Some Implications of Perturbative Approach
to $AdS$/CFT Correspondence

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Abstract

We show some implications of the approach to $AdS$/CFT correspondence based on Type IIB string in the flat space-time with D3-branes proposed in our previous paper. We discuss a correspondence for high energy scattering amplitudes of $\mathcal{N} = 4$ super-Yang-Mills proposed recently. We also discuss $AdS$/CFT correspondence at finite temperature. Our approach provides clear understanding of these issues.

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1 Introduction

In our previous paper [1], we proposed how to understand the reason why $AdS$/CFT correspondence holds. Our discussion there was based on the perturbative Type IIB string in the flat space-time with D3-branes introduced as boundaries of the worldsheets. We mainly discussed the relation [2][3] between Wilson loops in $N = 4$ super-Yang-Mills (SYM) in four dimensions and minimal surfaces in the $AdS_5 \times S^5$ space-time, and showed that the correspondence is a consequence of an approximate symmetry which exists in the worldsheet theory, if the large $N$ limit is taken with the 't Hooft coupling $\lambda$ kept finite but large, and if the worldsheets relevant for evaluating the Wilson loops and the minimal surfaces are restricted within a region near D3-branes.

In this paper, we show that our approach is useful for understanding the other issues discussed in the literature. The first example, discussed in section 2, is on the calculation of high energy scattering amplitudes of the SYM, which was proposed in [4] and recently confirmed on the strong coupling side at the level of four-gluon scattering in [5]. It will be shown that this proposal is also justified by the scale invariance in a similar way to the case of the Wilson loop. The second example is on $AdS$/CFT correspondence at finite temperature, discussed in section 3. The discussions on Wilson loops can be carried out also in this case. From the point of view of the perturbative string with D3-branes, the issue on the entropy can be understood rather clearly.

2 High energy scattering amplitudes

Let us consider a scattering process of $n$ massless open string states living on parallel $N$ D3-branes. At a low energy scale, the dynamics of open strings is governed by $N = 4$ SYM in four dimensions. The scattering amplitude can be calculated from worldsheets with a number of boundaries on which $n$ vertex operators for the massless states are attached. We only consider the case in which all the vertex operators are attached to a single boundary of the worldsheet. Let us take the large $N$ limit with the 't Hooft coupling $\lambda = g_s N$ fixed. Due to taking this limit, closed string handles of the worldsheets are suppressed, and therefore, the relevant topology of the worldsheets is that of the disk with boundaries. These worldsheets correspond to planar ribbon graphs of the SYM. Let $(t^{a_k})_{ij}$ be the color-dependent factor of the $k$-th vertex operator. Then the amplitude considered here is proportional to $\text{Tr}(t^{a_1} t^{a_2} \cdots t^{a_n})$ which is the only factor depending on the gauge group.

Suppose that one of the D3-branes is separated from the other $N - 1$ D3-branes which are on top of each other, and all of the external open string states are on the separated D3-brane. The restriction of the external states only results in choosing a trivial prefactor, instead of a generic $\text{Tr}(t^{a_1} t^{a_2} \cdots t^{a_n})$, and therefore, it is easy to deduce the generic amplitude from this special one. We can also restrict the sum over worldsheets so that the boundaries, on which there is no vertex operator, are on the coincident D3-branes. This restriction results in assigning $N - 1$, not $N$, to each color index loop, and therefore, this leads to a modification of the amplitude by an amount of order $\frac{1}{N}$, which is negligible in the large
Figure 1: Worldsheet configuration for the regularized amplitude.

 Due to the separation of the D3-brane, the strips of the string corresponding to the outermost loop are stretched between the D3-branes with a finite width, and therefore, it is a massive state that is propagating along this loop. The situation is depicted in figure 1. The figure 2 shows the corresponding Feynman diagram.

The introduction of mass in this manner is expected to make the amplitude well-defined in the IR region. In terms of the $\mathcal{N} = 4$ SYM, what we have done is the following. We first give a nonzero background to one of the scalar fields as

$$
\begin{bmatrix}
1 & \ldots & n & n+1 & \ldots & N \\
\vdots & \ddots & & & & \\
n & & \mu & & & \\
n+1 & & & 0 & & \\
\vdots & & & \ddots & & \\
N & & & & 0 & \\
\end{bmatrix},
$$

(2.1)

where the situation depicted in figure 1 corresponds to $n = 1$, and one can consider a situation for generic $n$. Here we assume $n \ll N$, and consider only planar diagrams. We further assume that the colors of external particles are taken from the $n \times n$ matrices. Therefore, in the double line notation, the index of the outermost index loop runs from 1 to $n$. Furthermore we can constrain the other index loops to run from $n+1$ to $N$. This constraint does not affect the amplitude if $n \ll N$. In this way, we give non-zero mass to all the propagators that belong to the outermost loop as in fig.2.

In order to illustrate the mechanism that mass of the outermost propagators regularizes the IR divergence, we consider the following integral which corresponds to the Feynman diagram depicted in figure 3

$$
I = \int d^4k_1 \frac{1}{k_1^2 (p_1 - k_1)^2} \frac{1}{(p_2 + k_1)^2} \int d^4k_2 \frac{1}{k_2^2 (p_1 - k_1 - k_2)^2} \frac{1}{(p_2 + k_1 + k_2)^2}.
$$

(2.2)

Here we assume $k^2 \neq 0$ for simplicity. If $p_1^2 = p_2^2 = 0$, the integral of $k_2$ gives a term of order $\log(k_1^2)$, and
massive states

Figure 2: Feynman diagram corresponding to the worldsheet in figure 1.

the $k_1$ integral is IR divergent. The regularization we are discussing is to give mass to all propagators in the outermost loop. However, it is sufficient to consider only one propagator in this simple case. Actually, if we give mass $\mu$ to the outer gluon, $I$ is regularized to

$$I_{\text{reg}} = \int \frac{1}{k_1^2 + \mu^2} \frac{1}{(p_1 - k_1)^2} \frac{1}{(p_2 + k_1)^2} \left( p_1 - k_1 - k_2 \right)^2 \left( p_2 + k_1 + k_2 \right)^2. \quad (2.3)$$

The $k_1$ integral converges this time, although the $k_2$ integral gives $\log(k_1^2)$. The point is that the mass of an outer propagator makes the whole diagram IR finite.

Note that the mass introduced for the IR regularization must be much smaller than the string scale, and therefore, the separation among D3-branes must be very small. Note also that the momenta of the external states are also much smaller than the string scale so as not to produce massive string states in the scattering process, although we call this process a “high energy” scattering.

To relate the scattering amplitude to a classical worldsheet configuration, let us make the following change of variables in the worldsheet theory,

$$\partial_\alpha X^\mu = \epsilon_{\alpha\beta} \partial_\beta X_D^\mu, \quad (2.4)$$

$$X^I = X_D^I, \quad (2.5)$$

for $\mu = 1, \ldots, 4$ and $I = 6, \ldots, 10$. This is nothing but the T-duality transformation of the worldsheet variables. The transformation of the fermionic variables is defined as

$$S^a = S_D^a, \quad (2.6)$$

$$\tilde{S}^a = M^{ab} \tilde{S}_D^b. \quad (2.7)$$

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$^3$ As far as we know, there is no proof for the justification of the regularization procedure described above. It is very interesting to prove that our procedure works for generic amplitudes.
Figure 3: A Feynman diagram which could have an IR divergence.

We have employed the Green-Schwarz formalism and taken the light-cone gauge. Since we consider worldsheets in the flat space-time, this transformation obviously preserves the worldsheet action. The boundary condition for $X^\mu$ turns into the Dirichlet boundary condition since (2.4) implies

$$\partial_\sigma X^\mu = \partial_\tau X^\mu_D,$$

(2.8)
at the boundaries. As a result, the D3-branes in the original setup turn into D-instantons. It is interesting to notice that the constant modes $x^\mu_D$ of $X^\mu_D$ are not fixed, since the boundary condition is $\partial_\tau X^\mu_D = 0$. Therefore, $x^\mu_D$ should be integrated in the worldsheet path-integral which indicates that the D-instantons are distributed uniformly along $x^\mu$-directions.

It is very interesting to compare this situation with the large $N$ reduction of the SYM. In the reduced model, the momentum integrals for Feynman diagrams correspond to integrals of diagonal elements of matrices [6]. If the reduced model is regarded as an effective theory of D-instantons, then the diagonal elements of the matrices dictate the positions of the D-instantons. This chain of correspondences also implies that the integration of the positions of the D-instantons is necessary in the calculation of the high energy scattering amplitude.

In [5], classical solutions for the worldsheet are discussed in $AdS_5 \times S^5$ background, and then the T-dual transformation is performed to obtain explicit expressions of the solutions. This corresponds, in our perturbative approach, to performing an anisotropic scale transformation of [1] to go to the gravity region, and then perform the above transformation. In this paper, we proceed through a different way; we perform the above T-dual transformation first, and then perform a scale transformation defined below.

It should be noted that the transformation we would like to perform is not exactly the T-duality transformation, since the $x^\mu$-directions are non-compact. The analysis below is thus valid only when we restrict ourselves with the worldsheets which are topologically a disk with boundaries. Inclusion of handles, corresponding to considering a finite $N$ case, would require a more complicated analysis. In the following, however, we call this the T-duality transformation, which would probably make no confusion.
From now on, we consider disk worldsheets in the presence of \( N \) D-instantons. As mentioned above, this system is equivalent to the original system including D3-branes only in the large \( N \) limit.

We consider the scale transformation discussed in [1] in this D-instanton system. The transformation properties of the T-dual variables can be easily derived from the original one. As will be shown below, it is an isotropic scale transformation in this case. However, this transformation will turn out to be an approximate symmetry in a similar sense to [1]. As a result, we will verify that the high energy scattering amplitude of \( \mathcal{N} = 4 \) SYM can be calculated by a specific configuration of the worldsheet in \( AdS_5 \), as is claimed in [5].

Recall that the scale transformation of the coordinate fields in the D3-brane setup is

\[
\delta X^i(\sigma) = M^{ij} X^j(\sigma), \quad \delta P^i(\sigma) = -M^{ij} P^j(\sigma), \quad \delta S^a(\sigma) = iM^{ab} \tilde{S}^b(\sigma), \quad \delta \tilde{S}^a(\sigma) = -iM^{ab} S^b(\sigma),
\]

where \( i, j, a, b \) run from 1 to 8, and the \( 8 \times 8 \) matrix \( M^{ij} \) and \( M^{ab} \) are defined as

\[
M^{ij} = \begin{bmatrix} -I_{4\times4} & 0 \\ 0 & I_{4\times4} \end{bmatrix}, \quad M^{ab} = (\gamma^1 \gamma^2 \gamma^3 \gamma^4)^{ab},
\]

where \( \gamma^i \) are the \( SO(8) \) gamma matrices.

The T-duality transformation (2.4)(2.5)(2.6)(2.7) provides the scale transformation of the T-dual variables as

\[
\delta X_D^i(\sigma) = X_D^i(\sigma), \quad \delta P_D^i(\sigma) = -P_D^i(\sigma), \quad \delta S_D^a(\sigma) = i\tilde{S}_D^a(\sigma), \quad \delta \tilde{S}_D^a(\sigma) = -iS_D^a(\sigma).
\]

Note that the transformation (2.4) exchanges the coordinates and the momenta which is clear in (2.8). In this T-dual situation, the scale transformation is indeed the ordinary isotropic scale transformation. One can easily obtain the scale transformation of the oscillators as

\[
\delta \alpha_n^i = -\tilde{\alpha}_{-n}^i, \quad \delta \tilde{\alpha}_n^i = -\alpha_{-n}^i, \quad \delta S_n^a = i\tilde{S}_{-n}^a, \quad \delta \tilde{S}_n^a = -iS_{-n}^a.
\]

Using these transformation rules, it is easy to show that the boundary state of a D-instanton

\[
|B\rangle = \exp\left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \alpha_{i-n}^i \tilde{\alpha}_{-n}^i - iS_{-n}^a \tilde{S}_n^a\right)\right]|B_0\rangle, \quad |B_0\rangle = (|i\rangle|i\rangle - i|\alpha\rangle|\tilde{\alpha}\rangle)|x^f = 0\rangle|p^\mu = 0\rangle \bigotimes_{n=1}^{\infty} |0_n\rangle
\]

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is invariant under the scale transformation. Note that

\[ |p^\mu = 0\rangle = \int d^4q \ |x^\mu = q^\mu\rangle \]

(2.25)
describes the uniform distribution of the D-instanton.

To show the existence of the scale invariance in the D-instanton case, let us consider the free energy in the D-instanton background defined as

\[ F(\lambda) = \sum_{n=0}^{\infty} \frac{F_n}{n!} \lambda^n, \]

(2.26)
where \( F_n \) contains the contributions from worldsheets with \( n \) boundaries.

It is easy to calculate the variation \( \delta S \) of the worldsheet action under the scale transformation, and it can be shown that the variation is a sum of vertex operators corresponding to the state \( |B_0\rangle \). Since the boundary state is invariant, the variation of \( F_n \) under the scale transformation is obtained by inserting the state \( |B_0\rangle \) on the worldsheet, that is,

\[ \delta F_n = F_n(|B_0\rangle). \]

(2.27)

To evaluate \( F_n(|B_0\rangle) \), consider a worldsheet path-integral \( F_{n+1}(z) \) with \( n+1 \) boundaries, one of whose boundary is placed at \( x^I = z^I \) with \( I = 5, \cdots, 10 \). Note that the \( N-1 \) D-instantons we have discussed so far are placed at \( x^I = 0 \). In other words, we place another set of D-instantons which are distributed parallel to the original D-instantons but their positions in \( x^I \)-directions are different. It is possible to
obtain $F_n(|B_0|)$ from $F_{n+1}(z)$ through an LSZ-like procedure:

$$F_n(|B_0|) = \int d^6z \Delta_z F_{n+1}(z), \quad (2.28)$$

where $\Delta_z$ is the Laplacian on $\mathbb{R}^6$. See figure 4 for an image of this procedure. The variation of the total free energy is therefore

$$\delta F(\lambda) = \int d^6z \Delta_z F(\lambda, z), \quad (2.29)$$

where

$$F(\lambda, z) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} F_{n+1}(z). \quad (2.30)$$

Since a single D-instanton is separated from the hypersurface along which the other D-instantons are distributed, the worldsheet is stretched in the region $0 \leq |x^I| = \sqrt{x^I x_I} \leq r$ where

$$r = \max\{l_s, h\} \quad (2.31)$$

and $h$ is the distance of the separated D-instanton from the hypersurface of the D-instantons. If we take $|z^I|$ to be larger than $r$, then the corresponding worldsheet has a thin tube connecting the boundary at $x^I = z^I$ and the body of the worldsheet. The tube represents the propagation of closed string states, and the dominant contribution comes from the massless propagation. As a result, $F(\lambda, z)$ behaves as $|z^I|^{-4}$ for large $|z^I|$. For the range $0 \leq |z^I| \leq r$, we assume that $F(\lambda, z)$ varies slowly with $|z^I|$.

Let us calculate the integral

$$I = \int d^6z \Delta_z f(z), \quad (2.32)$$

where

$$f(z) = \begin{cases} f(0), & (0 \leq |z^I| \leq r) \\ \frac{r^4}{|z^I|^4} f(0), & (|z^I| > r) \end{cases} \quad (2.33)$$

is a typical example of the function having the property assumed for $F(\lambda, z)$. It is easy to check that $I = -4 \text{vol}(S^5)r^4 f(0)$. Similarly, the RHS of (2.29) would be estimated as

$$\int d^6z \Delta_z F(\lambda, z) \sim r^4 C(\lambda, r) F(\lambda, z = 0), \quad (2.34)$$

where $C(\lambda, r)$ is assumed to be of order one. Noticing that $F_{n+1}(z = 0) = F_{n+1}$, we obtain

$$\delta F(\lambda) \sim r^4 C(\lambda, r) \partial_\lambda F(\lambda), \quad (2.35)$$

where

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} F_{n+1} = \partial_\lambda F(\lambda) \quad (2.36)$$

is used. In other words, the free energy transforms as

$$F(\lambda) \to F(\lambda + cr^4 C(\lambda, r)), \quad (2.37)$$

which indicates the existence of the scale invariance if $r \ll \lambda^{\frac{1}{4}}$. Note that the sum of the infinite number of worldsheets with boundaries is crucial for the existence of this scale invariance.
It should be pointed out that our calculation has been done with an appropriate analytic continuation of $\lambda$. At first, the free energy is defined as a power series of $\lambda$ as (2.26). This definition is valid when the effects from the D-instantons are small. However, to estimate the variation of the free energy, we assume the behavior of the sum $F(\lambda, z)$, not each $F_n(z)$, and therefore, this estimate should be valid even beyond the convergence radius of the perturbative series (2.26). The result that there exists a scale invariance if $r << \lambda^{1/4}$ agrees with the existence of an isometry of the corresponding metric when $\lambda$ is large. To see the importance of the analytic continuation, let us consider the metric of D-instantons distributed along a four-dimensional hypersurface [7]

$$ds^2 = H(\rho)^{1/2}(\eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2 + \rho^2 d\Omega_5^2),$$  \hspace{1cm} (2.38)

where

$$H(\rho) = 1 + \frac{C\lambda}{\rho^4}. \hspace{1cm} (2.39)$$

In the region $\rho << \lambda^{1/4}$, the metric (2.38) behaves as

$$ds^2 \sim \sqrt{C\lambda} \left[ \eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2 + \rho^2 d\Omega_5^2 \right],$$  \hspace{1cm} (2.40)

which is a metric on $\text{AdS}_5 \times S^5$. In this metric, the scale invariance is realized as the isometry

$$\delta x^\mu = x^\mu, \quad \delta \rho = \rho.$$ \hspace{1cm} (2.41)

From the perturbative string point of view, the metric (2.38) is given as a power series of $\lambda$. By summing them up, we obtain the non-trivial function $H(\rho)^{1/2}$. The possibility to have such a closed expression enables us to go beyond the convergence radius of the perturbative series where the isometry exists.

We consider a worldsheet which is relevant to a high energy scattering of open string states on the D3-branes. As is explained above, one of the D3-brane is separated from the other $N - 1$ D3-branes to implement the IR regularization, and all the external open string states are on the separated D3-brane.

We make a scale transformation in the T-dual picture which is isotropic and brings the worldsheet to a region far away from the D-instantons. The situation is depicted in figure [5]. If we take $\lambda$ to be large, then the worldsheet can be brought to a place far enough from the hypersurface of the D-instantons so that the presence of the D-instantons is represented by the curved background (2.38) while the scale invariance is still valid. In the region $\rho << \lambda^{1/4}$, the background (2.38) is approximated by $\text{AdS}_5 \times S^5$. Then, the classical solution of the string is the minimal surface obtained in [5]. The size of the minimal surface is determined by the momenta of the external open string states, which become larger and larger by the scale transformation. It should be noted that, before the scale transformation, the size of the worldsheet is smaller than the string scale, as mentioned at the beginning of this section. After the scale transformation, this classical solution will dominate the summation over worldsheets. Note that, since the scale transformation for the D-instanton system is isotropic, the ratio of the size of the minimal surface to the distance from the hypersurface does not change. This implies that the worldsheet always exists near the boundary ($\rho = 0$) of $\text{AdS}_5$. Notice that in the D-instanton case, the “near horizon” region corresponds to the region of the boundary of $\text{AdS}_5$, as can be seen from the metric (2.38). This is in contrast with the situation in [1] where D3-branes are placed at the center of the $\text{AdS}_5$ space-time from the gravity point of view.
Since the scale transformation is a symmetry as long as the worldsheet is restricted in the region $|x^I| \ll \lambda^{\frac{1}{4}}$, the classical action for the minimal surface provides the high energy scattering amplitude with which we started. In this way, the scale invariance found in [1] provides the argument which verifies the relation claimed in [5].

One may think that the approximate symmetry is broken by the presence of the non-trivial dilaton background

$$e^\Phi = H(\rho).$$

(2.42)

Recall that the dilaton coupling in the worldsheet action is a sub-leading order term of $\alpha'$. Since the actual expansion parameter is $\frac{\alpha'}{\pi R^2}$ where $R$ is a typical length scale of the target space-time, which is proportional to $\lambda^{\frac{1}{4}}$ in this case, the contribution from the dilaton background would be negligible if we take a large $\lambda$.

3 Finite temperature

Next, we consider D3-branes at finite temperature. This is realized by considering a Euclideanized D3-brane which wraps on a circle with the circumference $\beta$. AdS/CFT correspondence has also been considered in this case [8]. The argument on the scale transformation can be also carried out in the case of the finite temperature.

One of the crucial points of our perturbative approach is the transformation property of the boundary state of the wrapped D3-brane. Since we impose the anti-periodic boundary condition for space-time
fermions in the Euclidean time direction on which the D3-brane wraps, the fermionic worldsheet variables flip their signs as the string winds around the direction. The scale transformation for these winding sectors is the same as that for the zero-winding sector, except for the fact that the modings of $S^a$ and $\tilde{S}^a$ are half-integral if the winding number is an odd integer. Due to the presence of these winding sectors, the boundary state has the form

$$|B\rangle = \sum_{w \in \mathbb{Z}} |B; w\rangle,$$

where $w$ is the winding number, and

$$|B; 2k\rangle = \exp \left[ \sum_{n=1}^{\infty} \left( \frac{1}{n} M^{ij} \alpha_{-n}^i \tilde{\alpha}_{-n}^j - i M^{ab} S^a_{-n} \tilde{S}^b_{-n} \right) \right] |B_0; 2k\rangle,$$

$$|B; 2k + 1\rangle = \exp \left[ \sum_{n=1}^{\infty} \left( \frac{1}{n} M^{ij} \alpha_{-n}^i \tilde{\alpha}_{-n}^j - i M^{ab} S^a_{-n+\frac{1}{2}} \tilde{S}^b_{-n+\frac{1}{2}} \right) \right] |B_0; 2k + 1\rangle.$$

The term $|B; 2k\rangle$ has almost the same form with the supersymmetric boundary state [9], and the transformation property is the same except for the obvious scaling of the circumference $\beta$. It might look non-trivial to analyze the transformation property of the term $|B; 2k + 1\rangle$, due to the half-integral moding of the fermionic oscillators. However, one can check that the calculations can be done similarly with the case of $|B; 2k\rangle$, and obtain

$$\delta |B; 2k + 1\rangle \propto |B; 2k + 1\rangle.$$

Therefore, we can define the scale transformation so that the boundary state $|B\rangle$ is scale invariant up to the scaling of the circumference $\beta$. Then, AdS/CFT correspondence also follows at finite temperature case, as in [1]. Note that the length scale of the background, if exists, is also transformed by the scale transformation. This fact implies that, if one considers a Wilson loop realized as in [1], then it is related to a minimal surface placed at the outside of the event horizon of a black hole background, since the boundary of the worldsheet is always at the outside of the event horizon.

There are other researches in the case at finite temperature which discuss the entropy of $N = 4$ SYM and that of the AdS-Schwarzschild black hole [10][8]. Due to the limitation of the explicit calculations, the entropy is calculated in the classical gravity only when $\lambda$ is large, and in the SYM only when $\lambda$ is small. The corrections to these results have also been calculated in [11][12][13][14] which suggest that the entropy in the gravity region is smoothly interpolated to the entropy in the SYM region by varying $\lambda$. These are the arguments supporting that the SYM entropy and the black hole entropy are the same for any $\lambda$. In the following, we will show the coincidence of the entropy at large $\lambda$.

Our point of view on AdS/CFT correspondence is based on D3-branes in the flat space-time. The temperature is encoded in the radius of the Euclideanized time direction. The temperature-dependence thus comes from strings which wind around the time direction. Let us consider the free energy of strings in this case. In the large $N$ limit with $\lambda$ kept fixed, the winding closed strings with the genus $h \geq 1$ provide contributions of order $g_s^{2h-2} \sim \lambda^{2h-2} N^{2-2h}$ which is at most of order $N^0$. On the other hand, the winding open strings may provide contributions of order $N^2$. Therefore, in the large $N$ limit, the temperature-dependent part of the free energy is dominated by open strings. When the temperature is small, then only massless open string states contribute. As a result, the thermodynamical quantities
which are obtained from the temperature-dependence of the free energy, for example the energy and the entropy, are equal to those of the SYM in the limiting case mentioned above.

From an observer at the asymptotically flat region, the thermal D3-branes are regarded as a non-extremal black hole

\[ ds^2 = H(r)^{-\frac{1}{2}}(-f(r)dt^2 + d\vec{x}^2) + H(r)^{\frac{1}{2}}(f(r)^{-1}dr^2 + r^2d\Omega_5^2), \]

where

\[ H(r) = 1 + \frac{4\pi \lambda}{r^4}, \quad f(r) = 1 - \frac{r_0^4}{r^4}. \]

The constant \( r_0 \) is determined so that the Euclideanized version of (3.5) does not have a conical singularity. If \( \lambda \) is taken to be large, then \( r_0 \) can be written as

\[ r_0 = \frac{\pi \lambda^{\frac{4}{3}}}{\beta}. \]

Note that \( \beta \) here is the circumference of the Euclidean time circle at the asymptotically flat region. Therefore, this \( \beta \) coincides with the one appeared in the D-brane setup. In this way, we can equate the temperature of the gravity side with that of the gauge theory side.

The gravity description is valid when \( r_0 \) is large. If we take \( \lambda \) to be large, we can also take \( \beta \) to be large while keeping \( r_0 \) still large. This means that there exists a parameter region in which both the gravity description and the SYM description are valid. (The description in terms of free SYM cannot be valid in this region, of course.) The ADM mass of the black hole must be the same with the total energy of the SYM plus the contribution from the tension of the D3-branes. Since the entropy is obtained from the thermodynamic relation

\[ \frac{\partial S}{\partial E} = \frac{1}{T}, \]

we can conclude that the entropies of the SYM and the black hole must be the same, due to the coincidence of the energy and the temperature, although it is difficult to perform an explicit evaluation of the entropy of the SYM with large \( \lambda \).

The crucial points of our argument are that the temperature-dependence of the free energy is dominated by open strings in the large \( N \) limit, and that there is a region of parameters where both the gravity description and the SYM description are valid. It is also important that some of physical quantities, the energy and the temperature for example, in two descriptions can be compared directly with each other at the asymptotically flat region which is absent after taking the near-horizon limit.

4 Discussion

We have shown that our point of view of AdS/CFT correspondence, based on the perturbative Type IIB string theory with D3-branes, can be applied to some situations discussed in the literature of AdS/CFT correspondence. The amplitude of high energy scattering processes of the SYM can be related to classical worldsheet configurations in the AdS\(_5\) space obtained in [5] by the scale transformation, which clarifies
the reason why such a correspondence holds. Thermal properties of the SYM are also discussed in our point of view. It can be shown that the reduction of open string system on D3-branes to the SYM occurs even in the case of large $\lambda$, and some thermal properties of the SYM can be related to some gravitational quantities, the connection being made at the asymptotically flat region far from the D3-branes.

The successful applications of our point of view indicate that it would enable us to make further predictions for the correspondence which are not known so far. As long as the correspondence is based on our scale invariance, the coincidence of some quantities will have a firm footing. We hope to provide some new and non-trivial correspondences between gauge theory and gravity, possibly less supersymmetric, which could be checked by explicit calculations.

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