Hermitian structures on the product of Sasakian manifolds

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Abstract We investigate the curvature properties of a two-parameter family of Hermitian structures on the product of two Sasakian manifolds, as well as intermediate relations. We give a necessary and sufficient condition for a Hermitian structure belonging to the family to be Einstein and provide concrete examples.

Keywords Einstein · Hermitian structure · Sasakian manifold

Mathematics Subject Classification 53C25 · 53B35

1 Introduction

In [11], Tsukada worked on the isospectral problem with respect to the complex Laplacian for a two-parameter family of Hermitian structures on the Calabi-Eckmann manifold $S^{2p+1} \times S^{2q+1}$ including the canonical one. In this paper, we define a two-parameter family of almost Hermitian structures on the product manifold $M = M \times M'$ of a $(2p+1)$-dimensional Sasakian manifold $M$ and a $(2q+1)$-dimensional Sasakian manifold $M'$ similarly to the method used in [11], and show that any almost Hermitian structure on $M$ belonging to the two parameter family is integrable; thereby generalizing the result of the Calabi-Eckmann manifold by Tsukada [11, Proposition 3.1]. We shall further give a necessary and sufficient condition for a Hermitian manifold in the family to be Einstein (Theorem 1), which is the main result of the present paper. In the last section, we shall provide concrete examples of Einstein Hermitian structures on the Calabi-Eckmann manifolds $S^{2p+1} \times S^{2q+1}$ for all $(p, q) (p, q \geq 1)$, by making use of Theorem 1. However, we may also see that anyone
of these examples can not be weakly \(\ast\)-Einstein. Therefore, we may note that the fact “any compact Einstein Hermitian surface is weakly \(\ast\)-Einstein” is not valid for higher dimensional cases in general. The outline of these circumstances will be also given in the last section.

2 Preliminaries

In this section, we prepare some fundamental tools which we need in our arguments. Let \(\overline{M} = (\overline{M}, \overline{J}, \overline{g})\) be a \(2n(\geq 4)\)-dimensional almost Hermitian manifold with almost Hermitian structure \((\overline{J}, \overline{g})\). We denote by \(\overline{\nabla}, \overline{R}, \overline{\rho}\) and \(\overline{\tau}\) the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of \(\overline{M}\), respectively. The curvature tensor is defined by

\[
\overline{R} (\overline{X}, \overline{Y}) \overline{Z} = [\overline{\nabla}_{\overline{X}}, \overline{\nabla}_{\overline{Y}}] \overline{Z} - \overline{\nabla}_{[\overline{X}, \overline{Y}]} \overline{Z},
\]

for \(\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\overline{M})\), where \(\mathfrak{X}(\overline{M})\) denotes the Lie algebra of all smooth vector fields on \(\overline{M}\). The Ricci \(\ast\)-tensor \(\overline{\rho}\) of \(\overline{M}\) is defined by

\[
\overline{\rho}(\overline{X}, \overline{Y}) = \text{tr} (\overline{Z} \mapsto \overline{R}(\overline{X}, \overline{J}\overline{Z})\overline{J}\overline{Y})
= \frac{1}{2} \text{tr} (\overline{Z} \mapsto \overline{R}(\overline{X}, \overline{J}\overline{Y})\overline{J}\overline{Z})),
\]

(2.1)

for \(\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\overline{M})\). It is easily checked that the equality \(\overline{\rho} = \overline{\rho}\) holds on \(\overline{M}\) if \(\overline{M}\) is Kähler. We denote by \(\overline{\tau}\) the \(\ast\)-scalar curvature of \(\overline{M}\), which is the trace of the Ricci \(\ast\)-operator \(\overline{Q}\) defined by \(\overline{\rho} (\overline{Q}\overline{X}, \overline{Y}) = \overline{\rho}(\overline{X}, \overline{Y})\). A 4-dimensional almost Hermitian manifold is also called an almost Hermitian surface. We note that, for any almost Hermitian surface \(\overline{M}\), the Ricci and Ricci \(\ast\)-tensor are related by

\[
\overline{\rho}(\overline{X}, \overline{Y}) + \overline{\rho}(\overline{Y}, \overline{X}) - \{\overline{\rho}(\overline{X}, \overline{Y}) + \overline{\rho}(\overline{J}\overline{X}, \overline{J}\overline{Y})\} = \frac{\overline{\tau} - \overline{\tau}}{2} \overline{g}(\overline{X}, \overline{Y}),
\]

(2.2)

for \(\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\overline{M})\) [3,5,6]. A \(2n\)-dimensional almost Hermitian manifold \((\overline{M}, \overline{J}, \overline{g})\) is called a weakly \(\ast\)-Einstein manifold if the equality \(\overline{\rho} = \frac{\overline{\tau}}{2n} \overline{g}\) holds on \(\overline{M}\). Especially, if \(\ast\)-scalar curvature \(\overline{\tau}\) of a weakly \(\ast\)-Einstein manifold \(\overline{M}\) is constant, then \(\overline{M}\) is said to be \(\ast\)-Einstein. It is known that there exist weakly \(\ast\)-Einstein manifolds which are not \(\ast\)-Einstein [7,9]. We denote by \(\overline{\mathcal{N}}\) the Nijenhuis tensor of the almost complex structure \(\overline{J}\) defined by

\[
\overline{\mathcal{N}}(\overline{X}, \overline{Y}) = [\overline{J}\overline{X}, \overline{J}\overline{Y}] - [\overline{X}, \overline{Y}] - \overline{J}[\overline{J}\overline{X}, \overline{Y}] - \overline{J}[\overline{X}, \overline{J}\overline{Y}]
\]

(2.3)

for \(\overline{X}, \overline{Y} \in \mathfrak{X}(\overline{M})\). It is well-known that the almost complex structure \(\overline{J}\) is integrable if and only if the Nijenhuis tensor \(\overline{\mathcal{N}}\) vanishes identically on \(\overline{M}\) [8]. An almost Hermitian manifold \((\overline{M}, \overline{J}, \overline{g})\) with integrable almost complex structure \(\overline{J}\) is called a Hermitian manifold. It is well-known that the condition \(\overline{\mathcal{N}} = 0\) and the following condition (2.4) are equivalent:

\[
\overline{g}((\overline{\nabla}_{\overline{X}}\overline{J})\overline{Y}, \overline{Z}) - \overline{g}((\overline{\nabla}_{\overline{J}\overline{X}}\overline{J})\overline{J}\overline{Y}, \overline{Z}) = 0
\]

(2.4)

for any vector field \(\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\overline{M})\).

Next, we give a brief review of Sasakian manifolds. An almost contact metric manifold \(M = (M, \varphi, \xi, \eta, g)\) is called a contact metric manifold if it satisfies

\[
d\eta(X, Y) = g(X, \varphi Y),
\]
for any $X, Y \in \mathfrak{X}(M)$. Further, a normal contact metric manifold is called a Sasakian manifold. It is well-known that a Sasakian manifold is characterized as an almost contact metric manifold satisfying the condition

\begin{equation}
(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,
\end{equation}

for any $X, Y \in \mathfrak{X}(M)$ [2, Theorem 6.3]. On a $(2n + 1)$-dimensional Sasakian manifold $(M, \varphi, \xi, \eta, g)$, we have the following identities:

\begin{align*}
\nabla_X \xi &= -\varphi X, \quad (\nabla_X \eta)(Y) = -g(\varphi X, Y), \\
R(X, Y)\xi &= \eta(Y)X - \eta(X)Y, \\
\rho(\xi, X) &= 2n\eta(X),
\end{align*}

for any $X, Y \in \mathfrak{X}(M)$ [2]. For further investigation, we also prepare the following curvature identity on a Sasakian manifold [2, pp. 107]:

\begin{equation}
R(X, Y, \varphi Z, W) - R(\varphi Z, X, Y, W) \\
= -g(X, Y)g(\varphi Z, W) - 2g(Z, \varphi Y)g(X, W) + g(Z, \varphi X)g(Y, W),
\end{equation}

for $X, Y, Z, W \in \mathfrak{X}(M)$. From (2.7), we get

\begin{equation}
\sum_{i=1}^{2n+1} R(X, Y, \varphi e_i, e_i) = \sum_{i=1}^{2n+1} R(\varphi e_i, X, Y, e_i) = 3g(\varphi X, Y),
\end{equation}

for any orthonormal basis $\{e_1, \ldots, e_{2n+1}\}$ of $T_x M, x \in M$. Then, the left-hand side of (2.8) implies

\begin{equation}
\sum_{i=1}^{2n+1} R(X, Y, \varphi e_i, e_i) - \sum_{i=1}^{2n+1} R(\varphi e_i, X, Y, e_i) = 2 \sum_{i=1}^{2n+1} R(X, Y, \varphi e_i, e_i) \\
+ \sum_{i=1}^{2n+1} R(Y, \varphi e_i, X, e_i).
\end{equation}

Thus, from (2.8) and (2.9), we have

\begin{equation}
2 \sum_{i=1}^{2n+1} R(X, Y, \varphi e_i, e_i) + \sum_{i=1}^{2n+1} R(Y, \varphi e_i, X, e_i) = 3g(\varphi X, Y).
\end{equation}

From (2.10), we get also

\begin{equation}
2 \sum_{i=1}^{2n+1} R(Y, X, \varphi e_i, e_i) + \sum_{i=1}^{2n+1} R(X, \varphi e_i, Y, e_i) = 3g(\varphi Y, X).
\end{equation}

Thus, from (2.10) and (2.11), we have

\begin{equation}
4 \sum_{i=1}^{2n+1} R(X, Y, \varphi e_i, e_i) + \sum_{i=1}^{2n+1} R(Y, \varphi e_i, X, e_i) + \sum_{i=1}^{2n+1} R(\varphi e_i, X, Y, e_i) = 6g(\varphi X, Y),
\end{equation}

and hence,

\begin{equation}
\sum_{i=1}^{2n+1} R(X, Y, e_i, \varphi e_i) = -2g(\varphi X, Y).
\end{equation}
From (2.13), we have
\[
\sum_{i} R(X, \varphi Y, e_i, \varphi e_i) = -2\left( g(X, Y) - \eta(X)\eta(Y) \right),
\]
(2.14)
for any tangent vector \( X, Y \in T_x M, x \in M \). The equality (2.14) is useful in the calculation of the formula (3.34) in the next section.

### 3 Curvature formulas and main result

In this section, we define a two parameter family of almost Hermitian structures on the product of Sasakian manifolds and show the integrability. Further, we give a necessary and sufficient condition for a Hermitian structure belonging to the family to be Einstein one. Let \((M, \varphi, \xi, \eta, g)\) (resp. \((M', \varphi', \xi', \eta', g')\)) be a \((2p + 1)\)-dimensional Sasakian manifold (resp. a \((2q + 1)\)-dimensional Sasakian manifold). We denote by \(\nabla, R\) and \(\rho\) (resp. \(\nabla', R'\) and \(\rho'\)) the Riemannian connection, the curvature tensor and the Ricci tensor on \(M\) (resp. \(M'\)). Let \(\overline{M} = M \times M'\) be the product manifold of \(M\) and \(M'\). Then we define a Riemannian metric \(\overline{g} = \overline{g}_{a,b} (a, b \in \mathbb{R})\) on \(\overline{M}\) by
\[
\overline{g}_{a,b} = g + a(\eta \otimes \eta' + \eta' \otimes \eta) + (a^2 + b^2 - 1)\eta' \otimes \eta' + g' \tag{3.1}
\]
[11]. Further, we define an almost complex structure \(\overline{J} = \overline{J}_{a,b}(a, b \in \mathbb{R}, b \neq 0)\) on \(\overline{M}\) as follows.
\[
\overline{J}_{a,b}(X + X') = \varphi(X) - \left\{ \frac{a}{b} \eta(X) + \frac{a^2 + b^2}{b} \eta'(X') \right\} \xi
\]
\[
+ \varphi'(X') + \left\{ \frac{1}{b} \eta(X) + \frac{a}{b} \eta'(X') \right\} \xi',
\]
(3.2)
for any tangent vector \(X\) of \(M\) and any tangent vector \(X'\) of \(M'\) [11]. It is easily checked that \(\overline{J}^2 = -I\) holds and \((\overline{J}, \overline{g})\) is an almost Hermitian structure on \(\overline{M}\). Since \(\mathfrak{X}(M)\) and \(\mathfrak{X}(M')\) are regarded as the Lie subalgebra of \(\mathfrak{X}(\overline{M})\), we may note that (3.1) and (3.2) are rewritten respectively as follows:
\[
\overline{g}(X, Y) = g(X, Y), \quad \overline{g}(X', Y') = a\eta(X)\eta'(Y'),
\]
(3.3)
\[
\overline{J}(X) = \varphi(X) - \frac{a}{b} \eta(X)\xi + \frac{1}{b} \eta(X)\xi',
\]
\[
\overline{J}(X') = \varphi'(X') - \frac{a^2 + b^2}{b} \eta'(X')\xi + \frac{a}{b} \eta'(X')\xi',
\]
(3.4)
for \(X, Y \in \mathfrak{X}(M), X', Y' \in \mathfrak{X}(M')\). Let \(\overline{\nabla}, \overline{R}\) and \(\overline{\rho}\) be the Riemannian connection, the curvature tensor and the Ricci tensor of \(\overline{M}\), respectively, and \(X, Y, Z, W\) (resp. \(X', Y', Z', W'\)) be any smooth vector field on \(M\) (resp. \(M'\)). Then, from (3.3) and (3.4), by making use of (2.5) and (2.6), we have the following:
\[
\overline{g}(\overline{\nabla}_X Y, Z) = g(\nabla_X Y, Z), \quad \overline{g}(\overline{\nabla}_{X'} Y', Z') = -a\eta'(X')g(\varphi Y, Z),
\]
\[
\overline{g}(\overline{\nabla}_X Y, Z) = -a\eta'(Y')g(\varphi X, Z), \quad \overline{g}(\overline{\nabla}_{X'} Y', Z') = a\eta(\nabla_X Y)\eta'(Z'),
\]
\[
\overline{g}(\overline{\nabla}_X Y, Z) = a\eta'(\nabla'_{X'} Y')\eta(Z), \quad \overline{g}(\overline{\nabla}_{X'} Y', Z') = -a\eta(Y)g'(\varphi' X', Z'),
\]
By direct calculation using (3.3), (3.4) and (3.5), we get the following:

\[ \bar{g}(\bar{\nabla}_X \bar{J}) Y, Z) = \eta(Z) g(X, Y) - \eta(Y) g(X, Z), \]
\[ \bar{g}(\bar{\nabla}_X \bar{J}) Y, Z) = 0, \]
\[ \bar{g}(\bar{\nabla}_X \bar{J}) Y', Z) = b\eta(Y') g(\bar{\varphi} X, Z) - a \eta'(Y') (g(X, Z) - \eta(X) \eta(Z)), \]
\[ \bar{g}(\bar{\nabla}_X \bar{J}) Y', Z) = - a \eta(Z) (\eta'(X') \eta'(Y') - g'(X', Y')) + b \eta(Z) g'(\psi' X', Y'), \]
\[ \bar{g}(\bar{\nabla}_X \bar{J}) Y', Z) = 0, \]
\[ \bar{g}(\bar{\nabla}_X \bar{J}) Y', Z) = (a^2 + b^2) (g'(X', Y') \eta'(Z') - g'(X', Z') \eta'(Y')). \]

Then, from (3.4) and (3.6), we can check that

\[ \bar{g} ((\bar{\nabla}_X \bar{J}) \bar{Y}, \bar{Z}) - \bar{g} ((\bar{\nabla}_{\bar{J} \bar{X}}) \bar{J} \bar{Y}, \bar{Z}) = 0 \] (3.7)

holds for any \( \bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{M} \). Therefore, from (2.4), we see that the almost complex structure \( \bar{J} \) is integrable and hence, \( (\mathfrak{M}, \bar{J}, \bar{g}) \) is a Hermitian manifold, which generalizes the result of Tsukada [11, Proposition 3.1]. However, from (3.6), we also see that \( \mathfrak{M} \) is never Kähler. Now, we choose an orthonormal basis \( \{ e_1, \ldots, e_{2p}, \xi \} \) (resp. \( \{ e'_1, \ldots, e'_{2q}, \xi' \} \)) of the tangent space \( T_x \mathfrak{M} \) (resp. \( T_{X'} \mathfrak{M} \)) at each point \( x \in \mathfrak{M} \) (resp. \( x' \in \mathfrak{M} \)). Then we may easily check that \( \{ e_1, \ldots, e_{2p}, e'_1, \ldots, e'_{2q}, \xi, \xi' \} \) is an orthonormal basis of \( T_{(x, x')} \mathfrak{M} \), which will be useful in the forthcoming calculations of the present paper. From (3.3) and (3.5), by taking account of (2.5) and (2.6), we have the formulas for the curvature tensor of \( \mathfrak{M} \):

\[ \bar{g}(\bar{R}(X, Y) Z, W) = g(R(X, Y) Z, W), \]
\[ \bar{g}(\bar{R}(X', Y') Z, W) = -a \eta'(Y') (g(X, Z) \eta(W) - g(X, W) \eta(Z)), \]
\[ \bar{g}(\bar{R}(X, Y') Z, W) = 2a g'(\bar{\varphi}' X', Y') g(\bar{\varphi} Z, W), \]
\[ \bar{g}(\bar{R}(X, Y') Z, W) = a g(\bar{\varphi} X, Z) g'(\bar{\varphi}' Y', W') - a^2 \eta'(Y') \eta'(W') (g(X, Z) - \eta(X) \eta(Z)) \]
\[ -a^2 \eta(X) \eta(Z) (g'(Y', W') - \eta(Y') \eta'(W')), \]
\[ \bar{g}(\bar{R}(X', Y') Z', W) = a (a^2 + b^2) \eta(W') (\eta(X) g'(Y', Z') + \eta'(Y') g(X', Z)), \]
\[ \bar{g}(\bar{R}(X, Y') Z', W) = a \eta(R(X, Y) Z) \eta'(W'), \]
\[ \bar{g}(\bar{R}(X', Y') Z', W) = g'(\bar{R}'(X', Y') Z', W') \]
\[ +2(a^2 + b^2 - 1) \{ \eta'(X') \eta'(W') g'(Y', Z') - \eta'(Y') \eta'(W') g'(X', Z') \]
\[ -\eta'(X') \eta'(Z') g'(Y', W') + \eta'(Y') \eta'(Z') g'(X', W') \}
\[ -2(a^2 + b^2 - 1) \{ \eta'(X') \eta'(Z') g'(Y', W') - \eta'(Y') \eta'(Z') g'(X', W') \]
\[ +\eta'(Y') \eta'(W') g'(X', Z') - \eta'(X') \eta'(W') g'(Y', Z') \}
\[ +a^2 (a^2 + b^2 - 1) \{ 2g'(\bar{\varphi}' X', Y') g'(\bar{\varphi}' Z', W') \]
\[ +g'(\bar{\varphi}' X', Z') g'(\bar{\varphi}' Y', W') \]
\[ -g'(\bar{\varphi}' Y', Z') g'(\bar{\varphi}' X', W') \}. \]
From (3.8), by direct calculation, we have
\[
\tilde{\rho}(Y, Z) = \rho(Y, Z) + 2a^2 q \eta(Y) \eta(Z),
\]
\[
\tilde{\rho}(Y, Z') = 2a(p + q(a^2 + b^2))\eta(Y)\eta'(Z'),
\]
\[
\tilde{\rho}(Y', Z') = \rho'(Y', Z') - 2(a^2 + b^2 - 1)g'(Y', Z')
+ 2(pa^2 + a^2 + b^2 - 1 + q(a^2 + b^2 - 1)(a^2 + b^2 + 1))\eta'(Y')\eta'(Z'),
\]
for \(Y, Z \in \mathfrak{X}(M), Y', Z' \in \mathfrak{X}(M')\). Thus, from (3.3) and (3.9), we see that \((\overline{M}, \bar{g})\) is Einstein if and only if there is a constant \(\lambda\) satisfying the following conditions:
\[
\rho(Y, Z) + 2a^2 q \eta(Y) \eta(Z) = \lambda g(Y, Z),
\]
\[
2a(p + q(a^2 + b^2))\eta(Y)\eta'(Z') = a\lambda \eta(Y)\eta'(Z'),
\]
\[
\rho'(Y', Z') - 2(a^2 + b^2 - 1)g'(Y', Z')
+ 2(pa^2 + a^2 + b^2 - 1 + q(a^2 + b^2 - 1)(a^2 + b^2 + 1))\eta'(Y')\eta'(Z')
= \lambda(g'(Y', Z') + (a^2 + b^2 - 1)\eta'(Y')\eta'(Z')),\]
for \(Y, Z \in \mathfrak{X}(M), Y', Z' \in \mathfrak{X}(M')\). We see that (3.10) and (3.12) may be rewritten as the following, respectively.
\[
\rho(Y, Z) = \lambda g(Y, Z) - 2a^2 q \eta(Y) \eta(Z),
\]
\[
\rho'(Y', Z') = (\lambda + 2(a^2 + b^2 - 1))g'(Y', Z')
+ \{\lambda(a^2 + b^2 - 1) - 2(pa^2 + a^2 + b^2 - 1
+ q(a^2 + b^2 - 1)(a^2 + b^2 + 1))\}\eta'(Y')\eta'(Z'),\]
for \(Y, Z \in \mathfrak{X}(M), Y', Z' \in \mathfrak{X}(M')\). Here, by the assumption that \(M\) and \(M'\) are both Sasakian, we get
\[
\rho(Y, \xi) = 2p\eta(Y),
\]
\[
\rho'(Y', \xi') = 2q\eta'(Y'),
\]
for \(Y \in \mathfrak{X}(M), Y' \in \mathfrak{X}(M')\). Thus, from (3.13) and (3.15), we obtain
\[
2p\eta(Y) = (\lambda - 2a^2 q)\eta(Y).
\]
Similarly from (3.14) and (3.15), we have also
\[
2q\eta'(Y') = (\lambda(a^2 + b^2) - 2pa^2 - 2q(a^2 + b^2 - 1)(a^2 + b^2 + 1))\eta'(Y').
\]
Thus, from (3.16) and (3.17), we obtain
\[
\lambda = 2p + 2a^2 q,\]

and
\[
(a^2 + b^2)\lambda = 2pa^2 + 2q(a^2 + b^2)^2.\]
Thus, from (3.18) and (3.19), we get
\[
(p + a^2 q)(a^2 + b^2) = pa^2 + q(a^2 + b^2)^2,
\]
and hence
\[
pb^2 + (a^2 + b^2)a^2 q = q(a^2 + b^2)^2.\]
From (3.21), we have further
\[(p - (a^2 + b^2)q)b^2 = 0.\] (3.22)

If \(b \neq 0\), then from (3.22), we have
\[p = (a^2 + b^2)q.\] (3.23)

Further, we suppose that \(a \neq 0\) (under \(b \neq 0\)), then, from (3.11), we have
\[\lambda = 4p.\] (3.24)

Thus, in this case, from (3.18) and (3.24), we have
\[a^2q = p,\] (3.25)
and hence, taking account of (3.23), we have
\[b^2 = 0.\] (3.26)

But, this is a contradiction. So, it must follow that \(a = 0\). Thus, from (3.3), we have
\[\bar{g}(Y, Z') = 0,\] (3.27)
and
\[\bar{g}(Y', Z') = g'(Y', Z') + (b^2 - 1)\eta'(Y')\eta'(Z'),\] (3.28)
for \(Y \in \mathcal{X}(M), Y', Z' \in \mathcal{X}(M')\). Further, from (3.21), we have also
\[p = b^2q.\] (3.29)

From (3.18), we get
\[\lambda = 2p.\] (3.30)

From (3.13) and (3.14), taking account of (3.29) and (3.30), we have
\[\rho(Y, Z) = 2pg(Y, Z),\] (3.31)
\[\rho'(Y', Z') = 2(p + b^2 - 1)g'(Y', Z') - 2(b^2 - 1)(q + 1)\eta'(Y')\eta'(Z'),\] (3.32)
for \(Y, Z \in \mathcal{X}(M), Y', Z' \in \mathcal{X}(M')\). Further, from (3.9), we have also
\[\bar{\rho}(Y, Z') = 0,\] (3.33)
for \(Y \in \mathcal{X}(M), Z' \in \mathcal{X}(M')\).

From (3.31), we can easily check that the Einstein constant of any \((2p + 1)\)-dimensional Einstein Sasakian manifold is equal to \(2p\). Therefore, summing up the arguments above, we have the following.

**Theorem 1** Let \(M = (\mathcal{M}, \varphi, \xi, \eta, g)\) and \(M' = (\mathcal{M}', \varphi', \xi', \eta', g')\) be a \((2p + 1)\)-dimensional and a \((2q + 1)\)-dimensional Sasakian manifold respectively, and let \(\overline{M} = M \times M'\) be the product manifold of \(M\) and \(M'\). Then \(\overline{M} = (\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})\) is a Hermitian manifold equipped with the Hermitian structure \((\overline{g}, \overline{J})\) defined by (3.1) and (3.2). Furthermore, \(\overline{M} = (\overline{M}, \overline{g}, \overline{J})\) is Einstein if and only if \(a = 0\) and \(M\) is an Einstein Sasakian manifold and \(M'\) is an \(\eta\)-Einstein Sasakian manifold with the Ricci tensor \(\rho' = 2(p + \frac{p}{q} - 1)g' - 2(\frac{p}{q} - 1)(q + 1)\eta' \otimes \eta'\).
Remark 1 We see that the $\eta$-Einstein Sasakian manifold $M'$ in Theorem 1 is Einstein if $p = q$. Then $M$ is the Riemannian product of the same dimensional Einstein Sasakian manifolds $M$ and $M'$.

Next, we shall calculate the Ricci $\ast$-tensor $\tilde{\rho}^*$ of $\bar{M} = (M, \bar{g}, \bar{J})$. From (3.4) and (3.8), by making use of (2.14), we have

\[
\tilde{\rho}^*(X, Y) = (1 - 2aq)(g(X, Y) - \eta(X)\eta(Y)),
\]

\[
\tilde{\rho}^*(X, Y') = 0,
\]

\[
\tilde{\rho}^*(X', Y') = (1 - 2ap - (2q + 1)(a^2 + b^2 - 1))(g'(X', Y') - \eta'(X')\eta'(Y'))
\]

for $X, Y \in \mathfrak{X}(M), X', Y' \in \mathfrak{X}(M')$.

Remark 2 From (3.3) and (3.34), we see that $\bar{M} = (M, \bar{J}, \bar{g})$ is never weakly $\ast$-Einstein.

4 Examples

Let $S^{2p+1}$ be a $(2p + 1)$-dimensional unit sphere equipped with the canonical Sasakian structure $(\varphi, \xi, \eta, g)$ of constant sectional curvature 1. Then, it can be seen that $S^{2p+1} \times S^{2p+1} = (S^{2p+1} \times S^{2p+1}, J_{0,1}, \bar{g}_{0,1})$ is an Einstein Hermitian manifold (See Remark 1). In this section, we provide further concrete examples of Einstein Hermitian manifolds different from the above trivial one based on the result of Theorem 1. A Sasakian manifold with constant $\varphi$-holomorphic sectional curvature is called a Sasakian space form. First, we recall some fundamental facts concerning a Sasakian space form. Let $M' = (M', \varphi', \xi', \eta', g')$ be a $(2q + 1)(\geq 5)$-dimensional Sasakian space form with constant $\varphi$-holomorphic sectional curvature $c$. Then, it is known that the curvature tensor $R'$ is given by

\[
R'(X', Y')Z' = \frac{c + 3}{4}(g'(Y', Z')X' - g'(X', Z')Y')
\]

\[
+ \frac{c - 1}{4}(\eta'(X')\eta'(Z')Y' - \eta'(Y')\eta'(Z')X')
\]

\[
+ g'(X', Z')\eta'(Y')\xi' - g'(Y', Z')\eta'(X')\xi'
\]

\[
+ g'(\varphi'Y', Z')\varphi'X' - g'(\varphi'X', Z')\varphi'Y'
\]

\[
- 2g'(\varphi'X', Y')\varphi'Z')
\]

for $X', Y', Z' \in \mathfrak{X}(M')$ [2, 10]. Then, from (4.1), we get easily

\[
\rho'(Y', Z') = \frac{1}{2}(q(c + 3) + c - 1)g'(Y', Z') - \frac{q + 1}{2}(c - 1)\eta'(Y')\eta'(Z'),
\]

for $Y', Z' \in \mathfrak{X}(M')$.

Now, we shall introduce the following fact [2, p. 114].

Theorem 2 Let $(M, \varphi, \xi, \eta, g)$ be a $(2q + 1)(q > 1)$-dimensional Sasakian space form with constant $\varphi$-holomorphic sectional curvature $c$ and apply the following $D$-homothetic deformation

\[
\eta' = \alpha\eta, \quad \xi' = \frac{1}{\alpha}\xi, \quad \varphi' = \varphi, \quad \varphi' = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta.
\]

$\alpha$ being a positive constant, to the Sasakian structure $(\varphi, \xi, \eta, g)$. Then, $(M, \varphi', \xi', \eta', g')$ is a Sasakian space form with constant $\varphi$-holomorphic sectional curvature $c' = \frac{c + 3}{\alpha} - 3$.
Let \((S^{2q+1}, \varphi, \xi, \eta, g)\) be a \((2q + 1)\)-dimensional unit sphere equipped with the canonical Sasakian structure \((\varphi, \xi, \eta, g)\) of constant sectional curvature 1. Then, by applying \(D\)-homothetic deformation with \(\alpha = \frac{4}{\sqrt{q^2 + 1}}(c > -3)\) to the canonical Sasakian structure \((\varphi, \xi, \eta, g)\), we may obtain a Sasakian space form \((S^{2q+1}, \varphi', \xi', \eta', g')\) of constant \(\varphi\)-holomorphic sectional curvature \(c\) by virtue of Theorem 2. Now, for any positive integer \(p, q(p \neq q)\), we set

\[
c = \frac{4p}{q} - 3. \tag{4.3}
\]

Then, from (4.3), we may easily check that the both of following equalities

\[
\frac{1}{2}(q(c + 3) + c - 1) = 2\left(\frac{p + 1}{q} - 1\right), \tag{4.4}
\]

and

\[
c - 1 = 4\left(\frac{p}{q} - 1\right) \tag{4.5}
\]

hold. Thus, from Theorem 1 and (3.29), (4.2), (4.4), (4.5), we see that \((S^{2p+1} \times S^{2q+1}, \tilde{J}_{0, \sqrt{p/q}}, \tilde{g}_{0, \sqrt{q/p}})(p \neq q, q > 1)\) provides a non-trivial example of Einstein Hermitian manifold, where \(S^{2p+1}\) (resp. \(S^{2q+1}\)) is \((2p + 1)\)-dimensional (resp. \((2q + 1)\)-dimensional) unit sphere equipped with the canonical Sasakian structures of constant sectional curvature 1. Therefore, taking account of the Remarks 1, 2 and the examples provided in this section, we see that there exists a \(2n\)-dimensional compact Einstein Hermitian manifold for any integer \(n \geq 3\) which is not weakly \(\ast\)-Einstein. However, the situation concerning this fact is quite different in the 4-dimensional case. In fact, by applying the Riemannian version of the Goldberg–Sachs Theorem to a compact Einstein Hermitian surface \(M\) it follows that \(M\) is a locally conformal Kähler [1,6]. We may easily check that the Ricci \(\ast\)-tensor \(\rho^\ast\) is symmetric in a locally conformal Kähler surface since the Lee form is closed [3, (2.16)]. Thus, taking account of the identity (2.2) on any almost Hermitian surface, we see easily that \(M\) is also weakly \(\ast\)-Einstein.

**Remark 3** From (3.34), we see easily check that the \(\ast\)-scalar curvature \(\bar{\tau}^\ast\) of the Einstein Hermitian manifold \((S^{2p+1} \times S^{2q+1}, \tilde{J}_{0, \sqrt{p/q}}, \tilde{g}_{0, \sqrt{q/p}})(p \neq q, q > 1)\) is given by \(\bar{\tau}^\ast = 4q(1 - p + q)\). Thus, taking account of the Remark 1, we see that there exists a \(2n\)-dimensional compact Einstein Hermitian manifold with constant \(\ast\)-scalar curvature for any integer \(n \geq 3\) which is not Kähler, and this completes the assertion of [6].

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