Scalar hadrons in $AdS_5 \times S^5$.

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A holographic model is presented, which allows to describe scalar hadrons with an arbitrary number of constituents.

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I. INTRODUCTION.

The recent realization of the holographic principle as an AdS/CFT correspondence has provided a new theoretical tool for dealing with strong coupling gauge theories. To the traditional models based on lattice calculations, quark potential models, bag models, and others, now we have at our disposal phenomenological QCD models based on this correspondence. Although QCD is certainly not a conformal field theory, it approaches the conformal limit in the ultraviolet, and therefore the holographic description should be valid. On the other hand, confinement can be simulated by imposing appropriate boundary conditions in the holographic variable $z$ [1]. In the Hard Wall model, a sharp cutoff is imposed. Moreover, confinement can also be introduced by modifying the metric in order to mimic a confinement potential [2], in which case a soft cutoff is introduced through a dilaton field $\phi$ (Soft Wall model).

The mass spectra predicted by the Hard Wall model has the form $M \sim l$ at high angular momenta $l$, which differs from the quadratic dependence $M^2 \sim l$, expected within Regge phenomenology. The correct Regge behavior can be obtained by introducing a dilaton, which allows to retain the conformal metric and in practice introduces a soft cutoff that depends on the dilaton $\phi$. So the usual procedure is to begin with

$$S = \int d^4xdz \sqrt{g} e^{-\phi(z)} \mathcal{L}.$$  \hspace{1cm} (1)

The Hard Wall model corresponds to a constant dilaton in the region $z \leq \Lambda_{QCD}^{-1}$, tending to infinity for $z > \Lambda_{QCD}^{-1}$. The introduction of a soft cutoff avoids the appearance of ambiguities in the choice of boundary conditions. A convenient and usual choice is $\phi(z) = A z^2$ [1, 3, 4], which is consistent with the usual Regge behavior.

These models have been used to describe spectra of glueballs (scalars [2, 5, 6], and vectors [2, 4, 6]), mesons [3, 4], and baryons [6], getting satisfactory results, but the hadronic spectra should also contain exotic hadrons, like tetraquarks, pentaquarks and hybrids, which in general have not been included in holographic models. Here we will consider a model which in principle is able to describe these hadrons.

II. THE HOLOGRAPHIC MODEL.

We consider a scalar field with mass in an $AdS_5 \times S^5$ space with a dilaton field, whose action is given by

$$S = \int d^5x \sqrt{G} e^{-\phi} \frac{1}{2} [G^{MN} \partial_M \phi \partial_N \phi + m^2 \phi^2].$$  \hspace{1cm} (2)

Here the M,N indexes are divided in $\mu, \nu,$ which correspond to $AdS_5$ coordinates and run over 0 to 4, and i,j indexes that run over 5 to 9, and which are the compact space $S^5$ coordinates. The ten dimensional space considered is described by the metric

$$ds^2 = \frac{R^2}{z^2} (\eta_{\mu \nu} dx^\mu dx^\nu) + R^2 \Omega_{ij} dy^i dy^j,$$  \hspace{1cm} (3)

where $\eta_{\mu \nu} = \text{diag}(1,-1,-1,-1,-1)$, $R$ is the radius for the $AdS_5$ and $S^5$ space and $\Omega_{ij}$ is the metric for a unitary sphere in five dimensions. Beginning with (2), the equation for $\phi(x,y)$ is

$$\frac{1}{\sqrt{G} e^{-\phi}} \frac{\partial}{\partial x^M} (\sqrt{G} e^{-\phi} G^{MN} \frac{\partial}{\partial x^N} \phi) - m^2 \phi = 0.$$  \hspace{1cm} (4)

Using $\phi(x,y) = \phi(x) Y(y)$ it is possible to get an equation for the AdS part, namely

$$\frac{1}{\sqrt{g_{AdS}} e^{-\phi}} \frac{\partial}{\partial x^\mu} (\sqrt{g_{AdS}} e^{-\phi} g^{\mu \nu} \frac{\partial}{\partial x^\nu} \phi) - m^2_5 \phi = 0,$$  \hspace{1cm} (5)

where we have used

$$m^2_5 = m^2 + \lambda^2,$$  \hspace{1cm} (6)

which is the mass of the scalar field in $AdS_5$ and $\lambda^2 R^2 = \kappa (\kappa + 4)$. Notice that the reduction of $S^5$ induces an additional mass term that depends on $\kappa$, which is an integer.

Let us consider a mode propagating in the bulk, with the form

$$\phi(x) = e^{-i P \cdot x} f(z),$$  \hspace{1cm} (7)

where $\chi$ corresponds to coordinates in four dimensional space and $P^2 = M^2$. Using (2) in (5) we obtain an equation which can be used in order to obtain the hadronic spectra,

$$[z^2 \partial_z^2 - ((d-1)z + z^2 \partial_z \phi(z)) \partial_z + M^2 z^2 - m^2_5 R^2] f(z) = 0,$$  \hspace{1cm} (8)
written in a general form for an AdS space with \((d + 1)\) dimensions.

For \(d = 4\), the conformal dimension for scalar fields is

\[
\Delta = 2 + \sqrt{4 + m_5^2 R^2}.
\]

in \(AdS_5 \times S^5\). Notice that \(m_5\) depends of \(\kappa\) and \(m\). Let us see what happens in the QCD side, where the quark field (denoted by \(Q\)) has dimension 3/2, gluons (denoted by \(G\)) have dimension 2, and the operator that gives one unit of angular momentum has dimension 1. In this case, all \(O\) operators that describe hadrons have dimension

\[
[O] = \Delta_0 + l
\]

where \(\Delta_0\) corresponds to the contribution of the constituents to the dimension of the operator, and the derivatives, each with dimension one, give a total contribution 1 to \(\Delta_0\). The AdS / CFT correspondence tells us that operators with dimension \([O]\) in a conformal field theory, are related with fields with dimension \(\Delta\) in \(AdS_5 \times S^5\). This in practice provides a mapping between \(\kappa\) and the angular momentum \(l\), which is just phenomenological.

Of all possible cases in \([10]\), only some of them will be related to operators of the conformal theory. To be more precise, only some values for \(m_5^2 R^2\) will be of interest to us, and the selection of these values is given by solving the equation formed using \([9]\) and \([10]\),

\[
2 + \sqrt{4 + m_5^2 R^2} = \Delta_0 + l,
\]

and then

\[
m_5^2 R^2 = (\Delta_0 + l - 4)(\Delta_0 + l).
\]

In other words, only Klein - Gordon fields in \(AdS_5 \times S^5\) that satisfy \([12]\) can be used to describe scalar operators for the field theory. With the previous value for \(m_5^2 R^2\), equation \([8]\) can be expressed as

\[
[z^2 \partial_z^2 - 3z \partial_z + M^2 z^2 - (\Delta_0 + l - 4)(l + \Delta_0)]f(z) = 0,
\]

where \(\alpha = \sqrt{4 + (\Delta_0 + l - 4)(l + \Delta_0)}\).

The values for \(M\) are obtained using the condition \(f(z_0) = 0\), where \(z_0 = \frac{1}{\Lambda_{QCD}}\). Thus, if the \(n\)-th zero of \(J_\alpha(z)\) is \(\beta_{\alpha,n}\), the mass is

\[
M_{\alpha,n} = \beta_{\alpha,n} \Lambda_{QCD}.
\]

where the index \(n\) is related to the different radial excitations.

In Hard Wall models \(\Lambda_{QCD}\) usually depends on the index \(n\) \([10]\), because otherwise radial excitations are badly described \([3]\), and therefore it is necessary to consider a different \(\Lambda_{QCD}\) for each level. This can be corrected by demanding good Regge behavior for radial excitations, as we will see below.

For mesons we use \(\Lambda_{QCD} = 0.263\) GeV, and for glueballs this parameter is fixed by considering that in the lattice the lightest glueball has a mass of 1.61 GeV \([13]\), which gives \(\Lambda_{QCD} = 0.313\) GeV for scalar glueballs. The results are shown in Fig 1, where one can see that for small \(l\), exempting the pion, the agreement with data is quite good.

Since there is no experimental data for exotic hadrons or hybrids, it is not possible to fix \(\Lambda_{QCD}\), and therefore we will use both a minimum and maximum values for this parameter (0.1 GeV and 0.3 GeV). Then we get a range of mass values for exotics and hybrid hadrons, presented in Table 2 for \(l = 0\).

### III. SPECTRA OF SCALAR HADRONS IN A HARD WALL MODEL.

#### A. Hard Wall model

In this case equation \([13]\) can be expressed as

\[
[z^2 \partial_z^2 - 3z \partial_z + M^2 z^2 - (\Delta_0 + l - 4)(l + \Delta_0)]f(z) = 0,
\]

the normalizable solutions are

\[
f(z) = z^2 J_\alpha(M z),
\]

where \(\alpha = \sqrt{4 + (\Delta_0 + l - 4)(l + \Delta_0)}\).

The values for \(M\) are obtained using the condition \(f(z_0) = 0\), where \(z_0 = \frac{1}{\Lambda_{QCD}}\). Thus, if the \(n\)-th zero of \(J_\alpha(z)\) is \(\beta_{\alpha,n}\), the mass is

\[
M_{\alpha,n} = \beta_{\alpha,n} \Lambda_{QCD}.
\]

#### TABLE I: Relation between constituents in a hadron and the conformal dimension. We consider hadrons with \(n\) quarks (and/or antiquarks) and \(m\) gluons.

| \(\Delta_0\) | \((nQ)(mG)\) |
|----------------|----------------|
| 3              | (2Q)           |
| 4              | (2G)           |
| 5              | (2Q)(1G)       |
| 6              | (4Q)           |
| 7              | (2Q)(2G)       |
| 8              | (4G) ; (4Q)(1G)|
| 9              | (6Q) ; (2Q)(3G)|
| 10             | (4Q)(2G)       |
TABLE II: Range of values for the mass in GeV of hadrons with n quarks and/or antiquarks and m gluons. $M_{\text{min}}$ is calculated using $\Lambda_{QCD} = 0.1$ GeV and $M_{\text{max}}$ using $\Lambda_{QCD} = 0.3$ GeV. All cases considered here are for $l = 0$.

| (nQ)(mG) | $M_{\text{min}}$ | $M_{\text{max}}$ |
|----------|------------------|------------------|
| (2Q)(1G) | 0.877            | 2.631            |
| (4Q)     | 0.994            | 2.981            |
| (2Q)(2G) | 1.109            | 3.326            |
| (4Q) : (4Q)(1G) | 1.223          | 3.668            |
| (6Q) : (2Q)(3G) | 1.335          | 4.006            |
| (4Q)(2G) | 1.448            | 4.343            |

B. Mesons in an improved Hard Wall model.

Consider equation (13) with $\Delta_\theta = 3$.

\[ [z^2 \partial_z^2 - (d-1)z\partial_z + z^2 M^2 - (l-1)(l+3)]f(z) = 0, \quad (17) \]

with normalizable modes which are given by

\[ f(z) = z^2 J_{l+1}(zM). \quad (18) \]

The mass spectra are obtained from $\Phi(x, z_0 = 1/\Lambda_{QCD}) = 0$, and therefore masses are determined by:

\[ M_{1+l,n} = \beta_{1+l,n} \Lambda_{QCD}^{(n)}, \quad (19) \]

where $\beta_{1+l,n}$ are zeroes of Bessel function that appear in (18) and $n = 1, 2, \ldots$.

A string mode with a node in the $z$ coordinate corresponds to radial excitations, but as we just saw the Hard Wall model does not give good results in this case without changing $\Lambda_{QCD}$ for every level.

Nevertheless, values for $\Lambda_{QCD}$ can be obtained beginning with $\Lambda_{QCD}^{(1)}$, by demanding good Regge behavior. In fact, Regge Trajectories as functions of a radial quantum number $n_r$ (related to $n$ by $n_r = n - 1$) can be described by

\[ M^2(n_r, l) = An_r + M^2(0, l), \quad (20) \]

for fixed $l$.

An analysis of Regge trajectories for the radial excitations $^3P_2, ^1P_0$ and $^3F_2$ shows that the slope is constant, and its value is $A = 1.04 \pm 0.01$ GeV [10].

Using (19) and (20), with $\Lambda_{QCD}^{(n)} \rightarrow \Lambda_n$ we get

\[ (\beta_{1+l,n}\Lambda_n)^2 = An_r + (\beta_{1,1}\Lambda_1)^2, \quad (21) \]

and then

\[ \Lambda_n = \sqrt{An_r + (\beta_{1,1}\Lambda_1)^2}. \quad (22) \]

Using $\Lambda_1 = 0.263$ GeV, which is the value of $\Lambda_{QCD}$ used in [3], the other values for $\Lambda_n$, calculated for $l = 0$, are

\[ \Lambda_2 = 0.204 \text{ GeV}, \quad \Lambda_3 = 0.173 \text{ GeV}, \quad \Lambda_4 = 0.153 \text{ GeV}. \quad (23) \]

Using these values in (19) allows to obtain the scalar mesons masses, which appear in Fig. 2, showing that for small values of $l$ the improved model gives good results.

IV. SPECTRA OF SCALAR HADRONS IN A SOFT WALL MODEL.

Let us consider a Soft Wall model with a dilaton in its most general form. In this case equation (8) can be written as [13]

\[ [\partial_z^2 - ((\partial_z C(z))\partial_z + (M^2 - \frac{m^2 R^2}{z^2})]f(z) = 0, \quad (24) \]

where

\[ C(z) = (d-1) \ln(z) + \phi(z). \quad (25) \]

A correct choice of $\phi$ should be one which allows to get a hadronic spectra consistent with Regge trajectories. A usual choice for this function in $\phi(z) = A z^2$ [3, 4], but
a more general choice is \( \phi(z) = A z^2 + \delta \ln(z) \) \(^{14}\), and then \( C(z) \) has the form

\[
C(z) = A z^2 + B \ln(z),
\]

(26)

where \( B = \delta + d - 1 \).

Using

\[
f(z) = e^{-C(z)} \psi(z),
\]

(27)
equation (24) is transformed in a Schrödinger type equation

\[
- \partial_z^2 \psi + V(z) \psi = M^2 \psi,
\]

(28)

where

\[
V(z) = A(B-1) + \frac{B(B+2) + 4m^2 R^2}{4z^2} + A^2 z^2.
\]

(29)

Where \( m^2 R^2 \) is given by \(^{12}\). The potential in equation (29) should give a linear spectra in both \( n \) and \( l \), and for this purpose taking \( A \) constant is sufficient. With respect to \( B \), it has been shown that in order to get good Regge behavior, a dependence on angular momenta is necessary \(^{13}\), a result which is also present in our procedure. A specific form of \( B \), consistent with \( M^2 \sim (n+l+v) \) for all \( \Delta_0 \), is:

\[
B = \frac{2l + 3l^2 + 4\Delta_0 - 2l \Delta_0 - \Delta_0^2 - 2v + 8l v + 4v^2}{2(l+v)}.
\]

(30)

With this value for \( B \) we get a spectra given by:

\[
M^2 = 4A(n+l+v).
\]

(31)

In order to determinate \( A \) and \( v \), we use the \( \pi \) and \( \pi_2 \) masses. With this we get \( A = 0.346 \text{ GeV}^2 \) and \( v = 0.0142 \). Results for mesons appear in Fig. 3.

![Fig. 3: Scalar mesons spectra, calculated using a Soft Wall model with \( A = 0.346 \text{ GeV} \) and \( v = 0.0142 \).](image)

Considering that the slope in Regge trajectories is universal, the previous \( A \) value can also be used for glueball, TABLE III: Glueballs masses for different values of \( n \) and \( l \), measured in GeV.

| \( n \) | 0     | 1     | 2     | 3     |
|-------|-------|-------|-------|-------|
| 0     | 1.61  | 1.99  | 2.32  | 2.60  |
| 1     | 1.99  | 2.32  | 2.60  | 2.85  |
| 2     | 2.32  | 2.60  | 2.85  | 3.08  |
| 3     | 2.60  | 2.85  | 3.08  | 3.30  |

and therefore we only need to fix \( v \). For this we use lattice values for the lightest glueball \(^{13}\), and find \( v = 1.873 \). Results for glueballs appear in Table 3.

As can be seen, different values for \( v \) must be given for each kind of hadron considered, and therefore we did not make a table similar to Table II in this case.

V. CONCLUSIONS.

The present work shows how holographic models can be applied to the general problem of calculating hadronic spectra, considering even exotic hadrons and hybrids. These were excluded in previous works based on Hard and Soft Wall models, centered exclusively up to now on glueballs, mesons and baryons \(^\underline{3} \underline{4} \underline{5} \underline{6} \underline{7} \).

In relation to the parameters of the dilaton field in a Soft Wall model description, it is important that \( B \) depends on angular momenta in order to get a good Regge behavior, which is already present in \(^{30}\). This feature has also been mentioned in \(^{13}\).

At this moment, since the few parameters must be fixed by experimental information, it is not possible to make precise predictions, but we can establish a range for the masses, depending on a range for these parameters. In Hard Wall models the parameter is \( \Lambda_{QCD} \), for which a range of typical values is known. This allows us to get a range for the possible masses. In principle for Soft Wall models we can do the same, but the lack of knowledge about the \( v \) values does not allow to get a range for masses in this case.

One interesting consequence of this approach is that there must exist a maximum number for valence constituents inside hadrons.

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