Two-Parameters Bifurcation in Quasilinear Differential-Algebraic Equations

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ABSTRACT

In this paper, bifurcation of solution of quasilinear quasilinear DAE is eventually reducible to an ordinary differential equation (ODEs) and that this reduction so we canapply the classical bifurcation theory of the (ODEs). The taylor expansion applied to the reduced DAEs to prove that is equivalent to an ODE which is a normal form undergoes some non-degeneracy Conditions theorems given in this work deal with the saddle node, transcritical and pitchfork bifurcation with two parameters. Some illustrated examples are given to explain the idea of the paper.

Keywords: Differential Algebraic Equation; Quasilinear; Bifurcation.

INTRODUCTION

Nearly all DAEs arising in scientific or engineering problems are quasilinear. This article presents bifurcation in quasilinear differential algebraic equations (DAEs) differ from ordinary differential equations (ODEs). Over the years several approaches have been introduced for the study of local existence and uniqueness questions for DAEs. While they exhibit major technical differences and are based on different assumptions, all these approaches agree with the basic principle that a DAE is eventually reducible to an ODE and that this reduction should be done via a recursive process. The bifurcation in quasilinear parameterized DAEs form

\[ A(\mu, x) \dot{x} = G(\mu, x), \quad \mu \in \mathbb{R}^2, x \in \mathbb{R}^1, \]  

it will be investigated. Accordingly, we shall assume that for some open interval \( I \subset \mathbb{R} \) and open subset \( U \subset \mathbb{R}^n \)

the mappings \( A: U \times I \rightarrow \mathbb{R}^n \) and \( G: U \times I \rightarrow \mathbb{R}^n \) are of class \( C^\infty \). And proves a bifurcation theorem based on assumptions on the Taylor coefficients. Since we will impose conditions on these coefficients we will be able to show that the system undergoes saddle node, transcritical and pitchfork bifurcation that is a little more akin to bifurcation ODE.

A simple comparison of the areas of the sciences in which DAEs are involved with those in which examples of bifurcation in ODEs arise [1],[3] that bifurcation of periodic solutions occurs from \((0,0)\)and [4] Our exposition is based on Jepson, A. and Spence [2] and the references therein reveals a considerable overlap and suggests that an appropriate variant of the bifurcation theorem should be available in the DAE setting. It is important note that all theorems and conditions for bifurcation to be occurred in the reduced DAEs will be given in terms of \( A \) and \( G \) in (1.1) and this will be extension of the bifurcation theory to DAEs of index one.

In the index one case, our coal is it use the reduction of (1.1) to an ODE In with the reduction (1.1) method given in [5]

then apply classical bifurcation theory to the reduced ODEs.

This paper is organized as follows: Section 2 deals with the problem of reducing parameterized families of DAEs simultaneously. Since reduction of DAEs to ODE form leads to implicit rather than explicit ODEs, it is important to rephrase some of the hypotheses of the classical bifurcation theorems in that setting. This is done in section 3. The bifurcation theorems for quasilinear DAEs is proved in theorems (3.1,3.2,3.3) (for two-parameters) in Section 4 we will study of the behavior bifurcation.
and discuss their implementation in Maple.

2 Reduction of Parametrized DAEs [5]
The bifurcation in quasilinear parameterized DAEs form (1.1) will be investigated and all DAEs will be reduced to an equivalent parameterized ODEs. Then classical bifurcation theory can be applied. In the reduction process we will follow the method of reduction given in [5]. So the following theorem is an essential in our work, which summarizes the reduction of DAEs (1.1).

**Theorem 2.1.** [5]
Let $\tilde{x}, \tilde{\mu} \in W_1$ and let $\Phi = \text{id} \times \varphi: I \times U^{n} \rightarrow I \times U^{n}$ be a local $C^\infty$-parametrization of $W_1$ near $(\tilde{x}, \tilde{\mu})$. There exist an open subinterval $I(\mu)$ with $\tilde{\mu} \in I \times I$ and an open neighborhood $\mathcal{O}(\tilde{x}, \tilde{\mu})$ with the following property: For $\mu \in I$

, a $C^\infty$-mapping $x : J \rightarrow \mathbb{R}^n$ on an open interval $J \subset \mathbb{R}$ is a solution of the DAE (1.1) if and only if $x(t) = \phi(\mu, \xi(t)), \forall t \in J$, and $\xi : J \rightarrow U^{r}$ is a $C^1$ solution of the system $A_1(\mu, \xi)\dot{\xi} = G_1(\mu, \xi)$.

where $W_1 = (x, \mu) \in U^n \times I : G(x, \mu) \in \text{rge}G(x, \mu)$, and $A_1 : I \times I \times U^r \rightarrow \mathbb{R}(R^r, \text{rge}A(\tilde{\mu}, \tilde{x})) \ni \mathcal{E}(R^r)$, $G_1 : I \times I \times U^r \rightarrow \text{rge}A(\tilde{\mu}, \tilde{x}) \ni \mathcal{E}(R^r)$ are the $C^\infty$ mapping given by

\begin{align}
A_1(\mu, \xi) &:= \overline{P}A(\mu, \varphi(\mu, \xi))D\varphi(\mu, \xi), \\
G_1(\mu, \xi) &:= \overline{P}G(\mu, \varphi(\mu, \xi)),
\end{align}

and $\overline{P} \in \mathcal{E}(R^r)$ is an arbitrary linear projection onto $\text{rge}A(\mu, x) \ni \mathcal{E}(R^r)$. or Fixed $\mu \in I \times I$ and $x_1^\mu \in W_1^\mu$, the DAE (1.1) reduces to the form

$A_1(\mu, \xi)\dot{\xi} = G_1(\mu, \xi), \quad (2.3)$

The fixed but arbitrary $\mu = (\mu_1, \mu_2) \in I \times I$ are also given by the solutions of the non-parametrized DAE

$A(\mu, x)\dot{x} = G(\mu, x), \quad (2.4)$

where

$$
A = \begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix}
$$

For the sake of argument, assume that the DAE (1.1) with $\mu = \tilde{\mu}$ has index one at $\tilde{x} \in W_1^\mu$ (so that $(\tilde{\mu}, \tilde{x}) \in W_1$).

With the previous notation, this means that the operator $A_1(\mu, \xi) - G_1(\mu, \xi)$ has full rank $r_1$ and hence is invertible. By continuity $A_1(\mu, \xi)$ remains invertible for $(\mu, \xi) \in \mathbb{R} \times R^{r_1}$ and it thus follows from Theorem 2.1, that for the linearity of $(\tilde{\mu}, \tilde{x}) \in W_1$, the parameterized DAE (1.1) is equivalent to the explicit parameterized ODE:

$\dot{\xi} = A_1(\mu, \xi)^{-1}G_1(\mu, \xi), \xi \in U^{r_1}. \quad (2.5)$

To motivate the discussion in the next section, suppose also $\tilde{x} = 0, \tilde{\mu} = 0$

3 Local two-Parameter Bifurcations of Equilibrium Points
We will now consider the general quasilinear parameterized DAEs equation
A(\mu, x)\dot{x} = G(\mu, x), \quad \mu \in \mathbb{R}^2, x \in \mathbb{R}^1. \quad (3.1)

and \mu = (\mu_1, \mu_2), and prove a bifurcation theorems based on assumptions on the Taylor expansion of G. We assume that G(x, \mu) for all value of \mu, A(x, \mu) is independent of x and \mu.

3.1 Saddle-Node bifurcation

The saddle-node bifurcation can take place in any system and is, in fact, a very typical bifurcation to happen when a parameter is varied. Maybe because this bifurcation is so typical, it has a lot of other names. The saddle-node bifurcation is also called fold bifurcation, tangent bifurcation, limit point bifurcation, or turning point bifurcation.

from theorem (1.1) it follow that near (0,0) that DAEs (1.1) reduced to the system

\[ \dot{\xi} = A_1 (\mu, \xi) G_1 (\mu, \xi) \in U \subset \mathbb{R}^1. \]

A_1 : I \times I \times U \rightarrow f(\mathbb{R}^1, \text{regA}(0,0))

\[ U = f(\mathbb{R}^1, \text{regA}(0,0)) \] are of class \( C^\infty \).

the following theorem related to this kind of bifurcation.

**Theorem 3.1.** Consider one-dimensional quasilinear DAEs

\[ A(x, \mu)\dot{x} = G(x, \mu), \quad \mu \in \mathbb{R}^2, x \in \mathbb{R}^1 \quad (3.2) \]

where \( G \in \mathbb{R}^3 \) has at \( \mu = 0 \) the equilibrium \( x = 0 \) and \( \frac{\partial G}{\partial x}(0,0,0) \neq 0 \). Assume that the following two non-degeneracy conditions are satisfied:

(i) \( \frac{\partial^2 G}{\partial x^2}(0,0,0) \neq 0 \)

(ii) \( \frac{\partial G}{\partial y_1}(0,0,0) \neq 0 \& \frac{\partial G}{\partial y_2}(0,0,0) \neq 0 \)

then near (0,0,0), (3.2) is topologically equivalent to one of the following normal forms.

**Figure 1:** The normal form of a saddal- node bifurcation, where r ranges from \( \pi \) to \( \pi \) using mapleTheorem
\[ \dot{y} = \pm \alpha_1 \pm \alpha_2 + y^2 + O(y^3) \] (3.3)

**Proof.** According to the reduction processes mentioned in (Section 2) DAE will be reduced to ODEs:

\[ \xi = A_1(\xi, \mu)^{-1}G_1(\xi, \mu), \] (3.4)

where \( G_1(\xi, \mu) \) and \( A_1(\xi, \mu)^{-1} \) from theorem 2.1 reduced to \( G(x, \mu) \) and \( A(x, \mu) \). Then by Taylor expansion about (0,0,0) we have:

\[
\begin{align*}
G_1(\xi, \mu_1, \mu_2) &= G_1(0, 0, 0) + \frac{\partial G_1}{\partial \xi}(0, 0, 0)\xi + \frac{\partial^2 G_1}{\partial \mu_1 \partial \mu_2}(0, 0, 0)\mu_1 + \frac{\partial^2 G_1}{\partial \mu_2}(0, 0, 0)\mu_2 + \frac{\partial^2 G_1}{\partial \xi^2}\xi^2 + \cdots
\end{align*}
\]

\[ G_2(\xi, \mu_1, \mu_2) = G_2(0, 0, 0) + \frac{\partial G_2}{\partial \xi}(0, 0, 0)\xi + \frac{\partial^2 G_2}{\partial \mu_1 \partial \mu_2}(0, 0, 0)\mu_1 + \frac{\partial^2 G_2}{\partial \mu_2}(0, 0, 0)\mu_2 + \frac{\partial^2 G_2}{\partial \xi^2}\xi^2 + \cdots
\]

where.

\[ G_1(0, 0, 0) = 0, \frac{\partial G_1}{\partial \xi}(0, 0, 0) = 0, \frac{\partial^2 G_1}{\partial \xi^2}(0, 0, 0) \neq 0
\]

Next we remove the linear term w.r.t. \( \xi \) by introducing a new variable \( z \):

\[ \xi = z + \delta, \] (3.5)

where \( \delta \) is an unknown parameter; the inverse coordinate transformation is

\[ z = \xi - \delta. \]

Differentiate the direct transformation (3.5) that get:

\[ \frac{dz}{dt} A_1(z + \delta, 0, 0) = -\frac{\partial G_1}{\partial \mu_1}(\mu_1) + \frac{\partial^2 G_1}{\partial \mu_1 \partial \mu_2}(\mu_1)z + \frac{\partial^2 G_1}{\partial \mu_2}(\mu_1)z^2 + \frac{\partial^2 G_1}{\partial \xi^2}(\mu_1, \mu_2, z^2) + \cdots
\]

Therefore,

\[ \frac{dz}{dt} A_1(z + \delta, 0, 0) = \frac{\partial G_1}{\partial \mu_1}(\mu_1) + \frac{\partial^2 G_1}{\partial \mu_1 \partial \mu_2}(\mu_1)z + \frac{\partial^2 G_1}{\partial \mu_2}(\mu_1)z^2 + \frac{\partial^2 G_1}{\partial \xi^2}(\mu_1, \mu_2, z^2) + \cdots
\]

by removing the linear terms \( z^2(\frac{\partial^2 G_1}{\partial \mu_1 \partial \mu_2}(\mu_1)) \) and \( z\frac{\partial^2 G_1}{\partial \mu_1 \partial \mu_2}(\mu_1)z^2 + \frac{\partial^2 G_1}{\partial \xi^2}(\mu_1, \mu_2, z^2) + \cdots
\) which is required that:

\[ \delta_1(\mu) = \frac{\frac{\partial^2 G_1}{\partial \mu_1 \partial \mu_2}(\mu_1)}{\frac{\partial^2 G_1}{\partial \mu_2}(\mu_1)} \] and \[ \delta_1(\mu) = \frac{\frac{\partial^2 G_1}{\partial \mu_1 \partial \mu_2}(\mu_1)}{\frac{\partial^2 G_1}{\partial \xi^2}(\mu_1, \mu_2, z^2)} \]

Then the equation becomes:

\[ \frac{dz}{dt} = \beta_1z_1 + \beta_2z_2 + \frac{1}{2} \frac{\partial^2 G_1}{\partial \mu_2}(\mu_1)z^2
\]

Where
\[ \beta_1 \frac{\partial G_1}{\partial \mu_1} \mu_1^2 + \frac{\partial^2 G_1}{\partial z \partial \mu_1} \delta \mu_1 \frac{1}{2} \frac{\partial^2 G_1}{\partial z^2} \delta^2 \]

\[ \beta_2 \frac{\partial G_2}{\partial \mu_2} \mu_2^2 + \frac{\partial^2 G_2}{\partial z \partial \mu_2} \delta \mu_2 + \frac{1}{2} \frac{\partial^2 G_2}{\partial z^2} \delta^2 \]

Now consider as a new parameter \( \beta = (\beta_1(\mu), \beta_2(\mu)) \) and we have \( A_1(z + \delta, 0, 0)^{-1} \)

is independent of \( \xi \) and since \( \frac{\partial G}{\partial \mu_1} \neq 0 \) & \( \frac{\partial G}{\partial \mu_2} \neq 0 \) we can neglecting terms with

\( \mu_1^2 \) & \( \mu_2^2 \) respectively then we have:

\[ \beta_1 \approx \frac{\partial G_1}{\partial \mu_1} \mu_1 \quad \& \quad \beta_2 \approx \frac{\partial G_2}{\partial \mu_2} \mu_2 \]

So for \( \mu_1 > 0, \frac{\partial G_1}{\partial \mu_1} > 0 \), \( \beta_1 \) is increasing and \( \mu_2 > 0, \frac{\partial G_2}{\partial \mu_2} > 0 \), \( \beta_2 \) is increasing,

but if \( \frac{\partial G_1}{\partial \mu_1} < 0 \) & \( \frac{\partial G_2}{\partial \mu_2} < 0 \) then \( \beta_1, \beta_2 \) are decreasing when \( \mu_1 < 0, \mu_2 < 0 \) resp. Now let the following assumption

\[ y_1 = \begin{cases} \beta_1 & \text{if } \frac{\partial G_1}{\partial \mu_1} > 0 \\ \beta_2 & \text{if } \frac{\partial G_1}{\partial \mu_1} < 0 \end{cases} \]

Then the equation is:

\[ \frac{dy}{dz} \pm y_1 \gamma_2 \pm \frac{1}{2} \frac{\partial^2 G_1}{\partial z^2} z \]

Next assume \( y = \left| \frac{1}{2} \frac{\partial^2 G_1}{\partial z^2} \right| \) then we have:

\[ \frac{dy}{dt} \pm y_1 \left| \frac{1}{2} \frac{\partial^2 G_1}{\partial z^2} \right| \pm \frac{1}{2} \frac{\partial^2 G_1}{\partial z^2} + \frac{1}{2} \frac{\partial^2 G_1}{\partial z^2} z \]

Substituting \( z = \frac{y}{\frac{1}{2} \frac{\partial^2 G_1}{\partial z^2}} \) we get

\[ \frac{dy}{dt} \pm y_1 \left| \frac{1}{2} \frac{\partial^2 G_1}{\partial z^2} \right| \pm \frac{1}{2} \frac{\partial^2 G_1}{\partial z^2} + \frac{1}{2} \frac{\partial^2 G_1}{\partial z^2} y_2 \]

Suppose that \( \alpha_1 = y_1 \left| \frac{1}{2} \frac{\partial^2 G_1}{\partial z^2} \right| \) and \( \alpha_2 = y_2 \left| \frac{1}{2} \frac{\partial^2 G_1}{\partial z^2} \right| \) Then we get the normal form

\[ \dot{y} = \pm \alpha_1 \pm \alpha_2 + y^2 + O(y^3) \]

### 3.2 Trans-critical bifurcation

If two curves of fixed points intersect at the origin in the \( \mu - x \) plain, both existed on either side of \( \mu = 0 \) then the origin is called a transcritical bifurcation (TCB) point see[8].
Figure 2: The normal form of a Trans-critical bifurcation, where r ranges from $\pi$ to $\pi$ using maple.

**Theorem 3.2.** Consider one-dimensional quasilinear DAEs

$$A(x, \mu)x = G(x, \mu), \quad \mu \in \mathbb{R}^2, x \in \mathbb{R}^1$$  \hspace{1cm} (3.6)

where $G \in \mathbb{R}^3$ has at $\mu=0$ the equilibrium $x=0$, and $\frac{\partial G}{\partial x}(0,0,0) \neq 0$. Assume that the following two non-degeneracy conditions are satisfied:

(i) \( \frac{\partial^2 G}{\partial x^2}(0,0,0) \neq 0 \)

(ii) \( \frac{\partial^2 G}{\partial x \partial \mu_1}(0,0,0) = 0 \), \( \frac{\partial^2 G}{\partial x \partial \mu_2}(0,0,0) \neq 0 \)

then near $(0,0,0)$, (3.2) is topologically equivalent to the one of the following normal forms:

$$\dot{y} = \pm \alpha_1 y \pm \alpha_2 y \pm y^2 + o(y^3)$$ \hspace{1cm} (3.7)

**Proof.** According to the reduction processes mentioned in (Section 2) DAE will be reduced to ODEs:

$$\dot{\xi} = A_1(\xi, \mu)^{-1}G_1(\xi, \mu),$$ \hspace{1cm} (3.8)

where $G_1(\xi, \mu)$ and $A_1(\xi, \mu)^{-1}$ from theorem 2.1 reduced to $G(x, \mu)$ and $A(x, \mu)$. Then by Taylor expansion about $(0,0,0)$ we have:

$$G_1(\xi, \mu_1, \mu_2) = G_1(0,0,0) + \frac{\partial G_1}{\partial \xi}(0,0,0)\xi + \frac{\partial G_1}{\partial \mu_1}(0,0,0)\mu_1 + \frac{\partial G_1}{\partial \mu_2}(0,0,0)\mu_2 + \frac{\partial^2 G_1}{\partial \xi^2}(0,0,0)\frac{\xi^2}{2} + \frac{\partial^2 G_1}{\partial \xi \partial \mu_1}(0,0,0)\frac{\xi \mu_1}{2} + \frac{\partial^2 G_1}{\partial \xi \partial \mu_2}(0,0,0)\frac{\xi \mu_2}{2} + \frac{\partial^2 G_1}{\partial \mu_1^2}(0,0,0)\frac{\mu_1^2}{2} + \frac{\partial^2 G_1}{\partial \mu_2^2}(0,0,0)\frac{\mu_2^2}{2} + O(\mu_1^3, \mu_2^3, \xi^3)$$

where:

$$G_1(0,0,0) = 0, \quad \frac{\partial G_1}{\partial \xi}(0,0,0) = 0, \quad \frac{\partial^2 G_1}{\partial \xi^2}(0,0,0) \neq 0$$

As in the proof of Theorem (3.1) we get:
\[ \delta_1(\mu) = \frac{\sigma_1 G_1 - \rho}{\frac{\partial^2 G_1}{\partial \mu_1} \mu_1} \quad \text{and} \quad \delta_2(\mu) = \frac{\sigma_2 G_2 - \rho}{\frac{\partial^2 G_2}{\partial \mu_2} \mu_2} \]

Then the equation becomes:
\[ \frac{dz}{dt} = 1 + \frac{\partial^2 G_2}{\partial \mu_1} \mu_1 \frac{1}{2 \partial z^2} \]

Where
\[ \beta_1 \frac{\partial^2 G_1}{\partial z \delta \mu_1} \mu_1 \frac{1}{2 \partial z^2} \delta \]
\[ \beta_2 \frac{\partial^2 G_2}{\partial z \delta \mu_2} \mu_2 \frac{1}{2 \partial z^2} \delta \]

Now consider as a new parameter \( \beta = (\beta_1(\mu), \beta_2(\mu)) \) and we have \( A_1(z + \delta, 0, 0)^{-1} \)

is independent of \( \xi \) and since \( \frac{\partial G}{\partial \mu_1} \neq 0 \) \& \( \frac{\partial G}{\partial \mu_1} 
eq 0 \) we can neglecting terms with

\[ \mu_1^2 G_1, \mu_2^2 G_2 \]

respectively then we have:
\[ \beta_1 \frac{\partial^2 G_1}{\partial z \delta \mu_1} \mu_1 \frac{1}{2 \partial z^2} \Delta \]
\[ \beta_2 \frac{\partial^2 G_2}{\partial z \delta \mu_2} \mu_2 \frac{1}{2 \partial z^2} \Delta \]

So for \( \mu_1 > 0, \frac{\partial^2 G_1}{\partial z \delta \mu_1} > 0 \), \( \beta_1 \) is increasing and \( \mu_2 > 0 \), \& \( \frac{\partial^2 G_1}{\partial z \delta \mu_2} > 0 \), \( \beta_2 \)

is increasing, but if \( \frac{\partial^2 G_1}{\partial z \delta \mu_1} < 0 \) \& \( \frac{\partial^2 G_1}{\partial z \delta \mu_2} > 0 \) then \( \beta_1, \beta_2 \) are decreasing when \( \mu_1 < 0, \mu_1 < 0 \) resp. Now let the following assumption
\[ \gamma_1 = \begin{cases} \beta_1 & \text{if } \frac{\partial^2 G_1}{\partial z \delta \mu_1} > 0 \\ \beta_2 & \text{if } \frac{\partial^2 G_1}{\partial z \delta \mu_2} > 0 \\ \beta_1 & \text{if } \frac{\partial^2 G_1}{\partial z \delta \mu_1} < 0 \\ \beta_2 & \text{if } \frac{\partial^2 G_1}{\partial z \delta \mu_2} < 0 \end{cases} \]

Then the equation is:
\[ \frac{dz}{dt} = \gamma_1 \frac{\partial^2 G_1}{\partial z \delta \mu_1} \frac{1}{2 \partial z^2} z + \gamma_2 \frac{\partial^2 G_2}{\partial z \delta \mu_2} \frac{1}{2 \partial z^2} z^2 \]

Next assume \( y = \left[ \frac{1}{2 \partial z^2} \right] z \) then we have:
\[ \frac{dy}{dx} = \gamma_1 \frac{\partial^2 G_1}{\partial z \delta \mu_1} \frac{1}{2 \partial z^2} y + \gamma_2 \frac{\partial^2 G_2}{\partial z \delta \mu_2} \frac{1}{2 \partial z^2} y^2 \]

Substituting \( z = \frac{y^2}{2 \partial z^2} \) we get
\[ \frac{dy}{dx} = \gamma_1 \frac{\partial^2 G_1}{\partial z \delta \mu_1} \frac{1}{2 \partial z^2} y + \gamma_2 \frac{\partial^2 G_2}{\partial z \delta \mu_2} \frac{1}{2 \partial z^2} y^2 + \gamma_1 \frac{\partial^2 G_1}{\partial z \delta \mu_1} \frac{1}{2 \partial z^2} y^3 \]

Suppose that \( \alpha_1 = \gamma_1 \frac{1}{2 \partial z^2} y \) and \( \alpha_2 = \gamma_2 \frac{1}{2 \partial z^2} y^2 \) Then we get the normal form
\[ \dot{y} = \pm \alpha_1 y \pm \alpha_2 y \pm y^2 + \alpha y^3 \]

### 3.3 Pitchfork bifurcation

If two curves of fixed points intersect at the origin in the \( \mu - x \) plane and only one exists in both sides of \( \mu = 0 \), moreover, the other curve of fixed points lays entirely to one side of \( \mu = 0 \), then the origin is called a
Theorem 3.3. Consider one-dimensional quasilinear DAEs

\[ A(x, \mu)\dot{\xi} = G(x, \mu), \quad \mu \in \mathbb{R}^2, x \in \mathbb{R}^1 \]  

where \( G \in \mathbb{R}^3 \) has at \( \mu = 0 \) the equilibrium \( x = 0 \), and \( \frac{\partial G_1}{\partial x}(0,0,0) \neq 0 \). Assume that the following two non-degeneracy conditions are satisfied:

(i) \( \frac{\partial^2 G_1}{\partial x \partial \mu_1}(0,0,0) \neq 0 \)

(ii) \( \frac{\partial^2 G_1}{\partial x \partial \mu_2}(0,0,0) \neq 0 \)

then near \((0,0,0)\), (3.2) is topologically equivalent to the one of the following normal forms:

\[ \dot{y} = \pm \alpha_1 y \pm \alpha_2 y \pm y^3 + o(y^4) \] (3.10)

**Proof.** According to the reduction processes mentioned in (Section 2) DAE will be reduced to ODEs:

\[ \dot{\xi} = A_1(\xi, \mu)^{-1} G_1(\xi, \mu), \] (3.11)

where \( G_1(\xi, \mu) \) and \( A_1(\xi, \mu)^{-1} \) from theorem 2.1 reduced to \( G(x, \mu) \) and \( A(x, \mu) \). Then by Taylor expansion about \((0,0,0)\) we have:

\[ G_1(\xi, \mu_1, \mu_2) = G_1(0,0,0) + \frac{\partial G_1}{\partial \mu_1}(0,0,0) \xi_1 + \frac{\partial G_1}{\partial \mu_2}(0,0,0) \xi_2 + \frac{\partial^2 G_1}{\partial \mu_1^2}(0,0,0) \xi_1^2 + \frac{\partial^2 G_1}{\partial \mu_2^2}(0,0,0) \xi_2^2 + \frac{\partial^2 G_1}{\partial \mu_1 \partial \mu_2}(0,0,0) \xi_1 \xi_2 + \cdots \]

where,

\[ G_1(0,0,0) = 0, \quad \frac{\partial G_1}{\partial \xi_1}(0,0,0) = 0, \quad \frac{\partial G_1}{\partial \xi_2}(0,0,0) = 0 \]

From equation (3.11) we get:

\[ \frac{dz}{dt} = A_1(\xi + \delta, 0, 0)^{-1} \left[ \frac{\partial G_1}{\partial \mu_1}(\xi + \delta) \mu_1 + \frac{\partial G_1}{\partial \mu_2}(\xi + \delta) \mu_2 + \frac{\partial^2 G_1}{\partial \mu_1^2}(\xi + \delta) \mu_1^2 + \frac{\partial^2 G_1}{\partial \mu_2^2}(\xi + \delta) \mu_2^2 + \frac{\partial^2 G_1}{\partial \mu_1 \partial \mu_2}(\xi + \delta) \mu_1 \mu_2 \right] \]

Therefore,

\[ \frac{dz}{dt} = A_1(\xi + \delta, 0, 0)^{-1} \left[ \frac{\partial G_1}{\partial \mu_1} \mu_1 + \frac{\partial^2 G_1}{\partial \mu_1^2} \mu_1^2 + \frac{\partial^2 G_1}{\partial \mu_1 \partial \mu_2} \mu_1 \mu_2 + \frac{\partial^2 G_1}{\partial \mu_2^2} \mu_2^2 + \frac{\partial^2 G_1}{\partial \mu_1 \partial \mu_2} \mu_1 \mu_2 \right] + \cdots \]

and removing the linear terms \( \frac{\partial^2 G_1}{\partial \mu_1^2} \mu_1^2 \) and \( \frac{\partial^2 G_1}{\partial \mu_2^2} \mu_2^2 \) which is required that:

\[ \delta_1(\mu) = \frac{x \partial G_1}{\partial \mu_1} \mu_1 \text{ and } \delta_1(\mu) = \frac{x \partial G_1}{\partial \mu_2} \mu_2 \]

Then the equation becomes:

\[ \frac{dz}{dt} = A_1 \frac{\partial^2 G_1}{\partial \mu_1^2} \mu_1^2 + \cdots + \frac{\partial^2 G_1}{\partial \mu_2^2} \mu_2^2 + \cdots \]
Where
\[ \beta_1 \frac{\partial^2 G_1}{\partial z \partial \mu_1} \partial \mu_1 + \frac{1}{2} \frac{\partial^2 G_2}{\partial z^2} \delta^+ + \frac{\partial \gamma_1}{\partial x} \delta^+ = \frac{\partial^2 G_1}{\partial z \partial \mu_1} \delta^+ + \frac{\partial \gamma_1}{\partial x} \delta^+. \]

Now consider as a new parameter \( \beta = (\beta_1(\mu), \beta_2(\mu)) \) and we have \( A_1(z + \delta, 0, 0)^{-1} \) is independent of \( \xi \) and since \( \frac{\partial^2 G_1}{\partial z \partial \mu_1} \neq 0 \) \& \( \frac{\partial^2 G_1}{\partial z \partial \mu_2} \neq 0 \) we can neglecting terms with \( \mu_1^2 \delta \mu_2^2 \) respectively then we have:
\[ \beta_1 \simeq \frac{\partial^2 G_1}{\partial z \partial \mu_1} \mu_1 \beta_2 \simeq \frac{\partial^2 G_1}{\partial z \partial \mu_2} \mu_2. \]

So for \( \mu_1 > 0, \frac{\partial^2 G_1}{\partial z \partial \mu_1} > 0, \beta_1 \) is increasing and \( \mu_2 > 0, \frac{\partial^2 G_1}{\partial z \partial \mu_2} > 0, \beta_2 \) is increasing, but if \( \frac{\partial^2 G_1}{\partial z \partial \mu_1} < 0 \& \frac{\partial^2 G_1}{\partial z \partial \mu_2} < 0 \) then \( \beta_1, \beta_2 \) are decreasing when \( \mu_1 < 0, \mu_2 < 0 \) resp. Now let the following assumption
\[ \gamma_1 = \begin{cases} \beta_1 \text{if } \frac{\partial^2 G_1}{\partial z \partial \mu_1} > 0, \beta_2 \text{if } \frac{\partial^2 G_1}{\partial z \partial \mu_2} > 0, \\ \beta_1 \text{if } \frac{\partial^2 G_1}{\partial z \partial \mu_1} < 0, \beta_2 \text{if } \frac{\partial^2 G_1}{\partial z \partial \mu_2} < 0. \end{cases} \]

Then the equation is:
\[ \frac{dz}{dt} = \gamma_1 \frac{\partial G_1}{\partial x} x \frac{\partial G_1}{\partial z} z \pm \frac{1}{2} \frac{\partial^2 G_1}{\partial x^2} z^2. \]

Next assume \( y = \frac{1}{2} \frac{\partial^2 G_1}{\partial x^2} x \) then we have:
\[ \frac{dy}{dt} = \frac{1}{2} \frac{\partial^2 G_1}{\partial x^2} x \frac{\partial G_1}{\partial z} z \pm \gamma_2 z \frac{\partial G_1}{\partial z} z \pm \frac{1}{2} \frac{\partial^2 G_1}{\partial x^2} z^2. \]

Substituting \( z = \frac{y}{\gamma_2} \) we get:
\[ \frac{dy}{dt} = \gamma_1 \frac{1}{2} \frac{\partial G_1}{\partial x} x \frac{\partial G_1}{\partial z} z \pm \gamma_2 \frac{1}{2} \frac{\partial^2 G_1}{\partial x^2} z^2. \]

Suppose that \( \alpha_1 = \gamma_1 \frac{1}{2} \frac{\partial G_1}{\partial x} x \) and \( \alpha_2 = \gamma_2 \frac{1}{2} \frac{\partial^2 G_1}{\partial x^2} z^2 \) then we get the normal form
\[ \dot{y} = \pm \alpha_1 y \pm \alpha_2 y \pm y^3 + o(y^4) \]

### 4 Applications of DAEs

There are three types of one-zero-eigenvalue bifurcations: saddle-node, transcritical and pitchfork bifurcation.

Each one of them satisfies different genericity conditions; their bifurcation diagrams are also different. Without loss of generality let us assume that \( (x, \mu) = (0, 0) \) is point of one-dimensional parameterized dynamical system:

\[ A(\mu, x) \dot{x} = G(\mu, x), \quad \mu \in \mathbb{R}, x \in \mathbb{R}^n. \quad (4.1) \]

Now, for (4.1) to undergo a one-zero-eigenvalue bifurcation at \( (0, 0) \), the following conditions should be satisfied:
These conditions guarantee that the fixed point (0,0) is not hyperbolic. The natural environment for this kind of work are the computer algebra systems like Maple and Mathematica.

Their impact on dynamical systems studies is due to the fact that many calculations are too tedious for manual work, but do not challenge the computer resources.

In this paper we present algorithms for symbolical study of one parameter local bifurcations in guasilinear DAEs of equilibrium points and discuss their implementation in Maple.

\[ G(0,0) = 0, \quad G_x(0,0) = 0. \]  

(4.2)

Figure 3: The normal form of a Pitchfork bifurcation, where \( r \) ranges from \( \pi \) to \( -\pi \) using maple

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