ON THE MULTIPLICATIVE DECOMPOSITION OF HETEROGENEITY IN CONTINUOUS ASSEMBLAGES

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A system’s heterogeneity (equivalently, diversity) amounts to the effective size of its event space, and can be quantified using the Rényi family of indices (also known as Hill numbers in ecology or Hannah-Kay indices in economics), which are indexed by an elasticity parameter \( q \geq 0 \). Importantly, under these indices, the heterogeneity of a composite system (the \( \gamma \)-heterogeneity) is decomposable into heterogeneity arising from variation within and between component subsystems (the \( \alpha \) - and \( \beta \)-heterogeneities, respectively). Since the average heterogeneity of a component subsystem should not be greater than that of the pooled assemblage, we require that \( \alpha \leq \gamma \). There exists a multiplicative decomposition for Rényi heterogeneity of composite systems with discrete event spaces, but less attention has been paid to decomposition in the continuous setting. This paper therefore describes multiplicative decomposition of the Rényi heterogeneity for continuous mixture distributions under parametric and non-parametric pooling assumptions. We show that under non-parametric pooling (where \( \gamma \)-heterogeneity must typically be estimated numerically), the multiplicative decomposition holds such that \( \gamma \geq \alpha \) for all values of the elasticity parameter \( q > 0 \), and \( \beta \)-heterogeneity amounts to the discrete number of distinct mixture components in the system. Conversely, under parametric pooling (as in a Gaussian mixed-effects model), which facilitates efficient analytical computation of \( \gamma \)-heterogeneity, we show that the \( \gamma \geq \alpha \) condition holds only at \( q = 1 \). By providing conditions under which the decomposability axiom of heterogeneity measurement holds, our findings further advance the understanding of heterogeneity measurement in non-categorical systems.

1 INTRODUCTION

Measurement of heterogeneity is important across many scientific disciplines. Ecologists are interested in the heterogeneity of ecosystems’ biological composition (biodiversity; [Hooper et al. 2005]), economists are interested in the heterogeneity of resource ownership (wealth equality; [Cowell 2011]), and medical researchers and physicians are interested in the heterogeneity of diseases and their presentations [Nunes et al. 2020]. Using Rényi heterogeneity, also known as the Hill numbers [Hill 1973] or Hannah-Kay indices [Hannah and Kay 1977], one can measure a system’s heterogeneity as its effective number of distinct configurations.

The heterogeneity of an assemblage of systems is known as the \( \gamma \)-heterogeneity, and represents the heterogeneity arising from variation within and between constituent subsystems. It is generally required that the \( \gamma \)-heterogeneity be decomposable into these components. The heterogeneity due to variation within the assemblage’s subsystems is also known as \( \alpha \)-heterogeneity, and that due to variation between subsystems is also known as \( \beta \)-heterogeneity. Furthermore, one generally requires that \( \gamma \geq \alpha \), since it is counterintuitive that the heterogeneity of the overall assemblage should be less than that of its typical constituent subsystem. Jost [2007] introduced a multiplicative decomposition for discrete Rényi heterogeneity that satisfied this requirement. Unfortunately, there is no substantial guidance on decomposition for non-categorical Rényi heterogeneity. Therefore, our present work extends the multiplicative decomposition of Rényi heterogeneity to non-categorical systems, and provides conditions under which the \( \gamma \geq \alpha \) condition is satisfied.

In Section 2, we introduce decomposition of the Rényi heterogeneity in categorical and non-categorical systems. Specifically, we highlight that the most important decision guiding the availability of a decomposition is how one defines the distribution over the assemblage of pooled subsystems. We show that for non-parametrically pooled assemblages (i.e., finite mixture models, illustrated in Section 3), the \( \gamma \geq \alpha \) condition will hold for all values of the Rényi elasticity parameter \( q > 0 \), but that \( \gamma \)-heterogeneity will generally require numerical estimation. Section 4 introduces decomposition of Rényi heterogeneity under parametric assumptions on the pooled assemblage’s distribution. In this case, which amounts to a Gaussian mixed-effects model (as commonly implemented in biomedical meta-analyses), we show that \( \gamma \geq \alpha \) is guaranteed to hold only at \( q = 1 \). Finally, in Section 5, we discuss the implications of our findings and scenarios in which parametric or non-parametric pooling assumptions might be particularly useful.

2 BACKGROUND

2.1 Categorical Rényi Heterogeneity Decomposition

Consider an assemblage \( \mathcal{X} \) with abundance distribution \( \hat{p} \) that is a weighted composition of subsystems \( \mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \ldots, \mathcal{X}^{(K)} \) with corresponding probability distributions \( P = (p_{ij})_{i=1,2,...,K} \) and component weights \( w = (w_i)_{i=1,2,...,K} \). Although weights are generally taken to represent each component’s relative contribution to the abundance distribution in the pooled assemblage, they may be specified by the user to represent other measures of relative component importance. The Rényi heterogeneity for the \( i \)-th distribution is

\[
\Pi_q(p_i) = \left( \sum_{j=1}^{n} p_{ij}^q \right)^{1/q},
\]

The distribution over \( \mathcal{X} \) (the pooled assemblage) is

\[
\hat{p} = P^\top w,
\]

and the pooled assemblage’s heterogeneity—which also known as the total heterogeneity or \( \gamma \)-heterogeneity—is denoted \( \Pi^\gamma_q(\hat{p}) \).
The \( \gamma \)-heterogeneity is a composition of heterogeneity due to average within-group variation (the \( \alpha \)-heterogeneity, denoted \( \Pi_{\alpha} \)), and heterogeneity due to variation between groups (the \( \beta \)-heterogeneity, denoted \( \Pi_{\beta} \)).

Decomposition of the heterogeneity amounts to learning the deterministic function \( \Xi \) such that

\[
\Pi_{\gamma} = \Xi(\Pi_{\alpha}, \Pi_{\beta}).
\]

(3)

Jost [2007] noted that \( \Xi \) should satisfy the following properties:

1. The \( \alpha \) and \( \beta \) components are independent [Wilson and Shmida 1984]
2. The within-group heterogeneity is a lower bound on total heterogeneity: \( \Pi_{\alpha} \leq \Pi_{\gamma} \)
3. The \( \alpha \)-heterogeneity is a form of average heterogeneity over groups
4. The \( \alpha \) and \( \beta \) components are both expressed in numbers equivalent.

The last component, noting that \( \alpha \) and \( \beta \) heterogeneity must both be numbers equivalent, allows comparison of the with- and between-group heterogeneity values. These properties are satisfied by the multiplicative Jost [2007] decomposition

\[
\Pi_{\gamma} = \Pi_{\alpha} \Pi_{\beta}.
\]

(4)

where arguments were dropped for notational parsimony. Jost [2007] defined the \( \alpha \) component as

\[
\Pi_{\alpha} = \left\{ \begin{array}{ll}
\left( \sum_{i=1}^{K} w_i \sum_{j=1}^{n_i} p_{ij} \right)^{1/\alpha} & q \neq 1 \\
\exp\left\{ -\sum_{i=1}^{K} w_i \sum_{j=1}^{n_i} p_{ij} \log p_{ij} \right\} & q = 1
\end{array} \right.
\]

(5)

When \( w_i = w_j = 1/n \) for all \( (i, j) \), then \( \Pi_{\alpha} \geq \Pi_{\gamma} \) is guaranteed for all \( q \). However, one can easily show that Equation 4 is valid for \( w_i \neq w_j \) when \( q = 1 \).

2.2 Continuous Rényi Heterogeneity Decomposition

Consider a system defined on an \( n \)-dimensional continuous space \( X \subseteq \mathbb{R}^n \), which may be partitioned into subsystems \( \{X^{(i)}\}_{i=1,2,...,K} \) upon which there are defined probability densities

\[
\mathbf{f}(x) = \left\{ f_i(x) : \forall x \in X^{(i)}, i \in \{1, 2, \ldots, K\} \right\},
\]

with weights \( w = (w_i)_{i=1,2,\ldots,K} \). The pooled distribution on \( X \) is \( \bar{f}(x) = \sum_{i=1}^{K} w_i f_i(x) \). The continuous Rényi heterogeneity for the \( i \)th subsystem is

\[
\Pi_{\gamma}(f_i) = \left( \int_X f_i^q(x) \, dx \right)^{1/\gamma}.
\]

(6)

The corresponding heterogeneity over the pooled system (the \( \gamma \)-heterogeneity) is

\[
\Pi_{\gamma}(\bar{f}) = \left( \int_X \bar{f}^q(x) \, dx \right)^{1/\gamma},
\]

(7)

and the \( \alpha \)-heterogeneity is

\[
\Pi_{\alpha}(\mathbf{f}, w) = \left( \sum_{i=1}^{K} \frac{w_i^{1/\alpha}}{\sum_{k=1}^{K} w_k} \int_X f_i^q(x) \, dx \right)^{\frac{1}{\alpha}}.
\]

(8)

We now show that in the continuous case, Equation 4 holds such that the \( \alpha \)-heterogeneity is a lower bound on \( \gamma \) at \( q \rightarrow 1 \):

\[
- \sum_{i=1}^{K} w_i \int_X f_i(x) \log f_i(x) \, dx \leq - \int_X \bar{f}(x) \log \bar{f}(x) \, dx
\]

(9)

Letting \( H(x) = - \int g(x) \log g(x) \, dx \), we observe that Jensen’s inequality

\[
\mathbb{E}[H(f)] \leq H(\mathbb{E}[f]),
\]

(10)

demonstrates that \( \alpha \)-heterogeneity is a lower bound on \( \gamma \)-heterogeneity in the continuous case. When \( w_i = w_j \), this can be demonstrated for all \( q > 0 \):

\[
\frac{1}{K} \sum_{i=1}^{K} \int_X f_i^q(x) \, dx \leq \int_X \left( \frac{1}{K} \sum_{i=1}^{K} f_i(x) \right) \, dx.
\]

(11)

If \( f_i^q(x) \, dx \) is analytically tractable for all \( i \in \{1, 2, \ldots, K\} \), then a closed form expression for \( \Pi_{\alpha} \) will be available. If \( f_i^q(x) \, dx \) is also analytically tractable, then so will \( \Pi_{\beta} \). However, this will depend entirely on the functional form of \( \bar{f} \), and will rarely be the case when using real world data. In the majority of cases, \( \int_X f^q(x) \, dx \) will have to be computed numerically.

3 Rényi Heterogeneity Decomposition Under a Nonparametric Pooling Distribution

Components of a mixture distribution can be pooled non-parametrically, where each categorical component is assigned weight in accordance to its support in some data. An example is the Gaussian mixture model, whose likelihood is

\[
\bar{f}(x|\mu, \Sigma, w) = \sum_{i=1}^{K} w_i N(x|\mu_i, \Sigma_i),
\]

(12)

where \( 0 \leq w_i \leq 1 \) and \( \sum_i w_i = 1 \). The Rényi heterogeneity of a single \( n \)-dimensional multivariate Gaussian distribution is

\[
\Pi_{\gamma}(\Sigma) = \begin{cases} (2\pi)^{\frac{n}{2}} \sqrt{|\Sigma|} & q \notin \{0, 1, \infty\} \\ (2\pi)^{\frac{n}{2}} \sqrt{|\Sigma|} & q = 1 \\ (2\pi)^{\frac{n}{2}} \sqrt{|\Sigma|} & q = \infty \\ \text{Undefined} & q = 0 \end{cases}.
\]

(13)

A derivation for Equation 13 is provided in Appendix A. The \( \alpha \)-heterogeneity is the effective size of the domain per mixture component, computed as

\[
\Pi_{\alpha}(\Sigma_{1,K}) = \begin{cases} (2\pi)^{\frac{1}{2}} \left( \sum_{i=1}^{K} \frac{w_i \Sigma_i}{\| \Sigma_i \|^2} \right)^{\frac{1}{\alpha}} & q \notin \{0, 1\} \\ \exp\left\{ \frac{1}{n} \left( \sum_{i=1}^{K} w_i \log |2\pi \Sigma_i| \right) \right\} & q = 1 \\ \text{Undefined} & q = \infty \end{cases}.
\]

(14)
where \( \tilde{w}_i^q = \frac{w_i^q \cdot \tilde{f}_i}{\sum_{i=1}^K w_i^q \cdot \tilde{f}_i} \). The pooled (\( \gamma \)) heterogeneity is the effective size of the domain of the pooled data

\[
\Pi_\gamma^\gamma (\tilde{f}) = \left( \int_X \left( \sum_{i=1}^K w_i \mathcal{N}(x; \mu_i, \Sigma_i) \right)^q \, dx \right)^{\frac{1}{q}} ,
\]

(15)

and must be computed numerically. Consequently, the \( \beta \)-heterogeneity will yield the effective number of categorical components in the mixture distribution. Computation of \( \Pi_\gamma^\beta \) by Equation 4 will yield results whose accuracy will depend on the error of numerical integration. The efficiency of integrating high-dimensional distributions may also be problematic.

Figure 1 demonstrates the multiplicative decomposition of Rényi heterogeneity in a unidimensional Gaussian mixture model, where \( \gamma \)-heterogeneity was computed numerically, across varying separations of \( K \) mixture component locations. Note that the \( \beta \)-heterogeneity in this case represents the effective number of distinct components in the mixture distribution, and is bound between 1 (when all components overlap), and \( K \) (when all components are well separated). Further separating the mixture components beyond the point at which \( \beta \)-heterogeneity reaches \( K \) will yield no further increase in \( \beta \)-heterogeneity.

4 RÉNYI HETEROGENEITY DECOMPOSITION UNDER A PARAMETRIC POOLING DISTRIBUTION

An alternative approach to computing \( \tilde{f} \) is to assume it follows a parametric distribution. For example, one might assume that observations on \( X \) are globally Gaussian,

\[
f(x; \mu_*, \Sigma_*) = \mathcal{N}(x; \mu_*, \Sigma_*),
\]

(16)

where

\[
\mu_* = \mathbb{E}[\mu],
\]

(17)

\[
\Sigma_* = \mu_* \mu_*^\top + \sum_{i=1}^K w_i (\Sigma_i + \mu_i \mu_i^\top)
\]

(18)

are the pooled mean and covariance, respectively. The \( \gamma \)-heterogeneity may then be computed by submitting Equation 18 into Equation 13. Equation 14 continues to hold for the \( \alpha \)-heterogeneity. In this case, \( \beta \)-heterogeneity can be computed in closed-form. Figure 2 demonstrates the analytical decomposition of \( \gamma \)-heterogeneity into \( \alpha \) and \( \beta \) components for a mixture of 5 univariate Gaussians at varying levels of separation. Graphically, we observe that the minimal value of \( \Pi_\gamma^\beta \) is 1, when all distributions overlap perfectly. A more formal analysis shows indeed that \( \alpha \) is a lower bound on \( \gamma \) when \( q = 1 \).

**Theorem 4.1.** The \( \beta \)-heterogeneity at \( q = 1 \)

\[
\Pi_\gamma^\beta = \frac{\Pi_\gamma^\gamma}{\Pi_\gamma^\alpha} \geq 1
\]

(19)

for a system composed of \( K \in \mathbb{N}_{\geq 2} \) Gaussians with \( n \)-dimensional symmetric positive semidefinite covariance matrices \( \Sigma_{1:K} \) and means \( \mu_i = 0 \) \( \forall i \), where the pooled covariance matrix \( \Sigma_* \) is given by Equation 18, the pooled mean is \( \mu_* = 0 \), the \( \gamma \)-heterogeneity at \( q = 1 \) is

\[
\Pi_\gamma^\gamma (\Sigma_*) = (2\pi)^\frac{n}{2} |\Sigma_*|^{\frac{1}{2}},
\]

(20)

the \( \alpha \)-heterogeneity at \( q = 1 \) is

\[
\Pi_\gamma^\alpha (\Sigma_{1:K}, w) = \exp \left\{ \frac{n}{2} + \sum_{i=1}^K w_i \frac{1}{2} \log |2\pi \Sigma_i| \right\},
\]

(21)

and

\[
w = (w_i)_{i=1,2,\ldots,K} \mbox{ s.t. } 0 \leq w_i \leq 1, \sum_{i=1}^K w_i = 1
\]

(22)

are weights for each component Gaussian.

**Proof.** Substituting Equations 20 and 21 into 19 and simplifying yields

\[
\left| \sum_{i=1}^K w_i \Sigma_i \right|^\gamma \geq \prod_{i=1}^K |\Sigma_i|^{-\frac{\gamma}{n}}.
\]

(23)

Applying Minkowski’s determinant inequality to the left hand side

\[
\left| \sum_{i=1}^K w_i \Sigma_i \right|^\frac{\gamma}{n} \geq \sum_{i=1}^K w_i |\Sigma_i|^{-\frac{\gamma}{n}},
\]

(24)

and a property of power means stating that, for \( \xi > 0 \) and weights as per the conditions in Equation 22,

\[
\left( \sum_{i=1}^K w_i x_i^{-\xi} \right)^{\frac{1}{\xi}} \leq \prod_{i=1}^K x_i^{w_i} \leq \left( \sum_{i=1}^K w_i x_i^\xi \right)^{-\frac{1}{\xi}},
\]

(25)

yields

\[
\left| \sum_{i=1}^K w_i \Sigma_i \right|^\frac{\gamma}{n} \geq \sum_{i=1}^K w_i |\Sigma_i|^{-\frac{\gamma}{n}} \geq \prod_{i=1}^K |\Sigma_i|^{-\frac{\gamma}{n}}.
\]

(26)

Figure 3 provides a graphical counterexample showing that \( \Pi_\gamma^\beta \) is not always a lower bound on \( \Pi_\gamma^\gamma \) when \( q \neq 1 \) under Gaussian pooling. Thus, to satisfy the \( \gamma \geq \alpha \) condition, decomposition of Rényi heterogeneity in a Gaussian pooled assemblage must be done with the elasticity set to \( q = 1 \).

5 DISCUSSION

This paper provided approaches for multiplicative decomposition of heterogeneity in continuous assemblages, thereby extending the earlier work on discrete space heterogeneity decomposition presented by Jost [2007]. Two approaches were offered, dependent upon whether the distribution over the pooled assemblage is defined either parametrically or non-parametrically. Our results improve the understanding of heterogeneity measurement in non-categorical systems by providing conditions under which decomposition of heterogeneity into \( \alpha \) and \( \beta \) components conforms to the intuitive property that \( \gamma \geq \alpha \).

If one defines the pooled distribution over an assemblage non-parametrically, as in a finite mixture model, heterogeneity is decomposable such that \( \gamma \geq \alpha \) for all \( q > 0 \), and \( \beta \) may be interpreted
Fig. 1. Demonstration of the multiplicative decomposition of Rényi heterogeneity in Gaussian mixture models, where $\gamma$-heterogeneity is computed using numerical integration. Each row represents a different number of mixture components (from top to bottom: 2, 3, and 4 univariate gaussians with $\sigma = 0.5$, respectively). Each column shows a case in which the component locations are progressively further separated ($\max_i \mu_i - \min_i \mu_i$ distance from left to right: 0, 2, 4, 6). The $\alpha$-heterogeneity in all scenarios was $\approx 2.07$. The headings on each panel show the resulting $\gamma$ and $\beta$-heterogeneity values.

Fig. 2. Analytical decomposition of Rényi heterogeneity (at $q = 1$) for a mixture of 5 univariate Gaussians at varying levels of separation and width. The x-axes show the difference between the maximal and minimal component means. The y-axes show the Rényi heterogeneity. Lines are coloured according to the component widths.

as the discrete number of distinct mixture components in the assemblage (Sections 2.2 & 3). This has the advantage of conforming with the original discrete decomposition by Jost [2007], insofar as probability mass in the pooled assemblage is recorded only where it is observed in the data, and not elsewhere, as would be assumed under a parametric model of the pooled assemblage. Consequently, one achieves a more precise estimate of the size of the pooled assemblage’s base of support. The primary limitation arises from the need to numerically integrate the $\gamma$-heterogeneity, which can become prohibitively expensive in higher dimensions. Future work should investigate the error bounds on numerically integrated $\gamma$.

A more computationally efficient approach for decomposition of continuous Rényi heterogeneity is to assume that the pooled assemblage of subsystems has an overall parametric distribution. A common application for which this assumption is generally made is in mixed-effects meta-analysis [DerSimonian and Laird 1986]. An important departure from the non-parametric pooling approach of finite mixture models is that non-trivial probability mass may now be assigned to regions not covered by any of the constituent component distributions. From another perspective, one may appreciate that the non-parametric approach to pooling is insensitive to the distance between component distributions, and rather only measures the effective volume of event space to which component distributions assign probability. Conversely, assumption of the parametric distribution over the pooled assemblage (in the case of Section 4, a Gaussian) incorporates the distance between the component distributions into the calculation of $\gamma$-heterogeneity. This would be appropriate in scenarios where one assumes that the observed components undersamples the true distribution on the pooled assemblage. For example, in the case of mixed-effects meta-analysis, the available research studies for inclusion may differ significantly in terms of their means, but
one might assume that there is a significant probability of a new study yielding an effect somewhere in between. Specifying a parametric distribution over the pooled assemblage would capture this assumption.

One limitation of the present study is the use of a Gaussian model for the pooled assemblage distribution. This was chosen on account of (A) its prevalence in the scientific literature and (B) analytical tractability. Future work should expand these results to other distributions. Notwithstanding, we have demonstrated the decomposition of \( \gamma \) Rényi heterogeneity into its \( \alpha \) and \( \beta \) components for continuous assemblages. There are (broadly) two approaches, based on whether parametric assumptions are made about the pooled assemblage distribution. Under these assumptions applied to Gaussian mixture distributions, we provided conditions under which the criterion that \( \gamma \geq \alpha \) is satisfied. Future studies should evaluate this method as an alternative approach for the measurement of meta-analytic heterogeneity, and expand these results to other parametric distributions over the pooled assemblage.

### A DERIVATION OF RÉNYI HETEROGENEITY FOR MULTIVARIATE GAUSSIAN

The probability density function (pdf) of a multivariate Gaussian with mean \( \mu = (\mu_1, \ldots, \mu_n) \) and covariance matrix \( \Sigma = (\Sigma_{ij})_{i=1,2,\ldots,n} \)

\[
p(x|\mu, \sigma) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)}
\]  

(27)

Letting

\[
\psi(x) = e^{-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)},
\]

(28)

the Rényi heterogeneity is

\[
\Pi_q(p) = \left(2\pi\right)^{-\frac{nq}{2}} |\Sigma|^{-\frac{q}{2}} \int \psi^q(x) \, dx \left[\frac{1}{q-1}\right]^{1-q/\alpha}.
\]

(29)

Eigendecomposition of the covariance matrix is defined as \( \Sigma = U \Lambda U^{-1} \) such that \( U U^{-1} = I = U U^\top \), and where \( \Lambda \) is an \( n \times n \) matrix with eigenvalues \( \lambda_i \), \( i = 1, 2, \ldots, n \)

\[
\Pi_q(p) = \left(2\pi\right)^{-\frac{nq}{2}} |\Sigma|^{-\frac{q}{2}} \prod_{i=1}^{n} e^{-\frac{1}{2} \lambda_i \gamma_i^2} \, dy_i
\]

\[
= \left(2\pi\right)^{-\frac{q}{2}} |\Sigma|^{-\frac{q}{2}} \left(\prod_{i=1}^{n} \frac{(2\pi)^n}{q^n} \frac{1}{\lambda_i^{\frac{1}{q}}} \right) \right)^{\frac{1}{1-q}}
\]

\[
= \left(2\pi\right)^{-\frac{q}{2}} |\Sigma|^{-\frac{q}{2}} \left(\prod_{i=1}^{n} \frac{(2\pi)^n}{q^n} \right)^{\frac{1}{1-q}}
\]

\[
= \left(2\pi\right)^{-\frac{q}{2}} |\Sigma|^{-\frac{q}{2}} \left(\prod_{i=1}^{n} \frac{1}{\lambda_i^{\frac{1}{q}}} \right)^{\frac{1}{1-q}}
\]

We therefore denote the Rényi heterogeneity of a multivariate Gaussian as \( \Pi_q(\Sigma) \), since the covariance matrix is the only argument. Importantly, Equation 30 holds only at \( q \notin \{0, 1, \infty\} \). We can obtain the Rényi heterogeneity at the limiting points of elasticity \( \{0, 1, \infty\} \) as follows. For \( q = 1 \), we have

\[
\lim_{q \to 1} \log \Pi_q(p) = \lim_{q \to 1} \left(\frac{n}{2(q-1)} \log q + \frac{n}{2} \log(2\pi) + \frac{1}{2} \log |\Sigma| \right)
\]

\[
= \frac{n}{2} + \frac{n}{2} \log(2\pi) + \frac{1}{2} \log |\Sigma|,
\]

(31)

and therefore,

\[
\Pi_1(\Sigma) = (2\pi)^{\frac{1}{2}} \sqrt{|\Sigma|}.
\]

(32)

Using the same process, it is trivial to show that
\[ \Pi_\infty (\Sigma) = (2\pi)^{\frac{n}{2}} \sqrt{|\Sigma|}, \]  
(33)

and that \( \Pi_0(\Sigma) \) is undefined.

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