Abstract. A C*-algebra is said to be K-stable if its nonstable K-groups are naturally isomorphic to the usual K-theory groups. We study continuous $C(X)$-algebras, each of whose fibers are K-stable. We show that such an algebra is itself K-stable under the assumption that the underlying space $X$ is compact, metrizable, and of finite covering dimension.

Nonstable K-theory is the study of the homotopy groups of the unitary group of a C*-algebra. The study of these groups was initiated by Rieffel [18], who showed that, for an irrational rotation algebra $A$, the inclusion map from $A$ to $M_n(A)$ induces an isomorphism between the corresponding homotopy groups. In other words, the nonstable $K$-groups are naturally isomorphic to the usual $K$-theory groups of the algebra.

The theory was further explored by Thomsen [21], who used the notion of quasi-unitaries to profitably extend nonstable K-theory to encompass non-unital C*-algebras. Furthermore, he showed that this forms a homology theory, which allowed him to explicitly calculate these groups for certain C*-algebras. In particular, he showed that certain infinite dimensional C*-algebras (including the Cuntz algebras and simple infinite dimensional AF-algebras) satisfy the property enjoyed by the irrational rotation algebra mentioned above; a property he termed $K$-stability.

Since then, it has been proved (See Section 1.1) that a variety of interesting simple C*-algebras are $K$-stable. The goal of this paper is to enlarge this class of C*-algebras to include non-simple C*-algebras.

By the Dauns-Hoffmann theorem (See [14] or [19]), any non-simple C*-algebra may be represented as the section algebra of an upper semi-continuous C*-bundle over a compact space. If we assume that the underlying space $X$ is Hausdorff, then such an algebra carries a non-degenerate, central action of $C(X)$, and is called a $C(X)$-algebra. An interesting sub-class of $C(X)$-algebras are ones that come equipped with a natural continuity condition. These algebras, called continuous $C(X)$-algebras, are particularly tractable as one can often take phenomena that occur at each fiber and propagate them to understand local behaviour of the algebra. Using a compactness argument, one may even be able to understand global behaviour. It is this idea that we employ in this paper to prove our main theorem.

Theorem A. Let $X$ be a compact metric space of finite covering dimension, and let $A$ be a continuous $C(X)$-algebra. If each fiber of $A$ is K-stable, then $A$ is K-stable.

Note that this theorem applies, in particular, to continuous fields of Kirchberg algebras, which are widely studied from the point of view of classification ([3], [4]). Furthermore, we
should mention here that the assumptions on the space $X$ in the above theorem appear to be a common (and essential) hypothesis in many theorems that deal with continuous fields ([1], [3], [9]).

1. Preliminaries

1.1. Nonstable $K$-theory. We begin by reviewing the work of Thomsen of constructing the nonstable $K$-groups associated to a C*-algebra. For the proofs of all the facts mentioned below, the reader may refer to [21].

Let $A$ be a C*-algebra (not necessarily unital). Define an associative composition $\cdot$ on $A$ by

$$a \cdot b = a + b - ab$$

An element $a \in A$ is said to be quasi-invertible if $\exists b \in A$ such that

$$a \cdot b = b \cdot a = 0$$

We write $\hat{GL}(A)$ for the set of all quasi-invertible elements in $A$. An element $u \in A$ is said to be a quasi-unitary if

$$u \cdot u^* = u^* \cdot u = 0$$

We write $\hat{U}(A)$ for the set of all quasi-unitary elements in $A$. If $B$ is a unital C*-algebra, we write $GL(B)$ for the group of invertibles in $B$ and $U(B)$ for the group of unitaries in $B$.

**Lemma 1.1.** Let $B$ be a C*-algebra with unit 1 and $A$ be a closed, two-sided ideal in $B$. Then

$$\hat{GL}(A) = (1 - GL(B)) \cap A$$

$$\hat{U}(A) = (1 - U(B)) \cap A$$

and for $a, b \in A$, $a \cdot b = 1 - (1 - a)(1 - b)$.

If $A$ is a C*-algebra, then $A$ embeds as an ideal in its unitization $A^+$. By Lemma 1.1, it follows that both $\hat{GL}(A)$ and $\hat{U}(A)$ are groups under $\cdot$ given by Eq. (1). Furthermore, when equipped with the norm topology of $A$, $\hat{GL}(A)$ is open in $A$, $\hat{U}(A)$ is closed in $A$, and they both form topological groups.

**Lemma 1.2.** The function $r : \hat{GL}(A) \to \hat{U}(A)$ given by

$$r(a) := 1 - (1 - a)((1 - a^*)(1 - a))^{-1/2}$$

defines a strong deformation retract of $\hat{GL}(A)$ onto $\hat{U}(A)$. In particular, $\hat{GL}(A)$ is homotopy equivalent to $\hat{U}(A)$.

For elements $u, v \in \hat{U}(A)$, we write $u \sim v$ if there is a continuous function $f : [0, 1] \to \hat{U}(A)$ such that $f(0) = u$ and $f(1) = v$. We write $\hat{U}_0(A)$ for the set of $u \in \hat{U}(A)$ such that $u \sim 0$. Note that $\hat{U}_0(A)$ is a closed, normal subgroup of $\hat{U}(A)$. The next theorem of Thomsen, together with a result of Dold [5, Theorem 4.8], implies that, if $\varphi : A \to B$ is a surjective *-homomorphisms between two C*-algebras, then the induced map

$$\varphi : \hat{U}_0(A) \to \hat{U}_0(B)$$

is a Serre fibration. This fact will be used repeatedly throughout the paper.
Theorem 1.3. Let
\[ 0 \rightarrow J \xrightarrow{i} A \xrightarrow{\phi} D \rightarrow 0 \]
be a short exact sequence of C*-algebras. Then the sequences
\[ 0 \rightarrow i^{-1}(\hat{U}_0(A)) \xrightarrow{i} \hat{U}_0(A) \xrightarrow{\phi} \hat{U}_0(B) \rightarrow 0, \]
and
\[ 0 \rightarrow \hat{U}(J) \xrightarrow{i} \hat{U}(A) \xrightarrow{\phi} q(\hat{U}(B)) \rightarrow 0 \]
are both exact sequences of topological groups for which the map \( q \) admits continuous local sections.

For a C*-algebra \( A \), the suspension of \( A \) is defined to be
\[ SA := \{ f \in C([0,1], A) : f(0) = f(1) = 0 \} \cong C_0(\mathbb{R}) \otimes A \]
For \( n > 1 \), we set \( S^n A := S(S^{n-1} A) \). We then have

Lemma 1.4. For any C*-algebra \( A \),
\[ \pi_n(\hat{U}(A)) \cong \pi_0(\hat{U}(S^n A)) \]

Definition 1.5. The nonstable K-groups of a C*-algebra \( A \) are defined as
\[ k_n(A) := \pi_{n+1}(\hat{U}(A)), \quad \text{for } n = -1, 0, \ldots \]
By the above lemma and Bott periodicity, it follows [21, Proposition 2.6] that, for any C*-algebra \( A \),
\[ k_n(A \otimes K) \cong K_n(A) \]
where \( K \) denotes the C*-algebra of compact operators on a separable Hilbert space, and \( K_n(A) \) denotes the usual K-theory groups of \( A \). Motivated by this, Thomsen defines the notion of K-stability of a C*-algebra.

Definition 1.6. Let \( A \) be a C*-algebra and \( m \geq 2 \). Define \( \iota_m : M_{m-1}(A) \rightarrow M_m(A) \) by
\[ a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \]
We say that \( A \) is K-stable if \( (\iota_m)_* : k_n(M_{m-1}(A)) \rightarrow k_n(M_m(A)) \) is an isomorphism for all \( n = -1, 0, 1, 2 \ldots \) and all \( m = 2, 3, 4 \ldots \).

Remark 1.7. The following C*-algebras are known to be K-stable:
- If \( Z \) denotes the Jiang-Su algebra, then \( A \otimes Z \) is K-stable for any C*-algebra \( A \) [10]. In particular, every separable, approximately divisible C*-algebra is K-stable [22].
- Every irrational rotation algebra is K-stable [18].
- If \( O_n \) denotes the Cuntz algebra, then \( A \otimes O_n \) is K-stable for any C*-algebra \( A \) [21].
- If \( A \) is an infinite dimensional simple AF-algebra, then \( A \otimes B \) is K-stable for any C*-algebra \( B \) [21].
- If \( A \) is a purely infinite, simple C*-algebra, and \( p \) any non-zero projection of \( A \), then \( pAp \) is K-stable [23].

The next result we need deals with extensions. The first part of the theorem follows by the five lemma, while the second is [21, Theorem 3.11].

Theorem 1.8. Given a short exact sequence of C*-algebras
\[ 0 \rightarrow J \rightarrow A \rightarrow D \rightarrow 0 \]
(1) If $A$ and $D$ are $K$-stable, then so is $J$.
(2) If $J$ and $D$ are $K$-stable, then so is $A$.

We conclude this section with an important observation about $K$-stable $C^*$-algebras.

**Lemma 1.9.** If $A$ is $K$-stable, then for any $m \geq 2$, $\iota_m(\hat{U}(M_{m-1}(A)))$ is a strong deformation retract of $\hat{U}(M_m(A))$.

**Proof.** Note that $\hat{GL}(A)$ is an absolute neighbourhood retract by [15, Theorem 5]. Therefore, the pair $(\hat{GL}(M_m(A)), \iota_m(\hat{GL}(M_{m-1}(A))))$ has the homotopy extension property with respect to all spaces [15, Theorem 7]. If $A$ is $K$-stable, then $\iota_m: \hat{GL}(M_{m-1}(A)) \to \hat{GL}(M_m(A))$ is a weak homotopy equivalence. However, since $\hat{GL}(A)$ is an open subset of a normed linear space, $\hat{GL}(A)$ has the homotopy type of a CW-complex [13, Chapter IV, Corollary 5.5]. By Whitehead’s theorem [8, Theorem 4.5], it follows that $\iota_m$ is a homotopy equivalence.

Therefore, $\iota_m(\hat{GL}(M_{m-1}(A)))$ is a strong deformation retract of $\hat{GL}(M_m(A))$ by [8, Theorem 0.20]. Since the retractions $r: \hat{GL}(M_k(A)) \to \bar{U}(M_k(A))$ commute with the inclusion map $\iota_k$, we conclude that $\iota_m(\hat{U}(M_{m-1}(A)))$ is a strong deformation retract of $\hat{U}(M_m(A)))$. □

1.2. $C(X)$-algebras. Let $A$ be a $C^*$-algebra, and $X$ a compact Hausdorff space. We say that $A$ is a $C(X)$-algebra [11, Definition 1.5] if there is a unital $*$-homomorphism $\theta: C(X) \to ZM(A)$, where $ZM(A)$ denotes the center of the multiplier algebra of $A$.

If $Y \subset X$ is closed, the set $C_0(X,Y)$ of functions in $C(X)$ that vanish on $Y$ is a closed ideal of $C(X)$. By the Cohen factorization theorem [2, Theorem 4.6.4], $C_0(X,Y)A$ is a closed, two-sided ideal of $A$. The quotient of $A$ by this ideal is denoted by $A(Y)$, and we write $\pi_Y: A \to A(Y)$ for the quotient map (also referred to as the restriction map). If $Z \subset Y$ is a closed subset of $Y$, we write $\pi^Y_Z: A(Y) \to A(Z)$ for the natural restriction map, so that $\pi_Z = \pi^Y_Z \circ \pi_Y$. If $Y = \{x\}$ is a singleton, we write $A(x)$ for $A(\{x\})$ and $\pi_x$ for $\pi_{\{x\}}$. The algebra $A(x)$ is called the fiber of $A$ at $x$. For $a \in A$, write $a(x)$ for $\pi_x(a)$. For each $a \in A$, we have a map

$$\Gamma_a: X \to \mathbb{R} \text{ given by } x \mapsto \|a(x)\|$$

This map is, in general, upper semi-continuous [19, Proposition 1.2]. We say that $A$ is a continuous $C(X)$-algebra if $\Gamma_a$ is continuous for each $a \in A$.

If $A$ is a $C(X)$-algebra, we will often have reason to consider other $C(X)$-algebras obtained from $A$. At that time, the following result of Kirchberg and Wasserman will be useful.

**Theorem 1.10.** [12, Remark 2.6] Let $X$ be a compact Hausdorff space, and let $A$ be a continuous $C(X)$-algebra. If $B$ is a nuclear $C^*$-algebra, then $A \otimes B$ is a continuous $C(X)$-algebra whose fiber at a point $x \in X$ is $A(x) \otimes B$.

Finally, one fact that plays a crucial role in our investigation is that a $C(X)$-algebra may be patched together from quotients in the following way: Let $B, C,$ and $D$ be $C^*$-algebras, and $\delta: B \to D$ and $\gamma: C \to D$ be $*$-homomorphisms. We define the pullback of this system to be

$$A = B \oplus_D C := \{(b, c) \in B \oplus C : \delta(b) = \gamma(c)\}$$
This is describe by a diagram

(2) \[
\begin{array}{c}
A \xrightarrow{\phi} B \\
\parallel \downarrow \delta \\
C \xrightarrow{\gamma} D
\end{array}
\]

where \(\phi(b, c) = b\) and \(\psi(b, c) = c\).

**Lemma 1.11.** [3, Lemma 2.4] Let \(X\) be a compact Hausdorff space and \(Y\) and \(Z\) be two closed subsets of \(X\) such that \(X = Y \cup Z\). If \(A\) is a \(C(X)\)-algebra, then \(A\) is isomorphic to the pullback

\[
\begin{array}{c}
A \xrightarrow{\pi_Y} A(Y) \\
\parallel \downarrow \pi_{Y \cap Z} \\
A(Z) \xrightarrow{\pi_Y \circ \iota} A(Y \cap Z)
\end{array}
\]

1.3. **Notational Conventions.** We fix some notational conventions we will use repeatedly:
- If \(A\) is a \(C^*\)-algebra, we write \(\iota_A : A \to M_2(A)\) for the natural inclusion map. When there is no ambiguity, we write \(\iota\) for this map. Moreover, if \(\varphi : A \to B\) is a \(*\)-homomorphism between two \(C^*\)-algebras, then the induced map from \(\hat{U}(A)\) to \(\hat{U}(B)\) is also denoted by \(\varphi\).
- If \(A\) is a continuous \(C(X)\)-algebra, then \(M_2(A)\) is also a continuous \(C(X)\)-algebra with fibers \(M_2(A(x))\) by Theorem 1.10. We will often consider both simultaneously, so we fix the following convention: If \(Y \subset X\) is a closed set, we denote the restriction map by \(\eta_Y : M_2(A) \to M_2(A(Y))\), and write \(\iota_Y : A(Y) \to M_2(A(Y))\) for the natural inclusion map. If \(Y = X\), we simply write \(\iota\) for \(\iota_X\). Note that \(\eta_Y \circ \iota = \iota_Y \circ \pi_Y\). Once again, if \(Y = \{x\}\), we simply write \(\iota_x\) for \(\iota_{\{x\}}\).
- If \(A\) and \(B\) are two \(C^*\)-algebras, the symbol \(A \otimes B\) will always denote the minimal tensor product. If \(B = C_0(X)\) is commutative, we identify \(C_0(X) \otimes A\) with \(C_0(X,A)\), the space of continuous \(A\)-valued functions on \(X\) that vanish at infinity.

Finally, suppose \(f\) and \(g\) are two continuous paths in a topological space \(Y\). If \(f(1) = g(0)\), we write \(f \circ g\) for the concatenation of the two paths. If \(f\) and \(g\) agree at end-points, we write \(f \sim_h g\) if there is a path homotopy between them. Furthermore, we write \(\overline{f}\) for the path \(\overline{f}(t) := f(1 - t)\). If \(Y = \hat{U}(A)\) for some \(C^*\)-algebra \(A\), we write \(f \cdot g\) for the path \(t \mapsto f(t) \cdot g(t)\), and we write \(f^*\) for the path \(t \mapsto f(t)^*\).

2. **Main Results**

The goal of this section is to provide a proof of Theorem A. Before that, we mention the following result, which gave us reason to explore this question.

**Proposition 2.1.** Let \(B, C,\) and \(D\) be \(K\)-stable \(C^*\)-algebras, and let \(\delta : B \to D\) and \(\gamma : C \to D\) be \(*\)-homomorphisms. If either \(\delta\) or \(\gamma\) is surjective, then the pullback \(A = B \oplus_D C\) is also \(K\)-stable.
Proof. Consider the notation of Eq. (2), and assume without loss of generality that \( \delta \) is surjective. Then \( \psi \) is surjective, and we have a short exact sequence

\[
0 \to J_B \to B \xrightarrow{\delta} D \to 0
\]

where \( J_B = \ker(\delta) \). By Theorem 1.8, it follows that \( J_B \) is \( K \)-stable. Now note that the following sequence is exact

\[
0 \to J_B \xrightarrow{\iota} A \xrightarrow{\psi} C \to 0
\]

where \( \iota(b) := (b, 0) \). From Theorem 1.8, it now follows that \( A \) is \( K \)-stable. \( \square \)

Lemma 2.2. Let \( A \) be a \( K \)-stable \( C^* \)-algebra and \( X \) a locally compact Hausdorff space, then \( C_0(X) \otimes A \) is \( K \)-stable.

Proof. Suppose first that \( X \) is compact. For simplicity of notation, we write \( B \) for \( C(X) \otimes A \).

By Lemma 1.9, for each \( m \geq 2 \), there is a retraction \( r : \mathring{U}(M_m(A)) \to \mathring{U}(M_{m-1}(A)) \) and a homotopy \( F : [0, 1] \times \mathring{U}(M_m(A)) \to \mathring{U}(M_m(A)) \) such that \( F(0, v) = v \) and \( F(1, v) = r(v) \) for all \( v \in \mathring{U}(M_m(A)) \). Clearly, we may identify \( \mathring{U}(M_m(B)) \) with \( C(X, \mathring{U}(M_m(A))) \), where the latter denotes the space of continuous functions from \( X \) to \( \mathring{U}(M_m(A)) \), equipped with the uniform topology (which coincides with the compact-open topology). Therefore, we may define \( \mathring{r} : \mathring{U}(M_m(B)) \to \mathring{U}(M_{m-1}(B)) \) by

\[
\mathring{r}(f)(x) := r(f(x))
\]

and \( \mathring{F} : [0, 1] \times \mathring{U}(M_m(B)) \to \mathring{U}(M_m(B)) \) by

\[
\mathring{F}(s, f)(x) := F(s, f(x))
\]

It is easy to see that \( \mathring{r} \) is a retraction, and that \( \mathring{F} \) is continuous, and implements a homotopy between \( \iota_{M_{m-1}(B)} \circ \mathring{r} \) and \( \text{id}_{\mathring{U}(M_m(B))} \). Thus, \( B \) is \( K \)-stable.

If \( X \) is not compact, let \( X^+ \) denote its one-point compactification. We now have a short exact sequence \( 0 \to C_0(X) \otimes A \to C(X^+) \otimes A \to A \to 0 \). By the first part of the argument, \( C(X^+) \otimes A \) is \( K \)-stable, so the result follows from Theorem 1.8. \( \square \)

As a simple application of Proposition 2.1, we mention the following: A recursive subhomogeneous (RSH) algebra \([17, \text{Definition 1.1}]\) is an iterated pullback of \( C^* \)-algebras of the form \( C(X) \otimes M_n(\mathbb{C}) \), where \( X \) is compact and Hausdorff, where one map in each pullback diagram is a restriction map, and hence surjective. Note that each such algebra is nuclear (and hence exact). If \( A \) is a \( K \)-stable and \( B \) is an RSH algebra, it follows by \([16, \text{Theorem 3.9}]\) that \( A \otimes B \) is an iterated pullback of \( C^* \)-algebras of the form \( C(X) \otimes M_n(A) \). If \( A \) is \( K \)-stable, then \( M_n(A) \), and hence \( C(X) \otimes M_n(A) \) is also \( K \)-stable by Lemma 2.2. Therefore, by Proposition 2.1, we conclude that

Corollary 2.3. If \( A \) is \( K \)-stable and \( B \) is a recursive subhomogeneous \( C^* \)-algebra, then \( A \otimes B \) is \( K \)-stable.

Let \( X \) be a compact metric space of finite covering dimension, and let \( A \) be a continuous \( C(X) \)-algebra, each of whose fibers are \( K \)-stable. Under the assumption that each fiber is simple and semi-projective, \( A \) may be expressed as an inductive limit of iterated pullbacks by \([3, \text{Theorem 5.2}]\). By Proposition 2.1 and Lemma 2.2, each such pullback is itself \( K \)-stable.
Since the class of $K$-stable C*-algebras is closed under inductive limits (by [7, Proposition 4.4]), it follows that $A$ is itself $K$-stable.

It is this fact that gave us reason to consider the main question of this paper. The assumption that the fibers be semi-projective seemed extraneous, and we were able to avoid that by proving Lemma 2.6 and Lemma 2.9 below.

Let $A$ be a C*-algebra, and $c \in A$ a self-adjoint element. We define

$$\Lambda(c) := -\sum_{n=1}^{\infty} \frac{(ic)^n}{n!}$$

Observe that, in $A^+$, $\Lambda(c) = 1 - \exp(ic)$, so that $\Lambda(c) \sim 0$ in $\hat{U}(A)$ via the path $t \mapsto \Lambda(tc)$. The next lemma is implicit in [21, Lemma 1.7], but we spell it out since its proof is crucial to us.

**Lemma 2.4.** Let $a, b \in \hat{U}(A)$ such that $\|a - b\| < 2$, then $a \sim b$ in $\hat{U}(A)$

**Proof.** Consider $A$ as an ideal in $A^+$. If $\|a - b\| < 2$, then $d := a \cdot b^*$ satisfies

$$\|d\| = \|1 - (1 - a \cdot b^*)\| = \|1 - ((1 - a)(1 - b^*))\| = \|((1 - b) - (1 - a))(1 - b^*)\| \leq \|a - b\| < 2$$

Hence, $(1 - d)$ is a unitary in $A^+$, whose spectrum does not contain 1. Therefore, there is a continuous function $g : S^1 \to \mathbb{R}$ such that $g(1) = 0$ and $\exp(ig(x)) = x$ for all $x \in \sigma(1 - d)$.

Define $c := g(1 - d)$, then $c$ is self-adjoint and $a \cdot b^* = \Lambda(c)$. Thus $a = \Lambda(c) \cdot b \sim b$ in $\hat{U}(A)$.

The next lemma is a variant of [20, Exercise 2.8] for quasi-unitaries.

**Lemma 2.5.** Given $\epsilon > 0$, there is a $\delta > 0$ with the following property: If $A$ is a C*-algebra and $a \in A$ such that $\|a \cdot a^*\| < \delta$ and $\|a^* \cdot a\| < \delta$, then there is a quasi-unitary $u \in \hat{U}(A)$ such that $\|a - u\| < \epsilon$.

**Proof.** Fix $0 < \delta < \min\{\epsilon, 1\}$, and consider $A$ as an ideal in $A^+$. If $b := (1 - a)$, the hypothesis implies that $\|1 - bb^*\| = \|a \cdot a^*\| < \delta < 1$, and $\|1 - b^*b\| < 1$ as well. Hence, $bb^*$ and $b^*b$ are both invertible, and therefore $b$ and $b^*$ are also invertible. Thus, $|b| = (b^*b)^{1/2} \in GL(A^+)$, and $v := |b|^{-1} \in U(A^+)$.

Furthermore, $\sigma(b^*b) \subset (1 - \delta, 1 + \delta)$, so $\sigma(|b|) \subset ((1 - \delta)^{1/2}, (1 + \delta)^{1/2}) \subset (1 - \delta, 1 + \delta)$. Hence, $\|b - 1\| < \delta$, so that $u := 1 - v \in \hat{U}(A)$, and

$$\|u - a\| = \|v - b\| = \|v(1 - |b|)\| \leq \|1 - |b|\| < \delta < \epsilon$$

as required.

**Lemma 2.6.** Let $X$ be a compact, Hausdorff space and let $A$ be a continuous $C(X)$-algebra. Let $a \in \hat{U}(A)$ and $x \in X$ such that $a(x) \sim 0$ in $\hat{U}(A(x))$. Then there is a closed neighbourhood $Y$ of $x$ such that $\pi_Y(a) \sim 0$ in $\hat{U}(A(Y))$.

**Proof.** Since $\pi_x : \hat{U}(A) \to \hat{U}(A(x))$ is a fibration, there is a path $H : [0, 1] \to \hat{U}(A)$ such that $H(0) = 0$ and $H(1)(x) = a(x)$. Let $b := H(1)$, then since $A$ is a continuous $C(X)$-algebra,
there is a closed neighbourhood $Y$ of $x$ such that $\|\pi_Y(a - b)\| < 2$. It follows by Lemma 2.5 that $\pi_Y(a) \sim \pi_Y(b)$ in $\bar{U}(A(Y))$. Furthermore, the path $G : [0, 1] \to \bar{U}(A(Y))$ is of the form

$$G(t) = \pi_Y(b) \cdot \Lambda(t\pi_Y(c))$$

for some self-adjoint element $c \in A$ (since every self-adjoint element in $A(Y)$ lifts to a self-adjoint element in $A$). Since $a(x) = b(x)$, the proof of Lemma 2.4 in fact ensures that we may choose $c$ such that $c(x) = 0$. Therefore, $\pi_x \circ G(t) = a(x)$ for all $t \in [0, 1]$. Concatenating the paths $\pi_Y \circ H$ and $G$, we obtain a path connecting 0 to $\pi_Y(a)$ in $\bar{U}(A(Y))$. □

Our proof of Theorem A is by induction on the covering dimension of the underlying space. The next theorem is the base case, and it holds even if the space is not metrizable.

**Theorem 2.7.** Let $X$ be a compact Hausdorff space of zero covering dimension, and let $A$ be a continuous $C(X)$-algebra. If each fiber of $A$ is $K$-stable, then so is $A$.

**Proof.** If $A$ is a continuous $C(X)$-algebra, then so is every suspension of $A$. Furthermore, $(S^n A)(x) \cong S^n(A(x))$ by Theorem 1.10, and $S^n(A(x))$ is $K$-stable by Lemma 2.2. Hence, by Lemma 1.4, it suffices to show that, for each $m \geq 2$, the map

$$(\iota_m)_* : \pi_0(M_{m-1}(A)) \to \pi_0(M_m(A))$$

is an isomorphism. However, by Theorem 1.10, each $M_n(A)$ is also a continuous $C(X)$-algebra, with fibers $M_n(A(x))$, which is $K$-stable if $A(x)$ is $K$-stable. Therefore, suffices to show that the map

$$\iota_* : \pi_0(\bar{U}(A)) \to \pi_0(\bar{U}(M_n(A)))$$

is an isomorphism for each $n \geq 2$. For simplicity of notation, we fix $n = 2$.

We first consider injectivity. Suppose $a \in \bar{U}(A)$ such that $\iota(a) \sim 0$ in $\bar{U}(M_2(A))$. Then, for any $x \in X$,

$$\iota_x(a(x)) \sim 0 \text{ in } \bar{U}(M_2(A(x)))$$

Since $A(x)$ is $K$-stable, $a(x) \sim 0$ in $\bar{U}(A(x))$. By Lemma 2.6, there is a closed neighbourhood $Y_x$ of $x$ such that $\pi_{Y_x}(b) \sim 0$ in $\bar{U}(A(Y_x))$. Since $X$ is compact and zero dimensional, we obtain disjoint open sets $\{Y_{x_1}, Y_{x_2}, \ldots, Y_{x_n}\}$ which cover $X$. Then by Lemma 1.11,

$$A \cong A(Y_{x_1}) \oplus A(Y_{x_2}) \oplus \ldots \oplus A(Y_{x_n})$$

via the map $b \mapsto (\pi_{Y_{x_1}}(b), \pi_{Y_{x_2}}(b), \ldots, \pi_{Y_{x_n}}(b))$. Since $\pi_{Y_{x_i}}(a) \sim 0$ for each $1 \leq i \leq n$, it follows that $a \sim 0$, so $\iota_*$ is injective.

For surjectivity, choose $u \in \bar{U}(M_2(A))$, and we wish to construct a quasi-unitary $\omega \in \bar{U}(A)$ such that $u \sim \iota(\omega)$. To this end, fix $x \in X$. Then $u(x) \in \bar{U}(M_2(A(x)))$. Since $A(x)$ is $K$-stable, there exists $f_x \in \bar{U}(A(x))$ and a path $g_x : [0, 1] \to \bar{U}(M_2(A(x)))$ such that $g_x(0) = u(x)$ and $g_x(1) = \iota_x(f_x)$. Choose $e_x \in A$ such that $e_x(x) = f_x$ (Note that $e_x$ may not be a quasi-unitary).

Since the map $\eta_x : \hat{U}_0(M_2(A)) \to \hat{U}_0(M_2(A(x)))$ is a fibration, $g_x$ lifts to a path $G_x : [0, 1] \to \bar{U}(M_2(A))$ such that $G_x(0) = u$. Let $b_x := G_x(1)$, and so that $b_x(x) = \iota_x(e_x(x))$. Choose
δ > 0 so that conclusion of Lemma 2.5 holds for ε = 1. Since A is a continuous C(X)-algebra, there is a closed neighbourhood Y_x of x such that
\[ \|\eta_{Y_x}(b_x) - \eta_{Y_x}(\iota(e_x))\| < 1, \|\pi_{Y_x}(e_x^* e_x)\| < \delta, \text{ and } \|\pi_{Y_x}(e_x e_x^*)\| < \delta \]
By Lemma 2.5, there is a quasi-unitary d_x \in \widehat{U}(A(Y_x)) such that \( \|d_x - \pi_{Y_x}(e_x)\| < 1. \) Then
\[ \|\iota_{Y_x}(d_x) - \eta_{Y_x}(b_x)\| < 2 \]
By Lemma 2.4, \( \iota_{Y_x}(d_x) \sim \eta_{Y_x}(b_x) \) in \( \widehat{U}(A(Y_x)) \). Hence, \( \iota_{Y_x}(d_x) \sim \eta_{Y_x}(u) \). As before, since X is compact and zero-dimensional, we may choose disjoint, open sets \{Y_{x_1}, Y_{x_2}, \ldots, Y_{x_n}\} so that
\[ A \cong A(Y_{x_1}) \oplus A(Y_{x_2}) \oplus \ldots \oplus A(Y_{x_n}) \]
via the map \( a \mapsto (\pi_{Y_{x_1}}(a), \pi_{Y_{x_2}}(a), \ldots, \pi_{Y_{x_n}}(a)) \). Similarly,
\[ M_2(A) \cong M_2(A(Y_{x_1})) \oplus M_2(A(Y_{x_2})) \oplus \ldots \oplus M_2(A(Y_{x_n})) \]
via the map \( b \mapsto (\eta_{Y_{x_1}}(b), \eta_{Y_{x_2}}(b), \ldots, \eta_{Y_{x_n}}(b)) \). Therefore, \( \exists \omega \in \widehat{U}(A) \) such that \( \pi_{Y_{x_i}}(\omega) = d_{x_i} \)
for all \( 1 \leq i \leq n \). Furthermore, for each \( 1 \leq i \leq n \), \( \eta_{Y_{x_i}}(\iota(\omega)) = \iota_{Y_{x_i}}(d_{x_i}) \sim \eta_{Y_{x_i}}(u) \) in \( \widehat{U}(A(Y_{x_i})) \), so that \( \iota(\omega) \sim u \) in \( \widehat{U}(A) \), as required.

The next two lemmas help us to extend the above argument to higher dimensional spaces.

**Lemma 2.8.** Let X be a compact Hausdorff space, and A be a continuous C(X)-algebra. Let \( f_1 : [0, 1] \to \widehat{U}(A), i = 1, 2 \) be two paths, and \( x \in X \) be a point such that \( \pi_x \circ f_1 = \pi_x \circ f_2 \).

1. There is a closed neighbourhood Y of x and a homotopy \( H : [0, 1] \times [0, 1] \to \widehat{U}(A(Y)) \)
   such that \( H(0) = \pi_Y \circ f_1 \) and \( H(1) = \pi_Y \circ f_2 \).
2. If, in addition, \( f_1(0) = f_2(0) \) and \( f_1(1) = f_2(1) \), then the homotopy in part (1) may be chosen to be a path homotopy.

*Proof.* For the first part, note that \( C[0, 1] \otimes A \) is itself a continuous C(X)-algebra by Theorem 1.10. Hence, there is a closed neighbourhood Y of x such that \( \|\pi_Y \circ f_1 - \pi_Y \circ f_2\| < 2 \). The result now follows from Lemma 2.4.

For the second part, an examination of Lemma 2.4 shows that the homotopy is implemented by a path \( H : [0, 1] \times [0, 1] \to \widehat{U}(A(Y)) \) given by
\[ H(s, t) = \pi_Y(f_2(t)) \cdot A(sh(t)) \]
where \( h(t) = g(1 - \pi_Y(f_1(t)) \cdot \pi_Y(f_2(t)) e_x^* e_x) \) for an appropriate branch \( g : S^1 \to \mathbb{R} \) of the log function such that \( g(1) = 0 \). Since \( f_1 \) and \( f_2 \) agree at end-points, it follows that \( h(0) = h(1) = 0 \). Hence, \( H \) is a path homotopy.

**Lemma 2.9.** Let X be a compact Hausdorff space, A be a continuous C(X)-algebra, and \( x \in X \) be a point such that \( A(x) \) is K-stable. Let \( a \in \widehat{U}(A) \) be a quasi-unitary and \( F : [0, 1] \to \widehat{U}(M_2(A)) \) be a path such that
\[ F(0) = 0 \text{ and } F(1) = \iota(a) \]
Then, there is a closed neighbourhood Y of x and a path \( L_Y : [0, 1] \to \widehat{U}(A(Y)) \) such that
\[ L_Y(0) = 0, \quad L_Y(1) = \pi_Y(a) \]
and \( \iota_Y \circ L_Y \) is path homotopic to \( \eta_Y \circ F \) in \( \widehat{U}(M_2(A(Y))) \).
Proof. Let \( \iota_x : \hat{U}(A(x)) \to \hat{U}(M_2(A(x))) \) be the natural inclusion map. Since \( A(x) \) is \( K \)-stable, Lemma 1.9 implies that there is a continuous function \( r_x : \hat{U}(M_2(A(x))) \to \hat{U}(A(x)) \) such that \( \iota_x \circ r_x \sim \text{id}_{\hat{U}(M_2(A(x)))} \). Furthermore, since \( r_x \) is a retract, the function 

\[
F' := r_x \circ \eta_x \circ F
\]

is a path in \( \hat{U}(A(x)) \) such that \( F'(0) = 0 \) and \( F'(1) = a(x) \). Consider the commutative diagram

\[
\begin{array}{ccc}
\hat{U}(A) & \xrightarrow{\iota_A} & \hat{U}(M_2(A)) \\
\pi_x & & \eta_x \\
\hat{U}(A(x)) & \xrightarrow{\iota_x \circ r_x} & \hat{U}(M_2(A(x)))
\end{array}
\]

The map \( \pi_x : \hat{U}_0(A) \to \hat{U}_0(A(x)) \) is a fibration, so \( F' \) lifts to a path \( G_x : [0, 1] \to \hat{U}(A) \) such that \( G_x(0) = 0 \). Set \( b_x := G_x(1) \), then

\[
\iota_A \circ G_x : [0, 1] \to \hat{U}(M_2(A))
\]

is a path such that \( \iota_A \circ G_x(0) = 0 \) and \( \iota_A \circ G_x(1) = \iota_A(b_x) \). Furthermore,

\[
\eta_x \circ \iota_A \circ G_x = \iota_x \circ \pi_x \circ G_x = \iota_x \circ F' = \iota_x \circ r_x \circ \eta_x \circ F
\]

Since \( \iota_x(\hat{U}(A(x))) \) is a strong deformation retract of \( \hat{U}(M_2(A(x))) \), it follows that there is a path homotopy \( \widetilde{H} : [0, 1] \times [0, 1] \to \hat{U}(M_2(A(x))) \) such that, for all \( s, t \in [0, 1] \),

\[
\widetilde{H}(0, t) = \eta_x \circ F(t), \quad \widetilde{H}(1, t) = \eta_x \circ \iota_A \circ G_x(t)
\]

\[
\widetilde{H}(s, 0) = 0, \quad \widetilde{H}(s, 1) = \iota_x(a(x))
\]

The map \( \eta_x : \hat{U}_0(M_2(A)) \to \hat{U}_0(M_2(A(x))) \) is a fibration, and \( \eta_x \circ F \) has a lift, so \( \widetilde{H} \) lifts to a homotopy \( H_1 : [0, 1] \times [0, 1] \to \hat{U}(M_2(A)) \) such that

\[
H_1(0, t) = F(t), \quad \text{and} \quad \eta_x \circ H_1 = \widetilde{H}
\]

Similarly, \( \eta_x \circ \iota_A \circ G_x \) lifts to \( \iota_A \circ G_x \), so \( \widetilde{H} \) lifts to a homotopy \( H_2 : [0, 1] \times [0, 1] \to \hat{U}(M_2(A)) \) such that

\[
H_2(1, t) = \iota_A \circ G_x(t), \quad \text{and} \quad \eta_x \circ H_2 = \widetilde{H}
\]

Note that

\[
\eta_x \circ F(t) = \eta_x \circ H_2(0, t) \quad \text{and} \quad \eta_x \circ H_1(1, t) = \eta_x \circ \iota_A \circ G_x(t)
\]

Therefore, by Lemma 2.6 applied to the continuous \( C(X) \)-algebra \( C[0, 1] \otimes A \), there is a closed neighbourhood \( Y' \) of \( x \) and a homotopy \( H : [0, 1] \times [0, 1] \to \hat{U}(M_2(A(Y'))) \) such that

\[
H(0, t) = \eta_{Y'} \circ F(t), \quad H(1, t) = \eta_{Y'} \circ \iota_A \circ G_x(t), \quad \text{and} \quad \eta_{Y'} \circ H(s, 1) = \iota_x(a(x))
\]

and a path \( G_{Y'} : [0, 1] \to \hat{U}(A(Y')) \) such that \( G_{Y'}(0) = \pi_{Y'}(b_x), G_{Y'}(1) = \pi_{Y'}(a) \), and \( \pi_{Y'} \circ G_{Y'} \) is a constant path \( a(x) \) in \( \hat{U}(A(x)) \) (the last statement follows from the proof of Lemma 2.6). Furthermore, \( f : [0, 1] \to \hat{U}(M_2(A(Y'))) \) given by \( f(s) := H(1 - s, 1) \) is a path such that \( f(0) = \iota_{Y'} \circ \pi_{Y'}(b_x) \) and \( f(1) = \iota_{Y'} \circ \pi_{Y'}(a) \). Also,

\[
\eta_{Y'} \circ f(s) = \iota_x(a(x)) = \eta_{Y'} \circ \iota_{Y'} \circ G_{Y'}(s) \quad \forall s \in [0, 1]
\]
Thus, by Lemma 2.8 applied to \( f \) and \( \iota_{Y'} \circ G_{Y'} \), there is a closed neighbourhood \( Y \) of \( x \) such that \( Y \subset Y' \) and \( \eta_{Y}^{Y'} \circ \iota_{Y'} \circ G_{Y'} \sim_{h} \eta_{Y}^{Y'} \circ f \) in \( \hat{U}(M_{2}(A(Y))) \).

But \( \eta_{Y} \circ F \sim_{h} (\eta_{Y} \circ \iota_{A} \circ G_{x}) \bullet (\eta_{Y}^{Y'} \circ f) \), so \( \eta_{Y} \circ F \sim_{h} \iota_{Y} \circ L_{Y} \) where \( L_{Y} : [0,1] \to \hat{U}(A(Y)) \) is given by

\[
L_{Y} := (\pi_{Y} \circ G_{x}) \bullet (\pi_{Y}^{Y'} \circ G_{Y'})
\]

and \( L_{Y} \) satisfies the required conditions. \( \square \)

**Remark 2.10.** We are now in a position to prove Theorem A, but first, we need one important fact, which allows us to use induction: If \( X \) is a finite dimensional compact metric space, then covering dimension agrees with the small inductive dimension [6, Theorem 1.7.7]. Therefore, by [6, Theorem 1.1.6], \( X \) has an open cover \( \mathcal{B} \) such that, for each \( U \in \mathcal{B} \),

\[
\dim(\partial U) \leq \dim(X) - 1
\]

Now suppose \( \{U_{1}, U_{2}, \ldots, U_{m}\} \) is an open cover of \( X \) such that \( \dim(\partial U_{i}) \leq \dim(X) - 1 \) for \( 1 \leq i \leq m \), we define sets \( \{V_{i} : 1 \leq i \leq m\} \) inductively by

\[
V_{1} := \overline{U_{1}}, \text{ and } V_{k} := U_{k} \setminus \left( \bigcup_{i<k} U_{i} \right) \text{ for } k > 1
\]

and subsets \( \{W_{j} : 1 \leq j \leq m - 1\} \) by

\[
W_{j} := \left( \bigcup_{i=1}^{j} V_{i} \right) \cap V_{j+1}
\]

It is easy to see that \( W_{j} \subset \bigcup_{i=1}^{j} \partial U_{i} \), so by [6, Theorem 1.5.3], \( \dim(W_{j}) \leq \dim(X) - 1 \) for all \( 1 \leq j \leq m - 1 \).

**Proof of Theorem A.** Let \( X \) be a compact metric space of finite covering dimension, and let \( A \) be a continuous \( C(X) \)-algebra, each of whose fibers are \( K \)-stable. We wish to show that \( A \) is \( K \)-stable. As in the proof of Theorem 2.7, it suffices to show that the map

\[
\iota_{*} : \pi_{0}(\hat{U}(A)) \to \pi_{0}(\hat{U}(M_{2}(A)))
\]

is bijective. The proof is by induction on \( \dim(X) \), so by Theorem 2.7, we assume that \( \dim(X) \geq 1 \), and that \( A(Y) \) is \( K \)-stable for any closed subset \( Y \) of \( X \) such that \( \dim(Y) \leq \dim(X) - 1 \).

For injectivity, suppose \( a \in \hat{U}(A) \) such that \( \iota(a) \sim 0 \) in \( \hat{U}(M_{2}(A)) \). Let \( F : [0,1] \to \hat{U}(M_{2}(A)) \) be a path such that \( F(0) = \iota(a) \) and \( F(1) = 0 \). For \( x \in X \), by Lemma 2.9, there is a closed neighbourhood \( Y_{x} \) of \( x \) and a path \( L_{Y_{x}} : [0,1] \to \hat{U}(A(Y_{x})) \) such that

\[
L_{Y_{x}}(0) = \pi_{Y_{x}}(a), \quad L_{Y_{x}}(1) = 0
\]

and \( \iota_{Y_{x}} \circ L_{Y_{x}} \) is path homotopic to \( \pi_{Y_{x}} \circ F \) in \( \hat{U}(A(Y_{x})) \). By Remark 2.10, we may choose \( Y_{x} \) to be the closure of a basic open set \( U_{x} \) such that \( \dim(\partial U_{x}) \leq \dim(X) - 1 \). Since \( X \) is compact, we may choose a finite subcover \( \{U_{1}, U_{2}, \ldots, U_{m}\} \). Now define \( \{V_{1}, V_{2}, \ldots, V_{m}\} \) as in Remark 2.10. We observe that each \( V_{i} \) is a closed set such that \( \pi_{V_{i}}(a) \sim 0 \) in \( \hat{U}(A(V_{i})) \) since \( V_{i} \subset \overline{U_{i}} \) for all \( 1 \leq i \leq m \).
Note that $W_1 = V_1 \cap V_2$, and $\dim(W_1) \leq \dim(X) - 1$. By induction hypothesis, $A(W_1)$ is $K$-stable. Let $H_i : [0,1] \rightarrow \hat{U}(A(V_i))$, $i = 1, 2$ be paths such that

$$H_i(0) = \pi_{V_i}(a), H_i(1) = 0$$

and $\iota_{V_i} \circ H_i \sim_h \eta_{V_i} \circ F$. Let $S : [0,1] \rightarrow \hat{U}(A(W_1))$ be the path

$$S(t) := \pi_{W_1}^V(H_1(t)) \cdot \pi_{W_1}^V(H_2(t))$$

Note that $S(0) = S(1) = 0$, so $S$ is a loop, and

$$\iota_{W_1} \circ S = (\eta_{W_1}^V \circ \iota_{V_1} \circ H_1) \cdot (\eta_{W_1}^V \circ \iota_{V_2} \circ H_2^1) \sim_h \eta_{W_1} \circ F \circ (\eta_{V_1} \circ F)^* = 0$$

Hence, $\iota_{W_1} \circ S$ is null-homotopic. Since the map

$$(\iota_{W_1})_* : \pi_1(\hat{U}(A(W_1))) \rightarrow \pi_1(\hat{U}(M_2(A(W_1))))$$

is injective, it follows that $S$ is null-homotopic in $\hat{U}(A(W_1))$. Since the map $\pi_{W_1}^V : \hat{U}_0(A(V_2)) \rightarrow \hat{U}_0(A(W_1))$ is a fibration, it follows that $S$ has a lift $f : [0,1] \rightarrow \hat{U}(A(V_2))$ such that $f(0) = f(1) = 0$ and $\pi_{W_1}^V \circ f = S$. Define $H'_2 : [0,1] \rightarrow \hat{U}(A(V_2))$ be defined by

$$H'_2(t) := f(t) \cdot H_2(t)$$

Then $H'_2(0) = \pi_{V_2}(a), H'_2(1) = 0$, and

$$\pi_{W_1}^V \circ H_1 = \pi_{W_1}^V \circ H'_2$$

By Lemma 1.11, $A(V_1 \cup V_2)$ is a pullback

$$\begin{array}{ccc}
A(V_1 \cup V_2) & \longrightarrow & A(V_1) \\
\pi_{V_1 \cup V_2} & \downarrow & \pi_{V_1} \\
A(V_2) & \longrightarrow & A(W_1) & \pi_{W_1}^V
\end{array}$$

so we obtain a path $H : [0,1] \rightarrow \hat{U}(A(V_1 \cup V_2))$ such that $\pi_{V_1 \cup V_2} \circ H = H_1$ and $\pi_{V_1 \cup V_2} \circ H = H'_2$. In particular,

$$\pi_{V_1} \circ H(0) = \pi_{V_1}(a) \text{ and } \pi_{V_2} \circ H(0) = \pi_{V_2}(a)$$

so $H(0) = \pi_{V_1 \cup V_2}(a)$. Similarly, $H(1) = 0$, so that $\pi_{V_1 \cup V_2}(a) \sim 0$ in $\hat{U}(A(V_1 \cup V_2))$.

Now observe that $W_2 = (V_1 \cup V_2) \cap V_3$, and $\dim(W_2) \leq \dim(X) - 1$. Replacing $V_1$ by $V_1 \cup V_2$, and $V_2$ by $V_3$ in the earlier argument, we may repeat the earlier procedure. By induction on the number of elements in the finite subcover, we see that $a \sim 0$ in $\hat{U}(A)$. This completes the proof of injectivity of $\iota_*$. 

Now consider the surjectivity of $\iota_*$: Fix $u \in \hat{U}(M_2(A))$, and we wish to show that there is a quasi-unitary $\omega \in \hat{U}(A)$ such that $u \sim \iota(\omega)$. So fix $x \in X$. Then by $K$-stability of $A(x)$, there exists $f_x \in \hat{U}(A(x))$ such that $\eta_{x}(u) \sim \iota_{x}(f_x)$. As in the proof of Theorem 2.7, there is a closed neighbourhood $Y_x$ of $x$ and a quasi-unitary $d_x \in \hat{U}(A(Y_x))$ such that

$$\eta_{Y_x}(u) \sim \iota_{Y_x}(d_x)$$
As in the first part of the proof, we may reduce to the case where $X = V_1 \cup V_2$, and there are quasi-unitaries $d_{V_1} \in \hat{U}(A(V_1)), d_{V_2} \in \hat{U}(A(V_2))$ such that

$$\eta_{V_1}(u) \sim \iota_{V_1}(d_{V_1}) \text{ in } \hat{U}(M_2(A(V_i))), i = 1, 2$$

and if $W := V_1 \cap V_2$, then

$$\dim(W) \leq \dim(X) - 1$$

Fix paths $H_i : [0, 1] \to \hat{U}(M_2(A(V_i)))$ such that $H_1(0) = \iota_{V_1}(d_{V_1})$ and $H_1(1) = \eta_{V_1}(u)$, $H_2(0) = \eta_{V_2}(u)$ and $H_2(1) = \iota_{V_2}(d_{V_2})$ and consider the path $F : [0, 1] \to \hat{U}(M_2(A(W)))$ given by

$$F := (\eta_{V_1}^W \circ H_1) \bullet (\eta_{V_2}^W \circ H_2)$$

Then $F(0) = \iota_{V_1} \coprod \pi^V_{W_1}(d_{V_1})$ and $F(1) = \iota_{V_1} \coprod \pi^V_{W_1}(d_{V_2})$. By induction, $A(W)$ is $K$-stable, so by Lemma 1.9, there is a retraction $r_{V_1} : \hat{U}(M_2(A(W))) \to \hat{U}(A(W))$. Define $H := r_{V_1} \circ F$, then $H : [0, 1] \to \hat{U}(A(W'))$ is a path such that

$$H(0) = \pi^V_{W_1}(d_{V_1}) \text{, and } H(1) = \pi^V_{W_1}(d_{V_2})$$

The map $\pi^V_{W_1} : \hat{U}_0(A(V_2)) \to \hat{U}(A(W))$ is a fibration, so there is a path $H' : [0, 1] \to \hat{U}(A(V_2))$ such that

$$H'(1) = d_{V_2}, \text{ and } \pi^V_{W_1} \circ H' = H$$

Define $e_{V_2} := H'(0)$ so that

$$\pi^V_{W_2}(e_{V_2}) = \pi^V_{W_1}(d_{V_1})$$

By Lemma 1.11, $A = A(V_1 \cup V_2)$ is a pullback

$$\begin{array}{ccc}
A & \xrightarrow{\pi_{V_1}} & A(V_1) \\
\pi_{V_2} \downarrow & & \downarrow \pi_{V_2}^V \\
A(V_2) & \xrightarrow{\pi_{V_2}^W} & A(W)
\end{array}$$

so that $\omega := (d_{V_1}, e_{V_2})$ defines a quasi-unitary in $A$. We claim that $\iota(\omega) \sim u$ in $\hat{U}(A)$. To this end, define $H_2 : [0, 1] \to \hat{U}(M_2(A(V_2)))$ by

$$H_2 := H_2 \bullet (\iota_{V_2} \circ \overline{H'})$$

then $H_2(0) = \eta_{V_2}(u)$ and $H_2(1) = \iota_{V_2}(e_{V_2}) = \iota_{V_2}(\pi_{V_2}(\omega)) = \eta_{V_2}(\iota_{V_2}(\omega))$. So if $S : [0, 1] \to \hat{U}(M_2(A(W)))$ is given by

$$S(t) := \eta_{V_2}^W(H_2(t)) \cdot \eta_{V_1}^V(H_1(1 - t))$$

Then $S$ is a path with $S(0) = \eta_{W_1}(u) \cdot \eta_{W_1}(u) = 0$ and $S(1) = \iota_{V_2}^V(e_{V_2}) \cdot \eta_{V_1}^V(\iota_{V_1}(d_{V_1})) = 0$. Furthermore,

$$\eta_{V_1}^W \circ H_2 = (\eta_{V_1}^W \circ H_2) \bullet (\eta_{V_1}^W \circ \iota_{V_1} \circ \overline{H'}) = (\eta_{V_1}^W \circ H_2) \bullet (\iota_{V_1} \circ \pi_{V_1}^V \circ \overline{H'})$$

$$= (\eta_{V_1}^W \circ H_2) \bullet (\iota_{V_1} \circ \overline{F'}) = (\eta_{V_1}^W \circ H_2) \bullet (\iota_{V_1} \circ r_{V_1} \circ \overline{F'})$$

$$\sim_h (\eta_{V_1}^W \circ H_2) \bullet \overline{F'}$$

13
Hence,
\[
S \sim_h [(\eta_1^V \circ H_2) \cdot \mathcal{F}] \cdot (\eta_1^V \circ \mathcal{H})^* \\
= [(\eta_1^V \circ H_2) \cdot (\eta_1^V \circ \mathcal{H}_2)] \cdot (\eta_1^V \circ \mathcal{H}_1)^* \\
\sim_h (\eta_1^V \circ \mathcal{H}_1) \cdot (\eta_1^V \circ \mathcal{H}_1)^* = 0
\]
Hence, $S$ is null-homotopic. Once again, the map $\eta_1^V : \widetilde{U}_0(M_2(A(V_1)) \to \widetilde{U}_0(M_2(A(W)))$ is a fibration, so there is a loop $f : [0, 1] \to \widetilde{U}(M_2(A(V_1)))$ such that $f(0) = f(1) = 0$ and $\eta_1^V \circ f = S$. Define $H_4 : [0, 1] \to \widetilde{U}(M_2(A(V_1)))$ by
\[
H_4(t) := f(t) \cdot H_1(1 - t)
\]
Then $H_4(0) = \eta_1^V(u), H_4(1) = \iota(v_1) = \eta_1^V(\iota(\omega))$. Finally, by construction
\[
\eta_1^V \circ H_4 = \eta_1^V \circ H_3
\]
Therefore, the pair $(H_3, H_4)$ defines a path in $\widetilde{U}(M_2(A))$ connecting $u$ to $\iota(\omega)$. This concludes the proof that $\iota_*$ is surjective. \(\square\)

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