Chromomagnetic Catalysis of Chiral Symmetry Breaking and Color Superconductivity *

D. Ebert \(^b\), V.V. Khudyakov \(^*\), K. G. Klimenko \(^c\), H. Toki \(^b\), and V. Ch. Zhukovsky \(^*\)

\(^b\) Research Center for Nuclear Physics (RCNP), Osaka University, Ibaraki, Osaka 567, Japan
\(^c\) Institut für Physik, Humboldt-Universität zu Berlin, 10115 Berlin, Germany
\(^*\) Faculty of Physics, Department of Theoretical Physics, Moscow State University, 119899, Moscow, Russia

It is shown in the framework of an extended NJL model with two flavors that some types of external chromomagnetic field induce the dynamical chiral or color symmetry breaking even at weakest attraction between quarks. It is argued also that an external chromomagnetic field, simulating the chromomagnetic gluon condensate of the real QCD-vacuum, might significantly influence the color superconductivity formation.

I. INTRODUCTION

At the beginning of the last decade an exciting property of the homogeneous external magnetic field to dynamically generate the chiral symmetry breaking (CSB) even at the weakest attractive forces between fermions has been discovered \([1] - [7]\). Now it is well-known as magnetic catalysis effect (MCE).

First particular observations of MCE were done in \([1]\) on the basis of a \((2+1)\)-dimensional model with four fermion interaction. Then, it was shown that in 3D this effect is a model independent one, and the explanation for MCE was given in the framework of a dimensional reduction mechanism \([2]\). The investigation of MCE under the influence of different external factors and for the case of \((3+1)\)-dimensional models are given in \([3]\). Besides, this phenomenon finds interesting applications in cosmology \([5]\) and condensed matter physics \([6]\) (see also the reviews \([1]\) and references therein).

Later a similar property of the homogeneous external chromomagnetic field to dynamically generate the CSB has been found as well \([8] - [10]\). The physical essence of this effect is again the effective reduction of the space-time dimensionality in the presence of external chromomagnetic fields \([10]\). Recently, it was also shown in the framework of a Nambu – Jona-Lasinio (NJL) model that some types of chromomagnetic fields might induce the dynamical breaking of the color symmetry, thus catalyzing the appearance of color superconductivity \([11] - [13]\).

In accordance with modern knowledge, the QCD vacuum at low temperature and density is characterized by the confinement phenomenon, i.e. quarks and gluons are not observed, since they are confined into hadrons, and the color symmetry is not broken. Two nonperturbative features are inherent to the QCD vacuum in this phase. One is the nonzero value of the gluon condensate \(\langle FF\rangle \equiv \langle F_{\mu\nu}^a F^{a\mu\nu} \rangle\), where \(F_{\mu\nu}^a\) is the field strength tensor of the gluon fields. Another one is the nonzero chiral condensate \(\langle \bar{q}q \rangle\) which signals about CSB. At high temperatures the quark-gluon plasma phase is expected to exist. In this phase all symmetries of the QCD Lagrangian are restored, and quarks and gluons are elementary excitations of the theory. It was realized more than twenty years ago \([14]\), that at high densities (high values of the chemical potential \(\mu\)) the color superconducting (CSC) phase might exist. The CSC-vacuum is generated by the condensation of quark Cooper pairs, i.e. the vacuum expectation value of diquarks \(\langle qq \rangle\) is nonzero. Since quark Cooper pairing occurs in the color anti-triplet channel, the nonzero value of \(\langle qq \rangle\) means that, apart from the electromagnetic \(U(1)\) symmetry, the color \(SU_c(3)\) should be spontaneously broken down inside the CSC phase as well.

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The CSC phenomenon was investigated in the framework of the one-gluon exchange approximation in QCD \cite{15}, where the colored Cooper pair formation is predicted selfconsistently at extremely high values of the chemical potential \( \mu > \sim 10^9 \text{ MeV} \) \cite{16}. Unfortunately, such baryon densities are not observable in nature and not accessible in experiments (the typical densities inside the neutron stars or in the future heavy ion experiments correspond to \( \mu \sim 500 \text{ MeV} \)). The possibility for the existence of the CSC phase in the region of moderate densities was proved quite recently (see e.g. the papers \cite{17}—\cite{20} as well as the review article \cite{21} and references therein). In these papers it was shown on the basis of different effective theories for low energy QCD (instanton model, NJL model etc) that the diquark condensate \( \langle qq \rangle \) can appear already at a rather moderate baryon density \( \langle \mu \sim 400 \text{ MeV} \rangle \), which can possibly be detected in the future experiments on heavy ion-ion collisions.

In the framework of NJL models the CSC phase formation has generally been considered as a dynamical competition between diquark \( \langle qq \rangle \) and usual quark-antiquark condensation \( \langle \bar{q}q \rangle \). However, the real QCD vacuum is characterized in addition by the appearence of a gluon condensate \( \langle FF \rangle \) as well, which might change the generally accepted conditions for the CSC observation. As an effective theory for low energy QCD, the NJL model does not contain any dynamical gluon fields. As a consequence, the nonzero value of \( \langle FF \rangle \) cannot be generated dynamically in this scheme, but it can be mimicked with the help of external chromomagnetic fields. In particular, for a QCD-motivated NJL model with gluon condensate (i.e. in the presence of an external chromomagnetic field) and finite temperature, it was shown that a weak gluon condensate plays a stabilizing role for the behavior of the constituent quark mass, the quark condensate, meson masses and coupling constants for varying temperature \cite{22}.

The aim of the present talk is to discuss the influence of external conditions, such as the chemical potential and especially the gluon condensate (as modelled by external color gauge fields), on the phase structure of quark matter with particular emphasize of its CSC phase. To this end, we have extended our earlier analysis of the chromomagnetic generation of CSC at \( \mu = 0 \) \cite{11} to the case of a \((3+1)\)-dimensional NJL type model with finite chromomagnetic field and chemical potential presenting a generalization of the zero external field model of \cite{19}.

The talk is organized as follows. In Section II the extended NJL model under consideration is presented, and its effective potential (\( \equiv \) thermodynamic potential) at nonzero external chromomagnetic field and chemical potential is presented in the one-loop approximation. This quantity contains all the necessary informations about the quark and diquark condensates of the theory. It is well-known that the chemical potential is a factor, which promotes the generation of CSC. We argue that an external chromomagnetic field is another factor with similar properties. In order to prove this statement we first consider in Section III the simpler case with zero chemical potential. It is shown here that some types of the external chromomagnetic field can induce the transitions to the CSB or CSC phases even at weakest quark interaction (depending on the relation between couplings in \( \bar{q}q \) and \( qq \) channels). The combined influence of both chemical potential and external chromomagnetic field on the generation of \( \langle \bar{q}q \rangle \) and \( \langle qq \rangle \) condensates at physically meaningful values of coupling constants is considered in Section IV. It is shown there that the characteristics of the CSC phase significantly depend on the strength of the chromomagnetic field. Finally, section V contains a summary and discussion of the results.

II. THE MODEL AND THE EFFECTIVE POTENTIAL

Let us first give several (very approximative) arguments motivating the chosen structure of our QCD-inspired extended NJL model introduced below. For this aim, consider two-flavor QCD with nonzero chemical potential and color group \( SU_c(3) \) and decompose the gluon field
\( \mathcal{A}_a^\mu(x) \) into a condensate background (“external”) field \( A_\mu^a(x) \) and the quantum fluctuation \( a_\mu^a(x) \) around it, i.e. \( \mathcal{A}_a^\mu(x) = A_\mu^a(x) + a_\mu^a(x) \). By integrating in the generating functional of QCD over the quantum field \( a_\mu^a(x) \) and further “approximating” the nonperturbative gluon propagator by a \( \delta \)–function, one arrives at an effective local chiral four-quark interaction of the NJL type describing low energy hadron physics in the presence of a gluon condensate. Finally, by performing a Fierz transformation of the interaction term, one obtains a four-fermion model with \( (\bar{q}q) \)–and \( (qq) \)–interactions and an external condensate field \( A_\mu^a(x) \) of the color group \( SU_c(N_c) \) given by the following Lagrangian \(^{1}\)

\[
L = \bar{q}[\gamma^\nu(i\partial_\nu + gA_\nu^a(x)\frac{\lambda^a}{2}) + \mu\gamma^0]q + \frac{G_1}{2N_c}[(\bar{q}q)^2 + (\bar{q}i\gamma^5\tau q)^2] + \frac{G_2}{N_c}[\bar{q}c\varepsilon^b\gamma^5q][\bar{q}c\varepsilon^b\gamma^5qc], \tag{1}
\]

It is necessary to note that in order to obtain realistic estimates for masses of vector/axial-vector mesons and diquarks in extended NJL–type of models \(^{23}\), we have to allow for independent coupling constants \( G_1, G_2 \), rather than to consider them related by a Fierz transformation of a current-current interaction via gluon exchange. Clearly, such a procedure does not spoil chiral symmetry.

In \(^{1}\) \( g \) denotes the gluon coupling constant, \( \mu \) is the quark chemical potential, \( q_c = C\bar{q}^t \), \( \bar{q}_c = \bar{q}^tC \) are charge-conjugated spinors, and \( C = i\gamma^2\gamma^0 \) is the charge conjugation matrix (\( t \) denotes the transposition operation). In what follows we assume \( N_c = 3 \). Moreover, summation over repeated color indices \( a = 1, \ldots, 8; b = 1, 2, 3 \) and Lorentz indices \( \nu = 0, 1, 2, 3 \) is implied. The quark field \( q \equiv q_\alpha \) is a flavor doublet and color triplet as well as a four-component Dirac spinor, where \( i = 1, 2; \alpha = 1, 2, 3 \). (Latin and Greek indices refer to flavor and color indices, respectively; spinor indices are omitted.) Furthermore, we use the notations \( \lambda^a/2 \) for the generators of the color \( SU_c(3) \) group appearing in the covariant derivative as well as \( \tau \equiv (\tau^1, \tau^2, \tau^3) \) for Pauli matrices in the flavor space; \( (\varepsilon)_{ik} \equiv \varepsilon_{ik}; (\delta)_{\alpha\beta} \equiv \delta_{\alpha\beta} \) are totally antisymmetric tensors in the flavor and color spaces, respectively. Clearly, the Lagrangian \(^{1}\) is invariant under the chiral \( SU(2)_L \times SU(2)_R \) and color \( SU_c(3) \) groups.

Next, let us for a moment suppose that in \(^{1}\) \( A_\mu^a(x) \) is an arbitrary classical gauge field of the color group \( SU_c(3) \). (The following investigations do not require the explicit inclusion of the gauge field part of the Lagrangian). The detailed structure of \( A_\mu^a(x) \) corresponding to a constant chromomagnetic gluon condensate will be given below.

The linearized version of the model \(^{1}\) with auxiliary bosonic fields has the following form

\[
\bar{L} = \bar{q}[\gamma^\nu(i\partial_\nu + gA_\nu^a(x)\frac{\lambda^a}{2}) + \mu\gamma^0]q - \bar{q}(\sigma + i\gamma^5\bar{\sigma})q - \frac{N_c}{2G_1}(\sigma^2 + \bar{\sigma}^2) - \frac{N_c}{G_2}\Delta^{ab}\Delta^b - \Delta^{ab}[iq^t C\varepsilon^b\gamma^5q] - \Delta^{ab}[i\bar{q}\varepsilon^b\gamma^5C\bar{q}]. \tag{2}
\]

The Lagrangians \(^{1}\) and \(^{2}\) are equivalent, as can be seen by using the equations of motion for bosonic fields, from which it follows that

\[
\Delta^b \sim iq^t C\varepsilon^b\gamma^5q, \quad \sigma \sim \bar{q}q, \quad \bar{\sigma} \sim i\bar{q}\gamma^5\bar{q}. \tag{3}
\]

Clearly, \( \sigma \) and \( \bar{\sigma} \) fields are color singlets. Besides, the (bosonic) diquark field \( \Delta^b \) is a color antitriplet and a singlet under the chiral \( SU(2)_L \times SU(2)_R \) group. Note further that \( \sigma, \Delta^b \), are scalars, but \( \bar{\sigma} \) are pseudo-scalar fields. Hence, if \( \langle \sigma \rangle \neq 0 \), then chiral symmetry of the model

\(^{1}\)The most general four-fermion interaction would include additional vector and axial-vector \( (\bar{q}q) \) as well as pseudo-scalar, vector and axial-vector-like \( (qq) \)–interactions. For our goal of studying the effect of chromomagnetic catalysis for the competition of quark and diquark condensates, the interaction structure of \( (1) \) is, however, sufficiently general.
is spontaneously broken, whereas $\langle \Delta^b \rangle \neq 0$ indicates the dynamical breaking of both the color and electromagnetic symmetries of the theory.

In the one-loop approximation, the effective action for the boson fields is expressed through the path integral over quark fields:

$$\exp(i S_{\text{eff}}(\sigma, \vec{\pi}, \Delta^b, A^a_\mu)) = N' \int [dq][dq] \exp \left( i \int \tilde{L} \, d^4 x \right),$$  \hspace{1cm} (4)

where $N'$ is a normalization constant. Suppose that all the boson fields in (4), except $A^a_\mu$, do not depend on the space-time points. Since $S_{\text{eff}}$ is a function invariant under the chiral (flavor) as well as color and Lorentz groups, it is possible to find a frame in which $\Delta^1=\Delta^2=\pi^b=0$, i.e. $S_{\text{eff}} \equiv S_{\text{eff}}(\sigma, \Delta)$, where $\Delta \equiv \Delta^3$. Next, let us define the effective potential by the following relation $S_{\text{eff}}(\sigma, \Delta) \equiv -V_{\text{eff}}(\sigma, \Delta) \int d^4 x$. The global minimum point of $V_{\text{eff}}$ defines the vacuum expectation values of boson fields as well as the vacuum residual symmetry group. For example, if in this point $\Delta \equiv \langle \Delta \rangle \neq 0$, then $SU_c(3)$ is broken up to $SU_c(2)$, whose generators are the first three generators of initial $SU_c(3)$, and the CSC phenomenon is observed. Using pure symmetry arguments, it is easily shown that, if dynamical gluons were introduced into consideration, the three gluons living in the unbroken $SU_c(2)$ subgroup would stay massless, whereas the remaining five gluons would get masses. Correspondingly, in this frame the external chromomagnetic field $H^a$ can be represented in the following way: $H^a = H^a_1 + H^a_3$, where $H^a_1 = (H^1, H^2, H^3, 0, \ldots, 0)$, $H^a_3 = (0, 0, 0, H^4, \ldots, H^8)$. By analogy with ordinary superconductivity, it is expected that external chromomagnetic fields corresponding to massive gluons, i.e. external chromomagnetic fields of the form $H^a_3$, should be expelled from the CSC phase (Meissner effect). Moreover, sufficiently high values of such fields should destroy the CSC. However, our intuition tells us nothing about the action of external chromomagnetic fields of the form $H^a_3$ on the color superconducting state of the quark-gluon system.

In the present talk the influence of such external chromomagnetic fields of the form $H^a = (H^1, H^2, H^3, 0, \ldots, 0)$ on the phase structure of the NJL model is considered. Furthermore, due to the residual $SU_c(2)$ invariance of the vacuum, one can put $H^1 = H^2 = 0$ and $H^3 \equiv H$. Correspondingly, the gluon condensate which is mimicked by this external field has the value $\langle FF \rangle = 2H^2$. Next, some remarks about the structure of the external vector-potential $A^a_\mu(x)$ used in (4) are needed. From this moment on, we assume $A^a_\mu(x)$ in such a form that the only nonvanishing components of the corresponding field strength tensor $F^a_{\mu\nu}$ are $F^3_{12} = -F^3_{21} = H = \text{const}$. The above homogeneous chromomagnetic field can be generated by the following vector-potential

$$A^3_\mu(x) = (0, 0, H x^1, 0); \quad A^a_\mu(x) = 0 \quad (a \neq 3),$$  \hspace{1cm} (5)

which defines the well known Matinyan–Savvidy model of the gluon condensate in QCD [23]. In QCD the physical vacuum may be interpreted as a region splitted into an infinite number of domains with macroscopic extension [25]. Inside each such domain there can be excited a homogeneous background chromomagnetic field, which generates a nonzero gluon condensate $\langle FF \rangle$. (Averaging over all domains results in a zero background chromomagnetic field, hence color as well as Lorentz symmetries are not broken.) Recall, that in order to find condensates $\langle \sigma \rangle$ and $\langle \Delta \rangle$, we should calculate the effective potential whose global minimum point provides us with these quantities. The expression for the effective potential at $\mu \neq 0$, $H \neq 0$, $T = 0$ has the following form [11]–[13]:

$$V_{\mu}(\sigma, \Delta) = \frac{3\sigma^2}{2G_1} + \frac{3\Delta \Delta^*}{G_2} - \frac{\tilde{S}(\sigma, \Delta)}{v}, \quad v = \int d^4 x,$$  \hspace{1cm} (6)

where
\[ \exp(i\tilde{S}(\sigma, \Delta)) = \det \left[ (i\hat{\partial} - \sigma + \mu \gamma^0) \right] 
\cdot \det^{1/2} \left[ 4|\Delta|^2 + (-i\hat{\partial} - \sigma + \mu \gamma^0 - g\tilde{A}^3\sigma_3)(i\hat{\partial} - \sigma + \mu \gamma^0 + g\tilde{A}^3\sigma_3) \right] \]  
\]  

The operator under the first det-symbol in \((7)\) acts only in the flavor, coordinate and spinor spaces, whereas the operator under the second det-symbol acts in the two-dimensional color subspace, corresponding to the residual \(SU_c(2)\) symmetry of the vacuum, too. In \((7)\) \(\sigma_3 = \text{diag}(1,-1)\) is the matrix in the two-dimensional color space.

### III. Chromomagnetic Catalysis Effect; The Case \(\mu = 0, H \neq 0\)

The primary goal of the investigations in the present Section is to clarify the genuine role of the external chromomagnetic field in dynamical symmetry breaking. In particular, we bring special attention to the CSC generation. It is well-known that CSC is induced at sufficiently high values of the chemical potential \([15] - [21]\). In order to exclude its influence, we put here \(\mu = 0\) and study the phase structure of the NJL model \((1)\) at nonzero \(H\).

First of all let us study the \(H = 0\) case. Putting \(\mu\) and \(A_\nu\) equal to zero and taking into account the general formula \(\det O = \exp(\text{tr} \ln O)\) it is straightforwardly possible to perform the calculations of the determinants in \((8)\). As a result, we have

\[ V_0(\sigma, \Delta, \Delta^*) = \frac{3\sigma^2}{2G_1} + \frac{3|\Delta|^2}{G_2} - 8 \int \frac{d^3k}{(2\pi)^3} \sqrt{\sigma^2 + 4|\Delta|^2 + k^2} - 4 \int \frac{d^3k}{(2\pi)^3} \sqrt{\sigma^2 + k^2}. \tag{8} \]

This expression has ultraviolet divergences. Hence, we need to regularize it by cutting off the range of integration: \(|\vec{k}| \leq \Lambda\). As a result of integrations in the obtained relation, one can find instead of \((8)\) the following regularized expression:

\[ v_0(x, y) = \frac{3A}{2} x^2 + By^2 - \frac{1}{2} \sqrt{1 + x^2} - \frac{x^2}{4} F(x) - \sqrt{1 + x^2 + y^2} - \frac{x^2 + y^2}{2} F(\sqrt{x^2 + y^2}), \tag{9} \]

where the new notations are used:

\[ x = \frac{|\sigma|}{\Lambda}, \quad y = \frac{2|\Delta|}{\Lambda}, \quad A = \frac{\pi^2}{G_1\Lambda^2}, \quad B = \frac{3\pi^2}{4G_2\Lambda^2}, \quad V_0(\sigma, \Delta, \Delta^*) = \frac{\Lambda^4}{\pi^2} v_0(x, y), \]

\[ F(x) = \frac{\sqrt{1 + x^2} - x^2 \ln \frac{1 + \sqrt{1 + x^2}}{x}}{x}. \tag{10} \]

There are four different types of stationary points for the function \((9)\).

- Type I point: \((0, 0)\). It exists for all values of parameters \(A, B \geq 0\).
- Type II point: \((x_0, 0)\). It exists for \(0 \leq A \leq 1\).
- Type III point: \((0, y_0)\). It exists for \(0 \leq B \leq 1\).
- Type IV point: \((\tilde{x}_0, \tilde{y}_0)\). It is possible to show that this solution of the stationarity equations exists in the region \(\omega\) of the \((A, B)\) plane, where

\[ \omega = \{(A, B) : B \geq 0, B \leq A, 3A - 2B \leq 1\}. \tag{11} \]

Let us denote by \(v_1, v_2, v_3, v_4\) the values of the potential \((8)\) at the stationary points of type I,II,III,IV, correspondingly. In order to find the global minimum point (GMP) of the potential, we should compare the quantities \(v_1, ..., v_4\) and select the minimal one for each fixed value of
parameters $A, B$. As a result of such comparisons one can obtain the phase portrait of the model at $H = 0$, which is presented in Fig. 1. This figure shows the $(A, B)$-plane, which is divided into four regions (phases). These regions are denoted similarly to the stationary points, at which the GMP of the potential occurs. So, in the region I the GMP is the stationary point of type I, and there is the totally symmetric phase of the theory. In the region II GMP corresponds to the $\langle \bar{q}q \rangle \neq 0$, $\langle q\bar{q} \rangle = 0$, hence it is the CSB phase. The region III is the pure CSC phase, since for all the points from it only the diquark condensate is nonzero. Finally, in the region IV, which is the same as the domain $\omega (11)$, the mixed phase of the theory occurs, since in this case both condensates are nonzero: $\langle \bar{q}q \rangle \neq 0$, $\langle q\bar{q} \rangle \neq 0$.

We should also note that in the paper [26] the possibility for CSC at $\mu = 0$ was discussed in the framework of random matrix models at $H = 0$. Using general symmetry arguments, there is a strong constraint on the coupling constants, at which the CSC is forbidden, was obtained. In terms of the NJL model (1) this constraint means that at $B > A$ the existence of CSC is prohibited. Just the same result follows from our investigations (see Fig. 1).

Now let us study the influence of a nonzero external chromomagnetic field with vector-potential $V_H$ and at $\mu = 0$ on the phase structure of the model (1). In this case one can show from (7) that (for details see [4,11–13]):

$$V_H(\sigma, \Delta, \Delta^*) = \frac{3\sigma^2}{2G_1} + \frac{3|\Delta|^2}{2G_2} + \frac{gH}{4\pi^2} \int_0^\infty \frac{ds}{s^2} \exp(-s(\sigma^2 + 4|\Delta|^2)) \coth(gHs/2)$$

$$- 4 \int \frac{d^3k}{(2\pi)^3} \sqrt{\sigma^2 + k^2}. \quad (12)$$

The potential (12) is an ultraviolet divergent quantity. After regularization, it can be represented similar to the zero external field case, in the following form:

$$v_h(x, y) = v_0(x, y) - \frac{h^2}{2} \left\{ \zeta'(-1, z) - \frac{1}{2} [z^2 - z] \ln z + \frac{z^2}{4} \right\}, \quad (13)$$

where we have used the same notations as in [9], [10] as well as the new ones:

$$V_H(\sigma, \Delta, \Delta^*) = \frac{\Lambda^4}{\pi^2} v_h(x, y), \quad h = \frac{gH}{\Lambda^2}, \quad z = \frac{x^2 + y^2}{h}. \quad (14)$$

Besides, in [13] $\zeta'(-1, x) = d\zeta(\nu, x)/d\nu|_{\nu = -1}$, where $\zeta(\nu, x)$ -- is the generalized Riemann zeta-function.

Numerical and analytical investigations of the potential (13) result in the phase portrait of the model (1) at nonzero external field, depicted in Fig. 2 in terms of $A, B$. First of all one should note that the symmetric phase is absent, even for arbitrary small values of $H, G_1, G_2$ (large values of $A, B$). This is the so called chromomagnetic catalysis effect of dynamical symmetry breaking. Depending on the relation between $A$ and $B$, the external field (5) can induce CSB or CSC. The boundary between pure CSC and mixed phases is the line $3A - 2B = 1$. The boundary between IV and II phases is an $h$-dependent curve, which is depicted on the Fig. 2 for several values of $h$. The left and right boundaries of the region IV asymptotically coincide at $A, B \rightarrow \infty$. It is necessary to note that the mixed phase IV for arbitrary fixed $h$ is arranged inside the region $\Omega = \{(A, B) : 0 < 3A - 2B < 1\}$. Obviously, $\omega \subset \Omega$, i.e. under the influence of $H$ both CSC and mixed phases are spread. Moreover, it is possible to show that for an arbitrary fixed point $(A, B) \in \Omega$ there is the value $H_c(A, B)$ of the external chromomagnetic field, such that at $H > H_c(A, B)$ the point $(A, B)$ lies inside the phase IV.

It can easily be seen from our investigations, that the general constraint on coupling constants, which forbids the CSC and is valid at $H = 0$ (see [26]), is modified at $H \neq 0$. Indeed, at $H = 0$ the CSC is no longer generated at $B > A$ in the framework of model (1). However, at $H \neq 0$ it
is forbidden at $B > 3A/2$ only. This stronger restriction is based on the ability of the external chromomagnetic field to induce the CSC.

Finally, we should note that, as it was shown in [10,12], the non-abelian chromomagnetic fields, similar to the abelian ones of the type (5), are a good catalysts of CSB or CSC. As in the case of the ordinary magnetic catalysis effect, both the CSB or CSC generation by an external chromomagnetic field are due to the dimensional reduction mechanism. Notice further that, if chiral or color symmetry breaking is induced by some types of external chromomagnetic field at $A, B \gg 1$, then there exists a critical temperature $T_c \sim \sqrt{gH \exp(-C/gH)}$ (with $C$ some parameter depending on the coupling constant), at which the initial symmetry of the model is restored (see [10,12]).

**IV. THE CASE $\mu \neq 0, H \neq 0$**

In the present Section we consider the influence of nonzero values of the chemical potential and external chromomagnetic field on the competition between $\langle qq \rangle$- and $\langle \bar{q}q \rangle$ condensate generations. In this case it is possible to get from (6) the expression for the effective potential (see the paper [13] as well):

$$V_{H\mu}(\sigma, \Delta) = N_c \left( \frac{\sigma^2}{2G_1} + \frac{|\Delta|^2}{G_2} \right) - 2N_f \int \frac{d^3p}{(2\pi)^3} (N_c - 2) \left\{ E_p + \theta(\mu - E_p) \right\} -$$

$$- \frac{N_f gH}{8\pi^2} \sum_{n=0}^{\infty} dp_3 \alpha_n \left\{ \sqrt{(\varepsilon_n - \mu)^2 + 4|\Delta|^2} + \sqrt{(\varepsilon_n + \mu)^2 + 4|\Delta|^2} \right\},$$

where $E_p = \sqrt{p^2 + \sigma^2}$, $\varepsilon_n = \sqrt{gH n + p_3^2 + \sigma^2}$ and $\alpha_n = 2 - \delta_{0n}$. For convenience, relation (15) is written in terms of $N_f$ and $N_c$ even though in the following we will be concerned only with $N_f = 2$ and $N_c = 3$.

**Regularization.** First of all, let us subtract from (13) an infinite constant in order that the effective potential obeys the constraint $V_{H\mu}(0,0) = 0$. After this subtraction the effective potential still remains UV divergent. This divergency could evidently be removed by introducing a simple momentum cutoff $|\vec{p}| < \Lambda$. Instead of doing this, we find it convenient to use another regularization procedure. To this end, let us recall that all UV divergent contributions to the subtracted potential $V_{H\mu}(\sigma, \Delta) - V_{H\mu}(0,0)$ are proportional to powers of meson and/or diquark fields $\sigma, \Delta$. So, one can insert some momentum-dependent form factors in front of composite $\sigma$–and $\Delta$–fields in order to regularize the UV behaviour of integrals and sums.

It is clear by now that we are going to study the effects of an external chromomagnetic condensate field in the framework of the NJL-type model (1), which in addition to two independent coupling constants $G_1, G_2$ includes regularizing meson (diquark) form factors. Of course, it would be a very hard task to study the competition of DSB and CSC for arbitrary values of coupling constants $G_1, G_2$ and any form factors. Thus, in order to restrict this arbitrariness and to be able to compare our results (at least roughly) with other approaches, we find it convenient to investigate the phase structures of the model (1) at $H = 0$ and $H \neq 0$ only for some fixed values of $G_1, G_2$ and some simple expressions for meson/diquark form factors. We expect that qualitatively the obtained (integrated) results do not depend significantly on the chosen regularization procedures, including the momentum cutoff one.

Let us choose the form factors 2

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2The application of the smooth meson form factors (16) leads in a natural way to a suppression of higher Landau levels which is of particular use here. Hence, this regularization scheme is particularly suitable for the manifestation of the
\[ \phi = \frac{\Lambda^4}{(\Lambda^2 + \vec{p}^2)^2}, \quad \phi_n = \frac{\Lambda^4}{(\Lambda^2 + p_3^2 + gHn)^2}, \]  

which have to be included in the energy spectra by a corresponding multiplication of the \( \sigma \)-, \( \Delta \)-fields:

\[ E^r_p = \sqrt{\vec{p}^2 + \phi^2\sigma^2}, \quad \varepsilon^r_n = \sqrt{gHn + p_3^2 + \phi_n^2\sigma^2}, \quad |\Delta^2| \rightarrow \phi_n^2|\Delta^2|. \]

Let us denote the regularized expression for the thermodynamic potential as \( V_{\mu}^e(\sigma, \Delta) \). Notice that at \( H = 0 \) it formally coincides with the corresponding expression of Ref. \[18\] obtained for an NJL type model with instanton-induced four-fermion interactions and taken at zero temperature. In particular, by a suitable choice of coupling constants \( G_1, G_2 \), we will later “normalize” our phase portraits for \( H = 0 \) to the curves of this paper in order to illustrate the influence of a nonvanishing chromomagnetic field. \[1\] Despite the \( \Lambda \)-modification, the expression for \( V_{\mu}^e(\sigma, \Delta) \) contains yet UV-divergent integrals. However, as it was pointed out above, we shall numerically study the subtracted effective potential, i.e. the quantity \( V_{\mu}^e(\sigma, \Delta) - V_{\mu}^e(0, 0) \), which has no divergences.

**Numerical discussions.** Recall that we have chosen the form factors as in \[18\] in order to roughly normalize our numerical calculations at \( H \neq 0 \) on the results obtained at \( H = 0 \) in \[18\]. Comparing the effective potential \( V_{\mu}^e(\sigma, \Delta) \) at \( gH = 0 \) with the corresponding one from ref. \[18\] (denoting their respective diquark field and coupling constants by a tilde), we see that these quantities coincide if \( 2\Delta = \bar{\Delta}, G_1 = 2N_nG_1 \) and \( G_2 = N_nG_2 \). Using further their numerical ratio of coupling constants, we get in our case the following relation

\[ G_2 = 3G_1/8. \]  

Now, let us perform the numerical investigation of the global minimum point (GMP) of the potential \( V_{\mu}^e(\sigma, \Delta) \) for form factors and values of coupling constants as given by \[16\] and \[18\], respectively. We use three different values of cutoff: \( \Lambda = 0.6 \text{ GeV}, 0.8 \text{ GeV}, 1 \text{ GeV} \). Since the physics should not depend on \( \Lambda \), for each value of \( \Lambda \) the corresponding value of \( G_1 \) is selected from the requirement that the GMP of the function \( V_{\mu}^e(\sigma, \Delta) \) at \( \mu = H = 0 \) is at the point \( \sigma = 0.4 \text{ GeV}, \Delta = 0 \) in agreement with phenomenological results and \[18\]. (Then, the value of \( G_2 \) is fixed by the relation \[18\].) For example, \( G_1\Lambda^2 = 2N_n6.47 \) at \( \Lambda = 0.8 \text{ GeV} \), \( G_1\Lambda^2 = 2N_n6.16 \) at \( \Lambda = 1 \text{ GeV} \) etc.

First of all, it should be remarked that, as in paper \[18\] at \( gH = 0 \), a mixed phase of the model was not found for \( H \neq 0 \), i.e. for a wide range of parameters \( \mu, H \) we did not find a global minimum point of the potential \[18\], at which \( \sigma \neq 0, \Delta \neq 0 \). Since in the case under consideration the relation \[18\] corresponds to \( B = 2A \), this result does not contradict to the conclusion of the previous Section. (Recall, that at \( \mu = 0 \) the diquark condensation is prohibited in the region \( B > 3A/2 \).)

The results of our numerical investigations of the GMP of the effective potential are graphically represented in the Fig. 3. For each value of the cutoff \( \Lambda \) the phase portrait of the model consists of two phases II and III. The boundary between the two phases is practically \( \Lambda \)-independent and represents a first order phase transition curve. It is necessary to note also that

\(^3\)It is necessary to underline that in our case the meson/diquark form factors \[18\] mimic solutions of the BS-equation for some non-local four-fermion interaction arising from the one-gluon exchange approach to QCD. Contrary to this, the instanton-like form factor used in \[18\] has another physical nature. It appears as quark zero mode wave function in the presence of instantons \[18\].
for each of the above mentioned values of $\Lambda$ and for fixed value of $gH$ there is a critical chemical potential $\mu_c(H)$, at which the GMP is transformed from a point of type III to a symmetric point of type I. However, this phase transition is a significantly $\Lambda$-dependent one. Indeed, even in the simplest case with $H = 0$ we have $\mu_c(0) = 1$ GeV at $\Lambda = 0.6$ GeV, $\mu_c(0) = 1.3$ GeV at $\Lambda = 0.8$ GeV, $\mu_c(0) = 1.65$ GeV at $\Lambda = 1$ GeV. Hence, in the framework of the NJL model (1) such a phase transition is just an artefact of the regularization procedure, which agrees with the QCD results of [13,14] that CSC can exist at enormously high values of chemical potential $\mu > \sim 10^8$. By this reason, it is not indicated on the Fig. 3.

On the phase portrait of Fig. 3 $\mu$ and $gH$ are free parameters. However, as was emphasized in the Introduction, the external chromomagnetic field mimics the gluon condensate, so the value of $gH$ is some definite quantity. Here we should note that recent investigations yield the following value of the QCD gluon condensate at $T = \mu = 0$: $gH \approx 0.6$ GeV$^2$ [27]. In the paper [28] it was shown in the framework of a quark-meson model that at ordinary nuclear density $\rho_0$ the gluon condensate decreases by no more than six percent, compared with its value at zero density. At densities $3\rho_0$ the value of $\langle FF\rangle$ decreases by fifteen percent. This means that for values of the chemical potential $\mu < 1$ GeV the gluon condensate is a slowly decreasing function vs $\mu$. Taking in mind this circumstance, one can draw two important conclusions from our numerical analysis. Firstly, at $H = 0$ and $\mu = 0.4$ GeV there should exist the CSC phase (see [18]). However, if the real gluon condensate $gH \approx 0.5$ GeV$^2$ is taken into account at $\mu = 0.4$ GeV, and assuming that our results would remain valid also for more realistic condensate fields, this would seemingly render it more difficult to observe the CSC phase in heavy ion-ion experiments. Secondly, let us discuss some quantitative characteristics of the CSC phase at $\mu = 0.8$ GeV. As it follows from our numerical analysis, at $\mu = 0.8$ GeV, $gH = 0$, the GMP of the effective potential corresponds to the CSC phase with a stable diquark condensate $\Delta \approx 0.1$ GeV. However, assuming that the value of the gluon condensate $gH \approx 0.4$ GeV$^2$ would hold for the above nonvanishing chemical potential, one would get a value of the diquark condensate $\Delta \gtrsim 0.2$ GeV, which is significantly larger in magnitude, than at $gH = 0$ (this estimate was obtained for the case $\Lambda = 1$ GeV).

As a general conclusion, we see that taking into account an external chromomagnetic field at least in the form as considered in the model above, might, in principle, lead to remarkable qualitative and quantitative changes in the picture of the diquark condensate formation, obtained in the framework of NJL models at $H = 0$.

**V. SUMMARY AND CONCLUSIONS**

In the present talk the ability of external chromomagnetic fields to induce dynamical symmetry breaking (DSB) of chiral and color symmetry was studied in the framework of the extended NJL model (1) with attractive quark interactions in $qq$- and $\bar{q}q$-channels. Particular attention was paid to the CSC generation. First of all, in order to understand the genuine role of an external chromomagnetic field in the CSB or CSC phenomenon, we have removed the chemical potential from our consideration. The numerical analysis shows in this case (see Fig. 2) that even at sufficiently small values of coupling constants the external chromomagnetic field catalyzes the DSB of chiral and color symmetries (the chromomagnetic catalysis phenomenon). This effect is accompanied by an effective lowering of dimensionality in strong chromomagnetic fields, where the number of reduced units of dimensions depends on the concrete type of the field — a conclusion already made in the case of the CSB [10]. As was shown in [10,12], the phenomenon of $qq$- as well as $\bar{q}q$-condensation does exist for various (non-)abelian chromomagnetic field configurations in the weak coupling limit.

The possibility for vacuum CSC at $\mu = 0$ was also studied in the framework of random matrix models on the basis of general symmetry arguments [26]. There it was found a constraint on
the coupling constants in $qq$- and $\bar{q}q$-channels, at which the CSC is forbidden. We have, in particular, shown that the external chromomagnetic field modifies this constraint and reduces the region of coupling constants, in which the CSC cannot occur (see Section III).

Then, we have considered a more realistic case with nonzero chemical potential as well as with physically meaningful values of coupling constants (Section IV). It is well-known that in this case at rather moderate values of chemical potential ($\mu > 0.3$ GeV) the new CSC phase of QCD is predicted [17]-[21]. However, in these papers such nonperturbative feature of the real QCD vacuum as the nonzero gluon chromomagnetic condensate was not taken into account. In the present analysis in the framework of NJL model (1), the gluon condensate is simulated as an external chromomagnetic field, i.e. $\langle F^a F_a \rangle \equiv 2H^2$. Our numerical calculations show that for real values of the gluon condensate the CSC phase, in contrast with results of [17]-[21], cannot appear for low chemical potentials $0.3 \text{ GeV} < \mu < 0.6$ GeV. At larger values of $\mu$ the gluon condensate significantly modifies the value of the diquark condensate obtained at $H = 0$.

Thus, the main conclusion of our investigations is that the inclusion of an external chromomagnetic field might significantly change the picture of CSC formation, obtained at $H = 0$.

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**FIGURE CAPTIONS**

Fig.1. The phase \((A, B)\)-portrait of the model at \(H = 0\).

Fig.2. The phase \((A, B)\)-portrait of the model for several values of \(h = gH/\Lambda^2\).

Fig.3. The phase \((gH, \mu)\)-portrait of the model for several values of \(\Lambda\), which are indicated in GeV’s.
