^{i\theta} = -\frac{\pi}{4} - ix, \quad \tan^{-1} e^{-i\theta} = -\frac{\pi}{4} + ix, \quad -\cot \frac{\theta}{2} = \tanh x, \tag{3.9}
\]

\[
\frac{d\theta}{2\cos \theta} = -dx, \quad d\theta = \frac{2dx}{\cosh 2x}. \tag{3.10}
\]

The contribution from \( C_L \) is given by

\[
\int_{C_L} \frac{d\theta}{2\pi k} \left( 1 - e^{-k\tan^{-1} e^{i\theta}} \right) \left( \frac{1}{2\cos \theta} e^{-p\tan^{-1} e^{-i\theta}} \right) = -\int_{-\infty}^{\infty} \frac{dx}{2\pi k} \frac{1}{k} (1 - e^{\frac{\pi}{k} + ikx}) e^{\frac{\pi}{k} - ipx}
\]

\[
= -\frac{1}{k} \delta(p) + \frac{1}{k} e^{\frac{\pi}{k}} \delta(k - p). \tag{3.11}
\]
By adding (3.8) and (3.11), we can see that $\langle k|p \rangle$ is proportional to the $\delta$-function

$$\langle k|p \rangle = \mathcal{N}(k)\delta(k-p)$$

(3.12)

where $\mathcal{N}(k)$ is given by

$$\mathcal{N}(k) = \frac{2}{k} \sinh \frac{\pi k}{2}.$$  

(3.13)

From (3.8) and (3.11), $k$ can be regarded as an analogue of momentum along the lines $\pm \frac{n}{4} + iR$ on the $w$-plane.

We can introduce the normalized eigenvector $|\hat{k}\rangle$ of $K_1$ by

$$|\hat{k}\rangle = \mathcal{N}(k)^{-\frac{1}{2}}|k\rangle,$$

(3.14)

whose inner product is given by the $\delta$-function

$$\langle \hat{k}|\hat{p} \rangle = \delta(k-p).$$

(3.15)

In terms of this basis, we can write down the completeness relation

$$1 = \int_{-\infty}^{\infty} dk \langle \hat{k}|\hat{k} \rangle = \int_{-\infty}^{\infty} dk \mathcal{N}(k)^{-1} |k\rangle \langle k|.$$

(3.16)

As a consistency check of (3.16), let us prove the following relation

$$\langle z|w \rangle = \int_{-\infty}^{\infty} dk \langle z|E|\hat{k}\rangle \langle \hat{k}|E^{-1}|w \rangle$$

(3.17)

for arbitrary points $z, w$ on the unit disk. The left-hand-side is given by

$$\langle z|w \rangle = \sum_{n=1}^{\infty} z^n w^n = \frac{zw}{1-zw}.$$  

(3.18)

The right-hand-side is calculated as

$$\int_{-\infty}^{\infty} dk \langle z|E|\hat{k}\rangle \langle \hat{k}|E^{-1}|w \rangle = z\partial_z \int_{-\infty}^{\infty} dk \langle z|E^{-1}|k\rangle \mathcal{N}(k)^{-1} \langle k|E^{-1}|w \rangle$$

$$= \frac{z}{1+z^2} \int_{-\infty}^{\infty} dk \frac{k}{2 \sinh \frac{\pi k}{2}} e^{-k\tan^{-1}z} \frac{1}{k} (1 - e^{-k\tan^{-1}w})$$

$$= \frac{z}{1+z^2} \int_{0}^{\infty} dk \frac{k}{\sinh \frac{\pi k}{2}} \left[ - \sinh(k\tan^{-1}z) + \sinh(k\tan^{-1}z + k\tan^{-1}w) \right]$$

$$= \frac{z}{1+z^2} \left[ - \tan(\tan^{-1}z) + \tan(\tan^{-1}z + \tan^{-1}w) \right]$$

$$= \frac{zw}{1-zw}.$$  

(3.19)

Here we have used the formula

$$\int_{0}^{\infty} dx \frac{\sin ax}{\sinh bx} = \frac{\pi}{2b} \tan \frac{\pi a}{2b}, \quad (|\text{Re } a| < \text{Re } b).$$

(3.20)

Since the unit disk $|z| < 1$ is mapped to the strip $|\text{Re}(\tan^{-1}z)| < \pi/4$, the $k$-integral in (3.19) converges.
4. Proof of $f^{\text{HK}} = f^{\text{mid}}$

In this section, we prove the equivalence of $f^{\text{HK}}$ and $f^{\text{mid}}$ using the result in the previous section. We will show that the generating functions of $f^{\text{HK}}_n$ and $f^{\text{mid}}_n$ coincide:

$$\langle z | f^{\text{HK}} \rangle = \langle z | f^{\text{mid}} \rangle.$$  \hfill (4.1)

The generating function of $f^{\text{mid}}$ is easily found to be

$$\langle z | f^{\text{mid}} \rangle = \sum_{n=1}^{\infty} z^n \cos \frac{\pi n}{2} = \sum_{l=1}^{\infty} (-1)^l z^{2l} = \frac{z^2}{1 + z^2}. \hfill (4.2)$$

We assume $|z| < 1$ to make this summation converge. In order to calculate $\langle z | f^{\text{HK}} \rangle$, we have to know the explicit form of Neumann coefficients in the ghost sector. $\tilde{v}$ is given by \cite{21}

$$\tilde{v}_{2l} = \frac{2}{3} B_{2l}, \quad \tilde{v}_{2l-1} = 0, \hfill (4.3)$$  

where $B_{2l}$ is defined by

$$\left( \frac{1 + iz}{1 - iz} \right)^{2/3} = \exp \left( \frac{4i}{3} \tan^{-1} z \right) = \sum_{n=\text{even}} B_n z^n + i \sum_{n=\text{odd}} B_n z^n. \hfill (4.4)$$

The generating function of $\tilde{v}$ becomes

$$\langle z | \tilde{v} \rangle = \frac{2}{3} \cosh \left( \frac{4}{3} i \tan^{-1} z \right). \hfill (4.5)$$

The Neumann matrix $\tilde{M}$ in the ghost sector is related to the corresponding matrix $M$ in the matter sector as \cite{21}

$$\tilde{M} = -E \left( \frac{M}{1 + 2M} \right) E^{-1}. \hfill (4.6)$$

Thus, $f^{\text{HK}}$ can be written as

$$f^{\text{HK}} = \frac{1}{1 - M} \tilde{v} = EAE^{-1} \tilde{v}, \hfill (4.7)$$

where

$$A = \frac{1 + 2M}{1 + 3M}. \hfill (4.8)$$

Therefore, we can use the information of the spectrum of $M$ to calculate the generating function $\langle z | f^{\text{HK}} \rangle$. Plugging $M(k)$ of the form \cite{2.16} into (4.8), the eigenvalue of $A$ is found to be

$$A(k) = \frac{2 \cosh \frac{\pi k}{2} - 1}{2 \cosh \frac{\pi k}{2} - 2}. \hfill (4.9)$$
Using the completeness condition (4.10), $\langle z | f_{\text{HK}} \rangle$ can be rewritten as

$$
\langle z | f_{\text{HK}} \rangle = \int_{-\infty}^{\infty} dk \langle z | E | k \rangle \langle k | E^{-1} | f_{\text{HK}} \rangle
$$

$$
= z \partial_z \int_{-\infty}^{\infty} dk \langle z | E^{-1} | k \rangle N(k)^{-1} \langle k | E^{-1} | f_{\text{HK}} \rangle
$$

$$
= \frac{z}{1 + z^2} \int_{-\infty}^{\infty} dk \frac{e^{-k \tan^{-1} z}}{2 \sinh \frac{\pi k}{2}} k \langle k | E^{-1} | f_{\text{HK}} \rangle
$$

$$
= -\frac{z}{1 + z^2} \int_{0}^{\infty} dk \frac{\sinh(k \tan^{-1} z)}{\sinh \frac{\pi k}{2}} k \langle k | E^{-1} | f_{\text{HK}} \rangle.
$$

(4.10)

In the last step, we used the fact that $\langle k | E^{-1} | f_{\text{HK}} \rangle$ is an odd function of $k$ which follows from $C|k\rangle = -| -k \rangle$ and $C|f_{\text{HK}}\rangle = | f_{\text{HK}} \rangle$:

$$
\langle -k | E^{-1} | f_{\text{HK}} \rangle = -\langle k | C E^{-1} | f_{\text{HK}} \rangle = -\langle k | E^{-1} C | f_{\text{HK}} \rangle = -\langle k | E^{-1} | f_{\text{HK}} \rangle.
$$

(4.11)

From (4.7), the last factor in (4.11) is written as

$$
k\langle k | E^{-1} | \bar{\nu} \rangle = A(k) k\langle k | E^{-1} | \bar{\nu} \rangle.
$$

(4.12)

Therefore, the problem is reduced to the calculation of $k\langle k | E^{-1} | \bar{\nu} \rangle$. This quantity can be extracted from the generating function of $\bar{\nu}$ (4.14):

$$
k\langle k | E^{-1} | \bar{\nu} \rangle = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} k\langle k | E^{-1} | e^{i\theta} \rangle \langle e^{-i\theta} | \bar{\nu} \rangle
$$

$$
= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} (1 - e^{-k \tan^{-1} e^{i\theta}}) \frac{2}{3} \cosh \left( \frac{4}{3} \tan^{-1} e^{-i\theta} \right).
$$

(4.13)

By changing the integration variable as in the previous section, this integral can be written as

$$
k\langle k | E^{-1} | \bar{\nu} \rangle = \frac{8}{3} \int_{-\infty}^{\infty} \frac{dx}{2\pi \cosh 2x} \left[ 1 - \cosh \left( \frac{\pi k}{4} + ikx \right) \right] \cosh \left( \frac{4}{3} x + \frac{\pi i}{3} \right)
$$

$$
= \frac{8}{3} \int_{0}^{\infty} \frac{dx}{2\pi \cosh 2x} \left[ \cosh \left( \frac{4}{3} x \right) - \cosh \left( \frac{4}{3} x + ikx \right) \cosh \left( \frac{\pi k}{4} + \frac{\pi i}{3} \right) - \cosh \left( \frac{4}{3} x - ikx \right) \cosh \left( \frac{\pi k}{4} - \frac{\pi i}{3} \right) \right].
$$

(4.14)

By making use of the formula

$$
\int_{0}^{\infty} \frac{dx}{\cosh ax \cosh bx} = \frac{\pi}{2b} \cdot \frac{1}{\cos \frac{\pi a}{2b}}, \quad (|\text{Re} a| < \text{Re} b),
$$

(4.15)

the integral in (4.14) is evaluated as

$$
k\langle k | E^{-1} | \bar{\nu} \rangle = \frac{1}{3} \left[ 2 - \frac{\cosh \left( \frac{\pi k}{4} + \frac{\pi i}{3} \right)}{\cosh \left( \frac{\pi k}{4} + \frac{\pi i}{3} \right)} - \frac{\cosh \left( \frac{\pi k}{4} - \frac{\pi i}{3} \right)}{\cosh \left( \frac{\pi k}{4} + \frac{\pi i}{3} \right)} \right] = \frac{2}{2} \cosh \frac{\pi k}{2} - \frac{1}{2} \cosh \frac{\pi k}{2} - 1 = \frac{1}{A(k)}.
$$

(4.16)
Therefore,
\[ k\langle k|E^{-1}|f^{\text{HK}}\rangle = kA(k)\langle k|E^{-1}|\overline{v}\rangle = 1. \] (4.17)

Finally, (4.10) becomes
\[ \langle z|f^{\text{HK}}\rangle = -\frac{z}{1+z^2}\int_0^\infty dk\frac{\sinh(k\tan^{-1}z)}{\sinh\frac{k\pi}{2}} \]
\[ = -\frac{z}{1+z^2}\tan(\tan^{-1}z) = -\frac{z^2}{1+z^2} = \langle z|f^{\text{mid}}\rangle. \] (4.18)

Here we used the formula (3.20). This is valid since \(|\text{Re}(\tan^{-1}z)| < \pi/4\) for \(|z| < 1\). This completes the proof of (4.1).

We comment on the subtlety of (4.17). Naively, one might think that the following equation holds:
\[ k\langle k|E^{-1}|f^{\text{mid}}\rangle = k\langle k|E^{-1}|f^{\text{HK}}\rangle. \] (4.19)

However, the left-hand-side is ill-defined since \(\tan^{-1}(i) = i\infty\):
\[ k\langle k|E^{-1}|f^{\text{mid}}\rangle = \frac{k}{2}[f_k(i) + f_k(-i)] = 1 - \cosh(k\tan^{-1}(i)). \] (4.20)

By regularizing the last term of (4.20) with a parameter \(a < 1\)
\[ k\langle k|E^{-1}|f^{\text{mid}}\rangle = \lim_{a \to 1} \frac{k}{2}[f_k(ia) + f_k(-ia)] = 1 - \lim_{a \to 1} \cosh(k\tan^{-1}(ia)), \] (4.21)
we can show that this term does not contribute to the computation of \(\langle z|f\rangle\):
\[ \lim_{a \to 1} \int_0^\infty dk\frac{\sinh(k\tan^{-1}z)}{\sinh\frac{k\pi}{2}} \cosh\left(k\tan^{-1}(ia)\right) = \]
\[ = \lim_{a \to 1} \frac{1}{2} \int_0^\infty dk\frac{1}{\sinh\frac{k\pi}{2}} \left[ \sinh\left(k\tan^{-1}z + k\tan^{-1}(ia)\right) + \sinh\left(k\tan^{-1}z - k\tan^{-1}(ia)\right) \right] \]
\[ = \lim_{a \to 1} \frac{1}{2} \left[ \frac{z + ia}{1 - ia z} + \frac{z - ia}{1 + ia z} \right] = \lim_{a \to 1} \frac{(1-a^2)z}{1 + a^2 z^2} = 0. \] (4.22)

Therefore, (4.19) should be understood as an equation up to such an irrelevant term. Here we emphasize that \(\langle z|f^{\text{mid}}\rangle\) is well-defined and is equal to \(\langle z|f^{\text{HK}}\rangle\) without any ambiguity.

5. Index of Neumann matrix

As an application of our formalism, we define the index of Neumann matrix and compute it for some examples. For a general matrix \(X\) diagonalized on the basis \(\{|k\}\) with its eigenvalue \(X(k) = X(-k)\) an even-function of \(k\), the eigenvalue \(X(k) = X(-k)\) has a two-fold degeneracy
\[ |k, +\rangle = |k\rangle + C|k\rangle = |k\rangle - |k\rangle, \]
\[ |k, -\rangle = |k\rangle - C|k\rangle = |k\rangle + |k\rangle, \] (5.1)
where $|k, +\rangle$ ($|k, -\rangle$) is a twist-even (odd) eigenvector. Note that $X$ is twist-even $[X, C] = 0$ when $X(k) = X(-k)$. For $k = 0$, there is only one twist-odd eigenvector. Therefore, the index of $X$ defined by

$$I(X) = \text{Tr}(CX)$$

receives contribution only from $k = 0$. This can be thought of as a measure of the twist anomaly [13]. $I(X)$ can be calculated as

$$I(X) = \int_{-\infty}^{\infty} dk \langle \hat{k}|CX|\hat{k}\rangle = \int_{-\infty}^{\infty} dk \langle \hat{k}|C|\hat{k}\rangle X(k)$$

$$= -\int_{-\infty}^{\infty} dk \langle \hat{k}| - \hat{k}\rangle X(k) = -\int_{-\infty}^{\infty} dk \delta(2k) X(k) = -\frac{1}{2} X(0).$$

As expected, it depends only on the eigenvalue at $k = 0$.

For example, the index of $T_N$ is given by

$$I(T_N) = -\frac{1}{2} T_N(0) = \frac{1}{2} - \frac{1}{N}.$$  

Here we used (2.20). In the rest of this section, we check this prediction for some examples by directly computing the trace. We first consider the index of $B$ defined by [19]

$$B = \sum_{n=1}^{\infty} (-1)^n B_{nn} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{(2n)^2 - 1} = \frac{1}{4} \sum_{n=1}^{\infty} \left[ \frac{(-1)^{n-1}}{2n - 1} - \frac{(-1)^n}{2n + 1} \right] = \frac{1}{4}.\tag{5.8}$$

This agrees with the expected result

$$I(B) = -\frac{1}{2} B(0) = \frac{d}{dN} I(T_N) \bigg|_{N=2} = \frac{1}{4}.\tag{5.9}$$

Next we consider the index of $T_3 = M$. The diagonal element of $M$ is given by [21]

$$M_{nn} = -\frac{1}{3} \left[ 2 \sum_{k=0}^{n} (-1)^k A_k^2 - 1 - (-1)^n A_n^2 \right].\tag{5.10}$$
where $A_k$ is defined by

$$
\sum_{k=\text{even}} A_k z^k + i \sum_{k=\text{odd}} A_k z^k = \left( \frac{1 + iz}{1 - iz} \right)^{1/3} = \exp \left( \frac{2i}{3} \tan^{-1} z \right).
$$

The partial sum of the series $(-1)^n M_{nn}$ turns out to be

$$
\sum_{n=1}^{L} (-1)^n M_{nn} = \frac{1}{3} \left( 1 - \sum_{k=0}^{L} (-1)^k A_k^2 \right) \quad L = \text{even},
$$

$$
= \frac{1}{3} \sum_{k=0}^{L} (-1)^k A_k^2 \quad L = \text{odd}.
$$

This sum converges in the limit $L \to \infty$ since $\sum_{k=0}^{\infty} (-1)^k A_k^2 = 1/2$. This can be shown by integrating the generating function of $A_k$:

$$
\sum_{k=0}^{\infty} (-1)^k A_k^2 = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{2\pi} e^{2i \tan^{-1} e^{i\theta}} e^{2i \tan^{-1} e^{-i\theta}}
$$

$$
= \int_{C_R} \frac{d\theta}{2\pi} e^{\pi i} + \int_{C_L} \frac{d\theta}{2\pi} e^{-\pi i} = \frac{1}{2}.
$$

Therefore, the index of $M$ computed in this way agrees with the expected result (5.4)

$$
I(M) = \sum_{n=1}^{\infty} (-1)^n M_{nn} = \frac{1}{6}.
$$

It will be interesting to check (5.4) for general $N$ by directly computing the trace.

We comment on a deformation of the index. We can introduce a parameter $\beta$ in the definition of index, e.g.,

$$
I_\beta(X) = \text{Tr}(CX e^{-\beta K_2^2}).
$$

Naively, $I_\beta(X)$ is independent of $\beta$. However, it can be $\beta$-dependent as in the case of Witten index of a theory with continuous spectrum. It is interesting to study the $\beta$-(in)dependence of $I_\beta(X)$.

6. Norm of sliver

6.1 Eigenvalue density $\rho(k)$

In contrast with the finiteness of the index (5.2), the trace of $X$ itself is divergent in general. This can be seen as

$$
\text{Tr} X = \int_{-\infty}^{\infty} dk \langle \hat{k} | X | \hat{k} \rangle = \int_{-\infty}^{\infty} dk \langle \hat{k} | \hat{k} \rangle X(k) = \int_{-\infty}^{\infty} dk \delta(0) X(k).
$$

If we formally define the eigenvalue density $\rho(k)$ by

$$
\text{Tr} X = \int_{-\infty}^{\infty} dk \rho(k) X(k),
$$

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then $\rho(k)$ is independent of $k$ and divergent

$$\rho(k) = \langle k|k \rangle = \frac{\langle k|k \rangle}{N(k)} = \delta(0).$$

(6.3)

In the level truncation approximation, we can estimate the order of divergence of $\rho(k) = \rho(0)$ as follows:

$$\rho(k) = \langle k = 0|k = 0 \rangle \sim \frac{1}{N(0)} \sum_{n=1}^{L} \left( \frac{v_n^{(0)}}{V_n^{(0)}} \right)^2 \sim \frac{1}{\pi} \sum_{l=1}^{L/2} \frac{1}{2l - 1} \sim \frac{1}{2\pi} \log L \text{ (for large } L).$$

(6.4)

This expression agrees with the one obtained in [19]. We can also estimate $\rho(k)$ by computing the trace of $B$ in two different ways:

$$\langle k|k \rangle = \sum_{n=1}^{N(0)} \langle \langle n = 0|k \rangle \rangle \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \rho(k) B(k) \sim \sum_{n=1}^{L} B_{nn}$$

(6.5)

where

$$\int_{-\infty}^{\infty} dk \rho(k) B(k) = \rho(k) \int_{-\infty}^{\infty} dk \left( -\frac{\pi k}{2 \sinh \frac{\pi k}{2}} \right) = -\frac{\pi}{2} \rho(k),$$

$$\sum_{n=1}^{L} B_{nn} = -\sum_{n=1}^{L} \frac{n}{4n^2 - 1} \sim -\frac{1}{4} \log L.$$ 

(6.6)

This again reproduces (6.4).

Now it is natural to define the regularized trace $\text{Tr}_L$ by

$$\text{Tr}_L X = \log \int_{-\infty}^{\infty} \frac{dk}{2\pi} X(k).$$

(6.7)

Using this trace, the determinant of $X$ can be regularized as

$$\det X \sim \exp (\text{Tr}_L \log X) = L^{\gamma(X)},$$

(6.8)

where the exponent $\gamma(X)$ is defined by

$$\gamma(X) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \log X(k).$$

(6.9)

When the eigenvalue $X(k)$ is an even-function of $k$, $\gamma(X)$ can be written as

$$\gamma(X) = \int_{0}^{\infty} \frac{dk}{\pi} \log X(k) = \frac{2}{\pi^2} \int_{0}^{1} \frac{du}{u} \log X(\frac{u}{e^{-\frac{\pi}{2}}}).$$

(6.10)

### 6.2 Classical solution of VSFT

In this subsection, we review the classical solution $\Psi_0$ of VSFT found in [19]. $\Psi_0$ is given by

$$|\Psi_0\rangle = -N_mN_g \exp \left( -\frac{1}{2} a^\dagger C T a^\dagger + c^\dagger C T b^\dagger \right) c_1|0\rangle.$$

(6.11)
Here $T$ is defined by (2.19) and $\tilde{T}$ is defined by

$$\tilde{T} = \frac{1}{2M} \left( 1 + \tilde{M} - \sqrt{(1 - \tilde{M})(1 + 3\tilde{M})} \right) = -ETE^{-1}. \quad (6.12)$$

The matter part of $\Psi_0$ is the sliver state and the ghost part is conjectured to be the twisted sliver state [17]. $\mathcal{N}_m$ and $\mathcal{N}_g$ are the normalization constants given by

$$\mathcal{N}_m = \det^{D/2}(1 - M)(1 + T),$$
$$\mathcal{N}_g = \det^{-1}(1 - \tilde{M})(1 + \tilde{T}), \quad (6.13)$$

where $D$ ($= 26$) is the dimension of D-brane. The energy density of this solution is proportional to

$$\frac{(2\pi)^D}{V_D} \langle \Psi_0|Q|\Psi_0 \rangle = E_mE_g, \quad (6.14)$$

where $V_D$ is the volume of D-brane and

$$E_m = \det(1 - M)^{\frac{D}{4}}(1 + 3M)^{\frac{1}{4}D},$$
$$E_g = \det(1 - \tilde{M})^{-3/2} \det(1 + 3\tilde{M})^{-1/2}. \quad (6.15)$$

### 6.3 Calculation of various determinants

We can estimate the values of determinants in (6.13) and (6.15) by using the regularization (6.8). Let us compute the exponent $\gamma$ (6.10) for various matrices. One can show that $\gamma$’s of Neumann matrices are written as combinations of the following integrals:

$$I_- = \int_0^1 \frac{du}{u} \log(1 - u) = - \int_0^1 \frac{du}{u} \sum_{n=1}^{\infty} \frac{1}{n} u^n = -S_0 = -\frac{\pi^2}{6},$$
$$I_+ = \int_0^1 \frac{du}{u} \log(1 + u) = \int_0^1 \frac{du}{u} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} u^n = -S_1 = \frac{\pi^2}{12},$$
$$J_- = \int_0^1 \frac{du}{u} \log(1 - u + u^2) = \int_0^1 \frac{du}{u} \log(1 - e^{\frac{\pi}{3}i}u)(1 - e^{-\frac{\pi}{3}i}u)$$
$$= - \int_0^1 \frac{du}{u} \sum_{n=1}^{\infty} \frac{1}{n} u^n (e^{\frac{\pi}{3}ni} + e^{-\frac{\pi}{3}ni}) = -2S_1 = -\frac{\pi^2}{18},$$
$$J_+ = \int_0^1 \frac{du}{u} \log(1 + u + u^2) = \int_0^1 \frac{du}{u} \log(1 - e^{\frac{2\pi}{3}i}u)(1 - e^{-\frac{2\pi}{3}i}u)$$
$$= - \int_0^1 \frac{du}{u} \sum_{n=1}^{\infty} \frac{1}{n} u^n (e^{\frac{2\pi}{3}ni} + e^{-\frac{2\pi}{3}ni}) = -2S_2 = \frac{\pi^2}{9}, \quad (6.16)$$

where $S_a$ is defined by

$$S_a = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(\pi na) = \frac{\pi^2}{4} (1 - a^2) - \frac{\pi^2}{12}, \quad (0 \leq a \leq 2). \quad (6.17)$$
Substituting the eigenvalues (2.16) and (2.20) into the definition of $\gamma$ (5.10), the exponents of the matter Neumann matrices are found to be

$$
\gamma(1 + T) = \frac{2}{\pi^2} I_+ = -\frac{1}{3}, \quad \gamma(1 - T) = \frac{2}{\pi^2} I_+ = \frac{1}{6}
$$

$$
\gamma(1 - M) = \frac{2}{\pi^2}(2I_+ - J_+) = \frac{1}{9}, \quad \gamma(1 + 3M) = \frac{2}{\pi^2}(2I_+ - J_+) = -\frac{8}{9}
$$

$$
\gamma(1 + 2M) = \frac{2}{\pi^2}(J_+ - J_-) = -\frac{1}{3}, \quad \gamma(1 - TM) = \frac{2}{\pi^2}(I_+ - J_-) = -\frac{1}{18}.
$$

(6.18)

For the ghost part, using the relation

$$
\tilde{T} = -ETE^{-1}, \quad 1 - \tilde{M} = E \left( \frac{1 + 3M}{1 + 2M} \right) E^{-1}, \quad 1 + 3\tilde{M} = E \left( \frac{1 - M}{1 + 2M} \right) E^{-1},
$$

(6.19)

and assuming the property

$$
\det(EXT) = \det X,
$$

(6.20)

the exponents can be computed as

$$
\gamma(1 + \tilde{T}) = \gamma(1 - T) = \frac{1}{6},
$$

$$
\gamma(1 - \tilde{M}) = \gamma(1 + 3M) - \gamma(1 + 2M) = -\frac{5}{9},
$$

$$
\gamma(1 + 3\tilde{M}) = \gamma(1 - M) - \gamma(1 + 2M) = \frac{4}{9}.
$$

(6.21)

Combining these relations, the order of determinants (5.13) and (5.15) can be estimated:

$$
N_m \sim L^{-\frac{1}{3}D}, \quad N_g \sim L^{\frac{1}{3}D}, \quad E_m \sim L^{-\frac{1}{3}D}, \quad E_g \sim L^{\frac{1}{3}D}.
$$

(6.22)

In the limit $L \to \infty$, $N_m$ and $E_m$ tend to vanish. This behavior of the norm in the matter sector was observed in the numerical calculation [2]. On the other hand, the norm in the ghost sector diverges in the limit $L \to \infty$. In the early days of the study of VSFT, it was expected that the divergent factor coming from the ghost sector compensates the vanishing factor from the matter sector and in total the norm becomes finite [2]. However, our result (6.22) shows that the contribution from the ghost sector is not large enough to compensate the matter contribution. The total norm is still vanishing in the limit $L \to \infty$.

Recent discussion in [17] is that the action with the kinetic operator (1.1) is a singular description of the theory around the closed string vacuum and there is an infinite overall factor in front of the action due to a singular field redefinition. Therefore, this infinite factor would cancel the vanishing norm of the sliver as discussed in [17].

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