$S$-matrices and bi-linear sum rules of conserved charges in affine Toda field theories

S. Pratik Khastgir$^1$
Mehta Research Institute for Mathematics and Mathematical Physics, Chhatnag Road, Jhusi, Allahabad 211 019, INDIA.

Abstract

The exact quantum $S$-matrices and conserved charges are known for affine Toda field theories (ATFTs). In this note we report on a new type of bi-linear sum rules of conserved quantities derived from these exact $S$ matrices. They exist when there is a multiplicative identity among $S$-matrices of a particular ATFT. Our results are valid for simply laced as well as non-simply laced ATFTs. We also present a few explicit examples.

PACS: 11.10.Kk; 11.55.Ds; 02.20.Tw
Keywords: Integrable models; Toda field theory; Affine Lie algebra; $S$-matrices; Bi-linear sum rules

Affine Toda field theory$^2$ is a massive scalar field theory with exponential interactions in $1+1$ dimensions described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi^a \partial^{\mu} \phi^a - \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta \alpha_i \cdot \phi}. \quad (1)$$

The field $\phi$ is an $r$-component scalar field, $r$ is the rank of a compact semi-simple Lie algebra $G$ with $\alpha_i$; $i = 1, \ldots, r$ being its simple roots and $\alpha_0$ is the affine root. The Kac-Coxeter labels $n_i$ are such that $\sum_{i=0}^{r} n_i \alpha_i = 0$, with the convention $n_0 = 1$. The quantity, $\sum_{i=0}^{r} n_i$, is denoted by ‘$h$’ and known as the Coxeter number. ‘$m$’ is a real parameter.

$^1$email: pratik@mri.ernet.in
$^2$For an excellent review see Ref. [2]
setting the mass scale of the theory and $\beta$ is a real coupling constant, which is relevant only in quantum theory.

Integrable field theories (such as ATFTs) are characterised by an infinite set of conserved quantities. In ATFTs, it is well-known that these conserved quantities are related with the Cartan matrix of the associated finite Lie algebra (see eqn. (I)). In this note we report that in certain circumstances these conserved quantities satisfy interesting bi-linear sum rules, which we believe have never been encountered in theoretical physics.

Quantum $S$-matrices for all simply laced [3–8] as well as non-simply laced $\mathfrak{g}$ affine Toda theories are known. Based on the assumption that the infinite set of conserved quantities be preserved after quantisation, only the elastic processes are allowed and the multi-particle $S$-matrices are factorised into a product of two particle elastic $S$-matrices. A typical elastic, unitary $S$-matrix for a process $a+b \rightarrow a+b$ can be written as product of ratios of hyperbolic sines.

$$S_{ab}(\theta) = \prod_{x \in I_{ab}} \{x\}, \quad \{x\} = \frac{(x - 1)(x + 1)}{(x - 1 + B)(x + 1 - B)}.$$  \hspace{1cm} (2)

for some set of integers $I_{ab}$. The building block $(x)$ and the function $B(\beta)$ are given by

$$\sinh(\frac{x}{2} + \frac{i\pi}{2\theta} x), \quad \text{etc.} \sinh(\frac{x}{2} - \frac{i\pi}{2\theta} x), \quad B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1 + \beta^2/4\pi}.$$  \hspace{1cm} (3)

$\theta = \theta_a - \theta_b$ is the relative rapidity ($p_a \equiv (m_a \cosh \theta_a, m_a \sinh \theta_a)$), $h$ is the Coxeter number of the Lie algebra on which theory is based. The above $S$-matrices respect crossing symmetry and bootstrap principle [3]. Bootstrap equations will constrain both $S$-matrix elements, $S_{ab}(\theta)$, as well as eigenvalues, $q_s^a$, of the conserved charges acting on single particle states (defined by $Q_s |A_a(\theta)⟩ = q_s^a e^{\theta} |A_a(\theta)⟩$), with $A_a(\theta)$ denoting a particle type $a$). For the fusion process $a+b \rightarrow c$, bootstrap equation for the $S$-matrices is the following [10],

$$S_{dc}(\theta) = S_{da}(\theta - i\theta_{ac}) S_{db}(\theta + i\theta_{bc}),$$  \hspace{1cm} (4)

where $\bar{\theta}$s are certain angles defined for every triplet of particles possessing a non vanishing three-point coupling $C^{abc}$. For the conserved charges it has the following form (is in fact satisfied by the logarithmic derivative of the $S$-matrix [3]),

$$q_s^c = q_s^a e^{-i\theta_{ac}} + q_s^b e^{i\theta_{bc}}.$$  \hspace{1cm} (5)

$q_s^a = (-1)^{s+1} q_s^a$ is the effect of charge conjugation on the conserved charges. Nontrivial solutions to the conserved charge bootstrap only occur if the spin $s$ modulo $h$, is equal to an exponent of rank-$r$ group $G$ on which ATFT is based. Furthermore each of the $r$ particles of the simply laced theories is associated unambiguously with the spots on the unextended Dynkin diagram of $G$ and thus to the simple roots ($\alpha_i$) of the associated finite Lie algebra. Association happens in such a way that a vector made out of the conserved charges with spin $s$, i.e. $\vec{q}_s = (q_s^{\alpha_1}, q_s^{\alpha_2}, ..., q_s^{\alpha_r})$, forms an eigenvector of the Cartan matrix of $G$, with eigenvalue $2 - 2 \cos(\pi s/h)$ [3]. Thus,

$$C\vec{q}_s = \lambda_s \vec{q}_s, \quad \lambda_s = 2 - 2 \cos(\pi s/h)$$  \hspace{1cm} (6)
where \(C\) is the Cartan matrix \(C_{ij} = 2\alpha_i \cdot \alpha_j / \alpha_j^2\), \(i, j = 1, \ldots, r\). The complete list of eigenvectors for the various simply laced theories can be found in the Table-2 of Ref. [11]. The spin-1 charge vector gives the masses of the various particles (i.e. \(q_i^a = m_a\)). For non-simply laced theories these charges were calculated in the Ref. [12].

Before stating the results we would consider two examples. The following one is from \(e_6^{(1)}\) theory. The vector \(\bar{q}_s\) is given by (see Table 2 of Ref. [11].)

\[
\bar{q}_s = \begin{pmatrix}
q_1^s \\
q_2^s \\
q_3^s \\
q_4^s \\
q_5^s \\
q_6^s
\end{pmatrix}
= \begin{pmatrix}
\sin 11\theta_s \\
\sin 10\theta_s \\
\sin 8\theta_s - \sin 2\theta_s \\
\sin 3\theta_s \\
\sin 2\theta_s \\
\sin \theta_s
\end{pmatrix}
\text{ with } \theta_s = (\pi s/12).
\] (7)

The following identity exists in \(e_6^{(1)}\) [3]

\[
S_{Hh}(\theta) = S_{hl}(\theta)S_{hl}(\theta)S_{Ll}(\theta)
\] (8)

and we observe that,

\[
q_s^H q_s^h = q_s^h q_s^l + q_s^b q_s^l
\] (9)

for \(s = 1, 4, 5, 7, 8, 11\). For \(s = 1\), above equation becomes,

\[
m_H m_h = m_h m_l + m_l m_l.
\] (10)

Next we choose an example from non-simply laced theories. We consider the dual pair \((f_4^{(1)}, e_6^{(2)})\), Ref. [12],

\[
\bar{q}_s = \begin{pmatrix}
q_1^s \\
q_2^s \\
q_3^s \\
q_4^s \\
q_5^s \\
q_6^s
\end{pmatrix}
= \begin{pmatrix}
\sin \left(\frac{3s\pi}{H}\right) \sin \left(\frac{2s\pi}{H'}\right) \\
\sin \left(\frac{3s\pi}{H}\right) \sin \left(\frac{2s\pi}{H'}\right) \\
\sin \left(\frac{3s\pi}{H}\right) \sin \left(\frac{2s\pi}{H'}\right) \\
\sin \left(\frac{3s\pi}{H}\right) \sin \left(\frac{2s\pi}{H'}\right)
\end{pmatrix}, \text{ where } \frac{1}{H} + \frac{1}{H'} = \frac{1}{6} \text{ and } H = 12 + 3B.
\] (11)

In this case we have the following identity

\[
S_{33}(\theta) = S_{11}(\theta)S_{14}(\theta).
\] (12)

One can again observe,

\[
(q_s^3)^2 = (q_s^1)^2 + q_s^1 q_s^4 \text{ for } s = 1, 5 \text{ mod 6}.
\] (13)

The bi-linear mass relation in this case reads,

\[
m_3^2 = m_1^2 + m_1 m_4
\] (14)
In the above equation masses should be understood as the floating masses (defined in expression (4.1) of Ref. [12]) for the common quantum theory of the dual pair \((f^{(1)}_{4}, e^{(2)}_{6})\).

Observing these examples carefully we make the following proposition. To prove it we would need to make certain conjecture about the Fourier coefficients in the expansion of logarithmic derivatives of \(S\)-matrix.

**Proposition:**

If

\[
\prod_{a,b \in \{i,j\}} S_{ab}(\theta) = \prod_{a',b' \in \{i',j'\}} S_{a'b'}(\theta)
\]

for some sets \(\{i,j\}\) and \(\{i',j'\}\) then,

\[
\sum_{a,b \in \{i,j\}} q_s^a q_s^b = \sum_{a',b' \in \{i',j'\}} q_s^{a'} q_s^{b'}
\]

for every individual \(s\).

**Proof:** Starting with the logarithmic derivative of the \(S\)-matrix (Ref. [5]),

\[
T_{ab}(\theta) \equiv \frac{d}{d\theta} \ln S_{ab}(\theta)
\]

one can write eqn. (4) in terms of the quantities \(T\) to obtain,

\[
T_{dc}(\theta) = T_{da}(\theta - i\bar{\theta}_{ac}) + T_{db}(\theta + i\bar{\theta}_{bc}).
\]

Expanding \(T(\theta)\) as a Fourier series in \(\theta (S(\theta)\) is \(2\pi i\) periodic),

\[
T_{ab}(\theta) = \sum_{s=-\infty}^{\infty} t_{s}^{a} e^{s\theta},
\]

where quantities \(t_{s}^{a}\) are a set of coefficients which as a consequence of (17) must satisfy

\[
t_{s}^{dc} = t_{s}^{da} e^{-i\bar{\theta}_{ac}} + t_{s}^{db} e^{i\bar{\theta}_{bc}},
\]

very reminiscent of the eqn. (3). The unitarity and crossing symmetry conditions put the following restrictions on the coefficients \(t_{s}^{a}\).

\[
T_{ab}(\theta) = T_{ab}(-\theta) \Rightarrow t_{s}^{ab} = t_{-s}^{ab},
\]

and

\[
T_{ab}(i\pi - \theta) = -T_{ba}(\theta) \Rightarrow t_{s}^{ab} = (-1)^{s+1} t_{-s}^{ba}.
\]

From the form of eqn. (19) we are led to the following factorised form of the Fourier coefficients\(^5\),

\[
t_{s}^{ab} \equiv q_s^a q_s^b
\]

which is consistent with all the above constraints. The eqn. (19) will immediately follow from the eqn. (5).

\(^5\)There may be some multiplicative coefficient independent of \(a, b\) and \(s\) on the right hand side of the expression (22).
Now taking logarithmic derivative of both sides of the equation (15) we have,
\[
\sum_{a,b \in \{i,j\}} T_{ab}(\theta) = \sum_{a',b' \in \{i',j'\}} T_{a'b'}(\theta)
\]
\[
\sum_{a,b \in \{i,j\}} \sum_{s=-\infty}^{\infty} t_{ab}^s e^{s\theta} = \sum_{a',b' \in \{i',j'\}} \sum_{s=-\infty}^{\infty} t_{a'b'}^s e^{s\theta}
\]
\[
\sum_{s=-\infty}^{\infty} \left( \sum_{a,b \in \{i,j\}} t_{ab}^s \right) e^{s\theta} = \sum_{s=-\infty}^{\infty} \left( \sum_{a',b' \in \{i',j'\}} t_{a'b'}^s \right) e^{s\theta}
\]
\[
\sum_{a,b \in \{i,j\}} t_{ab}^s = \sum_{a',b' \in \{i',j'\}} t_{a'b'}^s.
\]
(23)

In the last equation we use (22) to arrive at the result (16).

To conclude we have shown an interesting relation between the S-matrices and the conserved charges. One may derive these results from the work of Niedermaier (Ref. [13]). But there author talks about the exact quantum charges of simply laced theories. Our results are valid for non-simply laced theories as well. Moreover we think that these bi-linear sum rules are not something sacred to the real coupling ATFTs, but would also work for other two-dimensional integrable models like sine-Gordon theory (which is actually an imaginary coupling \(a_1^{(1)}\) ATFT), massive Thirring model etc.. One of the main features of the these models is topological solitons. So, analogously we would like to think that the bi-linear sum rules would connect topological charges of the solitons when there are multiplicative identities present in soliton-soliton scattering matrix.

Acknowledgements

I would like to thank Prof. Ryu Sasaki for reading the manuscript carefully and for making valuable comments and suggestions at various stages of the work.

Appendix A:

List of few multiplicative identities of the S-matrix for different ATFTs.

i) \(a_r^{(1)}, d_r^{(1)}, (c_r^{(1)}, d_{r+1}^{(2)})\) and \((b_r^{(1)}, a_{2r-1}^{(2)})\):

\[
S_{22}(\theta) = S_{11}(\theta)S_{13}(\theta) \quad (A.1)
\]
\[
S_{23}(\theta) = S_{12}(\theta)S_{14}(\theta) \quad (A.2)
\]
\[
S_{24}(\theta) = S_{13}(\theta)S_{15}(\theta) \quad (A.3)
\]
\[
S_{33}(\theta) = S_{22}(\theta)S_{15}(\theta) = S_{11}(\theta)S_{13}(\theta)S_{15}(\theta) \quad (A.4)
\]
\[
S_{34}(\theta) = S_{23}(\theta)S_{16}(\theta) = S_{12}(\theta)S_{14}(\theta)S_{16}(\theta), \text{ etc.} \quad (A.5)
\]
ii) $e_6^{(1)}$:

$$S_{LL}(\theta) = S_{hl}(\theta)S_{ll}(\theta)$$

$$S_{HL}(\theta) = S_{hl}(\theta)S_{hl}(\theta) = S_{hl}(\theta)S_{hl}(\theta) = S_{hl}(\theta)S_{hl}(\theta) = S_{hl}(\theta)S_{hl}(\theta)$$

$$S_{HH}(\theta) = S_{hh}(\theta)S_{hh}(\theta) = S_{hh}(\theta)S_{hh}(\theta), \text{ etc.}$$

(A.6) 

(A.7) 

(A.8)

iii) $e_7^{(1)}$:

$$S_{33}(\theta) = S_{22}(\theta)S_{23}(\theta)$$

$$S_{54}(\theta) = S_{12}(\theta)S_{34}(\theta)$$

$$S_{16}(\theta) = S_{14}(\theta)S_{12}(\theta)S_{23}(\theta)$$

$$S_{11}(\theta)S_{77}(\theta) = S_{55}(\theta)S_{66}(\theta), \text{ etc.}$$

(A.9) 

(A.10) 

(A.11) 

(A.12)

iv) $e_8^{(1)}$:

$$S_{23}(\theta) = S_{16}(\theta)$$

$$S_{22}(\theta) = S_{11}(\theta)S_{12}(\theta)$$

$$S_{45}(\theta) = S_{17}(\theta)S_{23}(\theta)$$

$$S_{67}(\theta) = S_{15}(\theta)S_{25}(\theta)S_{34}(\theta), \text{ etc.}$$

(A.13) 

(A.14) 

(A.15) 

(A.16)

References

[1] A. V. Mikhailov, M. A. Olshanetsky and A. M. Perelomov, *Comm. Math. Phys.* 79 (1981) 473; G. Wilson, *Ergod. Th. and Dynam. Sys.* 1 (1981) 361; D. I. Olive and N. Turok, *Nucl. Phys.* B265 (1986) 469.

[2] E. Corrigan, *Recent developments in affine Toda quantum field theory*, Invited lecture at the CRM–CAP summer school ‘Particles and Fields 94’, Banff, Alberta, Canada, Durham preprint, DTP-94/55, hep-th/9412213.

[3] A. E. Arinshtein, V. A. Fateev and A. B. Zamolodchikov, *Phys. Lett.* B87 (1979) 389.

[4] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Phys. Lett.* B227 (1989) 411.

[5] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Nucl. Phys.* B338 (1990) 689; H. W. Braden and R. Sasaki, *Phys. Lett.* B255 (1991) 343; R. Sasaki and F. P. Zen, *Int. J. Mod. Phys.* 8 (1993) 115.

[6] P. Christe and G. Mussardo, *Nucl. Phys.* B330 (1990) 465; *Int. J. Mod. Phys.* A5 (1990) 4581.
[7] C. Destri and H. J. de Vega, *Phys. Lett.* B233 (1989) 336.

[8] T.R. Klassen and E. Melzer, *Nucl. Phys.* B338 (1990) 485.

[9] G. W. Delius, M. T. Grisaru and D. Zanon, *Nucl. Phys.* B382 (1992) 365.

[10] A. B. Zamolodchikov, *Int. J. Mod. Phys.* A 4 (1989) 4235.

[11] P. Dorey, *Nucl. Phys.* B358 (1991) 654.

[12] E. Corrigan, P. E. Dorey and R. Sasaki, *Nucl. Phys.* B408 (1993) 579.

[13] M. R. Niedermaier, *Nucl. Phys.* B424 (1994) 184.