On the hydrodynamic attractor of Yang-Mills plasma

Michał Spaliński\textsuperscript{a,b}

\textsuperscript{a}Physics Department, University of Białystok, Konstantego Ciołkowskiego 1L, 15-245 Białystok, Poland
\textsuperscript{b}National Centre for Nuclear Research, Hoża 69, 00-681 Warsaw, Poland

Abstract
There is mounting evidence suggesting that relativistic hydrodynamics becomes relevant for the physics of quark-gluon plasma as the result of nonhydrodynamic modes decaying to an attractor apparent even when the system is far from local equilibrium. Here we determine this attractor for Bjorken flow in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) using Borel summation of the gradient expansion of the expectation value of the energy momentum tensor. By comparing the result to numerical simulations of the flow based on the AdS/CFT correspondence we show that it provides an accurate and unambiguous approximation of the hydrodynamic attractor in this system. This development has important implications for the formulation of effective theories of hydrodynamics.

Keywords: Quark-gluon plasma, AdS/CFT

1. Introduction

Heavy-ion collision experiments and their phenomenological description have lead to the realization that relativistic hydrodynamics works very well rather far outside its traditionally understood domain of validity. Variants of Müller-Israel-Stewart (MIS) theory \cite{1, 2, 3} have successfully been applied in rather extreme conditions, which could hardly be assumed to be close to local equilibrium. Furthermore, model calculations exist where it is possible to study the emergence of universal, hydrodynamic behaviour and test to what extent an effective description in terms of hydrodynamics can match microscopic results \cite{4}. Such calculations were initially carried out in $\mathcal{N} = 4$ SYM using the AdS/CFT correspondence \cite{5, 6, 7}, but similar studies have since also been performed in models of kinetic theory \cite{8, 9, 10}. The conclusion from these investigations is that the domain of validity of a hydrodynamic description is delimited by the decay of nonhydrodynamic modes \cite{5, 6, 11, 12}. The outcome of this transition to hydrodynamics ("hydronization") is that the system reaches a hydrodynamic attractor \cite{13} which governs its subsequent evolution toward equilibrium. This attractor is a special solution to which generic histories decay exponentially, and do so well before local equilibrium sets in. It incorporates all orders of the hydrodynamic gradient expansion, and at sufficiently late times coincides with the predictions of relativistic Navier-Stokes theory. The existence of an attractor in this sense is a critically important issue for hydrodynamics, because it defines its very meaning. It has conceptual as well as practical implications for the formulation of hydrodynamic theories in general as well as for their application to the physics of quark-gluon plasma.

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Attractor behaviour was first identified explicitly in the differential equations of hydrodynamics [13, 14]. An outstanding problem is the determination of such attractors at the microscopic level [15]. The first calculations of this type were described by Romatschke [15], who found approximate attractor solutions in the context of kinetic theory and $N = 4$ SYM by scanning for the corresponding initial conditions. The purpose of this Letter is to argue that the Borel sum of the hydrodynamic gradient expansion provides a direct way of estimating the attractor. While at late times this calculation clearly must give the correct result (which coincides with the prediction of Navier-Stokes hydrodynamics) it is not obvious a priori that this calculation gives an accurate estimate at earlier times. We will however show explicitly that the result of Borel summation does indeed act as an attractor for histories of Bjorken flow simulated using techniques based on the AdS/CFT correspondence. This should be viewed in the context of the idea that higher orders of the gradient expansion may be relevant for real-world physics [16, 17, 18, 19].

An important point is that the hydrodynamic gradient expansion is the leading element of a transseries [13], and in general the higher order elements (“instanton sectors”) play an important role in defining the summation properly. These transseries sectors involve integration constants which need to be fixed. However, their contributions are exponentially suppressed and it is tempting to ignore them as a first approximation. Such an approach will definitely fail at sufficiently early times (before the exponential suppression sets in). However, we will see that it works fine for $\tau T > 0.3$, and this is enough to see that the result of the Borel sum acts as an attractor well before the Navier-Stokes approximation to hydrodynamics becomes accurate at $\tau T \approx 0.7$ [7].

A critical issue for Borel summation is the location of singularities of the analytic continuation of the Borel transform. These singularities reflect the spectrum of nonhydrodynamic modes – both at the microscopic level [18] and in hydrodynamics [13, 14]. An important testing ground for the feasibility and robustness of Borel summation of the gradient series of $N = 4$ SYM is the hydrodynamic theory proposed in [20], which we will refer to as HJSW. This theory extends Navier-Stokes hydrodynamics by adding degrees of freedom which mimic the least-damped nonhydrodynamic modes of $N = 4$ SYM plasma (known from calculations of quasinormal modes of black branes [21]). This results in the same leading singularities [14] as those identified at the microscopic level in Ref. [18]. This should be contrasted with BRSSS hydrodynamics [22], which instead involves only purely decaying modes.

In the case of BRSSS theory one cannot ignore the transseries sectors even as an approximation, because the analytically-continued Borel transform of the hydrodynamic series has branch-point singularities on the real axis (reflecting the purely-decaying MIS nonhydrodynamic mode) and this leads to a complex summation ambiguity. The addition of transseries sectors (which are constrained by resurgence relations [13, 14, 23]) resolves this ambiguity, but requires an integration constant (the transseries parameter) to be set correctly by comparing the result of the summation to the numerical calculation of the attractor. Luckily, this issue does not arise in $N = 4$ SYM, nor in HJSW hydrodynamics, because in these cases singularities of the analytic continuation of the Borel transform occur off the real axis. Thus, omitting the instanton sectors is a reasonable first approximation, which is what we focus on here.

As a way of determining the range of proper-time where the Borel sum can be expected to give an accurate estimate of the attractor we first calculate the Borel sum of the gradient expansion in the case of HJSW hydrodynamics, where it is easy to check the validity of the answer. The result
is unique, unambiguous, and coincides (even at rather early times) with the attractor determined directly from the hydrodynamic equations. This sets the stage for the main theme of this Letter: the Borel summation of the gradient series of $N = 4$ SYM. This is technically no more challenging than the calculation for HJSW theory, but its significance is that it provides an example of a hydrodynamic attractor obtained directly from a microscopic calculation. This result can only be fully appreciated by inspecting the behaviour of numerically simulated histories of boost-invariant expansion in $N = 4$ SYM. A very important point to note is that while the attractor coincides with first order hydrodynamics at late times, it turns out to be quite distinct from it even at moderate times. This has implications of foundational nature for relativistic hydrodynamics. A fuller discussion of this result and its ramifications can be found in the concluding section.

2. Bjorken flow

Throughout this paper we work with Bjorken flow [24], which imposes powerful simplifying symmetry constraints. We use proper time – rapidity coordinates $\tau, Y$ related to Minkowski lab-frame coordinates $t, z$ by $t = \tau \cosh Y$ and $z = \tau \sinh Y$ where $z$ is aligned along the collision axis. A system undergoing Bjorken flow has eigenvalues of the expectation value of the energy momentum tensor

$$T^{\mu\nu} = \text{diag}(E, P_L, P_T, P_T)$$

which are functions of the proper time $\tau$ alone. In a conformal theory, the conditions of tracelessness and conservation can be expressed as [25]

$$P_L = -E - \tau \dot{E}, \quad P_T = E + \frac{1}{2} \tau \dot{E}. \quad (2)$$

The departure of these quantities from the equilibrium pressure at the same energy density, $P \equiv E/3$, is a measure of how far a given state is from local equilibrium. This is conveniently captured by the pressure anisotropy

$$\mathcal{A} \equiv \frac{P_T - P_L}{P}$$

which we will study as a function not of the proper time $\tau$, but of the dimensionless “clock variable” $w \equiv T \tau$, where $T$ is the effective temperature (defined as the temperature of the equilibrium state with the same energy density). It is critically important to compare states of the system at different values of this dimensionless variable if we wish to see the attractor behaviour which is of central interest here.

3. The hydrodynamic attractor in hydrodynamics

Hydrodynamic theories are described by sets of nonlinear partial differential equations. The key simplification brought by the assumption of Bjorken flow is that the equations of hydrodynamics reduce to ordinary differential equations. For example, the evolution equation for the pressure
anisotropy in conformal BRSSS theory reads \[13, 4\]

\[ C_\tau \left( 1 + \frac{\mathcal{R}}{12} \right) \mathcal{A} + \left( \frac{C_{\tau s}}{3 w} + \frac{C_{d_1}}{8 C_\eta} \right) \mathcal{A}^2 = \frac{3}{2} \left( \frac{8 C_\eta}{w} - \mathcal{R} \right) \]  

(4)

where the prime denotes a derivative with respect to \( w \), and the dimensionless constants \( C_\eta, C_{\tau s}, C_{d_1} \) are transport coefficients (whose values in the case of \( \mathcal{N} = 4 \) SYM are known, see e.g. Ref. [4]). This equation is nonlinear, but it can be solved in powers of \( 1/w \): this is the hydrodynamic gradient expansion whose leading term reproduces the prediction of Navier-Stokes hydrodynamics. It also possesses an attractor, which can be determined numerically by setting initial conditions appropriately [13]. It is important to observe that the attractor becomes indistinguishable from the first order truncation of the gradient series only for \( w > 0.7 \). For smaller values of \( w \), the numerical solutions clearly decay to the attractor, not to the truncated gradient series.

The pressure anisotropy in HJSW theory satisfies a second order nonlinear ordinary differential equation, whose exact form can be found in Refs. [14, 4], and a similar analysis leads to the numerical determination of its attractor solution (to which we shall return shortly). The point we wish to make at this juncture is that we cannot proceed in the same way in \( \mathcal{N} = 4 \) SYM, because there we cannot write down a closed differential equation like Eq. (4). To find the attractor in this case one has to find another way. The approach explored in this Letter is to sum the hydrodynamic gradient expansion, whose leading 240 coefficients were obtained using the AdS/CFT correspondence in Ref. [18]. In the following we discuss the properties of the series and the summation, using HJSW theory as a testing ground.

4. Large order behaviour

In any conformal theory the general form of the gradient expansion of the pressure anisotropy for Bjorken flow is (see, e.g. Ref. [2])

\[ \mathcal{A}(w) = \sum_{n=1}^{\infty} a_n w^{-n}. \]  

(5)

The coefficients \( a_n \) have been calculated to high order in \( \mathcal{N} = 4 \) SYM [18], in kinetic theory [10, 4], as well as in various hydrodynamic theories [13, 14]. It is now well established that this series has a vanishing radius of convergence. In many cases, at large order \( n \) the coefficients grow in a way consistent with the Lipatov form [26]

\[ a_n \sim \frac{n!}{A^n}, \]  

(6)

where \( A \) is a real parameter. This formula implies linear behaviour of the ratio of neighbouring coefficients \( a_{n+1}/a_n \sim n/A \). For the case of the gradient series in BRSSS hydrodynamics this linear behaviour can be seen in the left-hand plot of Fig. 1. In the case of HJSW theory however the pattern is much more complex, as seen in the right-hand plot. The reason for this is that HJSW hydrodynamics instead of a single, purely damped nonhydrodynamic mode has a pair of modes with complex conjugate frequencies [14]. Furthermore, this signals that the hydrodynamic
The gradient expansion in this case is an element of a two-parameter transseries \cite{27} (while in BRSSS hydrodynamics the transseries involves only one parameter).

The analysis of Ref. \cite{14} can be used to find the following approximate formula describing the leading large $n$ behaviour

$$a_n \sim \frac{n!}{A^n} \cos \left( (n + \beta_R)\phi - \psi + \beta_I \log \left( \frac{A}{n + \beta_R} \right) \right)$$

for some real numbers $A, \phi, \psi, \beta_R, \beta_I$ (the $\psi$ appearing above is the phase of the Stokes constant of the transseries). Due to the oscillating factor this formula is not as useful as Eq. (6), but it does qualitatively capture the complex pattern in Fig. 1.

Importantly, if one plots the ratio of coefficients of the gradient expansion of $\mathcal{N} = 4$ SYM, calculated using the results in Ref. \cite{18}, one finds a picture very similar to the right-hand plot of Fig. 1. This happens because HJSW theory was constructed to reproduce the dominant nonhydrodynamic modes of $\mathcal{N} = 4$ SYM, which results in the close similarity of the large-order behaviour. This makes it a useful testbed for assessing the utility of Borel summation in this context, as explained below.

5. The attractor from Borel summation

The Borel transform of the gradient series removes the dominant factorial growth of the expansion coefficients:

$$\mathcal{B}a(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n!} \xi^n.$$
Figure 2: The attractor in HJSW theory, calculated numerically from the hydrodynamic equations (red curve), compared with the results of Borel summation (gray dots).

This series will typically define an analytic function within a disc around the origin in the complex \( \xi \) plane. The Borel sum of the series is defined by the Laplace transform:

\[
S\mathcal{A}(w) = w \int_C d\xi \, e^{-w\xi} \tilde{\mathcal{B}}\mathcal{A}(\xi),
\]

where \( \tilde{\mathcal{B}}\mathcal{A} \) is the analytic continuation of the Borel transform \( \mathcal{B}\mathcal{A}(\xi) \) (performed using Padé approximants) and \( C \) is a contour connecting 0 and \( \infty \).

The analytic continuation of \( \mathcal{B}\mathcal{A}(\xi) \) (performed using Padé approximants) necessarily contains singularities responsible for the vanishing radius of convergence of the original series. The singularities appearing in the cases of interest here have been discussed at length in the literature. For BRSSS theory one finds a branch point on the real axis \([13]\), which introduces a complex ambiguity in the Borel summation, given by the difference in the values obtained for Eq. (9) by integrating above and below the cut. As stressed earlier, this complication does not arise in the case of \( \mathcal{N} = 4 \) SYM \([18]\), or HJSW hydrodynamics \([14]\) where the branch points appear away from the real axis. This means that one can perform the integral in Eq. (9) by integrating over real values of \( \xi \) from zero to infinity. In practice, this integral has to be performed numerically for a set of values of \( w \).

To gauge the effectiveness of this method we begin with HJSW theory, which from this perspective offers qualitatively the same kind of challenge as \( \mathcal{N} = 4 \) SYM. The series can be calculated numerically to essentially arbitrarily high order \([14]\), but here we will use only the first 240 terms, to have a fair testing ground for the \( \mathcal{N} = 4 \) SYM case, where the cost of calculating the coefficients is much higher, and at this time only 240 are available \([18]\). As is clear from Fig. 2...
the Borel summation tracks the numerically determined attractor closely down to $w \approx 0.4$, and is still quite reasonable at $w \approx 0.3$.

6. The attractor of $\mathcal{N} = 4$ SYM

The procedure described above can readily be applied to the hydrodynamic gradient expansion of $\mathcal{N} = 4$ SYM using the results of Ref. [18], where the expansion coefficients of the energy density in powers of $\tau^{-2/3}$ were calculated up to order 240. Using equations (2) and (3), these results can be translated into coefficients of the pressure anisotropy (5). As mentioned earlier, their ratios qualitatively follow the pattern described by the approximate formula (7) (and seen in the lower plot in Fig. 1). These coefficients can be used to calculate the Borel transform and its analytic continuation (using a diagonal Pade approximant), which one can integrate numerically for a range of values of $w$ exactly as described above for the case of HJSW theory. The result is reproduced quite accurately by the rational function

$$A_0 = \frac{2530w - 276}{3975w^2 - 570w + 120}$$

for essentially all values of $w > 0$.

The attractor determined by Borel summation as described above can be expected to reliable down to $w \approx 0.3$, but its utility is best judged by comparing it with the results of numerical
holography simulations of Bjorken flow. A large number of such flow histories was studied in Ref. [7] following the earlier work of Refs. [6, 28]. The distinguished role played by the attractor is most prominently visible by considering values of $w$ for which it differs appreciably from the truncated gradient expansion. A selection of solutions which reach the attractor at such early times is plotted in Fig. 3 along with the rational fit $A_0$ given in Eq. (10). We see there the same kind of striking behaviour as seen at the level of hydrodynamics in Ref. [13].

In Ref. [7] the transition to hydrodynamics was defined in terms of the pressure anisotropy matching the truncated gradient expansion using the third order result from Ref. [29]. Each numerical solution followed the hydrodynamic prediction at sufficiently late times. The threshold was found to lie in a range of values of $w$ centered around $w_H = 0.65$, with a large pressure anisotropy $A(w_H) \approx 0.7$. However, given the results presented here it is tempting to think of hydronization in terms of reaching the attractor, which implies an even smaller value of $w_H$ and a correspondingly higher value of $A(w_H)$. For most of the histories shown in Fig. 3 the pressure anisotropy at hydronization exceeds 100%, showing that this observable can exhibit universal behaviour even in the highly nonequilibrium regime. It should however be stressed that this is not a claim concerning the behaviour of generic histories of the flow: the hydronization time depends on the initial conditions.

The calculation described above leaves a number of open problems. One of them is the explicit computation of leading transseries coefficients in the lowest instanton sectors. Such a calculation would make it possible to extend the range where the summation can be trusted to lower values of $w$. It would also allow us to verify that the resurgence relations written down in [14] connecting coefficients of different transseries sectors are satisfied. Another interesting problem to pursue would be a calculation of the $N = 4$ SYM attractor directly in the holographic representation. An attempt of this kind was recently made by Romatschke [15], who tried to find the special initial condition corresponding to the attractor solution, which he was then able to estimate by evolution using the bulk Einstein equations. The results presented in that paper are in good agreement with those presented here for $w > 0.4$, while at lower values of $w$ the method of Ref. [15] suggests that the true attractor may flatten out around $w = 0.4$ and for smaller values of $w$ lies somewhat below what is seen in Fig. 3.

7. Conclusions

The goal of hydrodynamics is to mimic universal, late-time behaviour of systems tending toward equilibrium [4]. The BRSSS philosophy [22], which can be seen as an incarnation of the effective field theory paradigm, tells us to match leading terms of the gradient expansion of hydrodynamics with the corresponding terms calculated in the underlying microscopic theory. The developments of the past couple of years suggest that it could make sense to set a more ambitious goal: to try to reproduce the attractor of the underlying theory at the level of hydrodynamics. While the attractor matches low orders of the gradient series at sufficiently late times, earlier on it is different, and the difference depends on the parameters of the theory. Even for $N = 4$ SYM the attractor is quite distinct from first (or second order) hydrodynamics. For QCD plasma, which almost certainly has a larger relaxation time, this distinction should be even more pronounced.
This could have important consequences for the interpretation of observables sensitive to early time dynamics.

At this time it would be most useful and interesting to find tractable examples which relax some of the technical elements which we have relied on, such as boost invariance and conformal symmetry. An important point will be to understand which observables reveal attractor behaviour. Any progress should be of interest not only in the context of quark-gluon plasma, but also for other areas of physics [30, 31].

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