On the Beauville form of the known irreducible symplectic varieties

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Abstract

We study the global geometry of the ten dimensional O’Grady irreducible symplectic variety. We determine its second Betti number, its Beauville form and its Fujiki constant.

Introduction

Irreducible symplectic varieties are simply connected compact Kähler manifolds with a unique, up to $\mathbb{C}^*$, global holomorphic two form and such that this two form is nondegenerate at each point. By Bogomolov’s decomposition theorem \cite{Bo}, they are fundamental in the classification of compact Kähler manifolds with torsion $c_1$.

Very few examples of irreducible symplectic varieties are known. For any positive integer $n$, Beauville exhibited two examples of dimension $2n$ \cite{Be83}: the Hilbert scheme $X^{[n]}$, parametrizing 0-dimensional subschemes of length $n$ on a K3 surface $X$, and the generalized Kummer variety $K^n(T)$ of a 2-dimensional torus $T$. More recently O’Grady, in \cite{OG99} and \cite{OG03}, found two new examples $\tilde{M}$ and $M$ of dimension six and ten. Up to now, each known example can be deformed into one of these examples.

For any irreducible symplectic variety $Y$, the group $H^2(Y, \mathbb{Z})$ is endowed with a deformation invariant integral primitive bilinear form $B_Y$, called the Beauville form. Related with the Beauville form there is a positive rational constant $c_Y$, called the Fujiki constant, which is a topological invariant of $Y$. The Beauville form and the Fujiki constant are fundamental invariants of an irreducible symplectic variety: in the cases of the Beauville examples they have been known since the paper \cite{Be83} and in the case of the six dimensional O’Grady example $\tilde{M}$ they were determined in \cite{Ra04}.

In this paper we deal with the remaining case: we determine the second Betti number of the ten dimensional O’Grady example $M$ and then compute...
In the following table we give the complete list of the Beauville forms and the Fujiki constants of all known irreducible symplectic varieties.

| Y         | dim(Y) | b2(Y) | cy    | (H^2(Y, Z), B_Y) |
|-----------|--------|-------|-------|------------------|
| X[n]      | 2n     | 23    | \frac{(2n)!}{n!2^n} | H^{\oplus 3} \oplus (-E_8^{\oplus 1}) \oplus (-2(n - 1)) |
| K^n(T)    | 2n     | 7     | \frac{12(n)!}{3^n} (n + 1) | H^{\oplus 3} \oplus (-2(n + 1)) |
| M         | 6      | 8     | 60    | H^{\oplus 3} \oplus (-2)^{\oplus 2} |
| M         | 10     | 24    | 945   | H^{\oplus 3} \oplus (-E_8^{\oplus 1}) \oplus \Lambda |

In this table the lattice \( H \) is the standard hyperbolic plane, the lattice \(-E_8\) is the unique negative definite even unimodular lattice of rank eight and \((i)\) is the rank 1 lattice generated by an element whose square is \( i \). Finally \( \Lambda \) is a rank 2 lattice whose associated matrix in a suitable basis is \[
\begin{pmatrix}
-6 & 3 \\
3 & -2
\end{pmatrix}.
\]

It is remarkable that both the O’Grady examples have the same Fujiki constants of Beauville examples of the same dimension: the Fujiki constant of \( \tilde{M}_6 \) equals the Fujiki constant of \( K_3(T) \) and the Fujiki constant of \( M \) equals the one of \( X[5] \).

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1 \quad b_2(M) = 24

In this paper \( X \) is a K3 surface such that \( Pic(X) = \mathbb{Z} < H > \) and \( H^2 = 2 \). We denote by \( M_{(a,b,c)} \) the Simpson moduli space of semistable sheaves on \( X \) with Mukai vector \((a, bc_1(H), c\eta) \in H^0(X, Z) \oplus H^2(X, Z) \oplus H^4(X, Z)\), where \( \eta \) is the fundamental form in \( H^4(X, Z) \).

If \( a = 0 \) and \( b > 0 \) the moduli space \( M_{(a,b,c)} \) parametrizes pure 1-dimensional sheaves on \( X \), hence there exists a regular morphism \( \Phi_{(a,b,c)} : M_{(a,b,c)} \to |bH| \) sending a sheaf to its Fitting subscheme (see [LP 93] end [Ei 95]). The general fiber of \( \Phi_{(a,b,c)} \) is easily described: if \( C \in |bH| \) is smooth then \( \Phi_{(a,b,c)}^{-1}(C) \) parametrizes push-forwards in \( X \) of line bundles of degree \( b^2 + c \) on \( C \). More precisely there is an isomorphism \( \Phi_{(a,b,c)}^{-1}(C) \simeq Pic b^2 + c(C) \).

If \( a = 0, b = 2 \) and \( c \) is odd, the moduli space \( M_{(a,b,c)} \) is an irreducible symplectic variety which can be deformed into the Hilbert scheme \( X[5] \) parametrizing 0-dimensional subschemes of length 5 on \( X \) (see [Y] and \( \Phi_{(a,b,c)} : M_{(a,b,c)} \to |2H| \) is a Lagrangian fibration by Matsushita’s theorem (see [Ma 01]). On the other hand \( M_{(0,2,2)} \) is an irreducible variety (see Theorem 4.4 of [KLS 05]) that admits a symplectic resolution \( \tilde{\pi} : \tilde{M}_{(0,2,2)} \to M_{(0,2,2)} \) and is birational to the 10-dimensional O’Grady example \( M \) (see Proposition (4.1.5) of [OG 99] and
Remark 1.1.9 of [Ra 04]. The composition $\Psi := \Phi_{(0,2,2)} \circ \overline{\pi} : \overline{M}_{(0,2,2)} \to |2H|$ gives also in this case a Lagrangian fibration on the irreducible symplectic variety $\overline{M}_{(0,2,2)}$.

By a theorem of Huybrechts (see Theorem 4.6 of [Hu 99]), $\overline{M}_{(0,2,2)}$ and $M$ are deformation equivalent, so they have the same Betti numbers. We are going to determine $b_2(\overline{M}_{(0,2,2)})$ by comparing the Lagrangian fibrations $\Phi := \Phi_{(0,2,1)}$ and $\Psi$ and then using results about the topology of $M_{(0,2,1)}$.

**Theorem 1.0.1.** The dimension of $H^2(M, \mathbb{Q})$ is 24.

This theorem is a consequence of the following three propositions. The first proposition compares open subsets of $M_{(0,2,2)}$ and $M_{(0,2,1)}$.

**Proposition 1.0.2.** Let $U \subset |2H|$ be the open subset parametrizing irreducible curves. The vector spaces $H^2(\Psi^{-1}(U), \mathbb{Q})$ and $H^2(\Phi^{-1}(U), \mathbb{Q})$ are isomorphic.

In the second proposition we determine the second Betti number of the open subvariety $\Phi^{-1}(U) \subset M_{(0,2,1)}$.

**Proposition 1.0.3.** The dimension of $H^2(\Phi^{-1}(U), \mathbb{Q})$ is 22.

In the third proposition we collect the required informations on the complement of $\Psi^{-1}(U)$ in $M$.

**Proposition 1.0.4.** Let $R \subset |2H|$ be the locus parametrizing reducible curves. The divisor $\Psi^{-1}(R) \subset M$ is the union of two irreducible divisors.

In the remaining part of this section, coefficients of singular cohomology groups are always rational and, for simplicity, we often omit them in the notation.

Before proving the three propositions we assume them to prove Theorem 1.0.1.

**Proof of Theorem 1.0.1** Let $Y$ be the singular locus of $\Psi^{-1}(R)$, the couple $(\overline{M}_{(0,2,2)} \setminus Y, \Psi^{-1}(U))$ induces the following exact sequence of cohomology groups with rational coefficients

$$H^2(\overline{M}_{(0,2,2)} \setminus Y, \Psi^{-1}(U)) \to H^2(\overline{M}_{(0,2,2)} \setminus Y) \to H^2(\Psi^{-1}(U)).$$

Let $N$ be the normal bundle of $\Psi^{-1}(R) \setminus Y$ in $\overline{M}_{(0,2,2)} \setminus Y$ and let $N^0$ be the complement of its zero section: by excision $H^2(\overline{M}_{(0,2,2)} \setminus Y, \Psi^{-1}(U)) = H^2(N, N^0)$, by Thom isomorphism $H^2(N, N^0) = H^0(\Psi^{-1}(R) \setminus Y)$ and by Proposition 1.0.3 we get $H^0(\Psi^{-1}(R) \setminus Y) = \mathbb{Q}^2$.

Since $H^2(\Psi^{-1}(U)) = H^2(\Phi^{-1}(U)) = \mathbb{Q}^{22}$ by Proposition 1.0.2 and Proposition 1.0.3 the previous exact sequence implies $b_2(\overline{M}_{(0,2,2)} \setminus Y) \leq 24$. Since $Y$ has codimension two in $\overline{M}_{(0,2,2)}$ we also get $b_2(\overline{M}_{(0,2,2)}) \leq 24$. Finally $b_2(M) = 24$ since $b_2(M) = b_2(\overline{M}_{(0,2,2)})$ and O’Grady already proved $b_2(M) \geq 24$ (see [OG 99]).
Remark 1.0.5. Under our assumption the linear system \(|H|\) induces a double covering \(f : X \to \mathbb{P}^2(\simeq |H|^\vee)\) ramified over a smooth sextic. In what follows we often use that \(f^*\) induces a bijection between plane conics and curves belonging to the linear system \(|2H|\).

Proof of Proposition 1.0.2 Denote by \(U_0 \subset U\) the open subset parametrizing curves whose singular loci are empty or consist of a unique nodal point. Since the complement of \(U_0\) in \(U\) has codimension two and the fibers of \(\Psi\) and \(\Phi\) are equidimensional (see [Ma 00]), the same property holds for the complements of \(\Psi^{-1}(U_0)\) and \(\Phi^{-1}(U_0)\) in \(\Psi^{-1}(U)\) and \(\Phi^{-1}(U)\) respectively. Hence Proposition 1.0.2 is equivalent to the equality \(b_2(\Psi^{-1}(U_0)) = b_2(\Phi^{-1}(U_0))\).

Let \(\Psi : \Psi^{-1}(U_0) \to U_0\) and \(\Phi : \Phi^{-1}(U_0) \to U_0\) be the restrictions of \(\Psi\) and \(\Phi\). The \(E_2^{p,q}\) terms of the associated Leray spectral sequences are \(H^p(R^q\Psi_{0*}(\mathbb{Q}))\) and \(H^p(R^q\Phi_{0*}(\mathbb{Q}))\).

Since the abutment of the Leray spectral sequence is the cohomology of the domain, Proposition 1.0.2 follows if we prove:

a) \(R^q\Psi_{0*}(\mathbb{Q})\) and \(R^q\Phi_{0*}(\mathbb{Q})\) are isomorphic for \(q \leq 2\).

b) For \(p + q = 2\), the \(E_2^{p,q}\) terms of both the spectral sequences survive to the \(E_\infty\) pages.

Let’s prove a). Let \(U_s \subset |2H|\) be the locus parametrizing smooth curves and let \(i : U_s \to U_0\) be the open inclusion. Statement a) is an obvious consequence of

1. For \(q \leq 2\) the sheaves \(i^*R^q\Psi_{0*}(\mathbb{Q})\) and \(i^*R^q\Phi_{0*}(\mathbb{Q})\) are isomorphic.

2. For \(q \leq 2\) the natural attachment maps \(R^q\Psi_{0*}(\mathbb{Q}) \to i_*i^*R^q\Psi_{0*}(\mathbb{Q})\) and \(R^q\Phi_{0*}(\mathbb{Q}) \to i_*i^*R^q\Phi_{0*}(\mathbb{Q})\) are isomorphisms.

In order to prove 1, we denote by \(\Psi_s : \Psi^{-1}(U_s) \to U_s\) and \(\Phi_s : \Phi^{-1}(U_s) \to U_s\) the restrictions of \(\Psi\) and \(\Phi\). Since \(R^q\Psi_{ss}(\mathbb{Q}) \simeq i_*R^q\Psi_{0*}(\mathbb{Q})\) and \(R^q\Phi_{ss}(\mathbb{Q}) \simeq i_*R^q\Phi_{0*}(\mathbb{Q})\) we are reduced to show that there exists an isomorphism between \(R^q\Psi_{ss}(\mathbb{Q})\) and \(R^q\Phi_{ss}(\mathbb{Q})\).

Let \(l \in |2H| \simeq \mathbb{P}^3\) be a general line, by the Zariski theorem the inclusion induces a surjection on fundamental groups \(\pi_1(l \cap U_s) \to \pi_1(U_s)\). Since \(\Psi_s\) and \(\Phi_s\) are smooth \(R^q\Psi_{ss}(\mathbb{Q})\) and \(R^q\Phi_{ss}(\mathbb{Q})\) are local systems, hence they are isomorphic if and only if their restrictions \(R^q\Psi_{ss}(\mathbb{Q})|_{l \cap U_s}\) and \(R^q\Phi_{ss}(\mathbb{Q})|_{l \cap U_s}\) to \(l \cap U_s\) are isomorphic. Hence, denoting by \(\Psi_l : \Psi^{-1}(l \cap U_s) \to l \cap U_s\) and \(\Phi_l : \Phi^{-1}(l \cap U_s) \to l \cap U_s\) the restrictions of \(\Psi\) and \(\Phi\) we want an isomorphism between \(R^q\Psi_{ls}(\mathbb{Q})\) and \(R^q\Phi_{ls}(\mathbb{Q})\). Since \(l \cap U_s\) parametrizes smooth curves the family \(\Psi_l : \Psi^{-1}(l \cap U_s) \to l \cap U_s\) is isomorphic to the degree 6 relative Picard group \(\text{Pic}^6(l \cap U_s) \to l \cap U_s\) of the family of curves parametrized by \(l \cap U_s\) and analogously \(\Phi_l\) can be identified with the degree 5 relative Picard group \(\text{Pic}^5(l \cap U_s) \to l \cap U_s\) of the same family of curves. The wanted isomorphism exists since \(\text{Pic}^6(l \cap U_s) \simeq \text{Pic}^5(l \cap U_s)\) over \(l \cap U_s\). This follows since the family of curves parametrized by \(l \cap U_s\) admits sections: any point in the base locus of \(l\) gives a section.
In order to prove 2 we need a general Lemma.

**Lemma 1.0.6.** Let \( g : X \to \Delta \) be a proper map with irreducible fibers, from a complex smooth surface \( X \) onto the open unit disk \( \Delta \subset \mathbb{C} \). Suppose that \( g \) has a unique critical point \( p \), suppose that \( p \) is non degenerate and \( g(p) = 0 \). Let \( \hat{g} : \overline{\text{Pic}}(X) \to \Delta \) be the compactification, by torsion free sheaves, of the degree \( i \) relative Picard group \( \text{Pic}^i(X) \). Let \( i : \Delta^* := \Delta \setminus 0 \to \Delta \) be the inclusion. For \( q \leq 2 \), the natural attachment map \( \alpha : R^q\hat{g}_*\mathbb{Q} \to i_*i^*R^q\hat{g}_*\mathbb{Q} \) is an isomorphism.

**Proof.** We only have to check that the map \( \Gamma(\alpha) : \Gamma(R^q\hat{g}_*\mathbb{Q}) \to \Gamma(i_*i^*R^q\hat{g}_*\mathbb{Q}) \) induced by \( \alpha \) on global sections is an isomorphism.

The vector space \( \Gamma(R^q\hat{g}_*\mathbb{Q}) \) is isomorphic to \( H^q(\overline{\text{Pic}}(X)) \) and the vector space \( \Gamma(i_*i^*R^q\hat{g}_*\mathbb{Q}) \) is isomorphic to the group \( H^q(\hat{g}^{-1}(\frac{1}{2}))^{\pi_1(\Delta^*)} \) of the \( q \)-cycles of the general fiber of \( \hat{g} \) that are invariant under the monodromy action of \( \pi_1(\Delta^*) \). Finally, using these identifications, the map \( \Gamma(\alpha) \) is just the map induced in cohomology by the inclusion \( \hat{g}^{-1}(\frac{1}{2}) \to \overline{\text{Pic}}(X) \).

The central fiber of \( \hat{g} \) is a normal crossings divisor (see [Se 00]) and \( \hat{g} \) is a semistable degeneration, hence the surjectivity of the map \( \Gamma(\alpha) \) is a consequence of the Clemens local invariant cycle theorem (see [Cl 77]). Since by retraction \( H^q(\overline{\text{Pic}}(X)) = H^q(\hat{g}^{-1}(0)) \), the injectivity of \( \Gamma(\alpha) \) follows if we prove that

\[
\dim(H^q(\hat{g}^{-1}(0))) \leq \dim(H^q(\hat{g}^{-1}(\frac{1}{2}))^{\pi_1(\Delta^*)}).
\]

The second term of this inequality can be computed by the Picard-Lefschetz formula. In fact, denoting by \( \delta \) the vanishing cycle, the Picard-Lefschetz formula says that the generator \( \gamma \) of \( \pi_1(\Delta^*) \) acts on \( H_1(\hat{g}^{-1}(\frac{1}{2})) \) sending \( \beta \) to \( A_\gamma(\beta) = \beta + \delta \cap \beta > \alpha \). Since the action of \( \gamma \) on \( H^q(\hat{g}^{-1}(\frac{1}{2})) \) is given by \( \wedge^q A_\gamma \) and the matrix of \( A_\gamma \) in a suitable basis is given by

\[
M = \begin{pmatrix}
1 & 1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & 0 & \ldots & \ldots & 1
\end{pmatrix},
\]

we have \( \dim(H^1(\hat{g}^{-1}(\frac{1}{2}))^{\pi_1(\Delta^*)}) = 2p - 1 \) where \( p \) is the genus of the general fiber of \( g \). A straightforward computation also yields \( \dim(H^2(\hat{g}^{-1}(\frac{1}{2}))^{\pi_1(\Delta^*)}) = 2p^2 - 3p + 2 \).

The first term of \( \mathbf{11} \) is determined by using the known description of the compactified Picard group of a curve with a node. The variety \( \hat{g}^{-1}(0) \) is obtained starting from a \( \mathbb{P}^1 \)-bundle over the Jacobian \( J \) of the normalization of \( g^{-1}(0) \). This \( \mathbb{P}^1 \)-bundle has two preferred sections and \( \hat{g}^{-1}(0) \) is obtained identifying the two sections by a translation on \( J \). It follows that \( \hat{g}^{-1}(0) \) is homeomorphic to a topologically locally trivial bundle over \( J \) whose fiber \( F \) is obtained from a
spheres by identifying two points. By the Leray spectral sequence we deduce
\[
\dim(H^1(\hat{g}^{-1}(0))) \leq \sum_{i+j=1} \dim(H^i(F) \otimes H^j(j)) = 2p - 1
\]
\[
\dim(H^2(\hat{g}^{-1}(0))) \leq \sum_{i+j=2} \dim(H^i(F) \otimes H^j(j)) = 2p^2 - 3p + 2.
\]

Since locally over small neighborhoods of points in $U^0 \setminus U^*$ the families $\Psi$ and $\Phi$ are homeomorphic to families of the form $\hat{g} \times id : \overline{Pic}(\mathcal{X}) \times \Delta^4 \to \Delta^5$ where $\hat{g} : \overline{Pic}(\mathcal{X}) \to \Delta$ is as in the previous lemma and $id : \Delta^4 \to \Delta^4$ is the identity, statement 2 follows from Lemma 1.0.6.

It remains to prove statement b). Its proof is a slight modification of the proof of the degeneration of the Leray spectral sequence of a smooth projective fibration. Since the same proof works for both $\Psi_0$ and $\Phi_0$, we deal explicitly only with the first case.

Let $d_{l+1}^{p,q} : E_{l+1}^{p,q} \to E_{l}^{p+l,q-l+1}$ be the differentials on the l-th page of the spectral sequence. We need to prove that $d_{3}^{2,0} = 0$ and $d_{2}^{0,q} = 0$ for $p + q \leq 2$. Obviously $d_{2}^{0,0} = 0$.

Now we prove $d_{2}^{2,1} = 0$. For any $\alpha \in H^2(\tilde{M}_{(0,2,2)}, \mathbb{Q})$ the cup product with the restrictions of $\wedge^j \alpha$ to the fibers of $\Psi_0$ induces morphisms $(\wedge^j \alpha)^q : R^q \Psi_0, \mathbb{Q} \to R^{q+2j} \Psi_0, \mathbb{Q}$. If $\alpha$ is the class of an ample divisor the irreducibility of the fibers of $\Psi_0$ implies that the morphism $(\wedge^j \alpha)^q$ is an isomorphism. On the other hand, for $q > 0$, the morphism $(\wedge^j \alpha)^q : R^q \Psi_0, \mathbb{Q} \to R^{q+10} \Psi_0, \mathbb{Q}$ is trivially 0. By Lemma 4.13 of [Vo 03] the differentials $d_{l}^{p,q}$ commute with the maps $H^p((\wedge j \alpha)^q) : H^p(R^q \Psi_0, \mathbb{Q}) \to H^p(R^{q+2j} \Psi_0, \mathbb{Q})$ induced in cohomology by the maps $(\wedge j \alpha)^q$, therefore we get $H^{p+2}((\wedge^j \alpha)^q) \circ d_{2}^{2,1} = d_{2}^{2,1} \circ H^p((\wedge^j \alpha)^q)$. It follows $d_{2}^{2,1} = 0$.

It remains to show that $d_{2}^{2,0} = 0$ and $d_{0}^{0,2} = 0$. By the global invariant cycle theorem (see Theorem 4.24 of [Vo 03]) any section in $H^0(R^2 \Psi_0, \mathbb{Q})$ is the image of a global cohomology class $\alpha \in H^2(\tilde{M}_{(0,2,2)}, \mathbb{Q})$: more precisely it is in the image of $H^0(i^*) \circ H^0((\wedge^1 \alpha)^0)$. By statement 2 the map $H^0(i^*) : H^0(R^2 \Psi_0, \mathbb{Q}) \to H^0(i^* R^2 \Psi_0, \mathbb{Q}) \simeq H^0(R^2 \Psi_0, \mathbb{Q})$ is an isomorphism: hence any section in $H^0(R^2 \Psi_0, \mathbb{Q})$ is in the image of $H^0((\wedge^1 \alpha)^0)$, for a certain $\alpha \in H^2(\tilde{M}_{(0,2,2)}, \mathbb{Q})$. Using again Lemma 4.13 of [Vo 03] we get the equality $H^2((\wedge^1 \alpha)^{-1}) \circ d_{2}^{2,0} = d_{2}^{2,0} \circ H^0((\wedge^1 \alpha)^0)$ and, since $d_{2}^{0,0} = 0$, we obtain $d_{2}^{2,0} = 0$. An analogous argument proves also $d_{0}^{0,2} = 0$.

Proof of Proposition 1.0.5. The Proposition is a consequence of the existence of an open smooth irreducible dense subset $V \subset \Phi^{-1}(R)$ such that $H^1(V, \mathbb{Q}) = 0$. In fact, assuming the existence of such a $V$, the couple $(\Phi^{-1}(U) \cup V, \Phi^{-1}(U))$ induces the exact sequence

\[
H^2(\Phi^{-1}(U) \cup V, \Phi^{-1}(U)) \to H^2(\Phi^{-1}(U) \cup V) \to H^2(\Phi^{-1}(U)) \cdots
\]
\[
\cdots \to H^3(\Phi^{-1}(U) \cup V, \Phi^{-1}(U)).
\]

Since \( V \) is smooth, by excision and Thom isomorphism we get \( H^i(\Phi^{-1}(U) \cup V, \Phi^{-1}(U)) = H^{i-2}(V) \) for \( i \geq 2 \). Hence \( H^3(\Phi^{-1}(U) \cup V, \Phi^{-1}(U)) = 0 \) and \( H^4(\Phi^{-1}(U) \cup V, \Phi^{-1}(U)) = \mathbb{Q} \) by irreducibility of \( V \). Since \( V \subset \Phi^{-1}(R) \) is dense, the complement of \( \Phi^{-1}(U) \cup V \) in \( M_{(0,2,1)} \) has codimension two, therefore \( H^2(\Phi^{-1}(U) \cup V) = H^2(M_{(0,2,9)}) = \mathbb{Q}^{23} \). Finally \( a \) is injective because its image contains the Chern class of the line bundle associated with the effective divisor \( \Phi^{-1}(R) \); hence \( H^2(\Phi^{-1}(U)) = \mathbb{Q}^{22} \).

In order to define \( V \) we denote by \( R^0 \subset R \) the locus parametrizing curves of the form \( C = C_1 \cup C_2 \), where \( C_1 \neq C_2 \), the singular locus of \( C \) consists of at most a nodal point and \( C_1 \cap C_2 \) is included in the smooth locus of both \( C_1 \) and \( C_2 \).

A concrete description of the inclusion \( R^0 \subset R \) is given by means of the map \( f : X \to \mathbb{P}^2 \) (see Remark 1.0.5). This map identifies \( R^0 \) with the set of pairs of distinct lines in \( \mathbb{P}^2 \) such that the intersection of each line with the branch locus \( S \) of \( f \) is either reduced or contains at most a unique double point and, in this case, the support of this double point does not belong to the intersection of the two lines.

The subvariety \( V \subset \Phi^{-1}(R^0) \) is defined as the locus parametrizing sheaves supported on a fixed curve in \( R^0 \).

**Lemma 1.0.7.** Let \( C = C_1 \cup C_2 \) be a curve in \( R^0 \). Denote by \( i : C \to X \) the inclusion of \( C \) and by \( i_1 : C_1 \to X \) and \( i_2 : C_2 \to X \) the inclusion of its components. Let \( F \) be a torsion free sheaf on \( C \) and suppose \( F = i_*G \in \Phi^{-1}(C) \). Then:

1) Up to exchange of \( C_1 \) and \( C_2 \), the sheaf \( F \) fits in an exact sequence of the form

\[
0 \to i_{1*}L_1 \to F \to i_{2*}L_2 \to 0
\]

where \( L_1 \) and \( L_2 \) are rank 1 torsion free sheaves whose degrees are one and two respectively,

2) For any non trivial extension of the form (2) the middle term \( F \) is a stable sheaf,

3) Fixing \( L_1 \) and \( L_2 \), two non trivial extensions of the form (2) have middle terms isomorphic if and only if they differ by a scalar multiplication,

4) Fixing \( L_1 \) and \( L_2 \), for any point \( p \) in \( C_1 \cap C_2 \) there exists a unique, up to \( \mathbb{C}^* \), non trivial extension of the form (2) such that the restriction \( G \) of \( F \) to \( C \) is not locally free at \( p \).
Proof. 1) Let $G_1$ and $G_2$ be the torsion free parts of the restrictions of $G$ to $C_1$ and $C_2$, then $F$ fits in an exact sequence of the form
\[ 0 \to F \to i_{1*}G_1 \oplus i_{2*}G_2 \to Q \to 0 \]
where $Q$ is a quotient of the schematic intersection of $C_1$ and $C_2$. Stability and $\text{ch}_2(F) = 1$ imply that either $\deg(G_1) = \deg(G_2) = 2$ and length($Q$) = 1 or \{deg($G_1$), deg($G_2$)\} = \{2, 3\} and length($Q$) = 2. Supposing $\deg(G_2) = 2$ and setting $G_2 = L_2$ and $L_1 : = \text{Ker}(F \to i_{2*}L_2)$ we get the sequence (2). 2) If $F$ were unstable, there would be a sheaf of the form $i_j*M$ with $\deg(M) = 2$ injecting into $F$. If $j = 1$ this would imply that $M$ is a subsheaf of $L_1$: absurd. If $j = 2$ then $M = L_2$ and the sequence splits. 3) By 2) $\text{End}(F) = \Gamma$. Since $\text{Hom}(F, i_{2*}L_2) = \text{End}(i_{1*}L_1) = \Gamma$ a diagram chase proves 3). 4) By the Grothendieck spectral sequence of $F$ to $\Phi$ and the extensions of the form (2) with $G$ is isomorphic to the structure sheaf of the schematic intersection of $C_1$ and $C_2$ is smooth and the differential of $\Phi$ at any point $p = F$ of $V$ is surjective: in fact since the restriction of $F$ to its support $C$ is a line bundle, such a differential is identified with the natural map $d : \text{Ext}^1(F, F) \to H^0(\text{Ext}^1(F, F)) \simeq H^0(N_C|_X)$ and this map is surjective because its cokernel is always included in $H^2(\text{Hom}^1(F, F))$ which is zero since $F$ is supported on a curve.

In order to prove the irreducibility of $V$ we show that there exists a $\mathbb{P}^1$-bundle over an irreducible base $b : P \to N$ having a surjective map, actually birational, to $\Phi^{-1}(R)$. More precisely, denoting by $g : M_{(0,1,0)} \times M_{(0,1,1)} \to |H|^2$ the product of the maps $\Phi_{(0,1,0)}$ and $\Phi_{(0,1,1)}$, defined at the beginning of Section 1, and denoting by $T \subset |H|^2$ the inverse image of $R^0 \subset R = \text{Sym}^2(|H|)$ in $|H|^2$, the base $N$ is $g^{-1}(T)$ and, keeping notation as in Lemma \ref{Lemma 1.0.4}, the fiber $b^{-1}(i_{1*}L_1, i_{2*}L_2)$ is naturally isomorphic to $\mathbb{P}(\text{Ext}^1(i_{1*}L_1, i_{2*}L_2))$.

Any point $p \in M_{(0,1,0)} \times M_{(0,1,1)}$ has an open neighborhood $U_p$ in the classical topology of the form $U_1 \times U_2$ such that each $X \times U_i$ is endowed with a tautological family $F_i$. Let $q_i : X \times U_1 \times U_2 \to X \times U_i$ and $q : X \times U_1 \times U_2 \to U_1 \times U_2$ be the projections. For any $U_p \subset g^{-1}(T)$ the sheaf $\mathcal{E}xt^1_q(q^*_2F_2, q^*_1F_1)$ is a rank 2 vector bundle. By 3) of Lemma \ref{Lemma 1.0.7} the fibers of the associated projective bundle $b_{U_p} : \mathbb{P}(\mathcal{E}xt^1_q(q^*_2F_2, q^*_1F_1)) \to U_p$ parametrize isomorphism classes of sheaves $F$ fitting in a non trivial extension of the form (2): it follows that the bundles $b_{U_p}$ can be glued to form a global $\mathbb{P}^1$-bundle $b : P \to N$. By 1) and 2) of Lemma \ref{Lemma 1.0.4} the natural modular map $\phi : P \to M_{(0,2,1)}$ surjects onto $\Phi^{-1}(R^0)$: hence the open subset $V \subset \Phi^{-1}(R^0)$ is irreducible.
Let $N^0 \subset N \subset M_{(0,1,0)} \times M_{(0,1,1)}$ be the open subset parametrizing pairs of sheaves whose restrictions to their supports are line bundles, set $P^0 := b^{-1}(N^0)$ and denote by $b^0 : P^0 \rightarrow N^0$ and by $\phi^0 : P^0 \rightarrow M_{(0,2,1)}$ the restrictions of $b$ and $\phi$. By 4) of Lemma 1.0.7 the locus $U \subset P^0$, parametrizing extensions of the form (2), whose middle terms have locally free restrictions to their supports, is the complement of a two section $D$.

The map $\phi^0$ induces a bijection, hence an isomorphism, between the smooth varieties $W$ and $V$. In fact, by 1) of Lemma 1.0.7 if $F = i_* G \in V$, then $F$ is the middle term in an exact sequence of the form (2); hence $\phi^0(W) = V$. Moreover since $G$ is a line bundle of degree 5, its restrictions to the components of $C$ have degree 2 and 3: hence the sheaf $i_2 L_2$ in the sequence (2) is unique quotient of $F$ belonging to $M_{(0,1,1)}$ and $i_1 L_1 \subset M_{(0,1,0)}$ is the associated kernel: it follows that $(\phi^0)^{-1}(F) \subset (b^0)^{-1}(i_1 L_1, i_2 L_2)$. Since, by 3) of Lemma 1.0.7 the map $\phi^0$ is injective on the fibers of $b^0$, the restriction of $\phi^0$ to $W$ is injective too.

We now show that $H^1(W) = H^1(V)$ is zero. Let $D_s$ be the smooth locus of $D$. The couple $(W \cup D_s, W)$ induces the long exact sequence

$$H^3(W \cup D_s) \rightarrow H^1(W) \rightarrow H^2(W \cup D_s, W) \rightarrow H^2(W \cup D_s).$$

The vector space $H^1(W \cup D_s)$ is zero. In fact the complement of $W \cup D_s$ in $P^0$ has codimension two, hence $H^1(W \cup D_s) = H^1(P^0)$ and since $P^0$ is a $\mathbb{P}^1$-bundle over $N^0$ its first cohomology group is trivial if $H^1(N^0) = 0$. The last equality holds since the complement of $N^0$ in the simply connected manifold $M_{(0,1,0)} \times M_{(0,1,1)}$ has codimension two. Indeed it is the union of $g^{-1}((H^2 \setminus T)$ and the locus $Y \subset M_{(0,1,0)} \times M_{(0,1,1)}$ parametrizing pairs of sheaves of the form $(i_1 L_1, i_2 L_2)$ where either $L_1$ or $L_2$ is not a line bundle. The subvariety $g^{-1}((H^2 \setminus T) \subset M_{(0,1,0)} \times M_{(0,1,1)}$ has codimension two since $|H^2 \setminus T$ has codimension two in $|H^2$ and by [Ma 10] the fibers of $g$ are equidimensional. The locus $Y \subset M_{(0,1,0)} \times M_{(0,1,1)}$ has codimension two since the same property holds for the subsets of $M_{(0,1,0)}$ and $M_{(0,1,1)}$ parametrizing sheaves which are not push-forwards of line bundles from curves in $|H|$.

Using the previous exact sequence, it remains to show the injectivity of $a$. The image of $a$ is the Chern class of the line bundle associated with the divisor $D_s$ and it is not zero since $D$ has degree 2 on the fibers of $b^0$. Since by excision and Thom isomorphism the dimension of $H^2(W \cup D_s, W)$ is the number of connected components of $D_s$ we need to prove that $D_s$ is connected or, equivalently, $D$ is irreducible.

Denote by $Z \subset X \times T \subset X \times |H|^2$ the incidence subvariety parametrizing triplets of the form $(p, C_1, C_2)$ where $p \in C_1 \cap C_2$. There exists a regular morphism $m : D \rightarrow Z$ given by sending a non trivial extension of the form (2), where $F = i_* G$ and $G$ is not locally free, to the triplet $(p, C_1, C_2)$ where $p \in C_1 \cap C_2$ is the unique point at which $G$ is not locally free. By 4) of Lemma 1.0.7 the fiber of $m$ over $(p, C_1, C_2)$ is isomorphic to $Pic^1(C_1) \times Pic^2(C_2)$ hence it is irreducible and of constant dimension 4: therefore the irreducibility of $D$ follows from the one of $Z$. Finally $Z$ is irreducible since the projection $p : Z \rightarrow |H|^2$ is a double covering and it is obtained from the double covering...
Proof. By \( H \) the irreducible divisor \( \tilde{B} \)jury map \( \tilde{\rho} \). The morphism \( \tilde{\rho} \) is injective and for general \( C = C_1 \cup C_2 \in R \), the stable locus of \( \Phi(0,2,2)(C) \) is a C*-bundle over \( Pic^3(C_1) \times Pic^3(C_2) \) (this can be proved as in the case \( C \in R(1) \cup R(2) \) in Proposition 2.1.4. of [Ra 04]).

2 A basis for \( H^2(M, \mathbb{Z}) \)

Following [OG 99] we denote by \( \tilde{\Sigma} \subset M \) the exceptional divisor of the desingularization map \( \tilde{\rho} : M \to M_{(2,0,-2)} \) and by \( \tilde{B} \subset M \) the strict transform of the irreducible divisor \( B \subset M_{(2,0,-2)} \) parametrizing not locally free sheaves. Let \( M^U \) be the Uhlenbeck compactification of the \( \mu \)-stable locus of \( M_{(2,0,-2)} \) and let \( \varphi : M_{(2,0,-2)} \to M^U \) be the functorial morphism. Let \( \mu : H^2(X, \mathbb{Z}) \to H^2(M^U, \mathbb{Z}) \) be the Donaldson morphism (see [Li 93], [Mo 93] and [FM 94]) and set \( \mu := \tilde{\rho} \circ \varphi \circ \mu \).

Theorem 2.0.8. The morphism \( \mu \) is injective and
\[
H^2(M, \mathbb{Z}) = \mu(H^2(X, \mathbb{Z})) \oplus \mathbb{Z}c_1(\tilde{\Sigma}) \oplus \mathbb{Z}c_1(\tilde{B}).
\]

Proof. By § 5 of [OG 99] \( \mu \) is injective and \( \mu(H^2(X, \mathbb{Z})) \oplus \mathbb{Z}c_1(\tilde{\Sigma}) \oplus \mathbb{Z}c_1(\tilde{B}) \subset H^2(M, \mathbb{Z}) \). We first prove that the submodule \( \mu(H^2(X, \mathbb{Z})) \) is saturated. In fact by [FM 94], VII.2.17 (see also formula 5.1 of [OG 99])
\[
\int_M \bigwedge_{i=1}^{10} \tilde{\mu}(\alpha) = \frac{10!}{5^2} \left( \int_X \bigwedge_{i=1}^{2} \alpha \right)^5.
\]

Polarizing this formula we get
\[
\int_M \bigwedge_{i=1}^{10} \tilde{\mu}(\alpha_i) = \frac{10!}{5^2} \sum_{\sigma \in S_{10}} \Pi_{i=1}^{5} \int_X \alpha_{\sigma(2i-1)} \wedge \alpha_{\sigma(2i)}.
\]

Let \( \alpha_1 \in H^2(X, \mathbb{Z}) \) be a primitive element, since \( (H^2(X, \mathbb{Z}), \wedge) \) is unimodular of dimension 24, there exist \( \alpha_2, \ldots, \alpha_{10} \in H^2(X, \mathbb{Z}) \) such that
\[
\int_X \alpha_j \wedge \alpha_k = 1 \quad \text{if} \quad \{j, k\} = \{2i - 1, 2i\}
\]
\[
\int_X \alpha_j \wedge \alpha_k = 0 \quad \text{if} \quad \{j, k\} \neq \{2i - 1, 2i\}.
\]

By Formula \( \mathbf{1} \) we have \( \int_M \bigwedge_{i=1}^{10} \tilde{\mu}(\alpha_i) = 1 \). Hence \( \tilde{\mu} \) sends primitive elements of \( H^2(X, \mathbb{Z}) \) to primitive elements of \( H^2(M, \mathbb{Z}) \), therefore \( \mu(H^2(X, \mathbb{Z})) \) is saturated.
We now prove that \( \bar{\mu}(H^2(X, \mathbb{Z})) \oplus \mathbb{Z}c_1(\Sigma) \oplus \mathbb{Z}c_1(B) \) is saturated by evaluating a basis on suitable homology classes. By Proposition 3.0.5 of [OG 99], there exists an open dense subset \( B^0 \subset B \) which is a \( \mathbb{P}^1 \)-bundle over the smooth locus of the symmetric product \( X^{(4)} \) and by the proof of Lemma 3.0.13 of [OG 99] the intersection \( \Sigma \cap B^0 \) is a three-section of this \( \mathbb{P}^1 \)-bundle. Since by [LS 05] the smooth variety \( M \) is the blow up of \( M_{(2,0,-2)} \) along \( \Sigma \), the strict transform \( \tilde{B} \) has an open subset which is a \( \mathbb{P}^1 \)-bundle and, denoting by \( \delta \) a fiber, we have \( \gamma \cdot \Sigma = 3 \). On the other hand \( \tilde{\Sigma} \) has an open dense subset \( \tilde{\Sigma}^0 \) which is \( \mathbb{P}^1 \)-bundle on the smooth locus of \( \Sigma \) and, denoting by \( \delta \) a fiber, we get \( \delta \cdot \tilde{B} = 1 \). Finally, since \( M \) has trivial canonical bundle, we also get \( \gamma \cdot \tilde{\Sigma} = -2 = \delta \cdot \tilde{B} \).

Let \( (\alpha_1, \ldots, \alpha_{22}) \) be a basis of \( H^2(X, \mathbb{Z}) \), since \( \bar{\mu}(H^2(X, \mathbb{Z})) \) is saturated there exist \( \beta_1, \ldots, \beta_{22} \in H^2(M, \mathbb{Z}) \) such that \( \det(\alpha_i(\beta_j))_{i,j \leq 22} = 1 \). Moreover \( \alpha_i(\delta) = \alpha_i(\gamma) = 0 \) for any \( i \), since \( \delta \) and \( \gamma \) are contracted by \( \phi \circ \tilde{\pi} \) (see Proposition (3.0.5) of [OG 99]). Therefore, denoting by \( M \) the evaluation matrix of \( (\alpha_1, \ldots, \alpha_{22}, c_1(\Sigma), c_1(B)) \) on \( (\beta_1, \ldots, \beta_{22}, [\delta], [\gamma]) \), we have

\[
\det(M) = \det \begin{pmatrix} c_1(\Sigma)([\delta]) & c_1(B)([\delta]) \\ c_1(\Sigma)([\gamma]) & c_1(B)([\gamma]) \end{pmatrix} = \det \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix} = 1
\]

which implies the statement.

\[\square\]

### 3 The Beauville form of M

In this section we determine the Beauville form and the Fujiki constant of \( M \). Before analyzing the case of \( M \), we recall the theorem due to Beauville and Fujiki (see [Be 83, Fu 87]) that defines the Beauville form and the Fujiki constant of an irreducible symplectic variety.

**Theorem 3.0.9.** Let \( Y \) be irreducible symplectic variety of dimension \( 2n \). There exist a unique indivisible bilinear integral symmetric form \( B_Y \in S^2(H^2(Y, \mathbb{Z}))^* \), called the Beauville form, and a unique positive constant \( c_Y \in \mathbb{Q} \), called the Fujiki constant, such that for any \( \alpha \in H^2(Y, \mathbb{C}) \)

\[
\int_Y \alpha^{2n} = c_Y B_Y(\alpha, \alpha)^n
\]

and for \( 0 \neq \omega \in H^0(\Omega_Y^2) \)

\[
B_Y(\omega + \overline{\omega}, \omega + \overline{\omega}) > 0.
\]

**Remark 3.0.10.** Formula (5) is Fujiki’s formula. We will use also its polarized form:

\[
\int_Y \alpha_1 \wedge \ldots \wedge \alpha_{2n} = \frac{c_Y}{2^n!} \sum_{\sigma \in S_{2n}} B(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}) \cdots B(\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)}).
\]

In the case of \( M \) we have the following theorem.
Theorem 3.0.11. Set $\Lambda := \mathbb{Z}(c_1(\Sigma)) \oplus \mathbb{Z}c_1(B) \subset H^2(M, \mathbb{Z})$. The direct sum decomposition

$$H^2(M, \mathbb{Z}) = \bar{\mu}(H^2(X, \mathbb{Z})) \oplus \perp \Lambda$$

is orthogonal with respect to $B_M$. The map $\bar{\mu} : (H^2(X, \mathbb{Z}), \wedge) \rightarrow (H^2(M, \mathbb{Z}), B_M)$ is an isometric embedding. The matrix of the Beauville form on $\Lambda$ is

\[
\begin{array}{|c|c|}
\hline
& c_1(\Sigma) & c_1(B) \\
\hline
2 & -6 & 3 \\
\hline
\end{array}
\]

In order to determine the Beauville form $B_M$ of $M$ we use the following fundamental property of the Donaldson morphism (see [Li 93], [Mo 93] and [FM 94]).

Let $Y$ be a smooth variety and let $\mathcal{F}$ be a coherent sheaf on $X \times Y$. Suppose that $\mathcal{F}$ is a flat family parametrizing semistable sheaves on $X$ with Mukai vector $(2, 0, -2)$. Let $\iota_\mathcal{F} : Y \rightarrow M_{(2,0,-2)}$ be the associated modular map and let $p$ and $q$ be the projections of $X \times Y$ on $X$ and $Y$ respectively. For any $\alpha \in H^2(X, \mathbb{Z})$,

$$\iota_\mathcal{F}^* \circ \varphi^* \circ \mu(\alpha) = q_*(c_2(\mathcal{F}) \wedge p^*(\alpha)).$$

Lemma 3.0.12. Let $\omega$ be a symplectic holomorphic two form on $X$.

1. $\int_M \bar{\mu}(\omega + \overline{\omega})^8 \wedge c_1(B) \wedge c_1(\Sigma) = 35 \cdot 9(\int_X (\omega + \overline{\omega})^2)^4$
2. $\int_M \bar{\mu}(\omega + \overline{\omega})^8 \wedge c_1(\Sigma)^2 = -35 \cdot 9 \cdot 2(\int_X (\omega + \overline{\omega})^2)^4$
3. $\int_M \bar{\mu}(\omega + \overline{\omega})^8 \wedge c_1(B)^2 = -35 \cdot 3 \cdot 2(\int_X (\omega + \overline{\omega})^2)^4$

Proof. Let $\mathcal{I}$ be the universal sheaf of ideals on $X \times X^{[2]}$ and let $\pi_i : X \times X^{[2]} \times X^{[2]} \rightarrow X \times X^{[2]}$ be the projection on the product of the first and the $i$-th factors. Denote by $\iota$ the modular map associated with the family $\pi_1^*(\mathcal{I}) \oplus \pi_2^*(\mathcal{I})$. By Formula (3)

$$\iota^* \circ \varphi^* \circ \mu(\alpha) = p_1^*(\alpha^{[2]}) + p_2^*(\alpha^{[2]})$$

where $p_j : X^{[2]} \times X^{[2]} \rightarrow X^{[2]}$ is the projection on the $j$-th factor and $\alpha^{[2]} \in H^2(X^{[2]}, \mathbb{C})$ is the class associated with $\alpha$, namely the pull-back from $X^{(2)}$ of the class whose pull-back to $X^{[2]}$ restricts to $\alpha$ on each component. It follows that

$$\int_{X^{[2]} \times X^{[2]}} (\iota^* \circ \varphi^* \circ \mu(\omega + \overline{\omega}))^8 = \int_{X^{[2]} \times X^{[2]}} ((p_1^* + p_2^*)(\omega^{[2]} + \overline{\omega}^{[2]}))^8 = 70 \int_{X^{[2]} \times X^{[2]}} (p_1^*(\omega^{[2]} + \overline{\omega}^{[2]}))^4 \wedge (p_2^*(\omega^{[2]} + \overline{\omega}^{[2]}))^4 = 70(\int_{X^{[2]}} (\omega^{[2]} + \overline{\omega}^{[2]})^4)^2 = 70 \cdot 9(\int_X (\omega + \overline{\omega})^2)^4.$$
Since $i$ is a double covering over $\Sigma$ and $\bar{B} \cap \bar{\Sigma}^0$ is a rational section of the restriction of $\varphi \circ \bar{\pi}$ to $\bar{\Sigma}^0$ we have $\int_B \mu(\omega + \bar{\omega})^8 \wedge c_1(B) \wedge c_1(\Sigma) = 35 \cdot 9(\int_X (\omega + \bar{\omega}))^4$. Since the normal bundle to $\Sigma$ has degree $-2$ on the fibers of $\bar{\Sigma}^0$, we also get $\int_B \mu(\omega + \bar{\omega})^8 \wedge c_1(\Sigma)^2 = -35 \cdot 9 \cdot 2(\int_X (\omega + \bar{\omega}))^4$. Analogously, since $B^0 \cap \Sigma$ is a rational three-section of the restriction of $\varphi \circ \bar{\pi}$ to $B^0$ and the normal bundle of $B$ has degree $-2$ on the general fiber, we get $\int_B \mu(\omega + \bar{\omega})^8 \wedge c_1(B)^2 = -35 \cdot 3 \cdot 2(\int_X (\omega + \bar{\omega}))^4$.

**Proof of Theorem 3.0.14** By equality (3) and Fujiki’s formula there exists $B \in \mathbb{Q}$ such that $B_M(\mu(\alpha), \mu(\alpha)) = a \int_X \alpha^2$ for any $\alpha \in H^2(X, \mathbb{Z})$ and $c_M = \frac{10!}{9^{2}}$. Since $\bar{B}$ and $\bar{\Sigma}$ are contracted by $\varphi \circ \bar{\pi}$, we get $\int_M c_1(\bar{B}) \wedge \mu(\alpha)^9 = \int_M c_1(\bar{\Sigma}) \wedge \mu(\alpha)^9 = 0$, hence $B_M(\mu(\alpha), c_1(\bar{B})) = B_M(\mu(\alpha), c_1(\bar{\Sigma})) = 0$. Using the polarized form of Fujiki’s Formula and item 1 of Lemma 3.0.12 we get

$$\int_M \mu(\omega + \bar{\omega})^8 \wedge c_1(\bar{B}) \wedge c_1(\bar{\Sigma}) = \frac{8! \cdot 2 \cdot c_M}{10!} B_M(c_1(\bar{B}), c_1(\bar{\Sigma})) B_M(\mu(\omega + \bar{\omega}), \mu(\omega + \bar{\omega})))^4 = 35 \cdot 9(\int_X (\omega + \bar{\omega}))^4.$$

Hence, comparing the second and the third term of this equation and replacing $c_M$ and $B_M(\mu(\omega + \bar{\omega}), \mu(\omega + \bar{\omega})))^4$ by their values,

$$B_M(c_1(\bar{B}), c_1(\bar{\Sigma})) = 3a.$$

Analogously we get

$$B_M(c_1(\bar{\Sigma}), c_1(\bar{\Sigma})) = -6a,$$

$$B_M(c_1(\bar{B}), c_1(\bar{B})) = -2a.$$

Since $B_M$ is integral and primitive and $B_M(\mu(\omega + \bar{\omega}), \mu(\omega + \bar{\omega}))) > 0$ we conclude $a = 1$ and $c_M = \frac{10!}{9^{2}}$.

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