In this study, we investigate the branching ratios of $B^0_s \rightarrow a_0(980)[\rightarrow K\overline{K}, \pi\eta]a_0(980)$, $B^0_s \rightarrow f_0(980)[\rightarrow \pi^+\pi^-, K^+K^-]f_0(980)$, and $B^{0*} \rightarrow f_0(500)[\rightarrow \pi^+\pi^-]f_0(500)$ decays in the pQCD approach, wherein the scalar mesons $a_0(980)$, $f_0(980)$, and $f_0(500)$ are regarded as the lowest-lying $q\overline{q}$ state. In the $SU(3)$ nonet, there exists a mixing between the scalars $f_0(980)$ and $f_0(500)$. Thus, we have considered the mixing effect in our calculations to obtain reliable data, and set the mixing angle $\theta$ as $[15^\circ, 82^\circ]$ and $[105^\circ, 171^\circ]$. Based on the isospin symmetry, we estimated the branching ratios of the $B^0_s \rightarrow f_0(980)[\rightarrow \pi^0\pi^0]f_0(980)$ and $B^{0*} \rightarrow f_0(500)[\rightarrow \pi^+\pi^-]f_0(500)$ decays. The branching ratios of the $B^0_s \rightarrow a_0(980)[\rightarrow K\overline{K}, \pi\eta]a_0(980)$ decays are much small, while those of the $B^0_s \rightarrow f_0(980)[\rightarrow \pi\pi, K\overline{K}]f_0(980)$ and $B^{0*} \rightarrow f_0(500)[\rightarrow \pi\pi]f_0(500)$ decays are at the order of $10^{-6} \sim 10^{-5}$, which can be tested in the LHCb and Belle II experiments, hopefully.

I. INTRODUCTION

The scalar mesons $a_0(980)$ and $f_0(980)$ have attracted immense attention since their discovery. These scalar mesons, as a key problem in the nonperturbative QCD [1], play a crucial role in understanding the chiral symmetry and confinement in the low-energy region. However, the mysterious internal structure of the scalar mesons $a_0(980)$ and $f_0(980)$ remains a puzzle, many related researches have been carried out accordingly. It has been raised that the scalar mesons $f_0(500)$, $K^+(700)$, $f_0(980)$, and $a_0(980)$ form a $SU(3)$ flavor nonet, whereas the scalars above 1 GeV, including $f_0(1370)$, $a_0(1450)$, $K^+(1430)$, and $f_0(1500)$, form a different nonet [2]. In Ref. [3], two scenarios have been proposed to describe the quark components of the light scalar mesons. According to the first scenario, the light scalar mesons contained in the first $SU(3)$ flavor nonet are the lowest-lying $q\overline{q}$ state. In the other scenario, the scalars in the second nonet are treated as the $q\overline{q}$ state, and the scalar mesons below or close to 1 GeV are considered to be the four-quark bound state. In addition, the presence of nonstrange and strange quark contents in $f_0(980)$ and $f_0(500)$ have been confirmed experimentally; thus, they can be regarded as a mixture of $s\overline{s}$ and $(u\overline{u}+d\overline{d})/\sqrt{2}$ [4]. The aforementioned studies provide a positive significance to explore the internal structure of the scalar mesons.

The perturbative QCD (pQCD) approach based on the $k_T$ factorization has been extensively used to study the decay of $B$ mesons [6–10]. It is well known that the QCD dynamics of the three-body decay is more complex than those of the two-body decay. In the pQCD approach, the three-body decay is usually simplified to a two-body decay by introducing two-hadron distribution amplitudes [5, 11], which can be called quasi-two-body decay, containing both resonance and nonresonance information. The dominant contributions are from the parallel motion region, where the invariant mass of the light meson pair is below $O(\Lambda M_B)$, and $\Delta M = M_B - m_k$ is the difference in mass between $B$ meson and $K$-quark. Hence, the pQCD factorization formula for the three-body decay amplitude of the $B$ meson is written as [12, 13]

$$A = H \otimes \phi_B \otimes \phi_{h_1} \otimes \phi_{h_2},$$

where the hard decay kernel $H$ can be calculated by using the perturbative theory. The nonperturbative inputs $\phi_B$, $\phi_{h_1}$, and $\phi_{h_2}$ are the distribution amplitudes of $B$ meson, $h_1h_2$ pair and $h_3$, respectively.

In the past two decades, several $B$ decays with a final state $a_0(980)$ or $f_0(980)$ have been observed in experiments [14–16], and the corresponding theoretical calculations have also attracted increased attention. In Ref. [17, 18], the $B_s \rightarrow f_0(980)$ transition form factors have been estimated, and the authors still predict the branching ratio of an interesting decay mode $B_s \rightarrow J/\psi f_0(980)$. At the same time, the light scalar mesons also can make contributions to the $B$ meson decays as intermediate resonance. For example, the LHCb Collaboration reported the $B^0 \rightarrow J/\psi K^+K^-$ decay with the $a_0(980)$ resonance [19], and the Belle Collaboration observed the $B^\pm \rightarrow K^\pm f_0(980) \rightarrow K^\pm \pi^+\pi^\mp$ where the scalar meson $f_0(980)$ was regarded as the intermediate resonance [20]. Recently, many works have been carried out to calculate the three-body decays of the $B$ meson with $a_0(980)$ or $f_0(980)$ resonance in the pQCD approach. In Ref. [21], the branching ratios of the $B \rightarrow J/\psi (K\overline{K}, \pi\eta)$ decays have been calculated with the contributions of the scalars $a_0(980)$ and $a_0(1450)$, in which the timelike form factors of $a_0(980)$ resonance and $a_0(1450)$ resonance are shaped by the Flatté model and the Breit-Wigner formula, respectively. The authors of Ref. [22], employing $a_0(980)$, $a_0(1450)$ and $a_0(1950)$ as resonances, analyze the quasi-two-body $B \rightarrow a_0(\rightarrow K\overline{K}, \pi\eta)h$

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decays within the two scenarios mentioned in the first paragraph. In Ref. [23] and Ref. [24], the branching ratios of the \( B \rightarrow K(R \rightarrow K^+K^-) \) and \( B_s \rightarrow V\pi\pi \) decays with \( f_0(980) \) resonance have been studied severally by considering the mixing of \( s\bar{s} \) and \( (u\bar{u} + d\bar{d})/\sqrt{2} \). Furthermore, the authors of Ref. [25] explore the branching ratios and \( CP \) violations of the two-body decay \( B_s^0 \rightarrow SS(a_0(980), f_0(980), f_0(500)) \) in the pQCD approach for the first time, and the branching ratios are at the order of \( 10^{-4} \sim 10^{-6} \) with a high probability to be tested experimentally in the future. Therefore, we further develop the work in Ref. [25]. In our calculation, we regard the scalar mesons \( a_0(980) \) and \( f_0(980) \) as the \( q\bar{q} \) state, just as mentioned in the first scenario, and predict the branching ratios of the quasi-two-body decays \( B_s^0 \rightarrow S[\rightarrow P_1 P_2]S \), where \( S \) denotes the light scalar mesons \( a_0(980), f_0(980) \) and \( f_0(500) \), and \( P_1 P_2 = \pi\eta, \pi\pi, \pi K \) is the final state meson pair. For the scalar mesons \( f_0(980) \) and \( f_0(500) \), we also adopt the corresponding mixing mechanism. The results presented in this paper can be validated in the LHCb and Belle II experiments in the near future.

The rest of this paper is organized as follows. In Section II, the theoretical framework of the pQCD, the wave functions involved in the calculations and the helicity amplitudes for the \( B_s^0 \rightarrow S[\rightarrow P_1 P_2]S \) decays are described. In Section III, the numerical results are presented and discussed. In Section IV, a summary of this work is provided. And the explicit formulas of all the helicity amplitudes are presented in the Appendix.

![Feynman diagrams](image)

**FIG. 1.** The Feynman diagrams for the \( B_s^0 \rightarrow S[\rightarrow P_1 P_2]S \) decays in pQCD. The symbol \( \bullet \) stands for the weak vertex, \( S \) means the scalar mesons, and \( P_1 P_2 \) denotes the final state meson pair, the corresponding relationship are \( a_0(\pi\eta, K\bar{K}), f_0(\pi\pi, K\bar{K}) \) and \( \sigma(\pi\pi) \).

### II. THE THEORETICAL FRAMEWORK AND HELICITY AMPLITUDES

#### A. The Wave Functions

The relevant weak effective Hamiltonian of the quasi-two-body \( B_s^0 \rightarrow S[\rightarrow P_1 P_2]S \) decays can be written as [26]

\(^1 a_0, f_0 \) and \( \sigma \) refers to \( a_0(980), f_0(980) \) and \( f_0(500) \) respectively in the following text
\[ \mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \{ V_{ub}V_{us}^* [C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)] - V_{tb}V_{ts}^* \sum_{i=3}^{10} C_i(\mu)O_i(\mu) \} \]  

with the Fermi constant \( G_F = 1.66378 \times 10^{-5} \text{ GeV}^{-2} \), the local four-quark operator \( O_i(\mu) \), the corresponding Wilson coefficient \( C_i(\mu) \), and \( V_{ub}V_{us}^* \) and \( V_{tb}V_{ts}^* \) are Cabibbo-Kobayashi-Maskawa (CKM) factors. The Feynman diagrams involved in this work are illustrated in Fig. 1.

Based on the light-cone coordinates, we let the \( B_s \) meson stay at rest, and choose the \( P_1P_2 \) meson pair and the final-state \( S \) movement along the direction of \( n = (1, 0, 0_T) \) and \( v = (0, 1, 0_T) \), respectively. So the \( B_s \) meson momentum \( p_B \), the total momentum \( p = p_1 + p_2 \) of the \( P_1P_2 \) meson pair, and the momentum \( p_3 \) of the final-state \( S \) are considered as

\[
\begin{align*}
p_B &= \frac{M_B}{\sqrt{2}} (1, 1, 0_T), \\
p &= \frac{M_B}{\sqrt{2}} (1 - r^2, \eta, 0_T), \\
p_3 &= \frac{M_B}{\sqrt{2}} (r^2, 1 - \eta, 0_T),
\end{align*}
\]

where \( M_B \) is the mass of \( B_s \), \( r = \frac{m_S}{M_B} \) is the mass ratio, \( m_S \) refers to the mass of the final-state \( S \). We think the variable \( \eta = \omega^2/(M_B^2 - m_S^2) \), and \( \omega \) is the invariant mass of the \( P_1P_2 \) meson pair, which satisfies the relation \( \omega^2 = p^2 \). Meanwhile, we define \( \xi = p_1^+ / p^+ \) as one of the \( P_1P_2 \) meson pair’s momentum fractions. Accordingly, the kinematic variables of other components in the meson pair can be expressed as

\[
\begin{align*}
p_1^- &= \frac{M_B}{\sqrt{2}} (1 - \zeta) \eta, \\
p_2^+ &= \frac{M_B}{\sqrt{2}} (1 - \zeta)(1 - r^2), \\
p_3^- &= \frac{M_B}{\sqrt{2}} \zeta \eta.
\end{align*}
\]

We adopt \( x_B, z, x_3 \) to indicate the momentum fraction of the light quark in each meson with the range from zero to unity. So the light quark’s momentum of the \( B_s(k_B), P_1P_2(k) \) and \( S(k_3) \) are read as

\[
\begin{align*}
k_B &= (0, \frac{M_B}{\sqrt{2}} x_B, x_B), \\
k &= (\frac{M_B}{\sqrt{2}} z(1 - r^2), 0, k_T), \\
k_3 &= (\frac{M_B}{\sqrt{2}} r^2 x_3, \frac{M_B}{\sqrt{2}} (1 - \eta)x_3, k_{3T}).
\end{align*}
\]

In this work, the wave function of the hadron \( B_s^0 \) can be given by Refs. [27–30]

\[ \Phi_{B_s^0} = \frac{i}{\sqrt{2N_c}} (\hat{\phi}_B + M_B) \gamma_5 \phi_{B_s}(x_B, b_B) \]  

where \( N_c = 3 \) is the color factor, and the distribution amplitude(DA) \( \phi_{B_s}(x_B, b_B) \) is expressed in the usual form, which is

\[ \phi_{B_s}(x_B, b_B) = N_B x_B^2 (1 - x_B)^2 \exp[-M_B^2 x_B^2 / (2\omega_b^2) - 1 / 2(\omega_b b_B)^2], \]

the factor \( N_B \) can be calculated by the normalization \( \int_0^1 \phi_{B_s}(x_B, b_B = 0) dx = f_B / (2 \sqrt{2N_c}) \) with the \( B_s^0 \) meson decay constant \( f_B \). And we select the shape parameter \( \omega_b = 0.50 \pm 0.05 \text{ GeV} \) [31].

For the light scalar mesons \( a_0(980) \) and \( f_0(980) \), the wave function can be found in Refs. [3, 32]
\[
\Phi_S(x_3) = \frac{1}{2\sqrt{2N_c}} \left[ \beta_3 \phi_S(x_3) + m_S \phi_S^0(x_3) + m_S (\mu - 1) \phi_S^T(x_3) \right]
\]

with the twist-2 distribution amplitude \(\phi_S\), which can be expanded by the Gegenbauer polynomials [3, 32]:

\[
\phi_S(x_3, \mu) = \frac{3}{\sqrt{2N_c}} x_3 (1 - x_3) \left\{ f_S(\mu) + \bar{f}_S(\mu) \sum_{m=1,3} B_m(\mu) C_m^{3/2}(2x_3 - 1) \right\},
\]

\(\phi_S^0\) and \(\phi_S^T\) are the twist-3 distribution amplitudes, their asymptotic forms can be written as:

\[
\phi_S^0(x_3, \mu) = \frac{1}{2\sqrt{2N_c}} \bar{f}_S(\mu),
\]

\[
\phi_S^T(x_3, \mu) = \frac{1}{2\sqrt{2N_c}} \bar{f}_S(\mu)(1 - 2x_3),
\]

where \(B_m\) is the Gegenbauer moment, \(C_m^{3/2}(2x_3 - 1)\) denotes the Gegenbauer polynomials, \(f_S\) and \(\bar{f}_S\) stand for the vector and scalar decay constants of the light scalar mesons \(a_0\) and \(f_0\), respectively. It is obvious that only the odd Gegenbauer moments are considered in the DA of \(\phi_S\), the even Gegenbauer coefficients \(B_m\) are suppressed because of the conservation of charge conjugation invariance or vector current, and we just notice the Gegenbauer moments \(B_1\) and \(B_3\) since the small contribution of higher order Gegenbauer moments can be ignored. On the basis of QCD sum rules with the default scale \(\mu = 1\) GeV, for the light scalar meson \(a_0\) [3, 32], we adopt the Gegenbauer moments \(B_1 = -0.93 \pm 0.10\) and \(B_3 = 0.14 \pm 0.08\), and the scalar decay constant value of which can be taken as \(f_{a_0} = 0.365 \pm 0.020\). Meanwhile, the Gegenbauer moments and scalar decay constants of the scalar meson \(f_0\) can be listed as [3, 32]:

\[
\bar{f}_S = \bar{f}_{f_0} = \bar{f}_{f_0} = 0.370 \pm 0.020 \ \text{GeV},
\]

\[
B_1^n = -0.78 \pm 0.08, B_3^n = 0.02 \pm 0.07,
\]

\[
B_1^{nT} = 0.8B_1^{nT}.
\]

Here, we take the same value of the two decay constants \(\bar{f}_{f_0}^n\) and \(\bar{f}_{f_0}^w\) [3]. In this article, we select the vector decay constants \(f_S = 0\) due to the discussions in the Ref. [25].

For the distribution amplitudes of the final-state \(P_1P_2\) meson pair, we adopt the consistent form of the S-wave pion pair [13, 33, 34]:

\[
\Phi_{P_1P_2}^S = \frac{1}{2\sqrt{2N_c}} \left[ \beta \phi_S(z, \zeta, \omega^2) + \omega \phi_S^0(z, \zeta, \omega^2) + \omega(\mu - 1) \phi_S^T(z, \zeta, \omega^2) \right],
\]

where the leading-twist distribution amplitude \(\phi_S\) and the twist-3 DAs \(\phi_S^0\) and \(\phi_S^T\) have similar expression forms as the corresponding twists of the light scalar meson obtained using the timelike form factor by replacing the original scalar decay constants. The asymptotic expression of the light-cone distribution amplitudes \(\phi_S^{(s,t)}\) are as follow [21, 33]:

\[
\phi_S(z, \zeta, \omega^2) = \frac{9F_S(\omega)}{\sqrt{2N_c}} a_2 z(1 - z)(1 - 2z),
\]

\[
\phi_S^0(z, \zeta, \omega^2) = \frac{F_S(\omega)}{2\sqrt{2N_c}} (1 - 2z),
\]

\[
\phi_S^T(z, \zeta, \omega^2) = \frac{F_S(\omega)}{2\sqrt{2N_c}} (1 - 2z).
\]

with the Gegenbauer moment \(a_2 = 0.3 \pm 0.1\) for \(a_0\) and \(a_2 = 0.3 \pm 0.2\) for \(f_0\) [19, 35]. \(F_S(\omega)\) is the timelike form factor, which can be described well in terms of the relative Breit-Wigner [36]. However, for the \(a_0\) (or \(f_0\)) resonance, the main decay channels are \(\pi\eta(\pi\pi)\) and \(KK\). The relative Breit-Wigner line shape cannot be well adapted to the timelike form factor of \(a_0\) and \(f_0\), because both of the \(a_0\) and \(f_0\) resonances are very close to the \(KK\) threshold, which greatly influences the resonance shape. As such, we choose the widely accepted prescription proposed by Flatté [37], which are given as [21]:
\[ F^{a_0}_S(\omega) = \frac{C_{a_0} m_{a_0}^2}{m_{a_0}^2 - \omega^2 - i(\rho_{\pi\eta}^2 + g_{KK}^2 \rho_{KK}^2)}, \quad (15) \]

for the \( a_0 \) resonance and

\[ F^{f_0}_S(\omega) = \frac{m_{f_0}^2}{m_{f_0}^2 - \omega^2 - i m_{f_0}(\rho_{\pi\pi}^2 + g_{KK}^2 \rho_{KK}^2 F^2_{KK})}, \quad (16) \]

for the \( f_0 \) resonance [24, 38, 39]. In the case of the timelike form factor of the \( a_0 \) resonance, \( C_{a_0} = |C_{a_0}| e^{i \phi_{a_0}} \) is the complex amplitude of the intermediate state \( a_0 \), with different values for the final states \( \pi \eta \) and \( K \bar{K} \). For the \( K \bar{K} \) channel, the magnitude \( |C_{KK}^a| = 1.07 \) and phase \( \phi_{a_0} = 82^\circ \) [40]. Then, the phase of the \( \pi \eta \) system is consistent with that of the \( K \bar{K} \) system, and the module of the magnitude satisfies the relation \( C_{a_0}^\pi \eta / C_{a_0}^KK = g_{a_0} \pi \eta / g_{a_0}KK \) according to the discussion in Ref. [21]. The definition of the strong coupling constants \( g_{a_0}KK (g_{a_0} \pi \eta) \) can be found in the literature [22, 41]. In this article, we take the coupling constants as \( g_{\pi \eta} = 0.324 \) GeV and \( g_{KK}^2 / g_{\pi \eta}^2 = 1.03 \) through the Crystal Barrel experiment [42]. The values of the constants \( g_{a_0}KK \) and \( g_{a_0} \pi \eta \) can be got with the relation \( g_{KK}^2 (g_{\pi \eta}) = g_{a_0}KK (g_{a_0} \pi \eta) / (4 \sqrt{\pi}) \). Meanwhile, we employ the coupling constants \( g_{\pi \eta} = 0.165 \pm 0.018 \) GeV and \( g_{KK}^2 / g_{\pi \eta}^2 = 4.21 \pm 0.33 \) for \( f_0 \) [36], and introduce the factor \( F_{KK} = e^{-\alpha q^2} \) into the timelike form factor \( F^{f_0}_S(\omega) \) to suppress the \( K \bar{K} \) contribution with the parameter \( \alpha = 2.0 \pm 1.0 \) GeV\(^{-2} \) [43]. In addition, the \( \rho \) factors are chosen as:

\[ \rho_{\pi \eta} = \sqrt{1 - \left( \frac{m_\pi - m_\eta}{\omega} \right)^2} \left[ 1 - \left( \frac{m_{a_0} + m_\pi}{\omega} \right)^2 \right], \quad (17) \]

\[ \rho_{\pi \pi} = \frac{2}{3} \sqrt{1 - \frac{4m_{\pi \pi}^2}{\omega^2}} + \frac{3}{2} \sqrt{1 - \frac{4m_{a_0}^2}{\omega^2}}, \quad (18) \]

\[ \rho_{KK} = \frac{1}{2} \left[ 1 - \frac{4m_{KK}^2}{\omega^2} \right] + \frac{1}{2} \left[ 1 - \frac{4m_{a_0}^2}{\omega^2} \right], \quad (19) \]

The shape of \( \sigma \) resonance can be well described by the Breit-Wigner model because it is a narrow intermediate resonance [44, 45]:

\[ F^{\sigma}_S(\omega) = \frac{C_\sigma m_{\sigma}^2}{m_{\sigma}^2 - \omega^2 - i m_\sigma \Gamma(\omega)}, \quad (20) \]

with the factor \( C_\sigma = 3.50 \) [43]. The energy-dependent width \( \Gamma(\omega) \) in the case of a scalar resonance decaying into pion pair can be parameterized as [33]:

\[ \Gamma(\omega) = \Gamma_0 \left( \frac{m_\sigma^2}{\omega^2} - \frac{4m_\sigma^2}{m_\sigma^2 - 4m_\pi^2} \right)^\frac{1}{2}, \quad (21) \]

where \( \Gamma_0 = 0.40 \) GeV is the width of the resonance.

In this paper, we take the mixing relation for \( f_0 - \sigma \) system [53],

\[ \begin{pmatrix} \sigma \\ f_0 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} f_n \\ f_s \end{pmatrix}, \quad (22) \]

with

\[ f_n = \frac{1}{\sqrt{2}}(\bar{u}d + \bar{d}u), f_s = s\bar{s}. \quad (23) \]
B. Helicity Amplitudes

The $\bar{B}_s^0 \to a_0 \to \pi^0 \eta, K \bar{K}|a_0$ decays only have annihilation Feynman diagrams, and their helicity amplitudes are written as

$$\mathcal{A}(\bar{B}_s^0 \to a_0^+ \to \pi^+ \eta, K^{+}\bar{K}^0|a_0^+) = \frac{G_F}{\sqrt{2}} V_{ub} V_{us}^* [(C_1 + \frac{1}{3} C_2) F_e^{LL} + C_2 (M_g^{LL})]$$

$$- V_{ub} V_{us}^* [(a_3 + a_5 + a_7 + a_9) F_e^{LL} + (a_3 + a_5 - \frac{1}{2} (a_7 + a_9)) F_e^{LL} + (C_4 - \frac{1}{2} C_1) M_g^{LL} + (C_4 + C_1) M_g^{LL}] + (C_6 + \frac{1}{2} C_8) (M_g^{SP} + (C_6 + C_8) M_g^{SP})]$$

$$= (24)$$

$$\mathcal{A}(\bar{B}_s^0 \to a_0^0 \to \pi^0 \eta, K^{+}\bar{K}^0|a_0^0) = \frac{G_F}{\sqrt{2}} V_{ub} V_{us}^* [(C_1 + \frac{1}{3} C_2) F_e^{LL} + C_2 (M_g^{LL})]$$

$$- V_{ub} V_{us}^* [(2a_3 + a_5) + \frac{1}{2} (a_7 + a_9)] (F_e^{LL} + F_e^{LL} + (2C_4 + \frac{1}{2} C_1) M_g^{LL} + (2C_4 - C_1) M_g^{LL}] + (2C_6 + \frac{1}{2} C_8) (M_g^{SP} + M_g^{SP})]$$

$$= (25)$$

$$\mathcal{A}(\bar{B}_s^0 \to a_0^{-} \to \pi^{0} \eta, K^{0} K^{-}|a_0^{-}) = \frac{G_F}{\sqrt{2}} V_{ub} V_{us}^* [(C_1 + \frac{1}{3} C_2) (F_e^{LL} + F_e^{LL}) + C_2 (M_g^{LL} + M_g^{LL})]$$

$$- V_{ub} V_{us}^* [(2a_3 + a_5) + \frac{1}{2} (a_7 + a_9)] (F_e^{LL} + F_e^{LL} + (2C_4 + \frac{1}{2} C_1) M_g^{LL} + (2C_4 - C_1) M_g^{LL}] + (2C_6 + \frac{1}{2} C_8) (M_g^{SP} + M_g^{SP})]$$

$$= (26)$$

where $F_e(F_e')$ stands for the contributions of the factorizable annihilation diagrams shown in Fig. 1(e1),(f1)(e2),(f2), and $M_g(M_g')$ comes from the nonfactorizable annihilation diagrams in Fig. 1(g1),(h1)(g2),(h2). In subsequent calculation, the function $F_\sigma(F_\sigma')$ denotes the contributions of the factorizable emission diagrams in Fig. 1(a1),(b1)(a2),(b2), and $M_\sigma(M_\sigma')$ comes from the nonfactorizable emission diagrams in Fig. 1(c1),(d1)(c2),(d2). The specific expressions for the aforementioned functions are presented in Appendix. The superscripts $LL, LR, SP$ represent the contributions of $(V-A)(V-A), (V-A)(V+A)$ and $(S-P)(S+P)$ vertices, respectively.

Based on the $f_0 - \sigma$ mixing scheme, the helicity amplitudes of the $\bar{B}_s^0 \to f_0 [\to \pi^+ \pi^-, K^+ K^-] f_0$ and $\bar{B}_s^0 \to \sigma [\to \pi^+ \pi^-] \sigma$ decays are given by

$$\mathcal{A}(\bar{B}_s^0 \to f_0 [\to \pi^+ \pi^-, K^+ K^-] f_0) = \sin^2 \theta A(\bar{B}_s^0 \to f_n [\to \pi^+ \pi^-, K^+ K^-] f_n)$$

$$+ \frac{1}{2} \sin 2 \theta A(\bar{B}_s^0 \to f_n [\to \pi^+ \pi^-, K^+ K^-] f_n)$$

$$+ \frac{1}{2} \sin 2 \theta A(\bar{B}_s^0 \to f_n [\to \pi^+ \pi^-, K^+ K^-] f_n)$$

$$= (27)$$

$$\mathcal{A}(\bar{B}_s^0 \to f_s [\to \pi^+ \pi^-, K^+ K^-] f_s) = \cos^2 \theta A(\bar{B}_s^0 \to f_s [\to \pi^+ \pi^-, K^+ K^-] f_s)$$
\[ A(B_s^0 \to \sigma \to \pi^+\pi^-) = \cos^2 \theta A(B_s^0 \to f_n \to \pi^+\pi^-) f_n \]
\[ -\frac{1}{2} \sin 2\theta A(B_s^0 \to f_s \to \pi^+\pi^-) f_s \]
\[ -\frac{1}{2} \sin 2\theta A(B_s^0 \to f_n \to \pi^+\pi^-) f_n \]
\[ + \sin^2 \theta A(B_s^0 \to f_s \to \pi^+\pi^-) f_s, \]

with

\[ A(B_s^0 \to f_s \to \pi^+\pi^-, K^+K^-) f_s = -G_F V_{tb} V_{ts}^* [(a_6 - \frac{1}{2} a_8) (F_{a}^{SP} + F_{c}^{SP} + F_{c}^{LL} + F_{c}^{SP}) \]
\[ + (a_3 + a_4 + a_5 - \frac{1}{2} (a_7 + a_9 + a_{10})) (F_{a}^{LL} + F_{c}^{LL}) \]
\[ + (C_3 + C_4 - \frac{1}{2} (C_9 + C_{10})) (M_{c}^{LL} + M_{c}^{SP} + M_{c}^{LL} + M_{c}^{SP}) \]
\[ + (C_5 - \frac{1}{2} C_7) (M_{c}^{LR} + M_{c}^{LR} + M_{c}^{LR}) \]
\[ + (C_6 - \frac{1}{2} C_8) (M_{c}^{SP} + M_{c}^{SP} + M_{c}^{SP} + M_{c}^{SP})], \]

\[ A(B_s^0 \to f_s \to \pi^+\pi^-, K^+K^-) f_s = \frac{G_F}{2} [V_{ub} V_{ts}^* C_2 M_{c}^{LL} \]
\[ - V_{tb} V_{ts}^* (2C_4 + \frac{1}{2} C_{10}) M_{c}^{LL} + (2C_6 + \frac{1}{2} C_8) M_{c}^{SP}]]. \]

Equation (30) is also applicable for $B_s^0 \to f_n \to \pi^+\pi^-, K^+K^-) f_n$ decay after replacing $M_{c}^{LL,SP}$ with $M_{c}^{LL',SP}$. Meanwhile, $B_s^0 \to f_n \to \pi^+\pi^-, K^+K^-) f_n$ decay has the same amplitude as $B_s^0 \to a_0 \to \pi^0\eta, K^+K^-) a_0$ decay.

### III. NUMERICAL RESULTS AND DISCUSSIONS

With the decay amplitudes $A$, the differential branching ratio for the $B_s^0 \to S \to P_1P_2 | S$ decays can be taken as

\[ \frac{d\bar{B}}{d\omega} = \tau|\bar{p}_1| \bar{p}_3 |[A]^2, \]

where $\tau$ is the $B$ meson lifetime, $|\bar{p}_1|$ and $|\bar{p}_3|$ respectively denote the magnitudes of momentum for one of the $P_1P_2$ meson pair and the scalar meson $S$

\[ |\bar{p}_1| = \frac{\lambda^{1/2}(\omega^2, m_{P_1}^2, m_{P_2}^2)}{2\omega}, \]
\[ |\bar{p}_3| = \frac{\lambda^{1/2}(M_B^2, m_s^2, \omega^2)}{2\omega}, \]

with the Källén function $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$.

In Table I, we present the input parameters used in the calculations, including the masses of the mesons, the decay constant and lifetime of the $B_s^0$ meson, and the Wolfenstein parameters of the CKM matrix elements [2, 46, 47].

By using the helicity amplitudes and the input parameters, we predict the $CP$-averaged branching fractions of the $B_s^0 \to S \to P_1P_2 | S$ decays in the pQCD approach, and make some comments on the results. In Table II, we present the branching fractions of the $B_s^0 \to a_0 \to \pi\eta, K\Xi | a_0$ decays. There are still many uncertainties in our calculation results. As shown in the Table II, we primarily consider four types of errors, namely the shape parameter of $B$ meson $\omega_b = 0.50 \pm 0.05$ GeV, the Gegenbauer moment $a_2 = 0.3 \pm 0.1$ for the $K\Xi(\pi\eta)$ pair, the Gegenbauer moments $B_1 = -0.93 \pm 0.10$ and $B_3 = 0.14 \pm 0.08$ for the
TABLE I. Input parameters of the $\bar{B}_s^0 \rightarrow S[\rightarrow P_1 P_2]|S$ decays

| Parameter                  | Value          |
|----------------------------|----------------|
| $M_B$                      | 5.367 GeV      |
| $m_b$                      | 4.2 GeV        |
| $f_B$                      | 227.2 ± 3.4 MeV|
| $\tau$                     | 1.509 ps       |
| $m_{a_0}$                  | 0.98 ± 0.02 GeV|
| $m_{f_0}$                  | 0.99 ± 0.02 GeV|
| $m_s$                      | 0.5 GeV        |
| $m_{f_n}$                  | 0.99 GeV       |
| $m_{f_s}$                  | 1.02 GeV       |
| $m_{f_0(1500)}$            | 1.50 GeV       |
| $m_{K^\pm}$                | 0.494 GeV      |
| $m_{K^0}$                  | 0.498 GeV      |
| $m_\eta$                   | 0.548 GeV      |
| $\lambda$                  | 0.22453 ± 0.00044 |
| $A$                        | 0.836 ± 0.015  |

TABLE II. Branching ratios for the $\bar{B}_s^0 \rightarrow a_0[\rightarrow \pi\eta, K\bar{K}]|a_0$ decays in the pQCD approach.

| Decay Modes                        | $B$                          |
|------------------------------------|------------------------------|
| $\bar{B}_s^0 \rightarrow a_0^0[\rightarrow \pi^0\eta]|a_0^0$ | $8.31^{+3.42}_{-2.26}(\omega_6)+^{6.06}_{-4.35}(a_2)+^{1.84}_{-0.61}(B)+^{0.93}_{-0.87}(\bar{f}_{a_0}) \times 10^{-8}$ |
| $\bar{B}_s^0 \rightarrow a_0^0[\rightarrow K+K^-]|a_0^0$ | $9.87^{+3.84}_{-2.61}(\omega_6)+^{6.76}_{-4.89}(a_2)+^{2.92}_{-1.34}(B)+^{1.07}_{-0.91}(\bar{f}_{a_0}) \times 10^{-9}$ |
| $\bar{B}_s^0 \rightarrow a_0^0[\rightarrow \pi^+\eta]|a_0^0$ | $8.58^{+4.62}_{-3.35}(\omega_6)+^{5.33}_{-4.05}(a_2)+^{1.01}_{-0.93}(B)+^{0.97}_{-0.91}(\bar{f}_{a_0}) \times 10^{-7}$ |
| $\bar{B}_s^0 \rightarrow a_0^0[\rightarrow K^+K^0]|a_0^0$ | $1.76^{+0.95}_{-0.81}(\omega_6)+^{1.06}_{-0.81}(a_2)-^{0.18}_{-0.18}(B)+^{0.19}_{-0.19}(\bar{f}_{a_0}) \times 10^{-7}$ |
| $\bar{B}_s^0 \rightarrow a_0^0[\rightarrow \pi^-\eta]|a_0^0$ | $8.76^{+4.66}_{-3.35}(\omega_6)+^{5.32}_{-4.06}(a_2)+^{1.11}_{-0.93}(B)+^{0.98}_{-0.93}(\bar{f}_{a_0}) \times 10^{-7}$ |
| $\bar{B}_s^0 \rightarrow a_0^0[\rightarrow K^-K^0]|a_0^0$ | $1.81^{+0.98}_{-0.78}(\omega_6)+^{1.06}_{-0.82}(a_2)+^{0.23}_{-0.18}(B)+^{0.20}_{-0.19}(\bar{f}_{a_0}) \times 10^{-7}$ |

scalar meson $a_0$, and the scalar decay constant $f_{a_0} = 0.365 ± 0.020$ GeV. We have neglected the uncertainties caused by the mass of $a_0$ and the Wolfenstein parameters $\lambda$, $A$, $\rho$, $\eta$ because they are typically very small. We notice that the main uncertainties come from the Gegenbauer moment in the wave function of the $K\bar{K}(\pi\eta)$ pair, thus, we look forward to obtaining more accurate experimental data in the future to reduce such errors.

Meanwhile, we can find the branching ratios of the $\bar{B}_s^0 \rightarrow a_0[\rightarrow \pi\eta]|a_0$ decays are much larger than that of the $\bar{B}_s^0 \rightarrow a_0[\rightarrow K\bar{K}]|a_0$ decays, which can be explained by the fact that the phase space for $K\bar{K}$ is suppressed. In the Ref. [22], the authors have predicted the branching ratios of the $\bar{B}_s^0 \rightarrow a_0[\rightarrow \pi\eta, K\bar{K}]h$ (h denote the pseudoscalar meson $\pi$ or $K$) decays in the pQCD approach and get the results as follows: $B(\bar{B}_s^0 \rightarrow a_0^0[\rightarrow K^-K^+]|\pi^0) = 0.74^{+0.01}_{-0.01} \times 10^{-6}$, $B(\bar{B}_s^0 \rightarrow a_0^0[\rightarrow \pi^0\eta]|\pi^0) = 0.74^{+0.01}_{-0.01} \times 10^{-6}$, these are comparable to our results because these decay modes have the same components in the quark model and only annihilation contributions. The branching ratio of $\bar{B}_s^0 \rightarrow a_0^0[\rightarrow \pi^0\eta]|a_0^0$ decay is much smaller than that of the $\bar{B}_s^0 \rightarrow a_0^0[\rightarrow \pi^0\eta]|\pi^0$ decay obviously, and the branching ratios of $\bar{B}_s^0 \rightarrow a_0^0[\rightarrow K^+K^-]|a_0^0$ decay and $\bar{B}_s^0 \rightarrow a_0^0[\rightarrow K^-K^+]|\pi^0$ decay exhibit the same relationship, the reason may be that the QCD dynamics of the final state mesons $a_0$ and $\pi^0$ are different. At the same time, the authors also concluded that the branching fractions of the $\pi\eta$ channel was 5 times larger than that of the $K\bar{K}$ channel with the resonance $a_0(980)$ in the Ref. [22]. Next, we will use our calculations to investigate the value of $\Gamma(a_0 \rightarrow K^+K^-)/\Gamma(a_0 \rightarrow \pi^0\eta)$ with the narrow-width approximation.

When the narrow-width approximation is considered, the branching ratio of the quasi-two-body decay can be written as

$$B(B \rightarrow M_1(R \rightarrow M_2M_3) \sim B(B \rightarrow M_1R) \times B(R \rightarrow M_2M_3),$$

with the resonance $R$. We can define a ratio $R_1$ as
\[ R_1 = \frac{\Gamma(a_0 \rightarrow K^+K^-)}{\Gamma(a_0 \rightarrow \pi^0\eta)} \]
\[ = \frac{\mathcal{B}(B_s^0 \rightarrow a_0^0a_1^0) \times \mathcal{B}(a_0^0 \rightarrow K^+K^-)}{\mathcal{B}(B_s^0 \rightarrow a_0^0a_1^0) \times \mathcal{B}(a_0^0 \rightarrow \pi^0\eta)} \]
\[ \simeq \frac{\mathcal{B}(B_s^0 \rightarrow a_0^0 \rightarrow K^+K^- a_0^0)}{\mathcal{B}(B_s^0 \rightarrow a_0^0 \rightarrow \pi^0\eta a_0^0)} \approx 0.12. \]  

After considering the isospin relation \( \Gamma(a_0 \rightarrow K^+K^-) = \Gamma(a_0 \rightarrow K\bar{K})/2 \), we obtain the relative partial decay width \( \Gamma(a_0 \rightarrow K\bar{K})/\Gamma(a_0 \rightarrow \pi^0\eta) \approx 0.24 \). The OBELIX Collaboration acquired \( \Gamma(a_0 \rightarrow K\bar{K})/\Gamma(a_0 \rightarrow \pi^0\eta) = 0.57 \pm 0.16 \) through the coupled channel analysis of \( \pi^+\pi^-\pi^0 \), \( K^+\pi^-\pi^0 \), and \( K^+K^-\pi^+ \) [48]. In Ref. [42], the authors calculated the branching ratio of the \( \rho\bar{\rho} \rightarrow a_0(980)\pi \rightarrow K\bar{K}\pi \) decay \((5.92^{+0.46}_{-1.04}) \times 10^{-4} \), and combined with the data \( \mathcal{B}(\rho\bar{\rho} \rightarrow a_0(980)\pi; a_0 \rightarrow \pi\eta) = (2.61 \pm 0.48) \times 10^{-3} \) in the annihilation channel \( a_0^0\pi^0\eta \) [49], the ratio of the partial widths was determined as \( \Gamma(a_0 \rightarrow K\bar{K})/\Gamma(a_0 \rightarrow \pi\eta) = 0.23 \pm 0.05 \). The WA102 Collaboration gained \( \Gamma(a_0 \rightarrow K\bar{K})/\Gamma(a_0 \rightarrow \pi\eta) = 0.166 \pm 0.01 \pm 0.02 \) from the decay of \( f_1(1285) \) [50]. The average relative partial decay widths is \( \Gamma(a_0 \rightarrow K\bar{K})/\Gamma(a_0 \rightarrow \pi\eta) = 0.183 \pm 0.024 \) given by the Particle Data Group [2]. It is obvious that our results are slightly larger than the average ratio, but are in agreement with the data in Ref. [42] within errors.

For the \( \overline{B}_s^0 \rightarrow f_0[\rightarrow \pi^+\pi^-] f_0 \) and \( \overline{B}_s^0 \rightarrow \sigma[\rightarrow \pi^+\pi^-] \sigma \) decays, the scalars \( f_0 \) and \( \sigma \) are not only regarded as \( s\sigma \), but also contain \( (u\bar{u} + d\bar{d})/\sqrt{2} \) in the quark model. The mixing angle \( \theta \) is introduced into the \( f_0 - \sigma \) mixing mechanism. In this case, the decay amplitudes of the \( \overline{B}_s^0 \rightarrow f_0[\rightarrow \pi^+\pi^-], K^+\pi^-\pi^0, K^+K^-\pi^0 \) and \( \overline{B}_s^0 \rightarrow \sigma[\rightarrow \pi^+\pi^-] \sigma \) decays consist of four parts, and the total amplitudes are related to these four parts by the mixing angle \( \theta \), which can be found in Eqs. (27) and (28). We then plot the branching fractions of \( \overline{B}_s^0 \rightarrow f_0[\rightarrow \pi^+\pi^-] f_0 \) and \( \overline{B}_s^0 \rightarrow \sigma[\rightarrow \pi^+\pi^-] \sigma \) decays dependent on the free parameter \( \theta \) in Fig. 2. The branching ratio of the \( \overline{B}_s^0 \rightarrow f_0[\rightarrow \pi^+\pi^-] f_0 \) decay obeys the cos law, as can be seen in Fig. 2(a), while the other channel’s contribution satisfies the sin law, as can be seen in Fig. 2(b). Considering both decays, the range of the mixing angle \( \theta \) can be set as \([15^\circ, \, 82^\circ]\) and \([105^\circ, \, 171^\circ]\) because the branching ratios of both decays are close to zero when \( \theta \) takes other values. This range is larger than that of the two-body decays \( \overline{B}_s^0 \rightarrow f_0 f_0 \) and \( \overline{B}_s^0 \rightarrow \sigma \sigma \), but still consistent with the data in the Refs. [51–53].

![Graph](image_url)

**FIG. 2.** (a) The branching ratio of \( \overline{B}_s^0 \rightarrow f_0[\rightarrow \pi^+\pi^-] f_0 \) decay on the mixing angle \( \theta \); (b) The branching ratio of \( \overline{B}_s^0 \rightarrow \sigma[\rightarrow \pi^+\pi^-] \sigma \) decay on the mixing angle \( \theta \).

The mixing angle \( \theta \) is not fixed for the \( f_0 - \sigma \) system, and the value may vary in different works depending on the study requirements [24, 54, 55]. In the current study, we take \( \theta = 30^\circ \) to make numerical calculation, and this value also satisfies the results presented by the LHCb Collaboration [45]. Taking the mixing angle into account, the averaged branching ratios for the \( \overline{B}_s^0 \rightarrow f_0[\rightarrow \pi^+\pi^-], K^+K^-\pi \) and \( \overline{B}_s^0 \rightarrow \sigma[\rightarrow \pi^+\pi^-] \sigma \) decays are presented in Table III.
As can be seen in Table III, the major uncertainties originate from the shape parameter \(\omega_b\) of the \(B\) meson’s wave function for both \(\overline{B}_s^0 \to f_0[\to \pi^+\pi^−, K^+K^−]f_0\) and \(\overline{B}_s^0 \to \sigma[\to \pi^+\pi^−]\sigma\) decays. We still can find the branching ratio for the \(\overline{B}_s^0 \to f_0[\to \pi^+\pi^−]f_0\) decay is larger than that of the \(\overline{B}_s^0 \to \sigma[\to \pi^+\pi^−]\sigma\) decay by one order of magnitude, we think the reason for this result may be that the scalar meson \(f_0\) has a greater mass than \(\sigma\). When we take the mixing angle \(\theta = 0°\), \(f_0\) is regarded as the pure \(\pi\pi\), and the branching ratio of \(\overline{B}_s^0 \to f_0[\to \pi^+\pi^−]f_0\) is about \(9.29 \times 10^{-5}\). Whereas for \(\overline{B}_s^0 \to \sigma[\to \pi^+\pi^−]\sigma\) decay, the branching ratio is very small with \(\theta = 0°\), and the value will increase a lot after considering the mixing of \(\pi\pi\). Therefore, \(\overline{B}_s^0 \to f_0[\to \pi^+\pi^−, K^+K^−]f_0\) makes the dominant contribution in the branching ratios. Numerical results of \(\overline{B}_s^0 \to f_0[\to \pi^0\pi^0]f_0\) and \(\overline{B}_s^0 \to \sigma[\to \pi^0\pi^0]\sigma\) decays are obtained with the isospin relationship \(B(f_0[\to \pi^+\pi^−])/B(f_0[\to \pi^0\pi^0]) = 2\). The magnitudes of the predicted branching ratios are at the order of \(10^{-5} \sim 10^{-6}\), we expect these results can be tested by the LHCb and Belle II experiments in the nearly future.

To compare with existing data and further discuss our calculations, we then use the narrow-width approximation to study the \(\overline{B}_s^0 \to f_0[\to \pi^+\pi^−, K^+K^−]f_0\) decays. The \(\overline{B}_s^0 \to f_0[\to \pi^+\pi^−]f_0\) and \(\overline{B}_s^0 \to f_0[\to K^+K^−]f_0\) decays have the same resonance, we can define a ratio \(R_2\) to describe the relationship between \(f_0 \to \pi^+\pi^−\) and \(f_0 \to K^+K^−\), which can be given as

\[
R_2 = \frac{B(f_0 \to K^+K^-)}{B(f_0 \to \pi^+\pi^-)}
\]

\[
= \frac{B(\overline{B}_s^0 \to f_0f_0) \times B(f_0 \to K^+K^-)}{B(\overline{B}_s^0 \to f_0f_0) \times B(f_0 \to \pi^+\pi^-)}
\]

\[
\approx \frac{B(\overline{B}_s^0 \to f_0[\to K^+K^-]f_0)}{B(\overline{B}_s^0 \to f_0[\to \pi^+\pi^-]f_0)} \approx 0.17,
\]

this ratio can be used to estimate the branching ratios for the \(f_0 \to \pi^+\pi^-\) and \(f_0 \to K^+K^-\) decays by using the formulas \(B(f_0 \to \pi^+\pi^-) = \frac{2}{\sqrt{3K_\pi^+\pi^-}}\) and \(B(f_0 \to K^+K^-) = \frac{RB_{\overline{B}_s^0}}{\sqrt{3K_\pi^+\pi^-}}\). So we can get

\[
B(f_0 \to \pi^+\pi^-) \approx 0.54,
\]

\[
B(f_0 \to K^+K^-) \approx 0.09.
\]

The BES Collaboration gained the relative branching ratios through the \(\psi(2S) \to \gamma\chi_{c0}\) decays, where \(\chi_{c0} \to f_0f_0 \to \pi^+\pi^−\pi^+\pi^-\) or \(\chi_{c0} \to f_0f_0 \to \pi^+\pi^-K^+K^-\) \([57, 58]\). Meanwhile, CLEO Collaboration has obtained \(B(f_0 \to K^+K^-)/B(f_0 \to \pi^+\pi^-) = (25 \pm 1)%\) and extracted \(B(f_0 \to \pi^+\pi^-) = (50 \pm 5)%\) by using the results of BES Collaboration \([59]\), and our calculations are still consistent with that in CLEO Collaboration.

In Fig. 3, we graph the differential branching ratios of the \(\overline{B}_s^0 \to a_0[\to K\bar{K}, \pi\eta]a_0\) decays on the invariant mass, the results of \(\overline{B}_s^0 \to a_0^0[\to K^+K^−, \pi^0\eta]a_0^0\) models are shown on the left and those of \(\overline{B}_s^0 \to a_0^-[\to K^-K^0, \pi^-\eta]a_0^+\) models are shown on the right. The differential branching ratios of \(\overline{B}_s^0 \to f_0[\to \pi^+\pi^-]f_0, \overline{B}_s^0 \to f_0[\to K^+K^-]f_0,\) and \(\overline{B}_s^0 \to \sigma[\to \pi^+\pi^-]\sigma\) decays on the \(\pi\pi\) or \(K\bar{K}\) invariant mass \(\omega\) are presented in Fig. 4 with red solid line, blue solid line and red dashed line, respectively. For the \(a_0\) resonance, the contributions of the \(K\bar{K}\) channel are much smaller than that of the \(\pi\eta\) channel, and for the \(d\overline{B}(\overline{B}_s^0 \to \sigma[\to \pi^+\pi^-]\sigma)/d\omega\) mode, we magnify the results tenfold for easy viewing. From the figures, it is clear that the
peak occurs around the resonance peak mass, and the majority of the branching ratios are concentrated around the resonance state, basically in the range of $[m_S - \Gamma_S, m_S + \Gamma_S]$. Here, we don’t plot the contributions of $B_s^0 \to a_0^+ [\to K^+ \bar{K}^0, \pi^+ \pi^-] a_0^-$ channels separately, because their results are very similar to the results of $B_s^0 \to a_0^- [\to K^- K^0, \pi^- \eta] a_0^+$ channels.

IV. SUMMARY

In this article, we predict the branching fractions of $B_s^0 \to a_0 [\to K \bar{K}, \pi \eta] a_0$, $B_s^0 \to f_0 [\to \pi^+ \pi^-, K^+ K^-] f_0$ and $B_s^0 \to \sigma [\to \pi^+ \pi^-] \sigma$ decays with the pQCD approach firstly, where the scalars are considered as the $q \bar{q}$ state in the first scenario. Our results show that: (1) For the $B_s^0 \to a_0 [\to K \bar{K}, \pi \eta] a_0$ decays, the largest branching ratio is $B(B_s^0 \to a_0^- [\to \pi^- \eta] a_0^-) = 8.76 \times 10^{-7}$, which is highly likely to be verified experimentally; (2) For the $B_s^0 \to f_0 [\to \pi^+ \pi^-, K^+ K^-] f_0$ and $B_s^0 \to \sigma [\to \pi^+ \pi^-] \sigma$ decays, we consider that the scalars $f_0$ and $\sigma$ contain $\pi \pi$ and $(\pi \pi + d \bar{d})/\sqrt{2}$ components in the quark model, so our calculations are carried out with the $f_0 - \sigma$ mixing scheme, when the value of the mixing angle $\theta$ is taken as $30^\circ$, our results are at the order of $10^{-6} \sim 10^{-5}$. Using the narrow-width approximation, we calculate the relative partial decay widths $\Gamma(a_0 \to K \bar{K})/\Gamma(a_0 \to \pi \eta)$ and the ratio $B(f_0 \to K^+ K^-)/B(f_0 \to \pi^+ \pi^-)$, which are in agreement with the existing experimental values. Our work
has positive implications for understanding the QCD behavior of the scalars, and we also expect that our calculations can be tested by the LHCb and Belle II experiments in the future.

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APPENDIX : FACTORIZATION FORMULAE

In this Section, we list the factorization formulas that are used in Eqs. (24)–(30). According to previous studies [38, 60], we can find the formulas for the factorization diagrams related to the decays $B \to S[P \to PP]V$. In our work, the formulas for the factorization diagrams for the case when the final-state scalars are emitted are roughly consistent with those shown in Ref. [60]. In contrast, for the factorization diagrams where the $PP$ meson pair is emitted or for the annihilation diagrams, the formulas are different from that in the existing works because of the difference in the wave functions between the final-state scalars and other mesons. So we recalculate the contributions of the Feynman diagrams. First, we give the contribution of the factorization diagrams in Fig. 1(a1) and (b1) with different currents, which are

1. $(V - A)(V - A)$

$$F_{a}^{LL} = -4 \pi C_{F} f_{s} M_{B}^{2} \int_{0}^{1} dx_{B} dz \int_{0}^{\infty} b_{B} b_{B} d b b_{B}(x_{B}, b_{B})$$

$$\left\{ \left[ -2 \sqrt{1 - r_{2}} \eta \eta_{ij}(2r_{b} + r_{2}(z - 2) - z)(\phi_{S}^{i}(z) - \phi_{S}^{j}(z)) + 4 \sqrt{1 - r_{2}} \eta r_{2} \phi_{S}^{i}(z) |\alpha_{s}(t_{a1})| h_{a1}(\alpha_{a1}, \beta_{a1}, b_{B}, b) \exp[-S_{B}(t_{a1}) - S(t_{a1})]S_{i}(z) \right] + 2(r_{2} - 1)(x_{B} - \eta) \phi_{S}(z) - 4 \sqrt{1 - r_{2}} \eta \eta_{ij} \phi_{S}^{i}(z) |\alpha_{s}(t_{b1})| h_{b1}(\alpha_{b1}, \beta_{b1}, b, b_{B}) \exp[-S_{B}(t_{b1}) - S(t_{b1})]S_{i}(x_{B} - \eta) \right\}$$

2. $(V - A)(V + A)$

$$F_{a}^{LR} = F_{a}^{LL}$$

3. $(S - P)(S + P)$

$$F_{a}^{SP} = 8 \pi C_{F} f_{s} M_{B}^{2} \int_{0}^{1} dx_{B} dz \int_{0}^{\infty} b_{B} b_{B} d b b_{B}(x_{B}, b_{B})$$

$$\left\{ \left[ (2r_{1}(1 - r_{2} + \eta) + 4(\eta_{ij} - 1)) \phi_{S}(z) - 2 \sqrt{1 - r_{2}} \eta \eta_{ij}(\phi_{S}^{i}(z) - \phi_{S}^{j}(z)) - 4 \sqrt{1 - r_{2}} \eta \right. \right.$$  

$$\times (r_{2}^{2} + z) \phi_{S}^{j}(z) + 8 \sqrt{1 - r_{2}} \eta r_{2} \phi_{S}^{j}(z) |\alpha_{s}(t_{a1})| h_{a1}(\alpha_{a1}, \beta_{a1}, b_{B}, b) \exp[-S_{B}(t_{a1}) - S(t_{a1})]S_{i}(z)$$

$$\left. + 2(r_{2} - 1)(x_{B} - \eta) \phi_{S}(z) - 4 \sqrt{1 - r_{2}} \eta (r_{2} - \eta_{ij}) \phi_{S}^{j}(z) \right. \times \alpha_{s}(t_{b1}) h_{b1}(\alpha_{b1}, \beta_{b1}, b, b_{B}) \exp[-S_{B}(t_{b1}) - S(t_{b1})]S_{i}(x_{B} - \eta) \right\}$$

with $\bar{x} = 1 - x_{i}, \eta = 1 - \eta$. $C_{F} = \frac{4}{3}$ stands for the color factor. We take $F_{a}^{LL} = F_{a}^{LR} = 0$ because of the vector decay constant’s small value. And the nonfactorization diagrams in Fig. 1 (c1) and (d1) give

1. $(V - A)(V - A)$

$$M_{c}^{LL} = \frac{4 \pi C_{F} M_{B}^{2}}{\sqrt{2} N_{c}} \int_{0}^{1} dx_{B} dz dx_{3} \int_{0}^{\infty} b_{B} b_{B} b_{3} b_{B3} \phi_{B}(x_{B}, b_{B})$$

$$\left\{ \left[ (2r_{1}(1 - r_{2} + \eta) + x_{3} \eta_{ij}) \phi_{S}(z) + 4 \sqrt{1 - r_{2}} \eta \bar{\eta}_{ij}(\phi_{S}^{i}(z) - \phi_{S}^{j}(z)) - 8 \sqrt{1 - r_{2}} \eta \bar{\eta} \phi_{S}^{i}(z) \right. \times \alpha_{s}(t_{a1}) h_{a1}(\alpha_{a1}, \beta_{a1}, b_{3}, b_{B}) \exp[-S_{B}(t_{a1}) - S(t_{a1})] - S_{3}(t_{a1})]$$

$$\left. + 2(r_{2} - 1)(x_{B} - \eta) \phi_{S}(z) - 4 \sqrt{1 - r_{2}} \eta \eta_{ij} \phi_{S}^{j}(z) \right. \times \alpha_{s}(t_{b1}) h_{b1}(\alpha_{b1}, \beta_{b1}, b, b_{B}) \exp[-S_{B}(t_{b1}) - S(t_{b1})] - S_{3}(t_{b1})] \right\}$$

(A.4)
\[
M^{LR}_e = \frac{4\pi C_F M_B^4}{\sqrt{2\pi N_c}} \int_0^1 dx_B dx_d x_3 \int_\infty^0 b_B b_B b_3 b_3 \phi_B(x_B, b_B) \\
\left\{ [4\sqrt{(1-r^2)}\eta(\bar{\pi} - x_B + \eta x_3)\phi_S^2(z) - \phi_S^2(x_3)] + 8\sqrt{(1-r^2)}\eta \bar{\pi} \right. \\
\times \phi_S^2(x_3)(\phi_S^2(z) - \phi_S^2(x_3)) - 8\sqrt{(1-r^2)}\eta(\bar{\pi} - x_3)\phi_S^2(z)(\phi_S^2(x_3) - \phi_S^2(x_3)) \\
-4r(\bar{\pi} + x_B + x_3)\phi_S^2(z) - \phi_S^2(x_3)) - 8r^2(\bar{\pi} + x_3)\phi_S^2(z)\phi_S^2(x_3) \\
\times \alpha_s(t_{e1}) h_{c1}(\alpha_{c1}, \beta_{c1}, b_3, b_3) \exp[-S_B(t_{e1}) - S(t_{e1}) - S_3(t_{e1})] \\
\left. + [4\sqrt{(1-r^2)}\eta(2 - x_B + z + x_3)\phi_S^2(z) - \phi_S^2(x_3)] + 8\sqrt{(1-r^2)}\eta \bar{\pi} \right\} \\
\times \alpha_s(t_{e1}) h_{d1}(\alpha_{d1}, \beta_{d1}, b_3, b_3) \exp[-S_B(t_{d1}) - S(t_{d1}) - S_3(t_{d1})] \right\} 
\]

(3) \( (S-P)(S+P) \)

\[
M^{SP}_e = -\frac{8\pi C_F M_B^4}{\sqrt{2\pi N_c}} \int_0^1 dx_B dx_d x_3 \int_\infty^0 b_B b_B b_3 b_3 \phi_B(x_B, b_B) \phi_S(x_3) \\
\left\{ [2\sqrt{(1-r^2)}\eta(\bar{\pi} + \eta x_3)\phi_S(z) + 2\sqrt{(1-r^2)}\eta \bar{\pi} \phi_S^2(z) - \phi_S^2(x_3)) \\
-4\sqrt{(1-r^2)}\eta^2(\bar{\pi} - x_3)\phi_S^2(z)\right. \\
\times \alpha_s(t_{e1}) h_{c1}(\alpha_{c1}, \beta_{c1}, b_3, b_3) \exp[-S_B(t_{e1}) - S(t_{e1}) - S_3(t_{e1})] \\
\left. + [2\sqrt{(1-r^2)}\eta \bar{\pi} - \eta \bar{\pi} \phi_S(z) - 2\sqrt{(1-r^2)}\eta \bar{\pi} \phi_S^2(z) + \phi_S^2(x_3)] \right\} \\
\times \alpha_s(t_{d1}) h_{d1}(\alpha_{d1}, \beta_{d1}, b_3, b_3) \exp[-S_B(t_{d1}) - S(t_{d1}) - S_3(t_{d1})] \right\} 
\]

The contributions of the factorizable annihilation diagrams in Fig. 1 (e1) and (f1) are

(1) \( (V-A)(V-A) \)

\[
F^{LL}_e = -2\pi C_F f_B M_B^4 \int_0^1 dz dx_3 \int_0^\infty b b b_3 b_3 \\
\left\{ [2\eta(x_3 \bar{\pi} - 1)\phi_S(z)\phi_S(x_3) - 4\sqrt{(1-r^2)}\eta(2 + x_3)\phi_S^2(z)\phi_S(x_3) + 4\sqrt{(1-r^2)}\eta^2 x_3 \phi_S^2(z)\phi_S^2(x_3) \\
\times \alpha_s(t_{e1}) h_{e1}(\alpha_{e1}, \beta_{e1}, b, b_3) \exp[-S_B(t_{e1}) - S(t_{e1})] S_3(x_3) \\
\right. \\
\left. + [2\sqrt{(1-r^2)}\eta \phi_S^2(z)\phi_S^2(x_3) - 4\sqrt{(1-r^2)}\eta^2 x_3 \phi_S^2(z)\phi_S^2(x_3) + 4\sqrt{(1-r^2)}\eta^2 r \phi_S^2(z)\phi_S^2(x_3) \\
\times \alpha_s(t_{f1}) h_{f1}(\alpha_{f1}, \beta_{f1}, b_3, b_3) \exp[-S_3(t_{f1}) - S(t_{f1})] S_3(z) \right\} 
\]

(2) \( (V-A)(V+A) \)

\[
F^{LL}_e = F^{LR}_e \tag{A.8} 
\]

(3) \( (S-P)(S+P) \)

\[
F_e^{SP} = -4\pi C_F f_B M_B^4 \int_0^1 dz dx_3 \int_0^\infty b b b_3 b_3 \\
\left\{ [2r(x_3 \bar{\pi} - 1 - \eta)\phi_S^2(x_3) - \phi_S^2(x_3)] - 4r(1 - x_3)\phi_S(z)\phi_T^2(x_3) + 4\sqrt{(1-r^2)}\eta \phi_S^2(z)\phi_S(x_3) \\
\times \alpha_s(t_{e1}) h_{e1}(\alpha_{e1}, \beta_{e1}, b, b_3) \exp[-S_B(t_{e1}) - S(t_{e1})] S_3(x_3) \\
\right. \\
\left. + [4r(\eta \bar{\pi} - 1)\phi_S(z)\phi_S^2(x_3) + 2\sqrt{(1-r^2)}\eta \phi_S^2(z)\phi_S^2(x_3) - \phi_S^2(z)] + 4\sqrt{(1-r^2)}\eta^2 \phi_S(x_3)\phi_S^2(z) \\
\times \alpha_s(t_{f1}) h_{f1}(\alpha_{f1}, \beta_{f1}, b_3, b) \exp[-S_3(t_{f1}) - S(t_{f1})] S_3(z) \right\} 
\]

The contributions of the nonfactorizable annihilation diagrams in Fig. 1 (g1) and (h1) are
\( M^{LL}_g = \frac{4\pi C_F M^4_b}{\sqrt{2 N_c}} \int_0^1 d\tau_B d\tau_3 d\tau_2 \int_0^\infty b_B b_{\tau_B} b_3 b_\tau \phi_B(x_B, b_B) \)

\[
\left\{ [4(1 + \eta) \eta^2 \phi_S(z) \phi_S(x_3) + 4\sqrt{1 - r^2} \eta \phi_S(z) \phi_S^T(x_3) - \phi_S^2(x_3) - \phi_S^2(x_3) + 4\sqrt{1 - r^2} \eta] \right. \\
\times (x_B + z - x_3 \eta)(\phi_S(z) \phi_S^T(x_3) - \phi_S^2(x_3)) \alpha_s(t_2) h_{g_1}(\alpha_{g_1}, \beta_{g_1}, b_B, b_3) \exp[-S_B(t_{g_1}) - S(t_{g_1}) - S_3(t_{g_1})] \\
+ [4r(3 - 2 \eta) + r_\tau] \phi_S(z) \phi_S^T(x_3) - 4\sqrt{1 - r^2} \eta (x_B - x_3 \eta) \phi_S^2(x_3) - \phi_S^2(x_3) + 16\sqrt{1 - r^2} \eta r \phi_S^2(x_3) - \phi_S^2(x_3)] \\
\times \alpha_s(t_1) h_{g_1}(\alpha_{h_1}, \beta_{h_1}, b_B, b_3) \exp[-S_B(t_{h_1}) - S(t_{h_1}) - S_3(t_{h_1})] \right\} 
\]

\[ M^{LR}_g = -\frac{4\pi C_F M^4_b}{\sqrt{2 N_c}} \int_0^1 d\tau_B d\tau_3 d\tau_2 \int_0^\infty b_B b_{\tau_B} b_3 b_\tau \phi_B(x_B, b_B) \)

\[
\left\{ [-4r(x_B + x_3 \eta - z - x_3 \eta)] \phi_S(z) \phi_S^T(x_3) + 8r \eta \phi_S(z) \phi_S^T(x_3) + 4\sqrt{1 - r^2} \eta (x_B + x_3 \eta + \eta) \right. \\
\times \phi_S(x_3) \phi_S^T(x_3) - \phi_S^2(x_3) + 8\sqrt{1 - r^2} \eta \phi_S(z) \phi_S^T(x_3) + 4\sqrt{1 - r^2} \eta \phi_S^2(x_3) - \phi_S^2(x_3)) \alpha_s(t_2) h_{g_1}(\alpha_{g_1}, \beta_{g_1}, b_B, b_3) \exp[-S_B(t_{g_1}) - S(t_{g_1}) - S_3(t_{g_1})] \\
+ [4r(x_B + x_3 \eta - z - x_3 \eta + \eta)] \phi_S(z) \phi_S^T(x_3) - 4\sqrt{1 - r^2} \eta \phi_S(z) \phi_S^T(x_3) - \phi_S^2(x_3) + 8\eta r \phi_S(z) \phi_S^T(x_3) \\
\times \alpha_s(t_1) h_{g_1}(\alpha_{h_1}, \beta_{h_1}, b_B, b_3) \exp[-S_B(t_{h_1}) - S(t_{h_1}) - S_3(t_{h_1})] \right\} 
\]

\[ M^{SP}_g = \frac{8\pi C_F M^4_b}{\sqrt{2 N_c}} \int_0^1 d\tau_B d\tau_3 d\tau_2 \int_0^\infty b_B b_{\tau_B} b_3 b_\tau \phi_B(x_B, b_B) \)

\[
\left\{ [2\eta (x_B + x_3 \eta - z - x_3 \eta)] \phi_S(z) \phi_S(x_3) + 2\sqrt{1 - r^2} \eta (x_B - z - x_3 \eta) \phi_S^T(x_3) - \phi_S^2(x_3) \phi_S^T(x_3) \right. \\
+ 2\sqrt{1 - r^2} \eta (x_B - z - x_3 \eta) \phi_S^2(x_3) \phi_S^T(x_3) - \phi_S^2(x_3) \phi_S^T(x_3) \right. \\
\times \phi_S(x_3) \phi_S^T(x_3) - \phi_S^2(x_3) \phi_S^T(x_3) - 2\sqrt{1 - r^2} \eta (x_B - x_3 \eta + \eta) \phi_S^2(x_3) - \phi_S^2(x_3) \phi_S^T(x_3) - \phi_S^2(x_3) - \phi_S^2(x_3) \phi_S^T(x_3) \right. \\
\times \alpha_s(t_1) h_{h_1}(\alpha_{h_1}, \beta_{h_1}, b_B, b_3) \exp[-S_B(t_{h_1}) - S(t_{h_1}) - S_3(t_{h_1})] \right\} 
\]

In Fig. 1 (a2) and (b2), we can see the meson pair will be factorized out, when taking the \((V - A)(V - A)\) and \((V - A)(V + A)\) currents into account, the S-wave meson pair cannot be emitted because of the charge conjugation invariance, so

\[ F_a^{LL} = F_a^{LR} = 0 \] (A.13)

\[ F_a^{SP} = 8\pi C_F \sqrt{\eta} M^4_b r \int_0^1 d\tau_B d\tau_3 d\tau_2 \int_0^\infty b_B b_{\tau_B} b_3 b_\tau \phi_B(x_B, b_B) \)

\[
\left\{ [2\eta (r_B - 2) \phi_S(x_3) - 2r (1 + \eta + x_3 \eta) \phi_S^T(x_3) - \phi_S^2(x_3)] - 4r \phi_S^T(x_3) + 8r \phi_S^2(x_3) \right. \\
\times \alpha_s(t_2) h_{g_2}(\alpha_{g_2}, \beta_{g_2}, b_B, b_3) \exp[-S_B(t_{g_2}) - S(t_{g_2}) - S_3(t_{g_2})] \\
+ [4r (x_B - x_3 \eta)] \phi_S^2(x_3) + 2\eta \phi_S(x_3) \alpha_s(t_2) h_{h_2}(\alpha_{h_2}, \beta_{h_2}, b_B, b_3) \exp[-S_B(t_{h_2}) - S(t_{h_2}) - S_3(t_{h_2})] \right\} 
\] (A.14)

The nonfactorizable diagrams in Fig. 1 (c2) and (d2) yield

\[ (1)(V - A)(V - A) \]
\[ M^{LL}_{e e} = \frac{4\pi e F M_{B}^4}{\sqrt{2N_c}} \int_{0}^{1} dx_B dz dx_3 \int_{0}^{\infty} b_B db_B bdb_B(x_B, b_B)\phi_S(z) \]
\[ \{ [4\sqrt{1 - r^2}] (z - x_B - \eta)\phi_S(x_3) + 4r(x_3 \bar{\eta} - \eta - \bar{\eta}) (\phi_S^T(x_3) - \phi_S^T(x_3)) + 8r(1 - x_B \bar{\eta})\phi_S^T(x_3) \}
\times \alpha_s(t_2) h_{c_2}(\alpha_c, \beta_c, b, b_B) \exp[-S_B(t_2) - S(t_2) - S_3(t_2)] \]
\[ + [4\sqrt{1 - r^2}] (\eta - \bar{\eta})\phi_S(x_3) + 4r(x_3 \bar{\eta} - \eta - \bar{\eta}) (\phi_S^T(x_3) - \phi_S^T(x_3)) + 8r(1 - x_B \bar{\eta})\phi_S^T(x_3) \}
\times \alpha_s(t_2) h_{d_2}(\alpha_d, \beta_d, b, b_B) \exp[-S_B(t_2) - S(t_2) - S_3(t_2)] \} \]
\[ (2)(V - A)(V + A) \]

\[ M^{LR}_{e e} = \frac{4\pi e F M_{B}^4}{\sqrt{2N_c}} \int_{0}^{1} dx_B dz dx_3 \int_{0}^{\infty} b_B db_B bdb_B(x_B, b_B) \]
\[ \{ [4\sqrt{1 - r^2}] (z - x_B - \eta)\phi_S(x_3) - 2r(\eta(x_B - z) + \eta - \bar{\eta})(\phi_S^T(x_3) - \phi_S^T(x_3)) + 4r(1 - r^2)\eta \phi_S^T(x_3) \}
\times \phi_S^T(z)(\phi_S^T(x_3) - \phi_S^T(x_3) - 4\sqrt{1 - r^2} \eta \phi_S^T(x_3) - \phi_S^T(x_3)) - 4\sqrt{1 - r^2} \eta \phi_S^T(x_3) = \phi_S(z) - \phi_S(z)) \]
\times \phi_S(x_3)(\phi_S^T(x_3) - \phi_S^T(x_3) - 4\sqrt{1 - r^2} \eta \phi_S^T(x_3) - \phi_S^T(x_3)) - 4\sqrt{1 - r^2} \eta \phi_S^T(x_3) = \phi_S(z) - \phi_S(z)) \]
\times \phi_S^T(x_3)(\phi_S^T(x_3) - \phi_S^T(x_3) - 4\sqrt{1 - r^2} \eta \phi_S^T(x_3) - \phi_S^T(x_3)) - 4\sqrt{1 - r^2} \eta \phi_S^T(x_3) = \phi_S(z) - \phi_S(z)) \]
\times \alpha_s(t_2) h_{d_2}(\alpha_d, \beta_d, b, b_B) \exp[-S_B(t_2) - S(t_2) - S_3(t_2)] \} \]
\[ (3) (S - P)(S + P) \]

\[ M^{SP}_{e e} = -\frac{8\pi e F M_{B}^4}{\sqrt{2N_c}} \int_{0}^{1} dx_B dz dx_3 \int_{0}^{\infty} b_B db_B bdb_B(x_B, b_B)\phi_S(z) \]
\[ \{ [2\sqrt{1 - r^2}](\eta + \eta)\phi_S(x_3) + 2r(\eta(x_B - z) + \eta - \bar{\eta})(\phi_S^T(x_3) - \phi_S^T(x_3)) + 4r(1 - r^2)\eta \phi_S^T(x_3) \}
\times \alpha_s(t_2) h_{c_2}(\alpha_c, \beta_c, b, b_B) \exp[-S_B(t_2) - S(t_2) - S_3(t_2)] \]
\[ + [2\sqrt{1 - r^2}](\eta + \eta)\phi_S(x_3) + 2r(\eta(x_B - z) + \eta - \bar{\eta})(\phi_S^T(x_3) - \phi_S^T(x_3)) + 4r(1 - r^2)\eta \phi_S^T(x_3) \}
\times \alpha_s(t_2) h_{d_2}(\alpha_d, \beta_d, b, b_B) \exp[-S_B(t_2) - S(t_2) - S_3(t_2)] \} \]
\[ \text{The contributions of the factorizable annihilation diagrams in Fig. 1 (c2) and (f2) are} \]
\[ (1)(V - A)(V + A) \]

\[ F^{LL}_{e e} = 2\pi e F f_B M_{B}^4 \int_{0}^{1} dz dx_3 \int_{0}^{\infty} bdb_B bdb_B \]
\[ \{ [-2\sqrt{1 - r^2}] \phi_S^T(x_3)\phi_S(z) + 4\sqrt{1 - r^2} \eta \phi_S^T(x_3)\phi_S^T(z) + 4\sqrt{1 - r^2} \eta \phi_S^T(x_3)\phi_S(z) \}
\times \alpha_s(t_2) h_{c_2}(\alpha_c, \beta_c, b, b_B) \exp[-S_B(t_2) - S(t_2) - S_3(t_2)] \]
\[ F_c^{'SP} = -4\pi C_F f_B M_B^4 \int_0^1 dz dx_3 \int_0^\infty b db db_b ddb_b \]

\[
\left\{ -2\sqrt{(1-r^2)}\bar{\eta}\bar{\phi} S(x_3)(\phi_S(z) - \phi_S(z)) - 4\sqrt{(1-r^2)}\eta\bar{\phi} S(x_3)\phi_S'(z) + 4(1+\eta)\bar{\phi} S(x_3)\phi_S'(z) \right\}
\times \alpha_s(t_2) h_{c_2}(\alpha_{t_2}, \beta_{c_2}, b, b) \exp[-S_3(t_{c_2}) - S(t_{c_2})]S_i(z)
\]

\[ + [-4\sqrt{(1-r^2)}\eta\bar{\phi} S(x_3)\phi_S'(z) + 2r(x_3\bar{\eta} + 2\eta)\phi S(x_3)(\phi_S(x_3) - \phi_T(x_3)) + 4(1-\eta)\phi S(x_3)\phi_T'(x_3)]
\times \alpha_s(t_3) h_{f_2}(\alpha_{t_2}, \beta_{f_2}, b, b) \exp[-S_3(t_2) - S(t_2)]S_1(x_3) \}

The contributions of the nonfactorizable annihilation diagrams in Fig. 1 (g2) and (h2) are

1. \((V - A)(V - A)\)

\[ M_g^{LL} = -4\pi C_F M_B^4 \int_0^1 dz dx_3 \int_0^\infty b db db_b ddb_b \phi_B(x_3, b_B) \]

\[
\left\{ 4\bar{\eta}((2-r)\eta + x_3\bar{\eta} - x_B)\phi S(z)\phi S(x_3) + 4\sqrt{(1-r^2)}\eta r(x_3\bar{\eta} + \eta - x_B)(\phi_T(x_3)\phi_S'(z))
\right.
\]

\[
- \phi_S(x_3)\phi_S'(z) + 4\sqrt{(1-r^2)}\eta r(x_3\bar{\eta} - \eta + x_B)(\phi_T(x_3)\phi_S'(z) - \phi_S(x_3)\phi_S'(z))
\]

\[
\times \alpha_s(t_2) h_{c_2}(\alpha_{t_2}, \beta_{c_2}, b, b) \exp[-S_3(t_{c_2}) - S(t_{c_2})]S_i(z)
\]

\[ + [4(2-r)\eta r(x_B - z - \bar{\eta}x_3)(\phi_T(x_3)\phi_S'(z))
\]

\[
- \phi_S(x_3)\phi_S'(z) - 4\sqrt{(1-r^2)}\eta r(x_B + z - \bar{\eta}x_3)(\phi_S'(x_3)\phi_S(z) - \phi_S(x_3)\phi_S'(z)) + 16\sqrt{(1-r^2)}\eta r
\]

\[
\times \phi_S(x_3)\phi_S'(z)) \alpha_s(t_3) h_{f_2}(\alpha_{t_3}, \beta_{f_2}, b, b) \exp[-S_3(t_{f_2}) - S(t_{f_2})]S_1(x_3)] \}

2. \((V - A)(V + A)\)

\[ M_g^{LR} = -4\pi C_F M_B^4 \int_0^1 dz dx_3 \int_0^\infty b db db_b ddb_b \phi_B(x_3, b_B) \]

\[
\left\{ 4\sqrt{(1-r^2)}\eta \bar{\eta} \phi S(x_3)(\phi_S'(z) - \phi_S'(z)) - 8\sqrt{(1-r^2)}\eta r(x_B - \eta - x_3\bar{\eta})\phi S(x_3)\phi_S'(z)
\right.
\]

\[
+ 4\sqrt{(1-r^2)}\eta r(x_B - \eta - x_3\bar{\eta})\bar{\phi} S(x_3)\phi S'(z) \alpha_s(t_2) h_{c_2}(\alpha_{t_2}, \beta_{c_2}, b, b)
\]

\[
\times \alpha_s(t_3) h_{f_2}(\alpha_{t_3}, \beta_{f_2}, b, b) \exp[-S_3(t_{f_2}) - S(t_{f_2})] + 4\sqrt{(1-r^2)}\eta r(x_B + \eta - x_3\bar{\eta})\phi S(x_3)(\phi_S'(z) - \phi_S'(z)) - 4\sqrt{(1-r^2)}\eta r(x_3\bar{\eta})
\]

\[
\times (x_B - \eta - x_3\bar{\eta})\phi S(x_3)\phi S'(z) + 8r \phi S(x_3)\phi S'(z) + 4r(1+\eta)\phi S(z)(\phi_S'(x_3) + \phi_T'(x_3))]
\]

\[
\times \alpha_s(t_2) h_{f_2}(\alpha_{t_2}, \beta_{f_2}, b, b) \exp[-S_3(t_{f_2}) - S(t_{f_2})]S_1(x_3)] \}

3. \((S - P)(S + P)\)

\[ M_g^{SP} = 8\pi C_F M_B^4 \int_0^1 dz dx_3 \int_0^\infty b db db_b ddb_b \phi_B(x_3, b_B) \]

\[
\left\{ -2\sqrt{(1-r^2)}\phi S(x_3)\phi S(z) + 2\sqrt{(1-r^2)}\eta r(x_3\bar{\eta} + x_B - \eta)(\phi_T(x_3)\phi_S'(z) - \phi_S(x_3)\phi_T'(z))
\right.
\]

\[
+ 2\sqrt{(1-r^2)}\eta r(-x_3\bar{\eta} + x_B - \eta)(\phi_T(x_3)\phi_S'(z) - \phi_S(x_3)\phi_T'(z)) \alpha_s(t_2) h_{c_2}(\alpha_{t_2}, \beta_{c_2}, b, b)
\]

\[
\times \alpha_s(t_3) h_{f_2}(\alpha_{t_3}, \beta_{f_2}, b, b) \exp[-S_3(t_{f_2}) - S(t_{f_2})] + [2\sqrt{(1-r^2)}\eta r(x_B + z - \bar{\eta}x_3)(\phi_T(x_3)\phi_S'(z) - \phi_S(x_3)\phi_T'(z))]
\]

\[
\times \alpha_s(t_2) h_{f_2}(\alpha_{t_2}, \beta_{f_2}, b, b) \exp[-S_3(t_{f_2}) - S(t_{f_2})]S_1(x_3)] \}

The hard functions \( h_i \) are derived from the Fourier transform with \( i = (a_1, ..., h_2) \), whose specific expression are
\[ h_i(\alpha, \beta, b_1, b_2) = h_1(\alpha, b_1) \times h_2(\beta, b_1, b_2), \]

\[
h_1(\alpha, b_1) = \begin{cases} 
K_0(\sqrt{\alpha} b_1), & \alpha > 0, \\
K_0(i \sqrt{-\alpha} b_1), & \alpha < 0, 
\end{cases} 
\]

\[
h_2(\beta, b_1, b_2) = \begin{cases} 
\theta(b_1 - b_2) I_0(\sqrt{\beta} b_2) K_0(\sqrt{\beta} b_1) + (b_1 \leftrightarrow b_2), & \beta > 0, \\
\theta(b_1 - b_2) I_0(i \sqrt{-\beta} b_2) K_0(i \sqrt{-\beta} b_1) + (b_1 \leftrightarrow b_2), & \beta < 0, 
\end{cases} 
\]

with the Bessel function \( J_0 \), the modified Bessel functions \( K_0 \) and \( I_0 \). The expressions for \( \alpha \) and \( \beta \) in the hard functions are

\[
\alpha_{(a1,b1)} = \beta_{(c1,d1)} = M_B^2 x_3 (1 - r^2)(x_B - \eta), \]

\[
\beta_{a1} = M_B^2 (1 - \overline{\eta}(z + r^2 \overline{\eta})), \\
\beta_{b1} = M_B^2 (1 - r^2)(x_B - \eta), \\
\alpha_{c1} = M_B^2 (\overline{\eta}(1 - r^2) + r^2 x_3)(x_B - \eta - x_3 \overline{\eta}), \\
\alpha_{d1} = M_B^2 (r^2(z - x_3) + \overline{\eta}(x_3 \overline{\eta} - \overline{\eta} B)), \\
\beta_{f1} = M_B^2 (z(1 - r^2) + r^2 \overline{\eta}_3)(x_B - \overline{\eta} B), \\
\alpha_{h1} = M_B^2 (1 + (\overline{\eta}(1 - r^2) + r^2 x_3)(x_B - \eta - x_3 \overline{\eta})), \\
\beta_{f1} = M_B^2 (z(1 - r^2) - 1), \\
\alpha_{c2} = M_B^2 (\overline{\eta}(1 - r^2) + r^2 x_3)(x_B - \eta - x_3 \overline{\eta}), \\
\alpha_{d2} = M_B^2 (1 - x_3 \overline{\eta})(r^2 x_3 + z(1 - r^2) - \overline{\eta} B), \\
\beta_{f2} = M_B^2 (1 - r^2 \overline{\eta}_3)(-\eta - x_3 \overline{\eta}), \\
\alpha_{g2} = M_B^2 (\overline{\eta}(1 - r^2) + r^2 x_3)(x_B - \eta - x_3 \overline{\eta}), \\
\alpha_{h2} = M_B^2 (1 + (z(1 - r^2) + r^2 \overline{\eta}_3)(x_B - \overline{\eta} B))
\]

The hard scales \( t_i(i = a1, \ldots, h2) \), which are taken to remove the large logarithmic radiative corrections, are given by

\[
t_{a1} = \text{Max} \left\{ \sqrt{\beta_{a1}}, 1/b_B, 1/b \right\}, \\
t_{b1} = \text{Max} \left\{ \sqrt{\beta_{b1}}, 1/b_B, 1/b \right\}, \\
t_{c1} = \text{Max} \left\{ \sqrt{\beta_{c1}}, \sqrt{\beta_{b1}}, 1/b_B, 1/b \right\}, \\
t_{d1} = \text{Max} \left\{ \sqrt{\alpha_{d1}}, \sqrt{\beta_{b1}}, 1/b_B, 1/b \right\}, \\
t_{e1} = \text{Max} \left\{ \sqrt{\beta_{e1}}, 1/b_B, 1/b \right\}, \\
t_{f1} = \text{Max} \left\{ \sqrt{\beta_{f1}}, 1/b_B, 1/b \right\}, \\
t_{g1} = \text{Max} \left\{ \sqrt{\alpha_{g1}}, \sqrt{\beta_{g1}}, 1/b_B, 1/b \right\}, \\
t_{h1} = \text{Max} \left\{ \sqrt{\alpha_{h1}}, \sqrt{\beta_{h1}}, 1/b_B, 1/b \right\}, \\
t_{a2} = \text{Max} \left\{ \sqrt{\beta_{a2}}, 1/b_B, 1/b \right\}, \\
t_{b2} = \text{Max} \left\{ \sqrt{\beta_{b2}}, 1/b_B, 1/b \right\}, \\
t_{c2} = \text{Max} \left\{ \sqrt{\alpha_{c2}}, \sqrt{\beta_{c2}}, 1/b_B, 1/b \right\}, \\
t_{d2} = \text{Max} \left\{ \sqrt{\alpha_{d2}}, \sqrt{\beta_{d2}}, 1/b_B, 1/b \right\}, \\
t_{e2} = \text{Max} \left\{ \sqrt{\beta_{e2}}, 1/b_B, 1/b \right\}, \\
t_{f2} = \text{Max} \left\{ \sqrt{\beta_{f2}}, 1/b_B, 1/b \right\}, \\
t_{g2} = \text{Max} \left\{ \sqrt{\alpha_{g2}}, \sqrt{\beta_{g2}}, 1/b_B, 1/b \right\}, \\
t_{h2} = \text{Max} \left\{ \sqrt{\alpha_{h2}}, \sqrt{\beta_{h2}}, 1/b_B, 1/b \right\}.
\]
The Sudakov exponents are defined as

\[ S_B = s(x_B \rho_{1}^+, \rho_B) + \frac{5}{3} \int_{1/b_B}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})) , \]

\[ S = s(zp^+, b) + s(\bar{z}p^+, \bar{b}) + 2 \int_{1/b}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})) , \]

\[ S_3 = s(x_3p_3^-, b_3) + s(\bar{x}_3p_3^-, \bar{b}_3) + 2 \int_{1/b_3}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})) , \]  

with the anomalous dimension of the quark \( \gamma_q = -\alpha_s/\pi \), and \( s(Q, b) \) is the Sudakov factor, which can be found in Ref. [61]. Meanwhile, the threshold resummation factor \( S_t(x) \) is taken from Ref. [62],

\[ S_t(x) = \frac{2^{1+2c} \Gamma\left(\frac{3}{2} + c\right)}{\sqrt{\pi} \Gamma(1+c)} |x(1-x)|^c , \]  

with the parameter \( c = 0.3 \).

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