A Reflection Model with a Radial Disk Density Profile

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Abstract

In this paper we present relxilldograd_nk, a relativistic reflection model in which the electron density of the accretion disk is allowed to have a radial power-law profile. The ionization parameter also has a nonconstant radial profile and is calculated self-consistently from the electron density and the emissivity. We show the impact of the implementation of the electron density gradient in our model by analyzing a NuSTAR spectrum of the Galactic black hole in EXO 1846–031 during its last outburst in 2019 and a putative future observation of the same source with Athena and eXTP. For the NuSTAR spectrum, we find that the new model provides a better fit, but there is no significant difference in the estimation of the model parameters. For the Athena+eXTP simulation, we find that a model without a disk density profile is unsuitable to test the spacetime metric around the compact object in the sense that modeling uncertainties can incorrectly lead to finding a nonvanishing deformation from the Kerr solution.

Unified Astronomy Thesaurus concepts: Astrophysical black holes (98); Kerr black holes (886); Accretion (14); X-ray astronomy (1810); Astronomical models (86)

1. Introduction

Blurred reflection features observed in the X-ray spectra of Galactic black holes and active galactic nuclei are thought to be produced by illumination of a cold accretion disk by a hot corona (Fabian et al. 1989; Reynolds & Nowak 2003; Risaliti et al. 2013; Bambi et al. 2021). For geometrically thin and optically thick accretion disks, the gas is in local thermal equilibrium and every point on the disk has a blackbody-like spectrum. The whole disk has a multitemperature blackbody-like spectrum, which is normally peaked in the soft X-ray band for stellar-mass black holes and in the UV band for the supermassive ones. The “corona” is some hotter gas near the black hole and the inner part of the accretion disk, but its exact morphology is not yet well understood. Thermal photons of the disk can thus inverse Compton scatter off free electrons in the corona. This process generates a continuum, which can often be approximated by a power-law spectrum with an exponential high-energy cutoff. The Comptonized photons can illuminate the disk: Compton scattering and absorption followed by fluorescent emission generate the reflection spectrum.

The reflection spectrum in the rest frame of the gas in the disk is characterized by narrow fluorescent emission lines below 7 keV, notably the iron Kα complex at 6.4 keV for neutral or weakly ionized iron and up to 6.97 keV for H-like iron ions, and the Compton hump peaked at 20–30 keV (Ross & Fabian 2005; García & Kallman 2010). The reflection photons detected far from the source are affected by relativistic effects during their propagation in the strong gravity region around the black hole (Fabian et al. 1989; Laor 1991; Dauser et al. 2010; Bambi 2017b). As a result, in the observed reflection spectrum of the source the originally narrow fluorescent emission lines become broadened and skewed.

Since the reflection spectrum is mainly generated from the inner part of the accretion disk, the analysis of the reflection features in the spectra of accreting black holes can be used to study the properties of the accreting material around the compact object, measure black hole spins, and even test fundamental physics in the strong gravity regime (Brenneman 2013; Reynolds 2014; Bambi 2017a; Bambi et al. 2021).

In the past ~10 yr, there have been significant efforts to develop models for the analysis of the observed relativistic reflection features in the spectra of accreting black holes and to understand the systematics affecting the measurements of the properties of these systems (see, e.g., Bambi et al. 2021, and references therein). As of now, there are four main relativistic reflection models used by the X-ray astronomy community: kyn (Dovčiak et al. 2004), reflkerr (Niedźwiecki & Życki 2008; Niedźwiecki et al. 2019), relxill (Dauser et al. 2013; García et al. 2014), and the more recent reltrans (Ingram et al. 2019). All these models employ, even if at different levels, the xillver model (García & Kallman 2010; García et al. 2013) for the calculation of the reflection spectrum in the rest frame of the gas in the disk and are today the most accurate models used in the analysis of relativistic reflection features. Another popular choice is the nonrelativistic reflection model relfionx (Ross & Fabian 2005) together with a relativistic convolution model, but in such a case it is not possible to account for the distribution of different emission angles (García et al. 2014; Tripathi et al. 2020).

Despite the significant advancements, there are still important simplifications in the current relativistic reflection models, and work is in progress to remove those simplifications affecting the final accuracy of the measurement of the properties of accreting black holes. relxill_nk (Bambi et al. 2017; Abdikamalov et al. 2019, 2020) is an extension of the relxill model to non-Kerr spacetimes and specifically designed to test the nature of accreting black holes (Cao et al. 2020).
2018; Zhang et al. 2019; Tripathi et al. 2019a, 2019b, 2021a, 2021b, 2021c). In this paper, we present a new flavor of relxill_nk. It is called relxilldgrad_nk and provides the possibility of modeling the electron density of the disk, \( n \), with a radial power-law profile. The ionization parameter of the disk, \( \xi \), defined as

\[
\xi = \frac{4\pi F_X}{n},
\]

where \( F_X \) is the X-ray flux illuminating the disk, is calculated self-consistently in relxilldgrad_nk from the local values of \( n \) and of the emissivity \( \epsilon \), which can be approximated to be proportional to \( F_X \).

With the exception of the last version of reltrans (Mastroserio et al. 2021) and a currently nonpublic version of relxill, all other reflection models assume that the electron density is constant over the whole disk. We note the importance of implementing an electron density gradient into our relativistic reflection models. There is indeed increasing evidence of the necessity of employing a reflection model with a radial ionization profile in the analysis of reflection features (see, e.g., Svoboda et al. 2012; Kammoun et al. 2019; Shreeram & Ingram 2020; Abdikamalov et al. 2021). However, a model with independent emissivity and ionization profiles is not self-consistent, because the two quantities are not independent but connected by Equation (1).

In relxilldgrad_nk we have thus two parameters describing the electron density profile, namely, the electron density at the inner edge of the disk and the power-law index of the density gradient, and one parameter for the ionization of the disk, which is the maximum value of the ionization parameter. The ionization profile is then calculated using Equation (1). We show the importance of this implementation by analyzing a NuSTAR observation of the black hole binary EXO 1846–031, as an example of an observation possible today, and a simulated observation of the same source with Athena and eXTP, to understand the impact of the disk density gradient for observations with the next generation of X-ray missions.

Our paper is organized as follows. In Section 2, we describe relxilldgrad_nk. In Section 3, we fit the NuSTAR spectrum of the black hole binary EXO 1846–031 with three different models to show the impact of the electron density gradient on the measurement of the properties of the system, and then we repeat the same analysis with a simulated observation of Athena and eXTP. We discuss our results in Section 4.

2. Radial Disk Density Profile

Up to very recently, relativistic reflection models employed an accretion disk with constant electron density and ionization parameter. Indeed, old attempts to add a radial ionization profile showed that observational data did not require any ionization gradient (T. Dauser & J. Garcia 2021, private communication), which was interpreted as due to the fact that the corona illuminated a small portion of the inner part of the accretion disk that could be approximated well by a one-zone ionization. Such an approximation was questioned in Svoboda et al. (2012) and, more recently, in Kammoun et al. (2019). The possibility of a nonconstant ionization parameter has been thus implemented in most reflection models and its relevance in spectral fitting with current data was shown in a few studies (see, e.g., Shreeram & Ingram 2020; Abdikamalov et al. 2021).

In relxill_nk, we have recently added the model relxillion_nk, which permits the user to model the radial ionization profile with a power law (Abdikamalov et al. 2021). The model has an extra parameter, the ionization gradient index \( \alpha \), and the ionization parameter is

\[
\xi(r) = \xi_{\text{in}} \left( \frac{R_{\text{in}}}{r} \right)^{\alpha},
\]

where \( \xi_{\text{in}} \) is the value of the ionization parameter at the inner edge of the accretion disk \( R_{\text{in}} \). However, a similar model is not self-consistent because one infers the ionization and the emissivity profiles from the fit as two independent quantities, while \( \xi \) and \( \epsilon \) are connected by Equation (1), where there is also the electron density \( n \). The same criticism applies to the radial ionization profiles of the other reflection models used in the literature to date. In this paper, we thus present relxilldgrad_nk, which is a new flavor of the relxill_nk package to solve this issue and to have a self-consistent model. relxilldgrad_nk employs the grid of the nonrelativistic reflection model xillverD, and the disk electron density profile is described by a power law

\[
n(r) = n_{\text{in}} \left( \frac{R_{\text{in}}}{r} \right)^{\alpha_n},
\]

where \( R_{\text{in}} \) is still the radial coordinate of the inner edge of the accretion disk, \( n_{\text{in}} \) is the value of the electron density at \( R_{\text{in}} \), and \( \alpha_n \) is the power-law index of the density gradient. Setting \( \alpha_n = 0 \), we recover the standard constant density profile, while for \( \alpha_n > 0 \) we have an accretion disk in which the electron density decreases as the radial coordinate \( r \) increases.

The ionization parameter at any point of the disk, \( \xi(r) \), is calculated self-consistently from the values of the electron density and the emissivity at that point of the disk as

\[
\xi(r) = \xi_{\text{max}} \left[ \frac{4\pi \epsilon(r)}{n(r)} \right]^{\alpha_n} \left( \frac{R_{\text{in}}}{r} \right)^{-\alpha_n},
\]

where \( \xi_{\text{max}} \) and \( \epsilon(r) \) are, respectively, the maximum value of the ionization parameter over the disk and the emissivity at the radial coordinate \( r \). Note the expression in square brackets on the right-hand side of Equation (4) is normalized with respect to its maximum value reached on the accretion disk. Depending on \( \alpha_n \), \( n(r) \), and \( \epsilon(r) \), the function \( \xi(r) \) may not have the maximum at \( R_{\text{in}} \).

In the calculation process, the emissivity profile for the entire disk is calculated first. Then, we divide the accretion disk into 50 annuli, and the values of the electron density and of the ionization parameter are calculated in the middle of each annulus using the already calculated emissivity profile values. Then, in each annulus, the reflection spectrum is extracted from the xillverD table according to the values of the density and ionization parameters of this annulus. Finally, the reflection components from each annulus are convolved using the transfer functions corresponding to that annulus to obtain the reflection spectrum far away from the source, and the total spectrum is obtained by summing all the convolved spectra together.

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4 Assuming a pointlike corona with a power-law spectrum, one finds \( F_X(r) \propto \left[ \frac{g(r)}{r} \right]^2 \epsilon(r) \) (see the Appendix for its derivation), where \( g = E_{\text{in}}/E_{\text{c}} \) is the redshift experienced by photons when they travel from the corona to the disk and \( \Gamma \) is the photon index of the power-law spectrum. If we do not specify the coronal geometry, we cannot calculate \( g(r) \) and we can only make the approximation \( F_X(r) \propto \epsilon(r) \).
In this section we show the impact of a radial electron density gradient on the reflection spectrum of a source. In all plots the inner edge of the accretion disk is assumed to be at the innermost stable circular orbit (ISCO), so \( R_{\text{in}} = R_{\text{ISCO}} \) in Equation (3). The constant electron density \( \alpha_n = 0 \) (blue curves) is compared with the spectra calculated for the density index \( \alpha_n = 1 \) (orange curves) and \( \alpha_n = 2 \) (black curves) for various values of the black hole spin parameter \( a_\ast \), the deformation parameter \( \alpha_1 \) of the Johannsen spacetime (which is the default metric in relxill\_nk and was proposed in Johannsen 2013, specifically for testing the Kerr metric with black hole electromagnetic data), the inclination angle \( \iota \), the electron density at the inner edge of the disk \( n_\text{in} \), and the maximum value of the ionization parameter \( \xi_{\text{max}} \). The value of \( \alpha_n \) has a larger impact on the shape of the spectrum below the iron line because that is the region of the fluorescent emission lines, and the energy and the strength of these lines are quite sensitive to the ionization parameter. Figure 5 shows the profiles of the electron density and of the ionization parameter for the systems in Figure 2 \( (a_\ast = 0.998, \; \log n_\text{in} = 18, \; \text{and} \; \log \xi_{\text{max}} = 3.1) \).  

### 3. Implications on Spectral Fitting

In this section we show the impact of a radial electron density profile on spectral fitting. First, we fit a NuSTAR spectrum of the Galactic black hole EXO 1846–031. Then, we simulate a simultaneous observation of the same source with Athena and eXTP.

#### 3.1. NuSTAR Spectrum of EXO 1846–031

EXO 1846–031 was discovered by EXOSAT on 1985 April 3 (Parmar & White 1985) and was later identified as a low-mass X-ray binary harboring a black hole candidate (Parmar et al. 1993). A second outburst was observed in 1994 (Zhang et al. 1994). A third and, to date, last outburst was first detected with MAXI in 2019 July (Negoro et al. 2019), and the source was then observed with several other instruments. NuSTAR observed EXO 1846–031 on 2019 August 3 (Observation ID 90501340002) and in what follows we analyze such data. Draghis et al. (2020) were the first to analyze this observation, and here we will follow their analysis. We note that the source was very bright during this observation and that previous studies found that the black hole has a spin parameter very close to 1 and that the viewing angle of the accretion disk is very high, two ingredients that maximize relativistic effects. This NuSTAR spectrum is quite simple, with a very broadened and prominent iron line. All these properties make this NuSTAR observation of EXO 1846–031 particularly suitable for testing fundamental physics from the analysis of the reflection features as well as for testing new reflection models. This observation was analyzed with relxill\_nk to test the Kerr metric in Tripathi et al. (2021b).

For the analysis, we use XSPEC v12.11.1 (Arnaud 1996), WILMS abundances (Wilms et al. 2000), and VERNER cross sections (Verner et al. 1996). The NuSTAR observation is 22 ks. The flux of the source does not show significant variability, and we can thus use the time-averaged spectrum for...
the spectral analysis. The unprocessed raw data are downloaded from the HEASARC website. HEASOFT module nupipeline is used to process the unfiltered data into cleaned event data. A circular region of 180° is taken around the center of the source to extract the spectrum. A background region of the same size is taken as far as possible from the source on the same detector. For the extraction of source and background spectra, another HEASOFT module, nuproducts, is used. The same module is used to generate the response matrix file and the ancillary file. We rebin the source spectra to have at least 30 counts per bin in order to use $\chi^2$ statistics.

If we fit the data with an absorbed power law, tbabsxpowerlaw in XSPEC language, we find the ratio between the data and the best-fit model shown in Figure 6. We clearly see a prominent and broadened iron line in the soft X-ray band and a Compton hump peaked at 20–30 keV, indicating that the spectrum has strong reflection features.

We fit the data with three reflection models: normal relxill_nk with constant ionization parameter and electron density over the whole disk, relxillion_nk with a radial ionization profile and a constant electron density, and relxilldgrad_nk with a radial electron density profile and a self-consistent ionization profile calculated from Equation (4). We note the key differences among the three models. relxill_nk and relxillion_nk use the xillver table, so the electron density is fixed to $10^{15}$ cm$^{-3}$ and the high-energy cutoff in the coronal spectrum $E_{\text{cut}}$ is free. When the ionization gradient index $\alpha_q$ in relxill_nk is set to zero, relxillion_nk reduces exactly to relxill_nk. On the other hand, relxilldgrad_nk uses the table of xillverD, where the electron density is free and the high-energy cutoff $E_{\text{cut}}$ is fixed to 300 keV. So relxilldgrad_nk reduces exactly to relxill_nk only when its density gradient index $\alpha_q$ is set to zero and, at the same time, we set $E_{\text{cut}} = 300$ keV in relxill_nk. In the three fits, we model the emissivity profile with a broken power law (three free parameters: the inner emissivity index $q_{\text{in}}$, the outer emissivity index $q_{\text{out}}$, and the breaking radius $R_B$), and we impose that the inner edge of the disk $R_{\text{in}}$ is at the ISCO radius. The black hole spin parameter $a_*$, the inclination angle of the disk $i$, the photon index $\Gamma$ and the high-energy cutoff $E_{\text{cut}}$ of the coronal spectrum, the iron abundance $A_{Fe}$, and the deformation parameter of the spacetime $\alpha_{13}$ are always free in the fits.

When we use relxill_nk, the XSPEC model is tbabsx(diskbb + relxill_nk + Gaussian).

This is the same fit as in Draghis et al. (2020), with the difference that Draghis et al. (2020) used relxill that assumes the Kerr metric and here we use relxill_nk with a free deformation parameter $\alpha_{13}$. diskbb is introduced to fit a weak thermal component of the accretion disk (Mitsuda et al. 1984), and the temperature of the inner edge of the disk and the normalization are free in the fit. Gaussian is necessary to fit an unresolved feature (but it only improves the fit, without any significant impact on the estimate of the model parameters). The width of the Gaussian and the reflection fraction $R_f$ in relxill_nk cannot be constrained simultaneously. In our fit, we allow the width of the Gaussian to vary in the range 1–100 eV (the best fit is around 70 eV, but the parameter is still unconstrained) and the reflection fraction is frozen to 1.
Interestingly, when we use relxillion\textsubscript{nk} and relxilldgrad\textsubscript{nk}, we do not need Gaussian and we can leave the reflection fraction free, as the fit itself requires a reflection fraction close to 1. Since relxilldgrad\textsubscript{nk} uses the table of xillverD, which is for a variable electron density \( n \) but assumes a constant high-energy cutoff in the coronal spectrum \( (E_{\text{cut}} = 300 \text{ keV}) \) in order to limit the size of the table, when we use relxilldgrad\textsubscript{nk} we freeze its reflection fraction to \(-1\) (no coronal spectrum in the output) and we add cutoffpl to describe the continuum and have a variable high-energy cutoff. In the end, the two XSPEC models are, respectively,

\begin{align*}
\text{tbabs} \times (\text{diskbb} & + \text{relxillion\textsubscript{nk}}), \\
\text{tbabs} \times (\text{diskbb} & + \text{relxilldgrad\textsubscript{nk}} + \text{cutoffpl}).
\end{align*}

The best-fit values of the three cases are reported in Table 1, left side. The data to best-fit model ratios are shown in the left panel of Figure 7. Figure 8 shows the constraints on the black hole spin parameter \( a_* \) and on the deformation parameter \( \alpha_{13} \) inferred from the three fits. We postpone the discussion of these results to Section 4.

### 3.2. Simulation with Athena and eXTP

In order to see the differences of the three models in the previous subsection for the next generation of X-ray missions, we simulate a simultaneous observation with Athena (Nandra et al. 2013) and eXTP (Zhang et al. 2016) of EXO 1846–031. We assume a 20 ks observation with the X-ray Integral Field Unit (X-IFU) for Athena and with the Large Area Detector (LAD) for eXTP. The best-fit values of the relxilldgrad\textsubscript{nk} model found in the previous subsection are used as input for the simulation, with the exception of the deformation parameter that is assumed to vanish: \( \alpha_{13} = 0 \) (Kerr metric). With such a prescription, we obtain about \( 3 \times 10^7 \) and \( 3 \times 10^8 \) counts for X-IFU and LAD, respectively. We fit the simulated spectra with the reflection models relxill\textsubscript{nk}, relxillion\textsubscript{nk}, and relxilldgrad\textsubscript{nk}. We note that now, when we use relxill\textsubscript{nk}, we do not need Gaussian and we can leave the reflection fraction free, as for the other two fits. The best-fit parameters and residuals for the simulation are shown, respectively, in Table 1, right side, and Figure 7, right panel. We do not show here the confidence level curves on the plane black hole spin parameter \( a_* \) versus deformation parameter \( \alpha_{13} \) because the 68%, 90%, and 99% confidence regions are too small, but the constraints on \( a_* \) and \( \alpha_{13} \) can be seen from Table 1. The discussion of these results is presented in the next section.

### 4. Discussion and Conclusions

In this paper, we have presented relxilldgrad\textsubscript{nk}, which is an extension of the relxill\textsubscript{nk} package in which the disk electron density profile is described by a power law and the ionization parameter is calculated self-consistently from the electron density and the emissivity. In order to see the impact of this implementation on the fits of reflection dominated spectra, in the previous section we have used relxill\textsubscript{nk}, relxillion\textsubscript{nk}, and relxilldgrad\textsubscript{nk} to fit a NuSTAR observation of EXO 1846–031 (as an example of current X-ray observation) and the simulated simultaneous observation Athena+eXTP of the same source (as the

![Figure 3. As in Figure 2, but for log \( n_{\text{in}} \) = 19.](image-url)
Concerning the fits of the NuSTAR spectrum, from the ratio plots (left panel in Figure 7), we do not see clear differences among the three models. If we consider the value of $\chi^2$, we see that the $\chi^2$ of the fit with relxill_nk is a bit higher than the other two. However, when we use relxill_nk we need to add a Gaussian and to freeze the reflection fraction to 1, while with the other two models these issues are not present: the Gaussian is not necessary and the data require a reflection fraction close to 1 (in the fit with relxilldgrad_nk, we do not have the reflection fraction, but we can estimate this quantity from the fluxes in the reflection and power-law components). In the end, we can argue that relxillion_nk and relxilldgrad_nk provide a better description of these data. If we compare relxillion_nk and relxilldgrad_nk, we see that the $\chi^2$ of relxillion_nk is slightly lower than that of relxilldgrad_nk. On the other hand, the fit with relxillion_nk recovers an unnaturally high iron abundance, suggesting a problem in the fit. We also note a discrepancy in the estimate of $E_{\text{cut}}$ between the two models. It is remarkable that the estimate of the black hole spin parameter, $a_*$, and of the Johannsen deformation parameter, $\alpha_{13}$, is consistent among the three models. In other words, the measurement of these two parameters does not seem to be affected by the profiles of the ionization parameter and of the electron density, at least for these high-quality NuSTAR data.

From the simulated observation of Athena+eXTP, we can learn something more. First, from the ratio plots in the right panel of Figure 7 we clearly see that relxill_nk is unsuitable to fit the data and there are important residuals in the iron line region. On the contrary, relxillion_nk can fit the data well. These conclusions are confirmed by the values of the reduced $\chi^2$. relxill_nk cannot fit these simulated data well, while both relxillion_nk and relxilldgrad_nk can (but in the case of relxilldgrad_nk it was expected because the simulation uses relxilldgrad_nk). This is quite an important point: even the high-quality data of Athena+eXTP may not be able to discriminate the two models from the quality of the fits.\(^5\) We still see the difference in the estimate of $E_{\text{cut}}$, as in the case of the NuSTAR spectrum. Last but not least, we see that the estimates of the black hole spin parameter are not very different among the three fits (so reliable spin measurements may still be obtained with a reflection model with a constant ionization parameter and electron density), while we find that neither relxill_nk nor relxillion_nk can get an accurate measurement of the deformation parameter $\alpha_{13}$. This is the most important finding of the present work. Since relxillion_nk provides a good fit, it may be challenging to test the Kerr metric with future X-ray data without additional constraints (e.g., study of the timing properties; see Ingram et al. 2019). Figure 9 shows the $\chi^2$ confidence level curves on the plane $\alpha_{13}$ versus $\alpha_n$ when we fit the simulated data with relxilldgrad_nk and we do not see any clear correlation between the two parameters. For a

\(^5\) Such a statement is based on the spectral analysis of our study. However, Athena and especially eXTP data will be excellent for timing analyses and modeling timing properties may well be able to break this degeneracy. For instance, Ingram et al. (2019) show that models with a constant ionization parameter and with an ionization gradient can have similar spectra but they can be distinguished from their different variability properties.
conclusive result on the possibility of testing the Kerr metric with Athena and eXTP, we presumably need a full model that has a density gradient and has $E_{\text{cut}}$ as a free parameter.

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Figure 5. Profiles of the electron density $n$ and of the ionization parameter $\xi$ for the astrophysical systems considered in Figure 2.

Appendix

Relation between $\epsilon$ and $F_X$

For simplicity, we assume that the corona is pointlike and that the photon number flux at the emission point in the corona is described by a power law

$$\frac{dN_{\epsilon}}{dt_{\epsilon}dE_{\epsilon}} = \begin{cases} K E_{\epsilon}^{-\Gamma} & \text{for } E_{\text{min}} < E < E_{\text{max}} \\ 0 & \text{otherwise} \end{cases},$$

(A1)

where $K$ is a constant, $\Gamma$ is the photon index, and the subindex $c$ reminds us that these quantities are evaluated at the emission point in the corona. The relation $F_X \propto g^{2-\Gamma} \epsilon$, where $g$ is the redshift experienced by photons when they travel from the corona to the disk, results from the fact that $\epsilon$ is defined as a specific flux while $F_X$ is a flux integrated over an energy range.

If we think of calculating $\epsilon$ and $F_X$ numerically, we can divide the accretion disk into annuli (radial bins). The annulus $i$ will have radial coordinate $r_i$ and width $\Delta r_i$. We can then fire photons from the pointlike corona to the disk and infer the ray number per radial bin in the disk, say, $N_i$. The ray number density of the radial bin $i$ is

$$n_i = \frac{N_i}{A_i \gamma_i},$$

(A2)

where $A_i$ and $\gamma_i$ are, respectively, the area of the radial bin $i$ and the Lorentz factor of the particles in the radial bin $i$ measured by the distant observer. The photon number is conserved, so $N = K E^{-\Gamma} \Delta t \Delta E$ is a constant along the photon path and can be associated with the number of photons for every ray. The
Table 1
Summary of the Analysis of the 2019 NuSTAR Observation of EXO 1846–031 and of the Athena + eXTP Simulation

| Model                  | NuSTAR | Athena + eXTP |
|------------------------|--------|---------------|
| relxill_nk             |        |               |
| $N_{\text{H}}$ [10^{22} cm^{-2}] | 10.8$^{+0.7}_{-0.5}$ | 8.4$^{+0.6}_{-0.4}$ |
| $kT_{\text{in}}$ [keV] | 0.434$^{+0.009}_{-0.010}$ | 0.387$^{+0.021}_{-0.027}$ |
| $\log E_{\text{cut}}$ [keV] | 3.518$^{+0.023}_{-0.027}$ | 3.298$^{+0.004}_{-0.005}$ |
| $\log \xi_{\text{min}}$ [erg cm s^{-1}] | 15$^*$ | 15$^*$ |
| $\alpha_q$             | 0.21  | 0.09          |
| $\Delta E_{\text{Fe}}$ | 8.35$^{+0.014}_{-0.013}$ | 1.03$^{+0.003}_{-0.004}$ |
| $R_0$                  | 1.92$^{+0.004}_{-0.005}$ | 2.045$^{+0.003}_{-0.005}$ |
| $\Gamma_k$             | 1.992 | 241.8$^{+11.6}_{-10.8}$ |
| Gaussian               |        |               |
| $E_{\text{line}}$ [keV] | 7.02$^{+0.05}_{-0.03}$ | 1.04939         |
| $\chi^2$/dof           | 2754.13/2594 | 23454.58/22844 |

Note. The reported uncertainties correspond to the 90% confidence level for one relevant parameter ($\Delta \chi^2 = 2.71$). $^*$ indicates that the value of the parameter is frozen in the fit. (P) means that the 90% confidence level reaches the upper/lower boundary of the parameter. If there is no upper/lower uncertainty, the best-fit value is stuck at the upper/lower boundary of the parameter. $q_{\text{in}}$ and $q_{\text{out}}$ are allowed to vary in the range 0 to 10. The maximum value of $a_\alpha$ is 0.998. The maximum value of log $\xi_{\text{min}}$ is 4.7.

Figure 7. Data to model ratio for the analysis of NuSTAR data (left panel; black crosses for FPMA and red crosses for FPMB) and the simulated observation with Athena + eXTP (right panel; red crosses for Athena and black crosses for eXTP) of EXO 1846–031.
energy density illuminating the disk at the radial bin $i$ is

$$
\mathcal{E}_i = E_d N n_i = E_d (KE_c^{-\Gamma} \Delta t_c \Delta E_c) n_i = g_i \Gamma KE_d^{-\Gamma+1} \Delta t_d \Delta E_d n_i, \tag{A3}
$$

where $g_i = \Delta t_c / \Delta t_d = E_d / E_c$ is the redshift factor of the radial bin $i$ between the emission point in the corona and the incident point in the disk and the subindex $d$ is used for the quantities on the disk. The emissivity profile of the radial bin $i$ can be written as

$$
\varepsilon_i \propto \frac{\Delta t_c}{\Delta E_d} = g_i \Gamma KE_d^{-\Gamma+1} n_i, \tag{A4}
$$

For the flux illuminating the disk, $F_x$, we can proceed as above and get the energy density illuminating the disk at the radial bin $i$, Equation (A3). If we integrate over the energy

$$
\int_{E_1}^{E_2} g_i \Gamma KE_d^{-\Gamma+1} \Delta t_d dE_d n_i.
$$

Note that $E_1 = g_i E_{\text{min}}$ and $E_2 = g_i E_{\text{max}}$ and Equation (A5) becomes

$$
\int_{E_1}^{E_2} g_i \Gamma KE_d^{-\Gamma+1} \Delta t_d dE_d n_i = g_i^2 \Delta t_d \Delta t_c n_i \left[ E_{\text{max}}^{2-\Gamma} - E_{\text{min}}^{2-\Gamma} \right]. \tag{A6}
$$

For the flux $F_X$, we divide the above expression by $\Delta t_d$ and we find (omitting the subindex $i$)

$$
F_X \propto g_i z^{-\Gamma} \varepsilon. \tag{A7}
$$

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**Figure 8.** Constraints on the black hole spin parameter $a_*$ and on the deformation parameter $\alpha_{13}$ obtained with rexlill_nk, relxillion_nk, and relxillgrad_nk from the analysis of the 2019 NuSTAR data of EXO 1846–031. The red, green, and blue curves correspond, respectively, to the 68%, 90%, and 99% confidence levels for two relevant parameters. The gray region is not included in our analysis because the spacetime is not regular there (see the discussion in Riaz et al. 2020 for more details).

**Figure 9.** Study of the possible correlation between the deformation parameter $\alpha_{13}$ and the gradient density index $\alpha_n$ from the simulated observation Athena +eXTP when we fit the data with relxillgrad_nk. The red, green, and blue curves represent, respectively, the 68%, 90%, and 99% confidence level limits for two relevant parameters.
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