Quantum discord as a resource for quantum cryptography

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Quantum discord is the minimal bipartite resource which is needed for a secure quantum key distribution, being a cryptographic primitive equivalent to non-orthogonality. Its role becomes crucial in device-dependent quantum cryptography, where the presence of preparation and detection noise (inaccessible to all parties) may be so strong to prevent the distribution and distillation of entanglement. The necessity of entanglement is re-affirmed in the stronger scenario of device-independent quantum cryptography, where all sources of noise are ascribed to the eavesdropper.

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Introduction.—One of the hot topics in the quantum information theory is the quest for the most appropriate measure and quantification of quantum correlations. For pure quantum states, this quantification is provided by quantum entanglement [1] which is the physical resource at the basis of the most powerful protocols of quantum communication and computation [2][3]. However, we have recently understood that the characterization of quantum correlations is much more subtle in the general case of mixed quantum states [4][5].

There are in fact mixed states which, despite being separable, have correlations so strong to be irreproducible by any classical probability distribution. These residual quantum correlations are today quantified by quantum discord [7], a new quantity which has been studied in several contexts with various operational interpretations and applications, including work extraction [8], quantum state merging [9, 10], remote state preparation [11], quantum metrology [12], discrimination of unitaries [13] and quantum channel discrimination [14].

In this paper, we identify the basic role of quantum discord in one of the most practical tasks of quantum information, i.e., quantum key distribution (QKD) [15]. The claim that quantum discord must be non-zero to implement QKD is intuitive. In fact, quantum discord and its geometric formulation are connected with the concept of non-orthogonality, which is the essential ingredient for quantum cryptography. That said, it is still very important to characterize the general framework where discord remains the only available resource for QKD. Necessarily, this must be a scenario where key distribution is possible despite entanglement being absent.

Here we show that this general scenario corresponds to device-dependent (or trusted-device) QKD, which encompasses all realistic protocols where the noise affecting the devices and apparatus of the honest parties is assumed to be trusted, i.e., not coming from an eavesdropper but from the action of a genuine environment. This can be preparation noise (e.g., due to imperfections in the optical switches/modulators or coming from the natural thermal background at lower frequencies [16][17]) as well as measurement noise and inefficiencies affecting the detectors (which could be genuine or even added by the honest parties [21][22]). Such trusted noise may be high enough to prevent the distribution and distillation of entanglement, but still a secure key can be extracted due to the presence of non-zero discord.

By contrast, if the extra noise in the apparatus is not trusted but considered to be the effect of side-channel attacks [24], then we have to enforce device-independent QKD [2]. In this more demanding scenario, quantum discord is still necessary for security but more simply becomes an upper bound to the coherent information. This means that secure key distribution becomes just a consequence of entanglement distillation.

Quantum discord.—Discord comes from different quantum extensions of the classical mutual information. The first is quantum mutual information, measuring the total correlations between two systems, A and B, and defined as I(A,B) := S(A) − S(A|B), where S(A) is the entropy of system A, and S(A|B) := S(AB) − S(B) its conditional entropy. The second extension is C(A|B) := S(A) − S_{\text{min}}(A|B), where S_{\text{min}}(A|B) is the entropy of system A minimized over an arbitrary measurement on B. This local measurement is generally described by a positive operator valued measure (POVM) \{M_y\}, defining a random outcome variable Y = \{y, p_y\} and collapsing system A into conditional states \rho_{A|y} [25]. Thus, we have

S_{\text{min}}(A|B) := \inf_{\{M_y\}} S(A|Y), \quad S(A|Y) = \sum_y p_y S(\rho_{A|y}),

where the minimization can be restricted to rank-1 POVMs [7].

The quantity C(A|B) quantifies the classical correlations between the two systems, corresponding to the maximal common randomness achievable by local measurements and one-way classical communication (CC) [26]. Thus, quantum discord is defined as the difference between total and classical correlations [5][7]

D(A|B) := I(A,B) − C(A|B) = S_{\text{min}}(A|B) − S(A|B) \geq 0.

An equivalent formula can be written by noticing that L(A|B) := −S(A|B) is the coherent information [23][24]. Then, introducing an ancillary system E which purifies

\[ D(A|B) := I(A,B) − C(A|B) = S_{\text{min}}(A|B) − S(A|B) \geq 0. \]
$\rho_{AB}$, we can apply the Koashi-Winter relation \[29\] and write $S_{\min}(A|B) = E_I(A,E)$, where the latter is the entanglement of formation between $A$ and $E$. Therefore

$$D(A|B) = I_e(A|B) + E_I(A,E) \geq \max\{0, I_e(A|B)\}. \tag{1}$$

It is important to note that $D(A|B)$ is different from $D(B|A)$, where system $A$ is measured. For instance, in classical-quantum states $\rho_{AB} = \sum_x p_x |x\rangle_A \otimes \rho_B(x)$, where $A$ embeds a classical variable via the orthonormal set $\{|x\rangle\}$ and $B$ is prepared in non-orthogonal states $\{|\rho_B(x)\rangle\}$, we have $D(B|A) = 0$ while $D(A|B) > 0$. By contrast, for quantum-classical states ($B$ embedding a classical variable), we have the opposite situation, i.e., $D(A|B) = 0$ and $D(B|A) > 0$.

**Device-dependent QKD protocols.** Any QKD protocol can be recast into a measurement-based scheme, where Alice sends Bob part of a bipartite state, then subject to classical-quantum states $\rho_{AB}$, where system $A$ is detected by another rank-1 POVM

$${}\langle x | \rho_{AB} | y \rangle = \sum_k p_{x|y} \rho_{A}(k) \otimes |k\rangle_B \langle k| \otimes |y\rangle_E \langle y|,$$

with Eve fully invisible, since her action is equivalent to an identity channel for Alice and Bob ($\rho_{ABE} = \rho_{AB}$).

Direct reconciliation fails since $\rho_{ABE}$ is symmetric under $B-E$ permutation, which means that Eve decodes Alice’s variable with the same accuracy of Bob. Reverse reconciliation also fails. Bob encodes $Y$ in the joint state $\rho_{ABE|y} = \sum_k p_{k|y} \rho_{A}(k) \otimes |k\rangle_B \langle k| \otimes |y\rangle_E \langle y|$, where $p_{k|y} := \langle k|M_y|k\rangle$. Then, Eve retrieves $K = \{k, p_{k|y}\}$ by a projective POVM, while Alice decodes a variable $X$ with distribution

$$p_{x|y} = \text{Tr}[M_x \rho_{A|y}] = \sum_k p_{x|k} p_{k|y}, \quad p_{x|k} := \text{Tr}[M_x \rho_{A}(k)].$$

This equation defines a Markov chain $Y \rightarrow K \rightarrow X$, so that $I(Y,K) \geq I(Y,X)$ by data processing inequality, i.e., Eve gets more information than Alice \[33\].

As expected, system $A$ sent through the channel must be quantum $D(A|a) > 0$ in order to have a secure QKD. Indeed, this is equivalent to sending an ensemble of non-orthogonal states. By contrast, the classicality of the private system $A$ is still acceptable, i.e., we can have $D(a|A) = 0$. In fact, we may build QKD protocols with preparation noise using classical-quantum states

$$\rho_{AB} = \sum_x p_x |x\rangle_A \otimes \rho_x(x), \tag{2}$$

after the previous process has been repeated many times, Alice and Bob publicly compare a subset of their data. If the error rate is below a certain threshold, they apply classical procedures of error correction and privacy amplification with the help of one-way CC, which can be either forward from Alice to Bob (direct reconciliation), or backward from Bob to Alice (reverse reconciliation). Thus, they finally extract a secret key at a rate $K \leq I(X,Y)$, which is denoted by $K(Y|X)$ in direct reconciliation and $K(X|Y)$ in reverse reconciliation.

To quantify these rates, we need to model Eve’s attack. The most general attack is greatly reduced if Alice and Bob perform random permutations on their classical data \[30\] \[31\]. As a result, Eve’s attack collapses into a collective attack, where each travelling system is probed by an independent ancilla. This means that Eve’s interaction can be represented by a two-system unitary $U_{ae}$ coupling system $a$ with an ancillary system $e$ prepared in a pure state \[32\]. The output ancilla $E$ is then stored in a quantum memory which is coherently measured at the end of the protocol (see Fig. 1). In this attack, the maximum information which is stolen on $X$ or $Y$ cannot exceed the Holevo bound.

**Non-zero discord is necessary.** Before analyzing the secret-key rates, we briefly clarify why discord is a necessary resource for QKD. Suppose that Alice prepares a quantum-classical state $\rho_{Aa} = \sum_k p_k \rho_{A}(k) \otimes |k\rangle_a \langle k|$ with $\{|k\rangle\}$ orthogonal, so that $D(A|a) = 0$. Classical system $a$ is perfectly clonable by Eve. This implies that the three parties will share the state \[33\]
whose local detection (on system $A$) prepares any desired ensemble of non-orthogonal signal states $\{\rho_a(x), p_x\}$. For instance, the classical-quantum signal state of two qubits $\rho_{AB} = (|0, 0\rangle A_0 |0, 0\rangle_2 + |1, \varphi\rangle_{A2} |1, \varphi\rangle_2)/2$, with $\{(0, 0), (1, \varphi)\}$ orthonormal and $\langle 0|\varphi\rangle \neq 0$, realizes the B92 protocol [35].

Secret-key rates.- Once we have clarified that non-zero input discord $D(A|a) > 0$ is a necessary condition for QKD, we now study the secret-key rates which can be achieved by device-dependent protocols. Our next derivation refers to the protocol of Fig. 1 and, more generally, to the scheme of Fig. 2 where Alice and Bob share an output state $\rho_{AB}$, where only part of the purification is accessible to Eve (system $E$), while the inaccessible part $P$ accounts for all possible forms of extra noise in Alice’s and Bob’s apparatus, including preparation noise and detection noise (quantum inefficiencies, etc..)

\[ K(A|a) \leq K(B|X) := I(B, X) - I(E, X) \tag{3} \]

The optimal forward-rate is defined by optimizing on Alice’s individual detections $K(A) := \sup_{M} I(A|a)$. We can write similar quantities in reverse reconciliation, where Bob’s variable $Y$ is the encoding to infer. The secret key rate is given by $K(X|Y) = I(X, Y) - I(E, Y)$, where $I(E, Y) = S(E) - S(E|Y)$ is Eve’s Holevo information on $Y$. Assuming a coherent detector for Alice, this rate is bounded by the backward DW rate

\[ K(X|Y) \leq K(A|Y) := I(A, Y) - I(E, Y) \tag{4} \]

which gives the optimal backward-rate $K(B|A)$ by maximizing on Bob’s individual detections $\{M\}$. Playing with system $P$, we can easily derive upper and lower bounds for the two optimal rates. Clearly, we get lower bounds $K_s \leq K$ if we assume $P$ to be accessible to Eve, which means to extend $E$ to the joint system $E = EP$ in previous Eqs. (3) and (4). By exploiting the purity of the global state $\Psi_{ABEP}$ and the fact that the encoding detections are rank-1 POVMs (therefore collapsing pure states into pure states), we can write the entropic equalities $S(AB) = S(E)$, $S(B|X) = S(E|X)$ and $S(A|Y) = S(E|Y)$. Then we easily derive

\[ K_s(B|A) = I_c(B|A), \quad K_s(A|B) = I_c(A|B) \]

where the right hand sides can be bounded using

\[ I(A, P|B) \leq I(A, P|B) = I(A|B) + I(B, P|A) \]

Thus, the optimal key rates satisfy the inequalities

\[ I(A|B) \leq K(\bullet) \leq I_c(A|B) + I(A, P|B), \quad (5) \]
\[ I_c(B|A) \leq K(\bullet) \leq I_c(B|A) + I(B, P|A) \quad (6) \]

where the right hand sides can be bounded using

\[ I(A, P|B), I(B, P|A) \leq I(AB, P) \leq 2\min\{S(P), S(AB)\} \]

According to Eqs. (5) and (6), key distribution can occur ($K \geq 0$) even in the absence of distillable entanglement ($I_s = 0$). It is now important to note the following facts:

(i) Device-dependent QKD is the only scenario where this is possible. In fact, only in the presence of trusted noise, i.e., $S(P) > 0$, we can have $K \geq I_s$ in the previous equations. Therefore device-dependent QKD is the only scenario where security may be achieved in the absence of distillable entanglement and, more strongly, in the complete absence of entanglement.

(ii) In device-dependent QKD, we can indeed build protocols which are secure ($K > 0$) despite entanglement being completely absent (in any form, distillable or bound). Clearly, this is possible as long as the minimal condition $D(A|a) > 0$ is satisfied. A secure protocol based on separable Gaussian states is explicitly shown in the supplementary material.

In general, there is an easy way to design device-dependent protocols which are secure and free of entanglement. Any prepare and measure protocol whose security is based on the transmission of non-orthogonal states

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FIG. 2: Output state from a device-dependent QKD protocol. Alice and Bob extract a secret-key by applying rank-1 POVMs on their local systems $A$ and $B$. Eve steals information from system $E$, while the extra system $P$ is inaccessible and completes the purification of the global state $\Psi_{ABEP}$. In direct reconciliation, Alice’s variable $X$ is the encoding to guess. The key rate is then given by $K(X|Y) = I(X, Y) - I(E, X)$, where $I(E, X) = S(E) - S(E|X)$ is the Holevo bound quantifying the maximal information that Eve can steal on Alice’s variable $X$. We can write an achievable upper bound if we allow Bob to use a quantum memory and a coherent detector. In this case, $I(X, Y)$ must be replaced by the Holevo quantity $I(B, X) = S(B) - S(B|X)$ and we get the forward Devetak-Winter (DW) rate. The optimal forward-rate is defined by optimizing on Alice’s individual detections $K(\bullet) := \sup_{M} I(A|a) K(B|X)$.

\[ K(Y|X) \leq K(B|X) := I(B, X) - I(E, X) \tag{3} \]

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\[ K_s(B|A) = I_c(B|A), \quad K_s(A|B) = I_c(A|B) \]

where the coherent information $I_c(A|B)$ and its reverse counterpart $I_c(B|A)$ quantify the maximal entanglement which is distillable by local operations and one-way CC, forward and backward, respectively.

It is also clear that we get upperbounds $K^* \geq K$ by assuming $P$ to be accessible to the decoding party, Alice or Bob, depending on the reconciliation. In direct reconciliation, we assume $P$ to be accessible to Bob, which means extending his system $B$ to $B = BP$ in Eq. (3). Using the equalities $S(AB) = S(E)$ and $S(B|X) = S(E|X)$, we get

\[ K^*(\bullet) = I_c(A|B) = I_c(B|A) + I(B, P|A) \]

Thus, the optimal key rates satisfy the inequalities

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In general, there is an easy way to design device-dependent protocols which are secure and free of entanglement. Any prepare and measure protocol whose security is based on the transmission of non-orthogonal states
\( \{ \rho_a(x), \rho_x \} \) can be recast into a device-dependent protocol, which is based on a classical-quantum state \( \rho_{AA} \) as in Eq. (2), whose classical part \( A \) is detected while the quantum part \( a \) is sent through the channel. This is as secure as the original one as long as the purification of the classical-quantum state is inaccessible to Eve. Thus in such assumption of trusted noise, any prepare and measure protocol has an equivalent discord-based representation, where non-zero discord guarantees security in the place of non-orthogonality.

Side-channels and device-independent QKD. – Let us consider the more demanding scenario where all sources of noise are untrusted. This means that the extra noise in Alice’s and Bob’s apparatus comes from side-channel attacks, i.e., system \( P \) in Fig. 2 is controlled by Eve. In this case, the secret-key rates are given by

\[
K(\blacklozenge) = I_c(A)I(B) - I_f(A, E) \leq D(A|B),
\]

so that QKD is equivalent to entanglement distillation. It is easy to check that quantum discord upperbounds these key rates. Applying Eq. (1) to Eq. (7), we obtain the cryptographic relations

\[
\begin{align*}
K(\blacklozenge) & = D(A|B) - E_f(A, E) \leq D(A|B), \\
K(\blacktriangleleft) & = D(B|A) - E_f(B, E) \leq D(B|A).
\end{align*}
\]

The optimal forward rate \( K(\blacklozenge) \), where Alice’s variable must be inferred, equals the difference between the output discord \( D(A|B) \), based on Bob’s detections, and the entanglement of formation \( E_f(A, E) \) between Alice and Eve. Situation is reversed for the other rate \( K(\blacktriangleleft) \). Note that quantum discord not only provides an upper bound to the key rates, but its asymmetric definition, \( D(A|B) \) or \( D(B|A) \), is closely connected with the reconciliation direction (direct \( \blacklozenge \) or reverse \( \blacktriangleleft \)).

Ideal QKD protocols. – In practical quantum cryptography, extra noise is always present, and we distinguish between device-dependent and device-independent QKD on the basis of Eve’s accessibility of the extra system \( P \). In theoretical studies of quantum cryptography, it is however common to design and assess new protocols by assuming no-extra noise in Alice’s and Bob’s apparatus (perfect state preparation and perfect detections).

This is an ideal scenario where system \( P \) of Fig. 2 is simply absent. For such ideal QKD protocols, the secret-key rates satisfy again Eqs. (7), (8), and (9), computed on the corresponding output states. Remarkably, the discord bound can be found to be tight in reverse reconciliation. In fact, as we show in the supplementary material, we can have \( K(\blacktriangleleft) = D(B|A) \) in an ideal protocol of continuous-variable QKD, where Alice transmits part of an Einstein-Podolsky-Rosen (EPR) state over a pure-loss channel, such as an optical fiber.

Conclusion and discussion. – Quantum discord can be regarded as a bipartite formulation of non-orthogonality, therefore capturing the minimal requisite for QKD. In this paper we have identified the general framework, device-dependent QKD, where discord remains the ultimate cryptographic primitive able to guarantee security in the place of quantum entanglement.

We have considered a general form of device-dependent protocol, where Alice and Bob share a bipartite state which can be purified by two systems: One system \( (E) \) is accessible to Eve, while the other \( (P) \) is inaccessible and accounts from the presence of trusted noise, e.g., coming from imperfections in the state preparation and/or the quantum detections. This is a scenario where the optimal key rate may outperform the coherent information and key distribution may occur in the complete absence of entanglement (in any form, distillable or bound) as long as discord is non-zero. As a matter of fact, any prepare and measure QKD protocol whose security is based on non-orthogonal quantum states can be recast into an entanglement-free device-dependent form which is based on a classical-quantum state, with non-zero discord transmitted through the channel.

This discord-based representation is secure as long as the extra system \( P \) is truly inaccessible to Eve, i.e., Alice’s and Bob’s private spaces cannot be accessed. Such a condition fails assuming side-channel attacks, where no noise can be trusted and \( P \) becomes part of Eve’s systems. In this case, the secret-key rates are again dominated by the coherent information, which means that entanglement remains the crucial resource for device-independent QKD. For both device-independent QKD and ideal QKD (where system \( P \) is absent), discord still represents an upper bound to the optimal secret-key rates achievable in direct or reverse reconciliation, with non-trivial cases where this bound becomes tight.

In conclusion, quantum discord is a necessary resource for secure QKD. This is particularly evident in device-dependent QKD where entanglement is a sufficient but not a necessary resource. Entanglement becomes necessary in device-independent and ideal QKD, where discord still provides an upper bound to the secret-key rates. Future work may involve the derivation of a direct mathematical relation between the amount of quantum discord in Alice and Bob’s output state and the optimal secret-key rates which are achievable in device-dependent QKD.

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\[ V_{AA} = \left( \mu I \begin{pmatrix} G & \mu \Gamma \end{pmatrix} \right), \]

where \( \mu \geq 1 \) and \( G \) is a diagonal correlation block which can be in one of the following forms

\[ G = \begin{pmatrix} g & \gamma \\ \gamma & g \end{pmatrix} : \equiv g I, \quad G = \begin{pmatrix} g & -g \\ g & -g \end{pmatrix} : = g Z. \]

Here the parameter \( g \) must satisfy \(|g| \leq \mu - 1\), so that \( V_{AA} \) is both physical and separable [32]. Apart from the singular case \( g = 0 \), this symmetric Gaussian state has always non-zero discord, i.e., \( D(A|a) = D(a|A) > 0 \) [33].

Mode \( a \) is sent through the channel, where Eve performs a collective Gaussian attack, whose most general description can be found in Ref. [34]. Assuming random permutations (so that quantum de Finetti applies), this is the most powerful attack against Gaussian protocols [31].

One of the canonical forms of this attack is the so-called ‘entangling cloner’ attack [31], where Eve uses a beam splitter with transmissivity \( \tau \) to mix the incoming mode \( a \) with one mode \( e \) of an EPR state \( \rho_{EE'} \) with CM

\[ V_{EE'} = \left( \begin{array}{cc} \omega I & \sqrt{\omega^2 - 1} Z \\ \sqrt{\omega^2 - 1} Z & \omega I \end{array} \right) := V(\omega), \quad (10) \]
where \( \omega \geq 1 \). One output mode \( B \) is sent to Bob, while the other output mode \( E \) is stored in a quantum memory together with the retained mode \( E' \). Such memory will be coherently detected at the end of the protocol.

In order to extract two correlated (complex) variables, \( X \) and \( Y \), Alice and Bob heterodyne their local modes \( A \) and \( B \). (Note that other protocols involving homodyne detection for one of the parties or even two homodynes may be considered as well.) One can easily check that Alice remotely prepares thermal states on mode \( a \). In fact, by heterodyning mode \( A \), the other mode \( a \) is collapsed in a Gaussian state \( \rho(a|X) = (1+\varepsilon)I \), where

\[ \varepsilon := \mu - 1 - \frac{g^2}{\mu + 1} \geq 0 \]

quantifies the thermalization above the coherent state. This conditional thermal state is randomly displaced in the phase space according to a bivariate Gaussian distribution with variance \( \mu - 1 - \varepsilon \) (so that the average input state on mode \( a \) is thermal with correct CM \( \nu(a) \)).

At the output of the channel, Bob’s average state is thermal with CM \( \nu_B I \), where \( \nu_B := \tau \mu + (1 - \tau) \omega \). By propagating the conditional thermal state \( \rho(a|X) \), we also get Bob’s conditional state \( \rho_B(X) \), which is randomly displaced and has CM \( \nu_B(X) I \), where

\[ \nu_B(X) := \tau (1 + \varepsilon) + (1 - \tau) \omega = \nu_B - \frac{\tau g^2}{\mu + 1}. \]

Therefore, we can easily compute Alice and Bob’s mutual information, which is equal to

\[ I(X,Y) = \log_2 \frac{\nu_B + 1}{\nu_B(X) + 1}. \]

The next step is the calculation of Eve’s Holevo information on Alice’s and Bob’s variables. We derive the global state of Alice, Bob and Eve, which is pure Gaussian with zero mean and CM

\[ V_{ABE} = \begin{pmatrix} \mu I & \sqrt{\tau} G & -\sqrt{1 - \tau} G & 0 \\ \sqrt{\tau} G & \nu_B I & \gamma I & \delta Z \\ -\sqrt{1 - \tau} G & \gamma I & \nu_B I & \kappa Z \\ 0 & \delta Z & \kappa Z & \omega I \end{pmatrix}, \]

where \( 0 \) is the \( 2 \times 2 \) zero matrix, and

\[ \nu_E := \tau \omega + (1 - \tau) \mu, \quad \gamma := \sqrt{\tau(1 - \tau)(\omega - \mu)}, \]

\[ \delta := \sqrt{1 - \tau} \sqrt{\omega^2 - 1}, \quad \kappa := \sqrt{\tau(\omega^2 - 1)}. \]

From this global CM, we extract Eve’s reduced CM \( V_{EE'} := V_E \) describing the two output modes \( E = EE' \) of the entangling cloner. This reduced CM has symplectic spectrum \([S1]\)

\[ \nu_E^\pm = \frac{\sqrt{\alpha^2 + 4 \beta \pm \alpha}}{2}, \]

where \( \alpha := (1 - \tau)(\mu - \omega) \) and \( \beta := \tau + (1 - \tau) \mu \omega \). The von Neumann entropy of Eve’s average state is then given by

\[ S(E) = h(\nu_E^+ + h(\nu_E^-)), \]

where

\[ h(x) := \frac{x + 1}{\log_2 x} \frac{1}{2} - \frac{x - 1}{\log_2 x - 1}. \]

By transforming the global CM under heterodyne detection \([S1]\), we compute Eve’s conditional CMs. First, we derive Eve’s CM conditioned to Bob’s detection

\[ V_{E|Y} = V_E - \frac{1}{\nu_B + 1} \begin{pmatrix} \gamma^2 I & \gamma \delta \sigma \Z \g \I \\ \gamma \delta \sigma \Z & \delta^2 \I \end{pmatrix}, \]

which has symplectic spectrum

\[ \nu_E^\pm = \frac{\mu + \beta}{1 + \mu \tau + (1 - \tau) \omega}. \]

Then, Eve’s CM conditioned to Alice’s detection is

\[ V_{E|X} = V_E - \frac{1 - \tau)g^2}{\mu + 1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \]

and has symplectic spectrum

\[ \nu_E^\pm = \frac{\sqrt{\theta^2 + 4 (\mu + 1) \phi \pm \theta}}{2 (\mu + 1)}, \]

where

\[ \theta := (1 - \tau)g^2 - (\mu + 1) \omega, \quad \phi := (\mu + 1) \beta - (1 - \tau) \omega g^2. \]

From the previous conditional spectra, we compute Eve’s conditional entropies

\[ S(E|X) = h(\nu_E^+ + h(\nu_E^-), \quad S(E|Y) = h(\nu_E^+), \]

and, therefore, we can derive the two Holevo quantities

\[ I(E, X) = S(E) - S(E|X) \quad \text{and} \quad I(E, Y) = S(E) - S(E|Y). \]

By subtracting these from Alice and Bob’s mutual information \( I(X, Y) \), we finally get the two key rates in direct and reverse reconciliation, i.e., \( K(Y|X) \) and \( K(X|Y) \).

It is easy to check the existence of wide range of parameters for which these two rates are strictly positive, so that Alice and Bob can extract a secret key despite the absence of entanglement (at the input state \( \rho_{AA} \), and, therefore, also at the output state \( \rho_{AB} \)). As an example, we may consider the maximum correlation value \( g = \mu - 1 \) for the separable Gaussian state \( \rho_{AB} \). We may take the large modulation limit \( \mu \to +\infty \), as typical in continuous variable QKD. In this case, we get the following asymptotical expression for Alice and Bob’s mutual information

\[ I(X, Y) \to \log_2 \frac{\omega}{1 + 3 \tau + (1 - \tau) \omega} + O(\mu^{-1}), \]

and the following asymptotical spectra

\[ \nu_E^\tau \to \frac{(1 - \tau) \mu + \tau \omega + O(\mu^{-1}),} \]

\[ \nu_E^\omega \to \omega + O(\mu^{-1}), \]

\[ \nu_E^\frac{1 + (1 - \tau) \omega}{\tau} + O(\mu^{-1}), \]

\[ \nu_E^{\xi \pm} \to \xi \pm + O(\mu^{-1}), \]
where
\[
\xi_{\pm} := \frac{\sqrt{(\omega + 3)^2 + \tau^2(\omega - 3)^2 - 2\tau(\omega^2 + 7)}}{2} \pm \frac{(1 - \tau)(\omega - 3)}{2}.
\]

Then, using the expansion \( h(x) \simeq \log_2(xe/2) + O(1/x) \) for large \( x \), we can write the two asymptotical rates
\[
K(Y|X) = R(\tau, \omega) + h(\xi_{\pm}) + h(\xi_{-}),
\]
\[
K(X|Y) = R(\tau, \omega) + h\left[\frac{1 + (1 - \tau)\omega}{\tau}\right],
\]
where we have introduced the common term
\[
R(\tau, \omega) := \log_2\frac{2\tau}{e(1 - \tau)[1 + 3\tau + (1 - \tau)\omega]} - h(\omega).
\]
As we can see from Fig. 3 there are wide regions of positivity for these rates.

![Fig. 3: Left panel. Rate \( K(Y|X) \) in direct reconciliation, as a function of channel transmissivity \( \tau \) and thermal variance \( \omega \). \( K \) is positive in the white area, while it is zero in the black area. Right panel. Rate \( K(X|Y) \) in reverse reconciliation, as function of \( \tau \) and \( \omega \). White area \( (K > 0) \) is wider at low \( \omega \).

In particular, for a pure loss channel \( (\omega = 1) \), the previous asymptotical rates simplify to the following
\[
K(Y|X) = \log_2\frac{\tau}{e(1 - \tau^2)} + h(3 - 2\tau),
\]
which is positive for any \( \tau > 0.693 \), and
\[
K(X|Y) = \log_2\frac{\tau}{e(1 - \tau^2)} + h\left(\frac{2}{\tau} - 1\right),
\]
which is positive for any \( \tau > 0.532 \).

**DISCORD BOUND CAN BE TIGHT**

Here we discuss a typical scenario where the optimal backward rate \( K(\bullet) \) of an ideal QKD protocol is exactly equal to the output discord \( D(B|A) \) shared by Alice and Bob. This happens in continuous variable QKD, where reverse reconciliation is important for its ability to beat the 3dB loss-limit affecting direct reconciliation [S1].

Consider an ideal QKD protocol which is based on the distribution of an EPR state \( \rho_{AB} \), with CM \( V_{AB} = V(\mu) \) defined according to Eq. (11) with \( \mu \geq 1 \). By performing a rank-1 Gaussian POVM on mode \( A \), Alice remotely prepares an ensemble of Gaussianly-modulated pure Gaussian states on the other mode \( \alpha \). For instance, heterodyne prepares coherent states, while homodyne prepares squeezed states. On average, mode \( \alpha \) is described by a thermal state with CM \( \mu \mathbf{I} \).

Suppose that signal mode \( \alpha \) is subject to a pure-loss channel. This means that Eve is using a beam splitter of transmissivity \( \tau \) mixing the signal mode with a vacuum mode \( e \). At the output of the beam splitter, mode \( B \) is detected by Bob, while mode \( E \) is stored in a quantum memory coherently detected by Eve (this is a collective entangling cloner attack with \( \omega = 1 \)).

Since the average state of mode \( \alpha \) is thermal and mode \( e \) is in the vacuum, no entanglement can be present between the two output ports \( B \) and \( E \) of the beam splitter. This implies that their entanglement of formation must be zero \( E_f(B,E) = 0 \) and, therefore, the optimal backward rate \( K(\cdot) \) must be equal to the discord \( D(B|A) \) of the Gaussian state \( \rho_{AB} \). Since this output state has CM
\[
V_{AB} = \left(\frac{\mu \mathbf{I}}{\sqrt{\mu^2 - 1}}\mathbf{Z} \frac{\sqrt{\tau(\mu^2 - 1)}\mathbf{Z}}{(\tau \mu + 1 - \tau)\mathbf{I}}\right),
\]
its discord is easy to compute and is equal to [S2]
\[
D(B|A) = h(\mu) - h[\tau + (1 - \tau)\mu].
\]
For large modulation \( (\mu \to +\infty) \), we have the asymptotic expression
\[
K(\bullet) = D(B|A) = \log_2\left(\frac{1}{1 - \tau}\right),
\]
which is positive for any \( 0 < \tau < 1 \). One can check that this rate can be achieved by heterodyne detections at Bob’s side (and coherent detection at Alice’s side).

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