A SIMPLE PROOF ON THE NON-EXISTENCE OF SHRINKING BREATHERS FOR THE RICCI FLOW

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Abstract. Suppose $M$ is a compact $n$-dimensional manifold, $n \geq 2$, with a metric $g_{ij}(x, t)$ that evolves by the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ in $M \times (0, T)$. We will give a simple proof of a recent result of Perelman on the non-existence of shrinking breather without using the logarithmic Sobolev inequality.

It is known that Ricci flow is a very powerful tool in understanding the geometry and structure of manifolds. In 1982 R. Hamilton [H1] first began the study of Ricci flow on a manifold. Suppose $M$ is a compact 3-dimensional manifold with a metric $g_{ij}(x)$ having a strictly positive Ricci curvature. R. Hamilton proved that if the metric $g_{ij}$ evolve by the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \quad (0.1)$$

with $g_{ij}(x, 0) = g_{ij}(x)$, then the evolving metric will converge modulo scaling to a metric of constant positive curvature. A similar result for compact 4-dimensional manifold with positive curvature operator was proved by R. Hamilton in the paper [H2]. By using a modification of the proof of Li-Yau Harnack inequality [LY] for the heat equations on manifolds R. Hamilton [H4] proved the Harnack inequality for the Ricci flow. Singularities of solutions of the Ricci flow was studied by R. Hamilton [H5] and G. Perelman [P1], [P2].

Ricci flow on non-compact manifolds was studied by W.X. Shi [S1], [S2], R. Hamilton [H3], and L.F. Wu [W1], [W2]. Existence and asymptotic behaviour of solutions of the Ricci flow equation on non-compact $\mathbb{R}^2$ was studied by S.Y. Hsu in the papers [Hs1–4].

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We refer the reader to the paper [H5] by R. Hamilton and the book [CK] B. Chow and D. Knopf for various recent results on the Ricci flow. One can also read the recent lecture notes by B. Chow [C] on Ricci flow.

A metric $g_{ij}(t)$ evolving by the Ricci flow in $M \times (0, T)$ is called a steady (shrinking, expanding respectively) breather if there exist $0 < t_1 < t_2 < T$ and $\alpha = 1$ ($0 < \alpha < 1$, $\alpha > 1$ respectively) and a diffeomorphism $\phi: M \to M$ such that $g_{ij}(t_2) = \phi^*(\alpha g_{ij}(t_1))$. As observed by G. Perelman [P1] if one considers Ricci flow as a dynamical system on the space of Riemannian metrics modulo diffeomorphism and scaling, then breathers correspond to periodic orbits for the Ricci flow. So it is interesting to know whether breather exists in a Ricci flow.

In the paper [P1] G. Perelman found two functionals for the Ricci flow which are monotone increasing with respect to time. G. Perelman then used these and logarithmic Sobolev inequality to proved that there is no expanding or shrinking breathers for the Ricci flow. However his proof of non-existence of shrinking breathers has some gaps and requires the existence of solution of some auxillary parabolic equation on a manifold with initial value a delta mass which is highly non-trivial. In this paper we will modify Perelman’s argument and give a simple proof of the non-existence of shrinking breathers without using the logarithmic Sobolev inequality.

The plan of the paper is as follows. In section 1 we will prove some technical lemmas. In section 2 we will fix the gaps in the proof of the monotonicity property of the $W$ functional in Perelman’s paper [P1]. We will also prove the non-existence of shrinking breathers.

We will assume that $M$ is a compact $n$-dimensional manifold, $n \geq 2$, with a metric $g(t) = (g_{ij}(\cdot, t))$ that evolves by the Ricci flow (0.1) in $M \times (0, T)$ for the rest of the paper.

Section 1

In this section we will establish some technical lemmas. We first recall a standard result (cf. Theorem 1.6.2 of [J]).

**Lemma 1.1.** Let $0 < t_1 < T$ and $f \in C^\infty(M \times (0, t_1))$. For any $t \in (0, t_1)$ there exist a smooth function $\psi^t_s(p) = \psi^t(p, s): M \times (0, t_1) \to M$ satisfying

$$\begin{cases} 
\frac{\partial}{\partial s} \psi^t_s(p, s) = -\nabla f(\psi^t_s(p, s), t) & \forall s \in (0, t_1), p \in M \\
\psi^t_s(p, 0) = p & \forall p \in M.
\end{cases} \quad (1.1)
$$

By an argument similar to the proof of Theorem 1.6.2 of [J] we have the following lemma.

**Lemma 1.2.** Let $0 < t_1 < T$ and $f \in C^\infty(M \times (0, t_1))$. For any $t_0 \in (0, t_1)$ there exist a smooth function $\phi_{t_0}(p, t) = \phi_{t_0, t}(p)$ such that $\phi_{t_0}: M \times (0, t_1) \to M$ and satisfies

$$\begin{cases} 
\frac{\partial}{\partial t} \phi_{t_0}(p, t) = -\nabla f(\phi_{t_0}(p, t), t) & \forall t \in (0, t_1), p \in M \\
\phi_{t_0}(p, t_0) = p & \forall p \in M.
\end{cases} \quad (1.2)
$$

If $t'_0 \in (0, t_1)$, then the map $\phi_{t_0, t'_0}: M \to M$ is a diffeomorphism with inverse $\phi_{t'_0, t_0}$. 

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Lemma 1.3. Let $0 < t_1 < T$, $t, t_0 \in (0, t_1)$, $\delta_0 = \min(t, t_1 - t)$, and let $f$, $\psi^i$, $\phi_{t_0, t}$ be as in Lemma 1.1 and Lemma 1.2. Let $p \in M$ and $x = (x_1, \ldots, x_n) : U \subset M \to \mathbb{R}^n$ be a local co-ordinate chart around $p_0 = \phi_{t_0, t}(p)$ for some open neighbourhood $U$ of $p_0$ such that $x(U) = B_{R_0}$ for some $R_0 > 0$ and $x(\phi_{t, t+h}(p_0))$, $x(\psi_h^i(p_0)) \in B_{R_0}$ for any $|h| \leq \delta_1$ for some constant $0 < \delta_1 \leq \delta_0$. Let $e(h) = x(\phi_{t, t+h}(p_0)) - x(\psi_h^i(p_0))$ for any $|h| \leq \delta_1$. Then there exists a constant $C > 0$ such that

$$|e(h)| + \max_{1 \leq k \leq n} \left| \frac{d e_k}{d h} \right| + \max_{1 \leq j, k \leq n} \left\{ \left( \frac{\partial}{\partial x_j} \right)_{p_0} e_k \right\} \leq C|h| \quad \forall |h| \leq \delta_1$$

(1.3)

where $e(h) = (e_k(h))_{k=1}^n$ in this local co-ordinate system and $|e(h)| = (\sum_{k=1}^n e_k(h))^2)^{1/2}$.

Proof. Without loss of generality we will abuse the notation and write $\phi_{t, t+h}(p_0)$, $\psi_h^i(p_0)$, instead of $x(\phi_{t, t+h}(p_0))$, $x(\psi_h^i(p_0))$, etc. and we will write $\partial/\partial x_j$ for $(\partial/\partial x_j)_{p_0}$. Let $\phi_{t, t+h}(p_0) = (\phi_{t, t+h}(p_0))_{k=1}^n$ and $\psi_h^i(p_0) = ((\psi_h^i)^k(p_0))_{k=1}^n$ in the local co-ordinate system $(x, U)$ and let

$$q(s) = s \phi_{t, t+h}(p_0) + (1 - s) \psi_h^i(p_0) \quad \forall 0 \leq s \leq 1.$$  

(1.4)

By (1.1) and (1.2),

$$\left| \frac{d e_k}{d h} \right| = \left| \frac{d}{d h} \phi_{t, t+h}^k(p_0) - \frac{d}{d h} (\psi_h^i)^k(p_0) \right|$$

$$= \left| -g^{kj}(\phi_{t, t+h}(p_0), t + h) \frac{\partial}{\partial x_j} f(\phi_{t, t+h}(p_0), t + h) 
+ g^{kj}(\psi_h^i(p_0), t) \frac{\partial}{\partial x_j} f(\psi_h^i(p_0), t) \right|$$

$$\leq |g^{kj}(\phi_{t, t+h}(p_0), t) \frac{\partial}{\partial x_j} f(\phi_{t, t+h}(p_0), t) - g^{kj}(\phi_{t, t+h}(p_0), t + h) \frac{\partial}{\partial x_j} f(\phi_{t, t+h}(p_0), t + h)|$$

$$+ \left| g^{kj}(\psi_h^i(p_0), t) \frac{\partial}{\partial x_j} f(\psi_h^i(p_0), t) - g^{kj}(\phi_{t, t+h}(p_0), t) \frac{\partial}{\partial x_j} f(\phi_{t, t+h}(p_0), t) \right|$$

$$\leq C|h| + \left| \int_0^1 \frac{d}{ds} \left( g^{kj}(q(s), t) \frac{\partial}{\partial x_j} f(q(s), t) \right) ds \right|$$

$$\leq C|h| + \left| \int_0^1 \frac{d}{ds} \left( g^{kj}(q(s), t) \frac{\partial}{\partial x_j} f(q(s), t) + g^{kj}(q(s), t) \frac{\partial^2}{\partial x_i \partial x_j} f(q(s), t) \right) ds \right|$$

$$\leq C(|h| + |e|) \quad \forall |h| \leq \delta_1, k = 1, 2, \ldots, n.$$  

(1.5)
Hence
\[
\left| \frac{d|e|^2}{dh} \right| \leq C(|e|^2 + h^2) \quad \forall |h| \leq \delta_1
\]
\[
\Rightarrow \left| \frac{d}{dh}(e^{-Ch}|e(h)|^2) \right| \leq C h^2 e^{-Ch} \quad \forall |h| \leq \delta_1
\]
\[
\Rightarrow |e(h)|^2 \leq C'h^2 \quad \forall |h| \leq \delta_1. \quad (1.6)
\]

Similarly
\[
\left| \frac{d}{dh} \left( \frac{\partial e_k}{\partial x_j} \right) \right| \leq C(|h| + |e|) \leq C|h| \quad \Rightarrow \left| \frac{\partial e_k}{\partial x_j} \right| \leq C|h| \quad \forall |h| \leq \delta_1, j, k = 1, 2, \ldots, n. \quad (1.7)
\]

By (1.5), (1.6), and (1.7) we get (1.3) and the lemma follows.

**Lemma 1.4.** Let $0 < t_1 < T$, $t_0 \in (0, t_1)$, and let $f, \phi_{t_0,t}$, be as in Lemma 1.2. Let
\[
\bar{g}(t) = \phi_{t_0,t}^*(g(t)) \quad \forall 0 < t < t_1.
\]
Then
\[
\frac{\partial}{\partial t} \bar{g}(t) = \phi_{t_0,t}^* \left( \frac{\partial}{\partial t} g(t) + L_V(t)(g(t)) \right) \quad \forall 0 < t < t_1 \quad (1.8)
\]
where $V(t) = -\nabla f(\cdot, t)$.

**Proof.** Let $p \in M$, $t \in (0, t_1)$, and let $\psi^t_s$ be as in Lemma 1.1. Let $(x, U)$, $\delta_1 > 0$, and $e(h) = (e_k(h))_{k=1}^n$ be as in Lemma 1.3. Let $X$ and $Y$ be two vector fields on $M$. Then there exist a constant $\delta_2 \in (0, \delta_1)$ and an open neighbourhood $V \subset U$ of the curve $t' \to \phi_{t_0,t'}(p)$, $t - \delta_2 \leq t' \leq t + \delta_2$, such that $x(V)$ is convex in $\mathbb{R}^n$ and the vector fields $d\phi_{t_0,t'}(X(p))$ and $d\phi_{t_0,t'}(Y(p))$ along the curve $t' \to \phi_{t_0,t'}(p)$, $t - \delta_2 \leq t' \leq t + \delta_2$, can be extended to two local vector fields $\bar{X}$ and $\bar{Y}$ on $V$. That is
\[
\begin{cases}
\bar{X}(\phi_{t_0,t'}(p)) = d\phi_{t_0,t'}(X(p)) & \forall t' \in (t - \delta_2, t + \delta_2) \\
\bar{Y}(\phi_{t_0,t'}(p)) = d\phi_{t_0,t'}(Y(p)) & \forall t' \in (t - \delta_2, t + \delta_2).
\end{cases}
\]

Let $p_0 = \phi_{t_0,t}(p)$ and
\[
E(h) = g(\phi_{t_0,t+h}(p), t)(d\phi_{t_0,t+h}(X(p)), d\phi_{t_0,t+h}(Y(p)))
\]
\[
- g(\psi^h_{t,s}(p_0), t)(d\psi^h_{t,s}(\bar{X}(p_0)), d\psi^h_{t,s}(\bar{Y}(p_0))).
\]

Let $\phi_{t,t+h} = (\phi_{t,t+h}^k)_{k=1}^n$ and $\psi^t_h = ((\psi^t_h)^k)_{k=1}^n$ in the local co-ordinate system $(x, U)$. We write
\[
\begin{align*}
\bar{X}(q) &= a^i(q) \frac{\partial}{\partial x_i} \\
\bar{Y}(q) &= b^i(q) \frac{\partial}{\partial x_i}
\end{align*}
\]
and let $q(s) = (q(s)^k)_{k=1}^n$ be given by (1.4). Since $\phi_{t_0,t+h} = \phi_{t,t+h} \circ \phi_{t_0,t}$ on $M$,

$$E(h) = g(\phi_{t,t+h}(p_0), t)(d\phi_{t,t+h} (\tilde{X}(p_0)), d\phi_{t,t+h}(\tilde{Y}(p_0)))$$

$$- g(\psi_h^0(p_0), t)(d\psi_h^0(\tilde{X}(p_0)), d\psi_h^0 (\tilde{Y}(p_0)))$$

$$= g_{ij}(\phi_{t,t+h}(p_0), t) \frac{\partial \phi_{t,t+h}^i}{\partial x_k}(p_0) \frac{\partial \phi_{t,t+h}^j}{\partial x_{k'}}(p_0)a^k(p_0)b^l(p_0)$$

$$- g_{ij}(\psi_h^0(p_0), t) \frac{\partial (\psi_h^0)^i}{\partial x_k}(p_0) \frac{\partial (\psi_h^0)^j}{\partial x_{k'}}(p_0)a^k(p_0)b^l(p_0)$$

$$= \int_0^1 \frac{d}{ds} g_{ij}(q(s), t) \frac{\partial q(s)^i}{\partial x_k} \frac{\partial q(s)^j}{\partial x_{k'}}a^k(p_0)b^l(p_0) ds$$

$$= e_l(h) \int_0^1 \frac{\partial g_{ij}}{\partial x_l}(q(s), t) \frac{\partial q(s)^i}{\partial x_k} \frac{\partial q(s)^j}{\partial x_{k'}}a^k(p_0)b^l(p_0) ds$$

$$+ \int_0^1 g_{ij}(q(s), t) \frac{\partial e_l}{\partial x_k} \frac{\partial q(s)^j}{\partial x_{k'}}a^k(p_0)b^l(p_0) ds$$

$$+ \int_0^1 g_{ij}(q(s), t) \frac{\partial q(s)^j}{\partial x_k} \frac{\partial e_j}{\partial x_{k'}}a^k(p_0)b^l(p_0) ds$$

$$= E_1(h) + E_2(h) + E_3(h) \quad \forall |h| \leq \delta_2. \quad (1.10)$$

Let

$$G_l = \frac{\partial g_{ij}}{\partial x_l}(q(s), t) \frac{\partial q(s)^i}{\partial x_k} \frac{\partial q(s)^j}{\partial x_{k'}}a^k(p_0)b^l(p_0) \quad \forall l = 1, 2, \ldots, n$$

$$H_i = g_{ij}(q(s), t) \frac{\partial q(s)^j}{\partial x_k}b^l(p_0) \quad \forall i = 1, 2, \ldots, n.$$ 

Then

$$\frac{d}{dh} E_1(h) = e_l(h) \int_0^1 \left( s \frac{d}{dh} \phi_{t,t+h}^m(p_0) + (1-s) \frac{d}{dh} (\psi_h^m(p_0)) \right) \frac{\partial G_l}{\partial x_m} ds$$

$$+ \left( \frac{d}{dh} e_l(h) \right) \int_0^1 G_l ds$$

$$= - e_l(h) \int_0^1 \left( s \nabla_m f(\phi_{t,t+h}(p_0), t+h) + (1-s) \nabla_m f((\psi_h^t(p_0), t) \right) \frac{\partial G_l}{\partial x_m} ds$$

$$+ \left( \frac{d}{dh} e_l(h) \right) \int_0^1 G_l ds$$
and

\[
\frac{d}{dh} E_2(h) = \int_0^1 \left( s \frac{d}{dh} \phi_{t+h}^m(p_0) + (1-s) \frac{d}{dh} (\psi_h^t)^m(p_0) \right) \frac{\partial H_i}{\partial x_m} \frac{\partial e_i}{\partial x_k} a_k^k(p_0) \, ds \\
+ \int_0^1 H_i \frac{d}{dh} \left( \frac{\partial e_i}{\partial x_k} \right) a_k^k(p_0) \, ds
\]

\[
= - \int_0^1 \left( s \nabla_m f(\phi_{t+h}(p_0), t+h) + (1-s) \nabla_m f((\psi_h^t)(p_0), t) \right) \frac{\partial H_i}{\partial x_m} \frac{\partial e_i}{\partial x_k} a_k^k(p_0) \, ds \\
+ \int_0^1 H_i \frac{d}{dh} \left( \frac{\partial e_i}{\partial x_k} \right) a_k^k(p_0) \, ds.
\]

Hence by Lemma 1.3,

\[
\left| \frac{d}{dh} E_1(h) \right| + \left| \frac{d}{dh} E_2(h) \right| \\
\leq C \left\{ |e(h)| + \max_{1 \leq k \leq n} \left| \frac{de_k}{dh} \right| + \max_{1 \leq j, k \leq n} \left[ \left| \frac{\partial e_k}{\partial x_j} \right| + \left| \frac{d}{dh} \left( \frac{\partial e_k}{\partial x_j} \right) \right| \right] \right\} \leq C|h| \quad \forall |h| \leq \delta_2
\]

\[
\Rightarrow \left. \frac{d}{dh} E_1(h) \right|_{h=0} = \left. \frac{d}{dh} E_2(h) \right|_{h=0} = 0.
\tag{1.11}
\]

Similarly,

\[
\left. \frac{d}{dh} E_3(h) \right|_{h=0} = 0.
\tag{1.12}
\]

By (1.10), (1.11), and (1.12),

\[
\left. \frac{d}{dh} E(h) \right|_{h=0} = 0.
\tag{1.13}
\]
Then

\[
\frac{d}{dt} f(p, t)(X, Y) = \frac{d}{dh} \Delta_{t \to h}(g(t + h))(p)(X, Y) \\
\Delta_{t \to h} = \left. \phi_{t \to h}^* (d \phi_{t \to h}(X(p)), d \phi_{t \to h}(Y(p))) \right|_{h=0} \\
\Delta_{t \to h} = \left. g(\phi_{t \to h}(p), t + h)(d \phi_{t \to h}(X(p)), d \phi_{t \to h}(Y(p))) \right|_{h=0} \\
\Delta_{t \to h} = \left. g(\phi_{t \to h}(p), t + h)(d \phi_{t \to h}(X(p)), d \phi_{t \to h}(Y(p))) \right|_{h=0} \\
\Delta_{t \to h} = \left. \frac{\partial}{\partial h} \Delta_{t \to h}(g(t))(\phi_{t \to h}(p))(X, Y) \right|_{h=0} \\
\Delta_{t \to h} = L_{V(t)}(g(t))(\phi_{t \to h}(p))(X, Y) + \phi_{t \to h}^* \left( \frac{\partial}{\partial t} g(t) \right) (p)(X, Y) \\
\Delta_{t \to h} = L_{V(t)}(g(t))(\phi_{t \to h}(p))(d \phi_{t \to h}(X), d \phi_{t \to h}(Y)) + \phi_{t \to h}^* \left( \frac{\partial}{\partial t} g(t) \right) (p)(X, Y) \\
\Delta_{t \to h} = \phi_{t \to h}^* \left( L_{V(t)}(g(t))(p)(X, Y) + \phi_{t \to h}^* \left( \frac{\partial}{\partial t} g(t) \right) (p)(X, Y) \right) \\
\Delta_{t \to h} = \phi_{t \to h}^* \left( \frac{\partial}{\partial t} g(t) + L_{V(t)}(g(t)) \right) (p)(X, Y) \\
\]
where $R(g)$ is the scalar curvature of $g$ and

\[ A(g, \tau) = \left\{ f \in C^\infty(M) : (4\pi\tau)^{-n/2} \int_M e^{-f} \, dV_g = 1 \right\}. \]  

(2.4)

We will first prove that $\mu(g, \tau)$ is well-defined.

**Lemma 2.1.** Let $g$ be a Riemannian metric on $M$ and $\tau_0 > 0$. Then there exist constants $0 < \delta < 1$, $C_1 > 0$, and $C_{\tau_0} > 0$ such that

\[ W(g, f, \tau) \geq (1 - \delta) \lambda_1 - \frac{\delta}{(4\pi\tau)^{n/2}} \| R(g) \|_{L^\infty(M)} - 4\delta \tau - C_{\tau_0} - \frac{C_1}{\tau^{n/2}} \log(4\pi\tau) \]  

(2.5)

holds for any $\tau \geq \tau_0$, $f \in A(g, \tau)$, where $\lambda_1$ is the first eigenvalue of the operator $R(g) - 4\Delta_g$. Hence $\mu(g, \tau) > -\infty$ is well-defined for any $\tau > 0$.

**Proof.** Without loss of generality we may assume that $n \geq 3$. Let $f \in A(g, \tau)$, $\Phi = e^{-f/2}$, and

\[ W(g, f, \tau) = (4\pi\tau)^{-n/2} \int_M \left\{ \tau (R(g)\Phi^2 + 4|\nabla \Phi|^2) - \Phi^2 \log \Phi^2 \right\} \, dV_g - n. \]  

Then

\[ W(g, f, \tau) = \overline{W(g, \Phi, \tau)} \]  

(2.6)

and

\[ (4\pi\tau)^{-n/2} \int_M \Phi^2 \, dV_g = 1. \]  

(2.7)

Let $\delta \in (0, 1)$. By (2.6) and (2.7),

\[ W(g, f, \tau) \geq \tau \int_M (R(g)\Phi^2 + 4|\nabla \Phi|^2) \, dV_g - \| R(g) \|_{L^\infty(M)} \int_M \Phi^2 \, dV_g - n \]

\[ \geq (1 - \delta) \tau \cdot \inf_{\psi \in C^\infty(M)} \left( \frac{\int_M (R(g)\psi^2 + 4|\nabla \psi|^2) \, dV_g}{\int_M \psi^2 \, dV_g} \right) - \frac{\delta \tau}{(4\pi\tau)^{n/2}} \| R(g) \|_{L^\infty(M)} + I(\Phi) - n \]

\[ \geq (1 - \delta) \tau \lambda_1 - \frac{\delta \tau}{(4\pi\tau)^{n/2}} \| R(g) \|_{L^\infty(M)} + I(\Phi) - n \]  

(2.8)

where $\lambda_1$ is the first eigenvalue of $R(g) - 4\Delta_g$ and

\[ I(\Phi) = \frac{4\delta \tau \int_M |\nabla \Phi|^2 \, dV_g - \int_M \Phi^2 \log \Phi^2 \, dV_g}{\int_M \Phi^2 \, dV_g}. \]
We will now use a modification of the technique of [R1], [R2], to control the term $I(\Phi)$. Choose $\varepsilon \in (0, 2/(n - 2))$. By the Jensen’s inequality, Sobolev inequality, and (2.7),

$$
4\delta \tau \int_M |\nabla \Phi|^2 dV_{\tilde{g}} - \int_M \Phi^2 \log \Phi^2 dV_{\tilde{g}} = 4\delta \tau \int_M |\nabla \Phi|^2 dV_{\tilde{g}} - \frac{1}{\varepsilon} \int M \Phi^2 \log \Phi^2 \varepsilon dV_{\tilde{g}} \\
\geq 4\delta \tau \int_M |\nabla \Phi|^2 dV_{\tilde{g}} - \frac{2 + 2\varepsilon}{\varepsilon} \log \|\Phi\|_{L^{2+2\varepsilon}(M, \tilde{g})} \\
\geq 4\delta \tau \int_M |\nabla \Phi|^2 dV_{\tilde{g}} - \frac{2 + 2\varepsilon}{\varepsilon} \log (C\|\Phi\|_{H^1(M, \tilde{g})})
$$

(2.9)

where

$$
\|\Phi\|_{L^q(M, \tilde{g})} = \left( \int_M \Phi^q dV_{\tilde{g}} \right)^{1/q}
$$

for any $q \geq 1$ and

$$
\|\Phi\|_{H^1(M, \tilde{g})} = \|\Phi\|_{L^2(M, \tilde{g})} + \|\nabla \Phi\|_{L^2(M, \tilde{g})}.
$$

Let

$$
\tilde{\Phi} = \frac{\Phi}{\|\Phi\|_{L^2(M, \tilde{g})}}.
$$

Then

$$
\|\tilde{\Phi}\|_{H^1(M, \tilde{g})} \geq \|\tilde{\Phi}\|_{L^2(M, \tilde{g})} = 1.
$$

(2.10)

By (2.7), (2.9), and (2.10), $\forall \tau \geq \tau_0$,

$$
I(\Phi) \geq 4\delta \tau \|\tilde{\Phi}\|_{H^1(M, \tilde{g})} - \frac{2 + 2\varepsilon}{\varepsilon} \log (C\|\tilde{\Phi}\|_{H^1(M, \tilde{g})}\|\Phi\|_{L^2(M, \tilde{g})}) - 4\delta \tau \\
\geq 4\delta \tau \|\tilde{\Phi}\|_{H^1(M, \tilde{g})} - \frac{2 + 2\varepsilon}{\varepsilon(4\pi\tau)^{n/2}} \log (C\|\tilde{\Phi}\|_{H^1(M, \tilde{g})}) - \frac{(1 + \varepsilon)n}{\varepsilon(4\pi\tau)^{n/2}} \log (4\pi\tau) - 4\delta \tau \\
\geq 4\delta \tau_0 \|\tilde{\Phi}\|_{H^1(M, \tilde{g})} - \frac{2 + 2\varepsilon}{\varepsilon(4\pi\tau_0)^{n/2}} \log (C\|\tilde{\Phi}\|_{H^1(M, \tilde{g})}) - \frac{(1 + \varepsilon)n}{\varepsilon(4\pi\tau)^{n/2}} \log (4\pi\tau) - 4\delta \tau \\
\geq C'_{\tau_0} - \frac{C_1}{\tau_0 n/2} \log (4\pi\tau) - 4\delta \tau
$$

(2.11)

where

$$
C'_{\tau_0} = \min_{y \geq 1} \left( 4\delta \tau_0 y - \frac{2 + 2\varepsilon}{\varepsilon(4\pi\tau_0)^{n/2}} \log (Cy) \right) > -\infty
$$

and

$$
C_1 = \frac{(1 + \varepsilon)n}{\varepsilon(4\pi)^{n/2}}.
$$

By (2.8) and (2.11) we get (2.5) with $C_{\tau_0} = C'_{\tau_0} - n$. By taking infimum over all function $f \in A(\tilde{g}, \tau)$ in (2.5) we get $\mu(\tilde{g}, \tau) > -\infty$ for any $\tau > 0$ and the lemma follows.
Corollary 2.2. Let \( \tilde{g} \) be a Riemannian metric on \( M \). Suppose the first eigenvalue of \( R(\tilde{g}) - 4\Delta \tilde{g} \) is positive. Then

\[
\lim_{\tau \to \infty} \mu(\tilde{g}, \tau) = \infty.
\]

Proof. This corollary is stated without proof in [P1]. We will give a short proof of it here. We fix \( \tau_0 > 0 \) and choose \( \delta \in (0, 1) \) sufficiently small such that

\[
\left[ (1 - \delta) \lambda_1 - \frac{\delta}{(4\pi\tau_0)^{\frac{2}{n}}} \| R(\tilde{g}) \|_{L^\infty(M)} - 4\delta \right] > 0.
\]

By Lemma 2.1 there exist constants \( C_1 > 0 \) and \( C_{\tau_0} > 0 \) such that (2.5) holds. Taking infimum over \( f \in A(\tilde{g}, \tau) \) in (2.5), we get

\[
\mu(\tilde{g}, \tau) \geq \left[ (1 - \delta) \lambda_1 - \frac{\delta}{(4\pi\tau_0)^{\frac{2}{n}}} \| R(\tilde{g}) \|_{L^\infty(M)} - 4\delta \right] \tau - C_{\tau_0} - \frac{C_1}{\tau^{n/2}} \log(4\pi\tau) \quad \forall \tau \geq \tau_0.
\]

Letting \( \tau \to \infty \) the corollary follows.

Lemma 2.3. Suppose \( 0 < t_1 < T \) and \( \overline{f} \in C^\infty(M \times (0, t_1)) \). Let \( \overline{g}(t) = (\overline{g}_{ij}(\cdot, t)) \) be an evolving metric on \( M \) which satisfies

\[
\frac{\partial}{\partial t} \overline{g}(t) = -2(R_{ij}(\overline{g}(t)) + \nabla_i^{\overline{g}(t)} \nabla_j^{\overline{g}(t)} \overline{f}) \quad \text{in } M \times (0, t_1)
\]

where \( \nabla_i^{\overline{g}(t)} \) is the covariant derivative with respect to the metric \( \overline{g}(t) \). Suppose

\[
\frac{\partial \overline{f}}{\partial t} = -\Delta_{\overline{g}} \overline{f} - R(\overline{g}) + \frac{n}{2\tau} \quad \text{in } M \times (0, t_1)
\]

where

\[
\tau = \tau(t) = t_0' - t
\]

for some constant \( t_0' > t_1 \). Then \( \forall t \in (0, t_1) \),

\[
\frac{d}{dt} W(\overline{g}(t), \overline{f}(\cdot, t), \tau) = \int_M 2\tau \left| R_{ij}(\overline{g}(t)) + \nabla_i^{\overline{g}(t)} \nabla_j^{\overline{g}(t)} \overline{f} - \frac{1}{2\tau} \overline{g}_{ij} \right|^2 (4\pi\tau)^{-n/2} e^{-\overline{f}} dV_{\overline{g}(t)}.
\]

Proof. This result is stated without proof in [P1]. For the sake of completeness we will give a simple proof of it here. Let the metric \( \tilde{g}(t) = (\tilde{g}_{ij}(t)) \) be given by

\[
\tilde{g}_{ij}(t) = \frac{\overline{g}_{ij}(t)}{4\pi\tau}.
\]
Then

\[
W(\mathbf{g}(t), \mathbf{f}(\cdot, t), \tau) = \frac{1}{4\pi} \mathcal{F}(\tilde{g}(t), \mathbf{f}(\cdot, t)) + \int_M (\mathbf{f}(p, t) - \bar{n} e^{-\mathbf{f}(p, t)}dV_{\tilde{g}(t)}(p))
\]

\[
\Rightarrow \frac{d}{dt}W(\mathbf{g}(t), \mathbf{f}(\cdot, t), \tau) = \frac{1}{4\pi} \frac{d}{dt} \mathcal{F}(\tilde{g}(t), \mathbf{f}(\cdot, t)) + \int_M \mathbf{f}_t(p, t)e^{-\mathbf{f}(p, t)}dV_{\tilde{g}(t)}(p)
\]

\[
+ \int_M (\mathbf{f}(p, t) - \bar{n}) \frac{\partial}{\partial t} (e^{-\mathbf{f}(p, t)}dV_{\tilde{g}(t)}(p))
\]

(2.16)

Now by (2.12) and (2.13),

\[
\frac{d}{dt} \left( e^{-\mathbf{f}}dV_{\tilde{g}(t)} \right) = \left( \frac{1}{2} \mathbf{g}^{ij}(\tilde{g}_{ij})_t - \mathbf{f}_t \right) e^{-\mathbf{f}}dV_{\tilde{g}(t)}
\]

\[
= \left\{ \frac{1}{2} (4\pi \tau) \mathbf{g}^{ij} \left( \frac{(\tilde{g}_{ij})_t}{4\pi \tau} + \mathbf{g}_{ij} \right) - \mathbf{f}_t \right\} e^{-\mathbf{f}}dV_{\tilde{g}(t)}
\]

\[
= \left\{ \mathbf{g}^{ij} \left( -(\mathbf{R}_{ij}(\mathbf{g}) + \nabla^g_i \nabla^g_j e^f) + \frac{\mathbf{g}_{ij}}{2\pi} \right) - \mathbf{f}_t \right\} e^{-\mathbf{f}}dV_{\tilde{g}(t)}
\]

\[
= 0.
\]

(2.17)

Hence by (2.16) and (2.17),

\[
\frac{d}{dt}W(\mathbf{g}(t), \mathbf{f}(\cdot, t), \tau) = \frac{1}{4\pi} \frac{d}{dt} \mathcal{F}(\tilde{g}(t), \mathbf{f}(\cdot, t)) + \int_M \mathbf{f}_t(p, t)e^{-\mathbf{f}(p, t)}dV_{\tilde{g}(t)}(p).
\]

(2.18)

By (2.17) and section 1.1 of [P1],

\[
\frac{1}{4\pi} \frac{d}{dt} \mathcal{F}(\tilde{g}(t), \mathbf{f}(\cdot, t)) = -\frac{1}{4\pi} \int_M <(\tilde{g}_{ij})_t, \mathbf{R}_{ij}(\mathbf{g}) + \nabla^g_i \nabla^g_j e^f>_{\tilde{g}} e^{-\mathbf{f}}dV_{\tilde{g}}
\]

\[
= -\frac{1}{4\pi} \int_M \mathbf{g}^{ij'}(\tilde{g}_{ij'})(\mathbf{R}_{ij}(\mathbf{g}) + \nabla^g_i \nabla^g_j e^f)e^{-\mathbf{f}}dV_{\tilde{g}}
\]

\[
= -(4\pi \tau)^{-\frac{3}{2}} \int_M [2\tau |(\mathbf{R}(\mathbf{g}) + \nabla^g_i \nabla^g_j f)|^2 - (\mathbf{R}(\mathbf{g}) + \Delta_{\mathbf{g}} e^f)]e^{-\mathbf{f}}dV_{\mathbf{g}}
\]

(2.19)

By (2.13), (2.18), and (2.19), we get (2.15) and the lemma follows.

**Lemma 2.4.** Let \( H_0 \in C^\infty(M) \) be such that \( \min_M H_0 > 0 \). Then for any \( 0 < t_1 < T \) there exists a unique solution \( H \in C^\infty(M \times [0, t]) \) of the problem

\[
\begin{cases}
H_t = -\Delta_{g(t)} H + R(g(t)) H & \text{in } M \times (0, t_1) \\
H(x, t_1) = H_0(x) & \text{in } M
\end{cases}
\]

(2.20)
satisfying the condition

$$H(x, t) \geq e^{-C_2(t_1-t)} \min_M H_0 > 0 \quad \text{in } M \times [0, t_1] \quad (2.21)$$

where $C_2 = \|R\|_{L^\infty(M \times [0, t_1])}$.

**Proof.** By Theorem 6 of [H1] there exists a unique smooth solution $H \in C^\infty(M \times [0, t_1])$ of (2.20). By continuity there exists $\delta_1 \in (0, t_1)$ such that $H(x, t) > 0$ on $M \times (t_1 - \delta_1, t_1]$.

Let

$$t_2 = \inf\{ t' > 0 : H(x, t) > 0 \quad \forall x \in M, t' < t \leq t_1 \}.$$  

Then $0 \leq t' \leq t_1 - \delta_1$. Suppose $t' > 0$. Let $s = t_1 - t$. Then

$$H_s = \Delta H - R(g(t))H \geq \Delta H - C_2 H \quad \forall (x, s) \in M \times (0, t_1 - t')$$

$$\Rightarrow \quad (e^{C_2s}H)_s \geq \Delta(e^{C_2s}H) \quad \forall (x, s) \in M \times (0, t_1 - t')$$

where $C_2 = \|R\|_{L^\infty(M \times [0, t_1])}$. By the maximum principle for parabolic equations,

$$e^{C_2s}H \geq \min_M H_0 \quad \forall (x, s) \in M \times [0, t_1 - t']$$

$$\Rightarrow \quad H(x, t) \geq e^{-C_2(t_1-t)} \min_M H_0 > 0 \quad \forall (x, t) \in M \times [t', t_1]. \quad (2.22)$$

Hence by continuity there exists a constant $\delta_2 \in (0, t')$ such that $H(x, t) > 0$ on $M \times (t' - \delta_2, t')$. This contradicts the maximality of $t'$. Hence $t' = 0$. Putting $t' = 0$ in (2.22) we get (2.21) and the lemma follows.

**Lemma 2.5.** Let $f_0 \in C^\infty(M)$. Then for any $0 < t_1 < T$, $t_0' > t_1$, there exists a solution $f \in C^\infty(M \times (0, t_1))$ of the problem

$$\begin{cases}
    f_t = -\Delta g(t)f + |\nabla f|^2 - R(g(t)) + \frac{n}{2\tau(t)} & \text{in } M \times (0, t_1) \\
    f(x, t_1) = f_0(x) & \text{in } M
\end{cases} \quad (2.23)$$

where $\tau(t) = t_0' - t$.

**Proof.** We will use a transform of [P1] to prove the lemma. Let

$$H_0(x) = (4\pi(t_0' - t_1))^{-n/2}e^{-f_0(x)}.$$  

Then $H_0 > 0$ on $M$. By Lemma 2.4 there exists a unique positive solution $H \in C^\infty(M \times (0, t_1))$ of (2.20). Let

$$f(x, t) = -\log[(4\pi\tau(t))^{n/2}H].$$

Then by (2.20) $f$ satisfies (2.23).
**Theorem 2.6.** For any \( t_0' > 0 \), \( \mu(g(t), t_0' - t) \) is a monotone increasing function of \( t \in (0, \min(t_0', T)) \). If \((M, g)\) is not a Ricci soliton, then \( \mu(g(t), t_0' - t) \) is a strictly monotone increasing function of \( t \in (0, \min(t_0', T)) \).

**Proof.** Let \( t_1 \in (0, \min(t_0', T)) \). By (2.6) and an argument similar to the proof in [R1], [R2], there exists a function \( f_0 \in \mathcal{C}^\infty(M) \) satisfying

\[
(4\pi\tau(t_1))^{-n/2} \int_M e^{-f_0} dV_{g(t_1)} = 1 \tag{2.24}
\]

and

\[
\mu(g(t_1), t_0' - t_1) = W(g(t_1), f_0, t_0' - t_1)
\]

where \( \tau(t) \) is given by (2.14). Let \( f \) be the solution of (2.23) given by Lemma 2.5. Choose \( t_0 \in (0, t_1) \). Let \( \phi_{t_0, t} \) be as in Lemma 1.2, \( \overline{g} \) be given by (1.8), and \( \overline{f}(p, t) = f(\phi_{t_0, t}(p), t) \). Then \( \overline{g} \) satisfies (1.9) with \( V(t) = -\nabla f(\cdot, t) \). By (2.24),

\[
(4\pi\tau(t_1))^{-n/2} \int_M e^{-\overline{f}(p, t_1)} dV_{\overline{g}(t_1)} = 1. \tag{2.25}
\]

By (1.9) and (0.1),

\[
\frac{\partial}{\partial t} \overline{g}(t) = \phi_{t_0, t}^* \left( -2R_{ij}(g(t)) - 2\nabla_i^g(t)\nabla_j^g(t) f \right) \quad \forall 0 < t < t_1
\]

\[
= -2(R_{ij}(\overline{g}(t)) + \nabla_i^\overline{g}(t)\nabla_j^\overline{g}(t) \overline{f}) \quad \forall 0 < t < t_1.
\]

Hence \( \overline{g} \) satisfies (2.12). By direct computation \( \overline{f} \) satisfies (2.13). Hence by Lemma 2.3 (2.15) holds. Thus

\[
W(\overline{g}(t_1), \overline{f}(\cdot, t_1), , t_0' - t_1) \geq W(\overline{g}(t), \overline{f}(\cdot, t), , t_0' - t) + E(t, t_1) \quad \forall 0 < t < t_1 \tag{2.26}
\]

where

\[
E(t, t_1) = \int_t^{t_1} \int_M 2\tau \left| R_{ij}(g(t)) + \nabla_i^g(t)\nabla_j^g(t) \overline{f} - \frac{1}{2\tau} \overline{g}_{ij} \right|^2 (4\pi\tau)^{-n/2} e^{-\overline{f}} dV_{g(t)} dt
\]

\[
= \int_t^{t_1} \int_M 2\tau \left| R_{ij}(g(t)) + \nabla_i^g(t)\nabla_j^g(t) f - \frac{1}{2\tau} g_{ij} \right|^2 (4\pi\tau)^{-n/2} e^{-f} dV_{g(t)} dt \geq 0
\]

with \( E(t, t_1) > 0 \) if \( g \) is not a Ricci soliton. Since the functional \( W \) is invariant under diffeomorphism, by (2.26) \( \forall 0 < t < t_1 \),

\[
W(g(t_1), f_0, t_0' - t_1) \geq W(g(t), f(\cdot, t), t_0' - t) + E(t, t_1)
\]

\[
\Rightarrow \quad \mu(g(t_1), t_0' - t_1) \geq W(g(t), f(\cdot, t), t_0' - t) + E(t, t_1). \tag{2.27}
\]

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By direct computation,
\[
\frac{d}{dt} \left( (4\pi \tau(t))^{-n/2} \int_M e^{-f} dV_g(t) \right) = 0
\]
\[
\Rightarrow (4\pi \tau(t))^{-n/2} \int_M e^{-f(p,t)} dV_g(t) = (4\pi \tau(t_1))^{-n/2} \int_M e^{-f_0(p)} dV_g(t_1) = 1 \quad \forall 0 < t < t_1.
\]
Hence \( f(\cdot, t) \in A(g(t), \tau(t)) \). Thus by (2.27),
\[
\mu(g(t_1), t_0' - t_1) \geq \mu(g(t), t_0' - t) + E(t, t_1).
\]
Since \( 0 < t < t_1 < \min(T, t_0') \) is arbitrary, the lemma follows.

**Theorem 2.7.** If \((M, g)\) is not a Ricci soliton, then there does not exist any shrinking breather for the manifold \( M \) with metric \( g \) evolving by the Ricci flow on \( M \times (0, T) \).

**Proof.** Suppose \((M, g)\) is not a Ricci soliton and there exists a shrinking breather. Then there exist constants \( \alpha \in (0, 1) \), \( 0 < t_1 < t_2 < T \), such that \((M, \alpha g(t_1))\) is diffeomorphic to \((M, g(t_2))\). Then
\[
\mu(\alpha g(t_1), \tau) = \mu(g(t_2), \tau) \quad \forall \tau > 0. \tag{2.28}
\]
Since \( W(\mathcal{F}, f, \tau) = W(\lambda \mathcal{F}, f, \lambda \tau) \) for any metric \( \mathcal{F} \) on \( M \) and \( \lambda > 0 \), for any metric \( \mathcal{F} \) on \( M \) we have
\[
\mu(\mathcal{F}, \tau) = \mu(\lambda \mathcal{F}, \lambda \tau) \quad \forall \lambda, \tau > 0. \tag{2.29}
\]
Hence by Theorem 2.6 and (2.29),
\[
\mu(\alpha g(t_1), \tau) = \mu(g(t_1), \tau/\alpha) < \mu(g(t_2), (\tau/\alpha) - (t_2 - t_1)). \tag{2.30}
\]
Let \( \tau = \alpha(t_2 - t_1)/(1 - \alpha) \). Then
\[
\frac{\tau}{\alpha} - (t_2 - t_1) = \tau. \tag{2.31}
\]
By (2.28), (2.30), and (2.31) we get a contradiction. Hence no shrinking breather exists.

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