The free and safe fate of symmetry non-restoration

Borut Bajc\textsuperscript{a,1}, Adrián Lugo\textsuperscript{b,2} and Francesco Sannino\textsuperscript{c,d,3}

\textsuperscript{a} J. Stefan Institute, 1000 Ljubljana, Slovenia
\textsuperscript{b} Instituto de Física de La Plata-CONICET, and Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, Argentina
\textsuperscript{c} CP3-Origins & the Danish IAS, University of Southern Denmark, Denmark
\textsuperscript{d} Dipartimento di Fisica, E. Pancini, Univ. di Napoli, Federico II and INFN sezione di Napoli. Complesso Universitario di Monte S. Angelo Edificio 6, via Cintia, 80126 Napoli, Italy

Abstract

We investigate the high temperature fate of four dimensional gauge-Yukawa theories featuring short distance conformality of either interacting or non-interacting nature. The latter is known as complete asymptotic freedom and, as templates, we consider non-abelian gauge theories featuring either two singlet scalars coupled to gauged fermions via Yukawa interactions or two gauged scalars with(out) fermions. For theories with interacting fixed points at short distance, known as asymptotically safe, we consider two calculable examples. Exploring the landscape of safe and free theories above we discover a class of complete asymptotically free theories for which symmetry breaks at arbitrary high temperatures. In its minimal form this class is constituted by a theory with two fundamental gauged scalars each gauged under an independent group.

\textsuperscript{1}borut.bajc@ijs.si
\textsuperscript{2}lugo@fisica.unlp.edu.ar
\textsuperscript{3}sannino@cp3.sdu.dk
1 Introduction

The phenomenon of symmetry non-restoration (for a review see for example [1,2]) has been first noticed by Weinberg [3] and then studied in detail by Mohapatra and Senjanović [4,5] who were the first ones to successfully apply the mechanism to phenomenology. Since then it has been employed in cosmology to address various issues like the monopole [6–8], the domain wall [9] and false vacuum problems. The phenomenon has also been invoked for other phenomena including baryogenesis [10–15] and inflation [16].

Symmetry non-restoration at high energy can occur also due to the concomitance of other mechanisms such as the presence of large charges that can induce either Bose-Einstein condensation or superconductivity. This mechanism has been used in the literature [17–24]. For example a large charge can still be realistically related to the yet to be experimentally determined neutrino lepton number [25–30].
Symmetry non-restoration at high temperature cannot occur in supersymmetry [31–33] unless we have flat directions [34,35] and/or nonzero fixed charge.

For non supersymmetric quantum field theories symmetry non-restoration has been tested via different methods in [36–41] for global symmetries and non-restoration for local symmetries have been investigated in [42]. The results seem to support the existence of symmetry non-restoration although these claims have been challenged in [43–47].

Analyses including generalisation to different space-time dimensions including $\epsilon$ dimensions away from four are summarised in Ref. [48–51]. More precisely: symmetry non-restoration at high temperature is possible also in lower [48,50,51] and non-integer dimensions [49].

A common feature of all the theories studied so far for symmetry non-restoration at high temperature is that these can be viewed as effective theories without a well defined ultraviolet completion. This fact implies that the arbitrary large temperature limit cannot be taken.

In this work we go one step beyond with respect to what has been done so far by analysing Weinberg’s symmetry non-restoration hypothesis within models that are well defined at short distance. These are, according to Wilson [52,53] and Weinberg [54] classification of well defined theories, of either asymptotically free or safe nature. Within these theories it is consistent to consider the infinite temperature limit. It is worth recalling that for these theories short scale conformality guarantees the existence of a well defined theory at high energy making them UV complete. Asymptotic safety for gauge-Yukawa theories was discovered in [55] with the corrections to the quantum potential presented in [56]. Interestingly, once asymptotic freedom is lost in the gauge-fermion sector, within perturbation theory, the fundamentality of the theory can only be reinstated via Yukawa interactions. This implies that elementary scalars are needed, for the first time, to tame the high energy behaviour of the theory. The discovery of asymptotic safe quantum field theories [55] has led to an ongoing number of theoretical [57–61] and phenomenological investigations [62–69], including the recent discovery of safe non-supersymmetric grand unified theories of [70] which naturally integrates and complements the supersymmetric story of [58].

For the issue of symmetry non-restoration at arbitrary high temperatures we consider, at first, the landscape of complete asymptotically free non-abelian gauge theories that feature either two singlet scalars coupled to gauged fermions via Yukawa interactions or two gauged scalars without Yukawa interaction.

The first model we encounter of complete asymptotically free theories for which symmetry breaks at arbitrary high temperatures is constituted by two gauged scalars transforming according to the fundamental representation of two distinct gauge groups with fermions also transforming in the fundamental representation and without Yukawa interactions.

To investigate the high-temperature fate of global symmetries for asymptotically safe theories we consider the Litim-Sannino model of [55] and one of its variations that has been used for perturbative safe extensions of the standard model [71]. We show that for these examples the safe quantum global symmetries are restored at high temperatures.
2 Complete asymptotically free theories at high temperature

Before embarking in our main quest, which is to investigate the symmetry (non)restoration phenomenon for complete asymptotically free quantum field theories we briefly summarise Weinberg’s (non-free) model mechanics. In its most minimal form the model features two scalars with the following quartic potential

\[ V = \frac{\lambda_1}{4} \phi_1^4 + \frac{\lambda_2}{4} \phi_2^4 - \frac{\lambda}{2} \phi_1^2 \phi_2^2 \]  

(2.1)

with discrete \( Z_2 \times Z_2 \) symmetry \( \phi_1 \rightarrow -\phi_1 \) and \( \phi_2 \rightarrow -\phi_2 \). For

\[ \lambda_{1,2} > 0 \quad \lambda^2 < \lambda_1 \lambda_2 \]  

(2.2)

the model is bounded from below. At high temperature the following correction arises: [3]:

\[ \Delta V_T = \frac{T^2}{24} \left( (3\lambda_1 - \lambda)\phi_1^2 + (3\lambda_2 - \lambda)\phi_2^2 \right) . \]  

(2.3)

With \( \lambda > 3\lambda_2 \) (but with \( \lambda_1 \) satisfying (2.2)) the field \( \phi_2 \) acquires a negative thermal mass squared at high temperature which yields a non-zero vev \( \langle \phi_2 \rangle \neq 0 \). Therefore in this case the second \( Z_2 \) breaks at sufficiently high temperatures. This theory is, however, not UV complete since the scalar couplings increase with the energy. Assuming a physical cutoff, for temperatures below this cutoff one therefore observes the phenomenon of symmetry non-restoration.

Because the theory is limited by a physical cutoff we cannot ask the relevant question of whether the symmetry remains broken at arbitrary high temperatures. This is exactly what our work wishes to achieve, i.e. what is the ultimate fate of the symmetry in a truly UV complete theory (up to gravity) at arbitrary large temperatures.

Here we analyse complete asymptotically free theories that are natural UV completions of the Weinberg’s model. These require the presence of gauge fields and the gauge sector to be asymptotically free given that it is this sector the one responsible to drive the Yukawa and scalar couplings to be asymptotically free as well.

We divide our theories in whether they feature gauge singlets or gauged scalars.

2.1 Symmetry restoration with singlet scalars

To start with we consider an \( SU(N_c) \) gauge group with \( N_f = N_{f_1} + N_{f_2} \) Dirac fermions in the fundamental representation coupled to the scalars \( \phi_{1,2} \) via the following \( Z_2 \times Z_2 \) symmetry (\( \phi_k \rightarrow -\phi_k, \psi_k \rightarrow i\gamma_5 \psi_k \)) preserving Yukawa terms:

\[ \mathcal{L}_Y = \phi_1 \sum_{i=1}^{N_{f_1}} y_{1i} \bar{\psi}_{1i} \psi_{1i} + \phi_2 \sum_{i=1}^{N_{f_2}} y_{2i} \bar{\psi}_{2i} \psi_{2i} \]  

(2.4)

Because we are searching for asymptotically free solutions we must have that \( \alpha_g \propto 1/t \) for large \( t = \log (\mu/\mu_0) \) with \( \mu \) the renormalisation scale and \( \mu_0 \) a reference scale. Complete asymptotic freedom requires that all couplings must vanish at infinity at least as fast as \( \alpha_g \) and therefore their scaling must be proportional to \( 1/t^a \) with \( a \geq 1 \). Additionally the requirement of a negative thermal mass given in (A.19), necessary for symmetry breaking,
implies that at least some scalar quartic couplings cannot decrease faster than the gauge
coupling, i.e. they must approach zero as $1/t^b$ with $b \leq 1$. Therefore, for the purpose
of our work, it is sufficient to investigate the fixed flow solution according to which all
couplings vanish at infinity as $1/t$ [72]. This observation greatly simplifies the following
analyses by transforming a set of non-linear and coupled first order ordinary differential
equations into a system of non-linear and coupled polynomial equations. In practice, by
defining ($g$ is the gauge coupling)

$$
\alpha_g = \frac{g^2}{(4\pi)^2}, \quad \alpha_{y_i} = \frac{y_i^2}{(4\pi)^2}, \quad \alpha_{\lambda_i} = \frac{\lambda_i}{(4\pi)^2}, \quad \alpha_{\lambda} = \frac{\lambda}{(4\pi)^2}
$$

(i = 1, 2) we will search for solutions of the asymptotic form

$$
\alpha_a = \frac{\tilde{\alpha}_a}{t}, \quad a = g, y_1, y_2, \lambda_1, \lambda_2, \lambda, \ldots
$$

with constant $\tilde{\alpha}_a$.

We are now ready to investigate the first relevant examples with singlets scalars and
then we will generalise the results to a wider class of theories.

2.1.1 $SU(N_c)$ with two singlet scalars and fundamental fermions

In this model, described in detail in Appendix B, we consider two singlet scalars coupled
through Yukawa interactions to $N_{f_1}$ ($N_{f_2}$) Dirac fermions in the fundamental representation
of $SU(N_c)$. We further allow for $N_{f_0}$ Dirac fermions in the fundamental representation of the
gauge group that are inert with respect to the scalars, i.e. do not possess Yukawa

couplings.

We now provide an elegant proof that at high temperature this theory, if complete
asymptotically free, cannot break any symmetry. Let us start with the thermal masses for
the scalars that at one loop read (B.18):

$$
m_i^2(T) = (4\pi)^2 \frac{T^2}{12 \log T} (3\tilde{\alpha}_{\lambda_i} - \tilde{\alpha}_\lambda + 2N_cN_{f_1}\tilde{\alpha}_{y_i}) ,
$$

written in terms of (2.6) couplings. It is sufficient to consider one of the two scalar masses
to be negative. Here we choose to be $m_1^2$ that requires

$$
\tilde{\alpha}_\lambda - 2N_cN_{f_1}\tilde{\alpha}_{y_1} > 3\tilde{\alpha}_{\lambda_1} > 0 .
$$

Under the assumption that there is a complete asymptotically free solution we have (B.14)

$$
-\tilde{\alpha}_{\lambda_1} = 18\tilde{\alpha}_{\lambda_1}^2 + 2\tilde{\alpha}_{y_1}^2 - 8N_cN_{f_1}\tilde{\alpha}_{y_1}^2 + 8N_cN_{f_1}\tilde{\alpha}_{y_1}\tilde{\alpha}_{\lambda_1} ,
$$

for the relevant scalar coupling as function of the other couplings. The general form of the
RGE equations can be found in the Appendix B.

Rewriting (2.9) as

$$
2 (\tilde{\alpha}_{\lambda_1}^2 - 4N_cN_{f_1}\tilde{\alpha}_{y_1}^2) + \tilde{\alpha}_{\lambda_i} + 18\tilde{\alpha}_{\lambda_1}^2 + 8N_cN_{f_1}\tilde{\alpha}_{y_1}\tilde{\alpha}_{\lambda_1} = 0 ,
$$

we notice that every term except the first one is positive. This means that to satisfy this
equation, the first term must be negative for the fixed flow solution to be possible. However,
since the first term can be rewritten as

$$
\tilde{\alpha}_{\lambda_1}^2 - 4N_cN_{f_1}\tilde{\alpha}_{y_1}^2 = (\tilde{\alpha}_\lambda - 2N_cN_{f_1}\tilde{\alpha}_{y_1}) (\tilde{\alpha}_\lambda + 2N_cN_{f_1}\tilde{\alpha}_{y_1}) + 4N_cN_{f_1} (N_cN_{f_1} - 1) \tilde{\alpha}_{y_1}^2 ,
$$
the simultaneous requirement of the presence of a negative mass squared term implies that also the first term is positive due to (2.8).

We have therefore shown that (2.10) cannot have a solution and that the symmetry must be restored for this model at high temperature once complete asymptotic freedom is enforced.

2.1.2 More general result for singlet scalars

Let us consider the more general scalar potential

\[ V = \frac{\lambda}{4} (\phi^T \phi)^2 - \frac{1}{2} (\phi^T \phi) \eta_{ij} \chi^i \chi^j + V(\chi) \]  (2.12)

where \( \phi \) is a real vector with \( d_\phi \) components. The global symmetry at the potential level over \( \phi \) is \( O(d_\phi) \). Under this group \( \phi \) transforms with a \( d_\phi \times d_\phi \) orthogonal matrix \( O \) of \( O(d_\phi) \) as:

\[ \phi' = O \phi \]  (2.13)

We further consider an arbitrary gauge group with Weyl fermions transforming according to an arbitrary gauge representation compatible with asymptotic freedom [73]. The Yukawa terms written directly in terms of the Weyl fermions read:

\[ L_{Yukawa} = \frac{1}{2} \bar{\psi}_i Y_a^{ij} \psi_j + h.c. + L_{Yukawa}(\chi, \psi') \]  (2.14)

Under the assumption that

\[ \psi' = U \psi , \quad \tilde{O}_a^b U_{ki} Y_{kl}^b U_{lj} = Y_{ij}^a \]  (2.15)

with \( \tilde{O} \) a rotation matrix that is part of a subgroup of \( O(d_\phi) \) and \( U \) a unitary transformation with \( i, j, k = 1, \ldots, N_f \) with \( N_f \) the number of Weyl matter fields, the Yukawa terms preserves the resulting subgroup of \( O(d_\phi) \). The information on which fermions couple to \( \phi \) is clearly hidden in the Yukawa matrix. The last unspecified Yukawa terms in (2.14) contain interactions of the \( \chi \) scalar fields with the Weyl fermions \( \psi' \) that are not coupled to \( \phi \). We now show that the thermal mass of \( \phi \) cannot be negative at high temperatures when the theory is required to be asymptotically free also in all couplings.

Let us consider the thermal mass

\[ m_{\phi}^2(T) = \frac{(4\pi)^2}{12 \log T} \left( (d_\phi + 2) \tilde{\lambda} - \tilde{\eta}_{kk} + Tr \left( \tilde{Y}_1^a \tilde{Y}_1^a \right) \right) \]  (2.16)

where we used

\[ Tr \left( Y^{a\dagger} Y^b \right) = \delta^{ab} Tr \left( \tilde{Y}_1^a \tilde{Y}_1^a \right) \]  (2.17)

and defined as usual

\[ \lambda = (4\pi)^2 \tilde{\lambda} / t , \quad \eta_{ij} = (4\pi)^2 \tilde{\eta}_{ij} / t \]  (2.18)

with \( \tilde{\lambda}, \tilde{\eta}_{ij}, \tilde{Y}^a \) constants.

In order not to restore the symmetry carried by the potential term and the Yukawa relative to \( \phi \) the thermal mass (2.16) must be negative. This implies:
\[ \tilde{\eta}_{kk} - Tr \left( \tilde{Y}_1^\dagger \tilde{Y}_1 \right) > (d_\phi + 2) \tilde{\lambda} > 0 \right. \quad (2.19) \]

Let us now compute the RGE for \( \tilde{\lambda} \) relative to achieving the fixed flow solution:

\[ 2 \left( \tilde{\eta}_{ij} \tilde{\eta}_{ij} - 2Tr \left( \tilde{Y}_1^\dagger \tilde{Y}_1 \right) \right) + 2(d_\phi + 8) \tilde{\lambda}^2 + \tilde{\lambda} + 4\lambda Tr \left( \tilde{Y}_1^\dagger \tilde{Y}_1 \right) = 0 \right. , \quad (2.20) \]

where we used

\[ Tr \left( Y^{a\dagger} Y^b Y^c Y^d \right) = A\delta_{ab}\delta_{cd} + B\delta_{ac}\delta_{bd} + C\delta_{ad}\delta_{bc} , \quad (2.21) \]

which follows from the symmetry properties of the Yukawa matrices (2.15).

To obtain a solution, the first term must be negative (all the others are positive). However, we have

\[ \tilde{\eta}_{ij} \tilde{\eta}_{ij} - 2Tr \left( \tilde{Y}_1^\dagger \tilde{Y}_1 \right) = \tilde{\eta}_{kk} - Tr \left( \tilde{Y}_1^\dagger \tilde{Y}_1 \right) \left( \tilde{\eta}_{kk} + Tr \left( \tilde{Y}_1^\dagger \tilde{Y}_1 \right) \right) \]

\[ + 2 \sum_{i<j} \tilde{\eta}_{ij} \tilde{\eta}_{ij} + \left( \left( Tr \left( \tilde{Y}_1^\dagger \tilde{Y}_1 \right) \right)^2 - 2Tr \left( \tilde{Y}_1^\dagger \tilde{Y}_1 \right)^2 \right) . \quad (2.22) \]

The first term on the right-hand-side is positive due to the assumption of the occurrence of a negative thermal mass squared (2.19), therefore the only possible negative term could be the last one. Since the above traces are invariant under unitary rotations of the Hermitian matrix \( \tilde{Y}_1^\dagger \tilde{Y}_1 \), we are free to consider the basis with diagonal

\[ \left( \tilde{Y}_1^\dagger \tilde{Y}_1 \right)_{ij} = \tilde{y}_{1i}^2 \delta_{ij} \quad (2.23) \]

so that (2.22) becomes

\[ \left( Tr \left( \tilde{Y}_1^\dagger \tilde{Y}_1 \right) \right)^2 - 2Tr \left( \tilde{Y}_1^\dagger \tilde{Y}_1 \right)^2 = \sum_{\mu} \dim \left( R_{\mu} \right) \left( \dim \left( R_{\mu}^1 \right) - 2 \right) \tilde{y}_{1\mu}^4 \]

\[ + 2 \sum_{\mu<\mu'} \dim \left( R_{\mu} \right) \dim \left( R_{\mu'}^1 \right) \tilde{y}_{1\mu}^2 \tilde{y}_{1\mu'}^2 \quad (2.24) \]

with \( \mu \) and \( \mu' \) running over the fermion representations. For non gauge singlet fermions we have \( \dim \left( R_{\mu} \right) \geq 2 \) and therefore the right hand side is positive. For gauge singlet fermions the only solution compatible with a UV well defined theory is the one for which the Yukawa coupling vanishes identically and therefore the previous equation does not apply.

Therefore there is no solution to the RGE for \( \tilde{\lambda} \). Or, in other words, if a fixed flow solution exists, it cannot have a negative thermal mass. The previous example with a \( Z_2 \) symmetry is included here by assuming the original symmetry to be simply a \( Z_2 \) for \( d_\phi = 1 \).

### 2.2 Exploring symmetry non-restoration with gauged scalars

So far we have shown that a great deal of gauge theories with scalar gauge singlets do not support symmetry non-restoration at arbitrary high temperatures. Does this phenomenon persists when considering gauged scalar fields? This is the question we will answer in
this section. We will find an example with the opposite behaviour, i.e. we will explicitly present a theory featuring two different gauge groups displaying simultaneously complete asymptotic freedom and symmetry non-restoration.

To motivate the introduction of a second gauge group we will first show that with a single gauge group symmetries will restore at arbitrary high temperatures with(out) fermionic matter fields.

Although the models in this section share some features with the ones investigated in [74] the main difference resides in the fact that we are interested in symmetry non-restoration at arbitrary high temperatures. This means that we investigate theories near their UV fixed point, while in [74] the authors concentrate on symmetry non-restoration occurring near interacting IR-fixed points.

2.2.1 $SU(N_c)$ with $N_s$ fundamental scalars

The $SU(N_c) \times SU(N_f) \times SU(N_s)$ symmetric Lagrangian is

$$
\mathcal{L} = -\frac{1}{2} Tr F_{\mu\nu} F^{\mu\nu} + Tr (\bar{Q} i D Q) + Tr (D^\mu S D_\mu S^\dagger) - v (Tr SS^\dagger)^2 - u Tr (SS^\dagger)^2
$$

(2.25)

with the fields transforming as

$$Q \sim (N_c, N_f, 1) \quad , \quad S \sim (N_c, 1, N_s)
$$

(2.26)

The scalar thermal mass at high temperature is

$$m_S^2(T) = (4\pi)^2 \frac{T^2}{24 \log T} \left(4(N_s N_c + 1)\tilde{\lambda}_1 + 4(N_s + N_c)\tilde{\lambda}_2 + 3\frac{N_c^2 - 1}{N_c}\tilde{\alpha}\right)
$$

(2.27)

where we introduced, following [75],

$$v = (4\pi)^2 \frac{\tilde{\lambda}_1}{t} \quad , \quad u = (4\pi)^2 \frac{\tilde{\lambda}_2}{t} \quad , \quad g^2 = (4\pi)^2 \frac{\tilde{\alpha}}{t}
$$

(2.28)

with constant $\tilde{\lambda}_{1,2}, \tilde{\alpha}$.

The positivity of (2.27) follows from boundedness arguments. In fact The $T = 0$ potential is bounded from below iff [75]

$$\tilde{\lambda}_2 \geq 0 \quad : \quad N_s\tilde{\lambda}_1 + \tilde{\lambda}_2 \geq 0
$$

(2.29)

$$\tilde{\lambda}_2 \leq 0 \quad : \quad \tilde{\lambda}_1 + \tilde{\lambda}_2 \leq 0
$$

(2.30)

Since

1. $\tilde{\lambda}_2 \geq 0$:

$$
(N_s N_c + 1)\tilde{\lambda}_1 + (N_s + N_c)\tilde{\lambda}_2 = \frac{1}{N_s} (N_s N_c + 1) \left( N_s\tilde{\lambda}_2 + \tilde{\lambda}_2 \right) + \left( N_c - \frac{1}{N_s} \right) \tilde{\lambda}_2 \geq 0
$$

(2.31)

\[\text{If } N_c = N_s = 4 \text{ one can add to the potential a new invariant } w \det S + w^* \det S^\dagger.\]
2. \( \bar{\lambda}_2 \geq 0: \)

\[
(N_s N_c + 1) \bar{\lambda}_1 + (N_s + N_c) \bar{\lambda}_2 = (N_s N_c + 1) \left( \bar{\lambda}_1 + \bar{\lambda}_2 \right) + (N_s - 1) (N_c - 1) \left| \bar{\lambda}_2 \right| \geq 0
\]

we can now conclude that the thermal mass is always positive

\[
m_2^2(T) > 0
\]

i.e. the symmetry is restored at high temperature.

2.2.2 \( SU(N_c) \) with two fundamental scalars

One of the problems of the previous model was that there was too much symmetry in the scalar potential. We now take the case of two scalars, \( N_s = 2 \), but instead of scalar \( SU(2) \) the symmetry of the potential will be just a discrete symmetry. We can take either a single \( Z_2 \) for even \( N_c = 2n \) or \( Z_2 \times Z_2 \) for odd \( N_c = 2n + 1 \).

In fact:

- \( Z_2 \subset Z_{2n} \) and for even \( N_c = 2n \) the center of \( SU(N_c) \) is \( Z_{2n} \) and so \( Z_2 \subset SU(N_c) \).
  - In other words, a common \( \vec{\varphi}_i \rightarrow -\vec{\varphi}_i \) is already present. So in this case there is only one extra \( Z_2 \) possible, say \( \vec{\varphi}_1 \rightarrow -\vec{\varphi}_1 \).
- for odd \( N_c = 2n + 1 \) there is no \( Z_2 \) subgroup of \( SU(N_c) \). In fact using the Levi-Civita tensor the invariant out of \( N_c = 2n + 1 \) fundamentals is possible. Here it is thus possible to have an extra \( Z_2 \times Z_2 \) symmetry for two fundamentals.

One way or another this means that each term of the potential can have only an even number of fundamentals \( \vec{\varphi}_2 \) and anti-fundamentals \( \vec{\varphi}_1^\ast \) and an even number of fundamentals \( \vec{\varphi}_2 \) and anti-fundamentals \( \vec{\varphi}_1^\ast \):

\[
V = \frac{\lambda_1}{2} (\vec{\varphi}_1^\ast \cdot \vec{\varphi}_1)^2 + \frac{\lambda_2}{2} (\vec{\varphi}_2^\ast \cdot \vec{\varphi}_2)^2 + \lambda_3 (\vec{\varphi}_1^\ast \cdot \vec{\varphi}_1)(\vec{\varphi}_2^\ast \cdot \vec{\varphi}_2) + \lambda_4 (\vec{\varphi}_1^\ast \cdot \vec{\varphi}_2)(\vec{\varphi}_2^\ast \cdot \vec{\varphi}_1) + \frac{\lambda_5}{2} (\vec{\varphi}_1^\ast \cdot \vec{\varphi}_1)^2 + \frac{\lambda_5^\ast}{2} (\vec{\varphi}_2^\ast \cdot \vec{\varphi}_2)^2
\]

with \( \lambda_{1,2,3,4} \) real and in general \( \lambda_5 \) complex.

By taking

\[
g^2 = \frac{16\pi^2 \tilde{\alpha}}{N_c t}, \quad \lambda_i = \frac{16\pi^2 \tilde{\lambda}_i}{N_c t}
\]

with constant \( \tilde{\alpha}, \tilde{\lambda}_i \), and

\[
\tilde{\lambda}_\pm = \frac{1}{2} \left( \tilde{\lambda}_1 \pm \tilde{\lambda}_2 \right)
\]

the solutions to the RGE (see the appendix C.3) are

\[
\tilde{\lambda}_+ = \frac{6\tilde{\alpha} - 1}{4}, \quad \tilde{\lambda}_3^2 + \tilde{\lambda}_5^2 = \frac{24\tilde{\alpha}^2 - 12\tilde{\alpha} + 1}{16}
\]
\[ \tilde{\lambda}_+ = \frac{6\tilde{\alpha} - 1 + a_+\sqrt{24\tilde{\alpha}^2 - 12\tilde{\alpha} + 1}}{4}, \quad a_+^2 = 1, \quad \tilde{\lambda}_3 = \tilde{\lambda}_- = 0 \] (2.38)

acceptable only for \( \tilde{\alpha} \geq (3 + \sqrt{3})/12 \).

The thermal potential at large \( N_c \) is

\[ \Delta V_T = (4\pi)^2 \frac{T^2}{24\log T} \left( 2 \left( \tilde{\lambda}_1 + \tilde{\lambda}_3 \right) (\vec{\varphi}_1^* \cdot \vec{\varphi}_1) + 2 \left( \tilde{\lambda}_2 + \tilde{\lambda}_3 \right) (\vec{\varphi}_2^* \cdot \vec{\varphi}_2) \right) \] (2.39)

The masses are quite symmetric and the search for symmetry restoration boils down to look for negative \( \tilde{\lambda}_3 = -|\tilde{\lambda}_3| \) which leads to a negative mass square for \( \tilde{\varphi}_1 \), i.e. a negative combination

\[ \frac{12\tilde{\alpha} - 1}{2} - 2 \left( \sqrt{\frac{24\tilde{\alpha}^2 - 12\tilde{\alpha} + 1}{16}} - |\tilde{\lambda}_3|^2 + |\tilde{\lambda}_3| \right) \] (2.40)

for

\[ \tilde{\alpha} \geq \frac{3 + \sqrt{3}}{12}, \quad 0 \leq |\tilde{\lambda}_3| \leq \sqrt{\frac{24\tilde{\alpha}^2 - a2\tilde{\alpha} + 1}{16}} \] (2.41)

The function (2.40) is minimised for

\[ |\tilde{\lambda}_3| = \frac{1}{\sqrt{2}} \sqrt{\frac{24\tilde{\alpha}^2 - 12\tilde{\alpha} + 1}{16}} \] (2.42)

which is however not enough for a negative mass square.

**2.2.3 SU(\( N_c_1 \)) \times SU(\( N_c_2 \)) with fundamental scalars:**

**Symmetry breaks at high temperatures**

The model we will study now is similar to the previous one, but now we have two simple groups, \( SU(N_c_1) \times SU(N_c_2) \), so that each \( \varphi_i \) is in a fundamentals representation of its \( SU(N_c_i) \) and a singlet under the other one. The most general potential is

\[ V = \frac{\lambda_1}{2} (\vec{\varphi}_1^* \cdot \vec{\varphi}_1)^2 + \frac{\lambda_2}{2} (\vec{\varphi}_2^* \cdot \vec{\varphi}_2)^2 - \lambda (\vec{\varphi}_1^* \cdot \vec{\varphi}_1) (\vec{\varphi}_2^* \cdot \vec{\varphi}_2) \] (2.43)

Defining first

\[ i = 1, 2 : \quad g_i^2 = \frac{16\pi^2\tilde{\alpha}_i}{N_{c_i}t}, \quad \lambda_i = \frac{16\pi^2\tilde{\lambda}_i}{N_{c_i}t} \] (2.44)

\[ \lambda = \frac{16\pi^2\tilde{\lambda}}{\sqrt{N_{c_1}N_{c_2}t}} \] (2.45)

with constant \( \tilde{\alpha}_i, \tilde{\lambda}_i, \tilde{\lambda} \), the thermal effective potential becomes at large \( N_c \) reads
\[
\Delta V_T = (4\pi)^2 \frac{T^2}{24 \log T} \left( \left( 2 \left( \tilde{\lambda}_1 - \sqrt{\frac{N_{c2}}{N_{c1}}} \tilde{\lambda} \right) + 3\tilde{\alpha}_1 \right) (\varphi_1^* \cdot \varphi_1) \\
+ \left( 2 \left( \tilde{\lambda}_2 - \sqrt{\frac{N_{c1}}{N_{c2}}} \tilde{\lambda} \right) + 3\tilde{\alpha}_2 \right) (\varphi_2^* \cdot \varphi_2) \right)
\] (2.46)

Introducing the new variables
\[
\tilde{\lambda}_\pm = \frac{1}{2} (\tilde{\lambda}_1 \pm \tilde{\lambda}_2), \quad \tilde{\alpha}_\pm = \frac{1}{2} (\tilde{\alpha}_1 \pm \tilde{\alpha}_2)
\]
(2.47)
on one finds the following solution\(^5\) of the RGE (see the appendix C.4):
\[
\tilde{\alpha}_- = 0 \quad (2.48) \\
\tilde{\lambda}_+ = \frac{6\tilde{\alpha}_+ - 1}{4} \quad (2.49) \\
\tilde{\lambda}_+^2 + \tilde{\lambda}_-^2 = \frac{1}{16} (24\tilde{\alpha}_+^2 - 12\tilde{\alpha}_+ + 1) \quad (2.50)
\]
valid for
\[
\tilde{\alpha}_+ \geq (3 + \sqrt{3})/12 \quad (2.51)
\]

5We will now prove that this solution supports symmetry non-restoration at arbitrary high temperatures.

Denoting by \(\mu_1^2\) the coefficient in front of \((\varphi_i^* \cdot \varphi_i)\) in the parenthesis on the right-hand-side of (2.46) we have
\[
\mu_1^2 = \frac{12\tilde{\alpha}_+ - 1}{2} + 2\tilde{\lambda}_- - 2\sqrt{\frac{N_{c2}}{N_{c1}}} \tilde{\lambda} \quad (2.52) \\
\mu_2^2 = \frac{12\tilde{\alpha}_+ - 1}{2} - 2\tilde{\lambda}_- - 2\sqrt{\frac{N_{c1}}{N_{c2}}} \tilde{\lambda} \quad (2.53)
\]
We are searching for positive
\[
\tilde{\lambda} = \left| \tilde{\lambda} \right| \quad (2.54)
\]
and, up to redefinitions of what is 1 and what is 2, we can take
\[
\tilde{\lambda}_- = -\sqrt{\frac{24\tilde{\alpha}_+^2 - 12\tilde{\alpha}_+ + 1}{16} - \left| \tilde{\lambda} \right|^2} \quad (2.55)
\]
so that
\[
\mu_1^2 = \frac{12\tilde{\alpha}_+ - 1}{2} - 2 \left( \sqrt{\frac{24\tilde{\alpha}_+^2 - 12\tilde{\alpha}_+ + 1}{16} - \left| \tilde{\lambda} \right|^2} + \sqrt{\frac{N_{c2}}{N_{c1}}} \left| \tilde{\lambda} \right| \right) \quad (2.56) \\
\mu_2^2 = \frac{12\tilde{\alpha}_+ - 1}{2} + 2 \left( \sqrt{\frac{24\tilde{\alpha}_+^2 - 12\tilde{\alpha}_+ + 1}{16} - \left| \tilde{\lambda} \right|^2} - \sqrt{\frac{N_{c1}}{N_{c2}}} \left| \tilde{\lambda} \right| \right) \quad (2.57)
\]

5The other possible solution \(\tilde{\alpha}_+ = 1/4, \tilde{\lambda}_+ = \frac{1}{8}, \left(\tilde{\lambda}_- - \frac{\sqrt{3}}{4} \tilde{\alpha}_-\right)^2 + \tilde{\lambda}_+^2 = \frac{1}{12} (48\tilde{\alpha}_-^2 - 1)\) describes a \(T = 0\) potential which is unbounded from below.
Minimising the expression for $\mu_1^2$ we obtain:

$$|\tilde{\lambda}|^2 = \frac{1 + \frac{24\tilde{\alpha}_+^2 - 12\tilde{\alpha}_+ + 1}{16}}{1 + \frac{N_{c2}}{N_{c1}}}$$

The minimised mass parameter

$$\mu_1^2 = \frac{12\tilde{\alpha}_+ - 1}{2} - 2\sqrt{\frac{24\tilde{\alpha}_+^2 - 12\tilde{\alpha}_+ + 1}{16}} \sqrt{1 + \frac{N_{c2}}{N_{c1}}}$$

can now be negative by a suitable choice of number of colours.

Let us now demonstrate that the previous solution leads to a bounded potential. The latter occurs if

$$\lambda_1\lambda_2 - \lambda^2 > 0$$

which can be rewritten first as

$$\tilde{\lambda}_+^2 - \tilde{\lambda}_-^2 - \tilde{\lambda}^2 > 0$$

and then as

$$\frac{(6\tilde{\alpha}_+ - 1)^2}{16} - \frac{24\tilde{\alpha}_+^2 - 12\tilde{\alpha}_+ + 1}{16} = \frac{12\tilde{\alpha}_+^2}{16}$$

which is indeed positive.

Finally, requiring equal gauge couplings in the large $N_{ci}$ limit

$$\tilde{\alpha}_1 = \tilde{\alpha}_2$$

means that the original not rescaled couplings satisfy the relation

$$N_{c1}g_1^2 = N_{c2}g_2^2$$

This is achieved by the following suitable choice of number of matter fermions:

$$\frac{N_{f1}}{N_{c1}} = \frac{N_{f2}}{N_{c2}}$$

Because of (2.51), they must satisfy

$$\frac{1}{2} \left( 2 + 3\sqrt{3} \right) \leq \frac{N_{f1}}{N_{c1}} < \frac{11}{2}$$

We arrive at the result, similar to Weinberg’s model, that only one thermal mass is negative.

### 2.2.4 The IR story of $SU(N_{c1}) \times SU(N_{c2})$ at nonzero temperature

Interestingly the model of the previous subsection can feature also an IR Banks-Zaks fixed point. This can be achieved by tuning the number of fermions to maintain both gauge
couplings equality and the occurrence of a perturbative IR fixed point. Once this is achieved the remaining equations for the IR fixed point values\textsuperscript{6} are:

\begin{align}
0 &= 2\lambda_1^2 + 2\lambda^2 - 6\alpha_1\lambda_1 + \frac{3}{2}\alpha_1^2 \\
0 &= 2\lambda_2^2 + 2\lambda^2 - 6\alpha_2\lambda_2 + \frac{3}{2}\alpha_2^2 \\
0 &= 2(\lambda_1 + \lambda_2)\lambda - 3(\alpha_1 + \alpha_2)\lambda
\end{align}

(2.67) \quad (2.68) \quad (2.69)

The solutions are

\begin{align}
\lambda_+ &= \frac{3}{2}\alpha_+ \\
\lambda_- + \lambda^2 &= \frac{3}{2}\alpha_+^2
\end{align}

(2.70) \quad (2.71)

if and only if

\[\alpha_- = 0\]

(2.72)

The thermal effective potential is

\[\Delta V_T = \frac{T^2}{24} \times \left( 2 \left( \lambda_1 - \sqrt{\frac{N_{c_2}}{N_{c_1}}}\lambda \right) + 3\alpha_+ \right) (\vec{\varphi}_1^* \cdot \vec{\varphi}_1) \]

\[+ \left( 2 \left( \lambda_2 - \sqrt{\frac{N_{c_1}}{N_{c_2}}}\lambda \right) + 3\alpha_+ \right) (\vec{\varphi}_2^* \cdot \vec{\varphi}_2)\]

(2.73)

so that the thermal masses are proportional to

\begin{align}
\mu_1^2 &= 6\alpha_+ + 2\lambda_- - 2\sqrt{\frac{N_{c_2}}{N_{c_1}}}\lambda \\
\mu_2^2 &= 6\alpha_+ - 2\lambda_- - 2\sqrt{\frac{N_{c_1}}{N_{c_2}}}\lambda
\end{align}

(2.74) \quad (2.75)

Searching again for the branch

\[\lambda = |\lambda|, \quad \lambda_- = -\sqrt{\frac{3}{2}\alpha_+^2 - |\lambda|^2}\]

(2.76)

we have first

\begin{align}
\mu_1^2 &= 6\alpha_+ - 2\left( \sqrt{\frac{3}{2}\alpha_+^2 - |\lambda|^2} + \sqrt{\frac{N_{c_2}}{N_{c_1}}} |\lambda| \right) \\
\mu_2^2 &= 6\alpha_+ + 2\left( \sqrt{\frac{3}{2}\alpha_+^2 - |\lambda|^2} - \sqrt{\frac{N_{c_1}}{N_{c_2}}} |\lambda| \right)
\end{align}

(2.77) \quad (2.78)

\textsuperscript{6}N.B. These values should not to be confused with the tilded $1/t$ coefficients used for the fixed flow solutions in the UV.
\( \mu_1^2 \) is minimised for
\[
|\lambda|^2 = \frac{1}{1 + N_{c1}/N_{c2}} \frac{3}{2} \alpha_+^2
\] (2.79)
so that the thermal mass
\[
\mu_1^2 = 6\alpha_+ - 2\sqrt{\frac{3}{2}} \alpha_+ \sqrt{1 + \frac{N_{c2}}{N_{c1}}}
\] (2.80)
is negative for (but still in the Veneziano limit \( N_{ci} \to \infty \))
\[
\frac{N_{c2}}{N_{c1}} > 5
\] (2.81)

Since
\[
\lambda_+^2 - \lambda_-^2 - \lambda^2 = \frac{3}{4} \alpha_+^2 > 0
\] (2.82)
the parameter choice describes a \( T = 0 \) potential which is bounded from below.

We have therefore found an example in which symmetry non-restoration occurs near an IR fixed point which is more minimal than the one presented in [74].

2.2.5 Another example of symmetry breaking at high \( T \): two adjoints in \( SU(N_{c1}) \times SU(N_{c2}) \)

This model is similar to the previous one, except that adjoint scalars are considered instead of fundamental scalars. The details are described in appendix C.5. The most general quartic potential is
\[
V = \frac{\lambda_1}{4} Tr \Sigma_1^4 + \frac{\lambda_2}{4} Tr \Sigma_2^4 + \frac{\lambda_1}{4} (Tr \Sigma_1^2)^2 + \frac{\lambda_2}{4} (Tr \Sigma_2^2)^2 - \frac{\lambda}{2} Tr \Sigma_1^2 Tr \Sigma_2^2
\] (2.83)

We redefine the couplings as
\[
\lambda_{1,2} = (4\pi)^2 \frac{\lambda_{1,2}}{N_{c1,2}} \times \frac{1}{t}, \quad \lambda_1,2 = (4\pi)^2 \frac{\lambda_{1,2}}{N_{c1,2}} \times \frac{1}{t}
\] (2.84)
\[
\lambda = (4\pi)^2 \frac{\lambda}{N_{c1}N_{c2}} \times \frac{1}{t}, \quad g_{1,2}^2 = (4\pi)^2 \frac{\lambda_{1,2}}{N_{c1,2}} \times \frac{1}{t}
\] (2.85)

with all tilded quantities constants, and eventually we will take the large \( N_{c1,2} \) limit.

As shown in [75], the potential (2.83) is bounded from below if the parameters satisfy the following inequalities:
\[
\lambda_i + \frac{\lambda_i}{k_i} > 0 \quad (1 \leq k_i \leq N_{ci}), \quad \left( \lambda_1 + \frac{\lambda_1}{k_1} \right) \left( \lambda_2 + \frac{\lambda_2}{k_2} \right) > \lambda^2
\] (2.86)

If \( \lambda_i' > 0 \), then it is enough to check the above for \( k_i = N_{ci} \), while if \( \lambda_i' < 0 \), \( k_i = 1 \) suffices. However, for large \( N_{ci} \), the second case is impossible, since
\[
\lambda_i + \lambda_i' > 0 \rightarrow \lambda_i' > 0
\] (2.87)
which is in contradiction with the original assumption of $\lambda_i' < 0$.

So the only possibility is just $\lambda_1 > 0$, $\lambda_2 > 0$:

$$\hat{\lambda}_1 + \hat{\lambda}_1' > 0 \ , \ \hat{\lambda}_2 + \hat{\lambda}_2' > 0 \ , \ (\hat{\lambda}_1 + \hat{\lambda}_1') (\hat{\lambda}_2 + \hat{\lambda}_2') > \hat{\lambda}^2$$  \hspace{1cm} (2.88)

The thermal mass is

$$V_T = (4\pi)^2 \frac{T^2}{48 \log T} \left( \left( \frac{\hat{\lambda}_1 + 2\hat{\lambda}_1'}{N_{c_2}} \hat{\lambda} + 12\hat{\alpha}_1 \right) Tr \Sigma_1^2 + \left( \frac{\hat{\lambda}_2 + 2\hat{\lambda}_2'}{N_{c_2}} \hat{\lambda} + 12\hat{\alpha}_2 \right) Tr \Sigma_2^2 \right)$$  \hspace{1cm} (2.89)

We provide here an existence proof for a negative thermal mass with parameters satisfying the boundedness of the potential constraint.

First one can show that only one sector would not work, as expected. This means that if $\hat{\lambda} = \hat{\alpha}_1 = \hat{\lambda}_2 = \hat{\lambda}_2' = 0$, there is no solution of the above fixed flow RG equations for real $\hat{\alpha}_1$, $\hat{\lambda}_1$, $\hat{\lambda}_1'$ assuming $\hat{\lambda}_1 + \hat{\lambda}_1' > 0$ (boundedness) and $\hat{\lambda}_1 + 2\hat{\lambda}_1' < 0$ (negative thermal mass).

However, a solution for bounded potential with negative thermal mass square exists for

$$\hat{\alpha}_1 = \hat{\alpha}_2 = \frac{2 + \sqrt{2}}{2} \hspace{1cm} (2.90)$$

$$\hat{\lambda}_1' = \hat{\lambda}_2' = 2 \hspace{1cm} (2.91)$$

$$\hat{\lambda}_1 = 12 \left( 2 + \sqrt{2} \right) - 26 \hspace{1cm} (2.92)$$

$$\hat{\lambda}_2 = 16 \hspace{1cm} (2.93)$$

$$\hat{\lambda} = \sqrt{120 \left( 2 + \sqrt{2} \right) - 392} \hspace{1cm} (2.94)$$

$$\frac{N_{c_2}}{N_{c_1}} = 16 \hspace{1cm} (2.95)$$

This is therefore another relevant example of symmetry non-restoration at arbitrary high temperature.

3 Asymptotic safety at high temperature

Another way to achieve a UV complete theory, up to gravity, is via the presence of an interacting ultraviolet fixed point in all couplings. In fact, one can have a combination of safe and free couplings for the model to be well defined at all scales.

Due to the fact that the discovery of asymptotically safe quantum field theory is relatively recent [55] the issue of symmetry non-restoration for this relevant class of models has never been investigated before.

We will consider here examples classified according to whether we can re-use part of the results and reasoning employed above for the complete asymptotically free theories or we need a separate in depth analysis of the safe model.

For the first class we consider theories structurally similar to the one considered above albeit with sufficient matter fields such that asymptotic freedom is lost while assuming that perturbative asymptotic safety occurs.

To transform the previous proof valid for asymptotically free theories to the equivalent potential asymptotically safe case we need to
• replace all tilded quantities with untilded ones;
• eliminate the log $T$ in the denominator of the thermal mass;
• replace the $16\pi^2d\alpha_i/dt$ in the left-hand-sides of the RGEs with a zero.

This means that in the theories investigated in the previous section, once asymptotic freedom is lost and potential asymptotic safety appears, symmetry restoration is a must.

3.1 Explicit examples of asymptotic safety

We now consider explicit constructions of asymptotically safe quantum field theories that cannot be reduced to the example above because they either have multiple gauge singlet scalar quartic terms or/and have gauged scalars. Interestingly we anticipate that in both examples the symmetry is restored at high temperature.

3.1.1 The Litim-Sannino (LS) model

The first model we consider here is the one put forward in [55] in which asymptotically safe quantum field theories and their structure was first discovered and understood. The Lagrangian reads:

$$
\mathcal{L} = -\frac{1}{2} Tr \left( F^{\mu\nu} F_{\mu\nu} \right) + Tr \left( \bar{Q} \gamma \partial Q \right) + Tr \left( \partial_{\mu} H^\dagger \partial^\mu H \right) 
+ y Tr \left( \bar{Q}_L H Q_R + \bar{Q}_R H^\dagger Q_L \right) - u Tr \left( H^\dagger H \right)^2 - v \left( Tr H^\dagger H \right)^2 ,
$$

(3.1)

with symmetry

$$
G = SU(N_C) \times SU(N_F) \times SU(N_F) \times U(1) ,
$$

(3.2)

under which the fields transform as

$$
Q_L \sim (N_C, N_F, 1, 1) , \quad (3.3)
$$

$$
Q_R \sim (N_C, 1, N_F, 1) , \quad (3.4)
$$

$$
H \sim (1, N_F, N_F, 0) . \quad (3.5)
$$

We assume the Veneziano limit, needed to ensure the rigorousness of the result

$$
N_F, N_C \rightarrow \infty , \quad \frac{N_F}{N_C} = \frac{11}{2} + \epsilon ,
$$

(3.6)

with $\epsilon \ll 1$ to control the size of the UV fixed point couplings that at the relevant order in perturbation theory read
\[ \alpha_g \equiv \frac{g^2 N_C}{(4\pi)^2} = \frac{26}{57} \epsilon + \mathcal{O}(\epsilon^2) , \quad (3.7) \]

\[ \alpha_y \equiv \frac{y^2 N_C}{(4\pi)^2} = \frac{4}{19} \epsilon + \mathcal{O}(\epsilon^2) , \quad (3.8) \]

\[ \alpha_h \equiv \frac{u N_F}{(4\pi)^2} = \frac{\sqrt{23} - 1}{19} \epsilon + \mathcal{O}(\epsilon^2) , \quad (3.9) \]

\[ \alpha_v \equiv \frac{v N_F^2}{(4\pi)^2} = -\frac{1}{19} \left( 2\sqrt{23} - \sqrt{20 + 6\sqrt{23}} \right) \epsilon + \mathcal{O}(\epsilon^2). \quad (3.10) \]

The \( T^2 \) term of the \( H \) mass square is

\[ m_T^2 = (4\pi)^2 \frac{T^2}{24} (8\alpha_h + 4\alpha_v + 2\alpha_y) \approx 9.7 \epsilon T^2 > 0 , \quad (3.11) \]

so that the symmetry is restored at high temperature. Therefore we arrive at the conclusion that the original model of an asymptotically safe quantum field theory is also safe with respect to global symmetries.

### 3.2 A gauged scalar variant of the LS model

Here we consider an interesting example featuring a two-scalar sector with one of the scalars being gauged while the full theory remains asymptotically safe [71]. This model allows for a relevant test of symmetry (non)restoration and the Lagrangian of the model reads:

\[
\mathcal{L} = -\frac{1}{2} Tr(F^\mu{}^\nu F_{\mu\nu}) + Tr(QiDQ) + Tr(\partial_\mu H^\dagger \partial^\mu H) + Tr(D_\mu \tilde{S}^\dagger D^\mu \tilde{S}) \\
+ \left( \frac{y}{\sqrt{2}} Tr(\tilde{Q}HQ) + h.c. \right) - u_2 Tr(H^\dagger H)^2 - u_1 (TrH^\dagger H)^2 \\
- w_2 Tr(\tilde{S}^\dagger \tilde{S})^2 - w_1 (Tr\tilde{S}^\dagger \tilde{S})^2 ,
\]

where the fields transform under the gauge and 3 global symmetries \( (N_S = N_C - 2) \)

\[ G = SU(N_C) \times SU(N_F)_L \times SU(N_F)_R \times SU(N_S) , \quad (3.13) \]

as

\[ Q \sim (N_C, N_F, 1, 1) , \quad (3.14) \]

\[ \tilde{Q} \sim (\overline{N_C}, 1, \overline{N_F}, 1) , \quad (3.15) \]

\[ H \sim (1, \overline{N_F}, N_F, 1) , \quad (3.16) \]

\[ \tilde{S} \sim (N_C, 1, 1, N_S) . \quad (3.17) \]

For small and positive

\[ \epsilon = \frac{N_F}{N_C} - \frac{11}{2} + \frac{N_S}{4N_C} \rightarrow \frac{N_F}{N_C} - \frac{21}{4} , \quad (3.18) \]
the following relations are satisfied \cite{71} at the UV fixed point:

\[ \alpha_g \equiv \frac{N_C g^2}{(4\pi)^2} = \frac{25}{18} \epsilon, \]  
\[ \alpha_y \equiv \frac{N_C y^2}{(4\pi)^2} = \frac{24}{25} \alpha_g, \]  
\[ \alpha_{u1} \equiv \frac{N^2_2 u_1}{(4\pi)^2} = \frac{-6\sqrt{22} + 3\sqrt{19} + 6\sqrt{22}}{100} \alpha_g, \]  
\[ \alpha_{u2} \equiv \frac{N_F u_2}{(4\pi)^2} = \frac{3}{25} \left( \sqrt{22} - 1 \right) \alpha_g, \]  
\[ \alpha_{w1} \equiv \frac{N^2_2 w_1}{(4\pi)^2} = \frac{3 \pm \sqrt{3 (4\sqrt{2} - 5)}}{16\sqrt{2}} \alpha_g, \]  
\[ \alpha_{w2} \equiv \frac{N_C w_2}{(4\pi)^2} = \frac{1}{16} \left( 2 - \sqrt{2} \right) \alpha_g. \]

Following the analysis of the LS case but now generalised to both scalars we arrive at

\[ m_T^2(H) = (4\pi)^2 \frac{T^2}{48} \left( 2\alpha_y + 16\alpha_{u2} + 8\alpha_{u1} \right) \approx 38.4 \epsilon T^2 > 0, \]  
\[ m_T^2(S) = (4\pi)^2 \frac{T^2}{24} \left( 8\alpha_{w2} + 4\alpha_{w1} + 3\alpha_y \right) \approx 37.2 \epsilon T^2 > 0. \]

This implies that no symmetries can be broken at high temperature.

## 4 Conclusions

In this paper we analysed Weinberg's symmetry non-restoration idea within UV complete theories of either asymptotically free or safe nature.

The reason why these are natural models to investigate is that only for UV complete theories it is consistent to consider the arbitrary large temperature limit.

Safe and free theories share short scale conformality that insures a well defined behaviour at arbitrary high energies. Because of this, they belong to a special subset of all possible quantum field theories. The remaining field theories should be considered as effective low energy descriptions that cannot be complete without quantum gravity possibly modifying their high energy behaviour. In any event, given the fact that we do not yet have a complete theory of quantum gravity, for these theories the symmetry non-restoration test cannot be performed at arbitrary high temperatures.

As complete asymptotically free templates we commenced our investigation with $SU(N_c)$ gauge-Yukawa theories featuring $N_f$ fundamental Dirac fermions and two singlet scalars coupled via Yukawa interactions to the fermions. We demonstrated that symmetry is restored for this class of asymptotically free theories. We then generalised the result to arbitrary (Weyl) fermion representations and to certain multiple singlet scalar theories. It was sufficient to demonstrate the incompatibility between the request of negative thermal mass squared for one of the scalars and the simultaneous need for its coupling to be asymptotically free.
We then moved to investigate the case of gauge scalars and have shown that high temperature symmetry non-restoration appeared for the case of two gauged scalars transforming according to the fundamental representation of two independent gauge sectors. Fermions in the fundamental representation were included as well but without Yukawa couplings.

We then moved to investigate the case of asymptotically safe theories starting by noticing that the symmetry restoration results discovered for the singlet scalars discussed above could be extended to potentially safe theories.

Two more relevant examples were investigated in the asymptotically safe scenario in which either multiple quartic scalar field terms were present in the Lagrangian [55] and/or some of the scalar were gauged [71]. In these models symmetries restore at high temperature.

As an interesting class of UV complete theories featuring symmetry non-restoration at arbitrary high temperatures we discovered the one featuring two gauged scalars, each in a fundamental representation of its own $SU(N_{ci})$ gauge group: for large enough ratios of colours, one scalar thermal mass can be negative.

So far we discussed UV complete theories before adding quantum gravity. We can imagine that a possible safe and free completion of the standard model occurs few orders of magnitude below the scale above which quantum gravity cannot be ignored. In this case our analysis still applies. It can even happen that quantum gravity is, per se, asymptotically free [76], and in this case we can ignore it.

The simplicity of the UV complete models discovered here featuring arbitrary high temperature symmetry non-restoration phenomenon invites for further theoretical and phenomenological investigations. For example, it would be interesting to investigate whether UV complete grand-unified theories of the Pati-Salam type exist and that can feature the phenomenon of symmetry non-restoration. Additionally there could be dark sectors that are gravitationally coupled to us that can be UV complete and feature early universe phase transitions from a symmetric to a broken one as the temperature increases.

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**Note added**

While we were completing the present work, a related paper appeared [74] in which explicit examples of Banks-Zaks type CFTs were considered in which symmetry nonrestoration occurred at nonzero temperature. Differently and in a complementary manner of [74] our work investigates, rather than theories around IR fixed points, models featuring either Gaussians (complete asymptotically free) or interacting (completely asymptotically safe) UV fixed points such that we can investigate the infinite temperature limit within a given UV complete quantum field theory.
A The 1-loop RG equations

In this section we summarise the relevant one loop RG equations used in the main text starting with the normalisation of the fields given by

\[ \mathcal{L}_{\text{kin}} = -\frac{1}{4} F_{\mu\nu}^A F^{A\mu\nu} + i \bar{\Psi} i D\Phi + \frac{1}{2} D^\mu \Phi^a D_\mu \Phi^a \] (A.1)

The gauge RG equation is

\[ (4\pi)^2 \beta_g \equiv (4\pi)^2 \mu \frac{dg}{d\mu} = -b_0 g^3 \] (A.2)

with

\[ b_0 = \frac{11}{3} T(G) - \frac{2}{3} T(F) - \frac{1}{6} T(S) \] (A.3)

where \( G, F, S \) stand for gauge bosons, Weyl fermions and real scalars, respectively, and \( T(R) \) is the Dynkin index of the representation \( R \), defined as

\[ \text{Tr} (T_A(R) T_B(R)) = T(R) \delta^{AB} \] (A.4)

In SU(\( N_c \)) we will need the following:

\[ T(\text{fundamental}) = \frac{1}{2}, \quad T(\text{adjoints}) = N_c \] (A.5)

The Yukawa RG equations for Dirac fermions \( \Psi_i \)

\[ \mathcal{L}_{\text{Yukawa}} = \sum_{i,j} Y_{ij}^a \bar{\Psi}_i \phi^a \Psi_j \] (A.6)

are [77] (\( \kappa = 1 \) for Dirac fermions and \( \kappa = 1/2 \) for Weyl fermions)

\[ (4\pi)^2 \beta_Y^a \equiv (4\pi)^2 \mu \frac{dY^a}{d\mu} = \frac{1}{2} \left( Y^b Y^{b|} Y^a + Y^a Y^{b|} Y^b \right) + 2 Y^b Y^{a|} Y^b
\]

\[ + \kappa Y^b \text{Tr} \left( Y^{b|} Y^a + Y^{a|} Y^b \right) - 3g^2 \left( C_2(F) Y^a + Y^a C_2(F) \right) \] (A.7)

where \( \phi^a \) are real scalars and

\[ (C_2(F))_{ij} = \sum_{kA} T^A_{ik} T^A_{kj} \] (A.8)

where the generators \( T^A \) are in the (in general reducible) representation of the fermions.

Here and in the following a repeated index gets summed (\( a, b \) over real scalars, \( \alpha \) over SU(\( N_c \)) generators, \( i, j, k \) over (bi-)spinors) even when the explicit sum is not written.

Notice that the Yukawa matrices in (A.6) are Hermitean by definition.

The scalar sector is defined by

\[ V = \frac{1}{4!} \lambda_{abcd} \phi_a \phi_b \phi_c \phi_d \] (A.9)

Following [78] we introduce the completely symmetric tensors
\[ \Lambda^2_{abcd} = \frac{1}{8} \sum_{\text{perm}} \lambda_{abcdef} \lambda_{efcd} \]  
(A.10)

\[ \Lambda^Y_{abcd} = \frac{1}{12} \sum_{\text{perm}} \text{Tr} \left( Y^{a\dagger} Y^e + Y^{c\dagger} Y^a \right) \lambda_{ebcd} \]  
(A.11)

\[ H_{abcd} = \frac{1}{4} \sum_{\text{perm}} \text{Tr} \left( Y^{a\dagger} Y^b Y^{c\dagger} Y^d \right) \]  
(A.12)

\[ \Lambda^S_{abcd} = \frac{1}{6} \sum_{\text{perm}} \sum_{A=1}^{N^2-1} \left( T^A(S) T^A(S) \right)_{ae} \lambda_{ebcd} \]  
(A.13)

\[ A_{abcd} = \frac{1}{8} \sum_{\text{perm}} \sum_{A,B=1}^{N^2-1} \left\{ T^A(S), T^B(S) \right\}_{ab} \left\{ T^A(S), T^B(S) \right\}_{cd} \]  
(A.14)

where the sum over "perm" means that we sum over all 4! permutations of the indices \( a, b, c \) and \( d \) so to make the left-hand sides completely symmetric in all indices. The matrices \( T^A(S) \) are the Hermitean SU(\( N_c \)) generators in the representation of the scalars. Since \( \phi^a \) are taken real, these generators are imaginary and anti-symmetric. For real representations of SU(\( N_c \)) this is automatic, while for complex representations one has to work out the form of these matrices. More precisely, they are found in the covariant derivative:

\[ D_\mu \phi^a = \partial_\mu \phi^a - ig W^A_\mu \left( T^A(S) \right)^a_b \phi^b \]  
(A.15)

For the case of more gauge couplings \( g_\alpha \) of gauge groups with generators \( T^A_\alpha \), one should remember that

\[ A_{abcd} \phi^a \phi^b \phi^c \phi^d = \left( M^2_W \right)^{AB} \left( M^2_W \right)^{ab} \]  
(A.16)

with the \( W \) mass

\[ \left( M^2_W \right)^{AB} = \frac{1}{2} \phi^a g_\alpha g_\beta \left\{ T^A_\alpha(S), T^B_\beta(S) \right\}_{ab} \phi^b \]  
(A.17)

The 1-loop RG equations then read [78]

\[ 16\pi^2 \frac{d\lambda_{abcd}}{dt} = \Lambda^2_{abcd} + 2\kappa \Lambda^Y_{abcd} - 8\kappa H_{abcd} - 3g^2 \Lambda^S_{abcd} + 3g^4 A_{abcd} \]  
(A.18)

Finally, at high temperature the thermal mass matrix is given by [3] (see also [79])

\[ m^2_{ab}(T) = \frac{T^2}{24} \left( \lambda_{abcdef} + 2\kappa \text{Tr} \left( Y^{a\dagger} Y^b + Y^{b\dagger} Y^a \right) + 6g^2 \left( T^A(S) T^A(S) \right)_{ab} \right) \]  
(A.19)

It is useful to rewrite the above formulae by multiplying the various quantities by constant \( \phi^a \phi^b \phi^c \phi^d / 4! \) and summing over the indices \( a, b, c, d \). We thus define
\[ V_{\Lambda^2} \equiv \Lambda_{abcd}^2 \frac{\phi^a \phi^b \phi^c \phi^d}{4!} = \frac{1}{2} \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \frac{\partial^2 V}{\partial \phi^c \partial \phi^d} \quad (A.20) \]

\[ V_{\Lambda^Y} \equiv 2\kappa \Lambda_{abcd}^Y \frac{\phi^a \phi^b \phi^c \phi^d}{4!} = \kappa \phi^a \partial Y (Y^a Y^e + Y^c Y^a) \frac{\partial V}{\partial \phi^e} \quad (A.21) \]

\[ V_H \equiv -8\kappa H_{abcd} \frac{\phi^a \phi^b \phi^c \phi^d}{4!} = -2\kappa \partial (Y^a Y^b Y^c Y^d) \phi^a \phi^b \phi^c \phi^d \quad (A.22) \]

\[ V_{\Lambda^S} \equiv -3g^2 \Lambda_{abcd}^S \frac{\phi^a \phi^b \phi^c \phi^d}{4!} = -3g^2 \phi^a (T^A(S)T^A(S))_{ae} \frac{\partial V}{\partial \phi^e} \quad (A.23) \]

\[ V_A \equiv 3g^4 A_{abcd} \frac{\phi^a \phi^b \phi^c \phi^d}{4!} = \frac{3}{8} g^4 \left( \phi^a \{ T^A(S), T^B(S) \}_{ab} \phi^b \right) \left( \phi^c \{ T^A(S), T^B(S) \}_{cd} \phi^d \right) \quad (A.24) \]

Eq. (A.18) can thus be written as

\[ 16\pi^2 \phi^a \phi^b \phi^c \phi^d \frac{d\lambda_{abcd}}{dt} = V_{\Lambda^2} + V_{\Lambda^Y} + V_H + V_{\Lambda^S} + V_A \quad (A.25) \]

while the equivalent of (A.19) is (for vanishing Yukawa)

\[ \Delta V(T) \equiv \frac{1}{2} m^2_{ab}(T) \phi^a \phi^b = \frac{T^2}{48} \left( 2 \frac{\partial^2 V}{\partial \phi^a \partial \phi^a} + 6g^2 \phi^a (T^A(S)T^A(S))_{ab} \phi^b \right) \quad (A.26) \]

### B \quad \textbf{SU}(N_c) with two singlet scalars and fundamental fermions

In this model the two singlet scalar couple through Yukawa couplings to \( N_{f_1} \) (\( N_{f_2} \)) Dirac fermions in the fundamental representation of \( SU(N_c) \). We further allow for \( N_{f_0} \) Dirac fermions in the fundamental representation of the gauge group that are inert with respect to the scalars, i.e. do not possess Yukawa couplings.

The gauge coupling 1-loop RGE is

\[ 16\pi^2 \frac{dg}{dt} = -b_0 g^3, \quad (B.1) \]

with

\[ b_0 = \frac{11}{3} N_c - \frac{2}{3} (N_{f_0} + N_{f_1} + N_{f_2}) . \quad (B.2) \]

The solution is

\[ \alpha_g = \frac{g^2}{(4\pi)^2} = \frac{\tilde{\alpha}_g}{t}, \quad (B.3) \]

with

\[ \tilde{\alpha}_g = \frac{1}{2b_0} . \quad (B.4) \]

The Yukawa RGE are
\[ 16\pi^2 \frac{dy_i}{dt} = \left(3 + 2N_cN_f, \right) y_i^3 - 3g^2 \frac{N_c^2 - 1}{N_c} y_i, \quad i = 1, 2. \] (B.5)

Assuming the ansatz

\[ \alpha_{y_i} = \frac{y_i^2}{(4\pi)^2} = \frac{\tilde{\alpha}_{y_i}}{t}, \] (B.6)

the fixed flow solution is given by

\[ \tilde{\alpha}_{y_i} = \frac{6N_c^2 - 1}{2(3 + 2N_cN_f)} \frac{\tilde{\alpha}_g - 1}{N_c} \] (B.7)

and have positive solutions only if the gauge coupling is big enough

\[ 6\tilde{\alpha}_g \frac{N_c^2 - 1}{N_c} - 1 > 0, \] (B.8)

which reduces to a constraint on the number of Dirac fermion fundamentals:

\[ \frac{22}{4}N_c - \frac{9}{2} \left( N_c - \frac{1}{N_c} \right) < N_{f_0} + N_{f_1} + N_{f_2} < \frac{22}{4}N_c. \] (B.9)

The RG equations for the scalar couplings are

\[ 16\pi^2 \frac{d\lambda_1}{dt} = 18\lambda_1^2 + 2\lambda^2 - 8N_cN_f_1y_1^4 + 8N_cN_f_1y_1^2\lambda_1 \] (B.10)
\[ 16\pi^2 \frac{d\lambda_2}{dt} = 18\lambda_2^2 + 2\lambda^2 - 8N_cN_f_2y_2^4 + 8N_cN_f_2y_2^2\lambda_2 \] (B.11)
\[ 16\pi^2 \frac{d\lambda}{dt} = -8\lambda^2 + 6\lambda(\lambda_1 + \lambda_2) + 4N_c(N_f_1y_1^2 + N_f_2y_2^2) \lambda \] (B.12)

The ansatz

\[ \alpha_{\lambda_1} = \frac{\lambda_1}{(4\pi)^2} = \frac{\hat{\alpha}_{\lambda_1}}{t}, \quad \alpha_{\lambda} = \frac{\lambda}{(4\pi)^2} = \frac{\hat{\alpha}_{\lambda}}{t} \] (B.13)

reduces the system of ODEs (B.10)-(B.12) to a system of algebraic equations

\[ -\hat{\alpha}_{\lambda_1} = 18\hat{\alpha}_{\lambda_1}^2 + 2\hat{\alpha}_{\lambda}^2 - 8N_cN_f_1\hat{\alpha}_{y_1}^2 + 8N_cN_f_1\hat{\alpha}_{y_1}\hat{\alpha}_{\lambda_1} \] (B.14)
\[ -\hat{\alpha}_{\lambda_2} = 18\hat{\alpha}_{\lambda_2}^2 + 2\hat{\alpha}_{\lambda}^2 - 8N_cN_f_2\hat{\alpha}_{y_2}^2 + 8N_cN_f_2\hat{\alpha}_{y_2}\hat{\alpha}_{\lambda_2} \] (B.15)
\[ -\hat{\alpha}_{\lambda} = -8\hat{\alpha}_{\lambda}^2 + 6\hat{\alpha}_{\lambda}(\hat{\alpha}_{\lambda_1} + \hat{\alpha}_{\lambda_2}) + 4N_c(N_f_1\hat{\alpha}_{y_1} + N_f_2\hat{\alpha}_{y_2}) \hat{\alpha}_{\lambda} \] (B.16)

To this we add (B.4) and (B.7). We look for strictly positive solutions for all 6 couplings \( \hat{\alpha}_{g,y_1,y_2,\lambda_1,\lambda_2,\lambda} \), with

\[ N_c > 1, \quad N_{f_0,1,2} \geq 0, \quad N_{f_1} > 0 \text{ or } N_{f_2} > 0 \] (B.17)

and \( N_{f_0} + N_{f_1} + N_{f_2} \) in the interval (B.9).

Once this is obtained one can compute the thermal mass for the scalar scalars:

\[ m_i^2(T) = (4\pi)^2 \frac{T^2}{12\log T} \left(3\tilde{\alpha}_{\lambda_i} - \tilde{\alpha}_{\lambda} + 2N_cN_f\tilde{\alpha}_{y_i}\right) \] (B.18)
It turns out that there are 1784 inequivalent (we do not count those obtained by $N_{f_1} \leftrightarrow N_{f_2}$) choices of colours and flavours which satisfy (B.17) and (B.9). However we are not only looking for fixed flow solutions, what we also need is that they lead to a negative thermal mass.

We will now prove in general that there are no solutions with symmetry non-restoration. Let it be $m^2(T) < 0$. To be so one needs

$$\tilde{\alpha}_\lambda - 2N_cN_{f_1}\tilde{\alpha}_{y_1} > 3\tilde{\alpha}_\lambda > 0$$ (B.19)

We can now rewrite (B.14) as

$$2\left(\tilde{\alpha}_\lambda^2 - 4N_cN_{f_1}\tilde{\alpha}_{y_1}^2\right) + \tilde{\alpha}_\lambda + 18\tilde{\alpha}_\lambda^4 + 8N_cN_{f_1}\tilde{\alpha}_{y_1}\tilde{\alpha}_\lambda = 0$$ (B.20)

All the terms except the first one are manifestly positive, so to satisfy the equation, the first term should be negative. However, the first term can be rewritten as

$$\tilde{\alpha}_\lambda^2 - 4N_cN_{f_1}\tilde{\alpha}_{y_1}^2 = (\tilde{\alpha}_\lambda - 2N_cN_{f_1}\tilde{\alpha}_{y_1}) (\tilde{\alpha}_\lambda + 2N_cN_{f_1}\tilde{\alpha}_{y_1}) + 4N_cN_{f_1}(N_cN_{f_1} - 1)\tilde{\alpha}_{y_1}^2$$ (B.21)

This is positive, since the last term is non-negative, while the first product is positive due to (B.19). Equation (B.20) thus cannot have a solution.

We conclude the section summarising the result for the model presented: there is no fixed flow solution once a negative thermal mass is assumed.

C  Gauged scalars

We consider in this appendix various examples of scalars in non-trivial representations of the gauge group.

C.1  SU(2) with two scalar triplets

First we take the two scalar fields as gauge SU(2) triplets, coupled each to one fermion SU(2) doublet ($N_{f_1} = N_{f_2} = 1$). To use almost all of the old results we still keep the $Z_2 \times Z_2$ discrete symmetry. There is now an extra quartic term:

$$V = \frac{\lambda_1}{4} (\bar{\varphi}_1 \cdot \varphi_1)^2 + \frac{\lambda_2}{4} (\bar{\varphi}_2 \cdot \varphi_2)^2 - \frac{\lambda_{11}}{2} (\bar{\varphi}_1 \cdot \varphi_1) (\bar{\varphi}_2 \cdot \varphi_2) - \frac{\lambda_{12}}{2} (\bar{\varphi}_1 \cdot \varphi_2)^2$$ (C.1)

Denoting

$$\phi = (\varphi_1, \varphi_2)$$ (C.2)

we compute the quartic couplings directly from the definition

$$V = \frac{\lambda_{abcd}}{4!} \phi^a \phi^b \phi^c \phi^d$$ (C.3)

i.e.

$$\lambda_{abcd} = \frac{\partial^4 V}{\partial \phi^a \partial \phi^b \partial \phi^c \partial \phi^d}, \quad a, b, c, d = 1, \ldots, 6$$ (C.4)
For the Yukawa term we take
\[ \mathcal{L}_{\text{Yukawa}} = \sum_{i=1}^{2} y_i \bar{\psi}_i \left( \frac{\tau}{2} \cdot \vec{\varphi}_i \right) \psi_i \]  
(C.5)

with \( \tau^A, A = 1, 2, 3 \) the Pauli matrices.

The (reducible) generators for the fermions (two fundamental representations of SU(2))
\[ \Psi = (\psi_1, \psi_2) \]  
(C.6)

are
\[ T^A = \frac{1}{2} \begin{pmatrix} \tau^A & 0 \\ 0 & \tau^A \end{pmatrix}, \quad A = 1, 2, 3 \]  
(C.7)

The fixed flow RGE are
\[ \ddot{\alpha}_g = 2b_0 \dot{\alpha}_g^2 \]  
(C.8)
\[ -\ddot{\alpha}_{y_i} = \frac{5}{2} \dot{\alpha}_{y_i}^2 - 9 \ddot{\alpha}_g \dot{\alpha}_{y_i} \]  
(C.9)
\[ -\ddot{\alpha}_{\lambda_i} = 22 \dot{\alpha}_{\lambda_i}^2 + 6 \dot{\alpha}_{\lambda_{11}}^2 + 4 \dot{\alpha}_{\lambda_{11}} \dot{\alpha}_{\lambda_{12}} + 2 \dot{\alpha}_{\lambda_{12}}^2 - \ddot{\alpha}_{y_i} + 4 \ddot{\alpha}_g \dot{\alpha}_{y_i} \]  
(C.10)
\[ \dot{\alpha}_{\lambda_{11}} = 6 \dot{\alpha}_g^2 + 24 \dot{\alpha}_g \ddot{\alpha}_{\lambda_{11}} + 8 \dot{\alpha}_{\lambda_{11}}^2 - 10 \ddot{\alpha}_{\lambda_{11}} (\ddot{\alpha}_{\lambda_1} + \ddot{\alpha}_{\lambda_2}) \]  
(C.11)
\[ \dot{\alpha}_{\lambda_{12}} = 6 \dot{\alpha}_g^2 + 24 \dot{\alpha}_g \ddot{\alpha}_{\lambda_{12}} + 16 \ddot{\alpha}_{\lambda_{11}} \ddot{\alpha}_{\lambda_{12}} \]  
(C.12)

The thermal mass square results
\[ m_i^2(T) = (4\pi)^2 \frac{T^2}{12 \log T} (\ddot{\alpha}_{y_i} + 5 \ddot{\alpha}_{\lambda_i} + 6 \ddot{\alpha}_g - 3 \ddot{\alpha}_{\lambda_{11}} - \ddot{\alpha}_{\lambda_{12}}) \]  
(C.13)

The gauge beta function is known,
\[ b_0 = \frac{22}{3} - \frac{2}{3} (N_f + 2) - \frac{2}{3} = \frac{16 - 2N_f}{3} \rightarrow \dot{\alpha}_g = \frac{3}{4(8 - N_f)} \]  
(C.14)

from where, to get \( \ddot{\alpha}_g > 1/9 \), see (C.9), we need
\[ 2 \leq N_f < 8 \]  
(C.15)

By explicit search on can find that there are no solutions of the fixed flow RGE for positive \( \ddot{\alpha}_g, \ddot{\alpha}_{y_{1,2}}, \ddot{\alpha}_{\lambda_{1,2}} \) and real \( \ddot{\alpha}_{\lambda_{11,12}} \).
C.2  \( SU(2) \) with one scalar singlet and one scalar triplet

We take now one adjoint scalar and one singlet scalar that couple to the fermions (again in the fundamental representation, \( N_f_1 = N_f_2 = 1 \)) with the following Yukawa term:

\[
\mathcal{L}_{\text{Yuk}} = y_1 \bar{\psi}_1 \phi_1 + y_2 \bar{\psi}_2 \left( \frac{r}{2} \cdot \vec{\phi}_2 \right) \psi_2 .
\]  \( \text{(C.16)} \)

Now the first scalar is singlet, the second is triplet. Obviously \( \lambda_{12} \) cannot appear now.

We will again call the remaining mixed constant \( \lambda_{11} = \lambda \) in this section.

The fixed flow RGE are now

\[
\tilde{\alpha}_g = 2b_0 \tilde{\alpha}_g^2 \]  \( \text{(C.17)} \)

\[-\tilde{\alpha}_{y_1} = 14\tilde{\alpha}_{y_1} - 9\tilde{\alpha}_g \tilde{\alpha}_{y_1} \]  \( \text{(C.18)} \)

\[-\tilde{\alpha}_{y_2} = \frac{5}{2} \tilde{\alpha}_{y_2} - 9\tilde{\alpha}_g \tilde{\alpha}_{y_2} \]  \( \text{(C.19)} \)

\[-\tilde{\alpha}_{\lambda_1} = 18\tilde{\alpha}_{\lambda_1} + 6\tilde{\alpha}_g^2 - 16\tilde{\alpha}_{y_1} + 16\tilde{\alpha}_{\lambda_1} \tilde{\alpha}_{y_1} \]  \( \text{(C.20)} \)

\[-\tilde{\alpha}_{\lambda_2} = 12\tilde{\alpha}_g - 24\tilde{\alpha}_g \tilde{\alpha}_{\lambda_2} + 2\tilde{\alpha}_g^2 - 2\tilde{\alpha}_{y_2} - 4\tilde{\alpha}_{\lambda_2} \tilde{\alpha}_{y_2} \]  \( \text{(C.21)} \)

\[\tilde{\alpha}_\lambda = 12\tilde{\alpha}_g \tilde{\alpha}_\lambda - 6\tilde{\alpha}_{\lambda_1} \tilde{\alpha}_\lambda + 8\tilde{\alpha}_g^2 - 10\tilde{\alpha}_g \tilde{\alpha}_{\lambda_2} - 8\tilde{\alpha}_\lambda \left( \tilde{\alpha}_{y_1} + \frac{1}{4} \tilde{\alpha}_{y_2} \right) \]  \( \text{(C.22)} \)

while the thermal masses are

\[
m_1^2(T) = (4\pi)^2 \frac{T^2}{12 \log T} (4\tilde{\alpha}_{y_1} + 3\tilde{\alpha}_{\lambda_1} - 3\tilde{\alpha}_\lambda) \]  \( \text{(C.23)} \)

\[
m_2^2(T) = (4\pi)^2 \frac{T^2}{12 \log T} (\tilde{\alpha}_{y_2} + 5\tilde{\alpha}_{\lambda_2} + 6\tilde{\alpha}_g - \tilde{\alpha}_\lambda) \]  \( \text{(C.24)} \)

The gauge beta function is

\[
b_0 = \frac{22}{3} - \frac{2}{3} (N_f_0 + 2) - \frac{1}{3} = \frac{17 - 2N_f_0}{3} \rightarrow \tilde{\alpha}_g = \frac{3}{2(17 - 2N_f_0)} \]  \( \text{(C.25)} \)

from where, to get \( \tilde{\alpha}_g > 1/9 \), see (C.18) or (C.19), we need

\[
2 \leq N_f_0 \leq 8 \]  \( \text{(C.26)} \)

We find only two solutions:

\[
N_f_0 = 8 : (\tilde{\alpha}_g, \tilde{\alpha}_{y_1}, \tilde{\alpha}_{y_2}, \tilde{\alpha}_{\lambda_1}, \tilde{\alpha}_{\lambda_2}, \tilde{\alpha}_\lambda) = (1.5, 0.893, 5.0, 0.518, 0.182, 0) \]  \( \text{(C.27)} \)

\[
N_f_0 = 8 : (\tilde{\alpha}_g, \tilde{\alpha}_{y_1}, \tilde{\alpha}_{y_2}, \tilde{\alpha}_{\lambda_1}, \tilde{\alpha}_{\lambda_2}, \tilde{\alpha}_\lambda) = (1.5, 0.893, 5.0, 0.518, 0.5, 0) \]  \( \text{(C.28)} \)

Since both have \( \tilde{\alpha}_\lambda = 0 \), symmetry is always restored at high enough \( T \).
C.3 $SU(N_c)$ with two scalar fundamentals

The potential is

$$V = \frac{\lambda_1}{2} (\bar{\varphi}_1 \cdot \varphi_1)^2 + \frac{\lambda_2}{2} (\bar{\varphi}_2 \cdot \varphi_2)^2 + \lambda_3 (\bar{\varphi}_1^* \cdot \varphi_1) (\varphi_2^* \cdot \varphi_2)$$

$$+ \lambda_4 (\bar{\varphi}_1^* \cdot \varphi_2) (\varphi_2^* \cdot \varphi_1) + \frac{\lambda_5}{2} (\varphi_1^* \cdot \varphi_2)^2 + \frac{\lambda_6}{2} (\varphi_2^* \cdot \varphi_1)^2$$

with $\lambda_{1,2,3,4}$ real and in general $\lambda_5$ complex.

The relation between the complex and the real basis is as usual

$$\varphi_{\alpha k} = \frac{1}{\sqrt{2}} (R_{\alpha k} + i I_{\alpha k}) \quad , \quad \alpha = 1, 2 \quad , \quad k = 1, \ldots, N_c$$

so that

$$\phi^a = (R^k_1, I^k_1, R^k_2, I^k_2)^T$$

We get

$$V_{\Lambda^2} = \frac{1}{2} \frac{\partial^2 V}{\partial \varphi^a \partial \varphi^b} \frac{\partial^2 V}{\partial \varphi^\alpha \partial \varphi^\beta}$$

$$= \sum_{\alpha,\beta=1}^2 \sum_{k,l=1}^{N_c} \frac{\partial^2 V}{\partial \varphi_{\alpha k} \partial \varphi_{\beta l} \partial \varphi_{\alpha k} \partial \varphi_{\beta l}^*} + \frac{\partial^2 V}{\partial \varphi_{\alpha k} \partial \varphi_{\beta l} \partial \varphi_{\beta l}^* \partial \varphi_{\alpha k}^*}$$

$$= \text{Tr} \left( M_1 M_1^\dagger + 2M_2 M_2^\dagger + M_3 M_3^\dagger + N_1 N_1^\dagger + 2N_2 N_2^\dagger + N_3 N_3^\dagger \right)$$

with

$$M_1 = (\lambda_1 (\bar{\varphi}_1 \cdot \varphi_1) + \lambda_3 (\bar{\varphi}_2 \cdot \varphi_2)) 1 + \lambda_4 \bar{\varphi}_1^* \otimes \varphi_1 + \lambda_5 \varphi_2 \otimes \bar{\varphi}_2$$

$$M_2 = (\lambda_4 (\bar{\varphi}_1 \cdot \varphi_2) + \lambda_5 (\bar{\varphi}_2 \cdot \varphi_1)) 1 + \lambda_3 \bar{\varphi}_2^* \otimes \varphi_2 + \lambda_5 \varphi_2 \otimes \bar{\varphi}_1$$

$$M_3 = (\lambda_2 (\bar{\varphi}_1 \cdot \varphi_2) + \lambda_3 (\bar{\varphi}_2 \cdot \varphi_1)) 1 + \lambda_4 \bar{\varphi}_1^* \otimes \varphi_1 + \lambda_2 \varphi_2 \otimes \bar{\varphi}_2$$

$$N_1 = \lambda_1 \bar{\varphi}_1 \otimes \varphi_1 + \lambda_5 \bar{\varphi}_2 \otimes \varphi_2$$

$$N_2 = \lambda_3 \bar{\varphi}_2 \otimes \varphi_2 + \lambda_5 \bar{\varphi}_2 \otimes \varphi_1$$

$$N_3 = \lambda_2 \bar{\varphi}_1 \otimes \varphi_1 + \lambda_5 \bar{\varphi}_1 \otimes \varphi_1$$

This gives

$$V_{\Lambda^2} = \left( (2N_c + 8) \lambda_1^2 + 2N_c \lambda_3^2 + 4\lambda_3 \lambda_4 + 2\lambda_4^2 + 2|\lambda_5|^2 \right) \frac{1}{2} (\bar{\varphi}_1 \cdot \varphi_1)^2$$

$$+ \left( (2N_c + 8) \lambda_2^2 + 2N_c \lambda_3^2 + 4\lambda_3 \lambda_4 + 2\lambda_4^2 + 2|\lambda_5|^2 \right) \frac{1}{2} (\bar{\varphi}_2 \cdot \varphi_2)^2$$

$$+ \left( 2(N_c + 1) (\lambda_1 + \lambda_2) \lambda_3 + 4\lambda_3^2 + 2(\lambda_1 + \lambda_2) \lambda_4 + 2\lambda_4^2 + 2|\lambda_5|^2 \right) (\bar{\varphi}_1 \cdot \varphi_1) (\bar{\varphi}_2 \cdot \varphi_2)$$

$$+ \left( (\lambda_1 + \lambda_2) \lambda_4 + 8\lambda_3 \lambda_4 + 2N_c \lambda_4^2 + (4 + 2N_c) |\lambda_5|^2 \right) (\bar{\varphi}_1 \cdot \varphi_1) (\bar{\varphi}_2 \cdot \varphi_2)$$

$$+ 2(\lambda_1 + \lambda_2 + 4\lambda_3 + 2(N_c + 1) \lambda_4) \lambda_5 \frac{1}{2} (\bar{\varphi}_1 \cdot \varphi_2)^2$$

$$+ 2(\lambda_1 + \lambda_2 + 4\lambda_3 + 2(N_c + 1) \lambda_4) \lambda_5 \frac{1}{2} (\bar{\varphi}_2 \cdot \varphi_1)^2$$

(C.39)
We easily find

\[
V_{A^a} = -3g^2 \phi^a (T^A(S)T^A(S))_{ab} \frac{\partial V}{\partial \phi^a} = -3N_c^2 - 1 \frac{\partial V}{\partial \phi^a} (\varphi^k \partial \varphi^k + \varphi^* \partial \varphi^*) = -6N_c^2 - 1 g^2 V \tag{C.40}
\]

Using

\[
\phi^a \{ T^A(S), T^B(S) \}_{ab} \phi^b = 2g^2 \varphi_{\alpha} \{ T^A, T^B \}_{ab} \varphi^{\alpha} \tag{C.41}
\]

and the usual

\[
(T^A)^a_b(T^A)^c_d = \frac{1}{2} \left( \delta^a_d \delta^c_b - \frac{1}{N_c} \delta^a_b \delta^c_d \right) \tag{C.42}
\]

we get

\[
V_A = \frac{3}{4} g^2 \frac{N_c^2}{N_c} \left( (\varphi_1 \cdot \varphi_1 + \varphi_2 \cdot \varphi_2) \right)^2
\]

\[
+ \frac{3}{4} g^2 \frac{N_c^2 - 4}{N_c} \left( (\varphi_1 \cdot \varphi_1)^2 + 2 (\varphi_1 \varphi_2) (\varphi_2 \varphi_1) + (\varphi_2 \cdot \varphi_2)^2 \right)
\]

\[
= \frac{3}{4} g^2 \frac{N_c^3 + N_c^2 - 4N_c + 2}{N_c} \left( (\varphi_1 \cdot \varphi_1)^2 + (\varphi_2 \cdot \varphi_2)^2 \right) \tag{C.43}
\]

\[
+ \frac{3}{2} g^2 \frac{N_c^2 + 2}{N_c} (\varphi_1 \cdot \varphi_1) (\varphi_2 \cdot \varphi_2) + \frac{3}{2} g^4 \frac{N_c^2 - 4}{N_c} (\varphi_1 \varphi_2) (\varphi_2 \varphi_1)
\]

By taking

\[
g^2 = \frac{16\pi^2 \hat{\alpha}}{N_c \tilde{\lambda}_i}, \quad \tilde{\lambda}_i = \frac{16\pi^2 \hat{\lambda}_i}{N_c \tilde{\lambda}_i} \tag{C.44}
\]

we get for constant \( \alpha, \lambda_i \) in the large \( N_c \) limit the following fixed flow RGEs:

\[
-\tilde{\lambda}_1 = 2\hat{\lambda}_1^2 + 2\hat{\lambda}_2^2 - 6\hat{\alpha} \hat{\lambda}_1 + \frac{3}{2} \hat{\alpha}^2 \tag{C.45}
\]

\[
-\tilde{\lambda}_2 = 2\hat{\lambda}_2^2 + 2\hat{\lambda}_3^2 - 6\hat{\alpha} \hat{\lambda}_2 + \frac{3}{2} \hat{\alpha}^2 \tag{C.46}
\]

\[
-\tilde{\lambda}_3 = 2 (\hat{\lambda}_1 + \hat{\lambda}_2) \hat{\lambda}_3 - 6\hat{\alpha} \hat{\lambda}_3 \tag{C.47}
\]

\[
-\tilde{\lambda}_4 = 2\hat{\lambda}_4^2 + 2 \left| \hat{\lambda}_5 \right|^2 - 6\hat{\alpha} \hat{\lambda}_4 + \frac{3}{2} \hat{\alpha}^2 \tag{C.48}
\]

\[
-\tilde{\lambda}_5 = 4\hat{\lambda}_4 \hat{\lambda}_5 - 6\hat{\alpha} \hat{\lambda}_5 \tag{C.49}
\]

The thermal potential is
\[
\Delta V_T = \frac{T^2}{48} \left( 2 \frac{\partial^2 V}{\partial \phi^a \partial \phi^a} + 6 \phi^a \left( T^A(S) T^A(S) \right)_{ab} \phi^b \right)
\]
\[
= \frac{T^2}{48} \sum_{i=1}^{2} \sum_{a=1}^{N_c} \left( 4 \frac{\partial^2 V}{\partial \varphi^a_i \partial \varphi^a_i} + 6 g^2 \frac{N_c^2 - 1}{N_c} \varphi^a_i \varphi^a_i \right)
\]  
\text{(C.50)}

Using (C.33) and (C.35)

\[
\sum_{i=1}^{2} \sum_{a=1}^{N_c} \frac{\partial^2 V}{\partial \varphi^a_i \partial \varphi^a_i} = Tr (M_1 + M_3)
\]
\[
= \left( (N_c + 1) \lambda_1 + N_c \lambda_3 + \lambda_4 \right) (\varphi^1 \cdot \bar{\varphi}^1) 
+ \left( (N_c + 1) \lambda_2 + N_c \lambda_3 + \lambda_4 \right) (\varphi^2 \cdot \bar{\varphi}^2)
\]  
\text{(C.51)}

At large \( N_c \)

\[
\Delta V_T = (4\pi)^2 \frac{T^2}{24 \log T} \left( 2 \left( \lambda_1 + \lambda_3 \right) + 3 \tilde{\alpha} \right) (\varphi^1 \cdot \bar{\varphi}^1) + \left( 2 \left( \lambda_2 + \lambda_3 \right) + 3 \tilde{\alpha} \right) (\varphi^2 \cdot \bar{\varphi}^2)
\]  
\text{(C.52)}

\text{C.4} \quad SU(N_{c1}) \times SU(N_{c2}) \text{ with two scalar fundamentals}

The model we will study now is similar to the previous one, but now we have two simple groups, \( SU(N_{c1}) \times SU(N_{c2}) \), so that each \( \varphi_i \) is in a fundamentals representation of its \( SU(N_{c1}) \) and a singlet under the other one. The most general potential is

\[
V = \frac{\lambda_1}{2} (\varphi^1 \cdot \bar{\varphi}^1)^2 + \frac{\lambda_2}{2} (\varphi^2 \cdot \bar{\varphi}^2)^2 - \lambda (\varphi^1 \cdot \bar{\varphi}^1) (\varphi^2 \cdot \bar{\varphi}^2)
\]  
\text{(C.53)}

As before we derive the various pieces of the RGE using (A.20), (A.23), (A.24):

\[
V_{A^2} = \left( 2N_{c1} + 8 \right) \lambda_1^2 + 2N_{c2} \lambda_3^2 \frac{1}{2} (\varphi^1 \cdot \bar{\varphi}^1)^2 
+ \left( 2N_{c2} + 8 \right) \lambda_2^2 + 2N_{c1} \lambda_3^2 \frac{1}{2} (\varphi^2 \cdot \bar{\varphi}^2)^2 
- 2 \left( N_{c1} \lambda_1 + N_{c2} \lambda_2 \right) \lambda + 2 \left( \lambda_1 + \lambda_2 \right) \lambda - 4 \lambda^2 \left( \varphi^1 \cdot \bar{\varphi}^1 \right) (\varphi^2 \cdot \bar{\varphi}^2)
\]  
\text{(C.54)}

\[
V_{A^s} = -6N_{c1}^2 - 1 g_1 \lambda_1^2 \left( \frac{1}{2} (\varphi^1 \cdot \bar{\varphi}^1)^2 - \frac{1}{2} (\varphi^2 \cdot \bar{\varphi}^2)^2 \right) 
-6N_{c2}^2 - 1 g_2 \lambda_2^2 \left( \frac{1}{2} (\varphi^2 \cdot \bar{\varphi}^2)^2 - \frac{1}{2} (\varphi^1 \cdot \bar{\varphi}^1)^2 \right)
\]  
\text{(C.55)}

\[
V_A = \frac{3}{4} g_1^2 N_{c1}^3 + N_{c1}^2 - 4N_{c1} + 2 \left( \varphi^1 \cdot \bar{\varphi}^1 \right)^2 
+ \frac{3}{4} g_2^2 N_{c2}^3 + N_{c2}^2 - 4N_{c2} + 2 \left( \varphi^2 \cdot \bar{\varphi}^2 \right)^2
\]  
\text{(C.56)}
Defining
\[ i = 1, 2 : \quad g_i^2 = \frac{16\pi^2\hat{\alpha}_i}{N_{c_i}t}, \quad \lambda_i = \frac{16\pi^2\hat{\lambda}_i}{N_{c_i}t} \]  \hspace{1cm} (C.57)
\[ \lambda = \frac{16\pi^2\hat{\lambda}}{\sqrt{N_{c_1}N_{c_2}t}} \]  \hspace{1cm} (C.58)
with constant we get for the RG equations at large \( N_{c_i} \)
\[ -\tilde{\lambda}_1 = 2\tilde{\lambda}_1^2 + 2\tilde{\lambda}_2^2 - 6\tilde{\alpha}_1\tilde{\lambda}_1 + \frac{3}{2}\tilde{\alpha}_1^2 \]  \hspace{1cm} (C.59)
\[ -\tilde{\lambda}_2 = 2\tilde{\lambda}_1^2 + 2\tilde{\lambda}_2^2 - 6\tilde{\alpha}_2\tilde{\lambda}_2 + \frac{3}{2}\tilde{\alpha}_2^2 \]  \hspace{1cm} (C.60)
\[ -\tilde{\lambda} = 2\left(\tilde{\lambda}_1 + \tilde{\lambda}_2\right)\tilde{\lambda} - 3\left(\tilde{\alpha}_1 + \tilde{\alpha}_2\right)\tilde{\lambda} \]  \hspace{1cm} (C.61)

The thermal effective potential
\[ \Delta V_T = \frac{T^2}{48} \sum_{i=1}^{2} \sum_{a=1}^{N_{a}} \left( 4 \frac{\partial^2 V}{\partial \varphi_i \partial \varphi_i^*} + 6g_i^2 N_{c_i}^2 - 1 \right) \left( \varphi_i \cdot \varphi_i^* \right) \]
becomes at large \( N_{c_i} \)
\[ \Delta V_T = (4\pi)^2 \frac{T^2}{24\log T} \left( \begin{array}{c} T^2 \left( 2\left(\tilde{\lambda}_1 - \sqrt{N_{c_1}}\tilde{\lambda}\right) + 3\tilde{\alpha}_1 \right) \left( \varphi_1 \cdot \varphi_1^* \right) \\
+ T^2 \left( 2\left(\tilde{\lambda}_2 - \sqrt{N_{c_2}}\tilde{\lambda}\right) + 3\tilde{\alpha}_2 \right) \left( \varphi_2 \cdot \varphi_2^* \right) \end{array} \right) \]  \hspace{1cm} (C.62)

C.5 \( SU(N_{c_1}) \times SU(N_{c_2}) \) with two scalar adjoints

We now present a model again with two simple gauge groups, \( SU(N_{c_1}) \times SU(N_{c_2}) \), and one adjoint for each gauge group. The potential is parametrised by
\[ V = \frac{\lambda_1}{4} Tr \Sigma_1^4 + \frac{\lambda_2}{4} Tr \Sigma_2^4 + \frac{\lambda_1}{4} (Tr \Sigma_1^2)^2 + \frac{\lambda_2}{4} (Tr \Sigma_2^2)^2 - \frac{\lambda_1}{2} Tr \Sigma_1^2 Tr \Sigma_2^2 \]  \hspace{1cm} (C.63)

The 1-loop corrections are (A.20), (A.23), (A.24):
\[ V_{\Lambda^2} = \frac{1}{8} \left( Tr \Sigma_1^4 \left( 12\lambda_1\lambda_1^2 + \lambda_1^2 \frac{2N_{c_1}^2 - 18}{N_{c_1}} \right) - Tr \Sigma_1^2 Tr \Sigma_2^2 \left( \lambda_1\lambda \left( 2N_{c_1}^2 + 2 \right) + \lambda_1^2 \frac{4N_{c_1}^2 - 6}{N_{c_1}} \right) \right) \]
\[ + Tr \Sigma_1^2 Tr \Sigma_2^2 \lambda_1^2 \]  \hspace{1cm} (C.64)
\[ + \frac{1}{8} \left( Tr \Sigma_2^4 \left( 12\lambda_2\lambda_2^2 + \lambda_2^2 \frac{2N_{c_2}^2 - 18}{N_{c_2}} \right) - Tr \Sigma_1^2 Tr \Sigma_2^2 \left( \lambda_2\lambda \left( 2N_{c_2}^2 + 2 \right) + \lambda_2^2 \frac{4N_{c_2}^2 - 6}{N_{c_2}} \right) \right) \]
\[ + Tr \Sigma_1^2 Tr \Sigma_2^2 \lambda_2^2 \]
\[ V_{\lambda^4} = -3g^2 N_{c_1} \left( \lambda'_1 T r \Sigma_1^4 + \lambda_1 \left( T r \Sigma_1^2 \right)^2 - \lambda T r \Sigma_1^2 T r \Sigma_2^2 \right) \]
\[ -3g^2 N_{c_2} \left( \lambda'_2 T r \Sigma_2^4 + \lambda_2 \left( T r \Sigma_2^2 \right)^2 - \lambda T r \Sigma_1^2 T r \Sigma_2^2 \right) \]
\[ (C.65) \]
\[ V_{\lambda^4} = 3g^4 \left( N_{c_1} T r \Sigma_1^4 + 3 \left( T r \Sigma_1^2 \right)^2 \right) + 3g^4 \left( N_{c_2} T r \Sigma_2^4 + 3 \left( T r \Sigma_2^2 \right)^2 \right) \]
\[ (C.66) \]

We redefine the constants as
\[ \lambda'_{1,2} = (4\pi)^2 \frac{\lambda'_{1,2}}{N_{c_{1,2}}} \times \frac{1}{T} \quad \lambda_{1,2} = (4\pi)^2 \frac{\lambda_{1,2}}{N_{c_{1,2}}} \times \frac{1}{T} \]
\[ (C.67) \]
\[ \lambda = (4\pi)^2 \frac{\lambda}{N_{c_1 N_{c_2}}} \times \frac{1}{T} \quad g_{1,2} = (4\pi)^2 \frac{g_{1,2}}{N_{c_{1,2}}} \times \frac{1}{T} \]
\[ (C.68) \]

with all tilded quantities constants, and eventually took the large \( N_{c_{1,2}} \) limit.

In the Veneziano limit the RGE are
\[ -\hat{\lambda}_1 = \frac{1}{2} \hat{\lambda}_1^2 + 2\hat{\lambda}_1 \hat{\lambda}_1' + \frac{3}{2} \hat{\lambda}_1^2 + \frac{1}{2} \hat{\lambda}^2 - 12\hat{\alpha}_1 \hat{\lambda}_1 + 36\hat{\alpha}_1^2 \]
\[ (C.69) \]
\[ -\hat{\lambda}_2 = \frac{1}{2} \hat{\lambda}_2^2 + 2\hat{\lambda}_2 \hat{\lambda}_2' + \frac{3}{2} \hat{\lambda}_2^2 + \frac{1}{2} \hat{\lambda}^2 - 12\hat{\alpha}_2 \hat{\lambda}_2 + 36\hat{\alpha}_2^2 \]
\[ (C.70) \]
\[ -\hat{\lambda} = \hat{\lambda} \left( \frac{1}{2} (\hat{\lambda}_1 + \hat{\lambda}_2) + \hat{\lambda}_1' + \hat{\lambda}_2' - 6 (\hat{\alpha}_1 + \hat{\alpha}_2) \right) \]
\[ (C.71) \]
\[ -\hat{\lambda}_1' = \hat{\lambda}_1^2 - 12\hat{\alpha}_1 \hat{\lambda}_1' + 12\hat{\alpha}_1^2 \]
\[ (C.72) \]
\[ -\hat{\lambda}_2' = \hat{\lambda}_2^2 - 12\hat{\alpha}_2 \hat{\lambda}_2' + 12\hat{\alpha}_2^2 \]
\[ (C.73) \]

The thermal mass is
\[ V_T = \frac{T^2}{48} \left( \left( \lambda_1 \left( N_{c_1}^2 + 1 \right) + \lambda_1' \frac{2 N_{c_1}^2 - 3}{N_{c_1}} \right) T r \Sigma_1^2 \right) \]
\[ \left( \lambda_2 \left( N_{c_2}^2 + 1 \right) + \lambda_2' \frac{2 N_{c_2}^2 - 3}{N_{c_2}} \right) T r \Sigma_2^2 \right) \]
\[ (C.74) \]

and becomes in the Veneziano limit
\[ V_T = (4\pi)^2 \frac{T^2}{48 \log T} \left( \left( \hat{\lambda}_1 + 2\hat{\lambda}_1' - \frac{N_{c_2} \hat{\lambda}}{N_{c_1}} + 12\hat{\alpha}_1 \right) T r \Sigma_1^2 + \left( \hat{\lambda}_2 + 2\hat{\lambda}_2' - \frac{N_{c_1} \hat{\lambda}}{N_{c_2}} + 12\hat{\alpha}_2 \right) T r \Sigma_2^2 \right) \]
\[ (C.75) \]

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