THE GEOMETRIC REALIZATION OF REGULAR PATH COMPLEXES VIA (CO-)HOMOLOGY

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Abstract. The aim of this paper is to give the geometric realization of regular path complexes via (co-)homology groups with coefficients in a ring $R$. Concretely, for each regular path complex $P$, we associate it with a $\Delta$-complex $S(P)$ and show that the (co-)homology groups of $P$ are isomorphic to those of $S(P)$ with coefficients in $R$. As a direct result we recognize path (co-)homology as Hochschild (co-)homology in case that $R$ is commutative. Analogues of the Eilenberg-Zilber theorem and Künneth formula are also showed for the Cartesian product and the join of two regular path complexes. In fact, we meanwhile improve some previous results which are covered by these conclusions in this paper.

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1. INTRODUCTION

S.-T.Yau et al. ([6]) studied the Shrödinger equation and further introduced the concept of quantum tunneling on graphs (or say, on complexes of graphs). These notions, which originally appeared in physics, can be viewed as the discrete forms of the related theories in physics. As a preparation of their work, Yau and his collaborators had previously studied the eigenfunction defined for graphs ([8],[5]), which is a generalization of the version for smooth Riemannian manifolds. Using this, one can study the nodes of a graph and to control the volume growth of a graph via estimate of some variables. It turns out to have a great amount of practical applications. Such theories of graphs are developed parallel to that of Riemannian geometry in many ways.

When considering digraphs (i.e., quivers), we can define their (co-)homology groups in a similar way as we have done in topology (see this paper or [9] for details). These (co-)homology groups can be used directly to study the relationship among some aspects of digraphs and their related theories, including the interaction between complexes of graphs and physics mentioned above. In this manner,
it is possible to give a geometric interpretation of eigenfunction, Schrödinger equation and quantum tunneling on digraphs following the geometric realization of digraphs given via the (co-)homology groups of the associated path complexes. This is the motivation for us to give the main results of this paper.

In this paper, we introduce and study the path (co-)homology of a path complex with coefficients in a unital ring $R$. The central idea is based on a topological approach. To be short, given any path complex $P$, the geometric realization $S(P)$ of $P$ as a $\Delta$-complex is obtained from a map $F_\Delta$ defined in an inductive process. Then for any unital ring $R$, we can recognizes path (co-)homology of $P$ as simplicial (co-)homology of $S(P)$ with coefficients in $R$. Concretely, we have the following:

**Main Theorem** (Theorem 3.10 and 3.11) Let $P$ be a path complex and $R$ be a ring, the map $F_\Delta$ induces isomorphisms $H_\ast(P) \cong H_\ast(S(P))$ and $H^\ast(P) \cong H^\ast(S(P))$.

**Main Theorem** (Theorem 3.10 and 3.11) Let $P$ be a path complex and $R$ be a ring, the map $F_\Delta$ induces isomorphisms $H_\ast(P) \cong H_\ast(S(P))$ and $H^\ast(P) \cong H^\ast(S(P))$.

Historically, in order to study the topological structure of directed graphs (digraphs) and further to classify them, there are many attempts to form a homological theory for digraphs. Among these approaches there are three of them being well-known: regarding a digraph as a special one-dimensional simplicial complex, considering all the cliques of a digraph as simplicies of the corresponding dimensions ($2$ $10$), or taking Hochschild cohomology of the path algebra of a digraph ($14$). But as it is commented in $9$ that all these approaches have their emphases and limitations. In view of this, the authors of $9$ introduced the notions of path complexes and path homology over a field (while their cohomology version can be found in $11$). This new (co-)homology theory for digraphs, including its sequel notions and results, not only shares many properties with the above approaches but also avoid many limitations. It is shown that one can use path homology to give a refined classification of digraphs via some homological invariants such as the dimensions of the homology groups, Euler characteristic and so on. On the other hand, as we can see that from $12$, path cohomology theory is a powerful tool when one deals with the algebraic aspect of simplicial cohomology, in fact it allows a delicate proof to the isomorphism obtained in $7$ without using Cohomology Comparison Theorem.

Furthermore, the classic Eilenberg-Zilber theorem and Künneth formula holding for the Cartesian product space of two topological spaces have also analogues in the theory of path homology. In fact, similar results hold for both of the Cartesian product and the join of two path complexes over a field ($9$).

Meanwhile, as we usually do in the theory of simplicial (co-)homology, it seems that there is no need to confine the coefficients of path (co-)homology in a field, since different coefficient rings usually induce different (co-)homology groups. For instance, when one ignores the orientation of a simplicial complex one should consider directly its simplicial homology groups with coefficients in $\mathbb{Z}_2$ instead of $\mathbb{Z}$.

So in this paper, we define and study the path (co-)homology with coefficients in a general ring $R$. Our main goal is to recognize the path (co-)homology as a geometric (co-)homology theory. This would give provide a new approach to study graph theory and interpret many of the results in $9$ $10$ $12$ in a more intuitive way. To be short, this is done by the geometric realization of path complexes as we mentioned at the beginning, the motivation is to view a path complex as a simplicial object. Moreover, given path complex $P$ over a commutative ring $R$, by this result and the results of $7$ $12$ one may also construct an associative algebra $A_{S(P)}$ over $R$ associated to $P$, which is very different from the related path algebra even for $P$ arising from a digraph, and by what we have in hand it is not surprising that one could prove that the Hochschild (co-)homology of $A_{S(P)}$ is isomorphic to path (co-)homology of $P$.

The paper is organized as follows. We set off after reviewing some definitions and notations in Section 2. In Section 3, we establish the correspondence of regular path complexes and $\Delta$-complexes (following $13$) which share the isomorphic (co-)homology groups with coefficients in general rings (see Theorems 3.11 and 3.12). This recognizes path (co-)homology as simplicial (co-)homology, and as we have pointed out that, it allows us to simplify proofs of many results in $9$ and $10$. As a first application, in Section 4, we further recognize path (co-)homology with coefficients in a commutative ring as Hochschild (co-)homology (see Theorems 4.2 and 4.4). Section 5 is dedicating to further applications of the obtained results. Analogues of the Eilenberg-Zilber theorem are obtained
in a unified way for the Cartesian product and the join of two regular path complexes (Theorems 5.8 and 5.11) over any commutative rings. These imply respectively two general Künneth formulae for path homology with coefficients in principle ideal domains. Note that not only this generalizes the previous result in [9] obtained for path complexes over a field \( K \), but also our proofs here go rather different from those given in [9].

2. Preliminaries

Throughout this paper, \( K \) denotes a field and \( R \) denotes a unital ring if not specified. We recall from [9] some notations and definitions in this section, though most of them are defined temporarily in the case where \( K \) is a field, as we shall see that they can be easily extended to the case when one replaces \( K \) by any ring \( R \).

2.1. Path complexes.

**Definition 2.1.** Let \( V \) be an arbitrary non-empty finite set whose elements will be called *vertices*. For any non-negative integer \( p \), an elementary \( p \)-path on a set \( V \) is any sequence \( \{i_k\}_{k=0}^{p} \) (or simply written as \( i_0 \cdots i_p \)) of \( p+1 \) vertices (needs not to be distinct) of \( V \). For any non-negative integer \( p \), an elementary \( p \)-path is understood to be an empty set. Furthermore, an elementary path \( i_0 \cdots i_p \) is said to be *non-regular* if \( i_{k-1} = i_k \) for some \( 1 \leq k \leq p \), and *regular* otherwise.

Denote by \( \Lambda_p = \Lambda_p(V; K) \) the \( K \)-linear space that consists of all formal linear combinations of all elementary \( p \)-paths with the coefficients from \( K \). The elements of \( \Lambda_p \) are called *\( p \)-paths* on \( V \), and an elementary \( p \)-path \( i_0 \cdots i_p \) as an element in \( \Lambda_p \) is written as \( e_{i_0 \cdots i_p} \). Obviously the basis in \( \Lambda_p \) is the family of all elementary \( p \)-paths, and each element \( v \) in \( \Lambda_p \) has the following form:

\[
  v = \sum_{i_0, \ldots, i_p \in V} v_{i_0 \cdots i_p} e_{i_0 \cdots i_p}
\]

where \( v_{i_0 \cdots i_p} \in K \). For any \( p \geq -1 \), consider the subspace of \( \Lambda_p \) spanned by the regular elementary paths: \( \mathcal{R}_p = \mathcal{R}_p(V; K) := \text{span}\{e_{i_0 \cdots i_p}; \ i_0 \cdots i_p \text{ is regular}\} \), whose elements are called *regular \( p \)-paths*.

For any \( p \geq 0 \), define the boundary operator \( \partial : \Lambda_p \to \Lambda_{p-1} \) as a linear operator that acts on elementary paths by

\[
  \partial e_{i_0 \cdots i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0 \cdots \hat{i}_q \cdots i_p}
\]

(2.1)

where the hat \( \hat{i}_q \) means omission of the index \( i_q \). Note that such boundary operators make \( \Lambda_\ast = \{\Lambda_p\} \) a chain complex (see [9] Lemma 2.4]). Similarly we can define the regular complex \( \mathcal{R}_\ast = \{\mathcal{R}_p\} \) consisting of regular elements and with natural boundary operators, i.e., those boundary operators are defined by the induced maps of \( \partial \) acting on the quotient space \( \Lambda_p \) over non-regular paths, and it is easy to check that \( \mathcal{R}_\ast = \{\mathcal{R}_p\} \) is a chain complex under such boundary operators (see [9] for details). Let \( V, V' \) be two finite sets, by definition, any map \( f : V \to V' \) gives rise to two natural morphisms \( \Lambda_\ast(V) = \Lambda_\ast(V') \) and \( \mathcal{R}_\ast(V) = \mathcal{R}_\ast(V') \).

The central concept in our study is the following.

**Definition 2.2.** A *path complex* over a finite set \( V \) is a non-empty collection \( P \) of elementary paths on \( V \) with the following property: for any \( n \geq 0 \), if \( i_0 \cdots i_n \in P \) then also the truncated paths \( i_0 \cdots i_{n-1} \) and \( i_1 \cdots i_n \) belong to \( P \). The elementary \( n \)-paths from \( P \) is denoted by \( F_n \). If all the paths in \( P \) are regular, then \( P \) is called a *regular* path complex.

Here is an example of path complex:

**Example 2.3.** Let \( V = \{0, 1, ..., 8\} \), and \( P \) be a path complex in which the elementary paths are given by:

- 0-paths: 0, 1, ..., 8
- 1-paths: 01, 02, 03, 04, 05, 07, 08, 12, 34, 35, 45, 67, 68, 78
- 2-paths: 012, 678, 034, 035, 045, 678
- 3-paths: 0345.

In fact, given a digraph \( G \), there is a natural way to associate it with a path complex \( P(G) \) whose...
vertices and elementary paths are decided by the digraph in the obvious way. As one can easily check that, the associated path complex of the following digraph is exactly $P$ (for more details please see [9 Example 3.3]).

![Figure 1: A digraph $G$ with $P(G) = P$.](image)

2.2. Path (co-)homology.

When a path complex $P$ is fixed, all the $n$-paths of the form $\sum_{i=1}^{s} r_i e_i$ with $s$ a finite integer and each $r_i \in K$, $e_i \in E$, are called allowed, otherwise are called non-allowed. The set of all allowed $n$-paths is denoted as $A_n(P) = \mathcal{A}_n(P; K)$. Furthermore, for any $n \geq 0$ we define $\Omega_n(P)$ as follows:

$$\Omega_n(P) = \Omega_n(P; K) := \{p_n | p_n \in \mathcal{A}_n(P) \text{ and } \partial(p_n) \in \mathcal{A}_{n-1}(P)\}.$$  

Apparently each $\Omega_n(P)$ is a $K$-module, namely a vector space over $K$. It is easy to verify that $\partial(\Omega_m(P)) \subseteq \Omega_{m-1}(P)$ and $\partial^2 = 0$, thus we obtain a chain complex of $K$-modules:

$$\Omega_*(P) = \Omega_*(P; K) := \cdots \to \Omega_n(P) \to \Omega_{n-1}(P) \to \cdots \to \Omega_0(P) \to \Omega_{-1}(P) = 0.$$  

Therefore, for any $n \geq 0$ we define the $n$-th path homology group of $P$ as $H_n(\Omega_*(P))$, or denoted shortly by $H_n(P)$. If the path complex $P$ is regular, which is the case we shall study in this paper, all the above definitions and notations have modified versions when one replaces the boundary operator by the modified boundary operator which is used to define $R_*$.  

The above definitions and notations also have dual versions. For any integer $p \geq -1$, denote by $\Lambda^p = \Lambda^p(V; K)$ the linear space of all $K$-valued functions on $(p + 1)$-multiplicative product $V^{p+1}$ of set $V$. Otherwise we set $\Lambda^{p-2} = \{0\}$. In particular, $\Lambda^0$ is the linear space of all $K$-valued functions on $V$, and $\Lambda^{-1}$ is the space of all $K$-values functions on $\Lambda^0 := \{0\}$, that is, one can identify $\Lambda^{-1}$ with $K$. The elements of $\Lambda^p$ are called $p$-forms on $V$, one can identify $\Lambda^p$ with the dual space of $\Lambda^p$ via the canonical identity $\Lambda^p \cong \text{Hom}_K(\Lambda^p, K)$. The boundary operator (2.1) should be replaced now by exterior differential $d : \Lambda^p \to \Lambda^{p+1}$ given by

$$\tag{2.2} (d\omega)_{i_1 \cdots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \omega_{i_1 \cdots \hat{i}_q \cdots i_{p+1}}$$

for any $\omega \in \Lambda^p$. Similarly we define the space of regular $p$-forms $\mathcal{R}^p = \mathcal{R}^p(V) := \text{Hom}_K(\mathcal{R}_p, K)$ (hereafter this means, any element in $\mathcal{R}^p$ always takes $\Lambda_p \setminus \mathcal{R}_p$, i.e., non-regular $p$-paths to 0). Given a path complex $P$, we define the space of allowed $p$-forms $\mathcal{A}^p(P) = \mathcal{A}^p(P; K) := \text{Hom}_K(\mathcal{A}_p(P), K)$, also denote

$$\mathcal{N}^p = \Lambda^p \setminus \mathcal{A}^p(P) \quad \text{and} \quad \mathcal{J}^p = \mathcal{N}^p + d\mathcal{N}^{p-1},$$

and define

$$\Omega^p(P) = \mathcal{A}^p(\mathcal{J}^p)$$

Actually, it follows from [9 Lemma 3.19] that $\Omega^p(P)$ is the dual space of $\Omega_n(P)$ while $d$ is dual to $\partial$. that is to say, one has

$$\tag{2.3} \Omega^p(P) \cong \text{Hom}_K(\Omega_n(P), K) \quad \text{and} \quad d \cong \text{Hom}_K(\partial, K).$$

It can be shown that $\{\Omega^p(P)\}$ amounts to a cochain complex with the differential operator given by (2.2), whereas the $n$-th path cohomology group of $P$ for any $n \geq 0$ is referred to the $n$-th cohomology group $H^n(\Omega^*(P))$ of this cochain complex, which is denoted shortly by $H^n(P)$.  

Hitherto, all the definitions are defined only for the case where $K$ is a field. But it is not hard to see that all of them can be easily carried over to the case when replacing $K$ by any ring $R$. To do this there is no need to change a word but replacing all $K$-vector spaces ($K$-modules) by $R$-modules. In this paper, we shall focus on this more general situation, that is, path (co-)homology is understood to be with coefficient in an associative unital ring $R$, and we shall omit “$R$” in the notation since there is no ambiguity.

3. The geometric realization of regular path complexes

In the rest of this paper, all path complexes are understood to be regular unless specified otherwise, and we shall give the geometric realization of regular path complexes in this section, i.e., we will construct the correspondence of regular path complexes and $\Delta$-complexes (see Definition 3.1 below).

Suppose we are given a path complex $P = \{P_i\}$ over a finite set $V = \{i_j\}$. The aim of this section is to construct a $\Delta$-complex $S(P)$ with good homological property. To do this we plan to associate each proper elementary $n$-path with an $n$-simplex $\Delta^n$ (see below for definition), after this we give the desired characteristic maps $\Delta^n \to S(P)$, and this process is demonstrated in several steps as follows.

3.1. Construction of the $\Delta$-complexes.

We start by deciding the possible $(n + 1)$-simplices with the given $n$-simplices as faces. Recall that a standard $n$-simplex $\Delta^n$ is an $n$-dimensional convex polyhedron in $\mathbb{R}^n$ containing the original point and $n$ points which are the $n$ standard basis vectors for $\mathbb{R}^n$. We enumerate the $n + 1$ vertices in order, say, $0, 1, \cdots, n$, and give each edge with two vertices $k, l$ ($0 \leq k < l \leq n$) the direction from $k$ to $l$. Similarly for any $k$-faces of $\Delta^n$, we assign to it a standard orientation in a well-known way (see also, for example, [13, p.233]). That is, any other ordering of the vertices obtained from an even time permutation of the original ordering $(0 < 1 < \cdots < n)$ is viewed as the same orientation of $\Delta^n$, which is said to be a canonical orientation of $\Delta^n$, otherwise we say that the ordering gives an opposite orientation to the canonical one. The same definition can be easily extended to any $\Delta$-complexes by characteristic maps, whose definition is given as follows:

**Definition 3.1 ([13]).** A $\Delta$-complex structure on a space $X$ is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with $n$ depending on the index $\alpha$, such that:

(i) The restriction $\alpha$ on the interior of $\Delta^n$ is injective, and each point of $X$ is in the image of exactly one such restriction of $\alpha$.

(ii) Each restriction of $\sigma_\alpha$ to a face of $\Delta^n$ is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of $\Delta^n$ with $\Delta^{n-1}$ by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

(iii) A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in $\Delta^n$ for each $\sigma_\alpha$.

For convenient we introduce the following definition.

**Definition 3.2.** An $n$-simplex with a canonical orientation is said to be an oriented $n$-simplex, and an $\Delta$-complex is said to be oriented if it consists only of the images of oriented simplices together with the collection of maps keeping the orientations.

In the sequel, we assume that all $n$-simplices (hence $\Delta$-complexes) are oriented if there is no special statement. In the construction of the characteristic maps, a useful operation here called retraction of an $n$-simplex is often needed. Now let us give the definition.

**Definition 3.3.** Let $n \geq 2$ be an integer and $\Delta_{i_0 \cdots i_n}$ be a standard $n$-simplex with ordering vertices set (or rather say, sequence) $V = \{i_j\}$ ($0 \leq j \leq n$). For any two vertices $i_k$ and $i_l$ with $k < l$, a map $\phi_{i_k i_l}$ is called a simple retraction along $i_k i_l$ provided that

(i) it takes $\Delta_{i_0 \cdots i_n}$ to be $\emptyset$ if $i_l = i_{k+1}$;

(ii) otherwise, $\phi_{i_k i_l}$ is a homotopic retraction which is a homeomorphism restricted on the interior of $\Delta_{i_0 \cdots i_n}$ such that

(1) it makes the oriented edge (as a 1-simplex) $i_k i_l$ be a vertex but left all other vertices and $(n - 1)$-faces which not contain $i_k$ and $i_l$ fixed;

(2) any two oriented edges having the directions from one common vertex but with different ending
vertices \(i_k\) and \(i_l\) or the inverse directions are retracted as one oriented edge.

We denoted by \(\Delta^{ab}_{i_0 \cdots i_n}\) the retracted \(n\)-simplex, and it can be embedded in \(\Delta_{i_0 \cdots i_n}\) by a natural map (denoted by \(\varepsilon_{i_k i_l}\)).

Keep the same notation as above. If the labels \(i_{t_1}, i_{t_2}, \cdots, i_{t_m}\) denote the essentially same one vertex in \(V\), then let \(V_l = \{i_{t_1}, i_{t_2}, \cdots, i_{t_m}\}\) be a subsequence of \(V\). Suppose further we have a family of simple retractions of \(\Delta_{i_0 \cdots i_n}: \phi_{i_{t_1} i_{t_2}}, \phi_{i_{t_2} i_{t_3}}, \cdots, \phi_{i_{t_{m-1}} i_{t_m}}\) respectively along \(i_{t_1} i_{t_2}, i_{t_2} i_{t_3}, \cdots, i_{t_{m-1}} i_{t_m}\).

Then by a successive process of retractions and inclusions we can define a map \(\phi_\Delta\) from \(\Delta_{i_0 \cdots i_n}\) to itself: \(\phi_\Delta^\Delta := \varepsilon_{i_{t_{m-1}} i_{t_m}} \circ \phi_{i_{t_{m-1}} i_{t_m}} \circ \cdots \circ \varepsilon_{i_{t_1} i_{t_2}} \circ \phi_{i_{t_1} i_{t_2}}\). The map \(\phi_\Delta\) is said to be a retraction in \(V_l\). We denote by \(\Delta_{\Delta l_{i_0 \cdots i_n}}\) the image of \(\phi_\Delta\), and call it a retracted \(n\)-simplex. It is not hard to see that \(\Delta_{\Delta l_{i_0 \cdots i_n}}\) is unique up to homotopy under different choices of the order of the vertices in \(V_l\).

For any \(n\)-simplex \(\delta_n: \Delta_{i_0 \cdots i_n} \rightarrow S_{i_0 \cdots i_n}\), we define naturally the retraction map \(\phi_\Delta\) of \(S_{i_0 \cdots i_n}\) in \(V_l\) by \(\phi_\Delta(S_{i_0 \cdots i_n}) = \delta_n(\Delta_{\Delta l_{i_0 \cdots i_n}})\), and we write \(\phi_\Delta(S_{i_0 \cdots i_n})\) for short as \(S_{i_0 \cdots i_n}\).

Furthermore, suppose we have a family of subsequences \(\{V_l\}\) (\(1 \leq l \leq k\)) of vertices in \(V\), where each \(V_l\) consists only of the same vertices, then we denote by \(\phi_{\{V_l\}}\) the composition map \(\phi_{V_k} \circ \cdots \circ \phi_{V_1}\), and denote \(S^{\{V_l\}}_{i_0 \cdots i_n} = \phi_{\{V_l\}}(S_{i_0 \cdots i_n})\)

**Example 3.4.** The following two figures demonstrate how one can obtain \(\Delta_{aba}^{aa}\) and \(\Delta_{aba}^{aabb}\) after a sequel of simple retractions. As we can see, the resulting \(\Delta\)-complexes are respectively a disk and a sphere.

![Figure 2: \(\Delta_{aba}^{aa}\) is obtained from \(\Delta_{aba}\) after a simple retraction along the edge aa.](image)

![Figure 3: \(\Delta_{aba}^{aabb}\) is obtained from \(\Delta_{aba}\) after two simple retractions.](image)

**Definition 3.5.** Let \(S\) be a \(\Delta\)-complex consisting of \(n\)-simplices \(\{S_i^\Delta\}\) and with vertices set \(V\). Consider all the possible \((n + 1)\)-simplices with all their vertices in \(V\). Each such \((n + 1)\)-simplex is called a lift of \(S\). Furthermore, any \(n\)-simplex is called admissible for \(S\) if either (1): the \(n\)-simplex is in \(\{S_i^\Delta\}\), or (2): the \(n\)-simplex is not in \(\{S_i^\Delta\}\), but it belongs to two different lifts as face. A lift of \(S\) is called an admissible lift if all its \(n\)-faces are admissible.

Now we can give the promised construction. Let \(P\) be a path complex, our strategy is to construct first an oriented \(\Delta\)-complex \(S(P)\) by an inductive process.

For 0-dimensional \(\Delta\)-complex. We begin by letting \(S_0(S(P))\) simply be \(P_0\), i.e., the set of allowed elementary 0-paths in \(P\) (namely some vertices in \(V\)).

For 1-dimensional \(\Delta\)-complex. To obtain \(S_1(S(P))\), for each allowed elementary 1-path \(e_{i_m i_n} \in P_1\), we define a 1-simplex \(\Delta_{i_m i_n}\) by assigning to it the direction from the vertex \(i_m\) to \(i_n\) as the canonical orientation, that is to say, any two elementary 1-paths give the same 1-simplex if and only if the two paths are the same. Going through all elements in \(P_1\), one gets a family of 1-simplices, after gluing the same vertices of all obtained 1-simplices, then put them together as \(S_1(S(P))\).

For 2-dimensional \(\Delta\)-complex. To construct \(S_2(S(P))\), we need to attach 2-simplices to \(S_1(S(P))\), and may do some retraction after that. For any elementary 2-path \(e_{i_m i_n i_l} \in P_2\), since it is regular,
there are two possibilities: (1) $i_a \neq i_b \neq i_c$; (2) $i_a = i_c \neq i_b$. For case (1), if the edge $i_a i_c$ is admissible for $\Delta_1(S(P))$, then we attach a 2-simplex $\Delta_{i_ai_i_c}$ along the 1-simplices $i_a i_b$ and $i_b i_c$ in $\Delta_1(S(P))$. For case (2), 1-simplices (edges) $i_a i_b$, $i_b i_c$ and $i_a i_c$ are retracted to be $i_a i_b$, $i_b i_c = i_b i_a$ and $\emptyset$. The retracted 2-simplex $\Delta_{i_ai_i_b}$ (a disk) is an admissible lift of $i_a i_b$, $i_b i_c \in \Delta_1(S(P))$, then we attach it along them as the characteristic map.

For $k$-dimensional $\Delta$-complex ($k \geq 2$). We continue this process by induction. Suppose for any $k \geq 2$ we have constructed $\Delta_k(S(P))$ (which is obtained from the elementary $k$-paths in $P(k)$). We denote by $\{\Delta_k'(S(P))\}$ the set of all $k$-simplices in $\Delta_k(S(P))$. For each elementary $(k+1)$-path $e_{i_0 \cdots i_{k+1}} \in P_{k+1}$, suppose the sequence $\{i_0, \ldots, i_{k+1}\}$ is divided into $\{V_0 = \{i_0, \ldots, i_l\}, \ldots, V_l = \{i_l, \ldots, i_{k+1}\}\}$ where each subsequence is a maximal subsequence containing only the same vertices in $\{i_0, \ldots, i_{k+1}\}$ (so some $V_s$ may be just one vertex). Consider all the admissible lifts (each one is decided uniquely up to homotopic equivalent) of $\{\Delta_k'(S(P))\}$. If they contain a retracted $(k+1)$-simplex of the form $\Delta_{i_0 \cdots i_{k+1}}^{V_0 \cdots V_l}$ where each $V_s$ is a subsequence (which may be empty) of $V_s$ for $0 \leq s \leq l$, we then associate $e_{i_0 \cdots i_{k+1}}$ with a retracted $(k+1)$-simplex $\Delta_{i_0 \cdots i_{k+1}}^{V_0 \cdots V_l}$ and attach $\Delta_{i_0 \cdots i_{k+1}}^{V_0 \cdots V_l}$ to $\Delta_k(S(P))$ along the corresponding (retracted) admissible $k$-faces. Otherwise there is nothing to be done. In this manner, one gets the $\Delta_{k+1}(S(P))$.

Due to the above procedure and by induction, we obtain the desired oriented $\Delta$-complex $S(P)$.

Suppose $\{e_i = e_{i_0 \cdots i_n}\}$ is a basis of elementary $n$-paths of $P$, for each $e_i$ by the above construction we can define a map

$$F_S : e_i \rightarrow \phi_{e_i}$$

where $\phi_{e_i}$ is the map

$$\phi_{e_i} : \Delta_{i_0 \cdots i_n} \rightarrow \Delta_{i_0 \cdots i_n}^{V_0 \cdots V_l}$$

defined above when $\Delta_{i_0 \cdots i_n}^{V_0 \cdots V_l}$ is an admissible lift, otherwise we set $\phi_{e_i}(\Delta_{i_0 \cdots i_n}) = \emptyset$. When $\phi_{e_i}(\Delta_{i_0 \cdots i_n}) \neq \emptyset$, by definition $\phi_{e_i}$ is in fact a characteristic map and we define $\partial_n \phi_{e_i}(\Delta_{i_0 \cdots i_n}) = \phi_{e_i} \partial_n(\Delta_{i_0 \cdots i_n})$.

Otherwise we define $\partial_n \phi_{e_i}(\Delta_{i_0 \cdots i_n}) = \emptyset$.

At last we denote $F_\Delta = \phi_{e_i} \circ F_S$ as the composition map

$$F_\Delta : e_i \rightarrow \Delta_{i_0 \cdots i_n}^{V_0 \cdots V_l}.$$ 

An elementary $n$-path $e_i$ is called admissible provided that $F_\Delta(e_i) \neq \emptyset$, i.e., it is an admissible lift. The map $F_\Delta$ in fact gives rise to a functor from the category of path complexes to that of $\Delta$-complexes (the proof is straightforward but we do not need this fact).

We use a simple example to illustrate the above construction.

**Example 3.6.** Consider the path complex $P$ defined as follows:

- 0-paths: 0, 1, 2, 3
- 1-paths: 01, 02, 03, 10, 12, 13
- 2-paths: 010, 012.

By definition and the result in Example 3.4, its geometric realization $\Delta(S(P))$ is a 2-dimensional $\Delta$-complex with a handle, which is drawn on Figure 4.

![Figure 4: The geometric realization $\Delta(S(P))$ of $P$.](image-url)

### 3.2. (Co-)homological isomorphism.

Now we are ready to prove the main result of this section.

**Proposition 3.7.** Let $P$ be a path complex, then every element in $\Omega_n(P)$ has the form of $\sum_{i=1}^{s} r_i e_i$ for some integer $s$ and $r_i \in R$, where each $e_i$ is an admissible elementary $n$-path.
Proof. We prove it by induction on \( n \). When \( n = 0 \) there is nothing to prove. Let \( n = 1 \), and let \( p_1 = \sum_{i=1}^{s} r_i e_i \in \Omega_1(P) \) with all \( e_i = e_{io_i1} \in A_1(P) \) are different and \( r_i \in R \). Then \( \partial_1(p_1) = \sum_{i=2}^{s} r_i (e_i - e_{io_i}) \in A_0(P) = P_0 \). It follows that for each \( e_{io_i1} \), if its 0-face (vertex) \( e_{io} \) or \( e_{i1} \) is not in \( P_0 \), then \( e_{io} \) or \( e_{i1} \) must be a 0-face of another elementary 1-path \( e_{ioj} \), which appears in the expression of \( p_1 \). That is to say, \( F_\Delta(e_{io1}) = \phi_1(\Delta_{io1}) \) is an admissible lift. Then the result holds for this case.

Suppose for \( n = l \) the result holds. Now let \( n = l + 1 \), and let \( p_{l+1} = \sum_{i=1}^{s} r_i e_i \in \Omega_{l+1}(P) \) with all \( e_i = e_{io\cdots i_{l+1}} \in A_{l+1}(P) \) are different and \( r_i \in R \). We need to show that each \( e_{io\cdots i_{l+1}} \) is admissible.

By the definition, one has that

\[
\partial_{l+1}(p_{l+1}) = \sum_{i=1}^{s} r_i \sum_{p=0}^{l+1} (-1)^p (e_{io\cdots i_p\cdots i_{l+1}})
\]

follows from \( \partial_{l+1}(p_{l+1}) \in \Omega_l(P) \). This implies that any \( e_{io\cdots i_p\cdots i_{l+1}} \notin A_l(P) \) must appear in at least two different summands of \( R(p) \), namely, each \( e_{io\cdots i_p\cdots i_{l+1}} \) appeared in \( R(p) \) is admissible. A similar discussion shows that each \( e_{io\cdots i_p\cdots i_{l+1}} \in A_l(P) \) in (3.2) is either admissible by the inductive hypothesis, or appears in a group of canceling summands and hence is also admissible. That is to say, all the elementary \( l \)-paths in (3.1) are admissible. Thus each \( F_\Delta(e_{io\cdots i_p\cdots i_{l+1}}) = \phi_1(\Delta_{io\cdots i_p\cdots i_{l+1}}) \) is an admissible lift for \( \Delta_l-1(S(P)) \). Hence they are all in \( \Delta_l(S(P)) \) by our construction. Now each \( F_\Delta(e_{io\cdots i_{l+1}}) = \phi_1(\Delta_{io\cdots i_{l+1}}) \) is an admissible lift for \( \Delta_l(S(P)) \) since all its \( l \)-faces are just \( \phi_1(\Delta_{io\cdots i_p\cdots i_{l+1}}) \) \((0 \leq p \leq l + 1)\), which are all admissible by the above discussion. This completes the proof.

Lemma 3.8. Given a path complex \( P \). Let \( e_n \in P_n \), if \( e_n \) is admissible, then \( F_\Delta(\partial_n(e_n)) = \partial_n(F_\Delta(e_n)) \).

Proof. Write \( e_n = e_{a_0a_1\cdots a_n} \), and denote \( V = \{a_i\} \). Note that \( \phi_\Delta(\Delta_{a_0a_1\cdots a_n}) \) is an admissible lift since \( e_n \) is admissible. Let us first compute \( F_\Delta(\partial_n(e_n)) \), recall that by definition one has

\[
F_\Delta(\partial_n(e_n)) = F_\Delta \left( \sum_{q=0}^{n} (-1)^q e_{a_0\cdots a_{q-1}a_qa_{q+1}\cdots a_n} \right)
\]

where two vertices \( a_{q-1}, a_{q+1} \) in each term are different, that is to say, any summands of non-regular elementary paths are viewed as 0.

By the definition of \( F_\Delta \) one has

\[
\partial_n(F_\Delta(e_n)) = \partial_n(\phi_\Delta(\Delta_{a_0\cdots a_n}))
\]

\[
= \phi_{e_n} \circ \partial_n(\Delta_{a_0\cdots a_n})
\]

\[
= \phi_{e_n} \left( \sum_{q=0}^{n} (-1)^q \Delta_{a_0\cdots a_{q-1}a_qa_{q+1}\cdots a_n} \right)
\]
where the second equality follows from that each \( \phi_{e_n}(\Delta_{a_0\cdots a_n}) \) is an admissible lift, and the third equality follows from that after applying \( \psi \), the terms \((-1)^q\Delta_{a_0\cdots a_{q-1}a_{q+1}\cdots a_n}\) with \( a_{q-1} = a_{q+1} \) vanish by the definition of retraction. Now comparing (3.3) and (3.4) we get the desired equality \( p \).

Lemma 3.9. Let \( p_n, q_n \in P_n \) be two admissible elementary \( n \)-paths. If \( F_\Delta(p_n) = F_\Delta(q_n) \), then \( p_n = q_n \).

Proof. Let \( p_n = e_{i_0}^{q_0}\cdots i_n^q \) and \( q_n = e_{i_0}^{q_0}\cdots i_n^q \) be two elementary \( n \)-paths such that \( F_\Delta(p_n) = F_\Delta(q_n) \).

From the vertices of \( F_\Delta(p_n) \) and \( F_\Delta(q_n) \) we get that the sets \( \{i_j^q\} \) and \( \{i_j^q\} \) have the same elements if we forget the repeated ones, we claim that also one has \( i_j^p = i_j^q \) for each \( j \). To show this, suppose \( l \) is the least number such that \( i_l^p \neq i_l^q \), then by what we have shown there must be an integer \( s \) such that it is the least number satisfying \( i_s^p = i_s^q \). Now if \( s > l \), suppose \( V_s \) is the subsequence of all the repeated vertices in \( \{i_r^p, \cdots, i_n^p\} \) then the retracted \( (n-l) \)-simplex \( \Delta_{i_r^p}\cdots i_n^p \) as an \( (n-l) \)-face in \( F_\Delta(p_n) \) could not appear in \( F_\Delta(q_n) \), hence a contradiction. So suppose \( s < l \), and that \( V_s \) is the subsequence of all the repeated vertices in \( \{i_r^p, \cdots, i_n^p\} \), then the retracted \( (n-s) \)-simplex \( \Delta_{i_r^p}\cdots i_n^p \) as an \( (n-s) \)-face of \( F_\Delta(p_n) \) could not appear in \( F_\Delta(q_n) \), again a contradiction, this completes the proof.

Now we can turn to the main result of this section, just keep in mind that hereafter, both of the path (co-)homology and simplicial (co-)homology should be read as (co-)homologies with coefficients in \( R \) if not specified.

Corollary 3.10. For a path complex \( P \), the above map \( F_\Delta : P \to S(P) \) gives a 1-1-correspondence of cycles between \( H_*(P) \) and \( H_*(S(P)) \).

Proof. Suppose \( c_n = F_\Delta(p_n) \) with \( p_n = \sum_{i=1}^{m \geq 2} r_i e_{a_1^0\cdots a_i^q} \) \((r_i \in R)\) such that each \( e_{a_1^0\cdots a_i^q} \) is an admissible elementary \( n \)-path. Let \( V_i = \{a_i^q\} \) and let \( \{V_i\} \) be the family of subsequence of repeated vertices in \( V_i \). By definition, we have:

\[
\partial_n(p_n) = \sum_{i=1}^{m \geq 2} r_i \left( \sum_{q=0}^{n} (-1)^q e_{a_1^0\cdots a_i^{q-1}a_i^q a_i^{q+1} \cdots a_n} \right) = \partial_n(c_n) = \sum_{i=1}^{m \geq 2} r_i \left( \partial_n(e_{a_1^0\cdots a_i^{q-1}}) \right) \tag{3.5}
\]

where the second equality follows from Lemma 3.8. Note that all the summands in the last equality of (3.6) are not empty since that any regular boundary paths of an admissible elementary path are obvious also admissible. Noticing that the vanishes of (3.5) and (3.6) are equivalent, since they involve exactly the same combinatorial computation by Lemma 3.9.

Now if \( c_n \) is an \( n \)-cycle in \( H_*(S(P)) \), then it follows from the construction of \( S(P) \) that we have \( c_n = F_\Delta(p_n) \), say, with \( p_n = \sum_{i=1}^{m \geq 2} r_i e_{a_1^0\cdots a_i^{q-1}} \) such that each \( F_\Delta(e_{a_1^0\cdots a_i^{q-1}}) \) is an admissible lift. It follows from \( \partial_n(c_n) = 0 \) that one has \( \partial_n(p_n) = 0 \) by what we have shown. So \( p_n \) is the desired \( n \)-cycle in \( H_n(P) \).
Conversely, for an \( n \)-cycle \( p_n = \sum_{i=1}^{m} r_i e_{a_i^0 a_i^1 \cdots a_i^n} \) in \( H_n(P) \). We claim that \( c_n = F_\Delta(p_n) \) is the desired \( n \)-cycle in \( H_n(S(P)) \). In fact, this follows from the discussion at the beginning of the proof if we can show that each elementary \( n \)-path \( e_{a_i^0 a_i^1 \cdots a_i^n} \) is admissible. But this is a direct result of Proposition 3.7 since \( p_n \) as an \( n \)-cycle is obvious in \( \Omega_n(P) \). The uniqueness of \( c_n \) follows from Lemma 3.9. Thus we obtain the desired 1-1-correspondence. \( \square \)

We conclude this section by the two main theorems as follows.

**Theorem 3.11.** Let \( P \) be a path complex and \( R \) be a ring, the map \( F_\Delta \) induces an isomorphism \( H_*(P) \cong H_*(S(P)) \).

**Proof.** Since any boundary is naturally a cycle, the theorem follows directly from Lemma 3.8 and Corollary 3.10. \( \square \)

Theorem 3.11 together with Mayer-Vietoris exact sequence can be used to simplify many proofs of results in Section 5 of [9]. Moreover, as a direct application, now let us sketch a new proof the main theorem of [10]. As we see in Example 2.3, one can associate each digraph \( P \) with a path complex \( P(G) \), hence it is reasonable to say the path (co-)homology of a digraph. Now for any finite simplicial complex \( S \), [10] gives a natural way to construct a finite cubical digraph \( G_S = (V,E) \). In details, the set \( V \) of vertices of \( G_S \) coincides with the set of all simplices from \( S \), and two simplices \( s, t \) are connected in \( G_S \) by a directed edge \( (s \rightarrow t) \in E \) if and only if \( s \supset t \) and \( \dim(s) = \dim(t) + 1 \).

![Figure 5: A simplicial complex S and its cubical digraph G_S.](image)

One can verify that (but we omit the details here) the geometric realization of \( P(G_S) \) is exactly the full barycentric subdivision \( B_S \) of \( S \), which obvious has the same simplicial homology groups as those of \( S \). Thus Theorem 3.11 implies [10, Theorem 5.1].

As the dual version of Theorem 3.11 we have

**Theorem 3.12.** Let \( P \) be a path complex and \( R \) be a unital ring. The map \( F_\Delta \) induces an isomorphism between the path cohomology of \( P \) and the simplicial cohomology of \( S(P) \), namely, we have an isomorphism \( H^*(P) \cong H^*(S(P)) \).

**Proof.** Recall that by definition, \( H^*(P) \) is defined as the homology group of the following cochain complex:

\[
\Omega^*(P;R) := \cdots \leftarrow \Omega^n(P;R) \leftarrow \Omega^{n-1}(P;R) \leftarrow \cdots \leftarrow \Omega^0(P;R) \leftarrow 0.
\]

Using the isomorphism \( \Omega^0(P;R) \cong \text{Hom}_R(\Omega_p(P;R),R) \) in (2.3) and the obvious isomorphism \( \Omega_p(P;R) \cong R \otimes \Omega_p(P;Z) \), by adjoint isomorphism \( \text{Hom}_R(R \otimes \Omega_p(P;Z),R) \cong \text{Hom}_Z(\Omega_p(P;Z),R) \) one has \( \Omega^0(P;R) \cong \text{Hom}_Z(\Omega_p(P;Z),R) \). That is to say, (2.3) and (3.7) imply a cochain isomorphism

\[
\Omega^*(P;R) \cong \text{Hom}_Z(\Omega_*(P;Z),R).
\]

On the other hand, by definition \( H^*(S(P)) \) is the homology groups of the cochain complex

\[
C^*(S(P);R) := \text{Hom}_Z(C_*(S(P);Z),R)
\]

where \( C_*(S(P);Z) \) is the simplicial chain complex of \( S(P) \). Now Theorem 3.11 asserts an isomorphism between the homology groups of two chain complexes of free abelian groups:

\[
F_\Delta : \Omega_*(P;Z) \rightarrow C_*(S(P);Z).
\]

Thus (3.8), (3.9) and (3.10) imply the desired isomorphism by [13, Corollary 3.4]. \( \square \)
At last let us make a comment on Theorems 3.10 and 3.11. As one may easily see that, for any complete digraph $G$ arising from an oriented $n$-simplex $\Delta^n$, $\Delta^n$ is the geometric realization of $P(G)$. One should also note that in usual, though $F_\Delta$ induces an isomorphism between the (co-)homologies of them, the realization map $F_\Delta$ itself does not induce a (co-)chain isomorphism between $\Omega_s(P(G))$ (resp. $\Omega^s(P(G))$) and $C_s(S(P(G);R)$ (resp. $C^s(S(P(G);R)$ except for $G$ with some strict conditions (for e.g. $G$ is complete, or of the case occurred in [3, Theorem 5.24]).

4. APPLICATION I: RELATIONSHIP WITH HOCHSCHILD (CO-)HOMOLOGY

Starting from here, we will give some applications of the main results obtained in Section 3. The aim of this section is to connect path (co-)homology with Hochschild (co-)homology.

To begin we recall from [10] that, for a finite simplicial complex $S$, one can associate it with a finite cubical digraph $G_S$ whose path homology is isomorphic to the simplicial homology of $S$ (c.f. [10] Theorem 5.1], see also the paragraph below Theorem 3.11). Furthermore, if $R = K$ is a field there exists an algebra $A_S$ such that the path cohomology of $G_S$ and the Hochschild cohomology of $A_S$ are isomorphic ([10, Corollary 5.2]). In fact, more than this, the restriction that the cubical path complex over a field can be removed, we shall show that the same result also holds true for path cohomology of any regular path complex with coefficients in a commutative unital ring.

Assume in the sequel that $R$ is a commutative unital ring. We shall associate a path complex $P$ with two associative unital algebras $A_{S(P)}$ and $\bar{A}_{S(P)}$ and consider the corresponding path (co-)homology and Hochschild (co-)homology. We recall from [7] that, for any (locally) finite simplicial complex $S$, one can define a digraph $E_S$ where the vertices are all simplices from $S$ and a couple $(s,t)$ is a directed edge if and only if $s \subseteq t$ (compare with the definition of cubical digraph stated above Figure 5). The $R$-algebra $A_S$ is defined as a set of all finite $R$-linear combinations of edges of $E_S$ with a multiplication given by the rule:

$$(s_1,t_1)(s_2,t_2) = \left\{ \begin{array}{ll} (s_1,t_2) & \text{if } t_1 = s_2; \\ 0 & \text{otherwise.} \end{array} \right.$$  

Moreover, if we replace the elements in $A_S$, i.e., all the edges of $E_S$ by arbitrary couples $(s,t)$ whenever $s,t$ are vertices in $E_S$, and keep the multiplication formula as above, then we obtain a new $R$-algebra different from $A_S$, which is denoted by $\bar{A}_S$. It is obvious that both of $A_S$ and $\bar{A}_S$ are free as $R$-modules.

The notion of Hochschild (co-)homology groups was first defined in [15] for the algebra over a field, and was later extended in [11] for the algebra over a commutative unital ring. Now we recall some of the definitions. Let $A$ be a $R$-algebra which is also a projective $R$-module and let $M$ be an $A$-bimodule. Denote $S_n(A)$ the $R$-module obtained by taking the $n$-fold tensor product of $A$ over $R$, which is easily checked to be $R$-projective. The Hochschild homology groups $HH_*(A,M)$ of $A$ with coefficients in $M$ are referred as to the homology groups of the complex $C_*(A,M) = M \otimes_K S_*(A)$ with differentiation

$$d_n(m \otimes \lambda_1 \otimes \cdots \otimes \lambda_n) = m\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_n + \sum_{i=1}^{n-1} (-1)^i m \otimes \lambda_1 \otimes \cdots \otimes \lambda_i \lambda_{i+1} \otimes \cdots \otimes \lambda_n$$

$$+ (-1)^n \lambda_1 \otimes \cdots \otimes \lambda_{n-1} \otimes m.$$

Similarly, set $C^n(A,M)$ as the set of all $R$-linear functions $f : S_n(A) \to M$. The Hochschild cohomology groups $HH^*(A,M)$ of $A$ with coefficients in $M$ are referred as to the cohomological groups of the cochain complex $C^*(A,M)$ with the coboundary operator $\delta : C^n(A,M) \to C^{n+1}(A,M)$ given by the following formula:

$$\delta f(x_1, x_2, \cdots, x_{n+1}) = x_1 f(x_2, \cdots, x_n) + \sum_{i=1}^{n} (-1)^i f(x_1, \cdots, x_i x_{i+1}, \cdots, x_n)$$

$$+ (-1)^{n+1} f(x_1, \cdots, x_n) x_n.$$  

In particular, if $M = A$, then we shall write them shortly by $HH_*(A)$ and $HH^*(A)$ as no confusion rises.

Now we can state the main result of [7], which plays a key role here.
Theorem 4.1. Let $R$ be a commutative unital ring, there above defined functor, $S \rightarrow A_S$, from Locally Finite Simplicial Complexes to Associative Unital $R$-algebras inducing an isomorphism

$$H^*(S; R) \cong HH^*(A_S)$$

between the simplicial cohomology of $S$ and the Hochschild cohomology of $A_S$. The isomorphism preserves the cup product.

This yields the following theorem.

Theorem 4.2. Let $P$ be a path complex and $R$ be a commutative unital ring, there exists an isomorphism $H^*(P) \cong HH^*(A_{S(P)})$.

Proof. The isomorphism follows from directly from Theorem 3.12 and Theorem 4.1. □

A weak conclusion similar to this was given in [10, Corollary 5.2] where the obtained isomorphism demands that $P$ is a cubical path complex. Note that Theorem 4.2 is generally true for any regular path complex.

To derive a similar result for path homology and Hochschild homology, we shall use the following result, which can be viewed as the dual version of the main part of Theorem 4.1. In the proof we shall use some results and notations in [12]. In fact, the results there reveal that the path (co-)homology theory is also a powerful tool when dealing with the algebraic aspect of the simplicial (co-)homology, as one obviously sees that it allows a direct proof of Theorem 4.1 by avoid using the diagram cohomology and the Cohomology Comparison Theorem (see [7] for details).

Given a finite simplicial complex $S$, consider the associated cubical digraph $G = G_S = (V, E)$ (see Figure 4 and the definition above it) and the path complex $P(G)$ which gives naturally a chain (cochain) complex $\Omega_*(G) = \Omega_*(P(G))$ ($\Omega^*(G) = \Omega^*(P(G))$) by definition. Also consider the product digraph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ where $\widetilde{V} = V \times V$ and $\widetilde{E}$ denotes the set of the edges $(i, j) \rightarrow (i', j')$ whenever $j \rightarrow i'$ is in $E$. One can also associate $P(\widetilde{G})$ with a chain complex $\widetilde{\Omega}_*(G) = \Omega_*(P(\widetilde{G}), R)$ and a cochain complex $\widetilde{\Omega}^*(G) = \Omega^*(P(\widetilde{G}), R)$, respectively. With all these in hand, we can now prove directly the mentioned dual result, also without involving diagram cohomology and Cohomology Comparison Theorem.

Proposition 4.3. Let $S$ be a finite simplicial complex, then there exists an isomorphism

$$H_*(S) \cong HH_*(A_S).$$

Proof. Let us recall the cochain map

\begin{equation}
C^*(A_S, A_S) \rightarrow C^*(A_S, \tilde{A}_S) \xrightarrow{\phi} \tilde{\Omega}^*(G) \xrightarrow{\varphi} C^*(B_S; R)
\end{equation}

constructed in [12] where $\varphi$ is a composition map of a sequel of quotient maps and isomorphism (see [12 (4)]) and the maps below it), and note that all the maps in (4.1) are also homomorphisms between cochain complexes of $R$-modules and preserve the cohomologies by the results of [12]. Also note that the isomorphism $\phi$ identifies $C^*(A_S, \tilde{A}_S)$ with $\tilde{\Omega}^*(G)$, whence $\phi$ identifies $C^*(A_S, A_S)$ with a subcomplex of $\tilde{\Omega}^*(G)$ (see the proofs of [12 Lemmas 5.1 and 5.2] for details), and it is obvious that both of $C^*(A_S, A_S)$ and $C^*(A_S, \tilde{A}_S)$ are free $R$-modules under this identification, hence all of $R$-modules in (4.1) are free. Thus applying the functor $\text{Hom}_R(\cdot, R_R)$ to sided-terms in (4.1) and by Universal Coefficient Theorem for cohomology (see, for example [17 3.6.5]), one has the following isomorphism:

\begin{equation}
H_n(\text{Hom}_R(C^*(A_S, A_S), R)) \cong H_n(\text{Hom}_R(C^*(B_S, R), R)),
\end{equation}

note that “cohomology” becomes “homology” here since $C^*(A_S, A_S)$ and $C^*(B_S, R)$ themselves are cochain complexes. On the one hand one computes that

\begin{equation}
\begin{aligned}
H_n(\text{Hom}_R(C^*(A_S, A_S), R)) &= H_n(\text{Hom}_R(\tilde{S}_*(A_S), A_S), R) \\
&\cong H_n(\text{Hom}_R(A_S, R_R) \otimes_R \tilde{S}_*(A_S)) \\
&\cong H_n(A_S \otimes_R \tilde{S}_*(A_S)) \\
&= HH_n(A_S, A_S)
\end{aligned}
\end{equation}
where the first isomorphism follows from \[1\] Proposition 5.2 since \(A_S\) is also finitely generated free giving that \(\tilde{S}_*\) is isomorphic. The second isomorphism follows from the obvious isomorphism \(\text{Hom}_R(A_S, R^n) \cong A_S\) of \(A_S\)-bimodules. On the other hand using \[1\] Proposition 5.2 again one computes that:

\[
H_n(\text{Hom}_R(C^*(B_S, R), R)) = H_n(\text{Hom}_R(\text{Hom}_\mathbb{Z}(C_*(B_S, \mathbb{Z}), R), R)) \\
\cong H_n(\text{Hom}_R(R, R) \otimes_\mathbb{Z} C_*(B_S, \mathbb{Z})) \\
\cong H_n(R \otimes_\mathbb{Z} C_*(B_S, \mathbb{Z})) \\
= H_n(B_S; R).
\]

Hence the result follows from (4.2), (4.3), (4.4) and the obvious isomorphism \(H_*(B_S; R) \cong H_*(S; R)\).

Now we conclude this section by the dual result of Theorem 4.2.

**Theorem 4.4.** Let \(P\) be a path complex and \(R\) be a commutative unital ring, there exists an isomorphism \(H_*(P) \cong \text{HH}_*(A_S(P))\).

**Proof.** Note that \(S(P)\) is a finite simplicial complex since \(P\) has a finite vertices. Let \(\tilde{S} = S(P)\), then the result follows immediately by Theorem 3.11 and Proposition 4.3.

---

5. Application II: Künneth formula of path homology

Now we turn to the functorial properties of path homology. Throughout this section, let \(R\) be a commutative ring.

Similar to simplicial homological theory, the authors of \[9\] defined the Cartesian product for two path complexes (see \[9\] Definition 7.3), and furthermore gave the analogues of the Eilenberg-Zilber theorem and Künneth formula for regular path homology with coefficients in a field \(K\).

In this section, we will give some geometric interpretation of these results in a more general setting, i.e., for path homology with coefficients in a commutative ring. Their proofs are based on the correspondence of the path complexes and \(\Delta\)-complexes presented in Section 3. With the same assumptions, we show that the pattern of these proofs can be used to give the Künneth formula for the join of two regular path complexes (see Definition 5.10 below).

Note that such two types of Künneth formula were obtained in \[9\] in different ways for regular path complexes over a field \(K\). Here we shall show that they could be obtained in essentially the same way.

### 5.1. The case of Cartesian product.

For our purpose, we shall first recall the construction of the simplicial cross product in terms of \(\Delta\)-complex. Given two standard simplices \(\Delta^m\) and \(\Delta^n\), let us first subdivide \(\Delta^m \times \Delta^n\) into simplices.

We label the vertices of \(\Delta^m\) as \(v_0, v_1, \ldots, v_m\) and the vertices of \(\Delta^n\) as \(w_0, w_1, \ldots, w_n\). Naturally, we label the \(mn\) vertices of \(\Delta^m \times \Delta^n\) as \((v_0, w_0), (v_0, w_1), (v_1, w_0), \ldots, (v_m, w_n)\). We now view the pairs \((i, j)\) with \(0 \leq i \leq m\) and \(0 \leq j \leq n\) as the vertices of an \(m \times n\) rectangle grid in \(\mathbb{R}^2\).

For any path \(\sigma\) formed by a sequence of \(m + n\) horizontal and vertical edges in the grid starting at the origin \((0, 0)\) and ending at \((m, n)\), we associate it with a standard \((m + n)\)-simplex \(\Delta^m \times \Delta^n\) whose vertices are \((v_{i_k}, w_{j_k})\) for all the \(k\)-th vertices \((i_k, j_k)\) of the path \(\sigma\), assigned the order "\(<\)" such that \((v_{i_k}, w_{j_k}) \prec (v_{i_l}, w_{j_l})\) for \(i_k \leq i_l, j_k < j_l\) or \(i_k < i_l, j_k \leq j_l\).

Define a linear map (which can be viewed as an inclusion) \(i_S : \Delta^m \times \Delta^n \to \Delta^m \times \Delta^n\) by sending the \(k\)-th vertex of \(\Delta^m \times \Delta^n\) to the vertex \((v_{i_k}, w_{j_k})\). When \(\sigma\) runs through all the possible \(m + n\) step-like paths in the grid from \((0, 0)\) to \((m, n)\), all the associated \((m + n)\)-simplices \(\Delta^m \times \Delta^n\) fit together by \(i_S\) nicely to form a \(\Delta\)-structure on \(\Delta^m \times \Delta^n\) (see, for e.g. \[13\] p.278 or \[9\] p.68).

Now we can define the cross product of simplices by the formula

\[
\Delta^m \otimes \Delta^n \xrightarrow{x} \sum_{\sigma} (-1)^{|\sigma|} \Delta^m \times \Delta^n
\]

where \(|\sigma|\) is the number of squares in the grid lying below the path \(\sigma\), and "\(\times\)" here means the cross product. The following boundary formula can be easily verified by a direct calculation.

\[
\partial(\Delta^m \times \Delta^n) = \partial(\Delta^m) \times \Delta^n + (-1)^m \Delta^m \times \partial(\Delta^n).
\]
In particular, let $X$ and $Y$ be $\Delta$-complexes, $\{\phi^n_m\} : \Delta^m \to X$ and $\{\phi^n_n\} : \Delta^n \to Y$ be the characteristic maps of $X$ and $Y$, respectively, we denote by $C_m(X; R)$ and $C_n(Y; R)$ the simplicial $n$-chain groups with coefficient $R$ of $X$ and $Y$ generated by all $\phi^n_m$ and $\phi^n_n$, respectively. Then (5.1) gives rise to the following definition of simplicial cross product

$$C_m(X; R) \otimes_R C_n(Y; R) \xrightarrow{\times} C_{m+n}(X \times Y; R)$$

by the formula

$$\phi^m_\alpha \times \phi^n_\beta = \sum_\sigma (-1)^{|\sigma|} (\phi^m_\alpha \times \phi^n_\beta) l_\sigma,$$

where $(\phi^m_\alpha \times \phi^n_\beta) l_\sigma$ means the map $(\phi^m_\alpha \times \phi^n_\beta)$ restricted on $l_\sigma(\Delta_{m+n})$. By (5.2) we have the boundary operators

$$\partial(\phi^m_\alpha \times \phi^n_\beta) = \partial(\phi^m_\alpha) \times \phi^n_\beta + (-1)^m \phi^m_\alpha \times \partial(\phi^n_\beta).$$

Lemma 5.1. The chain map $G : C_* (X; R) \otimes_R C_* (Y; R) \to C_* (X \times Y; R)$ induced by (5.3) is injective for any two $\Delta$-complexes $X$ and $Y$.

Proof. $G$ is a chain map that follows directly by the boundary formula of tensor products of two complexes and (5.4). To show that it is injective, assume $m + n = k$, let $\Delta^m = \Delta_{(v_0, v_m)}, \Delta^n = \Delta_{(v_m, v_n)}$, and $\Delta_{m+n} = \Delta_{(v_0, v_n) - (v_1, v_{j_1}) - \cdots - (v_m, v_n)}$, and let $f^j_m : \Delta^m \to X$ and $g^l_n : \Delta^n \to Y$ be respectively the characteristic maps of $X$ and $Y$. Let $w = \sum_{m+n=k} \sum_j c^j_{m,n}(f^j_m \otimes g^l_n)$, we need to show that $G(w) = 0$ implies all $c^j_{m,n} = 0$.

Indeed, by definition one has

$$G(w) = \sum_{m+n=k} \left( \sum_\delta (-1)^{|\delta|} \sum_j c^j_{m,n}(f^j_m \times g^l_n) l_\delta \right).$$

Note that the interiors of all $(f^j_m \times g^l_n) l_\delta(\Delta_{m+n})$ are different since all $f^j_m$ and $g^l_n$ are different characteristic maps of $X$ and $Y$, and it follows that the map $G(w)$ in $C_k(S(X) \times S(Y); R)$ is totally decided by the effect it acts on all $\Delta_{m+n}$. Therefore (5.5) vanishes and it implies that $c^j_{m,n}(f^j_m \times g^l_n) l_\delta = 0$ for each $\Delta_{m+n}$, or equivalently $c^j_{m,n} = 0$ since $f^j_m \times g^l_n$ as a sum of characteristic maps over $X \times Y$ is never a zero map. This finishes the proof. $\Box$

Next let us recall the definition of the cross product of two path complexes from [9] and compare it with the above definition.

Let $X, Y$ be two finite sets, and let $Z = X \times Y$. For two elementary $m$- and $n$-paths $e_{i_0 \cdots i_m} \in \mathcal{R}_m(X)$ and $e_{j_0 \cdots j_n} \in \mathcal{R}_n(Y)$, as we have already done to the cross product of simplices, again we have an $m \times n$ rectangle grid in $\mathbb{R}^2$ with the vertices $(i_0, j_0), (i_1, j_0), \cdots, (i_m, j_0)$, and let $f^j_m : \Delta^m \to X$ and $g^l_n : \Delta^n \to Y$ be respectively the characteristic maps of $X$ and $Y$. Let $w = \sum_{m+n=k} \sum_j c^j_{m,n}(f^j_m \otimes g^l_n)$, we need to show that $G(w) = 0$ implies all $c^j_{m,n} = 0$.

Indeed, by definition one has

$$G(w) = \sum_{m+n=k} \left( \sum_\delta (-1)^{|\delta|} \sum_j c^j_{m,n}(f^j_m \times g^l_n) l_\delta \right).$$

Note that the interiors of all $(f^j_m \times g^l_n) l_\delta(\Delta_{m+n})$ are different since all $f^j_m$ and $g^l_n$ are different characteristic maps of $X$ and $Y$, and it follows that the map $G(w)$ in $C_k(S(X) \times S(Y); R)$ is totally decided by the effect it acts on all $\Delta_{m+n}$. Therefore (5.5) vanishes and it implies that $c^j_{m,n}(f^j_m \times g^l_n) l_\delta = 0$ for each $\Delta_{m+n}$, or equivalently $c^j_{m,n} = 0$ since $f^j_m \times g^l_n$ as a sum of characteristic maps over $X \times Y$ is never a zero map. This finishes the proof. $\Box$

More generally, for any two path complexes $P(X)$, $P(Y)$, recall (see [9] Definition 7.3) that their Cartesian product $P(Z) = P(X) \boxtimes P(Y)$ are defined as the path complex over $Z$ with each elementary $k$-path of $P(Z)$ of the form $e_\sigma$, where the $k$ step-like path $e_\sigma$ comes from the above construction for any elementary $s$-path $e_\alpha = e_{i_0 \cdots i_s} \in P_s(X)$ and elementary $(k-s)$-path $e_\beta = e_{j_0 \cdots j_{k-s}} \in P_{k-s}(Y)$ with all $i_m \in X$ and $j_n \in Y$. Also recall that $A_n(Z)$ is the set of all $R$-linear combinations of elements in $P_n(Z)$ and $\partial_n(Z)$ is defined as the set $\{z_n | z_n \in A_n(Z) \text{ and } \partial_n(z_n) \in A_{n-1}(Z) \}$. $\square$

Lemma 5.2. $\Omega_s(X) \times \Omega_{k-s}(Y) \subseteq \Omega_k(Z)$.

Proof. It can be proved easily from the boundary formula for cross products. $\square$
The following lemma comes from [9 Proposition 7.12], whose proof can be carried over to the case where $R$ is a commutative ring without any change.

**Lemma 5.3.** Any path $w \in \Omega_*(Z)$ admits a representation

$$w = \sum_{e_x \in P(X), \ e_y \in P(Y)} c^{xy}(e_x \times e_y)$$

with finitely nonzero coefficients $c^{xy} \in R$ which are uniquely determined by $w$. Furthermore, the cross products $\{e_x \times e_y\}$ across all $e_x \in P(X)$ and $e_y \in P(Y)$ are linearly independent.

Our proof in the sequel depends on the following key lemma (comparing with [9 Theorem 7.15]).

**Lemma 5.4.** Any path $w \in \Omega_n(Z)$ can be written as a finite sum:

$$w = \sum_{k \leq n} \sum_{i} \left( p_i^k(X) \times q_{n-i}^k(Y) \right)$$

where $k$ runs a finite set, each $p_i^k(X) \in \Omega_i(X)$ and $q_{n-i}^k(Y) \in \Omega_{n-i}(Y)$.

**Proof.** Let $w$ be an $n$-path in $\Omega_n(Z)$, by Lemma 5.3 we can write it as a finite sum:

$$w = \sum_{i=s}^n \left( \sum_{e_x \in J_i(X), e_y \in J_{n-i}(Y)} c^{xy}(e_x \times e_y) \right)$$

where $0 \neq c^{xy} \in R$, $J_i(X) \subseteq P_i(X)$, $J_{n-i}(Y) \subseteq P_{n-i}(Y)$ and $s \geq 0$ is the lowest index of $e_x$ appeared in the expression. We do the proof by induction on the total number $a$ of the summands $c^{xy}(e_x \times e_y)$ in (5.6).

If $a = 1$, that is to say, $w = c^{xy}(e_x \times e_y) \in \Omega_n(Z)$ where $e_x \in P_s(X)$ and $e_y \in P_{n-s}(Y)$, $\partial(w) = c^{xy}(\partial(e_x) \times e_y + (-1)^s e_x \times \partial(e_y)) \in \delta_n(Z)$ and Lemma 5.2 imply that $\partial(e_x) \in \delta_{s-1}(X)$ and $\partial(e_y) \in \delta_{n-s-1}(Y)$. Let $k = s$, then $p_s = c^{xy} e_x$ and $q_{n-s} = e_y$ give the desired result.

Now suppose that for any $a < b$ (where $b > 1$) the result holds, we shall show that the result also holds for $a = b$. Write $w$ as in (5.6), suppose the total number of the summands is $b$. We rewrite $w$ as follows:

$$w = \sum_{i=s}^n \left( \sum_{e_x \in P_{n-i}(Y)} \left( \sum_{e_y \in J_i^b(X) \times e_y \in J_{n-i}(Y)} c^{xy}(e_x \times e_y) \right) \right)$$

(5.7)

$$= \sum_{i=s}^n \sum_{J_i^b(X)} \sum_{J_{n-i}(Y)} \left( \sum_{e_x \in J_i^b(X)} c^{xy}(e_x) \times \left( \sum_{e_y \in J_{n-i}(Y)} e_y \right) \right)$$

where each set $J_i^b(X) \subseteq P_i(X)$ is decided by $e_x$, and each set $J_{n-i}^b(Y) \subseteq P_{n-i}(Y)$ contains $e_y$ and is decided by $J_i^b(X)$. Note that $\cap J_{n-i}^b(Y) = \emptyset$, $\cup J_i^b(X) = J_i(X)$ and $\cup J_{n-i}^b(Y) = J_{n-i}(Y)$. It follows from $w \in \Omega_n(Z)$ and (5.7) that each sum

$$\sum_{e_x \in J_i^b(X)} c^{xy}(e_x) \in \delta_i(X) \quad \text{and} \quad \sum_{e_y \in J_{n-i}(Y)} e_y \in A_{n-i}(Y).$$

For the sake of simplicity, given any elementary $n$-path $e_n = e_0 \cdots n \in P_n(Y)$ and integer $0 \leq l \leq n$, we denote $e_{l(l-1)} = e_0 \cdots (l-1)(l+1) \cdots n$. On the other hand by (5.6) we can compute as follows:

$$\partial(w) = \sum_{e_y \in P_{n-s}(Y)} \left( \sum_{e_x \in J_i^b(X)} c^{xy}(e_x) \times e_y \right)$$

$$\quad + \sum_{J_i^b(X), J_{n-i}^b(Y)} \left( \sum_{e_x \in J_i^b(X)} \left( -1 \times c^{xy} \sum_{e_y \in J_{n-i}^b(Y)} \partial(e_y) \right) \right)$$

The expression above follows from:

1. $\cap J_{n-i}^b(Y) = \emptyset$, $\cup J_i^b(X) = J_i(X)$ and $\cup J_{n-i}^b(Y) = J_{n-i}(Y)$.
2. $w \in \Omega_n(Z)$ and $\delta(w) \in \Omega_{n-1}(Z)$. 
3. $\partial(w) \in \Omega_{n-1}(Z)$ and $\delta(w) \in \Omega_{n-2}(Z)$.
4. $\partial(w) \in \Omega_{n-2}(Z)$ and $\delta(w) \in \Omega_{n-3}(Z)$.
5. $\partial(w) \in \Omega_{n-3}(Z)$ and $\delta(w) \in \Omega_{n-4}(Z)$.

Therefore, $w$ has the desired properties.

**Theorem 7.15.** Let $X$ be a regular path complex. Then for any path $w \in \Omega_n(Z)$, the geometric realization $|\Omega_n(Z)|$ of $\Omega_n(Z)$ is homeomorphic to the $n$-dimensional simplicial complex $|\Omega_n(Z)|$.

**Proof.** The proof is similar to that of Lemma 5.4. We omit the details here. 

The proof is complete.
where $e$ is injective.

Thus by inductive hypothesis one has the desired result. \qed

By Proposition 5.7 and Lemma 5.1 one immediately has the following result.

**Corollary 5.5.** All the paths $e_x \in P(X)$ and $e_y \in P(Y)$ in Lemma 5.3 are admissible.

This corollary allows us to define an $R$-linear map

$$F^Z_R : \Omega_k(Z) \to C_k(S(X) \times S(Y); R)$$

by the formula

$$F^Z_R(e_x \times e_y) = \phi^x \times \phi^y$$

where $e_x$ and $e_y$ are admissible elementary paths in $P_1(X)$ and $P_{k-1}(Y)$, respectively, and the symbol `$\times$' means different cross products on the two sides of the equation.

**Lemma 5.6.** The map $F^Z_R : \Omega_k(Z) \to C_k(S(X) \times S(Y); R)$ is injective.
Proof. Let \( m + n = k \) and suppose that all the elementary paths in this proof are admissible, we need to show that for any non-zero \((m + n)\)-path \( w \in \Omega_{m+n}(Z) \) one always has \( F^Z_S(w) \neq 0 \). By Lemma 5.4 one can write

\[
w = \sum_{e_x \in P_m(X), \ e_y \in P_n(Y)} c^{xy}(e_x \times e_y).
\]

We claim that \( F^Z_S(w) = 0 \) implies all coefficients \( c^{xy} = 0 \). In fact, replace \( f^l \) and \( g^l \) by \( \phi_{e_x} \) and \( \phi_{e_y} \), respectively, as in the proof of Lemma 5.1 one has

\[
F^Z_S(w) = \sum_{e_x \in P_m(X), \ e_y \in P_n(Y)} c^{xy} \left( \sum_{\delta} (-1)^{|\delta|} (\phi_{e_x} \times \phi_{e_y}) l^{\delta} \right).
\]

Note that all \( l^{\delta}(\Delta^{m+n}) \) are different since all \( e_x \) and \( e_y \) are different from each other. Thus each map \( (\phi_{e_x} \times \phi_{e_y}) l^{\delta} \) as the characteristic map \( \Delta^{m+n} \to S(X) \times S(Y) \) has different images from each other since a retraction \( \phi \) is a homeomorphism on the interior of some standard simplex. That is to say, \( F^Z_S(w) = 0 \) if and only if all coefficients \( c^{xy} = 0 \), as asserted. \( \square \)

**Proposition 5.7.** The chain complex \( \Omega_*(Z) \) can be viewed as a subcomplex of the simplicial chain complex \( C_*(S(X) \times S(Y); R) \) via the map \( F^Z_S \).

**Proof.** By Lemma 5.6 it suffices to verify that the map \( F^Z_S \) is a chain map. Let

\[
w = \sum_{e_x \in P_m(X), \ e_y \in P_{n-1}(Y)} c^{xy}(e_x \times e_y)
\]

be an \( n \)-path in \( \Omega_n(Z) \). One computes that

\[
F^Z_S(\partial(w)) = F^Z_S\left( \sum_{e_x \in P_m(X), \ e_y \in P_{n-1}(Y)} c^{xy}(\partial(e_x) \times e_y + (-1)^i e_x \times \partial(e_y)) \right)
\]

\[
= \sum_{e_x \in P_m(X), \ e_y \in P_{n-1}(Y)} c^{xy} (\phi(\partial(e_x)) \times \phi(e_y) + (-1)^i \phi(e_x) \times \phi(\partial(e_y)))
\]

\[
= \sum_{e_x \in P_m(X), \ e_y \in P_{n-1}(Y)} c^{xy} \partial(\phi(e_x) \times \phi(e_y))
\]

\[
= \partial(F^Z_S(w))
\]

where the second equality follows from that any elementary path in the boundary of an admissible path is also admissible by Lemma 3.8 and the third equality follows by (5.4). It then finishes the proof. \( \square \)

We are ready to prove the first main result in this section:

**Theorem 5.8.** Let \( P(X) \) and \( P(Y) \) be two regular path complexes and \( R \) a commutative ring. Then for their Cartesian product \( P(Z) = P(X) \boxtimes P(Y) \) the following isomorphism of chain complexes holds:

\[
(5.10) \quad \Omega_*(X) \otimes_R \Omega_*(Y) \cong \Omega_*(Z)
\]

whose mapping is given by \( u \otimes v \mapsto u \times v \).

**Proof.** Let us first inspect the map

\[
F_S \otimes F_S : \Omega_*(X) \otimes_R \Omega_*(Y) \to C_*(X; R) \otimes_R C_*(Y; R)
\]

defined by the formula

\[
(F_S \otimes F_S)(u \otimes v) = F_S(u) \otimes F_S(v)
\]

for any \( u \in \Omega_m(X) \) and \( v \in \Omega_n(Y) \). By Proposition 3.6 and Lemma 3.8, as in the proof of Proposition 5.7, on can easily verify that \( F_S \otimes F_S \) is a chain map. Furthermore we shall show that \( F_S \otimes F_S \) is
injective. Suppose $u = \sum_{e_x \in P_n(X)} k^x e_x \in \Omega_m(X)$, $v = \sum_{e_y \in P_n(Y)} k^y e_y \in \Omega_n(Y)$ and $(F_S \otimes F_S)(u \otimes v) = 0$, then we have
\[ u \otimes v = \sum_{e_x \in P_m(X), e_y \in P_n(Y)} k^x k^y (e_x \otimes e_y). \]
and it follows from
\[ (F_S \otimes F_S)(u \otimes v) = \sum_{e_x \in P_m(X), e_y \in P_n(Y)} k^x k^y (\phi_{e_x} \otimes \phi_{e_y}) = 0 \]
that all coefficients $k^x k^y = 0$, since all $\phi_{e_x}$ and $\phi_{e_y}$ lay respectively in the bases of $C_m(X; R)$ and $C_n(Y; R)$ gives that all $\phi_{e_x} \otimes \phi_{e_y}$ lay in the basis of $C_m(X; R) \otimes C_n(Y; R)$. It follows that $u \otimes v = 0$, as desired.

Now we obtain a diagram
\[ \Omega_*(X) \otimes_R \Omega_*(Y) \xrightarrow{F_{1\Omega}} C_*(S(X) \times S(Y); R) \xrightarrow{H^S} \Omega_*(Z) \]
where $F_{1\Omega} = G \circ (F_S \otimes F_S)$ and $G$ is given by Lemma 5.1. It is obvious that $F_{1\Omega}$ is injective. We claim that $\text{Im} F_{1\Omega} = \text{Im} F^S_{F_S}$. Indeed, for any $w \in \Omega_*(Z)$, by Lemma 5.1 one can write $w = \sum_{p \times q} (p_x \times q_y)$ where $p_x$ goes through a subset $I \subset \Omega_*(X)$ and $q_y$ goes through a subset $J \subset \Omega_*(Y)$, let $w' = \sum_{p \times q} (p_x \times q_y)$ one immediately has $F_{1\Omega}(w') = F^S_{F_S}(w)$, that is, $\text{Im} F^S_{F_S} \subseteq \text{Im} F_{1\Omega}$. The inverse inclusion is obvious by Lemma 5.2. Thus the map $(F^S_{F_S})^{-1} \circ F_{1\Omega}$ has meaning and it gives the desired isomorphism 5.10 by sending $u \otimes v$ to $u \times v$. □

Künneth formula is used to compute the (co-)homology of a product space in terms of the (co-)homology of the factors. For path complexes over a field, a type of Künneth formula also holds (see [2], Theorem 5.8). In fact, for any principle ideal domain $R$, we have the following more general result.

**Corollary 5.9.** Let $P(X)$ and $P(Y)$ be two regular path complexes and $R$ a PID. Then, for each $n$, there holds a Künneth formula by the following natural splitting short exact sequence
\[ 0 \to \oplus_1 (H_i(X) \otimes_R H_{n-i}(Y)) \to H_n(Z) \to \oplus_i Tor^R_1 (H_i(X), H_{n-i-1}(Y)) \to 0. \]

**Proof.** By Theorem 5.8 one has
\[ H_n(Z) = H_n(\Omega_*(Z)) \cong H_n(\Omega_*(X) \otimes_R \Omega_*(Y)). \]
Note that each $A_n(X) (n \geq 0)$ is a finitely generated free $R$-module, thus $\Omega_*(X)$ is also a free $R$-module since $R$ is a PID. Therefore, the result follows from [3, Theorem 3B.5]. □

5.2. The case of join.

Comparing with the approach of by considering the Cartesian product, there is another way to derive the Künneth formula via an operation called the join (see Definition 5.10 below) of two regular path complexes. For path complexes over a field $K$, this is exactly what [2, Theorem 6.5] says. In fact the general result also holds when one replaces the field $K$ by any commutative ring $R$. Before we set off to prove this, let us do some preparation.

**Definition 5.10** ([2], Definition 6.1). Given two disjoint finite sets $X$, $Y$ and their path complexes $P(X)$, $P(Y)$, set $Z = X \times Y$ and define a path complex $P(Z)$ as follows: $P(Z)$ consists of all paths of the form $uv$ where $u \in P(X)$ and $v \in P(Y)$. The path complex $P(Z)$ is called a join of $P(X)$, $P(Y)$ and is denoted by $P(Z) = P(X) \ast P(Y)$.

Given any two paths $u \in P_{k-1}(X)$ and $v \in P_{n-1}(Y)$, it is easy to check that, the differential operator acting on the join $uv \in P_n(Z)$ is given by the following formula:
\[ \partial(uv) = \partial(u)v + (-1)^i u \partial(v). \]
For more properties and examples of the join of two regular path complexes the reader may refer to [2]. To prove the asserted Künneth formula, we proceed by a parallel way as that of proving Theorem 5.8. The key idea is to simply replace the symbol “×” of Cartesian product by the symbol “∗” of join in the previous proofs, and treat carefully the corresponding dimensions.
In details, suppose we are given two regular path complexes $P(X)$ and $P(Y)$, denote $P(Z)$ their join. We see that by definition $\Omega_{a-1}(X) \ast \Omega_{b-1}(Y) \subseteq \Omega_k(Z)$ (comparing with Lemma 5.2) and each path $w \in \Omega_a(Z)$ admits a representation
\[ w = \sum_{i=1}^{n} e_x \in P_{i-1}(X), e_y \in P_{n-i}(Y) \]
with finitely nonzero coefficients $e^{xy} \in R$, which are uniquely determined by $w$ since obviously $e_{xy} = e_x \ast e_y$ across all $e_x \in P(X)$ and $e_y \in P(Y)$ are $R$-linearly independent (comparing with Lemma 5.3). Now for the chain complex $\Omega$ with finitely nonzero coefficients
\[ \Omega \]
we see that by definition $\Omega$ consists of all elements of the form $e_{xy}$ where $e_x$ and $e_y$ are some elementary paths in $\Omega_{i-1}(X)$ and $\Omega_{n-i}(Y)$. Apparently $F$ is injective since all $e_{xy} = e_x \ast e_y$ are $R$-linearly independent.

Now we are done if we can show that the map $F: \Omega_i(X) \ast \Omega_j(Y) \to \Omega_k(Z)$ given by $u \otimes v \mapsto u \ast v$ for any $u \in \Omega_i(X)$ and $v \in \Omega_j(Y)$. One sees that the basis of $\Omega_i(X) \ast \Omega_j(Y)$ consists of all elements of the form $e_x \otimes e_y$ where $e_x$ and $e_y$ are some elementary paths in $\Omega_{i-1}(X)$ and $\Omega_{n-i}(Y)$. Apparently $F$ is injective since all $e_{xy} = e_x \ast e_y$ are $R$-linearly independent.

We now observe that $F$ is surjective. To see this one needs only to replace the symbol \textquotedblleft$\ast$\textquotedblright by \textquotedblleft$\ast\otimes\ast$\textquotedblright, $\Omega_i(X) P_i(X)$ and similarly $J_i(X)$ etc. by $\Omega_i(X) \otimes \Omega_{j-1}(Y)$, $P_i(X) \otimes \Omega_{j-1}(Y)$ and $J_i(X)$ etc., respectively, where $P_i(X) := P_{i-1}(X)$ and $J_i(X) := J_{i-1}(X)$. Then the proof still validates and this gives that $F$ is surjective.

**Remark 5.12.** If $R = K$ is a field, one immediately obtains \cite{9} Theorems 6.5 and 7.6 by Theorem 5.8, Corollary 5.9 and Theorem 5.11. But note that the proofs of \cite{9} Theorems 6.5 and 7.6] used the theory of vector spaces over a field, which no more works for $R$-modules when $R$ is a commutative ring, so our proofs of Theorems 5.8, 5.11 and the proof of \cite{9] Theorems 6.5 and 7.6] are the essentially different ones.

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