Generalized metallic pseudo-Riemannian structures

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Abstract

We generalize the notion of metallic structure in the pseudo-Riemannian setting, define the metallic Norden structure and study its integrability. We construct a metallic natural connection recovering as particular case the Ganchev and Mihova connection, which we extend to a metallic natural connection on the generalized tangent bundle. Moreover, we construct metallic pseudo-Riemannian structures on the tangent and cotangent bundles.

1 Introduction

For fixed positive integer numbers $p$ and $q$, the $(p,q)$-metallic number stands for positive solution of the equation $x^2 - px - q = 0$ and it is equal to

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$  

For particular values of $p$ and $q$, some important members of the metallic mean family are the followings: the Golden mean $\phi = \frac{1 + \sqrt{5}}{2}$ for $p = q = 1$, the Silver mean $\sigma_{Ag} = \sigma_{2,1} = 1 + \sqrt{2}$ for $q = 1$ and $p = 2$, the Bronze mean $\sigma_{Br} = \sigma_{3,1} = \frac{3 + \sqrt{13}}{2}$ for $q = 1$ and $p = 3$, the Subtle mean $\sigma_{s,1} = 2 + \sqrt{5} = \phi^2$ for $p = 4$ and $q = 1$, the Copper mean $\sigma_{Cu} = \sigma_{1,2} = 2$ for $p = 1$ and $q = 2$, the Nickel mean $\sigma_{Ni} = \sigma_{1,3} = \frac{1 + \sqrt{17}}{2}$ for $p = 1$ and $q = 3$ and so on.

Extending this idea to tensor fields, C.-E. Hrețcanu and M. Crasmareanu introduced the notion of metallic structure:

**Definition 1.1.** A $(1,1)$-tensor field $J$ on $M$ is called a metallic structure if it satisfies the equation:

$$J^2 = pJ + qI,$$
for \( p \) and \( q \) positive integer numbers, where \( I \) is the identity operator on \( C^\infty(TM) \). In this case, the pair \((M, J)\) is called a metallic manifold. Moreover, if a Riemannian metric \( g \) on \( M \) is compatible with \( J \), that is \( g(JX, Y) = g(X, JY) \), for any \( X, Y \in C^\infty(TM) \), we call the pair \((J, g)\) a metallic Riemannian structure and \((M, J, g)\) a metallic Riemannian manifold.

From the compatibility condition, we immediately get that a metallic Riemannian structure satisfies
\[
g(JX, JY) = pg(X, Y) + qg(X, JY),
\]
for any \( X, Y \in C^\infty(TM) \).

## 2 Metallic pseudo-Riemannian manifolds

The notion of metallic Riemannian manifold can be generalized to a metallic pseudo-Riemannian manifold. We pose the following:

**Definition 2.1.** Let \((M, g)\) be a pseudo-Riemannian manifold and let \( J \) be a \( g \)-symmetric \((1, 1)\)-tensor field on \( M \) such that \( J^2 = pJ + qI \), for some \( p \) and \( q \) real numbers. Then the pair \((J, g)\) is called a metallic pseudo-Riemannian structure on \( M \) and \((M, J, g)\) is called a metallic pseudo-Riemannian manifold.

Fix now a metallic structure \( J \) on \( M \) and define the associated linear connections as follows:

**Definition 2.2.** i) A linear connection \( \nabla \) on \( M \) is called a \( J \)-connection if \( J \) is covariantly constant with respect to \( \nabla \), namely \( \nabla J = 0 \).

ii) A metallic pseudo-Riemannian manifold \((M, J, g)\) such that the Levi-Civita connection \( \nabla \) with respect to \( g \) is a \( J \)-connection is called a locally metallic pseudo-Riemannian manifold.

The concept of integrability is defined in the classical manner:

**Definition 2.3.** A metallic structure \( J \) is called integrable if its Nijenhuis tensor field \( N_J \) vanishes, where \( N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y] \), for \( X, Y \in C^\infty(TM) \).

**Lemma 2.4.** If \((M, J, g)\) is a locally metallic pseudo-Riemannian manifold, then \( J \) is integrable.

**Proof.** We have:
\[
N_J(X, Y) = (\nabla_{JX} J)Y - (\nabla_{JY} J)X + J(\nabla_Y J)X - J(\nabla_X J)Y,
\]
for any \( X, Y \in C^\infty(TM) \). Then the statement.
Remark 2.5. Every pseudo-Riemannian manifold admits locally metallic pseudo-Riemannian structures, namely $J = \mu I$, where $\mu = \frac{p \pm \sqrt{p^2 + 4q}}{2}$ with $p^2 + 4q \geq 0$.

Definition 2.6. $J := \mu I$, where $\mu = \frac{p \pm \sqrt{p^2 + 4q}}{2}$ with $p^2 + 4q \geq 0$, is called a trivial metallic structure.

Definition 2.7. A metallic pseudo-Riemannian manifold $(M, J, g)$ such that the Levi-Civita connection $\nabla$ with respect to $g$ satisfies the condition

$$(\nabla_X J)Y + (\nabla_Y J)X = 0,$$

for any $X, Y \in C^\infty(TM)$, is called a nearly locally metallic pseudo-Riemannian manifold.

Proposition 2.8. A nearly locally metallic pseudo-Riemannian manifold $(M, J, g)$ such that $J^2 = p J + q I$ with $p^2 + 4q > 0$ is a locally metallic pseudo-Riemannian manifold if and only if $J$ is integrable.

Proof. For any $X, Y \in C^\infty(TM)$, we have:

$$N_J(X, Y) = (\nabla_{JX} J)Y - (\nabla_{JY} J)X + J(\nabla_Y J)X - J(\nabla_X J)Y =$$

$$= -(\nabla_Y J)X + (\nabla_X J)JY + J(\nabla_Y J)X - J(\nabla_X J)Y =$$

$$= -(\nabla_Y J^2 X) + 2(J \nabla_Y J)X + J^2 \nabla_Y X + (\nabla_X J^2 Y) - 2(J \nabla_X J)Y - J^2 \nabla_X Y =$$

$$= -2p(\nabla_Y J)X + 4J(\nabla_Y J)X = 2(2J - p I)(\nabla_Y J)X.$$

We observe that $\frac{p}{2}$ is not an eigenvalue of $J$ because $p^2 + 4q > 0$, thus we get that if $J$ is nearly locally metallic, then

$$N_J = 0 \iff \nabla J = 0$$

and the proof is complete. \qed

3 Metallic natural connection

Theorem 3.1. Let $(M, J, g)$ be a metallic pseudo-Riemannian manifold such that $J^2 = p J + q I$ with $p^2 + 4q \neq 0$. Let $\nabla$ be the Levi-Civita connection of $g$ and let $D$ be the linear connection defined by:

$$(3) \quad D := \nabla + \frac{2}{p^2 + 4q} J(\nabla J) - \frac{p}{p^2 + 4q} (\nabla J).$$

Then

$$\{\begin{align*}
DJ &= 0 \\
Dg &= 0.
\end{align*}\}$$

Proof. We have:
\[ DJ - JD = \nabla J - J\nabla + \frac{1}{p^2 + 4q}(2J\nabla J^2 - 2q\nabla J - p\nabla J^2 - 2J^2\nabla J + pJ^2\nabla + 2qJ\nabla) = \]
\[ = \nabla J - J\nabla + \frac{1}{p^2 + 4q}(2qJ\nabla - 2q\nabla J - p^2\nabla J - 2q\nabla J + p^2J\nabla + 2qJ\nabla) = \]
\[ = \nabla J - J\nabla + \frac{1}{p^2 + 4q}[(4q + p^2)J\nabla - (4q + p^2)\nabla J] = 0. \]
Moreover, for any \( X, Y, Z \in C^\infty(TM) \), we have:
\[ (DXg)(Y, Z) = X(g(Y, Z)) - g(DXY, Z) - g(Y, DXZ) = \]
\[ = \frac{1}{p^2 + 4q}[2g((\nabla_XJ)Y, JZ) - pg((\nabla_XJ)Y, Y) + 2g(JY, (\nabla_XJ)Z) - pg(Y, (\nabla_XJ)Z) + 2qX(g(Y, Z)) = \]
\[ = 0. \]
Then the proof is complete. \( \square \)

Definition 3.2. The linear connection \( D \) defined by (3) is called the metallic natural connection of \((M, J, g)\).

A direct computation gives the following expression for the torsion \( T^D \) of the natural connection \( D \):
\[ T^D(X, Y) = \frac{1}{p^2 + 4q}\{ (2J - pI)(\nabla_XJY - \nabla_JX) - (pJ + 2qI)[X, Y] \}, \]
for any \( X, Y \in C^\infty(TM) \).

Thus we get the following:

Proposition 3.3. Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold such that \( J^2 = pJ + qI \) with \( p^2 + 4q \neq 0 \). Then the torsion \( T^D \) of the natural connection \( D \) satisfies the following relation:
\[ T^D(JX, Y) + T^D(X, JY) - pT^D(X, Y) = (2J - pI)NJ(X, Y), \]
for any \( X, Y \in C^\infty(TM) \). In particular, if \( J \) is integrable, then:
\[ T^D(JX, Y) + T^D(X, JY) = pT^D(X, Y). \]

Remark 3.4. If \( p = 0, q = -1 \) and \( J \) is integrable, then the natural connection \( D \) coincides with the natural canonical connection defined by Ganchev and Mihova in [2].
4 Metallic Norden structures

Recall that a Norden manifold \((M, J, g)\) is an almost complex manifold \((M, J)\) with a neutral pseudo-Riemannian metric \(g\) such that \(g(JX, Y) = g(X, JY)\), for any \(X, Y \in C^\infty(TM)\). We can state:

**Proposition 4.1.** If \((M, J, g)\) is a Norden manifold, then for any real numbers \(a, b\),

\[
J_{a,b} := aJ + bI
\]

are metallic pseudo-Riemannian structures on \(M\).

**Proof.** We have:

\[
J_{a,b}^2 = 2bJ_{a,b} - (a^2 + b^2)I.
\]

Moreover, for any \(X, Y \in C^\infty(TM)\), we have:

\[
g(J_{a,b}X, Y) = g(X, J_{a,b}Y).
\]

Then the statement.

We remark that \(J = J_{1,0}\).

Also, from \(\nabla J_{a,b} = a\nabla J\) and \(N_{J_{a,b}} = a^2 N_J\), we get the following:

**Proposition 4.2.** Assume that \(a \neq 0\). Then:

1. \(J_{a,b}\) is integrable if and only if \(J\) is integrable.
2. \(J_{a,b}\) is locally metallic if and only if \(J\) is Kähler.
3. \(J_{a,b}\) is nearly locally metallic if and only if \(J\) is nearly Kähler.

Conversely, we have:

**Proposition 4.3.** If \((M, J, g)\) is a metallic pseudo-Riemannian manifold such that \(J^2 = pJ + qI\) with \(p^2 + 4q < 0\), then

\[
J^\pm := \pm(\sqrt{-p^2 - 4q}J - \frac{p}{\sqrt{-p^2 - 4q}}I)
\]

are Norden structures on \(M\) and \(J = aJ^\pm + bI\) with \(a = \pm(\frac{2}{\sqrt{-p^2 - 4q}})^{-1}\) and \(b = -\frac{p}{2}\).

**Proof.** We have:

\[
J_{\pm}^2 = \frac{1}{-p^2 - 4q}(4J^2 - 4pJ + p^2 I) = \frac{1}{-p^2 - 4q}(4qI + p^2 I) = -I.
\]
Moreover, for any $X, Y \in C^\infty(TM)$, we have:

$$g(J_X Y) = g(X, J_Y).$$

Finally, we have:

$$J = \pm \left( \frac{2}{\sqrt{-p^2 - 4q}} \right)^{-1} J - \frac{p}{2} J.$$

Then the statement.

We give the following definition:

**Definition 4.4.** Let $(M, J, g)$ be a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$. Then $J$ is called a **metallic Norden structure on** $M$ and $(M, J, g)$ is called a **metallic Norden manifold**.

## 5 Induced structures on $TM \oplus T^*M$

### 5.1 Generalized metallic pseudo-Riemannian structures

In [1] we introduced the notion of generalized metallic structure and generalized metallic Riemannian structure. We pose the following:

**Definition 5.1.** A pair $(\tilde{J}, \tilde{g})$ of a generalized metallic structure $\tilde{J}$ and a pseudo-Riemannian metric $\tilde{g}$ such that $\tilde{J}^2 = p\tilde{J} + qI$ with $p^2 + 4q < 0$, then $\tilde{J}$ is called a **generalized metallic pseudo-Riemannian structure**. If $\tilde{J}^2 = p\tilde{J} + qI$ with $p^2 + 4q < 0$, then $\tilde{J}$ is called a **generalized metallic Norden structure**.

Let $(M, J, g)$ be a Norden manifold and let $(\tilde{J}, \tilde{g})$ be the generalized Norden structure defined in [4]:

\[
\tilde{J} := \begin{pmatrix} J & 0 \\ \tilde{g} & -J^* \end{pmatrix}
\]

(5) \[\tilde{g}(X + \alpha, Y + \beta) := g(X, Y) + \frac{1}{2} g(JX, \tilde{g}_\alpha JY) + \frac{1}{2} g(\tilde{g}_\alpha JY, \tilde{g}_\alpha JY),\]

for any $X, Y \in C^\infty(TM)$ and $\alpha, \beta \in C^\infty(T^*M)$. Then $\tilde{J}$ defines the following family of generalized metallic Norden structures:

\[
\tilde{J}_{a,b} := a\tilde{J} + bI = \begin{pmatrix} aJ + bI & 0 \\ a\tilde{g} & -aJ^* + bI \end{pmatrix},
\]

where $a, b$ are real numbers, since

\[
\tilde{J}_{a,b}^2 = p\tilde{J}_{a,b} + qI
\]
with $p = 2b$ and $q = -(a^2 + b^2)$ and

$$\tilde{g}(\tilde{J}_{a,b}(\sigma), \tau) = \tilde{g}(\sigma, \tilde{J}_{a,b}(\tau)),$$

for any $\sigma, \tau \in C^\infty(TM \oplus T^*M)$.

We remark that, up to rescaling the metric, instead of $\tilde{J}_{a,b}$, we can consider the family:

$$\hat{J}_{a,b} := \left( \begin{array}{cc} aJ + bI & 0 \\ \flat_g & -aJ^* + bI \end{array} \right).$$

Moreover, if $(M, J, g)$ is a metallic pseudo-Riemannian manifold with $J^2 = pJ + qI$, for $p = 2b$ and $q = -(a^2 + b^2)$, we immediately have that

$$\hat{J} := \left( \begin{array}{cc} J & 0 \\ \flat_g & -J^* + pI \end{array} \right)$$

is a generalized metallic structure with:

$$\hat{J}^2 = p\hat{J} + qI.$$

If we assume $p^2 + 4q \neq 0$, generalizing (5), we define the pseudo-Riemannian metric:

$$\hat{g}(X + \alpha, Y + \beta) :=$$

$$= g(X, Y) + g(\sharp_g \alpha, \sharp_g \beta) + \frac{p}{p^2 + 4q} [g(X, \sharp_g \beta) + g(Y, \sharp_g \alpha)] - \frac{2}{p^2 + 4q} [g(JX, \sharp_g \beta) + g(JY, \sharp_g \alpha)] =$$

$$= g(X, Y) + g(\sharp_g \alpha, \sharp_g \beta) + \frac{p}{p^2 + 4q} (\alpha(Y) + \beta(X)) - \frac{2}{p^2 + 4q} (\alpha(JY) + \beta(JX)),$$

for any $X, Y \in C^\infty(TM)$ and $\alpha, \beta \in C^\infty(T^*M)$ and we have the following:

**Proposition 5.2.** Let $(M, J, g)$ be a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q \neq 0$. Then $(\hat{J}, \hat{g})$ is a generalized metallic pseudo-Riemannian structure with $\hat{J}$ given by (6) and $\hat{g}$ given by (7).

**Proof.** For any $X + \alpha, Y + \beta \in C^\infty(TM \oplus T^*M)$, we have:

$$\hat{g}(\hat{J}(X + \alpha), Y + \beta) = \hat{g}(JX + \flat_g(X), -J^*(\alpha) + p\alpha, Y + \beta) =$$

$$= g(JX, Y) + \beta(X) - g(\sharp_g J^*(\alpha), \sharp_g \beta) + pg(\sharp_g \alpha, \sharp_g \beta) +$$

$$+ \frac{p}{p^2 + 4q} [\beta(JX) + g(X, Y) - \alpha(JY) + p\alpha(Y)] - \frac{2}{p^2 + 4q} [p\beta(JX) + q\beta(X) +$$

$$+ g(X, JY) - qa(Y)]$$

and

$$\hat{g}(X + \alpha, \hat{J}(Y + \beta)) = g(X + \alpha, JY + \flat_g(Y) - J^*(\beta) + p\beta) =$$
= g(X, JY) + \alpha(Y) - g(\sharp g J^*(\beta), \sharp g \alpha) + p g(\sharp g \alpha, \sharp g \beta) +
+ \frac{p}{p^2 + 4q} [\alpha(JY) + g(X, Y) - \beta(JX) + p\beta(X)] - \frac{2}{p^2 + 4q} [p\alpha(JY) + q\alpha(Y) + 
+ g(Y, JX) - q\beta(X)].

Since \((M, J, g)\) is a metallic pseudo-Riemannian manifold and 
\(J^* = \flat g J^* g\) we get the
statement.

**Definition 5.3.** Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold. The pair
\((\hat{J}, \hat{g})\) defined by (6) and (7) is called the
**generalized metallic pseudo-Riemannian structure**
defined by \((J, g)\).

More generally, we can construct generalized metallic pseudo-Riemannian structures
by using a pseudo-Riemannian metric \(g\) on \(M\) and an arbitrary \(g\)-symmetric endomor-
phism \(J\) of the tangent bundle as in the followings.

**Theorem 5.4.** Let \(g\) be a pseudo-Riemannian metric on \(M\) and let \(J\) be an arbitrary
endomorphism of the tangent bundle which is \(g\)-symmetric. Then \((\hat{J}, \hat{g})\) is a generalized
metallic pseudo-Riemannian structure, where

\[
\hat{J} := \begin{pmatrix} J & (-J^2 + pJ + qI)_\sharp g \\ \flat g & -J^* + pI \end{pmatrix}
\]

and

\[
\hat{g}(X + \alpha, Y + \beta) := g(X, Y) + \frac{p^2 + 4q}{4} g(\sharp g \alpha, \sharp g \beta) + \frac{p}{4} (\alpha(Y) + \beta(X)) - \frac{1}{2} (\alpha(JY) + \beta(JX)),
\]

with \(p\) and \(q\) any real numbers.

Moreover, \(\hat{J}\) satisfies:

\[(\hat{J}(X + \alpha), Y + \beta) + (X + \alpha, \hat{J}(Y + \beta)) = p \cdot (X + \alpha, Y + \beta),\]

where

\[
(X + \alpha, Y + \beta) := -\frac{1}{2} (\alpha(Y) - \beta(X))
\]

is the natural symplectic structure on \(TM \oplus T^*M\).

**Proof.** A direct computation gives \(J^2 = p\hat{J} + qI\).

Moreover, for any \(X + \alpha \in C^\infty(TM \oplus T^*M)\), we have:

\[
\hat{J}(X + \alpha) = JX - J^2(\sharp g \alpha) + p J(\sharp g \alpha) + q\sharp g \alpha + b_g(X) - J^*(\alpha) + p\alpha
\]
and using the definition of $(\cdot, \cdot)$ we get the last statement.

Now, for any $X + \alpha, Y + \beta \in C^\infty(TM \oplus T^*M)$, we have:

$$\tilde{g}(\tilde{J}(X + \alpha), Y + \beta) = g(JX, Y) - g(J^2(\sharp_g \alpha), Y) + pg(J(\sharp_g \alpha), Y) +$$

$$+qg(\sharp_g \alpha, Y) + \frac{p^2 + 4q}{4}g(X, \sharp_g \beta) - \frac{p^2 + 4q}{4}g(\sharp_g J^*(\beta), \sharp_g \alpha) + \frac{p^2 + 4q}{4}pg(\sharp_g \alpha, \sharp_g \beta) +$$

$$+\frac{p}{4}[g(X, Y) - \alpha(JY) + p\alpha(Y) + \beta(JX) - \beta(J^2(\sharp_g \alpha)) + p\beta(J(\sharp_g \alpha)) + q\beta(\sharp_g \alpha)] -$$

$$-\frac{1}{2}[g(X, JY) - \alpha(J^2Y) + p\alpha(JY) + \beta(J^2X) - \beta(J^3(\sharp_g \alpha)) + p\beta(J^2(\sharp_g \alpha)) + q\beta(J(\sharp_g \alpha))]$$

and

$$\tilde{g}(\tilde{J}(X + \alpha), \tilde{J}(Y + \beta)) = g(JX, X) - g(J^2(\sharp_g \beta), X) + pg(J(\sharp_g \beta), X) +$$

$$+qg(\sharp_g \beta, X) + \frac{p^2 + 4q}{4}g(Y, \sharp_g \alpha) - \frac{p^2 + 4q}{4}g(\sharp_g J^*(\alpha), \sharp_g \beta) + \frac{p^2 + 4q}{4}pg(\sharp_g \beta, \sharp_g \alpha) +$$

$$+\frac{p}{4}[g(Y, X) - \beta(JX) + p\beta(X) + \alpha(JY) - \alpha(J^2(\sharp_g \beta)) + p\alpha(J(\sharp_g \beta)) + q\alpha(\sharp_g \beta)] -$$

$$-\frac{1}{2}[g(Y, JX) - \beta(J^2X) + p\beta(JX) + \alpha(J^2Y) - \alpha(J^3(\sharp_g \beta)) + p\alpha(J^2(\sharp_g \beta)) + q\alpha(J(\sharp_g \beta))].$$

Since $J$ is $g$-symmetric and $J^* = b_g J\sharp_g$ we get the statement. \qed

Remark 5.5. i) For $p = 0$, the structure $\tilde{J}$ is anti-calibrated with respect to $J^2$.

ii) In particular, for $p = 0$ and $q = 1$ we get the generalized product structure

$$\tilde{J}_p := \begin{pmatrix} J & -J^2 + I \sharp_g \\ b_g & -J^* \end{pmatrix}$$

and for $p = 0$ and $q = -1$ we get the generalized complex structure

$$\tilde{J}_c := \begin{pmatrix} J & -J^2 - I \sharp_g \\ b_g & -J^* \end{pmatrix}$$

which are both anti-calibrated.

iii) If $J$ is a metallic structure with $J^2 = pJ + qI$, then $\tilde{J} = \tilde{J}$. \qed

Remark 5.6. i) Notice that if $(J, g)$ is a metallic pseudo-Riemannian structure such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$, then $\tilde{J}^i$ obtained from $\hat{J}_{a,b}$ (beside $\hat{J}$) and given by:

$$(11) \quad \tilde{J}^i := \begin{pmatrix} -J + pI & 0 \\ b_g & J^* \end{pmatrix}$$

is a generalized metallic structure with:

$$\tilde{J}^2 = p\tilde{J}^i + qI.$$
Moreover, the two structures \( \hat{J} \) and \( \hat{J}' \) coincide with the ones obtained by considering first the Norden structures \( J_\pm \) induced by \( J \) and then defining the generalized metallic structures

\[
\hat{j}_{a,b} := \begin{pmatrix} aJ_\pm + bI & 0 \\ \frac{b}{\flat g} & -aJ^*_\mp + bI \end{pmatrix},
\]

where \( a = \pm \sqrt{-\frac{p^2 - 4q}{2}} \) and \( b = \frac{p}{2} \).

ii) The structure \( \hat{J}' \) defined by:

\[
\hat{j}' := \begin{pmatrix} -J + pI & (-J^2 + pJ + qI)\flat g \\ \frac{\flat g}{a} & J^* \end{pmatrix}
\]
is also a generalized metallic structure and for the particular case when \( J \) is metallic, it is precisely \( \hat{J}' \).

**Remark 5.7.** Let \( (M, g) \) be a pseudo-Riemannian manifold and let \( J \) be an arbitrary \( g \)-symmetric endomorphism of the tangent bundle. Then for any \( p \) and \( q \) real numbers with \( p^2 + 4q < 0 \):

\[
\hat{J}_\pm := \pm \left( \frac{2}{\sqrt{-p^2 - 4q}} \hat{J} - \frac{p}{\sqrt{-p^2 - 4q}} I \right)
\]
are generalized Norden structures with respect to the metric \( \hat{g} \).

### 5.2 Generalized metallic natural connection

Let \( (M, J, g) \) be a metallic pseudo-Riemannian manifold and let \( D \) be the metallic natural connection given by (3). We define:

\[
\hat{D} : C^\infty(TM \oplus T^*M) \times C^\infty(TM \oplus T^*M) \to C^\infty(TM \oplus T^*M)
\]

by:

\[
\hat{D}_{X+\alpha}(Y + \beta) := D_X Y + D_X \beta,
\]
for any \( X, Y \in C^\infty(TM) \) and \( \alpha, \beta \in C^\infty(T^*M) \) and we have:

**Theorem 5.8.** The linear connection \( \hat{D} \) satisfies the following conditions:

\[
\begin{cases}
\hat{D}\hat{J} = 0 \\
\hat{D}\hat{g} = 0.
\end{cases}
\]

Moreover, \( T^\hat{D}(X + \alpha, Y + \beta) = T^D(X,Y) \), for any \( X, Y \in C^\infty(TM) \) and \( \alpha, \beta \in C^\infty(T^*M) \), and \( \hat{D} \) is flat if and only if \( D \) is flat.
PROOF. From the definition of \( \hat{J} \) and from the properties of \( D \) we get:

\[
(\hat{D}_{X+\alpha} \hat{J})(Y + \beta) = \hat{D}_{X+\alpha}(JY + b_Y(Y)) - J^*(\beta + p\beta) = D_X JY + D_X(b_Y(Y) - J^*(\beta + p\beta) = J(D_X Y) + b_Y(D_X Y) - J^*(D_X \beta) + pD_X \beta = \hat{J}(D_X Y) + \hat{J}(D_X \beta) = \hat{J}(\hat{D}_{X+\alpha}(Y + \beta)),
\]

for any \( X, Y \in C^\infty(TM) \) and \( \alpha, \beta \in C^\infty(T^*M) \).

Moreover, from the definition of \( \hat{g} \) we get:

\[
X(\hat{g}(Y + \beta, Z + \gamma)) - \hat{g}(\hat{D}_{X+\alpha}(Y + \beta), Z + \gamma) - \hat{g}(Y + \beta, \hat{D}_{X+\alpha}(Z + \gamma)) =
\]

\[
= X(g(Y, Z) + g(\sharp_Y \beta, \sharp_Y \gamma) + \frac{p}{p^2 + 4q}(\beta(Z) + \gamma(Y)) - \frac{2}{p^2 + 4q}(\beta(JZ) + \gamma(JY)) - [g(D_X Y, Z) + g(\sharp_Y (D_X \beta), \sharp_Y \gamma) + \frac{p}{p^2 + 4q}((D_X \beta)Z + \gamma(D_X Y)) - \frac{2}{p^2 + 4q}((D_X \beta)(JZ) + \gamma(J(D_X Y)))] -
\]

\[
- [g(Y, D_X Z) + g(\sharp_Y \beta, \sharp_Y (D_X \gamma)) + \frac{p}{p^2 + 4q}((D_X \gamma)Y + \beta(D_X Z)) - \frac{2}{p^2 + 4q}((D_X \gamma)(JY) + \beta(J(D_X Z)))] = 0,
\]

for any \( X, Y, Z \in C^\infty(TM) \) and \( \alpha, \beta, \gamma \in C^\infty(T^*M) \).

If we define by:

\[
T^D(X + \alpha, Y + \beta) := \hat{D}_{X+\alpha}(Y + \beta) - \hat{D}_{Y+\beta}(X + \alpha) - [X + \alpha, Y + \beta]_D
\]

the torsion of \( \hat{D} \), where

\[
[X + \alpha, Y + \beta]_D := [X, Y] + D_X \beta - D_Y \alpha,
\]

we have:

\[
T^D(X + \alpha, Y + \beta) = T^D(X, Y).
\]

Also, if we define by:

\[
R^D(X + \alpha, Y + \beta)(Z + \gamma) := \hat{D}_{X+\alpha}\hat{D}_{Y+\beta}(Z + \gamma) - \hat{D}_{Y+\beta}\hat{D}_{X+\alpha}(Z + \gamma) - \hat{D}_{[X+\alpha,Y+\beta]_D}(Z + \gamma)
\]

the curvature of \( \hat{D} \), we have:

\[
R^D(X + \alpha, Y + \beta)(Z + \gamma) = R^D(X, Y)Z + R^D(X, Y)\gamma
\]

and we get the statement.

\[\square\]

**Definition 5.9.** The linear connection \( \hat{D} \) defined by (14) is called the generalized metallic natural connection of \((TM \oplus T^*M, \hat{J}, \hat{g})\).
6 Metallic pseudo-Riemannian structures on tangent and cotangent bundles

6.1 Metallic pseudo-Riemannian structures on the tangent bundle

Let \((M, g)\) be a pseudo-Riemannian manifold and let \(\nabla\) be a linear connection on \(M\). Then \(\nabla\) defines the decomposition into the horizontal and vertical subbundles of \(T(TM)\):

\[ T(TM) = \mathcal{T}^H(TM) \oplus \mathcal{T}^V(TM). \]

Let \(\pi: TM \to M\) be the canonical projection and \(\pi^*: T(TM) \to TM\) be the tangent map of \(\pi\). If \(a \in TM\) and \(A \in T_a(TM)\), then \(\pi^*(A) \in T_{\pi(a)}M\) and we denote by \(\chi_a\) the standard identification between \(T_{\pi(a)}M\) and its tangent space \(T_a(T_{\pi(a)}M)\).

Let \(\Psi^\nabla: TM \oplus T^*M \to T(TM)\) be the bundle morphism defined by:

\[ \Psi^\nabla(X + \alpha) := X_a^H + \chi_a(\sharp g(\alpha)), \]

where \(a \in TM\) and \(X_a^H\) is the horizontal lifting of \(X \in T_{\pi(a)}M\).

Let \(\{x^1, ..., x^n\}\) be local coordinates on \(M\), let \(\{\bar{x}^1, ..., \bar{x}^n, y^1, ..., y^n\}\) be respectively the corresponding local coordinates on \(TM\) and let \(\{X_1, ..., X_n, \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\}\) be a local frame on \(T(TM)\), where \(X_i = \frac{\partial}{\partial \bar{x}^i}\). We have:

\[ \Psi^\nabla \left( \frac{\partial}{\partial x^i} \right) = X_i^H \]

\[ \Psi^\nabla \left( dx^j \right) = g^{jk} \frac{\partial}{\partial y^k}. \]

Let \(J\) be an arbitrary \(g\)-symmetric endomorphism on the tangent bundle. For any \(p\) and \(q\) real numbers, let

\[ \tilde{J} := \begin{pmatrix} J & (-J^2 + pJ + qI)\sharp g \\ \sharp g & -J^* + pI \end{pmatrix} \]

be the generalized metallic pseudo-Riemannian structure defined by \((J, g)\) with the pseudo-Riemannian metric \(\tilde{g}\) defined by \((\sharp g)\). The isomorphism \(\Psi^\nabla\) allows us to construct a natural metallic structure \(\tilde{J}\) and a natural pseudo-Riemannian metric \(\tilde{g}\) on \(TM\) in the following way.

We define \(\tilde{J}: T(TM) \to T(TM)\) by

\[ \tilde{J} := (\Psi^\nabla) \circ \tilde{J} \circ (\Psi^\nabla)^{-1} \]

and the pseudo-Riemannian metric \(\tilde{g}\) on \(TM\) by

\[ \tilde{g} := ((\Psi^\nabla)^{-1})^*(\tilde{g}). \]
Proposition 6.1. \((TM, \bar{J}, \bar{g})\) is a metallic pseudo-Riemannian manifold.

Proof. From the definition it follows that \(\bar{J}^2 = p\bar{J} + qI\) and \(\bar{g}(\bar{J}X, Y) = \bar{g}(X, \bar{J}Y)\), for any \(X, Y \in C^\infty(T(TM))\).

In local coordinates, we have the following expressions for \(\bar{J}\) and \(\bar{g}\):

\[
\bar{J} \left( X^H_i \right) = J^k_i X^H_k + \frac{\partial}{\partial y^j}.
\]

\[
\bar{J} \left( \frac{\partial}{\partial y^j} \right) = (-J^2 + pJ + qI)_j^k X^H_k - J^k_j \frac{\partial}{\partial y^k} + p \frac{\partial}{\partial y^j}
\]

and

\[
\bar{g} \left( X^H_i, X^H_j \right) = g_{ij}
\]

\[
\bar{g} \left( X^H_i, \frac{\partial}{\partial y^j} \right) = \frac{p}{p^2 + 4q} g_{ij} - \frac{2}{p^2 + 4q} g_{ji} J^l_l,
\]

\[
\bar{g} \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = g_{ij}.
\]

Computing the Nijenhuis tensor of \(\bar{J}\), in the case when \(J\) is metallic with \(J^2 = pJ + qI\) and \(\nabla\) is the Levi-Civita connection of \(g\), we get:

\[
N_j \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 0
\]

\[
N_j \left( X^H_i, \frac{\partial}{\partial y^j} \right) = ((\nabla JX, J) X_j - J (\nabla X, J) X_j)_k^{} \frac{\partial}{\partial y^k}
\]

\[
N_j \left( X^H_i, X^H_j \right) = (N_j (X_i, X_j))_k^{} X^H_k -
\]

\[
- y^s \left( J^k_i J^l_j R^r_{khs} - J^k_i J^l_k R^r_{ij} + J^l_j J^k_i R^r_{hjs} + p J^r_i R^l_{ij} + q R^r_{ij} \right) \frac{\partial}{\partial y^r}.
\]

Therefore we can state the following:

Proposition 6.2. Let \((M, J, g)\) be a flat locally metallic pseudo-Riemannian manifold. If \(\nabla\) is the Levi-Civita connection of \(g\), then \((\bar{J}, \bar{g})\) is an integrable metallic pseudo-Riemannian structure on \(TM\).

6.2 Metallic pseudo-Riemannian structures on the cotangent bundle

Let \((M, g)\) be a pseudo-Riemannian manifold and let \(\nabla\) be a linear connection on \(M\). Then \(\nabla\) defines the decomposition into the horizontal and vertical subbundles of \(T(T^*M)\):

\[
T(T^*M) = T^H(T^*M) \oplus T^V(T^*M).
\]
Let \( \pi : T^*M \to M \) be the canonical projection and \( \pi_* : T(T^*M) \to TM \) be the tangent map of \( \pi \). If \( a \in T^*M \) and \( A \in T_a(T^*M) \), then \( \pi_*(A) \in T_{\pi(a)}M \) and we denote by \( \chi_a \) the standard identification between \( T^*_a(M) \) and its tangent space \( T_a(T^*_a(M)) \).

Let \( \Phi^\nabla : TM \oplus T^*M \to T(T^*M) \) be the bundle morphism defined by:
\[
\Phi^\nabla(X + \alpha) := X_a^H + \chi_a(\alpha),
\]
where \( a \in T^*M \) and \( X_a^H \) is the horizontal lifting of \( X \in T_{\pi(a)}M \).

Let \( \{x^1, ..., x^n\} \) be local coordinates on \( M \), let \( \{\tilde{x}^1, ..., \tilde{x}^n, y_1, ..., y_n\} \) be respectively the corresponding local coordinates on \( T^*M \) and let \( \{X_1, ..., X_n, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\} \) be a local frame on \( T(T^*M) \), where \( X_i = \frac{\partial}{\partial \tilde{x}^i} \). We have:
\[
\Phi^\nabla \left( \frac{\partial}{\partial x^i} \right) = X_i^H \quad \text{ and } \quad \Phi^\nabla \left( dx^j \right) = \frac{\partial}{\partial y_j}.
\]

Let \((M, g)\) be a pseudo-Riemannian manifold and let \( J \) be an arbitrary \( g \)-symmetric endomorphism on the tangent bundle. For any \( p \) and \( q \) real numbers, let
\[
\tilde{J} := \left( \begin{array}{cc}
J & (-J^2 + pJ + qI)\tilde{z}_g \\
\tilde{z}_g & -J^* + pI
\end{array} \right)
\]
be the generalized metallic pseudo-Riemannian structure defined by \( (J, g) \) with the pseudo-Riemannian metric \( \tilde{g} \) defined by \( \tilde{z}_g \). The isomorphism \( \Phi^\nabla \) allows us to construct a natural metallic structure \( \tilde{J} \) and a natural pseudo-Riemannian metric \( \tilde{g} \) on \( T^*M \) in the following way.

We define \( \tilde{\nabla} : T(T^*M) \to T(T^*M) \) by
\[
\tilde{\nabla} := (\Phi^\nabla) \circ \tilde{J} \circ (\Phi^\nabla)^{-1}
\]
and the pseudo-Riemannian metric \( \tilde{g} \) on \( T^*M \) by
\[
\tilde{g} := ((\Phi^\nabla)^{-1})^*(\tilde{g}).
\]

**Proposition 6.3.** \((T^*M, \tilde{J}, \tilde{g})\) is a metallic pseudo-Riemannian manifold.

**Proof.** From the definition it follows that \( \tilde{J}^2 = p\tilde{J} + qI \) and \( \tilde{g}(\tilde{J}X, Y) = \tilde{g}(X, \tilde{J}Y) \), for any \( X, Y \in C^\infty(T(T^*M)) \). \( \square \)
In local coordinates, we have the following expressions for \( \tilde{J} \) and \( \tilde{g} \):

\[
\begin{cases}
\tilde{J} (X_H^i) = J^k_i X_H^k + g_{ik} \frac{\partial}{\partial y_k} \\
\tilde{J} \left( \frac{\partial}{\partial y_j} \right) = (-J^2 + pJ + qI)^j_k g^{ij} \frac{\partial}{\partial y_k} - J^j_k \frac{\partial}{\partial y_k} + p \frac{\partial}{\partial y_j}
\end{cases}
\]

and

\[
\begin{cases}
\tilde{g} (X_H^i, X_H^j) = g_{ij} \\
\tilde{g} \left( \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) = \frac{p}{4} \delta_{ij} - \frac{1}{2} J^j_i \\
\tilde{g} \left( \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) = g^{ij}.
\end{cases}
\]

Computing the Nijenhuis tensor of \( \tilde{J} \), in the case when \( J \) is metallic with \( J^2 = pJ + qI \) and \( \nabla \) is the Levi-Civita connection of \( g \), we get:

\[
N_{\tilde{J}} \left( \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) = 0
\]

\[
N_{\tilde{J}} \left( X_H^i, \frac{\partial}{\partial y_j} \right) = ((\nabla_{JX_i} J) X_k - J (\nabla_{X_i} J) X_k)^j \frac{\partial}{\partial y_k}
\]

\[
N_{\tilde{J}} (X_H^i, X_H^j) = (N_J (X_i, X_j))^k X_H^k +
\]

\[
y_j \left( J^k_i J^k_j R_{kh}^l - J^k_j X^k R_{ijr}^l - J^r_j J^k_j R_{ikr}^l + pJ^k_i R_{ijk}^l + qR_{ij}^l \right) \frac{\partial}{\partial y_s}.
\]

Therefore we can state the following:

**Proposition 6.4.** Let \((M, J, g)\) be a flat locally metallic pseudo-Riemannian manifold. If \( \nabla \) is the Levi-Civita connection of \( g \), then \((\tilde{J}, \tilde{g})\) is an integrable metallic pseudo-Riemannian structure on \( T^* M \).

**Remark 6.5.** The metallic structures \( \bar{J} \) and \( \tilde{J} \) on the tangent and cotangent bundles respectively, satisfy:

\[
\bar{J} \circ (\Psi \nabla \circ (\Phi \nabla)^{-1}) = (\Psi \nabla \circ (\Phi \nabla)^{-1}) \circ \tilde{J}.
\]

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