1. Introduction

For bridge construction, various materials are used such as concrete, aluminium, composites; however, steel as one of the strongest materials is most popular. Support steel elements are main in most constructions. Support construction in girder bridges is bending element. Main support elements for girder bridges are designed from continuous rolled elements or complex welded elements.

Bending elements in bridge girders are replaced by axially loaded elements, connected one to another flexibly or rigidly. However, additional elements are necessary here – cross-beams to transfer uniformly distributed loads to girder joints in concentrated forces.

For a variety of bridge constructions, combined bridge constructions are to be mentioned; they contain both tensed, torsed and bended elements (Juozapaitis et al. 2008).

While analyzing system of bridges as constructional elements, the problem of separate element strength is encountered as well as the problem of separate element buckling. This problem is critical with longer elements.

Problems of buckling of compressed coloumn are discussed extensively (Duan, Wang 2008; Šapalas et al. 2005; Wang et al. 2004). However, problems of long bended element buckling are given rather less attention (Bradford 1990; Larue et al 2007; Plum, Svensson 1993; Samanta, Kumar 2006).

This article presents analysis of element stability for bridge constructions, as they suffer bending loads and external torsion.

2. Bending for long support double-T construction elements

During design and exploitation of bridge constructions, metal elements are suffering loads shown in Fig. 1.

Scheme of loading

\[
M_z = F \cdot z; M_{tor} = \frac{F (2z - L)}{8}
\]

Fig. 1. Loading thin-walled constructional elements under complicated loading

While analyzing long support construction elements loaded by one cross-bending force on free end analytical values are obtained (Bradford 1990). Apart from bending, buckling is observed. However, adding extra momentum of torsion, i.e. acting on the beam with bending moment \(M_z\) and torsion moment \(M_{tor}\), analytical values cannot be obtained. For obtaining this value first of all support double-T beam loaded by concentrated force \(F\) must be analyzed (Fig. 2).
Conditions of differential balance are presented as follows

\[ B_1 \frac{d^3 u}{dz^3} = -\varphi F_z, \]
\[ C \frac{du}{dz} - D \frac{h^2}{2} \frac{d^3 u}{dz^3} = F_z \frac{du}{dz} - F(\delta - u), \]  

(1)

where \( u \) – shift through turning; \( \varphi \) – angle of torsion; \( \delta \) – vertical shift; \( D \) – shelf stiffness \( D = \frac{1}{2} B_1 \); \( h \) – profile height; \( L \) – element length; \( B_1 = E h_y \) \((E \) – modulus of elasticity; \( I_y \) – min moment of inertia); \( C = G_{tor} \) \((G \) – shear modulus, \( G_{tor} \) – moment of inertia in torsion).

The solution presented in study Serna et al. (2006) for calculation of critical buckling moment \( M_{cr} \), is obtained with moment in one plane and requires estimating the empirical coefficients. Serna et al. (2006) presents the following formula for calculation of critical buckling moment:

\[ M_{cr} = \frac{C_k}{L} \left[ \frac{k_c}{I_y} \right] I_x \frac{EI_y}{\pi^2} + \left( \frac{k_c}{I_x} \right) \frac{EI_x}{\pi^2}, \]

(2)

where \( M_{cr} \) – critical bending moment; \( C_k \) – coefficient of lateral bending conditions at end supports; \( k_c \) – bending coefficient; \( k_w \) – warping coefficient.

Manual of Steel Construction: Load and Resistance Factor Design of American Institute of Steel Construction Inc (AISC) also requires many empirical coefficients for estimation of \( M_{cr} \) critical moment. Also the formula presented is applied in loading profiles with one load.

AISC presents the lateral–torsional buckling by the following Eq:

\[ M_{cr} = C_b \frac{\pi^2 EI_y}{L_y} \sqrt{\frac{k}{k_w}} \left[ \frac{I_x}{I_y} + \left( \frac{L_y}{L_x} \right)^2 \frac{G_{tor}}{EI_x} \right], \]

(3)

where \( L_y \) – lateral buckling length; \( C_b \) – moment gradient coefficient, obtained from the Eq

\[ C_b = 1.75 + 1.05 \left( \frac{M_1}{M_2} \right) + 0.3 \left( \frac{M_1}{M_2} \right)^2 \leq 2, \]

(4)

where \( M_1 \) and \( M_2 \) – absolute bending moment values at the ends of the beam. If \( M_1 = -M_2 \), then \( C_b = 1 \). The beam suffering several moments in one plane \( C_b \) is calculated as:

\[ C_b = \frac{12.5M_{max}}{2.5M_{max} + 3M_A + 4M_B + 3M_C}, \]

(5)

where \( M_{max} \) – absolute max moment; \( M_A \) – absolute moment in 1st quarter of the beam; \( M_B \) – absolute moment in middle part of the beam; \( M_C \) – absolute moment in 3rd quarter of the beam.

The British code for steelworks in buildings BS 5950-1:2000 Structural Use of Steelwork in Buildings. Code of Practice for Design. Rolled and Welded Sections incorporates a formulation very similar to that of AISC.

Lateral–torsional buckling of beams is presented in EN 1993-1-1 Eurocode 3: Design of Steel Structures. Part 1–1: General Rules and Rules for Buildings. \( M_{cr} \) is based on gross cross-sectional properties and takes into account the loading conditions, the real moment distribution and restraints:

\[ M_{cr} = \frac{k_c E I_y}{L_y^2} \left[ \frac{I_w}{I_y} + \frac{L^2 G_{tor}}{E I_y} \right], \]

(6)

where \( k_c \) and \( c \) – coefficients dependent on load type, place of loading and restraint conditions; \( I_w \) – constant of torsion.

For solving Eq (1) the energetic method is applied (Reddy 2002).

Therefore, estimating shelf curves for turning beam equal to \( u = 0.5q h \), side energy can be put as:

\[ (B_1 - 2D) \int_0^L \frac{d^2 u}{dz^2}^2 \, dz + D \int_0^L \left( \frac{d^2 u}{dz^2} + \frac{h d^2 \varphi}{2 dz^2} \right) \, dz + \]

\[ D \int_0^L \left( \frac{d^2 \varphi}{dz^2} - \frac{h d^2 \varphi}{2 dz^2} \right) \, dz = B_1 \int_0^L \left( \frac{d^2 u}{dz^2} \right) \, dz + \]

\[ D \int_0^L \left( \frac{d^2 \varphi}{dz^2} \right) \, dz. \]

(7)

Torsion energy is equal to

\[ C \int_0^L \left( \frac{d \varphi}{dz} \right) \, dz. \]

(8)

Action in buckling equals to

\[ F \int_0^L \left( \frac{d^2 u}{dz^2} \right) \, dz. \]

(9)

With energy equal to the action fulfilled the Eq can be put as

\[ F \int_0^L \left( \frac{d^2 u}{dz^2} \right) \, dz = B_1 \int_0^L \left( \frac{d^2 u}{dz^2} \right) \, dz + \]

\[ D \int_0^L \left( \frac{d^2 \varphi}{dz^2} \right) \, dz + C \int_0^L \left( \frac{d \varphi}{dz} \right) \, dz. \]

(10)
After eliminating \( \frac{d^2u}{dz^2} \) from Eq (1) is obtained
\[
\frac{F^2}{B_1} \int_0^L \phi^2 dz = C \left( \int_0^L \frac{d\phi}{dz} \right)^2 dz + \frac{1}{2} \int_0^L \left( \frac{d^2\phi}{dz^2} \right)^2 dz.
\]
Function \( \phi \) put as
\[
\phi = aL^2 - az^2,
\]
where \( a \) – constant.

After integration is obtained the following:
\[
\int_0^L \left( \frac{d\phi}{dz} \right)^2 dz = \frac{4}{3} a^2 L^3; \quad \int_0^L \phi^2 dz = \frac{8}{105} a^2 L^3; \quad \int_0^L \left( \frac{d^2\phi}{dz^2} \right)^2 dz = 4a^2 L; \quad \int_0^L \frac{d^2\phi}{dz^2} dz = \frac{4}{3} a^2 L^3.
\]

Therefore,
\[
\frac{F^2}{B_1} = \frac{8}{105} a^2 L^2 = C \frac{4}{3} a^2 L^3 + \frac{Dh^2}{2} 4a^2 L;
\]

thus
\[
F_{cr}^2 = 17.5B_1 \frac{CL^2 + 1.5Dh^2}{L^6};
\]

and
\[
F_{cr} = 4.18 \sqrt{\frac{B_1 (CL^2 + 1.5Dh^2)}{L^6}}.
\]

As given in classical and standard solutions, stability cases for bridge constructions are analyzed under simple loads. Studies of complicated loading are not yet made. Therefore, in cases with main bending moment loads that give torsion, extra torsional moment is also active, and it is an important problem in practice. For obtaining such values, applications of energy method must be analyzed.

3. Energetic method for critical buckling loads in case of bending and torsion

With additional torsion load and bending energy increases in value
\[
\frac{1}{2} \int_0^L \frac{M_{tor}^2}{C} dz,
\]
where \( M_{tor} \) – additional moment of torsion.

Action fulfilled increases in value
\[
\frac{1}{2} \int_0^L \frac{M_{tor}^2}{C} dz.
\]

Comparing action to energy is obtained the following:
\[
\frac{F^2}{B_1} \int_0^L \phi^2 dz + \frac{1}{2} \int_0^L \frac{M_{tor}^2}{C} dz = C \left( \int_0^L \frac{d\phi}{dz} \right)^2 dz + \frac{1}{2} \int_0^L \left( \frac{d^2\phi}{dz^2} \right)^2 dz + \frac{1}{2} \int_0^L \frac{M_{tor}^2}{C} dz;
\]

with
\[
\phi = aL^2 - az^2,
\]
where \( a \) – constant.

After integration the following equation is obtained:
\[
\frac{F^2}{B_1} = \frac{8}{105} a^2 L^2 - \frac{4}{3} a^2 L^3 C - \frac{Dh^2}{2} 4a^2 L + \frac{1}{2C} M_{tor}^2 L = 0.
\]

For estimating parameter \( a \), Eq of action fulfilled can be used
\[
F_{cr} \delta_{cr} = \frac{F_{cr}^2}{B_1} \int_0^L \phi^2 dz.
\]

Therefore,
\[
\delta_{cr} = \frac{8}{105B_1} F_{cr} a^2 L^2
\]

and
\[
a = \sqrt{\frac{105B_1 \delta_{cr} L^2}{8F_{cr}^2}}.
\]

Eq (24) presents the case of complicated loading as the beam is bended and turned with additional moment of torsion.

4. Experiment

For corroboration of theoretical solutions tests with double T profile No. 10 beam were performed in Fig. 3. Test results are given in Table. With the middle of parameter a value in Eq (24), having constant values of beam cross-section and length values \( F_{cr} \) and \( M_{tor,cr} \) can be predicted. Diagram of such dependences is given in Fig. 4.

5. Conclusions

1. Analysis of buckling values in thin-walled constructional elements shows that in case of both bending and buckling loads no analytical answers can be obtained.
2. In case of complicated buckling loads, the energetic method is suggested that allows solving problems with sufficient engineer error of 5% accuracy comparing with accurate analytical methods.
3. After calculating constants for buckling Eq in experimental critical bending forces and moments of torsion, while buckling can be predicted.
### Test results

| Cross-section | Length, m | Moment of torsion $M_{tor}$, Nm | Middle of experimental force $F_{cr}$, N | Variation ratio of experimental critical force, % | Middle of shift $\delta_{cr}$, m | Variation ratio of shift, % | Middle of parameter $a$ | Variation ratio of parameter $a$, % |
|---------------|-----------|---------------------------------|----------------------------------------|-----------------------------------------------|---------------------------------|----------------------------|-------------------------------|---------------------------------|
| 1 No. 10      | 4         | 10                              | 1910                                   | 6.776                                          | 0.036                           | 9.235                      | 0.023                         | 1.337                           |
|               | 4         | 30                              | 1798                                   | 7.256                                          | 0.034                           | 11.723                     | 0.023                         | 2.577                           |
|               | 4         | 50                              | 1658                                   | 7.586                                          | 0.031                           | 5.100                      | 0.023                         | 4.741                           |
|               | 3.5       | 10                              | 2422                                   | 6.196                                          | 0.018                           | 7.207                      | 0.023                         | 4.169                           |
|               | 3.5       | 30                              | 2314                                   | 6.966                                          | 0.017                           | 6.869                      | 0.023                         | 2.040                           |
|               | 3.5       | 50                              | 2170                                   | 7.027                                          | 0.016                           | 9.388                      | 0.023                         | 2.848                           |
|               | 3         | 10                              | 3370                                   | 6.267                                          | 0.008                           | 10.203                     | 0.023                         | 4.500                           |
|               | 3         | 30                              | 3196                                   | 7.780                                          | 0.008                           | 8.839                      | 0.023                         | 3.709                           |
|               | 3         | 50                              | 3004                                   | 7.052                                          | 0.007                           | 15.408                     | 0.023                         | 4.769                           |

### Fig. 3. Test diagram for supported beam

### Fig. 4. Dependence of critical buckling-bending forces and moment of torsion

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