Entanglement in the symmetric sector of \( n \) qubits

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We discuss the entanglement properties of symmetric states of \( n \) qubits. The Majorana representation maps a generic such state into a system of \( n \) points on a sphere. Entanglement invariants, either under local unitaries (LU) or stochastic local operations and classical communication (SLOCC), can be reduced.

Several entanglement measures have nevertheless been proposed (see [20] for a comprehensive review), and their behavior under state transformation studied. Important cases are given by those quantities which remain invariant under (stochastic) local operations and classical communication, noted (S)LOCC [21] . Stated as operations performed in the multiqubit Hilbert space \( \mathcal{H} \), the latter read \( \otimes_i M_i \), called local unitaries (LU) for LOCC (with \( M_i \) a unitary matrix), and invertible local operations (ILO) for SLOCC (\( M_i \) a matrix with non vanishing determinant).

One aims to find a complete set of such invariants that parameterizes the orbit space \( \mathcal{H}/ \otimes_i M_i \). Physically this means that states can only be obtained from each other with a local transformation (LU or ILO) if they share the same set of invariants. In the LOCC case, LU invariants can in principal be written as polynomial functions of the state component. However their number proliferates with \( n \), and finding explicit expressions becomes challenging; moreover their physical relevance is not necessarily obvious.

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The potential power of quantum information, either for cryptography and computation purpose, is largely based on the subtle concept of quantum entanglement. In a system composed of \( n \) two-level entities (qubits), a generic state is entangled, e.g. it cannot be written as a separable product of states belonging to each constitutive part. While it is rather easy to characterize entanglement for a 2-qubits system, the task of quantifying the amount of entanglement carried by the total system is very difficult, for increasing \( n \).

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unit sphere, is obtained by an inverse stereographic map of \( \{\alpha_i\} \rightarrow \{v_i\} \). The Majorana high spin spherical representation generalizes (although published earlier) the spin 1/2 Bloch sphere; it recently proved quite useful in the context of collective spin models.\(^{20}\)

**Symmetric LU Invariants (SLUI)**

A generic local (separable) unitary transformation acting on a set of \( n \) qubits can be written, up to a un-physical phase, in the form \( U = \otimes_i e^{i\sigma_i} \sigma_i \) with a collection of vectors \( h_i = 1 \) \( n \in \mathbb{R}^3 \). In the symmetric sector, we restrict to identical \( h_i \), leading to the simpler form

\[
U_s = e^{i h \cdot S}.
\]  

(2)

This implies that, in the symmetric sector, the only states that are LU equivalent correspond to sets of (unordered) Majorana zeroes which can be transformed into each other by a global rotation of their representative vectors \( v_i \rightarrow \tilde{v}_i = R v_i \); with \( R \) in SO(3). Moreover, one also expect equivalent entanglement measures for states that are related by an (anti-unitary) time reversal operation \( T = \otimes_{i=1}^n (i\sigma_y) \), where \( K \) is the complex conjugate operator in the computational basis \( K(\sum_{i,j,k=0,1} \tilde{t}_{i,j,k} |i,j,k\rangle) = (\sum_{i,j,k=0,1} \tilde{t}_{i,j,k} |i,j,k\rangle) \) and \( T^2 = (-1)^n \). Geometrically, this corresponds to an inversion \( v_i \rightarrow \tilde{v}_i = -v_i \).

An over-complete set of SLUI is obtained from symmetrized products of the inner-products \( v_{ij} = v_i \cdot v_j \), like for instance with the \( c_k \) coefficients of \( x^k \) in the symmetrized product \( \prod_{ij} (x - v_{ij}) = \sum c_k x^k \). It is instructive to relate them to the standard invariants for two and three qubits. We make use of density matrices \( \rho = |\Psi\rangle \langle \Psi| \) and eventually uses their partial trace, with indices in \( \rho \) indicating those parts which have not been traced out.

**The two-qubits case**

For 2 qubits, there is one entanglement invariant (if we disregard the trivial invariant \( \text{Tr}[\rho] = 1 \) for a normed state), which we express here with the single inner product \( v_{12} \). It can be given as the (equal) radius \( r_i \) of the partial Bloch sphere when tracing out one of the 2 subsystem. From \( r_i^2 = 2 \text{Tr}[\rho_i^2] - 1 \), one gets \( r_i = \frac{8(v_{12}^2 + 1)}{(v_{12}^2 + 3)^2} \).

Another most used form is the concurrence\(^{21}\) running from zero for a separable state to unity form maximally entangled EPR state. In the symmetric sector, it reads \( C = \frac{2}{v_{12}^2 + 3} - 1 \). Separable symmetric states corresponds to the case with the two identical Majorana points, while symmetric EPR corresponds to pairs of antipodal points \( (v_{12} = -1) \). The latter set is then given by the sphere \( S^2 \) with opposite points identified, the projective plane \( RP^2 \).

Note that a simple but careful analysis, not reproduced here, allows to extend the EPR case to the full Hilbert space (not only to the symmetric sector), and recover the well known \( RP^3 \) (\( \equiv \text{SO}(3) \)) EPR manifold.\(^{22}\)

**The three-qubits case**

A complete set of six independent LU invariant polynomials is known\(^{23,24}\). For a generic 3 qubit state,

\[
I_1 = \text{Tr}[\rho], \\
I_{1=2,3,4} = 2 \text{Tr}[\rho_i^2 - 1], \\
I_5 = \text{Tr}[3 (\rho_1 \otimes \rho_2) \cdot \rho_{12}] - \text{Tr}[\rho_1^2] - \text{Tr}[\rho_2^2] + \text{Tr}[\rho_{12}^2], \\
I_6 = \tau_3.
\]

Again, \( I_1 = 1 \) for a normed state. \( I_{2,3,4} \) are related to the radius of the (partial) Bloch balls of qubits \( (1, 2, 3) \) respectively, once the other two are traced out. \( I_5 \) is the Kempe invariant\(^{23}\) and \( I_6 \) the 3-tangle, which takes the form of a hyperdeterminant.\(^{24}\) Note that \( I_{1,...,6} \) are also invariant under a time reversal transformation. Restricted to the symmetric sector, these invariants explicitly read, with \( c_{0} = -v_{12}v_{13}v_{23}, c_{1} = v_{12}v_{13} + v_{12}v_{23} + v_{13}v_{23}, \) and \( c_{2} = -(v_{12} + v_{13} + v_{23}) \),

\[
I_{2,3,4} = \frac{-6c_0 + 18c_1 + (c_2 - 60)c_2 + 75}{9(c_2 - 3)^2}, \\
I_5 = \frac{1}{18(c_2 - 3)^3} \times [-9c_0 (c_2 - 9) + 459 + 27c_1 (c_2 - 5) + (c_2 - 24)c_2 (4c_2 - 21)], \\
I_6 = \frac{2(c_0 + c_1 + c_2 + 1)}{3(c_2 - 3)^2}.
\]

(3)

Using \( \theta_{i,j} = \arccos v_{i,j} \) as coordinate axes, and recalling that the set of Majorana points is not ordered, we can display the symmetric sector entanglement types inside the tetrahedron (OABC) shown in Fig.1\(^{11}\) Analyzing the subgroups of SO(3) that leave each representative state invariant one can characterize the manifold corresponding to each entanglement family (see Table I).

\[\text{Figure 1: Entanglement types, for symmetric 3-qubits space, in the } \theta_{i,j} \text{ space. Point } O \text{ corresponts to separable states (with coinciding 3 Majorana points), B and A to W and GHZ states respectively.}\]
Im-10, but now with metric sector invariant can also be parameterized as in transformations. Indeed, an ILO this problem to the classification of invariants of Möbius motion as simple manipulations, one parameterize this transformation as

\[ \frac{1}{2}\left(1 + \frac{2}{n}\right) \Theta \]

where \( \Theta \) is the minimum of \( I_5 \) within the class of symmetric states arising only for type B states. (**) Maximal 3-tangle states.

**Toward a determination of the unit 3-tangle manifold**

Symmetric GHZ states (with unit 3-tangle \( I_6 = 1 \)) correspond to the three Majorana points forming an equatorial triangle on an equatorial plane. The set of equatorial planes is the projective plane \( RP^2 \). Adding the triangles global rotation modulo \( 2\pi/3 \), the set of symmetric unit 3-tangle states inherits the geometry \( SO(3)/Z_3 \).

Using the above defined time reversal operator \( T \), we consider the operator \( Y(\theta) = (\cos \theta + \sin \theta T) \), whose inverse is \( Y(-\theta) \) (since \( T^2 = -1 \) for n odd); \( Y(\theta) \) is left unchanged under conjugation with a LU. Applying \( Y(\theta) \) onto a separable 3-qubit state, one gets interesting entangled states. Starting from a symmetric separable state, one proves that any symmetric GHZ state can be obtained as \( |\Psi\rangle = Y\left(\frac{\pi}{3}\right) |\alpha\rangle |\alpha\rangle |\alpha\rangle \). This shows that any symmetric GHZ state, in a separable state, maps a non symmetric separable state \( |\alpha_1\rangle |\alpha_2\rangle |\alpha_3\rangle \) onto a (non-symmetric) GHZ state, as can be verified by a direct check. One can show that these GHZ states form the manifold \( M = S^2 \times S^2 \times SO(3)/Z_3 \). In the case (yet unproved, but numerically plausible) that any generic unit 3-tangle GHZ can be sent to the symmetric space by a LU, this would prove that the full GHZ manifold is indeed \( M \). Note \( M \) differs by a factor \( Z_3 \) from that given in [25].

**Symmetric states SLOCC invariants**

A nice description of SLOCC invariant families was recently proposed for symmetric \( n \)-qubits states [25,26], which focuses on the number of different roots \( \alpha_i \) and their degeneracy. This allows a full classification for \( n = 2 \) or 3 but, as stressed by the authors, leaves continuous families of additional parameters for larger \( n \). Our aim here is to provide a closer look to this question, by mapping this problem to the classification of invariants of Möbius transformations. Indeed, an ILO \( A \) that leaves the symmetric sector invariant can also be parameterized as in [2], but now with \( \hbar \) being complex instead of real. Upon simple manipulations, one parameterize this transformation as

\[ A = e^{i\hbar \left( \frac{1}{\pi_1^2} S_+ S_+ - \frac{\beta_1 \phi_2}{\pi_2^2} S_- \right)} \]

where \( \beta_1, \beta_2, \hbar \in \mathbb{C} \). The action of this operator on a generic state in the coherent state basis is given by

\[ A\psi(\alpha) = \left[ \frac{\gamma^{-1}(\alpha - \beta_1) - \gamma(\alpha - \beta_2)}{(\beta_1 - \beta_2)} \right]^{2x} \hspace{1cm} \psi\left( \frac{\gamma^{-1} \beta_2 (\alpha - \beta_1) - \gamma \beta_1 (\alpha - \beta_2)}{\gamma^{-1}(\alpha - \beta_1) - \gamma(\alpha - \beta_2)} \right) \]

where \( \gamma = e^{\frac{i}{\hbar} \left( \frac{\beta_1 - \beta_2}{\pi_1^2 + \pi_2^2} \right)} \). Note that this transformation lets the wave function invariant for \( \alpha = \beta_1 \) and \( \alpha = \beta_2 \). The roots \( \alpha_i \) of the polynomial \( \psi(\alpha) \) transform according to the following Möbius Transformation (MT):

\[ \alpha_i \rightarrow \alpha_i' = \frac{\beta_2 \gamma - \beta_1 \gamma^{-1}}{(\gamma - \gamma^{-1})} \alpha_i + \gamma^{-1} \beta_2 - \gamma \beta_1 \]

Unitary transformations are recovered whenever \( \beta_1 = -\beta_2^{-1} \) and \( \hbar \in \mathbb{R} \), corresponding to the sub-class of elliptic MT. This mapping from ILO to MT is particularly interesting when looking to invariant quantities. Indeed, the latter are well known to preserve the “cross-ratio” of four (here complex) numbers:

\[ (\alpha, \alpha_j; \alpha_k, \alpha_l) = \frac{(\alpha_i - \alpha_k)(\alpha_j - \alpha_l)}{(\alpha_j - \alpha_k)(\alpha_i - \alpha_l)} \]

which therefore form the natural basis for SLOCC invariants. Note that permuting the roots \( \alpha \) in the ratio \( (\alpha, \alpha_j; \alpha_k, \alpha_l) = \lambda \) leads generically to the following 6 different values for the cross ratio out of the 24 permutations: \( \{\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, \frac{1-\lambda}{\lambda}\} \), belonging to distinct regions in the complex plane (Fig. 2).

**Figure 2:** Symmetries of the Cross Product. For a given set of four complex numbers, the 6 permutation related cross ratios belong to separate regions \( D_i \) labeled from 1 to 6 in the picture. The boundaries of the regions carry more symmetries, so one should for example only consider the black lines for region \( D_1 \). States associated with invariant on the boundary set, like the colored ones, are expected to display particular properties.

As discussed in [15], for \( n \) qubits, the symmetric SLOCC classes are parameterized by \( n - 3 \) continuous parameters. In terms of MT, this is nothing but the known

| States | Manifold | \( I_2 \) | \( I_5 \) | \( I_6 \) |
|--------|----------|--------|--------|--------|
| \( O \) | \( SO(3)/Z_3 \) | \( 0 \) | \( 1/4 \) | \( 1/3 \) |
| \( A \) | \( SO(3)/Z_3 \) | \( 0 \) | \( 1/9 \) | \( 2/9^{(*)} \) |
| \( B \) | \( SO(3)/Z_3 \) | \( 4/9 \) | \( 17/36 \) | \( 1/3 \) |

Table I: Manifold of the particular points \( O, A, B \) and \( C \) of Fig. 1 (*): 2/9 is the minimum of \( I_5 \) within the class of symmetric states arising only for type B states. (*) Maximal 3-tangle states.
property that a unique MT relate two sets of three distinct complex numbers, and that transformations involving $n$ complex numbers are parameterized by $n-3$ cross ratios. This immediately recovers the result that, for $n=3$, there are 3 SLOCC classes in the symmetric sector, labeled by the points $O$, $B$, and $A$ in Fig. 1. Separable states (point $O$), with the three roots $\alpha_i$ identical, $W$ states (point $B$) with two roots identical and the remaining (generic) states that can be mapped under SLOCC to the GHZ state (point $A$).

A complete set of SLOCC invariants (for any $n$) can be obtained by choosing 3 roots $\alpha_i$ ($i = 1, 2, 3$) in order to define the function $\lambda(z) = \frac{z - \alpha_1}{(z - \alpha_2)(z - \alpha_3)}$. The $n-3$ complex values $\lambda = \{\lambda_1, ..., \lambda_{n-3}\}$, where $\lambda_j = \lambda(\alpha_j)$ for each $\alpha_{j>3}$, form the SLOCC invariants. Since the ordering of the $n$ roots is arbitrary, there are in general $n!$ such sets: under a permutation $\Pi$ the cross ratios transform as $\lambda \rightarrow \lambda'(\Pi)$ where each $\lambda'_j(\Pi)$ is a rational function of the $\lambda_j$'s.

For $n = 4$, we noted the reduction to 6 independent transformations; the requirement that $\lambda = \lambda(\alpha_4) \in D_1$ fixes then a unique value of the SLOCC invariant. In Ref. 18 a state having four different roots was shown SLOCC equivalent to a state within the one-parameter family: $|\Psi(\mu)\rangle = |\text{GHZ}_4\rangle + \mu|D_4^{(2)}\rangle$ with $\mu \in \mathbb{C}\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\}$ (where $|\text{GHZ}_4\rangle = \frac{1}{\sqrt{6}}(|s = 2, m = 2\rangle + |s = 2, m = 0\rangle)$). Computing the cross ratio for this family one obtains the relation $\lambda = \frac{1}{2}(\sqrt{3}\mu + 1)$.

For $n > 4$, the set of permutation related cross ratios leads to complex geometrical patterns and the identification of a canonical domain analogous to $D_1$ is difficult (as an example for $n = 5$ all 5! transformations leads to inequivalent cross ratio sets). We therefore introduce a more symmetrical formulation of the invariant quantities, $I_k(\lambda) = \prod \lambda_j^{\lambda_1(\Pi)^k}$, which amounts to sum the $k^{th}$powers of the transformed cross ratios (say of $\lambda_1$) over the complete orbit of the permutation group. Back to $n = 4$, a non trivial symmetric invariant $I_2(\lambda)$ is obtained:

$$I_2(\lambda) = \frac{2(\lambda^6 + 1) - 6(\lambda^5 + \lambda) + 9(\lambda^4 + \lambda^2) - 8\lambda^3}{(\lambda - 1)^2\lambda^2} = -3 + \frac{27}{2}J(\lambda)$$

where $J(\lambda)$ is known as the Klein modular invariant. The next case is $n = 5$, where two independent invariants $I_2(\lambda_1, \lambda_2)$ and $I_4(\lambda_1, \lambda_2)$ can be generated by summing the cross ratios squares and fourth powers over the 120 permutations. Due to lack of space, the explicit form of the two invariants is not given here. When two Majorana roots are equal one can, without loss of generality, let $\lambda_1$ go to zero, in which case both invariants diverge, but we find again the Klein invariant in the following expression $\lim_{\lambda_1 \rightarrow 0} \frac{I_2(\lambda_1, \lambda_2)}{I_4(\lambda_1, \lambda_2)} = \frac{1}{8} - \frac{27}{2\sqrt{2}}J(\lambda_2)$, which allow to fully characterize the states having 3 or 4 different roots.

In conclusion we have explicitly constructed a set of entanglement invariants under LOCC and SLOCC for symmetric $n$-qubit states and given several examples for $n$ up to five. We also expect that this correspondence between ILO and Möbius Transformations, may find further possible experimental consequences. Indeed, a generic Möbius transform can be decomposed into elementary operations, such as translations, rotations, inversions and dilation. It would therefore be very interesting to perform such elementary operations by implementing suitable POVM’s within the symmetric sector.

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