DORMANT MIURA OPERS, TANGO STRUCTURES, AND THE BETHE ANSATZ EQUATIONS MODULO $p$

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Abstract. In the present paper, we study the Bethe ansatz equations and Miura opers in characteristic $p > 0$. Our study is based on a work by E. Frenkel, in which solutions to the Bethe ansatz equations are described in terms of Miura opers on the complex projective line. A result of the present paper provides its partial analogue in the case of dormant Miura PGL$_2$-opers. It is known that dormant generic Miura PGL$_2$-opers on an algebraic curve correspond bijectively to certain line bundles on the curve called Tango structures, which bring various sorts of pathological phenomena in positive characteristic (e.g., counter-examples to the Kodaira vanishing theorem). By applying our result, we can construct new examples of Tango structures by means of solutions to the Bethe ansatz equations modulo $p$.

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Introduction

In the present paper, we study the Bethe ansatz equations and Miura opers in characteristic $p > 0$. Recall (cf. [1], §3.2, (3.5)) that the Bethe ansatz equations for a simple finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ are the equations on the set of distinct complex numbers $z_1, \ldots, z_m$ ($m \geq 1$) of the form

$$\sum_{i=1}^r \frac{\langle \alpha_j, \tilde{\lambda}_i \rangle}{z_j - x_i} - \sum_{s \neq j} \frac{\langle \alpha_j, \tilde{\alpha}_s \rangle}{z_j - z_s} = 0 \quad (j = 1, \ldots, m),$$

where $\alpha_1, \ldots, \alpha_m$ are simple positive roots, $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m$ are the corresponding coroots, $x_1, \ldots, x_r$ ($r \geq 0$) are distinct complex numbers, and $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_r$ are dominant coweights of $\mathfrak{g}$. It is well-known that each solution to the Bethe ansatz equations (1) specifies an eigenvector of the
Hamiltonian of a certain spin model called the *Gaudin model*. It gives an effective way to solve the problem of simultaneous diagonalization of the Gaudin Hamiltonian. On the other hand, the Bethe Ansatz equations may be also interpreted as a “no-monodromy” condition on Miura opers. A *Miura oper* (cf. §1.3) is an oper equipped with additional data, or somewhat more precisely, a torsor over an algebraic curve equipped with two Borel reductions and a flat connection satisfying some conditions (including a certain transversality condition). For example, PGL₂-opers and Miura PGL₂-opers on an algebraic curve over $\mathbb{C}$ may be identified with, respectively, projective and affine connections on the associated Riemann surface. In a work by E. Frenkel (cf. [1] and [2]), a canonical correspondence between solutions to the Bethe ansatz equations and certain Miura opers with trivial monodromy on the complex projective line was constructed. Our study is based on this correspondence.

In the present paper, we consider the Bethe ansatz equations and the correspondence given by Frenkel in the case of characteristic $p > 0$. (For the previous study of the Bethe ansatz equations, we refer to [12].) Unfortunately, we are not able to apply the proofs of some results in [1] or [2] to our discussion because of the positive characteristic nature of the various objects involved. Indeed, on a flat connection in characteristic $p$, the property of having trivial monodromy is not sufficient to characterize the local triviality. Although our situation is rather different from the case over $\mathbb{C}$ in this sense, we can prove a partial analogue of Frenkel’s correspondence in positive characteristic (cf. Theorem 2.2.3). Moreover, solutions to the Bethe ansatz equations for $g = sl_2$ of a certain type turn out to correspond bijectively to *dormant* generic Miura PGL₂-opers on the projective line $\mathbb{P}$, i.e., generic Miura PGL₂-opers on $\mathbb{P}$ with vanishing $p$-curvature (cf. Corollary 2.3.2). (For the previous works concerning dormant Miura opers, we refer to [15] and [16].) On the other hand, according to a previous work by the author (cf. [15], Theorem A), dormant generic PGL₂-opers are in bijection with (pre-)Tango structures. A (pre-)Tango structure (cf. Definition 3.1.1 (i) or Definition 3.1.3 for the precise definition) is defined as a certain line bundle on an algebraic curve (or equivalently, a certain connection on the sheaf of 1-forms) and brings various sorts of pathological phenomena in positive characteristic (e.g., a counter-example to the Kodaira vanishing theorem). By combining the results mentioned above, we obtain the following assertion, which is the main result of the present paper.

**Theorem A.**

Let $l$ be a nonnegative integer and denote by $C^{l+1}(\mathbb{P})$ (cf. (18)) the set of ordered collections of $l+1$ distinct closed points of $\text{Spec}(k[x]) = \mathbb{P} \setminus \{\infty\}$. Given each $\mathbf{z} := (z_1, \cdots, z_{l+1}) \in C^{l+1}(\mathbb{P})$, we write $\text{Tang}(\mathbb{P}; \mathbf{z}; 1^{l+1})$ for the set of pre-Tango structures on the pointed projective line $(\mathbb{P}, \mathbf{z})$ of monodromy $1^{l+1} := (1, \cdots, 1) \in k^{l+1}$ (cf. (49) and (59)). Also, write $\text{PGL}_{2, BA}^{\infty, 0, \alpha^{l+1}}$ for the subset of $C^{l+1}(\mathbb{P})$ consisting of elements $\mathbf{z} := (z_1, \cdots, z_{l+1})$ satisfying the Bethe ansatz equations of the form (72), i.e., the equations (7) (divided by 2) in the case where $g = sl_2$, $r = 0$, and $\langle \alpha_j, \alpha_s \rangle = 2$ for any $j, s$. Then, there exists a canonical bijection

$$\prod_{\mathbf{z} \in C^{l+1}(\mathbb{P})} \text{Tang}(\mathbb{P}; \mathbf{z}; 1^{l+1}) \simeq \text{PGL}_{2, BA}^{\infty, 0, \alpha^{l+1}}.$$  

In particular, $(\mathbb{P}, \mathbf{z})$ admits a pre-Tango structure of monodromy $1^{l+1}$ if and only if the equality $f''(x) = 0$ holds, where $f(x) := \prod_{i=1}^{l+1} (x - z_i)$ (cf. Remark 2.3.3). Moreover, if $(\mathbb{P}, \mathbf{z})$ admits
a pre-Tango structure of monodromy $1^{lp+1}$, then it is uniquely determined and expressed as
$$\partial_x + \sum_{i=1}^{lp+1} \frac{1}{x-z_i} \text{ (under the identification } \Omega_{\mathbb{P}^1} \xrightarrow{\log/k} \mathcal{O}_{\mathbb{P}^1\setminus\{\infty\}} \text{ given by } dx \mapsto 1).$$

As a corollary of the above theorem, we can construct infinitely many examples of Tango curves, i.e., algebraic curves admitting a Tango structure (cf. Definition 3.1.1 (iii)). To the author’s knowledge, there are very few examples of Tango curves described explicitly at the present time. The major ones given by M. Raynaud (cf. [10], Example, or [6], Example 1.3) may be thought of as special cases of our construction. In particular, we provide other new Tango curves. Our result is described as follows.

**Theorem B (cf. Theorem 3.4.1).**

Let $(a, b)$ be a pair of positive integers with $\gcd(a, bp-1) = 1$. Denote by $Y$ the desingularization of the plane curve defined by the equation

$$y^{bp-1} = h(x),$$

where $h(x) \in k[x]$ denotes a monic polynomial of degree $ap$ satisfying that $\gcd(h(x), h'(x)) = 1$ and $h''(x) = 0$, and $(x, y)$ are an inhomogenous coordinate of the projective plane. Then, $Y$ is a Tango curve.

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1. **Opers and Miura opers**

In this section, we recall the definitions of an oper and a (generic) Miura oper.

1.1. **Algebraic groups and Lie algebras.**

Throughout the present paper, let us fix a prime $p$ and an algebraically closed field $k$ of characteristic $p$. Also, let $G$ be a connected simple algebraic group over $k$ of adjoint type satisfying the condition $(\ast)_G$ described as follows.

$$(\ast)_G : G \text{ is either equal to } \text{PGL}_n \text{ with } 1 < n < p \text{ or satisfies the inequality } p > 2 \cdot h, \text{ where } h \text{ denotes the Coxeter number of } G.$$  
Next, let us fix a maximal torus $T$ of $G$, a Borel subgroup $B$ of $G$ containing $T$. Write $N = [B, B]$, i.e., the unipotent radical of $B$, and write $W$ for the Weyl group of $(G, T)$. Denote by $g, b, n, \text{ and } t$ the Lie algebras of $G, B, N, \text{ and } T$ respectively (hence $t, n \subseteq b \subseteq g$). Denote by $\Phi^+$ the set of positive roots in $B$ with respect to $T$ and by $\Phi^-$ the set of negative roots. Also, denote by $\Gamma (\subseteq \Phi^+)$ the set of simple positive roots. In what follows, given each character $\varphi : T \to \mathbb{G}_m$ (resp., each cocharacter $\check{\varphi} : \mathbb{G}_m \to T$), we shall use, by abuse of
notation, the same notation \( \varphi \) (resp., \( \check{\varphi} \)) to denote its differential \( d\varphi \in \mathfrak{t}' \) (resp., \( d\check{\varphi} \in \mathfrak{t} \)). For each \( \alpha \in \Phi^+ \cup \Phi^- \), we write

\[
\mathfrak{g}^\alpha := \{ x \in \mathfrak{g} \mid \text{ad}(t)(x) = \alpha(t) \cdot x \text{ for all } t \in T \}.
\]

Then, \( \mathfrak{g} \) admits a canonical decomposition

\[
\mathfrak{g} = \mathfrak{t} \oplus \left( \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^\alpha \right) \oplus \left( \bigoplus_{\beta \in \Phi^-} \mathfrak{g}^\beta \right)
\]

(which restricts to a decomposition \( \mathfrak{b} = \mathfrak{t} \oplus \left( \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^\alpha \right) \)). In particular, each \( \mathfrak{g}^{-\alpha} \) (\( \alpha \in \Gamma \)) may be thought of as a subspace of \( \mathfrak{g} / \mathfrak{b} \) closed under the adjoint \( B \)-action. We shall write \( \check{\rho} \) for the cocharacter \( \mathbb{G}_m \to T \) (as well as its differential) defined as the sum \( \sum_{\alpha \in \Gamma} \check{\omega}_\alpha \), where each \( \check{\omega}_\alpha \) (\( \alpha \in \Gamma \)) denotes the fundamental coweight of \( \alpha \). In the following discussion, we shall fix a generator \( f_\alpha \) of \( \mathfrak{g}^{-\alpha} \) for each \( \alpha \in \Gamma \). Hence, we obtain \( p_{-1} := \sum_{\alpha \in \Gamma} f_\alpha \).

Denote by \( \mathfrak{t}_{\text{reg}} \) the set of regular elements in \( \mathfrak{t} \). It follows from [5], Chap. VI, Theorem 7.2, that

\[
\mathfrak{t}_{\text{reg}} = \left\{ \check{\lambda} \in \mathfrak{t} \mid \alpha(\check{\lambda}) \neq 0 \text{ for any root } \alpha \in \Phi^+ \cup \Phi^- \right\}.
\]

Also, denote by \( \mathfrak{t}^F_{\text{reg}} \) the set of \( F \)-invariant elements in \( \mathfrak{t}_{\text{reg}} \), where \( F \) denotes the absolute Frobenius morphism of \( \mathfrak{t}_{\text{reg}} \) viewed as a \( k \)-scheme.

### 1.2. Opers.

In this subsection, we recall the definitions of an oper. Let \( X \) be a connected proper smooth curve over \( k \). Given a reduced effective divisor \( D \) on \( X \), one can equip \( X \) with a log structure determined by \( D \) in the usual manner; we shall denote the resulting log scheme by \( X^{D,\log} \). (In particular, if \( D = 0 \), then \( X^{D,\log} \) coincides with \( X \).) Let \( G_0 \) be some algebraic group \( G_0 \) over \( k \), \( \mathcal{E} \) a (right) \( G_0 \)-torsor over \( X \). Given a \( k \)-vector space \( \mathfrak{h} \) equipped with a (left) \( G_0 \)-action, we shall write \( \mathfrak{h}_\mathcal{E} \) for the vector bundle on \( X \) associated with the relative affine space \( \mathcal{E} \times_{G_0} \mathfrak{h} := (\mathcal{E} \times_k \mathfrak{h})/G_0 \) over \( X \). We shall refer to any logarithmic connection on \( \mathcal{E} \) (with respect to the log structure of \( X^{D,\log} \)) as a \( D \)-log connection on \( \mathcal{E} \).

Let \( X \), \( D \) be as above, and \( G \) as in the previous subsection. Also, let \( \mathcal{E} \) be a (right) \( G \)-torsor over \( X \). The adjoint \( G \)-action on \( \mathfrak{g} \) gives rise to a vector bundle \( \mathfrak{g}_\mathcal{E} \). Next, suppose that we are given a \( D \)-log connection \( \nabla \) on \( \mathcal{E} \) and a \( B \)-reduction \( \mathcal{E}_B \) of \( \mathcal{E} \) (i.e., a \( B \)-torsor \( \mathcal{E}_B \) over \( X \) together with an isomorphism \( \mathcal{E}_B \times^B G \xrightarrow{\sim} \mathcal{E} \) of \( G \)-torsors). Let us choose, locally on \( X \), a \( D \)-log connection \( \nabla' \) on \( \mathcal{E} \) preserving \( \mathcal{E}_B \), and take the difference \( \nabla - \nabla' \), which forms a section of \( \mathfrak{g}_{\mathcal{E}_B} \otimes \Omega_{X^{D,\log}/k} \) \((\cong \mathfrak{g}_\mathcal{E} \otimes \Omega_{X^{D,\log}/k})\) determined by this section via projection is independent of the choice \( \nabla' \). Hence, these sections may be glued together to a global section of \( \left( \mathfrak{g}/\mathfrak{b} \right)_{\mathcal{E}_B} \otimes \Omega_{X^{D,\log}/k} \), denoted by \( \nabla/\mathcal{E}_B \).

By a \( G \)-oper on \( X^{D,\log} \) we mean a triple \( \mathcal{E}^\bullet := (\mathcal{E}, \nabla, \mathcal{E}_B) \), where \( \mathcal{E}, \nabla, \) and \( \mathcal{E}_B \) are as above such that the section \( \nabla/\mathcal{E}_B \) lies in the submodule \( \left( \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha} \right)_{\mathcal{E}_B} \otimes \Omega_{X^{D,\log}/k} \) and its image in \( \mathfrak{g}_{\mathcal{E}_B}^{\mathcal{E}_B} \otimes \Omega_{X^{D,\log}/k} \) (for each \( \beta \in \Gamma \)) via the projection \( \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha} \to \mathfrak{g}^{-\beta} \) specifies a nowhere vanishing section. Let \( \mathcal{E}_1^\bullet := (\mathcal{E}_1, \nabla_1, \mathcal{E}_{B,1}) \), \( \mathcal{E}_2^\bullet := (\mathcal{E}_2, \nabla_2, \mathcal{E}_{B,2}) \) be \( G \)-opers on \( X^{D,\log} \). Then, we refer to an isomorphism \( \mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2 \) compatible with both the respective connections and
B-reductions as an isomorphism of $G$-opers (from $\mathcal{E}_1^\bullet$ to $\mathcal{E}_2^\bullet$). Denote by

$$G\text{-Op}(X^{D\text{-log}})$$

the set of isomorphism classes of $G$-opers on $X^{D\text{-log}}$.

Next, let $\hat{O}$ be a complete discrete valuation ring over $k$, and write $\mathbb{D} := \text{Spec}(\hat{O})$ (i.e., the formal disc). The closed point of $\mathbb{D}$, denoted by $x_0$, defines a reduced effective divisor, and hence, defines a log structure on $\mathbb{D}$; we denote the resulting log scheme by $\mathbb{D}^{\log}$. As in the case of the entire $X^{D\text{-log}}$ discussed above, we have the definition of a $G$-oper on $\mathbb{D}$ (resp., $\mathbb{D}^{\log}$). Denote by

$$G\text{-Op}(\mathbb{D})$$

the set of isomorphism classes of $G$-opers on $\mathbb{D}$.

Let $\mathcal{E}^\bullet := (\mathcal{E}, \nabla, \mathcal{E}_B)$ be a $G$-oper on $\mathbb{D}^{\log}$. After choosing a uniformizer $t$ of $\hat{O}$ (which gives an isomorphism $k[[t]] \xrightarrow{\sim} \hat{O}$) and applying a suitable gauge transformation, $\nabla$ may be expressed as a log connection on the trivial $G$-torsor $\mathbb{D} \times G$ of the form

$$\partial_t + \frac{1}{t}(p_{-1} + u) + u(t)$$

for some $u \in \mathfrak{b}$, $u(t) \in \mathfrak{b}[[t]]$ (:= $\mathfrak{b}(k[[t]])$), where $\partial_t := \frac{d}{dt}$. In particular, the monodromy, in the sense of [13], Definition 1.6.1, of $\nabla$ at $x_0$ (with respect to this expression) is $p_{-1} + u$. Here, let

$$c := \mathfrak{g}/G$$

denote the GIT quotient of $\mathfrak{g}$ by the adjoint $G$-action and let

$$\chi : \mathfrak{g} \to c$$

denote the natural quotient. Then, it is verified that the element $\rho := \chi(p_{-1} + u) \in c(k)$ is independent of both the choice of $t$ and the expression (9) of $\nabla$. In this situation, we shall say that $\mathcal{E}^\bullet$ is of radius $\rho$. For each $\rho \in c(k)$, we shall write

$$G\text{-Op}(\mathbb{D}^{\log}, \rho)$$

for the set of isomorphism classes of $G$-opers on $\mathbb{D}^{\log}$ of radius $\rho$. As discussed at the end of [3], §9.1, there exists a canonical injection

$$G\text{-Op}(\mathbb{D}) \hookrightarrow G\text{-Op}(\mathbb{D}^{\log}; \chi(-\bar{\rho})),$$

obtained by applying suitable gauge transformations.

Remark 1.2.1.
Since the condition $(\ast)_G$ holds, it follows from a Chevalley’s theorem (cf. [7], Theorem 1.1.1; [5], Chap. VI, Theorem 8.2) that the composite $t \mapsto \mathfrak{g} \xrightarrow{\chi} c$ induces an isomorphism $t/W \xrightarrow{\sim} c$. In particular, an element $\lambda \in t$ satisfies the equality $\chi(\lambda) = \chi(-\bar{\rho})$ if and only if $\lambda = w(-\bar{\rho})$ for some $w \in W$.

Also, let $X$ be as before and $x := (x_i)_{i=1}^r$ ($r \geq 1$) be an ordered collection of distinct closed points of $X$. We shall write $D_x := \sum_{i=1}^r [x_i]$, where $[x_i]$ ($i = 1, \ldots, r$) denotes the reduced effective divisor on $X$ determined by $x_i$. For each $i \in \{1, \ldots, r\}$, denote by $\mathbb{D}_{x_i}$ the formal disc in $X$ around $x_i$. Let $\rho := (\rho_i)_{i=1}^r \in c(k)^r$. We shall say that a $G$-oper on $X^{D_{x}-\log}$ is of radii $\rho$.
if, for each \( i \in \{1, \cdots, r\} \), the \( G \)-oper on \( \mathbb{D}_{x_i} \) defined as its restriction is of radius \( \rho_i \). Denote by

\[
G\text{-Op}(X^{Dx^{\log}}; \rho)
\]

the subset of \( G\text{-Op}(X^{Dx^{\log}}) \) classifying \( G \)-opers of radii \( \rho \).

1.3. Generic Miura opers.

Next, let us recall the notion of a (generic) Miura oper on a curve with log structure (cf. \cite{15}, Definition 3.2.1). Let \( X \) be a connected proper smooth curve over \( k \) and \( D \) a reduced effective divisor on \( X \). A Miura \( G \)-oper on \( X \) (resp., on \( X^{D^{\log}} \)) is a quadruple \( \mathcal{E}^\bullet := (\mathcal{E}, \nabla, \mathcal{E}_B, \mathcal{E}_{B'}) \), where \( (\mathcal{E}, \nabla, \mathcal{E}_B) \) is a \( G \)-oper on \( X \) (resp., on \( X^{D^{\log}} \)) and \( \mathcal{E}_{B'} \) is another \( B \)-reduction of \( \mathcal{E} \) horizontal with respect to \( \nabla \). Also, for two Miura \( G \)-opers \( \mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet \), we shall refer to an isomorphism of \( G \)-opers from \( \mathcal{E}_1^\bullet \) to \( \mathcal{E}_2^\bullet \) compatible with the respective second \( B \)-reductions as an isomorphism of Miura \( G \)-opers (from \( \mathcal{E}_1^\bullet \) to \( \mathcal{E}_2^\bullet \)).

Let \( \mathcal{E}^\bullet := (\mathcal{E}, \nabla, \mathcal{E}_B, \mathcal{E}_{B'}) \) be a Miura \( G \)-oper on either \( X \) or \( X^{D^{\log}} \). By twisting the flag variety \( G/B \) by the \( B \)-torsor \( \mathcal{E}_B \), we obtain a proper scheme \( (G/B)_{\mathcal{E}_B} := \mathcal{E}_B \times^B (G/B) \) over \( X \). Here, denote by \( w_0 \) the longest element of \( W \). The Bruhat decomposition \( G = \bigsqcup_{w \in W} BwB \) gives rise to a decomposition

\[
(G/B)_{\mathcal{E}_B} = \bigsqcup_{w \in W} S_{\mathcal{E}_B,w}
\]

of \( (G/B)_{\mathcal{E}_B} \), where each \( S_{\mathcal{E}_B,w} \) denotes the \( \mathcal{E}_B \)-twist of \( Bw_0wB \); i.e., \( S_{\mathcal{E}_B,w} := \mathcal{E}_B \times^B (Bw_0wB) \). Note that the \( B \)-reduction \( \mathcal{E}_{B'} \) determines a section \( \sigma_{\mathcal{E}_B,\mathcal{E}_{B'}} : X \rightarrow (G/B)_{\mathcal{E}_B} \) of the natural projection \( (G/B)_{\mathcal{E}_B} \rightarrow X \). Given an element \( w \) of \( W \) and a point \( x \) of \( X \), we shall say that \( \mathcal{E}_B \) and \( \mathcal{E}_{B'} \) are in relative position \( w \) at \( x \) if \( \sigma_{\mathcal{E}_B,\mathcal{E}_{B'}}(x) \) belongs to \( S_{\mathcal{E}_B,w} \). In particular, if \( \sigma_{\mathcal{E}_B,\mathcal{E}_{B'}}(x) \) belongs to \( S_{\mathcal{E}_B,1} \), then we shall say that \( \mathcal{E}_B \) and \( \mathcal{E}_{B'} \) are in generic position at \( x \). Moreover, we shall say that a Miura \( G \)-oper \( \mathcal{E}^\bullet := (\mathcal{E}, \nabla, \mathcal{E}_B, \mathcal{E}_{B'}) \) is generic if \( \mathcal{E}_B \) and \( \mathcal{E}_{B'} \) are in generic position at any point of \( X \). We also have the various notions recalled here in the case where the entire \( X \) or \( X^{D^{\log}} \) is replaced by either \( \mathbb{D}, \mathbb{D}^\times \), or \( \mathbb{D}^{\log} \).

1.4. Exponent of Miura opers.

Let \( \mathcal{E}^\bullet := (\mathcal{E}, \nabla, \mathcal{E}_B, \mathcal{E}_{B'}) \) be a generic Miura \( G \)-oper on \( \mathbb{D}^{\log} \). After choosing a uniformizer \( t \) of \( \hat{\mathcal{O}} \) and applying a gauge transformation, \( \nabla \) may be expressed as a log connection on the trivial \( G \)-torsor \( \mathbb{D} \times G \) of the form

\[
\partial_t + \frac{1}{t} (p_{-1} + \lambda) + u(t)
\]

for some \( \lambda \in \mathfrak{t}, u(t) \in t[[t]] \) (cf. \cite{15}, Proposition 3.4.3, for the case where the underlying curve is globally defined and in positive characteristic). It is verified that \( \lambda \) is independent of both the choice of \( t \) and the expression \( \frac{1}{t} (p_{-1} + \lambda) \) of \( \nabla \). In this situation, we shall say that \( \mathcal{E}^\bullet \) is of exponent \( \lambda \). If a generic Miura \( G \)-oper \( \mathcal{E}^\bullet := (\mathcal{E}, \nabla, \mathcal{E}_B, \mathcal{E}_{B'}) \) on \( \mathbb{D}^{\log} \) is of exponent \( \lambda \), then the underlying \( G \)-oper \( (\mathcal{E}, \nabla, \mathcal{E}_B) \) is of radius \( \chi(p_{-1} + \lambda) \in \mathfrak{c}(k) \).
Remark 1.4.1.
Let \( \lambda \) and \( w \) be elements of \( t_{\text{reg}} \) and \( W \) respectively. Then, one may verify immediately that the equality \( \chi(w(\lambda)) = \chi(\hat{\lambda}) = \chi(p_{-1} + \lambda) \) holds (cf. Remark 1.2.1). In particular, if \( \hat{E}^\bullet \) is a generic Miura \( G \)-oper on \( D^{\log} \) of exponent \( \hat{\lambda} \), then the radius of its underlying \( G \)-oper coincides with \( \chi(\hat{\lambda}) \).

Remark 1.4.2.
By applying the gauge transformation by \( \hat{\rho} \circ t \in T(k[[t]]) \), we obtain a bijection from the set of isomorphism classes of Miura \( G \)-opers on \( D^{\log} \) to the set of isomorphism classes of Miura \( G \)-opers on \( D^{\log} \) of exponent \( -\hat{\rho} \). This bijection is compatible (under the equality \( \chi(-\hat{\rho}) = \chi(p_{-1} - \hat{\rho}) \) obtained in the above Remark) with the injection (13) via forgetting the data of second \( B \)-reductions of Miura \( G \)-opers.

Next, let \( x := (x_i)_{i=1}^r \) be an ordered collection of distinct closed points of \( X \), and let \( \hat{\lambda} := (\hat{\lambda}_i)_{i=1}^r \). We shall say that a Miura \( G \)-oper \( \hat{E}^\bullet := (E, \nabla, E_B, E_B') \) on \( X^{D_x} \) is of exponents \( \hat{\lambda} \) if, for each \( i \in \{1, \cdots, r\} \), the Miura \( G \)-oper on \( D^{\log} \) defined as its restriction is generic and of exponent \( \hat{\lambda}_i \). Denote by

\[
G\text{-MOp}(X^{D_x} \log; \hat{\lambda})_{\text{gen}}
\]

the set of isomorphism classes of generic Miura \( G \)-opers on \( X^{D_x} \log \) of exponents \( \hat{\lambda} \).

For each positive integer \( m \), we shall write

\[
C^m(x)
\]

for the set of ordered collections of \( m \) distinct closed points in \( X \setminus \{x_1, \cdots, x_r\} \). Given each \( w := (w_1, \cdots, w_m) \in W^m \), we shall write

\[
G\text{-MOp}(X^{D_x} \log; \hat{\lambda})_{\text{gen}, +w} := \prod_{z \in C^m(x)} G\text{-MOp}(X^{D(x,z)} \log; (\hat{\lambda}, w(\hat{\rho})))_{\text{gen}},
\]

where \( w(\hat{\rho}) := (w_i(-\hat{\rho}))_{i=1}^m \in t^m \).

For each \( z \in C^m(x) \), the injection (13) induces an injection

\[
G\text{-Op}(X^{D_x} \log, \chi(\hat{\lambda})) \rightarrow G\text{-Op}(X^{D(x,z)} \log; (\chi(\hat{\lambda}), \chi(\hat{\rho})^m)),
\]

where \( \chi(\hat{\lambda}) := (\chi(\hat{\lambda}_1), \cdots, \chi(\hat{\lambda}_r)) \in c(k)^r \) and \( \chi(\hat{\rho})^m := (\chi(-\hat{\rho}), \cdots, \chi(-\hat{\rho})) \in c(k)^m \). Given each \( w \) as above, we denote by

\[
G\text{-MOp}(X^{D_x} \log; \hat{\lambda})_{\text{triv}, +z;w}
\]

the inverse image of the image of (20) via the forgetting map

\[
G\text{-MOp}(X^{D(x,z)} \log; (\hat{\lambda}, w(-\hat{\rho})))_{\text{gen}} \rightarrow G\text{-Op}(X^{D(x,z)} \log; (\chi(\hat{\lambda}), \chi(\hat{\rho})^m))
\]

(cf. Remark 1.4.1). That is to say, the following square diagram is commutative and cartesian:

\[
\begin{array}{ccc}
G\text{-MOp}(X^{D_x} \log; \hat{\lambda})_{\text{triv} +z;w} & \rightarrow & G\text{-MOp}(X^{D(x,z)} \log; (\hat{\lambda}, w(\hat{\rho})))_{\text{gen}} \\
\downarrow & & \downarrow \\
G\text{-Op}(X^{D_x} \log, \chi(\hat{\lambda})) & \rightarrow & G\text{-Op}(X^{D(x,z)} \log; (\chi(\hat{\lambda}), \chi(\hat{\rho})^m)).
\end{array}
\]
If we write
\[ G-MOp(X^{x_{\log}}; \lambda)_{\text{triv.} + w} := \prod_{z \in C^m(x)} G-MOp(X^{Dx_{\log}}; \lambda)_{\text{triv.} + (z,w)}, \]
then the upper horizontal arrow in (23) in the case of each \( z \in C^m(x) \) gives an inclusion
\[ G-MOp(X^{Dx_{\log}}; \lambda)_{\text{triv.} + w} \hookrightarrow G-MOp(X^{Dx_{\log}}; \hat{\lambda})_{\text{gen.} + w}. \]
We shall consider \( G-MOp(X^{Dx_{\log}}; \lambda)_{\text{triv.} + w} \) as a subset of \( G-MOp(X^{Dx_{\log}}; \hat{\lambda})_{\text{gen.} + w} \) via this injection.

2. DORMANT MIURA OPERS AND THE BETHE ANSATZ EQUATIONS

In this section, we prove a partial analogue (cf. Theorem 2.2.3) of the correspondence given in [1] or [2] between solutions to the Bethe ansatz equations and certain Miura opers with trivial monodromy on the complex projective line. Moreover, we show (cf. Corollary 2.3.2) that any solution to the Bethe ansatz equations for \( g = \mathfrak{sl}_2 \) of a certain type turns out to come from dormant generic Miura PGL\(_2\)-opers, i.e., generic Miura PGL\(_2\)-opers with vanishing \( p \)-curvature.

2.1. DORMANT MIURA OPERS.

We shall show (cf. Proposition 2.1.2) that each dormant generic Miura oper specifies a Miura oper classified by the set \( G-MOp(X^{Dx_{\log}}; \lambda)_{\text{triv.} + (z,w)} \) (cf. (21)) defined in the previous section.

First, recall from [13], Definition 3.8.1, that a Miura \( G \)-oper \( \mathfrak{g}^{\mathfrak{d}} := (\mathfrak{e}, \nabla, \mathcal{E}_B, \mathcal{E}'_B) \) is dormant if \( \nabla \) has vanishing \( p \)-curvature (cf. e.g., [4], § 5.0, or [15], § 1.6, for the definition of \( p \)-curvature). Then, we verify the following assertion.

Proposition 2.1.1.

Let \( \lambda \) be an element of \( \mathfrak{t} \) such that there exists a dormant generic Miura \( G \)-oper \( \mathfrak{g}^{\mathfrak{d}} := (\mathfrak{e}, \nabla, \mathcal{E}_B, \mathcal{E}'_B) \) on \( \mathbb{D}^\log \) of exponent \( \lambda \). Then, \( \hat{\lambda} \) lies in \( \mathfrak{t}_{\text{reg}}^{\mathfrak{F}} \).

Proof. Denote by \( \mu \in \mathfrak{g} \) the monodromy (cf. [13], Definition 1.6.1) of \( \nabla \) at the closed point \( x_0 \), and consider a Jordan decomposition \( \mu = \mu_s + \mu_n \) with \( \mu_s \) semisimple and \( \mu_n \) nilpotent. Denote by \( \text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}) \) the adjoint representation of \( \mathfrak{g} \), which is injective and compatible with the respective restricted structures, i.e., \( p \)-power operations. One may find an isomorphism \( \text{End}(\mathfrak{g}) \cong \mathfrak{g}_{\text{dim}(\mathfrak{g})} \) of restricted Lie algebras which sends \( \alpha(\text{ad}(\mu)) = \alpha(\text{ad}(\mu_s) + \text{ad}(\mu_n)) \) to a Jordan normal form. Namely, \( \alpha(\text{ad}(\mu_s)) \) is diagonal and every entry of \( \alpha(\text{ad}(\mu_n)) \) except the superdiagonal is 0. Let us observe the following sequence of equalities:
\[ \alpha(\text{ad}(\mu_s)) + \alpha(\text{ad}(\mu_n)) = \alpha(\text{ad}(\mu_s + \mu_n)) = \alpha((\mu_s + [\mu_n])^p) \]
\[ = \alpha(\text{ad}(\mu_s + \mu_n))^p = (\alpha(\text{ad}(\mu_s)) + \alpha(\text{ad}(\mu_n)))^p, \]
where the second equality follows from the assumption that \( \nabla \) has vanishing \( p \)-curvature and \((-)^p \) denotes the \( p \)-power operation on \( \mathfrak{g} \) (cf. [13], § 3.2, Remark 3.2.2). By an explicit computation of \( (\alpha(\text{ad}(\mu_s)) + \alpha(\text{ad}(\mu_n)))^p \), (26) implies that \( \alpha(\text{ad}(\mu_n)) = 0 \) (hence \( \mu_n = 0 \)), namely,
μ is conjugate to some v ∈ t. On the other hand, μ is, by definition, conjugate to p_{-1} + \hat{λ}. It follows that p_{-1} + \hat{λ} is conjugate to v, and hence, \hat{λ} = w(v) for some w ∈ W. Since p_{-1} + \hat{λ}, as well as v, is regular, \hat{λ} (= w(v)) turns out to be regular. Moreover, by the equality \mu_n = 0, \(26\) reads the equality \( \alpha(\text{ad}(\mu)) = \alpha(\text{ad}(\mu))^p \). This implies the equality \( v^{[p]} = v \), or equivalently, \( v ∈ t_{\text{reg}} \). This completes the proof of the assertion. \(\square\)

Let \( X \) be a connected proper smooth curve over \( k \), \( x := (x_i)_{i=1}^r \) an ordered collection of distinct closed points of \( X \). Also, let \( \hat{λ} := (\hat{λ}_i)_{i=1}^r ∈ (t_{\text{reg}})^r \), \( w ∈ W^m \). Given each \( z ∈ C^m(x) \), we write

\[
G\text{-MOp}(X^{D(x,z)\text{-log}}; (\hat{λ}, w(-\hat{ρ})))_{\text{gen}}^{z_{\text{ax}}...}
\]

for the subset of \( G\text{-MOp}(X^{D(x,z)\text{-log}}; (\hat{λ}, w(-\hat{ρ})))_{\text{gen}} \) consisting of dormant generic Miura \( G \)-opers. (According to the above proposition, this subset is empty unless \( \hat{λ} ∈ (t_{\text{reg}})^r \).) Also, we write

\[
G\text{-MOp}(X^{D(x\text{-log}}; \hat{λ})_{\text{gen},+w}^{z_{\text{ax}}...} := \prod_{z ∈ C^m(x)} G\text{-MOp}(X^{D(x,z)\text{-log}}; (\hat{λ}, w(-\hat{ρ})))_{\text{gen}}^{z_{\text{ax}}...} .
\]

Then, the following assertion holds.

**Proposition 2.1.2.**

Let us keep the above notation. Then, the set \( G\text{-MOp}(X^{D(x,z)\text{-log}}; (\hat{λ}, w(-\hat{ρ})))_{\text{gen}}^{z_{\text{ax}}...} \) (for each \( z ∈ C^m(x) \)) is contained in \( G\text{-MOp}(X^{D(x\text{-log}}; \hat{λ})_{\text{triv},+(z,w)} \). In particular, we have an inclusion

\[
G\text{-MOp}(X^{D(x\text{-log}}; \hat{λ})_{\text{gen},+w}^{z_{\text{ax}}...} \hookrightarrow G\text{-MOp}(X^{D(x\text{-log}}; \hat{λ})_{\text{triv},+(z,w)} .
\]

**Proof.** Let \( w ∈ W \) and \( \hat{E}^{\bullet} := (E, \nabla, E_B, E_B^{\bullet}) \) a dormant generic Miura \( G \)-oper on \( D^{\text{log}} \) of exponent \( w(-\hat{ρ}) \). Let us choose a uniformizer \( t \) of \( O \) (inducing \( O \rightarrow k[[t]] \)). To complete the proof, it suffices to prove that \( (E, \nabla, E_B) \) becomes a Miura \( G \)-oper on \( D \) after a gauge transformation by some element of \( B(k((t))) \). Here, notice that by the assumption \( (\ast)_G \), one may obtain the exponential map \( : n → N \) given by \( [13] \), §1.4, (55). By means of this, one may apply an argument similar to the argument in the proof of \( [3] \), Proposition 9.2.1, to our positive characteristic case. Hence, after a gauge transformation by some element of \( B(k((t))) \), \( \nabla \) may be expressed as \( \hat{λ}l + p_{-1} + v(t) + \frac{v(t)}{t} \) for some \( v(t) ∈ b[[t]] \) and \( v ∈ n \). The mod \( t \) reduction of the \( p \)-curvature of \( \nabla \) is given by \( v^{[p]} - v \), which is equal to 0 because of the dormancy condition on \( \hat{E}^{\bullet} \). But, since \( v ∈ n \), we have \( v^{[p]} = 0 \), which implies that \( v = 0 \). Therefore, \( (E, \nabla, E_B) \) forms a \( G \)-oper on \( D \), as desired. \(\square\)

### 2.2. The Bethe ansatz equations I.

In this section, we consider the case where \( X \) is taken to be the projective line over \( k \), denoted by \( \mathbb{P} \). Denote by \( x \) the natural coordinate of \( \mathbb{P} \), i.e., \( \mathbb{P} \setminus \{∞\} = \text{Spec}(k[x]) \). Fix integers \( r, m \) with \( r ≥ 0, m ≥ 1 \). Let \( x := (x_1, \cdots, x_r, x_{r+1}) \) be an ordered collection of distinct closed points of \( \mathbb{P} \) with \( x_{r+1} = \infty \) and \( \hat{λ} := (\hat{λ}_1, \cdots, \hat{λ}_r, \hat{λ}_{r+1}) ∈ (t_{\text{reg}})^{r+1}, \hat{λ}' := (\hat{λ}_1', \cdots, \hat{λ}_m') ∈ (t_{\text{reg}})^m. \)
Proposition 2.2.1.
Let $z \in C^m(x)$. Then, the set $G\text{-MOp}(\mathbb{P}^{D(x,z)}_{\log}; (\lambda, \lambda'))$ is nonempty if and only if the equality

$$\sum_{i=1}^{r+1} (\lambda_i + \rho) + \sum_{j=1}^{m} (\lambda'_j + \rho) = 2\rho$$

holds. Moreover, if the equality (30) holds, then $G\text{-MOp}(\mathbb{P}^{D(x,z)}_{\log}; (\lambda, \lambda'))$ consists of a single element.

Proof. Let us write $\mathcal{E}^{\dagger}_{T,\log} := \Omega_{\mathbb{P}^{D(x,z)}_{\log}/k} \times_{G_{m,\rho}} T$ and $\mathcal{E}^{\dagger}_{T} := \Omega_{\mathbb{P}/k} \times_{G_{m,\rho}} T$, where for each line bundle $\mathcal{L}$ we denote by $\mathcal{L}^x$ the $\mathbb{G}_m$-torsor corresponding to $\mathcal{L}$. Recall from [15], Proposition 3.7.1, that there exists a canonical bijection

$$G\text{-Conn}(\mathbb{P}^{D(x,z)}_{\log}; (\lambda, \lambda')) \sim G\text{-MOp}(\mathbb{P}^{D(x,z)}_{\log}; (\lambda, \lambda'))_{\text{gen}},$$

where the left-hand side denotes the set of $D(x,z)$-log connections on $\mathcal{E}^{\dagger}_{T,\log}$ of monodromy $(\lambda, \lambda')$ (in the sense of [13], Definition 1.6.1).

First, suppose that there exists a $D(x,z)$-log connection $\nabla$ classified by $G\text{-Conn}(\mathbb{P}^{D(x,z)}_{\log}; (\lambda, \lambda'))$. One may find a unique connection $\nabla'$ on $\mathcal{E}^{\dagger}_{T}$ whose restriction to $\mathbb{P}^\circ := \mathbb{P}\setminus\{x_1, \ldots, x_{r+1}, z_1, \ldots, z_m\}$ is isomorphic to $\nabla|_{\mathbb{P}^\circ}$. $\nabla'$ is verified to have monodromy $(\lambda_1 + \rho, \ldots, \lambda_{r+1} + \rho, \lambda'_1 + \rho, \cdots, \lambda'_m + \rho)$, and must be expressed as

$$\partial_x + \sum_{i=1}^{r} \frac{\lambda_i + \rho}{x - x_i} + \sum_{j=1}^{m} \frac{\lambda'_j + \rho}{x - z_j},$$

where $\partial_x = \frac{d}{dx}$, via the trivialization of the $T$-torsor $\mathcal{E}^{\dagger}_{T}|_{\mathbb{P}\setminus\{\infty\}}$ given by $\Omega_{\mathbb{P}/k}|_{\mathbb{P}\setminus\{\infty\}} \sim \mathcal{O}_{\mathbb{P}\setminus\{\infty\}}; dx \mapsto 1$ (cf. [1], § 3.1, (3.1)). In particular, $G\text{-Conn}(\mathbb{P}^{D(x,z)}_{\log}; (\lambda, \lambda'))$ consists of a single connection. Moreover, according to the discussion in loc. cit., we have the equality

$$2\rho - \sum_{i=1}^{r} (\lambda_i + \rho) - \sum_{j=1}^{m} (\lambda'_j + \rho) = \lambda_{r+1} + \rho,$$

which is equivalent to the equality (30).

Conversely, if the equality (30) holds, then we obtain a unique connection $\nabla'$ on $\mathcal{E}^{\dagger}_{T}$ of the form (32). Moreover, there exists a unique $D(x,z)$-log connection $\nabla$ on $\mathcal{E}^{\dagger}_{T,\log}$ whose restriction to $\mathbb{P}^\circ$ is isomorphic to $\nabla|_{\mathbb{P}^\circ}$. Since $\nabla$ is verified to belong to $G\text{-Conn}(\mathbb{P}^{D(x,z)}_{\log}; (\lambda, \lambda'))$, the bijection (31) implies that $G\text{-MOp}(\mathbb{P}^{D(x,z)}_{\log}; (\lambda, \lambda'))$ is nonempty. This completes the proof of the assertion.

Let $\alpha := (\alpha_1, \cdots, \alpha_m) \in \Gamma^m$. For each $j = 1, \cdots, m$, we shall write $\alpha_j^W \in W$ for the simple reflection corresponding to $\alpha_j$. The coroot $\check{\alpha}_j$ corresponding to $\alpha_j$ satisfies the equality $\check{\alpha}_j = \alpha_j^W (-\rho) + \rho$. Write $-\lambda - \rho := (-\lambda_1 - \rho, \cdots, -\lambda_{r+1} - \rho)$ and $\alpha^W := (\alpha_1^W, \cdots, \alpha_m^W)$. Now,
suppose that the equality
\[ -\sum_{i=1}^{r+1} \tilde{\lambda}_i + \sum_{j=1}^{m} \tilde{\alpha}_j = 2\tilde{\rho} \]
(i.e., the equality \( (30) \) in the case where the data \((\tilde{\lambda}, \tilde{\lambda}')\) is taken to be \((-\tilde{\lambda} - \tilde{\rho}, \alpha^W(-\tilde{\rho}))\)) holds. The assignment from each \( z \in C^m(x) \) to a unique generic Miura \( G \)-oper on \( \mathbb{P}^{D(x,z)}_{\log} \) of exponents \((-\tilde{\lambda} - \tilde{\rho}, \alpha^W(-\tilde{\rho}))\) defines a bijection
\[ C^m(x) \cong G\text{-MOp}\left(\mathbb{P}^{D(x,z)}_{\log}; -\tilde{\lambda} - \tilde{\rho}\right)_{\text{gen} + \alpha^W}, \]
where the right-hand side contains \( G\text{-MOp}\left(\mathbb{P}^{D(x,z)}_{\log}; -\tilde{\lambda} - \tilde{\rho}\right)_{\text{triv} + \alpha^W} \) (cf. \( (25) \)). Now, denote by
\[ G\text{-BA}_{\tilde{\lambda}, \alpha}(\subseteq C^m(x)) \]
the set of elements \( z := (z_1, \ldots, z_m) \) of \( C^m(x) \) satisfying the equations
\[ \sum_{i=1}^{r} \frac{\langle \alpha_j, \tilde{\lambda}_i \rangle}{z_j - x_i} - \sum_{s \neq j} \frac{\langle \alpha_j, \tilde{\alpha}_s \rangle}{z_j - z_s} = 0 \quad (j = 1, \ldots, m) \]
(called the Bethe ansatz equations).

Remark 2.2.2.
We shall consider the case where \( G = \text{PGL}_2 \) and \( p \geq 3 \) (or equivalently, the condition \((*)_{\text{PGL}_2}\) is satisfied). Let us keep the above notation. We have that \( \tilde{\rho} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \), \( \Gamma = \{ \alpha \} \), and \( \tilde{\alpha} = 2\tilde{\rho} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). For each \( i \in \{1, \ldots, r\} \), let \( q_i \) be the element of \( k \) with \( \tilde{\lambda}_i = q_i \cdot \tilde{\rho} \). Then, \( (37) \) read the equations
\[ \sum_{i=1}^{r} \frac{q_i}{z_j - x_i} - \sum_{s \neq j} \frac{2}{z_j - z_s} = 0 \quad (j = 1, \ldots, m) \]
of values in \( k \).

A stronger assertion of the following theorem in the case where the base field \( k \) is replaced by \( \mathbb{C} \) was proved in \[1\], §3.2, Theorem 3.2. Also, Lemma \[2.2.4\] described later, which will be used in the proof of the following theorem, corresponds to \[1\], §2.6, Lemma 2.10. But, we describe it in the present paper in order to give its proof without any analytic argument unlike the proof in loc. cit..

Theorem 2.2.3.
(Recall the assumption that equality \( (37) \) holds.) The bijection \( (39) \) restricts to an injection
\[ G\text{-BA}_{\tilde{\lambda}, \alpha} \hookrightarrow G\text{-MOp}\left(\mathbb{P}^{D(x,z)}_{\log}; -\tilde{\lambda} - \tilde{\rho}\right)_{\text{triv} + \alpha^W}. \]
If, moreover, \( G = \text{PGL}_2 \) (and \( p \geq 3 \)), then this injection becomes a bijection. In particular, there exists a canonical injection
\[ \text{PGL}_2\text{-MOp}\left(\mathbb{P}^{D(x,z)}_{\log}; -\tilde{\lambda} - \tilde{\rho}\right)_{\text{gen} + \alpha^W} \hookrightarrow \text{PGL}_2\text{-BA}_{\tilde{\lambda}, \alpha} \]
(cf. Proposition \[2.1.2\]).
Proof. Let \( z := (z_1, \cdots, z_m) \) be an element of \( C^m(x) \) and denote by \( \mathcal{E}^\bullet := (\mathcal{E}, \nabla, \mathcal{E}_B, \mathcal{E}'_B) \) the Miura \( G \)-oper corresponding to the unique element of \( G\text{-MOp}(\mathbb{P}^D(x,x)_{\log}, -\lambda - \rho, \mathcal{A}^W(-\rho))_{\text{gen}} \). By considering the discussion in the proof of Proposition 2.2.1, we see that \( \nabla|_{\mathbb{P}\backslash \{\infty\}} \) may be expressed as

\[
\partial_x + p_{-1} - \sum_{i=1}^{r} \frac{\lambda_i}{x - x_i} + \sum_{j=1}^{m} \frac{\alpha_j}{x - z_j}
\]

under a suitable trivialization of \( \mathcal{E}|_{\mathbb{P}\backslash \{\infty\}} \). Let us fix \( j \in \{1, \cdots, m\} \). Also, consider the restriction of \( \nabla \) to \( \mathbb{D}_{z_j} \), which may be expressed, after choosing a uniformizer \( t \) at \( z_j \), as

\[
\partial_t + p_{-1} + \frac{\alpha_j}{t} + u_j(t)
\]

(\( \partial_t := \frac{d}{dt} \) for some \( u_j(t) \in t[t] \)). One verifies that the equality \( \langle \alpha_j, u_j(0) \rangle = 0 \) holds if and only if \( z \) satisfies the \( j \)-th equation in \((37)\). Thus, the assertion follows from the following lemma.

Lemma 2.2.4.

Let \( \alpha \in \Gamma \), and denote by \( \alpha^W \) the simple reflection corresponding to \( \alpha \). Choose a uniformizer \( t \) of \( \hat{\mathcal{O}} \). Also, let us take a \( G \)-oper \( \mathcal{E}^\bullet \) on \( \mathbb{D}^\log \) of the form \( (\mathbb{D} \times G, \nabla, \mathbb{D} \times B) \), where \( \nabla = \partial_t + p_{-1} + \frac{\alpha}{t} + u(t) \) (under the identification \( k[[t]] \cong \hat{\mathcal{O}} \) determined by \( t \)) for some \( u(t) \in t[t] \). Then, \( \langle \alpha, u(0) \rangle = 0 \) implies that \( \mathcal{E}^\bullet \) belongs to \( G\text{-Op}(\mathbb{D}) \) (cf. \((13)\)). Moreover, the inverse implication holds if \( G = \text{PGL}_2 \).

Proof. Suppose that \( \langle \alpha, u(0) \rangle = 0 \). Denote by \( e_\alpha \) the unique generator of \( \mathfrak{g}^\alpha \) such that \( \{f_\alpha, 2\alpha, e_\alpha\} \) forms an \( \mathfrak{sl}_2 \)-triple. Also, denote by \( \exp : \mathfrak{n} \rightarrow N \) the exponential map given by \((13)\), §1.4, (55). (We can construct this map because of the assumption \((*)_G\).) It follows from \((13)\), §1.4, Corollary 1.4.2, that after the gauge transformation by \( \exp(\frac{1}{t^2} \cdot e_\alpha)^{-1} \), \( \nabla \) becomes the connection \( \nabla' \) of the following from:

\[
\nabla' := \partial_t + \frac{d}{dt} \left( \frac{1}{t} \cdot e_\alpha \right) + \sum_{s=0}^{\infty} \frac{1}{s!} \cdot \text{ad} \left( \frac{1}{t} \cdot e_\alpha \right)^s \left( p_{-1} + \frac{\alpha}{t} + u(t) \right).
\]

Observe that \( \frac{d}{dt} \left( \frac{1}{t} \cdot e_\alpha \right) = -\frac{1}{t^2} \cdot e_\alpha \) and

\[
\text{ad} \left( -\frac{1}{t} \cdot e_\alpha \right) \left( p_{-1} + \frac{\alpha}{t} + u(t) \right) = -\frac{1}{t} \cdot \alpha + \frac{2}{t^2} \cdot e_\alpha - \frac{\langle \alpha, u(t) \rangle}{t} \cdot e_\alpha,
\]

\[
\text{ad} \left( -\frac{1}{t} \cdot e_\alpha \right)^2 \left( p_{-1} + \frac{\alpha}{t} + u(t) \right) = -\frac{2}{t^2} \cdot e_\alpha,
\]

\[
\text{ad} \left( -\frac{1}{t} \cdot e_\alpha \right)^l \left( p_{-1} + \frac{\alpha}{t} + u(t) \right) = 0
\]

\( (l = 3, 4, \cdots) \). Thus, we have the equality

\[
\nabla' = \partial_t + p_{-1} + u(t) - \frac{\langle \alpha, u(t) \rangle}{t} \cdot e_\alpha.
\]

Since \( \langle \alpha, u(0) \rangle = 0 \), the triple \( (\mathbb{D} \times G, \nabla', \mathbb{D} \times B) \), as well as \( \mathcal{E}^\bullet \), belongs to \( G\text{-Op}(\mathbb{D}) \). This completes the proof of the former assertion.
Next, we shall assume that \( \mathfrak{g} = \mathfrak{sl}_2 \), where \( \nabla \) can be expressed as
\[
\nabla = \partial_t + \left( \frac{1}{t} + u(t) \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{t} - u(t) \end{pmatrix} \right)
\]
with \( u(t) := \frac{1}{t} \cdot (\alpha, u(t)) \in \mathbb{K}[t] \). Suppose further that \( \nabla \) belongs to \( G\text{-Op}(\mathbb{D}) \). Then, there exists an element \( b \in B(k((t))) \) such that \( \nabla \) becomes, after the gauge transformation by \( b \), the connection \( \nabla_b = \partial_t + \begin{pmatrix} u(t) & w(t) \\ 0 & -v(t) \end{pmatrix} \) for some \( v(t), w(t) \in k[[t]] \). By an explicit computation, \( b \) turns out to belong to \( N(k((t))) \), i.e., \( b = \exp(\begin{pmatrix} 0 & -\beta(t) \\ 0 & 0 \end{pmatrix}) \), for some \( \beta(t) \in k((t)) \). Hence, we have
\[
\nabla_b := \partial_t + \frac{d}{dt} \left( \begin{pmatrix} 0 & -\beta(t) \\ 0 & 0 \end{pmatrix} \right) + \sum_{s=0}^{\infty} \frac{1}{s!} \cdot \text{ad} \left( \begin{pmatrix} 0 & \beta(t) \\ 0 & 0 \end{pmatrix} \right)^s \left( \begin{pmatrix} \frac{1}{t} + u(t) & 0 \\ 0 & -\frac{1}{t} - u(t) \end{pmatrix} \right)
\]
\[
= \partial_t + \begin{pmatrix} \frac{1}{t} + u(t) + \beta(t) & -\beta(t)' - 2 \cdot \beta(t) \cdot (\frac{1}{t} + u(t)) - \beta(t)^2 \\ 1 & -\frac{1}{t} - u(t) - \beta(t) \end{pmatrix}.
\]
By considering the \((1, 1)\)-th element of the matrix form of \( \nabla_b \), \( \beta(t) \) can be expressed as \( \beta(t) = -\frac{1}{t} + \beta_+(t) \) for some \( \beta_+(t) \in k[[t]] \). Hence, the \((1, 2)\)-th element of \( \nabla_b \) reads
\[
\beta(t)' - 2 \cdot \beta(t) \cdot \left( \frac{1}{t} + u(t) \right) - \beta(t)^2 = -\beta_+(t)' - 2 \cdot \beta_+(t) \cdot u(t) - \beta_+^2(t) + \frac{2}{t} \cdot u(t).
\]

Since it must belong to \( k[[t]] \), we see that \( \frac{2}{t} \cdot u(t) \in k[[t]] \), or equivalently, \( u(0) = 0 \). This completes the proof of the latter assertion. \( \square \)

**Remark 2.2.5.**

In [12], A. Varchenko studied the Bethe ansatz equations (for \( \mathfrak{g} = \mathfrak{sl}_2 \)) modulo \( p \) and showed that the Bethe vector corresponding to its solution is an eigenvector of the Gaudin Hamiltonian. By this result together with Thereom 2.2.3, we can construct eigenvectors by means of dormant generic Miura \( PGL_2 \)-opers. On the other hand, as explained in §3.1 later, dormant generic Miura \( G \)-opers correspond bijectively to (pre-)Tango structures. Thus, we also can say that a certain type of (pre-)Tango structures provides eigenvectors of the Gaudin Hamiltonian.

### 2.3. The Bethe ansatz equations II.

We show that the solutions of the Bethe ansatz equations of a certain type come from dormant Miura \( PGL_2 \)-opers. Let us keep the notation in the previous subsection. In what follows, for an element \( v \in \mathfrak{g} \) and a positive integer \( s \), we shall write
\[
v^s := (v, v, \cdots, v) \in \mathfrak{g}^s.
\]

**Proposition 2.3.1.**

The injection
\[
G\text{-MOp}(\mathbb{P}^{D_\mathfrak{g}^-\text{-log}}, (-\tilde{\rho})^{r+1})_{\text{gen}, +\alpha^w} \hookrightarrow G\text{-MOp}(\mathbb{P}^{D_\mathfrak{g}^-\text{-log}}, (-\tilde{\rho})^{r+1})_{\text{triv}, +\alpha^w}
\]
displayed in (44) of the case where the data \((X, \hat{\lambda}, w)\) is taken to be \((\mathbb{P}, (-\tilde{\rho})^{r+1}, \alpha^W)\) is an isomorphism.

**Proof.** We shall prove that (51) is surjective. Suppose that we are given a Miura \(G\)-oper \(\mathcal{E}_0^{\bullet}\) classified by \(G\)-MOp\((\mathbb{P}^{D_{x, \log}}; (-\tilde{\rho})^{r+1})_{\text{triv,+}+\alpha^W}\), i.e., by \(G\)-MOp\((\mathbb{P}^{D_{x, \log}}; (-\tilde{\rho})^{r+1})_{\text{triv,+}(z, \alpha^W)}\) for some \(z := (z_1, \cdots, z_m) \in C^m(x)\). According to the comment in Remark 1.4.2, it comes, via gauge transformation, from a Miura \(G\)-oper on \(\mathbb{P}^{D_{x, \log}}\) of exponent \(\chi(-\tilde{\rho})\). Moreover, since the underlying \(G\)-oper of \(\mathcal{E}_0^{\bullet}\) belongs to \(G\)-Op\((\mathbb{P}^{D_{x, \log}}; \chi(-\tilde{\rho}))\), its restriction to \(\mathbb{D}_{z_j}\) (for each \(j \in \{1, \cdots, m-1\}\)) belongs to \(G\)-Op\((\mathbb{D}_{z_j})\) \(\left(\subseteq G\text{-Op}(\mathbb{D}_{z_j}^{\log}; \chi(-\tilde{\rho}))\right)\). Hence, this \(G\)-oper comes from a \(G\)-oper \(\mathcal{E}_0^{\bullet}\) on \(\mathbb{P}^{D_{x, \log}}\) of exponent \(\chi(-\tilde{\rho})^2\). By [14], Corollary 2.6.2 and Theorem 3.2.2, (iv), \(\mathcal{E}_0^{\bullet}\) turns out to be dormant. It follows that \(\mathcal{E}_0^{\bullet}\) is dormant, and hence, belongs to \(G\)-MOp\((\mathbb{P}^{D_{x, \log}}; (-\tilde{\rho})^{r+1})_{\text{gen,+}+\alpha^W}\). This completes the proof of the assertion. \(\square\)

By combining the results of Theorem 2.2.3 and Proposition 2.3.1 we obtain Corollary 2.3.2 described as follows.

**Corollary 2.3.2.**

Let \(l\) be a nonnegative integer. Then, there exists a canonical bijection

\[
(51) \quad \text{PGL}_2\text{-BA}_{0^{r+1}, \alpha^W} \sim \text{PGL}_2\text{-MOp}(\mathbb{P}^{D_{x, \log}}; (-\tilde{\rho})^{r+1})_{Z_{\text{gen,+}+\alpha}}^{Z_{\text{gen,+}+\alpha^{l+1}}}
\]

\[
\left(= \prod_{z \in C^{l+1}(x)} \text{PGL}_2\text{-MOp}(\mathbb{P}^{D_{x, \log}}; ((-\tilde{\rho})^{r+1}, \tilde{\rho}^{l+1})_{\text{gen}}^{Z_{\text{gen,+}+\alpha^{l+1}}})\right)
\]

That is to say, for each \(z := (z_1, \cdots, z_{l+1}) \in C^{l+1}(x)\), there exists a dormant Miura PGL_2-oper on \(\mathbb{P}^{D_{x, \log}}\) of exponent \((-\tilde{\rho})^{r+1}, \tilde{\rho}^{l+1}\) if and only if \(z\) satisfies the Bethe ansatz equations of the form

\[
(52) \quad \sum_{s \neq j} \frac{1}{z_j - z_s} = 0 \quad (j = 1, \cdots, l+1).
\]

**Remark 2.3.3.**

We shall describe, in a somewhat simpler way, the condition \(\sum_{s \neq j} \frac{1}{z_j - z_s} = 0\) appearing in the above corollary. Let us consider

\[
(53) \quad f(x) := \prod_{j=1}^{l+1} (x - z_j) \in k[\tau].
\]
Since \( z_1, \ldots, z_{lp+1} \) are distinct, we have \( \gcd(f(x), f'(x)) = 1 \) (or equivalently, \( f'(z_j) \neq 0 \) for any \( j \in \{1, \ldots, lp + 1\} \)). Also, one verifies immediately that, for each \( j \in \{1, \ldots, lp + 1\} \),

\[
\frac{f''(z_j)}{f'(z_j)} = \sum_{s'=1}^{lp+1} \sum_{s'' \neq s'} \prod_{s'' \neq s'} (x - z_{s''}) |_{x = z_j} = 2 \cdot \sum_{s \neq j} \frac{1}{z_j - z_s}.
\]

On the other hand, (since \( f(z_j) = 0 \) for any \( j \in \{1, \ldots, lp + 1\} \)) the equality \( \frac{f''(z_j)}{f'(z_j)} = 0 \) holds for any \( j \in \{1, \ldots, lp + 1\} \) if and only if \( f''(x) = 0 \). Thus, we obtain an equivalence of conditions

\[
\sum_{s \neq j} \frac{1}{z_j - z_s} = 0 \quad (j \in \{1, \ldots, lp + 1\} \iff f''(x) = 0.
\]

For instance, if \( (a, b, c) \) is an element of \( k^3 \) with \( ab - c \neq 0 \) and \( z_1, \ldots, z_{p+1} \) are the roots of the polynomial \( f(x) := x^{p+1} + ax + bx + c \), then the equalities \( \sum_{s \neq j} \frac{1}{z_j - z_s} \) (\( j = 1, \ldots, p + 1 \)) are satisfied.

3. Tango structures

In this section, we recall the notion of a Tango structure and consider a bijective correspondence between Tango structures and solutions to the Bethe ansatz equations for \( g = \mathfrak{sl}_2 \). Also, by means of this correspondence, we construct new examples of Tango structures. In what follows, we suppose that \( p \geq 3 \).

3.1. Tango structures.

First, we recall the definition of a (pre-)Tango structure and the bijective correspondence between dormant Miura PGL\(_2\)-opers and them.

Let \( X \) be a connected proper smooth curve over \( k \) of genus \( g_X \) \((\geq 0)\). Denote by \( X^{(1)} \) the Frobenius twist of \( X \) relative to \( k \) and by \( F : X \to X^{(1)} \) the relative Frobenius morphism of \( X \). Also, denote by \( \mathcal{B}_{X/k} (\subseteq \Omega_{X/k}) \) the sheaf of locally exact 1-forms on \( X \) (relative to \( k \)). The direct image \( F_* (\Omega_{X/k}) \) forms a vector bundle on \( X^{(1)} \) of rank \( p \). \( \mathcal{B}_{X/k} \) may be considered, via the underlying homeomorphism of \( F \), as a subbundle of \( F_*(\Omega_{X/k}) \) of rank \( p - 1 \).

Now, let \( \mathcal{L} \) be a line subbundle of \( F_* (\mathcal{B}_{X/k}) \). Consider the \( \mathcal{O}_{X^{(1)}} \)-linear composite

\[
\mathcal{L} \hookrightarrow F_* (\mathcal{B}_{X/k}) \hookrightarrow F_*(\Omega_{X/k}),
\]

where the first arrow denotes the natural inclusion and the second arrow denotes the morphism obtained by applying the functor \( F_* (-) \) to the natural inclusion \( \mathcal{B}_{X/k} \to \Omega_{X/k} \). This composite corresponds to a morphism

\[
\xi_{\mathcal{L}} : F^*(\mathcal{L}) \to \Omega_{X/k}
\]
via the adjunction relation “$F^*(-) \to F_*(-)$”.

**Definition 3.1.1.**

(i) We shall say that $\mathcal{L}$ is a Tango structure on $X$ if $\xi_\mathcal{L}$ is an isomorphism.

(ii) A Tango curve is a connected proper smooth curve over $k$ admitting a Tango structure on it.

**Remark 3.1.2.**

Let $\mathcal{L}$ be a line subbundle of $\mathcal{B}_X/k$. Then, one verifies immediately that $\mathcal{L}$ is a Tango structure on $X$ if and only if $\mathcal{L}$ has degree $\frac{2g_X-2}{p}$. In particular, if $p \mid g_X - 1$, then there is no Tango structure on $X$. At the time of writing the present paper, the author does not obtain the complete list of $g$'s such that there is a Tango curve of genus $g$.

Next, we shall recall the definition of a pre-Tango structure in the sense of [15], Definition 5.3.1. Let $r$ be a nonnegative integer and $x := (x_1, \cdots, x_r)$ an ordered collection of distinct closed points of $X$. (We take $x := \emptyset$ if $r = 0$.) If there is no fear of confusion, we shall write $X^{\log}$ for $X^{D_{X^{\log}}}$.

The data $x$ induces, via base-change, a collection of closed points $x^{(1)} := (x_1^{(1)}, \cdots, x_r^{(1)})$ in $X^{(1)}$, which determines a log structure on $X^{(1)}$; denote by $X^{(1)\log}$ the resulting log structure. Moreover, let

$$C_{X^{\log}/k} : F_*(\Omega_{X^{\log}/k}) \to \Omega_{X^{(1)\log}/k}$$

be the Cartier operator of $X^{\log}/k$. That is to say, $C_{X^{\log}/k}$ is a unique $\mathcal{O}_{X^{(1)}}$-linear morphism whose composite with the injection $\Omega_{X^{(1)\log}/k} \to \Omega_{X^{(1)\log}/k} \otimes F_*(\mathcal{O}_X)$ induced by the natural injection $\mathcal{O}_{X^{(1)}} \to F_*(\mathcal{O}_X)$ coincides with the Cartier operator associated with $(\mathcal{O}_X, d)$ in the sense of [8], Proposition 1.2.4.

**Definition 3.1.3.**

A pre-Tango structure on $(X, x)$ is a $D_x$-log connection $\nabla$ on $\Omega_{X^{\log}/k}$ with vanishing $p$-curvature satisfying that $\text{Ker}(\nabla) \subseteq \text{Ker}(C_{X^{\log}/k})$. (Here, we consider $\text{Ker}(\nabla)$ as an $\mathcal{O}_{X^{(1)}}$-submodule of $F_*(\Omega_{X^{\log}/k})$ via $F_*$.)

**Remark 3.1.4.**

According to [15], Proposition 5.3.2, the set of pre-Tango structures on $(X, \emptyset)$ is in bijection with the set of Tango structures on $X$.

Indeed, let $\nabla$ be a pre-Tango structure on $(X, \emptyset)$. The $\mathcal{O}_{X^{(1)}}$-module $\text{Ker}(\nabla)$ is contained in $F_*(\mathcal{B}_X/k)$ (i.e., $F_*(\text{Ker}(C_X/k)))$. Since $\nabla$ gives an injection from $F_*(\Omega_{X/k})/\text{Ker}(\nabla)$ to $F_*(\Omega_{X/k}^2)$ (which implies that $F_*(\Omega_{X/k})/\text{Ker}(\nabla)$ is a vector bundle), the quotient $F_*(\mathcal{B}_X/k)/\text{Ker}(\nabla)$ (i.e., $F_*(\text{Ker}(\nabla))$) is verified to be a vector bundle. Namely, $\text{Ker}(\nabla)$ specifies a submodule of $F_*(\mathcal{B}_X/k)$. Moreover, the condition that $\nabla$ has vanishing $p$-curvature implies that $\xi_{\text{Ker}(\nabla)}$ is an isomorphism. Hence, $\text{Ker}(\nabla)$ specifies a Tango structure.

Conversely, let $\mathcal{L}$ be a Tango structure on $X$. The line bundle $F^*(\mathcal{L})$ has uniquely a connection determined by the condition that the sections in $F^{-1}(\mathcal{L})$ are horizontal. The connection $\nabla_{\mathcal{L}}$ corresponding, via $\xi_{\mathcal{L}}$, to this connection specifies a pre-Tango structure on $(X, \emptyset)$ (because
of the equality $F_*(\mathcal{B}_{X/k}) = F_*(\text{Ker}(C_{X/k}))$. One verifies that the assignments $\nabla \mapsto \text{Ker}(\nabla)$ and $\mathcal{L} \mapsto \nabla_{\mathcal{L}}$ constructed just above give the desired bijective correspondence.

Let us fix an ordered collection $l := (l_1, \cdots, l_r) \in \mathbb{F}_p^r$. (We take $l := \emptyset$ if $r = 0$.) Denote by
\[(59)\]
the set of pre-Tango structures on $(X, x)$ of monodromy $l$, i.e., whose monodromy at $x_i$ (for each $i \in \{1, \cdots, r\}$) is $l_i$.

**Proposition 3.1.5.**

(i) Let $m$ be a positive integer and $z := (z_1, \cdots, z_m) \in C^m(x)$. Then, there exists a canonical bijection
\[(60)\]
\[
\text{Tang}(X; (x, z); (l, (-1)^m)) \sim \text{Tang}(X; x; l).
\]

(ii) There exists a bijective correspondence between the set $\text{Tang}(X; x; (-1)^r)$ and the set of Tango structures on $X$.

**Proof.** We shall consider assertion (i). Let us take a pre-Tango structure $\nabla$ classified by $\text{Tang}(X; (x, z); (l, (-1)^m))$. Denote by $\nabla_{-z}$ the $D_\text{reg}(X; x, z)\text{-log}$ connection on $\Omega_{X_{\text{reg}}}$ obtained by restricting $\nabla$ via the inclusion $\Omega_{X_{\text{reg}}} \hookrightarrow \Omega_{X}$.

One verifies that the monodromy of $\nabla_{-z}$ at $z_j$ (for each $j \in \{1, \cdots, m\}$) coincides with 0. Hence, $\nabla_{-z}$ may be thought of as a $D_\text{reg}(X; x, z)\text{-log}$ connection. Moreover, since the equality
\[(61)\]
holds, $\nabla_{-z}$ specifies a pre-Tango structure classified by $\text{Tang}(X; x; l)$.

Conversely, let us take a pre-Tango structure $\nabla$ classified by $\text{Tang}(X; x; l)$; it may be thought of as a $D_\text{reg}(X; x, z)\text{-log}$ connection whose monodromy at $z_j$ (for each $j \in \{1, \cdots, m\}$) is 0. Then, there exists uniquely a $D_\text{reg}(X; x, z)\text{-log}$ connection $\nabla_{+z}$ on $\Omega_{X_{\text{reg}}}$ whose restriction to $\Omega_{X}$ coincides with $\nabla$. One verifies (from (61) again) that $\nabla_{+z}$ specifies a pre-Tango structure classified by $\text{Tang}(X; (x, z); (l, (-1)^m))$.

The assignments $\nabla \mapsto \nabla_{-z}$ and $\nabla \mapsto \nabla_{+z}$ constructed above give the desired bijective correspondence. This completes the proof of assertion (i).

Assertion (ii) follows from assertion (i) of the case where $m = 0$ together with the result mentioned in Remark 3.1.4.

**3.2. Proof of Theorem A.**

Now, we shall prove Theorem A, as follows. Recall from [15], Theorem A, that if $l := (l_1, \cdots, l_r) \in (\mathbb{F}_p^\times)^r$, then there exists a canonical bijection
\[(62)\]
\[
\text{Tang}(X; x; l) \sim \text{PGL}_2\text{-MOp}(X^{\text{reg}}; l_\text{reg})_{\text{gen}},
\]
where $\check{\rho} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ and $l_\check{\rho} := (l_1 \check{\rho}, \cdots, l_r \check{\rho}) \in (t_{\text{reg}})^r$. In particular, by considering the case where $(X, x, l)$ is taken to be $(\mathbb{P}, (\infty, x), (-1, 1^{l_{p+1}}))$ (for each $z \in C^{l_{p+1}}(\infty)$), we obtain a
bijection

\[(63) \quad \text{Tang}(\mathbb{P}; (\infty, z); (-1, 1^{l_p+1})) \xrightarrow{\sim} \text{PGL}_2\text{-MOp}(\mathbb{P}^{D_{(\infty, z)}\log}; (-\check{\rho}, \check{\rho}^{l_p+1}))^z_{\text{gen}}.\]

Also, it follows from Proposition 3.1.5 (of the case where \((X, m, l)\) is taken to be \((\mathbb{P}, 1, 1^{l_p+1})\)) that there exists a canonical bijection

\[(64) \quad \text{Tang}(\mathbb{P}; (\infty, z); (-1, 1^{l_p+1})) \xrightarrow{\sim} \text{Tang}(\mathbb{P}; z; 1^{l_p+1}).\]

The composition of the inverse of (63) and (64) determines a bijection

\[(65) \quad \text{Tang}(\mathbb{P}; z; 1^{l_p+1}) \xrightarrow{\sim} \text{PGL}_2\text{-MOp}(\mathbb{P}^{D_{(\infty, z)}\log}; (-\check{\rho}, \check{\rho}^{l_p+1}))^z_{\text{gen}}\]

(for each \(z \in C^{l_p+1}(\infty)\)). Thus, we have

\[(66) \quad \prod_{\mathbb{P} \in C^{l_p+1}(\infty)} \text{Tang}(\mathbb{P}; z; 1^{l_p+1}) \xrightarrow{\sim} \text{PGL}_2\text{-MOp}(\mathbb{P}^{D_{(\infty, z)}\log}; (-\check{\rho})^z_{\text{gen}},+_{\alpha^{l_p+1}})\]

The desired bijection (2) may be obtained from this bijection together with Corollary 2.3.2 of the case where \(r = 0\). This completes the proof of Theorem A.

3.3. Pull-back via tamely ramified coverings.

In what follows, we proceed to our discussion toward Theorem B. First, let us consider the pull-back of a pre-Tango structure via a tamely ramified covering. Suppose that we are given a collection of data

\[(67) \quad (Y, y, \pi),\]

where

- \(Y\) denotes another connected proper smooth curve over \(k\);
- \(y := (y_1, \cdots, y_s) (s > 1)\) denotes an ordered collection of distinct closed points of \(Y\);
- \(\pi : Y \to X\) denotes a tamely ramified covering such that \(\pi^{-1}(\bigcup_{i=1}^s \{x_i\}) = \bigcup_{j=1}^s \{y_j\}\) and \(\pi\) is étale away from \(\bigcup_{i=1}^s \{x_i\}\).

Denote by \(q : \{1, \cdots, s\} \to \{1, \cdots, r\}\) the map of sets determined by \(\pi(y_j) = x_{q(j)} (j = 1, \cdots, s)\). For each \(j \in \{1, \cdots, s\}\), denote by \(R_j\) the ramification index of \(\pi\) at \(y_j\) (hence \(p \nmid R_j\)). The morphism \(\pi\) extends to a log étale morphism \(\pi^\log : Y^D_{y^\log} \to X^D_{x^\log}\), and hence, the natural morphism \(\pi^*(\Omega_{X^D_{x^\log}/k}) \to \Omega_{Y^D_{y^\log}/k}\) is an isomorphism.

Now, let \(\nabla\) be an element of \(\text{Tang}(X; x; l)\). The pull-back of \(\nabla\) via \(\pi\) determines, under the isomorphism \(\pi^*(\Omega_{X^D_{x^\log}/k}) \xrightarrow{\sim} \Omega_{Y^D_{y^\log}/k}\), a \(D_y\)-log connection

\[(68) \quad \pi^*(\nabla) : \Omega_{Y^D_{y^\log}/k} \to \Omega_{Y^D_{y^\log}/k}^{\otimes 2}\]
on \(\Omega_{Y^D_{y^\log}/k}\). Here, observe that both the relative Frobenius morphisms and the Cartier operators of \(X \setminus \bigcup_{i=1}^s \{x_i\}\) and \(Y \setminus \bigcup_{i=1}^s \{y_i\}\) are compatible (in the evident sense) via \(\pi\). This implies that \(\text{Ker}(\pi^*(\nabla)) \subseteq \text{Ker}(C_{Y^D_{y^\log}/k})\), and hence, \(\pi^*(\nabla)\) specifies a pre-Tango structure on \((Y, y)\).
Proposition 3.3.1.
Write \( I' := (l_{q(1)} R_1, \ldots, l_{q(s)} R_s) \in \mathbb{P}^s \). Then, the assignment \( \nabla \mapsto \pi^*(\nabla) \) discussed above defines an injection

\[
(69) \quad \text{Tang}(X; x; I) \hookrightarrow \text{Tang}(Y; y; I').
\]

Proof. Since the injectivity may be verified immediately, it suffices to prove that the monodromy of the pre-Tango structure \( \pi^*(\nabla) \) at \( y_j \) (where \( \nabla \in \text{Tang}(X; x; I), j \in \{1, \ldots, s\} \)) is \( l_{q(j)} R_j \). For simplicity, we write \( x := x_q(j), y := y_j, R := R_j \), and \( l := l_{q(j)} \). The formally local description of \( \pi \) at \( y \) may be given by \( \hat{\mathbb{D}}_y := \text{Spec}(k[[t^{1/R}]]) \rightarrow \text{Spec}(k[[t]]) := \hat{\mathbb{D}}_x \) (corresponding to the natural inclusion \( k[[t]] \hookrightarrow k[[t^{1/R}]] \)), where \( t \) is a formal coordinate in \( X \) at \( x \). The restriction \( \nabla|_{\hat{\mathbb{D}}_y} \) of \( \pi^*(\nabla) \) to \( \hat{\mathbb{D}}_y \) may be expressed as \( \pi^*(\nabla)|_{\hat{\mathbb{D}}_y} = d + lR \cdot \frac{dt}{t^{1/R}} \). This implies that the monodromy of \( \pi^*(\nabla) \) at \( y \) is \( lR \), and hence, completes the proof of the assertion. \( \square \)

Example 3.3.2.
We shall consider the case where \( X = \mathbb{P} \). Let \( I := (l_1, \ldots, l_{r+1}) \) \( (r \geq 0) \) be an element of \( \mathbb{P}^{r+1}_p \) and \( x := (x_1, \ldots, x_{r+1}) \) an ordered collection of distinct \( r+1 \) closed points of \( \mathbb{P} \) with \( x_{r+1} := \infty \). Let us consider the desingularization \( Y \) of the plane curve defined by

\[
(71) \quad y^n = a \cdot \prod_{i=1}^{r} (x - x_i),
\]

where \( a \in k^\times \), \( p \nmid n \), and \((x, y)\) are an inhomogeneous coordinate of the projective plane \( \mathbb{P}^2 \). Denote by \( \pi: Y \rightarrow \mathbb{P} \) the projection given by \((x, y) \mapsto x \). Given \( i \in \{1, \ldots, r\} \), we shall write \( y_i \) for the unique point of \( Y \) over \( x_i \), i.e., \( \pi(y_i) = x_i \). (The ramification index of \( \pi \) at \( y_i \) for each \( i \in \{1, \ldots, r\} \) is \( n \). Also, let \( y_{r+1}, \ldots, y_{r+r'} \) \( (r' \geq 1) \) be the set of points of \( Y \) over \( x_{r+1} = \infty \). Write \( y := (y_1, \ldots, y_{r+r'}) \). According to \( [11] \), \( \S 6.3 \), Proposition 6.3.1, the ramification index of \( \pi \) at \( y_{r+i'} \) \( (\text{for each } i' \in \{1, \ldots, r'\}) \) is \( \frac{d}{d'} \), where \( d := \gcd(n, r) \). Moreover, the genus of \( Y \) is given by \( \frac{(n-1)(r-1)}{2} - \frac{\gcd(n, r)-1}{2} \).

Now, let us fix \( I := (l_1, \ldots, l_{r+1}) \in \mathbb{P}^{r+1}_p \) and write

\[
(72) \quad I' := (l_1n, l_2n, \ldots, l_{r}n, \frac{l_{r+1}n}{d}, \ldots, \frac{l_{r+r'}n}{d}) \in \mathbb{P}^{r+r'}_p
\]

(where each of the last \( r' \) factors is \( \frac{l_{r+1}n}{d} \)). Then, it follows from Proposition 3.3.1 that the assignment \( \nabla \mapsto \pi^*(\nabla) \) defines an injection

\[
(73) \quad \text{Tang}(\mathbb{P}; x; I) \hookrightarrow \text{Tang}(Y; y; I').
\]

In particular, \( \text{Tang}(Y; y; I') \) is nonempty unless \( \text{Tang}(\mathbb{P}; x; I) \) is empty.
3.4. Examples of Tango structures.

In this subsection, we construct examples of Tango structures by applying Theorem A proved above.

Let \( a, b \) be positive integers with \( \gcd(a, bp - 1) = 1 \) and \( h(x) \in k[x] \) a monic polynomial of degree \( ap \) with \( \gcd(h(x), h'(x)) = 1 \). Let
\[
(74) \quad Y
\]
be the smooth curve defined by the equation
\[
(75) \quad y^{bp-1} = h(x).
\]
(If \( a > b \), then the point at infinity is singular, and hence, we need to replace this plane curve by its desingularization to obtain a smooth curve.) Since \( \gcd(ap, bp - 1) = 1 \), there is only one point \( \infty \) at infinity in \( Y \) (cf. [9], §6.3, Proposition 6.3.1). Here, let us take an ordered collection \( z = (z_1, \cdots, z_{ap+1}) \) of elements of \( k \) (i.e., closed points in \( \text{Spec}(k[x]) = \mathbb{P} \setminus \{\infty\} \)) with \( z_{ap+1} = \infty \) satisfying the equality \( h(x) = \prod_{i=1}^{ap}(x - z_i) \). (The assumption \( \gcd(h(x), h'(x)) = 1 \) implies that \( z_i \neq z_{i'} \) if \( i \neq i' \).) Denote by \( \pi : Y \to \mathbb{P} \) the natural projection \( (x, y) \mapsto x \), which is tamely ramified. Also, denote by \( y := (y_1, \cdots, y_{ap+1}) \) the ordered collection of distinct points in \( Y \) determined uniquely by \( \pi(y_i) = z_i \) for any \( i = 1, \cdots, ap+1 \). (In particular, \( y_{ap+1} = \infty \).) Moreover, denote by \( \nabla \) the \( D_x \)-log connection on \( \Omega_{\mathbb{P}D_x^{\log}/k} \) defined as
\[
(76) \quad \nabla = \partial_x + \sum_{i=1}^{ap} \frac{1}{x - z_i}
\]
(under the identification \( \Omega_{\mathbb{P}D_x^{\log}/k}|_{\mathbb{P}\setminus\{\infty\}} \sim \mathcal{O}_{\mathbb{P}\setminus\{\infty\}} \) given by \( dx \mapsto 1 \)). By passing to the isomorphism \( \pi^*(\Omega_{\mathbb{P}D_x^{\log}/k}) \sim \Omega_{YD_y^{\log}/k} \) induced by \( \pi \), we obtain a \( D_y \)-log connection
\[
(77) \quad \pi^*(\nabla) : \Omega_{YD_y^{\log}/k} \to \Omega_Y^{\otimes 2}
\]
on \( \Omega_{YD_y^{\log}/k} \) defined to be the pull-back of \( \nabla \). In particular, we obtain an \( \mathcal{O}_{Y(1)} \)-submodule
\[
(78) \quad \text{Ker}(\pi^*(\nabla))
\]
of \( F_\bullet(\Omega_{YD_y^{\log}/k}) \), where \( F : Y \to Y^{(1)} \) denotes the relative Frobenius morphism of \( Y \) (relative to \( k \)). Then, the following assertion, which implies Theorem B, holds.

**Theorem 3.4.1** (cf. Theorem B).
*Suppose that \( h''(x) = 0 \). Then, \( \text{Ker}(\pi^*(\nabla)) \) forms a Tango structure on \( Y \). In particular, \( Y \) is a Tango curve.*

**Proof.** Let us fix an element \( \gamma \in k \) with \( h(\gamma) \neq 0 \). Consider the automorphism \( \iota \) of \( \mathbb{P} \) given by \( x \mapsto \frac{1}{x - \gamma} \). Then, \( \iota(z) := (\iota(z_1), \cdots, \iota(z_{ap+1})) \) are a collection of distinct points in \( \mathbb{P} \setminus \{\infty\} \). Since \( f(x) := \prod_{i=1}^{ap+1}(x - \iota(z_i)) \) coincides with \( h(\gamma)^{-1} \cdot h(x) \cdot x^{ap+1} \), we see that \( \gcd(f(x), f'(x)) = 1 \) and \( f''(x) = 0 \). It follows from Theorem A that the \( D_{(\iota(z))} \)-log connection on \( \Omega_{\mathbb{P}D_{(\iota(z))}^{\log}/k} \) given by \( \partial_x + \sum_{i=1}^{ap+1} \frac{1}{x - \iota(z_i)} \) forms a pre-Tango structure on \( (\mathbb{P}, \iota(z)) \) of monodromy \( 1^{ap+1} \). By pulling-back via \( \iota \), we obtain a pre-Tango structure on \( (\mathbb{P}, z) \) of monodromy \( 1^{ap+1} \). One verifies immediately that it coincides with \( \nabla \). Hence, (since \( bp - 1 = -1 \) mod \( p \)) the pull-back \( \pi^*(\nabla) \) forms a pre-Tango structure on \( (Z, z) \) of monodromy \((-1)^{ap+1} \) (cf. Proposition...
It follows from Proposition 3.1.5 (and its proof) that \( \text{Ker}(\pi^* (\nabla)) \) specifies a Tango structure, as desired.

Since the curve \( Y \) constructed above has genus \( g = \frac{(ap-1)(bp-2)}{2} \), it follows from Theorem 3.4.1 and [15], Theorem B, that the following assertion holds.

**Corollary 3.4.2.**

Let \( a, b \) be positive integers with \( \text{gcd}(a, bp-1) = 1 \), and write \( g := \frac{(ap-1)(bp-2)}{2} \). Denote by \( \mathfrak{T} \) the moduli stack classifying pairs \((X, L)\) consisting of a connected proper smooth curve \( X \) over \( k \) of genus \( g \) and a Tango structure \( L \) on \( X \). Then, \( \mathfrak{T} \) may be represented by a nonempty equidimensional smooth Deligne-Mumford stack over \( k \) of dimension \( 2g-2 + \frac{2g-2}{p} \) \((= abp^2 + (ab - 2a - b)p - 2a - b)\).

**Remark 3.4.3.**

A well-known example of a Tango structures can be found in some paper (cf. e.g., [10], Example, or [6], Example 1.3). This is constructed as follows. Let \( l \) be an integer with \( lp \geq 4 \). Also, let \( f(x) \) be a polynomial of degree \( l \) in one variable \( x \) and let \( Y \) be the plane curve defined by

\[
y^{lp-1} = f(x^p) - x.
\]

One verifies that \( Y \) is a smooth curve having only one point \( \infty \) at infinity and \( \Omega_{Y/k} = \mathcal{O}_Y(lp(lp-3) \cdot \infty) \). Then, the base-change of the line bundle \( \mathcal{O}(l(lp - 3) \cdot \infty) \) via the absolute Frobenius morphism of \( \text{Spec}(k) \) turns out to specify a Tango structure.

Notice that \( (f(x^p) - x)' = -1 \) (which implies that \( \text{gcd}(f(x^p) - x, (f(x^p) - x)') = 1 \)) and \( (f(x^p) - x)'' = 0 \). Hence, \( Y \) is a special case of Tango curves constructed preceding Theorem 3.4.1. On the other hand, the curve defined, e.g., by the equation \( y^{2p-1} = x^{2p} + x^{p+1} + ax^p + bx + c \) with \( c \neq b(a - b) \) is a Tango curve (by Theorem 3.4.1), which specifies a new example. Thus, one can obtain infinitely many new (explicit!) examples of Tango curves.

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