Thermodynamic description of (a)dS black holes in Born-Infeld massive gravity with a non-abelian hair

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We construct a new class of asymptotically (a)dS black hole solutions of Einstein-Yang-Mills massive gravity in the presence of Born-Infeld nonlinear electrodynamics. The obtained solutions possess a Coulomb electric charge, massive term and a non-abelian hair as well. We calculate the conserved and thermodynamic quantities, and investigate the validity of the first law of thermodynamics. Also, we investigate thermal stability conditions by using the sign of heat capacity through canonical ensemble. Next, we consider the cosmological constant as a thermodynamical pressure and study the van der Waals like phase transition of black holes in the extended phase space thermodynamics. Our results indicate the existence of a phase transition which is affected by the parameters of theory.

I. INTRODUCTION

General relativity (GR) of Einstein is one of the most successful theories in theoretical physics. It gave more insightful picture to understanding the gravity and solved some unanswered problems. Despite its amazing achievements to justify some phenomena, such as perihelion precession of Mercury, deflection of light, and gravitational redshift, there are still some unsolved problems in the universe. Among them, one can point out the hierarchy problem, the cosmological constant problem, and the late time accelerated expansion of the Universe. This shows that GR is not the final theory and it is logical to search for a more general and complete theory which be able to solve unanswered problems. GR is a theory which describes massless spin 2 particles \cite{1}. In order to generalize GR into a more effective theory, one can give mass to massless spin 2 particles and consider them as massive spin 2 particles. Such a theory is called massive theory of gravity.

One of the most well known theories of massive gravity is called dRGT model and has been introduced by de Rham, Gabadadze, and Tolley \cite{2,3}, which added a potential contribution to the Einstein-Hilbert action. This potential gives graviton a mass and modifies the dynamics of GR in the IR limit. Although the dRGT massive gravity is almost a successful model, the cosmological solutions do not admit flat FRW metric and theory exhibits a discontinuity at the flat FRW limit \cite{4,5} or the model meets instabilities \cite{6-8}. On the other hand, dRGT model has different modifications which are based on the definition of the reference metric. The most successful one has been introduced by Vegh \cite{9} with the motivation of breaking the translational symmetry. In other words, this model provides an effective bulk description in which momentum is not conserved anymore, and therefore, it includes holographic momentum dissipation. This property is what people needed to study physical systems in the context of gauge/gravity duality. In addition, it was shown that this model is ghost free and stable \cite{10}. The static black hole solutions and magnetic solutions in the presence of this model of massive gravity have been investigated in \cite{11-14} and \cite{15,16}, respectively. Moreover, the thermodynamic properties and van der Waals like phase transition of black holes have been studied \cite{12,17,20}. From the cosmological point of view, it has been shown that it is possible to remove the big bang singularity \cite{21}. In addition, the behavior of different holographic quantities have been investigated in \cite{9,22,20}.

On the other hand, existence of some limitations in the Maxwell theory motivate one to consider nonlinear electrodynamics (NED) \cite{27-35}. Moreover, it was shown that NED can remove both the big bang and black hole singularities \cite{36-41}. In addition, the effects of NED are important in superstrongly magnetized compact objects \cite{42-44}. Considering GR coupled to NED attracts attentions due to its specific properties in gauge/gravity coupling. Besides, NED theories are richer than the linear Maxwell theory and in some special cases they reduce to the Maxwell electrodynamics.

One of the most interesting NED theories has been introduced by Born and Infeld \cite{45,46} in order to remove the divergency of self energy of a point-like charge at the origin. The Lagrangian of Born-Infeld (BI) nonlinear gauge field

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is given by
\[
\mathcal{L}_{BI}(\mathcal{F}_M) = 4\beta^2 \left( 1 - \sqrt{1 + \frac{\mathcal{F}_M}{2\beta^2}} \right),
\]
where \(\beta\) is BI nonlinearity parameter, \(\mathcal{F}_M = F_{\mu
u} F^{\mu\nu}\) is the Maxwell invariant, \(F_{\mu
u} = 2\nabla_{[\mu} A_{\nu]}\) is the Faraday tensor, and \(A_\nu\) is the gauge potential. Using the expansion of this Lagrangian for large value of nonlinearity parameter, leads to the Maxwell linear Lagrangian
\[
\mathcal{L}_{BI}(\mathcal{F}_M) = -\mathcal{F}_M + \frac{\mathcal{F}^2_M}{8\beta^2} + O\left(\frac{1}{\beta^4}\right),
\]
in which we receive the Maxwell Lagrangian at \(\beta \to \infty\). BI NED arises in the low-energy limit of the open string theory \([47, 52]\). From the AdS/CFT correspondence point of view, it has been shown that, unlike gravitational correction, higher derivative terms of nonlinear electrodynamics do not have effect on the ratio of shear viscosity over entropy density \([53]\). Besides, NED theories make crucial effects on the condensation of the superconductor and its energy gap \([54, 55]\). GR in the presence of BI NED has been investigated for static black holes \([56–67]\), wormholes \([68–71]\), and superconductors \([53, 78–80]\). In addition, black hole solutions and their van der Waals like behavior in massive gravity coupled to BI NED have been studied in \([12, 81]\).

On the other hand, in addition to the Maxwell field, one can consider the non-abelian Yang-Mills (YM) field as matter source coupled to gravity. The presence of non-abelian gauge fields in the spectrum of some string models motivates us to consider them coupled to GR. In addition, the YM equations are present in the low energy limit of these models. Considering the YM field coupled to gravity violates the black hole uniqueness theorem and leads to hairy black holes. In such a situation, the field equations become highly nonlinear so that the early attempts for finding the black hole solutions in YM theory were performed numerically.

Nevertheless, Yasskin found the first analytic black hole solutions by using Wu-Yang ansatz \([82]\). Then, the black hole solutions in YM theory have been generalized to Gauss–Bonnet and Lovelock gravity in \([83–84]\) and \([85, 86]\), respectively. In addition, black holes have been investigated in non-Abelian generalization of BI NED in Einstein gravity \([87]\) and regular black holes have been obtained in \([88–91]\). Furthermore, hairy black holes coupled to YM field have been studied in \([92–94]\). Nonminimal Einstein-Yang-Mills (EYM) solutions have been investigated for regular black holes \([93, 96]\), wormholes \([97, 98]\), and monopoles \([99, 100]\). Thermodynamics and \(P–V\) criticality of EYM black holes in gravity’s rainbow have been explored in \([101]\). Besides, the solutions of EYM-dilaton theory have been considered in \([102–108]\). In addition, black holes and their van der Waals like phase transition in Gauss–Bonnet-massive gravity in the presence of YM field have been investigated in \([109]\).

The purpose of this paper is obtaining the exact black hole solutions of Einstein-Massive theory in the presence of YM and BI NED fields, and also, studying the thermal stability and phase transition of these black holes.

II. FIELD EQUATIONS AND BLACK HOLE SOLUTIONS

Here, we consider the following \((3+1)\)-dimensional action of EYM-Massive gravity with BI NED for the model
\[
I_G = -\frac{1}{16\pi} \int_M d^{3+1}x \sqrt{-g} \left( R - 2\Lambda + \mathcal{L}_{BI}(\mathcal{F}_M) - \mathcal{F}_{YM} + m^2 \sum_i c_i U_i(g, f) \right),
\]
where \(\mathcal{L}_{BI}(\mathcal{F}_M)\) and \(\mathcal{F}_{YM} = \text{Tr} \left( F_{(a)} F^{(a)(\mu
u)} \right)\) are, respectively, the Lagrangian of BI NED \([11]\) and the YM invariant. In addition, \(m\) is related to the graviton mass while \(f\) refers to an auxiliary reference metric which its components depend on the metric under consideration. Moreover, \(c_i\)'s are some free constants and \(U_i\)'s are symmetric polynomials of the eigenvalues of \(4 \times 4\) matrix \(K_{\mu\nu} = \sqrt{g^{\mu\sigma}} F_{\sigma\nu}\) which have the following forms
\[
\begin{align*}
U_1 &= |K|, \\
U_2 &= |K|^2 - |K|^2, \\
U_3 &= |K|^3 - 3 |K| |K|^2 + 2 |K|^3, \\
U_4 &= |K|^4 - 6 |K|^2 |K|^2 + 8 |K|^3 |K| + 3 |K|^2 |K|^2 - 6 |K|^4, \\
&\quad \vdots
\end{align*}
\]
where the rectangular bracket represents the trace of $K_{\mu}^\nu$. It is easy to obtain three tensorial field equations which come from the variation of action \( \mathcal{S} \) with respect to the metric tensor $g_{\mu\nu}$, the Faraday tensor $F_{\mu\nu}$, and the YM tensor $F_{\mu\nu}^{(a)}$ as

\[
G_{\mu\nu} + A g_{\mu\nu} = T^{M}_{\mu\nu} + T^{YM}_{\mu\nu} - m^2 \chi_{\mu\nu},
\]

where $\chi_{\mu\nu}$ can be written as

\[
\chi_{\mu\nu} = -\frac{c_1}{2} (U_1 g_{\mu\nu} - K_{\mu\nu}) - \frac{c_2}{2} (U_2 g_{\mu\nu} - 2U_1 K_{\mu\nu} + 2K_{\mu\nu}^2) - \frac{c_3}{2} (U_3 g_{\mu\nu} - 3U_2 K_{\mu\nu} + 6\delta_1 K_{\mu\nu}^2 - 6K_{\mu\nu}^3) - \frac{c_4}{2} (U_4 g_{\mu\nu} - 4U_3 K_{\mu\nu} + 12U_2 K_{\mu\nu}^2 - 24U_1 K_{\mu\nu}^3 + 24K_{\mu\nu}^4) + \ldots.
\]

In addition, the YM tensor $F_{\mu\nu}^{(a)}$ has the following form

\[
F_{\mu\nu}^{(a)} = 2\nabla_{[\mu} A_{\nu]}^{(a)} + f_{(b)(c)}^{(a)} A_{\mu}^{(b)} A_{\nu}^{(c)},
\]

where $A_{\mu}^{(b)}$ is the YM potential and the symbols $f_{(b)(c)}^{(a)}$'s denote the real structure constants of the 3-parameters YM gauge group $SU(2)$ (note: the structure constants can be calculated by using the commutation relation of the gauge group generators).

In order to obtain the spherically symmetric black hole solutions of EYM-Massive theory coupled to BI NED, we restrict attention to the following metric

\[
g_{\mu\nu} = \text{diag} \left[ -f(r), f^{-1}(r), r^2, r^2 \sin^2 \theta \right],
\]

with the following reference metric ansatz \( \mathcal{S} \)

\[
f_{\mu\nu} = \text{diag} \left[ 0, 0, c^2, c^2 \sin^2 \theta \right],
\]

where $c$ is an arbitrary positive constant. Using the metric ansatz \( \mathcal{S} \), $U_i$'s reduce to the following explicit forms \( \mathcal{S} \)

\[
U_1 = 2cr^{-1}, \quad U_2 = 2c^2 r^{-2}, \quad U_i = 0 \quad \text{for} \quad i \geq 3.
\]

Considering the field equations \( \mathcal{S} \) with the following radial gauge potential ansatz

\[
A_{\mu} = h(r) \delta_{\mu}^t,
\]

one can obtain the following differential equation

\[
\beta^2 r E'(r) + 2E(r) \left[ \beta^2 - E^2(r) \right] = 0,
\]

where $E(r) = -h'(r)$ and prime refers to $d/dr$. Solving Eq. \( \mathcal{S} \), we obtain

\[
E(r) = \frac{q}{r^2} \left( 1 + \frac{q^2}{\beta^2 r^4} \right)^{-1/2},
\]
where $q$ is an integration constant which is related to the total electric charge of the black hole. It is clear that in the limit $\beta \to \infty$, Eq. (16) tends to $q/r^2$, and therefore, the Maxwell electric field will be recovered.

Hereafter and for the sake of simplicity, we use the position dependent generators $t_{(r)}$, $t_{(\theta)}$, and $t_{(\varphi)}$ of the gauge group instead of the standard generators $t_{(1)}$, $t_{(2)}$, and $t_{(3)}$. The relation between the basis of $SU(2)$ group and the standard basis are

$$
t_{(r)} = \sin \theta \cos \nu \varphi t_{(1)} + \sin \theta \sin \nu \varphi t_{(2)} + \cos \theta t_{(3)},
\quad t_{(\theta)} = \cos \theta \cos \nu \varphi t_{(1)} + \cos \theta \sin \nu \varphi t_{(2)} - \sin \theta t_{(3)},
\quad t_{(\varphi)} = -\sin \nu \varphi t_{(1)} + \cos \nu \varphi t_{(2)},
$$

(17)

and it is straightforward to show that these generators satisfy the following commutation relations

$$
[t_{(r)}, t_{(\theta)}] = t_{(\varphi)}, \quad [t_{(\varphi)}, t_{(r)}] = t_{(\theta)}, \quad [t_{(\theta)}, t_{(\varphi)}] = t_{(r)}.
$$

(18)

In order to solve the YM field equations (6), just like the electromagnetic case, it is required to choose a gauge potential ansatz. Here, we are interested in the magnetic Wu-Yang ansatz of the gauge potential with the following nonzero components [95, 99]

$$
A_\theta^{(a)} = \delta_\theta^{(a)}, \quad A_\varphi^{(a)} = -\nu \sin \theta \delta_\theta^{(a)},
$$

(19)

where the magnetic parameter $\nu$ is a non-vanishing integer. It is easy to show that the chosen Wu-Yang gauge potential [19] satisfies the YM field equations (6). Using the YM tensor field (10) with Wu-Yang ansatz (19), one can show that the only non-vanishing component of the YM field is

$$
F_{\theta \varphi}^{(r)} = \nu \sin \theta.
$$

(20)

Considering the metric (11) with the electromagnetic (16) and YM fields (20), one can show that the only two different components of the field equations (4) are

$$
tt - \text{component} : e_{tt} = r f'(r) + f(r) - 1 + (\Lambda - 2\beta^2) r^2 - m^2 (c_1 r + c_2 r^2) + \frac{\nu^2}{r^2} + 2\beta \sqrt{q^2 + \beta^2 r^4} = 0,
$$

(21)

$$
\theta \theta - \text{component} : e_{\theta \theta} = \frac{r}{2} f''(r) + f'(r) + (\Lambda - 2\beta^2) r - m^2 c_1 - c_2^2 \frac{r^3}{\sqrt{q^2 + \beta^2 r^4}} = 0,
$$

(22)

Since there is one common unknown function in both $e_{tt}$ and $e_{\theta \theta}$ equations, it is expected to find that the mentioned field equations are not independent. After some manipulations, one can obtain the second order field equation by a suitable combination of first order one as

$$
e_{\theta \theta} = e_{tt} + \frac{1}{r} e_{tt}
$$

(23)

and therefore, the solutions of $e_{tt}$ with an integration constant satisfy $e_{\theta \theta}$ equation, directly. Solving Eq. (21), we can obtain the following metric function

$$
f(r) = 1 - m_0 r - \Lambda r^2 \frac{3}{2} + \frac{\nu^2}{r^2} + m^2 \frac{c_1 r^2 + 2 c_2 r^2}{2r} + \frac{2\beta^2 r^2}{3} (1 - \mathcal{H}_1),
$$

(24)

where $\mathcal{H}_1 = 2 F_1 \left( -\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}; -\frac{\nu^2}{r^2}, \frac{2\beta^2 r^2}{3} \right)$ is a hypergeometric function and $m_0$ is the only integration constant which is related to the total mass of black hole. Considering the obtained $f(r)$, one finds that the fourth term is related to the magnetic charge (hair), the fifth term is related to the massive gravitons, and finally, the last term comes from the nonlinearity of electric charge. It is notable to mention that for the massless graviton, $m = 0$, and linear electrodynamics, $\beta \to \infty$, the metric function (21) reduces to the EYM solution with Maxwell field, as we expected. Considering Eq. (24), it is clear that the asymptotical behavior of the solutions is adS (or dS) provided $\Lambda < 0$ (or $\Lambda > 0$).

In order to find the singularity of the solutions, one can obtain the Kretschmann scalar as

$$
R_{\mu \nu \lambda \kappa} R^{\mu \nu \lambda \kappa} = \frac{4}{r^4} \left[ 1 + f^2(r) - 2 f(r) + \left( \frac{2 f''(r)}{r} \right)^2 + \left( \frac{r^2 f''(r)}{2} \right)^2 \right],
$$

(25)
which by inserting (24), it is straightforward to show that the Kretschmann scalar has the following behavior
\[
\lim_{r \to 0} \left( R_{\mu \nu \lambda \kappa} R^{\mu \nu \lambda \kappa} \right) = \infty, \quad \lim_{r \to \infty} \left( R_{\mu \nu \lambda \kappa} R^{\mu \nu \lambda \kappa} \right) = \frac{8 \Lambda^2}{3}.
\] (26)

Equation (26) shows that there is an essential singularity located at the origin, \( r = 0 \). Moreover, the asymptotical behavior of the Kretschmann scalar for the large enough \( r \) confirms that the solutions are asymptotically (a)dS. Moreover, this singularity can be covered with an event horizon (for \( \Lambda < 0 \)), and therefore, one can interpret the singularity as a black hole (Fig. 1). As a final point of this section, we should note that the metric function can possess more than two real positive roots which this behavior is due to giving mass to the gravitons (see [12, 14] for more details).

III. THERMODYNAMICS

A. Conserved and thermodynamic quantities

Here, we first obtain the conserved and thermodynamic quantities of the black hole solutions, and then examine the validity of the first law of thermodynamics.

The Hawking temperature of the black hole on the event (outermost) horizon, \( r_+ \), can be obtained by using the definition of surface gravity, \( \kappa \),
\[
T = \frac{k}{2 \pi} = \frac{1}{2 \pi} \sqrt{- \frac{1}{2} \left( \nabla \mu \chi^\nu \right) \left( \nabla^\mu \chi^\nu \right)},
\] (27)
where \( \chi = \partial_t \) is the null Killing vector of the horizon. Thus, the temperature is obtained as
\[
T = \left. \frac{f'(r)}{4 \pi} \right|_{r = r_+} = \frac{1}{4 \pi r_+} \left[ 1 - \Lambda r_+^2 - \frac{\mu^2}{r_+^2} + m^2 \left( cc_1 r_+^2 + c^2 c_2 \right) + 2 \beta^2 r_+^2 \left( 1 - \sqrt{1 + \frac{q^2}{\beta^2 r_+^4}} \right) \right].
\] (28)

It is worthwhile to mention that fourth term of RHS of Eq. (28) does not depend on the horizon radius, and therefore, one can regard it as a constant background temperature, \( T_0 = \frac{m^2 cc_1}{4 \pi} \). As a result, we can investigate the solutions by using an effective temperature, \( \hat{T} = T - T_0 \).

The electric potential \( \Phi \), measured at infinity with respect to the horizon \( r_+ \), is obtained by
\[
\Phi = A_\mu \chi^\mu \bigg|_{r \to \infty} - A_\mu \chi^\mu \bigg|_{r = r_+} = \frac{q}{r_+} _2 F_1 \left( \frac{1}{2}, \frac{1}{4}; \frac{5}{4}; - \frac{q^2}{\beta^2 r_+^4} \right).
\] (29)

Since we are working in the context of Einstein gravity, the entropy of the black holes still obeys the so called area law. Therefore, the entropy of black holes is equal to one-quarter of the horizon area with the following explicit form
\[
S = \pi r_+^2.
\] (30)

In order to obtain the electric charge of the black hole, we use the flux of the electric field at infinity, yielding
\[
Q_E = q.
\] (31)

It was shown that by using the Hamiltonian approach, one can obtain the total mass \( M \) in the context of massive gravity as [11]
\[
M = \frac{m_0}{2},
\] (32)
where \( m_0 \) comes from the fact that \( f(r = r_+) = 0 \).

Now, we are in a position to check the validity of the first law of thermodynamics. To do so, we use the entropy (30), the electric charge (31), and the mass (32) to obtain mass as a function of entropy and electric charge
\[
M(S, Q_E) = \frac{1}{2} \left( \frac{S}{\pi} \right)^{3/2} \left[ \frac{\pi}{S} - \frac{\Lambda}{3} + \left( \frac{\pi \nu}{S} \right)^2 + \frac{2 \beta^2}{3} \left( 1 - H_3 \right) \right] + \frac{m^2}{4 \pi} \left( cc_1 S + 2 c^2 c_2 \sqrt{\frac{S}{\pi}} \right), \] (33)
where $\mathcal{H}_3 = 2F_1 \left( -\frac{1}{2}, -\frac{1}{4}; \frac{3}{4}; -\left( \frac{2\Omega}{m^2 c^2} \right)^2 \right)$. We consider the entropy ($S$) and electric charge ($Q_E$) as a complete set of extensive parameters, and define the temperature ($T$) and electric potential ($\Phi_E$) as the intensive parameters.
conjugate to them
\[ T = \left( \frac{\partial M}{\partial S} \right)_{Q_E} = \frac{1}{4\pi} \sqrt{\frac{\pi}{S}} \left[ 1 - \Lambda S - \frac{\pi^2}{S} + 2\beta^2 S \pi \left( 1 - H_3 \right) - \frac{4\beta^2 S^2}{3\pi} \left( \frac{dH_3}{dS} \right) + m^2 \left( \frac{c_1 \sqrt{\frac{S}{\pi}} + e^2 c_2}{} \right) \right], \tag{34} \]

\[ \Phi_E = \left( \frac{\partial M}{\partial Q_E} \right)_S = \sqrt{\frac{\pi}{S}} Q_E \left[ F_1 \left( \frac{1}{2}, \frac{1}{4}; \frac{5}{4}; -\frac{\pi Q_E}{\beta S} \right)^2 \right]. \tag{35} \]

Using Eqs. (30) and (31), one can easily show that the temperature \tag{34} and electric potential \tag{35} are, respectively, equal to Eqs. (28) and (29). Thus, these quantities satisfy the first law of thermodynamics
\[ dM = T dS + \Phi_E dQ_E. \tag{36} \]

On the other hand, the obtained black holes enjoy a global YM charge as well. In order to find this magnetic charge, we use the following definition
\[ Q_{YM} = \frac{1}{4\pi} \int \sqrt{F^{(a)}_{\theta\phi} F^{(a)}_{\theta\phi}} d\theta d\phi = \nu. \tag{37} \]

In order to complete the first law of thermodynamics in differential form \tag{36}, one can consider the YM charge as an extensive thermodynamic variable and introduce an effective YM potential conjugate to it as an intensive variable
\[ \Phi_{YM} = \left( \frac{\partial M}{\partial Q_{YM}} \right)_{S,Q_E} = \left( \frac{\partial M}{\partial \nu} \right)_{S,Q_E} \left/ \left( \frac{\partial Q_{YM}}{\partial \nu} \right)_{S,Q_E} \right\} = \nu \sqrt{\frac{\pi}{S}} = \frac{\nu}{r_+}. \tag{38} \]

which satisfies the first law of thermodynamics in a more complete way
\[ dM = T dS + \Phi_E dQ_E + \Phi_{YM} dQ_{YM}. \tag{39} \]

Regarding differential form of the first law, it is worth mentioning that this equation may be completed by other additional terms, such as \( V dP \) in the extended phase space. In order to check the validity of the existence of such terms, one should check the first law in a non-differential form, the so-called Smarr relation. After some manipulations, one can find that
\[ M = 2TS + \Phi_{YM} Q_{YM} - 2VP - B\beta - Cc_1, \tag{40} \]

where
\[ P = \frac{\Lambda}{8\pi}, \quad V = \left( \frac{\partial M}{\partial P} \right)_{S,Q_E,Q_{YM},c_1}, \quad B = \left( \frac{\partial M}{\partial \beta} \right)_{S,Q_E,Q_{YM},P,c_1}, \quad C = \left( \frac{\partial M}{\partial c_1} \right)_{S,Q_E,Q_{YM},P,\beta}, \tag{41} \]

which confirm that the existence of additional terms and leads to a more complete form of the first law of thermodynamics
\[ M = T dS + \Phi_{YM} dQ_{YM} + V dP + B d\beta + C dc_1. \tag{42} \]

**B. Thermal Stability**

In this section, we use the heat capacity for investigating thermal stability of the obtained black hole solutions. In this regard, one should consider the sign of heat capacity (its positivity and negativity) to study the stability conditions. The root of heat capacity (or temperature) represents a bound point. This point is a kind of border which is located between physical black holes related to the positive temperature and non-physical ones with negative temperature. On the other hand, in our case, both divergence points of the heat capacity indicate one thermal phase transition point where black holes jump from one divergency to the other one. Besides, the heat capacity changes sign at such divergence points. So, one can conclude that the divergence point is a kind of bound-like point which is located between unstable black holes with negative heat capacity and stable (or metastable) ones. Therefor, it is logical to say that the physical stable black holes are located everywhere that both the heat capacity and temperature are positive, simultaneously.
Here, we study thermal stability of the asymptotically adS solutions with \( \Lambda < 0 \). The heat capacity at constant both electric and YM charges is given by

\[
C_{Q,E,YM} = \frac{T}{\left( \frac{\partial^2 M}{\partial S^2} \right)_{Q,E,YM}},
\]

where \( T \) has been obtained in Eq. (25). Considering (30), (31) and (33), one can easily show that the denominator of heat capacity is

\[
\left( \frac{\partial^2 M}{\partial S^2} \right)_{Q,E,YM} = \frac{1}{8\pi^2 r_+^3} \left[ (\beta^2 - \Lambda) r_+^2 - 1 - m^2 c^2 c_2 + \frac{3
u^2}{r_+^2} + 2\beta q \left( 1 + \frac{\beta^2 r_+^4}{q^2} \right)^{-1/2} \left( 1 - \frac{\beta^2 r_+^4}{q^2} \right) \right].
\]

We recall that thermal stability criteria are based on the sign of heat capacity and it may change at root and divergence points. As a result, it is necessary to look for the root and divergence points of the heat capacity in first step. But unfortunately, because of the complexity of Eq. (13), it is not possible to obtain the root and divergencies of the heat capacity, analytically. So, we adopt the numerical analysis to obtain both bound and thermal phase transition points.

Before applying the numerical calculations, we are interested to clarify the general behavior of the heat capacity and temperature for the small and large black holes. For the fixed values of different parameters, there could exist two special \( r_+ \)'s, named \( r_{+\min} \) and \( r_{+\max} \). The small black holes and large black holes are located before \( r_{+\min} \) and after \( r_{+\max} \), respectively. The region of \( r_{+\min} < r_+ < r_{+\max} \) belongs to the intermediate black holes. Using the series expanding of (43), one obtains

\[
\begin{cases}
C_{Q,E,YM} = -\frac{2\pi}{3} r_+^2 + O\left( r_+^4 \right), & \text{for small } r_+, \\
T = -\frac{r_+^2}{4\pi r_+^3} + O\left( \frac{1}{r_+} \right), & \text{for large } r_+.
\end{cases}
\]

(45)

Considering Eq. (45), it is clear that for sufficiently small \( r_+ \), the heat capacity and temperature are negative, and therefore, we have unstable and non-physical black hole. Whereas from Eq. (46), we find that for large \( r_+ \), both heat capacity and temperature are positive and there exists stable and physical black hole. In other words, Eqs. (45) and (46) confirm that the small black holes (\( r_+ < r_{+\min} \)) are unstable and non-physical, whereas the large black holes (\( r_+ > r_{+\max} \)) are physical and enjoy thermal stability. It is notable that in a special case there is just one critical horizon radius, \( r_{+\crit} \). In this case, we have unstable black holes for \( r_+ < r_{+\crit} \) and stable ones for \( r_+ > r_{+\crit} \). However, it is not possible to identify this last property analytically, but we show it in Fig. 2 (see continues line).

Now, we back to numerical analysis of the heat capacity. Although we studied the general behavior of the heat capacity for the small and large black holes, the numerical calculations help us to classified the intermediate black holes (\( r_{+\min} < r_+ < r_{+\max} \)). However, we do not study all possible behaviors of the heat capacity (because they contain different cases due to lots of free parameters) and just take some interesting ones.

Figure 2 shows some different possibilities for the heat capacity. Clearly, this figure confirms that the small black holes are unstable (Eq. (45)) and large black holes are stable (Eq. (46)). According the numerical analysis, we find that the heat capacity contains (i) only one bound point, (ii) one bound point and two divergencies, and (iii) three bound points and two divergencies. In the first case, we have unstable and non-physical black holes before the bound point (\( r_{+\crit} \)), but after this point, stable and physical black holes are presented. It is worthwhile to recall that from Eqs. (45) and (46), we expected such behavior. In the second case, we have stable and physical solutions between the bound point and smaller divergency. There are physical and unstable black holes between two divergencies. It is notable to mention that the large black holes are stable and physical as well. As for the last case, there are stable and unstable solutions respectively before and after the larger divergency.
In addition, we investigate the effects of different parameters on the bound points and divergencies of the heat capacity for $\Lambda = c_1 = -1, q = \beta = \nu = c = 1, \text{ and } c_2 = 2$.

**Table I**: case (i): The root of heat capacity for $\Lambda = -1, c = 1, c_1 = -1, \text{ and } c_2 = 2$.

| $m$ | $\beta$ | $q$ | $\nu$ | $r_{+\text{crit}}$ |
|-----|---------|-----|-------|-----------------|
| 1.0 | 1.0     | 1.0 | 1.0   | 0.7705          |
| 1.1 | 1.0     | 1.0 | 1.0   | 0.7311          |
| 1.2 | 1.0     | 1.0 | 1.0   | 0.6901          |
| 1.0 | 2.0     | 1.0 | 1.0   | 0.8141          |
| 1.0 | 3.0     | 1.0 | 1.0   | 0.8255          |
| 1.0 | 1.0     | 2.0 | 1.0   | 1.0941          |
| 1.0 | 1.0     | 3.0 | 1.0   | 1.4391          |
| 1.0 | 1.0     | 1.0 | 2.0   | 1.2243          |
| 1.0 | 1.0     | 3.0 | 1.0   | 1.5904          |

**Table II**: case (ii): The root and divergencies of the heat capacity for $\Lambda = -1, c = 1, c_1 = -1, \text{ and } c_2 = 2$.

| $m$ | $\beta$ | $q$ | $\nu$ | $r_{+\text{min}}$ | $r_{+\text{max}}$ |
|-----|---------|-----|-------|------------------|------------------|
| 2.0 | 1.0     | 1.0 | 1.0   | 0.4142           | 0.6974           |
| 2.1 | 1.0     | 1.0 | 1.0   | 0.3913           | 0.6530           |
| 2.2 | 1.0     | 1.0 | 1.0   | 0.3706           | 0.6146           |
| 2.0 | 2.0     | 1.0 | 1.0   | 0.4602           | 0.7918           |
| 2.0 | 5.0     | 1.0 | 1.0   | 0.5111           | 0.8412           |
| 2.0 | 1.0     | 1.5 | 1.0   | 0.4580           | 0.7872           |
| 2.0 | 1.0     | 2.0 | 1.0   | 0.5203           | 0.9384           |
| 2.0 | 1.0     | 1.5 | 1.5   | 0.6391           | 1.0627           |
| 2.0 | 1.0     | 2.0 | 0.8702 | 1.4623           | 2.6076           |

**Table III**: case (iii): The root and divergence points of the heat capacity for $\Lambda = -1, c = 1, c_1 = -1, \text{ and } c_2 = 2$.

In addition, we investigate the effects of different parameters on the bound points and divergencies of the heat capacity for $\Lambda = -1, c = 1, c_1 = -1, \text{ and } c_2 = 2$. 
capacity in tables $I - III$. From table $I$, we find that the critical horizon radius, $r_{+\text{crit}}$, increases as the electric (magnetic) charge of black hole increases too. This could happen when the black hole absorb electric (magnetic) charge. As a result, the region of unstable black holes increases. When the nonlinearity parameter increases and the nonlinear theory tends to the Maxwell case, the critical horizon radius increases. On the contrary, $r_{+\text{crit}}$ is a decreasing function of the graviton mass ($m$). So, by increasing $m$, the region of unstable black holes decreases. Considering table $II$, it is clear that the smaller root ($r_{+\text{min}}$) and smaller divergency are decreasing functions of $m$, but the larger divergency ($r_{+\text{max}}$) increases as the massive parameter increases. In addition, we have found the same effects for $\beta$, $q$, and $\nu$, but opposite behavior is seen for $m$. Table $III$ shows that the smaller root ($r_{+\text{min}}$), the smaller divergency, and middle root decrease as the massive parameter increases, whereas the larger divergency and the larger root ($r_{+\text{max}}$) are increasing functions of $m$. Like case (ii), one can see the same behavior for $\beta$, $q$, and $\nu$. The smaller divergency, $r_{+\text{min}}$, and $r_{+\text{max}}$ are increasing functions of these parameters ($m$, $\beta$, $q$, and $\nu$), but the middle and larger divergency are decreasing functions of them. Based on these three tables, we conclude that the qualitative effects of $\beta$, $q$, and $\nu$ on the heat capacity are quite the same.

C. $P - V$ criticality in the extended phase space

It is well known that most of black holes can go under a van der Waals like phase transition when one considers the cosmological constant as a thermodynamic constant. In this section, we employ this analogy between the cosmological constant and pressure in the canonical ensemble (fixed $Q_{E}$, $Q_{YM}$, $\beta$, and $c_1$) to investigate the $P - V$ criticality and study phase transition of obtained black holes in extended phase space. Using the temperature given in Eq. (28) and the relation of $P = -\Lambda/8\pi$, it is straightforward to show that the equation of state is given by

$$P(r_+, \hat{T}) = \frac{\hat{T}}{2r_+} - \frac{1}{8\pi r_+^2} \left[ 1 - \frac{\nu^2}{r_+^2} + m^2c^2c_2 + 2\beta^2r_+^2 \left( 1 - \sqrt{1 + \frac{q^2}{\beta^2r_+^4}} \right) \right], \quad (47)$$

where $\hat{T} = T - \frac{m^2c}{4\pi}$ and we made this choice in order to have unique critical temperature (see appendix for more details). The thermodynamic volume is an extensive parameter which is conjugated to the pressure and has the following form

$$V = \left( \frac{\partial H}{\partial P} \right)_S, \quad (48)$$

where $H$ is the enthalpy of system. In this perspective, the total mass of black hole plays the role of enthalpy instead of internal energy due to the fact that the cosmological constant is not a fixed parameter anymore and it is actually a thermodynamic variable. Therefore, the thermodynamic volume is calculated as

$$V = \frac{4}{3} \pi r_+^3. \quad (49)$$

Hereafter, we use $r_+$ instead of $V$ as a thermodynamic variable since it is proportional to the specific volume [110, 111]. In order to study the phase transition of the black holes, we need to obtain the Gibbs free energy. In this extended phase space, one can determine the Gibbs free energy by using the following definition

$$G = H - TS = -\frac{2\pi r_+^3}{3} P + \frac{3\nu^2}{4r_+} + \frac{r_+}{4} \left( 1 + m^2c^2c_2 \right) - \frac{\beta^2r_+^3}{6} \left( 1 + 2\mathcal{H}_1 + 3\sqrt{1 + \frac{q^2}{\beta^2r_+^4}} \right), \quad (50)$$

where $\mathcal{H}_1 = \mathcal{H}_1(r = r_+)$. In addition, using the properties of inflection point

$$\left( \frac{\partial P(r_+, \hat{T})}{\partial r_+} \right)_{\hat{T} = \hat{T}_{c}, r_+ = r_{+c}} = \left( \frac{\partial^2 P(r_+, \hat{T})}{\partial r_+^2} \right)_{\hat{T} = \hat{T}_{c}, r_+ = r_{+c}} = 0, \quad (51)$$

and after some manipulations, we obtain the following equation

$$(1 + m^2c^2c_2) r_{+c}^2 - 6\nu^2 - 2q^2 \left( 3 + \frac{q^2}{\beta^2r_{+c}^4} \right) \left( 1 + \frac{q^2}{\beta^2r_{+c}^4} \right)^{-3/2} = 0. \quad (52)$$

Considering this equation, we find that it is not possible to obtain the critical horizon radius, $r_{+c}$, analytically. As a result, we will not be able to calculate, analytically, the other critical parameters as well. So, we use the numerical
FIG. 3: \( G - \hat{T}, \ r_+ - \hat{T}, \) and \( C_P - \hat{T} \) diagrams for \( P = P_c, \ \nu = 1, \ q = 2, \ \beta = 1, \ m = 3, \ c = 1, \) and \( c_2 = 2. \) The vertical dashed line represents the critical temperature, \( \hat{T}_c. \) The discontinuity is present in the first differential of the Gibbs free energy (due to existence of latent heat) in the middle diagram and there is a sharp spike in the right diagram at critical temperature.

FIG. 4: \( P - r_+, \ G - \hat{T}, \) and \( P - \hat{T} \) diagrams for \( \nu = 1, \ q = 2, \ \beta = 1, \ c = 1, \ c_2 = 2, \) and \( m = 3. \)

FIG. 5: \( G - \hat{T} \) and \( C_P - r_+ \) diagrams for \( P = 0.85 P_c, \ \nu = 1, \ q = 2, \ \beta = 1, \ c = 1, \ c_2 = 2, \) and \( m = 3. \) The path \( A - B \) indicates unstable black holes which is equivalence to the negative heat capacity between two divergencies. The path \( A - C \) (\( B - C \)) indicates metastable black holes which is equivalence to the positive heat capacity between the larger (smaller) divergence and \( C_2 (C_1). \) The SBH-LBH phase transition occurs at point \( C \) in \( G - \hat{T} \) diagram, and a jump between points \( C_1 \) and \( C_2 \) in \( C_P - r_+ \) diagram.
analysis in order to study the van der Waals like phase transition of the black holes. In addition, we use such numerical analysis for investigating the effects of different parameters on the critical quantities.

Paul Ehrenfest has categorized the phase transition of thermodynamical systems based on the discontinuity in derivatives of the Gibbs free energy. The order of a phase transition is the order of the lowest differential of the Gibbs free energy that shows a discontinuity at the critical temperature. Thus, in a first order phase transition, there exists a discontinuity in the first derivative of \( G \) (the entropy or volume). In this situation, the heat capacity, which is a second derivative of \( G \), at constant pressure is given by

\[
C_P = \hat{T} \left( \frac{\partial S}{\partial \hat{T}} \right)_P = \frac{8\hat{T}\pi^2 r_+^3 \sqrt{q^2 + \beta^2 r_+^4}}{2\beta r_+^2 (q^2 - \beta^2 r_+^4) + \sqrt{q^2 + \beta^2 r_+^4} \left[ 3\nu^2 + 2r_+^4 (4\pi P + \beta^2) - r_+^2 (1 + m^2 c^2 c_2) \right]} \tag{53}
\]

where shows a sharp spike. Clearly, Fig. 3 confirms that the black holes under consideration enjoy first order phase transition.

For instance, we plot \( P - r_+ \) isotherm, \( G - \hat{T} \), and \( P - \hat{T} \) diagrams for some fixed parameters to show the general phase transition behavior of the solutions (Fig. 4). Considering Fig. 4 we find that the obtained black holes have a van der Waals like phase transition between small black holes (SBH) and large black holes (LBH), and therefore, they enjoy a first order phase transition. In this figure, \( P - r_+ \) isotherms show SBH area on the left, SBH+LBH coexistence area in the middle, and LBH area on the right. The dotted curve is a boundary between the regions of SBH, SBH+LBH, and LBH in the \( P - r_+ \) diagram. For temperatures above the critical temperature, there is no physical distinction between SBH and LBH phases, and this area is denoted as the supercritical region. In addition, in \( G - \hat{T} \) diagram, the phase transition point is located at the cross point, where SBH+LBH are presented, and black holes always choose the lowest energy. Moreover, \( P - \hat{T} \) diagram indicates the coexistence line between SBH and LBH which terminates at the critical point. The critical point is located at the topmost of the coexistence line with \( P = P_c, r_+ = r_{+c}, \) and \( \hat{T} = \hat{T}_c \). If black hole crosses the coexistence line from left to right or top to bottom, the system goes under a first order phase transition from SBH to LBH. Above the critical point, SBH and LBH are physically indistinguishable which is denoted by supercritical region.

From the left panel of Fig. 5 one can see that the red dashed (solid green) line corresponds to the negative (positive) heat capacity at constant pressure, \( C_P \), in the right panel. In addition, the divergencies of \( C_P \) is indicated by two small black points \( A \) and \( B \) in \( G - \hat{T} \) diagram. The path bounded by these points is unconditionally unstable, but the paths \( A - C \) and \( B - C \) are metastable. Equivalently, in \( C_P \) diagram, the region between point \( C_1 \) (\( C_2 \)) and smaller (larger) divergency is metastable, and SBH-LBH phase transition does occur between \( C_1 \) and \( C_2 \). This figure shows that during the phase transition from SBH to LBH, the heat capacity of the system increases. In addition, Fig. 6 shows that the generalization of Einstein-Maxwell black holes into massive gravity and YM theory has significant

![FIG. 6: The coexistence curve of EYM-BI-Massive and Reissner-Nordström black holes for \( \nu = 1, q = 2, \beta = 1, c = 1, c_1 = 0, c_2 = 2, \) and \( m = 3 \).](image)
Effect on the Reissner-Nordström black holes. In this theory, the region of SBH and LBH increases, and therefore, there is van der Waals like phase transition for higher temperatures and pressures compare with Reissner-Nordström black holes.

\[ q \beta \nu m r_{+} T_{c} P_{c} \]

| q | \( \beta \) | \( \nu \) | \( m \) | \( r_{+} \) | \( T_{c} \) | \( P_{c} \) |
|---|---|---|---|---|---|---|
| 2.0 | 1.0 | 1.0 | 3.0 | 0.6374 | 2.5362 | 0.6900 |
| 2.1 | 1.0 | 1.0 | 3.0 | 0.6416 | 2.4855 | 0.6694 |
| 2.2 | 1.0 | 1.0 | 3.0 | 0.6460 | 2.4353 | 0.6492 |
| 2.0 | 1.2 | 1.0 | 3.0 | 0.6603 | 2.3547 | 0.5966 |
| 2.0 | 1.4 | 1.0 | 3.0 | 0.6893 | 2.1881 | 0.5106 |
| 2.0 | 1.0 | 1.1 | 3.0 | 0.7035 | 2.3138 | 0.5652 |
| 2.0 | 1.0 | 1.2 | 3.0 | 0.7708 | 2.1305 | 0.4718 |
| 2.0 | 1.0 | 1.0 | 3.1 | 0.6120 | 2.8472 | 0.8145 |
| 2.0 | 1.0 | 1.0 | 3.2 | 0.5888 | 3.1814 | 0.9538 |

Table IV: The effects of different parameters on the critical values of the horizon radius, temperature, and pressure for \( c = 1 \) and \( c_2 = 2 \).

In order to study the effects of different parameters on the critical points, we take table IV based on the numerical analysis. It is worthwhile to mention that by increasing the critical temperature and pressure, the region of SBH and LBH increases, and therefore, the region of phase transition increases too. From table IV, we find that the critical horizon radius is a decreasing function of massive parameter and the critical temperature and pressure are increasing functions of this parameter. Considering table IV, one can see opposite behavior for the other parameters such as \( q \), \( \beta \) and \( \nu \). In other words, the critical horizon radius is an increasing function of these parameters, whereas the critical temperature and pressure are decreasing functions of them.

IV. CONCLUSIONS

In this paper, we have obtained Einstein-Massive black hole solutions in the presence of YM and BI NED fields. We have also studied the geometric properties of the solutions and it was shown that there is an essential singularity at the origin which can be covered with an event horizon. In addition, we have calculated the conserved and thermodynamical quantities, and it was shown that even though the YM and BI NED fields modify the solutions, the first law of thermodynamics is still valid.

Moreover, we have studied thermal stability of the obtained black holes, and investigated the effects of different parameters on the stability conditions. We have found that the large black holes (\( r_{+} > r_{+\text{max}} \)) are physical and stable, whereas the small black holes (\( r_{+} < r_{+\text{min}} \)) are non-physical (\( T < 0 \)). Furthermore, we have classified the medium black holes (\( r_{+\text{min}} < r_{+} < r_{+\text{max}} \)) in Fig. 2 and investigated the effects of different parameters on thermal stability of these black holes in tables I – III.

In addition, we have considered the cosmological constant as thermodynamical pressure and it was shown that the obtained black holes enjoy the first order phase transition between SBH and LBH. Also, we have studied this kind of phase transition in the heat capacity diagram and specified the unstable and metastable phases of obtained black holes related to the negative and positive heat capacities, respectively. It was shown that during the phase transition from SBH to LBH, the heat capacity of system increases. We have seen that the generalization of Reissner-Nordström solutions into massive gravity and YM theory increases the critical temperature and pressure, and as a result, the region of SBH and LBH increases. Moreover, we have investigated the effects of different parameters on the critical points, and we found that the parameters \( q \), \( \beta \), and \( \nu \) have opposite effect on the critical points compared to massive parameter, \( m \).

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Appendix A: EYM-Maxwell black holes in massive gravity

Here, we give a brief study regarding $P - V$ criticality of EYM-Maxwell black holes in massive gravity. In order to find the related equation of state, one can use the expansion of the metric function (24) for large value of nonlinearity parameter, $\beta$, and follow the same procedure given in Sec. III C, which leads to

$$P(r_+, T) = \frac{T}{2r_+} - \frac{1}{8\pi r_+^3} \left[ 1 - \frac{q^2 + \nu^2}{r_+^3} + m^2 \left( r_+ c_1 + c_2^2 c_2 \right) \right]. \quad (A1)$$

Using the definition of inflection point (61), we can find the critical horizon radius, temperature, and pressure as follows

$$r_{+c} = \sqrt{\frac{6 (q^2 + \nu^2)}{1 + m^2 c_2^2 c_2}}, \quad (A2)$$

$$T_c = \frac{m^2 c_1}{4\pi} + \frac{\left[ 1 + m^2 c_2^2 c_2 \left( 2 + m^2 c_2^2 c_2 \right) \right]}{3\pi \sqrt{6 (q^2 + \nu^2) (1 + m^2 c_2^2 c_2)}}, \quad (A3)$$

$$P_c = \frac{1 + m^2 c_2^2 c_2 \left( 2 + m^2 c_2^2 c_2 \right)}{96\pi (q^2 + \nu^2)}. \quad (A4)$$

Considering equations mentioned above, we find that $T_c$ depends on $c_1$, but $r_{+c}$ and $P_c$ are independent from this parameter. This means that for the fixed values of $r_{+c}$ and $P_c$, there is infinite $T_c$ for the system depending on the value of $c_1$! So, in order to get rid of this situation, we define $\frac{m^2 c_1}{4\pi}$ as a background temperature, $T_0$, and rescale the critical temperature into

$$\hat{T}_c = T_c - T_0 = \frac{\left[ 1 + m^2 c_2^2 c_2 \left( 2 + m^2 c_2^2 c_2 \right) \right]}{3\pi \sqrt{(q^2 + \nu^2) (1 + m^2 c_2^2 c_2)}}, \quad (A5)$$

which shows a unique critical temperature.

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