Partial Differential Equations

The Fujita phenomenon in exterior domains under the Robin boundary conditions

Le phénomène de Fujita dans un domaine extérieur sous les conditions au bord de Robin

Jean-Francois Rault

LMPA FR 2956 CNRS, Université Lille Nord de France, 50, rue F. Buisson, B.P. 699, 62228 Calais cedex, France

1. Introduction

In an exterior domain $\Omega$ of $\mathbb{R}^N$ which boundary $\partial\Omega$ is of class $C^2$, we consider the following parabolic problem

$$\begin{cases}
\partial_t u = \Delta u + u^p & \text{in } \Omega \text{ for } t > 0, \\
\partial_n u + \alpha u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\
u(\cdot, 0) = \varphi & \text{in } \Omega,
\end{cases} \tag{1}$$

where $p > 1$ is a real number, $\alpha$ a non-negative continuous function on $\partial\Omega \times \mathbb{R}^+$ and $\varphi$ a continuous function in $\overline{\Omega}$. We aim to prove that the well-known Fujita phenomenon (see Refs. [2,5,6,8] and [10]) remains true under the Robin boundary conditions. Throughout, we shall assume that $\alpha$ is non-negative

$$\alpha \geq 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{2}$$

and, in order to deal with classical solutions, we need some regularity on $\alpha$

$$\alpha \in C(\partial\Omega \times \mathbb{R}^+). \tag{3}$$

To use the comparison method (see the truncation procedure described in Section 2 of [8]), we need

$$\varphi \in C(\overline{\Omega}), \quad 0 < \|\varphi\|_{\infty} < \infty, \quad \varphi \geq 0, \quad \lim_{|x| \to \infty} \varphi(x) = 0. \tag{4}$$
From [4], the unique classical solution of Problem (1) \( u \in C(\overline{\Omega} \times [0, T)) \cap C^2(\overline{\Omega} \times (0, T)) \) with maximal existence time \( T > 0 \) satisfies \( T = \infty \) and \( u \) is called a global solution or \( T < \infty \) and \( u \) blows up in finite time in the \( L^\infty \)-norm. In the second case, we have
\[
\limsup_{t \uparrow T, x \in \overline{\Omega}} u(x, t) = \infty.
\]

2. Main results

Using the comparison method described in [8], we just need to find some appropriate sub-solutions to prove the blow-up case \( 1 < p < 1 + 2/N \) and some adequate super-solution to obtain the global existence case \( p > 1 + 2/N \).

**Theorem 2.1.** Suppose that conditions (2), (3) and (4) are fulfilled. Then all non-trivial positive solutions of Problem (1) blow up in finite time for \( p \in (1, 1 + 2/N) \). Moreover, if \( N \geq 3 \), blow up also occurs for \( p = 1 + 2/N \).

**Proof.** Ab absurdo, suppose that there exists \( \alpha \) and a non-trivial \( \varphi \) satisfying the hypotheses above, and such that the solution \( u \) of Problem (1) with these parameters is global. By the truncation method and with the comparison principle from [3], we construct a global solution \( v \) of the following Dirichlet problem
\[
\begin{aligned}
\partial_t v &= \Delta v + v^p \quad \text{in} \quad \overline{\Omega} \times (0, +\infty), \\
v &= 0 \quad \text{on} \quad \partial \Omega \times (0, +\infty), \\
v(\cdot, 0) &= \varphi_0 \quad \text{in} \quad \overline{\Omega},
\end{aligned}
\]
where \( 0 \leq \varphi_0 \leq \varphi \) and \( \varphi_0 \in C_0(\overline{\Omega}) \). Thanks to the results of Bandle and Levine results [2] (see [1] for the one-dimensional case), we obtain a contradiction because the solution \( v \) must blow-up in finite time. If \( N \geq 3 \) and \( p = 1 + 2/N \), the contradiction holds with Mochizuki and Suzuki’s results [7] and [9]. \(\square\)

**Theorem 2.2.** Under hypotheses (2), (3) and (4), for \( N \geq 3 \) and
\[
p > 1 + \frac{2}{N},
\]
Problem (1) admits global non-trivial positive solutions for sufficiently small initial data \( \varphi \).

**Proof.** Consider the classical solution of the following Neumann problem
\[
\begin{aligned}
\partial_t v &= \Delta v + v^p \quad \text{in} \quad \overline{\Omega} \text{ for } t > 0, \\
\partial_n v &= 0 \quad \text{on} \quad \partial \Omega \text{ for } t > 0, \\
v(\cdot, 0) &= \varphi \quad \text{in} \quad \overline{\Omega}.
\end{aligned}
\]
By hypothesis (2), \( \alpha \geq 0 \) and by definition of \( v \), we obtain
\[
\partial_t v + \alpha v = \alpha v \geq 0 \quad \text{on} \quad \partial \Omega \text{ for } t > 0.
\]
Thus, \( v \) is clearly a super-solution of Problem (1). The comparison method previously used leads to \( 0 \leq u \leq v \) in \( \Omega \) and for \( t > 0 \), where \( u \) is a solution of (1) with the parameters \( \alpha \) and \( \varphi \). If the initial data \( \varphi \) is small enough, the function \( v \) is global, see Levine and Zhang’s results from [6]. Hence, the solution \( u \) cannot blow up in finite time, and it is a global solution. \(\square\)

The global existence theorem is true only for high dimensions \( N \geq 3 \) because Levine and Zhang’s results are no more satisfied in dimension 2. For this special case, we impose a restriction on the coefficient \( \alpha \): suppose that there exists a positive constant \( c > 0 \) such that
\[
\alpha \geq c \quad \text{on} \quad \partial \Omega \times \mathbb{R}^+.
\]
We use another super-solution, already used before by Bandle and Levine’s in [1].

**Theorem 2.3.** Let \( \alpha \) be a coefficient satisfying (3) and (5), \( \varphi \) an initial data with (4). For
\[
p > 1 + \frac{2}{N},
\]
Problem (1) admits global positive solutions for sufficiently small initial data \( \varphi \).
Proof. Let $U$ be the function defined in $\bar{\Omega} \times [0, \infty)$ by

$$U(x,t) = A(t+t_0)^{-\mu} \exp\left(-\frac{\|x\|^2_2}{4(t+t_0)}\right),$$

where $\mu = 1/(p-1)$, $t_0 > 0$ and $A > 0$ will be chosen below. A simple calculus of the derivatives leads to

$$\partial_t U(x,t) - \Delta U(x,t) \geq \frac{N-2\mu}{2(t+t_0)} U(x,t) \quad \text{in } \bar{\Omega} \times [0, \infty).$$

Since of $N-2\mu > 0$ by definition of $\mu$, and with $U^{p-1} \leq A^{p-1}(t+t_0)^{-1}$, we just need to choose $A > 0$ sufficiently small to obtain

$$\partial_t U - \Delta U \geq U^p \quad \text{in } \bar{\Omega} \times [0, \infty).$$

On the boundary $\partial \Omega$, hypothesis (5) gives

$$\partial_{\nu} U(x,t) + \alpha U(x,t) \geq \left(-\frac{x \cdot V(x)}{2(t+t_0)} + \alpha(x,t)\right) U(x,t) \geq \left(-\frac{x \cdot V(x)}{2(t+t_0)} + c\right) U(x,t).$$

As $\partial \Omega$ is compact, and with $t_0$ sufficiently big, this last term is non-negative. Thus, if $0 \leq \varphi \leq U(\cdot, 0)$ in $\bar{\Omega}$, the function $U$ is a global bounded super-solution.

In dimension one, an exterior domain does not exist, but we can consider the case $\Omega = \mathbb{R} \setminus [a, b]$ with $a < b$ two real numbers. We obtain:

**Theorem 2.4.** Let $\alpha$ be a coefficient satisfying (3) and $\varphi$ an initial data with (4). For

$$p > 3,$$

Problem (1) admits global positive solutions for sufficiently small initial data $\varphi$.

Proof. Use the same super-solution $U$ as in the previous proof. Modifications only appear in Eq. (6). Up to a translation, and thanks to the symmetry of the problem, we can treat only the case $\Omega = (0, \infty)$. On the boundary, we have $x = 0$ and $\partial_{\nu} U = 0$. Thus Eq. (6) is satisfied for all $\alpha \geq 0$.

**Remark 1.** As in Bandle and Levine's results [1], we can consider a more general non-linear reaction term of the form $t^q \|x\|_2^q u^p$. In this case, we prove that the Fujita exponent is $1 + (2 + 2q + s)/N$. If $p < 1 + (2 + 2q + s)/N$, then nontrivial positive solutions of

$$\begin{cases}
\partial_t u = \Delta u + t^q \|x\|_2^q u^p & \text{in } \bar{\Omega} \times (0, +\infty), \\
\partial_{\nu} u + \alpha u = 0 & \text{on } \partial \Omega \times (0, +\infty), \\
u(\cdot, 0) = \varphi & \text{in } \bar{\Omega},
\end{cases}$$

blow up in finite time. If $p > 1 + (2 + 2q + s)/N$, then there exist non-trivial global positive solutions if $\varphi$ is small enough.

**References**

[1] C. Bandle, H.A. Levine, Fujita type results for convective-like reaction diffusion equations in exterior domains, Z. Angew. Math. Phys. 40 (1989) 665–676.
[2] C. Bandle, H.A. Levine, On the existence and the nonexistence of global solutions of reaction–diffusion equations in sectorial domains, Trans. Amer. Math. Soc. 316 (1989) 595–622.
[3] J. vonBelow, C. De Coster, A qualitative theory for parabolic problems under dynamical boundary conditions, J. Inequal. Appl. 5 (2000) 467–486.
[4] A. Constantin, J. Escher, Global existence for fully parabolic boundary value problems, NoDEA Nonlinear Differential Equations Appl. 13 (2006) 91–118.
[5] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^1+\alpha$, J. Fac. Sci. Univ. Tokyo 13 (1966) 109–124.
[6] H.A. Levine, Q.S. Zhang, The critical Fujita number for a semilinear heat equation in exterior domains with homogeneous Neumann boundary values, Proc. Roy. Soc. Edinburgh Sect. A 130 (2000) 591–602.
[7] K. Mochizuki, R. Suzuki, Critical exponent and critical blow up for quasilinear parabolic equations, Israel J. Math. 98 (1997) 141–156.
[8] J.-F. Rault, The Fujita phenomenon in exterior domains under dynamical boundary conditions, Asymptot. Anal. 66 (2010) 1–8.
[9] R. Suzuki, Critical blow-up for quasilinear parabolic equations in exterior domains, Tokyo J. Math. 19 (1996) 397–409.
[10] F.B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, Israel J. Math. 38 (1981) 29–40.