Boolean Functions of Binary Type-II and Type-II/III Complementary Array Pair

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Abstract

The sequence pairs of length $2^m$ projected from complementary array pairs of Type-II of size $2^m$ and mixed Type-II/III and of size $2^{(m-1)} \times 2$ are complementary sequence pairs Type-II and Type-III respectively. An exhaustive search for binary Type-II and Type-III complementary sequence pairs of small lengths $2^m$ ($m = 1, 2, 3, 4$) shows that they are all projected from the aforementioned complementary array pairs, whose algebraic normal forms satisfy specified expressions. It’s natural to ask whether the conclusion holds for all $m$. In this paper, we proved that these expressions of algebraic normal forms determine all the binary complementary array pairs of Type-II of size $2^m$ and mixed Type-II/III of size $2^{(m-1)} \times 2$ respectively.

Keywords: Types I and II complementary pair, array, sequence, binary, Boolean function

1 Introduction

Golay complementary sequences were first introduced by Golay \cite{Golay} and had since found applications in diverse areas of digital communication such as channel measurement, synchronization, and power control for multi-carrier wireless transmission.

The binary Golay complementary sequences are known to exist for all lengths $2^a10^b26^c$ (where $a$, $b$ and $c$ are natural numbers). For Golay complementary sequences of length $2^m$, their algebraic normal forms (ANFs) are given by Davis and Jedwab \cite{Davis} in 1999. These sequences are called “standard”Golay

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complementary sequences. It is still open now whether there are non-standard binary Golay comple-
mentary sequences of length $2^m$.

One most powerful method to study Golay complementary sequences is Golay complementary arrays. 
The existence of a binary Golay complementary array of the given size was studied in [6]. In addition, 
it has been shown in [4] that all the known binary Golay complementary sequences of length $2^m$ can 
be obtained by a single binary Golay complementary array pair of dimension $m$ and size $2 \times 2 \times \cdots \times 2$ 
(or abbreviated as $2^{(m)}$).

The aforementioned Golay complementary sequence pairs (or array pairs) are referred to as Type-I 
complementary sequence pairs (or array pairs). Note that the Fourier spectrum of Type-I complemen-
tary polynomials are evaluated on the unit circle. By extending to evaluate them on the real axis 
and imaginary axis, respectively, Parker and Riera [10] proposed Type-II and Type-III complementary 
arrays. For more research on Type-II and Type-III complementary arrays, readers can refer to the 
literature [8, 1, 9, 11, 7].

The existence and construction of binary Types-II and Types-III complementary sequence pairs 
were further studied in [7]. In particular, it was shown that the length of Type-II complementary 
sequence must be a power of 2. Moreover, an exhaustive search in [7] for binary Type-II and Type-III 
complementary sequence pairs for length $n = 2^m$, $n = 2, 4, 8, 16$ shows that Type-II complementary 
sequence pairs of these lengths must satisfy specified ANFs respectively. Similar to the Type-I case, 
an open question left in [7] asked whether all the Type-II and Types-III complementary sequence pairs 
must satisfy these ANFs.

Until now, all the known binary Types I and II complementary sequence pairs of length $2^m$ can be 
obtained by binary Types I and II complementary array pairs of size $2^{(m)}$ respectively, all the known 
binary Type-III complementary sequence pairs of length $2^m$ can be obtained by binary complementary 
array pairs, being of Type-III for the first variable, and Type-II for the other $m-1$ variables [7]. We 
denote the complementary array pairs as mixed Type-II/III of size $2^{(m-1)} \times 2$ in this paper. Thus, to 
study whether the Types I-III complementary sequence pairs must satisfy specified ANFs, one should 
first answer whether the complementary array pairs of Types I-II of size $2^{(m)}$ and mixed Type-II/III 
of size $2^{(m-1)} \times 2$ must satisfy these specified ANFs. The spectrum of Walsh-Hadamard transform and 
Nega-Walsh-Hadamard transform of Types I-III complementary array pairs has been determined in [2]. 
In [12], we proved that all $q$-ary Type-I complementary array pairs of size $2^{(m)}$ must be standard. In 
this paper, we proved that the ANFs of all the binary complementary array pairs of Type-II of size 
$2^{(m)}$ and mixed Types-II/III of size $2^{(m-1)} \times 2$ are determined by the form shown in [7] respectively.

The rest of the paper is organized as follows. In Section 2 we introduce the Types I-III complemen-
tary sequences and arrays, and also introduce mixed Type-II/III arrays and projections from arrays to
sequences. In Section 3, our results for Type-II of size $2^{(m)}$ and mixed Type-II/III array pairs of size $2^{(m)} \times 2$ are proposed. We give the proof in Section 4 and conclude the unresolved problem in Section 5.

2 Preliminary

In this section, we introduce the basic notations and definitions of Types I-III complementary sequences and arrays and mixed Type-II/III array. We shall only study the binary case in this paper.

2.1 Type-I Complementary Sequence and Array

A binary sequence $f$ of length $L$ is defined as

$$f = (f(0), f(1), \ldots, f(L-1)),$$

where each entry $f(t)$ belongs to $\mathbb{Z}_2$ ($t \in \mathbb{Z}_L$).

**Definition 1** The aperiodic auto-correlation of binary sequence $f$ at shift $\tau$ ($-L < \tau < L$) is defined by

$$C_f(\tau) = \sum_t (-1)^{f(t+\tau)-f(t)},$$

where $(-1)^{f(t+\tau)-f(t)} := 0$ if $f(t+\tau)$ or $f(t)$ is not defined.

**Definition 2** A pair of sequences $\{f, g\}$ is called a Golay complementary pair if $C_f(\tau) + C_g(\tau) = 0$ for $\forall \tau \neq 0$. Each sequence in such a pair is called a Golay complementary sequence [5].

The generating function of a binary sequence $f(t)$ is given by

$$F(z) = \sum_{t \in \mathbb{Z}_L} (-1)^{f(t)} z^t. \quad (1)$$

Straightforward manipulation shows that

$$F(z) \cdot F(z^{-1}) = \sum_\tau C_f(\tau) z^\tau. \quad (2)$$

Then we have that $f(t)$ and $g(t)$ are Golay complementary sequences if and only if their generating functions $F(z)$ and $G(z)$ satisfy

$$F(z) \cdot F(z^{-1}) + G(z) \cdot G(z^{-1}) = 2L. \quad (3)$$

An $m$-dimensional binary array of size $2 \times 2 \times \cdots \times 2$ can be represented by a Boolean function

$$f(x) = f(x_1, x_2, \ldots, x_m) : \{0, 1\}^m \to \mathbb{Z}_2.$$
Definition 3 The aperiodic auto-correlation of an array \( f(x) \) at shift \( \tau = (\tau_1, \tau_2, \cdots, \tau_m) \), \((-1 \leq \tau_i \leq 1)\), is defined by

\[
C_f(\tau) = \sum_{x \in \{0, 1\}^m} (-1)^{f(x+\tau)-f(x)},
\]

where \((-1)^{f(x+\tau)-f(x)} := 0\) if \(f(x+\tau)\) or \(f(x)\) is not defined, and \(x+\tau\) is the element-wise addition of integers.

Definition 4 A pair of arrays \((f(x), g(x))\) is called a Golay complementary array pair if

\[
C_f(\tau) + C_g(\tau) = 0 \text{ for } \forall \tau \neq 0.
\]

Each array in such a pair is called a Golay complementary array [6].

The generating function of a binary array \(f(x)\) is given by

\[
F(z) = \sum_{x \in \{0, 1\}^m} (-1)^{f(x)}z_1^{\tau_1}z_2^{\tau_2}\cdots z_m^{\tau_m},
\]

where \(z = (z_1, z_2, \cdots, z_m)\).

A Golay complementary array can be alternatively defined from the generating functions. Denote \(z^{-1} = (z_1^{-1}, z_2^{-1}, \cdots, z_m^{-1})\), straightforward manipulation shows that

\[
F(z) \cdot F(z^{-1}) = \sum_{\tau} C_f(\tau)z_1^{\tau_1}z_2^{\tau_2}\cdots z_m^{\tau_m}.
\]

From this it follows that \(f(x)\) and \(g(x)\) are Golay complementary arrays if and only if their generating functions \(F(z)\) and \(G(z)\) satisfy

\[
F(z) \cdot F(z^{-1}) + G(z) \cdot G(z^{-1}) = 2^{m+1}.
\]

2.2 Types II and III Complementary Sequence and Array

In 2008, Parker [8, 10] proposed Type-II and Type-III complementary sequences (or arrays). Golay complementary sequence in Definition 2 (or satisfying (3)) is called Type-I complementary sequence, Golay complementary array in Definition 4 (or satisfying (6)) is called Type-I complementary array. Just as each Type-I complementary polynomial is naturally evaluated on the unit circle to yield its Fourier spectrum, Li et al. [7] showed that it is natural to evaluate Type-II and Type-III complementary polynomials on the real axis \(\mathbb{R}\) and imaginary axis \(\mathbb{I}\), respectively, to preserve the commutativity of conjugation in individual evaluations.
A pair of binary sequences \( \{f, g\} \) of length \( L \) is called a Type-II and Type-III complementary sequence pair respectively if their generating functions satisfy

\[
\text{Type-II sequence : } \frac{(F(z))^2 + (G(z))^2}{1 + z^2 + z^4 + \cdots + z^{2(L-1)}} = 2,
\]

(7)

and

\[
\text{Type-III sequence : } \frac{F(z) \cdot F(-z) + G(z) \cdot G(-z)}{1 - z^2 + z^4 - \cdots + (-1)^{L-1}z^{2(L-1)}} = 2.
\]

(8)

A pair of binary arrays \((f(x), g(x))\) of size \(2 \times 2 \times \cdots \times 2\) is called Type-II or Type-III complementary array pairs if their generating functions satisfy

\[
\text{Type-II array : } (F(z))^2 + (G(z))^2 = 2 \prod_{k=1}^{m}(1 + z_k^2),
\]

(9)

or

\[
\text{Type-III array : } F(z) \cdot F(-z) + G(z) \cdot G(-z) = 2 \prod_{k=1}^{m}(1 - z_k^2),
\]

(10)

respectively, where \(-z = (-z_1, -z_2, \cdots, -z_m)\).

2.3 Mixed Type-II/III Complementary Array and Projections

Suppose \(f(t)\) is a sequence of length \(L = 2^m\), \(f(x)\) is an array of size \(2^m\), \(F(z)\) and \(F(z)\) are their corresponding generating functions. The sequence \(f(t)\) is called \textit{projected} from the array \(f(x)\) by permutation \(\pi\) if

\[
f(t) = f(x),
\]

(11)

or equivalently,

\[
F(z) = F(z),
\]

(12)

where \(t = \sum_{k=1}^{m} x_k \cdot 2^{\pi(k)-1}\), and \(z_k = z^{2^{\pi(k)-1}}\), \(\pi\) is a permutation of \(\{1, 2, 3, \ldots, m\}\). Notice that the pair of generating functions \((F(z), G(z))\) projected from that of the Type-I (resp. Type-II) complementary array pair \((F(z), G(z))\) in (6) (resp. (9)) satisfy the condition of Type-I (resp. Type-II) complementary sequence pair in (3) (resp. (7)) for \(L = 2^m\). It is straightforward that the sequence pair projected from Type-I (resp. Type-II) complementary array pair by permutation \(\pi\) is a Type-I (resp. Type-II) complementary sequence pair.

However, the sequence pair projected from Type-III complementary array pair is not a Type-III complementary sequence pair. As [11] pointed out:
All these Type-III sequence pairs of length $2^m$ are projections of m-variable $2 \times 2 \times \cdots \times 2$ bipolar array pairs, being of Type-III for the first variable, and Type-II for the other $m - 1$ variables.

The aforementioned mixed type complementary array can be given as follows.

A pair of binary arrays $(f(x,x_0), g(x,x_0))$ is called a mixed Type-II/III complementary array pair of size $2^m \times 2$ if their generating functions satisfy

$$F(z, z_0) \cdot F(z, -z_0) + G(z, z_0) \cdot G(z, -z_0) = 2(1 - z_0^2) \prod_{k=1}^{m} (1 + z_k^2), \quad (13)$$

where $z_0$ is called the Type-III indeterminate, and $x_0$ is its corresponding variable, $z_k (1 \leq k \leq m)$ are called the Type-II indeterminates.

Notice that the pair of generating functions $(F(z), G(z))$ projected from that of the mixed Type-II/III complementary array pair $(F(z), G(z))$ in (13) satisfy the condition of Type-III complementary sequence pair in (8) for $L = 2^{m+1}$, if we restrict $z_0 = z$ and $z_k (1 \leq k \leq m)$ to be $z^{2^{\pi(k)}}$, where $\pi$ is a permutation of $\{1, 2, 3, \ldots, m\}$. This means that the sequence pair projected from mixed Type-II/III complementary array pair of size $2^m \times 2$ by a specific permutation is a Type-III complementary sequence pair.

### 2.4 Some Known Results

It has been proved that the length of Type-II complementary sequence must be a power of 2 [7]. In addition, an exhaustive search in [7] for binary Type-II complementary sequence pairs $(f(t), g(t))$ of length $L = 2^m$, $L = 2, 4, 8, 16$, reveals that they are all projected from the Type-II complementary array pairs given by

$$\begin{align*}
f(x) &= \sum_{1 \leq i < j \leq m} x_ix_j + \sum_{i=1}^{m} c_ix_i + c_0, \\
g(x) &= f(x) + \sum_{i=1}^{m} x_i + c',
\end{align*} \quad (14)$$

where $c' \in \mathbb{Z}_2$, $c_k \in \mathbb{Z}_2$ ($0 \leq k \leq m$). The projection can be expressed by

$$\begin{align*}
f(t) &= f(x), \\
g(t) &= g(x),
\end{align*} \quad (16)$$

where $t = \sum_{k=1}^{m} 2^{\pi(k)-1} \cdot x_k$, $\pi$ is a permutation of $\{1, 2, 3, \ldots, m\}$. 
An exhaustive search for the Type-III complementary sequence in \[ 7 \] reveals that, for length \( L = 2^{m+1} \) and \( L = 2, 4, 8, 16 \), all binary Type-III complementary sequence pairs, \((f(t), g(t))\), are projected via \([19]\) from the Type-III complementary array pairs of size \( 2^m \times 2 \) given by:

\[
\begin{align*}
    f_{II/III}(x, x_0) &= \sum_{1 \leq i < j \leq m} x_i x_j + x_0 \cdot \sum_{i=1}^{m} e_i x_i + \sum_{i=0}^{m} c_i x_i + c, \\
    g_{II/III}(x, x_0) &= f_{II/III}(x, x_0) + \sum_{i=1}^{m} x_i + e_0 x_0 + c',
\end{align*}
\]

where \( c, c' \in \mathbb{Z}_2, e_k, c_k \in \mathbb{Z}_2 (0 \leq k \leq m) \). The projection can be expressed by

\[
\begin{align*}
    f(t) &= f_{II/III}(x, x_0), \\
    g(t) &= g_{II/III}(x, x_0),
\end{align*}
\]

where \( t = \sum_{k=1}^{m} 2^{\pi(k)} \cdot x_k + x_0, \pi \) is a permutation of \( \{1, 2, 3, \ldots, m\} \).

\section{Main Results}

Based on the theoretical results of the lengths of the Type-II and Type-III complementary sequences and the exhaustive search for the Type-II and Type-III complementary sequence pairs of small lengths, Two open questions are proposed in \([7]\):

1. Prove that all bipolar Type-II complementary sequence pairs are constructed from primitive pair \((A = (1, 1), B = (1, -1))\) by an \( m \)-fold application of Construction \( G \), then a projection of the resulting \( m \)-variate Type-II complementary array pair back to a sequence pair.

2. Prove that all bipolar Type-III complementary sequence pairs of length \( 2^m \) can be constructed from primitive pair \((A = (1, 1), B = (1, 1))\) by an \( m \)-fold application of Construction \( G \), then a projection of the resulting \( m \)-variate Type-II/III complementary array pair back to a sequence pair.

From the viewpoint of array and sequence, the two open problems can be reorganized into two parts. First, all binary Type-II (mixed Type-II/III) complementary arrays must be of form \([14], [15]\) (resp. \([17], [18]\)). Second, all binary Type-II (resp. Type-III) complementary sequence pairs of length \( 2^m \) are projected from these Type-II (resp. Type-II/III) complementary array pairs. In this paper, we will prove that the first part is true.

**Theorem 1**

1. The array pair \((f(x), g(x))\) given in \([14], [15]\) form a Type-II complementary arrays of size \( 2 \times 2 \times \cdots \times 2 \).

2. Conversely, any Type-II complementary arrays \((f(x), g(x))\) of size \( 2 \times 2 \times \cdots \times 2 \) must be of form \([14]\) and \([15]\).
Theorem 2 1. The array pair \((f(x), g(x))\) given in (17) and (18) form a Type-II/III complementary arrays of size \(2^{(m)} \times 2\).

2. Conversely, any Type-II/III complementary arrays \((f(x), g(x))\) of size \(2^{(m)} \times 2\) must be of form (17) and (18).

4 Proof of Our Results

4.1 Proof of Theorem 1

Define \(F_{m+1}(z, z_{m+1})\) and \(G_{m+1}(z, z_{m+1})\) as the generating functions of arrays \(f_{m+1}(x, x_{m+1})\) and \(g_{m+1}(x, x_{m+1})\) of size \(2^{(m+1)}\). Denote \(f_{t}^{m}(x) = f_{m+1}(x, t)\) (resp. \(g_{t}^{m}(x) = g_{m+1}(x, t)\)) by the array of dimension \(m\) derived from \(f_{m+1}(x, x_{m+1})\) (resp. \(g_{m+1}(x, x_{m+1})\)) by restricting \(x_{m+1}\) to be \(t\) (\(t = 0\) or \(1\)). I.e.,

\[
f_{m+1}(x, x_{m+1}) = f_{0}^{m}(x)(1 - x_{m+1}) + f_{1}^{1}(x) \cdot x_{m+1},
\]

(20)

\[
g_{m+1}(x, x_{m+1}) = g_{0}^{m}(x)(1 - x_{m+1}) + g_{1}^{1}(x) \cdot x_{m+1}.
\]

(21)

Denote corresponding generating functions to be \(F_{0}^{m}(z)\) (resp. \(F_{1}^{m}(z)\)). It’s easy to verify that

\[
G_{m+1}(z, z_{m+1}) = G_{0}^{m}(z) + G_{1}^{m}(z) \cdot z_{m+1},
\]

(22)

\[
F_{m+1}(z, z_{m+1}) = F_{0}^{m}(z) + F_{1}^{m}(z) \cdot z_{m+1}.
\]

(23)

We would like to give the proof by applying the mathematical induction. It is known that Theorems 1 holds for \(m = 1, 2, 3\) and 4. Suppose Theorems 1 holds for \(m\).

Step 1. If \((f(x, x_{m+1}), g(x, x_{m+1}))\) are of form (14) and (15), i.e.,

\[
f(x, x_{m+1}) = \sum_{1 \leq k < j \leq m+1} x_{k}x_{j} + \sum_{k=1}^{m+1} c_{k}x_{k} + c_{0},
\]

(24)

and

\[
g(x, x_{m+1}) = f(x, x_{m+1}) + \sum_{k=1}^{m+1} x_{k} + c'.
\]

(25)
Then the restricted arrays are given by

\[
\begin{align*}
 f_0^m(x) &= \sum_{1 \leq k < j \leq m} x_k x_j + \sum_{k=1}^m c_k x_k + c_0, \\
 f_1^m(x) &= f_0^m(x) + \sum_{k=1}^m x_k + c_{m+1}, \\
 g_0^m(x) &= f_1^m(x) + c_{m+1} + c', \\
 g_1^m(x) &= f_0^m(x) + c_{m+1} + c' + 1.
\end{align*}
\]

So that \( G_0^m(z) = \pm F_1^m(z) \) and \( G_1^m(z) = \mp F_0^m(z) \). Since \((f_0^m(x), g_0^m(x))\) are of form (14) and (15), they form Type-II complementary arrays. Their generating functions \((F_0^m(z), G_0^m(z))\) must satisfy (9).

Based on (22) and (23),

\[
(F_{m+1}(z, z_{m+1}))^2 + (G_{m+1}(z, z_{m+1}))^2 = ((F_0^m(z))^2 + (G_0^m(z))^2) \cdot (1 + z_{m+1}^2)
\]

\[
= 2 \prod_{k=1}^{m+1} (1 + z_k^2),
\]

which meets the definition of Type-II complementary arrays (9).

**Step 2.** If \( F_{m+1}(z, z_{m+1}) \) and \( G_{m+1}(z, z_{m+1}) \) form a Type-II complementary array pair, according to (9),

\[
(F_{m+1}(z, z_{m+1}))^2 + (G_{m+1}(z, z_{m+1}))^2 = 2 \prod_{k=1}^{m+1} (1 + z_k^2).
\]

On the other hand, from (22)\(^2\) and (23)\(^2\) we have

\[
(F_{m+1}(z, z_{m+1}))^2 = (F_0^m(z))^2 + (F_1^m(z))^2 \cdot z_{m+1}^2
\]

\[
+ 2 F_0^m(z) \cdot F_1^m(z) \cdot z_{m+1},
\]

\[
(G_{m+1}(z, z_{m+1}))^2 = (G_0^m(z))^2 + (G_1^m(z))^2 \cdot z_{m+1}^2
\]

\[
+ 2 G_0^m(z) \cdot G_1^m(z) \cdot z_{m+1}.
\]

Expend the polynomial by the power of \( z_{m+1} \), compare the coefficients of 1, \( z_{m+1}^2 \) and \( z_{m+1} \) respec-
tively between (32)+(33) and (31), we have

\[(F_0^m(z))^2 + (G_0^m(z))^2 = 2 \prod_{k=1}^{m} (1 + z_k^2),\]  

(34)

\[(F_1^m(z))^2 + (G_1^m(z))^2 = 2 \prod_{k=1}^{m} (1 + z_k^2),\]  

(35)

\[2(F_0^m(z) \cdot F_1^m(z) + G_0^m(z) \cdot G_1^m(z)) = 0.\]  

(36)

Let

\[F_1^m(z) = K(z) \cdot G_0^m(z),\]  

(37)

where \(K(z)\) belongs to the field of fractions of the polynomial ring. According to (36), we have

\[G_1^m(z) = -K(z) \cdot F_0^m(z).\]  

(38)

Substituting \(F_1^m(z)\) and \(G_1^m(z)\) in (35) by (37) and (38), we have

\[(K(z))^2 \cdot ((F_0^m(z))^2 + (G_0^m(z))^2) = 2 \prod_{k=1}^{m} (1 + z_k^2).\]  

(39)

Compare (39) with (34), we get

\[K(z) = \pm 1.\]  

(40)

If \(K(z) = 1\), based on (37) and (38), it’s easy to know

\[f_1^m(x) = g_0^m(x), \quad g_1^m(x) = f_0^m(x) + 1,\]  

(41)

According to (34), \(F_0^m(z)\) and \(G_0^m(z)\) form a Type-II complementary array pair. Since Theorem 1 holds for \(m\), let

\[f_0^m(x) = \sum_{1 \leq k < j \leq m} x_k x_j + \sum_{k=1}^{m} c_k x_k + c_0,\]  

(42)

\[g_0^m(x) = f_0^m(x) + \sum_{k=1}^{m} x_k + c_{m+1},\]  

(43)

where \(c_k \in \mathbb{Z}_2\) (\(0 \leq k \leq m + 1\)). Then

\[f_{m+1}(x, x_{m+1}) = f_0^m(x)(1 - x_{m+1}) + f_1^m(x) \cdot x_{m+1}\]

\[= f_0^m(x) + x_{m+1} \cdot \left( \sum_{k=1}^{m} x_k + c_{m+1} \right)\]  

(44)

\[= \sum_{1 \leq k < j \leq m+1} x_k x_j + \sum_{k=1}^{m+1} c_k x_k + c_0,\]
\[ g_{m+1}(x, x_{m+1}) = g_m^0(x)(1 - x_{m+1}) + g_m^1(x) \cdot x_{m+1} \]
\[ = f_m^0(x) + \left( \sum_{k=1}^{m} x_k + c_{m+1} \right) (1 + x_{m+1}) + x_{m+1} \]
\[ = \sum_{1 \leq k < j \leq m+1} x_k x_j + \sum_{k=1}^{m+1} c_k x_k + \sum_{k=1}^{m+1} x_k + c_0 + c_{m+1}, \]

which are obviously of form (14) and (15). If \( k = -1 \), we can get the similar result.

Combine Steps 1 and 2, Theorem 1 holds for \( m + 1 \). By applying the mathematical induction, Theorem 1 holds for all \( m \geq 1 \).

### 4.2 Proof of Theorem 2

Define \( F_{\text{II/III}}(z, z_0) \) and \( G_{\text{II/III}}(z, z_0) \) as the generating functions of arrays \( f_{\text{II/III}}(x, x_0) \) and \( g_{\text{II/III}}(x, x_0) \), where \( z_0 \) is the Type-III indeterminate, \( z_k (1 \leq k \leq m) \) are the Type-II indeterminates.

Denote \( f_m^t(x) = f_{\text{II/III}}(x, t) \) (\( t = 0 \) or 1) (resp. \( g_m^t(x) = g_{\text{II/III}}(x, t) \)) by the array of dimension \( m \) derived from \( f_{\text{II/III}}(x, x_0) \) (resp. \( g_{\text{II/III}}(x, x_0) \)) by restricting \( x_0 \) to be \( t \) (\( t = 0 \) or 1). Thus,

\[ f_{\text{II/III}}(x, x_0) = f_m^0(x)(1 - x_0) + f_m^1(x) \cdot x_0, \] (46)

\[ g_{\text{II/III}}(x, x_0) = g_m^0(x)(1 - x_0) + g_m^1(x) \cdot x_0. \] (47)

Denote corresponding generating functions to be \( F_m^t(z) \) (\( t = 0 \) or 1) (resp. \( G_m^t(z) \)). It’s easy to verify that

\[ F_{\text{II/III}}(z, z_0) = F_m^0(z) + F_m^1(z) \cdot z_0, \] (48)

\[ G_{\text{II/III}}(z, z_0) = G_m^0(z) + G_m^1(z) \cdot z_0. \] (49)

**Step 1.** If \( (f_{\text{II/III}}(x, x_0), g_{\text{II/III}}(x, x_0)) \) are given by (17) and (18) respectively. Then the restricted
arrays are given by

\[
\begin{align*}
    f_m^0(x) &= \sum_{1 \leq i < j \leq m} x_i x_j + \sum_{i=1}^{m} c_i x_i + c, \\
    f_m^1(x) &= f_m^0(x) + \sum_{i=1}^{m} c_i x_i + c_0, \\
    g_m^0(x) &= f_m^0(x) + \sum_{i=1}^{m} x_i + c', \\
    g_m^1(x) &= f_m^1(x) + \sum_{i=1}^{m} x_i + e_0 + c'.
\end{align*}
\]

(50)  

(51)  

(52)  

(53)  

Since \((f_m^0(x), g_m^0(x))\) are of form [14] and [15], they form Type-II complementary arrays. Their generating functions \((F_m^0(z), G_m^0(z))\) must satisfy [9], i.e.,

\[
(F_m^0(z))^2 + (G_m^0(z))^2 = 2 \prod_{k=1}^{m} (1 + z_k^2),
\]

(54)  

Similarly,

\[
(F_m^1(z))^2 + (G_m^1(z))^2 = 2 \prod_{k=1}^{m} (1 + z_k^2),
\]

(55)  

Based on [49] and [48],

\[
\begin{align*}
    F_{II/III}(z, z_0) \cdot F_{II/III}(z, -z_0) + G_{II/III}(z, z_0) \cdot G_{II/III}(z, -z_0) \\
    &= ((F_m^0(z))^2 + (G_m^0(z))^2) + z_0^2 \cdot ((F_m^1(z))^2 + (G_m^1(z))^2) \\
    &= 2(1 - z_0^2) \prod_{k=1}^{m} (1 + z_k^2),
\end{align*}
\]

(56)  

which meets the definition of Type-II/III complementary arrays [9] of size \(2^{(m)} \times 2\).

**Step 2.** If \(F_{II/III}(z, z_0)\) and \(G_{II/III}(z, z_0)\) form a Type-II/III complementary array pair of size \(2^{(m)} \times 2\). According to [9],

\[
F_{II/III}(z, z_0) \cdot F_{II/III}(z, -z_0) + G_{II/III}(z, z_0) \cdot G_{II/III}(z, -z_0) = 2(1 - z_0^2) \prod_{k=1}^{m} (1 + z_k^2).
\]

(57)  

On the other hand, from [49] and [48] times their individual conjugates, we have

\[
F_{II/III}(z, z_0) \cdot F_{II/III}(z, -z_0) = (F_m^0(z))^2 - (F_m^1(z))^2 \cdot z_0^2,
\]

(58)
\[ G_{II/III}(z, z_0) \cdot G_{II/III}(z, -z_0) = (G_0^m(z))^2 - (G_1^m(z))^2 \cdot z_0^2. \] 

(59)

Expend the polynomial by the power of \( z_0 \), compare the coefficients of 1, \( z_0^2 \) and \( z_0 \) respectively between (58)+(59) and (57), we have

\[
(F_0^m(z))^2 + (G_0^m(z))^2 = 2 \prod_{k=1}^{m} (1 + z_k^2),
\]

(60)

\[
(F_1^m(z))^2 + (G_1^m(z))^2 = 2 \prod_{k=1}^{m} (1 + z_k^2),
\]

(61)

According to (9), \((F_0^m(z), G_0^m(z))\) and \((F_1^m(z), G_1^m(z))\) form Type-II complementary array pairs. According to Theorem 1, let

\[
\begin{align*}
\begin{cases}
  f_0^m(x) = \sum_{1 \leq i < j \leq m} x_i x_j + \sum_{i=1}^{m} c_i x_i + c_0, \\
g_0^m(x) = f(x) + \sum_{i=1}^{m} x_i + c',
\end{cases}
\end{align*}
\]

(62)

\[
\begin{align*}
\begin{cases}
  f_1^m(x) = \sum_{1 \leq i < j \leq m} x_i x_j + \sum_{i=1}^{m} e_i x_i + e_0, \\
g_1^m(x) = f(x) + \sum_{i=1}^{m} x_i + e',
\end{cases}
\end{align*}
\]

(63)

where \( c', e' \in \mathbb{Z}_2, \ c_i, e_i \in \mathbb{Z}_2 \) (0 \( \leq \) \( i \) \( \leq \) \( m \)). Then

\[
\begin{align*}
\begin{cases}
  f_{II/III}(x, x_0) = f_0^m(x)(1 - x_0) + f_1^m(x) \cdot x_0 \\
  = \sum_{1 \leq i < j \leq m} x_i x_j + x_0 \cdot \sum_{i=1}^{m} (e_i - c_i) x_i + \sum_{i=1}^{m} c_i x_i + (e_0 - c_0) x_0 + c_0,
\end{cases}
\end{align*}
\]

(66)

\[
\begin{align*}
\begin{cases}
  g_{II/III}(x, x_0) = g_0^m(x)(1 - x_0) + g_1^m(x) \cdot x_0 = f_{II/III}(x, x_0) + \sum_{i=1}^{m} x_i + (e' - c') x_0 + c',
\end{cases}
\end{align*}
\]

(67)

which are obviously of form (17) and (18).

5 Conclusion

In this paper, we proved that the algebraic normal forms of binary Type-II of size \( 2^m \) and mixed Type-II/III complementary array pairs of size \( 2^{(m-1)} \times 2 \) must satisfy specified expressions. The unresolved problem is that whether the Type-II and Type-III complementary sequence pairs of length \( 2^m \) must be projected from these array pairs. And we left it as an open problem.
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