On Adiabatic Limits and Rumin’s Complex

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July 24, 1994

Abstract

This paper shows that when the Riemannian metric on a contact manifold is blown up along the direction orthogonal to the contact distribution, the corresponding harmonic forms rescaled and normalized in the $L^2$-norms will converge to Rumin’s harmonic forms. This proves a conjecture in Gromov [11]. This result can also be reformulated in terms of spectral sequences, after Forman, Mazzeo-Melrose. A key ingredient in the proof is the fact that the curvatures become unbounded in a controlled way.

*Supported by the Ministry of Colleges and Universities of Ontario and the Natural Sciences and Engineering Research Council of Canada
1 Introduction

Rumin [15] constructed a differential complex adapted to a contact distribution, for which the Laplacians are sub-elliptic operators. In this paper we show how to arrive at this complex via adiabatic limits, using the ideas of Mazzeo-Melrose [13], and Witten [18].

Beginning with Witten’s work on adiabatic limits [18], there is a fair amount of work on the asymptotic behaviors of geometric-topological objects (e.g., harmonic forms, eta invariants, etc) associated with a family of Riemannian metrics on fiber bundles as the metrics become singular (see, for example, Cheeger [2]). In particular, Mazzeo-Melrose [13] studied those of harmonic forms and related them to spectral sequences (see also Forman [4]). In all these work an essential geometric assumption is that the curvatures of the metrics are uniformly bounded. In this paper we consider a different situation in which a Riemannian metric on a contact manifold is blown up along the direction orthogonal to the contact distribution. It is known that, despite that curvatures become unbounded, the Riemannian metric nevertheless converges to a Carnot-Caratheodory metric, and Gromov ([11], page 191-96) conjectured that the harmonic forms will converge to the corresponding objects associated with the Carnot-Caratheodory metrics, i.e., the Rumin’s harmonic forms. In this paper we will show that this is indeed the case if the harmonic forms are rescaled and normalized in the $L^2$-norms. A key ingredient in the proof is the fact that the curvatures become unbounded in a controlled way.

There is some interest to generalize Rumin’s theory to more general Carnot-Caratheodory spaces (see, for example, Gromov [11]). Some preliminary results in this direction can be found in [6], [7]. The results in this paper suggest that there is probably a different approach, namely that through the study of the adiabatic limits of harmonic forms and the associated “spectral sequence” $E^1_k$ (cf. §2, and Forman [4]). This is also related to the characteristic cohomology (cf. Bryant-Griffiths [3]) Vinogradov [17].
The results of this paper have been announced in [9].

2 Statement of Results

Let $M$ be a $(2m + 1)$-dimensional compact Riemannian manifold, $A$ a contact distribution. Let $B$ be the orthogonal distribution to $A$, so $TM = A \oplus B$. Write the Riemannian metric as $g = g_A \oplus g_B$. Consider a family of metrics $g_\epsilon = g_A \oplus \epsilon^{-2}g_B$. As $\epsilon \to 0$, the metric space $(M, g_\epsilon)$ converges in the sense of Gromov-Hausdorff to the Carnot-Caratheodory metric space $(M, g_A)$ (see, for example, Fukaya [3], Ge [8], Gromov [11]), in which the distance between two points is the minimum of lengths of curves tangent to $A$ joining the two points.

Let $\Omega^{p,q} = \Omega^p(A) \wedge \Omega^q(B)$. Decompose $d$ into

$$d = d^{2,-1} + d^{1,0} + d^{0,1}, \quad d^{a,b} : \Omega^{p,q} \to \Omega^{p+a,q+b}.$$  

Since $A$ is contact, $d^{2,-1}$ is not zero. This is the point of departure of this paper from Mazzeo-Melrose [13].

Equip $\Omega^{p,q}$ with the metric induced from $g_\epsilon$ (still to be denoted by $g_\epsilon$). Let $\Theta_\epsilon$ be the isometry (the rescaling map)

$$\Theta_\epsilon : (\Omega(M), g_\epsilon) \to (\Omega(M), g_1).$$

Define the normalized differential $d_\epsilon = \Theta_\epsilon \circ d \circ (\Theta_\epsilon)^{-1}$, then

$$d_\epsilon = \frac{1}{\epsilon} d^{2,-1} + d^{1,0} + \epsilon d^{0,1}.$$  

We will use “$\ast$” to denote the adjoint with respect to $g_1$.

Now Rumin’s complex (see Rumin [15]) can be written as

$$\mathcal{R}^k := \Omega^{k,0}/Im(d^{2,-1}), \quad k \leq m; \quad \mathcal{R}^k := \Omega^{k-1,1} \cap Ker(d^{2,-1}), k \geq m + 1,$$

with the induced differential

$$d_\epsilon = \pi d^{1,0} : \mathcal{R}^k \to \mathcal{R}^{k+1}, \quad k \neq m;$$

$$d_R = \pi(d^{0,1} - d^{1,0}(d^{2,-1})^{-1}d^{1,0}) : \mathcal{R}^m \to \mathcal{R}^{m+1},$$
where $\pi$ is the orthogonal projection $\Omega^k \to \mathcal{R}^k$. This is a sub-elliptic complex.

**Theorem 2.1** Assume $(M, g)$ is Heisenberg (cf. §3). Suppose $\omega_\epsilon, \|\omega_\epsilon\|_{L^2} = 1$, is a $d_\epsilon$-harmonic form,

$$d_\epsilon \omega_\epsilon = d_\epsilon^* \omega_\epsilon = 0,$$

(i.e. $\Theta_\epsilon^{-1} \omega_\epsilon$ is a harmonic form for $(M, g_\epsilon)$). Then as $\epsilon \to 0$, after passing to a subsequence,

$$\omega_\epsilon \to \omega_0 \neq 0 \quad \text{strongly in } L^2,$$

and $\omega_0$ is a Rumin’s harmonic form.

This result can also be reformulated in terms of spectral sequence, after Mazzeo-Melrose \[13\], Forman \[4\].

Fix a number $l$, one says a family of $k$-forms $\omega_\epsilon$ is of class $O(\epsilon^l)$, i.e. $f_\delta = O(\epsilon^l)$, if $\epsilon^{-l}\|\omega_\epsilon\|_{H^1}$ is uniformly bounded, where $H^1$ denotes the ordinary Sobolev space.

Define

$$E^k_l := \{ \omega_\epsilon \in \Omega^k, d_\epsilon \omega_\epsilon = O(\epsilon^{l-1}), \quad d_\epsilon^* \omega_\epsilon = O(\epsilon^{l-1}), \quad \|\omega_\epsilon\|_{L^2} = 1 \},$$

and set

$$\bar{E}^k_l := \text{linear span } \{ \text{the weakly limits of } \omega_\epsilon \text{ in } L^2 \text{ as } \epsilon \to 0 \text{, } \omega_\epsilon \in E^k_l \} \cap C^\infty(\Omega^k).$$

Obviously, for each $k$

$$\cdots \subset \bar{E}^k_2 \subset \bar{E}^k_1 \subset \bar{E}^k_0.$$

The following result says that most terms in the spectral sequence will degenerate at $E_2$ except those of degree either $m$ or $m + 1$, which degenerate at $E_3$. This may explain why $d_\mathcal{R}$ is a second order operator.

**Theorem 2.2** Suppose $(M, g)$ is Heisenberg (cf. §3).

(1) The terms in $E^1$ are

$$\bar{E}^k_1 = \mathcal{R}^k.$$
(2) The terms in $\tilde{E}^2$ are
\[
\begin{align*}
\tilde{E}^k_1 &= \{ \omega \in \mathcal{R}^k, d_{\xi} \omega = d_{\xi}^* \omega = 0 \}, \quad k \neq m, m + 1; \\
\tilde{E}^k_2 &= \{ \omega \in \mathcal{R}^k, (d^{k,0})^* \omega = 0 \}, \quad k = m; \\
\tilde{E}^k_2 &= \{ \omega \in \mathcal{R}^k, d^{1,0} \omega = 0 \}, \quad k = m + 1.
\end{align*}
\]

(3) The terms in $\tilde{E}^3$ are
\[
\begin{align*}
\tilde{E}^k_3 &= \tilde{E}^k_2, \quad k \neq m, m + 1; \\
\tilde{E}^k_3 &= \tilde{E}^k_4 = \cdots = \{ \omega \in \mathcal{R}^k, d_{\mathcal{R}} \omega = d_{\mathcal{R}}^* \omega = 0 \}, \quad k = m; \\
\tilde{E}^k_3 &= \tilde{E}^k_4 = \cdots = \{ \omega \in \mathcal{R}^k, d_{\xi} \omega = d_{\xi}^* \omega = 0 \}, \quad k = m + 1.
\end{align*}
\]

We shall use the following notations: $\| \cdot \|_{H^1_c}$ denotes the following weighted Sobolev’s norm
\[
\| \omega \|^2_{H^1_c} = \int_M \sum (D_{e_i} \omega, D_{e_i} \omega) d\nu,
\]
where $e_i$ is an orthonormal basis for $A$, and $\| \omega \|_{H^2_c}$ similarly.

3 Geometry of Heisenberg Manifolds

Let $v$ be a (local) unit tangent vector field spanning $B$, and $\xi$ the contact 1-form which satisfies $\xi(v) = 1$. If the metric $g_A$ can be written as $g_A(a, b) = d\xi(a, Jb), a, b \in A$, where $J$ is an endmorphism of $A$ satisfying $J^2 = -Id$, then we say $(M, g)$ is Heisenberg. Note that even though $v$ is in general only locally defined, this notion is well defined. Throughout the rest of this paper we assume that $(M, g)$ is Heisenberg.

We will use the following properties of Heisenberg manifold.

**Lemma 3.1** There is an orthonormal basis $e_1, \cdots, e_m, e_{m+1} := Je_1, \cdots, e_{2m} := Je_m$ for $A$ such that
\[
\begin{align*}
[e_i, e_j] &= 0 \quad \text{mod} \ (A), \\
[e_i, e_{m+j}] &= \delta_{ij} v \quad \text{mod} \ (A), \quad 1 \leq i, j \leq m.
\end{align*}
\]
Proof. This follows from the condition $g_A = dξ(·, J·), J^2 = -Id.$

The following property in fact holds for any contact manifold.

Lemma 3.2 Suppose $ξ$ is a contact 1-form and $x_0$ a fixed point on $M, v$ the vector field such that $i(v)ξ = 1, i(v)dξ = 0.$ There are vector fields $u_i, i = 1, · · · , 2m + 1, u_{2m+1} = v,$ such that

1. $u_i$ are linearly independent at $x_0;$
2. $u_i$ vanishes outside a small neighborhood for $i = 1, · · · , 2m;$
3. $Ł_{u_i}ξ = Ł_{u_i}dξ = 0,$ and $Ł_{u_i}$ preserves $Ω^{k,0}, i = 1, · · · , 2m + 1.$ (Here $Ł_u$ is the Lie derivative in the direction $u.$)

Proof. First note that $Ł_vξ = 0$ follows from $Ł_v = i(v)d + d i(v).

To choose $u_1, · · · , u_{2m},$ one takes a local coordinates $(x, z) ∈ R^{2n} × R$ near $x_0$ such that $ξ = dz − ρ$ where $ρ$ is a 1-form on $R^{2n}{x}$ and $v = ∂/∂z.$ Choose $2m$ functions $f_1, · · · , f_{2m}$ on $R^{2m}{x}$ with linearly independent $df_1, · · · , df_{2m}$ at $x_0$ such that $f_i$ vanishes outside a neighborhood. Let $H_f$ denote the Hamiltonian vector field of $f_i$ with respect to $dp.$ Define $u_i(x, z) = H_{f_i}(x) + a_i∂/∂z,$ where $a_i$ is determined from the equation $i(u_i)ξ = − f_i, i = 1, · · · , 2m.$ One easily verifies that $u_i$ thus defined satisfies all the requirements.

4 A priori Estimates

To prove Theorem 2.1 and Theorem 2.2, in this section we will derive some a priori estimates for the $H^1_k$-norm of $ω$ in terms of $(Δ_d, ω, ω)$ if $k ≠ m, m + 1,$ and for the $H^2_k$-norm of $ω$ if $k = m, m + 1.$ As the case of $k > m$ is similar to that of $k ≤ m,$ we will only consider the case $k ≤ m.$
We will use the following notations: If $L$ is an operator,

$$\Delta_L := L^* L + L L^*. $$

The letter $C$ denotes a generic positive number, $M$ a generic constant.

4.1 The case of $k$-forms ($k \neq m, m + 1$).

We have the following a priori estimates

**Theorem 4.1** For any $\omega = \alpha + \beta, \alpha \in \Omega^{k,0}, \beta \in \Omega^{k-1,1}$, we have

$$ (\Delta_d \omega, \alpha) \geq \frac{1}{\epsilon^2} \| (d^{2,-1})^* \alpha \|^2_{L^2} + \frac{m-k}{m} C \| \alpha \|^2_{H^1_0} + \epsilon^2 (D_v \alpha, D_v \alpha) - M(\omega, \omega),$$

$k \leq m - 1$;

and

$$ (\Delta_d \omega, \beta) \geq \frac{1}{\epsilon^2} \| (d^{2,-1})^* \beta \|^2_{L^2} + \frac{m-k+1}{m} C \| \beta \|^2_{H^1_0} + \epsilon^2 (D_v \beta, D_v \beta) - M(\omega, \omega),$$

$k \leq m$.

To prove this theorem, we need a few technical results.

**Lemma 4.2** The following operator

$$ Q := (d^{0,1})^* d^{1,0} + (d^{1,0})^* d^{0,1} + d^{0,1}(d^{1,0})^* + d^{1,0}(d^{0,1})^* $$

is a first-order linear differential operator.

**Proof.** We only need to prove that

$$ (d^{0,1})^* d^{1,0} + (d^{1,0}) (d^{0,1})^* $$

is a first order operator.
If \( e_i \) is an orthonormal basis for \( A, v \) for \( B \), then

\[
d^{1,0} = \sum e^i \wedge D_{e_i} + \text{0-order operator} ,
\]

\[
(d^{0,1})^* = i(v) D_v + \text{0-order operator} .
\]

So

\[
(d^{0,1})^* d^{1,0} + d^{1,0} (d^{0,1})^* = 
\]

\[
= \sum e^i \wedge D_{e_i} i(v) D_v + i(v) D_v e^i \wedge D_{e_i} + \text{1st order operator} 
\]

\[
= \sum e^i \wedge i(v) D_{e_i} D_v + i(v) e^i \wedge D_{e_i} D_v + \text{1st order operator} 
\]

\[
= \text{1st order operator} .
\]

Here we have used the fact that \( e^i \wedge i(v) + i(v) e^i \wedge = 0 \).

**Lemma 4.3** If \( \alpha, \beta \) are as in Theorem 4.1, then

\[
(\Delta_{(e^1, d^2, -1, +d^{1,0})} \alpha, \alpha) \geq \frac{1}{\epsilon^2} \| (d^{2,1})^* \alpha \|^2 + C \frac{m - k}{m} \| \alpha \|^2_{H^1_\epsilon} - M(\alpha, \alpha), \quad k \leq m - 1; 
\]

\[
(\Delta_{(e^1, d^2, -1, +d^{1,0})} \beta, \beta) \geq \frac{1}{\epsilon^2} \| d^{2,1} \beta \|^2_{L^2} + C \frac{m - k + 1}{m} \| \beta \|^2_{H^1_\epsilon} - M(\beta, \beta), \quad k \leq m. 
\]

**Proof.** We shall only prove the first inequality, as the second one can be proved similarly.

By counting the types of the differential forms, one has

\[
(\Delta_{(e^1, d^2, -1, +d^{1,0})} \alpha, \alpha) \geq \frac{1}{\epsilon^2} \| (d^{2,1})^* \alpha \|^2 + (\Delta_{d^{1,0}} \alpha, \alpha). 
\]

In terms of a local orthonormal basis \( e_i \) for \( A \), one can write

\[
d^{1,0} = \sum_{i=1}^{2m} e^i \wedge D_{e_i} + \text{0-order operator} ,
\]

\[
(d^{1,0})^* = \sum_{i=1}^{2m} i(e_i) \wedge D_{e_i} + \text{0-order operator} .
\]
\[ \Delta_{d.\alpha} = \sum e_i \wedge (e_j) De_i De_j + i(e_j) e_i \wedge De_i + 1\text{-st order operator in } e_i \]
\[ = - \sum De_i De_i - e_i \wedge i(e_j)(De_j De_i - De_i De_j) + 1\text{-st order operators in } e_i . \]

Now by Lemma 3.1, if \( i > j \),
\[ De_i De_j - De_j De_i \frac{1}{m} \delta_{i-m,j} \sum_{i=1}^{m} (De_{i+m} De_i - De_i De_{i+m}) + 1\text{st order operator in } De_i , \]
while if \( i < j \),
\[ De_i De_j - De_j De_i \frac{1}{m} \delta_{i+m,j} \sum_{i=1}^{m} (De_{i+m} De_i - De_i De_{i+m}) + 1\text{st order operator in } De_i . \]

So after an integration by parts,
\[ \int_M (e_i \wedge i(e_j)(De_i De_j \alpha - De_j De_i \alpha, \alpha) \]
\[ = \frac{2}{m} \int_M (i(e_i) De_{i+m} \alpha, i(e_{i+m}) De_j \alpha) - (i(e_i) De_j \alpha, i(e_{i+m}) De_{i+m} \alpha) \]
\[ + \text{terms of the form } (De_i \alpha, \alpha) \]
\[ \leq \frac{k}{m} \| \alpha \|^2_{H^{1}} + \text{terms of the form } (De_i \alpha, \alpha). \] (1)

This proves the lemma.

**Proof of Theorem 4.1.** We shall only prove the first inequality, as the second one can be proved similarly.

By a direct computation, one has
\[ (\Delta \omega, \alpha) = (D_{(e^{-1}d^{2,-1}d^{1,0})} \omega, \alpha) + \]
\[ + ((d^{2,-1})^{*} \alpha, (d^{0,1})^{*} \omega) + ((d^{2,-1})^{*} \omega, (d^{0,1})^{*} \alpha) \] (2)
\[ + ((d^{2,-1})^{*} \alpha, (d^{0,1})^{*} \omega) + ((d^{2,-1})^{*} \omega, (d^{0,1})^{*} \alpha) \] (3)
\[ + \epsilon (Q \omega, \alpha) + \epsilon^2 (\Delta_{d^{0,1}} \omega, \alpha). \] (4)

The first term (2) was considered in Lemma 4.3. By counting the types, the terms (3), (4) are zero. We only need to treat the remaining term (5).
Note that
\[(\Delta_{\partial^0} \omega, \alpha) = (D_v \alpha, D_v \alpha) + \int (\text{terms of the form } (D_v \omega, \alpha))\]

By the Schwartz inequality and the fact that \(Q\) (and \(Q^*\)) is a first order operator,
\[\epsilon(Q \omega, \alpha) = \epsilon(\omega, Q^* \alpha) > -\frac{\epsilon^2}{2} (\|\alpha\|_{H^1}^2 + \|D_v \alpha\|_{L^2}^2) - M\|\omega\|_{L^2}^2.\]

Substituting these inequalities into eq. (2)-(5), we prove the theorem.

### 4.2 The case of \(m, (m + 1)\)-forms.

If \(k = m\), the estimate for the derivatives of \(\alpha\) in Theorem 4.1 breaks down. So we need a different method to do the estimate.

Suppose \(\omega = \alpha + \beta, \alpha \in \Omega^{m,0}, \beta \in \Omega^{m-1,1}\), satisfies
\[
\begin{align*}
&d\epsilon \omega = \xi_1, \\
&d^*\epsilon \omega = \xi_2.
\end{align*}
\]
Let \(\gamma_1 = D_v \alpha, \gamma_2 = D_v \beta, \gamma = \gamma_1 + \gamma_2\). We first estimate the second order derivatives of \(\beta\).

Note that \(D_v\) commutes with \(d^{2-1}\) and with \((d^{2-1})^*\) modulo zero-order operators.

Thus, taking the derivative \(D_v\) of the eqs. (3), (4), one obtains
\[
\begin{align*}
&d\epsilon \gamma = D_v \xi_1 + (\text{0-order operator in powers of } \epsilon \text{ in powers of } \omega) \\
&\quad + \frac{1}{\epsilon} (\text{0-order operator in powers of } \epsilon) \omega, \\
&d^*\epsilon \gamma = D_v \xi_2 + (\text{0-order operator in powers of } \epsilon \text{ in powers of } \omega) \\
&\quad + \frac{1}{\epsilon} (\text{0-order operator in powers of } \epsilon) \omega.
\end{align*}
\]

Integrating
\[\int_M (\Delta_{\partial^0} \omega, \alpha),\]
as in the proof of Theorem 1.1, using the facts that \(\omega, \gamma_2\) are uniformly bounded in the \(L^2\)-norm (Theorem 4.1), one obtains
Lemma 4.4 Suppose \( k = m \). If \( \omega = \alpha + \beta \) satisfies eqs. (6), (7), then
\[
\|Dv\beta\|_{H^1}^2 \leq C\left( \frac{1}{\varepsilon^2} \|\omega\|_{L^2}^2 + \|\beta\|_{L^2}^2 + \|Dv\xi\|_{L^2}^2 \right).
\]

Now we estimate \( \|\alpha\|_{H^2} \). For this purpose we need to decompose \( \alpha = \alpha_1 + L\alpha_2 \), since \( \Omega^{m,0} = R^m \oplus L(\Omega^{m-2,0}) \), where \( \alpha_1 \in R^m \), \( L : \Omega^{m-2,0} \rightarrow \Omega^m \) is defined by \( L\alpha_2 = \alpha_2 \wedge d\xi \), and \( \alpha_1, L\alpha_2 \) are orthogonal. We will estimate the derivatives of \( \alpha_1 \) and \( L\alpha_2 \) separately. We first estimate the first order derivatives of \( L\alpha_2 \).

Lemma 4.5
\[
\|L\alpha_2\|_{H^2}^2 + \frac{1}{\varepsilon^2} \|(d^{2-1})^* \alpha_2\|_{L^2}^2 \leq C(\|\xi\|_{L^2}^2 + \|\omega\|_{L^2}^2).
\]

Proof. This is proved by integrating
\[
\int (L\alpha_2, \Delta_{d^\varepsilon} \omega)
\]
as in the proof of Theorem 4.1. The key point is that \( \Delta_{d^\varepsilon} \) is sub-elliptic on \( L(\Omega^{m-2,0}) \).

In fact, one has the following estimate which improves over that in (11)
\[
\int_M (e^i \wedge i(e_j)(D_{e_i}D_{e_j} - D_{e_j}D_{e_i})L\alpha_2, L\alpha_2)
= 2\frac{m}{m} \int_M (i(e_i)D_{e_{i+m}}L\alpha_2, i(e_{i+m})D_{e_i}L\alpha_2 - (i(e_i)D_{e_j}L\alpha_2, i(e_{i+m})D_{e_{j+m}}L\alpha_2)
+ \text{terms of the form } (D_{e_i}L\alpha_2, L\alpha_2)
\leq \frac{m-1}{m} \|L\alpha_2\|_{H^1}^2 + \text{terms of the form } (D_{e_i}(L\alpha_2), L\alpha_2),
\]
and hence one has
\[
(\Delta_{d^\varepsilon} \omega, L\alpha_2) \geq \frac{1}{\varepsilon^2} \|(d^{2-1})^* \alpha_2\|_{L^2}^2 + \frac{1}{m} C \|L\alpha_2\|_{H^1}^2 - M(\omega, \omega).
\]
We now estimate the second order derivatives of $L\alpha_{2}$. Let $u = u_{i}$ be as in Lemma 3.2, $i = 1, 2, \cdots, 2m + 1$. Take the Lie derivative of eqs. (6), (7) with respect to $u$, one has

$$d_{\epsilon} \varphi = \mathcal{L}_{u} \xi_{1} + (0\text{-order operator in powers of } \epsilon) \omega,$$

$$d_{\epsilon}^{*} \varphi = \mathcal{L}_{u} \xi_{2} + (0\text{-order operator in powers of } \epsilon) \omega,$$

where $\varphi := \mathcal{L}_{u} \omega$. Note that $\mathcal{L}_{u_{i}}$ preserves the decomposition $\Omega^{m,0} = \mathcal{R}^{m} \oplus L(\Omega^{m-2,0})$, hence, applying the same arguments as in the proof of Lemma 4.5 to the above equations, one obtains

**Lemma 4.6**

$$\|\mathcal{L}_{u_{i}} L\alpha_{2}\|_{H^{1}}^{2} \leq C(\|\mathcal{L}_{u_{i}} \xi\|_{L^{2}}^{2} + \|\xi\|_{L^{2}}^{2} + \|\omega\|_{L^{2}}^{2}), \quad i = 1, 2, \cdots, 2m + 1.$$

Then we have the following estimate on the second order derivatives of $L\alpha_{2}$.

**Corollary 4.7**

$$\sum_{i=1}^{2m} \|D_{e_{i}} L\alpha_{2}\|_{H^{1}}^{2} \leq C(\|\xi\|_{H^{1}}^{2} + \|\xi\|_{L^{2}}^{2} + \|\omega\|_{L^{2}}^{2}).$$

**Proof.** This follows from Lemma 4.6. In fact, by Lemma 4.6, every $x_{0} \in M$ has a neighborhood $U$ such that

$$\sum_{i=1}^{2m} \|D_{e_{i}} L\alpha_{2}\|_{H^{1}(U)}^{2} \leq C(\|\xi\|_{H^{1}}^{2} + \|\xi\|_{L^{2}}^{2} + \|\omega\|_{L^{2}}^{2} + \|L\alpha_{2}\|_{H^{1}}^{2}).$$

Then the corollary follows from a partition of unity and Lemma 4.3.

Now we estimate the derivatives of $\alpha_{1}$. Eliminating $\beta$ from eq. (6) by the fact that $d^{2-1}: \Omega^{m-2,1} \to \Omega^{m+1,0}$ is an isomorphism, one has

$$d_{R} \alpha_{1} = \pi((d^{2-1})^{-1} \xi_{1}^{1} + \frac{1}{\epsilon} \xi_{1}^{2}) - d_{R} L\alpha_{2}, \quad (8)$$
where $\xi_1 = \xi^1_1 + \xi^2_1, \xi^i \in \Omega^{m-i-1,i-1}, i = 1, 2$. From the $(1, 0)$-component of eq. (7) one obtains

$$d^{1,0}(d^{1,0})^* \alpha_1 = d^{1,0} (-\epsilon (d^{0,1})^* \beta + \xi^1_2) - d^{1,0}(d^{1,0})^* L \alpha_2,$$

(9)

where $\xi_2$ is decomposed into $\xi^1_2 + \xi^2_2$ as for $\xi^1_1$. By Rumin, $d^*_R d_R + (d^{1,0}d^{1,0})^2$ is hypoelliptic on $\mathcal{R}^m$. (However, note that $d^*_R d_R + (d^{1,0}d^{1,0})^2$ is not hypoelliptic on $L(\Omega^{m-2,0})$, which is the reason why we decompose $\alpha$. ) Hence, from the eqs. (8)-(9), plus the following estimates

$$\|d_R L \alpha_2\|_{L^2} \leq C(\sum_{i=1}^{2m} \|D_{e_i} L \alpha_2\|_{H^1} + \|L \alpha_2\|_{L^2}),$$

$$\|d^{1,0}(d^{1,0})^* L \alpha_2\|_{L^2} \leq C(\sum_{i=1}^{2m} \|D_{e_i} L \alpha_2\|_{H^1} + \|L \alpha_2\|_{L^2}),$$

which can be controlled by using Corollary 4.7, and

$$\|d^{1,0}(d^{0,1})^* \beta\|_{L^2} \leq C(\|D_v \beta\|_{H^1_k}^2 + \|\beta\|_{H^1_k}^2 + \|\beta\|_{L^2}^2),$$

which can be controlled by using Lemma 4.4, one obtains

Theorem 4.8 If $\omega = \alpha + \beta$ satisfies eqs. (7), (7), then

$$\|\alpha\|_{H^2_k}^2 \leq C(\|\omega\|_{L^2}^2 + \|\xi\|_{H^1}^2 + \frac{1}{\epsilon^2} \|\xi\|_{L^2}^2).$$

5 Proof of Theorem 2.1.

Much of the proof depends on the properties of $d^{2,-1}$, which we list now. These properties follow from a straight forward computation.

Lemma 5.1 (1) $d^{2,-1} : \Omega^{k-2,1} \to \Omega^{k,0}$ is an injection for $k \leq m - 1$, an isomorphism for $k = m$.

(2) $(d^{2,-1})^* : \Omega^{k,0} \to \Omega^{k-2,1}$ is an injection for $k \geq m + 2$, an isomorphism for $k = m + 1$. 

12
Let \( \omega = \alpha + \beta, \alpha \in \Omega^{k,0}, \beta \in \Omega^{k-1,1} \) be as in Theorem 2.1. Then \( d_{\epsilon} \omega = (d_{\epsilon})^* \omega = 0 \) is equivalent to

\[
\begin{align*}
\frac{1}{\epsilon} d^{2,-1} \beta + d^{1,0} \alpha &= 0, \\
& (10) \\
d^{1,0} \beta + \epsilon d^{0,1} \alpha &= 0, \\
& (11) \\
\frac{1}{\epsilon} (d^{2,-1})^* \alpha + (d^{1,0})^* \beta &= 0, \\
& (12) \\
(d^{1,0})^* \alpha + \epsilon (d^{0,1})^* \beta &= 0. \\
& (13)
\end{align*}
\]

**Lemma 5.2** Suppose \( Q \) is a first-order differential operator. If \( \| \omega_\epsilon \|_{L^2}, \| \epsilon Q \omega_\epsilon \|_{L^2} \) are uniformly bounded, then

\[ \epsilon Q \omega_\epsilon \to 0, \quad \text{weakly in } L^2. \]

**Proof.** We may choose a sub-sequence such that

\[ \epsilon Q \omega_\epsilon \to \gamma, \quad \text{weakly in } L^2. \]

We need only to prove \( \gamma = 0 \). Assume without loss of generality that \( \omega \to a \) weakly in \( L^2 \). Now choose a smooth \( k \)-form \( \mu \), then

\[ (\gamma, \mu)_{L^2} = \lim_{\epsilon \to 0} (\epsilon Q \omega_\epsilon, \mu)_{L^2} = \lim_{\epsilon \to 0} \epsilon (\omega_\epsilon, (Q)^* \mu)_{L^2} = 0, \]

so \( \gamma = 0 \).

**Proof of Theorem 2.1.** We will only prove the theorem for the case \( k \leq m \), as the case \( k > m \) is similar. We divide the proof into two cases: one for \( k \leq m - 1 \), the other \( k = m \).

(1) \( k \leq m - 1 \). By Theorem [T.1] we observe that both \( \alpha \) and \( \beta \) are uniformly bounded in \( H^1_\epsilon \), and \( \epsilon^{-1} d^{2,-1} \beta \) is uniformly bounded in \( L^2 \). By Lemma 5.1, this implies that \( \alpha \) converges to \( \alpha_0 \), after passing to a subsequence if necessary, and \( \beta \) to 0 strongly in \( L^2 \).
By Theorem 4.1, \( \epsilon \| d^{0,1} \beta \|_{L^2} \) is bounded. Then by Lemma 5.2, \( \epsilon d^{0,1} \beta \to 0 \) weakly in \( L^2 \). Then, from eq. (11), it follows that \( \alpha_0 \) satisfies

\[ d^{1,0} \alpha_0 = 0 \]

in the weak sense that for any \( \mu \in H^1_c \), \( (\alpha_0, (d^{1,0})^* \mu)_{L^2} = 0 \).

Similarly, from eq. (12) and Lemma 5.2 we have

\[ (d^{1,0})^* \alpha_0 = 0, \quad (d^{2,-1})^* \alpha_0 = 0 \]

in the weak sense. Now the theory of sub-elliptic operators implies that \( \alpha_0 \) is smooth and satisfies the Rumin’s Laplacian ( cf. Helffer-Nourrigat [12] ).

To conclude the proof, we note that \( \alpha_0 \neq 0 \), as \( \omega \) converges to \( \alpha_0 \) strongly in \( L^2 \), and \( \| \omega \|_{L^2} = 1 \). This proves the theorem for \( k < m \).

(2) If \( k = m \), then it follows from Theorem 4.1, Theorem 4.8, that \( \| \alpha \|_{H^2_c}, \| \beta \|_{H^1_c} \) are uniformly bounded. Moreover, as in the case \( k \leq m - 1 \), \( \beta \to 0 \) weakly in \( H^1_c \).

We may choose a subsequence of \( \alpha \) such that \( \alpha \to \alpha_0 \) weakly in \( H^2_c \).

It follows from eqs. (10), (11), that

\[ d_R \alpha = 0. \]

Hence \( \alpha_0 \) also satisfies the above equation in the weak sense that for any \( \mu \in H^2_c \), \( (\alpha_0, (d_R)^* \mu)_{L^2} = 0 \).

Now that \( \epsilon (d^{1,0})^* \beta, \epsilon (d^{0,1})^* \beta \) are uniformly bounded in \( L^2 \), by Lemma 5.2, \( \epsilon (d^{1,0})^* \beta \to 0, \epsilon (d^{0,1})^* \beta \to 0 \) weakly in \( L^2 \). Then it follows from eqs. (12), (13) that

\[ (d^{1,0})^* \alpha_0 = 0, \quad (d^{2,-1})^* \alpha_0 = 0 \]

in the weak sense. Thus \( \alpha_0 \) is a Rumin’s harmonic form.

At last note that since \( \omega \to \alpha_0 \) strongly in \( L^2 \), \( \alpha_0 \neq 0 \). This proves the theorem.
6 Proof of Theorem 2.2.

As before, we only consider the case $k \leq m$, as the case $k > m$ is similar.

First note that the equations $d_\omega = \xi_1$, $d_\omega = \xi_2$, are equivalent to

\begin{align}
\frac{1}{\epsilon} d_1^{2-1} \beta + d_1^{0} \alpha &= \xi_1^1, \\
\frac{1}{\epsilon} d_1^{1} \beta + \epsilon d_0^{0} \alpha &= \xi_1^2, \\
\frac{1}{\epsilon} (d_2^{2-1})^* \alpha + (d_1^{1})^* \beta &= \xi_2^1, \\
(d_1^{1})^* \alpha + \epsilon (d_0^{0})^* \beta &= \xi_2^2,
\end{align}

where $\xi_i = \xi_i^1 + \xi_i^2$, $\xi_i^j \in \Omega^{*,j-1}$.

(i) $l = 1$. Let $\omega_\epsilon = \alpha_\epsilon + \beta_\epsilon \in E_1^1$. We will study the limit of $\omega_\epsilon$ as $\epsilon \to 0$.

Suppose $k \leq m - 1$. By Theorem 4.1, we see that $\|\omega_\epsilon\|_{H_1^1}$, $\|\epsilon^{-1}(d_1^{2-1})^* \alpha_\epsilon\|_{L^2}$ and $\|\epsilon^{-1} d_2^{2-1} \beta_\epsilon\|_{L^2}$ are uniformly bounded. As in the proof of Theorem 2.1, this implies that after passing to a subsequence, $\omega \to \omega_0$ weakly in $H_1^1$, where $\omega_0 \in \Omega^{k,0} \cap Ker((d_2^{2-1})^*) = \mathcal{R}^k$. So $\bar{E}_1^1 \subset \mathcal{R}^k$ for $k \leq m - 1$.

We will prove that this inclusion relation also holds for $k = m$. If $k = m$, it follows from Theorem 4.1 that $\|\beta_\epsilon\|_{H_1^1}$, $\|\epsilon^{-1} d_2^{2-1} \beta_\epsilon\|_{L^2}$ are uniformly bounded. So $\beta_\epsilon \to 0$ weakly in $H_1^1$ by Lemma 5.1.

Also, since $\|\epsilon (d_1^{1})^* \beta_\epsilon\|_{L^2}$ is uniformly bounded (Theorem 4.1), by Lemma 5.2 $\epsilon (d_1^{1})^* \beta_\epsilon \to 0$ weakly in $L^2$. It follows from eq. (16) that the weak limit $\omega_0$ of $\omega_\epsilon$ in $L^2$ satisfies

\[(d_2^{2-1})^* \omega_0 = 0.\]

So $\bar{E}_1^1 \subset \mathcal{R}^m$.

Conversely, we will prove $\mathcal{R}^k \subset \bar{E}_k^1$ for $k \leq m$. This follows from the fact that if $\alpha_0 \in \Omega^{k,0}/Im(d_2^{2-1})$, $\|\alpha_0\|_{L^2} = 1$, then obviously $\alpha_0 \in E_k^1$ and hence $\alpha_0 \in \bar{E}_k^1$.

(ii) $l = 2$. The proof for the case $k \neq m$ is similar to that of Theorem 2.1 and will be omitted here.

Consider the case $k = m$. By Theorem 1.1, Theorem 1.8, $\|\alpha_\epsilon\|_{H_1^2}$, $\|\beta_\epsilon\|_{H_1^2}$ and $\|\epsilon^{-1} d_2^{2-1} \beta_\epsilon\|_{L^2}$ are uniformly bounded. Let $\alpha_0, \beta_0$ be the weaks limit of $\alpha_\epsilon, \beta_\epsilon$ in $H_1^2$,
$H^1_l$ respectively. First note that, as in the case $l = 1$, the weak limit $\omega_0 = \alpha_0 + \beta_0 \in R^m$.

Moreover, by Lemma 5.2, $\epsilon(d^{0,1})^* \beta_\epsilon \to 0$ weakly in $L^2$. So it follows from eq. (17) that $(d^{1,0})^* \alpha_0 = 0$. Thus $\bar{E}_m^2 \subset R^m \cap \text{Ker}(d^{1,0})^*$.

Conversely, if $\alpha_0 \in R^m \cap \text{Ker}(d^{1,0})^*$, then obviously $\alpha_0 \in \bar{E}_m^2$. So $R^m \cap \text{Ker}(d^{1,0})^* \subset \bar{E}_m^2$. This proves the case $l = 2$.

(iii) $l = 3$. The proof in this case is similar to that of Theorem 2.1 and will be omitted.

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