An Effective Non-Commutative Loop Quantum Cosmology

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Abstract. We construct a Non-Commutative extension of the Loop Quantum Cosmology effective scheme for the open FLRW model. We start from the holonomized Hamiltonian and implement a canonical non-commutativity among the matter degree of freedom and the holonomy variable, in the volume representation. We obtain a noncommutative extension of the modified Friedmann equation which arises in Loop Quantum Cosmology for a particular case of the theta deformation.

1. Introduction
The idea of Noncommutativity in spacetime is quite old (1947) [1]. It was proposed as an attempt to regularize Quantum Field Theory before the renormalization program was established. Due to its non-local behavior, Noncommutativity was quickly forgotten after renormalization proved to be successful. In the 1970’s M. Flato and co-workers proposed an alternative path to quantization [2], in which a deformation of the Poisson structure of classical phase space is performed and is encoded in the moyal \( \star \)-product [3] and generalizations of it (for a review see [4]). In the early 1980’s mathematicians led by A. Connes succeeded in formulating what they called Noncommutative Geometry [5], motivated by generalizing a classic theorem characterizing \( C^\ast \)-algebras.

Recently, the noncommutative paradigm has resurrected, mainly due to results in String Theory [6, 7], in which Yang-Mills theories in a noncommutative space arise in different circumstances as effective theories when taking certain simple limits (for instance, the low energy limit). This renewed interest has led to a deeper understanding, from the physical and mathematical points of view, of noncommutative field theory (for a review see [8]). It is believed that in a full quantum theory of gravity the continuum picture of spacetime would no longer be consistent at distances comparable to the Planck length \( \ell_p \sim 10^{-35} \text{cm} \), a quantization of spacetime itself could be in order. Furthermore, since quite ago, it has been argued [9] that measurements of position cannot be performed to better accuracies than the Planck length, since spacetime itself would be modified due to the energy required to perform such measurements.

A possible way to model this effects could be via an uncertainty relation for the spacetime coordinates of the form

\[
[x^i, x^j] = i\theta^{ij}
\]  

(1)
This commutation relation is the starting point of noncommutative field theory. Attempts to implement this idea have led to different proposals for a noncommutative theory of gravity [10]. Different incarnations of noncommutative gravity have a common feature, they are highly non-linear theories, which makes them very difficult to work with.

A possible way to study noncommutativity effects in the early universe was proposed by Garcia-Compean et al [11]. They implemented noncommutativity in configuration space (in contrast to noncommutative spacetime), but only after a symmetry reduction of spacetime had been imposed and the Wheeler-DeWitt quantization had been carried, giving rise to a noncommutative quantum cosmology (mathematically similar to noncommutative quantum mechanics [12]). Later, G. D. Barbosa and N. Pinto-Neto [13] introduced this minisuperspace noncommutativity already at the classical level. The idea is that, perhaps, this effective noncommutativity could incorporate novel effects and insights of a full quantum theory of the gravitational field, alongside with providing a simple framework for studying the implications of such possible noncommutative effects in the (early) universe through cosmological models. Since the publication of these two seminal investigations, some works along this line have been conducted. For instance, the noncommutativity of the Friedmann-Robertson-Walker cosmology has been studied, as well as some of the Bianchi Class A models [14]. Quantum black holes have also been investigated within this framework [15].

On the other hand, Loop Quantum Gravity (LQG) [16, 17] is an attempt to quantize the gravitational field taking seriously the lessons from General Relativity, that is, it aims at a full (non-perturbative) background independent quantization of General Relativity. Loop Quantum Cosmology (LQC) [18, 19] is the quantization of cosmological (symmetry reduced) models following closely the ideas and methods of LQG. In this way, the LQC of the FLRW and some of the Bianchi Class A models in the presence of a massless scalar field (employed as internal time) has been constructed [20, 21, 22, 23, 24, 25, 26], in particular, as a result of the underlying quantum geometry, it has been shown that the loop quantization of the FLRW models features a bouncing which enables the resolution of the cosmological singularity [27]. The LQC of the inhomogeneous Gowdy model has also been constructed [28]. As a result of loop quantization, the Wheeler-DeWitt equation is no longer a differential equation, but a difference equation, which is difficult to work with even in the simplest models. In order to extract physics, effective equations based on a geometrical formulation of Quantum Mechanics [29] have been employed to study the consequences of loop quantum corrections in cosmological models. For instance, the effective description of the FLRW models reproduces very well the behavior of the corresponding full loop quantization of such models. The present investigation aims at constructing a non-commutative effective scheme for the open FLRW model, in the presence of a scalar field.

The manuscript is organized as follows: In section II we introduce the loop variables for the open FLRW model with a free standard scalar field and the corresponding effective scheme. In section III we recall the Weyl-Wigner-Moyal correspondence and the construction of a Noncommutative Quantum Mechanics based on this correspondence. Section IV is devoted to construct a non-commutative model for the effective Loop Quantum Cosmology of the open FLRW model resembling the Noncommutative Quantum Mechanics of section III, along the lines of references [11, 13].

2. Connection Variables and Effective Dynamics
Here we recall the formulation of the open FLRW and Bianchi I models in the Ashtekar-Barbero variables, with a free massless scalar field. The Ashtekar-Barbero variables cast General Relativity in the form of a gauge theory, in which phase space is described by a $\mathfrak{su}(2)$ gauge connection, the Ashtekar-Barbero connection $A_{a}^{i}$ and its canonical conjugate momentum, the densitized triad $E_{i}^{a}$ (i, j, etc. denote internal $\mathfrak{su}(2)$ indices while a, b, etc. denote spatial indices).
These quantities are defined as

\[ A_i^a = \Gamma_i^a + \gamma K_i^a, \quad E_i^a = \sqrt{\gamma} e_i^a \]

(2)

where \( K_i^a = K_{ab} e_i^b \), with \( e_i^a \) and \( e_i^a \) the triad and co-triad, respectively (\( e_i^a e_j^a = \delta_i^j \), \( q_{ab} = \delta_{ij} e_i^a e_j^b \)); \( \Gamma_i^a \) is defined through the spin connection \( \omega_{ij}^a \) compatible with the triad (\( \nabla_a e_i^b + \omega_{aj}^i e_j^b = 0 \), \( \nabla_a \) being the usual spatial covariant derivative) by the relation \( \omega_{ij}^a = \epsilon_{ijk} \Gamma_k^a \) with \( \epsilon_{ijk} \) the totally antisymmetric symbol; \( \gamma \) a real constant called the Barbero-Immirzi parameter. The canonical pair has the following Poisson structure

\[ \{ A_i^a(x), E_j^b(y) \} = 8\pi G \gamma \delta_i^j \delta_i^a \delta(x, y) \]

(3)

with \( \delta(x, y) \) the Dirac delta distribution on the space-like hypersurface \( \Sigma \). In these variables, the Hamiltonian constraint takes the form

\[ C_{grav} = \left( \frac{1}{|E|} \epsilon_{ijk} \left[ F_i^{ab} - (1 + \gamma^2) \epsilon^{imn} K_m^a K_n^b \right] E^{aj} E^{bk} \right) \]

(4)

where \( F_i^{ab} = \partial_a A_i^b - \partial_b A_i^a + \epsilon_{ijk} A_j^a A_k^b \) is the curvature of the Ashtekar-Barbero connection and \( E \) the determinant of the densitized triad. While the full gravitational Hamiltonian is

\[ H_{grav} = \int d^3x N C_{grav} \]

(5)

2.1. Open FLRW model

for spatially flat homogeneous models the gravitational Hamiltonian reduces to [19]

\[ H_{grav} = -\gamma^2 N \int_V \frac{1}{|E|} \epsilon_{ijk} F_i^{ab} E^{aj} E^{bk} \]

(6)

where the integral is taken in a fiducial cell \( V \). When imposing also isotropy, the connection and triad can be described by parameters \( c \) and \( p \), respectively, defined by [19]

\[ A_i^a = eV_0 \tilde{o} e_i^a, \quad E_i^a = pV_0 \sqrt{\gamma} \tilde{o} e_i^a \]

(7)

where \( \tilde{o} q_{ab} \) is a fiducial flat metric (with which we can endow \( \Sigma \) due to the symmetries of the model) and \( \tilde{o} e_i^a \), \( \tilde{o} e_i^a \) are constant triad and co-triad compatible with \( \tilde{o} q_{ab} \); \( V_0 \) is the volume of \( V \) with respect to \( \tilde{o} q_{ab} \). These variables do not depend on the choice of the fiducial metric. The relation among these variables and the usual geometridynamical variables is

\[ c = V_0^{1/3} \gamma \hat{a}, \quad p = V_0^{2/3} a^2 \]

(8)

We have the canonical relations

\[ \{c, p\} = \frac{8\pi G \gamma}{3} \]

(9)

For convenience we perform the following change of variables

\[ \beta = \frac{c}{\sqrt{p}}, \quad V = p^{3/2} \]

(10)

with the canonical relations

\[ \{\beta, V\} = 4\pi G \gamma \]

(11)
In these variables the Hamiltonian (with a free massless scalar field as the matter content) takes the form \( N = 1 \)

\[
H = N \left( -\frac{3}{8\pi G \gamma^2} \beta^2 V + \frac{p^2_\phi}{2V} \right)
\]

(12)

The Holonomy correction due to Quantum Gravity effects is coded in the replacement [19]

\[
\beta \rightarrow \sin \frac{\lambda \beta}{\lambda}
\]

(13)

where \( \lambda^2 = 4\sqrt{3\pi G \ell^2} \) is the smallest eigenvalue of the area operator in the full Loop Quantum Gravity [17]. The resulting effective Hamiltonian is thus given by

\[
H_{\text{eff}} = -\frac{3}{8\pi G \gamma^2} \sin^2(\lambda \beta) V + \frac{p^2_\phi}{2V}
\]

(14)

The field equations are

\[
\dot{\beta} = 4\pi G \gamma \frac{\partial H_{\text{eff}}}{\partial V} = -\frac{3}{\gamma^2 \lambda} \sin^2(\lambda \beta) - 4\pi G \gamma \frac{p^2_\phi}{2V^2}
\]

\[
\dot{V} = -4\pi G \gamma \frac{\partial H_{\text{eff}}}{\partial \beta} = \frac{3}{\gamma^2 \lambda} V \sin(\lambda \beta) \cos(\lambda \beta)
\]

\[
\dot{\phi} = \frac{\partial H_{\text{eff}}}{\partial p_\phi} = \frac{p_\phi}{V}
\]

\[
\dot{p}_\phi = -\frac{\partial H_{\text{eff}}}{\partial \phi} = 0
\]

(15)-(18)

Since \( V = a^3, \frac{\dot{V}}{3V} = \frac{\dot{a}}{a} = H \), where \( H \) is the Hubble parameter. Then, taking into account the effective Hamiltonian constraint, \( H_{\text{eff}} = 0 \), and the field equation for \( V \) we have,

\[
H^2 = \left( \frac{\dot{V}}{3V} \right)^2 = \frac{1}{\gamma^2 \lambda^2} \left[ \frac{8\pi G \gamma^2 \lambda^2}{3} \frac{p^2_\phi}{2V^2} \left( 1 - \frac{8\pi G \gamma^2 \lambda^2}{3} \frac{p^2_\phi}{2V^2} \right) \right] = \frac{8\pi G}{3} \rho \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right)
\]

(19)

where \( \rho = \frac{p^2_\phi}{2V^2} = \frac{3}{8\pi G \gamma^2 \lambda^2} \sin^2(\lambda \beta) \) and \( \rho_{\text{max}} \) is the maximum value that the \( \rho \) can take in view of the effective Hamiltonian constraint, that is, \( \rho_{\text{max}} = \frac{3}{8\pi G \gamma^2 \lambda^2} \). The turning points of the volume function occur at \( \beta = \pm \frac{\pi}{2\lambda} \), which correspond to a bounce. The last equality is the modified Friedmann equation, which incorporates holonomy corrections due to Loop Quantum Gravity. In the limit \( \lambda \rightarrow 0 \) (no area gap) we recover the ordinary Friedmann equation \( H^2 = \frac{8\pi G}{3} \rho \).

The relational evolution of \( V \) in terms of \( \phi \) is given by

\[
\frac{dV}{d\phi} = \frac{dV}{dt} \frac{dt}{d\phi} = \frac{3}{\gamma \lambda} \sin(\lambda \beta) \cos(\lambda \beta) \frac{V}{p_\phi} = \sqrt{12\pi G V \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right)^{1/2}}
\]

(20)

where we have used the field equations for \( V \) and \( \phi \), and the effective Hamiltonian constraint.
3. Weyl-Wigner-Moyal Correspondence and Noncommutative Quantum Mechanics

The Weyl-Wigner-Moyal (WWM) correspondence [30] relates a phase space function $f$ with its operator analog $\hat{W}(f) = \hat{f}$, it is based on the following map [31]

$$\hat{f}(\hat{q}, \hat{p}) = \int \tilde{f}(\xi, \eta) e^{i\xi p_a q_a + i\eta p_b q_b} w(\xi, \eta) d\xi d\eta$$

(21)

where $\tilde{f}$ is the Fourier transform of $f$, $w$ a weight function, and $\dim(T^*Q) = 2l$; with $\hat{q}^a, \hat{p}_b$ satisfying the canonical Heisenberg algebra,

$$[\hat{q}^a, \hat{p}_b] = i\hbar \delta^a_b, \quad [\hat{q}^a, \hat{q}^b] = 0 = [\hat{p}_a, \hat{p}_b]$$

(22)

We define the $\star$-product with the help of this correspondence by

$$\hat{W}(f \star g) = \hat{W}(f) \cdot \hat{W}(g)$$

(23)

that is, $f \star g$ is the Weyl symbol of $\hat{f} \hat{g}$. For this case of the Heisenberg algebra (22) and in the symmetric ordering ($w = 1$) we have

$$f \star g = \exp\left(\frac{i\hbar}{2} P(f, g)\right) = fg + \sum_{r=1}^{\infty} \left(\frac{i\hbar}{2}\right)^r \frac{1}{r!} P^r(f, g) = f(x) \exp\left(\frac{i\hbar}{2} \partial_\mu \omega^{\mu\nu} \partial_\nu\right) g(x)$$

$$= fg + \frac{\hbar}{2} \{f, g\} + O(\hbar^2)$$

(24)

where $P^r(f, g) = \omega^{\mu_1 \nu_1} \cdots \omega^{\mu_r \nu_r} (\partial_{\mu_1 \cdots \mu_r} f)(\partial_{\nu_1 \cdots \nu_r} g)$ is the $r$th power of the Poisson bracket bidifferential operator, with $\omega^{\mu\nu}$ the components of the flat Poisson structure (??); $x^\mu$ are the phase space coordinates, denoted collectively as $x$, the first $n$ being the configuration coordinates $q^a$, the second $n$ being the momenta $p_a$. This is the Moyal $\star$-product [3], it replaces the ordinary pointwise multiplication in the algebra of functions defined in phase space. The Moyal $\star$-bracket $f \star g - g \star f$ is thus responsible for realizing the canonical Heisenberg algebra (22). Hence, (24) encodes a deformation of the classical phase space which yields the canonical Heisenberg algebra. This product is the cornerstone of deformation quantization [2, 4].

Now, consider the deformation of classical phase space which yields the algebra

$$[\hat{q}^a, \hat{p}_b] = i\hbar \delta^a_b; \quad [\hat{q}^a, \hat{q}^b] = i\theta \theta^{ab}; \quad [\hat{p}_a, \hat{p}_b] = i\kappa \kappa_{ab}$$

(25)

with $\theta^{ab}, \kappa_{ab}$ antisymmetric constant real matrices.

In light of the above prescription, we could encode this deformed Heisenberg algebra in a $\star$-product completely analogous to (24). The difference lies in the $\omega^{\mu\nu}$, which in this case would be of the form

$$i \begin{pmatrix} \theta \Theta & \hbar I \\ -\hbar I & \kappa K \end{pmatrix}$$

(26)

This quantization leads us to a Noncommutative Quantum Mechanics, rather than the ordinary Quantum Mechanis obtained above.

4. Effective Non-Commutative Loop Quantum Cosmology

In view of the above discussion, we would like to consider effects of the following deformed Poisson algebra

$$\{\beta, \phi\} = \theta, \quad \{\beta, V\} = 4\pi G\gamma$$

(27)
The above relations can be implemented by working with the shifted variables
\[ \beta^{nc} = \beta + a \theta p_\phi, \quad \phi^{nc} = \phi + b \theta V \]
where \( a \) and \( b \) satisfy the relation \( 4 \pi G \gamma b - a = 1 \). In deed, we have the algebra
\[ \{ \beta^{nc}, \phi^{nc} \} = \theta, \quad \{ \beta^{nc}, V \} = 4 \pi G \gamma \]
(29)
Since we are not actually performing a deformation of the symplectic structure (the algebra among the basic phase space variables \( \beta, \phi, V, p_\phi \)) is the same), the loop quantization of the original variables can be carried as usual, but with a slightly different Hamiltonian. Considering the steps above, we can therefore implement the effects of relation (28) in the effective Hamiltonian, which results in
\[ H_{eff}^{nc}^{\beta,\phi} = -\frac{3}{8 \pi G \gamma^2 \lambda^2} \sin^2(\lambda \beta^{nc})V + \frac{p_\phi^2}{2V} \]
(30)
The non-commutative effective field equations are
\[ \dot{\beta} = 4 \pi G \gamma \frac{\partial H_{eff}}{\partial V} = -\frac{3}{2\gamma} \sin^2(\lambda \beta^{nc}) - 4 \pi G \gamma \frac{p_\phi^2}{2V^2} \]
\[ \dot{V} = -4 \pi G \gamma \frac{\partial H_{eff}}{\partial \beta} = \frac{3}{2\gamma} V \sin(\lambda \beta^{nc}) \cos(\lambda \beta^{nc}) \]
\[ \dot{\phi} = \frac{\partial H_{eff}}{\partial p_\phi} = -\frac{3a \theta}{4 \pi G \gamma^2 \lambda} \sin(\lambda \beta^{nc}) \cos(\lambda \beta^{nc}) + \frac{p_\phi}{V} \]
\[ \dot{p}_\phi = -\frac{\partial H_{eff}}{\partial \phi} = 0 \]
(31) - (34)
In the limit \( \theta \to 0 \) we recover the commutative field equations. Due to the field equation for \( \phi \), we note that now the matter density \( \rho = \frac{\dot{\phi}^2}{2} \) is not given only by \( \frac{p_\phi^2}{2V} \), but by
\[ \rho^{nc} = \frac{1}{2} \left( -\frac{3a \theta}{4 \pi G \gamma^2 \lambda} \sin(\lambda \beta^{nc}) \cos(\lambda \beta^{nc}) + \frac{p_\phi}{V} \right)^2 \]
(35)
In order to construct the Friedmann equation we would need to obtain a relation for \( p_\phi \) in terms of \( \dot{\phi} \), but since the field equation for \( \dot{\phi} \) is now more involved, such relation can not be obtained; and so the Friedmann equation can not be constructed.

The relational evolution of \( V \) in terms of \( \phi \) is now given by
\[ \left( \frac{dV}{d\phi} \right)^2 = \left( \frac{3V}{\gamma \lambda} \right)^2 \sin^2(\lambda \beta^{nc}) \cos^2(\lambda \beta^{nc}) \left[ \frac{3 \theta}{8 \pi G \gamma^2 \lambda^2} \sin^2(\lambda \beta^{nc}) \cos^2(\lambda \beta^{nc}) + \frac{p_\phi}{V} \right]^{-2} \]
(36)
When taking \( \theta \to 0 \) this relation reduces to (20).

4.1. Addition of potential term
In the case of a scalar field with a potential term \( V(\phi) \) we have
\[ H_{eff}^{nc} = -\frac{3}{8 \pi G \gamma^2 \lambda^2} \sin^2(\lambda \beta^{nc}) V + \frac{p_\phi^2}{2V} + V(\phi^{nc})V \]
(37)
The non-commutative effective field equations are
\[ \dot{\beta} = 4 \pi G \gamma \frac{\partial H_{eff}}{\partial V} = -\frac{3}{2\gamma} \sin^2(\lambda \beta^{nc}) - 4 \pi G \gamma \frac{p_\phi^2}{2V^2} + 4 \pi G \gamma V(\phi^{nc}) + 4 \pi G \gamma b \theta V \frac{\partial V(\phi^{nc})}{\partial V} \]
\[ \dot{V} = -4 \pi G \gamma \frac{\partial H_{eff}}{\partial \beta} = \frac{3}{2\gamma} V \sin(\lambda \beta^{nc}) \cos(\lambda \beta^{nc}) \]
\[ \dot{\phi} = \frac{\partial H_{eff}}{\partial p_\phi} = -\frac{3a \theta}{4 \pi G \gamma^2 \lambda} V \sin(\lambda \beta^{nc}) \cos(\lambda \beta^{nc}) + \frac{p_\phi}{V} \]
\[ \dot{p}_\phi = -\frac{\partial H_{eff}}{\partial \phi} = -V \frac{\partial V(\phi^{nc})}{\partial \phi} \]
(38) - (41)
These field equations reduce to the commutative ones when taking \( \theta \to 0 \). The matter density \( \dot{\rho}^{nc} \) is given by

\[
\dot{\rho}^{nc} = \frac{1}{2} \left( -\frac{3a\theta}{4\pi G\gamma^2 \lambda} \sin(\lambda\beta^{nc}) \cos(\lambda\beta^{nc}) + \frac{p}{V} \right)^2 + V(\phi^{nc})
\] (42)

In the particular case in which \( a = 0 \) the whole noncommutativity is encoded in the shifted variable \( \phi^{nc} = \phi + b\theta V \), in this case we have the Hamiltonian

\[
H_{\text{eff}}^{nc} = \frac{3}{8\pi G\gamma^2 \lambda^2} \sin^2(\lambda\beta)V + \frac{p^2}{2V} + V(\phi^{nc})V
\] (43)

The non-commutative effective field equations are

\[
\dot{\beta} = 4\pi G\gamma \frac{\partial H_{\text{eff}}}{\partial V} = -\frac{3}{2\gamma^2 \lambda} \sin^2(\lambda\beta) - 4\pi G\gamma \frac{p^2}{2V^2} + 4\pi G\gamma \frac{b\theta V}{V} \frac{\partial V(\phi^{nc})}{\partial V}
\] (44)

\[
\dot{V} = -4\pi G\gamma V \frac{\partial H_{\text{eff}}}{\partial \phi} = \frac{3}{8\pi G\gamma^2 \lambda^2} V \sin(\lambda\beta) \cos(\lambda\beta)
\] (45)

\[
\dot{\phi} = \frac{\partial H_{\text{eff}}}{\partial p} = \frac{p}{V}
\] (46)

\[
\dot{p} = -\frac{\partial H_{\text{eff}}}{\partial \phi} = -V \frac{\partial V(\phi^{nc})}{\partial \phi}
\] (47)

These field equations reduce to the commutative ones when taking \( \theta \to 0 \). The matter density \( \dot{\rho}^{nc} \) is given by

\[
\rho^{nc} = \frac{p^2}{2V^2} + V(\phi^{nc})
\] (48)

In this particular case, it is possible to construct the Friedmann equation, which takes the form

\[
H^2 = \left( \frac{\dot{V}}{3V} \right)^2 = \frac{8\pi G^2}{3} \rho^{nc} \left( 1 - \frac{\rho^{nc}}{\rho_{\text{max}}} \right)
\] (49)

This equation reduces to the standard one by taking \( \theta \to 0 \).

The Klein-Gordon equation for the scalar field takes the form

\[
\ddot{\phi} = -3H\dot{\phi} - \frac{1}{V} \frac{\partial V(\phi^{nc})}{\partial V}
\] (50)

which reduces to the standard one by taking \( \theta \to 0 \). The relational evolution of \( V \) in terms of \( \phi \) is in this case the same as the commutative one since the field equations for \( V \) and \( \phi \) remain unchanged.

4.2. First order non-commutative quantum corrections

In the following we will neglect terms of \( O(\theta^2) \).

Expanding the Hamiltonian (30) in \( \lambda \), keeping only up to the leading term \( (\lambda^2) \), we have

\[
H_{\text{eff}}^{nc} = -\frac{3V}{8\pi G\gamma^2} (\beta^2 + 2a\theta \beta p) + \frac{V\lambda^2}{8\pi G\gamma^2} (\beta^4 + 4a\theta \beta^3 p) + \frac{p^2}{2V}
\] (51)

This Hamiltonian incorporates the first quantum correction as well as the first non-commutative correction. The first addend is composed of the classical term \( \beta^2 \) plus its first non-commutative
correction; whereas the second addend is composed of the first quantum correction ($\beta^4$ term) plus its first non-commutative correction.

The field equations are

\begin{align*}
\dot{\beta} &= 4\pi G \gamma \frac{\partial H_{\text{eff}}}{\partial \beta} = -\frac{3}{2\gamma} (\beta^2 + 2a\theta\beta p\phi) + \frac{\lambda^2}{2\gamma} (\beta^4 + 4a\theta\beta^3 p\phi) - 4\pi G \gamma \frac{p^2}{2V^2} \tag{52} \\
\dot{V} &= -4\pi G \gamma \frac{\partial H_{\text{eff}}}{\partial V} = \frac{3V}{\gamma} (\beta + a\theta p\phi) - \frac{2V\lambda}{\gamma} (\beta^3 + 3a\theta\beta^2 p\phi) \tag{53} \\
\dot{\phi} &= \frac{\partial H_{\text{eff}}}{\partial p\phi} = -\frac{3a\theta\beta V}{4\pi G^2 \gamma^2} + \frac{a\lambda^2 \theta \beta^3 V}{2\pi G^2 \gamma^2} + \frac{p\phi}{V} \tag{54} \\
\dot{p}_\phi &= -\frac{\partial H_{\text{eff}}}{\partial \phi} = 0 \tag{55}
\end{align*}

When taking $\theta \to 0$ these field equations reduce to the ones incorporating first order quantum corrections.

The energy density $\dot{\phi}^2$ is in this case given by

\begin{equation}
\rho = \frac{p^2}{2V^2} + a\theta \frac{\beta p\phi}{\pi G^2 \gamma^2} \left( \frac{\beta^2}{2} - \frac{3}{4} \right) \tag{56}
\end{equation}

Employing the Hamiltonian constraint we have

\begin{equation}
H^2 = \frac{1}{\gamma^2} \left( \beta^2 + 2a\theta\beta p\phi \right) - \frac{4\lambda^2}{3\gamma^2} (\beta^4 + 4a\theta\beta^3 p\phi) = \frac{8\pi G}{3} (\rho + \rho^\theta) \left( 1 - \frac{\rho + \rho^\theta}{\rho_{\text{max}}} \right) \tag{57}
\end{equation}

where $\rho^\theta = a\theta \left( \frac{3\beta p\phi}{4\pi G^2 \gamma^2} - \frac{\lambda^2 \beta^3 p\phi}{2\pi G^2 \gamma^2} \right)$.

Equation (57) is the Friedmann equation with first order non-commutative and loop quantum corrections. We note that when $\theta \to 0$ we obtain the leading term of the modified Friedmann equation (19).

5. Discussion

A simple noncommutative extension of the open FLRW loop quantum cosmology has been constructed, by introducing a theta deformation at the effective scheme of Loop Quantum Cosmology. This model thus incorporates effective corrections from both Loop Quantum Cosmology and Noncommutative Geometry. In the general case it was not possible to construct the corresponding Friedmann equation; however it was possible to construct it for the first order non-commutative corrections. Also, a noncommutative modified Friedmann equation could be constructed for the case in which the noncommutativity is encoded just in the matter degree of freedom. The physical implications of such noncommutative corrections are currently being investigated and will be reported elsewhere.

In the general case (eq. (30)) it is seen from the equation for $\dot{V}$ that a bounce occurs when $\beta = \frac{\pi}{\gamma} - a\theta p\phi$, moreover, at this value of $\beta$, the noncommutative density is equal to the value given by Effective Loop Quantum Cosmology, $\rho_{\text{max}}$. This means that the noncommutative extension constructed in this work does not modify the way in which a bounce is reached; however, the time at which a bounce occurs can be shifted by tuning the noncommutative parameter $\theta$. It is also seen from the equation for $\rho^\theta$, that this new energy density incorporates a geometric (gravitational) term, which could kill the usual matter term for particular values in the noncommutative parameter. Naturally, the same conclusions can be drawn from the first order approximation (eq. (51)). For the particular case in which the whole noncommutativity is encoded in the matter degree of freedom (eq. (43)) we observe that noncommutativity amounts to only modifying the potential term, and therefore there are no modifications to the Effective Loop Quantum Cosmology scheme; we can conclude that in order to obtain a real modification of Effective Loop Quantum Cosmology, noncommutativity should be implemented in the gravitational degrees of freedom. However, it is interesting to investigate if a cosmological constant term could arise when implementing noncommutativity only in the matter sector.
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