An Intrinsic Impulse Observability Criterion for Descriptor System *

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Abstract

Analyzing the order of unobservable impulse in descriptor system leads to a new testing criterion for impulse observability, both the statement and the proof of which use only the original system data.

Keywords: singular linear system; observability at infinity; impulse observability; Dirac delta distribution

AMS Subject Classifications: 93B05; 93B07; 93B10

1 Introduction

We consider the impulse observability of descriptor linear system [1], [2]

\[ \begin{align*}
E \dot{x}(t) &= Ax(t) \\
y(t) &= Cx(t)
\end{align*} \]  

(1.1)

where \( E, A \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{m \times n} \). Matrix \( E \) is singular, but matrix pencil \( sE - A \) is regular, i.e., \( \det(sE - A) \) is a nonzero polynomial on \( s \in \mathbb{C} \) [3]. Comparing with standard linear system \( (E = I, \) the identity matrix), descriptor one is featured by having impulse behavior. The underlying mathematics for this phenomenon is that the following initial value problem of differential-algebraic equation

\[ E \dot{x}(t) = Ax(t), \quad t \geq 0; \quad x(0) = w \]  

(1.2)

*Supported by Harbin Institute of Technology Science Foundation under grant HITC200712.
has no solution generally in the sense of classical differentiable function, and a generalized solution, as a mathematical model of the state response of the system to initial value, in the sense of distribution has to be adopted. It interprets the impulse behavior that the distributional solution may contain a linear combination of Dirac delta distribution $\delta(t)$ and its distributional derivatives $\delta^{(k)}(t)$, $k = 1, 2, \ldots$. This linear combination is called impulsive term in the state response for convenience in following. For details, see [2, pp. 16–22] and the references therein. More recent works about distributional solution are [4]–[8].

System (1.1) is called impulse observable (see, e.g., [2, p. 43], [9]–[13], [6]), if for arbitrarily given but unknown initial value $x(0)$, the impulsive term in state response $x(t)$, $t \geq 0$ can be uniquely determined out from the measured, therefore known, output response $y(t)$, $t \geq 0$.

A subtle point for impulse analysis in descriptor system is that explicit expressions of the impulsive terms in state and output responses directly using the system data, i.e., $\{E, A, C\}$ for system (1.1) discussed here, do not exist. As a result, the existing impulse observability criteria, mainly collected in [2, Theorem 2-3.4, p. 43] and the dual form of [1, Theorem 4.2, p. 20], and/or their proofs have to employ indirect data which comes from some transformations to the original system, e.g., slow-fast decomposition based on Weierstrass canonical form, descriptor system structure algorithm [1], etc. Besides causing numerical difficulties, to deal with which some condensed forms based on numerically reliable orthogonal matrix transformations are developed [14]–[19], such type of answer to a control theory problem can not satisfy a theoretical interest as well.

In this note, we are interested in obtaining an intrinsic impulse observability criterion in the sense that both its statement and its proof use the original system data directly. This concern is also motivated by seeking for a criterion with clear dynamical interpretation. The obtained new criterion is Theorem 2.4, the dynamical interpretation of which is the nonexistence of unobservable impulse of specific order, see Theorem 2.3. Main existing criteria for impulse observability in literature are special cases of our new one, see Corollaries 3.1 and 3.2.

2 Main Results

The frequency domain form of (1.1) with initial state $x(0) = w$ can be written as

$$\begin{bmatrix}
  sE - A \\
  C
\end{bmatrix} X(s) = \begin{bmatrix}
  Ew \\
  Y(s)
\end{bmatrix}.$$  \hspace{1cm} (2.1)
where $X, Y$ denote the Laplace transforms of $x, y$ respectively.

We use notations $\mathbb{R}^n(s)$ and $\mathbb{R}^n[s]$ to denote the set of $n$-dimensional rational fraction vectors and the set of $n$-dimensional polynomial vectors respectively. Note that any rational fraction can be uniquely expressed as the sum of a strictly proper fraction and a polynomial. In following for $F(s) \in \mathbb{R}^n(s)$ we always use $F_\lambda(s)$ and $F_P(s)$ to denote the strictly proper fraction part and polynomial part respectively for notation convenience.

For clarity we use the following Definition [7].

**Definition 2.1** Let $v_i \in \mathbb{R}^n$, $i = 0, 1, \ldots, p$ with $v_p \neq 0$. Then the distribution

$$\lambda(t) = \delta(t)v_0 + \delta^{(1)}(t)v_1 + \cdots + \delta^{(p)}(t)v_p$$

is called an $n$-dimensional impulse of order $p$, denoted by $\text{deg}(\lambda) = p$. The polynomial vector $\sum_{i=0}^{p} s^i v_i \in \mathbb{R}^n[s]$ in complex variable $s$ of degree $p$, which is Laplace transform of impulse, is called frequency domain form of impulse, or simply, impulse.

The Laplace inverse transform of a strictly proper fraction is a usual smooth function (a combination of exponential, triangular and polynomial functions of time variable $t$, see [20]). Therefore impulse observability of the system (1.1) means, in frequency domain language, that the polynomial part $X_P(s)$ in $X(s)$ can be uniquely determined out from $Y(s)$. We write this fact as definition of impulse observability for clarity, although “common” definition is not so stated [2, p. 43], [7], [9], [10], [13], [6].

**Definition 2.2** The system (1.1) is impulse observable, if the following equation

$$\begin{bmatrix} sE - A \\ C \end{bmatrix} X(s) = \begin{bmatrix} Ew \\ 0 \end{bmatrix}$$

(2.2)
on $(w, X(s)) \in \mathbb{R}^n \times \mathbb{R}^n(s)$ has no solution with $X_P(s)$ nonzero.

**Remark 2.1** In other words, from $Y(s) = CX(s) = 0$ one can conclude that $X(s)$ does not contain polynomial part (frequency domain form of impulse, see Definition 2.1).

**Remark 2.2** We adopt the equation viewpoint. Here a deliberate point is that both $w$ and $X(s)$ are seen undetermined simultaneously. It is this insight that results in an equal treating with $p_{-1}$ and $p_0, \cdots, p_r$ in (2.12) and the introducing of the matrix (2.8).
Theorem 2.1  The system (1.1) is not impulse observable, if and only if the following equation

$$\begin{bmatrix} sE - A \\ C \end{bmatrix} P(s) = \begin{bmatrix} Ev \\ 0 \end{bmatrix}$$

(2.3)
on $(v, P(s)) \in \mathbb{R}^n \times \mathbb{R}^n[s]$ has a solution with $P(s)$ nonzero.

Note the difference of $\mathbb{R}^n(s)$ and $\mathbb{R}^n[s]$.

**Proof.** The sufficiency is obvious by Definition 2.2 since $P(s) \in \mathbb{R}^n[s] \subset \mathbb{R}^n(s)$.

Necessity. By Definition 2.2, there exists $(w, X(s)) \in \mathbb{R}^n \times \mathbb{R}^n[s]$ with $X_P(s)$ nonzero such that

$$\begin{bmatrix} sE - A \\ C \end{bmatrix} (X_A(s) + X_P(s)) = \begin{bmatrix} Ew \\ 0 \end{bmatrix}.$$ 

Firstly, $0 = CX_A(s) + CX_P(s) \in \mathbb{R}^m(s)$ implies

$$CX_P(s) = 0.$$ (2.4)

Secondly, the limit $\lim_{s \to \infty} sX_A(s)$ exists, denoted by $q$, and moreover $\lim_{s \to \infty} (sE - A)X_A(s) = Eq$. Therefore

$$Ew = (sE - A)(X_A(s) + X_P(s)) = [(sE - A)X_A(s) - Eq] + [Eq + (sE - A)X_P(s)]$$

forms a decomposition of strictly proper fraction plus polynomial. The uniqueness of such decomposition implies $Ew = [Eq + (sE - A)X_P(s)]$, which is equivalent to

$$(sE - A)X_P(s) = E(w - q).$$ (2.5)

It follows from (2.4) and (2.5) that $(w - q, X_P(s)) \in \mathbb{R}^n \times \mathbb{R}^n[s]$ is a solution of (2.3). ■

We introduce the following technical notion.

**Definition 2.3** Let $(v, P(s)) \in \mathbb{R}^n \times \mathbb{R}^n[s]$ with $P(s)$ nonzero and $\deg(P(s)) = r$ be a solution of the equation (2.3). Then $v$ is called an unobservable impulsive initial state of the system (1.1), and $P(s)$ is called an unobservable impulse of order $r$ of the system (1.1).

**Lemma 2.1** The system (1.1) is impulse observable, if and only if it has no unobservable impulse of order $\leq n - 1$. 


Proof. It follows from the regularity of the pencil \( sE - A \) that the solution of (2.3), if exists, will be of deg\((P(s)) \leq n - 1.\]

**Theorem 2.2** Let \((v, P(s)) \in \mathbb{R}^n \times \mathbb{R}^n[s] \) be a solution of (2.3) with \( \text{deg}(P(s)) = r \geq 1 \) and write

\[
P(s) = p_0 - sp_1 + \cdots + (-s)^r p_r.
\]

Then \((p_{r-1}, p_r) \in \mathbb{R}^n \times \mathbb{R}^n[s] \) is a solution of (2.3) as well, where \( p_r \in \mathbb{R}^n[s] \) is of deg\(p_r = 0\) as a polynomial vector.

**Proof.** Combining (2.3) and (2.6), we have

\[
\sum_{i=0}^{r} \begin{bmatrix}
  s^{i+1}E - s^iA \\
  s^iC
\end{bmatrix} (-1)^i p_i = \begin{bmatrix}
  Ev \\
  0
\end{bmatrix}.
\]

Differentiating \( r \) times gives

\[
\begin{bmatrix}
  r!E - 0 \\
  0
\end{bmatrix} (-1)^{r-1} p_{r-1} + \begin{bmatrix}
  (r+1)!sE - r!A \\
  r!C
\end{bmatrix} (-1)^r p_r = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

and further

\[
\begin{bmatrix}
  sE - A \\
  C
\end{bmatrix} p_r = \begin{bmatrix}
  Ep_{r-1} \\
  0
\end{bmatrix}.
\]

Note that \((r+1)sEp_r = 0\) implies \(Ep_r = 0.\]

From an unobservable impulse \(P(s)\) of order \( r \) to initial value \( v \), Theorem 2.2 constructs an unobservable one \( p_r \) of order zero to initial value \( p_{r-1} \).

Form the following partitioned matrix

\[
O_k(E, A) = \begin{bmatrix}
  E & A \\
  E & \ddots \\
  \vdots & \ddots & A \\
  E & \ddots \\
  0 & C & \ddots \\
  \vdots & \ddots & \ddots \\
  0 & C
\end{bmatrix}
\]

for \( k = 2, \ldots, n + 1 \), where the other blocks not appearing are zero. It has \( k \) block columns and \( 2k - 1 \) block rows, and then has the size \((kn + (k-1)m) \times kn\).
Theorem 2.3 The system (1.1) has no unobservable impulse of order \( \leq r \) if and only if

\[
\text{rank}(O_{r+2}(E, A, C)) = n(r + 1) + \text{rank}(E). \tag{2.9}
\]

Proof. Let \( \text{deg}(P(s)) \leq r \) and write \( P(s) = p_0 - sp_1 + \cdots + (-s)^r p_r \). Then (2.3) is equivalent to the following \((r + 2) + (r + 1) = 2r + 3\) equations

\[
Ep_i + Ap_{i+1} = 0, \ i = -1, 0, \ldots, r, \tag{2.10}
\]

\[
Cp_i = 0, \ i = 0, \ldots, r \tag{2.11}
\]
on \((p_{-1}, p_0, \cdots, p_r) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n\), where we denote \( p_{-1} = v \) and \( p_{r+1} = 0 \) for notation convenience.

The group of equations (2.10) and (2.11) can be rewritten into the following matrix form

\[
O_{r+2}(E, A, C) \begin{bmatrix} p_{-1} \\ p_0 \\ \vdots \\ p_r \end{bmatrix} = 0. \tag{2.12}
\]

Therefore the system (1.1) has no \( r \) order unobservable impulse if and only if all solutions of (2.12) satisfy \( p_i = 0, \ i = 0, 1, \ldots, r \), i.e.,

\[
\left\{ \begin{bmatrix} p_{-1} \\ p_0 \\ \vdots \\ p_r \end{bmatrix} : O_{r+2}(E, A, C) \begin{bmatrix} p_{-1} \\ p_0 \\ \vdots \\ p_r \end{bmatrix} = 0 \right\} = \left\{ \begin{bmatrix} p_{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} : O_{r+2}(E, A, C) \begin{bmatrix} p_{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \right\}. \tag{2.13}
\]

It is easy to see that

\[
O_{r+2}(E, A, C) \begin{bmatrix} p_{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Ep_{-1}.
\]

Then by computing dimensions of solution spaces of linear equations, (2.13) gives

\[
n(r + 2) - \text{rank}(O_{r+2}(E, A, C)) = n - \text{rank}(E)
\]
and the result follows immediately.
Theorem 2.4 The following statements are equivalent:

1). The system (1.1) is impulse observable;
2). \( \text{rank}(O_{r+2}(E, A, C)) = n(r + 1) + \text{rank}(E) \) for some one \( r \in \{0, 1, \cdots, n - 1\} \);
3). \( \text{rank}(O_{r+2}(E, A, C)) = n(r + 1) + \text{rank}(E) \) for each one \( r \in \{0, 1, \cdots, n - 1\} \).

Proof. Follows from Lemma 2.1, Theorems 2.2 and 2.3. ■

Remark 2.3 Up till now, all statements and proofs use only the original system data, not involved in any transformation to the system.

3 Implying Existing Results

First, taking \( r = 0 \), the condition (2.9) gives the following consequence.

Corollary 3.1 The system (1.1) is impulse observable, if and only if \( \text{rank}(O_2(E, A, C)) = n + \text{rank}(E) \), i.e.,

\[
\text{rank} \begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = n + \text{rank}(E). \tag{3.1}
\]

Remark 3.1 The condition (3.1) is a well known criterion for impulse observability [2, Eq. (2-3.6), p. 44], which is featured by using the original system data. To our knowledge, it is the only criterion of this feature in literature. Existing proofs (see [2, p. 44], [13], etc.) rely on some decompositions. Now we see that it guarantees the nonexistence of unobservable impulse of zero order. In quantitative aspect, equation

\[
\begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} \begin{bmatrix} p^{-1} \\ p_0 \end{bmatrix} = 0
\]

provides total information about unobservable impulsive initial states of zero order.

Now we consider the Weierstrass canonical decomposition

\[
\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sE - A \\ C \end{bmatrix} S = \begin{bmatrix} sI_{n_1} - A_1 & 0 \\ 0 & sN - I_{n_2} \end{bmatrix}, \tag{3.2}
\]
where $T, S \in \mathbb{R}^{n \times n}$ are invertible, $n_1 + n_2 = n$ and $N$ is nilpotent with index $h$ (i.e., $N^{h-1} \neq 0$, but $N^h = 0$).

**Lemma 3.1** The condition (2.9) holds if and only if

$$\text{rank}(O_{r+2}(N, I_{n_2}, C_2)) = n_2(r + 1) + \text{rank}(N).$$

**Proof.** Straightforward. ■

**Lemma 3.2**

$$\text{rank}(O_{r+2}(N, I_{n_2}, C_2)) = n_2(r + 1) + \text{rank}\begin{bmatrix} N^{r+2} \\ C_2N \\ C_2N^2 \\ \vdots \\ C_2N^{r+1} \end{bmatrix}.$$  

**Proof.** Through a series of elementary row and column transformations, matrix $O_{r+2}(N, I_{n_2}, C_2)$ can be transformed into

$$\begin{bmatrix} 0 & I_{n_2} \\ 0 & \ddots \\ \vdots & \ddots & I_{n_2} \\ N^{r+2} & 0 \\ C_2N & 0 \\ \vdots & \ddots \\ C_2N^{r+1} & 0 \end{bmatrix}$$

and the result follows. ■

**Lemma 3.3** The condition (2.9) holds if and only if

$$\text{rank}\begin{bmatrix} N^{r+2} \\ C_2N \\ C_2N^2 \\ \vdots \\ C_2N^{r+1} \end{bmatrix} = \text{rank}(N). \quad (3.3)$$

**Proof.** Follows from Lemmas 3.1 and 3.2. ■
Corollary 3.2 The following statements are equivalent:

1). The system (1.1) is impulse observable;
2). Condition (3.3) holds for some one \( r \in \{0, 1, \ldots, n - 1\} \);
3). Condition (3.3) holds for each one \( r \in \{0, 1, \ldots, n - 1\} \).

Proof. Follows from Theorem 2.4 and Lemma 3.3 immediately. ■

Remark 3.2 When \( r \geq h - 2 \) with \( h \) the nilpotency index of \( N \), the rank criterion (3.3) gives the condition [2, Theorem 2-3.4, (iv)]; When \( r = 0 \), it gives another condition [2, Theorem 2-3.4, (v)]. We see that these two well known criteria are only two boundary cases of Corollary 3.2.

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