Information evolution in the interior of an axially symmetric BTZ black hole

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Abstract

In this paper, we consider an axially symmetric (2 + 1)–dimensional rotating Banados-Teitelboim-Zanelli (BTZ) black hole to investigate its interior information. First, we choose a largest space-like hyper-surface at \( r_v = 0.45 \) and calculate the maximal interior volume bound by it. We found the interior volume to increase with advance time \( v \). Similarly, the scalar quantum mode entropy is also found to increase with advance time. Next, considering two important assumptions, an evolution relation is obtained between the variation of scalar quantum mode entropy and Bekenstein Hawking entropy for an infinitesimal interval of time. In contrast to the evolution relation of higher dimensional black holes, the characteristic feature of this relation is its increase with extremely large increase in black hole mass. Moreover, this work extends the notion of black hole evaporation idea to lower space-time dimensions.

Keywords: BTZ black hole, interior volume and entropy, information paradox, evaporation.

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1 Introduction

In general relativity (GR) the notion of black hole interior volume depends on how the space-time is sliced. The choice of spacial hyper-surface and interchange of space-times coordinates are main problems in the investigation of interior volume. Parikh [1], found the interior volume of a stationary black hole as slicing invariant. Similarly, the thermodynamic volume of a black hole is studied as conjugate to cosmological constant [5–7]. The volume for dynamical black hole in terms of Kodama vector is proposed in [2, 3] and the vector volume is discussed in [4]. Recently, Christodoulou Rovelli (CR) [8] proposed the interior volume of a two-sphere $S^2$ as the volume of the largest space like spherically-symmetric hyper-surface bounded by $S^2$. This hyper-surface must be the largest one and lie on simultaneity surface. Conventionally, the volume bounded by such a sphere is $V = \frac{4}{3}\pi r^3$ (where $r$ is the radius). This means that a 3d space-like hyper-surface $\Sigma$ bounded the volume $V$.

In curved space-time, one can’t define the simultaneity surface so, a different technique is needed to define the interior volume of black hole. For this purpose a Penrose diagram is given in Fig. (1) below to choose a space-like hyper-surface enclosing maximum interior volume. From this figure, the increase in black hole interior volume is the contribution of part (2) of the hyper-surface. According to CR, the maximal interior volume of a sphere immersed in curved space-time is

$$V_\Sigma = 3\sqrt{3}\pi m^2 v; \tag{1}$$

Along this way, many authors defined the interior volume of different black holes for $D \geq 4$ [10–13, 15–17]. In all these analysis the interior volumes of black hole is found directly increasing with advance time $v$. Based on CR work, Baocheng Zhang [18, 19] considered the interior quantum modes of scalar field and investigated the entropy inside the black hole as

$$S_\Sigma = \frac{\pi^2}{45}T^3V_\Sigma; \tag{2}$$

From this Eq. (2), the quantum mode entropy is proportional to interior volume of the black hole and hence to advance time. This relation of advance time with interior volume and entropy has special role in affecting the statistical quantities and probing the information from the interior of black hole. According to Parikh [1], if one could investigate a relation between the interior and exterior entropy of black hole then we will be able to discuss the black hole evaporation. So, using the notion of interior entropy with two important assumptions of "black hole radiations as black body radiations" and "the emission rate as quasi-static", it is found that the rate of change of interior entropy is related to the Bekenstein Hawking entropy by a simple relation, for detail see [15, 20–22].

In Ref. [14], M. Zhang followed the CR work and found the interior volume of BTZ black hole increases with advance time. This work showed that the above stated discussion could be extended to $(2 + 1)$– dimensional BTZ black holes. So, in our recent work [31], we investigate the interior volume and quantum mode entropy of charged BTZ black hole also increases with advance time. Finally, the relation between the interior and exterior entropy is calculated as linear function of black hole mass. It gives a different notion in comparison with higher dimensional black holes. The main reason for this different result is its property of positive heat capacity.

In this work, we extend the above discussion to rotating BTZ black hole for investigating its interior information. The structure of this paper is such that in next section 2. we will discuss the metric for rotating BTZ black hole and its interior volume. In section 3, the quantum modes entropy for massless scalar field inside rotating BTZ black hole and its evolution relation
Figure 1: Figure shows the Penrose diagram of BTZ black hole formed under collapsed process. Here the green and red colors shows the event horizon and Cauchy horizon. A space-like hyper-surface is drawn form event horizon to the center of collapsed matter. We divided this hyper-surface into three parts: Part (1) is a null surface enclosed in a small circle. It has null volume. Part (2) is a stretched straight line extending from event horizon to the surface of collapsed matter. The volume enclosed by this part is claimed to increase with advance time [8] and part (3) is the part of hyper-surface in the interior of collapsed matter called limited surface with constant volume. So, part (1) has null volume, and part (3) has constant volume whereas, the increase in the black hole volume is the contribution of part (2).
with Bekenstein Hawking entropy will be discussed. After analyzing the final investigations, we will discuss the results and its comparison to those in high dimensional black holes in section 4.

2 Metric of rotating BTZ black hole and Interior volume

BTZ black hole is a solution of Einstein-Maxwell equations in \((2 + 1)\)-dimensional space-time, with negative cosmological constant and constant negative curvature \([23–25]\). According to Carlip \([27]\) the classical and quantum properties of \((2+1)\)-dimensional BTZ black hole shares many characteristics to \((3 + 1)\)-dimensional black holes. The metric of \((2 + 1)\)-dimensional rotating BTZ black hole is defined as

\[
\begin{align*}
    ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(N^\phi(r)dt + d\phi)^2;
    \end{align*}
\]

(3)

Where the lapse function and angular shift function are

\[
\begin{align*}
    f(r) &= -m + \frac{r^2}{l^2} + \frac{J^2}{4r^2}; \quad N^\phi(r) = -\frac{J}{2r^2}, (|J| \leq ml);
\end{align*}
\]

(4)

Here \(\Lambda = -\frac{1}{l^2}\) is the cosmological constant, \(\phi\) is the period with range \(0 \leq \phi \leq 2\pi\) for representation of black hole space-time, \(j, m, l\) are the azimuthal angular momentum, AdS mass and AdS radius corresponding to angular velocity \(\Omega(r)\) respectively. Since, the metric of rotating BTZ black holes has an azimuthal symmetry so, the angular momentum is conserved under coordinate transformation. The mass \(m\) and Bekenstein Hawking entropy \(S_{BH}\) of BTZ rotating black hole at horizon can be defined as \([23]\)

\[
\begin{align*}
    m &= \frac{r^2}{l^2} + \frac{J^2}{4r^2}; \quad S_{BH} = 2\pi r_+;
\end{align*}
\]

At horizon, the lapse function vanishes so,

\[
\begin{align*}
    r_\pm &= \sqrt{\frac{2m}{l^2}(1 \pm X)}, \quad X = \sqrt{1 - \left(\frac{J}{lm}\right)^2};
\end{align*}
\]

(5)

This Eq. (5) shows the coordinates singularities for rotating BTZ black hole occurs at \(\frac{J}{lm} = 1\). Using the lapse function, one can easily defines the temperature at horizon as

\[
\begin{align*}
    T = \frac{f''(r_+)}{4\pi} = \frac{mX}{\pi\sqrt{2l^2m(1 + X)}},
\end{align*}
\]

(6)

where the dash (’) represents the derivative with respect to \(r\). The angular velocity around the axis of rotation in its actual form can be defined as

\[
\begin{align*}
    d\phi = \Omega dt = \frac{J}{2r_+^2} dt = -N^\phi(r) dt;
\end{align*}
\]

(7)

Note that this definition of angular velocity is valid only in \((2 + 1)\)-dimensional space-time. The position of ergo-sphere is \(r_+ < r < r_s\), where Penrose process and Misnor super-radiation process
is active. The heat capacity due to constant angular momentum $J$ and constant angular velocity $C_\Omega$ are

$$C_J = T \left( \frac{\partial S}{\partial T} \right)_J = C_o \frac{X}{2 - X} \sqrt{\frac{1 + X}{2}}; \quad C_\Omega = T \left( \frac{\partial S}{\partial T} \right)_\Omega = 4\pi \sqrt{\frac{m(1 + X)}{2}};$$

(8)

where $C_o = 4\pi r_+$ is the heat capacity at $J = 0$. As $X < 1$ so, for BTZ black hole, heat capacity is always positive. The vacuum state has no traces of mass i.e. $m = 0 \Rightarrow J = 0$, so the metric is

$$ds^2 = -\left( \frac{r}{l} \right)^2 dt^2 + \left( \frac{r}{l} \right)^2 dr^2 + r^2 (N^\phi(r) dt + d\phi)^2;$$

(9)

From Eq. (3), the metric can be written as [27]

$$ds^2 = -(f(r)v^2 + 2\dot{v}r) d\lambda^2 + r^2 (N^\phi(r) dt + d\phi)^2;$$

(10)

As already stated that the shift function $N^\phi(r)$ is corresponding to angular velocity $\Omega$ so, it has direct affect on it. By using this notion, one can evolve maximum or minimum value of red-shift related to clock-wise or counter clock-wise angular velocity. This metric satisfies the ordinary field equation and shows a uniformly rotating BTZ black hole geometry, for more detail read Refs. [24,27,28]. BTZ black hole geometry in Eddington-Finkelstein coordinates can also be written as

$$ds^2 = -(f(r)v^2 + 2\dot{v}r) d\lambda^2 + r^2 (N^\phi(r) dt + d\phi)^2;$$

(11)

BTZ black hole is axially symmetric with conserved angular momentum $J$ so, any deviation from axial symmetry will have negligible effects at black hole horizon. In contrast, these forces will be more weak in the interior of black hole at $r = r_v$. Generally, the maximum interior volume bounded by the largest hyper-surface in the interior of BTZ black hole is

$$V_{\Sigma} = 2\pi v \sqrt{-r_\Sigma^2 f(r_v)};$$

(12)

Here the factor $\sqrt{-r_\Sigma^2 f(r_v)}$ guarantees the non-negativity of this expression under radical sign. Its maximum value for other than Schowarschid black hole can be easily evaluated by numerical method for detail see Ref. [13,35]. From this equation, the interior volume of black hole will be maximum for $r = r_v$, which we labeled as maximal hyper-surface ranging form $r_- < r < r_+$ and could be obtained by the maximization of polynomial $\sqrt{-r_\Sigma^2 f(r_v)}$ as

$$r_v = l^2 m \left( \sqrt{3X^2 + 1 + 2} \right) \sqrt{6};$$

(13)

The position of the maximal hyper-surface is numerically found at 0.45 as shown in Fig. (2) below.

So in case of very large $v$, the largest spherically symmetric spacelike hyper-surface is formed by a long stretched nearly constant radius at $r = r_v$ with $(0, r_s)$ as its end points. Using Eq. (13), one can get the maximal interior volume form Eq. (12). Which shows that the interior volume of BTZ black hole is proportional to advance time. This means that BTZ black hole has a large scope of storing information in its interior. As the quantum mode entropy of black hole is directly proportional to interior volume. So, the quantum mode entropy of BTZ black hole will also be proportional to advance time. This character of BTZ black hole could affect the statistical quantities inside it. This case will be discussed in next section.
3 Entropy of massless scalar field and its evolution relation with Bekenstein Hawking entropy

According to Refs [30,31], the interior entropy of massless scalar field $\Phi$ in the interior volume of BTZ black hole bound by the maximal hypersurface $r = r_v$ can be calculated by using WKB approximation. Where the massless scalar field is $\Phi = \exp[-iET]\exp[iI(\lambda, \phi)]$. So, using the Klein Gordon equation $\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\Phi) = 0$, we get

$$E^2 - \frac{1}{(-f(r)v^2 + 2\dot{v}r)}P_\lambda^2 - \frac{1}{r^2}P_\phi^2 = 0; \quad (14)$$

where $\partial_\lambda I = P_\lambda^2$, $\partial_\phi I = P_\phi^2$ are used. This equation can be written as

$$P_\lambda = \sqrt{-f(r)v^2 + 2\dot{v}r}\sqrt{E^2 - \frac{1}{r^2}P_\phi^2}; \quad (15)$$

Following Ref. [31], the quantum mode entropy for an inverse temperature $\beta = \frac{1}{T}$ is calculated as

$$S = \frac{3\zeta(3)V_{\Sigma}}{4\pi^2\beta^2}; \quad (16)$$

Using Eq. (12) along with (6), the quantum modes entropy in the interior of massless scalar field for rotating BTZ black hole is obtained as

$$S = \frac{3\zeta(3)\sqrt{-r_v^2f(r_v)}}{2\beta^2}v; \quad (17)$$

This Eq. (17) shows the entropy is also related linearly to advance time, which is an important feature of this entropy to affect the statistical quantities in the interior of BTZ black hole. To see this, let us consider two important assumptions, which could lead us to investigate the evaluation relation between quantum modes entropy and Bekenstein Hawking entropy. These assumptions are:
Black hole radiation as black body radiations: This assumption could guarantees the black hole temperature seen from infinity as event horizon temperature so, one can use the Boltzmann law for the emission of radiation \[18,22,33\]. In \((2+1)\) dimension space-time of rotating BTZ black hole the Boltzmann law can be written as

\[
\frac{dm}{dv} = -\sigma A T^3 \Rightarrow dv = -\frac{\beta^3 \gamma}{A} dm; \tag{18}
\]

where \(A = \pi l \sqrt{2m(X+1)}\) is the area of event horizon and the \(\beta\) value can be found from Eq. (6).

The rate of radiation emission from black hole as quasi-static process i.e. \(\frac{dm}{dv} \ll 1\). which means the evaporation process is slow but, Hawking temperature varies continuously. From this assumption the thermal equilibrium between the scalar field inside black hole and the event horizon is preserved in adiabatic process. It could guarantees the differential form of radiation emission at infinity small interval of time.

So, fixing these assumptions in Stephan Boltzmann law with the values of \(\beta\) and \(A\), we get the differential form of quantum mode entropy as

\[
\dot{S} = -\frac{3\zeta(3)\gamma \sqrt{-r_v^2 f(r_v)}}{2} \left( \frac{\beta}{A} \right) \dot{m}; \tag{19}
\]

where, the differential form is denoted by a dot (\(\dot{\}\)). As in our case the black hole angular momentum is conserved and no work is done by the black hole. So, the first law of black hole thermodynamics for a spherically symmetric uncharged black black hole can the First law of black hole thermodynamic as

\[
\dot{m} = \frac{\dot{S}_{BH}}{\beta}; \tag{20}
\]

Where \(S_{BH}\) is the horizon entropy of black hole. Using this equation in Eq. (19), we get a simplified relation between the interior and exterior entropy of black hole as

\[
\dot{S} = -\frac{3\zeta(3)\gamma \sqrt{-r_v^2 f(r_v)}}{2A} \dot{S}_{BH}; \tag{21}
\]

This equation gives a direct relation between the two type of entropy. As, we know that the quantum modes entropy is directly related to the interior volume of black hole, which could be maximized by the polynomial \(\sqrt{-r_v^2 f(r_v)}\). So, we call this entropy as maximal interior entropy. On other hand, the area \(A\) of black hole will be constant at \(r = r_v\). This mean that the relation between the interior and exterior entropy of black hole is a function of \(m\) as the angular momentum is a conserved quantity. This relation between two entropy can be simply written as

\[
\dot{S} = -\frac{4\gamma\zeta(3)}{3\pi} F(m, J) \dot{S}_{BH}; \tag{22}
\]

The negative sign in Eq. (22) shows the quantum modes entropy increases with advance time whereas, Bekenstein Hawking entropy decreases. The subscript \(J\) represents the conservation of angular momentum. The proportional function is

\[
F(m, J) = \frac{\sqrt{m \left( 3X^2 + \sqrt{3X^2 + 1} - 1 \right)}}{\sqrt{X + 1}}; \tag{23}
\]
The plot in Fig. (3) shows the evolution relation as a function of black hole mass. This plot is similar to the power function of a variable, when the power is fraction between 0 and 1. The plot shows that as the BTZ black hole mass \((m)\) increases from zero, the slope of curve also increases. At the beginning, the BTZ black hole mass is seem to be constant for some increase in evolution relation or slowly increasing but after acquiring certain mass limit, the evolution relation increases with increase in black hole mass continuously without any divergence.

4 Results and Discussion

Using the CR investigations [8], we analyzed \((2+1)\)– dimensional rotating BTZ black hole. We consider a uniformly rotating BTZ black hole with conserved angular momentum \(J\), an imaginary hypersurface is supposed to extend form the event horizon to the center of collapsed matter. This hyper-surface is divided in three parts. the first and third parts are simultaneously null and limited surfaces. The part of hyper-surface, which contributes to the increase in interior volume of black hole is stretched straight part as shown in Fig. (1). This hyper-surface is calculated by maximization of factor \(\sqrt{-r_v^2 f(r_v)}\) in Eq (13). Numerically, the position of this maximal hyper-surface is found at \(r_v = 0.45\) as shown in Fig. (2). Using this hyper-surface, the maximal interior volume is found linearly increasing with advance time.

Further, we consider the quantum modes of scalar field in the interior of black hole and calculated the entropy in Eq. (17) with some inverse temperature \(\beta\). It is also found to increase with advance time. From the relation of advance time with black hole’s interior volume and quantum modes entropy, we conclude that this property of BTZ black hole could change the statistical quantities in its interior. To check this, we consider two important assumptions as ”black hole radiations as black body radiations” and ”the emission rate of black hole radiations as quasi-static process”. The first assumption guaranteed us to use Boltzmann law, and the second assumption led us to investigate the differential form of interior quantum modes entropy. This differential form of interior quantum mode entropy gives an evolution relation with differential form of Bekenstein Hawking entropy as connection between interior and exterior entropy of BTZ black hole. The surprising feature of this evolution relation \(F(m, J)\) is that this evaluation relation is a
function of BTZ black hole mass $m$ as given in Eq. (23).

This evolution relation is further analysed numerically and a plot of $F(m, J)$ vs. $m$ is drawn in Fig. (3). From this plot, we see that at the initial stages, the function $F(m, J)$ seems to increase with some constant mass value. Which may be due to some thermodynamic effect. But after some time the evolution function increases with increase of black hole $m$ up to very large value. This means that the case of evaporation as discussed for higher dimensional black holes is not fulfilled here. The fact is that in contrast to these black holes, BTZ black holes have positive heat capacity and don’t evaporate in an asymptotically AdS spacetime. The curve of proportional relation could be divided into two stages. In the initial stage, the vertical portion of curve along $F(m, J)$ shows a constant mass and $J$. This means that in the initial stage the mass of black hole remains constant but as the black holes ages, its mass increases. This result in contrast to higher dimensional black holes is very surprising and supports the idea of BTZ black hole anti-evaporation.

The Physical significance of this work is the extension of interior volume and entropy notion to the lower space-time dimensions. So, this result will give a new background to the solution of information paradox problem in lower dimensional AdS space-time.

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