CESARO AVERAGES OF EULER-LIKE FUNCTIONS

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Abstract. By Euler-like function we mean a function defined on the positive integers and associating to \( n \) the product, over all primes \( p \) dividing \( n \), of 1 plus (or minus) the inverse of \( p \) to the power \( s \). We calculate the limit of the Cesaro means of these functions.

1. Origin of the problem

When work [1] was in progress, the author asked me about the Cesaro mean of the following rational function \( f \) of the natural \( n \)

\[
f(n) = \prod_{p|n} \left( 1 + \frac{1}{p^n} \right), \quad p \text{ prime},
\]

(the integer \( nf(n) \) being the number of straight lines through the origin in the torus \( \mathbb{Z}_2^n \), permuted by the action of \( \text{PSL}(2,\mathbb{Z}_n) \)). For \( n \) till \( 10^7 \), I found that the Cesaro mean of \( f \), equal to 1.5198177542107..., multiplied by \( \pi^2 \), becomes 14.9999999958.... To prove that this Cesaro average converges indeed to \( 15/\pi^2 \), I studied the Cesaro means of more general functions, that I call ‘Euler-like’, since the Euler function \( \phi(n) \) is equal to \( n \prod_{p|n} \left( 1 - \frac{1}{p} \right) \) (\( p \) prime). I hope these results can be applied to generalize some Arnold’s theorems in [1]. The idea of generalizing the present results to Dirichlet L-series was suggested by V.Timorin.

2. Two theorems

Theorem 1. For every natural \( n > 1 \), the limit for \( N \to \infty \) of the following Cesaro mean:

\[
\frac{1}{N} \sum_{m=1}^{N} \prod_{p|m} \left( 1 + \frac{1}{p^{n-1}} \right)
\]

is equal to

\[
\zeta(n)/\zeta(2n).
\]

\footnote{In fact, this statement holds for any complex number \( n \) with real part greater than 1.}
In particular, for $n = 2$,

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} \prod_{p | m} \left( 1 + \frac{1}{p^{n}} \right) = \zeta(2) = \frac{\pi^2}{6},
$$

Proof Remember that

$$
\prod_{p} \left( 1 - \frac{1}{p^{2n}} \right) = \frac{1}{\zeta(n)},
$$

where the product is extended to all primes.

Hence (see also [2], [3]),

$$
\prod_{p} \left( 1 + \frac{1}{p^{n}} \right) = \frac{\prod_{p} \left( 1 - \frac{1}{p^{2n}} \right)}{\prod_{p} \left( 1 - \frac{1}{p^{n}} \right)} = \frac{\zeta(n)}{\zeta(2n)}
$$

where the products are extended to all primes.

We have

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} \prod_{p | m} \left( 1 + \frac{1}{p^{n-1}} \right) = \lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} \left( \sum_{k \mid m} \frac{1}{k^{n-1}} \right)
$$

where $k$ is a natural square free (i.e., a natural number which is not divisible by the square of any prime number), and the $\sum_{k \mid m} \frac{1}{k^{n-1}}$ is extended to all $k$ dividing $m$.

The number of values of $m \leq N$ divisible by $k$ is $[N/k]$, approaching $N/k$ for for $N \to \infty$. Hence:

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} \left( \sum_{k \mid m} \frac{1}{k^{n-1}} \right) = \lim_{N \to \infty} \sum_{k \mid m} \frac{[N/k]}{k^{n-1}} = \sum_{k} \frac{1}{k^{n}}
$$

We write finally

$$
\sum_{k} \frac{1}{k^{n}} = \prod_{p} \left( 1 + \frac{1}{p^{n}} \right),
$$

since this last infinite product is the sum of all numbers which are powers of simple products of primes, as the sum over all $k$ at the first member of equality. The claim of the theorem is obtained using eq. (4).

We state now a similar theorem, with two proofs. The first proof holds only for naturals $n > 1$, while the second one holds for any $n$ with real part greater than 1.

**Theorem 2.** For every natural $n$, the limit for $N \to \infty$ of the following Cesaro mean is equal to the inverse of the $\zeta(n)$:

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} \prod_{p | m} \left( 1 - \frac{1}{p^{n-1}} \right) = \frac{1}{\zeta(n)}.
$$
First proof. Formula at the left member of (3) can be interpreted as the probability that \( n \) numbers \( m_1, m_2, \ldots, m_n \) randomly chosen in \( N \) (with uniform probability distribution) satisfy \( \gcd(m_1, m_2, \ldots, m_n) = 1 \).

Indeed, the probability that a natural number \( m \) be divisible by \( p \) is equal to \( \frac{1}{p} \). The probability that \( n \) natural numbers be all divisible by \( p \) equals \( \left(\frac{1}{p}\right)^n \), since the events (for different \( m \)) “\( m \) is divisible by the prime \( p \)” are independent.

The probability that, for a given \( p \), the \( n \) chosen numbers be not all divisible by \( p \) is \( 1 - \left(\frac{1}{p}\right)^n \). The probability that the \( n \) chosen numbers be not all divisible for any prime less or equal to \( N \) is \( \prod_p \left(1 - \frac{1}{p^n}\right) \), since the events (for different primes \( p \)) “the \( n \) chosen naturals are not all divisible by \( p \)” are independent. The last expression is the left member of (3).

We prove that the left member of (5) expresses the same probability as the left member of (3). Suppose now that \( n \) numbers be arbitrarily chosen in the interval \( I_N = [1, 2, \ldots, N] \) with uniform probability distribution. Let \( m \) be the maximum of them. The probability of choosing \( m \), as the probability of choosing any other number, is equal to \( \frac{1}{N} \). The probability that the \( n - 1 \) other numbers be not all divisible by a prime \( p \) dividing \( m \) equals

\[
P(p|m) = \prod_{p|m} (1 - q^{n-1}),
\]

where \( q \) is the probability that the number \( m \in I_N \) be divisible by \( p \):

\[q := p_{m\leq N}(p|m) = \left\lfloor \frac{N}{p} \right\rfloor / N.\]

The probability that the \( n \) chosen numbers in \( I_N \) be not all divisible by a common prime is the sum, over all the values \( m \) of the maximum among these numbers, of the probabilities that the other \( n - 1 \) numbers be not all divisible by the primes dividing \( m \), i.e.:

\[
\frac{1}{N} \sum_{m=1}^{N} \prod_{p|m} (1 - q^{n-1}).
\]

Since

\[
\lim_{N \to \infty} q = \frac{1}{p},
\]

the limit for \( N \to \infty \) of (6) is equal to the probability expressed by the left member of (3), and we thus obtain eq. (5).

Second proof. We have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} \prod_{p|m} \left(1 - \frac{1}{p^{n-1}}\right) = \lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} \left( \sum_{k \mid m} \frac{(-1)^{f_k}}{k^{n-1}} \right)
\]
where $k_\circ$ is square free and $f_k$ is the number of primes entering the factorization of $k_\circ$.

By the same arguments used in the proof of Theorem 1:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} \left( \sum_{k \mid m} \frac{(-1)^{f_k}}{k_{\circ}^{n-1}} \right) = \lim_{N \to \infty} \sum_{k \mid m} \frac{1}{N} \left( \sum_{k_{\circ}} (-1)^{f_k} \right) = \sum_{k_{\circ}} \frac{(-1)^{f_k}}{k_{\circ}^n},$$

and write finally

$$\sum_{k_{\circ}} \frac{(-1)^{f_k}}{k_{\circ}^n} = \prod_p \left( 1 - \frac{1}{p^n} \right).$$

The claim of the theorem is obtained using eq. (3). \hfill \Box

### 3. Generalization to Dirichlet L-series

Let $\chi$ be any Dirichlet character\footnote{A Dirichlet character is any function $\chi : \mathbb{Z} \to \mathbb{C}$ such that: 1) it is periodic: there exists a positive integer $k$ such that $\chi(n) = \chi(n + k)$ for all $n$; 2) $\chi(n) = 0$ iff $\gcd(n, k) > 1$; 3) it is multiplicative: $\chi(mn) = \chi(m)\chi(n)$ for all integers $m$ and $n$.} $s$ a complex number with real part greater than 0, and

$$L(\chi, s) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}$$

the corresponding Dirichlet L-series.

We recall that the square of a character is a character as well.

**Theorem 3.** The limit for $N \to \infty$ of the following Cesaro mean:

$$\frac{1}{N} \sum_{m=1}^{N} \prod_{p \mid m} \left( 1 - \frac{\chi(p)}{p^{s-1}} \right)$$

is equal to

$$\frac{1}{L(\chi, s)}.$$

The limit for $N \to \infty$ of the following Cesaro mean:

$$\frac{1}{N} \sum_{m=1}^{N} \prod_{p \mid m} \left( 1 + \frac{\chi(p)}{p^{s-1}} \right)$$

is equal to

$$\frac{L(\chi, s)}{L(\chi^2, 2s)}.$$

*Proof.*

We have only to remark that

$$L(\chi, s) = \frac{1}{\prod_p \left( 1 - \frac{\chi(p)}{p^s} \right)}.$$

since any character $\chi$ is multiplicative and the product is extended to all primes $p$. 

Hence

\[ \prod_p \left( 1 + \frac{\chi(p)}{p^s} \right) = \prod_p \left( 1 - \frac{\chi^2(p)}{p^{2s}} \right) = \frac{L(\chi, s)}{L(\chi^2, 2s)}. \]

Moreover

\[ \prod_p \left( 1 + \frac{\chi(p)}{p^s} \right) = \sum_{k \circlearrowleft \chi} \chi(k_0) \frac{k_0}{k_0^s}, \]

where the last sum is extended to all values of \( k_0 \) square free.

The limits for \( N \to \infty \) of (7) and (8) are then obtained the same way as the second proof of Theorem 2 and as the proof of Theorem 1, respectively.

References

[1] Arnold, V.I. Permutations, submitted to Uspekhi Matematicheskoi Nauki, 2008
[2] Hardy, G.H., Wright, E.M. An introduction to the theory of numbers. Oxford Clarendon Press 1960
[3] Montgomery, H.L., Vaughan, R.C. Multiplicative number theory. I: classical theory Publisher Cambridge, University Press, 2007