Hamilton-Jacobi formulation of the thermodynamics of Einstein-Born-Infeld-AdS black holes

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Abstract – A Hamilton-Jacobi formalism for thermodynamics was formulated by Rajeev (Ann. Phys., 323 (2008) 2265) based on the contact structure of the odd-dimensional thermodynamic phase space. This allows one to derive the equations of state of a family of substances by solving a Hamilton-Jacobi equation (HJE). In the same work it was applied to chargeless non-rotating black holes, and the use of Born-Infeld electromagnetism was proposed to apply it to charged black holes as well. This paper fulfills this suggestion by deriving the HJE for charged non-rotating black holes using the Born-Infeld theory and a negative cosmological constant. The most general solution of this HJE is found. It is shown that there exist solutions which are distinct from the equations of state of the Einstein-Born-Infeld-AdS (EBIAdS) black hole. The meaning of these solutions is discussed.

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Introduction. – General relativity predicts that black holes obey the laws of thermodynamics with the horizon area and the surface gravity playing the roles of entropy and temperature, respectively [1,2]. We may expect that in a quantum theory of gravitation these thermodynamic quantities such as mass and area are treated as operators. It is therefore reasonable to attempt to “quantize” thermodynamics. Another possible application of quantized thermodynamics is to systems where the thermal fluctuations are small, but the quantum fluctuations are not.

Thermodynamics can be formulated in terms of contact geometry [3], where the first law is used to define a contact structure and the equations of state pick out a Legendrian submanifold of the thermodynamic phase space. With the aim of quantizing thermodynamics, a quantization procedure for contact manifolds is established in [4]. Subsequently, a Hamilton-Jacobi formalism for thermodynamics is developed in [5].

In this formalism one extends a thermodynamic system of $n$ degrees of freedom into an $n$ parameter family; e.g., the ideal gas of a fixed particle number can be extended into the van der Waals family with the parameters $a$ and $b$. This family can be described as a hypersurface in the phase space, defined by the vanishing of a function which we take to be the Hamiltonian. Given the Hamiltonian function $H$, one can formulate a Hamilton-Jacobi equation (HJE), the characteristic curves of which correspond to the dynamics generated by $H$. The equations of state can be obtained in principle by solving the HJE.

In the same work the Hamilton-Jacobi formalism is applied to black holes of one thermodynamical degree of freedom, i.e., to electrically neutral non-rotating black holes. A negative cosmological constant is introduced to extend the Schwarzschild black hole into the Schwarzschild-AdS family. It is also proposed that the Born-Infeld action be used as a modification of the Einstein-Maxwell equations to describe the family of charged black holes (further modifications can be made to include the rotating ones as well). Following this suggestion we shall apply the Hamilton-Jacobi formalism to charged non-rotating black holes.

We first summarize Hamiltonian dynamics in contact geometry and how it is applied to thermodynamics in [5]. Next we review the non-rotating black-hole solutions in the Einstein-Born-Infeld (EBI) theory. We extract the thermodynamical quantities such as the mass and the surface gravity and find the hypersurface in the thermodynamical phase space which describes the EBIAdS family. Lastly, we write down the HJE and discuss its solutions.

Throughout this work, we work with units for which $c = G = 4\pi\epsilon_0 = 1$.

Hamilton-Jacobi formalism for thermodynamics.

Before describing the formalism that we shall use, we briefly review contact geometry and contact Hamiltonian
dynamics. The proofs of the claims stated here and further reading on the subject can be found in [6].

**Contact Hamiltonian dynamics.** A contact structure $\xi$ on a $(2n+1)$-dimensional manifold $\mathcal{M}$ is a codimension-one distribution which is maximally non-integrable. In other words it is (locally) the kernel of a 1-form $\alpha$ such that

$$\alpha \wedge (d\alpha)^n \neq 0. \quad (1)$$

The contact form $\alpha$ is not unique since $\ker(f\alpha) = \ker(\alpha)$ if $f$ is a non-vanishing function. The contact condition (1) is, however, independent of the choice of $\alpha$. For a fixed contact form $\alpha$ there is a unique vector field $R_\alpha$ called the *Reeb vector field* which satisfies $i_{R_\alpha} \alpha = 1$ and $i_{R_\alpha} d\alpha = 0$.

If a vector field $X$ generates a contactomorphism $\mathcal{M} \to \mathcal{M}$, i.e., a diffeomorphism which preserves the contact structure,

$$\mathcal{L}_X \alpha = \mu \alpha, \quad (2)$$

where $\mu$ is an arbitrary function, then $X$ is called a *contact vector field*. If the contact form $\alpha$ is fixed, there is a one-to-one correspondence between the smooth functions $F: \mathcal{M} \to \mathbb{R}$ and the contact vector fields $X$ on $\mathcal{M}$ which is given by

- $X \to F_X := i_X \alpha$,
- $F \to X_F$ defined as the unique solution of the equations $i_{X_F} \alpha = F$ and $i_{X_F} d\alpha = R_\alpha(F) \alpha - dF$.

In this context the function $F$ is called the generating function or the Hamiltonian and $X_F$ the corresponding Hamiltonian vector field. Thus, in close analogy to symplectic geometry, a Hamiltonian function $F$ defines a dynamics on $\mathcal{M}$ by the flow of $X_F$. Note, however, that

$$\mathcal{L}_{X_F} F = F R_\alpha(F), \quad (3)$$

so the Hamiltonian is in general not conserved. But if the initial value of zero is fixed, it remains zero.

As a final remark we note that if $L$ is a submanifold such that $TL \subset \xi$, then dim $L \leq n$. Such a submanifold of maximal dimension $n$ is called a *Legendrian submanifold*.

**Application to thermodynamics.** The phase space $\mathcal{M}$ of a thermodynamic system of $n$ degrees of freedom is $2n+1$ dimensional. For concreteness, take a gas with a fixed number of particles. The variables $U, S, V, T$ and $p$ can be taken as the coordinates on $\mathcal{M}$. The first law defines a contact form

$$\alpha = -dU + TdS - pdV, \quad (4)$$

whose kernel is the set of directions in which the energy is conserved. The equations of state define an $n$-dimensional submanifold, all of whose tangent vectors are annihilated by $\alpha$, i.e., a Legendrian submanifold:

$$U = U(S, V), \quad T = \left(\frac{\partial U}{\partial S}\right)_V, \quad p = \left(\frac{\partial U}{\partial V}\right)_S. \quad (5)$$

Note that once the fundamental relation $U = U(S, V)$ is given, the remaining $n$ equations of state follow automatically by the vanishing of $\alpha$.

In general, let

$$\alpha = d\mu + p_i dq^i \quad (6)$$

be a contact form on the thermodynamic phase space $\mathcal{M}$ with coordinates $u, q^1, \ldots, q^n, p_1, \ldots, p_n$ (see footnote $^1$). A family of substances can be described by allowing the fundamental relation $u = \Phi(q^1, \ldots, q^n)$ to depend on $n$ parameters $a_1, \ldots, a_n$ as well:

$$u = \Phi(q^1, \ldots, q^n, a_1, \ldots, a_n). \quad (7)$$

E.g., for the van der Waals family we have $U = U(S, V; a, b)$, where $a$ and $b$ are the van der Waals parameters.

Given the fundamental relation (7) and the other $n$ equations of state

$$p_i = -\frac{\partial \Phi}{\partial q^i}, \quad (8)$$

we can eliminate the parameters $a_1, \ldots, a_n$ to get a single relation between the $2n + 1$ coordinates:

$$F(u, q^1, \ldots, q^n, p_1, \ldots, p_n) = 0. \quad (9)$$

Given such a Hamiltonian $F$ for a family of substances, the equations of state may be recovered by solving the first-order PDE

$$F \left(\Phi, q^1, \ldots, q^n, \frac{\partial \Phi}{\partial q^1}, \ldots, \frac{\partial \Phi}{\partial q^n}\right) = 0. \quad (10)$$

A complete integral of this PDE will depend on $n$ parameters $a_1, \ldots, a_n$. The characteristic curves of this PDE are precisely the integral curves of the Hamiltonian vector field $X_F$ [5,7].

**The Einstein-Born-Infeld-AdS black hole.** – The EBI field equations with a cosmological constant $\Lambda$ may be derived from the action

$$S[e^a, A] = \frac{1}{16 \pi} \int (R^{ab} \wedge \ast e_{ab} - 2 \Lambda \ast 1) + \int \mathcal{L} \ast 1, \quad (11)$$

with the Born-Infeld Lagrangian [8]

$$\mathcal{L} = \frac{1}{4 \pi \lambda^2} \left(1 - \sqrt{1 - \lambda^2 X - \lambda^4 Y^2}\right), \quad (12)$$

where the quadratic invariants of the electromagnetic field are given by

$$X = \ast (F \wedge \ast F), \quad Y = \frac{1}{2} \ast (F \wedge F). \quad (13)$$

Here $R^{ab}$ are the curvature 2-forms of the Levi-Civita connection, $e^a$ are the orthonormal coframe 1-forms,

$^1$By Darboux’s theorem one can always find such coordinates for a given contact form $\alpha$.

$^2$We choose the number of parameters to be $n$ since this is going to allow us to define a Hamiltonian function, see [5].
Hamilton-Jacobi formulation of the thermodynamics of EBIAdS black holes

e_a = \eta_a e^b$ and $e_a \ldots e_a := e_a \land \ldots \land e_a$. The Born-Infeld parameter $\lambda$ is to be seen as a new fundamental constant of nature. Note that in the limit $\lambda \to 0$ the Born-Infeld Lagrangian $\mathcal{L}$ becomes the Maxwell Lagrangian $\mathcal{L} \to (8\pi)^{-1} * (F \land *F)$.

The variational field equations for the case with no cosmological constant are derived using the invariant tensor notation in [9]. The derivation with $\Lambda$ is analogous and hence we simply note the results. Variation with respect to the coframe $e$ leads to the EBI field equations

$$-\frac{1}{2}R^{bc} \land *e_{abc} + \Lambda * e_a = 8\pi \tau_a,$$

with the stress-energy 3-forms

$$\tau_a = M * e_a + N_r^{(\text{LED})},$$

where $M = \mathcal{L} - X \partial X - Y \partial Y, N = 8\pi \partial X \mathcal{L}$ and $r_a^{(\text{LED})}$ are the stress-energy 3-forms of linear (i.e., Maxwell) electrodynamics:

$$r_a^{(\text{LED})} = \frac{1}{8\pi} (i_a F \land *F - F \land i_a *F).$$

Variation with respect to the electromagnetic potential $A$ yields the field equation $dG = 0$ where

$$G = \frac{1}{4\pi \sqrt{\Delta}} (*F + \lambda^2 YF),$$

with $\Delta = 1 - \lambda^2 X - \lambda^2 Y^2$.

The static spherically symmetric solution without the cosmological constant was first found by Hoffmann [10] and then rediscovered by Demianski [11]. The solution in the presence of a cosmological constant was noted in [12]. Note that the action they use differs from ours by the absence of the $Y^2$ term. However, we shall see below that in the static case one has $Y = 0$, hence our results agree. The solution is given by

$$\begin{align*}
  g &= -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \\
  F &= \frac{Q}{\sqrt{r^2 + a^2}} dr \land dt,
\end{align*}$$

where

$$\begin{align*}
  f(r) &= -\frac{\Lambda}{3} r^2 + 1 - \frac{2M}{r} + \frac{2Q^2}{ar} h\left(\frac{r}{a}\right), \\
  h(x) &= \int_x^\infty dy (\sqrt{y^4 + 4} - y^2),
\end{align*}$$

$Q$ is the total electric charge, $a = (\Lambda|Q|)^{1/2}$ and $M$ is the total mass. For simplicity we shall work with a negative cosmological constant $\Lambda = -3l^{-2}$ as in [5]. Then $f$ can be expanded for $a \ll r$ as

$$f(r) = \frac{r^2}{l^2} + 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \left[1 - \frac{a^4}{20r^4} + O\left(\frac{a^6}{r^6}\right)\right].$$

Figure 1: Plot of the function $h(x)$ defined in (19). It satisfies $h(0) \approx 1.24$ and $\lim_{x \to \infty} h(x) = 0$.

In particular it is asymptotically AdS and in the limit $\lambda \to 0$ we recover the Reissner-Nordstr"om-AdS (RNAdS) black hole as expected.

To investigate the horizon structure we first note that

$$\lim_{r \to \infty} rf(r) = \infty \quad \text{and} \quad rf(r)|_{r=0} = -2M + \frac{2Q^2}{a} h(0)$$

$(h(0) \approx 1.24$ is finite, see fig. 1). We shall limit our attention to the case

$$2M < \frac{2Q^2}{a} h(0) \quad \text{and} \quad \lambda < 2Q,$$

which includes the RNAdS black hole. With the assumptions in (22), the function $rf(r)$ has exactly one minimum whose position we denote by $r_0 = r_0(\lambda, \Lambda, Q)$. Since $rf(r)$ is positive at $r = 0$ and $r = \infty$, it will have no zeros, a double zero or two zeros if $r_0 f(r_0)$ is positive, zero or negative, respectively. Hence from (19) we see that there is a critical mass

$$2M_c = \frac{r_0^3}{l^2} + r_0 + \frac{2Q^2}{a} h\left(\frac{r_0}{a}\right),$$

such that (see fig. 2) if:

1) $M < M_c$, there is no horizon. In that case there is a naked singularity at $r = 0$ where the Kretschmann scalar $K = 2 * (R_{ab}^e \land *R_{ab}^b)$ diverges.

2) $M = M_c$, we have an extremal black hole with one horizon.

3) $M > M_c$, there are two horizons.

Thus, the situation is completely analogous to the RN case.

If $l$ is small, it may not be possible to satisfy the condition $M > M_c$ simultaneously with (22), but it is possible if $l$ is sufficiently large. In the subsequent discussion we shall assume that $M > M_c$, which should cover the physically relevant cases. We denote the outer horizon radius by $r_H$.

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$^3$For the definition of the total mass in an asymptotically (A)dS spacetime, see [13]
The thermodynamical equations of state were given in [14] for the case with \( \Lambda = 0 \) and then in [15] for a non-zero \( \Lambda \). We also note that the equations of state of the EBIAdS family were examined in [16] using an extended phase space, where the Born-Infeld parameter and the cosmological constant are treated as thermodynamic variables. In the Hamilton-Jacobi formalism, however, they are parameters that specify a certain element of a family of substances, not thermodynamic degrees of freedom, hence we do not consider their variations in the first law. In what follows, we note the equations of state and derive them the HJE.

Using (19) and the fact that \( f(r_H) = 0 \) we can write the mass of the black hole as

\[
2M = \frac{r_H^3}{l^2} + r_H + \frac{2Q^2}{a} h \left( \frac{r_H}{a} \right), \tag{24}
\]

The surface gravity \( \kappa = (1/2) \frac{df}{dr} |_{r = r_H} \) is given by

\[
\kappa = \frac{r_H}{l^2} + \frac{M}{r_H^2} - \frac{Q^2}{a^2 r_H} h \left( \frac{r_H}{a} \right) = \frac{Q^2}{a^2 r_H} \left( \sqrt{r_H^4 + a^4} - r_H^2 \right), \tag{25}
\]

and the electrostatic potential on the horizon is

\[
\Phi = \int_{r_H}^{\infty} \frac{Q}{\sqrt{x^4 + a^4}} dx. \tag{26}
\]

Introducing the surface area \( A = 4\pi r_H^2 \) and using the identity

\[
h(x) = \frac{2}{3} \int_x^{\infty} \frac{dy}{\sqrt{y^4 + 1}} = \frac{2}{3} \left( \sqrt{x^4 + 1} - x \right), \tag{27}
\]

it is straightforward to verify the first law

\[
dM = \frac{\kappa}{8\pi} dA + \Phi dQ, \tag{28}
\]

which defines our contact structure. Furthermore, from the three equations (24)–(26) we can eliminate \( a \) and \( l \) to get a single relation between the variables \( M \), \( A \), \( \kappa \), \( Q \) and \( \Phi \):

\[
3M - \frac{\kappa A}{4\pi} - 2\Phi Q = \sqrt{\frac{A}{4\pi}}. \tag{29}
\]

We therefore see —using the first law— that the EBIAdS family is described by the hypersurface \( F(M, A, Q, p_A, p_Q) = 0 \) with the Hamiltonian

\[
F(M, A, Q, p_A, p_Q) = 3M - 2p_A A - 2p_Q Q - \sqrt{\frac{A}{4\pi}}. \tag{30}
\]

The HJE we get from this Hamiltonian is

\[
3M - 2\frac{\partial M}{\partial A} - 2 \frac{\partial M}{\partial Q} = \sqrt{\frac{A}{4\pi}}. \tag{31}
\]

This is a first-order linear PDE which is surprisingly nice and we can do even better than finding a particular complete integral. One can indeed show that the most general solution is

\[
2M = \sqrt{\frac{A}{4\pi}} + \left( \frac{A}{4\pi} \right)^{3/2} u \left( \frac{Q}{A/4\pi} \right), \tag{32}
\]

where \( u \) is an arbitrary function which must be fixed by a boundary condition. The complete integral corresponding to the actual equation of state (24) is given by the choice

\[
u(x) = \frac{1}{l^2} + \frac{2x^{3/2}}{\lambda^{1/2}} h \left( \frac{1}{\lambda^{1/2}} \right). \tag{33}
\]

It should be noted, however, that (24) is not the only complete integral of the PDE (31) as, e.g., the choice \( u(x) = l^{-2} + 2x^{3/2}\lambda^{-1/2} \) also yields a complete integral. It is not exactly clear what this non-uniqueness means, but at the very least it shows that one must be careful to use the HJE to get the equations of state of a family of substances.

**Conclusion.** — To study the thermodynamic HJE for charged black holes we have made the RN black hole into a two-parameter family by introducing a (negative) cosmological constant and replacing the Maxwell Lagrangian by the Born-Infeld one. Under the assumption that the Born-Infeld parameter \( \lambda \) is sufficiently small, the horizon structure of the resulting static black hole is quite similar to that of the RN one. By this we mean that there is a critical mass below which there is no horizon (fig. 2).

An element of the EBIAdS family has two thermodynamical degrees of freedom. Hence its phase space is five dimensional, which can be coordinatized by \( M \), \( \kappa \), \( A \), \( \Phi \) and \( Q \), and the two parameters \( \Lambda \) and \( \lambda \) specify the particular element. Using the three equations of state (24)–(26) we were able to eliminate the parameters \( \Lambda \) and \( \lambda \) to get a single relation between the phase space variables. This relation defines a hypersurface by the vanishing of a Hamiltonian function and therefore yields a HJE.
The HJE (31) we get for the EBIAdS family is linear and we can find an analytic expression for the most general solution. It is interesting to note that the solution is far more general than the equation of state of the EBIAdS black hole. A function must be specified by a boundary condition to get the actual equation of state, not just two constants of integration. As we mentioned, the precise meaning of the existence of these solutions that do not correspond to the actual equation of state is not clear. This may get clearer if it is understood whether and how the HJE is related to the quantization of black holes. In any case it provides a caveat against the use of the Hamilton-Jacobi formalism to determine the equations of state of a family of substances.

It should also be remarked that the above is not the only way of extending the RN black hole into a two-parameter family; one may think of parameters other than the cosmological constant and the Born-Infeld parameter. In fact, we can introduce \( n \) parameters to a system of \( n \) degrees of freedom in a completely arbitrary manner. It is possible that there is another choice of extension which is free of this ambiguity. Moreover, the question of finding a HJE for rotating black holes is still open.

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