EXPLICIT DESCENT OVER $X(3)$ AND $X(5)$

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Abstract. Let $E$ be an elliptic curve over a field $K$, and let $p$ denote either 3 or 5. In the case that all of the $p$-torsion points of $E$ are defined over $K$, we give an explicit description of $p$-descent on $E$ in two parts. The first part consists of an explicit map from the finite abelian group $E(K)/pE(K)$ to $Sel(E)[p]$, and the second part is an explicit identification of $Sel(E)[p]$ with models of principal homogeneous spaces of $E$ inside $\mathbb{P}^{p-1}$.

1. Introduction

This paper represents a complete solution, in a special case, to the problem of performing explicit $p$-descent on elliptic curves for the primes $p = 3$ and $p = 5$. Namely, we work with the assumption that all of the $p$-torsion points of the elliptic curve $E$ are defined over the field of definition of $E$. From this vantage point, an obstruction (called the ‘period-index obstruction’) to constructing smooth models of genus one curves in low-dimensional projective spaces is extremely concrete and in some sense is the only obstruction to performing descent. The general case, when there are no rationality assumptions on the $p$-torsion, also contains this obstruction and much more; we believe it is instructive to isolate this obstruction, i.e. to consider this a kind of ‘base case’ for a general method.

A few remarks are warranted. First, 2-descent over $\mathbb{Q}$ is completely known and implemented for the computer with no rationality assumptions on the 2-torsion points. References for 2-descent include John Cremona’s ‘mwrank’ program and [5] (see pages 84 and 85 for the manifestation of the period-index obstruction in the case of 2-descent), based on the methods explained in [2]. For complete 2-descent over a general number field see [12]. An algorithmic method for $p$-descent with no rationality assumption and which generalises to some higher genus cases can be found in [14].

Next, why do 3-descent if 2-descent is known? Loosely speaking, when there is nontrivial 2-torsion in $\mathbb{III}$, 2-descent is not effective, but 3-descent may be, if the 3-torsion of $\mathbb{III}$ is trivial. In other words the obstruction to the effectiveness of $p$-descent is $\mathbb{III}[p]$. Since $\mathbb{III}$ is conjecturally finite, descent will be ‘eventually effective.’ Certainly it is useful to have two or three primes completely understood, and maybe more if possible.

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Next, we know (by the Galois equivariance of the Weil pairing) that no elliptic curve defined over \( \mathbb{Q} \) satisfies the assumption that all of its 3 or 5 torsion is rational. Therefore we must be taking elliptic curves over larger number fields. However, given an elliptic curve \( E \) over \( \mathbb{Q} \), it is still worthwhile to 'base-change' to a number field \( K \) where the assumption does hold; performing descent over \( K \) may determine, for example, the Mordell-Weil rank of \( E \) over \( K \), thus bounding the Mordell-Weil rank of \( E \) over \( \mathbb{Q} \).

Work is currently in progress, involving the author and others, to explicitly perform 3-descent on \( E \) (in Weierstrass form) without any assumptions of rationality of the 3-torsion points of \( E \).

The paper is organised as follows: first we will give the general set-up, reviewing the mechanics of \( p \)-descent on elliptic curves and our approach, which will consist of two basic parts. Next we will work through the cases \( p = 3 \) and \( p = 5 \) separately.

2. The General Set-up

Fix a number field \( K \) and an elliptic curve \( (E, \mathcal{O}_E \subset E(K)) \) defined over \( K \), where \( \mathcal{O}_E \subset E(K) \) is the origin of the group law for \( E(K) \). Then the Mordell-Weil group \( E(K) \) is known to be a finitely generated abelian group, so we can write it as a product of a free part and a torsion part: \( E(K) = \mathbb{Z}^r \times E(K)_{\text{tors}} \). For a prime \( p \in \mathbb{N} \), we then know that \( E(K)/pE(K) = (\mathbb{Z}/p\mathbb{Z})^r \times E[p](K) \) is finite, and its dimension as an \( \mathbb{F}_p \)-vector space is \( r + \epsilon \), where \( \epsilon = 0, 1 \) or 2. A major and historical goal of number theorists is to compute \( r \). The theory of \( p \)-descent is one approach.

Indeed, ‘(The first) \( p \)-descent on \( E \’) can be described as an attempt to compute the finite group \( \text{Sel}(E)[p] \); this group sits in a commutative diagram whose rows are exact sequences and where each group on the top row is a subgroup of the corresponding group on the bottom row:

\[
\begin{array}{cccccc}
1 & \to & E(K)/pE(K) & \to & \text{Sel}(E)[p] & \to & \text{III}(E)[p] & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & E(K)/pE(K) & \to & H^1(K, E[p]) & \to & H^1(K, E) & \to & 1
\end{array}
\]

The elements of the torsion cohomological group \( H^1(K, E) \) have long been (abstractly) identified with (isomorphism classes of) principal homogeneous spaces for \( E \). A nontrivial class of \( H^1(K, E) \) is represented by a genus one curve with no points defined over \( K \) and whose Jacobian elliptic curve is isomorphic to \( E \). The right top group \( \text{III}(E) \) is a subgroup of \( H^1(K, E) \) and as such, an element of \( \text{III}(E) \) is characterized as a class represented by a curve \( C \) for which, for every place \( v \) of \( K \), there is a \( K_v \)-rational point of \( C \). A famous conjecture states that \( \text{III}(E) \) is finite.

If in fact \( \text{III}(E)[p] \) is trivial, then ‘knowing \( \text{Sel}(E)[p] \)’ is equivalent to ‘knowing \( E(K)/pE(K), \)’ which by the above discussion is very close to computing \( r \); indeed if one has a handle on the \( p \)-torsion of \( E \), then it is the same. If on the other hand
\( \mathrm{III}(E)[p] \) is non-trivial, then one proceeds to the second level of descent involving the following commutative diagram:

\[
\begin{array}{ccc}
1 & \rightarrow & E(K)/p^2 E(K) \rightarrow \, \text{Sel}(E)[p^2] \rightarrow \, \text{III}(E)[p^2] \rightarrow \, 1 \\
\downarrow & & \downarrow \\
1 & \rightarrow & E(K)/p E(K) \rightarrow \, \text{Sel}(E)[p] \rightarrow \, \text{III}(E)[p] \rightarrow \, 1
\end{array}
\]

The key point is that the right-most vertical arrow is ‘multiplication by \( p \),’ so if \( \text{III}[p^2] = \text{III}[p] \), then the image of the middle vertical map all comes from the left. In other words, we can think of the initial contribution from \( \text{III}[p] \) as noise which we’ve filtered out by going to the second level of descent. Notice if \( \text{III}(E) \) is finite, descent will eventually be effective.

Since we’ve already understood 2-descent, it might seem natural to next try to understand 4-descent. However, another approach to ‘filtering out’ the noise of \( \text{III}(E)[2] \) is to switch primes; that is, we hope that if \( \text{III}(E)[2] \) is non-trivial, than \( \text{III}(E)[3] \) will be trivial. Moreover, the actual computations grow very quickly; we therefore save ourselves work by restricting to \( p = 3 \).

A more refined description of the goal of \( p \)-descent is then to not only compute the middle group \( \text{Sel}(E)[p] \) but to piece together what part ‘comes from the left’ and what part ‘contributes to the right;’ thus descent breaks naturally into two basic parts. The first part is simply an explicit description of the first map. The second part is an explicit identification of elements of \( \text{Sel}(E)[p] \) with smooth models of the appropriate genus one curves inside projective space. The reason we want explicit models is that we can then actually look for ‘local points,’ that is, points over the field \( K_v \), and thus determine whether that curve contributes to the right.

Now assume \( K \) is large enough so that \( E[p](K) = E[p](\overline{K}) \), and choose an \( \mathbb{F}_p \) basis \( S, T \) of \( E[p](K) \) with fixed Weil pairing. Then the Galois cohomology group \( H^1(K, E[p]) \) can be identified with the group \( K^*/K^{*p} \times K^*/K^{*p} \) (see Lemma 3.1 on page 4 of \( \textbf{[1]} \)), and its subgroup \( \text{Sel}(E)[p] \) can be identified with a finite subgroup of \( K^*/K^{*p} \times K^*/K^{*p} \).

For the first part of descent (see Chapter X of \( \textbf{[1]} \)), we find a pair of rational functions \( (f_S, f_T) \) on \( E \) which, when evaluated at a point of \( E(K) \), give its image in \( H^1(G, E[p]) \cong K^*/K^{*p} \times K^*/K^{*p} \). Note this will depend on the chosen basis above. By Theorem 1.1d on page 278 of \( \textbf{[1]} \), it is sufficient for the functions \( f_S \) and \( f_T \) to satisfy \( \text{div}(f_S) = p \cdot (S) - p \cdot (O_E) \) and \( \text{div}(f_T) = p \cdot (T) - p \cdot (O_E) \) respectively; moreover, they can be chosen to satisfy \( f_S \circ [p] = g^p_S \) and \( f_T \circ [p] = g^p_T \) for some rational functions \( g_S \) and \( g_T \).

**Lemma 2.1.** The expansions of \( f_S \) and \( f_T \) with respect to a local parameter at \( O_E \) can be chosen to have leading coefficients which are perfect \( p \)th powers, i.e. so \( f_S = a + \ldots \) and \( f_T = b + \ldots \) where both \( a \) and \( b \) are perfect \( p \)th powers and where \( t \in O_{E,O_E} \) is a parameter in the local ring at \( O_E \). Here the ‘...’ refer to ‘higher order terms.’
Proof. First we prove that locally the expansions look like:

\[ f_S = \frac{a}{t^p} + \ldots, \quad g_S = \frac{c}{t} + \ldots, \quad \text{and} \quad t \circ [p] = d \cdot t + \ldots. \]

The first is because we know \( f_S \) has a pole of order \( p \) at \( O_E \), the second because \( g_S \) has a simple pole at \( O_E \); in fact \( \text{div}(g_S) = \sum_{p \in S} (P_i) - \sum_{p \in \mathcal{O}(Q_i)} \), and we get only one copy of \( \mathcal{O}_E \) on the right. Finally, \( t \circ [p] \) has a simple zero at \( O_E \) since \( [p]O_E = O_E \) and \( [p] \) is étale when \( \text{char}(K) \neq p \). We know that \( f_T \circ [p] = g_T^p \), and a quick calculation shows this is equivalent to \( a \) being a perfect \( p \)th power. \( \square \)

The background for the second part is to be found in [3], and is summarized as follows.

We want to find an efficient model of \( C \). However, there isn’t always a degree three line bundle on \( C \), so we can’t always model \( C \) as a smooth cubic in \( \mathbb{P}^2 \). The order of \( C \) as an element of \( H^1(K, E) \) (called the ‘period’ of \( C \)) is closely related to the smallest degree of a line bundle \( \mathcal{L} \) on \( C \) (called the ‘index’ of \( C \)). To explain this better, we will view the Selmer group \( \text{Sel}(E)[p] \) as a subgroup of the cohomology group \( H^1(K, E[\mathfrak{n}]) \). By Proposition 2.2 on page 3 of [9], elements of \( H^1(K, E[\mathfrak{n}]) \) correspond to diagrams \( C \rightarrow S \) where \( C \) is a period \( n \) principal homogeneous spaces of the elliptic curve \( E \) as above, and where \( S \) is a Brauer-Severi variety of dimension \( n - 1 \). These diagrams are twists of a fixed diagram \( E \rightarrow \mathbb{P}^{n-1} \), given by the divisor \( n \cdot O_E \). Under the natural map from \( H^1(K, E[p]) \) to \( H^1(K, E) \), the diagram \( C \rightarrow S \) goes to \( C \). The other forgetful map, sending \( C \rightarrow S \) to \( S \), is a quadratic map from \( H^1(K, E[p]) \) to \( H^1(K, \text{PGL}_p) \subset H^2(K, \mathcal{G}_m)[p] = \text{Br}(K)[p] \) (for this inclusion see p. 158 of [10]) and is called the period-index obstruction map. The obstruction is trivial exactly when \( S \) is isomorphic over \( K \) to \( \mathbb{P}^{p-1} \), and in this case we say that the diagram \( C \rightarrow S \) has its period equal to its index. In our situation we will make use of our identification \( H^1(K, E[p]) \cong K^*/K^{*p} \times K^*/K^{*p} \); then the period-index obstruction for an element \( (a, b) \in K^*/K^{*p} \times K^*/K^{*p} \) is (Proposition 3.4 on page 6 of [10]) its associated ‘Hilbert symbol’ \( (a, b)_{\text{Hib},p} \). In particular, \( (a, b)_{\text{Hib},p} \) is trivial exactly when \( b \) is in the image of the norm map from the field \( K(\alpha) \) to \( K \), where \( \alpha \) is chosen such that \( \alpha^p = a \).

A crucial fact that we will take advantage of is that the elements of \( \text{Sel}(E)[p] \), which a priori correspond to diagrams \( C \rightarrow S \), always have \( S \cong K \mathbb{P}^{p-1} \); in other words, diagrams \( C \rightarrow S \) where \( C \in \text{III}(E) \) always have trivial period-index obstruction (for a proof see the Remark on page 3 of [3]).

In this article we will perform the second part of descent not only for elements of \( \text{Sel}(E)[p] \) but for any element of \( H^1(K, E[p]) \) which has trivial period-index obstruction. Indeed we will start with a “trivialisation” of a Hilbert symbol (the data of an element \( \beta \in K[x]/(x^p - a) \) whose norm is \( a \)), and produce a model for the corresponding genus one curve under the isomorphism \( H^1(K, E[p]) \cong K^*/K^{*p} \times K^*/K^{*p} \).
3. Explicit Descent over $X(3)$

3.1. The image of $E(K)/3E(K)$. Let $K$ be a field whose characteristic is different from 3 and which contains a primitive 3rd root of unity $\zeta$. Let $E$ be an elliptic curve over $K$ with full 3-torsion given by the following equation:

$$E : X^3 + Y^3 + Z^3 + \lambda XYZ = 0.$$ 

Define the origin $O_E$ to be the point $(1; -1; 0)$ and a basis of 3-torsion points $< S, T >$ to be $S = (1; \zeta; 0), T = (1; 0; -1)$, where $\zeta \in K$ is a fixed cube root of unity.

By results in [8], there exist matrices in $PGL_3$ which act as “translation by $S$ and $T$” on $E$ and which we will denote by $M_S$ and $M_T$. We leave it to the reader to prove these matrices are $D_3$ and $M_{1,3}$ respectively, where $D_n$ is the diagonal matrix $diag(1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1})$ for some primitive $n$ root of unity $\zeta_n$, (here take $\zeta_3 = \zeta$) and where

$$M_{a,n} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
: & : & : & \vdots & : \\
0 & 0 & 0 & \ldots & 1 \\
a & 0 & 0 & \ldots & 0
\end{pmatrix}.$$

We will find a pair of rational functions $(f_S, f_T)$ on $E$ which when evaluated at a point of $E(K)$ gives its image in $H^1(G, E[3]) \cong K^*/K^{*3} \times K^*/K^{*3}$. By Lemma 2.1, the function $f_S$ needs to satisfy the condition that $\text{div}(f_S) = 3 \cdot (S) - 3 \cdot (O_E)$; this is satisfied trivially by the quotient of the hyperplane at $S$ by the hyperplane at $O_E$, since they are both flex points. The hyperplane at $O_E$ is easily seen to be given by $3X + 3Y - \lambda Z$, and the other hyperplanes are in fact translates of this one by the above matrices; this observation doesn’t save much time in $\mathbb{P}^2$, since hyperplanes are easy to compute, but will in the $n = 5$ case. Similarly $f_T$ is, up to a constant, a ratio of hyperplanes, and we get $f_S = \kappa_S \frac{3\zeta^2 X + \zeta Y - \lambda Z}{3X + 3Y - \lambda Z}$ and $f_T = \kappa_T \frac{3X - \lambda Y + 3Z}{3X + 3Y - \lambda Z}$ for some constants $\kappa_S$ and $\kappa_T$.

**Proposition 3.1.** $f_S = (\lambda^3 + 27) \frac{3\zeta^2 X + 3\zeta Y - \lambda Z}{3X + 3Y - \lambda Z}$ and $f_T = (\lambda^2 - 3\lambda + 9) \frac{3X - \lambda Y + 3Z}{3X + 3Y - \lambda Z}$.

**Proof.** By Lemma 2.1, we want then to compute the expansion of $\frac{3\zeta^2 X + 3\zeta Y - \lambda Z}{3X + 3Y - \lambda Z}$ at the origin $O_E = (1; -1; 0)$. Here we can work in affine coordinates by setting $X = 1$, since this is true locally. Then $F : 1 + Y^3 + Z^3 + \lambda YZ = 0$. Moreover, since $O_E$ takes on a non-zero value at the hyperplane at $S$, we can just evaluate there to get $(3\zeta^2 X + 3\zeta Y - \lambda Z)|_{O_E} = 3\zeta^2 - 3\zeta$. Note that $\zeta^2 - \zeta = \sqrt{-3}$, so $3\zeta^2 - 3\zeta = (-\sqrt{-3})^3$ is a cube in $K$. So in fact we can completely ignore this term. We’re left with

$$\frac{1}{3X + 3Y - \lambda Z}$$ 

which has a triple zero at $O_E$. In other words

$$3 + 3Y - \lambda Z \cong aZ^3 + \ldots,$$
where \( \cong \) signifies that we are working in \( \mathcal{O}_E,O_E \). Multiply the above by the function \( Y \), a nonzero function at \( O_E \), to get \( 3Y + 3Y^2 - \lambda Y Z \cong aY^3 + \ldots \); since \( -\lambda Y Z \cong 1 + Y^3 + Z^3 \) (we’re working modulo \( F \)) we substitute to get

\[
1 + 3Y + 3Y^2 + Y^3 + Z^3 = (1 + Y)^3 + Z^3 \cong aY^3 + \ldots.
\]

The function \( 1 + Y \) has a zero at \( O_E \), so it’s an alternative parameter for the local ring \( \mathcal{O}_{E,O_E} : 1 + Y \cong bZ + \ldots \) but we already have the tangent line equation which tells us that \( 3(1 + Y) \cong \lambda Z + \ldots \), i.e. \( b = \lambda/3 \). Replace \( (1 + Y)^3 \) above now by \( (\lambda/3)Z^3 \):

\[
Z^3(1 + (\lambda/3)^3) \cong aY^3.
\]

Evaluate at \( Y = -1 \) to get \( a = -(1 + (\lambda/3)^3) \). Our original constant then is \( 1/a \), which modulo cubes is seen to be \( \frac{1}{27 + \lambda^3} \). Finally, to normalize we want the function \( f_S \) to have a leading coefficient which is \( 1 \). In other words, we need to multiply the ratio of the two hyperplanes by the constant \( 27 + \lambda^3 \). For \( f_T \), we are already almost done- the only difference is the value of the numerator at the origin: \( (3X - \lambda Y + 3Z)|_{O_E} = 3 + \lambda \). The leading coefficient then is \( \frac{3 + \lambda}{27 + \lambda^3} = \frac{1}{3^2 - \lambda + 9} \). \( \square \)

### 3.2. Models of Genus One curves in \( \mathbb{P}^2 \)

Let \( E \) be an elliptic curve over the field \( K \). Assume \( \text{char}(K) \neq 3 \) and that \( E[3](K) = E[3](\mathbb{K}) \). Choose a basis \( \langle S, T \rangle \) of \( E[3] \) and identify \( H^1(K,E[3]) \) with \( K^*/K^{*3} \times K^*/K^{*3} \). Take \( (a,b) \in H^1(K,E[3]) \) whose corresponding Hilbert symbol is trivial. If both \( a \) and \( b \) are cubes, the element \( (a,b) \) represents \( E \). We will first assume that \( a \) is not a cube in \( K \); the other case will be dealt with at the end of this section. Define \( \alpha, \beta \in \mathbb{K} \) such that \( \alpha^3 = a \) and \( \beta = \sum_{i=0}^{2} \beta_i \alpha^i \) is such that \( \mathbb{N}_{K(\alpha)/K}(\beta) = b \). Moreover define \( \sigma \in G(K(\alpha)/K) \) to be the generator of that cyclic Galois group such that \( \sigma(\alpha) = \alpha \zeta \). Let \( Tr \) be the trace function of \( K(\alpha) \) to \( K \), and let \( u = \beta/\sigma^2(\beta) \).

**Theorem 3.2.** The curve \( (a,b) \) is given by

\[
(Tr(u) + \lambda)(a^2X^3 + aY^3 + Z^3) + 3Tr(\alpha u)(aX^2Z + aY^2X + Z^2Y) + 3Tr(\alpha^2u)(aX^2Y + Y^2Z + Z^2X) + 3(2aTr(u) - \lambda a)XYZ = 0.
\]

**Proof.** By results in \( \S \), the action of the 3-torsion points of \( E \) on a a genus one curve \( C \) embedded in \( \mathbb{P}^2 \) can be represented as automorphisms of \( \mathbb{P}^2 \), in other words as elements of \( \text{PGL}_3(K) \). Moreover, if \( C \) is represented in \( H^1(G,E[3]) \) by the pair \( (a,b) \) which depends on a chosen basis \( < S,T > \) then the determinants of the matrices representing “translation by \( S \)” and “translation by \( T \)” are \( a \) and \( b \) respectively. Moreover the Weil pairing is given by the commutator of lifts to \( \text{GL}_3(K) \). With that in mind, we will search for \( C \leftrightarrow (a,b) \) by finding cubics which are invariant under the action of standard matrices representing this 3-torsion action.

Define \( M_S \) to be the matrix \( M_{a,3} \) and \( M_T \) to be the matrix \( D_3 [\beta_0 I + \beta_1 M_S + \beta_2 M_S^2] \), if \( \beta = \beta_0 + \beta_1 \alpha + \beta_2 \alpha^2 \). The determinant of \( M_S \) is \( a \), the determinant of \( M_T \) is \( b \), and
the commutator $[M_S, M_T]$ is $\zeta I$. A cubic which is invariant under the action of $M_S$ but with no fixed points must be of the form

$$F = A(a^2X^3 + aY^3 + Z^3) + B(aX^2Z + aY^2X + Z^2Y) + C(aX^2Y + Y^2Z + Z^2X) + 3DXYZ = 0.$$ 

This is because $M_S$ acts linearly on the 10-dimensional space of cubics. There are 3 eigenspaces of dimensions 3, 3, and 4, and the first two have zeroes at the fixed points of $M_S$, whereas the last eigenspace does not and is generated by the above four cubics.

On the other hand we also insist that $F$ be invariant under the action of $M_T$. To ease computations we introduce the following notation: fix eigenvectors $v_i = (1, \alpha \zeta^i, \alpha^2 \zeta^{2i})$ of $M_S$. Then $M_S v_i = \alpha \zeta^i v_i$. The four coefficients $A, B, C,$ and $D$ are linear combinations of $F(v_0), F(v_1), F(v_2)$, and $T(v_0, v_1, v_2)$, where $T$ is the trilinear form associated to $F$, as follows:

$$\begin{pmatrix} F(v_0) \\ F(v_1) \\ F(v_2) \\ T(v_0, v_1, v_2) \end{pmatrix} = \begin{pmatrix} 3a^2 & 3a^{5/3} & 3a^{4/3} & 3a \\ 3a^2 & 3a^{5/3} & 3a^{4/3} & 3a \\ 3a^2 & 3a^{5/3} & 3a^{4/3} & 3a \\ 18a^2 & 0 & 0 & -9a \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}.$$

Now it is easy to see how $M_T$ acts on the $F(v_i)$:

$$F^{M_T}(v_i) = F(M_T v_i) = F(D(\beta_0 I + \beta_1 M_S + \beta_2 M_S^2) v_i) = F(D(\beta_0 + \beta_1 \alpha \zeta^i + \beta_2 \alpha^2 \zeta^{2i}) v_i) = F(\sigma^i(\beta) D v_i) = \sigma^i(\beta)^3 F(v_{i+1}).$$

Similarly,

$$T^{M_T}(v_0, v_1, v_2) = T(\beta v_1, \sigma(\beta)v_2, \sigma^2(\beta)v_0) = b T(v_0, v_1, v_2).$$

The fact that $F$ is invariant under the action of $M_T$ is equivalent to the projective point $P = (F(v_0); F(v_1); F(v_2); T(v_0, v_1, v_2))$ being fixed by $M_T$. We have seen that $M_T(P) = (\beta^3 F(v_1); \sigma(\beta)^3 F(v_2); \sigma^2(\beta)^3 F(v_0); b T(v_0, v_1, v_2))$. For some $\mu \neq 0$ we have $F(v_1) = \frac{\mu}{\beta^3} F(v_0)$ and $F(v_2) = \frac{\sigma^2(\beta)^3}{\mu} F(v_0)$. Moreover, the Jacobian of the above curve is (see [8]):

$$X^3 + Y^3 + \prod_{i=0}^2 F(v_i) Z^3 + T(v_0, v_1, v_2) XYZ = 0,$$

in other words

$$X^3 + Y^3 + \frac{F(v_0)^3 \sigma^2(\beta)^3}{\beta^3} Z^3 + T(v_0, v_1, v_2) XYZ = 0;$$

setting $F(v_0) = \frac{\beta}{\sigma(\beta)}$ (note that $F(v_0) \neq 0$ because $F$ has no fixed points under the action of $M_S$) and renaming $T(v_0, v_1, v_2) = \lambda$, the Jacobian is exactly $E$. Note that $\sigma(F(v_i)) = F(\sigma(v_i)) = F(v_{i+1})$, so $F(v_1) = \frac{\sigma(\beta)}{\beta}$ and $F(v_2) = \frac{\sigma^2(\beta)}{\sigma(\beta)}$. To finish the
proof, we need to invert the above matrix to find the coefficients $A, B, C, D$ in terms of the $F(v_i)'$s and $T$:

\[
\begin{pmatrix}
A \\
B \\
C \\
D
\end{pmatrix} = \frac{1}{2\alpha^2}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
3\alpha & 3\alpha\zeta & 3\alpha\zeta^2 & 0 \\
3\alpha^2 & 3\alpha^2\zeta^2 & 3\alpha^2\zeta & 0 \\
2a & 2a & 2a & -a
\end{pmatrix}
\begin{pmatrix}
F(v_0) \\
F(v_1) \\
F(v_2) \\
T(v_0, v_1, v_2)
\end{pmatrix}.
\]

We can ignore the factor of $\frac{1}{2\alpha^2}$, since we’re working projectively. Then $A = F(v_0) + F(v_1) + F(v_2) + T(v_0, v_1, v_2) = \frac{\beta}{\sigma(\beta)} + \frac{\sigma'(\beta)}{\beta} + \frac{\sigma^2(\beta)}{\beta} + \lambda = Tr(u) + \lambda$, and similarly for the other coefficients. □

**When a is a cube** The formula in Theorem 3.2 was computed assuming $a$ is not a cube in the base field $K$, but it holds in the case that $a$ is a cube as well. We can view the formula as holding over the algebra $K[\alpha]/(\alpha^3 - a)$, which is a field if $a$ is not a cube; in other words we view $\alpha$ as strictly a symbol. The norm function on this algebra takes an element $s = s_0 + s_1\alpha + s_2\alpha^2$ to the expression $N(s) = \prod_{i=0}^2(s_0 + s_1\alpha^i + s_2\alpha^2\zeta^i) = s_0^2 + s_1^2a + s_2^2a^2 - 3s_0s_1s_2a$. Similarly, the trace function has $Tr(s) = \sum_{i=0}^2(s_0 + s_1\alpha^i + s_2\alpha^2\zeta^i) = 3 \cdot s_0$, since $1 + \zeta + \zeta^2 = 0$. In other words, we can define the linear automorphism ‘$\sigma$’ to take $\alpha$ to $\zeta\alpha$, and $N(s) = s\sigma(s)\sigma^2(s)$. In the case of representing a curve corresponding to the pair $(a, b)$ for $a$ a cube, we may take $a = 1$ and we want to find $\beta = \beta_0 + \beta_1\alpha + \beta_2\alpha^2$ such that $N(\beta) = \beta\sigma(\beta)\sigma^2(\beta) = b$. There are many solutions: for example, we can take $\beta_0 = \frac{b+2}{3}$ and $\beta_1 = \beta_2 = \frac{b-1}{3}$ if those are nonzero quantities. Then as above we can formally compute $Tr(u) = (b - 1)^2/b + 3$, $Tr(\alpha u) = (b - 1)(b - \zeta^2)/b$, and $Tr(\alpha^2 u) = (b - 1)(b - \zeta)/b$.

4. **Explicit Descent over $X(5)$**

4.1. **The image of $E(K)/5E(K)$**. Let $K$ be a field whose characteristic is different from 5 and which contains a primitive 5th root of unity $\zeta$. Let $E = E_\lambda$ be the elliptic curve over $K$ given by the equations $\lambda x_i^2 + \lambda^2 x_{i-2}x_{i+2} - x_{i-1}x_{i+1}$ for $i$ between 0 and 4 and whose origin is given by $O_E = (0; \lambda; 1; -1; -\lambda)$.

**Remark.** The curves $E_\lambda$ for $n = 3, 5$ are also considered in [3], where the Cassels-Tate pairing is used to compute $S^{(n)}(E_\lambda/Q)$ for all non-cuspidal $\lambda \in Q$. Equations for $E_\lambda$ for $n = 5$ in Weierstrass form may be found in [3] or [4].

The curve $E$ as above has full 5-torsion over $K$ and the matrices which act as translation by a $S$ and $T$, for a basis $(S, T)$, are given by $D_5$ and $M_{1,5}$ respectively (see notation on page 5).

We will find a pair of rational functions $(f_S, f_T)$ on $E$ which when evaluated at a point of $E(K)$ gives its image in $H^1(G, E[5]) \cong K^*/K^{*5} \times K^*/K^{*5}$. By Lemma 2.3, the function $f_S$ can be chosen to satisfy $div(f_S) = 5 \cdot (S) - 5 \cdot (O_E)$; then we can
Proposition 4.2. The hypertangent planes at $E$ take $f_S$ to be a scalar multiple of the quotient of the hypertangent plane at $S$ by the hypertangent plane at $O$, since both $S$ and $O$ are hyperflex points (to see this, note that $E$ is a degree 5 curve in $\mathbb{P}^4_K$ and that the hyperplane $x_0 = 0$ goes through the points $i \cdot S$ for $i$ between 0 and 4, which means that the divisor giving the embedding $E \to \mathbb{P}^4$ is linearly equivalent to $(O_E) + (S) + (2S) + (3S) + (4S) \equiv 5 \cdot (O_E) \equiv 5 \cdot (S)$).

Proposition 4.1. The hypertangent plane at the origin of $E$ is given by

$$H_{O_E} : \alpha x_0 + \beta(x_1 + x_4) + \gamma(x_2 + x_3),$$

where $\alpha = \lambda^{10} - 14\lambda^5 - 1$, $\beta = -5\lambda^2(1 + 2\lambda^5)$, and $\gamma = 5\lambda^3(\lambda^5 - 2)$.

Proof. An easy calculation (in Maple for example) verifies that the above hyperplane intersects $E$ only at $O_E$. In order to find the above, we computed a local parameterization of the curve $E$ in the local ring $O_{E,O_E}$ using the equations for $E$ and Maple. □

To finish finding the equations $f_S$ and $f_T$, we simply find the hypertangent planes at $S$ and $T$ (these are translates of $H_{O_E}$ by the 5-torsion matrices $D_5$ and $M_{1,5}$) and evaluate them at $O_E$. We then scale $H_S/H_{O_E}$ by the appropriate function of $\lambda$ so that its leading coefficient in the expansion at $O_E$ is a perfect fifth power:

Proposition 4.2. The hypertangent planes at $S$ and at $T$ are given by

$$H_S : \alpha x_0 + \beta(x_1 + x_4) + \gamma(x_2 + x_3 + 2)$$

and

$$H_T : \alpha x_4 + \beta(x_0 + x_3) + \gamma(x_1 + x_2).$$

The rational functions $f_S$ and $f_T$ are given by

$$f_S = \frac{[(\lambda^2 + \lambda - 1)(\lambda^4 - 3\lambda^3 + 4\lambda^2 - 2\lambda + 1)(\lambda^4 + 2\lambda^3 + 4\lambda^2 + 3\lambda + 1)]}{5 \lambda (\lambda - \zeta^4)(\lambda^5 - 2)} \cdot \frac{H_S}{H_{O_E}};$$

$$f_T = \frac{\lambda^2(\lambda^4 - 3\lambda^3 + 4\lambda^2 - 2\lambda + 1)^2(\lambda^4 + 2\lambda^3 + 4\lambda^2 + 3\lambda + 1)}{(\lambda^2 + \lambda - 1)} \cdot \frac{H_T}{H_{O_E}}.$$

4.2. Models of Genus One curves in $\mathbb{P}^4$. Let $E$ be an elliptic curve over the field $K$. Assume $\text{char}(K) \neq 5$ and that $E[5](K) = E[5](\overline{K})$. Choose a basis $\langle S, T \rangle$ of $E[5]$ and identify $H^1(K, E[5])$ with $K^*/K^{*5} \times K^*/K^{*5}$. For $(a, b) \in H^1(K, E[5])$ whose corresponding Hilbert symbol is trivial, assume as in the previous section that $a$ is not a perfect fifth power in $K$, and define $\alpha, \beta \in \overline{K}$ such that $\alpha^5 = a$ and $\beta = \sum_{i=0}^4 \beta_i \alpha^i$ such that $N_{K(\alpha)/K}(\beta) = b$. Moreover define $\sigma \in G(K(\alpha)/K)$ to be the generator of that cyclic Galois group such that $\sigma(\alpha) = a^5$.

Theorem 4.3. There exists a unique 5-dimensional $K$-vector space of quadrics in the variables $x_i, i = 0, \ldots, 4$, denoted $V_{\alpha, \beta}$, satisfying the following:

- $V_{\alpha, \beta}$ defines a smooth degree 5 genus one curve $C_{\alpha, \beta}$ in $\mathbb{P}^4$. 

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Definition 4.2. Define the action of a matrix \( K \) is unique up to map defined over \( M \) has the effect of conjugating \( Q \) since we need to make sense of the nonzero quantities \( \alpha \zeta \).

\[ \blacksquare \]

We are actually taking a fixed quadric

\[ \text{Definition 4.1.} \] Define, for \( M \) of \( \text{PGL}_n(C) \) parameterizes diagrams \( C \rightarrow \mathbb{P}^{n-1} \) up to isomorphism, where an isomorphism is a map defined over \( K \) from \( C \) to \( C \) which extends to a \( K \)-map on \( \mathbb{P}^{n-1} \), i.e. an element of \( \text{PGL}_n(K) \). Then \( V_{\alpha, \beta} \) determines such a diagram, and the prescribed action of \( E[n] \) fixes the model: modifying a diagram \( C \rightarrow \mathbb{P}^{n-1} \) by an element of \( \text{PGL}_n(K) \) has the effect of conjugating \( M_S \) and \( M_T \). Since the group generated by \( M_S \) and \( M_T \) (a twist of the Heisenberg group) is its own centraliser, there is no non-trivial modification. \( \square \)

Our goal of this section is to determine \( V_{\alpha, \beta} \).

\[ \text{Definition 4.2.} \] Define the action of a matrix \( M \) on a function \( f \) to be such that \( f^M(x) = f(Mx) \). Then \( f^{M_1M_2} = (f^{M_1})^{M_2} \).

\[ \text{Lemma 4.4.} \] With notation as above, we can choose a quadric \( Q \in V_{\alpha, \beta} \) such that \( Q(v_0) = Q(v_1) \) and such that \( \langle Q^{M_i} \rangle_{i=0..4} \) forms a basis of \( V_{\alpha, \beta} \). Moreover, such a \( Q \) is unique up to \( K \)-scaling. Therefore to determine \( V_{\alpha, \beta} \) it suffices to determine \( Q \).

\[ \text{Remark.} \] We are actually taking a fixed quadric \( Q \) (not defined up to a scalar), since we need to make sense of the nonzero quantities \( Q(v_0) \) and \( Q(v_1) \).

\[ \text{Proof.} \] We can span \( V_{\alpha, \beta} \) by translates of any \( K \)-rational quadric \( Q \) by the action of powers of \( M_S \) since the eigenvalues of \( M_S \) acting on the space of quadrics are the fifth roots of \( a^2 \), not defined over \( K \). Note that \( Q(v_i) \neq 0 \) for all \( i \) since if so we would have \( Q^{M_S}(v_i) = Q(M_Sv_i) = 0 \) as well, in other words we would have a point on \( C \) fixed by \( M_S \), namely the projectivization of \( v_i \). Next, note that \( Q^{M_S}(v_0) = Q(M_Sv_0) = Q(\alpha v_0) = Q(\alpha^2v_0) = \alpha^2Q(v_0) \). Therefore if we replace \( Q \) by the quadric \( Q' = \sum_{i=0}^4 a_iQ^{M_S} \), for \( a_i \in K \), then \( Q'(v_0) = \sum_{i=0}^4 a_iQ^{M_S}(v_0) = (\sum_{i=0}^4 a_i\alpha^{2i})Q(v_0) \), and likewise \( Q'(v_1) = \sum_{i=0}^4 a_iQ^{M_S}(v_1) = (\sum_{i=0}^4 a_i\alpha^{2i} \zeta^{2i})Q(v_1) \). So in order to have \( Q'(v_0) = Q'(v_1) \), we need to find \( a = \sum_{i=0}^4 a_i\alpha^{2i} \) so that \( Q(v_0)/Q(v_1) = \sigma(a)/a \). This is possible by Hilbert’s Theorem 90, since \( Q(v_0)/Q(v_1) = Q(v_0)/\sigma(Q(v_0)) \) is in the kernel of the norm map. Next, \( Q' \) and its translates under \( M_S \) also generate \( V_{\alpha, \beta} \). The \( K \)-rational quadric \( Q' \) is nonzero since its value at \( v_0 \) is nonzero; by the above comment its
translates by powers of $M_S$ span $V_{a,b}$. Finally, such a $Q'$ is unique up to an element of $K$, since if we had both $Q$ and $Q'$ such that $Q(v_0)/Q(v_1) = Q'(v_0)/Q'(v_1) = 1$, we could write $Q' = \sum_{i=0}^4 a_i Q_{M_S}^i$ for $a_i \in K$ to get $Q'(v_0)/Q'(v_1) = a/\sigma(a) \cdot Q(v_0)/Q(v_1)$, i.e. we would have $a = \sigma(a)$, or in other words $a \in K$. □

Our newly defined goal is to find $Q$ as in Lemma 1.4. We will choose a matrix $M$ by which to modify $Q$ as in Lemma 1.4 so that the coefficients of $Q^M$ are easy to manipulate. Define $M = (v_0 v_1 v_2 v_3 v_4)$, a $5 \times 5$ matrix defined over $K(\alpha)$. Then

$$Q^M(x) = Q(Mx) = Q \left( \sum_{i=0}^4 v_i x_i \right) = \sum_{i=0}^4 Q(v_i) x_i^2 + \sum_{0 \leq i < j \leq 4} B(v_i, v_j) x_i x_j,$$

where $B(w, v) = Q(w + v) - Q(w) - Q(v)$. Define $M'_S = M^{-1} M_S M$ and $M'_T = M^{-1} M_T M$. Then we have $(Q^M)^{M_S} = (Q^{M_S})^M$ and $(Q^M)^{M'_T} = (Q^{M'_T})^M$. A calculation shows us $M'_S = \alpha D_5$ and $M'_T = M_{1,5} \cdot \text{diag}(\beta, \sigma(\beta), \sigma^2(\beta), \sigma^3(\beta), \sigma^4(\beta))$.

Next, note that by assumption $M_T$ fixes $V_{a,b}$. Moreover, since $M_T$ is defined over $K$, the image of $Q$ under the action by $M_T$ is again defined over $K$. Thus there exist numbers $\gamma_i \in K$ such that $Q^{M_T} = \sum_{i=0}^4 \gamma_i Q^{M'_S}$. Acting on that equation by $M$ we get (here let $Q' = Q^M$):

$$Q^{M'_T} = \sum_{i=0}^4 \gamma_i Q^{(M'_S)^i}.$$ 

Since we know the $M'_T$ and $M'_S$ explicitly, we can compute the left and right sides of this equation and compare them.

$$Q^{M'_T}(x) = \sum_{i=0}^4 Q(v_i) \sigma^{i-1}(\beta)^2 x_i^2 + \sum_{0 \leq i < j \leq 4} B(v_i, v_j) \sigma^{i-1}(\beta)\sigma^{j-1}(\beta) x_i x_j,$$

and

$$Q^{(M'_S)^i}(x) = \alpha^{2i} \left( \sum_{j=0}^4 Q(v_j) \zeta^{2ji} x_j^2 + \sum_{0 \leq j < k \leq 4} B(v_j, v_k) \zeta^{ji+k} x_j x_k \right).$$

**Proposition 4.5.** For every $i = 0, \ldots, 4$, $Q(v_i) = Q(v_0)$, $B(v_{i+1}, v_{i-1}) = B(v_1, v_4) \cdot \prod_{j=1}^4 \sigma^{j-1} \left( \frac{\beta^2}{\sigma(\beta)\sigma^2(\beta)} \right)$, and $B(v_{i+2}, v_{i-2}) = B(v_2, v_3) \cdot \prod_{j=1}^4 \sigma^{j-1} \left( \frac{\beta^2}{\sigma^2(\beta)\sigma^3(\beta)} \right)$.

**Proof.** We compare the coefficient of $x_i^2$ on both sides of the above equation. On the left, we get $Q(v_1)\beta^2$. On the right we get $\sum_{i=0}^4 \gamma_i \alpha^{2i} Q(v_0)$. Define $\gamma = \sum_{i=0}^4 \gamma_i \alpha^{2i}$, then we have $Q(v_1)\beta^2 = \gamma Q(v_0)$. On the other hand we have chosen $Q$ as in Lemma 1.4 so that $Q(v_0) = Q(v_1) \neq 0$. Therefore $\gamma = \beta^2$. We can determine the rest of the $Q(v_i)$'s now since $1 = \sigma(\frac{Q(v_1)}{Q(v_0)}) = \frac{Q(v_1)}{Q(v_0)}$ etc. so all of the $Q(v_i)$'s are equal to $Q(v_0)$.
Next, compare the coefficients of $x_1 x_4$; on the left we get $B(v_2, v_0) \sigma(\beta) \sigma^4(\beta)$ and on the right we get $\sum_{i=0}^{4} \gamma_i \alpha^i B(v_1, v_4)$. In other words we have $B(v_2, v_0) = \frac{\beta^2}{\sigma(\beta) \sigma^4(\beta)}$.

Acting by $\sigma$ on both sides gives $B(v_3, v_1) = \frac{\beta^2}{\sigma(\beta) \sigma^4(\beta)}$; so $B(v_3, v_1) = \frac{\beta^2}{\sigma(\beta) \sigma^4(\beta)} \cdot \sigma \left( \frac{\beta^2}{\sigma(\beta) \sigma^4(\beta)} \right)$. We continue in this way. Similarly, comparing coefficients of $x_2 x_3$ on both sides we get $B(v_3, v_4) = \frac{\beta^2}{\sigma(\beta) \sigma^4(\beta)}$ and we finish by acting on both sides by $\sigma$ and solving for $B(v_3, v_4)$.

Now we have only to determine the three unknowns $Q(v_0)$, $B(v_1, v_4)$, and $B(v_2, v_3)$; moreover, since we are actually working projectively, we only need to know two of them, or more precisely it is adequate to know the ratios $\frac{Q(v_0)}{B(v_1, v_4)}$ and $\frac{Q(v_0)}{B(v_2, v_3)}$. Define $E_\lambda$ to be the elliptic curve given by the equations $\lambda x_i^2 + \lambda^2 x_{i-2} x_{i+2} - x_{i-1} x_{i+1}$ for $i$ between 0 and 4 and whose origin is given by $O_E = (\lambda; -1; 1; -\lambda; 0)$.

**Theorem 4.6.** The Jacobian of $C_{\alpha, \beta}$ is $E_\lambda$, where

$$\lambda = -\frac{Q(v_0)}{B(v_2, v_3)} \frac{\sigma^3(\beta) \sigma^4(\beta)}{\beta \sigma(\beta)}.$$

**Proof.** By [3], Theorem 4.2 on page 37, the Jacobian of $C_{\alpha, \beta}$ is given as $E_A$ given by the quadrics $S_0 = x_0^2 - x_2 x_3 + x_1 x_4$, $S_1 = x_1^2 - x_0 x_2 + A x_3 x_4$, $S_2 = x_2^2 - x_1 x_3 - A x_0 x_4$, $S_3 = x_3^2 - x_0 x_1 - x_2 x_4$, and $S_4 = A x_4^2 + x_1 x_2 - x_0 x_3$ for a parameter $A = \prod_{i=0}^{4} \frac{Q(v_i)}{B(v_i, v_{i+1})}$.

A calculation using Proposition [4.3] shows that the above $A$ in this case is the fifth power of $\frac{Q(v_0)}{B(v_2, v_3)} \frac{\sigma^3(\beta) \sigma^4(\beta)}{\beta \sigma(\beta)}$. Finally, When $A$ is a perfect fifth power, say of $-\lambda$, the $K$-rational map $diag(1, -\lambda, \lambda, -1, \lambda^{-2})$ maps $E_A$ to $E_\lambda$. □

**Lemma 4.7.**

$$\frac{Q(v_0)^2}{B(v_1, v_4) B(v_2, v_3)} = -\frac{\beta^2 \sigma(\beta)}{\sigma^3(\beta) \sigma^4(\beta^2)}.$$

**Proof.** This follows from Lemma 4.3 on page 139 of [3] and using Proposition 4.5. Loosely speaking, this is a condition on the intersection of five quadrics to form a smooth genus one curve; note that five quadrics in general position do not intersect. □

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1This corrects a minus sign error in that paper, namely the coefficient of $x_4^2$ in $S_4$. 

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Corollary 4.8.

\[ \frac{B(v_1, v_4)}{Q(v_0)} = \lambda \cdot \frac{\sigma^4(\beta)}{\beta}. \]

We now have all of the coefficients of \( Q' \) in terms of \( Q(v_0) \). We can divide out by this non-zero term to get

Proposition 4.9. Let \( u_1 = \frac{\sigma^4(\beta)}{\beta} \) and \( u_2 = \frac{\sigma^4(\beta)\sigma^4(\beta)}{\beta\sigma(\beta)}. \) Then \( Q' \) is given by:

\[
\sum_{i=0}^{4} x_i^2 + \lambda \sum_{i=0}^{4} x_i x_{i+2} \cdot \sigma^{i+1} u_1 - \lambda^{-1} \sum_{i=0}^{4} x_i x_{i+1} \cdot \sigma^{i+3} u_2.
\]

Finally, since \( Q' = Q^M \), we can recover \( Q \) as \( Q^M^{-1} \). A calculation shows:

Theorem 4.10. Write

\[ Q = \sum_{0 \leq i \leq j \leq 4} a_{ij} x_i x_j. \]

Then the \( a_{ij} \) are given as follows. For \( i = j \) we have

\[ a_{ii} = \text{Tr} \left( \frac{1}{\alpha^{2i}} \right) + \lambda \text{Tr} \left( \frac{u_1}{\alpha^{2i}} \right) - \lambda^{-1} \text{Tr} \left( \frac{u_2}{\alpha^{2i}} \right) \]

and for \( i \neq j \) we have

\[ a_{ij} = \text{Tr} \left( \frac{2}{\alpha^{i+j}} \right) + \lambda \text{Tr} \left( \left( \zeta^{i-j} + \zeta^{j-i} \right) \frac{u_1}{\alpha^{i+j}} \right) - \lambda^{-1} \text{Tr} \left( \left( \zeta^{2i-2j} + \zeta^{2j-2i} \right) \frac{u_2}{\alpha^{i+j}} \right). \]

Remark. The question of solving “norm equations,” that is, developing an algorithm to find \( \beta \) as in the above theorem, has been studied extensively by Denis Simon [13], written up in the new book by Henri Cohen [4], and implemented in Pari.

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