WEAK GALERKIN FINITE ELEMENT METHODS FOR QUAD-CURL PROBLEMS

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Abstract. This article introduces a weak Galerkin (WG) finite element method for quad-curl problems in three dimensions. It is proved that the proposed WG method is stable and accurate in an optimal order of error estimates for the exact solution in discrete norms. In addition, an $L^2$ error estimate in an optimal order except the lowest orders $k = 1, 2$ is derived for the WG solution. Some numerical experiments are conducted to verify the efficiency and accuracy of our WG method and furthermore a superconvergence has been observed from the numerical results.

Key words. weak Galerkin, WG, finite element methods, quad-curl problem, polyhedral partition.

AMS subject classifications. Primary, 65N30, 65N15, 65N12, 74N20; Secondary, 35B45, 35J50, 35J35

1. Introduction. In this paper we are concerned with the development of a weak Galerkin (WG) finite element method for the quad-curl problem in three dimensions which seeks $u$ such that

\begin{align}
(\nabla \times)^4 u &= f, \quad \text{in } \Omega, \\
\nabla \cdot u &= 0, \quad \text{in } \Omega, \\
u \times n &= 0, \quad \text{on } \partial \Omega, \\
\nabla \times u \times n &= 0, \quad \text{on } \partial \Omega,
\end{align}

(1.1)

for a given $f$ defined on a bounded domain $\Omega \subset \mathbb{R}^3$.

The quad-curl problems arise in inverse electromagnetic scattering theory for nonhomogeneous media [9] and magneto-hydrodynamics equations [18]. Recently, some contributions have been made on the finite element methods for the quad-curl problems. The conforming finite element spaces for the quad-curl problem have been recently constructed in two dimensions (e.g. [7, 19]) and in three dimensions (e.g. [6, 10, 20]). [4, 18] proposed the nonconforming and low order finite element spaces for the quad-curl problems. [12, 13, 17] proposed the mixed methods for the quad-curl problems. [2] introduced a formulation using the Hodge decomposition for the quad-curl problems. [5] introduced a discontinuous Galerkin scheme. [11] proposed a novel weak Galerkin formulation using the conforming space for curl-curl problem as a nonconforming space for the quad-curl problem. [16] analyzed a

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posteriori error analysis for the quad-curl problems in two dimensions. [21] introduced a virtual element method for the quad-curl problems in two dimensions. [1] introduced a decoupled formulation for the quad-curl problems where the a priori and a posteriori error were analyzed.

In the literature, the existing WG methods for quad-curl problems proposed in [11] were curl-conforming and based on tetrahedral partitions. However, our WG method is not necessary to be curl-conforming and is based on any polyhedral partitions. Our WG numerical method (4.7)-(4.8) has provided an accurate and reliable numerical solution for the quad-curl system (1.1) in an optimal order of error estimates in discrete norms and in an optimal order of $L^2$ error estimates except the lowest two orders $k = 1, 2$. In addition, we have observed some superconvergence phenomena from numerical experiments.

The paper is organized as follows. Section 2 is devoted to the derivation of a weak formulation for the quad-curl system (1.1). Section 3 briefly introduces the discrete weak gradient operator and the discrete weak curl-curl operator. Section 4 is dedicated to the presentation of the weak Galerkin algorithm for the quad-curl problem and a discussion of the solution existence and uniqueness for the WG scheme. In Section 5, the error equations are derived for the WG scheme. Section 6 establishes an optimal order of error estimates in discrete norms for the WG approximation. In Section 7, the $L^2$ error estimate for the WG solution is established in an optimal order except the lowest two orders $k = 1, 2$ under some regularity assumptions. Section 8 demonstrates the numerical performance of the WG algorithm through some test examples.

We follow the standard notations for Sobolev spaces and norms defined on a given open and bounded domain $D \subset \mathbb{R}^3$ with Lipschitz continuous boundary. Denote by $\| \cdot \|_{s,D}$, $| \cdot |_{s,D}$ and $(\cdot, \cdot)_{s,D}$ the norm, seminorm and inner product in the Sobolev space $H^s(D)$ for any $s \geq 0$. The space $H^0(D)$ coincides with $L^2(D)$ (i.e., the space of square integrable functions), for which the norm and the inner product are denoted by $\| \cdot \|_D$ and $(\cdot, \cdot)_D$. When $D = \Omega$ or when the domain of integration is clear from the context, we shall drop the subscript $D$ in the norm and the inner product notation.

2. A Weak Formulation. Let $s > 0$ be an integer. We first introduce

$$H(\text{curl}^s; \Omega) = \{ u \in [L^2(\Omega)]^3 : (\nabla \times)^j u \in [L^2(\Omega)]^3, j = 1, \cdots, s \}$$

with the associated inner product $(u, v)_{H(\text{curl}^s; \Omega)} = (u, v) + \sum_{j=1}^s ((\nabla \times)^j u, (\nabla \times)^j v)$ and the norm \( \| u \|_{H(\text{curl}^s; \Omega)} = (u, u)^{\frac{1}{2}}_{H(\text{curl}^s; \Omega)} \). We further introduce

$$H_0(\text{curl}; \Omega) := \{ u \in H(\text{curl}; \Omega) : \nabla \times u = 0 \text{ on } \partial \Omega \},$$

$$H_0(\text{curl}^2; \Omega) := \{ u \in H(\text{curl}^2; \Omega) : \nabla \times u = 0 \text{ and } \nabla \times u \times n = 0 \text{ on } \partial \Omega \}.$$

We introduce

$$H(\text{div}; \Omega) = \{ u \in [L^2(\Omega)]^3 : \nabla \cdot u \in L^2(\Omega) \},$$

with the associated inner product $(u, v)_{H(\text{div}; \Omega)} = (u, v) + (\nabla \cdot u, \nabla \cdot v)$ and the norm \( \| u \|_{H(\text{div}; \Omega)} = (u, u)^{\frac{1}{2}}_{H(\text{div}; \Omega)} \). We further introduce

$$H(\text{div}^0; \Omega) = \{ u \in H(\text{div}; \Omega) : \nabla \cdot u = 0 \text{ in } \Omega \}.$$
Using the usual integration by parts, we are ready to propose the weak formulation of the quad-curl problem (1.1) as follows: Given \( f \in H(\text{div}^0; \Omega) \), find \((u; p) \in H_0(\text{curl}^2; \Omega) \times H_0^1(\Omega)\) such that

\[
\begin{align*}
((\nabla \times)^2 u, (\nabla \times)^2 v) + (v, \nabla p) &= (f, v), \\
(u, \nabla q) &= 0,
\end{align*}
\]

(2.1) \( \forall q \in H_0^1(\Omega) \).

**Theorem 2.1.** [1,2] Given \( f \in H(\text{div}^0; \Omega) \), the problem (2.1) has a unique solution \((u; p) \in H_0(\text{curl}^2; \Omega) \times H_0^1(\Omega)\). Furthermore, \( p = 0 \) and \( u \) satisfies

\[
\|u\|_{H(\text{curl}^2; \Omega)} \leq C\|f\|.
\]

### 3. Weak Differential Operators

The principal differential operators in the weak formulation (2.1) for the quad-curl problem (1.1) are the gradient operator \( \nabla \) and the curl-curl operator \((\nabla \times)^2\). We shall briefly review the discrete weak gradient operator [15,14] and define the discrete weak curl-curl operator.

Let \( T \) be a polyhedral domain with boundary \( \partial T \). A scalar-valued weak function on \( T \) refers to \( \sigma = \{\sigma_0, \sigma_b\} \) with \( \sigma_0 \in L^2(T) \) and \( \sigma_b \in L^2(\partial T) \). Here \( \sigma_0 \) and \( \sigma_b \) are used to represent the value of \( \sigma \) in the interior and on the boundary of \( T \). Note that \( \sigma_b \) may not necessarily be the trace of \( \sigma_0 \) on \( \partial T \). Denote by \( \mathcal{W}(T) \) the space of scalar-valued weak functions on \( T \):

\[
\mathcal{W}(T) = \{\sigma = \{\sigma_0, \sigma_b\} : \sigma_0 \in L^2(T), \sigma_b \in L^2(\partial T)\}.
\]

A vector-valued weak function on \( T \) refers to a triplet \( v = \{v_0, v_b, v_n\} \) where \( v_0 \) and \( v_b \) are used to represent the values of \( v \) in the interior and on the boundary of \( T \) and \( v_n \) represents the value of \( \nabla \times v \) on \( \partial T \). Note that \( v_b \) and \( v_n \) may not necessarily be the traces of \( v_0 \) and \( \nabla \times v_0 \) on \( \partial T \) respectively. Denote by \( V(T) \) the space of vector-valued weak functions on \( T \):

\[
V(T) = \{v = \{v_0, v_b, v_n\} : v_0 \in [L^2(T)]^3, v_b \in [L^2(\partial T)]^3, v_n \in [L^2(\partial T)]^3\}.
\]

The weak gradient of \( \sigma \in \mathcal{W}(T) \), denoted by \( \nabla_w \sigma \), is defined as a linear functional on \([H^1(T)]^3\) such that

\[
(\nabla_w \sigma, \psi)_T = -\langle \sigma_0, \nabla \cdot \psi \rangle_T + \langle \sigma_b, \psi \cdot n \rangle_{\partial T},
\]

for all \( \psi \in [H^1(T)]^3 \).

The weak curl-curl operator of any \( v \in V(T) \), denoted by \((\nabla \times)^2_w v\) is defined in the dual space of \( H(\text{curl}^2; T) \), whose action on \( q \in H(\text{curl}^2; T) \) is given by

\[
((\nabla \times)^2_w v, q)_T = (v_0, (\nabla \times)^2 q)_T - \langle v_b \times n, \nabla \times q \rangle_{\partial T} - \langle v_n \times n, q \rangle_{\partial T}.
\]

Denote by \( P_r(T) \) the space of polynomials on \( T \) with degree no more than \( r \).

A discrete version of \( \nabla_w \sigma \) for \( \sigma \in \mathcal{W}(T) \), denoted by \( \nabla_{w,r,T} \sigma \), is defined as a unique polynomial vector in \([P_r(T)]^3\) satisfying

\[
(\nabla_{w,r,T} \sigma, \psi)_T = -\langle \sigma_0, \nabla \cdot \psi \rangle_T + \langle \sigma_b, \psi \cdot n \rangle_{\partial T}, \quad \forall \psi \in [P_r(T)]^3,
\]

(3.3)
which, from the usual integration by parts, gives
\begin{equation}
\langle \nabla w, \sigma \rangle_T = \langle \nabla \sigma_0, \psi \rangle_T - \langle \sigma_0 - \psi, \sigma_b \cdot n \rangle_{\partial T}, \quad \forall \psi \in [P_r(T)]^3,
\end{equation}
provided that $\sigma_0 \in H^1(T)$.

A discrete version of $(\nabla \times)^2 w \mathbf{v}$ for $\mathbf{v} \in V(T)$, denoted by $(\nabla \times)^2 w, r, T \mathbf{v}$, is defined as a unique polynomial vector in $[P_r(T)]^3$ satisfying
\begin{equation}
(\nabla \times)^2 w, r, T \mathbf{v} = (\mathbf{v}_0, (\nabla \times)^2 \mathbf{q})_T - \langle \mathbf{v}_b \times n, \nabla \times \mathbf{q} \rangle_{\partial T} - \langle \mathbf{v}_n \times n, \mathbf{q} \rangle_{\partial T},
\end{equation}
for any $\mathbf{q} \in [P_r(T)]^3$.

4. Weak Galerkin Algorithm. Let $T_h$ be a finite element partition of the domain $\Omega \subset \mathbb{R}^3$ consisting of polyhedra that are shape-regular \cite{14}. Denote by $E_h$ the set of all faces in $T_h$ and $E_h^0 = E_h \setminus \partial \Omega$ the set of all interior faces. Denote by $h_T$ the meshsize of $T \in T_h$ and $h = \max_{T \in T_h} h_T$ the meshsize for the partition $T_h$.

For any given integer $k \geq 1$, denote by $W_k(T)$ the local discrete space of the scalar-valued weak functions given by
\[ W_k(T) = \{ \{ \sigma_0, \sigma_b \} : \sigma_0 \in P_k(T), \sigma_b \in P_k(e), e \subset \partial T \}. \]
Furthermore, denote by $V_k(T)$ the local discrete space of the vector-valued weak functions given by
\[ V_k(T) = \{ \{ \mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n \} : \mathbf{v}_0 \in [P_k(T)]^3, \mathbf{v}_b \in [P_k(e)]^3, \mathbf{v}_n \in [P_{k-1}(e)]^3, e \subset \partial T \}. \]
Patching $W_k(T)$ over all the elements $T \in T_h$ through a common value $\sigma_b$ on the interior interface $E_h^0$, we arrive at the following scalar-valued weak finite element space, denoted by $W_h$; i.e.,
\[ W_h = \{ \{ \sigma_0, \sigma_b \} : \{ \sigma_0, \sigma_b \} \mid T \in W_k(T), \forall T \in T_h \}, \]
and the subspace of $W_h$ with vanishing boundary values on $\partial \Omega$, denoted by $W_h^0$; i.e.,
\begin{equation}
W_h^0 = \{ \{ \sigma_0, \sigma_b \} \in W_h : \sigma_b = 0 \text{ on } \partial \Omega \}.
\end{equation}
Similarly, patching $V_k(T)$ over all the elements $T \in T_h$ through a common value $\mathbf{v}_b$ on the interior interface $E_h^0$, we arrive at the following vector-valued weak finite element space, denoted by $V_h$; i.e.,
\[ V_h = \{ \{ \mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n \} : \{ \mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n \} \mid T \in V_k(T), \forall T \in T_h \}, \]
and the subspace of $V_h$ with vanishing boundary values on $\partial \Omega$, denoted by $V_h^0$; i.e.,
\begin{equation}
V_h^0 = \{ \{ \mathbf{v}_0, \mathbf{v}_b, \mathbf{v}_n \} \in V_h : \mathbf{v}_b \times n = 0 \text{ and } \mathbf{v}_n \times n = 0 \text{ on } \partial \Omega \}.
\end{equation}

For simplicity of notation and without confusion, for any $\sigma \in W_h$ and $\mathbf{v} \in V_h$, denote by $\nabla w, \sigma$ and $(\nabla \times)^2 w \mathbf{v}$ the discrete weak actions $\nabla w, r, T \sigma$ and $(\nabla \times)^2 w, r, T \mathbf{v}$ computed by using (3.3) and (3.5) on each element $T$; i.e.,
\[ (\nabla w, \sigma) \mid T = \nabla w, r, T(\sigma) \mid T, \quad \sigma \in W_h, \]
\((\nabla \times)^2_w \mathbf{v} |_T = (\nabla \times)^2_{w,k-2,T}(\mathbf{v} |_T), \quad \mathbf{v} \in V_h.\)

For any \(\sigma, \lambda \in W_h\) and \(u, v \in V_h\), we introduce the following bilinear forms

(4.3) \(a(u, v) = \sum_{T \in T_h} a_T(u, v),\)

(4.4) \(b(u, \lambda) = \sum_{T \in T_h} b_T(u, \lambda),\)

(4.5) \(s_1(u, v) = \sum_{T \in T_h} s_{1,T}(u, v),\)

(4.6) \(s_2(\sigma, \lambda) = \sum_{T \in T_h} s_{2,T}(\sigma, \lambda),\)

where

\[
\begin{align*}
    a_T(u, v) &= ( (\nabla \times)^2_w u, (\nabla \times)^2_w v )_T, \\
    b_T(u, \lambda) &= ( u_0, (\nabla \times)_w \lambda )_T, \\
    s_{1,T}(u, v) &= h_T^{-3} (u_0 \times n - u_b \times n, v_0 \times n - v_b \times n)_{\partial T} \\
                    &\quad + h_T^{-1} (\nabla \times u_0 \times n - u_n \times n, \nabla \times v_0 \times n - v_n \times n)_{\partial T}, \\
    s_{2,T}(\sigma, \lambda) &= h_T^3 (\sigma_0 - \sigma_b, \lambda_0 - \lambda_b)_{\partial T}.
\end{align*}
\]

The following is the weak Galerkin scheme for the quad-curl problem (1.1) based on the variational formulation (2.1).

**Weak Galerkin Algorithm 4.1.** Given \(f \in H(div; \Omega)\), find \((u_h, p_h) \in V_h^0 \times W_h^0\), such that

(4.7) \(s_1(u_h, v_h) + a(u_h, v_h) + b(v_h, p_h) = (f, v_0), \quad \forall v_h \in V_h^0,\)

(4.8) \(s_2(p_h, q_h) - b(u_h, q_h) = 0, \quad \forall q_h \in W_h^0.\)

**Theorem 4.1.** The weak Galerkin finite element scheme (4.7)-(4.8) has a unique solution.

**Proof.** It suffices to prove that \(f = 0\) implies that \(u_h = 0\) and \(p_h = 0\) in \(\Omega\). To this end, taking \(v_h = u_h\) in (4.7) and \(q_h = p_h\) in (4.8) gives

\[
((\nabla \times)^2_w u_h, (\nabla \times)^2_w u_h) + s_1(u_h, u_h) + s_2(p_h, p_h) = 0.
\]

This yields

(4.9) \((\nabla \times)^2_w u_h = 0, \quad \text{in each } T,\)

(4.10) \(\nabla \times u_0 \times n = u_n \times n, \quad \text{on each } \partial T,\)

(4.11) \(u_0 \times n = u_b \times n, \quad \text{on each } \partial T,\)

(4.12) \(p_0 = p_b, \quad \text{on each } \partial T.\)

Using (4.9), (3.3), (4.10)-(4.11), and the integration by parts, we obtain

\[
\begin{align*}
0 &= ((\nabla \times)^2_w u_h, w)_T \\
 &= ((\nabla \times)^2 u_0, w)_T - \langle w, (u_n - \nabla \times u_0) \times n \rangle_{\partial T} + \langle \nabla \times w, (u_0 - u_b) \times n \rangle_{\partial T} \\
&= ((\nabla \times)^2 u_0, w)_T.
\end{align*}
\]
for any $w \in [P_{k-2}(T)]^3$. This gives $(\nabla \times)^2 u_0 = 0$ in each $T \in T_h$. It follows from (4.10) and (4.11) that $u_0 \times n$ and $\nabla \times u_0 \times n$ are continuous across the interior interface $\mathcal{E}_h^0$.

Thus, $u_0 \in H(\text{curl}^2; \Omega)$ and $(\nabla \times)^2 u_0 = 0$ in $\Omega$. Therefore, there exists a potential function $\phi$ such that $\nabla \times u_0 = \nabla \phi$ in $\Omega$. This gives

$$
(\nabla \psi, \nabla \phi) = \sum_{T \in T_h} (\nabla \times u_0, \nabla \phi)_T
= \sum_{T \in T_h} (u_0, \nabla \times \nabla \phi)_T + (\nabla \phi, \n \times u_0)_{\partial T}
= \sum_{T \in T_h} (\nabla \phi, \n \times u_b)_{\partial T}
= (\nabla \phi, \n \times u_b)_{\partial \Omega}
= 0,
$$

(4.13)

where we used the usual integration by parts. (4.11) and $n \times u_b = 0$ on $\partial \Omega$. This leads to $\psi = C$ in $\Omega$, and thus $\nabla \times u_0 = 0$ in $\Omega$. Furthermore, there exists a potential function $\psi$ such that $u_0 = \nabla \psi$ in $\Omega$.

From (4.12), (3.3) and (4.8), we have

$$
0 = \sum_{T \in T_h} (\nabla w q_0, u_0)_T
= \sum_{T \in T_h} (-q_0, \nabla \cdot u_0)_T + (q_0, u_0 \cdot n)_{\partial T}
= \sum_{T \in T_h} (-q_0, \nabla \cdot u_0)_T + \sum_{e \in \mathcal{E}_h^0} (q_0, [u_0 \cdot n]_e)_e,
$$

(4.14)

where $[u_0 \cdot n]$ is the jump of $u_0 \cdot n$ on edge $e \in \mathcal{E}_h^0$ and we used $q_0 = 0$ on $\partial \Omega$. Letting $q_0 = 0$ and $q_b = [u_0 \cdot n]$ in (4.13) yields that $[u_0 \cdot n]$ is continuous along the interior interface $e \in \mathcal{E}_h^0$. This follows that $u_0 \in H(\text{div}; \Omega)$. Taking $q_0 = \nabla \cdot u_0$ and $q_b = 0$ in (4.12) gives $\nabla \cdot u_0 = 0$ on each $T$ and further $\nabla \cdot u_0 = 0$ in $\Omega$ due to $u_0 \in H(\text{div}; \Omega)$. Recall that there exists a potential function $\psi$ such that $u_0 = \nabla \psi$ in $\Omega$. Hence, $\nabla \cdot u_0 = \Delta \psi = 0$ strongly holds true in $\Omega$ with the boundary condition $\nabla \psi \times n = u_0 \times n = 0$ on $\partial \Omega$. This implies that $\psi = C$ in $\Omega$. Thus, $u_0 = \nabla \psi = 0$ in $\Omega$. Using (4.10), (4.11) gives $u_b = 0$ and $u_n = 0$ in $\Omega$. Therefore, we obtain $u_b = 0$ in $\Omega$.

Using $u_b = 0$ gives $s_1(u_b, v_h) + a(u_b, v_h) = 0$ for any $v_h \in V^0_h$. It follows from the assumption $f = 0$ and (4.7) that $b(v_h, p_b) = 0$, which, together with (4.12) and (4.10) and the usual integration by parts, gives

$$
0 = b(v_h, p_b) = - \sum_{T \in T_h} (p_0, \nabla \cdot v_0)_T + (p_0, v_0 \cdot n)_{\partial T} = \sum_{T \in T_h} (\nabla p_0, v_0)_T.
$$

Letting $v_0 = \nabla p_0$ gives rise to $\nabla p_0 = 0$ on each $T \in T_h$; i.e., $p_0 = C$ on each $T \in T_h$. The fact that $p_0 = p_b$ on each $\partial T$ and $p_b = 0$ on $\partial \Omega$ give $p_0 = p_b = 0$ in $\Omega$ and further $p_b = 0$ in $\Omega$.

This completes the proof of the theorem. $\blacksquare$

Let $k \geq 1$. Let $Q_0$ be the $L^2$ projection operator onto $[P_k(T)]^3$. Analogously, for $e \subset \partial T$, denote by $Q_b$ and $Q_n$ the $L^2$ projection operators onto $[P_k(e)]^3$ and
This completes the proof of (4.15). For \( w \in [H(\text{curl}; \Omega)]^3 \), define the \( L^2 \) projection \( Q_h w \in V_h \) as follows
\[
Q_h w|_T = \{ Q_0 w, Q_h w, Q_n (\nabla \times w) \}.
\]
For \( \sigma \in H^1(\Omega) \), the \( L^2 \) projection \( Q_h \sigma \in W_h \) is defined by
\[
Q_h \sigma|_T = \{ Q_0 \sigma, Q_h \sigma \},
\]
where \( Q_0 \) and \( Q_h \) are the \( L^2 \) projection operators onto \( P_k(T) \) and \( P_k(\varepsilon) \) respectively. Denote by \( Q_h^{k-2} \) and \( Q_h^k \) the \( L^2 \) projection operators onto \( P_{k-2}(T) \) and \( P_k(T) \), respectively.

**Lemma 4.2.** The operators \( Q_h, Q_h, Q_h^k \) and \( Q_h^{k-2} \) satisfy the following commutative properties:

\[
\begin{align*}
(\nabla \times w)_w (Q_h w, q)_T &= (Q_0 w, (\nabla \times)^2 q)_T - (Q_h w \times n, \nabla \times q)_{\partial T} - (Q_n (\nabla \times w) \times n, q)_{\partial T} \\
&= (w, (\nabla \times)^2 q)_T - (w \times n, \nabla \times q)_{\partial T} - (\nabla \times w \times n, q)_{\partial T} \\
&= ((\nabla \times)^2 w, q)_T
\end{align*}
\]

This completes the proof of (4.15).

The proof of (4.16) can be found in [12, 14].

**5. Error Equations.** The goal of this section is to derive the error equations for the weak Galerkin method (4.7)-(4.8) for solving the quad-curl problem (1.1), which play a critical role in the forthcoming convergence analysis.

Let \( (u, p) \) be the solution of (2.1) and assume that \( u \in H(\text{curl}^4; \Omega) \). Then \( (u, p) \) satisfies

\[
\begin{align*}
((\nabla \times)^4 u, v) + (v, \nabla p) &= (f, v), \\
(\nabla \cdot u, q) &= 0,
\end{align*}
\]

for \( v \in [L^2(\Omega)]^3 \) and \( q \in L^2(\Omega) \). Let \( (u_h, p_h) \) be the WG solutions of (4.7) - (4.8). Define the error functions \( e_h \) and \( \epsilon_h \) by

\[
\begin{align*}
e_h &= \{ e_0, e_h, e_n \} = \{ Q_0 u - u_0, Q_h u - u_b, Q_n (\nabla \times u) - u_n \}, \\
\epsilon_h &= \{ \epsilon_0, \epsilon_b \} = \{ Q_0 p - p_0, Q_h p - p_b \}.
\end{align*}
\]

**Lemma 5.1.** Let \( u \in H(\text{curl}^4; \Omega) \) and \( (u_h, p_h) \in V_h^0 \times W_h^0 \) be the exact solution of quad-curl model problem (1.1) and the numerical solution arising from the WG
scheme \([4.7]\) respectively. The error functions \(e_h\) and \(\epsilon_h\) defined in \([4.3]-5.4\) satisfy the following error equations; i.e.,

\[(5.5)\] \(s_1(e_h, v_h) + a(e_h, v_h) + b(v_h, \epsilon_h) = s_1(Q_h u, v_h) + \ell_1(u, v_h), \quad \forall v_h \in V_h^0,\)

\[(5.6)\] \(-b(e_h, q_h) + s_2(\epsilon_h, q_h) = s_2(Q_h p, q_h) - \ell_2(u, q_h), \quad \forall q_h \in W_h^0.\)

Here

\[
\ell_1(u, v_h) = \sum_{T \in T_h} ((v_0 - v_b) \times n, \nabla \times (Q_h^{k-2} - I)((\nabla \times u)^2)_{\partial T}
+ ((\nabla \times v_0 - v_n) \times n, (Q_h^{k-2} - I)((\nabla \times u)^2))_{\partial T},
\]

\[
\ell_2(u, q_h) = \sum_{T \in T_h} (q_0 - q_b, (I - Q_h) u \cdot n)_{\partial T}.
\]

**Proof.** Using \([4.15]\), \([5.5]\) and the usual integration by parts, we have

\[
((\nabla \times)^2 w Q_h u, (\nabla \times)^2 v_h)_T
= (Q_h^{k-2} - I)((\nabla \times)^2 u)_{\partial T}
= (v_0, (\nabla \times)^2 Q_h^{k-2}((\nabla \times)^2 u)_{\partial T} - (v_b \times n, \nabla \times (Q_h^{k-2} - I)((\nabla \times u)^2))_{\partial T}
- (v_n \times n, Q_h^{k-2}((\nabla \times)^2 u))_{\partial T}
= (v_0 - v_b, \nabla \times (Q_h^{k-2} - I)((\nabla \times u)^2))_{\partial T}
+ ((\nabla \times v_0 - v_n) \times n, Q_h^{k-2}((\nabla \times)^2 u))_{\partial T}
= (v_0 - v_b, \nabla \times (Q_h^{k-2} - I)((\nabla \times u)^2))_{\partial T}
= ((\nabla \times v_0 - v_n) \times n, Q_h^{k-2}((\nabla \times)^2 u))_{\partial T}.
\]

Taking \(v = v_0\) in \((5.4)\) where \(v_h = \{v_0, v_b, v_n\} \in V_h^0\) and using the usual integration
by parts, we get

\[
\sum_{T \in T_h} ((\nabla \times)^2 u, (\nabla \times)^2 v_0)_T + ((\nabla \times)^3 u, (v_0 - v_b) \times n)_{\partial T}
+ ((\nabla \times)^2 u, \nabla \times v_0 \times n - v_n \times n)_{\partial T} + (\nabla p, v_0)_T = \sum_{T \in T_h} (f, v_0)_T,
\]

where we used the facts that

\[
\sum_{T \in T_h} ((\nabla \times)^2 u, v_n \times n)_{\partial T} = ((\nabla \times)^2 u, v_n \times n)_{\partial \Gamma} = 0,
\]

\[
\sum_{T \in T_h} ((\nabla \times)^3 u, v_b \times n)_{\partial T} = ((\nabla \times)^3 u, v_b \times n)_{\partial \Gamma} = 0.
\]

Substituting \((5.8)\) into \((5.7)\) gives

\[
((\nabla \times)^2 Q_h u, (\nabla \times)^2 v_h)
= (f - \nabla p, v_0) + ((v_0 - v_b) \times n, \nabla \times (Q_h^{k-2} - I)((\nabla \times)^2 u))_{\partial T}
+ ((\nabla \times v_0 - v_n) \times n, (Q_h^{k-2} - I)((\nabla \times)^2 u))_{\partial T}.
\]
It follows from (4.16) that

\[ b(v_h, Q_h p) = \langle \nabla_w (Q_h p), v_0 \rangle = \langle Q_h^k (\nabla p), v_0 \rangle = \langle \nabla p, v_0 \rangle. \]

Combining (5.9)-(5.10) gives

\[ s_1(Q_h u, v_h) + a(Q_h u, v_h) + b(v_h, Q_h p) = (f, v_0) + \langle (v_0 - v_h) \times n, \nabla \times (Q_h^k - I)((\nabla \times)^2 u) \rangle_{\partial T} \]
\[ + \langle (\nabla \times v_0 - v_h) \times n, (Q_h^{k-2} - I)((\nabla \times)^2 u) \rangle_{\partial T} + s_1(Q_h u, v_h). \]

Subtracting (4.7) from the above equation gives (5.5).

To derive (5.6), taking \( q = q_0 \) in (5.2) and using the usual integration by parts, we have

\[ 0 = -\sum_{T \in T_h} (u, \nabla q_0) + \sum_{T \in T_h} \langle u \cdot n, q_0 - q_b \rangle_{\partial T}, \]

where we used \( \sum_{T \in T_h} \langle u \cdot n, q_b \rangle_{\partial T} = 0 \). Using (3.3) and the usual integration by parts gives

\[ -b(Q_h u, q_h) = -\sum_{T \in T_h} \langle Q_0 u, \nabla_w q_h \rangle_T \]
\[ = \sum_{T \in T_h} \langle q_0, \nabla \cdot (Q_0 u) \rangle_T - \langle q_b, Q_0 u \cdot n \rangle_{\partial T} \]
\[ = \sum_{T \in T_h} -\langle \nabla q_0, Q_0 u \rangle_T + \langle q_0 - q_b, Q_0 u \cdot n \rangle_{\partial T} \]
\[ = \sum_{T \in T_h} -\langle \nabla q_0, u \rangle_T + \langle q_0 - q_b, Q_0 u \cdot n \rangle_{\partial T} \]
\[ = \sum_{T \in T_h} \langle q_0 - q_b, (Q_0 - I) u \cdot n \rangle_{\partial T}, \]

where we used (5.11) on the last line.

Subtracting (4.8) from the above equation completes the proof of (5.6).

This completes the proof of the lemma. \( \square \)

6. Error Estimates. For any \( v \in V_0^h \), we define the energy norm \( \|v\| \) as follows

\[ \|v\|^2 = \sum_{T \in T_h} \|\nabla \times_w v\|^2_T + s_1(v, v). \]

It is easy to check that \( \| \cdot \| \) is a semi-norm in \( V_0^h \), i.e.,

\[ \|v\|_1 = \|v\| + \left( \sum_{T \in T_h} \|\nabla \cdot v_0\|^2_T \right)^{\frac{1}{2}} + \left( \sum_{e \in E_0^h} h_e^{-1} \|[v_0 \cdot n]\|^2_e \right)^{\frac{1}{2}}. \]

For any \( q \in W_0^h \), we define the following norm

\[ \|q\|_0 = (s_2(q, q))^{\frac{1}{2}}. \]
Recall that $\mathcal{T}_h$ is a shape-regular finite element partition of the domain $\Omega$. For any $T \in \mathcal{T}_h$ and $\varphi \in H^1(T)$, the following trace inequality holds true [13]:

\begin{align}
\|\varphi\|^2_{\partial T} & \leq C(h_T^{-1}\|\varphi\|^2_T + h_T\|\varphi\|^2_{1,T}).
\end{align}

Furthermore, if $\varphi$ is a polynomial on $T$, the standard inverse inequality yields

\begin{align}
\|\varphi\|^2_{\partial T} & \leq Ch_T^{-2}\|\varphi\|^2_T.
\end{align}

**Lemma 6.1.** Let $k \geq 1$, and $s \in [1, k]$. Suppose $u \in [H^{k+1}(\Omega)]^3$ and $(\nabla \times)^2 u \in [H^k(\Omega)]^3$. Then, for $(v, q) \in V_h^0 \times W_h^0$, the following estimates hold true; i.e.,

\begin{align}
|s_1(Q_h u, v)| & \leq Ch^{s-1}_{\Omega}\|u\|_{s+1} s_1(v, v)^{\frac{1}{2}}, \\
|\ell_1(u, v)| & \leq Ch^{s-1}_{\Omega}\|\nabla \times u\|_{s} s_1(v, v)^{\frac{1}{2}}, \\
|\ell_2(u, q)| & \leq Ch^{s-1}_{\Omega}\|u\|_{s} \|q\||_{0}, \\
|s_2(Q_h p, q)| & = 0.
\end{align}

**Proof.** Using the Cauchy-Schwarz inequality, the trace inequality (6.3), gives

\begin{align}
|s_1(Q_h u, v)| & = \left| \sum_{T \in \mathcal{T}_h} h_T^{-3}\|Q_0 u - Q_0 u\|_{\partial T} + h_T^{-1}\|\nabla \times Q_0 u - \nabla \times Q_0 u\|_{\partial T} \right| \\
& \leq \left\{ \sum_{T \in \mathcal{T}_h} h_T^{-3}\|Q_0 u - u\|_{\partial T}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{T \in \mathcal{T}_h} h_T^{-1}\|\nabla \times (Q_0 u - u)\|_{\partial T}^2 \right\}^{\frac{1}{2}} s_1(v, v)^{\frac{1}{2}} \\
& \leq \left\{ \sum_{T \in \mathcal{T}_h} h_T^{-4}\|Q_0 u - u\|_{\partial T}^2 + h_T^{-2}\|Q_0 u - u\|_{1,T}^2 \right\}^{\frac{1}{2}} \\
& + \left( \sum_{T \in \mathcal{T}_h} h_T^{-2}\|\nabla \times (Q_0 u - u)\|_{\partial T}^2 + \|\nabla \times (Q_0 u - u)\|_{1,T}^2 \right)^{\frac{1}{2}} s_1(v, v)^{\frac{1}{2}} \\
& \leq Ch^{s-1}_{\Omega}\|u\|_{s+1} s_1(v, v)^{\frac{1}{2}}.
\end{align}

Using the Cauchy-Schwarz inequality, the trace inequality (6.3), gives

\begin{align}
|\ell_1(u, v)| \\
& = \sum_{T \in \mathcal{T}_h} ((v_0 - v_b) \times n, \nabla \times (Q_h^{k-2} - I)((\nabla \times)^2 u))_{\partial T} \\
& + ((\nabla \times v_0 - v_n) \times n, (Q_h^{k-2} - I)((\nabla \times)^2 u))_{\partial T} \\
& \leq \left\{ \sum_{T \in \mathcal{T}_h} h_T^2\|\nabla \times (Q_h^{k-2} - I)((\nabla \times)^2 u)\|_{\partial T}^2 \right\}^{\frac{1}{2}} \\
& + \left( \sum_{T \in \mathcal{T}_h} h_T\|\nabla \times (Q_h^{k-2} - I)((\nabla \times)^2 u)\|_{\partial T}^2 \right)^{\frac{1}{2}} s_1(v, v)^{\frac{1}{2}} \\
& \leq \left\{ \sum_{T \in \mathcal{T}_h} h_T^2\|\nabla \times (Q_h^{k-2} - I)((\nabla \times)^2 u)\|_{\partial T}^2 + h_T^2\|\nabla \times (Q_h^{k-2} - I)((\nabla \times)^2 u)\|_{1,T}^2 \right\}^{\frac{1}{2}} \\
& + \left( \sum_{T \in \mathcal{T}_h} \|\nabla \times (Q_h^{k-2} - I)((\nabla \times)^2 u)\|_{T}^2 + h_T^2\|\nabla \times (Q_h^{k-2} - I)((\nabla \times)^2 u)\|_{1,T}^2 \right)^{\frac{1}{2}} s_1(v, v)^{\frac{1}{2}} \\
& \leq Ch^{s-1}_{\Omega}\|\nabla \times u\|_{s} s_1(v, v)^{\frac{1}{2}}.
\end{align}
Similarly, using the Cauchy-Schwarz inequality, the trace inequality (6.3) gives

\[\ell_2(u, q) = \sum_{T \in T_h} \langle q_0 - q_0, (I - Q_0) u \cdot n \rangle_{\partial T}\]

\[\leq \left( \sum_{T \in T_h} h_T^2 \| q_0 - q_0 \|^2_{L^2(T)} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-3} \| (I - Q_0) u \cdot n \|^2_{L^2(T)} \right)^{\frac{1}{2}}\]

\[\leq \left( \sum_{T \in T_h} h_T^{-4} \| (I - Q_0) u \cdot n \|^2_T + h_T^{-2} \| (I - Q_0) u \cdot n \|^2_{L^2(T)} \right)^{\frac{1}{4}} \| q_0 \|_0\]

\[\leq Ch^{k-1} \| u \|_{s+1} \| q_0 \|_0.\]

Since \( p = 0 \), it is easy to obtain \( s_2(Q_h p, q) = 0. \)

**Theorem 6.2.** Let \( k \geq 1 \). Suppose that \( u \in [H^{k+1}(\Omega)]^3 \). The following error estimate holds

\[(6.9) \quad \| e_h \| + \| \epsilon_h \|_0 \leq Ch^{k-1}(\| u \|_{k+1} + \| (\nabla \times)^2 u \|_{k-1}).\]

**Proof.** Letting \( v_h = e_h \) in (5.5) and \( q_h = \epsilon_h \) in (5.6) and adding the two equations, we have

\[\| e_h \|^2 + \| \epsilon_h \|_0^2 = s_1(Q_h u, e_h) + s_2(Q_h p, \epsilon_h) + \ell_1(u, e_h) - \ell_2(u, \epsilon_h).\]

Using Lemma 6.1 completes the proof of the theorem. \( \square \)

**7. \( L^2 \) Error Estimates.** We consider an auxiliary problem of finding \((\phi, \xi)\) such that

\[(\nabla \times)^4 \phi + \nabla \xi = e_0, \quad \text{in } \Omega,\]

\[\nabla \cdot \phi = 0, \quad \text{in } \Omega,\]

\[\phi \times n = 0, \quad \text{on } \partial \Omega,\]

\[\nabla \times \phi = 0, \quad \text{on } \partial \Omega,\]

\[\xi = 0, \quad \text{on } \partial \Omega.\]

Let \( t_0 = \min\{k, 3\} \). We assume the regularity property holds true in the sense that \( \phi \) and \( \xi \) satisfy

\[(7.2) \quad \| \phi \|_{t_0+1} + \| (\nabla \times)^2 \phi \|_{t_0-1} + \| \xi \|_1 \leq C \| e_0 \|.\]

**Theorem 7.1.** Let \( k \geq 1 \) and \( t_0 = \min\{k, 3\} \). Suppose that \( u \in [H^{k+1}(\Omega)]^3 \). The following estimate holds

\[(7.3) \quad \| e_0 \| \leq C h^{t_0+k-2}(\| u \|_{k+1} + \| (\nabla \times)^2 u \|_{k-1}).\]

In other words, we have a sub-optimal order of convergence for \( k = 1, 2 \) and optimal order of convergence for \( k \geq 3 \).
Proof. Using the usual integration by parts, letting \( \mathbf{u} = \phi \) and \( \mathbf{v}_h = \mathbf{e}_h \) in (5.6), letting \( \mathbf{v}_h = Q_h\phi \) in (5.5), letting \( \mathbf{u} = \phi \) and \( q_h = \epsilon_h \) in (5.12), letting \( q_h = Q_h\xi \) in (5.6) and (4.16), we have

\[
\| \mathbf{e}_0 \|^2 \quad = \sum_{T \in T_h} (\nabla \times \phi, \mathbf{e}_0)_T \\
= \sum_{T \in T_h} ((\nabla \times)^2 \phi, (\nabla \times)^2 \mathbf{e}_0)_T + (\nabla \times \mathbf{e}_0, \mathbf{n} \times (\nabla \times)^2 \phi)_{\partial T}
+ \langle \mathbf{e}_0, \mathbf{n} \times (\nabla \times)^3 \phi \rangle_{\partial T} + \langle \mathbf{e}_0, Q_h^k \nabla \xi \rangle_T
\]

\[
= \sum_{T \in T_h} ((\nabla \times)^2 \phi, ((\nabla \times)^2 \mathbf{e}_0)_T - \langle \mathbf{e}_0 - \mathbf{e}_h \rangle \times \mathbf{n} \times Q_h^{k-1}((\nabla \times)^2 \phi))_{\partial T}
- \langle \mathbf{e}_0 - \mathbf{e}_h \rangle \times \mathbf{n} \times Q_h^{k-1}((\nabla \times)^2 \phi))_{\partial T}
+ (\nabla \times \mathbf{e}_0, \mathbf{n} \times (\nabla \times)^2 \phi)_{\partial T} + \langle \mathbf{e}_0, Q_h^k \nabla \xi \rangle_T
\]

\[
= -s_1(\mathbf{e}_h, Q_h\phi) - b(Q_h\phi, \epsilon_h) + s_1(Q_h\mathbf{u}, Q_h\phi) + \ell_1(\mathbf{u}, Q_h\phi)
+ \sum_{T \in T_h} \langle \mathbf{e}_0 - \mathbf{e}_h \rangle \times \mathbf{n} \times (Q_h^{k-1} - I)((\nabla \times)^2 \phi)_{\partial T}
- s_1(Q_h\mathbf{u}, Q_h\phi) + s_1(Q_h\mathbf{u}, Q_h\phi) + \ell_1(\mathbf{u}, Q_h\phi)
+ \sum_{T \in T_h} \langle \mathbf{e}_0 - \mathbf{e}_h \rangle \times (Q_0 - I)\phi \cdot \mathbf{n})_{\partial T}
- s_2(\epsilon_h, Q_h\xi) - s_2(Q_h\mathbf{p}, Q_h\xi) + \ell_2(\mathbf{u}, Q_h\xi)
\]

(7.4)

where we used \( \mathbf{e}_h \times \mathbf{n} = 0 \) and \( \mathbf{e}_h \times \mathbf{n} = 0 \) on \( \partial \Omega \).

Next, we shall estimate the terms on the last line of (7.4) one by one.

Recall that \( t_0 = \min \{ k, 3 \} \). Using (6.5) with \( \mathbf{v} = \mathbf{e}_h \) and \( \mathbf{u} = \phi \), we have

\[
|s_1(\mathbf{e}_h, Q_h\phi)| \leq C h^{t_0-1} \| \phi \|_{t_0+1} s_1(\mathbf{e}_h, \mathbf{e}_h)^{\frac{1}{2}} \leq C h^{t_0-1} \| \phi \|_{t_0+1} \| \mathbf{e}_h \|
\]

(7.5)

Using (6.3) with \( \mathbf{v} = Q_h\phi \), we have

\[
|s_1(Q_h\mathbf{u}, Q_h\phi)| \leq C h^{k-1} \| \mathbf{u} \|_{k+1} s_1(Q_h\phi, Q_h\phi)^{\frac{1}{2}}
\]

(7.6)
Note that
\[ s_1(Q_h \phi, Q_h \phi)^{\frac{3}{2}} \]
\[ \leq C \sum_{T \in T_h} h_T^{-2} \| Q_0 \phi \times n - Q_h \phi \times n \|_{\partial T}^2 \]
\[ + h_T^{-1} \| \nabla \times Q_0 \phi \times n - Q_n(\nabla \times \phi) \times n \|_{\partial T}^2 \]
\[ \leq C \sum_{T \in T_h} h_T^{-4} \| Q_0 \phi - \phi \|_{T}^2 + h_T^{-2} \| Q_0 \phi - \phi \|_{1,T}^2 \]
\[ + h_T^{-2} \| Q_0 \phi - \phi \|_{1,T}^2 + \| Q_0 \phi - \phi \|_{2,T}^2 \]
\[ \leq Ch^{t_0-1} \| \phi \|_{t_0+1}. \]

where we used trace inequality (6.3). Substituting (7.7) into (7.6) gives
\[ \| s_1(Q_h u, Q_h \phi) \| \leq Ch^{k-1} \| u \|_{k+1} h^{t_0-1} \| \phi \|_{t_0+1}. \]

Using (6.6) with \( v = Q_h \phi \) and (7.7), we have
\[ | \ell_1(u, Q_h \phi) | \leq Ch^{-k} \| (\nabla \times)^2 u \|_{k-1} s_1(Q_h \phi, Q_h \phi)^{\frac{3}{2}} \leq Ch^{-k} \| (\nabla \times)^2 u \|_{k-1} h^{t_0-1} \| \phi \|_{t_0+1}. \]

Using (6.7) with \( u = \phi \) and \( q_h = c_h \), we have
\[ | \sum_{T \in T_h} \langle c_0 - c_h, (Q_0 - I) \phi \cdot n \rangle_{\partial T} | \leq Ch^{t_0-1} \| \phi \|_{t_0+1} \| c_h \|_0. \]

Using (6.6) with \( u = \phi \) and \( v = e_h \), we have
\[ | \sum_{T \in T_h} \langle (e_0 - e_h) \times n, \nabla \times (Q_h^{k-1} - I)(\nabla \times)^2 \phi \rangle_{\partial T} \]
\[ + | \langle (\nabla \times e_0 - e_h) \times n, (Q_h^{k-1} - I)(\nabla \times)^2 \phi \rangle_{\partial T} | \]
\[ \leq Ch^{t_0-1} \| (\nabla \times)^2 \phi \|_{t_0-1} s_1(e_h, e_h)^{\frac{3}{2}} \leq Ch^{t_0-1} \| (\nabla \times)^2 \phi \|_{t_0-1} \| e_h \|. \]

Using the Cauchy-Schwartz inequality and trace inequality (6.3), we have
\[ s_2(c_h, Q_h \xi) = \sum_{T \in T_h} h_T^3 \| c_0 - c_h \|_{\partial T} \]
\[ \leq C \| c_h \|_0 \left( \sum_{T \in T_h} h_T^2 \| Q_0 \xi - \xi \|_{\partial T}^2 \right)^{\frac{3}{2}} \]
\[ \leq C \| c_h \|_0 \left( \sum_{T \in T_h} h_T^2 \| Q_0 \xi - \xi \|_{T}^2 + h_T^4 \| Q_0 \xi - \xi \|_{1,T}^2 \right)^{\frac{3}{2}} \]
\[ \leq Ch^2 \| \xi \|_1 \| c_h \|_0. \]

Using (5.3) with \( q = Q_h \xi \), we have
\[ s_2(Q_h p, Q_h \xi) = 0. \]
Using (6.7) with \( q = Q_h \xi \) and the trace inequality (5.3), we have

\[
\ell_2(u, Q_h \xi) \leq C h^{-1} \| u \|_{k+1} \| Q_h \xi \|_0 \leq C h^{-1} \| u \|_{k+1} \left( \sum_{T \in T_h} h_T^2 \| Q_0 \xi - Q_0 \xi \|_{H^1(T)}^2 \right)^{1/2} \\
\leq C h^{-1} \| u \|_{k+1} \left( \sum_{T \in T_h} h_T^2 \| Q_0 \xi - \xi \|_{H^1(T)}^2 \right)^{1/2} \\
\leq C h^{-1} \| u \|_{k+1} \left( \sum_{T \in T_h} h_T^2 \| Q_0 \xi - \xi \|_{L^2(T)}^2 + h_T^4 \| Q_0 \xi - \xi \|_{L^2(T)}^2 \right)^{1/2} \\
\leq C h^{-1} \| u \|_{k+1} h^2 \| \xi \|_1.
\]

(7.14)

Substituting (7.5)-(7.14) into (7.4) and using the regularity assumption (7.2) and the error estimate (6.9) gives

\[
\| e_0 \|^2 \leq C h \| u \|_{k+1} \| e_h \| + C h^{k-1} \| u \|_{k+1} h^{k-1} \| \phi \|_{k+1} \\
+ C h^{k-1} \| (\nabla \times)^2 u \|_{k+1} h^{k-1} \| \phi \|_{k+1} + C h^{k-1} \| \phi \|_{k+1} h^{k-1} \| e_h \|_0 \\
+ C h^{k-1} \| (\nabla \times)^2 \phi \|_{k+1} \| e_h \| + C h^{k-1} \| \xi \|_{k+1} \| e_h \|_0 \\
+ C h^{k-1} \| u \|_{k+1} h^2 \| \xi \|_1,
\]

which yields

\[
\| e_0 \|^2 \leq C h^{k+2-2} \| u \|_{k+1} + \| (\nabla \times)^2 u \|_{k-1} \| e_0 \|.
\]

This completes the proof of the theorem. \( \square \)

8. Numerical tests. In this section, we present some numerical results for the WG finite element method for solving the quad-curl problem analyzed in the previous sections. To this end, we shall solve the following quad-curl problem with non-homogeneous boundary conditions on an unit cube domain \( \Omega = (0,1)^3 \): Find an known \( \mathbf{u} \) such that

\[
(\nabla \times)^3 \mathbf{u} = \mathbf{f}, \quad \text{in} \ \Omega, \\
\nabla \cdot \mathbf{u} = 0, \quad \text{in} \ \Omega, \\
\mathbf{u} \times \mathbf{n} = \mathbf{g}_1, \quad \text{on} \ \partial \Omega, \\
\nabla \times \mathbf{u} \times \mathbf{n} = \mathbf{g}_2, \quad \text{on} \ \partial \Omega,
\]

where \( \mathbf{f}, \mathbf{g}_1 \) and \( \mathbf{g}_2 \) are calculated by the exact solution

\[
\mathbf{u} = \left( \begin{array}{c}
-2x^2 y^2 z \\
2x^2 y^3 z \\
-xy^2 z^2 (3x - 2)
\end{array} \right).
\]

We first compute the solution of (8.1) by the \( P_k \) weak Galerkin finite element method \((4.7)-(4.8)\) on uniform cubic grids shown in Figure 8.1. For simplicity of notations, we denote the WG finite element solution \((\mathbf{u}_h; p_h)\) by \( \{ P_k, P_k, P_{k-1}\} - P_{k-1} \) with \( \{ P_k, P_k\} - P_k \). In Table 8.1, we list the errors in various norms and the computed orders of convergence for \( P_2, P_3, P_4 \) and \( P_5 \) finite element solutions on uniform cubic grids. It seems we do have one order superconvergence in most cases in Table 8.1.
Fig. 8.1. The first three levels of uniform cubic grids used in Table 8.1.

Table 8.1
Error profiles and convergence rates on uniform cubic grids shown in Figure 8.1 for (8.1).

| level | $\|Q_h u - u_h\|$ | rate | $\|Q_h u - u_h\|$ | rate | $\|p_h\|$ | rate |
|-------|-----------------|------|-----------------|------|-----------|------|
|       | by the $\{P_2, P_3, P_1\}$-$P_1$ with $\{P_2, P_2\}$-P_2 WG method |      | by the $\{P_3, P_3, P_1\}$-$P_2$ with $\{P_3, P_3\}$-P_2 WG method |      | by the $\{P_4, P_3, P_1\}$-$P_3$ with $\{P_4, P_4\}$-P_3 WG method |      |
| 2     | 0.5263E+00      | 3.8  | 0.4078E+01      | 1.7  | 0.1742E-01| 3.9  |
| 3     | 0.3345E-01      | 4.0  | 0.1237E+01      | 1.7  | 0.1947E-02| 3.2  |
| 4     | 0.2151E-02      | 4.0  | 0.3739E+00      | 1.7  | 0.2231E-03| 3.1  |
|       | by the $\{P_3, P_3, P_1\}$-$P_2$ with $\{P_3, P_3\}$-P_2 WG method |      | by the $\{P_4, P_3, P_1\}$-$P_3$ with $\{P_4, P_4\}$-P_3 WG method |      | by the $\{P_5, P_5, P_1\}$-$P_3$ with $\{P_5, P_5\}$-P_3 WG method |      |
| 2     | 0.8522E-01      | 5.3  | 0.1512E+01      | 2.2  | 0.1117E-01| 4.0  |
| 3     | 0.2190E-02      | 5.3  | 0.2365E+00      | 2.7  | 0.4298E-03| 4.7  |
| 4     | 0.6360E-04      | 5.1  | 0.3444E-01      | 2.8  | 0.1841E-04| 4.5  |
|       | by the $\{P_4, P_4, P_1\}$-$P_3$ with $\{P_4, P_4\}$-P_3 WG method |      | by the $\{P_5, P_5, P_1\}$-$P_3$ with $\{P_5, P_5\}$-P_3 WG method |      | by the $\{P_6, P_6, P_1\}$-$P_3$ with $\{P_6, P_6\}$-P_3 WG method |      |
| 2     | 0.9472E-02      | 6.3  | 0.4145E+00      | 3.4  | 0.1476E-02| 4.2  |
| 3     | 0.1841E-03      | 5.7  | 0.3102E-01      | 3.7  | 0.1760E-04| 6.4  |
| 4     | 0.2992E-05      | 5.9  | 0.2162E-02      | 3.8  | 0.3430E-06| 5.7  |
|       | by the $\{P_5, P_5, P_1\}$-$P_3$ with $\{P_5, P_5\}$-P_3 WG method |      | by the $\{P_6, P_6, P_1\}$-$P_3$ with $\{P_6, P_6\}$-P_3 WG method |      | by the $\{P_7, P_7, P_1\}$-$P_3$ with $\{P_7, P_7\}$-P_3 WG method |      |
| 1     | 0.1010E-02      | 0.0  | 0.5724E-01      | 0.0  | 0.3493E-03| 0.0  |
| 2     | 0.1684E-04      | 5.9  | 0.3565E-02      | 4.0  | 0.5073E-05| 6.1  |
| 3     | 0.2580E-06      | 6.0  | 0.2204E-03      | 4.0  | 0.9438E-07| 5.7  |

Next we compute the solution of (8.1) again by the $P_k$ weak Galerkin finite element method but on uniform tetrahedral grids shown in Figure 8.2. In Table 8.2 we list the errors in various norms and the computed orders of convergence for $P_2$, $P_3$ and $P_4$ finite element solutions on uniform tetrahedral grids. It seems we do have one order superconvergence in most cases in Table 8.2.

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Fig. 8.2. The first three levels of uniform tetrahedral grids used in Table 8.2.

Table 8.2

| level | \|Q_h u - u_h\| rate | \|Q_h u - u_h\| rate | \|p_h\| rate |
|-------|-----------------|-----------------|-----------------|
|       | by the \{P_2, P_2, P_1\} \rightarrow P_1 with \{P_2, P_2\} \rightarrow P_2 WG method |       |       |
| 2     | 0.44E+00        | 3.9             | 0.357E+01       | 1.8             | 0.288E-01  | 3.8 |
| 3     | 0.279E-01       | 4.0             | 0.117E+01       | 1.6             | 0.193E-02  | 3.9 |
| 4     | 0.172E-02       | 4.0             | 0.451E+00       | 1.4             | 0.304E-03  | 2.7 |
|       | by the \{P_3, P_3, P_2\} \rightarrow P_2 with \{P_3, P_3\} \rightarrow P_3 WG method |       |       |
| 1     | 0.187E+01       | 0.0             | 0.480E+01       | 0.0             | 0.181E+00  | 0.0 |
| 2     | 0.551E-01       | 5.1             | 0.724E+00       | 2.7             | 0.764E-02  | 4.6 |
| 3     | 0.163E-02       | 5.1             | 0.128E+00       | 2.5             | 0.257E-03  | 4.9 |
|       | by the \{P_4, P_4, P_3\} \rightarrow P_3 with \{P_4, P_4\} \rightarrow P_4 WG method |       |       |
| 1     | 0.265E+00       | 0.0             | 0.888E+00       | 0.0             | 0.308E-01  | 0.0 |
| 2     | 0.411E-02       | 6.0             | 0.738E+01       | 3.6             | 0.632E-03  | 5.6 |
| 3     | 0.693E-04       | 5.9             | 0.721E-02       | 3.4             | 0.139E-04  | 5.5 |

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