The normal phase of an imbalanced Fermi gas

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(Dated: June 15, 2010)

Recent experiments on imbalanced Fermi gases have raised interest in the physics of an impurity immersed in a Fermi sea, the so-called Fermi polaron. In this letter, a simple theory is devised to describe dilute Fermi-polaron ensembles corresponding to the normal phase of an imbalanced Fermi gas. An exact formula is obtained for the dominant interaction between polarons, expressed solely in terms of a single polaron parameter. The physics of this interaction is identified as a signature of the Pauli exclusion principle.

PACS numbers: 03.75.Ss, 05.30.Fk, 32.80.Pj, 34.50.-s

Quasi-particles are generic emergent properties of many-body systems that simplify the description of complex interacting ensembles of particles. This concept is probably one of the most important in quantum physics since it lies at the foundation of fields as diverse as chemistry or band theory in solid state physics. Recently, experiments on spin imbalanced Fermi gases have highlighted once again its importance by showing that the main features of the phase diagram of these systems could be understood quantitatively from the properties of an impurity immersed in a Fermi sea of polarized atoms, the Fermi polaron \cite{4, 5, 9}. It was shown in particular that the quasi-particle arising from the interaction between the impurity and the surrounding Fermi gas could be described with great accuracy by assuming that a single particle-hole pair is excited \cite{4, 5}. The single-particle properties of the Fermi-polaron have been characterized experimentally and theoretically and are now well understood. For instance, at unitarity, where the scattering length between the majority and minority spins is infinite, the chemical potential of the majority \cite{4, 5, 9, 10}. Similarly, the effective mass is found to be close to the bare mass $m$, with $m^* = 1.20 m$ for recent experiment \cite{4, 11, 12}, close to the theoretical values obtained from variational or Monte-Carlo calculations \cite{4, 5, 8, 9}.

More generally, the Fermi-polaron is a good description of an impurity immersed in a Fermi sea around unitarity and for “attractive” ($a < 0$) interactions where $A$ and $m^*$ have also been calculated with great accuracy \cite{4, 8, 13}. Interestingly, the dressed impurity undergoes a transition from a fermionic polaron to a bosonic molecule at $1/k_{F1} a \sim 0.9$ \cite{4, 10, 14, 17}, where $k_{F1} = (6\pi^2 n_1)^{1/3}$ is the Fermi wave-vector of the majority species gas of density $n_1$. This transition reflects into the collective behavior of an ensemble of impurities. In particular, in the fermionic sector $1/k_{F1} a < 0.9$, pioneering Fixed Node Monte-Carlo simulations have shown that, for a small concentration of minority fermions, the equation of state of an imbalanced normal Fermi gas with two spin species noted $\sigma = 1, 2$ and densities $n_\sigma$ could be fitted by a Landau-Pomeranchuk law

$$E = E_{FG1} \left(1 + \frac{5A}{3} x + \frac{m}{m^*} x^{5/3} + F x^2\right), \quad (1)$$

where $x = n_2/n_1$, $E_{FG1}$ is the energy of a single-component (majority) Fermi gas with density $n_1$ and $F$ describes interactions between polarons \cite{4, 18}. $A$, $m^*$ and $F$ are functions of $1/k_{F1} a$. This Fermi liquid picture is supported by the absence of vortices in rotation experiments indicating a normal state \cite{4}. By contrast, it was noted recently that experimental data could be fitted with great accuracy by a grand-canonical equation of state

$$P = \frac{1}{15\pi^2} \left[ \left(\frac{2m}{\hbar^2}\right)^{3/2} \mu_1^{5/2} + \left(\frac{2m^*}{\hbar^2}\right)^{3/2} (\mu_2 - \mu_p)^{5/2} \right], \quad (2)$$

which apparently describes a mixture of two ideal Fermi gas of polarons and majority atoms \cite{4, 19}. However, the presence of a $\mu_1$ dependence of $\mu_p$ in the polaron part of the equation of state implies a coupling between the two gases, and the two equations of state can be reconciled by noting that expressed in the canonical ensemble, Eq. (2) indeed yields Eq. (1) with $F = 5A^2/9 \sim 0.2$ at unitarity, close to the Monte Carlo value $F \sim 0.14$ \cite{4}.

In this letter, we show that the equation of state of the normal phase follows the phenomenological expansion (1). Moreover, we argue that the relationship between $F$ and $A$ is exact and can be generalized to the full BEC-BCS crossover: indeed, we will show that that the parameter $F$ is solely a function of the single polaron chemical potential and is given by

$$F = \frac{5}{9} \left(\frac{d\mu_p}{dE_{FG1}}\right)^2, \quad (3)$$
where \( \mu_p \) is computed in the low impurity concentration limit where \( \mu_1 = E_{F1} = \hbar^2 k_{F1}^2 / 2m \). Finally, the study of the BCS (Bardeen-Cooper-Schrieffer) regime corresponding to small and negative value of \( a \) allows us to clarify the origin of the \( x^2 \) term in Eq. (1). We attribute it to a modification of the single-polaron properties due the Pauli blocking created by the presence of the minority Fermi sea and overruling density-mediated polaronic interactions 20 which contribute to the higher-order \( x^{7/3} \).

The starting point of our demonstration is the celebrated Luttinger sum-rule, stating that if a many-body fermionic system can be analytically connected to an ideal Fermi gas 27, then it possesses a Fermi surface where the momentum distribution is discontinuous and which encloses a volume depending on density only 21, 22. More quantitatively, the Fermi surface is the usual definition for the effective mass of a quasi-particle is the Fermi sea and overruling density-mediated polaronic interactions 19, 20 which contribute to the higher-order \( x^{7/3} \). To convert this equation of state in the canonical ensemble, we use the relationship

\[
\mu_1 = E_{F1} \left( 1 + x \frac{d\mu_p}{d\mu_1} \right)^{2/3} \\
\mu_2 = \mu_p + E_{F2}
\]  

(9) (10)

where \( E_{F2} = \hbar^2 k_{F2}^2 / 2m^* \) and we have neglected higher order terms in \( n_2 \) appearing when taking the derivative of \( m^* \) with \( \mu_1 \). Making use of the definition of the grand potential \( -PV = E - \sum_i \mu_i N_i \) we finally get Landau-Pomeranchuk law 11 with \( F \) given by 13.

We now verify that Eq. (3) is not altered when higher orders in Eq. 6 are included. Indeed, replacing \( \delta \mu_2 \) by its leading order expression, terms neglected in Eq. 6 give rise to a \( k_{F2}^2 \) contribution to \( \delta \mu_2(k_{F2}) \). From Gibbs-Duhem relation this gives rise to a term \( \propto (\mu_2 - \mu_p)^{7/2} \) in Eq. 2, hence a \( x^{7/3} \) contribution to the energy. For vanishing \( x \), this term is therefore negligible against \( x^2 \) and does not contribute to the value of \( F \): this argument proves that, provided analyticity conditions are fulfilled, Eq. (3) gives the exact value of \( F \).

We now provide evidence for the analyticity of \( \Sigma_2 \). To do so, we make use of a time ordered diagrammatic expansion of the self-energy illustrated in Fig. 1. 24 For each diagram, a line going forward (backward) in time is associated with \( \theta(\xi_{k,\sigma}) \) (\( \theta(-\xi_{k,\sigma}) \)), with \( \theta \) the Heaviside step function, and contributes to \( \xi_{k,\sigma} \) to the energy denominator. In addition, the incoming (outgoing) minority line contributes to \( \omega \) (\( -\omega \)). The main point of the argument is the negativity of \( \mu_p \), and of \( \mu_2 \) for small impurity concentration. Indeed, in this case, \( \xi_{k,\sigma} \) is always positive, which implies that the Heaviside functions associated with impurities traveling backward in time vanish. As a consequence, diagrams containing an impurity loop do not contribute to the self energy, and similarly, the inner part of the 'main' impurity line cannot travel back in time. The denominators are therefore always strictly positive and this absence of pole guarantees the analyticity of \( \Sigma_2 \). This can be interpreted physically by noting that the minority Fermi sea would be empty at these negative chemical potentials for vanishing interaction. The creation of minority fermions is therefore only triggered by interaction processes with the majority component.
The above ideas are best illustrated by going to the BCS weak coupling limit $a \rightarrow 0^-$, where exact perturbative calculations can be performed. The gas of fermions with two spin-species is described by the Hamiltonian

$$H = \sum_{k, \sigma} \varepsilon_k c_{k, \sigma}^\dagger c_{k, \sigma} + \frac{\hbar^2}{2m} \sum_{k, k', q} \varepsilon_{k+q} v_{k' - q, 2} c_{k', 2} c_{k, 1},$$

where $\varepsilon_k = \hbar^2 k^2 / 2m$, $V$ is a quantization volume, and $c_{k, \sigma}$ annihilates a fermion of spin $\sigma$ and momentum $k$.

The zero-range interaction potential in Eq. (11) suffers from ultraviolet divergences that are cured by imposing a cutoff $k_c$ in momentum space. The Lippmann-Schwinger formula then relates the bare coupling constant $g$ to the scattering length,

$$\frac{1}{g} = \frac{m}{4\pi\hbar^2 a} - \frac{1}{V} \sum_{k} \frac{2\varepsilon_k}{2\varepsilon_k},$$

Building on the Luttinger equation (4) relating $\mu_2$ and $k_F^2$ for the minority fermions, we wish to determine the equation of state $P(\mu_1)$ in the strongly imbalanced case with $\mu_1 > 0$ and $\mu_2 < 0$. The self-energy $\Sigma_2$ is calculated perturbatively in powers of $g$. In addition, Eq. (12) is used to expand the resulting expressions again in powers of $a$. The renormalizability of the model (14) imposes that ultraviolet divergences cancel out for each order in $a$, and the cutoff $k_c$ is eventually taken to infinity.

The first order is given by the usual Hartree diagram, $\Sigma_2^{(1)}(\omega, q) = (g/6\pi^2)(2m\mu_1/\hbar^2)^{3/2}$. We write the second order using the time ordered diagrams displayed in Fig. 1.

$$\Sigma_2^{(2)}(\omega, q) = \frac{g^2}{V^2} \sum_{k, k', q} \frac{\theta(\varepsilon_k)\theta(\varepsilon_{k+q} - \varepsilon_{k+q'} - \varepsilon_{k'} - \varepsilon_{k+q} - \varepsilon_{k+q'} - \varepsilon_{k+q'} - \varepsilon_{k+q} - \varepsilon_{k+q'})}{\omega - \varepsilon_{k+q} - \varepsilon_{k+q'} - \varepsilon_{k+q} - \varepsilon_{k+q'} - \varepsilon_{k+q} - \varepsilon_{k+q} - \varepsilon_{k+q'}} + \frac{g^2}{V^2} \sum_{k, k', q} \frac{\theta(-\varepsilon_k)\theta(-\varepsilon_{k+q'} - \varepsilon_{k+q} - \varepsilon_{k+q'} - \varepsilon_{k+q} - \varepsilon_{k+q'} - \varepsilon_{k+q} - \varepsilon_{k+q} - \varepsilon_{k+q})}{\omega - \varepsilon_{k+q} - \varepsilon_{k+q'} - \varepsilon_{k+q} - \varepsilon_{k+q'} - \varepsilon_{k+q} - \varepsilon_{k+q} - \varepsilon_{k+q} - \varepsilon_{k+q}}.$$ (13)

where the minority travels partially backward in time in the second term and always forward in the first one. As stated earlier, the negative minority chemical potential implies that $\varepsilon_{q+k} - \varepsilon_{k}$ is the second positive term of Eq. (13) thus vanishes in accordance with our general rule that backward travel is suppressed. Moreover, for the remaining first term in Eq. (13), the denominator does not vanish as long as $\omega < -\mu_2$, and the self-energy can be freely expanded with respect to $\mu_2$ and $q$ at $\omega = 0$.

Using the complete self-energy $\Sigma_2^{(1)} + \Sigma_2^{(2)}$, it is possible to calculate $\mu_2$ with the result

$$\mu_2 = \frac{2a}{\pi \hbar m} (2m\mu_1)^{3/2} + \frac{a^2}{\pi \hbar^2 m} (2m\mu_1)^2.$$ (14)

Using Eq. (4), we see that up to 3rd order included, the interaction parameter $F$ should read

$$F = \frac{20}{9} \left( \frac{k_{F1} a}{\pi} \right)^2 \left( 1 + \frac{k_{F1} a}{\pi} \right) + \ldots.$$ (15)

It is illuminating to check the weak coupling prediction (10) for the interaction by a direct calculation of the ground state energy using the standard Rayleigh-Schrödinger perturbation theory. We first discuss the energy of a single polaron $E_{pol}(q)$. The unperturbed state is then an impurity with momentum $q$ immersed in a Fermi sea of majority atoms. The first order correction to the energy is the mean-field correction $\mu_{11}$, while the next order correction involves the excitation of particle-hole pairs out of the majority Fermi sea. By definition of $E_{pol}(q)$ the energy of the system is given by

$$E_{pol}(q) = \frac{\hbar^2 q^2}{2m} + g_{n_1}^2 + \frac{g^2}{V^2} \sum_{q', q} \frac{1}{\varepsilon_{q' + \varepsilon_{q} - \varepsilon_{q'} - \varepsilon_{q}}},$$ (16)

where the majority momenta $q'$ and $k'$ satisfy the conditions $q < k_{F1}$ (i) and $k > k_{F1}$ (ii) imposed by Pauli exclusion principle.

We switch now to an ensemble of impurities, in which case two ideal Fermi gases with Fermi wavevectors $k_{F1}$ and $k_{F2}$ constitute the unperturbed ground state with energy $E_{FG,1} + E_{FG,2}$. The energy takes the form $E_{n_1, n_2} = E_{FG,1} + E_{pol}(q)$.

$$\bar{E} = E_{FG,1} + \Sigma_2 \frac{1}{n_2} + \frac{g^2}{V^2} \sum_{q', q} \frac{1}{\varepsilon_{q' + \varepsilon_{q} - \varepsilon_{q'} - \varepsilon_{q}}},$$ (17)

with the previous restrictions (i), (ii), complemented by $q < k_{F2}$ (iii), and $|q + q' - k'| > k_{F2}$ (iv), where the last two conditions are imposed by the Pauli exclusion principle in the presence of the minority Fermi seas. Except for the constraint (iv) $\bar{E}$ would simply be $\sum_{q < k_{F2}} E_{pol}(q)$ which constitute the energy of an ideal gas of polarons with a dispersion relation $E_{pol}(q)$. However, we can recover this term explicitly by expressing (iv) in terms of its complementary domain (v) $|q + q' - k| < k_{F2}$, in which case we can recast Eq. (17) as

$$\bar{E} = \sum_{q < k_{F2}} E_{pol}(q) - \frac{g^2}{V^2} \sum_{(i), (ii)} \frac{1}{\varepsilon_{q' + \varepsilon_{q} - \varepsilon_{q'} - \varepsilon_{q}}},$$ (18)

The first term in Eq. (18) corresponds to an ideal gas of polarons and contributes to the $x$ and $x^{5/3}$ scaling terms in Eq. (1), that is to $A$ and $m^*$. The second term describes the effect of Pauli blocking due to the minority Fermi sea on the formation of the polaron. A careful analysis of its behavior for low $k_{F2}$ shows that it scales as $x^2$ and thus gives the effective interaction $F$ between polarons.

The complete calculation of third order corrections is lengthy but straightforward. In the limit $x \ll 1$, one finds again Eq. (1) for the ground state energy together with an interaction parameter $F$ arising again from Pauli blocking and identical to Eq. (10).
The argument presented above makes a strong case for a $x^2$ interaction between polarons. However, noticing that s-wave interactions give a $x^2$ scaling and p-wave a subleading $x^{7/3}$, this may seem to contradict the fermionic nature of polarons. On the other hand, Fermi liquid theory does not forbid alike particles to interact, and the corresponding interaction is in fact not necessarily short-ranged. This paradox can be solved by noting that polarons have fermionic statistics at large distances and are composite objects at shorter distances. From this structure, they acquire an internal energy $\mu_p = A \mu_1$. This single polaron energy is held fixed in the grandcanonical ensemble and is not modified by the presence of other impurities. By contrast the internal energy depends on the minority concentration in the canonical ensemble through Pauli blocking which yields the $x^2$ interaction in Eq. (11). Based on these arguments, it is probably not surprising to find that $F$ is solely a function of the internal energy as given by Eq. (6). 

Finally, in Fig. 2 we compare our prediction Eq. (3) where $\mu_p$ is calculated using the variational scheme presented in [8] with the third order expansion Eq. (15) as well as Monte-Carlo data [18]. As expected, we observe that the perturbative expansion and the non perturbative result coincide for $a \to 0^-$. In the strongly interacting limit we observe that our result follows the same trend as the Monte-Carlo simulation, with in particular the presence of a maximum of $F$ close to $1/k_{F1} a \sim 0.5$.

In conclusion, we have demonstrated that in the low impurity concentration, the canonical equation of state of a spin imbalanced system could be described by a Landau Pomeranchuk energy. Quite surprisingly, we have shown that the interaction parameter $F$ was related to single impurity properties. Several extensions of this letter are worth exploring. From experimental data, it appears that Eq. (2) is valid on a wide range of impurity concentrations (up to $x = 0.5$ at unitarity). This surprisingly large validity domain remains to be understood by investigating higher orders or by making use of non-perturbative schemes. In fact, assuming further analyticity, the low density expansion performed here can in principle be extended to any order in $x$. The coefficients of the expansion are then expressed solely in terms of the single-polaron self-energy. Other open questions include the extension of our results to the one-dimensional situation [24, 25] and to the case of repulsive interactions [20].

We acknowledge R. Combescot, S. Giraud, S. Giorgini, C. Lobo, S. Nascimbène, N. Navon, A. Recati for stimulating discussions and we thank G. Bertain for providing us with the Monte-Carlo data. FC acknowledges support from EU (ERC Research grant FERLODIM), Région Ile de France (IFRAF) and Institut Universitaire de France.

\begin{thebibliography}{99}
\bibitem{1} M. Zwierlein, A. Schirotzek, C. Schunck, and W. Ketterle, Science \textbf{311}, 492 (2006).
\bibitem{2} G. Partridge, W. Li, R. Kamar, Y. Liao, and R. Hulet, Science \textbf{311}, 503 (2006).
\bibitem{3} S. Nascimbène, N. Navon, K. Jiang, F. Chevy, and C. Salomon, Nature \textbf{463}, 1057 (2010).
\bibitem{4} C. Lobo, A. Recati, S. Giorgini, and S. Stringari, Phys. Rev. Lett. \textbf{97}, 200403 (2006).
\bibitem{5} F. Chevy, Phys. Rev. A \textbf{74}, 063628 (2006).
\bibitem{6} N. Prokof’ev and B. Svistunov, Phys. Rev. B \textbf{77}, 020408 (2008).
\bibitem{7} A. Bulgac and M. McNeil Forbes, Phys. Rev. A \textbf{75} (2007).
\bibitem{8} R. Combescot, A. Recati, C. Lobo, and F. Chevy, Phys. Rev. Lett. \textbf{98}, 180402 (2007).
\bibitem{9} R. Combescot and S. Giraud, Phys. Rev. Lett. \textbf{101}, 050404 (2008).
\bibitem{10} A. Schirotzek, C.-H. Wu, A. Sommer, and M. W. Zwierlein, Phys. Rev. Lett. \textbf{102}, 230402 (2009).
\bibitem{11} Y. Shin, Phys. Rev. A \textbf{77}, 041603 (2008).
\bibitem{12} S. Nascimbène, N. Navon, K. Jiang, L. Tarruell, M. Teichmann, J. Mckeever, F. Chevy, and C. Salomon, Phys. Rev. Lett. \textbf{103}, 170402 (2009).
\bibitem{13} S. Pilati and S. Giorgini, Phys. Rev. Lett. \textbf{100}, 030401 (2008).
\bibitem{14} M. Punk, P. Dumitrescu, and W. Zwerger, Phys. Rev. A \textbf{80}, 053605 (2009).
\bibitem{15} C. Mora and F. Chevy, Phys. Rev. A \textbf{80}, 033607 (2009).
\bibitem{16} R. Combescot, S. Giraud, and X. Leyronas, EuroPhys. Lett. \textbf{88}, 60007 (2010).
\bibitem{17} F. Alzetto and X. Leyronas, Phys. Rev. A \textbf{81}, 043604 (2010).
\bibitem{18} G. Bertain for and S. Giorgini, Phys. Rev. A \textbf{79}, 013616 (2009).
\bibitem{19} N. Navon, S. Nascimbène, F. Chevy, and C. Salomon, Science Express, Published Online April 15 (2010).
\bibitem{20} L. Viverit, C. J. Pethick, and H. Smith, Phys. Rev. A \textbf{61}, 053605 (2000).
\bibitem{21} J. Luttinger and J. Ward, Phys. Rev. \textbf{118}, 1417 (1960).
\bibitem{22} S. Sachdev and K. Yang, Phys. Rev. B \textbf{73}, 174504 (2006).
\bibitem{23} J. Luttinger, Phys. Rev. \textbf{121}, 942 (1961).
\bibitem{24} G. Orso, Phys. Rev. Lett. \textbf{98}, 070402 (2007).
\bibitem{25} H. Hu, X.-J. Liu, and P. D. Drummond, Phys. Rev. Lett.
[26] S. Pilati, G. Bertaina, S. Giorgini, and M. Troyer, Arxiv preprint arXiv:1004.1169 (2010).

[27] We assume not too small temperature such that pairing and its non-analyticities can be neglected.

[28] By analogy with the ideal Fermi gas, and anticipating the result of the letter, we make the assumption that 
\[ \mu_2 - \mu_p \propto k_F^2. \]