COMPARISON OF INFORMATION STRUCTURES AND COMPLETELY POSITIVE MAPS

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Abstract. A theorem of Blackwell about comparison between information structures in classical statistics is given an analogue in the quantum probabilistic setup. The theorem provides an operational interpretation for trace-preserving completely positive maps, which are the natural quantum analogue of classical stochastic maps. The proof of the theorem relies on the separation theorem for convex sets and on quantum teleportation.

1. Introduction

Consider an observer with access to a quantum particle $S$ which is entangled with another quantum particle $N$. Let $\mathcal{H}_S$ and $\mathcal{H}_N$ be the corresponding Hilbert spaces and $\Phi$ the density operator over $\mathcal{H}_N \otimes \mathcal{H}_S$ that represents the state of the bipartite system. This paper considers the information that the observer can garner about $N$ via measurements over $S$. We call the triple $(\mathcal{H}_N, \mathcal{H}_S, \Phi)$ an information structure.

The approach follows Blackwell’s similar analysis in classical statistics ([3]). The analysis is comparative. Given a pair of information structures, $(\mathcal{H}_N, \mathcal{H}_S, \Phi)$ and $(\mathcal{H}_N, \mathcal{H}_T, \Psi)$, we seek for a condition under which $(\mathcal{H}_N, \mathcal{H}_S, \Phi)$ can be said to be more informative than $(\mathcal{H}_N, \mathcal{H}_T, \Psi)$. The concept of being ‘more informative’ is understood operationally, that is in terms of the payoffs that the observer can expect in a certain class of games. We consider the two scenarios that correspond to these structures: In the first scenario the observer has access to a particle $S$ such that the joint state of $N$ and $S$ is $\Phi$, and in the second scenario he has access to a particle $T$ such that the joint state of $N$ and $T$ is $\Psi$. In both scenarios, the observer is assumed to be engaged in some decision problem, or game, in which he has to choose an action depending on his estimation about the outcome of future measurements over $N$. We say that the information structure $(\mathcal{H}_N, \mathcal{H}_S, \Phi)$ is better then the information structure $(\mathcal{H}_N, \mathcal{H}_T, \Psi)$ if, for every possible game, the expected payoff for the observer in the first scenario is at least as good as the expected payoff in the second scenario.

Before delving into formal definitions, consider the following example. Assume that all particles have spin $\frac{1}{2}$, and that

$$\Phi = \frac{1}{2}(\ket{01}\bra{01} - \ket{10}\bra{01} - \ket{01}\bra{10} + \ket{10}\bra{10})$$

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is the density matrix of a singlet and that
\[
\Psi = \frac{1}{4}(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|)
\]
is the density matrix of two independent random states. Suppose for example that
the observer is involved in the following game. After performing measurements over
his particle, he has to guess the component of the spin of \(N\) along axis \(\hat{n}\) for some
fixed unit vector \(\hat{n} \in \mathbb{R}^3\). His payoff +1 for a correct guess and −1 for a wrong
guess. In the first scenario, when the observer has access to a particle \(S\) such that
the composite system of \(N\) and \(S\) is at state \(\Phi\), he can guarantee payoff +1 by
measuring the spin of \(S\) along \(\hat{n}\) (which is, with probability one, in the opposite
direction to the spin of \(N\).) On the other hand, in the second scenario, when the
observer has access to a particle \(T\) such that the composite system of \(N\) and \(T\) is
at state \(\Psi\), measuring \(T\) will give him no help, and, whatever strategy he uses for
his guess, his expected payoff is zero. In fact the situation described by \(\Phi\) is better
than the situation described by \(\Psi\), in the sense that in every “game” of this type
—a formal definition of game is given below— the observer can do better (in a weak
sense) in the former situation.

The advantage of \(\Phi\) over \(\Psi\) is also reflected in the fact that an observer who has
an access to a particle \(S\) such that the composite system of \(N\) and \(S\) is at state \(\Phi\)
can perform physical manipulations on \(S\) that transform the state of the bipartite
system to \(\Psi\). To do that, he applies \(S\) to the completely depolarizing map given by

\[
\rho \mapsto \frac{1}{4} \sum_{\mu=0}^{3} \sigma_\mu \rho \sigma_\mu,
\]
where
\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Essentially, Theorem 2 says that if \((\mathcal{H}_N, \mathcal{H}_S, \Phi)\) is better than \((\mathcal{H}_N, \mathcal{H}_T, \Psi)\) in
the sense that for all possible games it is better for the observer to play the game in
the scenario corresponding to \((\mathcal{H}_N, \mathcal{H}_S, \Phi)\), then the observer can perform physical
manipulations on \(S\) that transform \(\Phi\) into \(\Psi\).

As a second example, consider a pair of spin-\(\frac{1}{2}\) particles \(N\) and \(Q\) whose joint
state is given by
\[
\Upsilon = \Upsilon_{NQ} = \frac{1}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|).
\]
This is a separable state, corresponding to a mixing of two pure product states.
Assume that the observer, who as before has an access to the second particle \(Q\),
has to guess the spin of the first particle \(N\) along axis \(\hat{n}\) for some unit vector \(\hat{n}\),
with payoffs as before. If \(\hat{n} = \hat{z}\) (the z-axis) then the observer can he guarantee
payoff +1 by measuring the spin of \(Q\) along \(\hat{z}\) (which is, with probability one, in
the opposite direction to the spin of \(N\) along the same axis.) Thus, in terms of the
operational approach taken by this paper for measuring of information, \(\Upsilon\) is strictly
better then \(\Psi\). However, if \(\hat{n} = \hat{x}\) (the x-axis), one can verify that whatever strategy
the observer employs, his expected payoff under \(\Upsilon\) would be zero. Therefore \(\Phi\) is
better than \(\Upsilon\). Note that the observer can transform a pair of particles at state \(\Phi\)
to a pair of particles at state Υ by applying over S the map
\[ \rho \mapsto \frac{1}{2} \sigma_0 \rho \sigma_0 + \frac{1}{2} \sigma_3 \rho \sigma_3, \]
where \( \sigma_0 \) are given in (2), and he can transform a pair of particles at state Υ to Ψ by applying over Q the completely depolarizing map (11).

Note that in the last example, although Υ and Ψ are both separable, Υ is in some games strictly better than Ψ and is always at least as good as Ψ. Thus, the operational comparison of information structures is not just a comparison of the amount of entanglement between the two parts. Indeed quantum particles can be correlated without being entangled, and this correlation should also be taken into account when comparing information structures.

Finally we remark that the order ‘better’ over information structures is a partial order: There exists pairs of structures Φ and Ψ that are incomparable, that is for some games Φ can yield strictly higher payoffs than Ψ and for some games Ψ can yield strictly higher payoffs than Φ.

Section 2 introduces Blackwell’s Theorem in classical statistics. The quantum analogue is given in section 3 and proved in section 4. Section 5 discusses relationship to quantitative measure of correlation and Section 6 discusses the difference between the classical and quantum setup, which is related to the existence of positive maps which are not completely positive.

2. BLACKWELL’S THEOREM IN CLASSICAL STATISTICS

A classical information structure is given by a triple \((N, S, p)\) where \(N\) and \(S\) are finite sets and \(p\) is a (classical) distribution over \(N \times S\), i.e. \(p = (p_{n,s})_{n \in N, s \in S}\) such that \(p_{n,s} \geq 0\) and \(\sum_{n,s} p_{n,s} = 1\). We can think of \(N\) and \(S\) as the sets of possible states of two classical particles.

A game is given by a finite set \(A\) whose elements are called actions, each action \(a \in A\) corresponding to a payoff function \(M^a : N \rightarrow \mathbb{R}\). The game is played as follows: First a pair \((n, s) \in N \times S\) is randomly chosen according to \(p\). The observer sees \(s\) (the state of particle \(S\)) and then chooses an action \(a \in A\). The observer’s payoff is given by \(M^a(n)\) We assume that the observer is rational, i.e. that he chooses his action in accordance to a strategy that maximizes his expected payoff. Formally, a strategy is given by a partition \(P = \{P^a\}_{a \in A}\) of the set of signals (that is, \(P^a\) are disjoint subsets of \(S\) whose union is \(S\).) If the observer uses that strategy, then, if he sees the signal \(s\), he chooses the action \(a\) such that \(s \in P^a\). His payoff is given by
\[ \sum_a \sum_n \sum_{s \in P^a} p_{n,s} M^a(n). \]

A rational player chooses a strategy that maximizes this entity. His expected payoff is thus given by
\[ \max_P \sum_a \sum_n \sum_{s \in P^a} p_{n,s} M^a(n), \]
where the maximum ranges over all partitions \(P = \{P^a\}_{a \in A}\) of \(S\).

Let \((N, S, p)\) and \((N, T, q)\) be two information structures. \((N, S, p)\) is said to be better than \((N, T, q)\) if, for every game (that is, for every finite set \(A\) and every payoff functions \(\{M^a : N \rightarrow \mathbb{R}\}_{a \in A}\)), the expected payoff to the rational observer if the game is played over \((N, S, p)\) is at least as good as the expected payoff if
the game is played over \((N,T,q)\). Thus the partial order 'better' over information structures is defined in terms of games. Blackwell’s Theorem ([3], see also [4] for a recent survey) characterizes the same order in purely probabilistic terms:

**Theorem 1.** Let \((N,S,p)\) and \((N,T,q)\) be two classical information structures. Then \((N,S,p)\) is better than \((N,T,q)\) if and only if there exists a matrix \(F = (f_{s,t})_{s \in S, t \in T}\) such that \(f_{s,t} \geq 0\) and \(\sum_t f_{s,t} = 1\) for every \(s \in S\) (i.e. \(F\) is a stochastic matrix) and

\[
q_{n,t} = \sum_s p_{n,s} f_{s,t} \quad \text{for every } n, t.
\]

Note that every stochastic matrix \(F = (f_{s,t})_{s \in S, t \in T}\) corresponds to a linear transformation \(F : \mathbb{R}^S \to \mathbb{R}^T\) that transforms probability distributions over \(S\) to probability distributions over \(T\). We call \(F\) a (classical) stochastic map. If we think of \(p\) and \(q\) as elements of \(\mathbb{R}^N \times S\) and \(\mathbb{R}^N \times T\) then, using the natural isomorphisms \(\mathbb{R}^{N \times S} \leftrightarrow \mathbb{R}^N \otimes \mathbb{R}^S\) and \(\mathbb{R}^{N \times T} \leftrightarrow \mathbb{R}^N \otimes \mathbb{R}^T\), (3) can be written equivalently as

\[
q = (I \otimes F)p,
\]

where \(I\) stands for the identity map \(I : \mathbb{R}^N \to \mathbb{R}^N\).

In particular, a necessary condition for \((N,S,p)\) to be better than \((N,T,q)\) is that \(p\) and \(q\) induce the same marginal distributions over \(N\). For this reason, classical accounts of Blackwell’s Theorem usually define information structures using the conditional distribution of \(s\) given \(n\), and not in terms of the joint distribution as above. The two formulations are equivalent since, in classical probability, the joint distribution is uniquely determined by the marginal distribution of \(n\) and the conditional distribution of \(s\) given \(n\).

In statistical literature, the set \(S\) is viewed as a set of possible signals to the observer. The application of the stochastic map \(F\) in (4) is interpreted as simulation: If the observer receives a signal \(s\) he creates, or simulates, a new signal \(t\) from the set \(T\), distributed according to the \(s\)-th line of the matrix \(F\). If the distribution of \((n,s)\) was \(p\), the simulation process results in a new signal \(t\) such that the joint distribution of \((n,t)\) is \(q\). Since stochastic maps correspond to all the physical manipulation that can be performed over a classical particles, the physical meaning of the application of \(I \otimes F\) over \(p\) is that during the simulation process the observer performs manipulations only upon his part of the bipartite system.

### 3. Blackwell’s Theorem in Quantum Statistics

In the quantum probabilistic setup, an information structure is given by a triple \((\mathcal{H}_N, \mathcal{H}_S, \Phi)\), where \(\mathcal{H}_N\) and \(\mathcal{H}_S\) are two finite dimensional Hilbert Spaces and \(\Phi = \Phi_{NS}\) is a density operator over \(\mathcal{H}_N \otimes \mathcal{H}_S\), representing the state of a bipartite system of two particles \(N\) and \(S\), of which the observer can only access \(S\). Slightly abusing notations, we sometimes refer to the state \(\Phi\) as the information structure in cases where there should be no confusion to which particle in a pair of particles at state \(\Phi\) the observer has access.

There are two concepts which need clarification before we can formulate an analogue of Theorem 1 in quantum probability. First, we have to define the notion of game. Second, we have to find the appropriate analogue of stochastic maps.

We start with the second task. As mentioned above, a stochastic matrix corresponds to a linear mapping from classical probability distributions over one set
to classical probability distributions over another set. The first quantum analogue that comes to mind is a linear mapping that transforms density operators into density operators. These are sometimes called positive maps. But there is a crucial difference between stochastic maps in classical statistics and positive maps in quantum statistics. Whereas for every classical stochastic map $F$, $I \otimes F$ is also stochastic, there exist positive maps $E$ such that $I \otimes E$ is not positive (A well-known example is given by the transpose map.) We say that $E$ is completely positive, if $I \otimes E$ is positive over $H' \otimes H_S$ for every $H'$ where $I$ is the identity map over $H'$. For more information about completely positive map see, for example, [6].

We now turn to the definition of game in the quantum framework. Consider again the spin guessing game described in the introduction. It can be thought of as a game with two actions, namely ‘guess up’ and ‘guess down’. The observer, after performing measurements over his particle, has to choose one of these actions. If he chooses the first action, his payoff is $+1$ if the spin of $N$ is up, and $-1$ otherwise, i.e. the payoff is given by the observable $n \cdot \vec{\sigma}$ measured over $N$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. If, on the other hand, he chooses the second action (guess down) his payoff is given by the observable $-n \cdot \vec{\sigma}$.

Roughly speaking, a game is given by a finite set of actions, each action corresponding to some observable that determines the observer’s payoff should he choose that action. But in order to achieve the desired result, we need a more general setting, in which some auxiliary bipartite system $H_A \otimes H_B$ at a fixed state $\rho_{AB}$ is introduced as part of the game. The observer can perform his measurements on $S$ and $A$, and the payoff is determined by observables over $N$ and $B$. We call the system $H_A \otimes H_B$ the environment.

By introducing the environment we expand the set of games we look at. Game theoretically speaking, the environment has a natural interpretation: It represents random entities that, although independent of the information structure, can affect the observer’s payoff. In classical statistics, limiting the set of games to games without environment (as we did in Section 2) bears no consequences with regard to comparison of information structures. In quantum statistics, however, one must look at the larger set of games with environment in order to get a concept of comparison which is physically meaningful. The issue is related to the existence of positive maps which are not completely positive, which have no analogue in classical statistics. We return to it in Section 6.

Moving now to formal definitions, let $(\mathcal{H}_N, \mathcal{H}_S, \Phi_{NS})$ be an information structure. A game over this structure is given by $(\mathcal{H}_A, \mathcal{H}_B, \rho_{AB}, M^1, \ldots, M^k)$ where $\mathcal{H}_A, \mathcal{H}_B$ are finite dimensional Hilbert spaces, $\rho_{AB}$ is a density operator over $\mathcal{H}_A \otimes \mathcal{H}_B$ and $M^i = M^i_{NB}$ is an hermitian operator over $\mathcal{H}_N \otimes \mathcal{H}_B$ for each $i$ $(1 \leq i \leq k)$. $M^1, \ldots, M^k$ are called actions. Thus, the actions in the game correspond to observables over $\mathcal{H}_N \otimes \mathcal{H}_B$. The game is played as follows: First, the environment $\mathcal{H}_A \otimes \mathcal{H}_B$ is prepared at state $\rho_{AB}$ independently of the system $\mathcal{H}_N \otimes \mathcal{H}_S$. Then the observer can perform measurements on the particles $S$ and $A$. Using the information he gathered, he then chooses one action from the set of available actions $\{M^1, \ldots, M^k\}$. The observable corresponding to that action is then measured, and the numerical outcome of this measurement is the payoff to the observer in the game. Note that the observables $M^i$ need not commute, since only one of them is actually measured. We assume that the observer is rational, i.e. that he chooses
his action using a strategy that maximizes his expected payoff in the game. The following figure illustrates the role of the particles that are involved in the game.

Note that a strategy in the quantum framework involves the concept of measurement, which does not appear in the classical framework: Whereas the classical observer chose his action given the exact state of his particle, the quantum observer must first choose how to measure his particle and only then he chooses an action, given the outcome of the measurement. Formally, a strategy is given by a POVM measurement \( (D^1, \ldots, D^k) \) over \( H_S \otimes H_A \), i.e. a \( k \)-tuple of nonnegative operators over \( H_S \otimes H_A \) such that \( D^1 + \cdots + D^k = I \) (where \( I \) is the identity operator). If the observer uses this strategy, he performs this measurement and chooses action \( M^i \) if the outcome is \( i \). His expected payoff is given by

\[
\sum_{i=1}^k \text{tr}((\Phi_{NS} \otimes \rho_{AB}) \cdot (M^i_{NB} \otimes D^i_{SA})).
\]

We denote by \( R(\Phi_{NS}; \rho_{AB}, M^1, \ldots, M^k) \) the payoff to the observer under the best strategy:

\[
(5) \quad R(\Phi_{NS}; \rho_{AB}, M^1, \ldots, M^k) = \max_{(D^1, \ldots, D^k)} \sum_{i=1}^k \text{tr}((\Phi_{NS} \otimes \rho_{AB}) \cdot (D^i \otimes M^i)),
\]

where the maximum ranges over all strategies \( (D^1, \ldots, D^k) \) (that is, over all \( k \)-tuples \( (D^1, \ldots, D^k) \) of nonnegative operators over \( H_S \otimes H_A \) such that \( D^1 + \cdots + D^k = I \)).

We now turn to comparison of two information structures. Let \( \Phi_{NS} \) and \( \Psi_{NT} \) be two information structures. \( \Phi_{NS} \) is better than \( \Psi_{NT} \) if, for every bipartite system \( \rho_{AB} \), and every set \( \{M^1, \ldots, M^k\} \) of hermitian operators over \( H_N \otimes H_B \), one has

\[
R(\Phi_{NS}; \rho_{AB}, M^1, \ldots, M^k) \geq R(\Psi_{NT}; \rho_{AB}, M^1, \ldots, M^k),
\]

that is, the observer can gain in the situation corresponding to \( \Phi_{NS} \) at least as much as he can gain in the situation corresponding to \( \Psi_{NT} \). We prove the following theorem:

**Theorem 2.** Let \( \Phi = \Phi_{NS} \) and \( \Psi = \Psi_{NT} \) be two information structures. Then \( \Phi \) is better than \( \Psi \) if and only if there exists a completely positive trace preserving map \( E_S \) acting on \( S \) such that

\[
(6) \quad \Psi_{NT} = (I_N \otimes E_S)\Phi_{NS},
\]
where $I_N$ is the identity operation over $N$.

Note that completely positive trace preserving maps represent the physical manipulations that the observer can perform on particle $S$. Thus the theorem states that $\Phi$ is better than $\Psi$ if and only if the observer, starting from a pair of particles at state $\Phi$, can simulate a pair of particles at state $\Psi$ by manipulating only his particle. The ‘if’ part of the theorem is thus intuitively clear (and easily proved, see Section 4): if the observer can achieve some payoff $r$ in the situation corresponding to $\Psi$, he can achieve the same payoff in the situation corresponding to $\Phi$. To do that, he first simulates the situation $\Psi$ by manipulating $S$ and then applies the strategy that achieves $r$ in situation $\Psi$. The ‘only if’ part of the theorem says that existence of trace-preserving completely positive maps that transforms $\Phi$ to $\Psi$ is necessary for the information structure $\Phi$ to be better than $\Psi$ in the operational sense of allowing higher payoffs in games.

In particular, it follows from Theorem 2 that a necessary condition for $\Phi$ to be better than $\Psi$ is that $\text{tr}_S[\Phi] = \text{tr}_T[\Psi]$, where $	ext{tr}_S, \text{tr}_T$ are the partial traces over $S, T$ respectively. This means that the partial state of $N$ is the same in both structures. As remarked in Section 2, there is a similar necessary condition in the classical framework.

Theorem 2 characterizes the order ‘better’ over information structures (essentially states) as the order induced by the set of completely positive, state preserving maps. See Buscemi et al. [2] for a related partial order ‘cleaner’ among POVMs, which is induced by completely positive, identity-preserving maps.

4. Proof of Theorem 2

Before proving the theorem, we give another representation of strategies which is more convenient for the proof. Consider a strategy in a $k$-action game given by a POVM measurement $(D_1, \ldots, D_k)$. For a density operator $x$ over $\mathcal{H}_S \otimes \mathcal{H}_A$, the probability of getting outcome $i$ if we perform the measurement $(D_1, \ldots, D_k)$ on a particle at state $x$ is given by

$$\delta^i(x) = \text{tr}(x \cdot D_i).$$

Note that $\delta^i$ is a completely positive map from Hermitian operators over $\mathcal{H}_S \otimes \mathcal{H}_A$ to $1 \times 1$ matrices. Let $\delta(x) = (\delta^1(x), \ldots, \delta^k(x))$. Then, for every density operator $x$, $\delta(x)$ is an element of the simplex $\Delta_k = \{(p_1, \ldots, p_k) \mid p_i \geq 0, p_1 + \cdots + p_k = 1\}$ of (classical) probability distributions over the set of actions$^1$. Thus every strategy gives rise to a linear function from density operators to $\Delta_k$. The converse is also true, that is for every linear function $\delta$ that attaches an element in $\Delta_k$ for every density operator over $\mathcal{H}_S \otimes \mathcal{H}_A$ there corresponds a POVM measurement $(D_1, \ldots, D_k)$ such that (7) is satisfied.

Returning to the game defined by $\rho_{AB}$ and actions $M_1, \ldots, M_k$, recall that the payoff to the observer if he uses strategy $(D_1, \ldots, D_k)$ is $\sum_{i=1}^k \text{tr}((I_{NB} \otimes \delta^i_{SA})(\Phi_{NS} \otimes \rho_{AB}) \cdot (M_i \otimes D_i))$. Written in terms of $\delta^i$ this amount is given by $\sum_{i=1}^k \text{tr}((I_{NB} \otimes \delta^i_{SA})(\Phi_{NS} \otimes \rho_{AB}) \cdot M_i)$, where $I$ denotes the identity map, and subscripts of maps correspond to the particle over which they act. Thus $(I_{NB} \otimes \delta^i_{SA})(\Phi_{NS} \otimes \rho_{AB})$ is the density operator over $\mathcal{H}_N \otimes \mathcal{H}_B$ that represents the joint state of particles $N$.

$^1$In game theoretic literature elements of the simplex $\Delta_k$ are called mixed actions.
and \( B \) after the measurement, if the outcome was \( i \), multiplied by the probability to get outcome \( i \). The maximal possible payoff is given by

\[
R(\Phi_{NS}; \rho_{AB}, M^1, \ldots, M^k) = \max_{\delta} \sum_{i=1}^{k} \text{tr}((I_{NB} \otimes \delta_{SA}^i)(\Phi_{NS} \otimes \rho_{AB}) \cdot M^i),
\]

where the maximum ranges over all strategies \( \delta = (\delta^1, \ldots, \delta^n) \).

We now use this notation to prove the easy ‘if’ part of Theorem 2. Let \( \Phi_{NS} \) and \( \Psi_{NT} \) be two information structures. Assume that there exists a completely positive trace preserving map \( E_S \) acting on \( S \) such that \( \Psi_{NT} = (I_N \otimes E_S)\Phi_{NS} \). We prove that \( \Phi_{NS} \) is better than \( \Psi_{NT} \). Indeed, let \( \delta = (\delta^1, \ldots, \delta^n) \) be a strategy in the \( k \)-action game defined by the information structure \( \Psi_{NT} \). Thus \( \delta \) is a linear function from density operators over \( A \) to \( \Delta_k \). Let \( \delta \) be the map-composition of \( E_S \otimes I_B \) and \( \delta' = \delta \circ (E_S \otimes I_B) \). Since \( E_S \) is trace preserving and completely positive, it follows that \( E_S \otimes I_B \) is trace preserving and positive, and therefore \( \delta = (\delta^1, \ldots, \delta^n) \) is a strategy in a \( k \)-action game defined by the information structure \( \Phi_{NS} \).

Now for every environment \( \rho_{AB} \) and Hermitian operators \( M^1, \ldots, M^k \) over \( H_N \otimes H_B \), since \( \Psi_{NT} = (I_N \otimes E_S)\Phi_{NS} \), it follows that

\[
(I_{NB} \otimes \delta^i_{SA})(\Phi_{NS} \otimes \rho_{AB}) = (I_{NB} \otimes \delta^i_{TA})(\Psi_{NT} \otimes \rho_{AB}).
\]

In particular, it follows from (9) and (8) that

\[
\sum_{i=1}^{k} \text{tr}((I_{NB} \otimes \delta^i_{TA})(\Psi_{NT} \otimes \rho_{AB}) \cdot M^i) = \sum_{i=1}^{k} \text{tr}((I_{NB} \otimes \delta^i_{SA})(\Phi_{NS} \otimes \rho_{AB}) \cdot M^i) \leq R(\Phi_{NS}; \rho_{AB}, M^1, \ldots, M^k).
\]

And since this is true for every \( \delta \), if follows from (8) that

\[
R(\Psi_{NT}; \rho_{AB}, M^1, \ldots, M^k) \leq R(\Phi_{NS}; \rho_{AB}, M^1, \ldots, M^k).
\]

**Remark 3.** Note that in the proof, the only place we used the fact that \( E_S \) is completely positive (and not just positive) is to ensure that \( E_S \otimes I_B \) is positive. In particular, if there exist a positive trace-preserving map \( E_S \) such that \( \Psi = (I_N \otimes E_S)\Phi \) then (11) is still satisfied for every game with trivial environment (i.e. games in which \( \dim(H_B) = 1 \)). We return to this point in Section 6.

Turning to the second (‘only if’) part of Theorem 2, let \( \Phi = \Phi_{NS} \) and \( \Psi = \Psi_{NT} \) be two states such that \( \Phi \) is better than \( \Psi \). We construct a completely positive trace preserving map \( E_S \) acting on \( S \) such that \( \Psi_{NT} = (I_N \otimes E_S)\Phi_{NS} \). The main idea is that in order to create \( T \) from \( S \), the observer invokes a pair of fictitious agents, Alice and Bob. Alice has access to \( S \) and she wants to send Bob the information encoded in the state of system \( T \). To achieve this, they carry the standard teleportation protocol, only that Alice, instead of measuring the system \( T \), performs an alternative measurement on \( S \) with the same effect. The existence of such an alternative measurement follows from the fact that \( \Phi \) achieves higher payoff than \( \Psi \) in every game, and, in particular, in games whose environment is the auxiliary state that is used in the quantum teleportation scheme.

**Step 1: Application of Quantum Teleportation**
We start by recalling the quantum teleportation protocol. Let $H_T$ be a finite dimensional Hilbert space. Assume that Alice wants to send a particle, at a (possibly mixed) state $x$ that lives in $H_T$, to Bob. To do so, they use a certain bipartite quantum state $\rho_{AB}$ such that $H_T = H_B$ of which Alice holds the subsystem $H_A$ and Bob holds the subsystem $H_B$. Consider the state $x_T \otimes \rho_{AB}$. Alice performs on $H_T \otimes H_A$ the von-Neumann measurement corresponding to a certain basis $|\psi^i\rangle, \ldots, |\psi^k\rangle$. If she gets outcome $i$, Bob performs a certain unitary operation $U_i$ over the system $H_B$. This sets the system $H_B$ in state $x$.

For an Hermitian operator $r$ over $H_T \otimes H_A$ let $\delta^i_{TA}(r) = \langle \psi^i | r | \psi^i \rangle$. For an Hermitian operator $y$ over $H_B$ we let $\eta^i(y) = U_i^* y U_i$. Using these notations, we can summarize validity of the protocol with the following equation:

$$
(\sum_i \delta^i_{TA} \otimes \eta^i_B) (x_T \otimes \rho_{AB}) = x_T,
$$

For every state $x_T$. As usual, subscripts of maps correspond to the systems on which they act. Note that the state $\rho_{AB}$ that appears in (12) is the specific (maximally entangled) state that is used in the teleportation protocol. The magic also operates if the original particle was entangled with another particle with corresponding Hilbert space $H_N$. We summarize this with the following proposition:

**Proposition 4.** There exist a state $\rho_{AB}$, a linear function $\delta = (\delta^1, \ldots, \delta^k)$ from density operators over $H_T \otimes H_A$ to $\Delta_k$ and trace-preserving completely positive maps $\eta^i$ over $H_B$ such that for every density operator $\Psi_{NT}$ over $H_N \otimes H_T$ one has

$$
(\mathcal{I}_N \otimes \sum_i (\delta^i_{TA} \otimes \eta^i_B))(\Psi_{NT} \otimes \rho_{AB}) = \Psi_{NT}.
$$

**Step 2 - Application of Separation Theorem**

The function $\delta = (\delta^1_{TA}, \ldots, \delta^k_{TA})$ that appears in equation (13) corresponds to a strategy in a $k$-action game over the information structure $\Psi_{NT}$ with environment $\rho_{AB}$. We claim that there exists a strategy $\delta_{SA} = (\delta^1_{SA}, \ldots, \delta^k_{SA})$ for the information structure $\Phi_{NS}$ such that, for every $1 \leq i \leq k$,

$$
(\mathcal{I}_{NB} \otimes \delta^i_{SA})(\Phi_{NS} \otimes \rho_{AB}) = (\mathcal{I}_{NB} \otimes \delta^i_{TA})(\Psi_{NT} \otimes \rho_{AB}),
$$

where $\rho_{AB}$ is the state that appears in Proposition 1. Indeed, consider the set $C$ of all $k$-tuples

$$
(\mathcal{I}_{NB} \otimes \delta^1_{SA})(\Phi_{NS} \otimes \rho_{AB}), \ldots, (\mathcal{I}_{NB} \otimes \delta^k_{SA})(\Phi_{NS} \otimes \rho_{AB})
$$

for some strategy $\delta_{SA} = (\delta^1_{SA}, \ldots, \delta^k_{SA})$. This is a convex compact set in the linear space of all $k$-tuples of Hermitian operators over $H_N \otimes H_B$.

If $((\mathcal{I}_{NB} \otimes \delta^1_{TA})(\Psi_{NT} \otimes \rho_{AB}), \ldots, (\mathcal{I}_{NB} \otimes \delta^k_{TA})(\Psi_{NT} \otimes \rho_{AB}))$ is outside $C$ then by the separation theorem for convex sets there exist a hyperplane that separates it from $C$, i.e. there exist Hermitian operators $(M^1, \ldots, M^k)$ over $H_N \otimes H_B$ such that for every strategy $\delta_{SA} = (\delta^1_{SA}, \ldots, \delta^k_{SA})$,

$$
\sum_i \text{tr}((\mathcal{I}_{NB} \otimes \delta^i_{SA})(\Phi \otimes \rho) \cdot M^i) < \sum_i \text{tr}((\mathcal{I}_{NB} \otimes \delta^i_{TA})(\Psi \otimes \rho) \cdot M^i).
$$
Consider the game with set of actions \( \{M^1, \ldots, M^k\} \) and environment \( \rho_{AB} \). By (15) and (8) we have

\[
R(\Phi_{NS}; \rho_{AB}, M^1, \ldots, M^k) < \sum_i \text{tr}\left( (I_{NB} \otimes \delta_{SA}^i)(\Psi_{NT} \otimes \rho_{AB}) \cdot M^i \right) \leq R(\Psi_{NT}; \rho_{AB}, M^1, \ldots, M^k),
\]

contradicting the assumption that \( \Phi_{NS} \) is better than \( \Psi_{NT} \). Thus there exists a strategy \( \tilde{\delta}_{SA} = (\tilde{\delta}_{SA}^1, \ldots, \tilde{\delta}_{SA}^k) \) satisfying (14).

**Step 3 - Constructing \( E_S \)**

It follows from (14) that, for every \( 1 \leq i \leq k \),

\[
(I_N \otimes \tilde{\delta}_{SA}^i \otimes \eta_B^i)(\Phi_{NS} \otimes \rho_{AB}) = I_N \otimes \delta_{TA}^i \otimes \eta_B^i(\Psi_{NT} \otimes \rho_{AB}).
\]

From (16) and (17) we get

\[
(I_N \otimes \sum_i (\tilde{\delta}_{SA}^i \otimes \eta_B^i))(\Phi_{NS} \otimes \rho_{AB}) = \Psi_{NT}.
\]

Let \( E_S \) be the map defined over the system \( H_S \) by

\[
E_S : x_S \mapsto \sum_i (\tilde{\delta}_{SA}^i \otimes \eta_B^i)(x_S \otimes \rho_{AB}).
\]

This is a trace preserving, completely positive map. From (16) and (18) we get

\[
(I_N \otimes E_S)(\Phi_{NS}) = \Psi_{NT},
\]

as desired.

5. **Quantitative Measure of Information**

Recall the useful definition of mutual information of a bipartite state. If \( \rho_{XY} \) is a bipartite state, then \( I(\rho_{XY}) \) is the real number defined by

\[
I(\rho_{XY}) = S(\rho_X) + S(\rho_Y) - S(\rho_{XY}),
\]

where \( S(\rho) \) is von-Neumann’s entropy of \( \rho \). This is the quantum analogue of mutual information of random variables in classical statistics. The following lemma is an immediate consequence of Theorem 2. Its classical analogue is well known.

**Lemma 5.** Let \( \Phi = \Phi_{NS} \) and \( \Psi = \Psi_{NT} \) be two information structures. If \( \Phi \) is better than \( \Psi \) then \( I(\Phi) \geq I(\Psi) \).

**Proof.** By Theorem 2 there exists a trace preserving completely positive map \( \mathcal{E} \) such that \( \Psi = (I \otimes \mathcal{E})\Phi \). Since local quantum operation cannot increase mutual information, (see, for example, [6] Section 11.4.2) it follows that \( I(\Phi) \geq I(\Psi) \). \( \square \)

The converse, however, is not true (neither in classical statistics), as the following example shows.

Consider the bipartite state \( \Upsilon \) given by

\[
\Upsilon = \frac{1}{2}(|01\rangle \langle 01| + |10\rangle \langle 10|).
\]

The information structure was already discussed in Section 11. We consider games in which the observer, who has access to the second particle, has to guess the spin of the first particle along the \( \hat{n} \) axis for some \( \hat{n} \in \mathbb{R}^3 \), with payoff +1 for correct guess and −1 for incorrect guess. The maximal possible payoff is +1 if \( \hat{n} = \hat{z} \) and 0 if \( \hat{n} = \hat{x} \).
Let $\Upsilon' = (H \otimes H)\Upsilon(H \otimes H)$ where 
\[
H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.
\]
When playing the same games over $\Upsilon'$, the observer maximal payoff is $+1$ if $\hat{n} = \hat{x}$ and $0$ if $\hat{n} = \hat{z}$. Therefore it follows that even though $I(\Upsilon) = I(\Upsilon') = 1$, neither is better than the other in Blackwell’s sense: for some games it is better to play over $\Upsilon$ and for some games it is better to play over $\Upsilon'$.

6. Games with Trivial Environment

Theorems 1 and 2 have a similar purpose: to provide an operational interpretation for the specific map. Both uses the concept of game to formulate the operational aspect of. The main conceptual difference between is the additional introduction of environment. Two notes are in order. First, the class of quantum games that was defined in Section 3 includes as a subset games with trivial environment, that is games for which $\dim(H_A) = \dim(H_B) = 1$. This games are the natural analogue of the classical games that were considered in Section 2. Similarly, one can expand the set of classical games to games with classical environment. However, in the classical framework, the concept of games with environment is superfluous for comparison of information structures because of the following claim:

**Claim 6.** if $(N,S,p)$ and $(N,T,q)$ are two classical information structures such that the payoff over $(N,S,p)$ is at least as good as the payoff over $(N,T,q)$ for every classical game with trivial environment, then the payoff of $(N,S,p)$ is at least as good as the payoff over $(N,T,q)$ also for games with non-trivial environment.

The claim follows from Theorem 1 and from the fact that if $F$ is a classical stochastic map then so is $F \otimes I$. We do not prove it formally here, in order to avoid the notational encumbrance of classical environment.

In the quantum world, however, there is no analog for Claim 6. There exist information structures $\Phi$ and $\Psi$ such that the payoff over $\Phi$ is at least as good as the payoff over $\Psi$ for every game with trivial environment, but not for every game with non-trivial environment. This is why we had to explicitly define the order relation ‘better’ in Section 3 using games with environment. The construction of such a pair $\Phi$ and $\Psi$ of information structures, which we now describe, is based on the existence of positive maps which are not completely positive.

Let $H_N = H_S = H_T$ be three $n$-dimensional Hilbert space. Let $m = n^2$ and $\rho_1, \ldots, \rho_m$ a set of $n \times n$ density matrices which are linearly independent in the linear space of all $n \times n$ Hermitian matrices. Let $\Phi = \frac{1}{m}(\rho_1 \otimes \rho_1 + \cdots + \rho_m \otimes \rho_m)$ and $\Psi = \frac{1}{m}(\rho_1 \otimes \rho_1^t + \cdots + \rho_m \otimes \rho_m^t)$, where $\rho_j^t$ is the transpose of $\rho_j$. There exists only one linear map $E$ from Hermitian operators to Hermitian operators such that $\Psi = I \otimes E(\Phi)$, and this map is given by $E(H) = H^t$, which is not completely positive. In particular, it follows from Theorem 2 that there exists a game (with no-trivial environment) for which $\Psi$ offers higher payoff than $\Phi$. On the other hand, since $E$ is positive and $\Psi = I \otimes E(\Phi)$, it follows from the Remark that for every game with trivial environment the payoff under $\Phi$ is at least as good as the payoff under $\Psi$.

**Remark 7.** One may conjecture that if $\Phi$ and $\Psi$ are two structures such that the payoff under $\Phi$ is at least as good as the payoff under $\Psi$ for every game with trivial
environment, then there exist a positive (but not necessarily completely positive) trace preserving map $\mathcal{E}$ such that $\Psi = (I \otimes \mathcal{E})\Phi$. However, I do not know whether this is true.

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