Abstract: Starting from certain 3D non-abelian dual systems, we discuss a number of related dual systems in 2D, some of which are obtained by dimensional reduction. The dualities relate massive scalar and vector fields, and may be relevant for string theory in the context of massive type IIA supergravity. Supersymmetric extensions of the models are also presented.

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1 Introduction

Duality between different description of one and the same physical system has a long history \[1\]. Recently string theory has focused the interest on two dimensional scalar-scalar duality \[2\], where also non-abelian duality has played a role \[3\]. Non-abelian generalizations of other types of duality has been hard to come by, although there are some examples, such as the 3D non-abelian vector-vector duality of \[1, 5\]. In this paper we take these latter results as our starting point for an investigation of some non-abelian dualities in 2D. The non-abelian dualities we present relate vectors to vectors in 3D and massive scalar and vector models in 2D. We envisage possible applications to string theory in the context of massive type IIA supergravity \[6, 7\]. E.g., the massive D2-brane has been shown to be dual to a dimensional reduction of the M2-brane coupled to an auxiliary vector field \[8\] via a vector-vector duality. Our results would be relevant if one wanted to extend these considerations to include non-abelian fields and/or go to strings using double dimensional reduction.

The usual 2D non-abelian duality has a geometrical interpretation and generalization in terms of Poisson-Lie duality \[9\]. We believe that our 2D-results cannot be framed in that language.

It is also hoped that the present results contribute to the understanding of the rich structure of 2D field theory.

We have chosen to present the models in terms of scalar and vector fields. We could have made use of the local 2D equivalence

\[ A_a = \partial_a \varphi + \epsilon^b_a \partial_b \chi \]

to write the actions in terms of scalar fields only. Barring nonlocal field redefinitions, these actions would be higher order in derivatives, however.

The plan of the paper is as follows: In section 2 we introduce a short-hand notation for duality systems that will facilitate the presentation later. In section 3 we give a quick review of the 3D results in \[1, 5\]. Section 4 contains new 2D dualities that have no 3D counterpart, along with their supersymmetric versions. In section 5 we give the direct dimensional reduction of the 3D results collected in section 3. This is followed by some brief conclusions and an appendix containing our notation and conventions as well as some of the more cumbersome expressions for 2D actions.
Figure 1: The parent action $S(A, B)$ is (classically) equivalent to both $S(A)$ and $S(B)$ showing that $S(A)$ and $S(B)$ are dual to each other.

2 Notation

In what follows we will present a number of equivalences between different models, abelian, non-abelian and supersymmetric in two and three dimensions. To make the presentation as clear as possible we introduce the following succinct notation: $S(A)$ denotes an action with an abelian symmetry group acting on the field $A$ in the argument. The action for a non-abelian theory is denoted $I(A)$ and supersymmetry is indicated by bold face, e.g. $S(\Gamma)$ is an abelian supersymmetric action for the superfield $\Gamma$. We further introduce the notation $(S(A), S(B))[S(A, B)]$ to denote a duality system (DS), i.e. the dual pair of (abelian) actions $S(A)$ and $S(B)$ with parent action $S(A, B)$ as illustrated in Fig. 1. This notation will also be used when one or two of the entries do not exist or are not known. E.g. $(S(A), S(B))[S(A, B)] \to (I(A), \cdot )[\cdot ]$ means that $S(A)$ has a non-abelian extension $I(A)$, but that its parent action and dual are lacking.

We have collected the rest of our notation and conventions in Appendix A.
3 3D Vector-Vector Duality

In this section we collect some known 3D results. The primary actions are

\[ I_1(B, V) = \text{Tr} \int d^3x \left\{ \frac{1}{2} m^2 B_a B^a - \frac{1}{4} m \varepsilon^{abc} B_a [\nabla_b (V) B_c] \right\} + I(V), \]  

\[ I_1(A) = \text{Tr} \int d^3x \left\{ -\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} m \Omega(A) \right\}, \]  

\[ I(A, B) = \text{Tr} \int d^3x \left\{ \frac{1}{2} m^2 B_a B^a + \frac{1}{2} m \varepsilon^{abc} B_a F_{bc}(A) + \frac{1}{2} m \Omega(A) \right\}, \]

where \( A, B \) and \( V \) are fields in the adjoint representation of some group \( G \), \( F_{ab}(A) \) and \( \Omega(A) \) denotes a field strength and a Chern-Simons term, respectively, \( \nabla_b (V) \) is a covariant derivative and \( I(V) \) is some action for the spectator field \( V \). The action \( I_1(B, V) \) was first discussed in [10] and its abelian version \( I_1(B, V) \rightarrow S(B, V) \) is the self-dual model. The action \( I_1(A) \) was first presented in [11] and its abelian version \( I_1(A) \rightarrow S(A) \) is the topologically massive model. With \( I(A, B) \rightarrow S(A, B) \) and \( V = 0 \), the DS \( (S(A); S(B)) [S(A, B)] \) was displayed in [12].

The non-abelian generalization of this was subsequently investigated in [4] and was extended to include supersymmetry in [5]. To briefly recapitulate these developments we also need a second non-abelian generalization of \( S(B, V) \) namely

\[ I_2(B, V) = I_1(B, V) - \frac{m}{12} \text{Tr} \int d^3x \varepsilon^{abc} B_a B_b B_c. \]  

In [4] the DS \( (I_2(A, V), J_2(B, V)) [I_2(\bar{A}, B, V)] \) was derived. Here the parent action is

\[ I_2(\bar{A}, B, V) = \frac{1}{2} \text{Tr} \int d^3x \left\{ m^2 B_a B^a - \frac{1}{2} m \varepsilon^{abc} B_a \left[ \nabla_b (V) B_c + \frac{1}{3} B_a B_c \right] \right. \]

\[ \left. + m \Omega(\bar{A}) \right\} + I(V). \]

It is invariant under two sets of gauge transformations with parameters \( \Lambda \) and \( \Sigma \) respectively,

\[ (i) \quad \delta_\Lambda B_a = [B_a, \Lambda], \quad \delta_\Lambda V_a = \nabla_a (V) \Lambda, \quad \delta_\Lambda \bar{A}_a = 0, \]

\[ (ii) \quad \delta_\Sigma B_a = 0, \quad \delta_\Sigma V_a = 0, \quad \delta_\Sigma \bar{A}_a = \nabla_a (\bar{A}) \Sigma. \]
The model $I_2(A, V)$ dual to $I_2(B, V)$ is

$$I_2(A, V) = \frac{1}{2} \int d^3x \left\{ -\frac{1}{4} F_a^A(A) G_{aAbB}(A - V)^* F^{bB}(A) + m \Omega(A) \right\} + I(V), \quad (7)$$

where

$$(G^{-1})^a_{\quad A b} \equiv \left[ \eta^{ab} \delta^A_B + (1/2m) \epsilon^{abc}(A - V)_C f^{ABC} \right]. \quad (8)$$

Here $A, B, \ldots$ are Lie-algebra indices and $f^{ABC}$ are the structure constants.

The action $I_2(A, V)$ is invariant under

(i) $\delta_A A_a = -\frac{1}{2} [B_a, A]$, $\delta_A V_a = \nabla_a(V) A$,

(ii) $\delta_\Sigma A_a = \nabla_a(\Sigma) \equiv \nabla_a(A) \Sigma + \frac{1}{2} [B_a, \Sigma]$, $\delta_\Sigma V_a = 0$, \quad (9)\n
where, e.g.

$$[B_a, \Lambda]^A = -\frac{1}{2m} \epsilon^{bde} F_{de} G_{bdeab} \Lambda^D f^{BDA}. \quad (10)$$

This duality is somewhat unusual in that it also involves a shift $\bar{A} = A + \frac{1}{2} B$ before eliminating $B$ to get $I_2(A, V)$.

The introduction of the modified $I_2(B, V)$ was seen in [4] to be necessary for a straightforward dualization. The abelian $\text{DS} (S(B), S(A))[S(A, B)]$ has two direct non-abelian (partial) generalizations: $(\bar{I}(B), I_1(A)) \text{[I}(A, B)]$ and $(I_1(B, V), \cdot) [\cdot]$. More explicitly, the non-abelian topologically massive model of [11] has $I(A, B)$ as parent action which leads to dual action $\bar{I}(B)$. However $\bar{I}(B)$ is non-local (it is found in a Fock-Schwinger gauge), and is not equivalent to $I_1(B, V)$. For the action $I_1(B, V)$ of [10], on the other hand, no parent action and dual could be found.

All these results have their $N = 1$ supersymmetric counterpart. To state them we first give the supersymmetrization of the various actions found by letting $I_1(B, V) \to I(\Psi, V)$, $I_1(A) \to I_1(\Gamma)$, $I(A, B) \to I(\Psi, \Gamma)$, $I_2(B, V) \to I_2(\Psi, V)$, $I_2(\bar{A}, \bar{B}, V) \to I_2(\bar{\Gamma}, \Psi, V)$ where

$$I_1(\Psi, V) = -\frac{1}{2} \text{Tr} \int d^3x d^2 \theta \left[ m^2 \Psi^\alpha \Psi_\alpha + \frac{1}{4} m \Psi^\beta \nabla^\alpha \nabla_\beta \Psi_\alpha \right] + I(V)(11)$$
\[
\begin{align*}
I_1(\Gamma) &= \text{Tr} \int d^3x d^2\theta \left[ \frac{1}{2} W^\alpha W_\alpha + m \Omega(\Gamma) \right], \\
I(\Psi, \Gamma) &= \frac{1}{2} \text{Tr} \int d^3x d^2\theta \left[ -m^2 \Psi^\alpha \Psi_\alpha + 2m \left(\Psi^\alpha W_\alpha(\Gamma) + \Omega(\Gamma)\right) \right], \\
I_2(\Psi, V) &= I_1(\Psi, V) - \frac{m}{24} \text{Tr} \int d^3x d^2\theta \nabla^\alpha \Psi^\beta \{\Psi_\alpha, \Psi_\beta\}, \\
I_2(\Gamma, \Psi, V) &= I_2(\Psi, V) + \text{Tr} \int d^3x d^2\theta m \Omega(\Gamma).
\end{align*}
\]

Here \(\Psi, V\) and \(\Gamma\) are spinorial superfields in the adjoint representation of the gauge group \(G\), \(W_\alpha\) the superfield strength, \(\Omega\) the super Chern-Simons term and \(\nabla_\alpha\) super Yang-Mills covariant derivatives. There is also an action \(I_2(\Gamma, V)\), the supersymmetric generalization of \(I_2(A, V)\), which we shall not need. The results of [5] are as follows. First one has the supersymmetric abelian \(DS(\mathbf{S}(\Psi), \mathbf{S}(\Gamma))[\mathbf{S}(\Psi, \Gamma)]\). The corresponding non-abelian systems are \((I_2(\Psi, V), I_2(\Gamma, V))[I_2(\Gamma, \Psi, V)]\) and \((\mathbf{I}, I_1(\Gamma))[I(\Psi, \Gamma)]\) where \(\mathbf{I}\) is a non-local supersymmetric extension of \(I(B)\). It is not equivalent to \(I_1(\Psi, V)\) and furthermore had to be found by abandoning superspace and calculating in components. Finally again \((I_1(\Psi, V), \cdot)[\cdot]\) was not possible to complete.

This concludes our summary of the 3D results which we collect in table [5].

4 New 2D Non-Abelian Duality

In this section we present a new non-abelian vector-vector duality in 2D derived from the 3D action (3). Dimensionally reduced this action becomes

\[
I(A, a, B, b) = \text{Tr} \int d^2x \left\{ \frac{1}{2} m^2 B_a B^a - \frac{1}{2} m^2 b^2 + \frac{1}{2} m b F(A) \\
+ ma(F(A) + \epsilon^{ab} \nabla_a(A) B_b) \right\},
\]

where the scalar fields \(a\) and \(b\) arise in dimensional reduction according to \(A_a \rightarrow (A_a, a)\) and \(B_a \rightarrow (B_a, b)\), and \(F(A) \equiv \epsilon^{ab} F_{ab}\). As mentioned in the previous section, the 3D action is a parent action for \(I_1(A)\) and for a non-local \(B\)-action that is found after breaking gauge invariance. To find a local \(B\)-action one has to add an \(\epsilon B B B\)-term. In 2D the situation is different. Without the 3D-Lorentz invariance to respect there is a larger freedom. We
|                  | Bosonic                                                                 | Supersymmetric                                           |
|------------------|--------------------------------------------------------------------------|----------------------------------------------------------|
| **Abelian**      | \[
\begin{pmatrix}
  S(B) \\
  S(A)
\end{pmatrix}
_{S(A,B)} \quad \text{\textit{S}}(\Gamma)
\]                                             | \[
\begin{pmatrix}
  S(\Psi) \\
  S(\Gamma)
\end{pmatrix}
_{S(\Psi,\Gamma)}
\]                                             |
| **Non-Abelian**  | \[
\begin{pmatrix}
  I_2(B,V) \\
  I_2(A,V)
\end{pmatrix}
_{I_2(A,B,V)} \quad \text{\textit{non-local}}
\]                                             | \[
\begin{pmatrix}
  I_2(\Psi,V) \\
  I_2(\Gamma,V)
\end{pmatrix}
_{I_2(\Psi,\Gamma,V)}
\]                                             |
|                  | \[
\begin{pmatrix}
  \text{\textit{non-local}} \\
  I_1(A)
\end{pmatrix}
_{I_1(A,V)}
\]                                             | \[
\begin{pmatrix}
  \text{\textit{non-local}} \\
  I_1(\Gamma)
\end{pmatrix}
_{I_1(\Psi,\Gamma)}
\]                                             |
|                  | \[
\begin{pmatrix}
  I_1(B, V)
\end{pmatrix}
_{I_1(\Psi, V)}
\]                                             | \[
\begin{pmatrix}
  I_1(\Psi,V)
\end{pmatrix}
_{I_1(\Psi,\Gamma)}
\]                                             |

Table 1: The duality systems in 3D.
may choose to integrate out both $B_a$ and $b$ to find $I(A, a)$, the dimensional reduction of $I_1(A)$:

$$I(A, a) = \text{Tr} \int d^2x \left\{ -\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} \nabla_a (A) a \nabla^a (A) a + ma F(A) \right\}, \quad (17)$$

or we may integrate out either $B_a$ or $b$ by itself yielding actions which we denote by $I(A, a, b)$ and $I(A, a, B)$ respectively. Similarly, we may now contemplate integrating out the vector $A_a$ by itself, leaving the scalar $a$, i.e., the third 3D vector component, in the action. This can be done along the lines of standard 2D non-abelian duality for scalar fields. The $A_a$ field equations give

$$A_a^A = (M^{-1})^{AB} h_a^B \quad (18)$$

where

$$(M^{-1})^{AB} M^{BC} = \delta^{AC}, \quad (19)$$

$$M^{AB} \equiv (\frac{1}{2} b + a)^C f^{CAB}, \quad (20)$$

and

$$h_a^B \equiv \frac{1}{2} f^{BCD} a^C B_a^D - \partial_a (\frac{1}{2} b + a)^B \quad (21)$$

Here $A, B, ..$ are again adjoint group indices and $f^{ABC}$ are the Lie-algebra structure constants. The corresponding action, dual to $I(A, a, b)$ (or, equivalently to $I(A, a)$), is;

$$I(B, b, a) = \int d^2x \left\{ \frac{1}{2} m^2 B_a^A B^a A - \frac{1}{2} m^2 b^A b^A + m e^{ab} h_a^A (M^{-1})^{AB} h_b^B + m e^{ab} a^A \partial_a B_b^A \right\}. \quad (22)$$

Despite the non-covariant appearence, the action (22) is gauge invariant. It represents a theory with an anti-symmetric piece added to the group metric in the $B^2$-term, non-linear $\sigma$-model target-space torsion type coupling for

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4In fact, $B_a$ is an auxiliary field, so it might be most natural to always integrate it out. This can always be done at any stage below.
the scalars involving also the vector $B$-field, and non-linear scalar vector interaction terms. In agreement with the 3D result, integrating out the scalar $a$ to yield a local theory is not possible.

Having thus described the 2D DS $(I(A, a, b), I(B, b, a))[I(A, a, B, b)]$, we now turn to its supersymmetric version. In this subsection we write out the vector and spinor indices explicitly, using the “± notation” introduced in Appendix [A.2].

The supersymmetrization of $I(A, a, B, b)$ is given by the 2D version of $I(Ψ, Γ)$ in (13),

$$I(Ψ, Γ) = \text{Tr} \int d^2x d^2θ \left\{ i m^2 Ψ_+ Ψ_- + i m \nabla_+(Ψ_−H - 2mH^2) \right\},$$

where again $Ψ$ is a (2D) spinor superfield and $H$ is the $Γ$ field-strength introduced in Appendix [A.2]. To see how the dualization works, we perform the $θ$ integration and find the component action. We define the component fields using projections. The components of $Γ$ are, (in a Wess-Zumino gauge),

$$a \equiv \sqrt{2} H |, \quad A_{+/−} \equiv −i D_{±Γ} | − \frac{1}{2} (Γ_{±Γ} |),$$

$$λ_{±} \equiv W_{±} | = ± \nabla_± H |, \quad F(A) ≡ 2\sqrt{2} i \nabla_+ \nabla_- H |,$$

where $|$ denotes “the $θ$-independent part of”. The components of $Ψ$ are

$$ζ_{±} \equiv Ψ_{±} |, \quad B_{+/−} \equiv ±i \nabla_± Ψ_{±} |, \quad b \equiv −i \sqrt{2} \nabla_+(Ψ_−) |,$$

$$S \equiv −i \sqrt{2} \nabla_+ Ψ_{−} |, \quad χ_{±} \equiv ±i \nabla_±^2 Ψ_{±} | ± i \nabla_± \nabla_± Ψ_{±} |.$$

Using these definitions we derive the following component action

$$I(A, a, λ, B, b, ζ, χ, S)$$

$$= 2\text{Tr} \int d^2x \left\{ m^2 \left( −B_+ B_− − \frac{1}{2} b^2 + \frac{1}{2} S^2 + i(ζ_− D_+ ζ_+ − ζ_+ D_− ζ_−) \right) \right.$$

$$− 2iχ_− ζ_+ + \frac{1}{\sqrt{2}} [a, ζ_−] ζ_+ \right) + 4i mλ−λ_+ + mF(A)$$

$$− \frac{ma}{\sqrt{2}} D_{[+B=]} − 2imχ_+[λ−] + \frac{1}{2} mbF(A) \right\}. \quad (25)$$
We recognize the bosonic action $I(A, a, B, b)$ in (25), plus supersymmetric completions. Integrating out the fields $B, b, \zeta, \chi$ and $S$ we get

\[
I(A, a, \lambda) = 2I(A, a) + 2\text{Tr} \int d^2x \left\{ \lambda_- D_+ \lambda_- - \lambda_+ D_- \lambda_+ + \frac{1}{\sqrt{2}}[a, \lambda_-] \lambda_+ + 4i m \lambda_- \lambda_+ \right\},
\]  
(26)

whereas by varying $A$ and $\lambda$ we arrive at

\[
I(a, B, b, \zeta, \chi, S) = 2I(B, b, a) + 2\text{Tr} \int d^2x \left\{ m^2 (\zeta_- D_+ \zeta_- - \zeta_+ D_- \zeta_+) + \frac{1}{\sqrt{2}}[a, \zeta_-] \zeta_+ - 2i \chi[-\zeta_+] + \frac{1}{2} S^2 \right\} + im \chi_+ \chi_- .
\]  
(27)

The dualities found in this section are presented in table 2.

Table 2: The new non-abelian dualities in 2D.

| Non-Abelian | Supersymmetric |
|-------------|---------------|
| $I(A, a, B, b)$ | $I(a, B, \zeta, \chi, S)$ |

5 3D (self-dual) non-abelian duality from 2D point of view

So far we have discussed dualities of the reduced theory that have no correspondence in 3D. However, we may also mimic the three dimensional non-abelian dualization. To this end we reduce $I_2(\bar{A}, B, V)$ to 2D:

\[
I_2(\bar{A}, \bar{a}, B, b, V, v) = \text{Tr} \int d^2 x \left\{ \frac{1}{2} m^2 B_a B^a - \frac{1}{2} m^2 b^2 - \frac{1}{2} m b_{e a b} \left( \nabla_a (V) B_b + \frac{1}{2} B_a B_b \right) \right\}
\]
\begin{equation}
+\frac{1}{4} m \epsilon^{ab} B_a[v, B_b] + m \bar{\alpha} F(\bar{A}) \bigg) + I(V, v) \bigg). \tag{28}
\end{equation}

The action is invariant under two sets of gauge transformations

\begin{align*}
(i) \quad & \delta_{\Lambda} B_a = [B_a, \Lambda], \quad \delta_{\Lambda} b = [b, \Lambda], \\
& \delta_{\Lambda} V_a = \nabla_a (V) \Lambda, \quad \delta_{\Lambda} v = [v, \Lambda], \\
& \delta_{\Lambda} \bar{A}_a = 0, \quad \delta_{\Lambda} \bar{a} = 0.
\end{align*}

\begin{align*}
(ii) \quad & \delta_{\Sigma} B_a = 0, \quad \delta_{\Sigma} b = 0, \\
& \delta_{\Sigma} V_a = 0, \quad \delta_{\Sigma} v = 0, \\
& \delta_{\Sigma} \bar{A}_a = \nabla_a (\bar{A}) \Sigma, \quad \delta_{\Sigma} \bar{a} = [\bar{a}, \Sigma] \tag{29}
\end{align*}

which are the dimensional reduced version of eq. (3).

Variation with respect to \( \bar{a} \) implies that \( \bar{A}_a \) is pure gauge and can be dropped from the action. We then get an action \( I_2(B, b, V, v) \) which is the 3D action \( I_2(B, V) \) reduced to 2D. If we shift \( (\bar{A}_a, \bar{a}) = (A_a, a) + \frac{1}{2} (B_a, b) \) in (28) we get the equivalent action,

\begin{equation}
I_2(A, a, B, b, V, v) = \text{Tr} \int d^2x \left\{ \frac{1}{2} m^2 B_a B^a - \frac{1}{2} m^2 b^2 + \frac{1}{2} m b F(A) \right. \\
+ m \epsilon^{ab} B_a \nabla_b (A) a + \frac{1}{2} m (a - v) \epsilon^{ab} B_a B_b \\
+ m b \epsilon^{ab} (A_a - V_a) B_b + m a F(A) \bigg) + I(V, v). \tag{30}
\end{equation}

From the equations of motions for \( B_a \) and \( b \) we deduce

\begin{align*}
B^A G^{-1}_{aA} b^B &= k^{bB}, \tag{31} \\
b^A &= \frac{1}{2m} F^A (A) + \frac{1}{2m} \epsilon^{ab} (A_a^B - V_a^B) B_b^C f^{ABC}, \tag{32}
\end{align*}

where

\begin{align*}
k^{aA} &\equiv -\frac{1}{4m^2} \epsilon^{ab} F^C (A) \nabla_b a^A - \frac{1}{4m^2} \epsilon^{ab} (A_a^B - V_a^B) F^C (A) f^{ABC}, \tag{33} \\
G^{-1}_{aA} &\equiv \delta_{AB} \eta_{ab} + \frac{1}{2m} \epsilon_{ab} f_{ABC} (a^C - v^C) \\
- \frac{1}{4m^2} \epsilon_{ac} \epsilon_{bd} (A^{CC} - V^{CC}) (A^{DD} - V^{DD}) f^E_A f^F_B f^{DE} . \tag{34}
\end{align*}
Using these equations we can eliminate $B$ and $b$ in (30). We then obtain an action dual to $I_2(B, b, V, v)$,

\[
I_2(A, a, V, v) = \int d^2x \left\{ -\frac{1}{2} m^2 k^a G_{aD}(A - V, a - v) \right\} + I(V, v),
\]

containing two different gauge fields. The gauge transformations are now

(i) $\delta_A a = -\frac{1}{2} [B_a, \Lambda]$, $\delta_A a = -\frac{1}{2} [b, \Lambda],$

$\delta_A V_a = \nabla_a(V) \Lambda$, $\delta_A v = [v, \Lambda],$

(ii) $\delta_\Sigma A_a = \bar{\nabla}_a \Sigma \equiv \nabla_a(A) \Sigma + \frac{1}{2} [B_a, \Sigma],$

$\delta_\Sigma a = [\bar{a}, \Sigma] \equiv [a, \Sigma] + \frac{1}{2} [b, \Sigma],$

$\delta_\Sigma V_a = 0$, $\delta_\Sigma v = 0,$

(36)

where, e.g.

\[
[B^a, \Lambda]^C = G^{aB} k_{BD} \Lambda^B f^{ABC}.
\]

(37)

Thus we have the DS $(I_2(A, a, V, v), I_2(B, b, V, v))[I_2(\bar{A}, \bar{a}, B, b, V, v)]$ in analogy with the 3D result.

We now turn to the supersymmetric generalization of the self-dual theory. The procedure follows closely that of the bosonic theory above and is essentially the same as in 3D. The proliferation of fields and terms in the actions makes the expressions a bit difficult to appreciate, though, and for this reason we have chosen to be a bit sketchy below, emphasizing mainly the structure.

The two dimensional analogue of the first order action $I_2(\bar{\Gamma}, \Psi, V)$ is

\[
I_2(\Psi, \bar{H}, V, v) = -\frac{1}{2} \text{Tr} \int d^2 x d^2 \theta \left[ 2im^2 \Psi_- \Psi_+ + \frac{m}{4} \nabla(X \Psi_-) \nabla(X \Psi_-) 
\right.
\]

\[
+ \frac{m}{2} \{\Psi_+, \Psi_-\} \nabla(X \Psi_-) - \frac{m}{6} (\nabla(X \Psi_-) \Psi_+^2 
\]

\[
- \frac{m}{6} (\nabla(X \Psi_+) \Psi_-^2 + \frac{m}{2} \{\Psi_+, \Psi_-\} V + 4m \bar{H}^2 \right] + I(V, v)
\]

(38)
where \( \mathcal{V} \) is defined analogously to \( H \) but with the difference that the spinorial gauge field is \( V \) instead of \( \Gamma \). Integrating out the Grassmann variables we find the component action (52), given in the appendix A.3. A shift \((\bar{A}_a, \bar{a}) = (A_a, a) + \frac{1}{2}(B_a, b)\) allows us to eliminate the \((B_a, b)\) multiplet and find a complicated system consisting of the fields contained in the spinorial gauge multiplet \( \Gamma \) and the rest of the fields contained in the spinorial superfield \( \Psi \) multiplet which was not integrated out.

6 Conclusions

In this paper we have studied the reduction to 2D of certain 3D systems. We have seen that the 3D non-abelian dualities reduce to 2D non-abelian dualities between massive scalar and vector fields and we have presented new non-abelian dualities in the 2D systems that have no 3D counterpart. Further we have given the supersymmetrizations of these relations.

This investigation arose out of a wish to see if the 3D non-abelian dualities could lead to new types of non-abelian 2D dualities, possibly extending the set of \( T \)-dualities for strings. It was hoped that the 2D-equivalence \( A_a = \partial_a \varphi + \epsilon_a^b \partial_b \lambda \) in conjunction with nonlocal field redefinitions would allow for such an application. The reduced systems turned out to be much too unwieldy, however. As they stand, the 2D dualities presented involve massive scalar and vector fields and one has to look elsewhere for applications. It is then interesting to note that there are other extended objects within string/M-theory that do involve massive dualities. In particular, the duality between a massive D2-brane and a dimensionally reduced M2-brane coupled to an auxiliary vector field exemplifies this [8]. These models involve Born-Infeld terms in place of our \( B^2 \)-terms, but we may speculate that, as usual, a non-abelian generalization would start from the first term in a series expansion. This would directly lead to some of our 3D actions. Our 2D-results would then be relevant for the double dimensional reduction of these models.

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A Notation and Conventions

$a, b, \ldots$ and $\alpha, \beta, \ldots$ denote Lorentz vector and spinor indices, respectively. We do not discriminate between 3D and 2D indices since they appear in different sections. In $D = 2$ we will also use $a \in \{+, -\}$ and $\alpha \in \{+, -\}$. Our metric has signature $(+, -,-)$ and $(+,-)$ in $D = 3$ and $D = 2$. We (anti)symmetrize without combinatorial factors, e.g. $A_{(aB_b)} \equiv A_a B_b + A_b B_a$. All our fields $A = A^B T^B$ transform in the adjoint representation of some Lie group $G$ whose Lie algebra generators satisfy

$$\text{Tr}(T^A T^B) = \delta^{AB}, \quad [T^A, T^B] = f^{ABC} T^C \quad (39)$$

Covariant derivatives act according to

$$\left(\nabla_a (A B_b)\right)^A = \partial_a B_b^A + f^{ABC} A^B_a B^C_b \quad (40)$$

The field strength in $D = 3$ is $F_{ab} (A) = \partial_a A_b - \partial_b A_a + [A_a, A_b]$ and in $D = 2$ $F = \partial_+ A_+ - \partial_- A_- + [A_+, A_-]$. The dual field strength is $*F^a \equiv \epsilon^{abc} F_{bc}$ in 3D and $*F^a \equiv \epsilon^{ab} F_b$ in 2D. The (3D) Chern-Simons term is

$$\Omega(A) = \epsilon^{abc} A_a \left( F_{bc} - \frac{2}{3} A_b A_c \right) = 2 \epsilon^{abc} A_a \left( \partial_b A_c + \frac{2}{3} A_b A_c \right) . \quad (41)$$

A.1 3D-superspace

In superspace we replace vector indices by pairs of spinor indices according to

$$V_{\alpha \beta} = \frac{1}{\sqrt{2}} (\gamma^a)_{\alpha \beta} V_a , \quad (42)$$

where $\{(\gamma^a)_{\alpha \beta}, (\gamma^b)_\beta^\nu\} = 2 \sigma^a \delta_\alpha^\nu$. We raise and lower indices using $C_{\alpha \beta} = -C^{\alpha \beta} = \sigma^2$, e.g. $\Psi^\alpha = C^{\alpha \beta} \Psi_\beta$; $\psi_\alpha = \Psi^\beta C_\beta^\alpha$. These are essentially the conventions of [13]. Letting $D_\alpha$ be the flat superspace covariant derivatives,
the Yang-Mills covariant derivatives \( \nabla_{\alpha} = D_{\alpha} - i\Gamma_{\alpha} \) and \( \nabla_{\alpha\beta} = \partial_{\alpha\beta} - i\Gamma_{\alpha\beta} \) satisfy

\[
\{\nabla_{\alpha}, \nabla_{\beta}\} = 2i\nabla_{\alpha\beta}, \quad \text{(43)}
\]

\[
[\nabla_{\alpha}, \nabla_{\beta\gamma}] = C_{\alpha(\beta} W_{\gamma)}, \quad \text{(44)}
\]

\[
\{\nabla^{\alpha}, W_{\alpha}\} = 0, \quad \text{(45)}
\]

where \( W_{\alpha} \) is the super Yang-Mills spinorial field strength

\[
W_{\alpha} = \frac{1}{2} D_{\beta} D^{\alpha} \Gamma_{\beta} - \frac{i}{2} \{\Gamma_{\beta}, D_{\beta} \Gamma_{\alpha} \} - \frac{1}{6} \{\Gamma_{\beta}, \Gamma_{\alpha}\}. \quad \text{(46)}
\]

A convenient representation of the Clifford algebra is \((\gamma^a)_\alpha^\beta = (\sigma^2, -i\sigma^1, i\sigma^3)\).

### A.2 2D-superspace

We reduce to 2\(D\) by letting \((\gamma^a)_\alpha^\beta = (\sigma^2, -i\sigma^1)\) and introducing \(\gamma^5 \equiv \gamma\) with \((\gamma^5)_\alpha^\beta = \sigma^3\). A vector in 2\(D\) spinor notation is now given by a (symmetric) \(\gamma\)-traceless pair of indices \(V_{\alpha\beta} = V_{\beta\alpha}\); \(\gamma^{\alpha\beta} V_{\alpha\beta} = 0\), and a 3\(D\) vector reduces according to

\[
V_{\alpha\beta} \rightarrow V_{\alpha\beta} + \frac{i}{\sqrt{2}} \gamma_{\alpha\beta} v. \quad \text{(47)}
\]

The algebra thus becomes

\[
\{\nabla_{\alpha}, \nabla_{\beta}\} = 2i\nabla_{\alpha\beta} + 2i\gamma_{\alpha\beta} H, \quad \text{(48)}
\]

defining the 2\(D\) Yang-Mills field strength scalar superfield \(H\). Using projection operators \(P_\pm \equiv \frac{1}{2}(1 \pm \gamma)\) we may rewrite this in “\(\pm\)-notation” as

\[
\nabla_\pm \equiv D_\pm - i\Gamma_\pm, \quad \text{(49)}
\]

\[
\nabla^2_\pm = \pm i\nabla_+ =, \quad \text{(50)}
\]

\[
\{\nabla_+, \nabla_-\} = 2H, \quad \text{(51)}
\]

which is the form of the 2\(D\) superspace Yang-Mills algebra we need.
A.3 2D-supersymmetric actions

The component action of the supersymmetric self-dual massive action is

\[
I_2(\bar{A}, \bar{a}, \lambda, B, b, S, \zeta, \chi, V, v) = \text{Tr} \int d^2x \left\{ \frac{1}{2} m^2 B_a B^a - \frac{1}{2} m^2 b^2 - \frac{1}{2} m \epsilon^{ab} b(\nabla_a(V) B_b + \frac{1}{2} B_a B_b) + \frac{1}{4} m \epsilon^{ab} B_a [v, B_b] + m \bar{a} F(\bar{A}) + \frac{1}{2} m^2 S^2 + m^2 (i \bar{\zeta} \nabla(V) \zeta + \frac{i}{2 \sqrt{2}} \bar{\zeta} \gamma[v, \zeta] - 2 \bar{\chi} \zeta) + 2m \left( \bar{\lambda} \lambda - \frac{1}{4} \bar{\chi} \chi \right) - \frac{1}{2} m ([S, \bar{\zeta}] - i [B^a, \bar{\zeta}] \gamma_a + [b, \zeta] \gamma) \left( \eta + \frac{1}{2} \chi \right) + \frac{1}{4} m F(V) \bar{\zeta} \gamma \zeta + \frac{1}{2} m \epsilon^{ab} (\nabla_b(V) v)(\bar{\zeta} gamma) + \frac{1}{8} m \epsilon^{ab} B_a [v, \bar{\zeta} \gamma_b \zeta] + \frac{1}{8} m \epsilon^{ab} B_a \nabla_b (V) \bar{\zeta} \gamma \zeta \right\} + I(V, v). \tag{52}
\]

After performing the shift \((\bar{A}_a, \bar{a}) = (A_a, a) + \frac{1}{2} (B_a, b)\) the component action transforms into

\[
I_2(A, a, \lambda, B, b, S, \zeta, \chi, V, v) = \text{Tr} \int d^2x \left\{ \frac{1}{2} m^2 B_a B^a - \frac{1}{2} m^2 b^2 + \frac{1}{2} m \epsilon^{ab} B_a B_b + m \epsilon^{ab} (A_a - V_a) B_b + ma F(A) + \frac{1}{2} m (a - v) \epsilon^{ab} B_a B_b + m b \epsilon^{ab} (A_a - V_a) B_b + ma F(A) + \frac{1}{2} m^2 S^2 + m^2 (i \bar{\zeta} \nabla(V) \zeta + \frac{i}{2 \sqrt{2}} \bar{\zeta} \gamma[v, \zeta] - 2 \bar{\chi} \zeta) + 2m \left( \bar{\lambda} \lambda - \frac{1}{4} \bar{\chi} \chi \right) - \frac{1}{2} m ([S, \bar{\zeta}] - i [B^a, \bar{\zeta}] \gamma_a + [b, \zeta] \gamma) \left( \eta + \frac{1}{2} \chi \right) + \frac{1}{4} m F(V) \bar{\zeta} \gamma \zeta + \frac{1}{2} m \epsilon^{ab} (\nabla_b(V) v)(\bar{\zeta} gamma) + \frac{1}{8} m \epsilon^{ab} B_a [v, \bar{\zeta} \gamma_b \zeta] + \frac{1}{8} m \epsilon^{ab} B_a \nabla_b (V) \bar{\zeta} \gamma \zeta \right\} + I(V, v). \tag{53}
\]

We may now eliminate \(B_a\) and \(b\) using the equations of motion to find a very complicated system that we spare the reader.
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