Higgs potential in $S_3$ invariant model
for quark/lepton mass and mixing

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Abstract

We analyzed the $S_3$ invariant Higgs potential with $S_3$ singlet and doublet Higgs. We obtained a relation $\left(\frac{|v_1|}{|v_2|}\right)^2 = -\sin 2\phi_2/\sin 2\phi_1$ from this $S_3$ invariant Higgs potential, where $v_1$, $v_2$ and $\phi_1$, $\phi_2$ are vacuum expectation values and phases of $S_3$ doublet Higgs, respectively. This relation could be satisfied exactly by the results $|v_1|/|v_2| = 0.207$, $\phi_1 = -74.9^\circ$ and $\phi_2 = 0.74^\circ$ obtained from the previous our work analyzing the quark/lepton mass and mixing in $S_3$ invariant Yukawa interaction. Furthermore, the relation $v_S \sim v_D = \sqrt{|v_1|^2 + |v_2|^2} = 174$GeV is obtained and then the coupling strength of Higgs to top quark $g_{H_{tt}} = m_t/v_S$ is altered as by a factor $\sqrt{2}$ from the standard value.

Introduced the $S_3$ doublet Higgs, FCNC are produced in tree level. Predicted branching ratios for rare decays $\mu^- \rightarrow e^-e^+e^-$, $K^0_L \rightarrow \mu^+\mu^-$ etc., induced by the FCNC are sufficiently below the present experimental upper bounds.

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I. INTRODUCTION

The existence of the Higgs bosons which are the origin for Higgs mechanism producing the masses of all matter and gauge fields is expected to be discovered in LHC \cite{1, 2}. We analyzed the problem of the origin of quark/lepton mass and mixing using a $S_3$ invariant model \cite{3, 4}, where quark and lepton flavors and Higgs fields are considered to be governed by the $S_3$ symmetry. In our model, weak bases of flavors $(u, c)$, $(d, s)$, $(e, \mu)$, $(\nu_e, \nu_\mu)$ are assumed as $S_3$ doublet and $t, b, \tau, \nu_\tau$ are $S_3$ singlet and further there are assumed $S_3$ doublet Higgs $(\Phi_1, \Phi_2)$ and $S_3$ singlet Higgs $\Phi_S$. Constructing the $S_3$ invariant Yukawa interactions, we could explain the quark sector mass hierarchy and mixing $V_{CKM}$ including phases of CP violation. In the leptonic sector, assuming the see-saw mechanism \cite{5} with the Majorana masses, we could explain the tri-bimaximal-like character \cite{6–8} of neutrino mixing $V_{MNS}$ without imposing any other symmetry restriction than $S_3$ symmetry.

This minimal $S_3$ extension of the Higgs fields in flavor or generation space could play an important role in explaining the quark/lepton mass and mixing \cite{3, 4}. The ratio of vacuum expectation values $|v_1|, |v_2|$ of Higgs doublet $\Phi_1, \Phi_2$ is estimated to be not 1 and rather small, $|v_1|/|v_2| = 0.207$, and this ratio can explain the Cabibbo angle. The phases $\phi_1$ and $\phi_2$ of Higgs doublet $\Phi_1$ and $\Phi_2$ are the origins of CP violation, and these values are estimated as $\phi_1 = -74.9^\circ$ and $\phi_2 = 0.74^\circ$. In this paper, we construct the Higgs potential as $S_3$ invariant, and then investigate whether this Higgs potential could satisfy the above results or not.

We use the most general $S_3$ invariant Higgs potential adopted by many authors \cite{9–14}, assuming a hierarchy among the quartic coupling strengths of Higgs fields. A relation $(|v_1|/|v_2|)^2 = -\sin 2\phi_2/\sin 2\phi_1$ is obtained from the stationary condition in our $S_3$ invariant Higgs potential. The relation could be satisfied exactly by the result $|v_1|/|v_2| = 0.207$, $\phi_1 = -74.9^\circ$ and $\phi_2 = 0.74^\circ$. Furthermore the relation $v_S \sim v_D = \sqrt{|v_1|^2 + |v_2|^2} = 174$GeV, which should be compared to the standard value $v = 246$GeV, is obtained and then the coupling strength of Higgs $H_S$ to $t$ quark $g_{HSt} = m_t/v_S$, is altered as by a factor $\sqrt{2}$ from the standard value. Introduced the $S_3$ doublet Higgs, flavor changing neutral current (FCNC) are produced in tree level. We analyze the branching ratios for rare decays $\mu^- \rightarrow e^-e^+e^-$, $K_L^0 \rightarrow \mu^+\mu^-$ etc., induced by the FCNC, which are predicted to be sufficiently below the experimental upper bound obtained from PDG data \cite{15}.

2
II. $S_3$ Invariant Model for Quark/Lepton Mass and Mixing

First, we review our $S_3$ invariant model for quark/lepton mass and mixing [3, 4]. We assumed that the generations of quarks and leptons (charged leptons and Dirac neutrinos) and further Higgs fields are the irreducible representation of $S_3$ symmetry group,

$$S_3 \text{ doublet : } f_D^{L,R} = (f_1^{L,R}, f_2^{L,R})^T, \quad \Phi_D = (\Phi_1, \Phi_2)^T, \quad f_1 = u, d, \nu_e, e, \quad f_2 = c, s, \nu_\mu, \mu, \quad f_S = t, b, \nu_\tau, \tau.$$ 

(1)

In the $SU(2)_L$ gauge space,

$$SU(2)_L \text{ doublets : } \Phi_D = (\Phi_D^+, \Phi_0^0)^T, \quad \Phi_S = (\Phi_S^+, \Phi_0^0)^T,$$

$$Q^L_1 = (u_L, d_L)^T, \quad Q^L_2 = (c_L, s_L)^T, \quad Q^L_3 = (t_L, b_L)^T,$$

$$L^L_1 = (\nu_\mu_L, \mu_L)^T, \quad L^L_2 = (\nu_\tau_L, \tau_L)^T,$$

$$SU(2)_L \text{ singlets : } d^R_1 = d_R, \quad d^R_2 = s_R, \quad d^R_3 = b_R, \quad u^R_1 = u_R, \quad u^R_2 = c_R, \quad u^R_3 = t_R,$$

$$l^R_1 = \nu_\mu, \quad l^R_2 = \nu_\tau, \quad l^R_3 = \nu_\mu, \quad l^R_2 = \nu_\tau, \quad l^R_3 = \nu_\mu, \quad l^R_2 = \nu_\tau.$$

As a standard model, Yukawa interaction for Dirac mass of flavors is expressed as

$$- \mathcal{L}_D = \sum_{i,j,k=1,2,3} \Gamma_{ijk}^d \overline{\Phi}_j d_k^R + \Gamma_{ijk}^e \overline{\Phi}_j e_k^R + \Gamma_{ijk}^f \overline{\Phi}_j f_k^R + \Gamma_{ijk}^e \overline{\Phi}_j \epsilon_{ijk} \nu_k^R + h.c.,$$

(3)

where $\Gamma_{ijk}^f$ are interaction strengths and $\epsilon$ is the $2 \times 2$ antisymmetric tensor in the $SU(2)_L$ gauge space. In our model, we assumed that the Yukawa interaction Eq. (3) is $S_3$ invariant.

Under our present analysis considering the effects caused from the neutral Higgs $\Phi_0^S$ and $\Phi_0^0$, the Yukawa interactions are expressed as

$$- \mathcal{L}_D^{d,l} = \Gamma_{d,l}^{d,l} \overline{f}_D f_S^R \Phi_0^S + \Gamma_{d,l}^{d,l} \overline{f}_D f_D^R \Phi_0^0 + \Gamma_{d,l}^{d,l} [(\overline{f}_D^T f_S^R + \overline{f}_D^T f_D^R) \Phi_0^0 + (\overline{f}_D^T f_D^R - \overline{f}_D^T f_D^R) \Phi_0^D] + h.c.,$$

(4)

for down-type quark and charged lepton,

$$- \mathcal{L}_D^{u,\nu} = \Gamma_{u,\nu}^{u,\nu} \overline{f}_D f_S^R \Phi_0^0 + \Gamma_{u,\nu}^{u,\nu} \overline{f}_D f_D^R \Phi_0^0 + \Gamma_{u,\nu}^{u,\nu} [(\overline{f}_D^T f_S^R + \overline{f}_D^T f_D^R) \Phi_0^0 + (\overline{f}_D^T f_D^R - \overline{f}_D^T f_D^R) \Phi_0^D] + h.c.,$$

(4)

for up-type quark and Dirac neutrino.

These mass Lagrangians are almost similar to those in literature analyzing the $S_3$ invariant model [9, 13, 16, 18], where $\Gamma_{d,l}^{d,l} = \overline{f}_D^T \Phi_0^0 f_S^R + \overline{f}_D^T \Phi_0^0 f_D^R$ and $\Gamma_{u,\nu}^{u,\nu} = \overline{f}_D^T \Phi_0^0 f_S^R + \overline{f}_D^T \Phi_0^0 f_D^R$.
the physical Higgs fields $H$ and the expectation values of Higgs fields $\Phi$ in gauge space. The mass Lagrangian and mass matrices are obtained on the vacuum consideration. First we can recognize that the coupling constants $\Gamma^f_{D1,3,4}$ are considered to have the hierarchy as $\Gamma^f_S > \Gamma^f_{D1,3,4}$, because $\Gamma^f_S$ represents the coupling strength for the coupling of all $S$ singlet fields as $f^0_S f^R_S \Phi^0_S$, on the other hand, $\Gamma^f_{D1,3,4}$ do the coupling strengths for the coupling of $S$ singlet and doublet fields as $f^L_D f^R_D \Phi^0_D$, $f^L_D f^R_D \Phi^0_S$ or $f^R_D f^L_D \Phi^0_D$. Second, although there is considered to be a difference between the coupling strengths $\Gamma^f_{D3}$ and $\Gamma^f_{D4}$ for the couplings as $f^L_D \Phi^0_S f^R_S$ and $f^L_D \Phi^0_D f^R_D$, the difference is considered to be very small compared to $\Gamma^f_{D3}, \Gamma^f_{D4}$, that is, $\Gamma^f_{D3}, \Gamma^f_{D4} > \Gamma^f_{D3} - \Gamma^f_{D4}$, then we can assume that $\Gamma^f_{D3} = \Gamma^f_{D4}$.

We express the Higgs fields $\Phi^0_i$ in Eq. (4) by the vacuum expectation values $v_i$'s and the physical Higgs fields $H_i$'s as

$$\Phi^0_S = \frac{1}{\sqrt{2}}(v_S + H_S),$$

$$\Phi^0_1 = \cos \alpha \Phi_D e^{i\phi_1} = \frac{1}{\sqrt{2}}(v_1 + H_1) = \cos \alpha \frac{1}{\sqrt{2}}(v_D + H_D)e^{i\phi_1},$$

$$\Phi^0_2 = \sin \alpha \Phi_D e^{i\phi_2} = \frac{1}{\sqrt{2}}(v_2 + H_2) = \sin \alpha \frac{1}{\sqrt{2}}(v_D + H_D)e^{i\phi_2},$$

where we set $\Phi^0_S$ to be real because we can always make the phase of $\Phi_S$ zero by a rotation in gauge space. The mass Lagrangian and mass matrices are obtained on the vacuum expectation values of Higgs fields $\Phi^0_i$, and are expressed as

$$-L^f_D = \overline{f^L} M_f f^R + h.c., \quad f = d, u, l, \nu,$$

$$M_{d,l} = \begin{pmatrix}
\mu_1^{d,l} + \mu_2^{d,l} e^{i\phi_2} & \lambda \mu_1^{d,l} e^{i\phi_1} & \lambda \mu_3^{d,l} e^{i\phi_1} \\
\lambda \mu_2^{d,l} e^{i\phi_1} & \mu_1^{d,l} - \mu_2^{d,l} e^{i\phi_2} & \mu_3^{d,l} e^{i\phi_2} \\
\lambda \mu_3^{d,l} e^{i\phi_1} & \mu_3^{d,l} e^{i\phi_2} & \mu_0^{d,l}
\end{pmatrix},$$

$$M_{u,\nu} = \begin{pmatrix}
\mu_1^{u,\nu} + \mu_2^{u,\nu} e^{-i\phi_2} & \lambda \mu_2^{u,\nu} e^{-i\phi_1} & \lambda \mu_3^{u,\nu} e^{-i\phi_1} \\
\lambda \mu_2^{u,\nu} e^{-i\phi_1} & \mu_1^{u,\nu} - \mu_2^{u,\nu} e^{-i\phi_2} & \mu_3^{u,\nu} e^{-i\phi_2} \\
\lambda \mu_3^{u,\nu} e^{-i\phi_1} & \mu_3^{u,\nu} e^{-i\phi_2} & \mu_0^{u,\nu}
\end{pmatrix},$$

4
where we used the following parameterizations,

\[
\begin{align*}
\mu_0^f &= \Gamma_S^f \frac{v_S}{\sqrt{2}}, \\
\mu_1^f &= \Gamma_{D_1}^f \frac{v_S}{\sqrt{2}}, \\
\mu_2^f &= \Gamma_{D_2}^f \frac{|v_2|}{\sqrt{2}} = \Gamma_{D_2}^f \sin \alpha \frac{v_D}{\sqrt{2}}, \\
\mu_3^f &= \Gamma_{D_3}^f \frac{|v_2|}{\sqrt{2}} = \Gamma_{D_3}^f \sin \alpha \frac{v_D}{\sqrt{2}}, \\
\lambda &= \frac{|v_1|}{|v_2|} = \cot \alpha.
\end{align*}
\]  

For neutrino mass, we assume the very large Majorana masses, and from these Majorana masses one can get the very small neutrino masses through the see-saw mechanism \[5\].

We assume that the Majorana mass is constructed from only right handed neutrino \(\nu_R^D = (\nu_{R1}^D, \nu_{R2}^D)^T\) and \(\nu_S^R\) as to be \(S_3\) invariant and then has no Higgs fields \[16\],

\[
\mathcal{L}_M = \frac{1}{2} \Gamma_S^M (\nu_S^R)^T C^{-1} \nu_S^R + \frac{1}{2} \Gamma_D^M (\nu_D^R)^T C^{-1} \nu_D^R + h.c. \\
= \frac{1}{2} (\nu_R^T C^{-1} M_M \nu_R + h.c., \\
M_M = \begin{pmatrix} M_M & 0 & 0 \\
0 & M_1 & 0 \\
0 & 0 & M_0 \end{pmatrix},
\]

where \(C\) is a charge conjugation matrix.

From the numerical analyses explaining the masses of quarks and KM mixing matrix containing the CP-violation effects, we can get the following numerical results for 11 parameters \(\mu_i^f, \lambda\) and \(\phi_i\) \[4\]:

\[
\begin{align*}
\mu_0^d &= 4.20 \pm 0.12 \text{GeV}, \quad \frac{\mu_1^d}{\mu_0^d} = 0.0120 \pm 0.0030, \quad \frac{\mu_2^d}{\mu_0^d} = -0.0136 \pm 0.0004, \\
\frac{\mu_3^d}{\mu_0^d} &= \pm (0.0282 \pm 0.0008), \\
\mu_0^u &= 171.3 \pm 2.3 \text{GeV}, \quad \frac{\mu_1^u}{\mu_0^u} = 0.00369 \pm 0.00003, \quad \frac{\mu_2^u}{\mu_0^u} = -0.00378 \pm 0.00003, \\
\frac{\mu_3^u}{\mu_0^u} &= \mp (0.0127 \pm 0.0007) \quad (\text{opposite sign to that of the ratio } \mu_3^d/\mu_0^d), \\
\lambda &= 0.207 \pm 0.004, \quad \phi_1 = -(74.9 \pm 0.8)^\circ, \quad \phi_2 = (0.74 \pm 0.31)^\circ.
\end{align*}
\]  

From the numerical analysis of charged lepton masses and neutrino mixing , we can get the
following numerical results for 10 parameters $\mu_f^i$ and $M_i$[4];

$$
\mu_0^f = 1776.84 \pm 0.17\text{MeV}, \quad \frac{\mu_0^1}{\mu_0^0} = 0.0308 \pm 0.0007, \quad \frac{\mu_0^2}{\mu_0^0} = -(0.0307 \pm 0.0017), \\
\frac{\mu_0^3}{\mu_0^0} = -0.0233 \sim 0.0233, \\
\mu_0^\nu \approx 73.3\text{GeV}, \quad \frac{\mu_0^\nu}{\mu_0^0} = 0.035 \sim 0.038, \quad \frac{\mu_0^\nu}{\mu_0^0} = -0.001 \sim -0.007,
$$

(10)

$$
\frac{\mu_0^\nu}{\mu_0^0} = \pm (0.005 \sim 0.023), \\
M_1 \approx 1.6 \times 10^{11}\text{GeV}, \quad M_0 \approx 10^{14}\text{GeV}.
$$

From these numerical analyses, we can confirm that $\frac{\mu_0^f}{\mu_0^0} = \frac{\Gamma_f^f}{\Gamma_f^S} \approx O(0.01)$ for all flavors, then there is a hierarchy between $\Gamma_f^f$ and $\Gamma_f^S$; $\Gamma_f^S \gg \Gamma_f^D$ as mentioned above. The result $\lambda = 0.207(\alpha = 78.3^\circ)$ predicts the hierarchy between $|v_1|$ and $|v_2|$. In almost literature analyzing the flavor mass and mixing using $S_3$ symmetry [9, 14, 16–18], authors assume that $|v_1| = |v_2|$. It should be investigated by the analysis for Higgs potential of $S_3$ symmetry whether there is a hierarchy as our result $|v_1|/|v_2| = 0.207$ or not as $|v_1| = |v_2|$ assumed by other authors. From the result $|\mu_0^f| \sim |\mu_0^\nu|$ as shown in Eqs. (9), (10) and $|\Gamma_f^D| \sim |\Gamma_f^D|$, which may be considered to be suitable because $|\Gamma_f^D|$ and $|\Gamma_f^D|$ are coupling strengths for $f_D f_D \Phi_S$ and $f_D f_D \Phi_D$, respectively, it is recognized that there is an equality of magnitude for $v_S$ and $v_D$; $v_S \approx v_D$, by observing that $v_S \approx |v_2| = \sin \alpha v_D$ and $\sin \alpha = 0.98$. Thus, from the quark and lepton mass and mixing analysis [4], the vacuum expectation values and phases of Higgs fields are restricted as

$$
v_S \approx v_D, \quad \lambda = \frac{|v_1|}{|v_2|} = \cot \alpha = 0.207(\alpha = 78.3^\circ), \quad \phi_1 = -74.9^\circ, \quad \phi_2 = 0.74^\circ.
$$

(11)

The purpose of our present work is to investigate whether these results for Higgs fields are confirmed or not in $S_3$ invariant Higgs potential.
III. $S_3$ INVARIANT HIGGS POTENTIAL

The most general $S_3$ invariant Higgs potential composed of quadratic and quartic terms of Higgs fields is the following form \[9\,14,\]

$$V = -\mu_D^2(\Phi_1^0 \Phi_1^0 + \Phi_2^0 \Phi_2^0) - \mu_S^2 \Phi_S^0 \Phi_S^0$$

$$+ A(\Phi_S^0 \Phi_S^0)^2 + B(\Phi_S^0 \Phi_S^0)(\Phi_1^0 \Phi_1^0 + \Phi_2^0 \Phi_2^0) + C(\Phi_1^0 \Phi_1^0 + \Phi_2^0 \Phi_2^0)^2$$

$$+ D(\Phi_1^0 \Phi_1^0 - \Phi_2^0 \Phi_2^0)^2 + E[\Phi_S^0(\Phi_1^0 \Phi_2^0 + \Phi_2^0 \Phi_1^0) + \Phi_2(\Phi_1^0 \Phi_1^0 - \Phi_2^0 \Phi_2^0)] + h.c.]$$

$$+ F\{[(\Phi_1^0 \Phi_1^0)(\Phi_1^0 \Phi_1^0) + (\Phi_2^0 \Phi_2^0)(\Phi_2^0 \Phi_2^0)] + F'[((\Phi_S^0 \Phi_S^0)^2 + (\Phi_1^0 \Phi_1^0)^2 + \Phi_2^0 \Phi_2^0)^2 + h.c.]$$

$$+ G\{(\Phi_1^0 \Phi_1^0 - \Phi_2^0 \Phi_2^0)^2 + (\Phi_1^0 \Phi_1^0 + \Phi_2^0 \Phi_2^0)^2\},$$

(12)

where we disregarded the charged Higgs part $\Phi_i^+$ in gauge Higgs doublets $\Phi_i = (\Phi_i^+, \Phi_i^0)$, because we do not consider the effects induced from these charged Higgs $\Phi_i^+$, in this analysis. In present our analysis, we assume that the coupling constant $E$ describing the strength of the coupling between $\Phi_S^0$ and $(\Phi_D^0)^3$ is negligible small, because all other quartic couplings are composed of the pairs $\Phi_S^0 \Phi_S^0$ and $\Phi_D^0 \Phi_D^0$. Authors of literature \[9\] assumed that the potential is symmetric under the reflection $R: \Phi_S^0 \rightarrow -\Phi_S^0$, and then they settled $E = 0$.

Using the parameterization Eq. (5), the potential in Eq. (12) on the vacuum expectation values of Higgs fields can be expressed as

$$V = -\frac{1}{2}\mu_S^2 v_S^2 - \frac{1}{2}\mu_D^2 v_D^2 + \frac{1}{4}Av_S^4 + \frac{1}{4}B'v_S^2 v_D^2 + \frac{1}{4}C'v_D^4,$$

$$B' = B + F + 2F'(\cos^2 \alpha \cos 2\phi_1 + \sin^2 \alpha \cos 2\phi_2),$$

$$C' = C + G - (D + G) \sin^2 2\alpha \sin^2(\phi_1 - \phi_2).$$

(13)

From this, the following stationary conditions are obtained,

$$\frac{\partial V}{\partial \alpha} = v_D^2 \sin 2\alpha \left\{ \frac{1}{2}v_S^2 F'(- \cos 2\phi_1 + \cos 2\phi_2) - v_D^2(D + G) \cos 2\alpha \sin^2(\phi_1 - \phi_2) \right\} = 0,$$

(14)

$$\frac{\partial V}{\partial \phi_1} = v_D^2 \left\{ -v_S^2 F' \cos^2 \alpha \sin 2\phi_1 - \frac{1}{4}v_D^2(D + G) \sin 2\alpha \sin 2(\phi_1 - \phi_2) \right\} = 0,$$

(15)

$$\frac{\partial V}{\partial \phi_2} = v_D^2 \left\{ -v_S^2 F' \sin^2 \alpha \sin 2\phi_2 + \frac{1}{4}v_D^2(D + G) \sin 2\alpha \sin 2(\phi_1 - \phi_2) \right\} = 0,$$

(16)

$$\frac{\partial V}{\partial v_S} = v_S \left( -\mu_S^2 + Av_S^2 + \frac{1}{2}B'v_D^2 \right) = 0,$$

(17)

$$\frac{\partial V}{\partial v_D} = v_D \left( -\mu_D^2 + C'v_D^2 + \frac{1}{2}B'v_S^2 \right) = 0.$$
From Eqs. (15) and (16), we can obtain a relation for \( \alpha \), \( \phi_1 \) and \( \phi_2 \),

\[
\frac{\cos^2 \alpha}{\sin^2 \alpha} = -\frac{\sin 2\phi_2}{\sin 2\phi_1}.
\] (19)

Using Eqs. (15) and (16), the Eq. (14) is satisfied automatically, then the constraints (14), (15) and (16) give the only one relation (19) independent of the coupling constants \( D \), \( G \), \( F' \). Then this relation is the first restriction obtained from the \( S_3 \) invariant Higgs potential. This relation can be satisfied exactly by the result (11) obtained in numerical analysis of quark/lepton mass and mixing [4]. In fact using the numerical result (11), the left and right hand sides of relation (19) are given as follows,

\[
\frac{\cos^2 \alpha}{\sin^2 \alpha} = (0.207)^2 = 0.043,
-\frac{\sin 2\phi_2}{\sin 2\phi_1} = -\frac{\sin(2 \times 0.74^\circ)}{\sin(2 \times (-74.9^\circ))} = 0.051.
\]

Thus the \( S_3 \) invariant Higgs potential could produce the relation between Cabibbo angle \( \approx \lambda = \cot \alpha \) and the CP violation phases \( \phi_1 \) and \( \phi_2 \) which were decided by the quark/lepton mass and mixing through the \( S_3 \) invariant Yukawa interaction. The authors [9] assuming \( \cot \alpha = 1 \) that is, \(|v_1| = |v_2|\), has settled the angles \( \phi_1 \), \( \phi_2 \) as \( \phi_1 + \phi_2 = 0 \) obtained from the Eq. (19), and analyzed the lepton mass and mixing.

From Eqs. (17) and (18), the values of \( v_S \) and \( v_D \) are obtained as

\[
v_S^2 = \frac{4C''\mu_S^2 - 2B'\mu_D^2}{4AC'' - B'^2}, \quad v_D^2 = \frac{4A\mu_D^2 - 2B'\mu_S^2}{4AC'' - B'^2}, \quad 4AC'' - B'^2 > 0.
\] (20)

The third relation is obtained from the condition minimizing the potential \( V \). This relation is satisfied if there is a hierarchy \( A, C' \gg B' \) between these coupling strengths. This hierarchy is recognized from the fact that the coupling constants \((A, C')\) are the strengths for product of pair \( \Phi^0_S \Phi^0_S \) or \( \Phi^0_D \Phi^0_D \), on the other hand the coupling constant \( B' \) is the strength for product of different representation pair \( \Phi^0_S \Phi^0_S \) and \( \Phi^0_D \Phi^0_D \). Because \( v_S^2 \) and \( v_D^2 \) are positive, then the following relation must be satisfied among parameters \( A, B', C', \mu_D^2, \mu_S^2 \),

\[
\frac{B'}{2A} < \frac{\mu_D^2}{\mu_S^2} < \frac{2C'}{B'}.
\]

And further, from the hierarchy \((A, C') \gg B'\), the second restriction is obtained,

\[
\frac{\mu_D}{\mu_S} \text{ is not so far from 1.}
\] (21)

From the numerical result \( v_S \approx v_D \) obtained in analysis of quark/lepton mass and mixing, and from Eq. (20), the relation

\[
\frac{\mu_D^2}{\mu_S^2} \approx \frac{2C' + B'}{2A + B'},
\]
is obtained. Using the Eq. (21) and above result, we can predict a relation for the coupling strengths \( A \) and \( C' \) as
\[
\frac{C'}{A} \text{ is not so far from 1,}
\] (22)
this is the third restriction.

From the Lagrangian for the coupling between Higgs fields and gauge fields, one takes the relation as
\[
\sqrt{v_S^2 + |v_1|^2 + |v_2|^2} = \sqrt{v_S^2 + v_D^2} = \frac{2m_W}{g} = 246\text{GeV},
\] (23)
where \( g \) is the electroweak coupling. If we set the assumption \( v_S = v_D \), we can get the values for \( v_S \) as
\[
v_S = v_D = \frac{1}{\sqrt{2}} \times 246\text{GeV} = 174\text{GeV}.
\] (24)

Rewriting the \( \Phi_0' \)'s in the Higgs potential (12) by the expression of \( \Phi_0' \)'s in Eq. (5) and regarding the coefficients of the product of Higgs fields \( H_i H'_i, (i, i' = S, D) \), we obtain the mass matrix for Higgs fields \( H_{S,D} \),
\[
(H_S, H_D) \begin{pmatrix} 2A v_S^2 & B' v_S v_D \\ B' v_S v_D & 2C' v_D^2 \end{pmatrix} \begin{pmatrix} H_S \\ H_D \end{pmatrix}.
\] (25)
This is diagonalized approximately in the assumption \( \frac{B'^2 v_S^2 v_D^2}{(A v_S^2 - C' v_D^2)^2} \ll 1 \) as
\[
\begin{pmatrix} H_a \\ H_b \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} H_S \\ H_D \end{pmatrix}, \quad \tan \beta = \frac{B' v_S v_D}{2(A v_S^2 - C' v_D^2)}
\] (26)
\[
\begin{align*}
m_{H_a}^2 & \approx 2A v_S^2 + \frac{B'^2 v_S^2 v_D^2}{2(A v_S^2 - C' v_D^2)} \approx 2A v_S^2, \\
m_{H_b}^2 & \approx 2C' v_D^2 - \frac{B'^2 v_S^2 v_D^2}{2(A v_S^2 - C' v_D^2)} \approx 2C' v_D^2.
\end{align*}
\]

In the following discussion, we pursue an analysis in the assumption \( \tan \beta \ll 1 \), then in the approximation for sates \( H_a, H_b \) as
\[
H_a \approx H_S, \quad H_b \approx H_D.
\]
IV. $g_{Hff}$ AND FLAVOR CHANGING NEUTRAL CURRENTS (FCNC)

Introduced the $S_3$ doublet Higgs, flavor changing neutral currents (FCNC) are produced in tree level, and strengths of $g_{Hff}$ are changed from the standard model prediction. A prediction of the strengths of $g_{Hff}$ would be very important in the present status where the Higgs production rate and the branching ratio for these decays are observed in experiments \[1, 2\]. Coupling strengths of $g_{Hff}$ and FCNC are obtained from Yukawa interaction (4), inserted the $\Phi_i$’s containing the physical Higgs fields $H_i$’s expressed in Eq. (5), as

$$\sum_{i=S,1,2} [g_{H_i f f'}] |H_i| = V_{d,l}^\dagger \left[ \begin{pmatrix} \mu_1^{d,l} & 0 & 0 \\ 0 & \mu_1^{d,l} & 0 \\ 0 & 0 & \mu_0^{d,l} \end{pmatrix} \right] \frac{H_S}{v_S} + \left( \begin{pmatrix} 0 & \lambda \mu_2^{d,l} & \lambda \mu_3^{d,l} \\ \lambda \mu_2^{d,l} & 0 & 0 \\ \lambda \mu_3^{d,l} & 0 & 0 \end{pmatrix} \right) e^{i \phi_1} \frac{|H_1|}{|v_1|}$$

$$+ \left( \begin{pmatrix} \mu_2^{d,l} & 0 & 0 \\ 0 & -\mu_2^{d,l} & \mu_3^{d,l} \\ 0 & \mu_3^{d,l} & 0 \end{pmatrix} \right) e^{i \phi_2} \frac{|H_2|}{v_2} U_{d,l}, \quad (f, f') = (d, s, b) \text{ or } (e, \mu, \tau), \quad (27)$$

$$\sum_{i=S,1,2} [g_{H_i f f'}] |H_i| = V_{u}^\dagger \left[ \begin{pmatrix} \mu_1^u & 0 & 0 \\ 0 & \mu_1^u & 0 \\ 0 & 0 & \mu_0^u \end{pmatrix} \right] \frac{H_S}{v_S} + \left( \begin{pmatrix} 0 & \lambda \mu_2^u & \lambda \mu_3^u \\ \lambda \mu_2^u & 0 & 0 \\ \lambda \mu_3^u & 0 & 0 \end{pmatrix} \right) e^{-i \phi_1} \frac{|H_1|}{|v_1|}$$

$$+ \left( \begin{pmatrix} \mu_2^u & 0 & 0 \\ 0 & -\mu_2^u & \mu_3^u \\ 0 & \mu_3^u & 0 \end{pmatrix} \right) e^{-i \phi_2} \frac{|H_2|}{v_2} U_{u}, \quad f, f' = u, c, t. \quad (28)$$

Where, $V_f$ and $U_f$ are bi-unitary matrices diagonalizing $M_f$ in Eq. (6), as

$$V_f^\dagger M_f U_f = \text{diag}[m_{f1}, m_{f2}, m_{f3}], \quad f = d, l, u.$$

In Eqs. (27), (28), because $\mu_0 \gg \mu_1, \mu_2, \mu_3, [3, 3]$ element are scarcely altered by diagonalization, then $\mu_0^f = m_f, \quad f = t, b, \tau$. Thus, we can get the predictions,

$$g_{H_{sff}} = \frac{m_f}{v_S}, \quad g_{H_{1ff}} = 0, \quad g_{H_{2ff}} = 0, \quad v_S = 174 \text{GeV}, \quad f = t, b, \tau, \quad (29)$$

which are compared to the standard model predictions,

$$g_{H_{ff}} = \frac{m_f}{v}, \quad v = 246 \text{GeV}, \quad f = t, b, \tau. \quad (30)$$
Next, we estimate the coupling strengths of the FCNC in our model using the values Eqs. (9) and (10) for parameters obtained in our previous work, and get the following results:

\[
\begin{pmatrix}
  g_{H_{1dd}} & = \\
  \begin{pmatrix}
  -0.00028 e^{0.3^\circ i} & 0.000076 e^{92.2^\circ i} & 0.00022 e^{-74.6^\circ i} \\
  -0.000076 e^{90.9^\circ i} & 0.00030 e^{0.1^\circ i} & 0.00066 e^{-179.7^\circ i} \\
  -0.00022 e^{-75.5^\circ i} & 0.00066 e^{-179.3^\circ i} & 0.0241 e^{-0.0^\circ i}
  \end{pmatrix}
\end{pmatrix}, \quad d, d' = d, s, b
\]

\[
\begin{pmatrix}
  g_{H_{1uu}} & = \\
  \begin{pmatrix}
  -0.00028 e^{0.3^\circ i} & 0.000076 e^{92.2^\circ i} & 0.00022 e^{-74.6^\circ i} \\
  -0.000076 e^{90.9^\circ i} & 0.00030 e^{0.1^\circ i} & 0.00066 e^{-179.7^\circ i} \\
  -0.00022 e^{-75.5^\circ i} & 0.00066 e^{-179.3^\circ i} & 0.0241 e^{-0.0^\circ i}
  \end{pmatrix}
\end{pmatrix}, \quad u, u' = u, c, t
\]
The decay ratio of the FCNC induced process \( \mu^- \to e^- e^+ e^- \),
\( \tau^- \to e^- e^+ e^- \), \( \tau^- \to \mu^- \mu^- e^- \), \( \tau^- \to \mu^- e^- e^- \), \( \tau^- \to \mu^- e^- e^- \).

The decay ratio of the FCNC induced process \( \mu^- \to e^- e^+ e^- \) (Fig. 1(a)) to the process \( \mu^- \to \nu_\mu e^- \bar{\nu}_e \) is calculated in neglecting the terms \( O((m_e/m_\mu)^2) \) and neutrino mixing for the process \( \mu^- \to \nu_\mu e^- \bar{\nu}_e \), as

\[
\frac{\Gamma(\mu^- \to e^- e^+ e^-)}{\Gamma(\mu^- \to \nu_\mu e^- \bar{\nu}_e)} = \frac{1}{24} \left| \sum_i \bar{g}_{H_{i\mu\nu}} g_{H_{ee}} \left( \frac{m_W}{m_{H_i}} \right)^2 \right|^2 
\approx \frac{1}{24} \left| \sum_i \bar{g}_{H_{i\mu\nu}} g_{H_{ee}} \right|^2 \left( \frac{m_W}{m_{H_S}} \right)^4, \quad \text{for } m_{H_S} \approx m_{H_1} = m_{H_2},
\]

where \( g_{H_{i\mu\nu}'} = g_{H_{i\mu\nu}} / g / 2 \sqrt{2} \).

For the process \( \tau^- \to e^- \mu^+ \mu^- \), which has two processes as shown in Fig. 1(b), the decay ratio \( \Gamma(\tau^- \to e^- \mu^+ \mu^-) / \Gamma(\tau^- \to \nu_\tau \mu^- \nu_\mu) \) is expressed as

\[
\frac{\Gamma(\tau^- \to e^- \mu^+ \mu^-)}{\Gamma(\tau^- \to \nu_\tau \mu^- \nu_\mu)} \approx \frac{1}{24} \left| \sum_i \bar{g}_{H_{i\tau\mu}} g_{H_{i\mu\nu}} + \sum_i \bar{g}_{H_{i\tau\mu}} g_{H_{i\mu\nu}} \right|^2 \left( \frac{m_W}{m_{H_S}} \right)^4.
\]
The numerical results for the ratio of these decay widths to ordinary weak decay widths are estimated assuming the Higgs mass value is $m_{H_S} = 120$GeV and using the experimental data [15]. These are tabulated in Table I.

Next, we analyze the semileptonic decays induced in the FCNC of quarks,

$$K_{L,S}^0 \to e^+ e^-, \quad K_{L,S}^0 \to \mu^+ \mu^-, \quad K_{L,S}^0 \to e^\mp \mu^\pm, \quad D^0 \to e^+ e^-, \quad D^0 \to \mu^+ \mu^-, \quad D^0 \to e^\mp \mu^\pm.$$  

(37)

The diagram for these processes, for example, the process $K^0 \to e^+ e^-$ is expressed as in Fig. 2. We assume the following coupling between scalar current and $K^0$ meson state as

$$J^{(K^0)}(x) = i\sqrt{2}f_K \frac{1}{\sqrt{2p_0 V}} e^{ip_ux^u}, \quad f_K = m_K f_K,$$

(38)

where $f_K$ is defined in the coupling between weak current and $K^+$ mesons as

$$J^{(K^+)}_{\mu}(x) = i\sqrt{2}f_K p_\mu \frac{1}{\sqrt{2p_0 V}} e^{ip_\mu x^\mu}, \quad f_K = \tan \theta_C f_\pi,$$

here, $\theta_C$ is Cabibbo angle, $\tan \theta_C = 0.23$, and $f_\pi = 91$MeV. The ratio of the FCNC induced process, for example $K_{L}^0 \to e^+ e^-$, to the process $K^+ \to e^+ \nu_e$ is calculated, neglecting the
TABLE I: Theoretical ratios for the FCNC induced decays and experimental data [15].

| Processes                                      | Theoretical ratios | Experimental data[15] |
|------------------------------------------------|--------------------|----------------------|
| $\Gamma(\mu^- \rightarrow e^- e^+ e^-)/\Gamma(\mu^- \rightarrow \nu_\mu e^- \bar{\nu}_e)$ | $7.5 \times 10^{-13}$ | $< 1.0 \times 10^{-12}$ |
| $\Gamma(\tau^- \rightarrow e^- e^+ e^-)/\Gamma(\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e)$ | $5.9 \times 10^{-13}$ | $< 1.5 \times 10^{-7}$ |
| $\Gamma(\tau^- \rightarrow \mu^+ e^- e^-)/\Gamma(\tau^- \rightarrow \nu_\tau \mu^- \bar{\nu}_\mu)$ | $9.5 \times 10^{-12}$ | $< 8.6 \times 10^{-8}$ |
| $\Gamma(\tau^- \rightarrow \mu^- e^+ e^-)/\Gamma(\tau^- \rightarrow \nu_\tau \mu^- \bar{\nu}_\mu)$ | $1.5 \times 10^{-11}$ | $< 1.0 \times 10^{-7}$ |
| $\Gamma(\tau^- \rightarrow e^+ \mu^- e^-)/\Gamma(\tau^- \rightarrow \nu_\tau \mu^- \bar{\nu}_\mu)$ | $1.6 \times 10^{-13}$ | $< 9.8 \times 10^{-8}$ |
| $\Gamma(\tau^- \rightarrow e^- \mu^+ e^-)/\Gamma(\tau^- \rightarrow \nu_\tau \mu^- \bar{\nu}_\mu)$ | $9.1 \times 10^{-14}$ | $< 1.6 \times 10^{-7}$ |
| $\Gamma(\tau^- \rightarrow \mu^- \mu^+ e^-)/\Gamma(\tau^- \rightarrow \nu_\tau \mu^- \bar{\nu}_\mu)$ | $2.8 \times 10^{-15}$ | $< 1.2 \times 10^{-7}$ |
| $\Gamma(K_L^0 \rightarrow e^- e^-)/\Gamma(K^+ \rightarrow e^+ \nu_e)$ | $1.7 \times 10^{-9}$ | $1.4(1 \pm 0.67_{0.44}) \times 10^{-7}$ |
| $\Gamma(K_L^0 \rightarrow e^- e^-)/\Gamma(K^+ \rightarrow e^+ \nu_e)$ | $1.3 \times 10^{-5}$ | $< 7.9 \times 10^{-2}$ |
| $\Gamma(K_L^0 \rightarrow e^+ \mu^-)/\Gamma(K^+ \rightarrow \mu^+ \nu_\mu)$ | $8.9 \times 10^{-13}$ | $< 1.8 \times 10^{-12}$ |
| $\Gamma(K_L^0 \rightarrow \mu^+ \mu^-)/\Gamma(K^+ \rightarrow \mu^+ \mu^-)$ | $1.3 \times 10^{-16}$ | $< 2.6 \times 10^{-9}$ |
| $\Gamma(K_S^0 \rightarrow \mu^+ \mu^-)/\Gamma(K^+ \rightarrow \mu^+ \nu_\mu)$ | $1.0 \times 10^{-12}$ | $< 7.0 \times 10^{-5}$ |
| $\Gamma(D^0 \rightarrow e^- e^-)/\Gamma(D^+ \rightarrow \mu^+ \nu_\mu)$ | $1.2 \times 10^{-9}$ | $< 5.3 \times 10^{-4}$ |
| $\Gamma(D^0 \rightarrow e^- e^-)/\Gamma(D^+ \rightarrow \mu^+ \nu_\mu)$ | $6.6 \times 10^{-6}$ | $< 1.7 \times 10^{-3}$ |
| $\Gamma(D^0 \rightarrow \mu^+ \mu^-)/\Gamma(D^+ \rightarrow \mu^+ \nu_\mu)$ | $1.7 \times 10^{-9}$ | $< 9.3 \times 10^{-4}$ |

term $O(m_e^2/m_K^2)$ and assuming $m_{H_S} \approx m_{H_1} = m_{H_2}$, as

$$\Gamma(K_L^0 \rightarrow e^- e^-) \approx \frac{m_e^2}{2m_K^2} \left| \frac{1}{\sqrt{2}} \sum_i \left[ \bar{g}_{H,ds} + \bar{g}_{H,sd} \right] \bar{g}_{H,ee} \right|^2 \left( \frac{m_W}{m_{H_S}} \right)^4. \quad (39)$$

For the ratio of process $K_L^0 \rightarrow \mu^+ \mu^-$ to that of $K^+ \rightarrow \mu^+ \nu_\mu$, the kinematics are altered slightly from above as

$$\frac{\Gamma(K_L^0 \rightarrow \mu^+ \mu^-)}{\Gamma(K^+ \rightarrow \mu^+ \nu_\mu)} \approx \frac{m_\mu^2}{2m_\mu^2} \left( 1 - \frac{4m_e^2}{m_K^2} \right)^{3/2} \frac{1}{\sqrt{2}} \sum_i \left[ \bar{g}_{H,ds} + \bar{g}_{H,sd} \right] \bar{g}_{H,\mu\mu} \left( \frac{m_W}{m_{H_S}} \right)^4. \quad (39)$$

For the $K_S^0$ decays, the term $\left[ \bar{g}_{H,ds} + \bar{g}_{H,sd} \right]$ is replaced as $\left[ \bar{g}_{H,ds} - \bar{g}_{H,sd} \right]$. As shown in the numerical result $g_{H,ide}$ in Eq. (31), $g_{H,ids} \approx -g_{H,isd}$, thus the $K_L^0$ decay/$K_S^0$ decay ratio becomes very small, that is indicated in experimental data [15]. The numerical estimations for the ratios of these decay widths are given by assuming the Higgs mass value $m_{H_S} =$
120GeV, and the estimated results and the experimental data \cite{15} are shown in Table I. Regarding that our estimated results satisfy the experimental data very well, we can say that our present $S_3$ invariant model is considered to be fully realistic and reasonable model.

V. CONCLUSION

We constructed our $S_3$ invariant Higgs potential adopting the most general $S_3$ invariant Higgs potential \cite{9–14} and assuming a hierarchy between the coupling constants of $S_3$ singlet and doublet Higgs field quartic products. We obtained the relation 
\[
(\frac{|v_1|}{|v_2|})^2 = \cot^2 \alpha = -\sin 2\phi_2 / \sin 2\phi_1,
\]
where $v_1$, $v_2$ are vacuum expectation values of $S_3$ doublet Higgs and $\phi_1$, $\phi_2$ are phases of $S_3$ doublet Higgs, from the stationary condition in our $S_3$ invariant Higgs potential. This relation could be satisfied exactly by the results $|v_1|/|v_2| = 0.207$, $\phi_1 = -74.9^\circ$ and $\phi_2 = 0.74^\circ$ obtained from the quark/lepton mass and mixing analyses in the $S_3$ invariant Yukawa interaction \cite{4}. Furthermore, we obtained the relation $v_S \sim v_D = \sqrt{v_1^2 + v_2^2} = 174\text{GeV}$, which should be compared to the standard value $v = 246\text{GeV}$. This value affects the coupling strength of Higgs $H_S$ to $f$ quark expressed as $g_{H_S ff} = m_f / v_S$, which is altered as by a factor $\sqrt{2}$ to the standard value. Introduced the $S_3$ doublet Higgs, FCNC are produced in tree level. We estimated the branching ratios for rare decays $\mu^- \to e^- e^+ e^-$, $\tau^- \to \mu^-\mu^+\mu^-$, $\cdots$, $K_{L,S}^0 \to e^+ e^-$, $K_{L,S}^0 \to \mu^+ \mu^-$, $\cdots$ induced by the FCNC, using the values of strength for FCNC estimated in our model. The estimated branching ratios satisfy satisfactorily the upper bound obtained from experimental data \cite{15}. Thus we can say that our present $S_3$ invariant model for Yukawa interaction and Higgs potential is a fully realistic and reasonable model.

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