KIRILLOV-RESHETIKHIN CRYSTALS, ENERGY FUNCTION AND THE COMBINATORIAL R-MATRIX

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Abstract. We study the polytope model for the affine type $A$ Kirillov-Reshetikhin crystals and prove that the action of the affine Kashiwara operators can be described in a remarkable simple way. Moreover, we investigate the combinatorial $R$-matrix on a tensor product of polytopes and characterize the map explicitly on the highest weight elements. We further give a formula for the local energy function and provide an alternative proof for the perfectness. We determine for any dominant highest weight element $\Lambda$ of level $\ell$ the elements $b_A, b^A$ involved in the definition of perfect crystals and give an explicit description of the ground-state path in the tensor product of polytopes.

1. Introduction

Let $\mathfrak{g}$ be an affine Kac–Moody algebra and $U'(\mathfrak{g})$ be the quantized universal enveloping algebra corresponding to the derived algebra $\mathfrak{g}'$, called the quantum affine algebra. For finite-dimensional $U'(\mathfrak{g})$-modules $V$ and $V'$, such that the tensor product $V \otimes V'$ is irreducible and $V, V'$ have crystal bases $B, B'$ there exists a unique map $R$ from $B \otimes B'$ to $B' \otimes B$ commuting with any Kashiwara operators $\tilde{e}_l, \tilde{f}_l$ (see [10]). This map is called the combinatorial $R$-matrix. Moreover, there exists a $\mathbb{Z}$-valued function on $B \otimes B'$ which is defined through a combinatorial rule (see (4.2)) and is called the local energy function. Both functions play an important role in the affine crystal theory. The global energy function is defined (see Definition 4.1) on a tensor product $B_1 \otimes \cdots \otimes B_N$, where $B_j$ is the crystal basis of a finite-dimensional $U'(\mathfrak{g})$-module $V_j$, through the combinatorial $R$-matrix and the local energy function and is an important grading used in the theory of generalized Kostka polynomials (see [23, 24]). The calculation of the combinatorial $R$-matrix or the energy function is done for certain families of crystals in [20, 23, 24].

A certain subclass of finite–dimensional irreducible modules for $U'(\mathfrak{g})$, that gained a lot of attraction during the last decades, are the so-called Kirillov-Reshetikhin modules $W^{i,m}$ where $i$ is a node in the classical Dynkin diagram and $m$ is a positive integer [14]. The modules $W^{i,m}$ have distinguished properties among finite-dimensional modules of quantum affine algebras. One of such properties is that Kirillov–Reshetikhin modules were conjectured to admit a crystal bases $B^{i,m}$ (see [5, Conjecture 2.1]) and this was proven for type $A_n^{(1)}$ in [7] and for all non-exceptional cases in [21]. The combinatorial structure of $B^{i,m}$ was clarified in [2, 16, 24] by exploiting the existence of a map $\sigma$ on $B^{i,m}$ which is the analogue of the Dynkin diagram automorphism on the level of crystals. Another such property is that KR-crystals were conjectured to be perfect [5, 4], which is a technical condition, if and only if $m$ is a multiple of a particular constant $c_i$. Perfect crystals are used to give a path realization of crystal bases of integrable highest weight modules for the quantum algebra $U(\mathfrak{g})$ in terms of semi-infinite tensor product of perfect crystals [7]. The highest weight element in this semi-infinite tensor product is called the ground-state path. The prefectness of these crystals was proven for all non-exceptional types in [3].

D.K. was partially sponsored by the “SFB/TR 12-Symmetries and Universality in Mesoscopic Systems”.
In this paper we investigate the polytope realization of affine type $A$ Kirillov-Reshetikhin crystals developed in [16] and consider the natural question of finding explicit formulas for the combinatorial $R$-matrix and the energy function in terms of this realization.

Let $A \otimes B$ be a highest weight element in $B^{r_1,s_1} \otimes B^{r_2,s_2}$, then our first result yields the image of $A \otimes B$ under the combinatorial $R$-matrix (see Theorem 4.1). It is remarkable that the map $R$ behaves almost like the identity map in the sense that it preserves the entries of $A$ and $B$ and changes only the “shape”.

Our second result deals with the computation of the energy function. For an arbitrary element $A \otimes B$ contained in $B^{r_1,s_1} \otimes B^{r_2,s_2}$ we associate recursively elements $A \otimes B = A_0 \otimes B_0, A_1 \otimes B_1, \ldots, A_k \otimes B_k$, where $k = 0$ if $A \otimes B$ is a highest weight element and otherwise depends on the rank of the Lie algebra and $r_2$ (for a precise definition see (4.4)). Our second result shows that the energy of $A \otimes B$ is given up to a sign by the sum over the entries below the $r_2$-th row of $A_k$ (see Theorem 4.2). Furthermore, we consider the problem of obtaining the explicit elements $b_\Lambda$ and $b^\Lambda$ involved in the definition of perfect crystals and the explicit description of the ground-state path. In the language of polytopes these elements can be easily determined and are described in Theorem 5.2 where moreover an alternative proof for the perfectness is provided.

Another essential part of this paper, which is motivated by a work of Kwon [17], where a combinatorial model for type $A_n^{(1)}$ is given in terms of the RSK correspondence, is to simplify the affine crystal structure of $B^{i,m}$ given in [16] via the promotion operator. Our results give a very explicit and easy way to calculate the classical and affine crystal structure of $B^{i,m}$ even by hand. The simple formulas for the affine operators, the combinatorial $R$-matrix and the energy function let the author expect that similar calculations can be done for types $B_n^{(1)}$ and $C_n^{(1)}$ by using the polytopes from [1]. This will be part of forthcoming work.

The paper is organized as follows. In Section 2 we introduce the main notions and review some general facts about crystals, in particular we recall the realization of affine type $A$ Kirillov-Reshetikhin crystals via polytopes developed in [16]. In Section 3 we recall the notion of Nakajima monomials and prove that the affine crystal structure on the polytope is remarkable simple. In Section 4 we determine the classical highest weight elements in the tensor product of polytopes and calculate the image of the combinatorial $R$-matrix on such elements. We also obtain the value of the energy function for arbitrary elements in $B^{r_1,s_1} \otimes B^{r_2,s_2}$. Finally, in Section 5 we give an alternative proof for the perfectness of $B^{i,m}$ and determine explicitly the unique elements $b_\Lambda, b^\Lambda$ for any level $\ell$ dominant integral weight $\Lambda$. Further we describe the ground-state path of weight $\Lambda$.

Acknowledgements. The author would like to thank the referee of [16] for showing strong interest in the descriptions of the combinatorial $R$-matrix and the energy function in terms of this realization.

2. Notations and review of crystal theory

2.1. Crystal basis and abstract crystals. Crystal theory provides a combinatorial way to study the representation theory of quantum algebras. In this section we review the theory of crystal bases introduced by Kashiwara in [9] and fix the main notation. For an indeterminate element $q$ and an affine Lie algebra $g$ with index set $I$ we denote by $U_q^{+}(g)$ the corresponding quantum algebra without derivation. We denote further by $U_q(g)$ the corresponding quantum algebra with derivation and by $U_q(g_0)$ the quantum algebra of the classical subalgebra $g_0$ (with index set $I_0$) of $g$. A remarkable theorem of Kashiwara implies that every integrable highest weight module $V(\lambda)$ for $U_q(g)$ (resp. $U_q(g_0)$) has a crystal basis and hence a corresponding crystal $B(\lambda)$. For the finite-dimensional modules of quantum algebras of classical type Kashiwara and Nakashima described $B(\lambda)$ in terms of so-called Kashiwara–Nakashima tableaux [12], the analogue of semi–standard tableaux. For
alternative descriptions of $B(\lambda)$ we refer to a series of papers [8, 13, 15, 18]. All these crystal graphs are subject to certain properties, which leads to the definition of abstract crystals. For us, an abstract crystal is a nonempty set $B$ together with maps
\[
\tilde{\epsilon}_l, \tilde{\varphi}_l : B \to B \cup \{0\}, \text{ for } l \in I \text{ (resp. } I_0) \\
\epsilon_l, \varphi_l : B \to \mathbb{Z}, \text{ for } l \in I \text{ (resp. } I_0) \\
wt : B \to P \text{ (resp. } P_0),
\]
which satisfy some conditions. Here $P$ (resp. $P_0$) is the weight lattice associated to $\mathfrak{g}$ (resp $\mathfrak{g}_0$). The maps $\tilde{\epsilon}_l$ and $\tilde{\varphi}_l$ are Kashiwara’s crystal operators and $wt$ is the weight function. For quantum algebras of simply-laced Kac–Moody algebras Stembridge gave a set of local axioms characterizing the set of crystals of representations in the class of all crystals [25]. Moreover, by using Littelmann’s path model Stembridge proved that these axioms hold in all cases, simply-laced or not.

2.2. Tensor product of crystals and regular crystals. Suppose that we have two abstract crystals $B_1$, $B_2$, then we can construct a new crystal which is as a set nothing but $B_1 \times B_2$. This crystal is denoted by $B_1 \otimes B_2$ and the Kashiwara operators are given as follows:
\[
\tilde{f}_l(b_1 \otimes b_2) = \begin{cases} 
(f_l b_1) \otimes b_2, & \text{if } \epsilon_l(b_1) \geq \varphi_l(b_2) \\
b_1 \otimes (f_l b_2), & \text{if } \epsilon_l(b_1) < \varphi_l(b_2).
\end{cases}
\]
\[
\tilde{\epsilon}_l(b_1 \otimes b_2) = \begin{cases} 
(\tilde{\epsilon}_l b_1) \otimes b_2, & \text{if } \epsilon_l(b_1) > \varphi_l(b_2) \\
b_1 \otimes (\tilde{\epsilon}_l b_2), & \text{if } \epsilon_l(b_1) \leq \varphi_l(b_2).
\end{cases}
\]
Further, one can describe explicitly the maps $wt, \varphi_l$ and $\epsilon_l$ on $B_1 \otimes B_2$, namely:
\[
wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)
\]
\[
\varphi_l(b_1 \otimes b_2) = \max\{\varphi_l(b_1), \varphi_l(b_1) + \varphi_l(b_2) - \epsilon_l(b_1)\}
\]
\[
\epsilon_l(b_1 \otimes b_2) = \max\{\epsilon_l(b_2), \epsilon_l(b_1) + \epsilon_l(b_2) - \varphi_l(b_2)\}.
\]
We say that a crystal $B$ is regular if for each subset $J$ with $|J| = 2$ each $J$-component of $B$ is isomorphic to the crystal of an integrable $U_q(\mathfrak{g}_J)$-module, where $\mathfrak{g}_J$ is the Kac–Moody algebra associated to the Cartan matrix $A_J = (a_{ij})_{i,j \in J}$.

2.3. Kirillov-Reshetikhin crystals. The theory of crystal bases can likewise be defined in the setting of $U_q(\mathfrak{g})$ modules, respecting that crystal bases might not always exist. A certain class of finite-dimensional $U_q(\mathfrak{g})$ modules, where the existence of crystal bases is proven for the non-exceptional types [21], are the so-called Kirillov–Reshetikhin modules $W^{i,m}$, where $i$ is a node in the classical Dynkin diagram and $m$ is a positive integer. We recall the realization of Kirillov–Reshetikhin crystals for type $A_n^{(1)}$ from [16]. Recall that the classical positive roots are all of the form
\[
\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \text{ for } 1 \leq i \leq j \leq n.
\]
We denote by $\tilde{B}^{i,m}$ be the set of all pattern

\[
\begin{array}{cccccccc}
a_{1,i} & a_{2,i} & \cdots & a_{i-1,i} & a_{i,i} \\
a_{1,i+1} & a_{2,i+1} & \cdots & a_{i-1,i+1} & a_{i,i+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{1,n} & a_{2,n} & \cdots & a_{i-1,n} & a_{i,n}
\end{array}
\]
filled with non-negative integers, such that $\sum_{s=1}^{n} a_{\beta(s)} \leq m$ for all sequences $(\beta(1), \ldots, \beta(n))$ satisfying the following: $\beta(1) = (1, i), \beta(n) = (i, n)$ and if $\beta(s) = (p, q)$ then the next element in the sequence is either of the form $\beta(s + 1) = (p, q + 1)$ or $\beta(s + 1) = (p + 1, q)$.

Further we define the weight function by
\begin{equation}
\text{wt}(A) = m\omega_i - \sum_{1 \leq p \leq i, 1 \leq q \leq n} a_{p,q} \alpha_{p,q}.
\end{equation}

The maps $\epsilon_l$ and $\varphi_l$ are given as follows
\begin{equation}
\varphi_l(A) = \begin{cases} 
  m - \sum_{j=1}^{i-1} a_{j,i} - \sum_{j=i}^{n} a_{i,j}, & \text{if } l = i \\
  \sum_{j=1}^{p_l(A)} a_{j,l-1} - \sum_{j=1}^{p_{l-1}(A)} a_{j,l}, & \text{if } l > i \\
  \sum_{j=1}^{n} a_{l+1,j} - \sum_{j=p_l(A)+1}^{n} a_{l,j}, & \text{if } l < i
\end{cases}
\end{equation}
\begin{equation}
\epsilon_l(A) = \begin{cases} 
  a_{i,i}, & \text{if } l = i \\
  \sum_{j=q_l(A)}^{i-1} a_{j,i} - \sum_{j=q_{l-1}(A)+1}^{i} a_{j,l-1}, & \text{if } l > i \\
  a_{l,j} - \sum_{j=q_{l-1}(A)+1}^{n} a_{l+1,j}, & \text{if } l < i
\end{cases}
\end{equation}
where
\begin{equation}
p_{l+}(A) = \min \left\{ 1 \leq p \leq i \mid \sum_{j=1}^{p} a_{j,l-1} + \sum_{j=p}^{i} a_{j,l} = \max_{1 \leq q \leq i} \left\{ \sum_{j=1}^{q} a_{j,l-1} + \sum_{j=q}^{i} a_{j,l} \right\} \right\}
\end{equation}
\begin{equation}
q_{l+}(A) = \max \left\{ 1 \leq p \leq i \mid \sum_{j=1}^{p} a_{j,l-1} + \sum_{j=p}^{i} a_{j,l} = \max_{1 \leq q \leq i} \left\{ \sum_{j=1}^{q} a_{j,l-1} + \sum_{j=q}^{i} a_{j,l} \right\} \right\}
\end{equation}
\begin{equation}
p_{l-}(A) = \max \left\{ i \leq p \leq n \mid \sum_{j=i}^{p} a_{t,j} + \sum_{j=p}^{n} a_{t+1,j} = \max_{i \leq q \leq n} \left\{ \sum_{j=i}^{q} a_{t,j} + \sum_{j=q}^{n} a_{t+1,j} \right\} \right\}
\end{equation}
\begin{equation}
q_{l-}(A) = \min \left\{ i \leq p \leq n \mid \sum_{j=i}^{p} a_{t,j} + \sum_{j=p}^{n} a_{t+1,j} = \max_{i \leq q \leq n} \left\{ \sum_{j=i}^{q} a_{t,j} + \sum_{j=q}^{n} a_{t+1,j} \right\} \right\}.
\end{equation}

The Kashiwara operators are defined by
\begin{equation}
\tilde{f}_l A = \begin{cases} 
  \text{replace } a_{i,i} & \text{by } a_{i,i} + 1, & \text{if } l = i \\
  \text{replace } a_{p_{l+}(A),l-1} & \text{by } a_{p_{l+}(A),l-1} - 1 & \text{and } a_{p_{l+}(A),l} & \text{by } a_{p_{l+}(A),l} + 1, & \text{if } l > i \\
  \text{replace } a_{t,p_{l+}(A)} & \text{by } a_{t,p_{l+}(A)} + 1 & \text{and } a_{t+1,p_{l+}(A)} & \text{by } a_{t+1,p_{l+}(A)} - 1, & \text{if } l < i
\end{cases}
\end{equation}
\begin{equation}
\tilde{e}_l A = \begin{cases} 
  \text{replace } a_{i,i} & \text{by } a_{i,i} - 1, & \text{if } l = i \\
  \text{replace } a_{q_{l-}(A),l-1} & \text{by } a_{q_{l-}(A),l-1} + 1 & \text{and } a_{q_{l-}(A),l} & \text{by } a_{q_{l-}(A),l} - 1, & \text{if } l > i \\
  \text{replace } a_{t,q_{l-}(A)} & \text{by } a_{t,q_{l-}(A)} - 1 & \text{and } a_{t+1,q_{l-}(A)} & \text{by } a_{t+1,q_{l-}(A)} + 1, & \text{if } l < i
\end{cases}
\end{equation}

In order to define the affine operators a map $pr : \overline{B}^{i,m} \rightarrow \overline{B}^{i,m}$ is defined algorithmically (see \cite[Section 5.1]{[16]}), which is the analogue of the cyclic Dynkin diagram automorphism $i \mapsto i + 1 \mod (n + 1)$ on the level of crystals. The following theorem gives a realization of the Kirillov–Reshetikhin crystals via polytopes.
**Theorem 2.1.** [16] The polytope $\tilde{B}^{i,m}$ with classical crystal structure (2.4a) and (2.4b) and affine crystal structure
\[ f_0 := pr^{-1} \circ f_1 \circ pr, \] and $\bar{e}_0 := pr^{-1} \circ \bar{e}_1 \circ pr$
is isomorphic to the Kirillov–Reshetikhin crystal $B^{i,m}$.

3. **Simplified affine crystal structure on $B^{i,m}$**

In this section we define a remarkable simple affine structure on $B^{i,m}$ and keep the explicit classical crystal structure from (2.4a) and (2.4b). We prove that $B^{i,m}$ is a regular abstract crystal with respect to the “new” Kashiwara operators.

**Definition 3.1.** Let
\[ \varphi_0(A) = a_{1,n} \text{ and } \epsilon_0(A) = m - \sum_{j=1}^{n} a_{1,j} - \sum_{j=2}^{n} a_{j,n}. \]

Define $f_0A = 0$ (resp. $e_0A = 0$) if $\varphi_0(A) = 0$ (resp. $\epsilon_0(A) = 0$) and otherwise let
\[ f_0A = \text{ replace } a_{1,n} \text{ by } a_{1,n} - 1 \]
\[ e_0A = \text{ replace } a_{1,n} \text{ by } a_{1,n} + 1 \]

The maps $f_0$ and $e_0$ are obviously mutually inverse and we have
\[ \varphi_0(A) - \epsilon_0(A) = a_{1,n} - m + \sum_{j=1}^{n} a_{1,j} + \sum_{j=2}^{n} a_{j,n} = \langle m\omega_i, \alpha_i^\vee \rangle + \sum_{j=1}^{n} a_{1,j} + \sum_{j=1}^{n} a_{j,n} = \langle \text{wt}(A), \alpha_i^\vee \rangle. \]

**Lemma 3.1.** The polytope $B^{i,m}$ together with the maps from (2.1), (2.2a), (2.2b), (2.4a), (2.4b) and Definition 3.1 is an abstract crystal.

**Proof.** The property $\varphi_l(A) - \epsilon_l(A) = \langle \text{wt}(A), \alpha_i^\vee \rangle$ for $l = 0, \ldots, n$ follows from (3.1) and [16, Theorem 3.8]. The rest is straightforward. \hfill \Box

3.1. **Nakajima monomials and regularity of $B^{i,m}$**

In order to prove that $B^{i,m}$ is a regular crystal we shall recall the notion of Nakajima monomials. The set of Nakajima monomials can be understood as a translation of the geometrical realization of crystals provided by Nakajima [19]. Nakajima has shown that there exists a crystal structure on the set of irreducible components of a lagrangian subvariety of the quiver variety. For $i \in I$ and $n \in \mathbb{Z}$ we consider monomials in the variables $Y_i(n)$, i.e. we obtain the set of Nakajima monomials $\mathcal{M}$ as follows:
\[ \mathcal{M} := \left\{ \prod_{i \in I, n \in \mathbb{Z}} Y_i(n)^{y_{i}(n)} \mid y_{i}(n) \in \mathbb{Z} \text{ vanish except for finitely many } (i, n) \right\} \]

With the goal to define the crystal structure on $\mathcal{M}$, we take some integers $c = (c_{i,j})_{i \neq j}$ such that $c_{i,i} + c_{j,i} = 1$. Let now $M = \prod_{i \in I, n \in \mathbb{Z}} Y_i(n)^{y_{i}(n)}$ be an arbitrary monomial in $\mathcal{M}$ and $l \in I$, then we set:
\[ \text{wt}(M) = \sum_{i} \left( \sum_{n} y_{i}(n) \right) \omega_i \]
\[ \varphi_l(M) = \max \left\{ \sum_{k \leq n} y_{k}(n) \mid n \in \mathbb{Z} \right\}, \quad \epsilon_l(M) = \max \left\{ -\sum_{k > n} y_{k}(n) \mid n \in \mathbb{Z} \right\} \]
and
\[ n'_f = \min \left\{ n \mid \varphi_l(M) = \sum_{k \leq n} y_{k}(n) \right\}, \quad n'_e = \max \left\{ n \mid \epsilon_l(M) = -\sum_{k > n} y_{k}(n) \right\}. \]
The Kashiwara operators are defined as follows:

\[
\tilde{f}_l M = \begin{cases} 
A_l(n_l) - 1 M, & \text{if } \varphi_l(M) > 0 \\
0, & \text{if } \varphi_l(M) = 0
\end{cases}
\]

\[
\tilde{e}_l M = \begin{cases} 
A_l(n_l) M, & \text{if } \epsilon_l(M) > 0 \\
0, & \text{if } \epsilon_l(M) = 0,
\end{cases}
\]

whereby

\[ A_l(n) := Y_l(n)Y_l(n + 1) \prod_{i \neq l} Y_i(n + c_{i,l})^{(\varphi_i^+, \alpha_i)}. \]

**Remark 3.1.** A priori the crystal structure depends on \( c \), hence we will denote this crystal by \( \mathcal{M}_c \). But it is easy to see that the isomorphism class of \( \mathcal{M}_c \) does not depend on this choice. In the literature \( c \) is often chosen as \( c_{ij} = \chi(i \leq j) \) or \( c_{ij} = \chi(i \geq j) \) with \( \chi \) the indicator function.

The following result is due to Kashiwara [11].

**Proposition 3.1.** Let \( M \) be a monomial in \( \mathcal{M} \), such that \( \tilde{e}_l M = 0 \) for all \( l \in I \). Then the connected component of \( \mathcal{M} \) containing \( M \) is isomorphic to \( B(\text{wt}(M)) \).

The previous proposition will provide a method to prove a simple affine structure on \( B^{i,m} \) (cf. Theorem 3.1 and Corollary 3.1). Another method to show this is to verify that the classical crystal structure of [16] and [17] coincide, where the classical crystal structure in [17] is obtained from the tensor product of \( B(\omega_l)/s \) and is therefore not very explicit. We decided to use the previous proposition, since the classical crystal structures seem quite challenging to compare.

**Theorem 3.1.** The polytope \( B^{i,m} \) is a regular crystal.

**Proof.** Let \( J = \{ k, l \} \) be a subset of \( \{ 0, \ldots, n \} \). For \( k, l \neq 0 \) it is shown in [16, Theorem 4.5.] that each \( J \)-component of \( B^{i,m} \) is isomorphic to the crystal of an integrable \( U_q(\mathfrak{g}_J) \)-module. Hence it remains to prove the statement for \( J_1 = \{ 0, 1 \} \) and \( J_n = \{ 0, n \} \), since the Kashiwara operators \( f_0 \) and \( \tilde{f}_l \) (resp. \( e_0 \) and \( \tilde{e}_l \)) commute for all \( l \neq 1, n \). The proof for \( J_1 = \{ 0, 1 \} \) and \( J_n = \{ 0, n \} \) proceed very similar and therefore we present only the evidence for \( J_1 = \{ 0, 1 \} \). Let \( A \in B^{i,m} \) and cancel all arrows in \( B^{i,m} \) with color \( s \neq 0, 1 \) and denote the remaining connected graph containing \( A \) by \( Z_1(A) \). We define a map : \( Z_1(A) \cup \{ 0 \} \rightarrow \mathcal{M} \cup \{ 0 \} \) which maps \( 0 \) to \( 0 \) and an arbitrary element \( B = (b_{p,q}) \in Z_1(A) \) to

\[
Y_1(1) \sum_{n=1}^{\infty} b_{i,n}^{-m} \prod_{k=0}^{n-i} Y_1(k)^{b_{i,n-k}} \prod_{k=0}^{n-i} Y_2(k)^{b_{j,n-k}} Y_2(k+1)^{-b_{1,n-k}}.
\]

By Proposition 3.1 it is enough to prove that the above map \( \Psi \) is a strict crystal morphism. We claim

\[ n^2_f = n - p^1(B), \quad n^2_e = n - q^1(B). \]

The assumption \( p^1(B) > n - n^2_f \) implies

\[ b_{1,n^2_f+1} + \cdots + b_{1,p^1(B)} \geq b_{2,n^2_f} + \cdots + b_{2,p^1(B)-1} \]

and thus

\[ \sum_{j=n^2_f+1}^{n} b_{1,j} - \sum_{j=n^2_f}^{n} b_{2,j} \geq \sum_{j=p^1(B)+1}^{n} b_{1,j} - \sum_{j=p^1(B)}^{n} b_{2,j}. \]

By applying the above inequality by \((-1)\) we arrive at a contradiction to the minimality of \( n^2_f \).
The assumption \( p^\perp_j(B) < n - n_j^2 \) implies
\[
b_{1,p^\perp_j(B)+1} + \cdots + b_{1,n^2_j} < b_{2,p^\perp_j(B)} + \cdots + b_{2,n^2_j-1}
\]
and thus
\[
\sum_{j=p^\perp_j(B)+1}^n b_{1,j} - \sum_{j=p^\perp_j(B)}^n b_{2,j} < \sum_{j=n_j^2+1}^n b_{1,j} - \sum_{j=n_j^2}^n b_{2,j}.
\]

Again by applying the above inequality by \((-1)\) we arrive at a contradiction to the definition of \( \varphi_2(\Psi(B)) \). Hence \( n_j^2 = n - p^\perp_j(B) \). A similar calculation as above shows \( n_e^2 = n - q^\perp_e(B) \). Consequently we obtain with (3.2)
\[
\varphi_1(B) = \sum_{j=p^\perp_j(B)}^n b_{2,j} - \sum_{j=p^\perp_j(B)+1}^n b_{1,j} = \varphi_2(\Psi(B)),
\]
\[
e_1(B) = \sum_{j=1}^n b_{1,j} - \sum_{j=1}^n b_{2,j} = e_2(\Psi(B)).
\]

and if \( \tilde{f}_2(\Psi(B)) \neq 0, \tilde{e}_2(\Psi(B)) \neq 0 \)
\[
\tilde{f}_2(\Psi(B)) = Y_1(1)^{\sum_{j=1}^n b_{j,n-m}} \prod_{k=0}^{n-i} Y_1(k)^{b_{1,n-k}} \prod_{k=0}^{n-i} Y_2(k)^{b_{2,n-k}} Y_2(k+1)^{-b_{1,n-k}}
\]
\[
\times Y_2(n - p^\perp_j(B))^{-1} Y_2(n - p^\perp_j(B) + 1)^{-1} Y_1(n - p^\perp_j(B)) = \Psi(\tilde{f}_1 B).
\]
\[
\tilde{e}_2(\Psi(B)) = Y_1(1)^{\sum_{j=1}^n b_{j,n-m}} \prod_{k=0}^{n-i} Y_1(k)^{b_{1,n-k}} \prod_{k=0}^{n-i} Y_2(k)^{b_{2,n-k}} Y_2(k+1)^{-b_{1,n-k}}
\]
\[
\times Y_2(n - q^\perp_e(B)) Y_2(n - q^\perp_e(B) + 1) Y_1(n - q^\perp_e(B))^{-1} = \Psi(\tilde{e}_1 B).
\]
For \( l = 1 \) we get
\[
\varphi_1(\Psi(B)) = \max \left\{ b_{1,n}, b_{1,n} + \sum_{j=1}^n b_{j,n-m}, b_{1,n} + b_{1,n-1} + \sum_{j=1}^n b_{j,n-m}, \ldots \right\} = b_{1,n}, \quad n_j^1 = 0
\]

and
\[
e_1(\Psi(B)) = \max \left\{ -b_{1,i}, -b_{1,i} - b_{1,i+1}, \ldots, -\sum_{j=i}^{n-2} b_{1,j}, m - \sum_{j=i}^{n-1} b_{1,j} - \sum_{j=1}^n b_{j,n} \right\} = e_0(B), \quad n_e^1 = 0.
\]

Therefore, when \( \tilde{f}_1 \Psi(B) \neq 0, \tilde{e}_1 \Psi(B) \neq 0 \) we obtain
\[
\tilde{f}_1 \Psi(B) = Y_1(1)^{\sum_{j=1}^n b_{j,n-m}} \prod_{k=0}^{n-i} Y_1(k)^{b_{1,n-k}} \prod_{k=0}^{n-i} Y_2(k)^{b_{2,n-k}} Y_2(k+1)^{-b_{1,n-k}}
\]
\[
\times Y_1(0)^{-1} Y_1(1)^{-1} Y_2(1) = \Psi(f_0 B)
\]
\[
\tilde{e}_1 \Psi(B) = Y_1(1)^{\sum_{j=1}^n b_{j,n-m}} \prod_{k=0}^{n-i} Y_1(k)^{b_{1,n-k}} \prod_{k=0}^{n-i} Y_2(k)^{b_{2,n-k}} Y_2(k+1)^{-b_{1,n-k}}
\]
\[
\times Y_1(0) Y_1(1) Y_2(1)^{-1} = \Psi(e_0 B),
\]
which finishes the proof.

As a corollary we obtain a remarkable simple affine crystal structure and an explicit classical crystal structure on the Kirillov-Reshetikhin crystal $B^{i,m}$.

**Corollary 3.1.** We have

$$f_0 = pr^{-1} \circ f_0 \circ pr = f_0, \quad \tilde{e}_0 = pr^{-1} \circ \tilde{e}_0 \circ pr = e_0.$$  

**Proof.** The proof follows from Theorem 3.1, [16, Theorem 4.5] and [22, Lemma 2.6].

4. **Combinatorial R-matrix and the energy function**

4.1. **Combinatorial R-matrix.** Let $B_1$ and $B_2$ be two affine crystals with generators $v_1$ and $v_2$ such that the tensor product $B_1 \otimes B_2$ is connected and $v_1 \otimes v_2$ lies in a one-dimensional weight space. The combinatorial R-matrix (see [6, Section 4]) is the unique affine crystal isomorphism

$$\sigma : B_1 \otimes B_2 \xrightarrow{\sim} B_2 \otimes B_1.$$  

We consider two Kirillov-Reshetikhin crystals $B^{r_1,s_1}, B^{r_2,s_2}$ from Section 2 with the simplified crystal structure proven in Section 3. By weight consideration we must have $\sigma(v_1 \otimes v_2) = v_2 \otimes v_1$, where the generator $v_j \in B^{r_j,s_j}$ is the unique element with zero entries. In the following we determine the combinatorial $R$-matrix on the classical highest weight vectors of $B^{r_1,s_1} \otimes B^{r_2,s_2}$. The highest weight elements are described in the following lemma. Let $s = \min\{s_1, s_2\}$, $r = \min\{r_1, r_2\}$, $\tilde{r} = \max\{r_1, r_2\}$ and $\tilde{k} = \min\{r - 1, n - \tilde{r}\}$.

**Lemma 4.1.** The set of highest weight elements is in bijection to

$$\left\{ (a_0, a_1, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^{k+1} \ | \ 0 \leq a_k \leq a_{k-1} \leq \cdots \leq a_0 \leq s \right\}.$$  

To be more precise, if $A \otimes B \in B^{r_1,s_1} \otimes B^{r_2,s_2}$ is a highest weight element then we have

$$B = 0 \text{ and } a_{r,s} = 0 \text{ for all } (r, s) \notin \{(r, \tilde{r}), (r - 1, \tilde{r} + 1), \ldots, (r - k, \tilde{r} + k)\}$$  

and

$$0 \leq a_{r-k, \tilde{r}+k} \leq \cdots \leq a_{r, \tilde{r}} \leq s.$$  

**Proof.** We prove the lemma for $r_1 \leq r_2$, since the case $r_1 \leq r_2$ proceeds similarly. The tensor product property from Section 2 and $\tilde{e}_l(A \otimes B) = 0$ implies

$$\tilde{e}_l B = 0 \text{ for all } l = 1, \ldots, n \text{ and } \tilde{e}_l A = 0 \text{ for all } l \neq r_2 \text{ and } \epsilon_{r_2}(A) \leq s_2.$$  

Hence $B = 0$. We claim the following:

**Claim.** For $A \in B^{r_1,s_1}$ with $\epsilon_j(A) = 0$ for all $j \neq r_2$ we have

$$a_{r,s} = 0 \text{ for all } (r, s) \notin \{(r_1, r_2), (r_1 - 1, r_2 + 1), \ldots, (r_1 - k, r_2 + k)\}.$$  

We proof the claim by upward induction on $n$, where the initial step ($n = 1$) is obviously true. The property $\epsilon_j(A) = 0$ for all $j \neq r_2$ requires

$$a_1, r_1 = \cdots = a_{r_1, r_1} = a_{r_1, r_1+1} = \cdots = a_{r_1, r_2} = \cdots = a_{r_1, n} = 0.$$  

Moreover, it is easy to see that $a_{r,s} = 0$ for all $r_1 \leq s \leq r_2$ and $1 \leq r \leq r_1 - 1$. Hence the $r_1 = 1$ case is done, so let $r_1 > 1$. We denote by $\tilde{A}$ the element obtained from $A$ by removing the last column. We consider $\tilde{A}$ as an element in $B^{r_1-1,s_1}$ for $A^{(1)}_{n-1}$. We shall prove

$$\epsilon_j(\tilde{A}) = 0 \text{ for all } j \neq r_2 + 1,$$

(4.1)
where $\epsilon_{r_2}(\tilde{A}) = 0$ follows from the above observation. Assume $\epsilon_j(\tilde{A}) > 0$ for some \( j \neq r_2, r_2 + 1 \) and \( j \geq r_1 \). Since \( j \neq r_2 + 1 \) we have $a_{r_1,j-1} = 0$ and thus

$$0 < \epsilon_j(\tilde{A}) \leq \epsilon_j(\tilde{A}) + a_{r_1,j} = \epsilon_j(A),$$

which is a contradiction to $\epsilon_j(A) = 0$. Now assume $\epsilon_j(\tilde{A}) > 0$ for some \( j \) with \( j < r_1 \). Then

$$0 < \epsilon_j(\tilde{A}) = \epsilon_j(\tilde{A}) + a_{j,r_1} - a_{j+1,r_1} = \epsilon_j(A) = 0,$$

which is once more a contradiction and therefore we get (4.1). By induction we obtain that $a_{r,s} = 0$ for all $\{(r_1 - 1, r_2 + 1), \ldots, (r_1 - k, r_2 + k)\}$, which finishes the proof of the claim.

Since $A \in B^{r_1,s_1}$ and $\epsilon_{r_2}(A) = a_{r_1,r_2} \leq s_2$ we get $a_{r_1,r_2} \leq s$. The property $a_{r_1-s,r_2+s} < a_{r_1-s-1,r_2+s+1}$ would imply $\epsilon_{r_2+s+1}(A) > 0$ and thus $0 \leq a_{r_1-k,r_2+k} \leq \cdots \leq a_{r_1,r_2} \leq s$. Now it remains to prove that all elements with the property from Lemma 4.1 are highest weight elements, which is straightforward. □

The next theorem describes the image of $\sigma$ restricted to the set of highest weight vectors in $B^{r_1,s_1} \otimes B^{r_2,s_2}$. The isomorphism $\sigma$ on arbitrary elements is therefore reduced to a classical problem.

**Theorem 4.1.** Let $A \otimes B$ be a classical highest weight element in $B^{r_1,s_1} \otimes B^{r_2,s_2}$. Then $\sigma(A \otimes B) = \tilde{A} \otimes B$ is the unique highest weight element in $B^{r_2,s_2} \otimes B^{r_1,s_1}$ with

$$\tilde{B} = 0,$$

and $\tilde{a}_{r-j,\tilde{r}+j} = a_{r-j,\tilde{r}+j}$ for all $j = 0, 1, \ldots, k$.

**Proof.** Again we provide the proof only for $r_1 \leq r_2$. The combinatorial R-matrix is a crystal isomorphism and thus commutes with all operators $\tilde{e}_l, \tilde{f}_l$. This implies that $\sigma(A \otimes B)$ is again a classical highest weight element. By Lemma 4.1 we have

$$\tilde{a}_{r,s} = 0,$$

and $0 \leq \tilde{a}_{r_1-k,r_2+k} \leq \cdots \leq \tilde{a}_{r_1,r_2} \leq s$. We prove by downward induction on $j$ that $\tilde{a}_{r_1-j,r_2+j} = a_{r_1-j,r_2+j}$ for all $j = k, k-1, \ldots, 0$. We start with $j = k$ and suppose $k = n - r_2 \leq r_1 - 1$. Since $r_1 + r_2 - n - 1 \neq r_1, r_2$ we have

$$\varphi_{r_1+r_2-n-1}(A \otimes 0) = \max \{a_{r_1+r_2-n,n}, a_{r_1+r_2-n,n} + \varphi_{r_1+r_2-n-1}(0) - \epsilon_{r_1+r_2-n-1}(A)\}$$

$$= \max \{a_{r_1+r_2-n,n}, a_{r_1+r_2-n,n} + 0 - 0\} = a_{r_1+r_2-n,n}$$

$$= \varphi_{r_1+r_2-n-1}(\tilde{A} \otimes 0)$$

$$= \max \{\tilde{a}_{r_1+r_2-n,n}, \tilde{a}_{r_1+r_2-n,n} + \varphi_{r_1+r_2-n-1}(0) - \epsilon_{r_1+r_2-n-1}(\tilde{A})\} = \tilde{a}_{r_1+r_2-n,n}.$$ 

For $k = r_1 - 1 < n - r_2$ we get similarly as above with $r_1 + r_2 \neq r_1, r_2$

$$\varphi_{r_1+r_2}(A \otimes 0) = \max \{a_{1,r_1+r_2-1}, a_{1,r_1+r_2-1} + \varphi_{r_1+r_2}(0) - \epsilon_{r_1+r_2}(A)\}$$

$$= \max \{a_{1,r_1+r_2-1}, a_{1,r_1+r_2-1} + 0 - 0\} = a_{1,r_1+r_2-1}$$

$$= \varphi_{r_1+r_2}(\tilde{A} \otimes 0)$$

$$= \max \{\tilde{a}_{1,r_1+r_2-1}, \tilde{a}_{1,r_1+r_2-1} + \varphi_{r_1+r_2}(0) - \epsilon_{r_1+r_2}(\tilde{A})\} = \tilde{a}_{1,r_1+r_2-1}. $$
which finishes the initial step. Now let \( j < k \). Since \( r_2 + j + 1 \neq r_1, r_2 \) (otherwise \( j + 1 = 0 \) or \( j + 1 = r_1 - r_2 \leq 0 \)) we obtain by using the induction hypothesis
\[
\varphi_{r_2+j+1}(A \otimes 0) = \max \{ a_{r_1-j,r_2+j} - a_{r_1-j-1,r_2+j+1}, \varphi_{r_2+j+1}(A) + \varphi_{r_2+j+1}(A) + \varphi_{r_2+j+1}(A) \} \\
= \max \{ a_{r_1-j,r_2+j} - a_{r_1-j-1,r_2+j+1}, a_{r_1-j,r_2+j} - a_{r_1-j-1,r_2+j+1} + 0 - 0 \} \\
= a_{r_1-j,r_2+j} - a_{r_1-j-1,r_2+j+1} \\
= \varphi_{r_2+j+1}(A \otimes 0) \\
= \max \{ \widetilde{a}_{r_1-j,r_2+j} - \widetilde{a}_{r_1-j-1,r_2+j+1}, \varphi_{r_2+j+1}(A) + \varphi_{r_2+j+1}(A) - \epsilon_{r_2+j+1}(A) \} \\
= \widetilde{a}_{r_1-j,r_2+j} - \widetilde{a}_{r_1-j-1,r_2+j+1} = \widetilde{a}_{r_1-j,r_2+j} - a_{r_1-j-1,r_2+j+1},
\]
which finishes the proof of the theorem.

We have seen that the map \( \sigma \) behaves on the classical highest weight elements almost like the identity map. To find a formula for \( \sigma \) on arbitrary elements seems to be quite challenging, which is illustrated in the following example.

**Example 4.1.** We consider the Lie algebra \( A_1^{(1)} \) and the tensor product of Kirillov-Reshetikhin crystals \( B^{1,s_1} \otimes B^{1,s_2} \) with \( s_1 \leq s_2 \). For \( A \otimes B \in B^{1,s_1} \otimes B^{1,s_2} \) we denote by \( \sigma(A) \otimes \sigma(B) \) the image of \( A \otimes B \) in \( B^{1,s_2} \otimes B^{1,s_1} \). The study of the maps \( \varphi_1 \) and a simple weight consideration determines \( \sigma(A), \sigma(B) \). We shall consider three cases:

- Let \( A + B \leq s_1 \), then \( \sigma(A) = A, \sigma(B) = B \).
- Let \( s_1 < A + B \leq s_2 \), then \( \sigma(A) = 2A - s_1 + B, \sigma(B) = s_1 - A \).
- Let \( s_2 < A + B \), then \( \sigma(A) = A + s_2 - s_1, \sigma(B) = s_1 - s_2 + B \).

Below are the crystal graphs of \( B^{1,1} \otimes B^{1,3} \) and \( B^{1,3} \otimes B^{1,1} \) respectively:

![Crystal Graphs](image-url)

The study of the map \( \sigma \) on arbitrary elements for higher rank Lie algebras shows that we need considerably more case differentiations.
Example 4.2. Consider the Lie algebra $A_l^{(1)}$ and the affine crystal $B^{4,s_1} \otimes B^{5,s_2}$. Then

\[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & a \\
0 & b & 0 & 0 \\
0 & c & 0 & 0
\end{array}\otimes\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & a \\
0 & b & 0 & 0 \\
0 & c & 0 & 0
\end{array} \rightarrow \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\]  

4.2. Energy function. We recall the definition of the energy function from [6, 22]. There exists a function called the local energy function $H = H_{B_1,B_2} : B_1 \otimes B_2 \rightarrow \mathbb{Z}$ unique up to global additive constant, such that

\[H(e_l(b_1 \otimes b_2)) = H(b_1 \otimes b_2) + \begin{cases} -1 & \text{if } l = 0 \text{ and LL} \\ 1 & \text{if } l = 0 \text{ and RR} \\ 0 & \text{otherwise} \end{cases}\]

where LL (resp. RR) means that $\tilde{e}_0$ acts on both $b_1 \otimes b_2$ and $\sigma(b_1 \otimes b_2)$ on the first (resp. second) tensor factor. We consider the tensor product of KR-crystals and normalize $H$ by requiring $H(0 \otimes 0) = 0$.

Definition 4.1. For $B = B^{r_1,s_2} \otimes \cdots \otimes B^{r_N,s_N}$ and $1 \leq i < j \leq N$ set

\[H_{j,i} = H_i \sigma_{i+1} \cdots \sigma_{j-1},\]

where $\sigma_i$ and $H_i$ act on the $i$-th and $(i + 1)$-th tensor factor. The energy function is defined as

\[D_B = \sum_{1 \leq i < j \leq N} H_{i,j}.\]

Remark 4.1. In the definition of the energy function for tensor products of KR-crystals of arbitrary type the so-called $D$-function $D : B^{r_i,s_i} \rightarrow \mathbb{Z}$ is involved, which is constant on all classical components. Since the KR-crystals for type $A_l^{(1)}$ are classically irreducible $D$ is the constant function $0$.

The aim of the rest of this section is to give an explicit formula for the local energy function. Together with this formula and Theorem 4.1 we obtain a formula for the energy function $D_B$.

Proposition 4.1. Let $A \otimes B$ be a highest weight element in $B^{r_1,s_1} \otimes B^{r_2,s_2}$. Then

\[H(A \otimes B) = -\sum_{1 \leq p \leq r_1 \atop r_1 \leq q \leq n} a_{p,q}.\]

In particular, the local energy function is the negative sum over the entries of $A$.

Proof. We assume $r_1 \leq r_2$. We know by Lemma 4.1 that the only possible non-zero entries of $A$ are given by

\[a_{r_1,r_2}, a_{r_1-1,r_2+1}, \ldots, a_{r_1-k,r_2+k}.\]

We prove the statement by induction on $\sum_{j=0}^{k} a_{r_1-j,r_2+j}$. When $\sum_{j=0}^{k} a_{r_1-j,r_2+j} = 0$ we have by normalization $H(0 \otimes 0) = 0$ and the statement is true. So suppose that the sum is greater than zero and let $j$ be maximal with $a = a_{r_1-j,r_2+j} \neq 0$. In the following we exploit the simplified crystal structure from Section 3. It is easy to see that there exists a tuple $(i_1, \ldots, i_p)$ with $i_p \notin \{0, r_1, r_2\}$ and $\varphi_0(\bar{f}_{i_p} \cdots \bar{f}_{i_1} A) \geq a$. For instance we can choose the tuple

\[((r_1 - j - 1)^{a}, (r_1 - j - 2)^{a}, \ldots, 1^{a}, (r_2 + j + 1)^{a}, (r_2 + j + 2)^{a}, \ldots, n^{a})\].
Let \( \tilde{A} = \tilde{f}_0^a \tilde{f}_p \cdots \tilde{f}_{i_1} A \), which is the element obtained from \( A \) by replacing \( a \) by 0. Especially \( \tilde{A} \otimes 0 \) is a highest weight element and by induction we get
\[
H(\tilde{A} \otimes 0) = - \sum_{1 \leq p \leq r_1} a_{p,q} + a.
\]
Consequently, we obtain (recall \( i_p \notin \{0, r_1, r_2\} \))
\[
H(A \otimes 0) = H(\tilde{e}_0^a \tilde{f}_0^a \tilde{f}_p \cdots \tilde{f}_{i_1} (A \otimes 0)) = H(\tilde{e}_0^a (\tilde{A} \otimes 0)) = H(\tilde{e}_0^{a-1} (\tilde{A} \otimes 0)) + \begin{cases} -1 & \text{if LL} \\ 1 & \text{if RR} \\ 0 & \text{otherwise}. \end{cases}
\]
In order to prove the proposition we shall show that \( \tilde{e}_0 \) acts on \( \tilde{e}_0^{a-t} (\tilde{A} \otimes 0) \) and \( \sigma(\tilde{e}_0^{a-t} (\tilde{A} \otimes 0)) \) on the first tensor factor for all \( 1 \leq t \leq a \). Assume we have that the second tensor factor in \( \sigma(\tilde{e}_0^{a-t} (\tilde{A} \otimes 0)) \) is zero, i.e.

\[
\sigma(\tilde{e}_0^{a-t} (\tilde{A} \otimes 0)) = \sigma(\tilde{e}_0^{a-t} \tilde{A}) \otimes 0.
\]
Then we get
\[
e_0(\tilde{e}_0^{a-t} \tilde{A}) = s_1 - a + t > \varphi_0(0) = 0
\]
and
\[
e_0(\sigma(\tilde{e}_0^{a-t} \tilde{A}) \otimes 0) = e_0(\sigma(\tilde{e}_0^{a-t} \tilde{A})) + s_1 = e_0(\tilde{e}_0^{a-t} (\tilde{A} \otimes 0)) = s_1 - a + t + s_2.
\]
\[
\implies e_0(\sigma(\tilde{e}_0^{a-t} \tilde{A})) = s_2 - a + t > \varphi_0(0) = 0,
\]
which gives the statement. Hence it remains to prove (4.3). We set \( \tilde{\tilde{A}} \otimes 0 = \tilde{\tilde{e}}_{j_p} \cdots \tilde{\tilde{e}}_{j_1} \tilde{e}_0^{a-t} (\tilde{A} \otimes 0) \), where \( (j_1, \ldots, j_r) \) is the tuple
\[
(n^{a-t}, (n-1)^{a-t}, \ldots, (r_2 + j + 1)^{a-t}, 1^{a-t}, \ldots, (r_1 - j - 2)^{a-t}, (r_1 - j - 1)^{a-t}).
\]
In particular \( \tilde{\tilde{A}} \otimes 0 \) is a highest weight element and \( \tilde{\tilde{A}} \) is obtained from \( A \) by replacing \( a \) by \( a - t \).

By Theorem 4.1 we obtain that \( \sigma(\tilde{\tilde{A}} \otimes 0) \) is of the form \( \sigma(\tilde{\tilde{A}}) \otimes 0 \). Furthermore,
\[
\sigma(\tilde{e}_0^{a-t} (\tilde{A} \otimes 0)) = \tilde{\tilde{f}}_{j_1} \cdots \tilde{\tilde{f}}_{j_r} \sigma(\tilde{\tilde{A}} \otimes 0).
\]
Thus \( \sigma(\tilde{e}_0^{a-t} (\tilde{A} \otimes 0)) \) is of the form \( \sigma(\tilde{e}_0^{a-t} \tilde{A}) \otimes 0 \), since \( j_p \notin \{r_1, r_2\} \) for all \( p \in \{1, \ldots, r\} \). The proof for \( r_1 \geq r_2 \) proceeds similarly.

In the remaining part of this subsection we determine the local energy function on arbitrary elements \( A \otimes B \in B^{r_1,s_1} \otimes B^{r_2,s_2} \). We define a map \( + : \mathbb{Z} \rightarrow \mathbb{Z} \geq 0, x \mapsto x_+ \), where \( x_+ = x \), if \( x > 0 \) and \( x_+ = 0 \) otherwise.

Further we define elements \( A^s_r = ((a^s_r)_{p,q}) \), \( B^s_r = ((b^s_r)_{p,q}) \) for \( 0 \leq s \leq n - r_2 \) and \( 0 \leq r \leq r_2 + s \) as follows:

(4.4) For \( s = 0 \):
\[
A^0_r = A, \quad B^0_r = B,
\]
\[
A^0_r = \tilde{e}^0_r (e_r (A^0_r-1) - \varphi_r (B^0_{r-1})) + A^0_{r-1}, \quad B^0_r = \tilde{e}^0_r (B^0_{r-1}) B^0_{r-1}.
\]

For \( s > 0 \):
\[
A^s_r = \begin{cases} A^s_{r-1} & \text{if } r = 0 \\ A^s_{r-1} - \tilde{e}^s_r (e_r (A^s_{r-1}) - \varphi_r (B^s_{r-1})) + A^s_{r-1} & \text{if } 1 \leq r \leq r_2 - 1 \\ \tilde{e}^s_r (e_{2r_2 + s - r} (A^s_{r-1}) - \varphi_2r_2 + s - r (B^s_{r-1})) + A^s_{r-1} & \text{if } r_2 \leq r \leq r_2 + s 
\end{cases}
\]
\[ B^s_r = \begin{cases} 
 B^s_{r_2+s-1} & \text{if } r = 0 \\
 e_{r_2}(B^s_{r_2-s-1}) & \text{if } 1 \leq r \leq r_2 - 1 \\
 e_{r_2+s-r}(B^s_{r_2-s}) & \text{if } r_2 \leq r \leq r_2 + s
\end{cases} \]

**Theorem 4.2.** Let \( s_1 \leq s_2 \) and \( A \otimes B \in B^{s_1,s_1} \otimes B^{s_2,s_2} \). Then we have

\[
H(A \otimes B) = - \sum_{1 \leq p \leq r} \sum_{s=0}^{n-r_2} (\epsilon_{r_2}(A^s_{r_2+s-1}) - \varphi_{r_2}(B^s_{r_2+s-1}))_+.
\]

**Proof.** The local energy function is by definition constant on the classical components of \( B^{s_1,s_1} \otimes B^{s_2,s_2} \). Hence by Proposition 4.1 it is enough to show that the highest weight element in the classical component of \( A \otimes B \) is of the form \( A \otimes 0 \), where the sum over all entries of \( A \) is

\[
\sum_{1 \leq p \leq r} a_{p,q} - \sum_{s=0}^{n-r_2} (\epsilon_{r_2}(A^s_{r_2+s-1}) - \varphi_{r_2}(B^s_{r_2+s-1}))_+.
\]

We assume \( r_1 \leq r_2 \), since the proof for \( r_1 \geq r_2 \) proceeds similarly. We claim the following:

**Claim.** The entries in the first \( s + 1 \) rows of \( B^s_{r_2+s} \) are zero, i.e. \( (b^s_{r_2+s})_{p,q} = 0 \) for all \( 1 \leq p \leq r_2, r_2 \leq q \leq r_2 + s \). Especially we have \( B^n_{n-r_2} = 0 \).

We proof the claim by induction on \( s \) and start with \( s = 0 \). By the definition of the elements \( B^s_r \) (see (4.4)) we get

\[ B^0_{r_2} = e_{r_2}(B^0_{r_2-1}) \cdots e_2(B^0_1) e_1(B^0_0) B. \]

It follows \( (b^0_{r_2})_{1,r_2} = 0 \), since \( e_1(B^0_1) = e_1(e^1_1(B) B) = 0 \) and \( (b^0_{r_2})_{2,r_2} = 0 \) since \( e_2(B^0_2) = e_2(e_2(B^0_2) B^0_1) = 0 \). The action of \( e_2 \) on \( B^0_1 \) preserves the first column which implies \( (b^0_{r_2})_{1,r_2} = 0 \). By repeating the above arguments we obtain for \( B^0_j \)

\[ (b^0_j)_{1,r_2} = (b^0_j)_{2,r_2} = \cdots = (b^0_j)_{j,r_2} = 0, \]

which shows the initial step. Now suppose that the first \( s \) rows are zero in \( B^s_0 = B^{s-1}_{r_2+s-1} \), i.e. \( (b^s_{r_2+s-1})_{p,q} = 0 \) for all \( 1 \leq p \leq r_2, r_2 \leq q \leq r_2 + s - 1 \). Again by (4.4)

\[ B^s_{r_2+s} = e_{r_2}(B^s_{r_2+s-1}) \cdots e_{r_2+s-1}(B^s_{r_2-1}) e_{r_2+s}(B^s_{r_2-s}) e_{r_2+s-1}(B^s_{r_2-2}) \cdots e_2(B^s_1) e_1(B^s_0) B^s_0. \]

We obtain as above \( (b^s_{r_2+s})_{1,r_2+s} = 0 \) since the first \( s \) rows are zero and \( e_1(B^s_1) B^s_0 = e_1(B^s_1) = 0 \). A similar consideration as above shows therefore

\[ (b^s_{r_2-1})_{1,r_2+s} = (b^s_{r_2-1})_{2,r_2+s} = \cdots = (b^s_{r_2-1})_{r_2-1,r_2+s} = 0 \]

and hence \( B^s_{r_2-1} \) is of the form.
Applying the operators
\[ e_{r_2+1}(B_{r_2+s-2}^s) \cdots e_{r_2+s-1}(B_{r_2}^s) e_{r_2+s}(B_{r_2-1}^s) \]
means that we move the entry \( (b_r^s)_{r_2, r_2+s} \) (red star in Figure 1) along the last column to the upper right corner. Finally we apply the remaining operator \( e_{r_2}(B_{r_2+s-1}^s) \) and obtain the desired property for \( B_{r_2+s}^s \).

Now it is easy to see that the element \( A \otimes B \) and \( A_{n-r_2}^n \otimes B_{n-r_2}^n = A_{n-r_2}^n \otimes 0 \) are in the same classical component and thus \( H(A \otimes B) = H(A_{n-r_2}^n \otimes 0) \). In order to prove the theorem we shall show
\[ H(A_{n-r_2}^n \otimes 0) = - \sum_{1 \leq p \leq r_1 \atop r_2 \leq q \leq n} a_{p,q} + \sum_{s=0}^{n-r_2} (\epsilon_{r_2}(A_{r_2+s-1}^s) - \varphi_{r_2}(B_{r_2+s-1}^s))_+ . \]

So consider the element \( A_{n-r_2}^n \otimes 0 \). There is a sequence \( (i_1, \ldots, i_l) \) such that \( e_{i_1} \cdots e_{i_l}(A_{n-r_2}^n \otimes 0) \) is a highest weight element. Suppose that there exists an integer \( k, 1 \leq k \leq l \), with \( r_2 = i_k \) and let \( k \) be minimal with this property. The condition
\[ \epsilon_{r_2}(e_{i_k-1} \cdots e_{i_l}(A_{n-r_2}^n \otimes 0)) \leq s_1 \leq s_2 = \varphi_{r_2}(0) \]
yields \( e_{i_k} \cdots e_{i_l}(A_{n-r_2}^n \otimes 0) = 0 \), which is a contradiction. Thus \( i_k \neq r_2 \) for all \( k \in \{1, \ldots, l\} \).

It means that the sum \( \sum_{1 \leq p \leq r_1 \atop r_2 \leq q \leq n} (a_{n-r_2})_{p,q} \) is stable under the action with \( e_{i_1} \cdots e_{i_l} \), because the only possibility to change the sum is to apply the operator \( e_{r_2} \). It follows on the one hand that \( e_{i_1} \cdots e_{i_l}(A_{n-r_2}^n \otimes 0) \) is a highest weight element, i.e all entries above the \( r_2 \)-th row are zero by Lemma 4.1 and on the other hand the sum over the entries below is preserved. By Proposition 4.1 we have
\[ H(A_{n-r_2}^n \otimes 0) = - \sum_{1 \leq p \leq r_1 \atop r_2 \leq q \leq n} (a_{n-r_2})_{p,q} . \]

Thus it remains to prove
\[ \sum_{1 \leq p \leq r_1 \atop r_2 \leq q \leq n} (a_{n-r_2})_{p,q} = \sum_{1 \leq p \leq r_1 \atop r_2 \leq q \leq n} a_{p,q} - \sum_{s=0}^{n-r_2} (\epsilon_{r_2}(A_{r_2+s-1}^s) - \varphi_{r_2}(B_{r_2+s-1}^s))_+ . \]

Note that
\[ A_{n-r_2}^n = \epsilon_{r_2} \left( r_2(A_{n-1-r_2}^{n-r_2}) - \varphi_{r_2}(B_{n-1-r_2}^{n-r_2}) \right) + \epsilon_{r_2+1} \left( r_2+1(A_{n-2-r_2}^{n-r_2}) - \varphi_{r_2+1}(B_{n-2-r_2}^{n-r_2}) \right) + \cdots + \epsilon_{n} \left( r_2(A_{r_2-1}^{n-r_2}) - \varphi_{n}(B_{r_2-1}^{n-r_2}) \right) + \epsilon_{r_2-1} \left( r_2-1(A_{r_2-2}^{n-r_2}) - \varphi_{r_2-1}(B_{r_2-2}^{n-r_2}) \right) + \cdots + \epsilon_{1} \left( r_2(A_{r_2-1}^{n-r_2}) - \varphi_{1}(B_{r_2-1}^{n-r_2}) \right) + A_{n-1-r_2}^{n-(r_2+1)} . \]
$A_{n-1}^{n-(r_2+1)} = e_{r_2} \left( \epsilon_{r_2}(A_{n-2}^{n-(r_2+1)}) - v_{r_2}(B_{n-2}^{n-(r_2+1)}) \right) + \sum_{i=1}^{n-1} e_i \left( \epsilon_i(A_{n-2}^{n-(r_2+1)}) - \phi_i(B_{n-2}^{n-(r_2+1)}) \right) + \cdots + e_1 \left( \epsilon_1(A_{n-2}^{n-(r_2+1)}) - \phi_1(B_{n-2}^{n-(r_2+1)}) \right) + A_{n-2}^{n-(r_2+2)}.$

\[ A_0^r = e_{r_2} \left( \epsilon_{r_2}(A_{r_2-1}^0) - v_{r_2}(B_{r_2-1}^0) \right) + \cdots + e_2 \left( \epsilon_2(A_1^0) - v_2(B_1^0) \right) + \epsilon_1(A_0^0) - \varphi(A_0^0) + A. \]

Hence the sum over the entries in the last $n - r_2 + 1$ rows of $A_n^{n-r_2}$ is given by (4.5), since the only possibility to decrease the sum $\sum_{1 \leq p \leq r_2, q \leq n} a_{p,q}$ is to apply $e_{r_2}$. 

**Example 4.3.** Let $r_1 = r_2 = n$. The energy of $A \otimes B$ is given by

\[
H(A \otimes B) = \sum_{j=1}^{n} a_{j,n+} + \left( a_{n,n+} + \left( a_{n-1,n+} + \cdots + \left( a_{2,n+} + \left( a_{1,n} - \varphi_1(B) \right) + \varphi_2(B) \right) - \cdots - \varphi_{n-1}(B) \right) + \varphi_n(B) \right).
\]

5. Perfect crystals and the ground-state path

**5.1. Perfect crystals.** We recall the notion of a perfect crystal first introduced in [6]. Let $c = \sum_{i=1}^{n} a_i^\vee \alpha_i^\vee$ be the canonical central element associated to $\mathfrak{g}$ and $P^+ = \{ \Lambda \in P \mid \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \}$ be the set of dominant integral weights. The level of $\Lambda \in P^+$ is defined as $\text{lev}(\Lambda) := \Lambda(c)$. For $\ell \in \mathbb{Z}_{\geq 0}$ let

\[ P^+_\ell = \{ \Lambda \in P^+ \mid \text{lev}(\Lambda) = \ell \}. \]

**Definition 5.1.** A $U'_q(\mathfrak{g})$-crystal $B$ is called a perfect crystal of level $\ell > 0$, if the following conditions are satisfied:

1. $B$ is isomorphic to the crystal graph of a finite-dimensional $U'_q(\mathfrak{g})$-module.
2. $B \otimes B$ is connected.
3. there exists a classical weight $\lambda_0 \in P_0$ such that
   \[ \text{wt}(B) \subseteq \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\geq 0} \alpha_i, \]
   and there is a unique element in $B$ of weight $\lambda_0$.
4. For any $b \in B$, we have $\text{lev} \left( \sum_{i \in I} \epsilon_i(b) \Lambda_i \right) \geq \ell$
5. For all $\Lambda \in P^+_\ell$, there exist unique elements $b_\Lambda, b^\Lambda$, such that
   \[ \sum_{i \in I} \epsilon_i(b_\Lambda) \Lambda_i = \Lambda = \sum_{i \in I} \varphi_i(b^\Lambda) \Lambda_i. \]

**Example 5.1.**

1. Let $\mathfrak{g} = A_2^{(1)}$, the $U'_q(\mathfrak{g})$-crystal

\[
\begin{array}{c}
A & \xrightarrow{1} & B & \xrightarrow{2} & C \\
\end{array}
\]

is a perfect crystal of level 1.
(2) Let $\mathfrak{g} = C_2^{(1)}$, the $U_q'(\mathfrak{g})$-crystal

$$
\begin{array}{c}
A & \rightarrow & 1 & \rightarrow & B & \rightarrow & 2 & \rightarrow & C & \rightarrow & 1 & \rightarrow & D \\
& & & & & & & & & & & & 0
\end{array}
$$

is not a perfect crystal of level 1 because we may have $b_{\Lambda_1} = B$ or $D$.

5.2. Path realization. Perfect crystals are of particular importance because one can give a path realization of affine highest weight crystals via perfect crystals. Let $B$ be a perfect crystal of level $\ell$ and $\Lambda = \sum_{i \in I} a_i \Lambda_i$ be a dominant integral weight with $\text{lev}(\Lambda) = \ell$. The crystal graph $B(\Lambda)$ associated to the affine Lie algebra $\mathfrak{g}$ can be realized as follows.

Let

$$
\Lambda_0 = \Lambda, \quad \Lambda_{k+1} = \sum_{i \in I} \epsilon_i(b_i^{\Lambda_k}) \Lambda_i; \quad b_k = b_i^{\Lambda_k}.
$$

The sequence

$$
P_\Lambda = (b_k)_{k=0}^{\infty} = \cdots b_k \otimes b_{k-1} \otimes \cdots \otimes b_0
$$

is called the ground-state path of weight $\Lambda$ and a sequence

$$
P = (p_k)_{k=0}^{\infty} = \cdots p_k \otimes p_{k-1} \otimes \cdots \otimes p_0
$$

with the property $p_k \in B$ for all $k$ and $p_k = b_k$ for all $k >> 0$ is called a $\Lambda$-path. We have the following important theorem from [7].

**Theorem 5.1.** There exists an isomorphism of crystals

$$
\Psi : B(\Lambda) \rightarrow P(\Lambda), \quad u_\lambda \mapsto P_\Lambda
$$

where $P(\Lambda)$ is the set of all $\Lambda$-path in $B$.

Due to a result of [3] the Kirillov-Reshetikhin crystal $B^{i,\ell}_{\nu}$ is perfect for all non-exceptional types if $\ell$ is a multiple of $c_i$ (see Figure 4 in [22]). We give an alternative proof for the perfectness for $A_n^{(1)}$ by using the model from [16] and describe for any $\ell$ and $\Lambda \in P_{\ell}^+$ the elements $b_\Lambda, b_\Lambda^2 \in B^{i,\ell}$.

**Theorem 5.2.** The Kirillov-Reshetikhin crystal $B^{i,\ell}_{\nu}$ is perfect of level $\ell$ and for $\Lambda = \sum_{i \in I} a_i \Lambda_i \in P_{\ell}^+$ we have

$$
b^{\Lambda} = \begin{array}{cccc}
a_{i+1} & a_{i+2} & \cdots & a_{2i} \\
a_{i+2} & a_{i+3} & \cdots & a_{2i+1} \vdots & \vdots & \ddots & \vdots \\
a_{n-i+1} & a_{n-i+2} & \cdots & a_{n-1} \\
a_{n-i+2} & a_{n-i+3} & \cdots & a_n \\
\vdots & \vdots & \ddots & \vdots \\
a_n & a_0 & \cdots & a_{i-3} \\
a_0 & a_1 & \cdots & a_{i-2} \\
\end{array}
$$

$$
b_\Lambda = \begin{array}{cccc}
a_1 & a_2 & \cdots & a_{i-1} \\
a_2 & a_3 & \cdots & a_i \\
\vdots & \vdots & \ddots & \vdots \\
a_{n-i} & a_{n-i+1} & \cdots & a_{n-2} \\
a_{n-i+1} & a_{n-i+2} & \cdots & a_{n-1} \\
\end{array}
$$

**Proof.** We check stepwise the properties from Definition 5.1, where (1) and (3) are obviously true. In order to prove that $B^{i,\ell}_{\nu} \otimes B^{i,\ell}_{\nu}$ is connected we show that any highest weight element in the tensor product is connected to $0 \otimes 0$, i.e. from each highest weight element there is a path in the crystal graph of $B^{i,\ell}_{\nu} \otimes B^{i,\ell}_{\nu}$ to $0 \otimes 0$. So let $A \otimes 0$ be an arbitrary highest weight element. We prove
the claim by induction over the sum over the entries of $A$. If $A = 0$, there is nothing to prove. So let $j$ be maximal with $a_{i-j,i+j} \neq 0$. Then we consider the element

$$\tilde{A} \otimes 0 = f_{0}^{a_{i,j}} e_{1}^{a_{i,j}} \cdots e_{i-j-1}^{a_{i,j}} e_{i-j}^{a_{i,j}} \cdots e_{i+j}^{a_{i,j}} \cdots (A \otimes 0),$$

which is a highest weight element and has by induction the desired property. Hence $A \otimes 0$ has the desired property and condition (2) is proven. Now let $A \in B^{i,\ell}$ be an arbitrary element. Note (5.1)

$$\epsilon_{l}(A) = \begin{cases} a_{i,i}, & \text{if } l = i \\ \sum_{j=q_{l}^{i}(A)-1}^{q_{l}^{i}(A)} a_{j,l} - \sum_{j=q_{l}^{i}(A)+1}^{q_{l}^{i}(A)+1} a_{j,l-1}, & \text{if } l > i \\ a_{i,l}, & \text{if } l < i \end{cases}$$

Thus

$$\sum_{j=0}^{n} \epsilon_{j}(A) = (\ell - \sum_{j=1}^{n} a_{1,j} - \sum_{j=2}^{n} a_{j,n}) + \sum_{j=1}^{n} \epsilon_{j}(A) \geq (\ell - \sum_{j=1}^{n} a_{i,j}) + \sum_{j=i}^{n} \epsilon_{j}(A) \geq \ell,$$

which shows property (4). The last property can be deduced from the following observation. Let $b_{A}$ as above and let $\text{lev}(\Lambda) = \sum_{j \in I} a_{j} = \ell$, then

$$\epsilon_{0}(b_{A}) = \ell - \sum_{j \neq 0} a_{j} = a_{0}, \quad \epsilon_{i}(b_{A}) = a_{i}.$$

Moreover, for the $j$-th and $(j+1)$-th column $(j < i)$ we get $q_{l}^{i}(b_{A}) = i$ and thus $\epsilon_{j}(A) = a_{j}$ and for the $(j+1)$ and $j$-th row $(j > i)$ of $b_{A}$ we get $q_{l}^{j}(b_{A}) = i$ and thus $\epsilon_{j}(A) = a_{j}$. A similar calculation for $b_{A}$ proves that both $b_{A}$ and $b_{A}$ have the desired property. It remains to prove that $b_{A}$ is unique (the uniqueness for $b_{A}$ proceeds similarly). Let $A = (a_{p,q})$ be another element with $\sum_{j \in I} \epsilon_{j}(A) \Lambda_{i} = \Lambda$. Since $\sum_{j \in I} \epsilon_{j}(A) = \ell$ we get with (5.1) and (5.2)

$$\epsilon_{l}(A) = \begin{cases} a_{i,i}, & \text{if } l = i \\ a_{i,l}, & \text{if } l > i \\ \sum_{j=i}^{n} a_{l,j} - \sum_{j=i}^{n-1} a_{l+1,j}, & \text{if } l < i \end{cases},$$

which forces

$$p_{l}^{i}(A) = n \forall l = 1, \ldots, i - 1 \text{ and } q_{l}^{i}(A) = i \forall l = i + 1, \ldots, n.$$ 

It follows $\epsilon_{i}(A) = a_{i,i} = a_{i}, \epsilon_{i+1}(A) = a_{i+1}, \epsilon_{i+2}(A) = a_{i+2}, \ldots, \epsilon_{n}(A) = a_{n}$ and therefore the last column of $A$ is the same as the last column of $b_{A}$. Our aim is to prove that the $(i-1)$-th column is also the same. Since $q_{l}^{i}(A) = i$ we get $a_{i-1,n} \leq a_{n-1}$ and since $p_{l}^{i}(A) = n$ we get $a_{i-1,n} \geq a_{n-1}$. By repeating this argument with the remaining indices we obtain $a_{i-1,n} = a_{n-1}, a_{i-1,n-1} = a_{n-2}, \ldots, a_{i-1,1} = a_{i}$. Moreover, $\epsilon_{i-1}(A) = \sum_{j=i}^{n} a_{i-1,j} - \sum_{j=i}^{n-1} a_{i,j} = a_{i-1,i} = a_{i-1}$. Consequently the $(i-1)$-th column of $A$ and $b_{A}$ coincide. By repeating the same method with the $(i-2)$-th and $(i-1)$-th column we obtain that the $(i-2)$-th column of $A$ is the same as the $(i-2)$-th column of $b_{A}$. We repeat this procedure until we get $A = b_{A}$. \hfill \Box

Finally we can describe easily the ground-state path in $B^{i,\ell}$. We identify $\Lambda = \sum_{j \in I} a_{j} \Lambda_{j}$ with the tuple $(a_{0}, a_{1}, \ldots, a_{n})$. Then

$$\Lambda_{0} = (a_{0}, a_{1}, \ldots, a_{n}), \quad \Lambda_{1} = (a_{i}, a_{i+1}, \ldots, a_{n}, a_{0}, \ldots, a_{i-1}), \ldots$$

It means that $\Lambda_{k+1}$ arises from $\Lambda_{k}$ by cutting the tuple $\Lambda_{k}$ at the $i$-th position into two tuples $(a_{0}, \ldots, a_{i-1})$ and $(a_{i}, \ldots, a_{n})$ and gluing both pieces in reverse order.
\[ \Lambda_k = (a_0, a_1, \ldots, a_n) \Rightarrow (a_0, \ldots, a_{i-1}) \Rightarrow (a_i, \ldots, a_n) \Rightarrow \Lambda_{k+1} = (a_i, a_{i+1}, \ldots, a_n, a_0, \ldots, a_{i-1}) \]

The ground state path is then given by
\[ p_\Lambda = (b_k)_{k=0}^\infty = \cdots b_k \otimes b_{k-1} \otimes \cdots \otimes b_0, \]
where \( b_k = b^{\Lambda_k} \) with \( \Lambda_k \) as in (5.3) and \( b^{\Lambda_k} \) as in Theorem 5.2.

Example 5.2.

1. If \( i = n \), then
\[ \Lambda_0 = (a_0, \ldots, a_n), \quad \Lambda_1 = (a_0, a_n, \ldots, a_{n-1}), \ldots, \Lambda_n = (a_1, \ldots, a_n, a_0), \quad \Lambda_{n+1} = \Lambda_0 \]
and
\[ b_0 = \begin{array}{cccc} a_0 & a_1 & \cdots & a_{n-1} \\ a_0 & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_{n-1} \end{array}, \quad b_1 = \begin{array}{cccc} a_n & a_0 & \cdots & a_{n-2} \\ a_n & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_0 & \cdots & a_{n-2} \end{array}, \quad b_n = \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n \end{array} \]

2. For \( i = n-1 \) and \( n \) is odd, say \( n = 2j-1 \) we get
\[ \Lambda_0 = (a_0, \ldots, a_n), \quad \Lambda_k = (a_{n-(2k-1)}, a_{n-(2k-2)}, \ldots, a_n, a_0, \ldots, a_{n-2k}), \quad 1 \leq k \leq j-1 \]
\[ b_0 = \begin{array}{cccc} a_n & a_0 & \cdots & a_{n-3} \\ a_0 & a_1 & \cdots & a_{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_{n-3} \end{array}, \quad b_k = \begin{array}{cccc} a_{n-2k} & a_{n-(2k-1)} & \cdots & a_{0} \\ a_{n-(2k-1)} & a_{n-(2k-2)} & \cdots & a_{1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2k} & a_{n-2k-3} & \cdots & a_{n-2k-2} \end{array} \]

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