An obstacle problem for a class of Monge–Ampère type functionals

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\begin{abstract}
In this paper we study an obstacle problem for Monge–Ampère type functionals, whose Euler–Lagrange equations are a class of fourth order equations, including the affine maximal surface equation and Abreu’s equation.
\end{abstract}

1. Introduction

Free boundary and obstacle problems for partial differential equations have been studied extensively in the past decades. For Monge–Ampère equations, obstacle problems were studied in \cite{7,16,18} among others, and related free boundary problems were studied in \cite{1,6,12}. In this paper we consider

an obstacle problem for the functional...
\[ J_\alpha(u) = \begin{cases} \int_\Omega \left[ \det D^2 u \right]^\alpha - \alpha \int_\Omega f u, & \alpha > 0 \text{ and } \alpha \neq 1, \\ \int_\Omega \log \det D^2 u - \int_\Omega f u, & \alpha = 0, \end{cases} \]  

(1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and \( f \in L^\infty(\Omega) \). For simplicity, we denote the nonlinear part of the functional (1.1) by \( A_\alpha(u) \), see (2.4). We would like to study the maximization problem

\[ J_\alpha(u) = \sup \{ J_\alpha(v): v \in S[\varphi, \psi] \}, \]  

(1.2)

where \( S[\varphi, \psi] \) is the class of functions

\[ S[\varphi, \psi] = \{ u \in C(\overline{\Omega}): u \text{ convex, } u|_{\partial \Omega} = \varphi, Du(\Omega) \subset D\varphi(\overline{\Omega}), u \geq \psi \text{ in } \Omega \}, \]  

(1.3)

\( \varphi \) is a smooth, uniformly convex function defined on a neighborhood of \( \overline{\Omega} \), \( \psi \) is an obstacle function, and \( Du(\Omega) \) represents the image of the subgradients of \( u \) at all points \( x \in \Omega \).

The Euler–Lagrange equations of (1.1) are a class of fourth order equations, that is,

\[ U^{ij} w_{ij} = f, \]  

(1.4)

where \( (U^{ij}) \) is the cofactor matrix of the Hessian \( D^2 u \), and \n
\[ w = \left[ \det D^2 u \right]^{-(1-\alpha)}, \quad \alpha \geq 0. \]  

(1.5)

When \( \alpha = \frac{1}{n+2} \), Eq. (1.4) is the affine mean curvature equation and the functional (2.4) is the affine area functional. When \( \alpha = 0 \), Eq. (1.4) is Abreu's equation arising from the study of Calabi's extremal metrics on toric Kähler manifolds [8–11].

Due to their importance in geometry, variational problems of (1.1) have attracted much interest in recent years. In the case of \( \alpha = \frac{1}{n+2} \), the variational problem without obstacle is the graph case of affine Plateau problem [23,24], raised by Calabi and Chern. The case of \( \alpha = 0 \) has been treated in [25]. The obstacle problem of affine maximal surfaces was first introduced in [19]. In this paper, we obtain:

**Theorem 1.1.** Suppose \( n = 2 \), \( 0 \leq \alpha \leq \frac{1}{n+2} \), and \( f \in L^\infty(\Omega) \). Let \( \varphi \) be a smooth, uniformly convex function in \( \Omega \). If \( \psi \) is a convex function in \( \Omega \) satisfying \( \psi < \varphi \) on \( \partial \Omega \), then there exists a unique maximizer of (1.2) which is strictly convex and \( C^{1,\gamma} \) in \( \Omega \) for some \( \gamma \in (0, 1) \). Furthermore, if \( \psi \) is uniformly convex in \( \Omega \), then the maximizer of (1.2) is \( C^{1,1} \) in \( \Omega \).

We remark that in higher dimensions, the problem is more complicated since Lemma 4.1 does not hold. Furthermore, in the case of \( \alpha = 0 \), the interior estimate in Lemma 2.3 remains open when \( n > 2 \). We will consider the higher dimensional cases and more general forms of the Monge–Ampère type functionals with \( f = f(x, u, Du) \) in our forthcoming work.

This paper is organized as follows: In Section 2 we recall some preliminary results that will be used in subsequent sections. In addition, we show that how the functionals and equations change under a rotation in \( \mathbb{R}^{n+1} \) and obtain the a priori determinant estimates under the rotation transform, where the functionals have more general forms (2.15). In Section 3 we show that the maximizer of \( J_\alpha \) can be approximated by a sequence of smooth maximizers of appropriate penalized functionals. In Section 4 we prove that the maximizer is strictly convex by an observation in [21,24]. The proof of Theorem 1.1 is contained in Section 5, where the \( C^{1,\alpha} \) and \( C^{1,1} \) regularities are obtained, respectively.
2. Preliminaries

2.1. Monge–Ampère measure

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $u$ be a convex function in $\Omega$. The normal mapping of $u$, $N_u$, is a set-valued mapping defined as follows. For any point $x \in \Omega$, $N_u(x)$ is the set of slopes of supporting hyperplanes of $u$ at $x$, that is,

$$N_u(x) = \{ p \in \mathbb{R}^n : u(y) \geq u(x) + p \cdot (y - x), \forall y \in \Omega \}. \quad (2.1)$$

For any Borel set $E \subset \Omega$, $N_u(E) = \bigcup_{x \in E} N_u(x)$. If $u$ is $C^1$, the normal mapping $N_u$ is exactly the gradient mapping $Du$.

From the normal mapping we define the Monge–Ampère measure $\mu[u]$ by

$$\mu[u](E) = |N_u(E)| \quad (2.2)$$

for any Borel set $E \subset \Omega$, where the right hand side is the Lebesgue measure of $N_u(E)$. If $u$ is $C^2$ smooth, we have $\mu[u] = (\det D^2u) \, dx$. In the non-smooth case, the Monge–Ampère measure $\mu[u]$ is a Radon measure, and is weakly continuous with respect to the convergence of convex functions, namely if a sequence of convex functions $\{u_i\}$ converges to a convex function $u$ in $L^\infty_{loc}$, then for any open set $E \subset \Omega$,

$$\mu[u](E) \leq \liminf_{i \to \infty} \mu[u_i](E). \quad (2.3)$$

2.2. Existence and uniqueness of maximizer

Note that the functional $J_\alpha$ in (1.1) is well defined on the set of $C^2$-smooth, convex functions. To study the maximization problem, we extend the functional $J_\alpha$ to the set $S[\varphi, \psi]$ in (1.3), which is closed under the locally uniform convergence of convex functions. It is clear that the linear part in $J_\alpha$ is naturally defined. It suffices to extend the nonlinear part $A_\alpha$ to $S[\varphi, \psi]$. If $u$ is a convex function, $u$ is almost everywhere twice-differentiable, i.e., the Hessian matrix $(D^2u)$ exists almost everywhere. Denote the extended Hessian matrix by $\partial^2 u(x) = D^2 u(x)$ when $u$ is twice-differentiable at $x \in \Omega$ and $\partial^2 u(x) = 0$ otherwise. As a Radon measure, $\mu[u]$ can be decomposed into a regular part and a singular part as follows,

$$\mu[u] = \mu_r[u] + \mu_s[u].$$

It was proved in [21] that the regular part $\mu_r[u]$ can be given explicitly by $\mu_r[u] = \det \partial^2 u \, dx$ and hence $\det \partial^2 u$ is a locally integrable function. Therefore for any $u \in S[\varphi, \psi]$, we can define

$$A_\alpha(u) = \begin{cases} \int_{\Omega} [\det \partial^2 u]^\alpha, & \alpha > 0, \\ \int_{\Omega} \log \det \partial^2 u, & \alpha = 0. \end{cases} \quad (2.4)$$

Lemma 2.1. Suppose $0 \leq \alpha \leq \frac{1}{n+2}$. $J_\alpha$ is upper semi-continuous, bounded and concave in $S[\varphi, \psi]$. It follows that there exists a unique maximizer $u_0$ of (1.2).

Proof. The proof for the cases $\alpha = \frac{1}{n+2}$ and $\alpha = 0$ can be found in [21,26], respectively. One can check that the proof also holds for $0 < \alpha < \frac{1}{n+2}$. $\square$
2.3. Estimates for classical solutions

We include the following a priori estimates in [20,21], which will be needed in subsequent sections, see also [9,25] for the case of $\alpha = 0$. Consider the equation

$$U_{ij} w_{ij} = f \quad \text{in} \quad \Omega,$$

$$w = \left[ \det D^2 u \right]^{\alpha - 1}, \quad (2.5)$$

where $(U_{ij})$ is the cofactor matrix of the Hessian matrix $D^2 u$, and $\alpha \in (0, 1)$ is a constant.

Lemma 2.2. Let $u \in C^4(\Omega) \cap C^{0,1}(\overline{\Omega})$ be a convex solution of $(2.5)$ with $u = 0$ on $\partial \Omega$. Then for any $x \in \Omega$, we have the a priori estimate

$$\det D^2 u(x) \leq C, \quad (2.6)$$

where $C$ depends only on $n$, $\alpha$, $\text{dist}(x, \partial \Omega)$, $\sup_{\Omega}(-u)$, $\sup_{\Omega} |Du|$, and $\sup_{\Omega} f$.

Remark 2.1. In Lemma 2.2, the constant $C$ is independent of $\inf_{\Omega} f$. Hence it is independent of $f$ if $f \leq 0$. By Lemma 3.2, the maximizer $u_0$ of $J_\alpha$ can be locally approximated by smooth solutions of $(2.5)$, and thus Lemma 2.2 still holds for non-smooth maximizers. When $\alpha = \frac{1}{n+2}$, the estimate $(2.6)$ was previously proved in [21].

Remark 2.2. If $n = 2$, the assumption $u = 0$ on $\partial \Omega$ in Lemma 2.2 can be removed [21].

To prove that $\det D^2 u$ has a positive lower bound, we consider the Legendre transform $u^*$ of $u$, which is a convex function defined in the domain $\Omega^* = N_u(\Omega)$, given by

$$u^*(y) = \sup \{ x \cdot y - u(x) : x \in \Omega \}. \quad (2.7)$$

If $u$ is strictly convex near $\partial \Omega$, $u$ can be recovered from $u^*$ by the same transform. If $u$ is $C^2$ smooth at $x$, $y = Du(x)$ and $\det D^2 u(x) \neq 0$, then the Hessian matrix $D^2 u(x)$ is the inverse of the Hessian matrix $D^2 u^*(y)$, and

$$\det D^2 u(x) = \left[ \det D^2 u^*(y) \right]^{-1}. \quad (2.8)$$

In particular, if $u$ is a maximizer of the functional $J_\alpha$, $u^*$ is a maximizer of the dual functional

$$J_\alpha^*(u) = \begin{cases} \int_{\Omega^*} \left[ \det D^2 u^* \right]^{1-\alpha} dy - \alpha \int_{\Omega^*} f(Du^*)(yDu^* - u^*) \det D^2 u^* dy, & \alpha > 0 \text{ and } \alpha \neq 1, \\ - \int_{\Omega^*} \det D^2 u^* \log \det D^2 u^* dy - \int_{\Omega^*} f(Du^*)(yDu^* - u^*) \det D^2 u^* dy, & \alpha = 0. \end{cases} \quad (2.9)$$

Therefore, if $u^*$ is smooth, it satisfies the equation

$$U_{ij}^* w_{ij}^* = \begin{cases} -\frac{\alpha}{1-\alpha} f(Du^*) \det D^2 u^*, & \alpha > 0 \text{ and } \alpha \neq 1, \\ -f(Du^*) \det D^2 u^*, & \alpha = 0. \end{cases} \quad (2.10)$$

where $U_{ij}^*$ is the cofactor matrix of $D^2 u^*$ and
\[ w^* = \begin{cases} \det D^2 u^* - \alpha, & \alpha > 0 \text{ and } \alpha \neq 1, \\ -\log \det D^2 u^*, & \alpha = 0. \end{cases} \] (2.11)

By a similar argument to that of Lemma 2.2, we have the following result [20,21,25].

**Lemma 2.3.** Let \( u^* \) be a smooth convex solution of (2.10) in \( \Omega^* \) in dimension 2, \( u^* = 0 \) on \( \partial \Omega^* \). Then for any \( y \in \Omega^* \), we have the a priori estimate

\[ \det D^2 u^*(y) \leq C, \] (2.12)

where \( C \) depends only on \( \alpha, \text{dist}(y, \partial \Omega^*), \sup_{\Omega^*} |u^*|, \sup_{\Omega^*} |Du^*|, \) and inf \( f \).

By (2.8) and (2.12), we have \( \det D^2 u \geq C \) has a positive lower bound. Note that the estimate depends on inf \( f \), but is independent of sup \( f \).

Once the determinant \( \det D^2 u \) is bounded by Lemmas 2.2, 2.3, from (2.5) we also have the Hölder continuity of \( \det D^2 u \) by Caffarelli–Gutiérrez’s Hölder continuity for linearized Monge–Ampère equation [5] (Remark, page 456). Then we have the \( W^{2,p} \) and \( C^{2,\alpha} \) regularity for \( u \) by Caffarelli’s \( W^{2,p} \) and \( C^{2,\alpha} \) estimates for Monge–Ampère equation [3], see also [15]. Higher regularity then follows from the standard elliptic regularity theory [13]. One can also see Section 4 in [20] for the application of Caffarelli–Gutiérrez’s theory.

Therefore, we have the following Hölder and Sobolev space estimates.

**Theorem 2.1.** Let \( u \in C^4(\Omega) \) be a locally uniformly convex solution of (2.5).

(i) Assume \( f \in L^\infty(\Omega) \). Then we have the estimate

\[ \|u\|_{W^{4,p}(\Omega')} \leq C, \] (2.13)

for any \( p > 1 \) and \( \Omega' \subset \Omega \), where the constant \( C \) depends on \( n, p, \sup_{\Omega} |f|, \text{dist}(\Omega', \partial \Omega), \) and the modulus of convexity of \( u \).

(ii) Assume \( f \in C^\alpha(\Omega) \) for some \( \alpha \in (0, 1) \). Then

\[ \|u\|_{C^{4,\alpha}(\Omega')} \leq C, \] (2.14)

where \( C \) depends on \( n, \alpha, \|f\|_{C^\alpha(\Omega)}, \text{dist}(\Omega', \partial \Omega), \) and the modulus of convexity of \( u \).

Therefore, to prove the regularity of the maximizer \( u_0 \) in Lemma 2.1, it suffices to prove, in view of Lemmas 2.2, 2.3 and Theorem 2.1, that (a) the maximizer \( u_0 \) can be approximated by smooth solutions to Eq. (2.5) and (b) it is strictly convex. We will prove (a) and (b) in Sections 3 and 4, respectively.

### 2.4. Rotations in \( \mathbb{R}^{n+1} \)

In order to establish the estimate of the modulus of convexity, we need to treat convex functions as graphs in \( \mathbb{R}^{n+1} \), and rotate the graphs in \( \mathbb{R}^{n+1} \). When \( \alpha = 1/(n+2) \), the affine maximal surface equation (1.4) is invariant under unimodular transformations in \( \mathbb{R}^{n+1} \). But this is not true for other \( \alpha \). It has been proved in [25] that for \( \alpha = 0 \), under the rotations in \( \mathbb{R}^{n+1} \), Eq. (1.4) changes in a proper way such that the determinant estimate in Lemma 2.2 still holds.

For our purpose, we consider a more general functional

\[ J_\alpha(u) = A_\alpha(u) - \int_{\Omega} F(x,u) \, dx, \] (2.15)
where $A_\alpha$ is in (2.4), $F(x, t)$ is a function on $\Omega \times \mathbb{R}$. Let $u$ be a locally critical point of the functional $J_\alpha$, thus it satisfies (1.4) with the inhomogeneous term $f = F_t := \frac{\partial F}{\partial t}$.

Consider the rotation $Z = TX$, given by $z_1 = -x_{n+1}$, $z_{n+1} = x_1$, $z_i = x_i$ for $2 \leq i \leq n$. Assume the graph of $u$, $G_u = \{(x, u(x)) : x \in \Omega\}$, can be represented by a convex function $z_{n+1} = v(z_1, \ldots, z_n)$ in $z$-coordinates over a domain $\hat{\Omega}$. Following the computation in [26], $v$ is a locally critical point of

$$
\hat{J}_\alpha(v) = \hat{A}_\alpha(v) = \int_{\hat{\Omega}} F(v, z_2, \ldots, z_n, -z_1),
$$

where

$$
\hat{A}_\alpha(v) = \begin{cases} 
\int_{\hat{\Omega}} \det D^2 v)^\alpha|v_1|^{1-(n+2)\alpha} dz, & \alpha > 0, \\
\int_{\hat{\Omega}} \log \det D^2 v - \frac{n+2}{2} \log(v_1^2)\log|v_1^2|^\frac{1}{2} dz, & \alpha = 0.
\end{cases}
$$

When $\alpha > 0$, by computing the Euler equation, we can obtain the corresponding equation for $v$, that is,

$$
\alpha v_1^{1-\alpha(n+2)} v^{ij}(d^{\alpha-1})_{ij} + (1 - \alpha)\alpha(n + 2)(1 - \alpha(n + 2)) v_1^{\alpha(n+2)-1} v_{11} d^{\alpha} v^{ij}
$$

$$
+ (1 - \alpha(n + 2))(2\alpha - 2)v_1^{-\alpha(n+2)}(d^{\alpha})_1 = F_t,
$$

or equivalently, denoting $\lambda = 1 - \alpha(n + 2)$,

$$
V^{ij}(d^{\alpha-1})_{ij} = g + F_t,
$$

where $(V^{ij})$ is the cofactor matrix of $(v^{ij})$, $d = \det D^2 v$ and

$$
g = 2\lambda(1 - \alpha)d^{\alpha} v^{ij}v_{ij}1 v_1^{1} - (1 - \alpha)(n + 2)\lambda d^{\alpha} v_{11}^{v_1^{1}},
$$

$$
F_t = \alpha^{-1} F_t, \quad F_t = \frac{\partial F}{\partial t}(v, z_2, \ldots, z_n, -z_1).
$$

When $\alpha = 0$, by a similar computation we obtain (2.18) with $F_t = F_t/v_1$.

2.5. A priori estimates

In this subsection, we obtain the a priori determinant estimates under the rotation transform $Z = TX$. Let $v$ be a smooth solution of (2.18) satisfying

$$
v \geq 0, \quad v \geq z_1, \quad v_1 \geq 0,
$$

and $v(0)$ is as small as we want such that for the positive constant $s$ and $h$ in $(0, 1/2)$, $\hat{\Omega}_{s, h}$ is a nonempty open set, where

$$
\hat{\Omega}_{s, h} = \{ z : v(z) < sz_1 + h \}.
$$

Later on, in the proof of strict convexity, we will consider a sequence of smooth solutions converging to a limit function, which satisfies (2.19) and whose graph contains two line segments $\{(z_1, 0, 0)$:
\(-1 \leq z_1 \leq 0\) and \((z_1, 0, z_1): 0 \leq z_1 \leq 1\). The function \(v\) in this subsection can be regarded as one element in such an approximation sequence.

Set \(\hat{\nu} := v - sz_1 - h\), then \(\tilde{\Omega}_{s,h} = \{z: \hat{\nu}(z) < 0\}\) and \(\hat{\nu}\) satisfies

\[
\hat{\nu}^{ij}(\hat{d}^{\alpha - 1})_{ij} = \hat{g} + \hat{\kappa},
\]

where \((\hat{\nu}^{ij})\) is the cofactor matrix of \((\hat{\nu}^{ij})\), \(\hat{d} = \text{det} D^2 \hat{\nu}\) and

\[
\hat{g} = 2\lambda (1 - \alpha) \hat{d}^\alpha \frac{\hat{\nu}^{ij} \hat{\nu}_{ij}}{\hat{v} + s} - (1 - \alpha)(n + 2)\lambda \hat{d}^\alpha \frac{\hat{\nu}_{11}}{\hat{v} + s}^2,
\]

\[
\hat{\kappa} = \frac{\alpha}{s (\hat{v} + s)^\alpha}, \quad \hat{\kappa} = \frac{\partial F}{\partial t}(\hat{v} + sz_1 + h, z_2, \ldots, z_n, -z_1).
\]

**Lemma 2.4.** Assume \(0 \leq \alpha \leq \frac{1}{n+2}\). Let \(\hat{v}\) be a smooth solution of (2.21) in \(\tilde{\Omega}_{s,h}\) and \(\hat{v} = 0\) on \(\partial \tilde{\Omega}_{s,h}\). Then for any \(z \in \tilde{\Omega}_{s,h}\), we have the a priori estimate

\[
\text{det} D^2 \hat{v}(z) \leq C,
\]

where \(C\) depends only \(n, \alpha, \text{dist}(z, \partial \tilde{\Omega}_{s,h}), \sup_{\tilde{\Omega}_{s,h}} |\hat{v}|, \sup_{\tilde{\Omega}_{s,h}} |D\hat{v}|\) and \(\sup \hat{\kappa}_t\).

**Proof.** When \(\alpha = \frac{1}{n+2}\), the estimate (2.22) easily follows from the affine invariant property. Note that in this case, \(\lambda = 0\) and \(\hat{g}\) in (2.21) vanishes. The case of \(\alpha = 0\) was contained in [25]. Here we give a proof for the remaining case \(0 < \alpha < \frac{1}{n+2}\) as follows. Let

\[
\eta = \log w - \beta \log(-\hat{v}) - A|D\hat{v}|^2,
\]

where \(w = \hat{d}^{\alpha - 1}\), and \(\beta, A\) are positive constants to be determined later. Since \(\eta \to +\infty\) on \(\partial \tilde{\Omega}_{s,h}\), it attains a minimum at some point \(z_0 \in \tilde{\Omega}_{s,h}\). At \(z_0\), we then have

\[
0 = \eta_i = \frac{w_i}{w} - \frac{\eta \hat{v}_i}{\hat{v}} - 2A \hat{v}_k \hat{v}_{ki},
\]

\[
0 \leq \eta_{ij} = \left[\frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} - \frac{\beta \hat{v}_{ij}}{\hat{v}^2} + \frac{\beta \hat{v}_i \hat{v}_j}{\hat{v}^2} - 2A \hat{v}_k \hat{v}_{kj} - 2A \hat{v}_k \hat{v}_{kj}\right]
\]

as a matrix. Since \(w = \text{det} D^2 \hat{v}\)^{\alpha - 1}, we have

\[
\hat{\nu}^{ij} \hat{v}_{kij} = (\log \text{det} D^2 \hat{v})_k = \frac{1}{\alpha - 1} \frac{w_k}{w},
\]

where \((\hat{\nu}^{ij}) = \hat{d}^{-1}(V^{ij})\) is the inverse of \(D^2 \hat{v}\). We may assume that \(\hat{d} > 1\), otherwise the proof is done. Hence,

\[
\frac{\hat{\nu}^{ij} w_{ij}}{w} = \frac{\hat{g} + \hat{\kappa}_t}{\hat{d}^\alpha} \leq -2\lambda \frac{w_1}{w}(\hat{v} + s)^{-1} - (1 - \alpha)(n + 2)\lambda \frac{\hat{\nu}_{11}}{\hat{v} + s}^2 + \frac{\sup \hat{\kappa}_t}{\alpha(\hat{v} + s)^\alpha}.
\]
Therefore, we obtain
\begin{align*}
0 \leq & \hat{v}^{ij} \eta_{ij} \\
& \leq \frac{\sup \hat{F}_t}{\alpha(\hat{v}_1 + s)^2} + \frac{\lambda(\alpha - 1)(n + 2)\hat{v}_{11}}{(\hat{v}_1 + s)^2} - 4A\sum_{k=1}^{n} \frac{\hat{v}_{1k}\hat{v}_k}{\hat{v}_1 + s} - 2\lambda \beta \frac{\hat{v}_1}{(\hat{v}_1 + s)^2} \\
& \quad - \beta n \frac{\hat{v}}{\hat{v}} - \left(2A \hat{\Delta} \hat{v} - \frac{4A^2\alpha}{1 - \alpha} \hat{v}_{ij}\hat{v}_i\hat{v}_j\right) - \left(4A\beta - \frac{2A\beta}{1 - \alpha}\right) \frac{|D\hat{v}|^2}{\hat{v}} - \left(\beta^2 - \beta\right) \frac{\hat{v}_{ij}\hat{v}_i\hat{v}_j}{\hat{v}}.
\end{align*}
(2.28)

with the choice of \(\beta > 1\) and \(A\) small enough such that
\begin{equation}
A \frac{\hat{v}}{2} \hat{\Delta} \hat{v} \geq \frac{4A^2\alpha}{1 - \alpha} \hat{v}_{ij}\hat{v}_i\hat{v}_j + CA^2\hat{v}_{11},
\end{equation}
(2.29)
where \(C\) is a constant depending only on \(n, \alpha\) and \(|D\hat{v}|\). Observing that
\begin{equation}
\frac{\hat{v}_1}{(\hat{v}_1 + s)^2} = \frac{1}{\hat{v}} - \frac{s}{(\hat{v}_1 + s)^2},
\end{equation}
(2.30)
by choosing \(\beta\) large enough such that
\begin{equation}
(-\hat{v})(\hat{v}_1 + s)^{1 - \lambda} \sup \hat{F}_t \leq 2s\alpha\beta,
\end{equation}
(2.31)
we have
\begin{equation}
-\beta(n + 2\lambda) \frac{\hat{v}}{\hat{v}} - A \frac{\hat{v}}{2} \hat{\Delta} \hat{v} - \left(4A\beta - \frac{2A\beta}{1 - \alpha}\right) \frac{|D\hat{v}|^2}{\hat{v}} \geq 0,
\end{equation}
(2.32)
which implies
\begin{equation}
(-\hat{v}) \hat{\Delta} \hat{v} \leq C.
\end{equation}
(2.33)

It follows that \(\eta(z) \geq \eta(z_0) \geq -C\) and so (2.22) holds. \(\Box\)

3. Approximations

Let \(u_0\) be the maximizer of (1.2). In this section, we prove that \(u_0\) can be approximated by a sequence of smooth solutions of Eq. (2.5). The approximation enables us to apply the a priori estimates in Section 2. For Monge–Ampère equations or general second order equations, one can obtain the approximation from a perturbation of the equation. However, the perturbation does not work for fourth order equations because of the lack of maximum principle. We will construct the approximation using a penalty method to the functionals. We also need to deal with the difficulty coming from the obstacle.
3.1. Obstacle approximation

Let $u_0$ be the maximizer of $J_\alpha$ in $S[\varphi, \psi]$. We construct a sequence of penalized functionals whose maximizers do not touch the obstacle and approximate $u_0$ locally. Let $\Omega' \subset \Omega$ be a subdomain and assume that $u_0$ is equal to an affine function $\ell_0$ on $\partial \Omega'$ but $u_0 \neq \ell_0$ in $\Omega'$. Denote

$$S_0 = \{ v \in C(\overline{\Omega'}): v \text{ convex, } u_0 \leq v \leq \ell_0 \text{ in } \overline{\Omega'} \}. \quad (3.1)$$

Obviously any function in $S_0$ is equal to $\ell_0$ on $\partial \Omega'$.

**Lemma 3.1.** Let $0 \leq \alpha \leq \frac{1}{n+2}$. Suppose that $u_0$ is equal to an affine function $\ell_0$ on $\partial \Omega'$ and $u_0 < \ell_0$ in $\Omega'$. Then there exists a sequence of functions $\{u_i\}$ in $S_0$ such that each $u_i$ is the maximizer of the functional

$$J_\alpha^i(v) = J_\alpha(v, \Omega') - \int_{\Omega'} G_i(x, v), \quad v \in S_0 \quad (3.2)$$

and $u_i \to u_0$ as $i \to \infty$, where $J_\alpha(\cdot, \Omega')$ is the restriction of $J_\alpha$ on $\Omega'$ and $G_i(x, t)$ is a smooth, convex function monotone decreasing in $t$. Furthermore, there is no obstacle for $u_i$ in $\Omega'$, i.e., $u_0(x) < u_i(x) < \ell_0(x), x \in \Omega'$.

**Proof.** By subtracting an affine function we assume that $u_0$ vanishes on $\partial \Omega'$, namely $\ell_0 \equiv 0$, and $S_0$ denotes the set of convex functions $v$ satisfying $u_0 \leq v \leq 0$. First, we consider a penalized problem.

Define

$$G(x, t) = (-u_0)^{n+1}(t - u_0(x))^{-n} \quad \text{for } t > u_0(x), \ x \in \Omega'. \quad (3.3)$$

We notice that $G$ is smooth, convex, and monotone decreasing as a function of $t$. One can see that the set $\{v \in S_0: J_{\alpha, g}(v) > -\infty\} \neq \emptyset$. In fact, $\bar{v} = \frac{1}{2} u_0$ belongs to this set. It is straightforward to see that $J_\alpha(\bar{v}, \Omega') > -\infty$. Moreover,

$$\int_{\Omega'} G(x, \bar{v}) \leq 2^n \int_{\Omega'} (-u_0) < \infty.$$ 

This implies that $J_{\alpha, g}(\bar{v}) > -\infty$, thus the set $\{v \in S_0: J_{\alpha, g}(v) > -\infty\} \neq \emptyset$.

It is clear that $J_{\alpha, g}$ is still concave, upper semi-continuous and bounded from above. Hence there is a unique maximizer $v_g$ to the problem

$$\sup\{ J_{\alpha, g}(v): v \in S_0 \}. \quad (3.4)$$

We claim that for any $x \in \Omega'$,

$$0 > v_g(x) > u_0(x). \quad (3.5)$$
Indeed, when \( \alpha \in (0, \frac{1}{π+2}] \) by direct computation one has

\[
J_{α, g}(εu₀) - J_{α, g}(0) = e^{nα} \int_{Ω'} \det[D^2u₀]^q - εα \int_{Ω'} fu₀ + \left(1 - (1 - ε)^{-n}\right) \int_{Ω'} (-u₀).
\]

Hence \( J_{α, g}(εu₀) > J_{α, g}(0) \) when \( ε > 0 \) is sufficiently small. Namely \( v ≡ 0 \) cannot be a maximizer.

When \( α = 0 \), it is obvious that \( v ≡ 0 \) cannot be a maximizer. By convexity we see that \( v_0 < 0 \) in \( Ω' \).

Next we show that \( v_0 > u₀ \) in \( Ω' \). Indeed, if there is a point \( x₀ ∈ Ω' \) such that \( v_0(x₀) = u₀(x₀) \), by convexity the graphs of \( v_0 \) and \( u₀ \) are bounded by the cone \( K \) and the hyperplane \( P \), where \( K \) has the vertex at \( (x₀, u₀(x₀)) \) and passes through \( (\partial Ω', 0) \), and \( P \) is the support plane of \( u₀ \) at \( x₀ \). Then we have \( |v_0(x) - u₀(x)| \leq C|x - x₀| \). Hence by (3.3) the definition of \( G(x, t) \),

\[
\int_{Ω'} G(x, v_0(x)) \geq C \int_{Ω'} |x - x₀|^{-n} = \infty. \tag{3.6}
\]

That is, \( v_0 \) cannot be a maximizer. Hence (3.5) holds.

Replacing \( G \) by \( ε_lG \) for a sequence \( ε_l \rightarrow 0 \), accordingly there exists a sequence of maximizers \( v_{ε_l} \) to (3.4). Since \( u₀ \) is itself a maximizer, we have \( v_{ε_l} \rightarrow u₀ \) as \( ε_l \rightarrow 0 \) by the concavity of the functional \( J_{α} \).

Hence, the sequence \( u_i \) can be chosen from \( v_{ε_l} \). \( \square \)

**Remark 3.1.** If \( u_i \) is smooth, it satisfies the equation

\[
L[u] = f + g_i \quad \text{in } Ω', \tag{3.7}
\]

where \( L \) is the operator in (1.4), and \( g_i = \frac{∂}{∂t} G_i(x, t) \) at \( t = u_i(x) \). In the later proof of strict convexity, we will need the upper bound estimate for the determinant of \( D^2u₀ \) which depends on \( \sup f \).

Since \( g_i < 0 \) in the above approximation, the estimate in Section 2 still applies when turning to the sequence \( u_i \).

**Remark 3.2.** In fact, when studying the strict convexity in Section 4, the approximation in Lemma 3.1 applies on a sub-level set \( Ω' \) of \( u₀ \) at a point \( \hat{x} ∈ Ω \), given by \( Ω' = \{x ∈ Ω: u₀(x) < ℓ(x) + h\} \), where \( ℓ \) is an affine function satisfying \( ℓ(\hat{x}) = u₀(\hat{x}) \) and \( h \) is a small positive constant such that \( Ω' ∈ Ω \). Then by Lemma 3.1 one can obtain an approximation sequence \( \{u_i\} \) converging to \( u₀ \) on \( Ω' \).

Moreover, at the point \( \hat{x} ∈ Ω \) where \( u₀ \) touches the obstacle \( ψ \), namely \( u₀(\hat{x}) = ψ(\hat{x}) \), from the assumption \( ψ < φ \) on \( \partial Ω \) one can construct \( Ω' \) by \( Ω' = \{x ∈ Ω: u₀(x) < ℓ(x) + h\} \), where \( ℓ \) is a support function of \( ψ \) at \( \hat{x} \) and \( h \) is small positive constant such that \( Ω' ∈ Ω \).

### 3.2. Smooth approximation

Let \( u \) be the maximizer of (3.4) and let \( φ = u₀ \). From the obstacle approximation, \( u \) is also the maximizer of (3.2) over the set

\[
S[φ, Ω'] = \{v ∈ C(\overline{Ω'}): \text{v convex, } v|_{∂Ω'} = φ|_{∂Ω'}, Dv(Ω') \subset Dφ(\overline{Ω'})\}. \tag{3.8}
\]

Note that from the construction of (3.3), (3.2) is only defined for function \( v \) such that \( v > u₀ \) in \( Ω' \).

One may also define formally \( G(x, t) = +∞ \) when \( t < u₀(x), x ∈ Ω' \). In this subsection, we prove that \( u \) can be approximated by smooth solutions of

\[
U^{ij} w^{ij} = \tilde{f}(x, u), \tag{3.9}
\]
Lemma 3.2. Let $u$ be the maximizer of (3.4). Suppose $\partial \Omega'$ is Lipschitz continuous. Then there exists a sequence of smooth solutions to Eq. (3.9) converging locally uniformly to the maximizer $u$.

To prove the approximation, first we recall the existence and regularity of solutions of the following second boundary value problem [21]. Let $B = B_R(0)$ be a ball such that $\Omega' \subseteq B_{R-1}(0)$. Let $\phi$ be a convex, uniformly Lipschitz function in a neighborhood of $\Omega'$. One can extend $\phi$ to $B$ such that $\phi$ is convex in $B$ and $\phi = c^+$ on $\partial B$ for a large constant $c^+$. Let

$$H(t) = (1 - t^2)^{-2n}$$

be a nonnegative smooth function in the interval $(-1, 1)$. When $|t| > 1$, we can formally define $H(t) = +\infty$. Extend the function $\tilde{f}$ in (3.9) to $B$ such that

$$\tilde{f} = \begin{cases} \tilde{f}(x, u) & \text{in } \Omega', \\ H'(u - \phi(x)) & \text{in } B \setminus \Omega'. \end{cases}$$

(3.11)

Lemma 3.3. Suppose $\partial \Omega'$ is Lipschitz continuous. Then there is a uniformly convex solution $u \in W^{4,p}_0(B) \cap C^{0,1}(\overline{B})$ (for all $p < \infty$) with $\det D^2u \in C^0(\overline{\Omega'})$ of the boundary value problem

$$U^{ij}w_{ij} = \tilde{f}(x, u) \quad \text{in } B,$$
$$u = \bar{\phi}(\varepsilon c^+) \quad \text{on } \partial B,$$
$$w = 1 \quad \text{on } \partial B.$$

(3.12)

The existence and regularity of solutions of (3.12) was previously obtained in [21,24] for $\alpha = \frac{1}{n+2}$, and [25] for $\alpha = 0$. The crucial ingredient is to establish

$$|\tilde{f}(x, u)| \leq C$$

(3.13)

for some constant $C > 0$ independent of solution $u$. Once $\tilde{f}$ is bounded, the regularity and existence of solutions follow easily from [21]. The global $C^{4,\alpha}$ regularity was recently proved in [23]. Following the argument in [21], one can check that the proof works for all $\alpha \in [0, \frac{1}{n+2}]$.

Now, we show that the maximizer of $J_{\alpha, g}(u)$ can be approximated by smooth solutions to Eq. (3.9).

Proof of Lemma 3.2. Let $\phi = u_0$, which is convex and uniformly Lipschitz in a neighborhood of $\Omega'$, one can choose a large constant $c^+$ and extend $\phi$ to $B = B_R$ such that $\phi$ is convex in $B$, $\bar{\phi} \in C^{0,1}(\overline{B})$ and $\phi = c^+$ is constant on $\partial B$. Replacing $\phi$ by $\phi + (|x| - R + \frac{1}{2})^2$, where

$$\left(|x| - R + \frac{1}{2}\right)_+ = \max\left\{|x| - R + \frac{1}{2}, 0\right\},$$

we also assume that $\phi$ is uniformly convex in $[x \in \mathbb{R}^n : R - \frac{1}{2} < |x| < R]$. Consider the second boundary value problem (3.12) with

$$\tilde{f}_j(x, u) = \begin{cases} \tilde{f}(x, u) & \text{in } \Omega', \\ H_j'(u - \phi) & \text{in } B \setminus \Omega'. \end{cases}$$

(3.14)
where \( H_j(t) = H(4^j t) \) and \( H \) is defined by (3.10). By Lemma 3.3 there is a solution \( u_j \) to (3.12) with \( \tilde{\phi} = \phi \) and \( \tilde{f} = \tilde{f}_j \). By our definition of the penalty function \( H_j \), \( u_j \) satisfies

\[
|u_j - \phi| \leq 4^{-j}, \quad x \in B \setminus \Omega'.
\]

By the convexity, \( u_j \) sub-converges to a convex function \( u \) in \( B \) as \( j \to \infty \). Note that \( u = \phi \) in \( B \setminus \Omega' \).

Hence, \( u \in S[\phi, \Omega'] \) when restricted in \( \Omega' \). Using a similar argument as in [24] and [25], one can show that \( u \) is the maximizer of (3.2) over the set (3.8). By the uniqueness of maximizer, we obtain \( u = u_0 \).

\section{Strict convexity}

In this section, we prove the strict convexity of \( u_0 \) in dimension two. Let \( G_0 \) be the graph of \( u_0 \). If \( u_0 \) is not strictly convex, then \( G_0 \) contains a line segment. Let \( \ell(x) \) be a tangent function of \( u_0 \) at the segment and denote by \( C = \{ x \in \Omega : u_0(x) = \ell(x) \} \) the contact set. The set \( C \subset \mathbb{R}^2 \) is bounded and convex.

We say a point \( x_0 \in \partial U \) is an extreme point of a bounded convex domain \( U \subset \mathbb{R}^n \) if there is a hyperplane \( P \) such that \( \{x_0\} = P \cap \partial U \), namely the intersection \( P \cap \partial U \) is the single point \( x_0 \). We divide our discussion into the following two cases:

(a) \( C \) has an extreme point \( x_0 \), which is an interior point of \( \Omega \);
(b) All extreme points of \( C \) lie on \( \partial \Omega \).

We will rule out the possibility of both cases, and thus \( u_0 \) is strictly convex. The basic observation is that a convex function with a bounded Monge–Ampère measure is differentiable at any point on its graph, not lying on a line segment joining two boundary points [2]. In dimension two, recall the following

\begin{lemma}
(See [21].) Suppose \( u \) is a nonnegative convex function in a domain \( \Omega \subset \mathbb{R}^2 \). The origin \( 0 \in \Omega \) is an interior point. \( u \) satisfies \( u > 0 \) on \( \partial \Omega \), \( u(0) = 0 \) and \( u(x_1, 0) \geq |x_1| \). Then the Monge–Ampère measure \( \mu[u] \) cannot be a bounded function.
\end{lemma}

\subsection{Strict convexity I}

First we rule out the possibility that \( G_0 \) contains a line segment with one endpoint in the interior of \( \Omega \).

\begin{lemma}
\( C \) contains no extreme points in the interior of \( \Omega \).
\end{lemma}

\textbf{Proof.} The proof is by contradiction arguments as in [20,25]. Without loss of generality, we may assume that \( \ell(x_0) = 0 \), the origin is an extreme point of \( C \) and the segment \( \{(x_1, 0) : 0 \leq x_1 \leq 1\} \subset C \). From the approximation argument, we can choose a sequence of functions \( \{u_k\} \) converging to \( u_0 \) such that \( u_k \) is a solution of (3.7). Let \( G_k \) be the graph of \( u_k \). Then \( G_k \) converges in the Hausdorff distance to \( G_0 \).

For \( \varepsilon > 0 \) small enough, let

\[
\ell_\varepsilon = -\varepsilon x_1 + \varepsilon, \quad \text{and} \quad \Omega_\varepsilon = \{u_0 < \ell_\varepsilon\}.
\]
Let $T_\varepsilon$ be a coordinate transformation that normalizes the domain $\Omega_\varepsilon$ such that $B_{1/n}(x^*) \subset T_\varepsilon(\Omega_\varepsilon) \subset B_1(x^*)$ where $x^*$ is the centroid of $\Omega_\varepsilon$, see for instance [14] for a proof of the existence of such a transformation. Define

$$u_\varepsilon(x) = \frac{1}{\varepsilon} u_0(T_\varepsilon^{-1}(x)), \quad u_{k,\varepsilon} = \frac{1}{\varepsilon} u_k(T_\varepsilon^{-1}(x)), \quad x \in \tilde{\Omega}_\varepsilon,$$  

(4.3)

where $\tilde{\Omega}_\varepsilon = T_\varepsilon(\Omega_\varepsilon)$ is normalized. After this transformation we have the following observations:

(i) The equation $U^{ij} w_{ij} = f$ with $w = [\det D^2 u]^\alpha - 1$, $0 \leq \alpha \leq \frac{1}{4}$, will become

$$\tilde{U}^{ij} \tilde{w}_{ij} = \tilde{f},$$  

(4.4)

where $\tilde{U}^{ij}$ is the cofactor of $D^2 \tilde{u}$.

In fact, since $T_\varepsilon$ normalizes $\Omega_\varepsilon$, $|T_\varepsilon|^{-1} \leq |\Omega_\varepsilon| \leq C$. Therefore, $\tilde{f} \to 0$ as $\varepsilon \to 0$.

(ii) Denote by $\tilde{g}_0$ and $\tilde{g}_{k,\varepsilon}$ the graphs of $u_\varepsilon$ and $u_{k,\varepsilon}$, respectively. Taking $k \to \infty$, it is clear that $u_{k,\varepsilon} \to u_\varepsilon$ and $\tilde{g}_{k,\varepsilon}$ converges in the Hausdorff distance to $\tilde{g}_0$. Then taking $\varepsilon \to 0$, we have that the domain $\tilde{\Omega}_\varepsilon$ sub-converges to a normalized domain $\tilde{\Omega}$ and $u_\varepsilon$ sub-converges to a convex function $\tilde{u}$ defined in $\tilde{\Omega}$. We also have $\tilde{g}_0$ sub-converges in the Hausdorff distance to a convex surface $\tilde{g}_0 \in \mathbb{R}^3$.

(iii) By a linear transformation of coordinates, since the equation is invariant under unimodular affine transformation of $\{y_1, y_2\}$ coordinates, we may assume that the convex surface $\tilde{g}_0$ satisfies

$$\tilde{g}_0 \subset \{y_1 \geq 0\} \cap \{y_3 \geq 0\}$$  

(4.5)

and $\tilde{g}_0$ contains two segments

$$\{(0, 0, y_3): 0 \leq y_3 \leq 1\}, \quad \{(y_1, 0, 0): 0 \leq y_1 \leq 1\}.$$  

(4.6)

Hence, by (i)-(iii) we can assume that there is a sequence of solutions $\tilde{u}_k$ of

$$U^{ij} w_{ij} = \varepsilon_k f \quad \text{in} \quad \tilde{\Omega}_k,$$  

(4.7)

where $w = [\det D^2 u]^\alpha - 1$, and $\varepsilon_k \to 0$ such that the normalized domain $\tilde{\Omega}_k$ converges to $\tilde{\Omega}$, $\tilde{u}_k$ converges to $\tilde{u}$ and the graph of $\tilde{u}_k$, denoted by $\tilde{g}_k$, converges in the Hausdorff distance to $\tilde{g}_0$.

Note that in $y$-coordinates, $\tilde{g}_0$ is not a graph of a function near the origin. By adding some linear function to $\tilde{u}_k$ and $\tilde{u}$ and making a rotation of coordinates in $\mathbb{R}^3$, i.e., $z_i = R_{ij} y_j$, where $(R_{ij})$ is a $3 \times 3$ rotation matrix, $\tilde{g}_k, \tilde{g}_0$ can be represented by $z_3 = v_k(z_1, z_2)$, $z_3 = v(z_1, z_2)$, respectively [25]. Moreover, $v_k$ is a solution of the equation given in Section 2.4 near the origin, $v$ satisfies

$$v \geq \frac{1}{2} |z_1|, \quad \text{and} \quad v(z_1, 0) = \frac{1}{2} |z_1|.$$  

(4.8)

As we know that $\tilde{g}_k$ converges in the Hausdorff distance to $\tilde{g}_0$, in the new coordinates, $v_k$ converges locally uniformly to $v$. Let $\tilde{C} = \{(z_1, z_2): v(z_1, z_2) = 0\}$, and

$$L = \{(z_1, z_2, 0): (z_1, z_2) \in \tilde{C}\}$$  

(4.9)

in $z$-coordinates. $L$ could be a single point (Case I) or a segment on $z_2$-axis (Case II).
Case I: In this case, \( v \) is strictly convex at \((0, 0)\). The strict convexity implies that \( Dv \) is bounded on the sub-level set \( S_{h,v}(0) \) for small \( h > 0 \). Hence, by locally uniform convergence, \( Dv_k \) are uniformly bounded on \( S_{h/2,v_k}(0) \). By Lemma 2.4, we have the determinant estimate

\[
\det D^2 v_k \leq C
\]  
(4.10)

near the origin, where the constant \( C \) is uniform with respect to \( k \). By the weak continuity of Monge--Ampère measure, \( \mu[v] \leq C \) near the origin. The contradiction follows by Lemma 4.1.

Case II: In this case, \( L \) is a segment, we may also assume that 0 is an endpoint of \( L \), i.e., \( \tilde{C} = \{(0, z_2) : -1 \leq z_2 \leq 0\} \).

Define the linear function

\[
\ell_\varepsilon(z) = \delta_\varepsilon z_2 + \varepsilon
\]  
(4.11)

and \( \omega_\varepsilon = \{z \colon v(z) \leq \ell_\varepsilon\} \), where \( \delta_\varepsilon, \varepsilon \) are chosen such that \( \varepsilon \delta_\varepsilon^{-1} \to 0 \) as \( \varepsilon \to 0 \). By taking the similar transformations and normalizations as in (4.2), (4.3) with respect to \( z_2 \) direction, one can reduce Case II to Case I. The proof is then finished.

4.2. Strict convexity II

Next, we rule out the possibility of case (b) that all extreme points of \( C \) lie on the boundary \( \partial \Omega \). Recall the definition of \( C \) in (4.1), and define the set \( T = \{x \in \Omega \colon u(x) = \psi(x)\} \), where \( \psi \) is the obstacle.

**Lemma 4.3.** Let \( u_0 \in S[\phi, \psi] \) be the maximizer. The obstacle \( \psi \) is a convex function in \( \Omega \) satisfying \( \psi < \phi \) on \( \partial \Omega \). If all extreme points of \( C \) lie on the boundary \( \partial \Omega \), then \( \text{dist}(\overline{C}, T) > c_0 \) for some positive constant \( c_0 \).

**Proof.** This follows obviously from the convexity of \( \psi \) and definitions of \( C \) and \( T \).

**Lemma 4.4.** Assume that \( \phi \) is uniformly convex in a neighborhood of \( \Omega \). Then \( G_0 \) contains no line segments with both endpoints on \( \partial G_0 \).

**Proof.** By Lemma 4.3, we can restrict our discussion on \( \Omega' = N_C \cap \Omega \) such that \( \text{dist}(\Omega', T) > c_0 \) for some positive constant \( c_0 \) and \{extreme points of \( C \)\} \( \subset \partial \Omega' \cap \partial \Omega \), where \( N_C \) is a neighborhood of \( C \). Let \( u_0 \) be the maximizer of \( J_\alpha \) and

\[
\tilde{S}[u_0, \Omega'] := \{v \in C(\overline{\Omega'}): v \text{ convex, } v_{\partial \Omega'} = u_0, N_v(\Omega') \subset N_{u_0}(\overline{\Omega'})\}.
\]  
(4.12)

Note that since \( \text{dist}(\Omega', T) > c_0 \), when restricting on \( \Omega' \), \( u_0 \) is naturally a maximizer of \( J_\alpha \) over \( \tilde{S}[u_0, \Omega'] \) without obstacle. Therefore, we can apply a similar local approximation in [24] as follows:

**Claim.** There exists a sequence of smooth, uniformly convex solutions \( u_m \in W^{4,p}(\Omega') \) (\( \forall p < \infty \)) of

\[
U^{ij}w_{ij} = f + \beta_m \chi_{D_m} \text{ in } \Omega'
\]  
(4.13)

such that

\[
|u_m - u| \to 0 \text{ uniformly in } \Omega',
\]  
(4.14)
where \( D_m = \{ x \in \Omega': \text{dist}(x, \partial \Omega') < 2^{-m} \} \), \( \chi \) is the characteristic function, and \( \beta_m > 0 \) is a constant. Furthermore, we can choose \( \beta_m \) sufficiently large (\( \beta_m \to \infty \) as \( m \to \infty \)) such that for any compact, proper subset \( K \subset N_{u_0}(\Omega') \),

\[ K \subset N_{u_m}(\Omega') \tag{4.15} \]

provided \( m \) is sufficiently large, where \( N_{u} \) is the normal mapping introduced in Section 2.

The proof of the claim is contained in [24] for the case \( \alpha = \frac{1}{n+2} \), see also [25] for the case \( \alpha = 0 \). The idea is similar to the proof of Lemma 3.2. But instead of considering the second boundary value problem with inhomogeneous term (3.14), we consider a weighted one

\[ f_{m,j} = \begin{cases} f + \beta_m \chi_{D_m} & \text{in } \Omega', \\ H_j'(u - u_0) & \text{in } B_R \setminus \Omega' \end{cases} \tag{4.16} \]

where \( H_j(t) = H(4^j t) \) given by (3.10), \( B_R \) is a large ball enclosing \( \Omega' \). By Lemma 3.3, there is a solution \( u_{m,j} \) satisfying

\[ |u_{m,j} - u_0| \leq 4^{-j}, \quad x \in B_R \setminus \Omega'. \tag{4.17} \]

By the convexity, \( u_{m,j} \) sub-converges to a convex function \( u_m \) as \( j \to \infty \) and \( u_m = u_0 \) in \( B_R \setminus \Omega' \). Note that \( u_m \in S[u_0, \Omega'] \) when restricted in \( \Omega' \), therefore, \( u_m \) converges to a convex function \( u_\infty \) in \( S[u_0, \Omega'] \) as \( m \to \infty \). Similarly, one can show that \( u_\infty \) is the maximizer of \( J_\alpha \) over the set \( S[u_0, \Omega'] \). By the uniqueness of maximizer, we have \( u_\infty = u_0 \) and obtain the claim. See [24, 25] for more details.

Now, suppose that \( \ell \) is a line segment in \( G_0 \) with both end points on \( \partial G_0 \). By subtracting a linear function, we assume that \( u_0 \geq 0 \) and \( \ell \) lies in \( \{ x_3 = 0 \} \). From the definition of \( \Omega' \), we also have \( \ell \subset \Omega' \) with both end points on \( \partial \Omega' \cap \partial \Omega \). By a translation and a dilation of the coordinates, we may assume furthermore that

\[ \ell = \{(0, x_2, 0) : -1 \leq x_2 \leq 1\} \tag{4.18} \]

with the endpoints \((0, \pm 1) \in \partial \Omega' \cap \partial \Omega \).

Since \( \varphi \) is smooth in a neighborhood of \( \Omega \) and \( u_0 = \varphi \) on \( \partial \Omega \), it follows

\[ u_0(x) = \varphi(x) \leq \frac{C}{2} |x_1|^2, \quad x \in \partial \Omega' \cap \partial \Omega. \tag{4.19} \]

By the convexity of \( u_0 \),

\[ u_0(x) \leq \frac{C}{2} |x_1|^2, \quad x \in \Omega'. \tag{4.20} \]

Consider the Legendre transform \( u^*_0 \) of \( u_0 \) in \( \Omega^* = D \varphi(\Omega) \), given by

\[ u^*_0(y) = \sup \{ x \cdot y - u_0(x), x \in \Omega \}, \quad y \in \Omega^*. \tag{4.21} \]

Since both endpoints \((0, \pm 1) \in \partial \Omega' \cap \partial \Omega \), by the uniform convexity of \( \varphi \), \( 0 \notin D \varphi(\partial \Omega) \). Hence \( 0 \in \Omega^* \) is an interior point. By (4.19), (4.20) we have
Therefore, \( \det D^2 u_0^* \) is not bounded from above near the origin by Lemma 4.1.

But on the other hand, by the a priori estimate in Lemma 2.3, \( \det D^2 u_0^* \) must be bounded. Indeed, consider the Legendre transform \( u_m^* \) of \( u_m \). By the approximations (4.14), (4.15), and (2.10), \( u_m^* \) satisfies the equation

\[
U^{ij} w_{ij}^* = -f_m(Du^*) \det D^2 u^* \quad \text{in} \quad \Omega^*_m, 
\]

where \( f_m = f + \beta_m \chi_{D_m} \) and

\[
\Omega^*_m = \{ y \in \Omega^*: \text{dist}(y, \partial \Omega^*) > \varepsilon_m \}
\]

with \( \varepsilon_m \to 0 \) as \( m \to \infty \). By the growth estimates (4.22) and (4.23), \( u_0^* \) is strictly convex at 0, the set \( \{ u_0^* < h \} \) is strictly contained in \( \Omega^* \) provided \( h > 0 \) is small. Note that \( u_m^* \) converges to \( u_0^* \). By Lemma 2.3 we have the estimate

\[
\det D^2 u_m^* \leq C_1
\]

near the origin in \( \Omega^* \). Note also that in Lemma 2.3, the constant \( C_1 \) depends on \( \inf f \) but not on \( \sup f \). In other words, the large constant \( \beta_m \) in (4.13) does not affect the bound \( C_1 \). Therefore, sending \( m \to \infty \), we obtained

\[
\det D^2 u_0^* \leq C
\]

near the origin. This is in contradiction with the assertion that \( \det D^2 u_0^* \) is not bounded from above near the origin. \( \square \)

5. Regularity

We can now give the proof of Theorem 1.1, which is divided into two parts:

5.1. \( C^{1,\alpha} \) regularity

Assume that \( \psi \) is convex and satisfies \( \psi < \varphi \) on \( \partial \Omega \). Let \( u \) be the maximizer of (2.4) and \( G_u \) be the graph of \( u \) over \( \Omega \). From Section 4 we know \( G_u \) is strictly convex. The \( C^{1,\alpha} \) estimate for strictly convex solutions of Monge–Ampère equations was obtained by Caffarelli [4]. Here we adopt a similar argument from [22].

For an arbitrary point on \( G_u \), by choosing appropriate coordinates and a rotation in \( \mathbb{R}^3 \), we assume it is the origin and \( G_u \subset \{ x_3 > 0 \} \), and near the origin \( G_u \) is the graph of a strictly convex function \( u \).

**Lemma 5.1.** There exist positive constants \( \alpha, \beta, \) and \( C \) such that

\[
C^{-1} |x|^{1+\beta} \leq u(x) \leq C|x|^{1+\alpha} \quad \text{near the origin.} \tag{5.1}
\]
**Proof.** Denote $S^0_h = \{ x \in \Omega : u(x) < h \}$. By the strict convexity, $S^0_h \subset \Omega$ when $h > 0$ is small. We point out that the proof of strict convexity in Section 4 implies that $u$ is $C^1$ smooth. In fact, if $u$ is not $C^1$ at some point, by a rotation of axes we assume $G_u \subset \{ x_3 = a|x_1| \}$ for some constant $a > 0$. Let $L$ be the intersection of $G_u$ with $\{ x_3 = 0 \}$. $L$ could be a single point or a segment on $x_2$-axis. From the proof of Lemma 4.2, by a contradiction argument, we can rule out the possibility of both cases, which implies that $G_u$ is $C^1$ smooth. Hence we have

$$\text{dist}(S^0_{h/2}, \partial S^0_h) \geq C_1, \quad (5.2)$$

or equivalently,

$$u(\theta x) \geq \frac{1}{2} u(x) \quad (5.3)$$

for any $x \in \partial S^0_h$, where $\theta = 1 - \frac{1}{2} C_1$. As $h$ is any small constant, it follows that for any $x$ near the origin,

$$u(x) \geq 2^{-k} u(\theta^{-k} x) \quad (5.4)$$

provided $\theta^{-k} x \in \Omega$. Hence we obtain the first inequality in (5.1) with $\beta$ given by $\theta^{1+\beta} = 1/2$.

To prove the second inequality, we claim that there exists a constant $\sigma > 0$ such that for any small $h > 0$ and any $x \in \partial S^0_h$,

$$u\left(\frac{1}{2}x\right) < \frac{1-\sigma}{2} u(x). \quad (5.5)$$

Define $\alpha$ by $1-\sigma = 2^{-\alpha}$. Then for any $x \in \partial S^0_h$ and any $t \in (\frac{1}{2^{1+\alpha}}, \frac{1}{2})$,

$$u(tx) \leq 2^{-k}(1-\sigma)^k u(x)$$

$$= (2^{-k})^{1+\alpha} u(x)$$

$$\leq 2t^{1+\alpha} u(x). \quad (5.6)$$

Hence $u \in C^{1,\alpha}$.

Inequality (5.5) follows from (5.3) as proved in [22]. For the reader’s convenience, we include it here. Consider the convex function $g(t) = u(tx)$, $t \in [-1, 1]$. Replacing $g$ by $g/g(1)$, we may assume that $g(1) = 1$. Let $\psi(t) = g(t + \frac{1}{2}) - g'(\frac{1}{2})t - g(\frac{1}{2})$. Then $\psi(0) = 0$, $\psi \geq 0$. If $g(\frac{1}{2}) > \frac{1-\sigma}{2}$, by convexity we have $1 + \varepsilon \geq g'(\frac{1}{2}) \geq 1 - \varepsilon$ and $\psi(-\frac{1}{2}) \leq \varepsilon$. Applying (5.3) to $\psi$, we have $\psi(-\frac{1}{2}) \leq 2\psi(-\frac{1}{2}) \leq 2\varepsilon$. Hence $g(-\frac{1}{2}\theta^{-1} + \frac{1}{2}) < 0$ when $\varepsilon < \frac{1-\sigma}{\sqrt{2}}$, we reach a contradiction as $u \geq 0$. \hfill $\Box$

We remark that the estimate (5.1) was also obtained in [17] for strictly $c$-convex solutions of general Monge–Ampère equations arising in the optimal transportation by a duality argument.

### 5.2. $C^{1,1}$ regularity

Assume that $\psi$ is uniformly convex. Denote $T = \{ x \in \Omega : u(x) = \psi(x) \}$ and $F = \Omega - T$. Let $G_T, G_F$ be the graph of $u$ over $T, F$, respectively. For any point $p \in \partial G_F$, we may choose a proper coordinate system such that $p$ is the origin; and by a rotation in $\mathbb{R}^3$, we may also assume that $\{ x_3 = 0 \}$ is a tangent plane of $G_\psi$. Therefore, $\psi(0) = 0$, $D\psi(0) = 0$, $u \geq \psi$ and $\psi$ is uniformly convex.
Lemma 5.2. Assume that $\psi$ is uniformly convex. There exist two positive constants $C_1, C_2 > 0$ such that

$$C_1 |x|^2 \leq u(x) \leq C_2 |x|^2. \quad (5.7)$$

Proof. The first inequality follows from the uniform convexity of $\psi$. That is

$$u(x) \geq \psi(x) \geq C_1 |x|^2$$

as $\{x_3 = 0\}$ is the tangent plane of $G_\psi$ at the origin.

For the second inequality, suppose by contradiction that it is not true, then there is a sequence of points $x_k$ with $|x_k| \to 0$ such that $u(x_k) \geq 2^k |x_k|^2$. We claim that

$$|N_u(E_{\varepsilon_k})| \geq C 2^{k/2} \varepsilon_k^{n/2} \quad (5.8)$$

where $\varepsilon_k = u(x_k)$, $E_{\varepsilon} = \{x \in \Omega: u(x) < \varepsilon\}$. To prove (5.8), by a rescaling

$$u \to \varepsilon_k^{-1} u, \quad x \to \varepsilon_k^{-1/2} x,$$

we may assume $\varepsilon = 1$. Let $v$ be a convex function defined on the entire $\mathbb{R}^2$ such that $v(0) = 0$, $\nabla v = \mathbf{1}$ on $\partial E_1 = \partial \{u < 1\}$, and $v$ is homogeneous of degree 1. Then the graph of $v$ is a convex cone with vertex at the origin. By the convexity of $u$ we have

$$N_v(E_1) \subset N_u(E_1).$$

By the first inequality (5.7), we have

$$N_v(E_1) \supset B_{C_1^{1/2}}(0),$$

the ball of radius $C_1^{1/2}$. By the assumption that $1 = v(x_k) = u(x_k) > 2^k |x_k|^2$, the slope of $v$ at $x_k$ is greater than $2^{k/2}$. Hence there exists a point $\hat{\mathbf{p}} \in N_v(E_1)$ such that $|\hat{\mathbf{p}}| \geq 2^{k/2}$. Finally noting that $N_v(E_1) = N_v(\mathbb{R}^2)$ is a convex set as $v$ is a convex cone, we obtain

$$|N_v(E_1)| \geq C C_1^{(n-1)/2} |\hat{\mathbf{p}}| \geq C 2^{k/2}.$$

By rescaling back, we then obtain $|N_u(E_{\varepsilon_k})| \geq C 2^{k/2} \varepsilon_k^{n/2}$.

On the other hand, by the first inequality in (5.7) we have $|E_\varepsilon| \leq C \varepsilon^{n/2}$. Hence by the determinant estimate in Section 2.5 we have

$$|N_u(E_{\varepsilon_k})| = \int_{E_{\varepsilon_k}} \det D^2 u \leq C \varepsilon_k^{n/2}.$$

When $k$ is sufficiently large, we reach a contradiction. $\square$

Corollary 5.1. There is no line segment on $G_F$ with an endpoint on $\partial G_F$.

Now we prove the second part of Theorem 1.1.

Theorem 5.1. Suppose that $\psi$ is uniformly convex. Then $u$ is $C^{1,1}$ smooth in a neighborhood of $\partial F$. 
Proof. When $\alpha = \frac{1}{n+2}$, the $C^{1,1}$ regularity was obtained in [19] for enclosed convex hypersurfaces with maximal affine area, where the affine invariant property plays a crucial role. But for general $0 < \alpha < \frac{1}{n+2}$, we need to rotate the graph $\mathcal{G}$ in $\mathbb{R}^3$ and use the a priori determinant estimates in Section 2. Note that the dimension two is needed in the proof of strict convexity, see Lemmas 4.2 and 4.4.

Let $p = (p_1, p_2, p_3)$ be a point on $\mathcal{G}_F$, close to $\partial \mathcal{G}_F$. Let $\delta = \text{dist}(p, \partial \mathcal{G}_F)$ (Euclidean distance). Choosing a proper coordinate system we suppose the origin is a point on $\partial \mathcal{G}_F$ and $|p| = \delta$. By a rotation transform, suppose furthermore that $G_0^F \subset \{x_3 > 0\}$, and near the origin $u$ satisfies (5.7).

Let $u_\delta(x) = \delta^{-2} u(\delta x)$ and let $p_\delta = (\frac{p_1}{\delta}, \frac{p_2}{\delta}, \frac{p_3}{\delta})$. Then by (5.7),

$$C_1 |x|^2 \leq u_\delta(x) \leq C_2 |x|^2.$$  \hspace{1cm} (5.9)

From Section 4, $u_\delta$ is strictly convex near $p_\delta$. The a priori estimates, Lemma 2.4 in Section 2 and the approximation in Section 3 enable us to apply the argument in Lemma 5.2 to $u_\delta$, we then infer that there exist constants $C_1, C_2 > 0$ such that

$$C_1 I \leq D^2 u_\delta(p) \leq C_2 I$$

for any $\bar{p}$ near $p_\delta$, where $I$ is the unit matrix. The constants $C_1$ and $C_2$ are independent of $\delta$. By our rescaling, $D^2 u(p) = D^2 u_\delta(p_\delta)$. Hence the second derivatives of $u$ are uniformly bounded near $\partial F$. This completes the proof. \hfill \Box

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