A Geometry of Multimodal Systems

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Abstract

Multimodal normal incestual systems are investigated in terms of multiple categories. The different sorted composition of operators are exhibited as 2-cells in multiple categories built up from 2-categories giving rise to different axioms. Subsequently, coherence results are proved pointing the connections with (usual and mixed) Distributive Laws. This is given as a geometrical description of certain axioms inside various systems with a number of necessity and possibility operators.

Keywords: Inclusion Multimodal Logic, Geach Axiom, McKinsey Axiom, Multiple Category of Cubical Type, Distributive Laws.

1. Introduction

In this paper we face a categorical perspective about axioms in Multimodal Logic systems, that is, those dealing with a number of modalities. We give a characterization of some fragments of the known as Geach and McKinsey axioms.

Following previous studies on (uni- or bi-) modal systems, we make use of monads and comonads as modal operators with the addition of a number of Distributive Laws. We must consider not just the interaction between several different monads but also the interaction between both monads and comonads by using four different kinds of Distributive Laws (two normal and two entwining). Particularly, we work with comonads as necessity operators \(\Box\) and monads as possibility operators \(\Diamond\) (see [Bierman-de Paiva, Kobayashi]) together with certain natural transformations allowing the construction of composed modalities as a new modal operator. These natural transformations are Distributive Laws for \(\Box\), Distributive Laws for \(\Diamond\), Mixed Distributive Laws from a comonad \(\Box\) to a monad \(\Diamond\) and Mixed Distributive Laws from a monad \(\Diamond\) to a comonad \(\Box\) in the form respectively:

\[
\Box_a \Box_b \rightarrow \Box_b \Box_a \quad \Diamond_a \Diamond_b \rightarrow \Diamond_b \Diamond_a \quad \Box_a \Box_b \rightarrow \Box_b \Box_a \quad \Box_a \Diamond_b \rightarrow \Diamond_b \Box_a
\]

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The studies quoted above make use of a propositional language performing conjunction, disjunction and implication. In the spirit of [Dosen-Petric], and since we are only interested in the interaction among modalities, we do not mention nor the ambient category over which these modalities are defined (namely, a bicartesian closed category) neither the way in which the different endofunctors expressing modalities preserve that bicartesian closed structure (see [Bierman-de Paiva, Kobayashi] for these matters). On the other hand, we do not consider negation to occur in a modality.

In [Dosen-Petric] a proof-theoretical approach is considered to introduce categories whose objects are the same modalities and the deductions (arrows) deal with these modalities. Our study is quite similar since we consider 2-categories where the 1-cells are monads, comonads or both alternating and the 2-cells are deductions. Relying on [Grandis-Paré], we consider multiple categories of cubical type whose 1-cells are, respectively, monads and comonads for necessity and possibility operators while 2-cells are Distributive Laws in two different forms: those living in $\text{Mnd}(\text{Mnd}(C))$ and those living in $\text{Cmd}(\text{Cmd}(C))$. The category of quintets appear as an inspiring example because of its form. We then proceed to add different sorted arrows in new directions which are identified with the different modalities of the system. This gives a new point of view of the well known 3-categories of Distributive Laws for monads and comonads, obtaining new categories in multiple form for Distributive Laws, namely $\text{DMnd}$ and $\text{DCmd}$.

Multiple categories of cubical type and its symmetric variant are found to be an appropriate setting to obtain a description of Multimodal systems, in a geometric fashion, based on sets of axioms such as those in Geach or McKinsey form. For that we consider two applications of Ehresmann’s category of quintets, a 2-categorical instance of double category from which we build up certain multiple categories of modalities by adding more axis to the cells. They seem to be well suited for Modal systems to express interaction among the different modalities.

The interaction of possibility and necessity is more complicated and requires defining carefully the directions to ensure full interaction systems. We define in Section 6 $\text{Ent}$ as the analogous of the 3-categories of Mixed Distributive or Entwining Laws here in a multiple cubical way.

Following this line we obtain a number of axioms for Multimodal systems restricted to an order in the indexing. That is, Distributive Laws $d_{ ij}^M$ for which $i \geq j$. In particular, we obtain the known as Persistency axiom for $\mathcal{M} = \Box, \Diamond$ with the above-mentioned constraint. To develop a wider set of axioms we introduce some transposition functions (2-cycles in the permutation group) to make all axis permute and allow all modalities interact. That is, we need to consider a symmetric version of $\text{DMnd}$ and $\text{DCmd}$, increasing drastically the number of axioms available by permuting all axis in the cubes of $\text{DMnd}$ and $\text{DCmd}$. Subsequently, we do the same in Section 6 for $\text{Ent}$. For all multiple categories defined we prove a Coherence Lemma stating that the construction made is consistent, that is, that the endofunctors living in them behave as expected.

Section 2 offers a brief review of the Multimodal systems considered in the sequel. In Section 3 a multiple category made of comonads and Distributive Laws between them is introduced, based on the Ehresmann’s category of quintets, to perform
systems with necessity operators. In this Section a knowledge of these concepts is suposed (the contents about comonads and Distributive Laws can be found in [Street] and those about multiple categories in [Grandis-Paré]). The symmetric counterpart is defined in Section 4, its introduction is justified by enlarging the set of axioms at our disposal in the non-symmetric setting. Section 5 has an analogous content than that of 3 and 4 but based on monads for possibility operators. In Section 6 some interaction Laws are introduced for monads and comonads in two different ways (relying on [Power-Watanabe] for this matter) giving rise to the category of Distributive Laws in all forms: Ent. Finally, the symmetric version of Ent is given as that setting including the greatest number of axioms. In section 7 a summary of the axioms (in the terminology of [Baldoni]) obtained from every 2-category is given.

2. Inclusion Modal Logics

We introduce some axioms that give rise to multimodal systems generalizing many existing temporal, dynamic and epistemic modal systems.

Definition 1. A multimodal system is said to be non-homogeneous if not all modal operators belong to the same system.

An interaction axiom is that axiom producing dependent operators. In particular:

Definition 2. A modal inclusion system is that characterized by sets of logical axioms in the form

\[ \Box_{t_1} \ldots \Box_{t_n} A \rightarrow \Box_{s_1} \ldots \Box_{s_m} A \]

for \( n > 0, m \geq 0 \).

A Multimodal system is said to be normal if it satisfies \( K_i \)-formulas in the form

\[ K_i : \Box_i (A \rightarrow B) \rightarrow (\Box_i A \rightarrow \Box_i B) \]

Example 1. \( K_n, T_n, K4_n, S4_n \).

We extend the systems with \( K_i \) by including axioms in the form \( \Box_{t_1} \ldots \Box_{t_n} A \rightarrow \Box_{s_1} \ldots \Box_{s_m} A \) for \( n > 0, m \geq 0 \) where \( t_1, \ldots, t_n, s_1, \ldots, s_m \) belong to a certain language which we call \( Mod \).

This class of axioms is included into the one defined in [Catach]. We proceed now to add some symbols to the indexing language \( Mod \):

- a binary operator \( \cup \) (non-deterministic choice)
- a binary operator ; (sequential composition)
- \( \epsilon \) (neutral element for composition).

With them we are able to create more labels for modal operators having also the rule:

If \( A \) is a proposition in a certain language and \( i \in Mod \) then \( \Box_i A \) is also a proposition in that language.
Definition 3. In [Catach] the incestual modal logic is the class of normal modal logic containing the axioms:

\[ 
\square_i A \leftrightarrow A \quad \square_i j A \leftrightarrow \square_i \square_j A \quad \square_{i,j} A \leftrightarrow \square_i A \land \square_j A 
\]

In [Baldoni] it is defined for \( i, j \in \text{Mod} \) a set of incestual axioms by including possibility operators \( \Diamond \) in the form:

\[ 
G^{a,b,c,d} : \Diamond_a \square_b A \rightarrow \square_c \Diamond_d A 
\]

with \( a, b, c, d \in \text{Mod} \). The axiom \( G^{a,b,c,d} \), a generalization of the known as Geach Axiom, is precisely what we take as the one for which we will construct our model.\(^3\)

This paper develops from a categorical point of view certain fragments of generalized Geach axioms in the forms \( G^{\epsilon,b,c,\epsilon} \) (with no necessity operators), \( G^{a,\epsilon,\epsilon,d} \) (with no possibility operators) and \( G^{a,b,b,a} \) as well as axioms \( \square_a \Diamond_b A \rightarrow \Diamond_b \square_a A \), which are in McKinsey form.

3. The cubical categories \( DCmd(C) \)

We make use of the concept of monad and comonad for possibility and necessity operators respectively as in [Dosen-Petric]. However, we need to perform the (many different) interaction rules between the comonads which are involved in every axiom. So an extensive use of Distributive Laws for monads and Distributive Laws for comonads has to be considered in the context of \( Mnd(C) \) and \( Cmd(C) \), the 2-categories of monads and comonads. For that, rather than considering the 3-categories of Distributive Laws for monads and comonads over a 2-category, namely \( Mnd(Mnd(C)) \) and \( Cmd(Cmd(C)) = Mnd(Mnd(C_{op})_{op}) \), introduced in [Street] (see [Chikhladze] for a more recent treatment), we will make use of the language of multiple categories to consider coposition of Distributive Laws because of its clarity to illustrate the interaction between modalities in a geometric fashion.

As known we can compose monads and comonads separately whenever we have a Distributive Law at our disposal. We show in this section which the form of a multiple composition of comonads is for the possibility operators. In the 2-categorical point of view of [Street] a Distributive Law in the 2-categorical language appears as an instance of a comonad functor.

Multiple categories, as introduced in [Grandis-Paré], are the generalization of cubical categories for which the arrows can be thought of having different shape. In our case this will mean to increase the length in which a certain modal operator appears in sequences as those referred in Definition 3. Some sequences will be

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1 However, in our modelling we will not make use of non-deterministic choice operator.
2 Baldoni’s is included into Catach by \( \{G^{\epsilon,b,c,\epsilon}\} \subset \{G^{a,b,c,d}\} \) as seen below.
3 It should be noticed that axioms in the form \( \square_{t_1} ... \square_{t_n} A \rightarrow \square_{s_1} ... \square_{s_m} A \) for \( n > 0, m \geq 0 \) where \( t_1, ... t_n, s_1, ..., s_m \in \text{Mod} \) are in the form \( G^{a,b,c,d} \) by taking \( a = e, b = t_1; ...; t_n, c = s_1; ...; s_m, d = e \) while axioms \( \Diamond_{t_1} ... \Diamond_{t_n} A \rightarrow \Diamond_{s_1} ... \Diamond_{s_m} A \) are obtained by taking \( a = t_1; ...; t_n, b = e, c = e, d = s_1; ...; s_m \).
the domain of arrows relating the length of a modal composite with some cells in a certain direction. We will make use of these sequences in order to give a geometrical point of view about multimodal interaction.

**Definition 4.** A multiple category \( C \) of cubical type is a multiple set of cubical type (see [Grandis-Paré]) with components \( C_i \) whose elements are \( i \)-cells subject to the following data:

1. given two composable \( i \)-cells \( a, b \) (that is: such that \( \partial_i^-(a) = \partial_i^-(b) \) for \( i \in I \) ) we have the \( i \)-composition, denoted by \( a +_i b \), satisfying:
   \[
   \partial_i^-(a +_i b) = \partial_i^-(a) \quad \partial_i^+(a +_i b) = \partial_i^+(b) \\
   \partial_j^p(a +_i b) = \partial_j^p(a) +_i \partial_j^p(b) \quad e_j(a +_i b) = e_j(a) +_i e_j(b) \text{ for } i \neq j
   \]

2. for \( j \notin i \) we have a new category with objects in \( C_i \) , arrows in \( C_{ij} \) (standing for \( C_i \cup j \)) and new faces \( \partial_j^p \), identities \( e_j \) and composition \( +_j \)

3. for \( i < j \) we have the middle four-interchange rule
   \[
   (a +_i b) +_j (c +_i d) = (a +_j c) +_i (b +_j d)
   \]

An \( n \)-cell in a multiple category of cubical type will then be called an \( n \)-cube whenever \( n \) is the \( n \)th ordinal \( \{0, ..., n-1\} \). In the following sections all multiple categories are with no mention of cubical type.

We consider double categories over 2-categories. There exist several different ways to get a double category from a 2-category (see [? ]), our approach takes the double category of quintets \( QC \) of Ehresmann as an inspiring example. By endowing a double category, in Ehresmann’s form, with \( n \) arrows in different directions which turn out to be comonads, we will construct an \( n \)-dimensional multiple category of cubical type.

**Definition 5.** Given a 2-category \( C \) we denote by \( QC \) that double category whose objects are those of \( C \) , whose horizontal and vertical arrows are the 1-cells of \( C \) and whose squares are the 2-cells \( a : K \circ F \to G \circ J \) in

\[
\begin{array}{ccc}
\cdot & \overset{F}{\rightarrow} & \cdot \\
J & \downarrow & K \\
\cdot & \underset{G}{\rightarrow} & \cdot
\end{array}
\]

It is precisely from \( QC \) that we take the point of view of a double cell (a square) as a 2-cell with the same 2-categorical object in all four nodes, obtaining the 2-categorical description of Distributive Laws between comonads from Section 3. That is, given a 2-category \( C \) we consider a double category for which the diagonal 2-cells are Distributive Laws for comonads in the form \( a : N_2 N_1 \to N_1 N_2 \), where we identify \( F \) and \( G \) with \( N_1 \) and \( V \) and \( V \) with \( N_2 \).

**Definition 6.** Let \( DCmd_2(C) \) be the full subcategory of \( QC \) for which all 1-cells are comonads and all diagonal 2-cells are Distributive Laws between them.
$DCmd_2(C)$, as a double category, is endowed with horizontal maps in a square

\[
\begin{array}{ccc}
N_2N_1C & \xrightarrow{aC} & N_1N_2C \\
\downarrow & & \downarrow a \\
N_2N_1u & \xrightarrow{a} & N_1N_2u \\
\end{array}
\]

for $C$ an object and $u : C \to C$ a 1-cell in $C$.

From that double category we describe a multiple category $DCmd_n(C)$ of Distributive Laws between $n$ comonads by endowing $DCmd_2(C)$ with axis in more directions, as done in [Grandis-Paré]. Those axis will play the role of modalities acting over the objects of $C$, for that we identify the directions of the axis with the indexing of the modalities.

Let us remark that, while $Cmd(Cmd(C_{op})_{op})$ from [Street] is a 2-category of Distributive Laws between comonads, $DCmd_n(C)$ is a multiple category containing a geometrical (cubical) account of the different compositions of Distributive Laws that can be considered. In fact, $DCmd_2(C)$ can be seen as the image of $Cmd(Cmd(C_{op})_{op})$ through the known as functor of quintets

\[Q : 2-Cat \to Dbl\]

where $Dbl$ denotes the category of all double categories.

We now give a description of $DCmd_3(C)$. Let $C$ be a 2-category and $C$ an object in $C^{\geq}$

1. $DCmd_0(C)$ is the category whose objects are the objects of $C$
2. $DCmd_0(C), DCmd_1(C), DCmd_2(C)$ are the categories whose objects are comonads over $C$ in the horizontal ($\square_0$), diagonal ($\square_1$) and vertical ($\square_2$) directions respectively endowed with one degeneracy and two faces for $i = 0, 1, 2$
3. $DCmd_{01}(C), DCmd_{12}(C), DCmd_{02}(C)$ are the categories whose objects are, from left to right, squares together with three 2-cells $d_{10} : \square_0\square_1 \to \square_1\square_0, d_{21} : \square_2\square_1 \to \square_1\square_2$ and $d_{20} : \square_2\square_0 \to \square_0\square_2$

\[\text{Although it is not our concern in this paper, } C \text{ should have a bicartesian closed underlying category to get conjunctions, disjunctions and implications between the objects of } C \text{ as well as the requirement that all comonads are symmetric monoidal closed (see [Bierman-de Paiva] for this matter).}\]
respectively behaving as Distributive Laws according to the definition given in Section 4 and two degeneracies and four faces

\[ e_i : DCmd_i(C) \to DCmd_{ij}(C) \quad e_j : DCmd_i(C) \to DCmd_{ij}(C) \]
\[ \partial^\alpha_i : DCmd_{ij}(C) \to DCmd_j(C) \quad \partial^\alpha_j : DCmd_{ij}(C) \to DCmd_i(C) \]

for \( i, j = 0, 1, 2 \) such that \( i < j \) and \( \alpha = 0, 1 \).

4. \( DCmd_{012}(C) \) is a category whose objects are 3-cubes each face of which comes with a diagonal 2-cell in it.

For the comonads \( \Box_0, \Box_1, \Box_2 \) defined in the same object \( C \) in the 2-category \( C \), and the directions

the 3-cells in \( C_{012} \) are cubes as given at left, each face containing a Distributive Law as given at right:

Now we describe how \( i; j \) subindexes are defined from a cubical set structure performing \textit{concatenation}. With them we express the multiple relations between the modalities in cubical form. Every subset of a multi-index set of ordinals \( \mathbf{n} \) can

\[ ^5 \text{Faces and degeneracies act respectively as \textit{erasing} and \textit{introducing} a new modality in the system.} \]
be seen as a subindex for a comonad. Concatenation of modalities are describes by paths of edges of hypercubes whose dimension is that of the number of modalities we are dealing with. This is geometrically expressed by chains of hypercubes for which every kind of arrow refers to a different modality by being oriented in a different direction.\footnote{In \cite{Dosen-Petric} there is an interpretation of modal logics with one and two operators in terms of relations between the length of a composite of these modalities.}

Our modalities (and the nodes of the sets into a multiple cubical category) will then be \(\square_i \ldots \square_n = \square_{i_1} \ldots \square_{i_n}\) for \(i_1, \ldots, i_n \in \mathbb{N}\) such that \(i_1 < \ldots < i_n\).\footnote{From a logical perspective, subindexes in the form of a concatenation (a semicolon chain) describes the path to reach a certain place of observation or place of knowledge.}

We now show how can one compose Distributive Laws for comonads over a 2-category in \(\mathcal{DCmd}(\mathcal{C})\). These compositions are expressed here as a concatenation of 2-cubes in a multiple category of cubical type. We make use of the hypercube notation for compositions as introduced in \cite{Grandis-Paré}.

For instance, directed compositions in \(\mathcal{DCmd}_2(\mathcal{C})\), given the 0, 1, 2 axis as above, are
\[
\begin{align*}
d^2_{10} + 0 d^2_{10} &= d^2_{10(00)} \\
d^2_{10} + 1 d^2_{10} &= d^2_{11(10)} \\
d^2_{20} + 0 d^2_{20} &= d^2_{20(00)}
\end{align*}
\]
where we express \(d^2_{i(j,j)}\) for \(d^2_{i(j; j)}\). In \(\mathcal{DCmd}_4(\mathcal{C})\) we have, for \(i, j \in \mathbb{N}\) such that \(i \geq j\):
\[
\begin{align*}
d^2_{ij} + j d^2_{ij} &= d^2_{i(jj)} \\
d^2_{ij} + i d^2_{ij} &= d^2_{i(ij)}
\end{align*}
\]

Expressing \((d \mid d')\) and \((\frac{d}{d'})\) for \(d + 0 d'\) and \(d + 1 d'\) respectively in \(\mathcal{DCmd}_2(\mathcal{C})\) we have for

\[
\begin{array}{c}
N_2 \downarrow d^2_{N_2 N_1} \downarrow \uparrow d^2_{N_2 N_1'} \\
& \uparrow N_1' \downarrow \downarrow N_2 \\
N_2' \downarrow d^2_{N_2' N_1} \downarrow \uparrow d^2_{N_2' N_1'} \\
& \uparrow N_1' \downarrow \downarrow N_2'
\end{array}
\]

horizontal and vertical compositions
\[
(d^2_{N_2 N_1} \mid d^2_{N_2 N_1'}) = N'_1 d^2_{N_2 N_1} \cdot d^2_{N_2 N_1'} N_1 \\
\left(d^2_{N_2 N_1'} N_1 \right) d^2_{N_2 N_1} = d^2_{N_2 N_1} N'_2 \cdot N_2 d^2_{N_2' N_1}
\]

both being strict thanks to the 2-categorical structure of \(\mathcal{C}\). Finally, we have the \textbf{four-middle interchange rule}:
\[
\begin{align*}
\left(d^2_{N_2 N_1} \mid d^2_{N_2 N_1'} \right) &= \left(d^2_{N_2 N_1} \mid d^2_{N_2 N_1'} \right) \\
\left(d^2_{N_2 N_1} \mid d^2_{N_2 N_1'} \right) &= \left(d^2_{N_2 N_1} \mid d^2_{N_2 N_1'} \right)
\end{align*}
\]
for single Distributive Laws

The following is our first coherence Lemma for the interaction of n necessity modalities.

**Lemma 1.** Every composition of 2-cells of the form $d^\Box$ in $DCmd_n(C)$ is a Distributive Law.

**Proof.** Take $i, j, k \in n$ such that $k \geq j \geq i$. Having already identities and associativity, for

![Diagram](image)

we can compose

1. $(d^\Box_{k_i} \mid d^\Box_{k_j}) : \Box_k(\Box_j \Box_i) \rightarrow (\Box_j \Box_i) \Box_k$ defined as $\Box_j d^\Box_{k_i} \cdot d^\Box_{k_j} \Box_i$ and for which we have, according to the 2-categorical definition of Distributive Law of [Street], a pair

   $((C, \Box_j \Box_i), (\Box_k, (d^\Box_{k_j} \mid d^\Box_{k_i})))$

This is based on two Distributive Laws $d^\Box_{k_j}$ and $d^\Box_{k_i}$ seen as comonad functors:

(a) as $d^\Box_{k_j}$ and $d^\Box_{k_i}$ are Distributive Laws we have the following commuting diagrams

![Diagrams](image)

[8]

We also have some singular instances of 2-cubes such as

![Images](image)

They are instances of the so-called *special iso-cells* in [Grandis-Pare] and give rise to the $K_1$ axioms characterizing the normal multimodal systems (see Definition 2).
and then

commutes.

(b) On the other hand from

standing for \( \Box_j d_{k_1}^i \) and \( d_{k_j}^i \), we obtain

for the second condition of a comonad functor where the square \((\alpha)\) commutes by naturality of composition.
(c) We need to show also that there exists a comonad natural transformation
\[ \varepsilon : (\Box_k, (d^k_{ij} \mid d^k_{ki})) \to 1 \]
from comonad natural transformations \( \varepsilon_k : (\Box_k, d^k_{ij}) \to 1 \) and \( \varepsilon'_k : (\Box_k, d^k_{ki}) \to 1 \) subject to the commuting squares
\[
\begin{array}{c}
\Box_k \Box_j \xrightarrow{\varepsilon_k \Box j} \Box_j \\
d^k_{ij} \downarrow \quad \downarrow 1 \\
\Box_j \Box_k \xrightarrow{\Box j \varepsilon_k} \Box j
\end{array} 
\quad 
\begin{array}{c}
\Box_k \Box_i \xrightarrow{\varepsilon'_k \Box i} \Box_i \\
d^k_{ki} \downarrow \quad \downarrow 1 \\
\Box_i \Box_k \xrightarrow{\Box i \varepsilon'_k} \Box i
\end{array}
\]
from which we get
\[
\begin{array}{c}
\Box_k \Box_j \Box_i \xrightarrow{(\varepsilon_k \Box j) \Box_i} \Box_j \Box_i \\
(d^k_{ij} \mid d^k_{ki}) \downarrow \quad \downarrow 1 \\
\Box_j \Box_k \Box_i \xrightarrow{\Box j \varepsilon_k \Box i} \Box j \Box_i
\end{array}
\]
commuting for \( \varepsilon \).

(d) Finally, for duplication \( \delta \) we have duplications
\[ \delta^k_j : (\Box_k, d^k_{ij}) \to (\Box_k \Box_k, (d^k_{ij} \mid d^k_{ki})) \] and \( \delta^k_i : (\Box_k, d^k_{ki}) \to (\Box_k \Box_k, (d^k_{ij} \mid d^k_{ki})) \)
in the form
\[ \delta^k_j : (\Box_k, d^k_{ij}) \to (\Box_k \Box_k, d^k_{ij} \Box_k \Box_k d^k_{ij}) \] and \( \delta^k_i : (\Box_k, d^k_{ki}) \to (\Box_k \Box_k, d^k_{ki} \Box_k \Box_k d^k_{ki}) \)
respectively for which
\[
\begin{array}{c}
\Box_k \Box_j \xrightarrow{\delta^k_j \Box j} \Box_k \Box_k \Box_j \\
d^k_{ij} \downarrow \quad \downarrow d^k_{ij} \Box_k \Box_k d^k_{ij} \\
\Box_j \Box_k \xrightarrow{\Box j \delta} \Box_j \Box_k \Box_k
\end{array} 
\quad 
\begin{array}{c}
\Box_k \Box_i \xrightarrow{\delta^k_i \Box i} \Box_k \Box_k \Box_i \\
d^k_{ki} \downarrow \quad \downarrow d^k_{ki} \Box_k \Box_k d^k_{ki} \\
\Box_i \Box_k \xrightarrow{\Box i \delta^k_i} \Box_i \Box_k \Box_k
\end{array}
\]
commute. From them we get a duplication
\[ \delta : (\Box_k, (d^k_{ij} \mid d^k_{ki})) \to (\Box_k \Box_k, (d^k_{ij} \mid d^k_{ki})) \cdot (\Box_k, (d^k_{ij} \mid d^k_{ki})) \]
in the form
\[ \delta : (\Box_k, (d^k_{ij} \mid d^k_{ki})) \to (\Box_k \Box_k, ((\Box_k d^k_{ij} \mid d^k_{ki})) \cdot (\Box_k, (d^k_{ij} \mid d^k_{ki}))) \]
and a commuting pasting square

2. We had analogous diagrams for vertical compositions

\[
\left( \frac{d^2_{lj}}{d^2_{ki}} \right) : \Box_l \Box_i \Box_i \to \Box_l \Box_i \Box_i
\]

seen as comonad functors

\[
\left( (C, \Box_k \Box_i), (\Box_k, \left( \frac{d^2_{lj}}{d^2_{ki}} \right) ) \right)
\]

and given by \( d^2_{li} \Box_k \cdot \Box_l d^2_{ki} \). That is,

(a) from

we get

\[
\left( \frac{d^2_{lj}}{d^2_{ki}} \right)
\]
(b) and from

we can construct the following commuting diagram

for the second condition of a comonad functor where the square \((\beta)\) commutes by naturality of composition.

For \(\varepsilon\) and \(\delta\) we compute as above.

We are in this way obtaining a Distributive Law for three modalities in the form

\[
\left( \frac{d_{N_2N_1}^3}{d_{N_3N_1}^3} \right) = \left( \frac{d_{N_2N_1}^3}{d_{N_3N_1}^3} \mid d_{N_3}^3 \right) : (N_3N_2)N_1 \rightarrow N_1(N_3N_2)
\]
4. The symmetric version

To get the expressivity of the different Multimodal systems we need all modalities interact. This is obtained after considering hypercubes where a number of Distributive Laws exist in such a way that every chain of modalities becomes a new modality in its own right. For that we consider transposition functors between the different multiple sets underlying $DCmd_n(C)$, as known from the Symmetric Group Theory, to the permutations among all axis in the hypercube.

**Definition 7.** A multiple category of symmetric cubical type is a multiple category of cubical type with an assigned action of the symmetric group $S_n$ on each set $X_i$ for $i$ a multi-index with length $n$ generated by transpositions $s_i : X_i \rightarrow X_i$ switching the $i-$ and $(i-1)-$ axis for $i = 1, ..., n-1$.

We denote by $S_n$ the $n$-symmetric group. Since $s_i$, as 2-cycles, generate all elements in $S_n$ we have all permutations of coordinates available to combine the different modalities. For example, for the case of $n = 2$ we can permute all axis in every square and every cube giving rise to, for instance, to

\[
\begin{array}{c}
\text{·} \quad 1 \rightarrow \quad 2 \\
\downarrow \quad \downarrow \\
\text{·} \quad 0 \rightarrow \quad 0 \\
\end{array}
\]

obtained by composition of coordinate systems

\[
\begin{array}{c}
\text{·} \quad 0 \rightarrow \quad 2 \\
\downarrow \quad \downarrow \\
\text{·} \quad 1 \rightarrow \quad 1 \\
\end{array}
\]

in the direction of axis 1. That is, in the notation of [Grandis-Paré] for oriented compositions and making use of sequential composition of modalities from Section 1:

\[d_{01}^{01} + d_{02}^{02} = d_{0(21)}^{01}\]

For the calculation done we note also that, in the presence of transpositions, we do not have just Distributive Laws in the form $d_{ji}^{ji}$ for $j \geq i$ but a Distributive Law for every pair of indices no matter which one the greater is. When ordering the subindexes according to the order in $n$ we get an indexing normal form.

We now show how transpositions interact with composition in different directions:

\[
s_i(a +_i b) = s_i(a) +_{i-1} s_i(b) \quad s_i(a +_{i-1} b) = s_i(a) +_i s_i(b) \\
s_i(a + j b) = s_i(a) +_j s_i(b) \quad \text{for } j \neq i-1, i
\]

that is, for $n = 3$ we have three modalities 0, 1, 2 and two transpositions $s_1, s_2$ such that

\[
\begin{align*}
s_1(a +_0 b) &= s_1(a) +_1 s_1(b) & s_2(a +_0 b) &= s_2(a) +_0 s_2(b) \\
s_1(a +_1 b) &= s_1(a) +_0 s_1(b) & s_2(a +_1 b) &= s_2(a) +_2 s_2(b) \\
s_1(a +_2 b) &= s_1(a) +_2 s_1(b) & s_2(a +_2 b) &= s_2(a) +_1 s_2(b)
\end{align*}
\]

\[\text{For } n<2 \text{ we have no transpositions so we work from now on with } n \geq 2.\]
which means for example

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
2 & a + 1 \ b & 2
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
2 & a & 2
\end{pmatrix} + \begin{pmatrix}
1 & 0 \\
0 & 0 \\
2 & b & 2
\end{pmatrix}
\]

That is:

\[s_1(□2□1□1 \to □1□1□2) = □2□0□0 \to □0□0□2\]

Definitions 8. Let SD C m d_{n}(C) be the multiple category of symmetric cubical type generated by adding transpositions to D C m d_{n}(C).

Transpositions act over Distributive Laws in D C m d_{n}(C) by switching the comonads. Therefore, whenever \(n = 2\), we get from a 2-cell as the one at left a 2-cell as the one at right:

\[
\begin{array}{c}
□_1 \\
\downarrow \alpha_1 \\
□_0 \\
\downarrow β_1 \\
□_0
\end{array}
\]

\[
\begin{array}{c}
□_1 \\
\downarrow \alpha_0 \\
□_0 \\
\downarrow β_0 \\
□_0
\end{array}
\]

by an action of the only transposition \(s_1\). That is, since a 2-cell Distributive Law is a square, \(s_1\) acting over a square is the same thing as acting over a Distributive Law and we have \(s_1(d_{i,j}^{(1)}) = d_{i,j}^{(1)}\).

In general:

\[
s_k(d_{i,j}^{(1)}) = \begin{cases} 
\begin{pmatrix}
\alpha^{(1)} \\
\beta^{(1)}
\end{pmatrix}_{i,j} & \text{whenever } i = k, j \neq k, k - 1 \\
\begin{pmatrix}
\alpha^{(1)} \\
\beta^{(1)}
\end{pmatrix}_{i,j} & \text{whenever } i = k - 1, j \neq k, k - 1 \\
\begin{pmatrix}
\alpha^{(1)} \\
\beta^{(1)}
\end{pmatrix}_{j-1} & \text{whenever } j = k, i \neq k, k - 1 \\
\begin{pmatrix}
\alpha^{(1)} \\
\beta^{(1)}
\end{pmatrix}_{j+1} & \text{whenever } j = k - 1, i \neq k, k - 1 \\
\begin{pmatrix}
\alpha^{(1)} \\
\beta^{(1)}
\end{pmatrix}_{i,j} & \text{whenever } j = k, i = k - 1 \\
\begin{pmatrix}
\alpha^{(1)} \\
\beta^{(1)}
\end{pmatrix}_{i,j} & \text{otherwise}
\end{cases}
\]

for \(i, j, k \in \mathbb{N}\) such that \(j \leq i\). Similarly, for composed indexing we had

\[s_1(d_{2(1,1)}) = d_{2(0,0)}\]

We now show how can one compose Distributive Laws in SD C m d_{n}(C) over a 2-category. They are, for different indices \(\alpha, \gamma, \beta\) in \(\mathbb{N}\) and \(0 < j < n - 1:\

\[
\begin{aligned}
d_{\alpha,\beta}^{(1)} + (\alpha - 1) d_{\gamma,\beta}^{(1)} &= d_{(\alpha\gamma),\beta}^{(1)} \\
d_{\alpha,\beta}^{(1)} + 0 d_{\alpha,\gamma}^{(1)} &= d_{\alpha,\gamma}^{(1)} \\
d_{\alpha,\beta}^{(1)} + j d_{\alpha,\beta}^{(1)} &= \begin{cases} 
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}_{(\alpha\beta),\beta} & \text{for } \partial_j^\alpha(d_{\alpha,\beta}^{(1)} + j d_{\alpha,\beta}^{(1)}) = \beta \\
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}_{(\alpha\beta),\beta} & \text{for } \partial_j^\beta(d_{\alpha,\beta}^{(1)} + j d_{\alpha,\beta}^{(1)}) = \alpha
\end{cases}
\end{aligned}
\]

Taking into account (following Dosen-Petric) that a modality is a finite (possibly empty) sequence of the modal operators of necessity and possibility, to get a multimodal system as that of Baldoni in the style of cubical sets we point that
every level in this construction contains one more modality than the former and 
correspond to each category $DCmd_i(C)$. They are connected through the face and 
degeneration functions whose action erases or adds one modality respectively, while 
the empty modality gives all objects that can be formed in a bicartesian closed 
category as the analogues of conjunction, disjunction and implication.

We have for example

$$e_2 : DCmd_{i\{1\{2\}}(C) \to DCmd_i(C)$$

such that $e_2(\text{NC}) = \text{N’NC}$ for:

- $C$ an object in the 2-category $C$
- $N = \Box_0, \Box_1$ or a chain of possibly several $\Box_0$ and $\Box_1$ and
- $N’ = \Box_0, \Box_1, \Box_2$ or a chain of possibly $\Box_0$, $\Box_1$ and $\Box_2$.

5. The cubical categories $SDMnd(C)$

We now turn to consider an $n$-dimensional multiple category of cubical type by 
endowing a double category with $m$ arrows in different directions which turn out to 
be monads.

**Definition 9.** Let $DMnd_2(C)$ be the full subcategory of $QC$ for which all 1-cells 
are monads and all diagonal 2-cells are Distributive Laws between them.

$DMnd_2(C)$ is a double category which is also endowed with horizontal maps

$$aC : M_2M_1C \to M_1M_2C$$

in a square

$$\begin{array}{ccc}
M_2M_1C & \xrightarrow{aC} & M_1M_2C \\
M_2M_1u & \downarrow & M_1M_2u \\
M_2M_1C & \xrightarrow{aC} & M_1M_2C
\end{array}$$

for $C$ an object and $u : C \to C$ a 1-endocell in $C$. It should be noticed that, in this 
case, the diagonal 2-cells are written $d^{\Box}_{12}$ (switching the subindices from the case of 
$d^{\Box}_{21}$ for the same square).

From the double category considered, we can now define $DMnd_m(C)$ as the $m$- 
dimensional multiple category of cubical type of Distributive Laws between monads 
over a 2-category. $DMnd_2(C)$ can be seen as the image of $Mnd(Mnd(C))$ through 
the functor of quintets $Q$. We begin by showing how can one compose Distributive 
Laws between monads in a cell

$$\begin{array}{cccc}
&M_1 & & M'_1 \\
M_2 & \xleftarrow{d^{\Box}_{M_1M_2}} & d^{\Box}_{M'_1M_2} & \xrightarrow{d^{\Box}_{M_1M'_2}} M_2 \\
&M_2 & \xleftarrow{d^{\Box}_{M'_1M_2}} & d^{\Box}_{M'_1M'_2} & \xrightarrow{d^{\Box}_{M'_1M_2}} M_2 \\
&M'_1 & & M'_1
\end{array}$$
we can compose these Distributive Laws horizontal and vertically denoted respectively

\[(d_{M_1M_2} \diamond d_{M_1M_2}^\diamond) = M_1d_{M_1M_2}^\diamond d_{M_1M_2}^\diamond M_2 \quad (d_{M_1M_2}^\diamond M_2 \cdot d_{M_1M_2}^\diamond M_2) = d_{M_1M_2}^\diamond M_2 \cdot d_{M_1M_2}^\diamond M_2\]

where both compositions are strict thanks to the 2-categorical structure of \(C\).

The four-middle interchange rule has the form:

\[\left(\frac{d_{M_1M_2}^\diamond d_{M_1M_2}^\diamond}{d_{M_1M_2}^\diamond d_{M_1M_2}^\diamond} \right) = \left(\frac{d_{M_1M_2}^\diamond d_{M_1M_2}^\diamond}{d_{M_1M_2}^\diamond d_{M_1M_2}^\diamond}\right)\]

To prove that horizontal and vertical compositions are again Distributive Laws for monads we have just to dualize the diagrams from the 2-categorical definition in Section 3.

**Lemma 2.** Every composition of 2-cells of the form \(d_{\circ}^\diamond\) in \(DMnd_{m}(C)\) is a Distributive Law.

By adding transpositions we had, as in Section 4, a multiple category of symmetric cubical type \(SDMnd_{m}(C)\). We have an analogous definition for transposed Distributive Laws:

\[s_k(d_{ij}^\circ) = \begin{cases} 
  d_{i-1,j}^\circ & \text{whenever } i = k, j \neq k, k-1 \\
  d_{i+1,j}^\circ & \text{whenever } i = k-1, j \neq k, k-1 \\
  d_{i,j-1}^\circ & \text{whenever } j = k, i \neq k, k-1 \\
  d_{i,j+1}^\circ & \text{whenever } j = k-1, i \neq k, k-1 \\
  d_{ij}^\circ & \text{whenever } j = k, i = k-1 \\
  d_{ij}^\circ & \text{otherwise}
\end{cases}\]

for \(i, j, k \in m\) such that \(j \leq i\). Similarly, for composed indexing we had

\[s_1(d_{2(11)}^\diamond) = d_{2(00)}^\diamond\]

The possibility operators, denoted by \(\diamond\), in a sense dual to necessity, are performed in our setting by dualizing most of the structure given up to now. With the same indexing language \(Mod\) a set of incestual axioms in the form \(G^{n,b,c,d}\) : \(\diamond \square_k A \rightarrow \square_k \diamond d A\) with \(b = c = \epsilon\) will develop the fragment of \(G^{n,e,e,d}\) for which we consider just diamonds.

6. The cubical categories \(SEnt(C)\)

We now consider an \(n\)-dimensional multiple category of cubical type by endowing a double category with \(n\) arrows in different directions corresponding to \(\lceil n/2 \rceil\) monads and \(\lfloor n+1/2 \rfloor\) comonads. We need then to make use of entwining Laws all over them in the form of the following definitions (see [Power-Watanabe]).
**Definition 10.** Given a monad $M$ and a comonad $N$ over a 2-category $C$ we say that the 2-cell $e : MN \to NM$ is an **entwining Law of** $M$ **over** $N$ or a mixed **Distributive Law** from a monad $M$ to a comonad $N$ if the following diagrams commute:

\[
\begin{array}{ccc}
MMN & \xrightarrow{\mu_N} & MN \\
MMC & \xrightarrow{\eta_M} & NMM \\
MN & \xrightarrow{\epsilon_N} & NM \\
NMN & \xrightarrow{\delta_M} & MNC
\end{array}
\]

By dualizing this definition we obtain the definition of an **entwining Law of a comonad** $N$ **over a monad** $M$ or a mixed **Distributive Law** from a comonad $N$ to a monad $M$. We denote them respectively $d^{\square\diamond}$ and $d^{\diamond\square}$.

**Definition 11.** Let $Ent_2(C)$ be the full subcategory of $QC$ for which all horizontal 1-cells are comonads, all vertical 1-cells are monads and all diagonal 2-cells are Distributive Laws $d^{\square\diamond}$ between them.

Entwining Laws of type $d^{\square\diamond}$ in it are diagonal 2-cells giving rise to double categories which are also endowed with horizontal maps $d^{\square\diamond}C : MNC \to NMC$ in squares

\[
\begin{array}{ccc}
MNC & \xrightarrow{d^{\square\diamond}C} & NMC \\
MNC & \xrightarrow{d^{\diamond\square}C} & NMC
\end{array}
\]

for $C$ an object, $u : C \to C$ a 1-endocell in $C$ and $M$ and $N$ a monad and a comonad over $C$ respectively.

From the double category considered we can define $Ent_n(C)$ as the multiple category of cubical type whose nodes are the objects in a 2-category and whose underlying category is bicartesian closed. We establish the following conventions: axis in it indexed by odd numbers in $n$ are monads, axis indexed by even numbers in $n$ are comonads and the 2-cells between them are denoted $d_{ij}$ and, for $i > j$, identified as:

\[
d_{ij} = \begin{cases} 
    d^{\square\diamond}_{ij} & \text{whenever } i, j \text{ are even} \\
    d^{\diamond\square}_{ij} & \text{whenever } i \text{ is odd, } j \text{ is even} \\
    d^{\square\diamond}_{ij} & \text{whenever } i \text{ is even, } j \text{ is odd} \\
    d^{\diamond\square}_{ij} & \text{whenever } i, j \text{ are odd}
\end{cases}
\]

for $i, j \in n$ such that $i \geq j$.

\[^{10}\text{Notice that we do not still have 2-cells in the form } d_{ij} \text{ with } j > i.\]
Where we denote the mixed Distributive Laws as
\[ d_{ij}^{\diamond} : \diamond_i \diamond_j \rightarrow \square_j \diamond_i \]
\[ d_{ij}^{\square} : \square_i \diamond_j \rightarrow \diamond_j \square_i \]

For instance, the different Distributive Laws for our system in the model of four axis

\[ \cdot \]
\[ \rightarrow \]
\[ \rightarrow \]
\[ \rightarrow \]
\[ \rightarrow \]
\[ \rightarrow \]
\[ \rightarrow \]

are \( d_{10}^{\diamond} \), \( d_{20}^{\diamond} \), \( d_{30}^{\diamond} \), \( d_{31}^{\diamond} \) and \( d_{12}^{\square} \).

We give a description of \( Ent_n(\mathcal{C}) \) by showing composition of several Distributive Laws

\[ (d_{MN}^{\diamond} | d_{MN'}^{\diamond}) = N' \cdot d_{MN}^{\diamond} \cdot d_{MN'}^{\diamond} \cdot N \]
\[ \left( \frac{d_{MN}^{\diamond}}{d_{MN'}} \cdot d_{MN}^{\diamond} \cdot d_{MN'}^{\diamond} \cdot N \right) = d_{M'N}^{\diamond} \cdot M' \cdot d_{M'N}^{\diamond} \]

where both compositions are strict thanks again to the 2-categorical structure of \( \mathcal{C} \).

The \textit{four-middle interchange rule} has the form:

\[ \left( \frac{d_{MN}^{\diamond}}{d_{NM'}} \cdot d_{MN}^{\diamond} \cdot d_{MN'}^{\diamond} \cdot N \right) = \left( \frac{d_{MN}^{\diamond} | d_{MN'}^{\diamond} | d_{MN'}^{\diamond} | d_{MN}^{\diamond} \cdot N}{d_{MN}^{\diamond} | d_{MN'}^{\diamond} | d_{MN'}^{\diamond} | d_{MN}^{\diamond} \cdot N} \right) \]

\textbf{Lemma 3.} Every composition of Distributive Laws of the form \( d_M^M \) in \( Ent_n(\mathcal{C}) \) is a Distributive Law in the form \( d_M^M \) for \( M = \square, \diamond, \square \diamond \) or \( \square \diamond \).

\textbf{Proof.} We consider for example two entwining Laws \( d_{j_1i_1}^{\diamond} \) and \( d_{j_1i_2}^{\diamond} \) with \( i_1, j_1, i_2, j_2 \in \).
for which the following diagrams commute:

\[ i_1 \otimes j_1 \rightarrow i_2 \otimes j_2 \]
\[ i_1 \otimes j_2 \rightarrow i_2 \otimes j_1 \]

We then have two commutative diagrams

\[ i_2 \otimes j_2 \rightarrow i_2 \otimes j_2 \]
\[ i_2 \otimes j_1 \rightarrow i_2 \otimes j_1 \]

by considering the products \( d_{j_1 i_2} \otimes j_1 \) and \( i_2 d_{j_1 i_2} \).

Now we paste them to get

\[ i_2 \otimes j_2 \rightarrow i_2 \otimes j_2 \]
\[ i_2 \otimes j_1 \rightarrow i_2 \otimes j_1 \]

where diagrams \((\delta), (\zeta)\) and \((\gamma)\) commute trivially.

With which we deduce that \( (d_{j_1 i_1} \otimes d_{j_1 i_2}) \) behaves as an entwining Law according to Definition 10.
By adding transpositions we have, as in Section 4, a multiple category of symmetric cubical type $SEnt_n(C)$ whose Distributive Laws are for $i, j \in \mathbb{N}^{11}$

$$d_{ij} = \begin{cases} d_{ij} & \text{whenever } i, j \text{ are even} \\ d_{ij} & \text{whenever } i \text{ is odd, } j \text{ is even} \\ d_{ij} & \text{whenever } i \text{ is even, } j \text{ is odd} \\ d_{ij} & \text{whenever } i, j \text{ are odd} \end{cases}$$

7. Sets of axioms

We list sets of incestual axioms that can be performed in the different multiple categories of (symmetric and non-symmetric) cubical type. For that we make use of the terminology given in [Baldoni].

Incestual interaction axioms generated in $DCmd(C)$ with $a = d = \epsilon$ for $G_{a,b,c,d}^{n}$

1. Reflexivity axiom for $G_{a,b,c,d}^{n}: \Box A \rightarrow A$
2. Transitivity axiom for $G_{a,b,c,d}^{n}: \Box A \rightarrow \Box A$
3. Restricted Persistency axiom for $G_{a,b,c,d}^{n}: \Box \Box A \rightarrow \Box \Box A$.

Incestual interaction axioms generated in $SDCmd(C)$ with $a = d = \epsilon$ for $G_{a,b,c,d}^{n}$

1. General Persistency axiom for $G_{a,b,c,d}^{n}: \Box A \rightarrow \Box A$
2. Composition axiom for $G_{a,b,c,d}^{n}: \Box \Box A \rightarrow \Box A$.

Incestual interaction axioms generated in $DMnd(C)$ with $a = c = \epsilon$ for $G_{a,b,c,d}^{n}$

1. Reflexivity axiom for $G_{a,b,c,d}^{n}: A \rightarrow \Diamond A$
2. Transitivity axiom for $G_{a,b,c,d}^{n}: \Diamond A \rightarrow \Diamond A$
3. Restricted Persistency axiom for $G_{a,b,c,d}^{n}: \Diamond \Diamond A \rightarrow \Diamond \Diamond A$.

Incestual interaction axioms generated in $SDMnd(C)$ with $a = c = \epsilon$ for $G_{a,b,c,d}^{n}$

1. General Persistency axiom for $G_{a,b,c,d}^{n}: \Diamond A \rightarrow \Diamond A$
2. Composition axiom for $G_{a,b,c,d}^{n}: \Diamond \Diamond A \rightarrow \Diamond A$.

Incestual interaction axioms generated in $Ent(C)$ for $G_{a,b,c,d}^{n}$

1. Seriality axiom for $G_{a,b,c,d}^{n}: \Box A \rightarrow \Diamond A$
2. Composition axiom for $G_{a,b,c,d}^{n}: \Diamond \Box A \rightarrow \Diamond \Diamond A$
3. Axiom for $G_{a,b,c,d}^{n}: \Diamond \Box A \rightarrow \Diamond \Box A$.

Incestual interaction axioms generated in $SEnt(C)$ not in the form $G_{a,b,c,d}^{n}$

1. McKinsey axiom: $\Box \Diamond A \rightarrow \Diamond \Box A$
2. $\Box A \rightarrow \Diamond \Box A$.

$SEnt_2(C)$ contains both the images of $Mnd(Cmd(C))$ and $Cmd(Mnd(C))$ through the functor of quintets $Q$ as well as the images of $Mnd(Mnd(C))$ and $Cmd(Cmd(C))$. 

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