On the integrable inhomogeneous Myrzakulov I equation

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Abstract

By using the prolongation structure theory proposed by Morris, we give a (2+1)-dimensional integrable inhomogeneous Heisenberg Ferromagnet models, namely, the inhomogeneous Myrzakulov I equation. Through the motion of space curves endowed with an additional spatial variable, its geometrical equivalent counterpart is also presented.

PACS: 02.30.Ik, 02.40.Hw, 75.10.Hk

KEYWORDS: inhomogeneous Heisenberg ferromagnet model, (2+1)-dimensional integrable equation, motion of space curve

1. Introduction

The Heisenberg Ferromagnet (HF) equation,

\[ S_t = S \times S_{xx}, \tag{1} \]

which describes the motion of the magnetization vector of the isotropic ferromagnets is an important integrable system. The corresponding integrable inhomogeneous HF equation was derived by Balakrishnan [1],

\[ S_t = (fS \times S_x)_x + hS_x, \tag{2} \]

where the linear functions \( f \) and \( h \) take \( f = \mu_1x + \nu_1 \) and \( h = \mu_2x + \nu_2 \), and the coefficients \( \mu_i \) and \( \nu_i \), \( i = 1,2 \), are constants. On the basis of the prolongation structure theory of Wahlquist and Estabrook [2], the integrable deformations of the (inhomogeneous) HF model have been studied in Refs.[3,4].

The (2+1)-dimensional integrable extensions of a (1+1)-dimensional integrable equation have been of interest. For the HF equation (1), several (2+1)-dimensional integrable extensions have been constructed [5]. One of its important integrable extensions is given by

\[ S_t = (S \times S_y + uS)_x, \]

\[ u_x = -S \cdot (S_x \times S_y), \tag{3} \]

that is the Myrzakulov I (M-I) equation [5-7]. The M-I equation (3) is geometrical and gauge equivalent to the well-known (2+1)-dimensional focusing nonlinear Schrödinger equation \( NLS^+ \) [6]

\[ i\psi_t - \psi_{xx} - v\psi = 0, \quad v_x = 2\partial_y|\psi|^2, \tag{4} \]
where $\psi$ is a complex function. Recently Morris’s prolongation structure theory for nonlinear evolution equation in two spatial dimensions [8] has been successfully applied to analyze equation (3) in Ref.[9]. It is noted that this prolongation structure method is very simple and effective in the investigation of M-I equation. For the inhomogeneous HF equation (2), its (2+1)-dimensional integrable extensions have not been studied so far. The purpose of this paper is to apply Morris’s prolongation structure method to construct the (2+1)-dimensional integrable extension of equation (2).

2. The inhomogeneous integrable M-I equation

In order to investigate (2+1)-dimensional integrable extension of Eq.(2), we introduce a new vector $E(S, S_x, S_y)$ and the functions $f(x)$ and $g(x)$ into (3),

\[
S_t = \{fS \times S_y + guS\}_x + E,
\]

\[
= fS \times S_{xy} + fS_x \times S_y + f_x S \times S_y + gu_x S + guS_x + g_x u S + E,
\]

\[
u_x = -S \cdot (S_x \times S_y),
\]

(5)

where $S \cdot S = 1$ and $S \cdot E = 0$. Multiplying Eq. (5) by $S$ and using the restricting relation, $S \cdot S_t = S \cdot S_x = S \cdot S_y = 0$, we have $f = g$ and $f_x = g_x = 0$. It implies that $f$ and $g$ should be the constants. Let us take $f = 1$ and

\[
E = \rho(x)S_x + \nu(x)S_y + \mu(x)S \times S_x,
\]

(6)

where the functions $\rho(x)$, $\mu(x)$ and $\nu(x)$ need to be determined. Thus equation (5) can be rewritten as

\[
S_t = S \times S_{xy} + S_x \times S_y + u_x S + uS_x + \rho(x)S_x + \nu(x)S_y + \mu(x)S \times S_x,
\]

\[
u_x = -S \cdot (S_x \times S_y),
\]

(7)

As done in Ref.[9], we first consider the prolongation structure of (7) when $S_t = 0$. Setting $W = S_x$, $T = S_y$ and taking $S, T, W,$ and $u$ as the new independent variables, we can define the following set of two forms,

\[
\alpha_a = dS_a \wedge dx - T_a dy \wedge dx,
\]

\[
\alpha_{a+3} = dS_a \wedge dy - W_a dx \wedge dy,
\]

\[
\alpha_{a+6} = (W \times T)_a dx \wedge dy + (S \times dT)_a dy \wedge dy + S_a du \wedge dy
\]

\[
+ uW_a dx \wedge dy + (\rho W_a + \nu T_a + \mu (S \times W)_a) dx \wedge dy,
\]

\[
\alpha_{10} = du \wedge dy + S \cdot (W \times T) dx \wedge dy,
\]

\[
\alpha_{a+10} = dT_a \wedge dy + dW_a \wedge dx,
\]

\[
\alpha_{14} = (T \cdot W) dx \wedge dy + S_a \cdot dT_a \wedge dy,
\]

(8)
where \( a = 1, 2, 3 \), such that they constitute a closed ideal \( I = \{ \alpha_i, i = 1, 2, \cdots, 14 \} \). Then we extend the above ideal \( I \) by adding to it a set of one forms,

\[
\Omega^k = d\xi^k + F_k(x, y, S, T, W, u)\xi^k dx + G_k(x, y, S, T, W, u)\xi^k dy, \quad k = 1, 2, \cdots, n, \quad (9)
\]

where \( \xi^k \) is prolongation variable. In terms of the prolongation condition, \( d\Omega^k \subset \{ I, \Omega^k \} \), we obtain the following set of partial differential equations for \( F^k \) and \( G^k \),

\[
\frac{\partial F^k}{\partial T_a} = \frac{\partial F^k}{\partial u} = 0, \quad \frac{\partial G^k}{\partial W_a} = 0,
\]

\[
- \frac{\partial F^k}{\partial S_a} T_a + \frac{\partial G^k}{\partial S_a} W_a - \frac{\partial G^k}{\partial u} S \cdot (W \times T) + \left( \frac{\partial G^k}{\partial T_a} - \frac{\partial F^k}{\partial W_a} \right) \left\{ [S \times (W \times T)]_a \right\}
\]

\[
- S_a (T \cdot W) + u(S \times W)_a + (\rho S \times W + \nu S \times T + \mu S \times (S \times W))_a \}
\]

\[
- [F, G]^k + \frac{\partial G^k}{\partial x} - \frac{\partial F^k}{\partial y} = 0, \quad (10)
\]

where \([F, G]^k = \sum_{i=1}^{n} F_i \frac{\partial G^k}{\partial y^i} - \sum_{i=1}^{n} G_i \frac{\partial F^k}{\partial y^i} \). By solving (10), we have the following solution,

\[
F = \lambda \sum_{i=1}^{3} S_i X_i, \quad G = (\rho + u) \sum_{i=1}^{3} S_i X_i + \sum_{i=1}^{3} (S \times T)_i X_i, \quad \mu = \nu = 0, \quad (11)
\]

where \( X_i, i = 1, 2, 3 \), depend only on the prolongation variables \( \xi^k \) and have the commutation relation of the \( su(2) \) Lie algebra.

Let us turn now to define a set of 3-form \( \overline{\alpha}_i \) as follows,

\[
\overline{\alpha}_a = dS_a \wedge dx \wedge dt - T_a dy \wedge dx \wedge dt,
\]

\[
\overline{\alpha}_{a+3} = dS_a \wedge dy \wedge dt - W_a dx \wedge dy \wedge dt,
\]

\[
\overline{\alpha}_{a+6} = (W \times T)_a dx \wedge dy \wedge dt + (S \times dT)_a dy \wedge dt + S_adu \wedge dy \wedge dt
\]

\[
+ uW_a dx \wedge dy \wedge dt - dS_a dx \wedge dy + (\rho W_a + \nu T_a + \mu (S \times W)_a) dx \wedge dy \wedge dt,
\]

\[
\overline{\alpha}_{10} = du \wedge dy \wedge dt + S \cdot (W \times T) dx \wedge dy \wedge dt,
\]

\[
\overline{\alpha}_{a+10} = dT_a \wedge dy \wedge dt + dW_a \wedge dx \wedge dt,
\]

\[
\overline{\alpha}_{14} = (T \cdot W) dx \wedge dy \wedge dt + S_a \cdot dT_a \wedge dy \wedge dt,
\]

where \( a = 1, 2, 3 \), such that they constitute a closed ideal. When these two forms are null, we recover (7). Then we introduce the following two forms,

\[
\overline{\Omega}^k = \Omega^k \wedge dt + H^k_{ij} \xi^i dx \wedge dy + (A^k_j dx + B^k_j dy) \wedge d\xi^j, \quad k = 1, 2, \cdots, n, \quad (13)
\]

where the matrices of \( A \) and \( B \) depend on the variables \( (x, y, t) \) and the form of \( \Omega^k \) is given by (9), in which \( \lambda \) depends on the variables \( (x, y, t) \). It is easily shown that

\[
d\bar{\Omega}^k = \sum_{i=1}^{14} g^{ki} \overline{\alpha}_i + \sum_{j=1}^{n} \zeta^k_j \wedge \overline{\alpha}_j, \quad (14)
\]
provided that the matrix $H$ is given by

$$ H = GA - FB + A_y - B_x $$

(15)

and

$$ dH \wedge dx \wedge dy - \frac{\partial G}{\partial T_a}(S \times dS)_a \wedge dx \wedge dy - \lambda_y S_a dx \wedge dy \wedge dt 
+ \rho_x S_a dx \wedge dy \wedge dt - A_t G dx \wedge dy \wedge dt + B_t F dx \wedge dy \wedge dt = 0. $$

(16)

Substituting the expressions (11) of $F$ and $G$ into (15) and (16), we obtain

$$ A = 0, \quad B = \frac{1}{\lambda} I, $$

(17)

and

$$ \lambda_t = -\lambda \lambda_y + \lambda \rho_x, \quad \lambda_x = 0. $$

(18)

From Eq. (18), we note that the function $\rho(x)$ take the following expression,

$$ \rho(x) = \mu_3 x + \nu_3, $$

(19)

where the coefficients $\mu_3$ and $\nu_3$ are constants. Thus the integrable inhomogeneous M-I equation is

$$ S_t = \{S \times S_y + uS\}_x + (\mu_3 x + \nu_3)S_x $$

$$ u_x = -S \cdot (S_x \times S_y), $$

(20)

On imposing the reduction $\partial_y = \partial_x$, equation (20) reduces to the (1+1)-dimensional inhomogeneous HF equation (2) in which $f = 1$. By restricting (13) on the solution manifold, we obtain the Lax representation of equation (20)

$$ \xi_x = -F|_{X_i} = -\frac{i\lambda}{2} \sum_{i=1}^{3} S_i \sigma_i \xi, $$

$$ \xi_t = -\frac{1}{B} \xi_y - \frac{1}{B} G|_{X_i} = -\frac{1}{2} \sum_{i=1}^{3} \left[ (\rho + u) S_i \sigma_i + (S \times T)_i \sigma_i \right] \xi, $$

(21)

where $\sigma_i, i = 1, 2, 3$, are Pauli matrices, and the spectral parameter satisfies the nonlinear equation (18)

3. Geometrical equivalent counterpart

By associating with the motion of Euclidean space curves endowed with an extra spatial variable, Myrzakulov et al. showed that the M-I equation (8) is geometric equivalent to the (2+1)-dimensional $NLS^+ [6]$. In order
to give the geometrical equivalent counterpart of (20), we fist give a brief review on the motion of a curve. In general, the Serret-Frenet equation associated with a curve is given by

\[
\begin{align*}
t_s &= k\mathbf{n}, \\
\mathbf{b}_s &= -\tau\mathbf{n}, \\
\mathbf{n}_s &= \tau\mathbf{b} - kt, \\
\end{align*}
\]  

(22)

where \(k\) and \(\tau\) are the curvature and torsion of the curve, respectively. This equation can also be rewrite as

\[
\begin{align*}
\mathbf{N}_s &= -\psi\mathbf{t}, \\
\mathbf{t}_s &= \frac{1}{2}(\psi^*\mathbf{N} + \psi\mathbf{N}^*), \\
\end{align*}
\]  

(23)

where \(\mathbf{N} = (\mathbf{n} + i\mathbf{b})exp(i \int_{-\infty}^{s} ds'\tau), \psi = kexp(i \int_{-\infty}^{s} ds'\tau)\) and the new frame \(\mathbf{t},\mathbf{N}\) and \(\mathbf{N}^*\) satisfies the relations

\[
\begin{align*}
\mathbf{N} \cdot \mathbf{N}^* &= 2, \\
\mathbf{N} \cdot \mathbf{t} = \mathbf{N}^* \cdot \mathbf{t} = \mathbf{N} \cdot \mathbf{N} &= 0. \\
\end{align*}
\]  

(24)

The temporal variation of \(\mathbf{t},\mathbf{N}\) and \(\mathbf{N}^*\) may be expressed as

\[
\begin{align*}
\mathbf{N}_t &= \alpha\mathbf{N} + \beta\mathbf{N}^* + \gamma\mathbf{t}, \\
\mathbf{t}_t &= \lambda\mathbf{N} + \mu\mathbf{N}^* + \nu\mathbf{t}. \\
\end{align*}
\]  

(25)

Multiplying Eq. (25) by \(\mathbf{N}\) and \(\mathbf{t}\) and using the relation (24), we have

\[
\begin{align*}
\mathbf{N}_t &= iR\mathbf{N} + \gamma\mathbf{t}, \\
\mathbf{t}_t &= -\frac{1}{2}(\gamma^*\mathbf{N} + \gamma\mathbf{N}^*), \\
\end{align*}
\]  

(26)

where \(R(s,t)\) is real. Using the compatibility condition, \(\mathbf{N}_{ts} = \mathbf{N}_{st}\), we get

\[
\begin{align*}
\psi_t + \gamma_s - iR\psi &= 0, \\
R_s &= \frac{1}{2}i(\gamma^*\psi - \gamma\psi). \\
\end{align*}
\]  

(27)

If the auxiliary function \(\gamma\) and \(R\) can be expressed in terms of \(\psi\) and its spatial derivations, then equation (27) will provide an evolution equation for the spatial and temporal variation of the curvature and torsion of the curve as expressed through \(\psi\).

Let us identify \(\mathbf{S}\) with the tangent of a Euclidean space curve and endow the moving curve with an additional spatial variable \(y\). Thus equation (20) becomes

\[
\begin{align*}
\mathbf{t}_t &= \mathbf{t}_x \times \mathbf{t}_y + \mathbf{t} \times \mathbf{t}_{xy} + u_x \mathbf{t} + u \mathbf{t}_x + \rho \mathbf{t}_x, \\
\end{align*}
\]  

(28)

where the subscript \(x\) denotes the arc length parameter. On imposing the compatibility condition, \(\mathbf{t}_{xy} = \mathbf{t}_{yx}\), \(\mathbf{n}_{xy} = \mathbf{n}_{yx}\) and \(\mathbf{b}_{xy} = \mathbf{b}_{yx}\), we obtain the y-part equation of the tangent, normal and binormal vectors

\[
\begin{align*}
\mathbf{t}_y &= -\frac{u_x}{\kappa} \mathbf{b} + \partial_x^{-1}(\kappa_y - \frac{\tau u_x}{\kappa})\mathbf{n}, \\
\mathbf{n}_y &= (u + \partial_x^{-1}r_y)\mathbf{b} - \partial_x^{-1}(\kappa_y - \frac{\tau u_x}{\kappa})\mathbf{t}, \\
\mathbf{b}_y &= -(u + \partial_x^{-1}r_y)\mathbf{n} + \frac{u_x}{\kappa} \mathbf{t}, \\
\end{align*}
\]  

(29)
Substituting (29) and (22) into (28), we have
\[ t_t = \frac{1}{2} \left[ (-i\psi_y + \rho\psi)N + (i\psi_y^* + \rho\psi^*)N^* \right]. \] (30)
Comparing (30) with (26), we get
\[ \gamma = -(\rho\psi^* + i\psi^*_y), \] (31)
then substituting \( \gamma \) into (27), we obtain the following integrable evolution equation
\[ i\psi_t - \psi_{xy} - i(\rho\psi)_x - R\psi = 0, \quad R_x = \frac{1}{2} \partial_y |\psi|^2. \] (32)
It is the inhomogeneous extension of (11). The Lax representation of (32) is given by
\[ \Phi_x = U\Phi, \quad \Phi_t = V\Phi + \lambda\Phi_y, \] (33)
in which
\[ U = \begin{pmatrix} i\lambda/2 & \psi/2 \\ -\psi^*/2 & -i\lambda/2 \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{1}{2}R + \frac{1}{2}\lambda\rho & \frac{1}{2}\psi_y + \frac{1}{2}\rho\psi \\ -\frac{1}{2}\psi_y^* + \frac{1}{2}\rho\psi^* & \frac{1}{2}R - \frac{1}{2}\lambda\rho \end{pmatrix}, \] (34)
and the spectral parameter satisfies Eq. (18).

4. Summary

We have investigated a possible integrable inhomogeneous M-I equation by using Morris's prolongation structure theory. Under the reduction \( \partial_y = \partial_x \), the (2+1)-dimensional integrable equation which we obtained in this paper, i.e., Eq. (24), reduces to a special case of inhomogeneous HF equation (2). It has been noted that there exist several (2+1)-dimensional integrable extensions for the HF equation (1), such as M-VIII, Ishimori and M-IX equations [5-7, 11]. Therefore, whether there are more general (2+1)-dimensional integrable inhomogeneous extensions of (3) is a question for the future.

Acknowledgements

We would like to express our thanks to Moningside Center, CAS, part of the works was done when we were joining the workshop of Math-Phys there. We are also very grateful to Prof. R. Myrzakulov for his interest and helpful discussions. This work is partially supported by German(DFG)-Chinese(NSFC) Exchange Programme 446CHV113/231, NKBRC (2004CB318000), Beijing Jiao-Wei Key project (KZ200310028010), NSF projects (10375038 and 90403018) and SRF for ROCS(222225), SEM.

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