Tensor product representations of the quantum double of a compact group

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Abstract

We consider the quantum double $D(G)$ of a compact group $G$, following an earlier paper. We use the explicit comultiplication on $D(G)$ in order to build tensor products of irreducible $*$-representations. Then we study their behaviour under the action of the $R$-matrix, and their decomposition into irreducible $*$-representations. The example of $D(SU(2))$ is treated in detail, with explicit formulas for direct integral decomposition (‘Clebsch–Gordan series’) and Clebsch–Gordan coefficients. We point out possible physical applications.

1 Introduction

Over the last decade quantum groups have become an important subject of research both in mathematics and physics, see a.o. the monographs [8], [14], [15] and [17]. Of special importance are those quantum groups which are quasi-triangular Hopf algebras, and thus have a universal $R$-element satisfying the quantum Yang–Baxter equation. Via the QYBE there is a connection with the braid group and thus with the theory of invariants of links and 3-manifolds. In the physical context quantum groups play an important role in the theory of integrable lattice models, conformal field theory (Wess–Zumino–Witten models for example) and topological field theory (Chern–Simons theory).

Drinfel’d [11] has introduced the notion of the quantum double $D(\mathcal{A})$ of a Hopf algebra $\mathcal{A}$. His definition (rigorous if $\mathcal{A}$ is finite dimensional, and formal otherwise) yields a quasi-triangular Hopf algebra $D(\mathcal{A})$ containing $\mathcal{A}$ as a Hopf subalgebra. For $\mathcal{A}$ infinite dimensional, various
rigorous definitions for the quantum double or its dual have been proposed, see in particular Majid \cite{17} and Podles’ and Woronowicz’ \cite{20}.

An important mathematical application of the Drinfel’d double is a rather simple construction of the ‘ordinary’ quasi-triangular quantum groups (i.e. $q$-deformations of universal enveloping algebras of semisimple Lie algebras and of algebras of functions on the corresponding groups), see for example \cite{8} and \cite{17}.

In physics the quantum double has shown up in various places: in integrable field theories \cite{6}, in algebraic quantum field theory \cite{18}, and in lattice quantum field theories. For a short summary of these applications, see \cite{12}. Another interesting application lies in orbifold models of rational conformal field theory, where the physical sectors in the theory correspond to irreducible unitary representations of the quantum double of a finite group. This has been constructed by Dijkgraaf, Pasquier and Roche in \cite{9}. Directly related to the latter are the models of topological interactions between defects in spontaneously broken gauge theories in 2+1 dimensions. In \cite{2} Bais, Van Driel and De Wild Propitius show that the non-trivial fusion and braiding properties of the excited states in broken gauge theories can be fully described by the representation theory of the quantum double of a finite group. For a detailed treatment see \cite{23}.

Both from a mathematical and a physical point of view it is interesting to consider the quantum double $D(G)$ of the Hopf $*$-algebra of functions on a (locally) compact group $G$, and to study its representation theory. For $G$ a finite group, $D(G)$ can be realized as the linear space of all complex-valued functions on $G \times G$. Its Hopf $*$-algebra structure, which rigorously follows from Drinfel’d’s definition, can be given explicitly. In \cite{16} and in the present paper we take the following approach to $D(G)$ for $G$ (locally) compact:

We realize $D(G)$ as a linear space in the form $C_c(G \times G)$, the space of complex valued, continuous functions of compact support on $G \times G$. Then the Hopf $*$-algebra operations for $G$ finite can be formally carried over to operations on $C_c(G \times G)$ for $G$ non-finite (formally because of the occurrence of Dirac delta’s). Finally it can be shown that these operations formally satisfy the axioms of a Hopf $*$-algebra.

In \cite{16}, we focussed on the $*$-algebra structure of $D(G)$, and we derived a classification of the irreducible $*$-representations (unitary representations). In the present paper, where we restrict ourselves to the case where $G$ is compact, we address questions about ‘braiding’ and ‘fusion’ properties of tensor product representations of $D(G)$, for which the comultiplication and the $R$-matrix are explicitly needed. We envisage physical applications in nontrivial topological theories such as (2+1)-dimensional quantum gravity, and higher dimensional models containing solitons \cite{4}. In view of these and other applications we present our results on representation theory not just abstractly, but quite explicitly.

The outline of the paper is as follows. In section 2 we specify the Hopf $*$-algebra structure of $D(G)$. We then turn to the irreducible unitary representations in section 3, where we first recall a main result of \cite{14}, concerning the classification of these representations. We give a definition of their characters, and compare the result to the case of finite $G$.

An outstanding feature of quasi-triangular Hopf algebras is that their non-cocommutativity is controlled by the $R$-element. Together with the explicit expression for the comultiplication this results in interesting properties of tensor products of irreducible $*$-representations of $D(G)$. In section 4 we define such a tensor product representation, and specify the action of the quantum double. In section 5 we give the action of the universal $R$-matrix on tensor product states (‘braiding’) on a formal level. The rather non-trivial Clebsch–Gordan series of irreducible $*$-representations (‘fusion rules’) are discussed in section 6. They are calculated indirectly, namely, via direct projection of states, and the comparison of squared norms. This direct projection
results in a very general method to construct the Clebsch–Gordan coefficients of a quantum double in case orthogonal bases can be given for the representation spaces. Finally, section 6 treats the example of $G = SU(2)$ in detail.

2 The Hopf algebra structure of $D(G)$

Drinfel’d [10] has given a definition of the quantum double $D(A)$ of a Hopf algebra $A$. Write $A^o$ for the dual Hopf algebra to $A$ with opposite comultiplication. Then $D(A)$ is a quasi–triangular Hopf algebra, it is equal to $A \otimes A^o$ as a linear space, and it contains $A \otimes 1$ and $1 \otimes A^o$ as Hopf subalgebras. If $A$ is moreover a Hopf *-algebra then $D(A)$ naturally becomes a Hopf *-algebra. This definition of the quantum double is only rigorous if $A$ is finite dimensional.

If $G$ is a compact group and $C(G)$ the Hopf *-algebra of continuous complex values functions on $G$, then instead of $D(C(G))$ we will write $D(G)$ for the quantum double of $C(G)$. For $G$ a finite group we have

$$D(G) \simeq C(G) \otimes \mathbb{C}[G] \simeq C(G \times G)$$

(2.1)
as linear spaces. Also in the case of a finite group it is possible to write down the formulas for the Hopf *-algebra operations and the universal $R$-element of $D(G)$, both in the formulation with $D(G) = C(G) \otimes \mathbb{C}[G]$ (see [9]) and with $D(G) = C(G \times G)$. In the last picture the formulas may typically involve a summation over the group or a Kronecker delta on $G$. They suggest analogous formulas for $G$ arbitrarily compact, by simply replacing the summation over $G$ by integration w.r.t. the normalised Haar measure on $G$, and replacing the Kronecker delta by the Dirac delta. This way we obtain the following definitions, where $F,F_1,F_2 \in C(G \times G)$ and $x,y,x_1,y_1,x_2,y_2 \in G$:

Multiplication:

$$(F_1 \cdot F_2)(x,y) := \int_G F_1(x,z) F_2(z^{-1}x,z^{-1}y) \, dz.$$  

(2.2)

*-*operation:

$$F^*(x,y) = F(y^{-1}xy,y^{-1}).$$  

(2.3)

Unit element

$$1(x,y) = \delta_e(y),$$  

(2.4)

Comultiplication:

$$(\Delta F)(x_1,y_1;x_2,y_2) = F(x_1x_2,y_1) \delta_e(y_1^{-1}y_2)$$  

(2.5)

Counit:

$$\epsilon(F) = \int_G F(e,y) \, dy$$  

(2.6)

Antipode

$$(S(F))(x,y) = F(y^{-1}x^{-1}y,y^{-1})$$  

(2.7)

Universal $R$-element:

$$R(x_1,y_1;x_2,y_2) = \delta_e(x_1y_2^{-1}) \delta_e(y_1).$$  

(2.8)

Note that due to the occurring Dirac delta’s the unit element in fact does not lie inside $D(G)$. Similarly, the comultiplication doesn’t map into $D(G) \otimes D(G)$ (not even into the topological completion $\mathcal{C}(C(G \times G) \otimes C(G \times G)) \simeq C(G \times G \times G \times G)$), and furthermore the $R$-element doesn’t lie inside $D(G) \otimes D(G)$. In practice this does not pose a serious problem as we will always formally integrate over these Dirac delta’s, nevertheless we still have to be careful in
dealing with the resulting expressions, because it can happen that the Dirac delta is partially fulfilled, giving rise to infinities.

With the above operations $C(G \times G)$ formally becomes a quasi-triangular Hopf $*-$algebra called $\mathcal{D}(G)$. For the case of a finite group $G$ this holds rigorously, which is clear just by the quantum double construction. However, for the case of general compact $G$ we have to verify that Eqs. (2.2)–(2.8) do indeed satisfy all axioms of a quasi-triangular Hopf $*-$algebra.

In [16] it was observed that $C(G \times G)$ with Eqs. (2.2) and (2.3) is a $*-$algebra, and furthermore the irreducible unitary representations of this $*-$algebra were studied and classified. In the present paper we will consider tensor products and braiding properties of these irreducible $*-$representations (from now on mostly referred to as ‘irreps’) by using the comultiplication and the $R$-element.

### 3 Irreducible representations

We recapitulate the contents of Corollary 3.10, one of the main results of [16]. Throughout, when we speak of a compact group (or space), we tacitly assume that it is a separable compact Hausdorff group (or space).

**Definition 3.1** Let $G$ be a compact group, and $\text{Conj}(G)$ the collection of conjugacy classes of $G$ (so the elements of $\text{Conj}(G)$ are the sets of the form $\{xgx^{-1}\}_{x \in G}$ with $g \in G$). For each $A \in \text{Conj}(G)$ choose some representative $g_A \in A$, and let $N_A$ be the centralizer of $g_A$ in $G$. For each $\alpha \in \widetilde{N_A}$ (the set of equivalence classes of irreducible unitary representations of $N_A$) choose a representative, also denoted by $\alpha$, which is an irreducible unitary representation of $N_A$ on some finite dimensional Hilbert space $V_\alpha$. Also, let $dz$ be the normalised Haar measure on $G$. For measurable functions $\phi : G \rightarrow V_\alpha$ such that for all $h \in N_A$ it holds that

$$\phi(gh) = \alpha(h^{-1})\phi(g) \quad \text{for almost all } g \in G,$$

we put

$$\|\phi\|^2 := \int_G \|\phi(z)\|_{V_\alpha}^2 dz. \tag{3.2}$$

Now $L^2_\alpha(G,V_\alpha)$, which is the linear space of all such $\phi$ for which $\|\phi\| < \infty$ divided out by the functions with norm zero, is a Hilbert space.

The elements of $L^2_\alpha(G,V_\alpha)$ can also be considered as $L^2$-sections of a homogeneous vector bundle over $G/N_A$. The space $L^2_\alpha(G,V_\alpha)$ is familiar as the representation space of the representation of $G$ which is induced by the representation $\alpha$ of $N_A$.

**Theorem 3.2** For $A \in \text{Conj}(G)$ and $\alpha \in \widetilde{N_A}$ we have mutually inequivalent irreducible $*-$representations $\Pi^A_\alpha$ of $\mathcal{D}(G) = C(G \times G)$ on $L^2_\alpha(G,V_\alpha)$ given by

$$\left(\Pi^A_\alpha(F)\phi\right)(x) := \int_G F(xgAx^{-1},z)\phi(z^{-1}x) \, dz, \quad F \in \mathcal{D}(G) \tag{3.3}$$

These representations are moreover $\|\cdot\|_1$-bounded (see for this notion formula (33) in [14]). All irreducible $\|\cdot\|_1$-bounded $*-$representations of $\mathcal{D}(G)$ are equivalent to some $\Pi^A_\alpha$.

In fact, a much more general theorem holds (see Theorem 3.9 in [16]), namely for the representation theory of so-called transformation group algebras $C(X \times G)$, where the compact group $G$ acts continuously on the compact set $X$, instead of the conjugation action of $G$ on $G$. 


We may even assume $G$ and $X$ to be locally compact, under the extra condition of countable separability of the $G$-action. Then we have to consider $C_c(X \times G)$ and use a quasi-invariant measure on $G/N_A$.

Also, the rest of the Hopf algebra structure of $D(G)$, in particular the comultiplication, will survive for the case of noncompact $G$ as long as $G$ acts on itself by conjugation. It would be interesting to extend the results of this paper to this case of (special) noncompact $G$.

An interesting issue in representation theory is the character of an irrep. For the case of a finite group $G$ such characters have been derived in \cite{9}. For our case, where irreps are generally infinite dimensional, the operator $\Pi_A \alpha(F)$ will not be trace class for all $F \in D(G)$, so we restrict ourselves to the case of a Lie group $G$ and $C^\infty$-functions on $G \times G$. In this paper we will only state the formula for the characters of irreps of the quantum double. The proof for it, the orthogonality of the characters, and the related subject of harmonic analysis, will be given in a forthcoming paper.

**Theorem 3.3** Let $\chi_\alpha$ denote the character of the irreducible $\ast$-representation $\alpha$ of $N_A$. For an irreducible $\ast$-representation $\Pi_A^\alpha$ of the quantum double $D(G)$ the character is given by

$$\chi_A^\alpha(F) = \int_G \int_{N_A} F(zg_A z^{-1}, zn z^{-1}) \chi_\alpha(n) \, dn \, dz, \quad F \in C^\infty(G \times G).$$

(3.4)

Let us check the connection with the case of a finite group $G$. As discussed in \cite{16} for a finite group $G$ there is a linear bijection $D(G) = C(G) \otimes C[G] \leftrightarrow C(G \times G)$:

$$f \otimes x \mapsto ((y, z) \mapsto f(y) \delta_x(z))$$

(3.5)

Taking $f = \delta_g$ as function on (finite) $G$, we obtain

$$\chi_A^\alpha(\delta_g \otimes x) = \int_G \int_{N_A} \delta_g(zg_A z^{-1}) \delta_x(zn z^{-1}) \chi_\alpha(n) \, dn \, dz,$$

(3.6)

which indeed coincides with the definition of the character in \cite{16}.

**4 Tensor products**

In section \cite{3} we have recapitulated the classification of the irreducible $\ast$-representations of the quantum double $D(G)$. With the coalgebra structure of $D(G)$ that we have derived in section \cite{3} we can now consider tensor products of such representations.

Let $\Pi_A^\alpha$ and $\Pi_B^\beta$ be irreducible $\ast$-representations of $D(G)$. For the representation space of the tensor product representation we take the Hilbert space of vector-valued functions on $G \times G$ as follows: for measurable functions $\Phi : G \times G \to V_\alpha \otimes V_\beta$ such that for all $h_1 \in N_A, h_2 \in N_B$ it holds that

$$\Phi(xh_1, yh_2) = \alpha(h_1^{-1}) \otimes \beta(h_2^{-1}) \Phi(x, y), \quad \text{for almost all } (x, y) \in G \times G$$

(4.1)

we put

$$\|\Phi\|^2 := \int_G \int_G \|\Phi(x, y)\|_{V_\alpha \otimes V_\beta}^2 \, dx \, dy.$$

(4.2)

Now the space $L^2_{\alpha,\beta}(G \times G, V_\alpha \otimes V_\beta)$ is defined as the linear space of all such $\Phi$ for which $\|\Phi\| < \infty$, divided out by the functions of norm zero. Note that this space is the completion
of the algebraic tensor product of $L^2_{\alpha}(G, V_{\alpha})$ and $L^2_{\beta}(G, V_{\beta})$. By Eq.(3.3) the tensor product representation $\Pi^A_{\alpha} \otimes \Pi^B_{\beta}$ becomes formally:

$$\left(\left(\Pi^A_{\alpha} \otimes \Pi^B_{\beta}\right)(F) \Phi\right)(x, y) := \left(\left(\Pi^A_{\alpha} \otimes \Pi^B_{\beta}\right)(\Delta F) \Phi\right)(x, y)$$

$$= \int_G \int_G \Delta F(xg_A x^{-1}, z_1; yg_B y^{-1}, z_2) \Phi(z_1 x, z_1^{-1} y) dz_1 dz_2.$$

Then it follows by substitution of Eq.\((2.5)\) and by formally integrating the Dirac delta function that

$$\left(\left(\Pi^A_{\alpha} \otimes \Pi^B_{\beta}\right)(F) \Phi\right)(x, y) = \int_G F(xg_A x^{-1} y g_B y^{-1}, z) \Phi(z^{-1} x, z^{-1} y) dz. \quad (4.3)$$

It is easy to see that this is indeed a representation of $\mathcal{D}(G)$: there is the covariance property, as given in Eq.\((1.3)\), and the homomorphism property can be readily checked. The functions of the form

$$\Phi(x, y) = \phi^A_{\alpha}(x) \otimes \phi^B_{\beta}(y) \in V_{\alpha} \otimes V_{\beta} \quad (4.4)$$

(with $\phi^A_{\alpha}$ and $\phi^B_{\beta}$ basis functions of the representation spaces for $\Pi^A_{\alpha}$ and $\Pi^B_{\beta}$ respectively) span a dense subspace of $L^2_{\alpha\beta}(G \times G, V_{\alpha} \otimes V_{\beta})$. The positive–definite inner product then reads

$$\langle \Phi_1, \Phi_2 \rangle := \int_G \int_G \langle \phi^A_{\alpha}(x), \phi^A_{\alpha}(x) \rangle_{V_{\alpha}} \langle \phi^B_{\beta}(y), \phi^B_{\beta}(y) \rangle_{V_{\beta}} dx dy. \quad (4.5)$$

This tensor product representation now enables us to further analyse two important operations which are characteristic for quasi-triangular Hopf algebras, namely ‘braiding’ and ‘fusion’. They will turn up in several applications of these algebras \([3]\).

5 Braiding of two representations

Let us investigate the action of the universal $R$-element in the aforementioned tensor product representation. A simple formal calculation with use of Eqs.\((2.3)\) and \((1.3)\) yields

$$\left(\left(\Pi^A_{\alpha} \otimes \Pi^B_{\beta}\right)(R) \Phi\right)(x, y) = \int_G \int_G \delta_\epsilon(xg_A x^{-1} z^{-1}) \delta_\epsilon(w) \Phi(w^{-1} x, z^{-1} y) dw dz$$

$$= \Phi(x, x g_A^{-1} y^{-1} z). \quad (5.1)$$

The braid operator $\mathcal{R}$ is an intertwining mapping between $\Pi^A_{\alpha} \otimes \Pi^B_{\beta}$ on $V_{\alpha} \otimes V_{\beta}$ and $\Pi^B_{\beta} \otimes \Pi^A_{\alpha}$ on $V_{\beta} \otimes V_{\alpha}$ given by

$$\mathcal{R}^{AB}_{\alpha\beta} \Phi := \left(\sigma_L \circ \left(\Pi^A_{\alpha} \otimes \Pi^B_{\beta}\right)(R)\right) \Phi \quad (5.2)$$

where

$$(\sigma_L \Phi)(x, y) := \sigma(\Phi(y, x)), \quad \sigma(v \otimes w) := w \otimes v, \quad v \in V_{\alpha}, w \in V_{\beta}, \quad (5.3)$$

so it interchanges the representations $\Pi^A_{\alpha}$ and $\Pi^B_{\beta}$. Hence

$$\left(\mathcal{R}^{AB}_{\alpha\beta} \Phi\right)(x, y) = \sigma_L \left(\left(\Pi^A_{\alpha} \otimes \Pi^B_{\beta}\right)(R)\Phi(x, y)\right) = \sigma \left(\Phi(y, y g_B^{-1} x^{-1} z)\right). \quad (5.4)$$

To make sure that Eq.\((2.3)\), being derived from a formally defined $R$-element Eq.\((2.8)\), yields the desired intertwining property for $\mathcal{R}^{AB}_{\alpha\beta}$, one can derive this property directly from Eqs.\((5.4)\) and \((1.3)\). Then we must show that

$$\left(\mathcal{R}^{AB}_{\alpha\beta} \left(\Pi^A_{\alpha} \otimes \Pi^B_{\beta}\right)(F) \Phi\right)(x, y) = \left(\left(\Pi^B_{\beta} \otimes \Pi^A_{\alpha}\right)(F) \left(\mathcal{R}^{AB}_{\alpha\beta} \Phi\right)\right)(x, y). \quad (5.5)$$
The right hand side of this equation gives
\[
\int_G F(xg^{-1}y^{-1}g_Ay^{-1}, z) \left( R_{\alpha\beta}^A \Phi \right) (z^{-1}x, z^{-1}y) \, dz = \\
\int_G F(xg^{-1}y^{-1}g_Ay^{-1}, z) \sigma \left( \Phi(z^{-1}y, z^{-1}y^{-1}A^{-1}x^{-1}) \right) \, dz.
\]
(5.6)
which is obviously equal to the left hand side of Eq.(5.5), using Eq.(5.4) and Eq.(4.3).

6 Tensor product decomposition

Another general question is the decomposition of the tensor product of two irreducible representations into irreducible representations:
\[
\Pi_A^\alpha \otimes \Pi_B^\beta \cong \bigoplus_{C,\gamma} N_{\alpha\beta C}^{AB\gamma} \Pi_C^\gamma,
\]
(6.1)
where we suppose that such a tensor product is always reducible. For finite $G$ tensor products of irreps of $D(G)$ indeed decompose into a direct sum over single irreps. For compact $G$ the direct sum over the conjugacy class label $C$ has to be replaced by a direct integral,
\[
\Pi_A^\alpha \otimes \Pi_B^\beta \cong \bigoplus_{\gamma} \int \oplus N_{\alpha\beta C}^{AB\gamma} \Pi_C^\gamma \, d\mu(C)
\]
(6.2)
where $\mu$ denotes an equivalence class of measures on the set of conjugacy classes, but the multiplicities must be the same for different measures in the same class, see for instance the last Conclusion in [1] for generalities about direct integrals. Recall that two (Borel) measures $\mu$ and $\nu$ are equivalent iff they have the same sets of measure zero [1]. By the Radon–Nikodym theorem, $\mu$ and $\nu$ are equivalent iff $\mu = f_1 \nu$, $\nu = f_2 \mu$ for certain measurable functions $f_1, f_2 \geq 0$. If one considers specific states and/or their norms (so elements of specific Hilbert spaces), it is required to make a specific choice for the measure. But if one only compares equivalence classes of irreps, like we do in the Clebsch–Gordan series in Eq.(6.2), the exact measure on Conj($G$) is not of importance, only its equivalence class.

Our aim is to determine the measure $\mu$ (up to equivalence) and the multiplicities $N_{\alpha\beta C}^{AB\gamma}$ of this Clebsch–Gordan series for $D(G)$. In physics these $N_{\alpha\beta C}^{AB\gamma}$ are often referred to as ‘fusion rules’, as for example in [9] for the case of $G$ a finite group.

In ordinary group theory the multiplicities can be determined using the characters of representations. Recall that for a continuous group $H$ with irreducible representations $\pi^a, \pi^b, \pi^c, ...$ and characters $\chi^a, \chi^b, \chi^c$ the number of times that $\pi^c$ occurs in the $\pi^a \otimes \pi^b$ is given by
\[
n_{ab}^c = \int_{h \in H} \chi^c(h) \chi^a(h) \chi^b(h) \, dh.
\]
(6.3)
Thus a direct computation of the multiplicities requires an integration over the group. For the quantum double this approach is not very attractive, and we have to take an alternative route. Furthermore, the direct decomposition of the character of a tensor product of irreps into a direct sum/integral over characters of single irreps is problematic, since the tensor product character is not trace class, while the single characters are.

The rigorous approach we will take is to look at the decomposition in more detail, in the sense that we consider the projection of a state in $\Pi_A^\alpha \otimes \Pi_B^\beta$ onto states in the irreducible
components \( \Pi^C_\gamma \). Subsequently we compare the squared norm of the tensor product state with the direct sum/integral of squared norms of the projected states. This will lead to an implicit equation for the multiplicities \( N_{\alpha\beta\gamma} \). The projection involves the construction of intertwining operators from the tensor product Hilbert space to Hilbert spaces of irreducible representations. This construction is described in the next subsection, and the intertwining operators are given in Theorem 6.11. If orthonormal bases are given for the Hilbert spaces of irreducible representations this means we can derive the Clebsch–Gordan coefficients for the quantum double. In section \( \text{?} \) we will work this out explicitly for the case \( G = SU(2) \).

Since the proof of Theorem 6.10 is quite lengthy, in the following paragraph we first give a brief outline of the procedure we will follow.

To prove isometry between the Hilbert space of a tensor product representation and a direct sum of Hilbert spaces of irreducible representations we must construct an intertwining mapping \( \rho \) from the first space, whose elements are functions of two variables with a certain covariance property, to the second space (=direct sum of spaces), whose elements are functions of one variable with a similar covariance property. From Eq. (4.1) one can see that the conjugacy class label \( C \) of the representation to which \( \Phi \) must be mapped depends on the ‘relative difference’ \( \xi \) between the entries \( (y_1, y_2) \) of \( \Phi \). This \( \xi \) is the variable that remains if \( (y_1, y_2) \) and \( (z_1y_1, z_2y_2) \) are identified for all \( z \in G \) and for all \( n_1 \in N_A \) and \( n_2 \in N_B \). So \( \xi \) is an element of the double coset \( G_{AB} = N_A\backslash G/N_B \) we have introduced before. Eq. (6.7) in Proposition 6.3 shows how \( C \) depends on \( \xi \).

In Proposition 6.4 we give a map \( F_1 \) which constructs a function \( \phi \) on \( G \) out of a function \( \Phi \) on \( G \times G \). The action of \( D(G) \) on \( \phi \) depends on the possible ‘relative differences’ \( \xi \) between the entries of \( \Phi \), which is why we say that \( \phi \) also depends on \( \xi \). Therefore we introduce the function spaces of Eqs. (6.12) and (6.13).

Lemma 6.3 shows that the squared norm of \( \Phi \) equals the direct integral over \( \xi \) of the squared norm of \( \phi \), and thus that the map \( F_1 \) is an isometry of Hilbert spaces. One can also think of \( \xi \) as a label on \( \phi \), which distinguishes its behaviour under the action of \( D(G) \), which is in fact shown by Lemma 6.6. These two lemmas together provide the map \( \Pi^A_\alpha \otimes \Pi^B_\beta \rightarrow f^{\pi} \Pi^C_\gamma d\mu(\xi) \) from the tensor product representation to a direct integral over ‘single’ (not yet irreducible) representations.

Subsequently we must decompose these representations \( \Pi^C_\omega(\xi) \) into irreducible representation \( \Pi^C_\gamma \). Comparing the covariance properties before and after \( \rho \) we find the restriction on the set \( \gamma \) may be chosen from, which is given in Eq. (6.36).

Eq. (6.49) gives the isometry of a Hilbert space from the direct integral of Hilbert spaces we constructed before (via the map \( F_1 \)) into the direct sum of Hilbert spaces of irreducible representations \( \Pi^C_\gamma \).

The combination of these two steps in the tensor product decomposition is summarised in Theorem 6.10. Finally we compare the squared norms before and after the mapping \( \rho \), and arrive at Eq. (6.62), which gives us an implicit formula for the multiplicities. The degeneracy of the irreducible representation \( \Pi^C_\gamma \) depends on two things: firstly, the possible non-injectivity of the map \( \xi \rightarrow \gamma \), which is taken into account by the integration over \( N_A\backslash G/N_B \) with measure \( dp^C(\xi) \). And secondly by the dimension \( d_\gamma \) of \( V_\gamma \). We now turn to the explicit proof.

To start with, fix the conjugacy classes \( A \) and \( B \), and also the irreducible unitary representations \( \alpha \in \hat{N}_A \) and \( \beta \in \hat{N}_B \) with representation spaces \( V_\alpha \) and \( V_\beta \) of finite dimensions \( d_\alpha = \dim V_\alpha \) and \( d_\beta = \dim V_\beta \) respectively. The set \( \text{Conj}(G) \) of conjugacy classes of \( G \) forms a partitioning of \( G \). Therefore it can be equipped with the quotient topology, which is again compact Hausdorff and separable. In Definition 6.1 we had already chosen some representative \( g_A \in A \) for each
$A \in \text{Conj}(G)$. We will need the following assumption about this choice:

**Assumption 6.1** The representatives $g_A \in A$ can be chosen such that the map $A \mapsto g_A : \text{Conj}(G) \to G$ is continuous.

In fact, we will make this particular choice. The assumption means that the map from $G$ to $G$, which assigns to each $g \in G$ the representative in its conjugacy class, is continuous. For $G$ a compact connected Lie group we can make a choice of representatives $g_A$ in agreement with Assumption 6.1 as follows. Let $T$ be a maximal torus in $G$, let $T_r$ be the set of regular elements of $T$ (i.e. those elements for which the centraliser equals $T$), let $K$ be a connected component of $T_r$, and let $\overline{K}$ be the closure of $K$ in $T$. Take $g_A$ to be the unique element in the intersection of the conjugacy class $A$ with $\overline{K}$. See for instance reference [7].

Define

$$G_{AB} := N_A \backslash G/N_B$$

(6.4)

to be the collection of double cosets of the form $N_A y N_B$, $y \in G$. Then $G_{AB}$ also forms a partitioning of $G$ which can be equipped with the quotient topology from the action of $G$ (compact Hausdorff and separable). Now also choose for each $\xi \in G_{AB}$ some representative $y(\xi) \in \xi$. We will need the following assumption for this choice of representative:

**Assumption 6.2** The representatives $y(\xi) \in \xi$ can be (and will be) chosen such that the map $\xi \mapsto y(\xi) : G_{AB} \to G$ is continuous.

In other words, the map from $G$ to $G$ which assigns to each $g \in G$ the representative in the double coset $N_A y N_B$ is continuous. For $SU(2)$ a choice of representatives $y(\xi)$ in agreement with Assumption 6.2 will be given in section [7.3].

For $\xi \in G_{AB}$ define the conjugacy class $C(\xi) \in \text{Conj}(G)$ by

$$g_A y(\xi) g_B y(\xi)^{-1} \in C(\xi).$$

(6.5)

Then the map

$$\lambda_{AB} : \xi \mapsto C(\xi) : G_{AB} \to \text{Conj}(G)$$

(6.6)

is continuous. Note that the image of $\lambda_{AB}$ depends on the values of $A$ and $B$, but that $G_{AB}$ only depends on $N_A$ and $N_B$, so not on the precise values of the conjugacy class labels.

**Proposition 6.3** (a) We can choose a Borel map $\xi \mapsto w(\xi) : G_{AB} \to G$ such that

$$g_A y(\xi) g_B y(\xi)^{-1} = w(\xi) g_{C(\xi)} w(\xi)^{-1}$$

(6.7)

(b) We can choose a Borel map $x \mapsto (n_1(x), n_2(x)) : G \to N_A \times N_B$ such that

$$x = n_1(x) y(N_A x N_B) n_2(x)^{-1}$$

(6.8)

**Proof** (a) The map

$$(w, C) \mapsto wg_C w^{-1} : G \times \text{Conj}(G) \to G$$

(6.9)

is continuous (by Assumption 6.1) and surjective. By Corollary 4.3 there exists a Borel map $x \mapsto (w_x, C_x) : G \to G \times \text{Conj}(G)$ such that $x = w_x g_{C_x} w_x^{-1}$. Now take $x = g_A y(\xi) g_B y(\xi)^{-1}$ for $\xi \in G_{AB}$, then from Eq. (6.5) it follows that $C_x = C(\xi)$. The map $\xi \mapsto x$ is continuous by Assumption 6.2, the map $x \mapsto w_x$ is Borel. Put $w(\xi) := w_x$, then $\xi \mapsto w(\xi)$ is Borel, and Eq. (6.7) is satisfied.

(b) The map

$$(n_1, n_2, \xi) \mapsto n_1 y(\xi) n_2^{-1} : N_A \times N_B \times G_{AB} \to G$$

(6.10)
is continuous (by Assumption 6.2) and surjective. By Corollary 6.3 there exists a Borel map
\[ x \mapsto (n_1(x), n_2(x), \xi(x)) : G \to N_A \times N_B \times G_{AB} \]
such that \( x = n_1(x)y(\xi(x))n_2(x)^{-1} \). Then \( \xi(x) = N_AxN_B \), and thus Eq. (6.8) is satisfied.

Let \( Z \) be the center of \( G \), then \( Z \subset N_A \) and \( Z \subset N_B \). By Schur’s lemma \( \alpha(z) \) and \( \beta(z) \) will be a scalar for \( z \in Z \). Define the character \( \omega \) of \( Z \) by
\[ \alpha(z) \otimes \beta(z) =: \omega(z)\text{id}_{V_\alpha \otimes V_\beta}, \quad z \in Z. \tag{6.11} \]

With this character we now define the linear spaces

\[
\begin{align*}
\text{Fun}_{\alpha,\beta}(G \times G, V_\alpha \otimes V_\beta) &:= \{ \Phi : G \times G \to V_\alpha \otimes V_\beta \mid \Phi(un_1^{-1}, vm_2^{-1}) = \alpha(n_1) \otimes \beta(n_2) \Phi(u, v) \} \\
\forall n_1 \in N_A, n_2 \in N_B, u, v \in G \} \\
\text{Fun}_\omega(G \times G_{AB}, V_\alpha \otimes V_\beta) &:= \{ \phi : G \times G_{AB} \to V_\alpha \otimes V_\beta \mid \phi(xz^{-1}, \xi) = \omega(z)\phi(x, \xi), \\
& \text{for } z \in Z \} \tag{6.13}
\end{align*}
\]

We will also need the following sets:

\[
\begin{align*}
G_o &:= \{ x \in G \mid \text{if } n_1 \in N_A, n_2 \in N_B \text{ and } n_1xn_2^{-1} = x \text{ then } n_1 = n_2 \in Z \} \tag{6.14} \\
(G \times G)_o &:= \{ (u, v) \in G \times G \mid u^{-1}v \in G_o \} \\
(G_{AB})_o &:= \{ \xi \in G_{AB} \mid y(\xi) \in G_o \} \tag{6.15}
\end{align*}
\]

They have the following properties, which can be easily verified:

(a) If \( x \in G_o, n_1 \in N_A, n_2 \in N_B \) then \( n_1xn_2^{-1} \in G_o \).
(b) \( x \in G_o \iff y(N_AxN_B) \in G_o \).
(c) If \( x \in G_o, m_1, n_1 \in N_A, n_2, m_2 \in N_B \) then
\[
m_1xm_2^{-1} = n_1xn_2^{-1} \Rightarrow \exists z \in Z \text{ such that } m_1 = n_1z, m_2 = n_2z \tag{6.17}
\]
(d) If \( \xi \in (G_{AB})_o \) then \( \exists z \in Z \text{ such that } n_1(y(\xi)) = n_2(y(\xi)) = z \).
(e) If \( (u, v) \in (G \times G)_o \), and \( m_1 \in N_A, m_2 \in N_B \) then \( \exists z \in Z \text{ such that } \)
\[
n_1(m_1u^{-1}vm_2^{-1}) = zm_1n_1(u^{-1}v), \quad n_2(m_1u^{-1}vm_2^{-1}) = zm_2n_2(u^{-1}v) \tag{6.18}
\]

The next Proposition is the first step in the tensor product decomposition. Roughly speaking, we will consider the functions \( \Phi \) in Eq. (6.12) as elements of the tensor product representation space. After restriction to \( (G \times G)_o \) these functions \( \Phi \) can be rewritten in a bijective linear way as functions \( \phi \) in Eq. (6.13), restricted to \( G \times (G_{AB})_o \). The action of \( D(G) \) on \( \Phi \) affects both arguments of \( \Phi \) (according to Eq. (1.3)), but the corresponding action on \( \phi \) only affects its first argument, as we will see in Lemma 6.6. The second argument will in fact be directly related to the conjugacy class part of the label \( (C(\xi), \gamma) \) of a ‘new’ irreducible representation of \( D(G) \), and thus we will prove that the tensor product representation space is isomorphic to a direct integral of representation spaces of \( \Pi^C_{\omega}(\xi) \), where \( \Pi^C_{\omega}(\xi) \) is not yet irreducible.

**Proposition 6.4** There is a linear map
\[
F_1 : \Phi \mapsto \phi : \text{Fun}_{\alpha,\beta}(G \times G, V_\alpha \otimes V_\beta) \to \text{Fun}_\omega(G \times G_{AB}, V_\alpha \otimes V_\beta) \tag{6.19}
\]
given by
\[
\phi(x, \xi) := \Phi(xw(\xi)^{-1}, xw(\xi)^{-1}y(\xi)), \quad x \in G, \xi \in G_{AB}. \tag{6.20}
\]
This map, when considered as a map

\[ \Phi \mapsto \phi : \text{Fun}_{\alpha,\beta}(G \times G), V_{\alpha} \otimes V_{\beta} \rightarrow \text{Fun}_{\omega}(G \times (G_{AB}), V_{\alpha} \otimes V_{\beta}), \]  

(6.21)
is a linear bijection with inversion formula \( F_2 : \phi \mapsto \Phi \) given by

\[ \Phi(u,v) = \alpha(n_1(u^{-1}v)) \otimes \beta(n_2(u^{-1}v)) \phi(un_1(u^{-1}v)w(N_{Au}^{-1}vN_B), N_{Au}^{-1}vN_B). \]  

(6.22)

Proof

(i) Let \( \phi \) be defined in terms of \( \Phi \in \text{Fun}_{\alpha,\beta}(G \times G, V_{\alpha} \otimes V_{\beta}) \) by Eq. (6.20). The covariance condition of \( \phi \) w.r.t. \( Z \) follows because, for \( z \in Z \),

\[ \phi(xz^{-1}, \xi) = \Phi(xz^{-1}w(\xi)^{-1}, xz^{-1}w(\xi)^{-1}y(\xi)) = \Phi(xw(\xi)^{-1}z^{-1}, xw(\xi)^{-1}y(\xi)z^{-1}) = \alpha(z) \otimes \beta(z) \Phi(xw(\xi)^{-1}, xw(\xi)^{-1}y(\xi)) = \omega(z) \phi(x, \xi). \]

Moreover, \( \phi \) restricted to \( G \times (G_{AB}), V_{\alpha} \otimes V_{\beta} \) only involves \( \Phi \) restricted to \( (G \times G), V_{\alpha} \otimes V_{\beta} \), since for \( \xi \in (G_{AB}), V_{\alpha} \otimes V_{\beta} \) we have that \( (xw(\xi)^{-1}, xw(\xi)^{-1}y(\xi)) \in (G \times G), V_{\alpha} \otimes V_{\beta} \).

(ii) \( F_1 \) is injective because

\[ ((F_2 \circ F_1) \Phi)(u,v) = \alpha(n_1(u^{-1}v)) \otimes \beta(n_2(u^{-1}v)) \phi(un_1(u^{-1}v), un_1(u^{-1}v)y(N_{Au}^{-1}vN_B)) = \Phi(un_1(u^{-1}v)n_1(u^{-1}v)^{-1}, un_1(u^{-1}v)y(N_{Au}^{-1}vN_B)n_2(u^{-1}v)^{-1}) = \Phi(u, uu^{-1}v) = \Phi(u,v) \]

and thus \( F_2 \circ F_1 = \text{id} \). (Here it is not yet necessary to restrict \( (u,v) \) to \( (G \times G), V_{\alpha} \otimes V_{\beta} \).)

(iii) Let \( \Phi \) be defined in terms of \( \phi \in \text{Fun}_{\alpha,\beta}(G \times (G_{AB})), V_{\alpha} \otimes V_{\beta} \) by Eq. (6.22). The covariance condition of \( \Phi \) w.r.t. \( N \times N \) follows because, for \( m_1 \in N_A, m_2 \in N_B \) and \( (u,v) \in (G \times G), V_{\alpha} \otimes V_{\beta} \)

\[ \Phi(um_1^{-1}, vm_2^{-1}) = \alpha(n_1(m_1^{-1}vm_2^{-1})) \otimes \beta(n_2(m_1^{-1}vm_2^{-1})) \]

\[ \phi(um_1^{-1}n_1(m_1^{-1}vm_2^{-1})w(N_{Au}^{-1}vN_B), N_{Au}^{-1}vN_B) = (\alpha(z) \otimes \beta(z))(\alpha(m_1) \otimes \beta(m_2))(\alpha(n_1(u^{-1}v)) \otimes \beta(n_2(u^{-1}v))) \]

\[ \phi(uzn_1(u^{-1}v)w(N_{Au}^{-1}vN_B), N_{Au}^{-1}vN_B) = (\alpha(m_1) \otimes \beta(m_2))(\alpha(n_1(u^{-1}v)) \otimes \beta(n_2(u^{-1}v))) \]

\[ \phi(un_1(u^{-1}v)w(N_{Au}^{-1}vN_B), N_{Au}^{-1}vN_B) = \alpha(m_1) \otimes \beta(m_2) \Phi(u,v) \]

for some \( z \in Z \), where we have used property (e) from above.

(iv) \( F_1 \) is surjective (or: \( F_2 \) is injective) because for \( (\xi, x) \in G \times (G_{AB}), V_{\alpha} \otimes V_{\beta} \)

\[ ((F_1 \circ F_2) \phi)(x, \xi) = \alpha(n_1(y(\xi))) \otimes \beta(n_2(y(\xi))) \phi(xw(\xi)^{-1}n_1(y(\xi))w(\xi), \xi) = \alpha(z) \otimes \beta(z) \phi(xz, \xi) = \phi(x, \xi) \]

for some \( z \in Z \), where we have used property (d) from above. This concludes the proof. \( \square \)

Define a Borel measure \( \mu \) such that

\[ \int_G f(N_{Ay}N_B) \, dy = \int_{G_{AB}} f(\xi) \, d\mu(\xi) \]  

(6.23)

for all \( f \in C(G_{AB}) \). The measure \( \mu \) has support \( G_{AB} \).

We will now specialise the map \( F_1 \) from Eq. (5.14) to the \( L^2 \)-case.

\[ F_1 : \Phi \mapsto \phi : L^2_{\alpha,\beta}(G \times G, V_{\alpha} \otimes V_{\beta}) \rightarrow L^2_{\omega}(G \times G, V_{\alpha} \otimes V_{\beta}). \]  

(6.24)
Here the first $L^2$-space is defined as the representation space of a tensor product representation, see Eqs.\((4.1)\) and \((4.2)\), and the second $L^2$-space is defined as the set of all measurable $\phi : G \times G_{AB} \to V_{\alpha} \otimes V_{\beta}$ satisfying, for all $z \in Z$ that $\phi(xz^{-1}, \xi) = \omega(z)\phi(x, \xi)$ almost everywhere, and such that
\[
\|\phi\|^2 := \int_{G_{AB}} \int_{x \in G} \|\phi(x, \xi)\|^2 \, dx \, d\mu(\xi) < \infty,
\] (6.25)
with almost equal $\phi$'s being identified.

**Lemma 6.5** Let $\Phi \in \text{Fun}_{\alpha, \beta}(G \times G, V_{\alpha} \otimes V_{\beta})$ and let $\phi$ be given by Eq.\((6.20)\). If $\Phi : G \times G \to V_{\alpha} \otimes V_{\beta}$ is moreover Borel measurable then $\phi : G \times G_{AB} \to V_{\alpha} \otimes V_{\beta}$ is Borel measurable, and
\[
\int_{\xi \in G_{AB}} \int_{x \in G} \|\phi(x, \xi)\|^2 \, dx \, d\mu(\xi) = \int_G \int_G \|\Phi(u, v)\|^2 \, du \, dv
\] (6.26)
In particular, the map $F_1 : \Phi \mapsto \phi$ is an isometry of the Hilbert space $L^2_{\alpha, \beta}(G \times G, V_{\alpha} \otimes V_{\beta})$ into (not necessarily onto!) the Hilbert space $L^2_{\alpha, \beta}(G \times G_{AB}, V_{\alpha} \otimes V_{\beta})$.

**Proof** It follows from Eq.\((6.20)\) and Proposition \((6.3)\)\(\text{a}\) that $\phi$ is Borel measurable if $\Phi$ is Borel measurable. The left hand side of Eq.\((6.26)\) equals
\[
\int_{G \times G_{AB}} \left( \int_G \|\Phi(xw(\xi)^{-1}xw(\xi)^{-1}y(\xi))\|^2 \, dx \right) \, d\mu(\xi) = \int_G \int_{GAB} \|\Phi(u, uy(\xi))\|^2 \, du \, d\mu(\xi) = \int_G \int_G \|\Phi(u, uy(N_A v N_B))\|^2 \, du \, dv = \int_G \left( \int_G \|\Phi(u, un_1(v)\, un_2(v))\|^2 \, du \right) \, dv = \int_G \int_G \|\Phi(u, uv)\|^2 \, du \, dv = \int_{G \times G} \|\Phi(u, v)\|^2 \, du \, dv \quad \Box
\]
Subsequently we can show how the map $F_1$ transfers the action of $D(G)$ on $\Phi$ to an action of $D(G)$ on $\phi$:

**Lemma 6.6** Let $\Phi \in L^2_{\alpha, \beta}(G \times G, V_{\alpha} \otimes V_{\beta}), F \in D(G)$ and
\[
\Psi := (\Pi^A_\alpha \otimes \Pi^B_\beta)(F)\Phi.
\] (6.27)
Let $\phi$ be defined in terms of $\Phi$ and $\psi$ in terms of $\Psi$ via Eq.\((6.20)\). Then
\[
\psi(x, \xi) = \int_G F(xgC(\xi)x^{-1}, w) \, \phi(w^{-1}x, w) \, dw
\] (6.28)

**Proof**
\[
\psi(x, \xi) = \left( (\Pi^A_\alpha \otimes \Pi^B_\beta)(F)\Phi \right) (xw(\xi)^{-1}, xw(\xi)^{-1}y(\xi)) = \int_G F(xw(\xi)^{-1}g_A y(\xi)g_B y(\xi)^{-1} w(\xi)x^{-1}, w) \, \Phi(w^{-1}xw(\xi)^{-1}, w^{-1}xw(\xi)^{-1}y(\xi)) \, dw = \int_G F(xgC(\xi)x^{-1}, w) \, \phi(w^{-1}x, \xi) \, dw \quad \Box
\]
For $C \in \text{Conj}(G)$ define a $*$-representation $\Pi^C_\omega$ of $D(G)$ on $L^2_{\omega}(G, V_{\alpha} \otimes V_{\beta})$ as follows:
\[
(\Pi^C_\omega(F)\phi)(x) := \int_G F(xgCx^{-1}, w) \, \phi(w^{-1}x) \, dw
\] (6.29)
$F \in D(G), \phi \in L^2_{\omega}(G, V_{\alpha} \otimes V_{\beta})$
This has the same structure as the defining formula for the representation \( \Pi^A_\alpha \) as given in Eq. (3.3), but the covariance condition on the functions \( \phi \) in Eq. (3.2) is weaker, because it only involves right multiplication of the argument with respect to \( z \in \mathbb{Z} \). Eq. (6.28) can also be formulated as:

\[
\psi(x, \xi) = \left( \Pi^{C(\xi)}_\omega (F) \phi(\cdot, \xi) \right)(x)
\]

which clearly shows that Lemmas 6.3 and 6.6 form the first step in a direct integral decomposition of the representation \( \Pi^A_\alpha \otimes \Pi^B_\beta \) into irreducible representations. We will also need the following

**Assumption 6.7** The complement of \( G_\alpha \) has measure zero in \( G \).

This implies that the complement of \( (G \times G)_\circ \) has measure zero in \( G \times G \), and the complement of \( (G_{AB})_\circ \) has measure zero in \( G_{AB} \). For \( G = SU(n) \) or \( U(n) \) this Assumption will be satisfied if \( A \) and \( B \) are conjugacy classes for which \( g_A \) and \( g_B \) are diagonal matrices with all diagonal elements distinct (so they are regular elements of the maximal torus \( T \) consisting of diagonal matrices). Then \( N_A = N_B = T \), and \( G_\circ \) certainly contains all \( g = (g_{ij}) \in G \) which have only nonzero off-diagonal elements, so for which \( g_{ij} \neq 0 \) if \( i \neq j \). Clearly, Assumption 6.7 is then satisfied.

**Corollary 6.8** The ‘isometry into’ of Lemma 6.3 can be narrowed down to an ‘isometry onto’, namely:

The map \( \mathcal{F}_1 : \Phi \mapsto \phi \) is an isometry of the Hilbert space \( L^2_{\alpha,\beta}((G \times G)_\circ; V_\alpha \otimes V_\beta) \) onto the Hilbert space \( L^2_\omega(G \times (G_{AB})_\circ; V_\alpha \otimes V_\beta) \).

The second step in the decomposition of the tensor product representation \( \Pi^A_\alpha \otimes \Pi^B_\beta \) is the decomposition of the representation \( \Pi^{C(\xi)}_\omega \) into irreducible components \( \Pi^C_\gamma \). In other words, to decompose the action of \( \mathcal{D}(G) \) on \( L^2_\omega(G \times G_{AB}; V_\alpha \otimes V_\beta) \) as given by Eq. (6.28) or Eq. (6.30). For the moment suppose that \( \xi \) can be fixed in Eq. (6.30). Comparison of Eq. (6.30) and Eq. (6.29) with Eq. (3.3) then shows that essentially we have to decompose \( L^2_\omega(G) \) \(^{1}\) as a direct sum of Hilbert spaces \( L^2_\gamma(G, V_\gamma) \) (possibly with multiplicity) on which \( \mathcal{D}(G) \) acts by the irreducible representation \( \Pi^{C(\xi)}_\gamma \), with \( \gamma \in \hat{N}_C(\xi) \).

For \( \phi \in L^2_\omega(G), C \in \text{Conj}(G), \gamma \in \hat{N}_C, d_\gamma := \dim V_\gamma \), and \( i, j = 1, \ldots, d_\gamma \) put

\[
\phi^{C,\gamma}_{ij}(x) := \int_{N_C} \gamma_{ij}(n) \phi(xn) \, dn, \quad x \in G,
\]

where we have chosen an orthonormal basis of \( V_\gamma \). By construction, for \( n \in N_C \) we have that

\[
\phi^{C,\gamma}_{ij}(xn) = \sum_{k=1}^{d_\gamma} \gamma_{ik}(n^{-1}) \phi^{C,\gamma}_{kj}(x).
\]

For each \( j = 1, \ldots, d_\gamma \) the vector \( \phi^{C,\gamma}_{ij}(x) \) takes values in \( V_\gamma \), the label \( i \) denoting the component. Thus

\[
\left( \phi^{C,\gamma}_{ij} \right)_{i=1,\ldots,d_\gamma} \in L^2_\gamma(G, V_\gamma).
\]

Also, for \( F \in \mathcal{D}(G) \)

\[
\left( \left( \Pi^{C}_\omega(F) \phi \right)_{ij} \right)_{i=1,\ldots,d_\gamma} = \Pi^{C}_\gamma(F) \left( \phi^{C,\gamma}_{ij} \right)_{i=1,\ldots,d_\gamma}.
\]

\(^{1}\)The fact that the elements of \( L^2_\omega(G) \) should map to \( V_\alpha \otimes V_\beta \) is not important for this argument.
Thus we must take observation that of the Hilbert space \(L\) which is intertwining between the representations \(\Pi_{\gamma}^C\) that
\[
\int Z \gamma_{ij}(nz) \omega(z^{-1}) dz \phi(xn) dn = \left( \int Z \gamma(z) \omega(z^{-1}) dz \right) \phi_{ij}^{C;\gamma}(x). \quad (6.35)
\]
Thus we must take \(\gamma\) to be an element of
\[
\left( \hat{N}_C \right)_{\omega} = \{ \gamma \in \hat{N}_C | \gamma|_Z = \omega \, \text{id} \}. \quad (6.36)
\]
From the Peter-Weyl theorem applied to the function \(n \mapsto \phi(xn)\) with \(x \in G\) we can derive that
\[
\int_G \| \phi(x) \|^2 dx = \sum_{\gamma \in (\hat{N}_C)_{\omega}} d_{\gamma} \sum_{i,j=1}^{d_{\gamma}} \int_G \| \phi_{ij}^{C;\gamma}(x) \|^2 dx. \quad (6.37)
\]
Thus as a continuation of the maps in Proposition 6.4 we have an isometry
\[
G_1 : \phi \mapsto \left( \sqrt{d_{\gamma}} \left( \phi_{ij}^{C;\gamma} \right)_{i=1,...,d_{\gamma}} \right)_{\gamma \in (\hat{N}_C)_{\omega}, j=1,...,d_{\gamma}} \quad (6.38)
\]
of the Hilbert space \(L_{\omega}^2(G)\) into the direct sum of (degenerate) Hilbert spaces
\[
\bigoplus_{\gamma \in (\hat{N}_C)_{\omega}} \left( L_{\gamma}^2(G, V_{\gamma}) \right)^{d_{\gamma}} \quad (6.39)
\]
which is intertwining between the representations \(\Pi_{\gamma}^C\) and \(\bigoplus_{\gamma \in (\hat{N}_C)_{\omega}} d_{\gamma} \Pi_{\gamma}^C\) of \(D(G)\).

From the existence of an inversion formula we can see that the map \(G_1\) is even an isometry onto. To that aim, fix \(\gamma \in \left( \hat{N}_C \right)_{\omega}\) and take \((\psi_{ij})_{i,j=1,...,d_{\gamma}} \in \left( L_{\gamma}^2(G, V_{\gamma}) \right)^{d_{\gamma}},\) i.e. \(\psi_{ij} \in L^2(G)\) for \(i,j = 1,...,d_{\gamma}\) and
\[
\psi_{ij}(xn) = \sum_{k=1}^{d_{\gamma}} \gamma_{ik}(n^{-1}) \psi_{kj}(x), \quad n \in N_C. \quad (6.40)
\]
The map
\[
G_2^\gamma : \psi \mapsto \phi : \left( L_{\gamma}^2(G, V_{\gamma}) \right)^{d_{\gamma}} \rightarrow L_{\omega}^2(G) \quad (6.41)
\]
is defined by
\[
\phi(x) := d_{\gamma} \sum_{k=1}^{d_{\gamma}} \psi_{kk}(x). \quad (6.42)
\]
Then indeed \(\phi(xz^{-1}) = \omega(z) \phi(x)\) with \(z \in Z\). Furthermore we have that \(G_1 \circ G_2 = \text{id}\), since for \(\delta \in \left( \hat{N}_C \right)_{\omega}\) and \(\phi\) given by Eq. (6.42) we have
\[
\phi_{ij}^{C;\delta}(x) = d_{\gamma} \sum_{k=1}^{d_{\gamma}} \int_{N_C} \delta_{ij}(n) \psi_{kk}(xn) dn = d_{\gamma} \sum_{k,l=1}^{d_{\gamma}} \left( \int_{N_C} \delta_{ij}(n) \overline{\gamma_{lk}(n)} dn \right) \psi_{lk}(x)\]
\[
= d_{\gamma} \sum_{k,l=1}^{d_{\gamma}} \begin{cases} \psi_{ij}(x), & \delta = \gamma \\ 0, & \delta \neq \gamma. \end{cases} \quad (6.43)
\]
We want to apply the above decomposition of \( L^2_\omega(G) \) to our case of \( L^2_\omega(G \times G_{AB}, \mathcal{V}_\alpha \otimes \mathcal{V}_\beta) \).

A slight problem occurs since in Eq. (6.30) we had fixed \( \xi \), which is not allowed in an \( L^2 \)-space. For varying \( \xi \) we will have varying \( C(\xi) \) and hence varying \( N_C(\xi) \) and \( \left( \mathring{N}_C(\xi) \right)_\omega \). In order to keep this under control we make the following

**Assumption 6.9** Conjugacy \( \text{Conj}(G) \) splits as a disjoint union of finitely many Borel sets \( \text{Conj}_p(G) \), on each of which \( N_C \) does not vary with \( C \).

For \( G = SU(n) \) or \( U(n) \) this assumption certainly holds, because we can take the representatives \( g_C = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \) with \( \theta_1 \leq \theta_2 \leq \ldots \leq \theta_n < \theta_1 + 2\pi \). Then \( N_C \) only depends on the partition of the set \( \{1, \ldots, n\} \) induced by the equalities or inequalities between the \( \theta_j \)'s.

We would like to know whether the assumption holds for general compact connected Lie groups \( G \). Let \( T \) be a maximal torus in \( G \). For any conjugacy class \( A \) in \( G \) take the representative \( g_A \) uniquely as an element \( t \in \mathcal{T} \subset T \) (see after Assumption 3.1). Van den Ban [5] has described the centraliser of \( t \) in \( G \). From [5] we conclude that the possible centraliser subgroups form a finite collection. This can be seen as follows. Let \( g_C \) be the complexified Lie algebra of \( G \), let \( \Sigma \) be the root system of \( T \) in \( g_C \), and let \( g_\alpha \) be the root space for \( \alpha \in \Sigma \). Let \( W \) be the Weyl group of the root system \( \Sigma \), which can also be realized as the quotient group \( W = \text{normaliser}_G(\mathring{N}_t)/T \). Let \( t \in T \). Then the centraliser of \( t \) in \( G \) is completely determined by the two sets (each a finite subset of a given finite set):

\[
\Sigma(t) := \{ \alpha \in \Sigma \mid \text{Ad}(t)X = X \text{ for } X \in g_\alpha \}, \quad W(t) := \{ w \in W \mid wtw^{-1} = t \}. \tag{6.44}
\]

This also shows that, for \( t_0 \in T \), the set \( \{ t \in T \mid \Sigma(t) = \Sigma(t_0), \; W(t) = W(t_0) \} \) is Borel. Thus Assumption 6.9 is satisfied if \( G \) is a compact connected Lie group. Note that the Lie algebra of the centralizer of \( t \) in \( G \) is determined by \( \Sigma(t) \) (see for instance Ch. V, Proposition (2.3) in [7]). For determining the centralizer itself, we need also \( W(t) \). This can be seen (cf. [5]) by using the so-called Bruhat decomposition for a suitable complexification \( G_C \) of \( G \).

Put \( N_C = N_p \) if \( C \in \text{Conj}_p(G) \) and \( G_{AB,p} := \{ \xi \in G_{AB} \mid C(\xi) \in \text{Conj}_p(G) \} \). Similarly to Eq. (5.31) for any \( \phi \in L^2(G \times G_{AB}, \mathcal{V}_\alpha \otimes \mathcal{V}_\beta) \) we define

\[
\phi_{ij}^{p,\gamma}(x, \xi) := \int_{N_p} \gamma_{ij}(n) \phi(xn, \xi) dn, \quad x \in G, \xi \in G_{AB,p}, \gamma \in \left( \mathring{N}_p \right)_\omega, \; i, j = 1, \ldots, d_\gamma. \tag{6.45}
\]

with of course the same right covariance as Eq. (6.32). Because \( \phi \) now maps to \( \mathcal{V}_\alpha \otimes \mathcal{V}_\beta \) we can say that

\[
\left( \phi_{ij}^{p,\gamma} \right)_{i=1,\ldots,d_\gamma} \in L^2(G \times G_{AB,p}, \mathcal{V}_\alpha \otimes \mathcal{V}_\beta \otimes \mathcal{V}_\gamma) \tag{6.46}
\]

where again \( i \) denotes the component in \( V_\gamma \). Eq. (3.34) can now be generalised to

\[
\left( \left( \Pi^C(\xi)(F) \phi(., \xi) \right)_{ij}^{p,\gamma} \right)_{i=1,\ldots,d_\gamma} = \Pi^C(\xi)(F) \left( \phi_{ij}^{p,\gamma}(., \xi) \right)_{i=1,\ldots,d_\gamma}, \quad \xi \in G_{AB,p}. \tag{6.47}
\]

Corresponding to Eq. (6.37) we now have the isometry property

\[
\int_G \int_{G_{AB}} \| \phi(x, \xi) \|^2 dx \mu(\xi) = \sum_p \sum_{\gamma \in (\mathring{N}_p)_\omega} d_\gamma \sum_{i,j=1}^{d_\gamma} \int_G \int_{G_{AB,p}} \| \phi_{ij}^{p,\gamma}(x, \xi) \|^2 dx \mu(\xi) \tag{6.48}
\]

and the isometry from Eq. (6.38) now becomes the isometry

\[
\mathcal{G}_1 : \phi \mapsto \left( \sqrt{d_\gamma} \left( \phi_{ij}^{p,\gamma} \right)_{i=1,\ldots,d_\gamma} \right)_{p,\gamma \in (\mathring{N}_p)_\omega, j=1,\ldots,d_\gamma}, \tag{6.49}
\]
of the Hilbert space \( L^2_\omega(G \times G_{AB}, V_\alpha \otimes V_\beta) \) into the direct sum of Hilbert spaces

\[
\bigoplus_p \bigoplus_{\gamma \in (\mathbb{N}_p)_\omega} \left( L^2_\gamma(G \times G_{AB,p}, V_\alpha \otimes V_\beta \otimes V_\gamma) \right)^{d_\gamma}. 
\]

(6.50)

This isometry is intertwining between the direct integral of representations

\[
\int_{G_{AB}} \Pi^C_\omega \, d\mu(\xi) \quad \text{and} \quad \bigoplus_p \bigoplus_{\gamma \in (\mathbb{N}_p)_\omega} \int_{G_{AB,p}} d_\alpha d_\beta d_\gamma \Pi^C_\gamma \, d\mu(\xi) 
\]

(6.51)

of \( D(G) \). Keep in mind that only the equivalence class of the measure \( \mu \) matters in a direct integral of representations, as above.

Again, to show that \( G_1 \) is indeed an isometry into, we construct the inverse:

\[
G_2^{P,\gamma} : \psi \mapsto \phi : L^2_\gamma(G \times G_{AB,p} : V_\alpha \otimes V_\beta \otimes V_\gamma) \rightarrow L^2_\omega(G \times G_{AB}, V_\alpha \otimes V_\beta) 
\]

(6.52)

by

\[
\phi(x, \xi) := d_\gamma \sum_{k=1}^{d_\gamma} \psi_{kk}(x, \xi). 
\]

(6.53)

Then \( G_1 \circ G_2^{P,\gamma} = \text{id} \), which can be shown in the same way as under Eq.(6.42).

We now combine step one and step two in the procedure described above. The decomposition of the tensor product representation is then given by the intertwining isometry \( \rho := G_1 \circ F_1 \), and its inverse is given by \( F_2 \circ G_2^{P,\gamma} \). (The latter acting on \( L^2_\gamma(G \times G_{AB,p}, V_\alpha \otimes V_\beta \otimes V_\gamma)^{d_\gamma}, \) )

Thus we have determined the Clebsch–Gordan series from Eq.(6.2)

\[
\Pi^A_\alpha \otimes \Pi^B_\beta \simeq \int_{G_{AB}} \bigoplus_{\gamma \in \mathbb{N}_C} d_\alpha d_\beta d_\gamma \Pi^C_\gamma \, d\mu(\xi), 
\]

(6.54)

with \( \mu \) an equivalence class of measures. More precisely, we have to take the variation of \( N_C(\xi) \) with \( \xi \) into account, which splits the direct integral over \( \xi \):

\[
\Pi^A_\alpha \otimes \Pi^B_\beta \simeq \bigoplus_p \bigoplus_{\gamma \in (\mathbb{N}_p)_\omega} \int_{G_{AB,p}} d_\alpha d_\beta d_\gamma \Pi^C_\gamma \, d\mu(\xi) 
\]

(6.55)

Combining \( F_1 \) from Eq.(6.20) and \( G_1 \) from Eq.(6.49) we see that a \( \Phi \in L^2_{\alpha,\beta}(G \times G : V_\alpha \otimes V_\beta) \) is taken to an ‘object’ in the direct sum/integral of Hilbert spaces

\[
\bigoplus_p \bigoplus_{\gamma \in (\mathbb{N}_p)_\omega} \int_{G_{AB,p}} L^2_\gamma(G \times G_{AB,p}, V_\alpha \otimes V_\beta \otimes V_\gamma)^{d_\gamma} \, d\mu(\xi) 
\]

(6.56)

This object depends on \( \xi \in G_{AB} \), which determines the class label \( C \) of the (irreducible) representation \( \Pi^C_\gamma \) which occurs in the decomposition. It has an index \( i \) denoting the component of the vector (with tensor products of vectors in \( V_\alpha \otimes V_\beta \) as its entries) in \( V_r \) to which a group element \( x \) is mapped, an index \( p \) which denotes the Borel set in \( \text{Conj}(G) \), which in turn determines the set \( (\mathbb{N}_p)_\omega \) to which the label \( \gamma \) of the \( D(G) \)-representation must belong. Finally, the object has an index \( j \) indicating the degeneracy of the irreducible representation \( \Pi^C_\gamma \). The ‘vector of tensor products of vectors’ means that each component in \( V_\gamma \) of the object in fact depends on the full vector in \( V_\alpha \otimes V_\beta \) to which \( \Phi \) maps a pair \((x_1, x_2) \in G \times G\). We can ‘dissect’ the isometry \( \rho \) according to the way it maps the components of \( \Phi \) to components of the object described above, this results in the following
Theorem 6.10 Let $\Pi^A, \Pi^B$ be irreducible $\ast$-representations of $D(G)$, and let $p$ label the finitely many Borel sets in $\text{Conj}(G)$, on each of which $N_C$ does not vary with $C$. Take $\xi \in \mathcal{G}_{AB,p}$ and $\gamma \in (\hat{N}_p)_\omega$. Then, for each $k = 1, \ldots, d_\alpha$ and $l = 1, \ldots, d_\beta$ and $i, j = 1, \ldots, d_\gamma$ a mapping

$$
\rho^\xi_{\gamma,k,l,j} : L^2_{\alpha\beta}(G \times G, V_\alpha \otimes V_\beta) \to L^2_\gamma(G, V_\gamma)
$$

(6.57)

intertwining the representations $\Pi^A \otimes \Pi^B$ and $\Pi^C(\xi)$ is given by

$$
\left( \rho^\xi_{\gamma,k,l,j} \Phi \right)_i(x) := \left( \phi^{p,\gamma}_{ij}(x, \xi) \right)_{k,l}
\int_{N_{C(\xi)}} \gamma_{ij}(n) \Phi_{kl}(x n w(\xi)^{-1}, x n w(\xi)^{-1} y(\xi)) \, dn.
$$

(6.58)

An implicit expression for the fusion rules (multiplicities) can now also be obtained by comparing the squared norms before and after the action of $\rho$ on $\Phi$. To solve this, we also define a Borel measure $\nu$ on $\text{Conj}(G)$ such that

$$
\int_{G_{AB}} F(C(\xi)) \, d\mu(\xi) = \int_{\text{Conj}(G)} F(C) \, d\nu(C)
$$

(6.59)

for all $F \in C(\text{Conj}(G))$. The measure $\nu$ has support $\lambda_{AB}(G_{AB})$. By Theorem A.3 there exists for almost each $C \in \text{Conj}(G)$ a Borel measure $p^C$ on $G_{AB}$ such that

$$
\int_{G_{AB}} f(\xi) \, d\mu(\xi) = \int_{\xi \in \text{Conj}(G)} \left( \int_{\xi \in G_{AB}} f(\xi) \, dp^C(\xi) \right) \, d\nu(C)
$$

(6.60)

for each $f \in C(G_{AB})$. If the mapping $\lambda_{AB}$ is injective (like in the case of $G = SU(2)$, as we will discuss in the next section) then the above simplifies to

$$
\int_{G_{AB}} f(\xi) \, d\mu(\xi) = \int_{I_{AB}} f(\lambda_{AB}^{-1}(C)) \, d\nu(C),
$$

(6.61)

where $I_{AB}$ is the image of $G_{AB}$ under $\lambda_{AB}$.

Combining Eqs. (6.26) and (6.48) the isometry property which contains the implicit expression for the multiplicities now reads

$$
\int_{G} \int_{G} \| \Phi(u, v) \|^2 \, du \, dv =
$$

(6.62)

$$
d_\alpha d_\beta \sum_{p} \sum_{k=1}^{d_\alpha} \sum_{l=1}^{d_\beta} \sum_{\gamma \in (\hat{N}_p)_\omega} \int_{\text{Conj}(p)} \left( \int_{G_{AB,p}} \sum_{i=1}^{d_\gamma} \int_{G} \| \rho^\xi_{\gamma,k,l,j}(\Phi)_i(y) \|^2 \, dy \right) \, dp^C(\xi) \, d\nu(C).
$$

Eq. (6.62) can be written more compactly as:

$$
\| \Phi \|^2 = d_\alpha d_\beta \sum_{p} \sum_{k=1}^{d_\alpha} \sum_{l=1}^{d_\beta} \int_{\text{Conj}(p)} \left( \sum_{\gamma \in (\hat{N}_p)_\omega} \int_{G_{AB,p}} \| \rho^\xi_{\gamma,k,l,j}(\Phi) \|^2 \, dp^C(\xi) \right) \, d\nu(C). \quad (6.63)
$$
If $\lambda_{AB}$ is injective then Eq.(6.63) simplifies to

$$\|\Phi\|^2 = d_\alpha d_\beta \sum_p \sum_k \sum_l \int_{I_{AB,p}} \left( \sum_{\gamma \in (\hat{N}_p)_\omega} d_\gamma \sum_{j=1} d_{\gamma,j} \|\rho_{\gamma,k,l,j}^{-1}(C)\Phi\|^2 \right) d\nu(C) \quad (6.64)$$

with $I_{AB,p} = \lambda_{AB}(G_{AB,p})$. Note that the measures no longer stand for equivalence classes of measures, but for specific measures, since we are comparing (squared norms of) vectors in Hilbert spaces. The measure $\nu$ may involve a nontrivial Jacobian from the mapping $\lambda_{AB}$.

The multiplicities $N_{\alpha\beta\gamma}^{AB}$ can now more or less be extracted from Eq.(6.62) or Eq.(6.63), that is, we can conclude the following:

(i) $N_{\alpha\beta\gamma}^{AB} = 0$ if $C \not\in \lambda_{AB}(G_{AB})$

(ii) $N_{\alpha\beta\gamma}^{AB} = 0$ if $\gamma \not\in \hat{N}_\omega$

(iii) if $N_{\alpha\beta\gamma}^{AB} \neq 0$ then $N_{\alpha\beta\gamma}^{AB} = d_\alpha d_\beta d_\gamma$

(iv) the inner product on $V^C_\gamma$ will depend nontrivially on $A$ and $B$ according to the Jacobian of the mapping $\lambda_{AB}$ and its non-injectivity, which is reflected in the measure $p^C(\xi)$.

7 Explicit results for $G = SU(2)$

To illustrate the above aspects of tensor products of irreducible representations we will now consider the case of $G = SU(2)$. We will only discuss the decomposition of a ‘generic’ tensor product representation and give explicit formulas for the Clebsch–Gordan coefficients in this case. Some applications and the treatment of more special tensor products will be discussed elsewhere [3].

In [16] we have given the classification of the irreducible unitary representations of $D(SU(2))$. For application of the main result of this paper (the decomposition of the tensor product of such representations into single representations) we first need to establish the notation and parametrisation of elements of $SU(2)$. In this section we use the conventions of Vilenkin [22], because this book contains a complete and explicit list of formulas which are needed in our analysis. For the Wigner functions we use the notation of Varshalovich et al [21] (especially chapter 4).

7.1 Parametrisation and notation

To specify an $SU(2)$-element we use both the Euler angles $(\phi, \theta, \psi)$, and the parametrisation by a single rotation angle $r$ around a given axis $\hat{n}$. In the Euler–angle parametrisation each $g \in SU(2)$ can be written as

$$g = g_\phi a_\theta g_\psi \quad (7.1)$$

with

$$g_\phi = \begin{pmatrix} e^{\frac{1}{2}i\phi} & 0 \\ 0 & e^{-\frac{1}{2}i\phi} \end{pmatrix}, \quad a_\theta = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \quad (7.2)$$

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \quad -2\pi \leq \psi \leq 2\pi. \quad (7.3)$$

The diagonal subgroup consists of all elements $g_\phi$, and is isomorphic to $U(1)$.

The conjugacy classes of $SU(2)$ are denoted by $C_r$ with $0 \leq r \leq 2\pi$. The representative of $C_r$ can be taken to be $g_r$, so in the diagonal subgroup. Then Assumption 6.1 which states that the map of the set of conjugacy classes of $G$ to $G$ itself (i.e. the map to representatives) can be
chosen to be continuous is satisfied. For \( r = 0 \) and \( 2\pi \) the centralizer \( N_0 = N_{2\pi} = SU(2) \), for the other conjugacy classes the centralizer \( N_r = U(1) \).

Let \( 0 < r < 2\pi \). Then \( C_r \) clearly consists of the elements

\[
g(r, \theta, \phi) := g_{\phi}a g_r a^{-1} g_{\phi}^{-1}.
\]

If we take the generators of \( SU(2) \) in the fundamental representation to be

\[
\tau_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}
\]

and define the unit vector

\[
\hat{n}(\theta, \phi) := (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)
\]

then we can also write the element \( g(r, \theta, \phi) \) as

\[
g(r, \theta, \phi) = \exp(ri \, \hat{n}(\theta, \phi) \cdot \vec{\tau}) = 1 \cos \frac{r}{2} + \hat{n} \cdot \vec{\tau} i \sin \frac{r}{2}
\]

This means that there is a 1–1 correspondence between \( \hat{n}(\theta, \phi) \) and the cosets \( g_{\phi}a g_r N_r \). In other words, the mapping \( \hat{n}(\theta, \phi) \mapsto g(r, \theta, \phi) : S^2 \to C_r \) is bijective from the unit sphere \( S^2 \) in \( \mathbb{R}^3 \) onto the conjugacy class \( C_r \).

### 7.2 Irreducible representations

Next we consider the ‘generic’ irreducible unitary representations of \( D(SU(2)) \), i.e. for the case \( r \neq 0, 2\pi \). The other cases will be treated elsewhere [3]. The centralizer representations will be denoted by \( n \in \frac{1}{2} \mathbb{Z} \) (so not the elements themselves as we did in the sections before, when we discussed the general case). The irreducible unitary representations of \( N_r \) are the 1-dimensional representations

\[
n : g_{\zeta} \mapsto e^{in \zeta}, \quad -2\pi \leq \zeta \leq 2\pi, \quad n \in \frac{1}{2} \mathbb{Z}.
\]

For the generic representations \( \Pi^r_n \) of \( D(SU(2)) \) the representation space is

\[
V^r_n = \{ \phi \in L^2(SU(2), \mathbb{R}/2\pi) \mid \phi(gg_{\zeta}) = e^{-in \zeta} \phi(g), \quad -2\pi \leq \zeta \leq 2\pi \}.
\]

An orthogonal basis for \( V^r_n \) is given by the Wigner functions \( D^j_{mn} \), where the label \( n \) is fixed. A thorough treatment of the Wigner functions as a basis of functions on \( SU(2) \) can be found in [24]. For \( g \in SU(2) \) parametrised by the Euler angles as in Eq.(7.1) the Wigner function \( D^j_{mn} \) corresponding to the \( m, n \)-th matrix element in the \( j \)-th irreducible representation takes the value

\[
D^j_{mn}(g) = e^{-im \phi} P^j_{mn} (\cos \theta)e^{-in \psi},
\]

where \( P^j_{mn} \) can be expressed in terms of Jacobi polynomials. For all \( g_{\zeta} = e^{i\zeta} \in U(1) \) we have that

\[
D^j_{mn}(xg_{\zeta}) = e^{-in \zeta} D^j_{mn}(x).
\]

This shows indeed that the set \( \{ D^j_{mn} \mid n \text{ fixed}, j \in \frac{1}{2} \mathbb{N}, j \geq n, -j \leq m \leq j \} \) has the right covariance property. The Wigner functions form a complete set on \( SU(2) \), so the aforementioned set forms a basis for a Hilbert space corresponding to an irreducible unitary representation of \( D(SU(2)) \), with fixed centraliser representation \( n \) and arbitrary conjugacy class \( 0 < r < 2\pi \).
In other words, the Hilbert spaces for irreducible unitary representations with the same $n$ and different $r$ are equivalent, and thus can be spanned by identical bases. Recall that the $r$-dependence of the representation functions $\phi \in V_n^r$ is only reflected in the action of $\mathcal{D}(SU(2))$ on $V_n^r$:

$$(\Pi_n^r(F)\phi)(y) = \int_{SU(2)} F(yg_r y^{-1}, x)\phi(x^{-1}y) \, dx, \quad \phi \in V_n^r,$$

Strictly speaking, we should label the (basis) vectors of $V_n^r$ by $r$ as well, then an arbitrary state in a generic representation is written as

$$r\phi_n(x) = \sum_{j>n-j \leq m \leq j} c_{jm} rD_{mn}^j(x), \quad x \in G.$$  

(Note that the sum over $j$ is infinite.) However, since we will always specify which representation $\Pi_n^r$ we are dealing with, we will omit the $r$-label on the functions.

By Eq.(7.4) the character $\chi_n^r$ of a generic representation $\Pi_n^r$ is given by

$$\chi_n^r(F) = \int_{SU(2)} \int_{U(1)} F(zg_r z^{-1}, zg_\zeta z^{-1}) e^{imc} d\zeta \, dz, \quad F \in C^\infty(SU(2) \times SU(2)).$$

### 7.3 Clebsch–Gordan series

First we will determine the decomposition of the tensor product of two generic representations $\Pi_n^r_1$ and $\Pi_n^r_2$ as in Eq.(6.54). It will turn out that $p$ takes only one value, corresponding to generic $r_3$, and that the map $\lambda_{r_1,r_2}$ is injective. We have to determine the image $I_{r_1,r_2}$ of $\lambda_{r_1,r_2}$, the equivalence class of the measure $\nu$, and the set $\left\{ \tilde{N}_r \right\}_r$. Since the centraliser representations $n_1, n_2, n_3$ are one-dimensional we see that the nonvanishing multiplicities $N_{n_1 n_2 n_3}^r = 1$.

We choose $y(\theta) := a_\theta$ as a representative for the double coset $N_{r_1} a_\theta N_{r_2}$, which is an element of $G_{r_1 r_2} = U(1) \backslash SU(2) / U(1)$. Then Assumption 6.2, stating that the representatives of the double cosets can be chosen in a continuous way, is satisfied. Eq.(6.7), which for this case determines $r_3(\theta)$ and $w(\theta)$, now reads

$$g_{r_1} a_\theta g_{r_2} a_\theta^{-1} = w(\theta) g_{r_3(\theta)} w^{-1}(\theta).$$

By computing the trace of the left-hand side of Eq.(7.15) we find for $r_3 = r_3(\theta)$ that

$$\cos \frac{r_3}{2} = \cos \frac{r_1}{2} \cos \frac{r_2}{2} - \cos \theta \sin \frac{r_1}{2} \sin \frac{r_2}{2},$$

which gives us the mapping $\lambda_{r_1,r_2}$ from Eq.(6.6):

$$\lambda_{r_1,r_2}(U(1)a_\theta U(1)) = 2 \arccos(\cos \frac{1}{2} r_1 \cos \frac{1}{2} r_2 - \cos \theta \sin \frac{1}{2} r_1 \sin \frac{1}{2} r_2).$$

Thus the mapping $\lambda_{r_1,r_2} : G_{r_1 r_2} \to [0, 2\pi]$ is injective with image

$$I_{r_1 r_2} = [\max(r_1 - r_2, \min(r_1 + r_2, 4\pi - (r_1 + r_2))].$$

Now we compute the measures $\mu$ and $\nu$ from Eqs.(6.50) and (6.60). The measure $\mu$ on $G_{r_1 r_2}$ follows from

$$\int_{SU(2)} f(g) \, dg = \frac{1}{2} \int_0^\pi f(a_\theta) \sin \theta \, d\theta$$

(7.19)
for a function \( f \in C(G_{r_1 r_2}) \), and thus
\[
d\mu(\theta) = \frac{1}{2} \sin \theta \, d\theta.
\] (7.20)

The Borel measure \( \nu \) on the set of conjugacy classes can be derived via
\[
\int_{\pi_0} F(\lambda_{r_1 r_2}(U(1)U(1))) \, d\mu(\theta) = \int_{I_{r_1 r_2}} F(r_3) \, d\nu(r_3)
\] (7.21)
for an \( F \in C(\text{Conj}(SU(2))) \). With formula (7.17) it follows that
\[
d\nu(r_3) = \begin{cases} 
\sin \frac{r_3}{2} \frac{1}{4 \sin \frac{r_1}{2} \sin \frac{r_2}{2}} \, dr_3, & |r_1 - r_2| \leq r_3 \leq \min(r_1 + r_2, 4\pi - (r_1 + r_2)) \\
0, & \text{otherwise}
\end{cases}
\] (7.22)

We conclude that the nongeneric conjugacy classes \( r_3 = 0 \) and \( r_3 = 2\pi \) have \( \nu \)-measure zero in \( I_{r_1 r_2} \). We also see that the measure \( d\nu(r_3) \) is equivalent with the measure \( dr_3 \) on \( I_{r_1 r_2} \).

7.4 Clebsch–Gordan coefficients

We will now explicitly construct the mapping \( \rho \) from Eq.(6.58), successively applying the steps of section 6. We can compute \( w(\theta) = g_{\phi_w} a_{\theta_w} \) by first rewriting Eq.(7.15) as
\[
(1 \cos \frac{r_1}{2} + i \tau_1 \sin \frac{r_1}{2})(1 \cos \frac{r_2}{2} + i(\cos \theta \tau_1 + \sin \theta \tau_2) \sin \frac{r_2}{2}) = 1 \cos \frac{r_3}{2} + i \hat{n}_w \cdot \vec{\tau} \sin \frac{r_3}{2}
\] (7.25)
in view of Eqs.(7.4), (7.6), (7.7)), and then comparing coefficients of \( \tau_1, \tau_2, \tau_3 \) on both sides. This yields
\[
\hat{n}_w(\theta) = \begin{pmatrix} 
\cos \theta_w \\
\sin \theta_w \cos \phi_w \\
\sin \theta_w \sin \phi_w 
\end{pmatrix}
= \frac{1}{\sin \frac{r_3}{2}} \begin{pmatrix} 
\sin \frac{r_3}{2} \cos \frac{r_3}{2} + \cos \theta \cos \frac{r_3}{2} \sin \frac{r_3}{2} \\
\sin \theta \cos \frac{r_3}{2} \sin \frac{r_3}{2} \\
\sin \theta \sin \frac{r_3}{2} \sin \frac{r_3}{2}
\end{pmatrix}.
\] (7.26)

It follows from Eqs.(7.15) and (7.26) that \( \theta_w \) and \( \phi_w \) depend continuously on \( \theta \), even for \( r_1 = r_2 \), in which case the right hand side of Eq.(7.26) tends to
\[
\begin{pmatrix} 
0 \\
\cos \frac{r_3}{2} \\
\sin \frac{r_3}{2}
\end{pmatrix}
\] (7.27)
as \( \theta \uparrow \pi \), hence \( \theta_w \rightarrow \frac{\pi}{2}, \phi_w \rightarrow \frac{r_3}{2} \). Thus the Borel map from Proposition 6.3 (a) can be chosen continuously.
The first step in the tensor product decomposition is the construction of the map $F_1$ from Corollary 6.8. The isometry
\[ F_1 : L^2_{n_1,n_2}(SU(2) \times SU(2)) \rightarrow L^2_\epsilon(SU(2) \times [0,\pi]), \quad \epsilon = (n_1 + n_2) \mod \mathbb{Z} \tag{7.28} \]
is given by
\[ \phi(x,\theta) = \Phi(xw(\theta)^{-1},xw(\theta)^{-1}a_\theta). \tag{7.29} \]
For the inversion formula $F_2$ we need a choice for the Borel map from Proposition 6.3 (b). It follows straightforwardly from the Euler angle parametrisation: write $x \in SU(2)$ as $x = g_{\phi_x}a_{\theta_x}g_{\psi_x}S$ with $0 \leq \theta_x \leq \pi$, $0 \leq \phi_x < 2\pi$, $-2\pi \leq \psi_x < 2\pi$. Put $y(U(1)xU(1)) := a_{\theta_x}$ and $n_1(x) := g_{\phi_x}$, $n_2(x) := g_{\psi_x}$. Then $F_2 : \phi \mapsto \Phi$ is given by
\[ \Phi(u,v) = e^{in_1\phi_n-1u}\overline{e^{in_2\phi_n-1v}}\phi(ug_{\phi_n-1u},\theta_{u-1v}) \tag{7.30} \]
with $u^{-1}v \in SU(2)_o$, and
\[ SU(2)_o = \left\{ \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \in SU(2) \mid \alpha,\beta \neq 0 \right\}. \tag{7.31} \]
Assumption 6.7, stating that the complement of $G_o$ has measure zero in $G$, is satisfied for this case.

The second step in the tensor product decomposition is given by the isometry $G_1$ from Eq.(6.38)
\[ G_1 : L^2_\epsilon(SU(2) \times [0,\pi]) \rightarrow \bigoplus_{n_3 \in (n_1+n_2) \mod \mathbb{Z}} L^2_{n_3}(SU(2) \times I_{r_1,r_2}). \tag{7.32} \]
Assumption 6.3 about $\text{Conj}(SU(2))$ is satisfied, because there are two sets in $\text{Conj}(SU(2))$ with distinct centralisers: the set $p_0 = \{ r = 0, r = 2\pi \} = \mathbb{Z}$ with centraliser $SU(2)$, and the set $p_1 = \{ r \in (0,2\pi) \}$ with centraliser $U(1)$. From Eq.(7.22) we see that the set $p_0$ will give no contribution in the decomposition of the squared norm of the tensor product state, because for $r_3 = 0,2\pi$ the measure $\nu(r_3)$ on the conjugacy classes is zero. Therefore we only need to compute Eq.(7.43) for $p = p_1$:
\[ \phi^{p_1,n_3}(x,\theta) = \int_{U(1)} e^{in_3\zeta} \phi(xg_\zeta,\theta) d\zeta, \quad n_3 \in (n_1+n_2) \mod \mathbb{Z} \tag{7.33} \]
with the $U(1)$ over which we integrate embedded in $SU(2)$, so $-2\pi \leq \zeta \leq 2\pi$, and the Haar measure $d\zeta$ appropriately normalised. The isometry property of Eq.(6.48) now becomes
\[ \int_{SU(2)} \int_0^\pi |\phi(x,\theta)|^2 dx d\mu(\theta) = \sum_{n_3 \in (n_1+n_2) \mod \mathbb{Z}} \int_{I_{r_1,r_2}} \int_{SU(2)} |\phi^{p_1,n_3}(x,r_3)|^2 dx d\nu(r_3). \tag{7.34} \]
The inverse mapping $G_2^{p_1}$ reads
\[ \phi(x,\theta) = \sum_{n_3 \in (n_1+n_2) \mod \mathbb{Z}} \phi^{p_1,n_3}(x,\theta). \tag{7.35} \]
This results in the mapping $\rho$ intertwining the representations
\[ \Pi^{p_1}_{n_1} \otimes \Pi^{p_2}_{n_2} \quad \text{and} \quad \bigoplus_{n_3 \in (n_1+n_2) \mod \mathbb{Z}} \int_{I_{r_1,r_2}} \Pi^{p_3}_{n_3} d\nu(r_3) \tag{7.36} \]
We calculate the components of mapping ρ as given in Eq. (7.37). The labels \(i, j, k, l\) can be ignored, because \(V_{n_1}, V_{n_2}, V_{n_3}\) are one-dimensional.

\[
\left( \rho^\theta_{n_3} \Phi \right)(x) = \int_{U(1)} e^{im\zeta} \Phi(xg_\zeta w(\theta)^{-1}, xg_\zeta w(\theta)^{-1}a_\theta) d\zeta. \tag{7.37}
\]

The Clebsch–Gordan series from Eq. (7.24) is contained in

\[
\int_{SU(2)} \int_{SU(2)} \|\Phi(u, v)\|^2 du \, dv = \sum_{n_3 \in (n_1 + n_2)} \text{mod} \int_{I_{r_1, r_2}} \left( \int_{SU(2)} |(\rho^\theta_{n_3} \Phi)(x)|^2 dx \right) \, dv(r_3), \tag{7.38}
\]

where we have replaced the \(\theta\)-dependence by \(r_3\)-dependence, because the map \(\lambda_{r_1, r_2} : G_{r_1, r_2} \to \text{Conj}(SU(2))\) is injective, see Eq. (7.17).

If we now choose an explicit basis for the representation spaces we can explicitly calculate the Clebsch–Gordan coefficients of \(D(SU(2))\). For the orthogonal bases we take the Wigner functions \(D^i_{mnmn}\) as explained under Eq. (7.9).

We will use the notation and definition of the Clebsch–Gordan coefficients of \(SU(2)\) as given in [21], chapter 8. Thus

\[
D^i_{m_1n_1}(g)D^j_{m_2n_2}(g) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m,n=-j}^{j} C^{jm}_{j_1m_1j_2m_2} C^{jn}_{j_1n_1j_2n_2} D^i_{mn}(g). \tag{7.39}
\]

The Clebsch–Gordan coefficients \(C^{jm}_{j_1m_1j_2m_2}\) are equal to zero if \(m \neq m_1 + m_2\). So

\[
D^i_{m_1n_1}(g)D^j_{m_2n_2}(g) = \sum_{j} C^{j(m_1+m_2)}_{j_1m_1j_2m_2} C^{j(n_1+n_2)}_{j_1n_1j_2n_2} D^i_{(m_1+m_2)j_1j_2}(g), \tag{7.40}
\]

where the primed summation over \(j\) runs from \(\text{max}(|j_1 - j_2|, |m_1 + m_2|, |n_1 + n_2|)\) to \((j_1 + j_2)\). In the tensor product representation \(\Pi_{n_1}^r \otimes \Pi_{n_2}^r\) we consider the basis function

\[
\Phi = D^i_{m_1n_1}(g) \otimes D^j_{m_2n_2}(g) : (y_1, y_2) \mapsto D^i_{m_1n_1}(y_1) D^j_{m_2n_2}(y_2), \quad j_1 \geq n_1, -j_1 \leq m_1 \leq j_1, i = 1, 2 \tag{7.41}
\]

The mapping \(\rho\) from Eq. (7.37) takes this basis function to a linear combination of basis functions of a single irreducible unitary representation \(\Pi_{n_3}^r\):

\[
\left( \rho^\theta_{n_3} \Phi \right)(x) = \int_{U(1)} e^{im\zeta} \Phi(xg_\zeta w(\theta)^{-1}, xg_\zeta w(\theta)^{-1}a_\theta) d\zeta
\]

\[
= \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m,n=-j}^{j} C^{jm}_{j_1m_1j_2m_2} C^{jn}_{j_1n_1j_2n_2} D^i_{m_1n_1}(a_\theta) \times
\]

\[
\int_{U(1)} e^{im\zeta} \sum_{r,s=-j}^{j} D^i_{mr}(x) D^j_{rs}(g_\zeta) D^j_{sp}(w(\theta)^{-1}) d\zeta \tag{7.42}
\]

\[
= \sum_{j} \left\{ \sum_{p_2} C^{j(m_1+m_2)}_{j_1m_1j_2m_2} C^{j(n_1+p_2)}_{j_1n_1j_2p_2} D^i_{m_1n_1}(a_\theta) \overline{D^j_{(m_1+p_2)n_1}(w(\theta))} D^i_{(m_1+m_2)n_3}(x) \right\}
\]

where the primed summation over \(p_2\) runs from \(\text{max}((-j - n_1), -j_2)\) to \(\text{min}((j - n_1), j_2)\).

This shows how \(\Phi \in V_{n_1}^{r_1} \otimes V_{n_2}^{r_2}\) can be decomposed into single Wigner functions with a fixed label \(n_3\), which form a basis of \(V_{n_3}^{r_3}\). The coefficients between the large brackets \{\} now

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indeed are the generalised Clebsch–Gordan coefficients for the quantum double group of $SU(2)$. Clearly they depend on the representation labels, so on $(r_1, n_1), (r_2, n_2)$ and $(r_3, n_3)$, where $r_3$ corresponds one–to–one to the double coset $\theta$. They also depend on the specific ‘states’ labeled by the $j_1, m_1$, etc., just as one would expect. Note that $a_\theta$ and $w(\theta)$ are needed to implement the dependence on $\theta$. We can denote these Clebsch–Gordan coefficients by

$$\langle (r_1, n_1)j_1m_1, (r_2, n_2)j_2m_2 \mid (r_3, n_3)jm \rangle := \sum_{p_2} C_{j_1m_1j_2m_2}^{jm} C_{j_1m_1j_2p_2}^{(n_1+p_2)} D_{p_2n_2}^{j_2}(a_\theta) D_{(n_1+p_2)n_3}^{jm}(w(\theta))$$

(7.43)

with $r_3 = \lambda_{r_1r_2}(\theta)$. These coefficients are zero if $m \neq m_1 + m_2$. Also, they are zero if $n_3 \neq (n_1 + n_2) \mod \mathbb{Z}$, so $n_3$ must be integer if $n_1 + n_2$ integer, and half integer if $n_1 + n_2$ half integer. Thus we can write

$$\left(\rho_{n_3}^{r_3} \left(D_{m_1n_1}^{j_1} \otimes D_{m_2n_2}^{j_2}\right)\right)(x) = \sum_{m=-j}^{j} \sum_{m=m-n}^{m+n} \langle (r_1, n_1)j_1m_1, (r_2, n_2)j_2m_2 \mid (r_3, n_3)jm \rangle D_{m_3}^{jm}(x).$$

(7.44)

The isometry property of $\rho$ can now be calculated even more explicitly. The left hand side of Eq.(7.38) gives

$$\int_{SU(2)} \int_{SU(2)} D_{m_1n_1}^{j_1}(y_1) D_{m_2n_2}^{j_2}(y_2) D_{m_1n_1}^{j_1}(y_1) D_{m_2n_2}^{j_2}(y_2) dy_1 dy_2 = \frac{1}{2j_1 + 1} \frac{1}{2j_2 + 1}.$$  

(7.45)

For the right hand side of Eq.(7.38) we find

$$\sum_{n_3} \int_{I_{r_1r_2}} \int_{SU(2)} \left| \rho_{n_3}^{-1} \left(D_{m_1n_1}^{j_1} \otimes D_{m_2n_2}^{j_2}\right)(y) \right|^2 dy d\nu(r_3),$$

(7.46)

where $I_{r_1r_2}$ given by Eq.(7.18), and the measure $d\nu(r_3)$ by Eq.(7.22). Substituting Eq.(7.42) and Eq.(7.43) yields

$$\sum_{n_3} \int_{I_{r_1r_2}} \int_{SU(2)} \left( \sum_{j} \sum_{m} \langle (r_1, n_1)j_1m_1, (r_2, n_2)j_2m_2 \mid (r_3, n_3)jm \rangle D_{m_3}^{jm}(y) \right) \times

\left( \sum_{j'} \sum_{m'} \langle (r_1, n_1)j_1m_1, (r_2, n_2)j_2m_2 \mid (r_3, n_3)jm' \rangle D_{m_3}^{jm'}(y) \right) dy d\nu(r_3).$$

(7.47)

The integration over $y$ can be performed, and thus the isometry property of the mapping $\rho$ reads

$$\sum_{n_3} \int_{I_{r_1r_2}} \sum_{j} \frac{1}{2j_1 + 1} \left| \langle (r_1, n_1)j_1m_1, (r_2, n_2)j_2m_2 \mid (r_3, n_3)(m_1 + m_2) \rangle \right|^2 d\nu(r_3) = \frac{1}{2j_1 + 1} \frac{1}{2j_2 + 1}.$$  

(7.48)

More generally, if we start with the identity of inner products which is immediately implied by Eq.(7.38), we obtain

$$\sum_{n_3} \sum_{j} \frac{1}{2j_1 + 1} \int_{I_{r_1r_2}} \langle (r_1, n_1)j_1m_1, (r_2, n_2)j_2(m - m_1) \mid (r_3, n_3)jm \rangle \times

\langle (r_1, n_1)j'_1m'_1, (r_2, n_2)j'_2(m - m'_1) \mid (r_3, n_3)jm \rangle \left(2j_1 + 1\right)

(7.49)
This means that the Clebsch–Gordan coefficients \((7.43)\) for \(\mathcal{D}(SU(2))\), built from Wigner functions and Clebsch–Gordan coefficients for \(SU(2)\), satisfy interesting orthogonality relations, suggesting the existence of a ‘new’ kind of special functions.

Remember that the \(a_\theta\) and \(w(\theta)\) given in Eqs.(7.2) and (7.26) are the choices we made for the Borel mappings \(y(\xi)\) and \(w(\xi)\) in Assumption 6.2 and Proposition 6.3 which uniquely depend on \(r_3\) according to Eq.(7.15). It is now clear that the choice of representatives in the double coset (so the mapping \(\xi \rightarrow y(\xi)\) of Assumption 6.2), and the choice of Borel map \(\xi \rightarrow w(\xi)\) of Proposition 6.3 do not affect the fusion rules: for \(a_\theta \mapsto g_\phi a_\theta g_\psi\) and \(w(\theta) \mapsto g_\phi w(\theta) g_\psi\) the Clebsch–Gordan coefficients from Eq.(7.43) only change by a phase factor \(e^{i(n_1\phi-n_2\psi+n_3\zeta)}\), and thus the orthonormality relations of Eq.(7.49) do not change.

This concludes our discussion of the fusion rules of \(\mathcal{D}(SU(2))\).

8 Conclusion

In this paper we have focussed on the co-structure of the quantum double \(\mathcal{D}(G)\) of a compact group \(G\) and have used it to study tensor products of irreducible representations. We have explicitly constructed a projection onto irreducible components for tensor product representations, which of course has to take into account the (nontrivial) multiplication. By subsequently using the Plancherel formula (i.e. by comparing squared norms) we found an implicit formula for the multiplicities, or Clebsch–Gordan series. Also, we have given the action of the universal \(R\)-matrix of \(\mathcal{D}(G)\) on tensor product states. For the example of \(G = SU(2)\) we calculated the Clebsch–Gordan series and coefficients explicitly. In a forthcoming article we will expand further on the quantum double of \(SU(2)\), in particular the behaviour of its representations under braiding and fusion. These results also will enable us to describe the quantum properties of topologically interacting point particles, as in \(\tilde{ISO}(3)\) Chern–Simons theory, see \([3]\).

A Some measure theoretical results

In this appendix we have collected some measure theoretical results which have been used in section 6.

Theorem A.1 (Kuratowski’s theorem, see for instance Parthasarathy, \([1^3],\) Ch. I, Corollary 3.3)
If \(E\) is a Borel subset of a complete separable metric space \(X\) and \(\lambda\) is a one-one measurable map of \(E\) into a separable metric space \(Y\) then \(\lambda(E)\) is a Borel subset of \(Y\) and \(\lambda : E \rightarrow \lambda(E)\) is a Borel isomorphism.

Theorem A.2 (Theorem of Federer & Morse \([1^4]\), see also \([1^3]\), Ch. I, Theorem 4.2)
Let \(X\) and \(Y\) be compact metric spaces and let \(\lambda\) be a continuous map of \(X\) onto \(Y\). Then there is a Borel set \(B \subset X\) such that \(\lambda(B) = Y\) and \(\lambda\) is one-to-one on \(B\).

The set \(B\) is called a Borel section for \(\lambda\). Since the continuous image of a compact set is compact, we can relax the conditions of Theorem A.2 by not requiring surjectivity of \(\lambda\). Then \(\lambda(B) = \lambda(X)\). By Theorem A.1 the mapping \(\lambda|_{B} : B \rightarrow \lambda(X)\) is a Borel isomorphism. Let \(\psi : \lambda(X) \rightarrow B\) be the inverse of \(\lambda|_{B}\). We will also call the mapping \(\psi\) a Borel section for \(\lambda\). We conclude:
Corollary A.3 Let $X$ and $Y$ be compact metric spaces and let $\lambda$ be a continuous map of $X$ to $Y$. Then there is a Borel map $\psi : \lambda(X) \to X$ such that $\lambda(\psi(y)) = y$ for all $y \in \lambda(X)$ and $\psi(\lambda(X))$ is a Borel set in $X$.

Theorem A.4 (isomorphism theorem, see for instance [14], Ch. I, Theorem 2.12)
Let $X_1$ and $X_2$ be two complete separable metric spaces and let $E_1 \subset X_1$ and $E_2 \subset X_2$ be two Borel sets. Then $E_1$ and $E_2$ are Borel isomorphic if and only if they have the same cardinality. In particular, if $E_1$ is uncountable, $X_2 := \mathbb{R}$ and $E_2$ is an open interval, then $E_1$ and $E_2$ are Borel isomorphic.

Next we discuss conditional probability, although we will not deal with probabilistic interpretations. Our reference here is Halmos [13], §48. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, i.e. sets $X$ and $Y$ with $\sigma$-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\lambda : X \to Y$ be a measurable map. Let $\mu$ be a probability measure on $(X, \mathcal{A})$. Define a probability measure $\nu$ on $(Y, \mathcal{B})$ by the rule

$$\nu(B) := \mu(\lambda^{-1}(B)), \quad B \in \mathcal{B}. \quad (1.1)$$

By the Radon-Nikodym theorem there exists for each $A \in \mathcal{A}$ a $\nu$-integrable function $p_A$ on $Y$ such that

$$\mu(A \cap \lambda^{-1}(B)) = \int_B p_A(y) \, d\nu(y), \quad B \in \mathcal{B}. \quad (1.2)$$

Then $p_A(y)$ is called the conditional probability of $A$ given $y$. Note that the functions $p_A$ are not unique. For fixed $A$, two choices for $p_A$ can differ on a set of $\nu$-measure zero. We will write

$$p^\mu(y)(A) := p_A(y), \quad y \in Y, \quad A \in \mathcal{A}. \quad (1.3)$$

Then $p^\mu$ behaves in certain respects like a measure on $(X, \mathcal{A})$, but it may not be a measure. If $f$ is a $\mu$-integrable function on $X$ then, by the Radon-Nikodym theorem there exists a $\nu$-integrable function $e_f$ on $Y$ such that, for every $B \in \mathcal{B}$,

$$\int_{\lambda^{-1}(B)} f(x) \, d\mu(x) = \int_B e_f(y) \, d\nu(y). \quad (1.4)$$

Theorem A.5 If $\lambda$ is a measurable map from a probability space $(X, \mathcal{A}, \mu)$ to a measurable space $(Y, \nu)$, and if the conditional probabilities $p_A(y)$ can be determined such that $p^\mu$ is a measure on $(X, \mathcal{A})$ for almost every $y \in Y$, then

$$e_f(y) = \int_X f(x) \, dp^\mu(x) \quad \text{for } y \text{ almost everywhere on } Y \text{ w.r.t. } \nu. \quad (1.5)$$

In particular, if $X$ is an open interval in $\mathbb{R}$, or more generally a complete separable metric space, then $p_A(y)$ can be determined such that $p^\mu$ is a measure on $(X, \mathcal{A})$ for almost every $y \in Y$, and Eq.$(1.4)$ will hold with $e_f(y)$ given by Eq.$(1.3)$.

This theorem follows from Halmos [13], pp. 210–211, items (5) and (6) together with the above Theorem A.4.
Theorem A.5 greatly simplifies if $X$ is a complete separable metric space and, moreover, $\lambda$ is injective. Then

$$
p^\mu(A) = p_A(y) = \chi_{\lambda(A)}(y) = \begin{cases} 
0, & y \not\in \lambda(X), \\
def_{\lambda^{-1}(y)}(A), & y \in \lambda(X)
\end{cases}
$$

$$
e_f(y) = \begin{cases} 
0, & y \not\in \lambda(X), \\
f(\lambda^{-1}(y)), & y \in \lambda(X)
\end{cases}
$$

$$
\int_{\lambda^{-1}(B)} f(x) \, d\mu(x) = \int_{B \cap \lambda(X)} f(\lambda^{-1}(y)) \, d\nu(y)
$$

(1.6)

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