NP-completeness of anti-Kekulé and matching preclusion problems

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Abstract

Anti-Kekulé problem is a concept of chemical graph theory to measure stability of the molecule by precluding its Kekulé structure. Matching preclusion and conditional matching preclusion problems were proposed as measures of robustness in the event of edge failure in interconnection networks. It is known that matching preclusion problem on bipartite graphs is NP-complete. In this paper, we first show that conditional matching preclusion problem and anti-Kekulé problem are NP-complete, respectively. We then generalize matching preclusion problem to $s$-restricted matching preclusion problem and prove its NP-completeness. Moreover, we give some sufficient conditions to compute $s$-restricted matching preclusion numbers of regular graphs. As applications, $s$-restricted matching preclusion numbers of complete graphs, hypercubes and hyper Petersen networks are determined.

Key words: Matching preclusion; Conditional matching preclusion; Anti-Kekulé; $s$-restricted matching preclusion; NP-complete; Hypercube

1 Introduction

Let $G = (V, E)$ be a simple and connected graph. Let $N(v)$ be the set of neighbors of a vertex $v$ and $d(v) = |N(v)|$, the degree of $v$. The minimum degree of $G$ is denoted by $\delta(G)$. A matching $M$ of $G$ is a set of pairwise nonadjacent edges

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of $G$. The vertices of $G$ incident to the edges of $M$ are called saturated by $M$; the others are unsaturated. A matching in $G$ of maximum cardinality is called a maximum matching. The cardinality of a maximum matching is called the matching number of $G$, denoted by $\nu(G)$. A perfect matching is a matching of cardinality $|V(G)|/2$. An almost perfect matching is a matching covering all but one vertex of $G$. Let $F$ be a set of edges of $G$. If $G - F$ has neither a perfect matching nor an almost perfect matching, then we call $F$ a matching preclusion set of $G$. The matching preclusion number of $G$, denoted by $mp(G)$, is the minimum cardinality over all matching preclusion sets of $G$. A matching preclusion set of a graph $G$ with even order is trivial if all its edges are incident to a vertex of $G$. Based on the definition, we set $mp(G) = 0$ if $G$ has neither a perfect matching nor an almost perfect matching. For other standard graph notations and terminologies not defined here please refer to [2].

In organic molecule graphs, perfect matchings correspond to Kekulé structures, which play an important role in analyzing resonant energy and stability of hydrocarbon compounds. In [28], Vukičević and Trinajstić proposed the anti-Kekulé number of a connected graph $G$ as the smallest number of edges such that after deleting these edges of $G$ the resulting graph remains connected but has no Kekulé structure (perfect matching). For convenience, we call such a set of edges of $G$ an anti-Kekulé set. Anti-Kekulé numbers of some chemical graphs were studied, such as a hexagonal system [4], the infinite triangular, rectangular and hexagonal grids [27], cata-condensed benzenoids [20], fullerene graphs [33] and cata-condensed phenylenes [35].

The concept of matching preclusion was first introduced by Brigham et al. [3] as a measure of robustness of interconnection networks under the condition of edge failure. In the same paper, the authors showed that it will be more robust under edge failure if each vertex has a special matching vertex at any time, and they also determined the matching preclusion number of Petersen graph, complete graphs $K_n$, complete bipartite graphs $K_{n,n}$ and hypercubes. Recently, matching preclusion numbers for Cayley graphs generated by transposition trees and $(n, k)$-star graphs [12], HL-graphs and recursive circulant $G(2^m, 4)$ [25], tori (including related Cartesian product graphs) [13], burnt pancake graphs [19], binary de Bruijn graphs [22], $n$-grid graphs [6] and data center network [24] have been determined.

In large networks failure is inevitable, but it is unlikely that all the edges incident to a common vertex are all faulty simultaneously. Thus, it is meaningful to consider matching preclusion of a graph with some restriction on the order of components after edge deletion. Motivated by this, Cheng et al. [9] considered conditional matching preclusion set (resp. number) of a graph $G$. The conditional matching preclusion number of a graph $G$, denoted by $mp_1(G)$, is the minimum number of edges whose deletion leaves the resulting graph with no isolated vertices and without a perfect matching or an almost perfect matching. This problem is solved for complete graphs, complete bipartite graphs and hypercubes [9], arrangement graphs [10], Cayley graphs generated by 2-trees and hyper Petersen networks [11].
HL-graphs \[26\], \(k\)-ary \(n\)-cubes \[31\], balanced hypercubes \[23\] and cube-connected cycles \[21\].

By above, we know that anti-Kekulé numbers, matching preclusion numbers and conditional matching preclusion numbers of many graphs have already been determined. Here a natural question is: what’s the complexity of these problems on general graphs? In this paper, we solve this question.

The rest of the paper is organized as follows. In Section 2, we obtain NP-completeness of conditional matching preclusion problem and anti-Kekulé problem. In Section 3, we generalize matching preclusion problem to \(s\)-restricted matching preclusion problem and prove its NP-completeness. In Section 4, we present some sufficient conditions to calculate \(s\)-restricted \((s \geq 2)\) matching preclusion number of regular graphs. Applications of determining \(s\)-restricted matching preclusion numbers of complete graphs, hypercubes and hyper Petersen networks are given in Section 5. Finally, we conclude this paper in Section 6.

2 NP-completeness of conditional matching preclusion problem and anti-Kekulé problem

Let \(G = (U \cup V, E)\) be a bipartite graph with matching number \(\nu(G)\). In \[34\], Zenklusen et al. proposed a blocker for \(G\) as a minimal set of edges whose removal results in matching number of \(G\) smaller than \(\nu(G)\). They also defined the minimum blocker problem (MBP) as follows. Given \(G\) and a positive integer \(k\), does there exist an edge subset \(B\) of \(E\) with \(|B| \leq k\) such that \(B\) is a blocker for \(G\)? Additionally, NP-completeness of MBP was proved in the same paper. In \[20\], the authors studied a special case of MBP, called minimum blocker perfect matching problem (MBPMP), where \(G\) is a bipartite graph with a perfect matching. They proved that MBPMP is NP-complete.

Lemma 1 \[20\]. MBPMP is NP-complete.

MBPMP is the same as matching preclusion problem of a graph with perfect matching. Thus, matching preclusion problem on bipartite graphs is also NP-complete.

We shall prove the NP-completeness of conditional matching preclusion problem. We present the decision problem of conditional matching preclusion problem as follows.

*Conditional matching preclusion problem:*

*Instance:* A nonempty graph \(G = (V, E)\) having a perfect matching and a positive integer \(k\).

*Question:* Does there exist a subset \(B \subseteq E\) with \(|B| \leq k\) such that \(G - B\) has neither isolated vertices nor perfect matchings?
Fig. 1. The graph $G'$ constructed from $G$ for proving NP-completeness of conditional matching preclusion problem.

**Theorem 2.** Conditional matching preclusion problem on bipartite graphs is NP-complete.

**Proof.** Obviously, conditional matching preclusion problem is in NP, because we can check in polynomial time whether a set of edges is a conditional matching preclusion set. We prove the NP-completeness of conditional matching preclusion problem by reducing MBPMP to it in polynomial time.

Let $G = (U \cup V, E)$ be a bipartite graph with bipartition $U$ and $V$ such that $|U| = |V| = t$. Suppose that $G$ has a perfect matching. Let $u_1, u_2, \ldots, u_t$ (resp. $v_1, v_2, \ldots, v_t$) denote the vertices in $U$ (resp. $V$). The graph $G' = (U' \cup V', E')$ is constructed from $G$ as follows (see Fig. 1): $U' = U \cup \{u', u''\}$, $V' = V \cup \{v', v''\}$, where $u'$ and $v'$ are new added vertices. $E' = E \cup \{u'v : v \in V\} \cup \{uv' : u \in U\} \cup \{u'v', u''v', u''v', uv''\}$. Note that $u'$, $v'$, $u''$ and $v''$ form a 4-cycle in $G'$. For convenience, we denote $u''v''$ and $u'v'$ by $e$ and $e'$, respectively.

In the following, we shall show that $G$ has a matching preclusion set of cardinality no greater than $k$ if and only if $G'$ has a conditional matching preclusion set of cardinality no greater than $k + 1$.

**Necessity.** Suppose that $B$ is a matching preclusion set of $G$ with $|B| \leq k$. Then $G - B$ has no perfect matchings. Let $B' = B \cup \{e\}$. We will prove that $B'$ is a conditional matching preclusion set of $G'$. Since $u'$ joins to each vertex in $V$ and $v'$ joins to each vertex in $U$, $G' - B'$ is connected. If $G' - B'$ has a perfect matching $M$, noting that $e \in B'$, then $u'v'', u''v' \in M$. Thus, $G - B$ has a perfect matching $M \setminus \{u'v'', u''v'\}$, a contradiction. Hence, $B'$ is a conditional matching preclusion set of $G'$ with $|B'| \leq k + 1$.

**Sufficiency.** Suppose that $B'$ is a conditional matching preclusion set of $G'$ such
that \(|B'| \leq k + 1\). Thus, \(G' - B'\) has no perfect matchings, and each component of \(G' - B'\) contains at least two vertices. Let \(B = B' \cap E(G)\). We consider the following two cases:

**Case 1:** \(e \in B'\). Then \(|B| \leq k\). Since each component of \(G' - B'\) contains at least two vertices, \(u'v'', u''v' \not\in B'\). If \(G - B\) has a perfect matching \(M\), then \(M \cup \{u'v'', u''v'\}\) is a perfect matching of \(G' - B'\), a contradiction. Thus, \(B\) is a matching preclusion set of \(G\).

**Case 2:** \(e \not\in B'\). If \(B\) is a matching preclusion set of \(G\) with \(|B| \leq k\), we are done. If \(B\) is not a matching preclusion set of \(G\), then \(G - B\) has a perfect matching \(M\). For convenience, we denote the edges in \(M\) by \(u_i v_i\) with \(1 \leq i \leq t\). We claim that \(e' \in B'\). If not, then \(M \cup \{e, e'\}\) is a perfect matching of \(G' - B'\), a contradiction. Also, we claim that one of \(v'u_i\) and \(u'v_i\) is in \(B'\) for each \(1 \leq i \leq t\). If not, then \(M \cup \{e\} \cup \{u'v_i, v'u_i\} \setminus \{u_i v_i\}\) is a perfect matching of \(G' - B'\), a contradiction. Thus, \(t + 1 + |B| \leq |B'| \leq k + 1\), which implies that \(t \leq k\). Since \(G\) is bipartite, \(\Delta(G) \leq t \leq k\). It follows that \(G\) has a trivial matching preclusion set with cardinality no greater than \(k\).

We now consider the remaining case. If \(B\) is a matching preclusion set of \(G\) such that \(|B| = k + 1\). Then \(B' \subseteq E(G)\). Let \(M\) be a maximum matching of \(G - B\). We claim that \(|M| \leq t - 2\). Otherwise, \(|M| = t - 1\). Suppose that \(u_i (1 \leq i \leq t)\) and \(v_j (1 \leq j \leq t)\) are the only two vertices in \(G\) unsaturated by \(M\). Then \(M \cup \{u_i v', u'v_j, e\}\) is a perfect matching of \(G' - B'\), a contradiction, and the claim holds. Let \(e''\) be an arbitrary edge in \(B\) and \(B_1 = B \setminus \{e''\}\). Then \(|B_1| = k\) and \(\nu(G - B_1) \leq \nu(G - B) + 1 \leq t - 1\). Thus, \(B_1\) is a matching preclusion set of \(G\) with \(|B_1| \leq k\). This completes the proof. \(\square\)

Now we give the decision problem of anti-Kekulé problem as follows.

**Anti-Kekulé problem:**

**Instance:** A nonempty graph \(G = (V, E)\) having a perfect matching and a positive integer \(k\).

**Question:** Does there exist a subset \(B \subseteq E\) with \(|B| \leq k\) such that \(G - B\) is connected and \(G - B\) has no Kekulé structure (perfect matching)?

**Corollary 3.** Anti-Kekulé problem on bipartite graphs is NP-complete.

**Proof.** We adopt all the notations defined in Theorem\(^2\). If we replace the condition of \(G' - B'\) having no singletons by \(G' - B'\) being connected, then the proof is just a rewrite of Theorem\(^2\). \(\square\)
3 NP-completeness of $s$-restricted matching preclusion problem

Recently, Wheeler et al. [32] generalized the matching preclusion problem and introduced the concept of generalized matching preclusion in bipartite graphs, yielding a hierarchy of stronger matching preclusion properties in bipartite graphs.

We generalize the matching preclusion problem in a quite different flavor. It is known that after deleting a conditional matching preclusion set $F$ from a graph $G$, the resulting graph $G - F$ has neither perfect matching nor almost perfect matching, and each component of $G - F$ has at least two vertices. Motivated by $k$-restricted edge-connectivity [15,16], it is interesting to ask how the value of matching preclusion number changes if each component of $G - F$ contains at least $s + 1$ vertices, where $s$ is a nonnegative integer. This is closely related to the changing and unchanging of invariants studied in many areas [17,18].

Let $s$ be a nonnegative integer and $F$ an edge subset of $G$. If $G - F$ has neither perfect matching nor almost perfect matching, and each component of $G - F$ has at least $s + 1$ vertices, then $F$ is called an $s$-restricted matching preclusion set of $G$. The $s$-restricted matching preclusion number, denoted by $mp_s(G)$, is the minimum cardinality over all $s$-restricted matching preclusion sets of $G$. It is suitable to make a convention that $s \leq |V(G)| - 1$ for a given graph $G$. Based on the definition, we set $mp_s(G) = 0$ if $G$ has neither perfect matching nor almost perfect matching. We leave $mp_s(G)$ undefined if an $s$-restricted matching preclusion set does not exist, that is, we can not delete edges to satisfy the two conditions in the definition.

It is noticeable that 0-restricted matching preclusion problem is equivalent to matching preclusion problem, 1-restricted matching preclusion problem is the conditional matching preclusion problem and $(|V(G)| - 1)$-restricted matching preclusion problem is the anti-Kekulé problem. Thus, for notation consistence, we still use $mp(G)$ to denote $mp_0(G)$. Note that an $(s + 1)$-restricted matching preclusion set of $G$ is a special $s$-restricted matching preclusion set of $G$, the following proposition is obvious.

**Proposition 4.** Let $G$ be a nontrivial graph. In addition, let $s$ be a positive integer. Then $mp(G) \leq mp_1(G) \leq mp_2(G) \leq \cdots \leq mp_s(G)$.

To characterize the complexity of $s$-restricted matching preclusion problem, we present the following decision problem.

$s$-restricted matching preclusion problem:

**Instance:** A nonempty bipartite graph $G = (V, E)$ having a perfect matching, a positive integer $k$ and a positive integer $s$.

**Question:** Does there exist a set $B \subseteq E$ with $|B| \leq k$ such that $G - B$ has no perfect matching and each component of $G - B$ has at least $s + 1$ vertices.

**Theorem 5.** $s$-restricted matching preclusion problem on bipartite graphs is NP-complete.
Proof. We shall keep the definitions and notations introduced in Theorem 2. As mentioned earlier, matching preclusion is 0-restricted matching preclusion, whose NP-completeness has already been obtained. So we may assume that $s \geq 1$ in the remaining proof. We shall prove the NP-completeness of $s$-restricted matching preclusion problem by reducing MBPMP to it in polynomial time.

In the following, we shall show that $G$ has a matching preclusion set of cardinality no greater than $k$ if and only if $G'$ has an $s$-restricted matching preclusion set of cardinality no greater than $k + 1$.

Necessity. Suppose that $B$ is a matching preclusion set of $G$ with $|B| \leq k$. Then $G - B$ has no perfect matchings. Let $B' = B \cup \{e\}$. Since $u'$ joins to each vertex in $V$ and $v'$ joins to each vertex in $U$, $G' - B'$ is connected and has no perfect matchings. Thus, $B'$ is an $s$-restricted matching preclusion set of $G'$ with $|B'| \leq k + 1$.

Sufficiency. Suppose that $B'$ is an $s$-restricted matching preclusion set of $G'$ such that $|B'| \leq k + 1$. Thus, $G' - B'$ has no perfect matchings, and each component of $G' - B'$ contains at least $s + 1$ vertices. Let $B = B' \cap E(G)$. We consider the following two cases:

Case 1: $e \in B'$. Then $|B| \leq k$. Since each component of $G' - B'$ contains at least $s + 1$ vertices, $u'v'', u''v' \notin B'$. If $G - B$ has a perfect matching $M$, then $M \cup \{u''v'', u''v'\}$ is a perfect matching of $G' - B'$, a contradiction. Thus, $B$ is a matching preclusion set of $G$.

Case 2: $e \notin B'$. If $B$ is a matching preclusion set of $G$ with $|B| \leq k$, we are done. If $B$ is not a matching preclusion set of $G$, then $G - B$ has a perfect matching $M$. For convenience, we denote the edges in $M$ by $u_iv_i$ with $1 \leq i \leq t$. We claim that $e' \in B'$. If not, then $M \cup \{e, e'\}$ is a perfect matching of $G' - B'$, a contradiction. Also, we claim that at least one of $v'u_i$ and $v'u_i$ is in $B'$ for each $1 \leq i \leq t$. If not, then $M \cup \{e\} \cup \{u_iv_i, v'u_i\} \setminus \{u_iv_i\}$ is a perfect matching of $G' - B'$, a contradiction. Thus, $t + 1 + |B| \leq |B'| \leq k + 1$, which implies that $t \leq k$. Since $G$ is bipartite, we have $\Delta(G) \leq t \leq k$. It follows that $G$ has a trivial matching preclusion set with cardinality no greater than $k$.

We now consider the remaining case. If $B$ is a matching preclusion set of $G$ such that $|B| = k + 1$, then $B' \subseteq E(G)$. Let $M$ be a maximum matching of $G - B$. We claim that $|M| \leq t - 2$. Otherwise, $|M| = t - 1$. Suppose that $u_i$ and $v_j$ $(1 \leq i, j \leq t)$ are the only two vertices in $G$ unsaturated by $M$. Then $M \cup \{u_iv', v'u_j, e\}$ is a perfect matching of $G' - B'$, a contradiction. Thus, the claim holds. Let $e''$ be an arbitrary edge in $B$ and $B_1 = B \setminus \{e''\}$. Then $|B_1| = k$ and $\nu(G - B_1) \leq \nu(G - B) + 1 \leq t - 1$. Thus, $B_1$ is a matching preclusion set of $G$ with $|B_1| \leq k$. This completes the proof. 

$\square$
4 \ s\text{-restricted matching preclusion number of interconnection networks}

By the above section, it is difficult to determine \( s\text{-restricted matching preclusion numbers of general graphs} \), so it is meaningful to determine \( s\text{-restricted matching preclusion number for some famous interconnection networks} \). In what follows, we will give some sufficient conditions for determining \( s\text{-restricted (} s \geq 2 \text{)} matching preclusion numbers of regular graphs} \).

Cheng et al. [9] discussed the basic obstruction to a perfect matching or an almost perfect matching in the resulting graph with no isolated vertices. For a graph without isolated vertices, they showed that a basic obstruction to a perfect matching will be the existence of a path \( uvw \), where the degree of \( u \) and \( v \) are 1, respectively. We define \( v_e(G) = \min\{d_G(u) + d_G(v) - 2 - y_G(u,v): \text{there exists a vertex } w \text{ such that } uvw \text{ is a 2-path}\} \), where \( d_G(.) \) is the degree function and \( y_G(u,v) = 1 \) if \( u \) and \( v \) are adjacent and 0 otherwise.

Lemma 6 [9]. Let \( G \) be a graph with an even number of vertices. Suppose every vertex in \( G \) has degree at least three. Then \( mp_1(G) \leq v_e(G) \).

If \( mp_1(G) = v_e(G) \), then \( G \) is called \textit{conditionally maximally matched}. And the optimal solution of the form induced by \( v_e(G) \) is called a \textit{trivial conditional matching preclusion set}.

Cheng et al. [8] showed that there exists a close relationship between \( mp_1(G) \) and super edge-connectivity of \( G \). A graph \( G \) is \textit{maximally edge-connected} if the edge-connectivity of \( G \) is \( \delta(G) \). A maximally edge-connected graph \( G \) is called \textit{super edge-connected} if the deletion of at most \( \delta(G) \) edges results in either a connected graph or exactly two connected components, one of which is a singleton. A graph is \textit{super} \( m\text{-edge-connected of order } q \) if the deletion of at most \( m \) edges, the resulting graph is either connected or it has one big component together with a number of small components with at most \( q \) vertices in total. A set of edges \( F \) in a connected graph \( G \) is called a \textit{g-extra edge cut} if \( G - F \) is disconnected and each remaining component of \( G - F \) contains at least \( g \) vertices. The \textit{g-extra edge-connectivity of} \( G \), denoted by \( \lambda_g(G) \), is the minimum number of edges over all minimum \( g\text{-extra edge cuts of } G \). By convention, \( \lambda_1(G) \) and \( \lambda_2(G) \) are denoted by \( \lambda(G) \) and \( \lambda'(G) \), respectively. Therefore, \( G \) is \textit{super} - \( \lambda' \); if every minimum \( 2\text{-extra edge cut isolates one edge of } G \). For \( k\text{-regular bipartite graphs} \), Cheng [8] presented sufficient conditions for graphs to be conditionally maximally matched as follows.

Theorem 7 [8]. Let \( G \) be a \( k\text{-regular bipartite graph that is super } (3k - 6)\text{-edge-connected of order } 2 \). Then \( mp_1(G) = 2k - 2 \).

Naturally, we need the condition \( k \geq 3 \) since this ensures the resulting graph has no isolated vertices after some edges are deleted. So we assume that \( k \geq 3 \) in the remaining paper. Mirroring the above theorem, we obtain the following result.
**Theorem 8.** Let $G$ be a $k$-regular bipartite graph that is super $(3k - 6)$-edge-connected of order 2. Then $mp_s(G) = 2k - 2$ for all integers $s \geq 1$.

**Proof.** Clearly, we need only to consider $s \geq 2$. By Proposition 4 and Theorem 7, we have $mp_s(G) \geq mp_1(G) = 2k - 2$ for all integers $s \geq 2$. It remains to show that $mp_s(G) \leq 2k - 2$ for all integers $s \geq 2$. Let $(U, V)$ be the bipartition of $G$, where $|U| = |V|$. In addition, let $uvw$ be any 2-path in $G$. Without loss of generality, suppose $u, v \in U$ and $w \in V$. We shall show that the trivial conditional matching preclusion set obtained by $uvw$ (all the edges incident to $u$ or $v$ but not $w$), denoted by $F$, is also an $s$-restricted matching preclusion set of $G$.

If $k = 3$, we shall show that $G - F$ is connected. For any $e \in F$, let $F' = F \setminus \{e\}$. Obviously, $G - F'$ contains neither isolated vertices nor isolated edges. Since $|F'| = 3$ and $3k - 6 = 3$ when $k = 3$, $G - F'$ is connected. If $G - F$ is connected, we are done. So we assume that $G - F$ is disconnected, that is, $F$ is a minimal edge cut of $G$. Let $x$ be a neighbor of $w$ ($x \neq u, v$). Let $C_1$ and $C_2$ be two components of $G - F$, we may assume that $u, v, w, x \in V(C_1)$. Let $A = U \cap (V(C_1) \setminus \{u, v, x\})$ and $B = V \cap (V(C_1) \setminus \{w\})$. Since each vertex in $A$ and $B$ has degree 3, by Handshaking Lemma, we have $3|A| + d_{C_1}(u) + d_{C_1}(v) + d_{C_1}(x) = 3|B| + d_{C_1}(w)$, that is, $3|A| + 5 = 3|B| + 3$, which is a contradiction. Thus, $F$ is an $s$-restricted matching preclusion set of $G$.

If $k \geq 4$, the degree of each vertex (except $u$ and $v$) in $G - F$ is at least two. Then there exists no component of $G - F$ containing at most two vertices. Therefore, $G - F$ is connected since $G$ is $(3k - 6)$-edge-connected of order 2 and $3k - 6 \geq 2k - 2$. Thus, $F$ is an $s$-restricted matching preclusion set of $G$. 

We need some more definitions. An independent set in a graph $G$ is a set of vertices no two of which are adjacent. The cardinality of a maximum independent set in $G$ is called the independent number of $G$ and is denoted by $\alpha(G)$. Let $X \subseteq V(G)$, we define $\gamma_G(X)$ as the set of edges with both ends in $X$. (The subscript $G$ will be omitted if the context is clear.) Moreover, we define $\zeta(G, p, q) = \min\{\alpha(H)|H$ is an induced subgraph of $G$ with $p$ vertices and at most $q$ edges\}.

For non-bipartite regular graphs, Cheng et al. [11] obtained the following two theorems.

**Theorem 9 [11].** Let $G = (V, E)$ be a $k$-regular graph of even order, where $k \geq 3$. Suppose that $G$ contains a triangle, and $G$ is $k$-edge-connected and super $(3k - 8)$-edge-connected of order 2. Moreover, assume that either $|\gamma_G(X)| > 2k - 4$ for every $X \subseteq V$ of size $|X| = \frac{|V| + 2}{2}$, or $\alpha(G) < \zeta(G, \frac{|V| - 2}{2}, 2k - 8)$. If $k = 3$, it is additionally required that $G$ is super $(3k - 7)$-edge-connected of order 2. Then $mp_1(G) = 2k - 3$.

**Theorem 10 [11].** Let $G = (V, E)$ be a $k$-regular graph of even order, where $k \geq 3$. Suppose that $G$ is triangle-free, and $G$ is $k$-edge-connected and super $(3k - 6)$-edge-connected of order 2. Moreover, either $|\gamma_G(X)| > 2k - 3$ for every $X \subseteq V$ of size $|X| = \frac{|V| + 2}{2}$, or $\alpha(G) < \zeta(G, \frac{|V| - 2}{2}, 2k - 6)$. Then $mp_1(G) = 2k - 2$. 

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We generalize the two statements above to $s$-restricted matching preclusion problem as follows.

**Theorem 11.** Let $G = (V, E)$ be a $k$-regular graph of even order, where $k \geq 3$. Suppose that $G$ contains a triangle, and $G$ is $k$-edge-connected and super $(3k - 8)$-edge-connected of order 2. Let $s \geq 2$ be any integer. If $mp_1(G) = 2k - 3$, then $mp_s(G) = 2k - 3$.

**Proof.** By Proposition 4 and Theorem 9, we have $mp_s(G) \geq 2k - 3$ for all integers $s \geq 2$. It suffices to find an $s$-restricted matching preclusion set $F$ with $|F| = 2k - 3$. Let $uwv$ be a triangle of $G$ and let $F$ be the set of edges incident to $u$ and $v$ but not $w$. Clearly, $uv \in F$. Thus, $F$ is a trivial conditional matching preclusion set. Then $G - F$ has no perfect matchings. For convenience, let $F' = F \setminus \{uw\}$. We consider the following two cases.

**Case 1.** $k = 3$. Then $|F| = 3$ and $|F'| = 2$. Since $G$ is 3-edge-connected, $G - F'$ is connected. Note that $uwv$ is also a triangle in $G - F'$, by further deleting $uv$ from $G - F'$, so it is obvious that $G - F$ is connected. Therefore, $F$ is an $s$-restricted matching preclusion set.

**Case 2.** $k \geq 4$. Then there exist no components of $G - F'$ containing at most two vertices since the degree of each neighbor of $u$ or $v$ is at least two in $G - F'$. Since $2k - 4 \leq 3k - 8$ whenever $k \geq 4$, combining $G$ is super $(3k - 8)$-edge-connected of order 2, $G - F'$ is connected. So $G - F$ is connected. Therefore, $F$ is an $s$-restricted matching preclusion set with $|F| = 2k - 3$. This completes the proof. \hfill \Box

**Theorem 12.** Let $G = (V, E)$ be a $k$-regular graph of even order, where $k \geq 3$. Suppose that $G$ is triangle-free, $k$-edge-connected and super $(3k - 6)$-edge-connected of order 2. Let $s \geq 2$ be any integer. If $mp_1(G) = 2k - 2$, then $mp_s(G) = 2k - 2$.

**Proof.** By Proposition 4 and Theorem 10, we have $mp_s(G) \geq 2k - 2$ for all integers $s \geq 2$. It suffices to show that $mp_s(G) \leq 2k - 2$. It suffices to give an $s$-restricted matching preclusion set $F$ with $|F| = 2k - 2$. Let $uwv$ be a 2-path of $G$ and let $F$ be the set of edges incident to $u$ and $v$ but not $w$. Since $G$ is triangle-free, $uv \notin F$. Thus, $F$ is a trivial conditional matching preclusion set. Then $G - F$ has no perfect matchings. We consider the following two cases.

**Case 1.** $k = 3$. Then $|F| = 4$. If $G - F$ is connected, then $F$ is an $s$-restricted matching preclusion set, we are done. So we assume that $G - F$ is disconnected. Let $A = N_G(u) \cup N_G(v) \setminus \{w\}$. Then $2 \leq |A| \leq 4$. We consider the following subcases.

**Subcase 1.1.** $|A| = 2$. We may assume that $A = \{x, y\}$. Thus, $d_{G-F}(x) = d_{G-F}(y) = 1$, and $xy \notin E(G)$. So there exists a neighbor $u'$ (resp. $v'$) of $x$ (resp. $y$) in $G - F$. We have $d_{G-F}(u') = 3$ and $d_{G-F}(v') = 3$. Therefore, there exist no components of $G - F$ containing at most two vertices. Let $e \in F$ be any edge and let $F' = F \setminus \{e\}$. Obviously, there exist no components of $G - F'$ containing at most two vertices. Since $G$ is super $(3k - 6)$-edge-connected of order 2 and $|F'| = 3$, $G - F'$ is connected. So $F$ is a minimal edge cut of $G$. Let $C_1$ and $C_2$ be two...
components of \( G - F \), we may assume that \( u, w, v \in V(C_1) \) and \( x, y, u', v' \in V(C_2) \). Since \( d(w) = 3 \), there exists a neighbor \( w' \) (\( w' \neq u, v \)) of \( w \) in \( C_1 \). Thus, \( w \) is a cut vertex in \( G \). It implies that there exist at most two edge-disjoint paths from \( w \) to \( x \) in \( G \), which contradicts the fact that \( G \) is 3-edge-connected. It follows that \( G - F \) is connected.

**Subcase 1.2.** \( |A| = 3 \). We may assume that \( A = \{x, y, z\} \). In addition, we assume that \( xu, xv \in E(G) \). Since \( G \) is triangle-free, \( xy, xz \not\in E(G) \). It implies that \( x \) (resp. \( y, z \)) has a neighbor of degree 3 (not \( w \)) in \( G - F \). Then there exist no components of \( G - F \) containing at most two vertices. By a similar argument of the proof of Subcase 1.1, we know that \( G - F \) is connected.

**Subcase 1.3.** \( |A| = 4 \). We may assume that \( A = \{x, y, x', y'\} \). In addition, we assume that \( xu, yu, x'v, y'v \in E(G) \). Since \( G \) is triangle-free, \( xy, x'y' \not\in E(G) \). So each vertex in \( A \) has degree 2 in \( G - F \), indicating that there exist no components of \( G - F \) containing at most two vertices. Again, we know that \( G - F \) is connected.

Therefore, \( F \) is an \( s \)-restricted matching preclusion set with \( |F| = 4 \).

**Case 2.** \( k \geq 4 \). Noting the vertex degree of each neighbor of \( u \) or \( v \) is at least two in \( G - F \), then there exist no components of \( G - F \) containing at most two vertices. Since \( 2k - 2 \leq 3k - 6 \) whenever \( k \geq 4 \), combining \( G \) is super \((3k - 6)\)-edge-connected of order 2, \( G - F \) is connected. Therefore, \( F \) is an \( s \)-restricted matching preclusion set with \( |F| = 2k - 2 \). This completes the proof. \( \square \)

**5 Applications**

In this section, as applications of Theorems 8, 11 and 12 obtained in Section 4, we shall determine the \( s \)-restricted matching preclusion numbers of complete graphs, hypercubes and hyper Petersen networks.

**5.1 Complete graphs**

**Theorem 13** [9]. Let \( n \geq 4 \) be even. Then

\[
mp_1(K_n) = \begin{cases} 
\frac{(n^2 + 2n)}{8} & \text{if } n \in \{4, 6, 8\}, \\
2n - 5 & \text{if } n \geq 10.
\end{cases}
\]

We need the following lemma.

**Lemma 14** [30]. Let \( G \) be a complete graph with order at least four. Then \( G \) is super-\( \lambda' \).

Since \( K_n \) (\( n \geq 4 \)) is super-\( \lambda' \), we have \( \lambda'(K_n) = 2n - 4 \). The following result is straightforward.

**Lemma 15.** Let \( n \geq 4 \) be an integer. Then the complete graph \( K_n \) is super \((2n - 4)\)-edge-connected of order 2.
Similar to Theorem 13, we have the following result.

**Theorem 16.** Let \( n \geq 4 \) be even and let \( s \geq 2 \) be an integer. Then

\[
mp_s(K_n) = \begin{cases} 
\frac{n^2 + 2n}{8} & \text{if } n \in \{4, 6, 8\}, \\
2n - 5 & \text{if } n \geq 10.
\end{cases}
\]

**Proof.** By Proposition 4 and Theorem 13, it suffices to present an \( s \)-restricted matching preclusion set of size \( mp_1(K_n) \). Let \( F \) be a conditional matching preclusion set of \( K_n \) with \( |F| = mp_1(K_n) \). Since \( K_n \) is super \((2n - 4)\)-edge-connected of order \( 2n \), combining \( mp_1(K_n) < 2n - 4 \) whenever \( n \geq 4 \), we have that \( K_n - F \) is connected. Then \( F \) is also an \( s \)-restricted matching preclusion set of \( K_n \). \( \square \)

### 5.2 Hypercubes

The hypercube \( Q_n \) is a well-known topology for parallel computing. Any vertex \( v \) of \( Q_n \) is denoted by an \( n \)-bit binary string \( v_1v_2\cdots v_n \), where \( v_i \in \{0, 1\} \), for all \( i, 1 \leq i \leq n \). Two vertices of \( Q_n \) are adjacent if and only if their binary strings differ in exactly one bit position.

To compute the \( s \)-restricted matching preclusion number of the hypercube, we need the following results.

**Theorem 17** [36]. Let \( n \geq 2 \) be a positive integer. Then \( \lambda_3(Q_n) = 3n - 4 \).

**Theorem 18** [9]. Let \( n \geq 3 \) be an integer, then \( mp_1(Q_n) = 2n - 2 \).

**Theorem 19.** Let \( n \geq 3 \). For all integers \( s \geq 2 \), then \( mp_s(Q_n) = 2n - 2 \).

**Proof.** It is known that \( Q_n \) is bipartite and \( n \)-regular. We only need to verify the connectivity condition in Theorem 8. By Theorem 17 \( \lambda_3(Q_n) = 3n - 4 \). It means that after deletion of at most \( 3n - 5 \) edges in \( Q_n \), the resulting graph is either connected or contains one big component and some small components having two vertices in total. So \( Q_n \) is super \((3n - 6)\)-edge-connected of order \( 2 \). Hence, \( mp_s(Q_n) = 2n - 2 \). \( \square \)

### 5.3 Hyper Petersen networks

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs. The *Cartesian product* of \( G_1 \) and \( G_2 \) is the graph \( G_1 \times G_2 \) whose vertex set is \( V_1 \times V_2 \) and whose edge set is the set of all pairs \((u_1, u_2)(v_1, v_2)\) such that either \( u_1v_1 \in E_1 \) and \( u_2 = v_2 \), or \( u_2v_2 \in E_2 \) and \( u_1 = v_1 \). Das et al. [5] introduced the hyper Petersen networks \( PN_n \) for \( n \geq 3 \) as a kind of hypercube-like interconnection network. Some additional properties of \( PN_n \) can be found in [11]. \( PN_n \) is recursively defined as follows: \( PN_3 = P \), where \( P \) is the Petersen graph, and \( PN_n = Q_{n-3} \times P \) for \( n \geq 4 \). Obviously, \( PN_n \) is \( n \)-regular and has \( 10 \times 2^{n-3} \) vertices.
Lemma 20 \[7\]. \( mp_1(P) = 3 \).

Lemma 21. If \( s \geq 2 \), then \( mp_s(P) = 3 \).

**Proof.** Let \( F \) be a conditional matching preclusion set of \( P \) with \( |F| = 3 \). It is easy to see that the Petersen graph is super edge-connected. Then \( P - F \) is connected. So \( F \) is also an \( s \)-restricted matching preclusion set. By Proposition \[7\] \( mp_s(P) = 3 \). \( \square \)

Theorem 22 \[11\]. If \( n \geq 4 \), then \( mp_1(PN_n) = 2n - 2 \).

Theorem 23. If \( n \geq 4 \), then \( mp_s(PN_n) = 2n - 2 \).

**Proof.** Obviously, \( PN_n \) is \( n \)-regular and triangle-free. By the proof of Theorem 6.2 in \[11\], it can be known that \( PN_n \) is super edge-connected and super \((3n - 6)\)-edge-connected of order 2 for \( n \geq 3 \). By Theorems \[12\] and \[22\] we have \( mp_s(PN_n) = 2n - 2 \). \( \square \)

6 Conclusions

The MBPMP problem, arising in the structural analysis of differential-algebraic systems, is the same as matching preclusion problem for measuring robustness of interconnection networks. It is known that MBPMP is NP-complete, thus matching preclusion problem on bipartite graphs is also NP-complete. By reducing MBPMP to conditional matching preclusion problem in polynomial time, we prove its NP-completeness. As a corollary, we prove NP-completeness of anti-Kekulé problem. We generalize matching preclusion and conditional matching preclusion to \( s \)-restricted matching preclusion and obtain its NP-completeness.

To calculate the \( s \)-restricted \((s \geq 2)\) matching preclusion numbers of graphs, we present some sufficient conditions for regular graphs and, for example, obtain \( s \)-restricted matching preclusion numbers for complete graphs, hypercubes and hyper Petersen networks. It is interesting to study the \( s \)-restricted matching preclusion numbers for general graphs. Additionally, the complexity of \( s \)-restricted matching preclusion problem on graphs with restricted conditions, say maximum degree, should be further studied.

References

[1] A. Al-Ayyoub, K. Day, Comparative study of product networks, J. Paral. Distrib. Comput. 62 (2002) 1–18

[2] J.A. Bondy, U.S.R. Murty, Graph theory, Springer, New York, 2007

[3] R.C. Brigham, F. Harary, E.C. Bolin, J. Yellen, Perfect-matching preclusion, Congr. Numer. 174 (2005) 185–192
[4] J. Cai, H. Zhang, On the anti-Kekulé number of a hexagonal system, MATCH Commun. Math. Comput. Chem. 69 (3) (2013) 733–754

[5] S. Das, S. Öhring, A. Banerjee, Embeddings into hyper Petersen network: yet another hypercube-like interconnection topology, VLSI Design 2 (1995) 335–351

[6] Q. Ding, H. Zhang, Hui Zhou, Matching preclusion for n-grid graphs, Discrete Appl. Math. 243 (2018) 194–206

[7] E. Cheng, R. Connolly, C. Melekian, Matching preclusion and conditional matching preclusion problems for the folded Petersen cube, Theor. Comput. Sci. 576 (2015) 30–44

[8] E. Cheng, P. Hu, R. Jia, L. Lipták, Matching preclusion and conditional matching preclusion for bipartite interconnection networks I: Sufficient conditions, Networks 59 (2012) 349–356

[9] E. Cheng, L. Lesniak, M.J. Lipman, L. Lipták, Conditional matching preclusion sets, Inform. Sci. 179 (2009) 1092–1101

[10] E. Cheng, M.J. Lipman, L. Lipták, D. Sherman, Conditional matching preclusion for the arrangement graphs, Theor. Comput. Sci. 412 (2011) 6279–6289

[11] E. Cheng, M.J. Lipman, L. Lipták, Matching preclusion and conditional matching preclusion for regular interconnection networks, Discrete Appl. Math. 160 (2012) 1936–1954

[12] E. Cheng, L. Lipták, Matching preclusion for some interconnection networks, Networks 50 (2)(2007) 173–180

[13] E. Cheng, L. Lipták, Matching preclusion and conditional matching preclusion problems for tori and related Cartesian products, Discrete Appl. Math. 12 (2012) 1699–1716

[14] A.H. Esfahanian, Generalized measures of fault tolerance with application to n-cube networks, IEEE Trans. Compt. 38 (1989) 1586–1591

[15] J. Fàbrega, M.A. Fiol, Extraconnectivity of graphs with large girth, Discrete Math. 127 (1994) 163–170

[16] J. Fàbrega, M.A. Fiol, On the extraconnectivity of graphs, Discrete Math. 155 (1996) 49–57

[17] F. Harary, Conditional connectivity, Networks 13 (1983) 347–357

[18] F. Harary, Maximum versus minimum invariants for graphs, J. Graph Theory 7 (1983) 275–284
[19] X. Hu, H. Liu, The (conditional) matching preclusion for burnt pancake graphs, Discrete Appl. Math. 161 (2013) 1481–1489
[20] M. Lacroix, A.R. Mahjoub, S. Martin, C. Picouleau, On the NP-completeness of the perfect matching free subgraph problem, Theor. Comput. Sci. 423 (2012) 25–29
[21] Q. Li, W. Shiu, H. Yao, Matching preclusion for cube-connected cycles, Discrete Appl. Math. 190-191 (2015) 118–126
[22] R. Lin, H. Zhang, Matching preclusion and conditional edge-fault Hamiltonicity of binary de Bruijn graphs, Discrete Appl. Math. 233 (2017) 104–117
[23] H. Lü, X. Li, H. Zhang, Matching preclusion for balanced hypercubes, Theor. Comput. Sci. 465 (2012) 10–20
[24] H. Lü, T. Wu, Super edge-connectivity and matching preclusion of data center networks, Discrete Math. Theor. Comput. Sci. 21 (4) (2019) #2
[25] J. Park, Matching preclusion problem in restricted HL-graphs and recursive circulant $G(2^m, 4)$, J. KIISE 35 (2)(2008) 60–65
[26] J. Park, S. Son, Conditional matching preclusion for hypercube-like interconnection networks, Theor. Comput. Sci. 410 (2009) 2632–2640
[27] D. Velja, D. Vukičević, The anti-Kekulé number of the infinite triangular, rectangular and hexagonal grids, Glas. Mate. 43 (2008) 243–252
[28] D. Vukičević, N. Trinajstić, On the anti-forcing number of benzenoids, J. Math. Chem. 42 (2007) 575–583
[29] D. Vukičević, N. Trinajstić, On the anti-Kekulé number and anti-forcing number of cata-condensed benzenoids, J. Math. Chem. 43 (2008) 719–726
[30] S. Wang, S. Lin, C. Li, Sufficient conditions for super $k$-restricted edge connectivity in graphs of diameter 2, Discrete Math. 309 (2009) 908–919
[31] S. Wang, R. Wang, S. Lin, J. Li, Matching preclusion for $k$-ary $n$-cubes, Discrete Appl. Math. 158 (2010) 2066–2070
[32] Z. Wheeler, E. Cheng, D. Ferranti, L. Liptak, K. Nataraj, Generalized matching preclusion in bipartite graphs, Theory Appl. Graphs 5 (2018) Article 1
[33] Q. Yang, D. Ye, H. Zhang, Y. Lin, The anti-Kekulé number of fullerene graphs, MATCH Commun. Math. Comput. Chem. 67 (2) (2012) 281–288
[34] R. Zenklusen, B. Ries, C. Picouleau, D. Werra, M.-C. Costa, C. Bentz, Blockers and transversals, Discrete Math. 309 (2009) 4306–4314

[35] Q. Zhang, H. Bian, E. Vumar, On the anti-Kekulé number and anti-forcing number of cata-condensed phenylenes, MATCH Commun. Math. Comput. Chem. 65 (3) (2011) 799–806

[36] Q. Zhu, J.-M Xu, On restricted edge connectivity and extra edge connectivity of hypercubes and foled hypercubes, J. Univ. Sci. Technol. China 36 (2006) 246–253