RATIONALITY PROPERTIES OF
UNIPOTENT REPRESENTATIONS

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INTRODUCTION

0.1. Let $k$ be an algebraic closure of a finite field $F_q$ with $q$ elements. Let $G$ be a connected simple algebraic group of adjoint type over $k$ with a fixed $F_q$-rational structure; let $F : G \to G$ be the corresponding Frobenius map. The fixed point set $G^F$ is a finite group. Let $W$ be the Weyl group of $G$. For $w \in W$ let $R_w$ be the character of the virtual representation $R^1(w)$ of $G^F$ defined in [DL, 1.5]. (The definition of $R_w$ is in terms of $l$-adic cohomology but in fact $R_w$ has integer values and is independent of $l$, see [DL, 3.3].) An irreducible representation $\rho$ of $G^F$ over $\mathbb{C}$ is said to be unipotent if its character $\chi_\rho : G^F \to \mathbb{C}$ occurs with $\neq 0$ multiplicity in $R_w$ for some $w \in W$ (see [DL, 7.8]). Let $U$ be the set of isomorphism classes of unipotent representations of $G^F$. Let $U_\mathbb{Q} = \{ \rho \in U | \chi_\rho(g) \in \mathbb{Q} \; \forall g \in G^F \}$. Let $U_\mathbb{Q}$ be the set of all $\rho \in U$ such that $\rho$ is defined over $\mathbb{Q}$ (that is, it can be realized by a $\mathbb{Q}[G^F]$-module). We have $U_\mathbb{Q} \subset U$.

Unless otherwise specified, we assume that $G$ is split over $F_q$. The following is one of our results.

Theorem 0.2. We have $U_\mathbb{Q} = \tilde{U}_\mathbb{Q}$.

0.3. We will also show (see 1.12) that, if $G$ is of type $A, B, C$ or $D$, then $U_\mathbb{Q} = U$. (The analogous statement is false for exceptional types.) The rationality of certain unipotent cuspidal representations connected with Coxeter elements has been proved in [L1]. The method of [L1] has been extended in [L3] (unpublished) to determine explicitly $U_\mathbb{Q}$ in the general case (including non-split groups). The case where $G$ is non-split of type $A$ has been also considered by Ohmori [Oh] by another extension of the method of [L1].

Our study of rationality of unipotent representations is based on the statement that a given unipotent representation appears with multiplicity 1 in some (possibly virtual) representation $R$ defined using $l$-adic cohomology and then using the Hasse principle. In the first method (that of [L3]), $R$ is a particular intersection cohomology space of a variety; see Sec.1. In the second method (which applies

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only in the cuspidal case), \( R \) will be one of the \( R_w \) above; see Sec.2. In one case (\( G = SO_5 \) with \( q \) odd), we give an elementary approach to rationality (without using the Hasse principle); see Sec.3.

1. First method

1.1. Let \( p \) be the characteristic of \( F_q \). For any prime number \( l \neq p \), we choose an imbedding of the field \( \mathbb{Q}_l \), the \( l \)-adic numbers, into \( \mathbb{C} \). This allows us to regard any representation of \( G^F \) over \( \mathbb{Q}_l \), as one over \( \mathbb{C} \). Let \( X \) be the flag manifold of \( G \); let \( F : X \rightarrow X \) be the map induced by \( F : G \rightarrow G \). For \( w \in W \) let \( O_w \) be the set of all \((B, B') \in X \times X\) that are in relative position \( w \). As in [DL], for any \( w \in W \), let \( X_w \) be the subvariety of \( X \) consisting of all \( B \in X \) such that \((B, F(B)) \in O_w \); let \( \tilde{X}_w \) be the closure of \( X_w \) in \( X \). Then \( X_w, \tilde{X}_w \) are stable under the conjugation action of \( G^F \) on \( X \). Hence for any \( j \in \mathbb{Z} \) there is an induced action of \( G^F \) on the \( l \)-adic cohomology with compact support \( H^j_c(X_w, \mathbb{Q}_l) \) and on the \( l \)-adic intersection cohomology \( H^j(\tilde{X}_w, \mathbb{Q}_l) \). (Note that \( \tilde{X}_w \) has pure dimension \( l(w) \) where \( l : W \rightarrow \mathbb{N} \) is the length function.) Recall that \( R_w \) is the character of the virtual representation \( \sum_{j \in \mathbb{Z}} (-1)^j H^j_c(X_w, \mathbb{Q}_l) \) of \( G^F \).

**Lemma 1.2.** Let \( \rho \in \mathcal{U} \). There exists \( x \in W \) and \( j \in [0, l(x)] \) such that \( \rho \) appears with multiplicity 1 in the \( G^F \)-module \( H^j(\tilde{X}_x, \mathbb{Q}_l) \).

The proof is based on results of [L2]. For any \( x \in W \) let \( A_x \) be the virtual representation of \( W \) defined in [L2, p.154,156]. For any virtual representation \( E \) of \( W \) we set \( R_E = |W|^{-1} \sum_{w \in W} \text{tr}(w, E) R_w \). (A \( \mathbb{Q} \)-valued class function on \( G^F \).) Thus, \( R_{A_x} \) is defined. Let \( a : W \rightarrow \mathbb{N} \) be as in [L2, p.178]. Assume that

(a) \( x \in W \) is such that \( \rho \) appears with multiplicity 1 in \( (-1)^{l(x)-a(x)} R_{A_x} \).

Then from [L2, 6.15, 6.17(i), 5.13(i)] we deduce that \( \rho \) appears with multiplicity 1 in \( H^{l(x)-a(x)}(\tilde{X}_x, \mathbb{Q}_l) \). (Actually, in the references given, \( q \) is assumed to be sufficiently large; but this assumption is removed later in [L2].) Thus, to prove the lemma it is enough to show that (a) holds for some \( x \in W \). Now in [L2], the multiplicities of any unipotent representation in \( (-1)^{l(x)-a(x)} R_{A_x} \) have been explicitly described for many \( x \). (See for example the tables in [L2, p.304-306] for types \( E_8, F_4 \) and the results in [L2, Ch.9] for classical types.) In particular, we see that (a) holds for some \( x \in W \).

**Lemma 1.3.** Let \( \rho \in \tilde{\mathcal{U}}_{\mathbb{Q}} \). Let \( l \) be a prime number invertible in \( k \). Let \( x, j \) be as in 1.2.

(a) \( \rho \) is defined over \( \mathbb{Q}_l \).

(b) If \( j \) is even then \( \rho \) is defined over \( \mathbb{R} \). If \( j \) is odd then \( \rho \) is not defined over \( \mathbb{R} \).

(c) If \( j \) is even then \( \rho \in \mathcal{U}_{\mathbb{Q}} \).

Clearly, (a) follows from 1.2. We prove (b). Let \( c \in H^2(\tilde{X}_w, \mathbb{Q}_l) \) be the Chern class of an ample line bundle on \( \tilde{X}_w \) (we ignore Tate twists); we may assume that this line bundle is the restriction of a line bundle on \( X \). Since \( G^F \) acts
trivially on $H^2(X, \mathbb{Q}_l)$ it follows that $c$ is $G^F$-stable. Hence the map $\mathbf{H}^j(\tilde{X}_x, \mathbb{Q}_l) \to H^{2j(x)-j}(\tilde{X}_x, \mathbb{Q}_l)$ given by $\xi \mapsto c^{j(x)-j}\xi$ is compatible with the $G^F$-action. This map is an isomorphism, by the Hard Lefschetz Theorem [BBD, 5.4.10]. Let $(\cdot, \cdot) : H^j(\tilde{X}_x, \mathbb{Q}_l) \times H^{2j(x)-j}(\tilde{X}_x, \mathbb{Q}_l)$ be the Poincaré duality pairing. (We again ignore Tate twists.) Then $\xi, \xi' \mapsto (\xi, c^{j(x)-j}\xi')$ is a $(-1)^j$-symmetric, non-singular, $G^F$-invariant bilinear form $H^j(\tilde{X}_x, \mathbb{Q}_l) \times H^j(\tilde{X}_x, \mathbb{Q}_l) \to \mathbb{Q}_l$. This restricts to a $(-1)^j$-symmetric, $G^F$-invariant bilinear form on the $\rho$-isotypic part of $H^j(\tilde{X}_x, \mathbb{Q}_l)$, which is non-singular, since $\rho$ is isomorphic to its dual (recall that $\rho \in \tilde{\mathcal{U}}_{\mathbb{Q}}$). This $\rho$-isotypic part is isomorphic to $\rho$ and (b) follows. Under the assumption of (c), we see from (a),(b), using the Hasse principle for division algebras with centre $\mathbb{Q}$ that $\rho$ is defined over $\mathbb{Q}$. (The Hasse principle is applicable even when information is missing at one place, in our case at $p$-adic numbers.) The lemma is proved.

**Lemma 1.4.** Let $\rho \in \tilde{\mathcal{U}}_{\mathbb{Q}}$. Let $x, j$ be as in 1.2. Then $j$ is even.

It is known [L2] that the parity of an integer $j$ such that $\rho$ appears with non-zero multiplicity in $H^j(\tilde{X}_x, \mathbb{Q}_l)$ for some $x \in W$ is an invariant of $\rho$. Moreover, $j$ is even except if $G$ is of type $E_7$ and $\rho$ is one of the two unipotent cuspidal representations of $G$ or $G$ is of type $E_8$ and $\rho$ is a component of the representation induced by one of the two unipotent cuspidal representations of a parabolic of type $E_7$. (See [L2, Ch.11].) In these exceptional cases, we have $\rho \notin \tilde{\mathcal{U}}_{\mathbb{Q}}$, as one sees using [L2, 11.2]. The lemma is proved.

1.5. Now Theorem 0.2 follows immediately from 1.3(c) and 1.4.

1.6. Let $\mathcal{X}$ be the set of all triples $(\mathcal{F}, y, \sigma)$ where $\mathcal{F}$ is a “family” [L2, 4.2] of irreducible representations of $W$ (with an associated finite group $\mathcal{G}_\mathcal{F}$, see [L2, Ch.4]), $y$ is an element of $\mathcal{G}_\mathcal{F}$ defined up to conjugacy and $\sigma$ is an irreducible representation of the centralizer of $y$ in $\mathcal{G}_\mathcal{F}$ defined up to isomorphism. For $(\mathcal{F}, y, \sigma) \in \mathcal{X}$, let $\lambda_{y, \sigma}$ be the scalar by which $y$ acts on $\sigma$ (a root of 1). Let $\mathcal{X}_1$ be the set of all $(\mathcal{F}, y, \sigma) \in \mathcal{X}$ such that $|\mathcal{F}| \neq 2$ and $\lambda_{y, \sigma} = \pm 1$. If $q$ is a square, let $\mathcal{X}_2$ be the set of all $(\mathcal{F}, y, \sigma) \in \mathcal{X}$ such that $|\mathcal{F}| = 2, y = 1$. If $q$ is not a square, let $\mathcal{X}_2 = \emptyset$. In any case, $\mathcal{X}_2$ is empty unless $G$ is of type $E_7$ or $E_8$. Let $\mathcal{X}_Q = \mathcal{X}_1 \cup \mathcal{X}_2$.

In [L2, 4.23], $\mathcal{X}$ is put in a bijection

$$(\mathcal{F}, y, \sigma) \leftrightarrow \rho_{\mathcal{F}, y, \sigma}$$

with $\mathcal{U}$.

**Lemma 1.7.** Assume that $\rho = \rho_{\mathcal{F}, y, \sigma}$, $\rho' = \rho_{\mathcal{F}', y', \sigma'}$, where $(\mathcal{F}, y, \sigma) \in \mathcal{X}_Q$, $(\mathcal{F}', y', \sigma') \in \mathcal{X}$ are distinct. Then there exists $x \in W$ such that $\rho, \rho'$ have different multiplicities in the $G^F$-module $(-1)^{j(x)-a(x)}R_{A_x}$.

As mentioned in the proof of 1.2, the multiplicities of various unipotent representations have been explicitly computed in [L2] for many $x \in W$. From this the lemma follows easily.
Lemma 1.8. Let $\rho = \rho_{\mathcal{F}, y, \sigma}$, where $(\mathcal{F}, y, \sigma) \in X_{\mathbb{Q}}$. Then $\rho \in \tilde{U}_{\mathbb{Q}}$.

Let $\gamma \in \text{Gal} (\mathbb{C}/\mathbb{Q})$. Then $\gamma (\chi_{\rho}) = \chi_{\rho'}$ for some $\rho' \in \mathcal{U}$. Since the character of $(-1)^{(x-a)} R_{\mathcal{A}_x}$ is integer valued, it is fixed by $\gamma$. (Here $x$ is any element of $W$.) Hence $\rho, \rho'$ have the same multiplicity in $(-1)^{(x-a)} R_{\mathcal{A}_x}$. From 1.7 it follows that $\rho = \rho'$. Thus, $\gamma (\chi_{\rho}) = \chi_{\rho}$ for any $\gamma \in \text{Gal} (\mathbb{C}/\mathbb{Q})$, so that $\chi_{\rho}$ has rational values. The lemma is proved.

Lemma 1.9. Let $\rho = \rho_{\mathcal{F}, y, \sigma}$, where $(\mathcal{F}, y, \sigma) \not\in X_{\mathbb{Q}}$. Then $\rho \not\in \tilde{U}_{\mathbb{Q}}$.

Assume first that $\lambda_{y, \sigma} \neq \pm 1$. Then $\lambda_{y, \sigma} \notin \mathbb{Q}$ hence there exists $\gamma \in \text{Gal} (\mathbb{C}/\mathbb{Q})$ such that $\gamma (\lambda_{y, \sigma}) \neq \lambda_{y, \sigma}$. Using the interpretation of $\lambda_{y, \sigma}$ given in [L2, 11.2], it follows that $\gamma (\chi_{\rho}) \neq \chi_{\rho'}$. Hence $\rho \not\in \tilde{U}_{\mathbb{Q}}$. Next assume that $\lambda_{y, \sigma} = \pm 1$. Then $|\mathcal{F}| = 2$. Moreover, if $q$ is a square, then $y \neq 1$. Let $\sigma'$ be the character of $G_{\mathcal{F}} = \mathbb{Z}/2\mathbb{Z}$ other than $\sigma$. Let $\rho' = \rho_{\mathcal{F}, y, \sigma'}$. If $y \neq 1$, then by the results of [L1], $\chi_{\rho}$ is carried to $\chi_{\rho'}$ by an element of $\text{Gal} (\mathbb{C}/\mathbb{Q})$ that takes $\sqrt{-q}$ to $-\sqrt{-q}$. If $y = 1$, then by the known construction of representations of Hecke algebras in terms of $W$-graphs, $\chi_{\rho}$ is carried to $\chi_{\rho'}$ by an element of $\text{Gal} (\mathbb{C}/\mathbb{Q})$ that takes $\sqrt{q}$ to $-\sqrt{q}$. Hence again $\rho \not\in \tilde{U}_{\mathbb{Q}}$.

Proposition 1.10. Under the bijection $X \leftrightarrow \mathcal{U}$ in 1.6(a), the subset $\tilde{U}_{\mathbb{Q}}$ of $\mathcal{U}$ corresponds to the subset $X_{\mathbb{Q}}$ of $X$.

This follows immediately from 1.9, 1.10.

Combining this proposition with 0.2, we obtain:

Corollary 1.11. Under the bijection $X \leftrightarrow \mathcal{U}$ in 1.6(a), the subset $U_{\mathbb{Q}}$ of $\mathcal{U}$ corresponds to the subset $X_{\mathbb{Q}}$ of $X$.

If $G$ is of type $A, B, C$ or $D$, then for any family $\mathcal{F}$ we have $|\mathcal{F}| \neq 2$ and the group $G_{\mathcal{F}}$ is a elementary abelian 2-group hence $\lambda_{y, \sigma} = \pm 1$ for any $(\mathcal{F}, y, \sigma) \in X$. Thus, we have $X_{\mathbb{Q}} = X$ and we obtain:

Corollary 1.12. If $G$ is of type $A, B, C$ or $D$, then $U_{\mathbb{Q}} = U$.

1.13. In this subsection we assume that $G$ is non-split. The analogues of Lemmas 1.2, 1.3 continue to hold but that of Lemma 1.4 does not. (It does in type $D$ but not in type $A$.) Also, if $G$ is non-split of type $D$, then $U_{\mathbb{Q}} = U$. If $G$ is non-split of type $A$ we have $\tilde{U}_{\mathbb{Q}} = U$ but $U_{\mathbb{Q}} \neq U$ in general.

2. Second method

2.1. Let $n \in \mathbb{N}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_a$ be a sequence of integers such that

\begin{equation}
\sum_{i} \lambda_i = n + \left( \frac{a}{2} \right).
\end{equation}
We define a virtual representation \([\lambda_1, \lambda_2, \ldots, \lambda_a]\) of the symmetric group \(S_n\) as follows. If \(0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_a\), then \([\lambda_1, \lambda_2, \ldots, \lambda_a]\) is the irreducible representation of \(S_n\) corresponding to the partition \(\lambda_1 \leq \lambda_2 - 1 \leq \cdots \leq \lambda_a - a + 1\) of \(n\), as in [L2, p.81]. If \(\lambda_1, \lambda_2, \ldots, \lambda_a\) are in \(N\) and are distinct, then
\[
[\lambda_1, \lambda_2, \ldots, \lambda_a] = \text{sgn}(\sigma)[\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(a)}]
\]
where \(\sigma\) is the unique permutation of \(1, 2, \ldots, a\) such that \(\lambda_{\sigma(1)} < \lambda_{\sigma(2)} < \cdots < \lambda_{\sigma(a)}\). If \(\lambda_1, \lambda_2, \ldots, \lambda_a\) are not distinct, or if at least one of them is \(< 0\), we set \([\lambda_1, \lambda_2, \ldots, \lambda_a] = 0\). From the definition we see easily that \([\lambda_1, \lambda_2, \ldots, \lambda_a] = [0, \lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_a + 1]\) for any sequence of integers \(\lambda_1, \lambda_2, \ldots, \lambda_a\) such that (a) holds.

**Lemma 2.2.** Let \(\lambda_1, \lambda_2, \ldots, \lambda_a\) be a sequence of integers such that 2.1(a) holds. Let \(w = (k)w' \in S_k \times S_{n-k} \subset S_n\) where \((k)\) denotes a \(k\)-cycle in \(S_k\) and \(w' \in S_{n-k}\). We have
\[
(a) \quad \text{tr}(w, [\lambda_1, \lambda_2, \ldots, \lambda_a]) = \sum_{i=1}^{a} \text{tr}(w', [\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i - k, \lambda_{i+1}, \ldots, \lambda_a]).
\]
If the \(\lambda_i\) are not distinct or if at least one of them is \(< 0\) then both sides of (a) are 0. We may assume that \(0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_a\). In this case, (a) can be seen to be equivalent to Murnaghan’s rule, see [W].

**2.3.** For \(n \geq 0\) let \(W_n\) be the group of all permutations of \(1, 2, \ldots, n, n', \ldots, 2', 1'\) which commute with the involution \(i \leftrightarrow i'\) for \(i = 1, \ldots, n\). (We have \(W_0 = \{1\}\).) Given two sequences of integers \(\lambda_1, \ldots, \lambda_a\) and \(\mu_1, \mu_2, \ldots, \mu_b\) such that
\[
(a) \quad \sum_i \lambda_i + \sum_i \mu_i = n + \binom{a}{2} + \binom{b}{2},
\]
we define a virtual representation
\[
(b) \quad \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ \mu_1 & \mu_2 & \cdots & \mu_b \end{bmatrix}
\]
of \(W_n\) as follows. If the \(\lambda_i\) are not distinct or if the \(\mu_i\) are not distinct or if at least one of the \(\lambda_i\) or \(\mu_i\) is \(< 0\) we define (b) to be 0. Assume now that the \(\lambda_i \in N\) are distinct, and that the \(\mu_i \in N\) are distinct. Then \(r, \tilde{r}\) defined by
\[
\sum_i \lambda_i = r + \binom{a}{2}, \quad \sum_i \mu_i = \tilde{r} + \binom{b}{2}
\]
satisfy $r, \tilde{r} \in \mathbb{N}, r + \tilde{r} = n$. We identify $W_r \times W_\tilde{r}$ with a subgroup of $W_n$ as in [L2, p.82]. The virtual representation $[\lambda_1, \lambda_2, \ldots, \lambda_a] \otimes [\mu_1, \mu_2, \ldots, \mu_b]$ of $S_r \times S_\tilde{r}$ may be regarded as a virtual representation of $W_r \times W_\tilde{r}$ via the obvious projection $W_r \times W_\tilde{r} \to S_r \times S_\tilde{r}$ (see [L2, p.82]). We tensor this with the one dimensional character of $W_r \times W_\tilde{r}$ which is the identity on the $W_r$-factor and is the restriction of $\chi : W_n \to \{\pm 1\}$ (see [L2, p.82]) on the $W_\tilde{r}$-factor. Inducing the resulting virtual representation from $W_r \times W_\tilde{r}$ to $W_n$, we obtain the virtual representation (b) of $W_n$. Note that if $\lambda_1 < \lambda_2 < \cdots < \lambda_a$ and $\mu_1 < \mu_2 < \cdots < \mu_b$ then this is an irreducible representation; if $\sigma$ is a permutation of $1, 2, \ldots, a$ and $\sigma'$ is a permutation of $1, 2, \ldots, b$ then
\[
\left[ \begin{array}{c}
\lambda_{\sigma(1)} & \lambda_{\sigma(2)} & \cdots & \lambda_{\sigma(a)} \\
\mu_{\sigma'(1)} & \mu_{\sigma'(2)} & \cdots & \mu_{\sigma'(b)}
\end{array} \right] = \text{sgn}(\sigma)\text{sgn}(\sigma') \left[ \begin{array}{c}
\lambda_1 & \lambda_2 & \cdots & \lambda_a \\
\mu_1 & \mu_2 & \cdots & \mu_b
\end{array} \right].
\]

From the definition we see easily that
\[
\left[ \begin{array}{c}
\lambda_1 & \lambda_2 & \cdots & \lambda_a \\
\mu_1 & \mu_2 & \cdots & \mu_b
\end{array} \right] = \left[ \begin{array}{c}
0 & \lambda_1+1 & \lambda_2+1 & \cdots & \lambda_a+1 \\
0 & \mu_1+1 & \mu_2+1 & \cdots & \mu_b+1
\end{array} \right].
\]

**Lemma 2.4.** Let $\lambda_1, \lambda_2, \ldots, \lambda_a$ and $\mu_1, \mu_2, \ldots, \mu_b$ be two sequences of integers such that 2.3(a) holds. Let $w = (2k) \times w' \in W_k \times W_{n-k} \subset W_n$ where $0 < k \leq n$, $(2k)$ denotes an element of $W_k$ whose image under the obvious imbedding $W_k \subset S_{2k}$ is a $2k$-cycle and $w' \in W_{n-k}$ has no cycles of length $2k$ as an element of $S_{2n-2k}$. We have
\[
\text{tr}(w, \left[ \begin{array}{c}
\lambda_1 & \lambda_2 & \cdots & \lambda_a \\
\mu_1 & \mu_2 & \cdots & \mu_b
\end{array} \right]) = \sum_{i=1}^{a} \text{tr}(w', \left[ \begin{array}{c}
\lambda_1 & \lambda_2 & \ldots & \lambda_{i-1} & \lambda_i-k & \lambda_{i+1} & \cdots & \lambda_a \\
\mu_1 & \mu_2 & \cdots & \mu_{i-1} & \mu_i-k & \mu_{i+1} & \cdots & \mu_b
\end{array} \right]) - \sum_{i=1}^{a} \text{tr}(w', \left[ \begin{array}{c}
\lambda_1 & \lambda_2 & \ldots & \lambda_{a} \\
\mu_1 & \mu_2 & \cdots & \mu_{i-1} & \mu_i-k & \mu_{i+1} & \cdots & \mu_b
\end{array} \right]).
\]

This follows from Lemma 2.2, using the definitions.

**2.5.** Let $m \in \mathbb{N}$ and let $n = m^2 + m$. Let $w_m \in W_n$ be an element whose image under the imbedding $W_n \subset S_{2n} = S_{2(m^2+m)}$ is a product of cycles
\[
(4)(8)(12)\ldots(4m).
\]
Let
\[
(\text{a}) \quad \lambda_1 < \lambda_2 < \cdots < \lambda_{m+1} \quad \text{and} \quad \mu_1 < \mu_2 < \cdots < \mu_{m}
\]
be two sequences of integers such that $\lambda_1, \lambda_2, \ldots, \lambda_{m+1}, \mu_1, \mu_2, \ldots, \mu_{m}$ is a permutation of $0, 1, 2, 3, \ldots, 2m$. Then 2.3(a) holds (with $a = m + 1, b = m$ and $n = m^2 + m$). Consider the property
\[
(\ast) \quad \lambda_i + \lambda_j \neq 2m \quad \text{for any} \quad i \neq j \quad \text{and} \quad \mu_i + \mu_j \neq 2m \quad \text{for any} \quad i \neq j.
\]
Lemma 2.6. In the setup of 2.5, if \((*)\) holds, then
\[
\text{tr}(w_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = (-1)^{(m^2+m)/2}.
\]

If \((*)\) does not hold, then \(\text{tr}(w_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = 0.\)

We argue by induction on \(m\). The result is clear when \(m = 0\). Assume now that \(m > 0\). We can assume that \(w_m = (4m)w_{m-1} \in W_{2m} \times W_{n-2m} \subset W_n\) where \(w_{m-1} \in W_{n-2m}\) is defined in a way similar to \(w_m\). We apply 2.4 with \(w = w_m, k = 2m, w' = w_{m-1}\). Note that in the formula in 2.4, at most one term is non-zero, namely the one in which \(k = 2m\) is substracted from the largest entry \(\lambda_i\) or \(\mu_i\) (the other terms are zero since they contain some \(< 0\) entry). We are in one of the four cases below.

Case 1. \(2m = \lambda_{m+1}, 0 = \mu_1\).

Using 2.4, we have
\[
A = \text{tr}(w_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = \text{tr}(w_{m-1}, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ 0 & \mu_2 & \ldots & \mu_m \end{bmatrix})
\]
\[
= (-1)^m \text{tr}(w_{m-1}, \begin{bmatrix} 0 & \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ \mu_2 & \mu_3 & \ldots & \mu_m \end{bmatrix}) = (-1)^m \text{tr}(w_{m-1}, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{m-1} \\ \mu_2 & \mu_3 & \ldots & \mu_m \end{bmatrix}).
\]

Now the induction hypothesis is applicable to
\[
(a) \quad \lambda_1 - 1 < \lambda_2 - 1 < \cdots < \lambda_m - 1 \quad \text{and} \quad \mu_2 - 1 < \mu_3 - 1 < \cdots < \mu_m - 1
\]

instead of 2.5(a). (Clearly, 2.5(a) satisfies \((*)\) if and only if \((a)\) satisfies the analogous condition). Hence, if 2.5(a) satisfies \((*)\), then
\[
A = (-1)^m (-1)^{(m^2-m)/2} = (-1)^{(m^2+m)/2}
\]
as required. If 2.5(a) does not satisfy \((*)\), then \(A = (-1)^m 0 = 0\), as required.

Case 2. \(2m = \lambda_{m+1}, 0 = \lambda_1\).

Using 2.4, we have
\[
\text{tr}(w_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = \text{tr}(w_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \ldots & \lambda_m \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix})
\]

and this is 0 since 0 appears twice in the top row.

Case 3. \(2m = \mu_0, 0 = \lambda_1\).

Using 2.4, we have
\[
A = \text{tr}(w_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = -\text{tr}(w_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \ldots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix})
\]
\[
= (-1)^m \text{tr}(w_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \ldots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix})
\]
\[
= (-1)^m \text{tr}(w_{m-1}, \begin{bmatrix} \lambda_2 & \ldots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}).
\]
Now the induction hypothesis is applicable to
(b) $\lambda_2 - 1 < \cdots < \lambda_m - 1 < \lambda_{m+1} - 1$ and $\mu_1 - 1 < \mu_2 - 1 < \cdots < \mu_{m-1} - 1$

instead of 2.5(a). (Clearly, 2.5(a) satisfies (*) if and only if (b) satisfies the analogous condition.) Hence, if 2.5(a) satisfies (*), then

$$A = (-1)^m (-1)^{(m^2-m)/2} = (-1)^{(m^2+m)/2}$$

as required. If 2.5(a) does not satisfy (*), then $A = (-1)^m 0 = 0$, as required.

Case 4. $2m = \mu_m, 0 = \mu_1$.

Using 2.4, we have

$$\text{tr}(w_m, \begin{bmatrix} \lambda_1 & \ldots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = -\text{tr}(w_{m-1}, \begin{bmatrix} \lambda_1 & \ldots & \lambda_m \lambda_{m+1} \\ 0 & \mu_2 & \ldots & 0 \end{bmatrix})$$

and this is 0 since 0 appears twice in the bottom row. The lemma is proved.

Lemma 2.7. In the setup of 2.6, if (*) holds, then

$$\sharp(k \in \{1, 2, \ldots, m\}; \mu_k = \text{even}) = (m^2 + m)/2 \mod 2.$$

Since (*) holds, the left hand side is equal to the number of pairs

$$(0, 2m), (1, 2m-1), (2, 2m-2), \ldots, (m-1, m+1)$$

in which both components are even. This equals $m/2$ if $m$ is even and $(m+1)/2$ if $m$ is odd. Hence it is has the same parity as $m(m+1)/2$. The lemma is proved.

2.8. Let $m \in \mathbb{N}, m \geq 1$ and let $n = m^2$. Let $w'_m \in W_n$ be an element whose image under the imbedding $W_n \subset S_{2n} = S_{2m^2}$ is a product of cycles

$$(2)(6)(10) \ldots (4m-2).$$

Let

(a) $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ and $\mu_1 < \mu_2 < \cdots < \mu_m$

be two sequences of integers such that $\lambda_1, \ldots, \lambda_m, \mu_1, \mu_2, \ldots, \mu_m$ is a permutation of $0, 1, 2, \ldots, 2m - 1$. Then 2.3(a) holds (with $a = b = m$ and $n = m^2$).

Let

(b) $N = \sharp(k \in \{1, 2, \ldots, m\}; \mu_k \geq m)$.

Consider the property

(**) $\lambda_i + \lambda_j \neq 2m - 1$ for any $i \neq j$ and $\mu_i + \mu_j \neq 2m - 1$ for any $i \neq j$. 

Lemma 2.9. In the setup of 2.8, if (**) holds, then
\[ \text{tr}(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = (-1)^{N+m(m-1)/2}. \]

If (**) does not hold, then \[ \text{tr}(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = 0. \]

We argue by induction on \( m \). The result is clear when \( m = 1 \). Assume now that \( m > 1 \). We can assume that \( w'_m = (4m - 2)w'_{m-1} \in W_{2m-1} \times W_{n-2m+1} \subset W_n \) where \( w'_{m-1} \in W_{n-2m+1} \) is defined in a way similar to \( w'_m \). We apply 2.4 with \( w = w'_m, k = 2m - 1, w' = w'_{m-1} \). Note that in the formula in 2.4, at most one term is non-zero, namely the one in which \( k = 2m - 1 \) is substracted from the largest entry \( \lambda_i \) or \( \mu_i \) (the other terms are zero since they contain some < 0 entry).

We are in one of the four cases below.

*Case 1.* \( 2m - 1 = \lambda_m, 0 = \mu_1 \).

Using 2.4, we have
\[
A = \text{tr}(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = \text{tr}(w'_{m-1}, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{m-1} & 0 \\ 0 & \mu_2 & \ldots & \mu_m \end{bmatrix})
\]
\[
= (-1)^{m-1} \text{tr}(w'_{m-1}, \begin{bmatrix} 0 & \lambda_1 & \lambda_2 & \ldots & \lambda_{m-1} \\ 0 & \mu_2 & \ldots & \mu_m \end{bmatrix})
\]
\[
= (-1)^{m-1} \text{tr}(w'_{m-1}, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_{m-1} \\ \mu_2 & \ldots & \mu_{m-1} \end{bmatrix}).
\]

Now the induction hypothesis is applicable to
\[(a) \quad \lambda_1 - 1 < \lambda_2 - 1 < \cdots < \lambda_{m-1} - 1 \quad \text{and} \quad \mu_2 - 1 < \mu_3 - 1 < \cdots < \mu_{m-1} - 1 \]
instead of 2.8(a). (Clearly, 2.8(a) satisfies (**) if and only if (a) satisfies the analogous condition). Let \( N' \) be defined as \( N \) in 2.8(b), in terms of (a). Then \( N' = N \). If 2.8(a) satisfies (**), then
\[
A = (-1)^{m-1}(-1)^{(m-1)(m-2)/2}(-1)^N' = (-1)^{m-1/2}(-1)^N
\]
as required. If 2.8(a) does not satisfy (**), then \( A = (-1)^{m-1}0 = 0 \), as required.

*Case 2.* \( 2m - 1 = \lambda_m, 0 = \mu_1 \).

Using 2.4, we have
\[
\text{tr}(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = \text{tr}(w'_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \ldots & \lambda_{m-1} & 0 \\ \mu_1 & \mu_2 & \ldots & \mu_{m-1} \end{bmatrix})
\]
and this is 0 since 0 appears twice in the top row.

*Case 3.* \( 2m - 1 = \mu_m, 0 = \lambda_1 \).

Using 2.4, we have
\[
A = \text{tr}(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = -\text{tr}(w'_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \ldots & \lambda_m \\ \mu_1 & \mu_2 & \ldots & \mu_{m-1} \end{bmatrix})
\]
\[
= (-1)^{m} \text{tr}(w'_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \ldots & \lambda_m \\ \mu_1 & \mu_2 & \ldots & \mu_{m-1} \end{bmatrix})
\]
\[
= (-1)^{m} \text{tr}(w'_{m-1}, \begin{bmatrix} \lambda_2 & \ldots & \lambda_{m-1} \\ \mu_2 & \ldots & \mu_{m-1} \end{bmatrix}).
\]
Now the induction hypothesis is applicable to

\[(b) \quad \lambda_2 - 1 < \lambda_3 - 1 < \cdots < \lambda_m - 1 \quad \text{and} \quad \mu_1 - 1 < \mu_2 - 1 < \cdots < \mu_{m-1} - 1\]

instead of 2.8(a). (Clearly, 2.8(a) satisfies (**) if and only if (b) satisfies the analogous condition.) Let \(N'\) be defined as \(N\) in 2.8(b), in terms of (b). Then \(N' = N - 1\). If 2.8(a) satisfies (**), then

\[A = (-1)^m(-1)^{m-2}(m-1)/2(-1)^{N'} = (-1)^m(m-1)/2(-1)^N\]

as required. If 2.8(a) does not satisfy (**), then \(A = (-1)^m0 = 0\), as required.

**Case 4.** \(2m - 1 = \mu_m, 0 = \mu_1\).

Using 2.4, we have

\[\text{tr}(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{bmatrix}) = -\text{tr}(w'_{m-1}, \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ 0 & \mu_2 & \ldots & 0 \end{bmatrix})\]

and this is 0 since 0 appears twice in the bottom row. The lemma is proved.

**Lemma 2.10.** Assume that we are in the setup of 2.9, that (**)) holds and that \(m = 2m'\) for some integer \(m' > 0\). Then

\[(a) \quad \sharp(k \in \{1, 2, \ldots, m\}; \mu_k \geq m) - \sharp(k \in \{1, 2, \ldots, m\}; \mu_k \text{ even}) = m' \mod 2,
\[(b) \quad \sharp(k \in \{1, 2, \ldots, m\}; \mu_k \text{ even}) = N + m(m-1)/2 \mod 2.\]

Among the \(m'\) pairs \((0, 4m' - 1), (2, 4m' - 3), \ldots, (2m' - 2, 2m' + 1)\) there are, say, \(\alpha\) pairs with the first component of form \(\lambda_i\) and second component of form \(\mu_j\) and \(\beta\) pairs with the first component of form \(\mu_j\) and second component of form \(\lambda_i\). Clearly, \(\alpha + \beta = m'\). Among the \(m'\) pairs

\[(1, 4m' - 2), (3, 4m' - 4), \ldots, (2m' - 1, 2m')\]

there are, say, \(\gamma\) pairs with the first component of form \(\lambda_i\) and second component of form \(\mu_j\) and \(\delta\) pairs with the first component of form \(\mu_j\) and second component of form \(\lambda_i\). Clearly, \(\gamma + \delta = m'\). From the definitions we have

\[\sharp(k \in \{1, 2, \ldots, 2m'\}; \mu_k \geq 2m') = \alpha + \gamma,\]

\[\sharp(k \in \{1, 2, \ldots, 2m'\}; \mu_k \text{ even}) = \beta + \gamma.\]

Hence the left hand side of (a) is equal to \(\alpha + \gamma - (\beta + \gamma) = \alpha - \beta\), which has the same parity as \(\alpha + \beta = m'\). This proves (a). Now (b) follows from (a) since \(m' = 2m'(2m' - 1)/2 \mod 2\). The lemma is proved.
Proposition 2.11. Assume that $G$ in 0.1 is of type $B_n$ or $C_n$ where $n = m^2 + m, m \in \mathbb{N}, m \geq 1$. We identify the Weyl group $W$ of $G$ with $W_n$ (see 2.3) in the standard way. Let $w = w_m$, see 2.5. Let $\rho$ be the unique unipotent cuspidal representation of $G^F$. Then $\rho$ appears with multiplicity 1 in $R_w$.

For any subset $J$ of cardinal $m$ of $I = \{0, 1, 2, \ldots, 2m\}$ let $E_J = [\lambda_1 \lambda_2 \ldots \lambda_{m+1}]$ (an irreducible representation of $W$) where $\mu_1 < \mu_2 < \cdots < \mu_m$ are the elements of $J$ in increasing order and $\lambda_1 < \lambda_2 < \cdots < \lambda_{m+1}$ are the elements of $I - J$ in increasing order; let $f(J) = \sharp (j \in J | j \text{ even}).$ By [L2, 4.23], the multiplicity of $\rho$ in $R_w$ is

(a) \[ 2^{-m} \sum_J (-1)^{f(J)} \text{tr}(w_m, E_J) \]

where $J$ runs over all subsets of $I$ of cardinal $m$. Using 2.6 and 2.7 we see that (a) equals $2^{-m} \sharp (J; J \cap (2m - J) = \emptyset) = 1.$ The proposition is proved.

Proposition 2.12. Assume that $G$ in 0.1 is of type $D_n$ where $n = m^2, m = 2m', m' \in \mathbb{N}, m' \geq 1$. We identify the Weyl group $W$ of $G$ with the subgroup of $W_n$ consisting of all permutations $w \in W_n$ such that

$$\sharp (k \in \{1, 2, \ldots, n\}; w(k) \in \{1', 2', \ldots, n'\})$$

is even. (A subgroup of index 2.) Let $w = w'_m$, see 2.8. (We have $w'_m \in W$.) Let $\rho$ be the unique unipotent cuspidal representation of $G^F$. Then $\rho$ appears with multiplicity 1 in $R_w$.

For any subset $J$ of cardinal $m$ of $I = \{0, 1, 2, \ldots, 2m - 1\}$ let $E_J$ be the restriction of

$$[\lambda_1 \lambda_2 \ldots \lambda_{m+1}]$$

denotes uniquely $w$ up to conjugacy.

\[ \frac{\lambda_1}{\mu_1} \frac{\lambda_2}{\mu_2} \cdots \frac{\lambda_m}{\mu_m} \]

from $W_n$ to $W$ (an irreducible representation of $W$) where $\mu_1 < \mu_2 < \cdots < \mu_m$ are the elements of $J$ in increasing order and $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ are the elements of $I - J$ in increasing order; let $f(J) = \sharp (j \in J | j \text{ even}).$ Note that $E_J = E_{I - J}$ and $f(J) = f(I - J).$ By [L2, 4.23], the multiplicity of $\rho$ in $R_w$ is

(a) \[ 2^{-m} \sum_J (-1)^{f(J)} \text{tr}(w_m, E_J) \]

where $J$ runs over all subsets of $I$ of cardinal $m$. Using 2.9 and 2.10 we see that (a) equals $2^{-m} \sharp (J; J \cap (2m - J) = \emptyset) = 1.$ The proposition is proved.

We return to the general case.

Theorem 2.13. Let $\rho$ be a unipotent cuspidal representation of $G^F$. There exists $w \in W$ such that $\rho$ appears with multiplicity 1 in $R_w$.

In view of 2.11, 2.12, we may assume that $G$ is of exceptional type. If $\rho \in \mathcal{U}$ is cuspidal, we have $\rho_{F, y, \sigma}$ where $F$ is independent of $\rho$; we will write $\rho_{y, \sigma}$ instead of $\rho_{F, y, \sigma}$; for the pairs $(y, \sigma)$ we will use the notation of [L2, 4.3]. For $w \in W$ we denote by $|w|$ the characteristic polynomial of $w$ in the reflection representation of $W$. (A product of cyclotomic polynomials $\Phi_{y, \sigma}$) In the cases that appear below, $|w|$ determines uniquely $w$ up to conjugacy.
Other. Hence they contain $\rho$. The multiplicity of $\rho$ in $p$ is inspired by an argument of Ohmori [Oh]. Let $x$ be an element of minimal length in $W$ such that $\rho$ appears with odd multiplicity in $R_x$. (Such $x$ exists by 2.13.)

The multiplicity of $\rho$ in $\sum_j (-1)^j H^j(\bar{X}_x, Q_l)$ is equal to the multiplicity of $\rho$ in $\sum_j (-1)^j H^j_c(\bar{X}_x, Q_l)$ plus an integer linear combination of the multiplicities of $\rho$ in $\sum_j (-1)^j H^j_c(\bar{X}_{x'}, Q_l)$ for various $x'$ of strictly smaller length than $x$ (these multiplicities are even, by the choice of $x$). It follows that the multiplicity of $\rho$ in $\sum_j (-1)^j H^j(\bar{X}_x, Q_l)$ is odd. Recall that $\bar{X}_x$ has pure dimension $l(x)$. By Poincaré duality, the $G^F$-modules $H^j(\bar{X}_x, Q_l)$, $H^{2l(x)-j}(\bar{X}_x, Q_l)$ are dual to each other. Hence they contain $\rho$ with the same multiplicity (recall that $\rho$ is self-dual.)

Type $E_6$.

$|w| = \Phi_{12} \Phi_3$: $\rho_{g_3, \theta^\pm 1}$.

Type $E_7$.

$|w| = \Phi_{18} \Phi_2$: $\rho_{g_2, 1}, \rho_{g_2, \epsilon}$.

Type $E_8$.

$|w| = \Phi_{30}$: $\rho_{g_5, \zeta^j}, j = 1, 2, 3, 4; \rho_{g_6, \theta^\pm 1}$.

$|w| = \Phi_2$: $\rho_{g_4, i^\pm 1}$.

$|w| = \Phi_{18} \Phi_6$: $\rho_{g_3, \theta^\pm 1}$.

$|w| = \Phi^2_{12}$: $\rho_{g_2, \epsilon}$.

$|w| = \Phi_{12} \Phi^2_2$: $\rho_{g_2, -\epsilon}$.

$|w| = \Phi_6$: $\rho_{1, \lambda^4}$.

Type $F_4$.

$|w| = \Phi_{12}$: $\rho_{g_3, \theta^\pm 1}, \rho_{g_4, i^\pm 1}$.

$|w| = \Phi_6$: $\rho_{g_2, \epsilon}$.

$|w| = \Phi_8$: $\rho_{g_2, \epsilon}$.

$|w| = \Phi^2_4$: $\rho_{1, \lambda^3}$.

Type $G_2$.

$|w| = \Phi_6$: $\rho_{g_3, \theta^\pm 1}, \rho_{g_2, \epsilon}$.

$|w| = \Phi_3$: $\rho_{1, \lambda^2}$.

In each case, one can compute the multiplicity of $\rho_{y, \sigma}$ in $R_w$ for $w$ in the same row, using [L2, 4.23]; for the computation we need the character table of $W$ and the explicit entries of the non-abelian Fourier transform [L2, p.110-113]. The result in each case is 1. This completes the proof.

**Theorem 2.14.** (a) Assume that $\rho \in \tilde{U}_Q$ is cuspidal. Then $\rho \in U_Q$.

(b) If $G$ is of type $B, C$ or $D$ and $\rho$ is a unipotent cuspidal representation then $\rho \in U_Q$.

We prove (a). Let $w \in W$ be such that $\rho$ appears with multiplicity 1 in $R_w$ (see 2.13). Since $R_w$ is the character of a virtual representation defined over $Q_l$, it follows that $\rho$ is defined over $Q_l$. (Here $l$ is any prime $\neq p$.) Using the Hasse principle it is then enough to show that $\rho$ is defined over $R$. The following argument is inspired by an argument of Ohmori [Oh]. Let $x$ be an element of minimal length in $W$ such that $\rho$ appears with odd multiplicity in $R_x$. (Such $x$ exists by 2.13.)
It follows that the multiplicity of $\rho$ in $H^l(x)(X, Q)$ is odd. Now $H^l(x)(X, Q)$ admits a $(-1)^l$-symmetric, non-degenerate, $G^F$-invariant $Q$-bilinear form with values in $Q$. (Actually, $l(x)$ is even, by 1.4.) Since $\rho$ is self-dual and has odd multiplicity in $H^l(x)(X, Q)$, an argument in [Oh] shows that $\rho$ itself (regarded as a $Q[G^F]$-module) admits a symmetric, non-degenerate, $G^F$-invariant $Q$-bilinear form with values in $Q$. It follows that $\rho$ is defined over $R$. This proves (a).

In the setup of (b), $\rho$ is unique up to isomorphism hence it is automatically in $U_Q$. Thus, (a) is applicable and $\rho \in U_Q$. The theorem is proved.

2.15. In this subsection we assume that $G$ is non-split. The analogue of 2.13 continues to hold for $G$. But the analogue of 2.14(a) fails if $G$ is non-split of type $A$.

2.16. A statement like 2.13 was made without proof in [L2, p.356] (for not necessarily split $G$). In that statement, the assumption that $\rho$ is cuspidal was missing. That assumption is in fact necessary, as 2.17(ii) below (for $G$ of type $C_4$) shows.

Lemma 2.17. (i) Let $\varepsilon : W_2 \times W_2 \to \{\pm 1\}$ be a character. Then

\[ \text{tr}(w, \text{ind}_{W_2 \times W_2}^W(\varepsilon)) \in 2\mathbb{Z} \]

for all $w \in W$.

(ii) Let $E = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (an irreducible representation of $W_4$). Then $R_E$ is of the form $\chi_{\rho}$ for some $\rho \in \mathcal{U}$. The multiplicity of $\rho$ in $R_w$ is even for any $w \in W = W_4$.

The residue class mod 2 of the left hand side of (a) is clearly independent of the choice of $\varepsilon$. Hence to prove (a) we may assume that $\varepsilon = 1$. Let $\pi : W_4 \to S_4$ be the canonical homomorphism. We have $\text{tr}(w, \text{ind}_{W_2 \times W_2}^W(1)) = \text{tr}(\pi(w), \text{ind}_{S_2 \times S_2}(1))$. But if $y \in S_4$, then $\text{tr}(y, \text{ind}_{S_2 \times S_2}^W(1))$ is 6 if $y = 1$, is 2 if $y$ has order 2 and is 0, otherwise; in particular, it is even for any $y$. This proves (i).

In (ii), the multiplicity of $\rho$ in $R_w$ is $\text{tr}(w, E)$ that is, the left hand side of (a) for a suitable $\varepsilon$. Hence it is even by (i). The lemma is proved.

3. An example in $SO_5$

3.1. In this section we assume that $p \neq 2$ and that $G = SO(V)$ where $V$ is a 5-dimensional $k$-vector space with a fixed $F_q$-rational structure and a fixed non-degenerate symmetric bilinear form $(,) \ defined \ over \ F_q$. Let $C$ be the set of all $g \in G$ such that $g = su = us$ where $-s \in O(V)$ is a reflection and $u \in SO(V)$ has Jordan blocks of sizes 2, 2, 1. Then $C$ is a conjugacy class in $G$ and $F(C) = C$. A line $L$ in $V(F_q)$ is said to be of type 1 if $(x, x) \in F_q^2 - 0$ for any $x \in L - \{0\}$ and of type $-1$ if $(x, x) \in F_q - F_q^2$ for any $x \in L - \{0\}$. Let $\mathcal{L}_1$ (resp. $\mathcal{L}_{-1}$) be the set of lines of type 1 (resp. $-1$) in $V(F_q)$. For $\varepsilon, \delta \in \{1, -1\}$, let $C^\varepsilon, \delta$ be the set of all $g \in C^F$ such that the line $L$ in $V(F_q)$ such that $g|_L = 1$ is in $\mathcal{L}_{\varepsilon}$ and any line $L$ in $V(F_q)$ such that $g|_L = -1$ and $(L, L) \neq 0$ is in $\mathcal{L}_{\delta}$. Then $C^\varepsilon, \delta$ is a conjugacy class.
of $G^F$ and $C^F$ is union of the four conjugacy classes $C^{1,1}, C^{1,-1}, C^{-1,1}, C^{-1,-1}$. We define a class function $\phi : G^F \to \mathbf{Z}$ by $\phi(g) = 2\delta q$ if $g \in C^\epsilon, \delta$ and $\phi(g) = 0$ if $g \in G - C^F$. (This is the characteristic function of a cuspidal character sheaf on $G$.)

Let $O_+$ (resp. $O_-$) be the stabilizer in $G$ of a 4-dimensional subspace of $V$ defined over $F_q$ on which $(,)$ is non-degenerate and split (resp. non-split). Let $\det : O_+ \to \{\pm 1\}$ (resp. $\det : O_- \to \{\pm 1\}$) be the unique nontrivial homomorphism of algebraic groups. The restriction of $\det$ to $O^F_+$ or $O^F_-$ is denoted again by $\det$. Consider the virtual representation

$$\Phi = \text{ind}^{G^F}_{O^F_+}(1) - \text{ind}^{G^F}_{O^F_-}(\det) - \text{ind}^{G^F}_{O^F_+}(\det) + \text{ind}^{G^F}_{O^F_-}(\det)$$

of $G^F$. For $g \in G^F$ we have

$$\text{tr}(g, \Phi) = 2\sharp(L \in \mathcal{L}_1; g|_L = -1) - 2\sharp(L \in \mathcal{L}_{-1}; g|_L = -1).$$

It follows easily that $\text{tr}(g, \Phi) = \phi(g)$.

Let $\theta$ be the unique unipotent cuspidal representation of $G^F$. Then $\theta$ appears with multiplicity 1 in $\phi$. It follows that $\theta$ appears with multiplicity 1 in $\Phi$. Since the character of $\theta$ is $\mathbf{Q}$-valued and $\Phi$ is a difference of two representations defined over $\mathbf{Q}$, it follows that $\theta$ is defined over $\mathbf{Q}$. Thus we have proved the rationality of $\theta$ without using the Hasse principle.

3.2. Assume now that $q = 3$. Then $SO(V)$ is isomorphic to a Weyl group $W$ of type $E_6$ while $O^F_+$ is isomorphic to a Weyl group of type $F_4$ and $O^F_-$ is isomorphic to a Weyl group of type $A_5 \times A_1$ (imbedded in the standard way in the $W$). Now $\theta$ corresponds to the 6-dimensional reflection representation of $W$ (Kneser). Its restriction to the Weyl group of type $F_4$ contains no one dimensional invariant subspace while its restriction to the Weyl group of type $A_5 \times A_1$ splits into a 5-dimensional irreducible representation and a non-trivial 1 dimensional representation. Since $\theta$ has multiplicity 1 in $\Phi$ (see 3.1) it follows that $\theta$ has multiplicity 1 in $\text{ind}^{G^F}_{O^F_-}(\det)$.

References

[BBD] A. A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, Asterisque 100 (1982).

[DL] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, Ann. Math 103 (1976), 103-161.

[L1] G. Lusztig, Coxeter orbits and eigenspaces of Frobenius, Inv. Math. 38 (1976), 101-159.

[L2] G. Lusztig, Characters of reductive groups over a finite field, Ann. Math. Studies 107, Princeton Univ. Press, 1984.

[L3] G. Lusztig, lecture at the U.S.-France Conference on Representation Theory, Paris 1982, unpublished.

[Oh] Z. Ohmori, The Schur indices of the cuspidal unipotent characters of the finite unitary groups, Proc. Japan Acad. A (Math. Sci.) 72 (1996), 111-113.

[W] H. Weyl, The Classical Groups, Princeton Univ. Press.

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