Some upper bounds on ordinal-valued Ramsey numbers for colourings of pairs

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Abstract
We study Ramsey’s theorem for pairs and two colours in the context of the theory of \(\alpha\)-large sets introduced by Ketonen and Solovay. We prove that any 2-colouring of pairs from an \(\omega^{300n}\)-large set admits an \(\omega^n\)-large homogeneous set. We explain how a formalized version of this bound gives a more direct proof, and a strengthening, of the recent result of Patey and Yokoyama (Adv Math 330: 1034–1070, 2018) stating that Ramsey’s theorem for pairs and two colours is \(\forall \Sigma^0_2\)-conservative over the axiomatic theory RCA\(_0\) (recursive comprehension).

Keywords Ramsey’s theorem · Paris–Harrington principle · \(\alpha\)-Large sets · Proof theory · Reverse mathematics

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Introduction

The work described in this paper is mostly finite combinatorics. Much of the motivation, on the other hand, comes from logic.

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We contribute to the study of bounds on Ramsey’s theorem for pairs in a setting where the pairs being coloured always come from a finite subset of \( \mathbb{N} \), but the “size” of that subset can be described by a countable, possibly infinite ordinal rather than just by a natural number, and it depends not only on the cardinality of the set but also on which specific numbers it contains. More concretely, we use the framework of \( \alpha \)-large sets originally due to Ketonen and Solovay [11], in which, for instance:

- a set \( X \subseteq \mathbb{N} \) is \( n \)-large, for \( n \in \mathbb{N} \), exactly if \( X \) has at least \( n \) elements,
- a nonempty set \( X \) is \( \omega \)-large if \( X \setminus \{ \min X \} \) is \( \min X \)-large, that is, if \( X \) has strictly more than \( \min X \) elements,
- a nonempty set \( X \) is \( \omega^2 \)-large if \( X \setminus \{ \min X \} \) can be split into \( \min X \) many sets \( X_1, \ldots, X_{\min X} \) such that \( \max X_i < \min X_{i+1} \) and each \( X_i \) is \( \omega \)-large,

and so on. A precise definition of \( \alpha \)-largeness in the case that will matter to us, namely when \( \alpha < \omega^\omega \), is given in Sect. 1 below.

Our main aim is to obtain a good upper bound on the size of a set \( X \) guaranteeing that each 2-colouring of pairs from \( X \) will have an \( \omega^n \)-large homogeneous set, for \( n \in \mathbb{N} \).

This sort of work can be viewed simply as a particular kind of finite combinatorics: essentially, the study of bounds on Ramsey numbers that happen to take ordinal values rather than finite ones. Among the papers developing Ramsey theory in the context of \( \alpha \)-largeness—e.g. [1–3,13,21,22]—many do in fact focus on the purely combinatorial side of things. However, the original motivation for studying \( \alpha \)-largeness was the desire to understand the combinatorial reasons for the (un)provability of various statements in strong axiom systems.

For example, the seminal work of [11] was motivated by the desire to understand why a strengthening of finite Ramsey’s theorem known as the Paris–Harrington theorem [16] is unprovable in Peano Arithmetic, an axiom system that has the same strength as finite set theory (set theory without the axiom of infinity; see [10] for a precise discussion). The Paris–Harrington theorem can be formulated as saying:

for every \( n, k \in \mathbb{N} \) there exists a finite set \( X \subseteq \mathbb{N} \) with \( \min X \geq n \) such that every colouring of unordered \( n \)-tuples from \( X \) with \( k \) colours has an \( \omega \)-large homogeneous set.

The methods of [11] led to reasonably precise bounds on the size of a set needed to guarantee the existence of \( \omega \)-large homogeneous sets for colourings of \( n \)-tuples (for an appropriate number of colours \( k \)). Any function providing an upper bound on this size has to grow so fast that Peano Arithmetic will be unable to prove that the function has a value on every argument.

Our work is also inspired by a question from logic. It follows from a general-purpose result on colourings of \( n \)-tuples [2, Theorem 5] that

\[
\omega^{\omega^{n-2}} \rightarrow (\omega^n)_2^2.
\] (1)

That is, every 2-colouring of pairs from an \( \omega^{\omega^{n-2}} \)-large set has an \( \omega^n \)-large homogeneous set. It has been known for some time that determining whether this upper bound
is more or less tight would have important consequences for a longstanding open problem about the class of number-theoretic and finite-combinatorial statements that can be proved using infinite Ramsey’s theorem for pairs (see e.g. [19, Question 4.4] or [15, Question 2] for the question and e.g. [4–7] for some important related work). Recently, Patey and the second author [17] solved that open problem by showing that (1) is not tight. However, the argument in [17] was non-constructive and required a detour via infinite combinatorics and the set-theoretic technique of forcing; as a consequence, it did not give any specific bound.

Our main theorem here is

$$\omega^{300n} \rightarrow (\omega^n)_2^2.$$  \hspace{1cm} (2)

This is more or less tight, at least in the sense that it is impossible to get the left-hand side down from $$\omega^{O(n)}$$ to $$\omega^{(1+o(1))n}$$ ([13]; see the Remark at the end of Sect. 2). Moreover, our arguments use only relatively basic finite-combinatorial tools, which means that they can be formalized in axiomatic theories of modest strength. In effect, we obtain a new, significantly more direct proof of the main result of [17]: any simple enough statement provable using infinite Ramsey’s theorem for pairs and two colours can also be proved in the axiomatic theory RCA₀, which corresponds to a form of “computable mathematics” and (unlike infinite Ramsey’s theorem) is too weak to imply the existence of any non-computable subsets of $$\mathbb{N}$$. In fact, we also obtain some improvements of that result, which provide additional information concerning the proof-theoretic properties of Ramsey’s theorem.

The paper consists of three sections. In Sect. 1, we provide the necessary definitions and background. In Sect. 2, we prove the main theorem. Those two sections involve no logic beyond elementary facts about small infinite ordinals. The connections to logic are explained in Sect. 3.

**Notational conventions** We identify the number $$n$$ with the set $$\{0, \ldots, n-1\}$$. Most of the time, we index finite sets and sequences starting from 0, so that, for instance, an $$n$$-element set might be written as $$\{x_0, \ldots, x_{n-1}\}$$. Vertical lines, as in $$|X|$$, may stand for either the cardinality of a finite set or the length of a finite sequence; the intended meaning should be clear from the context.

We use interval notation to denote the appropriate intervals in the natural numbers: for instance, $$[2, 4]$$ is $$\{2, 3, 4\}$$ and $$[2, 4)$$ is $$\{2, 3\}$$.

The set of unordered $$n$$-tuples of elements of a set $$X$$ is denoted by $$[X]^n$$. For $$X \subseteq \mathbb{N}$$, we identify each element of $$[X]^n$$ with the ordered $$n$$-tuple enumerating it in increasing order.

The notation $$X \subseteq_{\text{fin}} Y$$ means “$$X \subseteq Y$$ and $$X$$ is finite”.

1 \textbf{\(\alpha\)-Largeness and Ramsey \(\alpha\)-largeness}

We fix a primitive recursive notation for ordinals below $$\omega^{\omega \omega}$$ by writing them in Cantor normal form: $$\alpha = \sum_{i<k} \omega^{n_i}$$ where $$n_i \in \mathbb{N}$$ and $$n_0 \geq \cdots \geq n_{k-1}$$. 

Let \( \alpha = \sum_{i<k} \omega^{n_i} \) and \( \beta = \sum_{i<k'} \omega^{n_i} \). We write \( \beta \succeq \alpha \) if \( \alpha = 0 \) or if \( mk' \geq n_0 \).

If \( \beta \succeq \alpha \), we can define the sum of \( \beta \) and \( \alpha \) as \( \beta + \alpha = \sum_{i<k+k'} \omega^i \) where \( t_i = m_i \) for \( i < k' \) and \( t_{j+k'} = n_j \) for \( j < k \). In what follows, we only consider sums of this form. We let \( \beta > \alpha \) if there is \( i \leq k, k' \) such that \( n_j = m_j \) for any \( j < i \) and \( (n_i < m_i \)
or\( i = k < k') \). By definition, \( \beta \succeq \alpha \) implies \( \beta \geq \alpha \).

We write \( 1 \) for \( \omega^0 \), and \( \omega^0 \cdot k \) for \( \sum_{i<k} \omega^i \). With this notation, one can write \( \alpha < \omega^\omega \) as \( \alpha = \omega^n \cdot k_n + \cdots + \omega^0 \cdot k_0 \), and put \( \text{MC}(\alpha) = \max\{k_n, \ldots, k_0\} \) (MC stands for the maximal coefficient of \( \alpha \)).

For \( \alpha < \omega^\omega \) and \( \omega \in \mathbb{N} \), define \( \alpha[m] = 0 \) if \( \alpha = 0 \), \( \alpha[m] = \beta \) if \( \alpha = \beta + 1 \), and \( \alpha[m] = \beta + \omega^n \cdot m \) if \( \alpha = \beta + \omega^n \) for some \( n \geq 1 \). By definition, \( m \leq n \) implies \( \alpha[m] \leq \alpha[n] \).

The following definition combines a fundamental concept from [11] with a variant from [17].

**Definition 1.1 (Largeness)** Let \( \alpha < \omega^\omega \), and let \( n, k, m \in \mathbb{N} \).

1. A set \( X = \{x_0 < \cdots < x_{\ell-1}\} \subseteq \mathbb{N} \) is called \( \alpha \)-large if \( \alpha[x_0] \cdots [x_{\ell-1}] = 0 \). In other words, any finite set is 0-large, and \( X \) is \( \alpha \)-large if
   - \( X \setminus \{\text{min } X\} \) is \( \beta \)-large if \( \alpha = \beta + 1 \),
   - \( X \setminus \{\text{min } X\} \) is \( (\beta + \omega^n \cdot \text{min } X) \)-large if \( \alpha = \beta + \omega^n \).
2. A set \( X \subseteq_{\text{fin}} \mathbb{N} \) is said to be \( RT^n_k \)-\( \alpha \)-large if for any \( P : [X]^n \rightarrow k \), there exists \( Y \subseteq X \) such that \( Y \) is \( P \)-homogeneous and \( \alpha \)-large.

With this terminology, the Paris–Harrington theorem can be rephrased as: “for every \( n, k \in \mathbb{N} \) there exists \( X \subseteq_{\text{fin}} \mathbb{N} \) with \( \text{min } X \geq n \) which is \( RT^n_k \)-\( \omega \)-large”.

The above definition of \( \omega^n \)-largeness causes minor issues if \( \text{min } X \) is a very small number—for instance, the set \( \{0\} \) ends up being \( \omega^n \)-large for every \( n \). To avoid this and simplify the notation, we will always consider finite sets \( X \subseteq_{\text{fin}} \mathbb{N} \) satisfying \( \text{min } X \geq 3 \). We will first check several basic properties.

**Lemma 1.2** Let \( \alpha, \beta < \omega^\omega \) and \( m \in \mathbb{N} \). If \( \alpha \leq \beta \) and \( \text{MC}(\alpha) < m \), then \( \alpha[m] \leq \beta[m] \).

**Proof** The case \( \alpha = \beta \) is trivial, so we assume \( \alpha < \beta \). Write \( \beta = \beta' + \omega^n \). If \( \alpha \leq \beta' \), then \( \alpha[m] \leq \beta' \leq \beta[m] \). Otherwise, \( n \geq 1 \) and there exists \( \gamma \leq \beta' \) such that \( \alpha = \beta' + \gamma \) and \( \gamma < \omega^n \). Since \( \text{MC}(\alpha) < m \), we also have \( \text{MC}(\gamma) < m \), thus \( \gamma < \omega^n \cdot m = \omega^n[m] \). Therefore, we obtain \( \alpha[m] \leq \alpha < \beta' + \omega^n \cdot m = \beta[m] \).

**Lemma 1.3** Let \( \alpha < \omega^\omega \) and \( X, Y \subseteq_{\text{fin}} \mathbb{N} \) where \( X = \{x_0 < \cdots < x_{\ell-1}\} \) and \( Y = \{y_0 < \cdots < y_{\ell'-1}\} \) for \( \ell \leq \ell' \). Assume that \( y_i \leq x_i \) for each \( i < \ell \) and that \( X \) is \( \alpha \)-large. Then \( Y \) is \( \alpha \)-large.

In particular, if \( X \) is \( \alpha \)-large and \( X \subseteq Y \), then \( Y \) is \( \alpha \)-large.

**Proof** We will show the following by induction on \( i \):

for any \( i < \ell \), there exists \( j_i < \ell \) such that \( j_i \geq i \) and \( \alpha[y_0] \cdots [y_{j_i}] = \alpha[x_0] \cdots [x_{j_i}] \).

The base case, which corresponds to \( i = -1 \), is the trivial statement \( \alpha = \alpha \).
Assume $\beta := \alpha[y_0] \ldots [y_i] = \alpha[x_0] \ldots [x_j]$ and $i + 1 < \ell$. If $\beta = 0$, put $j_{i+1} = \max\{j_i, i + 1\}$. If $\beta = \beta' + 1$, then $\beta[y_{i+1}] = \beta[x_{j_{i+1}}]$, so put $j_{i+1} = j_i + 1$. Note that $x_{j_{i+1}}$ must exist, because $\alpha[x_0] \ldots [x_j] = \beta \neq 0 = \alpha[x_0] \ldots [x_{\ell-1}]$.

If $\beta = \beta' + \omega^n$ for some $n \geq 1$, then

$$\beta[x_{j_{i+1}}] = \beta' + \omega^{n-1} \cdot (x_{j_{i+1}}) = \beta[y_{i+1}] + \omega^{n-1} \cdot (x_{j_{i+1}} - y_{i+1}).$$

Since $\beta[x_{j_{i+1}}] \ldots [x_{\ell-1}] = 0$, we have $\omega^{n-1} \cdot (x_{j_{i+1}} - y_{i+1})[x_{j_{i+2}}] \ldots [x_{\ell-1}] = 0$.

(Otherwise, $\beta[x_{j_{i+1}}] \ldots [x_{\ell-1}] = \beta[y_{i+1}] + \omega^{n-1} \cdot (x_{j_{i+1}} - y_{i+1})[x_{j_{i+2}}] \ldots [x_{\ell-1}] > 0$.) Let $\ell_{i+1}$ be the smallest $j$ such that $\omega^{n-1} \cdot (x_{j_{i+1}} - y_{i+1})[x_{j_{i+2}}] \ldots [x_j] = 0$. We then have $\beta[x_{j_{i+1}}] \ldots [x_{j_{i+1}}] = \beta[y_{i+1}]$.

The $\ell_{i+1}$ we get from the inductive argument for $i = \ell - 1$ must equal $\ell - 1$. This means we have $\alpha[y_0] \ldots [y_{\ell-1}] = \alpha[x_0] \ldots [x_{\ell-1}] = 0$. \qed

For a given $\alpha$-large set $X = \{x_0 < \ldots < x_{\ell-1}\} \subseteq \text{fin} \ N$, take the minimum $i < \ell$ such that $\alpha[x_0] \ldots [x_i] = 0$ and define $X|\alpha$ to be the set $\{x_0, \ldots, x_i\}$. (Thus, $X|\alpha$ is the smallest $\alpha$-large initial segment of $X$.)

**Lemma 1.4** Let $\alpha = \alpha_{k-1} + \cdots + \alpha_0 < \omega^\omega$ where $\alpha_{k-1} \geq \cdots \geq \alpha_0 \neq 0$. Then, a set $X \subseteq \text{fin} \ N$ is $\alpha$-large if and only if there is a partition $X = X_0 \cup \cdots \cup X_{k-1}$ such that

$$\max X_i < \min X_{i+1} \text{ and } X_i \text{ is } \alpha_i\text{-large.}$$

**Proof** Let $X = \{x_0 < \cdots < x_{\ell-1}\}$ be $\alpha$-large. By Lemma 1.3, we can assume without loss of generality that $X = X|\alpha$. For each $i < k$, let $X_i$ be the set

$$(X|\alpha)|\{\alpha_i + \cdots + \alpha_0\}) \setminus (X|\{\alpha_{i-1} + \cdots + \alpha_0\})).$$

One checks by induction on $i$ that $X_i$ equals $(X|(X_0 \cup \cdots \cup X_{i-1}))|\alpha_i$. It follows that $\max X_i < \min X_{i+1}$ and $X_i$ is $\alpha_i$-large.

Conversely, if $X = X_0 \cup \cdots \cup X_{k-1}$ such that $\max X_i < \min X_{i+1}$ and $X_i$ is $\alpha_i$-large, put $Y_i = X_i|\alpha_i$. Then, $Y = Y_0 \cup \cdots \cup Y_{k-1}$ is $\alpha$-large by the definition, and thus $X$ is $\alpha$-large by Lemma 1.3. \qed

As mentioned in the Introduction, Ketonen and Solovay [11] use $\alpha$-largeness to analyze the Paris–Harrington theorem and to clarify its relationship to hierarchies of fast-growing functions. In the process, they prove the following result concerning $\text{RT}_k^2 \cdot \omega$-largeness.

**Theorem 1.5** (Ketonen–Solovay [11], Lemma 6.4) Let $n \geq 2$. If $X \subseteq \text{fin} \ N$ is $\omega^{n+4}$-large and $\min X \geq 3$, then it is $\text{RT}_k^2 \cdot \omega$-large.

We will give a new proof of this theorem in Sect. 2.2.

Theorem 1.5 and its generalization to $\text{RT}_k^m$ proved in [11] only deal with the question how much $\alpha$-largeness is guaranteed to imply $\text{RT}_k^m \cdot \omega$-largeness, that is, the existence of an $\omega$-large homogeneous set for any given colouring. Our target is a generalization of the case $m = k = 2$ to bounds implying $\text{RT}_k^m \cdot \omega^n$-largeness for larger $n \in \mathbb{N}$. As already mentioned, even though this sort of work is purely combinatorial, much of the motivation comes from the study of the proof-theoretic strength of infinite Ramsey’s theorem for pairs. We discuss this in more detail in Sect. 3.

Our main result is as follows.
Theorem 1.6 If $X \subseteq \mathbb{N}$ is $\omega^{300n}$-large and $\min X \geq 3$, then $X$ is $\text{RT}^2_2$-$\omega^n$-large.

2 Calculation

In this section, we prove Theorem 1.6.

To simplify our calculations, we only consider “sparse enough” finite sets. A set $X$ with $\min X \geq 3$ is said to be exp-sparse if for any $x$, $y \in X$, if $x < y$ then $4^x < y$. We also say that $X$ is $\alpha$-sparse if for any $x$, $y \in X$ with $x < y$, the interval $(x, y]$ is $\alpha$-large. Trivially, any subset of an $\alpha$-sparse set is $\alpha$-sparse. By an easy calculation, one checks that any $\omega^3$-sparse set is exp-sparse: $y > 2x$ whenever $(x, y]$ is $\omega$-large, $y > x2^x$ whenever $(x, y]$ is $\omega^2$-large, and $y > 2^{2^x}$ (where there are $x$ applications of the exponential function) whenever $(x, y]$ is $\omega^3$-large.

Lemma 2.1 Let $n, m \in \mathbb{N}$. If $X \subseteq \mathbb{N}$ is $(\omega^{n+m} + 1)$-large and $\min X \geq 3$, then there exists $Y \subseteq X$ such that $Y$ is $\omega^n$-large and $\omega^m$-sparse. In particular, if $X \subseteq \mathbb{N}$ is $(\omega^{n+3} + 1)$-large and $\min X \geq 3$, then there exists $Y \subseteq X$ such that $Y$ is $\omega^n$-large and exp-sparse.

Proof We will show the following slightly stronger condition by induction on $n$:

if $X \subseteq \mathbb{N}$ is $(\omega^{n+m} + 1)$-large and $\min X \geq 3$, then there exists $Y \subseteq X \setminus \{\max X\}$ such that $Y$ is $\omega^n$-large and $Y \cup \{\max X\}$ is $\omega^m$-sparse.

For the case $n = 0$, let $X$ be $(\omega^m + 1)$-large and take $Y = \{\min X\}$. Then $Y$ is $\omega^0$-large, i.e. 1-large, and it follows from Lemma 1.3 and the $(\omega^m + 1)$-largeness of $X$ that if $\min X$, $\max X$ is $\omega^m$-sparse.

We turn to the case $n \geq 1$. If $X$ is $(\omega^{n+m} + 1)$-large, then $X \setminus \{\min X\}$ is $\omega^{n+m}$-large, thus there exist $X_0, \ldots, X_{k-1}$ such that

- $X = \{\min X, \min(X \setminus \{\min X\})\} \cup X_0 \cup \cdots \cup X_{k-1}$,
- $k = \min(X \setminus \{\min X\}) \geq 1 + \min X$,
- $\max X_i < \min X_{i+1}$,
- each $X_i$ is $\omega^{n+m-1}$-large.

Put $x_i = \max X_i$. By the induction hypothesis applied to the sets $\{x_i\} \cup X_{i+1}$ for $0 \leq i \leq k-2$, there exist $Y_0, \ldots, Y_{k-2}$ such that $Y_i \subseteq \{x_i\} \cup X_{i+1} \setminus \{x_{i+1}\}$, $Y_i$ is $\omega^{n-1}$-large and $Y_i \cup \{x_{i+1}\}$ is $\omega^m$-sparse. Now we can check that $Y = \{\min X\} \cup Y_0 \cup \cdots \cup Y_{k-2}$ is $\omega^n$-large and $Y \cup \{\max X\}$ is $\omega^m$-sparse. □

The following lemma means that if a large set $X$ is 2-coloured, we can always choose a “majority” colour without losing too much of its largeness. This fact underlies most of the constructions in the core part of our proof, as presented in Sect. 2.1. The lemma follows from the more general [1, Theorem 1], but our proof is very simple and—crucially for our purposes—involves no use of transfinite induction.

Lemma 2.2 For each $n \in \mathbb{N}$, the following holds.

1. If $X = Y_0 \cup Y_1 \subseteq \mathbb{N}$ is $\omega^n$ - 2-large and exp-sparse, then $Y_0$ is $\omega^n$-large or $Y_1$ is $\omega^n$-large.
2. If $X = Y_0 \cup Y_1 \subseteq \mathbb{N}$ is $\omega^n \cdot (4k)$-large and exp-sparse, then $Y_0$ is $\omega^n \cdot k$-large or $Y_1$ is $\omega^n \cdot k$-large.

**Proof** First, we show that 1. implies 2. for each $n \in \mathbb{N}$. If $X$ is $\omega^n \cdot (4k)$-large, then there exists a partition $X = X_0 \cup X_1 \cup \cdots \cup X_{2k-1}$ such that $\max X_i < \min X_{i+1}$ and $X_i$ is $\omega^n \cdot 2$-large. Then, by 1., at least one of $Y_0 \cap X_i$ and $Y_1 \cap X_i$ is $\omega^n$-large for each $i < 2k$. Depending on which case happens for at least half the $i$'s, at least one of $Y_0 \cap X$ and $Y_1 \cap X$ must be $\omega^n \cdot k$-large.

We now show 1., and thus also 2., by induction on $n$. The case $n = 0$ is trivial, so assume $n \geq 1$. Let $X = Y_0 \cup Y_1 \subseteq \mathbb{N}$ be $\omega^n \cdot 2$-large and exp-sparse. Take a partition $X = X_0 \cup X_1$ so that $\max X_0 < \min X_1$ and $X_0, X_1$ are both $\omega^n$-large. If $X_0 \subseteq Y_0$ or $X_0 \subseteq Y_1$, we are done. Otherwise, there are $c_0, c_1 \in X_0$ such that $c_0 \in Y_0$ and $c_1 \in Y_1$. Put $c = \max\{c_0, c_1\}$. Then, by exp-sparseness, $4c < \min X_1$, hence $X_1 \setminus \{\min X_1\}$ is $\omega^{n-1} \cdot (4c)$-large. By 2. of the induction hypothesis, at least one of $Y_0 \cap X_1$ and $Y_1 \cap X_1$ is $\omega^{n-1} \cdot c$-large. Thus, at least one of $\{c_0\} \cup (Y_0 \cap X_1) \subseteq Y_0$ and $\{c_1\} \cup (Y_1 \cap X_1) \subseteq Y_1$ is $\omega^n$-large.

### 2.1 The grouping principle

In this subsection, we consider the notion of grouping, introduced in [17, Section 7] as a useful tool in the analysis of Ramsey’s theorem for pairs. We will obtain an upper bound on the largeness of a set needed to guarantee the existence of sufficiently large groupings.

**Definition 2.3** (Grouping) Let $\alpha, \beta < \omega^\omega$. Let $X \subseteq \mathbb{N}$ and let $P : [X]^2 \rightarrow 2$ be a colouring. A finite family (sequence) of finite sets $\langle F_i \subseteq X : i < \ell \rangle$ is said to be an $(\alpha, \beta)$-grouping for $P$ if

1. $\forall i < j < \ell \ \max F_i < \min F_j$,
2. for any $i < \ell$, $F_i$ is $\alpha$-large,
3. $\{\max F_i : i < \ell\}$ is $\beta$-large, and,
4. $\forall i < j < \ell \ \forall x, x' \in F_i \ \forall y, y' \in F_j \ [P(x, y) = P(x', y')]$.

Moreover, $\langle F_i \subseteq X : i < \ell \rangle$ is said to be a strong $(\alpha, \beta)$-grouping for $P$ if the fourth condition is replaced with

4’. $\exists c < 2 \ \forall i < j < \ell \ \forall x \in F_i \ \forall y \in F_j \ [P(x, y) = c]$.

The intuition is that each $F_i$ is a “group” and that the colour of a pair consisting of representatives of two distinct groups depends only on the groups, not on the representatives. We say that a set $X \subseteq \mathbb{N}$ admits an $(\alpha, \beta)$-grouping if for any colouring $P : [X]^2 \rightarrow 2$, there exists an $(\alpha, \beta)$-grouping for $P$. Our target theorem in this subsection is the following.

**Theorem 2.4** Let $n, k \in \mathbb{N}$. If $X \subseteq \mathbb{N}$ is $\omega^{n+6k}$-large and exp-sparse, then $X$ admits an $(\omega^n, \omega^k)$-grouping.

To obtain a grouping, we need to stabilize the colour between elements of any two fixed groups. We first show how to stabilize the colour between one set and each
individual element of another set. This will have to be done both “from below” and “from above”.

**Lemma 2.5** Let \( X \subseteq \omega \) be \( \omega_1 \)-large and \( \exp_s \)-sparse, and let \( c \in \mathbb{N} \) such that \( 4^c \leq \min X \). Then, we have the following.

1. For any \( W \subseteq \omega \) such that \( |W| \leq c \) and \( \max W < \min X \) and for any colouring \( P : [W \cup X]^2 \to 2 \), there exists \( Y \subseteq X \) such that \( Y \) is \( \omega^n \)-large and \( P(w, y) = P(w', y') \) for any \( w, w' \in W \) and \( y, y' \in Y \).

2. For any \( W \subseteq \omega \) such that \( |W| \leq c \) and \( \max X < \min W \) and for any colouring \( P : [X \cup W]^2 \to 2 \), there exists \( Y \subseteq X \) such that \( Y \) is \( \omega^n \)-large and \( P(y, w) = P(y', w) \) for any \( w \in W \) and \( y, y' \in Y \).

**Proof** We only show 1., as the proof of 2. is virtually identical. Since \( X \) is \( \omega_1 \)-large and \( 4^c \leq \min X \), we know that \( X \setminus \{ \min X \} \) is \( \omega^n \cdot 4^c \)-large. Let \( Y_0 = X \setminus \{ \min X \} \). Without loss of generality, we may assume that \( |W| = c \). Enumerating \( W \) as \( \{ w_i : i < c \} \), we can construct a sequence \( Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_c \) so that \( Y_i \) is \( \omega^n \cdot 4^{c-i} \)-large and \( \forall y, y' \in Y_{i+1} (P(w_i, y) = P(w_i, y')) \). Indeed, Lemma 2.2 guarantees that at least one of \( \{ y \in Y_i : P(w_i, y) = 0 \} \) or \( \{ y \in Y_i : P(w_i, y) = 1 \} \) can be chosen as \( Y_{i+1} \). Take \( Y_c \) as the desired set \( Y \).

Next, we obtain a constant-length grouping.

**Lemma 2.6** Let \( X \subseteq \omega \) be \( \omega_{n+3} \)-large and \( \exp_s \)-sparse, and let \( d \in \mathbb{N} \) such that \( d \leq \min X \). Then, \( X \) admits an \((\omega^n, d)\)-grouping.

**Proof** Fix a colouring \( P : [X]^2 \to 2 \). We will construct an \((\omega^n, d)\)-grouping for \( P \).

First, we stabilize the colour from below in the sense of Lemma 2.5. By the assumption that \( d \leq \min X \), we know that \( X \setminus \{ \min X \} \) is \( \omega^{n+2} \cdot d \)-large. Take a partition \( X \setminus \{ \min X \} = X_0 \sqcup \cdots \sqcup X_{d-1} \) so that \( \max X_i < \min X_{i+1} \) and \( X_i \) is \( \omega^{n+2} \)-large. Put \( Y_0 = X_0 \), and for \( i \geq 1 \) take \( Y_i \subseteq X_i \) so that \( Y_i \) is \( \omega^{n+1} \)-large and \( P(x, y) = P(x', y') \) for any \( x \in X \cap [0, \max X_{i-1}] \) and \( y, y' \in Y_j \). This can be done using Lemma 2.5.1 with \( W = X \cap [d, \max X_{i-1}] \) and \( c = \max X_{i-1} \), because \( 4^{\max X_{i-1}} < \min X_i \) by the \( \exp_s \)-sparseness of \( X \). Then, \( \langle Y_i : i < d \rangle \) is a family of \( \omega^{n+1} \)-large sets such that for any \( 0 \leq i < j < d \) and for any \( x \in Y_i, y, y' \in Y_j \), we have \( P(x, y) = P(x', y') \).

Now, we stabilize the colour from above. Note that \( 4^d \leq \min Y_i \) for each \( i < d \), because \( d \leq \min X < \min Y_i \) and all \( Y_i \) are subsets of \( X \) which is \( \exp_s \)-sparse. Put \( Z_{d-1} = Y_{d-1} \), and for \( i < d - 1 \) take \( Z_i \subseteq Y_i \) so that \( Z_i \) is \( \omega^n \)-large and \( P(z, x) = P(z', x) \) for any \( x \in \{ \min Y_j : i < j < d \} \) and any \( z, z' \in Z_i \). This can be done using Lemma 2.5.2 with \( W = \{ \min Y_j : i < j < d \} \) and \( c = d - i - 1 \). Then, \( \langle Z_i : i < c \rangle \) is a family of \( \omega^n \)-large sets, and for any \( 0 \leq i < j < d \) and any \( x, x' \in Z_i, y, y' \in Z_j \), we have \( P(x, y) = P(x, \min Y_j) = P(x', \min Y_j) = P(x', y') \). Thus, \( \langle Z_i : i < c \rangle \) is an \((\omega^n, c)\)-grouping for \( P \).

By applying Lemma 2.6 twice, we obtain a grouping of \( \omega \)-large length.

**Lemma 2.7** Let \( X \subseteq \omega \) be \( \omega_{n+6} \)-large and \( \exp_s \). Then, \( X \) admits an \((\omega^n, \omega)\)-grouping.
Proof Fix a colouring $P : [X]^2 \to 2$. By Lemma 2.6, since $2 \leq \min X$, there is an $(\omega^{n+3}, 2)$-grouping $\langle Y_0, Y_1 \rangle$ for $P$. Again by Lemma 2.6, since $\max Y_0 < \min Y_1$, there is an $(\omega^n, \max Y_0)$-grouping $\langle Z_i : i < \max Y_0 \rangle$ for $P$ with $Z_i \subseteq Y_1$ for each $i$. One can easily check that $\langle Y_0, Z_0, \ldots, Z_{\max Y_0} \rangle$ is an $(\omega^n, \omega)$-grouping for $P$. \hfill \Box

Finally we prove Theorem 2.4 by using the previous lemma repeatedly.

Proof of Theorem 2.4 We prove the statement by induction on $k$. The case $k = 0$ is trivial, and the case $k = 1$ is Lemma 2.7. Assume that $k \geq 2$ and let $X \subseteq_{\text{fin}} \mathbb{N}$ be $\omega^{n+6k}$-large and exp-sparse. Fix a colouring $P : [X]^2 \to 2$. By Lemma 2.7, there is an $(\omega^n, \omega^{k-1})$-grouping $\langle Y_i : i \leq \ell \rangle$ for $P$. Since $\max Y_i : i \leq \ell$ is $\omega$-large, we know that $\ell \geq \max Y_0$. By the induction hypothesis, for each $1 \leq i \leq \ell$ there is an $(\omega^n, \omega^{k-1})$-grouping $\langle Z^i_j : j \leq m_i \rangle$ for $P$ such that $Z^i_j \subseteq Y_i$ for each $j$. Since $\max Z^i_j : j \leq m_i$ is $\omega^{k-1}$-large for any $1 \leq i \leq \ell$, the set

$$\{\max Y_0 \} \cup \{\max Z^i_j : j \leq m_i, 1 \leq i \leq \ell\}$$

is $\omega^k$-large. One can check that $\langle Y_0, Z^1_0, \ldots, Z^1_{m_1}, \ldots, Z^\ell_0, \ldots, Z^\ell_{m_\ell} \rangle$ is an $(\omega^n, \omega^k)$-grouping for $P$. \hfill \Box

2.2 Proof of Theorem 1.5

In this subsection, we give a simple proof of Theorem 1.5. The proof is still based on the original idea in [11], but the calculation is simplified. We include the details to make the paper more self-contained and because in Sect. 3 we will have to discuss the axioms required for the argument to go through.

For a given $P : [X]^2 \to n$ and $x \in X$, the hereditarily minimal prehomogeneous (h.m.p.h.) sequence $\sigma_x$ is the finite increasing sequence of elements of $X$ defined as follows:

$$\sigma_x(0) = \min X,$$

$$\sigma_x(i + 1) = \min\{y \in X : y > \sigma_x(i) \land \forall j \leq i P(\sigma_x(j), x) = P(\sigma_x(j), y)\},$$

stop this construction when $\sigma_x(i) = x$.

One can easily check the following from the definition.

- For any $i < j < k < |\sigma_x|$, $P(\sigma_x(i), \sigma_x(j)) = P(\sigma_x(i), \sigma_x(k))$.
- $\sigma_x(i) = y < x$ if and only if $\sigma_y = \sigma_x|_{i+1} \neq \sigma_x$. In particular, any nonempty initial segment of $\sigma_x$ has the form $\sigma_y$ for some $y < x$.

For a given colour $c < n$, let $\ho(\sigma_x, c) = \{\sigma_x(i) : i < |\sigma_x| - 1 \land P(\sigma_x(i), x) = c\}$. The set $\ho(\sigma_x, c) \cup \{x\}$ is $P$-homogeneous with colour $c$. We let $\text{col}(\sigma_x)$ be the set $\{c < n : \ho(\sigma_x, c) \neq \emptyset\}$. Clearly, $\sigma_x \subseteq \sigma_y$ implies $\text{col}(\sigma_x) \subseteq \text{col}(\sigma_y)$. For $x \in X \setminus \{\min X\}$, we write $\sigma_x^-$ to denote the longest initial segment $\sigma_y \subseteq \sigma_x$ such that $\text{col}(\sigma_y) \subseteq \text{col}(\sigma_x)$. Note that this definition would not make sense for $x = \min X$, because $\text{col}(\sigma_{\min X}) = \emptyset$. 

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Lemma 2.8 Let, $n \geq 2$, $X \subseteq \mathbb{N}$ and let $P : [X]^2 \to n$ be a colouring. Then we have the following.

1. For any $m \in \mathbb{N}$, $|\{x \in X : |\sigma_x| \leq m\}| \leq n^m$.
2. For any $x \in X$ and $c \in \text{col}(\sigma_x)$, min $ho(\sigma_x, c) \leq \sigma_x^- (|\sigma_x^-| - 1)$.

**Proof** By the definition of h.m.p.h. sequences, if $\sigma_y = \sigma_x^- (y)$ and $\sigma_z = \sigma_x^- (z)$, then $P(x, y) \neq P(x, z)$. Thus, for any $x \in X$, there are at most $n$-many $y$’s in $X$ such that $y > x$, $\sigma_y \supseteq \sigma_x$ and $|\sigma_y| = |\sigma_x| + 1$. Hence the size of $\{x \in X : |\sigma_x| \leq m\}$ is at most $1 + n + \cdots + n^{m-1} \leq n^m$, which gives 1.

For a given $x \in X$, put $y = \max\{\min ho(\sigma_x, c) : c \in \text{col}(\sigma_x)\}$. Then $\text{col}(\sigma_y) \subseteq \text{col}(\sigma_x)$. Thus, $\sigma_y \subseteq \sigma_x^-$, and we have 2. \hfill $\square$

**Proof of Theorem 1.5** Let $X_0 \subseteq \mathbb{N}$ be $\omega^{n+4}$-large and $\min X_0 \geq 3$. Then one can find $X \subseteq X_0$ which is $(\omega^n + 1)$-large, $\omega^3$-sparse and such that $\min X > n$. Indeed, $X_0' = X_0 \backslash \{\min X_0\}$ is at least $(\omega^{n+3} \cdot 3)$-large. Put $X_1' = X_0' \backslash \omega^{n+3}$, $X_2' = (X_0' \setminus X_1') \omega^{n+3}$ and $X_3' = (X_0' \setminus X_1' \cup X_2') \omega^{n+3}$. Note that $|X_1'| > n$ and that $\{\min X_2', \max X_2'\}$ is $\omega^3$-sparse. Moreover, $\{\min X_2', \max X_2'\}$ is $(\omega^{n+3} + 1)$-large, so by Lemma 2.1 it contains an $\omega^n$-large $\omega^3$-sparse subset $X''$. We can take $X = \{\min X_2' \cup \min X''\}$ as the desired set.

Now we show that $X$ chosen as above is $RT^2_n$-$\omega$-large by way of contradiction. Assume that $P : [X]^2 \to n$ is a colouring with no $\omega$-large homogeneous set. Write $X = \{x_0 < \cdots < x_{\ell - 1}\}$. Let $\sigma_i := \sigma_{x_i}$ be the h.m.p.h. sequence determined by $P$ and $x_i$. For each $1 \leq d \leq n$, we say that $i < \ell$ is $d$-critical if $|\sigma(i)| = d$ and $\sigma_i^- \neq \sigma_j^-$ for each $j < i$. For $1 \leq i < \ell$ and $1 \leq d \leq n$, define an ordinal $\gamma_i^d < \omega^n$ as follows.

If no $j < i$ is $d$-critical, put $\gamma_i^d = 0$. Otherwise, take the largest $d$-critical number $j_0 \leq i$ and let $m_{i, 1}^d = |\{k \leq i : \text{col}(\sigma_k)| = d\}|$, $m_{i, 2}^d = |\{k \leq i : k \text{ is } (d + 1)\text{-critical}\}|$ (where $m_{i, 2}^d = 0$ for $d = n$); then put $\gamma_i^d = \omega^{n-d} \cdot (x_{j_0} - m_{i, 1}^d - m_{i, 2}^d)$.

**Claim** If there is a $d$-critical number $j \leq i$, then $\gamma_i^d > 0$.

**Proof of Claim.** Let $j_0 \leq i$ be the largest $d$-critical number $\leq i$; since $d \geq 1$, we know that $j_0 > 0$. Note that for any $k \leq i$ such that $|\text{col}(\sigma_k)| = d$, we have $\sigma_k^- = \sigma_j^-$ for some $j \leq j_0$ (if not, there would be a $d$-critical number bigger than $j_0$) and therefore also $\sigma_k^- = \sigma_j$ for some $j < j_0$; this implies $\sigma_k^- (|\sigma_k^-| - 1) \leq x_{j_0 - 1}$. Fix $k \leq i$ such that $|\text{col}(\sigma_k)| = d$. Then, for any $c \in \text{col}(\sigma_k)$, $\text{min ho}(\sigma_k, c) \leq \sigma_k^- (|\sigma_k^-| - 1) \leq x_{j_0 - 1}$, where the first inequality follows from Lemma 2.8.2. Since $\text{ho}(\sigma_k, c) \cup \{x_k\}$ is $P$-homogeneous and thus not $\omega$-large, we have $|\text{ho}(\sigma_k, c) \cup \{x_k\}| \leq x_{j_0 - 1}$, and hence $|\sigma_k| \leq nx_{j_0 - 1}$. Therefore, by Lemma 2.8.1, we have $m_{i, 1}^d \leq n^{x_{j_0 - 1}}$. If $k, k' \leq i$ are both $(d + 1)$-critical, then $\sigma_k^- \neq \sigma_k'^- \text{ and } |\text{col}(\sigma_k^+)| = |\text{col}(\sigma_k'^+) = d$. Thus, $m_{i, 2}^d \leq m_{i, 1}^d \leq n^{x_{j_0 - 1}}$. Finally, since $X$ is $\omega^3$-sparse and $x_{j_0 - 1} > n$, one can easily check that $x_{j_0} > 2n^{x_{j_0 - 1}} \geq m_{i, 1}^d + m_{i, 2}^d$. This completes the proof of the claim.

Now, define $\gamma_0^d = \omega^n$ and $\gamma_i = \gamma_1^d + \cdots + \gamma_i^d$ for $i = 1, \ldots, \ell - 1$. Note that 1 is $1$-critical, because $|\text{col}(\sigma_1)| = 1$ and $\sigma_1^- = \emptyset$ while $\sigma_0^-$ does not exist. Thus, by the Claim, $\gamma_i > 0$ for any $i < \ell$.

For $i < \ell - 1$, consider the difference between $\gamma_i$ and $\gamma_{i+1}$. Let $d = |\text{col}(\sigma_{i+1})|$. There are two cases:
• if \( i + 1 \) is \( d \)-critical, then \( \gamma_{i+1} \) is obtained from \( \gamma_i \) by removing one \( \omega^{n-(d-1)} \) and adding at most \( x_{i+1} \)-many \( \omega^{n-d} \)'s (note that if \( i > 0 \), then \( d > 1 \) and \( \gamma_i^{d-1} > 0 \) because there must be a \( (d-1) \)-critical \( j \leq i \)),
• if \( i + 1 \) is not \( d \)-critical, then \( \gamma_{i+1} \) is obtained from \( \gamma_i \) simply by removing one \( \omega^{n-d} \).

In either case, \( \gamma_{i+1} \leq \gamma_i [x_{i+1}] \). Note also that \( \text{MC}(\gamma_i) < x_{i+1} \). This lets us check by induction that \( \gamma_i \leq \gamma_0 [x_1] \ldots [x_i] \) for any \( 1 \leq i < \ell \). Indeed, \( \gamma_i \leq \gamma_0 [x_1] \ldots [x_i] \) and \( \text{MC}(\gamma_i) < x_{i+1} \) implies that \( \gamma_{i+1} \leq \gamma_i [x_{i+1}] \leq \gamma_0 [x_1] \ldots [x_i][x_{i+1}] \) by Lemma 1.2. Since \( \gamma_0 = (\omega^n+1)[x_0] \), we have \( 0 < \gamma_i \leq (\omega^n+1)[x_0] \ldots [x_i] \) for any \( i < \ell \). However, \( (\omega^n+1)[x_0] \ldots [x_{\ell-1}] = 0 \) since \( X \) is \( (\omega^n+1) \)-large. This implies \( 0 < \gamma_{\ell-1} \leq 0 \), which is a contradiction. \( \square \)

### 2.3 Decomposition of Ramsey’s theorem for pairs

Recall that \( [X]^2 \) is identified with the set of ordered pairs from \( X \) in which the second element is strictly greater than the first. A colouring \( P : [X]^2 \to 2 \) is said to be transitive if both \( P^{-1}(0) \) and \( P^{-1}(1) \) are transitive relations on \( X \). In other words, for a transitive \( P \), if \( x < y < z \) and \( P(x, y) = P(y, z) \), then \( P(x, z) \) must have the same value as well.

Using this notion, \( \text{RT}_2^2 \) can be decomposed as \( \text{RT}_2^2 = \text{EM} + \text{ADS} \) where

• EM: for any colouring \( P : [\mathbb{N}]^2 \to 2 \), there exists an infinite set \( H \subseteq \mathbb{N} \) such that \( P \) is transitive on \([H]^2\),
• ADS: for any transitive colouring \( P : [\mathbb{N}]^2 \to 2 \), there exists an infinite set \( H \subseteq \mathbb{N} \) such that \( H \) is \( P \)-homogeneous.

EM and ADS were originally introduced as combinatorial principles about ordered graphs and linear orders, respectively; see [4,9,14]. We consider a similar decomposition for \( \text{RT}_2^2 \)-largeness.

**Definition 2.9** Let \( \alpha < \omega^\omega \).

1. A set \( X \subseteq_\text{fin} \mathbb{N} \) is EM-\( \alpha \)-large if for any colouring \( P : [X]^2 \to 2 \), there exists \( Y \subseteq X \) such that \( P \) is transitive on \([Y]^2\) and \( Y \) is \( \alpha \)-large.
2. A set \( X \subseteq_\text{fin} \mathbb{N} \) is ADS-\( \alpha \)-large if for any transitive colouring \( P : [X]^2 \to 2 \), there exists \( Y \subseteq X \) such that \( Y \) is \( P \)-homogeneous and \( Y \) is \( \alpha \)-large.

We prove Theorem 1.6 by combining appropriate upper bounds for EM-\( \alpha \)-largeness and ADS-\( \alpha \)-largeness.

**Theorem 2.10** If \( X \subseteq_\text{fin} \mathbb{N} \) is \( \omega^{36n} \)-large and exp-sparse, then it is EM-\( \omega^n \)-large.

Note that [17, Lemma 7.2] essentially says that for every \( n \) there is an \( m \) such that an \( \omega^m \)-large set is EM-\( \omega^n \)-large. Theorem 2.10 strengthens this by providing a concrete upper bound on \( m \), which is possible thanks to Theorem 2.4.

**Proof** We follow the proof of [17, Lemma 7.2], replacing the use of [17, Lemma 7.1] by Theorem 2.4. It is enough to show that if \( X \) is \( \omega^{36(n-1)+6} \)-large and exp-sparse then it is EM-\( \omega^n \)-large. We prove this by induction on \( n \).
Theorem 2.11 If $X \subseteq \mathbb{N}$ be $\omega^{36(n-1)+6}$-large. Fix $P : |X|^2 \to 2$. By Theorem 2.4, there exists an $(\omega^{36(n-2)+6}, \omega^6)$-grouping $\langle Y_i : i \leq \ell \rangle$ for $P$. Applying Theorem 1.5 to the $\omega^6$-large set $\{ Y_i : i \leq \ell \}$, we get an $(\omega^{36(n-2)+6}, \omega)$-subgrouping $\langle Y_{ij} : j \leq \ell' \rangle$ which is strong, i.e., there is a fixed colour $c$ such that for any $x, y$ from different groups, $P(x, y) = c$. By the induction hypothesis, for each $j \leq \ell'$ there is some $\omega^{n-1}$-large $Z_j \subseteq Y_{ij}$ such that $P$ is transitive on $[Z_j]^2$. Since $\max Z_0 \leq \max Y_{i0} \leq \ell'$, the set $H = \{ \max Z_0 \} \cup \bigcup_{1 \leq j \leq \ell'} Z_j$ is $\omega^n$-large. Moreover, by construction, $P$ is transitive on $[H]^2$. □

Theorem 2.11 is a reformulation of [17, Lemma 4.4]. The proof below is still based on the idea of the original proof.

Proof Let $X \subseteq \mathbb{N}$ be an $\omega^{4n+4}$-large set with $\min X \geq 3$. Assume towards a contradiction that $X$ is not $\text{ADS-}\omega^n$-large. Thus, there is a transitive colouring $P : |X|^2 \to 2$ without an $\omega^n$-large homogeneous set. Given $x, y \in X$ with $x < y$, say that an interval $[x, y]$ is $(i, \alpha)$-long if $P(x, y) = i$ and there exists an $\alpha$-large set $H \subseteq [x, y] \cap X$ such that $x, y \in H$ and $H$ is $P$-homogeneous with colour $i$. Define a new colouring $Q : |X|^2 \to 4n$ as follows:

$$Q(x, y) = \begin{cases} 4k & \text{if } [x, y] \text{ is } (0, \omega^k)\text{-long but not } (0, \omega^k+1)\text{-long}, \\ 4k+1 & \text{if } [x, y] \text{ is } (0, \omega^k+1)\text{-long but not } (0, \omega^{k+1})\text{-long}, \\ 4k+2 & \text{if } [x, y] \text{ is } (1, \omega^k)\text{-long but not } (1, \omega^k+1)\text{-long}, \\ 4k+3 & \text{if } [x, y] \text{ is } (1, \omega^k+1)\text{-long but not } (1, \omega^{k+1})\text{-long}, \end{cases}$$

where $0 \leq k < n$. Since there is no $\omega^n$-large $P$-homogeneous set, $Q$ is well-defined. By Theorem 1.5, there exists an $\omega$-large $Q$-homogeneous set $H \subseteq X$. Write $H = \{ x_0, \ldots, x_m \}$ where $x_0 < \cdots < x_m$. By $\omega$-largeness, $m \geq x_0$.

We now claim that $Q(x_0, x_1) \neq Q(x_0, x_m)$, which is a contradiction with the homogeneity of $H$ for $Q$. The proof of the claim splits into four cases depending on $Q(x_0, x_1)$. Consider for instance the case where $[x_0, x_1]$, and thus each of $[x_i, x_{i+1}]$, is $(0, \omega^k+1)$-long but not $(0, \omega^{k+1})$-long. For each $i \leq m-1$, let $H_i$ be the $(\omega^k+1)$-large $Q$-homogeneous subset of $[x_i, x_{i+1}]$ whose existence follows from the assumption that $[x_i, x_{i+1}]$ is $(0, \omega^k+1)$-long. Let $H = \bigcup_{i < m} H_i$. Note that $x_0 \in H_0$ and $x_i = \max H_{i-1} = \min H_i$ for $1 \leq i \leq m-1$; in particular, $H_{i-1} \cap H_i \neq \emptyset$. Thus, by the transitivity of $P$, the set $H$ is $Q$-homogeneous with colour 0. Moreover, $m \geq x_0$ and $H = \{ x_0 \} \cup \bigcup_{i < m} (H_i \setminus \{ x_i \})$ implies that $H$ is $\omega^{k+1}$-large. Hence, $[x_0, x_m]$ is $(0, \omega^{k+1})$-long, which implies $Q(x_0, x_1) \neq Q(x_0, x_m)$. The other cases are similar or easier. □

Proof of Theorem 1.6 We show that if $X \subseteq \mathbb{N}$ is $(\omega^{(4n+4)36+3} + 1)$-large, then it is $\text{RT}_2^2$-$\omega^n$-large. Fix a colouring $P : |X|^2 \to 2$. First, using Lemma 2.1, take some $X_0 \subseteq X$ which is $\omega^{(4n+4)36}$-large and exp-sparse. Next, using Theorem 2.10, take $X_1 \subseteq X_0$ such that $X_1$ is $\omega^{4n+4}$-large and $P$ is transitive on $|X_1|^2$. Finally, Theorem 2.11 gives $Y \subseteq X_1$ which is $\omega^n$-large and $P$-homogeneous. □
Remark One may obtain slightly better bounds for some of the theorems and lemmas above. For example, in Lemma 2.6, if $d = 2$ then we only need $X$ to be $\omega^{n+2}$-large, because we only need to shrink $X_1$ in the first stage of the proof and $Y_0 = X_0$ in the second stage. This could actually be used to obtain a slightly better upper bound ($\omega^{n+5\ell}$-largeness) in Theorem 2.4 but such small improvements are not particularly important from our perspective.

On the other hand, the bound in Theorem 2.4 cannot be reduced to $\omega^{n+o(n)}$-largeness. Indeed, Kotlarski et al. [13, Theorem 5.4] showed that if a set $X$ is $RT^2_\omega$-$\omega^n$-large, then it is $\omega^{2n}$-large.

3 Finite consequences of Ramsey’s theorem for pairs

In this section, we explain the relevance of Theorem 1.6 to logic, or more specifically to proof theory. Ramsey-theoretic principles are well-known to display interesting behaviour with respect to provability in axiomatic theories. For example, as discussed in the Introduction, the Paris–Harrington theorem is a statement from finite Ramsey theory that cannot be proved using only the means of finite set theory. In contrast, it was recently proved in [17] that infinite Ramsey’s theorem restricted to the case of colourings of pairs is in a certain sense proof-theoretically “tame”. Theorem 1.6 makes it possible to give a more direct proof of that result and, in fact, to strengthen it.

To understand the proofs in this section, the reader will need some familiarity with axiomatic theories of first- and second-order arithmetic and their models—see [8,20] for details. The following very brief review will hopefully suffice for understanding the statements of the results. The language of second-order arithmetic has two types of variables: first-order variables $x, y, z, \ldots$ or $k, \ell, n, \ldots$ to stand for natural numbers (which can also be used to code other finite objects, such as finite subsets of $\mathbb{N}$) and second-order variables to stand for subsets of $\mathbb{N}$ (which can also be used to code relations on $\mathbb{N}$). A formula in this language is $\Sigma^0_n$ if it has no second-order quantifiers and consists of at most $n$ first-order quantifiers (beginning with $\exists$) followed by a formula in which all quantifiers have to be bounded, i.e. have the form $\exists x < y$ or $\forall x < y$. The dual class of formulas beginning with $\forall$ is called $\Pi^0_n$, while $\forall \Sigma^0_n$ stands for the class of formulas consisting of universal (possibly second-order) quantifiers followed by a $\Sigma^0_n$ formula. $\mathsf{RCA}_0$ is an axiomatic theory in this language which has: (a) some basic axioms specifying that $\mathbb{N}$ is a discrete ordered semiring, (b) the $\Delta^0_1$-comprehension axiom, which states that for every computable property $R$ of natural numbers (as given by an appropriate syntax) the set $\{n \in \mathbb{N} : R(n)\}$ exists, and (c) the $\Sigma^0_1$-induction axiom, which allows the use of mathematical induction for any property expressed by a $\Sigma^0_1$ formula (which in fact means: for any recursively enumerable property). $\mathsf{RCA}_0$ may be viewed as embodying the methods of “computable mathematics”. For each $n$, any $\forall \Sigma^0_n$ statement provable in $\mathsf{RCA}_0$ is provable in the weaker theory $\Pi^0_n$, which only has axioms of type (a), (c). $\mathsf{EFA}$ (Elementary Function Arithmetic) is an even weaker theory in which mathematical induction can only be used for properties defined without using any unbounded quantifiers; to counteract this weakness, $\mathsf{EFA}$
has to include an additional axiom guaranteeing the basic properties of the exponential function on \( \mathbb{N} \), including its totality, which means that \( 2^n \) exists for every \( n \in \mathbb{N} \).

The main result of [17] concerns the theory \( \text{WKL}_0 + \text{RT}^2_2 \), which is obtained by adding a weak version of König’s Lemma and a natural statement of Ramsey’s theorem for pairs and two colours to \( \text{RCA}_0 \).

**Theorem** [17, Theorem 7.4] \( \text{WKL}_0 + \text{RT}^2_2 \) is \( \forall \Sigma^0_2 \)-conservative over \( \text{RCA}_0 \). That is, each \( \forall \Sigma^0_2 \) statement provable in \( \text{WKL}_0 + \text{RT}^2_2 \) is already provable in \( \text{RCA}_0 \).

The combinatorial core of the proof of this theorem in [17] is contained in the following result about \( \alpha \)-largeness. Here and below, ordinals smaller than \( \omega^\omega \) are represented in \( \text{RCA}_0 \) by letting the number coding \( \langle n_0, \ldots, n_{k-1} \rangle \) stand for \( \sum_{i < k} \omega^{n_i} \).

**Proposition** [17, Proposition 7.7] For every natural number \( n \) there exists a natural number \( m \) such that \( \text{RCA}_0 \) proves: for every \( X \subseteq \text{fin} \mathbb{N} \) with \( \min X \geq 3 \), if \( X \) is \( \omega^m \)-large, then \( X \) is \( \text{RT}^2_2 \)-\( \omega^n \)-large.

However, the proof of [17, Theorem 7.4] does not work with [17, Proposition 7.7] directly, but instead makes use of an intermediate notion of “density”. Moreover, even though [17, Proposition 7.7] is a statement of finite combinatorics, its proof involves a major detour through an infinitary principle (cf. [17, Section 6]). Our proof of Theorem 1.6 is considerably more direct and it is readily seen to give the following stronger version of [17, Proposition 7.7]:

**Corollary 3.1** \( \text{RCA}_0 \) (and, in fact, the weaker theory \( \text{EFA} \)) proves the following: for every \( n \in \mathbb{N} \) and every \( X \subseteq \text{fin} \mathbb{N} \) with \( \min X \geq 3 \), if \( X \) is \( \omega^{300n} \)-large, then \( X \) is \( \text{RT}^2_2 \)-\( \omega^n \)-large.

**Proof** An inspection of the arguments in Sects. 1 and 2 (including the proof of Theorem 1.5 as presented in Sect. 2.2) reveals that they only make use of elementary manipulations of finite combinatorial objects such as finite sets, finite trees and Cantor Normal Forms, and of the usual principle of mathematical induction applied to properties that can be expressed using bounded quantifiers, possibly with exponentially large bounds. These tools are available within \( \text{EFA} \). (A different proof of Theorem 1.5 in EFA was recently given by Pelupessy [18].)

Crucially, none of the arguments involve transfinite induction up to \( \omega^\omega \) (which is not available in \( \text{RCA}_0 \)) or mathematical induction for \( \Sigma^0_1 \) or \( \Pi^0_1 \) properties whose definitions require unbounded quantifiers (this would be available in \( \text{RCA}_0 \) but not in \( \text{EFA} \)). Regarding the second point, note that all apparent uses of \( \Pi^0_1 \)-induction—as in, for instance, the proof of Theorem 2.4, where we seem to be using induction for a statement quantifying over all \( X \subseteq \text{fin} \mathbb{N} \)—can be replaced by bounded induction: for any given \( X \), the universal quantifier in the induction property can be restricted to range over subsets of \( X \).

The extra strength provided by Corollary 3.1 can be used to obtain a strengthening of [17, Theorem 7.4], by means of a relatively simple proof that avoids the concept of density. To express the strengthening, let \( \text{WO}(\alpha) \), for \( \alpha < \omega^\omega \), denote the statement that there is no infinite descending sequence of ordinals starting from \( \alpha \). The following lemma lists some basic properties of ordinals below \( \omega^\omega \) provable within \( \text{RCA}_0 \). The properties are well-known and their easy proofs seem to be part of the folklore.
Lemma 3.2 The following are provable within RCA₀.

1. For any $\alpha < \omega^\omega$, WO(\alpha) if and only every set of ordinals smaller than $\alpha$ has a minimum element.
2. For any $\alpha < \omega^\omega$, WO(\alpha) if and only if any infinite set contains an $\alpha$-large subset.
3. For any $m \in \mathbb{N}$, WO($\omega^m$) implies WO($\omega^{2m}$).

**Remark** The above lemma implies that for each fixed $\alpha < \omega^\omega$, RCA₀ proves WO(\alpha). In contrast, RCA₀ is unable to prove the general statement “WO(\alpha) holds for every $\alpha < \omega^\omega$”. Thus, the so-called proof-theoretic ordinal of RCA₀—roughly, the supremum of the ordinals whose well-foundedness is provable in the theory—is exactly $\omega^\omega$. It follows from the results of [17] that the proof-theoretic ordinal of WKL₀ + RT² is also $\omega^\omega$.

Theorem 3.3 WKL₀ + RT² is conservative over RCA₀ with respect to sentences of the form:

$$\forall \alpha < \omega^\omega (\text{WO}(\alpha) \rightarrow \varphi(\alpha))$$

where $\varphi$ is $\forall \Sigma^0_2$.

Note that the class of sentences considered in Theorem 3.3 is strictly larger than the one in [17, Theorem 7.4] because WO(\alpha) is not a $\forall \Sigma^0_2$ statement (it is actually $\forall \Sigma^0_3$).

**Proof** (In this argument, we follow the notational conventions of [17], using the symbol $\omega$ to denote the smallest infinite ordinal as formalized in RCA₀ and reserving $\omega$ for the set of actual (standard) natural numbers.)

Let $\varphi(\alpha) \equiv \forall X \forall x \exists y \exists z \varphi_0(X[z], x, y, z, \alpha)$, where $\varphi_0$ is $\Sigma^0_0$, be a $\forall \Sigma^0_2$-formula such that RCA₀ does not prove $\forall \alpha < \omega^\omega (\text{WO}(\alpha) \rightarrow \varphi(\alpha))$. Take a countable nonstandard model $(M, S) \models \text{RCA}_0 + \exists \alpha < \omega^\omega (\text{WO}(\alpha) \land \neg \varphi(\alpha))$. There exist $A \in S$ and $a, \alpha \in M$ such that

$$(M, S) \models \alpha < \omega^\omega \land \text{WO}(\alpha) \land \forall y \exists z \neg \varphi_0(A[z], a, y, z, \alpha).$$

Take some $c \in M \setminus \{0\}$ such that $\alpha < \omega^c$ and $\text{WO}(\omega^c)$ holds in $(M, S)$. (If $\alpha = \omega^c + \beta$, then Lemma 3.2 part 3. lets us take $c := c_0 + 1$.) Also take some $b \in M$ which is greater than each of $a, c$, and the code for $\alpha$.

By $\Sigma^0_1$-induction—or even by the weaker principle known as $\Sigma^0_1$-collection—for every $x$ there is another number $x'$ such that $\forall y < x \exists z < x' \neg \varphi_0(A[z], a, y, z, \alpha)$. Thus, we can define a sequence of numbers by recursion in the following way:

$$x_0 = b,$$

$$x_{i+1} = \min\{x > x_i : \forall y < x_i \exists z < x \neg \varphi_0(A[z], a, y, z, \alpha)\},$$

with $\Sigma^0_1$-induction in $(M, S)$ guaranteeing that $x_i$ exists for each $i \in M$. By $\Delta^0_1$-comprehension, the set $Y = \{x_i : i \in M\}$ belongs to $S$. Moreover, $Y$ is infinite in $(M, S)$.
By Lemma 3.2 parts 2. and 3., every infinite set contains an $\omega^\omega$-large finite subset for each $n \in \omega$. It follows that $Y$ has an $\omega^\omega$-large $M$-finite subset for each $n \in \omega$. By overspill, there exists an $M$-finite set $X \subseteq Y$ which is $\omega^{300^d+c}$-large for some $d \in M \setminus \omega$.

Let $\{E_i\}_{i \in \omega}$ be an enumeration of all $M$-finite sets which are not $\omega^c$-large, and $\{P_i\}_{i \in \omega}$ be an enumeration of all $M$-finite 2-colourings of pairs from $[0, \max X]$. We will construct an $\omega$-length sequence of $M$-finite sets $X = X_0 \supseteq X_1 \supseteq \cdots$ such that for each $i \in \omega$, the set $X_i$ is $\omega^{300^d-i+c}$-large, the colouring $P_i$ is constant on $[X_{2i+1}]^2$, and $[\min X_{2i+2}, \max X_{2i+2}] \cap E_i = \emptyset$.

To achieve this, we do the following for each $i \in \omega$. At stage $2i+1$ of the construction, we take $X_{2i+1} \subseteq X_{2i}$ such that $P_i$ is constant on $[X_{2i+1}]^2$. Assuming $X_{2i}$ was $\omega^{300^d-2i+c}$-large, Corollary 3.1 lets us choose $X_{2i+1}$ so that it is $\omega^{300^d-2i-1+c}$-large. Then, at stage $2i + 2$, consider the colouring $Q : [X_{2i+1}]^2 \to 2$ such that $Q(x, y) = 0$ if and only if $E_i \cap \{x, y\} = \emptyset$. Again by Corollary 3.1, we can take $X_{2i+2} \subseteq X_{2i+1}$ such that $Q$ is constant on $[X_{2i+2}]^2$ and $X_{2i+2}$ is $\omega^{300^d-2i-2+c}$-large. $X_{2i+2}$ is in particular $(\omega^c + 1)$-large, so if the colour of $Q$ on $[X_{2i+2}]^2$ was 1, then by Lemma 1.3 the set $E_i$ would be $\omega^c$-large. Therefore, the colour of $Q$ on $[X_{2i+2}]^2$ must be 0, which implies $[\min X_{2i+2}, \max X_{2i+2}] \cap E_i = \emptyset$.

Now, let $I = \sup \{\min X_i : i \in \omega\} \subseteq M$. The even-numbered stages of our construction ensure that $I$ is a cut in $M$ and that $X_j \cap I$ is unbounded in $I$ for each $j \in \omega$ (consider the case where $E_i$ is a singleton set). They also ensure that that any set $E \in \mathrm{Cod}(M/I)$ which is unbounded in $I$ has an $\omega^c$-large subset. To see this, assume $E$ has no $\omega^c$-large subset and take an $M$-finite set $\hat{E}$ such that $E = \hat{E} \cap I$. By overspill, there exists $e \in M \setminus I$ such that $\hat{E} \cap \{0, e\}$ has no $\omega^c$-large subset, but then $\hat{E} \cap \{0, e\} = E_i$ for some $i \in \omega$ and so by construction $E = \hat{E} \cap I = E_i \cap I$ must be bounded in $I$.

It follows in particular that $I$ is a semi-regular cut—that is, for every $e \in I$, any $E \in \mathrm{Cod}(M/I)$ which is unbounded in $I$ has an $M$-finite subset with at least $e$ elements. By standard arguments, this implies $(I, \mathrm{Cod}(M/I)) \models \mathrm{WO}(\omega^c)$.

On the other hand, the odd-numbered stages ensure that $(I, \mathrm{Cod}(M/I)) \models \mathrm{RT}_2^2$. To see this, let $P : [I]^2 \to 2$ be a function in $\mathrm{Cod}(M/I)$. Then $P = P_i \cap I$ for some $i \in \omega$. Hence $P$ is constant on $[X_{2i+1} \cap I]^2$, and $X_{2i+1} \cap I \in \mathrm{Cod}(M/I)$ is an infinite set in $I$.

Finally, since $X \cap I$ is unbounded in $I$, so is $Y \cap I$. Thus, we have

$$(I, \mathrm{Cod}(M/I)) \models \forall y \exists z \neg \varphi_0((A \cap I)[z], a, y, z, \alpha),$$

and hence $(I, \mathrm{Cod}(M/I)) \models \neg \varphi(\alpha)$. We have $(I, \mathrm{Cod}(M/I)) \models \mathrm{WO}(\alpha)$ because $\alpha < \omega^c$. Therefore, $\mathrm{WKL}_0 + \mathrm{RT}_2^2$ does not prove $\forall \alpha < \omega^\omega (\mathrm{WO}(\alpha) \to \varphi(\alpha))$. $\square$

The following consequence of Theorem 3.3 states, intuitively speaking, that $\mathrm{RT}_2^2$ does not imply any new closure properties of ordinals below $\omega^\omega$ compared to $\mathrm{RCA}_0$.

**Corollary 3.4** For any primitive recursive function $p : \omega^\omega \to \omega^\omega$ (defined on codes of ordinals), if $\mathrm{WKL}_0 + \mathrm{RT}_2^2$ proves

$$\forall \alpha < \omega^\omega (\mathrm{WO}(\alpha) \to \mathrm{WO}(p(\alpha))),$$
then $\text{RCA}_0$ proves the same statement.

As a special case, $\text{WKL}_0 + \text{RT}_2^2$ does not prove $\forall x \ (\text{WO}(\omega^x) \rightarrow \text{WO}(\omega^{2^x}))$, as this is not provable within $\text{RCA}_0$. Actually, already the construction in the proof of Theorem 3.3 can be arranged so that the constructed model satisfies $\text{WO}(\omega^c) \land \neg \text{WO}(\omega^{2^c})$.

Another strengthening of [17, Theorem 7.4]—in fact, the original motivation for Corollary 3.1—concerns proof lengths. Corollary 3.1 can be used to obtain the theorem below, which states that $\text{WKL}_0 + \text{RT}_2^2$ has no significant proof speedup for proofs of $\forall \Sigma^0_2$ sentences over $\text{RCA}_0$. This answers Question 9.5 of [17] in the negative.

**Theorem 3.5** There is a polynomial-time computable mapping which, given a proof $p$ of a $\forall \Sigma^0_2$ sentence $\varphi$ in $\text{WKL}_0 + \text{RT}_2^2$ as input, returns a proof $p'$ of $\varphi$ in $\text{RCA}_0$ as output. In particular, the size of $p'$ is at most polynomially larger than the size of $p$.

Proving Theorem 3.5 requires a more extensive development of the logical framework. The proof will be provided in the forthcoming paper [12].

**References**

1. Bigorajska, T., Kotlarski, H.: A partition theorem for $\alpha$-large sets. Fund. Math. **160**(1), 27–37 (1999)
2. Bigorajska, T., Kotlarski, H.: Some combinatorics involving $\xi$-large sets. Fund. Math. **175**(2), 119–125 (2002)
3. Bigorajska, T., Kotlarski, H.: Partitioning $\alpha$-large sets: some lower bounds. Trans. Am. Math. Soc. **358**(11), 4981–5001 (2006)
4. Bovykin, A., Weiermann, A.: The strength of infinitary Ramseyan principles can be accessed by their densities. Ann. Pure Appl. Logic **168**(9), 1700–1709 (2017)
5. Cholak, P.A., Jockusch, C.G., Slaman, T.A.: On the strength of Ramsey’s theorem for pairs. J. Symb. Log **66**(1), 1–15 (2001)
6. Chong, C.T., Slaman, T.A., Yang, Y.: The metamathematics of Stable Ramsey’s theorem for pairs. J. Am. Math. Soc. **308**, 121–141 (2017)
7. Chong, C.T., Slaman, T.A., Yang, Y.: The inductive strength of Ramsey’s theorem for pairs. Adv. Math. **27**(3), 863–892 (2014)
8. Hájek, P., Pudlák, P.: Metamathematics of First-Order Arithmetic. Springer, Berlin, XIV+460 pages (1993)
9. Hirschfeldt, D.R., Shore, R.A.: Combinatorial principles weaker than Ramsey’s theorem for pairs. J. Symb. Log. **72**(1), 171–206 (2007)
10. Kaye, R., Wong, T.L.: On interpretations of arithmetic and set theory. Notre Dame J. Formal Logic **48**(4), 497–510 (2007)
11. Ketonen, J., Solovay, R.: Rapidly growing Ramsey functions. Ann. Math. (2) **113**(2), 267–314 (1981)
12. Kołodziejczyk, L., Wagner, T.L., Yokoyama, K.: Ramsey’s theorem for pairs, collection, and proof size. Preprint, available at arXiv:2005.06854
13. Kotlarski, H., Piekart, B., Weiermann, A.: More on lower bounds for partitioning $\alpha$-large sets. Ann. Pure Appl. Logic **147**(3), 113–126 (2007)
14. Lerman, M., Solomon, R., Towsner, H.: Separating principles below Ramsey’s theorem for pairs. J. Math. Log. **13**(02), 1350007 (2013)
15. Montalbán, A.: Open questions in reverse mathematics. Bull. Symb. Log. **17**(3), 431–454 (2011)
16. Paris, J., Harrington, L.: A mathematical incompleteness in Peano arithmetic. In Barwise, J. (ed.) Handbook of Mathematical Logic, volume 90 of Stud. Logic Found. Math., pp. 1133–1142. North-Holland, Amsterdam (1977)
17. Patey, L., Yokoyama, K.: The proof-theoretic strength of Ramsey’s theorem for pairs and two colors. Adv. Math. **330**, 1034–1070 (2018)
18. Pelupessy, F.: On $\alpha$-largeness and the Paris-Harrington principle in $\text{RCA}_0$ and $\text{RCA}_0^*$. Available at arXiv:1611.08988
19. Seetapun, D., Slaman, T.A.: On the strength of Ramsey’s theorem. Notre Dame J. Formal Logic 36(4), 570–582 (1995)

20. Simpson, S.G.: Subsystems of Second Order Arithmetic. Perspectives in Mathematical Logic. Springer-Verlag, 1999. XIV + 445 pages; Second Edition, Perspectives in Logic, Association for Symbolic Logic, Cambridge University Press, 2009, XVI+ 444 pages

21. De Smet, M., Weiermann, A.: Partitioning $\alpha$-large sets for $\alpha < \varepsilon_0$. Available at arXiv:1001.2437

22. Weiermann, A.: A classification of rapidly growing Ramsey functions. Proc. Am. Math. Soc. 132(2), 553–561 (2004)

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