A COMPLETELY MONOTONIC FUNCTION INVOLVING
DIVIDED DIFFERENCES OF PSI AND POLYGAMMA
FUNCTIONS AND AN APPLICATION

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Abstract. A function involving the divided differences of the psi function and
the polygamma functions is proved to be completely monotonic. As an appli-
cation of this result, the monotonicity and convexity of a function originated
from establishing the best upper and lower bounds in Kershaw’s inequality is
deduced.

1. Introduction

Recall [5] that a function \( f \) is said to be completely monotonic on an interval \( I \)
if \( f \) has derivatives of all orders on \( I \) and \( (-1)^nf^{(n)}(x) \geq 0 \) for \( x \in I \) and \( n \geq 0 \).
For information about the history, applications and recent developments on the
completely monotonic function, please refer to the expository article [5] and the
references therein.

The Kershaw’s inequality [4] states that

\[
\left( x + \frac{s}{2} \right)^{1-s} \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left( x - \frac{1}{2} + \sqrt{s + \frac{1}{4}} \right)^{1-s},
\]

for \( 0 < s < 1 \) and \( x \geq 1 \), where \( \Gamma \) denotes the classical Euler’s gamma function and
the middle term in (1) is a special case of the Wallis’ function \( \frac{\Gamma(x+p)}{\Gamma(x+q)} \) for \( x + p > 0 \)
and \( x + q > 0 \). It is clear that inequality (1) can be rearranged as

\[
\frac{s}{2} < \frac{\Gamma(x+1)}{\Gamma(x+s)}^{1/(1-s)} - \frac{1}{4} - \frac{1}{2}.
\]

Let \( s \) and \( t \) be nonnegative numbers and \( \alpha = \min\{s, t\} \). Define

\[
z_{s,t}(x) = \begin{cases} 
\frac{\Gamma(x+t)}{\Gamma(x+s)}^{1/(t-s)} - x, & s \neq t \\
e^{\psi(x+s)} - x, & s = t
\end{cases}
\]

in \( x \in (-\alpha, \infty) \). Standard differentiating and simplifying yields

\[
z'_{s,t}(x) = \left[ z_{s,t}(x) + x \right] \frac{\psi(x+t) - \psi(x+s)}{t-s} - 1,
\]

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is complete.

The functions \( \Theta_{s,t} \) are completely monotonic in \( (s,t) \). It is clear from (5) and (6) that

\[
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\]

Note that, among other things, the positivity of the function \( \Delta_{0,0}(x) = [\psi'(x)]^2 + \psi''(x) \) in (8) has been verified in [1].

As a straightforward application of Theorem 1, the monotonicity and convexity of the function \( z_{s,t}(x) \) is obtained.

**Theorem 2** ([2] [3] [7]). The function \( z_{s,t}(x) \) in \( (-\alpha, \infty) \) is either convex and decreasing for \(|t-s| < 1\) or concave and increasing for \(|t-s| > 1\).

**2. Proofs of theorems**

The basic tool of this paper is the following lemma.

**Lemma 1.** Let \( f(x) \) be defined in an infinite interval \( I \). If \( \lim_{x \to \infty} f(x) = 0 \) and \( f(x) - f(x + \varepsilon) > 0 \) for any given \( \varepsilon > 0 \), then \( f(x) > 0 \) in \( I \).

**Proof.** By induction, for any \( x \in I \), we have

\[
f(x) > f(x + \varepsilon) > f(x + 2\varepsilon) > \cdots > f(x + k\varepsilon) \to 0
\]

as \( k \to \infty \). The proof of Lemma 1 is complete. □
2.1. **Proof of Theorem 1.** It is well known that for any positive integer \( n \in \mathbb{N} \) the psi function \( \psi(x) \) and the polygamma or multigamma functions \( \psi^{(n)}(x) \) have the following integral expressions

\[
\psi(x) = \ln x + \int_0^\infty \left[ \frac{1}{u} - \frac{1}{1-e^{-u}} \right] e^{-ux} \, du \tag{10}
\]

and

\[
\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{u^n}{1-e^{-u}} e^{-ux} \, du. \tag{11}
\]

Using \( \psi^{(i-1)}(x+1) = \psi^{(i-1)}(x) + \frac{(-1)^{i-1}(i-1)!}{x^i} \) for \( i \in \mathbb{N} \) and \( x > 0 \) and direct computing gives

\[
\Theta_{s,t}(x) - \Theta_{s,t}(x+1) = \left\{ \left[ \psi(x + t) + \psi(x + t + 1) \right] - \left[ \psi(x) + \psi(x + 1) \right] \right\} \times \left\{ \left[ \psi(x + s) + \psi(x + s + 1) \right] - \left[ \psi(x) + \psi(x + 1) \right] \right\}
+ (t-s) \left\{ \left[ \psi'\left(x + t\right) - \psi'\left(x + t + 1\right) \right] - \left[ \psi'\left(x + s\right) - \psi'\left(x + s + 1\right) \right] \right\}
= \left\{ \frac{[\psi(x + t + 1) + \psi(x + t)] - [\psi(x + s + 1) + \psi(x + s)]}{t-s} \right\}
- \frac{2x + s + t}{(x+s)(x+t)} \left( \frac{t-s}{x+s}(x+t) \right) \Delta_{s,t}(x) \frac{(t-s)^2}{(x+s)(x+t)}
\]

and

\[
\Lambda_{s,t}(x) - \Lambda_{s,t}(x+1) = \frac{1}{t-s} \left( \frac{1}{x+s} + \frac{1}{x+s+1} - \frac{1}{x+t} - \frac{1}{x+t+1} \right)
- \frac{2x + s + t}{(x+s)(x+t+s^2+t^2+s+t)}
= \frac{1 - (s-t)^2}{(x+s)(x+s+1)(x+t)(x+t+1)}.
\]

Since \( \lim_{x \to -\infty} \Lambda_{s,t}^{(i)}(x) = 0 \) for any nonnegative integer \( i \) by (10) and (11), and the function \( \frac{\Lambda_{s,t}(x)-\Lambda_{s,t}(x+1)}{1-(s-t)^2} \) is completely monotonic, that is,

\[
(-1)^i \frac{\left[ \Lambda_{s,t}(x) - \Lambda_{s,t}(x+1) \right]^{(i)}}{1-(s-t)^2} = \frac{(-1)^i \Lambda_{s,t}^{(i)}(x) - (-1)^i \Lambda_{s,t}^{(i)}(x+1)}{1-(s-t)^2} \geq 0,
\]

in \((-\alpha, \infty)\), then \( (-1)^i \Lambda_{s,t}^{(i)}(x) \geq 0 \) follows from Lemma 1. This means the function \( \frac{\Lambda_{s,t}(x)}{1-(s-t)^2} \) is completely monotonic in \((-\alpha, \infty)\).

Since the function \( \frac{(t-s)^2}{(x+s)(x+t)} \) is completely monotonic and a product of two completely monotonic functions is also completely monotonic, then the function \( \Theta_{s,t}(x) - \Theta_{s,t}(x+1) \frac{\Lambda_{s,t}(x)}{1-(s-t)^2} \) is completely monotonic in \((-\alpha, \infty)\) by considering (13), which is equivalent to

\[
(-1)^k \left[ \Theta_{s,t}(x) - \Theta_{s,t}(x+1) \right]^{(k)} \frac{\Lambda_{s,t}(x)}{1-(s-t)^2} \geq 0
\]

for nonnegative integer \( k \). Further, from \( \lim_{x \to -\infty} \Theta_{s,t}^{(k)}(x) = 0 \) for nonnegative integer \( k \), which can be deduced by utilizing (10) and (11), and Lemma 1, it is concluded
that \((-1)^k \Theta_{s,t}^{(k)}(x) \geq 0\) for any nonnegative integer \(k\). This implies \((-1)^k \Theta_{s,t}^{(k)}(x) \geq 0\) if and only if \(|t - s| \leq 1\). Therefore, the functions \(\Theta_{s,t}(x)\) for \(|t - s| < 1\) and \(-\Theta_{s,t}(x)\) for \(|t - s| > 1\) are completely monotonic in \((-\alpha, \infty)\).

Since \(\Theta_{s,t}(x) = (t - s)^2 \Delta_{s,t}(x)\), the function \(\Delta_{s,t}(x)\) has the same monotonicity property as \(\Theta_{s,t}(x)\) in \((-\alpha, \infty)\). The proof of Theorem 1 is complete.

### 2.2. Proof of Theorem 2

By Theorem 1, it is easy to see that \(\Theta_{s,t}(x) \geq 0\) and \(\Delta_{s,t}(x) \geq 0\) in \((-\alpha, \infty)\) if and only if \(|t - s| \leq 1\). Then \(z_{s,t}''(x) \geq 0\) for \(|t - s| \leq 1\) follows from formula (9). The convexity and concavity of the function \(z_{s,t}(x)\) is proved.

In [6], the inequality
\[
\exp[(s - r)\psi(s)] > \frac{\Gamma(s)}{\Gamma(r)} > \exp[(s - r)\psi(r)]
\]
(14)
for \(s > r > 0\) was obtained, which is equivalent to
\[
\max \left\{ e^{\psi(s)}, e^{\psi(r)} \right\} > \left[ \frac{\Gamma(s)}{\Gamma(r)} \right]^{1/(s-r)} > \min \left\{ e^{\psi(s)}, e^{\psi(r)} \right\}
\]
for any positive numbers \(s > 0\) and \(t > 0\). This implies
\[
z_{s,t}'(x) = \left[ \frac{\Gamma(x + t)}{\Gamma(x + s)} \right]^{1/(t-s)} \frac{\psi(x + t) - \psi(x + s)}{t - s} - 1
\]
(15)
\[
< e^{\psi(x+t)} \frac{\psi(x + t) - \psi(x + s)}{t - s} - 1
\]
\[
= e^{\psi(x+t)} \psi'(x + \xi) - 1 < \psi'(x + t) e^{\psi(x+t)} - 1
\]
and
\[
z_{s,t}'(x) > e^{\psi(x+s)} \frac{\psi(x + t) - \psi(x + s)}{t - s} - 1
\]
(16)
\[
= e^{\psi(x+s)} \psi'(x + \xi) - 1
\]
\[
> \psi'(x + s) e^{\psi(x+s)} - 1,
\]
if assuming \(t > s > 0\) without loss of generality, where \(\xi \in (s, t)\).

By inequality
\[
\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}
\]
(17)
for \(x > 0\), we obtain
\[
x \psi'(x) e^{-1/x} < \psi'(x) e^{\psi(x)} < x \psi'(x) e^{-1/2x}
\]
(18)
for \(x > 0\). Using the asymptotic representation
\[
\psi'(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \cdots
\]
(19)
as \(x \to \infty\) yields
\[
\lim_{x \to \infty} \left[ x \psi'(x) e^{-1/x} \right] = 1 \quad \text{and} \quad \lim_{x \to \infty} \left[ x \psi'(x) e^{-1/2x} \right] = 1.
\]
Hence,
\[
\lim_{x \to \infty} \left[ \psi'(x) e^{\psi(x)} \right] = 1.
\]
(20)
Combining (21) with (15) and (16) leads to
\[
\lim_{x \to \infty} z'_{s,t}(x) \leq \lim_{x \to \infty} \left[ \psi'(x + t)e^{\psi(x + t)} \right] - 1 = \lim_{x \to \infty} \left[ \psi'(x + t)e^{\psi(x + t)} \right] - 1 = 0
\]
and
\[
\lim_{x \to \infty} z'_{s,t}(x) \geq \lim_{x \to \infty} \left[ \psi'(x + s)e^{\psi(x + s)} \right] - 1 = \lim_{x \to \infty} \left[ \psi'(x + s)e^{\psi(x + s)} \right] - 1 = 0.
\]
Thus, it is concluded that \(\lim_{x \to \infty} z'_{s,t}(x) = 0\).

Since \(z''_{s,t}(x) \geq 0\) in \(x \in (-\alpha, \infty)\) for \(|t - s| \leq 1\), then the function \(z'_{s,t}(x)\) is increasing/decreasing in \(x \in (-\alpha, \infty)\) for \(|t - s| \leq 1\). Thus, it follows that \(z'_{s,t}(x) \leq 0\) and \(z_{s,t}(x)\) is decreasing/increasing in \(x \in (-\alpha, \infty)\) for \(|t - s| \leq 1\). The monotonicity of the function \(z_{s,t}(x)\) is proved.

References

[1] N. Batir, Some new inequalities for gamma and polygamma functions, J. Inequal. Pure Appl. Math. 6 (2005), no. 4, Art. 103. Available online at http://jipam.vu.edu.au/article.php?sid=577.

[2] Ch.-P. Chen, Monotonicity and convexity for the gamma function, J. Inequal. Pure Appl. Math. 6 (2005), no. 4, Art. 100. Available online at http://jipam.vu.edu.au/article.php?sid=574.

[3] N. Elezović, C. Giordano and J. Pečarić, The best bounds in Gautschi’s inequality, Math. Inequal. Appl. 3 (2000), 239–252. Available online at http://www.mia-journal.com/contents.asp?number=10.

[4] D. Kershaw, Some extensions of W. Gautschi’s inequalities for the gamma function, Math. Comp. 41 (1983), 607–611.

[5] F. Qi, Certain logarithmically N-alternating monotonic functions involving gamma and q-gamma functions, RGMIA Res. Rep. Coll. 8 (2005), no. 3, Art. 5. Available online at http://rgmia.vu.edu.au/v8n3.html.

[6] F. Qi, Monotonicity results and inequalities for the gamma and incomplete gamma functions, Math. Inequal. Appl. 5 (2002), no. 1, 61–67. RGMIA Res. Rep. Coll. 2 (1999), no. 7, Art. 7, 1027–1034. Available online at http://rgmia.vu.edu.au/v2n7.html.

[7] F. Qi, B.-N. Guo, and Ch.-P. Chen, The best bounds in Gautschi-Kershaw inequalities, Math. Inequal. Appl. 9 (2006), no. 3, 427–436. RGMIA Res. Rep. Coll. 8 (2005), no. 2, Art. 17. Available online at http://rgmia.vu.edu.au/v8n2.html.

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