Deciding the existence of quasi weak near unanimity terms in finite algebras

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We show that for a fixed positive integer $k$ one can efficiently decide if a finite algebra $A$ admits a $k$-ary weak near unanimity operation by looking at the local behavior of the terms of $A$. We also observe that the problem of deciding if a given finite algebra has a quasi Taylor operation is solvable in polynomial time by looking, essentially, for local quasi Siggers operations.

Key words: computational complexity, Maltsev condition, Taylor term, weak near unanimity, local to global

1 INTRODUCTION

Maltsev conditions, the functional equations that have a solution in a given algebra, serve as useful lens through which to view the behavior of algebras and varieties. Classically, various properties of congruence lattices of varieties are equivalent to Maltsev conditions \[1\]. More recently, height 1 Maltsev conditions turned out to describe the complexity of non-uniform Constraint Satisfaction Problem \[2\].

Given a finite algebra and a fixed Maltsev condition, we would like to decide if the algebra has a term satisfying the Maltsev condition. At the moment, the only known widely applicable test for such a condition is to check whether the Maltsev condition holds locally and then, hopefully, put together the local
pieces into a term or terms that satisfy the Maltsev condition globally. This “local to global” method works for a wide spectrum of Maltsev conditions in idempotent algebras [6, 8, 10] (however, see [9] for a case where local terms are not enough to construct a global one).

In this paper we show that the local to global method works when deciding whether the input finite algebra $A$ admits a $k$-ary quasi weak near unanimity operation ($k$-qWNU) for $k$ fixed. In this situation, we can check for local $k$-qWNU operations in time polynomial in the size of $A$ and thus we obtain an efficient algorithm for deciding if a given algebra has a $k$-qWNU operation. We also explain how the related problem of deciding if an input algebra has a quasi Taylor operation can be solved by a local to global algorithm.

The “quasi” Maltsev conditions do not force the operations in question to be idempotent and our result does not require that the input algebra be idempotent. This is in contrast to the numerous examples of idempotent Maltsev conditions that are EXPTIME-complete to decide in general finite algebras [6, 8]. We speculate that the “local to global” construction will remain useful for general algebras as long as we avoid the need to construct an idempotent operation out of non-idempotent basic operations. In particular we believe that the local to global method should work for many height 1 Maltsev conditions other than having a qWNU or a quasi Taylor operation.

2 PRELIMINARIES

An $n$-ary operation is a mapping $f : A^n \rightarrow A$. An algebra $A$ consists of non-empty set $A$, called the universe or domain of $A$, and a set of basic operations $f_i$ where $i$ ranges over some index set $I$. In this paper we will only consider algebras with finite universes.

A term is a valid formal expression that describes an operation as a composition of basic operation and variable symbols. Examples of terms are “$f(x, g(y, y, x), h(z))$” or “$x$.” For a formal definition of terms, see [4, Definition 10.1]. If the symbols in a term $t$ are the basic operations of an algebra $A$, we can evaluate $t$ in $A$ and obtain an operation. Operations that we can construct in this way are called the term operations of $A$. We will call the set of all term operations of $A$ the clone of $A$. If the clone of $A$ contains a particular kind of operation, we will often simply say that $A$ has this kind of operation.

An identity or an equation is the statement $t \approx u$ where $t$ and $u$ are terms. The identity $t \approx u$ holds in an algebra $A$ if $t$ and $u$ evaluate to the same operation in $A$ (that is, the identity holds for all values of the input variables).
A strong Maltsev condition is a finite system of identities. An algebra \( A \) satisfies a strong Maltsev condition \( \Sigma \) if we can choose for each operation symbol in \( \Sigma \) some operation in \( A \) so that all the identities of \( \Sigma \) hold. A strong Maltsev condition is of height 1 if both the left and the right hand side of all identities in the Maltsev condition contain exactly one term symbol (i.e. both sides are of the form “term(variables)”).

An operation \( t \) on a set \( A \) is idempotent if the identity \( t(x, \ldots, x) \approx x \) holds. An algebra \( A \) is idempotent if each of its basic operations is idempotent.

A \( k \)-ary operation \( w \) is called a weak near unanimity (WNU) operation if it is idempotent and satisfies the chain of identities

\[
w(y, x, \ldots, x, x) \approx w(x, y, \ldots, x, x) \approx \cdots \approx w(x, x, \ldots, x, y).
\]

If we drop the idempotence requirement, we will call \( w \) a quasi weak near unanimity (qWNU).

An operation is a Taylor operation if it is idempotent and satisfies a system of identities of the form

\[
t(x, ?, ?, \ldots, ?) \approx t(y, ?, ?, \ldots, ?) \\
t(?, x, ?, \ldots, ?) \approx t(? ,y, ?, \ldots, ?) \\
\vdots \\
t(?, ?, ?, \ldots, x) \approx t(? ,?, ?, \ldots, y),
\]

where \( x, y \) are variables and the question marks stand for some choice of \( x \)'s and \( y \)'s. As before, if we do not require \( t \) to be idempotent, we will call \( t \) a quasi Taylor operation.

It is a classic result by W. Taylor [14, Corollary 5.2 and 5.3] that an idempotent variety \( V \) interprets into the trivial algebra on two elements if and only if \( V \) does not have a Taylor term operation. Obviously, a \( k \)-WNU operation is a special case of a Taylor operation. Much less obviously, it turns out that having a (quasi) Taylor operation implies having a (quasi) weak near unanimity operation of some arity as well as a specific arity 4 (quasi) Taylor term. For idempotent algebras this was shown in [11, 13]. We state this result as a variant of [2, Theorem 1.4] here (in particular, we replace cyclic terms with the weaker WNU terms):

**Theorem 1.** Let \( A \) be a finite algebra. Then the following are equivalent:

a) \( A \) has a quasi Taylor term,
b) there exists \( k \geq 2 \) such that \( A \) has a \( k \)-qWNU term,

c) \( A \) has a quasi Siggers term, i.e. an arity 4 term \( s \) satisfying the identity

\[
s(r, a, r, e) \approx s(a, r, e, a).
\]

Moreover, if \( A \) is idempotent, the “quasi” qualifier can be dropped in the first two points and \( s \) is idempotent in the last point.

The following lemma will allow us to transfer properties between general and idempotent algebras. This is not really a new result; rather, it is a different way to express some ideas present in [2] and [5, Lemma 6.4] in a style similar to the construction of minimal algebras in the Tame Congruence Theory [7].

**Lemma 2.** Let \( A \) be a finite algebra with the universe \( A \). Then there exists a finite idempotent algebra \( B \) with universe \( B \subseteq A \) such that:

a) There exists a unary term operation \( \alpha \) of \( A \) whose image is \( B \) and such that \( \alpha \circ \alpha = \alpha \),

b) for each strong Maltsev condition \( \Sigma \) of height 1 we have that \( A \) satisfies \( \Sigma \) if and only if \( B \) satisfies \( \Sigma \), and

c) for each operation \( t \) in the clone of term operations of \( A \) there exists a unary term operation \( \tau \) of \( A \) such that \( \tau \circ t \) restricted to \( B \) lies in the clone of term operations of \( B \).

**Proof.** Let \( \alpha \) be a unary term operation of \( A \) such that the image of \( \alpha \) is inclusion minimal among all images of unary operations in the clone of \( A \). Denote the image set of \( \alpha \) by \( B \). Since \( \alpha \) restricted to \( B \) needs to be a permutation (or else \( \alpha^2 \) would have a smaller image than \( B \)), we can replace \( \alpha \) by its suitable power so that \( \alpha \) restricted to \( B \) is the identity map. This gives us the identity \( \alpha \circ \alpha = \alpha \).

We construct the algebra \( B \) on \( B \) as follows: For each term operation \( t \) of \( A \) consider the operation \( u = \alpha \circ t \) restricted to \( B \). If \( u \) is idempotent, we make it into a basic operation of \( B \). By definition \( B \) is then an idempotent algebra.

Let us now prove \( c \). Let \( t \) be an operation in the clone of \( A \). Consider the mapping \( \beta : B \to B \) given by \( \beta(b) = \alpha(t(b, b, \ldots, b)) \). From the minimality of \( B \), we get that the image of \( \beta \) needs to be exactly \( B \) and that \( \beta \) is a permutation. Let \( n \) be such that \( \beta^{n+1} \) is the identity mapping. Then the operation

\[
u(x_1, \ldots, x_n) = \beta^n \circ \alpha \circ t(x_1, \ldots, x_n)
\]
restricted to $B$ is a basic operation of $B$. Taking $\tau = \beta^n \circ \alpha$ gives us part (c).

To prove the “only if” part of item (b), let $\Sigma$ be a strong Mal’tsev condition of height 1 and let $t_1, \ldots, t_k$ be operations from the clone of operations of $A$ that satisfy $\Sigma$. We know that for each $t_i$ there is a $\tau_i$ such that $\tau_i \circ t_i$ restricted to $B$ lies in the clone of $B$. Moreover, from the proof of part (c) we see that $\tau_i$ depends only on the mapping $t_i(x, x, \ldots, x)$. From this it follows that we can choose $\tau_1, \ldots, \tau_k$ so that when $t_i$ and $t_j$ appear in one identity in $\Sigma$ we must have $\tau_i = \tau_j$. This accomplished, it is straightforward to verify that the operations $\tau_1 \circ t_1, \tau_2 \circ t_2, \ldots, \tau_k \circ t_k$ satisfy $\Sigma$ and the implication is proved.

Finally, let us prove the “if” part of item (b). Let $\Sigma$ be a strong Mal’tsev condition of height 1. Assume that $B$ has operations $t_1, \ldots, t_k$ that satisfy $\Sigma$. Then it is easy to verify that composing $t_1, \ldots, t_k$ with $\alpha$ from the inside (that is, considering operations of the form $t_i(\alpha(x_1), \alpha(x_2), \ldots)$) gives us operations from the clone of operations of $A$ that also satisfy $\Sigma$.  

A subset $B$ of $A$ is a subuniverse of $A$ if it is closed under the basic operations of $A$ (equivalently, under all term operations of $A$). The subuniverse generated by $X \subseteq A$ is the smallest subuniverse of $A$ that contains the set $X$. It is easy to see that $a$ lies in the subuniverse of $A$ generated by $X$ if and only if there is a term operation of $A$ that outputs $a$ when applied to the list of elements of $X$.

The $n$-th power of $A$ is the algebra with universe $A^n$ and operations $f_i$ acting coordinate-wise on $A^n$. A subuniverse of a power of $A$ is called a subpower of $A$.

A tuple of elements of $A$ is a member of the set $A^n$. To emphasize that $\overline{\mathbf{v}}$ is a tuple, we will sometimes put a bar above it. In Section 3 we will use a notation such as “$(\overline{\mathbf{v}}, e)$” to denote a tuple, whose prefix is $\overline{\mathbf{v}}$, followed by the sequence of elements from $\overline{\mathbf{v}}$ and finally the element $e \in A$.

An $n$-ary relation $R$ over a set $A$ is a subset of $A^n$. A relation is admissible for (or compatible with) $A$ if $R$ is a subuniverse of $A^n$; in other words if $R$ is closed under the term operations of $A$ applied coordinate-wise.

A set $A$ together with a binary relation $E \subseteq A^2$ is called a directed graph (digraph). When examining digraphs, the elements of $A$ are called vertices and a pair $(u, v) \in E$ is called the edge from $u$ to $v$. A loop in a digraph is an edge of the form $(u, u)$ for some $u \in A$.

A walk from $u$ to $v$ in a digraph $E$ is a sequence of vertices and edges of the form

$$u = w_1, e_1, w_2, e_2, \ldots, e_{k-1}, w_k = v$$

such that for each $i$ the edge $e_i \in E$ is either $(w_i, w_{i+1})$ (a forward edge) or
(\(w_{i+1}, w_i\)) (a backward edge).

The algebraic length of a walk is the number of its forward edges minus the number of its backward edges. (Algebraic length is not uniquely defined for walks with loops, but this will not be a concern for our purposes.) A directed cycle of length \(k\) is a walk from some \(u\) to the same vertex \(u\) that consists of \(k\) forward edges and no backward edges.

A digraph has algebraic length 1 if there exists \(u \in A\) and a walk from \(u\) back to \(u\) of algebraic length 1. In particular, a digraph has algebraic length 1 if it contains two directed cycles of lengths \(n\) and \(n - 1\) that share a vertex.

A digraph is smooth if for each \(a \in A\) there exists \(b, c \in A\) such that \((a, b), (c, a) \in E\).

We will need the following theorem. As an aside, we note that the theorem has since its publication inspired many variants and generalizations, often called “loop lemmas.”

**Theorem 3** ([1, Theorem 3.5]). *If a smooth digraph (on a finite set) has algebraic length 1 and admits a Taylor polymorphism then it contains a loop.*

Here “admits a Taylor polymorphism” means that the relation \(E\) is compatible with some algebra \(A\) that has a Taylor term operation.

In our proof we will need the following corollary of Theorem 3. We include a proof here because we have not found it in the published literature; we do not claim originality (the result was known, e.g., to M. Olšák [12]).

**Corollary 4.** *If a smooth digraph (on a finite set) has algebraic length 1 and admits a quasi Taylor polymorphism then it contains a loop.*

**Proof.** Denote the digraph in question by \((A, E)\). Let \(A\) be an algebra with a quasi Taylor term such that \(E\) is admissible for \(A\). Let \(B\) be the idempotent algebra for \(A\) from Lemma 2 and let \(\alpha\) be a unary term operation of \(A\) whose image is \(B\) from the same Lemma.

We know that \(A\) satisfies some quasi Taylor identity \(\Sigma\). Since this particular identity is a strong Maltsev condition, it follows that \(B\) also satisfies \(\Sigma\). Since \(B\) is idempotent, it has a Taylor term.

Consider the digraph with the edge relation \(F = E \cap B^2\) on \(B\). Since the operations of \(B\) are just restrictions of (some) operations from the clone of \(A\), it follows that \(F\) is an admissible relation for \(B\). Moreover, \(F\) has algebraic length 1, which we can prove by taking the \(\alpha\)-image of any closed walk of algebraic length 1 in \((A, E)\).

Since the digraph \((B, F)\) admits a Taylor polymorphism, Theorem 3 gives us a loop in \(F\). Given that \(F \subseteq E\), it follows that there is a loop in \(E\) as
3 DECIDING THE EXISTENCE OF A K-QWNU FOR K FIXED

In this section we will define and provide an algorithm for two computational problems. In both of the problems the input algebra is given by a list of tables of its (finitely many) basic operations. The sum of the sizes of these tables will be denoted by $\|A\|$. We will assume that the input algebra has at least one basic operation so that $\|A\| \geq |A|$.

**Definition 5.** Define HAS-$k$-WNU-IDEMP to be the following decision problem:

**INPUT:** An idempotent algebra $A$ (on a finite set with finitely many basic operations).

**QUESTION:** Does $A$ have a $k$-ary weak near unanimity operation?

Define HAS-$k$-qWNU to be the following decision problem:

**INPUT:** An algebra $A$ (on a finite set with finitely many basic operations).

**QUESTION:** Does $A$ have a $k$-ary quasi weak near unanimity operation?

Note that in both problems the number $k$ is not a part of the input. Indeed, the running time of our algorithm will depend exponentially on $k$ and we do not know if there is a polynomial time algorithm if $k$ is allowed to be a part of the input (even if $k$ were written in the unary number system). Also, both problems are trivial if $k = 1$, so we will assume that $k \geq 2$ in the rest of this section.

Note also that in HAS-$k$-WNU-IDEMP we demand that the input algebra be idempotent. We do not know the complexity of HAS-$k$-WNU-IDEMP should we drop the requirement that $A$ be idempotent, but we guess that the problem is hard.

**Observation 6.** The problem HAS-$k$-WNU-IDEMP reduces to HAS-$k$-qWNU.

**Proof.** Let $A$ be an idempotent algebra. Since a WNU is just an idempotent qWNU, it follows that $A$ has a $k$-WNU if and only if it has a $k$-qWNU. Therefore, we can just run HAS-$k$-qWNU with the input $A$ to solve HAS-$k$-WNU-IDEMP. \qed
Given Observation 6, it is enough to find a polynomial time algorithm for HAS-\(k\)-qWNU. Our strategy for that will be to show that having local \(k\)-qWNU terms implies that the input algebra actually has a \(k\)-qWNU term. However, in order to use Corollary 4 we will need to show that the algebra in question has a Taylor term first.

**Definition 7.** We say that an algebra \(A\) has **local quasi Taylor operations** if for every \(a, b \in B\) there exists a term operation \(t_{a,b}\) of \(A\) such that we can replace the question marks below by either \(a\) or \(b\) so that the following equalities hold

\[
\begin{align*}
t_{a,b}(a, ?, ?, \ldots, ?) &= t_{a,b}(b, ?, ?, \ldots, ?) \\
t_{a,b}(?, a, ?, \ldots, ?) &= t_{a,b}(?, b, ?, \ldots, ?) \\
&\vdots \\
t_{a,b}(?, ?, ?, \ldots, a) &= t_{a,b}(?, ?, ?, \ldots, b).
\end{align*}
\]

If in addition each \(t_{a,b}\) can be chosen to be idempotent we say that \(A\) has **local Taylor operations**.

It turns out that the local to global principle works for (quasi) Taylor operations.

**Lemma 8.** Let \(A\) be a finite idempotent algebra with local Taylor operations. Then \(A\) has a Taylor operation.

**Proof.** We use the fact that for a finite idempotent algebra \(A\), having a Taylor term is equivalent to there not being a two element algebra in \(HS(A)\) (the class of homomorphic images of subalgebras of \(A\)) whose term operations are projection maps. This was first established by A. Bulatov and P. Jeavons [3, Proposition 4.14]; see also [7, Lemma 9.4 and Theorem 9.6] and [15, Proposition 3.1].

So, suppose to the contrary, that \(HS(A)\) contains a two element algebra; let this algebra be the quotient of \(B\) of \(A\) by the congruence \(\theta\). Take two elements \(r, s \in B\) with \((r, s) \notin \theta\). Let \(t_{r,s}\) be a local Taylor term for \(r\) and \(s\) in \(A\).

By assumption, the term \(t_{r,s}\) on \(B/\theta\) is a projection map, say onto its first variable (without loss of generality). But then we have (the question marks stand for one of \(r, s\) according to the first local Taylor equality for \(t_{r,s}\))

\[
\begin{align*}
s/\theta &= t_{r,s}(s/\theta, ?, ?, \ldots, ?)/\theta \\
&= t_{r,s}(r, ?, ?, \ldots, ?)/\theta = t_{r,s}(r/\theta, ?, ?, \ldots, ?)/\theta = r/\theta,
\end{align*}
\]
a contradiction.

**Corollary 9.** Let $A$ be a finite algebra with local quasi Taylor operations. Then $A$ has a quasi Taylor operation.

**Proof.** Let $A$ be a finite algebra with local Taylor terms, but no quasi Taylor operation. Let $B$ be the algebra from Lemma 2 for $A$. Since $A$ and $B$ satisfy the same strong height 1 Mal'tsev conditions, if we show that $B$ has a Taylor operation, we shall get that $A$ has a quasi Taylor operation.

Given Lemma 8, it is enough to show that $B$ has local Taylor terms. But that is easy: Let $b, c \in B$ and let $t_{b,c}$ be the local quasi Taylor term for $b, c$ in $A$. By part (c) of Lemma 2 there exists a $\tau$ such that $\tau \circ t_{b,c}$ restricted to $B$ is a term operation of $B$. Applying $\tau$ to the both sides of the equalities

\[
t_{b,c}(a, ?, ?, \ldots, ?) = t_{b,c}(b, ?, ?, \ldots, ?)
\]

\[
t_{b,c}(?, a, ?, \ldots, ?) = t_{b,c}(?, b, ?, \ldots, ?)
\]

\[
\vdots
\]

\[
t_{b,c}(?, ?, ?, \ldots, a) = t_{b,c}(?, ?, ?, \ldots, b).
\]

gives us the equalities

\[
\tau \circ t_{b,c}(a, ?, ?, \ldots, ?) = \tau \circ t_{b,c}(b, ?, ?, \ldots, ?)
\]

\[
\tau \circ t_{b,c}(?, a, ?, \ldots, ?) = \tau \circ t_{b,c}(?, b, ?, \ldots, ?)
\]

\[
\vdots
\]

\[
\tau \circ t_{b,c}(?, ?, ?, \ldots, a) = \tau \circ t_{b,c}(?, ?, ?, \ldots, b).
\]

Since $\tau \circ t_{b,c}$ is idempotent by the choice of $\tau$, we see that $\tau \circ t_{b,c}$ is a local quasi Taylor operation for $b, c$ in $B$.

**Definition 10.** Let $n \in \mathbb{N}$, $k \geq 2$. An algebra $A$ has $n$-local $k$-qWNU terms if for every choice of $n$-tuples $\overline{r}, \overline{s} \in A^n$ there is a $k$-ary term operation $t_{\overline{r}, \overline{s}}$ of $A$ such that

\[
t_{\overline{r}, \overline{s}}(\overline{r}, \overline{r}, \ldots, \overline{r}) = t_{\overline{r}, \overline{s}}(\overline{r}, \overline{s}, \ldots, \overline{r}) = \cdots = t_{\overline{r}, \overline{s}}(\overline{r}, \overline{r}, \ldots, \overline{s}),
\]

where the term operation $t_{\overline{r}, \overline{s}}$ is applied coordinate-wise to the given $n$-tuples. We call such a term an $n$-local $k$-qWNU term of $A$ for $\overline{r}$ and $\overline{s}$.

Observe that $n$-local $k$-qWNU terms are a special case of local quasi Taylor terms.
It is elementary to show that $A$ has $n$-local $k$-qWNU terms if and only if for each pair of $n$-tuples $\mathbf{r}, \mathbf{s} \in A^n$ (which we will write as column tuples) the subuniverse $R$ of $A^{kn}$ generated by the column vectors of the matrix

$$
\begin{pmatrix}
\mathbf{r} & \mathbf{r} & \ldots & \mathbf{r} & \mathbf{r} \\
\mathbf{r} & \mathbf{r} & \ldots & \mathbf{r} & \mathbf{r} \\
\vdots & & & & \\
\mathbf{r} & \mathbf{r} & \ldots & \mathbf{r} & \mathbf{r} \\
\mathbf{r} & \mathbf{r} & \ldots & \mathbf{r} & \mathbf{r}
\end{pmatrix}
$$

contains a tuple of the form $(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u})$ for some $\mathbf{u} \in A^n$.

Furthermore, large enough local $k$-qWNUs are actually global: For an an algebra $A$ with the universe $A$ having $|A|^2$-local $k$-qWNU terms is the same thing as having a $k$-qWNU term. All that remains is to bridge the gap between $1$-local and $|A|^2$-local $k$-qWNU terms.

**Lemma 11.** Let $A$ be a finite algebra and let $n \geq 1$. If $A$ has $n$-local $k$-qWNU terms then $A$ also has $(n+1)$-local $k$-qWNU terms.

**Proof.** Take $\mathbf{r}, \mathbf{s} \in A^n$ and $c, d \in A$. We want to show that the subpower of $A^{(n+1)k}$ generated by the columns of the matrix

$$
\begin{pmatrix}
\mathbf{r} & \mathbf{r} & \ldots & \mathbf{r} & \mathbf{r} \\
c & d & \ldots & d & d \\
\mathbf{r} & \mathbf{r} & \ldots & \mathbf{r} & \mathbf{r} \\
d & c & \ldots & d & d \\
\vdots & & & & \\
\mathbf{r} & \mathbf{r} & \ldots & \mathbf{r} & \mathbf{r} \\
d & d & \ldots & c & d \\
\mathbf{r} & \mathbf{r} & \ldots & \mathbf{r} & \mathbf{r} \\
d & d & \ldots & d & c
\end{pmatrix}
$$

contains a tuple of the form $(\mathbf{u}, e, \mathbf{u}, e, \ldots)$ for some $\mathbf{u} \in A^n$ and $e \in A$. By the $n$-local $k$-WNU property, we find that there is an $\mathbf{u} \in A^n$ and elements $b_1, \ldots, b_n \in A$ such that

$$(\mathbf{u}, b_1, \mathbf{u}, b_2, \mathbf{u}, b_3, \ldots, \mathbf{u}, b_n) \in R.$$ 

Inspired by the tuple above, let us consider the relation $S$ that we get from $R$ by the following formula

$$S = \{(x_1, \ldots, x_n) : \exists \mathbf{u} \in A^n, (\mathbf{c}, x_1, \mathbf{c}, x_2, \ldots, \mathbf{c}, x_n) \in R\}.$$
It is easy to verify that $S$ is a subpower of $A$ (either from the definition or by observing that $S$ is defined from the admissible relation $R$ using a primitive positive formula). We know that $(b_1, \ldots, b_n) \in S$. Moreover, since we can permute the generators of $R$, it follows that if $(x_1, \ldots, x_n) \in S$ and $\pi$ is a permutation of $[n]$, then $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in S$. As an intermediate step in our proof, we will show that $S$ contains a tuple of the form $(e, e, \ldots, e, f)$ for some $e, f \in A$.

We define the following digraph $G$: The vertex set of $G$ is 
\[ V = \{(v_1, \ldots, v_{n-2}) : \exists y, z \in A, (v_1, \ldots, v_{n-2}, y, z) \in S\} \]
and the edge relation is 
\[ E = \\{(v_1, v_2, \ldots, v_{n-2}), (v_2, v_3, \ldots, v_{n-1}) : \exists z \in A, (v_1, v_2, \ldots, v_{n-2}, v_{n-1}, z) \in S\} \].

We want to show that $G$ contains a loop, since any loop of $G$ will witness that there is a tuple in $S$ of the form $(e, e, \ldots, e, f)$ for some $e, f \in A$. To find a loop, we want to apply Corollary 4 to $G$. To do that we need to prove that $G$ is a smooth digraph of algebraic length 1 that admits a quasi Taylor polymorphism.

Let $t$ be a quasi Taylor term operation of $A$. As with $S$, it is easy to verify that both $V$ and $E$ are subpowers of $A$; hence $t$ applied coordinate-wise is a quasi Taylor polymorphism of $G$.

To see that $G$ is smooth, consider some $(v_1, \ldots, v_{n-2}) \in V$. By definition, there are $y, z$ such that $(v_1, \ldots, v_{n-2}, y, z) \in S$. By definition of $E$, we immediately get that there is an edge from $(v_1, \ldots, v_{n-2})$ to $(v_2, \ldots, v_{n-2}, z)$. Since $S$ is invariant under permutations, we also have $(z, v_1, \ldots, v_{n-2}, y) \in S$. Hence, there is an edge in $G$ from $(z, v_1, \ldots, v_{n-3})$ to $(v_1, \ldots, v_{n-2})$, concluding the proof of smoothness of $G$.

To show that $G$ has algebraic length 1, we find two cycles of lengths $n - 1$ and $n$ that start at the vertex $(b_1, \ldots, b_{n-2})$. Using $(b_1, \ldots, b_n) \in S$, we get the directed cycle of length $n$ in $G$ with vertices 
\[ (b_1, \ldots, b_{n-2}), (b_2, \ldots, b_{n-1}), (b_3, \ldots, b_n), \ldots, (b_n, b_1, \ldots, b_{n-3}) \].

Since $S$ is invariant under permutations, we know that for each $k \in \mathbb{N}$ we also have 
\[ (b_{1+k}, b_{2+k}, \ldots, b_{n-1+k}, b_n) \in S \]
where the addition is modulo \( n - 1 \). This gives us the direct cycle of length \( n - 1 \) in \( G \) with vertices
\[
(b_1, \ldots, b_{n-2}), (b_2, \ldots, b_{n-1}), (b_3, \ldots, b_{n-1}, b_1), \ldots, (b_{n-1}, b_1, \ldots, b_{n-3}).
\]

Taken together, these two cycles imply that the algebraic length of \( G \) is 1 and thus, by Theorem 3, \( G \) contains a loop and so \( S \) contains a tuple of the form \((e, \ldots, e, f)\). The symmetry of \( S \) gives us that \( S \) contains all tuples of the form \((e, e, \ldots, e, f, e, \ldots, e)\).

Since \( A \) has \( n \)-local \( k \)-qWNU operations, it follows that \( A \) has a 1-local \( k \)-qWNU operations. Applying the 1-local \( k \)-qWNU operation \( t_{e,f} \) to \( S \), we conclude that \( S \) contains a constant tuple \((g, g, \ldots, g)\) for some \( g \in A \). From this, it follows that \( R \) contains a tuple of the form \((\tau, g, \tau, g, \ldots, \tau, g)\), showing that \( A \) has \((n + 1)\)-local \( k \)-WNU terms and we are done.

**Theorem 12.** The problem HAS-\( k \)-qWNU is solvable in polynomial time.

**Proof.** By Lemma 11 we just need to test if \( A \) has 1-local \( k \)-qWNU terms. As noted just after Definition 10, this amounts to testing if for each \( r, s \in A \) the subpower of \( A \) generated by the \( k \)-tuples
\[
(s, r, r, \ldots, r), (r, s, r, \ldots, r), \ldots, (r, r, r, \ldots, s)
\]
contains a constant tuple. By [6, Proposition 6.1], we can generate this subpower (for \( r, s \) fixed) by an algorithm whose run-time is \( O(m\|A\|^k) \). Here \( m \) is the largest arity of a basic operation of \( A \). Since this test needs to be performed for each pair of elements \((r, s)\) from \( A^2 \), we conclude that testing for a \( k \)-qWNU term can be carried out by an algorithm whose run-time is \( O(m|A|^2\|A\|^k) \), which is polynomial in \( \|A\| \).

### 4 DECIDING QUASI TAYLOR TERMS

We end our paper by an observation that connects our result to a related problem of deciding if an input algebra has a (quasi) WNU operation.

**Definition 13.** Define HAS-TAYLOR-IDEEMP to be the following decision problem:

**INPUT:** An idempotent algebra \( A \) (on a finite set with finitely many basic operations).

**QUESTION:** Does \( A \) have a Taylor term?
Define HAS-QTAYLOR to be the following decision problem:

INPUT: An algebra $A$ (on a finite set with finitely many basic operations).

QUESTION: Does $A$ have a quasi Taylor term?

Note that by Theorem 1, the “yes” instances of HAS-TAYLOR-IDEMP are exactly those idempotent algebras that have some WNU operation and the “yes” instances of HAS-QTAYLOR are exactly algebras with some qWNU operations. However, HAS-QTAYLOR is a different problem than the problem of deciding whether the variety generated by $A$ omits type 1. Omitting type 1 is equivalent to having a Taylor term and this problem is EXPTIME-complete [6, Corollary 9.3].

The problem HAS-TAYLOR-IDEMP is known to be in P [6, Theorem 6.3]. However, should we drop the requirement that the input algebra be idempotent, HAS-TAYLOR-IDEMP would become EXPTIME-complete as mentioned in the previous paragraph. We will show by combining several known results that HAS-QTAYLOR lies the class P.

Observation 14. A finite algebra $A$ has a quasi Taylor operation if and only if for every $a, b \in A$ the subalgebra of $A^4$ generated by the columns of the matrix

$$
\begin{pmatrix}
    a & b & b & b \\
    b & a & b & b \\
    b & b & a & b \\
    b & b & b & a
\end{pmatrix}
$$

contains a tuple of the form $(q, q, r, r)$ for some $q, r \in A$.

Proof. Assume first that $A$ has a quasi Taylor operation. By Theorem 1 the algebra $A$ also has a quasi Siggers operation $s$ that satisfies the identity $s(r, a, r, e) \approx s(a, r, e, a)$. It is easy to verify that from this identity it follows

$$
s(a, b, b) \approx s(b, a, b),
$$

$$
s(b, b, a) \approx s(b, b, a).
$$

Therefore, given $a, b \in A$ we can apply $s$ to the columns of the matrix

$$
\begin{pmatrix}
    a & b & b & b \\
    b & a & b & b \\
    b & b & a & b \\
    b & b & b & a
\end{pmatrix}
$$

13
to get a tuple whose first two and last two entries are identical.

In the other direction, the terms \( s_{a,b} \) that witness the existence of a tuple \((q, q, r, r)\) in the subpower of \( A \) generated by the columns of

\[
\begin{pmatrix}
a & b & b & b \\
b & a & b & b \\
b & b & a & b \\
b & b & b & a \\
\end{pmatrix}
\]

are local quasi Taylor terms. Therefore \( A \) has a quasi Taylor operation by Corollary 9.

\[\square\]

**Corollary 15.** The following algorithm solves the problem HAS-QTAYLOR in time polynomial in \( \|A\| \): For each \( a, b \in A \) examine the subpower \( R_{a,b} \) generated by the columns of the matrix

\[
\begin{pmatrix}
a & b & b & b \\
b & a & b & b \\
b & b & a & b \\
b & b & b & a \\
\end{pmatrix}
\]

If for some \( a, b \) the relation \( R_{a,b} \) does not contain a tuple of the form \((q, q, r, r)\), answer “no.” Else answer “yes.”

**Proof.** The correctness of the algorithm follows from Observation 14 while the analysis of the running time is similar to that done in the proof of Theorem 12.

\[\square\]

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