On fixed points of quantum gravity

Daniel Litim

School of Physics and Astronomy
University of Southampton, Southampton SO17 1BJ, U.K.
and CERN, Theory Group, CH – 1211 Geneva 23

Abstract. We study the short distance behaviour of euclidean quantum gravity in the light of Weinberg’s asymptotic safety scenario. Implications of a non-trivial ultraviolet fixed point are reviewed. Based on an optimised renormalisation group, we provide analytical flow equations in the Einstein-Hilbert truncation. A non-trivial ultraviolet fixed point is found for arbitrary dimension. We discuss a bifurcation pattern in the spectrum of eigenvalues at criticality, and the large dimensional limit of quantum gravity. Implications for quantum gravity in higher dimensions are indicated.

INTRODUCTION

Gravitational interactions at distances sufficiently large compared to the Planck length are described by the classical theory of general relativity. At smaller length scales, quantum effects are expected to become important. The quantisation of general relativity, however, still poses problems. It is known since long that four-dimensional quantum gravity is perturbatively non-renormalisable, meaning that an infinite number of parameters have to be fixed to renormalise standard perturbation theory. One may wonder whether a quantum theory of gravity in terms of the metric degrees of freedom can exist as a well-defined, non-trivial and cutoff-independent local theory down to arbitrarily small distances. It is generally believed that the above requirements imply the existence of a non-trivial ultraviolet (UV) fixed point under the renormalisation group (RG), governing the short-distance physics. Then it would suffice to adjust a finite number of parameters, ideally taken from experiment, to make the theory asymptotically safe. The corresponding short distance fixed point action would then provide a valid microscopic starting point to access classical general relativity as a “low energy phenomenon” of a local quantum field theory in the metric field.

For quantum gravity, this asymptotic safety scenario has been introduced by Weinberg [1]. In the vicinity of two dimensions, a non-trivial fixed point has been identified within perturbation theory, to leading [1]-[3] and subleading order [4] in $\epsilon = d - 2 \ll 1$. In the last couple of years, non-perturbative renormalisation group studies have been performed in the four-dimensional case. A non-trivial fixed point has been detected within various renormalisation group studies, also including higher dimensional operators or non-interacting scalar, vector and matter fields [5]-[16]. Additional indications for the existence of a non-trivial fixed point in four dimensions have been provided through lattice simulations within both Regge’s simplicial lattice formulation [17] and the causal dynamical triangulations approach, e.g. [18].

From a renormalisation group point of view, the “critical” dimension of quantum
gravity is $d_{cr} = 2$.\footnote{This is different from most particle physics theories, where the critical dimension of the relevant couplings is $d = 4$, e.g. QED, QCD.} In two dimensions, the gravitational coupling has vanishing canonical dimension, and standard perturbation theory is applicable. In turn, for any $d > d_{cr}$ the gravitational coupling has negative mass dimension, indicating that the theory is perturbatively non-renormalisable. Hence, one expects that the local renormalisation group properties of quantum theories of gravity for different dimensions with $d > d_{cr}$ share qualitative properties. We conclude that the case of four dimensions, from a quantum gravity point of view, is by no means distinguished. Continuity in the dimension, together with the indications for a non-trivial fixed point in four dimensions, suggest that a non-trivial fixed point should persist, to the least, in some vicinity of four dimensions. We emphasize that this heuristic line of reasoning is solely based on local RG properties of the theory, and insensitive against global properties within specific dimensions.

In this contribution, we discuss the asymptotic safety scenario in the context of quantum gravity. Based on a wilsonian renormalisation group, we provide unique analytical fixed point solution in the Einstein-Hilbert truncation for any dimensions $d > d_{cr}$ \cite{12}. The approach is related to the integrating-out of momentum modes from a path integral representation of the theory \cite{19}, amended by an appropriate optimisation \cite{20, 21}. Consequently, the reliability of results based on optimised flows is enhanced \cite{22}. Results and implications for quantum gravity in higher dimensions are discussed.

**ASYMPTOTIC SAFETY**

We recall a few general requirements and implications of the asymptotic safety scenario and a non-trivial fixed point in quantum gravity \cite{1}. First of all, the asymptotic safety scenario relies on the existence of a (non-trivial) fixed point at short distances. This generalises a pattern observed for perturbatively renormalisable theories, which often are related to a non-interacting UV fixed point, e.g. asymptotic freedom in QCD. Secondly, it is mandatory that the short-distance fixed point is connected with the long-distance behaviour of the theory by a well-defined renormalisation group trajectory. Elsewise the putative fixed point would remain disconnected from the known physics at large distances. Finally, it is required that the UV fixed point displays at most a finite number of (infrared) unstable directions. Elsewise, the predictive power is spoiled, because an infinite number of unstable directions would require the fine-tuning of infinitely many parameters in order to reach the IR limit. Then, the fixed point together with the RG trajectory serve as a fundamental definition of the theory.

We proceed with a discussion of the renormalisation group flow for the gravitational coupling $G$ in $d$ dimensions. We introduce the renormalised dimensionless coupling as $g = \mu^{d-2}Z_G^{-1}(\mu) G$ and the graviton wave function renormalisation factor $Z_G(\mu)$, normalised as $Z_G(\mu_0) = 1$ at $\mu = \mu_0$. The momentum scale $\mu$ denotes the renormalisation scale, and can be thought of as an energy scale $E$, or momentum transfer $p$, or, more formally, as some wilsonian momentum cutoff $k$ as used below. The graviton anomalous dimension, given by $\eta = -\mu \partial_\mu \ln Z_G$, is a non-trivial function of $g$ and other couplings
parametrising the theory. Then the RG flow reads

$$\mu \partial_\mu g = (d - 2 + \eta) g.$$  \hspace{1cm} (1)

From (1) we conclude that the gravitational coupling displays a non-interacting (gaussian) fixed point $g = 0$. This entails $\eta = 0$, i.e. classical scaling. On the other hand, non-trivial RG fixed points, if they exist, correspond to the implicit solutions of

$$\eta = 2 - d.$$  \hspace{1cm} (2)

Hence, a non-trivial fixed point of quantum gravity in $d > 2$ implies a negative integer value for the graviton anomalous dimension, precisely counter-balancing the canonical dimension of $G$.

Integer values for anomalous dimensions are well-known from other gauge theories at criticality: in the $d$-dimensional abelian higgs theory, for example, the abelian charge $\epsilon^2$ has mass dimension $[\epsilon^2] = 4 - d$, whence $\beta_{\epsilon^2} = (d - 4 + \eta) \epsilon^2$. In three dimensions, the theory displays a non-perturbative infrared fixed point at $\epsilon^2 \neq 0$, leading to $\eta = 1$ [23]. The fixed point belongs to the universality class of standard superconductors with the charged scalar field describing the Cooper pair. The integer value $\eta = 1$ implies that the magnetic field penetration depth and the Cooper pair correlation length scale with the same universal critical exponent at the phase transition [23, 24]. In turn, scalar theories at criticality, not protected by a local gauge symmetry, often display non-integer anomalous dimensions characterising the universality class.

In quantum gravity, a non-trivial fixed point behaviour leads to two important implications. First of all, the dimensionful gravitational coupling constant scales as $G(\mu) \rightarrow g_*/\mu^{d-2}$ at the fixed point. In the case of an ultraviolet fixed point for diverging of $\mu$, the coupling $G$ becomes arbitrarily small. This indicates that gravity might become asymptotically free at short distances, similar to QCD. Conversely, in case of an infrared fixed point for vanishing $\mu$, the dimensionful coupling grows large. This behaviour implies non-trivial long distance modifications of gravity, e.g. [25]. Secondly, the scalar part of the renormalised graviton propagator scales $\sim p^{-2+\eta}$. Here we have identified $\mu$ with the momentum scale $p$. At an UV fixed point, this leads to an increased momentum decay $\sim p^{-d}$, which, in position space, corresponds to a logarithmic behaviour $\sim \ln(|x-y|/\mu)$ of the propagator, reminiscent from two dimensions. Hence, the integer value of (2), on the level of the graviton propagator, implies a dimensional crossover from $d$-dimensional behaviour in the perturbative regime to an effectively two-dimensional behaviour in the vicinity of a non-trivial fixed point. A similar result is found for the spectral dimension of quantum gravity space-times at short distances [26, 18].

**REnormalisation Group**

Whether or not the non-trivial fixed point is realised in quantum gravity can only be assessed by an explicit renormalisation group study. We apply a wilsonian renormalisation group, based on a momentum cutoff for the propagating degrees of freedom (for reviews,
It is based on a scale-dependent action functional $\Gamma_k$ of the mean gravitational field $\langle g_{\mu \nu} \rangle_k$, where $k$ denotes a wilsonian momentum cutoff scale. The wilsonian flow equation describes the change of $\Gamma_k$ under an infinitesimal variation of $k$. By construction, $\Gamma_k$ comprises all quantum fluctuations down to the momentum scale $q^2 \approx k^2$. The flow interpolates between some microscopic action at short distances $k \to \infty$ and the full quantum effective action at large distances $k \to 0$. Diffeomorphism invariance under local coordinate transformations is controlled by modified Ward identities [5], similar to those known for non-Abelian gauge theories [27]. In its modern formulation, the flow with respect to the logarithmic scale parameter $t = \ln k$ is given by

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)}} R_k \partial_t R_k. \tag{3}$$

Here, the trace stands for a momentum integration and a sum over indices and fields, and $R_k$ denotes an appropriate wilsonian momentum cutoff at momentum scale $q^2 \approx k^2$. For quantum gravity, we consider the flow (3) for $\Gamma_k$ in the Einstein-Hilbert truncation

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^d x \sqrt{g} \left[ -R(g) + 2\lambda_k \right] + \text{classical gauge fixing}, \tag{4}$$

retaining the volume element and the Ricci scalar as independent operators. In (4) we have introduced the gravitational coupling constant $G_k$ and the cosmological constant $\lambda_k$. The Ansatz (4) differs from the standard Einstein-Hilbert action in Euclidean dimensions by the fact that the gravitational coupling and the cosmological constant have turned into running couplings. We introduce dimensionless renormalised gravitational and cosmological constants $g_k$ and $\lambda_k$ as

$$g_k = k^{d-2} G_k \equiv k^{d-2} Z_{N,k}^{-1} \tilde{G}, \quad \lambda_k = k^{-2} \lambda_k. \tag{5}$$

and $Z_{N,k}$ denotes the wave function renormalisation factor for the Newtonian coupling. Their flows follow from (3) by an appropriate projection onto the operators in (4). For explicit constructions of a momentum cutoff in the background field gauge with gauge fixing parameter $\alpha$, see [5, 7]. Closely related flows have also been derived in [13] within a proper-time approximation [28]. Here, we use the approach of [5, 7] for the tensorial structure of the momentum cutoff in addition with an optimised momentum cutoff [21] for its scalar part. This leads to an analytical flow equation

$$\partial_t \lambda \equiv \beta_\lambda = \frac{P_1}{P_2 + 4(d+2)g} \quad \partial_t g \equiv \beta_g = \frac{(d-2)g P_2}{P_2 + 4(d+2)g}, \tag{6}$$

$$P_1 = -16\lambda^3 + 4\lambda^2 (4 - 10d \lambda - 3d^2 \lambda + d^3 \lambda) + 4\lambda (10d \lambda + d^2 \lambda - d^3 \lambda - 1) + d(2 + d)(d - 16 \lambda + 8d \lambda - 3) \lambda \tag{7}$$

$$P_2 = 8(\lambda^2 - \lambda - d \lambda) + 2. \tag{8}$$

For convenience, a factor $1/\alpha$ is absorbed into the definition (5), and a factor $c_d = (4\pi)^{d/2-1} \Gamma(d/2 + 2)$ is absorbed into the definition of $g$. Furthermore, we have performed the limit $\alpha \to \infty$ in (6), where the results take their simplest form. The flow for
arbitrary $\alpha$ is given in [15]. The flow of $g$ vanishes identically in two dimensions. For the anomalous dimension, we find

$$\eta = \frac{(d+2)g}{g - g_{\text{bound}}}, \quad g_{\text{bound}}(\lambda) = \frac{(1 - 2\lambda)^2}{2(d-2)}.$$  \hfill (9)

The anomalous dimension diverges at $g = g_{\text{bound}}(\lambda)$, which limits the domain of validity. For all real $\lambda$ and $d > 2$, we have $g_{\text{bound}} \geq 0$. The anomalous dimension vanishes for $g = 0$, and in $d = 2$ dimensions.

**ANALYTICAL FIXED POINTS**

Next we identify the non-trivial fixed points of $\beta_g(\lambda_s, g_s) = 0 = \beta_\lambda(\lambda_s, g_s)$. From $P_2 = 0$, [8], we deduce that $g_s = g_{\text{bound}}(\lambda_s) \times (d-2)/(2d)$. With (9), we conclude that the gravitational coupling fixed point is positive for any real fixed point $\lambda_s$. Inserting this result into (7), equating $P_1 = 0$, and factoring-out a common factor $(1 - 2\lambda)^2$ leads to a simple quadratic equation with two real solutions, as long as $d \geq (1 + \sqrt{17})/2 \approx 2.56$.

The physical fixed point obeys $\lambda < \frac{1}{2}$ and reads

$$\lambda_s = \frac{d^2 - d - 4 - \sqrt{2d(d^2 - d - 4)}}{2(d-4)(d+1)}, \quad g_s = \frac{\Gamma(d/2 + 2) (\sqrt{d^2 - d - 4} - \sqrt{2d})^2}{(4\pi)^{1-d/2} 2(d-4)^2(d+1)^2}. \hfill (10)$$

Here, we have reinserted the factor $c_d$ in $g$. The solution (10) is continuous and well-defined for all $d \geq 2.56$ and becomes complex for lower dimensions. For general $\alpha < \infty$, the fixed point extends down to $d = 2$.

Fixed points are non-universal quantities and may depend on unphysical parameters. In turn, the rates at which small perturbations about the fixed point grow with scale are universal. These are denoted as $-\theta$, or $-1/\nu$ in the statistical physics literature, and given by the eigenvalues of the stability matrix at criticality. In four dimensions, and reinserting $c_d$, we find $\theta = \theta' + i\theta''$, with

$$\lambda_s = \frac{1}{4}, \quad g_s = \frac{3\pi}{8}, \quad \theta' = \frac{5}{3}, \quad \theta'' = \frac{\sqrt{167}}{3}, \quad |\theta| = \frac{8}{\sqrt{3}}. \hfill (11)$$

This compares well with the lattice result $\nu \approx 1/3$ of [17]. In Fig. 1, the trajectory connecting the UV fixed point with the gaussian one is given, together with the running couplings along the separatrix. The complex eigenvalue is reflected by a rotating trajectory at the UV fixed point. It originates from a strong mixing of the scaling of the volume operator and the Ricci scalar. Along the separatrix, the anomalous dimension displays a cross-over from a perturbatively small value towards the non-trivial fixed point value (2). The fixed point and the flow pattern persists for arbitrary gauge fixing parameter.
The fixed point (10), and the corresponding solution for arbitrary gauge fixing parameter (12), is found independently of the dimension. The fixed point is unique, real, positive, and continuously connected to the perturbatively known fixed point in two dimensions. This is quite remarkable, also in view of the fact that the Einstein-Hilbert truncation is expected to be more sensitive to higher dimensional operators in higher than in lower dimensions. For a detailed study of the general cutoff and gauge fixing independence of this fixed point, see [16].

Here, we point out three noteworthy aspects of the fixed point for general dimension. For more details, see [15]. First of all, we notice a non-trivial bifurcation in the eigenvalue spectrum, as a function of dimension. For $2 < d_{\text{low}} < d < d_{\text{up}} < \infty$, the universal eigenvalues of the stability matrix are complex, and real otherwise (see Tab. 1 for numerical values). Complex eigenvalues indicate that the operators $\sqrt{g}$ and $\sqrt{g}R$ scale with a similar strength, and that they are subject to strong mixing effects, parametrised by the off-diagonal elements of the Jacobi matrix. This is the case in four dimensions. In turn, real eigenvalues indicate that the scaling behaviour at the fixed point is dominated by the volume element, while the Ricci scalar remains subleading. This behaviour is well-known in the vicinity of two dimensions. The new result is that a similar behaviour persists for sufficiently large dimension. The large dimensional limit, in this light, shares an important similarity with the limit of small dimensions. If this structure persists in the full theory, we expect that an expansion about two dimensions (in inverse dimension) has a finite radius of convergence, given by the lower (upper) critical dimension.

Secondly, it is possible to perform the large dimensional limit. For any $\alpha \in [0, 1]$ (and similarly for other $\alpha$), one finds the universal eigenvalues $\theta_1 = d^3/156$ and $\theta_2 = 24d/13$, related to the scaling of the $\sqrt{g}$ and $\sqrt{g}R$ operator, respectively. The index $\nu =


TABLE 1. Lower and upper critical dimensions as functions of the gauge fixing parameter. The data is obtained using a wilsonian momentum cutoff with tensorial momentum structure as in [8] (for $\alpha = 1_A$) or [7] (for general $\alpha \geq 0$), and a scalar momentum structure as in [21].

| $\alpha$ | 0    | $\frac{1}{2}$ | 1    | $1_A$ | 2    | $\infty$ |
|---------|------|---------------|------|-------|------|---------|
| $d_{\text{low}}$ | 2.900 | 2.901 | 2.872 | 2.756 | 2.765 | 2.562 |
| $d_{\text{up}}$  | 23.727 | 23.672 | 24.282 | 23.985 | 17.394 | 21.381 |

$1/\theta_2$ agrees very well with the result $\nu = 1/(2d)$ as obtained previously for vanishing cosmological constant $\lambda = 0$ [12]. This should also be compared with $\nu = 1/(d - 1)$ based on geometrical considerations [29], with $\nu \approx 1.9/d$ based on an extrapolation of low dimensional lattice results [30], and $\nu = 1/(d - 2)$ as obtained within a perturbative expansion about two dimensions [2, 3].

Finally, a non-trivial fixed point of quantum gravity in higher dimensions is of immediate interest for phenomenological applications. It has been suggested that the fundamental Planck scale may be as low as the electroweak scale, if gravity propagates in a $d = 4 + n$ dimensional “bulk” with $n$ “extra” spatial dimensions, while standard model particles only propagate in a four-dimensional “brane” [31, 32]. Then high-energy particle physics experiments at LHC are potentially sensitive to the fundamental Planck scale and quantum gravitational effects. Up to now, phenomenological effects due to higher dimensional gravity have been studied using classical propagators and vertices, amended by an ultraviolet cutoff, e.g. [33]. In turn, a fixed point scenario as described here is ultraviolet finite and would not require an additional UV regularisation. It will be interesting to identify physical observables most strongly sensitive to the above scenario.

CONCLUSIONS

We have discussed basic implications of an asymptotic safety scenario for quantum gravity. It has been emphasized that the case of four space-time dimensions, from a renormalisation group point of view, is not distinguished. This is reflected by the non-trivial fixed point structure in the Einstein Hilbert theory, which displays an UV fixed point for arbitrary dimension, e.g. [10]. Maximal reliability in the present truncation is guaranteed by the underlying optimisation. It was pointed out that the large dimensional limit – in view of its scaling properties at criticality – shares qualitative similarities with the low dimensional limit, the vicinity of $d = 2$. In contrast, in the neighbourhood of four dimensions, scaling at criticality is characterised by strong operator mixing. This has lead to an interesting bifurcation pattern in the eigenvalue spectrum.

In four dimensions and below, our results are fully consistent with previous renormalisation group studies. The fixed point is remarkably stable, with only a mild dependence on the gauge fixing parameter. The phase diagram is equally robust. Furthermore, it is encouraging that the qualitative picture achieved so far is backed-up by lattice simulations. The fixed point consistently extends to higher dimensions, a region which previously has not been accessible. Hence it is likely that the fixed point exists in the full
theory. We expect that the analytical form of the flow, crucial for the present analysis, is equally useful in extended truncations. If the above picture persists in these cases, quantum gravity exists as a well-defined local quantum field theory in the metric field down to arbitrarily short distances.

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REFERENCES

1. S. Weinberg, in General Relativity: An Einstein centenary survey, Eds. S.W. Hawking and W. Israel, Cambridge University Press (1979), p. 790.
2. R. Gastmans, R. Kallosh and C. Truffin, Nucl. Phys. B 133 (1978) 417.
3. S. M. Christensen and M. J. Duff, Phys. Lett. B 79 (1978) 213.
4. T. Aida and Y. Kitazawa, Nucl. Phys. B 491 (1997) 427 [hep-th/9609077].
5. M. Reuter, Phys. Rev. D 57 (1998) 971 [hep-th/9605030].
6. W. Souma, Prog. Theor. Phys. 102 (1999) 181 [hep-th/9907027].
7. O. Lauscher and M. Reuter, Phys. Rev. D 65 (2002) 025013 [hep-th/0108040].
8. O. Lauscher and M. Reuter, Class. Quant. Grav. 19 (2002) 483 [hep-th/0110021].
9. M. Reuter and F. Saueressig, Phys. Rev. D 66 (2002) 065016 [hep-th/0110054].
10. R. Percacci and D. Perini, Phys. Rev. D 67 (2003) 081503 [hep-th/0207033].
11. P. Forgacs and M. Niedermaier, JHEP 0212 (2002) 066 [hep-th/0207143].
12. D. F. Litim, Phys. Rev. Lett. 92 (2004) 201301 [hep-th/0312114].
13. A. Bonanno and M. Reuter, JHEP 0502 (2005) 035 [hep-th/0410191].
14. R. Percacci, hep-th/0511177.
15. D. F. Litim, to appear.
16. P. Fischer, D. F. Litim, hep-th/0602203, Physics Letters B (in print).
17. H. W. Hamber, Phys. Rev. D 61 (2000) 124008, [hep-th/9912246].
18. J. Ambjorn, J. Jurkiewicz and R. Loll, Phys. Rev. Lett. 95 (2005) 171301 [hep-th/0505113].
19. J. Berges, N. Tetradis and C. Wetterich, Phys. Rept. 363 (2002) 223 [hep-ph/0005122].
20. D. F. Litim and J. M. Pawlowski, hep-th/9901063.
21. D. F. Litim, Phys. Lett. B 486 (2000) 92 [hep-th/0005245].
22. D. F. Litim, Phys. Rev. D 64 (2001) 105007 [hep-th/0103195].
23. D. F. Litim, Int. J. Mod. Phys. A 16 (2001) 208 [hep-th/0104221].
24. I. F. Herbut and Z. Tesanovic, Phys. Rev. Lett. 76 (1996) 4588 [cond-mat/9605185].
25. E. Bentivegna, A. Bonanno and M. Reuter, JCAP 0401 (2004) 001 [astro-ph/0303150].
26. O. Lauscher and M. Reuter, JHEP 0510 (2005) 050 [hep-th/0508202].
27. M. Reuter and C. Wetterich, Phys. Rev. D 56 (1997) 7893 [hep-th/9708051].
28. F. Freire, D. F. Litim and J. M. Pawlowski, Phys. Lett. B 495 (2000) 256 [hep-th/0009110].
28. D. F. Litim and J. M. Pawlowski, Phys. Lett. B 516 (2001) 197 [hep-th/0107020], Phys. Rev. D 65 (2002) 081701 [hep-th/0111119], Phys. Rev. D 66 (2002) 025030 [hep-th/0202188].
29. H. W. Hamber and R. M. Williams, Phys. Rev. D 70 (2004) 124007 [hep-th/0407039].
30. H. W. Hamber and R. M. Williams, hep-th/0512003.
31. N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 429 (1998) 263 [hep-ph/9803315].
32. I. Antoniadis, Phys. Lett. B 246 (1990) 377.
33. G. F. Giudice, R. Rattazzi and J. D. Wells, Nucl. Phys. B 544 (1999) 3 [hep-ph/9811291].
       G. F. Giudice, T. Plehn and A. Strumia, Nucl. Phys. B 706 (2005) 455 [hep-ph/0408320].