Implications of CP asymmetry parameter $\sin 2\beta$ on structural features of texture specific mass matrices

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Abstract

In the context of Fritzsch-like texture 4 zero Hermitian quark mass matrices, we have attempted to find an ‘exact’ formula for $\sin 2\beta$ wherein the dependence of $\beta$ on the quark masses and the elements of the quark mass matrices is visible in a simple and clear manner. This has been achieved keeping in mind the strong hierarchy of the quark masses and assuming the weak hierarchy amongst the elements of the mass matrices. This ‘exact’ formula represents a vast improvement over the leading order formula based on strong hierarchy of the elements of the mass matrices. Apart from showing the compatibility of texture 4 zero mass matrices with the present value of $\sin 2\beta$ and other CKM parameters, we find interesting conclusions regarding the structural features of the mass matrices.

Keywords: Mass matrices, Quark mixing, CP asymmetry parameter $\sin 2\beta$.
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1 Introduction

The last few years have seen a precise measurement of $\sin 2\beta$, characterizing CP asymmetry $a_{\psi K_s}$ in the $B_d^0 \rightarrow \psi K_s$ decay, as well as of other well measured Cabibbo-Kobayashi-Maskawa (CKM) [1] matrix elements $V_{us}, V_{cb}, V_{ud}$. Based on these precise values, several phenomenological analyses [2]-[5] have allowed us to conclude that the single CKM phase looks to be a viable solution of CP violation not only in the case of K-decays but also in the context of B-decays, at least to the leading order. Interestingly, it has also been shown [6] that the present value of $\sin 2\beta$ along with unitarity and other well measured CKM parameters leads to an almost precise value of $V_{ub}$ and CP violating phase $\delta$, two
important parameters yet to be measured precisely. Keeping in mind that the parameter $\sin 2\beta$ also provides vital clues to the structural features of texture specific mass matrices, comprising of hierarchy and phases of the elements of the mass matrices, several authors [7]-[14] have explored its implications for these. In particular, using assumption of ‘strong hierarchy’ of the elements of the mass matrix, having its motivation in the hierarchy of the quark mixing angles, the following leading order relationships between the various elements of the mixing matrix and quark masses have been obtained in [7]-[13],

$$
\frac{|V_{ub}|}{|V_{cb}|} = \sqrt{\frac{m_u}{m_c}}, \quad \frac{|V_{td}|}{|V_{ts}|} = \sqrt{\frac{m_d}{m_s}}, \quad |V_{us}| = \sqrt{\frac{m_d}{m_s} e^{i\phi} - \frac{m_u}{m_c}}. \quad (1)
$$


Following Particle Data Group (PDG) [2] definition, these further give the expression for $\beta$ in the ‘strong hierarchy’ case, e.g.,

$$
\beta \equiv \arg \left[ -\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*} \right] = \arg \left[ 1 - \sqrt{\frac{m_u m_s}{m_c m_d}} e^{-i\phi} \right]. \quad (2)
$$

Unfortunately, the value of $\sin 2\beta$ predicted by the above formula is in quite disagreement with its present precisely known value. In particular, with the present values of input quark masses and by giving full variation to phase $\phi$, the maximum value of $\sin 2\beta$ comes out to be 0.5, which is in sharp conflict with its present PDG 2008 [2] value 0.681 ± 0.025. Attempts have been made to resolve this conflict [11]- [13], however without getting into detailed and comprehensive analysis of the issues involved in the formulation of the above equations. Further, it may be added that this issue escapes explicit attention of some of the recent analyses [15, 16]. Therefore, a closer look at the whole issue is very much desirable.

It may be further noted that Fritzsch-like texture 4 zero mass matrices are known to be compatible with specific models of GUTs [14, 17, 18], Abelian family symmetries [19], as well as describe the neutrino oscillation data quite well [20]. It may also be kept in mind that the mixing patterns of quarks and neutrinos are quite different, e.g., in the case of neutrinos, neither the mixing angles nor the neutrino masses show any hierarchy, this being in sharp contrast to the distinct hierarchy shown by quark masses and mixing angles. These distinct features of quark and neutrino mixings may have a constraining effect on mass matrices, particularly in case if these have a common origin due to quark lepton unification hypothesis [21]. In this context, the question of compatibility of texture 4 zero Fritzsch-like mass matrices with precisely known $\sin 2\beta$ warrants a closer scrutiny as it may provide vital clues regarding the structural features, including the hierarchy and the phase structure of the elements of the mass matrices.

Before reaching at any firm conclusion in this regard, we have to address several issues. A precisely known $\sin 2\beta$ perhaps requires a careful look at the relations arrived at in equations (1) and (2), which were derived in the absence of precise knowledge regarding the parameter $\sin 2\beta$. It seems that while arriving at the above relations one of the key assumption used is that hierarchical mixing angles are reproduced by ‘strongly hierarchical’ mass matrices, having bearing on the structural features of the mass matrices. This
assumption requires a careful and detailed scrutiny in the era of precision measurements of CKM parameters.

In this context, one may note that the hierarchy followed by mixing angles is $s_{13} < s_{23} < s_{12}$, whereas the quark masses follow a somewhat stronger hierarchy, e.g., $m_u \ll m_c \ll m_t$ and $m_d < m_s \ll m_b$. The situation gets further complicated when one notes that the mixing angles are not proportional to the quark masses, in fact as can be seen from equation (1), they involve square roots of the ratios of the masses with certain phase factors between the up and down quark sectors. This suggests that in case we have to deal with a precisely known $\sin 2\beta$, then one has to be careful in invoking a particular hierarchy between the elements of the mass matrices. Further, in view of the dependence of $\sin 2\beta$ on the ratios of the small quark masses, it becomes essential to find an exact relationship involving small quark masses as well as the phases involved in the mass matrices which have bearing on the hierarchy of the elements of the mass matrices. Furthermore, one may wonder whether only one phase is sufficient to support the data or one requires two phases as are available in the case of texture 4 zero Fritzsch-like Hermitian mass matrices.

From an analysis of texture 4 zero mass matrices it is very easy to check that the issues raised above are not easy to answer, e.g., in case we relax the ‘strong hierarchy’ conditions on the elements of the mass matrices then this immediately makes the task of relating the CKM matrix elements to the ratio of masses, given in equation (1), a complicated affair. In principle, it is an easy task to diagonalize exactly the texture 4 zero Fritzsch-like Hermitian mass matrices, however in practice the expressions of the CKM matrix elements so obtained are quite lengthy, resulting in an expression for $\beta$ having complicated dependence on quark masses, phases and free parameters of quark mass matrices. Such an expression, although exact, does not allow an insight into the role of phases, hierarchy of the elements of the mass matrices or on the contributions of the non leading terms. The first step in this direction is to develop a formula for $\sin 2\beta$ which allows one to not only go beyond the leading order but also to study the structural features, e.g., the phases and the hierarchy of the elements of the mass matrices.

The purpose of the present paper on the one hand is to develop an exact expression for $\sin 2\beta$ in terms of quark masses, phases and free parameters of the texture specific mass matrices. On the other hand, we would like to study, in detail, the implications of such a formula on the compatibility of texture 4 zero mass matrices with $\sin 2\beta$. In particular, keeping in mind the recent refinements of the CKM matrix elements, we would like to investigate in detail the implications of the exact formula on the structural features of the mass matrices such as the hierarchy of the elements of the mass matrices and their phase structures. We would also like to examine the relationship between the earlier formula of $\sin 2\beta$ and the present one derived here.

The detailed plan of the paper is as follows. In Section (2), we detail the essentials of the formalism regarding the texture specific mass matrices as well as the derivation of the formula for $\sin 2\beta$. Inputs used in the present analysis and the discussion of the calculations and results have been given in Section (3). Finally, Section (4) summarizes our conclusions.
2 Texture specific mass matrices and the formula for \( \sin 2\beta \)

To fix the notations and conventions as well as to facilitate the understanding of the relationship of the present work with the earlier attempts, we detail some of the essentials of the formalism. To begin with, we define the modified Fritzsch-like matrices, e.g.,

\[
M_i = \begin{pmatrix}
0 & A_i & 0 \\
A_i^* & D_i & B_i \\
0 & B_i^* & C_i
\end{pmatrix}, \quad i = U, D,
\]

(3)

\( M_U \) and \( M_D \), respectively corresponding to the mass matrix in the up sector and the down sector. It may be noted that each of the above matrix is texture 2 zero type with \( A_i = |A_i|e^{i\alpha_i} \) and \( B_i = |B_i|e^{i\beta_i} \). The phases of the elements of the mass matrices \( A_i, B_i, C_i, D_i \) and their relative magnitudes characterize the structural features of the mass matrices. A strongly hierarchical mass matrix would imply \( |A_i| \ll |D_i| \ll |B_i| < C_i \), whereas a weaker hierarchy of the mass matrix implies \( |A_i| < D_i \ll |B_i| \ll C_i \). For the purpose of numerical work, one can conveniently take the ratio \( D_i/C_i \sim 0.01 \) characterizing strong hierarchy whereas \( D_i/C_i \gtrsim 0.1 \) implying weak hierarchy. This can be understood by expressing these parameters in terms of the quark masses, in particular \( D_U/C_U \sim 0.01 \) implies \( C_U \sim m_t \) and \( D_D/C_D \sim 0.01 \) leads to \( C_D \sim m_b \).

The mass matrices \( M_U \) and \( M_D \) can be exactly diagonalized, for details we refer the reader to [22]. To facilitate diagonalization, the complex matrix \( M_i \) can be expressed in terms of the real matrix \( M_i^r \) which can be diagonalized by the orthogonal transformation, for example,

\[ M_i^{\text{diag}} = O_i^T M_i^r O_i, \]

(4)

where

\[ M_i^{\text{diag}} = \text{diag}(m_1, -m_2, m_3), \]

(5)

the subscripts 1, 2 and 3 referring respectively to \( u, c \) and \( t \) for the up sector and \( d, s \) and \( b \) for the down sector. The negative sign before \( m_2 \) is only for the convenience of calculations, without having physical significance.

The exact diagonalizing transformation \( O_i \) is expressed as

\[
O_i = \begin{pmatrix}
\sqrt{\frac{m_2 m_3 (C_i - m_1)}{(m_3 - m_1)(m_2 + m_1)C_i}} & \sqrt{\frac{m_2 m_3 (C_i + m_2)}{C_i(m_2 + m_1)(m_3 + m_2)}} & \sqrt{\frac{m_1 m_2 (m_3 - C_i)}{C_i(m_3 + m_2)(m_3 - m_1)}} \\
\sqrt{\frac{m_1 (C_i - m_1)}{(m_3 - m_1)(m_2 + m_1)}} & -\sqrt{\frac{m_2 (C_i + m_2)}{(m_3 + m_2)(m_3 - m_1)}} & \sqrt{\frac{m_3 (C_i - m_1)}{C_i(m_3 + m_2)(m_3 - m_1)}} \\
-\sqrt{\frac{m_1 (m_3 - C_i)(C_i + m_2)}{C_i(m_3 - m_1)(m_2 + m_1)}} & \sqrt{\frac{m_2 (C_i - m_1)(m_3 - C_i)}{C_i(m_3 + m_2)(m_2 + m_1)}} & \sqrt{\frac{m_3 (C_i - m_1)(C_i + m_2)}{C_i(m_3 + m_2)(m_3 - m_1)}}
\end{pmatrix}.
\]

(6)

It may be noted that while finding the above diagonalizing transformation \( O_i \) one has the freedom to choose several equivalent possibilities of phases. Similarly, while
normalizing the diagonalized matrix to quark masses, one again has the freedom to choose the phases for the quark masses. Out of these several possibilities, we arrive at the above mentioned expression for $O_i$ by considering the phase of $m_2$ to be negative, facilitating the diagonalization process as well as the construction of the CKM matrix.

The CKM mixing matrix $V_{\text{CKM}}$ which measures the non-trivial mismatch between diagonalizations of $M_U$ and $M_D$ can be obtained using $O_{U(D)}$ through the relation

$$V_{\text{CKM}} = O_U^T P_U P_D^\dagger O_D.$$

Explicitly, the elements of the CKM mixing matrix can be expressed as

$$V_{lm} = O_{l1} O_{1m} e^{-i\phi_1} + O_{l2} O_{2m} + O_{3m} e^{i\phi_2},$$

where the subscripts $l$ and $m$ run respectively over $u, c, t$ and $d, s, b$ and $\phi_1 = \alpha_U - \alpha_D$, $\phi_2 = \beta_U - \beta_D$.

Using the above equation, the elements of the CKM mixing matrix can be easily found, e.g.,

$$V_{cd} = \sqrt{\frac{m_u m_t (-D_u + m_u + m_t)}{C_u (m_u + m_c) (m_c + m_t)}} \frac{m_s m_b (-D_d + m_b - m_s)}{C_d (m_b - m_d) (m_s + m_d)} e^{-i\phi_1}$$

$$- \sqrt{\frac{m_c (-D_u + m_u + m_t)}{(m_u + m_c) (m_c + m_t)}} \frac{m_d (-D_d + m_b - m_s)}{(m_b - m_d) (m_s + m_d)}$$

$$- \sqrt{\frac{m_c (-D_u + m_t - m_c) (D_u - m_u + m_c)}{C_u (m_u + m_c) (m_t + m_c)}} \times$$

$$\sqrt{\frac{m_d (D_d - m_d + m_s) (-D_d + m_d + m_b)}{C_d (m_b - m_d) (m_s + m_d)}} e^{i\phi_2}.$$  

The other elements $V_{cb}, V_{td}$ and $V_{tb}$, also required to be known to evaluate $\beta$, can also be obtained similarly. In case we use the above complicated expression for $V_{cd}$ as well as similar expressions of the other elements to evaluate $\sin 2\beta$, we find that these would yield a long and complicated formula from which it would be difficult to understand the implications on the phases and other parameters of the mass matrices. To derive a simple and informative formula, we first rewrite the diagonalizing transformation $O_i$ keeping in mind $m_3 \gg m_2 \gg m_1$ and the element of the mass matrix $C_i \gg m_1$, which is always valid without any dependence on the hierarchy of the elements of the mass matrices. It may be mentioned that this approximation induces less than a fraction of a percentage
error in the numerical results. The structure of \( O_i \) can be simplified and expressed as

\[
O_i = \begin{pmatrix}
1 & \zeta_{1i} \sqrt{\frac{m_t}{m_2}} & \zeta_{2i} \sqrt{\frac{m_t m_u}{m_2^2}} \\
\zeta_{3i} \sqrt{\frac{m_t}{m_2}} & -\zeta_{1i} \zeta_{3i} & \zeta_{2i} \\
-\zeta_{1i} \zeta_{2i} \sqrt{\frac{m_t}{m_2}} & \zeta_{2i} & \zeta_{1i} \zeta_{3i}
\end{pmatrix},
\tag{10}
\]

where the three parameters \( \zeta_{1i}, \zeta_{2i}, \zeta_{3i} \), with \( i \) denoting \( U \) and \( D \) are given by

\[
\zeta_{1i} = \sqrt{1 + \frac{m_2}{C_i}}, \quad \zeta_{2i} = \sqrt{1 - \frac{C_i}{m_3}}, \quad \zeta_{3i} = \sqrt{\frac{C_i}{m_3}}.
\tag{11}
\]

Making use of this equation, along with relation (8), we obtain the following elements needed to evaluate \( \beta \)

\[
V_{cd} = \zeta_{1U} \sqrt{\frac{m_u}{m_c}} e^{-i\phi_1} - \sqrt{\frac{m_d}{m_s}} \left[ \zeta_{1U} \zeta_{3U} \zeta_{3D} + \zeta_{2U} \zeta_{1D} \zeta_{2D} e^{i\phi_2} \right],
\tag{12}
\]

\[
V_{cb} = \frac{\zeta_{1U} \zeta_{2D}}{\zeta_{3D}} \sqrt{\frac{m_u m_d m_s}{m_c m_b}} e^{-i\phi_1} - \left( \zeta_{1U} \zeta_{3U} \zeta_{2D} - \zeta_{2U} \zeta_{1D} \zeta_{3D} e^{i\phi_2} \right),
\tag{13}
\]

\[
V_{td} = \frac{\zeta_{2U} \zeta_{3D}}{\zeta_{3U}} \sqrt{\frac{m_u m_c m_s}{m_t m_b}} e^{-i\phi_1} + \sqrt{\frac{m_d}{m_s}} \left[ \zeta_{2U} \zeta_{3U} \zeta_{3D} - \zeta_{1U} \zeta_{3U} \zeta_{1D} \zeta_{2D} e^{i\phi_2} \right],
\tag{14}
\]

\[
V_{tb} = \frac{\zeta_{2U} \zeta_{2D}}{\zeta_{3U} \zeta_{3D}} \sqrt{\frac{m_u m_c m_d m_s}{m_t^2 m_b^2}} e^{-i\phi_1} + \left[ \zeta_{2U} \zeta_{2D} + \zeta_{1U} \zeta_{3U} \zeta_{1D} \zeta_{2D} e^{i\phi_2} \right].
\tag{15}
\]

A general look on the above elements clearly shows that the above relations are not only more compact but also more useful to view the dependence of these CKM matrix elements on the quark masses and phases. Using these elements, after some non trivial algebra, one arrives at the following expression of \( \beta \), wherein its dependence on the quark masses and the elements of the quark mass matrices is visible in a simple and clear manner, e.g.,

\[
\beta \equiv \arg \left[ -\frac{V_{cd} V_{cb}^*}{V_{td} V_{tb}^*} \right] = \arg \left[ \left( 1 - \frac{m_u m_s}{m_c m_d} e^{-i(\phi_1 + \phi_2)} \right) \left( \frac{1 - r_2 e^{i\phi_2}}{1 - r_1 e^{i\phi_2}} \right) \right],
\tag{16}
\]

where the parameters \( r_1 \) and \( r_2 \) can be expressed in terms of the quark masses and the elements of the quark mass matrices via the relations,

\[
r_1 = \frac{\zeta_{1U} \zeta_{3U} \zeta_{1D} \zeta_{2D}}{\zeta_{2U} \zeta_{3D}} \quad \text{and} \quad r_2 = \frac{\zeta_{1U} \zeta_{3U} \zeta_{2D}}{\zeta_{2U} \zeta_{1D} \zeta_{3D}}.
\tag{17}
\]

The relationship derived by us, given in equation (16), is an ‘exact’ formula emanating from texture 4 zero mass matrices, incorporating both the phases. This formula has several
interesting aspects. Apart from clearly underlying the dependence of small quark masses and the phases \( \phi_1 \) and \( \phi_2 \), it also clearly establishes the modification of the earlier formula. It is very easy to check that the earlier formula can be easily deduced from the present one by using the strong hierarchy assumption which essentially translates to \( \zeta_{1D} \approx \zeta_{1U} \approx 1 \), further implying \( r_1 = r_2 \), leading to the term \( \left( \frac{1 - r_1 e^{i\phi_2}}{1 - r_1 e^{i\phi_1}} \right) \) becoming 1. The phase of the earlier formula can be obtained by identifying \( \phi_1 + \phi_2 \) as \( \phi \), taking values from 0 to \( 2\pi \).

It also needs to be re-emphasized that while arriving at the above ‘exact’ relationship, we have considered the hierarchy of the quark masses, e.g., \( m_t \gg m_u \) and \( m_b \gg m_d \) as well as have used \( m_3 \gg m_2 \gg m_1 \) and the element of the mass matrix \( C_i \gg m_1 \), these being valid in both weak and strong hierarchy cases. It may also be added that the formula remains valid for both the weak hierarchy of the elements of the mass matrices given by \( |A_i| < D_i \lesssim |B_i| \lesssim C_i \) as well as for the strong hierarchy assumption \( |A_i| \ll D_i \lesssim |B_i| < C_i \). Interestingly, the modification to the earlier formula contributes only when \( \phi_2 \neq 0 \), implying thereby that both the phases of the mass matrices might play an important role in achieving the agreement with data.

### 3 Calculations and results

In order to investigate the implications of the formula given in equation (16) on the structural features of the mass matrices and the CKM parameters, as a first step we find the range of \( \sin 2\beta \) predicted by the above formula by using the latest inputs. In this regard, we have adopted the following ranges of quark masses [23] at the energy scale of \( M_Z \), e.g.,

\[

t_u = 0.8 - 1.8 \text{ MeV}, \quad t_d = 1.7 - 4.2 \text{ MeV}, \quad s = 40.0 - 71.0 \text{ MeV}, \\

c = 0.6 - 0.7 \text{ GeV}, \quad b = 2.8 - 3.0 \text{ GeV}, \quad t = 169.5 - 175.5 \text{ GeV}.
\]

With these inputs and the ‘exact’ formula given in equation (16), we have evaluated \( \sin 2\beta \) by giving full variation to the phases \( \phi_1 \) and \( \phi_2 \), the parameters \( D_U \) and \( D_D \) have been given wide variation in conformity with the natural hierarchy of the elements of the mass matrices e.g., \( D_i < C_i \) for \( i = U, D \). The \( \sin 2\beta \) so evaluated comes out to be

\[
\sin 2\beta = 0.4105 - 0.7331.
\]

Interestingly, we find that the above value is inclusive of its experimental range \( 0.681 \pm 0.025 \). This clearly indicates that the exact formula which includes weak hierarchy as well as additional phase factors plays a crucial role in bringing out reconciliation between texture 4 zero mass matrices and the present precise value of \( \sin 2\beta \).

Before we get into examining the detailed implications of \( \sin 2\beta \) on structural features of the mass matrices, it is perhaps desirable to check the compatibility of texture 4 zero mass matrices with other precisely known parameters of the CKM phenomenology. To this end, along with the latest experimental value of \( \sin 2\beta \), we have imposed the following
PDG 2008 [2] constraints given by

\[ |V_{us}| = 0.2255 \pm 0.0019, \quad |V_{cb}| = (41.2 \pm 1.1)10^{-3}, \quad |V_{ub}| = 0.0035 \pm 0.0002. \quad (20) \]

As mentioned earlier, we have considered the value of $|V_{ub}|$ obtained recently [6] using only the unitarity of the elements of the CKM matrix and current sin2$\beta$ value. Keeping in mind the above mentioned constraints and using the current values of quark masses given in equation (18), we have evaluated the entire CKM matrix at 1$\sigma$ C.L. by using equations (12)-(15) and the other corresponding expressions for the remaining CKM matrix elements, e.g.,

\[ V_{CKM} = \begin{pmatrix}
0.9738 - 0.9747 & 0.2236 - 0.2274 & 0.0033 - 0.0037 \\
0.2234 - 0.2273 & 0.9729 - 0.9739 & 0.0401 - 0.0423 \\
0.0068 - 0.0103 & 0.0390 - 0.0417 & 0.9991 - 0.9992
\end{pmatrix}. \quad (21) \]

A general look at the matrix reveals that the ranges of CKM elements obtained here are quite compatible with those obtained by recent global analyses. In particular, the ranges found here are in good agreement with those emerging from global fits by PDG 2008 [2], UTfit [3], CKMfitter [4] and HFAG [5].

After having shown the compatibility of texture 4 zero mass matrices with sin2$\beta$ as well as the recent ranges of the elements of the CKM matrix, we investigate the implications of the formula derived in equation (16) on the structural features of mass matrices. In particular, we examine the constraints imposed on the ratio $D_i/C_i$ for $i = U, D$, characterizing hierarchy, as well as on the phases $\phi_1$ and $\phi_2$ of the mass matrices. As a first step, using exact relation obtained earlier, we investigate the role of hierarchy by plotting sin2$\beta$ against the ratio $D_D/C_D$ in figure 1. Several interesting conclusions follow from the graph. It can be easily noted that when $D_D/C_D < 0.02$, we are not able to reproduce any point within the 1$\sigma$ range of sin2$\beta$, even after giving full variation to all the other parameters. It may be of interest to mention that the earlier attempts [7]-[13] had considered a value of $D_B/C_B \lesssim 0.02$, thereby resulting in the incompatibility of texture 4 zero mass matrices with sin2$\beta$. From the figure it can be easily checked that only for $D_D/C_D > 0.05$, full range of sin2$\beta$ is reproduced. This clearly shows that as we deviate from strong hierarchy characterized by the ratio $D_D/C_D \sim 0.01$ towards weak hierarchy given by $D_D/C_D \gtrsim 0.1$, we are able to reproduce the results. It may be mentioned that although the graph has been plotted for $D_D/C_D$ up to 0.4, however the same pattern is followed up to $D_D/C_D \sim 0.6$, beyond which the basic structure of the mass matrix is changed. It may be added that the corresponding graph of $D_U/C_U$ is also very much similar. One would also like to emphasize that the agreement between sin2$\beta$ and higher values of $D_i/C_i$ does not spoil the overall agreement of texture 4 zero mass matrices with the CKM matrix derived earlier. This brings out an extremely important point as the conventional belief was that the hierarchical quark mixing angles can be reproduced only by strong hierarchy mass matrices. Further, we believe that this point would provide strong impetus for quark-lepton unification at the GUTs scale.

Coming to the issue of phases $\phi_1$ and $\phi_2$ of the mass matrices, it may be pointed out
that our analysis yields $\phi_1$ from $70^\circ$ - $90^\circ$ and $\phi_2$ takes values from $3^\circ$ - $9^\circ$ in order to achieve compatibility with the CKM matrix derived earlier. Most of the earlier analyses considered only one phase $\phi_1$ whereas the phase $\phi_2$ was assumed to be zero. However, as already mentioned, considering phase $\phi_2$ to be zero immediately leads to incompatibility of texture 4 zero mass matrices with $\sin 2\beta$, therefore a detailed analysis pertaining to the role of phase $\phi_2$ is in order. To this end, in figure 2 we have plotted a graph of $\sin 2\beta$ versus angle $\phi_2$. A close look at the graph reveals several interesting points. In particular, the graph clearly illustrates the crucial role played by the phase $\phi_2$ in bringing out agreement of texture 4 zero mass matrices and the precisely known $\sin 2\beta$. It is interesting to emphasize that despite giving full variation to other parameters, for $\phi_2 = 0^\circ$ we are not able to reproduce $\sin 2\beta$ within the experimental range.

After having realized the role of weak hierarchy, characterized by the ratio $D_i/C_i$, as well as the phase $\phi_2$ being non zero, for describing the present value of $\sin 2\beta$, we attempt to assess the quantitative role of these parameters in improving its value. To this end, we re-express $\beta$ as

$$\beta = \beta_1 + \beta_2,$$

with

$$\beta_1 = \tan^{-1}\left(1 - \sqrt{\frac{m_u m_s}{m_c m_d}} e^{-i(\phi_1 + \phi_2)}\right), \quad \beta_2 = \tan^{-1}\left(\frac{(\zeta_{1D}^2 - 1)r_2 \sin \phi_2}{1 + \zeta_{1D}^2 r_2^2 - (\zeta_{1D}^2 + 1)r_2 \cos \phi_2}\right).$$

It may be noted that $\beta_1$ corresponds to the contribution given by the expression 22, with $\phi_1 + \phi_2$ being identified as $\phi$, whereas $\beta_2$ represents additional contribution coming from retention of non leading terms as well as due to weak hierarchy and additional phase factors.

In Table 1, corresponding to the phase $\phi_2$ values from $0^\circ$ - $12^\circ$, we have presented values of $\sin 2\beta$, $\beta_1$ and $\beta_2$ for some typical values of $D_D/C_D$ to illustrate the role of these in achieving a quantitative fit. It may be mentioned that the purpose here is not to give a systematic interdependence of various parameters, rather to give an idea about the amount of contribution of phase $\phi_2$ and weak hierarchy towards $\sin 2\beta$. From the table, one finds that for $\phi_2 = 0^\circ$ the corresponding value of $\beta_2$ is zero, leading to a small value of $\sin 2\beta$. However, as $\phi_2$ increases up to $\sim 3^\circ$, the corresponding values of $\sin 2\beta$ are within experimental limits. As $\phi_2$ increases further up to $\sim 10^\circ$, one finds that $\sin 2\beta$ values still remain within the experimental range, this being in agreement with the observations of figure 2. One may wonder why a small change in phase $\phi_2$ leads to a relatively large contribution to $\sin 2\beta$. This can be understood from the exact relationship between the parameter $\beta_2$ and the phase $\phi_2$ given in equation 23, showing that $\beta_2$ is represented by ratio of two very small numbers, signifying that even a small change in the value of $\phi_2$ can produce reasonable contribution to $\beta_2$. Further, from the table some light is also shed on the relative importance of hierarchy and the phase $\phi_2$. In particular, one finds that corresponding to lower values of the phase $\phi_2$, the ratio $D_D/C_D$ acquires somewhat higher values as compared to the ones obtained by increasing $\phi_2$.  

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4 Summary and conclusions

To summarize, in the context of Fritzsch-like texture 4 zero Hermitian quark mass matrices, we have found an ‘exact’ formula for $\sin^2 \beta$ wherein the dependence of $\beta$ on the quark masses and the elements of the quark mass matrices is visible in a simple and clear manner. This has been done keeping in mind $m_3 \gg m_2 \gg m_1$ and $C_i \gg m_1$ as well as the weak hierarchy of the elements of the mass matrices given by $|A_i| < D_i \lesssim |B_i| \lesssim C_i$. The ‘exact’ formula found here represents a vast improvement over the usual formula based on strong hierarchy of the elements of the mass matrices. Besides clearly underlying the compatibility of texture 4 zero mass matrices in the case of weak hierarchy, the formula also explains why in the strong hierarchy case we are unable to obtain the value of $\sin 2\beta$.

A detailed analysis based on the present formula as well as by using other well measured CKM matrix elements shows that the texture 4 zero Hermitian mass matrices are compatible with recent results emerging from global fits by PDG 2008 [2], UTfit [3], CKMfitter [4] and HFAG [5]. Further, the formula clearly provides a detailed insight into the phase structure and the hierarchy of the elements of the mass matrices. In fact, we find that both the phases $\phi_1$ and $\phi_2$ are required to fit the data, with $\phi_1$ ranging from 70° - 90°, whereas $\phi_2$ taking values from 3° - 9°. In conclusion, we would like to state that we can reproduce hierarchical mixing angles even with weakly hierarchical mass matrices which may have vital implications for quark-lepton unification hypothesis.

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Figure 1: Plot showing variation of CP violating parameter $\sin 2\beta$ versus hierarchy characterizing ratio $D_D/C_D$

Figure 2: Plot showing variation of CP violating parameter $\sin 2\beta$ versus the phase $\phi_2$
Table 1: Some of the values of $\sin 2\beta$, $\beta_1$ and $\beta_2$ obtained by varying $\phi_2$ from $0^\circ - 12^\circ$. The angles $\beta_1$ and $\beta_2$ are in degrees.

| $\phi_2$ | $D_D/C_D$ | $\beta_1$ | $\beta_2$ | $\sin 2(\beta_1 + \beta_2)$ |
|----------|-----------|-----------|-----------|-----------------------------|
| 0        | 0.6649    | 12.12     | 0.00      | 0.4106                      |
| 1        | 0.3506    | 12.78     | 3.90      | 0.5499                      |
| 2        | 0.2666    | 12.73     | 6.40      | 0.6192                      |
| 3        | 0.2179    | 12.68     | 7.92      | 0.6587                      |
| 4        | 0.1728    | 12.62     | 8.30      | 0.6671                      |
| 5        | 0.1538    | 12.56     | 8.74      | 0.6769                      |
| 6        | 0.1309    | 12.49     | 9.17      | 0.6861                      |
| 7        | 0.1134    | 12.44     | 9.52      | 0.6937                      |
| 8        | 0.0717    | 13.01     | 9.31      | 0.7026                      |
| 9        | 0.0559    | 13.03     | 9.06      | 0.6969                      |
| 10       | 0.0404    | 12.95     | 8.03      | 0.6686                      |
| 11       | 0.0327    | 12.85     | 6.75      | 0.6320                      |
| 12       | 0.0216    | 12.77     | 6.10      | 0.6121                      |