Gravitational instability and star formation in disk galaxies

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ABSTRACT

We present a general star formation law where star formation rate depends upon efficiency $\alpha$, timescale $\tau$ of star formation, gas component $\sigma_g$ of surface mass density and a real exponent $n$. A given exponent $n$ determines $\tau$ which however yields the corresponding star formation rate. Current nominal Schmidt exponent $n_s$ for our model is $2 < n_s < 3$. Based on a gravitational instability parameter $Q_A$ and another dimensionless parameter $f_p \equiv (P/G\sigma_c^2)^{1/2}$, where $P =$ pressure, $\sigma_c =$ column density of molecular clouds, we suggest a general equation for star formation rate which depends upon relative competence of the two parameters for various physical circumstances. We find that $Q_A$ emerges to be a better parameter for star formation scenario than Toomre Q-parameter. Star formation rate in the solar neighbourhood is found to be in good agreement with values inferred from previous studies. Under closed box approximation model, we obtain a relation between metallicity of gas and the efficiency of star formation. Our model calculations of metallicity in the solar neighbourhood agree with earlier estimates. We conclude that metallicity dispersion for stars of same age may result due to a change in efficiency through which different sample stars were processed. For no significant change of metallicity with age, we suggest that all sample stars were born with almost similar efficiency.

Key words: galaxies: general – galaxies: ISM – galaxies: evolution – stars: formation

1 INTRODUCTION

It was realized by Kenicutt (1989) that there occurs non-linear increase in the star formation rate near the threshold surface density corresponding to $Q$-parameter. This shows in fact why rate of star formation $R$ is large in the spiral arms of several galaxies. For example, in M 51 and NGC 6946 (Rydbeck, Hjalmarson & Rydbeck 1985; Lord 1987; Tacconi-Garman 1988), gas densities in the spiral arms are larger by a factor of two indicating deviations in the usual power law exponent ($n > 2$) of Schmidt (1959, 1963). In fact, star formation in many spiral galaxies under extreme conditions of low gas density and low disk self–gravity present a challenge to all current theories for disk star formation (Ferguson et al. 1996). New star formation laws have therefore been proposed (see e.g., Talbot & Arnett 1975; Dopita 1985; Wyse 1986; Silk 1987; Köppen & Frölich 1997).

However, a global star formation law has been put to doubt (Hunter & Gallagher 1986). For a general star formation scenario one may refer to Zinnecker (1984); Zinnecker (1989 and references therein). Thus, many interesting variants on the simple star formation laws include e.g. self-propagating star formation (stochastic) in Gerola & Seiden (1978); Seiden & Gerola (1982); Dopita (1985); Coziol (1996); self-propagating star formation in Arimoto (1989) and Hensler & Burkert (1990a, 1990b); star formation bursts (stochastic) in Matteucci & Tosi (1985). Krügel & Tutukov (1993) and Tutukov & Krügel (1995) have used one–zone dynamical code without radial dependence of the variables to study the conditions for bursts of star formation. In the later paper, using one–zone code, they studied other types of burst of star formation in a galactic nucleus different from periodic bursts. Further, surface gas density threshold for star formation has been discussed in Kenicutt (1989).

Wyse & Silk (1989) have discussed an extended Schmidt model with $R$ dependence on surface gas density $\sigma_g$ and local angular frequency $\Omega(r)$ for both atomic and molecular gases respectively with $n = 1$ and $n = 2$. Wang & Silk (1994) have presented recently a self-consistent model (considering total gas surface density) for global star formation based on the gravitational instability parameter $Q < 1$ by Toomre (1964). In the solar neighbourhood, the model agrees with (i) the observed star formation rate; (ii) the metallicity distribution among G–dwarfs, and (iii) the age metallicity relation for F–dwarfs. The model may be compared for star formation rate in galactic disks with Schmidt law with an exponent of about 2. Star formation rate depends also on the epicyclic frequency. A natural cut-off for $Q = 1$ in the star formation rate results. However, their analysis is heav-
ily based on $Q < 1$ criterion which has been put to question in relation to non-radial instabilities in the galactic disks that may play more fundamental role when magnetic field supported by azimuthal gas motions (thus resulting thermal instability not at all related to $Q$) is taken into consideration (Elmegreen 1993). We note (see e.g. Figures 4 and 6 of Wang & Silk 1994) that the star formation does proceed in the regions where $Q \geq 1$. A natural question to ask is: how does star formation occur when $Q \geq 1$ and consequently the system has attained the state of gravitational equilibrium? We attempt here to precisely answer this question and provide a scenario to circumvent this natural cut-off in the star formation process (see Sect. 3 in the text for details).

The fact that star formation occurs via gravitational instability was also suggested by Fall & Efstathiou (1980). The $Q$-regulation near its threshold value has been discussed by Dopita (1985) and Silk (1992). Silk (1995) has argued that local self-regulation of star formation may help explain the initial mass function of stars and that global self-regulation can account for the rate of star formation. Effect of environment on the gas content and rotation curves of disk may play a crucial role in determining star formation rates and histories.

Elmegreen (1995) has discussed critical column densities for gravitational instabilities and for cooling to diffuse cloud temperatures. It has been shown that the fundamental scale for star formation in the outer regions of galaxies (in spiral arms) and in the resonance rings are related to the local unstable length. Since critical gas density for gravitational instability scales as local density, inner regions of galaxies have higher star formation rate beyond threshold density.

The consideration of magnetic field changes velocity dispersion by a factor of two pushing $Q > 1$, (i.e. stable region). Incorporating this with the fact that there occurs shear instability of magnetised gas in the azimuthal direction, one is stimulated to think that $Q < 1$ may not be the only criterion for cloud formation which leads to star formation. An alternative suggestion for cloud formation because of energy dissipation accompanied by shear instability thus leading to star formation (even if $Q > 1$) has been given (Elmegreen 1991a, 1993, and below for details). Macroscopic thermal instabilities and various cloud formation mechanisms are reviewed in Elmegreen (1991b). We assume that instability parameter suggested by Elmegreen (1993), i.e. $Q_A < 1$ (instead of $Q < 1$) is the criterion which determines occurrence of significant cloud formation instabilities. A natural consequence of our analysis is that star formation proceeds in the regions where one has $Q \geq 1$. It may be noted that in these regions (system being gravitationally stable) an altogether different cloud formation mechanism (leading to star formation) as suggested by Elmegreen (1993) is asked for. We shall present subsequently the evidence in support of our assumption. The outline of the paper is as follows: we give a general law for star formation rate in Section 2. In Section 3, we suggest a general equation for star formation rate which depends upon two fundamental parameters $Q_A$ and $f_p$ (defined in the text). We also give a comparison of star formation rate in the solar neighbourhood and timescale of gas depletion. Variations in the star formation rate and metallicity distribution in the solar neighbourhood are discussed in Section 4. The Section 5 gives discussions and a resume of our results.

## 2 STAR FORMATION RATE

We write the star formation rate in the form

$$R^\alpha = \alpha (\sigma_g/\tau)^n$$  \hspace{1cm} (1)

where $\alpha =$ efficiency of star formation which also depends upon $n$, $\tau =$ timescale of star formation, $\sigma_g =$ surface density of gas composed of atomic and molecular components, $n =$ an exponent. Clearly, $\tau^{-1}$ is related to growth rate of instability (Goldreich & Lynden-Bell 1965). A review of recent observations of the history of star formation and its relevance to galaxy formation and evolution has been discussed by Kennicutt (1996). For the evolution of the global star formation history measured from the Hubble Deep Field one may refer to Connew et al. (1997). Gravitational instability of galactic disks has also been studied by Elmegreen (1979), Cowie (1981), Ikeuchi, Habe & Tanaka (1984) and Bizyaev (1997). However, gravitational instabilities in the presence of turbulence are discussed in Bonazzola et al. (1987) and Leorat, Passot & Pouquet (1990). It is found that supersonic turbulence may be strong enough (in some cases) to hold the Jeans criterion for gravitational instability. As a result, it may stop gravitational collapse. In this scenario, star formation takes place in molecular cloud complexes at places where the turbulence evolves to subsonic phase. In the present analysis we do not aim to discuss the instability criteria and their relevance to star formation (which are certainly interesting topics of research at present), instead we aim to obtain a general star formation law with small number of adjustable parameters. We assume neither infall nor radial flow in the disk. We consider gravitational instability due to axisymmetric perturbations (for non-axisymmetric case, one may refer to Goldreich & Lynden-Bell 1965) with magnetic field in the azimuthal direction which gives rise to shear instability in a magnetised gas. Groth rate of instability is now expressed as

$$\omega^2 = k^2 \nu_{eff}^2 - 2\pi G \sigma_g k + \kappa^2$$  \hspace{1cm} (2)

where $k$ is the wave number and $\kappa$ is the epicyclic frequency, $\nu_{eff}$ is the effective velocity dispersion for ambient Alfven speed such that

$$\nu_{eff} = \left( v^2 \gamma_{eff} + v^2_{\Lambda eff} \right)^{1/2}$$  \hspace{1cm} (3)

$v$ being the velocity dispersion without magnetic field and

$$\gamma_{eff} = \frac{\gamma \omega - \omega_c}{\omega + \omega_c (3 - s)}.$$  \hspace{1cm} (4)

Here $\gamma =$ ratio of two specific heats, $\omega_c =$ cooling rate (see e.g. Elmegreen 1993 for details). The parameter $Q$ is written as $Q \equiv \kappa v_{eff} / \pi G \sigma_g$. Gravitational instability requires that both $Q < 1$ and $k$ are smaller than a critical value

$$k_{cr} = \frac{\pi G \sigma_g}{\nu_{eff}^2 \left( 1 + (1 - Q^2)^{1/2} \right).}$$  \hspace{1cm} (5)

Due to thermal instability, if $\gamma_{eff}$ reaches large negative values (such that $\nu_{eff}^2 < 0$), it implies no critical (or minimum) wavelength for gravitational perturbation in the radial direction. This makes $Q^2 < 0$. However, we do have

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a maximum wavelength of the perturbation. Thus, equation (2) shows the absence of Q-threshold for azimuthal instability which means that all Q-values provide unstable growth. Q-threshold may appear only if \( \gamma_{\text{eff}}(\omega) \) becomes a constant. Therefore, for the present treatment, we demand \( Q_A \equiv 2\sqrt{2}A\nu_{\text{eff}}/\pi G\sigma_g < 1 \) for growth of gravitational instability but we are well aware that thermal and shear instabilities (along azimuthal direction) are capable of determining cloud formation leading to star formation even if \( Q > 1 \). This amounts to replacing \( \kappa \) by \( 2\sqrt{2}A \) in the original \( Q \) (a scale transformation for Keplerian disk).

Maximum of \( \omega^2 \) occurs at

\[
\kappa_{\text{max}} = 2\sqrt{2}A/\nu_{\text{eff}}Q_A
\]

(\( A = \) Oort shear constant), which provides maximum growth rate as (Wang & Silk 1994)

\[
|\omega_{\text{max}}| = \frac{2\sqrt{2}A(1 - Q_A^2)^{1/2}}{Q_A}.
\]

(7)

Since, \( \tau \simeq |\omega_{\text{max}}|^{-1} \), one gets from equations (1) and (7)

\[
R^n = \frac{\alpha(2\sqrt{2}A)^n\sigma_g^n(1 - Q_A^2)^{n/2}}{Q_A^n}.
\]

(8)

Following Wang & Silk (1994), we define a function \( f_c = \frac{\sigma_g}{\sigma_c} \), \( \sigma_c = \) column density of individual molecular clouds. But, however, the relation between individual cloud formation and star formation is complicated. Even cloud formation process is not well known. The assumption that star formation rate results due to gravitational instability naturally demands for its relation with cloud formation process. Elmegreen (1990) has shown that gravitational instabilities generally form giant molecular clouds faster than due to random collisions. Cloud formation followed by star formation in the interstellar medium is certainly not the purpose of our investigation. Under the assumption that only gravitational instability is predominant, small cloud collisions may lead to large molecular clouds wherein star formation ensues. It is then natural to think that within an order of magnitude cloud formation timescale or equivalently cloud collision timescale and growth rate of local instability timescale are similar. With this scenario, Wang & Silk (1994) derive the expression for collision time between two clouds. We, thus, make use of their result and write collision time between two clouds as

\[
t_{\text{coll}}^{-1} = \frac{\sigma_g(2\sqrt{2}A)}{\sigma_cQ_A}.
\]

(9)

In view of the above, \( t_{\text{coll}}^{-1} \sim \omega_{\text{max}} \), and one gets

\[
Q_A \sim (1 - f_c^2)^{1/2}.
\]

(10)

It is to be noted that this may not reflect the general property of the interstellar medium, e.g. other types of instabilities namely thermal instability and Parker instability might also contribute and affect the timescale of star formation (subsequently other physical quantities). Making use of eq. (10) into eq. (8), star formation rate is now expressed as

\[
R^n = \frac{\alpha(2\sqrt{2}A)^n\sigma_g^n f_c^n}{(1 - f_c^2)^{n/2}}.
\]

(11)

Eventually, in this form eq. (11) now assumes the conversion from column density to density using the galactic scale height. Let us write equation (11) in the form

| Distance (kpc) | A (km s\(^{-1}\)kpc\(^{-1}\)) | log A | \( \sigma_g \) (\( M_\odot \)pc\(^{-2}\)) | log\( \sigma_g \) |
|---------------|-----------------|-------|----------------|----------|
| 1             | 105             | 2.0212 | 100            | 2.0000   |
| 2             | 30              | 1.4771 | 3              | 0.4771   |
| 3             | 20.9            | 1.3202 | 5              | 0.6990   |
| 4             | 19.7            | 1.2945 | 10             | 1.0000   |
| 5             | 19.1            | 1.2820 | 10.5           | 0.4212   |
| 6             | 18.2            | 1.2601 | 10.2           | 1.0086   |
| 7             | 17.2            | 1.2355 | 10             | 1.0000   |
| 10            | 13.8            | 1.1399 | 7              | 0.8551   |
| 12            | 11.5            | 1.0607 | 5              | 0.6990   |
| 14            | 9.6             | 0.9823 | 4              | 0.6021   |
| 16            | 7.9             | 0.8976 | 3              | 0.4771   |
| 18            | 6.5             | 0.8129 | 2              | 0.3010   |
| 20            | 5.44            | 0.7356 | 1              | 0.0000   |

\[
\frac{\partial \ln R}{\partial \ln \sigma_g} = n + n \frac{\partial \ln A}{\partial \ln \sigma_g} + n \frac{\partial}{\partial \ln \sigma_g} \left[ \ln \left( \frac{f_c}{(1 - f_c^2)^{1/2}} \right) \right]
\]

or

\[
n_s = \frac{\partial \ln R}{\partial \ln \sigma_g} = 2 + \frac{\partial \ln A}{\partial \ln \sigma_g} + \frac{f_c^2}{1 - f_c^2}.
\]

(12)

where \( n_s \) stands for the nominal Schmidt exponent. The second term in equation (12) appears because for spiral waves epicyclic frequency is expressed through

\[
\kappa = \kappa_0(\sigma_g/\sigma_0)^{1/2}
\]

(13)

and shear constant \( A \) is

\[
A = A_0(2 - \sigma_g/\sigma_0).
\]

(14)

Non-axisymmetric gravitational perturbation of a magnetic gaseous disk has been discussed by Elmegreen (1987) who has obtained eq. (13) and eq. (14). Here, \( A_0 \) and \( \sigma_0 \) represent equilibrium values of the shear rate and the surface mass density (see also Waller & Hodge 1991). It is easy to see that for vanishing shear constant, equation (12) reduces to equation (19) of Wang & Silk (1994, hereafter WS). It may be regarded as generalised version of WS equation in the sense that there is an additional term on the right side which is certainly non-zero. We calculate the second term on the right of equation (12), i.e., \( \partial \log A/\partial \log \sigma_g \sim 0.54 \), using least squares method. Data reported in Table 1 have been taken from Einasto (1979) and WS. Since mostly \( f_c \) is very small compared to unity (see e.g. Table 5) for present Galactic disk, we conclude that the nominal Schmidt exponent \( n_s \) for our model corresponds to \( 2 < n_s < 3 \) for the Galaxy. For usual Schmidt law \( n_s \) lies between 1 and 2. The other normal spiral galaxies of the Galaxy type are supposed to follow the same signature.

### 3 GENERAL EQUATION FOR STAR FORMATION

We suggest, that two fundamental parameters (Elmegreen 1993) which determine star formation may be put in the form

\[
R^n = \alpha a^n + \beta (af\rho)^n
\]

(15)
Integrate equation (18) to obtain
\[
\frac{t^n}{\tau_0^n} + \frac{t^n}{\tau_0^n} f_P^n = - \left(\frac{(1 - f_0^2 f_0^2)^{n/2}}{f_0^2 f_0^2} \right) - \sin^{-1}\left(f_0^n f_0^n\right) + \text{constant}. \tag{21}
\]
At \( t = 0 \), \( \sigma_0 = \sigma_i (f_0 = 1) \), one obtains the value of constant in equation (21) as
\[
\text{constant} = \left(\frac{1 - f_0^2 f_0^2}{f_0^2 f_0^2}\right) + \sin^{-1}\left(f_0^n f_0^n\right). \tag{22}
\]
Now, equation (21) becomes
\[
(1 + f_0^n) \frac{t^n}{\tau_0^n} = - \left(\frac{1 - f_0^2 f_0^2}{f_0^2 f_0^2}\right)^{n/2} - \sin^{-1}\left(f_0^n f_0^n\right) + \sin^{-1}\left(f_0^n f_0^n\right) \tag{23}
\]
where we have put \( \tau_0^n = \tau_0^n = \tau^n \). The contribution of second parameter may be observed on the right hand side of equation (23). For \( f_0 = 0 \) and \( n = 1 \), equation (23) reduces (except for a minus sign) to the equation derived by WS (cf. eq. (23) in WS). When \( f_0 \ll 1 \) near the centre of the disk, we find
\[
\sigma_0^n \sim \frac{\sigma_0^n}{1 + f_0}(\frac{1}{r})^n. \tag{24}
\]
Toward the centre, \( A \) increases which shows that gas surface density decreases (\( \tau \) varies inversely with \( A \)). Large values of \( f_0 \) for diffuse clouds again guarantees the depletion of gas in the centre. For outer parts of disk, \( f_0 \sim 1 \) and after expanding various terms in equation (23) and neglecting higher order terms, we get the following
\[
\sigma_0^n = \sigma_0^n \left[ (1 + f_0^n) f_0^n \left(\frac{1}{r}\right)^n + 1 \right]. \tag{25}
\]
In view of large values of \( \tau \) and \( f_0 \), the gas density scales as the initial one.

We now proceed to obtain critical column densities based on \( \kappa \) and that on new parameter \( Q_A \). We write
\[
\sigma_{cr., \kappa} = \frac{2\kappa v_{eff}}{\pi G}, \quad \sigma_{cr., A} = \frac{2\sqrt{2A} v_{eff}}{\pi G}. \tag{26}
\]
Assume a rotation curve of the form \( V \propto \tau^\mu (\mu = 0 \text{ for flat curve}) \). One gets
\[
\sigma_{cr., \kappa} = \frac{1}{\sqrt{1 + \mu}}. \tag{27}
\]
It is found that for \( \mu = 0 \), both densities agree. But for large \( \mu \) (i.e. departures from flatness), \( \sigma_{cr., A} \) becomes smaller than \( \sigma_{cr., \kappa} \). For example, for \( M \text{~33,} \mu = 0.3 \) (Newton 1980) which yields
\[
\sigma_{cr., A} = 0.61 \sigma_{cr., \kappa}. \tag{28}
\]
Observations of \( \sigma_0 \) (Wilson, Scoville & Rice 1991) for this galaxy are better explained if one takes \( \sigma_{cr., A} \) as threshold density rather than \( \sigma_{cr., \kappa} \) (see also Elmegreen 1993). Thus, \( Q_A \) emerges as a better parameter relating to disk instabilities than Toomre Q-parameter for star formation. This is also supported by the ratio of the two threshold densities (see e.g. Table 2).

Table 2 shows that for highly non-linear region of rotation velocity, threshold density based on \( Q_A \) is lowered (relatively) favouring instability for star formation. On the other
hand, threshold density based on Q-parameter is higher (by about one order) in this region. This shows that Q-parameter is relatively less efficient to favour star formation. We, therefore, conclude that in the non-linear regime of rotation velocity curve, the Q-parameter is approximately 10% less effective than Q parameter for triggering the process.

We have computed the ratio Q_A/Q with radial distances from the centre shown in Table 2. In fact, for a disk radius below 30 kpc deviations in the two parameters become significant which shows the relative merit of Q_A parameter over Q-parameter to keep track of the physical process like star formation and other nuclear activity as well.

Table 2. Variation of ratio of threshold densities with index $\mu$

| $\mu$  | $\sigma_{cr, A}/\sigma_{cr, s}$ |
|--------|---------------------------------|
| 0.005  | 0.99                            |
| 0.05   | 0.93                            |
| 0.10   | 0.86                            |
| 0.15   | 0.80                            |
| 0.20   | 0.73                            |
| 0.30   | 0.61                            |
| 0.40   | 0.51                            |
| 0.50   | 0.41                            |
| 0.60   | 0.32                            |
| 0.70   | 0.23                            |
| 0.80   | 0.12                            |
| 0.90   | 0.07                            |

Table 3. Radial variation of $Q_A/Q$ for the Galaxy

| $r$ (kpc) | $A$ $(10^{-16.5} s^{-1})$ | $B$ $(10^{-16.5} s^{-1})$ | $\kappa$ $(10^{-16.5} s^{-1})$ | $Q_A/Q$ |
|-----------|--------------------------|--------------------------|-------------------------------|---------|
| 1         | 105                      | 62                       | 203.5                         | 1.5     |
| 2         | 30                       | 55                       | 136.8                         | 0.6     |
| 3         | 20.9                     | 44.5                     | 107.9                         | 0.5     |
| 4         | 19.7                     | 34.1                     | 85.7                          | 0.6     |
| 5         | 19.1                     | 26.1                     | 68.7                          | 0.8     |
| 6         | 18.2                     | 20.1                     | 55.5                          | 0.9     |
| 7         | 17.2                     | 15.6                     | 45.2                          | 1.1     |
| 8         | 15.8                     | 10.0                     | 36.2                          | 1.7     |
| 9         | 12.4                     | 5.5                      | 19.8                          | 1.7     |
| 10        | 8.9                      | 3.6                      | 12.9                          | 1.7     |
| 11        | 6.5                      | 3.3                      | 11.4                          | 1.6     |
| 12        | 5.44                     | 3.11                     | 10.3                          | 1.5     |
| 13        | 3.91                     | 2.40                     | 7.1                           | 1.2     |
| 14        | 1.59                     | 1.53                     | 4.4                           | 1.0     |
| 15        | 1.06                     | 1.00                     | 2.9                           | 1.0     |

We assume constant IMF in the solar neighbourhood (Miller & Scalo 1979; Scalo 1986) and take the following input parameters: initial surface density $\sigma_{i, \odot} \approx \sigma_{g, \odot} + \sigma_{s, \odot} \approx 50M_{\odot}pc^{-2}$ (Kuijken & Gilmore 1989; Bahcall, Flynn & Gould 1992), $\sigma_{g} \approx 10M_{\odot}pc^{-2}$ (McKee 1990), $f_g \sim 0.2$, $f \sim 0.05$ (Elmegreen 1993), $t = \text{age of the Galaxy} = 15 \text{Gyr}$, $\alpha = 0.1$ (Myers et al. 1986), Oort shear constant $A = 15 \text{km s}^{-1} \text{kpc}^{-1}$ (Kerr & Lynden-Bell 1986), $\delta = 0.3$ (Miller & Scalo 1979; Scalo 1986). For $n = 1$, we get timescale of star formation as $\tau = 0.38 \text{Gyr}$. We calculate $f_{ci, \odot}$ using equation (23) as $f_{ci, \odot} \sim 0.10$. After substituting these values into equation (11), we get star formation rate as $R = 3.8M_{\odot}pc^{-2}Gyr^{-1}$. This is in agreement with Scalo (1986) who infers $R \sim (1 - 4)M_{\odot}pc^{-2}Gyr^{-1}$ within an uncertainty factor of about 3. For $n = 2$, we get $\tau = 0.12 \text{Gyr}$, $f_{ci, \odot} \sim 0.05$ and star formation rate $R = 6.0M_{\odot}pc^{-2}Gyr^{-1}$.

We find that our model with $n = 1$ provides star formation rate which is in good agreement with inferred rate in the solar neighbourhood. It is to be remarked that our models are sensitive enough to efficiency $\alpha$ introduced in equation (1) which however is determined by star formation timescale $\tau$. We note that even if the efficiency drops by 10% (i.e. when the value of $\alpha$ becomes of the order of 0.01) model with $n = 2$ gives same star formation rate as model with $n = 1$ and $\alpha = 0.1$. Parametric freedom for $\alpha$ and $f_P$ even when $Q_A \geq 1$ (i.e. non-gravitational instabilities are dominant) provide a general star formation scenario. Our model thus presents a generalisation of WS model with a dependence of star formation rate on Oort shear constant $A$. In contrast to WS, we find continuous (in the sense of Q-values) star formation rate obeying a similar but however different criterion (i.e. $Q_A < 1$) of gravitational instability.
for gaseous disks. In fact, competitive nature of two terms in equation (15) helps one to visualize the essence of continuity in the star formation process. We discuss the scenario in more details in Sect. 3.3.

3.2 TIMESCALE OF GAS DEPLETION

For a particular \( n \), we get from equations (1) and (16) as

\[
\frac{d\sigma^n_g}{\sigma_0^n} = -\alpha (1 - \delta)^n t^{-n} \sigma^n g dt^n.
\]

Integrate equation (29) to obtain

\[
\ln \sigma^n_g = -\alpha (1 - \delta)^n t^{-n} t + \text{constant}.
\]

At \( t = 0 \), \( \sigma^n_g (r, t) = \sigma^n_g (r, 0) \) which yields

\[
\sigma^n_g (r, t) = \sigma^n_g (r, 0) \exp \left[ -\alpha (1 - \delta)^n t^{-n} t \right]
\]

(see also Lynden-Bell 1975; Güsten & Mezger 1983). Now we can write e-folding time as

\[
t^n_d = \frac{1}{\alpha (1 - \delta)^n t^{-n}}.
\]

For our input parameters, the depletion time \( t_d \) for model \( n = 1 \) is \( t_d \approx 5.4 \) Gyr and for model \( n = 2 \), \( t_d \approx 0.54 \) Gyr. Model with \( n = 2 \) has 10% depletion time as compared to \( n = 1 \). For an age of 15 Gyr of the Galaxy model with \( n = 1 \) implies that present gas fraction is \( \approx 10\% \) of its initial value assuming that there is little variation over the last 5 Gyr (Dopita 1985, 1987).

3.3 THE \( f_P \)-PARAMETER AND STAR FORMATION

The \( f_P \)-parameter introduced in equation (15) requires further analysis as regards the process of star formation. It is dimensionless and measures the fraction of diffuse clouds to self-gravitating clouds. Low values of \( f_P \) (\( f_P \sim 0.01 \)) means that clouds are dense and self-gravitating. In this case, physics of star formation is largely determined by the first term in equation (15). If, however, \( f_P \sim 100 \) as for example in the inner Galaxy where pressure becomes high (Elmegreen & Elmegreen 1987, Polk et al. 1988, see also Vogel, Kulkarni & Scoville 1988 for M 51), as a result diffuse molecular clouds collide and cool leading to large mass cloud formation. Nevertheless, this does not mean that such regions result into large star-forming clouds. In fact, gravitational instabilities are more efficient (as compared to diffuse cloud collision) to produce large mass star-forming clouds. But, then, in this case local energy dissipation occurs through diffuse cloud collisions (Elmegreen 1989). A major difficulty for gravitational instability triggered star formation appears when \( Q_A \) and \( f_P \) both are large. In this case, only thermal instability is responsible for switching on star formation process. Murray & Lin (1989) have stressed the dominating role of thermal instability over gravitational instability for a proto-globular cluster where fragmentation (into protostars) is initiated by the former. Low \( f_P \) values may also result when pressure becomes low (i.e. in the outer spiral arms of galaxies where gravity is not significant to form large molecular clouds) and star formation proceeds via shear instability. This instability does not depend upon \( Q_A \). Still, \( Q_A \) has to be relatively small to guarantee unstable radial motion which in turn facilitates dense cloud formation. Hence, the new parameter \( f_P \) appearing in equation (15) also explains star formation scenario plausibly in the spiral arms and outer parts of galaxies.

We suggest that two competitive parameters (\( Q_A \) and \( f_P \)) may be understood to ultimately decide star formation (in view of observed continuity of the process from centre to arm for the Galaxy) as follows. We propose a relation

\[
f_P = 10^{1 - Q_A} \quad \text{for} \quad Q_A < 1
\]

\[
f_P = 10^{1 + Q_A} \quad \text{for} \quad Q_A \geq 1.
\]

From equations (33), we find that \( Q_A \) changes continuously from its value \( - (1 + \log f_P) \) to \( - (1 + \log f_P) \) respectively from gravitational instability region (\( f_P \sim 0.01 \)) to non-gravitational instability region (\( f_P \sim 100 \)) and star formation is supposed to occur from near the centre to outer spiral arms in contrast to earlier suggestion (Quirk 1972) who favours (necessarily) gravitational instability to trigger the process.

4 VARIATION IN THE STAR FORMATION RATE

We assume cloud mass density in the solar neighbourhood, \( \sigma_c \sim 100 M_\odot \text{pc}^{-2} \), as constant (Larson 1981). We further assume \( f_c \sim 0.01 \) at \( d = 1 \) kpc (since \( \sigma_g \) at 1 kpc is \( \sim 100 M_\odot \text{pc}^{-2} \) which makes \( f_c = 1 \) yielding an infinite \( R \)) to keep star formation reasonably large in the model. Components of surface densities (data taken from Einasto 1979; Lacey & Fall 1985 and WS) are given in Table 4 and plotted as Figure 2 at various distances from the Galactic centre.

We infer from Figure 2 that the Einasto model shows \( \sigma_c \sim r^{-0.8} \) dependence for \( r \leq 6 \) kpc and deviates for \( r > 6 \) kpc (see also e.g. Kundt 1990 for a variant in the Galactic mass distribution). Now, we aim to discuss the variation of star formation rate normalised to that in the solar neighbourhood and therefore calculate \( R/R_{\odot} \) (see e.g. Table 5) using data from Einasto (1979) and plot as Figure 3 at

| d (kpc) | \( \log \sigma_i (M_\odot \text{pc}^{-2}) \) | \( \log \sigma_q (M_\odot \text{pc}^{-2}) \) |
|--------|---------------------------------|---------------------------------|
| 0.001  | 5.5                             |                                 |
| 0.01   | 4.4                             |                                 |
| 1.0    | 3.0                             | 2.0                             |
| 2.0    | 3.0                             | 0.5                             |
| 3      | 2.8                             | 0.7                             |
| 4      | 2.7                             | 1.0                             |
| 5      | 2.5                             | 1.0                             |
| 6      | 2.4                             | 1.0                             |
| 7      | 2.2                             | 1.0                             |
| 10     | 1.7                             | 0.9                             |
| 12     | 1.4                             | 0.7                             |
| 14     | 1.0                             | 0.6                             |
| 16     | 0.7                             | 0.5                             |
| 18     | 0.4                             | 0.3                             |
| 20     | 0.1                             | 0.0                             |
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Figure 2. Radial variation of total surface density $\sigma_i (M_\odot pc^{-2})$ and gas surface density $\sigma_g (M_\odot pc^{-2})$ for the Galaxy. The solid line represents the data taken from Lacey & Fall (1985) and WS; the dotted line represents the data from Einasto (1979).

Table 5. Radial variation of star formation rate for the Galaxy

| d (kpc) | A ($10^{-16.5} s^{-1}$) | $\sigma_i (M_\odot pc^{-2})$ | $f_c$ | $R/R_\odot$ |
|---------|-------------------------|-----------------------------|-------|-------------|
| 1       | 105                     | 1016.3                      | 0.01  | 78.6        |
| 2       | 30                      | 851.1                       | 0.03  | 56.5        |
| 3       | 20.9                    | 633.9                       | 0.05  | 48.8        |
| 4       | 19.7                    | 452.9                       | 0.10  | 66.0        |
| 5       | 19.1                    | 318.4                       | 0.11  | 49.5        |
| 6       | 18.2                    | 222.3                       | 0.10  | 29.9        |
| 7       | 17.2                    | 154.9                       | 0.10  | 19.7        |
| 10      | 13.8                    | 50.9                        | 0.07  | 3.6         |
| 12      | 11.5                    | 23.6                        | 0.05  | 1.0         |
| 14      | 9.6                     | 10.7                        | 0.04  | 0.3         |
| 16      | 7.9                     | 4.9                         | 0.03  | 0.1         |
| 18      | 6.5                     | 2.3                         | 0.02  | 0.02        |
| 20      | 5.44                    | 1.2                         | 0.01  | 0.005       |

Figure 3. Star formation rates normalized to its value in the solar neighbourhood. The data are based on Einasto (1979); Lacey & Fall (1985) and WS.

various Galactocentric distances. From Figures 2 and 3 we infer that star formation rate varies like gas component of surface density.

A minimum in $\sigma_g$ occurs at $\sim 3$ kpc where we also observe a minimum in the star formation rate. Thereafter, $\sigma_g$ increases again and reaches a maximum at $\sim 4$ kpc where we observe corresponding increase and maximum in $R/R_\odot$. Our model may be applied to a general dependence of $\sigma_g$ on the exponent $n$. A given value of $n$ determines the timescale of star formation ($\tau$) which however yields the corresponding $R$. For $n = 1$, our model agrees with WS model but we obtain larger Schmidt exponent (see e.g. equation (12)).

The star formation rates inferred from (i) pulsar data (Lyne, Manchester & Taylor 1985), (ii) from observations of supernova remnants (Guibert, Lequeux & Viallefond 1978), and (iii) Lyman-continuum photon observations from H II regions (Güsten & Mezger 1983) are consistent from our model at all radial distances. For example, higher rate of star formation traced by Lyman-continuum near 4 kpc agrees with our model calculations. This is evident by the maximum in Figure 3 at 4 kpc from the Galactic centre. In view of comments (Wyse & Silk 1989) regarding higher star formation rates of Güsten & Mezger than those given by Scuto (1988) (viz. these estimates may be higher by an order of magnitude) and also the fact that it does not match with star formation profile obtained by other techniques (Rana & Wilkinson 1986), our values are apparently better tuned.

We assume $\tau = 0.45$ Gyr. Using parameters as described in Section 3.1, we calculate efficiency $\alpha$ of star formation as a function of distance from the galactic centre. It is interesting to observe that $\alpha$ changes in the solar neighbourhood. Small values of $\alpha$ at 1 kpc may be understood to arise because of shear instability which removes growth of perturbations. Star formation can proceed if the self-gravitational collapse time becomes shorter than the shear time ($\sim 0.01$ Gyr). However, relatively large value of $\alpha$ out to 10 kpc does not lead to large star formation rate $R/R_\odot$ (see Figure 3) due to paucity of gas. In fact, density $\sigma_i$ drops below the observed value (Wilson, Scoville & Rice 1991) of critical density at 14 kpc where we expect turn-off of star formation due to gravitational instability. It is also supported by significant depletion of gas at this distance (see e.g. Figure 2). The striking feature of our result is that $\alpha$ changes in the solar neighbourhood (an efficiency gradient $\sim 0.0057 kpc^{-1}$) by an amount $\sim 0.02$. It seems thus natural to think that the efficiency gradient is responsible for radial abundance gradients which are reported in many disk galaxies (Edmunds & Pagel 1984; Diaz & Tosi 1984; Tosi & Diaz 1985). The fact that
metallicity gradients may be due to changes in efficiency of star formation was suggested previously by Lacey & Fall (1985). We aim to confirm this suggestion from our calculations too. There is hardly a need to invoke radial flows (see discussion in Scalo 1988) in this scenario.

4.1 METALLICITY GRADIENT VS EFFICIENCY GRADIENT

Following Pagel & Patchett (1975) (see also e.g. Pagel & Edmunds 1981) model of chemical evolution of the Galaxy in the solar neighbourhood, we define $\xi$, a mass ratio in the form of long-lived stars, and $p$ as the yield of heavy elements which represents mass ejected per unit mass of long-lived stars (cf. Searle & Sargent 1972; Talbot & Arnett 1973a). For our model $\sigma_s = \xi \sigma_i$, $\sigma_g = (1 - \xi) \sigma_i$, $\sigma_s = \xi/(1 - \xi) \sigma_g$ and

$$\frac{d\sigma_s}{dt} = -\alpha \left(\frac{\sigma_g}{\tau}\right) = \left(\frac{\xi}{1 - \xi}\right) \frac{d\sigma_g}{dt}. \quad (34)$$

Equation (34) gives

$$\frac{d}{dt} (\ln \sigma_g) = -\alpha (1 - \xi) \frac{t}{\tau} + \text{constant}. \quad (35)$$

Integration of equation (35) yields

$$\ln \sigma_g = -\frac{(1 - \xi)}{\xi} \frac{t}{\tau} + \text{constant}. \quad (36)$$

At $t = 0$, $\sigma_g = \sigma_i$, hence constant $= \ln \sigma_i$. Thus, equation (36) takes the form

$$\frac{\sigma_g}{\sigma_i} = (1 - \xi) \exp \left[-\frac{(1 - \xi)}{\xi} \frac{t}{\tau}\right] \cdot \quad (37)$$

Therefore,

$$\tau(t) = -\frac{(1 - \xi)}{\xi} \ln(1 - \xi) t. \quad (38)$$

Metallicity $Z$ is expressed as (Pagel & Patchett 1975)

$$Z = p \ln \left[\frac{1}{1 - \xi}\right] = \frac{p(1 - \xi)}{\xi} \frac{t}{\tau}. \quad (39)$$

We see that $\tau$ is now a function of time and is given by equation (38). Time evolution of $Z$ may be written as

$$\frac{dZ}{dt} = \frac{p}{\tau} = \frac{p\xi}{\alpha(1 - \xi)\ln(1 - \xi)}\tau. \quad (40)$$

From equation (39) we infer that $Z$ varies linearly both with time and efficiency of star formation. We assume that $\xi \approx 0.8$ (Talbot & Arnett 1973a) and $p \approx 0.7Z_\odot$ (Wang & Silk 1993) to calculate $Z$ in the solar neighbourhood. For an efficiency of $\alpha \approx 0.07$ we find $Z \approx 1.13Z_\odot$ for solar age. This is in agreement with the plot of metallicity in the solar neighbourhood by Wyse & Silk (1989, cf. Figure 2b). The present model thus provides time evolution of metallicity which however depends upon efficiency of star formation.

For an efficiency run of $\alpha \approx 0.07, 0.08, 0.09, 0.10$, we find $Z/Z_\odot \approx 1.23, 1.17, 1.13, 1.09$ respectively which is independent of Galactic age provided of course the parameters $p$ and $\xi$ do not change with time. In other words, disk aging alters $\tau(t)$ such that $t/\tau(t)$ remains constant (for a fixed $\alpha$) and hence there occurs no change in $Z/Z_\odot$. Our calculations show that metallicity decreases with increase of $\alpha$ at a given age. In the solar neighbourhood, this may be understood due to paucity of gas favouring relatively low star formation at large distances (see Figure 3) and therefore low metal production (see also Friel & Janes 1993). We note that the observed run of metallicity of G-K dwarfs in our Galaxy is very sensitive to chemical composition of stars of same age (Tinsley 1975). Janes & McClure (1972) have suggested enhancement in the dispersion due to chemical inhomogeneities in the Galaxy (Talbot & Arnett 1973b). The structure of Galactic disk and the presence of population gradients are given in Ferrini et al. (1994). For a radial distribution of abundances in galaxies one may refer to Mollié et al. (1996). They have also discussed the chemical evolution of solar neighbourhood, see e.g. Pardi et al. (1995).

However, as remarked by Tinsley (1975) that the observed dispersion (see also Hearnsaw 1972 for dispersion) in metallicity for stars of same age may result either partly due to chemical inhomogeneities (of interstellar medium) or due to causes altogether different, essentially favours this analysis. We find that metallicity dispersion for stars of same age may be due to variation of efficiency $\alpha$ through which different sample stars were processed. This confirms the assumption of Rama & Wilkinson (1986) that metallicity dispersion is due to stellar processing only. It is found that $\alpha$ depends upon star formation rate and also the gas component of surface density $\sigma_g$. We conclude that $\alpha$ predominantly determines the observed dispersion and plays a key role towards metal enrichment or otherwise of the interstellar medium.

At various disk ages (at a given radial distance), there occurs change in $\alpha$ which causes metallicity dispersion. We notice that $\alpha$ also changes at various distances from the Galactic centre which results in spatial metallicity gradients. One immediately finds that apparent metallicity dispersions with either age or distance depend upon $\alpha$. The $[O/H]$ vs age plot (Wyse & Silk 1987, see Figure 2d; see also Carlberg et al. 1985) shows hardly significant metallicity gradient at all disk ages (cf. Friel & Janes 1993). We suggest that all sample stars might have followed evolution with almost similar efficiency. Thus, the important result of this analysis is the confirmation of the suggestion by Lacey & Fall (1985) and Richtler (1995) regarding metallicity gradients. For a comprehensive treatment of radial abundance gradients in spiral disks and age-metallicity relation in different stellar populations one may refer to Edvardsson et al. (1993); Pagel (1994). An interesting modern analysis of kinematics and abundance distribution for our Galaxy has been given by Gilmore, Wyse & Kuijken (1989). Matteucci (1996) has reviewed exhaustively the evolution of the abundances of heavy elements in gas and stars (indicating observational and theoretical constraints) in galaxies of different morphological types. After a similar work by Tinsley (1980), the article provide a good document of the progress in the understanding of the physical processes regulating the chemical evolution of galaxies. Formation and evolution of our Galaxy is also discussed. For a review on abundance ratios and Galactic chemical evolution see McWilliam (1997). Chemical evolution of solar neighbourhood according to the standard infall model using SN II data are summarized in Thomas et al. (1998).
5 DISCUSSIONS

We have studied that the suggestion by Elmegreen (1993) regarding star formation appears more robust than Q-criterion. It is because unless \( Q \leq 1 \), gravitational instability does not permit star formation. However, when \( Q > 1 \), the system becomes gravitationally stable and consequently star formation via large cloud formation is not feasible. A natural question which one would ask is: how does star formation proceed when \( Q \) enters the stable regime? This infact led to an alternative criterion for cloud formation (discussed in the text) leading to star formation as originally suggested by Elmegreen (1993). Accordingly, when magnetic field is taken into consideration, velocity dispersion changes and thus \( Q \) is pushed to the stable regime. At this stage, non-gravitational instabilities (e.g. thermal instability, shear instability) dominate over gravitational instability. We infer from Figure 1 that dependence of \( Q_A/Q \) on distance from the Galactic centre describes the relative merit of \( Q_A \)-parameter over \( Q \)-parameter beyond 6 kpc. Observations of \( \sigma_p \) for M 33 are in better agreement with theory when one regards \( Q_A \) as the gravitational instability parameter (see e.g. Wilson, Scoville & Rice 1991). It is found that both \( Q_A \) and \( Q \) parameters agree beyond 30 kpc.

We have obtained generalised version of WS equation (see e.g. Sect. 2 equation (12) and Sect. 3 equation (23)) in the sense that (i) there is an additional non-zero term in equation (12) and (ii) in view of equation (15), one arrives at a natural rescue from cut-off criterion for star formation. We also show that the nominal Schmidt exponent \( n_s \) is given by \( 2 < n_s < 3 \) in our model. We suggest a general equation (e.g. equation (15)) for the star formation rate consisting of two terms: the first term dominates when \( Q < 1 \) and \( f_P \ll 1 \); the second term dominates when \( Q \geq 1 \) and \( f_P \ll 1 \). Apparently, the relative competence of either of these terms determines star formation scenario as discussed in Sect. 3 at all radial distances. Virtually, \( Q < 1 \) (or \( Q_A < 1 \)) is not an absolute criterion for star formation. For our model with \( n = 1 \), we get star formation rates which are in good agreement with values inferred by Scalo (1986). We find that our models are sensitive enough to efficiency \( \alpha \) and timescale \( \tau \) of star formation. A given exponent \( n \) determines \( \tau \) which however yields the corresponding star formation rate \( R \).

We suggest that essentially the efficiency gradient is the cause for radial abundance gradients which are reported in many disk galaxies. Under the approximation of closed box model, we have derived time evolution of \( \tau(t) \) and also the metallicity \( Z(t) \). Both \( \tau(t) \) and \( Z(t) \) are functions of \( \alpha, p \) (the yield of heavy element) and mass ratio \( \xi \). We notice hardly any metallicity change as disk ages which however reflects that stellar processing occurs at a fixed \( \alpha \). Metallicity dispersion for stars of same age may be caused due to variation in \( \alpha \). We conclude that \( \alpha \) is predominantly responsible for metallicity dispersion and also for metal enrichment of interstellar medium. A simple model as discussed above provides some important characteristics of our Galactic disk. Although, as suggested by Tinsley (1980), the star formation is a complicated function of several physical parameters, e.g. gas density, gas sound speed, shock frequency, shock strength, gas rotation, shear constant \( A \), magnetic field, gas metal abundance, and background star density. It is however difficult to predict the actual dependence of \( R \) on these parameters. One therefore studies some form of \( R \) and its consequent effect on chemical and photometric evolution. Finally, the model predictions are compared with observations.

We note that star formation rate was probably higher in the central part of the disk of our Galaxy at an early epoch of evolution. It is to be remarked that hydrodynamical simulations of the formation and evolution of a galaxy may be performed incorporating our model formulation of star formation rate and metallicity. Model predictions when compared with observations of other galaxies would speak of its robustness and proof.

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