A STABILITY RESULT FOR THE STEKLOV LAPLACIAN EIGENVALUE PROBLEM WITH A SPHERICAL OBSTACLE

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Abstract. In this paper we study the first Steklov-Laplacian eigenvalue with an internal fixed spherical obstacle. We prove that the spherical shell locally maximizes the first eigenvalue among nearly spherical sets when both the internal ball and the volume are fixed.

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1. INTRODUCTION

Let $\Omega_0 \subset \mathbb{R}^n$, $n \geq 2$, be an open, bounded, connected set, with Lipschitz boundary such that $B_r \subseteq \Omega_0$, where $B_r$ is the open ball of radius $r > 0$ centered at the origin. Let us set $\Omega := \Omega_0 \setminus B_r$, then we study the following Steklov-Dirichlet boundary eigenvalue problem for the Laplacian:

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u u = 0 & \text{on } \partial B_r, \\
\nu u = \sigma(\Omega) u & \text{on } \partial \Omega_0
\end{cases} \quad (1.1)$$

where $\nu$ is the outer unit normal to $\partial \Omega_0$. The study of the first eigenvalue of problem (1.1) leads to the following minimization problem:

$$\sigma_1(\Omega) = \min_{\substack{w \in H^1_{\nu, \partial B_r}(\Omega) \\
w \neq 0}} \frac{\int_{\partial B_r} |Dw|^2 \, dx}{\int_{\partial \Omega_0} w^2 \, d\mathcal{H}^{n-1}}, \quad (1.2)$$

where $H^1_{\nu, \partial B_r}(\Omega)$ is the set of Sobolev functions on $\Omega$ that vanish on $\partial B_r$ (for the precise definition see Section 2). Notice also that the value $\sigma_1(\Omega)$ is the optimal
constant in the Sobolev-Poincaré trace inequality:

\[ \sigma_1(\Omega) \|w\|_{L^2(\partial \Omega_0)} \leq \|Dw\|_{H^1_{\text{in}}(\Omega)}. \]  

(1.3)

In this paper we treat the following shape optimization issue:

**Which sets maximize** \( \sigma_1(\cdot) \) **among sets containing the fixed ball** \( B_r \) **and having prescribed measure?**

We partially solve the problem of the optimality of \( \sigma_1 \), restricting our study to nearly spherical sets, that are set whose boundary can be parametrized on the sphere by means of a Lipschitz function with a small \( W^{1, r} \)-norm. The main result of the paper is the following.

**Main Theorem.** Let \( \Omega = \Omega_0 \setminus B_r \), with \( \Omega_0 \) a nearly spherical set. Then

\[ \sigma_1(\Omega) \leq \sigma_1(A_{r, R}), \]  

(1.4)

where \( A_{r, R} = B_R \setminus B_r \), with \( R > r > 0 \), is the spherical shell with the same volume as \( \Omega \). Moreover the equality in (1.4) holds if and only if \( \Omega \) is a spherical shell.

So, we study the optimal shape for \( \sigma_1(\Omega) \) when both the volume of the domain and the radius of the internal ball are fixed. We also find some counterexamples showing that when only a volume constraint holds, then \( \sigma_1 \) is not upper bounded, hence we cannot speak about optimality.

In order to prove the main Theorem, we obtain a stability result in quantitative form. In Theorem 3.8, we find \( K = K(n, |\Omega|) > 0 \), such that

\[ \sigma_1(A_{r, R}) \geq \sigma_1(\Omega) \left( 1 + K(n, |\Omega|) \int_{S^{n-1}} v^2(\xi) \, d\mathcal{H}^{n-1} \right). \]

When \( r = 0 \) and \( \Omega_0 \) is connected, the problem becomes the Steklov eigenvalue problem introduced by Steklov. Optimal upper bounds for classical Steklov eigenvalues have been proved by several authors. When the domain is simply connected, a Weinstock inequality (\[14\] for \( n = 2 \) and \[2\] for higher dimensions) holds. This means that, among convex sets with prescribed perimeter, the maximum for the first Steklov Laplacian eigenvalue is reached by the ball. On the other hand, in [1], the author proved that the ball is the maximum for the same eigenvalue keeping the volume fixed.

In the class of sets of the form \( B_R(x_0) \setminus B_r \), with \( B_R(x_0) \) being a ball containing in \( B_r \), the maximizer of \( \sigma_1 \) is the spherical shell, that is the annulus when the balls are concentric (see [8, 13]).

A natural way to prove the result in (1.4) is in finding the right test function for the Rayleigh quotient in (1.2), in order to obtain the sought spectral inequality. This approach works for other mixed boundary condition eigenvalue problems on perforated domains, e.g when using the so-called web functions (see [4] and the references therein). In particular, in [11] it is proved that the first eigenvalue of the \( p \)-Laplacian with external Robin and internal Neumann boundary conditions
is maximum on spherical shells, when the volume and the external perimeter are fixed. In [12] can be found the original proof for $p = 2$ in the bidimensional case. In [5] the authors prove that the first eigenvalue of the $p$-Laplacian with external Neumann and internal Robin boundary conditions is maximum on spherical shells when the volume and the internal $(n - 1)$-quermassintegral are fixed. See [10] for the original proof in the plane and for $p = 2$.

We use the solution $z$ of (1.2) on the spherical shell to introduce the *weighted volume* $V(\Omega)$ and the *weighted perimeter* $P(\Omega)$:

$$V(\Omega) := \int_{\Omega} |\nabla z|^2 \, dx,$$
$$P(\Omega) := \int_{\partial\Omega_0} z^2 \, dx.$$

So we have

$$\sigma_1(\Omega) \leq \frac{V(\Omega)}{P(\Omega)},$$

with equality at least in the case $\Omega = A_{r,R}$, where $|\Omega| = |A_{r,R}|$. Unfortunately, using the web-function testing method in the Rayleigh quotient (1.2), we do not obtain the correct inequality:

$$\frac{V(\Omega)}{P(\Omega)} \leq \frac{V(A_{r,R})}{P(A_{r,R})}. \tag{1.5}$$

In this paper, we restrict our study to the class of sets $\Omega = \Omega_0 \setminus B_r$, with $\Omega_0$ nearly spherical. We use some classical stability results for isoperimetric problems (see e.g. [9]) to parametrize the outer boundary of $\Omega$ and then we perform a Taylor expansion (as in [7]) to translate the sought spectral inequality into the Poincaré inequality (1.3).

The outline of the paper follows. In Section 2 we give some properties on the mixed Steklov-Dirichlet eigenvalue problem we are dealing with. In Section 3 we prove the main Theorem.

2. The Eigenvalue Problem

Let $R > r > 0$, throughout this paper, we denote $B_r := \{x \in \mathbb{R}^n : |x| < r\}$ the ball centered at the origin with radius $r > 0$; $A_{r,R}$ the spherical shell $B_R \setminus B_r$ and

$$\mathcal{A}_r := \left\{ \Omega = \Omega_0 \setminus B_r : \Omega_0 \subset \mathbb{R}^n \text{ open, bounded, connected,} \right\}.$$

with Lipschitz boundary, s.t. $B_r \subseteq \Omega_0$.

Furthermore, we denote by $H^{n-1}$ the $(n - 1)$-dimensional Hausdorff measure and by $|\cdot|$ the Lebesgue measure in $\mathbb{R}^n$.

Since we are studying a Steklov eigenvalue problem with a spherical obstacle, we need to introduce the definition of a closed subspace of $H^1(\Omega)$, that incorporates
the Dirichlet boundary condition on $\partial B_r$. We denote the set of Sobolev functions on $\Omega$ that vanish on $\partial B_r$ by

$$H^1_{\partial B_r}(\Omega),$$

that is (see [6]) the closure in $H^1(\Omega)$ of the set of test functions

$$C^\infty_0(\Omega) := \{ u|_\Omega \mid u \in C^\infty_0(\mathbb{R}^n), \text{spt}(u) \cap \partial B_r = \emptyset \}.$$

2.1. Eigenvalues and Eigenfunctions. We are dealing with the following boundary eigenvalue problem:

$$\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
u u &= 0 \quad \text{on } \partial B_r, \\
\nu u &= \sigma(\Omega) u \quad \text{on } \partial \Omega_0
\end{align*}$$

(2.1)

where $\nu$ is the outer normal to $\partial \Omega_0$. We give now the definitions and some geometric properties of eigenvalues and eigenfunctions of problem (2.1).

**Definition 2.1.** The real number $\sigma(\Omega)$ and the function $u \in H^1_{\partial B_r}(\Omega)$ are, respectively, called eigenvalue of (2.1) and eigenfunction associated to $\sigma(\Omega)$, if and only if

$$\int_\Omega Du D\varphi \, dx = \sigma(\Omega) \int_{\partial \Omega_0} w \varphi \, d\mathcal{H}^{n-1},$$

for every $\varphi \in H^1_{\partial B_r}(\Omega)$.

Furthermore, the first eigenvalue is variationally characterized by

$$\sigma_1(\Omega) = \min_{w \in H^1_{\partial B_r}(\Omega) \setminus \{0\}} J[w],$$

(2.2)

where

$$J[w] := \frac{\int_{\Omega} |Du|^2 \, dx}{\int_{\partial \Omega_0} w^2 \, d\mathcal{H}^{n-1}}.$$  \hspace{1cm} (2.3)

We point out that the condition of being orthogonal to constants in $L^2(\partial \Omega)$ is not required, unlike the classical Steklov eigenvalue (when $r = 0$).

The following ensures the existence of minimizers of problem (2.2).

**Proposition 2.2.** Let $r > 0$ and $\Omega \in \mathcal{A}_r$, then there exists a function $u \in H^1_{\partial B_r}(\Omega)$ achieving the minimum in (2.2) and satisfying problem (2.1). Moreover, $u$ is positive (or negative) in $\Omega$.

**Proof.** Let $u_k \in H^1_{\partial B_r}(\Omega)$ be a minimizing sequence of (2.2) such that $\|u_k\|_{L^2(\partial \Omega_0)} = 1$. Since the minimum in (2.2) is positive, then there exists a constant $C > 0$ such that $J[u_k] \leq C$ for every $k \in \mathbb{N}$ and therefore $\|Du_k\|_{L^2(\Omega)} \leq C$. Moreover, a Poincaré inequality in $H^1_{\partial B_r}(\Omega)$ holds and this implies that $\{u_k\}_{k \in \mathbb{N}}$ is a bounded sequence in $H^1_{\partial B_r}(\Omega)$. Therefore, there exist a subsequence, still denoted by $u_k$, 

and a function \( u \in H^1_{\partial B_r}(\Omega) \) with \( \|u\|_{L^2(\partial \Omega)} = 1 \), such that \( u_k \to u \) strongly in \( L^2(\Omega) \), hence also almost everywhere, and \( Du_k \to Du \) weakly in \( L^2(\Omega) \). By the compactness of the trace operator, \( u_k \) converges strongly to \( u \) in \( L^2(\partial \Omega) \) and almost everywhere on \( \partial \Omega \) to \( u \). Then, by weak lower semicontinuity we have

\[
\lim_{k \to +\infty} J[u_k] \geq J[u].
\]

Hence the existence of a minimizer \( u \in H^1_{\partial B_r}(\Omega) \) follows. Moreover, the fact that

\[
J[u] = J[|u|]
\]

implies that any eigenfunction must have constant sign on \( \Omega \). So, by Harnack inequality, \( u \) is strictly positive on \( \Omega \). □

Now we state the simplicity of the first eigenvalue of (2.1).

**Proposition 2.3.** Let \( r > 0 \) and \( \Omega \in A_r \), then the first eigenvalue \( \sigma_1(\Omega) \) of (2.1) is simple, that is all the associated eigenfunctions are scalar multiple of each other.

**Proof.** Let \( u, \tilde{u} \) be two non trivial weak solutions of the problem (2.1). Since, by Proposition 2.2, we can assume that \( \tilde{u} \) is positive in \( \Omega \), then it is clear that

\[
\int_{\Omega} \tilde{u} \, dx \neq 0.
\]

So, we can find a real constant \( \chi \) such that

\[
\int_{\Omega} \left( u - \chi \tilde{u} \right) \, dx = 0. \tag{2.4}
\]

Since \( u - \chi \tilde{u} \) is still a solution of the problem (2.1), then it is also non-negative (or non-positive) in \( \Omega \). Therefore, (2.4) implies that \( u = \chi \tilde{u} \) in \( \Omega \) and the simplicity of \( \sigma_1(\Omega) \) follows. □

It is worth noticing that the first nontrivial eigenvalue for the classical Steklov-Laplacian problem (when \( r = 0 \)) on \( B_R \) is \( 1/R \) and the corresponding eigenfunctions are the coordinate axis \( x_i \), for \( i = 1, \ldots, N \). This means that the first nontrivial eigenvalue has multiplicity \( N \) and this makes a huge difference with problem (2.1), for which we proved that the simplicity holds.

On the other hand, it is easy to verify that both have the same scaling property:

\[
\sigma(t\Omega) = \frac{1}{t} \sigma(\Omega), \quad \forall t \in \mathbb{R}. \tag{2.5}
\]

The first attempts to study the optimal shape of problem (2.1) has been done on spherical shells, i.e. when \( \Omega_0 = B_R \), for \( R > r > 0 \). We recall from [13], the
explicit expression of the first eigenfunction on the spherical shell $A_{r,R}$:

$$z(\rho) = \begin{cases} 
\ln \rho - \ln r & \text{for } n = 2 \\
\left( \frac{1}{r^{n-2}} - \frac{1}{\rho^{n-2}} \right) & \text{for } n \geq 3,
\end{cases} \quad (2.6)$$

with $\rho = |x|$. This function is radial, positive, strictly increasing and it is associated to the following eigenvalue:

$$\sigma_1(A_{r,R}) = \begin{cases} 
\frac{1}{R \log \left( \frac{R}{r} \right)} & \text{for } n = 2 \\
\frac{n-2}{R \left( \frac{2}{n} \right)^{(n-1)/n}} & \text{for } n \geq 3.
\end{cases} \quad (2.7)$$

It is worth noting that, since problem (2.1) and the classical Steklov ($r = 0$) have the same scaling property (2.5), then the shape functional $\Omega \mapsto |\Omega|^\frac{1}{n} \sigma(\Omega)$ is scaling invariant, as in the classical case.

2.2. A first upper bound. We show an upper bound for $\sigma_1$ depending only by the dimension $n$, the measure of $\Omega$ and by the radius of the internal ball $r$.

**Proposition 2.4.** Let $r > 0$ and $\Omega \in A_r$, then

$$\sigma_1(\Omega) \leq \frac{2}{n \omega_n \left( \left( \frac{|\Omega|}{2 \omega_n} + r^n \right)^{1/n} - r \right)} |\Omega|^{1/n}. \quad (2.9)$$

**Proof.** Let $\bar{R} > 0$ be such that $|A_{r,\bar{R}}| = |\Omega|/2$, then $\bar{R}$ depends only by the dimension $n$, the measure $|\Omega|$ and $r$, that is

$$\bar{R} = \left( \frac{|\Omega|}{2 \omega_n} + r^n \right)^{1/n}. \quad (2.8)$$

Consider the function

$$\varphi(x) = \begin{cases} 
|x| - r & \text{if } r \leq |x| \leq \bar{R}; \\
\bar{R} - r & \text{if } |x| \geq \bar{R}.
\end{cases}$$

We distinguish now two cases. Firstly, we assume that $B_{\bar{R}} \subseteq \Omega_0$, i.e. $d := \text{dist}(\bar{c}B_{\bar{R}}, \partial \Omega_0) > 0$. By using (2.8) as test function in the Rayleigh quotient (1.5) and by the isoperimetric inequality, we obtain

$$\sigma_1(\Omega) \leq \frac{|\Omega|}{(\bar{R} - r)^2 P(\Omega_0)} \leq \frac{1}{n \omega_n \left( \bar{R} - r \right)^2} |\Omega|^{\frac{1}{n}}. \quad (2.9)$$

We consider now the case $d = 0$, that is when the ball $B_{\bar{R}}$ is not strictly contained in $\Omega_0$. Therefore, we divide the boundary of $\Omega_0$ in the two sets $\partial^{int}\Omega_0$ and $\partial^{ext}\Omega_0$.
that live, respectively, inside and outside of \( B_R \). Using the test function \( \varphi \) in the Rayleigh quotient \( (2.3) \), we have
\[
\sigma_1(\Omega) \leq \frac{|\Omega|}{\int_{\partial \Omega_0} |\varphi|^2 \, d\mathcal{H}^{n-1}} \leq \frac{|\Omega|}{(R - r)^2 \int_{\partial^{ext} \Omega_0} 1 \, d\mathcal{H}^{n-1}}. \tag{2.10}
\]
We recall that a relative isoperimetric inequality with supporting set \( B_R \) holds (see as a reference e.g. \( [3] \)):
\[
\mathcal{H}^{n-1}(\partial^{ext} \Omega_0) \geq n \left( \frac{\omega_n}{2} \right)^{1/n} \left( \frac{|\Omega_0|}{2} \right)^{1-n}. \tag{2.11}
\]
By using \( (2.11) \) in \( (2.10) \), we have
\[
\sigma_1(\Omega) \leq \frac{2}{n\omega_n^\frac{1}{n}(R - r)^2} |\Omega|^\frac{1}{n}. \tag{2.12}
\]
The conclusion follows by observing that the upper bound \( (2.12) \) is greater than \( (2.9) \). \( \square \)

We remark that, when a volume constraint for \( \Omega \) holds, then the upper bound is still finite, when \( r \to 0 \). On the other hand, when \( r \to \infty \), the first eigenvalue cannot be upper bounded. This, together with other examples we are giving in the rest of this Section, motivates the study the optimality of \( \sigma_1 \) when another constraint holds, besides the volume one.

### 2.3. Volume constraint on the spherical shells

In this paper we deal with geometric properties of the first eigenvalue of \( (2.1) \). We look for shapes minimizing \( \sigma_1(\Omega) \), when both \( \omega \), the volume of \( \Omega \), and the radius \( r \) of the internal ball are fixed. We show that, even among the spherical shells, \( \sigma_1 \) cannot be upper bounded when only a volume constraint holds.

Let us consider the spherical shell \( A_{r,R} \) with the volume constraint:
\[
|A_{r,R}| = \omega_n (R^n - r^n) = \omega.
\]
We show that both in bidimensional case and in higher dimension, \( \sigma_1 \) is not upper bounded in the class of spherical shells of fixed volume. Let \( n = 2 \), then \( R = \left( r^2 + \frac{\omega}{\pi} \right)^{\frac{1}{2}} \) and, by \( (2.7) \), we have
\[
\sigma_1(A_{r,R}) = \frac{1}{\left( r^2 + \frac{\omega}{\pi} \right)^{\frac{1}{2}} \log \left( 1 + \frac{\omega}{\pi r} \right)^{\frac{1}{2}}} = \frac{2}{r \left( 1 + \frac{\omega}{\pi r} \right)^{\frac{1}{2}} \log \left( 1 + \frac{\omega}{\pi r} \right)}.
\]
Hence for \( r \) big enough:
\[
\sigma_1(A_{r,R}) \approx \frac{2}{r \left( 1 + \frac{\omega}{2\pi r} \right)} \approx \frac{2\pi r}{\omega \left( 1 + \frac{\omega}{2\pi r} \right)}.
\]
and so
\[ \lim_{r \to +\infty} \sigma_1(A_{r,R}) = +\infty. \]

Let \( n \geq 3 \), then
\[ R = (r^n + \frac{\omega}{\omega_n})^{\frac{1}{n}} \]
and
\[ \sigma_1(A_{r,R}) = \frac{n - 2}{r \left( 1 + \frac{\omega}{\omega_n r^n} \right)^{\frac{1}{n}} \left( 1 + \frac{\omega}{\omega_n r^n} \right)^{\frac{2}{n}} - 1} \]
\[ = \frac{n - 2}{r \left( 1 + \frac{\omega}{\omega_n r^n} \right)^{\frac{1}{n}} - \left( 1 + \frac{\omega}{\omega_n r^n} \right)^{\frac{2}{n}}} \]

Again, if \( r \) is big
\[ \sigma_1(A_{r,R}) \approx \frac{n - 2}{r \left( 1 - \frac{1}{n} \frac{\omega}{\omega_n r^n} - 1 - \frac{1}{n} \frac{\omega}{\omega_n r^n} \right)} = \frac{n \omega_n r^{n-1}}{\omega}. \]

and hence again
\[ \lim_{r \to +\infty} \sigma_1(A_{r,R}) = +\infty. \] (2.13)

Further, it is clear that, in any dimension, we have
\[ \lim_{r \to 0^+} \sigma_1(A_{r,R}) = 0. \] (2.14)

The limiting results (2.13) and (2.14) motivate the fact that it is not sufficient to fix the volume to study the first eigenvalue \( \sigma_1 \). Indeed, when \( r \) is too big, it is not possible to find an upper bound, and, on the other hand, when \( r \) is too small, the eigenvalue is trivial. We remark that, in the class of sets of the form \( B_{R}(x_0) \setminus B_r \) with \( B_{R}(x_0) \) being a ball containing in \( B_r \), the maximizer of \( \sigma_1 \) is the spherical shell (see [8]).

2.4. Spherical shell with fixed difference between radii. It is clear now that we cannot study the shape optimization for \( \sigma_1 \) when only a volume constraint holds. On the other hand, it could be interesting to understand if we can study the shape optimization for double connected domains, when only one geometric quantity is fixed. Here, e.g., we briefly study the behavior of the spherical shell when the distance between the radii is fixed. Let \( d \) be a positive real number such that
\[ R - r = d, \]
so that \( R = r + d \) and \( \frac{R}{r} = 1 + \frac{d}{r} \).

If \( n = 2 \), then for \( r \) big enough, we have
\[ \sigma_1(A_{r,R}) = \frac{1}{(r + d) \log \left( 1 + \frac{d}{r} \right)} \approx \frac{r}{rd + d^2}. \]
and hence
\[ \lim_{r \to +\infty} \sigma_1(A_{r,R}) = \frac{1}{d}. \]

If \( n \geq 3 \), we have
\[
\sigma_1(A_{r,R}) = \frac{n - 2}{(r + d) \left[ (1 + \frac{2}{n})^{n-2} - 1 \right]}
\approx \frac{n - 2}{(r + d) \left[ 1 + (n - 2)\frac{2}{r} - 1 \right]} = \frac{r}{rd + d^2},
\]
and hence
\[ \lim_{r \to +\infty} \sigma_1(A_{r,R}) = \frac{1}{d}. \]

Furthermore, in any dimensions, we have
\[ \lim_{r \to 0^+} \sigma_1(A_{r,R}) = 0. \]

The case of \( r \) small is again trivial. On the other hand, \( \sigma_1 \) is upper bounded for any value of \( R \) by the reciprocal of the difference between the radii \( d \). The fact that a uniform upper bounds holds for spherical shells when only the difference between the radii is fixed, suggests that could be interesting to study the shapes minimizing \( \sigma_1 \) in the class of double connected sets when only the width is fixed.

3. Main result

In this section we prove that the spherical shell is a local maximizer for the first eigenvalue of (2.1) among nearly spherical sets with fixed volume, containing \( B_r \), for a fixed value \( r > 0 \). Firstly, we give the definition of nearly spherical sets.

**Definition 3.1.** Let \( n \geq 2 \). An open, bounded set \( \Omega_0 \subset \mathbb{R}^n \) with \( 0 \in \Omega_0 \) is said a nearly spherical set parametrized by \( v \) if there exists \( v \in W^{1,\infty}(\mathbb{S}^{n-1}) \) such that
\[
\partial \Omega_0 = \left\{ y \in \mathbb{R}^n : y = R\xi(1 + v(\xi)), \xi \in \mathbb{S}^{n-1} \right\},
\]
where \( R \) is the radius of the ball having the same measure of \( \Omega_0 \) and \(||v||_{W^{1,\infty}} \leq 1\). The volume of a nearly spherical set is given by
\[
|\Omega_0| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} (1 + v(\xi))^n \, d\mathcal{H}^{n-1}.
\]

The class of nearly spherical sets has a peculiar importance in shape optimization theory, in particular for stability results for spectral inequalities. In this paper, we are considering sets \( \Omega = \Omega_0 \setminus B_r \) belonging to \( \mathcal{A}_r \) with \( r > 0 \), with \( \Omega_0 \) nearly spherical. Now, we are in position to state the main Theorem of this article.
Theorem 3.2. Let \( n \geq 2 \), \( r > 0 \), \( \omega > 0 \) and let \( R > r \) be such that \( |A_{r,R}| = \omega \). There exists \( \varepsilon = \varepsilon(n,r,\omega) > 0 \) such that, for any \( \Omega = \Omega_{0}\setminus\Omega_{r} \) belonging to \( A_{r} \), with \( \Omega_{0} \) nearly spherical set parametrized by \( v \) such that \( ||v||_{W^{1,\infty}} \leq \varepsilon \) and \( |\Omega| = \omega \), then

\[
\sigma_{1}(\Omega) \leq \sigma_{1}(A_{r,R}). \tag{3.2}
\]

Moreover the equality in (3.2) holds if and only if \( \Omega \) is a spherical shell.

Let us remark that, in order to have \( B_{r} \subset \Omega_{0} \), we need to require that \( \varepsilon \leq 1 - r/R \) to verify that \( |y| \geq r \), that is \( R(1 + v(\xi)) \geq r \). Moreover, we observe that, since all the quantities involved are translation invariant, the result in Theorem 3.2 holds also among nearly spherical sets with fixed volume and containing a fixed internal ball.

Recalling the explicit expression (2.5) of the first eigenfunction \( z \) on the spherical shell \( A_{r,R} \), we define the weighted volume and the weighted perimeter as:

\[
V(\Omega) := \int_{\Omega} |\nabla z|^{2} \, dx,
\]

\[
P(\Omega) := \int_{\partial\Omega} z^{2} \, dx.
\]

Furthermore, to simplify the notations, we set, for \( n = 2 \),

\[
h_{R}(t) = (\ln(tR) - \ln r)^{2}, \tag{3.3}
\]

\[
f_{R}(t) = \frac{h_{R}'(t)}{2R} = \frac{\sqrt{h_{R}(t)}}{(tR)}, \tag{3.4}
\]

and for \( n \geq 3 \)

\[
h_{R}(t) = \left( \frac{1}{t^{n-2}} - \frac{1}{(tR)^{n-2}} \right)^{2}, \tag{3.5}
\]

\[
f_{R}(t) = \frac{h_{R}'(t)}{2R} = \frac{n-2}{(tR)^{n-1}} \left( \frac{1}{t^{n-2}} - \frac{1}{(tR)^{n-2}} \right), \tag{3.6}
\]

where \( R \) is the radius of the ball with the same volume of \( \Omega_{0} \) and \( t \geq \frac{r}{R} \).

Now, we write the Rayleigh quotient (2.3) using the parametrization in (3.1).

Lemma 3.3. Let \( n \geq 2 \), \( r > 0 \), \( \omega > 0 \) and let \( R > r \) be such that \( |A_{r,R}| = \omega \). For any \( 0 < \varepsilon < 1 - r/R \) and for any \( \Omega = \Omega_{0}\setminus\Omega_{r} \) belonging to \( A_{r} \), with \( \Omega_{0} \) nearly spherical set parametrized by \( v \) such that \( ||v||_{W^{1,\infty}} \leq \varepsilon \) and \( |\Omega| = \omega \), then

\[
\sigma_{1}(\Omega) \leq \frac{V(\Omega)}{P(\Omega)} = \frac{\int_{S^{n-1}} f_{R}(1 + v(\xi))(1 + v(\xi))^{n-1} \, d\mathcal{H}^{n-1}}{\int_{S^{n-1}} h_{R}(1 + v(\xi))(1 + v(\xi))^{n-1} \sqrt{1 + \frac{||\nabla v(\xi)||^{2}}{(1 + v(\xi))^{2}}} \, d\mathcal{H}^{n-1}}. \tag{3.7}
\]

Moreover if \( \Omega = A_{r,R} \), then equality holds in (3.7) and \( \sigma_{1}(A_{r,R}) = \frac{f_{R}(1)}{h_{R}(1)}. \)
Proof. From the variational characterization (2.2) of \( \sigma_1(\Omega) \), we have

\[
\sigma_1(\Omega) \leq \frac{V(\Omega)}{P(\Omega)} = \frac{\int_{\Omega} |\nabla z|^2 \, dx}{\int_{\partial \Omega} \frac{\partial z}{\partial n} \nu \, d\mathcal{H}^{n-1}} = \frac{\int_{\partial \Omega} \frac{\partial z}{\partial n} \nu \, d\mathcal{H}^{n-1}}{\int_{\partial \Omega} \nu \, d\mathcal{H}^{n-1}}.
\]

The conclusion follows using the change of variables in (3.1). \( \square \)

We recall the following result, whose proof can be found in [9].

**Lemma 3.4.** Let \( n \geq 2 \) and \( R > 0 \). There exists a constant \( C = C(n) > 0 \) such that for any \( 0 < \varepsilon < 1 \) and for any \( v \) parametrizing a nearly spherical set \( \Omega_0 \) such that \( \|v\|_{W^{1,\infty}} \leq \varepsilon \) and \( |\Omega_0| = |B_R| \), then

\[
(1 + v)^{n-1} - \left(1 + (n - 1)v + (n - 2)(v^2)\right) \leq C\varepsilon v^2 \quad \text{on} \quad S^{n-1},
\]

\[
1 + \frac{V^2}{2} - \sqrt{1 + \frac{|V|^2}{(1 + v)^2}} \leq C\varepsilon \left(v^2 + |V|^2\right) \quad \text{on} \quad S^{n-1},
\]

\[
\frac{n}{2} - \frac{n-1}{2} \int_{S^{n-1}} v^2 d\mathcal{H}^{n-1} \leq C\varepsilon \|v\|_{L^2}^2.
\]

As a consequence of the analyticity of \( h_R \) and \( f_R \), defined in (3.3)-(3.4)-(3.5)-(3.6), the following Lemma holds.

**Lemma 3.5.** Let \( n \geq 2 \) and \( 0 < r < R \). There exists \( K = K(n, r, R) > 0 \) such that for any \( 0 < \varepsilon < 1 \) and for any \( v \) parametrizing a nearly spherical set \( \Omega_0 \) such that \( \|v\|_{W^{1,\infty}} \leq \varepsilon \) and \( |\Omega_0| = |B_R| \), then

\[
|h_R(1 + v) - h_R(1) - h'_R(1)v - h''_R(1)\frac{v^2}{2}| \leq K\varepsilon v^2 \quad \text{on} \quad S^{n-1},
\]

\[
|f_R(1 + v) - f_R(1) - f'_R(1)v - f''_R(1)\frac{v^2}{2}| \leq K\varepsilon v^2 \quad \text{on} \quad S^{n-1}.
\]

Furthermore, the following Poincaré inequality holds.

**Lemma 3.6.** (Poincaré inequality) Let \( n \geq 2 \) and \( R > 0 \), then there exists a positive constant \( C = C(n) \) such that for any \( 0 < \varepsilon < 1 \) and for any function \( v \) parametrizing a nearly spherical set \( \Omega_0 \) such that \( \|v\|_{W^{1,\infty}} \leq \varepsilon \) and \( |\Omega_0| = |B_R| \), then

\[
\|\nabla v\|_{L^2}^2 \geq (n - 1)(1 - C\varepsilon)\|v\|_{L^2}^2.
\]

**Proof.** The function \( v \in L^2(S^{n-1}) \) admits a harmonic expansion in the sense that there exists a family of \( n \)-dimensional spherical harmonics \( \{H_j(\xi)\}_{j \in \mathbb{N}} \) such that

\[
v(\xi) = \sum_{j=0}^{+\infty} c_j H_j(\xi), \quad \xi \in S^{n-1} \quad \text{with} \quad \|H_j\|_{L^2(S^{n-1})} = 1,
\]
where
\[ c_j = \langle v, H_j \rangle_{L^2(S^{n-1})} = \int_{S^{n-1}} v(\xi) H_j(\xi) d\mathcal{H}^{n-1}. \]
and \( H_j \) satisfying
\[ \Delta_{S^{n-1}} H_j = j(j + n - 2) H_j, \quad \forall \ j \in \mathbb{N}, \]
where \( \Delta_{S^{n-1}} \) is the Laplace-Beltrami operator. Furthermore the following identities hold true
\[
\|v\|_{L^2(S^{n-1})}^2 = \sum_{j=0}^{\infty} c_j^2, \tag{3.8}
\]
\[
\|\nabla v\|_{L^2(S^{n-1})}^2 = \sum_{j=1}^{\infty} j(j + n - 2) c_j^2. \tag{3.9}
\]
Since \( H_0 = (n\omega_n)^{-\frac{1}{2}} \), we have
\[
|c_0| = (n\omega_n)^{-\frac{1}{2}} \left| \int_{S^{n-1}} v(\xi) d\mathcal{H}^{n-1} \right| \leq (n\omega_n)^{-\frac{1}{2}} \left| \int_{S^{n-1}} v^2(\xi) d\mathcal{H}^{n-1} \right| \left( \frac{n-1}{2} + C\varepsilon \right) = C\varepsilon \|v\|_{L^2},
\]
where the constant \( C \) has been renamed. Using this estimate, by (3.8) and (3.9), we have
\[
\|v\|_{L^2} = \sum_{j=0}^{\infty} c_j^2 = c_0^2 + \sum_{j=1}^{\infty} c_j^2 \leq C\varepsilon \|v\|_{L^2}^2 + \sum_{j=1}^{\infty} c_j^2,
\]
and
\[
\|\nabla v\|_{L^2} = \sum_{j=1}^{\infty} j(j + n - 2) c_j^2 \geq (n-1) \sum_{j=1}^{\infty} c_j^2 \geq (n-1)(1-C\varepsilon) \|v\|_{L^2}^2,
\]
which concludes the proof. \( \square \)

Now we give a key estimate for the main Theorem.

**Proposition 3.7.** Let \( n \geq 2, \ r > 0, \ \omega > 0 \) and let \( R > r \) be such that \( |A_{r,R}| = \omega \). There exist two positive constants \( K > 0 \) and \( 0 \leq \varepsilon_0 < 1 - r/R \), depending on \( n, \ r \) and \( \omega \) only, such that for any \( 0 < \varepsilon < \varepsilon_0 \), for any \( \Omega = \Omega_0 \setminus B_r \) belonging to \( A_r \), with \( \Omega_0 \) nearly spherical set parametrized by \( v \) such that \( \|v\|_{W^{1,\infty}} \leq \varepsilon \) and \( |\Omega| = \omega \),
then

$$V(\Omega^0)P(\Omega) - P(\Omega^0)V(\Omega) =$$

$$= f_R(1) \int_{\mathbb{S}^{n-1}} h_R(1 + v(\xi))(1 + v(\xi))^{n-1} \sqrt{1 + \frac{\|\nabla v(\xi)\|^2}{(1 + v(\xi))^2}} d\mathcal{H}^{n-1}$$

$$- h_R(1) \int_{\mathbb{S}^{n-1}} f_R(1 + v(\xi))(1 + v(\xi))^{n-1} d\mathcal{H}^{n-1} \geq K \int_{\mathbb{S}^{n-1}} v^2 d\mathcal{H}^{n-1}. \quad (3.10)$$

**Proof.** Using Lemmata 3.4, 3.5, 3.6 we have

$$f_R(1) \int_{\mathbb{S}^{n-1}} h_R(1 + v(\xi))(1 + v(\xi))^{n-1} \sqrt{1 + \frac{\|\nabla v(\xi)\|^2}{(1 + v(\xi))^2}} d\mathcal{H}^{n-1}$$

$$- h_R(1) \int_{\mathbb{S}^{n-1}} f_R(1 + v(\xi))(1 + v(\xi))^{n-1} d\mathcal{H}^{n-1} \geq$$

$$\geq \int_{\mathbb{S}^{n-1}} v (f_R(1)h_R'(1) - f_R'(1)h_R(1)) d\mathcal{H}^{n-1}$$

$$+ \int_{\mathbb{S}^{n-1}} \frac{\|\nabla v\|^2}{2} d\mathcal{H}^{n-1} - \varepsilon K_1 \|\nabla v\|^2_{L^2}, \quad (3.11)$$

where $K_1$ is a positive constant. Let us set

$$Q_{1}(t) := f_R(t)h_R'(t) - f_R'(t)h_R(t),$$

$$Q_{2}(t) := f_R(t)h_R''(t) - f_R''(t)h_R(t),$$

$$Q_{3}(t) := f_R(t)h_R(t),$$

In order to show (3.10), we need to prove

1. $Q_1(1) > 0$,
2. $Q_3(1) > 0$,
3. $(n - 1) [Q_1(1) + Q_3(1)] + Q_2(1) > 0$. 
Indeed, when (1), (2), (3) hold, then, by using Lemmata 3.4 and 3.6, the last term in (3.11) can be estimated as

\[ Q_1(1) \int_{S_{n-1}} v \, d\mathcal{H}^{n-1} + (2(n - 1)Q_1(1) + Q_2(1)) \int_{S_{n-1}} \frac{v^2}{2} \, d\mathcal{H}^{n-1} + Q_3(1) \int_{S_{n-1}} \frac{\nabla v^2}{2} \, d\mathcal{H}^{n-1} - \varepsilon K_1 \| \nabla v \|_{L^2}^2 \]

\[ \geq - \frac{n - 1}{2} Q_1(1) \int_{S_{n-1}} v^2 \, d\mathcal{H}^{n-1} - \varepsilon K_2 \| v \|_{L^2}^2 + \left( (n - 1)Q_1(1) + \frac{Q_2(1)}{2} \right) \int_{S_{n-1}} v^2 \, d\mathcal{H}^{n-1} + \frac{n - 1}{2} Q_3(1) \int_{S_{n-1}} v^2 - \varepsilon K_3 \| v \|_{L^2}^2 - \varepsilon K_4 \| \nabla v \|_{L^2}^2 \]

\[ = \frac{1}{2} (\| Q_1(1) + Q_3(1) \| v \|_{L^2}^2 - \varepsilon K_2 \| v \|_{L^2}^2 - \varepsilon K_3 \| v \|_{L^2}^2 - \varepsilon K_1 \| \nabla v \|_{L^2}^2 \]

\[ \geq K \| v \|_{L^2}^2 - \varepsilon K_4 \| v \|_{W^{1,2}(S_{n-1})}^2, \]

where we denoted \( K = \frac{1}{2} \{(n - 1)[Q_1(1) + Q_3(1)] + Q_2(1)\} > 0 \) and \( K_4 = \max\{K_1, K_2, K_3\}. \)

The proof concludes by choosing \( \varepsilon \) small enough.

It remains to prove (1), (2), (3) by distinguishing the bidimensional from the higher dimensional case. We note that

\[ Q_1(t) = \int_{R(t)} f^2_R(t) \left[ \frac{h_R(t)}{f_R(t)} \right]' = 2Rf^2_R(t) \left[ \frac{h_R(t)}{f_R(t)} \right]', \quad (3.12) \]

and

\[ Q_2(t) = Q_1'(t) = \left[ \frac{f^2_R(t)}{f_R(t)} \right]' \left[ \frac{h_R(t)}{f_R(t)} \right]' + f^2_R(t) \left[ \frac{h_R(t)}{f_R(t)} \right]''. \quad (3.13) \]

**Case 1.** Let be \( n = 2. \) We observe that

\[ \frac{h_R(t)}{f_R(t)} = Rt(\ln(t)R - \ln r), \]

is positive and strictly increasing, since it is a product of two strictly increasing positive functions. Hence \( Q_1(t) > 0 \) and in particular

\[ Q_1(1) = \frac{h_R(1)}{R} \left( \sqrt{h_R(1)} + 1 \right) > 0. \]

Moreover, it is clear that

\[ Q_3(1) = \frac{h_R(1)\sqrt{h_R(1)}}{R} > 0. \]

Let us now calculate all the terms in (3.13) and evaluate them for \( t = 1. \) We have
and
\[
\left[ \frac{h_R(t)}{f_R(t)} \right]''_{t=1} = R \left( \sqrt{h_R(t) + 1} \right)_{t=1} = R \left( \sqrt{h_R(1) + 1} \right) > 0,
\]
\[
\left[ \frac{h_R(t)}{f_R(t)} \right]''_{t=1} = \left( \frac{R}{t} \right)_{t=1} = R > 0
\]

and
\[
f_R^2(1) = \frac{h_R(1)}{R^2} > 0,
\]
\[
\left[ f_R^2(t) \right]'_{t=1} = \left[ \frac{2R}{(tR)^3} \left( \sqrt{h_R(t)} - h_R(t) \right) \right] = \frac{2}{R^2} \left( \sqrt{h_R(t)} - h_R(1) \right).
\]

Summing up, estimate (3) follows by
\[
Q_1(1) + Q_3(1) + Q_2(1) = \frac{h_R(1)\sqrt{h_R(1)}}{R} + \frac{h_R(1)}{R} + \frac{h_R(1)\sqrt{h_R(1)}}{R} + \frac{2\sqrt{h_R(1)}}{R} - 2\frac{h_R(1)\sqrt{h_R(1)}}{R} + \frac{h_R(1)}{R} = \frac{2}{R}(h_R(1) + \sqrt{h_R(1)}) > 0.
\]

**Case 2.** For \( n \geq 3 \), from (3.12) we have
\[
\frac{h_R(t)}{h_R'(t)} = \frac{(tR)^{n-1}}{2(n-2)R} \left( \frac{1}{r^{n-2}} - \frac{1}{(tR)^{n-2}} \right),
\]
that is a strictly increasing function, since it is product of two strictly increasing and positive functions. Hence \( Q_1(t) > 0 \) and, in particular
\[
Q_1(1) = \frac{(n-1)(n-2)}{R^{n-1}} h_R(1) \sqrt{h_R(1)} + \frac{2(n-2)^2}{R^{2n-3}} h_R(1) > 0.
\]
Moreover, it is easily seen that
\[
Q_3(1) = \frac{n-2}{R^{n-1}} h_R(1) \sqrt{h_R(1)} > 0.
\]
Eventually, we have
\[
Q_2(1) = \frac{(n-2)^3}{R^{3n-3}} \sqrt{h_R(1)} - \frac{(n-1)^2(n-2)}{R^{n-1}} h_R(1) \sqrt{h_R(1)} + \frac{(n-1)(n-2)^2}{R^{2n-2}} h_R(1),
\]
and therefore, it follows that \((n-1) [Q_1(1) + Q_3(1)] + Q_2(1) > 0.\)

We use the previous result to give a stability result in a quantitative form.

**Theorem 3.8.** Let \( n \geq 2, r > 0, \omega > 0 \) and let \( R > r \) be such that \( |A_{r,R}| = \omega \). There exist two positive constants \( K > 0 \) and \( 0 \leq \varepsilon_0 < 1 - r/R \), depending on \( n, r \) and \( \omega \) only, such that for any \( 0 < \varepsilon < \varepsilon_0 \), for any \( \Omega = \Omega_0 \setminus \overline{B_r} \) belonging to \( A_r \),
with $\Omega_0$ nearly spherical set parametrized by $v$ such that $\|v\|_{W^{1,\infty}} \leq \varepsilon$, and $|\Omega| = \omega$, then

$$\sigma_1(A_{r,R}) \geq \sigma_1(\Omega) \left( 1 + K(n, r, \omega) \int_{S^{n-1}} v^2 d\mathcal{H}^{n-1} \right).$$

**Proof.** From Proposition 3.7 we know that there exists $K > 0$ such that

$$P(A_{r,R})P(\Omega) \left( \frac{V(A_{r,R})}{P(A_{r,R})} - \frac{V(\Omega)}{P(\Omega)} \right) \geq n\omega_n K \int_{S^{n-1}} v^2 d\mathcal{H}^{n-1}.$$

Then, we have

$$\sigma_1(A_{r,R}) = \frac{V(A_{r,R})}{P(A_{r,R})} \geq \frac{V(\Omega)}{P(\Omega)} + \frac{n\omega_n K \int_{S^{n-1}} v^2 d\mathcal{H}^{n-1}}{P(A_{r,R})P(\Omega)}$$

$$= \frac{V(\Omega)}{P(\Omega)} \left( 1 + \frac{n\omega_n K \int_{S^{n-1}} v^2 d\mathcal{H}^{n-1}}{P(A_{r,R})V(\Omega)} \right)$$

$$= \frac{V(\Omega)}{P(\Omega)} \left( 1 + \frac{K \int_{S^{n-1}} v^2 d\mathcal{H}^{n-1}}{h_R(1) \int_{S^{n-1}} f_R(1 + v(\xi))(1 + v(\xi))^{n-1} d\mathcal{H}^{n-1}} \right)$$

$$\geq \frac{V(\Omega)}{P(\Omega)} \left( 1 + \frac{K \int_{S^{n-1}} v^2 d\mathcal{H}^{n-1}}{n\omega_n 2^{n-1} h_R(1) f_R(2)} \right) \geq \sigma_1(\Omega) \left( 1 + K \int_{S^{n-1}} v^2 d\mathcal{H}^{n-1} \right),$$

where the second inequality follows by the fact that $\|v\|_{W^{1,\infty}(S^{n-1})} \leq \varepsilon < 1$ and by the monotonicity of $f_R(\cdot)$. □

Eventually, the main result (Theorem 3.2) easily follows by Theorem 3.8. Moreover, if $\Omega = A_{r,R}$, then the function $v$ parametrizing the outer boundary is constantly equal to zero and equality in (3.2) holds.

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