Explicit Actions for Electromagnetism with Two Gauge Fields
with Only one Electric and one Magnetic Physical Fields

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Abstract

We extend the work of Mello et al. based in Cabbibo and Ferrari concerning the description of electromagnetism with two gauge fields from a variational principle, i.e. an action. We provide a systematic independent derivation of the allowed actions which have only one magnetic and one electric physical fields and are invariant under the discrete symmetries $P$ and $T$. We conclude that neither the Lagrangian, nor the Hamiltonian, are invariant under the electromagnetic duality rotations. This agrees with the weak-strong coupling mixing characteristic of the duality due to the Dirac quantization condition providing a natural way to differentiate dual theories related by the duality rotations (the energy is not invariant). Also the standard electromagnetic duality rotations considered in this work violate both $P$ and $T$ by inducing Hopf terms (theta terms) for each sector and a mixed Maxwell term. The canonical structure of the theory is briefly addressed and the magnetic gauge sector is interpreted as a ghost sector.

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1 Introduction and Discussion of Results

The seminal works of Dirac [1] introduced the famous charge quantization relation $eg = n$ which is obtained in the presence of both electric and magnetic poles (charges). The existence of both electric and magnetic charges raised the problem of a variational description of electromagnetism from an action that could actually contain explicitly both types of charges. Also it is widely accepted that in order to achieve that goal one must consider a description in terms of gauge fields which minimally couple to both currents, so necessarily we need to consider the existence of two distinct gauge fields, $A$ that couples to ordinary electric currents and $C$ that couples to magnetic current [2–6]. One possible approach first considered by Cabbibo and Ferrari [2] is to consider two physical gauge fields $A$ and $C$. Although this approach preserves both time-space isotropy and Lorentz invariance has the drawback of the inexistence of experimental observable effects of the second gauge field. Another approach have been to consider mechanisms that starting from a theory with two gauge fields give us only one physical gauge field, either by considering solutions (constraints) for the second gauge field [3–6] (this approach has the drawback of not preserving space isotropy or not preserving Lorentz invariance) or by considering a very massive second gauge field [7]. Yet another very simple approach is to consider electromagnetism as an effective theory of an extended theory with two gauge fields such that one gauge field is fixed by the second gauge field obeying the equations of motion [8].

In Mello et al. [9] it is build for the first time an explicit action for electromagnetism with two gauge fields based in the work of Cabbibo and Ferrari [2]. In here we build a similar lower order action with two gauge fields $A$ and $C$ of the gauge group $U(1) \times U(1)$. In order to accomplish it we take an independent approach of the original work [9] by studying in detail and systematically the desired properties of such an action. First we note that due to the different nature of $A$ and $C$ under the discrete symmetries of parity $P$ and time inversion $T$ [10, 11], standard electromagnetic duality [10, 12] violates $P$ and $T$ symmetries. So it is desirable that under an electromagnetic duality transformation our action gains terms that explicit violate these symmetries 1. Secondly we demand that there are only one electric and one magnetic physical fields. Implicitly this assumption means that the group charge flux of each of the $U(1)$’s is of the same nature of the topological flux of the other $U(1)$ group. The action suggested coincides (up to a sign choice) with the one of [9] and consists of two Maxwell terms with opposite relative sign, one for each of the gauge fields and a topological cross Hopf term that mixes both gauge sectors allowing for the desired characteristics,

$$S_{\text{Max}}^I = - \int_M \left[ \frac{\sqrt{-g}}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{\sqrt{-g}}{4g^2} G_{\mu\nu} G^{\mu\nu} - \frac{\dot{e}}{4e g} e^{\mu\nu\rho\delta} F_{\mu\nu} G_{\rho\delta} + \frac{1}{e} (A_\mu - \dot{e} \hat{C}_\mu) J^\mu_e - \frac{1}{g} (\dot{e} C_\mu + \hat{A}_\mu) J^\mu_g \right],$$

with $\dot{e} = \pm 1$ corresponding to the two physical fields

$$E^i = \frac{1}{e} F^{0i} - \frac{\dot{e}}{2g} e^{0ijk} G_{jk}$$

$$B^i = \frac{\dot{e}}{g} G^{0i} + \frac{1}{2e} e^{0ijk} F_{jk}$$

However the Maxwell terms of each of the gauge sectors have opposite sign, this has no consequences at classical level but at quantum level allows negative energy solutions which clearly violates causality. There are two approaches to overcome this problem. We can consider the $C$ field to be a ghost, this means that upon quantization it has the opposite spin-statistics relations than the one of standard fields and therefore it has anti-commutation relations [15], such kind of theories both with a matter and a ghost sector were introduced in cosmology by Linde [16]. Alternatively we can consider some mechanism that allows for a classical treatment of the $C$ field, as examples we have in cosmology the Phantom matter models [17] and a dynamical symmetry breaking mechanism [8] that allows a effective electric description of the theory. Also compatible with this last mechanism we can give a vacuum-expectation-value to the $C$ field that renders an effective Proca mass to the standard photon, the $A$ field [18, 19].

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1This argument is not completely closed once there are ways of implementing duality rotations that preserve $P$ and $T$ symmetries [13, 14].
2 Electromagnetic Duality

The study of theories with two gauge fields were first considered by Cabbibo and Ferrari [2]. More recently several studies addressed electromagnetic duality with two gauge fields, namely in [11] an explicit electromagnetic duality in terms of the gauge fields is presented. Here we review these results.

2.1 The Original Duality

The generalized Maxwell equations with both Electric and Magnetic currents [10] read

\[ \nabla \cdot E = \rho_e \]
\[ \nabla \cdot B = \rho_g \]
\[ \dot{B} + \nabla \times E = -J_g \]
\[ E - \nabla \times B = -J_e. \]

This equation obey the well known electromagnetic duality which rotates the electric and magnetic fields and currents [12]

\[ \begin{align*}
    E & \rightarrow \cos(\theta)E + \sin(\theta)B \\
    B & \rightarrow -\sin(\theta)E + \cos(\theta)B \\
    J_e & \rightarrow \cos(\theta)J_e + \sin(\theta)J_g \\
    J_g & \rightarrow -\sin(\theta)J_e + \cos(\theta)J_g
\end{align*} \]

where \( J = (\rho, J) \) stand for the 4-vector current densities.

2.2 Duality with two Gauge Fields

In order to build an action for electromagnetism with magnetic monopoles it is necessary to consider two \( U(1) \) gauge fields which minimally couple to the external electric and magnetic current densities. By introducing gauge fields one is led to the question whether the above duality can be extended to a duality of gauge fields instead of the electric and magnetic fields (i.e. the gauge field connections). By considering that both gauge fields have true physical degrees of freedom it is possible to elevate the duality to a transformation of those gauge fields as have been shown in [11]. In [11] the electric and magnetic fields are defined as

\[ E^i = \frac{1}{2e} F^0i - \frac{1}{4g} \epsilon^{ijk} G_{jk} \]
\[ B^i = \frac{1}{2g} G^0i + \frac{1}{4e} \epsilon^{ijk} F_{jk} \]

where \( F = dA \) and \( G = dC \) are the gauge connections of the gauge fields \( A \) and \( C \). In section 3.2 we will properly discuss the physical field definitions, for the moment being we use these definitions which can be found in the literature. The electromagnetic duality reads now

\[ \begin{align*}
    \frac{1}{e} F^0i - \frac{1}{2g} \epsilon^{ijk} G_{jk} & \rightarrow \cos(\theta) \left( \frac{1}{e} F^0i - \frac{1}{2g} \epsilon^{ijk} G_{jk} \right) + \sin(\theta) \left( \frac{1}{g} G^0i + \frac{1}{2e} \epsilon^{ijk} F_{jk} \right) \\
    \frac{1}{g} G^0i + \frac{1}{2e} \epsilon^{ijk} F_{jk} & \rightarrow -\sin(\theta) \left( \frac{1}{e} F^0i - \frac{1}{2g} \epsilon^{ijk} G_{jk} \right) + \cos(\theta) \left( \frac{1}{g} G^0i + \frac{1}{2e} \epsilon^{ijk} F_{jk} \right)
\end{align*} \]

There are two ways to implement these transformations, either in terms of each \( U(1) \) gauge sectors independently or mixing both gauge sectors. If we consider each sector independently we obtain the standard
electromagnetic transformations for each of the connections $F$ and $G$

$$
F^0_i \rightarrow \cos(\theta) F^0_i + \sin(\theta) \frac{1}{2} \epsilon^{ijk} F_{jk} \\
F_{jk} \rightarrow \sin(\theta) \frac{1}{2} \epsilon^{ijk} F_{ik} + \cos(\theta) F_{jk} \\
G^0_i \rightarrow \cos(\theta) G^0_i + \sin(\theta) \frac{1}{2} \epsilon^{ijk} G_{jk} \\
G_{jk} \rightarrow \sin(\theta) \frac{1}{2} \epsilon^{ijk} G_{ik} + \cos(\theta) G_{jk}
$$

(5)

These transformations are not compatible with a transformation of the gauge fields because the $(0i)$ components transform differently from the components $(ij)$.

If we consider mixing between both sectors we can rewrite the electromagnetic duality in terms of the gauge fields or respective connections [11]

$$
\begin{align*}
F & \rightarrow \cos(\theta) F + \sin(\theta) \frac{e}{g} G \\
G & \rightarrow -\sin(\theta) \frac{g}{e} F + \cos(\theta) G \\
A & \rightarrow \cos(\theta) A + \sin(\theta) \frac{e}{g} C \\
C & \rightarrow -\sin(\theta) \frac{g}{e} A + \cos(\theta) C
\end{align*}
$$

(6)

There is a very simple argument to choose the second kind of duality (6) and exclude the possibility of the transformations (5). Let us consider the Lorentz gauge (or Lorentz condition) for both gauge fields $\partial_\mu A^\mu = \partial_\mu C^\mu = 0$ and assume regular gauge fields (meaning without discontinuities) such that the Bianchi identities are obeyed $\epsilon^{\mu\nu\rho\delta} \partial_\nu \partial_\rho A_\delta = \epsilon^{\mu\nu\rho\delta} \partial_\nu \partial_\rho C_\delta = 0$. Then the Maxwell equations (1) read simply [11]

$$
\Delta A^\mu = J^\mu_e \\
\Delta C^\mu = J^\mu_g
$$

(7)

where the Laplacian is $\Delta = \partial_\mu \partial^\mu$. Taking in account the duality transformations for the current densities expressed in (2) we conclude straight away that only (6) correctly transform the Maxwell equations for these particular standard conditions. Here particular means that the gauge choice is not unique, we could have some other gauge fixing prescription and generally we can have discontinuities on the gauge fields such that the Bianchi identity is not obeyed everywhere. As a example there are the cases of the Dirac string [1] or equivalently the nontrivial fiber-bundle of Wu and Yang [20]. However regular gauge fields describe most of physical applications and must therefore be a possible choice. There is however a serious problem concerning these equations, the two $U(1)$ gauge fields are completely decoupled and we obtain two different interactions corresponding to each of the gauge fields instead of only one as in standard electromagnetism. Our main aim in the remaining of this work is how to obtain one only interaction described by two physical gauge fields.

So we have reviewed how to elevate electromagnetic duality of the Maxwell equations in terms of the electric and magnetic fields to a electromagnetic duality in terms of the gauge fields. Next we will briefly describe how the discrete symmetries act on the several fields and how electromagnetic duality breaks parity and time inversion.

### 2.3 Discrete Symmetries: $P$ and $T$ Violation

We proceed to resume the known results for parity $P$ and time inversion $T$ for the electromagnetic physical quantities. The remaining discrete symmetry is Charge Conjugation $C$ and plays no role in the following discussion.

Parity ($P$) stands for the inversion of spatial coordinates and time-inversion ($T$) stands for the inversion of
the time coordinate. Under these discrete symmetries the fields and current densities transform as \[10\]

\[
P : \quad x^i \to -x^i \quad T : \quad t \to -t
\]

\[
E^i \to -E^i \quad E^i \to +E^i
\]

\[
B^i \to +B^i \quad B^i \to -B^i
\]

\[
\rho_e \to +\rho_e \quad \rho_e \to +\rho_e
\]

\[
J^i_e \to -J^i_e \quad J^i_e \to -J^i_e
\]

\[
\rho_g \to -\rho_g \quad \rho_g \to -\rho_g
\]

\[
J^i_g \to +J^i_g \quad J^i_g \to +J^i_g
\]

Electric and magnetic fields transform differently under \( P \) and \( T \) being respectively vectors and pseudo-vectors. Accordingly also the electric and magnetic currents have the same properties \[10\]. Then necessarily the gauge fields \( A \) and \( C \) also have to transform accordingly as vectors and pseudo-vectors \[11\]. The most straightforward way to show this is by considering an action for electromagnetism such that the electric and magnetic current densities are minimally coupled to the gauge fields \( A \) and \( C \) respectively (we will return to this discussion later). Demanding invariance of the action under \( P \) and \( T \) imposes the gauge field \( C \) to transform as a pseudo-vector. We note that the field definitions \((3)\) agree with these results. Then for the two gauge fields and respective gauge connections we have the discrete transformations

\[
P : \quad A^0 \to +A^0 \quad T : \quad A^0 \to +A^0
\]

\[
A^i \to -A^i \quad A^i \to -A^i
\]

\[
C^0 \to -C^0 \quad C^0 \to -C^0
\]

\[
C^i \to +C^i \quad C^i \to +C^i
\]

\[
F^{0i} \to -F^{0i} \quad F^{0i} \to +F^{0i}
\]

\[
F^{ij} \to +F^{ij} \quad F^{ij} \to -F^{ij}
\]

\[
G^{0i} \to +G^{0i} \quad G^{0i} \to -G^{0i}
\]

\[
G^{ij} \to -G^{ij} \quad G^{ij} \to +G^{ij}
\]

We want now to show that neither \( P \) nor \( T \) are maintained by the standard duality rotations \((2)\) or equivalently \((6)\). Here we consider duality as a global transformation independent of space-time coordinates such that the angle \( \theta \) is an exterior parameter to the theory used in the redefinition of the fields. Therefore it does not depend on the space-time coordinates and transform as a scalar with respect to the discrete symmetries \( P \) and \( T \).

We can see explicitly that the duality transformations mix vector with pseudo-vectors such that

\[
P : \quad \tilde{E} = \cos(\theta)E + \sin(\theta)B
\]

\[
\to -\cos(\theta)E + \sin(\theta)B
\]

\[
\tilde{B} = -\sin(\theta)E + \cos(\theta)B
\]

\[
\to \sin(\theta)E + \cos(\theta)B
\]

4
Clearly $\tilde{E}$ and $\tilde{B}$ are not transformed to $-\tilde{E}$ and $-\tilde{B}$ under parity as they should. The same argument follows for $T$

$$
T: \quad \tilde{E} = \cos(\theta)E + \sin(\theta)B
$$

$$
\rightarrow \cos(\theta)E - \sin(\theta)B
$$

$$
\tilde{B} = -\sin(\theta)E + \cos(\theta)B
$$

$$
\rightarrow -\sin(\theta)E - \cos(\theta)B
$$

(11)

and the redefined fields do not transform correctly under $T$. The current duality transformations (2) behaves in the same way.

Charge conjugation $C$ is not a space-time symmetry, it exchanges particles with antiparticles. At classical level this is simply equivalent to change the sign of the current densities and it is preserved by electromagnetic duality.

So, to summarize, at the level of single fields, the electromagnetic duality preserves $C$, $PT$ and $CPT$ while it violates $P$, $T$, $CP$ and $CT$.

The issue of $P$ and $T$ violation by the existence of dyons with both electric and magnetic charge can be found in [10]. As for $P$ and $T$ violation by electromagnetic duality is discussed in [13]. The argument is generic and applicable to the original duality transformations (2) independently of considering a gauge field description of electromagnetism. Also we point out that upon redefinition of the fields one may as well redefine $P$ and $T$, but in order to do so one would be changing the space-time interpretation of the discrete symmetries and necessarily redefining the action of the Lorentz group. This could be interpreted then as an extended duality of space-time. An alternative interesting construction is to consider $\theta$ to be a pseudo-scalar [13], in this way we manage to obtain a duality that preserves the discrete symmetries. Also it is possible to gauge the duality by considering $\theta = \theta(x)$ to be an additional gauge parameter [14, 21] (in this works the duality rotations constitute one further distinct $U(1)$ group).

Here we are considering $\theta$ to be a parameter exterior to the theory that transforms as a scalar, then, although the discrete symmetries violations are not explicit in the equations of motion, at the level of the action (a Lagrangian formulation of the theory) they will be explicit. As we will see in detail electromagnetic duality induces $P$ and $T$ violating terms.

In addition we will demand that there is only one electric and one magnetic physical gauge fields. This requirement is going to reduce the allowed actions.

### 3 Gauge Sector

In this section we will build a $U(1) \times U(1)$ gauge action such that the physical electric and magnetic fields are identified with the definitions (3). In order to do so one expects that the group charge flux of each of the $U(1)$’s is coupled to the topological charge flux of the other $U(1)$. It is also desirable that a classical description of electromagnetism preserves both parity $P$ and time inversion $T$ (see for instance [10] for a discussion on this topic). So we are further demanding our action to be invariant under these discrete symmetries. In addition, and from the discussion on the last section, we expect that under an electromagnetic rotation our action explicitly gains terms that violate $P$ and $T$. This will be the case.
3.1 Possible Actions

Let us consider all the possible lower order terms that are Lorentz and gauge invariant. First we list the lower order terms containing the gauge connections \( F \) and \( G \) which are invariant under \( P \) and \( T \)

\[
\mathcal{L}_{\text{Maxwell}_F} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}
\]

\[
\mathcal{L}_{\text{Maxwell}_G} = -\frac{1}{4g^2} G_{\mu\nu} G^{\mu\nu}
\]  \hspace{1cm} (12)

\[
\mathcal{L}_{\text{Hopf}_{FG}} = -\frac{1}{4eg} \epsilon^{\mu\nu\rho\delta} F_{\mu\nu} G_{\rho\delta}
\]

The last term is a cross Hopf term (or theta term). To show that it is invariant let us rewrite the expression as \( \mathcal{L}_{\text{Hopf}} = 2\epsilon^{0ij}(F_{0i}G_{jk} + G_{0i}F_{jk}) \), then we see from (9) that \( F_{0i} \) and \( G_{0i} \) always transform in the same way as \( G_{jk} \) and \( F_{jk} \) (respectively) such that \( \mathcal{L}_{\text{Hopf}_{FG}} \) is invariant under any of the discrete symmetries \( P \) and \( T \).

The remaining possible lower order terms which are Lorentz and gauge invariant are not invariant under \( P \) and \( T \). They are the cross Maxwell term and the usual Hopf (or theta) terms for each of the gauge sectors

\[
\mathcal{L}_{\text{Maxwell}_G} = -\frac{1}{4eg} F_{\mu\nu} G^{\mu\nu}
\]

\[
\mathcal{L}_{\text{Hopf}_F} = -\frac{1}{4e^2} \epsilon^{\mu\nu\rho\delta} F_{\mu\nu} F_{\rho\delta}
\]

\[
\mathcal{L}_{\text{Hopf}_G} = -\frac{1}{4g^2} \epsilon^{\mu\nu\rho\delta} G_{\mu\nu} G_{\rho\delta}
\]  \hspace{1cm} (13)

To show that they are not invariant under \( P \) and \( T \) we note that, from equation (9), \( (F_{0i}, F_{ij}) \) and \( (G_{0i}, G_{ij}) \) transform in the opposite way under \( P \) and \( T \) such that the cross Maxwell term transforms as \( \mathcal{L}_{\text{Maxwell}_G} \rightarrow -\mathcal{L}_{\text{Maxwell}_G} \). Concerning the Hopf terms we note that \( F_{0i} \) and \( G_{0i} \) transform in the opposite way than \( F_{ij} \) and \( G_{ij} \) (respectively) under \( P \) and \( T \) such that \( \mathcal{L}_{\text{Hopf}_G} \rightarrow -\mathcal{L}_{\text{Hopf}_G} \) and \( \mathcal{L}_{\text{Hopf}_G} \rightarrow -\mathcal{L}_{\text{Hopf}_G} \). This is a known feature of such terms which have been extensively studied to explain \( CP \)-violation, both in Abelian and Non-Abelian gauge theories (see for instance [15, 22] and references therein).

We have listed all the possible lower order candidate terms to build our action. We also need to study how these several candidate terms behave under electromagnetic duality:

\[
\mathcal{L}_{\text{Maxwell}_F} \rightarrow \cos^2(\theta)\mathcal{L}_{\text{Maxwell}_F} + \sin^2(\theta)\mathcal{L}_{\text{Maxwell}_G} + 2\cos(\theta)\sin(\theta)\mathcal{L}_{\text{Maxwell}_FG}
\]

\[
\mathcal{L}_{\text{Maxwell}_G} \rightarrow \sin^2(\theta)\mathcal{L}_{\text{Maxwell}_F} + \cos^2(\theta)\mathcal{L}_{\text{Maxwell}_G} - 2\cos(\theta)\sin(\theta)\mathcal{L}_{\text{Maxwell}_FG}
\]

\[
\mathcal{L}_{\text{Hopf}_{FG}} \rightarrow \sin \theta \cos \theta \left( \mathcal{L}_{\text{Hopf}_{FG}} - \mathcal{L}_{\text{Hopf}_F} \right) + \cos^2 \theta - \sin^2 \theta \mathcal{L}_{\text{Hopf}_F}
\]  \hspace{1cm} (14)

\[
\mathcal{L}_{\text{Maxwell}_F} \rightarrow \sin \theta \cos \theta \left( \mathcal{L}_{\text{Maxwell}_G} - \mathcal{L}_{\text{Maxwell}_F} \right) + \cos^2 \theta - \sin^2 \theta \mathcal{L}_{\text{Maxwell}_F}
\]

\[
\mathcal{L}_{\text{Hopf}_F} \rightarrow \cos^2(\theta)\mathcal{L}_{\text{Hopf}_F} + \sin^2(\theta)\mathcal{L}_{\text{Hopf}_G} + 2\cos(\theta)\sin(\theta)\mathcal{L}_{\text{Hopf}_{FG}}
\]

\[
\mathcal{L}_{\text{Hopf}_G} \rightarrow \sin^2(\theta)\mathcal{L}_{\text{Hopf}_F} + \cos^2(\theta)\mathcal{L}_{\text{Hopf}_G} - 2\cos(\theta)\sin(\theta)\mathcal{L}_{\text{Hopf}_{FG}}
\]

We are now ready to build an action that describes electromagnetism with two gauge fields. Demanding the action to be \( P \) and \( T \) invariant we are left only with the terms listed in (12). So we conclude that the most standard action that explicitly depends on two gauge fields must be a combination of \( \mathcal{L}_{\text{Maxwell}_F} \) and \( \mathcal{L}_{\text{Maxwell}_G} \). We will call this action the minimal action [13, 23, 24]

\[
S_{\text{Min}+} = -\int_M \sqrt{-g} \left[ \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4g^2} G_{\mu\nu} G^{\mu\nu} \right]
\]  \hspace{1cm} (15)

\[
S_{\text{Min} -} = -\int_M \sqrt{-g} \left[ \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4g^2} G_{\mu\nu} G^{\mu\nu} \right]
\]  \hspace{1cm} (16)
We note that from the electric and magnetic fields definition (3) both Maxwell terms must have the same numerical factor (up to the relative sign). The standard would be to consider both with the same sign in order to have the same quantum structure in both sectors, however for completeness we consider both cases. These actions imply the existence of two electric and two magnetic fields as we will discuss in detail in section 3.2. Instead we expect to have only one electric and one magnetic field such that the group charge flux of one \( U(1) \) is of the same nature of the flux of topological charge of the other \( U(1) \) as implied by the field definitions (3). As a weaker but valid argument we note that the pure gauge sectors are completely decoupled, a priori one would expect that some sort of mixing (meaning coupling) between the two sectors exist that, at least, accomplishes the coupling of topological flux with group charge fluxes. We consider these argument as a drawback of the minimal actions.

From the above arguments we are further considering the remaining allowed term that preserves \( T \) and \( P \), the cross Hopf term. We call these actions the maximal actions

\[
S_{\text{Max},+}^\ell = -\int_M \left[ \frac{\sqrt{-g}}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\sqrt{-g}}{4g^2} G_{\mu\nu} G^{\mu\nu} - \frac{\hat{\ell}}{4eg^4} e^{\mu\nu\rho\delta} F_{\mu\nu} G_{\rho\delta} \right] \tag{17}
\]

\[
S_{\text{Max},-}^\ell = -\int_M \left[ \frac{\sqrt{-g}}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{\sqrt{-g}}{4g^2} G_{\mu\nu} G^{\mu\nu} - \frac{\hat{\ell}}{4eg^4} e^{\mu\nu\rho\delta} F_{\mu\nu} G_{\rho\delta} \right] \tag{18}
\]

The cross Hopf term couples the flux of the group charges \( (F^{\alpha_1} \text{ and } G^{\alpha_1}) \) with the flux of the topological charge \( (G_{ij} \text{ and } F_{ij} \text{ respectively}) \) of the two different \( U(1) \)'s. As we are going to show in the next section 3.2 the only action that can be defined using only one electric and one magnetic field is \( S_{\text{Max},-}^\ell \) as given by (18). \( \hat{\ell} = \pm 1 \) sets the relative sign of the Hopf term and will be relevant in the definition of the physical fields as we will show in detail.

We also note that when both Maxwell terms have the same sign the minimal action \( S_{\text{Min},-}^\ell \) as given by (15) is invariant under electromagnetic duality, so for an action of this form we have elevated the duality to a symmetry [14,21]. Also the respective Hamiltonian will be duality invariant. However this is not necessarily a good feature. As we already pointed out, standard duality does not preserve either \( P \) or \( T \) and this fact is not explicit on the action \( S_{\text{Min},-}^\ell \), neither can be on a duality invariant action. This argument is not completely close because one can consider the duality angle parameter \( \theta \) to be a pseudo-scalar [13]. However another physical argument is that due to Dirac quantization condition [1] \( (eg = n) \) we have that the \( \mathcal{A} \) field obeys a weak coupling regime while the \( C \) field obeys a strong coupling regime. Then we expect that the energy (Hamiltonian) not to be conserved under a duality rotation. For these reasons a duality symmetric action does not look like a good choice. The remaining actions, the maximal actions (18) and (17) and the minimal action (16), are not invariant under duality but they are \( P \) and \( T \) invariant. Furthermore a duality transformation does not preserve \( P \) and \( T \) invariance as can explicitly be seen from (14). This is actually a good feature, duality explicitly breaks \( P \) and \( T \) at the level of the action as expected from (10) and (11) and the respective Hamiltonians are not invariant under duality.

Finally the as we show in detail in the next subsection, only \( S_{\text{Max},-}^\ell \) is compatible with the existence of only one electric and one magnetic physical fields.

### 3.2 Physical Electric and Magnetic Fields

Due to have introduced a second gauge field \( \mathcal{C} \) we have now twice the degrees of freedom than usual electromagnetism. Accordingly we expect to have as well a new interaction such that generally we have two electric fields and two magnetic fields. From a theoretical point of view this is standard, each of the gauge fields carry a different kind of interaction. Nevertheless we are led to the question if both definitions are physical fields or not. Here we will show that for both the minimal actions (15) and (16) and the maximal action (17) we have indeed four physical fields (two electric and two magnetic), while for the maximal action (18) we have only two physical fields (one electric and one magnetic).

Let us consider the generic definitions of electric and magnetic fields corresponding to the gauge fields \( \mathcal{A} \) and
We note that the definitions of electric and magnetic fields for $C$ are reversed to the ones of the $A$ field and for reasons that will become clear in the remaining of this section we consider a minus sign in the definition of $E_C$. Both in order to define the electric and magnetic fields accordingly to (3) and to preserve the properties of the fields in relation to the discrete symmetries, i.e. the electric field is a vector and the magnetic field is a pseudo-vectors. Then we take the following linear combinations of the above definitions (19)

$$E_i^+ = \frac{1}{2} (E_A^i + E_C^i) = + \frac{1}{2} F^{0i} - \frac{1}{4g} \epsilon^{ijk} G_{jk}$$

$$B_i^+ = \frac{1}{2} (B_A^i + B_C^i) = + \frac{1}{2g} G^{0i} + \frac{1}{4e} \epsilon^{ijk} F_{jk}$$

$$E_i^- = \frac{1}{2} (E_A^i - E_C^i) = + \frac{1}{2} F^{0i} + \frac{1}{4g} \epsilon^{ijk} G_{jk}$$

$$B_i^- = \frac{1}{2} (B_A^i - B_C^i) = - \frac{1}{2g} G^{0i} + \frac{1}{4e} \epsilon^{ijk} F_{jk}.$$  

(20)

To consider these combination is the approach of several authors that consider only $E_+$ and $B_+$ as physical fields [11, 23]. As already explained in the introduction the main motivation is to achieve a generalized description of electromagnetism with both electric and magnetic particles [2]. Also these combinations are used to implement an explicit electromagnetic duality between the two sectors in terms of two distinct gauge fields as we explained in section 2.

For what follows we will need the identities

$$\frac{1}{2e^2} F^{0i} F_{0i} = \frac{1}{2} (E_+^i + E_-^i) (E_+^i + E_-^i) = \frac{1}{2} (E_+^i E_+^i + E_-^i E_-^i + 2 E_+^i E_-^i)$$

$$\frac{1}{4e^2} F^{ij} F_{ij} = - \frac{1}{2} (B_+^i + B_-^i) (B_+^i + B_-^i) = \frac{1}{2} (- B_+^i B_+^i - B_-^i B_-^i - 2 B_+^i B_-^i)$$

$$\frac{1}{2g^2} G^{0i} G_{0i} = \frac{1}{2} (B_+^i - B_-^i) (B_+^i - B_-^i) = \frac{1}{2} (+ B_+^i B_+^i + B_-^i B_-^i - 2 B_+^i B_-^i)$$

$$\frac{1}{4g^2} G^{ij} G_{ij} = - \frac{1}{2} (- E_+^i + E_-^i) (- E_+^i + E_-^i) = \frac{1}{2} (- E_+^i E_+^i - E_-^i E_-^i + 2 E_+^i E_-^i)$$

$$\frac{1}{2e g} \epsilon^{0ij} F_{0i} G_{jk} = (E_+^i + E_-^i) (E_+^i + E_-^i) = - E_+^i E_+^i + E_-^i E_-^i$$

$$\frac{1}{2e g} \epsilon^{0ij} G_{0i} F_{jk} = (B_+^i + B_-^i) (B_+^i - B_-^i) = + B_+^i B_+^i - B_-^i B_-^i$$

where the minus sign and the factor of 2 in the second and forth lines are due to the contraction of the indices of the antisymmetric tensor, i.e. $\epsilon^{0ijk} \epsilon_{0jk} i^i = - 2 \delta^{ii'}$.

Let us consider both the minimal actions (15) and (16) and the maximal actions (17) and (18) and rewrite the respective Lagrangians in terms of the above combinations (20) using the identities (21). We obtain that

$$\mathcal{L}_{\text{Min+}} = - 2 (E_+^i E_-^i - B_+^i B_-^i)$$

$$\mathcal{L}_{\text{Min-}} = - (E_+^i E_-^i + E_-^i E_+^i - B_+^i B_+^i - B_-^i B_-^i)$$

$$\mathcal{L}_{\text{Max+}} = - 2 (E_+^i E_-^i - B_+^i B_-^i) - \epsilon (E_+^i E_+^i - E_-^i E_-^i - B_+^i B_+^i + B_-^i B_-^i)$$

$$\mathcal{L}_{\text{Max-}} = - 2 (E_+^i E_-^i - B_+^i B_-^i).$$  

(22)
We readily conclude that the only action that can be written in terms of only two fields (gauge connections) in terms of the new 2-form and rewriting the Lagrangians in terms of the redefined fields (22) is equivalent to rewriting the Lagrangian. In order to analyze the maximal action (18) we note that the above procedure of redefinition of fields (20) is essentially the same as the one of standard electromagnetism, therefore we have a very strong indication that, for this Lagrangian, the physical fields are

\[ E^i = \frac{1}{e} F^{0i} - \frac{\hat{\epsilon}}{2g} \epsilon^{0ijk} G_{jk} \]

\[ B^i = \frac{\hat{\epsilon}}{g} G^{0i} + \frac{1}{2e} \epsilon^{0ijk} F_{jk} . \]

This is only possible for the maximal Lagrangian \( \mathcal{L}_{\text{Max}} \).

To show that this is indeed the case we will formalize this argument and analyze the four possible actions. Let us compute the equations of motion for the actions and check which fields appear in them. We will properly discuss how to couple each type of current densities to both the gauge fields in the next section, for the moment being let us assume the standard minimal coupling

\[ S_{\text{Sources, Min} \pm} = - \frac{1}{e} \int A_\mu J^\mu_{\pm} \pm \frac{1}{g} \int C_\mu J^\mu_g \]

where the \( \pm \) correspond to the relative sign between the Maxwell terms.

For the minimal actions we have that the equations of motion are

\[
\begin{align*}
\frac{1}{e} \partial_\mu F^{\mu\nu} &= J^\nu_e \\
\frac{1}{g} \partial_\mu G^{\mu\nu} &= J^\nu_g \\
\n\n\end{align*}
\]

\[
\begin{align*}
\n\n\)

The electric and magnetic equations are completely decoupled and we have two electric and two magnetic fields. Also in addition to this equation we have the Bianchi identities for each gauge field. There is a way to couple both sector by using non homogeneous Bianchi identities, for that consider non regular gauge fields such that we have the respective Bianchi identities \( dF = \ast J_g \) and \( dG = \ast J_e \). Then by an appropriate combination of the equations of motion with the Bianchi identities we obtain \( d(\ast F - G) = \ast J_e - \ast J_g \) and \( d(F + \ast G) = \ast J_g + \ast J_e \) which correspond to the generalized Maxwell equations (1) with the current densities changed from \( J_e \rightarrow J_e - J_g \) and \( J_g \rightarrow J_g + J_e \). Here \( \ast \) denotes the usual Hodge duality operation and we used form notation for compactness. There are two drawbacks for this approach, first the current densities are no longer the ones which minimally couple to the gauge fields at the level of the action and secondly, the identification of the topological charge fluxes with the group charge fluxes of different gauge groups is imposed (by hand) not emerging naturally from the action. These problems are solved by using the maximal action with opposite signs for the Maxwell terms as given by (18).

In order to analyze the maximal action (18) we note that the above procedure of redefinition of fields (20) and rewriting the Lagrangians in terms of the redefined fields (22) is equivalent to rewriting the Lagrangian in terms of the new 2-form gauge connections [13]

\[
(F^\pm)^{\mu\nu} = \frac{1}{2} (F \pm \hat{\epsilon} \ast G)^{\mu\nu} = \frac{1}{2e} F^{\mu\nu} \pm \frac{\hat{\epsilon}}{4g} \epsilon^{\mu\nu\rho\delta} G_{\rho\delta} \\
(F^\pm)^{\nu\mu} = \frac{1}{2} (F \pm \hat{\epsilon} \ast G)^{\nu\mu} = \frac{1}{2e} F^{\nu\mu} \mp \frac{\hat{\epsilon}}{4g} \epsilon^{\mu\nu\rho\delta} G_{\rho\delta}.
\]
or their Hodge duals

\[
(G_1^+)_{\mu\nu} = - \ast (F_1^+)_{(\mu\nu)} = \frac{\hat{\epsilon}}{2e} G^{\mu\nu} + \frac{1}{4g} \epsilon^{\mu\nu\rho\delta} F_{\rho\delta}
\]

\[
(G_1^-)_{\mu\nu} = - \ast (F_1^-)_{(\mu\nu)} = \frac{\hat{\epsilon}}{2e} G^{\mu\nu} - \frac{1}{4g} \epsilon^{\mu\nu\rho\delta} F_{\rho\delta}.
\]

(27)

Then the maximal Lagrangian (18) is rewritten in both equivalent expressions as

\[
L_{\text{Max}}^\pm = - (F_\pm^+)_{\mu\nu} (F_\pm^+)_{\mu\nu} + (G_\pm^+)_{\mu\nu} (G_\pm^+)_{\mu\nu}.
\]

(28)

where we used the Hodge duality property $**G = -G$ for 2-forms $G$ in Lorentzian 4D manifolds. This is basically the reason why in (19) we defined $E_\pm^\pm = -\epsilon^{ijk}G_{jk}$ with a minus sign [11, 23]. We note that these two ways of rewriting are algebraically equivalent. However physically they have an important meaning, we can have both a electric and a magnetic description of the theory. This is seen in the equations of motion. Upon variation of the maximal action with respect to $A$ and $C$ we obtain

\[
\partial_\mu (F_\pm^+)_{\mu\nu} = J_\nu^E
\]

\[
\partial_\mu (G_\pm^+)_{\mu\nu} = \hat{\epsilon} J_\nu^g
\]

(29)

which indeed correspond to the generalize Maxwell equations (1) and are expressed only in terms of the fields $E$ and $B$ as given by (23). So these must be the physical fields! This is only possible for the maximal action. As for $L_{\text{Max}}^\pm$ corresponding to the action (17) this construction is not possible, we obtain that

\[
L_{\text{Max}}^\pm = - \frac{1}{2} (F_\pm^+)_{\mu\nu} (F_\pm^+)_{\mu\nu} + \frac{1}{2} (F_\pm^+)_{\mu\nu} (F_\pm^+)_{\mu\nu} - (F_\pm^+)_{\mu\nu} (F_\pm^-)_{\mu\nu}
\]

(30)

such that we need two distinct gauge connections in order to define it, hence as expected four physical fields. One must be careful with the way we couple the source to both gauge fields depending on the choice of $\hat{\epsilon}$ due to the definitions of the physical fields and the current sign in the second line of (29). We will discuss this issue in detail in the next section 4.

There is a subtlety here. The reader may by now be recalling the Bianchi identities (or homogeneity conditions for Abelian gauge fields) on the gauge connection and claiming that as usual for topological terms the variation

\[
\delta L_{\text{Hopf}EFG} = \frac{1}{2e} \epsilon^{\mu\nu\rho\delta} (\partial_\mu G_\delta\rho \delta A_\mu + \partial_\nu F_\rho\delta \delta C_\mu)
\]

(31)

should be always null and does not contribute to the equations of motion. This is true for regular fields, however as already mentioned in the first section and in the analysis of the minimal actions, if non-regular gauge fields are allowed then this contribution to the equations of motion is not null everywhere and must be taken in account and (29) are actually the correct ones. By discontinuities we mean that $\partial_\mu \partial_\nu C \neq \partial_\nu \partial_\mu C$. Allowing for corrections to the Bianchi identities allows for the inclusion of magnetic charge in standard electromagnetism (with only one $U(1)$ gauge field) and is in the basis of the original construction that originates the Dirac string [1] or the equivalent non-trivial fiber-bundle of Wu and Yang [20]. We present this argument only to show that algebraically (29) are correct, we don’t need to necessarily have these discontinuities to describe both electric and magnetic charge as long as we work with two distinct gauge fields. However we show in [8] that in order to have effective theories obtained from the maximal action only with one gauge field we still have discontinuities, but the discontinuities will be present on the extra field (instead of the physical field of the effective theory as in [1, 20]).

An important result here is that for the maximal action the topological fluxes of one $U(1)$ are identified with the charge fluxes of the other $U(1)$ as desired for the existence of only one electric and one magnetic physical fields. We must stress that this does not imply that we are constraining the fundamental fields $A$ and $C$, we are maintaining the same degrees of freedom. We have 4 physical degrees of freedom (2 for each of the gauge fields $A$ and $C$) which are still maintained in the electric and magnetic fields (again 2 for each of the fields $E$ and $B$). In standard electromagnetism with only one gauge field there is only 2 degrees of freedom. The interpretation in terms of the fields is quite interesting. For each of the $U(1)$ fields the 2 physical degrees of freedom correspond to the transverse modes while the longitudinal modes are not physical and do not
constitute physical degrees of freedom. When combining the gauge connections as in (26) the degrees of freedom of the second gauge field $C$ are combined with the degrees of freedom of the original gauge field $A$ in such a way that they play the role of two Longitudinal modes of the gauge field $A$, simply we have now two longitudinal modes instead of a single one as is usual in theories with massive photons. These degrees of freedom constitute here physical degrees of freedom and are due to the inclusion of a second $U(1)$ gauge group.

Our discussion would not be complete without discussing the canonical variables. We do so next and also briefly the expression for the Hamiltonians corresponding to the minimal and maximal actions.

3.3 Canonical Variables and Hamiltonian Formulation

The canonical momenta for the minimal actions (15) and (16) are

$$\pi_{A,\text{Min}}^i = \frac{1}{e^2} F_{0i} = \frac{1}{e} F_{Ai}$$

$$\pi_{C,\text{Min}}^i = \pm \frac{1}{g^2} G_{0i} = \pm \frac{1}{g} B_C^i$$

where the $\pm$ refers respectively to $\mathcal{L}_{\text{Min}_+}$ (the + sign) and $\mathcal{L}_{\text{Min}_-}$ (the − sign). This means that the canonical momenta are each of the $U(1)$ group charge fluxes. The Hamiltonian depends on both gauge sectors but each of them are completely decoupled

$$H_{\text{Min}_\pm} = \frac{1}{2} \left( e^2 \pi_{A,\text{Min}}^i \pi_{A,\text{Min}}^i + \frac{1}{2} e^2 F_{ij} F_{ij} \right)$$

$$\pm \frac{1}{2} \left( g^2 \pi_{C,\text{Min}_\pm}^i \pi_{C,\text{Min}_\mp}^i + \frac{1}{2} g^2 G_{ij} G_{ij} \right)$$

such that the Hilbert space factorizes into states carrying charge fluxes of both gauge sectors. The topological charge fluxes are present only trough the potential $F_{ij} F_{ij}$ and $G_{ij} G_{ij}$ as in standard electromagnetism. So basically we have two distinct copies of standard electromagnetism and no interaction terms between the two sectors.

The canonical momenta for the maximal action (17) and (18) are

$$\left( \pi_{A,\text{Max}_\pm}^i \right)^i = \frac{1}{e} F_{0i} \pm \frac{\hat{\epsilon}_{ikl}}{2e} \tilde{e}^{ijk} G_{jk} = \frac{2}{e} F_{0i}$$

$$\left( \pi_{C,\text{Max}_\pm}^i \right)^i = \pm \frac{1}{g^2} G_{0i} - \frac{\hat{\epsilon}_{ikl}}{2e} \tilde{e}^{ijk} F_{jk} = \frac{2}{g} B_{C+}^i$$

where the $\pm$ refers respectively to $\mathcal{L}_{\text{Max}_+}$ (the + sign) and $\mathcal{L}_{\text{Max}_-}$ (the − sign). In the electric field the subscript $(\pm \hat{\epsilon})$ means the product of $\pm 1$ by $\hat{\epsilon}$. The canonical momenta coincide up to constants with the physical electric and magnetic fields, this is a good indication that indeed, also at quantum level, we can have the correct identifications between group charge and topological charge fluxes from the opposite $U(1)$’s.

After a straight forward computation we obtain the following Hamiltonians

$$H_{\text{Max}_+}^\hat{\epsilon} = \frac{e^2}{2} \left( \pi_{A,\text{Max}_+}^i + \frac{\hat{\epsilon}_{ikl}}{2e} \tilde{e}^{ijk} G_{jk} \right) \left( \pi_{A,\text{Max}_+}^i - \frac{\hat{\epsilon}_{ikl}}{2e} \tilde{e}^{ijk} G_{jk} \right)$$

$$+ \frac{g^2}{2} \left( \pi_{C,\text{Max}_+}^i + \frac{\hat{\epsilon}_{ikl}}{2g} \tilde{e}^{ijk} F_{jk} \right) \left( \pi_{C,\text{Max}_+}^i - \frac{\hat{\epsilon}_{ikl}}{2g} \tilde{e}^{ijk} F_{jk} \right)$$

$$+ \frac{\hat{\epsilon}_{ikl}}{2e} \tilde{e}^{ijk} \left( \pi_{A,\text{Max}_+}^i G_{jk} + \pi_{C,\text{Max}_+}^j F_{jk} \right) - \frac{3}{4e^2} F_{ij} F_{ij} + \frac{5}{4g^2} G_{ij} G_{ij}$$

$$H_{\text{Max}_-}^\hat{\epsilon} = \frac{e^2}{2} \left( \pi_{A,\text{Max}_-}^i + \frac{\hat{\epsilon}_{ikl}}{2e} \tilde{e}^{ijk} G_{jk} \right) \left( \pi_{A,\text{Max}_-}^i - \frac{\hat{\epsilon}_{ikl}}{2e} \tilde{e}^{ijk} G_{jk} \right)$$

$$+ \frac{g^2}{2} \left( \pi_{C,\text{Max}_-}^i + \frac{\hat{\epsilon}_{ikl}}{2g} \tilde{e}^{ijk} F_{jk} \right) \left( \pi_{C,\text{Max}_-}^i - \frac{\hat{\epsilon}_{ikl}}{2g} \tilde{e}^{ijk} F_{jk} \right)$$

$$+ \frac{\hat{\epsilon}_{ikl}}{2e} \tilde{e}^{ijk} \left( \pi_{A,\text{Max}_-}^i G_{jk} + \pi_{C,\text{Max}_-}^j F_{jk} \right) - \frac{3}{4e^2} F_{ij} F_{ij} + \frac{5}{4g^2} G_{ij} G_{ij}$$
and

\[ \mathcal{H}_{\text{Max}}^i = \frac{e^2}{2} \left( \pi_{\text{A,Max}}^i + \frac{\dot{e}}{2e} \epsilon^{ijk} G_{jk} \right) - \frac{g^2}{2} \left( \pi_{\text{C,Max}}^i + \frac{\dot{\epsilon}}{2e} \epsilon^{ijk} F_{jk} \right) + \frac{\dot{\epsilon}}{2e} \epsilon^{ijk} (\pi_{\text{A,Max}}^i G_{jk} - \pi_{\text{C,Max}}^i F_{jk}) + \frac{5}{4e^2} F_{ij} F_{ij} - \frac{5}{4g^2} G_{ij} G_{ij}. \]  

(36)

The first lines of both these equations are interpreted as usual with \( a_+^i a_-^i \) where \( a_\pm \) are creation and annihilation operators of electric excitations and the second lines correspond to \( b_+^i b_-^i \) where \( b_\pm \) are creation and annihilation operators of magnetic excitations. The third lines contain a generalized angular momenta term between the two gauge sectors and the potentials \( F_{ij} F_{ij} \) and \( G_{ij} G_{ij} \). We note that the potential terms have non-standard factors and opposite signs in both Hamiltonians. In particular the factors for the potentials in \( \mathcal{H}_{\text{Max}}^i \) as given in (35) have different weights (i.e., besides the opposite sign have different numerical factors independently of the coupling constants) while they have the same numerical weight for \( \mathcal{H}_{\text{Max}}^i \) as given in (36). This is also a good indication that the Maximal action (18) is the correct one since there is no reason for the potentials \( F_{ij} F_{ij} \) and \( G_{ij} G_{ij} \) having different numerical weights (besides the coupling constants).

The Hilbert space is not generally factorizable, the states should only be factorizable for states that have null eigenvalues of the generalized angular momenta.

The main problem in quantizing this theory is that the \( b_+^i b_-^i \) has the opposite sign (than the standard fields) and, using the usual commutation relations for the \( C \) field, makes the existence of negative energy states possible. In order to solve this issue the standard way out is to consider anti-commutation relations for the \( C \) field, which makes the existence of negative energy states possible. In this case we are in presence of a ghost field [16], not a standard boson. An alternative approach is to consider some mechanism that allows to quantize only the electric sector as done in [26]. As such examples we have Phantom matter in cosmology [17] where such fields are considered at classical level (i.e., we may consider them to be a collective field, meaning a statistical effective field) in inflationary models. Also we can consider a dynamical symmetry breaking mechanism [8], a possible application is considered in [18, 19] as a way to generate a Proca mass for the usual photon.

We are not discussing any further the quantization procedure here.

### 4 Inclusion of Current Densities

In here we are analysing in detail the current densities coupling to both the gauge fields for the Maximal action (18) and accordingly derive the Lorentz force with both gauge fields.

#### 4.1 Current Coupling Terms

Concerning the inclusion of currents let us consider the standard action

\[ S_{\text{Sources,Max}} = - \int_M \left[ \frac{1}{e} A_\mu J_\mu - \frac{\epsilon}{g} C_\mu J_\mu \right] \]  

(37)

where we have the \( \dot{\epsilon} \) correctly sets the current sign in the generalized Maxwell equations (29). This action is both \( P \) and \( T \) invariant but it is not invariant under electromagnetic duality rotation. Under a duality rotation we effectively couple each current density with both gauge fields obtaining violating terms.

From the discussion of the last section we concluded that the physical electric and magnetic fields are given by (23). So each of the currents need to couple in some way to both \( U(1) \) gauge fields. The question is how to do it maintaining \( P \) and \( T \) symmetries and having the variation of the action with respect to space-time coordinates holding the Lorentz force defined in terms of the fields (3). We note that (37) is not enough since it holds that we would have two Lorentz forces, one for each \( U(1) \)'s in terms of the decoupled fields as given in (19).
The way out is to consider the dual fields $\tilde{A}$ and $\tilde{C}$ defined in terms of the original gauge fields by the differential equations
\[
\begin{align*}
\tilde{F} &= *F \quad \Leftrightarrow \quad d\tilde{A} = *dA \\
\tilde{G} &= *G \quad \Leftrightarrow \quad d\tilde{C} = *dC 
\end{align*}
\]
(38)
where again $*$ denotes the Hodge duality operation. We note that the dual fields have only longitudinal modes, so by dual we mean that we are exchanging transverse modes in $A$ and $C$ by longitudinal modes in $\tilde{A}$ and $\tilde{C}$. So the extra action for the current densities read
\[
S_{\text{Dual Sources,Max.}} = + \int_M \left[ \frac{1}{g} \tilde{C}_\mu J^\mu + \frac{1}{e} \tilde{A}_\mu J^\mu \right] 
\]
(39)
Both terms are $P$ and $T$ invariant because $\tilde{A}$ and $\tilde{C}$ are, respectively, a pseudo-vector and a vector due to (38). Again the sign choice is not arbitrary, we already fixed it in order to obtain the correct Lorentz forces. Electromagnetic duality couples both current densities with both gauge fields $A$ and $C$ such that it induces $P$ and $T$ violating terms. For this action we indeed have that the group charges of each $U(1)$ (given by the $J$’s) are coupled to the topological charges of the other $U(1)$ (in terms of $\tilde{A}$ and $\tilde{C}$). This is what is expressed in the definition of the dual fields as given by (38). Also there are a couple of very important points we must address. These terms do not contribute to the equations of motion of the gauge fields. The reason is that due to (38) we exchange transverse with longitudinal modes in the definitions of $\tilde{A}$ and $\tilde{C}$ and that the current densities only carry transverse modes. Let us be more precise the variation of a term $A_\mu X^\mu$ reads
\[
\frac{\delta \tilde{A}_\mu}{\delta A_\nu} X^\mu = \left( \frac{\delta \tilde{F}_{\alpha\beta}}{\delta A_\mu} \right)^{-1} \frac{\delta \tilde{F}_{\alpha\beta}}{\delta A_\mu} \frac{\delta \tilde{F}_{\delta\rho}}{\delta A_\nu} X^\mu 
\]
(40)
\[
= 8 \epsilon_{\mu \nu \alpha \beta} (\partial_\alpha) \partial_\beta X^\mu 
\]
Now considering the gauge invariance condition (continuity condition) for current densities $d_* J = \partial_\mu J^\mu = 0$ we obtain that the currents are given in terms of a regular anti-symmetric 2-tensor $\phi$ (a 2-form) as
\[
J^\mu = \epsilon^{\mu \nu \rho \lambda} \partial_\rho \phi_{\nu \lambda} + \epsilon^\mu
\]
(41)
where $c_\mu$ is a constant. This same result is already expressed in [25, 26]. We note that the above expression is obtained from the Hodge decomposition of the current densities $J = d\phi + *d\phi + c$. Then replacing this expression for $X = J$ in the above action variation (40) we have the derivatives in $\mu$ and $\delta$ contracted with the anti-symmetric tensor. Therefore we obtain a null variation. We note that although we may generally consider non-regular fields, we cannot consider non-regular current densities, the continuity equation for currents $\partial_\mu J^\mu = 0$ is demanded everywhere for gauge invariance, while for the gauge fields $F$ and $G$ are gauge invariant independently of $A$ and $C$ being regular or not (as long as the gauge transformation parameter is regular, well understood). The second point to stress is that for regular gauge fields this term is a total derivative, however for non-regular gauge fields it is not. So by admitting the existence of non-regular gauge fields the term is present in the action and cannot be integrated to the boundary.

To clarify we give a explicit example. Let us rewrite the first term of the above expression (39) in terms of $\phi$ as given in (41) as
\[
S_\phi = -\frac{\epsilon}{g} \int_M G_{\mu\nu} \phi^{\mu\nu} 
\]
(42)
being as usual $G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. One can notice that for regular fields we would integrate by parts obtaining $S_\phi = -\epsilon/g \int_M \partial_\mu C_\nu \phi^{\mu\nu} = 0$ because $\partial_\mu \phi^{\mu\nu} = 0$. However take as an example of a non-regular gauge field $C_1 = H(x)$ and all the remaining components null, $C_0 = C_2 = C_3 = 0$. Here $H(x)$ is the Heaviside function (also known as unit step function). Then the above action reads
\[
S_\phi = -\frac{\epsilon}{g} \int_M \delta(x_2) \phi^{21} = -\frac{\epsilon}{g} \int dtdx^1 dx^3 dx^4 \phi^{21} \neq 0.
\]
(43)
Clearly we are not allowed to integrate by parts for non-regular gauge fields. However when computing the equations of motion for $S_\phi$ we obtain upon a functional derivation on $C_\mu$ the null contribution for the equations of motion $\partial_\nu \phi^{\mu\nu} = 0$ as desired.
As a last remark we note that adding a current carrying both electric and magnetic charges (corresponding to a dyon) we obtain an explicit $P$ and $T$ violation

$$S_{Mix \ Sources} = - \int_M \left( \frac{1}{e} A_\mu - \frac{1}{g} C_\mu \right) J_{eg}^\mu.$$  

(44)

This violation is independent of electromagnetic duality by the simple fact that $J_{eg}$ must be a combination both of a vector and a pseudo-vector. So we are assuming that we have no dyons, meaning particles with both electric and magnetic charge. If they do exist then $P$ and $T$ are not valid symmetries [10].

In the next subsection we derive the Lorentz force checking that we actually have the usual expression but with the electric and magnetic fields defined as in (3).

4.2 Lorentz Force and the Physical Fields

In order to derive the Lorentz force consider the Lagrangian for a relativistic classical electron with charge $-e$ described by the current density $J_{e}^\mu = -e(1, \dot{x})$

$$\mathcal{L}_{\text{Lorentz}} = -m \gamma^{-1} \left( \frac{1}{e} A_\mu - \frac{1}{g} C_\mu \right) J_{e}^\mu$$  

(45)

where the first term accounts for the rest mass and as usual $\gamma^{-1} = \sqrt{1 - \dot{x}^2}$. We have set $c = 1$. Varying this action with respect to the coordinates $x_i$ is equivalent to the Euler–Lagrange equations and we obtain after a straightforward computation that

$$\frac{dp_i}{dt} = +e \left[ \left( \frac{1}{e} F^{0i} - \frac{\dot{\epsilon}}{g} G^{0i} \right) + \dot{x}_j \left( \frac{1}{e} F^{ij} - \frac{\dot{\epsilon}}{g} G^{ij} \right) \right]$$  

$$= +e \left[ E^i + \epsilon^{ijk} \dot{x}_j B_k \right].$$  

(46)

Where we used the definition of the dual fields $\tilde{G}$ as given in (38) and $E^i$ and $B_i$ are given by (23).

If instead we consider the Lagrangian for a relativistic classical magnetic monopole with charge $+g$ and current given by $J_{g}^\mu = +g(1, \dot{x})$ we obtain

$$\mathcal{L}_{\text{Lorentz}} = -m \gamma^{-1} \left( \frac{1}{g} C_\mu + \frac{1}{e} A_\mu \right) J_{g}^\mu.$$  

(47)

Then we obtain

$$\frac{dp_i}{dt} = +g \left[ \left( \frac{1}{g} G^{0i} + \frac{1}{e} F^{0i} \right) + \dot{x}_j \left( \frac{1}{g} G^{ij} + \frac{1}{e} F^{ij} \right) \right]$$  

$$= +g \left[ B_i - \epsilon^{ijk} \dot{x}_j E_k \right].$$  

(48)

Where again we used the definition of the dual fields $\tilde{F}$ as given in (38) and $E^i$ and $B_i$ are given by (23).

We note that here we considered a positive magnetic charge with rest energy positive, for that reason we obtain a plus sign in the definition of the Lorentz force.

We note that both Lorentz forces are duality invariant [13].

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