Simultaneous packing and covering in the Euclidean plane II

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Dedicated to Professor Edmund Hlawka on the occasion of his 90th birthday

Abstract

In 1950, C.A. Rogers introduced and studied the simultaneous packing and covering constants for a convex body and obtained the first general upper bound. Afterwards, they have attracted the interests of many authors such as L. Fejes Tóth, S.S. Ryshkov, G.L. Butler, K. Böröczky, H. Horváth, J. Linhart and M. Henk since, besides their own geometric significance, they are closely related to the packing densities and the covering densities of the convex body, especially to the Minkowski-Hlawka theorem. However, so far our knowledge about them is still very limited. In this paper we will determine the optimal upper bound of the simultaneous packing and covering constants for two-dimensional centrally symmetric convex domains, and characterize the domains attaining the upper bound.

MSC: primary 52C17; secondary 11H31

Keywords: packing density; covering density; Minkowski-Hlawka theorem; affinely regular hexagon; affinely regular octagon

1. Introduction

In 1950, C.A. Rogers introduced and studied two constants $\gamma(K)$ and $\gamma^*(K)$ for an $n$-dimensional convex body $K$. Namely, $\gamma(K)$ is the smallest positive number $r$ such that there is a translative packing $K + X$ satisfying $E^n = rK + X$, and $\gamma^*(K)$ is the smallest positive number $r^*$ such that there is a lattice packing $K + \Lambda$ satisfying $E^n = r^*K + \Lambda$. In some references, they are called the simultaneous packing and covering constants for the convex body. Clearly, these numbers are closely related to the packing densities and the covering densities of the convex body, especially to the Minkowski-Hlawka theorem.

In 1970 and 1978, S.S. Ryškov and L. Fejes Tóth independently introduced and investigated two related numbers $\rho(K)$ and $\rho^*(K)$, where $\rho(K)$ is the largest positive number $r$ such that one can put a translate of $rK$ into every translative...
packing $K + X$, and $\rho^*(K)$ is the largest positive number $r^*$ such that one can put a translate of $r^*K$ into every lattice packing $K + \Lambda$.

Clearly, for every convex body $K$ we have

$$\gamma(K) \leq \gamma^*(K)$$

and

$$\rho(K) \leq \rho^*(K).$$

As usual, let $C$ denote an $n$-dimensional centrally symmetric convex body. Then, we also have

$$\gamma(C) = \rho(C) + 1$$

and

$$\gamma^*(C) = \rho^*(C) + 1.$$

Let $B^n$ denote the $n$-dimensional unit ball. Just like the packing density problem and the covering density problem, to determine the values of $\gamma(B^n)$ and $\gamma^*(B^n)$ is important and interesting. However, so far our knowledge about $\gamma(B^n)$ and $\gamma^*(B^n)$ is very limited. We list the main known results in the following table.

| $n$ | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|
| $\gamma^*(B^n)$ | $\frac{\sqrt{3}}{3}$ | $\frac{\sqrt{2}}{2}$ | $\sqrt{2} \sqrt{3(\sqrt{3} - 1)}$ | $\frac{\sqrt{3}}{3} + \frac{\sqrt{13}}{6}$ |
| Author | Böröczky [1] | Horváth [12] | Horváth [13] |

Let $\delta(K)$ and $\delta^*(K)$ denote the maximal translative packing density and the maximal lattice packing density of $K$, respectively. A fundamental problem in Packing and Covering is to determine if

$$\delta(K) = \delta^*(K)$$

holds for every convex body. It is easy to see that $\gamma^*(C) \geq 2$ will imply

$$\delta(C) \geq 2\delta^*(C),$$

which will give a negative answer to the previous problem. On the other hand, if $\gamma^*(C) \leq 2 - k$ holds for a positive constant $k$ and for every centrally symmetric convex body $C$, then the Minkowski-Hlawka theorem can be improved to

$$\delta^*(C) \geq \frac{1}{(2 - k)^n}.$$  (2)

In 1950, C.A. Rogers [18] discovered a constructive method by which he deduced

$$\gamma^*(C) \leq 3$$

In 1950, C.A. Rogers [18] discovered a constructive method by which he deduced
for all $n$-dimensional centrally symmetric convex bodies. In 1972, by mean value techniques developed by C.A. Rogers and C.L. Siegel, the above upper bound was improved by G.L. Butler [3] to

$$\gamma^*(C) \leq 2 + o(1).$$

This result is fascinating, because it gives hope to both (1) and (2).

In two and three dimensions, as one can imagine, the situation is much better. In 1978, based on an ingenious idea of I. Fáry [6], J. Linhart [15] proved that

$$\gamma(K) = \gamma^*(K) \leq \frac{3}{2}$$

holds for every two-dimensional convex domain and the upper bound is attained only by triangles. However, just like the packing density problem, to determine the best upper bound for $\gamma^*(C)$ turns out to be much more challenging. Recently C. Zong [22] and [23] obtained

$$\gamma^*(C) \leq 1.2$$

for all two-dimensional centrally symmetric convex domains and

$$\gamma^*(C) \leq 1.75$$

for all three-dimensional centrally symmetric convex bodies. Needless to say, neither of them is optimal. In this paper we will prove the following theorem.

**Theorem.** For every two-dimensional centrally symmetric convex domain $C$ we have

$$\gamma(C) = \gamma^*(C) \leq 2(2 - \sqrt{2}) \approx 1.17157 \cdots,$$

where the second equality holds if and only if $C$ is an affinely regular octagon.

**Remark 1.** To determine the above upper bound has been listed as an open problem in several references, for example in Brass, Moser and Pach [2], Linhart [15], Zong [22] and [23].

**Remark 2.** The identity

$$\gamma(C) = \gamma^*(C)$$

was proved by J. Linhart [15] and C. Zong [22]. We restate it here just for completion.

**Remark 3.** Let $\theta(K)$ and $\theta^*(K)$ denote the least translative covering density and the least lattice covering density of $K$, respectively. In the plane it was proved by L. Fejes Tóth that

$$\theta(C) = \theta^*(C) \leq \frac{2\pi}{3\sqrt{3}},$$

where the second equality holds if and only if $C$ is an ellipse. It was proved by C.A. Rogers [19] that

$$\delta(K) = \delta^*(K)$$
holds for every two-dimensional convex domain $K$ in 1951. However, to find the optimal lower bound for $\delta(C)$ is still a challenging open problem (see Reinhardt [17], Mahler [16] or Brass, Mosser and Pach [2]). Nevertheless, it has been proved that neither ellipses nor affinely regular octagons can attain the optimal lower bound.

2. Several Basic Lemmas

Let $\partial(K)$ and $\text{int}(K)$ denote the boundary and the interior of $K$, respectively. As usual, we call a convex body regular if for every point $x \in \partial(K)$ there is a unique tangent hyperplane and every tangent plane touches its boundary at a single point. For convenience, in the rest of this paper $C$ always means a two-dimensional centrally symmetric convex domain. Now let us introduce several basic lemmas which will be useful in our proof.

**Lemma 1 (Mahler [16]).** If $\pm v_1, \pm v_2$ and $\pm v_3$ are the six vertices of an affinely regular hexagon inscribed in $C$, then $C + \Lambda$ is a lattice packing of $C$, where

$$
\Lambda = \{ 2z_1v_1 + 2z_2v_2 : z_i \in \mathbb{Z} \}.
$$

**Lemma 2 (Eggleston [5]).** For every convex body there is a sequence of regular convex bodies which converges to the convex body in the sense of Hausdorff metric.

Let $v_1, v_2, \ldots, v_6$ be the six vertices (in anti-clock order) of a centrally symmetric hexagon $H$ which is inscribed in $C$, let $m_i$ denote the midpoint of $v_i, v_{i+1}$, and let $m_i^*$ denote the point in the direction of $m_i$ and on the boundary of $C$. Then, for $i = 1, 2$ and 3, we define

$$
f_i(v_1) = \frac{\|0, m_i^*\|}{\|0, m_i\|},
$$

where $\|x, y\|$ denotes the Euclidean distance between $x$ and $y$.

**Lemma 3 (Zong [21]).** For any $x \in \partial(C)$, we can choose five points $x_2, x_3, x_4, x_5$ and $x_6$ from $\partial(C)$ such that they together with $x$ are the six vertices of an affinely regular hexagon. When $C$ is regular and $x$ moves along $\partial(C)$, we can choose the points such that all $f_1(x)$, $f_2(x)$ and $f_3(x)$ are continuous functions of $x$.

**Proof.** In fact, [21] only contains a proof for the first part of this lemma. Here let us outline a proof for the second part.
Assume that $C$ is regular and, without loss of generality, we assume further that $xx_2x_3x_4x_5x_6$ is a regular hexagon with $x = (1,0)$, as shown in Figure 1. Let $\epsilon$ be a positive number, let $x'$ be a point on the boundary of $C$ such that $\angle x'ox = \epsilon$, let $\Gamma_2$ and $\Gamma_3$ denote the straight lines which are parallel with $ox'$ and pass $x_2$ and $x_3$, respectively.

![Figure 1](image)

Clearly, when $\epsilon$ is sufficiently small, $\Gamma_2$ intersects $\partial(C)$ at two points $x_2$ and $x'_3$, $\Gamma_3$ intersects $\partial(C)$ at two points $x_3$ and $x'_2$, and both $\|x_2, x'_2\|$ and $\|x_3, x'_3\|$ are small. In addition, then the three directions $xx'$, $x_2x'_2$ and $x_3x'_3$ are approximately the tangent directions of $C$ at $x, x_2$ and $x_3$, respectively. Thus, comparing triangles $oxx'$ and $x_2x_3x'_3$ with $x_3x_2x'_2$ and $ox_4x'_4$ (where $x'_4 = -x'$), respectively, by convexity and elementary geometry we get

$$\|x'_2, x_3\| < \|o, x'\| < \|x_2, x'_3\|$$

when $\epsilon$ is sufficiently small. Therefore $\partial(C)$ has two points $x'_2$ and $x'_3$ between $\Gamma_2$ and $\Gamma_3$ such that $x'_3x'_2$ is parallel with $ox'$ and

$$\|x'_3, x'_2\| = \|o, x'\|.$$

Taking $x'_4 = -x'$, $x'_5 = -x'_2$ and $x'_6 = -x'_3$, it is easy to see that $x'_2x'_3x'_4x'_5x'_6$ is an affinely regular hexagon. Since both $x'_2$ and $x'_3$ continuously depend on $x'$, all $f_1(x)$, $f_2(x)$ and $f_3(x)$ are continuous functions of $x$. The lemma is proved.

**Remark 4.** Without the regularity assumption the second part of the lemma will not be true. For example, when

$$C = \{ (x, y) : |x| \leq 1, |y| \leq 1 \},$$

the corresponding function $f_1(x)$ is not continuous at $x = (1,0)$.  

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Lemma 4 (Zong [22]). Let $f_1(v_1), f_2(v_1)$ and $f_3(v_1)$ be the numbers defined above Lemma 3. Then we have

$$\gamma^*(C) \leq \max\{f_1(v_1), f_2(v_1), f_3(v_1)\}.$$ 

3. A Proof for the Theorem

For convenience, let $L(x, y)$ denote the straight line passing two points $x$ and $y$, and write

$$\alpha = 2(2 - \sqrt{2}).$$

To make the complicated proof more transparent, we divide it into three parts.

Assertion I. For every two-dimensional centrally symmetric convex domain $C$, there is a corresponding inscribed affinely regular hexagon $v_1v_2\cdots v_6$ satisfying

$$f_1(v_1) \geq f_2(v_1) = f_3(v_1).$$

Proof. First let us consider the case that $C$ is regular. By Lemma 3, all $f_1(x)$, $f_2(x)$ and $f_3(x)$ are continuous functions of $x \in \partial(C)$. Therefore,

$$f(x) = \min_{i=1,2} \{f_i(x)\} - f_3(x)$$

is also a continuous function of $x \in \partial(C)$. If, without loss of generality,

$$f_i(x_1) > f_3(x_1), \quad i = 1, 2,$$

hold at some point $x_1 \in \partial(C)$ and $x_1x_2x_3x_4x_5x_6$ is the corresponding affinely regular hexagon inscribed in $C$, then we get

$$f(x_1) = \min_{i=1,2} \{f_i(x_1)\} - f_3(x_1) > 0$$

and

$$f(x_2) = f_3(x_1) - f_1(x_1) < 0.$$ 

Therefore, there are two suitable points $v, v_1 \in \partial(C)$ satisfying

$$f(v) = \min_{i=1,2} \{f_i(v)\} - f_3(v) = 0$$

and

$$f_1(v_1) \geq f_2(v_1) = f_3(v_1).$$

In the general case, by Lemma 2, there is a sequence of regular centrally symmetric convex domains $C_1, C_2, \cdots$ which converges to $C$ in the sense of the Hausdorff-metric. Assume that

$$H_i = v_i^1v_i^2v_i^3v_i^4v_i^5v_i^6$$
is an affinely regular hexagon inscribed in $C_i$ and satisfying
\[ f_1(v_i^1) \geq f_2(v_i^1) = f_3(v_i^1). \]

Then, by Blaschke’s selection theorem, there is a subsequence of the sequence $H_1, H_2, \cdots$ which converges to an affinely regular hexagon $v_1v_2v_3v_4v_5v_6$ which is inscribed in $C$ and satisfies
\[ f_1(v_1) \geq f_2(v_1) = f_3(v_1). \]

Thus, Assertion I is proved.

**Assertion II.** For each two-dimensional centrally symmetric convex domain $C$ there is a corresponding lattice $\Lambda$ such that $C + \Lambda$ is a packing and $\alpha C + \Lambda$ is a covering in $E^2$.

**Proof.** Let $v_1v_2\cdots v_6$ be the hexagon obtained in Assertion I. For convenience, we write
\[ \kappa = f_1(v_1), \quad \lambda = f_2(v_1) \]
and define
\[ \Lambda_1 = \{2z_2v_2 + 2z_3v_3 : z_i \in \mathbb{Z}\}. \]

By Lemma 1, it follows that $C + \Lambda_1$ is a lattice packing.

If $\kappa < \alpha$, then by Lemma 4 we can get
\[ \gamma^*(C) \leq \kappa < \alpha. \]

Thus, from now on we assume that $\kappa \geq \alpha$. 

![Figure 2](image-url)
As it is shown in Figure 2, without loss of generality, we assume that \(v_1v_2v_3v_4v_5v_6\) is a regular hexagon with \(v_2 = (\sqrt{3}/2, 1/2)\) and \(v_3 = (0, 1)\). By a routine computation based on elementary geometry it can be shown that the equation of \(L(\alpha v_3, \alpha m_2)\) is

\[ y - \alpha = \frac{3\lambda - 4}{\sqrt{3}\lambda} x \]

and therefore

\[ L(\alpha v_3, \alpha m_2) \cap L(v_3, 2v_2) = \left( \frac{\sqrt{3}\lambda(\alpha - 1)}{4 - 3\lambda}, 1 \right). \tag{3} \]

Let \(p\) denote the midpoint of \(v_2w\) and let \(p^*\) denote the boundary point of \(C + 2v_2\) in the direction from \(2v_2\) to \(p\). By symmetry we have

\[ \frac{\|2v_2, p^*\|}{\|2v_2, p\|} = \frac{\|o, m_1^*\|}{\|o, m_1\|} = \kappa. \]

Therefore, by a routine computation it can be deduced that

\[ [v_3, 2v_2] \cap \partial(\alpha C + 2v_2) = \left( \sqrt{3} \left( 1 - \frac{1}{2} \alpha \kappa \right), 1 \right), \tag{4} \]

where \([x, y]\) denotes the segment between \(x\) and \(y\). Then, by (3) and (4) it follows that \(\alpha C + \Lambda_1\) will be a covering of \(E^2\) and therefore

\[ \gamma^*(C) < \alpha \]

if

\[ \sqrt{3} \left( 1 - \frac{1}{2} \alpha \kappa \right) < \frac{\sqrt{3}\lambda(\alpha - 1)}{4 - 3\lambda}. \tag{5} \]

By convexity it is easy to see from Figure 2 that \(\lambda \leq 4/3\). Then it can be easily verified that (5) holds whenever \(\kappa > \sqrt{2}\) or \(\kappa \geq \lambda > \alpha\). Thus, in the rest of the proof we assume that

\[ 1 \leq \lambda \leq \alpha, \tag{6} \]

\[ \alpha \leq \kappa \leq \sqrt{2} \tag{7} \]

and

\[ \frac{4 - 3\lambda}{\lambda} \geq \frac{\alpha - 1}{1 - \frac{1}{2} \alpha \kappa}. \tag{8} \]
As it is shown in Figure 3, without loss of generality, we assume that \( v_1 v_2 v_3 v_4 v_5 v_6 \) is a regular hexagon with \( v_2 = (\sqrt{3}/2, 1/2) \) and \( v_3 = (0, 1) \). Since \( \|o, m_1^*\|/\|o, m_1\| = \kappa \) and \( \|o, m_2^*\|/\|o, m_2\| = \lambda \), by a routine computation we get that the equations of \( L(v_3, m_2^*) \) and \( L(v_2, m_1^*) \) are

\[
y - 1 = \frac{3\lambda - 4}{\sqrt{3}\lambda} x \tag{9}
\]

and

\[
y = \frac{x - \frac{4\sqrt{3}\kappa}{\sqrt{3}(1 - \kappa)}}{\sqrt{3}(1 - \kappa)} \tag{10}
\]

respectively.

Since \( \|o, m_1^*\|/\|o, m_1\| \geq \alpha \), there is a point \( p \) between \( m_1 \) and \( m_1^* \) satisfying

\[
\frac{\|o, m_1^*\|}{\|o, p\|} = \alpha. \tag{11}
\]

Then, there are two points \( q_1 \in L(v_1, m_1^*) \) and \( q_2 \in L(v_2, m_1^*) \) such that \( p \in L(q_1, q_2) \) and \( L(q_1, q_2) \) is parallel with \( L(v_1, v_2) \). It follows by a routine computation that

\[
p = \left( \frac{\sqrt{3}\kappa}{2\alpha}, 0 \right)
\]

and

\[
q_2 = \left( \frac{\sqrt{3}\kappa}{2\alpha}, \frac{\kappa(\alpha - 1)}{2(\kappa - 1)\alpha} \right).
\]
Let $u_2$ denote the midpoint of $v_3q_2$. It is easy to see that
$$u_2 = \left( \frac{\sqrt{3}\kappa}{4\alpha}, \frac{\kappa(\alpha - 1)}{4(\kappa - 1)\alpha} + \frac{1}{2} \right)$$

and the equation of $L(o, u_2)$ is
$$y = \left( \frac{\kappa(\alpha - 1)}{4(\kappa - 1)\alpha} + \frac{1}{2} \right) \frac{4\alpha}{\sqrt{3}\kappa} x.$$  \hfill (12)

Let $u^*_2$ and $u^\star_2$ denote the intersections of $L(o, u_2)$ with $L(v_3, m_2^*)$ and $L(v_2, m_1^*)$, respectively. By (9), (10) and (12), we can get
$$u^*_2 = \left( \frac{\alpha - 1}{\sqrt{3}(\kappa - 1)} + \frac{2\alpha}{\sqrt{3}\kappa} + \frac{4 - 3\lambda}{\sqrt{3}\lambda} \right)^{-1}, y^*$$
and
$$u^\star_2 = \left( \frac{\sqrt{3}\kappa^2}{2\alpha(3\kappa - 2)}, y^* \right),$$
where the $y$-coordinates of both $u^*_2$ and $u^\star_2$ are not necessary for our purpose. Thus, we get
$$\frac{\|o, u^*_2\|}{\|o, u_2\|} = \frac{4\alpha}{\kappa} \left( \frac{\alpha - 1}{\kappa - 1} + \frac{2\alpha}{\kappa} + \frac{4 - 3\lambda}{\lambda} \right)^{-1}$$
and
$$\frac{\|o, u^\star_2\|}{\|o, u_2\|} = \frac{2\kappa}{3\kappa - 2}.$$

For convenience, we write
$$f(\kappa, \lambda) = \frac{4\alpha}{\kappa} \left( \frac{\alpha - 1}{\kappa - 1} + \frac{2\alpha}{\kappa} + \frac{4 - 3\lambda}{\lambda} \right)^{-1}$$
and
$$g(\kappa) = \frac{2\kappa}{3\kappa - 2}.$$

Next, we proceed to show $f(\kappa, \lambda) \leq \alpha$ and $g(\kappa) \geq \alpha$. It follows by (8) that
$$f(\kappa, \lambda) \leq \frac{4\alpha}{\kappa} \left( \frac{\alpha - 1}{\kappa - 1} + \frac{2\alpha}{\kappa} + \frac{\alpha - 1}{1 - \frac{3}{2}\alpha\kappa} \right)^{-1}.$$

By routine computations, it is easy to see that
$$\frac{4\alpha}{\kappa} \left( \frac{\alpha - 1}{\kappa - 1} + \frac{2\alpha}{\kappa} + \frac{\alpha - 1}{1 - \frac{3}{2}\alpha\kappa} \right)^{-1} \leq \alpha$$
is equivalent with
$$\frac{2\alpha - 4}{\kappa} + \frac{\alpha - 1}{\kappa - 1} + \frac{\alpha - 1}{1 - \frac{3}{2}\alpha\kappa} \geq 0.$$
and, substituting α by $2(2 - \sqrt{2})$,
\[
\frac{(\sqrt{2} - 1)(3 - \sqrt{2})^2}{\kappa(\kappa - 1)(1 - (2 - \sqrt{2})\kappa)} \geq 0,
\]
which is clearly true under the assumption of (6) and (7). Thus, we get
\[
f(\kappa, \lambda) = \frac{\|o, u_2^*\|}{\|o, u_2\|} \leq \alpha, \quad (13)
\]
where the equality holds if and only if
\[
\left\{ \begin{array}{l}
\kappa = \frac{2}{3 - \sqrt{2}} \approx 1.261203875 \cdots \\
\lambda = \frac{2 + \sqrt{2}}{3 - \sqrt{2}} \approx 1.09383632 \cdots
\end{array} \right. \quad (14)
\]
on the other hand, it is easy to see that $g(\kappa)$ is a decreasing function of $\kappa$ when $\kappa$ satisfies (7). Thus, we have
\[
g(\kappa) = \frac{\|o, u_2^*\|}{\|o, u_2\|} \geq \frac{2}{3 - \sqrt{2}} > \alpha. \quad (15)
\]
As it is shown in Figure 4, let $u_2', u_3', u_5'$ and $u_6'$ be the four points satisfying
\[
\frac{\|o, u_i'\|}{\|o, u_i\|} = \alpha, \quad i = 2, 3, 5, 6. \quad (16)
\]
Then $m_1^* u_2^* u_3^* m_1^* u_5^* u_6^*$ is an affinely regular hexagon. For convenience, we write

$$H = \text{conv}\{m_1^*, u_2^*, u_3^*, m_1^*, u_5^*, u_6^*\},$$

$$C' = \text{conv}\{C, u_2^*, u_3^*, u_5^*, u_6^*\}$$

and

$$\Lambda_2 = \{2z_1 m_1^* + 2z_2 u_2^* : z_i \in Z\}.$$ 

By (13), (15) and convexity it follows that

$$\{m_1^*, u_2^*, u_3^*, m_1^*, u_5^*, u_6^*\} \subset \partial(C').$$

Thus, by Lemma 1, $C' + \Lambda_2$ is a packing and therefore $C + \Lambda_2$ is a packing too. On the other hand, it follows by (16) that $\alpha H + \Lambda_2$ is a tiling of $E^2$ and therefore $\alpha C + \Lambda_2$ is a covering of $E^2$. Thus, we get

$$\gamma^*(C) \leq \alpha = 2(2 - \sqrt{2}).$$ \hspace{1cm} (17)

Assertion II is proved.

**Assertion III.** The equality

$$\gamma^*(C) = 2(2 - \sqrt{2})$$

holds if and only if $C$ is an affinely regular octagon.

**Proof.** Let $P_8$ denote an affinely regular octagon. It was proved by Linhart [15] and Zong [22] that

$$\gamma^*(P_8) = 2(2 - \sqrt{2}).$$

On the other hand, if $D$ is a two-dimensional centrally symmetric convex domain satisfying

$$\gamma^*(D) = 2(2 - \sqrt{2}),$$ \hspace{1cm} (18)

we proceed to show that it must be an affinely regular octagon.

First of all, by reexamining the proof of Assertion II, especially (13) and the construction to prove (17), it is not hard to see that (18) holds only if the corresponding $\kappa$ and $\lambda$ satisfy (14).

Second, using the notation in Figure 3, we claim that

$$v_2 m_1^* \subset \partial(D)$$ \hspace{1cm} (19)

and

$$v_3 u_2^* \subset \partial(D).$$ \hspace{1cm} (20)

If, on the contrary, (19) does not hold, then

$$(1 + \epsilon) q_2 \in \text{int}(D)$$
holds with small positive number \( \epsilon \). For convenience, we write

\[ q'_2 = (1 + \epsilon)q_2. \]

Then there is a point \( q'_1 \in (q_1, v_1) \) such that the midpoint \( p' \) of \( q'_1q'_2 \) is in \( (p, m^*_1) \), where \((x, y)\) denotes the open segment between \( x \) and \( y \). Let \( b_2 \) denote the midpoint of \( v_3q'_2 \), let \( b_6 \) denote the midpoint of \( v_6q'_1 \) and let \( b_i^* \) denote the point on the boundary of \( D \) and in the direction of \( b_i \). By elementary geometry and convexity one can deduce that

\[
\frac{\|o, m^*_1\|}{\|o, p'\|} < \frac{\|o, m^*_1\|}{\|o, p\|} = \alpha, \\
\frac{\|o, b_2\|}{\|o, b_2\|} < \frac{\|o, u_2\|}{\|o, u_2\|} = \alpha \tag{21}
\]

and

\[
\frac{\|o, b_6\|}{\|o, b_6\|} < \frac{\|o, u_6\|}{\|o, u_6\|} = \alpha.
\]

For example, (21) can be deduced from the elementary geometry illustrated by Figure 5.

![Figure 5](image_url)

Then, by a construction similar to that in the proof of (17) we get

\[ \gamma^*(D) < 2(2 - \sqrt{2}), \]

which contradicts (18). Thus (19) is proved. The relation (20) can be shown in a similar way.

Finally, let us complete the proof of the assertion based on the next figure.
Let $w_2$ denote the intersection of $L(v_3, m_3^*)$ and $L(v_2, m_1^*)$, and let $w_4$, $w_6$ and $w_8$ be the points defined similarly as shown in Figure 6. It is easy to verify that $m_1^* w_2 v_3 w_4 m_2^* w_6 v_6 w_8$ is an affinely regular octagon. In addition, in this case we have $v_3 = (0, 1)$, $v_2 = (\sqrt{3}/2, 1/2)$, $m_1^* = (\sqrt{3}/(3-\sqrt{2}), 0)$, $u_2^* = \left(\frac{\sqrt{3}}{2(3-\sqrt{2})}, \frac{3-\sqrt{2}}{2}\right)$ and $w_2 = \left(\frac{\sqrt{3}}{3\sqrt{2}-2}, \frac{\sqrt{3}}{2}\right)$. We proceed to show

$$D = m_1^* w_2 v_3 w_4 m_2^* w_6 v_6 w_8. \quad (22)$$

If $D$ is not the octagon and let $w'_2$ and $w'_4$ denote the intersections of $\partial(D)$ with $ow_2$ and $ow_4$, respectively. In addition, we write

$$\rho = \|o, w_2\|/\|o, w'_2\|,$$

$$\sigma = \|o, w_4\|/\|o, w'_4\|$$

and assume that $\rho \geq \sigma$. Based on the coordinates of $v_2$, $u_2^*$ and $w_2$, by a routine computation we get

$$1 \leq \sigma \leq \rho \leq \frac{4\sqrt{2} + 2}{7}. \quad (23)$$

Let $w'_3$ be the point defined by $ow'_2/ow'_3 = \alpha$, let $\Gamma$ denote the straight line passing $w'_3$ and parallel with $L(o, w_8)$, let $s_1$ and $s_2$ denote the intersections of $\Gamma$ with $L(v_2, m_1^*)$ and $L(v_3, m_2^*)$, respectively, let $t^*$ denote the midpoint of $w'_2 s_2$, let $t$ denote the intersection of $L(o, t^*)$ with $L(v_3, w_4)$, and finally define

$$h(\rho, \sigma) = \|o, t\|/\|o, t^*\|.$$

Based on (23), it can be verified by routine arguments and computations that

$$s_1 \in v_2 m_3^* \subset \partial(D),$$
s_2 \in v_3u_2^* \subset \partial(D),
\|w_2^*, s_1\| = \|w_2^*, s_2\|
and t \in \partial(D). Especially, by routine but complicated computations we get
\[ h(\rho, \sigma) = \left( \frac{\sqrt{2} + 1}{2} - \frac{2 + \sqrt{2}}{4\rho} + \frac{1}{2\sigma} \right)^{-1}. \]

Then, by (23) we get
\begin{align*}
h(\rho, \sigma) & \leq \left( \frac{\sqrt{2} + 1}{2} - \frac{2 + \sqrt{2}}{4\rho} + \frac{1}{2\rho} \right)^{-1} \\
& = \left( \frac{\sqrt{2} + 1}{2} - \frac{\sqrt{2}}{4\rho} \right)^{-1} \\
& \leq 2(2 - \sqrt{2}),
\end{align*}
where the final equality holds if and only if \rho = \sigma = 1. Then, (22) follows from Lemma 4. Assertion III is proved.

As a conclusion of Assertion II and Assertion III the theorem is proved.

4. Three Further Remarks

Remark 5. Let \lambda_i(C, \Lambda) denote the i-th successive minimum of C with respect to a lattice \Lambda, and let \mu_i(C, \Lambda) denote the i-th covering minimum of C with respect to \Lambda (see Gruber and Lekkerkerker [10] and Kannan and Lovász [14], respectively). As a corollary of the theorem we get
\[ \min_{\Lambda} \frac{\mu_2(C, \Lambda)}{\lambda_1(C, \Lambda)} \leq 2(2 - \sqrt{2}), \]
where the equality holds if and only if C is an affinely regular octagon.

Remark 6. It is well known (see L. Fejes Tóth [8]) that
\[ \frac{\theta^*(C)}{\delta^*(C)} \leq \frac{4}{3} \approx 1.3333 \cdots \]
holds for every two-dimensional centrally symmetric convex domain. However, although our theorem is optimal, it only can produce
\[ \frac{\theta^*(C)}{\delta^*(C)} \leq \min_{\Lambda} \left( \frac{\mu_2(C, \Lambda)}{\lambda_1(C, \Lambda)} \right)^2 \leq (3 - 2\sqrt{2}) \approx 1.37258 \cdots. \]
The reason for this phenomenon is the optimal covering lattice of a regular octagon is not homothetic to its optimal packing lattice.

Remark 7. Let $m_2(C)$ denote the Steiner ratio of the Minkowski plane determined by a two-dimensional centrally symmetric convex domain $C$. It is known (see Cieslik [4]) that

$$m_2(C) \leq \frac{3}{4} \gamma^*(C).$$

Thus, we have

$$m_2(C) \leq \frac{3}{2 + \sqrt{2}}.$$

Acknowledgements. In 1996, I learned this problem from Professor C.A. Rogers when I was a visitor at University College London. In 2003, when I published two papers ([22] and [24]) on this problem, I received an offprint of [15] from Professor J. Linhart. Clearly, he was not aware of Rogers and Butler’s papers on this topic when he published [15], just like my unawareness of his paper. Fortunately, our papers have almost no important overlap. I am very grateful to Professor Rogers for driving my attention to this problem and to Professor Linhart for sending me his related papers. For some helpful comments on this paper, I am obliged to Professor Martin Henk.

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