Residual dimension of nilpotent groups: correction to
"Effective separability of finitely generated nilpotent
groups"

Mark Pengitore

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Abstract

The functions $F_{G}(n)$ and $\text{Conj}_{G}(n)$ measure the asymptotic behavior of residual finiteness and conjugacy separability for a finitely generated group $G$, respectively. In previous work [7], the author claimed a characterization for $F_{G}(n)$ and lower asymptotic bounds for $\text{Conj}_{G}(n)$ when $G$ is a finitely generated nilpotent group. However, a counterexample to the characterization of $F_{G}(n)$ for finitely generated nilpotent groups was communicated to the author which also had consequences to the lower asymptotic bound provided for $\text{Conj}_{G}(n)$. Subsequently, the statements of the asymptotic characterization of $F_{G}(n)$ and the asymptotic lower bound for $\text{Conj}_{G}(n)$ along with their proofs are incorrect. In this article, we provide correct lower asymptotic bounds for $F_{G}(n)$ and asymptotic lower bounds for $\text{Conj}_{G}(n)$ when $G$ is a finitely generated nilpotent group.

1 Introduction

A group $G$ is residual finite if for each nontrivial element $x \in G$, there exists a surjective group morphism $\phi : G \to Q$ to a finite group such that $\phi(x) \neq 1$. A group $G$ is conjugacy separable if for all non-conjugate pairs of elements $x, y \in G$, there exists a surjective group morphism $\phi : G \to Q$ to a finite group such that $\phi(x)$ and $\phi(y)$ are non-conjugate. When $G$ comes equipped with a finite generating subset $S$, we are able to quantify residual finiteness of $G$ with the function $F_{G,S}(n)$. The value of $F_{G,S}(n)$ is the maximum order of a finite group needed to distinguish a nontrivial element from the identity as one varies over nontrivial elements of word length at most $n$. Similarly, we quantify conjugacy separability of $G$ with the function $\text{Conj}_{G,S}(n)$ whose value is the maximum order of a finite group needed to distinguish the conjugacy classes of two non-conjugate elements as one varies over pairs of non-conjugate elements of word length at most $n$. Since the dependence of $F_{G,S}(n)$ and $\text{Conj}_{G,S}(n)$ on $S$ is mild, we suppress the generating subset throughout the introduction.

In previous work [7], we claimed an effective characterization of $F_{N}(n)$ and an asymptotic lower bound for $\text{Conj}_{N}(n)$ when $N$ is an infinite, finitely generated nilpotent group as seen in the following theorems. Note that the numbering and any unexplained terminology comes from [7].
Let \( N \) be an infinite, finitely generated nilpotent group. Then there exists a \( \psi_{RF}(N) \in \mathbb{N} \) such that \( F_N(n) \approx (\log(n))^{\psi_{RF}(N)} \). Additionally, we may compute \( \psi_{RF}(N) \) given a basis for \( \gamma_c(N/T(N)) \) where \( c \) is the step length of \( N/T(N) \).

Let \( N \) be an infinite, finitely generated nilpotent group. Suppose that \( N \) is not virtually abelian. There exists a \( \psi_{lower}(N) \in \mathbb{N} \) such that \( n^{\psi_{lower}(N)} \leq \text{Conj}_N(n) \). Additionally, one can compute \( \psi_{lower}(N) \) given a basis for \( \gamma_c(N/T(N)) \) where \( c \) is the step length of \( N/T(N) \).

Khalid Bou-Rabee communicated to us a counterexample to [7, Theorem 1.1]. To be specific, he gave us an example of a torsion free, finitely generated nilpotent group \( N \) where the asymptotic behavior of \( F_N(n) \) produced in [7], while no longer sharp, is still correct. Similarly, we have that the upper bound for \( \text{Conj}_N(n) \) from [7] is still correct.

The purpose of this article is to provide a correct lower asymptotic bound for \( F_N(n) \) and a correct asymptotic lower bound for \( \text{Conj}_N(n) \) along with correct proofs when \( N \) is an infinite, finitely generated nilpotent group. We also provide conditions for when the upper asymptotic bound of [7] Theorem 1.1] for \( F_N(n) \) can be improved and introduce a class of examples of finitely generated nilpotent groups where the asymptotic behavior of \( F_N(n) \) can be computed precisely.

Before we start, we introduce some notation. For two nondecreasing functions \( f, g : \mathbb{N} \to \mathbb{N} \), we say that \( f \approx g \) if there exists a constant \( C > 0 \) such that \( f(n) \leq C g(n) \). We say that \( f \leq g \) if \( f \approx g \) and \( g \not\approx f \). We say that \( f \approx g \) if \( f \leq g \) and \( g \leq f \). For a group \( G \), we denote \( \gamma_i(G) \) as the \( i \)-th step of the lower central series of \( G \). Finally, for a finitely generated nilpotent group \( G \), we denote \( h(G) \) as its Hirsch length and define \( T(G) \) as the subgroup generated by finite order elements.

1.1 Residual finiteness

We start with the case of infinite, finitely generated nilpotent groups \( N \) where \( N/T(N) \) is abelian.

**Theorem 1.1.** Let \( N \) be an infinite, finitely generated nilpotent group such that \( N/T(N) \) is abelian. Then

\[ F_N(n) \approx \log(n). \]

The situation is more interesting when \( N/T(N) \) has step length \( c > 1 \) as seen in the following theorem.

**Theorem 1.2.** Let \( N \) be an infinite, finitely generated nilpotent group such that \( N/T(N) \) has step length \( c > 1 \). There exists a natural number \( \dim_{RFL}(N) \) such that \( \dim_{RFL}(N) \geq c + 1 \) and where

\[ (\log(n))^{\dim_{RFL}(N)} \lesssim F_N(n). \]

For this theorem, we need to find an infinite sequence of elements \( \{x_i\}_{i=1}^{\infty} \) such that the minimal finite order group \( Q_i \) where there exists a surjective group morphism \( \phi_i : N \to Q_i \) satisfying \( \phi_i(x_i) \neq 1 \) has order approximately \( (\log(||x_i||))^{\dim_{RFL}(N)} \). We start the search for this sequence by
demonstrating that we may assume that the nilpotent group \( N \) is torsion free. We then find an element \( x \in N \), which we call a residual key, such that there exists an infinite sequence of natural numbers \( \{m_i\}_{i=1}^{\infty} \) for which our desired sequence of elements is given by \( \{x^{m_i}\}_{i=1}^{\infty} \). In order to find residual keys, we introduce a notion of \( \mathbb{F}_p \)-dimension associated to any primitive, central element \( x \) of \( N \), denoted \( \dim_{\mathbb{F}_p}(N, x) \), which measures the difficulty of separating \( x \) from the identity with a surjective group morphism to a finite \( p \)-groups. Letting \( \text{RP}_{N,x,i} \) be the set of prime numbers where the \( \dim_{\mathbb{F}_p}(N, x) = i \), we take the minimal \( i_0 \) such that \( |\text{RP}_{N,x,i_0}| = \infty \) and denote it as \( \dim_{\text{RFL}}(N, x) \). The natural number \( \dim_{\text{RFL}}(N, x) \) captures the lower asymptotic behavior of separating powers of \( x \) from the identity with a surjective group morphism to a finite \( p \)-group as one varies the prime number \( p \). By maximizing over all primitive elements in the isolator of \( \gamma(N) \), we obtain the value \( \dim_{\text{RFL}}(N) \). Primitive elements \( x \) in the isolator of \( \gamma(N) \) where \( \dim_{\text{RFL}}(N, x) = \dim_{\text{RFL}}(N) \) are defined to be residual keys.

For the next theorem, see [4,7] for the definition of tame residual dimension. The constant \( \psi_{\text{RF}}(N) \) was originally defined in [7] Definition 3.3 and its specific value can be found in [6]. The following theorem provides a general upper bound for residual finiteness and gives a collection of groups for which the upper asymptotic behavior for residual finiteness can be strengthened relative to the bound found in [7] Theorem 1.1.

**Theorem 1.3.** Let \( N \) be an infinite, finitely generated nilpotent group. Then

\[
\text{F}_N(n) \lesssim (\log(n))^{\psi_{\text{RF}}(n)}.
\]

Now suppose that \( N \) has tame residual dimension. Then there exists a natural number \( \dim_{\text{RFU}}(N) \) satisfying \( \dim_{\text{RFU}}(N) \leq \psi_{\text{RF}}(N) \) and a nondecreasing function \( \Theta_N(n) \) such that

\[
\text{F}_N(n) \lesssim (\Theta_N(n) \log(\Theta_N(n)) n)^{\dim_{\text{RFU}}(N)} \lesssim (\log(n))^{\psi_{\text{RF}}(N)}.
\]

In particular, if \( (\Theta_N(n) \log(\Theta_N(n)) n)^{\dim_{\text{RFU}}(N)} \lesssim (\log(n))^{\psi_{\text{RF}}(N)} \), then \( \text{F}_N(n) \) grows strictly slower than what is predicted by [7] Theorem 1.1.

We observe that the first statement of the above theorem is the upper bound produced for \( \text{F}_N(n) \) in [7] Theorem 1.1 for the class of infinite, finitely generated nilpotent groups, and we claim no originality in including it in this note. We restate this theorem and give its proof for two reasons. The first reason to collect all of the author’s work on effective residual finiteness of finitely generated nilpotent groups in one place. The second reason is to show the similarities and differences between the proof of the original upper bound and the proof of improved upper bound in the case of finitely generated nilpotent groups that have tame residual dimension. In particular, we want to highlight how the geometry of finite \( p \)-quotients of a finitely generated nilpotent group changes as we vary the prime.

Tame residual dimension for an infinite, finitely generated nilpotent group says for any primitive element \( x \) in the isolator of \( \gamma(N) \) and fixed value \( 1 \leq i \leq h(N) \) that the sets \( \text{RP}_{N,x,i} \) either have density 0 or have a subset \( F_{N,x,i} \) with positive density. We let \( i_0 \) be the minimal index such that there exists a subset \( F_{N,x,i_0} \subseteq \text{RP}_{N,x,i_0} \) having positive density. We then apply the Prime Number theorem relative to the set \( F_{N,x,i_0} \) to separate powers of \( x \) from the identity. By maximizing over primitive elements in the isolator of \( \gamma(N) \), we obtain the value \( \dim_{\text{RFU}}(N) \).

For the definition of accessible residual dimension, see [4,14].
Theorem 1.4. Let $N$ be an infinite, finitely generated nilpotent group such that $N/T(N)$ has step length $c > 1$, and suppose that $N$ has accessible residual dimension. Then there exists a natural number $\dim_{\text{RFU}}(N)$ such that $c + 1 \leq \dim_{\text{RFU}}(N) \leq \psi_{\text{RF}}(N)$ and where

$$F_{N}(n) \approx (\log(n))^{\dim_{\text{RFU}}(N)}.$$ 

Torsion free, finitely generated nilpotent groups with accessible residual dimension are nilpotent groups that have tame residual dimension which satisfy an extra condition for primitive elements $x$ in the isolator of $\gamma(N)$ where $\dim_{\text{RFU}}(N,x) > 1$. When $\dim_{\text{RFU}}(N,x) > 1$, we assume that the sets $RP_{N,x,i}$ for $1 \leq i < \dim_{\text{RFU}}(N,x)$ are finite. When $N$ only has tame residual dimension, we just have that these sets have 0 density which may still have infinite cardinality. Thus, a nilpotent group having accessible residual dimension is a stronger assumption than a nilpotent having tame residual dimension. This assumption allows us to essentially ignore these prime numbers when separating powers of $x$ from the identity. The interest in doing so is that they provide examples of prime numbers $p$ where $\dim_{F_p}(N,x) < \dim_{\text{RFU}}(N,x)$, and thus, we want to avoid these primes when computing a lower bound for $F_{N}(n)$ using the residual key $x$. As a natural consequence of the definition of accessible residual dimension, we have that $\dim_{\text{RFU}}(N,x) = \dim_{\text{RFU}}(N,x)$ for all primitive elements $x$ in the isolator of $\gamma(N)$. Subsequently, $\dim_{\text{RFU}}(N) = \dim_{\text{RFU}}(N)$ whose common value we define to be $\dim_{\text{ARF}}(N)$.

1.2 Conjugacy separability

For this last theorem, refer to [8] for the definition of $\dim_{\text{Conj}}(N)$.

Theorem 1.5. Let $N$ be an infinite, finitely generated nilpotent group that is not virtually abelian, and suppose that $N^{[T(N)]}$ step length $c$. Then there exists a natural number $\dim_{\text{Conj}}(N)$ such that $c + 1 \leq \dim_{\text{Conj}}(N) \leq \psi_{\text{RF}}(N)$ and where

$$n^{(c-1)\dim_{\text{Conj}}(N)} \gtrsim \text{Conj}_N(n).$$

For the lower bounds of $\text{Conj}_N(n)$, we need to find an infinite sequence of non-conjugate elements $x_i$ and $y_i$ satisfying the following. The minimal finite group $Q_i$ where there exists a surjective group morphism $\psi_i : N \rightarrow Q_i$ such $\psi(x_i)$ and $\psi(y_i)$ are non-conjugate has order approximately $\max\{||x_i||,||y_i||\}^{\dim_{\text{Conj}}(N)}$. The main concept used in finding this sequence is known as the admissible 4-tuple. These admissible 4-tuples $(g, m, a, b)$ contain the data of a primitive element $g$ in the isolator of $\gamma(N)$, a natural number $m$, an element $a$ of $\gamma_{-1}(N)$, and an element $b$ in $N$ such that $g^m = [a, b]$. For prime numbers $p$ where $[a, b] \notin N^p$, we introduce a $\mathbb{F}_p$-dimension to $(g, m, a, b)$, denoted $\dim_{\text{Conj},\mathbb{F}_p}(g, m, a, b)$, that measures the difficulty separating the conjugacy classes of $a^p[a, b]$ and $a^p[a, b]^2$ with surjective group morphisms to finite $p$-groups. As a natural consequence of conjugacy in the integral Heisenberg group, we have that the images of $a^p[a, b]$ and $a^p[a, b]^2$ under surjective group morphisms to finite $q$-groups are conjugate when $q$ is a prime number distinct from $p$, and subsequently, the minimal finite group which separates the conjugacy classes of $a^p[a, b]$ and $a^p[a, b]^2$ will be a finite $p$-group. Letting $\text{RCP}_{N,[g,m,a,b],i}$ be set of prime numbers $p$ where $\dim_{\text{Conj},\mathbb{F}_p}(g, m, a, b) = i$, there exists a maximal index $1 \leq i_0 \leq h(N)$ such that $|\text{RCP}_{N,[g,m,a,b],i_0}| = \infty$. We denote this value as $\dim_{\text{Conj}}(g, m, a, b)$ and we obtain the
value \( \dim_{\text{Conj}}(N) \) by maximizing the value \( \dim_{\text{Conj}}(g,m,a,b) \) over all such admissible 4-tuples \((g, m, a, b)\). The admissible 4-tuples \((g, m, a, b)\) which attain this maximum are known as conjugacy keys, and these conjugacy keys give us the necessary sequence of non-conjugate elements. In particular, the sequences are given by \( a^p [a, b] \) and \( a^p [a, b]^2 \) for \( p \in \text{RCP}_{N,(g,m,a,b),10} \).

### 1.3 Plan of paper

In §2 we introduce necessary background and conventions for this paper. §4 exposnds on the example provided to by Khalid Bou-Rabee. §6 introduces and defines the constants \( \dim_{\text{RFL}}(N) \), \( \dim_{\text{RFU}}(N) \), and \( \dim_{\text{RF}}(N) \) for infinite, finitely generated nilpotent groups. §5, §6 and §7 provide proofs of Theorem 1.2, Theorem 1.3, and Theorem 1.4, respectively. §8 introduces admissible 4-tuples and defines the constant \( \dim_{\text{Conj}}(N) \). §9 provides a proof of Theorem 1.5, §10 computes \( F_{\text{H(Z)}}(n) \) and \( \text{Conj}_{\text{H(Z)}}(n) \) where \( \text{H(Z)} \) is the integral Heisenberg group. Finally, §11 finishes with some open questions.

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# 2 Background

## 2.1 Notation and conventions

We let \( \text{lcm} \{r_1, \cdots, r_m\} \) be the lowest common multiple of \( \{r_1, \cdots, r_m\} \subseteq \mathbb{Z} \) with the convention that \( \text{lcm}(a) = |a| \) and \( \text{lcm}(a, 0) = 0 \). We let \( \text{gcd}(r_1, r_2) \) be the greatest common divisor of the integers \( r_1 \) and \( r_2 \) with the convention that \( \text{gcd}(r, 0) = |r| \). We write \( \mathbb{F}_p \) to be the finite field of \( p \) elements, and denote \( \mathbb{P} \) as the set of prime numbers.

We denote \( \|g\|_S \) to be the word length of \( g \) in \( G \) with respect to the finite generating subset \( S \), and when the subset \( S \) is clear from context, we will write \( \|g\| \). We denote the identity of \( G \) as \( 1 \), denote \( \{1\} \) as the trivial group, and write \( G^* = G \setminus \{1\} \). We let \( \text{Ord}_G(x) \) be the order of \( x \) as an element of \( G \) and denote the cardinality of a finite group \( G \) as \( |G| \). For a normal subgroup \( H \trianglelefteq G \), we set \( \pi_H : G \to G/H \) to be the natural projection and write \( \bar{x} = \pi_H(x) \) when \( H \) is clear from context. When given a nonempty subset \( X \subseteq G \), we denote \( \langle X \rangle \) as the subgroup generated by the elements of \( X \). We denote \( [x, y] = x^{-1}y^{-1}xy \). For \( x_1, \cdots, x_k \), we define \( [x_1, \cdots, x_k] = [x_1, [x_2, \cdots, x_k]] \) where \( [x_2, \cdots, x_k] \) is inductively defined. For nonempty subsets \( A, B \subseteq G \), we define \( [A, B] \) as the subgroup generated by the subset \( \{[a, b] \mid a \in A, b \in B\} \). We write \( x \sim y \) when there exists an element \( g \in G \) such that \( g^{-1} x g = y \) and denote the conjugacy class of \( x \) as an element of \( G \) as \( [x]_G \).

We denote the center of \( G \) as \( Z(G) \) and denote \( \gamma(G) \) as the \( i \)-th term of the lower central series of \( G \). For a finitely generated nilpotent group \( N \), we denote \( h(N) \) as its Hirsch length and \( T(N) \) as the subgroup generated by finite order elements. For natural numbers \( m \), we let \( G^m = \langle g^m \mid g \in G \rangle \) and write \( \pi_m \) for the natural projection \( \pi_m : G \to G/G^m \).
2.2 Residual finiteness

Following [1], we define the depth function $D_G : G^* \to \mathbb{N} \cup \{\infty\}$ of the finitely generated group $G$ as

$$D_G(g) = \min \{|Q| \mid \varnothing : G \to Q, |Q| < \infty, \text{ and } \varnothing(g) \neq 1\}$$

with the understanding that $D_G(g) = \infty$ if no such finite group exists.

**Definition 2.1.** Let $G$ be a finitely generated group. We say that $G$ is residually finite if $D_G(g) < \infty$ for all $g \in G^*$.

For a residually finite, finitely generated group $G$ with finite generating subset $S$, we define the associated complexity function $F_{G,S} : \mathbb{N} \to \mathbb{N}$ as

$$F_{G,S}(n) = \max \{D_G(g) \mid g \in G^* \text{ and } |g|_S \leq n\}.$$ 

For any residually finite, finitely generated group $G$ with finite generating subsets $S_1$ and $S_2$, we have that $F_{G,S_1}(n) \approx F_{G,S_2}(n)$ (see [1, Lemma 1.1]). Henceforth, we will suppress the choice of finite generating subset.

2.3 Conjugacy Separability

For the group $G$, we define the conjugacy depth function $CD_G(x,y) : (G \times G) \setminus \{(x,y) \mid x \sim y\} \to \mathbb{N} \cup \{\infty\}$ as

$$CD_G(x,y) = \min \{|Q| \mid \text{ there exists } \varnothing : G \to Q \text{ such that } \varnothing(x) \sim \varnothing(y) \text{ and where } |Q| < \infty\}$$

with the understanding that $CD_G(x,y) = \infty$ if the images of $x$ and $y$ are conjugate in every surjective group morphism from $G$ to any finite group.

**Definition 2.2.** Let $G$ be a finitely generated group. We say that $G$ is conjugacy separable if $CD_G(x,y) < \infty$ for all non-conjugate pairs of elements of $G$.

We now quantify conjugacy separability with the following complexity function. Let $G$ be a conjugacy separable, finitely generated group with a finite generating subset $S$. Define $\text{Conj}_{G,S} : \mathbb{N} \to \mathbb{N}$ as

$$\text{Conj}_{G,S}(n) = \max \{CD_G(x,y) \mid x \sim y, |x|_S, |y|_S \leq n\}.$$ 

For any conjugacy separable, finitely generated group $G$ with finite generating subsets $S_1$ and $S_2$, we have that $\text{Conj}_{G,S_1}(n) \approx \text{Conj}_{G,S_2}(n)$. The proof of this statement is similar to [1, Lemma 1.1]; thus, from here on, we will suppress the choice of finite generating subset.

Since every nontrivial element is not conjugate to the identity, it follows that every conjugacy separable group is residually finite. Moreover, we have that $F_G(n) \lesssim \text{Conj}_G(n)$.

We finish this subsection with the following technical lemma.

**Lemma 2.3.** Let $G$ and $H$ be conjugacy separable, finitely generated groups, and suppose that $H$ is a subgroup of $G$. If $x, y \in H$ are two non-conjugate elements of $G$, then $CD_H(x,y) \leq CD_G(x,y)$.

**Proof.** Let $\varnothing : G \to Q$ be a surjective group morphism to a finite group such that $\varnothing(x) \sim \varnothing(y)$ and where $|Q| = CD_G(x,y)$. Thus, restricting $\varnothing$ to $H$ gives a surjective group morphism $\varnothing|_H : H \to \varnothing(H)$ such that $\varnothing|_H(x) \sim \varnothing|_H(y)$. By definition, we have that $CD_H(x,y) \leq |\varnothing(H)| \leq |Q| = CD_G(x,y)$. \qed
2.4 Nilpotent groups

For more details of the following discussion, see \([5, 8]\). We define \(\gamma_i(G) = G\) and inductively define \(\gamma_i(G) = [\gamma_{i-1}(G), G]\). We call the subgroup \(\gamma_i(G)\) the \(i\)-th term of the lower central series of \(G\).

**Definition 2.4.** Let \(G\) be a finitely generated group. We say that \(G\) is a nilpotent group if \(\gamma_k(G) = \{1\}\) for some natural number \(k\). We say that \(G\) is a nilpotent group of step length \(c\) if \(c\) is the smallest natural number such that \(\gamma_{c+1}(G) = \{1\}\). If the step size is unspecified, we simply say that \(G\) is a nilpotent group.

For a subgroup \(H \leq N\) of a nilpotent group, we define the isolator of \(H\) in \(N\) as

\[
\sqrt[\gamma]{H} = \left\{ g \in N \mid \exists k \in \mathbb{N} \text{ such that } g^k \in H \right\}.
\]

\(\sqrt[\gamma]{H}\) is a subgroup of \(N\) for all \(H \leq N\) when \(N\) is a torsion free, finitely generated nilpotent group. Additionally, if \(H \leq N\), then \(\sqrt[\gamma]{H} \leq N\). As a natural consequence, \(N/\sqrt[\gamma]{H}\) is torsion free. When the group \(N\) is clear from context, we will simply write \(\sqrt[\gamma]{H}\).

The **torsion subgroup** of a finitely generated nilpotent group \(N\) is defined as

\[
T(N) = \{ g \in N \mid \text{Ord}_N(g) < \infty \}.
\]

It is well known that \(T(N)\) is a finite characteristic subgroup of \(N\). Moreover, if \(N\) is an infinite, finitely generated nilpotent group, then \(N/T(N)\) is a torsion free, finitely generated nilpotent group.

This next lemma will be useful in the studying lower bounds for conjugacy separability of finitely generated nilpotent groups.

**Lemma 2.5.** Let \(N\) be a torsion free, finitely generated nilpotent group of step size \(c > 1\). If \(z \in (\gamma_c(N))^\bullet\), then there exist elements \(a \in \gamma_{c-1}(N)\) and \(b \in N\) such that \([a, b] = z\).

**Proof.** Let \(X = \{x_j\}_{j=1}^m\) be a finite generating subset of \(N\), and since every subgroup of a finitely generated nilpotent group is finitely generated, there exists a finite generating subset, say \(Y = \{y_j\}_{j=1}^k\), for \(\gamma_{c-1}(N)\). \([5\text{ Lemma 1.6}]\) implies that the set \(\{[x_i, y_j] \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq k\}\) normally generates \(\gamma_c(N)\). Thus, \([5\text{ Lemma 2.1}]\) implies that there exist elements \(\{g_{s, \ell}\}_{s=1, \ell=1}^{\alpha, \beta}\) such that

\[
z = \prod_{s=1}^\alpha \prod_{\ell=1}^\beta g_{s, \ell}^{-1} [x_i, y_j] g_{s, \ell} = \prod_{s=1}^\alpha \prod_{\ell=1}^\beta [x_i, y_j] = \prod_{s=1}^\alpha \prod_{\ell=1}^\beta x_i y_j = \prod_{s=1}^\ell \prod_{\ell=1}^\beta x_i y_j \quad \Box.
\]

When given a torsion free, finitely generated nilpotent group, we may refine the series given by \(\{\sqrt[\gamma]{N}\}_{i=1}^c\) to obtain a cyclic series \(\{H_i\}_{i=1}^{h(N)}\) where \(H_i/H_{i-1} \cong \mathbb{Z}\) for all \(i\). The number of terms in this series is known as the Hirsch length of \(N\) and is denoted \(h(N)\). The Hirsch length can be computed as \(h(N) = \sum_{i=1}^c \text{rank}_\mathbb{Z}(\gamma_i(N) / \gamma_{i+1}(N))\). We choose \(x_1 \in N\) such that \(H_1 \cong \langle x_1 \rangle\), and for each \(2 \leq i \leq h(N)\), we choose \(x_i \in H_i\) such that \(H_i \cong \pi_{H_{i-1}}(x_i)\). Any generating subset chosen in this way will be called a **Mal’stev basis**. Via \([6\text{ Lemma 8.3}]\), we may represent each element
of $N$ with respect to this generating subset uniquely as $g = \prod_{i=1}^{h(N)} x_i^{\alpha_i}$ where $\alpha_i \in \mathbb{Z}$. The values \{\alpha_i\}_{i=1}^{h(N)} are referred to as the Mal’stev coordinates of $g$. Whenever we reference a Mal’tsev basis, we suppress reference to the series \{H_i\}_{i=1}^{h(N)}.

The following proposition and its proof can be originally found in [7, Lemma 3.8]. It relates the Mal’stev coordinates of an element $g$ to its word length with respect to the generating subset given by the Mal’tsev basis.

**Proposition 2.6.** Let $N$ be a torsion free, finitely generated nilpotent group of step length $c$ with a Mal’stev basis \{\alpha_i\}_{i=1}^{h(N)}. Suppose that $g = \prod_{i=1}^{h(N)} x_i^{\alpha_i}$ is a nontrivial element where $\|g\| \leq n$. For each $1 \leq i \leq c$, we define $M_i = \sqrt{\gamma_i(N)}$, and for each $1 \leq i \leq h(N)$, we define $t_i$ as the minimal natural number where $\pi_{M_i}(x_i) = 1$ and $\pi_{M_{i+1}}(x_i) \neq 1$. Then $|\alpha_i| \leq C n^h$ for all $i$ where $C > 0$ is some constant.

**Proof.** We proceed by induction on step length, and since the base case of abelian groups is evident, we may assume that $N$ has step length $c > 1$. We observe that $\|\pi_{M_i}(g)\| \leq n$ and that the image of the set \{\alpha_i\}_{i=1}^{h(N)} in $N/M_c$ is a Mal’tsev basis for $N/M_c$. Therefore, the inductive hypothesis implies that there exists a constant $C_1 > 0$ such that $|\alpha_i| \leq C_1 n^{\ell_i}$. For each $1 \leq i \leq h(N)$, there exists a minimal natural number $\ell_i$ such that $x_i^{\ell_i} \in \gamma_i(N)$. In particular, we may write $\alpha_i = s_i \ell_i + r_i$ where $0 \leq r_i < \ell_i$. We let $M = \max \{\ell_i \mid 1 \leq i \leq h(N)\}$.

To proceed, we will demonstrate that we may assume that $g \in H_c$. We have that

$$|s_i \ell_i| \leq |s_i \ell_i + r_i - r_i| \leq |\alpha_i| + |r_i| \leq C_1 n^{\ell_i} + M \leq C_2 n^{\ell_i}$$

for $h(H_c) + 1 \leq i \leq h(N)$ where $C_2 > 0$ is some constant. Thus, $|s_i| \leq C_2 n^{\ell_i}$ for all $i$. By [4, 3.B2], we have that $\|x_i^{s_i \ell_i}\| \approx |s_i|^{1/\ell_i}$. Therefore, there exists a constant $C_{3,1} > 0$ such that $\|x_i^{s_i \ell_i}\| \leq C_{3,1} |s_i|^{1/\ell_i}$. In particular, we have that

$$\|x_i^{s_i \ell_i + r_i}\| \leq \|x_i^{s_i \ell_i}\| + \|x_i^{r_i}\| \leq C_{3,1} |s_i|^{1/\ell_i} + M \leq M C_2^{1/\ell_i} C_{3,1} n.$$

Hence, by setting $C_4 = \max \left\{MC_2^{1/\ell_i} C_{3,1} \mid h(H_c) + 1 \leq i \leq h(N) \right\}$, we may write $\|x_i^{s_i \ell_i}\| \leq C_4 n$. Note that $g h^{-1} \in \sqrt{\gamma_i(N)}$ and that $\|g h^{-1}\| \leq \|g\| + \|h^{-1}\| \leq C_5 n$ for some constant $C_5 > 0$ which gives our claim.

By passing to the quotient $N/K_i$ where $K_i = \langle x_j \rangle_{j=1, j \neq i}$, it is straightforward to see that $\|x_i^{s_i \ell_i}\| \leq C_5 n$ for each $i$. Using the same arguments as above, [4, 3.B2] implies that $|s_i| \leq C_{6,i} n^c$ for some constant $C_{6,i} > 0$ for each $1 \leq i_0 \leq h(H_c)$. Thus, we may write

$$|\alpha_i| = |s_i \ell_i + r_i| = |s_i| |\ell_i| + |r_i| \leq M + M C_{6,i} n^c \leq (M + 1) C_{6,i} n^c.$$

Letting $C_7 = \max \left\{C_{1,6,1}, \cdots, C_{1,6,h(H_c)} \right\}$, we have by construction that $|\alpha_i| \leq C_7 n^6$ for all $i$. $\square$

### 2.5 The integral Heisenberg group

Let

$$H(\mathbb{Z}) = \{x,y,z \mid [x,y] = z \text{ and } z \text{ is central} \}.$$
$H(\mathbb{Z})$ is known as the integral Heisenberg group. We may represent each element $g \in H(\mathbb{Z})$ uniquely as $g = x^{a_1} y^{a_2} z^{a_3}$ where $|a_1|, |a_2| \leq C \|g\|$ and $|a_3| \leq C (\|g\|)^2$ for some constant $C > 0$. In particular, we have that $h(H(\mathbb{Z})) = 3$ and that $\{x, y, z\}$ is a Mal’tsev basis for $H(\mathbb{Z})$. Moreover, we may write

$$[g]_{H(\mathbb{Z})} = \left\{ x^{a_1} y^{a_2} z^{\tau \gcd(a_1, a_2) + a_3} \mid \tau \in \mathbb{Z} \right\}.$$  

We note that the abelianization of $H(\mathbb{Z})$ is isomorphic to $\mathbb{Z}^2$ where $H(\mathbb{Z})/\gamma_2(H(\mathbb{Z})) \cong \langle \bar{x}, \bar{y} \rangle$. We also note that $\gamma_2(H(\mathbb{Z})) \cong Z(H(\mathbb{Z})) \cong \langle z \rangle$. Thus, $H(\mathbb{Z})$ has step length 2.

We finish with the following proposition which is originally from [3, Proposition 7.9].

**Proposition 2.7.** Let $p$ be a prime number greater than 2, and suppose that $\varphi : H(\mathbb{Z}) \to \mathbb{Q}$ is a surjective group morphism to a $q$-group where $q$ is a some prime number distinct from $p$. Then $\varphi(x^p z) \sim \varphi(x^p z^2)$.

**Proof.** We may write $[x^p z]_{H(\mathbb{Z})} = \{ x^p z^{p+1} \mid t \in \mathbb{Z} \}$. Let $q^k = \text{Ord}_Q(\varphi(z))$. Given that $\gcd(p, q^k) = 1$, there exist integers $a, b$ such that $ap + bq^k = 1$. Thus, we have that

$$\varphi(x^p z) \sim \varphi(x^p z^{ap+1}) = \varphi(x^p z^{1-bq^k+1}) = \varphi(x^p z^2). \quad \square$$

### 2.6 Density

For $A \subset \mathbb{P}$, we define the density of $A$ in $\mathbb{P}$ as

$$\delta(A) = \lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{|\mathbb{P} \cap \{1, \ldots, n\}|}$$

when the limit exists.

### 3 A counterexample to [7, Proposition 4.10] and [7, Theorem 1.1]

The following example was communicated to us by Khalid Bou-Rabee. Let $G$ be the torsion free, finitely generated nilpotent group given by

$$G = \{ x, y, w, z, u, v \mid [x, y] = [w, z] = 1, [x, w] = [y, z] = u, [x, z] = v, [y, w] = v^{-1}, u \text{ and } v \text{ are central} \}.$$  

We start by listing some basic facts about $G$. The set $S = \{ x, y, w, z, u, v \}$ is a Mal’tsev basis for $G$ from which it follows that $h(G) = 6$. Additionally, we have that the abelianization of $G$ is given by $G/\gamma_2(G) \cong \{ \bar{x}, \bar{y}, \bar{w}, \bar{z} \}$ and that the center is given by $Z(G) \cong \langle u, v \rangle$. Finally, we observe that $G$ has step length 2 and that $\gamma_2(G) = Z(G)$.

The main tool used in [7] is the following proposition. We first introduce a definition.

**Definition 3.1.** Let $N$ be a torsion free, finitely generated nilpotent group of step length $c$ with a Mal’tsev basis $\{ x_i \}_{i=1}^{h(N)}$. For a vector $\vec{a} = (a_i)_{i=1}^{\ell}$ where $a_i$ is a natural number such that $1 \leq a_i \leq h(N)$ for all $i$, we denote $[x_{\vec{a}}] = [x_{a_1}, \ldots, x_{a_\ell}]$. We call $[x_{\vec{a}}]$ a simple commutator of weight $\ell$ with respect to $\vec{a}$, and since $N$ is a nilpotent group, we have that all simple commutators of weight...
greater than $c$ are trivial. We denote $W_k(N, \{x_i\})$ to be the set of nontrivial simple commutators in $\{x_i\}$ of weight $k$, and let $W(N, \{x_i\})$ to be the set of nontrivial commutators.

We may write $[x_{\delta}] = \prod_{i=1}^{h(N)} [x_{\delta}]$. Let

$$B(N, \{x_i\}) = \text{lcm}\{\mid \delta_{i,j} \mid 1 \leq i \leq h(N), \delta_{i,j} \neq 0, \text{ and } [x_{\delta}] \in W(N, \{x_i\}) \}.$$ 

See [7, Proposition 4.10] for where the following proposition is originally found.

**Proposition 3.2.** Let $N$ be a torsion free, finitely generated nilpotent group with a Mal’tsev basis $\{x_i\}_{i=1}^{h(N)}$. Let $\varphi: N \rightarrow Q$ be a surjective group morphism to a finite $p$-group where $p > B(N, \{x_i\})$. Suppose that $\varphi([x_{\delta}]) \neq 1$ for all $[x_{\delta}] \in W(N, \{x_i\}) \cap Z(N)$. Also, suppose that $\varphi(x_i) \neq \varphi(x_j)$ for all $x_i, x_j \in Z(N)$ where $i \neq j$. Then $\varphi(x_i) \neq \varphi(x_j)$ for $1 \leq i < j \leq h(N)$. Finally, $|Q| \geq p^{h(N)}$.

We note that $W(G, S) \cap Z(G) = \{u, v, u^{-1}\}$ and that $B(G, S) = 1$. The following proposition produces infinitely many prime numbers $p$ where there exists a finite $p$-quotient $Q_p$ of $G$ such that the hypotheses of Proposition 3.2 are satisfied and where $|Q| = p^4$. Since Proposition 3.2 predicts $|Q| \geq p^{h(N)}$.

Before starting, we introduce some notation. Let $E = \{p \in \mathbb{P} \mid 4 \text{ divides } p - 1\}$. For $p \in E$, we let $\{a_p, b_p\}$ be the two distinct solutions to the equation $T^2 + 1 \equiv 0 \mod p$. Finally, we let $A_p$ and $B_p$ be the normal closures of the subgroups $\langle x^{a_p} y \rangle$ and $\langle x^{b_p} y \rangle$ in $G$, respectively.

**Proposition 3.3.** If $p \in E$, then $\pi_p(A_p) \cap Z(G/G^p) \cong \mathbb{F}_p$ and $\pi_p(B_p) \cap Z(G/G^p) \cong \mathbb{F}_p$. Also, $|G/G^p \cdot A_p| = |G/G^p \cdot B_p| = p^4$ and $Z(G/G^p \cdot A_p) \cong Z(G/G^p \cdot B_p) \cong \mathbb{F}_p$. Additionally, we have that $\pi_p(A_p) \cap \pi_p(B_p) \cong \{1\}$ and $\langle \pi_p(A_p), \pi_p(B_p) \rangle \cong Z(G/G^p)$. We finish with $\pi_{G^p \cdot A_p}(u), \pi_{G^p \cdot A_p}(v) \neq 1$, $\pi_{G^p \cdot B_p}(u), \pi_{G^p \cdot B_p}(v) \neq 1$, $\pi_{G^p \cdot A_p}(u) \neq \pi_{G^p \cdot A_p}(v)$, and $\pi_{G^p \cdot B_p}(u) \neq \pi_{G^p \cdot B_p}(v)$.

**Proof.** For the first statement, it is sufficient to prove that $|G/G^p \cdot A_p| = p^4$ and that $Z(G/G^p) \cap \pi(A_p) \cong \mathbb{F}_p$. By direct calculation, we have that $A_p \cong \langle x^{a_p} y, u^{a_p} v^{-1}, u v^{a_p} \rangle$. Subsequently, $A_p \cap Z(G) \cong \langle u v^{a_p} \rangle$. We have that $(u v^{a_p})^{-1} = u^{-a_p} v^{-1} = u v^{a_p}$ mod $G^p$. Thus, $\pi_p(A_p) \cap Z(G/G^p) \cong \langle \pi_p(u v^{a_p}) \rangle \cong \mathbb{F}_p$. We note that each element $G/G^p \cdot A_p$ can be written uniquely as $\pi_{G^p \cdot A_p}(x^{a_p} y, z^{a_p} v^{a_p})$ where $0 \leq a_p, a_p, \alpha_p, \alpha_p < p$; thus, the second paragraph after [6, Definition 8.2] implies that $|G/G^p \cdot A_p| = p^4$. Moreover, we have that $\pi_2(G/G^p \cdot A_p) \cong Z(G/G^p \cdot A_p)$. Hence, $Z(G/G^p \cdot A_p) \cong \mathbb{F}_p$.

For the next step, we note that $\pi_p(A_p) \cong \langle u v^{a_p} \rangle$ and that $\pi_p(B_p) \cong \langle u v^{b_p} \rangle$. Since $a_p \neq b_p$ mod $p$, we have that $u v^{a_p} \neq u v^{b_p}$ mod $G^p$. Suppose for a contradiction that there exists a natural number $\ell$ such that $(u v^{a_p})^\ell = u v^{b_p}$ mod $G^p$. Since $(u v^{a_p})^\ell = u^{\ell a_p} v^{a_p}$, we must have that $\ell \equiv 1 \mod p$ and $\ell a_p \equiv b_p \mod p$. Since $\ell a_p \equiv a_p \mod p$, we have that $a_p \equiv b_p \mod p$ which is a contradiction. In particular, $\pi_p(A_p) \cap \pi_p(B_p) = \{1\}$; hence, $\langle \pi_p(A_p), \pi_p(B_p) \rangle \cong \mathbb{F}_p \times \mathbb{F}_p$. Since $Z(G/G^p) \cong \mathbb{F}_p \times \mathbb{F}_p$, it follows that $\langle \pi_p(A_p), \pi_p(B_p) \rangle \cong Z(G/G^p)$.

The remaining two statements are evident. \hfill $\square$

[7, Theorem 1.1] predicts that $F_n(n) \approx (\log(n))^5$, and the following proposition provides a counterexample.

**Proposition 3.4.** $F_4(n) \preceq (\log(n))^4$.
Theory [9, 1.2] implies that there exists a prime number. Therefore, D

The purpose of this section is to define the constants dim

\[ \phi = \psi \]

Thus, \( D_G(g) \leq C_1 n^2 \) for some constant \( C_1 > 0 \).

Suppose that \( \pi_{2(G)}(g) \neq 1 \). \cite{11} Corollary 2.3 implies that there exists a surjective group morphism \( \phi : G/\pi_2(G) \rightarrow P \) to a finite group where \( |P| \leq C_1 C_2 \log(C_1 C_2 n) \) such that \( \phi(\pi_{2(G)}(g)) \neq 1 \).

Therefore, \( D_G(g) \leq C_1 C_2 \log(C_1 C_2 n) \).

Now suppose that \( \pi_{2(G)}(g) = 1 \). That implies \( \alpha, \alpha_y, \alpha_w, \alpha_c = 0 \). In particular, we may write \( g = u^{\alpha_y} v^{\alpha_c} \). Let \( E, A_p, \) and \( B_p \) be defined as above. We may without loss of generality assume that \( \alpha_y \neq 0 \). Since Chebotarev’s Density Theorem implies that \( \delta(E) > 0 \), the Prime Number Theory \cite{9, 1.2} implies that there exists a prime number \( p \in E \) such that \( p \mid \alpha_y \) and where \( p \leq C_1 \log(C_3 |\alpha_y|) \) for some constant \( C_3 > 0 \).

Therefore, there exists a constant \( C_4 > 0 \) such that \( p \leq C_4 \log(C_n n) \).

Proposition \cite{3, 3} implies that \( \pi(A_p) \cap Z(G/G^p) \cong \mathbb{F}_p \) and that \( \pi(N_p) \cap Z(G/G^p) \cong \mathbb{F}_p \). Since \( Z(G/G^p) \cong \mathbb{F}_p \times \mathbb{F}_p \), we may assume that \( \pi_p(g) \notin \pi(A_p) \).

Thus, \( \pi_{G_p}(g) \neq 1 \), and Proposition \cite{3, 3} implies that \( |G/G^p \cdot A_p| = p^4 \).

Hence, there exists a constant \( C_5 > 0 \) such that \( D_G(g) \leq C_5 \left( \log(C_5 n) \right)^4 \). Hence, \( F_G(n) \lesssim (\log(n))^4 \).

\[ \square \]

## 4 Residual dimension

The purpose of this section is to define the constants \( \text{dim}_{RFL}(N) \), \( \text{dim}_{RFL}(N) \), and \( \text{dim}_{ARF}(N) \) for a torsion free, finitely generated nilpotent group \( N \). In order to do so, we introduce a notion of \( F_p \)-dimension for finite \( p \)-groups.

**Definition 4.1.** Let \( Q \) be a finite \( p \)-group. The \( F_p \)-dimension of \( Q \) is the natural number \( \text{dim}_{F_p}(Q) \) given by \( |Q| = p^{\text{dim}_{F_p}(Q)} \).

For any finite \( p \)-group \( Q \), there exists a normal series \( \{ P_i \} \) where \( P_i \cong \mathbb{F}_p \) and where \( P_i/P_{i-1} \cong \mathbb{F}_p \) for \( i > 1 \). By choosing a generating subset \( \{ x_i \} \) compatible with the normal series \( \{ P_i \} \), we have that any element of \( Q \) may be written uniquely as \( \prod x_i^{t_i} \) where \( 0 \leq t_i < p \) for all \( i \). In particular, our notion of \( F_p \)-dimension for finite \( p \)-groups mirrors Hirsch length for torsion free, finitely generated nilpotent groups.

We now demonstrate a relationship between \( \text{dim}_{F_p}(Q) \) and the step length of \( Q \) when \( Q \) is a finite \( p \)-group.

**Lemma 4.2.** If \( Q \) is an abelian finite \( p \)-group, then \( \text{dim}_{F_p}(Q) \geq 1 \). If \( Q \) has step length \( c > 1 \), then \( \text{dim}_{F_p}(Q) \geq c + 1 \).

**Proof.** Since the first statement is clear, we may assume that \( Q \) has step length \( c > 1 \). We prove the second statement by induction on step length. For the base case, we may assume that \( Q \) has step length 2. There exist elements \( x, y \in Q \) such that \( [x, y] \neq 1 \). Since \( Q \) has step length 2, we have that \( [x, y] \in [Q, Q] \leq Z(Q) \). Consider the group \( H = \langle x, y, [x, y] \rangle \). Since each element in \( H \) can be written uniquely as \( x^i y^j [x, y]^k \) for natural numbers \( 0 \leq i < \text{Ord}_Q(x), 0 \leq j < \text{Ord}_Q(y), \) and \( 0 \leq k < \text{Ord}_Q([x, y]) \), we observe that the second paragraph after \cite{6} Definition 8.2] implies that \( |H| = \text{Ord}_Q(x) \cdot \text{Ord}_Q(y) \cdot \text{Ord}_Q([x, y]) \geq p^3 \).

Thus, \( \text{dim}_{F_p}(H) \geq 3 \). Since \( |H| \mid |Q| \), we have that \( \text{dim}_{F_p}(H) \leq \text{dim}_{F_p}(Q) \). Thus, \( \text{dim}_{F_p}(Q) \geq 3 \).
Now suppose that $Q$ has step length $c > 2$. By induction, $\dim_{\mathbb{F}_p}(Q/\gamma_i(Q)) \geq c$, and since $\gamma_i(Q)$ is abelian, we have that $\dim_{\mathbb{F}_p}(\gamma_i(Q)) \geq 1$. In particular, $|Q| = |Q/\gamma_i(Q)||Q| \geq p^{c-1}$. Thus, $|Q| \geq p^{c+1}$, and subsequently, $\dim_{\mathbb{F}_p}(Q) \geq c + 1$. □

The following lemma implies for any prime number $p$ that we may separate a primitive central element $x$ in a torsion free, finitely generated nilpotent group $N$ from the identity with a surjective group morphism to a finite $p$-group. For this lemma, we say that an infinite order element $g$ of a torsion free finitely generated group $G$ is primitive if whenever there exists an element $h \in G$ and a non-zero integer $m$ such that $h^m = g$, then either $g = h$ or $g = h^{-1}$. In particular, if $G$ is a torsion free, finitely generated abelian group with a primitive element $z$, there exists a generating basis $\{x_i\}_{i=1}^{h(G)}$ for $G$ as a $\mathbb{Z}$-module such that $x_1 = z$.

**Lemma 4.3.** Let $N$ be a torsion free, finitely generated nilpotent group, and let $z \in Z(N)$ be a primitive element. Let $p$ be a prime number. There exists a surjective group morphism $\phi : N \to Q$ to a finite $p$-group $Q$ such that $\phi(z) \neq 1$ and where $\dim_{\mathbb{F}_p}(Q) \leq h(N)$.

**Proof.** Let $\{x_i\}_{i=1}^{h(N)}$ be a Mal’tsev basis for $N$. We may write $z = \prod_{i=1}^{h(N)} x_i^{\alpha_i}$, and since $z$ is a primitive element, there exists an index $i_0$ such that $\alpha_{i_0} \neq 0 \mod p$. Since $\pi_p(Z(N)) = \prod_{i=1}^{h(N)} \mathbb{Z}/p\mathbb{Z}$, we have that each element of $\pi_p(N/N^p)$ may be written uniquely as $\prod_{i=1}^{h(N)} x_i^{\beta_i}$ where $0 \leq \beta_i < p$. Thus, we have that $\pi_p(z) \neq 1$. One last observation is that $|N/N^p| = p^{h(N)}$ as desired. □

The above lemma implies that we are able to find a surjective group morphism from $N$ to a finite $p$-group of minimal $\mathbb{F}_p$-dimension where the image of $z$ is not trivial. Thus, we have the following definition.

**Definition 4.4.** Let $N$ be a torsion free, finitely generated nilpotent group with a prime number $p$. Let $z \in Z(N)$ be a primitive element. Proposition 4.3 implies that there exists a surjective group morphism $\phi : N \to P$ to a finite $p$-group such that $\phi(z) \neq 1$ and where

$$\dim_{\mathbb{F}_p}(P) = \min \{ \dim_{\mathbb{F}_p}(Q) \mid \exists \phi : N \to Q \text{ that satisfies Proposition 4.3 for } z \}.$$ 

We refer to $\phi : N \to P$ as an admissible $p$-surjection of $N$ with respect to $z$. We call $\dim_{\mathbb{F}_p}(P)$ the residual $\mathbb{F}_p$-dimension of $N$ with respect to $z$ and denote it as $\dim_{\mathbb{F}_p}(N,z)$. Proposition 4.3 implies that $\dim_{\mathbb{F}_p}(N,z) \leq h(N)$ for all primitive, central elements $z$ of $N$ and prime numbers $p$.

### 4.1 Lower residual dimension

For a torsion free, finitely generated nilpotent group $N$ with a primitive element $z \in Z(N)$, the value $\dim_{\mathbb{F}_p}(N,x)$ may attain any natural number between 1 and $h(N)$ as we vary the prime number $p$. Therefore, we introduce some notation. For each $1 \leq i \leq h(N)$, we define

$$\text{RP}_{N,z,i} \overset{\text{def}}{=} \{ p \in \mathbb{P} \mid \dim_{\mathbb{F}_p}(N,z) = i \}.$$ 

We call $\text{RP}_{N,z,i}$ the set of residual prime numbers of $N$ with respect to $z$ of dimension $i$.

Suppose that $N$ is a torsion free, finitely generated nilpotent group of step length $c$ with a primitive element $z \in \sqrt[12]{\gamma(N)}$. We now introduce a natural number that measures the lower asymptotic complexity of separating $z$ from the identity with surjective group morphisms to finite $p$-groups as we vary over all prime numbers $p$.  

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**Definition 4.5.** Let $N$ be a torsion free, finitely generated nilpotent group of step length $c$ with a primitive element $z \in \sqrt{\gamma_c(N)}$. There exists a minimal natural number $t_0$ where $1 \leq t_0 \leq h(N)$ such that $|RP_{N,z,t_0}| = \infty$. We call the natural number $t_0$ the lower residual dimension of $z$ in $N$ and denote it as $\dim_{\text{RFL}}(N,z)$. We call $LR_{N,z} = RP_{N,z,t_0}$ the set of prime numbers that realize the lower residual dimension of $z$.

By maximizing over primitive elements in $\sqrt{\gamma_c(N)}$, we obtain a natural invariant associated to any torsion free, finitely generated nilpotent group which gives the degree of logarithmic growth for a lower bound of residual finiteness.

**Definition 4.6.** Let $N$ be a torsion free, finitely generated nilpotent group of step length $c$. Let
\[
\dim_{\text{RFL}}(N) \overset{\text{def}}{=} \max \left\{ \dim_{\text{RFL}}(N,z) \mid z \in \sqrt{\gamma_c(N)} \text{ is primitive} \right\}.
\]

We call $\dim_{\text{RFL}}(N)$ the lower residual dimension of $N$. For an infinite, finitely generated nilpotent group $N$, we denote $\dim_{\text{RFL}}(N) = \dim_{\text{RFL}}(N/T(N))$.

When $N$ is a torsion free, finitely generated nilpotent group, there exists a primitive element $x \in \sqrt{\gamma_c(N)}$ such that $\dim_{\text{RFL}}(N,x) = \dim_{\text{RFL}}(N)$. Any such element will be called a residual key.

### 4.2 Upper residual dimension

This subsection gives conditions on when the upper asymptotic bound for $F_{N}(n)$ can be improved to be better than that of [7, Theorem 1.1].

**Definition 4.7.** Let $N$ be a torsion free, finitely generated nilpotent group of step length $c$, and suppose that $x \in \sqrt{\gamma_c(N)}$ is a primitive element. Suppose for each $1 \leq i \leq h(N)$ that either $\delta(RP_{N,x,i}) = 0$ or that there exists a subset $F_{N,x,i} \subset RP_{N,x,i}$ such that $\delta(F_{N,x,i}) > 0$. We then say that $N$ has tame residual dimension at $x$. We define the upper residual dimension of $N$ at $x$ as the minimal index $i_0$, denoted $\dim_{\text{RFL}}(N,x)$, such that there exists a subset $F_{N,x,i_0} \subset RP_{N,x,i_0}$ where $\delta(F_{N,x,i_0}) > 0$. We write $UR_{N,x} = F_{N,x,i_0}$ and call $UR_{N,x}$ the set of prime numbers that realize the upper residual dimension of $N$ at $x$.

Now suppose that $N$ has tame residual dimension at all primitive elements $x \in \sqrt{\gamma_c(N)}$. We denote the upper residual dimension of $N$ as
\[
\dim_{\text{RFU}}(N) \overset{\text{def}}{=} \max \left\{ \dim_{\text{RFU}}(N,x) \mid x \in \sqrt{\gamma_c(N)} \text{ such that } x \text{ is primitive} \right\}.
\]

When $N$ is an infinite, finitely generated nilpotent where $N/T(N)$ has tame residual dimension at every primitive element $x \in \sqrt{\gamma_c(N/T(N))}$ where $c$ is the step length of $N/T(N)$, we denote $\dim_{\text{RFU}}(N) = \dim_{\text{RFU}}(N/T(N))$.

Let $N$ be a torsion free, finitely generated nilpotent group of step length $c$ with a finite generating subset $S$. Associated to each primitive element $x \in \sqrt{\gamma_c(N)}$, there exists a minimal natural number $DC_{N}(x) > 0$ satisfying the following. If $k$ is a natural number, then there exists a prime number $p \in UR_{N,x}$ such that $p \leq DC_{N}(x) \log(DC_{N}(x)k)$ and where $p \mid k$. For a torsion free, finitely
generated nilpotent group $N$ of step length $c$ with a finite generating subset $S$, we define a function $\text{PDC}_{N,S} : \mathbb{N} \to \mathbb{N}$ as

$$\text{PDC}_{N,S}(n) \overset{\text{def}}{=} \max_{\|x\|_{S} \leq n} \left\{ \text{DC}_{N}(x) \left| x \in \sqrt{\gamma_{c}(N)} \text{ is primitive} \right. \right\}.$$  

We have the following lemma.

**Lemma 4.8.** Let $N$ be a torsion free, finitely generated nilpotent group of step length $c$ with finite generating subsets $S_1$ and $S_2$. Then $\text{PDC}_{N,S_1}(n) \approx \text{PDC}_{N,S_2}(n)$.

**Proof.** There exists a constant $C > 0$ such that any element in $S_1$ can be written in terms of at most $C$ elements of $S_2$. Hence, we have the following inequality:

$$\text{PDC}_{N,S_1}(n) = \max_{\|x\|_{S_1} \leq n} \left\{ \text{DC}_{N}(x) \left| x \in \sqrt{\gamma_{c}(N)} \text{ is primitive} \right. \right\} \leq \max_{\|x\|_{S_2} \leq Cn} \left\{ \text{DC}_{N}(x) \left| x \in \sqrt{\gamma_{c}(N)} \text{ is primitive} \right. \right\} = \text{PDC}_{N,S_2}(Cn).$$  

Therefore, $\text{PDC}_{N,S_1}(n) \preceq \text{PDC}_{N,S_2}(n)$. A similar argument shows that $\text{PDC}_{N,S_1}(n) \succeq \text{PDC}_{N,S_2}(n)$. Thus, $\text{PDC}_{N,S_1}(n) \approx \text{PDC}_{N,S_2}(n)$.  

The following lemma forms the basis for the definition of tame residual dimension.

**Lemma 4.9.** Let $A$ be a torsion free, finitely generated abelian group, and let $x$ be a primitive element. Then $\text{RP}_{A,x,1} = \mathbb{P}$.

**Proof.** Let $p$ be prime number. Since $x$ is primitive, there exists a generating basis $\{z_i\}_{i=1}^{h(A)}$ for $A$ such that $z_1 = x$. Letting $B = \langle z_i \rangle_{i=2}^{h(A)}$, we note that $A/B \cong \mathbb{Z}$ and that $\pi_B(x) \neq 1$. By taking the natural map $\varphi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ given by reduction modulo $p$, one can see that $\varphi \circ \pi_B(x) \neq 1$. Since $\dim_{\mathbb{F}_p}(\mathbb{Z}/p\mathbb{Z}) = 1$, we have that $\dim_{\mathbb{F}_p}(A,x) = 1$, and thus, $p \in \text{RP}_{A,x,1}$ as desired.  

**Definition 4.10.** Suppose that $N$ is a torsion free, finitely generated nilpotent group of step length $c$. If $N$ is abelian, we have by Lemma 4.9 that $N$ has tame residual dimension at all primitive elements. Thus, we say that all torsion free, finitely generated abelian groups have tame residual dimension. If $N$ has step length $c > 1$, we say that $N$ has tame residual dimension if inductively $N/\sqrt{\gamma_{c}(N)}$ has tame residual dimension, $N$ has tame residual dimension at all primitive elements $x \in \sqrt{\gamma_{c}(N)}$, and

$$(\text{PDC}_{N}(n) \log(\text{PDC}_{N}(n) n))^{\dim_{\text{gr}}(N)} \preceq (\log(n))^{\psi_{\text{RF}}(N)}$$

where $\psi_{\text{RF}}(N)$ is the constant from [7, Theorem 1.1].

If $N$ is an infinite, finitely generated nilpotent group such that $N/T(N)$ has tame residual dimension, we say that $N$ has tame residual dimension.

For a torsion free, finitely generated nilpotent group $N$ of step length $c$ with a finite generating subset, we define

$$\vartheta_{N,S}(n) \overset{\text{def}}{=} \max_{1 \leq i \leq c} \left\{ \text{PDC}_{N/\sqrt{\gamma_{c}(N)},S}(n) \right\}.$$
When $N$ is an infinite, finitely generated nilpotent group with a finite generating subset $S$, we set $\mathcal{O}_{N,S}(n) \stackrel{\text{def}}{=} \mathcal{O}_{N/T(N),S}(n)$.

As an immediate application of Lemma 4.8, we have the following lemma.

**Lemma 4.11.** Let $N$ be an infinite, finitely generated nilpotent group with finite generating subsets $S_1$ and $S_2$. Then $\mathcal{O}_{N,S_1}(n) \approx \mathcal{O}_{N,S_2}(n)$.

Thus, whenever we reference the functions $PDC_{N,S}(n)$ and $\mathcal{O}_{N,S}(n)$, we will suppress the choice of generating subset.

We now relate the upper residual dimension of a torsion free, finitely generated nilpotent group to torsion free quotients of lower step length.

**Proposition 4.12.** Let $N$ be a torsion free, finitely generated nilpotent group of step length $c$, and suppose that $N$ has tame residual dimension. If $N$ is abelian, then

$$(\mathcal{O}_{N}(n) \log(\mathcal{O}_{N}(n)n))^{\dim_{RFU}(N)} \approx \log(n).$$

Otherwise, letting $M = \sqrt{\gamma(N)}$, we then have that

$$(\mathcal{O}_{N/M}(n) \log(\mathcal{O}_{N/M}(n)n))^{\dim_{RFU}(N/M)} \preceq (\mathcal{O}_{N}(n) \log(\mathcal{O}_{N}(n)n))^{\dim_{RFU}(N)}$$

**Proof.** The first statement is an easy consequence of Lemma 4.9 and the observation that $\mathcal{O}_{N}(n) < C$ for some constant $C > 0$ when $N$ is a torsion free, finitely generated abelian group. Thus, we may assume that $N$ has step length $c > 1$. The definition of $\mathcal{O}_{N}(n)$ implies that is sufficient to show that $\dim_{RFU}(N) \geq \dim_{RFU}(N/M)$. If $\dim_{RFU}(N/M) = 1$, then there is nothing to show. Thus, we may assume that $\dim_{RFU}(N/M) > 1$.

Let $a \in \gamma(N/M)$ be a primitive element. There exists a primitive element $b \in \sqrt{\gamma(N)}$ such that $\pi_M(b) = a$. Since $b \in \sqrt{\gamma(N)}$, there exists a natural number $s$ such that $b^s \in \gamma(N)$. Thus, there exists some primitive element $g \in N$ such that $[b^s, g]$ is a nontrivial element of $\gamma(N)$. Hence, there exists some primitive element $x \in \gamma(N)$ and a natural number $k$ such that $x^k = [b^s, g]$. Let $p \in UR_{N,x}$ be a prime number that does not divide $k$, and let $\psi : N \to Q$ be an admissible $p$-surjection of $N$ with respect to $x$. Since $\text{Ord}_Q(\phi(x)) \nmid k$, we have that $\phi(x^k) \neq 1$. In particular, $\phi([b^s, g]) \neq 1$. Therefore, $\phi(b) \not\in \gamma(Q)$. Thus, we have an induced homomorphism $\tilde{\phi} : N/M \to Q/\gamma(Q)$ such that $\tilde{\phi}(\pi_M(b)) \neq 1$. Therefore, $\tilde{\phi}(a) \neq 1$. By definition, we have that $\dim_{RFU}(N/M, a) \leq \dim_{RFU}(Q) = \dim_{RFU}(N,x)$. Consider $A_i = \text{RP}_{N,M,a,i} \cap UR_{N,x}$ for $1 \leq i \leq \dim_{RFU}(N,x)$. The previous inequality implies that $\cup_{i=1}^{\dim_{RFU}(N,x)} A_i$. If $\delta(\text{RP}_{N,M,a,i}) = 0$ for each $i$, then we would have that $\delta(A_i) = 0$. In particular, we would have that $\delta(U_{N,x}) = \sum_{i=1}^{\dim_{RFU}(N,x)} \delta(A_i) = 0$ which is a contradiction. Thus, there exists an index $1 \leq i_0 \leq \dim_{RFU}(N/M,a)$ where there exists a subset $F_{N/M,a,i_0} \subseteq \text{RP}_{N,M,a,i_0}$ such that $\delta(F_{N/M,a,i_0}) > 0$. Thus, $\dim_{RFU}(N/M,a) \leq \dim_{RFU}(N,x) \leq \dim_{RFU}(N)$. This inequality holds for all primitive elements of $\sqrt{\gamma(N/M)}$; thus, we have that $\dim_{RFU}(N/M) \leq \dim_{RFU}(N)$.

We finish by giving an explicit inequality that relates the values $\dim_{RFU}(N)$ and $\dim_{RFU}(N)$ for a torsion free, finitely generated nilpotent group $N$.

**Proposition 4.13.** Suppose that $N$ is a torsion free, finitely generated nilpotent group that has tame residual dimension. Then $\dim_{RF}(N) \leq \dim_{RFU}(N)$. 

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Proof. Let $x$ be a primitive nontrivial element of $\sqrt{\gamma(N)}$. Since $\delta(UR_{N,x}) > 0$, we have that $|UR_{N,x}| = \infty$. Thus, $\dim_{RFL}(N,x) \leq \dim_{RFU}(N,x)$. Thus, we have that $\dim_{RFL}(N,x) \leq \dim_{RFU}(N)$. Since this inequality holds for all primitive elements of $\sqrt{\gamma(N)}$, the definition of $\dim_{RFL}(N)$ implies that $\dim_{RFL}(N) \leq \dim_{RFU}(N)$.

4.3 Accessible residual dimension

For a torsion free, finitely generated nilpotent group $N$ that has tame residual dimension, one may be interested in when

$$(\log(n))^{\dim_{RFL}(N)} \approx (\varnothing_N(n) \log(\varnothing_N(n)n))^{\dim_{RFU}(N)}$$

In this case, we would be able to obtain a precise asymptotic characterization of the growth of residual finiteness of a finitely generated nilpotent group. Therefore, we introduce the following definition and proposition.

**Definition 4.14.** Let $N$ be a torsion free, finitely generated nilpotent group of step length $c$ that has tame residual dimension. Let $x \in \sqrt{\gamma(N)}$ be a primitive element such that $\dim_{RFL}(N,x) > 1$. We say that $N$ has low dimension residual vanishing at $x$ if $|RP_{N,x,i}| < \infty$ for $1 \leq i < \dim_{RFU}(N,x)$. If $\dim_{RFU}(N,x) = 1$, we will always say that $N$ has low dimension residual vanishing at $x$. Suppose that $N$ has step length $c = 1$. Since Lemma 4.9 implies that all torsion free, finitely generated abelian groups have low dimension residual vanishing for all primitive elements $x$, we will say that $N$ has accessible residual dimension. Suppose that $N$ has step length $c > 1$. Suppose that inductively $N/\sqrt{\gamma(N)}$ has accessible residual dimension, $N$ has low dimension residual vanishing at all primitive elements $x \in \sqrt{\gamma(N)}$, and $PDC_N(n) \gtrsim C$ for some constant $C > 0$. We then say that $N$ has accessible residual dimension.

If $N$ is an infinite, finitely generated nilpotent group such that $N/T(N)$ has accessible residual dimension, we say that $N$ has accessible residual dimension.

**Proposition 4.15.** Let $N$ be a torsion free, finitely generated nilpotent group that has accessible residual dimension. Then $\dim_{RFL}(N) = \dim_{RFU}(N)$ and

$$(\log(n))^{\dim_{RFL}(N)} \approx (\varnothing_N(n) \log(\varnothing_N(n)n))^{\dim_{RFU}(N)}.$$

Proof. By definition of accessible residual dimension, we have that $\varnothing_N(n) \gtrsim C$. Thus, we need only show for all primitive elements $z \in \sqrt{\gamma(N)}$ that $\dim_{RFL}(N,z) = \dim_{RFL}(N,z)$. Since our statement is evident when $\dim_{RFU}(N,z) = 1$, we may assume that $\dim_{RFU}(N,z) > 1$. Since $\delta(UR_{N,x}) > 0$, we have that $|RP_{N,c,dim_{RFU}(N,z)}| = \infty$. By definition, we have that $|RP_{N,z,i}| < \infty$ for $1 \leq i \leq \dim_{RFU}(N,z) - 1$. Thus, $\dim_{RFL}(N,z) = \dim_{RFU}(N,z)$; hence, $\dim_{RFL}(N) = \dim_{RFU}(N)$.

The above proposition allows us to be able to associate a natural number to any infinite, finitely generated nilpotent group with accessible residual dimension that captures the polynomial in logarithm degree of growth for residual finiteness.
**Definition 4.16.** Let $N$ be a torsion free, finitely generated nilpotent group that has accessible residual dimension. Proposition 4.13 implies that $\dim_{\text{RFU}}(N) = \dim_{\text{RFU}}(N)$. We denote their common value as $\dim_{\text{ARF}}(N)$ and call this value the accessible residual dimension of $N$. When $N$ is an infinite, finitely generated nilpotent group with accessible residual dimension, we set $\dim_{\text{ARF}}(N) \equiv \dim_{\text{ARF}}(N/T(N))$.

### 4.4 Residual finiteness of nilpotent groups with torsion

Before proceed to the upper and lower bounds for residual finiteness of finitely generated nilpotent groups, we have the following proposition. This proposition and its proof are originally found in [2] Proposition 4.4. It relates the effective residual finiteness of an infinite, finitely generated nilpotent group to its torsion free quotient.

**Proposition 4.17.** Let $N$ be an infinite, finitely generated nilpotent group. Then $F_N(n) \approx F_{N/T(N)}(n)$.

**Proof.** We proceed by induction on the order of $T(N)$, and note that the base case is clear. Hence, we may assume that $|T(N)| > 1$. We have that the group morphism $\pi_{Z(T(N))} : N \to N/Z(T(N))$ is surjective with kernel given by $Z(T(N))$ which is a finite central subgroup. Since finitely generated nilpotent groups are linear, [2] Lemma 2.4 implies that $F_N(n) \approx F_{N/Z(T(N))}(n)$. Since $(N/Z(T(N)))/T(N/Z(T(N))) \cong N/T(N)$, the induction hypothesis implies that $F_N(n) \approx F_{N/T(N)}(n)$.

With the above proposition, we may prove the following theorem.

**Theorem 4.1.** Let $N$ be an infinite, finitely generated nilpotent group such that $N/T(N)$ is abelian. Then $F_N(n) \approx \log(n)$.

**Proof.** Proposition 4.17 implies that $F_N(n) \approx F_{N/T(N)}(n)$. We also have that [1] Corollary 2.3 implies $F_{N/T(N)}(n) \approx \log(n)$. Therefore, $F_{N/T(N)}(n) \approx \log(n)$.

### 5 Lower bounds for residual finiteness of finitely generated nilpotent groups

Before proceeding to the lower bound, we give a lower bound for $\dim_{\text{RFL}}(N)$ in terms of the step length of $N$ when $N$ is a torsion free, finitely generated nilpotent group.

**Proposition 5.1.** Let $N$ be a torsion free, finitely generated nilpotent group of step length $c > 1$. Then $\dim_{\text{RFL}}(N) \geq c + 1$.

**Proof.** Let $z \in \sqrt{\chi}(N)$ be a primitive element. Thus, there exists some natural number $k$ such that $z^k \in \chi(N)$. Proposition 4.5 implies that there exist elements $x \in \chi_{-1}(N)$ and $y \in N$ such that $z^k = [x,y]$. Let $\varphi : N \to Q$ be an admissible $p$-surjection of $N$ with respect to $z$ where $p$ is a prime number does not divide $k$. Since $p \nmid k$, we have that $\gcd(k,p) = 1$. In particular, $\langle \varphi(z^k) \rangle = \langle \varphi(z) \rangle$. Since $\varphi(z^k) \neq 1$ and $z^k \in \chi(N)$, $Q$ has the same step length as $N$, and thus, Lemma [4.2] implies that $\dim_{F_p}(Q) \geq c + 1$. 
Let \( A \) be the set of prime numbers that divide \( k \), and let \( B = \bigcup_{i=0}^{h(N)} \text{RP}_{N,i} \). We note that if \( p \in \mathbb{P} \setminus A \), then the above claim implies that \( p \notin B \). Since \( A \) is finite, we must have that \( B \) is infinite and that \( \text{RP}_{N,z} \) is finite for \( 1 \leq t < c \). Thus, there exists a minimal index \( i_0 \) such that \( i_0 \geq c + 1 \) and where \( \text{RP}_{N,z,i_0} \) is infinite. That implies \( \text{dim}_{\text{REFL}}(N,z) \geq c + 1 \). By definition, \( \text{dim}_{\text{REFL}}(N) \geq \text{dim}_{\text{REFL}}(N,z) \geq c + 1 \). Hence, \( \text{dim}_{\text{REFL}}(N) \geq c + 1 \).

We now proceed to the main result of this section.

**Theorem 1.2** Let \( N \) be an infinite, finitely generated nilpotent group such that \( N/T\langle N \rangle \) has step length \( c > 1 \). There exists a natural number \( \text{dim}_{\text{REFL}}(N) \) such that \( \text{dim}_{\text{REFL}}(N) \geq c + 1 \) and where

\[
(\log(n))^{\text{dim}_{\text{REFL}}(N)} \lesssim F_N(n).
\]

**Proof.** We start by assuming that \( N \) is torsion free; hence, \( N \) has step length \( c > 1 \). Let \( x \in \sqrt{\chi}(N) \) be a residual key of \( N \), and let \( k \) be the minimal natural number such that \( x^k \in \chi(N) \).

We may choose a finite generating subset \( \{x_i \}_{i=1}^{h(N)} \) for \( N \) such that \( x_1 = x \). Proposition 5.1 implies that \( \text{dim}_{\text{REFL}}(N,x) \geq c + 1 \), and the definition of \( \text{dim}_{\text{REFL}}(N,x) \) implies that the set \( A_{N,x} = \bigcup_{i=1}^{h(N)-1} \text{RP}_{N,x,i} \) is finite. Thus, we may write \( A_{N,x} = \{q_1 < q_2 < \cdots < q_r \} \) where \( q_i \) are prime numbers for all \( i \). Let \( \{p_j\}_{j=1}^\infty \) be an enumeration of the set \( \{p \in L\mathbb{R}_{N,x} \mid p > \max \{q_i,k\} \} \), and let \( m_j = (\text{lcm} \{1, \cdots, p_j - 1\})^{\text{dim}_{\text{REFL}}(N)+1} \). We claim that \( D_N(x^{km_j}) \approx (\log(\|x^{km_j}\|))^{\text{dim}_{\text{REFL}}(N)} \). [4]

3.B2 implies that \( \|x^{km_j}\| \approx m_j^{1/c} \), and additionally, the Prime Number Theorem [9, 1.2] implies that \( \log(m_j) \approx p_j \). Subsequently, \( \log(\|x^{km_j}\|) \approx p_j \), and thus, \( (\log(\|x^{km_j}\|))^{\text{dim}_{\text{REFL}}(N)} \approx p_j^{\text{dim}_{\text{REFL}}(N)} \).

Hence, we have two tasks. We first need to demonstrate that there exists a surjective group morphism \( \phi : N \to P \) to a finite group \( P \) such that \( |P| = p_j^{\text{dim}_{\text{REFL}}(N)} \) and where \( \phi(x^{km_j}) \neq 1 \). Secondly, we need to demonstrate that if given a surjective group morphism \( \phi : N \to Q \) to a finite group where \( |Q| < p_j^{\text{dim}_{\text{REFL}}(N)} \), then \( \phi(x^{km_j}) = 1 \).

Let \( \psi_j : N \to P_j \) be an admissible \( p_j \)-surjection of \( N \) with respect to \( x \). By definition, \( \psi_j(x) \neq 1 \) and \( \text{dim}_{\text{REFL}}(P_j) = \text{dim}_{\text{REFL}}(N) \). Therefore, \( |P_j| = p_j^{\text{dim}_{\text{REFL}}(N)} \) and \( \text{Ord}_P(\psi_j(x)) = p_j^{t} \) for some \( 1 \leq t \leq \text{dim}_{\text{REFL}}(N) \). Since \( p_j \nmid km_j \), we have that \( \psi_j(x^{km_j}) \neq 1 \) as desired.

Now suppose that \( \phi : N \to Q \) is a surjective group morphism to a finite group where \( |Q| < p_j^{\text{dim}_{\text{REFL}}(N)} \). If \( \phi(x) = 1 \), then \( \phi(x^{km_j}) = 1 \). Therefore, we may assume that \( \phi(x) \neq 1 \). [5] Theorem 2.7 implies we may assume that \( |Q| = q^k \) where \( q \) is some prime number. We have a number of possibilities.

**Case 1:** \( q^k < p_j \).

By construction, \( |Q| \mid m_j \), and since the order of an element divides the order of the group, we have that \( \text{Ord}_Q(\phi(x)) \mid m_j \). Therefore, \( \phi(x^{km_j}) = 1 \).

**Case 2:** \( q < p_j \) and \( p_j < q^k < p_j^{\text{dim}_{\text{REFL}}(N)} \).

There exists a natural number \( v \) such that \( q^v < p_j < q^{v+1} \). Thus, we have that

\[
q^v \text{dim}_{\text{REFL}}(N) < p_j^{\text{dim}_{\text{REFL}}(N)} < q^{(v+1)\text{dim}_{\text{REFL}}(N)}.
\]
Subsequently, we may write $\lambda = \nu \ell + r$ where $\ell \leq \dim_{\text{RFL}}(N)$ and $0 \leq r < \nu$. By assumption, $q^\nu < p_j$. Therefore, we must have that $q^\nu \mid \lcm\{1, \ldots, p_j - 1\}$ and thus,

$$(q^\nu)^\ell \mid (\lcm\{1, \ldots, p_j - 1\})^{\dim_{\text{RFL}}(N)}.$$  

Hence, $q^\lambda \mid m_j$. Since the order of $\varphi(x)$ divides $|Q|$, we have that $\varphi(x) = 1$.

**Case 3:** $q > p_j$ and $\dim_{\psi_q}(N, x) \geq \dim_{\text{RFL}}(N)$.

Since $\varphi(x) \neq 1$, we have that $\lambda \geq \dim_{\psi_q}(N, x)$. In particular, we have that

$$|Q| = q^\lambda \geq q^{\dim_{\psi_q}(N, x)} \geq q^{\dim_{\text{RFL}}(N)} \geq p_j^{\dim_{\text{RFL}}(N)}.$$  

Hence, we may disregard this case.

**Case 4:** $q > p_j$ and $\dim_{\psi_q}(N, x) < \dim_{\text{RFL}}(N)$.

By definition, $q \in A_{N, x}$; however, by the choice of prime numbers $p_j$ we have that $p_j > q$, which is a contradiction. Therefore, this case is not possible, and we may ignore it.

Therefore, $D_N(x^{km_j}) \approx (\log(\|x^{km_j}\|))^{\dim_{\text{RFL}}(N)}$, and thus, $(\log(n))^{\dim_{\text{RFL}}(N)} \lesssim F_N(n)$. Additionally, Proposition 5.1 implies that $\dim_{\text{RFL}}(N) \geq c + 1$.

When $N$ is an infinite, finitely generated nilpotent group where $|T(N)| > 1$, we have by the above arguments that $(\log(n))^{\dim_{\text{RFL}}(N/T(N))} \lesssim F_{N/T(N)}(n)$. We also have that $\dim_{\text{RFL}}(N/T(N)) \geq c + 1$ where $c$ is the step length of $N/T(N)$. Proposition 4.17 implies that $F_N(n) \approx F_{N/T(N)}(n)$ and moreover, it follows from definition that $\dim_{\text{RFL}}(N) = \dim_{\text{RFL}}(N/T(N))$. Therefore, $\dim_{\text{RFL}}(N) \geq c + 1$ and $(\log(n))^{\dim_{\text{RFL}}(N)} \lesssim F_N(n)$.

\[\square\]

6 Upper bounds for residual finiteness for finitely generated nilpotent groups

The main goal of this section is to prove the following theorem.

**Theorem 1.3** Let $N$ be an infinite, finitely generated nilpotent group such that $N/T(N)$. Then

$$F_N(n) \lesssim (\log(n))^{\psi_{\text{RF}}(N)}.$$  

Now suppose that $N$ has tame residual dimension. Then there exists a natural number $\dim_{\text{RFU}}(N)$ satisfying $\dim_{\text{RFU}}(N) \leq \psi_{\text{RF}}(N)$ and a nondecreasing function $\varrho_N(n)$ such that

$$F_N(n) \lesssim (\varrho_N(n) \log(\varrho_N(n)))^{\dim_{\text{RFU}}(N)} \lesssim (\log(n))^{\psi_{\text{RF}}(N)}.$$  

In particular, if $(\varrho_N(n) \log(\varrho_N(n)))^{\dim_{\text{RFU}}(N)} \lesssim (\log(n))^{\psi_{\text{RF}}(N)}$, then $F_N(n)$ grows strictly slower than what is predicted by [27] Theorem 1.1.
In order to make sense of this statement, we need to define the constant $\psi_{RF}(N)$ found in [7] Theorem 1.1. We start with the following proposition which is originally found in [7 Proposition 3.1].

**Proposition 6.1.** Let $N$ be a torsion free, finitely generated nilpotent group, and suppose that $z$ is a primitive central element. There exists a normal subgroup $H \trianglelefteq N$ such that $N/H$ is a torsion free, finitely generated nilpotent group where $\langle \pi_H(z) \rangle \cong Z(N/H)$.

**Proof.** We proceed by induction on Hirsch length to produce a normal subgroup $H \trianglelefteq N$ such that $\langle \pi_H(z) \rangle \cong Z(N/H)$ and where $N/H$ is a torsion free, finitely generated nilpotent group. Since the statement is evident for when $N \cong \mathbb{Z}$, we may suppose that $h(N) > 1$. If $h(Z(N)) = 1$, then as in the base case, the statement is evident. Therefore, we may assume that $h(Z(N)) > 1$.

There exists a generating basis $\{x_i\}_{i=1}^{h(Z(N))}$ for $Z(N)$ such that $x_1 = z$. If consider the subgroup given by $M = \langle x_i \mid i \leq 2 \leq h(Z(N)) \rangle$, induction implies that there exists a normal subgroup $H/M \trianglelefteq N/M$ such that $(N/M)/(H/M)$ is a torsion free, finitely generated nilpotent group such that $\langle \pi_{H/M}(\pi_M(z)) \rangle \cong Z(N/H)/(H/M)$. The third isomorphism theorem implies that $(N/M)/(H/M) \cong N/H$, and subsequently, $Z(N/H) \cong Z((N/M)/(H/M))$. Therefore, it is evident that $H \trianglelefteq N$ is our desired normal subgroup. \qed

With the above proposition, we introduce the following definition.

**Definition 6.2.** Let $N$ be a torsion free, finitely generated nilpotent group of step length $c$ with a primitive element $x \in \sqrt{\gamma_c(N)}$. Proposition 6.1 implies that the value

$$\dim_{RF}(N,x) \overset{\text{def}}{=} \min \{ h(N/H) \mid H \text{ satisfies Proposition 6.1 for } x \}$$

is bounded by $h(N)$. We refer to the value $\dim_{RF}(N,x)$ as the real residual dimension of $N$ with respect to $x$. There exists a primitive element $z \in \sqrt{\gamma_c(N)}$ such that

$$\dim_{RF}(N,z) \overset{\text{def}}{=} \max \left\{ \dim_{RF}(N,x) \mid x \in \sqrt{\gamma_c(N)} \text{ is a primitive nontrivial element} \right\}.$$  

We refer to $\dim_{RF}(N,z)$ as the real residual dimension of $N$ and denote it as $\psi_{RF}(N)$. When $N$ is an infinite, finitely generated nilpotent group, we denote $\psi_{RF}(N) \overset{\text{def}}{=} \psi_{RF}(N/T(N))$.

For a torsion free, finitely generated nilpotent group $N$, Lemma 2.5 implies that the given definition and [7] Definition 3.3] are equivalent. If $N$ additionally satisfies $h(Z(N)) = 1$, then one can see that the definition of $\psi_{RF}(N)$ implies that $\psi_{RF}(N) = h(N)$.

With the above in mind, we now compare the values $\dim_{RFL}(N)$ and $\psi_{RF}(N)$ for torsion free, finitely generated nilpotent groups.

**Proposition 6.3.** Let $N$ be a torsion free, finitely generated nilpotent group that has tame residual dimension. Then $\dim_{RFL}(N) \leq \psi_{RF}(N)$.

**Proof.** If $h(Z(N)) = 1$, we have that $\psi_{RF}(N) = h(N)$. Thus, it follows that $\dim_{RFL}(N) \leq \psi_{RF}(N)$. Hence, we may assume that $h(Z(N)) > 1$. Let $x \in \sqrt{\gamma_c(N)}$ be a primitive element. Proposition 6.1 implies there exists a normal subgroup $H \trianglelefteq N$ such that $h(N/H) = \dim_{RF}(N,x)$. Letting $p$
be a prime number, we have that $\pi_{N^p H}(x) \neq 1$ and that $\dim_{F_{p^i}}(N/N^{p^i} H) = \dim_R(N, x)$. By definition, we have that $\dim_{F_{p^i}}(N, x) \leq \dim_R(N, x)$. Since $h(Z(N)) > 1$, we have that $\text{RP}_N.N_i = \emptyset$ for $\dim_{F_{p^i}}(N, x) < i \leq h(N)$. Thus, there exists an index $i_0$ where $1 \leq i_0 \leq \psi_{RF}(N)$ such that $UR_{N,x} = \text{RP}_{N,x,i_0}$. Therefore, $\dim_{RF}(N, x) \leq \psi_{RF}(N)$. Since this is true independent of primitive elements in $\sqrt{\gamma}(N)$, we have that $\dim_{RF}(N) \leq \psi_{RF}(N)$.

We have the following technical proposition.

**Proposition 6.4.** Let $N$ be a torsion free, finitely generated nilpotent group. If $N$ is abelian, then $\psi_{RF}(N) = 1$. If $N$ has step size $c > 1$, then $\psi_{RF}(N/\sqrt{\gamma}(N)) \leq \psi_{RF}(N)$.

**Proof.** Let $M = \sqrt{\gamma}(N)$. Let $g$ be a primitive element in $\gamma(N)$. There exists a primitive element $x \in \gamma(N)$ such that $\pi_M(x) = g$. Subsequently, there exists a natural number $m$ such that $x^m \in \gamma(N)$. Thus, there exists an element $y \in N$ such that $[x^m, y]$ is a nontrivial element in $\gamma(N)$. Hence, there exists a primitive element $z \in \gamma(N)$ and a natural number $k$ such that $z^k = [x^m, y]$. Let $H \trianglelefteq N$ satisfy Proposition 6.1 for $z$ such that $\dim_R(N, z) = h(N/H)$. By definition, $\pi_H(z) \neq 1$, and thus, $\pi_H(x^m) \neq 1$. Since $\pi_H(x^m) \notin Z(N/H)$ and $\pi_H(M) \leq Z(N/H)$, we have that $\pi_H(x^m) \notin \pi_H(M)$. In particular, $\pi_H(x) \notin \pi_H(M)$. Let $K/M \leq N/M$ satisfy Proposition 6.1 for $\pi_M(x)$ where $\dim_{RF}(N/M, \pi_M(x)) = h(N/M)/(K/M)$. By direct calculation, we have that $H \leq K$; thus, we have that $h(N/H) \geq h(N/M)/(K/M))$. Since $\dim_{RF}(N/M, g) \leq \dim_{RF}(N, g)$, we may write $\dim_{RF}(N/M, g) \leq \dim_{RF}(N)$. Since $g$ is an arbitrary primitive element of $\sqrt{\gamma}(N/M)$, we have by definition that $\dim_{RF}(N/M) \leq \dim_{RF}(N)$.

We now proceed to the proof of Theorem 1.3.

**Proof.** Let us first assume that $N$ is a torsion free, finitely generated nilpotent group of step length $c$. We proceed with the proof of the first statement.

Let $S = \{x_i\}_{i=1}^{h(N)}$ be a Mal’cev basis, and for simplicity, let $M = \sqrt{\gamma}(N)$. Let $g = \prod_{i=1}^{h(N)} x_i^{a_i}$ be a nontrivial element of word length at most $n$.

We proceed by induction on step length, and observe that the base case follows from [1] Corollary 2.3. Thus, we may assume that $N$ has step length $c > 1$.

If $\pi_M(g) \neq 1$, then induction implies that there exists a surjective group morphism to a finite group $\varphi : N/M \to Q_1$ such that $\varphi(g) \neq 1$ and where $|Q_1| \leq C_1 (\log(C_1 n))^{\dim_{RF}(N/M)}$ for some constant $C_1 > 0$. Proposition 4.12 implies that $\dim_{RF}(N/M) \leq \dim_{RF}(N)$. Since $\varphi \circ \pi_M : N \to Q_1$ satisfies $\varphi \circ \pi_M(g) \neq 1$, we have that $D_N(g) \leq C_1 (\log(C_1 n))^{\dim_{RF}(N)}$.

Now suppose that $\pi_M(g) = 1$. That implies that we may write $g = \prod_{i=1}^{h(M)} x_i^{a_i}$, and since $\|g\| \leq n$, Lemma 2.6 implies that there exists a constant $C_2 > 0$ such that $|a_i| \leq C_2 n^c$ for all $i$. There exists a primitive element $z = \prod_{i=1}^{h(M)} x_i^{\beta_i}$ and a nonzero integer $m$ so that $z^m = g$. Let $H \trianglelefteq N$ satisfy Proposition 6.1 for $z$ in $N$ such that $\dim_R(N, z) = h(N/H)$. Since $\{x_1, \ldots, x_{h(M)}\}$ are central elements, we have for all $i$ that $\beta_i m = a_i$. In particular, we have that $|m| \leq C_2 n^c$. By the Prime Number Theorem, there exists a constant $C_3 > 0$ such that $p \leq C_3 \log(C_3 n)$ and where $p \nmid m$. By construction, $\pi_{H, N^p}(g) = \pi_{H, N^p}(z^m) \neq 1$. Thus, there exists a constant $C_4 > 0$ such that

$$D_N(g) \leq |N/H \cdot N^p| = p^{\dim_R(N, x)} \leq \psi_{RF}(N) \leq C_4 (\log(C_4 n))^{\psi_{RF}(N)}.$$
Since all possibilities have been covered, we have that $F_N(n) \preceq (\log(n))^{\psi_{RF}(N)}$ when $N$ is an arbitrary, torsion free, finitely generated nilpotent group.

We now assume that $N$ is a torsion free, finitely generated nilpotent group with tame residual dimension. As before, we may proceed by induction on step length, and observe that the base case follows from Corollary 2.3. Thus, we may assume that $c > 1$. If $\pi_M(g) \neq 1$, then by the inductive hypothesis applied to $N/M$, there exists a surjective group morphism $\psi : N/M \to Q_1$ such that $\psi(\pi_M(g)) \neq 1$ and where

$$|Q_1| \leq (\Theta_N(C_5 n) \log(C_5 \Theta_{N/M}(C_5 n))^{\dim_{RFU}(N/M)}.$$

Proposition 4.12 implies that there exists a constant $C_6 > 0$ such that

$$(\Theta_{N/M}(n) \log(\Theta_{N/M}(n)))^{\dim_{RFU}(N/M)} \preceq (C_6 \Theta_N(C_6 n) \log(\Theta_N(C_6 n))^{\dim_{RFU}(N)},$$

and thus, there exists some constant $C_7 > 0$ such that

$$D_N(g) \leq (\Theta_N(C_7 n) \log(\Theta_N(C_7 n))^{\dim_{RFU}(N)}.$$

Therefore, we may assume that $\pi_M(g) = 1$. That implies we may write $g = \prod_{i=1}^{h(M)} x_i^{\alpha_i}$. Proposition 2.6 implies that there exists a constant $C_8 > 0$ such that $|\alpha_i| \leq C_8 n^c$ for all $1 \leq i \leq h(M)$. There exists a primitive element $z \in \sqrt{\mathcal{N}}$ such that $z^m = g$ for some nonzero integer $m$. Writing $z = \prod_{i=1}^{h(M)} x_i^{\beta_i}$ and noting that $\{x_1, \ldots, x_{h(M)}\}$ are central elements, we have that $\beta_i m = \alpha_i$ for all $i$. We have that $|m| \leq |m\beta_i| = |\alpha_i| \leq C_8 n^c$. Since $\delta(UR_{N,z}) > 0$, the Prime Number Theorem implies that there exists a prime number $p \leq UR_{N,z}$ such that $\log(\mathcal{D}_N(z)|m|)$ and where $p \nmid |m|$. Let $\psi : N \to P$ be an admissible $p$-sujection of $N$ with respect to $z$. We have that $|P| \leq p^{\dim_{RFU}(N)}$ and that $\psi(z) \neq 1$. Since $p \nmid m$, we have that $\text{Ord}_p(\psi(z)) \nmid |m|$. In particular, $\psi(g) = \psi(z^m) \neq 1$. Thus, there exists a constant $C_9 > 0$ such that $p \leq \mathcal{D}_N(z) \log(C_9 \mathcal{D}_N(n))$. Since $||z|| \leq n$, we have that $\mathcal{D}_N(z) \leq \Theta_N(n)$. Subsequently, $D_N(g) \leq (\Theta_N(n) \log(C_8 \Theta_N(n)))^{\dim_{RFU}(N)}$. Thus, $F_N(n) \preceq (\Theta_N(n) \log(\Theta_N(n)n))^{\dim_{RFU}(N)}$.

Now suppose that $N$ is an infinite, finitely generated nilpotent group where $|T(N)| > 1$. By definition, $\psi_{RF}(N) = \psi_{RF}(N/T(N))$. Thus, we have that $F_{N/T(N)}(n) \preceq (\log(n))^{\psi_{RF}(N)}$ by the above arguments. Proposition 4.17 implies that $F_N(n) \approx F_{N/T(N)}(n) \preceq (\log(n))^{\psi_{RF}(N)}$.

Now suppose that $N$ is an infinite, finitely generated nilpotent group that has tame residual dimension where $|T(N)| > 1$. As before, we note that $\dim_{RFU}(N) = \dim_{RFU}(N/T(N))$ by definition. The above arguments and Proposition 4.17 imply that we may write $F_N(n) \approx F_{N/T(N)}(n) \preceq (\Theta_N(n) \log(\Theta_N(n)n))^{\dim_{RFU}(N)}$.

\section{Residual finiteness of finitely generated nilpotent groups with accessible residual dimension}

We now prove Theorem 1.4.
Theorem [1.4] Let $N$ be an infinite, finitely generated nilpotent group such that $N/T(N)$ has step length $c > 1$, and suppose that $N$ has accessible residual dimension. Then there exists a natural number $\dim_{\text{ARF}}(N)$ such that $c + 1 \leq \dim_{\text{ARF}}(N) \leq \psi_{\text{RF}}(N)$ and where

$$F_N(n) \approx (\log(n))^{\dim_{\text{ARF}}(N)}.$$  

Proof. By Theorem [1.3] we have that $F_{N/T(N)}(n) \lesssim (\Theta_N(n) \log(\Theta_N(n)))^{\dim_{\text{RFU}}(N/T(N))}$ and that $\dim_{\text{RFU}}(N/T(N)) \leq \psi_{\text{RF}}(N/T(N))$. Theorem [1.2] implies that $(\log(n))^{\dim_{\text{RFU}}(N/T(N))} \lesssim F_{N/T(N)}(n)$ and that $c + 1 \leq \dim_{\text{RFU}}(N/T(N))$. Proposition [4.17] implies that $(\log(n))^{\dim_{\text{ARF}}(N)} \approx (\Theta_N(n) \log(\Theta_N(n)))^{\dim_{\text{ARF}}(N)}$

and that $\dim_{\text{ARF}}(N/T(N)) = \dim_{\text{RFU}}(N/T(N)) = \dim_{\text{RFU}}(N/T(N))$. Subsequently, $F_{N/T(N)}(n) \approx (\log(n))^{\dim_{\text{ARF}}(N/T(N))}$. Proposition [4.17] implies that $F_N(n) \approx F_{N/T(N)}(n)$, and by definition, we have that $\dim_{\text{ARF}}(N) = \dim_{\text{ARF}}(N/T(N))$ and $\psi_{\text{RF}}(N) = \psi_{\text{RF}}(N/T(N))$. Subsequently, $c + 1 \leq \dim_{\text{ARF}}(N) \leq \psi_{\text{RF}}(N)$; hence, $F_N(n) \approx (\log(n))^{\dim_{\text{ARF}}(N)}$.

\[\Box\]

8 Residual Conjugacy Dimension

We use the above notions of $\mathbb{F}_p$-dimension of finite $p$-groups and conjugacy in the integral Heisenberg group to study effective conjugacy separability of finitely generated nilpotent groups.

8.1 Admissible 4-tuples and conjugacy

Definition 8.1. Let $N$ be a torsion free, finitely generated nilpotent group of step length $c > 1$. Let $g \in \sqrt{\gamma_1}(N)$ be a primitive element. There exists a natural number $m$ such that $g^m \in \gamma_1(N)$. Additionally, Proposition [2.1] implies that there exist elements $a \in \gamma_{-1}(N)$ and $b \in N$ such that $[a,b] = g^m$. We call $(g,m,a,b)$ an admissible 4-tuple.

One can then see that $(a,b,[a,b])$ is isomorphic to the 3-dimensional integral Heisenberg group. Let $p$ be a prime number such that $[a,b] \notin N^p$. Since $\pi_p(a^p[a,b])$ and $\pi_p(a^p[a,b]^2)$ are non-equal central elements in $N/N^p$, Proposition [2.7] for all surjective group morphisms $\psi : N \to Q$ to finite $q$-groups where $q$ is a prime number distinct from $p$ that $\psi(a^p[a,b]) \sim \psi(a^p[a,b]^2)$. Thus, we have a natural notion of $\mathbb{F}_p$-dimension associated to the 4-tuple $(g,m,a,b)$ which measures the difficulty of separating the conjugacy classes of $a^p[a,b]$ and $a^p[a,b]^2$ using surjective group morphisms to finite $p$-groups. We denote

$$\dim_{\text{Conj,\,F}_p}(g,m,a,b) \overset{\text{def}}{=} \min \left\{ \dim_{\mathbb{F}_p}(Q) \mid \exists \varphi : N \to Q \text{ s.t. } \varphi(a^p[a,b]) \sim \varphi(a^p[a,b]^2) \right\}.$$  

Definition 8.2. We call $\dim_{\text{Conj,\,F}_p}(g,m,a,b)$ the residual $\mathbb{F}_p$-conjugacy dimension of $(g,k,a,b)$. When $[a,b] \in N^p$, we define $\dim_{\text{Conj,\,F}_p}(g,k,a,b) = 0$.

When $\dim_{\text{Conj,\,F}_p}(g,m,a,b) \neq 0$, it is easy to see that $\dim_{\mathbb{F}_p}(N,g) \leq \dim_{\text{Conj,\,F}_p}(g,m,a,b) \leq h(N)$.
For $0 \leq i \leq h(N)$, we define

$$RCP_{N,(g,m,a,b),i} \overset{\text{def}}{=} \{ p \mid \dim_{\text{Conj}} F_p (g,m,a,b) = i \}.$$ 

We call the collection $RCP_{N,(g,m,a,b),i}$ the set of residual conjugacy primes of $N$ with respect to $(g,m,a,b)$ of dimension $i$. Since the set of prime numbers $p$ where $[a,b] \in N^p$ is finite, we have that $|RCP_{N,(g,m,a,b),0}| < \infty$. There exists a maximal index $i_0$ such that $|RCP_{N,(g,m,a,b)}| = \infty$. We denote this value as $\dim_{\text{Conj}} (g,m,a,b) = i_0$ and call it the residual conjugacy dimension of $(g,m,a,b)$. We define the collection $LC_{N,(g,m,a,b)} = RCP_{N,(g,m,a,b),\dim_{\text{Conj}}(g,m,a,b)}$ as the set of prime numbers that realize the residual conjugacy dimension of $(g,m,a,b)$. For each $p \in LC_{N,(g,m,a,b)}$, there exists a surjective group morphism $\varphi : N \to Q$ to a finite $p$-group such that $\dim_{\text{Conj}} (N) = \dim_{\text{F}_p(Q)}$ and $\varphi (a^p [a,b]) \approx \varphi (a^p [a,b]^2)$. We say that $\varphi : N \to Q$ realizes the residual $\mathbb{F}_p$-conjugacy dimension of $N$ with respect to the admissible 4-tuple $(g,m,a,b)$.

Finally, we denote

$$\dim_{\text{Conj}} (N) \overset{\text{def}}{=} \max \{ \dim_{\text{Conj}} (g,m,a,b) \mid (g,m,a,b) \text{ is an admissible 4-tuple} \}.$$ 

We call $\dim_{\text{Conj}} (N)$ the residual conjugacy dimension of $N$. By definition, there exists an admissible 4-tuple $(g,m,a,b)$ such that $\dim_{\text{Conj}} (N) = \dim_{\text{Conj}} (g,m,a,b)$. We call such an admissible 4-tuple $(g,m,a,b)$ a conjugacy key.

When $N$ is an infinite, finitely generated nilpotent group where $|T(N)| > 1$, we have that $N^{[T(N)\ell]}$ is a torsion free, finitely generated nilpotent group. In this case, we set $\dim_{\text{Conj}} (N) = \dim_{\text{Conj}} (N^{[T(N)\ell]})$.

We finish this section with the following.

**Proposition 8.3.** Let $N$ be a torsion free, finitely generated nilpotent group of step length $c > 1$ that is not virtually abelian. Then $c + 1 \leq \dim_{\text{RFL}} (N) \leq \dim_{\text{Conj}} (N) \leq \psi_{\text{RF}} (N)$.

**Proof.** Let $(g,m,a,b)$ be an admissible 4-tuple. Proposition 6.1 implies that there exists a normal subgroup $H \trianglelefteq N$ such that $h(N/H) = \dim_{\mathbb{R}} (N,g)$ and where $N/H$ is a torsion free, finitely generated nilpotent group satisfying $\langle \pi_h (g) \rangle \approx Z(N/H)$. Let $p \in LC_{N,(g,m,a,b)}$. By definition, we have that $[a,b] \notin N^p$; thus, $\pi_{N^p,H}([a,b]) \neq 1$. Subsequently, $\pi_{N^p,H} (a^p [a,b])$ and $\pi_{N^p,H} (a [a,b]^2)$ are non-equal central elements in $N/N^p \cdot H$. That implies $\pi_{N^p,H} (a^p [a,b]) \approx \pi_{N^p,H} (a^p [a,b]^2)$. By definition, we have that $\dim_{\text{Conj},\mathbb{F}_p} (g,m,a,b) \leq \dim_{\mathbb{F}_p} (N/N^p : H)$ for all prime numbers $p$ where $[a,b] \notin N^p$, we have that $\dim_{\text{Conj}} (g,m,a,b) \leq \psi_{\text{RF}} (N)$. By maximizing over all admissible 4-tuples, we may write $\dim_{\text{Conj}} (N) \leq \psi_{\text{RF}} (N)$.

Now suppose that $g \in \sqrt{\gamma (N)}$ is a primitive element. Lemma 2.3 implies that there exists a natural number $m$ and elements $a \in \gamma_{-1} (N)$ and $b \in N$ such that $g^m = [a,b]$. In particular, we have that $(g,m,a,b)$ is an admissible 4-tuple. If $p$ is a prime number such that $[a,b] \notin N^p$, then $\dim_{\text{Conj},\mathbb{F}_p} (g,m,a,b) \neq 0$; moreover, $\dim_{\mathbb{F}_p} (N,g) \leq \dim_{\text{Conj},\mathbb{F}_p} (g,m,a,b)$. Since this equality holds for all but finitely many prime numbers, it follows from definition that $\dim_{\text{RFL}} (N,g) \leq \dim_{\text{Conj}} (g,m,a,b) \leq \dim_{\text{Conj}} (N)$. Since $g$ is an arbitrary primitive element of $\sqrt{\gamma (N)}$, we have that $\dim_{\text{RFL}} (N) \leq \dim_{\text{Conj}} (N)$. Proposition 5.1 implies that $c + 1 \leq \dim_{\text{RFL}} (N)$. Therefore, $c + 1 \leq \dim_{\text{RFL}} (N) \leq \dim_{\text{Conj}} (N) \leq \psi_{\text{RF}} (N)$.

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9 Proof of Theorem 1.5

In this section, we prove the following theorem.

Theorem 1.5 Let $N$ be an infinite, finitely generated nilpotent group that is not virtually abelian, and suppose that $N^{[T(N)]}$ has step length $c$. Then there exists a natural number $\dim_{\text{Conj}}(N)$ such that $c+1 \leq \dim_{\text{Conj}}(N) \leq \psi_{RF}(N)$ and where

$$n^{(c-1)\dim_{\text{Conj}}(N)} \lesssim \text{Conj}_{N}(n).$$

Proof. We first assume that $N$ is a torsion free, finitely generated nilpotent group. Since $N$ is not virtually abelian, we must have that $N$ has step length $c > 1$. Let $S$ be a finite generating subset of $N$, and let $(g,k,a,b)$ be a conjugacy key of $N$. [4, 3.B2] implies that $||a^p [a,b]||_S, ||a^p [a,b]^2||_S \approx p^{1/(c-1)}$. For prime numbers $p \in \text{LC}_N(g,m,a,b)$, we have that $\pi_p(a^p [a,b])$ and $\pi_p(a^p [a,b]^2)$ are non-equal central elements of $N/N^p$. Thus, $a^p [a,b] \sim a^p [a,b]^2$. Let $\varphi_p : N \to Q_p$ realize the residual $\mathbb{F}_p$-conjugacy dimension of $N$ with respect to $(g,m,a,b)$. We have that $\varphi_p(a^p [a,b]) \sim \varphi_p(a^p [a,b]^2)$, and we may write

$$(||a^p [a,b]||_S)^{(c-1)\dim_{\text{Conj}}(N)}, (||a^p [a,b]^2||_S)^{(c-1)\dim_{\text{Conj}}(N)} \approx \left(p^{1/(c-1)}\right)^{(c-1)\dim_{\text{Conj}}(N)} = p^{\dim_{\text{Conj}}(Q_p)}.$$ 

We claim that

$$\text{CD}_N(a^p [a,b], a^p [a,b]^2) = (||a^p [a,b]||_S)^{(c-1)\dim_{\text{Conj}}(N)}, (||a^p [a,b]^2||_S)^{(c-1)\dim_{\text{Conj}}(N)}.$$ 

To proceed, we need to demonstrate if given a surjective group morphism $\varphi : N \to Q$ to a finite group where $|Q| < |Q_p|$, then $\varphi(a^p [a,b]) \sim \varphi(a^p [a,b]^2)$. [5, Theorem 2.7] implies that we may assume that $Q$ is a $q$-group where $q$ is a prime number. Additionally, if $\varphi([a,b]) = 1$, then $\varphi(a^p [a,b]) = \varphi(a^p [a,b]^2)$. Thus, we may assume that $\varphi([a,b]) \neq 1$. Since $\varphi([a,b]) \neq 1$, it follows that $\varphi(a), \varphi(b) \neq 1$.

Suppose that $q \neq p$. Restricting $\varphi$ to $\langle a, [a,b] \rangle$ gives us a group morphism $\psi : H(\mathbb{Z}) \to Q$. Proposition 2.7 implies that there exists an element $g \in \langle a, [a,b] \rangle$ such that $\psi(g a^p [a,b] g^{-1}) = \psi(a^p [a,b]^2)$. In particular, $\varphi(a^p [a,b]) \sim \varphi(a^p [a,b]^2)$. Therefore, we may assume that $p = q$. By assumption, $\dim_{\text{Conj}}(Q) < \dim_{\text{Conj}}(Q_p) = \dim_{\text{Conj}}(F_p(g,m,a,b))$, and thus, we have that $\varphi(a^p [a,b]) \sim \varphi(a^p [a,b]^2)$ as desired. Therefore, $\text{CD}_N(a^p [a,b], a^p [a,b]^2) = p^{\dim_{\text{Conj}}(Q_p)}$. In particular, we have that $n^{(c-1)\dim_{\text{Conj}}(N)} \lesssim \text{Conj}_{N}(n)$.

We now assume that $N$ is an infinite, finitely generated nilpotent group that is not virtually abelian. Let $(g,m,a,b)$ be a conjugacy key of $N^{[T(N)]}$. Using the conjugacy key $(g,m,a,b)$, we will construct an infinite sequence of elements in $N^{[T(N)]}$ that are non-conjugate in $N$ where we may apply the above arguments to. Given that $\pi_{T(N)}(N^{[T(N)]})$ is a finite index subgroup of $N/T(N)$, we have that $Z(\pi_{T(N)}(N^{[T(N)]})) \cong \pi_{T(N)}(N^{[T(N)]}) \cap Z(N/T(N))$. Thus, $\pi_{T(N)}([a,b])$ is a nontrivial element of $Z(N/T(N))$. Hence, the set $E = \{ p \in \text{LC}_N(g,m,a,b) \mid \pi_{T(N)}([a,b]) \notin (N/T(N))^p \}$ is an infinite set. Consider the elements given by $a^p [a,b]$ and $a^p [a,b]^2$ for $p \in E$. We claim that $a^p [a,b] \sim a^p [a,b]^2$ in $N$. We have that $\pi_{T(N),N^p}(a^p [a,b]) = \pi_{T(N),N^p}(a^p [a,b]^2)$. Since $\pi_{T(N)}([a,b]) \in Z(N/T(N))$, the elements given by $\pi_{T(N),N^p}(a^p [a,b])$ and $\pi_{T(N),N^p}(a^p [a,b]^2)$ are non-equal central elements of $N/T(N) \cdot N^p$. Thus, we have our claim.
Let $S$ and $S'$ be generating subsets of $N$ and $N_{T(N)}$, respectively. Since $\|a^p [a, b]\|_S \approx \|a^p [a, b]\|_{S'}$ and $\|a^p [a, b]^2\|_S \approx \|a^p [a, b]^2\|_{S'}$, we have by prior arguments that $\|a^p [a, b]\|_{S'}, \|a^p [a, b]^2\|_{S'} \approx p^{1/(c-1)}$ where $c$ is the step length of $N_{T(N)}$. We also have that $\text{CD}_{N_{T(N)}}(a^p [a, b], a^p [a, b]^2) = p^{\dim_{\text{Conj}}(N_{T(N)})}$. In particular,

$$\text{CD}_{N_{T(N)}}(a^p [a, b], a^p [a, b]^2) \approx (\|a^p [a, b]\|_S)^{(c-1) \cdot \dim_{\text{Conj}}(N_{T(N)})} (\|a^p [a, b]^2\|_S)^{(c-1) \cdot \dim_{\text{Conj}}(N_{T(N)})}.$$ 

Lemma 2.3 implies that $\text{CD}_{N_{T(N)}}(a^p [a, b], a^p [a, b]^2) \leq \text{CD}_N(a^p [a, b], a^p [a, b]^2)$. Thus,

$$(\|a^p [a, b]\|_S)^{(c-1) \cdot \dim_{\text{Conj}}(N_{T(N)})} (\|a^p [a, b]^2\|_S)^{(c-1) \cdot \dim_{\text{Conj}}(N_{T(N)})} \lesssim \text{CD}_N(a^p [a, b], a^p [a, b]^2).$$

By definition, $\dim_{\text{Conj}}(N) = \dim_{\text{Conj}}(N_{T(N)})$, and therefore, $n^{(c-1) \cdot \dim_{\text{Conj}}(N)} \lesssim \text{Conj}_N(n)$. 

10 Effective separability of the Heisenberg group

In this last section, we explicitly compute effective residual finiteness and conjugacy separability for the integral Heisenberg group. We start with the following lemma.

**Lemma 10.1.** Let $p$ be prime number. Then $\dim_{\varphi_p}(H(\mathbb{Z}), z) = 3$.

**Proof.** Under the group morphism $\pi_p : H(\mathbb{Z}) \to H(\mathbb{Z})/(H(\mathbb{Z}))^p$, we have that $\pi_p(z) \neq 1$ and $\dim_{\varphi_p}(H(\mathbb{Z})/(H(\mathbb{Z}))^p) = 3$. Therefore, $\dim_{\varphi_p}(H(\mathbb{Z}), z) \leq 3$. Now suppose that $\varphi : H(\mathbb{Z}) \to Q$ is a surjective group morphism to a finite $p$-group where $\varphi(z) \neq 1$. Since $\varphi(z) \neq 1$, we have that $\varphi(x), \varphi(y) \neq 1$, and in particular, $\varphi(x), \varphi(y) \notin Z(Q)$. Thus, $Q$ is not abelian. Since all groups of order $p$ and $p^2$ are abelian, we have that $\text{Ord}(Q) \geq p^3$. In particular, $\dim_{\varphi_p}(Q) \geq 3$. Thus, $\dim_{\varphi_p}(H(\mathbb{Z}), z) = 3$.

With the above lemma, we now calculate $F_{H(\mathbb{Z})}(n)$.

**Proposition 10.2.** $F_{H(\mathbb{Z})}(n) \approx (\log(n))^3$.

**Proof.** Since $h(Z(H(\mathbb{Z}))) = 1$, it is straightforward to see that $\psi_{\text{RF}}(H(\mathbb{Z})) = 3$. Additionally, we note that the only primitive elements of $Z(H(\mathbb{Z}))$ are given by $z$ and $z^{-1}$. Since $\dim_{\varphi_p}(H(\mathbb{Z}), z) = 3$ for all prime numbers $p$, we have that $\text{RP}_{H(\mathbb{Z}), z, 3} = \mathbb{P}$. Moreover, it follows that $\mathcal{O}_{H(\mathbb{Z})}(n) \leq C$ for some constant $C > 0$. Since $\text{RP}_{H(\mathbb{Z}), z^{-1}, 3} = \text{RP}_{H(\mathbb{Z}), z, 3} = \mathbb{P}$, we have that $H(\mathbb{Z})$ has accessible residual dimension. Since $H(\mathbb{Z})$ has step length $2$, we have by Theorem 1.4 that $3 \leq \dim_{\text{ARF}}(H(\mathbb{Z})) \leq 3$ and that $F_{H(\mathbb{Z})}(n) \approx (\log(n))^3$. Thus, $F_{H(\mathbb{Z})}(n) \approx (\log(n))^3$. 

We finish with the computation of $\text{Conj}_{H(\mathbb{Z})}(n)$.

**Proposition 10.3.** $\text{Conj}_{H(\mathbb{Z})}(n) \approx n^3$.

**Proof.** As noted in the calculation $F_{H(\mathbb{Z})}(n)$, we have that $\psi_{\text{RF}}(H(\mathbb{Z})) = 3$. Since $H(\mathbb{Z})$ has step length $2$, Proposition 8.3 implies that $\dim_{\text{Conj}}(H(\mathbb{Z})) = 3$. Theorem 1.5 implies that $n^3 \lesssim \text{Conj}_{H(\mathbb{Z})}(n)$. 

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For the upper bound, we let \( g \) and \( h \) be two non-conjugate elements of word length at most \( n \). If \( \pi_{p,(\mathbb{Z})}(g) \neq \pi_{p,(\mathbb{Z})}(h) \), we have that \( \pi_{p,(\mathbb{Z})}(g) \sim \pi_{p,(\mathbb{Z})}(h) \) since \( \mathbb{H}/\mathbb{G}(\mathbb{Z}) \) is abelian. Theorem 1.2 implies that there exists a constant \( C_1 > 0 \) and a surjective group morphism \( \varphi : \mathbb{H}/\mathbb{G}(\mathbb{Z}) \to Q \) such that \( \varphi(\pi_{p,(\mathbb{Z})}(g h^{-1})) \neq 1 \) and where \( |Q| \leq C_1 \log(C_1 n) \). In particular, \( \varphi(\pi_{p,(\mathbb{Z})}(g)) \) and \( \varphi(\pi_{p,(\mathbb{Z})}(h)) \) are non-equal central elements of \( Q \). Thus, \( \varphi(\pi_{p,(\mathbb{Z})}(g)) \sim \varphi(\pi_{p,(\mathbb{Z})}(h)) \) and \( CD_{\mathbb{H}(\mathbb{Z})}(g,h) \leq C_1 \log(C_1 n) \).

Now suppose that \( \pi_{p,(\mathbb{Z})}(g) = \pi_{p,(\mathbb{Z})}(h) \). In particular, we may write \( g = x^{a_1} y^{a_2} z^{a_3} \) and \( h = g^a \) where \( |a_1|, |a_2| \leq C_2 n \) and \( |a| \leq C_2 n^2 \) for some constant \( C_2 > 0 \). Let \( p^\omega \) be a prime power that divides \( \gcd(|a_1|,|a_2|) \), but does not divide \( t \). We claim that \( \pi_{p^\omega}(g) \sim \pi_{p^\omega}(h) \), and for a contradiction, suppose otherwise. Thus, there exists an element \( a \in \mathbb{H}(\mathbb{Z}) \) such that \( \pi_{p^\omega}(a^{-1} g a) = \pi_{p^\omega}(g^a) \). By the explicit description of the conjugacy class of \([g]\), we have that \( t \in \{ \gcd(|a_1|,|a_2|) k | k \in \mathbb{Z} \} \mod p^\omega \). Thus, there exist integers \( \ell, s \) such that \( t = \ell \gcd(|a_1|,|a_2|) + sp^\omega \). Since \( p^\omega | \gcd(|a_1|,|a_2|) \), we have that \( p^\omega | t \) which is a contradiction. Therefore, \( \pi_{p^\omega}(g) \sim \pi_{p^\omega}(h) \).

When \( \gcd(|a_1|,|a_2|) \neq 0 \), we have that \( p^\omega \leq \gcd(|a_1|,|a_2|) \leq C_2 n \). In this case, we have that \( CD_{\mathbb{H}(\mathbb{Z})}(g,h) \leq C_3 n^3 \). When \( \gcd(|a_1|,|a_2|) = 0 \), the Prime Number Theorem implies there exists some prime number \( p \) such that \( p \nmid t \) and where \( p \leq C_3 \log(C_1 |t|) \) for some constant \( C_3 > 0 \). In particular, we have in this case that \( CD_{\mathbb{H}(\mathbb{Z})}(g,h) \leq C_4 (\log(C_4 n)) \) for some constant \( C_4 > 0 \). Taking everything together, we have that \( \text{Conj}_{\mathbb{H}(\mathbb{Z})}(n) \lesssim n^3 \), and thus, \( \text{Conj}_{\mathbb{H}(\mathbb{Z})}(n) \approx n^3 \).

### 11 Further Questions

There are some questions of interest to us regarding the asymptotic behavior of residual finiteness for a general infinite, finitely generated nilpotent groups. The first question regards whether we can extend our methods to all finitely generated nilpotent groups.

**Question 1.** Let \( N \) be an infinite, finitely generated nilpotent group. Does \( N \) have tame residual dimension? Does every infinite, finitely generated nilpotent group that has tame residual dimension have accessible residual dimension?

If the above questions have positive answers, then we would have a complete asymptotic characterization of \( F_N(n) \) for all finitely generated nilpotent groups. Currently we have no approach to these questions.

It may be the case that these questions do not have positive answers. A natural question to then ask is the following.

**Question 2.** Does there exist a finitely generated nilpotent group \( N \) such that \( F_N(n) \not\approx (\log(n))^k \) for all natural numbers \( k \)? Does there exist a finitely generated nilpotent group \( N \) where \( F_N(n) \approx (\log(n))^r \) where \( r \) is a non-integer rational number greater than 1?

The last question to possibly look into would be to see if the methods introduced here could be adapted to the more general situation of polycyclic groups.

**Question 3.** Let \( N \) be a polycyclic group that is not virtually nilpotent. Does there exist a natural number \( k \) such that \( F_N(n) \approx n^k \)?
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