Complete linearization of a mixed problem to the Maxwell–Bloch equations by matrix Riemann–Hilbert problems

Vladimir Kotlyarov

Institute for Low Temperature Physics 47, Lenin ave, 61103 Kharkiv, Ukraine

E-mail: kotlyarov@ilt.kharkov.ua

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Abstract

Considered in this paper, the Maxwell–Bloch (MB) equations became known after Lamb (1967 Phys. Lett. A 25 181–2; 1971 Rev. Mod. Phys. 43 99–144; 1973 Phys. Rev. Lett. 31 196–199; 1974 Phys. Rev. A 9 422–30). Ablowitz, Kaup and Newell (1974 J. Math. Phys. 15 1852–58) proposed the inverse scattering transform (IST) to the MB equations for studying a physical phenomenon known as the self-induced transparency. A description of general solutions to the MB equations and their classification was done by Gabitov, Zakharov and Mikhailov (1985 Teor. Mat. Fiz. 63 11–31). In particular, they gave an approximate solution of the mixed problem to the MB equations in the domain \( x, t \in (0, L) \times (0, \infty) \) and, on this basis, a description of the phenomenon of superfluorescence. It was emphasized in Gabitov et al (1985 Teor. Mat. Fiz. 63 11–31) that the IST method is non-adopted for the mixed problem. Authors of the mentioned papers have developed the IST method in the form of the Marchenko integral equations. We propose another approach for solving the mixed problem to the MB equations in the quarter plane. We use a simultaneous spectral analysis of both the Lax operators and matrix Riemann–Hilbert (RH) problems. Firstly, we introduce compatible solutions of the corresponding Ablowitz–Kaup–Newell–Segur equations and then we suggest such a matrix RH problem which corresponds to the mixed problem for MB equations. Secondly, we generalize this matrix RH problem, prove a unique solvability of the new RH problem and show that the RH problem (after a specialization of jump matrix) generates the MB equations. As a result we obtain solutions defined on the whole line and studied in Ablowitz (1974 J. Math. Phys. 15 1852–58) and Gabitov et al (1985 Teor. Mat. Fiz. 63 11–31), solutions to the mixed problem studied below in this paper and solutions with periodic (finite-gap) boundary conditions. The type of solution is defined by specialization of the conjugation contour and jump matrix. Suggested matrix RH problems will be useful for studying the long time/long distance (\( x \in \mathbb{R}_+ \))
asymptotic behavior of solutions to the MB equations using the Deift–Zhou method of steepest decent.

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1. Introduction

The Maxwell–Bloch (MB) equations arise in different physical problems. The most significant applications of this system deal with the problem of the propagation of an electromagnetic wave (ultrashort optical pulse) in a resonant medium with distributed two-level atoms. In particular, there is the problem of self-induced transparency, the laser problems of the quantum amplifier and superfluorescence. The system of the MB equations can be written in the form

\[ \mathcal{E}_t + \mathcal{E}_x = \langle \rho \rangle, \]

\[ \rho_t + 2i\lambda \rho = \mathcal{N} \mathcal{E}, \]

\[ \mathcal{N}_t = -\frac{1}{2}(\mathcal{E}^* \rho + \mathcal{E} \rho^*). \]

Here \( \mathcal{E} = \mathcal{E}(t, x) \) is the complex electric field envelope, so that the field in the resonant medium is

\[ \mathbf{E}(t, x) = \mathcal{E}(t, x) e^{i\Omega_1(x-t)} + \mathcal{E}^*(t, x) e^{-i\Omega_1(x-t)}. \]

Subindexes mean partial derivatives in \( t \) and \( x \), and \( * \) means a complex conjugation. \( N = N(t, x, \lambda) \) and \( \rho = \rho(t, x, \lambda) \) are entries of the density matrix of a quantum two-level atom subsystem. The parameter \( \lambda \) is the deviation of the transition frequency of the given two-level atom from the mean frequency \( \Omega_1 \). The angular brackets mean averaging

\[ \langle \rho \rangle = \Omega \int_{-\infty}^{\infty} n(\lambda) \rho(t, x, \lambda) \, d\lambda \]

with the given ‘weight’ function \( n(\lambda) \), such that

\[ \int_{-\infty}^{\infty} n(\lambda) \, d\lambda = \pm 1. \]

The weight function \( n(\lambda) \) characterizes the inhomogeneous broadening, i.e. a form of the spectral line. It is the difference between the initial populations of the upper and lower levels. If \( n(\lambda) > 0 \), then an unstable medium is considered (the so-called quantum laser amplifier). If \( n(\lambda) < 0 \), then a stable medium is considered (the so-called attenuator).

The mixed problem of the MB equations is defined by initial and boundary conditions

\[ \mathcal{E}(0, x) = \mathcal{E}_0(x), \quad 0 < x < L \leq \infty, \]

\[ \rho(0, x, \lambda) = \rho_0(x, \lambda), \]

\[ \mathcal{N}(0, x, \lambda) = \mathcal{N}_0(x, \lambda), \]

\[ \mathcal{E}(t, 0) = \mathcal{E}_m(t), \quad 0 < t < \infty, \]

\[ \mathcal{E}_m(0) = \mathcal{E}_0(0). \]
We assume that entering in the medium \(0 < x < L\) pulse \(E_m(t)\) is smooth and fast decreasing. Initial functions \(E_0(x)\) and \(\rho_0(x, \lambda)\) are also smooth for \(0 < x < L\) and \(\lambda \in \mathbb{R}\). Function \(N_0(x, \lambda)\) is defined by \(\rho_0(x, \lambda)\):

\[
N_0(x, \lambda) = \sqrt{1 - |\rho_0(x, \lambda)|^2}, \quad 0 < x < L \leq \infty. \tag{1.9}
\]

Here we choose the positive branch of the square root and to find (1.9) we use the second and third equations of (1.1)–(1.3). They give

\[
\frac{\partial}{\partial t}(N^2(t, x, \lambda) + |\rho(t, x, \lambda)|^2) = 0,
\]

and one can put

\[N^2(t, x, \lambda) + |\rho(t, x, \lambda)|^2 \equiv 1.\]

In what follows we also put \(\Omega = 1\) in (1.4).

The Lax pair for the MB system was first found in [5] by using the results of [1–4] (see also [6, 7] and [8, 9]). It was shown that (1.1)–(1.3) are equivalent to the over-determined linear system, known as Ablowitz–Kaup–Newell–Segur (AKNS) equations:

\[
w_t + i\lambda \sigma_3 w = -H(t, x) w, \tag{1.10}
\]

\[
w_x - i\lambda \sigma_3 w + iG(t, x, \lambda) w = H(t, x) w, \tag{1.11}
\]

where \(\sigma_3, H(t, x), G(t, x, \lambda)\) are the matrices

\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H(t, x) = \frac{1}{2} \begin{pmatrix} 0 & \mathcal{E}(t, x) \\ -\mathcal{E}^*(t, x) & 0 \end{pmatrix},
\]

\[
G(t, x, \lambda) = \frac{1}{4} \text{p.v.} \int_{-\infty}^{\infty} \begin{pmatrix} N(t, x, s) & \rho(t, x, s) \\ \rho^*(t, x, s) & -N(t, x, s) \end{pmatrix} \frac{n(s)}{s - \lambda} \, ds,
\]

and the symbol p.v. denotes the Cauchy principal value integral. Differential equations (1.10) and (1.11) are compatible if and only if \(\dot{\mathcal{E}}(t, x), \rho(t, x, \lambda)\) and \(N(t, x, \lambda)\) satisfy equations (1.1)–(1.3) (see [7]). As shown in [4], \(\rho(t, x, \lambda)\) and \(N(t, x, \lambda)\) are related to the fundamental solutions of (1.10). Indeed, let \(\Phi(t, x, \lambda)\) be a solution of (1.10) such that \(\det \Phi(t, x, \lambda) \equiv 1\). Then it is easy to show that \(F(t, x, \lambda) = \Phi(t, x, \lambda) \sigma_3 \Phi^\dagger(t, x, \lambda)\), where \(\Phi^\dagger\) is the Hermitian-conjugated to \(\Phi\), satisfies equation

\[
F_t + [i\lambda \sigma_3 + H, F] = 0, \quad F(t, x, \lambda) = \begin{pmatrix} \mathcal{N}(t, x, \lambda) & \rho(t, x, \lambda) \\ \rho^*(t, x, \lambda) & -\mathcal{N}(t, x, \lambda) \end{pmatrix}, \tag{1.12}
\]

which is a matrix representation of equations (1.2) and (1.3). It is well-known that the fundamental solution of equation (1.10) is entire in \(\lambda\). Therefore the matrix \(F(t, x, \lambda) = \Phi(t, x, \lambda) \sigma_3 \Phi^\dagger(t, x, \lambda)\) is smooth in \(\lambda \in \mathbb{R}\) if the initial matrix \(F(0, x, \lambda)\) is smooth too.

In what follows we will use the \(x^+\)-equations (upper/lower bank equations instead of \(x\)-equation):

\[
w_x - i\lambda \sigma_3 w + iG_{+}(t, x, \lambda) w = H(t, x) w, \tag{1.13}
\]

where

\[
G_{+}(t, x, \lambda) = \frac{1}{4} \int_{-\infty}^{\infty} \frac{F(t, x, s) n(s)}{s - \lambda + i0} \, ds = \frac{1}{4} \text{p.v.} \int_{-\infty}^{\infty} \frac{F(t, x, s) n(s)}{s - \lambda} \, ds \pm \frac{\pi i}{4} F(t, x, \lambda) n(\lambda).
\]

Thus we have new Lax pairs (\(t\)- and \(x^+\)-equations and \(t\)- and \(x^-\)-equations) for the MB equations. Equations (1.10) and (1.13) (as well as (1.10) and (1.11)) are compatible if and only
if $E(t, x), \rho(t, x, \lambda)$ and $N(t, x, \lambda)$ satisfy equations (1.1)-(1.3). This known fact is proved below (see section 2).

The main goal of this paper is to show that the inverse scattering transform (IST) is applicable to the mixed problem for MB equations using a simultaneous spectral analysis of the compatible differential equations (1.10) and (1.13). We develop the IST method in the form of matrix Riemann–Hilbert (RH) problems in the complex $z$-plane and give an integral representation for $E(t, x)$ through the solution of a singular integral equation, which is equivalent to the matrix RH problem. To produce the RH problem we first suppose that the solution of the mixed problem does exist and then, on this basis, introduce appropriate compatible solutions of the AKNS equations (section 2), transition matrices and spectral data of the problem (section 3). The spectral data are defined through given initial and boundary conditions for MB equations by solving direct scattering problems for AKNS equations with $t = 0$ and $x = 0$ respectively. After, we suggest such a matrix RH problem which corresponds to the mixed problem for MB equations (section 4). Further, we give a formulation of a more general matrix RH problem, to prove a unique solvability of the new RH problem.

After specialization of a jump matrix we prove that the RH problem generates the AKNS linear equations for the MB equations with a given inhomogeneous broadening (section 5). Thus, this RH problem generates different solutions to the MB equations. There are solutions defined on the whole line and studied in [5] and [6], solutions to the mixed problem (1.4)-(1.8) with vanishing boundary conditions and periodic (finite-gap) boundary conditions, etc. The kind of solution is defined by specialization of the conjugation contour and jump matrix. In particular, theorems 5.1 and 5.2 guarantee unique solvability of the mixed problem. Suggested matrix RH problems will be useful for studying the long time/long distance ($x \in \mathbb{R}^+$) asymptotic behavior of solutions to the MB equations using the nonlinear method of steepest descent.

Our approach differs from those which were proposed in [19] for initial boundary problems to integrable nonlinear equations. Described in [19] problems remain (in general) nonlinearized because the proposed methods rely on nonlinear integro-differential equations; only special cases are linearized. Another situation takes place for the Goursat problem to the MB equations and some other integrable equations (see [10] and references therein). In this case $t$ and $x$ are light-cone coordinates while we consider MB equations in laboratory coordinates. In particular, for the MB equations, the corresponding Goursat problem gives linear ODEs for coefficients of the RH problem. In our approach all auxiliary problems are also linear. The author of [10] uses the scattering problem and the RH problem based on $t$-equation only, while we simultaneously exploit scattering problems for the $t$-equation (1.10) as well as for $x^\pm$-equations (1.13). The author of [10] works with the $x$-equation (1.11) in which the spectral parameter is real. He uses this equation for defining the ‘evolution’ of scattering data in $x$, but not for analytical construction of the RH problem. The main novelty of our method is the possibility of working with the complex spectral parameter in $x^\pm$-equations (1.13) and using corresponding solutions for the construction of a new type of RH problem. We develop the method, initially proposed in [11–14] and also in [15–18], which leads to essentially different RH problems than those appearing in classical scattering problems when we deal with one Lax operator. This method uses the simultaneous spectral analysis of the Lax operators with a complex spectral parameter. The method allows us to formulate such a matrix RH problem which proves that the mixed problem is completely linearizable. In our mind, the success of linearization was obtained due to the use of the mentioned (three) Lax operators and because we deal with the system of the partial differential equation of the first order.

We will use the following notations: if $A$ denotes matrix $A = (A[1], A[2])$, then vectors $A[1], A[2]$ denote the first and the second columns of matrix $A$. We also set $[A, B] = AB - BA$. 


2. Compatible solutions of the AKNS equations

Let

\[ W_t = U(t, x, \lambda)W, \quad (2.1) \]
\[ W_x = V(t, x, \lambda)W \quad (2.2) \]

be a system of matrix differential equations. Equations (2.1), (2.2) are compatible if \( W_{tx} \equiv W_{xt} \) for any solution of these equations, i.e. the following condition holds:

\[ U_t - V_x + UV - VU = 0. \quad (2.3) \]

**Lemma 2.1.** Let \( U(t, x, \lambda) \) and \( V(t, x, \lambda) \) be defined and smooth for all \( t, x, \lambda \in \mathbb{R} \) and satisfy the condition (2.3). Let \( W(t, x, \lambda) \) satisfy \( t \)-equation (2.1) for all \( x \), and let \( W(t_0, x, \lambda) \) satisfy \( x \)-equation (2.2) for some \( t = t_0, t_0 \leq \infty \). Then \( W(t, x, \lambda) \) satisfies \( x \)-equation for all \( t \). The same result is true if one changes \( t \) into \( x \) and vice versa.

**Proof.** Let \( \hat{W}(t, x, \lambda) = W_t - V(t, x, \lambda)W \). Then \( \hat{W}(t, x, \lambda) \) solves equation (2.1). Indeed

\[ \hat{W}_t = U(t, x, \lambda)\hat{W} + (U_t - V_x + [U, V])W = U(t, x, \lambda)\hat{W}. \]

Matrices \( W, \hat{W} \) are solutions of the same equation. Therefore they are linear dependent, i.e. \( \hat{W}(t, x, \lambda) = W(x, t, \lambda)C(x, \lambda) \). Since \( \hat{W}(t_0, x, \lambda) = 0 \), then \( C(x, \lambda) \) and \( \hat{W}(t, x, \lambda) \) are identically equal to zero. Lemma 2.1 is proved.

Due to (1.10), (1.11) and (1.13) we have three AKNS systems. The \( t \)-equation (1.10) is defined by matrix \( U(t, x, \lambda) = -i\lambda \sigma_3 - H(t, x) \) while the \( x \)-equation (1.11) and upper/lower bank \( x^\pm \)-equations (1.13) are defined by matrices:

\[ V(t, x, \lambda) = \begin{pmatrix} i\lambda \sigma_3 + H(t, x) - iG(t, x, \lambda) \\ V(t, x, \lambda) = V(t, x, \lambda) \pm \frac{\pi \sin(\lambda)}{4} F(t, x, \lambda). \end{pmatrix} \]

The compatibility condition of \( t \)-equation (1.10) and \( x \)-equation (1.11) takes the form of (2.3).

The compatibility condition of \( t \)-equation (1.10) and \( x^\pm \)-equations (1.13) is as follows:

\[ U_t - V_t^\pm + [U, V^\pm] = U_t - V_t + [U, V] \pm \frac{\pi \sin(\lambda)}{4} F(t, x, \lambda) = 0. \]

If \( F_t + [i\lambda \sigma_3 + H, F] = 0 \) (in view of equation (1.12)), then the new compatibility conditions coincide with (2.3). Conversely, if \( U_t - V_t + [U, V] = 0 \), then \( F_t + [i\lambda \sigma_3 + H, F] = 0 \) as well as \( U_t - V_t^\pm + [U, V^\pm] = 0 \). Each of them gives the same MB equations. The upper/lower bank equation (1.13) (in contrast with \( x \)-equation (1.11)) will allow us to obtain such vector-solutions of the AKNS equations, which have analytic continuation to the upper and lower complex \( z \)-plane \( (z = \lambda + iv) \). We will use them for a construction of the matrix RH problem, which gives a solution of the mixed problem to the MB equations.

**Theorem 2.1.** Let \( \mathcal{E}(t, x), N(t, x, \lambda), \rho(t, x, \lambda) \) be a smooth solution to the MB equations (1.1)–(1.3) defined for all \( t, x \in \mathbb{R}^+ \) and \( \lambda \in \mathbb{R} \). Let \( \mathcal{E}(t, 0) \) be as fast decreasing as \( t \to +\infty \) and let \( \rho(0, x, \lambda) \equiv 0 \) for \( x > L \) or is also as fast decreasing as \( x \to +\infty \) (if \( L = \infty \)). Then there exists two pairs of compatible solutions \( Y^\pm(t, x, \lambda) \) and \( Z^\pm(t, x, \lambda) \) of the AKNS equations (1.10), (1.13) such that (see figure 1)

\[ Y^\pm(t, x, \lambda) = W^\pm(t, x, \lambda)\Phi(t, \lambda), \quad (2.4) \]
\[ Z^\pm(t, x, \lambda) = \Psi(t, x, \lambda)w^\pm(x, \lambda). \quad (2.5) \]
Here $W^\pm(t, x, \lambda)$ satisfies $x^\pm$-equation (1.13) for all $t$, $W(t, 0, \lambda) = I$, and $\Phi(t, \lambda)$ satisfies t-equation (1.10) with $x = 0$ under the Jost asymptotic condition:

$$\lim_{t \to \infty} \Phi(t, \lambda) e^{i\lambda t \sigma_3} = I.$$  

Function $\Psi(t, x, \lambda)$ satisfies t-equation (1.10) under the initial condition $\Psi(0, x, \lambda) = I$, and $w^\pm(x, \lambda)$ satisfies $x^\pm$-equation (1.13) with $t = 0$ under the initial condition $w^\pm(L, \lambda) = e^{i\eta_\pm(\lambda) \sigma_3}$, where

$$\eta_\pm(\lambda) = \lambda - 1 \int_{-\infty}^{\infty} \frac{n(s) \text{d}s}{s - \lambda + 0^+}.$$  

If $L = \infty$ then the condition at $x = L$ changes with the Jost asymptotic condition:

$$\lim_{x \to \infty} w^\pm(x, \lambda) e^{-i\eta_\pm(\lambda) \sigma_3} = I.$$

**Proof.** If matrices $W^\pm(t, x, \lambda)$, $\Phi(t, \lambda)$ and $\Psi(t, x, \lambda)$, $w^\pm(x, \lambda)$ do exist then, due to lemma 2.1, the products $W^\pm(t, x, \lambda) \Phi(t, \lambda)$ and $\Psi(t, x, \lambda) w^\pm(x, \lambda)$ are compatible solutions of the AKNS equations (1.10), (1.13). The existence of these matrices will be given in the next lemmas. \[ \square \]

**Lemma 2.2.** Let $E(t, 0) = E^m(t)$ be smooth and as fast decreasing as $t \to \infty$. Then the Jost solution $\Phi(t, \lambda)$ has an integral representation:

$$\Phi(t, \lambda) = e^{-i\lambda t \sigma_3} + \int_t^\infty \hat{K}(t, \tau) e^{-i\lambda \tau \sigma_3} \text{d}\tau, \quad \text{Im} \lambda = 0.$$  

The kernel $\hat{K}(t, \tau)$ satisfies the symmetry condition $\hat{K}^*(t, \tau) = \Lambda \hat{K}(t, \tau) \Lambda^{-1}$ with matrix $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and it is defined by the Goursat problem:

$$\begin{align*}
\sigma_3 \hat{K}_t(t, \tau) + \hat{K}_x(t, \tau) \sigma_5 &= -\sigma_3 H(t, 0) \hat{K}(t, \tau) \\
\sigma_3 \hat{K}(t, t) - \hat{K}(t, \tau) \sigma_3 &= \sigma_3 H(t, 0).
\end{align*}$$

The kernel $\hat{K}(t, \tau)$ is smooth and as fast decreasing as $t + \tau \to \infty$.

The proof of this lemma is well-known (see [20]). Due to this integral representation, vector columns $\Phi[1](t, \lambda)$ and $\Phi[2](t, \lambda)$ of the matrix $\Phi(t, \lambda) = (\Phi[1](t, \lambda), \Phi[2](t, \lambda))$ have analytic continuations $\Phi[1](t, z)$ and $\Phi[2](t, z)$ to the lower and upper half-planes of the complex $z$-plane respectively. The next asymptotic formulas are valid:

- $\Phi[1](t, z) e^{i\nu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(z^{-1})$, $\text{Im} z \leq 0$, $z \to \infty$,
The solvability of the Volterra integral equation (2.9) and the smoothness of the solution with respect to \( t \) and \( x \) can be easily proved by using the method of successive approximations. Solutions \( W^\pm(t, x, \lambda) \) are not independent. One can easily verify that \( W^+(t, x, \lambda) \) and \( W^-(t, x, \lambda) \) have the above asymptotic behavior.

**Lemma 2.3.** Let \( n(\lambda) \) be a Hölder function for \( \lambda \in \mathbb{R} \) and let \( E(t, x), N(t, x, \lambda), \rho(t, x, \lambda) \) be smooth functions for \( t, \lambda \in \mathbb{R} \) and \( x \in \mathbb{R}_+ \). Then the solution \( W^\pm(t, x, \lambda) \) can be represented in the form:

\[
W^\pm(t, x, \lambda) = e^{i\lambda x \sigma^1} \chi^\pm(t, x, \lambda),
\]

where \( \chi^\pm(t, x, \lambda) \) is the unique solution of the Volterra integral equation

\[
\chi^\pm(t, x, \lambda) = I + \int_0^t e^{-i\lambda y} (H(t, y) - iG_\pm(t, y, \lambda)) e^{i\lambda x} \chi^\pm(t, y, \lambda) \, dy.
\]

The solutions \( W^\pm(t, x, \lambda) \) are smooth in \( t \) and \( x \) and related to each other by the formula:

\[
W^-(t, x, \lambda) = \sigma_2 (W^+(t, x, \lambda))^* \sigma_2, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}
\]

and have an analytic continuations \( W(t, x, \lambda) \) to the upper \( \mathbb{C}_+ \) and \( \overline{W}(t, x, z) \) to the lower \( \mathbb{C}_- \) complex \( z \)-plane respectively. Moreover, \( \overline{W}(t, x, z) e^{-izx} \) is continuous, bounded in \( \mathbb{C}_- \cup \mathbb{R} \) and has the following asymptotics:

\[
\overline{W}(t, x, z) e^{-izx} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(z^{-1}), \quad \text{Im} z \leq 0, \quad z \to \infty,
\]

and matrix \( W(t, x, z) e^{izx} \) is continuous, bounded in \( \mathbb{C}_- \cup \mathbb{R} \) and has the following asymptotics:

\[
W(t, x, z) e^{izx} = \begin{pmatrix} e^{izx} & 0 \\ 0 & 1 \end{pmatrix} + O(z^{-1}), \quad \text{Im} z \geq 0, \quad z \to \infty.
\]

**Proof.** The solvability of the Volterra integral equation (2.9) and the smoothness of the solution can be evidenced by the method of successive approximations. Solutions \( W^\pm(t, x, \lambda) \) are not independent. One can easily verify that \( W^+(t, x, \lambda) \) and \( W^-(t, x, \lambda) \) have the above asymptotic behavior.

The previous lemmas and formulas (2.4), (2.7)–(2.9) imply the following properties of the matrix \( Y^\pm(t, x, \lambda) = (Y^\pm[1](t, x, \lambda) \quad Y^\pm[2](t, x, \lambda)) \):

(i) \( Y^\pm(t, x, \lambda) \) satisfy the \( t- \) and \( x^\pm \)-equations;

(ii) \( \det Y^\pm(t, x, \lambda) \equiv 1, \quad \lambda \in \mathbb{R} \);

(iii) the map \( (x, t) \mapsto Y^\pm(t, x, \lambda) \) is smooth in \( t \) and \( x \);

(iv) vector column \( Y^-[1](t, x, \lambda) \) has analytic continuation \( \overline{Y}[1](t, x, z) \) for \( z \in \mathbb{C}_- \), which is continuous and bounded in \( z \in \mathbb{C}_- \cup \mathbb{R} \) and

\[
\overline{Y}[1](t, x, z) e^{izx-ix^3} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(z^{-1}), \quad z \to \infty;
\]

(v) vector column \( Y^+[2](t, x, \lambda) \) has analytic continuation \( Y[2](t, x, z) \) for \( z \in \mathbb{C}_+ \), which is continuous and bounded in \( z \in \mathbb{C}_+ \cup \mathbb{R} \) and

\[
Y[2](t, x, z) e^{-izx+ix^3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(z^{-1}), \quad z \to \infty.
\]
**Lemma 2.4.** Let $E(t, x)$ be smooth functions in the domain of its definition. Then the function $\Psi(t, x, \lambda)$ has an integral representation:

$$\Psi(t, x, \lambda) = e^{-i\lambda t} + \int_t^\tau \hat{L}(t, \tau, x) e^{-i\lambda \tau} \, d\tau.$$  \hfill (2.10)

The kernel $\hat{L}(t, x, \lambda)$ is smooth, satisfies the symmetry condition $\hat{L}^*(t, x, \lambda) = \Lambda \hat{L}(t, x, \lambda) \Lambda^{-1}$ with matrix $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and is defined by the Goursat problem:

$$\sigma_1 \hat{L}_x(t, x, \lambda) + \hat{L}_t(t, x, \lambda) \sigma_3 = -\sigma_3 H(t, x) \hat{L}(t, x, \lambda)$$

$$\sigma_2 \hat{L}_x(t, -t, x) - \hat{L}_t(t, -t, x) \sigma_3 = -\sigma_3 H(t, x),$$

$$\sigma_2 \hat{L}_x(t, -t, x) + \hat{L}_t(t, -t, x) \sigma_3 = 0.$$  

The proof is well-known (see [15]). This lemma gives that $\Psi(t, x, \lambda)$ has an analytic continuation $\psi(t, x, z)$ for all $z \in \mathbb{C}$. Moreover, $\Psi(t, x, z) e^{iz}$ is continuous, bounded in $\mathbb{C}_+ \cup \mathbb{R}$ and

$$\psi(t, x, z) e^{iz} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2iz} \end{pmatrix} + O(z^{-1}), \quad |z| \to \infty,$$  \hfill (2.11)

and $\psi(t, x, z) e^{-iz}$ is continuous, bounded in $\mathbb{C}_- \cup \mathbb{R}$ and

$$\psi(t, x, z) e^{-iz} = \begin{pmatrix} e^{-2iz} & 0 \\ 0 & 1 \end{pmatrix} + O(z^{-1}), \quad |z| \to \infty.$$  \hfill (2.12)

If $H(t, x) \equiv 0$ and $F(t, x, \lambda) \equiv \sigma_3$, then $x^\perp$-equation has the exact solution $e^{i\eta_\perp(t,x)}$, where $\eta_\perp(\lambda)$ is defined in (2.6). These functions are the boundary values of the sectorially analytic function

$$\eta(z) = z - \frac{1}{4} \int_{-\infty}^{\infty} \frac{n(s)}{s - z} \, ds$$

for $z \in \mathbb{C}_\perp$. Across the real line it has the jump:

$$\eta(\lambda + i0) - \eta(\lambda - i0) = -\frac{\pi i}{2} n(\lambda) \quad \lambda \in \mathbb{R}.$$  

Furthermore, for $z = \lambda + iv$,

$$\text{Im} \eta(z) = v \left( 1 - \frac{1}{4} \int_{-\infty}^{\infty} \frac{n(s) \, ds}{(s - \lambda)^2 + v^2} \right) = v(1 - I(\lambda, v)).$$

If $n(\lambda) < 0$ then $\text{sign} \text{Im} \eta(z) = \text{sign} \text{Re} z$. In the case $n(\lambda) > 0$,

$$\frac{\text{Im} \eta(z)}{\text{Im} z} > 0 \quad \text{if} \quad 1(\lambda, v) < 1; \quad \frac{\text{Im} \eta(z)}{\text{Im} z} < 0 \quad \text{if} \quad 1(\lambda, v) > 1.$$

For $n(\lambda) = \delta(\lambda)$ this integral equals to $1/4(\lambda^2 + v^2) = 1/|2z|^2$ and equation $I(\lambda, v) = 1$ defines the circle $|z| = 1/2$. In a general position when $n(\lambda) > 0$, there exists a curve, which is the boundary of a domain $D$ containing the origin of the complex $z$-plane. This curve is defined by equation: $\text{Im} \eta(z) = 0$ if $\text{Im} z \neq 0$. It is symmetric with respect to the real $\lambda$-axis, because $\eta^*(z^*) = \eta(z)$. We denote it as $\gamma \cup \overline{\gamma}$, where $\gamma$ lies in $\mathbb{C}_+$, and $\overline{\gamma}$ in $\mathbb{C}_-$. Thus there is such a domain $D = D^+ \cup D^-$ that

$$\text{sign} \text{Im} \eta(z) = \begin{cases} -\text{sign} \text{Im} z, & z \in D = D^+ \cup D^-, \\ \text{sign} \text{Im} z, & z \in (\mathbb{C}_+ \setminus D^+) \cup (\mathbb{C}_- \setminus D^-). \end{cases}$$

Domain $D$ may be bounded and unbounded as well. Indeed, let

$$n(\lambda) = \begin{cases} \frac{1}{2\varepsilon}, & |\lambda| \leq \varepsilon, \\ 0, & |\lambda| > \varepsilon. \end{cases}$$
Then the curve \( \gamma \) (\( \text{Im} \eta(z) = 0, \text{Im} z \neq 0 \)) is described by equation:

\[
\lambda = \lambda(v) = \sqrt{\epsilon^2 - v^2 + \frac{2\epsilon v}{\tan 8\epsilon v}},
\]

where the square root is positive. If \( \epsilon \) is positive and sufficiently small then there exists \( \delta > 0 \) such that

\[
\sqrt{\epsilon^2 - v^2 + \frac{1}{4 + \delta}} < \lambda(v) < \sqrt{\epsilon^2 - v^2 + \frac{1}{4}},
\]

i.e. finite curve \( \gamma \) together with interval \([\lambda_-, \lambda_+] (\lambda_\pm = \pm \sqrt{1/4 + \epsilon^2})\) bound domain \( D^+ \). An example of the unbounded domain \( D \) gives a physical and simple model with the Lorentzian line shape:

\[
n(\lambda) = \frac{l}{\pi \lambda^2 + l^2}, \quad l > 0.
\]

Then

\[
\eta(z) = z - \frac{1}{4} \int_{-\infty}^{\infty} \frac{n(\lambda)}{\lambda - z} \, d\lambda = \begin{cases} 
\frac{1}{4(z + i\epsilon)}, & z \in \mathbb{C}_+ \\
\frac{1}{4(z - i\epsilon)}, & z \in \mathbb{C}_-
\end{cases}
\]

and curve \( \gamma \) is defined by the equation:

\[
\lambda^2 = \frac{v + l}{4\epsilon} - (v + l)^2.
\]

This equation yields that \( \lambda_\pm(0) = \pm \infty \), i.e. \( (\lambda_-, \lambda_+) = \mathbb{R} \), and \( v_{\text{max}} = (\sqrt{1 + l^2} - l)/2 \). Thus, domain \( D^+ \) is unbounded along the real line because \( \lambda(v) \) is unbounded when \( v \to 0 \). Domain \( D^- \), being complex conjugated to \( D^+ \), is unbounded also.

If \( F(0, x, \lambda) = \sigma_3 \) for \( x > L \) we put

\[
\hat{H}^\pm(x, \lambda) = H(0, x) - i G_\pm(0, x, \lambda) - g_\pm(\lambda) \sigma_3 ), \quad 0 \leq x \leq L,
\]

\[
g_\pm(\lambda) = \frac{1}{4} \int_{-\infty}^{\infty} \frac{n(s) \, ds}{s - \lambda \mp i\epsilon}.
\]

**Lemma 2.5.** Let \( n(\lambda) \) be a Hölder function and as fast decreasing as \( \lambda \to \pm \infty \). Let \( E(0, x) = E_0(x), N(0, x, \lambda) = N_0(x, \lambda), \rho(0, x, \lambda) = \rho_0(x, \lambda) \) be smooth for \( \lambda \in \mathbb{R} \), \( 0 \leq x \leq L \) and as fast decreasing as \( x \to \infty \) when \( L = \infty \). Then \( w^\pm(x, \lambda) \) can be represented in the form:

\[
w^\pm(x, \lambda) = \hat{w}^\pm(x, \lambda) e^{\hat{h}_\pm(x, \lambda) \sigma_3}, \tag{2.14}
\]

where \( \hat{w}^\pm(x, \lambda) \) is the unique solution of the Volterra integral equation

\[
\hat{w}^\pm(x, \lambda) = I - \int_x^L e^{\hat{h}_\pm(x, \lambda) \sigma_3 y} \hat{H}_\pm(y, \lambda) \hat{w}^\pm(y, \lambda) e^{-\hat{h}_\pm(x, \lambda) \sigma_3 y} \, dy. \tag{2.15}
\]

The solutions \( w^\pm(x, \lambda) \) are smooth in \( t \) and \( x \), related to each other by the formula:

\[
w^-(x, \lambda) = \sigma_2 (w^+(x, \lambda))^* \sigma_2.
\]

If \( n(\lambda) < 0 \) then the first column \( w^+[1](x, \lambda) \) has an analytic continuation \( w[1](x, z) \) to the upper \( \mathbb{C}_+ \) complex half-plane and the second column \( w^-[2](x, \lambda) \) has an analytic continuation \( \overline{w}[2](x, z) \) to the lower \( \mathbb{C}_- \) half-plane. The corresponding analytic vector column \( w[1](x, z) e^{-\hat{h}[1](z) x} \) and \( \overline{w}[2](x, z) e^{\hat{h}(z) x} \) are continuous in \( \mathbb{C}_\pm \cup \mathbb{R} \) and have the following asymptotics:
If \( n(\lambda) > 0 \) then the first column \( w^+[1](x, \lambda) \) and the second column \( w^-[2](x, \lambda) \) have analytic continuations into domains \( \mathbb{C}_+ \setminus D^+ \) and \( \mathbb{C}_- \setminus D^- \) respectively with the same asymptotic behavior, while the first column \( w^-[1](x, \lambda) \) and the second column \( w^+[2](x, \lambda) \) have analytic (bounded) continuations \( \overline{w^}[1](x, z) \) and \( w[2](x, z) \) to the domains \( D^- \) and \( D^+ \) respectively.

The proof can be done in the same way as for lemma 2.3.

Lemmas 2.4, 2.5 and formulas (2.5), (2.10)–(2.12), (2.14), (2.15) imply the following properties of the matrix \( Z^+[t, x, \lambda) = (Z^+[1](t, x, \lambda), Z^+[2](t, x, \lambda)) \):

(i) \( Z^+[t, x, \lambda) \) satisfy the \( t^- \) and \( x^- \)-equations;
(ii) \( \det Z^+[t, x, \lambda) \equiv 1 \), \( \lambda \in \mathbb{R} \);
(iii) the map \((t, x) \mapsto Z^+[t, x, \lambda) \) is smooth in \( t \) and \( x \);
(iv) vector column \( Z^+[1](t, x, \lambda) \) has an analytic continuation \( Z[1](t, x, z) \) for \( z \in \mathbb{C_+} \) if \( n(\lambda) < 0 (z \in \mathbb{C_+} \setminus D^+) \) if \( n(\lambda) > 0 \), which is continuous in the closure of \( \mathbb{C}_+ (\mathbb{C}_+ \setminus D^+) \) and

\[
Z[1](t, x, z) e^{-i\chi_1(x, \lambda) z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(z^{-1}), \quad z \to \infty.
\]

(v) vector column \( Z^-[2](t, x, \lambda) \) has an analytic continuation \( \overline{Z}[2](t, x, z) \) for \( z \in \mathbb{C_-} \) if \( n(\lambda) < 0 (z \in \mathbb{C_-} \setminus D^-) \) if \( n(\lambda) > 0 \), which is continuous in the closure of \( \mathbb{C}_- (\mathbb{C}_- \setminus D^-) \) and

\[
\overline{Z}[2](t, x, z) e^{-i\chi_2(x, \lambda) (z - 1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(z^{-1}), \quad z \to \infty.
\]

If \( n(\lambda) > 0 \) then

(vi) vector column \( Z^-[1](t, x, \lambda) \) has an analytic continuation \( \overline{Z}[1](t, x, z) \) to the domain \( D^- \), and vector column \( Z^+[2](t, x, \lambda) \) has an analytic continuation \( \overline{Z}[2](t, x, z) \) to the domain \( D^+ \), where both of the analytic continuations are continuous up to the boundary of \( D^- \) and \( D^+ \) respectively.

To prove (iv) and (v) we use asymptotic relations (2.11), (2.12) and the following one (where signs \( \pm \) are omitted for convenience)

\[
w(t, x, \lambda) \sim \begin{pmatrix} e^{i\chi_1(x, \lambda)} & 0 \\ 0 & e^{-i\chi_2(x, \lambda)} \end{pmatrix} + \begin{pmatrix} \chi_{11}(x, \lambda) & e^{i\chi_1(x, \lambda)} \\ \chi_{21}(x, \lambda) & e^{-i\chi_2(x, \lambda)} \end{pmatrix} \begin{pmatrix} \chi_{12}(x, \lambda) & e^{-i\chi_1(x, \lambda)} \\ \chi_{22}(x, \lambda) & e^{i\chi_2(x, \lambda)} \end{pmatrix} \chi_{12}(x, \lambda).
\]

Here \( \chi_{11}(x, \lambda), \chi_{21}(x, \lambda) = O(\lambda^{-1}) + O(\lambda^{-1} e^{2iLx}) \) and \( \chi_{12}(x, \lambda), \chi_{22}(x, \lambda) = O(\lambda^{-1}) + O(\lambda^{-1} e^{-2iLx}) \) as \( \lambda \to \pm \infty \) (\( 0 \leq x \leq L < \infty \)). If \( L = \infty \) then \( \chi_{ij}(x, \lambda) = O(\lambda^{-1}) \) as \( \lambda \to \pm \infty \). Indeed, for the first vector column, we have

\[
Z[1](t, x, z) e^{i\chi_1(x, \lambda) z} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\chi_1(x, \lambda) z} + O(z^{-1}) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \chi_{11}(x, \lambda) \\ \chi_{21}(x, \lambda) \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(z^{-1}).
\]

For the second vector column there is

\[
\overline{Z}[2](t, x, z) e^{-i\chi_2(x, \lambda) (z - 1)} \sim \left[ \begin{pmatrix} e^{-2i\chi_1(x, \lambda)} \\ 0 \\ 0 \end{pmatrix} e^{(z - 1) \chi_1(x, \lambda)} + O(z^{-1}) \right] \left[ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \chi_{12}(x, \lambda) \\ \chi_{22}(x, \lambda) \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(z^{-1}).
\]

Due to analytic continuations of the first vector column to the upper half-plane and second vector column to the lower half-plane we have \( \chi_{11}(x, \lambda), \chi_{21}(x, \lambda) = O(z^{-1}) \) for \( \text{Im}\ z \geq 0 \) and \( \chi_{22}(x, \lambda), \chi_{22}(x, \lambda) = O(z^{-1}) \) for \( \text{Im}\ z \leq 0 \) as \( z \to \infty \).
Pairs of vectors \((\overline{\Phi}[1](t, x, z), \overline{Z}[2](t, x, z))\) and \((Z[1](t, x, z), Y[2](t, x, z))\) will allow us to define below the suitable matrix RH problem in the case \(n(\lambda) < 0\). In the case \(n(\lambda) > 0\) we will use additionally more vectors \((\overline{Z}[1](t, x, z), \overline{\Phi}[1](t, x, z))\) and \((Y[2](t, x, z), Z[2](t, x, z))\).

3. Transition matrices and spectral functions

Now we consider a dependence between solutions of the AKNS equations, corresponding transition matrices and spectral functions. First of all, since \(Y^\pm(t, x, \lambda)\) and \(Z^\pm(t, x, \lambda)\) are solutions of the \(t\)- and \(x\)-equations \((1.10), (1.13)\), they are linear dependent. So there exist transition matrices \(T^\pm(\lambda)\), independent of \(x\) and \(t\), such that

\[
Y^\pm(t, x, \lambda) = Z^\pm(t, x, \lambda)T^\pm(\lambda), \quad \text{Im} \lambda = 0.
\]  

They are equal to

\[
T^\pm(\lambda) = (w^\pm(0, \lambda))^{-1}\Phi(0, \lambda), \quad \text{Im} \lambda = 0
\]

and have the following structure:

\[
T^\pm(\lambda) = \begin{pmatrix}
\pi^\pm(\lambda) & b^\pm(\lambda) \\
-b^\pm(\lambda) & a^\pm(\lambda)
\end{pmatrix}.
\]

Matrices \(w^\pm(0, \lambda)\) have a similar structure

\[
w^\pm(0, \lambda) = \hat{w}^\pm(0, \lambda) = \begin{pmatrix}
a^\pm(\lambda) & -\overline{b}^\pm(\lambda) \\
\beta^\pm(\lambda) & \sigma^\pm(\lambda)
\end{pmatrix}
\]

and define the spectral functions of the \(x^\pm\)-equation. Since the coefficients of the \(x^\pm\)-equation are taken for the fixed time \(t = 0\), the spectral functions are uniquely defined by given initial functions \(\mathcal{E}(0, x), \rho(0, x, \lambda)\) and \(\mathcal{N}(0, x, \lambda)\). They are not independent because the solutions \(w^\pm(x, \lambda)\) of the \(x^\pm\)-equations are related to each other by \(w^-(x, \lambda) = \sigma_2(w^+(x, \lambda))^*\sigma_2\), that yields the following symmetry properties:

\[
\overline{\sigma}^\pm(\lambda) = (\alpha^\mp(\lambda))^*,
\overline{\beta}^\pm(\lambda) = -(\beta^\mp(\lambda))^*.
\]

The matrix

\[
\Phi(0, \lambda) = \begin{pmatrix}
\overline{A}(\lambda) & B(\lambda) \\
-\overline{B}(\lambda) & A(\lambda)
\end{pmatrix}
\]

gives the spectral functions of the \(t\)-equation with \(x = 0\). They are uniquely defined by the boundary condition \(\mathcal{E}(t, 0)\). The analogous conjugation condition \(\Phi(t, \lambda) = \sigma_2(\Phi^*(t, \lambda))^*\sigma_2\) means the following symmetry properties:

\[
\overline{A}(\lambda) = A^*(\lambda), \quad \overline{B}(\lambda) = B^*(\lambda).
\]

Finally we have that the spectral functions, generated by the transition matrix \(T^\pm(\lambda)\), are defined by initial and boundary conditions. This matrix satisfies the analogous symmetry property:

\[
T^-(\lambda) = \sigma_2(T^+(\lambda))^*\sigma_2,
\]

which is equivalent to the following ones:

\[
\overline{\pi}^\pm(\lambda) = (\alpha^\mp(\lambda))^*,
\overline{\beta}^\pm(\lambda) = (\beta^\mp(\lambda))^*.
\]  

(3.2)

Due to Fokas [11] we call all of these functions the spectral functions.
At the conjugated points
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Entries of the matrix \( T^\pm(\lambda) \) are
\[
\begin{align*}
\bar{a}^\pm(\lambda) &= \det[Y^\pm[1](t, x, \lambda), Z^\pm[2](t, x, \lambda)] \\
\bar{b}^\pm(\lambda) &= \det[Y^\pm[1](t, x, \lambda), Z^\pm[1](t, x, \lambda)] \\
b^\pm(\lambda) &= \det[Y^\pm[2](t, x, \lambda), Z^\pm[2](t, x, \lambda)] \\
a^\pm(\lambda) &= \det[Z^\pm[1](t, x, \lambda), Y^\pm[2](t, x, \lambda)].
\end{align*}
\]

If \( n(\lambda) < 0 \) these relations show that \( a^\pm(\lambda) \) has an analytic continuation \( a(z) \) for \( z \in \mathbb{C}_+ \),
and \( a^-(\lambda) \) has an analytic continuation \( \bar{a}(z) \) for \( z \in \mathbb{C}_- \) \((\bar{a}(z) = a^*(z*))\). Functions \( b^\pm(\lambda) \)
and \( \bar{b}^\pm(\lambda) \) are defined for \( \lambda \in \mathbb{R} \) only. Determinant of \( T^\pm(\lambda) \equiv 1 \) for \( \text{Im} \lambda = 0 \). The spectral functions have the following asymptotics:
\[
\begin{align*}
\bar{a}(z) &= 1 + O(z^{-1}), \quad \text{Im } z \leq 0, \quad |z| \to \infty. \\
b^\pm(\lambda) &= O(\lambda^{-1}), \quad \lambda \to \pm \infty, \\
a^\pm(\lambda) &= 1 + O(z^{-1}), \quad \text{Im } z \geq 0, \quad |z| \to \infty.
\end{align*}
\]

If \( a(z) \) have zeroes \( z_j \in \mathbb{C}_+ \) \((j = 1, 2, \ldots, p)\) then
\[
a(z_j) = \det[Z[1](t, x, z_j), Y[2](t, x, z_j)] = 0.
\]

Hence vector columns of the determinant are linear dependent:
\[
Y[2](t, x, z_j) = \gamma_j Z[1](t, x, z_j), \quad \gamma_j = \frac{B(z_j)}{a(z_j)} = \frac{A(z_j)}{\bar{b}(z_j)} \quad (3.3)
\]

At the conjugated points \( z_j^\ast \in \mathbb{C}_- \) \((j = 1, 2, \ldots, p)\)
\[
\bar{a}(z_j^\ast) = \det[\bar{Y}[1](t, x, z_j^\ast), \bar{Z}[2](t, x, z_j^\ast)] = 0.
\]

Therefore
\[
\bar{Y}[1](t, x, z_j^\ast) = \bar{\gamma}_j \bar{Z}[2](t, x, z_j^\ast), \quad \bar{\gamma}_j = \frac{B^*(z_j^\ast)}{a^*(z_j^\ast)} = \frac{A^*(z_j^\ast)}{\bar{b}^*(z_j^\ast)} = \gamma_j^\ast. \quad (3.4)
\]

If \( n(\lambda) > 0 \) we have other analytic properties. Namely, \( a^+ (\lambda) \) has an analytic continuation \( a(z) \) for \( z \in \mathbb{C}_+ \setminus D^+ \),
and \( a^- (\lambda) \) has an analytic continuation \( \bar{a}(z) \) for \( z \in \mathbb{C}_- \setminus D^- \)
\((\bar{a}(z) = a^*(z*))\). The function \( \bar{b}^+ (\lambda) \) has an analytic continuation \( \bar{b}(z) \) for \( z \in D^- \),
and \( b^+ (\lambda) \) has an analytic continuation \( b(z) \) for \( z \in D^+ \) \((b^*(z^*) = \bar{b}(z))\). The spectral functions have
the same asymptotics as above.

If \( a(z) \) have zeroes \( z_j \in \mathbb{C}_+ \setminus D^+ \) \((j = 1, 2, \ldots, p)\) then
\[
a(z_j) = \det[Z[1](t, x, z_j), Y[2](t, x, z_j)] = 0.
\]

Hence vector columns of the determinant are linear dependent:
\[
Y[2](t, x, z_j) = \gamma_j Z[1](t, x, z_j), \quad \gamma_j = \frac{B(z_j)}{a(z_j)} = \frac{A(z_j)}{\bar{b}(z_j)} \quad (3.5)
\]

At the conjugated points \( z_j^\ast \in \mathbb{C}_- \setminus D^- \) \((j = 1, 2, \ldots, p)\)
\[
\bar{a}(z_j^\ast) = \det[\bar{Y}[1](t, x, z_j^\ast), \bar{Z}[2](t, x, z_j^\ast)] = 0.
\]

Therefore
\[
\bar{Y}[1](t, x, z_j^\ast) = \bar{\gamma}_j \bar{Z}[2](t, x, z_j^\ast), \quad \bar{\gamma}_j = \frac{B^*(z_j^\ast)}{a^*(z_j^\ast)} = \frac{A^*(z_j^\ast)}{\bar{b}^*(z_j^\ast)} = \gamma_j^\ast. \quad (3.6)
\]
Being analytic in the domain $D_+$ the function $b(z)$ can have zeroes at the points $\hat{\zeta}_j \in D_+$ ($j = 1, 2, \ldots, q$). Hence

$$Y[2](t, x, \hat{\zeta}_j) = \hat{\gamma}_j Z[2](t, x, \hat{\zeta}_j), \quad \hat{\gamma}_j = \frac{B(\hat{\zeta}_j)}{A(\hat{\zeta}_j)} = \frac{A(\hat{\zeta}_j)}{\beta(\hat{\zeta}_j)}.$$ (3.7)

At the conjugated points $\hat{\zeta}_j^* \in D^-$ ($j = 1, 2, \ldots, q$)

$$\bar{Y}[1](t, x, \hat{\zeta}_j^*) = \bar{\gamma}_j \bar{Z}[1](t, x, \hat{\zeta}_j^*), \quad \bar{\gamma}_j^* = \frac{B^*(\hat{\zeta}_j^*)}{A^*(\hat{\zeta}_j^*)} = \frac{A^*(\hat{\zeta}_j^*)}{\beta^*(\hat{\zeta}_j^*)} = \gamma_j^*.$$ (3.8)

4. Matrix Riemann–Hilbert problems

In this section we give a reconstruction of the solution of the mixed problem to the MB equations in terms of the associated matrix RH problem.

In the case $n(\lambda) < 0$ we put

$$\mathbf{\hat{\Phi}}(t, x, z) = \begin{pmatrix} \bar{Z}[1](t, x, z) & Y[2](t, x, z) \end{pmatrix} a(z), \quad z \in \mathbb{C}_+,$$

$$= \begin{pmatrix} \bar{Y}[1](t, x, z) & \bar{Z}[2](t, x, z) \end{pmatrix}, \quad z \in \mathbb{C}_-.$$ (4.1)

Scattering relations (3.1) gives:

$$\frac{Y^-[1](t, x, \lambda)}{a^-(\lambda)} = Z^-[1](t, x, \lambda) - \bar{\tau}^-(\lambda) Z^-[2](t, x, \lambda),$$

$$\frac{Y^+[2](t, x, \lambda)}{a^+(\lambda)} = r^+(\lambda) Z^+[1](t, x, \lambda) + Z^+[2](t, x, \lambda),$$

where

$$\bar{\tau}^-(\lambda) = \frac{\bar{\alpha}(\lambda)}{\bar{\sigma}(\lambda)} = \frac{\alpha^{-}(\lambda) \bar{B}(\lambda) + \beta^{+}(\lambda) \bar{A}(\lambda)}{\alpha^{-}(\lambda) \bar{A}(\lambda) - \beta^{+}(\lambda) \bar{B}(\lambda)},$$

$$r^+(\lambda) = \frac{b^+(\lambda)}{a^+(\lambda)} = \frac{\sigma^+(\lambda) B(\lambda) + \beta^+(\lambda) A(\lambda)}{\alpha^+(\lambda) A(\lambda) - \beta^+(\lambda) B(\lambda)}.$$

Using these relations we find that

$$\det \mathbf{\hat{\Phi}}(t, x, \lambda + i0) = \det \mathbf{\hat{\Phi}}(t, x, \lambda - i0) = 1, \quad \text{Im} \lambda = 0.$$

Since $\mathbf{\Phi}_+(t, x, \lambda) := \mathbf{\hat{\Phi}}(t, x, \lambda + i0)$ and $\mathbf{\Phi}_-(t, x, \lambda) := \mathbf{\hat{\Phi}}(t, x, \lambda - i0)$ satisfy the $t$-equation, we have

$$\mathbf{\hat{\Phi}}_-(t, x, \lambda) = \mathbf{\hat{\Phi}}_+(t, x, \lambda) J_0(x, \lambda),$$

where the unimodular matrix

$$J_0(x, \lambda) = \left( \begin{array}{c} Z^+[1](0, x, \lambda) & \frac{Y^+[2](0, x, \lambda)}{a^+(\lambda)} \\ \frac{Y^-[1](0, x, \lambda)}{a^-(\lambda)} & Z^-[2](0, x, \lambda) \end{array} \right)^{-1} \left( \begin{array}{c} Y^-[1](0, x, \lambda) \bar{Z}^-[2](0, x, \lambda) \end{array} \right).$$

All entries

$$Z^+[1](0, x, \lambda) = W^+(0, x, \lambda) w^+[1](0, \lambda),$$

$$Y^+[2](0, x, \lambda) = W^+(0, x, \lambda) \Phi[2](0, \lambda),$$

$$Y^-[1](0, x, \lambda) = W^-(0, x, \lambda) \Phi[1](0, \lambda),$$

$$Z^-[2](0, x, \lambda) = W^-(0, x, \lambda) w^-[2](0, \lambda).$$
of these matrices are known by initial and boundary conditions. Indeed, since \( W^\pm (0, x, \lambda) = w^\pm (x, \lambda) (w^\pm (0, \lambda))^{-1} \) then
\[
\begin{align*}
Z^+ [1] (0, x, \lambda) &= w^+ (x, \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
Y^+ [2] (0, x, \lambda) &= w^+ (x, \lambda) \begin{pmatrix} b^+ (\lambda) \\ a^+ (\lambda) \end{pmatrix}, \\
Y^- [1] (0, x, \lambda) &= w^- (x, \lambda) \begin{pmatrix} \sigma^- (\lambda) \\ b (\lambda) \end{pmatrix}, \\
Z^- [2] (0, x, \lambda) &= w^- (x, \lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

The matrix \( J_0 (x, \lambda) \) can be written in the form:
\[
J_0 (x, \lambda) = (S^+ (\lambda))^{-1} (w^+ (x, \lambda))^{-1} w^- (x, \lambda) S^- (\lambda)
\] (4.2)

where
\[
\begin{align*}
S^+ (\lambda) &= \begin{pmatrix} 1 & r^+ (\lambda) \\ 0 & 1 \end{pmatrix}, & r^+ (\lambda) &= \frac{b^+ (\lambda)}{a^+ (\lambda)}, \\
S^- (\lambda) &= \begin{pmatrix} 1 & r^- (\lambda) \\ -\sigma^- (\lambda) & 0 \end{pmatrix}, & r^- (\lambda) &= \frac{b (\lambda)}{a (\lambda)}
\end{align*}
\] (4.3)

are spectral matrices defined by boundary \( E_m (t) \) and initial \( E_0 (x), \rho_0 (x, \lambda), N_0 (x, \lambda) \) conditions. They are unimodular matrices. Matrix \( w^\pm (x, \lambda) \) is the Jost solution of the \( x^\pm \)-equation (with \( t = 0 \)):
\[
\begin{align*}
\frac{d}{dx} w^\pm (x, \lambda) &= (i \lambda \sigma_3 - i G_L (0, x, \lambda) + H (0, x)) w^\pm (x, \lambda), \\
u^\pm (L, \lambda) e^{-i L \eta (x) \sigma_3} = I.
\end{align*}
\] (4.5) (4.6)

It is defined by given initial functions \( E_0 (x), \rho_0 (x, \lambda), N_0 (x, \lambda) \) only. The jump matrix \( J_0 (x, \lambda) \) can be rewritten in the form:
\[
J_0 (x, \lambda) = (K^+ (x, \lambda))^{-1} K^- (x, \lambda),
\] (4.7)

where \( K^\pm (x, \lambda) \) obeys the same (4.5) but with another (than (4.6)) boundary condition:
\[
K^\pm (L, \lambda) = e^{i L \eta (\lambda) \sigma_3} S^\pm (\lambda).
\] (4.8)

The following symmetry properties
\[
\begin{align*}
S^- (\lambda) &= \sigma_2 (S^+ (\lambda))^* \sigma_2, \\
w^- (x, \lambda) &= \sigma_2 (w^+ (x, \lambda))^* \sigma_2, \\
K^- (x, \lambda) &= \sigma_2 (K^+ (x, \lambda))^* \sigma_2
\end{align*}
\]
are fulfilled. These properties are equivalent to the following ones:
\[
\begin{align*}
(S^+ (\lambda))^\dagger &= (S^- (\lambda))^\dagger, \\
(w^+ (x, \lambda))^\dagger &= (w^- (x, \lambda))^\dagger, \\
(K^+ (x, \lambda))^\dagger &= (K^- (x, \lambda))^\dagger
\end{align*}
\]
where \( \dagger \) means the Hermitian conjugation. It is very important to emphasize that the jump matrix \( J_0 (x, \lambda) \) is obtained by solving the linear problems. Indeed, matrices \( K^\pm (x, \lambda) \) are solutions of the linear problem (4.5), (4.8) and matrices \( S^+ (\lambda) \) (4.3) and \( S^- (\lambda) \) (4.4) are defined by the Jost solutions of the linear AKNS equations. Moreover, the \( t \)-equation is solved when \( x = 0 \) and, hence, it is completely defined by the known boundary condition \( E_m (t) \); \( x \)-equation is solved when \( t = 0 \) and, hence, it is completely defined by known initial functions \( E_0 (x), \rho_0 (x, \lambda) \) and \( N_0 (x, \lambda) \).
Lemma 4.1. For any \( x \in \mathbb{R}_+ \) and \( \lambda \in \mathbb{R} \) the jump matrix
\[
J_0(x, \lambda) = (K^-(x, \lambda))^\dagger K^-(x, \lambda)
\]
is positive defined.

Proof. Indeed, the scalar product
\[
(J_0(x, \lambda)\xi, \xi) = (K^-(x, \lambda)\xi, K^-(x, \lambda)\xi)
\]
is positive for any \( \xi \in \mathbb{C}^n, \xi \neq 0 \). Suppose the contrary, i.e. there exists \( \xi_0 = \xi_0(x, \lambda) \neq 0 \) such that \( K^-(x, \lambda)\xi_0(x, \lambda) = 0 \). Since \( \det K^-(x, \lambda) = 1 \), the vector function \( \xi_0(x, \lambda) \) is equal zero identically. Hence \( (J_0(x, \lambda)\xi, \xi) > 0 \).

Lemma 4.2. For any fixed \( x \in \mathbb{R}_+ \) the jump matrix \( J_0(x, \lambda) \) has the following asymptotic behavior:
\[
J_0(x, \lambda) = I + O(\lambda^{-1}), \quad \lambda \to \pm \infty.
\]

Proof. The matrix \( K := K^-(x, \lambda) \) satisfies the equation:
\[
K'_i = (i\lambda\sigma_3 + H(0, x) - iG_-(0, x, \lambda))K.
\]
The Hermitian conjugation matrix \( K^\dagger \) satisfies the following:
\[
K'^\dagger = K^\dagger (-i\lambda\sigma_3 + H^\dagger(0, x) + iG^\dagger_-(0, x, \lambda)).
\]
Since \( H^\dagger(0, x) = -H(0, x) \) and \( G^\dagger_-(0, x, \lambda) = G_+(0, x, \lambda) \), the last equation takes the form:
\[
K'^\dagger = K^\dagger (-i\lambda\sigma_3 - H^\dagger(0, x) + iG_+(0, x, \lambda)).
\]
Taking into account the definition of \( J_0(x, \lambda) \), we find
\[
J'_0 = iK^\dagger(G_+(0, x, \lambda) - G_-(0, x, \lambda))K = -\frac{\pi n(\lambda)}{2} K^\dagger F(0, x, \lambda)K.
\]
Since \( F(0, x, \lambda) \) and \( K = K^-(x, \lambda) \) are bounded, and \( n(\lambda) = o(\lambda^{-1}) \) as \( \lambda \to \pm \infty \), we have that \( J_0(x, \lambda) = \hat{J}(\lambda) + o(\lambda^{-1}) \) as \( \lambda \to \pm \infty \). Then \( \hat{J}(\lambda) = J_0(0, \lambda) + o(\lambda^{-1}) \) and \( J_0(0, \lambda) = K^+(0, \lambda)K^-(0, \lambda) = I + O(\lambda^{-1}) \) as \( \lambda \to \pm \infty \). Hence \( \hat{J}(\lambda) = I + O(\lambda^{-1}) \) as \( \lambda \to \pm \infty \).

Now we put
\[
M(t, x, z) = \Phi(t, x, z) e^{i(\zeta - x\eta(z))\theta}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
where \( \eta(z) \) is defined by (2.13). Then \( M(t, x, z) \) is a solution of the following RH problem RH\(_{\eta} \):

- **RH1**: \( M(t, x, z) \) is analytic (a(z) \( \neq 0 \)) or meromorphic (a(z) has finite number of zeroes) in \( z \in \mathbb{C} \setminus \mathbb{R} \) and continuous up to the real \( \lambda \)-axis;

- **RH2**: if \( a(z_j) = \overline{a(z_j)} = 0, j = 1, 2, \ldots, p \) then \( M(t, x, z) \) has poles at points \( z = z_j \), \( z = z_j^* \) (\( j = 1, 2, \ldots, p \)), and corresponding residues satisfy relations:
\[
\text{res}_{z=z_j^*} M[2](t, x, z) = m_j e^{-2i(t, x, z_j^*)} M[1](t, x, z_j),
\]
\[
\text{res}_{z=z_j^*} M[1](t, x, z) = m_j^* e^{-2i(t, x, z_j^*)} M[2](t, x, z_j^*),
\]
where \( m_j = \gamma_j/\alpha_j(z_j), m_j^* = \overline{\gamma}_j/\overline{\alpha}(z_j^*) \), and the numbers \( \gamma_j, \overline{\gamma}_j \) are defined in (3.3) and (3.4);
and

\[ J(t, x, \lambda) = e^{-i(\lambda t - x_0(\lambda))\sigma_3}, \quad \lambda \in \mathbb{R}, \]

where \( J_0(x, \lambda) \) is defined by the formula (4.7) and equations (4.5), (4.8); \( J_0(x, \lambda) \) is analytic \( z \in \mathbb{C} \setminus \mathbb{D}_+ \). Where \( \mathbb{D}_+ \) is the right half-plane and \( \mathbb{D}_- \) is the left half-plane. The asymptotic behavior of \( J(t, x, \lambda) \) is much more complicated compared with explicit exponential dependence in \( t \).

Analytical properties follow from (4.1), and residue relations arise from (3.3) and (3.4). The jump matrix \( J_0(x, \lambda) \) in (4.10) was described in (4.2)–(4.8) and lemmas 4.1 and 4.2. The asymptotic behavior of \( M(t, x, \lambda) \) follows from sections 2 and 3. We emphasize again that the jump matrix \( J(t, x, \lambda) \) is a result of the solving of linear problems only. But its dependence in \( x \) is much more complicated compared with explicit exponential dependence in \( t \).

When \( n(\lambda) > 0 \) we introduce another matrix

\[
\tilde{\Phi}(t, x, z) = \begin{pmatrix} Z[1](t, x, z) & Y[2](t, x, z) \\ \bar{Y}[1](t, x, z) & \bar{Z}[2](t, x, z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{D}_+,
\]

where \( \bar{\Phi}(t, x, z) \) is defined by the formula (4.1) and \( \Phi(t, x, z) \) is analytic \( z \in \mathbb{C} \setminus \mathbb{D}_+ \). Where \( \mathbb{D}_+ \) is the right half-plane and \( \mathbb{D}_- \) is the left half-plane.

In the same way, using residue relations (3.5)–(3.8), we obtain a meromorphic matrix RH problem. But for the sake of simplicity we assume that \( a(z) \neq 0 \) in the domain \( \mathbb{C}_+ \setminus \mathbb{D}_+ \) and \( \bar{b}(z) \neq 0 \) in the domain \( \mathbb{D}_+ \), i.e. we consider a regular RH problem. In this case matrix \( M(t, x, \lambda) = \tilde{\Phi}(t, x, z) e^{i\theta(\lambda x, \lambda)} \sigma_3 \) satisfies the next items.

- **RH1:** \( M(t, x, \lambda) \) is analytic \( z \in \mathbb{C} \setminus \Sigma \), \( \Sigma := \mathbb{R} \cup \gamma \cup \bar{\gamma} \) and continuous up to the contour \( \gamma \).
- **RH2:** \( M_-(t, x, z) = M_+(t, x, z) J(t, x, z), \quad z \in \Sigma, \)

\[
J(t, x, z) = e^{-i(\lambda t - x_0(\lambda))\sigma_3} J_0(x, \lambda) e^{i(\lambda t - x_0(\lambda))\sigma_3}, \quad \lambda \in \mathbb{R},
\]

where

\[
J_0(x, \lambda) = (K^+(x, \lambda))^{-1} K^-(x, \lambda), \quad z = \lambda \in \mathbb{R}_D = \mathbb{R} \cap \mathbb{D}_- \cup \mathbb{D}_+,
\]

\[
J_0(z) = \begin{pmatrix} b(z) & 0 \\ -\bar{a}(z) & a(z) \end{pmatrix}, \quad z \in \gamma,
\]

\[
= \begin{pmatrix} 1 & -\bar{b}(z) \\ \bar{b}(z) & 0 \end{pmatrix}, \quad z \in \bar{\gamma}
\]

- **RH3:** \( \det J(t, x, z) \equiv 1 \) for \( z \in \Sigma \);
- **RH4:** \( M(t, x, z) = \mathbf{I} + O(z^{-1}), \quad |z| \to \infty. \)
Here $K_0^+(x, \lambda) = w^+(x, \lambda)S_0^+(\lambda)$, $K_0^-(x, \lambda) = w^-(x, \lambda)S_0^-(\lambda)$ and

\[
S_0^+(\lambda) = \begin{pmatrix} 1 & 0 \\ a^+(\lambda) & b^+(\lambda) \end{pmatrix}, \quad S_0^-(\lambda) = \begin{pmatrix} 1 & 0 \\ \frac{\bar{a}^-}{\bar{b}^-}(\lambda) & \frac{\bar{b}^-}{\bar{a}^-}(\lambda) \end{pmatrix}.
\]

Functions $w^+(x, \lambda)$ and $w^-(x, \lambda)$ are defined in (4.5) and (4.6).

Note that the matrix $J_0(x, z)$ depends on $x$ for $z = \lambda \in \mathbb{R}$, and matrix $J_0(z)$ is independent on $x$ (and on $t$ as well) for $z \in \gamma \subset \mathbb{C}^+$ and $z \in \bar{\gamma} \subset \mathbb{C}^-$. Thus we prove the following.

**Theorem 4.1.** Let $E(t, x), N(t, x, \lambda)$ and $\rho(t, x, \lambda)$ be the solution of the mixed problem (1.4)–(1.8) to the MB equations (1.1)–(1.3). Then there exists the matrix $M(t, x, z)$ which is the solution of the meromorphic RH problem RH1–RH4 if $n(\lambda) < 0$ or the regular (under additional conditions $a(z) \neq 0$ and $b(z) \neq 0$) RH problem RRH1–RRH4 if $n(\lambda) > 0$. A complex electric field envelope $E(t, x)$ is defined by relation:

\[
E(t, x) = -\lim_{z \to \infty} 4izM_{12}(t, x, z),
\] (4.11)

and $N(t, x, \lambda)$ and $\rho(t, x, \lambda)$ can be found from linear equations (1.2) and (1.3) by already known $E(t, x)$.

**Proof.** Formula (4.11) follows from (1.10) and RH4. Indeed, substituting (4.9) into equation (1.10), we find

\[
M_1 + iz[\sigma_3, M] + HM = 0.
\] (4.12)

Using RH4 we put

\[
M(t, x, z) = I + \frac{m(t, x)}{z} + o(z^{-1}),
\]

where

\[
m(t, x) = \lim_{z \to \infty} z(M(t, x, z) - I).
\]

These asymptotics and equation (4.12) give

\[
H(t, x) = -i[\sigma_3, m(t, x)]
\]

and hence

\[
E(t, x) = -4im_{12} = -\lim_{z \to \infty} 4izM_{12}(t, x, z).
\]

Thus the mixed problem on the finite interval $0 < x < L$ (or half-line $0 < x < \infty$) for the MB equations is completely linearizable.

**5. More general matrix Riemann–Hilbert problems**

Now we prove that any RH problem like RRH1–RRH4 generates a solution to the MB equations. From here and below we will consider a more general construction. Let the oriented
contour $\Sigma$ contain the real line $\mathbb{R}$, the circle $\Gamma$ of sufficiently large radius and some finite arcs $\gamma_j \cup \bar{\gamma}_j$ ($j = 1, 2, \ldots, p$), which are symmetric over the real line. Thus

$$\Sigma = \mathbb{R} \cup \Gamma \cup \bigcup_{j=1}^{p} \gamma_j \cup \bar{\gamma}_j.$$  

If $n(\lambda) > 0$ then the contour $\Sigma$ includes additionally the closed oval $\gamma \cup \bar{\gamma}$ where $\text{Im}\, \eta(z) = 0$. Such contours appear when we deal with periodic initial data or/and periodic boundary conditions. The large circle $\Gamma$ allows us to bypass difficulties connected with a discrete spectrum and spectral singularities (cf [22], [18]). The contour $\Sigma$ has the following orientation: real line $\mathbb{R}$ is oriented from the left to the right, the circle $\Gamma$ and oval $\gamma \cup \bar{\gamma}$ are oriented clockwise, the arcs $\gamma_j \cup \bar{\gamma}_j$ are oriented updown. Then the formulation of a regular matrix RH problem is as follows.

Find $2 \times 2$ matrix $M(t, x, z)$ such that

- $R1$: $M(t, x, z)$ is analytic in $z \in \mathbb{C} \setminus \Sigma$ and bounded up to the contour $\Sigma$;
- $R2$: $M_{-}(t, x, z) = M_{+}(t, x, z)J(t, x, z), \quad z \in \Sigma$;
- $R3$: $\text{det} J(t, x, z) \equiv 1$ for $z \in \Sigma$;
- $R4$: $M(t, x, z) = I + O(z^{-1}), \quad |z| \to \infty$.

Let the contour $\Sigma$ and jump matrix $J(t, x, z)$ satisfy the Schwarz reflection principle:

- The contour $\Sigma$ is symmetric over the real axis $\mathbb{R},$
- $J^{-1}(x, t, z) = J^*(x, t, z^*)$ for $z \in \Sigma$ and $\text{Im}\, z \neq 0$,

where $^*$ and $\dagger$ are Hermitian and complex conjugations, respectively.

Furthermore,

- the jump matrix $J(t, x, \lambda)$ for $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$ has a positive definite real part and the following asymptotic behavior:

$$J(t, x, \lambda) = I + O(\lambda^{-1}), \quad \lambda \to \pm \infty.$$  

**Theorem 5.1.** Let the jump matrix $J(t, x, z)$ satisfy the Schwarz reflection principle, have a positive definite real part and $I - J(t, x, \cdot) \in L^2(\Sigma) \cap L^\infty(\Sigma)$. Then for any fixed $t, x \in \mathbb{R}$, the regular RH problem $R1, R2, R3, R4$ has a unique solution $M(t, x, z)$.

**Proof.** Existence. Let $t$ and $x$ be fixed. We look for the solution $M(t, x, z)$ of the RH problem in the form

$$M(t, x, z) = I + \frac{1}{2\pi i} \int_{\Sigma} \frac{P(t, x, s)[I - J(t, x, s)]}{s - z} \, ds, \quad z \notin \Sigma. \quad (5.1)$$  

The Cauchy integral (5.1) provides all properties of the RH problem (see [21]) if and only if the matrix $Q(t, x, \lambda) := P(t, x, \lambda) - I$ satisfies the singular integral equation

$$Q(t, x, z) - Q[Q](t, x, z) = R(t, x, z), \quad z \in \Sigma. \quad (5.2)$$  

The singular integral operator $Q$ and the right-hand side $R(t, x, z)$ are as follows:

$$Q[Q](t, x, z) := \frac{1}{2\pi i} \int_{\Sigma} \frac{Q(t, x, s)[I - J(t, x, s)]}{s - z^*} \, ds, \quad R(t, x, z) := \frac{1}{2\pi i} \int_{\Sigma} \frac{I - J(t, x, s)}{s - z^*} \, ds.$$  

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We consider this integral equation in the space $L^2(\Sigma)$ of 2 × 2 matrix complex valued functions $Q(z) := Q(t, x, z), z \in \Sigma$. The norm of $Q \in L^2(\Sigma)$ is given by

$$||Q||_{L^2(\Sigma)} = \left( \int_{\Sigma} \text{tr}(Q^*(z)Q(z)) \, dz \right)^{1/2} = \left( \sum_{j,l=1}^{2} \int_{\Sigma} |Q_{jl}(z)|^2 \, dz \right)^{1/2}.$$ 

The operator $K$ is defined by the jump matrix $J(t, x, z)$ and the generalized function

$$\frac{1}{s - z_+} = \lim_{z \to z, z \in \text{ad}_{+}} \frac{1}{s - z}.$$ 

Furthermore, since the jump matrix $J(t, x, \lambda)$ has a positive definite real part, when $\lambda \in \mathbb{R}$, then theorem 9.3 from [22, p 984] guarantees the $L^2$ invertibility of the operator $\text{Id} - K$ (Id is the identity operator). The function $R(t, x, z)$ belongs to $L^2(\Sigma)$ because $I - J(t, x, z) \in L^2(\Sigma)$ when $z \in \Sigma$, and the Cauchy operator

$$C_+[f](z) := \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s - z_+} \, ds = \frac{f(z)}{2} + \text{p.v.} \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s - z} \, ds$$

is bounded in the space $L^2(\Sigma)$ [23]. Therefore, the singular integral equation (5.2) has a unique solution $Q(t, x, z) \in L^2(\Sigma)$ for any fixed $t, x \in \mathbb{R}$, and the formula (5.1) gives the solution of the above RH problem.

**Uniqueness.** A sketch of the proof is as follows. Since $\det J(t, x, z) \equiv 1$ one can find that $\det M(t, x, z) \equiv 1$ by repeating step by step the proof of the theorem 7.18 from [21, p 194–8]. Hence the matrix $M^{-1}(t, x, z)$ exists, is analytic in $z \in \mathbb{C} \setminus \Sigma$ and bounded up to the contour $\Sigma$. Let us now suppose that there is another matrix $M(t, x, z)$, which solves the given RH problem. Then

$$\tilde{M}_-(t, x, z)M^{-1}_-(t, x, z) = \tilde{M}_+(t, x, z)J(t, x, z)J^{-1}(t, x, z)M^{-1}_+(t, x, z)$$

$$= \tilde{M}_+(t, x, z)M^{-1}_+(t, x, z).$$

Since $\tilde{M}(t, x, z)$ and $M^{-1}(t, x, z)$ are bounded up to the contour $\Sigma$, end points and points of self-intersection are removable singularities. Hence the matrix $\tilde{M}(t, x, z)M^{-1}(t, x, z)$ is analytic in $z \in \mathbb{C}$ and tends to identity the matrix as $z \to \infty$. By Liouville’s theorem $\tilde{M}(t, x, z)M^{-1}(t, x, z) \equiv I$ and therefore $\tilde{M}(t, x, z) \equiv M(t, x, z)$, i.e. the matrix $M(t, x, z)$ is unique. 

Now we specialize the jump matrix $J(t, x, z)$ in such a way that the corresponding RH problem solution $M(t, x, z)$ generates exactly the MB equations. Other specializations of the jump matrix $J(t, x, z)$ give other nonlinear integrable equations.

**Theorem 5.2.** Let inhomogeneous broadening $n(\lambda)$ be smooth and fast decreasing for $\lambda \in \mathbb{R}$ and

$$\eta(z) = z - \frac{1}{4} \int_{-\infty}^{\infty} \frac{n(s)}{s - z} \, ds, \quad \int_{-\infty}^{\infty} n(s) \, ds = \pm 1.$$ 

Let $\Phi(t, x, z) := M(t, x, z) e^{-i(z - \chi(t,z))(\eta)}$, where $M(t, x, z)$ is the solution of the regular RH problem $R1–R4$ with the jump matrix $J(t, x, z)$ such that

- $\det J(t, x, z) \equiv 1$;
- $J(t, x, z) = \begin{cases} e^{-i(\sigma - x\chi(t,z))(\eta)} \hat{J}(z) e^{i(z - \chi(t,z))(\eta)}, & z \in \Sigma \setminus \mathbb{R} \quad (\text{Im } z \neq 0), \\ e^{-i(\sigma - x\chi(t,x))(\eta)} \hat{J}(x, \lambda) e^{i(\lambda - \chi(t,x))(\eta)}, & z = \lambda \in \mathbb{R}; \end{cases}$
- $\hat{J}(z)$ is independent on $t, x$ and satisfies the Schwarz reflection principle;
- $\hat{J}(x, \lambda)$ is independent on $t$ and has a positive definite real part for any $x \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$;
- $\hat{J}(x, \lambda) = I + O(\lambda^{-1}), \quad \lambda \to \pm \infty$. 

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If $M(t, x, z)$ is absolutely continuous (smooth) in $t$ and $x$, then $\Phi(t, x, z)$ ($z \in \mathbb{C}_\pm$) satisfies the AKNS equations:

$$
\Phi_t = -(iz\sigma_3 + H(t, x))\Phi, \quad \Phi_x = (iz\sigma_3 + H(t, x) - iG(t, x, z))\Phi
$$

almost everywhere (point-wise) with respect to $t$ and $x$. These equations become (1.10) and (1.13) as $z = \lambda \pm i0$. Matrix $H(t, x)$ is given by

$$
H(t, x) = -i[\sigma_3, m(t, x)], \quad m(t, x) = \frac{1}{\pi} \int_{\Sigma} \left( I + Q(t, x, z) \right) (J(t, x, z) - I) \, dz,
$$

where $Q(t, x, z)$ is the unique solution of a singular integral equation (5.2). The matrix $F(t, x, \lambda)$ is Hermitian and has the structure:

$$
F(t, x, \lambda) = \begin{pmatrix}
N(t, x, \lambda) & \rho(t, x, \lambda) \\
\rho(t, x, \lambda)^* & -N(t, x, \lambda)
\end{pmatrix}.
$$

It is reconstructed as the following jump:

$$
\frac{\pi n(\lambda)}{2} F(t, x, \lambda) = \Phi_+(t, x, z) \Phi_+(t, x, z)^{-1}|_{z=\lambda+i0} - \Phi_-(t, x, z) \Phi_-(t, x, z)^{-1}|_{z=\lambda-i0}.
$$

**Proof.** The matrix $\Phi(t, x, z)$ is analytic in $z \in \mathbb{C} \setminus \Sigma$ and has the jump across $\Sigma$:

$$
\Phi_-(t, x, z) = \Phi_+(t, x, z) \hat{J}(x, z),
$$

where $\hat{J}(x, z)$ is independent on $t$. This relation implies:

$$
\frac{\partial \Phi_-(t, x, z)}{\partial t} \Phi_+(t, x, z)^{-1} = \frac{\partial \Phi_+(t, x, z)}{\partial t} \Phi_+(t, x, z)^{-1}
$$

for $z \in \Sigma$. This relation means that the matrix logarithmic derivative $\Phi_t(t, x, z) \Phi_{-1}(t, x, z)$ is analytic (entire) in $z \in \mathbb{C}$. Indeed, matrices $M(t, x, z)$, $M^{-1}(t, x, z)$ and the derivative (in $t$) $M_t(t, x, z)$ are analytic in $z \in \mathbb{C} \setminus \Sigma$ and the Cauchy integral (5.1) gives the following asymptotic formulas:

$$
M(t, x, z) = I + \frac{m_+(t, x)}{z} + O(z^{-2}), \quad \frac{dM(t, x, z)}{dz} = \frac{dm_+(t, x)}{dz} + O(z^{-2}), \quad z \to \infty.
$$

Hence:

$$
\Phi_t(t, x, z) \Phi^{-1}(t, x, z) = -iz\sigma_3 + i[\sigma_3, m_+(t, x)] + O(z^{-1}) = -iz\sigma_3 + i[\sigma_3, m_-(t, x)] + O(z^{-1}), \quad z \to \infty.
$$

where $[A, B] := AB - BA$ and

$$
m_-(t, x) = m_+(t, x) = m(t, x) = \frac{1}{\pi} \int_{\Sigma} \left( I + Q(t, x, z) \right) (J(t, x, z) - I) \, dz.
$$

Since $\Phi_t(t, x, z) \Phi^{-1}(t, x, z)$ has no jump across $\Sigma$ and $M(t, x, z)$, $M^{-1}(t, x, z)$ are bounded up to the boundary, then the end points and points of self-intersection of the contour $\Sigma$ are removable singularities for $\Phi_t(t, x, z) \Phi^{-1}(t, x, z)$. Therefore, by Liouville’s theorem, this derivative is a polynomial:

$$
U(z) := \Phi_t(t, x, z) \Phi^{-1}(t, x, z) = -iz\sigma_3 - H(t, x),
$$

where $H(t, x) := -i[\sigma_3, m(t, x)] = \begin{pmatrix} 0 & q(t, x) \\ q(t, x)^* & 0 \end{pmatrix}$. Using the Schwarz symmetry of the jump matrix $J(t, x, z)$, we show (see [20]) that $U(z) = \sigma_2 U^*(z^*) \sigma_2$, where $\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This symmetry implies $H(t, x) = -H^*(t, x)$, i.e. $q(t, x) = -q^*(t, x)$ and we put $q(t, x) := \mathcal{E}(t, x)/2$. Formula (5.1) gives an integral representation for the electric field envelope

$$
\mathcal{E}(t, x) = \frac{2}{\pi} \int_{\Sigma} \left( I + Q(t, x, z) \right) [J(t, x, z) - I]_{12} \, dz.
$$

(5.3)
through the solution \( Q(t, x, z) \) of the singular integral equation (5.2). Thus \( \Phi(t, x, z) \) satisfies the equation (1.10), and a scalar function \( \mathcal{E}(t, x) \) is defined by (5.3).

In contrast with the previous case matrix logarithmic derivative \( \Phi_x(t, x, z) \Phi^{-1}(t, x, z) \) is analytic in \( z \in \mathbb{C}_\pm \) only. Indeed, since the matrix \( J(z) \) is independent on \( t \) and \( x \) for \( z \in \Sigma \setminus \mathbb{R} \), then this logarithmic derivative is continuous across the contour \( \Sigma \setminus \mathbb{R} \), while it is not continuous across the real line because \( J(x, \lambda) \) depends on \( x \). End points of the contour \( \Sigma \) are removable singularities because the matrices \( M(t, x, z) \) and \( M^{-1}(t, x, z) \) are bounded up the conjugation contour \( \Sigma \). Further, the asymptotic behavior at infinity gives

\[
\Phi_x(t, x, z) \Phi^{-1}(t, x, z) = iz\sigma_3 + H(t, x) + O(z^{-1}), \quad z \to \infty.
\]

Therefore we find that \( \Phi_x(t, x, z) \Phi^{-1}(t, x, z) - iz\sigma_3 - H(t, x) \) is represented as the Cauchy integral:

\[
\Phi_x(t, x, z) \Phi^{-1}(t, x, z) - iz\sigma_3 - H(t, x) = \frac{1}{4i} \int_{-\infty}^{\infty} \frac{F(t, x, s)n(s)}{s - z} \, ds, \quad z \not\in \mathbb{R},
\]

where \( F(t, x, \lambda) \) is some matrix, and factor \( 1/4i \) is chosen for convenience. Due to the reflection (symmetry) property of the jump matrix we find that \( F(t, x, \lambda) \) is Hermitian. Since \( \text{tr}(\Phi_x(t, x, \lambda \pm i0) \Phi^{-1}(t, x, \lambda \pm i0)) = (\det \Phi(t, x, \lambda \pm i0))'_{t} = 0 \) and \( \text{tr}\sigma_3 = \text{tr}H(t, x) = 0 \) then \( \text{tr}F(t, x, \lambda) = 0 \) and, hence, \( F(t, x, \lambda) \) has the structure:

\[
F(t, x, \lambda) := \begin{pmatrix} N(t, x, \lambda) & \rho(t, x, \lambda) \\ \rho^*(t, x, \lambda) & -N(t, x, \lambda) \end{pmatrix}.
\]

Thus we see that the matrix \( \Phi(t, x, z) \) satisfies two differential equations:

\[
\begin{align*}
\Phi_t &= U(t, x, z) \Phi_x \quad U(t, x, z) = -iz\sigma_3 - H(t, x) \\
\Phi_x &= V(t, x, z) \Phi_t \quad V(t, x, z) = iz\sigma_3 + H(t, x) - iG(t, x, z),
\end{align*}
\]

where

\[
G(t, x, z) = \frac{1}{4} \int_{-\infty}^{\infty} \frac{F(t, x, s)n(s)}{s - z} \, ds, \quad z \not\in \mathbb{R}.
\]

For real \( z = \lambda \in \mathbb{R} \) we have:

\[
\Phi_x = V_\pm(t, x, \lambda) \Phi_x \quad V_\pm(t, x, \lambda) = i\lambda\sigma_3 + H(t, x) - iG_\pm(t, x, \lambda),
\]

where \( G_\pm(t, x, \lambda) := G(t, x, \lambda \pm i0) \). In particular, the last equations give a reconstruction of the function \( F(t, x, \lambda) \):

\[
\frac{\pi n(\lambda)}{2} F(t, x, \lambda) = \Phi_x(t, x, \lambda + i0) \Phi^{-1}(t, x, \lambda + i0) - \Phi_x(t, x, \lambda - i0) \Phi^{-1}(t, x, \lambda - i0),
\]

where \( \Phi(t, x, z) = M(t, x, z) e^{-\frac{i(\pi - x\rho(z))\sigma_3}{4}} \) and \( M(t, x, z) \) is the solution of the regular RH problem R1–R4. The compatibility condition \( \Phi_{\sigma}(t, x, \lambda \pm i0) = \Phi_x(t, x, \lambda \pm i0) \) gives the identity in \( \lambda \):

\[
U_\pm(t, x, \lambda) = V^\pm(t, x, \lambda) + [U(t, x, \lambda), V^\pm(t, x, \lambda)] = 0.
\]

This identity is equivalent to

\[
H_{\sigma}(t, x) + H_\rho(t, x) = -\frac{1}{4} \int_{-\infty}^{\infty} \frac{[\sigma_3, F(t, x, s)]n(s)}{s - \lambda \mp i0} \, ds
\]

\[
= \frac{i}{4} \int_{-\infty}^{\infty} \frac{(F_{\sigma}(t, x, s) + [i\sigma_3 + H(t, x), F(t, x, s)]n(s)}{s - \lambda \mp i0} \, ds
\]

\[
= \frac{i}{4} \int_{-\infty}^{\infty} \frac{(F_{\rho}(t, x, s) + [i\sigma_3 + H(t, x), F(t, x, s)]n(s)}{s - \lambda \mp i0} \, ds
\]

\[
= \frac{i}{4} \int_{-\infty}^{\infty} \frac{(F_{\rho}(t, x, s) + [i\sigma_3 + H(t, x), F(t, x, s)]n(s)}{s - \lambda \mp i0} \, ds
\]
and it is possible if and only if the left and right-hand sides are equal zero, i.e.

\[ H_i(t, x) + H_o(t, x) = \frac{1}{4} \int_{-\infty}^{\infty} [\sigma_3, F(t, x, s)] \eta(s) \, ds = 0 \]

\[ F_i(t, x, \lambda) + [i \lambda \sigma_3 + H(t, x), F(t, x, \lambda)] = 0. \]

These matrix equations are equivalent to the MB equations (1.1)–(1.3). Thus we proved that the matrices \( \Phi(t, x, \lambda \pm i0) \) satisfy equations (5.4) and (5.6), which coincide with the AKNS system (1.10) and (1.13), and scalar functions \( E(t, x), N(t, x, \lambda) \) and \( \rho(t, x, \lambda) \) satisfy the MB equations (1.1)–(1.3). \( \square \)

6. Conclusions

It is proved that the mixed problem (1.4)–(1.8) to the Maxwell–Bloch equations (1.1)–(1.3) is linearizable completely by using the matrix Riemann–Hilbert problem RH1–RH4 or RRH1–RRH4. A more general Riemann–Hilbert problem R1–R4 generates a solution to the Maxwell–Bloch equations if the conjugation contour and jump matrix are satisfied by the Schwarz reflection principal and some special restrictions. Among generated solutions there are: solutions defined for \( t \in \mathbb{R} \) and \( x \in \mathbb{R}_+ \) and studied in [5, 6], solutions to the mixed problem (1.1)–(1.3), (1.4)–(1.8) \((t, x \in \mathbb{R}_+)\) with decreasing or periodic input pulse \( E(t, 0) \) and different initial functions \( E(0, x), N(0, x), \rho(0, x), \) etc. The kind of solution is defined by specialization of the conjugation contour and the jump matrix. Suggested matrix Riemann–Hilbert problems will be useful for studying the long time/long distance asymptotic behavior of solutions to the Maxwell–Bloch equations using the Deift–Zhou method of steepest decent.

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Appendix

In this section we consider the problem on the whole \( t \)-line studied in [5] and [6], where the Marchenko integral equations were used. We would like to formulate here the corresponding matrix RH problem. Let \( E(t, x) \) be vanishing as \( t \to -\infty \) in such a way that there exists the Jost solution

\[ \hat{\Psi}(t, x, \lambda) = e^{-i \sigma_3 \tau_3} + \int_{-\infty}^{t} \hat{L}(t, \tau, x) e^{-i \sigma_3 \tau_3} \, d\tau \]

of \( t \)-equation (1.10). Let \( N(t, x, \lambda) - \sigma_3 \) and \( \rho(t, x, \lambda) \) also be vanishing as \( t \to -\infty \). In this case we use the following pair of compatible solutions: \( \hat{Y}^\pm(t, x, \lambda) \) are the same above for the mixed problem, while \( Z^\pm(t, x, \lambda) \) we choose in another way:

\[ Z^\pm(t, x, \lambda) = \hat{\Psi}(t, x, \lambda) e^{i \eta_{\pm}(\lambda) \tau_3}. \]

Due to the lemma 2.1 \( Z^\pm(t, x, \lambda) \) is a compatible solution of (1.10) and (1.13) because, when \( N(t, x, \lambda) = \sigma_3 \) and \( \rho(t, x, \lambda) = 0 \), \( x^\pm \)-equation takes the form \( E_x = i \eta_{\pm}(\lambda) \sigma_3 E \) and we put \( E_x^\pm(t, x, \lambda) = e^{i \eta_{\pm}(\lambda) \tau_3}. \) Further, the vector column \( Z^+[1](t, x, \lambda) \) has an analytic continuation \( \hat{Z}[1](t, x, z) \) for \( z \in \mathbb{C}_+ \), and the vector column \( Z^-[2](t, x, \lambda) \) has an analytic continuation \( \hat{Z}[2](t, x, z) \) for \( z \in \mathbb{C}_- \). These vector columns have the same asymptotic behavior as \( z \to \infty. \)
The transition matrix $T(\lambda)$ is as follows:
\[ T^\pm(\lambda) = (\Phi^\pm(0, 0, \lambda))^{-1} \Phi(0, \lambda), \quad \text{Im} \lambda = 0 \]
and it has the following structure:
\[ T^\pm(\lambda) = \begin{pmatrix} a^\pm(\lambda) & b^\pm(\lambda) \\ -\bar{b}^\pm(\lambda) & a^\pm(\lambda) \end{pmatrix}. \]
This transition matrix is defined by $\mathcal{E}(t, 0)$ known on the whole $t$-line ($t \in \mathbb{R}$). Further, by the same way as before, we obtain the corresponding Riemann–Hilbert problem. This RH problem looks like the problem RH1–RH4. But, due to conditions that $N(t, x, \lambda) = \sigma_3$ and $\rho(t, x, \lambda)$ vanishing as $t \to -\infty$, the jump matrix $J_0(x, \lambda)$ can be found explicitly in terms of reflection coefficients. Indeed, for $t \to -\infty$ we find
\[ J_0(x, \lambda) = \begin{pmatrix} \nu^+[t, x, \lambda] & \nu^-[t, x, \lambda] \\ \nu^-[t, x, \lambda] & \nu^-[t, x, \lambda] \end{pmatrix}^{-1} \begin{pmatrix} \nu^-[t, x, \lambda] & 0 \\ \nu^-[t, x, \lambda] & \nu^-[t, x, \lambda] \end{pmatrix} \]
(\ref{eq:J0}).

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\[ J_0(x, \lambda) = \begin{pmatrix} \nu^+[t, x, \lambda] & \nu^-[t, x, \lambda] \\ \nu^-[t, x, \lambda] & \nu^-[t, x, \lambda] \end{pmatrix}^{-1} \begin{pmatrix} \nu^-[t, x, \lambda] & 0 \\ \nu^-[t, x, \lambda] & \nu^-[t, x, \lambda] \end{pmatrix} \]
(\ref{eq:J0}).

Thus for the problem on the whole $t$-line the jump matrix $J(t, x, \lambda)$ takes the explicit form:
\[ J(t, x, \lambda) = e^{-i(\lambda t - \eta_\lambda(x))\sigma_3} J_0(x, \lambda) e^{i(\lambda t - \eta_\lambda(x))\sigma_3} \]
\[ = \begin{pmatrix} 1 + |r^+(\lambda)|^2 e^{2i\lambda t} & \lambda_{\eta}(\lambda) e^{2i\lambda t} \\ -\lambda_{\eta}(\lambda) e^{-2i\lambda t} & 1 \end{pmatrix}. \]
Here we have used (\ref{eq:J0}). Since $\eta_{\pm} = \eta(\pm i0) = \lambda - I_0(\lambda) \mp \frac{1}{2} n(\lambda)$ where $I_0(\lambda) = \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{\lambda t} dt}{\lambda}$ then
\[ J(t, x, \lambda) = I + O(e^{\pi n(\lambda)x/2}) \to I, \quad x \to +\infty \]
for a long attenuator ($n(\lambda) < 0$), but it is unbounded ($O(e^{\pi n(\lambda)x})$) for a long amplifier ($n(\lambda) > 0$). Therefore the jump matrix is exponentially closed to the identity matrix if $n(\lambda) < 0$. Hence the main term of asymptotics contains solitons generated by a discrete spectrum, while a contribution of continuous spectrum is exponentially small. This justifies the phenomenon of self-induced transparency of attenuators. For the first time it was proved in \cite{5}.

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