The Eulerian Distribution on Involutions is Indeed Unimodal

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Abstract. Let $I_{n,k}$ (resp. $J_{n,k}$) be the number of involutions (resp. fixed-point free involutions) of $\{1, \ldots, n\}$ with $k$ descents. Motivated by Brenti’s conjecture which states that the sequence $I_{n,0}, I_{n,1}, \ldots, I_{n,n-1}$ is log-concave, we prove that the two sequences $I_{n,k}$ and $J_{n,k}$ are unimodal in $k$, for all $n$. Furthermore, we conjecture that there are nonnegative integers $a_{n,k}$ such that
\[
\sum_{k=0}^{n-1} I_{n,k} t^k = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1 + t)^{n-2k-1}.
\]
This statement is stronger than the unimodality of $I_{n,k}$ but is also interesting in its own right.

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1 Introduction

A sequence $a_0, a_1, \ldots, a_n$ of real numbers is said to be unimodal if for some $0 \leq j \leq n$ we have $a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$, and is said to be log-concave if $a_i^2 \geq a_{i-1} a_{i+1}$ for all $1 \leq i \leq n-1$. Clearly a log-concave sequence of positive terms is unimodal. The reader is referred to Stanley’s survey \cite{Stanley} for the surprisingly rich variety of methods to show that a sequence is log-concave or unimodal. As noticed by Brenti \cite{Brenti}, even though log-concave and unimodality have one-line definitions, to prove the unimodality or log-concavity of a sequence can sometimes be a very difficult task requiring the use of intricate combinatorial constructions or of refined mathematical tools.

Let $\mathfrak{S}_n$ be the set of all permutations of $[n] := \{1, \ldots, n\}$. We say that a permutation $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ has a descent at $i$ ($1 \leq i \leq n-1$) if $a_i > a_{i+1}$. The number of descents of $\pi$ is called its descent number and is denoted by $d(\pi)$. A statistic on $\mathfrak{S}_n$ is said to be Eulerian, if it is equidistributed with the descent number statistic. Recall that the polynomial
\[
A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{1+d(\pi)} = \sum_{k=1}^{n} A(n,k) t^k
\]
is called an Eulerian polynomial. It is well-known that the Eulerian numbers $A(n,k)$ ($1 \leq k \leq n$) form a unimodal sequence, of which several proofs have been published: such
as the analytical one by showing that the polynomial $A_n(t)$ has only real zeros \cite[p. 294]{3}, by induction based on the recurrence relation of $A(n, k)$ (see \cite[9]{9}), or by combinatorial techniques (see \cite[11]{11}).

Let $I_n$ be the set of all involutions in $\mathfrak{S}_n$ and $J_n$ the set of all fixed-point free involutions in $\mathfrak{S}_n$. Define

$$I_n(t) = \sum_{\pi \in I_n} t^{d(\pi)} = \sum_{k=0}^{n-1} I_{n,k} t^k,$$

$$J_n(t) = \sum_{\pi \in J_n} t^{d(\pi)} = \sum_{k=0}^{n-1} J_{n,k} t^k.$$

The first values of these polynomials are given in Table 1.

| $n$ | $I_n(t)$ | $J_n(t)$ |
|-----|----------|----------|
| 1   | 1        | 0        |
| 2   | $1 + t$  | $t$      |
| 3   | $1 + 2t + t^2$ | 0  |
| 4   | $1 + 4t + 4t^2 + t^3$ | $t + t^2 + t^4$ |
| 5   | $1 + 6t + 12t^2 + 6t^3 + t^4$ | 0  |
| 6   | $1 + 9t + 28t^2 + 28t^3 + 9t^4 + t^5$ | $t + 3t^2 + 7t^3 + 3t^4 + t^5$ |

As one may notice from Table 1 that the coefficients of $I_n(t)$ and $J_n(t)$ are symmetric and unimodal for $1 \leq n \leq 6$. Actually, the symmetries had been conjectured by Dumont and were first proved by Strehl \cite[12]{12}. Recently, Brenti (see \cite{5}) conjectured that the coefficients of the polynomial $I_n(t)$ are log-concave and Dukes \cite{5} has obtained some partial results on the unimodality of the coefficients of $I_n(t)$ and $J_{2n}(t)$. Note that, in contrast to Eulerian polynomials $A_n(t)$, the polynomials $I_n(t)$ and $J_{2n}(t)$ may have non-real zeros.

In this paper we will prove that for $n \geq 1$, the two sequences $I_{n,0}, I_{n,1}, \ldots, I_{n,n-1}$ and $J_{2n,1}, J_{2n,2}, \ldots, J_{2n,2n-1}$ are unimodal. Our starting point is the known generating functions of polynomials $I_n(t)$ and $J_n(t)$:

$$\sum_{n=0}^{\infty} I_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} \frac{t^r}{(1-u)^{r+1}(1-u^2)^{r(r+1)/2}}, \quad (1.1)$$

$$\sum_{n=0}^{\infty} J_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} \frac{t^r}{(1-u^2)^{r(r+1)/2}}, \quad (1.2)$$

which have been obtained by Désarménien and Foata \cite{4} and Gessel and Reutenauer \cite{8} using different methods. We first derive linear recurrence formulas for $I_{n,k}$ and $J_{2n,k}$ in the next section and then prove the unimodality by induction in Section 3. We end this paper with further conjectures beyond the unimodality of the two sequences $I_{n,k}$ and $J_{2n,k}$.
2 Linear recurrence formulas for \( I_{n,k} \) and \( J_{2n,k} \)

Since the recurrence formula for the numbers \( I_{n,k} \) is a little more complicated than \( J_{2n,k} \), we shall first prove it for the latter.

**Theorem 2.1.** For \( n \geq 2 \) and \( k \geq 0 \), the numbers \( J_{2n,k} \) satisfy the following recurrence formula:

\[
2nJ_{2n,k} = [k(k+1) + 2n - 2]J_{2n-2,k} + 2[(k-1)(2n-k-1) + 1]J_{2n-2,k-1} + [(2n-k)(2n-k+1) + 2n-2]J_{2n-2,k-2}.
\]

(2.1)

Here and in what follows \( J_{2n,k} = 0 \) if \( k < 0 \).

**Proof.** Equating the coefficients of \( u^{2n} \) in (1.2), we obtain

\[
\frac{J_{2n}(t)}{(1 - t)^{2n+1}} = \sum_{r=0}^{\infty} \frac{r(r+1)/2 + n - 1}{n} t^n.
\]

(2.2)

Since

\[
\left(\frac{r(r+1)/2 + n - 1}{n}\right) = \frac{r(r-1)/2 + r + n - 1}{n} \left(\frac{r(r+1)/2 + n - 2}{n - 1}\right),
\]

it follows from (2.2) that

\[
\frac{J_{2n}(t)}{(1 - t)^{2n+1}} = \frac{t^2}{2n} \left(\frac{J_{2n-2}(t)}{(1 - t)^{2n-1}}\right)'' + \frac{t}{n} \left(\frac{J_{2n-2}(t)}{(1 - t)^{2n-1}}\right)' + \frac{n-1}{n} \frac{J_{2n-2}(t)}{(1 - t)^{2n-1}},
\]

or

\[
J_{2n}(t) = \frac{t^2(1 - t)^2}{2n} J''_{2n-2}(t) + \left[\frac{(2n-1)t^2(1 - t)}{n} + \frac{t(1 - t)^2}{n}\right] J'_{2n-2}(t)
\]

\[
+ \left[\frac{(2n-1)t^2}{n} + \frac{(n-1)(1 - t)^2}{n}\right] J_{2n-2}(t)
\]

\[
= \frac{t^4 - 2t^3 + t^2}{2n} J''_{2n-2}(t) + \left[\frac{(2 - 2n)t^3}{n} + \frac{(2n - 3)t^2}{n} + \frac{t}{n}\right] J'_{2n-2}(t)
\]

\[
+ \left[\frac{(2n - 2)t^2 + t}{n} + \frac{n-1}{n}\right] J_{2n-2}(t).
\]

(2.3)

Equating the coefficients of \( t^n \) in (2.3) yields

\[
J_{2n,k} = \frac{(k-2)(k-3)}{2n} J_{2n-2,k-2} - \frac{(k-1)(k-2)}{n} J_{2n-2,k-1} + \frac{k(k-1)}{2n} J_{2n-2,k}
\]

\[
+ \frac{(2 - 2n)(k-2)}{n} J_{2n-2,k-2} + \frac{(2n - 3)(k-1)}{n} J_{2n-2,k-1} + \frac{k}{n} J_{2n-2,k}
\]

\[
+ (2n - 2) J_{2n-2,k-2} + \frac{1}{n} J_{2n-2,k-1} + \frac{n-1}{n} J_{2n-2,k}.
\]

After simplification, we obtain (2.1).
Theorem 2.2. For \( n \geq 3 \) and \( k \geq 0 \), the numbers \( I_{n,k} \) satisfy the following recurrence formula:

\[
nI_{n,k} = (k+1)I_{n-1,k} + (n-k)I_{n-1,k-1} + [(k+1)^2 + n - 2]I_{n-2,k} + [2k(n-k-1) - n + 3]I_{n-2,k-1} + [(n-k)^2 + n - 2]I_{n-2,k-2}.
\] (2.4)

Here and in what follows \( I_{n,k} = 0 \) if \( k < 0 \).

Proof. Extracting the coefficients of \( u^{2n} \) in (1.1), we obtain

\[
\frac{I_n(t)}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} t^r \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{r(r+1)/2 + k - 1}{k} \binom{r + n - 2k}{n - 2k}.
\] (2.5)

Let

\[
T(n,k) := \binom{x+k-1}{k} \binom{y-2k}{n-2k},
\]

and

\[
s(n) := \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k).
\]

Applying Zeilberger’s algorithm, the Maple package \texttt{ZeilbergerRecurrence(T,n,k,s,0..n)} gives

\[
(2x + y + n + 1)s(n) + (y + 1)s(n + 1) - (n + 2)s(n + 2) = 0
\]

i.e.,

\[
s(n) = \frac{y + 1}{n} s(n - 1) + \frac{2x + y + n - 1}{n} s(n - 2).
\] (2.6)

When \( x = r(r+1)/2 \) and \( y = r \), we get

\[
s(n) = \frac{r + 1}{n} s(n - 1) + \frac{r(r - 1) + 3r + n - 1}{n} s(n - 2).
\] (2.7)

Now, from (2.6) and (2.7) it follows that

\[
\frac{nI_n(t)}{(1-t)^{n+1}} = t \left( \frac{I_{n-1}(t)}{(1-t)^n} \right)' + \frac{I_{n-1}(t)}{(1-t)^n} + t^2 \left( \frac{I_{n-2}(t)}{(1-t)^{n-1}} \right)'' + 3t \left( \frac{I_{n-2}(t)}{(1-t)^{n-1}} \right)'
\]

\[
+ (n-1) \frac{I_{n-2}(t)}{(1-t)^{n-1}},
\]

or

\[
nI_n(t) = (t - t^2)I'_{n-1}(t) + [1 + (n-1)t]I_{n-1}(t) + t^2(1-t)^2I''_{n-2}(t)
\]

\[
+ t(1-t)[3 + (2n-5)t]I'_{n-2}(t) + (n-1)[1 + t + (n-2)t^2]I_{n-2}(t).
\] (2.8)
Comparing the coefficients of \( t^k \) in both sides of (2.8), we obtain
\[
nI_{n,k} = kI_{n-1,k} - (k-1)I_{n-1,k-1} + I_{n-1,k} + (n-1)I_{n-1,k-1} - 2(k-1)(k-2)I_{n-2,k-1} + (k-2)(k-3)I_{n-2,k-2} + 3kI_{n-2,k} + (2n-8)(k-1)I_{n-2,k-1} - (2n-5)(k-2)I_{n-2,k-2} + (n-1)I_{n-2,k} + (n-1)(n-2)I_{n-2,k-2},
\]
which, after simplification, equals the right-hand side of (2.4). □

Remark. The recurrence formula (2.6) can also be proved by hand as follows. It is easy to see that the generating function of \( s(n) \) is
\[
\sum_{n=0}^{\infty} s(n)u^n = (1 - u^2)^{-x}(1-u)^{-y-1}. \tag{2.9}
\]
Differentiating (2.9) with respect to \( u \) implies that
\[
\sum_{n=0}^{\infty} ns(n)u^{n-1} = \left( \frac{2ux}{1-u^2} + \frac{y+1}{1-u} \right)(1-u^2)^{-x}(1-u)^{-y-1},
\]
consequently,
\[
(1-u^2)\sum_{n=0}^{\infty} ns(n)u^{n-1} = [(2x+y+1)u + y+1](1-u^2)^{-x}(1-u)^{-y-1}
= [(2x+y+1)u + y+1]\sum_{n=0}^{\infty} s(n)u^n. \tag{2.10}
\]
Comparing the coefficients of \( u^{n+1} \) in both sides of (2.10), we obtain
\[
(n+2)s(n+2) - ns(n) = (2x+y+1)s(n) + (y+1)s(n+1),
\]
which is equivalent to (2.6).

Note that the right-hand side of (2.1) (resp. (2.4)) is invariant under the substitution \( k \to 2n - k \) (resp. \( k \to n - 1 - k \)), provided that the sequence \( I_{n-1,k} \) (resp. \( J_{2n-2,k} \)) is symmetric. Thus, by induction we derive immediately the symmetry properties of \( J_{2n,k} \) and \( I_{n,k} \) (see [4,8,12]).

Corollary 2.3. For \( n, k \in \mathbb{N} \), we have
\[
I_{n,k} = I_{n,n-1-k}, \quad J_{2n,k} = J_{2n,2n-k}.
\]

It would be interesting to find a combinatorial proof of the recurrence formulas (2.1) and (2.4), since such a proof could hopefully lead to a combinatorial proof of the unimodality of these two sequences.
3 Unimodality of the sequences $I_{n,k}$ and $J_{2n,k}$

The following observation is crucial in our inductive proof of the unimodality of the sequences $I_{n,k}$ ($0 \leq k \leq n - 1$) and $J_{2n,k}$ ($1 \leq k \leq 2n - 1$).

Lemma 3.1. Let $x_0, x_1, \ldots, x_n$ and $a_0, a_1, \ldots, a_n$ be real numbers such that $x_0 \geq x_1 \geq \cdots \geq x_n \geq 0$ and $a_0 + a_1 + \cdots + a_k \geq 0$ for all $k = 0, 1, \ldots, n$. Then

$$\sum_{i=0}^{n} a_i x_i \geq 0.$$

Indeed, the above inequality follows from the identity:

$$\sum_{i=0}^{n} a_i x_i = \sum_{k=0}^{n} (x_k - x_{k+1})(a_0 + a_1 + \cdots + a_k),$$

where $x_{n+1} = 0$.

Theorem 3.2. The sequence $J_{2n,1}, J_{2n,2}, \ldots, J_{2n,2n-1}$ is unimodal.

Proof. By the symmetry of $J_{2n,k}$, it is enough to show that $J_{2n,k} \geq J_{2n,k-1}$ for all $2 \leq k \leq n$. We proceed by induction on $n$. Clearly, the $n = 2$ case is obvious. Suppose the sequence $J_{2n-2,k}$ is unimodal in $k$. By Theorem 2.1 one has

$$2n(J_{2n,k} - J_{2n,k-1}) = A_0 J_{2n-2,k} + A_1 J_{2n-2,k-1} + A_2 J_{2n-2,k-2} + A_3 J_{2n-2,k-3},$$

(3.1)

where

$$A_0 = k^2 + k + 2n - 2, \quad A_1 = 4nk - 3k^2 - 6n + k + 6,$$

$$A_2 = 3k^2 + 4n^2 - 8nk - 5k + 12n - 4, \quad A_3 = 3k - k^2 + 4nk - 4n^2 - 8n.$$

We have the following two cases:

- If $2 \leq k \leq n - 1$, then

$$J_{2n-2,k} \geq J_{2n-2,k-1} \geq J_{2n-2,k-2} \geq J_{2n-2,k-3}$$

by the induction hypothesis, and clearly

$$A_0 \geq 0, \quad A_0 + A_1 = 2(k-1)(2n-k) + 4 \geq 0,$$

$$A_0 + A_1 + A_2 = (2n-k)^2 - 3k + 8n \geq 0, \quad A_0 + A_1 + A_2 + A_3 = 0.$$

Therefore, by Lemma 3.1 we have

$$J_{2n,k} - J_{2n,k-1} \geq 0.$$
• If \( k = n \), then
\[
J_{2n-2,n-1} \geq J_{2n-2,n} = J_{2n-2,n-2} \geq J_{2n-2,n-3}
\]
by symmetry and the induction hypothesis. In this case, we have \( A_1 = (n-2)(n-3) \geq 0 \) and thus the corresponding condition of Lemma 3.1 is satisfied. Therefore, we have
\[
J_{2n,n} - J_{2n,n-1} \geq 0.
\]
This completes the proof.

**Theorem 3.3.** The sequence \( I_{n,0}, I_{n,1}, \ldots, I_{n,n-1} \) is unimodal.

**Proof.** By the symmetry of \( I_{n,k} \), it suffices to show that \( I_{n,k} \geq I_{n,k-1} \) for all \( 1 \leq k \leq (n-1)/2 \). From Table 1, it is clear that the sequences \( I_{n,k} \) are unimodal in \( k \) for \( 1 \leq n \leq 6 \).

Now suppose \( n \geq 7 \) and the sequences \( I_{n-1,k} \) and \( I_{n-2,k} \) are unimodal in \( k \). Replacing \( k \) by \( k - 1 \) in (2.4), we obtain
\[
nI_{n,k-1} = kI_{n-1,k-1} + (n - k + 1)I_{n-1,k-2} + (k^2 + n - 2)I_{n-2,k-1}
+ 2(k - 1)(n - k) - n + 3[I_{n-2,k-2} + [(n - k + 1)^2 + n - 2]I_{n-2,k-3}.
\]
(3.2)
Combining (2.4) and (3.2) yields
\[
n(I_{n,k} - I_{n,k-1}) = B_0I_{n-1,k} + B_1I_{n-1,k-1} + B_2I_{n-1,k-2}
+ C_0I_{n-2,k} + C_1I_{n-2,k-1} + C_2I_{n-2,k-2} + C_3I_{n-2,k-3},
\]
where
\[
B_0 = k + 1, \quad B_1 = n - 2k, \quad B_2 = -(n - k + 1),
C_0 = (k + 1)^2 + n - 2, \quad C_1 = 2nk - 3k^2 - 2k - 2n + 5,
C_2 = n^2 - 4nk + 3k^2 + 4n - 2k - 5, \quad C_3 = -(n - k + 1)^2 - n + 2.
\]
Notice that \( I_{n-1,k} \geq I_{n-1,k-1} \geq I_{n-1,k-2} \) for \( 1 \leq k \leq (n-1)/2 \). By Lemma 3.1 we have
\[
B_0I_{n-1,k} + B_1I_{n-1,k-1} + B_2I_{n-1,k-2} \geq 0.
\]
(3.4)
It remains to show that
\[
C_0I_{n-2,k} + C_1I_{n-2,k-1} + C_2I_{n-2,k-2} + C_3I_{n-2,k-3} \geq 0, \quad \forall 1 \leq k \leq (n-1)/2.
\]
(3.5)
We need to consider the following two cases:

• If \( 1 \leq k \leq (n-2)/2 \), then
\[
I_{n-2,k} \geq I_{n-2,k-1} \geq I_{n-2,k-2} \geq I_{n-2,k-3}
\]
by the induction hypothesis, and
\[
C_0 = (k + 1)^2 + n - 2 \geq 0, \quad C_0 + C_1 = (2k - 1)(n - k - 1) + k + 3 \geq 0,
C_0 + C_1 + C_2 = (n - k + 1)^2 + n - 2 \geq 0, \quad C_0 + C_1 + C_2 + C_3 = 0.
\]
If \( k = (n - 1)/2 \), then by symmetry and the induction hypothesis,
\[
I_{n-2,k-1} \geq I_{n-2,k} = I_{n-2,k-2} \geq I_{n-2,k-3}.
\]
In this case, we have \( C_1 = (n - 3)(n - 7)/4 \geq 0 \) for \( n \geq 7 \).

Therefore, by Lemma 3.1 the inequality (3.5) holds. It follows from (3.3)–(3.5) that
\[
I_{n,k} - I_{n,k-1} \geq 0, \quad \forall 1 \leq k \leq (n - 1)/2.
\]
This completes the proof. \( \Box \)

4 Further remarks and open problems

Since \( I_{n,k} = I_{n,n-1-k} \), we can rewrite \( I_n(t) \) as follows:
\[
I_n(t) = \sum_{k=0}^{n-1} I_{n,k} t^k = \begin{cases} 
\sum_{k=0}^{n/2-1} I_{n,k} t^k (1 + t^{n-2k-1}), & \text{if } n \text{ is even}, \\
I_{n,(n-1)/2} t^{(n-1)/2} + \sum_{k=0}^{(n-3)/2} I_{n,k} t^k (1 + t^{n-2k-1}), & \text{if } n \text{ is odd}.
\end{cases}
\]

Applying the well-known formula
\[
x^n + y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (xy)^j (x+y)^{n-2j},
\]
we obtain
\[
I_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1 + t)^{n-2k-1}, \tag{4.1}
\]
where
\[
a_{n,k} = \begin{cases} 
\sum_{j=0}^{k} (-1)^{k-j} \frac{n-2j-1}{n-k-j-1} \binom{n-k-j-1}{k-j} I_{n,j}, & \text{if } 2k + 1 < n, \\
I_{n,k} + \sum_{j=0}^{k-1} (-1)^{k-j} \frac{n-2j-1}{n-k-j-1} \binom{n-k-j-1}{k-j} I_{n,j}, & \text{if } 2k + 1 = n.
\end{cases}
\]

The first values of \( a_{n,k} \) are given in Table 2 which seems to suggest the following conjecture.

**Conjecture 4.1.** For \( n \geq 1 \) and \( k \geq 0 \), the coefficients \( a_{n,k} \) are nonnegative integers.
Table 2: Values of $a_{n,k}$ for $n \leq 16$ and $0 \leq k \leq \lfloor (n-1)/2 \rfloor$.

| $k \setminus n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| 0               | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| 1               | 1 | 2 | 4 | 6 | 9 | 12 | 16 | 20 | 25 | 30  | 36  | 42  | 49  |    |    |    |
| 2               | 2 | 6 | 18 |39 |79 |141 |239 |379 |579 |849 |1211 |1680 |    |    |    |    |
| 3               |0 |18 |78 |272 |722 |1716 |3626 |7160 |13206|23263|    |    |    |    |    |    |
| 4               |20 |124 |668 |2560 |8360 |23536 |59824 |139457|    |    |    |    |    |    |    |    |
| 5               |32 |700 |44376|    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 6               |440 |5480 |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 7               |2176 |44376|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |

Since the coefficients of $t^k(1 + t)^{n-2k-1}$ are symmetric and unimodal with center of symmetry at $(n-1)/2$, Conjecture 4.1 is stronger than the fact that the coefficients of $I_n(t)$ are symmetric and unimodal. A more interesting question is to give a combinatorial interpretation of $a_{n,k}$. Note that the Eulerian polynomials can be written as

$$A_n(t) = \sum_{k=1}^{\lfloor(n+1)/2\rfloor} c_{n,k} t^k (1 + t)^{n-2k+1},$$

where $c_{n,k}$ is the number of increasing binary trees on $[n]$ with $k$ leaves and no vertices having left children only (see [11, 16, 17]).

We now proceed to derive a recurrence relation for $a_{n,k}$. Set $x = x(t) = t/(1 + t)^2$ and

$$P_n(x) = \sum_{k=0}^{\lfloor(n-1)/2\rfloor} a_{n,k} x^k.$$

Then we can rewrite (4.1) as

$$I_n(t) = (1 + t)^{n-1} P_n(x). \quad (4.2)$$

Differentiating (4.2) with respect to $t$ we get

$$I'_n(t) = (n - 1)(1 + t)^{n-2} P_n(x) + (1 + t)^{n-1} P'_n(x) x'(t), \quad (4.3)$$

$$I''_n(t) = (n - 1)(n - 2)(1 + t)^{n-3} P_n(x) + 2(n - 1)(1 + t)^{n-2} P'_n(x) x'(t) + (1 + t)^{n-1} P''_n(x) x'(t)^2 + (1 + t)^{n-1} P'_n(x) x''(t), \quad (4.4)$$

$$x'(t) = \frac{1 - t}{(1 + t)^3}, \quad x''(t) = \frac{2t - 4}{(1 + t)^4}. \quad (4.5)$$
Substituting (4.2)–(4.5) into (2.8), we obtain
\[
n(1+t)^{n-1}P_n(x)
\]
\[
= [1 + (2n - 2)t + t^2](1 + t)^{n-3}P_{n-1}(x) + t(1 - t)^2(1 + t)^{n-5}P'_{n-1}(x)
\]
\[
+ [-t^2 + 14t + 1](1 - t)^2 + (1 + 6t - 18t^2 + 6t^3 + t^4)n + 4t^2n^2](1 + t)^{n-5}P_{n-2}(x)
\]
\[
+ [3t(t^2 - 4t + 1)(1 - t)^2 + 4t^2(1 - t)^2n](1 + t)^{n-7}P'_{n-2}(x)
\]
\[
+ t^2(1 - t)^4(1 + t)^{n-9}P''_{n-2}(x).
\]
(4.6)

Dividing the two sides of (4.6) by \((1 + t)^{n-1}\) and noticing that \(t/(1 + t)^2 = x\), after a little manipulation we get
\[
nP_n(x) = [1 + (2n - 4)x]P_{n-1}(x) + (x - 4x^2)P'_{n-1}(x)
\]
\[
+ [(n - 1) + (2n - 8)x + 4(n - 3)(n - 4)x^2]P_{n-2}(x)
\]
\[
+ [3x + (4n - 30)x^2 + (72 - 16n)x^3]P'_{n-2}(x) + (x^2 - 8x^3 + 16x^4)P''_{n-2}(x).
\]

Extracting the coefficients of \(x^k\) yields
\[
na_{n,k} = a_{n-1,k} + (2n - 4)a_{n-1,k-1} + ka_{n-1,k} - 4(k - 1)a_{n-1,k-1}
\]
\[
+ (n - 1)a_{n-2,k} + (2n - 8)a_{n-2,k-1} + 4(n - 3)(n - 4)a_{n-2,k-2}
\]
\[
+ 3ka_{n-2,k} + (4n - 30)(k - 1)a_{n-2,k-1} + (72 - 16n)(k - 2)a_{n-2,k-2}
\]
\[
+ k(k - 1)a_{n-2,k} - 8(k - 1)(k - 2)a_{n-2,k-1} + 16(k - 2)(k - 3)a_{n-2,k-2}.
\]

After simplification, we obtain the following recurrence formula for \(a_{n,k}\).

**Theorem 4.2.** For \(n \geq 3\) and \(k \geq 0\), there holds
\[
na_{n,k} = (k + 1)a_{n-1,k} + (2n - 4k)a_{n-1,k-1} + [k(k + 2) + n - 1]a_{n-2,k}
\]
\[
+ [(k - 1)(4n - 8k - 14) + 2n - 8]a_{n-2,k-1} + 4(n - 2k)(n - 2k + 1)a_{n-2,k-2},
\]
(4.7)

where \(a_{n,k} = 0\) if \(k < 0\) or \(k > (n - 1)/2\).

Note that, if \(n \geq 2k + 3\), then
\[
(k - 1)(4n - 8k - 14) + 2n - 8 > 0
\]
for any \(k \geq 1\),

and so are the other coefficients in (4.7). Therefore, Conjecture 4.1 would be proved if one can show that \(a_{2n+1,n} \geq 0\) and \(a_{2n+2,n} \geq 0\).
Finally, from (4.1) it is easy to see that
\[
a_{2n+1,n} = (-1)^n I_{2n+1}(-1) = \sum_{k=0}^{2n} (-1)^{n-k} I_{2n+1,k},
\]
\[
a_{2n+2,n} = (-1)^n I'_{2n+2}(-1) = \sum_{k=1}^{2n+1} (-1)^{n+1-k} k I_{2n+2,k}.
\]
Thus, Conjecture 4.1 is equivalent to the \textit{nonnegativity} of the above two alternating sums.

Since \(J_{2n,k} = J_{2n,2n-k}\), in the same manner as \(I_{n}(t)\) we obtain
\[
J_{2n}(t) = \sum_{k=1}^{n} b_{2n,k} t^k (1 + t)^{2n-2k},
\]
where
\[
b_{2n,k} = \begin{cases} 
\sum_{j=1}^{k} (-1)^{k-j} \frac{2n-2j}{2n-k-j} \binom{2n-k-j}{k-j} J_{2n,j}, & \text{if } k < n, \\
J_{2n,k} + \sum_{j=1}^{k-1} (-1)^{k-j} \frac{2n-2j}{2n-k-j} \binom{2n-k-j}{k-j} J_{2n,j}, & \text{if } k = n.
\end{cases}
\]

Now, it follows from (2.2) that
\[
J_{2n,k} = \sum_{i=0}^{k} (-1)^{k-i} \binom{2n+1}{k-i} \left( i(i+1)/2 + n - 1 \right)
\]
is a polynomial in \(n\) of degree \(d := k(k+1)/2 - 1\) with leading coefficient \(1/d!\), and so is \(b_{2n,k}\). Thus, we have \(\lim_{n \to +\infty} b_{2n,k} = +\infty\) for any fixed \(k > 1\).

The first values of \(b_{2n,k}\) are given in Table 3, which seems to suggest

\textbf{Conjecture 4.3.} For \(n \geq 9\) and \(k \geq 1\), the coefficients \(b_{2n,k}\) are nonnegative integers.

Similarly to the proof of Theorem 4.2, we can prove the following result.

\textbf{Theorem 4.4.} For \(n \geq 2\) and \(k \geq 1\), there holds
\[
2nb_{2n,k} = [k(k+1) + 2n-2]b_{2n-2,k-1} + [2 + 2(k-1)(4n-4k-3)]b_{2n-2,k-1}
+ 8(n-k+1)(2n-2k+1)b_{2n-2,k-2}.
\]
where \(b_{2n,k} = 0\) if \(k < 1\) or \(k > n\).

Theorem 4.4 allows us to reduce the verification of Conjecture 4.3 to the boundary case \(b_{2n,n} \geq 0\) for \(n \geq 9\).

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Table 3: Values of $b_{2n,k}$ for $2n \leq 24$ and $1 \leq k \leq n$.

| $k \backslash 2n$ | 2   | 4   | 6   | 8   | 10  | 12  | 14  | 16  | 18  | 20  | 22  | 24  |
|------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1                | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| 2                | −1  | −1  | 0   | 2   | 5   | 9   | 14  | 20  | 27  | 35  | 44  |     |
| 3                | 3   | 12  | 36  | 91  | 201 | 409 | 728 | 1242| 2007| 3102|     |
| 4                | −7  | −10 | 91  | 652 | 2593| 7902| 20401| 46852| 98494|     |
| 5                | 25  | 219 | 1719| 10532| 50165| 191439| 639968|     |
| 6                | −65 | 249 | 11299| 422971| 1284008| 4376646| 18747924|     |
| 7                | 283 | 6366| 135545| 1737905| 15219292| 101116704|     |
| 8                | −583| 33188| 1372734| 24412940| 277963127|     |
| 9                | 4417| 300299| 16488999| 367507439|     |
| 10               | 1791| 3320211| 203698690|     |
| 11               |     | 133107| 36903128|     |
| 12               |     |     | 701785|     |

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