Singular Hermitian–Einstein monopoles on the product of a circle and a Riemann surface

Benoit Charbonneau and Jacques Hurtubise

October 19, 2009.

Abstract

In this paper, the moduli space of singular unitary Hermitian–Einstein monopoles on the product of a circle and a Riemann surface is shown to correspond to a moduli space of stable pairs on the Riemann surface. These pairs consist of a holomorphic vector bundle on the surface and a meromorphic automorphism of the bundle. The singularities of this automorphism correspond to the singularities of the singular monopole. We then consider the complex geometry of the moduli space; in particular, we compute dimensions, both from the complex geometric and the gauge theoretic point of view.\footnote{The authors wish to thank Tom Mrowka and Mark Stern for useful discussions, and Marco Gualtieri and Paul Norbury for comments on the first two versions of the paper. The second author is supported by NSERC and FQRNT. The authors can be reached respectively at [Math Dept, Duke University, Box 90320, Durham, NC 27708-0320, USA, benoit@alum.mit.edu], and [Dept of Math and Stat, McGill U., 805 Sherbrooke St. W, Montreal, Canada H3A 2K6, jacques.hurtubise@mcgill.ca].}

1 Introduction

In a recent paper about the geometric Langlands program, Kapustin and Witten [21] expound the idea that the moduli of singular monopoles on the product of a Riemann surface $\Sigma$ with an interval should mediate the Hecke transforms which play a part in the geometric Langlands correspondence. A particular case is when the product of the transforms gives back the original bundle, and in understanding this relationship, it is then natural to ask what one gets as a monopole moduli space when one closes the interval to a circle.

Another motivation to study monopole moduli spaces on such a product is obtained by specialising to when the surface is a torus. One of the main tools for understanding monopoles, and more generally anti-self-duality, has been the Nahm transform heuristic. It tells us that singular monopoles on $T^3 = S^1 \times T^2$ should correspond to instantons on some “dual” $\mathbb{R} \times T^3$. (While aspects of this correspondence have been elucidated in [5], the correspondence is still not completely proven.) Thus, even when studying smooth solutions to the anti-self-duality equations, one is led to consider configurations with singularities.

The monopoles under consideration in this paper are solutions to a generalisation of the Bogomolny equation

$$F_\nabla = *d_\nabla \phi$$

linking the curvature $F_\nabla$ of a unitary connection $\nabla$ on a Hermitian bundle $E$ over a Riemannian three-manifold $Y$ and a skew-Hermitian endomorphism $\phi$ of $E$ called a Higgs field. The generalisation is the addition, in the special case where $Y = S^1 \times \Sigma$, of a constant central term:

$$F_\nabla - iC_{E \Sigma} = *d_\nabla \phi.$$

When $Y$ is compact, global smooth solutions to the standard Bogomolny equation are quite trivial: the connection $\nabla$ must be flat and $\phi$ must be parallel. Over open manifolds, monopoles and their moduli have been extensively studied over the past twenty-five years, beginning with the case of $\mathbb{R}^3$. For $\mathbb{R}^3$, Hitchin in [13] constructed all monopoles for SU(2) using the twistor methods of Penrose, Ward and Atiyah. Using this construction, and its natural extension to other gauge groups, the moduli spaces were described in [10; 17; 19]. On hyperbolic space, a similar description was given by Atiyah in [1] (see also [20]). The $\mathbb{R}^3$-spaces have natural metrics [2]; for a unified discussion of Euclidean and hyperbolic cases, see [25]. Monopoles on $\mathbb{R}^2 \times S^1$ have been studied in [6] from the perspective of the Nahm transform.

The first study of singular monopoles (with Dirac-type singularities) is due to Kronheimer in [22] for Euclidean spaces. Pauly computed in [30] the virtual dimension of the moduli space of singular SU(2)-monopoles and in [31]
started the study of singular monopoles on the round three-sphere. More recently, Nash considered in [26] the twistor theory of singular hyperbolic SU(2)-monopoles. Norbury proved in [28] the existence and uniqueness of singular monopoles satisfying prescribed boundary conditions on an interval times a surface. The properties of certain moduli spaces of singular monopoles on $\mathbb{R}^3$ and $\mathbb{R}^2 \times S^1$ allowed Cherkis–Kapustin in [8; 9] and Cherkis–Hitchin in [7] to produce families of asymptotically locally flat gravitational instantons.

We shall restrict our attention in this paper to the gauge group $U(n)$. Our solutions have singularities, and we must fix the nature of the singularity. Fortunately, in our case, there is a fairly natural choice: we ask that near the singularity, the monopole, in essence, should decompose into a sum of Dirac monopoles. These boundary conditions were studied by Pauly [30], who shows, by exploiting the geometry of the Hopf fibration, that there are natural local lifts from $\mathbb{R}^3$ to $\mathbb{R}^4$ that tame the singularity.

The solutions to the Bogomolny equation we consider thus have Dirac-type singularities at fixed points $p_i = (t_i, z_i) \in S^1 \times \Sigma$, where the $z_i$ are distinct. The bundle $E$ is defined on $S^1 \times \Sigma \setminus \{p_1, \ldots, p_N\}$. Precise definitions of the allowed singularities are given below; for the moment, let $R_i$ be the geodesic distance, in $S^1 \times \Sigma$, to $p_i$. On a small sphere surrounding the singularity $p_i$, the bundle is a sum of line bundles of degree $k_{i1}, \ldots, k_{in}$ and near the singularity the monopoles will have Higgs field $\phi$ asymptotic to $(\sqrt{-1/2R_i}) \text{diag}(k_{i1}, \ldots, k_{in})$, where $k_i = (k_{i1}, \ldots, k_{in})$ is a sequence of integers, ordered so that $k_{i1} \geq \cdots \geq k_{in}$. We collect these sequences together as a sequence

\[ K = ((k_{i1}, z_1), \ldots, (k_{iN}, z_N)). \]

We also collect the length $T$ of the circle and the circle coordinates $t_i$ of the singularities in a vector

\[ \vec{t} = (t_1, \ldots, t_N, T). \]

Let $E_t$ the restriction to $\{t\} \times \Sigma$ of the bundle $E$ for $t \neq t_i$. Let the degree of $E_0$ be $k_0$. As one moves through the point $p_i$, the degree of $E_t$ changes by $\text{tr}(k_i) := \sum_j k_{ij}$; in particular, it must be that $\sum_i \text{tr}(k_i) = 0$.

As with other examples of moduli of solutions to the anti-self-duality equations and their reductions, we exploit the fact that the equations decompose into two components. The first component simply states that we are dealing with a holomorphic object; the second is variational in nature, and possesses a unique solution once one has a solution to the first component, as long as the holomorphic object is stable in a suitable sense. This general scheme often goes by the name of the Kobayashi–Hitchin correspondence. In our case, we obtain the following theorem.

**Theorem 1.1** The moduli space $M^{\vec{t}}_{k_0}(S^1 \times \Sigma, p_1, \ldots, p_N, \vec{k}_1, \ldots, \vec{k}_N)$ of $U(n)$ irreducible Hermitian–Einstein monopoles on $S^1 \times \Sigma$ with $E_0$ of degree $k_0$ and singularities at $p_j$ of type $\vec{k}_j$ maps bijectively to the space $M_{\vec{t}}(\Sigma, k_0, K, \vec{t})$ of $t$-stable holomorphic pairs $(\mathcal{E}, \rho)$ with

- $\mathcal{E}$ a holomorphic rank $n$ bundle of degree $k_0$ on $\Sigma$,
- $\rho$ a meromorphic section of $\text{Aut}(\mathcal{E})$ of the form $F_i(z) \text{diag}_j((z - z_i)^{k_{ij}})G_i(z)$ near $z_i$, with $F_i, G_i$ holomorphic and invertible, and such that $\det(\rho)$ has divisor $\sum_i \text{tr}(k_i)z_i$.

More generally, the reducible HE-monopoles correspond bijectively to $t$-polystable, but unstable, pairs.

The required notions, in particular of stability, are defined below.

Sections two through four of this paper are concerned with the proof of this theorem. The fifth section considers some examples. In the sixth and seventh sections, we use these ideas to consider monopoles on the product of an interval and a Riemann surface and on certain flat circle bundles over Riemann surfaces.

## 2 Definitions

### 2.1 The Bogomolny and HE–Bogomolny equations

Let $\Sigma$ be a Riemann surface, equipped with a Hermitian metric; let $z$ denote a coordinate on $\Sigma$. Let $S^1$ be the circle, equipped with metric such that its circumference is $T$. Let $t \in [0, T]$ be a coordinate on $S^1$, such that the $S^1$-metric is given by $dt^2$. We consider the three-fold $S^1 \times \Sigma$ equipped with the product metric, and as above, denote the submanifolds $\{t\} \times \Sigma$ by $\Sigma_t$. Let $p_1, \ldots, p_N$ be a collection of points on $S^1 \times \Sigma$; set $p_i = (t_i, z_i)$. We suppose that the $z_i$ are distinct, and that the $t_i$ are ordered $t_1 \leq \cdots \leq t_N$. We suppose the $t_i$ distinct and the origin of the circle chosen so that $t_i \neq 0$. The case where some $t_i$ are equal imposes no supplementary conceptual difficulty, but complicates the notation.
As above, we fix integers $k_{ij}, i = 1, \ldots, N, j = 1, \ldots, n$ with $\sum_{ij} k_{ij} = 0$. Now let $E$ be a Hermitian vector bundle of rank $n$ on $(S^1 \times \Sigma) \setminus \{p_1, \ldots, p_N\}$, with degree $k_0$ on $\Sigma_0$, and degree $\text{tr}(k_i) = \sum_j k_{ij}$ on small spheres surrounding the $p_i$. Let $E$ be equipped with a unitary connection $\nabla$ of curvature $F_\nabla$, and a skew-Hermitian section $\phi$ of $\text{End}(E)$ called the Higgs field. We say that $(E, \nabla, \phi)$ satisfies the Bogomolny equation if

$$F_\nabla = *\nabla \phi.$$  \hfill (3)

Let us recall how this equation is a reduction of the anti-self-duality equation in four dimensions. Extend the circle to a cylinder $S = R \times S^1$ with extra coordinate $s$, so that $w = s + it$ is a holomorphic coordinate on $S$. The product metric on $S^1 \times \Sigma$ extends to a product Kähler metric $g$ on $S \times \Sigma$ with Kähler form $\Omega$. Let $\pi: S \times \Sigma \to S^1 \times \Sigma$ be the projection, and let $\nabla = (\dfrac{\partial}{\partial s} + \pi^* \phi) ds + \pi^* \nabla$. The pair $(\nabla, \phi)$ satisfies the Bogomolny equation if and only if $\nabla$ satisfies the anti-self-dual (ASD) equation $*F_\nabla = -F_\nabla$. Another way to phrase this ASD equation more in tune with the complex structure is to split the curvature into bitype, and then to isolate the component $\Lambda F_\nabla \cdot \Omega$ of the curvature parallel to the Kähler form $\Omega$. The ASD equations become

$$F^{0,2}_\nabla = 0, \quad F^{2,0}_\nabla = 0, \quad \Lambda F_\nabla = 0.$$  \hfill (4)

The first two equations simply state that one has a holomorphic object, with a compatible Hermitian structure; the third, $\Lambda F_\nabla = 0$, is, as we shall see, variational in nature.

On a compact complex surface, these equations impose constraints on the first Chern class of the bundle. There are more general equations, the Hermitian–Einstein equations, that free us from this constraint. They are

$$F^{0,2}_\nabla = 0, \quad F^{2,0}_\nabla = 0, \quad \Lambda F_\nabla = iC \mathbf{I},$$  \hfill (4)

for an imaginary constant multiple $iC$ of the identity endomorphism. These equations can have non-trivial solutions on bundles of arbitrary degree.

In our case also, we find that the Bogomolny equations impose constraints, this time on the location of the singularities, that are too restrictive for our purposes. Just as the Bogomolny equations are reductions of the ASD equations, we have a reduced version of the Hermite–Einstein equation, which we call the Hermitian–Einstein–Bogomolny equation (or HE–Bogomolny for short). It is

$$F_\nabla - iC \mathbf{I} \omega_\Sigma = *\nabla \phi.$$  \hfill (5)

If $\nabla^{0,1}$ is the 0, 1 component of the covariant derivative along $\Sigma$, and $F_\Sigma \cdot \omega$ is the component of the curvature along $\Sigma$, and $\omega$ the Kähler form on $\Sigma$, these equations are

$$[\nabla^{0,1}_\Sigma, \nabla_t - i\phi] = 0,$$

$$F_\Sigma - \nabla_t \phi = iC \mathbf{I} \mathbf{E}.$$  \hfill (7)

### 2.2 The Dirac monopole

Our singularities are modeled on those of the Dirac monopole. We begin by considering this example in some detail. On $R^3$, one has spherical coordinates related to the Euclidean coordinates by $(t, x, y) = (R \cos \theta, R \cos \psi \sin \theta, R \sin \psi \sin \theta)$. The volume form is

$$d\mu_{R^3} = R^2 \sin \theta dR \wedge d\theta \wedge d\psi = -R^2 dR \wedge d(\cos \theta d\psi).$$

Consider the Hermitian line bundle $L_k$ over $R^3 \setminus \{(0, 0, 0)\}$ defined by the transition function $g_{\pi_0} = e^{ik\psi}$ on the complement of the t-axis, from $\theta \neq 0$ to $\theta \neq \pi$. Hence any section $\sigma$ of $L_k$ is given by two functions

$$\sigma_0: \{(t, x, y) \in R^3 \setminus \{(0, 0, 0)\} \mid \theta \neq 0\} \to C,$$

$$\sigma_\pi: \{(t, x, y) \in R^3 \setminus \{(0, 0, 0)\} \mid \theta \neq \pi\} \to C$$

subject to the relation $\sigma_\pi = g_{\pi_0} \sigma_0$.

Consider the connection $\nabla$ on $L_k$ defined by the connection matrices

$$A_0 = \frac{ik}{2} \left( 1 + \cos \theta \right) d\psi$$

$$A_\pi = \frac{ik}{2} \left( -1 + \cos \theta \right) d\psi$$

\hfill (8) \hfill (9)
Note that while $d\psi$ is not defined at $\theta = 0$ and $\theta = \pi$, the connection matrices $A_0$ and $A_\pi$ are smooth respectively at $\theta = \pi$ and $\theta = 0$.

Let

$$\phi = \frac{ik}{2R} \tag{10}$$

We have

$$d\phi = -\frac{ik}{2R^2} dR = *\left(\frac{ik}{2} d(\cos \theta d\psi)\right) = *F_\nabla,$$

so $(\nabla, \phi)$ satisfy the Bogomolny equation. Notice that, restricted to the sphere, we have

$$c_1(L_k) = \frac{i}{2\pi} \int_{S^2} F_\nabla = \frac{i}{2\pi} (-\frac{ik}{2}) \int_{S^2} d\mu_{S^2} = k.$$

We call this special solution to the Bogomolny equation the model Dirac monopole of charge $k$.

We now work out the $\bar{\partial}$ operator $\nabla^{0,1} = \frac{i}{2} (\nabla_x + i \nabla_y)$ and the change of gauge from a unitary to a “holomorphic gauge,” that is a non-unitary trivialisation by a section $\sigma$ with the same $(0,1)$ part. We can change the trivialisation to eliminate the $(0,1)$ part by applying a change of trivialisation $g_0$ that solves, in cylindrical coordinates $t, z, \psi$, the equation

$$\frac{\partial}{\partial r} \ln(g_0) = -\frac{k}{2} \frac{(t + \sqrt{t^2 + r^2})}{r\sqrt{t^2 + r^2}}.$$

A solution to this differential equation is

$$g_0 = (R - t)^{-\frac{k}{2}}. \tag{11}$$

In the new trivialisation, the connection form transformed by $(a \mapsto a - (dg)^{-1})$ is given by $-\frac{k}{2R} dt + \frac{k(R+t)}{2R^2} \bar{z} dz$. Hence, in this trivialisation $\nabla^{0,1} = \partial^{0,1}$, and $\nabla_t - i \phi = \bar{\partial}_t$. In this holomorphic gauge, the metric is given by

$$\left( g_0^* g_0 \right)^{-1} = (R - t)^k. \tag{12}$$

Similarly when $\theta \neq \pi$, after the change of trivialization,

$$g_\pi = (R + t)^{-\frac{k}{2}}, \tag{13}$$

the connection form is $-\frac{k}{2R} dt - \frac{k(R-t)}{2R^2} \bar{z} dz$. In this trivialisation again $\nabla^{0,1} = \partial^{0,1}$ and $\nabla_t - i \phi = \partial_t$. In this holomorphic gauge, the metric is given by

$$\left( g_\pi^* g_\pi \right)^{-1} = (R + t)^{-k}. \tag{14}$$

The two new trivialisations are related by

$$g_\pi g_\pi g_0^{-1} = z^k. \tag{15}$$

The operator $(\nabla_t - i \phi)$ appears already in the work of Hitchin [13] and was used there, as it will be here, in defining a “scattering map”; see below.
2.3 Monopoles and HE-Monopoles

Definition 2.1 Let $Y$ be a three-manifold, equipped with a metric, and $p$ be a point of $Y$. Let $R$ denote the geodesic distance to $p$. Let $(t, x, y)$ be coordinates centred at $p$ for which the metric is in these coordinates of the form $1 + O(R)$ as $R → 0$. Let $ψ, θ$ be, as above, angular coordinates on the spheres $R = c$, so that $R, ψ, θ$ provide standard spherical coordinates on a neighbourhood $B^3$ of $p$ defined by the inequality $R < c$. We say that a solution to the HE-Bogomolny equations $(E, ∇, φ)$ on $Y \setminus \{ p \}$ has a singularity of Dirac type, with weight $k = (k_1, \ldots, k_n)$ at $p$ if

- the restriction of the bundle $E$ to $B^3 \setminus \{ p \}$ is a sum of line bundles $L_{k_1} \oplus \cdots \oplus L_{k_n}$, and
- one can choose unitary trivialisations of $E$ over the two open subsets $θ ≠ 0$ and $θ ≠ π$ of $B^3$, with transition function $\text{diag}(e^{ik_1ψ}, \ldots, e^{ik_nψ})$, as above, in such a way that, in both trivialisations,

$$φ = \frac{i}{2R} \text{diag}(k_1, \ldots, k_n) + O(1),$$
$$∇(Rφ) = O(1).$$

A solution to the HE-Bogomolny equations that has singularities of Dirac type is called an HE-monopole.

The last condition tells us, via the HE–Bogomolny equations, that the curvature $F$ is $O(R^{-2})$.

As was first pointed out by Kronheimer [22] for $Y = \mathbb{R}^3$, and expanded upon by [30] for arbitrary three-manifolds, these conditions correspond to the connection matrices and curvature being bounded at the origin (modulo gauge) once they are “lifted” to $B^4$ via the Hopf map

$$B^4 → B^3$$

$$(w_1 = u_1 + iu_2, w_2 = u_3 + iu_4) → (t, x + iy) = (w_1\overline{w}_1 - w_2\overline{w}_2, 2w_1w_2)$$

This lift is performed as follows: one considers the trivial bundle $\hat{E} = B^4 \times \mathbb{C}^n$ equipped with the $S^1$-action:

$$(w_1, w_2, v_1, \ldots, v_n) → (e^{iθ}w_1, e^{-iθ}w_2, e^{ik_1θ}v_1, \ldots, e^{ik_nθ}v_n)$$

Then the $S^1$ quotient maps $\hat{E}$ to $E$, covering the map $π: B^4 → B^3$. Pauly [30] then shows that one can choose an appropriate metric above so that if (1) one defines a connection matrix over $B^4$ in a lifted trivialis over the complement of the origin as the lift $π∗∇ - ξ \otimes π∗φ$, where $ξ$ is a suitable one-form on $B^4$ (in the Euclidean case, $ξ = 2(-u_2du_1 + u_1du_2 - u_3du_3 + u_4du_4)$); in the general case, the form is modified by a term of order 2 at the origin) and (2) one applies a gauge transformation taking one from $S^1$-invariant trivialisations to one defined at the origin, one obtains:

Proposition 2.2 The lifting process gives a correspondence between

- Solutions to the Bogomolny equations on $B^3 \setminus \{ 0 \}$ with a Dirac type singularity at the origin, with weight $\vec{k}$.
- Anti-self-dual (ASD) connections on $\hat{E}$, invariant under the circle action, smooth away from the origin, and represented in a gauge in which the action is given as above by a connection matrix which is in $L^2_3$ (and so is continuous).

We note that the regularity one obtains above translates into $O(R^{-1})$ bounds for the connection matrices and for the Higgs field downstairs.

Remark 2.3 For the Dirac monopole, this process in fact simply has a Dirac monopole correspond to a flat connection upstairs; we note that since a flat connection is anti-self-dual in any metric upstairs, we have (locally) a Dirac monopole in any metric downstairs.

We will return to the HE–Bogomolny equations for this lifting process later on. In the mean time, we note that in the case we are considering the metric on $B^3$ is not generic. Indeed with a coordinate $z = x + iy$ on $Σ$, the metric on $\mathbb{R}^3$ is given by

$$g = α(z, \overline{z})dzd\overline{z} + dt^2$$

One can choose $z$ centred at the singularity, so that

$$α(z, \overline{z}) = 1 + z\overline{z}f(z, \overline{z})$$
The Bogomolny equations for $S^1 \times \Sigma$ are equivalent to time invariant anti-self-duality equations for $S^1 \times S^1 \times \Sigma$, with the metric $g + ds^2$, where $s$ denotes the natural coordinate on the extra circle. This metric is Kähler with respect to the complex structure with complex coordinates $s + it, z$. One can think of the lifts from $B^3$ to $B^4$ in terms of a “virtual” lift from $S^1 \times B^3$ to $B^4$, with the forms $dx, dy, dt$ lifting in the normal way and $ds$ replaced by $\xi$. This process works quite well, in that the lift preserves bitype for two-forms, the space of $(i, j)$ forms correspond to $(i, j)$ forms (when $i + j = 2$). In particular, if one uses the fact that $d\xi$ is of type $(1, 1)$, then “holomorphically integrable” connections below (that is, satisfying the equation $[\nabla_x - i\nabla_y, \nabla_t - i\phi] = 0$) correspond to holomorphically integrable connections upstairs.

2.4 The scattering map

**Definition 2.4** Let $(E, \nabla, \phi)$ be a solution to the Bogomolny (or HE–Bogomolny) equation on the product $I \times U$ of a interval $I$ with a possibly open Riemann surface $U$. Let $t_0 < t_1$ be two values of $t$ in $I$. The scattering map $R_{t_0, t_1}: E_{(t_0, x, y)} \to E_{(t_1, x, y)}$ is defined by taking for each $\sigma_0 \in E_{(t_0, x, y)}$, the unique solution $\sigma$ of $(\nabla_t - i\phi)\sigma = 0$ with $\sigma(t_0) = \sigma_0$; one then sets $R_{t_0, t_1}(\sigma_0) := \sigma(t_1)$.

If $(E, \nabla, \phi)$ is an HE-monopole on $S^1 \times \Sigma$, restricting the connection to the surfaces $\Sigma_t$ defines a $\bar{\partial}$-operator $\nabla_{\Sigma_t}^{0,1}$ on the surface, and so gives $E_t$ the structure of a holomorphic bundle. Now let $t < t'$. The fact that $[\nabla_{\Sigma_t}^{0,1}, \nabla_t - i\phi] = 0$ means that the scattering map defines a holomorphic isomorphism $R_{t, t'}$ from $(E_t)|_U$ to $(E_{t'})|_U$, for $U$ open, as long as the set $[t, t'] \times U$ does not contain one of the singular points $p_i$. In particular, for $t_i < t < t' < t_{i+1}$, the holomorphic bundles on $\Sigma_{t_i}, \Sigma_{t'}$ are globally isomorphic.

We want also to pass through singularities. To understand what the parallel transport does in that case, consider the model Dirac monopole of charge $k$. The scattering map $R_{-1,1}$ from $t = -1$ to $t = 1$ defined for $(x, y) \neq (0, 0)$ takes a particularly pleasant form. In the holomorphic trivialisations introduced in Section 2.2 for which $(\nabla_t - i\phi) = \partial_t$, it is simply given by the transition function $z^k$, as shown by Equation (15).

We now consider the asymptotics of the scattering map near a singularity for a general $U(n)$ HE-monopole on $S^1 \times \Sigma$. We do so in a trivialisation that satisfies both

$$\nabla_t - i\phi)\sigma = 0, \quad (\nabla_x + i\nabla_y)\sigma = 0.$$  \hspace{1cm} (20)

In particular, it is holomorphic along the surfaces $\Sigma_t$. As before, we call such a trivialisation a **holomorphic trivialisation**.

**Proposition 2.5** In holomorphic trivialisations at $t = \pm 1$, the scattering map is of the form

$$h(z)\text{diag}(z^{k_1}, \ldots, z^{k_n})g(z),$$

with $h$ and $g$ holomorphic and invertible, and the coordinate $z$ chosen so that the singularity is at $z = 0$.

**Proof:** We compare our monopole with a sum of $n$ Dirac monopoles with

$$E_0 = L_{k_1} \oplus \cdots \oplus L_{k_n}$$

$$\phi_0 = \frac{i}{2R}\text{diag}(k_1, \ldots, k_n)$$

There is then a natural monopole structure on $E_0^* \otimes E$ with connection $\nabla$ and Higgs fields $\phi$, and with weights $(k_i - k_j)$. In the trivialisations used in Definition 2.1, there is a natural identification of $E$ and $E_0$; under this identification, we choose the (holomorphic) section $S$ corresponding to the identity map along $t = 0$ (away from $z = 0$), and extended outwards by integrating $(\nabla_t - i\phi)|_U = 0$.

As explained above, solutions to Equation (20) correspond to holomorphic, $S^1$-invariant sections upstairs on $B^4$. Our section $S$ lies in the weight space of weight 0, and so it extends holomorphically and invertibly (since one can do the same for $S^{-1}$) to the origin.

One has the scattering map $\text{diag}(z^{k_1}, \ldots, z^{k_n})$ for $E_0$; applying $S$ gives the scattering map $S(1, z)\text{diag}(z^{k_1}, \ldots, z^{k_n})S(-1, z)^{-1}$ for $E$. If one changes the holomorphic trivialisations at $t = 1, -1$, one has the general form given above. \qed
3 Monopoles and stable pairs

3.1 From monopoles to a bundle pair

Let \((E, \nabla, \phi)\) be, as above, an HE-monopole. As we have noted, restricting the connection to the surfaces \(\Sigma_t\) defines a \(\partial\)-operator \(\nabla_{\Sigma_t}^{0,1}\), and so gives \(E_t\) the structure of a holomorphic bundle, denoted \(E_t\); simultaneously solving \((\nabla_t - i\phi)s = 0\) and \(\nabla_{\Sigma_t}^{0,1}s = 0\) defines a holomorphic isomorphism \(R_{t,t'}\) from \(E_t|_U\) to \(E_{t'}|_U\), for \(U\) open, as long as the set \([t, t']\times U\) does not contain any of the singular points \(p_i\). The discussions of the previous section can be summarised in:

**Proposition 3.1 (Kapustin–Witten [21])** The monopole restricted to the slice \(\Sigma_t\) defines a holomorphic \(\text{Gl}(n, \mathbb{C})\) bundle \(E_t\), away from the \(p_i\).

- If there are no singular time \(t_i\) between \(t\) and \(t'\), the scattering map \(R_{t,t'}: E_t \to E_{t'}\) is an isomorphism.
- If only one \(t_i\) lies between \(t\) and \(t'\), then \(c_1(E_{t'}) - c_1(E_t) = \text{tr}(\tilde{K}_i)\), and \(R_{t,t'}: E_t \to E_{t'}\) is a meromorphic bundle map which is an isomorphism away from \(z_i\), and there exist near \(z_i\) trivialisations of \(E_t, E_{t'}\) such that \(R_{t,t'}\) is given by \(\text{diag}((z - z_i)^{k_{i,1}}, \ldots, (z - z_i)^{k_{i,N}})\) in these trivialisations. (Here we abuse notation by letting \(z_i\) denote both a point and its coordinate.)
- More generally for all \(t, t'\), by composition, the scattering maps \(R_{t,t'}: E_t \to E_{t'}\) are meromorphic bundle maps which are isomorphisms away from the points \(z_1, \ldots, z_N\).

**Proof:** The proof is straightforward and follows from Proposition 2.5. \(\square\)

In particular, integrating around the circle, we have:

**Definition 3.2** The monodromy \(\rho_t\) of \(E_t\) is the map \(R_{t, t+T}\).

The monodromy \(\rho_t\) is a meromorphic endomorphism of \(E_t\). It has singularities near \(z_i\) of the form \(F(z)\text{diag}((z - z_i)^{k_{i,1}}, \ldots, (z - z_i)^{k_{i,N}})G(z)\), where \(F(z), G(z)\) are holomorphic and invertible.

**Definition 3.3** A bundle pair \((E, \rho)\) is the datum of a holomorphic bundle \(E\) on \(\Sigma\) and a meromorphic endomorphism \(\rho: E \to E\) such that \(\rho\) is an isomorphism outside of a finite set of points.

Thus \((E_t, \rho_t)\) is a bundle pair. For a bundle pair \((E, \rho)\), let us suppose that \(\rho\) fails to be regular at \(p\). If the rank of \(E\) is \(n\), Iwahori’s theorem (see [32, Chap. 8] and [18]) tells us that near the singular point \(p\), choosing a trivialisation, and a coordinate \(z\) centred at \(p\), one can find invertible holomorphic matrices \(F(z), G(z)\) and integers such that the map \(\rho\) in our trivialisation factors as

\[
\rho = F(z)\text{diag}(z^{\ell_1}, \ldots, z^{\ell_n})G(z),
\]

(22)

with the set \(\ell = \{\ell_1, \ldots, \ell_n\}\) as invariants.

In the case that concerns us here, we adapt the notation given in the introduction and define

**Definition 3.4** A bundle pair \((E, \rho)\) has type \(K = (\tilde{K}_1, z_1), \ldots, (\tilde{K}_N, z_N)\) if its non-regular points are the points \(z_1, \ldots, z_N\), and the map \(\rho\) is of the form \(F(z)\text{diag}(z^{\ell_1}, \ldots, z^{\ell_n})G(z)\), with \(F\) and \(G\) invertible, near \(z_j\), in coordinates \(z\) centred at \(z_j\).

We define the bundle pair associated to a monopole \((E, \nabla, \phi)\) as

\[
\mathcal{H}(E, \nabla, \phi) := (E_0, \rho_0).
\]

From what precedes, we see that for the monopoles we are studying, \(\mathcal{H}(E, \nabla, \phi)\) has type \(K\).
3.2 Constraints coming from the U(1) case

Let \( Y = (S^1 \times \Sigma) \setminus \{ p_1, \ldots, p_N \} \). If \((E, A, \phi)\) is a U(n) singular monopole on \( Y \) with a singularity of weight \( \vec{k}_j \) at \( p_j \), then \((\det(E), \text{tr}(A), \text{tr}(\phi))\) is a U(1) singular monopole on \( Y \) with a singularity of weight \( \text{tr}(\vec{k}_j) \) at \( p_j \). We shall see that there are constraints on U(1)-monopoles, in particular on the weights at the singularities and the location of the singular points; these restrictions propagate to arbitrary monopoles by taking traces.

We consider a U(1) monopole, that is, a triple \((E, \nabla, \phi)\), considered modulo gauge transformations:

1) \( \phi \) is a harmonic purely imaginary function \((d * d\phi = 0)\), of the form \( i\bar{k}_j/2r + O(1) \) near the points \( p_j \), and otherwise regular, \( r \) is the geodesic distance to the singularity
2) \( \nabla = d + A \) is a U(1)-connection satisfying \( dA = *d\phi \).

Note that once the location of the singular points and the charges \( k_j \) are fixed, the function \( \phi \) is unique up to a constant.

Let \( r \) be small, and let \( S^2_r(r) \) denote the sphere of radius \( r \) around \( p_j \), and let \( Y_r \) denote the subset of points in \( Y \) that are at least distance \( r \) from the singular set. Thus \( \partial Y_r = -\bigcup J S^2_r(r) \). We have

\[
\sum_{j=1}^N k_j = \sum_{j=1}^N \frac{i}{2\pi} \int_{S^2_r(r)} F_A
= -\frac{i}{2\pi} \int_{\partial Y_r} *d\phi
= -\frac{i}{2\pi} \int_{Y_r} d * d\phi = 0.
\]

By extension, all singular U(n) monopoles on \( Y \) must satisfy

\[
\sum_{j=1}^N \text{tr}(\vec{k}_j) = 0.
\]

This equation is also satisfied by HE-monopoles.

There are also constraints on the locations of the singularities. Since the monodromy \( \rho_t \) is a meromorphic bundle map, its determinant is a meromorphic function, hence its divisor is principal and therefore the locations of the singularities are constrained in the direction along \( \Sigma \). They are, in fact, also constrained in the circle direction.

**Proposition 3.5** For a singular U(n)-monopole, we must have

\[
\sum_{j=1}^N \text{tr}(\vec{k}_j) t_j = c_1(E|_{\{0\} \times \Sigma}) T.
\]

For a singular U(n) HE-monopole with constant \( C \), we must have

\[
\sum_{j=1}^N \text{tr}(\vec{k}_j) t_j = T \left( c_1(E|_{\{0\} \times \Sigma}) + \frac{Cn}{2\pi} \text{Vol}(\Sigma) \right).
\]

**Proof:** Let’s start with the U(1) case. Because we want \( *d\phi = F_A \), the integral of \( \frac{i}{2\pi} *d\phi \) on any compact 2-cycle must be an integer, indeed the first Chern class of the restriction of the bundle to that 2-cycle. This condition imposes constraints on the location of the singular points.

Let \( p_i = (t_i, z_i) \). Suppose for notational simplicity that none of the \( t_i \) are 0 and that the \( t_i \) are distinct, and let

\[
k_0 = \frac{i}{2\pi} \int_{\{0\} \times \Sigma} *d\phi.
\]

Moving \( t \) past the singular points \( t_1, \ldots, t_j \) in turn, the integral \( \frac{i}{2\pi} \int_{\{t_i\} \times \Sigma} *d\phi \) becomes \( k_0 + k_1 + \cdots + k_j \). On the other hand, one has \( 0 = \int_{\Sigma} \partial_\phi \) in the circle orthogonal to the surface, away from the singular points. Let us suppose that the length of the circle is \( T \), and set \( t_{n+1} = T, t_0 = 0 \). Integrating over \( S^1 \times \Sigma \) (removing small cylinders around the singular points and taking a limit), we find that

\[
0 = \sum_{i=1}^{N+1} \sum_{j=0}^{i-1} k_j(t_i - t_{i-1}) = \sum_{j=0}^{N+1} k_j(t_i - t_{i-1}) = \sum_{j=0}^{N} k_j(T - t_j) = k_0 T - \sum_{j=1}^{N} k_j t_j.
\]

Equation (24) follows easily. The general U(n) case follows by taking traces. \( \square \)
As a corollary of the proposition, the geometric interpretation for the constant $C$ is

$$ C = \left( -\frac{2\pi}{\text{Vol}(\Sigma)} \right) \left( \frac{c_1(E \mid \{t\} \times \Sigma) - \sum_{j=1}^{N} \text{tr}(\hat{k}_j) t_j}{n} \right). \quad (27) $$

Therefore $C$ is determined by the location of the singularities.

We pause for a few remarks that underline the strong parallels that exist between the monopole geometry and the complex geometry in the Abelian case. Let us first see how one can deform from a given solution, fixing the locus of the singularities.

For the monopoles, in the $U(1)$ case, once one has $\phi$, one has $dA$ and hence the connection $A$ up to a closed imaginary form. The equivalence relation on connections is modification by an imaginary exact form. Thus the deformations of $A$ are parameterised locally by $2g + 1$ real parameters, the dimension of $H^1(S^1 \times \Sigma, \mathbb{R})$. The free parameters correspond to the integrals of $A$ around cycles; one can modify $A$ by adding to it lifts of imaginary harmonic forms on the Riemann surface, and multiples of the form $idt$. In addition, one can add an imaginary constant to $\phi$: that is all the freedom one has, since the difference of any two $\phi$ with the same asymptotics is bounded, and so a constant, by elliptic regularity and the maximum principle. In short, there is modulo gauge transformations a $2g + 2$ real parameter space of pairs $(A, \phi)$ once one has fixed the singularities.

The monopole yields the complex geometric data of a line bundle on $\Sigma_0$, given by the restriction of the connection to $\Sigma_0$, and so an element of the Jacobian; it also gives the monodromy $\rho$, which is independent of the line bundle and which once the divisor is fixed is determined up to a non-zero complex constant. In short, one sees that the moduli space of complex data is a $\mathbb{C}^*$-bundle over the Jacobian, and so has the same dimension as our monopole moduli.

The constraints on the location of the singularities are similar also: the locations of the poles in the $t$-direction are determined in effect by the integral of $dt$ from some base point. In the direction of the Riemann surface $\Sigma$, the divisor $\sum j k_j z_j$ must be principal, and Abel's theorem requires that its image under the Abel map lie in the period lattice. These constraints on periods can also be seen to intervene in the construction of an Abelian monopole, in a way similar to the way they intervene in the classical proof of the existence of a holomorphic function in Abel's theorem (see e.g., [12, p.232]). Indeed, in both cases one builds a logarithmic derivative of the Higgs field or of the function; in both cases one first gets the right singularities, then adjusts so that one has integer periods. A complete proof for the Abelian monopoles would lead us into too long a digression, and in any case the general proof valid also in the non-Abelian case is the subject of this paper.

### 3.3 The stability conditions

We now define an appropriate notion of stability for our holomorphic objects. Set

$$ \vec{t} = (t_1, t_2, \ldots, t_N, T), \quad 0 < t_1 \leq t_2 \leq \cdots \leq t_N \leq T. \quad (28) $$

**Definition 3.6** The $\vec{t}$-degree $\delta_{\vec{t}}(E, \rho)$ of a bundle pair $(E, \rho)$ of singular type $K$ is defined by

$$ \delta_{\vec{t}}(E, \rho) = c_1(E) - \frac{\sum_{j=1}^{N} \text{tr}(\hat{k}_j) t_j}{T}. \quad (29) $$

The $\vec{t}$-slope of a bundle pair $(E, \rho)$ of rank $n$ is the quotient

$$ \mu_{\vec{t}}(E, \rho) = \delta_{\vec{t}}(E, \rho)/n. $$

**Definition 3.7** A bundle pair $(E, \rho)$ is $\vec{t}$-stable ($\vec{t}$-semi-stable) if any proper non-trivial $\rho$-invariant subbundle $E'$ satisfies $\mu_{\vec{t}}(E', \rho) < (\leq) \mu_{\vec{t}}(E, \rho)$. A bundle pair is $\vec{t}$-polystable if it is the sum of stable bundle pairs of equal $\vec{t}$-slope.

**Remark 3.8** The notion of $\vec{t}$-degree, and hence stability, is invariant under shifting the origin in the circle, as $\sum_j \text{tr}(\hat{k}_j) = 0$: it is also invariant as one moves through the singularities, as going through the point $p_j$ changes $c_1(E)$ by $\text{tr}(\hat{k}_j)$, but also shifts $t_j$ by $-T$. Indeed, if $(E, \rho)$ is obtained from an HE-monopole $(E, \nabla, \phi)$, then rewriting the degree as

$$ \delta_{\vec{t}}(E, \rho) = \frac{\sum_{j=1}^{N} (t_{j+1} - t_j)(c_1(E) + \sum_{i \leq j} \text{tr}(\hat{k}_i))}{T}, $$

one sees that $\delta_{\vec{t}}(E, \rho)$ is the average degree (average in $t$) of the restrictions to $\{t\} \times \Sigma$ of the bundle $E$. 

---

**Singular HE monopoles on the product of a circle and a Riemann surface** 9
In this paper we show the equivalence between \( \tilde{t} \)-polystable bundle pairs and HE-monopoles with Dirac singularities. We proceed inductively on the rank. We now prove it in one direction, showing that an HE-monopole yields a \( \tilde{t} \)-polystable pair. Note that stability is automatic in the case of rank one.

**Definition 3.9** A HE-monopole \((E', \nabla', \phi')\) with constant \(C'\) is a sub-HE-monopole of \((E, \nabla, \phi)\) (with constant \(C\)) if \(E'\) is a subbundle of \(E\) preserved by \(\nabla\) and \(\phi\) and if \(\nabla\) and \(\phi\) restricted to \(E'\) are \(\nabla'\) and \(\phi'\).

In particular, \(C = C'\).

**Proposition 3.10** A bundle pair \((E, \rho)\) on \(\Sigma\) corresponding through \(H\) to an HE-monopole \((E, \nabla, \phi)\) on \(S^1 \times \Sigma\) with constant \(C\) is \(\tilde{t}\)-polystable, and \(\tilde{t}\)-stable if \((E, \nabla, \phi)\) is irreducible, that is it admits no sub-HE-monopole.

**Proof:** Let \(V\) be a \(\rho\)-invariant proper subbundle of \(E\). We can choose this subbundle to be \(\tilde{t}\)-stable. By Proposition 4.2, it corresponds to an HE-monopole \((V, \nabla, \phi)\) with constant \(D\), and singularities of weight \(\tilde{t}_j\) at the point \(p_j\). We do not assume that \(V\) is a submonopole of \(E\), although it is clearly a subbundle of \(E\) by construction. The HE-monopole \((V^* \otimes E, \nabla, \phi)\) has constant \(C - D\), and has a global \(\rho\) invariant section \(s\).

Let \(\nabla_{\Sigma}\) denote the restriction of the connection to the Riemann surface, and \(\Delta_{\Sigma}\) the associated Laplacian. Using the identities

\[
\nabla_{\Sigma}^{1,0} \nabla_{\Sigma}^{0,1} = (\Delta_{\Sigma} + iF_{\Sigma})\omega
\]

\[
(\nabla_t + i\phi)(\nabla_t - i\phi) = \nabla_t^2 + \phi^2 - i\nabla_t,\phi
\]

we find that for the section \(s\) on \(\Sigma\),

\[
(C - D)|s|^2_{L^2} = \int_{S^1 \times \Sigma} \langle s, i(\nabla_t \phi - F_{\Sigma})s \rangle d\mu
\]

\[
\leq \int_{S^1 \times \Sigma} \big( \langle s, i\nabla_t s \rangle + |\nabla_t s|^2 + |\phi s|^2 \big) d\mu + \int_{S^1 \times \Sigma} \big( \langle s, -iF_{\Sigma} s \rangle + |\nabla_{\Sigma} s|^2 \big) d\mu
\]

\[
= \int_{S^1 \times \Sigma} \langle s, (i\nabla_t \phi - \phi^2 - \nabla_t^2) s + (-iF_{\Sigma} - \Delta_{\Sigma})s \rangle d\mu
\]

\[
= -\int_{S^1 \times \Sigma} \langle s, (\nabla_t + i\phi)(\nabla_t - i\phi)s + \omega^{-1}\nabla_{\Sigma}^{1,0}\nabla_{\Sigma}^{0,1}s \rangle d\mu + 0.
\]

The third step involves an integration by parts; one checks that this causes no difficulties at the singularities. Using the geometric interpretation of the HE constants given by Equation (27), we obtain \(\mu_{\tilde{t}}(V, \rho) \leq \mu_{\tilde{t}}(E, \rho)\) with equality only if the section \(s\) is covariant constant and intertwines the Higgs fields for \(E\) and \(V\), hence if \(V\) is a sub-HE-monopole. Since the orthogonal complement of \(V\) is then also a sub-HE-monopole of constant \(C\), its image \(V\) in \(E\) is also preserved by \(\rho\), and we can therefore by induction decompose \(E\) in a sum of \(\rho\)-invariant \(\tilde{t}\)-stable bundles of same slope and \((E, \rho)\) is \(\tilde{t}\)-polystable. The proof is now complete.

We note that in other situations of this type of correspondence (see, e.g., Lubke–Teleman [23]), the proof of this direction does not assume the converse for bundles of lower rank, but simply uses an integral of what looks like a second fundamental form. The approach used here allows us to simplify dealing with the asymptotics at the singularity.

### 4 Equivalence between stable pairs and monopoles

In this section, we prove Theorem 1.1 given in the Introduction.

**Theorem 4.1** Suppose that the points \(p_1, \ldots, p_N \in S^1 \times \Sigma\) project to \(N\) different points on \(\Sigma\). The map

\[
\mathcal{H}: \mathcal{M}_h^{ir}(S^1 \times \Sigma, p_1, \ldots, p_N, k_1, \ldots, k_N) \to \mathcal{M}_s(\Sigma, k_0, K, \tilde{t})
\]

\[
(E, \nabla, \phi) \mapsto (\mathcal{E}_0, \rho_0)
\]

(31)

between the moduli space of irreducible HE-monopoles and the moduli space of \(\tilde{t}\)-stable pairs described by Section 3.1 is a bijection. More generally, the reducible HE-monopoles correspond bijectively to \(\tilde{t}\)-polystable, but unstable, pairs.
The proof of surjectivity and injectivity are tackled separately by Propositions 4.2 and 4.7 below.

**Proposition 4.2**  
Given a $\tilde{t}$-stable pair $(\mathcal{E}, \rho)$ on $\Sigma$ of type $K = ((\tilde{k}_1, z_1), \ldots, (\tilde{k}_N, z_N))$ and the singular time data $0 < t_1 \leq \cdots \leq t_N \leq T$. There is a singular HE-monopole on $S^1 \times \Sigma$ of with Dirac-type singularities of weight $\tilde{k}_j$ at $p_j = (t_j, z_j)$ for which $\mathcal{H}(E, \nabla, \phi) = (\mathcal{E}, \rho)$.

**Proof:** The steps we follow in showing this proposition are as follows:

- We use $\rho$ to extend $\mathcal{E}$ to a bundle $E$ on $Y = S^1 \times \Sigma \setminus \{p_1, \ldots, p_N\}$, with the correct degrees on the spheres around the points $p_j$, that is holomorphic on all the slices $\Sigma_t$, and lifts to a holomorphic bundle $\tilde{E}$ on the complex manifold $X = S^1 \times Y$, subset of $\hat{X} = S^1 \times S^1 \times \Sigma$, invariant under the action of $S^1$ on the first factor $S^1$ of $X$.

- We have on $\tilde{E}$ a holomorphic structure; thus for any hermitian metric on the bundle, there is a unique unitary connection (the Chern connection) compatible with the holomorphic structure. We choose a hermitian metric on $\tilde{E}$ whose Chern connection around the $j$th singularity is that of the sum of Dirac monopoles of weights $\tilde{k}_j$.

- This metric serves as an initial metric for the heat flow of Simpson’s paper [34]. We take the limit as time goes to infinity to produce a Hermitian–Einstein connection on $\tilde{E}$, invariant under the action of $S^1$ on the first factor of $X$, and so descending to $Y$. This process gives us the connection we want on $X$, and, reducing to $Y$, our HE-monopole.

- Simpson’s theorem does not immediately give us the regularity we need at the singular points. To see that the singularities are indeed of Dirac type; we finish the proof by lifting locally on three-balls $B^3$ surrounding the singularities using the Hopf map $B^4 \to B^3$.

The first step is to extend $\mathcal{E}$ to a bundle on $Y$. We have supposed, without loss of generality, that none of the $t_j$ is zero. Consider the projection map $\pi$ from $\tilde{Y} = ((-T, T) \times \Sigma) \setminus \bigcup_j ((-T, t_j - T) \cup (t_j, T)) \times \{z_j\}$ to $\Sigma$; take the lift $\pi^* \mathcal{E}$ to $\tilde{Y}$. As $Y$ can be obtained from $\tilde{Y}$ by an identification $(t, z) \to (t + T, z)$, we define a bundle $E$ on $Y$ by identifying $(t, z, v), t \in (-T, 0), z \in \Sigma, v \in \mathcal{E}|_z$ to $(t + T, z, \rho(z)v)$. Since the holomorphic map $\rho$ decomposes, by hypothesis, into $h(z) \text{diag}(z^{k_j})g(z)$, in coordinates $z$ centred at the points $p_j$, the bundle one obtains is then indeed a sum $L_{k_{j1}} \oplus \cdots \oplus L_{k_{jn}}$ in a punctured neighborhood of $p_j$. Lifting to $X$, as the clutching functions are holomorphic, the result is a holomorphic bundle $\tilde{E}$. Since it is holomorphic, it has a $\bar{\partial}$ operator on it.

For the second step, we specify a hermitian metric $K$ on $\tilde{E}$. This metric specifies the Chern connection: one has $\nabla = d + ((\bar{\partial}K)K^{-1})^T$ in a holomorphic trivialisation. We need a metric $K$ whose associated connection is close to a solution of the Hermitian–Einstein equations, and has the singular behaviour that we want: we therefore choose hermitian metrics corresponding to lifts of a sum of Dirac monopoles in a neighbourhood of the singularities. As we also want our metric to be $S^1$-invariant on $X$, we define it on $E$ over $Y$.

We can choose in $\Sigma$ a small disk $D_0$, and disks $D_j$ surrounding the $z_j$, such that all these disks are mutually disjoint. We now choose over $\Sigma = E_0$ a metric $k_0$: trivialise $E$ over the complement of $D_0$, and with respect to this trivialisation, choose the metric $k_0 = 1$ on this complement, then extend to a metric $k_0$ over $D_0$. The curvature $F_0$ of the induced connection is then concentrated on $D_0$; its trace represents the first Chern class of $E = E_0$. Lift the bundle $E_0$, and the metric $k_0$, to $\tilde{Y}$. We note that as we vary $t$, the curvature $F_0$ can no longer represent the Chern class of $E_t$ as $t$ moves through the various singular points, as this Chern class changes. This problem is solved by glueing in Dirac monopoles in balls around the singular points, in such a way that after moving through the singular point $p_j$ some curvature is added into the disk $D_j$.

More explicitly, for $j = 1, \ldots, N$ let $C_j$ be disks properly included in $D_j$, and let $\epsilon$ be such that $4\epsilon < \min(t_1, t_2 - t_1, \ldots, t_N - t_{N-1}, T - t_N)$. We cover $Y$ by open sets

$$U_0 := ((-2\epsilon, t_N + 2\epsilon) \times \Sigma) \setminus \bigcup_j (t_j - \epsilon, t_N + 2\epsilon) \times C_j),$$

$$U_{N+1} := (t_N + \epsilon, T - \epsilon) \times \Sigma,$$

$$U_{j-} := ((t_j - 2\epsilon, t_j + 2\epsilon) \times D_j) \setminus ((t_j, t_j + 2\epsilon) \times \{z_j\}), \quad j = 1, \ldots, N,$$

$$U_{j+} := ((t_j - 2\epsilon, t_N + 2\epsilon) \times D_j) \setminus ((t_j + 2\epsilon, t_j) \times \{z_j\}), \quad j = 1, \ldots, N.$$

In the trivialisation given above over the complement of $D_0$, factor $\rho$ near each point $z_j$ as $\rho = h_j(z)\text{diag}(z^{k_j})g_j(z)$ with $g_j, h_j$ invertible, where the coordinate $z$ is centred at $z_j$ and is chosen so that the metric osculates at $z_j$ the
Euclidean metric with orthonormal coordinates \( t, \mathbb{R}(z), \Im(z) \). An equivalent construction of the bundle \( E \) is given by specifying the transition functions \((f_{a,b} \text{ over } U_a \cap U_b)\):

\[
\begin{align*}
  f_{0,j-} &= g_j, & f_{j-,j+} &= \text{diag}(z^{k_j}), & f_{0,j+} &= g_j\text{diag}(z^{k_j}), & f_{j+,N+1} &= h_j, \\
  f_{0,N+1} &= \rho \text{ over } (t_N + \epsilon, t_N + 2\epsilon), & f_{0,N+1} &= 1 \text{ over } (T - 2\epsilon, T - \epsilon).
\end{align*}
\]

Now note that the bundle and its transition functions are those for sums of Dirac monopoles on \( U_{j-} \) and \( U_{j+} \); we choose the hermitian metrics \( \text{diag}((R - t)^{k_j}) \) on \( U_{j-} \), and \( \text{diag}((R + t)^{k_j}) \) on \( U_{j+} \), which are compatible under the change of basis. In parallel, we have the metric lifted from \( E \) on \( U_{0,0} U_{N+1} \). Choosing a partition of unity, we patch all these metrics together over \( Y \), taking the non-trivial changes of trivialisations into account.

The metric \( K \) we have obtained can be lifted to a metric \( \bar{K} \) on \( \bar{E} \).

**Lemma 4.3** The pair \((\bar{E}, \bar{K})\) constructed above has the following properties

- \( \bar{E} \) is invariant under the action of \( S^1 \) on \( X \); this action complexifies to an action of \( \mathbb{C}^* \) over \( S^1 \times S^1 \times (\Sigma \setminus \{z_1, \ldots, z_N\}) \), with the action of the real element \( T \in \mathbb{R} \subset \mathbb{C}^* \) corresponding to \( \rho \)
- \( \bar{K} \) is invariant under the action of \( S^1 \) on \( \bar{E} \).
- In the neighborhood of the inverse image of the singular point \( p_j \), the pair \((\bar{E}, \bar{K})\) corresponds to a sum of Euclidean Dirac monopole of weight \( \bar{k}_j \).
- \((\bar{E}, \bar{K})\) satisfies a bound \(|\Lambda F_{\bar{K}}| \leq c < \infty \).

The first three properties follow by construction. For the fourth, we note that \( \Lambda F_{\bar{K}}^\perp \) would be 0 in a neighbourhood of the singular circles if the metric on \( X \) in this neighbourhood were Euclidean. Since we took coordinates osculating the Euclidean metric to second order, we still have the bound. We note that we could have glued in the Dirac monopoles in the \( S^1 \times \Sigma \) metric produced in Section 2.3 instead of the Euclidean ones, in which case \( \Lambda F_{\bar{K}}^\perp \) would indeed be zero in a neighbourhood of the singularities; we have used the Euclidean ones for explicitness.

We note that the curvature of the Chern connection is concentrated on \((S^1 \times S^1 \times D_0) \cup (\bigcup_{j=1}^N (S^1 \times (t_j - 2\epsilon, T) \times D_0))\).

We take \( K \) as the starting point for Simpson’s heat flow

\[
\begin{align*}
H^{-1} \frac{dH}{du} &= -i\Lambda F_{\bar{H}}^\perp, \\
H_0 &= K.
\end{align*}
\]

The asymptotic behaviour of this heat flow is governed by the following theorem.

**Theorem 4.4 (Simpson [34, Thm 1, p. 878, case \( \theta = 0 \))** Let \((X, \omega)\) satisfy certain conditions given below in Lemma 4.6, and suppose \( E \) is an \( S^1 \)-invariant bundle on \( X \) with \( S^1 \)-invariant metric \( K \) satisfying the assumption that \( \sup |\Lambda F_K| < c \). Suppose \( E \) is stable, in the sense that it arises from a stable pair on \( \Sigma \). Then there is a \( S^1 \)-invariant metric \( H \) with \( \det(H) = \det(K) \), \( H \) and \( K \) mutually bounded, \( \partial(K^{-1}H) \in L^2 \), and such that \( \Lambda F_{\bar{H}}^\perp = 0 \). In addition, if \( R \) is the distance to one of the singularities, \( R \cdot d(K^{-1}H) \) is bounded by a constant.

(The last sentence does not appear as part of Simpson’s statement of the theorem, but is given in a remark after his Lemma 6.4)

Since \( H_\infty \) is \( S^1 \)-invariant, we can quotient out and consider it on \( E \) (over \( Y \)). The equation \( \Lambda F_{\bar{H}}^\perp = 0 \) then becomes the HE-Bogomolny equation (5). The metric \( H_\infty \) is of course obtained as the limit \( \lim_{u \to -\infty} H_u \) of the heat flow.

Simpson uses a notion of stability slightly different from ours. His degree is

\[
\deg(E, K) = i \int_X \text{tr}(\Lambda F_K).
\]

**Lemma 4.5** The two notions of degree coincide: \( \deg(E, K) = T\delta_\mathcal{E}(\mathcal{E}, \rho) \). More explicitly,

\[
i \int_X \text{tr}(\Lambda F_K) = Tc_1(E_0) - \sum_{j=1}^N \text{tr}(\bar{k}_j)t_j.
\]

Furthermore, there a correspondence between holomorphic subpairs \((\mathcal{V}, \rho|_\mathcal{V})\) of \((\mathcal{E}, \rho)\) and \( S^1 \)-invariant holomorphic subbundles \( V \) of \( \bar{E} \), and for these also the notions of degree coincide.
Proof of Lemma 4.5. Indeed, for $\hat{E}$, the quantity tr$(\Lambda F_K)$ is equal to the lift of tr$(F_{\Sigma} - \nabla t \phi)$. Integrating tr$(F_{\Sigma})$ gives, as above, $Tc_1(E_0) - \sum_{j=1}^{N} \text{tr}(\hat{k_j} t_j)$; integrating $\nabla t \phi$, starting with the $t$-direction, gives 0.

We note that from the definition of $\hat{E}$, $\rho$-invariant subbundles $\mathcal{V}$ of $\mathcal{E}$ naturally give $S^1$-invariant subbundles $\hat{V}$ of $\hat{E}$; on the other hand, if a subbundle $\hat{V}$ of $\hat{E}$ is $S^1$-invariant, it is also $C^\ast$-invariant, and one can define a pair $(\mathcal{V}, \rho)$ by setting $\mathcal{V} = \hat{V}|_{\{0,0\} \times \Sigma}$ and using the time $T$ action of $C^\ast$ to define $\rho$. The proof of the equivalence of degrees for $\mathcal{V}$ and $\hat{V}$ goes through as for the bundle as a whole.

(Lemma 4.5)

Lemma 4.6 The manifold $X = S^1 \times ((S^1 \times \Sigma) \setminus \{p_1, \ldots, p_N\})$ satisfy the three conditions necessary to Simpson’s Theorem 4.4:

1. $X$ is a Kähler manifold of finite volume;
2. there is on $X$ a non-negative exhaustion function whose Laplacian is bounded;
3. there is an increasing function $a: [0, \infty) \to [0, \infty)$ with $a(0) = 0$ and $a(x) = x$ when $x > 1$, such that if $f$ is a bounded positive function on $X$ with $\Delta(f) \leq B$ then
   $$\sup_X |f| \leq C(B) a\left(\int_X |f|\right),$$
   and furthermore, if $\Delta(f) \leq 0$ then $\Delta(f) = 0$.

Proof of Lemma 4.6. The first condition is obviously satisfied by the construction of $X$.

To construct the non-negative exhaustion function on $X$ subject of the second condition, we first build a function $f$ on $Y$ whose Laplacian is bounded. The wanted exhaustion function is the pull back of $f$ to $X$ via the projection on $Y$. If the Laplacian of $f$ on $Y$ is bounded, then the Laplacian of the corresponding pull-backed $f$ on $X$ is also bounded.

Let $R_j$ be the geodesic distance in $S^1 \times \Sigma$ to the singularity $p_j$. In the Euclidean case, there is an obvious candidate for $f$: let $f$ be $1/R_j$ close to $p_j$ and extend it smoothly to the rest of $Y$. Since $1/R$ is harmonic in $\mathbb{R}^3$, the Laplacian of $f$ on $Y$ is obviously bounded. For other Riemann surfaces, $\Delta(1/R)$ has a term behaving like $1/R$ so we have to be careful and find a bounded function whose Laplacian kills that extra $1/R$ factor. In [29, Prop 3.2.2], Pauly proves that there is a harmonic function $f_j$ on a neighborhood of the singularity $p_j$ such that $f_j = 1/R_j + O(1)$. This function is exactly what we are looking for, and extending all the $f_j$ to $Y$ we find a function $f$ whose Laplacian is bounded. The obtained function $f$ is exactly the type of exhaustion function we are looking for.

Simpson proves in [34, Prop 2.2] that the third condition is fulfilled for the smaller space $X^o = (T^2 \times \Sigma) \setminus \bigcup_j T^2 \times \{z_j\}$. Since $X^o$ is dense in $X$, the condition is also fulfilled for $X$.

(Lemma 4.6)

In the case where $K$ is the metric we have carefully constructed above, the limiting $H = H_\infty$ this theorem gives us yields a solution on $Y$ to the HE-monopole equations. We need to understand why the corresponding HE-monopole has the desired Dirac monopole behavior at the poles. For this, as above, we can use the local construction exploited both by Kronheimer [22] and Pauly [30] and explained on page 5: one considers the quadratic map $\pi: B^4 \to B^3$ given by

$$\pi(w_1, w_2) = (t = |w_1|^2 - |w_2|^2, z = x + iy = 2w_1w_2, )$$

(33)

This defines a lift of forms $\pi^*$: as noted above in Section 2.3, if we add an extra variable $s$ to $B^3$, expanding to $S^1 \times B^3$, we can write the HE–Bogomolny equations for $\nabla, \phi$ as the HE equations for $\nabla = \nabla + \phi ds$ (recall that our complex coordinates are $t - is, z$); if we make the formal definition $\pi^* ds = \xi$, where $\xi$ as above is the $S^1$ invariant form

$$\xi = \frac{1}{i}(w_1 dw_1 - \bar{w}_1 dw_1 - w_2 dw_2 + \bar{w}_2 dw_2)$$

then the process used by Kronheimer and Pauly to smooth out the Dirac singularities associates to $\langle \nabla = d + A, \phi \rangle$ the “lift” on $B^4$: $\hat{\nabla} = \pi^* \nabla = \pi^* \nabla + \pi^* \phi \xi$; the curvatures are related by

$$F_\phi = \pi^* F_\phi + \pi^* \phi \xi.$$

Now we have for an HE-monopole on $B^3$ an equation given by asking that, after lifting to $S^1 \times B^3$, the projection of the curvature onto the space of self-dual two-forms, with kernel the space of anti-self-dual two-forms, take on a
specified value. We can ask what this projection corresponds to on \( B^4 \). Unlike Pauly, we do not modify our form \( \xi \) and the metric on \( B^4 \); we keep the standard Euclidean form and metric, and simply consider how the equation varies.

Keeping the formal lift \( \pi^*(\xi) \), we have the lifts (dropping the \( \pi^* \)):

\[
\begin{align*}
 dz &= 2(w_1 dw_2 + w_2 dw_1), & dt - ids &= 2\overline{w}_1 dw_1 - 2\overline{w}_2 dw_2, \\
 d\overline{z} &= 2(\overline{w}_1 d\overline{w}_2 + \overline{w}_2 d\overline{w}_1), & dt + ids &= 2w_1 \overline{d}w_1 - 2w_2 \overline{d}w_2.
\end{align*}
\]

Therefore the bitypes above and below correspond; in particular the spaces of \((2,0), (0,2)\) and \((1,1)\) forms upstairs and downstairs correspond.

Let us now look at the \((1,1)\) forms under this formal pullback. We have the lift of the Kähler form \( \pi^*(\Omega) = \frac{i\alpha}{2} dz \wedge d\overline{z} - dt \wedge \xi = 2\left[ (\alpha|w_2|^2 + |w_1|^2)dw_1 \wedge d\overline{w}_1 + (\alpha|w_1|^2 + |w_2|^2)dw_2 \wedge d\overline{w}_2 + (\alpha - 1)(w_2 \overline{w}_1 dw_1 \wedge d\overline{w}_2 + w_1 \overline{w}_2 dw_2 \wedge d\overline{w}_1) \right] \)

and the lift of the three anti-self-dual forms

\[
\begin{align*}
 \tilde{\epsilon}_1 &= \frac{1}{4} (dz \wedge d\overline{z} - \alpha(dt - i\xi) \wedge (dt + i\xi)) = (|w_2|^2 - |w_1|^2)(dw_1 \wedge d\overline{w}_1 - dw_2 \wedge d\overline{w}_2) \\
 &\quad + (1 - \alpha)(|w_1|^2 dw_1 \wedge d\overline{w}_2 + |w_2|^2 dw_2 \wedge d\overline{w}_1) \\
 &\quad + (1 + \alpha)(w_2 \overline{w}_1 dw_1 \wedge d\overline{w}_2 + w_1 \overline{w}_2 dw_2 \wedge d\overline{w}_1), \\
 \tilde{\epsilon}_2 &= \frac{1}{4} (dz \wedge (dt + i\xi)) = w_1 w_2 (dw_1 \wedge d\overline{w}_1 - dw_2 \wedge d\overline{w}_2) - w_2^2 dw_1 \wedge d\overline{w}_2 + w_1^2 dw_2 \wedge d\overline{w}_1, \\
 \tilde{\epsilon}_3 &= \frac{1}{4} (d\overline{z} \wedge (dt - i\xi)) = -\overline{w}_1 \overline{w}_2 (dw_1 \wedge d\overline{w}_1 - dw_2 \wedge d\overline{w}_2) - \overline{w}_1^2 dw_1 \wedge d\overline{w}_2 + \overline{w}_2^2 dw_2 \wedge d\overline{w}_1.
\end{align*}
\]

Dividing by \( 4(|w_1|^2 + |w_2|^2) \) the lift of Kähler form, we have

\[
\tilde{\Omega} := \frac{i}{2} dw_1 \wedge d\overline{w}_1 + dw_2 \wedge d\overline{w}_2 + \frac{i(\alpha - 1)}{2(|w_1|^2 + |w_2|^2)} (|w_1|^2 dw_2 \wedge d\overline{w}_2 + |w_2|^2 dw_1 \wedge d\overline{w}_1 + w_2 \overline{w}_1 dw_1 \wedge d\overline{w}_2 + w_1 \overline{w}_2 dw_2 \wedge d\overline{w}_1).
\]

Let \( \omega \) be the standard Kähler form for the Euclidean metric upstairs, let \( R^2 = |w_1|^2 + |w_2|^2 \) and let \( Q \) be the quadratic expression in \( w_1, \overline{w}_1 \) such that \( \tilde{\Omega} = \omega + (\alpha - 1)\frac{Q}{R^2} \).

Interestingly, the basis \( \{\tilde{\Omega}, \tilde{\epsilon}_1, \tilde{\epsilon}_2, \tilde{\epsilon}_3\} \) of \( \bigwedge^{1,1} \) is orthogonal for the usual Euclidean inner product on \( B^4 \), and for that inner product \( |\tilde{\Omega}|^2 = (\alpha^2 + 1) \). Therefore the projection operator on the linear subspace spanned by \( \tilde{\Omega} \) can be written

\[
P_{\tilde{\Omega}}(F) = \frac{\langle F, \tilde{\Omega} \rangle}{\langle \tilde{\Omega}, \tilde{\Omega} \rangle} \tilde{\Omega}.
\]

One can check that in fact,

\[
P_{\tilde{\Omega}}(F) = \frac{\langle F, \omega \rangle}{2} + \frac{\alpha - 1}{(\alpha^2 + 1)} \left( \frac{\langle F, Q \rangle}{R^2} \omega + \frac{\langle F, \omega \rangle Q}{R^2} + (\alpha - 1) \frac{\langle F, Q \rangle}{R^4} Q - \frac{(\alpha + 1)}{2} \langle F, \omega \rangle \omega \right).
\]

For \( k \in \mathbb{N} \), let \( p_k \) symbolically represents any homogeneous polynomial of degree \( k \) in the \( w_i, \overline{w}_i \) and let \( p_\infty \) be any smooth function. Therefore \( p_k + p_k = p_k \) and \( p_k p_\infty = p_k p_\infty \). Using this formalism, when \( \alpha \) is normalised as in Equation (19), we have \( \alpha = 1 + p_4 p_\infty \).

For \( k \) finite, notice that, if \( \partial \) represents any derivative with respect to \( w_i, \overline{w}_i \), we have \( \partial (\frac{p_k}{R^2}) = \frac{p_k + 1}{R^{2k+1}} \). Since \( \frac{p_k}{R^{2k}} \in C^0 \) if \( k - j > 0 \), we have \( \frac{p_k}{R^{2k}} \in C^{k-j-1} \). As

\[
P_{\tilde{\Omega}}(F) = p_\infty + p_\infty \frac{p_6}{R^2} + p_\infty \frac{p_{12}}{R^4},
\]

the coefficients of the projectors are in \( C^3 \).

Now the equation for HE connections below is

\[
P_{\tilde{\Omega}}(F_{\tilde{\Omega}}) = C_{1E} \Omega
\]
Lifting, the equation becomes

\[ P_\Omega (F_\varphi - \pi^* \phi d\xi) = \pi^*(CI_E \Omega). \]

We note that if \( \alpha \) is uniformly one (the Euclidean case), \( \Lambda(d\xi) = P_\omega (d\xi) = 0 \); for \( \alpha \) of the form \( 1 + w\overline{w} f(w, \overline{w}) \), \( \Lambda(d\xi) = P_\Omega (d\xi) \) is of the form (bounded)(quartic) near the origin.

The hermitian connection \( \nabla \) of interest to us is obtained from an initial hermitian connection \( \nabla_0 \), the Chern connection for a metric \( H_0 = K \), by keeping the same \((0,1)\) part and modifying the \((1,0)\) part so that \( \nabla \) is the Chern connection for a modified metric \( H_\infty = H_0 h \): this gives for the connection matrices (Simpson, lemma 3.1):

\[ A^{0,1} = A_0^{0,1}, A^{1,0} = A_0^{1,0} + h^{-1} \nabla_0^{1,0} h \]

In particular, for the Higgs field

\[ \phi = \phi_0 + \frac{i}{2} h^{-1} \nabla_{0,h} \]

For the \((1,1)\) component of the curvature, one has

\[ F^{1,1}_\varphi = F^{1,1}_\varphi + \nabla_0^{0,1} (h^{-1} \nabla_0^{1,0}) \]

Now lift this to \( B^4 \): one has the equation

\[ F_\varphi - \pi^* \phi d\xi = F_\varphi - \pi^* \phi_0 d\xi + \nabla_0^{0,1} (\pi^* h^{-1} \nabla_0^{1,0} \pi^* h) + \pi^* [h^{-1} (\pi^* \phi d\xi, \pi^* h)] \]

Since the original upstairs connection, a sum of the flat connections corresponding to Dirac monopoles, has zero curvature in the neighbourhood of the origin, we have in this neighbourhood

\[ F_\varphi - \pi^* \phi d\xi = -\pi^* \phi_0 d\xi + \nabla_0^{0,1} (\pi^* h^{-1} \nabla_0^{1,0} \pi^* h) + \pi^* [h^{-1} (\pi^* \phi d\xi, \pi^* h)] \]

We have the equation

\[ \Lambda(-\pi^* \phi_0 d\xi + \nabla_0^{0,1} (\pi^* h^{-1} \nabla_0^{1,0} \pi^* h) + \pi^* [h^{-1} (\pi^* \phi d\xi, \pi^* h)]) = \pi^*(CI_E \Omega). \]

When \( \alpha \) is uniformly one, the forms \( d\xi, \overline{d\xi} \) are anti-self-dual, and the equations reduce to

\[ \hat{D}(h) \equiv \Delta h - 2i \Lambda \nabla_0^{0,1} h^{-1} \nabla_0^{1,0} h = 0. \]  

(34)

When \( \alpha \) is not uniformly one, we have an elliptic equation \( \hat{D}(h) = 0 \), a deformation of the one above, whose coefficients are \( C^1 \) (taking into account the poles of \( \phi_0 \) and the behaviour of \( \Lambda(-\pi^* \phi_0 d\xi) \)).

Now let us recall that \( h \) is obtained as \( h_\infty \) from a heat flow \( h_u \) downstairs; \( h \) is smooth away from the singularities. Upstairs, \( h_u \) solves the heat equation \( \partial_u h_u = \hat{D}(h_u) \). Now take upstairs a four-ball around the singular point, mapping to a three-ball downstairs, and take as initial conditions for the heat flow \( \partial_u h_u = \hat{D}(h_u) \) the value \( h_0 = 1 \), and as boundary condition the Dirichlet condition \( h = h \). Applying the work of Donaldson [11], or simply again the results of Simpson, one obtains a \( C^1 \) solution \( h_u \), which is \( S^1 \)-invariant as the initial and boundary conditions are so, and which satisfies the same boundary conditions and initial conditions as \( h(t) \). One again has a limit \( h = h_\infty \), solution to \( \hat{D}(h) = 0 \).

Both \( h \) and \( \hat{h} \) descend to the three-ball, and solve the HE–Bogomolny equations there. One then can refer to the lemma in Simpson [34, p. 893], that tells us that one has uniqueness if solutions are bounded, which they are. Thus \( \hat{h} = h \), telling us that the global \( h \) given by Simpson’s result has the required smoothness at the singular points (as \( h \) does) to ensure that the Higgs field and its covariant derivative have the correct Dirac type singularities.

We now have a HE-monopole corresponding to our initial data; we now must check that it is unique.

**Proposition 4.7** Given two monopoles \( (E, \nabla, \phi) \) and \( (E', \nabla', \phi') \) yielding isomorphic holomorphic data. Then the two monopoles are isomorphic. Hence the map \( \mathcal{H} \) given by Equation (31) is injective.

**Proof:** Let \( \mathcal{H}(E, \nabla, \phi) = (E', \rho) \) and \( \mathcal{H}(E', \nabla', \phi') = (E', \rho') \). Since the holomorphic data are isomorphic, the associated holomorphic bundles \( E, E' \) are isomorphic by a holomorphic map \( \tau \), in a way that intertwines \( \rho \) and \( \rho' \). The same holds more generally for \( E_i, E_i' \): the map \( \tau \) thus also aligns the corresponding eigenspaces of \( \rho, \rho' \). One then has an isomorphism \( \tilde{\tau} \) from \( E \) to \( E' \) over \( S^1 \times \Sigma \). One can combine the two monopoles and get a monopole \( (E_\ast \otimes E', \nabla = -\nabla \otimes \mathbb{I} + \mathbb{I} \otimes \nabla', \phi = -\phi \otimes \mathbb{I} + \mathbb{I} \otimes \phi') \).
Consider $\hat{\tau}$ as a section of $E^* \otimes E'$. We already know that $\hat{\tau}$ is in the kernel of $\hat{\nabla}_{\Sigma}^{0,1}$ and $\hat{\nabla}_t - i\phi$. Using the identities (30), we find that

$$0 = -\int_{S^1 \times \Sigma} \langle \hat{\tau}, (\hat{\nabla}_t + i\phi)(\hat{\nabla}_t - i\phi)\hat{\tau} + \omega^{-1}\hat{\nabla}_{\Sigma}^{1,0}\hat{\nabla}_{\Sigma}^{0,1}\hat{\tau}\rangle d\mu$$
$$= \int_{S^1 \times \Sigma} \langle \hat{\tau}, (-\hat{\omega}^2 - \hat{\nabla}_t^2 - \hat{\Delta}_\Sigma)\hat{\tau}\rangle d\mu$$
$$= \int_{S^1 \times \Sigma} \langle \hat{\phi}t, \hat{\phi}\tau\rangle + \langle \hat{\nabla}_t\tau, \hat{\nabla}_t\tau\rangle + \langle \hat{\nabla}_{\Sigma}\hat{\tau}, \hat{\nabla}_{\Sigma}\hat{\tau}\rangle d\mu.$$ 

Hence $\hat{\tau}$ is covariant constant. As a map $E \to E'$, it intertwines the two Higgs fields. Hence the two monopoles are isomorphic. \qed

5 Moduli

5.1 HE-Monopoles on a three-torus

It turns out that the moduli space of stable pairs has already been extensively studied for curves $\Sigma$ of genus one in the context of integrable systems; see in particular [15; 16].

Definition 5.1 The pair $(E, \rho)$ is simple if any section of $\text{End}(E)$ commuting with $\rho$ is a multiple of the identity.

Proposition 5.2 If $(E, \rho)$ is $\tilde{t}$-stable, then it is simple.

The proof is the usual one: if it is not simple, then the eigenbundles and generalised eigenbundles of a section $\sigma$ that commutes with $\rho$ but is not a multiple of the identity occur as both quotients and subbundles of $E$, and are $\rho$-invariant. One has for any $\rho$-invariant subbundle $F$ and quotient $Q = E/F$, that $c_1(E) = c_1(F) + c_1(Q)$. On the other hand, if $\rho_F, \rho_Q$ denote the endomorphisms on $F, Q$ induced by $\rho$, the fact that $\det(\rho) = \det(\rho_F) \det(\rho_Q)$ tells us that $\text{tr}(k_j)(E) = \text{tr}(k_j)(F) + \text{tr}(k_j)(Q)$. One than has $\delta_j(E, \rho) = \delta_j(F, \rho_F) + \delta_j(Q, \rho_Q)$, so that the existence of a non-trivial eigenbundle or generalised eigenbundle indeed contradicts stability.

Remark that for a stable pair with non-zero $\rho$ to exist, the divisor $D = \sum_{j=1}^N \text{tr}(k_j)z_j$, of degree zero by Equation (23), must be a principal divisor since it is the divisor of the determinant of $\rho$.

The paper [15] describes the moduli of simple pairs $(E, \rho)$ for arbitrary complex reductive groups. We adapt the result here for $\text{Gl}(n, \mathbb{C})$. We begin by noting that there is a spectral curve $S^0 = S^0_{(E, \rho)}$ in $\Sigma \times \mathbb{C}$ defined by

$$\det(\rho(z) - \lambda I) = 0. \quad (35)$$

The curve $S^0$ extends to a closed curve $\Sigma$ in $\Sigma \times \mathbb{P}^1$; its intersection with $\Sigma \times \{0, \infty\}$ occurs over the divisor $\sum_j z_j$. Let $\text{Char}(K)$ denote the family of curves obtained from our simple pairs (recall the definition of $K$ from page 7); we do not describe it here with any thoroughness, referring instead to [15], except to note the fact that the orders $k_{jj'}$ of $\rho$ at the points $z_j$ constrain the intersection of the spectral curve at $z_j$ with $\Sigma \times \{0, \infty\}$. Let $D_+ = \sum_{k_{kk'} \geq 0} k_{kk'}z_j$. The family $\text{Char}(K)$ is the family of curves in the linear system $[\pi_1^*(\mathcal{O}(D_+)) \otimes \pi_2^*(\mathcal{O}(n))]$ satisfying these constraints.

For $k = (k_1, \ldots, k_n)$ with $k_1 \geq \cdots \geq k_n$, set

$$[k] = \sum_{a < b} k_a - k_b = \sum_{a=1}^{n-1} (k_a - k_{a+1})a(n-a). \quad (36)$$

Theorem 5.3 (Main result of [15], adapted for $\text{Gl}(n, \mathbb{C})$) Let $D$ be a principal divisor, and suppose that $K \neq 0$. The moduli space $\mathcal{M}_s(\Sigma, K, k_0)$ of simple pairs $(E, \rho)$ of type $K$, with $E$ of degree $k_0$ is smooth, of complex dimension $2 + \sum_i [k_i]$. It has a holomorphic symplectic structure, and the map

$$\mathcal{M}_s(\Sigma, K, k_0) \to \text{Char}(K) \quad (37)$$

is Lagrangian, with generic fibre a smooth compact Abelian variety.
5.2 Gauge theoretic dimensions

Thus, in the case of $M = S^1 \times T^2$, Theorem 1.1 relates the moduli space of HE-monopoles to a complex moduli space described by Theorem 5.3. In particular, we see that the moduli space for $K \neq 0$ has real dimension

$$\dim_{\mathbb{R}} \mathcal{M}_{k_0}(T^3, p_1, \ldots, p_N, k_1, \ldots, k_N) = 4 + 2 \sum_{j=1}^{N} [k_j].$$ (38)

The tangent space to the moduli space can also be understood from a gauge theoretic point of view. In this context first order deformations of our HE-monopoles (modulo gauge) correspond to the kernel of a complex

$$D^* + dB : \Omega^1(\text{ad}(E)) \oplus (\text{ad}(E)) \to \text{ad}(E) \oplus \Omega^2(\text{ad}(E))$$ (39)

where $D^*$ fixes the gauge infinitesimally, and $dB$ is the derivative of the Bogomolny equation. In our case of simple bundles, the cokernel of this complex is of constant real dimension 4, while the index is given by a result of Pauly, for a quite general three-manifold.

**Theorem 5.4 (Extension of Pauly’s result in [30] to higher order groups)**  Let $M$ be a compact, oriented, and connected Riemannian 3-manifold. Fix $N$ points $p_1, \ldots, p_N \in M$, and $N$ sequences $k_1, \ldots, k_N$ of $n$ integers. Then the real index of the deformation complex (39) of a HE-monopole $(E, \nabla, \phi)$ is $2 \sum_{j=1}^{N} [k_j]$.

The proof of Pauly, originally written for $SU(2)$, extends with little work to this more general case. By exhibiting parametrices, Pauly shows that despite the presence of singularities, the complex is Fredholm. This result relies on the fact that near the singularities, the asymptotic behaviour guarantees that a local lift from the three-ball to the four-ball using the Hopf fibration is non-singular. The index can then be obtained by an excision argument from the case with no singularities. In that (compact) case, since the dimension is odd, Atiyah–Singer’s index theorem tells us the index is 0. The contribution to the index given by the singularities translates the problem into one over the three-sphere, and Pauly then uses the lift to the four-sphere to transform the calculation into that of an $S^4$ equivariant index.

We note also that, in the case of the three-torus, when there are no singularities $(K = 0)$, we are in essence reduced to the flat (Abelian) case: the index is zero, and both kernel and cokernel are of constant real rank $4n$. This result is confirmed by a parameter count: $3n$ parameters for a flat connection, which is a representation of $\mathbb{Z}^3 = \pi_1(T^3)$ into (the maximal torus of) $U(n)$, and $n$ parameters for a constant Higgs field.

5.3 Higher genus

Returning to our complex descriptions of the moduli, many of the techniques used for studying the moduli space in [15] also apply to the case of HE-monopoles over $S^1 \times \Sigma$ with $\Sigma$ of higher genus, apart from the derivation of the Poisson structure and the existence of an integrable system.

We first note that our moduli space of stable pairs can be examined as a subspace of a space that has already been constructed. Consider the divisor $D_{\text{max}} = -\sum j \min_i (k_{ji}z_j)$. Then $\rho$ is a section of $\text{End}(E)(D_{\text{max}})$, and one can realise the space of our $(E, \rho)$ (for $t = (t_i)$ small) as a subvariety of the space of stable pairs consisting of a bundle and an endomorphism with poles at $D_{\text{max}}$. This moduli space $\mathcal{N}(k_0, D_{\text{max}})$ has been constructed in [27; 33]. We can thus study $\mathcal{M}(\Sigma, k_0, K, \ell)$ as a subvariety of $\mathcal{N}(k_0, D_{\text{max}})$.

A first step, however, is to check that at least some of our spaces $\mathcal{M}(\Sigma, k_0, K, \ell)$ are non-empty, provided again that the divisor $D$ is principal, which it must be as it represents $\det(\rho)$. Suppose that $D$ is non-zero. For each point $z_i$, choose a permutation $\sigma_i$ of $\{1, \ldots, n\}$ and set $D_j = \sum_i k_{ji} \sigma_i(z_i)$. Let $L$ be a line bundle on the curve, and set $L_j = L(D_1 + \cdots + D_j)$, $j = 1, \ldots, n$. There is a natural meromorphic map $\rho_{j+1} : L_j \to L_{j+1}$ with divisor $D_{j+1}$. In the same vein, the natural map $L \to L_1$ yields a map $\rho_1 : L_n \to L_1$ when premultiplied by $\det(\rho)^{-1}$. This map has divisor $D_1$. Let $E$ be the bundle $\oplus L_j$; it has degree $n \deg(L) + \sum_i (n - i + 1) \deg(D_i)$. Let $s$ be the residue modulo $n$ of this degree; as usual, it is just the value modulo $n$ that is of any importance.

**Proposition 5.5** Suppose that the divisor $D$ is principal and that the permutations are such that not all the degrees of $L_j$ are the same. The moduli space $\mathcal{M}(\Sigma, mn + s, K, \ell)$ is non-empty. Suppose in addition that one of the $L_j$ has degree $n$ greater than any of the others. Then one can also produce an element of $\mathcal{M}(\Sigma, mn + r, K, \ell)$ for any $r$. 
Proof: Consider the bundle \( E \) constructed above of degree \( mn + s \). Define the map \( \rho \) by

\[
\rho = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & \rho_1 \\
\rho_2 & 0 & 0 & \cdots & 0 & 0 \\
0 & \rho_3 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \rho_{n-1}
\end{pmatrix}
\]

(40)

It has the right polar structure. We can ask when the pair \((E, \rho)\) is stable. The determinant of \( \rho \) has divisor \( D \); away from the singular set of \( \rho \), the eigenvalues of \( \rho \), which are the \( n \)-th roots of \( \det(\rho) \), are distinct. Now consider a subbundle \( V \) of rank \( k \) that is \( \rho \)-invariant; locally, it is a sum of eigenbundles. The projection of \( V \) onto any sum \( L_{i_1} \oplus L_{i_2} \oplus \cdots \oplus L_{i_k} \) has generically non-zero determinant since it is a Vandermonde-type determinant of the eigenvalues involved. Globally, \( V \) is then a subsheaf of \( L_{i_1} \oplus L_{i_2} \oplus \cdots \oplus L_{i_k} \); this inclusion bounds the degree of \( V \). If we have chosen the \( L_{i_j} \) of smallest degree, \( V \) cannot destabilise.

To consider the case of general degree, one can take Hecke transforms \( E' \) of \( E \) along the eigenspaces of \( \rho \), obtaining subsheaves of \( E \) and reducing the degree by \( 1, 2, \ldots, n - 1 \) in turn. The result is still stable, as any \( \rho \)-invariant subbundle \( V' \) of \( E' \), thought of as a subsheaf of \( E \), then satisfies \( \deg(V')/\text{rank}(V') < (-n + \deg(E))/n \leq \deg(E')/n \).

One would like to get an idea of the Zariski tangent space of \( \mathcal{M}(\Sigma, j, K, \vec{\ell}) \), at the points of the moduli space we have just constructed. As in [15, Sec. 4], let us consider the subbundle

\[
\Delta_\rho := \{(a, b) \mid a + \text{Ad}_\rho(b) = 0\}
\]

(41)

of \( \text{End}(E) \oplus \text{End}(E) \). Denote the quotient by

\[
\text{ad}(E, \rho) := [\text{End}(E) \oplus \text{End}(E)]/\Delta_\rho.
\]

(42)

We get the short exact sequence

\[
0 \to \Delta_\rho \to [\text{End}(E) \oplus \text{End}(E)] \to \text{ad}(E, \rho) \to 0.
\]

(43)

The degree of \( \text{ad}(E, \rho) \) is computed by [15, Lemma 4.9] to be \( \sum_{i=1}^N [\vec{k}_i] \). As in [15, Cor. 4.3], we have the following proposition; see also [3; 4; 24].

**Proposition 5.6** The infinitesimal deformations of the pair \((E, \rho)\) in \( \mathcal{M}(\Sigma, j, K, \vec{\ell}) \) are naturally identified by the first hyper-cohomology of the complex (in degrees 0 and 1)

\[
\text{End}(E) \xrightarrow{\text{ad}} \text{ad}(E, \rho).
\]

(44)

Roughly, the first hypercohomology combines both \( H^1(\Sigma, \text{End}(E)) \), the deformations of the bundle, and \( H^0(\Sigma, \text{ad}(E, \rho)) \), the deformations of \( \rho \).

The dimension \( H^1 \) of the first hypercohomology is given by

\[
H^1 = H^0 + H^2 - \chi(\text{End}(E)) + \chi(\text{ad}(E, \rho)) = H^0 + H^2 + \sum_{i=1}^N [\vec{k}_i]
\]

If \((E, \rho)\) is simple, the dimension \( H^0 \) is 1, as in [15, Sec. 4]. The dual space to the second cohomology is given, again as in [15, Sec. 4], by the kernel of \( \text{ad}_\rho \); \( \Delta_\rho \oplus K \to \text{End}(E) \otimes K \), that is the sections of \( \text{End}(E) \otimes K \) that commute with \( \rho \). If \( \rho \) is generically regular, this kernel is generated away from the poles by the powers \( 1, \rho, \ldots, \rho^{n-1} \) of \( \rho \), tensored with \( K \). Globally, the sections of the kernel are given as expressions \( \sum_{i=0}^{n-1} a_i \rho^i \), where \( a_i \) is a form such that the product \( a_i \rho^i \) is holomorphic. In our case, the coefficient \( a_i \) must have divisor greater or equal to \( \max(-D_1 - D_2 - \cdots - D_i, -D_2 - D_3 - \cdots - D_{i+1}, \ldots, -D_n - D_1 - \cdots - D_{i-1}) \). For fairly general choices of divisors, if one has more than \( g \) points, this condition forces \( a_i = 0 \) for \( i > 0 \) and \( a_0 \) to be a holomorphic one-form; one then has \( \dim(H^2) = g \). This computation yields

\[
\dim(H^1) = (g + 1) + \sum_{i=1}^N [\vec{k}_i].
\]

When the elements of \( H^2 \) live in the trace component \( a_0 \), as it is the case here under our genericity assumptions on the divisors, it is shown in [15, Thm 4.13] that the deformations are unobstructed and the space is smooth.
Remark 5.7 If $E$ has trivial determinant and $\det(\rho) = 1$, one can consider the space of deformations that preserve this property; this situation corresponds to $SU(n)$ monopoles. (We note that this constrains the location of the singularities in the circle direction.) As the elements of $H^0, H^2$ lie in the trace component, the deformations keeping one in $SL(n)$ lie in a $\sum_{i=1}^{N}|k_i|$-dimensional space; the extra $g + 1$ parameters for $GL(n)$ correspond to rescaling $\rho$ (one parameter) and tensoring $E$ by a line bundle (g parameters).

5.4 Higher genus: SU(2), U(2)

The preceding results are only rather partial: as one can see, the combinatorics of the degrees is fairly complicated. We now consider the case of $SL(2)$-bundles, with endomorphisms of determinant one; these correspond to $SU(2)$-monopoles. Let us suppose given a pair $(E, \rho)$, with singularities of type $(k_i, -k_i), k_i > 0$ at points $z_i$. Let $D_+$ be the divisor $\sum_i k_i z_i$. By what is now a fairly standard construction, one has a spectral curve $\{(z, \lambda) \in \Sigma \times \mathbb{P}^1 \mid \det(z - \lambda I) = 0\}$ in $\Sigma \times \mathbb{P}^1$, giving a two sheeted cover of $\Sigma$. In addition, one can define what is generically a line bundle $L$ over the spectral curve; away from the poles of $\rho$, it is the quotient sheaf, defined over the total space of the bundle $O(D_+)$ over $\Sigma$ as

$$0 \to E \to L.$$

We note that as the spectrum is invariant under $\lambda \mapsto \lambda^{-1}$, the spectral curve is the pullback of the graph of the function $tr(\rho)$ under the map $f = \lambda + \lambda^{-1}$. One sees that the singular points of the endomorphism $\rho$ are located at the poles of $f$, and that the branch locus $B$ for projection of the spectral curve to $\Sigma$ is given by the inverse image of $\{-2, 2\}$. The curve is smooth if all the points of $B$ occur with multiplicity one. The curve has an involution $i$, and the determinant form on the bundle $E$ identifies $L^*$ with $i^*(L)(-B)$ (as in [14]), so that $L$ lies in a suitable Prym variety.

Conversely, given the spectral curve $S$ and the line bundle $L$ lying in the Prym variety, one can reconstruct $E$ as a push-down $\pi_*(E)$, and $\rho$ as the endomorphism on $E$ induced by multiplication by $\lambda$ on $L$. If $S$ is smooth, stability is automatic: any invariant subbundle corresponds to a subset of the eigenvalues away from the branch points, and going around these branch points permutes them, so that an invariant subbundle is necessarily the whole bundle. Thus

Proposition 5.8 An $SU(2)$-monopole on $S^1 \times \Sigma$, with singularities $K$, yields a pair $(f, L)$ where $f$ is a meromorphic function with polar divisor $D_+$, and $L$ is a sheaf over the double cover $S$ of $\Sigma$ branched over $B$, the locus defined by $f^{-1}(\{2, -2\})$. If all the points in the support of $B$ occur with multiplicity one in $B$, the spectral curve $S$ is smooth, and $L$ is a line bundle on $S$ belonging to the Prym variety of sheaves on $S$ satisfying $L^* \simeq i^*(L)(-B)$, where $i$ is the natural involution.

Conversely, given $(f, L)$, where $f$ has polar divisor $D$, if all the points in the support of $B$ occur with multiplicity one in $B$, then the double cover $S$ is smooth. If $L$ belongs to the Prym variety, then the pair $(f, L)$ corresponds to a singular $SU(2)$-monopole.

One can then count parameters. For $d_+ > 2g - 2$, the space of functions $f$ with divisor exactly $D_+$ has dimension $d_+ = 1 - g$, for $d_+ = \deg(D_+)$. If the locus $f^{-1}(\{2, -2\})$ consists of distinct points, the genus of $S$, from the Riemann–Hurwitz formula, is then $2g - 1 + d_+$, and so the Prym variety has dimension $g - 1 + d_+$; the $SU(2)$ moduli space thus has dimension $2d_+$ in all.

We now consider the more general case of a $GL(2, \mathbb{C})$ bundle, and a map $\rho$ with singularities $K = ((k_{1, +}, k_{1, -}), z_1), \ldots, ((k_N, z_N)$ with $k_{i, +} \geq k_{i, -}$; this pair correspond to a $U(2)$ (HE)-monopole. There is again the constraint imposes by the requirement that the divisor $\sum_i (k_{i, +} + k_{i, -})z_i$ be principal, as it must be the divisor of $\det(\rho)$. The same constructions give one a double cover $S$ of $\Sigma$, defined by the equation

$$\lambda^2 = tr(\rho)\lambda + \det(\rho) = 0.$$

As above, one has a sheaf $L$ over $S$, which is a line bundle if $S$ is smooth. $L$ no longer necessarily satisfies the Prym condition.

Again, one can count parameters. The function $tr(\rho)$ must have a divisor which is greater than $\sum_i k_{i, -}z_i$; if $d = -\sum_i k_{i, -} = \sum_i k_{i, +}$, then for $d > 2g - 2$ this gives $d + 1 - g$ parameters. The determinant is fixed, up to scale; this gives one extra parameter. On the line bundles, one has $2g - 1 + d$ parameters, giving $2d + g + 1$ parameters in all: roughly, the parameters for the $SL(2, \mathbb{C})$ case, plus a line bundle on $\Sigma$, plus a scale parameter for $\rho$. 
6 Monopoles on the product of a Riemann surface and an interval

One can use the result to consider the case of a monopole on the product \( I \times \Sigma \) of a Riemann surface and an interval \( I = [0, c] \). In this case, the relevant holomorphic data is a pair of holomorphic bundles \( E = E(0), E' = E(c) \) and a meromorphic automorphism \( \rho: E \rightarrow E' \). If \( z_i, t_i \) are the locations of our eventual monopole’s singularities, then \( \rho \) descends to \( \Sigma \); one uses \( z,t \) identifying \((\rho,E)\) with \((E',\rho)\) on \( \tilde{\Sigma} \), the transport from \( \tilde{\Sigma} \) itself, and so \( \rho \) descends to \( \tilde{\Sigma} \). One simply embeds the interval \( I \) in a circle \( C \), and places the extra poles (those of \( \sigma \)) in the complement of \( I \). One then applies the theorem for the circle, obtains a monopole on \( \Sigma \), and restricts to \( I \times \Sigma \).

The proof is fairly simple. If one allows sufficiently many poles, one can find a large number of \( \sigma \); the trick is to ensure that the result is \( \tilde{t} \)-stable. The simplest way to ensure stability is to arrange for there to be no \( \sigma \circ \rho \)-invariant subbundles at all, a more restrictive condition. As pointed out above, invariant subbundles are sums of generalised eigenspaces; if the spectral curve has a branch point that permutes all of these eigenspaces with no invariant subset, then we are done. Now, one can prescribe the behaviour of \( \sigma \circ \rho \) on any formal neighbourhood of a given point, providing one allows sufficiently many poles elsewhere; one then asks that this point be a branch point that permutes all the eigenspaces, which is ensured by prescribing the behaviour on a formal neighbourhood of sufficiently high order by letting \( \sigma \) be the composition of a well chosen Jordan form by \( \rho^{-1} \).

Corollary 6.2 There is a monopole with singularities at \( p_i, i = 1, \ldots, N \) corresponding to the holomorphic data \( E = E(0), E' = E(c), \rho: E \rightarrow E' \).

One simply embeds the interval \( I \) in a circle \( C \), and places the extra poles (those of \( \sigma \)) in the complement of \( I \). One then applies the theorem for the circle, obtains a monopole on \( C \times \Sigma \), and restricts to \( I \times \Sigma \).

The monopoles associated to a given set of holomorphic data are not unique. Indeed, one can complete by different maps \( \sigma, \sigma' \), and there is no reason why the monopoles one obtains by restricting to \( I \times \Sigma \) should be the same.

7 HE-Monopoles on a flat circle bundle

If one has a flat principal circle bundle \( Y \) over a Riemann surface \( \Sigma \), then any metric on \( \Sigma \) and any choice of length \( T \) of the circle yield a canonical metric on the circle bundle: at a point \( p \) on the circle bundle, one takes a local flat lift of \( \Sigma \) given by \( x(D) \) of the circle such that the original flat circle bundle over \( \Sigma \) is obtained by identifying \( (z, t) \) and \( (D(z), x(D) \cdot t) \).

Now pass to the universal covering \( \tilde{\Sigma} \rightarrow \Sigma \), and lift our circle bundle \( Y \) with flat connection to \( \tilde{Y} \rightarrow \tilde{\Sigma} \). One can choose a global flat section \( S \) on this universal cover, avoiding the lifts of the singularities. This section trivialises the circle bundle; let \( S \) correspond to \( t = 0 \). Let \( \mathcal{D} \) be the group of deck transformation of this covering. For each \( D \in \mathcal{D} \), one has an element \( x(D) \) of the circle such that the original flat circle bundle over \( \Sigma \) is obtained by identifying \( (z, t) \) and \( (D(z), x(D) \cdot t) \).

The restriction of the lift of an HE-monopole to \( S \) yields a holomorphic vector bundle \( \tilde{E} \) over \( \tilde{\Sigma} \). This bundle descends to \( \Sigma \); one uses \( \nabla_i - i\phi \) to define a parallel transport \( T(x(D)) \) from the fiber at \( (D(z), x(D) \cdot t) \) to \( (D(z), t) \). The holomorphic bundle \( E \) on \( \Sigma \) is obtained by composing the natural identification of \( \tilde{E}(z,t) \) and \( \tilde{E}(D(z),x(D)\cdot t) \) with the transport from \( \tilde{E}(D(z),x(D)\cdot t) \) to \( \tilde{E}(D(z), t) \). These identifications intertwine the parallel transport \( \tilde{\rho} \) mapping \( \tilde{E} \) to itself, and so \( \tilde{\rho} \) descends to \( \rho: E \rightarrow E \).

Let \( \tilde{z}_i \) be a lift of the point \( z_i \) to \( S \) and let \( t(\tilde{z}_i) \) denote the time one must flow from the section \( S \) to the lift of the point \( p_i \). If \( U \) is the union of \( \ell \) disjoint fundamental domains in \( S \) for the covering map to \( \Sigma \), we define a degree by:

\[
\delta_{\tilde{z},U}(E, \rho) = \frac{\sum_{\tilde{z}_j \in U} \text{tr}(\tilde{k}_j) t(\tilde{z}_j)}{T\ell}.
\] (45)
Let $\tilde{M}_t: \tilde{Y} \to \tilde{Y}$ denote the action of $\exp(2\pi it/T)$ on $\tilde{Y}$. Suppose now that our circle bundle is such that its lift to a finite cover $\Sigma'$ of $\Sigma$ is trivial. Choosing $U$ to be the union of fundamental domains for $\Sigma$ corresponding to one fundamental domain of $\Sigma'$, one has an equivalent to Remark 3.8:

**Lemma 7.2** If one translates $S$ to $\tilde{M}_t(S)$, the degree does not change.

**Theorem 7.3** Let $Y$ be a circle bundle that become trivial when lifted to a finite cover $\Sigma'$ of $\Sigma$. The moduli space $M_{k_0}(Y, p_1, \ldots, p_N, k_1, \ldots, k_N)$ of irreducible U(n) HE-monopoles on $Y$ with $E$ of degree $k_0$ and singularities at $p_j$ of type $\tilde{k}_j$ maps bijectively to the space $M(\Sigma, k_0, K, \tilde{t})$ of $(\tilde{t}, U)$-stable holomorphic pairs $(E, \rho)$ with

- $E$ a holomorphic rank $n$ bundle of degree $k_0$ on $\Sigma$,
- $\rho$ a meromorphic section of $\text{Aut}(E)$ of the form $F_j(z)\text{diag}(z^{k_j})G_j(z)$ near $z_j$, with $F_j, G_j$ holomorphic and invertible, and with $\det(\rho)$ having divisor $\sum_j \text{tr}(\tilde{k}_j)z_j$.

**Proof:** Let $\mathcal{D}$ be the group of deck transformations of the covering $\Sigma' \to \Sigma$. The proofs (and therefore statements) of Lemma 4.3 and Theorem 4.4 work even if we replace $S^1$ invariance by $S^1 \times \mathcal{D}$ invariance. In turn, Theorem 4.1 and its constituents Propositions 4.7 and 4.2 are also true even if we consider objects that are invariant under the action of $\mathcal{D}$. The proof is thus complete. □

When the circle bundle is not one whose lift to a finite cover is trivial, one expects that the appropriate definition of degree and stability is obtained by taking a limit of averages over larger and larger $U$.

**References**

[1] M. F. Atiyah, *Magnetic monopoles in hyperbolic spaces*, Vector bundles on algebraic varieties (Bombay, 1984), Tata Inst. Fund. Res. Stud. Math., vol. 11, Tata Inst. Fund. Res., Bombay, 1987, pp. 1–33.

[2] Michael Atiyah and Nigel Hitchin, *The geometry and dynamics of magnetic monopoles*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 1988.

[3] I. Biswas and S. Ramanan, *An infinitesimal study of the moduli of Hitchin pairs*, J. London Math. Soc. (2) **49** (1994), no. 2, 219–231.

[4] Francesco Bottacin, *Symplectic geometry on moduli spaces of stable pairs*, Ann. Sci. École Norm. Sup. (4) **28** (1995), no. 4, 391–433.

[5] Benoit Charbonneau, *From spatially periodic instantons to singular monopoles*, Comm. Anal. Geom. **14** (2006), no. 1, 183–214, arXiv:math.DG/0410561.

[6] Sergey Cherkis and Anton Kapustin, *Nahm transform for periodic monopoles and $N = 2$ super Yang–Mills theory*, Comm. Math. Phys. **218** (2001), no. 2, 333–371, arXiv:hep-th/0006050v2.

[7] Sergey A. Cherkis and Nigel J. Hitchin, *Gravitational instantons of type $D_k$*, Comm. Math. Phys. **260** (2005), no. 2, 299–317, arXiv:hep-th/0310084.

[8] Sergey A. Cherkis and Anton Kapustin, *Singular monopoles and gravitational instantons*, Comm. Math. Phys. **203** (1999), 713–728, arXiv:hep-th/9803160.

[9] Sergey A. Cherkis and Anton Kapustin, *Periodic monopoles with singularities and $N = 2$ super-QCD*, Comm. Math. Phys. **234** (2003), no. 1, 1–35, arXiv:hep-th/0011081.

[10] S. K. Donaldson, *Nahm’s equations and the classification of monopoles*, Comm. Math. Phys. **96** (1984), no. 3, 387–407.

[11] , *Boundary value problems for Yang–Mills fields*, J. Geom. Phys. **8** (1992), no. 1-4, 89–122.

[12] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978.

[13] Nigel J. Hitchin, *Monopoles and geodesics*, Comm. Math. Phys. **83** (1982), no. 4, 579–602.
[14] Benoit Charbonneau and Jacques Hurtubise, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126.

[15] J. C. Hurtubise and E. Markman, *Elliptic Sklyanin integrable systems for arbitrary reductive groups*, Adv. Theor. Math. Phys. **6** (2002), no. 5, 873–978 (2003), arXiv:math.AG/0203031.

[16] , *Surfaces and the Sklyanin bracket*, Comm. Math. Phys. **230** (2002), no. 3, 485–502, arXiv:math.AG/0107010.

[17] Jacques Hurtubise, *The classification of monopoles for the classical groups*, Comm. Math. Phys. **120** (1989), no. 4, 613–641.

[18] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of $p$-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 5–48.

[19] Stuart Jarvis, *Euclidean monopoles and rational maps*, Proc. London Math. Soc. (3) **77** (1998), no. 1, 170–192.

[20] Stuart Jarvis and Paul Norbury, *Compactification of hyperbolic monopoles*, Nonlinearity **10** (1997), no. 5, 1073–1092.

[21] Anton Kapustin and Edward Witten, *Electric-magnetic duality and the geometric Langlands program*, Commun. Number Theory Phys. **1** (2007), no. 1, 1–236, arXiv:hep-th/0604151.

[22] Peter B. Kronheimer, Master’s thesis, Oxford, 1986.

[23] Martin Lübke and Andrei Teleman, *The Kobayashi–Hitchin correspondence*, World Scientific Publishing Co. Inc., River Edge, NJ, 1995.

[24] Eyal Markman, *Spectral curves and integrable systems*, Compositio Math. **93** (1994), no. 3, 255–290.

[25] Oliver Nash, *A new approach to monopole moduli spaces*, Nonlinearity **20** (2007), no. 7, 1645–1675.

[26] , *Singular hyperbolic monopoles*, Comm. Math. Phys. **277** (2008), no. 1, 161–187.

[27] Nitin Nitsure, *Moduli space of semistable pairs on a curve*, Proc. London Math. Soc. (3) **62** (1991), no. 2, 275–300.

[28] Paul Norbury, *Magnetic monopoles on manifolds with boundary*, Trans. Amer. Math. Soc. (in press) (2009), arXiv:0804.3649.

[29] Marc Pauly, *Gauge theory in 3 and 4 dimensions*, Ph.D. thesis, Oxford University, 1996.

[30] , *Monopole moduli spaces for compact 3-manifolds*, Math. Ann. **311** (1998), no. 1, 125–146.

[31] , *Spherical monopoles and holomorphic functions*, Bull. London Math. Soc. **33** (2001), no. 1, 83–88.

[32] Andrew Pressley and Graeme Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1986, Oxford Science Publications.

[33] Carlos T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I and II*, Inst. Hautes Études Sci. Publ. Math., No. 79, 47–129 and No. 80, 5–79.

[34] , *Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), no. 4, 867–918.