THE COHOMOLOGY OF $p$-ADIC DISTRIBUTION REPRESENTATIONS

WEIBO FU

ABSTRACT. We give a generalization of Kostant’s theorem on Lie algebra cohomology of finite dimensional highest weight representations to some infinite dimensional cases over a $p$-adic family of highest weight distribution representations. For proving this, we develop a theory of eigen orthonormalizable Banach representations of $p$-adic torus over an affinoid algebra, and we construct eigen orthonormalizable weight completions of the distribution representations.

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1. INTRODUCTION

Let $p$ be a prime number, let $G$ be either the general linear group $\text{GL}_{2n}$ or the symplectic group $\text{Sp}_{2n}$ or a unitary group over $\mathbb{Z}_p$ with a chosen Borel pair $(B,T)$. We use $I$ to denote the Iwahori subgroup of $G(\mathbb{Z}_p)$ associated to $(B,T)$, consisting of matrices congruent to $B(F_p)$ modulo $p$.

Let $N$ be a unipotent subgroup of $G$ with Lie algebra $\mathfrak{u}$ of $N(\mathbb{Z}_p)$. It corresponds to a Levi decomposition $P = LN$ for a parabolic subgroup $P$ and Levi subgroup $L$ of $G$. Let $I_L := I \cap L(\mathbb{Z}_p)$ be the Iwahori subgroup of $L(\mathbb{Z}_p)$ for the Levi $L$. For an algebraic representation $V$ of $\text{GL}_n(\mathbb{Z}_p)$ and $\mathfrak{n} := \text{Lie}(N(\mathbb{Z}_p))$, the Lie algebra cohomology $H^*(\mathfrak{n}, V)$ has a representation structure of $L(\mathbb{Z}_p)$. Working over the complex numbers, Kostant’s work on Lie algebra cohomology $[\text{Kos61}]$ directly extends to this setting, and completely describes $H^*(\mathfrak{n}, V)$ in terms of

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irreducible representations of $I_L$ (as Lie algebra representations in [Kos61]) for any finite dimensional algebraic representation $V$ of $I \subset GL_n(V)$.

Let $\mathcal{W}$ be the weight space of $T(Z_p)$ over $Q_p$, parametrizing locally analytic characters of $T(Z_p)$. And let $\Omega$ be an affinoid subdomain of $\mathcal{W}$, there is a universal character $\chi_{\Omega} : T(Z_p) \to \mathcal{O}(\Omega)^\times$. There is a countable filtration $\{I^s\}$ of open normal subgroups for $I$ such that $I^s$ consists of matrices in $I$ congruent to the identity matrix modulo $p^s$. Let $N_I$ be the $Z_p$-points of unipotent radical of $B$ and

$$\text{Ind}_{\Omega}^s := \{ f : I \to \mathcal{O}(\Omega), f \text{ analytic on each } I^s \text{ coset},$$

$$f(g t n) = \chi_{\Omega}(t) f(g) \forall n \in N_I, t \in T(Z_p), g \in I \},$$

for some big enough $s$ depending on $\Omega$.

$\text{Ind}_{\Omega}^s$ is a Banach representation of $I$ over $\mathcal{O}(\Omega)$. Let $D_{\Omega}^s$ be the $\mathcal{O}(\Omega)$-linear Banach dual of $\text{Ind}_{\Omega}^s$. The so called distribution representation $D_{\Omega}^s$ interpolates $p$-adic analogue of Verma modules, and therefore finite dimensional presentations. This distribution representation plays a significant role in $p$-adic automorphic forms such as in [AS08, Urb11, AIP13, Han17].

Suppose $P$ is a Siegel parabolic subgroup of $G$ containing $B$ and whose unipotent radical $N$ is abelian. For example, if $G = GL_{2n}$, $B$ is chosen to be the upper triangular Borel and $P$ is chosen to be the block upper triangular parabolic with zero lower left $n \times n$ block:

$$\begin{bmatrix}
* & * \\
0 & *
\end{bmatrix}.$$

Let $w$ be a relative Weyl group element for $G$ with respect to $P$ of length $l(w)$. Let $N_w := N(Z_p) \cap wIw^{-1}$. Let $D_{w, \Omega}^s$ be the representation of $wIw^{-1}$ obtained from $D_{\Omega}^s$ by conjugation of $w$ (41). By choosing a representative in the class, we will consider certain $w$ such that the Iwahori group $I_L$ for $L$ is contained in $wIw^{-1}$. A useful example is that

$$w = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \in GL_4(Z_p),$$

Iwahori subgroups are associated to the standard upper triangular Borels of both $GL_2 \times GL_2 \subset GL_4$. In the present article, we aim to analyze cohomology of $D_{w, \Omega}^s$ with respect to an open subgroup of $N(Z_p)$ through the Lie algebra cohomology $H^*(n, D_{w, \Omega}^s)$. The cohomology $H^*(n, D_{w, \Omega}^s)$ admits a natural action of $I_L$. We extract a $I_L$-direct summand of this cohomology in Thm [9.1] which interpolates distribution representation of the Levi subgroup.

**Theorem 1.1.** There is a natural $I_L$-equivariant direct summand

$$i : D_{w, \Omega}^s \hookrightarrow H^*(n, D_{w, \Omega}^s)^{N_w}.$$

Here $D_{w, \Omega}^s$ is the $\mathcal{O}(\Omega)$-linear Banach dual of

$$\text{Ind}_{w, \Omega}^s := \{ f : I_L \to \mathcal{O}(\Omega), f \text{ analytic on each } I_L^s \text{ - coset},$$

$$f(g t n) = \chi_{\Omega}(w^{-1}tw) \cdot (w^\delta - \delta)(t) f(g) \forall n \in N_p, t \in T_p, g \in I_p \},$$

where $I_L^s$ is an open subgroup of $I_L$ depending on $s$ and $\delta$ is half sum of positive roots with respect to $(B, T)$. 
The direct summand $D_{w, \Omega}$ is a distribution module of $I_L$. In particular, it interpolates locally algebraic representations of $I_L$. Our result can be regarded as a generalization of Kostant’s theorem in a $p$-adic family. Here we give a sketch of proof in §9.

Using Chevalley–Eilenberg complex $\bigwedge^n n^* \otimes D_{w, \Omega}$ to compute the Lie algebra cohomology, we explicitly construct a $I_L$-equivariant inclusion $D_{w, \Omega} \hookrightarrow \bigwedge^l(u, D_{w, \Omega})_{N_w}$ when passing to cohomology. We prove this map induces an inclusion $D_{w, \Omega} \hookrightarrow H^l(w, D_{w, \Omega})_{N_w}$ when specializing to a generic weight in $\Omega$. The statement of Thm 1.1 is the best possible in the sense that for any other cohomological degree $* \neq l(w)$, $H^*(n, D_{w, \Omega})_{N_w}$ is “almost” vanishing generically by considering the infinitesimal character on $H^*(n, D_{w, \Omega})_{N_w}$.

And since we are dealing with Iwahori groups over a general $p$-adic field, we also need to establish some results for the weight spaces of the torus, which we think may be of independent interests.

If $E$ is a finite extension of $\mathbb{Q}_p$, $\sigma : E \hookrightarrow \mathbb{Q}_p$, and $\Gamma$ is a compact profinite $E$-analytic abelian group, we use $W_{\sigma}$ to denote the weight space parametrizing continuous/$E$-locally analytic characters of $\Gamma$, $W_{E, \sigma}^{\Gamma}$ to denote the weight space parametrizing $(\sigma, E)$-analytic locally analytic characters of $\Gamma$.

**Theorem 1.3.** There is a decomposition of the weight space $W_{\Gamma}$ in terms of $W_{E, \sigma}^{\Gamma}$:

$$
\prod_{\sigma} \times \prod_{\sigma : E \hookrightarrow K} W_{E, \sigma}^{\Gamma} \twoheadrightarrow W_{\Gamma},
$$

up to isogeny, i.e., $\prod_{\sigma}$ is surjective with the kernel being a zero-dimensional Zariski closed subspace. Moreover, points of the kernel correspond to smooth characters.

Our motivation for studying this problem is however derived from a global Langlands correspondence perspective.
We refer to $X^H$ as the symmetric space for a connected linear algebraic group $H$ over $F$. For example, if $H = \text{GL}_{n,F}$, one can take $X^H = \text{GL}_{n}(F_{\infty})/K_{\infty}\mathbb{R}^\times$ for $K_{\infty} \subset \text{GL}_{n}(F_{\infty})$ a maximal compact subgroup. If $F$ is CM with a totally real field $F^+$, $L := \text{Res}_{F/F^+}\text{GL}_{n,F}$ appears as a Levi of a Siegel parabolic subgroup $P$ of a quasi-split unitary group $G := U(n,n)_{F/F^+}$. $X^G$ has complex dimension $d = [F^+ : \mathbb{Q}]n^2$. Given a certain good level subgroup $\widetilde{K} \subset G(\mathbb{A}_{F^+}^\infty)$ and $K = \widetilde{K} \cap G(\mathbb{A}_{F^+}^\infty)$, the locally symmetric space $\overline{X}_{\widetilde{K}} = G(F^+) \backslash G(\mathbb{A}_{F^+}^\infty) \times X^G/\widetilde{K}$ admits the Borel-Serre compatification $\overline{X}_{\widetilde{K}}$ (BS73) and the Borel-Serre boundary $\partial X_{\widetilde{K}} := \overline{X}_{\widetilde{K}} - X_{\widetilde{K}}$.

There is a stratum $X_{\varphi,K}^P \subset \partial X_{\widetilde{K}}$, together with a torus fibration

$$X_{\varphi,K}^P \rightarrow X_{\varphi,K} = L(F^+) \backslash L(\mathbb{A}_{F^+}^\infty) \times X^L/K.$$ 

Given a representation $V$ of $\widetilde{K}$, there is a local system $\underline{V}$ on $X_{\varphi,K}$ associated to $V$. Consider the local system on $X_{\varphi,K}^P$ obtained by composition of pushforward and restriction via $X_{\varphi,K}^P \hookrightarrow \overline{X}_{\widetilde{K}} \leftarrow X_{\varphi,K}$, which is still denoted as $\underline{V}$. There is a Hecke-equivariant long exact sequence

\begin{align*}
\ldots \rightarrow H^i_c(X_{\varphi,K}, \underline{V}) \rightarrow H^i(\overline{X}_{\widetilde{K}}, \underline{V}) \rightarrow H^i(\partial X_{\varphi,K}, \underline{V}) \rightarrow \ldots ,
\end{align*}

and a Hecke-equivariant Leray spectral sequence

\begin{align*}
H^j(X_{\varphi,K}, R^i\pi_*\underline{V}) \Rightarrow H^{i+j}(X_{\varphi,K}^P, \underline{V}).
\end{align*}

If we take $V$ to be $\mathbb{D}_\Omega$, the Leray spectral sequence naturally leads one to analyze the sheaf $R^j\pi_*\mathbb{D}_\Omega$ on $X_{L,K}$. In this article, we also prove a comparison $\text{SS}$ between discrete group cohomology and the locally analytic cohomology introduced by Koh11 using the Koszul resolution. Let $N \simeq \mathbb{Z}_p^n = \bigoplus_{i=1}^n \mathbb{Z}_p e_i$ be a compact $p$-adic analytic abelian group, with Lie algebra $n$ over $\mathbb{Q}_p$. Let $\Gamma = \bigoplus_{i=1}^n \mathbb{Z}_p e_i \subset N$ be a finitely generated free abelian group, dense in $N$.

**Theorem 1.4.** There are natural isomorphisms

$$H^q(\Gamma, V) \simeq H^q(n, V)^N$$

for all $q \geq 0$ and any complete Hausdorff locally convex $\mathbb{Q}_p$-vector space $V$ with a separately continuous action of the distribution algebra $D(N, \mathbb{Q}_p)$.

As an application of the comparison theorem, we can relate the local systems $R^j\pi_*\mathbb{D}_\Omega$ on the locally symmetric space $X_{L,K}$ to

$$H^j_{an}(N_w, \mathbb{D}_{w,\Omega}) \simeq H^j(n, \mathbb{D}_{w,\Omega})^{N_w}.$$ 

The works of [AS08], [Urb11], [Han17] use overconvergent cohomology of locally symmetric space of a reductive group with the distribution representation coefficient to construct eigenvarieties. Among reductive groups, eigenvarieties for groups with discrete series are easiest to study. We refer the reader to [Urb11] for example. The quasi-split unitary group $G$ is such an example. But for a general $\text{GL}_{n,F}$, the eigenvarieties are rather mysterious, we do not even know their dimensions. To better understand these eigenvarieties, we may want to relate the overconvergent cohomology for $\text{GL}_{n,F}$ to the overconvergent cohomology for $G$.

**Theorem 1.1** and **Theorem 5.1** together provide the distribution representation coefficient of $\text{GL}_{n,F}$ as a direct summand of $R^j\pi_*\mathbb{D}_{\Omega}$ for suitable degree shift $j$. Therefore sequences (1), (2) realize this relation between $G$ and $L$ when we take
V to be $\mathbb{D}_p^3$. We hope our paper offers some foundations and stimulations in the future studies of this direction for functoriality of eigenvarieties between unitary groups and general linear groups.

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2. Notation

If $H$ is a quasi-split reductive group over a field $k$ which splits over a Galois extension $k'/k$ and $S$ is a maximal $k$-split torus in $H$. Let $T = Z_H(S) \subset H$ be the centralizer of $S$ as a maximal torus contained in a fixed Borel subgroup $B$, then we write

$$W(H, S) := N_H(S)/Z_H(S) = N_H(S)/T$$

for the relative Weyl group. It is a constant étale group scheme over $k$ by [Con17, Lem P.4.1], hence we will identify it with its $k$-points. Let

$$W(H, T) := N_H(T)/Z_H(T) = N_H(T)/T$$

be the Weyl group for $H$ as a étale group scheme over $k$. Note that these two notions coincide for split groups. In the quasi-split case, $W(H, S)$ is embedded into $W(H, T)$ as a étale subgroup scheme over $k$ and is furthermore identified with $W(H, T)(k)$. Length functions for either relative or absolute Weyl groups are defined using the Coxeter group structures for each of them and they are usually different functions when restricted to $W(H, S)$.

If $P \subset H$ is a parabolic subgroup which contains $B$ with its unipotent radical $N$, then there is a unique Levi subgroup $L \subset P$ which contains $T$. We write $W_P(H, S)$ and $W_P(H, T)$ for the corresponding Weyl groups of this Levi subgroup, which can be identified with subgroups of $W(H, S)$ and $W(H, T)$.

We can form

$$X^*(T) := \text{Hom}(T \times_k k', \mathbb{G}_m)$$

as the algebraic characters group of $T$. The absolute Weyl group $W(H, T)(k')$ acts on $X^*(T)$. Similarly, $W(H, S)$ acts on

$$X^*(S) := \text{Hom}(S, \mathbb{G}_m).$$

The absolute root system $\Delta(H, T) \subset X^*(T)$ and relative root system $\Delta(H, S) \subset X^*(S)$ are stable respectively under actions of $W(H, T)(k')$ and $W(H, S)$. With respect to $B$,

$$\Delta(H, T) = \Delta^+(H, T) \cup \Delta^-(H, T), \quad \Delta(H, S) = \Delta^+(H, S) \cup \Delta^-(H, S).$$

If $P$ is a parabolic subgroup of $H$ which contains $B$, then $B \cap L$ is a Borel subgroup of $L$ and we have similar notions $\Delta^{(\pm)}(L, T)$, $\Delta^{(\pm)}(L, S)$ for $L$. The sets

$$W^P(H, T) = \{ w \in W(H, T)(k) \subset W(H, T)(k') \mid w^{-1}(\Delta^+(L, T)) \subset \Delta^+(H, T) \}$$

$$W^P(H, S) = \{ w \in W(H, S) \mid w^{-1}(\Delta^+(L, S)) \subset \Delta^+(H, S) \}$$
are respectively parabolic quotients of Coxeter groups \( W(H, T)(k) \) and \( W(H, S) \) as sets of representatives for the quotients

\[
W_P(H, T)(k) \backslash W(H, T)(k) \text{ and } W_P(H, S) \backslash W(H, S).
\]

For \( w_1, w_2 \in W^P(H, S) \) (resp. \( W^P(H, T) \)), \( w_1 \geq w_2 \) if there exists \( w_1^\prime, w_2^\prime \in W_P(H, S) \) such that \( w_1^\prime w_1 \geq w_2^\prime w_2 \) by proposition 2.5.1 of [BB05], which defines a partial Bruhat order on \( W^P(H, S) \) (resp. \( W^P(H, T) \)). See also section §9.1.3 in [Har] for the use of such sets of representatives.

Let \( E \) be a \( p \)-adic local field with its ring of integers \( \mathcal{O}_E \), uniformizer \( \varpi \) and residue field \( k_E \). Let \( k = E \), \( k' = K \) for the discussion above. With respect to a previously fixed Borel subgroup containing a maximal torus \( T \subset B \subset H \), we use \( \Delta^+_H, \Delta_H \) to denote corresponding roots for the split root system \( \Phi(H) \) on \( H \). Suppose \( I \) is a parabolic subgroup of \( G \), and \( R \) is a maximal torus in \( I \). We write \( \Delta(I) = \Delta(I, R) \) for the unipotent radicals of \( B, T \) and \( \mathcal{O}_E \). Then \( \mathcal{O}_E \) is \( \mathcal{O}_E \) over \( \mathbb{Z}_p \). We write \( \mathcal{O}_E \) for the \( \mathcal{O}_E \)-algebraic integral model of \( G \). We also set \( \mathcal{O}_E := \mathcal{O}_E / p \mathcal{O}_E \) and \( \mathcal{O}_E / p \mathcal{O}_E \) for the unipotent radicals of \( B, T \) and \( \mathcal{O}_E \). We write \( I \) for the Iwahori subgroup

\[
I = \{ h \in \mathcal{O}_E \text{ with } h \text{ mod } p \in B(k) \}.
\]

Set \( N_I := N_B(\mathcal{O}_E) = N_B(\mathbb{Z}_p) \). For any integer \( s \geq 1 \), set

\[
B^s := \{ b \in B(\mathcal{O}_E), b \equiv 1 \text{ in } \mathcal{O}_E / \varpi^s \mathcal{O}_E \}, \quad \bar{B} := \{ b \in \bar{B}(\mathcal{O}_E), b \equiv 1 \text{ in } \mathcal{O}_E / \varpi^s \mathcal{O}_E \},
\]

\[
N_I^s := N_B(\mathcal{O}_E) \cap B^s = N_B(\mathbb{Z}_p) \cap B^s, \quad N_I^s := N_B(\mathcal{O}_E) \cap \bar{B} = N_B(\mathbb{Z}_p) \cap \bar{B}.
\]

Moreover, if there are a finite collection of unramified \( p \)-adic reductive groups and their reductive group schemes

\[
\{(H_1)/E_1, \cdots, (H_l)/E_l\}, \{(\mathcal{H}_1)/\mathcal{O}_{E_1}, \cdots, (\mathcal{H}_l)/\mathcal{O}_{E_1}\}
\]

with

\[
\{S_1, \cdots, S_l\}, \{B_1, \cdots, B_l\}, \{T_1, \cdots, T_l\},
\]

\[
\{I_1^s, \cdots, I_l^s\}, \{\bar{B}_1^s, \cdots, \bar{B}_l^s\}, \{N_1^s, \cdots, N_l^s\}, \{N_1, \cdots, N_l\}
\]
associated to them, we use \( I = I^1 = \prod_{1 \leq i \leq l} I_i \) to denote the Iwahori subgroup of \( \prod_{1 \leq i \leq l} H_i(E_i) \). We use \( \varpi_i \) and \( k_i \) to denote a choice of uniformizer and residue field for \( E_i \). In general, we use

\[
I^s = \prod_{1 \leq i \leq l} I^s_i, \quad I^s_{0,i} = \prod_{1 \leq i \leq l} I^s_{0,i}, \\
T^s = \prod_{1 \leq i \leq l} T^s_i, \quad \mathcal{B} = \prod_{1 \leq i \leq l} \mathcal{B}_i, \quad B_i = \prod_{1 \leq i \leq l} B_i, \\
\mathcal{N}_i = \prod_{1 \leq i \leq l} \mathcal{N}_i, \quad \mathcal{N}_i^s = \prod_{1 \leq i \leq l} \mathcal{N}_i^s
\]

to denote various subgroups of the \( p \)-adic Lie group \( \prod_{1 \leq i \leq l} H_i(E_i) \) and \( I = \mathcal{N}_i \times \mathcal{T^0} \times \mathcal{N}_i \) to denote the Iwahori decomposition similar to notions we introduced above. Similarly one defines

\[ W(H, S), \ W(H, T), \ W_P(H, S), \ W_P(H, T), \ W^P(H, S), \ W^P(H, T) \]
to be products of respective Weyl groups for each component.

For an element \( a \) in the finite rank abelian group \( \mathbb{Z}^d \), total degree \( |a| \) of \( a \) is defined to be the sum of the absolute value of coordinates of \( a \). There is a partial order relation \( a \geq a' \) if and only if \( a_i \geq a'_i \) for all \( i \).

We refer to [BGR84] and [Sch01] for basics in rigid analytic geometry and nonarchimedean functional analysis. For two local nonarchimedean field of characteristic zero \( E \subset K \) and a locally \( E \)-analytic manifold \( M \), we write \( C^{an}(M, K) \) for the space of \( K \)-valued \( E \)-analytic functions on \( M \). For any \( K \)-affinoid algebra \( R \), \( C^{an}(M, R) := C^{an}(M, K) \hat{\otimes} R \).

If \( E \subset K \) are finite extensions of \( \mathbb{Q}_p \) and \( G \) is a locally \( E \)-analytic group, let \( D(G, K) \) be the distribution algebra of \( G \), i.e., the strong dual of the space of locally analytic \( K \) valued functions ([ST02], [ST03]).

For defining the unitary group of our interests, we use the Hermitian form

\[ J_n = \begin{pmatrix} 0 & \Psi_n \\ -\Psi_n & 0 \end{pmatrix}, \]

where \( \Psi_n \) is the matrix with 1’s on the anti-diagonal and 0’s elsewhere.

3. Weight spaces

Let \( E \) be a finite extension of \( \mathbb{Q}_p \) with uniformizer \( \varpi \). Suppose \( T_{/E} \) is an unramified torus as generic fibre of a reductive group scheme \( \mathcal{G}_{/\mathcal{O}_E} \).

\[ T^\sigma := \{ t \in \mathcal{G}(\mathcal{O}_E), \ t \equiv 1 \text{ in } \mathcal{G}(\mathcal{O}_E/\varpi \mathcal{O}_E) \}. \]

In particular, \( T^0 = \mathcal{G}(\mathcal{O}_E) \) is a compact (profinite) \( E \)-analytic abelian group. In this section we give a product structure up to isogeny for the usual weight spaces of \( T^0 \) in terms of their locally \( \sigma \)-analytic subspaces for \( \sigma : E \to \mathbb{Q}_p \) running through all embeddings.

Let \( \Gamma \) be an \( E \)-analytic profinite abelian group containing an open subgroup \( (E\text{-analytic}) \) isomorphic to \( \mathcal{O}_E^d \) for some \( d \). Let \( \Gamma_0 \) denote the locally \( \mathbb{Q}_p \) analytic group obtained from \( \Gamma \) by restriction of scalars from \( E \) to \( \mathbb{Q}_p \). When \( \Gamma = T^0 \), we use \( T^0_0 \) to denote \( \Gamma_0 \). If \( X \) is a rigid space over a \( p \)-adic local field, let \( \mathcal{O}(X) \) denote the ring of rigid functions on \( X \). According to [Buz07], we say that a group homomorphism
We denote by $\mathcal{J}(X)$ (resp. $\mathcal{J}(X)^\times$) the image of $\lambda$ on rigid spaces over $\mathbb{Q}$, the induced map $\mathcal{J}(X)$ (resp. $\mathcal{J}(X)^\times$) is continuous.

By \cite[Lem 8.2]{Buz04}, we denote $\mathcal{J}_f$ as the quasi-separated rigid space over $\mathbb{Q}_p$ (also as a group object in the category of $\mathbb{Q}_p$-rigid spaces) represented the functor on rigid spaces over $\mathbb{Q}_p$ sending a $\mathbb{Q}_p$-rigid space $X$ to the set of continuous group homomorphisms $\Gamma \to \mathcal{J}(X)^\times$, i.e., $X$-points of $\mathcal{J}$ are given by $\mathcal{J}_f(X) = \text{Hom}_{cts}(\Gamma, \mathcal{J}(X)^\times)$.

We denote by $\lambda$ for both a $\mathbb{Q}_p$-point of $\mathcal{J}_f$ as well as the corresponding character of $\mathcal{J}(\mathcal{O}_E)$. Let $\mathcal{J}_{\mathcal{O}}$ denote $\mathcal{J}(\mathcal{O}_E)$. Given any such character $\lambda: \mathcal{J}(\mathcal{O}_E) \to \mathbb{Q}_p$, the image of $\lambda$ generates a subfield $K_{\lambda} \subset \mathbb{Q}_p$ finite over $\mathbb{Q}_p$.

For an $E$-affinoid algebra $R$, we say a continuous group homomorphism $\Gamma \to R$ (resp. $\Gamma \to R^\times$) locally $E$-analytic if the underlying function in $C^\infty(\Gamma_0, R)$ belongs to the image of $C^\infty(\Gamma, R)$. For a rigid space $X$ over $K$, we say a continuous group homomorphism $\Gamma \to \mathcal{J}(X)$ (resp. $\Gamma \to \mathcal{J}(X)^\times$) locally $E$-analytic if for all affinoid subdomains $U$ of $X$, the induced map is locally $E$-analytic. We use $\text{Hom}_{\mathbb{Q}_p}^E(\Gamma, \mathcal{J}(X)^\times)$ to denote the subgroup of locally $E$-analytic characters. Thus we get a subfunctor of $\mathcal{J}_f$. In the case that $\Gamma$ is $\mathcal{O}_E$, Schneider and Teitelbaum constructed a solution variety using a generalization of Amice’s $p$-adic Fourier transform. We prove this subfunctor is (relatively) represented by a closed Zariski subspace $\mathcal{J}_{\mathcal{O}}^\times$ and compare our results with results in \cite{ST01}.

\cite{ST01} works in the setting that $R$ is a complete field over $E$ and contained in $\mathbb{C}_p$. $\sigma$ was not needed.

Suppose $\mathcal{O}_E = \bigoplus_{i=1}^{[E: \mathbb{Q}_p]} \mathbb{Z}_p u_i$ for $u_i \in \mathcal{O}_E$, $1 \leq i \leq [E: \mathbb{Q}_p]$. We use $R^{n_0}$ to denote the power series ring in $n_0$ variables over $\mathbb{Z}_p$. Thus

\[
\log : R^{n_0} \to \mathbb{R} \quad r \mapsto \log(1 + r).
\]

**Lemma 3.1.** Let $\chi: \mathcal{O}_E \to R^\times$ be a continuous character. Then the followings are equivalent:

1. $\chi$ is locally $E$-analytic.
2. $\chi|_{x^n \mathcal{O}_E}$ is $E$-analytic for certain $n \in \mathbb{Z}_{\geq 0}$, i.e, this function is given by a function in the Tate algebra for $x^n \mathcal{O}_E$.
3. $\chi$ satisfies a set of “Cauchy-Riemann” equations, i.e.,

\[
\frac{\log(\chi(u_1))}{u_1} = \cdots = \frac{\log(\chi(u_i))}{u_i} = \cdots = \frac{\log(\chi(u_{[E: \mathbb{Q}_p]}))}{u_{[E: \mathbb{Q}_p]}}.
\]

**Proof.** The equivalence of the first two claims follows from the definition of $C^\infty(\mathcal{O}_E, R)$. By \cite[Lem 1]{Buz04}, $\chi(a) - 1$ is topologically nilpotent for any $a \in \mathcal{O}_E$, the log map makes sense for $\chi(a)$.

If $\chi$ is locally $E$-analytic, there exists $n_1 \in \mathbb{Z}_{\geq 0}$ such that $\chi|_{p^n \mathcal{O}_E}$ is $E$-analytic and is represented by an analytic function $f(x)$, $x$ as a variable for $\mathcal{O}_E \simeq p^n \mathcal{O}_E$. $f(0) = 1 \in R$ and $|f(x) - 1|_{R(x)} < \infty$ with respect to the norm of $R(x)$. Thus there exists $n_2 \in \mathbb{Z}_{\geq 0}$ such that $|f(x) - 1|_{R(p^{n_2} \mathcal{O}_E)}$ is sufficiently small (say, $< \frac{1}{p}$) with respect to the norm of $R(p^{n_2} \mathcal{O}_E)$. Hence $\log \circ \chi|_{p^{n_1+n_2} \mathcal{O}_E} : \mathcal{O}_E \to R$ is an $E$-analytic group homomorphism. It is direct to show that if $g(x) \in R(x)$ such that
\[ g(x + y) = g(x) + g(y) \text{ for any } x, y \in \mathcal{O}_E, \text{ then } g \text{ must be linear. } \log \circ \chi \big|_{p^{n_1+n_2}\mathcal{O}_E} \text{ is given by } g(x) = cx \text{ for certain } c \in R \text{ (} c \in 1 + R^{\infty} \text{ if } n_2 \text{ is sufficiently large). This means that } \frac{\log(x(p^{n_1+n_2}u_i))}{u_i} = p^{n_1+n_2}\frac{\log(\chi(u_i))}{u_i} \text{ is independent of } i \text{ (} 1 \leq i \leq [E : \mathbb{Q}_p] \text{) for sufficiently large } n_1 + n_2. \]

For any continuous character \( \chi \) satisfying the equations above, it suffices to restrict to \( p^n\mathcal{O}_E \) such that the image of \( \chi \) is contained in a sufficiently small neighbourhood of 1 for which \( \exp \circ \log \) makes sense and equals to the identity map. There exists \( r_i \in R^{\infty} \) for each \( 1 \leq i \leq [E : \mathbb{Q}_p] \) such that \( \chi(p^n u_i) = (1 + r_i)^{u_i} \), then the equations force \( r_i \) to all be equal to \( r \) for certain small enough \( r \). \( \chi(p^nx) = (1 + r)^x \) for any \( x \in \mathcal{O}_E \). And since \( r \) is sufficiently small, the coefficients \( \frac{r^n}{n!} \) go to 0, for which \( (1 + r)^x \) defines a rigid function on \( p^n\mathcal{O}_E \).

**Theorem 3.2.** The subfunctor associating to any \( E \)-rigid space \( X \)
\[ X \mapsto \text{Hom}_{\text{cts}}^E(\Gamma, \mathcal{O}(X)^\times) \subseteq \text{Hom}_{\text{cts}}(\Gamma, \mathcal{O}(X)^\times) \]
is relatively represented by a Zariski closed subgroup object \( \mathcal{W}_E \hookrightarrow \mathcal{W}_\Gamma \) in the category of \( E \)-rigid spaces.

**Proof.** By assumption \( \mathcal{O}_E^d \subseteq \Gamma \) as a open subgroup. By \[ \text{Buz04} \text{ Lem 2, (iv)}, \]
the natural transform corresponding to
\[ \text{Hom}_{\text{cts}}(\Gamma, \mathcal{O}(X)^\times) \to \text{Hom}_{\text{cts}}(\mathcal{O}_E^d, \mathcal{O}(X)^\times) \]
by restriction is represented by a finite étale map \( \mathcal{W}_\Gamma \to \mathcal{W}_{\mathcal{O}_E^d} \). Once \( \mathcal{W}_{\mathcal{O}_E^d} \hookrightarrow \mathcal{W}_{\mathcal{O}_E} \)
is constructed, we make \( \mathcal{W}_E \) as the fibre product \( \mathcal{W}_\Gamma \times_{\mathcal{W}_{\mathcal{O}_E^d}} \mathcal{W}_{\mathcal{O}_E} \to \mathcal{W}_\Gamma \) in the sense of rigid spaces, see for example, \[ \text{BGRSA} \text{ §9.3.5} \]. We obtain the universal character of \( \Gamma \) by composing
\[ \overline{\chi} : \Gamma \to \mathcal{O}(\mathcal{W}_\Gamma)^\times \to \mathcal{O}(\mathcal{W}_E)^\times. \]
We verify this immersion is the desired one. For any morphisms of \( E \)-rigid spaces \( X \to \mathcal{W}_E \to \mathcal{W}_E \), we have the commutative diagram
\[ \begin{CD}
\Gamma @>>> \mathcal{O}(\mathcal{W}_\Gamma)^\times @>>> \mathcal{O}(\mathcal{W}_E)^\times @>>> \mathcal{O}(X)^\times \\
\mathcal{O}_E^d @>>> \mathcal{O}(\mathcal{O}_E^d)^\times @>>> \mathcal{O}(\mathcal{W}_E)^\times \\
\end{CD} \]
\( \Gamma \to \mathcal{O}(X)^\times \) is locally \( E \)-analytic restricted to \( \mathcal{O}_E^d \) by representability of \( \mathcal{W}_{\mathcal{O}_E^d} \), hence locally \( E \)-analytic. On the other hand, for and \( E \)-rigid space \( X \) any locally \( E \)-analytic character \( \chi_X : \Gamma \to \mathcal{O}(X)^\times \), \( \chi_X \) is continuous and induced by a certain map \( X \to \mathcal{W}_\Gamma \). It is also locally \( E \)-analytic restricted to \( \mathcal{O}_E^d \), which gives another map \( X \to \mathcal{W}_{\mathcal{O}_E} \), hence induced by a map \( X \to \mathcal{W}_E \) from the universal property of fibre products. The fact that going from morphisms to characters are inverse to each other follows from the same statement for continuous characters of \( \Gamma \) and \( \mathcal{W}_\Gamma \).

For construction of \( \mathcal{W}_{\mathcal{O}_E^d} \hookrightarrow \mathcal{W}_{\mathcal{O}_E} \), it suffices to consider \( d = 1 \) as the immersion is really \( d \)-fold product of \( \mathcal{W}_{\mathcal{O}_E^d} \hookrightarrow \mathcal{W}_{\mathcal{O}_E} \). \( \mathcal{W}_{\mathcal{O}_E} \) is a \( [E : \mathbb{Q}_p] \)-fold product of open unit disks. For any affinoid subdomain \( U \) in \( \mathcal{W}_{\mathcal{O}_E} \), the “Cauchy-Riemann” equations in \[ \text{Lem}[\xi] \]
define rigid analytic functions on \( U \) since coefficients of log function going to infinity with a smaller rate of any positive linear growth and hence cut out a Zariski closed space \( U^E \). \( \mathcal{W}_{\mathcal{O}_E} \hookrightarrow \mathcal{W}_{\mathcal{O}_E} \) is constructed by gluing \( U^E \) for all affinoid
subdomain $U$ in $\mathcal{O}_E$. Then everything is reduced to Lem \ref{lem:factorization}. The fact that $\mathcal{F}_E$ is a subgroup object follows from that subfunctor is valued in the category of groups as well.

\begin{remark}
Our variety for $\mathcal{O}_E$ produces the same set of $R$ points when $R$ is a finite extension field of $E$ as the reduced variety constructed in \cite{ST01} does. Although our “Cauchy-Riemann” equations seem slightly different from theirs as they use infinitesimal “Cauchy-Riemann” differential equations.
\end{remark}

Let $K/E$ be a sufficiently large extension such that $K$ contains all Galois conjugates of $E$, with a fixed embedding $K \hookrightarrow \mathbb{Q}_p$. We make a base change for $\mathcal{F}_\Gamma$ from $\mathbb{Q}_p$ to $K$. By \cite{Buz07} Lem 8.2. (c)], $(\mathcal{F}_\Gamma)/K$ represents the functor on rigid spaces over $K$ sending a $K$-rigid space $X$ to the set of continuous group homomorphisms $\Gamma \to \mathcal{O}(X)_p^\times$, for which we still denote as $\mathcal{F}_\Gamma$. For any embedding $\sigma : E \hookrightarrow K \hookrightarrow \overline{\mathbb{Q}}_p$, it induces a closed embedding of

$$C^{an}(\Gamma, K) \hookrightarrow C^{an}(\Gamma_0, K)$$

by \cite{ST01} Lem 1.1]. In general for a $K$-affinoid algebra $R$, $\sigma$ induces an embedding

$$\sigma : C^{an}(\Gamma, R) \rightarrow C^{an}(\Gamma_0, R).$$

One can similarly define locally $(\sigma, E)$-analytic group homomorphisms (resp. characters) of $\Gamma$ valued in rigid functions of any rigid spaces over $E$. We use $\text{Hom}^{an}_E(\Gamma, \mathcal{O}(X)^\times)$ to denote the subgroup of locally $(\sigma, E)$-analytic characters. By Thm \ref{thm:homogeneity}, the subfunctor associating to any $K$-rigid space $X$

$$X \mapsto \text{Hom}^{an}_E(\Gamma, \mathcal{O}(X)^\times) \subseteq \text{Hom}_{cts}(\Gamma, \mathcal{O}(X)^\times)$$

is represented by $\mathcal{F}^{an}_\Gamma \hookrightarrow \mathcal{F}_\Gamma$ over $K$. The logarithmic functions again define rigid analytic functions on $\mathcal{F}_\Gamma$. And they cut out zero-dimensional Zariski subspaces $\mathcal{F}^{an}_E \rightarrow \mathcal{F}^{an}_\Gamma \rightarrow \mathcal{F}_\Gamma$ for any $\sigma : E \hookrightarrow K$ whose points correspond to smooth characters of $\Gamma$.

\begin{lemma}
$R$ is a $K$-affinoid algebra. Let $\chi : \mathcal{O}_E \rightarrow R^\times$ be a continuous character. There exists a constant $c_E$ depending only on $E$ such that if $|\chi(a)| < c_E$ for all $a \in \mathcal{O}_E$, then $\chi$ has a factorization $\chi = \prod_{\sigma : E \hookrightarrow K} \chi^\sigma$ such that $\chi^\sigma$ is locally $(\sigma, E)$-analytic for any $\sigma : E \hookrightarrow K$. Each $\chi^\sigma$ is unique up to a smooth character for the factorization.
\end{lemma}

\begin{proof}
Suppose $\chi(u_i) = 1 + r_i$ for all $1 \leq i \leq [E : \mathbb{Q}_p]$ and the set of embedding from $E \hookrightarrow K$ is $\{\sigma_1, \ldots, \sigma_{[E : \mathbb{Q}_p]}\}$. For $z_1, \ldots, z_{[E : \mathbb{Q}_p]} \in \mathbb{Z}_p$,

$$z = u_1 \cdot z_1 + \cdots + u_{[E : \mathbb{Q}_p]} \cdot z_{[E : \mathbb{Q}_p]} \in \mathcal{O}_E,$$

$$\sigma_i z = \sigma_i(u_1) \cdot z_1 + \cdots + \sigma_i(u_{[E : \mathbb{Q}_p]}) \cdot z_{[E : \mathbb{Q}_p]}.$$ 

Let $M := \begin{pmatrix} \sigma_1 u_1 & \sigma_1 u_{[E : \mathbb{Q}_p]} \\ \vdots & \vdots \\ \sigma_{[E : \mathbb{Q}_p]} u_1 & \sigma_{[E : \mathbb{Q}_p]} u_{[E : \mathbb{Q}_p]} \end{pmatrix}$, $c_E := |\text{det}(M)|p^{-1}$. Let $(p_1, \ldots, p_{[E : \mathbb{Q}_p]}) := (\log(1 + r_1), \ldots, \log(1 + r_{[E : \mathbb{Q}_p]})) \cdot M^{-1}$, then $|p_i| < p^{-1}$, $|\exp(p_i) - 1| < p^{-1}$ for any $1 \leq i \leq [E : \mathbb{Q}_p]$. Define $\chi^\sigma(z) := \exp(p_i)^{\sigma_i z}$ for any $z \in \mathcal{O}_E$, whose underlying functions are rigid functions (convergent power series) on $\mathcal{O}_E$ via $\sigma_i$ because of $|\exp(p_i) - 1| < p^{-1}$, $1 \leq i \leq [E : \mathbb{Q}_p]$.
Since $|r_1| < p^{-\frac{1}{r_1}}$, $\exp \circ \log(1 + r_1) = 1 + r_1$. Applying exponential function to both sides of 

$$ (p_1, \cdots, p_{[E: \mathbb{Q}_p]}) \cdot M = (\log(1 + r_1), \cdots, \log(1 + r_{[E: \mathbb{Q}_p]})) $$

shows values of $\chi$ and $\prod_{1 \leq i \leq [E: \mathbb{Q}_p]} \chi^{\sigma_i}$ coincide on $\{u_1, \cdots, u_{[E: \mathbb{Q}_p]}\}$, hence they are equal.

Uniqueness suffices to show that if $\prod_{1 \leq i \leq [E: \mathbb{Q}_p]} \chi^{\sigma_i} = 1$ for locally $(\sigma_i, E)$-analytic characters, then each $\chi^{\sigma_i}$ is smooth for $1 \leq i \leq [E: \mathbb{Q}_p]$. By passing to a small open subgroup, we may assume each $\chi^{\sigma_i}$ is $(\sigma_i, E)$-analytic and whose image is contained in a small enough neighbourhood of 1. By reasonings in the second paragraph of proof of Lem 3.1, $\log \circ \chi^{\sigma_i}$ is linear and represented by $c_i \sigma_i(z)$. Then $\sum_{i=1} c_i \sigma_i(z) = 0$ for any $z \in \mathcal{O}_E \Rightarrow c_i = 0$, implying smoothness of $\chi^{\sigma_i}$ for all $1 \leq i \leq [E: \mathbb{Q}_p]$.

\[ \square \]

**Corollary 3.5.** $R$ is a $K$-affinoid algebra. Let $\chi : \Gamma \to R^\times$ be a continuous character. Then there exists an open subgroup $\Gamma_0 \simeq \mathcal{O}_E^K$ such that $\chi|_{\Gamma_0}$ admits a unique decomposition

$$ \chi|_{\Gamma_0} = \prod_{\sigma : E \to K} \chi^\sigma_0 $$

of $(\sigma, E)$-analytic characters $\chi^\sigma : \mathcal{O}_E^K \simeq \Gamma_0 \to R^\times$, i.e., each $\chi^\sigma_0$ is represented by a rigid analytic function on $\mathcal{O}_E$ via $\sigma$.

**Proof.** Everything is reduced to Lem 3.4 except for the uniqueness. The uniqueness follows from the fact that as a $K$-affinoid algebra, $R$ only contains a finite number of roots of unity. Hence if $\Gamma_0$ is small enough, any smooth character of $\Gamma_0$ lifting to a character of $\Gamma$ must be trivial. \[ \square \]

**Theorem 3.6.** The composition of the maps using that $\mathcal{W}_\Gamma$ is a group object

$$ \prod_{\sigma} \prod_{\sigma : E \to K} \mathcal{W}_\Gamma^\sigma \hookrightarrow \prod_{\sigma : E \to K} \mathcal{W}_\Gamma^m \rightarrow \mathcal{W}_\Gamma $$

is an isogeny, i.e., $\prod \sigma$ is surjective with the kernel being a zero-dimensional Zariski closed subspace. Moreover, if we denote the kernel as $K_\Gamma$, then $K_\Gamma \hookrightarrow \prod_{\sigma : E \to K} \mathcal{W}_\Gamma^m$.

**Proof.** As in the proof of Thm 3.2 we have the pullback diagram

$$ K_\Gamma \quad \prod_{\sigma : E \to K} \mathcal{W}_\Gamma^\sigma \quad \prod_{\sigma : E \to K} \mathcal{W}_\Gamma^m \quad \mathcal{W}_\Gamma $$

and the bottom line is a $d$-fold product. It suffices to prove the case $\Gamma = \mathcal{O}_E$. For surjectivity, it boils down to show that for any $K$-affinoid algebra $R$ any a continuous character $\chi : \mathcal{O}_E \to R^\times$, $\chi = \prod_{\sigma : E \to K} \chi^\sigma$ for locally $(\sigma, E)$-analytic characters $\chi^\sigma$. $\mathcal{W}_{\mathcal{O}_E} \to \mathcal{W}_{\mathcal{O}_E}$ (resp. $\mathcal{W}_{\mathcal{O}_E}^m \to \mathcal{W}_{\mathcal{O}_E}^{m\sigma}$) is surjective since for any
$K$-affinoid algebra $R$ and a topologically nilpotent element $r \in R$, $(1 + r)^{1/2} - 1$ are again topologically nilpotent (one may need to enlarge the ring $R$ to include this element). If we prove this factorization exists when $\chi$ restricts to some $p^n\mathcal{O}_E$, $n \in \mathbb{Z}_{\geq 0}$, namely, $\chi|_{p^n\mathcal{O}_E} = \prod_{\sigma:E \to K} \chi^\sigma_{n}$, we pick $\chi^\sigma_{n} : \mathcal{O}_E \to R^\times$ lifting each $\chi^\sigma_{n}$, then $\chi^{-1} \prod_{\sigma:E \to K} \chi^\sigma_{n}$ is smooth, hence locally $(\sigma, E)$-analytic for any $\sigma$. By passing to a sufficiently small open subgroup as we do in Lem 3.1 we can assume the image of $\chi$ is contained in a small enough neighbourhood of 1 in $R$. Then everything is reduced to Lem 3.1.

Given an affinoid subdomain $\Omega \subset \mathcal{W}_T$, we write $\chi_\Omega : \mathcal{T}(\mathcal{O}_E) \to \mathcal{O}(\Omega)^\times$ for the universal (continuous) character it determines. In the case $T$ comes from a maximal torus of a reductive group $H$ with the relative Weyl group $W(H, S)$, we make the following notation. For a general weight valued in an affinoid algebra $R$,

\[ \lambda : \mathcal{T}(\mathcal{O}_E) \to R^\times, \lambda^w(t) := \lambda(w^{-1}tw), \]

\[ w \cdot \lambda(t) := \lambda^w(t) \cdot (w^\delta - \delta)(t) \]

for $w \in W(H, S)$ and all $t \in \mathcal{T}(\mathcal{O}_E)$.

4. Locally analytic distributions

Consider a reductive group $H$ over a $p$-adic local field $E$ along with various subgroups and an Iwahori subgroup $I$ defined in 2. Let $K/E$ be an unramified quadratic extension.

Suppose for each $s \geq 1$, there exists and we fix an analytic isomorphism $\psi^\flat_{s,K} : \mathcal{O}_E^{d_s} \times \mathcal{O}_K^{d_s} \simeq \mathcal{N}_I^{s}$, $d = d_e + 2d_k = \dim \mathcal{N}_I$ such that each coordinate corresponds to a negative root in $\Delta^-(H, S)$. An explicit analytic (or strict analytic in the sense of [ANOS §3.3]) structure for relevant groups will be given in 7. Namely, let

\[ x = \left( (x_E^1, \cdots, x_E^{d_E}), (x_K^1, \cdots, x_K^{d_K}) \right) \in \mathcal{O}_E^{d_E} \times \mathcal{O}_K^{d_K}, \]

for any $t \in \mathcal{T}(\mathcal{O}_E)$ and characters $\{ \delta^E_e, \cdots, \delta^{K-}_e, \delta^E_1, \cdots, \delta^{K-}_1 \}$,

\[ t \cdot x = tx^{-1}t = \left( (\delta^E_1(t)x_E^1, \cdots, \delta^{E-}_e(t)x_E^{d_E}), (\delta^{K-}_1(t)x_K^1, \cdots, \delta^{K-}_e(t)x_K^{d_K}) \right). \]

Choose an isomorphism $\tau_{K/E} : \mathcal{O}_K \simeq \mathcal{O}_E^{d_s}$, which amounts to an isomorphism $\psi^\sharp_{s,K} := \psi^\flat_{s,K} \circ (\text{id}, \tau_{K/E}^{-1}) : \mathcal{O}_E^{d_s} \simeq \mathcal{N}_I^{s}$.

We furthermore assume that the transition function between $\psi^\flat_I$ and $\psi^\sharp_{s}$ is given by

\[ (\psi^\flat_I)^{-1} \circ \psi^\sharp_{s} : \mathcal{O}_E^{d_s} \simeq \mathcal{N}_I^{s} \]

\[ (x_1, \cdots, x_d) \mapsto (\omega^{s-1}x_1, \cdots, \omega^{s-1}x_d). \]

We will verify this key assumption for our interested groups in 7.

We choose an isomorphism $\tau_E : \mathcal{O}_E \simeq \mathbb{Z}_p^{[E:Q_p]}$, which leads to an isomorphism $\psi^\sharp_{I,p} := \psi^\flat_{I,p} \circ (\tau_{E})^{-1} : \mathbb{Z}_p^{d_s} \simeq \mathcal{N}_I^{s}$.

\[ \text{One can certainly generalise this assumption. However, to save notation and words we decide not to do so. Also, the most complicated case of considerations will be the quasi-split unitary group for an unramified extension $K/E$ in 7 which satisfies the assumption.} \]
For example, if $H$ is a symplectic group over $E$, 
\[ d_k = 0, d = d_e = \dim \mathcal{N}_B = n^2. \]
If $H$ is a quasi-split unitary group with the skew anti-diagonal Hermitian form $\langle \cdot, \cdot \rangle$, 
\[ d_e = n, d_k = n(n - 1), d = d_e + 2d_k = \dim \mathcal{N}_B = n(2n - 1). \]

For Iwahori subgroup $I$ of a finite product of $p$-adic groups $\prod_{1 \leq i \leq l} H_i(E_i)$ considered in [2], we fix isomorphisms $\psi^l_{i,E_i} : \mathcal{O}^l_{E_i} \simeq \mathcal{N}_{I,i}$ with respect to root decomposition coordinates as well as $\iota_{E_i} : \mathcal{O}_{E_i} \simeq \mathbb{Z}_{p[E_i; \mathbb{Q}_p]}$, $\psi^l_{i,\mathbb{Q}_p} : \psi^l_{i,E_i} \circ \iota^{-1}_{E_i} : \mathbb{Z}_{p[E_i; \mathbb{Q}_p]} \simeq \mathcal{O}_{E_i}$ for $1 \leq i \leq l$ and $\psi^l_{i,p} : \prod_{1 \leq i \leq l} \psi^l_{i,\mathbb{Q}_p} \circ \iota^{-1}_{E_i} : \mathbb{Z}_{p} \simeq \mathcal{N}_{I}$ for $d = \sum_{1 \leq i \leq l} d_i[E_i : \mathbb{Q}_p]$ as above.

**Definition 4.1.** If $R$ is any $\mathbb{Q}_p$-Banach algebra and $s$ is a positive integer, the module $C^{s,an}(\mathcal{N}_I, R)$ of $s$-locally analytic $R$-valued functions on $\mathcal{N}_I$ is the $R$-module of continuous functions $f : \mathcal{N}_I \to R$ such that 
\[ f(x, \psi^s_{i,p}(z_1, \ldots, z_d)) : \mathbb{Z}_p^d \to R \]
is given by an element of the $m$-variables Tate algebra $T^l_R = R(z_1, \ldots, z_d)$ for any fixed $x \in \mathcal{N}_I$.

Let $\| \cdot \|_{T^l_R}$ denote the Gauss norm on the Tate algebra, the norm $\|f(x, \psi^s_{i,p})\|_{T^l_R}$ depends only on the image of $x$ in $\mathcal{N}_I^{l}/\mathcal{N}_I$, and the formula 
\[ \|f\|_{a} = \sup_{x \in \mathcal{N}_I} \|f(x, \psi^s_{i,p})\|_{T^l_R} \]
defines a Banach $R$-module structure on $C^{s,an}(\mathcal{N}_I, R)$, with respect to which the canonical inclusion $C^{s,an}(\mathcal{N}_I, R) \subset C^{s+1,an}(\mathcal{N}_I, R)$ is compact. For a proof of compactness, see for example, [Urb1] Lem 3.2.2.

For each $s \geq 1$ and $w \in \prod_{1 \leq i \leq l} W^P(H_i, S_i) = W^P(H, S)$, we define 
\[ \psi^s_{w,K} := \prod_{1 \leq i \leq l} \psi^s_{w_i,1,w_i^{-1},K_i} : \prod_{1 \leq i \leq l} \mathcal{O}^l_{E_i} \times \mathcal{O}^l_{K_i} \simeq w^{-1}\mathcal{N}_I^{l}, \]
\[ \psi^s_{w} := \prod_{1 \leq i \leq l} \psi^s_{w_i,1,w_i^{-1}} : \prod_{1 \leq i \leq l} \mathcal{O}_{E_i} \simeq w^{-1}\mathcal{N}_I^{l}, \]
\[ \psi^s_{w,p} := \psi^s_{w_1w^{-1},p} : \mathbb{Z}_p^d \simeq w^{-1}\mathcal{N}_I^{l}. \]
This is just the case where the usual Borel $B$ is replaced by $wBw^{-1}$. We have a parallel definition as above.

**Definition 4.2.** If $R$ is any $\mathbb{Q}_p$-Banach algebra and $s$ is a positive integer, the module $C^{s,an}(w\mathcal{N}_Iw^{-1}, R)$ of $s$-locally analytic $R$-valued functions on $w\mathcal{N}_Iw^{-1}$ is the $R$-module of continuous functions $f : w\mathcal{N}_Iw^{-1} \to R$ such that 
\[ f(x, \psi^s_{w,p}(z_1, \ldots, z_d)) : \mathbb{Z}_p^d \to R \]
is given by an element of the $d$-variables Tate algebra $T^l_R = R(z_1, \ldots, z_d)$ for any fixed $x \in w\mathcal{N}_Iw^{-1}$. 
Similarly we use \( \| \cdot \|_{T^w_{I,R}} \) to denote the Gauss norm on \( T^w_{I,R} \), and the norm
\[
\| f \|_s = \sup_{x \in w \mathcal{N}_{I,R}} \| f(x\psi_{w,p}) \|_{T^w_{I,R}}
\]
defines a Banach \( R \)-module structure on \( C^{s,an}(w\mathcal{N}_Iw^{-1}, R) \).

For the Iwahori subgroup \( I \) of \( \prod_{1 \leq i \leq l} H_i(E_i) \) and for a spherically complete field \( K \supset E_1, \ldots, E_l \) with respect to a nonarchimedean valuation extending the ones on \( E_1, \ldots, E_l, D(I, K) \) is the distribution algebra of the Iwahori group \( I \). Let \( \Omega \) be an irreducible Zariski closed subspace of an affinoid subdomain of \( \mathcal{H}_{T^0} = \prod_{1 \leq i \leq l} \mathcal{H}_{T^i} \).

For an analytic Banach \( \mathcal{O}([\Omega]) \)-module \( V \), where \( \mathcal{B}_I \) is the Borel subgroup of \( I \), \( T \) is the diagonal torus of \( I \) and \( N_I, \mathcal{N}_I \) are the corresponding unipotent and opposite unipotent of \( I \), we can define the locally analytic induction of \( V \) from \( \mathcal{B}_I \) to \( I \) as follows:
\[
\text{Ind}^I_{\mathcal{B}_I}(V) = \{ f : I \to V, f \text{ analytic on each } I^* \text{-coset, } f(gb) = b \cdot f(g) \forall b \in \mathcal{B}_I, g \in I \}.
\]

Recall \( A = \mathcal{O}(\Omega) \), for definition of \( V \)-valued analyticity, we mean \( f|_{I^*} \in C^{an}(i \cdot I^*, A) \hat{\otimes}_A V \) for every \( i \in I/I^* \) and \( C^{an}(i \cdot I^*, A) \) stands for the space of \( A \)-valued analytic functions on \( i \cdot I^* \), or in other words, copies of Tate algebras corresponding to \( i \cdot I^* \) as isomorphic to a finite copies of \( (\mathbb{Z}_p)^m \).

\[
\text{Ind}^I_{\mathcal{B}_I}(V) \cong C^{s,an}(\mathcal{N}_I, V) \cong C^{s,an}(\mathcal{N}_I, \mathcal{O}(\Omega)) \hat{\otimes}_{\mathcal{O}(\Omega)} V
\]
\[
f \mapsto f|_{\mathcal{N}_I},
\]
and we regard \( \text{Ind}^I_{\mathcal{B}_I}(V) \) as a Banach \( \mathcal{O}(\Omega) \)-module via pulling back the Banach module structure on \( C^{s,an}(\mathcal{N}_I, \mathcal{O}(\Omega)) \) under this isomorphism. The rule \( (f|_{\gamma})(g) = f(\gamma g) \) gives \( \text{Ind}^I_{\mathcal{B}_I}(V) \) the structure of a continuous right \( \mathcal{O}(\Omega)[I] \)-module.

We define the Banach dual, i.e., bounded \( \mathcal{O}(\Omega) \) linear functionals, as a Banach \( \mathcal{O}(\Omega)[\Delta^+_H] \)-module
\[
\mathcal{D}^I_{\mathcal{B}_I}(V) := \mathcal{L}_{\mathcal{O}(\Omega)}(\text{Ind}^I_{\mathcal{B}_I}(V), \mathcal{O}(\Omega)).
\]
The action of \( \mathcal{O}(\Omega)[\Delta^+_H] \) for \( \mathcal{D}^I_{\mathcal{B}_I}(V) \) is induced from that of \( \text{Ind}^I_{\mathcal{B}_I}(V) \). In particular, for a character \( \chi \) of \( T^0 \) valued in \( \mathcal{O}(\Omega)^\times \),
\[
\text{Ind}^I_{\mathcal{B}_I} \chi := \{ f : I \to \mathcal{O}(\Omega), f \text{ analytic on each } I^* \text{-coset, } f(gtn) = \chi(t)f(t) \forall n \in N_I, t \in T(\mathbb{Z}_p), g \in I \},
\]
\[
\mathcal{D}^I_{\mathcal{B}_I} \chi := \mathcal{L}_{\mathcal{O}(\Omega)}(\text{Ind}^I_{\mathcal{B}_I} \chi, \mathcal{O}(\Omega)).
\]
We define \( \chi_{\Omega} : T^0 \to \mathcal{O}(\Omega)^\times \) to be the universal (continuous) character of \( T^0 \) for \( \Omega \subset \mathcal{H}_{T^0} \). We define \( s[\Omega] \) as the minimal integer such that \( \chi_{\Omega}|_{T^0}^{[s]} \) satisfies Cor 33 (uniquely decomposes as a product of analytic characters) for all \( 1 \leq i \leq l \).

For any positive integer \( s \), the Weyl group element \( w \in W^F(H, T) \) and \( wIw^{-1} \), we make the following definitions.

\[
\text{Ind}^w_{\mathcal{B}_I} \chi := \text{Ind}^w_{\mathcal{B}_Iw^{-1}}(\chi) = \{ f : wIw^{-1} \to \mathcal{O}(\Omega), f \text{ analytic on each } wI^*w^{-1} \text{-coset, } f(gtn) = \chi(t)f(t) \forall n \in wN_Iw^{-1}, t \in T(\mathbb{Z}_p), g \in I \}.
\]
By the Iwahori decomposition for $wIw^{-1}$, restricting an element $f \in \text{Ind}^w_{w,\chi}$ to $wNfw^{-1}$ induces an isomorphism

$$\text{Ind}^w_{w,\chi} \simeq C^{\infty,an}(wNfw^{-1},\mathcal{O}(\Omega)) \quad f \mapsto f|_{wNfw^{-1}},$$

and we regard $\text{Ind}^w_{w,\chi}$ as a Banach $\mathcal{O}(\Omega)$-module via pulling back the Banach module structure on $C^{\infty,an}(wNfw^{-1},\mathcal{O}(\Omega))$ under this isomorphism. The rule $(f|\gamma)(g) = f(\gamma g)$ gives $\text{Ind}^w_{w,\chi}$ the structure of a continuous right $\mathcal{O}(\Omega)[wIw^{-1}]$-module. We define the Banach dual as a Banach $\mathcal{O}(\Omega)[wIw^{-1}]$-module

$$\mathcal{D}^\ast_{w,\chi} := \mathcal{L}_{\mathcal{O}(\Omega)}(\text{Ind}^w_{w,\chi},\mathcal{O}(\Omega)).$$

The action of $\mathcal{O}(\Omega)[\Delta_H^+]$ for $\mathcal{D}^\ast_{w,\chi}$ is induced from that of $\text{Ind}^w_{w,\chi}$. When $s \geq s[\Omega]$, we set $\text{Ind}^w_{w,\chi} := \text{Ind}^w_{w,\chi_{\Omega}}$ and $\mathcal{D}^\ast_{w,\Omega} := \mathcal{D}^\ast_{w,\chi_{\Omega}}$, where $\chi_{\Omega}(t) = \chi_{\Omega}(w^{-1}tw)$ for \(\forall t \in T(\mathcal{O}_E)\). In particular, we define $\mathcal{D}^\ast_{\Omega} := \mathcal{D}^\ast_{\text{id},\Omega}$.

5. $p$-adic Banach representations over $T_0$

In this section, $T_0$ is meant to be a $p$-adic compact torus. More specifically, $T_0 \subset (\mathcal{O}_E)^n$ is a closed subgroup for a $p$-adic local field $E$. In application we will set $T_0 \subset \prod_{1 \leq i \leq l} (\mathcal{O}_E)^{n_i}$ to be a compact open subgroup of finite index. For understanding the structure of the distribution modules better, we need to establish some general results on $T_0$-Banach representations.

Let $A$ be an affinoid algebra over finite extensions $K/\mathbb{Q}_p$, where $K$ contains all conjugates of $E$.

**Definition 5.1.** Let $\iota : S \subset \mathbb{Z}_{\geq 0}$ be a subset of $l$ copies of non-positive integers. For $\forall s \in S$, $V_s$ is a finite free $A$-module with unitary Banach representation structure of $T_0$, with norm $| \cdot |_s$. Moreover, the $T_0$ action on $V_s$ is via a locally analytic character $\chi_s : T_0 \rightarrow A^\times$. For different $s$, $\chi_s$ are different.

$$(\prod_{s \in S} V_s)^c := \{ v \in \prod_{s \in S} V_s \mid \forall \varepsilon > 0, \exists N > 0, \text{ s.t. } |(v)s|_s \leq \varepsilon \text{ if } \max_{1 \leq j \leq l} \iota(s)_j > N \}$$

$$(\prod_{s \in S} V_s)^b := \{ v \in \prod_{s \in S} V_s \mid \max_{s \in S} |(v)s|_s < \infty \}$$

We equip both with the norm $| \cdot | := \max_{s \in S} | \cdot |_s$. Thus $(\prod_{s \in S} V_s)^c$ and $(\prod_{s \in S} V_s)^b$ are unitary Banach representations of $T_0$ over $A$ and $V_s \hookrightarrow (\prod_{s \in S} V_s)^c, V_s \hookrightarrow (\prod_{s \in S} V_s)^b$ as the $\chi_s$ isotypic part of $(\prod_{s \in S} V_s)^c, (\prod_{s \in S} V_s)^b$. We call a $T_0$ Banach $A$-module convergent eigen orthonormalizable if it is isomorphic to some $(\prod_{s \in S} V_s)^c$, bounded eigen orthonormalizable if it is isomorphic to some $(\prod_{s \in S} V_s)^b$. 

Lemma 5.3. \((\prod_{s \in S} V_s)^c \hookrightarrow (\prod_{s \in S} V_s)^b\) as a closed subunitary \(T_0\)-Banach representation over \(A\) with the induced subnorm. The latter notion is dual to the first one, the exact meaning will be clear shortly afterwards. In the remaining section, \((\prod_{s \in S} V_s)^c\) is convergent eigen orthonormalizable and \((\prod_{s \in S} V_s)^b\) is bounded eigen orthonormalizable.

For two Banach \(A\)-modules \(M, N\), the \(A\)-module \(\mathcal{L}_A(M, N)\) of continuous \(A\)-linear homomorphisms from \(M\) to \(N\) is then also a Banach \(A\)-module. For a Banach \(A\)-module \(M\), we use \(M^*\) to denote \(\mathcal{L}(M, A)\), the Banach dual of \(M\).

For a given \((\prod_{s \in S} V_s)^c\) and assignment \(s \mapsto V_s\) for \(s \in S\), we define the bounded (resp. convergent) eigen orthonormalizable Banach representation \((\prod_{s \in S} V_s^*)^b\) (resp. \((\prod_{s \in S} V_s^*)^c\)) constructed above in Def 5.1 to be construction associated to the dual assignment \(s \mapsto V_s^*\) for \(s \in S\).

**Lemma 5.3.** \((\prod_{s \in S} V_s^*)^b \simeq \mathcal{L}_A((\prod_{s \in S} V_s)^c, A)\) as \(T_0\)-unitary Banach representation.

**Proof.** It is a standard fact in nonarchimedean functional analysis that the Banach dual of \(c_0\) is \(l^\infty\), where \(c_0\) is the space of sequences converging to 0 and \(l^\infty\) is the space of bounded sequences. Apply this fact to our \((\prod_{s \in S} V_s)^c\) and \((\prod_{s \in S} V_s)^b\) as \(c_0\) and \(l^\infty\). \(\square\)

**Remark 5.4.** In \((\prod_{s \in S} V_s)^b\), each \(s \mapsto V_s^*\) for the assignment with the natural dual norm. The dual of \((\prod_{s \in S} V_s)^b\) is not of the form \((\prod_{s \in S} V_s^*)^c\) as we know for Banach spaces over spherically complete fields, infinite dimensional Banach spaces are never reflexive.

In applications, \((\prod_{s \in S} V_s)^c\) is the \(s\)-analytic induction up to tensoring a finite dimensional vector space. And \((\prod_{s \in S} V_s)^b\) is the \(s\)-distribution dual to the \(s\)-analytic induction up to tensoring a finite dimensional vector space. We have the following simple statement.

**Lemma 5.5.** Assume \(A\) is an integral domain. If \(v \in (\prod_{s \in S} V_s)^b\) (resp. \((\prod_{s \in S} V_s)^c\)) is an eigenvector for the \(T_0\)-action, then \(v \in V_s\) for some \(s \in S\), the action is given by \(\chi_s\).

**Proof.** \(v = (v_s).\) If \(v_s \in V_s, v_{s'} \in V_{s'}\) are both non-zero for \(s \neq s', t \cdot v = \chi_s(t) \cdot v\) for any \(t \in T_0\) and a character \(\chi_v : T_0 \to A^s\). Thus

\[
(\chi_s(t) - \chi_{s'}(t)) \cdot v_s = (\chi_{s'}(t) - \chi_v(t)) \cdot v_{s'} = 0
\]

for any \(t \in T_0\). Since \(A\) is an integral domain and \(v_s, v_{s'}\) are non-zero, \(\chi_s = \chi_v = \chi_{s'}\), which is a contradiction. \(\square\)
Definition 5.6. We call a continuous linear map \( f : (\prod_{s \in S} V_s)^b \to (\prod_{s' \in S'} V_{s'})^b \) nice between two bounded eigen orthonormalizable \( A \)-modules if for any \( \hat{v} = (\hat{v}_s) \in (\prod_{s \in S} V_s)^b, \sum_{s \in S} (f|_{V_s}(\hat{v}_s))_{s'} \) converges for all \( s' \in S' \) and moreover we have
\[
(f(\hat{v}))_{s'} = \sum_{s \in S} (f|_{V_s}(\hat{v}_s))_{s'}, \text{for } f(\hat{v}) \in (\prod_{s' \in S'} V_{s'})^b, \forall s' \in S'.
\]

We will show that the differentials in the Chevalley–Eilenberg complex and Iwahori group action for \( \mathbb{D}_w^1 \) are nice.

Lemma 5.7. Let \( f : (\prod_{s_1 \in S_1} V_{s_1})^c \to (\prod_{s_2 \in S_2} V_{s_2})^c \) be a \( A \)-linear continuous map between two convergent eigen orthonormalizable \( A \)-modules as in Def 5.4. Then the dual \( A \)-linear map \( f^* : (\prod_{s_2 \in S_2} V_{s_2})^b \to (\prod_{s_1 \in S_1} V_{s_1})^b \) of \( f \) between bounded eigen orthonormalizable \( A \)-modules is nice.

Proof. Let \( \iota_1 : S_1 \subset \mathbb{Z}_{\leq 0}^1 \), \( \iota_2 : S_2 \subset \mathbb{Z}_{\leq 0}^2 \). For any finite free Banach \( A \)-module \( M \) with basis \( e_1, \ldots, e_m \), norm on \( M \) is equivalent to the norm with orthonormalizable basis \( e_1, \ldots, e_m \). For any \( s_1 \in S_1 \), we can assume that \( V_{s_1} \) has orthonormalizable basis \( e_{s_1}^1, \ldots, e_{s_1}^{m_{s_1}} \). Pick any \( \hat{v} = (\hat{v}_{s_2}) \in (\prod_{s_2 \in S_2} V_{s_2})^b, (f^*|_{V_{s_2}}(\hat{v}_{s_2}))_{s_1} \to 0 \) as \( \iota_2(s_2) \to \infty \) since for any \( \varepsilon > 0 \), there exists \( N_2 > 0 \) such that
\[
|((f^*|_{V_{s_2}}(\hat{v}_{s_2}))_{s_1}(e_{s_1}^i))| = |\hat{v}_{s_2}(f(e_{s_1}^i))_{s_2}| < \varepsilon
\]
whenever \( \iota_2(s_2) > N_2, 1 \leq i \leq m_{s_1} \), that means \( |(f^*|_{V_{s_2}}(\hat{v}_{s_2}))_{s_1}(v_{s_1})| < \varepsilon \) for all \( v_{s_1} \in V_{s_1} \), such that \( |v_{s_1}|_1 \leq 1 \) and \( |\iota_2(s_2)| > N_2 \). Hence \( \sum_{s_2 \in S_2} (f^*|_{V_{s_2}}(\hat{v}_{s_2}))_{s_1} \) exists and the reasoning above also shows that \( f^*(\check{v}_2)_{s_1} = \sum_{s_2 \in S_2} (f^*|_{V_{s_2}}(\hat{v}_{s_2}))_{s_1} \) for all \( s_1 \in S_1 \).

Lemma 5.8. Suppose \( T \) is an abelian group with a finite index subgroup \( T_0 \). Let \( (\prod_{s \in S} V_s)^c \) be convergent eigen orthonormalizable. Moreover, \( T \) acts on \( (\prod_{s \in S} V_s)^c \) compatibly with \( T_0 \). Then the induced action of \( T \) on \( (\prod_{s \in S} V_s)^b \simeq \mathcal{L}_A((\prod_{s \in S} V_s)^c, A) \) is given by
\[
(t \cdot \check{v}^*)_s = t \cdot \check{v}^*_s,
\]
for any \( t \in T, \check{v}^* \in (\prod_{s \in S} V_s)^b \). In particular, \( t : (\prod_{s \in S} V_s)^b \to (\prod_{s \in S} V_s)^b \) is a nice automorphism for any \( t \in T \).

Proof. Notice that the equality makes sense only if we know \( T \) maps \( V_s \) to \( V_s \) for any \( s \in S \), which is clear since \( T \) is abelian and the action is compatible with restriction to \( T_0 \).
\[
(t \cdot \check{v}^*)(v_s) = \check{v}^*(t^{-1} \cdot v_s) = \check{v}^*_s(t^{-1} \cdot v_s) = t \cdot \check{v}^*_s(v_s)
\]
for any \( t \in T, s \in S, v_s \in V_s \).
Lemma 5.9. Assume $A = K$. If $\tilde{V}$ is a closed sub $T_0$-Banach representation of $(\bigoplus_{s \in S} V_s)^c$, then there exists an assignment of subspaces $\tilde{V}_s \subset V_s$ to each $s \in S$, with the induced subspace norms, such that $\tilde{V} \simeq (\bigoplus_{s \in S} \tilde{V}_s)^c$. Moreover, $(\bigoplus_{s \in S} V_s)^c/\tilde{V} \simeq (\bigoplus_{s \in S} V_s/\tilde{V}_s)^c$ as $T_0$-Banach representations, where $(\bigoplus_{s \in S} V_s)^c$ is equipped with the complete quotient norm of $(\bigoplus_{s \in S} V_s)^c$ and $(\bigoplus_{s \in S} V_s/\tilde{V}_s)^c$ is the Banach representation associated to the assignment $s \mapsto V_s/\tilde{V}_s$ by our construction in Definition 5.5. Each $V_s/\tilde{V}_s$ is equipped with the complete quotient norm of $V_s$.

Proof. $\tilde{V} \cap \bigoplus_{s \in S} V_s$ is a sub representation of $\bigoplus_{s \in S} V_s$, which must be $\bigoplus_{s \in S} \tilde{V}_s$ for an assignment of subspaces $\tilde{V}_s \subset V_s$. We need to prove $\tilde{V} \simeq (\bigoplus_{s \in S} \tilde{V}_s)^c$ for such a sub assignment $s \mapsto \tilde{V}_s$. It is enough to prove $\tilde{V} \subset \bigoplus_{s \in S} V_s$. It suffices to prove that for $\forall v \in \tilde{V}$, $v = (v_s) \in (\bigoplus_{s \in S} V_s)^c$, if $0 \neq v_s \in V_s$, then $v_s \in \tilde{V}$. Given any $s \in S$, there are only finitely many $s' \in S$ such that $|v_{s'}| > |v_s|$. Without loss of generality, assume $|v_s| = |v|$, i.e., $v_s$ is the largest component vector of $v$. Suppose $T_0$ is topologically finitely generated by $\{t_1, \cdots, t_n\}$. By Lem 5.5, it suffices to inductively construct $\{v_1, \cdots, v_n\} \subset \tilde{V}$ such that $(v_s)_s = v_s$ and $t_j \cdot v_s = \chi_s(t_j)v_s$ for $j \leq i$.

Assume we have already constructed $\{v_1, \cdots, v_i\}$ ($i$ can be 0). Again we inductively construct a sequence $v_i^N (N \geq 0)$ converging to our desired $v_{i+1}$ with $v_i^0 := v_i$. Assume we have constructed $\{v_i^0, \cdots, v_i^N\}$. $i : S \leftrightarrow \mathbb{Z} \geq 0$ (recall that total degree $|s|$ of $s \in S$ is defined to be the sum of the absolute value of coordinates of $i(s)$). Set $S_i^N := \{s' \in S | (v_i^N)_{s'} \neq 0\}$. Now either $\chi_{s'}(t_{i+1}) = \chi_s(t_{i+1})$ for $s' \in S_i^N$, or we can pick an element $s_{N,i} \in S_i^N$ such that $|s_{N,i}|$ is minimal among those of all such elements satisfying the property that

$$|\chi_{s_{N,i}}(t_{i+1}) - \chi_s(t_{i+1})| \geq |\chi_{s'}(t_{i+1}) - \chi_s(t_{i+1})|$$

for $\forall s' \in S_i^N$. If the former case occurs, we set $v_{i+1} := v_i^N$ and stop constructing the sequence. Otherwise set

$$v_{i+1}^N := (\chi_{s_{N,i}}(t_{i+1}) - \chi_s(t_{i+1}))^{-1} \cdot (\chi_{s_{N,i}}(t_{i+1})v_i^N - t_{i+1} \cdot v_i^N),$$

$v_i^{N+1}$ satisfies that

$$\frac{(v_i^{N+1})_s}{v_i^{N+1} - v_s} = \frac{v_i^N - v_s}{v_i^N - v_s},$$

$$S_i^{N+1} \subset S_i^N \setminus \{s_{N,i}\} \quad (v_i^{N+1})_{s_{N,i}} = 0.$$

If $S_i^N$ is finite, then $v_{i+1}^N$ is obtained within finitely many steps. If $S_i^N$ is infinite, the latter cases always happen, we claim the existence of $\lim_{N \to \infty} v_i^N$. For any $s' \in S_i^0$ such that $\chi_{s'}(t_{i+1}) \neq \chi_s(t_{i+1})$, there exists $N' > 0$ such that $|s_{N,i}|$ is greater then the maximal coordinate of $|s|$ for all $N > N'$.

$$|v_i^{N+1}|_{s'} \leq |v_i^N| \cdot |(v_i^N)_{s'}|$$
for all $N > N'$, which means that $s'$ component of $v^N_i \mapsto v^{N+1}_i$ is a contraction for all $N > N'$. Such components clearly go to 0. For $s' \in S^0_l$ such that $\chi_{s'}(t_{l+1}) = \chi_{s}(t_{l+1}) = \chi_{s}(t_{j}) = (v^N_l)_{s'} = (v_l)_{s'}$ for any $N \geq 0$. Set $v_{l+1} := \lim_{N \to \infty} v^N_l$ for this case. We see from the construction $(v_{l+1}) = v_s, t_j \cdot v_{l+1} = \chi_s(t_l)v_{l+1}$ for $j \leq l + 1$, which completes the induction, hence the proof of the first part.

For the second part of the lemma, we construct the following natural map

$$\big(\prod_{s \in S} V_s\big)^c/\hat{V} \to \big(\prod_{s \in S} V_s/\tilde{V}_s\big)^c$$

$$(x_s) \mapsto (\tilde{\tau}_s), \ x_s \in V_s, \ \tilde{\tau}_s \in V_s/\tilde{V}_s.$$  

This map makes sense and is well defined thanks to the first part of the lemma. It is continuous since $|\tilde{\tau}_s| \leq |x_s|$ for each $s \in S$. Given $\tilde{\tau}_s \in V_s/\tilde{V}_s$, there exists $x_s^0 \in V_s$ such that $|x_s^0| = |\tilde{\tau}_s|$. We construct the inverse map

$$\big(\prod_{s \in S} V_s/\tilde{V}_s\big)^c \to \big(\prod_{s \in S} V_s\big)^c/\hat{V}$$

$$(\tilde{\tau}_s) \mapsto (x_s^0).$$

Note if $|x_s^0 + v_s^0| = |x_s^0|$ for some $v_s^0 \in \tilde{V}_s$, $|x_s^0| \leq \max(|x_s^0 + v_s^0|, |x_s^0|) = |\tilde{\tau}_s|$. Thus the inverse map is well defined. We know the two norms on both sides induced from the isomorphism are equivalent since one is stronger than another. To prove $(\prod_{s \in S} V_s)^c/\hat{V}$ and $(\prod_{s \in S} V_s/\tilde{V}_s)^c$ have the same norm, it is translated to prove for

$$\forall x \in \big(\prod_{s \in S} V_s\big)^c,$$

$$\inf_{v \in \tilde{V}} \sup_{s \in S} |x_s + v_s| = \sup_{s \in S} \inf_{v_s \in V_s} |x_s + v_s|.$$  

Fix a $v \in \tilde{V}$,

$$\sup_{s \in S} |x_s + v_s| \geq \sup_{s \in S} \inf_{v_s' \in V_s} |x_s + v_s'|,$$

which gives "≥". For each $s \in S$, choose $v_s^0 \in \tilde{V}_s$ minimizing $|x_s + v_s^0|$, then $|v_s^0| \leq |x_s|$ by the same non-Archimedean triangle inequality. Let $v^0 = (v_s^0)_{s \in S} \in \tilde{V} = (\prod_{s \in S} \tilde{V}_s)^c$,

$$\sup_{s \in S} |x_s + (v^0)_s| = \sup_{s \in S} |x_s + v_s^0| = \sup_{s \in S} \inf_{v_s \in V_s} |x_s + v_s|,$$

yielding "≤".

There is a completion operation on closed $T_0$-subrepresentations of $(\prod_{s \in S} V_s)^b$ (and $(\prod_{s \in S} V_s)^c$). We regard $(\prod_{s \in S} V_s)^c$ as a closed subrepresentation of $(\prod_{s \in S} V_s)^b$. By the previous lemma, $\hat{V} \cap (\prod_{s \in S} V_s)^c \simeq (\prod_{s \in S} \tilde{V}_s)^c$ for the assignment $s \mapsto \tilde{V}_s$. We set the completion of $\hat{V}$ to be $(\prod_{s \in S} \tilde{V}_s)^b$ for this assignment in the sense of Def[5.1].

Remark 5.10. It seems that in general $\hat{V}$ is not necessarily contained in $\tilde{V}$.
Lemma 5.11. Assume $A = K$. Let $\tilde{V}$ be a closed $T_0$-representation of $(\prod_{s \in S} V_s)^b$.

The followings are equivalent:

1. $\tilde{V} \subset \check{V}$.
2. For any $\tilde{v} = (\tilde{v}_s)_s \in \check{V}$, if $\tilde{v}_s \neq 0$, then $\tilde{v}_s \notin \check{V}$.

Proof. (2) $\Rightarrow$ (1) is obvious. (1) $\Rightarrow$ (2): If there exists $\tilde{v} = (\tilde{v}_s)_s \in \check{V}$ such that $\tilde{v}_s \neq 0$ and $\tilde{v}_s \notin \check{V}$. Then for $\tilde{V} = (\prod_{s \in S} \tilde{V}_s)^b$, we have $\tilde{v}_s \notin \check{V}_s$, which implies that $\tilde{v} = (\tilde{v}_s)_s \notin \check{V}$.

Lemma 5.12. Assume $A = K$. $V_1, V_2$ are both closed $T_0$-representations of $(\prod_{s \in S} V_s)^b$.

Suppose $p : (\prod_{s \in S} V_s)^b \to V_1$ is a continuous projection and $(V_1 \cap (\prod_{s \in S} V_s)^c) \oplus (V_2 \cap (\prod_{s \in S} V_s)^c) = (\prod_{s \in S} V_s)^c$, then

$$(\prod_{s \in S} V_s)^b = \check{V}_1 \oplus \check{V}_2.$$ 

Proof. $V_1, V_2$ are both closed in $(\prod_{s \in S} V_s)^b$, so are $V_1 \cap (\prod_{s \in S} V_s)^c, V_2 \cap (\prod_{s \in S} V_s)^c$ in $(\prod_{s \in S} V_s)^c$. By the Lem 5.9, $V_i \cap (\prod_{s \in S} V_s)^c = (\prod_{s \in S} V_{i,s})^c$ for the assignment $s \mapsto V_{i,s}$ ($i = 1, 2$) such that $V_{1,s} \oplus V_{2,s} = V_s$ for every $s \in S$. If $v \in \check{V}_1 \cap \check{V}_2$, its $s$-component $v_s \in V_{1,s} \cap V_{2,s} = 0$, so $v = \check{0}$, $\check{V}_1 \cap \check{V}_2 = 0$. And for any $v \in (\prod_{s \in S} V_s)^b$, $v_s = v_{1,s} + v_{2,s}$ for each $s \in S$ and unique $v_{1,s} \in V_{1,s}, v_{2,s} \in V_{2,s}$. Since $p$ is continuous, there exists a constant $c$ such that

$$|v_{1,s}| = |v_{1,s}|^S = |p(v_s)| = c \cdot |v_s|$$

$$|v_{2,s}| = |v_{2,s}|^S = |v_s - p(v_s)| = c \cdot |v_s|.$$ 

$|(v_{i,s})_s|$ are bounded since $|v_{i,s}|$ are bounded. $(v_{i,s})_s \in V_{i,s}$ for $i = 1, 2$.

For two bounded eigen orthonormalizable $(\prod_{s \in S} V_{1,s})^b, (\prod_{s \in S} V_{2,s})^b$, we define $\text{Hom}_{\text{nice}}((\prod_{s \in S} V_{1,s})^b, (\prod_{s \in S} V_{2,s})^b)$ to be the space of all nice maps from $(\prod_{s \in S} V_{1,s})^b$ to $(\prod_{s \in S} V_{2,s})^b$.

Lemma 5.13.

$\text{Hom}_{\text{nice}}((\prod_{s \in S} V_{1,s})^b, (\prod_{s \in S} V_{2,s})^b) \hookrightarrow \text{Hom}_{\text{cont}}((\prod_{s \in S} V_{1,s})^b, (\prod_{s \in S} V_{2,s})^b)$

is a closed $A$-linear subspace.

Proof. It is clear that from the definition of nice map that

$\text{Hom}_{\text{nice}}((\prod_{s \in S} V_{1,s})^b, (\prod_{s \in S} V_{2,s})^b)$
is a linear subspace. For closeness, let \( \{f_i\} \) be a sequence of nice maps between 
\[
(\prod_{s_1 \in S_1} V_{1,s_1})^b \quad \text{and} \quad (\prod_{s_2 \in S_2} V_{2,s_2})^b
\]
converging to 
\[
f \in \text{Hom}_{\text{cont}}((\prod_{s_1 \in S_1} V_{1,s_1})^b, (\prod_{s_2 \in S_2} V_{2,s_2})^b).
\]

Let \( \iota : S_1 \hookrightarrow \mathbb{Z}_{<0} \), total degree \(|s_1|\) of \( s_1 \in S_1 \) is sum of the absolute value of coordinates of \( \iota_1(s_1) \). For any nonzero \( v \in (\prod_{s_1 \in S_1} V_{1,s_1})^b, s_2 \in S_2, \varepsilon > 0 \), there exists \( N_0 > 0 \)
such that \(|f - f_n| < \varepsilon |v|^{-1}\) for all \( n \geq N_0 \), and there exists \( N_1 > 0 \) such that for any finite set \( \{s_{n_1}, \ldots, s_{n_m}\} \) satisfying \(|s_{n_k}| > N_1\), \( \sum_{1 \leq k \leq m} (f_{N_0}(v_{s_{n_k}}))_{s_2} < \varepsilon \) since \( \{f_i\} \) are nice. By non-Archimedean triangle inequality, \( \sum_{1 \leq k \leq m} (f(v_{s_{n_k}}))_{s_2} < \varepsilon \) for any finite set \( \{s_{n_1}, \ldots, s_{n_m}\} \) satisfying \(|s_{n_k}| > N_1\), \( \sum_{s_1 \in S_1} f(v_{s_1})_{s_2} \) converges for all \( s_2 \in S_2 \) and 
\[
\sum_{s_1 \in S_1} f(v_{s_1})_{s_2} = \lim_{n \to \infty} \sum_{s_1 \in S_1, |s_1| \leq n} f(v_{s_1})_{s_2} \\
= \lim_{n \to \infty} \lim_{i \to \infty} \sum_{s_1 \in S_1, |s_1| \leq n} f_i(v_{s_1})_{s_2} \\
= \lim_{n \to \infty} \lim_{i \to \infty} \sum_{s_1 \in S_1, |s_1| > n} f_i(v_{s_1})_{s_2} \\
= f(v)_{s_2}.
\]
The last equation holds since for any \( \varepsilon > 0 \), there exists \( N_1 > 0 \) as we chose before such that \( \sum_{s_1 \in S_1, |s_1| > N_1} f_i(v_{s_1})_{s_2} < \varepsilon \) independent of \( f_i \).

**Lemma 5.14.** Assume \( A = K \). The kernel of a \( T_0 \)-equivariant nice \( f : (\prod_{s_1 \in S} V_{1,s_1})^b \to (\prod_{s_2 \in S} V_{2,s_2})^b \) is also of the form \( (\prod_{s_1 \in S} V_{s_1})^b \). Moreover, \( \overline{\text{Im}(f)} \subset \overline{\text{Im}(f)} \).

**Proof.** The kernel of \( f \mid_{(\prod_{s_1 \in S} V_{1,s_1})^c} \) is of the form \( \prod_{s_1 \in S} \overline{V_{1,s_1}}^c \) by the Lem 5.39. Since \( f \) is nice, \( f((\prod_{s_1 \in S} \overline{V_{1,s_1}})^c) = f((\prod_{s_1 \in S} \overline{V_{1,s_1}})^b) = 0 \). Again by the niceness of \( f \), \( f(x) \neq 0 \) for \( x \notin (\prod_{s_1 \in S} \overline{V_{1,s_1}})^b \). For the inclusion, it suffices to prove 
\[
\overline{\text{Im}(f)} = (\prod_{s_2 \in S} \text{Im}(f)(V_{1,s_2}))^b.
\]
We have $\text{Im}(f)(V_{1,s_1}) \subset \text{Im}(f)$ for any $s_1 \in S$. The equality follows from $\text{Im}(f) \cap V_{2,s_2} = \text{Im}(f)(V_{1,s_2})$ for any $s_2 \in S$.

For a finite free $T_0$-eigen orthonormalizable $A$-module $M \simeq \bigoplus_{s_m \in S_M} M_{s_m}$ ($S_M$ can be chosen as a finite set), $M \otimes_A \left( \bigoplus_{s \in S} V_s \right)^c$ (resp. $M \otimes_A \left( \bigoplus_{s \in S} V_s \right)^b$) is again equipped with the same structure we will explain in the following. Moreover, the Banach norm on $M \otimes_A V$ is defined to be the norm for $\mathcal{L}_A(M^*, V)$ for a Banach space $V$, where $V$ can be $(\bigoplus_{s \in S} V_s)^c$, $(\bigoplus_{s \in S} V_s)^b$, $V_s$ in our setting.

More specifically, if $M \simeq \bigoplus_{m \in S_M} M_m$, $V_1 := \bigoplus_{m \in S_M, s \in S, s + m = l} M_m \otimes V_s$,

$$M \otimes_A \left( \bigoplus_{s \in S} V_s \right)^b \simeq \left\{ v \in \bigcap_{s \in S, s + m = l} V_{s+m} \mid \max_{s \in S, m \in S_M} |(v)_{s+m}|_{s+m} < \infty \right\},$$

$$\bigoplus_{m \in S_M} x_m \otimes v^m \mapsto \sum_{m \in S_M} (x_m \otimes (v^m))_{l-m},$$

where $L := S \times S_M / \{ (s, m) \sim (s', m') \} \text{ whenever } \chi_s \cdot \chi_m = \chi_{s'} \cdot \chi_{m'} \}$ and $s + m := (s, m) \in L$. For any $l \in L$ and $m \in S_M$, there is at most one $l - m$ in $S$ such that $(l - m, m) = l$. Similarly for $l \in L$ and $s \in S$, there is at most one $l - s$ in $S_M$ such that $(s, l - s) = l$. We can similarly define $M \otimes_A \left( \bigoplus_{s \in S} V_s \right)^c$.

**Lemma 5.15.** For a $A$-linear map $g : M_1 \rightarrow M_2$ between two finite free eigen orthonormalizable $A$-modules, if $f : \left( \bigoplus_{s_1 \in S_1} V_{1,s_1} \right)^b \rightarrow \left( \bigoplus_{s_2 \in S_2} V_{2,s_2} \right)^b$ is nice, then

$$M_1 \otimes \left( \bigoplus_{s_1 \in S_1} V_{1,s_1} \right)^b \xrightarrow{g \otimes f} M_2 \otimes \left( \bigoplus_{s_2 \in S_2} V_{2,s_2} \right)^b$$

is nice.

**Proof.** Let

$$M_1 \simeq \bigoplus_{s_{m_1} \in S_{M_1}} M_{s_{m_1}}, \quad M_2 \simeq \bigoplus_{s_{m_2} \in S_{M_2}} M_{s_{m_2}},$$

$$M_1 \otimes_A \left( \bigoplus_{s_1 \in S_1} V_{1,s_1} \right)^b \simeq \left( \bigoplus_{l_1 \in L_1} V_{l_1} \right)^b, \quad M_2 \otimes_A \left( \bigoplus_{s_2 \in S_2} V_{2,s_2} \right)^b \simeq \left( \bigoplus_{l_2 \in L_2} V_{l_2} \right)^b.$$

Here we use the same notation for $L_1, L_2$ as in the previous discussion before the current lemma.
For any eigen element \( m \in V_k \) for \( k \in S_{M_1} \) and \( \hat{v}_1 \in ( \prod_{s_1 \in S_1} V_{1,s_1} )^b \), \( g(m) \) can be expressed as \( g(m) = \sum_{k_2 \in S_{M_2}} m_{k_2} \), where \( m_{k_2} \in V_{k_2} \). We have

\[
(g \otimes f)(m \otimes \hat{v}_1)_{l_2} = ( (\sum_{k_2 \in S_{M_2}} m_{k_2} \otimes f(\hat{v}_1)) )_{l_2}
\]

\[
= ( \sum_{k_2 \in S_{M_2}} m_{k_2} \otimes f(\hat{v}_1)_{l_2-k_2} )_{l_2}
\]

\[
= \sum_{k_2 \in S_{M_2}} m_{k_2} \otimes \sum_{s_1 \in S_1} (f|_{V_{1,s_1}})(\hat{v}_1)_{l_2-k_2}
\]

\[
= \sum_{k_2 \in S_{M_2}} m_{k_2} \otimes \sum_{l_1 \in L_1} (f|_{V_{1,l_1}})(\hat{v}_1)_{l_2-k_2}
\]

\[
= ( \sum_{l_1 \in L_1} (g \otimes f)(m \otimes (\hat{v}_1)_{l_1-k}))_{l_2}.
\]

The fourth equality above is a substitution of variables from \( s_1 \) to \( l_1 - k \). \( \square \)

**Lemma 5.16.** Assume \( A = K \). Let \( f : ( \prod_{s_1 \in S_1} V_{1,s_1} )^b \rightarrow ( \prod_{s_2 \in S_2} V_{2,s_2} )^b \) be a nice map. If

\[
W_1 \hookrightarrow ( \prod_{s_1 \in S_1} V_{1,s_1} )^b, \quad W_2 \hookrightarrow ( \prod_{s_2 \in S_2} V_{2,s_2} )^b
\]

are closed \( A[T_0] \) subrepresentations inside \( ( \prod_{s_1 \in S_1} V_{1,s_1} )^b, ( \prod_{s_2 \in S_2} V_{2,s_2} )^b \) such that \( f(W_1) \subset W_2 \). Assume \( W_2 \subset \hat{W}_2 \), then \( f(\hat{W}_1) \subset \hat{W}_2 \).

**Proof.** We have \( \hat{W}_1 \simeq ( \prod_{s_1 \in S_1} W_{1,s_1} )^b, \hat{W}_2 \simeq ( \prod_{s_2 \in S_2} W_{2,s_2} )^b \). For \( \forall \hat{w}_1 \in \hat{W}_1, s_2 \in S_2 \)

\[
(f(\hat{w}_1))_{s_2} = \sum_{s_1 \in S_1} (f|_{V_{1,s_1}})(\hat{w}_1)_{s_1,s_2}.
\]

As \( W_1 \cap ( \prod_{s_1 \in S_1} V_{1,s_1} )^c \) being convergent eigen orthonormalizable by Lem 5.9

\[
(\hat{w}_1)_{s_1} \in W_1 \cap ( \prod_{s_1 \in S_1} V_{1,s_1} )^c \subset W_1.
\]

Each term \( (f|_{V_{1,s_1}})(\hat{w}_1)_{s_1,s_2} \) \( \in W_2 \) by Lem 5.11 as \( W_2 \subset \hat{W}_2 \). The sum converges in \( ( \prod_{s_2 \in S_2} V_{2,s_2} )^b \) and bounded uniformly for all \( s_2 \), thus in \( W_2 \). \( f(\hat{w}_1) \in \hat{W}_2. \) \( \square \)

6. Some auxiliary modules and their duals

We use the same notation as in 3 and let \( T^0 = \mathcal{F}(O_E) \) with \( T^s \subset T^0 \). In the remaining paper we view the right \( I \)-modules \( \text{Ind}_I^*(V) \) and \( \mathbb{D}_I^*(V) \) introduced in 3 as left \( I \)-modules in the usual way. Let \( \mathfrak{n}, \mathfrak{m} \) be the Lie algebra of \( N, N \) over \( \mathbb{Q}_p \). All the tensor products are defined over \( \mathbb{Q}_p \). Let \( A = \mathcal{O}(\Omega) \) for some irreducible Zariski closed subspace of an affinoid subdomain \( \Omega \subset \mathcal{W} \) (\( A \) is an integral domain).
We assume in this section $H/O_E = GL_n$, $Sp_{2n}$ or a unitary group associated to $J_n$ and an unramified extension $K/E$. For an Iwahori subgroup of $H(O_E)$,

$$I = \mathbb{N}_I \times T^0 \times N_I,$$

let $\Delta^+_I, \Delta^-_I$ denote positive roots and negative roots respectively viewed over $E$. Every factor of a product of different $I$ satisfies the assumption for $H$ as above. In particular, for each Iwahori factor, we use the coordinates of $\mathbb{N}_I$ as in the beginning of [4] which will be constructed for each case of $H$ in [7].

Assume that $A = O(\Omega)$ is an $E$-Banach algebra containing a large enough $p$-adic field such that $A$ contains all Galois conjugates of $E$ and the splitting field of $H$. For later applications, we introduce a Banach space of functions on $\mathbb{N}_I \simeq O_E^d$. For $x \in \mathbb{N}_I$, we use $x_1, \cdots, x_d$ to denote the coordinates of $x$ via the isomorphism above. A basis of

$$C_{I,\chi}^{s,1} := C^{s,1}(I, A) \subset C^{s,an}(\mathbb{N}_I, A) = \text{Ind}_{I,\chi}$$

as representation of $I$ for this case is given by

$$f_{g,\sigma}(x) := \begin{cases} (\sigma x_1)^{a_1} \cdots (\sigma x_d)^{a_d} & x \in g \cdot \mathbb{N}_I^s, \\ 0 & x \not\in g \cdot \mathbb{N}_I^s, \end{cases}$$

for $\sigma$ running through embedding from $E$ to $\mathfrak{o}_p$ and $a^\sigma$ running through $\mathbb{Z}_{\geq 0}^d$. Recall that there is a partial order relation $a \geq a'$ if and only if $a_i \geq a'_i$ for all $i$. $C_{I,\chi}^{s,1}(\mathbb{N}_I, A)$ is defined to be the subspace of $C^{s,an}(\mathbb{N}_I, A)$ whose elements can be expressed as the sum of $f_{g,\sigma}(x)$ with coefficients tending to zero. Or equivalently, $C_{I,\chi}^{s,1}(\mathbb{N}_I, A)$ consists of (globally) analytic functions on $\mathbb{N}_I$ multiplied by characteristic functions on $\mathbb{N}_I$-cosets.

In the case of unitary group for an unramified quadratic extension $K/E$, there is another basis which is eigenbasis for $T^s$ action with respect to the identification $\mathbb{N}_I \simeq O_E^{d_E} \times O_K^{d_K}$ which is equivalent to the basis above by

$$f_{g,\sigma}^s a_{E,K}(x) := \begin{cases} (\sigma x_1)^{a_{E,1}} \cdots (\sigma x_{d_E})^{a_{E,d_E}}, (\sigma x_1)^{a_{K,1}} \cdots (\sigma x_{d_K})^{a_{K,d_K}} & x \in g \cdot \mathbb{N}_I^s, \\ 0 & x \not\in g \cdot \mathbb{N}_I^s, \end{cases}$$

for $\sigma$ running through embedding from $K$ to $\mathfrak{o}_p$ and $a_{E,K}^\sigma = (a_{E,1}^\sigma, \cdots, a_{E,d_E}^\sigma)$ running through $\mathbb{Z}_{\geq 0}^{d_E}$, $a_{K,D}^\sigma = (a_{K,1}^\sigma, \cdots, a_{K,d_K}^\sigma)$ running through $\mathbb{Z}_{\geq 0}^{d_K}$.

To save spaces, we introduce the following notation:

$$\delta^E_+ := (\delta^E_-)^{-1}, \delta^K_+ := (\delta^K_-)^{-1},$$
$$\delta^E \pm (t) x_E := (\delta^E_1 \pm (t)x_E, \cdots, \delta^E_{d_E} \pm (t)x_E),$$
$$\delta^K \pm (t) x_K := (\delta^K_1 \pm (t)x_K, \cdots, \delta^K_{d_K} \pm (t)x_K),$$
$$\sigma \delta^E \pm (t) := \sigma \delta^E_1 \pm (t)a_{E,1}^\sigma \cdots \sigma \delta^E_{d_E} \pm (t)a_{E,d_E}^\sigma,$$
$$\sigma \delta^K \pm (t) := \sigma \delta^K_1 \pm (t)a_{K,1}^\sigma \cdots \sigma \delta^K_{d_K} \pm (t)a_{K,d_K}^\sigma.$$
since for any \( x \in g \cdot \mathbb{N}_t \) and any \( t \in T^s \),
\[
    t \cdot f_{g, \sigma}^\sigma(x) = f_{g, \sigma}^\sigma(t^{-1}x) = f_{g, \sigma}^\sigma((t^{-1}xt)^{-1}) = \chi(t)^{-1} \cdot f_{g, \sigma}^\sigma(t^{-1}xt) = \chi(t)^{-1} \cdot f_{g, \sigma}^\sigma(\delta^E(t)x_E, \delta^K(t)x_K) = \chi(t)^{-1} \cdot f_{g, \sigma}^\sigma(\delta^E(t)x_E, \delta^K(t)x_K)) = \chi(t)^{-1} \cdot f_{g, \sigma}^\sigma(\delta^E(t)x_E, \delta^K(t)x_K)).
\]
A general \( t \in T^0 \) permutes different \( f_{g, \sigma}^\sigma \) and \( f_{g', \sigma}^\sigma \) for \( g' = tgt^{-1} \) up to a scalar twist.

**Lemma 6.1.** The bases \( \{f_{g, \sigma}^\sigma\} \) and \( \{f_{g', \sigma}^\sigma\} \) generate the same subspace in \( C^s,an(\mathbb{N}_t, A) \).

**Proof.** Suppose \( t_{K/E} \) gives rise to \( O_K = O_Eu_1 \oplus O_Eu_2 \) for \( u_1, u_2 \in O_K \). By transitions of coordinates introduced in (4), \( x^i_K = u_1 \cdot x_{d_i + 2i - 1} + u_2 \cdot x_{d_i + 2i} \), \( \sigma x^i_K = \sigma u_1 \cdot \sigma x_{d_i + 2i - 1} + \sigma u_2 \cdot \sigma x_{d_i + 2i} \). Moreover, there are two lifts of embedding \( \tilde{\sigma}_1, \tilde{\sigma}_2 : K \to \mathbb{Q}_p \) for each \( \sigma : E \to \mathbb{Q}_p \). Then
\[
    \left( \begin{array}{c}
\tilde{\sigma}_1 x^i_K \\
\tilde{\sigma}_2 x^i_K
\end{array} \right)^{-1} = \left( \begin{array}{cc}
\tilde{\sigma}_1 u_1 & \tilde{\sigma}_1 u_2 \\
\tilde{\sigma}_2 u_1 & \tilde{\sigma}_2 u_2
\end{array} \right)^{-1} \left( \begin{array}{c}
\sigma x_{d_i + 2i - 1} \\
\sigma x_{d_i + 2i}
\end{array} \right)
\]
\( \tilde{\sigma}_1 u_1 \tilde{\sigma}_1 u_2)^{-1} \in \text{GL}_2(O_K) \) since \( K/E \) is unramified. Change of variables says that \( f_{g, \sigma}^\sigma \) (resp. \( f_{g', \sigma}^\sigma \)) is the sum of the first basis \( \{f_{g, \sigma}^\sigma\} \) (resp. \( \{f_{g', \sigma}^\sigma\} \)) with integral coefficients. \( \square \)

For Iwahori subgroup \( I \) of a finite product of \( p \)-adic groups \( \prod_{1 \leq i \leq l} H_i(E_i) \) considered in (4) let \( \chi : T^0 = \prod_{1 \leq i \leq l} T_i^0 \to A^\times \) be a locally analytic character, where
\( A = G(\Omega), \Omega \) is an irreducible Zariski closed subspace of an affinoid subdomain of the weight space of \( T^0 \). And recall that \( \chi_\Omega : T^0 \to A^\times \) is the universal character of \( T^0 \) for \( \Omega \). We keep assuming that \( A \) contains all Galois conjugates of \( E_i \) and the splitting field of \( H_i \) for \( 1 \leq i \leq l \).

We use variables \( x^i = (x^i_1, \ldots, x^i_{d_i}) \) to denote coordinates for \( \psi^1_{\mathfrak{l}_i} \) introduced in (4). \( C^s,1(\mathbb{N}_t, A) \) is defined to be the subspace of \( C^s,an(\mathbb{N}_t, A) \) whose elements can be expressed as sum of \( f_{g, \sigma}^\sigma \) with coefficients tending to zero, indexed by \( g \in \mathbb{N}_t^d/\mathbb{N}_f^d \approx \prod_{1 \leq i \leq l} \mathbb{N}_i^d/\mathbb{N}_f^d, \sigma = (\sigma_1, \ldots, \sigma_l) \) running through all possible embeddings
\[
\prod_{1 \leq i \leq l} E_i \to (\mathbb{Q}_p)^{d_i}, g^\sigma = \prod_{1 \leq i \leq l} a^\sigma_i \text{ running through } \prod_{1 \leq i \leq l} \mathbb{Z}_{\geq 0}^{d_i},
\]
\[
f_{g, \sigma}^\sigma(x) := \begin{cases} 
\prod_{1 \leq i \leq l} (\sigma_i x^i_1)^{a^\sigma_i} \cdots (\sigma_i x^i_{d_i})^{a^\sigma_i} & \text{if } x \in g \cdot \mathbb{N}_t \\
0 & \text{if } x \notin g \cdot \mathbb{N}_t
\end{cases}
\]
Like \( l = 1 \) case, there is another equivalent basis which is eigenbasis for \( T^s = \prod_{1 \leq i \leq l} T^s_i \) action by

\[
 f_{g; a_E^s, a_K^s}(x) := \begin{cases} 
 \prod_{1 \leq i \leq l} (\sigma x^s_{E_i})^{x_{E_i}, s} \cdots (\sigma x^s_{E_l})^{x_{E_l}, s} 
 \cdot \prod_{1 \leq i \leq l} (\sigma x^s_{K_i})^{x_{K_i}, s} \cdots (\sigma x^s_{K_l})^{x_{K_l}, s} & x \in g \cdot \mathcal{N}_I \\
 0 & x \not\in g \cdot \mathcal{N}_I,
\end{cases}
\]

for \( \sigma \) running through embedding from \( K \) to \( \mathbb{O}_p \) and \( a_E^s = \prod_{1 \leq i \leq l} (a_{E_i, 1}^s, \cdots, a_{E_i, d_{E_i}}^s) \)
running through \( \prod_{1 \leq i \leq l} \mathbb{Z}_{d_{E_i}}^2, a_K^s = \prod_{1 \leq i \leq l} (a_{K_i, 1}^s, \cdots, a_{K_i, d_{K_i}}^s) \) running through \( \mathbb{Z}_{d_{K_i}}^2 \).

We make the following notation:

\[
\sigma \delta^{E, \pm}(t) := \prod_{1 \leq i \leq l} \sigma \delta^{E_i, \pm}(t)^{x_{E_i}, s} \cdots \sigma \delta^{E_l, \pm}(t)^{x_{E_l}, s},
\]

\[
\sigma \delta^{K, \pm}(t) := \prod_{1 \leq i \leq l} \sigma \delta^{K_i, \pm}(t)^{x_{K_i}, s} \cdots \sigma \delta^{K_l, \pm}(t)^{x_{K_l}, s}.
\]

Then similar calculations show

\[
t \cdot f_{g; a_E^s, a_K^s} = \chi(t) \cdot \sigma \delta^{E, +}(t) \cdot \sigma \delta^{K, +}(t) \cdot f_{g; a_E^s, a_K^s}
\]

for any \( x \in g \cdot \mathcal{N}_I \) and any \( t \in T^s = \prod_{1 \leq i \leq l} T^s_i \). The proof of Lem [6.1] shows that these two bases define the same subspace \( C^{s, 1}(\mathcal{N}_I, A) \). For an analytic Banach \( \mathcal{O}(\Omega)[B_1] \)-module \( V \),

\[
\text{Ind}_1^0(V) \simeq C^{s, an}(\mathcal{N}_I, V)
\]

\[
f \mapsto f|_{\mathcal{N}_I}.
\]

Similarly we define

\[
C^{s, 1}(\mathcal{N}_I, V) := C^{s, 1}(\mathcal{N}_I, A) \otimes_A V \subset C^{s, an}(\mathcal{N}_I, A) \otimes_A V \simeq C^{s, an}(\mathcal{N}_I, V).
\]

If \( V \) is finite free eigen orthonormalizable with eigenbasis \( e_1, \cdots, e_m \) for \( T^0 \) (notions of being convergent or bounded eigen orthonormalizable coincide for finite rank case), we will see in Prop [6.2] this auxiliary module \( C^{s, 1}(\mathcal{N}_I, V) \) has the advantageous structure being convergent eigen orthonormalizable in Def [5.1] with the basis \( f_{g; a_E^s} \otimes e_i \). In particular, \( C^{s, 1}(\mathcal{N}_I, V) \) is orthonormalizable in the sense of [Buz07] with the basis \( f_{g; a_E^s} \otimes e_i \).

Pick a Weyl group element \( w = (w_1, \cdots, w_l) \in \prod_{1 \leq i \leq l} W(H_i, S_i) \) and set

\[
N_w := w I w^{-1} \cap \prod_{1 \leq i \leq l} N(E_i), \quad \mathcal{N}_w := w I w^{-1} \cap \prod_{1 \leq i \leq l} \mathcal{N}(E_i).
\]

For the \( w I w^{-1} \)-representation \( \text{Ind}^{s}_{w, \chi} = \text{Ind}^{s}_{w I w^{-1}}(\chi) \simeq C^{s, an}(w \mathcal{N}_I, w^{-1}) \) defined in §1 we can similarly define

\[
C^{s, 1}_{w, \chi} := C^{s, 1}(w \mathcal{N}_I w^{-1}, A) \subset C^{s, an}(w \mathcal{N}_I w^{-1}, A),
\]

with orthonormalizable basis \( f_{g; a_{E, K}^s} \) with respect to \( w \mathcal{N}_I w^{-1} \simeq \mathcal{O}_E \otimes \mathcal{O}_K \) and indexed by \( g \in w \mathcal{N}_I w^{-1} / w \mathcal{N}_I w^{-1} \simeq \mathcal{N}_I / \mathcal{N}_I \), embeddings \( \sigma \) and degrees \( a_{E, K}^s \).
We use $\sigma x^w(t)$ to denote 
\[
\prod_{1 \leq i \leq r} \sigma_i(\delta^{E_i'+}_i)^{w_i}(t_i)^{a_i^w} \cdot \prod_{1 \leq i \leq r} \sigma_i(\delta^{K_i'+}_i)^{w_i}(t_i)^{a_i^w}.
\]

Then similar calculations show 
\[
t \cdot f_{g \cdot \underline{w}, \underline{a}^w} = \chi^w(t)^{-1} \cdot \sigma x^w(t) \cdot f_{g \cdot \underline{w}, \underline{a}^w}
\]
for any $x \in g \cdot w N_f^s w^{-1}$ and any $t \in T^s$.

**Proposition 6.2.** Let $V$ be a finite free eigen orthonormalizable Banach $A[B_1]$-module. The $A[T]$-module $C^{<1}(\overline{N}_I, V)$ is convergent eigen orthonormalizable, i.e. $C^{<1}(\overline{N}_I, V)$ is isomorphic to a $(\prod V_s)^c$ in Def 5.7 as Banach $A[T^s]$ module with endowed norm. There is a natural continuous embedding of 
\[
C^{<1}(\overline{N}_I, V) \hookrightarrow C^{<1,\text{an}}(\overline{N}_I, V)
\]
of Banach $A$-modules with dense image.

We defer the proof to section 5.7.

**Remark 6.3.** We dualize $C^{<1}(\overline{N}_I, V)$ to get $\mathbb{D}_f^{<1}(V) := \mathcal{L}_A(C^{<1}(\overline{N}_I, V), A)$. As the image of $C^{<1}(\overline{N}_I, V)$ in $C^{<1,\text{an}}(\overline{N}_I, V)$ is dense by Prop 6.2, we have a continuous embedding of $\mathbb{D}_f^{<1}(V) \hookrightarrow \mathbb{D}_l^{<1}(V)$. In particular, 
\[
\mathbb{D}_{w, \chi} \hookrightarrow \mathbb{D}_{w, \chi}^{<1} := \mathcal{L}_A(C_{w, \chi}^{<1}, A).
\]
The motivation for introducing $\mathbb{D}_{w, \chi}^{<1}$ is that $\mathbb{D}_{w, \chi}$ itself is not bounded eigen orthonormalizable, and $\mathbb{D}_{w, \chi}^{<1}$ can be regarded as a completion of $\mathbb{D}_{w, \chi}$ with respect to $T^0$.

**Proposition 6.4.** Let $A = \mathcal{O}(\Omega)$ in the Def 5.1. The $A[T]$-module $\mathbb{D}_{w, \chi}^{<1} \simeq (\prod V_s)^b$ for some $S \subset \mathbb{Z}_{\geq 0}^m$ is bounded eigen orthonormalizable as $T$-unitary Banach representation.

**Proof.** Note that $\mathbb{D}_{w, \chi}^{<1} = \mathcal{L}_A(\text{Ind}_{w, \chi}^{<1}, A)$ and $\text{Ind}_{w, \chi}^{<1} \simeq C_{w, \chi}^{<1} \subset C^{<1,\text{an}}(w N_f w^{-1}, A)$, we see from proposition 6.2 $\text{Ind}_{w, \chi}^{<1} = \text{of the form } (\prod V_s)^c$. Now the conclusion follows from the Lem 5.3.

We give a basis of $\mathbb{D}_{w, \chi}^{<1}$ by 
\[
f_{g \cdot \underline{w}, \underline{a}^w} = \begin{cases} 
1 & g^{-1} g' \in w N_f w^{-1}, \sigma = \sigma' \text{ and } \underline{a}^w_E = \underline{a}'^w_E, \underline{a}^w_K = \underline{a}'^w_K \\
0 & \text{otherwise}
\end{cases}
\]
And by the Lem 5.3 $\mathbb{D}_{w, \chi}^{<1}$ is bounded eigen orthonormalizable, i.e., an element of $\mathbb{D}_{w, \chi}^{<1}$ can be expressed as a sum of $f_{g \cdot \underline{w}, \underline{a}^w}$, with bounded coefficients. And 
\[
t \cdot f_{g \cdot \underline{w}, \underline{a}^w} = \chi^w(t)^{-1} \cdot \sigma x^w(t) \cdot f_{g \cdot \underline{w}, \underline{a}^w}
\]
for $t \in T^s$.
Proposition 6.5. The weights appearing in $\mathbb{D}^{s,1}_{w,\chi}$, are exactly those of the form $\chi^w \cdot \mu^-$, where $\mu^-$ is an algebraic nonpositive weight with respect to $\prod_{1 \leq i \leq l} w_i \mathcal{N}_B w_i^{-1}$, i.e. $\mu^-$ can be written as a nonnegative integral linear combination of negative roots in $\cup_{1 \leq i \leq l} \Delta_{H,i}$ with respect of the Borel pairs $(w_i \mathcal{B} w_i^{-1}, w_i \mathcal{T}_i w_i^{-1})$ and $w_i \Delta_{H,i,p} = \Delta_{H,i} \cup w_i \Delta_{H,i,p}$ for $1 \leq i \leq l$. Moreover, the multiplicity of $\chi^w$ ($\mu^-$ trivial) in $\mathbb{D}^{s,1}_{w,\chi}$ equals to the index of $w \mathcal{N}_I w^{-1} \leq w \mathcal{N}_I w^{-1} = \prod_{1 \leq i \leq l} \mathcal{B}_i \mathcal{N}_B_i (\mathcal{O}_E / \mathcal{w}_i^{s-1} \mathcal{O}_E)$. Especially the highest weights correspond to $\prod_{1 \leq i \leq l} \mathcal{B}_i \mathcal{N}_B_i (\mathcal{O}_E / \mathcal{w}_i^{s-1} \mathcal{O}_E)$. 

Proof. The formula before Prop 6.5 combined with Lem 6.5 gives the proof. The highest weights correspond to $\mathfrak{p}_j = 0$ and arbitrary $g \in w \mathcal{N}_I w^{-1} / w \mathcal{N}_I w^{-1} \simeq \mathcal{N}_I / \mathcal{N}_I \simeq \prod_{1 \leq i \leq l} \mathcal{B}_i \mathcal{N}_B_i (\mathcal{O}_E / \mathcal{w}_i^{s-1} \mathcal{O}_E)$. □

Proposition 6.6. $C^{s,1}(\mathcal{N}_I, V)$ is $I_0^s$-stable, and it is a locally analytic $I_0^s$ representation. $\mathbb{D}^{s,1}_I(V)$ is a continuous $D(I_0^s, \mathbb{Q}_p)$ module, and $\mathbb{D}^{s,1}_I(V) \rightarrow \mathbb{D}^{s,1}_I(V)$ is $I_0^s$ equivariant. Moreover, any group element of $I_0^s$ induces a nice automorphism of $\mathbb{D}^{s,1}_I(V)$ in the sense of Def 5.6. In particular, these properties hold for $C^{s,1}_{w,\chi}$ as a dense $w \mathcal{F}_{s,w}^{-1}$-stable submodule of $C^{s,an}_{w,\chi}$ and $\mathbb{D}^{s,1}_{w,\chi} \hookrightarrow \mathbb{D}^{s,1}_{w,\chi}$.

We leave the proof to section §7, which reduces to the $\text{GL}_n$ case.

7. Some $p$-adic Banach analysis statements

In this section we include some statements about $p$-adic Banach analysis, mostly in terms of analysing coefficients and coordinates.

We start from proving the assumptions used in the beginning of §11 for $H_{/E}$. We call this choice of coordinates standard. In all the cases, our chosen Borel subgroup and Iwahori subgroup are implicit.

When $H = \text{GL}_n$: coordinates of $\mathcal{N}_I \times T^0$ correspond to entries of lower triangular matrices of radius $|\mathfrak{w}|^s$ and diagonal entries of standard $n$-tuples $\mathcal{O}_E^s$.

When $H = \text{Sp}_{2n}$, we use $\psi_T : (\mathcal{O}_E^s)^n \simeq T^0 \leftrightarrow I \hookrightarrow \text{GL}_{2n}(\mathcal{O}_E)$
$$(x_1^*, \cdots, x_n^*) \leftrightarrow \text{diag}(x_1^*, \cdots, x_n^*, (x_1^*)^{-1}, \cdots, (x_n^*)^{-1})$$
to denote a set of standard coordinates on $T^0$. We embed $\mathcal{H}(\mathcal{O}_E) \hookrightarrow \text{GL}_{2n}(\mathcal{O}_E)$ such that $\psi_T \times \psi_T : (\mathcal{O}_E^s)^n \simeq T^0 \hookrightarrow \mathcal{N}_I \times T^0 \hookrightarrow \text{GL}_{2n}(\mathcal{O}_E)$
$$(x_1, \cdots, x_{2n}) \leftrightarrow (x_1^*, \cdots, x_n^*) \leftrightarrow \text{diag}(x_1^*, \cdots, x_n^*, (x_1^*)^{-1}, \cdots, (x_n^*)^{-1})$$
where $X = 
$$
\begin{pmatrix}
 x_1 & \cdots & x_n \\
 \vdots & \ddots & \vdots \\
 x_{2n} & \cdots & x_{n+1}
\end{pmatrix}
$$. \quad D = \text{diag}(x_1^*, \cdots, x_n^*)$$. It is clear that there is a projection $p_{GL}$, realizing the embedding $i_{GL} : \mathcal{N}_I \hookrightarrow \mathcal{N}_{\text{GL}_{2n}}$ as a section of it with respect to the standard coordinates of $\mathcal{N}_{\text{GL}_{2n}}$ by simply
forgetting coordinates of lower triangular parts of both top left $n \times n$ block and bottom right $n \times n$ block.

When $H$ is a unitary group defined by an unramified quadratic extension $K/E$ ($\text{Gal}(K/E) \simeq \{1, c \}$) and Hermitian form

\[ J_n = \begin{pmatrix} 0 & \Psi_n \\ -\Psi_n & 0 \end{pmatrix}, \]

where $\Psi_n$ is the matrix with 1’s on the anti-diagonal and 0’s elsewhere: we use

\[ \psi_T : (\mathcal{O}_K^n)^n \simeq T^0 \leftrightarrow I \mapsto \text{GL}_{2n}(\mathcal{O}_K) \]

\[ \left( x_1^*, \ldots, x_n^* \right) \mapsto \text{diag}(x_1^*, \ldots, x_n^*, c(x_n^*)^{-1}, \ldots, c(x_1^*)^{-1}) \]

to denote a set of standard coordinates on $T^0$. We embed $\mathcal{H}(\mathcal{O}_E) \hookrightarrow \text{GL}_{2n}(\mathcal{O}_K)$ such that

\[ \psi_T^1 \times \psi_T : \mathcal{O}_E^1 \times (\mathcal{O}_K^n)^n \xrightarrow{\sim} \mathcal{N}_c^1 \times T^0 \hookrightarrow \text{GL}_{2n}(\mathcal{O}_K) \]

\[ \mathcal{E} \times (x_1^*, \ldots, x_n^*) \mapsto \left( \begin{array}{ccc} X & 0 \\ X'\Psi_nX & \Psi_n(tXc)^{-1}\Psi_n \end{array} \right), \]

where $\mathcal{E} = \left((x_1^1, \ldots, x_1^n), (x_1^1, \ldots, x_K^{n(n-1)})\right)$

\[ X = D + \mathcal{E}^s \cdot \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \frac{n(n-1)}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n^2-2n+1}{2} & \frac{n^2-2n+2}{2} & \cdots & 0 \end{pmatrix}, \]

\[ D = \text{diag}(x_1^*, \ldots, x_n^*) \]

\[ X' = \mathcal{E}^s(X'_d + X'_d + tX'^c), \]

where $X'_d = \text{diag}(x_1^d, \ldots, x_n^d)$

\[ X'_d = \begin{pmatrix} 0 & x_1^d & \cdots & x_n^d \\ 0 & 0 & \cdots & x_n^d \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \]

Choose any splitting $p_{K/E} : \mathcal{O}_K \rightarrow \mathcal{O}_E$, we use $\tilde{t}$ to denote a map from $n \times n$ matrices over $\mathcal{O}_K$ to $n \times n$ Hermitian matrices over $\mathcal{O}_E$

\[ \tilde{t} : M_{n \times n}(\mathcal{O}_K) \rightarrow H_{n \times n}(\mathcal{O}_K/\mathcal{O}_E) := \{ M \in M_{n \times n}(\mathcal{O}_K) | M = tM^c \} \]

\[ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \mapsto \begin{pmatrix} p_{K/E}(x_{11}) & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ c(x_{n1}) & \cdots & p_{K/E}(x_{nn}) \end{pmatrix}, \]

\[ i.e., M \mapsto \tilde{M} \text{ such that } \tilde{M}_{ij} = \begin{cases} M_{ij} & i < j \\ p_{K/E}(M_{ij}) & i = j \\ c(M_{ji}) & i > j \end{cases}. \]

We construct a projection $p_{\text{GL}}^\ast$ realizing the embedding $\mathcal{N}_I^\ast \hookrightarrow \mathcal{N}_{I\text{GL}_{2n}}^\ast$ as a section with respect to the
standard coordinates of \( \mathcal{N}_{I_{GL_{2n}}} \) as follows:

\[
p_{GL}^* : O_E^d \simto \mathcal{N}_{I_{GL_{2n}}} \rightarrow \mathcal{N}_{I}^{(\psi_1^{-1})} \rightarrow O_E^d
\]

We see coordinate functions of both \( i_s^* \) and \( p_{GL}^* \) are polynomials of \( \{x_E^1, \ldots, x_E^n, x_K, \ldots, x_K^{(n-1)}, c(x_K), \ldots, c(x_K^{(n-1)})\} \) with integral coefficients, hence polynomials of variables for \( O_E^d \).

We now give an explicit basis of \( C^{s,an}(\mathcal{N}_I, A) \) defined in 4. Suppose \( \iota_E \) gives rise to \( O_E = \bigoplus_{1 \leq i \leq |E| \cdot \mathbb{Q}_p} \mathbb{Z}_p u_i \), where \( u_1, \ldots, u_{|E| \cdot \mathbb{Q}_p} \in O_E \). For each \( g \in N_I/N_I^s \), we choose a lift of \( g \) as follows:

\[
(\bar{g}_1, \ldots, \bar{g}_d) := (\psi_1^{-1})^{-1}(g).
\]

We use \( G \subset N_I \) to denote this set of representatives of \( N_I/N_I^s \) with \( \bar{g} = g \).

Moreover, \( \psi_1 : (\bar{g}_1 + \bar{w}^{-1}O_E, \ldots, \bar{g}_d + \bar{w}^{-1}O_E) \simto g \cdot N_I \) through calculations for all cases. For \( x \in g \cdot N_I^s \), we change variables and let

\[
(\psi_1)^{-1}(x) = (\bar{g}_1 + \bar{w}^{-1}x_1, \ldots, \bar{g}_d + \bar{w}^{-1}x_d), \quad x_i = u_1 \cdot z_{i-1} + \cdots + u_{|E| \cdot \mathbb{Q}_p} \cdot z_{|E| \cdot \mathbb{Q}_p} \text{ for } 1 \leq i \leq d.
\]

Remark 7.1. We claim that a basis of \( C^{s,an}(\mathcal{N}_I, A) \) is given by

\[
f_{g,a}^*(x) := \begin{cases} z_1^{a_1} \cdots z_d^{a_{|E| \cdot \mathbb{Q}_p}} & x \in g \cdot N_I^s, \\ 0 & x \notin g \cdot N_I^s, \end{cases}
\]

which is indexed by \( g \in N_I/N_I^s \) and \( \mathbb{a} \in \mathbb{Z}_{\geq 0}^{d|E| \cdot \mathbb{Q}_p} \).

Proof. The rigid analytic functions \( C^{an}(\mathcal{N}_I^s, A) \) on \( N_I \) extend to s-analytic functions on \( N_I, C^{an}(\mathcal{N}_I, A) \rightarrow C^{s,an}(\mathcal{N}_I, A) \) by defining their values to be 0 on any other points. Then the following decomposition follows from definition:

\[
C^{s,an}(\mathcal{N}_I, A) = \bigoplus_{g \in N_I/N_I^s} g \cdot C^{an}(\mathcal{N}_I, A).
\]

It suffices to argue that for each \( g \in N_I/N_I^s \), \( \{f_{g,a}^* : a \in \mathbb{Z}_{\geq 0}^{d|E| \cdot \mathbb{Q}_p}\} \) is also an orthonormal basis of \( g \cdot C^{an}(\mathcal{N}_I, A) \). The \( O_E \) coordinates

\[
(x_1', \ldots, x_d') = (\psi_1)^{-1}(g^{-1} \psi_1 (\bar{g}_1 + \bar{w}^{-1}x_1, \ldots, \bar{g}_d + \bar{w}^{-1}x_d))
\]

of \( g \cdot N_I^s \) correspond to translation of coordinates on \( N_I^s \) by \( g \) in terms of \( (x_1, \ldots, x_d) \).

\[
x_i' = u_1 \cdot z_{i-1} + \cdots + u_{|E| \cdot \mathbb{Q}_p} \cdot z_{|E| \cdot \mathbb{Q}_p} \text{ for } 1 \leq i \leq d,
\]

a basis for \( g \cdot C^{an}(\mathcal{N}_I, A) \) is given by monomials of \( z_1', \ldots, z_d' \). Translation function of \( (\psi_1)^{-1} \circ \psi_1' \) is multiplication by \( \bar{w}^{-1} \). Direct calculations from explicit coordinates for all cases express \( (x_1', \ldots, x_d') \) in terms of polynomials of \( (x_1, \ldots, x_d) \) (resp. \( (x_1', \ldots, x_d') \)) with integral coefficients without constant terms. The same statement for transition functions between monomials of \( z_1', \ldots, z_d' \) and \( z_1, \ldots, z_d \) holds as well. \( \square \)
Proof of Proposition 6.2: Let \( e_1, \ldots, e_m \) be an eigenbasis of \( V \) for \( T^0 \) as free \( A \)-module, \( t \cdot e_i = \chi_i(t)e_i \) for any \( t \in T^0 \). For the first part, it suffices to show that any sum of \( f_{g,a}^\sigma \otimes e_i \) with converging to zero coefficients is a \( s \)-analytic function on \( \overline{N}_I \) and for any such sum,
\[
\sum_{g,a \in \mathfrak{g}} \lambda_g^a f_{g,a}^\sigma = 0 \iff \lambda_g^a = 0.
\]
We choose \( \tilde{g} \in \overline{N}_I \) lifting each \( g \),
\[
(\tilde{g}^1, \ldots, \tilde{g}^l) := (\psi_1)^{-1}(\tilde{g}), \quad \tilde{g}_i := (\tilde{g}_i^1, \ldots, \tilde{g}_i^l) \in \mathcal{O}_{E_i}^d.
\]
Suppose \( \iota_{E_i} \) gives rise to \( \mathcal{O}_{E_i} = \bigoplus_{1 \leq j \leq [E_i : \mathbb{Q}_p]} \mathbb{Z}_p u_j \). We choose coordinates \( x^i = \left(x_1^i, \ldots, x_{d_i}^i\right) \) on \( \mathcal{O}_{E_i}^d \), \( x = \left(x^1, \ldots, x^l\right) \) for \( \overline{N}_I = \prod_{1 \leq i \leq l} \mathcal{N}_I \) through \( \psi_i \) as in [23] and coordinates \( z_j^i \) for \( 1 \leq i \leq l, 1 \leq j \leq d_i, 1 \leq k \leq [E_i : \mathbb{Q}_p] \) such that
\[
x_i^j := (x_1^i, \ldots, x_{d_i}^i) = ((\tilde{g}_i^1 \cdot u_{1,1} + \cdots + u_{[E_i : \mathbb{Q}_p]} \cdot z_j^i), \ldots, (\tilde{g}_i^{d_i} \cdot u_{1,1} + \cdots + u_{[E_i : \mathbb{Q}_p]} \cdot z_j^i)).
\]
Apply Rem [23] to a product of Iwahori groups so an orthonormalizable basis for \( C^{s,\text{an}}(N_I, A) \) is given by
\[
f_{g,a}^\sigma(x) := \left\{ \begin{array}{ll}
\prod_{1 \leq i \leq l} \prod_{1 \leq j \leq d_i} \prod_{1 \leq k \leq [E_i : \mathbb{Q}_p]} (z_j^i)_{a_j,k} & x \in g \cdot \mathcal{N}_I \\
0 & x \notin g \cdot \mathcal{N}_I,
\end{array} \right.
\]
for \( g \) running through \( \mathcal{N}_I/N_I \), \( a \) running through \( \mathbb{Z}_{\geq 0}^{\sum_{1 \leq i \leq l} d_i \cdot [E_i : \mathbb{Q}_p]} \). For each, \( 1 \leq i \leq l \), we index the set of embedding of \( E_i \) to \( \mathbb{Q}_p \) as \( \{\sigma_1^i, \ldots, \sigma_{[E_i : \mathbb{Q}_p]}^i\} \). For each \( 1 \leq i \leq l, 1 \leq j \leq d_i, 1 \leq k \leq [E_i : \mathbb{Q}_p] \), there is a \( [E_i : \mathbb{Q}_p] \times [E_i : \mathbb{Q}_p] \) transition matrix
\[
M_{j,k}^i := \begin{pmatrix}
\sigma_1^i u_1^i & \cdots & \sigma_{[E_i : \mathbb{Q}_p]}^i u_{[E_i : \mathbb{Q}_p]}^i \\
\vdots & \ddots & \vdots \\
\sigma_{[E_i : \mathbb{Q}_p]}^i u_1^i & \cdots & \sigma_{[E_i : \mathbb{Q}_p]}^i u_{[E_i : \mathbb{Q}_p]}^i
\end{pmatrix}
\]
such that
\[
\begin{pmatrix}
\sigma_j^i u_1^i \\
\vdots \\
\sigma_j^i u_{[E_i : \mathbb{Q}_p]}^i
\end{pmatrix} = \sigma_j^{s_1} \begin{pmatrix}
\sigma_1^i u_1^i & \cdots & \sigma_{[E_i : \mathbb{Q}_p]}^i u_{[E_i : \mathbb{Q}_p]}^i \\
\vdots & \ddots & \vdots \\
\sigma_{[E_i : \mathbb{Q}_p]}^i u_1^i & \cdots & \sigma_{[E_i : \mathbb{Q}_p]}^i u_{[E_i : \mathbb{Q}_p]}^i
\end{pmatrix} \begin{pmatrix}
z_{j,1}^i \\
\vdots \\
z_{j,[E_i : \mathbb{Q}_p]}^i
\end{pmatrix} + \begin{pmatrix}
\sigma_j^i \tilde{g}_j^i \\
\vdots \\
\sigma_j^i \tilde{g}_{d_i}^i
\end{pmatrix}.
\]
Then we have
\[
f_{g,a}^\sigma = \sum_{g,a \in \mathfrak{g}} t_{g,a} f_{g,a}^\sigma,
\]
for integral transition coefficients \( t_{g,a} \). This shows that any convergent sum of \( f_{g,a}^\sigma \) belongs to \( C^{s,\text{an}}(\overline{N}_I, A) \).
\[
C^{s,\text{an}}(\overline{N}_I, V) \simeq \bigoplus_{i=1}^m C^{s,\text{an}}(\overline{N}_I, A) \otimes e_i,
\]
any convergent sum of \( f_{g,a}^\sigma \otimes e_i \) belongs to \( C^{s,\text{an}}(\overline{N}_I, V) \). The claim for density follows from the fact that \( \det(M_{j,k}^i) \neq 0 \), hence invertible over \( A \) for all \( 1 \leq i \leq m \).
l, 1 ≤ j ≤ d_i, 1 ≤ k ≤ [E_i : Q_p]. Now we form \((\prod_{s \in S} V_s)^c\) for the basis \(f_{\sigma, a_{\mathfrak{w}}, a_{\mathfrak{o}} r}^s \otimes e_i\) such that \((\prod_{s \in S} V_s)^c \rightarrow C^{s, an}(N, V)\) is \(T_s\) equivariant. We have the commutative diagram

\[
\begin{array}{ccc}
(\prod_{s \in S} V_s)^c & \longrightarrow & C^{s, an}(N, V) \\
\downarrow & & \downarrow \\
C^{s,1}(N, V) & \longrightarrow & C^{s, an}(N, V).
\end{array}
\]

The bottom arrow is \(T_s\)-equivariant and continuous since all \(f_{\sigma, a_{\mathfrak{w}}, a_{\mathfrak{o}} r}^s\) are in the unit ball of \(C^{s, an}(N, A)\) by the discussion above.

We claim the arrow \((\prod_{s \in S} V_s)^c \rightarrow C^{s,1}(N, V)\) is a bijection. Otherwise the kernel of the bottom arrow is nonempty, by the Lem \ref{lem:kernel} there exists a weight mapping to zero, i.e., a finite linear combination of \(f_{\sigma, a_{\mathfrak{w}}, a_{\mathfrak{o}} r}^s \otimes e_i\) equals to zero, which implies a finite sum of \(f_{\sigma, a_{\mathfrak{w}}, a_{\mathfrak{o}} r}^s\) equals to zero. The coefficients must be zero. We endow the Banach \(A\)-module structure to \(C^{s,1}_{w, \chi}\) from \((\prod_{s \in S} V_s)^c\).

Let \(K/E\) be a finite extension of \(p\)-adic local fields with the ring of integers \(\mathcal{O}_K/\mathcal{O}_E\) and uniformizers \(\varpi_K, \varpi_E\). We call an analytic function \(f \in K\langle z_1, \cdots, z_d\rangle\) on \((\mathcal{O}_K)^d\) \textit{overconvergent} if for \(f = \sum_{a \in \mathbb{Z}^d, |a| \geq 1} c_a z^a\), there exists \(\varepsilon > 1\) such that

\[
\lim_{|a| \to \infty} |c_a| \cdot \varepsilon^{|a|} < \infty,
\]

in which case \(f\) is said to be of rate \(\varepsilon\). This is equivalent to that \(f\) converges in a larger open ball than the unit ball in \(d\) dimensional space over an extension of \(K\), or \(f\) is represented by a function in a certain formal power series ring over \(\mathcal{O}_K\) with \(p\) inverted.

\textbf{Definition 7.2.} An analytic map \(g : (\mathcal{O}_K)^{d_1} \rightarrow (\mathcal{O}_K)^{d_2}\) is said to be overconvergent (resp. overconvergent with integral coefficients) if all the \(d_2\) coordinate functions are so. If \(d_1 = d_2\), \(g\) is bijective and \(g^{-1}\) is also overconvergent (resp. overconvergent with integral coefficients), \(g\) is called an overconvergent (resp. overconvergent with integral coefficients) isomorphism. Moreover, we say \(g\) is of rate \(\varepsilon\) if the coordinate functions are so.

\textbf{Remark 7.3.} Our notion of overconvergent isomorphism is stronger than the notion “strict isomorphism” in \cite{AS08}, where the authors only require the coordinate functions to be in the Tate algebra of radius 1.

\textbf{Lemma 7.4.} For two analytic maps \(g : (\mathcal{O}_K)^{d_1} \rightarrow (\mathcal{O}_K)^{d_2}\) and \(g' : (\mathcal{O}_K)^{d_2} \rightarrow (\mathcal{O}_K)^{d_3}\), if \(g\) is overconvergent with integral coefficients of rate \(\varepsilon\), then \(g' \circ g\) is overconvergent of rate \(\varepsilon\). overconvergent with integral coefficients map \(\mathcal{O}_K^{d_1} \rightarrow \mathcal{O}_K^{d_2}\) pulls back any rigid function (analytic power series with coefficients converging to 0) on \(\mathcal{O}_K^{d_2}\) to a rigid function on \(\mathcal{O}_K^{d_1}\).

\textbf{Proof.} These are derived by direct calculations. \(\square\)
We choose an isomorphism \( i_{K/E} : \mathcal{O}_K \simeq \mathcal{O}_E^{[K:E]} \). For an analytic map \( g : (\mathcal{O}_K)^{d_1} \to (\mathcal{O}_K)^{d_2}, i_{K/E} \) induces
\[
g : \mathcal{O}_K^{d_1} \xrightarrow{\quad} \mathcal{O}_K^{d_2}
\]
\[
(\iota_{K/E})^{d_1} \simeq \mathcal{O}_E^{[K:E]d_1} \xrightarrow{\quad} \mathcal{O}_E^{[K:E]d_2}
\]
\[
\tilde{g} : \mathcal{O}_E^{[K:E]d_1} \xrightarrow{\quad} \mathcal{O}_E^{[K:E]d_2}.
\]

**Proposition 7.5.** If \( g \) is overconvergent (resp. overconvergent with integral coefficients) of rate \( \varepsilon \), then so is \( \tilde{g} \) (resp. overconvergent with integral coefficients).

**Proof.** We use \( p_j : \mathcal{O}_K \to \mathcal{O}_E, 1 \leq j \leq [K : E] \) to denote the \( j \)-th projection to the \( j \)-th \( \mathcal{O}_E \) of the map \( i_{K/E} \). For any \( x \in \mathcal{O}_E, x = \omega_k \cdot y \) for some \( n \geq 0 \), and \( y \in \mathcal{O}_E \) such that \( |y| > |\omega_k| \). This map is \( \mathcal{O}_E \)-linear and thus has the property that \( |p_j(x)| \leq \frac{|x|}{|\omega_k|} \) for any \( x \in \mathcal{O}_E \). \( i_{K/E} \) gives a basis \( \{ e_1, \cdots, e_{[K:E]} \} \subset \mathcal{O}_K \) for \( \mathcal{O}_K \) such that \( z_i = z_1^{e_1} + \cdots + z_i^{e_{[K:E]}} \) for coordinate functions \( z_i \in \mathcal{O}_K \) and \( \{ z_1, \cdots, z_{[K:E]} \} \) of \( \mathcal{O}_E^{[K:E]} \) (1 \( \leq i \leq d_1 \)). Consider any coordinate function \( f : \mathcal{O}_K^{d_1} \to \mathcal{O}_K \) of \( g, f = \sum_{a \in \mathbb{Z}_{\geq 0}} c_a z_a \),
\[
p_j \circ f \circ (i_{K/E}^{-1})^{[K:E]d_1} = \sum_{a \in \mathbb{Z}_{\geq 0}} p_j(c_a) \prod_{1 \leq i \leq d_1} \sum_{j \in \mathbb{Z}_{[E]} \cdot |j| = a_i} z_i^{j} z_i^{-j}
\]
for some \( c_a \in \mathcal{O}_E \). The coefficient of each monomial is bounded by \( p_j(c_a) \leq \frac{|x|}{|\omega_k|} \). The same rate \( \varepsilon \) of staying bounded for \( f \) then works for the coordinate function \( p_j \circ f \circ (i_{K/E}^{-1})^{[K:E]d_1} \).

**Remark 7.6.** The overconvergent property of \( \tilde{g} \) does not depend on choices of \( i_{K/E} \) by the lemma above since linear isomorphisms are overconvergent.

For \( \mathcal{H}/\mathcal{O}_E = \text{GL}_n, \text{Sp}_{2n} \) or a unitary group associated to \( J_n \) and an unramified extension \( K/E \), and each \( s \geq 1 \), we use \( \iota^n_T \) to denote
\[
\iota^n_T : \mathcal{O}_E^s \text{ (resp. } \mathcal{O}_K^s) \simeq T^s
\]*
\[
(y_1, \cdots, y_n) \mapsto \psi_T(1 + \omega^s y_1, \cdots, 1 + \omega^s y_n).
\]

**Lemma 7.7.** Let \( \mathcal{H}/\mathcal{O}_E = \text{GL}_n, \text{Sp}_{2n} \) or a unitary group associated to \( J_n \) and an unramified extension \( K/E \). We equip \( \overline{N}_I \subset I \) the coordinates described in the beginning of this section. For any \( n \in 1 \), \( g \) induces an automorphism
\[
\tilde{g}_n : \mathcal{O}_E^d \underset{\psi_i}{\xrightarrow{\quad}} \overline{N}_I \quad \mapsto \quad I = \overline{N}_I \times T^0 \times N_1 \rightarrow \overline{N}_I \rightarrow \overline{N}_I \rightarrow \mathcal{O}_E^d
\]
\[
x \mapsto \quad g \cdot x \rightarrow \text{proj}_1(g \cdot x),
\]
where the second map is projection to the \( \overline{N}_I \) factor via the Iwahori decomposition. Let \( I_n^0 \) be the subgroup of \( I \) defined in \( \mathcal{I} \). Let \( \{ t_1, \cdots, t_l \} \) be a set of representatives of \( T^0/T^* \). For \( g = \pi_{0} \cdot t_{g} \cdot n_{g} \in I_{n}^{0}, \) for \( \pi_{0} \in \overline{N}_{I}, t_{g} \in T^{0} \) (with representative \( t_{0}, \) \( n_{g} \in N_{I} \)), \( g \) induces a map
\[
\tilde{g}_n : \mathcal{O}_E^d \underset{\psi_i}{\xrightarrow{\quad}} \overline{N}_I \quad \mapsto \quad I_0 \rightarrow T^0 \rightarrow T^0 \rightarrow T^* \rightarrow \mathcal{O}_E^d \text{ (resp. } \mathcal{O}_K^s \simeq \mathcal{O}_E^{2n})
\]
\[
x \mapsto \quad g \cdot x \rightarrow \text{proj}_2(g \cdot x) \rightarrow t_{g}^{-1} \text{proj}_2(g \cdot x),
\]
where \( \text{proj}_a \) is projection to the \( T^0 \) factor and the image lands in \( t_{g_0} \cdot T^s \).

Coordinate functions of \( \hat{g}_n \) and \( \hat{g}_t \) are represented by functions \( \frac{u}{\omega} \), with \( u \in \mathcal{O}_E[[(\omega x_1), \cdots, (\omega x_m)]] \). In particular, \( \hat{g}_n \) is an overconvergent with integral coefficients isomorphism of rate \( |\omega|^{-1} \) and coordinate functions of \( \hat{g}_t \) are overconvergent with integral coefficients of rate \( |\omega|^{-1} \).

Proof. We prove both claims for a fixed \( g \in I_0^0 \) at the same time, note that for the first claim for \( \hat{g}_n \), we just apply the case \( I_0^0 = I \). We first prove the case \( H = \text{GL}_m \).

If \( g \in \overline{N}_I \times T^0 = T^0 \times \overline{N}_I \), coordinate functions are linear polynomials with integral coefficients. By the Iwahori decomposition and Lem 7.4, it reduces to only consider

\[
g = \begin{pmatrix}
g_1 & \cdots & \cdots & g_m \\
1 & \cdots & \cdots & \cdots \\
\vdots & \ddots & & \vdots \\
0 & \cdots & 1 & 1
\end{pmatrix} \in N_I^s, \quad m = \sum_{1 \leq i \leq n-1} i = \frac{n(n-1)}{2}.
\]

Let \( x = \begin{pmatrix}
1 \\
\omega x_1 & 1 & 0 \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & 1 \\
\omega x_m & \cdots & x_{m,g} & 1
\end{pmatrix} \in \overline{N}_I \)

and \( x_{1,g}, \cdots, x_{m,g} \) be translated coordinate functions \( \hat{g} \) of \( \overline{N}_I \) with respect to left multiplication by \( g \).

\[
x_g = \begin{pmatrix}
1 \\
x_{1,g} & 1 & 0 \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & 1 \\
\omega x_m & \cdots & x_{m,g} & 1
\end{pmatrix} \in \overline{N}_I
\]

satisfying \( g \cdot x = x_g \cdot t \cdot n_I, \ t \in T^s, n_I \in \overline{N}_I \). \( \Rightarrow \)

\[
\begin{pmatrix}
1 \\
x_{1,g} & 1 & 0 \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & 1 \\
\omega x_m & \cdots & x_{m,g} & 1
\end{pmatrix} \cdot t \cdot n_I = \begin{pmatrix}
1 \\
g_1 & \cdots & \cdots & \cdots \\
\omega x_1 & 1 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & \omega x_m & 1
\end{pmatrix},
\]

\( g_i \in \mathcal{O}_E \) for \( 1 \leq i \leq m \), we implement Gram–Schmidt process to the result of right hand side to produce our desired \( x_{1,g}, \cdots, x_{m,g} \). Now we view all the coefficients as inside the formal power series ring of Tate algebra \( \mathcal{O}_E[[(\omega x_1), \cdots, (\omega x_m)]] \subset E(\omega x_1, \cdots, \omega x_m) \). Let

\[
\mathcal{J} = (\omega^a), \quad \mathfrak{m} = (\omega x_1, \cdots, \omega x_m) \subset \mathcal{O}_E[[(\omega x_1), \cdots, (\omega x_m)]].
\]
The matrix \( g \cdot x \) has the property that lower triangular entries are in \( m \), upper triangular entries are in \( \mathfrak{f} \) and diagonal entries are in \( 1 + m \mathfrak{f} \). The Gram–Schmidt process produces \( x_g, t, n_I \) such that entries of \( x_g \) are in \( m \), entries of \( t \) are in \( 1 + m \mathfrak{f} \) and entries of \( n_I \) are in \( \mathfrak{f} \). This is true since the right hand side is an upper unipotent matrix mod \( m \) and every time we want to invert an element, that element must be a diagonal element and always be in \( 1 + m \mathfrak{f} \). So we obtain all \( x_{1,g}, \ldots, x_{m,g} \in m \subseteq \mathcal{O}_E[[\{\omega x_1\}, \ldots, (\omega x_m)]] \subseteq E \langle x_1, \ldots, x_m \rangle \) and \( \frac{1 + m^{n-1}}{2} = m \), which translates to the statement that both \( \hat{g}_n \) and \( \hat{g}_t \) are overconvergent with integral coefficients.\(^2\)

For other cases, we have constructed embeddings \( i^\times_{\text{GL}} : \overline{N}_{I_{GL}_{2n}} \to \overline{N}_{I_{GL}_{2n}} \) and sections of projections \( \hat{p}^1_{\text{GL}} : \overline{N}_{I_{GL}_{2n}} \to \overline{N}_I \) at the beginning of this section. We have seen that coordinate functions of both \( \hat{p}^1_{\text{GL}} \) and \( \hat{p}^1_{\text{GL}} \) are polynomials of integral coefficients. Since \( i^\times_{\text{GL}} : I \to I_{GL_{2n}} \) is compatible with respect to the Iwahori decomposition, we have the following commutative diagram:

\[
\begin{array}{ccc}
\overline{\text{GL}}(g) : \overline{N}_{I_{GL}_{2n}} & \longrightarrow & I_{GL_{2n}} \\
\downarrow \text{id}_{\overline{N}_{I_{GL}_{2n}}} & & \downarrow \text{id}_{I_{GL_{2n}}} \\
\hat{g} : \overline{N}_I & \longrightarrow & I \\
\end{array}
\]

By Lem \([7,4]\) we conclude that \( \hat{g}_n = \hat{p}^1_{\text{GL}} \circ i^\times_{\text{GL}}(g) \circ \text{id}_{\overline{N}_{I_{GL}_{2n}}} \) is overconvergent with integral coefficients. Replace \( g \) by \( g^{-1}, g^{-1}_n = \hat{g}_n^{-1} \) is overconvergent with integral coefficients as well. The argument works similarly for \( \hat{g}_t \) as well. \( \square \)

**Proof of Proposition \([6,6]\).** By Lem \([5,7]\), if a prescribed group element induces a nice endomorphism of \( D_I^{-1}(V) \) if it stabilizes \( C^s_{\text{an}}(N_I, V) \). It remains to prove that \( C^s_{\text{an}}(N_I, A) \) is \( I_0^s \)-stable for \( I \) being a single factor of Iwahori \((l = 1 \text{ case})\), as any element of the eigenbasis we introduced is a product of eigenfunctions on \( N_I \) factors. Let \( \{t_1, \ldots, t_I\} \) be in Lem \([7,7]\) \( \chi_0(t_n) \in A^x \). For any \( f \in C^s_{\text{an}}(N_I, A) \) and \( g \in I_0^s \),

\[
(g^{-1} \cdot f)(x) = f(g \cdot x) = f(\hat{g}_n(x))\chi_0(t_n) \hat{g}_t(x).
\]

Since \( \hat{g}_n \) is an overconvergent with integral coefficients automorphism by Lem \([7,7]\) \( f(\hat{g}_n \cdot -) \in C^s_{\text{an}}(N_I, A) \) by Lem \([6,4]\). Recall the assumption on \( s \) in \([5,4]\) \((s \geq s(\Omega))\) and results in \([3,3]\) \( \chi_0|_{T^*} \) is uniquely decomposed as \( (\sigma, E) \)-analytic characters of \( i^*_{\text{GL}} : \mathcal{O}_E^\times (\text{resp. } \mathcal{O}_{E_0}^\times) \simeq T^*, \chi_0(\hat{g}_t \cdot -) \in C^s_{\text{an}}(N_I, A) \) as well.

The group multiplication \( I \times I \to I \) is analytic, and so is its compositions with \( \text{proj}_1, \text{proj}_2 \):

\[
(3) \quad I \times N_I \to I \to \overline{N}_I, \quad \hat{m}_T : I_0^s \times N_I \to I \to T^0.
\]

The first morphism \([3,3]\) induces the pullback \( C^a_{\text{an}}(N_I, A) \to C^a(I, \mathbb{Q}_p) \otimes C^a_{\text{an}}(N_I, A) \), therefore inducing \( C^s_{\text{an}}(N_I, A) \to C^a(I, \mathbb{Q}_p) \otimes C^s_{\text{an}}(N_I, A) \) since the space of characteristic functions on \( N_I \) cosets are preserved by left \( I \) translations. And by results in \([3,3]\) \( \chi_0 \) is \( \mathbb{Q}_p \)-analytic on \( T^* \), the pullback of this character via \( \hat{m}_T \) is represented by a function in \( C^a(I_0^s, \mathbb{Q}_p) \otimes C^s_{\text{an}}(N_I, A) \) by the same reason. By definition, the \( I_0^s \) action on \( C^s_{\text{an}}(N_I, A) \) is analytic. By \([7,17]\) Cor 5.1.9, \( D_I^{s,1}(V) \) is a continuous \( D(I_0^s, \mathbb{Q}_p) \) module. \( \square \)

\(^2\)The Gram–Schmidt process used here can be regarded as an analogue of Iwahori decomposition for the complete local ring \( \mathcal{O}_E[[\{\omega x_1\}, \ldots, (\omega x_m)]] \).
8. Koszul complex and a comparison of $N$ cohomology

Let $\Gamma \simeq \mathbb{Z}^n$ be a finitely generated torsion free abelian group. Let $N \simeq \mathbb{Z}_p^n$ be a compact $p$-adic analytic abelian group containing $\Gamma$ as a dense subgroup with the $\mathbb{Z}_p$ action via module structure of $\mathbb{Z}_p^n$. Let $n$ be its Lie algebra over $\mathbb{Q}_p$. We prove $H^*(\Gamma, \mathbb{D}_n^{\omega, \Omega}) \simeq H^*_n(N, \mathbb{D}_n^{\omega, \Omega}) \simeq H^*(n, \mathbb{D}_n^{\omega, \Omega})^N$ in the sense of Kohlhaase, and the second isomorphism can be deduced by applying a general result in [Koh11].

The orbits of this action are 1-parameter subgroups of $N$. Choose a basis of generators $e_1, \cdots, e_n$ of $\Gamma$, $\Gamma \subset \mathbb{Z}_p \cdot e_1 + \cdots + \mathbb{Z}_p \cdot e_n \subset N$ for a dense $\Gamma$, hence $N = \mathbb{Z}_p \cdot e_1 \oplus \cdots \oplus \mathbb{Z}_p e_n$. Therefore we assume $\Gamma < N$ is induced by the standard inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$ with (topological) generators $e_1, \cdots, e_n$.

Let $K$ be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{M}_N$ denote the category of complete Hausdorff locally convex $K$-vector spaces with the structure of a separately continuous $D(N, K)$-module, taking as morphisms all continuous $D(N, K)$-linear maps as in [Koh11].

Let $\mathcal{W}_N \simeq \mathcal{W}_N^n$ be the (continuous) weight space of $N$ over $K$. For any $z = (z_1, \cdots, z_n) \in \mathcal{W}_N(K)$, there is a locally $\mathbb{Q}_p$-analytic $K$-valued character $\kappa_z$ such that

$$\kappa_z(a) = \prod_{i=1}^n (1 + z_i)^{a_i}, \quad a = \sum_{i=1}^n a_i e_i \in N.$$  

So we have the embedding

$$\mathcal{W}_N(K) \hookrightarrow C^{an}(N, K)$$  

$$z \mapsto \kappa_z.$$  

Hence any linear form $\lambda \in D(N, K)$ gives rise to the function

$$F_\lambda(z) := \lambda(\kappa_z)$$  

on $\mathcal{W}_N(K)$ which is called the Fourier transform of $\lambda$. Moreover, $F_\lambda$ is a rigid function on $\mathcal{W}_N$ with coefficients in $K$. Let $\mathcal{O}(\mathcal{W}_N)$ denote the ring of all $K$-rigid functions on $\mathcal{W}_N$,

$$\mathcal{O}(\mathcal{W}_N) = \{ F(T_1, \cdots, T_n) = \sum_{m \in \mathbb{Z}_p^n} a_m T^m, \quad a_m \in K, \text{which converge on } \mathcal{W}_N/\mathbb{Q}_p(C_p). \}$$

We have the following multivariable Amice’s theorem stated as Thm 2.2. in [ST01].

**Theorem 8.1** (Amice). The Fourier transform

$$D(N, K) \xrightarrow{\simeq} \mathcal{O}(\mathcal{W}_N)$$  

$$\lambda \mapsto F_\lambda$$  

is an isomorphism of $K$-Fréchet algebra.

One can embed the usual group algebra $K[\Gamma] \subset K[N]$ as dense subalgebras of $D(N, K)$ viewing a group element as a Dirac distribution as in [ST02]. Under the Fourier transform,

$$e_i \mapsto 1 + T_i, \quad 1 \leq i \leq n,$$

$$K[\Gamma] \simeq K[T_1, \cdots, T_n, (1 + T_1)^{-1}, \cdots, (1 + T_n)^{-1}] \subset D(N, K).$$

**Lemma 8.2.** $(T_1, \cdots, T_n) \subset K[\Gamma] \subset \mathcal{O}(\mathcal{W}_N)$ form a regular sequence for both algebras.
Proof. For $K[\Gamma]$, the result follows from $K[T_1, \cdots, T_n, (1+T_1)^{-1}, \cdots, (1+T_n)^{-1}]/(T_1) \simeq K[T_2, \cdots, T_n, (1+T_2)^{-1}, \cdots, (1+T_n)^{-1}].$

For any $m = (m_2, \cdots, m_n) \in \mathbb{Z}_{\geq 0}^{n-1}$, we use $T_{m}^m$ to denote $T_2^{m_2} \cdots T_n^{m_n}$. We construct an algebra homomorphism

$$\mathcal{O}(\mathcal{W}_p) \to \mathcal{O}(\mathcal{W}_p^{n-1})$$

$$\sum_{i=0}^{+\infty} \sum_{m \in \mathbb{Z}_{\geq 0}^{n-1}} a_i, m T_i T_{m}^m \mapsto \sum_{m \in \mathbb{Z}_{\geq 0}^{n-1}} a_0, m T_{m}^m.$$

If we show that the kernel of this map is the principal ideal generated by $T_1$, then we prove the statement inductively since $\mathcal{O}(\mathcal{W}_p)$ is an integral domain. Suppose that $f \in \mathcal{O}(\mathcal{W}_p)$ maps to 0, then $\hat{f}$ is a power series with coefficients in $K$, it certainly converges on $\mathcal{W}_p(C_p)$ as $f$ converges on it. \qed

Let $R$ be a commutative ring and a sequence $x_1, \cdots, x_l$ of elements of $R$. There is a so called Koszul complex $K(x_1, \cdots, x_l)$ associated to such a sequence.

$$0 \to \bigwedge^l R^l \xrightarrow{d_l} \bigwedge^{l-1} R^l \to \cdots \to \bigwedge^0 R^l \xrightarrow{d_1} R \to R/(x_1, \cdots, x_l) \to 0,$$

where $R^l = \bigoplus_{i=1}^{l} R s_i$ and the differential $d_k$ is given by: for any $1 \leq i_1 < \cdots < i_k \leq l$,

$$d_k(s_{i_1} \wedge \cdots \wedge s_{i_k}) = \sum_{j=1}^{k} (-1)^{i_j+1} x_{i_j} s_{i_1} \wedge \cdots \wedge \hat{s}_{i_j} \wedge \cdots \wedge s_{i_k}.$$

A fundamental theorem for the Koszul complex is:

**Theorem 8.3 ([Mat89] Thm 16.1).** If $(x_1, \cdots, x_l)$ is a regular sequence of elements in $R$, then the Koszul complex $K(x_1, \cdots, x_l)$ is a free resolution of the quotient ring $R/(x_1, \cdots, x_l)$.

Now we are ready to state and prove the main theorem of this section.

**Theorem 8.4.** There are natural $K$-linear isomorphisms

$$H^q(\Gamma, V) \simeq H^q(n, V)^N$$

for all $q \geq 0$ and any object $V$ of $\mathcal{M}_N$.

Proof. $\mathbb{Z}[\Gamma] \hookrightarrow K[\Gamma]$ is flat since flatness is stable under base change $\mathbb{Z} \hookrightarrow K$, therefore

$$H^q(\Gamma, V) = \text{Ext}^q_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, V) \simeq \text{Ext}^q_{K[\Gamma]}(K, V).$$

Lem $\text{S.2}$ and $\text{S.3}$ enables us to use the Koszul complex $K(T_1, \cdots, T_n)/K[\Gamma]$ to represent $K$. One checks that

$$K(T_1, \cdots, T_n)/K[\Gamma] \otimes_{K[\Gamma]} D(N, K) \simeq K(T_1, \cdots, T_n)/D(N, K),$$

which again represents $K$ as a $D(N, K)$-module by Lem $\text{S.2}$ and Lem $\text{S.3}$. Then

$$\text{Ext}^q_{K[\Gamma]}(K, V) \simeq \text{Ext}^q_{D(N, K)}(K, V) \simeq H^q_{\text{an}}(N, V) \simeq H^q(n, V)^N$$

by $[\text{Koh11}]$ Thm 4.8, Thm 4.10. \qed
Remark 8.5. If moreover $V$ has a $A = \mathcal{O}(\Omega)$-module structure which commutes with the group action, then this isomorphism is as an $A$-linear isomorphism since $A$ action commutes with the $D(N, K)$ action.

9. $N$ Cohomology of $D_{w, \Omega}$

We keep the same notation as in the previous sections. Let $G_i$ be either $GL_{2n}$ or $Sp_{2n}$ over $E_i$ or the unitary group associated to the Hermitian form $J_n$ and an unramified extension $K_i/E_i$ with various subgroups and their derived compact $p$-adic groups as in $[\mathbb{A}]$ For distinguishing these $p$-adic groups associated with the different algebraic groups, we use subscripts to specify. For example, the Iwahori subgroup of $G_i(E_i)$ is denoted by $I_{G_i}$.

For the case of $GL_{2n}$ and $Sp_{2n}$, we specify each Borel $B_i$ as the upper triangular Borel subgroup, and each $P_i$ to be the standard Siegel parabolic whose lower left $n \times n$ quadrant is zero:

$$
\begin{bmatrix}
\ast & \ast \\
0 & \ast
\end{bmatrix}.
$$

For the unitary group case, $B_i$ and $P_i$ are taken to be the pullbacks of the upper triangular Borel and standard Siegel parabolic via the standard embedding $G_i \hookrightarrow \text{Res}_{K_i/E_i}GL_{2n}$.

We make several abbreviations for this section:

$$W_G := \prod_{i=1}^l W(R_{E_i/Q_p}G_i, R_{E_i/Q_p}T_i)(\mathbb{Q}_p),$$

$$W_L := \prod_{i=1}^l W(R_{E_i/Q_p}L_i, R_{E_i/Q_p}T_i)(\mathbb{Q}_p),$$

$$W^P := \prod_{i=1}^l W^P_{G_i, S_i}, \quad w = \prod_{i=1}^l (w_i) \in W^P,$$

$$N := \prod_{i=1}^l (N_i(E_i) \cap w_iI_{G_i}w_i^{-1}),$$

$$n_i := \text{Lie}(N_i(E_i))/E_i, \quad n := \text{Lie}(N)/Q_p = \bigoplus_{i=1}^l n_i.$$

As $T_{G_i}^s = T_{L_i}^s$, we use $T_{L_i}^s$ to denote both groups.

$$T^0 := \prod_{i=1}^l T_i^0, \quad I_G := \prod_{i=1}^l I_{G_i}, \quad I_G^s := \prod_{i=1}^l I_{G_i}^s,$$

$$I_{0,G}^s := \prod_{i=1}^l I_{0,G_i}, \quad N_G := \prod_{i=1}^l N_{G_i}, \quad N_G^s := \prod_{i=1}^l N_{G_i},$$

$$g_i := \text{Lie}(I_{G_i})/Q_p, \quad n_{b,i} := \text{Lie}(N_{G_i})/Q_p, \quad \bar{n}_{b,i} := \text{Lie}(\bar{N}_{G_i})/Q_p,$$

$$g := \prod_{i=1}^l g_i, \quad n_b := \prod_{i=1}^l n_{b,i}, \quad \bar{n}_b := \prod_{i=1}^l \bar{n}_{b,i}.$$
For the Levi \( L \), we define
\[
I_L := \prod_{i=1}^{l} I_{L_i}, \quad B_L := T^0 \cdot N_L, \quad I_{0,L}^s := \prod_{i=1}^{l} I_{0,L_i}^s, \quad 1 := \text{Lie}(I_L)/\mathbb{Q}_p,
\]
\[
N_L = \prod_{i=1}^{l} N_{L_i}, \quad \mathcal{N}_L = \prod_{i=1}^{l} \mathcal{N}_{L_i}, \quad \mathcal{N}_L := \mathcal{N}^L,
\]
\[
\mathcal{N}_+ := \prod_{i=1}^{l} w_i \mathcal{N}_{G_i}^s w_i^{-1} \cap N_i(E_i), \quad \mathcal{N}_- := \prod_{i=1}^{l} w_i \mathcal{N}_{G_i}^s w_i^{-1} \cap N_i(E_i).
\]
There are Iwahori decompositions
\[
I_G = \mathcal{N}_G \times T^0 \times N_G, \quad I_L = \mathcal{N}_L \times T^0 \times N_L.
\]

Let \( \Omega \) be an irreducible Zariski closed subspace of an affinoid subdomain of the weight space of \( T^0 \). The integral domain \( A = \mathcal{O}(\Omega) \) contains a big enough \( p \)-adic field \( K \) such that the weight decomposition for Lie algebra exists over \( K \). The Lie algebra \( \mathfrak{g} \otimes \mathbb{Q}_p \) decomposes as a direct sum of eigenspaces for \( T^0 \) action with (positive/negative) roots \( \Delta = \Delta^+ \cup \Delta^- \). Positive (resp. negative) roots correspond to eigenspaces of \( \mathfrak{n}_s \otimes \mathbb{Q}_p \) \( K \) (resp. \( \mathfrak{n}_s \otimes \mathbb{Q}_p \)).

We want to understand the structure of \( H^*(\mathfrak{n}, \mathbb{D}^s_{w,\Omega})^N \) as an \( I_L \) representation. And we establish the structure of the \( N \) cohomology of \( \mathbb{D}^s_{w,\Omega} \) via \( N \) cohomology of the auxiliary module \( \mathbb{D}^s_{w,\Omega} \). We have the following main theorem as below.

**Theorem 9.1.** For each \( w \in W^P \), let \( l(w) := \sum_{i=1}^{l} (E_i : \mathbb{Q}_p)|l(w_i) \), where the later \( l \) refers to the length function on absolute Weyl group \( W(G_s, T_i) \). The \( N \) cohomology \( H^l(w)(\mathfrak{n}, \mathbb{D}^s_{w,\Omega}) \simeq H^l(w)(\mathfrak{n}, \mathbb{D}^s_{w,\Omega})^N \) in degree \( l(w) \) admits a \( I_L \)-equivariant direct summand
\[
i : \mathbb{D}^s_{L,w,\Omega} \hookrightarrow H^l(w)(\mathfrak{n}, \mathbb{D}^s_{w,\Omega})^N.
\]

By Harish-Chandra homomorphism (for example [Kna86 Lem 8.17]), there is an infinitesimal character \( \theta^p_\Omega \) of \( Z(l) \) on \( \mathbb{D}^s_{L,w,\Omega} \).

We endow \( \mathcal{N}_L \) the analytic structure coming from \( GL_n \) case in \([\S 7] \), \( \mathcal{N}_L \) is isomorphic to copies of \( \mathbb{Z}_p \) as \( p \)-adic Lie groups, we use the canonical analytic structure on it.
Lemma 9.2. Fix a \( w \in W^P \), \( wN_Gw^{-1} \simeq \mathcal{N}^r_+ \times \mathcal{N}^L_+ \times \mathcal{N}^L_- \) as strict p-adic manifolds ([AS08 §3.3]) for any \( s \in \mathbb{Z}_{\geq 1} \).

Proof. It suffices to prove the statement for one group of interests \( G \) over \( E \). We apply [CGP15 Prop 2.1.8 (3), Prop 2.1.12] to the solvable group \( w\mathcal{N}_Bw^{-1} \) and the Siegel parabolic group \( P \) as \( P(\lambda) \). Together with the fact \( w\mathcal{N}_Gw^{-1} \cap L(E) = \mathcal{N}_L^r \) for \( w \in W^P(G, S) \), this gives us the desired bijection. Note that the (strict) analytic structures on both sides are compatible with the correlating schematic structures and the isomorphism in [CGP15 Prop 2.1.12] is schematic, the transition isomorphism is actually given by polynomials which is in the Tate algebra, hence strict in the sense of §3.3, [AS08]. □

We shall use Lem 9.2 to give a projection \( p_L \) from \( w\mathcal{N}_Gw^{-1} \) to \( \mathcal{N}_L \)

\[
\begin{array}{ccl}
\mathcal{N}_L & \xrightarrow{i_L} & w\mathcal{N}_Gw^{-1} \\
& \xrightarrow{p_L} & \mathcal{N}_L \end{array}
\]

such that \( p_L \circ i_L = \text{id} \).

We now construct \( i \) and \( p \) as claimed in Thm 9.1. First, we construct \( \tilde{i} \) and \( \tilde{p} \) as morphisms of \( \mathcal{O}(\Omega) \)-modules for \( l(w) \)-th term of the chain complex. There are natural \( I_L \) (resp. \( I_{0,L} \))-equivariant inclusions

\[
\begin{array}{ccc}
\tilde{i} : \mathcal{D}^s_{L,w,\Omega} & \hookrightarrow & \bigwedge^{l(w)} n^* \otimes \mathcal{D}^s_{w,\Omega} \\
\bigwedge^{l(w)} n^* \otimes \mathcal{D}^s_{w,\Omega} & \twoheadrightarrow & \bigwedge^{l(w)} n^* \otimes \mathcal{D}_{w,\Omega} \\
\tilde{p} : \mathcal{D}_{L,w,\Omega}^{s,1} & \hookrightarrow & \bigwedge^{l(w)} n^* \otimes \mathcal{D}_{w,\Omega}^{s,1} \\
\bigwedge^{l(w)} n^* \otimes \mathcal{D}_{w,\Omega}^{s,1} & \twoheadrightarrow & \bigwedge^{l(w)} n^* \otimes \mathcal{D}_{w,\Omega}^{s,1} \\
\end{array}
\]

which come from the natural \( I_L \) (resp. \( I_{0,L} \))-equivariant surjections by the Banach duality relation

\[
\begin{array}{ccc}
\tilde{i}^* : \bigwedge^{l(w)} n \otimes \text{Ind}^s_{w,\Omega} & \twoheadrightarrow & \text{Ind}_{L,w,\Omega}^s \\
\bigwedge^{l(w)} n \otimes \text{Ind}^s_{w,\Omega} & \twoheadrightarrow & \text{Ind}_{L,w,\Omega}^s \\
\end{array}
\]

We need two steps to construct the horizontal arrows above. Firstly, note that there is a natural \( I_L \)-equivariant surjection

\[
\text{Ind}^s_{w,\Omega} \twoheadrightarrow \text{Ind}^s_{L,w,\Omega} \\
f \mapsto f|_{I_L}
\]

where \( s \)-analytic functions restrict to \( s \)-analytic functions due to Lem 9.2 and [AS08 Prop 3.3.6].

Secondly, we need to use Mackey’s tensor product theorem: if \( V \) is a finite dimensional representation of \( I_L \),

\[
\phi : V \otimes \text{Ind}_{L,w,\Omega}^s \rightarrow \text{Ind}_{L,w,\Omega}^{s,an}(V \otimes \chi^w_{\Omega})
\]

\[
[\phi(v \otimes f)](g) = g^{-1}v \otimes f(g)
\]

is an \( I_L \)-equivariant linear isomorphism. Therefore there is a \( I_L \)-filtration on \( \bigwedge^{l(w)} n \otimes \text{Ind}_{L,w,\Omega}^s \), which comes from a filtration of \( \bigwedge^{l(w)} n \otimes Q_p K \) as \( B_L \) representations.
Lemma 9.3. If \( w \in W^P \), \( \bigwedge^{l(w)} n^*_p \) has a highest weight vector as a \( \mathbb{Q}_p \)-algebraic \( I_L \)-representation, in particular,
\[
\bigwedge^{l(w)} (n \cap w_\mathfrak{p}_w w^{-1})^* \subset \bigwedge^{l(w)} n^*_L \neq 0.
\]

The adjoint representation on the Lie algebra is algebraic, so do its wedge products, restrictions to subgroups and subquotients. Our Weyl group element \( w \) acts on \( \mathfrak{g} \) by conjugation and
\[
\dim_{\mathbb{Q}_p} \mathfrak{m}_w \cap w^{-1} n w = \sum_{i=1}^{l(w)} \dim_{\mathbb{Q}_p} \mathfrak{m}_{w_i} \cap w_{-1}^{-1} n w_i = \sum_{i=1}^{l(w)} [E_i : \mathbb{Q}_p] l(w_i) = l(w).
\]

The rest follows from the Kostant’s theorem on Lie algebra cohomology of \( \mathfrak{n} \) for trivial representation of \( \text{Lie}(I_L) \) when the base field is allowed to be non-algebraically closed (Kostant’s theorem applies since the involved representations of \( T^0 \) or Cartan subalgebra splits as sum of characters over \( K \)). See [Kos61, Car61, 4.2-4.4 of \( CO75 \)].

More specifically, The line \( \bigwedge^{l(w)} (n \cap w_\mathfrak{p}_w w^{-1})^* \) gives a highest weight of \( H^{l(w)} (n, 1)^N_L \) by [Kos61 Thm 5.14]. It should lift to a highest weight of \( \bigwedge^{l(w)} n^* \). Cartier further proves that \( \bigwedge^{l(w)} (n \cap w_\mathfrak{p}_w w^{-1})^* \) occurs in \( \bigwedge^{l(w)} n^*_L \) with multiplicity one. As \( w \in W^P \),
\[
\bigwedge^{l(w)} (n \cap w_\mathfrak{p}_w w^{-1})^* \subset \bigwedge^{l(w)} n^*_L \neq 0.
\]

For the last claim, note that \( \mathfrak{m} \simeq (n)^* \) as \( I_L \)-representations. Then the same argument applying to \( \bigwedge^{l(w)} \mathfrak{m} \) yields the claim. \hfill \Box

By Lem 9.3 the algebraic representation \( \bigwedge^{l(w)} n \otimes \mathbb{Q}_p K \) admits a filtration \( 0 = F_0 \subset F_1 \subset \cdots \subset F_m = \bigwedge^{l(w)} n \otimes \mathbb{Q}_p K \) as \( B_L \) representations such that \( \dim F_i = i \) and \( m = \dim_{\mathbb{Q}_p} (\bigwedge^{l(w)} n) \). The 1-dimensional quotient of the filtration \( F_m/F_{m-1} \) in the \( N_L \)-coinvariants is of algebraic weight \( -(\omega \delta - \delta) \). If we have a short exact sequence of finite dimensional \( B_L \) representations over \( K \)
\[
0 \to U \to V \to W \to 0,
\]
tensoring it with the rank 1 \( \mathcal{O}(\Omega) \)-module \( \chi^w_\Omega \), where \( B_L \) acts on it through \( T^0 \) via the character \( \chi^w_\Omega \), we arrive at
\[
0 \to U \otimes_K \chi^w_\Omega \to V \otimes_K \chi^w_\Omega \to W \otimes_K \chi^w_\Omega \to 0.
\]
By the left exactness of the induction we defined,

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Ind}_{L,\theta}^s(u \otimes \chi_{\Omega}^w) \\
\uparrow & & \uparrow \\
\text{Ind}_{L,\theta}^s(u \otimes \chi_{\Omega}^w) & \rightarrow & \text{Ind}_{L,\theta}^s(V \otimes \chi_{\Omega}^w) \\
\uparrow & & \uparrow \\
\text{Ind}_{L,\theta}^s(V \otimes \chi_{\Omega}^w) & \rightarrow & \text{Ind}_{L,\theta}^s(W \otimes \chi_{\Omega}^w)
\end{array}
\]

and the commutativity of the whole diagram follows from the definition of \(C^{s,1}\) and Prop 6.6.

Note that any \(K\)-linear section \(W \rightarrow V\) gives rise to \(\Theta(\Omega)\)-linear section \(W \otimes \chi_{\Omega}^w \rightarrow V \otimes \chi_{\Omega}^w\), which tells us the surjectivity of \(C^{s,an}(\overline{N}_L, V \otimes \chi_{\Omega}^w) \rightarrow C^{s,an}(\overline{N}_L, W \otimes \chi_{\Omega}^w)\). This shows the sequence is right exact as well.

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Ind}_{L,\theta}^s(u \otimes \chi_{\Omega}^w) \\
\uparrow & & \uparrow \\
\text{Ind}_{L,\theta}^s(V \otimes \chi_{\Omega}^w) & \rightarrow & \text{Ind}_{L,\theta}^s(W \otimes \chi_{\Omega}^w) \\
\uparrow & & \uparrow \\
\text{Ind}_{L,\theta}^s(W \otimes \chi_{\Omega}^w) & \rightarrow & 0
\end{array}
\]

Let \(U\) be \(F_{m-1}\), let \(V\) be \(F_m\), and let \(W\) be \(F_m/F_{m-1}\), we have

\[
\begin{array}{ccc}
\tilde{i}^* : \Lambda^{l(u)} n \otimes \text{Ind}_{w,\Omega}^s & \rightarrow & \text{Ind}_{L,\theta}^s((u\delta - \delta) \otimes \chi_{\Omega}^w) = \text{Ind}_{L,\theta}^s \\
\downarrow & & \downarrow \\
\tilde{i}^* : \Lambda^{l(u)} n \otimes \text{Ind}_{w,\Omega}^s & \rightarrow & \text{Ind}_{L,\theta}^s((u\delta - \delta) \otimes \chi_{\Omega}^w) = \text{Ind}_{L,\theta}^s
\end{array}
\]

By composing these two \(I_{L,\theta}\) (resp. \(I_{0,L,\theta}\))-equivariant natural surjections, we get our desired surjection \(\tilde{i}^*\), hence \(\tilde{i}\). From the arguments above one can see that this arrow carries \(\Lambda^{l(u)} n \otimes \text{Ind}_{w,\Omega}^s\) compatibly to \(\text{Ind}_{L,\theta}^s\).

Next, we construct \(\tilde{p}\). For weights in \(n\), we pick \(e_{\alpha}\) for an eigenvector corresponding to \(\alpha \in \Delta^+\), \(e := \wedge_{\alpha \in \Delta^+,w} e_{\alpha} \subset \Lambda^{l(u)} n\). By Lem 9.3 \(e\) is fixed by \(\overline{N}_L\).

As \(w \in W^p\), recall we identify

\[
\text{Ind}_{L,\theta,\Omega}^s \simeq C^{s,an}(\overline{N}_L, \Theta(\Omega)), \quad \text{Ind}_{w,\Omega}^s \simeq C^{s,an}(w\overline{N}_Gw^{-1}, \Theta(\Omega))
\]

as \(\Theta(\Omega)\) modules. \(T^0\) acts on \(w\overline{N}_Gw^{-1}\). For a function \(f\) on \(\overline{N}_L\), we define \(\tilde{f}\) to be pullback of \(f\) via the projection \(p_L : w\overline{N}_Gw^{-1} \rightarrow \overline{N}_L\) defined after Lem 9.2 between strict \(p\)-adic manifolds (Lem 9.2 together with [AS08] Prop 3.3.6) ensure that \(\text{Ind}_{L,\theta,\Omega}^s \rightarrow \text{Ind}_{w,\Omega}^s\).

By properly scaling \(e\),

\[
\begin{array}{ccc}
\tilde{p}^* : \text{Ind}_{L,\theta,\Omega}^s & \rightarrow & \Lambda^{l(u)} n \otimes \text{Ind}_{w,\Omega}^s \\
\downarrow & & \downarrow \\
\tilde{p}^* : \text{Ind}_{w,\Omega}^s & \rightarrow & \Lambda^{l(u)} n \otimes \text{Ind}_{w,\Omega}^s \\
\end{array}
\]

\[
f \mapsto e \otimes \tilde{f}
\]
which gives sections of \( \tilde{i}^* \) as a morphism of \( \mathcal{O}(\Omega) \) Banach modules: for \( \forall \bar{n} \in \mathbb{N}_L \), the projection of \( \bar{n}^{-1} e \otimes \tilde{f}(\bar{n}) \) equals to \( f(\bar{n}) \).

By dualizing \( \tilde{p}^* \) in the sense of Banach spaces, we obtain the sections \( \tilde{p} \) of \( \tilde{i} \),

\[
\tilde{p} : \bigwedge^{(w)} n^* \otimes \mathbb{D}^s_{w, \Omega} \longrightarrow \mathbb{D}^s_{L,w, \Omega}
\]

\[
\tilde{p} : \bigwedge^{(w)} n^* \otimes \mathbb{D}^{s,1}_{w, \Omega} \longrightarrow \mathbb{D}^{s,1}_{L,w, \Omega}.
\]

**Remark 9.4.** \( \tilde{p} \) is not \( I_L \) (resp. \( I_{0,L}^s \))-equivariant, although \( \tilde{i} \) is. The choice of the projection \( p_L \) is not really important for us.

We claim that \( \tilde{i} \) gives an \( I_L \) (resp. \( I_{0,L}^s \))-equivariant inclusion from \( \mathbb{D}^s_{L,w, \Omega} \) to the cohomology (Prop 9.6)

\[
i : \mathbb{D}^s_{L,w, \Omega} \longrightarrow H^{(w)}(n, \mathbb{D}^s_{w, \Omega})^N
\]

\[
i : \mathbb{D}^{s,1}_{L,w, \Omega} \longrightarrow H^{(w)}(n, \mathbb{D}^{s,1}_{w, \Omega})^N.
\]

Similarly, \( p \) is defined on cohomology and induced by \( \tilde{p} \). Currently we only know that horizontal arrows

\[
p : H^{(w)}(n, \mathbb{D}^s_{w, \Omega}) \longrightarrow \mathbb{D}^s_{L,w, \Omega}
\]

\[
p : H^{(w)}(n, \mathbb{D}^{s,1}_{w, \Omega}) \longrightarrow \mathbb{D}^{s,1}_{L,w, \Omega}.
\]

are sections as \( \mathcal{O}(\Omega) \)-modules. Let \( d_k \) be the chain maps of the Chevalley–Eilenberg complexes

\[
\bigwedge^n n^* \otimes \mathbb{D}^s_{w, \Omega} \xrightarrow{d_k} \bigwedge^{(k+1)} n^* \otimes \mathbb{D}^s_{w, \Omega}
\]

\[
\bigwedge^n n^* \otimes \mathbb{D}^{s,1}_{w, \Omega} \xrightarrow{d_k^i} \bigwedge^{(k+1)} n^* \otimes \mathbb{D}^{s,1}_{w, \Omega}.
\]

We will study the weights appearing in the bottom complex.

**Lemma 9.5.** The chain maps \( d_k^i \) of the Chevalley–Eilenberg complex

\[
\bigwedge^n n^* \otimes \mathbb{D}^{s,1}_{w, \Omega} \xrightarrow{d_k^i} \bigwedge^{(k+1)} n^* \otimes \mathbb{D}^{s,1}_{w, \Omega}
\]

are \( I_{0,L}^s \)-equivariant and nice. For any \( \delta \in D(I_{0,L}^s,K) \), the \( \delta \) action on \( \bigwedge^n n^* \otimes \mathbb{D}^{s,1}_{w, \Omega} \)

is nice.

**Proof.** The differentials are \( I_{0,L}^s \)-equivariant by the same argument before [CO75, Lem 2.1]. Pick the weight vectors basis \( e_1, \ldots, e_l \in n \) and \( e_1^*, \ldots, e_l^* \in n^* \) such that
$e_i^*(e_j) = \delta_{ij}$. $d_k^1 = \sum_{1 \leq \alpha \leq \ell} (-1)^\alpha \cdot d_{k,\alpha}^1$, where

$$d_{k,\alpha}^1 : \bigwedge^n n^* \otimes D_{w,\Omega}^{s,1} \rightarrow \bigwedge^{k+1} n^* \otimes D_{w,\Omega}^{s,1}$$

$e^* \otimes v \mapsto e^*_\alpha \wedge e^* \otimes e_\alpha \cdot v$.

We see $d_{k,\alpha}^1$ is $T^0$-equivariant, hence $d_k^1$. Note that $e_\alpha$ only shifts weights in $D_{w,\Omega}^{s,1}$ by $e_\alpha$ with a less or equal to 1 multiplier. More explicitly, if $v \in D_{w,\Omega}^{s,1}$ is a $T^*$-eigenvector with weight $\chi_v$ and $O(\Omega) \cdot v$ generates a direct summand of $\chi_v$ part of $D_{w,\Omega}^{s,1}$, $e_\alpha$ corresponds to weight $\alpha$, then $e_\alpha v = c(\alpha, v')v'$ for some $|c(\alpha, v')| \leq 1$ and a $T^*$-eigenvector $v'$, where $v'$ has weight $\chi_v \cdot \alpha$ and $O(\Omega) \cdot v'$ generates a direct summand of $\chi_v \cdot \alpha$ part of $D_{w,\Omega}^{s,1}$. By Lem 5.15 $d_{k,\alpha}^1$ are nice. And a sum of nice maps is nice, so is $d_k^1$.

For the second part, we know $D(wI_{0,G}^s w^{-1}, K)$ acts continuously on $D_{w,\Omega}^{s,1}$ by Prop 6.6. And a dense subalgebra $K[wI_{0,G}^s w^{-1}] \subset D(wI_{0,G}^s w^{-1}, K)$ ([ST02 Lem 3.1]) acts via nice endomorphisms on $D_{w,\Omega}^{s,1}$. So $D(wI_{0,G}^s w^{-1}, K)$ acts via nice endomorphisms on $D_{w,\Omega}^{s,1}$ by Lem 5.15 $e_\alpha \in D(wI_{0,G}^s w^{-1}, K)$ induces a nice endomorphism on $D_{w,\Omega}^{s,1}$. \hfill \qed

We define

$\Delta^+, w = \{ \alpha \in \Delta^+, |w|^{-1} \alpha \in \Delta^+ \}$, and $e^* := \wedge_{\alpha \in \Delta^+, w} e^*_\alpha$ as a weight vector of $\wedge l(w) n^*$.

**Proposition 9.6.** $\tilde{i}$ induces

$$i : D_{L,\omega,\Omega}^s \rightarrow H^l(w)(n, D_{w,\Omega}^s)^N$$

$\tilde{p}$ induces

$$p : H^l(w)(n, D_{w,\Omega}^s)^N \rightarrow D_{L,\omega,\Omega}^s,$$

$$p : H^l(w)(n, D_{w,\Omega}^{s,1}) \rightarrow D_{L,\omega,\Omega}^{s,1}.$$
The image of $\tilde{t}^* \circ \partial_{l(w)}$ is a $I_L$-subrepresentation of the space of $\mathcal{O}(\Omega)$-valued functions. It suffices to show that $\tilde{t}^* \circ \partial_{l(w)}(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{l(w)}+1} \otimes f)$ vanishes at identity of $I_L$ for all roots $e_{\alpha_1}, \ldots, e_{\alpha_{l(w)}+1}$ and $f \in \text{Ind}_{w,\Omega}^{\mathfrak{sl}(1)}$. By construction of $\tilde{t}^*$, this expression vanishes if \( \{e_{\alpha_1}, \ldots, e_{\alpha_{l(w)}+1}\} \) (up to scalar multiples) does not contain $\Delta^+, w$. Otherwise let $e_\alpha$ be $\{e_{\alpha_1}, \ldots, e_{\alpha_{l(w)}+1}\} - \Delta^+, w$. The value of $f|_{I_L}$ at identity is $N \cap w_N w^{-1}$-invariant since $I_L$ stabilizes $N$ and $n \cdot f(1) = f(n^{-1}) = f(1)$ for any $n \in N \cap w_N w^{-1}$. As $e_\alpha \in \text{Lie}(N \cap w_N w^{-1})$, the value of $f|_{I_L}$ at identity vanishes.

There is another interesting viewpoint to see the highest weight vectors map to zero. Under the embedding $i$, the highest weight vectors $\{f_{\alpha}^w, h \in \overline{N_L}/\overline{N_L}\}$ in $D_{L,w,\Omega}^{s,1}$ are identified with the combination of weight vectors $e^* \otimes f_{g,\Omega}^w$, where $e^* := \wedge_{\alpha \in \Delta^+, w} e_\alpha (\mathbb{H})$ and for any $g \in w_NGw^{-1}/w_NGw^{-1}$. These vectors correspond to the weight $w \cdot \chi_\Omega$, for which we write as $-\sum_{\alpha \in \Delta^+, w} \alpha + w_\Omega$ where $w_\Omega$ is the additive form of the multiplicative weight $\chi_\Omega^w$. It suffices to show $w \cdot \chi_\Omega$ is not a weight in $\wedge^{l(w)+1} n^* \otimes D_{w,\Omega}^{s,1}$. Weights appearing in $\wedge^l n^*$ are opposite of sums of $i$ distinct roots in $\mathfrak{n}$. By proposition Prop 6.5 and Lem 5.5, weights in $\bigwedge^l n^* \otimes D_{w,\Omega}^{s,1}$ are of the form

$$-\sum_{\alpha \in \Delta_i, w \Delta_i = \Delta^+, w \Delta_i \subset \Delta^+, \Delta_i \subset \Delta_N \cap \Delta^+} \alpha + \sum_{\alpha \in w \Delta^-} n_\alpha \alpha + w_\Omega,$$

where $\Delta_N \subset \Delta^+$ is the set of roots in $\mathfrak{n}$, $\Delta_i$ is a subset of $\Delta_N$ with $i$ elements, $n_\alpha$ are natural numbers. It remains to show:

$$-\sum_{\alpha \in \Delta_i} \alpha + \sum_{\alpha \in w \Delta^-} n_\alpha \alpha + w_\Omega = -\sum_{\alpha \in \Delta^+, w} \alpha + w_\Omega$$

$$\Rightarrow \forall n_\alpha = 0, i = l(w), \Delta_i = \Delta^+, w.$$

The equation is equivalent to

$$\sum_{\alpha \in \Delta^+, w - \Delta_i} \alpha + \sum_{\alpha \in w \Delta^-} n_\alpha \alpha = \sum_{\alpha \in \Delta_i - \Delta^+, w} \alpha.$$

Notice that $\Delta^+, w = \Delta^+ \cap w \Delta^+ \subset w \Delta^-$, thus $\Delta_i - \Delta^+, w \subset \Delta^+ - w \Delta^- \subset w \Delta^+$. The left hand side is a nonnegative sum of negative roots in $w \Delta^-$ and at the same time the right hand side is a nonnegative sum of positive roots in $w \Delta^+$. All coefficients in the above equation vanish. $\forall n_\alpha = 0, i = l(w), \Delta_i = \Delta^+, w$.

As we have already seen that $\tilde{p}$ gives a section of $i$. To show this part, it suffices to show $\tilde{p} \circ d_{l(w)-1} = 0$. It suffices to prove that for any $\delta \in D_{w,\Omega}^{s,1}$ (resp. $D_{w,\Omega}^{s,1}$) and any $e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{l(w)-1}}^*$, wedge of $l(w)-1$ roots in $\mathfrak{n}^*$, $\tilde{p} \circ d_{l(w)-1}(e_{\alpha_1}^* \wedge \cdots \wedge e_{\alpha_{l(w)-1}}^* \otimes \delta) = 0$. Pick the weight vectors basis $e_1, \ldots, e_l \in \mathfrak{n}$ and $e_1^*, \ldots, e_l^* \in \mathfrak{n}^*$ such that $e_i^*(e_j) = \delta_{ij}$. For any $f \in \text{Ind}_{l,\Omega}^{\mathfrak{sl}(1)}$,

$$\tilde{p} \circ d_{l(w)-1}(e_{\alpha_1}^* \wedge \cdots \wedge e_{\alpha_{l(w)-1}}^* \otimes \delta)(f) = d_{l(w)-1}(e_{\alpha_1}^* \wedge \cdots \wedge e_{\alpha_{l(w)-1}}^* \otimes \delta)(e \otimes \tilde{f})$$

$$= \sum_{1 \leq \alpha \leq l} (-1)^\alpha (e_{\alpha}^* \wedge e_{\alpha_1}^* \wedge \cdots \wedge e_{\alpha_{l(w)-1}}^*)(e) \otimes (e_{\alpha} \cdot \delta) \tilde{f}$$

$$= \sum_{1 \leq \alpha \leq l} c(\alpha) \cdot \delta(e_{\alpha} \cdot \tilde{f}),$$
for any \( d \wedge l \) vanishing on all non locally constant locally monomial functions for

Moreover, if and only if \( e_\alpha \) and \( e_{\alpha_1},\cdots,e_{\alpha_{n(w)-1}} \) correspond exactly to all elements in \( \Delta^{+,w} \) defined in [1] When this is the case, \( e_\alpha \in n \cap wNw^{-1} \subset \mathrm{Lie}(N) \). \( e_\alpha \cdot \tilde{f} = 0 \) since \( \tilde{f} \) is constant on \( NW_+ \) fibres.

It suffices to prove for any \( f \in \mathrm{Ind}_{L,w,\Omega} \) and \( n \in N \), \( i(n \cdot \tilde{p}(f) - \tilde{p}(f)) = 0 \). If \( \bar{n}_L \in N_L \), \( \bar{n}_L^{-1}n\bar{n}_L \in N \), we write \( \bar{n}_L^{-1}n\bar{n}_L = n_L \cdot n_b \) such that \( n_+ \in N_+ \), \( n_b \in N \cap wNw^{-1} \). Note that

\[
e \otimes n \cdot \tilde{f}(\bar{n}_L) = e \otimes \tilde{f}(n^{-1}\bar{n}_L) = e \otimes \tilde{f}((\bar{n}_Ln_+\bar{n}_L^{-1})n_Ln_b) = e \otimes \tilde{f}(\bar{n}_L),
\]

by the construction of \( \tilde{f} \).

\( \square \)

**Remark 9.7.** \( p \) is defined on \( H^{l(w)}(n, D^s_{w,\Omega}) \supset H^{l(w)}(n, D^s_{w,\Omega})^N \).

If \( \Omega' \) is closed in \( \Omega \), there is a specialization map \( D^s_{w,\Omega} \rightarrow D^s_{w,\Omega'} \) with respect to \( \mathcal{O}(\Omega) \rightarrow \mathcal{O}(\Omega') \). The analytic induction is compatible with base change \( \mathrm{Ind}_{w,\Omega} \otimes \mathcal{O}(\Omega' \supset \mathrm{Ind}_{w,\Omega'} \), yielding

\[
\mathcal{L}(\mathcal{O}(\Omega)\mathrm{Ind}_{w,\Omega}, \mathcal{O}(\Omega)) \rightarrow \mathcal{L}(\mathcal{O}(\Omega')\mathrm{Ind}_{w,\Omega'}, \mathcal{O}(\Omega')).
\]

It remains to prove \( p \) is \( I_o \)-equivariant for completing the proof of Thm 1.2

We define \( V^+, \mathcal{V}^+ := \ker p \) correspondingly for \( H^{l(w)}(n, D^s_{w,\Omega}), H^{l(w)}(n, D^s_{w,\Omega})^N \). For \( \forall a \in V^+, l \in I_o \), it suffices to show

\[
x := p(l \cdot a) = 0.
\]

If \( x \neq 0 \), there exists a \( f_0 \in \mathrm{Ind}_{w,\Omega}^s \) such that \( x(f_0) \neq 0 \). Then both \( \mathrm{supp}(f_0) \) and \( \mathrm{supp}(x(f_0)) \) are Zariski open dense in \( \Omega \). By viewing \( f_0 \in C^{s,an}(N_L, \mathcal{O}(\Omega)) \), vanishing of \( f_0 \) means vanishing of all coefficients of \( f_0 \), we have

\[
\mathrm{supp}(f_0) \cap \mathrm{supp}(x(f_0)) \subset \mathrm{supp}(x),
\]

where \( \mathrm{supp}(x) \) is a Zariski open dense subset of \( \Omega \). To prove \( x = 0 \), it suffices to find a dense subset of \( \Omega \) on which \( x \) vanishes.

We say a character \( \chi \) of \( T^0 \) is **generic** if \( \chi(w \cdot \chi)^{-1} \) are not locally \( (Q_p, \cdot) \)-algebraic for all \( w \in W_G \).

**Lemma 9.8.** If \( s \) is a weight in \( \wedge n^* \otimes D^s_{w,\lambda} \), set \( i(s) := d(s) - d(w \cdot \lambda) \), where \( d \) passes a character to Lie algebra. Then \( i(s) \) is nonpositive with respect to \( w\Delta^+ \) (Prop 6.2).

We endow the partial ordering on weights of \( \wedge n^* \otimes D^s_{w,\lambda} \) under the map \( i \) with respect to \( w\Delta^+ \) as well.

**Proof.** This is essentially in (1) of proof of proposition 9.6 which is a combination of Prop 6.3 and the fact that the difference of any weight in \( \wedge n^* \) and \( e^* = \wedge_{\alpha \in \Delta^+,w} e^*_\alpha \) is nonpositive with respect to \( w\Delta^+ \).

Viewing \( w\Delta^+ \) as positive roots, \( i \) maps any such \( s \) to non-positive span of them. Moreover, \( 0 \) must correspond to weights of the form \( c^* \otimes c \), where \( c^* = \wedge_{\alpha \in \Delta^+,w} e^*_\alpha \subset \wedge^{l(w)}n^* \) for \( e^*_\alpha \) being an eigenvector corresponding to \( \alpha \in \Delta^+ \) and \( c \) is a distribution vanishing on all non locally constant locally monomial functions for \( w\overline{N}_Gw^{-1} \).
Theorem 9.9. If $\lambda = \Omega$ is a generic weight, 
\[
V^{\perp,1}|_{\lambda} \simeq \text{Im}(d_{i(w)}^1)/\text{Im}(d_{i(w)}^1).
\]
Here $V^{\perp,1}|_{\lambda}$ (resp. $V^{\perp}|_{\lambda}$) is defined to be $\ker(p|_{\lambda} : H^{i(w)}(n, D_{w,\lambda}^{s,1}) \to D_{L,w,\lambda}^{s,1}$ (resp. $\ker(p|_{\lambda} : H^{i(w)}(n, D_{w,\lambda}^{s,1}) \to D_{L,w,\lambda}^{s,1}$).

We use Lem 5.9 and the infinitesimal character argument of [CO75] to prove Thm 9.9.

Proposition 9.10. For a generic weight $\lambda$, $\ker(d_{i(w)}^1) = \underbrace{D_{L,w,\lambda}^{s,1} \oplus \text{Im}(d_{i(w)}^1)}$.

Proof. We know that $d_{i(w)}^1$ is nice from Lem 9.8. Applying Lem 5.9 to $\ker(d_{i(w)}^1) \hookrightarrow \Lambda_{(w)}^{i(w)} n^* \otimes D_{w,\lambda}^{s,1}$, there exists $\iota : K \subseteq I_{\infty}$ such that $K \subseteq \text{non-positive combinations of } w \Delta^+$ in the weight lattice and $V_k$ for each $k \in K$ corresponding to $(\prod_k V_k)^b := \ker(d_{i(w)}^1)$. Let 
\[
V := (\prod_k V_k)^c/\tilde{V}. \text{ The Lie algebra action on } (\prod_k V_k)^b \text{ is locally finite, therefore preserving } (\prod_k V_k)^c.
\]

Let $\iota$ be as in Lem 9.8. We want to show $V = 0$ by contradiction. If $V \neq 0$, choose $s \in K$ such that $\iota(s)$ is maximal in the partial order set $\{\iota(s'), s' \in K| V_{s'} \neq 0\}$. Note that $V$ is a natural subquotient of $H^{i(w)}(n, D_{w,\lambda}^{s,1})$ as an $I_{\infty}$ representation, by [CO75 Cor 2.7], $V$ has the infinitesimal character of $Z(U(L))$ extending the infinitesimal character of $Z(U(g))$. Pick a non-zero element $v$ in $V \hookrightarrow V$. Since $\iota(s)$ is maximal in $\{\iota(s'), s' \in K| V_{s'} \neq 0\}$, $\text{Lie}(N_L)$ kills it. By the infinitesimal character restriction,
\[
\chi_s = \lambda' \cdot (w \cdot \lambda)
\]
for some $w' \in W_L$ considered as characters of Lie algebra of $T^0$. On the other hand, $\chi_s$ can be zero since all the weights which are of the form $e^s \otimes c$ all live in $(\prod_k V_k)^c$ part of $D_{L,w,\lambda}^{s,1}$, $(w \cdot \lambda) \times \chi_s^{-1}$ is a strictly dominant algebraic weight. Therefore we have $V = 0, \tilde{V} = (\prod_k V_k)^c$, and
\[
(\prod_k V_k)^c \subseteq \underbrace{D_{L,w,\lambda}^{s,1} \oplus \text{Im}(d_{i(w)}^1)}.
\]
which concludes
\[
\ker(d_{i(w)}^1) = (\prod_k V_k)^b = \underbrace{D_{L,w,\lambda}^{s,1} \oplus \text{Im}(d_{i(w)}^1)}.
\]

Proof of Theorem 2.3. Let $(\prod_k V_k)^b := \ker(d_{i(w)}^1)$ as in Prop 9.10. For $x \in (\prod_k V_k)^c$ such that $x \in V^{\perp,1}|_{\lambda}$, one can find a sequence $(a_n, b_n) \in D_{L,w,\lambda}^{s,1} \oplus
\[ \text{Im}(d^1_{l(w)}), \text{converging to } x \text{ by Lemma 5.12.} \text{ We have the following commutative diagram with continuous arrows} \]

\[ \begin{array}{ccc}
\mathbb{D}^{s,1}_{L,w,\lambda} & \xrightarrow{i} & \ker(d^1_{l(w)}) \\
\downarrow{\tilde{\iota}} & & \downarrow{\tilde{\varphi}} \\
H^l(w)(n, \mathbb{D}^{s,1}_{w,\Omega}) & \xrightarrow{\varphi} & \mathbb{D}^{s,1}_{L,w,\lambda}
\end{array} \]

\[ \tilde{\varphi}(x) = 0 \implies \lim_{n \to \infty} a_n = \tilde{\varphi}(a_n, b_n) \to 0. \]

\( \tilde{\varphi} \) is the dual map of \( \varphi^* \), hence nice by Lemma 5.12. It sends convergent series to convergent series. Since \( \tilde{\varphi} \circ \iota = \text{id} \) and both \( \tilde{\varphi}, \iota \) are continuous, \( \iota \) induces equivalent norms on \( \mathbb{D}^{s,1}_{L,w,\lambda} \).

This proves that

\[ (\prod_{k \in K} V_k)^c = (\iota(\mathbb{D}^{s,1}_{L,w,\lambda}) \cap (\prod_{k \in K} V_k)^c) \oplus (\text{Im}(d^1_{l(w)})) \cap (\prod_{k \in K} V_k)^c). \]

Now by applying Lemma 5.12 for \( \mathbb{D}^{s,1}_{L,w,\lambda} \) as \( V_1 = \text{Im}(d^1_{l(w)}) \) as \( V_2 \) and \( \ker(d^1_{l(w)}) \) as \( (\prod_{k \in S} V_k)^b \) in the lemma, we have

\[ \ker(d^1_{l(w)}) = \mathbb{D}^{s,1}_{L,w,\lambda} \oplus \text{Im}(d^1_{l(w)-1}), \quad V^\perp_{l,1}|_\lambda = \text{Im}(d^1_{l(w)-1})/\text{Im}(d^1_{l(i(w)-1)}). \]

\[ \square \]

**Lemma 9.11.** \( \widehat{\text{Im}(d^1_{l(w)-1})} \) is \( I^s_{0,L} \) stable.

**Proof.** The Chevalley–Eilenberg complex for \( \mathbb{D}^{s,1}_{L,w,\lambda} \) is \( I^s_{0,L} \)-equivariant as a complex of continuous \( K[I^s_{0,L}] \subset D(I^s_{0,L}, K) \) modules in the sense of Schneider–Teitelbaum by Proposition 6.6. \( \widehat{\text{Im}(d^1_{l(w)-1})} \) is \( I^s_{0,L} \) stable. By Proposition 6.6 and Lemma 5.15 any group element action of \( I^s_{0,L} \) is nice for \( \bigwedge^{l(w)} n^* \otimes \mathbb{D}^{s,1}_{w,\lambda} \). Then apply last part of Lemma 5.14 to \( d^1_{l(w)-1} \) and Lemma 5.10 to \( \widehat{\text{Im}(d^1_{l(w)-1})} \hookleftarrow \bigwedge^{l(w)} n^* \otimes \mathbb{D}^{s,1}_{w,\lambda} \). \( \square \)

**Remark 9.12.** It should be much easier to see \( \widehat{\text{Im}(d^1_{l(w)-1})} \) is stable under \( \text{Lie}(I^s_{0,L}) \) action. But by [Koh07, Lem 1.2.5, Prop 1.2.8], the universal enveloping algebra of \( \text{Lie}(I^s_{0,L}) \) is only dense in a closed subalgebra of \( D(I^s_{0,L}, K) \) while \( K[I^s_{0,L}] \) is dense in \( D(I^s_{0,L}, K) \) by [ST02, Lem 3.1].

**Proof of Theorem 9.1.** By Lemma 9.11 \( \widehat{\text{Im}(d^1_{l(w)-1})} \) is \( I^s_{0,L} \) stable. So \( V_{\perp,1} \) is \( I^s_{0,L} \) invariant. We have the following morphism of short exact sequences

\[ 0 \rightarrow V_{\perp} \rightarrow H^l(w)(n, \mathbb{D}^{s,1}_{w,\lambda}) \rightarrow \mathbb{D}^{s,1}_{L,w,\lambda} \rightarrow 0 \]

\[ 0 \rightarrow V_{\perp,1} \rightarrow H^l(w)(n, \mathbb{D}^{s,1}_{w,\lambda}) \rightarrow \mathbb{D}^{s,1}_{L,w,\lambda} \rightarrow 0. \]
Since the middle vertical arrow is $I_{0,L}^\ast$ equivariant and the rightmost arrow is injective, $\mathcal{V}^\perp$ is also $I_{0,L}^\ast$ stable. Let $x$ be the element defined below Rem. 9.7, $x|_{\lambda} = 0$ for all generic weights $\lambda \in \Omega$. Generic weights are dense. $x = 0$, $p$ is $I_{0,L}^\ast$-equivariant.

Lastly, if $U \hookrightarrow V$ is an embedding of characteristic zero representations of a group which splits over a subgroup of finite index, then $U$ is a direct summand. □

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Department of Mathematics, Princeton University, Princeton, NJ, USA.

Email address: wfu@math.princeton.edu