MATRIX FOURIER TRANSFORM IN DYNAMIC THEORY OF ELASTICITY OF PIECEWISE HOMOGENEOUS MEDIUM

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Abstract. The analytical solving dynamic problems of elasticity theory for piecewise homogeneous half-space is found. The explicit construction of direct and inverse Fourier’s vector transform with discontinuous coefficients is presented. The technique of applying Fourier’s vector transform with discontinuous coefficients for solving problems of mathematical physics in the heterogeneous environments is developed on an example of the dynamic problems of the elasticity theory.

Keywords: piecewise homogeneous medium, theory of elasticity, Fourier’s vector transform

Mathematics Subject Classification 2010:35N30 Overdetermined initial-boundary value problems; 35Cxx Representations of solutions; 65R10 Integral transforms.

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1. Introduction

The purpose of the mathematical theory of elasticity is to define the tension and deformations on border and inside the elastic body any form under all load conditions. Required values are functions of coordinates and time in dynamic problems of the theory of elasticity. Problems about oscillations of constructions and buildings are problems of this type of dynamic problems. Forms of oscillations and their possible changes, amplitudes of oscillations and their increase or decrease in the course of time, resonance modes, dynamic tension, methods of excitation and extinguish of oscillation and others, and also problems about distribution of elastic waves; seismic waves, and their influence on constructions and buildings, waves arising at explosions and blows, thermoelastic waves etc. are defined in the given problems.

Different representations of the solutions of the equilibrium equation through functions of tension are used when solving problems by the variable separation method. The required problem is taken to the solution of differential equations of a more simple structure with the help of such representations. Each functions of tension in these equations ”is not fastened” with others, but it enters into boundary conditions together with the others. A.F.Ulitko [7] has offered rather effective method of research of problems of mathematical physics - a method Eigen vector-valued functions. This method is the vector analogue of the Fourier method.

The method of integral transformations is also an analytical method of the decision of solution of problems theory of elasticity. The method of integral transformations we consider and develop in this article. we come to the most simple problem in space of images with the help of the integral transformations (Fourier, Laplace, Hankel, etc.). The finding of the formula of direct is the main difficulty in
solving of problems of this approach. Extensive enough bibliography of works on use of this method in problems of the theory of elasticity is resulted in J.S.Ufljand’s monography [2].

Problems of the theory of elasticity for heterogeneous bodies are of great practical interest. Lame coefficients are not constant in these problems. They are the functions of coordinates defining the field of elastic properties of bodies. Application of analytical methods is connected with considerable mathematical difficulties because there is no corresponding mathematical apparatus, when the tension-strain state of bodies of the complex configuration is researched.

Method of the vector integral transforms of Fourier is equivalent the method Eigen vector-valued functions, however, unlike the last it can to be applied successfully be used, applied to the solution of problems of the theory of elasticity in a piece-wise homogeneous medium. The theory of integral transforms of Fourier with piece-wise constant coefficients in a scalar case was studied by Ufljand J.S. [16], [17], Najda L.S. [11], Protsenko V. S [12], [13], Lenjuk M. P [8], [9], [10]. The vector variant of a method adapted for the solution of problems in piece-wise homogeneous medium is developed by the author in [2], [19]. Unknown tension in the boundary conditions and in the internal conditions of conjugation don’t commit splitting in a considered dynamic problem, so the application of the scalar integral transforms of Fourier with piece-wise constant coefficients does not lead to success. Method of the vector integral transforms of Fourier with discontinuous coefficients is used for its solution in the present work. Conformable theoretical bases of a method are presented in item 4 for granted. The necessary proofs by the method of contour under the scheme developed in [2] and [19]. The closed form solution of the dynamic problem found in the use of this method in item 4.

2. Problem statement

Let’s consider a problem about distribution of tension in an \( n+1 \)-layer elastic semi-infinite solid \( I^+_{n+1} \times R = \{ (x, y) : x \in I^+_n, y \in R \} \), where \( I^+_n = \bigcup_{i=1}^{n+1} (l_{i-1}, l_i) \). The vector of displacement \( \mathbf{u}_i \) has components \( u_i, v_i, 0 \) in the case of plane strain. If introduced two functions tension \( \varphi_i(x, y, t) \) and \( \psi_i(x, y, t) \), under the condition [14], functions are defined by the relations

\[
\begin{align*}
    u_i &= \frac{\partial \varphi_i}{\partial x} + \frac{\partial \psi_i}{\partial y}, \\
    v_i &= \frac{\partial \varphi_i}{\partial y} - \frac{\partial \psi_i}{\partial x},
\end{align*}
\]

than expressions for the component of pressure become [14]

\[
\begin{align*}
    \sigma_{ix} &= \lambda_i \Delta \varphi_i + 2\mu_i \left( \frac{\partial^2 \varphi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial x \partial y} \right), \\
    \sigma_{iy} &= \lambda_i \Delta \varphi_i + 2\mu_i \left( \frac{\partial^2 \varphi_i}{\partial y^2} - \frac{\partial^2 \psi_i}{\partial x \partial y} \right), \\
    \tau_{ixy} &= \mu_i \left( 2 \frac{\partial^2 \varphi_i}{\partial x \partial y} - \frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial y^2} \right),
\end{align*}
\]
where $\lambda_i, \mu_i$-elastic Lame constants. If to choose functions of tension $\varphi_i$ and $\psi_i$ in the form of solutions of a system of wave equations

\[
\frac{\partial^2 \varphi_i}{\partial t^2} = c^2_{1i} \Delta \varphi_i, \quad \frac{\partial^2 \psi_i}{\partial t^2} = c^2_{2i} \Delta \psi_i
\]

with zero initial conditions

\[
\varphi_i(x, y, 0) = 0, \quad \psi_i(x, y, 0) = 0, \quad \frac{\partial \varphi_i(x, y, 0)}{\partial t} = 0, \quad \frac{\partial \psi_i(x, y, 0)}{\partial t} = 0
\]

than the movement equations will be satisfied. The tension $p(y, t)$, changing with time, is applied on the border of the body. If tangent tension is equal to zero, than the boundary conditions become

\[
\sigma_{1x} = -p(y, t), \quad \tau_{1xy} = 0 \quad \text{as} \quad x = 0.
\]

Let the components of the vector of displacement $\mathbf{u}_i$ and the components of the tension tensor $\sigma_{ix}, \tau_{ixy}$ be continuous, we get internal boundary conditions, so-called conjugation conditions [5]:

\[
\mathbf{u}_i = u_{i+1}, \quad \mathbf{v}_i = v_{i+1}, \quad \sigma_{ix} = \sigma_{i+1x}, \quad \tau_{ixy} = \tau_{i+1xy}, \quad x = l_i, \quad i = 1, ..., n.
\]

3. Vector Fourier transform with discontinuous coefficients

Let’s develop the method of vector Fourier transform for the solution this problem. Let’s consider Sturm–Liouville vector theory [1] about a design bounded on the set of non-trivial solution of separate simultaneous ordinary differential equations with constant matrix coefficients

\[
\left( A^2_{2m} \frac{d^2}{dx^2} + \lambda^2 \mathbf{E} + \Gamma^2_{m} \right) y_m = 0, \quad q^2_m = \lambda^2 \mathbf{E} + \Gamma^2_{m}, \quad m = 1, n + 1
\]

on the boundary conditions.

\[
\left. \left( \alpha^0_{11} + \lambda^2 \delta^0_{11} \right) \frac{d}{dx} + \left( \beta^0_{11} + \lambda^2 \gamma^0_{11} \right) \right|_{x=l_0} = 0, \quad \|y_{n+1}\|_{x=\infty} < \infty
\]

and conditions of the contact in the points of conjugation of intervals

\[
\left( \alpha^k_{j1} + \lambda^2 \delta^k_{j1} \right) \frac{d}{dx} + \left( \beta^k_{j1} + \lambda^2 \gamma^k_{j1} \right) y_k = \left( \alpha^k_{j2} + \lambda^2 \delta^k_{j2} \right) \frac{d}{dx} + \left( \beta^k_{j2} + \lambda^2 \gamma^k_{j2} \right) y_{k+1},
\]

$x = l_k, \quad k = 1, n, \quad j = 1, 2, ..., \text{where}

\[
y_m(x, \lambda) = \begin{pmatrix}
    y_{1m}(x, \lambda) \\
    \vdots \\
    y_{rm}(x, \lambda)
\end{pmatrix}, \quad \|y_m\| = \sqrt{y^2_{1m} + ... + y^2_{rm}}, \quad m = 1, n + 1.
\]

Let for some $\lambda$ the considered the boundary problem has a non-trivial solution

\[
y(x, \lambda) = \sum_{k=1}^{n} \theta(x - l_{k-1}) \theta(l_k - x) y_k(x, \lambda) + \theta(x - l_n) y_{n+1}(x, \lambda).
\]

The number $\lambda$ is called an Eigen value in this case, and the corresponding decision $y(x, \lambda)$ is called Eigen vector-valued function.
\( \alpha_{11}^0, \beta_{11}^0, \gamma_{11}^0, \delta_{11}^0, \alpha_j^k, \beta_j^k, \gamma_j^k, \delta_j^k, A_j - (j = 1, 2; \ m = 1, n + 1; \ k = 1, n) \)

are matrixes of the size \( r \times r \). We will required invertible

\[
\text{det } M_{mk} \neq 0, \ \lambda \in [0, \infty)
\]

for matrixes

\[
M_{mk} = \begin{pmatrix}
\alpha_{1m} + \lambda^2 \delta_{1m}^k & \beta_{1m} + \lambda^2 \delta_{2m}^k \\
\beta_{2m} + \lambda^2 \delta_{2m}^k & \alpha_{2m} + \lambda^2 \delta_{2m}^k
\end{pmatrix}, \ m = 1, 2; \ k = 1, n.
\]

Matrixes \( A_m^2 \) and \( \Gamma_m^2 \), are is \( m = 1, n + 1 \) -positive-defined [6]. We denote

\[
\Phi_{n+1} (x) = e^{q_n + xi}; \quad \Psi_{n+1} (x) = e^{-q_n + xi}; \quad q_{n+1}^2 = A_{n+1}^{-2} (\lambda^2 E + \Gamma^2).
\]

Define the induction relations the others n-pairs a matrix-importance functions

\[
(\Phi_k, \Psi_k), \ k = 1, \ n:
\]

\[
\left[ \left( \alpha_{j1}^k + \lambda^2 \delta_{j1}^k \right) \frac{d}{dx} + \left( \beta_{j1}^k + \lambda^2 \gamma_{j1}^k \right) \right] (\Phi_k, \Psi_k) =
\]

\[
\left[ \left( \alpha_{j2}^k + \lambda^2 \delta_{j2}^k \right) \frac{d}{dx} + \left( \beta_{j2}^k + \lambda^2 \gamma_{j2}^k \right) \right] (\Phi_{k+1}, \Psi_{k+1}), \ k = 1, n, \ j = 1, 2.
\]

Let us introduce the following notation

\[
0 \Phi (\lambda) = \left[ \left( \alpha_{11}^0 + \lambda^2 \delta_{11}^0 \right) \frac{d}{dx} + \left( \beta_{11}^0 + \lambda^2 \gamma_{11}^0 \right) \right] \Phi_1 (x, \lambda) \bigg|_{x=l_0},
\]

\[
0 \Psi (\lambda) = \left[ \left( \alpha_{11}^0 + \lambda^2 \delta_{11}^0 \right) \frac{d}{dx} + \left( \beta_{11}^0 + \lambda^2 \gamma_{11}^0 \right) \right] \Psi_1 (x, \lambda) \bigg|_{x=l_0},
\]

\[
\Omega_k = \begin{pmatrix}
\Phi_k \\
\Psi_k
\end{pmatrix}, \ i = 1, n + 1.
\]

**Theorem 1.** The spectrum of the problem [7], [8], [9] is a continuous and fills all semi axis \((0, \infty)\). Sturm–Liouville theory \( r \) time is degenerate. To each Eigen value \( \lambda \) corresponds to exactly \( r \) linearly independent vector-valued functions. As the last it is possible to take \( r \) columns matrix-importance functions.

\[
u (x, \lambda) = \sum_{k=1}^{n} \theta (x - l_{k-1}) \theta (l_k - x) \ u_k (x, \lambda) + \theta (x - l_n) \ u_{n+1} (x, \lambda),
\]

\[
u_j (x, \lambda) = \Phi_j (x, \lambda) 0 \Phi_1^{-1} (\lambda) - \Psi_j (x, \lambda) 0 \Psi_1^{-1} (\lambda).
\]

That is

\[
y_m (x, \lambda) = \begin{pmatrix}
u_{1m} (x, \lambda) \\
\vdots \\
u_{rm} (x, \lambda)
\end{pmatrix}.
\]

Dual Sturm–Liouville theory consists in a finding of the non-trivial solution of separate simultaneous ordinary differential equations with constant matrix coefficients.

\[
A_m^2 \frac{d^2}{dx^2} + \lambda^2 E + \Gamma_m^2 y_m = 0, \ q_m^2 = \lambda^2 E + \Gamma_m^2, \ m = 1, n + 1
\]
on the boundary conditions
\[ (14) \quad \left. \left( \frac{d}{dx} y_k^*(\beta_{11}^0 + \lambda^2 \gamma_{11}^0) + y_k^* (\alpha_{11}^0 + \lambda^2 \delta_{11}^0) \right) \right|_{x=0} = 0, \quad \|y_{n+1}^*\| < \infty, \]

and conditions of the contact in the points of conjugation of intervals
\[ (15) \quad \left( -\frac{d}{dx} y_k^*, y_k \right) \left( \frac{\beta_{11}^k + \lambda^2 \gamma_{11}^k}{\beta_{21}^k + \lambda^2 \gamma_{21}^k} \right) \left( \frac{\alpha_{11}^k + \lambda^2 \delta_{11}^k}{\alpha_{21}^k + \lambda^2 \delta_{21}^k} \right) = \left( -\frac{d}{dx} y_{k+1}^*, y_{k+1} \right) \left( \frac{\beta_{12}^k + \lambda^2 \gamma_{12}^k}{\beta_{22}^k + \lambda^2 \gamma_{22}^k} \right) \left( \frac{\alpha_{12}^k + \lambda^2 \delta_{12}^k}{\alpha_{22}^k + \lambda^2 \delta_{22}^k} \right), \quad x = l_k, \quad k = 1, n. \]

The solution of the boundary value problem we write in the form of
\[ y^*(\xi, \lambda) = \sum_{k=2}^n \theta (\xi - l_{k-1}) \theta (l_k - \xi) y_k^*(\xi, \lambda) + \theta (l_1 - \xi) y_1^*(\xi, \lambda) + \theta (\xi - l_n) y_{n+1}^*(\xi, \lambda), \]

\[ y_m^*(\xi, \lambda) = ( y_{m1}^*(\xi, \lambda) \cdots y_{mr}^*(\xi, \lambda) ), \]

\[ \|y_m^*\| = \sqrt{(y_{1m}^*)^2 + \cdots + (y_{rm}^*)^2}, \quad m = 1, n + 1. \]

**Theorem 2.** The spectrum of the problem \[7, \, \[8, \, \[9\] is a continuous and fills semi-axis \((0, \infty)\). Sturm–Liouville theory \(r\) time is degenerate. To each Eigen value \(\lambda\) corresponds to exactly \(r\) linearly independent vector-valued functions. As the last it is possible to take \(r\) rows matrix-importance functions.

\[ u^*(x, \lambda) = \sum_{k=1}^n \theta (x - l_{k-1}) \theta (l_k - x) u_k^*(x, \lambda) + \theta (x - l_n) u_{n+1}^*(x, \lambda), \]

\[ u_j^*(x, \beta) = \begin{pmatrix} 0 \\ E \end{pmatrix} \Omega^{-1}_j(x, \beta) \begin{pmatrix} 0 \\ \Phi(\beta) \Psi(\beta) \end{pmatrix} A_j^{-2}, \]

That is
\[ (16) \quad y_j^*(\xi, \lambda) = ( u_{j1}^*(\xi, \lambda) \cdots u_{jr}^*(\xi, \lambda) ), \quad j = 1, r. \]

The existence of spectral functions \(u(x, \lambda)\) and the conjugate spectral function \(u^*(x, \lambda)\) allows to write a vector decomposition theorem on the set of \(I_1^+\).

**Theorem 3.** Let the vector-valued function \(f(x)\) is defined on \(I_1^+\) continuous, absolutely integrated and has the bounded total variation. Then for any \(x \in I_1^+\) true formula of decomposition
\[ f(x) = -\frac{1}{\pi} \int_0^\infty \int_{l_0}^\infty u^*(\xi, \lambda) f(\xi) d\xi + \left( \gamma_{11}^0 f_1(l_0) + \delta_{11}^0 f_1'(l_0) \right) + \sum_{k=1}^n \left( \phi_{11}^0(\lambda), \psi_{11}^0(\lambda) \right) \Omega_k^{-1}(l_k, \lambda) M_{k-1}^{-1}(\lambda) \]
\[ + \left( \gamma_{21}^k f_1(l_k) + \delta_{21}^k f_1'(l_k) \right) \left( f_{k+1}^*(l_k) - f_k^*(l_k) \right) \right) \lambda d\lambda. \]
The decomposition theorem allows to enter the direct and inverse matrix integral Fourier transform on the real semi axis with conjugation points:

\[
F_{n+} [f] (\lambda) = \int_{0}^{\infty} u^* (\xi, \lambda) f (\xi) \, d\xi + \\
\left( \gamma_{11}^{\delta} f_1 (0) + \delta_{11}^{\delta} f_1' (0) \right) + \sum_{k=1}^{n} \left( \phi_1^0 (\lambda), \psi_1^0 (\lambda) \right) \Omega_k^{-1} (l_k, \lambda) M_k^{-1} (\lambda).
\]

(18)

\[
\left\{ \left( \begin{array}{c} \gamma_{21}^k \\ \gamma_{22}^k \\ \delta_{21}^k \\ \delta_{22}^k \end{array} \right) \left( \begin{array}{c} f_{k+1} (l_k) \\ f_k (l_k) \\ f_{k+1}' (l_k) \\ f_k' (l_k) \end{array} \right) \right\} = \tilde{f} (\lambda),
\]

(19)

\[
F_{n+}^{-1} \left[ \tilde{f} \right] (x) = -\frac{1}{\pi i} \int_{0}^{\infty} \lambda u (x, \lambda) \tilde{f} (\lambda) \, d\lambda \equiv f (x),
\]

when

\[
f (x) = \sum_{k=1}^{n} \theta (l_k - x) \theta (x - l_{k-1}) f_k (x) + \theta (x - l_n) f_{n+1} (x).
\]

Let’s apply the obtained integral formulas for the solution of the problem of elasticity theory (1), (2), (2), (2). Let’s result the basic identity of integral transform of the differential operator

\[
B = \sum_{j=1}^{n} \theta (x - l_{j-1}) \theta (l_{j} - x) \left( A_j^2 \frac{d^2}{dx^2} + \Gamma_j^2 \right) + \theta (x - l_n) \left( A_{n+1}^2 \frac{d^2}{dx^2} + \Gamma_{n+1}^2 \right).
\]

Theorem 4. If vector-valued function

\[
f (x) = \sum_{k=1}^{n} \theta (x - l_{k-1}) \theta (l_k - x) f_k (x) + \theta (x - l_n) f_{n+1} (x),
\]

is continuously differentiated on set three times, has the limit values together with its derivatives up to the third order inclusive

\[
f_k^{(m)} (l_{k-1}) = f_k^{(m)} (l_{k-1} + 0), \quad m = 0, 1, 2, 3; \quad k = 1, \ldots, n + 1
\]

Satisfies to the boundary condition on infinity

\[
\lim_{x \to \infty} \left( u^* (x, \lambda) \frac{d}{dx} f (x) - \frac{d}{dx} u^* (x, \lambda) f (x) \right) = 0
\]

Satisfies to homogeneous conditions of conjugation (4), that basic identity of integral transform of the differential operator B hold

\[
F_{n+} \left[ B (f) \right] (\lambda) = -\lambda^2 \tilde{f} (\lambda) - \left\{ \left( \beta_{11}^0 f_1 (l_0) + \alpha_{11}^0 f_1' (l_0) \right) - \left( \gamma_{11}^0 A_{11}^2 f_1'' (l_0) + \delta_{11}^0 A_{11}^3 f_1'''' (l_0) \right) \right\}
\]

(20)

The proof of theorems 1, 2, 3, 4 is spent by a method of the method of contour integration. Similarly presented to work of the author [19].
4. THE SOLUTION OF DYNAMIC PROBLEMS OF THE THEORY OF ELASTICITY

Let’s apply on the variable $y$ Fourier transformation [4], and let’s apply on the variable $x$ the vector integral transforms of Fourier [18]. In the images of Fourier series in the variable $y$ the problem (1), (2), (3), (4) takes the form of the simultaneous equations

$$
\frac{\partial^2 \bar{\varphi}_i}{\partial t^2} = c_{1i}^2 \frac{\partial^2 \bar{\varphi}_i}{\partial x^2} - c_{2i}^2 \xi^2 \bar{\varphi}_i,
$$

(21)

$$
\frac{\partial^2 \bar{\psi}_i}{\partial t^2} = c_{2i}^2 \frac{\partial^2 \bar{\psi}_i}{\partial x^2} - c_{2i}^2 \xi^2 \bar{\psi}_i, t > 0, \quad l_{i-1} < x < l_i
$$

with initial conditions

$$
\bar{\varphi}_i(x, y, 0) = 0, \quad \bar{\psi}_i(x, y, 0) = 0,
$$

(22)

$$
\frac{\partial \bar{\varphi}_i(x, y, 0)}{\partial t} = 0, \quad \frac{\partial \bar{\psi}_i(x, y, 0)}{\partial t} = 0
$$

where $\bar{\varphi}_i, \bar{\psi}_i$ - images of Fourier series in the variable $y$ functions of tension

$$
\bar{\varphi}_i = \frac{1}{\sqrt{2\pi}} \int_0^\infty \varphi_i(x, y, t) e^{-j\xi y} dy, \quad \bar{\psi}_i = \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi_i(x, y, t) e^{-j\xi y} dy
$$

with boundary conditions

$$
\sigma_{1x} = \lambda_1 \frac{\partial^2 \bar{\varphi}_i}{\partial x^2} - \lambda_1 \xi^2 \bar{\varphi}_i + 2\mu_1 \left( \frac{\partial^2 \bar{\psi}_i}{\partial x^2} + j\xi \frac{\partial \bar{\psi}_i}{\partial x} \right) = -\tilde{\rho}(\xi, t), as \ x = 0
$$

(23)

$$
\tau_{1xy} = \mu_1 \left( 2j\xi \frac{\partial \bar{\varphi}_i}{\partial x} - \frac{\partial^2 \bar{\psi}_i}{\partial x^2} - \xi^2 \bar{\psi}_i \right) = 0, as \ x = 0
$$

with the internal conditions of conjugation

$$
\frac{\partial \bar{\varphi}_i}{\partial x} + j\xi \bar{\psi}_i = \frac{\partial \bar{\varphi}_{i+1}}{\partial x} + j\xi \bar{\psi}_{i+1},
$$

$$
\frac{\partial \bar{\psi}_i}{\partial x} = \frac{\partial \bar{\psi}_{i+1}}{\partial x}, as \ x = l_i
$$

$$
\lambda_1 \frac{\partial^2 \bar{\varphi}_i}{\partial x^2} - \lambda_1 \xi^2 \bar{\varphi}_i + 2\mu_1 \left( \frac{\partial^2 \bar{\psi}_i}{\partial x^2} + j\xi \frac{\partial \bar{\psi}_i}{\partial x} \right) =
$$

$$
= \lambda_{i+1} \frac{\partial^2 \bar{\varphi}_{i+1}}{\partial x^2} - \lambda_{i+1} \xi^2 \bar{\varphi}_{i+1} + 2\mu_{i+1} \left( \frac{\partial^2 \bar{\psi}_{i+1}}{\partial x^2} + j\xi \frac{\partial \bar{\psi}_{i+1}}{\partial x} \right) as \ x = l_i
$$

$$
\mu_i \left( 2j\xi \frac{\partial \bar{\varphi}_i}{\partial x} - \frac{\partial^2 \bar{\psi}_i}{\partial x^2} - \xi^2 \bar{\psi}_i \right) =
$$

(24)

$$
\mu_{i+1} \left( 2j\xi \frac{\partial \bar{\varphi}_{i+1}}{\partial x} - \frac{\partial^2 \bar{\psi}_{i+1}}{\partial x^2} - \xi^2 \bar{\psi}_{i+1} \right) as \ x = l_i
$$

Denote $c = \max_i \{c_{1i}, c_{2i}\}$. Let’s apply to a problem (21), (22), (23), (24) vector integral Fourier transform with discontinuous coefficients, defined by formulas (18) - (19). Let’s put in simultaneous equations (7)

$$
r = 2, \quad A_i^2 = \begin{pmatrix} c_{1i}^2 & 0 \\ 0 & c_{2i}^2 \end{pmatrix}, \quad \Gamma_i^2 = \begin{pmatrix} (c^2 - c_{1i}^2) \xi^2 & 0 \\ 0 & (c^2 - c_{2i}^2) \xi^2 \end{pmatrix}.\]
in boundary conditions \[8\] let’s consider

\[
\begin{align*}
\alpha_{11}^0 &= \begin{pmatrix} 0 & 2j\mu_1 \xi \\ 2j\mu_1 \xi & 0 \end{pmatrix}, \\
\beta_{11}^0 &= -\begin{pmatrix} \lambda_1 + 2\mu_1 & 0 \\ 0 & -\mu_1 \end{pmatrix} A_1^{-2} \Gamma_1^2 - \begin{pmatrix} 0 & \lambda_1 \xi^2 \\ \mu_1 \xi^2 & 0 \end{pmatrix}, \\
\gamma_{11}^0 &= -\begin{pmatrix} \lambda_1 + 2\mu_1 & 0 \\ 0 & -\mu_1 \end{pmatrix} A_1^{-2}, \quad \delta_{11}^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\end{align*}
\]

in the conditions of conjugation \[11\] we will put

\[
\begin{align*}
\alpha_{11}^k &= \begin{pmatrix} 0 & 2j\mu_k \xi \\ 2j\mu_k \xi & 0 \end{pmatrix}, \quad \beta_{11}^k = -\begin{pmatrix} \lambda_k + 2\mu_k & 0 \\ 0 & -\mu_k \end{pmatrix} A_k^{-2} \Gamma_k^2 - \begin{pmatrix} 0 & \lambda_k \xi^2 \\ \mu_k \xi^2 & 0 \end{pmatrix}, \\
\gamma_{11}^k &= -\begin{pmatrix} \lambda_k + 2\mu_k & 0 \\ 0 & -\mu_k \end{pmatrix} A_k^{-2}, \quad \delta_{11}^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\alpha_{12}^k &= \begin{pmatrix} 0 & 2j\mu_{k+1} \xi \\ 2j\mu_{k+1} \xi & 0 \end{pmatrix}, \quad \beta_{12}^k = -\begin{pmatrix} \lambda_{k+1} + 2\mu_{k+1} & 0 \\ 0 & -\mu_{k+1} \end{pmatrix} A_{k+1}^{-2} \Gamma_{k+1}^2 - \begin{pmatrix} 0 & \lambda_{k+1} \xi^2 \\ \mu_{k+1} \xi^2 & 0 \end{pmatrix}, \\
\gamma_{12}^k &= -\begin{pmatrix} \lambda_{k+1} + 2\mu_{k+1} & 0 \\ 0 & -\mu_{k+1} \end{pmatrix} A_{k+1}^{-2}, \quad \delta_{12}^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
\alpha_{2i}^k &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta_{2i}^k = \begin{pmatrix} 0 & j\xi \\ j\xi & 0 \end{pmatrix}, \\
\gamma_{2i}^k &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \delta_{2i}^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, 2.
\end{align*}
\]

Let’s apply to a problem \[21, 22, 23, 24\] transforms of Fourier \(F_{n+}\) on the variable \(x\). Using identity \[20\], we get Cauchy problem

\[
\begin{align*}
\frac{d^2}{dt^2} \left( \begin{pmatrix} \varphi \ 
\psi \end{pmatrix} \right) &= -c^2 \xi^2 \left( \begin{pmatrix} \varphi \\
\psi \end{pmatrix} \right) - \eta^2 \left( \begin{pmatrix} \varphi \\
\psi \end{pmatrix} \right) + \left( \begin{pmatrix} \tilde{p} (\xi, t) \\
0 \end{pmatrix} \right), \\
\left( \begin{pmatrix} \varphi \\
\psi \end{pmatrix} \right) (\xi, 0, 0) &= 0, \quad \frac{d}{dt}\left( \begin{pmatrix} \varphi \\
\psi \end{pmatrix} \right) (\xi, \eta, 0) = 0, 
\end{align*}
\]

Here denote

\[
\left( \begin{pmatrix} \varphi \\
\psi \end{pmatrix} \right) (\eta, \xi) = F_{n+}\left( \begin{pmatrix} \varphi \\
\psi \end{pmatrix} \right) (\eta),
\]

\[
\left( \begin{pmatrix} \varphi \\
\psi \end{pmatrix} \right) = \sum_{k=1}^{n} \theta (l_k - x) \theta (x - l_{k-1}) \left( \begin{pmatrix} \varphi_k \\
\psi_k \end{pmatrix} \right) + \theta (x - l_n) \left( \begin{pmatrix} \varphi_k \\
\psi_k \end{pmatrix} \right).
\]

Let’s result the solution of the problem \[25-29\]

\[
\left( \begin{pmatrix} \varphi \\
\psi \end{pmatrix} \right) (\eta, \xi, t) = \int_0^t \sin \left( \frac{c^2 \xi^2 + \eta^2 (t - \tau)}{\sqrt{c^2 \xi^2 + \eta^2}} \right) \left( \begin{pmatrix} \tilde{p} (\xi, \tau) \\
0 \end{pmatrix} \right) d\tau.
\]

Let’s apply the inverse Fourier transform on \(y\) and inverse integral transform of Fourier series \(F_{n+}^{-1}\) on the variable \(x\). Using \[19\], we get functions of tension \(\varphi_i, \psi_i\):

\[
\left( \begin{pmatrix} \varphi_i \\
\psi_i \end{pmatrix} \right) (x, y, t) = \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} H_1 (x, y - s, t - \tau) \left( \begin{pmatrix} p(s, \tau) \\
0 \end{pmatrix} \right) ds d\tau,
\]

The above equations are the solution to the problem, and the functions \(\varphi_i, \psi_i\) represent the tension functions.
when

\[ H_i(x, y - s, t - \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( -\frac{1}{j2\pi} \int_{0}^{\infty} e^{j\xi} u_j(x, \eta, \xi) \cdot \sin \left( \frac{\sqrt{c^2\xi^2 + \eta^2} (t - \tau)}{\sqrt{c^2\xi^2 + \eta^2}} \right) d\eta \right) d\xi. \]

The formula (27) takes the form

\[ \left( \begin{array}{c} \varphi \\ \psi \end{array} \right)(x, y, t) = \frac{-1}{j2\sqrt{2\pi}} \int_{0}^{t} \int_{y - c(t - \tau)}^{y + c(t - \tau)} H \left( x, \sqrt{(t - \tau)^2 - \frac{(y - s)^2}{c^2}} \right) \left( \begin{array}{c} p(s, \tau) \\ 0 \end{array} \right) ds d\tau, \]

in the case of a homogeneous environment, that is not the dependence of the \( \lambda_i, \mu_i \)-elastic Lama constants of \( i \), when \( J_0 \) is Bessel function [3],

\[ H(x, z) = \int_{0}^{\infty} Im \left( \frac{e^{jx\eta}}{j\eta(a_{11}^{01} + \eta^2 b_{11}^{01}) + (c_{11}^{01} + \eta^2 d_{11}^{01})} \right) J_0(\eta z) d\eta. \]

The expressions for the functions of tension allow to find components of the vector of displacements \( u_i, v_i, 0 \) and the components of the tension tensor \( \sigma_{ix}, \sigma_{iy}, \tau_{ixy} \) according to the formulae (1),(2).

Remark. The dynamic problem of the theory of elasticity for semi space was considered in the known monograph [15]. However, this problem was solved without initial conditions. The authors apply the Fourier transform of the time variable. It leads them to imprecision in the received formulas for the functions of tension. In our opinion the solution by the method of integral transforms of Fourier (15),(19) on a spatial variable also is more natural.

5. Conclusion

In the work the dynamic problem of elasticity theory are considered: the problem of oscillations of constructions and buildings, the problem of the propagation of elastic waves; thermo elastic waves. The method of integral transforms developed in solving problems. Using the integral transformation (Fourier, Laplace, Hankel) we came to a more simple task in the pattern space. Problem of elasticity theory for inhomogeneous bodies studied. These tasks are of great use in practice. The method of the vector integral transforms of Fourier with discontinuous coefficients used for the decision of problems of the theory of elasticity in a piecewise-homogeneous media. The solution of the dynamic problem in the analytical form found.

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