A generalization of Pappus chain theorem

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Abstract. We generalize Pappus chain theorem and give an analogue to this theorem.

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1. Introduction

Let $\alpha$, $\beta$ and $\gamma$ be circles with diameters $BC$, $CA$ and $AB$, respectively for a point $C$ on the segment $AB$. Pappus chain theorem says: if $\{\alpha = \delta_0, \delta_1, \delta_2, \cdots\}$ is a chain of circles whose members touch $\beta$ and $\gamma$, the distance between the center of the circle $\delta_n$ and the line $AB$ equals $2nr_n$, where $r_n$ is the radius of $\delta_n$ (see Figure 1). In this article we give a simple generalization of this theorem and show that if we consider a line passing through the centers of two circles in the chain instead of $AB$, a similar theorem still holds.

Figure 1.

2. A generalization of Pappus chain theorem

Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{\alpha, \beta, \gamma\}$ and $\{P_1, P_2, P_3\} = \{A, B, C\}$, where $P_3P_1$ and $P_1P_2$ are diameters of $\varepsilon_2$ and $\varepsilon_3$, respectively. We consider the chain of circles $C = \{\cdots, \delta_{-2}, \delta_{-1}, \varepsilon_1 = \delta_0, \delta_1, \delta_2, \cdots\}$ whose members touch the circles $\varepsilon_2$ and $\varepsilon_3$. Let $r_n$ be the radius of $\delta_n$. Pappus chain theorem is obtained in the case $i = 0$ in the following theorem (see Figure 2).

**Theorem 1.** If $D_i$ is the center of the circle $\delta_i \in C$ and $H_i(n)$ is the point of intersection of the line $P_iD_i$ and the perpendicular to $AB$ from $D_n$, the following relation holds.

\[(1) \quad |D_nH_i(n)| = 2|n - i|r_n.\]

*Proof.* We invert the figure in the circle with center $P_1$ orthogonal to $\delta_n$. Then $\delta_n$ and $P_1D_i$ are fixed and $\varepsilon_2$ and $\varepsilon_3$ are inverted to the tangents of $\delta_n$ perpendicular to $AB$. Let $F$ be the foot of perpendicular from $D_n$ to $AB$. Since $H_i(n)$ is the center of the image of $\delta_i$, we have $|H_i(n)F| = 2ir_n$, while $|D_nF| = 2nr_n$. Hence we get (1).

\[\square\]
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Figure 2: \( C = C_\alpha, \ i = 1, \ n = 3 \)

3. An analogue to Pappus chain theorem

Let \( a \) and \( b \) be the radii of the circles \( \alpha \) and \( \beta \), respectively. We use a rectangular coordinate system with origin \( C \) such that \( A \) and \( B \) have coordinates \((-2b,0)\) and \((2a,0)\), respectively. If \( \epsilon_1 = \alpha \), the chain is explicitly denoted by \( C_\alpha \). The chains \( C_\beta \) and \( C_\gamma \) are defined similarly. Let \( c = a + b \) and let \((x_n, y_n)\) be the center coordinate of the circle \( \delta_n \in C \). We have \( y_n = 2nr_n \) by Pappus chain theorem, and \( x_n \) and \( r_n \) are given in Table 1 [2, 3].

| Chain | \( x_n \) | \( r_n \) |
|-------|-----------|-----------|
| \( C_\alpha \) | \(-2b + \frac{bc(b + c)}{n^2a^2 + bc}\) | \( \frac{abc}{n^2a^2 + bc}\) |
| \( C_\beta \) | \(2a - \frac{ca(c + a)}{n^2b^2 + ca}\) | \( \frac{abc}{n^2b^2 + ca}\) |
| \( C_\gamma \) | \( \frac{ab(b - a)}{n^2c^2 - ab}\) | \( \frac{abc}{n^2c^2 - ab}\) |

Table 1: \( y_n = 2nr_n \)

Let \( l_{ij} \ (i \neq j) \) be the line passing through the centers of the circles \( \delta_i \) and \( \delta_j \) for \( \delta_i, \delta_j \in C \). It is expressed by the equations

\[
\begin{align*}
2(bc - a^2ij)x + a(b + c)(i + j)y - 2b(2a^2ij - c(b - c)) &= 0, \\
2(ca - b^2ij)x - b(c + a)(i + j)y + 2a(2b^2ij + c(c - a)) &= 0, \\
2(ab + c^2ij)x + c(a - b)(i + j)y - 2ab(a - b) &= 0
\end{align*}
\]

in the cases \( C = C_\alpha, \ C = C_\beta, \ C = C_\gamma \), respectively.

Let \( H_{ij}(n) \) be the point of intersection of the lines \( l_{ij} \) and \( x = x_n \) with \( y \)-coordinate \( h_{ij}(n) \). Let \( d_{ij}(n) = h_{ij}(n) - y_n \), i.e., \( d_{ij}(n) \) is the signed distance between the center of \( \delta_n \) and \( H_{ij}(n) \). The following theorem is an analogue to Pappus chain theorem (see Figure 3). It is also a generalization of [1].

**Theorem 2.** If \( i + j \neq 0 \), then \( d_{ij}(n) = f_{ij}(n)r_n \) holds, where

\[
f_{ij}(n) = \frac{2(n - i)(n - j)}{i + j}.
\]

**Proof.** We consider the chain \( C_\alpha \). By Table 1 and (2), we get

\[
h_{ij}(n) = \frac{2(n^2 + ij)abc}{(i + j)(n^2a^2 + bc)} = \frac{2(n^2 + ij)}{(i + j)}r_n.
\]
Therefore
\[ d_{i,j}(n) = h_{i,j}(n) - y_n = 2 \frac{(n^2 + ij)}{(i + j)} r_n - 2nr_n = \frac{2(n-i)(n-j)}{i+j} r_n. \]

The rest of the theorem can be proved in a similar way. □

Figure 3: \( C = C_\beta, \{i, j\} = \{0, 1\}, n = 2 \)

**Corollary 1.** If \( i = 0 \) in Theorem 2, the following statements hold.
(i) If \( j = \pm 1 \), \( d_{i,j}(n) = \pm 2n(n \mp 1)r_n \).
(ii) If \( j = \pm 2 \), \( d_{i,j}(n) = \pm n(n \mp 2)r_n \).

**Corollary 2.** \( d_{i,j}(n) - d_{i,j}(-n) = -4nr_n \) for any integers \( i, j, n \) with \( i \neq \pm j \).

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