On the use of the autonomous Birkhoff equations in Lie series perturbation theory

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Abstract

In this article, we present the Lie transformation algorithm for autonomous Birkhoff systems. Here, we are referring to Hamiltonian systems that obey a symplectic structure of the general form. The Birkhoff equations are derived from the linear first-order Pfaff-Birkhoff variational principle, which is more general than the Hamilton principle. The use of 1-form in the formulation of the equations of motion in dynamics makes the Birkhoff method more universal and flexible. Birkhoff’s equations have a tensorial character, so their form is independent of the coordinate system used. Two examples of normalization in the restricted three-body problem are given to illustrate the application of the algorithm in perturbation theory. The efficiency of this algorithm for problems of asymptotic integration in dynamics is discussed for the case where there is a need to use non-canonical variables in the phase space.

Keywords: Lie transformations, perturbation theory, averaging method, Birkhoff’s equations, restricted three-body problem, satellites dynamics, Pfaffian.

1 Introduction

Birkhoff’s autonomous equations are the Hamilton equations expressed in terms of non-canonical variables in phase space. We assume that a local coordinate transformation exists and is not explicitly dependent on time, and equations can always be reduced to canonical form (Darboux theorem; see e.g., [Arnold (1989)]). In this article, we show that the Birkhoff equations can be useful in problems of Celestial Mechanics, and in particular, in perturbation theory.

The construction of analytical models in high-order perturbation theory requires rather cumbersome calculations, which is practically impossible without modern computer algebra systems. The Lie transformation algorithm ([Hori(1966), Deprit(1969)]) is best suited for algebraic manipulations of this kind. Despite the seeming simplicity of the method, in the process of solving practical problems, it is usually required to perform a large number of transformations of different systems of variables. For example, theoretical constructions in problems of perturbation theory are more easily performed in canonical variables, but in practice it is not always convenient to use them.

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There are more serious problems as well. Hamiltonian and Lagrangian formulations of dynamics are associated with some constraints. In the former, the variables must be canonical, in the latter, transformations of variables can be done only in the configuration space.

In this context, a method which is based on Pfaffian or linear differential form is more flexible (Blimovitch (1942), Broucke (1978)). Pfaff’s equations or the 'associated Pfaff equations' were published by Pfaff (1815). However, in this publication, these equations were not considered in the quality of the analytical dynamic equations. These properties of equations were identified in full by Birkhoff (1927). Birkhoff also showed that the equations could be derived from a variational principle. For these reasons, Santilli (Santilli(1983)) proposed calling the equations 'Birkhoff equations'. Variational principle came to be called the Pfaff-Birkhoff integral variational principle. This terminology has now been adopted by a number of authors (e.g., Sun(2005), Zhang Xing-wu and et.al. (2002)). In our work, we use the study by Santilli (1983), which showed that the Birkhoff equations preserved the Lie algebra character of the canonical Hamilton equations, and that these equations have the most general symplectic structure in local coordinates.

The application of the Pfaffian or 1-form to the formulation of equations of motion in Celestial Mechanics was investigated by Broucke (1978), who showed that the method enabled the use of a very broad class of variables in constructing equations of motion - in particular, the classical Kepler elements. In this case, the equations of motion are the Lagrange planetary equations. The possibility of using Kepler’s elements as the orbital coordinates was considered by Abraham & Marsden (1978).

The concept, in accordance with which all operations of the Lie transformation algorithm for certain problems of Celestial Mechanics can be made in the Kepler elements, was first outlined by Kholshevnikov (1973). The Lie transformation algorithm, presented in the Kepler elements, was used to construct the analytical theory of Phoebe, the ninth satellite of Saturn (Boronenko & Shmidt (1990)).

This paper provides a general method for constructing the Lie transformation algorithm for the autonomous Birkhoff systems, and here we consider only the formal aspects of dynamics that have no relation to the issues of convergence or divergence of series and to the problems of equilibrium.

The paper is organized as follows.

Section 2 provides a description of the method. Here, we briefly describe the Lie transformation algorithm for Hamiltonian systems for the case where the solution of the problem requires a great number of approximations. We then introduce special variables in phase space and show the possibility of using the autonomous Birkhoff equations in the Lie series perturbation theory.

In Sect.3 we present two examples using the algorithm in problems of perturbation theory. In the first example, we consider the satellite case of the spatial restricted three-body problem. The averaging method based on Lie transformations of Birkhoffian system is proposed for the case where the expansions for short-period perturbations are presented in powers of $m$ (ratio of mean motions of Sun and satellite), but in closed form with respect to eccentricity and inclination.

In the second example, we present an analytical solution of the restricted three-body problem using the Delaunay arguments. Unlike the previous example, here we deal with
an explicit expression of the perturbing function in terms of the mean anomaly of the satellite. Therefore, all the expansions considered now include power series in the eccentricity of the satellite orbit. We applied a similar algorithm in an earlier work for the construction of an analytical theory of motion of Phoebe, the ninth satellite of Saturn [Boronenko & Shmidt (1990)]. In this paper we derived this algorithm in the context of the Birkhoff theory.

Section 4 contains discussion of the results and our conclusions.

2 Description of the method

2.1 Lie transformation algorithm

Here we provide a brief description of the Lie transformation method applied to canonical perturbation theory in the case of high-order computations. In this algorithm, all operations are based on the Lie series [Hori(1966), Ferraz-Mello(2007), Kholshevnikov(1985)].

Let us consider the Hamiltonian in the form of formal truncated series in $\tau$:

$$H(\eta) = H_{00}(\eta) + \tau H_{01}(\eta) + \cdots + \tau^m H_{0m},$$

where through $\eta$, two sets are denoted: $\eta_1, \ldots \eta_l$ are generalized coordinates and $\eta_{l+1}, \ldots \eta_n$ are generalized momenta. Thus, the dimension of the phase space is given by $n = 2l$. We assume that $H(\eta)$ is a smooth time-independent function, $\tau$ is a constant small parameter and $H_{00}$ is the Hamiltonian of an integrable system. The corresponding canonical equations have the form:

$$\dot{\eta}_i = \frac{\partial H(\eta, \tau)}{\partial \eta_i}, \quad \dot{\eta}_{i+l} = -\frac{\partial H(\eta, \tau)}{\partial \eta_i}, \quad i = 1, \ldots, l.\quad (2)$$

We introduce a transformation $\eta \rightarrow \tilde{\eta}$ that is determined by the Lie series. The concept of Lie series follows from the solution of the Cauchy problem, which can be formulated in the following way (see e.g., Kholshevnikov(1985)):

$$\dot{\eta}_i = \frac{\partial W(\eta, \tau)}{\partial \eta_i}, \quad \dot{\eta}_{i+l} = -\frac{\partial W(\eta, \tau)}{\partial \eta_i},
\eta_i|_{t=0} = \tilde{\eta}_i, \quad \eta_{i+l}|_{t=0} = \tilde{\eta}_{i+l}, \quad i = 1, \ldots, l.\quad (3)$$

where $W = W(\eta, \tau)$ is some analytic function in a neighborhood of the initial point $\tilde{\eta}_0 = (\tilde{\eta}_01, \ldots, \tilde{\eta}_0n)$. The solution of (3) at $t = \tau$ is determined by the Lie series

$$\eta_j = \sum_{m=0}^{\infty} \frac{\tau^m}{m!} L_W^m \tilde{\eta}_j = e^{\tau L_W} \tilde{\eta}_j, \quad j = 1, \ldots, n.\quad (4)$$

The general solutions (4) is a one-parameter group of canonical transformations. In accordance with the terminology adopted in the theory of Lie groups, the function $W$ is called the Lie generating function or the Lie generator of the group (see e.g., Ferraz-Mello(2007)).
In (4) the Lie generator represents the Poisson bracket of the form:

\[ L_W = \sum_{i=1}^{l} \left( \frac{\partial W}{\partial \tilde{\eta}_{i}} \frac{\partial}{\partial \eta_{i}} - \frac{\partial W}{\partial \eta_{i}} \frac{\partial}{\partial \tilde{\eta}_{i}} \right). \]  

Let us assume that the function \( W \) is represented as a truncated series:

\[ W(\eta) = \tau W_{01}(\eta) + \cdots + \tau^m W_{0n}. \]  

The transformation of the original Hamiltonian \( H(\eta) \) can then be performed using the following recursive algorithm:

\[
\tilde{H}_{0n} = \sum_{i=0}^{n} H_{i,n-i}, \quad n = 0, \ldots, m;
\]

\[
H_{i,n-i} = \frac{1}{i} \sum_{\rho=0}^{n-i} \{ H_{i-1,n-i-\rho}, W_{0,\rho+1} \}, \quad i \neq 0,
\]

where the expression in curly brackets is a Poisson bracket. We assume that the Hamiltonian is not dependent on time, and this allows us to write \( \tilde{H}(\tilde{\eta}) = H(\eta) \). A new Hamiltonian is defined as follows:

\[ \tilde{H} = \tilde{H}_{00} + \tau \tilde{H}_{01} + \cdots + \tau^m \tilde{H}_{0m}. \]  

The recursive algorithm (7) is easy to demonstrate with the following triangle:

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| H00 | H01 | H02 | H03 |   |   |
| H10 | H11 | H12 |   |   |   |
| H20 | H21 |   |   |   |   |
| H30 |   |   |   |   |   |

In the future, we will use the notation for the variables \( \eta \) without tilde, because in the Lie transformation algorithm, new or old variables are generally defined from the context.

It is known (see e.g., [Ferraz-Mello(2007)]) that Lie transformations have the following general properties. These transformations are defined as infinitesimal canonical transformations. To obtain the inverse transformation, we must reverse the sign of the function \( W \). Transformations of variables and functions of these variables are performed by the same algorithm (9).

The above-described method offers a systematic approach to the problem of the separation of variables in the differential equations of the perturbation theory. This algorithm is especially effective in combination with averaging methods in Hamiltonian systems. Such approaches to solving the problems of perturbation theory are well known in celestial mechanics – for example, the method of Delaunay, Von Zeipel, and Kolmogorov-Arnold. These methods use the averaging principle and the concept of dividing motion into slow drift and fast oscillations (see e.g., [Brumberg(1995)])

In developing similar algorithms, it is sometimes preferable to use the orbital elements or their functions as variables in the phase space. In the next section we will show that all operations of the Lie transformations for Hamiltonian systems can be performed with the help of some special variables, which are non-canonical.
2.2 Introduction of the special variables in phase space

Let us consider some function \( f_{ij}(\eta) \equiv H_{ij}(\eta) \) from triangle (9). The Lie transformation algorithms are reduced to the successive calculation of the Poisson brackets of the following form:

\[
L_k f_{ij} = \sum_{c=1}^{l} \left( \frac{\partial W_{0k}(\eta)}{\partial \eta_{c+t}} \frac{\partial f_{ij}(\eta)}{\partial \eta_c} - \frac{\partial W_{0k}(\eta)}{\partial \eta_c} \frac{\partial f_{ij}(\eta)}{\partial \eta_{c+t}} \right), \quad i + j = k, \tag{10}
\]

where \( W_{0k} \) are coefficients in the truncated series (6). It is assumed that all the functions \( W_{0k}(\eta) \) and \( f_{ij}(\eta) \) are sufficiently smooth.

Now let us suppose that the orbit in phase space is determined by the values of some suitable parameters \( \theta_1, \ldots, \theta_n \), which we accept as new variables. We then enter the following transformation:

\[
\eta_p = \eta_p(\theta_1, \ldots, \theta_n), \quad p = 1, 2, \ldots, n. \tag{11}
\]

It is assumed that transformations (11) are analytic in their region of definition. In addition, we assume that all transformations are regular, i.e., their Jacobian does not vanish:

\[
J = \frac{\partial(\eta_1, \ldots, \eta_n)}{\partial(\theta_1, \ldots, \theta_n)} \neq 0. \tag{12}
\]

As a consequence, the transformations (11) are invertible in their region of definition, with the inverse transformations given by:

\[
\theta_p = \theta_p(\eta_1, \ldots, \eta_n). \tag{13}
\]

After performing the substitution (11), the Lie generator (10) takes the form:

\[
L_k f_{ij}(\theta) = \sum_{p,s=1}^{n} \left( \sum_{c=1}^{l} \left( \frac{\partial \theta_p}{\partial \eta_{c+t}} \frac{\partial \theta_s}{\partial \eta_c} - \frac{\partial \theta_p}{\partial \eta_c} \frac{\partial \theta_s}{\partial \eta_{c+t}} \right) \frac{\partial W_{0k}(\theta)}{\partial \theta_s} \right) \frac{\partial f_{ij}(\theta)}{\partial \theta_p}, \tag{14}
\]

We introduce the Poisson brackets

\[
\{\theta_p, \theta_s\} = \sum_{c=1}^{l} \left( \frac{\partial \theta_p}{\partial \eta_{c+t}} \frac{\partial \theta_s}{\partial \eta_c} - \frac{\partial \theta_p}{\partial \eta_c} \frac{\partial \theta_s}{\partial \eta_{c+t}} \right). \tag{15}
\]

Now, we can rewrite the expression (14) as follows:

\[
L_k f_{ij}(\theta) = \sum_{p=1}^{n} \left( \left\{ \theta_p, \theta_s \right\} \frac{\partial W_{0k}(\theta)}{\partial \theta_s} \right) \frac{\partial f_{ij}(\theta)}{\partial \theta_p}, \tag{16}
\]

or

\[
L_k f_{ij}(\theta) = \sum_{p,s=1}^{n} \left( a_{ps}(\theta) \frac{\partial W_{0k}(\theta)}{\partial \theta_s} \right) \frac{\partial f_{ij}(\theta)}{\partial \theta_p}, \tag{17}
\]
where \( a_{ps}(θ) = \{θ_p, θ_s\} \) is a skew-symmetric matrix of Poisson brackets. The Lie generator can also be represented as follows:

\[
L_k f_{ij}(θ) = \sum_{p=1}^{n} W_{pk} \frac{∂f_{ij}}{∂θ_p}, \quad (k = i + j).
\] (18)

Here, \( W_{pk} \) are elements of the matrix \( Ψ = (W_{pk}), (p = 1, \ldots, n; k = 1, \ldots, m) \), and \( k \) is the order of operation.

Performing a similar substitution (13) in (2), after some transformations we obtain the following equations:

\[
\dot{θ}_p = \sum_{s=1}^{n} \{θ_p, θ_s\} \frac{∂B(θ, τ)}{∂θ_s}, \quad (p = 1, \ldots, n)
\] (19)

or

\[
\dot{θ}_p = \sum_{s=1}^{n} a_{ps}(θ) \frac{∂B(θ, τ)}{∂θ_s}, \quad (p = 1, \ldots, n),
\] (20)

where the function \( B(θ, τ) \) is the Hamiltonian expressed in terms of the variables \( θ \). The system of ordinary differential equations (19) coincides with the variational Lagrange equations in the general form \[Smart(1961)\]. Therefore, we can interpret our special variables, which can now include the Keplerian orbital elements.

If we then compare the right-hand sides of (16) and (19), we see that the form of the expressions

\[
\sum_{s=1}^{n} \{θ_p, θ_s\} \frac{∂W_{0k}(θ)}{∂θ_s} \quad \text{and} \quad \sum_{s=1}^{n} \{θ_p, θ_s\} \frac{∂B(θ, τ)}{∂θ_s}
\] (21)

coincides. Thus, the right-hand sides of equations (19) or (20) can be used to form the matrix \( Ψ \), if we know the generating function. Consequently, equations (19) and (20) can be adopted as basic equations; then, using the Lie generator (18), all calculations can be performed in the variables \( θ \) remaining within the framework of the Lie transformation theory for canonical systems. In the next section, we define the meaning of the obtained relationships.

### 2.3 Birkhoff’s equations

Consider an extended phase space with \( m = 2l + 1 \) dimensions, where \( l \) is the number of degrees of freedom of the system. Let us turn to the usual notations of canonical variables: \( q, p \). In the space under consideration, we then have the following set of generalized coordinates and generalized momenta: \( q (q_1, \ldots, q_l); p (p_1, \ldots, p_l) \). The additional variable is the time \( t \).

Let the function \( H(q, p, t) \) be the Hamiltonian of a dynamical system. For this system, we introduce the following 1-form (or Pfaffian):

\[
ω^1 = p dq − H dt.
\] (22)

It is known \[Arnold (1989)\] that in the extended phase space, the phase trajectories of a dynamical system with the Hamiltonian \( H(q, p, t) \) are the vortex lines of the form \( ω^1 \). If the variables used are non-canonical, then the Pfaffian (22) can be written in a large number of different forms \[Broucke (1978)\]. This means that we can write the equations of motion
in any suitable system of coordinates in the extended phase space with \( m \) dimensions. It follows, therefore, the ratio [Arnold (1989)]:

\[
p dq - H dt = \Theta_1 d\theta_1 + \ldots + \Theta_m d\theta_m, \tag{23}
\]

where \( \Theta_i = \Theta_i(\theta) \) are smooth functions, and \( \theta_m = t, \Theta_m = H(\theta) \). In addition, the function \( H(\theta) \) will be denoted by \( B \).

In the variables \( \theta \), the Pfaffian has the form

\[
\Phi^1 = \Theta_1 d\theta_1 + \ldots + \Theta_m \theta_m. \tag{24}
\]

The set

\[
P = (\Theta_1, \ldots, \Theta_m) \tag{25}
\]

is called the Pfaff vector of the considered dynamic system [Broucke (1978)]. If the Pfaffian (24) is an exact differential, then the curl of \( P \) is equal to zero. From here, we follow the dynamic equations:

\[
\sum_{j=1}^m \left( \frac{\partial \Theta_i}{\partial \theta_j} - \frac{\partial \Theta_j}{\partial \theta_i} \right) d\theta_j = \sum_{j=1}^m b_{ij} d\theta_j = 0, \quad i = 1, \ldots, m. \tag{26}
\]

These equations are considered in the work of [Broucke (1978)].

The Eq. (26) is represented in the phase space with \( m = 2l + 1 \) dimensions. However, further study of these equations is not convenient, since \( b_{ij} \) is the askew-symmetric matrix of odd order, and its determinant is equal to zero.

We can obtain the dynamic equations in the even-dimensional phase space using the Pfaff-Birkhoff variational principle. Let us define 1-form in phase space with \( n = 2l \) dimensions

\[
\omega^1 = \sum_{\nu=1}^n R_{\nu}(\theta) \, d\theta_{\nu} = \sum_{i=1}^n \Theta_i \, d\theta_i, \tag{27}
\]

where \( R_{\nu} \) is traditionally accepted designation of the Birkhoff functions (see e.g., [Santilli (1983)]).

In this work, we assume that a set of Birkhoff’s functions \( (R_1, \ldots, R_n) \) coincides with the elements \( (\Theta_1, \ldots, \Theta_{(m-1)}) \) of the Pfaff vector. In addition, for convenience of presentation, we denote the Birkhoff functions by a set: \( (\Theta_1, \ldots, \Theta_n) \).

In the introduced notations, the Pfaff-Birkhoff variational principle can be written as follows:

\[
\delta S = \delta \int_{t_1}^{t_2} \left( \sum_{j=1}^n \Theta_j \dot{\theta}_j - B \right) = 0. \tag{28}
\]

The integrand in (28) is a linear function of the derivatives \( \dot{\theta}_j \) with coefficients \( \Theta_j(\theta), B(\theta) \). By using the first-order variation \( \delta S = 0 \) with fixed end-point conditions \( \delta \theta_j(t_1) = 0, \delta \theta_j(t_2) = 0, \, j = 1, \ldots, n \), the autonomous Birkhoff equations can be obtained as follows [Santilli (1983)]:

\[
\sum_{p=1}^n \left( \frac{\partial \Theta_p}{\partial \theta_s} - \frac{\partial \Theta_s}{\partial \theta_p} \right) \dot{\theta}_p - \frac{\partial B}{\partial \theta_s} = 0, \quad (s = 1, \ldots, n; \, n = 2l). \tag{29}
\]
In (29), the function $B(\theta)$ is called the Birkhoffian, and

$$
\Omega_{ps} = \frac{\partial \Theta_s}{\partial \theta_p} - \frac{\partial \Theta_p}{\partial \theta_s}
$$

(30)
is called the Birkhoff tensor. Using the terminology of Santilli(1983), it is a covariant Birkhoff tensor.

The symplectic structure $\Omega$ in the space under consideration is defined as the external differential of the 1-form $\varpi$

$$
\Omega = d\varpi^1
$$

(31)
or

$$
\Omega = \sum_{p=1}^{n} \Omega_{ps} d\theta_s \wedge d\theta_p, \quad (s = 1, \ldots, n),
$$

(32)
where $\Omega_{ps}$ is defined by the ratio (30).

It can be shown that the matrix $(\Omega_{ps})$ coincides with the skew-symmetric matrix $(\omega_{ps})$ of the Lagrange brackets. From (23) and (11) we find

$$
\Theta_s = p_i \frac{\partial q_i}{\partial \theta_s} = \eta_{l+i} \frac{\partial \eta_i}{\partial \theta_s}.
$$

(33)

Next, we substitute

$$
\Theta_s = \eta_{l+i} \frac{\partial \eta_i}{\partial \theta_s}, \quad \Theta_p = \eta_{l+i} \frac{\partial \eta_i}{\partial \theta_p}
$$

(34)
to $(\Omega_{ps})$. The result is a skew-symmetric square matrix, each element of which represents the Lagrange bracket

$$
\omega_{ps} = \sum_{i=1}^{l} \left( \frac{\partial \eta_{l+i}}{\partial \theta_p} \frac{\partial \eta_i}{\partial \theta_s} - \frac{\partial \eta_{l+i}}{\partial \theta_s} \frac{\partial \eta_i}{\partial \theta_p} \right).
$$

(35)
We suppose that the dynamic system is non-singular, i.e.,

$$
\det(\Omega_{ps}) \neq 0.
$$

(36)
Thus, matrix $(\omega_{ps})$ is non-degenerate. Therefore there is an inverse matrix

$$
(a_{ps}) = (\omega_{ps})^{-1}.
$$

(37)
Matrix (37) is also a skew-symmetric square matrix, and each element of this matrix is the Poisson bracket. A detailed derivation can be found in Smart(1961). The matrix $(a_{ps})$ consists of the Poisson brackets of the following form:

$$
a_{ps} = \sum_{i=1}^{l} \left( \frac{\partial \theta_p}{\partial \eta_{l+i}} \frac{\partial \theta_s}{\partial \eta_i} - \frac{\partial \theta_p}{\partial \eta_i} \frac{\partial \theta_s}{\partial \eta_{l+i}} \right).
$$

(38)
From the above, it follows that the matrix $(a_{ps})$ defines the Birkhoff tensor, which is expressed in terms of Poisson brackets. In accordance with the terminology of Santilli(1983),
$a_{ps}$ is the contravariant Birkhoff tensor, also called the Lie tensor. Thus, Eq. (20) are the Birkhoff equations, which are presented in a contravariant form.

In this section, we have shown that the representation of the Lie generator for canonical systems in terms of the special variables of the phase space leads to its expression in terms of the Birkhoff tensor $a_{ps}$. It follows that all the components of the algorithm can be expressed using tensors. This makes it possible to use of the autonomous Birkhoff’s equations for solving of problems of dynamics in the non-canonical variables in the phase space.

3 Some applications

3.1 Closed form representation of the short-period perturbations in the motion of a satellite. The first example

We now consider the motion of a satellite under the action of gravity of the planet and the Sun, provided that all three bodies are material points. The satellite has an infinitely small mass, i.e., it has no gravitational effect on the other two bodies. The central body is a planet with mass $m_0$. The Sun, with mass $m'$, moves around the planet in a circular orbit located in the main coordinate plane. We use a rotating coordinate system that is the same as that used by Hill’s group in their studies of lunar theory. In the Delaunay canonical elements, $p = (L, G, H)$, $q = (l, g, h)$, the equations of motion have the form

\[
\dot{q}_k = \frac{\partial F}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial F}{\partial q_k}, \quad k = 1, 2, 3, \tag{39}
\]

where $F(p, q)$ is the Hamiltonian of the perturbed problem:

\[
F = F_{00} + \nu F_{01} + \nu^2 F_{02}, \tag{40}
\]

\[
F_{00} = -\frac{\mu^2}{2L^2}, \quad F_{01} = -H, \tag{41}
\]

\[
F_{02} = -\sum_{n=2}^{\infty} r^n \, P_n[\cos(S)]. \tag{42}
\]

In (40), (41), and (42) $\mu = G \, m_0$; $G$ is the constant of universal gravitation; $P_n$ are the Legendre polynomials (in this example, assume $n=2$), $\nu = n'$ is the small parameter, and $n'$ is the mean motion of the Sun. It is a constant formal parameter. The strength of the disturbances is characterized by the implicit parameter $m = n'/n$, where $n$ is the mean motion of the satellite, and $H$ is the Delaunay variable. In (41) and (42) it is taken into account that the canonical elements $p = (L, G, H)$ are variables of the action.

In the ratio (42)

\[
\cos(S) = \cos(f + g) \cos h - \sin(f + g) \sin h \cos i, \tag{43}
\]

where $f$ is the true anomaly of the satellite; $g = \omega$, $\omega$ determines the argument of periapsis of the satellite orbit; $h = \Omega - \lambda'$, $\Omega$ is the longitude of the ascending node of the satellite.
orbit, $\lambda'$ is the mean longitude of the Sun; $i$ is the inclination of the satellite orbit to the primary coordinate plane.

The first term $F_{00}$ in (40) is caused by the attraction of the planet in the absence of perturbations, $\nu F_{01}$ is a term that appears due to the use of a rotating coordinate system, and the third term $\nu^2 F_{02}$ is the disturbing function.

We consider the non-resonant case, and suppose that Hamiltonian is an analytic function of all variables and has period $2\pi$ in all angular variables. The Lee transformation method, in combination with the averaging of the disturbing function over the mean anomaly $l$ of the satellite, was chosen in order to eliminate terms of a short period from the Hamiltonian. The Pfaffian of this problem is given by

$$\Phi = L \, dl + G \, dg + H \, dh - F \, dt. \quad (44)$$

In this example, instead of the mean anomaly $l$, we use eccentric anomaly $u$ of the satellite in all expressions, in order to avoid an expansion in powers of eccentricity $e$ of the satellite orbit. All analytical expansions are carried out as the truncated series in $m = n'/n$, but the coefficients of these series are in closed form.

We now introduce the variables $\epsilon = (\alpha, \eta, \gamma, u, g, h)$ using the ratios:

$$L = \alpha \mu_1, \quad G = \alpha \mu_1 \eta, \quad H = \alpha \mu_1 \eta \gamma, \quad l = u - \sqrt{1 - \eta^2} \sin u, \quad g = g, \quad h = h, \quad (45)$$

where $\mu_1 = \sqrt{\mu}$; $\alpha = \sqrt{a}$, $a$ is the semi-major axis of the satellite orbit, $\eta = \sqrt{1 - e^2}$, and $\gamma = \cos i$. The Delaunay elements are presented in the form (45) for convenience of computation using analytical computer systems.

Pfaffian in the terms of the variables $\epsilon$ has the form:

$$\Phi = \alpha \mu_1 \frac{\eta}{\sqrt{1 - \eta^2}} \sin u \, d\eta + \alpha \mu_1 (1 - \sqrt{1 - \eta^2} \cos u) \, du + \alpha \mu_1 \eta \, dg + \alpha \mu_1 \eta \gamma \, dh - B(\epsilon) \, dt. \quad (46)$$

The coefficients of the differentials in (46) define the Pfaff vector $P$ (25) of the system. The Birkhoff functions, as we previously identified, are included in the Pfaff vector, and are given by the following set:

$$\left( 0, \alpha \mu_1 \frac{\eta}{\sqrt{1 - \eta^2}} \sin u, 0, \alpha \mu_1 (1 - \sqrt{1 - \eta^2} \cos u), \alpha \mu_1 \eta, \alpha \mu_1 \eta \gamma \right). \quad (47)$$

Next, we find the equations of motion by making use of (20) (Eq. (29) can be used to
verify the correctness of the output):

\[
\begin{align*}
\frac{d\alpha}{dt} &= -\frac{\alpha^2}{\mu_1 r} \frac{\partial B}{\partial u}, \\
\frac{d\eta}{dt} &= \frac{\alpha \eta}{\mu_1 r} \frac{\partial B}{\partial u} - \frac{1}{\mu_1 \alpha} \frac{\partial B}{\partial g}, \\
\frac{d\gamma}{dt} &= \frac{\mu_1 \alpha \eta}{\mu_1 \alpha \eta} \frac{\partial B}{\partial g} - \frac{1}{\mu_1} \frac{\partial B}{\partial h}, \\
\frac{du}{dt} &= \frac{\alpha^2}{\mu_1 r} \frac{\partial B}{\partial \alpha} - \frac{\alpha \eta}{\mu_1 \alpha} \frac{\partial B}{\partial \eta} + \frac{\alpha \eta \sin u}{\mu_1 r \sqrt{1 - \eta^2}} \frac{\partial B}{\partial g}, \\
\frac{dg}{dt} &= \frac{1}{\mu_1 \alpha} \frac{\partial B}{\partial \eta} - \frac{\alpha \eta \sin u}{\mu_1 r \sqrt{1 - \eta^2}} \frac{\partial B}{\partial u} - \frac{\gamma}{\mu_1 \alpha \eta} \frac{\partial B}{\partial \gamma}, \\
\frac{dh}{dt} &= \frac{1}{\mu_1 \alpha \eta} \frac{\partial B}{\partial \gamma},
\end{align*}
\]

(48)

where \( B(\epsilon) \) is the Birkhoffian (Hamiltonian expressed in terms of the variables \( \epsilon \)):

\[
B = B_{00} + \nu B_{01} + \nu^2 B_{02},
\]

(49)

\[
B_{00} = -\frac{\mu_1^2}{2\alpha^2}, \quad B_{01} = -\mu_1 \alpha \eta \gamma.
\]

(50)

In \( B_{02} \), \( B_{01} \) is the function \( B(\epsilon) \) expressed in terms of the variables \( \epsilon \). As shown by Birkhoff (1927), in the autonomous case the Birkhoffian is the integral of motion. The Birkhoff equations (48) can be written in general form as follows:

\[
\dot{\epsilon}_i = \sum_{j=1}^{6} a_{ij}(\epsilon) \frac{\partial B(\epsilon)}{\partial \epsilon_j}, \quad i = 1, 2, \ldots, 6,
\]

(51)

where \( a_{ij}(\epsilon) \) is a skew-symmetric matrix, obtained from (38):

\[
\begin{pmatrix}
0 & 0 & 0 & \frac{\alpha^2}{\mu_1 r} & 0 & 0 \\
0 & 0 & 0 & \frac{\alpha \eta}{\mu_1 r} & -\frac{1}{\mu_1 \alpha} & 0 \\
0 & 0 & 0 & \frac{\alpha \eta}{\mu_1 r} & -\frac{\mu_1 \alpha \eta}{\mu_1 \alpha \eta} & 0 \\
\frac{\alpha^2}{\mu_1 r} & -\frac{\alpha \eta}{\mu_1 r} & 0 & 0 & \frac{\alpha \eta \sin u}{\mu_1 r \sqrt{1 - \eta^2}} & 0 \\
0 & \frac{1}{\mu_1 \alpha} & -\frac{\gamma}{\mu_1 \alpha \eta} & -\frac{\alpha \eta \sin u}{\mu_1 r \sqrt{1 - \eta^2}} & 0 & 0 \\
0 & 0 & \frac{1}{\mu_1 \alpha \eta} & 0 & 0 & 0
\end{pmatrix}
\]

In addition, we introduce the generating function in the form:

\[
W(\epsilon) = \nu W_{01}(\epsilon) + \nu^2 W_{02}(\epsilon) + \cdots + \nu^k W_{0k}(\epsilon) + \ldots.
\]

(52)
The Lie generator is represented as follows[18]:

\[ L_k f_{ij}(\epsilon) = \sum_{p=1}^{6} W_{pk} \frac{\partial f_{ij}(\epsilon)}{\partial \epsilon_p}, \]  

(53)

where \( f_{ij}(\epsilon) \) are some analytical functions, \( W_{pk} \) are elements of the matrix \( \Psi = (W_{pk}) \), \((p = 1, \ldots, 6; k = 1, \ldots, 5)\), and \( k \) is the order of transformation. Each column of the matrix \( \Psi \) is determined by the ratio:

\[ W_{ik} = \sum_{j=1}^{6} a_{ij}(\epsilon) \frac{\partial W_{0k}}{\partial \epsilon_j}, \quad (i = 1, \ldots, 6). \]  

(54)

As an example, let us write the expression (54) explicitly for \( k \)-th order

\[ W_{1k} = -\frac{\alpha^2}{\mu_1 r} \frac{\partial W_{0k}}{\partial u}, \]

\[ W_{2k} = \frac{\alpha \eta}{\mu_1 r} \frac{\partial W_{0k}}{\partial u} - \frac{1}{\mu_1 \alpha} \frac{\partial W_{0k}}{\partial g}, \]

\[ W_{3k} = \frac{\gamma}{\mu_1 \alpha \eta} \frac{\partial W_{0k}}{\partial g} - \frac{1}{\mu_1 \alpha} \frac{\partial W_{0k}}{\partial h}, \]

\[ W_{4k} = \frac{\alpha^2}{\mu_1 r} \frac{\partial W_{0k}}{\partial \alpha} - \frac{\alpha \eta}{\mu_1 r} \frac{\partial W_{0k}}{\partial \eta} + \frac{\alpha \eta \sin u}{\mu_1 r \sqrt{1 - \eta^2}} \frac{\partial W_{0k}}{\partial g}, \]

\[ W_{5k} = \frac{1}{\mu_1 \alpha} \frac{\partial W_{0k}}{\partial \eta} - \frac{\alpha \eta \sin u}{\mu_1 r \sqrt{1 - \eta^2}} \frac{\partial W_{0k}}{\partial u} - \frac{\gamma}{\mu_1 \alpha \eta} \frac{\partial W_{0k}}{\partial \gamma}, \]

\[ W_{6k} = \frac{1}{\mu_1 \alpha \eta} \frac{\partial W_{0k}}{\partial \gamma}. \]  

(55)

Here, the functions \( W_{0k} \) are found from the homological equation [Ferraz-Mello(2007)]

\[ L_k B_{00} = B_{0k}^* - \Delta_k^*. \]  

(56)

The expression for \( \Delta_k^* \) will be determined later. The function \( B_{0k}^* \) is found from the relation

\[ B_{0k}^* = \frac{1}{2\pi} \int_0^{2\pi} \Delta_k^* dl = \frac{1}{2\pi} \int_0^{2\pi} \Delta_k^*(1 - \sqrt{1 - \eta^2 \cos(u)}) du. \]  

(57)

For a better understanding of the algorithm let us introduce the Lie triangle [9] in the form:

\[ B_{00} \quad B_{01} \quad B_{02} \quad B_{03} \quad B_{04} \ldots \]

\[ B_{10} \quad B_{11} \quad B_{12} \quad B_{13} \ldots \]

\[ B_{20} \quad B_{21} \quad B_{22} \ldots \]

\[ B_{30} \quad B_{31} \ldots \]

\[ B_{40} \ldots \]  

(58)
In (58), functions $B_0^0, B_0^1,$ and $B_0^2$ are the coefficients of the original Birkhoffian (49), and $B_{03} = B_{04} = B_{05} = \ldots = 0$.

The Birkhoffian of the transformed system is written as

$$B^* = B_{00}^* + \nu B_{01}^* + \nu^2 B_{02}^* + \nu^3 B_{03}^* + \nu^4 B_{04}^* + \nu^5 B_{05}^* + \ldots.$$  \hfill (59)

In the algorithm process, short-period terms appear at higher orders; therefore, we also find the functions $B_{03}^*, B_{04}^*, B_{05}^*, \ldots$.

The transformation $B(\epsilon) \to B^*(\epsilon)$ can be carried out using the following recursive algorithm:

$$\Delta_k = \sum_{i=0}^{k} B_{i,k-i}, \quad k = 0, \ldots, m;$$

$$B_{i,k-i} = \frac{1}{i} \sum_{\rho=0}^{k-i} L_{\rho+1} B_{i-1,k-1-\rho}, \quad i \neq 0.$$  \hfill (60)

The function $\Delta_k'$ is determined from the relation (56) as follows:

$$\Delta'_k = \Delta_k - L_k B_{00}.$$  \hfill (61)

Let us consider the first three normalization steps in detail. For the most compact form of writing, we assume $\gamma = 1, h = 0, g = \omega - \lambda'$ and use the ratio $e^2 + \eta^2 = 1$. That is, we consider a planar version of the problem.

*Oder 0:* $B_{00} = B_{00}^* = -\frac{\mu_1^2}{2\alpha^2}$.

*Oder 1:* If $B_{02}$ is neglected in (49), Eq. (48) are integrable, and the transformation is identical. Therefore, $W_{01} = 0$. All elements of the first column of the matrix $\Psi$ are equal to zero and $\nu B_{01}^* = -\nu \mu_1 \alpha \eta$.

*Oder 2:* At this point, we find the diagonal elements of the second order of the Lie triangle, and we then define the homological equation:

$$B_{20} = 0, \quad B_{11} = L_2 B_{00}, \quad \Delta'_2 = B_{20} + B_{02} = B_{02},$$

$$L_2 B_{00} = B_{02}^* - B_{02}.$$  \hfill (62)

Let us enter the expression for $B_{02}$. In the assumptions adopted above, the function $B_{02}$ is written as follows:

$$B_{02} = \nu^2 \left( \frac{1}{2} - \frac{3}{2} \cos^2 S \right), \quad \cos S = \cos(f + g).$$  \hfill (63)

Then, using the formulas of the theory of Keplerian motion

$$r = \alpha^2 (1 - e \cos u), \quad \sin f = \frac{\alpha^2}{r} \eta \sin u, \quad \cos f = \frac{\alpha^2}{r} (\cos u - e),$$

where $r$ is the distance, $\alpha$ the semi-major axis, $e$ the eccentricity, $\eta$ the eccentricity of the osculating orbit, $u$ the true anomaly, and $f$ the true anomaly of the averaged motion.
where $u$ is the eccentric anomaly, we obtain the expression for the disturbing function $\nu^2B_{02}$ in the form:

$$
\nu^2B_{02} = \frac{1}{16}\nu^2\alpha^4(4 + 2e^2 + 18\epsilon^2 \cos 2g - 3(-2 + e^2 + 2\eta) \cos(2g - 2u) + (-12e + 12\epsilon\eta) \cos(2g - u) - 8\epsilon \cos u + 2\epsilon^2 \cos 2u + (6 - 3\epsilon^2 + 6\eta) \cos(2g + 2u) + (-12e - 12\epsilon\eta) \cos(2g + u)).
$$

(64)

Averaged over mean anomaly $l$ of the satellite, the function $B_{02}^*$ is found from the relation

$$
B_{02}^* = \frac{1}{2\pi} \int_0^{2\pi} B_{02} dl = \frac{1}{2\pi} \int_0^{2\pi} B_{02}(1 - \epsilon \cos(u)) du.
$$

(65)

Finally, we have

$$
\nu^2B_{02}^* = \frac{\nu^2\alpha^4}{4} \left(1 - \frac{3}{2}\epsilon^2 + \frac{15}{2}\epsilon^2 \cos(2g)\right) = \frac{\nu^2\mu_1^2}{n^2\alpha^2} \frac{1}{4} \left(1 - \frac{3}{2}\epsilon^2 + \frac{15}{2}\epsilon^2 \cos(2g)\right).
$$

Here, the multiplier $\mu_1^2/\alpha^2$ has the dimension of energy.

We then find

$$
B_{00} = -\frac{\mu_1^2}{2\alpha^2}, \quad L_2B_{00} = -\frac{\alpha^2}{\mu_1\epsilon} \frac{\partial W_{02}}{\partial \alpha} \frac{\partial B_{02}}{\partial u} = -\frac{\mu_1}{\alpha\epsilon} \frac{\partial W_{02}}{\partial u}.
$$

Now, the homological equation is written as

$$
\frac{\mu_1}{\alpha\epsilon} \frac{\partial W_{02}}{\partial u} = B_{02} - B_{02}^*
$$

(66)

and

$$
W_{02} = \int \frac{\alpha^3}{\mu_1} (B_{02} - B_{02}^*)(1 - \epsilon \cos u) du.
$$

(67)

The function $W_{02}$ should be periodic in mean anomaly $l$. Therefore, we find

$$
C = \frac{1}{2\pi} \int_0^{2\pi} W_{02}(1 - \epsilon \cos u) du = \frac{15}{16\mu_1} \alpha^7 e^2 \eta \sin(2g), \quad W_{02} = W_{02} - C.
$$

The final expression for the function $\nu^2W_{02}$ has the form

$$
\nu^2W_{02} = \nu^2\alpha^7 \frac{W_{02}'}{\mu_1} = \left(\frac{\nu^2L}{n^2}\right) W_{02}',
$$

(68)

where $n = \frac{\mu_1}{\alpha^3}$ — mean motion of the satellite, $L = \mu_1\alpha$ — is an element of Delaunay, which determines the dimension of the generating function,

$$
W_{02}' = \frac{1}{32}(-30\epsilon^2 \eta \sin(2g) + (2\epsilon - \epsilon^3 - 2\epsilon\eta) \sin(2g - 3u) + (-6 - 3\epsilon^2 + 6\eta + 6\epsilon^2 \eta) \sin(2g - 2u) + (-16\epsilon + 6\epsilon^3) \sin u + (30\epsilon - 15\epsilon^3 - 30\epsilon\eta) \sin(2g - u) + 6\epsilon^2 \sin(2u) - \frac{2}{3}\epsilon^3 \sin(3u) + (-30\epsilon + 15\epsilon^3 - 30\epsilon\eta) \sin(2g + u) + (6 + 3\epsilon^2 + 6\eta + 6\epsilon^2 \eta) \sin(2g + 2u) + (-2\epsilon + \epsilon^3 - 2\epsilon\eta) \sin(2g + 3u)).
$$

(69)
In accordance with the assumptions made above for the planar version of the problem, we define the second column of the matrix $\Psi$:

$$
W_{12} = \frac{\alpha^2}{\mu_1 r} \frac{\partial W_{02}}{\partial u},
$$

$$
W_{22} = \frac{\alpha \eta}{\mu_1 r} \frac{\partial W_{02}}{\partial u} - \frac{1}{\mu_1 \alpha} \frac{\partial W_{02}}{\partial g},
$$

$$
W_{32} = \frac{\alpha^2}{\mu_1 r} \frac{\partial W_{02}}{\partial \alpha} - \frac{\alpha \eta}{\mu_1 r} \left( \frac{\partial W_{02}}{\partial \eta} - \frac{\eta}{e} \frac{\partial W_{02}}{\partial e} \right) + \frac{\alpha \eta \sin u}{\mu_1 r} \frac{\partial W_{02}}{\partial g},
$$

$$
W_{42} = \frac{1}{\mu_1 \alpha} \left( \frac{\partial W_{02}}{\partial \eta} - \frac{\eta}{e} \frac{\partial W_{02}}{\partial e} \right) - \frac{\alpha \eta \sin u}{\mu_1 r} \frac{\partial W_{02}}{\partial g}.
$$

(70)

The last step of this phase is to compute the function $B_{11}$

$$
B_{11} = L_2 B_{00} = -\frac{\alpha^2}{\mu_1 r} \frac{\partial W_{02}}{\partial u} \frac{\partial B_{00}}{\partial \alpha},
$$

$$
\nu^2 B_{11} = \frac{1}{16 \pi} \int_0^{2\pi} B_{12}'(1 - e \cos u) du = 0.
$$

(71)

Oder 3: In the third order, similar calculations lead to the following results:

$$
\Delta_3' = B_{30} + B_{21} + B_{12} + B_{03} - L_3 B_{00}; \quad B_{30} = 0, \quad B_{21} = 0, \quad B_{03} = 0;
$$

$$
\Delta_3' = B_{12}' = L_2 B_{01} = L_2 (-\mu_1 \alpha \eta) = -\frac{\alpha^2}{\mu_1 r} \frac{\partial W_{02}}{\partial u} \frac{\partial B_{01}}{\partial \alpha} + \left( \frac{\alpha \eta}{\mu_1 r} \frac{\partial W_{02}}{\partial u} - \frac{1}{\mu_1 \alpha} \frac{\partial W_{02}}{\partial g} \right) \frac{\partial B_{01}}{\partial \eta} = \frac{\partial W_{02}}{\partial g};
$$

$$
B_{03}^* = \frac{1}{2 \pi} \int_0^{2\pi} \Delta_3'(1 - e \cos u) du = 0; \quad L_3 B_{03} = B_{02}' - \Delta';
$$

$$
\frac{\mu_1}{\alpha r} \frac{\partial W_{03}}{\partial u} = \frac{\partial W_{02}}{\partial g}; \quad W_{03} = \int \frac{\alpha^3}{\mu_1} (1 - e \cos u) \Delta_3' du.
$$

15
We then define $W_{03}$ in the form
\[
\nu^3 W_{03} = \frac{\nu^3 L}{384 n^3} W'_{03},
\]
\[
W'_{03} = -36e^2(-22 + 9e^2) \sin(2g) +
4e(10(-1 + \eta + e^2)(-1 + 6\eta)) \sin(2g - 3u) +
(72 + 228 - 96e^4 - 72\eta - 264e^2\eta) \sin(2g - 2u) +
(6e^2 - 3e^4 - 6e^2\eta) \sin(2g - 4u) +
(-792e + 324e^3 + 792e\eta - 288e^3\eta) \sin(2g - u) +
(72 + 228e^2 - 96e^4 + 72\eta + 264e^2\eta) \sin(2g + 2u) +
(-792e + 324e^3 - 792e\eta + 288e^3\eta) \sin(2g + u) +
(-40e - 4e^3 - 40e\eta - 24e^3\eta) \sin(2g + 3u) +
(6e^2 - 3e^4 + 6e^2\eta) \sin(2g + 4u).
\] (72)

Next, we define the third column of the matrix $\psi$ by the scheme (70). The last step of this phase is to compute the function $B_{12}$:
\[
B_{12} = L_2 B_{01} + L_3 B_{00} = 0.
\]

Let us briefly discuss the calculations of the fourth order.

Function $\Delta'_{4}$ is defined as follows:
\[
\Delta'_{4} = B'_{13} + B_{22};
B'_{13} = L_2 B_{02} + L_3 B_{01};
B_{22} = L_2 B_{11}.
\]

For example, consider the operation $L_2 B_{11}$:
\[
L_2 B_{11}(\epsilon) = \sum_{p=1}^{4} W_{p2} \frac{\partial B_{11}(\epsilon)}{\partial \epsilon_p}.
\] (73)

In the process of computation of the derivatives $\frac{\partial B_{11}(\epsilon)}{\partial \epsilon_p}$ we must keep in mind that $e^2 + \eta^2 = 1$.

Evaluating expressions in the form
\[
W_{p2} \frac{\partial B_{11}(\epsilon)}{\partial \epsilon_p}
\] (74)
is rather cumbersome, but the algorithm is designed to minimize the number of such operations. It is easy to see that the expression (73) can be represented in the form of the generalized Poisson brackets (16), but in this case, the number of operations of type (74) is increased about twofold. Thus, the use of the matrix $\Psi$ minimizes the number of cumbersome operations.

We see from the relations (70) that the functions $W_{ik}$ have a multiplier $r^{-1}$. However, because in the process of averaging over the mean anomaly we have
\[
\left< \frac{\alpha^2}{r} \right> = 1, \left< \frac{\alpha^2}{r} \sin ju \right> = 0, j \geq 1,
\]
and expressions of the form \( (67) \) are integrated by using formula

\[
dl = \frac{r}{\alpha^2} du,
\]

negative powers of \( r \) do not appear in the final results.

The function \( \nu^4 B_{04}^* \) has the form:

\[
\nu^4 B_{04}^* = \nu^4 \frac{Q_{10}}{\mu_1^2} B_{04}^* = \frac{\nu^4}{n^4 \alpha^2} \mu_1^2 \frac{B_{04}^*}{n^4},
\]

\[
B_{04}^* = \frac{1}{16} \left( \frac{49}{4} - \frac{873}{4} e^2 + \frac{4347}{32} e^4 - \left( \frac{333}{4} e^2 - \frac{237}{8} e^4 \right) \cos 2g + \frac{615}{32} \cos 4g \right),
\]

and for the generating function we have

\[
\nu^4 W_{04} = \frac{\nu^4 L}{n^4} W'_{04},
\]

where the expression \( W'_{04} \) is quite cumbersome, and we do not present it here.

In the results of the normalization, we received a new Birkhoffian \( B^* \) in the new variables \( \tilde{e} \). Thus,

\[
B^* = -\frac{\mu_1^2}{2\alpha^2} - m \frac{\mu_1^2}{\alpha^2} \tilde{\eta} + m^2 \frac{\mu_1^2}{4\alpha^2} \left( 1 - \frac{3}{2} \tilde{e}^2 + \frac{15}{2} \tilde{e}^2 \cos 2\tilde{\gamma} \right) + m^3 * 0 +
\]

\[
m^4 \frac{\mu_1^2}{16\alpha^2} \left[ \frac{49}{4} - \frac{873}{4} \tilde{e}^2 + \frac{4347}{32} \tilde{e}^4 - \left( \frac{333}{4} \tilde{e}^2 - \frac{237}{8} \tilde{e}^4 \right) \cos 2\tilde{g} + \frac{615}{32} \cos 4\tilde{g} \right].
\]

In the example above, for the plane problem, we see that the functions \( B^* \) and \( W \) are presented in the form of truncated series in \( m = n'/n \), but the coefficients of these series are in closed form.

The solution for the spatial restricted three-body problem was obtained in the Mathematica package up to the fifth order in the small parameter \( m \). The obtained solution coincides with the result of \[17\].

The new system of equations averaged over \( l \) has the form

\[
\begin{align*}
d\tilde{\alpha} &= 0, & \frac{d\tilde{l}}{dt} &= \frac{1}{\mu_1} \frac{\partial B^*}{\partial \tilde{\alpha}} - \tilde{\eta} \frac{\partial B^*}{\mu_1 \tilde{\alpha} \tilde{\eta}}, \\
d\tilde{\eta} &= -\frac{1}{\mu_1} \frac{\partial B^*}{\partial \tilde{\eta}}, & \frac{d\tilde{\gamma}}{dt} &= \tilde{\gamma} \frac{\partial B^*}{\mu_1 \tilde{\alpha} \tilde{\eta}} - \frac{1}{\mu_1} \frac{\partial B^*}{\partial \tilde{\gamma}}, \\
d\tilde{g} &= \frac{1}{\mu_1} \frac{\partial B^*}{\partial \tilde{\gamma}} - \frac{\tilde{\gamma}}{\mu_1 \tilde{\alpha} \tilde{\eta}} \frac{\partial B^*}{\partial \tilde{\gamma}}, & \frac{d\tilde{h}}{dt} &= \frac{1}{\mu_1 \tilde{\alpha} \tilde{\eta}} \frac{\partial B^*}{\partial \tilde{\gamma}}.
\end{align*}
\]

(77)
Equations (77) were obtained from (48) using the following substitution:

\[
\frac{du}{dt} = \alpha^2 \frac{dl}{dt} - \alpha^2 \frac{\eta \sin u}{r} \sqrt{1 - \eta^2} \frac{d\eta}{dt}.
\]

The resulting system of equations has two degrees of freedom, since the first two equations are separated from the system.

Remark: The function \( B^* \) in the Appendix includes variables without tilde, because it is the result of calculations on a computer.

### 3.2 Representation of analytical solution of restricted three-body problem using the Delaunay arguments. The second example

The motion of the satellite due to the attraction of the central planet and the disturbing body \( S \) is considered by assuming that all the bodies are mass-points, and the satellite has infinitesimal mass, the central body has mass \( m_0 \) and the body \( S \) has mass \( M \).

For this task, we use the variables \( \epsilon = (\alpha, E, J, \Lambda, D, l, F, l') \). Variable \( \alpha \) was defined in the previous example; \( \Lambda \) is an auxiliary variable conjugate to the variable \( l' \) (mean anomaly of the disturbing body); \( D, l, F, l' \) are Delaunay’s basic arguments: \( D = \lambda - \lambda', F = \lambda - \Omega, l = \lambda - \pi \), where \( \lambda \) is the orbital longitude of the satellite, measured from the chosen direction of the \( x \)-axis of a rectangular coordinate system; \( \Omega \) and \( \pi \) are the longitude of the ascending node and pericenter of the satellite orbit, respectively; \( \lambda' \) is the mean longitude of the perturbing body. Variables \( E \) and \( J \) are determined by the formulas:

\[
E^2 = 2(1 - \sqrt{1 - e^2}), \quad J^2 = 4\sqrt{1 - e^2} \gamma^2, \quad \gamma = \sin(i/2), \quad (78)
\]

where \( e \) is the eccentricity satellite’s orbit, \( i \) is the inclination of the satellite orbit to the plane in which the body \( S \) moves.

Birkhoffian of this system is represented as follows:

\[
B = -\frac{\mu_1^2}{2\alpha^2} + \nu \Lambda + R, \quad (79)
\]

where \( \mu_1 \) was defined in the previous example, \( \nu \) is a mean motion of the disturbing body. The first term \( -\mu_1^2/2\alpha^2 \) in the expression (79) is due to the attraction of the planet in the absence of disturbances. The second term \( (\nu \Lambda) \) is introduced to eliminate the explicit dependence function \( B(\epsilon) \) from time, and \( R = \nu^2 S' \) is a disturbing function. \( S' \) has the form of a truncated series

\[
S' = \sum_{i,j} A^j_i \nu^{i_1} \alpha^{j_1} E^{j_2} J^{j_3} e^{j_4} \cos(sin)(i_1 D + i_2 l + i_3 F + i_4 l'), \quad (80)
\]

where \( A^j_i \) are numerical coefficients; \( \alpha' = \alpha/\sqrt{a'} \), \( a' \) and \( e' \) are the semi-major axis and the eccentricity of the orbit on which the disturbing body moves.

Thus, unlike the previous example, here the perturbing function is explicitly dependent on the mean satellite anomaly. It follows that all the expressions now include an expansion in powers of the eccentricity of the satellite orbit. These are the traditional expansions of
the perturbation theory of Celestial Mechanics. As mentioned above, the generalized Lie generator can be represented as the generalized Poisson bracket \( \{16\} \), thus preserving the invariant properties of Poisson brackets, without destroying the d’Alembert characteristics in the expansions of the perturbation theory.

Pfaffian of the problem can be written as

\[
\Phi = \alpha \mu_1 (1 - \frac{1}{2} E^2 - \frac{1}{2} J^2) d\,D + \frac{1}{2} \alpha \mu_1 E^2 d\,l + \frac{1}{2} \alpha \mu_1 J^2 d\,F + \frac{1}{2} \alpha \mu_1 (1 - \frac{1}{2} E^2 - \frac{1}{2} J^2) d\,l' - B\,dt. \tag{81}
\]

Birkhoff’s equations are represented in this case as follows:

\[
\begin{align*}
\frac{d\alpha}{dt} &= -\frac{1}{\mu_1} \left( \frac{\partial B}{\partial D} + \frac{\partial B}{\partial l} + \frac{\partial B}{\partial F} \right), \\
\frac{dE}{dt} &= \frac{1}{\mu_1 \alpha} \left( \frac{E \partial B}{2 \partial D} + \left( \frac{E}{2} - \frac{1}{E} \right) \frac{\partial B}{\partial l} + \frac{E \partial B}{2 \partial F} \right), \\
\frac{dJ}{dt} &= \frac{1}{\mu_1 \alpha} \left( \frac{J \partial B}{2 \partial D} + \frac{J \partial B}{2 \partial l} + \left( \frac{J}{2} - \frac{1}{J} \right) \frac{\partial B}{\partial F} \right), \\
\frac{d\Lambda}{dt} &= \frac{\partial B}{\partial D} - \frac{\partial B}{\partial l'^*}, \\
\frac{dD}{dt} &= \frac{1}{\mu_1} \left( \frac{\partial B}{\partial \alpha} - \frac{E \partial B}{2\alpha \partial E} - \frac{J \partial B}{2\alpha \partial J} \right) - \frac{\partial B}{\partial \Lambda}, \\
\frac{dl}{dt} &= \frac{1}{\mu_1} \left( \frac{\partial B}{\partial \alpha} + \left( \frac{1}{\alpha E} - \frac{E}{2\alpha} \right) \frac{\partial B}{\partial E} - \frac{J \partial B}{2\alpha \partial J} \right), \\
\frac{dF}{dt} &= \frac{1}{\mu_1} \left( \frac{\partial B}{\partial \alpha} - \frac{E \partial B}{2\alpha \partial E} + \left( \frac{1}{\alpha J} - \frac{J}{2\alpha} \right) \frac{\partial B}{\partial J} \right), \\
\frac{dl'}{dt} &= \frac{\partial B}{\partial \Lambda}. \tag{82}
\end{align*}
\]

Birkhoffian \( B'_{00} \) unperturbed motion we define as

\[
B'_{00} = -\frac{\mu^2}{2\alpha^2} + \nu \Lambda + \nu^2 R'_{00}. \tag{83}
\]

The expression \( R'_{00} \) includes only those terms of the secular part of the disturbing function that contain variables \( E \) and \( \Theta \) of no higher than second degree [Brouwer & Clemence (1961)]:

\[
R'_{00} = -\frac{1}{4} \alpha^4 - \frac{3}{8} \alpha^4 E^2 + \frac{3}{8} \alpha^4 J^2. \tag{84}
\]

With this choice of the unperturbed motion of the satellite, the averaging procedure involves all angular variables \( (D, l, F, l') \), i.e., the system is a non-degenerate. The frequencies associated with these variables are determined from the system of equations (82) using the substitution \( B = B'_{00} \).
The analytical expressions for the frequencies of this system are determined by formulas:

\[ \omega_1 = \frac{\mu_1}{\alpha^3} - \frac{\nu^2 \alpha^3}{\mu_1} \left(1 + \frac{9}{8} E^2 - \frac{9}{8} J^2\right) - \nu, \]
\[ \omega_2 = \frac{\mu_1}{\alpha^3} - \frac{\nu^2 \alpha^3}{\mu_1} \left(1 + \frac{9}{8} E^2 - \frac{9}{8} J^2\right) - \frac{3 \alpha^3}{4 \mu_1}, \]
\[ \omega_3 = \frac{\mu_1}{\alpha^3} - \frac{\nu^2 \alpha^3}{\mu_1} \left(1 + \frac{9}{8} E^2 - \frac{9}{8} J^2\right) + \frac{3 \alpha^3}{4 \mu_1}, \]
\[ \omega_4 = \nu. \]  

(85)

In this example, the homological equation is written in a general form as:

\[ 4 \sum_{j=1}^{4} \omega_j \frac{\partial W_{0k}}{\partial \varepsilon_{j+4}} = B_{0k}^* - B_{0k}, \]

(86)

where the functions \( W_{0k}, B_{0k}^* \) and \( B_{0k} \) have the same meaning as in the previous example.

The solution of Eq. (86) leads to the appearance of divisors of the following form:

\[ d = \left( (i_1 + i_2 + i_3) \mu_1 \alpha^{-3} + (i_1 + i_2 + i_3)(-\nu^2 \mu_1^{-1} \alpha^3) + (i_1 + i_2 + i_3)(-9/8 \nu^2 \mu_1^{-1} \alpha^3 E^2) + (i_1 + i_2 + i_3)(9/8 \nu^2 \mu_1^{-1} \alpha^3 J^2) + (i_1 - i_4) \nu + (i_2 - i_3)(3/4 \nu^2 \mu_1^{-1} \alpha^3) \right). \]

(87)

From the expression (87) we can obtain conditions for periods of various disturbances. Let us consider the trigonometric arguments

\[ i_1 D + i_2 L + i_3 F + i_4 l' \]

in the expression (80).

Mean anomaly \( l \) of the satellite is contained in \( D, L, F \); therefore the condition \( i_1 + i_2 + i_3 \neq 0 \) determines those terms of the perturbing function whose periods are commensurate with the period of orbital motion of a satellite around the planet. This period is denoted via \( P(l) \).

For the terms of the perturbing function whose periods are commensurate with the orbital period of a planet around the Sun \( (P(l')) \), the conditions are \( i_1 + i_2 + i_3 = 0, i_1 \neq i_4 \). These terms include the mean anomaly \( l' \) of the planet, but do not contain the mean anomaly \( l \) of a satellite.

The conditions for the long-period terms would be as follows: \( i_1 + i_2 + i_3 = 0, i_1 = i_4, i_2 \neq i_3 \). These terms contain neither \( l \) nor \( l' \).

The system under consideration is non-resonant; therefore, in our case, the perturbation function \( R \) may be decomposed into its \( P(l) \)-period, \( P(l') \)-period and long-period parts. Next, we can carry out the operation of averaging over different periods. Normalization of this kind has already been used by several authors [Hori(1963), Deprit(1971)].

In averaging the perturbing function over mean anomaly \( l \), we can take a Keplerian motion as the unperturbed motion of the satellite. Thus, we have from (83) that

\[ B'_{00} = B_{00}^{(l)} = -\frac{I_1^2}{2\alpha^2}. \]
and $\omega_1 = \omega_2 = \omega_3 = \mu_1 \alpha^{-3}$.

Therefore, the homological equation at this stage can be written as

$$-\frac{\mu_1}{\alpha^3} \left( \frac{\partial W_{0k}}{\partial D} + \frac{\partial W_{0k}}{\partial \hat{l}} + \frac{\partial W_{0k}}{\partial F} \right) = B^*_0 - B_{0k}. \quad (88)$$

This equation is easily solved. Function $B^*_0 - B_{0k}$ contains only those terms for which $i_1 + i_2 + i_3 \neq 0$. It follows that we can obtain the function $W_{0k}$ from the expression $B^*_0 - B_{0k}$ with the help of the substitution $\cos \rightarrow -\sin$, and then multiplying the result by

$$\frac{1}{(i_1 + i_2 + i_3)\mu_1 \alpha^{-3}}.$$

Next, we form the k-th column of the generating matrix $\Psi$, as described in the previous example. When the $\Psi$ matrix has been defined up to the desired order, we can begin the process of the transformation of variables.

For the transformation of variables, we enter on the first line of the triangular matrix the following information: $B_{00} = \epsilon, B_{01} = 0, \ldots, B_{0m} = 0$. We then use the above-referenced algorithm. To obtain the inverse transformation, we must reverse the signs of all elements of the $\Psi$ matrix to the opposite signs. The result is the following transformation of variables: $\epsilon \leftrightarrow \hat{\epsilon}$.

At the first step of normalization, we obtain the integral of motion $\hat{\alpha} = \text{const}$. It follows that in order to eliminate the $P^{(f)}$-terms from perturbing function, we can use the function $B'_{00}$ in the form

$$B'_{00} = B_{00}^{(f)} = \nu \Lambda.$$

In this case the frequencies are defined so that $\omega_1 = -\nu, \omega_4 = \nu$, and the homological equation is

$$\nu \left( \frac{\partial \hat{W}_{0k}}{\partial D} - \frac{\partial \hat{W}_{0k}}{\partial \hat{l}'} \right) = \hat{B}^*_0 - \hat{B}_{0k}. \quad (89)$$

It follows from expression (87) that in order to eliminate the long-period perturbations from the Birkhoffian, one can use the homological equation in the form:

$$\frac{3}{4} \nu^2 \alpha^3 \left( \frac{\partial \hat{W}_{0k}}{\partial \hat{l}} - \frac{\partial \hat{W}_{0k}}{\partial F} \right) = \hat{B}^*_0 - \hat{B}_{0k}. \quad (90)$$

Using this algorithm, we can obtain any desired order of transformation. As a result, Eq.
\( \frac{d\tilde{\alpha}}{dt} = 0, \)
\( \frac{d\tilde{E}}{dt} = 0, \)
\( \frac{d\tilde{J}}{dt} = 0, \)
\( \frac{d\tilde{\Lambda}}{dt} = 0, \)
\( \frac{d\tilde{D}}{dt} = \frac{1}{\mu_1} \left( \frac{\partial \tilde{B}}{\partial \tilde{\alpha}} - \frac{\tilde{E}}{2\tilde{\alpha}} \frac{\partial \tilde{B}}{\partial \tilde{E}} - \frac{\tilde{J}}{2\tilde{\alpha}} \frac{\partial \tilde{B}}{\partial \tilde{J}} \right) - \frac{\partial \tilde{B}}{\partial \tilde{\Lambda}} = \tilde{n} - \nu, \)
\( \frac{d\tilde{l}}{dt} = \tilde{n} + \frac{1}{\tilde{\alpha} \tilde{E}} \frac{\partial \tilde{B}}{\partial \tilde{E}}, \)
\( \frac{d\tilde{F}}{dt} = \tilde{n} + \frac{1}{\tilde{\alpha} \tilde{J}} \frac{\partial \tilde{B}}{\partial \tilde{J}}, \)
\( \frac{d\tilde{l}'}{dt} = \nu. \)  \( \tag{91} \)

The solutions are trivial:

\( \tilde{\alpha} = \alpha_0, \tilde{E} = E_0, \tilde{J} = J_0, \tilde{\Lambda} = \Lambda_0, \)
\( \tilde{D} = (\tilde{n} - \nu)t + D_0, \tilde{l} = (\tilde{n} - \frac{d\tilde{\pi}}{dt})t + l_0, \)
\( \tilde{F} = (\tilde{n} - \frac{d\tilde{\Omega}}{dt})t + F_0, \tilde{l}' = \nu t + l'_0. \)  \( \tag{92} \)

Here \( \alpha_0, E_0, J_0, D_0, l_0, F_0 \) are the constants of the analytical theory, which are the mean elements of the orbit of the satellite in the epoch \( t_0 \), \( l'_0 \) is the mean anomaly of the disturbing body in reference to the same moment \( t_0 \), \( \Lambda_0 \) is an auxiliary parameter, which is not present in the final expansions, and \( t \) is the time in Julian days from the epoch \( t_0 \). The mean motion of the longitude \( \tilde{n} \), of the longitude of the pericentre \( d\tilde{\pi}/dt \) and of the longitude of the node \( d\tilde{\Omega}/dt \) are represented by series of the form

\[ C_q = \sum_j K_j m^{j_1} \alpha_0^{j_2} E_0^{j_3} j_4^4 e^{j_5}. \]

The mean elements of the orbit of the satellite are obtained using the expansions for the inverse transformation of variables.

We have used a similar algorithm in an earlier work [Boronenko & Shmidt (1990)]. The literal solution of the restricted three-body problem, which the authors obtained up to the 11-th order with respect to the minor parameter \( m = \nu/n \), was applied to the investigation of motion of Phoebe, the ninth satellite of Saturn. In this article, we derived the algorithm in the context of the theory of Birkhoff. A more complete description of the solution of this problem can be found in [Boronenko & Shmidt (1990)].
4 Conclusion

In this article, we demonstrated the usefulness of the Lie transformation algorithm for Birkhoff systems, which are described by the equations of the following form:

\[ \dot{\theta}_p = \sum_{s=1}^{n} a_{ps}(\theta) \frac{\partial B(\theta)}{\partial \theta_s}, \quad (p = 1, \ldots, n), \]

where the function \( B(\theta) \) is the Hamiltonian expressed in the special variables \( \theta \) of the phase space. Indeed, Birkhoff’s autonomous equations are the Hamilton equations, which are represented in non-canonical variables in the phase space. However, we use the term ‘Birkhoffian’ because of certain physical differences with Hamiltonian, in that the matrix \( (a_{ps}) \) contains the time \( t \) via the variables \( \theta(t) \). A more detailed discussion can be found in [Santilli(1983)].

Tensor \( a_{ps} \) is the Birkhoff tensor, expressed in terms of the Poisson brackets. In accordance with the terminology of [Santilli(1983)], \( a_{ps} \) is the contravariant Birkhoff tensor, also called the Lie tensor.

In Sect. 2.2 we showed that the representation of the Lie generator for canonical systems in terms of the special variables of the phase space led to its expression through the Birkhoff tensor \( a_{ps} \). As shown by [Santilli(1983)], the Birkhoff tensor \( a_{ps} \) and its associated symplectic form \( \Omega \) preserve their Lie and symplectic character under arbitrary transformations. This allowed us to use the autonomous Birkhoff equations in the construction of the Lie series perturbation theory.

The basis of the algorithm is a generalized Lie generator, which we express in terms of the tensor \( a_{ps} \). To reduce the need for cumbersome operations of multiplication of series, we introduced the generating matrix \( \Psi \), which is defined with the help of the tensor \( a_{ps} \) and partial derivatives of the generating function. The matrix \( \Psi \) is also used for the direct and inverse coordinate transformations.

In this work, we have demonstrated the algorithm, based on a generalized Lie generator, using two examples from Celestial Mechanics.

In the first example, we considered the satellite case of the spatial restricted three-body problem, using an averaging method, based on a Lie transformation of the Birkhoffian system. The new Birkhoffian, averaged over mean anomaly of a satellite, was obtained in the form of series in \( m \) (ratio of mean motions of the Sun and satellite), but in closed form with respect to eccentricity and inclination. The accuracy of the analytical expansion is \( O(m^5) \). Our results were coincident with the result of [Hori(1966)] up to the fifth order [Boronenko (2010)].

In the second example, we represented an analytical solution of restricted three-body problem using the Delaunay arguments \( (D,l,F,l') \). Unlike the previous example, here we dealt with an explicit expression of the perturbing function in terms of the mean anomaly of the satellite. Therefore, all the considered expressions now included power series in the eccentricity of the satellite orbit. These are the traditional expansions of the perturbation theory of Celestial Mechanics. This example shows that the Lie generator, expressed in terms of the Birkhoff tensor, preserves the invariant properties of Poisson brackets. For example, the use of the generalized Lie generator does not destroy the d’Alembert characteristic in the series of perturbation theory. We used a similar algorithm in an earlier work for constructing an analytical theory of motion of Phoebe, the ninth satellite of Saturn [Boronenko & Shmidt (1990)]. In this paper, we derived the algorithm in accordance
with the theory Birkhoff: we introduced the Pfaffian, Birkhoff’s equations, and the generalized Lie generator for this problem. A more complete description of the problem can be found in [Boronenko & Shmidt (1990)].

The examples above show that the proposed algorithm does not violate the basic approaches of the standard Lie transformation theory, but it provides an efficient alternative in the case where there is a need to use the non-canonical variables ($\theta$) in phase space. Here, the Birkhoffian scheme provides a clear way in which to build the solution. It is important that all operations are performed only in the variables ($\theta$). As shown by the examples, the proposed algorithm does not increase the number of operations compared to the standard Lie transformation theory, but in the case of non-canonical variables, the generating matrix $\Psi$ allows the number of multiplications of large expressions to be reduce.

In addition to the technical characteristics of the algorithm, the properties of Birkhoff systems are useful for developing a common approach to the process of formulating a problem. The use of 1-forms allows us to expand the types of coordinate transformations, as the dynamic Pfaffians can be represented by a large number of different forms. Pfaffian does not change its form when a transformation of coordinates is made in the phase space with dimension $2n + 1$, i.e., in the extended phase space of Pfaff. Therefore, in our work, we define Birkhoff functions as a components of the Pfaff vector.

It should be noted that the Birkhoff tensor $\Omega_{ps}$ in covariant form coincides with the matrix of the Lagrange brackets. Using Keplerian elements as coordinates in phase space is a simple way of deriving the Lagrange planetary equations. In this case, the Birkhoff autonomous equations coincide with Lagrange’s planetary equations, and the method can be used for analytical integration of these equations.

Unlike the other Lie transformations algorithms for non-canonical systems (see e.g., [Nayfeh (2000)]), the Lie series transformations for the autonomous Birkhoff equations that we have considered here are derived from the Pfaff-Birkhoff variational principle, which is more general than the Hamilton principle. The use of 1-form in the formulation of equations of motion in dynamics renders the Birkhoff method more universal and flexible. The Birkhoff equations have a tensorial character; therefore, their form is independent of the coordinate system that is used.

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A The expression for the averaged Birkhoffian (the first example)

The resulting averaged Birkhoffian is represented as follows:
\[ B^* = B^*_{00} + B^*_{01} + B^*_{02} + B^*_{03} + B^*_{04} + B^*_{05}, \quad B^*_{00} = B_{00}, \quad B^*_{01} = B_{01}, \]

where

\[
B^*_{02} = \frac{1}{16} \nu^2 a^2 (((2 + 3e^2)(1 - 3\gamma^2 + 3(-1 + \gamma^2) \cos(2h)) - 15e^2 \cos(2g)(1 - \gamma^2 + (1 + \gamma^2) \cos(2h) + 30e^2 \gamma \sin(2g) \sin(2h)),
\]

\[ B^*_{03} = 0, \]

\[
B^*_{04} = \frac{1}{4096} \frac{\nu^4 a^2}{n^2} (8(47 + 282\gamma^2 + 63\gamma^4) + 63e^4(239 + 170\gamma^2 + 143\gamma^4) - 72e^2(377 + 190\gamma^2 + 209\gamma^4) + 2592 \cos(2h) + 168 \cos(4h) - 24e^2 \cos(2g)(1 - \gamma^2 + (1 + \gamma^2) \cos(2h) - (27(2 + e^2) + 5(78 - 37e^2) \gamma^2 + 5(-78 + 37e^2)(-1 + \gamma^2) \cos(2h) - 410e^2 \gamma \sin(2g) \sin(2h) + 3((56 \gamma^2(-2 + \gamma^2) - 1672e^2(-1 + \gamma^2)^2 + 100e^4(-1 + \gamma^2)) \cos(4h) + 205e^4 \cos(4g)(-3(-1 + \gamma^2)^2 + 4(-1 + \gamma^4) \cos(2h) - (1 + 6\gamma^2 + \gamma^4) \cos(4h)) + 16e^2 \gamma(27(2 + e^2) + 5(78 - 37e^2) \gamma^2) \sin(2g) \sin(2h) + 4 \cos(2h)(-8\gamma^2(20 + 7\gamma^2) + 152e^2(-13 + 2\gamma^2 + 11\gamma^4) - 7e^4(-199 + 56\gamma^2 + 143\gamma^4) + 20e^2(-78 + 37e^2) \gamma(-1 + \gamma^2) \sin(2g) \sin(2h))),
\]

\[
B^*_{05} = \frac{1}{128} \frac{\nu^4 a^2}{n^2} m \eta(\gamma(176 - 2775e^2 + 870e^4 + (212 - 1895e^2 + 675e^4) \gamma^2 - (212 - 1895e^2 + 675e^4)(-1 + \gamma^2) \cos(2h) + 4e^2(-101 + 17e^2) \gamma \cos(2g)(3 - 3\gamma^2 + (-1 + 3\gamma^2) \cos(2h)) - 8e^2(-101 + 17e^2)(-1 + 2\gamma^2) \sin(2g) \sin(2h)).
\]

The above expressions were checked by comparison with the results of \([\text{Hori}(1963)]\), under the condition \(\gamma = 1\) and \(h = 0\). Complete coincidence was found with the analytical expressions for the functions \(B^*_{02}, B^*_{03}, B^*_{04}\). For the function \(B^*_{05}\), only the secular part coincided. This discrepancy can be explained by the use of different methods (Lie transformations and von Zeipel method) in solving the problem.

Note. Expressions for the \(B^*_{0i}\) functions in the traditional form can be obtained in the Mathematica package using the function \(\text{Expand[TrigReduce}[B_{0i}]]\). Examples of these expressions for the planar version of the problem can be found in section 3.1

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