Non-Abelian Geometric Phases and Conductance of Spin-$\frac{3}{2}$ Holes

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(December 31, 2021)

Angular momentum $J = \frac{3}{2}$ holes in semiconductor heterostructures are showed to accumulate nonabelian geometric phases as a consequence of their motion. We provide a general framework for analyzing such a system and compute conductance oscillations for a simple ring geometry. We also analyze a figure-8 geometry which captures intrinsically nonabelian interference effects.

PACS numbers: 03.65.Bz, 73.23.-b

Introduction – A nondegenerate quantum state undergoing adiabatic evolution accumulates both a dynamical as well as a geometric (Berry’s) phase. The geometric phase is responsible for a wide array of interference phenomena, and has been measured in optics, with neutral beams, and by magnetic resonance. Most theoretical and experimental work has focussed on the theoretical and experimental work has focussed on the degenerate case of the solid angle subtended by the fictitious in-plane component of the field can be of

\[ \oint A \cdot d\lambda \]

where $A_{i}^{\alpha\beta} = -i \langle \alpha | \frac{\partial}{\partial \lambda_{\beta}} | \beta \rangle$ is the gauge potential matrix ($| \alpha \rangle$, $| \beta \rangle$ are adiabatic eigenstates), $\{ \lambda_{\beta}(t) \}$ is a set of slowly evolving parameters, and $P$ is the path ordering operator. Such a system may exhibit nonabelian effects in which e.g. one member of a multiplet evolves into another upon completion of a cycle in parameter space. One example of this phenomenon is in crystalline nuclear quadrupole resonance (NQR), since the quadrupole Hamiltonian $H = \frac{1}{2}I_{ijkl}I_{ijkl}$ is quadratic in the spin. When the electric field gradient tensor has cylindrical symmetry, the Hamiltonian can be taken to be $H = \hbar \omega Q(\hat{n} \cdot J)^{2}$, where $\hat{n}$ lies in the direction of the principal axis of $Q_{ij}$. The nonabelian gauge structure for this problem was discussed by Zee and measured in the ${ }^{35}$Cl by Zwanziger, Koenig, and Pines. Paths in which $\hat{n}(t)$ rotates about more than one axis are essential if intrinsically nonabelian aspects are to be captured.

In this paper we consider an alternative setting for an observation of the nonabelian geometric phase. We study angular momentum-$\frac{3}{2}$ holes confined to conducting loops embedded in a two-dimensional hole gas of a heterostructure. A hole’s momentum $p(t)$ (or coordinate $\phi(t)$ in a ring) acts as an adiabatically changing quantization axis for its angular momentum $J$. For motion around a ring, however, this amounts to a rotation about only one axis. To exhibit the nonabelian effects lurking here, we propose to effectively place the system in a rotating frame by imposing a static magnetic field $H$ in the plane of the ring. Intrinsically nonabelian interference effects are measurable in the conductance oscillations of the figure-8 device discussed below (see fig. 2). Another notable feature is that unlike the case studied in refs. 8, 9, both hole doublets manifest a nonabelian holonomy.

The coupling of $p$ to $J$, which arises naturally within a $\vec{k} \cdot \vec{p}$ treatment of conduction electron and valence hole states, is analogous to the spin-orbit interaction. It is qualitatively different, however, from the spin-orbit splitting of electron states. For electrons in zincblende crystals, the spin states are split because of the inversion asymmetry, which in a quantum well or heterostructure leads to a linear coupling between spin and momentum; another source of linear coupling is the asymmetry of the quantum well or heterojunction itself. The electron’s momentum then acts as an in-plane component of the magnetic field, and as the electron moves around a ring its spin quantization axis traverses the unit sphere at a colatitude $\theta = \tan^{-1}(H_{z}/\lambda p_{0})$, where $H_{z}$ is the physical magnetic field (oriented perpendicular to the plane of the ring), $p_{0}$ is the azimuthal component of the electron’s momentum, and $\lambda$ is a coupling constant. Although the spin-orbit coupling is by nature a relativistic effect, it is effectively enhanced in a crystalline environment, and the fictitious in-plane component of the field can be of
considerable magnitude [16], although the splitting of $\uparrow$ and $\downarrow$ states is still much less than the kinetic energy of the electrons.

This situation is quite different for holes in group IV or III-V semiconductors, which are characterized by a $4 \times 4$ matrix Luttinger Hamiltonian, acting on states in the $\Gamma_8$ representation of double groups of $T_d$ or $O_h$ [11,12,17]. The effective Hamiltonian for bulk holes contains a term $(p \cdot J)^2$, which distinguishes between light $(J^z = \pm \frac{1}{2})$ and heavy $(J^z = \pm \frac{3}{2})$ hole branches.

We next consider the nonabelian gauge potential matrix $A_{\alpha\beta}(t)$:

$$A_{\alpha\beta}(\phi) = -i \langle \bar{\alpha}(t) | \frac{d}{d\phi} | \bar{\beta}(t) \rangle$$

with $\Omega = g\mu_B H_z/\hbar$. This Hamiltonian is similar to that explored in refs. [18,19] in the context of $J = \frac{1}{2}$ nuclear quadrupole resonance.

We next compute the nonabelian gauge potential matrix $A_{\alpha\beta}(t)$:

$$A_{\alpha\beta}(\phi) = -i \langle \bar{\alpha}(t) | \frac{d}{d\phi} | \bar{\beta}(t) \rangle$$

with $| \bar{\alpha}(t) \rangle = V^\dagger(t) | \alpha \rangle$ is an adiabatic eigenstate of $\tilde{H}(t)$. The eigenstates of $\tilde{H}_0$, expressed in eigenstates of $J_z$, form two degenerate blocks ($\sigma = \pm$):

$$| 1, \sigma \rangle = u_\sigma | -\frac{3}{2} \rangle + v_\sigma | +\frac{3}{2} \rangle$$

$$| 2, \sigma \rangle = u_\sigma | +\frac{3}{2} \rangle + v_\sigma | -\frac{3}{2} \rangle$$

where $E_\sigma = \frac{3}{2}(J + K) + \sigma\sqrt{K^2 + \Xi^2 - K\Xi}$.

This gauge potential determines the adiabatic evolution of wavefunctions of holes. Finally, the $U(2)$ phase acquired by a state evolving according to the Hamiltonian $\mathcal{H} + \mathcal{H}'$ is

$$U(t_1, t_0) = e^{i\beta_t L} e^{-i\beta_t J_z/\hbar} A_{\alpha\beta}(t_0) \Lambda_\beta A^{\dagger}_{\alpha} e^{+i\beta_t J_z/\hbar}$$

where $\Lambda_\beta$ transforms from the $J^z$ eigenbasis to the basis of eq. [18]. $k_F$ is the Fermi momentum of the holes, $L$ is the distance traveled, and where the path ordering operator places earlier times to the right. Adiabaticity is satisfied provided $\omega, \Omega \ll \sqrt{K^2 + \Xi^2 - K\Xi}$.

**Conductance oscillations in a loop** – We next consider the transport of holes in the upper ($\sigma = +$) doublet through a ring confined to the two-dimensional hole gas [20]. The ring is connected to leads through two antipodally placed T-junctions, each described by the $S$-matrix [21].

$$S = \begin{pmatrix} r & \frac{1}{2} \sqrt{1 - r^2} & \frac{1}{2} \sqrt{1 - r^2} \\ \frac{1}{2} \sqrt{1 - r^2} & -\frac{1}{2} (1 + r) & \frac{1}{2} (1 - r) \\ \frac{1}{2} \sqrt{1 - r^2} & \frac{1}{2} (1 - r) & -\frac{1}{2} (1 + r) \end{pmatrix}$$

where $r$ is the reflection amplitude for a wave incident from the incoming lead, and where, for simplicity, we assume that $S$ is real and is diagonal in the basis of Eq. [3]. In Fig. 3 we plot the transmission probabilities $T_{\sigma\sigma'} = T_{\sigma'\sigma}$ – due to the nonabelian geometric phase.
an incoming hole in state $|1+\rangle$ may be transformed with probability $T_{12}$ to the state $|2+\rangle$. The conductance of the device is given by $G = \frac{2 e^2}{h} \sum_{\sigma, \sigma'} T_{\sigma \sigma'} = \frac{2 e^2}{h} T$ (since both degenerate levels are occupied in the incoming lead).

In our computations we assumed that holes are confined to a plane thickness 100 Å, Fermi energy (for holes) 2 meV, ring radius 1 μm, and the width of the ring and leads is 400 Å. This corresponds to a rotation frequency $\omega \approx 10^{10}$ Hz.

The $T_{\sigma \sigma'}$ are plotted for $k_F R = 0.370$ (mod $\pi$) – the qualitative results are roughly insensitive to this parameter – as a function of $\Omega/\omega$ for two values of $\vartheta$ at both weak ($r = 0.10$) and strong ($r = 0.85$) coupling of leads to ring. A ratio of $\Omega/\omega = 4$ corresponds to a field of roughly 1 Tesla. The resonances arise due to interference effects both abelian and nonabelian in origin – the intrinsically nonabelian effects manifested in $T_{12}$ are not possible to isolate in this geometry.

**Figure-8 device** – The device depicted in Fig. 2 probes nonabelian interference effects. Holes incident from the lead A may scatter into branch B of the figure-8 or else continue on to lead F. Neglecting the effect of the magnetic field on the contacts themselves, we assume a time reversal invariant (i.e., symmetric) $S$-matrix for the ABEF vertex of the form

$$
\begin{pmatrix}
A_{\text{out}} \\
B_{\text{out}} \\
E_{\text{out}} \\
F_{\text{out}}
\end{pmatrix} =
\begin{pmatrix}
u & v & 0 & t \\
v & -u & t & 0 \\
t & 0 & u & -v \\
t & 0 & -v & -u
\end{pmatrix}
\begin{pmatrix}
A_{\text{in}} \\
B_{\text{in}} \\
E_{\text{in}} \\
F_{\text{in}}
\end{pmatrix},
$$

where $A_{\text{in}}$ is a two-component vector describing the incident flux of holes in the upper doublet. The real parameters $t, u, v$ satisfy $t^2 + u^2 + v^2 = 1$ and are assumed to be the same for both states of the doublet. $t \approx 1$ corresponds to weak coupling between the leads and the figure-8, and $S_{1,3} = S_{2,4} = 0$ means that there is negligible backscattering between E and A and between B and F. The vertex at the center of the figure-8 is assumed to pass B to C and D to E with negligible scattering into other channels, i.e. the EDCB $S$-matrix corresponds to that of eq. (11) with $t = 1$. This ensures that holes which enter the figure-8 through branch B will execute a BDCE circuit before entering the lead F or being re-scattered into B. Such a scattering matrix for the BECD contact is realized when this contact is collimating, i.e. it conserves the momentum of holes. The conservation of momentum in the course of transmission through the contact holds if dimensions of the contact are larger than the wavelength of holes, so that diffraction effects are suppressed. Such contacts are technologically feasible and were studied in electron transport (for a review see [24]). Under these conditions, we may write the relation between B and E as

$$
E_{\text{in}} = e^{i k_F L} W_{\omega}(0, -\pi) W_{-\omega}(-\pi, \pi) W_{\omega}(\pi, 0) B_{\text{out}} (11)
$$

$$
E_{\text{out}} = e^{-i k_F L} W_{-\omega}(0, \pi) W_{\omega}(\pi, -\pi) W_{-\omega}(-\pi, 0) B_{\text{in}} .
$$

This result, in conjunction with eq. (10) determines the conductance of the figure-8 device. It is easy to see that when $\Omega = 0$ (no in-plane field), the gauge potential $A_{\omega}$ is diagonal and there are no nonabelian effects – the “quadrupole field” rotates only about the $z$ axis. This reduces the products of $W$-matrices in eq. (11) to the unit matrix, so that the only interference between the paths AF and ABCDEF is due to the difference in their lengths. In Fig. 3 resonances in the transmission probability for the figure-8 device are shown (we use parameters identical to those for the ring). We find pronounced oscillations with the variation of magnetic field. These oscillations are nonabelian effects, because the path executed by the holes corresponds to a zero net solid angle subtended by the effective quadrupole field [23], meaning that abelian effects are cancelled.
FIG. 3. Transport through the figure-8. The parameters of eq. (10) are related to $\eta$ by $t = \cos \eta$, $u = v = \frac{1}{\sqrt{2}} \sin \eta$.

We note that the interaction of holes moving in constrictions with localized holes or nuclei via spin-spin interactions leads to possibilities of observing nonabelian phases in nuclear (spin) resonance experiments. Yet another option is to study optical transitions of constricted holes.

We are grateful to Alex Pines for interesting discussions of non-abelian settings and to Richard Webb for very useful remarks about adiabatic contacts. We also thank P. Goldbart, A. Leggett, A. Manohar and L. J. Sham for discussions. YLG was supported by NSF under grants DMR91-57018 and DMR94-24511.

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