The Naimark dilated $\mathcal{PT}$–symmetric brachistochrone

Uwe Günter$^1$ and Boris F. Samsonov$^2$

$^1$Research Center Dresden-Rossendorf, POB 510119, D-01314 Dresden, Germany
$^2$Tomsk State University, 36 Lenin Avenue, 634050 Tomsk, Russia

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Introduction The quantum brachistochrone problem consists in finding a Hamiltonian $H$ which evolves a given initial state $|\psi_I\rangle$ into a given final state $|\psi_F\rangle$ in a minimal time $\tau$. Considering this problem for quantum mechanics with $\mathcal{PT}$–symmetric Hamiltonians (PTQM) Bender, Brody, Jones and Meister (BBJM) found the surprising result $[1]$ that the minimal evolution (passage) time $\tau_{PT}$ was less than the minimal time $\tau_h$ required for the evolution induced by a Hermitian Hamiltonian $[2]$. It could be made even arbitrary small $\tau_{PT} < 0$ in a strongly non-Hermitian regime $[1, 4]$. If this effect of a ‘faster than Hermitian’ evolution $[1]$ were experimentally realizable it would open a way to ultra-fast quantum computing processes $[3]$. A problem still unsolved in $[1]$ concerned the switching mechanism between the $\mathcal{PT}$–symmetric brachistochrone system and a conventional (von Neumann) quantum system necessary for an experimental implementation of the suggested ultra-fast quantum process.

As shown by Mostafazadeh $[6]$, an equivalence mapping $[5]$ between PTQM in the sector of unbroken $\mathcal{PT}$–symmetry and conventional quantum mechanics (CQM) leaves the passage time of a brachistochrone invariant $\tau_h = \tau_{PT}$. This implies that a vanishing passage time $\tau_{PT} \to 0$ in the $\mathcal{PT}$–symmetric system is necessarily connected with a vanishing distance between initial and finite state in the equivalent Hermitian system — an effect geometrically analyzed in $[8]$. In case of the Hermitian equivalent of the BBJM brachistochrone, initial and final states will nearly coincide (coincidence problem) so that the brachistochrone effect in such an interpretation would lose any physical relevance.

In this Letter, we propose a realization of the BBJM brachistochrone $[1]$ which resolves the switching problem between PTQM and CQM regimes $[1, 4]$ and which can be considered as starting point for a direct experimental implementation. The key idea consists in a reinterpretation of the BBJM brachistochrone as $\mathcal{PT}$–symmetric subsystem of a larger CQM system living in a higher-dimensional Hilbert space. For this purpose we use a Naimark dilatation (extension) technique $[4]$ as it is widely used in quantum information theory $[3]$. We will demonstrate that the resulting large system will have the structure of an entangled two-spin (two-qubit) system so that an experimental realization of the BBJM brachistochrone effect should be feasible, e.g., in a suitably designed system of entangled polarized photons $[10]$.

Technically, the construction of the large Hermitian system will be accomplished by a three-step procedure: (i) by building a suitable positive operator valued measure (POVM) $[4, 9, 11]$ over the nonorthogonal eigenstates of the $\mathcal{PT}$–symmetric Hamiltonian $H$ and its adjoint $H^\dagger$, (ii) by Naimark dilating (extending) $[9]$ this POVM into an orthogonal projector set in the higher dimensional Hilbert space and (iii) by constructing from it a corresponding Hermitian Hamiltonian $H = H^\dagger$ and a unitary evolution operator $U(t) = e^{-i\tau H}$.

BBJM brachistochrone The BBJM brachistochrone $[1]$ that we are going to Hermitianly dilate (extend) describes the evolution from an initial state $|\psi_I\rangle$ to a final state $|\psi_F\rangle$ governed by a $\mathcal{PT}$–symmetric Hamiltonian $H$ whose structure is chosen in such a way that the time $\tau$ required for the evolution becomes minimal. As shown in $[2]$ such a minimal-passage-time solution follows a minimal geodesic in projective Hilbert space and it is therefore located in the two-dimensional subspace $\mathcal{H}_2 = \mathbb{C}^2$ spanned by $|\psi_I\rangle$ and $|\psi_F\rangle$. In this $\mathcal{H}_2$ the $\mathcal{PT}$–symmetric Hamiltonian $H$ can be chosen as $[1, 4]$

$$H = E_0 I_2 + s \begin{pmatrix} i \sin(\alpha) & 1 \\ 1 & -i \sin(\alpha) \end{pmatrix}, \quad E_0, s \in \mathbb{R}, \quad (1)$$

where $\mathcal{P} = \sigma_z$ denotes the parity operator, $T$ is the antilinear operator of time reflection and complex conjugation $[12]$, $E_0$ denotes an irrelevant offset energy and $s$ a general scaling factor of the matrix. (As usual, $\sigma_x$, $\sigma_y$ and $\sigma_z$ are Pauli matrices.) The angle $\alpha \in (-\pi/2, \pi/2)$ characterizes the non-Hermiticity of the Hamiltonian: $H(\alpha = 0)$ is Hermitian, whereas in the limit $\alpha \to \pm \pi/2$ the Hamiltonian $H$ becomes strongly non-Hermitian and
We supplement them via vectors as \( \langle E \rangle \). Apart from the bi-orthonormality relation \( \Xi \), \( \Psi \) and \( \eta \) are identified with regard to the parameter sector of exact \( \mathcal{PT} \)-symmetry. The operator \( U(t) = e^{-itH} \) of the non-unitary evolution induced by \( H \) has the explicit form
\[
U(t) = \frac{e^{-itE_0}}{\cos(\alpha)} \left( \cos(y - \alpha) - i \sin(y) \right. \\
\left. - i \sin(y) \cos(y + \alpha) \right)
\]  
with \( y := \omega_0 t / 2 \) (we set \( \hbar = 1 \)). In the BBJM-brachistochrone setup, this \( U(t) \) is used to evolve an initial state \( |\psi_I\rangle = (1, 0)^T \) into a final state \( |\psi_F\rangle = \mu_F(0, 1)^T \), \( \mu_F := -ie^{-iE_0}\tau \). The time \( \tau \) required for this evolution follows from the condition \( y = \alpha + \pi / 2 \) as \( \tau = \frac{2\pi \pm 2}{\pi \cos(\alpha)} \) and tends for
\[
\alpha = \varepsilon - \pi / 2, \quad \varepsilon \to +0
\]
and fixed \( s \cos(\alpha) = \omega_0 / 2 \) to zero: \( \tau \to 0 \). In this way the evolution from \( |\psi_I\rangle \) to the orthogonal \( |\psi_F\rangle \) induced by the \( \mathcal{PT} \)-symmetric Hamiltonian \( H \) with eigenstates of fixed energy difference \( E_+ - E_- = \omega_0 \) appears faster than an evolution between these states induced by any Hermitian Hamiltonian with the same energy difference \( \omega_0 \) between its eigenstates. This is due to the fact that the evolution time \( \tau \) between orthogonal states in Hermitian systems is bounded larger than the Anandan-Aharanov lower bound \( \tau \geq \tau_h = \pi / \omega_0 \) [2, 13].

Before we embed the BBJM-brachistochrone into a larger Hermitian model, we briefly collect the required setup information. The eigenvectors \( E_{\pm}(\alpha) \) of \( H \) are normalized with regard to the \( \mathcal{PT} \) inner product \( (u, v) = \mathcal{PT} u \cdot v \) as \( (E_\pm, E_\pm) = \pm 1 \), \( (E_\pm, E_\mp) = 0 \) [12] and for \( \alpha \neq 0 \) they are nonorthogonal with regard to the standard inner product in the Hilbert space \( \mathbb{C} \): \( (E_\pm, E_\mp) \neq 0 \). We supplement them via \( H^\dagger(\alpha) = H(\alpha) \) with the eigenvectors \( |E_+(\alpha)\rangle \), \( |E_-(\alpha)\rangle \) of the adjoint operator \( H^\dagger \) and arrange them as columns in the matrices
\[
\Psi := \{ |E_+(\alpha)\rangle, |E_-(\alpha)\rangle \}, \quad \Xi := \{ |E_+(\alpha)\rangle, |E_-(\alpha)\rangle \},
\]
with \( \hat{E} := \text{diag}(E_+, E_-) \) the eigenvalue problems for \( H \) and \( H^\dagger \) take then the compact matrix form
\[
H \Psi = \Psi \hat{E}, \quad H^\dagger \Xi = \Xi \hat{E}.
\]

Apart from the bi-orthonormality relation \( \Xi^\dagger \Psi = I_2 \), it holds \( \Psi \Psi^\dagger H^\dagger = H \Psi \Psi^\dagger \), \( \Xi \Xi^\dagger H = H^\dagger \Xi \Xi^\dagger \) so that one identifies \( (\Psi \Psi^\dagger)^{-1} = \Xi \Xi^\dagger = \eta \) as metric operator in the pseudo-Hermiticity condition \( \eta H = H^\dagger \eta \). Additionally to its obvious Hermiticity, \( \eta = \eta^\dagger \) the metric can be suitably scaled to be an element of the hyperbolic (“boost”) sector of the complex orthogonal group \( SO(2, \mathbb{C}) \) [8].

\[
\eta = \frac{1}{\cos(\alpha)} \left( \begin{array}{cc} 1 & -i \sin(\alpha) \\ i \sin(\alpha) & 1 \end{array} \right) = \left( \begin{array}{cc} \cosh(\beta) & -i \sinh(\beta) \\ i \sinh(\beta) & \cosh(\beta) \end{array} \right) = e^{\beta \sigma_y}
\]

with parameter identification \( \sin(\alpha) = \tanh(\beta) \) and \( \rho(\alpha) = 1 / [\cosh(\beta)] \). As final ingredient we fix the notation for the one-to-one similarity mapping between the \( \mathcal{PT} \)-symmetric Hamiltonian \( H \) and its isospectral Hermitian counterpart \( h = h^\dagger \) [13], \( H = \rho^\dagger \rho \), \( H^\dagger = \rho \rho^{-1} \), \( \rho^2 = \eta \), as well as for the unitary eigenvector matrix \( \Phi \)
\[
h \Phi = \Phi \hat{E}, \quad \Phi^\dagger = \Phi^{-1}.
\]
The eigenvectors of \( H \) and \( H^\dagger \) can be regarded as \( -\alpha, i.e. \beta = -\beta \), mirror symmetrically distorted versions of the eigenvectors of the Hamiltonian \( h \)
\[
\Psi = \rho^{-1} \Phi, \quad \Xi = \rho \Phi, \quad \rho^{-1}(\beta) = \rho(-\beta).
\]

The orthogonal initial and final vectors \( |\psi_I\rangle \), \( |\psi_F\rangle \) in the BBJM-brachistochrone model on their turn can be considered as eigenstates of a Hermitian spin operator \( S_z = \sigma_z \) (a von-Neumann observable with orthogonal projector decomposition), whereas the \( \mathcal{PT} \)-symmetric (non-Hermitian) Hamiltonian \( H \) has nonorthogonal eigenvectors \( |E_\pm\rangle \) and is not a von-Neumann observable. Under the one-to-one equivalence mapping [7] from \( H \) to the Hermitian Hamiltonian \( h \) the spin operator \( S_z \) maps into a non-Hermitian operator \( s_z = \rho S_z \rho^{-1} \neq s_z \). Hence, the BBJM-brachistochrone system in both representations \( (H, S_z) \) and \( (h, s_z) \) contains operators which are not von-Neumann observables and therefore the system cannot be considered as fundamental.

Naimark dilation In order to give the BBJM system with evolution \( \psi(t) = U(t) \psi_I \) a meaning in CQM we embed it into a larger purely Hermitian system
\[
\hat{\psi}(t) = U(t) \hat{\psi}_I, \quad \hat{\psi}(t) = \left( \begin{array}{c} \psi(t) \\ \chi(t) \end{array} \right)
\]
with unitary evolution operator \( U(t) = [U^\dagger(t)]^{-1} \) and additional ancilla wave function component \( \chi(t) \). For this purpose we construct an auxiliary POVM [5, 8]
\[
\sum_{k=1}^{11} A_k = I_2 \text{ over the nonorthogonal eigenvectors of } H \text{ and its adjoint } H^\dagger \text{ with rank-one operators } A_{1,2} = f^2 |E_+(\alpha)\rangle \langle E_+(\alpha)| \text{ and } A_{3,4} = f^2 |E_-(\alpha)\rangle \langle E_-(\alpha)|. \quad \text{For symmetry reasons all } A_k \text{ are scaled with the same normalization factor:}
\]
\[
f := \sqrt{\frac{\cos(\alpha)}{2}} = \frac{1}{\sqrt{2 \cosh(\beta)}}.
\]
Next we note that any rank-one-type POVM built from a general vector set \( \{x_k\}_{k=1}^{n-N} \), \( x_k \in \mathbb{C}^N \) reads in matrix form \( \sum_{k=1}^{n-N} x_k x_k^\dagger = MM^\dagger = I_N \), \( x_k \in \mathbb{C}^N \), \( M := [x_1, x_2, \ldots, x_n] \in \mathbb{C}^{N \times n} \), where \( M \) describes a partial matrix isometry \( M : \mathbb{C}^n \rightarrow \mathbb{C}^N \) with \( n > N \). The lifting of this isometry to a unitary mapping \( V : \mathbb{C}^n \rightarrow \mathbb{C}^n \) is known as Naimark dilation (extension) \( \mathcal{E} \) and can be accomplished by a suitable embedding \( M \leftarrow V = \begin{bmatrix} M & L \end{bmatrix} \), \( L \in \mathbb{C}^{(n-N) \times n} \) with additional unitarity constraint
\[
VV^\dagger = V^\dagger V = I_n. \tag{12}
\]

For our \( C^2 \times 4 \) setup the matrix \( M \) can be read off from eqs. (5) and (9) as \( M = f[\Psi, \Xi] = f[\rho^{-1}, \rho, \Phi] \), where \( \Phi := \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix} \). The natural ansatz \( V = f \begin{pmatrix} \rho^{-1} & \rho \\ X & Y \end{pmatrix} \Phi \) together with the auxiliary condition \( MM^\dagger = f^2[\rho^2 + \rho^{-2}] = f^2[\eta + \eta^{-1}] = I_2 \) (following from (7), (8) and (11)) and the constraint (12) fix the nonsingular matrices \( X, Y \) up to an irrelevant unitary rotation as \( X = \rho, Y = -\rho^{-1} \) so that
\[
V := f \left[ \sigma_z \otimes \rho^{-1} + \sigma_y \otimes \rho \right] (I_2 \otimes \Phi) = f \left[ \sigma_z \otimes \Psi + \sigma_y \otimes \Xi \right]. \tag{13}
\]

The columns of \( V = [v_1, v_2, v_3, v_4] \) are formed by four orthogonal vectors \( v_k \in \mathcal{H}_4 = \mathbb{C}^4 \) which provide the desired Naimark dilations of the nonorthogonal eigenvectors of \( H \) and its adjoint \( H^\dagger \) living in \( \mathcal{H}_2 = \mathbb{C}^2 \). Additionally, they yield the embedding \( A_k \hookrightarrow P_k = v_k v_k^\dagger \) of the \( \mathcal{H}_2 \)-POVM into the ortho-projector set \( P_j P_k = \delta_{jk} P_k, \sum_k P_k = I_4 \) in \( \mathcal{H}_4 \).

We start the construction of a selfconsistent CQM in \( \mathcal{H}_4 \) by requiring that the original eigenvalue problems for \( H \) and \( H^\dagger \) are recovered when the model is restricted to the first two rows of \( V \). From relations (10) and an ansatz \( f[\mathcal{H}, \mathcal{H}^\dagger | \Xi] = f[\mathcal{F} \Xi, \Xi \mathcal{E} \Xi] = M \mathcal{E} \) the eigenvalue matrix \( \mathcal{E} \) for the dilated problem can be read off as \( \mathcal{E} := \begin{pmatrix} \mathcal{E} & 0 \\ 0 & \mathcal{E} \end{pmatrix} = I_2 \otimes \mathcal{E} \). This means that the corresponding dilated Hamiltonian \( \mathcal{H} \) will have the two eigenvalues \( E_{\pm} \) of \( H \) and its isospectral adjoint \( H^\dagger \) as double degenerate eigenvalues. The Hamiltonian \( \mathcal{H} \) itself can be constructed from eq. (5) and \( \mathcal{H} = \mathcal{V} \mathcal{E} \mathcal{V}^\dagger \) as \( \mathcal{H} = \mathcal{V} \mathcal{E} \mathcal{V}^\dagger \) so that
\[
\mathcal{H} = f^2 \left[ I_2 \otimes (H \eta y - \eta H) + i \sigma_y \otimes (H - H^\dagger) \right] =: I_2 \otimes \Lambda + i \sigma_y \otimes \Omega
\]
where
\[
\Lambda := f^2(H \eta y - \eta H) = E_0 I_2 + \frac{\omega_0}{2} \cos(\alpha) \sigma_x
\]
\[
\Omega := f^2(H - H^\dagger) = i \frac{\omega_0}{2} \sin(\alpha) \sigma_z.
\]
This \( \mathcal{H} \) is Hermitian by construction. In the Hermitian limit of the original \( \mathcal{P} \mathcal{T} \)-symmetric Hamiltonian \( H \), i.e. for \( \alpha = \beta = 0 \), it holds \( \eta = I_2 \) and \( \mathcal{H} \) reduces to \( \mathcal{H} = I_2 \otimes h - a \) — a trivially doubled \( h \). In contrast to the PTQM Hamiltonian \( H \) its dilation \( \mathcal{H} \) remains well defined also in the strongly non-Hermitian vanishing-passage-time regime \( \mathcal{E} \) where the matrix components of \( \mathcal{H} \) diverge for fixed \( \omega_0 \) as \( s \rightarrow \infty \). This regularization effect is due to the normalization factor \( f^2 \) induced in \( \mathcal{H} \) via the auxiliary POVM construction.

The \( \mathcal{H} \)-induced unitary evolution in \( \mathcal{H}_4 \) is governed by the operator \( \mathcal{U}(t) = e^{-i\mathcal{H} t} = V e^{-i\mathcal{E} \mathcal{V} \mathcal{V}^\dagger} \) which via (9), (7), (8), (11), \( \mathcal{U}(t) = e^{-i\mathcal{H} t} = \rho^{-1} e^{-it \rho} \) and \( y = \omega_0 t/2 \) can be represented as
\[
\mathcal{U}(t) = f^2 \left\{ I_2 \otimes \left[ \mathcal{U}(t) \eta y^{-1} + \eta \mathcal{U}(t) \right] + i \sigma_y \otimes \left[ \mathcal{U}(t) - \eta \mathcal{U}(t) \eta^{-1} \right] \right\} = (I_2 \otimes F + i \sigma_y \otimes G) = \begin{pmatrix} F & G \\ -G & F \end{pmatrix}
\]
\[
F := e^{-i\omega_0 t} \left[ I_2 \cos(y) - i \sigma_x \sin(y) \cos(\alpha) \right]
\]
\[
G := e^{-i\omega_0 t} \left[ i \sin(y) \sin(\alpha) \sigma_z \right]. \tag{14}
\]

Physically \( \mathcal{U}(t) \) describes the time evolution of the coupled brachistochrone-ancilla system (10) in a Hilbert space \( \mathcal{H}_4 = \mathcal{H}_2 \otimes \mathcal{H}_2 \) with \( \psi(t) \in \mathcal{H}_2 \) and \( \chi(t) \in \mathcal{H}_2 \). In order to exactly reproduce the \( \mathcal{H}_2 \)-evolution (3) of the BBJM-brachistochrone subsystem
\[
\chi(t) = U(t) \psi_I = \frac{e^{-i\omega_0 t}}{\cos(\alpha)} \left( \cos(y - \alpha) - i \sin(y) \right) \tag{15}
\]
the initial vector \( \chi_I \in \mathcal{H}_2 \) of the ancilla subsystem should be chosen appropriately. To obtain \( \chi_I \) we represent \( \psi(t) \in \mathcal{H}_4 \) as
\[
\phi(t) = \left( \begin{array}{c} \psi(t) \\ \chi(t) \end{array} \right) = e_+ \otimes \psi(t) + e_- \otimes \chi(t) \tag{16}
\]
with \( e_+ := (1, 0)^T \) and \( e_- := (0, 1)^T \), define \( P_\pm := e_\pm \otimes e_\pm \) and introduce the projectors \( P_\pm = P_\pm \otimes I_2 \) on the brachistochrone (\( P_+ \)) and the ancilla (\( P_- \)) subspace. The identification rule (10) takes then the form
\[
\mathcal{P}_+ \mathcal{H} \psi(t) = \mathcal{P}_+ \mathcal{U}(t) \psi_I = e_+ \otimes \psi(t) = e_+ \otimes U(t) \psi_I. \tag{17}
\]
After taking the time derivative one finds from
\[
\mathcal{P}_+ \mathcal{H} \phi(t) = e_+ \otimes [\Lambda \psi(t) + \Omega \chi(t)] = e_+ \otimes \mathcal{H} \psi(t)
\]
a synchronization link (12) between ancilla and brachistochrone evolution \( \chi(t) = \Omega^{-1}(H - \Lambda) \psi(t) = \eta \psi(t) \) as well as the explicit ancilla evolution \( \chi(t) = \frac{e^{i\omega_0 t}}{\cos(\alpha)} \left( \cos(y - \alpha) - i \sin(y) \right) \right. \), \( \chi_F = \eta \psi_F = -\frac{\mu}{\cos(\alpha)} \left( \frac{\sin(\alpha)}{i} \right) \right. \) with \( \mu = e^{-i\omega_0 t} \) an irrelevant phase factor and \( \psi_I = (1, 0)^T, \psi_F = \mu(0, 1)^T \).
Discussion  For a BBJM-brachistochrone in the vanishing passage time regime \(1\), the ancilla vectors \(\chi_I\) and \(\chi_F\) become collinear and their common denominator \(\cos(\alpha) \approx \varepsilon\) makes them highly dominant compared to \(\psi_I, F\). This \(\chi\)-dominance remains preserved for the normalized state vector \(\hat{\phi} := g \phi, \langle \phi | \phi \rangle = 1\), with \(g := \cos(\alpha) / \sqrt{2}\) and leads to a very small brachistochrone component \(|\psi(t)|^2 \approx \varepsilon^2 / 2\) compared to the ancilla component \(|g\chi(t)|^2 \approx 1 - \varepsilon^2 / 2\). As result the geodesic distance between the initial and final states in \(\mathcal{H}_4\) becomes small \(\delta_4 \approx 2 \arccos(|\langle \phi | \phi \rangle |) \approx 2\varepsilon\). This means that the original geodesic distance \(\delta = 2 \arccos(|\langle \psi | \psi \rangle |) = \pi\) between the initial and final states in the brachistochrone subsystem is strongly contracted by embedding the latter into the larger Hermitian \(\mathcal{H}_4\)-system. Geometrically, this follows from the fact that the geodesic distance is given by the angle spanned by the corresponding vectors on the Bloch sphere \(2\) and its generalization to higher dimensions. In the \(\mathcal{H}_2\)-subsystem the vectors \(\psi_I\) and \(\psi_F\) are antipodal and span an angle \(\delta = \pi\). Adding a much longer vector \(\chi\) orthogonal to \(\psi_I\) and \(\psi_F\) makes the resulting \(\hat{\phi}_I \approx (\psi_I, \chi)\) and \(\hat{\phi}_F \approx (\psi_F, \chi)\) almost collinear \(\delta_4 \to 0\) in \(\mathcal{H}_4\). In this way the dilated model reconciles the Aharonov-Anandan lower bound \(2, 13\) on minimal passage times in Hermitian systems with the vanishing passage time effect of the BBJM-brachistochrone \(1\) for orthogonal states in the subsystem. The embedding of the BBJM-system into a higher-dimensional Hilbert space can be regarded as a strength
tem one has to prepare an initial entangled state \(\hat{\alpha}\) to switch on the interaction Hamiltonian \(\mathcal{H}\) during the passage time \(\tau\). A wormhole connection of two distant points on a given lower dimensional manifold \(\mathcal{M}\) can be best visualized by embedding \(\mathcal{M}\) into a higher dimensional surrounding \(\mathcal{N} \supset \mathcal{M}\) so that the corresponding short distance in \(\mathcal{N}\) becomes obvious \(14\).

The representation \(10\) indicates on a natural interpretation of the obtained Hermitian system \(\hat{\phi}(t) = e^{-i t \mathcal{H}} \hat{\phi}_I\) as system of two entangled spin 1/2 particles, i.e. as a two-qubit system \(5\), with \(\Sigma_1 = \sigma_+ \otimes I_2\) and \(\Sigma_2 = I_2 \otimes \sigma_z\) as spin operators of the two spin-subsystems. In order to observe the BBJM-brachistochrone effect of the subsystem one has to prepare an initial entangled state \(\hat{\phi}_I = e^+ \otimes \psi_I + e^- \otimes \chi_I\), to switch on the interaction Hamiltonian \(\mathcal{H}\) during the passage time \(\tau\) (assumed as smaller than the lower passage time bound \(\tau_0 = \pi / \omega_0\)) and to evolve \(\hat{\phi}_I\) into the final state \(\hat{\phi}_F = e^{-i t \mathcal{H}} \hat{\phi}_I\). This final state has to be analyzed in a two-step measurement. In a first (instantaneous) \(\Sigma_1\)-measurement one selects (filters out) the up-component \(e^+\) of the first spin. This results in a state \(\hat{\phi}_F = (\hat{\phi}_F | \hat{\phi}_F \rangle)_{1/2} = e^+ \otimes \psi_F / |\psi_F| |\psi_F\rangle_{1/2}\) and separates the brachistochrone component from the ancilla component (connected with the down-component \(e^-\) of the first spin). In a subsequent \(\Sigma_2\)-measurement, one analyzes the spin-up and spin-down states of the brachistochrone component \(|\psi_F | \langle \psi_F |\psi_F\rangle_{1/2}|\) to recover the spin-flip effect from \(\psi_I = (1, 0)^T\) to \(\psi_F = \mu_F (0, 1)^T\).

A direct experimental test should be feasible with a suitably designed system of entangled photons passing an appropriately chosen arrangement of beam splitters, phase shifters, and mirrors as implementation of the unitary operator \(U(t) = e^{-i t \mathcal{H}}\) \(10\).

Conclusions We have demonstrated that the quantum brachistochrone for a system with \(\mathcal{P} T\)-symmetric Hamiltonian can be realized as subsystem of a larger Hermitian system living in a higher-dimensional Hilbert space. The Hermitian system (constructed by Naim dilating an auxiliary positive operator valued measure) has the structure of an entangled two-quibit system. This opens a way to direct experimental tests on the recently hypothesized ‘faster than Hermitian’ evolution in \(\mathcal{P} T\)-symmetric quantum systems.

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* Electronic address: u.guenther@fzd.de
† Electronic address: samsonov@phys.tsu.ru

[1] C. M. Bender, D. C. Brody, H. F. Jones and B. K. Meister, Phys. Rev. Lett. 98, 040403 (2007).
[2] D. J. Brody, J. Phys. A 36, 5587 (2003); D. J. Brody and D. W. Hook, J. Phys. A 39, L167 (2006).
[3] A. Carlini, A. Hosoya, T. Koike, and Y. Okudaira, Phys. Rev. Lett. 96, 060503 (2006).
[4] U. G" untner, I. Rotter and B. Samsonov, J. Phys. A 40, 8815 (2007).
[5] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information (CUP, Cambridge, 2000).
[6] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. 73, 1032 (1994).
[7] E. W. Hartmann, J. Phys. A 40, 7267 (2007).
[8] A. Mostafazadeh, Phys. Rev. Lett. 99, 170502 (2007).
[9] A. Mostafazadeh, J. Math. Phys. 43, 2814 (2002).
[10] U. G" untner and B. Samsonov, e-print arXiv:0709.0433.
[11] A. S. Holevo, Probabilistic and statistical aspects of quantum theory, (North-Holland, Amsterdam, 1982); A. S. Holevo, Statistical structure of quantum theory, (Springer, Berlin, 2001).
[12] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. 73, 58 (1994).
[13] A. Peres, Quantum theory: concepts and methods, (Kluwer, Dordrecht, 1993).
[14] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
[15] A. S. Holevo, Probabilistic and statistical aspects of quantum theory, (North-Holland, Amsterdam, 1982); A. S. Holevo, Statistical structure of quantum theory, (Springer, Berlin, 2001).
[16] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. 73, 58 (1994).
[17] A. Peres, Quantum theory: concepts and methods, (Kluwer, Dordrecht, 1993).
[18] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
[19] J. Anandan and Y. Aharonov, Phys. Rev. Lett. 65, 1697 (1990).
[20] M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. 61, 1446 (1988).
[21] Comparison with the relations \(9\) between the \(\mathcal{P} T\)-symmetric subsystem and its adjoint, i.e. with \(\Sigma = \pi\Psi\), shows that the ancilla \(\chi(t)\) can be interpreted as adjoint to BBJM brachistochrone.