Work-conjugacy of stress and strain tensors and its role in 3D hyperelastic isotropic material formulation

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In accordance with the framework of hyperelastic isotropic material description it appears possible to replace work-conjugate stress and strain quantities by alternative pairs of work-conjugate stress and strain quantities while other things left unaltered, i.e. based on a retained architecture of the strain energy density function. This replacement results in formally representing different kinds of material response behavior. This may be recognized from the various comparable forms of resulting uniaxial force versus axial stretch behavior in the representative 3D principal axes system. Clarifying and achieving comprehensive understanding of these observations is the main purpose of this paper.

1 Introduction

Local length measurement in a continuum solid body is fully enabled by use of the (right or left) Cauchy–Green tensors representing metric quantities based on the generalized Pythagoras (or Cosine) theorem. This permits the introduction of auxiliary strain tensors based on scalar power mappings (or alternative exponential mappings, etc.) in the principal axes system. The invariance property of the internal (virtual) work expression directly leads to establishing stress tensors, work-conjugate to these aforementioned strain tensors (section 2). The related principal invariants of any of these strain tensors may serve as distinguished arguments of the strain energy density function, characterizing hyperelastic isotropic material (section 3). The final example is based on homogeneous affine deformations due to uniaxial uniformly distributed end loading, with a strain energy density-function based on the specific Saint Venant Kirchhoff (SVK)-formula architecture (section 4).

2 Work-conjugacy of stress and strain tensors

The global deformation of a solid body is given by the topological mapping of the coordinates of the undeformed into those of the deformed configuration. The deformation gradient \( F \) represents the locally affine deformation around a material point. The \( F \)-column vectors are called grid vectors \( F_j \) which represent the tangent vectors to the deformed coordinate grid. Then grid vector cross products form the column vectors \( F_j \) of the area-related deformation gradient \( F^* \), see Eq. (1). Sixteen alternative stress quantities may be established by purely geometrical considerations, i.e. transformations of the differential force vector \( df \) and differential area vector \( da \), based on four distinct configurations: the undeformed one, two intermediate ones due to right or left polar decomposition of the deformation gradient and the deformed one \([1,2]\), designated by subscripts \( i, j \) \((= 0, 1, 2, -)\) put within squares. The virtual work per unit deformed volume is given by the product of Cauchy stress and virtual displacement gradient \( \delta \text{w} \) with respect to the deformed configuration. Replacing Cauchy stress by one of the alternative configuration-based transformed quantities and keeping the (virtual) work expression invariant directly leads to related work-conjugate virtual displacement gradients, see Eq. (4). It turns out that some of these transformed virtual displacement gradients may actually be interpreted as virtual strain quantities, i.e. for which associated total strain quantities are connected to related work-conjugate virtual displacement gradients, see Eq. (4).

\[
\begin{align*}
F_j &= \frac{\partial x}{\partial x_j}; & (F^*)_K &= F_I \times F_j; & C_{IJ} &= F_I \cdot F_J \quad \text{(1)} \\
C &= \sum_{\alpha} \lambda_\alpha^2 N_\alpha \otimes N_\alpha; & E_{(k)} &= \sum_{\alpha} E_{(k)\alpha} N_\alpha \otimes N_\alpha \quad \text{with} \quad E_{(k)\alpha} = \frac{1}{k} (\lambda_\alpha^k - 1) \quad \text{(2)} \\
df = \sigma \, da; & \quad df = \sigma \, da \quad \text{with} \quad \sigma = F^{-1} \sigma F^-1 \quad \text{(3)} \\
\delta W_{\text{int}} &= \int \sigma : \delta FF^{-1} \, dv = \int \delta \sigma : \delta_{\text{e}} \, dt \quad \text{with} \quad \delta_{\text{e}} = F^T \text{sym} (\delta FF^{-1}) \, F \quad \text{(4)}
\end{align*}
\]

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3 Isotropic hyperelasticity

The elastic potential function per unit mass \(\psi_m\) or per unit undeformed volume \(\psi = \psi_m/\rho_0\) serves as strain energy density function which in the case of material isotropy depends on the scalar principal stretches (their squares or generalized functions thereof) or the related strain invariants only. See Eq. (5) with \(\alpha = (1, 2, 3)\), summation convention does not apply. Material isotropy effects coaxiality of work-conjugate pairs of stress and strain. Constitutive calculations are reduced to scalar ones related to the principal axes-related quantities. The variation of the strain energy density function with respect to the (scalar) principal strains yields the (scalar) principal stresses, see Eq. (6). These principal-axes-related scalar quantities are upscale to full tensor quantities in a standard way, see Eq. (7).

\[
\begin{align*}
\psi_m &= \psi \left( E_{(k)} \right) = \psi \left( E_{(k)\alpha} \right) = \psi \left( E_{(k)1}, E_{(k)2}, E_{(k)3} \right) \\
\delta \psi &= \sum_\alpha S_{(k)\alpha} \frac{\partial E_{(k)\alpha}}{\partial \varepsilon_{\alpha}} \delta \varepsilon_{\alpha} = \sum_\alpha S_{(k)\alpha} \delta \varepsilon_{\alpha} \quad \text{with} \quad S_{(k)\alpha} = \frac{\partial \psi}{\partial E_{(k)\alpha}} \quad \text{and} \quad S_{(k)\alpha} = \frac{\partial E_{(k)\alpha}}{\partial E_{(k)\alpha}} \\
S_{(k)} &= \sum_\alpha S_{(k)\alpha} N_\alpha \otimes N_\alpha; \quad S = \sum_\alpha S_{(k)\alpha} N_\alpha \otimes N_\alpha
\end{align*}
\]

4 Example: Homogeneous deformation of a solid cube

A solid cube is uni-axially loaded by end forces \(N_1\), i.e. the resultants of uniformly distributed normal stresses \(\sigma_{11}\) over the end faces. The lateral faces are stress-free (Fig. 1). The strain energy density function \(\psi_{(k)}\) is based on the classical SVK-architecture with two (Lamé) material constants, depending on the invariants \(I_{E_{(k)}}\) and \(II_{E_{(k)}}\) or the principal strains \(E_{(k)\alpha}\) of the generalized strain \(E_{(k)}\) in a specific way, with \((k) = 2, 1, 2/3, 0\), Eq. (8). The coaxial work-conjugate stresses \(S_{(k)\alpha}\) result as linear functions of the \(E_{(k)\alpha}\), Eq. (9). Nondimensional load-stretch curves are shown in Fig. 2. Termination conditions are indicated by symbols (red in tension, blue in compression), crosses denoting zero-volume condition, circles denoting states of mechanical instability (snap-through). Varying the exponent \((k)\) causes wide variation of the load-stretch curves.

\[
\begin{align*}
\psi_{(k)} &= \mu II_{E_{(k)}} + \lambda \frac{1}{2} \left( I_{E_{(k)}} \right)^2 \\
&= \mu E_{(k)} : E_{(k)} + \lambda \frac{1}{2} \text{tr} \left( E_{(k)} \right)^2 \\
&= \mu \sum_\alpha \left( E_{(k)\alpha} \right)^2 + \lambda \frac{1}{2} \left( \sum_\alpha E_{(k)\alpha} \right)^2
\end{align*}
\]

\[
S_{(k)\alpha} = 2\mu E_{(k)\alpha} + \lambda \sum_\alpha E_{(k)\alpha}
\]

\[
S_{(k)} = 2\mu E_{(k)} + \lambda \text{tr} \left( E_{(k)} \right) 1
\]

5 Conclusion

The unique relationship of any (generalized) stress tensor, work-conjugate to a generalized strain tensor (e.g. based on the \((k)\)-exponent concept, Eq. (2)), to a physical stress tensor \(\sigma\) (Cauchy; or Piola-Kirchhoff \(P\) or \(S\)) may be established by equating the related (virtual) work expressions. In isotropic hyperelasticity these pairs of quantities appear only with their scalar principal values in constitutive calculations, i.e. due to their co-axiality. Strictly distinguishing the architecture of the \(\psi\)-expression and the type of introduced generalized strain quantity offers a promising possibility for comparing and cataloging the multitude of existing \(\psi\)-functions and representing them under a common roof in the future.

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