Exploring Timelike Region of QCD
Exclusive Processes in Relativistic Quark Model

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Abstract. We investigate the form factors and decay rates of exclusive $0^- \rightarrow 0^-$ semileptonic meson decays using the constituent quark model based on the light-front quantization. Our model is constrained by the variational principle for the linear plus Coulomb interaction motivated by QCD. Our numerical results are in a good agreement with the available experimental data.

One of the distinctive advantages in the light-front approach is the well-established formulation of various form factor calculations using the well-known Drell-Yan-West ($q^+=0$) frame [1]. In $q^+=0$ frame, only parton-number-conserving Fock state (valence) contribution is needed when the “good” components of the current, $J^+$ and $J_\perp=(J_x,J_y)$, are used [2]. For example, only the valence diagram shown in Fig. 1(a) is used in the light-front quark model (LFQM) analysis of spacelike meson form factors. Successful LFQM description of various hadron form factors can be found in the literatures [3–5].

However, the timelike ($q^2 > 0$) form factor analysis in the LFQM has been hindered by the fact that $q^+=0$ frame is defined only in the spacelike region ($q^2=q^+q^--q_\perp^2<0$). While the $q^+\neq0$ frame can be used in principle to compute the timelike form factors, it is inevitable (if $q^+\neq0$) to encounter the nonvalence diagram arising from the quark-antiquark pair creation (so called “$Z$-graph”). For example, the nonvalence diagram in the case of semileptonic meson decays is shown in Fig. 1(b). The main source of the difficulty, however, in calculating the nonvalence diagram(see Fig. 1(b)) is the lack of information on the black blob which should contrast with the white blob representing the usual light-front valence wave function. In fact, we noticed [2] that the omission of nonvalence contribution leads to a large deviation from the full results.

In this paper, we circumvent this problem by calculating the semileptonic processes in $q^+=0$ frame and then analytically continuing to the timelike region. The $q^+=0$ frame is useful because only valence contributions are needed. However, one needs to calculate the component of the current other than $J^+$ to obtain the form
FIGURE 1. The LFQM description of a electroweak meson form factor: (a) the usual light-front valence diagram and (b) the nonvalence(pair-creation) diagram. The vertical dashed line in (b) indicates the energy-denominator for the nonvalence contributions. While the white blob represents the usual light-front valence wave function, the modeling of black blob has not yet been made.

factor $f_-(q^2)$. Since $J^-$ is not free from the zero-mode contributions even in $q^+ = 0$ frame [6,7], we use $J_\perp$ instead of $J^-$ to obtain $f_-$. The key idea in our LFQM [5] for mesons is to treat the radial wave function as a trial function for the variational principle to the QCD-motivated Hamiltonian saturating the Fock state expansion by the constituent quark and antiquark. The spin-orbit wave function is uniquely determined by the Melosh transformation. We take the QCD-motivated effective Hamiltonian as the well-known linear plus Coulomb interaction given by

$$H_{q\bar{q}} = H_0 + V_{q\bar{q}} = \sqrt{m_q^2 + k^2} + \sqrt{m_{\bar{q}}^2 + k^2} + V_{q\bar{q}},$$

where

$$V_{q\bar{q}} = V_0 + V_{hyp} = a + br - \frac{4\kappa}{3r} + \frac{2\vec{S}_q \cdot \vec{S}_{\bar{q}}}{3m_qm_{\bar{q}}}\nabla^2 V_{\text{Coul}}.$$  

(2)

We take the Gaussian radial wave function $\phi(k^2) = N \exp(-k^2/2\beta^2)$ as our trial wave function to minimize the central Hamiltonian [5]. Since the string tension $\sigma=0.18$ GeV$^2$ and the constituent $u$ and $d$ quark masses $m_u = m_d = 0.22$ GeV are rather well known from other quark model analyses commensurate with Regge phenomenology [8], we take them as our input parameters. The model parameters of $a$, $\kappa$, and $\beta_{u\bar{d}}$ are determined by the variational principle using the masses of $\rho$ and $\pi$ [5,9]. It is very important to note that all other model parameters such as $m_c$, $m_b$, $\beta_{uc}$, $\beta_{ub}$, etc. are then uniquely determined by our variational principle as shown in [9]. More detailed procedure of determining the model parameters and ground state meson mass spectra can be found in [5,9].

Our predictions of the ground state meson mass spectra [9] are in a good agreement with the available experimental data. Furthermore, our model predicts the two unmeasured mass spectra of $^1S_0(b\bar{b})$ and $^3S_1(b\bar{s})$ systems as $M_{b\bar{b}}=9657$ MeV and $M_{b\bar{s}}=5424$ MeV, respectively. Our values of the decay constants [9] are also in
a good agreement with the results of lattice QCD [10] anticipating future accurate experimental data.

The matrix element of the current $j^\mu = \bar{q}_2 \gamma^\mu Q_1$ for $0^- (Q_1 \bar{q}) \to 0^- (q_2 \bar{q})$ decays can be parametrized in terms of two hadronic form factors as follows

$$\langle P_1 | q_2 \gamma^\mu Q_1 | P_1 \rangle = f_+(q^2) (P_1 + P_2)^\mu + f_-(q^2) (P_1 - P_2)^\mu,$$

$$= f_+(q^2) \left[ (P_1 + P_2)^\mu - \frac{M_1^2 - M_2^2}{q^2} q^\mu \right] + f_0(q^2) \frac{M_1^2 - M_2^2}{q^2} q^\mu,$$  \hspace{1cm} (3)

where $q^\mu = (P_1 - P_2)^\mu$ is the four-momentum transfer to the leptons and $m_1^2 \leq q^2 \leq (M_1 - M_2)^2$. The form factors $f_+$ and $f_0$ are related to the exchange of $1^-$ and $0^+$, respectively, and satisfy the following relations:

$$f_+(0) = f_0(0), \quad f_0(q^2) = f_+(q^2) + \frac{q^2}{M_1^2 - M_2^2} f_-(q^2).$$ \hspace{1cm} (4)

In the LFQM calculations presented in Ref. [11], the $q^+ \neq 0$ frame has been used to calculate the semileptonic decays in the timelike region. However, when the $q^+ \neq 0$ frame is used, the inclusion of the nonvalence contributions arising from quark-antiquark pair creation (see Fig. 1(b)) is inevitable and this inclusion may be very important for light-to-light and heavy-to-light decays. Nevertheless, the previous analyses [11] considered only valence contributions in $q^+ \neq 0$ frame neglecting nonvalence contributions. In this work, we circumvent this problem by calculating the processes in $q^+=0$ frame and analytically continuing to the timelike region. The $q^+=0$ frame is useful because only valence contributions are needed. However, one needs to calculate the component of the current other than $J^+$ to obtain the form factor $f_-(q^2)$. Since $J^-$ is not free from the zero-mode contributions even in $q^+ = 0$ frame [6], we use $J_\perp$ instead of $J_-$ to obtain $f_-$. In the $q^+=0$ frame, we obtain the form factors $f_+(q^2)$ and $f_-(q^2)$ using the matrix element of the “$+$” and “$\perp$”-components of the current, $J^\mu$, respectively, and then analytically continue to the timelike $q^2 > 0$ region by changing $q_\perp$ to $iq_\perp$ in the form factors.

**Light-to-light decays:** For $K_{l3}$ decays, the three form factor parameters, i.e., $\lambda_+, \lambda_0$ and $\xi_A$, have been measured using the following linear parametrization [12]:

$$f_\pm(q^2) = f_\pm(q^2 = m_1^2) \left( 1 + \lambda_\pm \frac{q^2}{M_{\pi^0}^2} \right),$$ \hspace{1cm} (5)

where $\lambda_{\pm,0}$ is the slope of $f_{\pm,0}$ evaluated at $q^2 = m_1^2$ and $\xi_A = f_-/f_+|_{q^2=m_1^2}$. Our predictions of the parameters for $K_{l3}$ decays in $q^+ = 0$ frame, i.e., $f_+(0)$, $\lambda_+$, $\lambda_0$, $\langle r^2 \rangle_{\pi^\pm} = 6 f_+(0)/f_+(0) = 6\lambda_+ / M_{\pi^+}^2$, and $\xi_A = f_-/f_+|_{q^2=m_1^2}$, are summarized in Table 1. Our result for the form factor $f_+$ at zero momentum transfer, $f_+(0) = 0.962$, is consistent with the Ademollo-Gatto theorem [13] and also in a good agreement with the result of chiral perturbation theory [14], $f_+(0) = 0.961 \pm 0.008$. Our results for other observables such as $\lambda_+$, $\xi_A$, and $\Gamma(K_{l3})$ are overall in a good agreement...
TABLE 1. Model predictions for the parameters of \( K_{13} \) decay form factors obtained from \( q^+ = 0 \) frame. The charge radius \( r_{\pi K} \) is obtained by \( \langle r^2 \rangle_{\pi K} = 6f'_+(q^2 = 0)/f_+(0) \). For comparison, we include the results (in square brackets) of the valence contribution in \( q^+ \neq 0 \) frame. The CKM matrix used in the calculation of the decay width (in units of \( 10^6 \text{ s}^{-1} \)) is \( |V_{us}| = 0.2205 \pm 0.0018 \) [12].

| Observables          | Our model | Other models | Experiment |
|----------------------|-----------|--------------|------------|
| \( f_+(0) \)         | 0.962[0.962] | 0.961 \pm 0.008 [14], 0.952 [15], 0.93 [16] |            |
| \( \lambda_+ \)      | 0.026[0.083] | 0.028 [15], 0.019 [16] | 0.0286 \pm 0.0022[\( K^+_{\pi} \)] |
| \( \lambda_0 \)      | -0.009[-0.017] | 0.0026 [15], -0.005 [16] | 0.004 \pm 0.007[\( K^+_{\pi} \)] |
| \( \xi_A \)          | -0.41[-1.10] | -0.28 [15], -0.28 [15] | -0.35 \pm 0.15[\( K^+_{\pi} \)] |
| \( \langle r^2 \rangle_{\pi K} \text{(fm)} \) | 0.56 | 0.57 [15], 0.48 [16] |            |
| \( \Gamma(K^0_{\pi}) \) | 7.36 \pm 0.12 |            | 7.7 \pm 0.5[\( K^0_{\pi} \)] |

with the experimental data [12]. For comparison, we also include the results (in square brackets of the second column of Table 1) of \( f_+(0) \), \( \lambda_+ \), \( \lambda_0 \), and \( \xi_A \) obtained from the valence contribution in \( q^+ \neq 0 \) frame. Even though the form factor \( f_+(0) \) in \( q^+ \neq 0 \) frame is free from the nonvalence contributions, its derivative at \( q^2 = 0 \), i.e., \( \lambda_+ \), receives the nonvalence contributions. Moreover, the form factor \( f_+(q^2) \) in \( q^+ \neq 0 \) frame is not immune to the nonvalence contributions even at \( q^2 = 0 \) [6]. Unless one includes the nonvalence contributions in the \( q^+ \neq 0 \) frame, one cannot really obtain reliable predictions for the observables such as \( \lambda_+ \), \( \lambda_0 \) and \( \xi_A \) for \( K_{13} \) decays.

In Fig. 2, we show the form factors \( f_+ \) obtained from both \( q^+ = 0 \) and \( q^+ \neq 0 \) frames for \( 0 \leq q^2 \leq (M_K - M_\pi)^2 \) region. As one can see in Fig. 2, the form factor \( f_+(q^2) \) obtained from \( q^+ = 0 \) frame is not linear functions of \( q^2 \) justifying Eq. (5) usually employed in the analysis of experimental data [12]. Note, however, that the \( f_+(q^2) \) obtained from only valence contribution in \( q^+ \neq 0 \) frame (dotted lines) does not exhibit the same behavior.

Heavy-to-light(heavy) decays: Our predicted decay rates for \( D \to K \) and \( D \to \pi \) are \( \Gamma(D^0 \to K^-e^+\nu_e) = 8.36|V_{cs}|^2 \times 10^{-2} \text{ ps}^{-1} \) and \( \Gamma(D^0 \to \pi^-e^+\nu_e) = 0.113|V_{cd}|^2 \text{ ps}^{-1} \), respectively. Using \( |V_{cs}| = 1.04 \pm 0.16 \) and \( |V_{cd}| = 0.224 \pm 0.016 \) [12], we obtain the branching ratio of \( Br(D \to K) = (3.75 \pm 1.16)\% \) and \( Br(D \to \pi) = (2.36 \pm 0.34) \times 10^{-3} \), while the experimental data are \( (3.66 \pm 0.18)\% \) for \( D \to K \) and \( (3.9 \pm 2.3 \pm 0.4) \times 10^{-3} \). Also, our predicted decay rates for \( B \to \pi \) and \( B \to D \) are \( \Gamma(B^0 \to \pi^-\ell^+\nu_\ell) = 8.16|V_{ub}|^2 \text{ ps}^{-1} \), \( \Gamma(B^0 \to D^-\ell^+\nu_\ell) = 9.39|V_{cb}|^2 \), respectively. Using \( |V_{ub}| = (3.3 \pm 0.4 \pm 0.7) \times 10^{-3} \) and \( |V_{cb}| = 0.0395 \pm 0.003 \) [12], we obtain \( Br(B \to \pi) = (1.40 \pm 0.34) \times 10^{-4} \) and \( Br(B \to D) = (2.28 \pm 0.20)\% \). Our results are quite comparable with the recent experimental data [12], \( (1.8 \pm 0.6) \times 10^{-4} \) for \( B \to \pi \) and \( (2.00 \pm 0.25)\% \) for \( B \to D \), within the given error range.

In the heavy quark limit \( M_{1(2)} \to \infty \), the form factor \( f_+(q^2) \) is reduced to the
universal Isgur-Wise (IW) function, \( \xi(v_1 \cdot v_2) = \left[ 2 \sqrt{M_1 M_2} / (M_1 + M_2) \right] f_+(q^2) \), where 
\( v_{1(2)} = P_{1(2)} / M_{1(2)} \). Our prediction of the slope \( \rho^2=0.8 \) of the IW function at the zero-recoil point defined as 
\( \xi(v_1 \cdot v_2) = 1 - \rho^2 (v_1 \cdot v_2 - 1) \) is quite comparable with the current world average \( \rho_{\text{avg}}=0.66\pm0.19 \) [12] extracted from exclusive semileptonic \( \bar{B}\rightarrow D \ell \bar{\nu} \) decay.

In conclusion, we analyzed the exclusive \( 0^+\rightarrow0^+ \) semileptonic meson decays using the LFQM constrained by the variational principle for the QCD-motivated effective Hamiltonian with the well-known linear plus Coulomb interaction [5,9]. The form factors \( f_\pm \) are obtained in \( q^+=0 \) frame and then analytically continued to the timelike region by changing \( q_\perp \) to \( iq_\perp \) in the form factors. The matrix element of the “\( \perp \)” component of the current \( J^\mu \) is used to obtain the form factor \( f_- \). Our model provided overall a good agreement with the available experimental data and the lattice QCD results for the transition form factors and branching ratios of the \( 0^-\rightarrow0^- \) semileptonic meson decays. Also it rendered a large number of predictions to the heavy meson mass spectra and decay constants [9]. We think that the success of our model hinges on the advantage of light-front quantization realized by the rational energy-momentum dispersion relation. It is crucial to calculate the “good” components of the current in the reference frame which deletes the complication from the nonvalence Z-graph contribution. The present work broadens the utility of the standard light-front frame à la Drell-Yan-West to the timelike form factor calculation. We anticipate further stringent tests of our model with more accurate data from future experiments and lattice QCD calculations.
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REFERENCES

1. S. D. Drell and T. M. Yan, Phys. Rev. Lett. 24, 181 (1970); G. West, Phys. Rev. Lett. 24, 1206 (1970); G. P. Lepage and S. J. Brodsky, Phys. Rev. D 22, 2157 (1980).
2. H.-M. Choi and C.-R. Ji, Phys. Rev. D 59, 034001 (1999).
3. P. L. Chung, F. Coester, and W. N. Polyzou, Phys. Lett. B 205, 545 (1988); W. Jaus, Phys. Rev. D 44, 2851 (1991); F. Cardarelli et al., Phys. Lett. B 332, 1 (1994); Phys. Rev. D 53, 6682 (1996).
4. H.-M. Choi and C.-R. Ji, Nucl. Phys. A 618, 291 (1997); ibid. 56, 6010 (1997).
5. H.-M. Choi and C.-R. Ji, Phys. Rev. D 59, 074015 (1999).
6. H.-M. Choi and C.-R. Ji, Phys. Rev. D 58, 071901 (1998).
7. S. J. Brodsky and D. S. Hwang, NPB 543, 239 (1998).
8. S. Godfrey and N. Isgur, Phys. Rev. D 32, 189 (1985); N. Isgur, D. Scora, B. Grinstein, and M. B. Wise, Phys. Rev. D 39, 799 (1989); D. Scora and N. Isgur, Phys. Rev. D 52, 2783 (1995).
9. H.-M. Choi and C.-R. Ji, “Light-Front Quark Model Analysis of Exclusive 0− → 0− Semileptonic Heavy Meson Decays”, to be published in Phys. Lett. B [hep-ph/9903496].
10. J. M. Flynn and C. T. Sachrajda, “Heavy Quark Physics From Lattice QCD”, to appear in Heavy Flavor (2nd edition) edited by A. J. Buras and M. Lindner (World Scientific, Singapore), hep-lat/9710057.
11. N.B. Demchuk, I. L. Grach, I. M. Narodetskii, and S. Simula, Phys. At. Nuclei 59, 2152 (1996); H.-Y. Cheng, C.-Y. Cheng, and C.-W. Hwang, Phys. Rev. D 55, 1559 (1997).
12. Particle Data Group, C. Caso et al., Eur. Phys. J. C 3, 1 (1998).
13. M. Ademollo and R. Gatto, Phys. Rev. Lett. 13, 264 (1964).
14. H. Leutwyler and M. Roos, Z. Phys. C 25, 91 (1984).
15. A. Afanasev and W. W. Buck, Phys. Rev. D 55, 4380 (1997).
16. D. Scora and N. Isgur, Phys. Rev. D 52, 2783 (1995).