GENERALIZED KÄHLER GEOMETRY AND THE PLURICLOSED FLOW

JEFFREY STREETS AND GANG TIAN

Abstract. In [16] the authors introduced a parabolic flow for pluriclosed metrics, referred to as pluriclosed flow. We also demonstrated in [17] that this flow, after certain gauge transformations, gives a class of solutions to the renormalization group flow of the nonlinear sigma model with \( B \)-field. Using these transformations, we show that our pluriclosed flow preserves generalized Kähler structures in a natural way. Equivalently, when coupled with a nontrivial evolution equation for the two complex structures, the \( B \)-field renormalization group flow also preserves generalized Kähler structure. We emphasize that it is crucial to evolve the complex structures in the right way to establish this fact.

1. INTRODUCTION

The purpose of this note is to show that the pluriclosed flow introduced in [16] preserves generalized Kähler geometry. This introductory section introduces the main results in a primarily mathematical context, while a physical discussion of the results appears in section 2. First we recall the concept of a generalized Kähler manifold.

Definition 1.1. A generalized Kähler manifold is a Riemannian manifold \((M^{2n}, g)\) together with two complex structures \(J^+, J^-\), each compatible with \(g\), further satisfying

\[
\begin{align*}
\partial \omega^+ &= -\partial \omega^- = H, \\
\partial H &= 0.
\end{align*}
\]

(1.1)

This concept first arose in the work of Gates, Hull, and Roček [5], in their study of \( N = (2, 2) \) supersymmetric sigma models. Later these structures were put into the rich context of Hitchin’s generalized geometric structures [10] in the thesis of Gualtieri [8] (see also [9]).

Recall that a Hermitian manifold \((M^{2n}, \omega, J)\) is pluriclosed if the Kähler form \(\omega\) satisfies \(i\partial \bar{\partial} \omega = d d^c \omega = 0\). Note that a generalized Kähler manifold \((M, g, J^+, J^-)\) consists of a pair of pluriclosed structures \((M, \omega^+, J^+)\) and \((M, \omega^-, J^-)\) whose associated metrics are equal and furthermore satisfy the first equation of (1.1), where \(\omega^\pm(\cdot, \cdot) = g(J^\pm \cdot, \cdot)\). The pluriclosed flow is the time evolution equation

\[
\frac{\partial}{\partial t} \omega = \partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det g.
\]

(1.2)

It follows from Theorem 1.2 in [16] that with \(\omega^\pm\) as initial metrics, we get solutions \(\omega^\pm(t)\) of (1.2) on \(M \times [0, T_\pm]\). Let \(g^\pm(t)\) be the Hermitian metric on \(M\) whose Kähler form is \(\omega^\pm(t)\). As in Theorem 6.5 of [17], we set

\[
X^\pm = \left(-J^\pm d^{*}_{g^\pm(t)} \omega^\pm(t)\right)^2.
\]

(1.3)

Date: January 6, 2012.
where $\sharp_\pm$ denotes the natural isomorphism from $T^*M$ onto $TM$ defined using $g_\pm(t)$. Further, let $\phi_\pm(t)$ denote the one-parameter family of diffeomorphisms generated by $X_\pm$, with $\phi_\pm(0) = \text{Id}$. Then our main theorem can be stated as follows:

**Theorem 1.2.** Let $(M, g, J_+, J_-)$ be a generalized Kähler manifold. With notations as above, one has that $\phi_+(t)^*(g_+(t)) = \phi_-(t)^*(g_-(t))$, and we denote this metric $g(t)$. Furthermore, $T_+ = T_- = T$, and $(M, g(t), \phi_+(t)^*J_+, \phi_-(t)^*J_-)$ is a family of generalized Kähler manifolds on $[0, T)$ with initial value $(M, g, J_+, J_-)$.

Hence, (1.2) preserves generalized Kähler structures. In [17], we found a striking relationship between solutions to (1.2) and the B-field renormalization group flow, and the proof of Theorem 1.2 makes essential use of this. The B-field renormalization group flow arises from physical considerations. Consider a pair $(g, H)$ of a Riemannian metric $g$ and closed three-form $H$ on a manifold $M$. The form $H$ is thought of as the field strength of a locally defined 2-form $B$ (i.e. $H = dB$). Given this data, and a dilaton $\Phi$ on $M$, one can associate a Lagrangian of maps of Riemann surfaces $f : (\Sigma, h) \to M$, called the worldsheet nonlinear sigma model action, given by

\[
S = -\frac{1}{2} \int_\Sigma \left[ |\nabla f|^2 + \frac{e^{2\Phi}}{\sqrt{h}} B_{ij} \partial_\alpha f^i \partial_\beta f^j - 2 \Phi R(h) \right] dV_h.
\]

We have suppressed a scaling parameter $\alpha'$ which is often included in this definition, see ([14] p. 111) for more detail on this action and what follows. Imposing cutoff independence of the associated quantum theory leads to first order renormalization group flow equations

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= -2 R_{cij} + \frac{1}{2} H_{ipq} H_j^{pq} \\
\frac{\partial}{\partial t} H &= \Delta_d H,
\end{align*}
\]

where $\Delta_d = -(dd^* + d^*d)$ is the Laplace-Beltrami operator. In general there is a dilaton evolution as well, but this decouples from the above system after applying a diffeomorphism gauge transformation ([12] pg. 6), and so is not directly relevant to the discussion here.

In view of results in [17], one can ask: Does (1.3) preserve generalized Kähler geometry? As it turns out, in the naive sense in which this question is usually asked, the answer is no. Specifically, given $(M^{2n}, g, J_\pm)$ a generalized Kähler manifold, one may ask whether the solution to (1.3) with initial condition $(g, d_\pm^*\omega_\pm)$ remains generalized Kähler in the sense that $g$ remains compatible with $J_\pm$ and the equations (1.1) hold. This is false in general. One has to evolve the complex structures appropriately so that they are compatible with $(M, g(t))$ and consequently, give rise to generalized Kähler structures. The corresponding evolution equation is not obvious at all, and would be quite difficult to guess directly from (1.3). The key insight comes from the pluriclosed flow and its relation to (1.5) established in [17].

The next theorem is a reformulation of Theorem 1.2 in terms of the B-field flow.

**Theorem 1.3.** Let $(M^{2n}, g, J_+, J_-)$ be a generalized Kähler structure. The solution to (1.3) with initial condition $(g, d_\pm^*\omega_\pm)$ remains a generalized Kähler structure in the following sense: There exists a parabolic flow of complex structures such that if $J_\pm(t)$ are its solutions with initial value $J_\pm$, then the triple $(g(t), J_+(t), J_-(t))$ satisfies the conditions of (1.7).
In fact, \( J_\pm(t) = (\phi_\pm^t)^*J_\pm \) for the one-parameter families of diffeomorphisms \( \phi_\pm^t \) which relate (1.2) to (1.5), as described in Theorem 1.2. However, it is unclear yet how to construct \( \phi_\pm^t(t) \) from (1.5) since (1.5) does not tell how to get \( J_\pm(t) \). A more precise statement of Theorem 1.3 is given below as Corollary 3.3.

We end the paper in section 4 with some structural results that must be satisfied for pluriclosed structures which evolve under (1.5) by homotheties, which we call static structures. In particular, we exhibit some properties showing that a static structure is automatically Kähler-Einstein, and give a complete classification in the case of non-Kähler complex surfaces.

**Theorem 1.4.** Let \((M^4, g, J)\) be a static pluriclosed structure and suppose \( b_1(M) \) is odd. Then \((M^4, J)\) is locally isometric to \( \mathbb{R} \times S^3 \) with the standard product metric. The universal cover of \((M, J)\) is biholomorphic to \( \mathbb{C}^2 \setminus \{(0,0)\} \), and \( M \) admits a finite sheeted cover \( \tilde{M} \) with fundamental group \( \mathbb{Z} \), specifically
\[
\pi_1(\tilde{M}) \cong \mathbb{Z} = \langle (z_1, z_2) \to (\alpha z_1, \beta z_2) \rangle \quad (1.6)
\]
where \( \alpha, \beta \in \mathbb{C} \), \( 1 < |\alpha| = |\beta| \).

**Acknowledgements:** The authors would like to thank Sergey Cherkis for his comments, and the referee for a careful reading and some helpful suggestions.

## 2. Physical Interpretation

The first order RG flow equations (1.5) are derived by imposing cutoff independence for the quantum field theory associated to a nonlinear sigma model. For the pure gravity model these equations were first derived by Friedan [3], yielding the Ricci flow for the order \( \alpha' \) approximation, while for the model including a skew-symmetric background field, these equations were derived in [1] (see also [4]). Recently, due partly to the mathematical breakthroughs of Perelman [13], this flow has garnered more interest in the mathematics and physics communities. In particular, in [12] the authors generalized Perelman’s \( \mathcal{F} \) functional to show that (1.5) is in fact the gradient flow of the lowest eigenvalue of a certain Schrödinger operator. This property is suggested by Zamolodchikov’s \( c \)-theorem [18], which implies the irreversibility of some RG flows. Furthermore, the first author showed in [15] that a certain generalization of Perelman’s entropy functional is monotone for (1.5).

In this paper we address a different issue related to the RG flow. Recall that, as exhibited in [5], when imposing \( N = (2, 2) \) supersymmetry, the equations (1.1) are induced on the target space of a 2-dimensional nonlinear sigma model, whose underlying sigma model action (1.4). A very natural question in this context is whether one can expect the supersymmetry equations to be preserved along the solution to the RG flow, when away from a fixed point. Our results show that the system of equations (1.5), will not in general preserve the \( N = (2, 2) \) supersymmetry equations (1.1). However, if one adds an evolution equation for the complex structures \( J_\pm \), specified in (3.25), then the renormalized coupling constants \( (g(t), H(t), J_\pm(t)) \), will define a supersymmetric model, for all cutoff scales. In fact it is clear from our proofs that the entire discussion is true for the weaker \( N = 2 \) supersymmetry equations.

Our derivation of the evolution equation for \( J_\pm \) comes from recognizing special diffeomorphism gauges relating solutions to (1.5) to solutions of (1.2), where half of the supersymmetry equations (1.1) are clearly preserved with respect to a fixed complex structure.
Thus the evolution equations for $J_{\pm}$ come from the action of the gauge group, hence it is unlikely one could modify the sigma model action (1.4) to derive these equations. This makes equation for $J_{\pm}$ in (3.25) all the more surprising and mysterious. Understanding the physical meaning of the evolution for $J_{\pm}$ therefore remains an interesting open problem. We also remark that our results only apply to the order $\alpha'$ approximation of the renormalization group flow. It remains an interesting open problem to ask whether higher order approximations, or even the full RG flow, preserve $N = (2, 2)$ supersymmetry in the sense we have described here.

Finally, Theorem 1.4 can be thought of as a “No-Go” theorem for certain string vacua. In particular, we have given a complete classification of supersymmetric solutions to (1.5) which evolve purely by homothety on non-Kähler surfaces. In the end only a restricted class of Hopf surfaces can possibly admit solutions to these equations. Other structural results on these vacua in arbitrary dimension appear in section 4. An interesting further problem is to classify solutions to the RG flow which evolve entirely by the action of the diffeomorphism group.

3. PROOF OF MAIN THEOREMS

Proof of Theorem 1.2 Consider the Hermitian manifold $(M^{2n}, g, J_+)$. By (1.1), this is a pluriclosed structure, i.e.

\[(3.1) \quad dd^c_+ \omega_+ = 0.\]

By (16 Theorem 1.2), there exists a solution to (1.2) with initial condition $\omega_+$ on $[0, T)$ for some maximal $T \leq \infty$. Call this one-parameter family of Kähler forms $\omega_+(t)$, and define $\omega_-(t)$ analogously as the solution to (1.2) on the complex manifold $(M, J_-)$ with initial condition $\omega_-$. Next consider the time-dependent vector fields

\[(3.2) \quad X^\pm = \left(-J_{\pm} d^c_{\pm} \omega_{\pm}\right)^{\pm},\]

and let $\phi_{\pm}(t)$ denote the one-parameter family of diffeomorphisms of $M$ generated by $X^\pm$, with $\phi^0_{\pm} = \text{Id}$. Theorem 1.2 in [17] implies that $(\phi_+(t)^* g_+(t), \phi_+(t)^* (d^c_+ \omega_+(t)))$ is a solution to (1.5) with initial condition $(g, d^c_+ \omega_+)$. Likewise, we have a solution $(\phi_-(t)^* g_-(t), \phi_-(t)^* (d^c_- \omega_-(t)))$ to (1.5) with initial condition $(g, d^c_- \omega_-)$. However, if we let $(\tilde{g}(t), \tilde{H}(t))$ denote this latter solution, we observe that

\[(3.3) \quad \frac{\partial}{\partial t} \tilde{g}_{ij} = -2\tilde{R}_{ij} + \frac{1}{2} \tilde{H}_{iplq} \tilde{H}^{pq} = -2\tilde{R}_{ij} + \frac{1}{2} \left( -\tilde{H}_{iplq} \right) \left( -\tilde{H}^{pq} \right), \]

i.e. $(\tilde{g}(t), -\tilde{H}(t))$ is a solution to (1.5) with initial condition $(g, -d^c_- \omega_-)$. By (1.1), we see that $(\phi_+(t)^* g_+(t), \phi_+(t)^* (d^c_+ \omega_+(t)))$ and $(\phi_-(t)^* g_-(t), -\phi_-(t)^* (d^c_- \omega_-(t)))$ are two solutions of (1.5) with the same initial condition. Using the uniqueness of solutions of (1.5) ([13 Proposition 3.3]), we conclude that these two solutions coincide, and call the resulting one-parameter family $(g(t), H(t))$. 

Next we want to identify the two complex structures with which \( g \) remains compatible. We observe by that for arbitrary vector fields \( X, Y, \)

\[
g(\phi_+(t)^*J_+X, \phi_+(t)^*Y) = g(\phi_-(t)^{-1}\cdot J_\pm \cdot \phi_+(t)^*X, \phi_+(t)^{-1}\cdot J_\pm \cdot \phi_+(t)^*Y) \\
= [\phi_+(t)^{-1}\cdot g](J_\pm \cdot \phi_+(t)^*X, J_\pm \cdot \phi_+(t)^*Y) \\
= g_\pm (J_\pm \cdot \phi_+(t)^*X, J_\pm \cdot \phi_+(t)^*Y) \\
= [\phi_+(t)^*g_\pm](X, Y) \\
= g(X, Y).
\]

Therefore \( g(t) \) is compatible with \( \phi_\pm(t)^*J_\pm(t) \). Denote these two time dependent complex structures by \( \tilde{J}_\pm \). It follows that \( \tilde{\omega}_\pm = \phi_\pm(t)^*\omega_\pm \). Next we note by naturality of \( d \) that

\[
\begin{align*}
\tilde{d}_\pm^\ast \tilde{\omega}_\pm(X, Y, Z) &= -[d\tilde{\omega}_\pm](\tilde{J}_\pm X, \tilde{J}_\pm Y, \tilde{J}_\pm Z) \\
&= -[d\phi_\pm(t)^*\omega_\pm](\phi_\pm(t)^{-1}\cdot J_\pm \cdot \phi_+(t)^*X, \cdots) \\
&= [\phi_+(t)^*(-d\omega_\pm)](\phi_+(t)^{-1}\cdot J_\pm \cdot \phi_+(t)^*X, \cdots) \\
&= -d\omega_\pm(J_\pm \cdot \phi_+(t)^*X, \cdots) \\
&= d_\pm^\ast\omega_\pm(\phi_+(t)^*X, \cdots) \\
&= \phi_+(t)^*(d_\pm^\ast\omega_\pm)(X, Y, Z) \\
&= \pm H(X, Y, Z).
\end{align*}
\]

It follows that

\[
\tilde{d}_\pm^\ast \tilde{\omega}_+ = -\tilde{d}_\pm^\ast \tilde{\omega}_- = H, \quad dH = 0,
\]

showing that the triple \((g(t), \tilde{J}_+(t), \tilde{J}_-(t))\) is generalized Kähler for all time. This finishes the proof of Theorem \ref{thm:pluriKahler} \qed

To prove Theorem \ref{thm:pluriKahler2} we need to find the evolution equation for \( J_\pm(t) \). Note that our curvature convention is that \((\nabla^2_{e_\iota e_j} - \nabla^2_{e_j e_\iota})e_k = R_{i\jmath k}^l e_l\).

**Proposition 3.1.** Let \((M^{2n}, \tilde{g}(t), J)\) be a solution to the pluriclosed flow. Let \( \phi_t \) be the one parameter family of diffeomorphisms generated by \(-Jd_\tilde{g}^\ast\omega\) with \( \phi_0 = \text{Id} \), and let \( g(t) = \phi_t^\ast(\tilde{g}(t)), J(t) = \phi_t^\ast(J) \). Then

\[
\frac{\partial}{\partial t} J_k^l = (\Delta J)^l_k - [J, g^{-1}\text{Rc}]^l_k \\
- J_p^s D^s J_k^l D_p J_s^l - J_p^s D^s J_k^l J_p D_s J_s^l + J_p^s D^s J_k^l J_p D_p J_s^l + J_p^s D^s J_k^l D_p J_s^l \\
- J_p^s D_p J_k^l D^s J_s^l + J_p^s D_p J_k^l D^s J_s^l - J_p^s D^s J_k^l D_p J_s^l.
\]

**Proof.** It suffices to compute the time derivative of \( J \) at \( t = 0 \). First we note

\[
\left(\frac{\partial}{\partial t} J(t)\right)|_{t=0} = \frac{\partial}{\partial t} (\phi_t^\ast J)|_{t=0} = \mathcal{L}_{X(0)} J.
\]

One may compute

\[
(\mathcal{L}_X J)(Y) = [X, JY] - J[X, Y].
\]
In coordinates this reads
\begin{equation}
(\mathcal{L}_X J)_k^i = J_p^i \partial_p X^p - J_p^i \partial_p X^i + X^p \partial_p J_p^i.
\end{equation}
Furthermore, since the Levi-Civita connection $D$ is torsion-free we have
\begin{equation}
(\mathcal{L}_X J)_k^i = J_p^i D_k X^p - J_p^i D_p X^i + X^p D_p J_k^i.
\end{equation}
We next observe a formula for the vector field $X$.
\begin{equation}
X^p = -g^{pq} J_q^r (d^r \omega)_r = -J_p^i D^s J_s^i.
\end{equation}
Thus
\begin{equation}
(\mathcal{L}_X J)_k^i = - J_p^i D_k \left( J_p^i D^s J_s^i \right) + J_p^i D_p \left( J_k^i D^s J_s^i \right) - J_p^i D^s J_s^i D_p J_k^i
\end{equation}
\begin{equation}
= D_k D^s J_s^i - J_p^i J_k^i \partial_p D^s J_s^i + J_p^i J_k^i D_p D^s J_s^i + J_p^i J_k^i D^s J_s^i - J_p^i J_k^i D^s J_s^i D_p J_k^i
\end{equation}
\begin{equation}
= D^s D_k J_s^i + g^{uv} \left( R_{uv}^i \partial_p - R_{ukp}^i J_k^i \right)
+ J_p^i J_k^i D^s D_p J_s^i + J_p^i J_k^i g^{uv} \left( R_{uv}^i \partial_p - R_{upq}^i J_s^i \right)
- J_p^i J_k^i D_p J_s^i + J_p^i J_k^i D^s J_s^i - J_p^i J_k^i D^s J_s^i D_p J_k^i.
\end{equation}
We will simplify this expression using the vanishing of the Nijenhuis tensor of $J$. Recall
\begin{equation}
N(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY].
\end{equation}
In coordinates we may express
\begin{equation}
N_{jk}^i = J_j^p \partial_p J_k^i - J_k^p \partial_p J_j^i - J_j^q \partial_q J_k^i + J_k^q \partial_q J_j^i.
\end{equation}
Again since $D$ is torsion-free we may express
\begin{equation}
N_{jk}^i = J_j^p D_p J_k^i - J_k^p D_p J_j^i - J_j^q D_q J_k^i + J_k^q D_q J_j^i.
\end{equation}
Thus since $J$ is integrable we may conclude
\begin{equation}
0 = D^k \left( J_k^i N_{jk}^i \right)
= D^k \left[ J_k^i \left( J_j^p D_p J_k^i - J_k^p D_p J_j^i - J_j^q D_q J_k^i + J_k^q D_q J_j^i \right) \right]
= D^k J_k^i J_k^j - D^k D_k J_k^j + D^k \left[ J_i^j J_k^p D_p J_k^i - J_i^j J_k^p D_p J_k^i \right].
\end{equation}
Plugging this into the first term of (3.13) we conclude
\begin{equation}
(\mathcal{L}_X J)_k^i = D^s D_s J_k^i - D^s \left[ J_v^i \left( J_k^p D_p J_v^i - J_k^p D_p J_v^i \right) \right] + J_p^i J_i^j D^s D_p J_k^j
+ g^{uv} \left( R_{uv}^i \partial_p - R_{ukp}^i J_k^i \right)
- J_p^i J_k^i D_p J_k^i + J_p^i J_k^i D^s J_k^i
- J_p^i J_k^i D^s J_k^i D_p J_k^i.
\end{equation}
Using the skew-symmetry of $J$ one has

$$J^l_i J^p_s D^s D^*_p J^i_k = \frac{1}{2} J^l_i J^p_s (D^s D^*_p - D^*_p D^s) J^i_k$$

(3.19)

$$= \frac{1}{2} J^l_i J^p_s g^{st} (R_{ptk}^m J^m_i - R_{ptm}^i J^m_k)$$

$$= -\frac{1}{2} g^{st} \left( R_{ptk}^l J^p_s + J^l_i J^p_s J^m_k R_{ptm}^i \right).$$

Plugging this into (3.18) yields

$$\frac{\partial}{\partial t} J^l_k = (\Delta J)^l_k - \frac{1}{2} g^{uv} \left( R_{puk}^l J^p_u + J^l_i J^p_u J^m_k R_{pvm}^i \right)$$

$$+ g^{uv} \left( R_{ukp}^l J^p_v - R_{ukp}^l J^p_v - J^l_i J^p u R_{amu}^l - J^l_i J^p u R_{upq}^l \right)$$

$$- J^p_k D^s J^l_i D_p J^i_k - J^p_k D^s D_p J^i_k + J^p_k D^s J^l_i J^i_k + J^p_k D^s J^i_k D_p J^l_i$$

$$- J^p_k D_p J^l_i D^s J^i_k + J^p_k D_p J^l_i D^s J^i_k - J^p_k D^s J^l_i D_p J^i_k.$$  

Next we observe the simplification

$$g^{uv} \left( J^l_i J^p u R_{pvm}^i + 2 J^p_k J^l_i J^q u R_{upq}^l \right) = g^{vu} J^p i J^m_k (R_{pvm}^i + 2 R_{vmp}^i)$$

(3.21)

$$\scriptstyle{\hfill\hfill}= g^{vu} J^p i J^m_k (R_{pvm}^i + R_{vmp}^i + R_{mpv}^i)\hfill\hfill= 0.$$  

Likewise

$$g^{uv} \left( R_{puk}^l J^p v - 2 R_{ukp}^l J^p v \right) = g^{uv} J^p u \left( R_{puk}^l + 2 R_{ukp}^l \right)$$

(3.22)

$$\scriptstyle{\hfill\hfill}= g^{uv} J^p u \left( R_{puk}^l + R_{ukp}^l + R_{kpv}^l \right)\hfill\hfill= 0.$$  

Finally we note that

(3.23)

$$g^{uv} \left( R_{ukv}^l J^p p - J^p_k R_{up}^l \right) = J^p_k R c^l_p - R c^p_k J^l_p = [J, g^{-1} R] c^l_k$$

Plugging these simplifications into (3.18) yields the result.  

With this proposition in hand we can add an equation to the $B$-field flow system to yield a new system of equations which preserves the generalized Kähler condition. Specifically, given a Riemannian manifold $(M^n, g)$ and $J \in \text{End}(TM)$, let

$$\mathcal{R}(J)^i_k = [J, g^{-1} R] c^i_k$$

(3.24)

$$\mathcal{Q}(DJ)^i_k = - J^p_k D^s J^l_i D_p J^i_s - J^p_k D^s D_p J^i_s J^l_i J^i_k + J^p_k D^s J^l_i D_p J^i_s + J^p_k D^s J^l_i D_p J^i_s$$

$$- J^p_k D_p J^l_i D^s J^i_s + J^p_k D_p J^l_i D^s J^i_s - J^p_k D^s J^l_i D_p J^i_s.$$
Now consider the system of equations for an a priori unrelated Riemannian metric $g$, three-form $H$, and tangent bundle endomorphisms $J_\pm$:

\[
\frac{\partial}{\partial t}g_{ij} = -2Rc_{ij} + \frac{1}{2}H_{ipq}H_{j}^{pq} \tag{3.26}
\]

\[
\frac{\partial}{\partial t}H = \Delta dH, \tag{3.25}
\]

\[
\frac{\partial}{\partial t}J_{\pm} = \Delta J_{\pm} + \mathcal{R}(J_{\pm}) + \mathcal{Q}(DJ_{\pm}).
\]

**Theorem 3.2.** Let $M^n$ be a smooth compact manifold. Let $g_0 \in \text{Sym}^2(T^*M)$ be a Riemannian metric, $H_0 \in \Lambda^3(T^*M)$, $(J_\pm)_0 \in \text{End}(TM)$. There exists $T > 0$ and a unique solution to (3.26) on $[0, T)$ with initial condition $(g_0, H_0, (J_\pm)_0)$.

**Proof.** The proof is by now standard and we only give a sketch. For a metric $g$ consider the vector field $X^k_g = g^{ij} \left( \Gamma^k_{ij} - (T^0)^k_{ij} \right)$. Now consider the gauge-fixed system

\[
\frac{\partial}{\partial t}g_{ij} = -2Rc_{ij} + \frac{1}{2}H_{ipq}H_{j}^{pq} + (L_{X_g}g)_{ij},
\]

\[
\frac{\partial}{\partial t}H = \Delta dH + L_{X_g}H, \tag{3.26}
\]

\[
\frac{\partial}{\partial t}J_{\pm} = \Delta J_{\pm} + \mathcal{R}(J_{\pm}) + \mathcal{Q}(DJ_{\pm}) + L_{X_g}J_{\pm}.
\]

Let $\mathcal{O}(g, H, J_\pm)$ denote the total differential operator representing the right hand sides of (3.26). A calculation shows that the principal symbol of the linearized operator of $\mathcal{O}$ is elliptic. More specifically,

\[
[\sigma \mathcal{O}] (\xi) \begin{pmatrix}
\delta g \\
\delta H \\
\delta J_+ \\
\delta J-
\end{pmatrix} = \begin{pmatrix}
|\xi|^2 \text{Id} & 0 & 0 & 0 \\
* & |\xi|^2 \text{Id} & 0 & 0 \\
* & 0 & |\xi|^2 \text{Id} & 0 \\
* & 0 & 0 & |\xi|^2 \text{Id}
\end{pmatrix} \begin{pmatrix}
\delta g \\
\delta H \\
\delta J_+ \\
\delta J-
\end{pmatrix}. \tag{3.27}
\]

It follows from standard results that there is a unique solution to (3.26) on some maximal time interval $[0, T)$. Moreover, if we let $\phi_t$ denote the one-parameter family of diffeomorphisms generated by $-X_t$ satisfying $\phi_0 = \text{Id}$, it follows that $(\phi_t^* (g(t)), \phi_t^* (H(t)), \phi_t^* (J_\pm (t)))$ is a solution to (3.25). Finally, the proof of uniqueness is the same as that for Ricci flow, where one uses that in the modified gauge the diffeomorphisms $\phi_t$ satisfy the harmonic map heat flow equation. For more detail see ([2] pg. 117).

We now give a corollary to this discussion which is a more precise statement of Theorem 1.3.

**Corollary 3.3.** Let $(M^{2n}, g, J_\pm)$ be a compact generalized Kähler manifold. The solution to (3.25) with initial condition $(g, d_+^\omega, J_\pm)$ remains a generalized Kähler structure in that $g$ is compatible with $J_\pm$, $H = \pm d_+^\omega$, and (1.3) holds at all time the solution exists.

**Proof.** Let $(g(t), H(t))$ be the solution to (1.3) with initial condition $(g, d_+^\omega)$. We showed in Theorem 1.3 that $(g(t), H(t), J_\pm (t))$ is a generalized Kähler structure for all times, where $J_\pm (t) = \phi_\pm (t)^* J_\pm$ and $\phi_\pm$ denote the one-parameter families of diffeomorphisms.
used above. However, from Proposition 3.1 we have that $J_{\pm}$ are solutions of
\begin{equation}
\frac{\partial}{\partial t}J_{\pm} = \Delta J_{\pm} + \mathcal{R}(J_{\pm}) + \mathcal{Q}(DJ_{\pm}).
\end{equation}
It follows that $(g(t), H(t), J_{\pm}(t))$ is the unique solution to (3.25) with initial condition $(g, dc^+\omega_+, J_{\pm})$, and the result follows.

4. The structure of static metrics

In this section we will collect some results on static pluriclosed solutions to (1.5). We begin with some general definitions.

Definition 4.1. Let $(M^{2n}, \omega, J)$ be Hermitian manifold with pluriclosed metric, and let $H = dc^\ast\omega$. We say that $\omega$ is a $B$-field flow soliton if there exists a vector field $X$ and $\lambda \in \mathbb{R}$ such that
\begin{equation}
\text{Rc} - \frac{1}{4}H^2 + L_Xg = \lambda g,
\end{equation}
\begin{equation}
-\frac{1}{2}\Delta dH + L_XH = \lambda H.
\end{equation}
The form $\omega$ is called $B$-field flow static, or simply static for short, if (4.1) is satisfied with $X = 0$.

Proposition 4.2. Let $(M^{2n}, \omega, J)$ be a static structure. Then
\begin{itemize}
    \item If $\lambda = 0$ then $d^\ast H = 0$ and $b_1(M) \leq 2n$
    \item If $\lambda < 0$ then $g$ is Kähler, i.e. $H = 0$.
    \item If $\lambda > 0$ then $|\pi_1(M)| < \infty$.
    \item If $\lambda \neq 0$ then $[H] = 0$.
\end{itemize}

Proof. We note that
\begin{equation}
\int_M |d^\ast H|^2 = \int_M (dd^\ast H, H) = 2\lambda \int_M |H|^2.
\end{equation}
The first part of the first statement and second statement immediately follow. Now note that
\begin{equation}
\text{Rc} = \lambda g + \frac{1}{4}H^2.
\end{equation}
Since $H^2$ is positive semidefinite, we conclude that if $\lambda > 0$ then $\text{Rc} > 0$. It follows from the Bonnet-Meyers Theorem that $|\pi_1(M)| < \infty$. Also, if $\lambda = 0$, the Bochner argument yields the bound $b_1(M) \leq 2n$. Finally, since $H$ is closed, if $\lambda \neq 0$ we conclude that $H = \frac{1}{4}\Delta dH = \frac{1}{4}dd^\ast H$ and hence $[H] = 0$. \hfill \Box

Corollary 4.3. Let $(M^4, \omega, J)$ be a static structure, and suppose $M$ is not of Kähler type, i.e. $b_1$ is odd. Then $\lambda = 0$.

Proof. Since the manifold $M$ does not admit Kähler metrics, the second statement of Proposition 4.2 rules out $\lambda < 0$. Likewise, if $\lambda > 0$ then by the third statement of Proposition 4.2 we conclude that $|\pi_1(M)| < \infty$, so that $b_1(M) = 0$, contradicting that $b_1$ is odd. Thus $\lambda = 0$. \hfill \Box

Proposition 4.4. Let $(M^4, \omega, J)$ be a static structure and suppose $M$ is of Kähler type and $\lambda = 0$. Then $\omega$ is Kähler-Einstein.
Proof. It follows from the first statement of Proposition 4.2 that $H$ is harmonic. It follows from $\partial \bar{\partial}$-lemma that $H$ is exact, and hence $H$ vanishes and so the metric is Kähler-Einstein. \hfill \Box

**Proposition 4.5.** Suppose $(M^4, g, J)$ is a static structure with $\lambda = 0$. Then $D\theta = 0$, that is, the Lee form is parallel with respect to the Levi-Civita connection.

Proof. As noted above, if $\lambda = 0$ then $Rc = \frac{1}{4}H^2 \geq 0$. Also, we have $d^*H = 0$. But in the case of complex surfaces, $\theta = \ast H$, therefore $\theta$ is harmonic. It then follows from the Bochner technique that $\theta$ is parallel. \hfill \Box

Finally we give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** It follows from Corollary 4.3 that $\lambda = 0$. Thus from Proposition 4.5 we conclude that $D\theta = 0$. Recalling that the pluri closed flow equations and $B$-field flow equations differ by the Lie derivative of the vector dual to $\theta$, we conclude that in fact $\omega$ is static for the pluri closed flow (see [17] Theorem 6.5), i.e.

$$\partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det g = 0. \quad (4.4)$$

One can check (see [16] Proposition 3.3, [11] Proposition 3.3) that this is the same as

$$S - Q = 0 \quad (4.5)$$

where $S = \text{tr}_\omega \Omega$ is the curvature endomorphism associated to the Chern connection, and

$$Q_{ij} = g^{klm}T_{ikm}^j, \quad (4.6)$$

where $T$ is the torsion of the Chern connection. In the case of surfaces, one has ([16] Lemma 4.4) that $Q = \frac{1}{2} |T|^2 \omega$. Therefore the metric defines a Hermitian-Einstein connection on $TM$. It follows from ([7] Theorem 2) that $(M, g, J)$ is either Kähler-Einstein or $g$ is locally isometric, up to homothety, to $\mathbb{R} \times S^3$. It follows from [6] that the manifold is a Hopf surface. More specifically, it follows from ([6] III Lemma 11) that the fundamental group takes the form claimed in the theorem. \hfill \Box

**Remark 4.6.** Indeed each Hopf surface described in Theorem 1.4 admits static metrics, given by the metric $\frac{1}{\rho^2} \partial \bar{\partial} \rho^2$, where $\rho = \sqrt{z_1 \bar{z}_1 + z_2 \bar{z}_2}$.

**References**

[1] C.G. Callan, D. Friedan, E.J. Martinec, M.J. Perry, *Strings in background fields*, Nuc. Phys. B262 (1985) 593-609.
[2] B. Chow, P. Lu, L. Ni *Hamilton's Ricci Flow*, Lectures in Contemporary Mathematics, American Mathematical Society, Providence, RI 2005.
[3] D. Friedan, *Nonlinear models in 2 + $\epsilon$ Dimensions* Ann. Phys. 163, 318-419 (1985).
[4] D. Friedan, E.J. Martinec, S. Shenker, *Conformal invariance, supersymmetry and string theory* Nuc. Phys. B271 (1986), 93-165.
[5] S.J. Gates, C.M. Hull, M. Roček, *Twisted multiplets and new supersymmetric nonlinear sigma models*, Nuc. Phys. B 248 (157-186), 1984.
[6] P. Gauduchon, *Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$*, J. Reine Angew. Math. 469 1-50.
[7] P. Gauduchon, S. Ivanov, *Einstein-Hermitian surfaces and Hermitian Einstein-Weyl structures in dimension 4*. Math. Z. 226, 317-326 (1997).
[8] M. Gualtieri *Generalized Complex Geometry*. D. Phil. thesis, Oxford University, 2003, arXiv:math/0401221v1.
[9] M. Gualtieri, *Generalized Kähler Geometry*, arXiv:1007.3485
[10] N. Hitchin, *Generalized Calabi-Yau manifolds* Q.J. Math, 54(3), 281-308, 2003, arXiv:math/0209099.
[11] S. Ivanov, G. Papadopoulos, *Vanishing theorems and string backgrounds*, Class. Quantum Grav. 18(2001) 1089-1110.
[12] T. Oliynyk, V. Suneeta, E. Woolgar, *A gradient flow for worldsheet nonlinear sigma models*, Nucl. Phys. B739 (2006), 441-458.
[13] G. Perelman, *The entropy formula for Ricci flow and its geometric applications*, arxiv:0211159
[14] J.G. Polchinski, *String Theory* Vol. I (Cambridge 1998).
[15] J. Streets, *Regularity and expanding entropy for connection Ricci flow*, J. Geom. Phys. 58 (2008), 900-912.
[16] J. Streets, G. Tian, *A parabolic flow of pluriclosed metrics*, Int. Math. Res. Not. 16 (2010), 3101-3133.
[17] J. Streets, G. Tian, *Regularity results for pluriclosed flow*, arxiv:1008.2794
[18] A.B. Zamolodchikov “Irreversibility” of the flux of the renormalization group in a 2D field theory. JETP Lett. 43, p. 730-732, 1986.

Rowland Hall, University of California, Irvine, Irvine, CA 92617
E-mail address: jstreets@uci.edu

Beijing University, China and Princeton University, Princeton, NJ 08544
E-mail address: tian@math.princeton.edu