Discrete solitons in graphene metamaterials

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We study nonlinear properties of multilayer metamaterials created by graphene sheets separated by dielectric layers. We demonstrate that such structures can support localized nonlinear modes described by the discrete nonlinear Schrödinger equation, and its solutions are associated with stable discrete plasmon solitons. We also analyze the nonlinear surface modes in truncated graphene metamaterials being a nonlinear analogue of surface Tamm states.

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I. INTRODUCTION

Graphene is known to exhibit remarkable physical properties including a strong optical response related to its surface conductivity and dependence on graphene’s chemical potential⁴,⁵. At certain frequencies, doped graphene behaves like a metal, and it can support p-polarized surface plasmon polaritons due to the coupling of the electromagnetic field to the electron excitations³–⁶.

As was shown recently, graphene is a strongly nonlinear material⁷–¹³. In particular, several nonlinear effects associated with a self-action correction to graphene’s conductivity have been predicted recently¹⁴–¹⁷. In order to increase effective nonlinearity of photonic structures with graphene, a natural idea is to use graphene multilayers which, depending on different wavelength regimes, may possess the basic properties of photonic crystals and metamaterials¹⁸–²⁰.

One of the remarkable general properties of nonlinear systems is their ability to support nonlinear localized modes—self-trapped states or solitons which can propagate over long distances without changing their shape due to a balance between nonlinearity and dispersion (or diffraction). A special kind of solitons, the so-called discrete solitons, appear as intrinsic localized modes in homogeneous periodic physical systems, such as nonlinear atomic chains²¹,²², Bose-Einstein condensates loaded into optical lattices²³,²⁴, arrays of nonlinear optical waveguides²⁵, and semiconductor-dielectric periodic nanostructures²⁶. If compared to continuous localized waves, the discrete solitons possess a number of additional properties such as the Peierls-Nabarro barrier²⁷ and staggering transformation²⁸. In plasmonics, discrete solitons were studied in metal-dielectric multilayer structures²⁹–³³, arrays of nanowires³⁴–³⁷, and arrays of nanoparticles³⁸,³⁹.

Less than a decade ago, an interesting type of discrete solitons, surface solitons, was predicted theoretically⁴⁰ and then observed experimentally⁴¹,⁴². It is sustained by the boundary between a periodic structure and an uniform medium (although the maximum of soliton can be either exactly at the interface⁴⁰ or at some distance from it⁴³), i.e. a surface soliton can be considered as a nonlinear analogue of surface Tamm states⁴⁴,⁴⁵.

In this paper, we study nonlinear graphene-based multilayer structures and demonstrate that, similar to metal-dielectric metamaterials, they can be described by the discrete nonlinear Schrödinger (NLS) equation and support nonlinear localized modes in the form of discrete solitons, see Fig. 1(a). We also analyze such modes near

Figure 1. (Color online) Geometry of the problem: a multilayer structure is composed of graphene sheets separated by dielectric layers with permittivity ε and thickness d. Red curves show example profiles of the plasmonic solitons: (a) a discrete soliton in an infinite structure, and (b) surface soliton in a truncated metamaterial. Shown is the absolute value of the tangential electric field component.
the surfaces and predict the existence of nonlinear surface modes being a nonlinear analogue of surface Tamm states, as shown schematically in Fig. 1(b).

The paper is organized as follows. In Sec. II, we derive the nonlinear current response of graphene to an external harmonic electric field. In Sec. III, we obtain the discrete nonlinear Schrödinger equation and describe the properties of discrete solitons. Section IV is devoted to the study of surface solitons, localized in the vicinity of the terminated layer of the graphene metamaterial.

II. NONLINEAR CURRENT IN GRAPHENE

In this section, for the sake of completeness and clarity, we derive a Kerr-type nonlinear correction to the graphene conductivity, considered earlier in Refs.7,8 for the ballistic regime.

We consider a 2D doped graphene monolayer, placed parallel to the plane xy. Also we admit that a time-dependent external electric field is applied to graphene. For definiteness, the electric field is supposed to be directed along the x axis, i.e. $\vec{E} = [E(t), 0, 0]$. In principle, a temporal dependence of $E(t)$ can have an arbitrary form, although in the calculations below it is considered to be of the form $E(t) = E_0 \exp(-i\omega t) + c.c.$, where $E_0$, $\omega$ are the amplitude and frequency.

In the classical frequency range, $\hbar \omega \lesssim E_F$, in the relaxation time approximation, graphene charge-carriers transport properties are governed by the Boltzmann kinetic equation written for the electrons

$$\frac{\partial f(\vec{k},t)}{\partial t} - \frac{e}{\hbar} \vec{E} \frac{\partial f(\vec{k},t)}{\partial \vec{k}} = -\gamma \left[ f(\vec{k},t) - f_0(\vec{k}) \right], \quad (1)$$

where $f(\vec{k},t)$ is the nonequilibrium distribution function, $f_0(\vec{k})$ is the equilibrium Fermi-Dirac distribution function, and $\gamma$ is the inverse relaxation time. Equation (1) can be solved analytically, and its exact solution at $t \gg 1/\gamma$ is given by10,46

$$f(\vec{k},t) = e^{-\gamma t} \int_{-\infty}^{t} dt' e^{\gamma t'} f_0[k_x + H(t,t'),k_y], \quad (2)$$

where

$$H(t,t') = \pi \int_0^t E(t'' )dt'' = -\frac{e}{\gamma \hbar} [E_0 \exp(-i\omega t) - E_0 \exp(i\omega t) - E_0 \exp(-i\omega t') + E_0 \exp(i\omega t')],$$

and the overbars stand for complex conjugation.

The induced 2D current in graphene is expressed through the function $f(\vec{k},t)$ as

$$\vec{j} = -\frac{e}{(2\pi)^2} \int d\vec{k} f(\vec{k},t) \frac{\partial \epsilon(\vec{k})}{\hbar \partial \vec{k}}, \quad (3)$$

where $\epsilon(\vec{k}) = v_F \hbar \sqrt{k_x^2 + k_y^2}$ is the Dirac cone spectrum of charge carriers in graphene, $v_F$ is the Fermi velocity, and the factor 4 is due to the spin and valley degeneracy. For degenerate electrons at zero temperature (and, consequently, the step-like Fermi-Dirac distribution function $f_0(\vec{k}) = \Theta \left[ E_F - \epsilon(\vec{k}) \right]$, we obtain

$$j_x = -\frac{e v_F}{\pi^2} \gamma e^{-\gamma t} \int_{-\infty}^{t} dt' e^{\gamma t'} I(t,t'), \quad (4)$$

where

$$I(t,t') = \int d\vec{k} \frac{k_x}{\sqrt{k_x^2 + k_y^2}} \times \Theta \left[ E_F - v_F \hbar \sqrt{k_x^2 + k_y^2} \right] = \int_0^{k_F} k' dk' \int_0^{2\pi} d\varphi \frac{k' \cos \varphi - H(t,t')}{\sqrt{(k' \cos \varphi - H(t,t'))^2 + k'^2 \sin^2 \varphi}}$$

$E_F = v_F \hbar k_F$ is the Fermi energy ($k_F$ is the Fermi wavevector), $\Theta (x)$ is the Heavyside function, and change of variables, $k_x = k' \cos \varphi - H(t,t')$, $k_y = k' \sin \varphi$.

After integration with respect to $k'$, expression (5) can be represented in the form

$$I(t,t') = \int_0^{2\pi} \left\{ \sqrt{k_F^2 - 2k_F H(t,t') H(t,t')} H^2(t,t') \left[ \frac{k_F \cos \varphi}{2} - H(t,t') \left( 1 - \frac{3}{2} \cos^2 \varphi \right) \right] + H^2(t,t') \left( 1 - \frac{3}{2} \cos^2 \varphi \right) + \frac{3}{2} H^2(t,t') \left( \cos^3 \varphi - \cos \varphi \right) \ln \frac{\sqrt{k_F^2 - 2k_F H(t,t') H(t,t')} + H(t,t') \cos \varphi}{H(t,t') (1 - \cos \varphi)} \right\} d\varphi.$$}

After the expansion with respect to $H(t,t')$ (up to the third order), the integral (5) is reduced to

$$I(t,t') = -k_F \pi H(t,t') + \frac{\pi}{8k_F} H^3(t,t'). \quad (6)$$

Finally, substituting Eq. (6) into Eq. (4) and performing integration, we obtain

$$j_x = \sigma_0 \frac{4E_F}{\pi^2} \frac{E_0 \exp(-i\omega t) - E_0 \exp(i\omega t)}{\gamma - i \omega - \frac{1}{\gamma - 2i \omega} (\gamma + 4\omega)} + c.c.,$$

where $\sigma_0 = e^2/4\hbar$ is the conductivity quantum. Note,
in Eq. (7) we write out only the terms with the time dependence \( \sim \exp(\pm i\omega t) \), while the terms corresponding to the third harmonic are omitted.

In the limit \( \omega/\gamma \gg 1 \), Eq. (7) can be written as

\[
j_x = i \left[ \nu^{(1)} - \nu^{(3)} |E_0|^2 \right] E_0 \exp(-i\omega t),
\]

where

\[
\nu^{(1)} = \sigma_0 \frac{4E_F}{\pi\hbar\omega}, \quad \nu^{(3)} = \sigma_0 \frac{9e^2v_F^2}{2\pi\hbar^2\omega^3}.
\]

Below, we use this result, obtained as seen by free-space light normally incident on a graphene layer, for the effective nonlinear conductivity of surface plasmons propagating along graphene layers, assuming the additional correction due to the in-plane wavevector \( k \) to be small, which is well-justified if \( k_x/(\omega/\epsilon) \ll 300 \).

## III. DISCRETE SOLITONS

Now we consider a periodic multilayer graphene stack, consisting of an infinite number of parallel graphene layers arranged at equal distances \( d \) from each other at the planes \( z = md, m = (-\infty, \infty) \), inside a dielectric medium with permittivity \( \epsilon \). In this case, the electric \( \vec{E} \) and magnetic \( \vec{H} \) fields are governed by Maxwell’s equations

\[
\begin{align*}
\text{rot} \vec{E} &= i\omega\mu_0 \vec{H}, \\
\text{div} \vec{E} &= \frac{\rho}{\varepsilon_0}, \\
\text{rot} \vec{H} &= -i\omega\varepsilon_0 \vec{E} + \vec{J}, \\
\text{div} \vec{H} &= 0,
\end{align*}
\]

where \( \varepsilon_0, \mu_0 \) are free-space permittivity and permeability, and \( \vec{J}, \rho \) are full three-dimensional (3D) current and charge densities, respectively, given by

\[
\vec{J} = \sum_{m=-\infty}^{\infty} J^{(m)}_x \delta(z-md), \quad \rho = \sum_{m=-\infty}^{\infty} \varrho^{(m)}_x \delta(z-md),
\]

(9)

where \( J^{(m)}_x \) and \( \varrho^{(m)}_x \) are 2D current and charge densities in \( m \)-th graphene layer. In all above equations the time-dependence \( \exp(-i\omega t) \) is implied.

The electric and magnetic fields can be expressed through scalar \( \varphi \) and vector \( \vec{A} \) potentials as

\[
\begin{align*}
\vec{E} &= -\text{grad} \varphi + i\omega \vec{A}, \\
\vec{H} &= \frac{\text{rot} \vec{A}}{\mu_0}.
\end{align*}
\]

(10)

These relations, jointly with the Lorentz gauge

\[
\begin{align*}
\text{div} \vec{A} - (i\omega/\epsilon^2) \varphi &= 0,
\end{align*}
\]

(11)

result in inhomogeneous Helmholtz equations for both scalar and vector potentials

\[
\begin{align*}
\Delta \varphi + \frac{\omega^2}{\epsilon^2} \varphi &= -\frac{\rho}{\varepsilon_0}, \\
\Delta \vec{A} + \frac{\omega^2}{\epsilon^2} \vec{A} &= -\mu_0 \vec{J}.
\end{align*}
\]

(12)

We assume the electromagnetic field uniform along the \( y \) direction, \( \partial/\partial y \equiv 0 \), and propagating in the \( x \) direction, \( \vec{A}, \vec{J}_x, \varphi \sim \exp(ik_x x) \). Under these assumptions, Eq. (13) can be solved by using a standard Green function formalism. Accordingly, a general solution of Eq. (13) has the form

\[
A_x(z) = -\mu_0 \int_{-\infty}^{\infty} dz' G(z-z') J_x,
\]

(14)

where

\[
G(z) = \frac{-\exp(-p|z|)}{2p}, \quad p = \sqrt{k_x^2 - \omega^2\epsilon/\epsilon^2}
\]

is the one-dimensional (1D) Green function being a solution of the equation

\[
\left( \frac{d^2}{dz^2} - p^2 \right) G(z) = \delta(z)
\]

with the boundary conditions, \( G(\pm \infty) = 0 \), denoting the evanescent character of waves (when \( k_x^2 > (\omega/\epsilon^2) \epsilon \) and \( \text{Re}(p) > 0 \), or absence of waves coming from \( z = \pm \infty \) for traveling waves, when \( k_x^2 < (\omega/\epsilon^2) \epsilon \) and \( \text{Im}(p) < 0 \). Substituting Eq. (9) into Eq. (14) and using the properties of Delta-functions, we obtain

\[
A_x(z) = \frac{\mu_0}{2p} \sum_{m=-\infty}^{\infty} j^{(m)} x \exp(-p|z-md|).
\]

(15)

Due to the 2D nature of currents in graphene layers \( A_x \equiv 0 \), while the vector potential components \( A_x \) and \( A_y \) describe \( p- \) and \( s- \)polarized waves, correspondingly. Further we will concentrate on the \( p- \)polarized waves only. Thus, using the Lorentz gauge (11), we can express the \( x \)-component of the electric field through \( A_x \) as

\[
E_x(z) = \frac{\epsilon^2 p^2}{i\omega \epsilon} A_x(z).
\]

(15)

After substitution this relation into Eq. (8), \( A_x \) can be represented in the form

\[
A_x(z) = \frac{p}{2\omega \varepsilon_0 \epsilon} \sum_{m=-\infty}^{\infty} \left[ \nu^{(1)} - \nu^{(3)} \frac{\epsilon^4 p^4}{\omega^2 \epsilon^2} |A_x(md)|^2 \right] \times
\]

\[
A_x(md) \exp(-p|z-md|).
\]

(16)

Alternatively, Eq. (16) can be rewritten in the form of the stationary discrete nonlinear Schrödinger equation

\[
A_x([n+1] d) + A_x([n-1] d) - 2A_x(nd) \cosh(pd) = - \frac{p}{\omega \varepsilon_0 \epsilon} \left[ \nu^{(1)} - \nu^{(3)} \frac{\epsilon^4 p^4}{\omega^2 \epsilon^2} |A_x(nd)|^2 \right] A_x(nd) \sinh(pd)
\]

(17)

for \( n \in (-\infty, \infty) \).

The linear counterpart (when \( \nu^{(3)} = 0 \)) of the discrete nonlinear Schrödinger equation (17) defines the linear spectrum. Domains of allowed frequencies (where
in the linear case the wave propagation is possible) are parametrized by the real Bloch wavevector $q$ [such that $A_x (nd) = A_x (0) \exp (i q d)]$. As a result, the equation
\[
\cos (q d) = \cosh (p d) - \frac{p}{2 \omega \epsilon_0 \varepsilon} \sinh (p d) \quad (18)
\]
determines the propagating bands of the spectrum $\omega = \Omega \left( k_x, q \right)$ ($l \geq 1$ is the number of band), which are depicted in Figs. 2(a) and 3(a) by black color, see, e.g., Ref. 19.

Although generally the nonlinear Eq. (17) possesses an infinite number of solutions \(^{28}\), here we concentrate on the properties of the fundamental bright solitons, bifurcating from the edge of the allowed band of the spectrum. To describe the solitons properties, we introduce a soliton norm as
\[
P = \sum_{m=-\infty}^{\infty} |E_x (md)|^2 = \frac{c^4 p^4}{\omega^2 \varepsilon^2} \sum_{m=-\infty}^{\infty} |A_x (md)|^2.
\]

The fundamental mode of discrete soliton is depicted in Fig. 2. Due to effectively defocusing nonlinearity (positive cubic term in Eq. (17), bright solitons [see Fig. 2(a)] bifurcate from the low-frequency boundary of the first band $\Omega_1 (k_x, q)$ (which corresponds to the phase shift $q d = \pi$ between oscillations in adjacent graphene layers) and exist in the semi-infinite gap $\omega \leq \Omega_1 (k_x, \pi/d)$. Since in this region $k_x > \omega \varepsilon^{1/2}/c$, this type of solitons is characterized by the evanescent waves in the dielectric between the graphene layers, and these solitons will be further referred to as \textit{plasmonic solitons}. For fixed $k_x$ [Fig. 2(b)] the soliton norm $P$, being zero at the band edge $\omega = \Omega_1 (k_x, \pi/d)$, initially grows up to values $\sim 10^{11} V^2/m^2$, but after that decreases and attains zero at zero frequency. At the same time, the frequency defines the degree of soliton localization, as follows from the comparison of Figs. 2(d)–2(f). Thus, in the vicinity of the band edge $\Omega_1 (k_x, q)$ the soliton is delocalized – its electric field is distributed over a large number of graphene layers [Fig. 2(d)]. When frequency is gradually detuned from the band edge, the soliton becomes more localized – its electric field is either distributed over a few graphene layers [Fig. 2(e)], or effectively concentrated in the vicinity of one graphene layer, as shown in Fig. 2(f). It should be underlined that the soliton inherits the properties of a Bloch wave at the band edge from which it bifurcates: signs of the electric field tangential components at adjacent graphene layers are opposite (staggered soliton). For fixed frequency $\omega$ [Fig. 2(c)] the soliton norm increases monotonically with increasing $k_x$.

Equation (17) possesses two approximate types of solutions. The first type, so-called \textit{continuum limit}, is valid for low amplitude solutions. To obtain this solution, we use the ansatz $A_x (nd) = \epsilon (-1)^n \psi (\zeta)$ with $\epsilon$ being a small parameter, and $\zeta = \epsilon n$. As a result, function $\psi (\zeta)$ satisfies the nonlinear Schrodinger equation
\[
\frac{d^2 \psi}{d \zeta^2} + \rho (3) \frac{c^4 p^4}{\omega^2 \varepsilon_0 \varepsilon^3} \sinh (p d) \psi^3 (x) = \frac{2 \cosh (\beta) - 2}{\epsilon^2} \psi (x), \quad (19)
\]
which is parameterized by parameter $\beta$ such that
\[
\cosh (\beta) = 2 \frac{p}{2 \omega \epsilon_0 \varepsilon} \sinh (p d) - \cosh (p d). \quad (20)
\]
The parameter $\beta$ can be formally considered as the imaginary part of the Bloch wavevector $q = (\pi + i \beta)/d$ (note, inside the gap the Bloch wavevector is complex that in the linear case corresponds to the evanescent wave). Using exact solution of Eq. (19), we now approximate solu-
tion of Eq. (17) in the continuum limit

\[ A_x (nd) = \sqrt{\frac{2 \omega^3 \varepsilon_0 \varepsilon_3}{c^4 p \nu \phi (\beta) \sinh (pd)}} \frac{(-1)^n \sqrt{2 \cosh (\beta) - 2}}{\cosh \left( \sqrt{2 \cosh (\beta) - 2n} \right)}. \]

Consequently, the soliton norm in the continuum limit can be expressed as

\[ P = \frac{2 \omega \varepsilon_0 \varepsilon}{p \nu (\beta) \sinh (pd)} \sum_{n=-\infty}^{\infty} \frac{1}{\cosh^2 \left( \sqrt{2 \cosh (\beta) - 2n} \right)} \approx \frac{4 \omega \varepsilon_0 \varepsilon \sqrt{2 \cosh (\beta) - 2}}{p \nu (\beta) \sinh (pd)}. \]

In the last equation the summation has been replaced by the integration. As seen from Fig. 2, the continuum approximation (depicted by blue dash-and-dot line) is valid in the narrow domain in the vicinity of band edge \( \Omega_1 (k_x, \pi/d) \) [more specifically, in domains 1.95 meV \( \lesssim \omega \lesssim 1.997 \text{ meV} \) in Fig. 2(b) and 0.0236 \( \mu\text{m}^{-1} \lesssim k_x \lesssim 0.0245 \mu\text{m}^{-1} \) in Fig. 2(c)].

The other type of approximate solutions, so-called anti-continuum limit, is valid far from the band edge \( \Omega_1 (k_x, \pi/d) \) (deeply in the gap). Hence, introducing scaled dimensionless variables

\[ a_n = \left( \frac{\nu \beta \sinh (pd)}{2 \omega \varepsilon_0 \varepsilon_3 (2 \cosh (\beta))} \right)^{1/2} A_x (nd), \]

and taking into account Eq. (20), we obtain

\[ \frac{a_{n+1} + a_{n-1}}{2 \cosh (\beta)} + a_n - a_n^3 = 0. \]

As a result, when \( \beta \rightarrow \infty \), \( a_n \) become independent and acquire one of the three values: \( a_n = -1 \), \( a_n = 0 \), and \( a_n = +1 \). In this limit, the fundamental mode [see, e.g., Fig. 2(e)] corresponds to the case, where \( a_n = \delta_{n,0} \). This case allows for the approximate analytical continuation valid for large values of \( \beta \)

\[ a_0 = 1 - \frac{1}{4 \cosh^2 (\beta)}, \]

\[ a_1 = a_{-1} = -\frac{1}{2 \cosh (\beta)} + \frac{1}{8 \cosh^3 (\beta)}, \]

\[ a_2 = a_{-2} = -\frac{1}{4 \cosh (\beta)}. \]

As a result, the soliton norm can be represented in the form

\[ P = \frac{\varepsilon_0 \omega}{p \nu (\beta) \sinh (pd)} \left[ a_0^2 + 2a_1^2 + 2a_2^2 \right] = \left[ \frac{\varepsilon_0 \omega}{p \nu (\beta) \sinh (pd)} \frac{1}{2 \cosh (\beta)} + \frac{7}{8 \cosh^3 (\beta)} \right]. \]

As seen from Fig. 2, anti-continuum limit approximation (depicted by green dashed line) well describes the solution in the domains 0 \( \lesssim \omega \lesssim 1.95 \text{ meV} \) in Fig. 2(b) and \( k_x \gtrsim 0.0245 \mu\text{m}^{-1} \) in Fig. 2(c).

Solitons can also exist in the upper (finite) gaps of the spectrum. Notice, that in those gaps \( k_x < \omega c^{1/2} / c \), and solitons are characterized by propagating waves in the dielectric between graphene layers (this type of solitons will be further referred to as photonic solitons). An example of photonic solitons is shown in Fig. 3. Photonic solitons are characterized by considerably larger solitons norms \( P \) if compared to the plasmionic ones [soliton norm is of order of 500 MV\(^2/m^2\) in Fig. 3(a) and 0.1 MV\(^2/m^2\) in Fig. 2(a)]. Photonic solitons bifurcate from the upper edge of the gap – the soliton norm, being zero at the high-frequency boundary of the gap \( \Omega_1 (k_x, \pi/d) \), is increased, when frequency is decreased [see Fig. 3(b)]. The decrease of frequency also leads to the growth of the soliton amplitude [compare Figs. 3(c,d,e)]. At the same time, photonic solitons are considerably wider than plasmionic ones, and at large amplitudes they become two-hump [Figs. 3(d) and 3(e)]. This happens due to the fact that, by contrast to plasmionic solitons, for photonic solitons local maxima and minima of the electromagnetic field are generally not located at graphene layers.
Figure 4. (Color online) Dependence of surface soliton norm \( P \) (in MV^2/m^2) upon frequency \( \omega \) for fixed value \( k_x = 0.05 \mu m^{-1} \) [panel (a)], or upon frequency \( k_x \) for fixed value \( \omega = 1 \) meV [panel (b)]; (c)–(e) Soliton spatial profiles for \( k_x = 0.05 \mu m^{-1} \) and \( \omega = 1.89 \) meV [panel (c)], \( \omega = 1.6 \) meV [panel (d)], or \( \omega = 0.52 \) meV [panel (e)]. The parameters of panels (c), (d) and (e) correspond to points A, B and C in panel (a), respectively.

**IV. DISCRETE SURFACE SOLITONS**

Finally, we consider a semi-infinite array of graphene layers, arranged at equal distances \( d \) from each other at planes \( z = md, m = 0, \infty \), as shown in Fig. 1(b). In other words, graphene layers are embedded inside a semi-infinite dielectric medium at \( z \geq 0 \), while at \( z < 0 \) there are no layers. The 3D current and charge density for this semi-infinite array can be written as

\[
\vec{J} = \sum_{m=0}^{\infty} j^{(m)} \delta(z - md), \quad \rho = \sum_{m=0}^{\infty} \phi^{(m)} \delta(z - md),
\]

and the solution of the wave equation (13) has [in full analogy with Eq. (16)] the form

\[
A_x(z) = \frac{p}{2\omega_0 \varepsilon_0} \sum_{m=0}^{\infty} \left[ \mu^{(1)} - \mu^{(3)} \frac{\varepsilon_1 p^4}{\omega^2 z^2} \right] A_x(md) \exp(-p|z - md|),
\]

or

\[
A_x([n + 1]d) + \frac{p}{2\omega_0 \varepsilon_0} \sum_{m=0}^{\infty} \left[ \mu^{(1)} - \mu^{(3)} \frac{\varepsilon_1 p^4}{\omega^2 z^2} \right] A_x(md) \exp(-p|z - md|),
\]

Properties of plasmonic surface solitons are summarized in Fig. 4. The principal difference between the cases of surface and the bulk solitons is the nonexistence of the low-amplitude surface soliton in the vicinity of the band edge \( \Omega_1 (k_x, \pi/d) \) [compare, e.g., Figs. 4 (a) and 2 (b), as well as Figs. 4(b) and 2(c)]. More specifically, there exists an end-point of the spectrum, at which the fundamental mode bifurcates with the other type of the surface soliton mode, for details see, e.g., Ref. 47. In the vicinity of the end-point of the spectrum soliton norm \( P \) achieves a local minimum. At the same time, from the comparison of Figs. 4(c-e) it follows that, similar to the case of bulk solitons, lower frequencies correspond to more localized solitons (when the power is mostly concentrated at the graphene layer, truncating the photonic crystal).

It is also worth noting that the principal difference between linear and nonlinear cases is the possibility to have the nonlinear surface state (namely, surface soliton) in the uniform structure (semi-infinite array of *equally* doped graphene layers, placed at *equal* distances from each other, and embedded into the *uniform* dielectric medium), while in the linear case the existence of surface state is possible only in the nonuniform structure – it is necessary to have either the defect of the periodicity at the surface of photonic crystal\(^{45}\), the defect of graphene doping at surface, or to truncate the photonic crystal with the dielectric, characterized by the dielectric constant, different from that of the medium inside the photonic crystal.

**V. CONCLUSION**

We have analyzed nonlinear graphene-based multilayer metamaterials and demonstrated that they can support spatially localized nonlinear modes in the form of discrete plasmon solitons. We have described the properties of this novel class of discrete solitons, including the dependence of their parameters on graphene conductivity. We have also predicted the existence of nonlinear surface modes in the form of discrete surface solitons.

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1. R. R. Nair, P. Blake, A. N. Grigorenko, K. S. Novoselov, T. J. Booth, T. Stauber, N. M. R. Peres, and A. K. Geim, Science 320, 1308 (2008).
2. F. Bonaccorso, Z. Sun, T. Hasan, and A. C. Ferrari, Nature Photon. 4, 618 (2010).
3. J. Chen, M. Badioli, P. Alonso-Gonzales, S. Thongratanasiri, F. Huth, J. Osmond, M. Spasenovic, A. Centeno, A. Pesquera, P. Godignon, A.Z. Elorza, N. Camara, F. J.G. de Abajo, R. Hillenbrand, and F.H.L. Koppens, Nature 487, 77 (2012).
4. Z. Fei, A. S. Rodin, G.O. Andreev, W. Bao, A.S. McLeod, M. Wagner, L.M. Zhang, Z. Zhao, G. Dominguez, M. Thiemens, M.M. Fogler, A.H. Castro-Neto, C.N. Lau, F. Keilmann, and D.N. Basov, Nature 487, 82 (2012).
5. A. N. Grigorenko, M. Polini, and K. S. Novoselov, Nature Photon. 6, 749 (2012).
6. X. Luo, T. Qiu, W. Lu, and Z. Ni, Mater. Sci. Eng., R Rep. 74, 351 (2013).
7. S. A. Mikhailov, Europhys. Lett. 79, 27002 (2007).
8. S. A. Mikhailov and K. Ziegler, J. Phys.: Condens. Mat. 20, 384204 (2008).
9. E. Hendry, P. Hale, J. Moger, A. Savchenko, and S. A. Mikhailov, Phys. Rev. Lett. 105 (2010).
10. M. M. Glazov and S. D. Ganichev, Phys. Rep. 535, 101 (2014).
11. S.-Y. Hong, J. I. Dadap, N. Petrone, P.-C. Yeh, J. Hone, and R. M. Osgood, Phys. Rev. X 3, 021014 (2013).
12. J. L. Cheng, N. Vermeulen, and J. E. Sipe, New J. Phys. 16, 053014 (2014).
13. J. L. Cheng, N. Vermeulen, and J. E. Sipe, Opt. Express 22, 15868 (2014).
14. M. L. Nesterov, J. Bravo-Abad, A. Y. Nikitin, F. J. Garcia-Vidal, and L. Martin-Moreno, Laser Photonics Rev. 7, L7 (2013).
15. D. A. Smirnova, A. V. Gorbach, I. V. Iorsh, I. V. Shadrivov, and Y. S. Kivshar, Phys. Rev. B 88, 045443 (2013).
16. M. Gullans, D. E. Chang, F. H. L. Koppens, F. J. Garcia de Abajo, and M. D. Lukin, Phys. Rev. Lett. 111, 247401 (2013).
17. D. A. Smirnova, I. V. Shadrivov, A. I. Smirnov, and Y. S. Kivshar, Laser Photonics Rev. 8, 291 (2014).
18. B. Wang, X. Zhang, F. J. Garcia-Vidal, X. Yuan, and J. Teng, Phys. Rev. Lett. 109, 073001 (2012).
19. Y. V. Bludov, N. M. R. Peres, and M. I. Vasilevskiy, J. Opt. 15, 114004 (2013).
20. I.V. Iorsh, I.S. Mukhin, I.V. Shadrivov, P.A. Belov, and Yu.S. Kivshar, Phys. Rev. B 87, 075416 (2013).
21. A. J. Sievers and S. Takeno, Phys. Rev. Lett. 61, 970 (1988).
22. N. Kuroda, Y. Wakabayashi, M. Nishida, N. Wakabayashi, M. Yamashita, and N. Matsushita, Phys. Rev. Lett. 79, 2510 (1997).
23. A. Trombettoni and A. Smerzi, Phys. Rev. Lett. 86, 2353 (2001).
24. G. L. Alfimov, P. G. Kevrekidis, V. V. Konotop, and M. Salerno, Phys. Rev. E 66, 046608 (2002).
25. D. N. Christodoulides and R. I. Joseph, Opt. Lett. 13, 794 (1988).
26. A. V. Gorbach and D. V. Skryabin, Phys. Rev. A 79, 053812 (2009).
27. Y. S. Kivshar and D. K. Campbell, Phys. Rev. E 48, 3077 (1993).
28. G. Alfimov, V. Brazhnyi, and V. Konotop, Physica D: Nonlinear Phenomena 194, 127 (2004).
29. Y. Liu, G. Bartal, D. A. Genov, and X. Zhang, Phys. Rev. Lett. 99, 153901 (2007).
30. Y. Kou, F. Ye, and X. Chen, Phys. Rev. A 84, 033855 (2011).
31. Y. Kou, F. Ye, and X. Chen, Opt. Lett. 37, 3822 (2012).
32. Y. Kou, F. Ye, and X. Chen, Opt. Lett. 38, 1271 (2013).
33. A. Marini, A. V. Gorbach and D. V. Skryabin, Opt. Lett. 35, 3532 (2010).
34. F. Ye, D. Mihalache, B. Hu, and N. C. Panoiu, Phys. Rev. Lett. 104, 106802 (2010).
35. F. Ye, D. Mihalache, B. Hu, and N. C. Panoiu, Opt. Lett. 36, 1179 (2011).
36. M. G. Silveirinha, Phys. Rev. B 87, 235115 (2013).
37. D. E. Fernandes and M. G. Silveirinha, Photonics and Nanostructures - Fundamentals and Applications (2014).
38. R. E. Noskov, P. A. Belov, and Y. S. Kivshar, Phys. Rev. Lett. 108, 093901 (2012).
39. R. Noskov, P. Belov, and Y. Kivshar, Sci. Rep. 2, 873 (2012).
40. K. G. Makris, S. Suntsov, D. N. Christodoulides, G. I. Stegeman, and A. Hache, Opt. Lett. 30, 2466 (2005).
41. S. Suntsov, K. G. Makris, D. N. Christodoulides, G. I. Stegeman, A. Hache, R. Morandotti, H. Yang, G. Salamo, and M. Sorel, Phys. Rev. Lett. 96, 063901 (2006).
42. C. R. Rosberg, D. N. Neshev, W. Krolikowski, A. Mitchell, R. A. Vicencio, M. I. Molina, and Yu. S. Kivshar, Phys. Rev. Lett. 97, 083901 (2006).
43. M. I. Molina, R. A. Vicencio, and Y. S. Kivshar, Opt. Lett. 31, 1693 (2006).
44. I. E. Tamm, Z. Phys. 76, 849 (1934).
45. D. Smirnova, P. Buslaev, I. Iorsh, I. V. Shadrivov, P. A. Belov, and Y. S. Kivshar, Phys. Rev. B 89, 245414 (2014).
46. N. M. R. Peres, Yu. V. Bludov, J. E. Santos, Antti-Pekka Jauho, M. I. Vasilevskiy, Phys. Rev. B 90, 125425 (2014).
47. Y. V. Bludov and V. V. Konotop, Phys. Rev. E 76, 046604 (2007).