A conservative semi-Lagrangian method for oscillation-free computation of advection processes

Masato Ida

Satellite Venture Business Laboratory, Gunma University, 1–5–1 Tenjin-cho, Kiryu-shi, Gunma 376–8515, Japan
E-mail: ida@vbl.gunma-u.ac.jp

Abstract

The semi-Lagrangian method using the hybrid-cubic-rational interpolation function [M. Ida, Comput. Fluid Dyn. J. 10 (2001) 159] is modified to a conservative method by incorporating the concept discussed in [R. Tanaka et al., Comput. Phys. Commun. 126 (2000) 232]. In the method due to Tanaka et al., not only a physical quantity but also its integrated quantity within a computational cell are used as dependent variables, and the mass conservation is completely achieved by giving a constraint to a fourth-order polynomial used as an interpolation function. In the present method, a hybrid-cubic-rational function whose optimal mixing ratio was determined theoretically is employed for the interpolation, and its derivative is used for updating the physical quantity. The numerical oscillation appearing in results by the method due to Tanaka et al. is sufficiently eliminated by the use of the hybrid function.

Key words: Numerical method, Conservative method, Semi-Lagrangian, Interpolation, Cubic function, Rational function, Convexity preserving
PACS: 02.60.Cb, 02.70.-c, 47.11.+j

1 Introduction

The semi-Lagrangian method is a category of numerical methods for partial differential equations including an advection (or convection) term, and has been used mainly for atmospheric and geographic problems [1,2]. In the semi-Lagrangian approach, the advection of a physical quantity is solved as a transportation and interpolation problem on fixed grids. The trajectory of the Lagrangian invariant that will reach a grid point is computed backwardly by
the time integration of the advection velocity, and the invariant at the departure point, which is determined by an interpolation within a computational cell, is transported to the grid point. One can use a CFL-free time step in this approach. Since there are some well-established methods for calculating the trajectory such as the Runge-Kutta methods, recent studies on this approach are focused on the construction of interpolation functions.

One of problems of this approach is the difficulty of constructing a completely conservative scheme. There are only a few studies attacking to this problem. In Ref. [3] Priestley proposed a quasi-conservative semi-Lagrangian method by coupling a flux-corrected transport method and a high-order interpolation. In Ref. [4], Leonard et al. proposed a conservative, explicit algorithm achieving stable high-CFL computations, by using an integral variable of a Lagrangian invariant as the dependent variable, and showed results for one-dimensional advection at constant velocity by coupling the algorithm with many interpolation schemes. In Ref. [5], Manson and Wallis modified the QUICKEST scheme [6] so that the stable high-CFL computation is achieved. Applications of their method to pure advection problems in non-uniform flow fields are shown in Ref. [7]. In Ref. [8] Lin and Rood discussed the use of readymade Eulerian schemes (the MUSCL [9] and PPM [10] schemes) modified to achieve stable computations even with a large time step.

Recently, Tanaka et al. proposed a way to solve this difficulty. In Ref. [11,12], they modified the CIP method, which is a non-conservative semi-Lagrangian method using the Hermite cubic interpolation function [13,14], into a conservative one by employing the cell-integrated quantity of a Lagrangian invariant as an additional dependent variable. A 4th- or 2nd-order polynomial is used as an interpolation function, and is constructed using a physical quantity, its first spatial derivative (only for the 4th-order one), and its cell-integrated quantity. The conservation of total mass is achieved by updating the cell-integral quantity in a conservation sense. The CIP method and its conservative variant using the 2nd-order polynomial (the CIP-CSL2 method) are briefly reviewed in the next section.

The aim of this paper is to propose a variant of the CIP-CSL2 method by employing the hybrid-cubic-rational (HCR) interpolation function introduced recently by the author [15]. The hybrid function is constructed with a combination of the cubic polynomial and a rational function, and is proposed to obtain oscillation-free results having higher resolution than those given by the rational method [16] being a variant of the CIP. In the results using the CIP-CSL2 method one can sometimes observe numerical oscillation caused by the use of the classical polynomial [12]. As is well known, the numerical oscillation becomes a serious problem for a certain examples such as the propagation of shock waves and the multiphase flow problems including large density jump. In Ref. [17] the use of the rational function employed in the ra-
tional method is suggested to overcome the numerical oscillation. The author expects that combining the hybrid function with the concept of the CIP-CSL can derive an oscillation-free, conservative semi-Lagrangian method that has higher resolution than that by the rational method. In Sec. 3 we show that the hybrid method can easily be modified into a conservative one, and in Sec. 4 we demonstrate the accuracy and stability of the present method. In this paper, we discuss only the case of CFL number \( \leq 1 \); however, we believe that the present method is applicable to a high-CFL condition by incorporating the concept discussed in Ref. [12].

2 CIP method and its conservative variants

The CIP method [13,14] is a semi-Lagrangian solver for an advection equation,

\[
\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} = 0,
\]

where \( f \) is a dependent variable and \( u \) is the advection velocity. The fact that the solution of this equation can be represented as

\[
f(x, t + \Delta t) = f(X(x, t), t)
\]

allows us to solve Eq. (1) as an interpolation problem. Where \( X \) is the trajectory of fluid particle that locates at \( x \) at the time \( t + \Delta t \),

\[
X(x, t) = x + \int_{t}^{t-\Delta t} u(X(x, \tau), \tau) d\tau.
\]

In the CIP method, the dependent variable \( f \) is interpolated by the Hermite cubic function,

\[
C(X) = f_i + d_i(X - x_i) + C2_i(X - x_i)^2 + C3_i(X - x_i)^3,
\]

constructed using both \( f_i \) and its derivative \( d_i (= \partial f/\partial x|_i) \) defined at each grid points \( x_i \), where \( i = 1, 2, \cdots, N, \) \( N \) is the number of grids, and \( C2_i \) and \( C3_i \), are determined from the condition of continuity [13]. Equation (1) and its spatial derivative, i.e.,

\[
\frac{\partial f(x, t + \Delta t)}{\partial x} = \frac{\partial X(x, t)}{\partial x} \frac{\partial f(X(x, t), t)}{\partial X}.
\]
are used to update \( f_i \) and \( d_i \), respectively. In the case where we assume the first-order accuracy in time for example, Eqs. (2) and (4), respectively, are reduced to

\[
f(x, t + \Delta t) = f(x - u(x, t) \Delta t, t),
\]

\[
\frac{\partial f(x, t + \Delta t)}{\partial x} = \left[ 1 - \frac{\partial u(x, t)}{\partial x} \right] \Delta t \frac{\partial f(x - u(x, t) \Delta t, t)}{\partial x},
\]

where \( \Delta t \) is the time interval during a computational step. In the case of \( u_i < 0 \), those equations are represented using the Hermite cubic function as

\[
f_{i+1} = f_i + d_i \xi + C2_i \xi^2 + C3_i \xi^3,
\]

\[
d_{i+1} = d_i + 2C2_i \xi + 3C3_i \xi^2,
\]

where

\[
C2_i = -(d_{i+1} + 2d_i - 3S_{i+1/2})/h,
\]

\[
C3_i = (d_{i+1} + d_i - 2S_{i+1/2})/h^2,
\]

\[
S_{i+1/2} = (f_{i+1} - f_i)/h,
\]

\[
\xi = -u_i \Delta t,
\]

\( h \) is the grid width, and the superscript \( n \) denotes the time \( t = n \Delta t \).

Basically, the CIP-CSL2 method is a solver for the equation of continuity,

\[
\frac{\partial f}{\partial t} + \frac{\partial uf}{\partial x} = 0.
\]

In this method, a quadratic function of

\[
Q_i(X) = f_i + 2q1_i(X - x_i) + 3q2_i(X - x_i)^2
\]

is used to interpolate the physical quantity \( f \), where the coefficients \( q1_i \) and \( q2_i \) are determined, for example in the case of \( u_i < 0 \), from the constrains of

\[
Q_i(x_{i+1}) = f_{i+1} \quad \text{and} \quad \int_{x_i}^{x_{i+1}} Q_i(x) \, dx/h = \rho_{i+1/2},
\]
\[ q_1 i = -(f_{i+1} + 2f_i - 3\rho_{i+1/2})/h, \quad (13) \]
\[ q_2 i = (f_{i+1} + f_i - 2\rho_{i+1/2})/h^2, \quad (14) \]

where \( \rho \) is the cell-integrated average of \( f \) within a computational cell, defined at the cell centers, and is used as a dependent variable in this method (We should note that the definition of \( \rho \) is slightly different from that in Ref. [11]). \( \rho \) is updated so that the total mass is conserved. The mass flux flowing out from the cell \([x_i, x_{i+1}]\) through the point \( x_i \) during \( \Delta t \) can be calculated by

\[
\Delta \rho_i h = \int_{x_i}^{x_i - u_i \Delta t} Q_i(x) \, dx = f_i \xi + q_1 i \xi^2 + q_2 i \xi^3,
\]

where we assume again the first-order accuracy in time. Using this quantity, \( \rho \) is updated as follows:

\[ \rho_{i+1/2}^{n+1} = \rho_{i+1/2}^n + \Delta \rho_{i+1}^n - \Delta \rho_i^n. \quad (15) \]

The use of this formula allows us the complete conservation of the total mass, i.e., the total sum of \( \rho \).

The quantity \( f \) is updated by a time splitting technique [11]. We solve Eq. (11) by splitting into two phases as

\[
\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} = 0,
\]

\[
\frac{\partial f}{\partial t} = - \frac{\partial u}{\partial x} f.
\]

The former one shows the advection of \( f \) and, therefore, is solved as an interpolation problem by using Eq. (12) as

\[ f_{i}^{n+1} = f_i^n + 2q_1 i^n \xi + 3q_2 i^n \xi^2. \quad (16) \]

The latter one represents the change of \( f \) due to the compression or expansion and is solved in general by a finite difference technique [11,13].

As was pointed out already [12], the coefficients of the first- and second-order terms in Eq. (16) correspond to those in Eq. (8) by replacing \( f \) and \( \rho \) in Eq. (16)
with $d$ and $S$, respectively. Furthermore, Eq. (15) also can be rewritten into a form like the conventional CIP. By introducing a variable $D$ defined as

$$D_i = \int_{x_0}^{x_i} f(x) \, dx \left( = \sum_{j=0}^{i-1} \int_{x_j}^{x_{j+1}} Q_j(x) \, dx \right),$$

one obtains

$$D_{i}^{n+1} = D_{i}^{n} + \Delta \rho_{i}^{n} h = D_{i}^{n} + f_{i}^{n} \xi + q1_{i}^{n} \xi^2 + q2_{i}^{n} \xi^3, \quad (17)$$

and

$$\rho_{i+1/2}^{n} = (D_{i+1}^{n} - D_{i}^{n})/h,$$

where we assume $\Delta \rho_{0}^{n} = 0$, namely no transfer of mass occur at the boundary $x_0$. Those equations reproduce Eq. (15), and replacements of $D$ and $f$ in Eq. (17) with $f$ and $d$, respectively, yield Eq. (7) (Note that the former replacement reduces $\rho$ to $S$). This result suggests us that the variants of the CIP method such as the rational method [16] and the HCR method [15] can be modified to a conservative one only by those replacements.

3 A conservative method by the hybrid-cubic-rational interpolation

In Ref. [15], the author proposed a numerical solver for the advection equation by employing both the cubic [13] and rational [16] functions. In the method, the following combination of those functions is used as an interpolation function:

$$F(X) = \alpha R(X) + (1 - \alpha) C(X), \quad (18)$$

where $R$ and $C$ show the rational and the cubic functions, respectively, and $\alpha$ denotes the weighting parameter whose range is $\alpha \in [0, 1]$. Equation (18) is reduced to the rational function for $\alpha = 1$ and to the cubic one for $\alpha = 0$. The value of $\alpha$ is determined theoretically so as to be the minimum of the values to which the convexity-preserving condition [16] is satisfied. Where the convexity-preserving condition is expressed as that $F_{xx}(X) > 0$ for a concave data of $d_i < S_{i+1/2} < d_{i+1}$ or $F_{xx}(X) < 0$ for a convex data of $d_i > S_{i+1/2} > d_{i+1}$ is satisfied for the region of $X \in [x_i, x_{i+1}]$. The resulting formulae are

$$F(k) = f_i + d_i h k + (G1_i + G2_i) k^2, \quad (19)$$
\[
\frac{\partial F(k)}{\partial x} = d_i + \left[ G1_i \frac{Q_i + D_i}{D_i} + 2G2_i + (1 - \alpha)(Q_i - D_i) \right] \frac{k}{h}, \quad (20)
\]

where

\[ G1_i = \alpha P_i^2 / D_i, \quad G2_i = (1 - \alpha)(2P_i - D_i), \]

\[ D_i = Q_i + (P_i - Q_i)k, \]

\[ \alpha = \frac{M_i(M_i - 2)}{M_i(M_i - 2) + 1}, \]

\[ M_i = \text{max}[2, \text{max}(\frac{Q_i}{P_i}, \frac{P_i}{Q_i})] \]

with

\[ P_i = (S_{i+1/2} - d_i)h, \quad Q_i = (d_i + 1 - S_{i+1/2})h, \]

and

\[ k = \frac{\xi}{h} = -u_i \Delta t / h. \]

For the case of \( u_i > 0 \), we need replacements of \( i + 1 \to i - 1 \) and \( h \to -h \). This interpolation function provides oscillation-free results, unlike the cubic one, and higher resolution of solution than that by the conventional rational function [16]. Same as the conventional CIP, this method is constructed using a physical quantity \( f_i \) and its first spatial derivative \( d_i \); thus this method may be modified into a conservative method only by the replacements as discussed in the last section. Though the explicit definition and use of the cell-integrated average \( \rho \) may be more convenient in an actual application, we now use \( D_i \), instead of \( \rho \), and adapt the replacements of

\[
\begin{cases} 
  f \to D, \\
  d \to f 
\end{cases} \quad (21)
\]

for demonstration. Namely, the subroutine “HCR(\( f, d \))” of the hybrid method whose parameters are \( f \) and \( d \) is used as a function of \( D \) and \( f \), i.e., HCR(\( D, f \)). The initial condition is set as follows. At first, \( f \) is initialized using an arbitrary function \( G(x) \), which expresses the initial profile of \( f \), as

\[ f_i^0 = G(x_i). \]
Next, the initial value of $D$ is calculated using a recurrence procedure of

\begin{equation}
D^0_0 = 0,
D^0_i = D^0_{i-1} + h(f^0_i + f^0_{i-1})/2, \quad \text{for } i = 1, 2, \ldots, N.
\end{equation}

Here the last term of Eq. (22) means the approximated integration of $f$ over a cell between $x_{i-1}$ and $x_i$.

In the next section we show some numerical results using those procedures.

4 Numerical experiments

Example 1:

The first example is the linear propagation of square waves. In this example we use $u = 1$, $h = 1$ and

\[ G(x_i) = \begin{cases} 
-1, & \text{for } 13 \leq i \leq 21, \\
1, & \text{for } 40 \leq i \leq 48, \\
0, & \text{elsewhere}. 
\end{cases} \]

Thus, the width of the square waves is $9h$. Figures 1 and 2 show the results using CFL = 0.2 at $n = 200$ and $n = 2000$, respectively. Here we show results using four types of method, the HCR, CIP (the case of $\alpha = 0$), rational ($\alpha = 1$), and modified rational methods. In the modified rational method [18], a switching technique, which is represented as

\[ \gamma = \begin{cases} 
1, & d_i \cdot d_{i+1} < 0, \\
0, & \text{otherwise}, 
\end{cases} \]

is added to the conventional rational method to select the interpolation function. The rational function is adopted in the case of $\gamma = 1$ and the cubic one in the case of $\gamma = 0$. It was pointed out that this technique sometimes breaks the preservation of the convexity of solution while this improves the dissipation property of the rational method [15]. In the present example, this technique is adapted by using $f$, not by $d$, because of the replacements. In the figures, we see clearly that the HCR method provides most accurate results, which are less diffusive than those by the conventional rational method.
Fig. 1. Linear propagation of square waves by (a): the HCR, (b): CIP, (c): rational, and (d): modified rational methods with the replacements of Eq. (21). Those results are with CFL = 0.2 at \( n = 200 \). The circles and the solid lines show the numerical and theoretical results, respectively.

Fig. 2. Same as Fig. 1 but \( n = 2000 \).

and less oscillatory than those by the CIP and modified rational methods. In Table 1, we show the calculated values at \( n = 200 \) around the point \( i = 12 \) at which the left discontinuity of the negative pulse locates. The calculated data by the CIP and modified rational methods are positive near the left side of the discontinuity and become minimums at \( i = 14 \), which are smaller than \(-1\). On the contrary, the data by the HCR and conventional rational methods decrease monotonously as \( i \) increases, and are not less than \(-1\). Those results mean that only the HCR and conventional rational methods can keep
Fig. 3. Same as Fig. 2 but the replacements of Eq. (21) are not adapted; namely those schemes are solved as non-conservative methods.

| $i$  | Present | CIP     | Rational | Mod. Rat. |
|------|---------|---------|----------|-----------|
| 4    | 0       | -0.000014| 0        | 0         |
| 5    | 0       | -0.000986| -0.000004| 0         |
| 6    | 0       | -0.001887| -0.00037 | 0.000019  |
| 7    | -0.000001| 0.004413| -0.000345| 0.000223  |
| 8    | -0.000044| 0.024674| -0.002738| 0.000387  |
| 9    | -0.001716| 0.032729| -0.018069| -0.000109 |
| 10   | -0.052075| -0.052964| -0.09232 | -0.04052  |
| 11   | -0.305191| -0.304522| -0.32068 | -0.277075 |
| 12   | -0.681895| -0.665011| -0.67146 | -0.647653 |
| 13   | -0.954887| -0.955764| -0.90682 | -0.951461 |
| 14   | -0.999656| -1.058063| -0.982846| -1.059713 |
| 15   | -0.999996| -1.029841| -0.997188| -1.031324 |
| 16   | -0.999997| -0.999959| -0.998634| -1.000665 |

Table 1
Calculated values of the results corresponding to those shown in Fig. 1.

For comparison, we shall show results without the replacement of Eq. (21). Figure 3 shows the results at $n = 2000$ given by using the same initial condition as the previous one. In this case, we set the initial value of the first derivative as $d_i^0 = 0$ at all points. As was pointed out in Ref. [12], the CIP-CSL2 and CIP methods (the CIP with and without the replacements, respectively) give quite similar results each other. However, another methods provide obviously different results by whether the replacements are employed or not. It is interesting to point out that the result by the HCR method with the replacements is less diffusive than that by without the replacements, while we cannot explain this
result for the moment.

Example 2:

Next, we solve the problem used in Ref. [16] to compare the performance of the conventional rational method with those of the MUSCL [9] and PPM [10] schemes (See Fig. 1 in the paper). In this example, we use $h = 1$ and $CFL = 0.2$. The initial value of $f$ is set to

$$G(x_i = i) = \begin{cases} 
(i - 20)/11, & \text{for } 20 \leq i < 31, \\
1 - (i - 31)/20, & \text{for } 31 \leq i < 41, \\
1/2, & \text{for } 41 \leq i < 60, \\
1, & \text{for } 60 \leq i < 80, \\
0, & \text{elsewhere},
\end{cases}$$

and the velocity, assumed as constant in time and space, is set to $u = 1$. Figures 4 and 5 show the results at $n = 440$, given by the four types of methods, with and without the replacements, respectively. In Ref. [16] it was concluded that the conventional rational method gives better representation of the sharp corner at the top of the triangular wave than those by the MUSCL and PPM methods. From the present result, however, we know that the HCR method and its conservative version achieve still better representation of that than the conventional rational method. The calculated maximum values of $f$ at the corner are 0.935 (conservative HCR), 0.937 (HCR), 0.916 (conservative rational), and 0.923 (rational). Furthermore, we know that the conservative HCR method gives almost equivalent (or slightly better) resolution of the square wave to that by the PPM, which gives the best one among those by the schemes discussed in Ref. [16].

Example 3:

Now we show results to a nonlinear problem in which the inviscid Burgers’ equation,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0,$$  \hspace{1cm} (23)

is selected as the governing equation [11,12]. This equation can be solved directly by the conservative methods discussed in this paper. For the calculation
of $u$, we can use the procedures for $f$ because Eq. (23) is an advection equation (here $f$ means that after the replacements). For the calculation of $D$ ($= \int u \, dx$), we need to rewrite Eq. (23) into a conservation form as

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2/2)}{\partial x} = 0. \quad (24)$$

From this formula we found that the transportation velocity $u$ used in the procedures for $D$ should be replaced with $u/2$ [11].
Fig. 6. Result of inviscid Burgers’ equation at $t = 100$ by (a): the HCR method with the replacements, (b): the CIP-CSL2 method, and (c): the HCR method without the replacements. The parameters are $h = 1$ and $\Delta t = 0.1$. The solid lines show results by the first-order upwind method with $h = 1/5$ and $\Delta t = 0.1/5$.

Fig. 7. Maximum and minimum values in the results of inviscid Burgers’ equation, as a function of time. Those results are by (a): the present and (b): CIP-CSL2 methods with $h = 1$ and $\Delta t = 0.1$, and by (c): the first-order upwind method with $h = 1/5$ and $\Delta t = 0.1/5$. 
The initial condition and the grid width are set to

\[ G(x) = 0.5 + 0.4 \cos(2\pi x/100) \]

and \( h = 1 \). Figure 6 shows results at \( t = 100 \) with \( \Delta t = 0.1 \). In this figure, the solid line shows the result by the first-order upwind method using the conservation formula (24) with \( h = 1/5 \) and \( \Delta t = 0.1/5 \). Same as the result by the CIP-CSL2, that by the conservative HCR method represents the correct position of the shock wave. Figure 6(c) shows the result by the HCR method without the replacements. This result is obviously incorrect. Those results prove that the HCR method has gained complete conservation by the replacements. Figure 7 shows the maximum and minimum values of \( u \) as a function of time. In the result by the CIP-CSL2 we can observe unphysical strong oscillation due to the numerical dispersion of a method using a classical polynomial interpolation; however, such the oscillation cannot be seen in the result by the present method. This result allows us to consider that the present method is applicable to a shock problem even without any artificial viscosity, and is oscillation free even for the nonlinear problem.

**Example 4:**

Now we shall try an experimental study which uses further replacements. We already know that the CIP method provides an almost equivalent result whether the replacements (21) are adapted or not. If the variables \( D \) and \( f \) are replaced with those integrated once more, what result is obtained? Figure 8 shows results to Example 2 obtained by replacing \( D^0_i \) and \( f^0_i \) with

\[ E^0_i \equiv \int_{x_0}^{x_i} D^0 dx \quad \text{and} \quad D^0_i \]

and by updating those variables with the HCR and conventional methods. The parameters are same as those used in Example 2. \( f_i \) shown in Fig. 8 is given by \( f_i = (E_{i+1} - 2E_i + E_{i-1})/h^2 \). Despite the adaptation of two times of the replacements, the CIP method provides an almost equivalent result to those given without or with one-time replacements. However, one can observe some differences in the results using another methods from the previous ones. The numerical oscillation around the discontinuities of the square wave appears even in the results obtained by using the HCR and the conventional rational methods. In the result by the modified rational method, the oscillation can be observed not only at the top of the square wave but also at the bottom of it.
5 An additional comment

It is known that when no artificial viscosity is adapted, a numerical method for an advection equation based on the non-conservative formula cannot give correct propagation speed of shock in the region where the direction of the advection velocity changes spatially. In Ref. [11] the use of high-order polynomial that covers two cells are discussed for solving this problem. Now we present an alternative method to overcome this difficulty.

Here we use again the Burgers’ equation (23) as the governing equation. Introducing \( U \equiv u + C \), where \( C \) is a constant, reduces Eq. (23) to

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} - C \frac{\partial U}{\partial x} = 0.
\]

This is also the Burgers’ equation, but has a linear advection term (the last term of left hand side). We can solve this equation by splitting into two stages. First, we solve \( \partial U/\partial t + U \partial U/\partial x = 0 \) and update \( U \) during the time interval \( \Delta t \). Next, we shift the position of \( U \) by \( -C \Delta t \). If \( C \) is determined so that \( U_i > 0 \) or \( U_i < 0 \) for all \( i \), we may be able to calculate the position of shock correctly. In the case where we set \( C \) so that the condition of \( m |C\Delta t| = h \) (where \( m \) is a positive integer) is satisfied, the shift step can be solved only by one substitution like \( U_{i+1} = U_i \) per \( m \) computational steps. This procedure is convenient since we need no additional interpolation function or difference equation; however, further discussions should be required for the practical application.
6 Conclusion

In this paper we have derived a conservative variant of the hybrid-cubic-rational method for an advection equation. It was proved by some numerical experiments that the hybrid method is oscillation free even when it is modified to a conservative one. Interestingly, the conservative hybrid method gives less-diffusive representation of the discontinuities of the square wave than that by the non-conservative one. We should try to make this result clear theoretically.

We expect that combining the present method with the hybrid interpolation-extrapolation method, which realizes the discontinuous representation of the density interface [19], yields a powerful tool for a semi-Lagrangian computation of multiphase flows like those treated in Refs. [20,21].

References

[1] A. Staniforth, J. Côté, Mon. Wea. Rev. 119 (1991) 2206.
[2] D. R. Durran, Numerical methods for wave equations in geophysical fluid dynamics, Springer, Sec. 6, p.303.
[3] A. Priestley, Mon. Wea. Rev. 121 (1993) 621.
[4] B. P. Leonard, A. P. Lock, M. K. Macvean, Int. J. Num. Meth. Heat Fluid Flow 5 (1995) 341.
[5] J. R. Manson, S. G. Wallis, Commun. Num. Meth. Eng. 11 (1995) 1039.
[6] B. P. Leonard, Comput. Meth. Appl. Mech. Eng. 19 (1979) 59.
[7] J. R. Manson, S. G. Wallis, Adv. Env. Res. 1 (1997) 98.
[8] S. Lin, R. B. Rood, Mon. Wea. Rev. 124 (1996) 2046.
[9] B. Van Leer, J. Comput. Phys. 32 (1979) 101.
[10] P. Colella, P. R. Woodward, J. Comput. Phys. 54 (1984) 174.
[11] R. Tanaka, T. Nakamura, T. Yabe, Comput. Phys. Commun. 126 (2000) 232.
[12] T. Yabe, R. Tanaka, T. Nakamura, F. Xiao, Mon. Wea. Rev. 129 (2001) 332.
[13] T. Yabe, T. Aoki, Comput. Phys. Commun. 60 (1991) 219.
[14] T. Yabe, T. Ishikawa, P. Y. Wang, T. Aoki, Y. Kadota, F. Ikeda, Comput. Phys. Commun. 66 (1991) 233.
[15] M. Ida, Comput. Fluid Dyn. J. 10 (2001) 159.
[16] F. Xiao, T. Yabe, T. Ito, Comput. Phys. Commun. 93 (1996) 1.
[17] R. Tanaka, T. Nakamura, T. Yabe, in: Proc. of 13th Symp. on Comput. Fluid Dyn., Tokyo, Japan, 1999 (in Japanese).

[18] F. Xiao, T. Yabe, G. Nizam, T. Ito, Comput. Phys. Commun. 94 (1996) 103.

[19] M. Ida, Comput. Phys. Commun. 132 (2000) 44.

[20] S. Y. Yoon, T. Yabe, Comput. Phys. Commun. 119 (1999) 149.

[21] M. Ida, Y. Yamakoshi, Jpn. J. Appl. Phys. 40 (2001) 3846.