Active-to-Absorbing Phase Transition Subjected to the Velocity Fluctuations in the Frozen Limit Case

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Abstract—The directed bond percolation process is studied in the presence of compressible velocity fluctuations with long-range correlations. We discuss a construction of a field theoretic action and a way of obtaining its large scale properties using the perturbative renormalization group. The most interesting results for the frozen velocity limit are given.

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INTRODUCTION

Non-equilibrium continuous phase transitions [1] have been and still constitute an object of intense research activity. Underlying dynamic laws are responsible for a more diverse behaviour in contrast to the equilibrium models. One of the most prominent examples is the directed bond percolation [2, 3] process that can be interpreted as a simple spreading process of an infection disease [4]. A distinguished feature is the presence of non-equilibrium phase transition between the active and absorbing state. There the correlation length diverges and large scale properties become independent on the microscopic details. It is well known that phase transitions are quite sensitive to additional disturbances such as quenched disorder [5] or long-range interactions [6]. From practical point of view this might be a reason why there are not so many experimental realizations for the percolation process [7].

In this paper we assume that the effect of environment can be simulated by advective velocity fluctuations with finite correlation in time and compressibility taken into account. Following the works [8, 9] we consider a generalization of the original Kraichnan model where the advection-diffusion problem of non-interacting admixture has been studied. As some progress in this direction has already been made [10–15], our main aim here is to elucidate in detail the case of frozen velocity limit [9, 16]. The main theoretical tool is the field-theoretic approach [17–19] with subsequent Feynman diagrammatic technique and renormalization group (RG) approach, which allows us to determine the large-scale behaviour, which is of special interest from macroscopic point of view.

The paper is organized as follows. In Sec. 1, we introduce a field-theoretic version of the problem and in Sec. 4 we present an analysis of possible regimes, which correspond to the fixed points (FP) of the RG equations.

1. FIELD-THEORETIC MODEL

Dynamic properties of the directed bond percolation process [2, 3] are governed by the following Langevin equation

\[ \partial_t \psi = D_0 (\nabla^2 - \tau_0) \psi - \lambda_0 D_0 \psi^2 / 2 + \eta, \]  

(1)

where the field \( \psi(t, x) \) is the density of percolating agents, \( \partial_t = \partial / \partial t \) is the time derivative, \( \nabla^2 = \partial / \partial x \) is the Laplace operator, \( D_0 \) is the diffusion constant, \( \lambda_0 \) is the coupling constant and \( \tau_0 \) controls a deviation

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from the criticality. The noise field $\eta(t, x)$ stems from the coarse graining procedure and mimics the influence of fast degrees of freedom. It is assumed that $\eta$ is a random Gaussian variable with zero mean and correlator in the following form
\[
\langle \eta(t_1, x_1) \eta(t_2, x_2) \rangle = g_0 D_0 \psi(t_1, x_1) \delta(t_1 - t_2) \delta^{(d)}(x_1 - x_2),
\]
where $\delta^{(d)}(x)$ is the $d$-dimensional delta function. The multiplicative factor $\psi$ in Eq. (2) ensures that the fluctuations cease in the absorbing state $\psi = 0$.

An advection process can be introduced in the standard fashion [20, 21] using the replacement of time derivative \( \partial_t \) by the Lagrangian derivative \( \mathbf{V}_t \equiv \partial_t + (\mathbf{v} \cdot \nabla) \). A basic idea of the Kraichnan model [8, 9] is that the velocity field $\mathbf{v}$ is a random Gaussian variable with zero mean and a translationally invariant correlator [9] conveniently given in the Fourier representation as follows
\[
\langle v_i v_j \rangle_0 (\omega, k) = |P^k_j + \alpha Q^k_j|D_\sigma(\alpha, k),
\]
\[
D_\sigma(\alpha, k) = \frac{g_{10}^{10} u_{10}^{10} D_0^{10} k^{4-d-y-\eta}}{\alpha^2 + u_{10}^{10} D_0^{10} (k^{2-\eta})^2}.
\]
Here, $P^k \equiv \delta_{ij} - k_i k_j \cdot k^2$ is a transverse and $Q^k \equiv k_i k_j \cdot k^2$ a longitudinal projection operator. Further, the function $D_{\sigma}$ is known as a kernel, where $k = |k|$ and a positive parameter $\alpha > 0$ can be interpreted as a deviation [9] from the incompressibility condition $\nabla \cdot \mathbf{v} = 0$. The coupling constant $g_{10}$ and the exponent $\gamma$ describe the energy spectrum [8, 9] of the velocity fluctuations. The constant $u_{10} > 0$ and the exponent $\eta$ are related to the characteristic frequency of the mode with wavelength $k$. The kernel function for the correlator (3) contains special cases: rapid-change model, frozen velocity ensemble and turbulent advection (see [8, 9]). Our aim here is to analyze the case of the frozen velocity limit, which is obtained in the limit $u_{10} \to 0$. It bears resemblance to a model of random walks in a random environment with long-range correlations [16]. In this case the correlator (3) takes the form
\[
\langle v_i v_j \rangle_0 (\omega, k) \to |P^k_j + \alpha Q^k_j|g_0 D_0 k^{2-d-\gamma}.
\]

We easily observe that velocity correlator does not depend on the time difference ($t - t'$). The name “frozen” is motivated by the detailed analysis [8], where it was shown that in this limit the correlation time of the velocity field $t_v(k)$ as a function of momentum scale $k$ is much larger than the correlation time of the advecting scalar quantity $t_v(k)$ and effectively $t_v$ can be set to infinity. Hence from the point of view of $\psi$ field the velocity fluctuations are not evolving in time, hence “frozen”.

The stochastic problem (1) and (3) is amenable to the full machinery of the quantum field theory methods such as Feynman diagrammatic expansion and renormalization group approach. The latter can be done in a straightforward fashion following the well-established theory [19] and constitutes our main theoretical tool. According to a general theorem [19], the stochastic problem is equivalent to the field theoretic model of a doubled set of fields $\tilde{\psi}, \psi$ with De Dominici-Jansen functional written in a compact form as follows
\[
\tilde{\mathcal{F}}[\tilde{\psi}, \psi, \mathbf{v}] = \tilde{\mathcal{F}}[\nabla \cdot \mathbf{v} + D_0 \tau_0] + \psi \partial_t \psi + \nu D_0^{-1} \partial_t^2 + \psi^2,
\]
where we used a condensed notation in which integrals over the spatial variable $x$ and the time variable $t$ and summation over repeated index are implicitly assumed. For example, the first term on the right-hand side in (5) stands for $\int dt \int d^d x \tilde{\psi}(t, x) \partial_t \psi(t, x)$. The last two terms in the action (5) have been added in order to ensure renormalizability of the model [13, 15]. Their presence can be tracked down to the broken Galilean invariance of the velocity ensemble and assumed compressibility. These considerations directly lead to the more involved numerical analysis of the flow equations than previously considered models [10–13].

2. RG ANALYSIS

A crucial task of statistical physics is to determine properties of correlation functions (in quantum field theory known as Green functions) for the given theory. In statistical field theory correlation functions can be given in the form of functional (path) integrals and further using perturbation theory it is possible to rewrite them in series of Feynman diagrams [1, 18, 19]. The functional formulation provides a convenient theoretical framework suitable for applying methods of quantum field theory.

If a model under consideration is multiplicatively renormalizable, the correlation between ultraviolet (UV) and infrared (IR) singularities for a logarithmic theory can be exploited [18, 19] and a non-trivial information about large-scale (macroscopic) behavior can be extracted.

A proper renormalization procedure is needed for the elimination of UV divergences. There are various renormalization prescriptions applicable for such problem, each with its own advantages [18]. In this work we employ the minimal subtraction (MS) scheme. UV divergences manifest themselves in the form of poles in the small expansion parameters and the minimal subtraction scheme is characterized by discarding all finite parts of the Feynman graphs in the
calculation of the renormalization constants. In the vicinity of critical points large fluctuations on all spatio-temporal scales dominate the behaviour of a system, which in turn results into the divergences in Feynman graphs. Resulting RG functions satisfy certain differential equations and their analysis provides us with an efficient computational technique for an estimation of universal quantities.

In order to apply the dimensional regularization for a computation of renormalization constants, an analysis of possible superficial divergences has to be performed [17, 19]. For translationally invariant systems, it is sufficient to analyze only 1-particle irreducible (1PI) graphs. In contrast to static models, dynamic models [1, 19] contain two independent scales: a frequency scale \(d_Q^k\) and a momentum scale \(d_Q^{\omega}\) for each quantity \(Q\). The corresponding dimensions are found using the standard normalization conditions

\[
d^k = -d^k = 1, \quad d^{\omega} = 0,
\]

\[
d^{\omega} = d^{\omega} = 0, \quad d^{\omega} = d^{\omega} = 1
\]

(6)
together with a condition field-theoretic action to be a dimensionless quantity. Using values \(d_Q^k\) and \(d_Q^{\omega}\), the total canonical dimension \(d_Q\) reads \(d_Q = d_Q^k + 2d_Q^{\omega}\). Its form is obtained from a comparison of IR most relevant terms \((\partial, \sim \nabla^2)\) in the free part of the action (5). The total dimension \(d_Q\) for the dynamical models plays the same role as the conventional (momentum) dimension does in static problems. The dimensions of all quantities for the model are summarized in Table 1. It follows that the model is logarithmic (when coupling constants are dimensionless) at \(\varepsilon = y = \eta = 0\), and the UV divergences are in principle realized as poles in these parameters. The total canonical dimension of an arbitrary 1-irreducible Green function is given by the relation

\[
d_\Gamma = d_\Gamma^k + 2d_\Gamma^{\omega} = d + 2 - \sum_{\phi} N_\phi d_\phi,
\]

(7)

\(\phi \in \{\bar{\psi}, \psi, \nu, \psi\}\).

Superficial UV divergences, whose removal requires counterterms, could be present only in those functions \(\Gamma\) for which \(d_\Gamma = d_\Gamma|_{\varepsilon=y=\eta=0}\) is a non-negative integer [19].

Dimensional analysis should be augmented by certain additional considerations. In dynamical models of the type (5), all the 1-irreducible diagrams without the fields \(\bar{\psi}\) vanish, and it is sufficient to consider the functions with \(N_\phi \geq 1\). As it was shown in [11] the rapidity symmetry (12) requires also \(N_\phi \geq 1\) to hold. Using these considerations together with the relation (7), possible UV divergent structures are expected only for the 1PI Green functions listed in Table 2.

A starting point of the perturbation theory is a free part of the action given by the expression

\[
\mathcal{J}_{\text{free}}[\bar{\psi}, \psi, \nu] = \bar{\psi}[\nabla_\Gamma + D_\Gamma(\tau_0 - \nabla^2)]\psi + \nu D^{-1}_\Gamma \psi 2 ,
\]

(8)

From free part we have to construct an inverse operator, whose components give us propagators. In graphical terms, they are then represented as lines in Feynman diagrams, whereas the non-linear terms correspond to vertices connected by lines. The non-linear (interaction) part of the action reads

\[
\mathcal{J}_{\text{int}}[\bar{\psi}, \psi, \nu] = D_\lambda\bar{\psi}[\psi - \bar{\psi}\bar{\psi}\psi] + \nu D^{-1}_\Gamma \psi
\]

(9)

As calculations show, the only poles to the one-loop order are of two types: either \(1/\varepsilon\) or \(1/y\). This simple picture pertains only to the lowest orders in a perturbation scheme. In higher order terms, poles in the form of general linear combinations in \(\varepsilon\) and \(y\) are to be expected.

The propagators are presented in the wave-number-frequency representation, which is for the translationally invariant systems the most convenient way for doing explicit calculations. The bare propagators are easily read off from the Gaussian part of the model given by (5). Their graphical representation is depicted

Table 1. Canonical dimensions of the bare fields and bare parameters for the model (5)

| \(Q\) | \(\bar{\psi}, \bar{\psi}\) | \(\nu\) | \(D_0\) | \(\tau_0\) | \(g_{10}\) | \(\lambda_0\) | \(u_{10}\) | \(u_{20}, a_0, \alpha\) |
|-----|------------------|--------|--------|--------|--------|--------|--------|------------------|
| \(d_Q^k\) | \(d/2\) | \(-1\) | \(-2\) | \(2\) | \(y\) | \(\varepsilon/2\) | \(\eta\) | \(0\) |
| \(d_Q^{\omega}\) | \(0\) | \(1\) | \(1\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(d_Q\) | \(d/2\) | \(1\) | \(0\) | \(2\) | \(y\) | \(\varepsilon/2\) | \(\eta\) | \(0\) |

Table 2. Canonical dimensions for the (1PI) divergent Green functions of the model

| \(\Gamma_{1-1}\) | \(\Gamma_{\bar{\psi}\psi}\) | \(\Gamma_{\bar{\psi}\psi\psi}\) | \(\Gamma_{\bar{\psi}\psi^2}\) | \(\Gamma_{\bar{\psi}\psi^2}\) | \(\Gamma_{\bar{\psi}\psi^2}\) |
|-------------|------------------|------------------|------------------|------------------|------------------|
| \(d_\Gamma\) | \(2\) | \(1\) | \(\varepsilon/2\) | \(\varepsilon/2\) | \(0\) |
| \(\delta_\Gamma\) | \(2\) | \(1\) | \(0\) | \(0\) | \(0\) |
in Fig. 1. The corresponding algebraic expressions can be easily read off and in the frequency-momentum representation are given by the expressions

$$\langle \psi \bar{\psi} \rangle_0 = \langle \psi \bar{\psi} \rangle^*_0 = \frac{1}{-i\omega + D_0(k^2 + \tau_0)}; \quad (10)$$

$$\langle \psi \nu \rangle_0 = \left[ D^{\nu}_{\bar{\nu}} + \alpha g_g \pi g_{10} D_0^3 k^{2-d-y} \delta(\omega) \right].$$

The interaction vertices from the nonlinear part of the action in -model in critical statics \[18, 19\] due to fluctuations effects. As such it is analogous to the mass term in $\phi^4$-model in critical statics \[3\], which is not captured by the dimensional regularization. It just represents a shift in critical probability due to fluctuations effects. As such it is analogous to the mass term in $\phi^4$-model in critical statics \[18, 19\].

The renormalization prescription (15) together with the renormalization of fields

$$\psi = Z_\psi \tilde{\psi}, \quad \bar{\psi} = Z_{\bar{\psi}} \tilde{\bar{\psi}}, \quad \nu = Z_\nu \tilde{\nu}$$

is sufficient for obtaining a fully renormalized theory. Thus the total renormalized action for the renormalized fields $\psi_R \equiv \{\tilde{\psi}, \tilde{\bar{\psi}}, \tilde{\nu}\}$ can be written in a compact form

$$\int \frac{d^d \psi_R}{(2\pi)^d} = \int \frac{d^d \tilde{\psi}}{(2\pi)^d} \left[ Z_\psi \partial_\mu \tilde{\psi} - \frac{1}{2} \nabla \cdot \tilde{\nu} \right]$$

The relations between renormalization constants follow directly from the action (17)

$$Z_1 = Z_\psi Z_{\bar{\psi}}, \quad Z_2 = Z_\psi Z_{\bar{\psi}} Z_D, \quad Z_3 = Z_{\bar{\psi}} Z_\psi Z_D \psi, \quad Z_4 = Z_{\bar{\psi}} Z_\psi \bar{\psi}, \quad Z_5 = Z_{\bar{\psi}} Z_\psi \bar{\psi}, \quad Z_6 = Z_{\bar{\psi}} Z_\psi \bar{\psi},$$

$$Z_7 = Z_{\bar{\psi}} Z_\psi Z_D \psi, \quad Z_8 = Z_{\bar{\psi}} Z_\psi Z_D^2 \psi.$$

The theory is made UV finite through the appropriate choice of RG constants $Z_1, ..., Z_8$. Afterwards the relations (18) yield the corresponding RG constants for the fields and parameters appearing in relations (15).

According to the general theory of RG approach \[19\], the nonlocal term in action (17) should not be
renormalized. Then from the inspection of the kernel function (3), two additional relations

\[ 1 = Z_{\phi} Z_{D}, \quad 1 = Z_{\phi} Z_{\phi} Z_{D} Z_{v}^{-2} \]  

(19)

follow, which have to be satisfied to all orders in the perturbation scheme.

3. CALCULATION OF THE RENORMALIZATION CONSTANTS

In order to perform one-loop renormalization of the model, altogether twenty Feynman graphs have to be analyzed. For the two-point Green functions \( \Gamma_{\psi\psi} \), we have

\[ \Gamma_{\phi\phi} : \frac{1}{2} \]

(20)

The perturbation expansions for the vertex functions contain these graphs

\[ \Gamma_{\phi\phi} : \frac{1}{2} \]

(21)

\[ \Gamma_{\phi\phi} : \frac{1}{2} \]

(22)

\[ \Gamma_{\phi\phi} : \frac{1}{2} \]

(23)

\[ \Gamma_{\phi\psi} : \frac{1}{2} \]

(24)

In these equations, we have explicitly stated a symmetry coefficient [19] of the corresponding graph. The numerical contributions arising from variational derivatives with respect to the external fields are included in the contribution of a given graph. Note that in the language of Feynman graphs the need for the term \( \propto \bar{\psi} \psi v^2 \) can be traced out to the presence of the second Feynman graph in (23), which does not vanish due to finite correlation time property of the velocity propagator (4).

The computation of the diverging parts of the Feynman graphs follows the standard methods of dimensional regularization [17, 19] and the 1-loop results are

\[
Z_1 = 1 + \frac{g_1 a (1-a)}{y} + \frac{g_2}{4\varepsilon}, \quad Z_2 = 1 - \frac{g_1}{4y} [3 - \alpha] + \frac{g_2}{8\varepsilon},
\]

\[
Z_3 = 1 + \frac{g_1 a (1-a)}{y} + \frac{g_2}{2\varepsilon}, \quad Z_4 = 1 + \frac{g_1}{4(1 + u_1)} [\alpha - 6u_2] + \frac{g_2}{4\varepsilon},
\]

\[
Z_5 = 1 + \frac{g_1 a (1-a)}{4y} [1 + 2(1-a)(2a-1)] - \frac{g_1 u_2}{4ay} [3 - \alpha + 2\alpha a] + \frac{g_2 (4a - 1)}{8a\varepsilon},
\]

\[
Z_6 = 1 - \frac{g_1 a (1-a)}{y} [1 - 3a] + \frac{g_2}{\varepsilon}, \quad Z_7 = 1 - \frac{g_1 a a}{y} [3a - 2] + \frac{g_2}{\varepsilon},
\]

\[
Z_8 = 1 + \frac{g_1}{2y} \left[ \alpha(2a - 2a^2 + 1) - \frac{\alpha a (1-a)}{u_2} - u_2(3 + \alpha) \right] + \frac{g_2}{2\varepsilon}.
\]
The ubiquitous geometric factor stemming from the angular integration is included into the renormalized charges $g_1$ and $g_2$ via the following redefinitions

$$\frac{g_1 S_d}{2(2\pi)^d} \rightarrow g_1, \quad \frac{g_2 S_d}{2(2\pi)^d} \rightarrow g_2,$$

where

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

is the surface area of the unit sphere in the $d$-dimensional space and $\Gamma(x)$ is Euler’s Gamma function.

It can be proven [15] that the field theoretic action (5) is closed with respect to the renormalization group analysis and no other interactions than that already involved in (5) are generated. In contrast to the standard $\varepsilon$-expansion in $\varphi^4$-theory we deal here with the three parameter expansion $(\varepsilon, y, \eta)$, which are considered to be of the same order of magnitude. Here as usual $\varepsilon = 4 - d$ is a deviation from the upper critical dimension $d_c = 4$ [3].

### 4. SCALING REGIMES

The beta-functions, which express the flows of parameters under the RG transformation [17], are defined through

$$\beta_s = \mu \partial_\mu g|_{0}.$$

Applying this definition to relations (15) yields

$$\beta_{s_1} = g_1 (-y + 2y_D - 2y_c),$$
$$\beta_{s_2} = g_2 (-\varepsilon - \gamma_s),$$
$$\beta_{a_1} = a_1^{-1} (-\eta + \gamma_D),$$

According to the general statement of the RG theory [17, 19], a possible IR asymptotic behaviour can be associated with an existence of the fixed point (FP) of the beta-functions. All fixed points are zeroes of beta-functions of the model, i.e.,

$$\beta_{s_1}(g^*) = \beta_{s_2}(g^*) = \beta_{a_1}(g^*) = 0,$$

where $g^*$ stands for an entire set of charges $\{g_1^*, g_2^*, u_1, u_2, a\}$. In what follows, the asterisk will always refer to coordinates of some fixed point. Whether the given FP could be realized in physical systems (IR stable) or not (IR unstable) is determined by eigenvalues of the matrix $\Omega = \{\Omega_{ij}\}$ with the elements

$$\Omega_{ij} = \frac{\partial \beta_i}{\partial g_j},$$

where $\beta_i$ is a full set of beta-functions and $g_j$ is the full set of charges $\{g_1, g_2, u_1, u_2, a\}$. For the IR stable FP the real parts of the eigenvalues of the matrix $\Omega$ have to be strictly positive. In general, these conditions determine a region of stability for the given FP in terms of $\varepsilon, \eta$, and $y$.

In order to obtain the RG equation, one can exploit a fact that the bare Green functions are independent of the momentum scale $\mu$ [17]. Applying the differential operator $\mu \partial_\mu$ at the fixed bare quantities leads to the following equation for the renormalized Green function $G_R$

$$\{\mathcal{D}_R + N_\varphi \gamma_\varphi + N_\varphi \gamma_\varphi + N_\varphi \gamma_\varphi\} G_R(e, \mu, \ldots) = 0,$$

where $G_R$ is a function of the full set $e$ of renormalized counterparts to the bare parameters $e_0 = \{D_0, \tau_0, u_0, u_20, g_{10}, g_{20}, a_0\}$, the reference mass scale $\mu$ and other parameters, e.g. spatial and time variables. The RG operator $\mathcal{D}_R$ is given by

$$\mathcal{D}_R \equiv \mu \partial_\mu + \sum g_\beta \partial g_\beta - \gamma_D \mathcal{D}_D - \gamma_t \mathcal{D}_t,$$

where $g \in \{g_1, g_2, u_1, u_2, a\}$, $\mathcal{D}_x = x \partial_x$ for any variable $x$, stands for fixed bare parameters and $\gamma_x$ are the so-called anomalous dimensions of the quantity $x$ defined as

$$\gamma_x \equiv \mu \partial_\mu \ln Z_{x,0}.$$

It turns out that for some fixed points the computation of the eigenvalues of the matrix (31) is cumbersome and rather unpractical. In those cases it is possible to obtain information about the stability from analyzing RG flow equations [19]. The parameter $s$ is a scaling parameter and we are interested in the behaviour of running charges in the IR limit $s \rightarrow 0$. The evolution of invariant charges is dictated by the equation

$$\nabla_s \mathcal{G} = \beta(\mathcal{G}).$$

Now we analyze obtained fixed points’ structure. First, we explicitly list analytical expressions for the coordinates of the fixed points. For the brevity have introduced a new parameter $a^\prime$ via the relation

$$a^\prime = (1 - 2a)^2.$$

In the studied frozen velocity limit [9] the fixed point coordinate of the charge $u^*_1$ is zero. The RG analysis reveals existence of eight possible regimes listed in Table 2. Only three points FP1, FP2 and FP3 are IR stable and hence possibly observable on the macroscopic scale.

The fixed point FP8 corresponds to the line of possible fixed points determined by the following system of equations

$$g_1^* (1 - 2u^*_2) = \frac{2y}{3}, \quad g_1^* (\alpha a^* - 3) = \frac{2y}{3} (\alpha - 3).$$
Further, the coordinates of two fixed points FP3 and FP4 are given by following expressions

\[ g^*_1 = \frac{-4}{(\alpha - 6)(\alpha - 12\alpha - 180)} \times [(\alpha^2 - 12\alpha - 72)e + 3(2\alpha - 2\alpha^2 + 54)y + 9A], \]

\[ g^*_2 = \frac{2[(2\alpha - 2\alpha^2 + 54)y + 36e + 3A]}{(\alpha - 12\alpha - 180)}, \]

\[ u^*_2 = \frac{4(\alpha - 3)e + (42 - 25\alpha)y + A}{8(\alpha - 6)e - 48(\alpha - 3)y}, \]

where \( A \) is

\[ A = [-8(\alpha^2 - 9\alpha + 12)e]y + (49\alpha^2 - 372\alpha + 1764)y^2 + 144e^2]^{1/2}. \]

The plus sign in (37) refers to the point FP3, whereas the minus sign for FP4.

The physical interpretation of the stable regimes is the following: the fixed point FP1 describes the free (Gaussian) theory for which all interactions are irrelevant and no scaling and universality is expected. The point FP2 is a standard nontrivial percolation regime [4] with irrelevant advection. The last point FP3 embodies a nontrivial regime for which both velocity and percolation interactions are relevant and corresponds to a new universality class. Because for points FP1 and FP2 the velocity field is irrelevant, their boundaries are not affected by the value of the parameter \( \alpha \). On the other hand the stability region of FP3 heavily depends on it. For incompressible case \( \alpha = 0 \) the boundaries between the regions can be computed exactly and can be read out from Fig. 2a. From (3) one can expect that the parameter \( \eta \) formally drops out of the theory. However, RG analysis shows that \( \eta \) affects the boundaries of fixed points. Note that in the blank space other regimes, e.g. rapid-change model or turbulent diffusion, can be realized.

With increasing value of parameter \( \alpha \) situation becomes more involved. Numerical analysis shows that up to \( \alpha = 6 \) the region of FP3 spreads to the origin, which can be seen on Fig. 2 for different values of \( \alpha \). Wherever it was possible, we have given the explicit form of boundary conditions. The main observation is that compressibility in cooperation with percolation interactions can lead to a stabilization of the nontrivial regime.

For \( \alpha \geq 6 \) the lower boundary of the regime FP3 can be computed exactly and is given by the following expression

\[ y = \frac{6(\alpha - 3)e + [\alpha(2\alpha - 21) - 54]\eta - 3\sqrt{D}}{(\alpha - 3)(\alpha + 3)}, \]

where \( D \) is the quadratic polynomial in the expansion parameters \( \eta, e \)

\[ D = [1764 + \alpha(49\alpha - 372)]\eta^2 - 4(\alpha - 42)(\alpha - 3)e\eta - 12(\alpha - 3)e^2. \]

This situation is depicted on Fig. 3. With increasing value of \( \alpha \) the regime FP3 is stable also in the region with negative values of \( \epsilon \) and the bottom boundary is given by Eq. (38). As FP3 can be IR stable also for \( \epsilon < 0 \), which corresponds to the space dimensions \( d > d_c = 4 \), we conclude that velocity fluctuations can...
destroy the expected mean-field behaviour. In the case of the pure irrotational (curl-free) field $\alpha \to \infty$ the bottom boundary becomes a line $y = 2\eta$.

From the theoretical point of view a special interest is paid to the three-dimensional Kolmogorov regime \[21\], which is obtained for the choice $y = 2\eta = 8/3$ and $\varepsilon = 1$. From Fig. 3 it follows this regime lies precisely on the boundary of the regime FP3. According to detailed numerical analysis \[15\] this regime competes with another nontrivial regime and exhibits non-universal behavior, i.e. dependence on the initial value of bare parameters.

5. CONCLUSIONS

In this paper we have studied percolation spreading in the presence of frozen velocity field with long-range correlations. The field theoretic formulation of the model has been formulated and the consequences following from its multiplicative renormalizability have been discussed. We have found that depending on the values of a spatial dimension $d = 4 - \varepsilon$, scaling exponents $y$ and $\eta$, which describe statistics of velocity fluctuations, the model exhibits various universality classes. Some of them are already well-known: the Gaussian (free) fixed point, a directed percolation without advection and the last one which corresponds

![Phase diagram obtained for $\alpha = 12$ (a) and in the limit of pure irrotational velocity field $\alpha \to \infty$ (b).](image)

Table 3. List of all fixed points obtained in the frozen velocity limit. The value of the charge $u_1^*$ is equal 0 for all points. NF stands for Not Fixed, i.e., for the given FP the corresponding value of a charge coordinate could not be unambiguously determined. Coordinates for FP3 and FP4 are too cumbersome to be given and are discussed in a text.

| FP   | $g_1^*$ | $g_2^*$ | $u_2^*$ | $a^*$ |
|------|---------|---------|---------|-------|
| FP1  | 0       | 0       | NF      | NF    |
| FP2  | 0       | $\frac{2\varepsilon}{3}$ | 0       | 0     |
| FP3  | $\frac{\alpha}{2(\alpha - 3)}$ | $\frac{\alpha}{2(\alpha - 3)}$ | $\frac{\alpha}{2(\alpha - 3)}$ | 0     |
| FP4  | $\frac{2\varepsilon}{9}$ | $\frac{2\varepsilon}{9}$ | $\frac{2\varepsilon}{9}$ | 0     |
| FP5  | $\frac{2\varepsilon}{9}$ | $\frac{2\varepsilon}{9}$ | $\frac{2\varepsilon}{9}$ | 0     |
| FP6  | $\frac{2\varepsilon}{2\alpha - 9}$ | $\frac{4[3\varepsilon + 2\varepsilon(\alpha - 6)]}{2\alpha - 9}$ | 1       | $\frac{\varepsilon(12 - \alpha) + 5\varepsilon(\alpha - 6)}{\alpha(\varepsilon - y)}$ |
| FP7  | $\frac{-2[6\varepsilon + 5\varepsilon(\alpha - 3)]}{3(9 + \alpha)}$ | $\frac{3[6\varepsilon + 5\varepsilon(\alpha - 1)]}{6\varepsilon + 5\varepsilon(\alpha - 3)}$ | $\frac{18\varepsilon - (\alpha - 6)(\alpha - 3)y}{\alpha(6\varepsilon + 5(\alpha - 3)y}$ | 0     |
| FP8  | NF      | 0       | NF      | NF    |
to new universality classes, for which mutual interplay between advection and percolation is relevant. As has been shown [22] the main drawback of the model is that anomalous scaling behavior is destroyed when $\alpha$ and $y$ are large enough. Therefore only relatively small values of $\alpha$ are admissible. In order to investigate the case of strong compressibility and clarify the role of compressibility one should try to apply more sophisticated model for the velocity fluctuations, e.g. one considered in [23–25].

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