DADE’S ORDINARY CONJECTURE IMPLIES THE
ALPERIN-MCKAY CONJECTURE

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Abstract. We show that Dade’s ordinary conjecture implies the Alperin-McKay
conjecture. We remark that some of the methods can be used to identify a canonical
height zero character in a nilpotent block.

Dade proved in [4] that his projective conjecture [4, 15.5] implies the Alperin-
McKay conjecture. Navarro showed in [11, Theorem 9.27] that the group version of
Dade’s ordinary conjecture implies the McKay conjecture. We show here that Dade’s
ordinary conjecture [3, 6.3] implies the Alperin-McKay conjecture. Let \( p \) be a prime
number.

Theorem 1. If Dade’s ordinary conjecture holds for all \( p \)-blocks of finite groups, then
the Alperin-McKay conjecture holds for all \( p \)-blocks of finite groups.

The proof combines arguments from Sambale [17] and formal properties of chains
of subgroups in fusion systems from [7]. Let \( (K, \mathcal{O}, k) \) be a \( p \)-modular system. We
assume that \( k \) is algebraically closed, and let \( \bar{K} \) be an algebraic closure of \( K \). By a
character of a finite group, we will mean a \( \bar{K} \)-valued character. For a finite group \( G \)
and a block \( B \) of \( \mathcal{O}G \), let \( \text{Irr}(B) \) denote the set of irreducible characters of \( G \) in
the block \( B \), and let \( \text{Irr}_0(B) \) denote the set of irreducible height zero characters of \( G \) in
\( B \). For a central \( p \)-subgroup \( Z \) of \( G \) and a character \( \eta \) of \( Z \), let \( \text{Irr}_0(B|\eta) \) denote the
subset of \( \text{Irr}_0(B) \) consisting of those height zero characters which cover the character
\( \eta \). The following lemma is implicit in [17].

Lemma 2. Let \( P \) be a finite \( p \)-group, let \( \mathcal{F} \) be a saturated fusion system on \( P \) and let
\( Z \leq Z(\mathcal{F}) \). Suppose that \( \eta \) is a linear character of \( P \). There exists a linear character
\( \hat{\eta} \) of \( P \) such that \( \hat{\eta}|_Z = \eta|_Z \) and \( \text{foc}(\mathcal{F}) \leq \text{Ker}(\hat{\eta}) \).

Proof. First consider the case that \( \eta|_Z \) is faithful. Then \( Z \cap [P, P] = 1 \). Hence
by [5] Lemma 4.3, \( \text{foc}(\mathcal{F}) \cap Z = 1 \). The result is now immediate. Now suppose
\( Z_0 = \text{Ker}(\eta|_Z) \) and let \( \mathcal{F} = \mathcal{F}/Z_0 \). By the previous argument, applied to \( P/Z_0 \) and
there exists a character \( \hat{\eta} \) of \( P/Z_0 \) such that \( \hat{\eta}|_{Z/Z_0} = \eta|_{Z/Z_0} \) and \( \text{foc}(\hat{F}) \leq \text{Ker}(\hat{\eta}) \). Denote also by \( \hat{\eta} \) the inflation of \( \hat{\eta} \) to \( P \). Then \( \hat{\eta} \) has the required properties since \( \text{foc}(\hat{F}) = \text{foc}(\hat{F})Z_0/Z_0 \).

The following result is a special case of a result due to Murai; we include a proof for convenience.

**Lemma 3 (cf. [9, Theorem 4.4]).** Let \( G \) be a finite group, \( B \) be a block of \( OG \), and \( P \) a defect group of \( B \). Let \( Z \) be a central \( p \)-subgroup of \( G \) and let \( \eta \) be an irreducible character of \( Z \) such that \( \text{Irr}_0(B|\eta) \neq \emptyset \). Then \( \eta \) extends to \( P \).

**Proof.** By replacing \( K \) by a suitable finite extension we may assume that \( K \) is a splitting field for all subgroups of \( G \). Let \( i \in B^P \) be a source idempotent of \( B \) and let \( V \) be a \( KG \)-module affording an element of \( \text{Irr}_0(B|\eta) \). Then \( n := \dim_K(iV) \) is prime to \( p \). Since \( i \) commutes with \( P \), \( iv \) is a \( KP \)-module via \( x \cdot iv = ixv \), where \( x \in P, v \in V \). Let \( \rho : P \to \text{GL}_n(K) \) be a corresponding representation and let \( \delta : P \to K^\times \) be the determinantal character of \( \rho \). Then \( \delta|_Z = \eta^n \). The result follows since \( n \) is prime to \( p \).

**Lemma 4.** Let \( G \) be a finite group, let \( B \) be a block of \( OG \) with a defect group \( P \), and let \( Z \) be a central \( p \)-subgroup of \( G \). Then \( |\text{Irr}_0(B)| \) equals the product of \( |\text{Irr}_0(B|1_Z)| \) with the number of distinct linear characters \( \eta \) of \( Z \) which extend to \( P \).

**Proof.** Let \( \mathcal{F} = \mathcal{F}_{(P,e_P)}(G,B) \) be the fusion system of \( B \) with respect to a maximal \( B \)-Brauer pair \((P,e_P)\), and let \( \eta \) be a linear character of \( Z \) which extends to \( P \). Since \( Z \leq Z(\mathcal{F}) \), by Lemma 2 there exists a linear character \( \hat{\eta} \) of \( P \) such that \( \hat{\eta}|_Z = \eta \) and \( \text{foc}(\hat{F}) \leq \text{Ker}(\hat{\eta}) \). By the properties of the Broué-Puig \( * \)-construction \([1],[16] \) the map \( \chi \mapsto \hat{\chi} \) is a bijection between \( \text{Irr}_0(B|1_Z) \) and \( \text{Irr}_0(B|\eta) \). The result follows by Lemma 3.

Slightly strengthening the terminology in \([10] \), we say that a pair \((G,B)\) consisting of a finite group \( G \) and a block \( B \) of \( OG \) is a minimal counterexample to the Alperin-McKay conjecture if \( B \) is a counterexample to the Alperin-McKay conjecture and if \( G \) is such that first \( |G/Z(G)| \) is smallest possible and then \( |G| \) is smallest possible.

**Proposition 5.** Let \((G,B)\) be a minimal counterexample to the Alperin-McKay conjecture. Then \( O_p(G) = 1 \).

**Proof.** By a result of Murai \([10] \), we have that \( Z := O_p(G) \) is central in \( G \). Let \( P \) be a defect group of \( B \) and let \( \bar{C} \) be the block of \( ON_G(P) \) in Brauer correspondence with \( B \). By Lemma 4 \( |\text{Irr}_0(B)| = |\text{Irr}_0(C)| \) if and only if \( |\text{Irr}_0(B)| = |\text{Irr}_0(C)| \) where \( B \) (respectively \( C \)) is the block of \( OG/Z \) (respectively \( ON_G(P)/Z \)) dominated by \( B \) (respectively \( C \)). The result follows since \( N_{G/Z}(P/Z) = N_G(P)/Z \) and \( B \) and \( \bar{C} \) are in Brauer correspondence.
Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $P$, and let $\mathcal{C}$ be a full subcategory of $\mathcal{F}$ which is upwardly closed; that is, if $Q$, $R$ are subgroups of $P$ such that $Q$ belongs to $\mathcal{C}$ and if $\text{Hom}_\mathcal{F}(Q, R)$ is nonempty, then also $R$ belongs to $\mathcal{C}$. Drawing upon notation and facts from [7, §], $S_\sigma(\mathcal{C})$ is the category having as objects nonempty chains $\sigma = Q_0 < Q_1 < \cdots < Q_m$ of subgroups $Q_i$ of $P$ belonging to $\mathcal{C}$ such that $m \geq 0$ and $Q_i$ is normal in $Q_m$, for $0 \leq i \leq m$. Morphisms in $S_\sigma(\mathcal{C})$ are given by certain ‘obvious’ commutative diagrams of morphisms in $\mathcal{F}$; see [7, 2.1, 4.1] for details. With this notation, the length of of a chain $\sigma$ in $S_\sigma(\mathcal{C})$ is the integer $|\sigma| = m$.

The chain $\sigma$ is called fully normalised if $Q_0$ is fully $\mathcal{F}$-normalised and if either $m = 0$ or the chain $\sigma_{\geq 1} = Q_1 < Q_2 < \cdots < Q_m$ is fully $N_\mathcal{F}(Q_0)$-normalised. Every chain in $S_\sigma(\mathcal{C})$ is isomorphic (in the category $S_\sigma(\mathcal{C})$) to a fully normalised chain. There is an involution $n$ on the set of fully normalised chains which fixes the chain of length zero $P$ and which sends any other fully normalised chain $\sigma$ to a fully normalised chain $n(\sigma)$ of length $|\sigma| \pm 1$. This involution is defined as follows. If $\sigma = P$, then set $n(\sigma) = \sigma$. If $\sigma = Q_0 < Q_1 < \cdots < Q_m$ is a fully normalised chain different from $P$ such that $Q_m = N_\sigma(P)$, then define $\sigma$ by removing the last term $Q_m$; if $Q_m < N_\sigma(P)$, then define $\sigma$ by adding $N_\sigma(P)$ as last term to the chain. Then $n(\sigma)$ is fully normalised, and $n(n(\sigma)) = \sigma$. Denote by $[S_\sigma(\mathcal{C})]$ the partially ordered set of isomorphism classes of chains in $S_\sigma(\mathcal{C})$, and for each chain $\sigma$ by $[\sigma]$ its isomorphism class. We have a partition

$$[S_\sigma(\mathcal{C})] = \{[P]\} \cup \mathcal{B} \cup n(\mathcal{B}) ,$$

where $\mathcal{B}$ is the set of isomorphism classes of fully normalised chains $\sigma$ satisfying $|n(\sigma)| = |\sigma| + 1$. The following Lemma is a very special case of a functor cohomological statement [7, Theorem 5.11].

**Lemma 6.** With the notation above, let $f : [S_\sigma(\mathcal{C})] \to \mathbb{Z}$ be a function on the set of isomorphism classes of chains in $S_\sigma(\mathcal{C})$ satisfying $f([\sigma]) = f([n(\sigma)])$ for any fully normalised chain $\sigma$ in $S_\sigma(\mathcal{C})$. Then

$$\sum_{[\sigma] \in [S_\sigma(\mathcal{C})]} (-1)^{|\sigma|} f([\sigma]) = f([P]) .$$

**Proof.** The hypothesis on $f$ implies that the contributions from chains in $\mathcal{B}$ cancel those from chains in $n(\mathcal{B})$, whence the result.

**Proposition 7.** Let $G$ be a finite group such that $O_p(G) = 1$, and let $B$ be a block of $OG$ with nontrivial defect groups. Suppose that Dade’s ordinary conjecture holds for $B$ and that the Alperin-McKay conjecture holds for any block of any proper subgroup of $G$. Then the Alperin-McKay conjecture holds for the block $B$.

**Proof.** Let $(P, e)$ be a maximal $B$-Brauer pair, and denote by $\mathcal{F}$ the associated fusion system on $P$. For $d$ a positive integer, denote by $k_d(G, B)$ the number of ordinary irreducible characters in $B$ of defect $d$. If $p^d = |P|$, then $k_d(G, B)$ is the number of height zero characters, and if $p^d > |P|$, then $k_d(G, B) = 0$. 


Let $C$ be the full subcategory of $F$ consisting of all nontrivial subgroups of $P$. We briefly describe the standard translation process between chains in a fusion system of a block and the associated chains of Brauer pairs. The map sending a chain $\sigma = Q_0 < Q_1 < \cdots < Q_m$ in $S_{\sigma}(C)$ to the unique chain of nontrivial $B$-Brauer pairs $\tau = (Q_0, e_0) < (Q_1, e_1) < \cdots < (Q_m, e_m)$ contained in $(P, e)$ induces a bijection between isomorphism classes of chains in $S_{\sigma}(C)$ and the set of $G$-conjugacy classes of normal chains of nontrivial $B$-Brauer pairs (cf. [7, 2.5]). If $\sigma$ is fully normalised, then the corresponding chain of Brauer pairs $\tau = (Q_0, e_0) < (Q_1, e_1) < \cdots < (Q_m, e_m)$ has the property that $e_\tau = e_m$ remains a block of $N_G(\tau)$, and by [7, 5.14], $N_P(\sigma) = N_P(\tau)$ is a defect group of $e_\tau$ as a block of $N_G(\tau)$. Denote by $n(\tau)$ the chain of Brauer pairs corresponding to $n(\sigma)$.

Let $d > 0$ such that $p^d = |P|$. Define a function $f$ on $S_{\sigma}(C)$ by setting

$$f([\sigma]) = k_d(N_G(\tau), e_\tau)$$

for any fully normalised chain $\sigma$ and corresponding chain $\tau$ of Brauer pairs. If $N_P(\sigma)$ is a proper subgroup of $P$, then $f([\sigma]) = 0$, and if $N_P(\sigma) = P$, then $f([\sigma])$ is the number of height zero characters of the block $e_\tau$ of $N_G(\tau)$. Dade’s ordinary conjecture for $B$, reformulated here in terms of chains of Brauer pairs, asserts that $k_d(G, B)$ is equal to the alternating sum

$$\sum_{[\sigma] \in S_{\sigma}(C)} (-1)^{|\sigma|} f([\sigma]).$$

The passage between formulations in terms of normalisers of chains of Brauer pairs rather than normalisers of chains of $p$-subgroups is well-known; see e.g. [6, 4.5], [15].

If $|n(\sigma)| = |\sigma| + 1$, then setting $H = N_G(\tau)$, we have $N_G(n(\sigma)) = N_H(N_P(\tau), e_{n(\tau)})$; that is, $(N_P(\tau), e_{n(\tau)})$ is a maximal $(H, e_\tau)$-Brauer pair. By the assumptions, the Alperin-McKay conjecture holds for the block $e_\tau$ of $H$. This translates to the equality $f([\sigma]) = f([n(\sigma)])$. That is, the function $f$ satisfies the hypotheses of Lemma [6]. Thus the above alternating sum is equal to $f([P])$, which by definition is $k_d(N_G(P, e), e)$, and thus the Alperin-McKay conjecture holds for $B$. 

Theorem 1 follows now immediately from combining Propositions 5 and 7.

**Remark 8.** By work of Dade [2] and Okuyama and Wajima [12], the Alperin-McKay conjecture holds for blocks of finite $p$-solvable groups. G. R. Robinson pointed out that Proposition 5 yields another short proof of this fact.

**Remark 9.** Let $G$ be a finite group, $B$ a block algebra of $OG$, $(P, e_P)$ a maximal $(G, B)$-Brauer pair with associated fusion system $F$ on $P$, and let $Z$ be a central $p$-subgroup of $G$. Let $\eta$ be a linear character of $Z$, and suppose that $\eta$ extends to a linear character $\hat{\eta}$ of $P$ satisfying $\text{foc}(F) \leq \text{Ker}(\hat{\eta})$. The proof of Lemma 4 is based on the fact that the $*$-construction $\chi \mapsto \hat{\eta} * \chi$ yields a bijection $\text{Irr}(B|1_Z) \to \text{Irr}(B|\eta)$.
There is some slightly more structural background to this. For $\chi \in \text{Irr}(B)$, denote by $e(\chi)$ the corresponding central primitive idempotent in $K \otimes_O B$. Set

$$
e_1 = \sum_{\chi \in \text{Irr}_0(B)\{1\}} e(\chi), \quad \ne = \sum_{\chi \in \text{Irr}(B)\eta} e(\chi).$$

Identify $B$ to its image in $K \otimes_O B$. Multiplying $B$ by the central idempotents $e_1$ and $e_\eta$ in $K \otimes_O B$ yields the two $O$-free $O$-algebra quotients $Be_1$ and $Be_\eta$ of $B$. By [8, Theorem 1.1], there is an $O$-algebra automorphism $\alpha$ of $B$ which induces the identity on $k \otimes_O B$ and which acts on $\text{Irr}(B)$ as the map $\chi \to \hat{\eta} \star \chi$. Thus the extension of $\alpha$ to $K \otimes_O B$ sends $e_1$ to $e_\eta$ and hence induces an $O$-algebra isomorphism

$$Be_1 \cong Be_\eta.$$ 

We conclude this note with an observation regarding canonical height zero characters in nilpotent blocks, based in part on some of the above methods.

Let $G$ be a finite group, $B$ a block algebra of $OG$, $P$ a defect group of $B$ and $i \in B^P$ a source idempotent of $B$. Denote by $\mathcal{F}$ the fusion system of $B$ on $P$ determined by the choice of $i$. Suppose that $K$ is a splitting field for all subgroups of $G$. For $V$ a finitely generated $O$-free $B$-module, denote by

$$\Delta_{V,P,i} : P \to O^\times$$

the map sending $u \in P$ to the determinant of the $O$-linear automorphism of $iV$ induced by the action of $u$ on $V$ (this makes sense since all elements in $P$ commute with $i$). By standard properties of determinants, this map depends only on the $(B^P)^{\times}$-conjugacy class of $i$ and the isomorphism class of the $K \otimes_O B$-module $K \otimes_O V$. Thus if $V$ affords a character $\chi \in \text{Irr}(B)$, we write $\Delta_{\chi,P,i}$ instead of $\Delta_{V,P,i}$.

**Proposition 10.** With the notation above, let $\chi \in \text{Irr}(B)$ and $\eta \in \text{Irr}(P/\text{soc}(P))$. Regard $\eta$ as a linear character of $P$. We have

$$\Delta_{\eta \star \chi,P,i} = \eta^{\chi(i)} \Delta_{\chi,P,i}.$$ 

**Proof.** The statement makes sense as the value of $\chi$ on an idempotent is a positive integer. Let $V$ be an $O$-free $OG$-module affording $\chi$. By [8, Theorem 1.1] there exists an $O$-algebra automorphism $\alpha$ of $B$ such that the module $V^\alpha$ (obtained from twisting $V$ by $\alpha$) affords $\eta \star \chi$ and such that $\alpha(\alpha(i)) = \eta(\alpha)$$ for all $u \in P$. Since in particular $\alpha(i) = i$, it follows that

$$\Delta_{V^\alpha,P,i}(u) = \Delta_{V,P,i}(\eta(u)i)$$

for all $u \in P$. The result follows as $\text{rank}_O(iV) = \chi(i)$. \hfill \square

Denote by $\text{Irr}'(B)$ the set of all $\chi \in \text{Irr}(B)$ such that $\Delta_{\chi,P,i}$ is the trivial map (sending all elements in $P$ to $1$). Set $\text{Irr}_0(B) = \text{Irr}'(B) \cap \text{Irr}_0(B)$. The maximal local pointed groups on $B$ are $G$-conjugate. Thus if $P'$ is any other defect group of $B$ and $i' \in B^{P'}$ a source idempotent, then there exist $g \in G$ and $c \in (B^{P'})^{\times}$ such that $P' = gPg^{-1}$ and $i' = cgig^{-1}c^{-1}$. Therefore the map $\Delta_{V,P,i}$ is trivial if and only if the map
\( \Delta_{V,P',i} \) is trivial, and hence the sets \( \text{Irr}'(B) \) and \( \text{Irr}'_0(B) \) are independent of the choice of \( P \) and \( i \). The following is immediate.

**Proposition 11.** The sets \( \text{Irr}'(B) \) and \( \text{Irr}'_0(B) \) are invariant under any automorphism of \( G \) which stabilises \( B \).

The next result shows that if \( B \) is nilpotent, then \( \text{Irr}'_0(B) \) consists of a single element.

**Proposition 12.** Suppose that \( B \) is nilpotent. Then \( |\text{Irr}'_0(B)| = 1 \). Moreover, if \( p \) is odd, then the unique element of \( \text{Irr}'_0(B) \) is the unique \( p \)-rational height zero character in \( B \).

**Proof.** Let \( \chi \in \text{Irr}_0(B) \). Since \( i \) is a source idempotent of \( B \), \( \chi(i) \) is prime to \( p \) (see [13]). Hence if \( \eta, \zeta \) are linear characters of \( P \), then \( \eta^{\chi(i)} = \zeta^{\chi(i)} \) implies that \( \eta = \zeta \). Since \( B \) is nilpotent, we have that \( \text{foc}(\mathcal{F}) = [P,P] \) and \( |\text{Irr}_0(B)| = |P : [P,P]| \). Thus, by Proposition [10] the map \( \chi \mapsto \Delta_{\chi,P,i} \) is a bijection from \( \text{Irr}_0(B) \) to \( \text{Irr}(P/[P,P]) \). This proves the first assertion.

Suppose that \( p \) is odd. Let \( \chi_0 \) be the unique \( p \)-rational character in \( \text{Irr}_0(B) \). Let \( W(k) \) be the ring of Witt vectors in \( \mathcal{O} \). By the structure theory of nilpotent blocks (see [14]) there exists a \( W(k)G \)-module \( V \) affording \( \chi_0 \). Since the source idempotent \( i \) can be chosen to be in \( W(k)G \), we have that \( \Delta_{\chi,P,i} \) takes values in \( W(k) \). Since \( p \) is odd, it follows that the trivial character of \( P \) is the unique linear character of \( P \) which takes values in \( W(k) \). \( \square \)

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