COMPLEXITY AND INTEGRABILITY IN 4D BI-RATIONAL MAPS WITH TWO INVARIANTS
Giorgio Gubbiotti, Nalini Joshi, Thi Tran, Claude-Michel Viallet

To cite this version:
Giorgio Gubbiotti, Nalini Joshi, Thi Tran, Claude-Michel Viallet. COMPLEXITY AND INTEGRABILITY IN 4D BI-RATIONAL MAPS WITH TWO INVARIANTS. Asymptotic, Algebraic and Geometric Aspects of Integrable Systems, SANYA TSIMF, 2018, Sanya, China. hal-02445007

HAL Id: hal-02445007
https://hal.archives-ouvertes.fr/hal-02445007
Submitted on 19 Jan 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
COMPLEXITY AND INTEGRABILITY IN 4D BI-RATIONAL MAPS WITH TWO INVARIANTS

GIORGIO GUBBIOTTI, NALINI JOSHI, DINH THI TRAN, AND CLAUDE-MICHEL VIALLET

Abstract. In this letter we give fourth-order autonomous recurrence relations with two invariants, whose degree growth is cubic or exponential. These examples contradict the common belief that maps with sufficiently many invariants can have at most quadratic growth. Cubic growth may reflect the existence of non-elliptic fibrations of invariants, whereas we conjecture that the exponentially growing cases lack the necessary conditions for the applicability of the discrete Liouville theorem.

1. Introduction

Bi-rational maps in two dimensions have played a crucial role in the study of integrable discrete dynamical systems since the seminal paper of [29] and the introduction of the QRT mappings in [33,34]. Elliptic curves and rational elliptic surfaces proved to be one of the main tools in understanding the geometry behind this kind of integrability, see [11,37,39]. In this letter we give examples of higher-order maps whose properties go beyond those of the two-dimensional maps, and show that the geometry of elliptic fibrations is no longer sufficient to explain their behaviour.

Up to now the QRT mappings appear to describe almost totality of the known integrable examples in dimension two. With some notable exceptions [11,43], no general framework exists for higher order maps. A generalization of the QRT scheme [33,34] in dimension four was given in [6]. Certain maps obtained in [6] were shown in [17] to be autonomous reductions of members of $\varphi$-Painlevé hierarchies (multiplicative equations in Sakai’s scheme [37]). Since hierarchies are known also for the additive discrete Painlevé equations [9], it is clear that the cases considered in [6] cannot exhaust all the possible integrable autonomous maps in four dimensions, as already shown in [18]. We mention that other examples of discrete mappings of higher orders were produced either by periodic or symmetry reduction of integrable partial difference equations [23,28,32,40] or as Kahan-Hirota-Kimura discretization [19,20] of continuous integrable systems [7,8,30,31].

In this letter, we focus on the study of integrability properties of autonomous recurrence relations. Here a autonomous recurrence relation is given by bi-rational map of the complex projective space into itself:

$$\varphi: \mathbb{C}P^n \to \mathbb{C}P^n,$$

Date: August 16, 2018.

2010 Mathematics Subject Classification. 37F10; 14J30.
where \( n > 1 \). We take \( [x] = [x_1 : x_2 : \cdots : x_{n+1}] \) and \( [x'] = [x_1' : x_2' : \cdots : x_{n'+1}'] \) to be homogeneous coordinates on \( \mathbb{CP}^n \). In this language non-autonomous recurrence relations of order \( n \) are represented by bi-rational maps \( \Phi: \mathbb{CP}^{n+1} \to \mathbb{CP}^{n+1} \), where one of the variables \( x_k \), for \( k \in \{1, \ldots, n + 1\} \) fixed, in affine coordinates has a linear evolution. Moreover we recall that a bi-rational map is a rational map \( \phi: V \to W \) of algebraic varieties \( V \) and \( W \) such that there exists a map \( \psi: W \to V \), which is the inverse of \( \phi \) in the dense subset where both maps are defined [35].

Integrability for autonomous recurrence relations (discrete equations) can be characterized in different ways. In the continuous case, for finite dimensional systems, integrability is usually understood as the existence of a “sufficiently” high number of first integrals, i.e. of non-trivial functions constant along the solution of the differential system. In the Hamiltonian setting a characterization of integrability was given by Liouville [24]. In the case of maps (1) the analogue of first integrals are the invariants. To be more precise we state the following:

**Definition 1.** An invariant of a bi-rational map \( \phi: \mathbb{CP}^n \to \mathbb{CP}^n \) is a homogeneous function \( I: \mathbb{CP}^n \to \mathbb{C} \) such that it is left unaltered by action of the map, i.e.

\[
\phi^* (I) = I,
\]

where \( \phi^* (I) \) means the pullback of \( I \) through the map \( \phi \), i.e. \( \phi^* (I) = I (\phi ([x])) \).

For \( n > 1 \), an invariant is said to be non-degenerate if:

\[
\frac{\partial I}{\partial x_1} \frac{\partial I}{\partial x_n} \neq 0.
\]

Otherwise an invariant is said to be degenerate.

In what follows we will concentrate on a particular class of invariants:

**Definition 2.** An invariant \( I \) is said to be polynomial, if in the affine chart \([x_1 : \cdots : x_n : 1]\) the function \( I \) is a polynomial function.

A polynomial invariant in the sense of definition 2 written in homogeneous variables is always a rational function homogeneous of degree 0. The form of the polynomial invariant in homogeneous coordinates is then given by:

\[
I ([x]) = \frac{I' ([|x|])}{t^d}, \quad d = \deg I' ([|x|]),
\]

where \( \deg \) is the total degree.

To better characterize the properties of these invariants we introduce the following:

**Definition 3.** Given a polynomial function \( F: \mathbb{CP}^n \to V \), where \( V \) can be either \( \mathbb{CP}^n \) or \( \mathbb{C} \), we define the degree pattern of \( F \) to be:

\[
dp F = (\deg x_1, \deg x_2, F, \ldots, \deg x_n, F).
\]

**Example 1.** Consider the following map in \( \mathbb{CP}^2 \):

\[
\varphi: [x : y : t] \mapsto [-y(x^2 - t^2) + 2axt^2 : x(x^2 - t^2) : t(x^2 - t^2)]
\]

This map is known as the McMillan map [24] and possesses the following invariant:

\[
t^4 I_{McM} = x^2y^2 + (x^2 + y^2 - 2axy)t^2
\]

\(^1\)Bi-rational maps in \( \mathbb{CP}^1 \) are just Möbius transformations so everything is trivial.
We have $d\rho I_{\text{McM}} = (2, 2)$, i.e. it is a bi-quadratic polynomial. We also note that the invariant of a QRT map $I_{\text{QRT}}$, which is a generalization of the McMillan map $I$, is the ratio of two bi-quadratic in the dynamical variables of $\mathbb{C}P^2$. Hence QRT mappings leave invariant a pencil of curves of degree pattern $(2, 2)$.

**Example 2.** The invariants of the maps presented in [6], $I_{\text{CS}}$, are are ratios of bi-quadratic in all the four dynamical variables of $\mathbb{C}P^4$, i.e. ratios of polynomial of degree pattern $(2, 2, 2, 2)$. In this sense the classification of [6] is an extension of the one in [33, 34].

Finally we will consider invariants are not of the most general kind, but satisfy the following condition

**Definition 4.** We say that a invariant $I: \mathbb{C}P^n \to \mathbb{C}$ is symmetric if it is left unaltered by the following involution:

$$\iota: [x_1 : x_2 : \cdots : x_n : x_{n+1}] \to [x_n : x_{n-1} : \cdots : x_1 : x_{n+1}],$$

i.e. $\iota^* (I) = I$.

We then have the following characterization of integrability for autonomous recurrence relations:

(i) **Existence of invariants** A $n$-dimensional map is (super)integrable if there exists $n-1$ invariants.

(ii) **Liouville integrability** [4, 25] A $n$-dimensional map (in affine coordinates) is integrable if it preserves a Poisson structure of rank $2r$ and $n - 2r = n - r$ functionally independent invariants in involution with respect to this Poisson structure. In affine coordinates $w = (w_{n-1}, \ldots, w_0) = [w_{n-1} : \cdots : w_0 : 1]$ we say that a map $\varphi: w \mapsto w'$ is called a Poisson map of rank $2r \leq n$ if there is a skew-symmetric matrix $J(w)$ of rank $2r$ satisfies the Jacobian identity

$$\sum_{i=1}^{n} \left( J_{ij} \frac{\partial J_{jk}}{\partial w_{l-1}} + J_{ij} \frac{\partial J_{kl}}{\partial w_{l-1}} + J_{jk} \frac{\partial J_{il}}{\partial w_{l-1}} \right) = 0, \quad \forall i, j, k. \tag{9}$$

and

$$d\varphi J(w) d\varphi^T = J(w'),$$

where $d\varphi$ is the Jacobian matrix of the map $\varphi$, see [6, 27]. The Poisson bracket of two smooth functions $f$ and $g$ is defined as

$$\{f, g\} = \nabla f \cdot J (\nabla g)^T, \tag{11}$$

where $\nabla f$ is the gradient of $f$. We can easily see that $\{w_{l-1}, w_{j-1}\} = J_{ij}$. We note that in the case where the Poisson structure has full rank, i.e. $n = 2r$, we only need $n/2$ invariants which are in involution. In this case the Poisson matrix is invertible, and its inverse is called a symplectic matrix. A symplectic matrix give rise to a symplectic structure.

(iii) **Existence of a Lax pair** [22] A $n$-dimensional map is integrable if it arises as compatibility condition of an overdetermined linear system.

(iv) **Low growth condition** [2, 12, 42] A $n$-dimensional bi-rational map is integrable if the degree of growth of the iterated map $\varphi^k$ is polynomial with respect
to the initial conditions \([x_0]\). Integrability is then equivalent to the vanishing of the algebraic entropy:

\[
\varepsilon = \lim_{k \to \infty} \frac{1}{k} \log \deg_{[x_0]} \varphi^k.
\]

Algebraic entropy is a measure of the complexity of a map, analogous to the one introduced by Arnol’d [1] for diffeomorphisms. In this sense growth is given by computing the number of intersections of the successive images of a straight line with a generic hyperplane in complex projective space [12].

We underline that the above list is not meant to be completely exhaustive of all the possible definitions of integrability. Since we are focused on autonomous recurrence relations we chose to cover only the most used definition for these ones. Additional definitions of integrability have been proposed by other for non-autonomous systems.

**Remark 1.** In principle, the definition of algebraic entropy in equation (12) requires us to compute all the iterates of a bi-rational map \(\varphi\) to obtain the sequence \(\{d_k = \deg_{[x_0]} \varphi^k\}_{k=0}^{\infty}\). Fortunately, for the majority of applications the form of the sequence can be inferred by using generating functions [21]:

\[
g(z) = \sum_{n=0}^{\infty} d_k z^k.
\]

A generating function is a predictive tool which can be used to test the successive members of a finite sequence. It follows that the algebraic entropy is given by the logarithm of the smallest pole of the generating function, see [14, 15].

**Remark 2.** The condition of Liouville integrability [4, 25, 41] is stronger than the existence of integrals. Indeed, for a map, being measure preserving and preserving a Poisson/symplectic structure are very strong conditions. However, they lead to a great drop in the number of invariants needed for integrability. The same can be said for the existence of a Lax pair, since it is well known that a well posed Lax pair gives all the invariants of the system through the spectral relations. Finally, the low growth condition means that the complexity of the map is very low, and it is known that invariants help in reducing the complexity of a map. Indeed the growth of a map possessing invariants cannot be generic since the motion is constrained to take place on the intersection of hypersurfaces defined by the integrals. For maps in \(\mathbb{CP}^2\), it was proved in [10] that the growth can be only bounded, linear, quadratic or exponential. Linear cases are trivially integrable in the sense of invariants. We note that for polynomial maps, it was already known from [12] that the growth can be only linear or exponential. It is known that QRT mappings and other maps with invariants in \(\mathbb{CP}^2\) possess quadratic growth [11], so the two notions are actually equivalent for large class of integrable systems.

Now we discuss briefly the concept of duality for rational maps, which was introduced in [35]. Let us assume that our map \(\varphi\) possesses \(L\) independent invariants, i.e. \(I_j\) for \(j \in \{1, \ldots, L\}\). Then we can form the linear combination:

\[
H = \alpha_1 I_1 + \cdots + \alpha_L I_L.
\]

For an unspecified autonomous recurrence relation

\[
[x_1 : x_2 : \cdots : x_{n+1}] \mapsto [x'_1 : x_1 : \cdots : x_{n}]
\]
we can write down the invariant condition for $H$ (14):

$\hat{H}(x'_1, [x]) = H([x']) - H([x]) = 0.$

Since we know that $[x'] = \varphi([x])$ is a solution of (15) we have the following factorization:

$\hat{H}(x'_1, [x]) = A(x'_1, [x]) B(x'_1, [x]).$

We can assume without loss of generality that the map $\varphi$ corresponds to the annihilation of $A$ in (17). Now since $\deg_x \hat{H} = \deg_x H$ and $\deg_x \hat{H} = \deg_x H$ we have that if $\deg_x H, \deg_x H > 1$ the factor $B$ in (17) is non-constant. In general, since the map $\varphi$ is bi-rational, we have the following equalities:

$\deg B_{x'_1} = \deg_x \hat{H} - \deg_x A = \deg_x H - 1,$

$\deg B_{x_n} = \deg_x \hat{H} - \deg_x A = \deg_x H - 1.$

Therefore we have that if $\deg_x H, \deg_x H > 2$, the annihilation of $B$ does not define a bi-rational map in general, but an algebraic one. However when $\deg_x H, \deg_x H = 2$ the annihilation of $B$ defines a bi-rational projective map. We call this map the dual map and we denote it by $\varphi \vee$.

Remark 3. We note that in principle for $\deg_x H = \deg_x H = d > 2$, more general factorizations can be considered:

$\hat{H}(x'_1, [x]) = \prod_{i=1}^{d} A_i(x'_1, [x]),$

but we will not consider this case here.

Now assume that the invariants (and hence the map $\varphi$) depends on some arbitrary constants $I_i = I_i([x]; a_i)$, for $i = 1, \ldots, M$. Choosing some of the $a_i$ in such a way that there remains $M$ arbitrary constants and such that for a subset $a_{i_k}$ we can write equation (14) in the following way:

$H = a_{i_1} J_1 + a_{i_2} J_2 + \cdots + a_{i_K} J_{i_{i_K}},$

where $J_i = J_i([x])$, $i = 1, 2, \ldots, K$ are new functions. Then using the factorization (17) we have that the $J_i$ functions are invariants for the dual maps.

Remark 4. It is clear from equation (20) that even though the dual map is naturally equipped with some integrals, it is not necessarily equipped with a sufficient number of integrals to claim integrability. In fact there exist examples of dual maps with any possible behaviour, integrable, superintegrable and non-integrable [16,18].

In a recent paper [18], the authors considered the autonomous limit of the second member of the $dP_1$ and $dP_{11}$ hierarchies [9]. We will denote these equations as $dP_1^{(2)}$ and $dP_{11}^{(2)}$ equations. These $dP_1^{(2)}$ and $dP_{11}^{(2)}$ equations are given by autonomous recurrence relations of order four, and showed to be integrable according to the algebraic entropy approach. They showed that both maps possess two invariants, one of degree pattern $1, 3, 3, 1$ and one of degree pattern $2, 4, 4, 2$. Using these invariants, they showed that the dual maps of the $dP_1^{(2)}$ and $dP_{11}^{(2)}$ equations are integrable according to the algebraic entropy test and moreover, produced some

---

2We remark that this assertion is possible because we are assuming that all the invariants are non-degenerate. It is easy to see that degenerate invariants can violate this property.
integrals, showing that these dual maps were actually superintegrable. Finally they gave a scheme to construct autonomous recurrence relations with the assigned degree pattern (1, 3, 3, 1) associated with \( I_{\text{low}} \) and (2, 4, 4, 2) associated with \( I_{\text{high}} \) and they provided some new examples out of this construction.

In a forthcoming paper \cite{16} we consider the problem of finding all fourth order bi-rational maps \( \varphi: [x: y: z: u: t] \mapsto [x': y': z': u': t'] \) possessing a polynomial a symmetric invariant \( I_{\text{low}} \) such that \( dp_{\text{low}} = (1, 3, 3, 1) \) where the only non-zero coefficients are those appearing in the (1, 3, 3, 1) integral of both the \( dP_{I}^{(2)} \) and \( dP_{II}^{(2)} \) equation, and such that \( \varphi \) possesses a polynomial symmetric invariant \( I_{\text{high}} \) such that \( dp_{\text{high}} = (2, 4, 4, 2) \). The two invariants \( I_{\text{low}} \) and \( I_{\text{high}} \) are assumed to be functionally independent and non-degenerate. Within this class we have found the known \( dP_{I}^{(2)} \) and \( dP_{II}^{(2)} \) equations as well as new examples of maps with these properties.

In this letter we will present in detail four particular examples of this class. In Section 2 we will discuss two pairs of main-dual maps. We will discuss the integrability property of these maps in light of their invariants and of their growth. We will present maps possessing two invariants and integrable according to the algebraic entropy test with cubic growth. This implies that another rational invariant cannot exist. Indeed, the orbits of superintegrable maps with rational invariant are confined to elliptic curves and the growth is at most quadratic \cite{3, 13}. From this general statement follows that a four-dimensional map with cubic growth can possess at most two rational first integrals. We note that some examples of cubic growth were already presented in \cite{18}. However, it was pointed out that these examples can be deflated to lower dimensional maps with quadratic growth. This also holds for our maps, i.e. we can deflate them to integrable maps in lower dimension. Furthermore, we will present a map with two invariants and exponential growth, that is non-integrable according to the algebraic entropy test. We discuss some possible reasons why this map is non-integrable even though it possesses two invariants. In the final Section, we will give some conclusions and an outlook on the future perspectives of this approach.

2. Notable examples

In this section we discuss two pairs of maps, which arise as part of a systematic classification to be presented in \cite{16}. The interest in these particular maps arises since the relation between the invariants and the growth properties is non-trivial. In both cases the main maps possess two functionally independent invariants, but one has cubic degree growth, and the other one has exponential degree growth. In both cases, the degree growth property of the dual maps reflect the growth of the main map. However, we note that not always the degree growth of the dual map reflects the main map one \cite{15}.

2.1. (P.i) and its dual map (Q.i). Consider the map \([x] \mapsto \varphi_{i}([x]) = [x']\) given as follows:

\[
\begin{align*}
x' &= -\{ut^{2}(x+z) + uz^{2}\}y + t^{2}\mu uz + (x+z)^{2}y^{2}\}d - at^{4}, \\
y' &= x^{2}d(t^{2}\mu + xy), \quad z' = yxd(t^{2}\mu + xy), \\
u' &= zxd(t^{2}\mu + xy), \quad t' = txd(t^{2}\mu + xy).
\end{align*}
\]

This map depends on four parameters \(a, d\) and \(\mu, \nu\).
From the construction in [16] we know that the map (P.i) possesses the following invariants:

\[
\begin{align*}
I_{\text{low}}^{P.i} &= at^4yz + d \left[ \nu y^2z^2 - yz(ux - uz - xy)\mu \right] t^2 \\
&- y^2z^2d(ux - xy - yz - uz), \quad (21a) \\
\end{align*}
\]

\[
\begin{align*}
I_{\text{high}}^{P.i} &= \left( [uz + xy - yz]\mu - \nu yz \right) at^6 \\
&+ \left[ yz(xy + yz + uz)\alpha + d\mu^2(uz + xy - yz)^2 \\
&+ 2d\mu\nu yz(ux - yz) - d\nu^2y^2z^2 \right] t^4 \\
&+ \left[ 2dzyz(uz + xy - yz)(xy + yz + uz)\mu + 2d\nu y^2z^2\nuux \right] t^2 \\
&+ d\nu^2y^2z^2(xy + yz + uz)^2. \quad (21b)
\end{align*}
\]

Moreover, the map (P.i) has the following degrees of iterates:

\[
\{ d_n \}_{n=1}^{\infty} = 1, 4, 12, 28, 86, 130, 188, 260, 348, 452, 576, 720, 886, 1074, 1288, 1528, 1796, 2092, \ldots
\]

The generating function of the sequence (22) is given by:

\[
g_{P.i}(s) = \frac{s^7 - 3s^6 + s^5 - s^4 + 3s^3 + 3s^2 + s + 1}{(s + 1)(s^2 + 1)(s - 1)^4}.
\]

Due to the presence of \((s - 1)^4\) in the denominator we have that the growth of the map (P.i) is fitted by a cubic polynomial. As discussed in the Introduction this means at once that the map is integrable according to the algebraic entropy test and that another rational invariant cannot exist. This simple observation on the degree of growth also shows that the geometry of the orbits of the map (P.i) is nontrivial, and goes beyond the existence of elliptic fibrations.

The dual map \([x] \mapsto \varphi_i^\vee ([x]) = [x']\) of (P.i) is given by:

\[
\begin{align*}
x' &= \beta(2xy - 2yz + uz)\mu + (\beta\nu - \alpha)y(x - z) \mu t^2 \\
&+ \beta y(z^2y - x^2y + uz^2) \\
y' &= x^2\beta(t^2\mu + xy), \quad z' = yx\beta(t^2\mu + xy), \\
u' &= zx\beta(t^2\mu + xy), \quad t' = tx\beta(t^2\mu + xy). \\
\end{align*}
\]

This map depends on three parameters \(\alpha, \beta\), and \(\mu, \nu\). The parameters \(\mu\) and \(\nu\) are shared with the main map (P.i).

The main map (P.i) possesses two integrals and depends on \(a\) and \(d\) whereas the dual map (Q.i) do not depend on them. Then according to (20) we can write down the invariants for the dual map (Q.i) as:

\[
\alpha I_{\text{low}}^{Q.i} + \beta I_{\text{high}}^{Q.i} = a I_{\text{low}}^{Q.i} + d I_{\text{high}}^{Q.i},
\]

Therefore, we obtain the following expressions:

\[
\begin{align*}
t^4 I_{\text{low}}^{Q.i} &= \left[ yz\alpha + (\mu xy - yz\mu - y\nu z + \mu uz)\beta \right] t^2 \\
&+ \beta yz(xy + yz + uz), \quad (25a)
\end{align*}
\]
We remark that the invariant (25a) has degree pattern (1, 2, 2, 1) which differs from \( dp^{P,i} \). The map (Q.i) has the following degrees of iterates:

\[
\{d_{n}\}_{Q,i} = 1, 4, 12, 26, 48, 78, 118, 170, 234, 312, 406, 516, 644, 792 \ldots
\]

with generating function:

\[
s_{Q,i}(s) = \frac{(s^3 - 2s^2 - 1)(s^3 - s^2 - s - 1)}{(s^2 + s + 1)(s - 1)^4}.
\]

This means that the dual map is integrable according to the algebraic entropy test with cubic growth, just like the main map.

Therefore, the pair of main-dual maps (P,i) and (Q,i) consists of two integrable equations with non-standard degree of growth. However, as remarked above the degree pattern of the invariants of the maps (P,i) and (Q,i) differ slightly.

We now consider the maps (P,i) and (Q,i) in affine coordinates which are given by

\[\varphi : (w_3, w_2, w_1, w_0) \mapsto (w_4, w_3, w_2, w_1),\]

where

\[(AP,i)\]

\[w_4 = \frac{N_1}{d w_4 (w_2 w_3 + \mu)},\]

\[(AD,i)\]

\[w_4 = \frac{N_2}{\beta w_3 (w_2 w_3 + \mu)},\]

with

\[(29)\]

\[N_1 = -d w_0 w_1^2 w_2 + w_1^2 w_2^2 + 2 w_1 w_2^2 w_3 + w_2^2 w_3^2 + \mu w_0 w_1 + \nu w_1 w_2 + \nu w_2 w_3 - \alpha,\]

\[(30)\]

\[N_2 = \beta w_0 w_1^2 w_2 + \beta w_0 w_1 + \beta w_1^2 w_2^2 + (\alpha - 2\beta \mu - \beta \nu) w_1 w_2 - \beta w_2^2 w_3^2 + (2\beta \mu + \beta \nu - \alpha) w_2 w_3.\]

Invariants for these maps are obtained from \( I_{\text{low}} \) and \( I_{\text{high}} \) respectively by taking \( t = 1, u = w_0, z = w_1, y = w_2, \) and \( x = w_3 \).

We note that when a Poisson structure has the full rank, using equation (10), one gets

\[
\big(\text{det}(d \varphi)\big)^2 = \frac{\text{det} (J(\varphi))}{\text{det} (J(\varphi))}.
\]

This implies that the map \( \varphi \) is either volume or anti-volume preserving.

We recall that a map \( \varphi \) is called (anti)volume preserving if there is a function \( \Omega(w) \) such that the following volume form is preserved

\[
\Omega(w) \ d w_0 \wedge d w_1 \wedge \ldots \wedge d w_{n-1} = \pm \Omega(w') \ d w'_0 \wedge d w'_1 \wedge \ldots \wedge d w'_{n-1}.
\]
Thus, we can write
\[ \frac{\partial (w'_0, w'_1, \ldots, w'_{n-1})}{\partial (w_0, w_1, \ldots, w_{n-1})} = \pm \frac{\Omega(w)}{\Omega(w')} \]
where the left hand side is the determinant of the Jacobian matrix of the map \( \varphi \).

In [5] it was proved that if a map in \( n \)-dimension is (anti) volume preserving and possesses \( n-2 \) invariant, then we can construct an (anti) Poisson structure of rank 2 from these invariants. However, these invariants turn out to be Casimirs (functions that Poisson commute with all other functions) of this Poisson bracket. Therefore, in order to have Liouville integrability we need an extra invariant apart from the known \( n-2 \) invariants if we want to use use Poisson structures constructed this way. In other words, the map is super integrable. Thus, to discuss about Liouville integrability of the maps (AP.i) and (AQ.i) we need to find either the third invariant or a Poisson bracket of rank 4. We do not have that information for these maps but we can show they reduce to three dimensional Liouville integrable maps via a process called inflation [18]. This process will preserve the integrals, and in dimension three, two integrals are sufficient to claim integrability in the general sense as discussed in the Introduction.

It is easy to check that the maps (AP.i) and (AQ.i) are volume and anti-volume preserving, respectively, with respect to the same volume form:
\[ \Omega = w_1 w_2 (w_1 w_2 + \mu). \]

We now construct the (anti) Poisson structures for these two maps following [5]. We consider the dual multi-vector of the volume form
\[ \tau = m \frac{\partial}{\partial w_0} \wedge \frac{\partial}{\partial w_1} \wedge \frac{\partial}{\partial w_2} \wedge \frac{\partial}{\partial w_3}, \]
where \( m = 1/\Omega \). A degenerate Poisson structure for the map (AP.i) and a degenerate anti-Poisson structure for the map (AQ.i) are given by the following contraction
\[ J = \tau \mid dI_{\text{low}} \mid dI_{\text{high}}, \]
where \( I_{\text{low}} \) and \( I_{\text{high}} \) are invariants for these maps in affine coordinates. Since these (anti) Poisson structures are quite big, we do not present them here.

Remark 5. The Poisson structures which can be constructed using the method of [5] are degenerate and cannot be used to explain the integrability of the two maps (AP.i) and (AQ.i).

We also note that the maps (AP.i) and (AQ.i) can be reduced to three dimensional maps using a deflation \( v_i = w_i w_{i+1} \). The recurrences for these maps are denoted by (DP.i) and (DQ.i) and are given as follows
\begin{align*}
\text{(DP.i)} & \quad d\mu (v_0 + v_3) + dv (v_1 + v_2) + d (v_0 v_1 + v_1^2 + 2v_1 v_2 + v_2^2 + v_2 v_3) + a = 0, \\
\text{(DQ.i)} & \quad \beta \mu (-v_0 + 2 \beta v_1 - 2 \beta v_2 + v_3) + (\beta v - \alpha) (v_1 - v_2) \\
& \quad + \beta (-v_0 v_1 - v_1^2 + v_2^2 + v_2 v_3) = 0.
\end{align*}
Each of the maps (DP.i) and (DQ.i) has two functionally independent invariants which can be obtained directly from \( I_{\text{low}} \) and \( I_{\text{high}} \) even though they live in a different space. One can check that the map (DP.i) and (DQ.i) are anti-volume preserving and volume preserving with \( \Omega = v_1 + \mu \). Therefore, we can construct
their (anti) Poisson structure using the three dimensional version of (39). Using
the following invariant from $I_{\text{low}}$ for (DP,i)

$$(37) \quad I^{P,i}_{\text{low}} = d\mu v_0 v_1 - d\mu v_0 v_2 + d\mu v_1 v_2 + d\nu v_1^2 + d\nu v_0 v_1 v_2 + d\nu^2 v_1^2 + d\nu v_1 v_2 + \alpha v_1$$

we have found that the map (DP,i) has an anti-Poisson structure given by

$$J^{P,i}_{12} = d(v_1 - v_0), \quad J^{P,i}_{2,3} = d(v_1 - v_2)$$

$$J^{P,i}_{13} = \frac{d\mu v_0 - d\mu v_2 - 2d\nu v_1 - 2d\nu v_0 v_1 + d\nu^2 v_1^2 - 2d\nu v_2 - \alpha v_1}{\mu + v_1}.$$  

Similarly, for the map (DQ,i) we obtain the invariant

$$(38) \quad I^{PQ,i}_{\text{low}} = \beta \mu v_0 - \beta \mu v_1 + \beta \mu v_2 - \nu \beta v_1 + \beta v_0 v_1 + \beta v_1^2 + \beta v_1 v_2 + \alpha v_1,$$

and the corresponding Poisson structure

$$(39) \quad J^{Q,i} = \begin{bmatrix} 0 & \beta & \beta (\mu + \nu - v_0 - 2v_1 - v_2) - \alpha \\ -\beta & 0 & \frac{\mu + v_1}{\beta} \\ -\beta (\mu + \nu - v_0 - 2v_1 - v_2) - \alpha & -\beta & 0 \end{bmatrix}.$$  

For these constructions, $I^{P,i}_{\text{low}}$ and $I^{Q,i}_{\text{low}}$ are Casimirs for their associated (anti) Poisson structures. Their second (anti) Poisson structures can be obtained from the invariant $I_{\text{high}}$, but we do not present here as they are quite big.

It is important to note that the (anti) Poisson structures of (AP,i) and (AQ,i) under inflation give us the trivial Poisson structures for (DP,i) and (DQ,i), i.e. $J = 0$, where 0 is the zero matrix. On the other hand, from the common factor that appears in the Poisson structure of (AP,i), we have found that there exists an anti-invariant $K^{P,i}$ for this map, i.e. $K^{P,i}(w) = -K^{P,i}(w')$, where

$$(40) \quad K^{P,i} = 2d \left(w_2 w_4 w_0 + w_2^2 w_4 + w_1 w_3^2 w_1 + \mu w_0 w_1 - \mu w_1 w_2 + \mu w_2 w_3 + \nu w_1 w_2\right) + a.$$  

However, $K^{P,i}$ is functionally dependent with $I^{P,i}_{\text{low}}$ and $I^{P,i}_{\text{high}}$ through the relation

$$(41) \quad (K^{P,i})^2 - 4d I^{P,i}_{\text{high}} - 8d \nu I^{P,i}_{\text{low}} = a^2.$$  

Using this anti-invariant, we obtain the following anti-invariant for the map (DP,i)

$$(42) \quad K^{DP,i} = 2d \mu v_0 - 2d \mu v_1 + 2d \mu v_2 + 2d \nu v_1 + 2d v_0 v_1 + 2d v_1^2 + 2d v_1 v_2 + a.$$  

Therefore, using this anti-invariant, we get a Poisson structure for (DP,i) as follows (after factoring out a constant term)

$$(43) \quad J^{P,i}_{2} = \begin{bmatrix} 0 & 1 & \frac{\mu + \nu - v_0 - 2v_1 - v_2}{\mu + v_1} \\ -1 & 0 & \frac{\mu + v_1}{1} \\ -\frac{\mu + \nu - v_0 - 2v_1 - v_2}{\mu + v_1} & -1 & 0 \end{bmatrix}.$$  

We can check directly that the invariants inherited from the affine map (AP,i) are in involution with respect to the Poisson structure (43). In the sense of the definition given in the Introduction, this means that the reduced maps (DP,i) and (DQ,i) are Liouville integrable.
(44a) \[ t^5 \text{P}_{\text{low}} = (x-z)(u-y)(t^2 + z^2 \mu)(\mu y^2 + t^2), \]
(44b) \[ t^5 \text{P}_{\text{high}} = \left[(x-z)^2y^4 + y^2z^4 - 2yz^4u + u^2z^4\right]\mu^2 + 2t^2 \left[(x^2 - 2xz + 2z^2)y^2 - 2yz^2u + u^2z^2\right]\mu + t^4 \left(z^2 + u^2 + x^2 + y^2 - 2uy - 2xz\right). \]

Moreover, the map \( \text{P}_{\text{ii}} \) has the following degrees of iterates:

(45) \[ \{d_n \}_{\text{P}_{\text{ii}}} = 1, 3, 9, 21, 45, 93, 189, 381, 765, 1533 \ldots \]

with generating function:

(46) \[ \eta_{\text{P}_{\text{ii}}}(s) = \frac{1 + 2s^2}{(2s - 1)(s - 1)}. \]

This means that despite the existence of the two invariants (44) the map \( \text{P}_{\text{ii}} \) is non-integrable according to the algebraic entropy test: its entropy is positive and given by \( \epsilon = \log 2 \).

Therefore we have that the map \( \text{P}_{\text{ii}} \) is an example of non-integrable admitting two invariants.

Again following [3] we can produce a Poisson structure of rank 2 for \( \text{P}_{\text{ii}} \) as the affine version of \( \text{P}_{\text{ii}} \) is volume preserving with \( \Omega = (1 + \mu w_1^2)(1 + \mu w_2^2) \), where we have taken \( t = 1, u = w_0, z = w_1, y = w_2, \) and \( x = w_3 \). By the construction, the two invariants (44) become Casimir functions for it, so again the existence of such Poisson structure do not imply any form of Liouville integrability. However, we notice that there are common factors appear at every non-zero entries of this structure. Thus, we have found the following anti-invariant for the map \( \text{P}_{\text{ii}} \) using these common factors:

\[
K_{\text{P}_{\text{ii}}} = \left[ \mu(w_0w_1^2 - w_1w_2^2 - w_1w_2^2 + w_2^2w_3) + w_0 - w_1 - w_2 + w_3 \right] \times \\
\left[ \mu(w_0w_1^2 - w_1w_2^2 + w_1w_2^2 - w_2^2w_3) + w_0 + w_1 - w_2 - w_3 \right] \\
= F_1 F_2
\]

This suggests that we should check each factor of \( K_{\text{P}_{\text{ii}}} \) to see whether they are (anti) invariants of \( \text{P}_{\text{ii}} \). By direct calculation we can see that the first factor \( F_1 \)
Remark is non-integrable in the sense of the algebraic entropy.

Using the formula (36)

\[ I^{\text{P.ii}}_{\text{high}} - F_1^2 + 2I^{\text{P.ii}}_{\text{low}} = 0, \text{ and } I^{\text{P.ii}}_{\text{high}} - F_2^2 - 2I^{\text{P.ii}}_{\text{low}} = 0. \]

Therefore, the map (P.ii) actually has two invariants of degrees \((1, 2, 2, 1)\) and \((1, 3, 3, 1)\). Nevertheless, despite the existence of such invariants the map (P.ii) is non-integrable in the sense of the algebraic entropy.

Remark 7. We can use \(F_1\) and \(F_2\) to construct an anti-Poisson structure for (P.ii) using the formula (36)

\[ J_{1,2} = -1, \quad J_{2,3} = 1, \quad J_{3,4} = -1 \]

\[ J_{1,3} = \frac{2\mu w_1 (w_2 - w_0)}{\mu u_1^2 + 1}, \quad J_{2,4} = -\frac{2\mu w_2 (w_3 - w_1)}{\mu u_2^2 + 1} \]

\[ J_{1,4} = -\frac{\mu^2 w_1 w_2 [4 (w_0 w_1 - w_0 w_3 + w_2 w_3) - 3 w_1 w_2] + \mu (w_1^2 + w_2^2) + 1}{(\mu u_1^2 + 1)(\mu u_2^2 + 1)} \]

We have checked that \(F_2\) and \(I^{\text{P.ii}}_{\text{low}}\) are in involution with respect to this anti-Poisson structure. A Poisson structure can be obtained by multiplying this anti-Poisson structure with the anti-invariant \(F_1\).

The dual map \([x] \mapsto \varphi^*_n ([x]) = [x']\) of (P.ii) is given as follows:

\[ x' = \alpha \left[ (x^2 - z^2) y + uz^2 \right] \mu + t^2 \alpha u + \beta y^2 (x - z) \mu + t^2 \beta (x - z), \]

(Q.ii)

\[ y' = \alpha x (t^2 + \mu x^2), \quad z' = \alpha y (t^2 + \mu x^2), \]

\[ u' = \alpha z (t^2 + \mu x^2), \quad t' = \alpha t (t^2 + \mu x^2). \]

This map depends on three parameters \(\alpha, \beta\) and \(B\). The parameter \(\mu\) is shared with the main map (P.ii).

Since the main map (P.ii) possess two integrals depending only on one parameter \(\mu\) then according to (20) we can write down only a single first integral for the dual map (Q.ii):

\[ J^{\text{Q.ii}} = \alpha J^{\text{P.ii}}_{\text{high}} + \beta J^{\text{P.ii}}_{\text{low}}. \]

The first integral (50) has degree pattern \((2, 4, 4, 2)\).

We have then that the dual map (Q.ii) has the following fast-growing degrees of iterates:

\[ \{d_n\}^{\text{Q.ii}} = 1, 3, 9, 21, 45, 93, 189, 381, 765, 1533, 3069, \ldots \]

The growth (51) is clearly exponential and its generating function:

\[ g^{\text{Q.ii}}(s) = \frac{1 + 2s^2}{(2s - 1)(s - 1)}, \]

confirms this showing that the algebraic entropy is positive and equal to \(\varepsilon = \log 2\).

This means that the dual map is non-integrable with same rate of growth as the main map. In this case we can show that the map is anti-volume preserving with the same measure as the main map (P.ii). Moreover, we proved that the map (Q.ii) do not possesses any addition integral up to order 14. Therefore at the present stage we cannot construct a Poisson structure using the method of (5).
3. Conclusions and outlook

In this letter, we gave some examples of fourth order bi-rational maps with two invariants possessing interesting degree growth properties. These examples come from our forthcoming classification of all the fourth-order autonomous recurrence relations possessing two invariants in a given class of degree patterns [16].

The first pair of bi-rational maps is given by the map $P.i$ and its dual $Q.i$ and consists of integrable maps with cubic growth. The interest in maps with cubic growth arises from geometrical considerations: maps with polynomial but higher than quadratic growth, can arise only in dimension greater than two [10] and prove, in the case of superintegrable maps, the existence of non-elliptic fibrations of invariant varieties [2]. The interest in maps with this type of growth arose recently following the examples given in [18] and we expect them to lead to many new and interesting geometric structures.

The second pair of fourth order bi-rational maps given by the map $P.ii$ and its dual $Q.ii$, consists of non-integrable maps with exponential growth. There are various possible reasons why the map $P.ii$ is non-integrable despite possessing two invariants. To claim integrability with two invariants according to the discrete Liouville theorem [4,25,41] we need to prove that the map has a symplectic structure and that the two invariants commute with respect to this symplectic structure. Hence, either the map $P.ii$ does not admit any symplectic structure, or the map $P.ii$ admits only symplectic structures such that the two integrals do not commute. Since, usually, from a set of non-commuting integrals it is possible to find a set of functionally independent commuting integrals we are more leaned to conjecture that equation $P.ii$ is devoid of a non-degenerate Poisson structure.

Work is in progress to characterize the surfaces generated by the invariants in both integrable and non-integrable cases. We expect this to give new results in the geometric theory of integrable systems.

References

[1] V. I. Arnol’d. Dynamics of complexity of intersections. Bol. Soc. Bras. Mat., 21:1–10, 1990.
[2] M. Bellon and C-M. Viallet. Algebraic entropy. Comm. Math. Phys., 204:425–437, 1999.
[3] M. P. Bellon. Algebraic entropy of birational maps with invariant curves. Lett. Math. Phys., 50(1):79–90, 1999.
[4] M. Bruschi, O. Ragnisco, P. M. Santini, and G-Z. Tu. Integrable symplectic maps. Physica D, 49(3):273 – 294, 1991.
[5] G. B. Byrnes, F. A. Haggar, and G. R. W. Quispel. Sufficient conditions for dynamical systems to have pre-symplectic or pre-implctic structures. Physica A, 272:99 –129, 1999.
[6] H. W. Capel and R. Sahadevan. A new family of four-dimensional symplectic and integrable mappings. Physica A, 289:80–106, 2001.
[7] E. Celledoni, R. I. McLachlan, B. Owren, and G. R. W. Quispel. Geometric properties of Kahan’s method. J. Phys. A: Math. Theor., 46(2):025201, 2013.
[8] E. Celledoni, R. I. McLachlan, B. Owren, and G. R. W. Quispel. Integrability properties of Kahan’s method. J. Phys. A: Math. Theor., 47(36):365202, 2014.
[9] C. Cresswell and N. Joshi. The discrete first, second and thirty-fourth Painlevé hierarchies. J. Phys. A: Math. Gen., 32:655–669, 1999.
[10] J. Diller and C. Favre. Dynamics of bimeromorphic maps of surfaces. Amer. J. Math., 123(6):1135–1169, 2001.
[11] J.J. Duistermaat. Discrete Integrable Systems: QRT Maps and Elliptic Surfaces. Springer Monographs in Mathematics. Springer New York, 2011.
[12] G. Falqui and C-M. Viallet. Singularity, complexity, and quasi-integrability of rational mappings. Comm. Math. Phys., 154:111–125, 1993.
[13] M. Kh. Gizatullin. Rational $g$-surfaces. Izv. Akad. Nauk SSSR Ser. Mat., 44:110–144, 1980.
[14] B. Grammaticos, R. G. Halburd, A. Ramani, and C-M. Viallet. How to detect the integrability of discrete systems. J. Phys A: Math. Theor., 42:454002 (41 pp), 2009. Newton Institute Preprint NI09060-DIS.
[15] G. Gubbiotti. Integrability of difference equations through algebraic entropy and generalized symmetries. In D. Levi, R. Verge-Rebelo, and P. Winternitz, editors, Symmetries and Integrability of Difference Equations: Lecture Notes of the Abecedarian School of SIDE 12, Montreal 2016, CRM Series in Mathematical Physics, chapter 3, pages 75–152. Springer International Publishing, Berlin, 2017.
[16] G. Gubbiotti, N. Joshi, D. T. Tran, and C-M. Viallet. Integrability properties of a class of 4d bi-rational maps with two invariants, 2018. In preparation.
[17] M. Hay. Hierarchies of nonlinear integrable $q$-difference equations from series of Lax pairs. J. Phys. A: Math. Theor., 40:10457–10471, 2007.
[18] N. Joshi and C-M. Viallet. Rational maps with invariant surfaces, 2017. preprint on arXiv:1706.00173.
[19] W. Kahan. Unconventional numerical methods for trajectory calculations, 1993. Unpublished lecture notes.
[20] K. Kimura and R. Hirota. Discretization of the Lagrange top. J. Phys. Soc. Japan, 69:3193–3199, 2000.
[21] S. K. Lando. Lectures on Generating Functions. American Mathematical Society, 2003.
[22] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. Comm. Pure Appl. Math., 21(5):467–490, 1968.
[23] D. Levi and P. Winternitz. Continuous symmetries of difference equations. J. Phys. A Math. Theor., 39(2):R1–R63, 2006.
[24] J. Liouville. Note sur l’intégration des équations différentielles de la Dynamique, présentée au Bureau des Longitudes le 29 juin 1853. J. Math. Pures Appl., 20:137–138, 1855.
[25] S. Maeda. Completely integrable symplectic mapping. Proc. Jap. Ac. A, Math. Sci., 63:198–200, 1987.
[26] E. M. McMillan. A problem in the stability of periodic systems. In E. Britton and H. Odabasi, editors, A tribute to E.U. Condon, Topics in Modern Physics, pages 219–244. Colorado Assoc. Univ. Press., Boulder, 1971.
[27] P. J. Olver. Applications of Lie Groups to Differential Equations. Springer-Verlag, Berlin, 1986.
[28] V. G. Papageorgiou, F. W. Nijhoff, and H. W. Capel. Integrable mappings and nonlinear integrable lattice equations. Phys. Lett. A, 147(2):106–114, 1990.
[29] R. Penrose and C. A. B. Smith. A quadratic mapping with invariant cubic curve. Math. Proc. Camb. Phil. Soc., 89:89–105, 1981.
[30] M. Petrera, A. Pfadler, and Yu. B. Suris. On integrability of Hirota-Kimura type discretizations: Experimental study of the discrete Clebsch system. Exp. Math., 18:223–247, 2009.
[31] M. Petrera and Yu. B. Suris. On the Hamiltonian structure of Hirota-Kimura discretization of the Euler top. Math. Nachr., 283(11):1654–1663, 2010.
[32] G. R. W. Quispel, H. W. Capel, V. G. Papageorgiou, and F. W. Nijhoff. Integrable mappings derived from soliton equations. Physica A, 173(1):243–266, 1991.
[33] G. R. W. Quispel, J. A. G. Roberts, and C. J. Thompson. Integrable mappings and soliton equations. Phys. Lett. A, 126:419, 1988.
[34] G. R. W. Quispel, J. A. G. Roberts, and C. J. Thompson. Integrable mappings and soliton equations II. Physica D, 34(1):183 – 192, 1989.
[35] G. W. R. Quispel, H. R. Capel, and J. A. G. Roberts. Duality for discrete integrable systems. J. Phys. A: Math. Gen., 38(18):3965, 2005.
[36] J. A. G. Roberts and D. Jogia. Birational maps that send biquadratic curves to biquadratic curves. J. Phys. A Math. Theor., 48:08FT02, 2015.
[37] H. Sakai. Rational surfaces associated with affine root systems and geometry of the Painlevé Equations. Comm. Math. Phys., 220(1):165–229, 2001.
[38] I. R. Shafarevich. Basic Algebraic Geometry 1, volume 213 of Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, Heidelberg, New York, 2 edition, 1994.
[39] T. Tsuda. Integrable mappings via rational elliptic surfaces. J. Phys. A: Math. Gen., 37:2721, 2004.
[40] P. van der Kamp and G. W. R. Quispel. The staircase method: integrals for periodic reductions of integrable lattice equations. *J. Phys. A: Math. Theor.*, 43:465207, 2010.

[41] A. P. Veselov. Integrable maps. *Russ. Math. Surveys*, 46:1–51, 1991.

[42] A. P. Veselov. Growth and integrability in the dynamics of mappings. *Comm. Math. Phys.*, 145:181–193, 1992.

[43] C-M. Viallet, B. Grammaticos, and A. Ramani. On the integrability of correspondences associated to integral curves. *Phys. Lett. A*, 322:186-93, 2004.

School of Mathematics and Statistics F07, The University of Sydney, NSW 2006, Australia

E-mail address: giorgio.gubbiotti@sydney.edu.au
E-mail address: nalini.joshi@sydney.edu.au
E-mail address: dinhthi.tran@sydney.edu.au

LPTHE, UMR 7589 Centre National de la Recherche Scientifique & UPMC Sorbonne Université, 4 place Jussieu, 75252 Paris Cedex 05, France

E-mail address: claude.viallet@upmc.fr