Conservative Cascade of Kinetic Energy in Compressible Turbulence

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ABSTRACT

The physical nature of compressible turbulence is of fundamental importance in a variety of astrophysical settings. We investigate the question: “At what scales does the mechanism of pressure-dilatation operate?” and present the first direct evidence that mean kinetic energy cascades conservatively beyond a transitional “conversion” scale range despite not being an invariant of the dynamics. We use high-resolution 1024\textsuperscript{3} subsonic and transonic simulations. The key quantity we measure is the pressure-dilatation cospectrum, $E^{PD}(k)$, where we show that it decays at a rate faster than $k^{-4}$ in wavenumber at least the subsonic and transonic regimes. This is sufficient to imply that mean pressure-dilatation acts primarily at large scales and that kinetic and internal energy budgets statistically decouple beyond a transitional scale range. However, we observe that small-scale dynamics remains highly compressible locally in space and that the statistical decoupling in the energy budgets is unrelated to the existence of a subsonic scale range. Our results suggest that an extension of Kolmogorov’s inertial-range theory to compressible turbulence is possible.

Key words: hydrodynamics – methods: numerical – turbulence

Online-only material: color figures

1. INTRODUCTION

Turbulence plays a critical role in essentially all astrophysical systems that involve gas dynamics. For example, interstellar turbulence is thought to be driven on large scales by differential galactic rotation or supernovae explosions and dissipated at the smallest scales by microphysical processes. It is widely believed that energy is transferred from the largest scales down to the dissipation scales through a cascade process. Measurements of interstellar scintillation (Armstrong et al. 1995) in the warm interstellar medium (ISM) show a power-law spectrum similar to Kolmogorov’s $-5/3$ scaling over five decades in scale. Yet turbulent motions in the ISM are subsonic/transonic in the hot/warm medium and reach up to Mach numbers of 20 in cold molecular clouds, such that the flow is often compressible (Elmegreen & Scalo 2004). Similarly, high-resolution subsonic and transonic simulations of the intracluster medium between galaxies (Xu et al. 2009, 2010) reveal Kolmogorov-like spectra. High-resolution simulations of the ISM, with a Mach number of 6 (Kritsuk et al. 2007), reveal velocity spectra with a $-2$ slope and structure functions with power-law scaling. Such observations suggest that a cascade process is taking place in these systems. Yet it is not clear how Kolmogorov’s 1941 phenomenology, which forms the cornerstone for our understanding of incompressible turbulence, can carry over to such flows which exhibit significant compressibility effects. Several recent studies have addressed the problem of compressible turbulence numerically (e.g., Porter et al. 1998; Kritsuk et al. 2007; Schmidt et al. 2008) and theoretically (e.g., Erlebacher et al. 1990; Falkovich et al. 2010), but a physical description equivalent to that of Kolmogorov’s still eludes us.

The idea of a cascade itself is without physical basis since kinetic energy (KE) is not a global invariant of the inviscid dynamics. Recently, Aluie (2011a, 2011c) showed how the assumption that pressure-dilatation cospectrum decays fast enough rigorously implies that mean KE cascades conservatively despite not being an invariant. The assumption entails that the mechanism of pressure-dilatation acts primarily at the largest scales and vanishes on average at smaller scales. Below, we provide evidence in support of this assumption.

The outline of this Letter is as follows. Section 2 describes our numerical simulations and Section 3 presents our coarse-graining method for analyzing scale interactions. Section 4 discusses the significance of the pressure-dilatation cospectrum and the assumption we are testing. Our main results are in Section 5 followed by discussions in Sections 6–8. The Letter concludes with Section 9.

2. NUMERICAL SIMULATIONS

Table 1 summarizes the simulations we conducted by solving
\begin{equation}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}
\end{equation}

\begin{equation}
\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla P + \rho \mathbf{F}, \tag{2}
\end{equation}

\begin{equation}
\partial_t (\rho e) + \nabla \cdot (\rho e \mathbf{u}) = -P \nabla \cdot \mathbf{u}, \tag{3}
\end{equation}

supplemented with an equation of state (EOS). Here, $\mathbf{u}$ is the velocity, $\rho$ is the density, $P$ is the pressure, $e = P/[(\gamma - 1)\rho]$ is the internal energy per unit mass for heat capacity ratio $\gamma$, and $\mathbf{F}$ is an external acceleration field stirring the fluid. Runs I and III are of isothermal forced turbulence. Runs II and IV are of decaying turbulence in an ideal gas (Table 1).

The domain is a periodic box $[0, 2\pi)^3$. Runs I and III start with a uniform density field $\rho(x, t = 0) = 1$. The forcing is adapted from Cho & Ryu (2009), $\mathbf{F}(x, t) = \sum_{j=-1}^{22} \hat{\mathbf{F}}(k_j) \exp(i k_j \cdot x) + c.c.$ Fourier amplitudes, $\hat{\mathbf{F}}(k_j)$, have a solenoidal component perpendicular to $k_j$ and, in Run I, also have a compressive component parallel to $k_j$. On average, the
components have equal magnitude with a flat spectrum. The 22 forced wavevectors, \( k_j \), are nearly isotropically distributed over \( 2 \leq |k_j| \leq \sqrt{12} \). Mach number, \( M_t = u_{rms}/c_{th} \), for a sound speed \( c_{th} \), is subsonic in Run I and transonic in Run III. Initial conditions for decaying Runs II and IV are taken from the last output of Runs I and III, respectively. Data from these runs comprise about 30 evenly spaced time snapshots starting just after the initial time until mean KE reaches approximately 10% of its initial value.

We use a central finite-volume scheme on overlapping cells (Liu et al. 2007) to solve Equations (1) and (2) in conservative form. Instead of solving Equation (3), we solve the total energy equation. Details of the algorithm are documented in Li (2008, 2010).

Figure 1 plots spectra of solenoidal and compressive components of the velocity from Runs III and IV, both of which exhibit putative power-law scaling. Here, \( \mathbf{u}(x) = \mathbf{u}^*(x) + \mathbf{u}'(x) \), where \( \mathbf{u}' \) is solenoidal and \( \mathbf{u}^* \) is irrotational.

### 3. ANALYZING NONLINEAR SCALE INTERACTIONS

The key method we use is a “coarse-graining” approach inspired by Large Eddy Simulation modeling of turbulence (Leonard 1974; Germano 1992; Meneveau & Katz 2000). The approach was developed by Eyink (1995, 2005) to analyze scale interactions in flow fields. It was further refined and utilized by Eyink & Aluie (2009) and Aluie & Eyink (2009) and extended to magnetohydrodynamic (Aluie & Eyink 2010), geophysical (Aluie & Kurien 2011), and compressible (Aluie 2011a, 2011b, 2011c) flows. The method is simple: for any field \( \mathbf{a}(x) \), a “coarse-grained” field which contains modes at length scales \( \gg \ell \) is

\[
\overline{\mathbf{a}}(x) = \int d^3 r G(\ell)(x) \mathbf{a}(x + \mathbf{r}),
\]

where \( G(\mathbf{r}) \) is a normalized window function, such as a Gaussian, \( G(\mathbf{r}) = (1/\sqrt{2\pi})e^{-r^2/2} \). \( G(\ell) = \ell^{-3}G(\ell/\ell) \) is a rescaled window with its main “mass” in a ball of diameter \( \ell \). Operation (4) may be interpreted as a local space average.

From the equation of \( \mathbf{a}(x) \), coarse-grained equations can be written of the evolution of \( \overline{\mathbf{a}}(x) \) at every \( x \) in space and at any instant of time. Furthermore, the coarse-grained equations describe flow at scales \( \gg \ell \), for arbitrary \( \ell \), therefore allowing the simultaneous resolution of dynamics both in scale and in space.

#### 3.1. Analyzing Compressible Flows

Aluie (2011a, 2011b) showed how a density-weighted decomposition, first introduced by Favre (1969), can be employed to extend the coarse-graining approach to compressible turbulence. A Favre filtered field is weighted by density,

\[
\bar{f}_\ell(x) \equiv \frac{\rho \bar{f}_\ell(x)}{\rho(x)}. \tag{5}
\]

Hereafter, we take the liberty of dropping the subscript \( \ell \) whenever there is no risk of ambiguity. From large-scale continuity and momentum equations (see Aluie 2011a, 2011b), it is straightforward to derive a KE budget for the large scales:

\[
\frac{\partial}{\partial t} \frac{\rho(x)}{2} + \nabla \cdot \mathbf{J} \ell = -\Pi_\ell - \Lambda_\ell + \bar{P} \nabla \cdot \bar{\mathbf{u}} + \epsilon^{inj}. \tag{6}
\]

Here, \( \mathbf{J}(x) \) is the spatial transport flux of large-scale KE and \( \epsilon^{inj}(x) \) is the energy injected due to forcing. Definitions of these terms can be found in Aluie (2011a, 2011b), \( -\bar{P} \nabla \cdot \bar{\mathbf{u}} \) is the large-scale pressure-dilatation, where the dilatation is \( \nabla \cdot \mathbf{u} \), and \( \Pi_\ell(x) + \Lambda_\ell(x) \) is the subgrid scale (SGS) KE flux to scales \( < \ell \),

\[
\Pi_\ell(x) = -\bar{P} \nabla \cdot \bar{\mathbf{u}} \tau(\mathbf{u}, \mathbf{u}), \tag{7}
\]

\[
\Lambda_\ell(x) = \frac{1}{\rho} \nabla \bar{\mathbf{P}} \cdot \tau(\mathbf{u}, \mathbf{u}). \tag{8}
\]

Here, \( \tau(\mathbf{u}, \mathbf{u}) \equiv (\mathbf{u} \cdot \mathbf{u}) - \bar{\mathbf{u}} \bar{\mathbf{u}} \) in expression (8) is subgrid mass flux and \( \bar{\mathbf{P}} \) in expression (7) is subgrid stress from the eliminated scales \( < \ell \). Equation (6) describes the dynamics at scales \( \gg \ell \), for arbitrary \( \ell \), at every \( x \) and at every time.

SGS flux comprises deformation work, \( \Pi_\ell \), and baropycnal work, \( \Lambda_\ell \), which are discussed in detail in Aluie (2011b). These
represent the only two processes capable of direct transfer of KE across scales. Pressure-dilatation, \(-P\nabla \cdot \mathbf{u}\), does not contain modes at scales \(<\ell\) (or a moderate multiple thereof). Therefore, pressure-dilatation cannot participate in the inter-scale transfer of KE and only contributes to the conversion of large-scale KE into internal energy. This was a crucial observation made in Aluie (2011a) on which the following analysis is based.

4. PRESSURE-DILATATION COSPECMTR

Kinetic and internal energy budgets couple through two mechanisms. One is viscous dissipation which was proved in Aluie (2011b) to be confined to the smallest scales \(\ell \leq \ell_{\mu}\) (the dissipation scale range). Therefore, large-scale KE in Equation (6) does not couple to internal energy via viscous dynamics for \(\ell > \ell_{\mu}\). The second mechanism is pressure-dilatation, \(-P\nabla \cdot \mathbf{u}\), which exchanges kinetic and internal energy via compression and rarefaction.

It was shown in Aluie (2011a) that if the pressure-dilatation cospectrum,

\[
E_{PD}(k) \equiv \sum_{k-0.5<|k|<k+0.5} \hat{P}(k)\hat{\nabla} \cdot \mathbf{u}(-k), \tag{9}
\]

decays fast enough in wavenumber (where \(\hat{f}(k)\) is the Fourier transform of \(f(x)\)):

\[
|E_{PD}(k)| \leq C u_{rms} P_{rms} (kL)^{-\beta}, \quad \beta > 1, \tag{10}
\]

then mean pressure-dilatation exchanges mean kinetic and internal energy over a transitional “conversion” scale range of limited extent. Here, \(C\) is a dimensionless constant and \(L\) is an integral scale. At smaller scales beyond the conversion range, mean kinetic and internal energy budgets statistically decouple, giving rise to an inertial range over which mean KE undergoes a conservative cascade. Figure 2, which we discuss below, provides the first empirical evidence in support of assumption (10).

The idea behind assumption (10) is straightforward and rests on the convergence of a series or an integral at infinity. In the limit of a large Reynolds number, assumption (10) implies that mean large-scale pressure-dilatation, \(PD(\ell) \equiv -\langle P\nabla \cdot \mathbf{u} \rangle_{\ell}\), asymptotes to a finite constant, \(\theta \equiv -\langle P\nabla \cdot \mathbf{u} \rangle\), as \(\ell \to 0\). Here, \(\langle \ldots \rangle\) is a volume average. In other words, \(PD(\ell)\) acting at scales \(>\ell\) converges and becomes independent of \(\ell\) at small enough scales:

\[
\lim_{\ell \to 0} PD(\ell) = \lim_{K \to \infty} \sum_{0<k<K} E_{PD}(k) = \theta. \tag{11}
\]

Note that \(PD(\ell)\) in Equation (11) (also shown in Figure 3) is a cumulative quantity representing the contribution from all wavenumbers \(k \leq K = \ell^{-1}\), whereas spectra \(E_{PD}(k)\) and \(E_{\nu}(k)\) in Figures 1 and 2 are density functions. Convergence of \(PD(\ell)\) in Equation (11) expresses the mean decoupling between large-scale kinetic and internal balances. Such decoupling is statistical and does not imply that small scales evolve according to incompressible dynamics. However, while small-scale compression and rarefaction can still take place pointwise, they yield a vanishing contribution to the space average.

We denote the smallest wavenumber at which such statistical decoupling occurs by \(K_{e}\). Over the ensuing wavenumber range, \(K_{e} < K < K_{\mu}\) (where \(K_{\mu} = \ell^{-1}_{\mu}\)), net pressure-dilatation does not play a role. For \(\ell = K^{-1}\) over this wavenumber range, budget (6) of large-scale KE becomes

\[
\langle \Pi_{t} + \Lambda_{t} \rangle = \langle \epsilon_{\text{inj}} \rangle - \theta \tag{12}
\]

in a steady state and after space averaging. If \(\langle \epsilon_{\text{inj}} \rangle\) is localized to the largest scales as discussed in Aluie (2011b), then \(\langle \Pi_{t} + \Lambda_{t} \rangle\) will be a constant, independent of \(\ell\).

A constant SGS flux implies that mean KE cascades conservatively to smaller scales, despite not being an invariant of the governing dynamics. In particular, KE can only reach dissipative scales via the SGS flux, \(\Pi_{t} + \Lambda_{t}\), through a conservative scale-local cascade process. We are therefore justified in calling wavenumber range \(K_{e} < K < K_{\mu}\) the inertial range of compressible turbulence. The existence of this inertial range in such flows warrants expectations that spectra with power-law scalings should exist and that observation of them is evidence of a turbulent cascade process similar (although not necessarily identical) to that in incompressible flows.

5. RESULTS

We now present evidence from Runs I–IV in support of assumption (10). Figure 2 shows \(|E_{PD}(k)|\) decaying at a rate significantly faster than \(k^{-1}\) in all four cases, well in agreement with condition (10). We note that pressure-dilatation cospectrum from Run III at \(M_{t} = 1.25\) is shallower than that from Run I at \(M_{t} = 0.44\). We also note significant differences at the highest wavenumbers where the kink from Run III is due to a change in sign in \(E_{PD}(k)\) as a function of \(k\). Checking whether the shallowing of cospectra at increasing
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$M_t$ persists requires further simulations at higher $M_t$ and also at higher Reynolds numbers, which should indicate if the observed scaling is asymptotic in wavenumber or if it is due to contamination from small-scale numerical artifacts. In Section 7, we will argue that it is unlikely for the pressure-dilatation cospectrum to have asymptotic scaling $\sim k^{-1}$ or shallower for it would imply that the mean rate of kinetic-energy conversion becomes unbounded with higher resolution (or Reynolds number). Cospectra from unforced Runs II and IV exhibit similar scaling, albeit with discernible differences at the highest wavenumbers. Inset of Figure 2 shows that $M_t$ of Run IV is larger than $M_t = 0.4$ of forced Run I over a substantial part of the time interval analyzed, while $M_t$ of Run II is less than 0.4. Yet, $\langle E^{pd}(k) \rangle$ from Run I is steeper than cospectra from both decaying Runs II and IV suggesting that differences in the EOS are playing a role more important than $M_t$. We caution, however, that both Runs II and IV are subsonic and such trends might not hold at higher $M_t$.

To further illustrate the main idea of this Letter, we analyzed SGS flux terms, $(\Pi_k + \Lambda_k)$, in budget (6). Figure 3 shows that $PD(\ell)$ tends to a constant beyond a transitional conversion range. Since $PD(\ell)$ is a cumulative quantity representing contributions from all wavenumbers $k < K = \ell^{-1}$, its convergence to a constant beyond $K \approx 10$ implies that smaller scales with $k > 10$ give a negligible contribution to pressure-dilatation. The SGS flux also becomes approximately constant beyond this conversion range, implying that KE is cascading conservatively (without “leakage” to/from internal energy) until the dissipation range is reached.

6. THE ROLE OF SHOCKS

The physical picture we are advancing might seem counterintuitive at first. After all, a hallmark of compressible turbulence is shock formation and soundwave generation. Such phenomena involve compression and rarefaction at all scales and are not restricted to small wavenumbers.

The existence of such phenomena poses no contradiction since our results concern global pressure-dilatation, $-(P^{\nabla \cdot u})$, and not the pointwise quantity. While we expect very large pressure-dilatation values in the vicinity of small-scale shocks, our results indicate that such a contribution will vanish after space averaging due to cancellations between compression and rarefaction regions. For example, linear (small-amplitude) soundwaves yield zero mean pressure-dilatation, $-(P^{\nabla \cdot u}) = 0$. Figure 4 shows that values of pointwise pressure-dilatation at small scales, $-(P^{\nabla \cdot u}(x) - P^{\nabla \cdot u}(x))$, are an order of magnitude more intense relative to large-scale pressure-dilatation, $-(P^{\nabla \cdot u}(x))$, as expected. The distribution of $-(P^{\nabla \cdot u}(x) - P^{\nabla \cdot u}(x))$ has heavy tails implying spatially rare but intense two-way exchange between kinetic and internal energy. Furthermore, the small-scale distribution has positive skewness implying that conversion from kinetic into internal energy through compression occupies less volume and is more intense relative to conversion from internal into KE through rarefaction. Such observations indicate that pointwise small-scale dynamics are far from being incompressible, an issue we discuss more in Section 8.

Despite the larger intensity of conversion at small scales, its global contribution, $-(P^{\nabla \cdot u} - P^{\nabla \cdot u}) = 2.0 \times 10^{-6}$, is negligible compared to mean large-scale conversion, $PD(\ell) = 3.7 \times 10^{-5}$, consistent with the plateau of $PD(\ell)$ in Figure 3. The origin of such cancellations at small scales may be understood through the following plausible argument (Aluie 2011a). While pressure in $-(P^{\nabla \cdot u})$ derives most of its contribution from the largest scales, $P^{\nabla \cdot u}$ is dominated by the smallest scales in the flow. Therefore, pressure varies slowly in

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4 Unlike $PD(\ell)$ which only involves wavenumbers $k < K = \ell^{-1}$, the SGS flux $(\Pi_k + \Lambda_k)$ is a hybrid quantity involving both wavenumbers $\ell < K = \ell^{-1}$ (large-scale terms) and $k > K = \ell^{-1}$ (subgrid terms) as definitions (7) and (8) show.

5 Pressure should converge and not diverge, $(P^{\nabla \cdot u}) \to (\text{const.}) < \infty$, as we resolve successively smaller scales. This would imply that small scales yield a negligible contribution.
Figure 4. For $\ell^{-1} = K = 20$ in Figure 3 from Run II, top two panels visualize pointwise pressure-dilatation at large scales (top left), $P_{\ell} \nabla \cdot u(x)$, and the residual from small scales (top right), $-P_{\ell} \nabla \cdot u(x) + P_{\ell} \nabla \cdot u_{\infty}(x)$, in a 1024$^2$ cross-section. Bottom two panels show the distributions of large-scale (bottom left) and small-scale (bottom right) pressure-dilatation. Dashed curves are for reference and show unnormalized Gaussian distributions with zero mean.

(A color version of this figure is available in the online journal.)

space, primarily at scales $\sim L$, while $\nabla \cdot u$ varies much more rapidly, primarily at scales $\ell_\mu \ll L$, leading to decorrelation between the two factors.

7. SURPRISING YET EXPECTED

While our results, such as plots of PD($K$) in Figure 3, might seem unexpected at some level, they are also inevitable from a physical standpoint. There are only a few possibilities for the asymptotic behavior of the series, PD($K$) = $\sum_{k<K} E^{PD}(k)$, at large $K$.

One is that the series diverges with $K$ to $\pm \infty$. This would imply that for a sequence of simulations at increasingly higher resolution (smaller viscosity) in which all other flow parameters are held fixed, the mean rate of kinetic-internal energy conversion, $\theta = \sum_{\ell \in \ell} E^{PD}(k)$, increases without bound. This would occur despite having asymptotically constant (finite) values of kinetic and internal energy. Therefore, we maintain that such a possibility is unphysical.

Another possibility is that PD($K$) oscillates in $K$ without flattening to a constant at high wavenumbers. While we do not know of a physical argument precluding this scenario, we also cannot imagine what could cause such persistent oscillations at arbitrarily large $K$ (linear sound waves do not contribute to PD($K$)). Furthermore, there is no indication of such oscillations in Figure 3.

The only remaining possibility is that the series, PD($K$), converges and asymptotes to a constant at large $K$ (possibly zero, which is a constant). This implies that there is a range of small scales, beyond the conversion range, over which no net exchange between kinetic and internal energy occurs.

8. SONIC SCALE AND THE CONVERSION RANGE

In compressible turbulence, there is a prevalent notion of a sonic scale, $\ell_s = K_s^{-1}$, at which $\delta u(\ell_s) \sim c_{\text{th}}$. The belief is that at smaller scales, $\ell < \ell_s$, turbulence becomes subsonic and the dynamics approaches that of an incompressible flow. Underlying this expectation is a questionable assumption that Mach number is the sole parameter characterizing compressibility. We remark that significant fluctuations in density and large dilatation, $\nabla \cdot u$, can occur in low speed flows such as mixing of fluids with different densities or in turbulent combustion. It is also known that compressibility effects become progressively more important with increasing shear despite keeping $M_s$ fixed (Sarkar 1995). To investigate whether dynamics at subsonic scales is approximately incompressible, the following question needs to be addressed. In a supersonic flow, will subsonic scales, $\ell < \ell_s$, evolve identically to those in a subsonic simulation at lower resolution? In other words, is it possible to simulate a supersonic flow using the compressible equations at large scales coupled with incompressible Euler at subsonic scales?

It might be argued that the transition we observe in Figure 3 from the conversion range to the inertial range occurs at the sonic scale, $K_s$. However, we note that the transition occurs in all four cases we study, including subsonic Runs I and II in which $\ell_s$ is larger than the domain size. Furthermore, we observe that pressure-dilatation, a physical process inherently
due to compressibility, is most intense at small scales as shown in Figure 4. This indicates that the local small-scale dynamics are far from incompressible, i.e., the equation which describes and governs the evolution of the small-scale flow is that of the fully compressible Euler.

9. CONCLUSIONS AND DISCUSSION

In summary, we investigated the question “At what scales does the mechanism of pressure-dilatation operate?” The importance of this question to the validity of the concept of a conservative KE cascade was not appreciated before. We provided the first empirical evidence from subsonic and transonic simulations that pressure-dilatation takes place at large scales, on average. At smaller scales beyond a transitional “conversion” scale range, mean kinetic and internal energy budgets statistically decouple and KE can only reach dissipation scales via a conservative cascade.

The existence of such a conservative cascade in compressible turbulence justifies expectations that spectra with power-law scalings should exist in such flows as recently proposed, for example, by Kritsuk et al. (2007). Furthermore, it suggests that observations of power-law spectra in astrophysical systems are indeed evidence of a turbulent cascade process similar (although not necessarily identical) to that in incompressible flows. However, we make no conclusions regarding the power-law exponents of spectra in this study.

This work leads us to some new questions such as: Is the scaling of cospectra we observe in Figure 2 reflecting asymptotic scaling in the limit of high resolution (i.e., free of contamination from numerical artifacts)? How will the asymptotic scaling change at even higher Mach numbers? We invite future studies to investigate these questions and to try to reproduce our results.

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