Horizon-Free Reinforcement Learning for Latent Markov Decision Processes

Runlong Zhou* Ruosong Wang† Simon S. Du‡

October 24, 2022

Abstract

We study regret minimization for reinforcement learning (RL) in Latent Markov Decision Processes (LMDPs) with context in hindsight. We design a novel model-based algorithmic framework which can be instantiated with both a model-optimistic and a value-optimistic solver. We prove an \( \tilde{O} \left( \sqrt{M S A K} \right) \) regret bound where \( M \) is the number of contexts, \( S \) is the number of states, \( A \) is the number of actions, \( K \) is the number of episodes, and \( \Gamma \leq S \) is the maximum transition degree of any state-action pair. The regret bound only scales logarithmically with the planning horizon, thus yielding the first (nearly) horizon-free regret bound for LMDP. Key in our proof is an analysis of the total variance of alpha vectors, which is carefully bounded by a recursion-based technique. We complement our positive result with a novel \( \Omega \left( \sqrt{M S A K} \right) \) regret lower bound with \( \Gamma = 2 \), which shows our upper bound minimax optimal when \( \Gamma \) is a constant. Our lower bound relies on new constructions of hard instances and an argument based on the symmetrization technique from theoretical computer science, both of which are technically different from existing lower bound proof for MDPs, and thus can be of independent interest.

1 Introduction

One of the most popular model for Reinforcement Learning (RL) is Markov Decision Process (MDP), in which the transitions and rewards are dependent only on current state and agent’s action. In standard MDPs, the agent has full observation of the state, so the optimal policy for the agent also only depends on states (called a history-independent policy). There is a line of research on MDPs, and the minimax regret and sample complexity guarantees have been derived.

Another popular model is Partially Observable MDPs (POMDPs) in which the agent only has partial observations of states. Even though the underlying transition is still Markovian, the lower bound for sample complexity has been proven to be exponential in state and action sizes. This is in part because the optimal policies for POMDPs are history-dependent.

In this paper we focus on a middle group between MDP and POMDP, namely Latent MDP (LMDP). An LMDP can be viewed as a collection of MDPs sharing the same state and action spaces, but the transitions and rewards may vary across them. Each MDP has a probability to be sampled at the beginning of each episode, and it will not change during the episode. The agent needs to find a policy which works well on these MDPs in an average sense. Empirically, LMDPs can be used for a wide variety of applications [Yu et al., 2020, Iakovleva et al., 2020, Finn et al., 2018, Ramamoorthy et al., 2013, Doshi-Velez and Konidaris, 2016, Yao et al., 2018]. In general, there exists no policy that is optimally on every single MDP simultaneously, so this task is definitely harder than MDPs. On the other hand, LMDP is a special case of POMDP because for each MDP, the unobserved state is static in each episode and the observable state is just the state of MDP.

*University of Washington. Email: vectorzh@cs.washington.edu
†University of Washington. Email: ruosongw@cs.washington.edu
‡University of Washington. Email: ssdu@cs.washington.edu
Unfortunately, for generic LMDPs, there exists exponential sample complexity lower bound [Kwon et al., 2021], so additional assumptions are needed to make the problem tractable. In this paper, we consider the setting that after each episode ends, the agent will get the context on which MDP it played with. Such information is often available. For example, in a maze navigation task, the location of the goal state can be viewed as the context.

In this setting, Kwon et al. [2021] obtained an \( \tilde{O}(\sqrt{MS^2AHK}) \) regret upper bound where \( M \) is the number of contexts, \( S \) is the number of states, \( A \) is the number of actions, \( H \) is the planning horizon, and \( K \) is the number of episodes. They did not study the regret lower bound.\(^1\) To benchmark this result, the only available bound is \( \tilde{\Theta}(\sqrt{SAK}) \) from standard MDP by viewing MDP as a special case of LMDP.

Comparing these two bounds, we find significant gaps: ① Is the dependency on \( M \) in LMDP necessary? ② The bound for MDP is (nearly) horizon-free (no dependency on \( H \)), is the polynomial dependency on \( H \) in LMDP necessary? ③ The dependency on the number of states is \( \sqrt{S} \) for MDP but the bound in Kwon et al. [2021] for LMDP is \( S \).

In this paper, we resolve the first two questions and partially answer the third.

1.1 Main contributions and technical novelties

We obtain the following new results:

- **Near-optimal regret guarantee for LMDPs.** We present an algorithm framework for LMDPs with context in hindsight. This framework can be instantiated with a plug-in solver for planning problems. We consider two types of solvers, one model-optimistic and one value-optimistic, and prove their regret bound to be \( \tilde{O}(\sqrt{MTSAK}) \) where \( \Gamma \leq S \) is the maximum transition degree of any state-action pair. Compared with the result in Kwon et al. [2021], ours only requires the total reward to be bounded whereas they required a bounded reward for each step. Furthermore, we improve the \( H \)-dependence from \( \sqrt{H} \) to logarithmic, making our bound (nearly) horizon-free. Lastly, our bound scales with \( \sqrt{SAK} \), which is strictly better than \( S \) in their bound.

The main technique of our model-optimistic algorithm is to use a Bernstein-type confidence set on each position of transition dynamics, leading to a small Bellman error. The main difference between our value-optimistic algorithm and Kwon et al. [2021]’s is that we use a bonus depending on the variance of next-step values according to Bennett’s inequality, instead of using Bernstein’s inequality. It helps propagate the optimism from the last step to the first step, avoiding the \( H \)-dependency. We analyse these two solvers in a unified way, as their Bellman error are of the same order.

- **New regret lower bound for LMDPs.** We obtain a novel \( \Omega(\sqrt{MSAK}) \) regret lower bound for LMDPs. This regret lower bound shows the dependency on \( M \) is necessary for LMDPs. Notably the lower bound also implies \( \tilde{O}(\sqrt{MTSAK}) \) upper bound is optimal up to a \( \sqrt{\Gamma} \) factor. Furthermore, our lower bound holds even for \( \Gamma = 2 \), which shows our upper bound is minimax optimal for a class of LMDPs with \( \Gamma = O(1) \).

Our proof relies on new constructions of hard instances, different from existing ones for MDPs [Domingues et al., 2021]. In particular, we use a two-phase structure to construct hard instances (cf. Figure 1). Furthermore, the previous approaches for proving lower bounds of MDPs do not work on LMDPs. For example, in the MDP instance of Domingues et al. [2021], the randomness comes from the algorithm and the last transition step before entering the good state or bad state. In an LMDP, the randomness of sampling the MDP from multiple MDPs must also be considered. Such randomness not only dilutes the value function by averaging over each MDP, but also divides the pushforward measure (see Page 3 of Domingues et al. [2021]) into \( M \) parts. As a result, the \( M \) terms in KL divergence in Equation (2) of Domingues et al. [2021] and that in Equation (10) cancels out — the final lower bound does not contain \( M \). To overcome this, we adopt the symmetrization technique from theoretical computer science. This novel technique is helps generalize the bounds from a single-party result to a multiple-party result, which may give rise to a tighter lower bound.

\(^1\)Their original bound is \( \tilde{O}(\sqrt{MS^2AH^2K}) \) with the scaling that the reward from each step is bounded by 1. We rescale the reward to be bounded by \( 1/H \) in order to make the total reward from each episode bounded by 1, which is the setting we consider.
2 Related work

LMDPs. As shown by Steimle et al. [2021], in the general cases, optimal policies for LMDPs are history dependent and P-SPACE hard to find. This is different from standard MDP cases where there always exists an optimal history-independent policy. However, even finding the optimal history-independent policy is NP-hard [Littman, 1994]. Chades et al. [2012] provided heuristics for finding the optimal history-independent policy.

Kwon et al. [2021] investigated the sample complexity and regret bounds of LMDPs. Specifically, they presented an exponential lower-bound for general LMDPs without context in hindsight, and then they derived an algorithm with polynomial sample complexity and sub-linear regret for two special cases (with context in hindsight, or δ-strongly separated MDPs).

LMDP has been studied as a type of multi-task RL [Taylor and Stone, 2009, Brunskill and Li, 2013, Liu et al., 2016, Hallak et al., 2015]. It has been applied to model combinatorial optimization problems [Zhou et al., 2022]. There are also some related studies such as model transfer [Lazaric, 2012, Zhang and Wang, 2021] and contextual decision processes [Jiang et al., 2017]. In empirical works, LMDP has has wide applications in multi-task RL [Yu et al., 2020], meta RL Iakovleva et al. [2020], Finn et al. [2018], latent-variable MDPs [Ramamoorthy et al., 2013] and hidden parameter MDPs [Doshi-Velez and Konidaris, 2016, Yao et al., 2018].

Regret Analysis for MDPs. LMDPs are generalizations of MDPs, so some previous approaches to solving MDPs can provide insights. There is a long line of work on regret analysis for MDPs [Azar et al., 2017, Dann et al., 2017, 2019, Zanette and Brunskill, 2019, Zhang et al., 2021a]. In this paper, we focus on time-homogeneous, finite horizon, undiscounted MDPs whose total reward is upper-bounded by 1. Recent work showed in this setting the regret can be (nearly) horizon-free for tabular MDPs [Wang et al., 2020], Zhang et al. [2021a, 2021b], Ren et al. [2021]. Importantly these results indicate RL may not be more difficult than bandits in the minimax sense. More recent work generalized the horizon-free results to other MDP problems [Zhang et al., 2021b, Kim et al., 2021, Tarbouriech et al., 2021, Zhou and Gu, 2022]. However, all existing work with horizon-free guarantees only considered single-environment problems. Ours is the first horizon-free guarantee that goes beyond MDP.

Neu and Pike-Burke [2020] summarized up the “optimism in the face of uncertainty” (OFU) principle in RL. They named two types of optimism: ① model-optimistic algorithms construct confidence sets around empirical transitions and rewards, and select the policy with the highest value in the best possible models in these sets. ② value-optimistic algorithms construct upper bounds on the optimal value functions, and select the policy which maximizes this optimistic value function. Our paper follows their idea and provide one algorithm for each type of optimism.

3 Problem setup

In this section, we give a formal definition of Latent Markov Decision Processes (Latent MDPs).

Notations. For any event $\mathcal{E}$, we use $\mathbb{I}[\mathcal{E}]$ to denote the indicator function, i.e., $\mathbb{I}[\mathcal{E}] = 1$ if $\mathcal{E}$ holds and $\mathbb{I}[\mathcal{E}] = 0$ otherwise. For any set $X$, we use $\Delta(X)$ to denote the probability simplex over $X$. For any positive integer $n$, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. For any probability distribution $P$, we use $\text{supp}(P) = |P|_0$ to denote the size of support of $P$, i.e., $\sum_x \mathbb{I}[P(x) > 0]$. There are three ways to denote a $d$-dimensional vector (function): suppose $p$ is any parameter, $x(p) = (x_1(p), x_2(p), \ldots, x_d(p))$ if the indices are natural numbers, $x(i|p) = (x(i_1|p), x(i_2|p), \ldots, x(i_d|p))$ and $x(p|d) = (x(p_1), x(p_2), \ldots, x(p_d))$ if the indices are from the set $I = \{i_1, i_2, \ldots, i_d\}$. For any number $q$, we use $x^q$ to denote the vector $(x_1^q, x_2^q, \ldots, x_d^q)$. For two $d$-dimensional vectors $x$ and $y$, we use $x^\top y = \sum_i x_i y_i$ to denote the inner product. If $x$ is a probability distribution, we use $V(x, y) = \sum_i x_i(y_i - x^\top y)^2 = x^\top (y^2) - (x^\top y)^2$ to denote the empirical variance. We use $\ell = 2 \ln \left(\frac{2MSAHK}{\delta}\right)$ as a log term where $\delta$ is the confidence parameter.

3
3.1 Latent Markov Decision Process

Latent MDP [Kwon et al., 2021] is a collection of finitely many MDPs \( M = \{ M_1, M_2, \ldots, M_M \} \) where \( M = |M| \). All the MDPs share state set \( S \), action set \( A \) and horizon \( H \). Each MDP \( M_m = (S, A, H, \nu_m, P_m, R_m) \) has its own initial state distribution \( \nu_m \in \Delta(S) \), transition model \( P_m : S \times A \rightarrow \Delta(S) \) and a deterministic reward function \( R_m : S \times A \rightarrow [0, 1] \). Let \( w_1, w_2, \ldots, w_M \) be the mixing weights of MDPs such that \( w_m > 0 \) for any \( m \) and \( \sum_{m=1}^M w_m = 1 \).

Denote \( S = |S|, A = |A| \) and \( \Gamma = \max_{m,s,a} \text{supp}(P_m(|s,a)) \). \( \Gamma \) can be interpreted as the maximum degree of each transition, which is a quantity our regret bound depends on. Note we always have \( \Gamma \leq S \). In previous work, Lattimore and Hutter [2012] assumes \( \Gamma = 2 \), and Fruit et al. [2020] also has a regret bound that scales with \( \Gamma \).

In the worst case, the optimal policy of an LMDP is history-dependent and is PSPACE-hard to find (Corollary 1 and Proposition 3 in Steimle et al. [2021]). Aside from computational difficulty, storing a history-dependent policy needs a space which is exponentially large, so it is generally impractical. In this paper, we seek to provide a result for any fixed policy class \( \Pi \). For example, we can have \( \Pi \) to be the set of all history-independent deterministic policies to alleviate the space issue. Following previous work [Kwon et al., 2021], we assume access to oracles for planning and optimization. See Section 4 for the formal definitions.

We consider an episodic, finite-horizon and undiscounted reinforcement learning problem on LMDPs. In this problem, the agent interacts with the environment for \( K \) episodes. At the start of every episode, one MDP \( M_m \in M \) is randomly chosen with probability \( w_m \). Throughout the episode, the true context is hidden. The agent can only choose actions based on the history information up until current time. However, at the end of each episode (after \( H \) steps), the agent gets revealed the true context \( m \). This permits an unbiased model estimation for the LMDP. As in Cohen et al. [2020], the central difficulty is to estimate the transition, we also focus on learning \( P \) only. For simplicity, we assume that \( w, \nu \) and \( R \) are known to the agent, because they can be estimated easily. The assumption of deterministic rewards is also for simplicity. Our analysis can be extended to unknown and bounded-support reward distributions.

3.2 Value functions, Q-functions and alpha vectors

By convention, the expected reward of executing a policy on any MDP can be defined via value function \( V \) and Q-function \( Q \). Since for MDPs there is always an optimal policy which is history-independent, \( V \) and \( Q \) only need the current state and action as parameters.

However, these notations fall short of history-independent policies under the LMDP setting. The full information is encoded in the history, so here we use a more generalized definition called alpha vector (following notations in Kwon et al. [2021]). For any time \( t \geq 1 \), let \( h_t = (s,a,r)_{t-1:s} \) be the history up until time \( t \). Define \( H_t \) as the set of histories observable at time step \( t \), and \( H := \cup_{t=1}^H H_t \) as the set of all possible histories. We define the alpha vectors \( \alpha_m^\pi(h) \) for \( (m,h) \in [M] \times H \) as follows:

\[
\alpha_m^\pi(h) := \mathbb{E} \left[ \sum_{t=1}^H R_m(s_{t'}, a_{t'}) \mid M_m, \pi, h_t = h \right],
\]

\[
\alpha_m^\pi(h, a) := \mathbb{E} \left[ \sum_{t=1}^H R_m(s_{t'}, a_{t'}) \mid M_m, \pi, (h_t, a_t) = (h, a) \right].
\]

The alpha vectors are indeed value functions and Q-functions on each individual MDP.

Next, we introduce the concepts of belief state to show how to do planning in LMDP. Let \( b_m(h) \) denote the belief state over \( M \) MDPs corresponding to a history \( h \), i.e., the probability of the true MDP being \( M_m \) conditioned on observing history \( h \). We have the following recursion:

\[
b_m(s) = \frac{w_m \nu_m(s)}{\sum_{m'=1}^M w_m' \nu_m'(s)} \quad \text{and} \quad b_m(ha's') = \frac{b_m(h)P_m(s'|s,a)1[r = R_m(s,a)]}{\sum_{m'=1}^M b_m(h)P_m'(s'|s,a)1[r = R_m'(s,a)]}.
\]
The value functions and Q-functions for LMDP is defined via belief states and alpha vectors:

\[ V^\pi(h) := b(h)^\top \alpha^\pi(h) \quad \text{and} \quad Q^\pi(h, a) := b(h)^\top \alpha^\pi(h, a). \]

Direct computation (see Appendix B.1) gives

\[
V^\pi(h) = \sum_{a \in A} \pi(a|h) \left( b(h)^\top R(s, a) + \sum_{s' \in S} \sum_{m' = 1}^M b_{m'}(h) P_{m'}(s'|s, a) \mathbb{1}[r = R_{m'}(s, a)] V^\pi(h a r s') \right). 
\]

So planning in LMDP can be viewed as planning in belief states. For the optimal history-dependent policy, we can select

\[
\pi(h) = \arg \max_{a \in A} \left( b(h)^\top R(s, a) + \sum_{s' \in S} \sum_{m' = 1}^M b_{m'}(h) P_{m'}(s'|s, a) \mathbb{1}[r = R_{m'}(s, a)] V^\pi(h a r s') \right). 
\]

using dynamic programming in descending order of h’s length.

### 3.3 Performance measure.

We use cumulative regret to measure the algorithm’s performance. The optimal policy is \( \pi^* = \arg \max_{\pi \in \Pi} V^\pi \), which also does not know the context when interacting with the LMDP. Suppose the agent interacts with the environment for \( K \) episodes, and for each episode \( k \) a policy \( \pi^k \) is played. The regret is defined as

\[ \text{Regret}(K) := KV^* - \sum_{k=1}^{K} V^{\pi^k}. \]

### 4 Main algorithms and results

In this section, we present two algorithms, and show their minimax regret guarantee. The first is to use a Bernstein confidence set on transition probabilities, which was first applied to SSP in Cohen et al. [2020] to derive a horizon-free regret. This algorithm uses a bi-level optimization oracle: for the inner layer, an oracle is needed to find the optimal policy inside \( \Pi \) under a given LMDP; for the outer layer, an oracle finds the best transition inside the confidence set which maximizes the optimal expected reward. The second is to adapt the Monotoic Value Propagation (MVP) algorithm [Zhang et al., 2021a] to LMDPs. This algorithm requires an oracle to solve an LMDP with dynamic bonus: the bonuses depends on the variances of the next-step alpha vector. Both algorithms enjoy the following regret guarantee.

**Theorem 1.** For both the Bernstein confidence set for LMDP (Algorithm 1 combined with Algorithm 2) and the Monotonic Value Propagation for LMDP (Algorithm 1 combined with Algorithm 3), with probability at least \( 1 - \delta \), we have that

\[
\text{Regret}(K) = O \left( \sqrt{MSAHK \ln \left( \frac{MSAHK}{\delta} \right)} + MS^2A \ln^2 \left( \frac{MSAHK}{\delta} \right) \right).
\]

As we have discussed, our result improves upon Kwon et al. [2021], and has only logarithmic dependency on the planning horizon \( H \). We also have a lower order which scales with \( S^2 \). We note that even in the standard MDP setting, it remains a major open problem how to obtain minimax optimal regret bound with no lower order term [Zhang et al., 2021a].

Below we describe the details of our algorithms.
The Bernstein confidence set is constructed as below:

$$\mathcal{P}^{k+1} = \left\{ \tilde{P} : \left| \tilde{P}_{m} - \tilde{P}^{k}_{m}(s'|s,a) \right| \leq 2 \sqrt{\frac{\tilde{P}^{k}_{m}(s'|s,a) \delta}{N^{k}_{m}(s,a)}} + \frac{5\delta}{N^{k}_{m}(s,a)}, \forall (m,s,a,s') \in [M] \times S \times A \times S \right\}. $$

Notice that we do not change the reward function, so we still have the total reward of any trajectory upper-bounded by 1.
**Policy solver.** The policy solver is in Algorithm 2. It solves a two-step optimization problem on Line 2: for the inner problem, given a transition model $\tilde{P}$ and all other known quantities $w, \nu, R$, it needs a planning oracle to find the optimal policy; for the outer problem, it needs to find the optimal transition model. For planning, we can use the method presented in Equation (1). For the outer problem, we can use Extended Value Iteration as in Auer et al. [2008], Fruit et al. [2020], Filippi et al. [2010], Cohen et al. [2020]. For notational convenience, we denote the alpha vectors and value functions calculated by $\tilde{P}^k$ and $\pi^k$ as $\tilde{r}^k$ and $\tilde{V}^k$.

| Algorithm 2 Solver-L-Bernstein |
|----------------------------------|
| 1: Construct $P^{k+1}$ using Equation (2). |
| 2: Find $\tilde{P}^{k+1} \leftarrow \max_{P \in P^{k+1}} \left( \max_{\pi \in \Pi} V^\pi_P \right)$. |
| 3: Find $\pi^{k+1} \leftarrow \max_{\pi \in \Pi} V^\pi_{\tilde{P}^{k+1}}$. |
| 4: Return: $\pi^{k+1}$. |

| 4.2 Monotonic Value Propagation for LMDP |

We introduce a value-optimistic approach by calculating a variance-dependent bonus. This technique was originally used to solve standard MDPs [Zhang et al., 2021a].

**Optimistic LMDP construction.** The optimistic model contains a bonus function, which is inductively defined using the next-step alpha vector. In episode $k$, or any policy $\pi$, assume the alpha vector for any history with length $t+1$ is calculated, then for any history $h$ with length $t$, the bonus is defined as follows:

$$B^k_m(h, a) := \max \left\{ 4 \sqrt{\text{supp} \left( \tilde{P}^k_m(\cdot, a) \right) \lor \left( \tilde{P}^k_m(\cdot, a), \tilde{r}^k_m(har) \right) \lor \frac{16S_l}{N^k_m(s, a)}}, \frac{16S_l}{N^k_m(s, a)} \right\},$$

where $r = R_m(s, a)$. Next, the alpha vector of history $h$ is:

$$\tilde{\alpha}^\pi_m(h) := \min \left\{ R_m(s, a) + B^k_m(h, a) + \tilde{P}^k_m(\cdot, a) \lor \tilde{r}^k_m(har), 1 \right\}, \text{ where } a = \pi(h).$$

Finally, the value function is:

$$\tilde{V}^\pi := \sum_{m=1}^M \sum_{s \in S} w_m \nu_m(s) \tilde{\alpha}^\pi_m(s).$$

**Policy solver.** The policy solver is in Algorithm 3. It finds the policy maximizing the optimistic value, with a dynamic bonus function depending on the policy itself. This solver is tractable if we only care about deterministic policies in $\Pi$. This restriction is reasonable because for the original LMDP there always exists an optimal policy which is deterministic. Further, according to the proof of Lemma 13, we only need a policy which has optimistic value no less than that of the optimal value. Thus, there always exists an exhaustive search algorithm for this solver, which enumerates each action at each history.

| Algorithm 3 Solver-L-MVP |
|---------------------------|
| 1: Use the optimistic model defined in Equation (3), Equation (4) and Equation (5). |
| 2: Find $\pi^{k+1} \leftarrow \max_{\pi \in \Pi} \tilde{V}^\pi$. |
| 3: Return: $\pi^{k+1}$. |
5 Regret lower bound

In this section, we present a regret lower bound for the unconstrained policy class, i.e., when $H$ contains all possible history-dependent policies.

First, we note that this lower bound cannot be directly reduced to solving $M$ MDPs (with the context revealed at the beginning of each episode). Because simply changing the time of revealing the context results in the change of the optimal policy and its value function.

At a high level, our approach is to transform the problem of context in hindsight into a problem of essentially context being told beforehand, while not affecting the optimal value function. To achieve this, we can use a small portion of states to encode the context at the beginning, then the optimal policy can extract information from them and fully determine the context.

After the transformation, we can view the LMDP as a set of independent MDPs, so it is natural to leverage results from MDP lower bounds. Intuitively, since the lower bound of MDP is $\sqrt{SAK}$, and each MDP is assigned roughly $\frac{K}{M}$ episodes, the lower bound of LMDP is $M\sqrt{SA \cdot \frac{K}{M}} = \sqrt{MSAK}$. To formally prove this, we adopt the symmetrization technique from the theoretical computer science community [Phillips et al., 2012, Woodruff and Zhang, 2014, Fischer et al., 2017, Vempala et al., 2020]. When an algorithm interacts with an LMDP, we can focus on each MDP, while viewing the interactions with other MDPs as irrelevant – we hard code the other MDPs into the algorithm, deriving an algorithm for an MDP. In other words, we can insert an MDP into any of the $M$ positions, and they are all symmetric to the algorithm’s view. So, the regret can be averaged over to each MDP.

The main theorem is shown here, before we introduce the construction of LMDP instances. Its proof is placed in Appendix B.4.

**Theorem 2.** Assume that $S \geq 6, A \geq 2$ and $M \leq \lfloor \frac{S}{2} \rfloor$. For any algorithm $\pi$, there exists an LMDP $M_{\pi}$ such that, for $K \geq \Omega(M^2 + RSA)$, its expected regret in $M_{\pi}$ after $K$ episodes satisfies

$$R(M_{\pi}, \pi, K) := \mathbb{E} \left[ \sum_{k=1}^{K} (V^* - V^k) \mid M_{\pi}, \pi \right] = \Omega \left( \sqrt{MSAK} \right).$$

Several remarks are in the sequel. ① This is the first regret lower bound for LMDPs with context in hindsight. To the best of our knowledge, the introduction of the symmetrization technique is novel to the construction of lower bounds in the field of RL. ② This lower bound matches the minimax regret upper bound (Theorem 1) up to logarithm factors, because in the hard instance construction $\Gamma = 2$. For general cases, our upper bound is optimal up to a $\sqrt{\Gamma}$ factor. ③ There is a limitation of $M$, which could be at most $\lfloor \frac{S}{2} \rfloor$, though an exponentially large $M$ is not practical.

5.1 Hard instance construction

Since $M \leq \lfloor \frac{S}{2} \rfloor$, we can always find an integer $d_1$ such that $d_1 \leq \frac{S}{2}$ and $M \leq d_1!$. Since $S \geq 6$ and $d_1 \leq \frac{S}{2}$, we can always find an integer $d_2$ such that $d_2 \geq 1$ and $2^{d_2} - 1 \leq S - d_1 - 2 < 2^{d_2+1} - 1$. We can construct a two-phase structure, each phase containing $d_1$ and $d_2$ steps respectively.

The hard instance uses similar components as the MDP instances in Domingues et al. [2021]. We construct a collection of LMDPs $\mathcal{C} := \{M(\ell', \alpha^*) : (\ell', \alpha^*) \in [L]^M \times [A]^M\}$, where we define $L := 2^{d_2-1} = \Theta(S)$. For a fixed pair $(\ell', \alpha^*) = ((\ell'_1, \ell'_2, \ldots, \ell'_m), (a'_1, a'_2, \ldots, a'_m))$, we construct the LMDP $M(\ell', \alpha^*)$ as follows.

5.1.1 The LMDP layout

All MDPs in the LMDP share the same logical structure. Each MDP contains two phases: the encoding phase and the guessing phase. The encoding phase contains $d_1$ states, sufficient for encoding the context because $M \leq d_1!$. The guessing phase contains a number guessing game with $C := LA = \Theta(SA)$ choices. If the agent makes a correct choice, it receives an expected reward slightly larger than $\frac{s}{2}$. Otherwise, it receives an expected reward of $\frac{s}{2}$. 

8
5.1.2 The detailed model

Now we give more details about our construction. Figure 1 shows an example of the model with $M = 2$, $S = 11$, arbitrary $A \geq 2$ and $H \geq 6$.

**States.** The states in the encoding phase are $e_1, \ldots, e_{d_1}$. The states in the guessing phase are $s_1, \ldots, s_N$ where $N = \sum_{i=0}^{d_2-1} 2^i = 2^{d_2} - 1$. There is a good state $g$ for reward and a terminal state $t$. All the unused states can be ignored.

**Transitions.** The weights are equal, i.e., $w_m = \frac{1}{M}$. We assign a unique integer in $m \in [M]$ to each MDP as a context. Each integer $m$ is uniquely mapped to a permutation $\sigma(m) = (\sigma_1(m), \sigma_2(m), \ldots, \sigma_{d_1}(m))$. Then the initial state distribution is $\nu_m(e_{\sigma_1(m)}) = 1$. The transitions for the first $d_1$ steps are: for any $(m, a) \in [M] \times A$,

$$P_m(e_{\sigma_i(m)} \mid e_{\sigma_{i-1}(m)}, a) = 1, \forall 1 \leq i \leq d_1 - 1; \quad P_m(s_1 \mid e_{\sigma_{d_1}(m)}, a) = 1.$$  

This means, in the encoding phase, whatever the agent does is irrelevant to the state sequence it observes.

The guessing phase is a binary tree which we modify from Section 3.1 of Domingues et al. [2021] (here we equal each action $a$ to an integer in $[A]$): for any $(m, a) \in [M] \times A$,

$$P_m(s_{2i+(a \mod 2)} \mid s_{i}, a) = 1, \forall 1 \leq i \leq 2^{d_2-1} - 1.$$
For the tree leaves $\mathcal{L} = \{ s_\ell : 2^{d_\ell - 1} \leq \ell \leq 2^{d_\ell} - 1 \}$ (notice that $|\mathcal{L}| = L$), we construct: for any $(m, \ell, a) \in [M] \times \mathcal{L} \times \mathcal{A}$,

\[ P_m(t \mid s_\ell, a) = \frac{1}{2} - \varepsilon \mathbb{1}[\ell = \ell^*_m, a = a^*_m], \quad P_m(g \mid s_\ell, a) = \frac{1}{2} + \varepsilon \mathbb{1}[\ell = \ell^*_m, a = a^*_m]. \]

Recall that we denote $C = LA$ as the effective number of choices. The agent needs to first find the correct leaf by inputting its binary representation correctly, then choose the correct action.

The good state is temporary between the guessing phase and the terminal state: if the agent is at $g$ and makes any action, it enters $t$. The terminal state is self-absorbing. For any $(m, a) \in [M] \times \mathcal{A}$,

\[ P_m(t \mid g, a) = 1, \quad P_m(t \mid t, a) = 1. \]

All the unmentioned probabilities are 0. Clearly, this transition model guarantees that $\text{supp}(P_m(\cdot \mid s, a)) \leq 2$ for any pair of $(m, s, a) \in [M] \times \mathcal{S} \times \mathcal{A}$.

**The rewards.** The only non-zero rewards are $R_m(g, a) = 1$ for any $(m, a) \in [M] \times \mathcal{A}$. Since this state-action pair is visited at most once in any episode, this reward guarantees that in a single episode the cumulative reward is either 0 or 1.

### 6 Conclusion

In this paper, we present two different RL algorithms (one model-optimistic and one value-optimistic) for LMDPs with context in hindsight, both achieving $O(\sqrt{MSAK})$ regret. This is the first (nearly) horizon-free regret bound for LMDP with context in hindsight. We also provide a regret lower bound for this setting, which is $\Omega(\sqrt{MSAK})$. In this lower bound, $\Gamma = 2$, so the upper bound is minimax optimal for the subclass of LMDPs with constant $\Gamma$. One future direction is to obtain a minimax regret bound for LMDPs for the $\Gamma = \Theta(S)$ case. For example, can we derive a regret lower bound of $\Omega(\sqrt{M^2S^2K})$? On the other hand, it is also possible to remove the $\sqrt{\Gamma}$ in our upper bound. We believe this will require properties beyond the standard Bellman-optimality condition for standard MDPs.

### References

[1] Peter Auer, Thomas Jaksch, and Ronald Ortner. Near-optimal regret bounds for reinforcement learning. *Advances in neural information processing systems*, 21, 2008.

[2] Mohammad Gheshlaghi Azar, Ian Osband, and Rémi Munos. Minimax regret bounds for reinforcement learning. In *ICML*, 2017.

[3] Emma Brunskill and Lihong Li. Sample complexity of multi-task reinforcement learning. *ArXiv*, abs/1309.6821, 2013.

[4] I. Chades, Josie Carwardine, Tara G. Martin, Sam Nicol, Régis Sabbadin, and Olivier Buffet. Mmondps: A solution for modelling adaptive management problems. In *AAAI*, 2012.

[5] Alon Cohen, Haim Kaplan, Y. Mansour, and Aviv Rosenberg. Near-optimal regret bounds for stochastic shortest path. In *ICML*, 2020.

[6] Christoph Dann, Tor Lattimore, and Emma Brunskill. Unifying pac and regret: Uniform pac bounds for episodic reinforcement learning. In *NIPS*, 2017.

[7] Christoph Dann, Lihong Li, Wei Wei, and Emma Brunskill. Policy certificates: Towards accountable reinforcement learning. In *International Conference on Machine Learning*, pages 1507–1516. PMLR, 2019.
Omar Darwiche Domingues, Pierre Ménard, Emilie Kaufmann, and Michal Valko. Episodic reinforcement learning in finite mdps: Minimax lower bounds revisited. In Vitaly Feldman, Katrina Ligett, and Sivan Sabato, editors, Proceedings of the 32nd International Conference on Algorithmic Learning Theory, volume 132 of Proceedings of Machine Learning Research, pages 578–598. PMLR, 16–19 Mar 2021. URL https://proceedings.mlr.press/v132/domingues21a.html.

Finale Doshi-Velez and George Konidaris. Hidden parameter markov decision processes: A semiparametric regression approach for discovering latent task parametrizations. In IJCAI: proceedings of the conference, volume 2016, page 1432. NIH Public Access, 2016.

Sarah Filippi, Olivier Cappé, and Aurélien Garivier. Optimism in reinforcement learning and kullback-leibler divergence. In 2010 48th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pages 115–122. IEEE, 2010.

Chelsea Finn, Kelvin Xu, and Sergey Levine. Probabilistic model-agnostic meta-learning. Advances in neural information processing systems, 31, 2018.

Orr Fischer, Shay Gershtein, and Rotem Oshman. On the multiparty communication complexity of testing triangle-freeness. In Proceedings of the ACM Symposium on Principles of Distributed Computing, pages 111–120, 2017.

Ronan Fruit, Matteo Pirotta, and Alessandro Lazaric. Improved analysis of ucr12 with empirical bernstein inequality. arXiv preprint arXiv:2007.05456, 2020.

Assaf Hallak, Dotan Di Castro, and Shie Mannor. Contextual markov decision processes. ArXiv, abs/1502.02259, 2015.

Ekaterina Iakovleva, Jakob Verbeek, and Karteek Alahari. Meta-learning with shared amortized variational inference. In International Conference on Machine Learning, pages 4572–4582. PMLR, 2020.

Nan Jiang, Akshay Krishnamurthy, Alekh Agarwal, John Langford, and Robert E Schapire. Contextual decision processes with low bellman rank are pac-learnable. In International Conference on Machine Learning, pages 1704–1713. PMLR, 2017.

Yeoneung Kim, Insoon Yang, and Kwang-Sung Jun. Improved regret analysis for variance-adaptive linear bandits and horizon-free linear mixture mdps. arXiv preprint arXiv:2111.03289, 2021.

Jeongyeol Kwon, Yoathan Efroni, Constantine Caramanis, and Shie Mannor. Rl for latent mdps: Regret guarantees and a lower bound, 2021.

Tor Lattimore and Marcus Hutter. Pac bounds for discounted mdps. In International Conference on Algorithmic Learning Theory, pages 320–334. Springer, 2012.

Alessandro Lazaric. Transfer in reinforcement learning: A framework and a survey. In Reinforcement Learning, 2012.

Michael L Littman. Memoryless policies: Theoretical limitations and practical results. In From Animals to Animats 3: Proceedings of the third international conference on simulation of adaptive behavior, volume 3, page 238. Cambridge, MA, 1994.

Yao Liu, Zhaohan Daniel Guo, and Emma Brunskill. Pac continuous state online multitask reinforcement learning with identification. In AAMAS, 2016.

Andreas Maurer and Massimiliano Pontil. Empirical bernstein bounds and sample-variance penalization. In COLT, 2009.

Gergely Neu and Ciara Pike-Burke. A unifying view of optimism in episodic reinforcement learning. Advances in Neural Information Processing Systems, 33:1392–1403, 2020.
Jeff M Phillips, Elad Verbin, and Qin Zhang. Lower bounds for number-in-hand multiparty communication complexity, made easy. In *Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms*, pages 486–501. SIAM, 2012.

Subramanian Ramamoorthy, MM Hassan Mahmud, Benjamin Rosman, and Pushmeet Kohli. Latent-variable mdp models for adapting the interaction environment of diverse users, 2013.

Tongzheng Ren, Jialian Li, Bo Dai, Simon S Du, and Sujay Sanghavi. Nearly horizon-free offline reinforcement learning. *Advances in neural information processing systems*, 34:15621–15634, 2021.

Lauren N Steimle, David L Kaufman, and Brian T Denton. Multi-model markov decision processes. *IISE Transactions*, 53(10):1124–1139, 2021.

Jean Tarbouriech, Runlong Zhou, Simon S Du, Matteo Pirotta, Michal Valko, and Alessandro Lazaric. Stochastic shortest path: Minimax, parameter-free and towards horizon-free regret. volume 34, 2021.

Matthew E Taylor and Peter Stone. Transfer learning for reinforcement learning domains: A survey. *Journal of Machine Learning Research*, 10(7), 2009.

Santosh S Vempala, Ruosong Wang, and David P Woodruff. The communication complexity of optimization. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1733–1752. SIAM, 2020.

Ruosong Wang, Simon S Du, Lin F Yang, and Sham M Kakade. Is long horizon reinforcement learning more difficult than short horizon reinforcement learning? *arXiv preprint arXiv:2005.00527*, 2020.

David P Woodruff and Qin Zhang. An optimal lower bound for distinct elements in the message passing model. In *Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms*, pages 718–733. SIAM, 2014.

Jiayu Yao, Taylor Killian, George Konidaris, and Finale Doshi-Velez. Direct policy transfer via hidden parameter markov decision processes. In *LLARLA Workshop, FAIM*, volume 2018, 2018.

Tianhe Yu, Deirdre Quillen, Zhanpeng He, Ryan Julian, Karol Hausman, Chelsea Finn, and Sergey Levine. Meta-world: A benchmark and evaluation for multi-task and meta reinforcement learning. In *Conference on robot learning*, pages 1094–1100. PMLR, 2020.

Andrea Zanette and Emma Brunskill. Tighter problem-dependent regret bounds in reinforcement learning without domain knowledge using value function bounds. In *ICML*, 2019.

Chicheng Zhang and Zhi Wang. Provably efficient multi-task reinforcement learning with model transfer. *ArXiv*, abs/2107.08622, 2021.

Zihan Zhang, Simon S Du, and Xiangyang Ji. Nearly minimax optimal reward-free reinforcement learning. *arXiv preprint arXiv:2010.05901*, 2020.

Zihan Zhang, Xiangyang Ji, and Simon Shaolei Du. Is reinforcement learning more difficult than bandits? a near-optimal algorithm escaping the curse of horizon. In *COLT*, 2021a.

Zihan Zhang, Jiaqi Yang, Xiangyang Ji, and Simon S Du. Improved variance-aware confidence sets for linear bandits and linear mixture mdp. *Advances in Neural Information Processing Systems*, 34:4342–4355, 2021b.

Zihan Zhang, Xiangyang Ji, and Simon Du. Horizon-free reinforcement learning in polynomial time: the power of stationary policies. In *Conference on Learning Theory*, pages 3858–3904. PMLR, 2022.

Dongruo Zhou and Quanquan Gu. Computationally efficient horizon-free reinforcement learning for linear mixture mdps. *arXiv preprint arXiv:2205.11507*, 2022.
Runlong Zhou, Yuandong Tian, Yi Wu, and Simon S Du. Understanding curriculum learning in policy optimization for solving combinatorial optimization problems. *arXiv preprint arXiv:2202.05423*, 2022.
Appendix

A Technical lemmas

Lemma 3 (Anytime Azuma, Theorem D.1 in Cohen et al. [2020]). Let $(X_n)_{n=0}^\infty$ be a martingale difference sequence with respect to the filtration $(\mathcal{F}_n)_{n=0}^\infty$ such that $|X_n| \leq B$ almost surely. Then with probability at least $1 - \delta$,

$$\left| \sum_{i=1}^n X_i \right| \leq B\sqrt{n \ln \frac{2n}{\delta}}, \quad \forall n \geq 1.$$

Lemma 4 (Bennett’s Inequality, Theorem 3 in Maurer and Pontil [2009]). Let $Z, Z_1, \ldots, Z_n$ be i.i.d. random variables with values in $[0, b]$ and let $\delta > 0$. Define $V[Z] = \mathbb{E}[(Z - \mathbb{E}[Z])^2]$. Then we have

$$\mathbb{P} \left[ \left| \sum_{i=1}^n Z_i \right| > \sqrt{n \ln \left( \frac{2b}{\delta} \right) + \frac{b \ln(2/\delta)}{n}} \right] \leq \delta.$$

Lemma 5 (Theorem 4 in Maurer and Pontil [2009]). Let $Z, Z_1, \ldots, Z_n$ (n $\geq 2$) be i.i.d. random variables with values in $[0, b]$ and let $\delta > 0$. Define $\tilde{Z} = \frac{1}{n}Z$, and $\tilde{V}_n = \frac{1}{n} \sum_{i=1}^n (Z_i - \tilde{Z})^2$. Then we have

$$\mathbb{P} \left[ \left| \sum_{i=1}^n Z_i \right| > \sqrt{\frac{2\tilde{V}_n \ln(2/\delta)}{n-1} + \frac{7b \ln(2/\delta)}{3(n-1)}} \right] \leq \delta.$$

Lemma 6 (Lemma 30 in Tarbouriech et al. [2021]). Let $(M_n)_{n\geq0}$ be a martingale such that $M_0 = 0$ and $|M_n - M_{n-1}| \leq c$ for some $c > 0$ and any $n \geq 1$. Let $\text{Var}_k = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[(M_k - M_{k-1})^2|\mathcal{F}_{k-1}]$ for $n \geq 0$, where $\mathcal{F}_k = \sigma(X_4, \ldots, M_k)$. Then for any positive integer $n$ and $\delta \in (0, 2(\ln 2/c^2)^{1/2})$, we have that

$$\mathbb{P} \left[ \left| M_n \right| \geq 2\sqrt{2\text{Var}_n(\ln(2/c^2)) + \ln(2/\delta)} \right] \leq \delta.$$

Lemma 7 (Lemma 11 in Zhang et al. [2021a]). Let $\lambda_1, \lambda_2, \lambda_4 \geq 0$, $\lambda_3 \geq 1$ and $\lambda' = \log_2 \lambda_1$. Let $a_1, a_2, \ldots, a_{i'}$ be non-negative reals such that $a_i \leq \lambda_1$ and $a_i \leq \lambda_2 \sqrt{a_{i+1} + 2^{i+1} \lambda_3 + \lambda_4}$ for any $1 \leq i \leq i'$. Then we have that $a_{i'} \leq \max\{\lambda_2 + \sqrt{\lambda_2^2 + \lambda_4^2}, \lambda_2 \sqrt{8\lambda_3 + \lambda_4}\}$.

B Skipped proofs

B.1 Omitted calculation

Here we give the details for omitted calculations.

$$V^\pi(h) = \sum_{a \in A} \pi(a|h) \left( b(h)^\top R(s, a) + \sum_{s' \in \mathcal{S}, r \in \mathcal{M}} \sum_{m=1}^M b_m(h)P_m(s'|s, a)\mathbb{I}[r = R_m(s, a)]\alpha_m^*(\text{hars}') \right)$$

$$= \sum_{a \in A} \pi(a|h) \left( b(h)^\top R(s, a) + \sum_{s' \in \mathcal{S}, r \in \mathcal{M}} \sum_{m=1}^M b_m(h)P_m(s'|s, a)\mathbb{I}[r = R_m(s, a)] \sum_{m=1}^M b_m(\text{hars}')\alpha_m^*(\text{hars}') \right)$$

$$= \sum_{a \in A} \pi(a|h) \left( b(h)^\top R(s, a) + \sum_{s' \in \mathcal{S}, r \in \mathcal{M}} b_m(h)P_m(s'|s, a)\mathbb{I}[r = R_m(s, a)]V^\pi(\text{hars}') \right).$$

$$= \underbrace{Q^\pi(h, a)}_{-Q^\pi(h, a)}.$$
B.2 Unified analyses of Algorithm 1, Algorithm 2 and Algorithm 3

In this subsection, we present the proof of Theorem 1 by showing each step. However, when encountered with some lemmas, the proofs of lemmas are skipped and deferred to Appendix B.3.

Good events. The entire proof depends heavily on the good events defined below in Definition 8. They show that the estimation of transition probability is very close to the true value. We show in Lemma 9 that they happen with a high probability.

Definition 8 (Good events). For every episode $k$, define the following events:

$$\Omega_1^k := \{ (m, s, a, s') \in [M] \times S \times A \times S, \left| \hat{P}_m^k - P_m \right|(s'|s, a) \leq 2 \sqrt{\frac{\sup \left( \hat{P}_m^k \left( \cdot \middle| s, a \right), \alpha \right)}{N_m^k(s, a)}} + \frac{5t}{N_m^k(s, a)} \}.$$  \hspace{1cm} (6)

$$\Omega_2^k := \{ (m, s, a, s') \in [M] \times S \times A \times S, \left| \hat{P}_m^k - P_m \right|(s'|s, a) \leq \sqrt{\frac{2P_m(s'|s, a)}{N_m^k(s, a)}} + \frac{t}{N_m^k(s, a)} \}.$$  \hspace{1cm} (7)

Further, define $\Omega_1 := \cap_{k=1}^K \Omega_1^k$ and $\Omega_2 := \cap_{k=1}^K \Omega_2^k$.

Lemma 9. $\mathbb{P}[\Omega_1], \mathbb{P}[\Omega_2] \geq 1 - \delta$.

Assume that good events hold, then we have the following useful property:

Lemma 10. Conditioned on $\Omega_1$, we have that for any $(m, s, a, k) \in [M] \times S \times A \times [K]$, and any $S$-dimensional vector $\alpha$ such that $\|\alpha\|_\infty \leq 1$,

$$\left| \hat{P}_m^k - P_m \right|(s|a) \leq 2 \sqrt{\sup \left( \hat{P}_m^k \left( \cdot \middle| s, a \right), \alpha \right)} + \frac{5S_t}{N_m^k(s, a)}.$$ Similarly, conditioned on $\Omega_2$, we have that,

$$\left| \hat{P}_m^k - P_m \right|(s|a) \leq \sqrt{\frac{2\sup \left( P_m^k \left( \cdot \middle| s, a \right), \alpha \right)}{N_m^k(s, a)}} + \frac{S_t}{N_m^k(s, a)}.$$  \hspace{1cm}

Trigger property. Let $K$ be the set of indexes of episodes in which no update is triggered. By the update rule, it is obvious that $|K| \leq MSA(\log_2(HK) + 1) \leq MSA_\ell$. Let $t_0(k)$ be the first time an episode is triggered in the $k$-th episode if there is an update in this episode and otherwise $H + 1$. Define $X_0 = \{(k, t_0(k)) : k \in K\}$ and $X = \{(k, t) : k \in K, t_0(k) + 1 \leq t \leq H\}$. We will study quantities multiplied by the trigger indicator $I[(k, t) \notin X]$, which we denote using an extra “$\sim$”.

We will encounter a special type of summation, so we state it here.

Lemma 11. Let $\{w_k^t \geq 0 : (k, t) \in [K] \times [H]\}$ be a group of weights, then

$$\sum_{k=1}^K \sum_{t=1}^H I[(k, t) \notin X] \leq 3MSA_\ell, \quad \sum_{k=1}^K \sum_{i=1}^H \sqrt{\frac{w_k^t I[(k, t) \notin X]}{N_m^k(s_i^t, a_i^t)}} \leq \sqrt{3MSA_\ell \sum_{k=1}^K \sum_{i=1}^H w_k^t I[(k, t) \notin X]}.$$  \hspace{1cm}

B.2.1 Optimism

As a standard approach, we need to show that both Algorithm 2 and Algorithm 3 have optimism in value functions.

For Algorithm 2, it is straightforward. For each episode $k$, we choose the optimistic transition $\hat{P}_k$ with the maximum possible value. Lemma 9 shows that with probability at least $1 - \delta$, $\Omega_1$ holds, hence the true transition $P$ is inside the confidence set $\hat{P}_k$ for all $k \in [K]$. Therefore, $V^k \geq V^\star$.

Algorithm 3 relies on an important function introduced by Zhang et al. [2021a], so we cite it here:
Lemma 12 (Adapted from Lemma 14 in Zhang et al. [2021a]). For any fixed dimension $D$ and two constants $c_1, c_2$ satisfying $c_1^2 \leq c_2$, let $f : \Delta([D]) \times \mathbb{R}^D \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ with $f(p, v, n, \iota) = pv + \max \left\{ c_1 \sqrt{\frac{\langle v, p \rangle}{n}}, c_2 \right\}$. Then for all $p \in \Delta([D])$, $\|v\|_x \leq 1$ and $n, \iota > 0$,

1. $f(p, v, n, \iota)$ is non-decreasing in $v$, i.e.,
   \[ \forall v, v' \text{ such that } \|v\|_x, \|v'\|_x \leq 1, v \leq v', \text{ it holds that } f(p, v, n, \iota) \leq f(p, v', n, \iota); \]

2. $f(p, v, n, \iota) \geq pv + \frac{\kappa}{2} \sqrt{\frac{\langle v, p \rangle}{n}} + \frac{\kappa}{2} \frac{v}{n}$.

Due to the complex structure of LMDP, we cannot prove the strong optimism in Zhang et al. [2021a]. This is because in LMDP, the optimal policy cannot maximize all alpha vectors simultaneously, hence the optimal alpha vectors are not unique. As Algorithm 2, we can only show the optimism at the first step, which is stated in Lemma 13.

Lemma 13 (Optimism of Algorithm 3). Algorithm 3 satisfies that: Conditioned on $\Omega_1$, for any episode $k \in [K]$, $\tilde{V}^k \geq V^*$.

B.2.2 Regret decomposition

We introduce the Bellman error here. It contributes to the main order term in the regret.

Lemma 14 (Bellman error). Both Algorithm 2 and Algorithm 3 satisfy the following Bellman error bound: Conditioned on $\Omega_1$ and $\Omega_2$, for any $(m, h, a, k) \in [M] \times \mathcal{H} \times \mathcal{A} \times [K],$

\[
\alpha^k_m(h, a) - R_m(s, a) - P_m(\cdot|s, a)^\top \alpha^k_m(ha \cdot) \leq \min \{ \beta^k_m(h, a), 1 \},
\]

where $r = R_m(s, a)$ and

\[
\beta^k_m(h, a) = \sqrt{\frac{\Gamma V(P_m(\cdot|s, a), \alpha^k_m(ha \cdot))}{N_m^k(s, a)}} + 30S_t \frac{S_k}{N^k_m(s, a)}.
\]

Throughout the proof, we denote $\tilde{\beta}^k_m = \beta^k_m(h^k_t, a^k_t)I[(k, t) \notin \mathcal{X}]$.

Assume that optimism holds, then it is more natural to bound $\tilde{V}^k - V^*$ instead of $V^* - V^*$, because the underlying policies are the same for the former case. With simple manipulation, we decompose the regret into $X_1$ the Monte Carlo estimation term for the optimistic value, $X_2$ the Monte Carlo estimation term for the true value, $X_3$ the model estimation error, $X_4$ the Bellman error (main order term), and $|\mathcal{X}^c|$ the correction term for $1((k, t) \notin \mathcal{X})$.

\[
\text{Regret}(K) = \sum_{k=1}^{K} \left( V^* - V^* \right) \leq \sum_{k=1}^{K} \left( \tilde{V}^k - V^* \right)
\]

\[
= \sum_{k=1}^{K} \left( \tilde{V}^k - \alpha_m^k(s^k_t) \right) + \sum_{k=1}^{K} \left( \alpha_m^k(s^k_t) - \sum_{t=1}^{H} \tilde{r}_t \right) + \sum_{k=1}^{K} \left( \sum_{t=1}^{H} \tilde{r}_t - V^* \right)_{X_2}
\]

\[
\leq X_1 + X_2 + \sum_{k=1}^{K} \sum_{t=1}^{H} \left( \alpha_{\Pi^k_m}(h^k_t) - r_t - P_m(\cdot|s^k_t, a^k_t)^\top \alpha_{\Pi^k_m}(h^k_t a^k_t r^k_t) \right) I[(k, t) \notin \mathcal{X}) \leq \tilde{\beta}^k_m
\]
\[ + \sum_{k=1}^{K} \sum_{t=1}^{H} (P_m^k (\cdot | s^k_t, a^k_t) \top \tilde{\alpha}^k_{m^k} (h^k_t a^k_t r^k_t) - \tilde{\alpha}^k_{m^k} (h^k_{t+1})) \]
\[ + \sum_{k=1}^{K} \sum_{t=1}^{H} P_m^k (\cdot | s^k_t, a^k_t) \top \hat{\alpha}^k_{m^k} (h^k_t a^k_t r^k_t) (\mathbb{I}[(k, t) \notin \mathcal{X}] - \mathbb{I}[(k, t + 1) \notin \mathcal{X}]) \]
\[ \lessapprox X_1 + X_2 + X_3 + \sum_{k=1}^{K} \sum_{t=1}^{H} \hat{\tilde{\alpha}}^k_{m^k} (s^k_t, a^k_t) \top \tilde{\alpha}^k_{m^k} (h^k_t a^k_t r^k_t) \]
\[ \lessapprox X_1 + X_2 + X_3 + X_4 + |\mathcal{K}|, \]
where (i) is by \((k, 1) \in \mathcal{X}\) so \(\hat{\alpha}^k_{m^k} (s^k_t) = \hat{\alpha}^k_{m^k} (s^k_t)\); (ii) follows by Lemma 14 and checking the difference between \(\mathbb{I}[(k, t) \notin \mathcal{X}]\) and \(\mathbb{I}[(k, t + 1) \notin \mathcal{X}]\); (iii) is from the fact that \(\tilde{\alpha}^k \leq 1\), and the definition of \(t_0(k)\) and \(\mathcal{K}\).

**B.2.3 Bounding each term**

We start from the easier terms \(X_1\) and \(X_2\).

**Lemma 15.** With probability at least \(1 - \delta\), we have that \(X_1 \leq \sqrt{K_\ell}\).

**Lemma 16.** With probability at least \(1 - \delta\), we have that \(X_2 \leq \sqrt{K_\ell}\).

\(X_3\) is a martingale difference sequence. However, if we want to avoid polynomial dependency of \(H\), we cannot apply the Azuma’s inequality which scales as \(\sqrt{H}\). Instead, we use a variance-dependent martingale bound, and this changes \(X_3\) into a lower-order term of \(X_4\).

**Lemma 17.** With probability at least \(1 - \delta\), we have that \(X_3 \leq 2\sqrt{2X_4} + 5\delta\).

Here we show the bound for \(X_4\) and its proof first, next we prove Theorem 1. When bounding \(X_4\), we faced with another quantity \(X_5\), which is the summation of variances. We do not bound \(X_5\) explicitly. Instead, we derive a relation between \(X_5\) and \(X_4\) (Lemma 19), so finally we solve an inequality of \(X_4\).

**Lemma 18.** Conditioned on \(\Omega_1\) and \(\Omega_2\), with probability at least \(1 - \delta\), we have that \(X_4 \leq 46\sqrt{M}S^2AK_\ell + 947MS^2A\ell^2\).

**Proof.** From Lemma 14 and Lemma 11, we have that
\[ X_4 \leq 13 \sqrt{M}TS\ell^2 \sum_{k=1}^{K} \sum_{t=1}^{H} \mathbb{V} (P_m^k (\cdot | s^k_t, a^k_t), \hat{\alpha}^k_{m^k} (h^k_t a^k_t r^k_t)) \mathbb{I}[(k, t) \notin \mathcal{X}] + 90MS^2A\ell^2. \]

Applying Lemma 19, using \(\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}\), and loosening the constants, we have the following inequality:
\[ X_4 \leq 23\sqrt{M}TS\ell^2 + 209MS^2A\ell^2 + 23\sqrt{M}TS\ell^2 \cdot \sqrt{X_4}. \]
Since \(x \leq a + b\sqrt{x}\) implies \(x \leq b^2 + 2a\), we finally have
\[ X_4 \leq 46\sqrt{M}TS\ell^2 + 947MS^2A\ell^2. \]

This completes the proof. \(\square\)

We use the technique of higher-order variance expansion used by [Zhang et al., 2021a] to draw the relation between \(X_5\) and \(X_4\).

**Lemma 19.** Conditioned on \(\Omega_1\) and \(\Omega_2\), with probability at least \(1 - \delta\), we have that \(X_5 \leq 3(K + X_4) + 83\delta\).
B.2.4 Proof of Theorem 1

Finally, we are able to prove the main theorem.

Proof. From Lemma 15, Lemma 16, Lemma 17 and property of $K$, we have that, with probability at least $1 - 3\delta$, 

$$\text{Regret}(K) \leq 2\sqrt{K}t + 2\sqrt{2X_4t} + 5t + X_4 + MSAt.$$ 

Plugging in Lemma 18, using $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$, we finally have 

$$\text{Regret}(K) \leq 68\sqrt{MS^2AKt} + 1041MS^2At^2$$

holds with probability at least $1 - 6\delta$ (using Lemma 9). Rescaling $\delta \leftarrow \frac{\delta}{6}$ completes the proof. \(\square\)

B.3 Proof of the lemmas used in the minimax regret guarantee

Lemma 9. $\mathbb{P}[\Omega_1], \mathbb{P}[\Omega_2] \geq 1 - \delta$.

Proof. From Lemma 5 we have that, for any fixed $(m, s, a, s', k) \in [M] \times S \times A \times S \times [K]$ and $2 \leq N_m^k(s, a) \leq HK$, 

$$\mathbb{P} \left[ \left( \hat{P}_m^k - P_m \right) (s'|s, a) \right] > \frac{2\alpha'}{N_m^k(s, a) - 1} + \frac{7\alpha'}{3(N_m^k(s, a) - 1)} \leq \frac{\delta}{M \cdot S \cdot A \cdot S \cdot K \cdot HK},$$

where $\alpha' = \ln \left( \frac{2MS^2AK^2}{\delta} \right) \leq \frac{1}{2}$. From $\frac{1}{x} \leq \frac{2}{x}$ when $x \geq 2$ ($N_m^k(s, a) = 1$ is trivial), and applying union bound over all possible events, we have that $\mathbb{P}[\cap_{k=1}^K \Omega_m^k] \geq 1 - \delta$.

From Lemma 4 we have that, for any fixed $(m, s, a, s', k) \in [M] \times S \times A \times S \times [K]$ and $1 \leq N_m^k(s, a) \leq HK$, 

$$\mathbb{P} \left[ \left( \hat{P}_m^k - P_m \right) (s'|s, a) \right] > \frac{2\alpha'}{N_m^k(s, a) - 1} + \frac{7\alpha'}{3(N_m^k(s, a) - 1)} \leq \frac{\delta}{M \cdot S \cdot A \cdot S \cdot K \cdot HK}.$$ 

By taking a union bound over all possible events, we have that $\mathbb{P}[\cap_{k=1}^K \Omega_m^k] \geq 1 - \delta$. \(\square\)

Lemma 10. Conditioned on $\Omega_1$, we have that for any $(m, s, a, k) \in [M] \times S \times A \times [K]$, and any $S$-dimensional vector $\alpha$ such that $\|\alpha\|_\infty \leq 1$, 

$$\left| \left( \hat{P}_m^k - P_m \right) (\cdot | s, a)^\top \alpha \right| \leq 2\sqrt{\sup \left( \hat{P}_m (\cdot | s, a) \right) \vee \left( \hat{P}_m (\cdot | s, a), \alpha \right)} \cdot \frac{5S_t}{N_m^k(s, a)}.$$ 

Similarly, conditioned on $\Omega_2$, we have that, 

$$\left| \left( \hat{P}_m^k - P_m \right) (\cdot | s, a)^\top \alpha \right| \leq \sqrt{2\sup \left( P_m (\cdot | s, a) \right) \vee \left( P_m (\cdot | s, a), \alpha \right)} \cdot \frac{S_t}{N_m^k(s, a)}.$$ 

Proof. We fix the episode number $k$ and omit it for simplicity. 

$$\left| \left( \hat{P}_m - P_m \right) (\cdot | s, a)^\top \alpha \right|$$ 

$$\sup_{s' \in S} \left| \left( \hat{P}_m - P_m \right) (s'|s, a) \left( \alpha(s') - \hat{P}_m (\cdot | s, a)^\top \alpha \right) \right|$$
\[ \leq \sum_{s \in \mathcal{S}} \left| \hat{P}_m - P_m \right| (s'|s, a) \left| \alpha(s') - \hat{P}_m(\cdot|s, a)^\top \alpha \right| \]
\[ \overset{(\cdot)}{=} \sum_{s \in \mathcal{S}} \left( 2 \sqrt{\frac{\hat{P}_m(s'|s, a)}{N_m(s, a)}} + \frac{5l}{N_m(s, a)} \right) \left| \alpha(s') - \hat{P}_m(\cdot|s, a)^\top \alpha \right| \]
\[ \leq 2 \sqrt{\frac{l}{N_m(s, a)}} \sum_{s \in \mathcal{S}} \mathbb{1}_{\hat{P}_m(s'|s, a) > 0} \sqrt{\frac{\hat{P}_m(s'|s, a)}{N_m(s, a)}} \left| \alpha(s') - \hat{P}_m(\cdot|s, a)^\top \alpha \right| + \frac{5Sl}{N_m(s, a)} \]
\[ \overset{(\cdot)}{=} 2 \sqrt{\frac{l}{N_m(s, a)}} \sum_{s \in \mathcal{S}} \mathbb{1}_{\hat{P}_m(s'|s, a) > 0} \cdot \sum_{s \in \mathcal{S}} \hat{P}_m(s'|s, a) \left( \alpha(s') - \hat{P}_m(\cdot|s, a)^\top \alpha \right)^2 + \frac{5Sl}{N_m(s, a)} \]
\[ = 2 \sqrt{\frac{\text{supp} \left( \hat{P}_m(\cdot|s, a) \right)}{N_m(s, a)}} \sum_{s \in \mathcal{S}} \mathbb{1}_{\hat{P}_m(\cdot|s, a) > 0} \cdot \sum_{s \in \mathcal{S}} \hat{P}_m(s'|s, a) \left( \alpha(s') - \hat{P}_m(\cdot|s, a)^\top \alpha \right)^2 + \frac{5Sl}{N_m(s, a)}, \]

where (i) comes from that \( P_m(\cdot|s, a)^\top \alpha \) is a constant and \( \hat{P}, P \) are two distributions; (ii) is by the definition of \( \Omega^k_t \) (Equation (6)) and \( \| \alpha \|_X \leq 1 \); (iii) is from the Cauchy-Schwarz inequality. The second part is similar. \( \square \)

**Lemma 11.** Let \( \{w^k_t \geq 0 : (k, t) \in [K] \times [H]\} \) be a group of weights, then
\[ \mathbb{E}_{(k, t) \sim \delta \mathcal{K}} \sum_{k=1}^K \sum_{t=1}^H \frac{\mathbb{1}[(k, t) \notin \mathcal{X}]}{N^{k}_{m}(s^{k}_t, a^{k}_t)} \leq 3MSA, \quad \mathbb{E}_{(k, t) \sim \delta \mathcal{K}} \sum_{k=1}^K \sum_{t=1}^H \sqrt{\frac{w^k_t \mathbb{1}[(k, t) \notin \mathcal{X}]}{N^{k}_{m}(s^{k}_t, a^{k}_t)}} \leq \sqrt{3MSA} \sum_{k=1}^K \sum_{t=1}^H w^k_t \mathbb{1}[(k, t) \notin \mathcal{X}]. \]

**Proof.** From Algorithm 1 and the definition of \( \mathcal{K} \), we have that for any \( i \in \mathbb{N}, (m, s, a) \in [M] \times \mathcal{S} \times \mathcal{A} \),
\[ \sum_{k=1}^K \sum_{t=1}^H \mathbb{1}[(m^k, s^k_t, a^k_t) = (m, s, a), N^k_m(s, a) = 2^i, (k, t) \notin \mathcal{X}] \leq \begin{cases} 2, & i = 0, \\ 2^i, & i \geq 1. \end{cases} \]

So
\[ \sum_{k=1}^K \sum_{t=1}^H \frac{\mathbb{1}[(k, t) \notin \mathcal{X}]}{N^{k}_{m}(s^{k}_t, a^{k}_t)} = \sum_{(m, s, a) \in [M] \times \mathcal{S} \times \mathcal{A}} \sum_{i=0}^{\log_2(HK)} \sum_{k=1}^K \sum_{t=1}^H \mathbb{1}[(m^k, s^k_t, a^k_t) = (m, s, a), N^k_m(s, a) = 2^i, (k, t) \notin \mathcal{X}] \leq \sum_{(m, s, a) \in [M] \times \mathcal{S} \times \mathcal{A}} \left( 2 + \sum_{i=1}^{\log_2(HK)} 1 \right) \leq 3MSA. \]

Therefore,
\[ \sum_{k=1}^K \sum_{t=1}^H \sqrt{\frac{w^k_t \mathbb{1}[(k, t) \notin \mathcal{X}]}{N^{k}_{m}(s^{k}_t, a^{k}_t)}} \overset{(\cdot)}{=} \sum_{k=1}^K \sum_{t=1}^H \sqrt{\frac{w^k_t \mathbb{1}[(k, t) \notin \mathcal{X}]}{N^{k}_{m}(s^{k}_t, a^{k}_t)}} \frac{\mathbb{1}[(k, t) \notin \mathcal{X}]}{N^{k}_{m}(s^{k}_t, a^{k}_t)} \]
\[ \overset{(\cdot)}{=} \sqrt{\left( \sum_{k=1}^K \sum_{t=1}^H w^k_t \mathbb{1}[(k, t) \notin \mathcal{X}] \right) \left( \sum_{k=1}^K \sum_{t=1}^H \frac{\mathbb{1}[(k, t) \notin \mathcal{X}]}{N^{k}_{m}(s^{k}_t, a^{k}_t)} \right)} \]
\[ \leq \sqrt{3MSA} \sum_{k=1}^K \sum_{t=1}^H w^k_t \mathbb{1}[(k, t) \notin \mathcal{X}], \]

where (i) is by the property of indicator function; (ii) is by the Cauchy-Schwarz inequality. \( \square \)
Lemma 13 (Optimism of Algorithm 3). Algorithm 3 satisfies that: Conditioned on $\Omega_1$, for any episode $k \in [K]$, $\hat{V}^k \geq V^\pi$.

Proof. We first argue that for any policy $\pi$ and any episode $k$, we have that $\hat{V}^k_{M_k} \geq V^\pi$. Throughout the proof, the episode number $k$ is fixed and omitted in any superscript. We proceed the proof for $h$ in the order $H_H, H_{H-1}, \ldots, H_1$, using induction. Recall that for any $h \in H_{t+1}$ we define $\alpha_m(h, a) = \alpha_m(h, a) = 0$. Now suppose for time step $t$, we already have $\bar{\alpha}_m(h', a) \geq \alpha_m(h', a)$ for any $h' \in H_{t+1}$, then $\bar{\alpha}_m(h') = \bar{\alpha}_m(h', \pi(h')) \geq \alpha_m(h', \pi(h')) = \alpha_m(h')$. For any $h \in H_t$,

$$
\alpha_m(h, a) = \min \left\{ R_m(s, a) + B_m(h, a) + \hat{P}_m(|s, a)\top \bar{\alpha}_m(h, a), 1 \right\} = \min \left\{ R_m(s, a) + \max \left\{ 4 \sqrt{\frac{\sup \left( \hat{P}_m(|s, a) \bigvee \left( \hat{P}_m(|s, a), \bar{\alpha}_m(h, a) \right) \right) N_m(s, a)}{N_m(s, a)}, 16 \right\} \right\} 
$$

$$
\alpha_m(h, a) \geq \min \left\{ R_m(s, a) + f \left( \hat{P}_m(|s, a), \alpha_m(h, a), N_m(s, a), \epsilon \right), 1 \right\} \geq \min \left\{ R_m(s, a) + f \left( \hat{P}_m(|s, a), \alpha_m(h, a), N_m(s, a), \epsilon \right), 1 \right\} \geq \min \left\{ R_m(s, a) + \hat{P}_m(|s, a)\top \alpha_m(h, a) \right\} + 2 \sqrt{\frac{\sup \left( \hat{P}_m(|s, a) \bigvee \left( \hat{P}_m(|s, a), \alpha_m(h, a) \right) \right) N_m(s, a)}{N_m(s, a)} + \frac{8 \epsilon}{N_m(s, a)}, 1 \right\} \geq \min \left\{ R_m(s, a) + P_m(|s, a)\top \alpha_m(h, a), 1 \right\} = \alpha_m(h, a),
$$

where (i) is by taking $r = R_m(s, a)$; (ii) is by recognizing $c_1 = 4 \sqrt{\sup \left( \hat{P}_m(|s, a) \right) / \epsilon}$; $c_2 = 16 \epsilon$ in Lemma 12, which satisfy $c_1^2 \leq c_2^2$; (iii) and (iv) come by successively applying the first and second property in Lemma 12; (v) is an implication of Lemma 10, conditioning on $\Omega_1$ and taking $\alpha = \alpha_m(h, a)$.

The proof is completed by the fact that $\pi^k = \arg \max_{\pi \in \Pi} \hat{V}^k$, hence $\hat{V}^k \geq V^\pi \geq V^\pi$.

Lemma 14 (Bellman error). Both Algorithm 2 and Algorithm 3 satisfy the following Bellman error bound: Conditioned on $\Omega_1$ and $\Omega_2$, for any $(m, h, a, k) \in [M] \times H \times A \times [K]$,

$$
\hat{\alpha}_m(h, a) - R_m(s, a) - P_m(|s, a)\top \hat{\alpha}_m(h, a) \leq \min \{ \beta_m^k(h, a) \}, \tag{8}
$$

where $r = R_m(s, a)$ and

$$
\beta_m^k(h, a) = \epsilon \sqrt{\frac{\Gamma V (P_m(|s, a), \hat{\alpha}_m^k(h, a) \top) N_m(s, a)}{N_m(s, a)} + \frac{30 \epsilon}{N_m(s, a)}},
$$

where $r = R_m(s, a)$ and
Proof. Here we decompose the Bellman error in a generic way. We use \( \tilde{P} \) for the transition and \( B \) for the bonus used in the optimistic model. For Algorithm 2, \( B = 0 \); while for Algorithm 3, \( \tilde{P} = \hat{P} \).

The upper bound of 1 is trivial. Fix any \( (m, h, a, k) \in [M] \times \mathcal{H} \times \mathcal{A} \times [K] \), then

\[
\begin{align*}
\Theta &= R_m(s, a) + B_m^k(h, a) + \tilde{P}_m^k(\cdot | s, a) \Gamma \hat{\alpha}_m^k(h_{\text{har}}) - R_m(s, a) - P_m(\cdot | s, a) \Gamma \hat{\alpha}_m^k(h_{\text{har}}) \\
&= B_m^k(h, a) + \left( \tilde{P}_m^k - P_m \right)(\cdot | s, a) \Gamma \hat{\alpha}_m^k(h_{\text{har}}).
\end{align*}
\]

Next we proceed in two ways.

For Algorithm 2, we utilize Lemma 20 and a similar argument as Lemma 10. It gives

\[
\Theta_{\text{Bernstein}} \leq 4 \sqrt{\frac{\text{supp}(P_m(\cdot | s, a)) \mathcal{V}(P_m(\cdot | s, a), \alpha_m \hat{\alpha}_m^k(h_{\text{har}})) \iota}{N_m^k(s, a)}} + \frac{30S_I}{N_m^k(s, a)}.
\]

For Algorithm 3, we plug in the definition of \( B \) and use Lemma 10. It gives

\[
\Theta_{\text{MVP}} \leq 4 \sqrt{\frac{\text{supp}(\hat{P}_m^k(\cdot | h, a)) \mathcal{V}(\hat{P}_m^k(\cdot | h, a), \hat{\alpha}_m^k(h_{\text{har}})) \iota}{N_m^k(s, a)}} + \frac{2 \text{supp}(P_m(\cdot | s, a)) \mathcal{V}(P_m(\cdot | s, a), \hat{\alpha}_m^k(h_{\text{har}})) \iota}{N_m^k(s, a)} + \frac{17S_I}{N_m^k(s, a)}.
\]

Next we bound \( \mathcal{V}(\hat{P}_m^k(\cdot | h, a), \hat{\alpha}_m^k(h_{\text{har}})) \).

\[
\begin{align*}
\mathcal{V}(\hat{P}_m^k(\cdot | h, a), \hat{\alpha}_m^k(h_{\text{har}})) &= \sum_{s' \in S} \hat{P}_m^k(s' | s, a) \left( \hat{\alpha}_m^k(h_{\text{har}}) - \hat{P}_m^k(\cdot | s, a) \Gamma \hat{\alpha}_m^k(h_{\text{har}}) \right)^2 \\
&\leq \sum_{s' \in S} \hat{P}_m^k(s' | s, a) \left( \hat{\alpha}_m^k(h_{\text{har}}') - P_m(\cdot | s, a) \Gamma \hat{\alpha}_m^k(h_{\text{har}}') \right)^2 \\
&\leq \sum_{s' \in S} \left( \frac{2}{S} P_m(s' | s, a) + \frac{2 \iota}{N_m^k(s, a)} \right) \left( \hat{\alpha}_m^k(h_{\text{har}}') - P_m(\cdot | s, a) \Gamma \hat{\alpha}_m^k(h_{\text{har}}') \right)^2 \\
&\leq \frac{3}{2} \mathcal{V}(P_m(\cdot | s, a), \hat{\alpha}_m^k(h_{\text{har}}')) + \frac{2S_I}{N_m^k(s, a)}.
\end{align*}
\]

where (i) is by that \( z^* = \sum_i p_i x_i \) minimizes \( f(z) = \sum_i p_i (x_i - z)^2; \) (ii) is by \( \Omega_2 \). Finally, using \( \sqrt{x + y} \leq \sqrt{x} + \sqrt{y} \) and \( \text{supp}(\hat{P}_m^k(\cdot | s, a)) \leq \text{supp}(P_m(\cdot | s, a)) \), we have

\[
\Theta_{\text{MVP}} \leq 7 \sqrt{\frac{\text{supp}(P_m(\cdot | s, a)) \mathcal{V}(P_m(\cdot | s, a), \hat{\alpha}_m^k(h_{\text{har}})) \iota}{N_m^k(s, a)}} + \frac{23S_I}{N_m^k(s, a)}.
\]

Therefore, setting

\[
\beta_m^k(h, a) = 7 \sqrt{\mathcal{V}(P_m(\cdot | s, a), \hat{\alpha}_m^k(h_{\text{har}})) \iota} + \frac{30S_I}{N_m^k(s, a)}
\]

completes the proof. \( \square \)
Lemma 15. With probability at least $1 - \delta$, we have that $X_1 \leq \sqrt{Kt}$. 

Proof. By definition, $V^k = \sum_{m=1}^M \sum_{s \in S} w_m v_m(s) \alpha_m^k(s)$. Thus, $\alpha_m^k(s^k)$ is a random variable with mean $V^k$. Also, $\alpha_m^k(s^k)$ is measurable with respect to $\hat{U}^{k-1}$. Using Lemma 3 and $|\alpha_m^k(s^k) - V^k| \leq 1$, we have
\[
P \left[ X_1 > \sqrt{Kt} \right] \leq \delta.
\]
This completes the proof.

Lemma 16. With probability at least $1 - \delta$, we have that $X_2 \leq \sqrt{Kt}$. 

Proof. By definition, $X_2 = \sum_{k=1}^K \left( \sum_{t=1}^H r_t^k - V^\pi^k \right)$. From Monte Carlo simulation, $E \left[ \sum_{t=1}^H r_t^k \right] = V^\pi^k$. Also, $\sum_{t=1}^H r_t^k$ is measurable with respect to $\hat{U}^{k-1}$. Using Lemma 3 and $|\sum_{t=1}^H r_t^k - V^\pi^k| \leq 1$, we have
\[
P \left[ \sum_{k=1}^K \left( \sum_{t=1}^H r_t^k - V^\pi^k \right) > \sqrt{Kt} \right] \leq \delta.
\]
This completes the proof.

Lemma 17. With probability at least $1 - \delta$, we have that $X_3 \leq 2\sqrt{2X_4} + 5t$. 

Proof. Observe that $I[(k, t + 1) \notin \mathcal{X}] \leq I[(k, t) \notin \mathcal{X}]$, so
\[
X_3 \leq \sum_{k=1}^K \sum_{t=1}^H \left( P_m^k(\cdot | s_t^k, a_t^k) \hat{\alpha}_m^k(h_t^k a_t^k r_t^k) - \hat{\alpha}_m^k(h_{t+1}^k) \right) I[(k, t) \notin \mathcal{X}].
\]
This is a martingale. By taking $c = 1$ in Lemma 6, we have
\[
P \left[ X_3 > 2\sqrt{2X_4} + 5t \right] \leq \delta.
\]
This completes the proof.

Lemma 19. Conditioned on $\Omega_1$ and $\Omega_2$, with probability at least $1 - \delta$, we have that $X_5 \leq 3(K + X_4) + 83t$. 

Proof. For any non-negative integer $d$, define
\[
F(d) := \sum_{k=1}^K \sum_{t=1}^H \left( P_m^k(\cdot | s_t^k, a_t^k) \hat{\alpha}_m^k(h_t^k a_t^k r_t^k) \right)^{2^d} - \left( \hat{\alpha}_m^k(h_{t+1}^k) \right)^{2^d} I[(k, t) \notin \mathcal{X}],
G(d) := \sum_{k=1}^K \sum_{t=1}^H \left( P_m^k(\cdot | s_t^k, a_t^k) \hat{\alpha}_m^k(h_t^k a_t^k r_t^k) \right)^{2^d} I[(k, t) \notin \mathcal{X}].
\]
Then $X_5 = G(0)$. Direct computation gives that
\[
G(d) = \sum_{k=1}^K \sum_{t=1}^H \left( P_m^k(\cdot | s_t^k, a_t^k) \hat{\alpha}_m^k(h_t^k a_t^k r_t^k) \right)^{2^d+1} - \left( \hat{\alpha}_m^k(h_{t+1}^k) \right)^{2^d+1} I[(k, t) \notin \mathcal{X}]
\]
\[
\leq \sum_{k=1}^K \sum_{t=1}^H \left( P_m^k(\cdot | s_t^k, a_t^k) \hat{\alpha}_m^k(h_t^k a_t^k r_t^k) \right)^{2^d+1} - \left( \hat{\alpha}_m^k(h_{t+1}^k) \right)^{2^d+1} I[(k, t) \notin \mathcal{X}] + \left( \hat{\alpha}_m^k(h_{H+1}^k) \right)^{2^d+1} _{t=0}
\]
\[
+ \sum_{k=1}^K \sum_{t=1}^H \left( \hat{\alpha}_m^k(h_{t+1}^k) \right)^{2^d+1} - \left( P_m^k(\cdot | s_t^k, a_t^k) \hat{\alpha}_m^k(h_t^k a_t^k r_t^k) \right)^{2^d+1} I[(k, t) \notin \mathcal{X}] - \left( \hat{\alpha}_m^k(s_t^k) \right)^{2^d+1} _{t=0}
\]
22
Taking a union bound over $\Omega$.

Conditioned on Lemma 20.

where (i) uses "(iii) is by the assumption that reward within an episode is upper-bounded by 1 and the definition of $X_4$.

For a fixed $d$, $F(d)$ is a martingale. By taking $c = 1$ in Lemma 6, we have

$$\Pr \left[ F(d) > 2\sqrt{2G(d)(\log_2(HK) + \ln(2/\delta)) + 5(\log_2(HK) + \ln(2/\delta))} \right] \leq \frac{\delta}{\log_2(HK) + 1}.$$ 

Taking a union bound over $d = 1, 2, \ldots, \log_2(HK)$, we have that with probability at least $1 - \delta$,

$$F(d) \leq 4\sqrt{F(d + 1) + 2^{d+1}(K + X_4)} + 10\epsilon.$$ 

From Lemma 7, taking $\lambda_1 = HK$, $\lambda_2 = 4\sqrt{7}$, $\lambda_3 = K + X_4$, $\lambda_4 = 10\epsilon$, we have that

$$F(1) \leq \max\{(4\sqrt{t} + \sqrt{26\epsilon})^2, 8\sqrt{2(K + X_4)} + 10\epsilon\} \leq K + X_4 + 83\epsilon,$$

where (i) uses $\sqrt{x+y} \leq \frac{x+y}{2}$ and $\max\{x, y\} \leq x+y$ for $x, y \geq 0$. Hence

$$X_5 = G(0) \leq F(1) + 2(K + X_4) \leq 3(K + X_4) + 83\epsilon.$$ 

This completes the proof. 

Lemma 20. Conditioned on $\Omega_1$, we have that for any $(k, m, s, a, s') \in [K] \times [M] \times S \times A \times S$,

$$\left| P_m(s'|s, a) - \hat{P}_m^k(s'|s, a) \right| \leq 4\sqrt{\frac{P_m(s'|s, a)\epsilon}{N_m^k(s, a)}} + \frac{30\epsilon}{N_m^k(s, a)}.$$ 

Proof. From $\Omega_1$ we have

$$\hat{P}_m^k(s'|s, a) \leq 2\sqrt{\frac{\hat{P}_m^k(s'|s, a)\epsilon}{N_m^k(s, a)}} + \frac{5\epsilon}{N_m^k(s, a)} + P_m(s'|s, a).$$ 

This is a quadratic inequality in $\sqrt{\hat{P}_m^k(s'|s, a)}$. Using the fact that $x^2 \leq ax + b$ implies $x \leq a + \sqrt{b}$ with $a = 2\sqrt{\frac{\hat{P}_m^k(s'|s, a)}{N_m^k(s, a)}}$, $b = \frac{5\epsilon}{N_m^k(s, a)} + P_m(s'|s, a)$, and $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$, we have

$$\sqrt{\hat{P}_m^k(s'|s, a)} \leq \sqrt{P_m(s'|s, a)} + 5\sqrt{\frac{\epsilon}{N_m^k(s, a)}}.$$ 

23
Substituting this into \( \Omega \) we have

\[
\left| P_m(s'|s, a) - \hat{P}_m^k(s'|s, a) \right| \leq 2 \sqrt{\frac{P_m(s'|s, a)}{N_m^k(s, a)}} + \frac{15\varsigma}{N_m^k(s, a)}.
\]

From the construction of \( \hat{P}_m^k \) we also have

\[
\left| \hat{P}_m^k(s'|s, a) - \hat{P}_m^k(s'|s, a) \right| \leq 2 \sqrt{\frac{P_m(s'|s, a)}{N_m^k(s, a)}} + \frac{15\varsigma}{N_m^k(s, a)}.
\]

Therefore, from triangle inequality we have the desired result. \( \square \)

**B.4 Proof of the regret lower bound**

**Theorem 2.** Assume that \( S \geq 6, A \geq 2 \) and \( M \leq \left\lfloor \frac{S}{2} \right\rfloor \). For any algorithm \( \pi \), there exists an LMDP \( \mathcal{M}_\pi \) such that, for \( K \geq \tilde{\Omega}(M^2 + \text{MSA}) \), its expected regret in \( \mathcal{M}_\pi \) after \( K \) episodes satisfies

\[
R(\mathcal{M}_\pi, \pi, K) := \mathbb{E} \left[ \sum_{k=1}^{K} (V^* - V^k) \right| \mathcal{M}_\pi, \pi] = \Omega \left( \sqrt{\text{MSAK}} \right).
\]

**Proof:** We need to introduce an alternative regret measure for an MDP based on simulating an LMDP algorithm. Let \( \mathcal{M}(m, \ell^*, a^*) \) be an MDP which contains an encoding phase with permutation \( \sigma(m) \), and a guessing phase with correct answer \( (\ell^*, a^*) \). Given any LMDP algorithm \( \pi \), a target position \( m \) and a pair of LMDP configuration \( (\ell^*, a^*) \), we can construct an MDP algorithm \( \pi(m, \ell^*, a^*) \) as in Algorithm 4.

**Algorithm 4** \( \pi(m, \ell^*, a^*) \): an algorithm for an MDP.

1. **Input:** an MDP \( \mathcal{M}(m, \ell^*, a^*) \); an LMDP algorithm \( \pi \); a pair of LMDP configuration \( (\ell^*, a^*) \); specify exactly one: a simulation episode budget \( K \) or a target interaction episode \( K_m \).
2. Initialize actual interaction counter \( K_m \leftarrow 0 \).
3. for \( k = 1, 2, \ldots \) do
4. Randomly choose \( m' \sim \text{Unif}(M) \).
5. if \( m^k \neq m \) then
6. Use \( \pi \) to interact with the \( m^k \)th MDP of \( \mathcal{M}(\ell^*, a^*) \).
7. else
8. Use \( \pi \) to interact with \( \mathcal{M}(m, \ell^*, a^*) \).
9. \( K_m \leftarrow K_m + 1 \).
10. end if
11. if \((K \text{ is specified and } k = K) \text{ or } (\overline{K}_m \text{ is specified and } K_m = \overline{K}_m)\) then
12. Break.
13. end if
14. end for

This algorithm admits two types of training: (1) When \( K \) is specified, it returns after \( K \) episodes, regardless of how many times it interacts with the target MDP; (2) When \( \overline{K}_m \) is specified, it does not return until it interacts with the MDP for \( \overline{K}_m \) times, regardless of how many episodes elapse.

Let \( V^* \) and \( V^k \) be the optimal value function and the value function of \( \pi(m, \ell^*, a^*) \) under the MDP \( \mathcal{M}(m, \ell^*, a^*) \). The alternative regret for MDP (corresponding to (1)) is:

\[
\tilde{R}(\mathcal{M}(m, \ell^*, a^*), \pi(m, \ell^*, a^*), K) := \mathbb{E} \left[ \sum_{k=1}^{K} \mathbb{I}[m^k = m](V^* - V^k) \right| \mathcal{M}(m, \ell^*, a^*), \pi(m, \ell^*, a^*)] .
\]
Roughly, this is a regret for \( K_m \) episodes, though \( K_m \) is stochastic.

In our hard instances, the MDPs in the LMDP can be considered separately. So \( V^* = \frac{1}{M} \sum_{m=1}^{M} V^*_{m} \), where \( V^*_{m} \) is the optimal value function of (which is equal to the value function of the optimal policy applied to) the \( m \)th MDP. According to Monte-Carlo sampling,

\[
R(M(\ell^*, a^*), \pi, K) = E \left[ \sum_{k=1}^{K} (V^*_{m_k} - V^*_{m_k}) \right] \mid M(\ell^*, a^*), \pi
\]

\[
= \sum_{m=1}^{M} E \left[ \sum_{k=1}^{K} [1[m^k = m](V^*_{m_k} - V^*_{m_k})] \right] \mid M(\ell^*, a^*), \pi
\]

\[
= \sum_{m=1}^{M} \hat{R}(M(m, \ell^*_m, a^*_m), \pi(m, \ell^*, a^*), K).
\]

The last step is because the behavior of “focusing on the \( m \)th MDP in the LMDP” and “using the simulator” are the same. Denote \( K_m \) as the number of episodes spent in the \( m \)th MDP, which is a random variable. According to Lemma 4,

\[
P \left[ \left| K_m - \frac{1}{M} \right| > \sqrt{\frac{2 \gamma}{K}} \left( 1 - \frac{1}{\gamma} \right) \ln \left( \frac{2MC^M}{\delta} \right) + \ln \left( \frac{2MC^M}{\delta} \right) \right] \leq \delta MC^M.
\]

which implies

\[
P \left[ \left| K_m - \frac{K}{M} \right| > \sqrt{2K} \ln \left( \frac{2MC}{\delta} \right) + M \ln \left( \frac{2MC}{\delta} \right) \right] \leq \delta MC^M.
\]

When \( K > (6 + 4\sqrt{2})M^2 \ln \left( \frac{2MC}{\delta} \right) \), we have \( \sqrt{2K} \ln \left( \frac{2MC}{\delta} \right) + M \ln \left( \frac{2MC}{\delta} \right) < \frac{K}{2M} \). By a union bound over all possible hard instances \( M(\ell^*, a^*) \in C \) and all indices \( m \in [M] \), the following event happens with probability at least \( 1 - \delta \):

\[
\mathcal{E} := \left\{ K_m \geq \frac{K}{2M} \text{ for all } M(\ell^*, a^*) \in C \text{ and } m \in [M] \right\}.
\]

Now look into Equation (8), (11) and (12) of Domingues et al., 2021. For any \( K' \geq SA \) and any fixed encoding number \( m \), we have that

\[
\frac{1}{C} \sum_{\ell', a'} R(M(m, \ell', a'), \pi(m, \ell', a'), K') \geq \frac{1}{4\sqrt{2}} \left( 1 - \frac{1}{C} \right) \sqrt{C}K' \geq \frac{\sqrt{C}K'}{8\sqrt{2}}.
\]

when set \( \varepsilon = \frac{1}{4\sqrt{2}} \left( 1 - \frac{1}{C} \right) \sqrt{C}K' \). The desired value of \( K' \) is \( \frac{K}{2M} \) according to \( \mathcal{E} \).

We study the cases when we use \( \pi(m, \ell', a^*) \) to solve \( M(m, \ell^*_m, a^*_m) \) with a target interaction episode \( K_m = \frac{K}{2M} \). The regret is \( R(M(m, \ell^*_m, a^*_m), \pi(m, \ell^*, a^*), \frac{K}{2M}) \) (this is the regret of MDPs).

- The \( \frac{K}{2M} \)-th interaction with the \( m \)th MDP comes before the \( K \)th simulation episode. This case happens under \( \mathcal{E} \). The regret of this part is denoted as \( R^+ \).

- Otherwise. This case happens under \( \bar{\mathcal{E}} \). The regret of this part is denoted as \( R^- \). Since the regret of a single episode is at most 1, we have that \( R^- < \frac{4K}{2M} \).

Now we study the cases when we use \( \pi(m, \ell', a^*) \) to solve \( M(m, \ell^*_m, a^*_m) \) with a simulation episode budget \( K \). The alternative regret for MDP is \( \hat{R}(M(m, \ell^*_m, a^*_m), \pi(m, \ell^*, a^*), K) \).
The $\frac{K}{2M}$th interaction with the $m$th MDP comes before the $K$th simulation episode. This case happens under $\mathcal{E}$. The regret of this part is denoted as $\tilde{R}^+$. Since the regret of a single episode is at least 0, and in this case $K_m \geq \frac{K}{2M}$, we have $\tilde{R}^+ \geq R^+$. 

Otherwise. This case happens under $\mathcal{E}$. The regret of this part is denoted as $\tilde{R}^- \geq 0$.

Using the connection between $R^+$ and $\tilde{R}^+$, we have:

$$
\frac{1}{C^M} \sum_{\ell', a'} R(\mathcal{M}(\ell', a'), \pi, K)
= \frac{1}{C^M} \sum_{\ell', a'} \tilde{R}(\mathcal{M}(m, \ell_m^*, a_m^*), \pi(m, \ell', a^*), K)
\geq \frac{1}{C^M-1} \sum_{\ell', a'} \frac{1}{C} \sum_{\ell_m^*, a_m^*} \tilde{R}^+(\mathcal{M}(m, \ell_m^*, a_m^*), \pi(m, \ell', a^*), K)
\geq \frac{1}{C^M-1} \sum_{\ell', a'} \left( \frac{R - R^-}{C (M, \ell_m^*, a_m^*), \pi(m, \ell', a^*), \frac{K}{2M}} \right)
= \left( \frac{\sqrt{CK}}{16\sqrt{2M}} - \frac{\delta K}{2M} \right)
\geq \frac{\sqrt{MCK}}{16\sqrt{2}} - \frac{\delta K}{2},
$$

where in (i) we use $\bar{a}_{-m}$ to denote the positions other than $m$ in $\bar{a}$; (ii) is by setting $K' = \frac{K}{2M}$ in Equation (9). Set $\delta := \frac{\sqrt{MCK}}{16\sqrt{2}}$, then we have that

$$
\max_{\ell', a'} R(\mathcal{M}(\ell', a'), \pi, K) \geq \frac{1}{C^M} \sum_{\ell', a'} R(\mathcal{M}(\ell', a'), \pi, K) > \frac{\sqrt{MCK}}{32\sqrt{2}} = \Omega\left(\sqrt{MSAK}\right).
$$

This holds when $K \geq (6 + 4\sqrt{2})M^2 \ln \left(\frac{2MC}{\delta}\right)$ and $K' \geq SA$. It then reduces to

$$
K \geq \Omega(M^2 \text{poly}(\log(M, S, A)) + MSA).
$$

This completes the proof.