APPLICATIONS OF THE AFFINE STRUCTURES ON THE
TEICHMÜLLER SPACES

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Abstract. We prove the existence of global sections trivializing the Hodge bundles on
the Teichmüller spaces of Calabi–Yau manifolds, as well as a global splitting property
of these Hodge bundles. We also prove that a compact Calabi–Yau manifold cannot be
deformed to its complex conjugate. These results answer certain open questions in the
subject. A general result about the period map to be bi-holomorphic from the Hodge
metric completion space of the Teichmüller space of Calabi–Yau type manifolds to their
period domains is proved and applied to the cases of K3 surfaces and cubic fourfolds.

Introduction

This paper is motivated by the certain open questions mentioned in the above abstract
brought to us by Professors Bong Lian and Si Li. As applications of our recent results
about the affine structure on the Teichmüller spaces of Calabi–Yau manifolds, we answer
these questions affirmatively. We also prove a general result about the period map to be
both injective and surjective from the Hodge metric completion space of the Teichmüller
spaces of Calabi–Yau type manifolds to their period domains.

In our recent papers [4, 5], we studied the period maps and the Hodge metric completion
spaces on the Teichmüller spaces of polarized and marked Calabi–Yau and Calabi–Yau
type manifolds, respectively. Our original motivation of considering the Hodge metric
completion space of the Teichmüller spaces is to prove the global Torelli theorem on the
Teichmüller space, which asserts the injectivity of the period map on the Teichmüller
space of polarized and marked Calabi–Yau or Calabi–Yau type manifolds. Such Torelli
problem has been studied for a long time. In [40] Weil reformulated the Torelli problem
for Riemann surfaces with polarization. Andreotti proved Weil’s version of the Torelli
problem in [1]. In 1960s and 70s, Griffiths [7, 8] and Deligne [6] developed the general
theory of variations of Hodge structures, which re-shaped the theory of Torelli problems
in terms of period domains and period maps. The global Torelli problem for K3 surfaces
was conjectured by Weil in [41]. The proof of it was given by Shafarevich and Piatetski-
Shapiro in [26], Looijenga in [20], and Burns-Rapoport in [2]. Todorov [33] and Siu [30]
were able to show the surjectivity of the period mapping for K3 surfaces. Furthermore,
Voisin in [39] and Looijenga in [21] proved the global Torelli theorem for moduli spaces of
cubic fourfolds, the moduli space and period maps of cubic fourfolds are also well studied
in Laza [16, 17]. A version of global Torelli theorem for marked hyperkähler manifolds
was recently proved by Verbitsky [36]. Surjectivity of the period map from certain moduli
spaces for hyperkähler manifolds was proved by Huybrechts and Verbitsky in [13, 36].
There are also many works on local Torelli theorems and generic Torelli theorems.
In [4], we have proved a global Torelli theorem on the Teichmüller space of polarized and marked Calabi–Yau manifolds. Our method works without change for more general Calabi–Yau manifolds such as hyperkähler manifolds, as long as their Teichmüller spaces are smooth. In [5], we adapted similar argument from [4] to give a proof of the global Torelli theorem for the Teichmüller space of polarized and marked Calabi–Yau type manifolds, which is a more general kind of polarized and marked projective manifolds, including cubic fourfolds.

The key property we proved in [4, 5] will be recalled in Theorem 2.2 in §2 in this paper. Based on this theorem, we were able to construct the holomorphic affine structure on the Teichmüller spaces as well as the Hodge metric completion space and thus the injectivity of the period map on the Teichmüller space. We will give a brief review of the definitions of basic concepts in §1, and give a sketch of the construction of holomorphic affine structures on the Teichmüller spaces in §2. In §3, we first show the construction of the Hodge metric completions space of the Teichmüller space, and finally conclude a global Torelli theorem on the Teichmüller space. One may refer to [4] and [5] for more details.

In this paper we use our result to study period maps and Hodge bundles on the Teichmüller spaces of polarized and marked Calabi–Yau and Calabi–Yau type manifolds.

In Section 4.1, we will first prove a simple but general result about when the extended period map is a bi-holomorphic map, and apply it to give a simple proof of the surjectivity of the extended period map on Hodge metric completion of the Teichmüller space of polarized and marked K3 surfaces and cubic fourfolds. More precisely, let $\mathcal{T}^H$ denote the Hodge metric completion space of the Teichmüller space of a polarized and marked Calabi–Yau type manifold, and $D$ be its corresponding period domain. In [4] we have proved that $\mathcal{T}^H$ is smooth complex affine manifold if $\mathcal{T}$ is smooth. Then in Section 4.1 we prove the following theorem for polarized and marked Calabi–Yau and Calabi–Yau type manifolds,

**Theorem 0.1.** If $\dim \mathcal{T}^H = \dim D$, then the extended period map $\Phi^H : \mathcal{T}^H \to D$ is surjective.

Note that in the case of Calabi–Yau type manifolds, we require that the Teichmüller spaces are smooth, as in the cases of Calabi–Yau manifolds. In particular, the above theorem applies to the cases of K3 and cubic fourfolds, and more generally to hyperkähler manifolds. We note that similar results for the moduli spaces of K3 surfaces and cubic fourfolds were first proved in [30, 33] and [21] respectively.

In Section 4.2 we let $F^k$ with $0 \leq k \leq n$ denote the Hodge bundles on $\mathcal{T}^H$, and prove the following result for polarized and marked Calabi–Yau and Calabi–Yau type manifolds,

**Theorem 0.2.** All the Hodge bundles $F^k$ over $\mathcal{T}^H$ are trivial bundles, and the trivialization can be obtained by the canonical sections in (9).

These canonical sections are explicitly constructed in Section 4.2 by using unipotent matrices in our proof of the affine structure on $\mathcal{T}^H$.

Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi–Yau manifolds. Let $p \in \mathcal{T}$ be a base point with the corresponding Calabi–Yau manifolds $M_p$. Let $\varphi(\tau)$ denote
the Kuranishi family of Beltrami differential for the local deformation of complex structure of a Calabi–Yau manifold \( M_p \), and \([\Omega^c_p(\tau)] = [\exp(\tau) \omega_p]\) denotes the local canonical family of holomorphic \((n,0)\)-classes around the base point \( p \) in \( T \). Let \( T^H \) denote the Hodge metric completion of \( T \), then we have the following theorem for Calabi–Yau manifolds,

**Theorem 0.3.** There is a global holomorphic section \( s_0 \) of the Hodge bundle \( F^n \) on \( T^H \) which extends the local canonical holomorphic section \([\Omega^c_p(\tau)]\).

In Section 4.3 we still consider the Teichmüller space of polarized and marked Calabi–Yau manifolds and let \( p \in T \) be the base point. Let \( F^k_p \) denote the fiber of the Hodge bundle \( F^k \) at \( p \) for any point \( p \in T^H \), and prove the following theorem for polarized and marked Calabi–Yau manifolds and Calabi–Yau type manifolds.

**Theorem 0.4.** For any two points \( p \) and \( q \) in \( T^H \) and \( 1 \leq k \leq n \), we have that \( H^n(M, \mathbb{C}) = F^k_p \oplus F^{n-k+1}_q \).

Finally we let \( M_q \) denote a compact polarized and marked Calabi–Yau manifold with complex structure \( J \), and \( \overline{M}_q \) denote the corresponding polarized and marked Calabi–Yau manifold with conjugate complex structure \( -J \), then we have the following theorem

**Theorem 0.5.** If \( q \neq q' \) are two distinct points in \( T \), then \( M_q \neq \overline{M}_q \).

Here \( M_q \neq \overline{M}_q \) means that the two polarized and marked Calabi–Yau manifolds can not be identical in \( T \). Equivalently this implies that \( M_q \) can not be deformed to its complex conjugate manifold \( \overline{M}_q \). We remarked such type of problems have been studied, and that one may find results of similar type of problems in [15]. Theorem 0.3 and Theorem 0.5 answer some long-standing questions in the subject of Calabi–Yau manifolds.

We would like to thank Professors Si Li and Bong Lian for stimulating discussions and many helpful suggestions.

## 1. Period maps on the Teichmüller spaces

This section is a review of basic definitions of Calabi–Yau and Calabi–Yau type manifolds, as well as their moduli and Teichmüller spaces and period maps. Basic results about the moduli spaces and Teichmüller spaces of polarized and marked Calabi–Yau and Calabi–Yau type manifolds are briefly recalled in Section 1.1 and Section 1.2. In Section 1.3 basic properties of period maps are also reviewed for reader’s convenience.

### 1.1. Moduli space and Teichmüller space

Let \( M \) be a projective manifold with \( \dim_{\mathbb{C}} M = n \). Let \( L \) be an ample line bundle over \( M \). We call the pair \((M, L)\) a polarized projective manifold.

The *moduli space* \( \mathcal{M} \) of polarized complex structures on a given differential manifold \( X \) is a complex analytic space consisting of biholomorphically equivalent pairs \((M, L)\) of complex structures and ample line bundles. Let us denote by \([M, L]\) the point in \( \mathcal{M} \) corresponding to a pair \((M, L)\), where \( M \) is a complex manifold diffeomorphic to \( X \) and \( L \) is an ample line bundle on \( M \). If there is a biholomorphic map \( f \) between \( M \) and \( M' \) with \( f^*L' = L \), then \([M, L] = [M', L'] \in \mathcal{M} \). One may also look at [25] and [37] for more details about the construction of moduli spaces of polarized projective manifolds.

Let \((M, L)\) be a polarized projective manifold. For any integer \( m \geq 3 \), we call a basis of the quotient space \((H^n(M, \mathbb{Z})/\text{Tor})/m(H^n(M, \mathbb{Z})/\text{Tor})\) a level \( m \) structure on...
the polarized projective manifold manifold. Let $\mathcal{Z}_m$ be the moduli space of polarized projective manifolds with level $m$ structure.

In this paper, we assume that there exists a $m_0$, such that for $m \geq m_0$, $\mathcal{Z}_m$ is a connected quasi-projective smooth complex manifold with a versal family of projective manifolds with level $m$ structures,

$$\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m,$$

containing $M$ as a fiber and polarized by an ample line bundle $\mathcal{L}_{\mathcal{Z}_m}$ on $\mathcal{X}_{\mathcal{Z}_m}$. This assumption is true for a large class of interesting manifolds, including Calabi–Yau manifolds, and many complete intersections of Calabi–Yau type manifolds, which are our main objects to study.

A polarized and marked projective manifold is a triple consisting of a projective manifold $M$, an ample line bundle $L$ over $M$, and a basis $\{\gamma_1, \cdots, \gamma_{h^n}\}$ of the integral middle homology group modulo torsion, $H_n(M, \mathbb{Z})/\text{Tor}$. We now define the Teichmüller space $\mathcal{T}$ to be a complex analytic space consisting of biholomorphically equivalent triples of $(M, L, \{\gamma_1, \cdots, \gamma_{h^n}\})$. To be more precise, for two triples $(M, L, \{\gamma_1, \cdots, \gamma_{h^n}\})$ and $(M', L', \{\gamma'_1, \cdots, \gamma'_{h^n}\})$, if there exists a biholomorphic map $f : M \to M'$ with

$$f^*L' = L,$$

$$f^*\gamma'_i = \gamma_i \quad \text{for} \quad 1 \leq i \leq h^n,$$

then $[M, L, \{\gamma_1, \cdots, \gamma_{h^n}\}] = [M', L', \{\gamma'_1, \cdots, \gamma'_{h^n}\}] \in \mathcal{T}$. For simplicity we use $[M, L, \gamma]$ to denote the triple $[M, L, \{\gamma_1, \cdots, \gamma_{h^n}\}]$. By this definition, we know that the Teichmüller space $\mathcal{T}$ is a covering space of $\mathcal{Z}_m$ with the covering map denoted by $\pi_m : \mathcal{T} \to \mathcal{Z}_m$. We then have the pull-back family $\pi : U \to \mathcal{T}$ of $\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$.

**Proposition 1.1.** Assume the moduli space $\mathcal{Z}_m$ is smooth for $m \geq m_0$, then the Teichmüller space $\mathcal{T}$ is a smooth and connected complex manifold and the family

$$\varphi : U \to \mathcal{T},$$

containing $M$ as a fiber, is local Kuranishi at each point of $\mathcal{T}$.

**Proof.** Since there is a natural covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$ for any $m \geq m_0$ by the definition of $\mathcal{T}$, the Teichmüller space $\mathcal{T}$ is a smooth and connected complex manifold, as $\mathcal{Z}_m$ is a connected smooth complex manifold. Since the family $\mathcal{X}_{\mathcal{Z}_m} \to \mathcal{Z}_m$ is a versal family at each point of $\mathcal{Z}_m$ and that $\pi_m$ is locally biholomorphic, the pull-back family via $\pi_m$ is also versal at each point of $\mathcal{T}$. By the definition of local Kuranishi family, we get that $U \to \mathcal{T}$ is local Kuranishi at each point of $\mathcal{T}$. It is easy to see that $\mathcal{T}$ is independent of $m$ by using the versal property and the simply connectedness of $\mathcal{T}$. \qed

1.2. Calabi–Yau and Calabi–Yau type manifolds. The basic objects we are considering in [4] and [5] are polarized and marked Calabi–Yau manifolds and Calabi–Yau type manifolds, respectively.

**Definition 1.2.** A compact projective manifold $M$ of complex dimension $n \geq 2$ is called Calabi–Yau, if it has a trivial canonical bundle and satisfies $H^i(M, \mathcal{O}_M) = 0$ for $0 < i < n$.

**Definition 1.3.** A compact simply connected projective manifold $M$ of complex dimension $n$ is called a Calabi–Yau type manifold, if it satisfies the following:
Theorem 1.7. Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi–Yau type manifolds. Assume that $\mathcal{Z}_m$ is smooth for some $m \geq m_0$, then $\mathcal{T}$ is a simply connected smooth complex manifold. In particular, the Teichmüller space of polarized and marked Calabi–Yau manifold is a simply connected smooth complex manifold.

We refer the reader to [3, Theorem 2.5] for the proof. Moreover, one may conclude that the Teichmüller space is actually the universal cover of the smooth moduli space $\mathcal{Z}_m$ from the above theorem.

We also remark that if $\mathcal{Z}_m$ is smooth for any $m \geq m_0$, then we can define the Teichmüller space as the universal cover of $\mathcal{Z}_m$ for any fixed $m \geq m_0$. In fact, let $m_1$ and $m_2$ be two different integers, and $\mathcal{U}_1 \to \mathcal{T}_1$, $\mathcal{U}_2 \to \mathcal{T}_2$ be two versal families constructed via level $m_1$ and level $m_2$ structures respectively as above, and both of which contain $M$ as a fiber. By using the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ are simply connected and the definition of versal family, we have a biholomorphic map $\hat{f} : \mathcal{T}_1 \to \mathcal{T}_2$, such that the versal family $\mathcal{U}_1 \to \mathcal{T}_1$ is the
pull back of the versal family $U_2 \to T_2$ by $f$. Thus these two families are biholomorphic to each other. Moreover, using this alternate definition, one can see that the Teichmüller space is the deformation of the complex structure on the polarized and marked Calabi–Yau manifolds or Calabi–Yau type manifolds. We actually used this alternate definition of the Teichmüller space of Calabi–Yau manifolds in $[4]$.

1.3. Period domain and period map on the Teichmüller space. We will give a general review about Hodge structures and period domain. One may refer to $[27] \; \S 3$ for more details. For a polarized and marked Calabi–Yau manifold or Calabi–Yau type manifold $M$ with background smooth manifold $X$, we identify the basis of $H_n(M, \mathbb{Z})/\text{Tor}$ to a lattice $\Lambda$ as in $[31]$. This gives us a canonical identification of the middle dimensional de Rham cohomology of $M$ to that of the background manifold $X$, that is, $H^n(M) \cong H^n(X)$, where the coefficient ring can be $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Since the polarization $L$ is an integer class, it defines a map

$$L : H^n(X, \mathbb{Q}) \to H^{n+2}(X, \mathbb{Q}), \quad A \mapsto L \wedge A.$$  

We denote by $H^n_{pr}(X) = \ker(L)$ the primitive cohomology groups where, the coefficient ring can also be $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. We let $H^{k,n-k}_{pr}(M) = H^{k,n-k}(M) \cap H^n_{pr}(M, \mathbb{C})$ and denote its dimension by $h^{k,n-k}$. We have the Hodge decomposition $H^n_{pr}(M, \mathbb{C}) = H^n_{pr,0}(M) \oplus \cdots \oplus H^n_{pr,n}(M)$. The Poincaré bilinear form $Q$ on $H^n_{pr}(X, \mathbb{Q})$ is defined by

$$Q(u, v) = (-1)^{\frac{n(n-1)}{2}} \int_X u \wedge v$$

for any $d$-closed $n$-forms $u, v$ on $X$. Let $f^k = \sum_{i=k}^n h^{i,n-i}$, $f^0 = m$, and $F^k = F^k(M) = H^0_{pr,0}(M) \oplus \cdots \oplus H^{k,n-k}_{pr}(M)$, from which we have the decreasing filtration $H^n_{pr}(M, \mathbb{C}) = F^0 \supset \cdots \supset F^n$. We know that

(1) \quad $\dim_{\mathbb{C}} F^k = f^k$,

(2) \quad $H^n_{pr}(X, \mathbb{C}) = F^k \oplus F^{n-k+1}$, and $H^{k,n-k}_{pr}(M) = F^k \cap F^{n-k}$.

In terms of the Hodge filtration, the Poincaré bilinear form satisfies the Hodge-Riemann relations

(3) \quad $Q(F^k, F^{n-k+1}) = 0$, \quad and

(4) \quad $Q(Cv, \tau) > 0$ if $v \neq 0$,

where $C$ is the Weil operator given by $Cv = (\sqrt{-1})^{2k-n} v$ for $v \in H^{k,n-k}(M)$. The period domain $D$ for polarized Hodge structures with data $[1]$ is the space of all such Hodge filtrations

$$D = \left\{ F^n \subset \cdots \subset F^0 = H^n_{pr}(X, \mathbb{C}) \mid [1], [3] \; \text{and} \; [4] \; \text{hold} \right\}.$$  

The compact dual $\tilde{D}$ of $D$ is $\tilde{D} = \left\{ F^n \subset \cdots \subset F^0 = H^n_{pr}(X, \mathbb{C}) \mid [1] \; \text{and} \; [3] \; \text{hold} \right\}$. The period domain $D \subset \tilde{D}$ is an open subset. We note that the conditions $[3]$ and $[4]$ imply the identities in $[2]$.

Remark 1.8. We remark the notation change for the primitive cohomology groups. For simplicity, we will use $H^n(M, \mathbb{C})$ and $H^{k,n-k}(M)$ to denote the primitive cohomology groups $H^n_{pr}(M, \mathbb{C})$ and $H^{k,n-k}_{pr}(M)$ respectively. Moreover, we will use cohomology to mean primitive cohomology in the rest of the paper.
Then the period map $\Phi : T \rightarrow D$ is defined by assigning to each point in $T$ the Hodge structure of the corresponding fiber. The period map has several good properties, and one may refer to [38, Chapter 10] for details. Among them, one of the most important is the following Griffiths transversality: the period map $\Phi$ is a holomorphic map and its tangent map satisfies

$$\Phi_*(v) \in \bigoplus_{k=1}^n \text{Hom}\left(\frac{F^k_p}{F^{k+1}_p}, \frac{F^{k-1}_p}{F^k_p}\right) \quad \text{for any} \quad p \in T \quad \text{and} \quad v \in T^{1,0}_p \mathcal{T}$$

with $F^{n+1}_p = 0$, or equivalently, $\Phi_*(v) \in \bigoplus_{k=0}^n \text{Hom}(F^k_p, F^{k-1}_p)$.

### 2. Affine structure on the Teichmüller space

This section reviews our construction of the affine structure on the Teichmüller spaces of polarized and marked Calabi–Yau and Calabi–Yau type manifolds. We will first give a general review about Hodge structures and period domain from Lie group point of view. One may refer to [9] and [27] for more details. Let us simply denote $\mathcal{D}$ of polarized and marked Calabi–Yau and Calabi–Yau type manifolds.

Let us consider the nilpotent Lie subalgebra $n_+ := \bigoplus_{k \geq 1} g^{-k,k}$. Then one gets the holomorphic isomorphism $g/b \cong n_+$. Since $D$ is an open set in $\mathcal{D}$, we have the following relation:

$$T^{1,0}_{o,h} D = T^{1,0}_{o,h} \mathcal{D} \cong b \oplus g^{-1,1}/b \hookrightarrow g/b \cong n_+.$$
We take the unipotent group $N_+ = \exp(n_+)$. Then $N_+ \simeq \mathbb{C}^d$ for some $d$. We remark that with a fixed base point, we can identify $N_+$ with its unipotent orbit in $\bar{D}$ by identifying an element $c \in N_+$ with $[c] = cB$ in $\bar{D}$; that is, $N_+ = N_+(\text{base point}) \simeq N_+B/B \subseteq \bar{D}$.

Let us introduce the notion of an adapted basis for the given Hodge decomposition or the Hodge filtration. For any $p \in \mathcal{T}$ and $f^k = \dim F^k_p$ for any $0 \leq k \leq n$, we call a basis
\[
\xi = \{\xi_0, \xi_1, \ldots, \xi_N, \ldots, \xi_{f_{k+1}}, \ldots, \xi_{f_{k-1}}, \ldots, \xi_{f_2}, \ldots, \xi_{f_0}, \xi_{f_{n-1}}\}
\]
of $H^n(M_p, \mathbb{C})$ an adapted basis for the given Hodge decomposition
\[
H^n(M_p, \mathbb{C}) = H_p^{n,0} \oplus H_p^{n-1,1} \oplus \cdots \oplus H_p^{1,n-1} \oplus H_p^{0,n},
\]
if it satisfies $H_p^{k,n-k} = \text{Span}_\mathbb{C} \{\xi_{k+1}, \ldots, \xi_{f_k-1}\}$ with $\dim H_p^{k,n-k} = f^k - f^{k+1}$. We call a basis
\[
\zeta = \{\zeta_0, \zeta_1, \ldots, \zeta_N, \ldots, \zeta_{f_{k+1}}, \ldots, \zeta_{f_{k-1}}, \ldots, \zeta_{f_2}, \ldots, \zeta_{f_0}, \zeta_{f_{n-1}}\}
\]
of $H^n(M_p, \mathbb{C})$ an adapted basis for the given filtration
\[
F^n \subseteq F^{n-1} \subseteq \cdots \subseteq F^0
\]
if it satisfies $F^k = \text{Span}_\mathbb{C} \{\zeta_0, \ldots, \zeta_{f_k-1}\}$ with $\dim F^k = f^k$. Moreover, unless otherwise pointed out, the matrices in this paper are $m \times m$ matrices, where $m = f^0$. The blocks of the $m \times m$ matrix $T$ is set as follows: for each $0 \leq \alpha, \beta \leq n$, the $(\alpha, \beta)$-th block $T^{\alpha,\beta}$ is
\[
T^{\alpha,\beta} = [T_{ij}(\tau)]_{f^{-\alpha+n+1} \leq i \leq f^{-\alpha+n-1}, f^{-\beta+n+1} \leq j \leq f^{-\beta+n-1}};
\]
where $T_{ij}$ is the entries of the matrix $T$, and $f^{n+1}$ is defined to be zero. In particular, $T = [T^{\alpha,\beta}]$ is called a block lower triangular matrix if $T^{\alpha,\beta} = 0$ whenever $\alpha < \beta$.

Let us fix an adapted basis $\{\eta_0, \ldots, \eta_m\}$ for the Hodge decomposition of the base point $p \in \mathcal{T}$, then elements in $N_+$ can be realized as nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix. By viewing $N_+$ as a subset of $\bar{D}$ with the fixed base point, we define
\[
\tilde{T} = \Phi^{-1}(N_+).
\]
Let us still denote the restriction map $\Phi|\mathcal{T}$ by $\Phi$. We first prove the following important proposition by using structure theory for the Lie groups and Lie algebras.

**Proposition 2.1.** The restriction map $\Phi : \tilde{T} \to N_+$ is bounded in $N_+ \simeq \mathbb{C}^d$ with respect to the Euclidean metric on $N_+$.

The Euclidean metric on $N_+$ is induced from the Hodge metric on $D$ at the reference point $p$ through the identification $T_pD \simeq n_+$. The main idea of the proof of this theorem is as follows. One takes Cartan decomposition $g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, where the Lie subgroup $K \subseteq G_\mathbb{R}$ corresponding to $\mathfrak{k}_0$ is a maximal compact subgroup in $G_\mathbb{R}$. We first realize that the simple real Lie algebra $g_0$ in our case has a Cartan subalgebra $\mathfrak{h}_0 \subseteq \mathfrak{k}_0$. Thus $g_0$ is of the first category (cf. [28], Theorem 8, pp 422). Then using the result of Lemma 3 in [20] about the real semisimple Lie algebra of first category that the maximal abelian subspace in $\mathfrak{p}_0$ can be decomposed using the noncompact root vectors. This decomposition along with the property that the period map is horizontal allows us to show that $\Phi(\tilde{T})$ sits in a polydisc, a product of discs of radius 1 up to normalization, in $N_+$. This argument is a slight extension of Harish-Chandra’s proof of his famous embedding theorem of the Hermitian symmetric domains as bounded domains in complex Euclidean spaces. One
affine structures on $T$ holomorphic affine structure on $T$
coordinate cover

Remark
complex affine manifold.

basis to the Hodge decomposition,
where $(6)$

Theorem 2.2.
The image of the period map $\Phi : \mathcal{T} \to D$ is in $N_+ \cap D$.

With the above theorem, we have $\mathcal{T} = \Phi^{-1}(N_+)$. Thus for any $q \in \mathcal{T}$, $\Phi(q) \in N_+$ is a nonsingular block lower triangular matrix with identity diagonal blocks. Therefore, we can define a holomorphic map

$$\tau : \mathcal{T} \to \mathbb{C}^N \cong H_p^{n-1,1}, \quad q \mapsto (1,0)$-block of the matrix $\Phi(q) \in N_+$,

that is, $\tau(q) = (\tau_1(q), \tau_2(q), \cdots, \tau_N(q)) = ([\Phi(q)]_{10}, [\Phi(q)]_{20}, \cdots, [\Phi(q)]_{N0})$.

If we define the following projection map with the fixed base point and its fixed adapted basis to the Hodge decomposition,

$$(6) \quad P : N_+ \cap D \to H_p^{n-1,1} \cong \mathbb{C}^N, \quad F \mapsto (\eta_1, \cdots, \eta_N) F^{(1,0)} = F_{10} \eta_1 + \cdots + F_{N0} \eta_N,$$

where $F^{(1,0)}$ is the $(1,0)$-block of the unipotent matrix $F$. Then $\tau = P \circ \Phi : \mathcal{T} \to \mathbb{C}^N$.

Let us review the definition of complex affine manifolds. One may refer to [22, pp 215] for more details.

Definition 2.3. Let $M$ be a complex manifold of complex dimension $n$. If there is a coordinate cover $\{(U_i, \varphi_i) ; i \in I\}$ of $M$ such that $\varphi_{ik} = \varphi_i \circ \varphi_k^{-1}$ is a holomorphic affine transformation on $\mathbb{C}^n$ whenever $U_i \cap U_k$ is not empty, then $\{(U_i, \varphi_i) ; i \in I\}$ is called a complex affine coordinate cover on $M$ and it defines a holomorphic affine structure on $M$.

By the local Torelli theorem for Calabi–Yau manifolds (cf. [32] and [35]) or Calabi–Yau type manifolds and the definition of holomorphic affine structure, we can conclude the main theorem of this section,

Theorem 2.4. The holomorphic map $\tau = (\tau_1, \cdots, \tau_N) : \mathcal{T} \to \mathbb{C}^N$ defines a local coordinate around each point $q \in \mathcal{T}$. Thus the map $\tau$ itself gives a global holomorphic coordinate for $\mathcal{T}$ with the transition maps all identity maps. In particular, the global holomorphic coordinate $\tau : \mathcal{T} \to \mathbb{C}^N$ defines a holomorphic affine structure on $\mathcal{T}$. Therefore, $\mathcal{T}$ is a complex affine manifold.

Remark 2.5. We remark that the construction of the holomorphic affine structure on the Teichmüller space depends on the choices of the base point. In fact, fix another base point $p' \in \mathcal{T}$, and we analogously define another coordinate map $\tau' : \mathcal{T} \to \mathbb{C}^N$, which gives a holomorphic affine structure on $\mathcal{T}$. Then in general $\tau$ and $\tau'$ define different holomorphic affine structures on $\mathcal{T}$ in the following sense: if $\varphi : \mathbb{C}^N \to \mathbb{C}^N$ is the holomorphic map satisfying $\tau = \varphi \circ \tau'$, then $\varphi$ is not an affine map in general.
3. Hodge metric completion space of Teichmüller space and a global Torelli theorem

This section contains a review of our extension of the affine structure to the Hodge metric completion $\mathcal{T}^H$ of the Teichmüller space $\mathcal{T}$, as well as its consequences including a global Torelli theorem for polarized and marked Calabi–Yau and Calabi–Yau type manifolds.

Let us denote the period map on the smooth moduli space by $\Phi_{Z_m} : \mathcal{Z}_m \to D/\Gamma$, where $\Gamma$ denotes the global monodromy group which acts properly and discontinuously on $D$. Then $\Phi : \mathcal{T} \to D$ is the lifting of $\Phi_{Z_m} \circ \pi_m$, where $\pi_m : \mathcal{T} \to \mathcal{Z}_m$ is the universal covering map. There is Hodge metric $h$ on $D$, which is a complete homogeneous metric and is studied in [9]. For both Calabi–Yau and Calabi–Yau type manifolds, both $\Phi_{Z_m}$ and $\Phi$ are locally injective. Thus the pull-backs of $h$ on $\mathcal{Z}_m$ and $\mathcal{T}$ are both well-defined Kähler metrics, and they are still called Hodge metrics.

Let $\mathcal{Z}_m^H$ be the Hodge metric completion of the smooth moduli space $\mathcal{Z}_m$ and let $\mathcal{T}_m^H$ be the universal cover of $\mathcal{Z}_m^H$ with the universal covering map $\pi_m^H : \mathcal{T}_m^H \to \mathcal{Z}_m^H$. We compactify $\mathcal{Z}_m$ to a smooth projective manifold $\mathcal{Z}_m \cup \mathcal{Z}_m^\circ$ so that $\mathcal{Z}_m \backslash \mathcal{Z}_m$ is normal crossing divisors by Hironaka resolution of singularity. It is easy to see that $\mathcal{Z}_m^H$ is a connected and complete smooth submanifold in $\mathcal{Z}_m \cup \mathcal{Z}_m^\circ$ with codim$_{\mathcal{Z}_m}(\mathcal{Z}_m^H \backslash \mathcal{Z}_m) \geq 1$. Thus $\mathcal{T}_m^H$ is a connected and simply connected complete smooth complex manifold. We also obtain the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{T} & \overset{i_m}{\longrightarrow} & \mathcal{T}_m^H \\
\downarrow \pi_m & & \downarrow \pi_m^H \\
\mathcal{Z}_m & \overset{i}{\longrightarrow} & \mathcal{Z}_m^H \\
\end{array}
$$

where $\Phi_{Z_m}^H$ is the continuous extension of the period map $\Phi_{Z_m} : \mathcal{Z}_m \to D/\Gamma$, $i$ is the inclusion map, $i_m$ is a lifting of $i \circ \pi_m$, and $\Phi_{Z_m}^H$ is a lifting of $\Phi_{Z_m}^H \circ \pi_m^H$, and we fix a suitable choice of $i_m$ and $\Phi_{Z_m}^H$ such that $\Phi = \Phi_{Z_m}^H \circ i_m$. Let us denote $\mathcal{T}_m := i_m(\mathcal{T})$ and the restriction map $\Phi_m = \Phi_{Z_m}^H |_{\mathcal{T}_m}$, then we also have $\Phi = \Phi_m \circ i_m$. Moreover, it is easy to see that $\Phi_m$ is also bounded by Theorem 2.1. With these notations, we can conclude that the image $\mathcal{T}_m$ equals to the preimage $(\pi_m^H)^{-1}(\mathcal{Z}_m)$. Therefore, $\mathcal{T}_m$ is a connected open submanifold in $\mathcal{T}_m^H$ and codim$_{\mathcal{T}_m}(\mathcal{T}_m^H \backslash \mathcal{T}_m) \geq 1$.

Recall that in [2] we have fixed a base point $p \in \mathcal{T}$ and an adapted basis $\{\eta_0, \cdots, \eta_{m-1}\}$ for the Hodge decomposition of the base point $\Phi(p) \in D$. With the fixed base point in $D$, we can identify $N_+$ with its unipotent orbit in $D$. Then applying Riemann extension theorem on the bounded map $\Phi_m : \mathcal{T}_m \to N_+ \cap D$, we may conclude

Lemma 3.1. The map $\Phi_m^H$ is a bounded holomorphic map from $\mathcal{T}_m^H$ to $N_+ \cap D$.

With the fixed the base point $p \in \mathcal{T}$ and the fixed adapted basis $\{\eta_0, \cdots, \eta_{m-1}\}$ for the Hodge decomposition of $\Phi(p)$, we can analogously define the holomorphic map

$$\tau_m^H = P \circ \Phi_m : \mathcal{T}_m^H \to \mathbb{C}^N.$$
where \( P \) is the projection map defined in (6). Moreover, we also have \( \tau = P \circ \Phi = P \circ \Phi^H \circ \imath_m = \tau^H \circ \imath_m \). Then using the property that the holomorphic map \( \tau \) defines the holomorphic affine structure on \( T \), we can show that

**Theorem 3.2.** The holomorphic map \( \tau^H_m : T^H_m \to \mathbb{C}^N \) is a local embedding, and it defines a global holomorphic affine structure on \( T^H_m \).

Now using the completeness of \( T^H_m \) and the holomorphic affine structure on it, we can show that any two points in \( T^H_m \) can be joined by a straight line. Then the injectivity of \( \tau^H_m \) follows by contradiction: if \( \tau^H_m(p) = \tau^H_m(q) \) for some \( p \neq q \), then affineness of \( \tau^H_m \) would imply that \( \tau^H_m \) was a constant map on the line connecting \( p \) and \( q \). However, this contradicts the local injectivity of \( \tau^H_m \). Moreover, as \( \tau^H_m = P \circ \Phi^H_m \), where \( P \) is the projection map and \( \Phi^H_m \) is a bounded map, we may conclude that \( \tau^H_m \) is bounded as well.

To conclude, we have the following theorem.

**Theorem 3.3.** For any \( m \geq 3 \), the holomorphic map \( \tau^H_m : T^H_m \to \mathbb{C}^N \) is an injection. In particular, the completion space \( T^H_m \) is a bounded domain in \( \mathbb{C}^N \).

**Corollary 3.4.** The holomorphic map \( \Phi^H_m : T^H_m \to \mathbb{N} + \cap \mathcal{D} \) is also an injection.

Using the above corollary and the completeness of \( T^H_m \), we can conclude the following proposition, which shows that the definition of \( T^H_m \) is independent of the choice of the level \( m \) structure.

**Proposition 3.5.** For any \( m, m' \geq 3 \), the complete complex manifolds \( T^H_m \) and \( T^H_{m'} \) are biholomorphic to each other.

**Definition 3.6.** We define the complete complex manifold \( T^H = T^H_0 \) the holomorphic map \( i_T : T \to T^H \) by \( i_T = i_m \), and the extended period map \( \Phi^H : T^H \to \mathcal{D} \) by \( \Phi^H = \Phi^H \) for any \( m \geq 3 \). In particular, with these new notations, we have the commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{i_T} & T^H \\
\downarrow{\pi_m} & & \downarrow{\pi^H_m} \\
\mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}^H_m \\
\end{array}
\]

\[
\begin{array}{ccc}
\Phi^H & & D \\
\downarrow{\Phi^H_m} & & \downarrow{\pi_D} \\
\mathcal{Z}^H_m & \longrightarrow & D/\Gamma. \\
\end{array}
\]

**Remark 3.7.** By the definition, we can immediately conclude that the complex manifold \( T^H \) is a complex affine manifold, which is a bounded domain in \( \mathbb{C}^N \).

**Proposition 3.8.** The map \( i_T : T \to T^H \) is an embedding. In particular, \( T^H \) the completion space of \( T \) with respect to the Hodge metric and the extended period map \( \Phi^H : T^H \to \mathbb{N} + \cap \mathcal{D} \) is a holomorphic injection.

To show \( i_T : T \to T^H \) is an embedding, we first note that \( T_0 := T_m = i_m(T) \) is also well-defined. Then it is not hard to show that \( i_T : T \to T_0 \) is a covering map. Moreover, we prove that \( i_T : T \to T_0 \) is actually one-to-one by showing that the fundamental group of \( T_0 \) is trivial. Here the markings of the Calabi–Yau manifolds or the Calabi–Yau type manifolds and the simply connectedness of \( T \) come into play substantially.

Since \( \Phi = \Phi^H \circ i_T \) with both \( \Phi^H \) and \( i_T \) embeddings, we get the global Torelli theorem for the period map from the Teichmüller space to the period domain as follows.
Theorem 3.9 (Global Torelli theorem). The period map \( \Phi : \mathcal{T} \rightarrow D \) is injective.

Using the completeness of \( \mathcal{T}^H \) and the injectivity of \( \Phi^H \), together with the function \( f : D \rightarrow \mathbb{R} \) which is constructed in Theorem 8.1 in \[9\], we can construct a plurisubharmonic exhaustion function \( \mathcal{T}^H \). This shows that \( \mathcal{T}^H \) is a bounded domain of holomorphy in \( \mathbb{C}^N \). Moreover, the existence of the Kähler-Einstein metric follows directly from a theorem of Mok–Yau in \[24\].

Theorem 3.10. The Hodge metric completion space \( \mathcal{T}^H \) is a bounded domain of holomorphy in \( \mathbb{C}^N \); thus there exists a complete Kähler–Einstein metric on \( \mathcal{T}^H \).

4. Applications

This section contains several applications of the results reviewed in the previous sections. In Section 4.1 we prove a general result for the extended period map to be a bi-holomorphic map from \( \mathcal{T}^H \), the Hodge metric completion of the Teichmüller space \( \mathcal{T} \) of polarized and marked Calabi–Yau and Calabi–Yau type manifolds to the corresponding period domain; and apply this result to the cases of K3 surfaces and cubic fourfolds. We hope to find more interesting examples in our subsequent work. In Section 4.2 we construct explicit holomorphic sections of the Hodge bundles on \( \mathcal{T}^H \), which trivialize those Hodge bundles. In particular, for Calabi–Yau manifolds, a global holomorphic section of holomorphic \((n,0)\)-classes on \( \mathcal{T}^H \) is constructed, which coincides with explicit local Taylor expansion in the affine coordinates at any base point \( p \) in \( \mathcal{T}^H \). Finally in Section 4.3 we prove a global splitting property for the Hodge bundles, as well as a theorem proves that a compact polarized and marked Calabi–Yau manifold with complex structure \( J \) can not be deformation equivalent to a polarized and marked Calabi–Yau manifolds with conjugate complex structure \( -J \).

4.1. Surjectivity of the period map on the Teichmüller space. In this section we use our results on the Hodge completion space \( \mathcal{T}^H \) to give a simple proof of the surjectivity of the period maps of K3 surfaces and cubic fourfolds. First we have the following general result for polarized and marked Calabi–Yau manifolds and Calabi–Yau type manifolds,

Theorem 4.1. If \( \dim \mathcal{T}^H = \dim \mathcal{T} = \dim D \), then the extended period map \( \Phi^H : \mathcal{T}^H \rightarrow D \) is surjective.

Proof. Since \( \dim \mathcal{T}^H = \dim D \), the property that \( \Phi^H : \mathcal{T}^H \rightarrow D \) is an local isomorphism shows that the image of \( \mathcal{T}^H \) under the extended period map \( \Phi^H \) is open in \( D \). On the other hand, the completeness of \( \mathcal{T}^H \) with respect to Hodge metric implies that the image of \( \mathcal{T}^H \) under \( \Phi^H \) is close in \( D \). As \( \mathcal{T}^H \) is not empty and that \( D \) is connected, we can conclude that \( \Phi^H(\mathcal{T}^H) = D \). \( \square \)

It is well known that for K3 surfaces, which are two dimensional Calabi–Yau manifolds, we have \( \dim \mathcal{T}^H = \dim \mathcal{T} = \dim D = 19 \); for cubic fourfolds, they are Calabi–Yau type manifolds. One knows that both K3 and cubic fourfolds have smooth Teichmüller spaces, and \( \dim \mathcal{T}^H = \dim \mathcal{T} = \dim D = 20 \). Thus applying the above theorem, we can easily conclude that

Corollary 4.2. Let \( \mathcal{T}^H \) be the Hodge metric completion space of the Teichmüller space for polarized and marked K3 surfaces or cubic fourfolds. Then the extended period map \( \Phi^H : \mathcal{T}^H \rightarrow D \) is surjective.
In fact, among all the Calabi–Yau or Calabi–Yau type projective hypersurfaces, K3 surfaces and cubic fourfolds are the only two satisfying the condition that the dimensions of the Teichmüller space and the period domain are the same. It would be interesting to find such examples for complete intersections in weighted projective spaces and compact homogeneous manifolds.

**Remark 4.3.** Let $\mathcal{T}$ be the Teichmüller space of polarized and marked hyperkähler manifolds, $H^2_{pr}(M, \mathbb{C})$ the degree 2 primitive cohomology group, and $D$ the period domain of weight two Hodge structures on $H^2_{pr}(M, \mathbb{C})$. Then our method can be applied without change to prove that the period map from $\mathcal{T}$ to $D$ is also injective. Furthermore, let $\mathcal{T}^H$ be the Hodge completion of $\mathcal{T}$ with respect to the Hodge metric induced from the homogeneous metric on $D$, then the extended period map from $\mathcal{T}^H$ to $D$ is also surjective. This follows from the same argument of above theorem. See [36] and [13] for different injectivity and surjectivity results for hyperkähler manifolds.

### 4.2. Global holomorphic sections of the Hodge bundles

In this section we prove the existence and study the property of global holomorphic sections of the Hodge bundles $\{F^k\}_{k=0}^n$ over Hodge completion space $\mathcal{T}^H$ of Teichmüller space of polarized and marked Calabi–Yau and Calabi–Yau type manifolds. The main ingredient of proofs in this section is Theorem 2.2.

Recall that we have fixed a base point $p \in \mathcal{T}$ and an adapted basis $\{\eta_0, \cdots, \eta_{m-1}\}$ for the Hodge decomposition of the base point $\Phi(p) \in D$. With the fixed base point in $D$, we can identify $N_+$ with its unipotent orbit in $\bar{D}$ by identifying an element $c \in N_+$ with $[c] = cB$ in $\bar{D}$.

On one hand, as we have fixed an adapted basis $\{\eta_0, \cdots, \eta_{m-1}\}$ for the Hodge decomposition of the base point. Then elements in $G_\mathbb{C}$ can be identified with a subset of the nonsingular block matrices. In particular, the set $N_+$ with its unipotent orbit in $\bar{D}$, elements in elements in $N_+$ can be realized as nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix. Namely, for any element $\{F^k\}_{k=0}^n \in N_+ \subseteq \bar{D}$, there exists a unique nonsingular block lower triangular matrices $A(o) \in G_\mathbb{C}$ such that $(\eta_0, \cdots, \eta_{m-1})A(o)$ is an adapted basis for the Hodge filtration $\{F^k\}_{k=0}^n \in N_+$ that represents this element in $N_+$. Similarly, any elements in $B$ can be realized as nonsingular block upper triangular matrices in $G_\mathbb{C}$. Moreover, as $\bar{D} = G_\mathbb{C}/B$, we have that for any $U \in G_\mathbb{C}$, which is a nonsingular block upper triangular matrix, $(\eta_0, \cdots, \eta_{m-1})A(o)U$ is also an adapted basis for the Hodge filtration $\{F^k\}_{k=0}^n$. Conversely, if $(\zeta_0, \cdots, \zeta_{m-1})$ is an adapted basis for the Hodge filtration $\{F^k\}_{k=0}^n$, then there exists a unique $U \in G_\mathbb{C}$ such that $(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})A(o)U$.

For any $q \in \mathcal{T}$, let us denote the Hodge filtration at $q \in \mathcal{T}$ by $\{F^k_q\}_{k=0}^n$ and we have that $\{F^k_q\}_{k=0}^n \in N_+ \cap D$ by Theorem 2.2. Thus there exists a unique nonsingular block lower triangular matrices $\tilde{A}(q)$ such that $(\eta_0, \cdots, \eta_{m-1})\tilde{A}(q)$ is an adapted basis for the Hodge filtration $\{F^k_q\}_{k=0}^n$.

On the other hand, for any adapted basis $\{\zeta_0(q), \cdots, \zeta_{m-1}(q)\}$ for the Hodge filtration $\{F^k_q\}_{k=0}^n$ at $q$, we know that there exists an $m \times m$ transition matrix $A(q)$ such that

$$(\zeta_0(q), \cdots, \zeta_{m-1}(q)) = (\eta_0, \cdots, \eta_{m-1})A(q).$$

Moreover, we set the blocks of $A(q)$ as in [3] and denote the $(i, j)$-th block of $A(q)$ by $A^{ij}(q)$. 
According to the above discussion, as both \((\eta_0, \cdots, \eta_{m-1}), \tilde{A}(q)\) and \((\eta_0, \cdots, \eta_{m-1})A(q)\) are adapted bases for the Hodge filtration for \(\{F^k_q\}_{k=0}^n\), there exists a \(U \in G_C\) which is a block nonsingular upper triangular matrix such that

\[
(\eta_0, \cdots, \eta_{m-1})\tilde{A}(q)U = (\eta_0, \cdots, \eta_{m-1})A(q).
\]

Therefore, we conclude that

\[(8) \quad \tilde{A}(q)U = A(q),\]

where \(\tilde{A}(q)\) is a nonsingular block lower triangular matrix in \(G_C\) with all the diagonal blocks equal to identity submatrix, while \(U\) is a block upper triangular matrix in \(G_C\).

However, according to basic linear algebra, we know that a nonsingular matrix \(A(q) \in G_C\) have the decomposition of the type in [3] if and only if the principal submatrices \([A^{i,j}(q)]_{0 \leq i,j \leq n-k}\) are nonsingular for all \(0 \leq k \leq n\).

To conclude, by Theorem 2.2 we have that \(\Phi(q) \in N_+\) for any \(q \in T\). Therefore, for any adapted basis \((\zeta_0(q), \cdots, \zeta_{m-1}(q))\), there exists a nonsingular block matrix \(A(q) \in G_C\) with \(\det[A^{i,j}(q)]_{0 \leq i,j \leq n-k} \neq 0\) for any \(0 \leq k \leq n\) such that

\[
(\zeta_0(q), \cdots, \zeta_{m-1}(q)) = (\eta_0, \cdots, \eta_{m-1})A(q).
\]

Let \(\{F^k_q\}_{k=0}^n\) be the reference Hodge filtration at the base point \(p \in T\). For any point \(q \in T^H\) with the corresponding Hodge filtrations \(\{F^k_q\}_{k=0}^n\), we define the following maps

\[
P^k_q : F^k_q \to F^k_p \quad \text{for any} \quad 0 \leq k \leq n
\]

to be the projection map with respect to the Hodge decomposition at the reference point \(p\).

**Lemma 4.4.** For any point \(q \in T^H\) and \(0 \leq k \leq n\), the map \(P^k_q : F^k_q \to F^k_p\) is an isomorphism. Furthermore, \(P^k_q\) depends on \(q\) holomorphically.

**Proof.** We have already fixed \(\{\eta_0, \cdots, \eta_{m-1}\}\) as an adapted basis for the Hodge decomposition of the Hodge structure at the base point \(p\). Thus it is also the adapted basis for the Hodge filtration \(\{F^k_p\}_{k=0}^n\) at the base point. For any point \(q \in T\), let \(\{\zeta_0, \cdots, \zeta_{m-1}\}\) be an adapted basis for the Hodge filtration \(\{F^k_q\}_{k=0}^n\) at \(q\). Let \([A^{i,j}(q)]_{0 \leq i,j \leq n} \in G_C\) be the transition matrix between the bases \(\{\eta_0, \cdots, \eta_{m-1}\}\) and \(\{\zeta_0, \cdots, \zeta_{m-1}\}\) for the same vector space \(H^n(M, \mathbb{C})\). Then we have showed that \([A^{i,j}(q)]_{0 \leq i,j \leq n-k}\) is nonsingular for all \(0 \leq k \leq n\).

On the other hand, the submatrix \([A^{i,j}(q)]_{0 \leq i,j \leq n-k}\) is the transition matrix between the bases of \(F^k_q\) and \(F^k_p\) for any \(0 \leq k \leq n\), that is

\[
(\zeta_0(q), \cdots, \zeta_{f^{k-1}}(q)) = (\eta_0, \cdots, \eta_{m-1})[A^{i,j}(q)]_{0 \leq i,j \leq n-k} \quad \text{for any} \quad 0 \leq k \leq n,
\]

where \((\zeta_0(q), \cdots, \zeta_{f^{k-1}}(q))\) and \((\eta_0, \cdots, \eta_{m-1})\) are the bases for \(F^k_q\) and \(F^k_p\) respectively. Thus the matrix of \(P^k_q\) with respect to \(\{\eta_0, \cdots, \eta_{f^{k-1}}\}\) and \(\{\zeta_0, \cdots, \zeta_{f^{k-1}}\}\) is the first \((n - k + 1) \times (n - k + 1)\) principal submatrix \([A^{i,j}(q)]_{0 \leq i,j \leq n-k}\) of \([A^{i,j}(q)]_{0 \leq i,j \leq n}\). Now since \([A^{i,j}(q)]_{0 \leq i,j \leq n-k}\) for any \(0 \leq k \leq n\) is nonsingular, we conclude that the map \(P^k_q\) is an isomorphism for any \(0 \leq k \leq n\).

From our construction, it is clear that the projection \(P^k_q\) depends on \(q\) holomorphically. \(\square\)
Now we are ready to construct the global holomorphic sections of Hodge bundles over \( \mathcal{T}^H \). For any \( 0 \leq k \leq n \), we know that \( \{ \eta_0, \eta_1, \ldots, \eta_{f^k-1} \} \) is an adapted basis of the Hodge decomposition of \( F^k_p \) for any \( 0 \leq k \leq n \). Then we define the sections

\[
s_i : \mathcal{T}^H \to F^k, \quad q \mapsto (P^{[k]}_q)^{-1}(\eta_i) \in F^k_q \quad \text{for any } 0 \leq i \leq f^k - 1.
\]

Lemma 4.4 implies that \( \{ (P^{[k]}_q)^{-1}(\eta_i) \}_{i=0}^{f^k-1} \) form a basis of \( F^k_q \) for any \( q \in \mathcal{T}^H \). In fact, we have proved the following theorem for polarized and marked Calabi–Yau and Calabi–Yau type manifolds.

**Theorem 4.5.** For all \( 0 \leq k \leq n \), the Hodge bundles \( F^k \) over \( \mathcal{T}^H \) are trivial bundles, and the trivialization can be obtained by \( \{ s_i \}_{0 \leq i \leq f^k - 1} \) which is defined in (9).

**Remark 4.6.** In particular, the section \( s_0 : \mathcal{T}^H \to F^n \) is a global nowhere zero section of the Hodge bundle \( F^n \) for Calabi–Yau manifolds.

By using the local deformation theory for Calabi–Yau manifolds in [34], Todorov constructed a canonical local holomorphic section of the line bundle \( F^n \) over the local deformation space of a Calabi–Yau manifold. In fact, let \( \Omega_p \) be a holomorphic \((n,0)\)-form on the central fiber \( M_p \) of the family. Then there exists a coordinate chart \( \{ U_p, (\tau_1, \ldots, \tau_N) \} \) around the base point \( p \) and a basis \( \{ \varphi_1, \ldots, \varphi_N \} \) of harmonic Beltrami differentials \( H^{0,1}(M_p, T^{1,0}M_p) \), such that

\[
\Omega_p^c(\tau) = e^{\varphi(\tau)} \cdot \Omega_p,
\]

is a family of holomorphic \((n,0)\)-forms over \( U_p \). We can assume this local coordinate chart is the same as the affine coordinates at \( p \) we constructed, which can be achieved simply by taking \( p \) as the base point in our construction in §2 of the affine structure on \( \mathcal{T} \). The Kuranishi family of Beltrami differentials \( \varphi(\tau) \) satisfies the integrability equation which is solvable for Calabi–Yau manifolds by the Tian-Todorov lemma,

\[
\overline{\partial} \varphi(\tau) = \frac{1}{2} [\varphi(\tau), \varphi(\tau)],
\]

and the Taylor expansion \( \varphi(\tau) = \sum_{i=1}^{N} \varphi_i \tau_i + O(|\tau|^2) \) converges for \( |\tau| \) small by classical Kodaira-Spencer theory. We have

**Lemma 4.7.** Let \( \Omega_p^c(\tau) \) be a canonical family defined by (10). Then we have the following section of \( F^n \) over \( U_p \),

\[
[\Omega_p^c(\tau)] = [\Omega_p] + \sum_{i=1}^{N} \tau_i [\varphi_i \cdot \Omega_p] + A(\tau),
\]

where \( \{ [\varphi_i \cdot \Omega_p] \}_{i=1}^{N} \) give a basis of \( H^{n-1,1}(M_p) \), and \( A(\tau) = O(|\tau|^2) \in \bigoplus_{k=2}^{n} H^{n-k,k}(M_p) \) denotes terms of order at least 2 in \( \tau \).

**Proof.** Details of the proof of this lemma can be found in [4] Page 12–14, or [19 Proposition 5.1]. In fact one can directly check the following formula

\[
e^{-\varphi(\tau)} \cdot \left( d(e^{\varphi(\tau)} \cdot \Omega_p) \right) = \overline{\partial} \Omega_p + \partial (\varphi(\tau) \cdot \Omega_p).
\]
The construction of the Kuranishi family \( \varphi(\tau) \) implies that \( \partial(\varphi(\tau)_*\Omega_p) = 0 \), and the fact that \( \Omega_p \) is holomorphic on \( M_p \) implies \( J\Omega_p = 0 \). So the right hand side of formula \((12)\) is equal to 0. Then by replacing the de Rham differential operator \( d \) on the left hand side by \( \partial_\tau + \overline{\partial}_\tau \) on fiber \( M_\tau \), we get \( (\partial_\tau + \overline{\partial}_\tau)(e^{\varphi(\tau)}_*\Omega_p) = 0 \). Note that \( e^{\varphi(\tau)}_*\Omega_p \) is a \((n,0)\) form on \( M_\tau \), and \( \partial_\tau(e^{\varphi(\tau)}_*\Omega_p) = 0 \), we get
\[
\overline{\partial}_\tau(e^{\varphi(\tau)}_*\Omega_p) = 0.
\]
Therefore \( e^{\varphi(\tau)}_*\Omega_p \) is a holomorphic \((n,0)\)-form on the Calabi–Yau manifold \( M_\tau \). The Taylor expansion \((11)\) follows from the corresponding Taylor expansion of \( \varphi(\tau) \).

Using the same notation as in Lemma \ref{lem:18} we are ready to prove the following theorem for Calabi–Yau manifolds,

**Theorem 4.8.** Choose \([\Omega_p] = \eta_0\), then the section \( s_0 \) of \( F^n \) is a global holomorphic extension of the local section \([\Omega_p^c(\tau)]\).

**Proof.** Because both \( \eta_0 \) and \([\Omega_p^c(\tau)]\) are holomorphic sections of \( F^n \), we only need to show that \( s_0|_{U_p} = [\Omega_p^c(\tau)] \). In fact, from the expansion formula \((11)\), we have that for any \( q \in U_p \)
\[
P^n_q([\Omega_p^c(\tau(q))]) = [\Omega_p^c] = \eta_0.
\]
Therefore, \([\Omega_p^c(\tau(q))]) = (P^n_q)^{-1}(\eta_0) = s_0(q) \) for any point \( q \in U_p \).

As an example, if we consider the Teichmüller space of polarized and marked hyperkähler manifolds and the weight two variation of Hodge structure of hyperkähler manifolds, then the Taylor series \((11)\) is a finite degree polynomial and converges globally on \( T^H \). More precisely we have the following

**Example 4.9.** Let \( T^H \) be the Hodge completion of Teichmüller space of a polarized and marked hyperkähler manifold, and \((\tau_1, \cdots, \tau_N) \) be global affine coordinates with respect to the reference point \( p \) and an orthonormal basis \( \{\eta_1, \cdots, \eta_N\} \) of \( H^1_{pr}(M_p) \), then
\[
[\Omega_p^c(\tau)] = [\Omega_p] + \sum_{i=1}^{19} \tau_i \eta_i + \left( \frac{1}{2} \sum_{i=1}^{19} \tau_i^2 \right) [\Omega_p],
\]
is a global holomorphic section of \( F^2 \) over \( T^H \). In fact, in this case, \( T^H \) is bi-holomorphic to \( D \) given by the period map as discussed in Section 4.1, and the affine structure on \( T^H \) is induced from the affine structure on \( D \) by the Harish-Chandra embedding of \( D \) into the complex Euclidean space. The global affine coordinates on \( T^H \) is induced by the Harish-Chandra embedding.

### 4.3. A global splitting property of the Hodge bundles.

In section \ref{sec:12} we proved that the Hodge bundles \( \{F^k\}_{k=0}^n \) over the Hodge metric completion space of the Teichmüller space of polarized and marked Calabi–Yau and Calabi–Yau type manifolds are trivial bundles by directly constructing global trivializations. In particular, because the Hodge bundle \( F^0 \) is a trivial bundle over \( T^H \), we have a trivialization
\[
F^0 = T^H \times H^n(M, \mathbb{C}).
\]

Then for any sub-bundle \( V \subset F^0 \), the fiber \( V_q \) at a point \( q \in T^H \) will be considered as a subspace of \( H^n(M, \mathbb{C}) \), which does not depend on the point \( q \in T^H \).
In this section, we directly construct global defined anti-holomorphic vector bundles $\tilde{F}^k$ over $T^H$, such that for any point $q \in T$, the vector space $H^n(M, \mathbb{C})$ splits as

$$H^n(M, \mathbb{C}) = F^k_p \oplus \overline{F}^k_q$$

for any $q \in T^H$, where $p$ is the base point in $T^H$. Then as an application of our method, in Theorem 4.13 we also prove that any two fibers of the versal family $\mathcal{U} \to T$ can not be conjugate manifolds of each other.

The construction of vector bundles $\tilde{F}^k$ is again based on Lemma 4.10 in fact, we have the following equivalent lemma,

**Lemma 4.10.** For any $q \in T^H$ and $1 \leq k \leq n$, we have that $H^n(M, \mathbb{C}) = F^k_p \oplus \overline{F}^{n-k+1}_q$.

**Proof.** Firstly, the decomposition $H^n(M, \mathbb{C}) = F^k_p \oplus \overline{F}^{n-k+1}_q$ follows from the definition of the Hodge structure for any $0 \leq k \leq n$. Secondly Lemma 4.12 implies that $P^{n-k+1}_q : F^k_p \to F^{n-k+1}_q$ is an isomorphism for any $q \in T^H$ and any $0 \leq k \leq n$. Therefore $F^k_p \cap \overline{F}^{n-k+1}_q = \{0\}$ as the projection from $F^k_p$ to $F^{n-k+1}_q$ is a zero map.

On the other hand, $\dim F^k_p + \dim \overline{F}^{n-k+1}_q = \dim F^k_p + \dim \overline{F}^{n-k+1}_p = \dim H^n(M, \mathbb{C})$, so we have that

$$H^n(M, \mathbb{C}) = F^k_p \oplus \overline{F}^{n-k+1}_q.$$ 

Because the reference point $p$ is an arbitrary prefixed point on $T$, and the Hodge filtration at each point does not depend on the choice of the reference point, Lemma 4.10 actually implies,

**Corollary 4.11.** For any different points $q$ and $q'$ on $T$, and $1 \leq k \leq n$, we have $H^n(M, \mathbb{C}) = F^k_q \oplus \overline{F}^{n-k+1}_{q'}$.

Let us write $\tilde{F}^k = \overline{F}^{n-k+1}$, for each $0 \leq k \leq n$. Then we have proved the following result,

**Theorem 4.12.** The vector bundles $\{\tilde{F}^k\}_{k=0}^n$ are globally defined anti-holomorphic vector bundles over $T^H$ such that

$$H^n(M, \mathbb{C}) = F^k_p \oplus \tilde{F}^k_q$$

for any $q \in T^H$.

Now we let $M$ be a complex manifold with background differential manifold $X$ and complex structure $J : T_X \to T_X$, then the complex conjugate manifold $\overline{M}$ is a complex manifold with the same background differential manifold $X$ and with conjugate complex structure $-J$. In fact, $M$ and its complex conjugate manifold $\overline{M}$ satisfy the relation $T^{1,0}M = T^{0,1}\overline{M}$ and $T^{0,1}M = T^{1,0}\overline{M}$.

Problems regarding deformation inequivalent complex conjugated complex structures have been studied, for example one may find interesting results in [15]. We will apply our results to study such problem for polarized and marked Calabi–Yau manifolds. In fact, another interesting application of Corollary 4.11 is that a polarized and marked Calabi–Yau manifold $M$ can not be connected to its complex conjugate manifold $\overline{M}$ by deformation of complex structure. For any point $q$ in $T$, let $M_q$ denote the fiber of the versal family $\mathcal{U} \to T$ at point $q$. Then we have the following theorem,
Theorem 4.13. If \( q \neq q' \) are two different points on \( T \), then \( M_q \neq \overline{M}_q \).

Proof. We prove this theorem by contradiction. Suppose \( M_q = \overline{M}_q \), and let \( \Omega \) be an holomorphic \((n,0)\) form on \( M_q \), then \( \overline{\Omega} \) is a holomorphic \((n,0)\) form on \( M_q = \overline{M}_q \). Therefore the fibers of Hodge bundles on the two points satisfy \( F^n_q = F^n_{q'} \subset F^1_{q'} \), and

\[
H^n(M, \mathbb{C}) \neq F^n_q \oplus F^1_{q'}.
\]

But this contradicts to Corollary 4.11, so \( M_q \neq \overline{M}_q \) as desired. \( \square \)

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