Transformation optics with artificial Riemann sheets

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Abstract. The two original versions of ‘invisibility’ cloaks (Leonhardt 2006 Science 312 1777–80 and Pendry et al 2006 Science 312 1780–2) show perfect cloaking but require unphysical singularities in material properties. A non-Euclidean version of cloaking (Leonhardt 2009 Science 323 110–12) was later presented to address these problems, using a very complicated non-Euclidean geometry. In this work, we combine the two original approaches to transformation optics into a more general concept: transformation optics with artificial Riemann sheets. Our method is straightforward and can be utilized to design new kinds of cloaks that can work not only in the realm of geometric optics but also using wave optics. The physics behind this design is similar to that of the conformal cloak for waves. The resonances in the interior region make the phase delay disappear and induce the cloaking effect. Numerical simulations confirm our theoretical results.

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Since the early pioneering work on ‘invisibility shields’ [1, 2], the field of transformation optics has become a focus of intense activity for physicists, electronic engineers and materials scientists [3, 4]. Two approaches to this topic have prevailed: that devised by Pendry et al [2] provides a general platform for understanding the principles of controlling the electromagnetic field and a powerful tool for designing novel devices [3, 4], while that by Leonhardt [1] presents a specific tool—optical conformal mapping—which emphasizes further refractive index profile operations and leads to the design of isotropic cloaking. It has not been clear, however, if these two approaches are congruent.

In this work, we present a unified approach to transformation optics, which contains both the methods [1, 2] as its special cases. In particular, we first propose new multi-valued mappings with artificial Riemann sheets based on [2], and then apply further refractive index profile operations following [1]. This shows that Leonhardt’s approach can be extended beyond conformal mapping when combined with that of Pendry et al.

In the beginning, we briefly summarize the two important works [1, 2]. In [1], Leonhardt firstly uses Zhukowski conformal mapping to build up the relationship between virtual space (figure 1(a)) and physical space (figure 1(b)). This is a multi-valued mapping: two Riemann sheets in virtual space are mapped on to one physical space. By expanding a finite line (in yellow, mathematically called a ‘branch cut’) into a (yellow) circle, the upper sheet is mapped to the exterior region of physical space (outside the circle). The lower one is then mapped to the interior region (inside the circle). There are no spatial discontinuities during such a mapping. Therefore, when the ray impinges at the branch cut, it will propagate from one sheet to another (the two lines with arrows). In physical space, it will propagate from the exterior region to the interior one. By applying a specific refractive index profile in the lower sheet (or, correspondingly, to the interior region) that brings the rays back to the upper sheet, a conformal cloak can be designed. In [2], Pendry and co-workers use the push-forward mapping (a mathematical term) by expanding one point (yellow dot, figure 1(c)) into a (yellow) circle (figure 1(d)), inside which the cloaking region is created. The ray in virtual space is a straight line (red, with arrow), while in physical space it will go around the cloaking region and then propagate in a straight line (red curve with arrows). To make the analysis more elegant, we create an artificial ‘Riemann sheet’ in the lower part (the ‘branch cut’ is now a point, figure 1(c)), which is related to the cloaking region. Therefore, both of these descriptions can be regarded as multi-valued mappings.

Let us discuss another example. The folded geometry mapping of illusion optics [7–9] enables the appearance of one arbitrary object to be transformed into another. Such a mapping is in fact an opposite of push-forward mapping and can also be rewritten in a similar form with three artificial Riemann sheets. For example, the first sheet is a full space without a circular region. The branch cut now is a (yellow) circle (figure 1(e)). It is mapped on to the same region in physical space (figure 1(f)). The second sheet is a concentric circular region, with two circles as branch cuts. The first one is joined to the one in the first sheet (the yellow circle). The second one is a larger circle (in purple). This sheet is mapped on to a smaller concentric circular region in physical space. The third sheet is a large circular region with the outer boundary as the branch cut, and is mapped on to the inner circular region in physical space. When the ray enters the second sheet, it will propagate in opposite directions from those in the first and the third sheet—whence the term ‘folded geometry mapping’. The overlapping regions in virtual spaces can be used to design cloaks at a distance (the cloaking device does not need to surround the
Figure 1. Virtual spaces and physical spaces for different mappings. (a) Virtual space of the Zhukowski mapping [1]. (b) Physical space of the Zhukowski mapping [1]. (c) Virtual space of the push-forward mapping. (d) Physical space of the push-forward mapping [2]. (e) Virtual space of the folded geometry mapping. (f) Physical space of the folded geometry mapping [8].
object but just set nearby). We see thus that multi-valued mappings provide a versatile tool for designing various perfect cloaks.

However, the above cloaks have either material singularities or negative refractive indexes, rendering the devices capable of working only at one single frequency. Non-Euclidean cloaks [10] have been proposed to address this problem in the geometric optics realm. Later it is proved [11] that they are able to work in the wave-optics for a series of eigenfrequencies. However, non-Euclidean geometry is too complicated to design other types of cloaks. In this paper, we will see that when the two original ideas meet, a new and straightforward method can be obtained to design various types of cloaks that have similar functionalities as non-Euclidean cloaks.

We now introduce two new kinds of multi-valued mappings. The first is based on an elliptical coordinate system \((u, v, z)\), related to the Cartesian system \((x, y, z)\) by \(x = f \cosh u \cos v, y = f \sinh u \sin v, z = \text{const}, f = 1\) (arbitrary unit). In virtual space (see figure 2(a)), we need two sets of elliptical coordinates, \((u', v', z')\) and \((u'', v'', z'')\). In physical space (figure 2(b)), we use \((u, v, z)\). The mapping is as follows. Firstly, the elliptical region (surrounded by a black ellipse) is mapped on to a shell with two elliptical boundaries. The outer boundary is the same as that in virtual space (in the elliptical coordinate system, it is surrounded by a black ellipse) is mapped on to a shell with two elliptical boundaries. The second mapping is based on the Cartesian coordinate system. In virtual space, we need two sets of elliptical coordinates, \((u', v', z')\) and \((u'', v'', z'')\). In physical space, \((u, v, z)\) are related to the Cartesian system \((x, y, z)\).

The second mapping is based on the Cartesian coordinate system. In virtual space, we need two sets of Cartesian coordinates, \((x', y', z')\) and \((x'', y'', z'')\). In physical space, we use \((x, y, z)\). Firstly, the elliptical region (surrounded by the black ellipse with the equation \(x'^2/a^2 + y'^2 = 1\), \(a = 1\) (arbitrary unit), figure 2(c)) is mapped on to an elliptical region without an inner circular part (surrounded by the yellow circle with the equation \(x^2 + y^2 = a^2\) in figure 2(d), which is the image of the line \(y' = 0\), and \(-a \leq x' \leq a\) in figure 2(c)). Secondly, we inherit the second part of the Zhukowski mapping, i.e. the second sheet is mapped on to the inner circular region here. What is worth noting is that the branch cut at the first sheet is not the same as that of the second sheet, but has a relationship of \(y' = y'' = 0\) and \(x'' = 2x'\). If the ray comes from the +y direction (the solid green line in figure 2(c)) and crosses the branch cut at the point \((x_0', 0)\) in
Figure 2. Two mappings with artificial Riemann sheets in virtual space. Panels (a) and (b) are virtual space and physical space through the mapping on an elliptical coordinate system. Panels (c) and (d) are virtual space and physical space through the mapping on a Cartesian coordinate system.

the first sheet, it will continue to propagate in the second sheet from the point \((x'' = 2x'_0, 0)\) and continue to infinity (the solid purple line in figure 2(c)). In physical space, it will be bent before entering into the inner core and be absorbed eventually, as we have explained in the first part of this paper. The detailed mapping here is written as follows,

\[
x = x', \quad y = y', \quad z = z' \quad \text{for} \quad \frac{x^2}{a^2} + \frac{y^2}{4a^2} \geq 1; \tag{2a}
\]

\[
x = x', \quad y = \frac{y'}{2} + \sqrt{a^2 - x'^2}, \quad z = z' \quad \text{for} \quad \frac{x^2}{a^2} + \frac{y^2}{4a^2} < 1 \quad \text{and} \quad x^2 + y^2 \geq a^2; \tag{2b}
\]

\[
x'' = x \left(1 + \frac{a^2}{x^2 + y^2}\right), \quad y'' = y \left(1 - \frac{a^2}{x^2 + y^2}\right), \quad z'' = z \quad \text{for} \quad x^2 + y^2 < a^2. \tag{2c}
\]

A key benefit of this kind of general mapping is that we can easily obtain its inverse form explicitly and apply further refractive index profile operations for cloaking designs. For

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Figure 3. The ray trajectories in virtual space and the cloaking wave effect in physical space. (a) The ray trajectories in virtual space of the first mapping, the two kissing black circles are PECs. (b) The field pattern for the cloak based on the first mapping. In the simulation, we choose $u_1 = 0.9$, $u_2 = 1.5$ and the wavelength $\lambda = \frac{2\pi f}{\sqrt{l(l+1)}}$ with $l = 5$. (c) The ray trajectories in virtual space of the second mapping, the two kissing black circles are PECs. (d) The field pattern for the cloak based on the second mapping. In the simulation, the wavelength is chosen as $\lambda = \frac{4\pi a}{\sqrt{l(l+1)}}$ with $l = 10$.

example, following [12], we apply two kissing mirrored Maxwell’s fish-eye lenses in the second sheets of both the above mappings to design new kinds of cloaks that can hide perfect electric conductors (PECs) in the wave optics realm.

The reason why we choose the mapping introduced in [12] is that (i) the cloaking region is a PEC; (ii) the whole device has a symmetric structure; (iii) the maximum values of the material parameters will not be too large for practical implementation.

Let us now consider the first mapping. If we add the above profile in the second sheet, all the rays crossing the branch cut will enter the second sheet, propagate in close orbits and will come back to the first sheet again (see figure 3(a)). Physically, the rays will propagate around the PEC in the inner elliptical region (see figure 3(b)) and appear as though nothing is there. (For detailed material parameters in each region, see Methods). In wave optics, as we
Figure 4. The distributions of the material parameters of the cloak in figure 3(b). Panels (a) $\mu_u$, (b) $\mu_v$, and (c) $\varepsilon_z$.

have shown in [13, 14] for a conformal cloak, the device here is also expected to work well at one of the eigenfrequencies of the mirrored Maxwell’s fish-eye lenses. We plot the field pattern for the cloak illuminated by a transverse electric (TE) polarized wave. Figure 3(b) shows that the cloaking functionality is pretty good. As the mapping here is not singular, the cloak does not own any singular material parameters (see in figures 4(a)–(c)) and can work for a series of eigenfrequencies. In particular, the maximum value of $\varepsilon_z$ here is less than 8, while for the original non-Euclidean cloak, the maximum value of $\varepsilon_z$ is about 180 [11].

Similarly, if we perform the same operation to the second sheet of the second mapping, an acceptable cloaking effect can also be achieved. (For material parameters, see again Methods.) As shown in figure 3(c), if we add two kissing mirrored Maxwell’s fish-eye lenses (each twice the size of those in the first mapping), all the rays entering from the first sheet will propagate in closed trajectories and will come back to the first sheet again. For a particular eigenfrequency, the cloak also works well (figure 3(d)). Notice that the outer part of this cloaking is just a one-dimensional cloak, while the inner part is exactly the same as that in [13]. Therefore, such a cloak is in fact a combination of the two concepts: the push-forward mapping [2] and the further refractive index profile operations [1]. There are some small ripples of the wavefronts outside the cloaks in figures 3(b) and (d). Such imperfection comes from the impedance mismatching at the artificial branch cuts.
In summary, we have proposed a new general concept for transformation optics. It employs multi-valued mappings with artificial Riemann sheets for designing new kinds of cloaks and incorporates the pioneering cloaks [1, 2] as special cases. We note that there is a design of cloaking without superluminal propagation [15], and the method therein can also be cataloged into the current case with the artificial branch cut built up on the bipolar coordinate system. As the non-Euclidean cloak [10] is also designed on the bipolar coordinate system, we feel that it might also be possible to incorporate the non-Euclidean cloak using the current method if we change the further refractive index profile with some other anisotropic and inhomogeneous materials. Such a concept also has mathematical importance as it enables the construction of artificial Riemann sheets. It may be used to design other novel devices, in view of the recent progress of multi-valued conformal mappings [16, 17].

1. Methods

In this section, we provide the material parameters of the above two cloaks. As we focus on TE polarization, only $\mu$ and $\varepsilon_z$ are involved.

The first one is written in the elliptical coordinate system (the detailed calculation method can be found in [18]).

$$\mu_u = \frac{u_2 - u_1}{u_2}, \quad \mu_v = \frac{u_2}{u_2 - u_1}, \quad \varepsilon_z = \frac{u_2}{u_2 - u_1} \quad \text{for} \quad u_1 \leq u < u_2; \quad (3a)$$

$$\mu_u = u_1 \exp(-u), \quad \mu_v = \frac{1}{u_1 \exp(-u)}, \quad \varepsilon_z = \frac{n'^2}{u_1 \exp(-u)} \quad \text{for} \quad u < u_1 \quad \text{but outside the PEC,} \quad (3b)$$

where

$$n' = \begin{cases} \frac{2}{1 + (\cosh(\ln(u_1/u)) - \cos(v) + 1) + (\sinh(\ln(u_1/u)) - \sin(v))^2} & -\frac{\pi}{2} < v < \frac{\pi}{2} \\ \frac{2}{1 + (\cosh(\ln(u_1/u)) - \cos(v) - 1) + (\sinh(\ln(u_1/u)) - \sin(v))^2} & \text{else} \end{cases}. \quad (4)$$

It should be emphasized that all the above material parameters are not singular.

The second one is written in the Cartesian coordinate system. For $\frac{x^2}{a^2} + \frac{y^2}{a^2} < 1$ and $x^2 + y^2 \geq a^2$,

$$\tilde{\mu} = \begin{bmatrix} 2 & -2|x| \sqrt{1-x^2} \\ -2|x| \sqrt{1-x^2} & 1/2 + 2x^2 \end{bmatrix}, \quad \varepsilon_z = 2 \quad (5)$$

for the region $x^2 + y^2 < a^2$ but outside the PEC,

$$\mu = 1, \quad \varepsilon_z = n''^2, \quad (6)$$

where

$$n'' = \begin{cases} 1 - \frac{1}{(x+y)^2} & 1 + \frac{2(x+y)^2}{2} & x < 0 \\ 1 - \frac{1}{(x+y)^2} & \frac{2}{1 + \frac{2(x+y)^2}{2}} & x > 0 \end{cases}. \quad (7)$$

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Author contributions H C conceived the idea. L X did the theoretical calculations and the numerical simulations. Both the authors wrote the manuscript.

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