Measurement catastrophe and ballistic spread of charge density with vanishing current

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Abstract
One of the features of many-body quantum systems with Hilbert-space fragmentation are stationary states manifesting quantum jamming. It was recently shown that these are ‘states with memory’, in which, e.g. measuring a localised observable has everlasting macroscopic effects. We study such a measurement catastrophe with an example that stands out for its clarity. We show in particular that at late times the expectation value of a charge density becomes a nontrivial function of the ratio between distance and time notwithstanding the corresponding current approaching zero.

Keywords: quantum jamming, generalised hydrodynamics, local quantum measurement, ballistic spreading, Hilbert space fragmentation

(Some figures may appear in colour only in the online journal)

1. Introduction

Complexity is behind the most intriguing phenomena observed in many-body systems: phases of matter, spontaneous symmetry breaking, relaxation, etc. Obtaining a clear understanding of such phenomena requires simplifications that lead to the conception of toy models, among which a significant role is played by exactly solvable ones. Despite the fact that these models (which are generally integrable) are clearly special, phenomena identified in them are always potentially observable, although not necessarily exactly as predicted. Prethermalisation is a remarkable example of this kind: integrable systems are known to relax differently than generic ones [1–5], and yet such special dynamics leave a mark also on the relaxation of generic systems [6], which can exhibit prethermalisation plateaux [7].
Even working in the framework of integrability can present formidable complications that undermine our qualitative understanding. This is one of the motivations behind the recent interest in identifying the ‘simplest’ interacting integrable systems. In this respect, a spin-$1/2$ chain model with a three-site interaction termed ‘dual folded XXZ model’ was recently highlighted [8–10]. It corresponds to a special point of a rather complicated integrable system called Bariev model [11, 12], and is equivalent to the celebrated Heisenberg XXZ model in the limit of large anisotropy. The dual folded XXZ model was recently solved by two of us from scratch by means of a coordinate Bethe Ansatz that has been adapted to the special symmetries emerging in the large-anisotropy limit of the XXZ model [8]. One particular consequence of the emergent symmetries is the splitting of the Hilbert space into an exponentially large number of dynamically disconnected sectors—a phenomenon recently dubbed ‘Hilbert-space fragmentation’ [13–15]. A striking feature found in fragmented models, such as the folded XXZ one, is the existence of an exponentially large sector of jammed states in which particles are stuck and cannot move [16–21]. In [22] it was shown that such states have memory: the effect of a localised perturbation or measurement does not fade away but rather becomes visible at macroscopic scales for arbitrarily long times. This is similar to the memory effects observed when perturbing ground states with spontaneously broken symmetries [23, 24] (see also [24–27] for a perturbation local in the fermionic basis), where however the dynamics are effectively constrained in a finite-dimensional space. In the setting of [22], instead, an exponentially large number of states are involved. Considering how dramatic the change in the state is, it seems appropriate to refer to it as a ‘measurement catastrophe’. In particular, this effect was reported in a class of states that allow for an exact analytic analysis but are, however, not simple enough to make us gain a clear understanding of the phenomenon. This paper is first of all a solution to that problem.

We consider a family of jammed states for which time evolution after a spin flip can be solved exactly. We focus on observables whose expectation values can be predicted with minimal information about the initial state. Among them are the densities of two conserved charges, magnetisation and staggered magnetisation. They are affected by the spin flip in a macroscopic way and will be used to monitor the measurement catastrophe. This sets the example herein apart from the one considered in [22], where instead, in the long time limit, the densities of conserved charges and their currents become independent of their positions. The central part of the paper will be devoted to devising a physical picture that could reconcile the nontrivial profile of the (staggered) magnetisation at the Euler/ballistic scale with the fact that the corresponding currents vanish in the limit of infinite time, both observations following from our exact solution.

We will present a set of simple rules that allows us to predict the asymptotic behaviour of any spin solely from the knowledge of the initial configuration of few neighbouring spins and parities of the positions of spins up between the origin and the spin under consideration. We will then show that the asymptotic state of the system is locally jammed and that the currents of the magnetisation and of the staggered magnetisation, as well as of all other conventional conserved charges of the folded XXZ model, vanish in it. By computing the statistics of the particles’ positions, we will show that the macroscopic reorganisation of the spin configuration coincides with small correlated microscopic fluctuations of particles (spins up) that do not exceed two lattice sites. Finally, we will study the entanglement properties of a special class of weakly interacting initial jammed states, which allow for the preparation of correlated spin pairs in an antiferromagnetic background.
1.1. Overview of the model

The dual folded XXZ model describes the time evolution of states in the large-anisotropy limit of the Heisenberg model. It is obtained in the ‘folded picture’, which was defined in [8] as a representation of time evolution with Hamiltonians that have a large coupling constant. Based on the strong coupling expansion of [28], the ‘folded picture’ specifically refers to a decomposition of time evolution into a periodic high-frequency evolution of operators and a ‘gentle’ evolution of the state with a time-independent Hamiltonian. At the leading order in the strong coupling expansion, the dual folded XXZ Hamiltonian reads

$$H = J \sum_{\ell} \left( \frac{1 - \sigma_{\ell+1}^x}{2} \sigma_{\ell+1}^x \sigma_{\ell+2}^x + \sigma_{\ell}^y \sigma_{\ell+2}^y \right).$$

This Hamiltonian is integrable and describes constrained hopping on a lattice, summarised by the following local dynamical rule: $|\uparrow\downarrow\downarrow\rangle \leftrightarrow |\downarrow\uparrow\rangle$, i.e. a spin up can exchange with two adjacent spins down upon the action of the Hamiltonian’s local terms. Due to a two-site hopping, the parities of the positions of spins up are preserved by the dynamics, allowing us to define two species of particles, $\#\uparrow$ and $\#\downarrow$, living on the ‘macrosites’ composed of two neighbouring spins. We label the particles by the parity of the position of the spin up that represents them, i.e. $\#\uparrow$ corresponds to 1 and $\#\downarrow$ to 0. The dynamics now preserves the configuration of particle labels as seen, for example, in the following sequence of hoppings:

$$
\begin{array}{cccccccc}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow
\end{array}
$$

The presence of multiple species of stable particles typically signals that an integrable model is solvable via a complicated nested Bethe Ansatz. And indeed, the dual folded XXZ model corresponds to a strong-repulsion regime of the two-component Bariev model [11, 12], which is of that type. Nevertheless, at the special point described by equation (1) the nested structure of the Bariev model is reduced to such a degree that Bethe equations can be solved explicitly. Perhaps the most striking remnant of the nested structure is the conservation of the configuration, which results in the breakdown of one-site shift invariance into two-site shift invariance.

The constrained hopping leads to a rich Hilbert-space structure combined with an exceptional simplicity of the Bethe equations. Hilbert space is split into sectors where the model is almost free, with a diagonal scattering matrix in which only one diagonal element is different from $-1$ and equal to $-e^{i\pi}$, $\pi$ being the difference between the momenta of the scattering particles. This bears a remarkable resemblance to the scattering phase modification factor in hard-rod deformed models (see, e.g. [29, 30]). Such a structure enables exact analytical and numerical analysis of various phenomena that have recently been in the spotlight: Hilbert-space fragmentation [17], pre-relaxation [31], generalised hydrodynamics [9], spin transport phenomena [32], the effect of non-abelian symmetries, time-translation symmetry breaking [10], and the macroscopic effects of local measurements [22]. Most of these phenomena are in fact accessible in sectors that allow for an even simpler description; we mention, for example, the non-interacting sectors, where only one species of particles is present, or the exponentially large jammed sector, where, in the basis of eigenstates of all
A jammed state fragments of the form \( \cdots \uparrow \downarrow \cdots \) are prohibited and no dynamics takes place: a spin up can only exchange with two adjacent spins down, no dynamics is possible in the latter sector.

The local conserved charges of the dual folded XXZ model have been classified in [8]. Their place in the Yang–Baxter integrability structure of the model [33–35] was described in [36, 37]. We mention that the fragmentation of the Hilbert space in the folded XXZ model could be related to the existence of additional non-abelian conservation laws, potentially associated with the emergence of the integrable analogue (see [38–40]) of quantum many-body scars [15, 41–46]. These charges are expected to be pseudolocal [47, 48], but not necessarily with a quasilocal density [49, 50].

To conclude the overview of the model, we stress that the only ingredient needed in this paper is the dynamical rule \( \sigma^z \), we will not exploit at all the Bethe Ansatz solution [8] of the model.

2. Initial state and spin-flip protocol

Applying an operator to a single spin is an example of a local perturbation. If the operator flips the spin this perturbation can be interpreted as a double projective measurement. For instance, one possible outcome of the subsequent measurements of operators \( \sigma^x \) and \( \sigma^z \) on the state \( | \uparrow \rangle \) of a single spin is

\[
| \uparrow \rangle \langle \uparrow | \xrightarrow{\sigma^x} | \rightarrow \rangle \langle \rightarrow | \xrightarrow{\sigma^z} | \downarrow \rangle \langle \downarrow |
\]

where, in each step, the result of the measurement has been read off the measurement device and the state of the spin has collapsed accordingly. The probability of such an outcome is irrelevant for our discussion.

Consider now a many-body system, e.g. a spin chain, whose time evolution is generated by some Hamiltonian \( H \). If the initial state is an eigenstate, \( H | \Psi \rangle = E | \Psi \rangle \), the local perturbation on top of such a state typically initiates a nontrivial time evolution. Still, in general it is reasonable to expect its effects to fade away with time: in the large-time limit, the state of the model is indistinguishable from the initial unperturbed eigenstate \( | \Psi \rangle \) for what concerns any local observable. Here we will consider an exception in which a local perturbation on top of a certain class of eigenstates of the Hamiltonian \( (1) \) leads to everlasting effects affecting a number of spins that diverges with time. To this end we will consider a protocol that consists of the following steps:

(a) We prepare the system in a jammed product state with spins aligned along the z-axis (in either direction).

In a jammed state fragments of the form \( \cdots \uparrow \downarrow \cdots \) are prohibited and no dynamics takes place: a spin up can only exchange with two adjacent spins down. Following the notations of [8] spins up correspond to particles belonging to two species that are determined by the parity of their positions: a spin up represents a particle of species \( b = 0 \) (\( b = 1 \)) if its position is even (odd). More formally, a spin up at site \( \ell \) corresponds to a particle of species \( b = 2 \lceil \ell/2 \rceil - \ell \in \{0, 1\} \), where \( \lceil \cdot \rceil \) is the rounding-up to the nearest integer. We define \( \ell' = \lceil \ell/2 \rceil \) as the particle’s macrosite—a pair of neighbouring sites where the particle is located. A jammed state is then completely characterised by the configuration \( \underline{\ell} = (\ldots, \ell_n, b_{n+1}, \ldots) \) of spins up as follows:

\[
| \underline{\ell} \rangle = \prod_{j} \sigma_{2\ell' - b_j}^+ | \downarrow \cdots \downarrow \rangle,
\]
where macrosites $\ell'_j$ satisfy the recurrence relation
\[
\ell'_{j+1} = \ell'_j + 1 - b_j (1 - b_{j+1}).
\] (5)

The latter encodes the fact that the particles are packed as closely as possible. Specifically, it implies that whenever a particle of species 1 is followed by that of species 0 the two particles occupy the same macrosite, e.g. $\cdots \uparrow \downarrow \uparrow \cdots$ (the macrosite is underlined). This prevents state fragments of the form $\cdots \uparrow \downarrow \downarrow \uparrow \cdots$, which violate the jamming condition.

(b) **We flip a spin up that was close to a spin down (otherwise the state remains jammed).**

As discussed above, this step can be interpreted as a result of a double projective measurement—see equation (3). The spin flip destroys a particle. We choose its index to be 0, so that the state now reads
\[
Y_j \leq -\sigma + 2 \ell'_j - b_j Y_j \geq \sigma + 2 \ell'_j - b_j.
\] (6)

Recurrence (5) can now be solved with respect to the position of the spin flipped. First we note that either $b_{j-1}$ or $b_{j+1}$ should be equal to $b_{j}$, indeed we flipped one of the spins up adjacent to a spin down and the position of the other spin up must have the same parity. We choose the macrosites in such a way that at $\ell'_0 = 0$ there are two spins down: as a result we either have $b_{j-1} = b_{j} = 0$ or $b_{j-1} = b_{j} = 1$. In fact, since we removed a particle (a spin up), it is convenient to redefine the index of the particles, $j$, so as to fill the hole at $j = 0$. We choose the convention $j_{\text{new}} = j_{\text{old}} + \theta (j_{\text{old}} < 0)$ and then solve the recurrence (5) to find
\[
\ell'_j = j - \theta (j \leq 0) + \left\{ \begin{array}{ll}
- \sum_{m=1}^{j-1} b_m^{(0)} (1 - b_{m+1}^{(0)}), & \text{if } j > 0 \\
\sum_{m=1}^{j} (1 - b_{j-m}^{(0)}) b_{j-m}^{(0)}, & \text{if } j \leq 0.
\end{array} \right.
\] (7)

Here, $b_m^{(0)}$ is the configuration obtained by removing $b_0$ from $b$ ($b_m^{(0)} = b_{j-m}$ for $j < 0$) and the last term in equation (7) counts the number of particle pairs that are situated between the position of the spin flip and the $j$th particle, and occupy a single macrosite each (these are the pairs containing a particle of species 1 followed by that of species 0). We indicate this state by
\[
|\Psi (t = 0)\rangle = |0; \tilde{b}^{(0)}\rangle,
\] (8)

where $|n; c\rangle$ denotes a state which is jammed everywhere except between the spins up that represent the particles $c_n$ and $c_{n+1}$, where either two or three consecutive spins down are present. We call this excess of spins down ‘impurity’. The fragment containing the particles $c_n$ and $c_{n+1}$ surrounding an impurity of three spins down could, for instance, be of the form
\[
\begin{array}{cccccccc}
\cdots & \uparrow & \downarrow & \downarrow & \downarrow & \uparrow & \cdots \\
1 & 1 & 1 & 1 & 0 & 1 & 0 \\
\end{array}
\] or
\[
\begin{array}{cccccccc}
\cdots & \uparrow & \uparrow & \downarrow & \downarrow & \downarrow & \uparrow & \cdots \\
1 & 1 & 1 & 1 & 0 & 0 & 1 \\
\end{array}
\] (9)

where the spins belonging to the same macrosite are underlined for the purpose of distinguishing odd and even sites of the chain. In this example we have $c_n = c_{n+1} = 1$ in the first case and $c_n = c_{n+1} = 0$ in the second. The choice of the macrosites (and therefore of the species of the particles that surround the impurity) will not be relevant for the discussion that follows.
(c) We let the state to time evolve with \( H \).
While the folded XXZ model is interacting, time evolution is particularly simple in the sector of our initial state. As shown in appendix A.1, this sector is spanned by states \( |n; b^{(0)}⟩ \), in which the particles’ macrosites read

\[
\ell_j^{(n)} = j - \ell(j \leq n) + \begin{cases} 
- \sum_{m=1}^{j-1} b_m^{(0)} (1 - b_{m+1}^{(0)}), & \text{if } j > 0 \\
(1 - b_{j-m}^{(0)}) b_{-m}^{(0)}, & \text{if } j \leq 0.
\end{cases}
\]  

They are obtained from equation (7) by noting that, when the impurity jumps across a particle, the macrosite of the latter shifts by 1 in the opposite direction due to the two-site hopping rule \( |↑↓⟩ \leftrightarrow |↓↑⟩ \). For example, for \( n > 0 \) the impurity had to jump \( n \)-times towards the right, displacing particles with indices \( 0 < j \leq n \) for one macrosite towards the left: this is represented by the changed argument of the step function \( \ell \) in equation (10).

Note that the configuration of particles’ species has not changed during this process, which is why the last term of equation (10) is the same as in equation (7).

3. Exact solution to the dynamics

The configuration of particles in the state is preserved under the action of the Hamiltonian (1). As a result, the impurity can be interpreted as a fermion freely hopping in the background \( b^{(0)} \) of particles, its dynamics being exactly solvable. As shown in appendix A.1, the state at time \( t \) reads

\[
|Ψ(t)⟩ = \sum_n (-i)^n J_n(τ)|n; b^{(0)}⟩, \quad τ = 4Jt,
\]

where \( J_n(x) \) are Bessel functions. Note that this equation describes the evolution of the state \( c_0^0|\text{vac}⟩ \) under a non-interacting (tight-binding) Hamiltonian

\[
H_b = 2J \sum_j c_j^† c_{j+1} + \text{h.c.},
\]

provided that we substitute \( |n; b^{(0)}⟩ \) by \( c_n^0|\text{vac}⟩ \). In this respect, the fermionic vacuum \( |\text{vac}⟩ \) corresponds to a jammed background, while the impurity acts as a single fermion created on top of it.

Let us denote the expectation value \( ⟨Ψ(t)|A|Ψ(t)⟩ \) by \( ⟨A⟩_t \), for any operator \( A \) at arbitrary time \( t \). For the expectation values of operators \( D \) that are diagonal in the basis of states \( |n; b^{(0)}⟩ \), in which all spins are aligned along the \( z \)-axis, we have

\[
⟨D⟩_t = \sum_n |J_n(τ)|^2 ⟨n; b^{(0)}|D|n; b^{(0)}⟩.
\]

Perhaps the simplest example of such an operator is \( D = σ^z_j \). Note that, when \( D^2 = 1 \), like for a string of Pauli matrices \( σ^z_j \), the expectation values \( ⟨n; b^{(0)}|D|n; b^{(0)}⟩ \) can be either 1 or \(-1\).

We remind the reader that the tight-binding Hamiltonian (12) is connected by a Jordan–Wigner transformation to the XX model with Hamiltonian \(-J \sum_j (σ^x_j σ^x_{j+1} + σ^y_j σ^y_{j+1})\). The subtle difference between the dynamics described in our work and the ones in the XX model, in which the initial state has a single spin up, is hidden in the mapping (10) between the positions in the chain and those in the configuration. As will be shown in the following, this
mapping has striking effects on the dynamics of local observables, such as the $z$-component of the spin.

### 3.1. Macrosite fluctuations of particles

Our first aim is to investigate the microscopic dynamics triggered by a spin flip. To this end we adopt the Lagrangian perspective of following the motion of particles. A particle is characterised by its species and its macro-position. The species is fixed, since it is determined by the parity of the particle’s position and the latter is preserved by the dynamics (a spin up can only jump by two sites). The macro-position, on the other hand, is a dynamical variable.

In the following we compute the statistics of the particles’ macro-positions. To that aim, let us consider the macrosite (10) corresponding to the particle labelled by $j$ (i.e. the $j$th spin up). This can be thought of as the eigenvalue of an operator that is diagonal in the basis of states $|n; k(0)\rangle$, in which it assumes values $\theta(n > j) + c(j)$, where $c(j)$ denotes its state-independent part—cf equation (10). Since the operator is diagonal, we can use equation (13) to compute the characteristic function $\langle e^{ik\ell_j} \rangle$, which defines the probability distribution of the particle’s macro-position. We find

$$\langle e^{ik\ell_j} \rangle = e^{ikc(j)} \left[ e^{ik} - 1 \right] f(j, t) + 1, \tag{14}$$

where we introduced the auxiliary quantity

$$f(x, t) := \sum_n \theta(x > n) |J_n(\tau)|^2 \sim \frac{1}{2} + \frac{1}{\pi} \arcsin \left( \frac{x}{\tau} \right). \tag{15}$$

The last equality is asymptotic—it holds on large space-time scales (see equation (A.16) in appendix A.4 for the derivation).

The characteristic function (14), in particular, implies

$$\text{Prob}(\ell_j') = \left\{ \begin{array}{ll} 1 - f(j, t) & \ell_j' = c(j) \\ f(j, t) & \ell_j' = c(j) + 1, \end{array} \right. \tag{16}$$

as well as

$$\langle \ell_j' \rangle = f(j, t) + c(j), \quad \langle \ell_j'^2 \rangle = f(j, t) - f(j, t)^2, \tag{17}$$

where $\langle AB \rangle := \langle AB \rangle - \langle A \rangle \langle B \rangle$, denotes the connected correlation function. Since $|f(x, t)| \leq 1$, after the spin flip the particles experience fluctuations of order $O(1)$ around their average macro-positions. The fluctuations of different particles are moreover coherent, as can be inferred from the following non-negative equal-time connected correlation function

$$\langle \ell_m' \ell_n' \rangle = \theta(m \geq n) f(n, t) + \theta(n > m) f(m, t) - f(m, t) f(n, t), \tag{18}$$

which can again be obtained from equation (13).

As we will see in the following, the microscopic fluctuations of particles’ positions described here coexist with a large-scale dynamics in which the block of spins whose $z$-components are irremediably changed as a result of the initial perturbation spreads linearly in time.

### 3.2. Local magnetisation

We now adopt an Eulerian point of view and study the properties of the state at a given position or ray. The Hamiltonian commutes with $\prod_{j} \sigma_{jz}$ and the initial state is an eigenstate of that operator, therefore $\langle \sigma_{jz}^2 \rangle_t = 0$. The local magnetisation along the $z$-axis instead
exhibits a nontrivial ballistic-scale profile. Indeed, as shown in appendices A.3 and A.4, the asymptotic behaviour of the local magnetisation along the $z$-axis is determined by the following rules:

\[
\langle \sigma_{\ell} \rangle_t \sim \begin{cases} 
1, & |\cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \rangle \lor |\cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \rangle \\
\frac{2}{\pi} \arcsin \left( \frac{\alpha(\ell)}{n} \right), & |\cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \rangle \lor |\cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \rangle \\
-\frac{2}{\pi} \arcsin \left( \frac{\alpha(\ell)}{n} \right), & |\cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \rangle \lor |\cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \rangle \\
-1, & |\cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \rangle \lor |\cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \rangle
\end{cases}
\]  

(19)

Here, $|\cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \rangle$ and $|\cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots \rangle$ stand for fragments of the initial state in which $\ell$ is positive or negative, respectively: $\cdot$ signifies the impurity. By $x(\ell)$ we have denoted the label of the particle in the initial configuration, represented by the spin up in the site $\ell$ or, if the latter is occupied by a spin down, in the adjacent site. For large $|\ell|$ we can compute $x(\ell)$ from

\[
\frac{1}{\ell} \int_0^{x(\ell)} d\xi(n) = \frac{1}{2},
\]

(20)

where $\xi(n)$ is the average of $1 - b_j^0(1 - b_{j+1}^0)$ over a large-enough interval of particle labels $n$ that includes $n$: for example, $\xi(n) = 3/4$ for the fragment $\ldots, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, \ldots$ in the sequence of particle species. To interpret equation (20) we note that $1 - b_j^0(1 - b_{j+1}^0)$ is the macro-distance between the particles $j$ and $j+1$: it is equal to 1, except if the two particles are of species 1 and 0, respectively, in which case they occupy the same macrosite (the configuration is jammed). The equation thus simply states that the macrosite of the spin up with label $x(\ell)$ (i.e. the spin up at site $\ell$) is $\ell/2$. More formally, it follows from equation (10) by noting that $\ell(\ell) \sim \ell/2$, and by assuming that $\ell$ and $x(\ell)$ are large—see appendix A.4.

The remarkable feature of equation (19) is that the asymptotic value of the local magnetisation at site $\ell$ is determined solely by a configuration of three consecutive spins around that site. The only sites in which the spins are asymptotically not aligned along the $z$-axis are those that start in one of the following spin configurations: $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots$, $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots$, and $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdots$. Their common feature is the existence of both species of particles, which was shown in [8] to signal the presence of interaction: the only nontrivial scattering phase of the dual folded XXZ model is associated with a scattering process between two particles of species 1 and 0, which can occupy the same macrosite. As shown in figure 1, indeed, initial states with a single species of particles do not exhibit nontrivial ballistic profiles.

### 3.3. Emergence of a locally quasi-jammed state (LQJS)

Since the spin flip at time $t = 0$ creates an excess of spins down, for $t > 0$ the state is not jammed anymore. However, as we will demonstrate shortly, a weaker condition

\[
P_{\perp\perp}(\ell, t) = \frac{1 - \sigma_\ell^+ - \sigma_{\ell+1}^+}{2} \rightarrow_0^\infty \forall \ell,
\]

(21)

holds. The quantity $P_{\perp\perp}$ corresponds to the probability that in the state at time $t$ the adjacent spins $\ell$ and $\ell + 1$ simultaneously point downwards; thus, in a jammed state it is equal to zero. The condition (21) was used in [22] to define LQJSs, in which the probability for a particle to move approaches zero, and hence at long time after the spin flip the state becomes locally jammed in each finite subsystem.
Figure 1. Panel (a) shows $\langle \sigma_z^\ell \rangle_t$ as a function of the ballistic ray coordinate $\ell/(Jt)$. The initial state (immediately after the spin flip) is $|\cdots \uparrow \uparrow \downarrow \uparrow \uparrow \downarrow \uparrow \downarrow \downarrow \uparrow \uparrow \cdots \rangle$. Blue points correspond both to numerical implementation of equation (13) with $D = \sigma_z^\ell$, as well as to the DMRG data (they are in perfect agreement). The orange line corresponds to the asymptotic behaviour (19), in which $x(\ell) = 2\ell/3$, as follows from equation (20). Panel (b) shows $\langle \sigma_z^\ell \rangle_t$ for $Jt = 10$ for a state with one species of particles, i.e. an noninteracting state. For the initial configuration we chose a Néel state with one of the spins up flipped down $|\cdots \uparrow \downarrow \uparrow \downarrow \cdots \uparrow \uparrow \cdots \rangle$. Panel (c) shows $\langle \sigma_z^\ell \rangle_t$ for $Jt = 10$ in the state given in Panel (b) with complementary flipped up spin at $\ell = 10$. Two blue points describe the spins not aligned with the $z$-axis.

Expressed in the basis of states $|n; b_0^0 \rangle$, in which the projectors of the form $1/2(1 - \sigma_z^\ell)$ are diagonal, the expectation value used in the LQJS condition reads

$$P_{\downarrow \downarrow}(\ell, t) = \sum_n |J_n(\tau)|^2 \left\langle n; b_0^0 | \frac{1 - \sigma_z^\ell}{2} | n; b_0^0 \right\rangle \left\langle n; b_0^0 | \frac{1 - \sigma_z^{\ell+1}}{2} | n; b_0^0 \right\rangle,$$

(22)

where equation (13) has been used. Let us now call $j(\ell)$ the index of the particle represented by a spin up at site $\ell$, or at one of the neighbouring sites if the spin at $\ell$ is down. For $|n - j(\ell)| > n_0$, with $n_0$ some finite positive integer, the matrix elements of $\sigma_z^\ell$ and $\sigma_z^{\ell+1}$ have simple asymptotic forms derived in appendix A.3: both matrix elements are determined by the initial configuration of four neighbouring spins including the two in sites $\ell$ and $\ell+1$. For $\ell > 0$ we report the expectation values $\langle n; b_0^0 | \frac{1}{2}(1 - \sigma_z^\ell) | n; b_0^0 \rangle$ and $\langle n; b_0^0 | \frac{1}{2}(1 - \sigma_z^{\ell+1}) | n; b_0^0 \rangle$ for all possible configurations of the four spins in table 1. Observing that their product is always zero, we conclude that $P_{\downarrow \downarrow}(\ell, t)$, as expressed in equation (22), is a finite sum containing possibly only the terms for which $|n - j(\ell)| \leq n_0$, that is, the terms to which the asymptotic formulas do not apply. Specifically, $P_{\downarrow \downarrow}(\ell, t)$ is a finite sum of squared Bessel functions that decay as $1/t$ and it thus follows that the LQJS condition is satisfied up to corrections of order $O(1/t)$, a feature corroborated by exact numerics—see figure 2 (an exact analytical calculation in a weakly interacting scenario is presented in section 3.5).

3.4. Local conservation laws and currents

In our analysis so far we have not used the integrable structure of the dual folded XXZ model. One of the hallmarks of the model’s integrability is the existence of infinitely many local...
Table 1. The expectation values of the projectors \( \frac{1}{2} (1 - \sigma_z^\ell) \) and \( \frac{1}{2} (1 - \sigma_z^{\ell+1}) \) in the states \( |n; b \rangle^{(0)} \) depend only on the initial configuration of the four spins on sites \( \ell, \ell + 1, \ell + 2, \) and \( \ell + 3 \). Specifically, spins \( \ell, \ell + 1, \ell + 2 \) (resp. \( \ell + 1, \ell + 2, \ell + 3 \)) are required to determine the asymptotic value of the \( z \)-component of the spin \( \ell \) (resp. \( \ell + 1 \)): equations (A.11) and (A.12) are used for this purpose. For \( |n - j(\ell)| > n_0 \), with \( n_0 \) a finite positive integer, the products of these expectation values are always zero.

| \( n - j(\ell) > n_0 \) | \( \frac{1}{2} (1 - \sigma_z^\ell) \) | \( \frac{1}{2} (1 - \sigma_z^{\ell+1}) \) |
| --- | --- | --- |
| \( \uparrow \uparrow \uparrow \uparrow \) | 1 | 0 |
| \( \uparrow \uparrow \uparrow \downarrow \) | 0 | 1 |

| \( n - j(\ell) < -n_0 \) | \( \frac{1}{2} (1 - \sigma_z^\ell) \) | \( \frac{1}{2} (1 - \sigma_z^{\ell+1}) \) |
| --- | --- | --- |
| \( \uparrow \uparrow \downarrow \downarrow \downarrow \) | 0 | 0 |
| \( \uparrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \) | 1 | 1 |

Figure 2. Rescaled jamming condition profile \( J t P_{↓↓}(\ell, t) \) for \( J t = 25 \) (blue) and \( J t = 50 \) (orange) in the initial state \( |\cdots \uparrow \uparrow \downarrow \uparrow \uparrow \downarrow \uparrow \downarrow \cdots \rangle \). The DMRG data confirm the \( O(1/t) \) approach towards an LQJS: \( J t P_{↓↓}(\ell, t) \) forms a profile with a scale invariant envelope, conjectured to be \( a/\sqrt{1 - \zeta/v_\pm^2} \), where \( a \approx 0.16, v_\pm = \pm 8 J / (1 + 2 \langle S_z \rangle_0) = \pm 6 \langle S_z \rangle_0 \) is the total magnetisation in the initial state)—see [22].

conservation laws, whose charge densities satisfy continuity equations together with associated currents. In fact, the spin along the \( z \)-axis, considered in section 3.2, is related to the macrosite densities of two such local conservation laws,

\[
S^+ = \sum_{\ell} \frac{1}{2} (\sigma_{2\ell-1}^z + \sigma_{2\ell+1}^z), \quad S^- = \sum_{\ell} \frac{1}{2} (\sigma_{2\ell}^z - \sigma_{2\ell-1}^z),
\]  

(23)
charges have a nontrivial profile in the ballistic scaling limit corrections. This argument can however be applied only if the charge density is a (piece-wise) continuous grained position in the chain, computed from equation (19).

\[
\left\langle \frac{\sigma_{1,0} + \sigma_{1,1}}{2} \right\rangle,
\begin{align*}
1, & \quad \frac{1}{\pi} \arccos \left( -\frac{\bar{\lambda}}{\pi} \right), \\
\frac{1}{\pi} \arccos \left( \frac{\bar{\lambda}}{\pi} \right), & \quad \frac{1}{\pi} \arccos \left( -\frac{\bar{\lambda}}{\pi} \right), \\
\frac{1}{\pi} \arccos \left( \frac{\bar{\lambda}}{\pi} \right), & \quad \frac{1}{\pi} \arccos \left( -\frac{\bar{\lambda}}{\pi} \right), \\
0, & \quad \frac{1}{\pi} \arccos \left( \frac{\bar{\lambda}}{\pi} \right),
\end{align*}
\]

\[
\sim \begin{cases} 
1, & \quad \frac{1}{\pi} \arccos \left( -\frac{\bar{\lambda}}{\pi} \right), \\
\frac{1}{\pi} \arccos \left( \frac{\bar{\lambda}}{\pi} \right), & \quad \frac{1}{\pi} \arccos \left( -\frac{\bar{\lambda}}{\pi} \right), \\
\frac{1}{\pi} \arccos \left( \frac{\bar{\lambda}}{\pi} \right), & \quad \frac{1}{\pi} \arccos \left( -\frac{\bar{\lambda}}{\pi} \right), \\
0, & \quad \frac{1}{\pi} \arccos \left( \frac{\bar{\lambda}}{\pi} \right), \\
1, & \quad \frac{1}{\pi} \arccos \left( -\frac{\bar{\lambda}}{\pi} \right),
\end{cases}
\]

for the magnetisation, and

\[
\left\langle \frac{\sigma_{1,0} - \sigma_{1,1}}{2} \right\rangle,
\begin{align*}
-\frac{1}{\pi} \arcsin \left( \frac{\bar{\lambda}}{\pi} \right), & \quad \frac{1}{\pi} \arcsin \left( -\frac{\bar{\lambda}}{\pi} \right), \\
\frac{1}{\pi} \arcsin \left( \frac{\bar{\lambda}}{\pi} \right), & \quad \frac{1}{\pi} \arcsin \left( -\frac{\bar{\lambda}}{\pi} \right), \\
\frac{1}{\pi} \arcsin \left( \frac{\bar{\lambda}}{\pi} \right), & \quad \frac{1}{\pi} \arcsin \left( -\frac{\bar{\lambda}}{\pi} \right), \\
-1, & \quad 1,
\end{align*}
\]

\[
\sim \begin{cases} 
-\frac{1}{\pi} \arcsin \left( \frac{\bar{\lambda}}{\pi} \right), & \quad \frac{1}{\pi} \arcsin \left( -\frac{\bar{\lambda}}{\pi} \right), \\
\frac{1}{\pi} \arcsin \left( \frac{\bar{\lambda}}{\pi} \right), & \quad \frac{1}{\pi} \arcsin \left( -\frac{\bar{\lambda}}{\pi} \right), \\
\frac{1}{\pi} \arcsin \left( \frac{\bar{\lambda}}{\pi} \right), & \quad \frac{1}{\pi} \arcsin \left( -\frac{\bar{\lambda}}{\pi} \right), \\
-1, & \quad 1,
\end{cases}
\]

for the staggered magnetisation. In both cases \(\bar{\lambda}(\ell') \sim x(2\ell') \sim x(2\ell' - 1)\) is the coarse-grained position in the chain, computed from equation (20). It corresponds to the label of the particle living on the macrosite \(\ell'\) (since the equations are asymptotic, the precise site of the spin up representing the particle is irrelevant).

The currents of the magnetisation and of the staggered magnetisation are obtained from the time derivative of \(\sigma_j^z\) in the Heisenberg picture

\[
\frac{d}{dt} \sigma_j^z(t) \bigg|_{t=0} = -J D_{\ell, \ell+2} \frac{\sigma_{x+1}^z - \sigma_{x}^z}{2} + J D_{\ell-2, \ell} \frac{\sigma_{x}^z - \sigma_{x-1}^z}{2},
\]

where \(D_{\ell, m} = \sigma_j^z\sigma_m^z - \sigma_j^z\sigma_m^z\). In a jammed state, in which any spin down is surrounded by two spins up, each term on the right-hand side vanishes. We can therefore conclude that the two currents approach zero in any LQJS in which the jamming condition is satisfied up to \(O(1/t)\) corrections.

The reader could be surprised at this result as we showed earlier that the corresponding charges have a nontrivial profile in the ballistic scaling limit \(x \sim J t\). In these situations, using the continuity equation, one could infer that also the differences of currents should be functions of \(x/t\)

\[
\partial_t \sigma_j^z \left( \frac{x}{t} \right) + \partial_x j(x, t) = 0 \quad \Rightarrow \quad j(x, t) - j(0, t) = \int_0^{x/t} d\zeta q'(\zeta) \zeta.
\]

Here, \(q\) is the expectation value of the charge density and \(j\) that of the corresponding current. This argument can however be applied only if the charge density is a (piece-wise) continuous
Figure 3. Spin current $q^-_\ell (\ell, t)$ and higher charge $q^+_\ell (\ell, t)$ for $Jt = 25$ (blue points) and $Jt = 50$ (orange points), obtained from the DMRG data for the initial state \( | \cdots \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \uparrow \downarrow \uparrow \uparrow \uparrow \downarrow \cdots \rangle \). In the large-time limit both profiles vanish as $1/t$, whereas the charges $q^+_\ell$ and $q^-_\ell$ are strictly zero during the time evolution.

function of $x/t$. In our case we have an additional microscopic dependence on the lattice position $q = q(\ell, t)$: the charge expectation value depends on the pattern of spins around the site $\ell$—cf equation (19). As we move from site to site, $q$ jumps between four smooth asymptotic curves that can be recognised only on large (ballistic) scales. The continuity equation on the chain then reads

\[
\partial_t q(\ell, t) = -j(\ell + 1, t) + j(\ell, t)
\]

and, due to the microscopic dependence on $\ell$, it now becomes possible to have a nontrivial ballistic profile for the charge notwithstanding the corresponding current approaching zero. The macroscopic reconfiguration of the spin profiles after a localised perturbation is thus not a typical transport phenomenon: as shown in section 3.1, the positions of the spins up only experience 'collective' fluctuations of magnitude $O(1)$. The phenomenon is somewhat reminiscent of the travelling surface waves in which the particles of water undergo only a short-scale oscillatory motion. The distinguishing and perhaps most surprising feature of the phenomenon shown here is that it is a result of an instantaneous perturbation.

Finally we note that the expectation values of the remaining (known) local conservation laws (reported in [8, 9]) asymptotically vanish. This follows from the thermodynamic Bethe Ansatz description, developed in [9]. In the thermodynamic limit, the charge and current expectation values per macrosite can be expressed respectively as

\[
q(x, t) = \int dp \, \rho_{x,t}(p) Q(p), \quad j(x, t) = \int dp \, \rho_{x,t}(p) v_{x,t}(p) Q(p) + \text{const.},
\]

where $\rho_{x,t}(p)$ is the quasi particle density in real and momentum space (also called ‘root density’), $v_{x,t}(p)$ is the quasiparticle velocity, and $Q(p)$ is the charge carried by a quasi particle with momentum $p$. In particular, $Q(p)$ are periodic functions of the momentum. Now, it can be shown that, in a jammed state of an infinite system the particle density $\rho_{x,t}$ is constant as a function of the momentum, while the product $\rho(p)v(p)$ does not depend on the state [9]. For example, the fact that $\rho_{x,t}$ is constant simply means that in a jammed state particles fully occupy all of the available momenta (both real and momentum space are densely populated).
As a consequence, in LQJSs, such as the ones emerging in our setting, the expectation values of all charges for which $Q(p)$ is periodic function of the momentum vanish, while their currents approach trivially constant profiles (see also figure 3).

### 3.5. Example: exact results in a weakly interacting scenario

In this section we consider an example of a spin flip performed on a weakly interacting initial state, in which local magnetisation, jamming condition, and entanglement properties can be computed exactly for all times $t$. In particular, exact calculation of entanglement properties allows us to identify both classical and pure quantum contributions to the two-point spin correlation functions.

We consider the initial jammed state obtained from the Néel state by flipping up a sequence of $M$ spins on the macrosites $m', m' + 1, \ldots, m' + M - 1$ (we assume $m' > 0$):

$$|\Psi(0^-)\rangle = |\cdots \downarrow \uparrow \downarrow \cdots \uparrow \downarrow \cdots \uparrow \downarrow \cdots \rangle.$$  \hspace{1cm} (30)

Note that for this specific state $M = \sum_j b_j (1 - b_{j+1})$ counts the number of particle pairs, in which a particle of species 1 is followed by that of species 0. Such pairs of particles occupy one macrosite each and a nontrivial scattering phase is associated with their scattering process. This is why a nonzero sum $\sum_j b_j (1 - b_{j+1})$ is associated with interaction [8]. Since, however, the sum is finite in the state (30) and we are in the thermodynamic limit, the state can be considered weakly interacting.

By flipping the spin in position 0 according to the protocol described in section 2, we end up with the state

$$|\Psi(0)\rangle = |\cdots \downarrow \uparrow \downarrow \cdots \downarrow \uparrow \cdots \rangle,$$  \hspace{1cm} (31)

which is no longer jammed, but evolves in time towards a LQJS. Indeed, in appendix B we show

$$P_{\downarrow \downarrow}(\ell, t) = \begin{cases} J_{[\ell/2]}^2(\tau), & [\ell/2] < m' - 1 \\ J_{-[\ell/2] - 2}^2(\tau), & m' - 1 \leq [\ell/2] < m' + M - 2 \\ J_{[\ell/2] + M}^2(\tau), & [\ell/2] \geq m' + M - 2, \end{cases}$$  \hspace{1cm} (32)

where $[\[ \cdot \]]$ and $[\cdot]$ denote the rounding up and down, respectively. We stress that this is an exact result: the equation describes the full time evolution of $P_{\downarrow \downarrow}(\ell, t)$. Specialising to large times we now note that, since the squared Bessel functions asymptotically decay as $1/t$, the jamming condition is satisfied up to $O(1/t)$ corrections, as already predicted in section 3.3.

The relative simplicity of the initial state also allows us to obtain exact space-time dependence of the local magnetisation (see appendix B.1 for details):

$$\langle \sigma_{2\ell' - 1}^z \rangle_t = \begin{cases} 2f(m' + 2M - 2, \tau) - 1, & \ell' = m' + M - 1 \\ 1 - 2f(m', \tau), & \ell' = m' - 1 \\ 1 - 2f_{m' - 2\ell' - 1}^2(\tau) - 2J_{m' - 2\ell'}^2(\tau), & m' \leq \ell' \leq m' + M - 2 \\ -1, & \text{otherwise}, \end{cases}$$  \hspace{1cm} (33)
Figure 4. Magnetisation profile $\langle \sigma_z^\ell \rangle_t$ for the initial state described in equation (31) with $m' = 9$ and $M = 10$. Colours of the points correspond to different fixed times $Jt = 10$ (green) and $Jt = 20$ (blue). The spins at sites $2m' - 3$ and $2m' + 2M - 3$ are the only ones for which the $z$-components of the magnetisation are not asymptotically equal to $\pm 1$. Instead, as time $t$ is increased, they move towards the zero ray (time increases while the lattice positions are finite and fixed) along the trajectories (orange dashed lines) given in equation (35).

\[
\langle \sigma_z^\ell \rangle_t = \begin{cases} 
1 - \frac{2\mathcal{F}_j^2}{\tau}, & \ell' < m' - 1 \\
1 - \frac{2\mathcal{F}_j^2}{\tau}, & \ell' > m' + M - 2 \\
1 - \frac{2\mathcal{F}_j^2}{\tau} - 2\mathcal{F}_j^2, & m' - 1 \leq \ell' \leq m' + M - 2,
\end{cases}
\]

(34)

where $f(x, t)$ is given in equation (15). By symmetry we again have $\langle \sigma_x^\ell \rangle_t = 0$ and $\langle \sigma_y^\ell \rangle_t = 0$ for the other components of local magnetisation. This is due to the fact that the initial state is an eigenstate of $Q_j \sigma_z^j$, which is a symmetry of the Hamiltonian. Note that, except for the sites $2m' - 3$ and $2m' + 2M - 3$, which are included in the first two cases in equation (33) above, the long-time limit of the expectation value of $\sigma_z$ is either $1$ or $-1$. The asymptotic behaviours of the $z$-components of the spins in sites $2m' - 3$ and $2m' + 2M - 3$ are instead

\[
\langle \sigma_z^{2m' - 3} \rangle_t \sim \frac{2}{\pi} \arcsin \left( \frac{m'}{\tau} \right), \quad \langle \sigma_z^{2m' + 2M - 3} \rangle_t \sim \frac{2}{\pi} \arcsin \left( \frac{m'}{\tau} \right) \left[ 1 + \frac{1}{1 + \frac{M - 1}{m'}} \right]
\]

(35)

and are shown in figure 4. These observations suggest that, in the infinite-time limit, the reduced density matrix of any spin with site index $j \notin \{2m' - 3, 2m' + 2M - 3\}$ is pure: it corresponds either to $|\uparrow \rangle \langle \uparrow |$ or $|\downarrow \rangle \langle \downarrow |$. The reduced density matrix of each of the two spins in sites $2m' - 3$ and $2m' + 2M - 3$ is instead maximally mixed in the infinite-time limit. Indeed, since $m'$ and $M$ are finite and fixed, also the $z$-components of magnetisation given in equation (35) vanish, therefore the reduced density matrix describing the sites $2m' - 3$ and $2m' + 2M - 3$ is proportional to the identity. In particular, this implies that the spins in these sites are maximally entangled with the rest of the system (the entanglement entropy of their reduced density matrices is maximal).

We can go further and compute the two-point correlation functions (reported in appendix B.2). In particular, we find the large-time limit of the correlation $\langle \sigma_z^{2m' - 3} \sigma_z^{2m' + 2M - 3} \rangle_t$
Figure 5. Spin-spin entanglement for an initial state with $m' = 12$, $M = 9$ at different times, rescaled by $\log(Jt)/(Jt)^2$. The numbers on both axes refer to spin indices: the element $(i,j)$ of the plot characterises the entanglement between the spins at the sites $i$ and $j$. The plot is symmetric w.r.t. the bisector, i.e. the line containing the points $(i,i)$, since the entanglement between the spins $i$ and $j$ equals that between the spins $j$ and $i$. The blue (orange) lines refer to the spin $2m' - 1$ (resp. $2m' + 2M - 5$), i.e. the one close to the start (resp. end) of the domain wall of spins up. They delimit the nonzero entanglement regions. Panels (a)–(c) show the spread of the entanglement before the domain of spins up is reached by the light cone, during the spread of the entanglement through the domain, and after the edge of the light cone has passed through the domain, respectively.

to be equal to $-1$, i.e. the two spins are asymptotically maximally correlated. One might be tempted to conclude that these two spins are entangled with one another even in the infinite-time limit, i.e. that they are in a cat-state. Instead, it can be shown that their entanglement goes to zero as $\log(Jt)/(Jt)^2$, as does the entanglement between any other two spins. This can be seen by computing the ‘entanglement of formation’ [51, 52] $E$ between two given spins:

$$E(\hat{\rho}) = h \left( \frac{1 + \sqrt{1 - C(\hat{\rho})}}{2} \right).$$

(36)

Here, $\hat{\rho}$ is the reduced density matrix of the two spins, $h(x) = -x\log_2 x - (1 - x)\log_2(1 - x)$, while $C(\hat{\rho})$ is the so-called concurrence, defined as

$$C(\hat{\rho}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$

(37)

where $\lambda_j$ are the square roots of the eigenvalues of $\hat{\rho}(\sigma^y \otimes \sigma^y)\hat{\rho}^*(\sigma^y \otimes \sigma^y)$ in descending order, and $\hat{\rho}^*$ is the complex conjugate of $\hat{\rho}$. Even though $\hat{\rho}(\sigma^y \otimes \sigma^y)\hat{\rho}^*(\sigma^y \otimes \sigma^y)$ is not necessarily self-adjoint, its eigenvalues are real and non-negative, since it is a product of two positive semi-definite matrices.

Note that the concurrence (and hence the entanglement) is automatically zero whenever the density matrix is diagonal because the matrix $\hat{\rho}(\sigma^y \otimes \sigma^y)\hat{\rho}^*(\sigma^y \otimes \sigma^y)$ has two pairs of degenerate positive eigenvalues and hence $C(\hat{\rho}) = 0$. One can show that the off-diagonal elements of the two-spin reduced density matrix $\hat{\rho}$ go to zero as $1/t$ for large times, thus lifting the degeneracy of the matrix $\hat{\rho}(\sigma^x \otimes \sigma^x)\hat{\rho}^*(\sigma^x \otimes \sigma^x)$ by a term of order $O(1/t)$ at finite times. Using this in equation (36) and expanding for large $t$ we observe that the entanglement of formation approaches zero as $\log(Jt)/(Jt)^2$. Nonetheless, the two spins at sites $2m' - 3$ and
2m′ + 2M − 3 are special, since they are maximally correlated: in the infinite-time limit their reduced density matrix reads
\[ \hat{\rho}_{t \to \infty} \to \frac{1}{2} \left| \downarrow \downarrow \right\rangle \left\langle \downarrow \downarrow + \left| \downarrow \uparrow \right\rangle \left\langle \downarrow \uparrow \right|, \]
which is a classical mixture.

In general, the spread of the rescaled spin–spin entanglement with time is shown in figure 5. At \( t = 0 \) there are no entangled spins (the initial state is a product state). For \( t > 0 \) the region containing entangled spins grows linearly in time, consistently with the light-cone dynamics. At first the entanglement spreads from the initial position of the impurity in all directions—cf figure 5(a)—with the maximal velocity \( 8J \). When the edge of the light-cone reaches macro-site \( m \), a ‘leakage’ of the entanglement between the wedges occurs through a corridor, in which only the next-nearest neighbour spins become entangled—cf figures 5(b) and (c). We observe that the couples of spins \((i, j)\) with \( i, j < 2m′ − 1 \) are not affected by the presence of the domain wall in the initial state: their behaviour is the same as if the initial state were a Néel state. Instead, the entanglement properties of the pairs of spins \((i, j)\) with \( i, j > 2m′ + M − 5 \) are shifted: their entanglement corresponds to that in the Néel state in which the impurity is initially located at macro-site \( −M \). Finally, in our case, all the spins outside those two regions are decoupled unless they are next-nearest neighbours, which is not the case when the dynamics start from the Néel state.

4. From the dual folded XXZ to the XXZ model

The dual folded XXZ model was originally envisioned as an effective model describing the large-anisotropy limit of the Heisenberg XXZ spin-1/2 chain
\[ H_{\text{XXZ}}(\Delta) = J \sum_{\ell} \sigma^x_{\ell} \sigma^x_{\ell+1} + \sigma^y_{\ell} \sigma^y_{\ell+1} + \Delta \sigma^z_{\ell} \sigma^z_{\ell+1}. \]
In this respect it is natural to wonder how well it acts as such. Specifically, is there a sign of the measurement catastrophe studied herein and in \[22\] in the XXZ model with large anisotropy \( \Delta \)?

To answer this question we need to consider the relation between the two models. In order to obtain the dual folded XXZ model from the Heisenberg XXZ spin-1/2 chain one needs to work in the ‘folded picture’ based on the large coupling expansion, the coupling being the anisotropy parameter \( \Delta \). In addition, one then has to perform a duality transformation. The only feature of the folded picture that we should have in mind is a unitary transformation that acts on the initial state and contributes \( O(1/\Delta) \) corrections: since \( \Delta \) is large, we will neglect it. What remains, in order to map our protocol to the corresponding one in the Heisenberg XXZ model, is therefore to consider the inverse of the duality transformation. The latter maps \( \sigma^z_{\ell} \rightarrow \sigma^z_{\ell+1} \), so it essentially acts on the initial state as follows:
\[ \cdots \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \cdots \mapsto \cdots \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \cdots. \]
Time-evolving this initial state with the Heisenberg Hamiltonian \( H_{\text{XXZ}}(\Delta) \) is now expected to yield a scaling profile of \( \langle \sigma^z_{\ell} \sigma^z_{\ell+1} \rangle \), converging, as \( \Delta \) is increased, to the one of \( \langle \sigma^z_j \rangle \) in the dual folded XXZ model. This is indeed observed in numerical simulations based on tensor network techniques—cf figure 6, and supports the validity of the strong-coupling expansion described in \[8\]. Besides that, transforming back to XXZ helps us see why the local measurement could
Figure 6. Panel (a) shows correlation function $\langle \sigma_z^{\ell} \sigma_z^{\ell+1} \rangle_t$ in the Heisenberg XXZ model (blue and orange points) compared to the magnetization of the folded model equation (19) (black line). Data points quickly converge to the prediction of the folded model for a relatively large values of $\Delta$, whereas for small anisotropy they exhibit a disagreement with the prediction during the whole time evolution. Panel (b) shows the difference $\langle \sigma_z^{\ell} \rangle_{\text{folded}} - \langle \sigma_z^{\ell} \sigma_z^{\ell+1} \rangle_{\text{XXZ}}$ outside of the light cone as a function of time. For short times the expectation values in both models linearly deviate from each other. In the inset we plot a slope $\alpha$ of this difference as a function of the anisotropy $\Delta$. Data points are fitted with $a\Delta^{-2} + b\Delta^{-3}$, where $a \approx 0.241$, $b \approx 1.31$.

have such a relevant effect: in the XXZ setting local measurement creates a domain-wall structure. If the two halves of the chain prepared in a domain wall state had different integrals of motion, the theory of generalised hydrodynamics \cite{53–55} would predict nontrivial ballistic profiles. This is not the case here. One could then wonder whether we are seeing the effects of quasi-conserved operators that are exactly conserved only in the large-anisotropy limit. So far, our investigations have been inconclusive, so the question is still open.

5. Discussion

The simplicity of the calculations carried out in this paper is rather surprising in view of the fact that we are dealing with a genuinely interacting model. It goes even beyond the simplicity of the Bethe equations of the dual folded XXZ model \cite{8}. It seems rather related to the setting considered—that of locally perturbed jammed states, in which particles are closely packed. Such a scenario can be mapped into the problem of a free particle moving in a jammed background. We have exploited this mapping to solve the measurement catastrophe discussed in \cite{22}, i.e. the macroscopic change of spin profiles after a localised perturbation. We have established that no genuine spin transport occurs: the particles’ positions exhibit small fluctuations around their expectation values, and the current of the local magnetisation is asymptotically zero. This is a result of the emergent scale separation in which the smooth asymptotic scaling of the sublattice local magnetisation coexists with an underlying discontinuous dependence of $\langle \sigma_z^{\ell} \rangle$ on the position $\ell$. Interactions play a key role in the setting we studied: the phenomenon is not observed if particles do not interact. On the other hand, a finite number of interacting particles is enough to trigger the effect. In particular, perturbing a weakly interacting initial state enables isolation of classically correlated pairs of spins.

Jammed states typically arise in kinetically constrained models, and it remains unclear whether the measurement catastrophe described in this work is a universal feature of such models \cite{56}.

During the finalisation of this work and in the attempt to unveil the physical mechanism behind the measurement catastrophe, one of us has realised that an analogous sensitivity to
localised perturbations can be present also after global quenches []. In that case the existence of infinitely many ‘semilocal charges’ was identified as crucial. That mechanism, however, does not seem to account for the phenomenon discussed in this paper, which appears to be more related to the exceptionality of the jammed sector, consisting of states with memory. Indeed, to the best of our knowledge the dual folded Hamiltonian only has a single semilocal charge, first identified in [8], which corresponds to the total spin along the $z$-axis when mapping back to the XXZ Heisenberg model. That charge has a nontrivial profile in some of the examples that we considered, but it is irrelevant in others, including in the example investigated in [22].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Proofs

These sections contain the proofs of the equations stated in the main part of the text.

A.1. Time evolution of the state and $\langle \sigma_z^\ell \rangle_t$

Recall that the state $|n; b(0)\rangle$ is defined in such a way that the impurity lies on the immediate right-hand side of the particle (spin up) with index $n$. The state is jammed everywhere, except at the position of the impurity, which consists of either two or three consecutive spins down. The width of the impurity, i.e. the number of spins down that it contains, is not preserved under the action of the Hamiltonian. Nevertheless, the types of particles represented by spins up are well defined and their sequence is preserved, as shown in [8] (e.g. the particle corresponding to a spin up at the site $\ell$ is of the type $b = 2 [\ell/2] - \ell$).

Let us define the impurity creation operator $c_n^\dagger$ and the reference state $|\Psi_i(0)\rangle = |n; b(0)\rangle$, so that

$$c_n^\dagger |\Psi_i(0)\rangle = |n; b(0)\rangle. \tag{A.1}$$

Since the background $b(0)$ is preserved under the action of the Hamiltonian (1), the impurity effectively behaves as a freely hopping fermion. States $|n; b(0)\rangle$ span an invariant subspace, on which the dynamics can be readily solved. Employing a diagonalisation procedure for a free hopping Hamiltonian in the thermodynamic limit, we obtain

$$e^{-itH}c_\ell^\dagger |\Psi_i(0)\rangle = \sum_n \int \frac{dk}{2\pi} e^{-ik(\ell-\ell_0)} e^{i\epsilon(k)\tau} |n; b(0)\rangle,$$

$$= \sum_n (-i)^{n-\ell} J_n (\tau \ell) |n; b(0)\rangle \tag{A.2}$$

for the evolution of the state $|\ell; b(0)\rangle \equiv c_\ell^\dagger |\Psi_i(0)\rangle$. Here, the energy dispersion relation reads $\epsilon(k) = 4J \cos k$, and $J_n(x)$ are Bessel functions. In particular, the evolution of the initial state $|\Psi(0)\rangle = |0; b(0)\rangle \equiv c_0^\dagger |\Psi_i(0)\rangle$ in the protocol described in the main text reads
\[ |\Psi(t)\rangle = e^{-iH_0 t}|0; b^{(0)}\rangle = \sum_n (-i)^n J_n(\tau)|n; b^{(0)}\rangle. \quad (A.3) \]

Consider now the expectation value \( \langle \sigma^\ell_z \rangle_t = \langle \Psi(t)|\sigma^\ell_z|\Psi(t)\rangle \), where \( \ell \) is the site on the lattice. We have
\[
\langle \sigma^\ell_z \rangle_t = \sum_{n,m} \int \frac{dk}{2\pi} \int \frac{dp}{2\pi} e^{ikx - im\phi} e^{i[\varepsilon(k) - \varepsilon(p)]t} \langle m; b^{(0)}|\sigma^\ell_z|n; b^{(0)}\rangle. \quad (A.4) \]

The operator \( \sigma^\ell_z \) is diagonal in the basis \( |n; b^{(0)}\rangle \), therefore the only non-vanishing matrix elements are \( \langle n; b^{(0)}|\sigma^\ell_z|n; b^{(0)}\rangle \), whence
\[
\langle \sigma^\ell_z \rangle_t = \sum_n |J_n(\tau)|^2 \langle n; b^{(0)}|\sigma^\ell_z|n; b^{(0)}\rangle. \quad (A.5) \]

In equation (A.5) \( \sigma^\ell_z \) can be substituted by any operator \( D \), diagonal in the basis of states \( |n; b^{(0)}\rangle \). For observables \( A \) that are not diagonal in this basis, we instead have
\[
\langle A_\ell \rangle_t = \sum_{n_1, n_2} e^{i\pi(n_1 - n_2) J_{n_1}(\tau) - J_{n_2}(\tau)} \langle n_1; b^{(0)}|A|n_2; b^{(0)}\rangle. \quad (A.6) \]

A.2. Equal and non-equal time correlations

Let \( \langle \bullet \rangle_\tau \equiv \langle \bullet \rangle_{\tau=0} \) denote the average in the initial state \( |\Psi(0)\rangle = |0; b^{(0)}\rangle \). In this section we report different-time correlation functions of operators that are diagonal in the basis of states \( |n; b^{(0)}\rangle \). In particular, the general two-point correlation function of two diagonal operators \( D_1 \) and \( D_2 \) reads
\[
\langle D_1(t_1)D_2(t_2) \rangle = \sum_{m,n} J_m(\tau_1)J_{m-n}(\tau_1 - \tau_2)J_n(\tau_2) \times \langle m; b^{(0)}|D_1|m; b^{(0)}\rangle \langle n; b^{(0)}|D_2|n; b^{(0)}\rangle, \quad (A.7) \]

where \( \tau_{1,2} = 4J_1t_{1,2} \). At equal times the correlation function becomes
\[
\langle D_1(t)D_2(t) \rangle = \sum_n |J_n(\tau)|^2 \langle n; b^{(0)}|D_1|m; b^{(0)}\rangle \langle n; b^{(0)}|D_2|n; b^{(0)}\rangle, \quad (A.8) \]

while at \( t_2 = 0 \) it factorises
\[
\langle D_1(t)D_2 \rangle = \left( \sum_n |J_n(\tau)|^2 \langle m; b^{(0)}|D_1|m; b^{(0)}\rangle \right) \langle 0; b^{(0)}|D_2|0; b^{(0)}\rangle = \langle D_1 \rangle \langle D_2 \rangle. \quad (A.9) \]

A.3. Matrix elements of \( \sigma^\ell_z \)

In this section we compute the diagonal matrix elements \( \langle n; b^{(0)}|\sigma^\ell_z|n; b^{(0)}\rangle \). First we consider the case when the site \( \ell \) is situated on the right-hand side of the initial spin flip. Here it is most convenient to increase \( n \) in the state \( |n; b^{(0)}\rangle \) from its value \( n = 0 \) in the initial state so that the impurity passes through the site \( \ell \) from the left- towards the right-hand side.

In the second part we will compute the matrix elements in a way more suited to the case when the site \( \ell \) is on the left-hand side of the initial spin flip. Here we will decrease \( n \) from its value \( n = 0 \) in the initial state, so that the impurity will pass from the left- to the right-hand side of the site \( \ell \).
Both ways of computing the matrix element are essentially equivalent, since the process of moving the impurity is reversible. The asymptotic result of the calculation here is finally reported in appendix A.4.

A.3.1. Increasing \( n \): the right-moving impurity

First we consider the case of increasing \( n \), so that the impurity passes from the left- to the right-hand side of the site \( \ell \). The local Hamiltonian density

\[
\hat{h}_{\ell, \ell+1, \ell+2} = \mathcal{J} \frac{1 - \sigma_{\ell+1}^z}{2} \left( \sigma_{\ell}^x \sigma_{\ell+2}^x + \sigma_{\ell}^y \sigma_{\ell+2}^y \right)
\]

(A.10)

acts on three subsequent sites. The impurity is identified with two or three consecutive spins down: once it reaches the left-hand side of the site \( \ell \), two spins down occupy the sites \( \ell - 2 \) and \( \ell - 1 \). Three subsequent actions of the local Hamiltonian density are then required for it to pass to the right-hand side of the site \( \ell \). Specifically, the actions are those of \( h_{\ell-2, \ell-1, \ell} \), \( h_{\ell-1, \ell, \ell+1} \), and \( h_{\ell, \ell+1, \ell+2} \) (only two steps are needed, for example, when a spin down already occupies the \( \ell \)th site). To determine the matrix element of \( \sigma_\ell^z \) we thus need to consider eight processes, determined by the spins on sites \( \ell, \ell + 1 \), and \( \ell + 2 \) on the right-hand side of the right-moving impurity. Three of these processes do not occur, since the state is jammed everywhere but at the position of the impurity itself: we can not have \( \uparrow \downarrow \downarrow \), \( \downarrow \downarrow \uparrow \), or \( \downarrow \downarrow \downarrow \) on sites \( \ell, \ell + 1 \), and \( \ell + 2 \), while the impurity is on the left-hand side of the site \( \ell \). The remaining five processes that determine the matrix elements \( \langle \hat{n}; \hat{b}(0) | \sigma_\ell^z | \hat{n}; \hat{b}(0) \rangle \) are presented in figure A1, which serves as a proof of the following rules determining the matrix element:

\[
\langle \hat{n}; \hat{b}(0) | \sigma_{\ell>0}^z | \hat{n}; \hat{b}(0) \rangle = \begin{cases} 
1 - 2\delta_{n,j}\, - 2\delta_{n,j+1}, & |0; \hat{b}(0)\rangle = |\ldots \uparrow(0) \bullet^{(1)} \ldots \uparrow(j) \uparrow \ldots \rangle \\
1 - 2\delta_{n,j}, & |0; \hat{b}(0)\rangle = |\ldots \uparrow(0) \bullet^{(1)} \ldots \uparrow(j) \downarrow \ldots \rangle \\
1 - 2\delta_{n \geq j}, & |0; \hat{b}(0)\rangle = |\ldots \uparrow(0) \bullet^{(1)} \ldots \uparrow(j) \downarrow \ldots \rangle \\
1 - 2\delta_{n \leq j}, & |0; \hat{b}(0)\rangle = |\ldots \uparrow(0) \bullet^{(1)} \ldots \downarrow(j) \uparrow \ldots \rangle \\
-1, & |0; \hat{b}(0)\rangle = |\ldots \uparrow(0) \bullet^{(1)} \ldots \downarrow(j) \downarrow \ldots \rangle.
\end{cases}
\]

(A.11)

Here, \( j \) is the index of the particle represented by a spin up at the site \( \ell \), or, if the latter is occupied by a spin down, at the site \( \ell+1 \). The impurity is represented by \( \bullet \).

A.3.2. Decreasing \( n \): the left-moving impurity

Let us now consider the case of decreasing \( n \), in which the impurity passes from the right- to the left-hand side of the site \( \ell \). The rules proven in figure A2 now read

\[
\langle \hat{n}; \hat{b}(0) | \sigma_{\ell<1}^z | \hat{n}; \hat{b}(0) \rangle = \begin{cases} 
1 - 2\delta_{n,j+1} - 2\delta_{n,j-1}, & |0; \hat{b}(0)\rangle = |\ldots \uparrow(0) \bullet^{(1)} \ldots \uparrow(j) \uparrow \ldots \rangle \\
1 - 2\delta_{n,j-1}, & |0; \hat{b}(0)\rangle = |\ldots \uparrow(0) \bullet^{(1)} \ldots \uparrow(j) \downarrow \ldots \rangle \\
1 - 2\delta_{n \geq j+1}, & |0; \hat{b}(0)\rangle = |\ldots \uparrow(0) \bullet^{(1)} \ldots \downarrow(j+1) \uparrow \ldots \rangle \\
1 - 2\delta_{n \leq j+1}, & |0; \hat{b}(0)\rangle = |\ldots \uparrow(0) \bullet^{(1)} \ldots \downarrow(j+1) \downarrow \ldots \rangle \\
-1, & |0; \hat{b}(0)\rangle = |\ldots \uparrow(0) \bullet^{(1)} \ldots \downarrow(j+1) \downarrow \ldots \rangle.
\end{cases}
\]

(A.12)
The five processes that determine the rules (A.11) for the right-moving impurity. Dark blue and orange colours represent spins up and down, respectively: two or three consecutive spins down (orange boxes) constitute an impurity, which starts on the left-hand side of the site $\ell$. The white dot represents the spin up corresponding to the particle with index $j_R$. The consequent steps (actions of the local Hamiltonian densities) follow in the downwards direction, and the site $\ell$ is framed.

Here, $j_L$ is the index of the particle represented by a spin up at the site $\ell$, or, if the latter is occupied by a spin down, at the site $\ell - 1$.

### A.4. Asymptotic profiles of magnetisation

In the previous section we have established the rules for the computation of the matrix elements of $\sigma_z^\ell$. With this let us return to its expectation value (A.5). For the asymptotic profiles only large values of $n$ and large particle labels are relevant, whence the rules (A.11) and (A.12) reduce to

$$\langle n; \hat{b}^{(0)} | \sigma_z^\ell | n; \hat{b}^{(0)} \rangle \sim \begin{cases} 1, & \text{if } n \geq x(\ell), \\ 1 - 2\theta(n \geq j_R), & \text{if } n \geq j_R, \\ 1 - 2\theta(n \leq j_R), & \text{if } n \leq j_R, \\ -1, & \text{otherwise} \end{cases}$$

(A.13)
Figure A2. The five processes that determine the rules (A.12) for the left-moving impurity. Dark blue and orange colours represent spins up and down, respectively; two or three consecutive spins down (orange boxes) constitute an impurity, which now starts on the right-hand side of the site \( \ell \). The white dot represents the spin up corresponding to the particle with index \( j_L \). The consequent steps (actions of the local Hamiltonian densities) follow in the downwards direction, and the site \( \ell \) is framed.

Here, \( \hat{\xi} \) stands for either a spin up or a spin down, while \( x(\ell) \) corresponds to the index of the particle represented by the spin up at the site \( \ell \) (or the neighbouring site). Asymptotically we compute it by observing \( \ell_{x(\ell)}' \sim \ell/2 \) and using equation (10). For example, for large \( x(\ell) > 0 \) one has

\[
\ell_{x(\ell)}' = x(\ell) - \theta(x(\ell) \leq n) - \sum_{m=1}^{x(\ell)-1} b_m^{(0)} (1 - b_{m+1}^{(0)}) - \frac{\ell}{2},
\]

where any terms of magnitude \( O(1) \) can be neglected. For large \( |\ell| \) we can thus compute \( x(\ell) \) from the integral equation.
\[ \frac{1}{\ell} \int_0^{\tau(t)} \, dn \xi(n) = \frac{1}{2}, \quad \text{(A.15)} \]

where \( \xi(n) \) is the average \( 1 - b_j^{(0)} (1 - b_{j+1}^{(0)}) \) around the particle with index \( n \).

Using the asymptotic rules (A.13) we now readily obtain equation (19). Consider, for example, the second rule in equation (A.13):

\[ \sum_n \langle n; h^{(0)} | \sigma_i | n; h^{(0)} \rangle | J_n(\tau) \rangle^2 = 1 - 2 \sum_n \theta(n \geq x(\ell)) | J_n(\tau) \rangle^2 \]
\[ = 1 - 2 \int \frac{dk}{2\pi} \int \frac{dp}{2\pi} \sum_{n \geq x(\ell)} e^{-i(k-p)n - i\ell (\cos k - \cos p)} \]
\[ = 1 - 2 \int \frac{dk}{2\pi} \int \frac{dp}{2\pi} \frac{e^{-i(k-p)x(\ell) - i\ell (\cos k - \cos p)}}{1 - e^{-i(k-p)}} \]
\[ \sim 1 - \frac{1}{\pi} \int dp \theta \left( \sin p - \frac{x(\ell)}{r} \right) = \frac{2}{\pi} \arcsin \left( \frac{x(\ell)}{r} \right). \quad \text{(A.16)} \]

**Appendix B. Correlation functions in the weakly-interacting case**

**B.1. The idea behind the computation**

In this section we show the idea behind the exact computation of any correlation function for the weakly-interacting scenario discussed in the main text, in which the unperturbed state at time \( t = 0^- \) is

\[ |\Psi(0^-)\rangle = |\cdots \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \cdots \rangle. \quad \text{(B.1)} \]

Essentially, the sums over \( n_1 \) and \( n_2 \), e.g. in equation (A.6), are split in such a way that the indices referring to eigenstates with the same characteristics are grouped together. In particular, we can distinguish four different classes of eigenstates \( |n; h^{(0)}\rangle \):

- \( n < m' \): the macrosite \( n \) is in the configuration \( \downarrow \downarrow \), the macrosites \( m', m' + 1, \ldots, m' + M - 1 \) are each in the configuration \( \uparrow \uparrow \), all the other macrosites are in the configuration \( \downarrow \uparrow \);
- \( n = m' + 2a, a \in \{0, 1, \ldots, M - 1\} \): the macrosite \( m' + a - 1 \) is in the configuration \( \uparrow \uparrow \), the macrosites \( m' - 1, m', \ldots, m' + a - 2, m' + a + 1, \ldots, m' + M - 1 \) are each in the configuration \( \uparrow \uparrow \) (note that the latter set is empty for \( M = 1 \)), all the other macrosites are in the configuration \( \downarrow \uparrow \);
- \( n = m' + 2b + 1, b \in \{0, 1, \ldots, M - 2\} \): (note that this set is empty for \( M = 1 \)) the macrosite \( m' + b \) is in the configuration \( \downarrow \downarrow \), the macrosites \( m' - 1, m', \ldots, m' + b - 1, m' + b + 1, \ldots, m' + M - 1 \) are each in the configuration \( \uparrow \uparrow \), all the other macrosites are in the configuration \( \downarrow \uparrow \);
- \( n \geq m' + 2M - 1 \): the macrosite \( n - M \) is in the configuration \( \downarrow \downarrow \), the macrosites \( m' - 1, m', \ldots, m' + M - 2 \) are each in the configuration \( \uparrow \uparrow \), all the other macrosites are in the configuration \( \downarrow \uparrow \).
Using equation (A.5), we obtain
\[
\langle \sigma^z_{2\ell'-1} \rangle = \sum_{\alpha=0}^{M-2} \left| J_{m'+2\alpha} (\tau) \right|^2 \left( 2\chi_{[m'-1,m'+\alpha-1]}(\ell') + 2\chi_{[m'+\alpha+1,m'+M-1]}(\ell') - 1 \right)
\]
\[
+ \sum_{b=0}^{M-2} \left| J_{m'+2b+1} (\tau) \right|^2 \left( 2\chi_{[m'-1,m'+b-1]}(\ell') + 2\chi_{[m'+b+1,m'+M-1]}(\ell') - 1 \right)
\]
\[
+ \sum_{n=-\infty}^{m'-1} |J_n(\tau)|^2 \left( 2\chi_{[m',m'+M-1]}(\ell') - 1 \right)
\]
\[
+ \sum_{n=m'+2M-2}^{m'+1} |J_n(\tau)|^2 \left( 2\chi_{[m'-1,m'+M-2]}(\ell') - 1 \right),
\]
(2.2)
for the local magnetisation at the odd sites. Here, \(\chi_{[a,b]}(x)\) is the characteristic function on the interval \([a,b]\), i.e. it is equal to one for \(x \in [a,b]\) and zero otherwise. A straightforward calculation now yields
\[
\langle \sigma^z_{2\ell'-1} \rangle = \begin{cases} 
2f(m' + 2M - 2, t) - 1, & \ell' = m' + M - 1 \\
1 - 2f(m', t), & \ell' = m' - 1 \\
1 - 2|J_{m'-2\ell'-1}(\tau)|^2 - 2|J_{m'-2\ell'}(\tau)|^2, & \ell' \in \{m', m'+1, \ldots, m'+M-2\} \\
-1, & \text{otherwise}, 
\end{cases}
\]
(2.3)
where \(f(x,t)\) is defined in equation (15).

Let us now consider an example of a non-diagonal operator, and compute it using equation (A.6). The operators \(\sigma^x\) and \(\sigma^y\) have zero expectation value, since the states \(\{n_1; b^{(0)}\}\) and \(\{n_2; b^{(0)}\}\) in equation (A.6) contain the same number of spins up. To give an example of a non-trivial non-diagonal operator, we need to consider correlation functions involving an even number of \(\sigma^x\) and \(\sigma^y\), such as \(\langle \sigma^x_{2\ell'} \sigma^y_{2\ell'} \rangle\), for \(d \in \mathbb{Z}^+\). In order to get a non-zero contribution from a generic term of the sum, the overall action of the spin-flips on \(\{n_2; b^{(0)}\}\) should not change the background. By dividing the sum over \(n_2\) in the sets that we introduced above, the only options are
\[
\sigma^x_{2\ell'} \sigma^y_{2\ell'} \langle n_2; b^{(0)} \rangle \rightarrow \begin{cases} 
\delta_{n_1,\ell'-d} \delta_{\ell', b^{(0)}} + \delta_{n_1,\ell'} \chi_{(-\infty,m')}(\ell') |\ell' - d; b^{(0)}\rangle, & n_2 < m' \\
\delta_{n_1,\ell'-d} \delta_{\ell', M+b^{(0)}} + \delta_{n_1,\ell'} \chi_{[m'+d,M-\infty)}(\ell') |\ell'+M-d; b^{(0)}\rangle, & n_2 \geq m' + 2M - 2 \\
\delta_{\ell', m'+2a+1} |\ell', m'+2a+1; b^{(0)}\rangle, & n_2 = m' + 2a \\
\delta_{\ell', m'+2a} |\ell', m'+2a; b^{(0)}\rangle, & n_2 = m' + 2a + 1. 
\end{cases}
\]
(2.4)
where \(a \in \{1,2,\ldots, M-2\}\) and we reported only the cases in which the background is unchanged (whence we used \(\leftarrow\rightarrow\) instead of \(\Rightarrow\)). By inserting back such result into the initial expression, we finally have
\[
\langle \sigma^x_{2\ell'} \sigma^y_{2\ell'} \rangle = 2 \cos \left( \frac{\pi}{2} d \right) \left( \theta(\ell' < m') J_{\ell'}(\tau) J_{\ell'-d}(\tau) + \theta(\ell' \geq m' + M + d - 2) J_{\ell'+M}(\tau) J_{\ell'+M-d}(\tau) \right).
\]
(2.5)
All the other one- and two-point functions are obtained analogously.
B.2. List of the two-point correlation functions

\begin{align*}
\langle \sigma_{1e}^{\alpha} \sigma_{2m'-1}^{\beta} \rangle &= 0, \\
\langle \sigma_{2m'}^{\alpha} \sigma_{2m'-1}^{\beta} \rangle &= 0, \\
\langle \sigma_{2m'}^{\alpha} \sigma_{2m'-1}^{\beta} \rangle &= 2 \cos \left( \frac{\pi}{2} (\beta - \alpha + d) \right) \\
&\quad \times \left[ \theta(\ell' < m') J_{\ell'}(\tau) J_{\ell'-d}(\tau) \\
&\quad + \theta(\ell' \geq m' + M + d - 2) J_{\ell'+1,M}(\tau) J_{\ell'+M-\ell'(d-1)}(\tau) \\
&\quad + \delta_{\ell,d}(\ell' \leq m' \leq m' + M - 2) J_{\ell'-m'-1}(\tau) \right] J_{\ell'-m'}(\tau), \\
\langle \sigma_{2m'}^{\alpha} \sigma_{2m'-1}^{\alpha} \rangle &= 2(\ell' - 1)^{\delta_{\ell,d}(\ell' \leq m' \leq m' + M - 1) J_{\ell'-2m' + 1}(\tau) \\
&\quad \times (\tau) J_{\ell'-m'}(\tau), \\
\langle \sigma_{2m'}^{\alpha} \sigma_{2m'-1}^{\beta} \rangle &= 1 - 2 \delta_{\ell',m'+M-1}(\ell' - d) \left( \begin{array}{c} J_{\ell'-m'-d} \quad (\ell' - m' + 2M - 2, \ell') + 2 \delta_{\ell',m'-1,\ell'-d} \\
\end{array} \right) \\
&\quad \times \left( (-1 + f(m',\tau) + \theta(d < M)(2(m',\tau - 1)) \\
&\quad - 2 \delta_{\ell',m'+M-1}(f(m',2M-2,\ell') \\
&\quad + 2 \theta(d < M)) \left( \begin{array}{c} f(m',\tau) \quad (f(m',\tau) + 2 \delta_{\ell',m'-1,\ell'-d} \\
\end{array} \right) \\
&\quad \times \left( f(m',2M-2,\tau) - f(m',\tau) \right) \\
&\quad - 2 \theta(d > M)) \left( \begin{array}{c} \theta(m' \leq m' + M - 1) \\
\end{array} \right) + \theta(m' + d \leq m' + \ell' + d + M - 1) \\
&\quad - 2 \theta(d < M)) \left( \begin{array}{c} \theta(m' \leq m' + d - 1) \\
\end{array} \right) + 2 \theta(m' \leq \ell' \leq m' + M + 2d - 2) \\
&\quad - 2 \theta(m' + d - 1 \leq \ell' \leq m' + M + 2d - 2) \\
&\quad \times \left( \begin{array}{c} \delta_{\ell',m'-2d+2} + \delta_{\ell',m'-2d+2}(\tau) \\
\end{array} \right) \\
&\quad \times \left( \begin{array}{c} \delta_{\ell',m'-2d+1}(\tau) + \delta_{\ell',m'-2d+1}(\tau) \\
\end{array} \right), \\
\langle \sigma_{2m'}^{\alpha} \sigma_{2m'-1}^{\alpha} \rangle &= 1 - 2 \theta(\ell' - d < m') \left( \begin{array}{c} J_{\ell',m'-d}(\tau) - 2 \theta(\ell' < m') \right) J_{\ell'}(\tau) \\
&\quad - 2 \theta(\ell' \geq m' + M - 2) J_{\ell'+M-\ell'-(\delta-1)}(\tau) \\
&\quad - 2 \theta(\ell' \geq m' + M - 2) J_{\ell'+M-\ell'-(\delta+1)}(\tau) \\
&\quad - 1 \leq \ell' < m' + M - 2) \left( \begin{array}{c} J_{\ell'-m'-2d-2}(\tau) + 2 \delta_{\ell',m'-1,\ell'-d}(\tau) \\
\end{array} \right) \\
&\quad - 2 \theta(m' - 1 \leq \ell' < m' + M - 2) J_{\ell'-m'-1}(\tau) - 2 \theta(\ell' < m' + M - 1) J_{\ell'-m'-1}(\tau) \\
&\quad - 2 \theta(m' \leq \ell' < m' + M - 1) J_{\ell'-m'-1}(\tau), \\
\langle \sigma_{2m'}^{\alpha} \sigma_{2m'-1}^{\beta} \rangle &= -1 + 2 \theta(a' < m') \left( \begin{array}{c} J_{\ell'}(\tau) + 2 \theta(a' \geq m' \\
\end{array} \right) J_{\ell'}(\tau) + M - 2) J_{\ell'-m'+3}(\tau) \\
&\quad + 2 \theta(m' - 1 \leq a' \leq m' + M - 3) J_{\ell'-m'-2}(\tau) \\
&\quad + 2 \theta(m' \leq a' \leq m' + M - 2) J_{\ell'-m'-1}(\tau) \\
&\quad + 2 \delta_{\ell',m'-1}(1 - f(m',\tau)) - 2 \theta(a' \geq m' + M - 2) \\.
\end{align*}
(C.1)

\[ \times \delta_{E', m', M-1} \]
\[ \times [(m' - 2, t) - 20(a' < m')\delta_{E', \tau}(\tau)] \]
\[ + \delta(m' < \ell' < m' + M - 2) | - \delta_{m' - 2\ell' - 1}, \tau(\tau) \]
\[ - \delta_{m' - 2\ell' - 1}(\tau) - 20(a' < m')\delta_{E', \tau}(\tau) \]
\[ - 20(a' > m' + M - 2)| \delta_{E', \tau}(\tau) \]
\[ + 2\delta_{E', m', \tau} \delta_{m' - 2\ell' - 1}(\tau) + 2\delta_{\ell', m', \tau} \delta_{m' - 2\ell' - 1}(\tau) \]
\[ - 4\delta(m' - 1 \leq \ell' \leq m' + M - 1) \]
\[ \times [(m' - 1 < a' < m' + M - 3)| \delta_{E', \tau}(\tau) \]
\[ + \delta(m' < a' < m' + M - 2)| \delta_{E', \tau}(\tau) \]
\[ \times J_{\alpha, \beta} \]
\[ \alpha, \beta \in \{1, 2\}, \sigma^1 = \sigma^x, \sigma^2 = \sigma^y, \sigma^j = \{0, 1\}, \tau = 4Jt \text{ and } d \geq 1. \]

**Appendix C. Details on numerical simulations**

Numerical calculations are performed using C++ Itensor library [57]. We use the time evolving block decimation (TEBD) as a time-evolution scheme, where \( e^{-i\delta t H} \) for the folded model is approximated by Trotter-Suzuki gates [9, 58–60]. In order to obtain higher approximation, we represent the Hamiltonian (1) as a sum

\[
H = \sum_\ell h_{3\ell, 3\ell+1, 3\ell+2} + \sum_\ell h_{3\ell+1, 3\ell+2, 3\ell+3} + \sum_\ell h_{3\ell+2, 3\ell+3, 3\ell+4}
= H_1 + H_2 + H_3,
\]

with commuting terms within each \( H_\alpha \). The time evolution operator reads

\[
e^{-i\delta t (H_1 + H_2 + H_3)} = e^{-i\frac{\delta t}{2} H_1} e^{-i\frac{\delta t}{2} H_2} e^{-i\frac{\delta t}{2} H_3} e^{-i\delta t H_2} e^{-i\delta t H_1} + \mathcal{O} (\delta t^3),
\]

where the time step is chosen as \( \delta t = 0.01 J^{-1} \). Numerical simulations for the XXZ model are performed using the matrix-product-based approximation to \( e^{-i\delta t H} \) of the third order, denoted by \( W^3 \) in [61, 62]. As a time step in this case we chose \( \delta t = 0.001 J^{-1} \) to reduce the potential errors from the large-anisotropy term.

The level of matrix-product state truncation is determined by some maximal value of the discarded weight, which we chose to be \( \delta \lambda = 10^{-12} \). It turns out that for the most of the calculations in the folded XXZ model the maximal bond dimension is \( \chi_{\text{max}} = 3 \) during the whole time evolution, meanwhile the bipartite entanglement entropy \( S_{\alpha\beta} \) forms a nontrivial scaling profile in the ballistic limit—see figure C1.
Figure C1. Bipartite entanglement entropy $S_{vN}(\ell,t)$ for $t = 25$ (blue points) and $t = 50$ (orange points). The inhomogeneity in the initial state $\ket{\cdots \uparrow \uparrow \uparrow \uparrow \downarrow \uparrow \downarrow \downarrow \uparrow \uparrow \uparrow \uparrow \downarrow \cdots}$ spreads with time and forms a nontrivial ballistic profile. The inset shows the entanglement entropy $S_{vN}(\ell = 0,t)$ in the centre of the system. Oscillations vanish with time, while the entanglement reaches its limiting value.

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