Abstract. In this paper we settle Fock-Goncharov’s duality conjecture for cluster varieties associated to their moduli spaces of G-local systems on a punctured surface G with boundary data, when G is a group of type A_2, namely SL_3 and PGL_3. Based on Kuperberg’s A_2-web, we introduce the notion of A_2-laminations on G defined as certain A_2-webs with integer weights. We introduce coordinate systems for A_2-laminations, and show that A_2-laminations satisfying a congruence property are geometric realizations of the tropical integer points of the cluster X-moduli space X_{SL_3,G}. Per each such A_2-lamination, we construct a regular function on the cluster X-moduli space X_{PGL_3,G}. We show that these functions form a basis of the rank of all regular functions. For a proof, we develop SL_3 classical trace map for any triangulated bordered surface with marked points, and a state-sum formula for it. Moreover, we propose generalization to higher ranks.

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1. Introduction

1.1. Background on Fock-Goncharov duality conjecture. Let G be a punctured surface, i.e. a compact oriented surface of genus g ≥ 0 minus n ≥ 1 punctures. We say G is triangulable if g = 0.
and \( n \geq 4 \), or if \( g \geq 1 \) and \( n \geq 1 \) (so we exclude the three-punctured sphere; see §2.1). Let \( G \) be an algebraic group. A \( G \)-local system \( \mathcal{L} \) on \( \mathcal{S} \) can be thought of as a right principal \( G \)-bundle on \( \mathcal{S} \) together with a flat \( G \)-connection. An isomorphism class of \( \mathcal{L} \) is captured by the monodromy representation \( \pi_1(\mathcal{S}) \rightarrow G \) which is a group homomorphism defined up to conjugation in \( G \). Hence the moduli space \( \mathcal{L}_{G,\mathcal{S}} \) of \( G \)-local systems on \( \mathcal{S} \) is identified with the \( G \)-representation variety for \( \mathcal{S} \)

\[
\mathcal{L}_{G,\mathcal{S}} = \text{Hom}(\pi_1(\mathcal{S}), G)/G,
\]

which is a stack. The ring of regular functions \( \mathcal{O}(\mathcal{L}_{G,\mathcal{S}}) \) can be viewed as the coordinate ring of the \( G \)-character variety for \( \mathcal{S} \). Fock and Goncharov defined \cite{FG06} two related moduli stacks

\[ \mathcal{A}_{G,\mathcal{S}} \quad \text{and} \quad \mathcal{I}_{G,\mathcal{S}}, \]

where \( \mathcal{A}_{G,\mathcal{S}} \) parametrizes the decorated \( G \)-local systems, while \( \mathcal{I}_{G,\mathcal{S}} \) parametrizes the framed \( G \)-local systems. To briefly recall the definitions, consider \( \mathcal{S} \) as being given by a compact oriented surface minus \( n \) open discs, so that punctures now become boundary circle components. Choose a Borel subgroup \( B \) of \( G \), and let \( B := G \times B \) be the flag variety. For a \( G \)-local system \( \mathcal{L} \) on \( \mathcal{S} \), let \( \mathcal{L}_B := \mathcal{L} \times_G B \) be the associated flag bundle. A framing on \( \mathcal{L} \) is the choice \( \beta \) of a flat section of the restriction \( \mathcal{L}_B|_{\partial \mathcal{S}} \) of \( \mathcal{L}_B \) to the boundary of \( \mathcal{S} \).

**Definition 1.1** (\cite{FG06}). A framed \( G \)-local system on \( \mathcal{S} \) is the pair \((\mathcal{L}, \beta)\) of a \( G \)-local system \( \mathcal{L} \) on \( \mathcal{S} \) together with a framing on \( \mathcal{L} \). Let \( \mathcal{I}_{G,\mathcal{S}} \) be the moduli stack parametrizing framed \( G \)-local systems on \( \mathcal{S} \).

For our case, define a decorated \( G \)-local system and its moduli space \( \mathcal{A}_{G,\mathcal{S}} \), analogously, with \( B \) being replaced by the maximal unipotent subgroup \( U := [B, B] \). For the case when \( G \) is of type \( A_1 \), Fock and Goncharov showed \cite{FG06} that \( \mathcal{A}_{SL_2,\mathcal{S}} \) and \( \mathcal{I}_{PGL_2,\mathcal{S}} \) recover the decorated Teichmüller space and the enhanced (or ‘holed’) Teichmüller space of the surface \( \mathcal{S} \) respectively. For higher rank groups \( G \), these spaces can be viewed as providing models of higher Teichmüller spaces.

The present paper concerns the case when \( G \) is of type \( A_2 \), or more precisely the spaces \( \mathcal{A}_{SL_3,\mathcal{S}} \) and \( \mathcal{I}_{PGL_3,\mathcal{S}} \). Pivotal in the study of these spaces are Fock-Goncharov’s special coordinate systems \cite{FG06} (1.1)

\[
\mathcal{A}_{SL_3,\mathcal{S}} \rightarrow (\mathbb{G}_m)^N \quad \text{and} \quad \mathcal{I}_{PGL_3,\mathcal{S}} \rightarrow (\mathbb{G}_m)^N
\]

which are birational maps, associated to each choice of an ideal triangulation \( \Delta \) of \( \mathcal{S} \), which is a maximal collection of mutually disjoint simple arcs in \( \mathcal{S} \) running between punctures of \( \mathcal{S} \) (when \( \mathcal{S} \) is viewed as a punctured surface again), dividing \( \mathcal{S} \) into ideal triangles (we assume the valence of \( \Delta \) at each puncture is at least 3). Remarkable fact is that for two ideal triangulations, the coordinate change maps are positive rational, not involving any subtraction, and moreover, they follow patterns called cluster mutations appearing in the theory of cluster algebras and cluster varieties \cite{FZ07,FZ10}. To elaborate, for an ideal triangulation \( \Delta \), define the 3-triangulation \( Q_\Delta \) of \( \Delta \) as a quiver obtained by gluing the quivers associated to triangles of \( \Delta \), as depicted in Fig.1. So \( Q_\Delta \) will have two nodes lying on each edge of \( \Delta \) and one node lying in the interior of each triangle of \( \Delta \), while the arrows are as in Fig.1 for each triangle. Let

\[
\mathcal{V}(Q_\Delta) = \text{the set of all nodes of the 3-triangulation quiver } Q_\Delta \text{ of the triangulation } \Delta.
\]

The Fock-Goncharov coordinates (eq.(1.1)), for each of \( \mathcal{A}_{SL_3,\mathcal{S}} \) and \( \mathcal{I}_{PGL_3,\mathcal{S}} \), are enumerated by \( \mathcal{V}(Q_\Delta) \).

![Figure 1. 3-triangulation quiver, for one triangle](image-url)

In the theory of cluster algebras and varieties, there is a certain combinatorial process called the quiver mutation at a node of a quiver, transforming a quiver into another one according to some rule (Def.2.12). When ideal triangulations \( \Delta \) and \( \Delta' \) differ only by one edge \( e \), we say that they are related by a flip, and it is known that the quiver \( Q_{\Delta'} \) can be obtained from \( Q_\Delta \) by a sequence of four quiver mutations, first at the two nodes lying in the edge \( e \) of \( \Delta \), then at the two nodes lying in...
the interiors of the two triangles of $\Delta$ having $e$ as a side (Lem. 2.19). Per each quiver mutation, there are \textit{cluster $A$-mutation} and \textit{cluster $A'$-mutation} which are certain coordinate change formulas for the coordinate functions associated to the nodes (Def. 2.13). Fock and Goncharov \cite{FG06} showed that their coordinates on $\mathcal{A}_{SL_3,\mathcal{E}}$ and on $\mathcal{X}_{PGL_3,\mathcal{E}}$ associated to $\Delta$ and $\Delta'$ indeed transform as the corresponding composition of four cluster mutations (Prop. 2.20). One can summarize this result as having constructed birational equivalences

$$\mathcal{A}_{SL_3,\mathcal{E}} \rightarrow \mathcal{A}_{Q_{\Delta}} \quad \text{and} \quad \mathcal{X}_{PGL_3,\mathcal{E}} \rightarrow \mathcal{X}_{|Q_{\Delta}|}$$

from the moduli spaces $\mathcal{A}_{SL_3,\mathcal{E}}$ and $\mathcal{X}_{PGL_3,\mathcal{E}}$ to the cluster varieties $\mathcal{A}_{Q_{\Delta}}$ and $\mathcal{X}_{|Q_{\Delta}|}$, which are abstract schemes constructed by gluing split algebraic tori $(\mathbb{G}_m)^N = (\mathbb{G}_m)^{Q_{\Delta}}$ along the cluster mutation maps, where $|Q_{\Delta}|$ denotes the quiver mutation equivalence class of $Q_{\Delta}$. One of the important conjectures set out by Fock-Goncharov in \cite{FG06} is the following, which they proposed as a generalization of their SL$_2$-PGL$_2$ result. We first define the necessary notions.

**Definition 1.2.** The cluster coordinate charts of $\mathcal{A}_{SL_3,\mathcal{E}}$ in eq. (1.1) associated to ideal triangulations $\Delta$ of $\mathcal{E}$ are related by positive rational maps, so it makes sense to consider the set $\mathcal{A}_{SL_3,\mathcal{E}}(\mathbb{P})$ of points valued in a semi-field $\mathbb{P}$, which is a set with addition and multiplication, where multiplication makes $\mathbb{P}$ an abelian group, and addition is merely associative, commutative, and distributive for the multiplication. In particular, this applies for the semi-field of tropical integers $\mathbb{Z}^t$, which is $\mathbb{Z}$ as a set, and the tropical addition of $a, b$ is $\max(a, b)$ while the tropical multiplication of $a, b$ is $a + b$.

It is known that, as a set, $\mathcal{A}_{SL_3,\mathcal{E}}(\mathbb{P})$ is in bijection with $\mathbb{P}^n$.

**Definition 1.3.** Let $L(\mathcal{X}_{PGL_3,\mathcal{E}})$ be the ring of all rational functions on $\mathcal{X}_{PGL_3,\mathcal{E}}$ that are regular with respect to all the cluster coordinate charts of $\mathcal{X}_{PGL_3,\mathcal{E}}$ in eq. (1.1) associated to ideal triangulations $\Delta$ of $\mathcal{E}$, i.e. universally Laurent polynomial functions for all ideal triangulations.

**Conjecture 1.4** (Fock-Goncharov duality conjecture for SL$_3$-PGL$_3$; \cite{FG06}). There exists a canonical map

$$\mathbb{I} : \mathcal{A}_{SL_3,\mathcal{E}}(\mathbb{Z}^t) \rightarrow L(\mathcal{X}_{PGL_3,\mathcal{E}})$$

satisfying favorable properties.

A kind of implicit prerequisite conjecture is:

**Conjecture 1.5** (Conjectural geometric model of tropical integer points of $\mathcal{A}_{SL_3,\mathcal{E}}$; \cite{FG06}). There is a natural geometric model for the set $\mathcal{A}_{SL_3,\mathcal{E}}(\mathbb{Z}^t)$.

There has been attempts for Conjecture 1.5 e.g. by Ian Le \cite{L16} (‘higher’ laminations) and by Goncharov-Shen \cite{GS15} (top-dimensional components of ‘surface affine Grassmannian’ stack), but these are not as direct and intuitive as Fock-Goncharov’s answer \cite{FG06} for the SL$_2$ case $\mathcal{A}_{SL_2,\mathcal{E}}(\mathbb{Z}^t)$ by \textit{integral laminations} on $\mathcal{E}$, which are certain collections of simple closed curves in $\mathcal{E}$ with integer weights. More importantly, a good answer must also immediately help answering Conjecture 1.4 but Conjecture 1.4 has been elusive.

In the present paper we settle the above two conjectures to a large extent. In particular, we provide a geometrically intuitive model for $\mathcal{A}_{SL_3,\mathcal{E}}(\mathbb{Z}^t)$, and for each $\ell \in \mathcal{A}_{SL_3,\mathcal{E}}(\mathbb{Z}^t)$ we explicitly construct a universally Laurent function on $\mathcal{X}_{PGL_3,\mathcal{E}}$, and prove some important properties. In fact, $L(\mathcal{X}_{PGL_3,\mathcal{E}})$ is replaced by $\mathcal{O}(\mathcal{X}_{PGL_3,\mathcal{E}})$ and $\mathcal{O}_c(\mathcal{X}_{PGL_3,\mathcal{E}})$ which we believe are more correct target spaces, where

$$\mathcal{O}_c(\mathcal{X}_{PGL_3,\mathcal{E}}) := \text{the ring of all functions on } \mathcal{X}_{PGL_3,\mathcal{E}} \text{ that are regular for all cluster } \mathcal{X}'\text{-charts,}$$

which may be different from $L(\mathcal{X}_{PGL_3,\mathcal{E}})$ as there are cluster $\mathcal{X}'$-charts that are not associated to ideal triangulations $\Delta$ (see §2.3); this better suits the theory of cluster varieties, e.g. we have $\mathcal{O}_c(\mathcal{X}_{PGL_3,\mathcal{E}}) \cong \mathcal{O}(\mathcal{X}_{|Q_{\Delta}|})$.

Meanwhile, there is a more general version of Conjecture 1.4 for cluster varieties for any quiver $Q$ (or more generally for any skew-symmetric integral matrix $\varepsilon$), which was solved in the celebrated paper by Gross-Hacking-Keel-Kontsevich \cite{GHKK18}. They showed that if a quiver $Q$ satisfies certain combinatorial condition, then there exists a canonical map

$$\mathbb{I} : \mathcal{A}_{Q}(\mathbb{Z}^t) \rightarrow \mathcal{O}(\mathcal{X}_{|Q_{\Delta}|})$$

for the cluster varieties $\mathcal{A}_{Q}$ and $\mathcal{X}_{|Q_{\Delta}|}$, satisfying favorable properties. It was shown by Goncharov-Shen \cite{GS18} that the 3-triangulation quiver $Q = Q_{\Delta}$ satisfies the condition of \cite{GHKK18}, hence proving the
existence of a \( SL_3 \)-PGL\(_3 \) duality map as being sought for in Conjecture 1.4. However, the construction in \cite{GHKK18} is quite combinatorial and uniform for all quivers \( Q \), not giving special geometric meaning for quivers coming from surfaces. Actual computation of their functions \( I(t) \in \mathcal{O}(\mathcal{Z}_Q) \) require enormous amount of combinatorics in large dimensional Euclidean spaces, and to find a geometric meaning of the resulting (universally) Laurent polynomials is a big important challenge. To the knowledge of the author, even for the \( SL_2 \)-PGL\(_2 \) case, i.e. when \( Q \) is the 2-triangulation \cite{PG06} (Def. 6.15) of an ideal triangulation \( \Delta \) of \( \mathcal{S} \), Gross-Hacking-Keel-Kontsevich’s duality map \cite{GHKK18} has not been proved to match Fock-Goncharov’s map \( I \) \cite{FG06}, except for a couple of surfaces \( \mathcal{S} \).

On the other hand, our duality map constructed in the present paper is down to earth. Of course a very natural conjecture would then be to compare our map with Gross-Hacking-Keel-Kontsevich’s, but that will be quite difficult.

1.2. \( A_2 \)-webs and \( A_2 \)-laminations. One of the major objects to tackle is \( \mathcal{O}(\mathcal{Z}_{PGL_3,\mathcal{S}}) \), the ring of regular functions on the moduli stack \( \mathcal{Z}_{PGL_3,\mathcal{S}} \). We will see step by step in the present paper how this is closely related to \( \mathcal{O}(\mathcal{Z}_{SL_3,\mathcal{S}}) \), the coordinate ring of the \( SL_3 \)-character variety for \( \mathcal{S} \), which has been studied in relation to higher rank versions of so-called skein algebras of the surface \( \mathcal{S} \). We first briefly recall the \( SL_2 \)-PGL\(_2 \) story. For each unoriented closed curve \( \gamma \) in \( \mathcal{S} \), there is a natural regular function \( f_\gamma \) on \( \mathcal{Z}_{SL_2,\mathcal{S}} \) given by the \textit{trace of monodromy} along \( \gamma \), namely, whose value at the point of \( \mathcal{Z}_{SL_2,\mathcal{S}} \) represented by a monodromy homomorphism \( \rho: \pi_1(\mathcal{S}) \to SL_2 \) is defined as
\[
(1.2) \quad f_\gamma([\rho]) = \text{tr}(\rho(\gamma)),
\]
where in the right hand side \( \gamma \) is given an arbitrary orientation. Then \( f_\gamma \) is well-defined because the trace is invariant under conjugation, and under taking inverse in \( SL_2 \). Due to the matrix identity \( (\text{tr}A)(\text{tr}B) = \text{tr}(AB) + \text{tr}(AB^{-1}) \) in \( SL_2 \), these trace-of-monodromy functions \( f_\gamma \) satisfy the relation
\[
f_\gamma f_\gamma = f_\gamma_1 + f_\gamma_2, \quad \text{when } \gamma, \gamma', \gamma_1, \gamma_2 \text{ look like } \begin{array}{c} \gamma \\gamma' \\gamma_1 \\gamma_2 \end{array} \begin{array}{c} \gamma_1 \\gamma_2 \end{array} \text{ in a small disc.}
\]
Since a \textit{commutative Kauffman bracket skein algebra} is a free associative algebra generated by closed curves in \( \mathcal{S} \) up to isotopies mod out by the \textit{skein relations} which model the above relation, one obtains an algebra homomorphism from the skein algebra to \( \mathcal{O}(\mathcal{Z}_{SL_3,\mathcal{S}}) \), which had been known to be an isomorphism \cite{P76} \cite{S01}. By using skein relations repeatedly, any element \( \mathcal{O}(\mathcal{Z}_{SL_3,\mathcal{S}}) \) can be expressed as linear combination of products \( f_{\gamma_1} \cdot \ldots \cdot f_{\gamma_m} \), where \( \gamma_1, \ldots, \gamma_m \) are mutually disjoint simple loops, hence forming a multicurve or an example of a \textit{lamination}. For our \( SL_3 \)-PGL\(_3 \) situation, let \( \gamma \) be an oriented closed curve in \( \mathcal{S} \). Still, the trace of monodromy function \( f_\gamma \) on \( \mathcal{Z}_{SL_3,\mathcal{S}} \) is defined by the formula eq. (1.2), and it is known \cite{P76} \cite{S01} that they generate \( \mathcal{O}(\mathcal{Z}_{SL_3,\mathcal{S}}) \), but the algebraic relations among them are different from the case of \( SL_2 \). In particular, oriented multicurves do not yield a basis \( \mathcal{O}(\mathcal{Z}_{SL_3,\mathcal{S}}) \), and one needs to consider certain tri-valent oriented graphs on the surface, called the \textit{webs}, first studied by Kuperberg \cite{K96} in the context of invariant theory, which is kind of directly relevant to our situation which can be viewed as a surface version of invariant theory. Webs for groups of other Dynkin types can be considered (A1 type yielding the unoriented curves), and the corresponding (higher) versions of skein algebras have been studied, notably by Sikora and collaborators \cite{S01} \cite{S05} \cite{SW07}. For our purposes, we take the \( A_2 \)-webs and the corresponding \( A_2 \)-\textit{skein algebras}, which are extensively investigated recently in Frohman-Sikora \cite{FS20} and Higgins \cite{H20}. We only use the commutative versions, which we somewhat simplified for this section.

**Definition 1.6** (\cite{K96} \cite{S01} \cite{SW07} \cite{FS20}). An \( A_2 \)-\textit{web} \( W \) in a punctured surface \( \mathcal{S} \) is a union of any finite number of isotopy classes of oriented loops in \( \mathcal{S} \) and/or oriented tri-valent (connected) graphs such that each tri-valent vertex is either sink or source. The (commutative) \( A_2 \)-\textit{skein algebra} \( S(\mathcal{S}) \) is a free \( \mathbb{Z} \)-module generated by \( A_2 \)-webs mod out by the \( A_2 \)-\textit{skein relations} in Fig 2 where the product of elements of \( S(\mathcal{S}) \) is given by union.

![Diagram](image.png)

**Figure 2.** \( A_2 \)-skein relations, drawn locally (\( \emptyset \) means empty); the regions bounded by 1-gon, 2-gon, 4-gon in (S1), (S2), (S3) are contractible.
It is known [P76] [S01] that there is a ring isomorphism
\[ \Phi : S(S) \to \mathcal{O}(L_{SL_3}, S) \] (1.3)
sending each oriented loop \( \gamma \) to the trace-of-monodromy function \( f_\gamma \). So a basis of \( \mathcal{O}(L_{SL_3}, S) \) can be obtained from a basis of \( S(S) \); there is a nice basis consisting of the so-called non-elliptic \( A_2 \)-webs.

**Definition 1.7** ([K96] [SW07]). An \( A_2 \)-web in a punctured surface \( S \) is non-elliptic if it has no self-intersection other than (possibly) the tri-valent vertices and does not bound a contractible region bounded by a loop or by two or four oriented edges like in (S1), (S2), (S3).

The first main definition of the present paper is the following.

**Definition 1.8** (\( A_2 \)-lamination). An \( A_2 \)-lamination \( \ell \) in a punctured surface \( S \) is a non-elliptic \( A_2 \)-web \( W(\ell) \) in \( S \) together with integer weights on the (connected) components of \( W(\ell) \), subject to the following conditions and equivalence relations:

1. the weight of a component containing a tri-valent vertex is 1;
2. the weight of a component is non-negative unless the component is a peripheral loop, i.e. a small loop surrounding a puncture of \( S \);
3. an \( A_2 \)-lamination with one of the components having weight 0 is equivalent to the \( A_2 \)-lamination with this component removed;
4. an \( A_2 \)-lamination with two of the components being isotopic and with weights \( a, b \) is equivalent to the \( A_2 \)-lamination with one of these two components removed and the weight \( a + b \) given on the other.

Let \( \mathcal{A}_L(S; \mathbb{Z}) \) be the set of all \( A_2 \)-laminations.

The negative weights for peripheral loops will be used to compensate the difference between \( \mathcal{O}(\mathcal{X}) \) and \( \mathcal{O}(\mathcal{Z}) \), like in Fock-Goncharov’s \( SL_2 \)-PGL\(_2 \) duality. Crucial in the study of \( A_2 \)-laminations is a special coordinate system, associated to an ideal triangulation \( \Delta \) of \( S \). We make use of Frohman-Sikora’s coordinates [FS20] and Douglas-Sun’s coordinates [DS20] for non-elliptic \( A_2 \)-webs. Here we only recall Douglas-Sun’s, multiplied by \( \frac{1}{3} \). Let \( \Delta \) be an ideal triangulation of \( S \), and let \( \hat{\Delta} \) be a split ideal triangulation of \( S \) for \( \Delta \), obtained from \( \Delta \) by adding one more arc \( e' \) in \( S \) per each arc \( e \) of \( \Delta \), where \( e' \) is isotopic to but disjoint with \( e \) and such that \( \Delta' \) is a mutually disjoint collection [BW11]. Then \( \hat{\Delta} \) divides \( S \) into ideal triangles and ideal biangles, where a biangle of \( \hat{\Delta} \) is associated to each arc of \( \Delta \), and a triangle \( \ell \) of \( \hat{\Delta} \) to each triangle \( \ell \) of \( \Delta \).

**Figure 4.** Pyramids \( H_d \) in a triangle

**Definition 1.9** ([FS20]). A non-elliptic \( A_2 \)-web \( W \) in a triangulable punctured surface \( S \) is in a canonical position with respect to a split ideal triangulation \( \hat{\Delta} \) of \( S \) for a triangulation \( \Delta \), if
(1) for each triangle \( t \) of \( \Delta \), \( W \cap \hat{t} \) is an \( A_2 \)-web in \( \hat{t} \) that is canonical, i.e., consists of some number of left turn or right turn \( \text{corner arcs} \), each of which connects two distinct sides of \( \hat{t} \), and/or a single degree \( d \) pyramid \( H_d \) for some \( d \in \mathbb{Z} \), where \( H_0 = \emptyset \), and some examples of \( H_d \) for nonzero \( d \)'s are as depicted in Fig.4 (see [DS20] for precise description of \( H_d \));

(2) for each biangle \( B \) of \( \hat{\Delta} \), \( W \cap B \) is an \( A_2 \)-web in \( B \) that is a minimal crossbar web, i.e., when the orientations are forgotten, is homeomorphic to union of some number of simple arcs connecting two distinct sides of \( B \), called strands, and some number of simple arcs connecting two adjacent strands, called crossbars, where intersections of strands and crossbars are transverse double and in the interior of \( B \), such that in between any two adjacent strands there is no consecutive crossbars that form a contractible \( 4 \)-gon as in (S3) of Fig.2, and that under a homeomorphism of the biangle \( B \) to \( \mathbb{R} \times [0,1] \) (the two sides going to \( \mathbb{R} \times \{0\} \) and \( \mathbb{R} \times \{1\} \) each strand is of the form \( \{c\} \times [0,1] \) (i.e., vertical) and each crossbar is of the form \( [c_1,c_2] \times \{a\} \) (i.e., horizontal).

For example, \( \leftarrow \) is a minimal crossbar web in a biangle, with 4 strands and 3 crossbars; as an \( A_2 \)-web, it has 14 vertices, where 6 among them are tri-valent, and 13 edges.

**Definition 1.10** (Douglas-Sun coordinates of non-elliptic \( A_2 \)-webs; [DS20]). Let \( W \) be a non-elliptic \( A_2 \)-web in a triangulable punctured surface \( \mathcal{S} \), with a chosen split ideal triangulation \( \hat{\Delta} \) (for \( \Delta \)). Put \( W \) into a canonical position with respect to \( \hat{\Delta} \) by isotopy. For each triangle \( t \) of \( \Delta \), for the nodes \( v \) of the 3-triangulation quiver \( Q_\Delta \) living in \( t \), define \( a_v(W) \) as the sum of coordinates at \( v \) of the components of \( W \cap \hat{t} \), as defined in Fig.5.

![Figure 5. Tropical coordinates for elementary \( A_2 \)-webs in a triangle](image)

**Definition 1.11** (tropical coordinates for \( A_2 \)-laminations). Let \( \ell \) be an \( A_2 \)-lamination in a triangulable punctured surface \( \mathcal{S} \), represented by an \( A_2 \)-web \( W \). Let \( W_1, \ldots, W_m \) be the components of \( W \), with weights \( k_1, \ldots, k_m \) respectively. For each node \( v \in V(Q_\Delta) \), define the tropical coordinate of the \( A_2 \)-lamination \( \ell \) at \( v \) as the weighted sum of Douglas-Sun coordinates \( a_v(W) := \sum_{i=1}^{m} k_i a_v(W_i) \).

We also give explicit formulas for \( a_v(\ell) \) in terms of Frohman-Sikora coordinates of \( A_2 \)-webs. We prove:

**Proposition 1.12** (well-definedness and image of tropical coordinates of \( A_2 \)-laminations; Prop.3.30 Prop.3.34). For an ideal triangulation \( \Delta \) of a punctured surface \( \mathcal{S} \), the tropical coordinate map

\[
a_\Delta : \{ A_2 \text{-laminations in } \mathcal{S} \} \rightarrow (\frac{1}{2}\mathbb{Z})^{V(Q_\Delta)}
\]

is well-defined and is a bijection onto the set of all balanced elements of \( (\frac{1}{2}\mathbb{Z})^{V(Q_\Delta)} \), where an element \( (a_v) \in (\frac{1}{2}\mathbb{Z})^{V(Q_\Delta)} \) is said to be balanced if for each triangle \( t \) of \( \Delta \), if we denote by \( e_1, e_2, e_3 \) the sides of \( t \) and by \( v_{e_1,1}, v_{e_1,2}, v_{e_2,1}, v_{e_2,2}, v_{e_3,1}, v_{e_3,2} \) denote the nodes of \( Q_\Delta \) lying on the sides of \( t \) appearing clockwise this order (with \( v_{e_\alpha,*} \) on \( e_\alpha \)), then \( d_\alpha := \sum_{\alpha=1}^{3} a_{v_{e_\alpha,1}} - \sum_{\alpha=1}^{3} a_{v_{e_\alpha,2}} \) belongs to \( \mathbb{Z} \).

**Proposition 1.13** (tropical coordinates change under tropical \( \mathcal{A} \)-mutations; Prop.3.35). Let \( \Delta, \Delta' \) be ideal triangulations of a punctured surface \( \mathcal{S} \) related by a flip. Then the coordinates \( a_\Delta \) and \( a_{\Delta'} \) are related by the sequence of tropicalized versions of the cluster \( \mathcal{A} \)-mutations relating the cluster \( \mathcal{A} \)-charts of \( \mathcal{A}_{\mathcal{T}_{\Delta} \mathcal{S}} \), associated to \( \Delta, \Delta' \).

Prop.1.13 is a consequence of the corresponding statement for Douglas-Sun coordinates of non-elliptic \( A_2 \)-webs, which will be proved in an upcoming paper of Douglas-Sun [DS2].
Definition 1.14. For an ideal triangulation $\Delta$ of a punctured surface $\mathcal{S}$, an $A_2$-lamination $\ell$ in $\mathcal{S}$ is said to be $\Delta$-congruent if the tropical coordinates $a_v(\ell)$, $v \in V(\mathcal{Q})$, are all integers.

We say $\ell$ is congruent if it is $\Delta$-congruent for all $\Delta$.

Proposition 1.15 (congruence is independent on triangulations; Prop. [3.41]). For any two ideal triangulations $\Delta$ and $\Delta'$ of a punctured surface $\mathcal{S}$, an $A_2$-lamination $\ell$ in $\mathcal{S}$ is $\Delta$-congruent iff it is $\Delta'$-congruent.

Prop. 1.15 would be an easy corollary of Prop 1.13. But since Prop 1.13 depends on an upcoming work [DS2], we provide a proof of Prop 1.15 independent of Prop 1.13. Note that, although the statement of Prop 1.15 itself is of topological and combinatorial nature, our proof of it heavily uses the proof of our main result (Thm. 1.16), which is heavily algebraic. Anyways, consequently we have

Theorem 1.16 (geometric model of tropical integer points of $A_{\Delta}^0(\mathcal{S})$: Thm. [5.39]). Let $\mathcal{S}$ be a triangulable punctured surface. The tropical coordinate maps $k^p$ for ideal triangulations $\Delta$ of $\mathcal{S}$ provide bijections

$$\{ \text{congruent } A_2\text{- laminations in } \mathcal{S} \} \rightarrow \mathbb{Z}^{V(\mathcal{Q})}$$

which, under change of ideal triangulations, transform by sequence of tropical versions of cluster $\mathcal{A}$-mutations for the corresponding cluster $\mathcal{A}$-charts of $A_{\Delta}^0(\mathcal{S})$. Therefore we have the identification

$$\{ \text{congruent } A_2\text{- laminations in } \mathcal{S} \} \leftrightarrow A_{\Delta}^0(\mathcal{S})(\mathbb{Z}^t).$$

This is our solution to Conjecture [1.3]. The full content of Thm. 1.16 depends on Prop 1.13 and even without [DS2] we have at least a weaker version of Thm. 1.16 due to our Prop 1.15.

1.3. $A_2$ duality map: main theorem. For each congruent $A_2$-lamination $\ell$ in $\mathcal{S}$, we should now describe our regular function $\mathbb{I}(\ell)$ on $A_{PGL_3,\mathcal{S}}$. We do this through several steps. First, we let

$$A_{\ell}^0(\mathcal{S};\mathbb{Z}) := \text{the set of all } A_2\text{- laminations in } \mathcal{S} \text{ with non-negative weights.}$$

Since $A_{\ell}^0(\mathcal{S};\mathbb{Z})$ is in bijection with the set of all non-elliptic $A_2$-webs in $\mathcal{S}$, it embeds into the $A_2$-skein algebra $S(\mathcal{S})$ as a basis, hence from eq. 1.13 we get a map

$$A_{\ell}^0(\mathcal{S};\mathbb{Z}) \rightarrow \mathcal{O}(Z_{SL_3,\mathcal{S}})$$

which is injective and whose image forms a basis of $\mathcal{O}(\mathcal{L}_{SL_3,\mathcal{S}})$. Pullback of the natural frame-forgetting regular map

$$F : \mathcal{L}_{SL_3,\mathcal{S}} \rightarrow \mathcal{L}_{SL_3,\mathcal{S}}$$

yields a map

$$F^* : \mathcal{O}(\mathcal{L}_{SL_3,\mathcal{S}}) \rightarrow \mathcal{O}(\mathcal{L}_{SL_3,\mathcal{S}}).$$

The gap between (the image under $F^*$ of) $\mathcal{O}(\mathcal{L}_{SL_3,\mathcal{S}})$ and $\mathcal{O}(\mathcal{L}_{SL_3,\mathcal{S}})$ is filled in by the regular functions on $\mathcal{L}_{SL_3,\mathcal{S}}$ reading the framing data at punctures as follows (as in [FG06]). Namely, for a framed SL3-local system on $\mathcal{S}$, the monodromy along a peripheral loop surrounding a puncture $p$ yields an element of a Borel subgroup of SL3, and by reading the semi-simple part one gets an element of the Cartan group $H := B/U$ of SL3. Choosing B to be the subgroup of all upper triangular matrices, H is isomorphic to the subgroup of all diagonal matrices. Monodromy is defined only up to conjugation, so from monodromy alone we really get an element of $H/W$ where $W$ is the Weyl group. However, the framing data pins down an element of $H$ indeed, and we get a regular map

$$\pi_p : \mathcal{L}_{SL_3,\mathcal{S}} \rightarrow H,$$

for each puncture $p$; we elaborate more on this process in the main text. Hence one obtains

$$\mathcal{O}(\mathcal{L}_{SL_3,\mathcal{S}}) \cong \mathcal{O}(\mathcal{L}_{SL_3,\mathcal{S}}) \otimes \mathcal{O}(H/W^{P}) \mathcal{O}(H^P)$$

as done in [FG06], where $P$ is the set of all punctures of $\mathcal{S}$. In the map $\pi_p$, further reading the three diagonal entries of $H$, we get three monomial regular functions $(\pi_p)_i : \mathcal{L}_{SL_3,\mathcal{S}} \rightarrow \mathbb{G}_m$, $i = 1, 2, 3$, which fill in the gap between the SL3-character variety and the SL3 $\mathcal{L}$-moduli space.
Definition 1.17. For a triangulable surface \( S \), define the map
\[
\mathbb{I}_{\text{SL}_3} : \mathcal{A}_H(S; \mathbb{Z}) \rightarrow \mathcal{O}(\mathcal{X}_{\text{SL}_3}, \mathcal{S})
\]
as follows. For \( \ell \in \mathcal{A}_H(S; \mathbb{Z}) \), write \( \ell = \ell_1 \cup \cdots \cup \ell_m \) as disjoint union of \( A_2 \)-laminations with weighted single-component \( A_2 \)-webs. Let
\[
\mathbb{I}_{\text{SL}_3}(\ell_i) := \begin{cases} 
((\pi_0 p)_1)^{k_i} & \text{if } \ell_i \text{ is a peripheral loop going counterclockwise around } p \text{ with weight } k_i, \\
n^{-k_i} & \text{if } \ell_i \text{ is a peripheral loop going clockwise around } p \text{ with weight } k_i,
\end{cases}
\]
\[F^e(\mathbb{I}_{\text{SL}_3}(\ell_i))\]
otherwise,
and let \( \mathbb{I}_{\text{SL}_3}(\ell) := \mathbb{I}_{\text{SL}_3}(\ell_1) \cdots \mathbb{I}_{\text{SL}_3}(\ell_m) \).

We combine the results and arguments above to prove:

Proposition 1.18 (Prop[4.18]). \( \mathbb{I}_{\text{SL}_3} \) is injective, and its image forms a basis \( \mathcal{O}(\mathcal{X}_{\text{SL}_3}, \mathcal{S}) \).

Our original interest is the space \( \mathcal{X}_{\text{PGL}_3, \mathcal{S}} \), for which the cluster \( \mathcal{X} \)-coordinate variables \( X_v, v \in \mathcal{V}(Q_{\Delta}) \) are defined. In our final answer, for each \( \ell \in \mathcal{A}(\mathcal{S}; \mathbb{Z}) \), we construct a Laurent polynomial function in the variables \( X_v, v \in \mathcal{V}(Q_{\Delta}) \). As a tool to relate \( \mathcal{X}_{\text{PGL}_3, \mathcal{S}} \) and \( \mathcal{X}_{\text{SL}_3, \mathcal{S}} \), we make use of the evaluations at the positive-real semi-field and the field of real numbers: let
\[
\mathcal{X}_{\text{PGL}_3, \mathcal{S}}^{+} := \mathcal{X}_{\text{PGL}_3, \mathcal{S}}(\mathbb{R}_{>0}) = \text{the higher Teichmüller space},
\]
so that \( \mathcal{X}_{\text{PGL}_3, \mathcal{S}}^{+} \) is defined. In our final answer, for each \( \ell \in \mathcal{A}_H(S; \mathbb{Z}) \) and \( \Delta \), we shall construct a Laurent function of the evaluations at the positive-real semi-field and the field of real numbers: let
\[
\Psi : \mathcal{X}_{\text{PGL}_3, \mathcal{S}}^{+} \rightarrow C^\infty(\mathcal{X}_{\text{PGL}_3, \mathcal{S}}^{+})
\]
by using Fock-Goncharov’s basic monodromy matrices \([\text{FG06}]\), suitably normalized. Namely, given positive real numbers \( (X_v)_{v} \in (\mathbb{R}_{>0})^{\mathcal{V}(Q_{\Delta})} \), we construct a point of \( \mathcal{X}_{\text{PGL}_3, \mathcal{S}}^{+}(\mathbb{R}) \), i.e. a framed \( \text{SL}_3(\mathbb{R}) \)-local system of \( \mathcal{S} \). Monodromy along each loop in \( \mathcal{S} \) is constructed by the product of the basic monodromy matrices associated to small elementary pieces of this loop, where each basic monodromy matrix is an element of \( \text{SL}_3(\mathbb{R}) \) whose entries depend on \( (X_v)_{v} \); also the framing data can be constructed from this process. In particular, by composing eq.\((1.4)\) and the pullback of eq.\((1.5)\) we obtain a map
\[
\mathbb{I}_{\text{PGL}_3}^{+} : \mathcal{A}_H(S; \mathbb{Z}) \rightarrow C^\infty(\mathcal{X}_{\text{PGL}_3, \mathcal{S}}^{+})
\]
\[
\text{For each } \ell \in \mathcal{A}_H(S; \mathbb{Z}), \text{ the expression of } \mathbb{I}_{\text{PGL}_3}^{+}(\ell) \text{ as a Laurent polynomial in } \{X_{v}^{1/3} | v \in \mathcal{V}(Q_{\Delta})\} \text{ for any chosen ideal triangulation } \Delta \text{ has the unique highest Laurent monomial in the natural partial ordering, and it is } \prod_{v \in \mathcal{V}(Q_{\Delta})} X_{v}^{-\alpha_{\ell}(v)}, \text{ with coefficient 1.}
\]

Proposition 1.20 (congruence of terms; Prop[4.19]). For \( \ell \in \mathcal{A}_H(S; \mathbb{Z}) \), we have
\[
\mathbb{I}_{\text{PGL}_3}^{+}(\ell) \in \left( \prod_{v \in \mathcal{V}(Q_{\Delta})} X_{v}^{-\alpha_{\ell}(v)} \right) \mathcal{Z}[\{X_{v}^{\pm 1} | v \in \mathcal{V}(Q_{\Delta})\}].
\]
In particular, for a congruent \( A_2 \)-laminations \( \ell \), \( \mathbb{I}_{\text{PGL}_3}^{+}(\ell) \) is a Laurent polynomial in the (positive-real evaluations of) usual cluster \( \mathcal{X} \)-variables \( X_v \)'s, \( v \in \mathcal{V}(Q_{\Delta}) \). As a matter of fact, proving these innocent-looking assertions was much more of a challenge than it looked. For this we developed a whole set of new machinery throughout the entire section \([\text{FG06}]\) which we call the \( \text{SL}_3 \) classical trace.

Definition 1.21. For a triangulable generalized marked surface \( \mathcal{S} \) possibly with boundary, define the (commutative) stated \( A_2 \)-skew algebra \( \mathcal{S}_s(\mathcal{S}) \) as in Def[5.4] using \( A_2 \)-webs \( \mathcal{W} \) in \( \mathcal{S} \) that can have endpoints in \( \partial \mathcal{S} \), together with a state \( s : \partial \mathcal{W} \rightarrow \{1, 2, 3\} \). For an ideal triangulation \( \Delta \) of \( \mathcal{S} \), let \( Z_{\Delta} := \mathcal{Z}[[Z_v^{\pm 1} | v \in \mathcal{V}(Q_{\Delta})]] \) be the classical cube root Fock-Goncharov algebra (Def[5.2]).

Proposition 1.22 (SL3 classical trace map; Prop[5.6]). There exists a family of ring homomorphisms
\[
\text{Tr}_{\Delta, \mathcal{S}} : \mathcal{S}_s(\mathcal{S}) \rightarrow Z_{\Delta}
\]
for each triangulable generalized marked surface \( \mathcal{S} \) and its ideal triangulation \( \Delta \), satisfying favorable properties, e.g. restricting to \( \mathbb{I}_{\text{PGL}_3}^{+} \) on the \( A_2 \)-webs not containing peripheral loops, with \( X_v = Z_v^3 \).
The SL$_3$ classical trace maps behave well under cutting along an arc of $\Delta$, and this cutting property yields a state-sum type formula for computing the values of $Tr_{\Delta;\mathcal{E}}$. This map is interesting in its own right, and can be viewed as a surface generalization of the classical version of Reshetikhin-Turaev operator invariant for $\mathfrak{sl}_2$ [RT90], and as a classical SL$_3$ analog of Bonahon-Wong’s quantum SL$_2$ trace [BW11]. We believe our results in [3] will lay a foundation of a full version of quantum SL$_3$ trace, which was partially dealt with only for oriented loops in [D20] and [G17], and which will be obtained in an upcoming work [L]. Coming back to our situation, what makes the computation of the result of Higgins [H20], who proved that his version of stated $\Delta$-web of $\mathfrak{sl}_3$, so that the computation boils down essentially to computation of Reshetikhin-Turaev type invariants $(\text{main theorem}:\text{SL}_3)$.

Moreover, we also perform the computation of the effect on $\mathcal{I}^+_\text{PGL}_3(\ell)$ of a single mutation at every possible node of $Q_\Delta$, which too was quite heavy a calculation (§3.5).

**Proposition 1.23** (effect of single mutation; Prop 4.26, Cor 4.27). For any ideal triangulation $\Delta$ of a punctured surface $\mathcal{E}$, any $\Delta$-congruent $A_2$-lamination $\ell$, and any node $v$ of the 3-triangulation quiver $Q_\Delta$ of $\Delta$, if we write $\mathcal{I}^+_\text{PGL}_3(\ell)$ as a function in the (cube-root) coordinate functions $(X'_v | v' \in \mathcal{V}(Q))$ for the cluster $\mathcal{X}$-chart of $\mathcal{I}_{\text{PGL}_3;\mathcal{E}}$ (with quiver $Q' = \mu_v(Q_\Delta)$) obtained from the chart for $\Delta$ by mutating at $v$, it is a Laurent polynomial in $(X'_v | v' \in \mathcal{V}(Q'))$.

Using these propositions, together with Shen’s result $\mathcal{O}(\mathcal{I}_{\text{PGL}_3;\mathcal{E}}) = \mathcal{O}(\mathcal{I}_{\text{PGL}_3;\mathcal{E}})$ [S20] (Prop 4.24), and a theorem of Gross-Hacking-Keel [CHK15] (Prop 4.25) stating that if a regular function on a cluster $\mathcal{X}$-chart stays regular after all possible single mutations then it is regular for all cluster $\mathcal{X}$-charts, we are able to prove the main theorem of the present paper:

**Theorem 1.24** (main theorem: SL$_3$-PGL$_3$ duality map; Thm 4.23). Let $\mathcal{E}$ be a triangulable punctured surface. The above process using $\mathcal{I}^+_\text{PGL}_3$ yields a map

$$\mathcal{I} : \mathcal{A}_{\text{PGL}_3}(Z') \to \mathcal{O}(\mathcal{I}_{\text{PGL}_3;\mathcal{E}})$$

satisfying

1. $\mathcal{I}$ is injective, and the image of $\mathcal{I}$ forms a basis of $\mathcal{O}(\mathcal{I}_{\text{PGL}_3;\mathcal{E}}) = \mathcal{O}(\mathcal{I}_{\text{PGL}_3;\mathcal{E}})$, where we refer to this basis as an $A_2$-bangle basis;
2. for each congruent $A_2$-lamination $\ell \in \mathcal{A}_{\text{PGL}_3}(Z') \subset \mathcal{A}_L(\mathcal{E};\mathbb{Z})$ in $\mathcal{E}$ and any ideal triangulation $\Delta$, $\mathcal{I}(\ell)$ is a Laurent polynomial in the cluster $\mathcal{X}$-variables $\{X_v | v \in \mathcal{V}(Q_\Delta)\}$ with integer coefficients, with the unique highest term $\prod_{v \in \mathcal{V}(Q_\Delta)} X_v^{a_v(\ell)}$ where $(a_v(\ell))_v$ are the unique tropical coordinates constructed in the present paper;
3. If $\ell$ consists of peripheral loops only, then for each ideal triangulation $\Delta$, $\mathcal{I}(\ell) = \prod_{v \in \mathcal{V}(Q_\Delta)} X_v^{a_v(\ell)}$;
4. The structure constants of this basis are integers, i.e. for each $\ell, \ell' \in \mathcal{A}_{\text{PGL}_3}(Z')$, we have

$$\mathcal{I}(\ell) \mathcal{I}(\ell') = \sum_{\ell'' \in \mathcal{A}_{\text{PGL}_3}(Z')} c(\ell, \ell'; \ell'') \mathcal{I}(\ell'')$$

where $c(\ell, \ell'; \ell'') \in \mathbb{Z}$ and $c(\ell, \ell'; \ell'')$ are zero for all but at most finitely many $\ell''$.

As mentioned, we obtain a proof of Prop 1.15 during our proof of Thm 1.24. This main theorem, Thm 1.24 settles Conjecture 1.4 as promised. One may write the domain and codomain of this map $\mathcal{I}$ as $\mathcal{A}_{Q_3}(Z')$ and $\mathcal{O}(\mathcal{X}_{Q_3})$ in terms of the cluster varieties. We discuss further research topics in §6, including a proposal for higher rank generalization.

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2.1. Generalized marked surface and ideal triangulation. We first recall some basic definitions about the surfaces which will be the base manifolds for moduli spaces. We mostly adapt conventions used by Lê [17] [18].

Definition 2.1 [17] [18](•) A generalized marked surface \((\Sigma, P)\) consists of an oriented compact smooth surface \(\Sigma\) with possibly-empty boundary \(\partial \Sigma\) and a non-empty finite subset \(P\) of \(\Sigma\). We always require that each component of \(\partial \Sigma\) contains at least one point of \(P\). Two generalized marked surfaces \((\Sigma_1, P_1)\) and \((\Sigma_2, P_2)\) are isomorphic if there exists an orientation-preserving diffeomorphism \(\Sigma_1 \to \Sigma_2\) mapping \(P_1\) onto \(P_2\).

• The elements of \(P\) are called marked points. A marked point that lies in the interior \(\Sigma := \Sigma \setminus \partial \Sigma\) of \(\Sigma\) is called a puncture of \((\Sigma, P)\).

• A generalized marked surface \((\Sigma, P)\) is called a punctured surface if \(\partial \Sigma = \emptyset\).

• Each component of \((\partial \Sigma) \setminus P\) is called a boundary arc of \((\Sigma, P)\).

For a generalized marked surface \((\Sigma, P)\), we often let \(\mathcal{G} := \Sigma \setminus P\), and refer to \(\mathcal{G}\) as the generalized marked surface, by identifying it with the data \((\Sigma, P)\), by abuse of notation. Note  
\[ \partial \mathcal{G} = (\partial \Sigma) \setminus P, \quad \mathcal{G} = \Sigma \setminus P. \]

A crucial ingredient is an ideal triangulation of the surface \(\mathcal{G}\).

Definition 2.2 [17] [18]. Let \((\Sigma, P)\) be a generalized marked surface, and \(\mathcal{G} = \Sigma \setminus P\).

• A component of \(\mathcal{G}\) in \(\Sigma\), or an ideal arc in \(\mathcal{G}\), is the image of an immersion \(\alpha : [0, 1] \to \Sigma\) such that \(\alpha([0, 1]) \subset P\) and \(\alpha|_{(0, 1)}\) is an embedding. The elements of \(\alpha([0, 1])\) are called endpoints, and the subset \(\alpha(0, 1)\) is called the interior of this \(P\)-arc. Two \(P\)-arcs are said to be isotopic if they are isotopic within the class of \(P\)-arcs. We often identify \(\alpha\) with its image \(\alpha([0, 1])\) in \(\Sigma\), or even with \(\alpha((0, 1))\) in \(\mathcal{G}\).

• We say that \((\Sigma, P)\), or \(\mathcal{G}\), is triangulable if it is none of the following:
  - monogon, i.e. \(\Sigma\) is diffeomorphic to a closed disc, and \(P\) consists of a single point on \(\partial \Sigma\);
  - biangle, i.e. \(\Sigma\) is diffeomorphic to a closed disc, and \(P\) consists of two points on \(\partial \Sigma\);
  - sphere with one or two punctures, i.e. \(\Sigma\) is diffeomorphic to the sphere \(S^2\), and \(|P| \leq 2\).

• When \((\Sigma, P)\) is triangulable, a \(P\)-triangulation of \(\Sigma\), or an ideal triangulation of \(\mathcal{G}\), is defined as a collection \(\Delta\) of \(P\)-arcs in \(\Sigma\) such that
  - (IT1) interiors of any two members of \(\Delta\) are disjoint;
  - (IT2) no two members of \(\Delta\) are isotopic;
  - (IT3) \(\Delta\) is maximal among the collections satisfying (IT1) and (IT2).

Two ideal triangulations are isotopic if they are related by a simultaneous isotopy of their members, within the class of ideal triangulations.

We assume (by applying an isotopy if necessary) that each constituent arc of \(\Delta\) that is isotopic to a boundary arc of \(\mathcal{G}\) is indeed a boundary arc of \(\mathcal{G}\). The constituent arcs of \(\Delta\) that are not boundary arcs are called internal arcs of \(\Delta\). Constituent arcs of \(\Delta\) are often called edges of \(\Delta\).

Remark 2.3. The monogon and biangle will play crucial roles later in the present paper.

Let \(\Delta\) be an ideal triangulation of a triangulable generalized marked surface \((\Sigma, P)\). Let \(\tilde{t}\) be a connected component of the complement \(\Sigma \setminus (\bigcup_{e \in \Delta} e)\). The closure \(t\) of \(t\) in \(\Sigma\) is called an ideal triangle of \(\Delta\). Let  
\[ \mathcal{F}(\Delta) := \text{the set of all ideal triangles of } \Delta. \]

Note that \(t \setminus \tilde{t}\) is union of two or three ideal arcs in \(\Delta\), which are called sides of \(t\). In case \(t\) has only two sides, \(t\) is called self-folded. In the present paper, we do not allow ideal triangulation having a self-folded ideal triangle. In fact, we only use ideal triangulations satisfying a somewhat stronger condition.
Definition 2.4 (from [FG06]). An ideal triangulation $\Delta$ of a triangulable generalized marked surface $(\Sigma, \mathcal{P})$ is regular if for each puncture $p$ of $(\Sigma, \mathcal{P})$, the valence of $\Delta$ at $p$ is at least 3, where the valence of $\Delta$ at $p$ is the total number of arcs of $\Delta$ meeting $p$ counted with multiplicity, where the multiplicity of an arc is 1 if $p$ is exactly one of the two distinct endpoints of the arc and is 2 if both endpoints of the arc coincide with $p$.

We do not require that every ideal triangulation of $\mathcal{S}$ be regular, as done in [FG06] for some statement relevant to our situation. Rather, we require that $\mathcal{S}$ admits at least one regular ideal triangulation. Especially, our main theorem will be on punctured surface, hence we need the following observation:

Lemma 2.5. Every triangulable punctured surface except for the sphere with three punctures admits at least one regular ideal triangulation.

Proof. Let’s use induction. Suppose that a triangulable punctured surface $(\Sigma, \mathcal{P})$, where $\Sigma$ is a compact oriented surface of genus $g$ and $|\mathcal{P}| = n$, admits a regular ideal triangulation $\Delta$. Let $\mathcal{P}' := \mathcal{P} \cup \{x\}$ where $x$ is a point lying in the interior of some ideal triangle $t$ of $\Delta$. Then, by adding to $\Delta$ three $\mathcal{P}'$-arcs, each connecting $x$ and a vertex marked point $v \in \mathcal{P}$ of $t$, one obtains an ideal triangulation $\Delta'$ of $(\Sigma, \mathcal{P}')$. The valence of $\Delta'$ at $x$ is 3, and the valence of $\Delta'$ at each $p \in \mathcal{P}$ is at least the valence of $\Delta$ at $p$ hence is at least 3. So $\Delta'$ is regular. Hence, for each genus $g$ surface $\Sigma$, it suffices to prove the statement for $\mathcal{P}$ with minimal possible $|\mathcal{P}|$. For genus 0 surface, i.e. sphere, when $|\mathcal{P}| = 4$, one can easily find a regular triangulation. For genus $g \geq 1$ surface $\Sigma$ with $|\mathcal{P}| = 1$, it is well known that $(\Sigma, \mathcal{P})$ admits an ideal triangulation $\Delta$, and that any ideal triangulation of it has $6g - 6 + 3 = 6g - 3$ arcs. So the unique $\mathcal{P}$ has valence $2(6g - 3) \geq 3$, hence $\Delta$ is regular. $\blacksquare$

Basic constructions of Fock-Goncharov’s higher Teichmüller theory [FG06] make use of the choice of a regular ideal triangulation $\Delta$ of a generalized marked surface $\mathcal{S}$. A key point is then to assure certain compatibility under changes of ideal triangulations. One common strategy is to focus on certain elementary changes called flips, which change an ideal triangulation by only one edge at a time.

Definition 2.6. We say that two ideal triangulations of a generalized marked surface $\mathcal{S}$, defined up to isotopy, are related by a flip if they differ by exactly one edge.

If $\Delta$ and $\Delta'$ are related by a flip, then we have a natural bijection between $\Delta$ and $\Delta'$ as sets, through which we identify the two sets. If the only differing edge is labeled by $e$ (both in $\Delta$ and $\Delta'$), we say that this flip is a flip at the edge $e$.

One thing to keep in mind is:

Throughout the paper, by an ideal triangulation we mean a regular ideal triangulation.

In particular, when we refer to a triangulable generalized marked surface, we mean one that admits at least one regular ideal triangulation, and we will only consider flips between regular ideal triangulations.

2.2. Moduli spaces of G-local systems on surface. We recall Fock-Goncharov’s versions [FG06] of moduli spaces of G-local systems on a triangulable generalized marked surface $\mathcal{S}$, where $G$ is a split reductive algebraic group. A G-local system $L$ on $\mathcal{S}$ may be understood as a right principal $G$-bundle on $\mathcal{S}$ together with a flat $G$-connection on it.

Definition 2.7 ($L$-moduli space; [FG06]). Let $\mathcal{L}_{G, \mathcal{S}}$ be the moduli stack parametrizing all isomorphism classes of $G$-local systems on $\mathcal{S}$.

A G-local system $L$ induces a group homomorphism $\pi_1(\mathcal{S}) \to G$ defined up to conjugation by an element of $G$, which is called a monodromy representation of $L$, and which in fact determines the isomorphism class of $L$. Hence we have a natural identification

$$\mathcal{L}_{G, \mathcal{S}} \cong \text{Hom}(\pi_1(\mathcal{S}), G)/G$$

where the right hand side is the quotient of $\text{Hom}(\pi_1(\mathcal{S}), G)$ by the action of $G$ by conjugation.

Choose any Borel subgroup $B$ of $G$, and let $U := [B, B]$ be a maximal unipotent subgroup of $G$. Let $B = G/B$ be the flag variety for $G$. For a G-local system $L$ on $\mathcal{S}$, denote by

$$L_B := L \times_G B \quad \text{and} \quad L_A := L/U$$

the associated flag bundle and principal affine bundle on $\mathcal{S}$, respectively; each of these associated bundles is also naturally equipped with a flat connection induced by $L$. 


The present paper concerns $G = \text{PGL}_3$ and $\text{SL}_3$. For $G = \text{PGL}_3$ we may choose $B$ to be the subgroup of all upper triangular matrices in $\text{PGL}_3$, and then for $G = \text{SL}_3$ we choose $U$ to be the subgroup of all upper triangular matrices with all diagonal entries being 1.

To describe an extra boundary data, it is more convenient to deal with a holed surface, instead of a punctured surface. We fix a notation, which is similar but slightly different from what is in [FG06].

**Definition 2.8** (holed surface). Let $(\Sigma, \mathcal{P})$ be a generalized marked surface, with $\mathcal{S} = \Sigma \setminus \mathcal{P}$. For each puncture $p$ of $(\Sigma, \mathcal{P})$, choose a neighborhood $N_p$ of $p$ in $\Sigma$ diffeomorphic to an open disc. Let

$$\tilde{\mathcal{S}} := \Sigma \setminus \bigcup_{p : 	ext{puncture of } (\Sigma, \mathcal{P})} N_p$$

be a holed surface for $\mathcal{S}$. Each boundary component of $\tilde{\mathcal{S}}$ that is diffeomorphic to a circle is called a hole of $\tilde{\mathcal{S}}$, and other boundary components of $\tilde{\mathcal{S}}$ are called boundary arcs of $\tilde{\mathcal{S}}$.

The boundary-orientation on a hole of $\tilde{\mathcal{S}}$ is the orientation induced by the counterclockwise orientation along the boundary of the disc $N_p$ (hence is ‘clockwise’ when viewed from the interior of $\tilde{\mathcal{S}}$).

Holes of $\tilde{\mathcal{S}}$ correspond to punctures of $\mathcal{S}$. One can view $\tilde{\mathcal{S}}$ as being embedded as a subspace of $\mathcal{S}$, onto which $\mathcal{S}$ deformation retracts. In particular, the inclusion $\tilde{\mathcal{S}} \rightarrow \mathcal{S}$ naturally induces isomorphism $\pi_1(\tilde{\mathcal{S}}) \rightarrow \pi_1(\mathcal{S})$ and hence an identification of $\mathcal{L}_{G,\tilde{\mathcal{S}}}$ with $\mathcal{L}_{G,\mathcal{S}}$. We will use $\mathcal{S}$ and $\tilde{\mathcal{S}}$ interchangeably. In particular, a hole of $\tilde{\mathcal{S}}$ is sometimes regarded as the oriented loop in $\mathcal{S}$, oriented according to the boundary-orientation as in Def. 2.8. The boundary arcs of $\tilde{\mathcal{S}}$ are naturally identified with boundary arcs of $\mathcal{S}$.

**Definition 2.9** ($\mathcal{X}$-moduli space: [FG06] [A19]). Let $G = \text{PGL}_3$. A framing for a $G$-local system $\mathcal{L}$ on $\mathcal{S}$ is a flat section $\beta$ of the restriction of $\mathcal{L}_B$ to $\partial \mathcal{S}$. A pair $(\mathcal{L}, \beta)$ is called a framed $G$-local system on $\mathcal{S}$. Two framed $G$-local systems $(\mathcal{L}_1, \beta_1)$ and $(\mathcal{L}_2, \beta_2)$ are isomorphic if there is an isomorphism $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ of $G$-local systems whose induced map $(\mathcal{L}_1)_B \rightarrow (\mathcal{L}_2)_B$ sends $\beta_1$ to $\beta_2$. Let $\mathcal{X}_{G,\mathcal{S}}$ be the moduli stack parametrizing all isomorphism classes of framed $G$-local systems on $\mathcal{S}$.

**Definition 2.10** ($\mathcal{A}$-moduli space: [FG06] [A19]). Let $G = \text{SL}_3$. A decoration for a $G$-local system $\mathcal{L}$ on $\mathcal{S}$ is a flat section $\alpha$ of the restriction of $\mathcal{L}_A$ to $\partial \mathcal{S}$. A pair $(\mathcal{L}, \alpha)$ is called a decorated $G$-local system on $\mathcal{S}$. Two decorated $G$-local systems $(\mathcal{L}_1, \alpha_1)$ and $(\mathcal{L}_2, \alpha_2)$ are isomorphic if there is an isomorphism $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ of $G$-local systems whose induced map $(\mathcal{L}_1)_B \rightarrow (\mathcal{L}_2)_B$ sends $\alpha_1$ to $\alpha_2$. Let $\mathcal{A}_{G,\mathcal{S}}$ be the moduli stack parametrizing all isomorphism classes of decorated $G$-local systems on $\mathcal{S}$.

The moduli spaces $\mathcal{X}_{G,\mathcal{S}}$ and $\mathcal{A}_{G,\mathcal{S}}$ can be defined for other groups $G$. For general cases, the definition of $\mathcal{A}_{G,\mathcal{S}}$ is more complicated; see [FG06] [A19].

### 2.3. Cluster atlasses

Let $\mathcal{S}$ be a triangulable generalized marked surface. Fock and Goncharov [FG06] constructed special coordinate systems for $\mathcal{A}_{\text{SL}_3,\mathcal{S}}$ and $\mathcal{X}_{\text{PGL}_3,\mathcal{S}}$ respectively, per each choice of an ideal triangulation $\Delta$ of $\mathcal{S}$. They showed that, upon each change of ideal triangulations, the coordinates transform according to the mutation formulas appearing in the theory of cluster varieties. We first recall and define some basic notions needed in this theory.

**Definition 2.11.** By a (labeled) quiver $Q$ we mean a directed graph without cycles of length 1 or 2, with a labeling on vertices. Its vertices are called nodes of $Q$ and depicted as hollow circles with labels, while its oriented edges are called arrows of $Q$ and depicted for example as $v \rightarrow w$. Denote by $\mathcal{V}(Q)$ and $\mathcal{E}(Q)$ the set of all nodes and the set of all arrows of $Q$.

The signed adjacency matrix of a quiver $Q$ is the $\mathcal{V}(Q) \times \mathcal{V}(Q)$ matrix $\varepsilon_Q = \varepsilon$ whose $(v, w)$-th entry is

$$\varepsilon_{vw} = \varepsilon_{wv} = (\text{number of arrows from } v \text{ to } w) - (\text{number of arrows from } w \text{ to } v).$$

In the present paper, the $(\alpha, \beta)$-th entry of a matrix refers to the entry in the $\alpha$-th row and $\beta$-th column.

The set $\mathcal{E}(Q)$ of arrows should be understood as a multiset of elements of $\mathcal{V}(Q) \times \mathcal{V}(Q)$. So, two quivers $Q$ and $Q'$ defined on a same set of nodes $\mathcal{V}$ are identified if $\varepsilon_Q = \varepsilon_{Q'}$. Note that, for a fixed set $\mathcal{V}$ of nodes, the correspondence $Q \leftrightarrow \varepsilon_Q$ is a bijection between the set of all quivers and the set of all $n \times n$ skew-symmetric integer matrices.
We need to recall a certain transformation rule for quivers.

**Definition 2.12.** Let $Q$ be a quiver with the set of nodes $V$ and the signed adjacency matrix $\varepsilon$. Let $k \in V$. The quiver mutation $\mu_k$ at the node $k$ transforms the quiver $Q$ into another quiver $Q' = \mu_k(Q)$ whose set of nodes is $V$ and the signed adjacency matrix $\varepsilon'$ is defined as

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i, j\}, \\ \varepsilon_{ij} + \frac{1}{2}(\varepsilon_{ik}\varepsilon_{kj}) + |\varepsilon_{ik}|\varepsilon_{kj} & \text{if } k \notin \{i, j\}. \end{cases}$$

As is well-known, the quiver mutation can be described combinatorially as follows. First, from $Q$, reverse the orientations of all arrows starting or ending at $k$. Second, for each pair of an incoming arrow $i \rightarrow k$ and an outgoing arrow $j \rightarrow k$ at the node $k$, add the arrow $i \rightarrow j$ (i.e. ‘complete the 3-cycle through $k$’). Finally, remove cycles of length 2, until there are none. Then the resulting quiver is $\mu_k(Q)$.

To characterize Fock-Goncharov’s special coordinate systems on the moduli spaces $\mathcal{A}_{SL_3,\mathbb{C}}$ and $\mathcal{I}_{PGL_3,\mathbb{C}}$, we establish some terminology, based on [FG06].

**Definition 2.13.** Let $\mathcal{S}$ be an irreducible stack or a scheme. A cluster $\mathcal{A}$-chart on $\mathcal{S}$ is a pair $(Q, \psi)$, where $Q$ is a (labeled) quiver and

$$\psi : \mathcal{S} \rightarrow (\mathbb{G}_m)^{|V(Q)|}$$

is a birational map, providing a rational coordinate system for $\mathcal{S}$. Denote by $A_i$ the coordinate function for the node $i \in V(Q)$, which is called a cluster $\mathcal{A}$-variable for this chart.

We say that a cluster $\mathcal{A}$-chart $(Q, \psi)$ is related to another cluster $\mathcal{A}$-chart $\mu_k(Q, \psi) = (Q', \psi')$ by the cluster $\mathcal{A}$-mutation at the node $k \in V(Q)$ if $Q' = \mu_k(Q)$ holds, so that we have an identification of $V(Q)$ and $V(Q')$, and the coordinate functions $A_i'$ for $\psi'$ are related to those $A_i$ of $\psi$ as

$$A_i' = \begin{cases} A_i & \text{if } i \neq k, \\ A_k^{-1}(\prod_j A_j^{\varepsilon_{ik}})^+ + \prod_j A_j^{-\varepsilon_{jk}} & \text{if } i = k, \end{cases}$$

where $[a]^+$ is the positive part of a real number $a$:

$$[a]^+ := \begin{cases} a & \text{if } a \geq 0, \\ 0 & \text{if } a < 0. \end{cases}$$

Two cluster $\mathcal{A}$-charts $(Q, \psi)$ and $(Q', \psi')$ are identified if each coordinate function $A_i$ for the former coincides with the corresponding coordinate function $A_i'$ for the latter, on an open dense subset.

**Definition 2.14.** A cluster $\mathcal{A}$-atlas on an irreducible stack or a scheme $\mathcal{S}$ is a collection $\mathcal{C}$ of cluster $\mathcal{A}$-charts on $\mathcal{S}$ such that each two members of $\mathcal{C}$ are related by a finite sequence of members of $\mathcal{C}$ such that each pair of consecutive members are related by a cluster $\mathcal{A}$-mutation.

Note that cluster $\mathcal{A}$-mutation can also be used as a tool to construct from a cluster $\mathcal{A}$-chart another cluster $\mathcal{A}$-chart. So, starting from any cluster $\mathcal{A}$-atlas, by adding all cluster $\mathcal{A}$-charts obtained by applying cluster $\mathcal{A}$-mutations, one gets a uniquely determined maximal cluster $\mathcal{A}$-atlas.

**Definition 2.15.** Let $\mathcal{S}$ be an irreducible stack or a scheme. A cluster $\mathcal{X}$-chart on $\mathcal{S}$ is a pair $(Q, \psi)$, where $Q$ is a (labeled) quiver and

$$\psi : \mathcal{S} \rightarrow (\mathbb{G}_m)^{|V(Q)|}$$

is a birational map, providing a rational coordinate system for $\mathcal{S}$. Denote by $X_i$ the coordinate function for the node $i \in V(Q)$, which is called a cluster $\mathcal{X}$-variable for this chart.

We say that a cluster $\mathcal{X}$-chart $(Q, \psi)$ is related to another cluster $\mathcal{X}$-chart $\mu_k(Q, \psi) = (Q', \psi')$ by the cluster $\mathcal{X}$-mutation at the node $k \in V(Q)$ if $Q' = \mu_k(Q)$ holds, so that we have an identification of $V(Q)$ and $V(Q')$, and the coordinate functions $X_i'$ for $\psi'$ are related to those $X_i$ of $\psi$ as

$$X_i' = \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i(1 + X_k^{-\text{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i \neq k, \end{cases}$$

where $\text{sgn}(a)$ means the sign of a real number:

$$\text{sgn}(a) = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$
**Definition 2.16.** Define the notion of cluster $\mathcal{X}$-atlas similarly as for cluster $\mathcal{A}$-atlas.

One of the major objects of study in the theory of cluster varieties is the following.

**Definition 2.17** (Ring of cluster $\mathcal{X}$-regular functions; \[FG06\].) Let $\mathcal{X}$ be an irreducible stack or a scheme, equipped with a chosen cluster $\mathcal{X}$-atlas. Let $\mathcal{O}_a(\mathcal{X})$ the ring of all cluster $\mathcal{X}$-regular functions on $\mathcal{X}$, i.e. the rational functions on $\mathcal{X}$ that are regular on each cluster $\mathcal{X}$-chart belonging to the maximal cluster $\mathcal{X}$-atlas determined by the given cluster $\mathcal{X}$-atlas.

Observe that a rational function defined on a cluster $\mathcal{X}$-chart $(Q, \psi)$ is regular on this chart if it can be written as a Laurent polynomial in the coordinate functions $X_i$, $i \in \mathcal{V}(Q)$, for this chart. So a cluster $\mathcal{X}$-regular function is often referred to as universally ($\mathcal{X}$-)Laurent (polynomial) functions. Note that the similarly defined ring of all cluster $\mathcal{A}$-regular functions coincides with the notion of the so-called upper cluster algebra. However, in the present paper, only the cluster $\mathcal{X}$-regular functions will be dealt with.

Fock and Goncharov showed \[FG06\] that the moduli spaces $\mathcal{A}_{SL_3, \mathcal{S}}$ and $\mathcal{A}_{PGL_3, \mathcal{S}}$ exhibit a cluster $\mathcal{A}$-atlas and a cluster $\mathcal{X}$-atlas respectively, with some special quivers $Q = Q_\Delta$ associated to ideal triangulations $\Delta$ of $\mathcal{S}$.

**Definition 2.18** (\[FG06\].) Let $\Delta$ be an ideal triangulation of a triangulable generalized marked surface $\mathcal{S}$. Let $Q_\Delta$ be the quiver, which may be drawn on the surface $\mathcal{S}$, whose nodes and arrows are defined as follows.

For each edge $e$ of $\Delta$, there are two nodes of $Q_\Delta$ lying in the interior of $e$. For each triangle $t$ of $\Delta$, there is one node of $Q_\Delta$ lying in the interior of $t$.

For each triangle of $\Delta$, consider a quiver as depicted in Fig. 7, consisting of three small counterclockwise cycles of length 3. The quiver $Q_\Delta$ is obtained by gluing these quivers for all triangles of $\Delta$.

The quiver $Q_\Delta$ is called the 3-triangulation of $\mathcal{S}$ associated to the ideal triangulation $\Delta$.

Although the construction of $Q_\Delta$ made use of the surface $\mathcal{S}$, the resulting quiver $Q_\Delta$ can be considered as an abstract quiver. One property of these quivers is that when we flip an ideal triangulation to another one, the corresponding quivers are related by a sequence of quiver mutations.

**Lemma 2.19** (flip as four quiver mutations; \[FG06\].) Let $\Delta$ and $\Delta'$ be ideal triangulations of a triangulable generalized marked surface $\mathcal{S}$ that are related by the flip at an edge labeled by $e$. Let $k_1, k_2$ be the two nodes of $Q_\Delta$ lying on the edge $e$ of $\Delta$, and $k_3, k_4$ be the nodes of $Q_\Delta$ lying in the interiors of the two ideal triangles of $\Delta$ having $e$ as one of their sides. Then

\[ Q_{\Delta'} = \mu_{4\mu_{3\mu_{2}}}(Q_\Delta). \]

This lemma is straightforward to check, and is partly depicted in Fig. 6.

**Figure 6.** Sequence of mutations for a flip at an edge, transforming $Q_\Delta$ to $Q_{\Delta'}$.

We now state Fock-Goncharov’s special atlases.

**Proposition 2.20** (Cluster atlases for Fock-Goncharov moduli spaces; \[FG06\].) Let $\mathcal{S}$ be a triangulable punctured surface.

- For each ideal triangulation $\Delta$ of $\mathcal{S}$, there exists a cluster $\mathcal{A}$-chart $(Q_\Delta, \psi_{\Delta})$ of $\mathcal{A}_{SL_3, \mathcal{S}}$, such that these charts are contained a cluster $\mathcal{A}$-atlas, so that if two triangulations $\Delta, \Delta'$ are related by a flip, then
the corresponding cluster \( \mathscr{A} \)-charts are related by the sequence \( \mu_4 \mu_3 \mu_2 \mu_1 \) of four cluster \( \mathscr{A} \)-mutations as appearing in eq. (2.4).

• For each ideal triangulation \( \Delta \) of \( \mathfrak{S} \), there exists a cluster \( \mathcal{X} \)-chart \( (Q_\Delta, \psi_\Delta) \) of \( \mathcal{X}_{\text{PGL}_3, \mathfrak{S}} \), such that these charts are contained a cluster \( \mathcal{X} \)-atlas, so that if two triangulations \( \Delta, \Delta' \) are related by a flip, then the corresponding cluster \( \mathcal{X} \)-charts are related by the sequence \( \mu_4 \mu_3 \mu_2 \mu_1 \) of four cluster \( \mathcal{X} \)-mutations as appearing in eq. (2.4).

For explicit construction of these charts, see [FG06, D20]. What we will do recall later is the reconstruction map for the above cluster \( \mathcal{X} \)-charts of \( \mathcal{X}_{\text{PGL}_3, \mathfrak{S}} \); namely, given the cluster \( \mathcal{X} \)-coordinates for an ideal triangulation, we will reconstruct a framed \( \text{PGL}_3 \)-local system on \( \mathfrak{S} \).

Before going to the next section, we give a couple more remarks on a stack \( \mathscr{S} \) equipped with a cluster \( \mathscr{A} \)- or a cluster \( \mathcal{X} \)-atlas. When one focuses only on the coordinate change formulas, this cluster structure on \( \mathscr{S} \) is completely determined by the quiver \( Q \) for any of the cluster chart chosen. As a matter of fact, Given any abstract quiver \( Q \) with \( N \) nodes, or any skew-symmetric exchange matrix \( \varepsilon = \varepsilon_Q \), one may start from the affine scheme \( (\mathbb{G}_m)^N \) associated to \( Q \), package it as the data of a seed torus \( (Q, (\mathbb{G}_m)^N) \), then by mutating at a node \( k \) construct another seed torus \( (\mu_k(Q), (\mathbb{G}_m)^N) \) which is glued to the original seed along the cluster \( \mathscr{A} \)- or cluster \( \mathcal{X} \)-mutation map. Staring from one seed torus, one can mutate in \( N \) directions to get \( N \) seed tori to glue, then mutate at nodes of these new seed tori, etc. Gluing all such seed tori, one obtains the so-called cluster \( \mathscr{A} \)-variety \( \mathscr{A}_Q = \mathscr{A}(Q) \), and the cluster \( \mathcal{X} \)-variety \( \mathcal{X}_Q = \mathcal{X}(Q) \), where \( |Q| \) means the mutation equivalence class of a quiver \( Q \).

These cluster varieties are schemes defined abstractly and combinatorially, and having a cluster atlas of a stack \( \mathcal{S} \) provides a birational map from \( \mathcal{S} \) to the corresponding cluster variety. Many properties and questions on \( \mathcal{S} \) related to the chosen cluster structure can be translated to those on the corresponding abstract cluster varieties. For example, \( \mathcal{O}_\mathcal{A}(\mathcal{S}) \) is isomorphic to the ring of all regular functions on the corresponding cluster variety.

Observe that cluster mutation coordinate change formulas involve only multiplication, division, and addition, but not subtraction. This allows us to consider the set of points valued in a semi-field, not just in a field. A semi-field \( (\mathbb{P}, +, \odot) \) means a set \( \mathbb{P} \) equipped with two binary operations \( + \) and \( \odot \), where \( (\mathbb{P}, \odot) \) is an abelian group, where \( + \) is required only to be associative and commutative, so that \( + \) and \( \odot \) satisfy the distributive law. Of our particular interest are two examples of semi-fields:

\( \mathbb{R}_{>0} \) : semi-field of positive-real numbers, with usual addition and usual multiplication of real numbers

\( \mathbb{Z}^t \) : semi-field of tropical integers, where \( \mathbb{Z}^t = \mathbb{Z} \) as a set, and \( a + b := \max(a, b) \) and \( a \odot b := a + b \).

What will play important roles are \( \mathscr{A}_{\text{GL}_3, \mathfrak{S}}(\mathbb{Z}^t) \) and \( \mathcal{X}_{\text{PGL}_3, \mathfrak{S}}(\mathbb{R}_{>0}) \). We note that, in general, for a stack \( \mathcal{S} \) equipped with a cluster atlas, or for a cluster variety \( \mathcal{S} \), and for a semi-field \( \mathbb{P} \), the set \( \mathcal{S}(\mathbb{P}) \) can be understood as being obtained by gluing the sets \( \mathbb{P}^N \) along the tropicalized version of the cluster mutation formulas, namely by replacing the operations \( +, \odot, \div \) by \( +, \odot, \odot \) (where \( \odot \) is the inverse operation of the tropical multiplication \( \odot \)). Note also that, unlike the general case of fields, these gluing maps between \( \mathbb{P}^N \) are bijections, so that \( \mathcal{S}(\mathbb{P}) \) is in fact \( \mathbb{P}^N \) as a set.

3. \( A_2 \)-webs and \( A_2 \)-laminations on a surface

3.1. \( A_2 \)-webs and \( A_2 \)-skein algebra. A web on a surface is a certain oriented graph on the surface \( \mathfrak{S} \). Since Kuperberg [K96] introduced it for the case when the surface is a disc, for a diagrammatic calculus for representation theory of the (quantized) Lie algebra of rank up to two, it has been extensively studied by many authors, being generalized in several directions. In particular, the \( A_2 \)-type webs are generalized to corresponding objects living in a thickened surface \( \mathfrak{S} \times (-1, 1) \), leading to the definition of a \( SU_3 \)-skein algebra [S05, FS20]. We start from the following definition, taken from [K96, S05, FS20] and modified to fit our purpose.

**Definition 3.1.** Let \( (\Sigma, \mathcal{P}) \) be a generalized marked surface, and let \( \mathfrak{S} = \Sigma \setminus \mathcal{P} \). An \( A_2 \)-web \( W \) in \( (\Sigma, \mathcal{P}) \) (or in \( \mathfrak{S} \)) consists of

• a finite subset of \( \partial \mathfrak{S} \), whose elements are called external vertices; when there is no confusion, we refer to them as endpoints of \( W \), and let \( \partial W \) be the set of all endpoints of \( W \);

• a finite subset of \( \mathfrak{S} \), whose elements are called internal vertices:
• a finite set of non-closed oriented smooth curves in \( \mathcal{S} \) ending at points in external or internal vertices, whose elements are called (oriented) edges of \( W \);
• a finite set of closed oriented smooth curves in \( \mathcal{S} \), whose elements are called (oriented) loops of \( W \), subject to the following conditions:

(W1) each external vertex \( v \) is 1-valent, i.e. exactly one edge of \( W \) ends at \( v \) and this edge meets \( v \) once;
(W2) each internal vertex \( v \) is either a 3-valent sink or a 3-valent source, i.e. exactly three edges of \( W \) end at \( v \), and the orientations of them are either all toward \( v \) or all outgoing from \( v \);
(W3) each self-intersection of the union of all members of \( W \) except for internal vertices, called a crossing of \( W \), is a transverse double intersection lying in the interior of \( \mathcal{S} \);
(W4) there are at most finitely many crossings.

We depict the external and the internal vertices of \( W \) by bullets •. We allow the empty \( A_2 \)-web \( \emptyset \).

**Definition 3.2.** Let \( \mathcal{S} \) be a generalized marked surface.

• An isotopy of \( A_2 \)-webs in \( \mathcal{S} \) is an isotopy within the class of \( A_2 \)-webs in \( \mathcal{S} \). Two \( A_2 \)-webs in \( \mathcal{S} \) are said to be equivalent if they are related by a sequence of isotopies and the following moves:

(M1) Reidemeister moves I \( \leftrightarrow \), II \( \leftrightarrow \), and III \( \leftrightarrow \), with all possible orientations;
(M2) the web Reidemeister move \( \leftrightarrow \), with all possible orientations (according to Def 3.1);
(M3) the boundary exchange move \( \leftrightarrow \), with all possible orientations, where the horizontal blue line is boundary.

• Let \( \mathcal{R} \) be a commutative ring with unity \( 1 \). The (commutative) \( A_2 \)-skein algebra \( S(\mathcal{S}; \mathcal{R}) \) is the free \( \mathcal{R} \)-module with the set of all equivalence classes of \( A_2 \)-webs in \( \mathcal{S} \) as a free basis, mod out by the following \( A_2 \)-skein relations \( (S1), (S2), (S3) \) and \( (S4) \) in Fig.2

• For an equivalence class of \( A_2 \)-webs \( W \) in \( \mathcal{S} \), the corresponding element of the \( A_2 \)-skein algebra \( S(\mathcal{S}; \mathcal{R}) \) is denoted by \([W]\) and is called an \( A_2 \)-skein.

The following special class of \( A_2 \)-webs are important.

**Definition 3.3 ([K96 SW07 FS20]).** Let \( \mathcal{S} \) be a generalized marked surface.

• An \( A_2 \)-web \( W \) in \( \mathcal{S} \) is said to be non-elliptic if all of the following hold:

(NE1) \( W \) has no crossings;
(NE2) none of the loops of \( W \) is a contractible loop in \( \mathcal{S} \);
(NE3) none of the components of the complement in \( \mathcal{S} \) of the union of all edges of \( W \) is a contractible region bounded by either two or four edges of \( W \) (as appearing in the first term of \( (S2) \) or \( (S3) \)).

• A non-elliptic \( A_2 \)-web \( W \) is weakly reduced if it contains none of \( \begin{array}{c}
\begin{array}{c}
\text{•} \\
\text{•}
\end{array}
\end{array} \) and \( \begin{array}{c}
\begin{array}{c}
\text{•}
\end{array}
\end{array} \), and is reduced if furthermore it contains none of \( \begin{array}{c}
\begin{array}{c}
\text{•} \text{•}
\end{array}
\end{array} ; \) in these pictures, the blue line is boundary, the edges can be given all possible orientations (according to Def 3.1), and the boundary 2-gon, 3-gon and 4-gon are contractible.

**Remark 3.4.** For a punctured surface \( \mathcal{S} \), any non-elliptic \( A_2 \)-web is reduced.

**Proposition 3.5 ([SW07 FS20]).** Let \( \mathcal{S} \) be a punctured surface. The set of all \( A_2 \)-skeins for non-elliptic \( A_2 \)-webs form a basis of \( S(\mathcal{S}) \).

This can be viewed as an \( A_2 \) analog of the so-called bangle basis of the usual (Kaufmann bracket) skein algebra (of type \( A_1 \)); we will elaborate in the next section.

3.2. \( A_2 \)- laminations. **For \( A_1 \)-type theory**, Fock and Goncharov [FG06] introduced certain versions of laminations on a surface \( \mathcal{S} \), where a lamination is defined as a collection of mutually non-intersecting simple unoriented curves equipped with weights, where a constituent curve is either closed or ends at components of \( \partial \mathcal{S} \), and weights are rational or integer numbers. For \( A_2 \)-type theory, here based on reduced non-elliptic \( A_2 \)-webs we propose a generalization of Fock-Goncharov’s integral \( A \)- laminations which were based on just simple curves, or \( A_1 \)-webs. Basic idea is to consider reduced non-elliptic \( A_2 \)-webs with weights given on its (connected) components. The weights are required to be non-negative integers, except for the special cases as in [FG07] for the \( A_1 \)-webs, which we call peripheral.
Definition 3.6. Let $(\Sigma, P)$ be a generalized marked surface, and $\mathcal{S} := \Sigma \setminus P$.

- A simple loop in $\mathcal{S}$ is called a peripheral loop if it bounds a region homeomorphic to a disc with one puncture in the interior. If the corresponding puncture is $p \in P$, we say that this peripheral loop surrounds $p$.

- A peripheral arc in $\mathcal{S}$ is a simple curve $e$ in $\mathcal{S}$ that ends at points of $\partial \mathcal{S}$ and that bounds a region homeomorphic to an upper-half disc with one puncture on the boundary, i.e. $e$ is homotopic in $\Sigma$ (rel endpoints) to a simple arc $e'$ lying in $\partial \Sigma$ such that $e'$ contains exactly one point of $P$, say $p$; in this case, we say this peripheral arc surrounds $p$.

- Peripheral loops and peripheral arcs are referred to as peripheral curves.

Note that an oriented peripheral curve is an example of a non-elliptic $A_2$-web.

Definition 3.7. Let $W$ be an $A_2$-web in a generalized marked surface $\mathcal{S}$. We define components of $W$ as follows. First, each loop of $W$ is a component of $W$, and each edge of $W$ whose two endpoints are both external vertices of $W$ is a component. A union of a collection $C$ of at least two edges of $W$ is called a component if

(C1) for any two distinct edges $e$ and $e'$ of $C$, there is a sequence of edges $e_1, \ldots, e_n$ such that $e_1 = e$, $e_n = e'$, and $e_i$ meets $e_{i+1}$ at an internal vertex of $W$ for each $i = 1, \ldots, n - 1$;

(C2) $C$ is maximal among the collections satisfying (C1).

Remark 3.8. Each component of $W$ is an $A_2$-web on its own.

The following is the first main definition of the present paper.

Definition 3.9 ($A_2$-laminations). Let $\mathcal{S}$ be a generalized marked surface.

- An (integral) $A_2$-lamination $\ell$ in $\mathcal{S}$ consists of the equivalence class of a reduced non-elliptic $A_2$-web $W = \overline{W(\ell)}$ in $\mathcal{S}$ and the assignment of an integer weight to each component of $W(\ell)$, subject to the following conditions and equivalence relation:

  (L1) the weight for each component of $W(\ell)$ containing an internal vertex is 1;

  (L2) the weight for each component of $W(\ell)$ that is not a peripheral curve is non-negative;

  (L3) an $A_2$-lamination containing a component of weight zero is equivalent to the $A_2$-lamination with this component removed;

  (L4) an $A_2$-lamination with two of its components being homotopic with weights $a$ and $b$ is equivalent to the $A_2$-lamination with one of these components removed and the other having weight $a + b$.

Let $\mathcal{A}_2(\mathcal{S}; \mathbb{Z})$ be the set of all (integral) $A_2$-laminations in $\mathcal{S}$.

Let $\mathcal{A}_{2,0}^d(\mathcal{S}; \mathbb{Z})$ be the set of all (integral) $A_2$-laminations in $\mathcal{S}$ with no negative weights.

Lemma 3.10. An $A_2$-lamination in $\mathcal{A}_{2,0}^d(\mathcal{S}; \mathbb{Z})$ can be represented by an $A_2$-lamination whose weights are all 1. This gives a bijection

$$\mathcal{A}_{2,0}^d(\mathcal{S}; \mathbb{Z}) \leftrightarrow \{\text{reduced non-elliptic } A_2\text{-webs in } \mathcal{S}\}. \quad \blacksquare$$

Crucial in the study of $A_2$-webs and $A_2$-laminations is a coordinate system for them. The coordinate system which we will construct requires the choice of an ideal triangulation $\Delta$ of the surface $\mathcal{S}$. We first isotope an $A_2$-web to be in a minimal position with respect to $\Delta$.

Definition 3.11 ([FS20]). Let $\mathcal{S}$ be a generalized marked surface, and let $\Delta$ be a collection of mutually non-intersecting ideal arcs in $\mathcal{S}$. A non-elliptic $A_2$-web $W$ in a triangulable punctured surface $\mathcal{S}$ is said to be in a minimal position with respect to $\Delta$ of $\mathcal{S}$ if the cardinality of the intersection $W \cap \Delta$ equals the minimum of the cardinality of $W' \cap \Delta$ among all non-elliptic $A_2$-webs $W'$ in $\mathcal{S}$ isotopic to $W$.

For our convenience, we may assume that a non-elliptic $A_2$-web in a minimal position with respect to $\Delta$ meets edges of $\Delta$ transversally. In fact, putting into a minimal position with respect to $\Delta$ is not sufficient for the purpose of constructing our coordinates, and we need a further tidying-up process; we use a result obtained in [FS20]. In the end, we would like our $A_2$-web in each triangle $t$ of $\Delta$ to be a disjoint union of elementary pieces; namely, peripheral arcs of $t$ called corner arcs in $t$, or special webs having internal vertices called pyramids $H_d$ for $d \in \mathbb{Z} \setminus \{0\}$, some examples of which are depicted in Fig. 4. In particular, $H_d$ has $|d|$ external vertices on each side of $t$, and $H_{-d}$ can be obtained from $H_d$ by reversing the orientation of all edges of $H_d$. 
By looking at these pictures, we believe that the readers can deduce the definition of $H_d$ for each $d \in \mathbb{Z} \setminus \{0\}$; see [FS20, §10] for a precise recipe for constructing $H_d$, and also see [DS20] where $H_d$ is called a honeycomb-web. One way of understanding the construction is first to consider the nodes of a $(|d|+1)$-triangulation quiver of Fock-Goncharov [FG06] (Def.6.15) one example being the 3-triangulation we saw in Def.2.18 (which has $1 + 2 + \cdots + (|d| + 2) - 3 = \frac{(|d|+2)(|d|+3)}{2} - 3$ nodes), and for each triple of nodes forming an ‘upside-down’ small triangle add one more node in the middle of this small triangle and add three edges as $\bigtriangleup$ if $d > 0$ and as $\bigtriangleup$ if $d < 0$. In particular, one can observe that there are $|d|^2$ internal 3-valent vertices in the interior of $t$.

**Definition 3.12 (FS20).** Let $t$ be a triangle, viewed as a generalized marked surface diffeomorphic to a closed disc with three marked points on the boundary.

- For $d \in \mathbb{Z} \setminus \{0\}$, the $A_2$-web $H_d$ described above is a degree $d$ pyramid in $t$. Let $H_0 = \emptyset$.
- A single-component non-elliptic $A_2$-web in $t$ consisting of an edge connecting the 1-valent vertices lying in two distinct sides of $t$ (i.e. a peripheral arc in $t$) is called a corner arc in $t$.
- An $A_2$-web in $t$ is canonical if it is a disjoint union of one $H_d$ for some $d \in \mathbb{Z}$ and some number of (possibly none of) corner arcs.

We note that it is not always possible to isotope a reduced non-elliptic $A_2$-web in a triangulated surface so that it is canonical in each triangle of $\Delta$. One remedy is to fatten each edge of $\Delta$ to a biangle, and push some 3-valent vertices into the biangles.

**Definition 3.13 (BW11).** Let $\Delta$ be a triangulation of a triangulable generalized marked surface $\mathcal{G} = \Sigma \setminus \mathcal{P}$. For each edge $e$ of $\Delta$, choose a $\mathcal{P}$-arc $e'$ in $\Sigma$ isotopic to $e$ as $\mathcal{P}$-arcs, such that $\Delta := \Delta \cup \{e' : e \in \Delta\}$ is a mutually non-intersecting collection of $\mathcal{P}$-arcs. We call $\tilde{\Delta}$ a split ideal triangulation of $\mathcal{G}$ for the triangulation $\Delta$. The closure (in $\Sigma$ or $\tilde{\Delta}$) of a connected component of the complement of (the union of members of) $\tilde{\Delta}$ in $\Sigma$ is called a (ideal) triangle of $\tilde{\Delta}$ if it is bounded by three edges of $\tilde{\Delta}$, and a (ideal) biangle of $\tilde{\Delta}$ if it is bounded by two edges of $\tilde{\Delta}$.

Edges of $\Delta$ are in bijection with the biangles of $\tilde{\Delta}$, and triangles of $\Delta$ are in bijection with the triangles of $\tilde{\Delta}$. Each triangle and biangle of $\tilde{\Delta}$ may be viewed as a generalized marked surface on its own, in a natural way. In particular, a biangle $\Delta$ (Def.2.2) can be considered as being a generalized marked surface diffeomorphic to a closed disc, with two marked points on the boundary, with no punctures.

We now recall from [FS20] some special classes of $A_2$-webs in a biangle. To match our convention used in a later section, we change some notations and definitions, and re-interpret a little bit.

**Definition 3.14 (FS20, §9).** Let $B$ be a biangle, viewed as a generalized marked surface. For integer $n \geq 1$, consider a mutually disjoint collection $\mathcal{H}$ of $n$ strands, each of which is a simple curve connecting the two sides of $B$. Choose a possibly-empty mutually disjoint finite collection of crossbars, each of which is a simple curve connecting two adjacent strands, such that each intersection of crossbars and strands are transverse double intersection in the interior of $B$, and that under a homeomorphism of the biangle $B$ to $\mathbb{R} \times [0,1]$ (the two sides going to $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$) each strand is of the form $[c] \times [0,1]$ (i.e. vertical) and each crossbar is of the form $[c_1, c_2] \times \{a\}$ (i.e. horizontal). Union of all $n$ strands together with all of these chosen crossbars is called a crossbar graph of index $n$.

- A crossing-less $A_2$-web $W$ in $B$ is called a crossbar $A_2$-web (in $B$) of index $n$ if the union of all its components, with orientations forgotten, is a crossbar graph of index $n$.
- The signature of a crossbar $A_2$-web $W$ is the map $\partial W \to \{\text{sink, source}\}$, which records the sink-source information of $W$ at each external vertex of $W$.
- A minimal crossbar $A_2$-web in $B$ (of index $n$) is a crossbar $A_2$-web in $B$ (of index $n$) that is non-elliptic (i.e. does not contain a contractible 4-gon as in (S3) of Fig.2).

**Remark 3.15.** Not every crossbar graph admits a crossbar $A_2$-web structure.

By convention, a crossbar $A_2$-web of index 0 means the empty $A_2$-web, which is a minimal crossbar $A_2$-web. In the above definition, one observes that a crossbar $A_2$-web is minimal if there are no two ‘consecutive’ crossbars. It is easy to observe that a minimal crossbar $A_2$-web is weakly reduced.
Definition 3.17 ([FS20]). A weakly reduced non-elliptic $A_2$-web $W$ in a triangulable generalized marked surface $\mathcal{S}$ is said to be canonical with respect to a split ideal triangulation $\Delta$ of $\mathcal{S}$ if:

(CW1) for each triangle $\hat{\ell}$ of $\Delta$, the intersection $W \cap \hat{\ell}$ is a canonical $A_2$-web in $\hat{\ell}$ (Def. 3.12);
(CW2) for each biangle $B$ of $\Delta$, the intersection $W \cap B$ is a minimal crossbar $A_2$-web in $B$ (Def. 3.14).

An $A_2$-lamination in $\mathcal{S}$ is said to be canonical with respect to $\Delta$ if the underlying $A_2$-web is canonical with respect to $\Delta$.

Lemma 3.18 ([FS20]). Let $\Delta$ be a split ideal triangulation of a triangulable generalized marked surface $\mathcal{S}$. Any weakly reduced non-elliptic $A_2$-web $W$ in $\mathcal{S}$ is isotopic to a weakly reduced non-elliptic $A_2$-web in $\mathcal{S}$ that is canonical with respect to $\Delta$.

A useful observation:

Lemma 3.19. Any canonical $A_2$-web in a triangle is a reduced non-elliptic $A_2$-web in that triangle, when the triangle is viewed as a generalized marked surface.

Corollary 3.20. Let $\ell$ be a $A_2$-lamination in a triangulable generalized marked surface $\mathcal{S}$ that is represented by a weighted reduced non-elliptic $A_2$-web $W$ in $\mathcal{S}$ that is canonical with respect to a split ideal triangulation $\Delta$. For each triangle $\hat{\ell}$ of $\Delta$, $\ell \cap \hat{\ell}$ is an $A_2$-lamination in $\hat{\ell}$ (represented by weighted $A_2$-web $W \cap \hat{\ell}$), when $\hat{\ell}$ is viewed as a generalized marked surface on its own.

3.3. Frohman-Sikora coordinates and Douglas-Sun coordinates for $A_2$-webs. In the present subsection we recall two coordinate systems for non-elliptic $A_2$-webs in $\mathcal{S}$ with respect to an ideal triangulation $\Delta$ of $\mathcal{S}$, one by Frohman-Sikora [FS20] and the other by Douglas-Sun [DS20]. We modified the notations to fit our purpose.

Definition 3.21 ([FS20]). Let $t$ be a triangle, viewed as a generalized marked surface. Let $e_1, e_2, e_3$ be the sides of $t$, appearing clockwise in $\partial t$. Let $W$ be a canonical $A_2$-web in $t$ (Def. 3.12).

For each side $e_\alpha$, let $e_{\text{out}, \alpha}(W)$ be the number of external vertices of $W$ that are sinks and lie on $e_\alpha$, and $e_{\text{in}, \alpha}(W)$ be the number of external vertices of $W$ that are sources and lie on $e_\alpha$. These six numbers are called intersection coordinates of $W$.

A corner arc of $W$ is said to be left turn if it starts at a vertex in $e_\alpha$ and terminates at a vertex in $e_{\alpha+1}$ (where $e_4 = e_1$), and right turn otherwise. Let

$$r_\alpha(W) = (\text{number of left turn corner arcs of } W) - (\text{number of right turn corner arcs of } W),$$

which is called the rotation number of $W$.

Definition 3.22 ([FS20]). Let $\hat{\Delta}$ be a split ideal triangulation of a triangulable generalized marked surface $\mathcal{S}$. Let $W$ be a weakly reduced non-elliptic $A_2$-web in $\mathcal{S}$ that is canonical with respect to $\hat{\Delta}$. For each triangle $\hat{\ell}$ of $\hat{\Delta}$, consider the intersection coordinates and the rotation number for the $A_2$-web $W \cap \hat{\ell}$ in the triangle $\hat{\ell}$. These numbers are the Frohman-Sikora coordinates of $W$ with respect to $\hat{\Delta}$.

Each edge $e$ of $\hat{\Delta}$ is a side of a unique triangle of $\hat{\Delta}$, say $\hat{\ell}$; denote by $e_{\text{out}, \alpha}(W)$ and $e_{\text{in}, \alpha}(W)$ the intersection coordinates of $W \cap \hat{\ell}$ at this side $e$. Write $r_{\alpha}(W) := r_{\alpha}(W \cap \hat{\ell})$. If $e$ and $e'$ are edges of $\hat{\Delta}$ forming a biangle, then it is easy to see from Lem. 3.16 (MC2) that $e_{\text{out}, \alpha}(W) = e_{\text{in}, \alpha}(W)$ and $e_{\text{out}, \alpha}(W) = e_{\text{out}, \alpha}(W)$. So, one can think of the intersection coordinates to be assigned to edges of $\Delta$, instead of edges of $\hat{\Delta}$, and hence one may also say that the Frohman-Sikora coordinates are defined with respect to the ideal triangulation $\Delta$ instead of $\hat{\Delta}$; we might be using $\Delta$ and $\hat{\Delta}$ interchangeably in
this respect. The following asserts that these coordinates indeed form a coordinate system, and is one of the two main results of [FS20].

**Proposition 3.23** ([FS20]). Let $\hat{\Delta}$ be a split ideal triangulation of a triangulable generalized marked surface $\mathcal{S}$. Let $W$ be a weakly reduced non-elliptic $A_2$-web in $\mathcal{S}$, not necessarily canonical with respect to $\hat{\Delta}$.

(FS1) Define Frohman-Sikora coordinates of $W$ with respect to $\hat{\Delta}$ by using any weakly reduced non-elliptic $A_2$-web $W'$ in $\mathcal{S}$ that is equivalent to $W$ and is canonical with respect to $\hat{\Delta}$. Then these coordinates are well-defined, i.e. do not depend on the choice of $W'$;

(FS2) If $W$ is reduced, the Frohman-Sikora coordinates of $W$ with respect to $\hat{\Delta}$ completely determine $W$ up to equivalence, i.e. two reduced non-elliptic $A_2$-webs with same Frohman-Sikora coordinates are equivalent.

This coordinate system is geometrically intuitive, and gives an injection

$$\{\text{equivalence classes of reduced non-elliptic } A_2\text{-webs in } \mathcal{S}\} \longrightarrow (\mathbb{Z}_{\geq 0})^{\hat{\Delta}} \times (\mathbb{Z}_{\geq 0})^{\mathcal{S}}_F(\Delta),$$

where $\mathcal{S}(\Delta)$ is the set of all ideal triangles of $\Delta$. This coordinate map is not surjective, so one may want to study the structure of the image set; see [FS20] for a discussion.

We now recall another set of coordinates studied by Douglas-Sun [D20] [DS20], which better suits our purposes. These works [D20] [DS20] deal only with punctured surfaces, and we allow $\mathcal{S}$ to have boundary. Their coordinates are parametrized by the nodes of the 3-triangulation quiver $Q_\Delta$ (Def. 2.18).

**Definition 3.24** (Douglas-Sun [DS20]). Let $\mathcal{S}$ be a triangulable generalized marked surface, $\Delta$ an ideal triangulation of $\mathcal{S}$ and $\hat{\Delta}$ a split ideal triangulation for $\Delta$. Let $Q_\Delta$ be the 3-triangulation quiver for $\Delta$ (Def. 2.18). Define the Douglas-Sun coordinate map

$$\left(\begin{array}{c}
\text{equivalence classes of weakly reduced non-elliptic } A_2\text{-webs in } \mathcal{S}\end{array}\right) \longrightarrow \left(\begin{array}{c}
\frac{1}{3}\mathbb{Z}_{\geq 0}\end{array}\right)^{(Q_\Delta)},$$

for $\prod \in (\mathbb{Z}_{\geq 0})^{\mathcal{S}}_F(\Delta)$, given, for a weakly reduced non-elliptic $A_2$-web $W$ in $\mathcal{S}$, by the following number per each node of $Q_\Delta$.

Let $W'$ be any weakly reduced non-elliptic $A_2$-web in $\mathcal{S}$ that is equivalent to $W$ and is canonical with respect to $\hat{\Delta}$. Let $t$ be a triangle of $\Delta$, and $\hat{t}$ be the triangle of $\hat{\Delta}$ corresponding to $t$, so that $W' \cap \hat{t}$ is a canonical $A_2$-web in $\hat{t}$. The coordinates of $W$ for these nodes $V(Q_\Delta) \cap t$ are defined as the coordinates of the $A_2$-web $W' \cap \hat{t}$ in $\hat{t}$ for these nodes, given as follows.

We require that the coordinates are additive for $t$, in the sense that for $v \in V(Q_\Delta) \cap t$, if $W_1,W_2$ are disjoint $A_2$-webs in $\hat{t}$, then the coordinate of $W_1 \cap W_2$ for $v$ equals the sum of the coordinates of $W_1$ and $W_2$ for $v$. Then it suffices to define the coordinates for a corner arc $A_2$-web in $\hat{t}$ and for a pyramid $A_2$-web $\hat{H}_\Delta$ in $\hat{t}$, which are given in Fig. 5.

In fact, the original Douglas-Sun coordinates defined in [DS20] are 3 times the ones depicted in Fig. 5 hence are integers. The reason why we use the $\frac{1}{3}$-scaled version will be justified by our main result.

**Proposition 3.25** ([DS20]). In case when $\mathcal{S}$ is a punctured surface, the above coordinate system yields a well-defined injection as in eq. (3.1).

Note that, for a punctured surface $\mathcal{S}$, the condition of a non-elliptic $A_2$-web being weakly reduced or reduced is redundant. In view of the relationship between Frohman-Sikora coordinates and Douglas-Sun coordinates as shall be seen in the next subsection, Prop. 3.25 can be enhanced to reduced non-elliptic $A_2$-webs in a triangulable generalized marked surface.

In [DS20], the image of eq. (3.1) is studied in detail. As mentioned in [DS20], this coordinate system is inspired by the degrees of the highest term of a (sought-for) canonical regular function on $\mathcal{X}_{PGL_3}$ associated to each $A_2$-web $W$, and this idea goes back to Xie [X13]. In a sense, the results of the present paper will fully justify this idea. Even without the result of the present paper, one can study some remarkable properties of the Douglas-Sun coordinate systems, a crucial one being the behavior under flip of an ideal triangulation.

**Proposition 3.26** (coordinate change formula for Douglas-Sun coordinates; [DS2]). Let $\Delta$ and $\Delta'$ be ideal triangulations of a triangulable punctured surface $\mathcal{S}$ related to each other by a flip at an edge. Let $W$ be a non-elliptic $A_2$-web in $\mathcal{S}$. The Douglas-Sun coordinates $(a_v)_v \in (\frac{1}{3}\mathbb{Z}_{\geq 0})^{V(Q_{\Delta'})}$ and $(a'_v)_{v'} \in (\frac{1}{3}\mathbb{Z}_{\geq 0})^{V(Q_{\Delta'})}$ of $W$ with respect to $\Delta$ and $\Delta'$ are related by the sequence of tropical cluster
\(A_2\)-mutations with respect to the sequence of mutations associated to a flip. To be more precise, if we label the nodes of \(Q_\Delta\) and \(Q_{\Delta'}\) for triangles having the flipped arc as a side as in Fig[4], then
\[
a_{v_3}' = -a_{v_3} + \max(a_{v_2} + a_{v_{12}}, a_{v_7} + a_{v_8}), \quad a_{v_4}' = -a_{v_4} + \max(a_{v_7} + a_{v_{11}}, a_{v_5} + a_{v_{12}})
\]
\[
a_{v_5}' = -a_{v_5} + \max(a_{v_1} + a_{v_4}', a_{v_6} + a_{v_9}'), \quad a_{v_12}' = -a_{v_{12}} + \max(a_{v_3}' + a_{v_{10}}, a_{v_4}' + a_{v_9}).
\]
Nodes \(v\) in \(Q_\Delta\) other than \(v_3, v_4, v_7, v_{12}\) in Fig[4] are naturally in bijection with nodes \(v'\) in \(Q_{\Delta'}\). To be more precise, if we label the nodes of \(A\)-mutations with respect to the sequence of mutations associated to a flip. To be more precise, if we label the nodes of \(Q_\Delta\) and \(Q_{\Delta'}\) for triangles having the flipped arc as a side as in Fig[4], then
\[
a_{v_3}' = -a_{v_3} + \max(a_{v_2} + a_{v_{12}}, a_{v_7} + a_{v_8}), \quad a_{v_4}' = -a_{v_4} + \max(a_{v_7} + a_{v_{11}}, a_{v_5} + a_{v_{12}})
\]
\[
a_{v_5}' = -a_{v_5} + \max(a_{v_1} + a_{v_4}', a_{v_6} + a_{v_9}'), \quad a_{v_12}' = -a_{v_{12}} + \max(a_{v_3}' + a_{v_{10}}, a_{v_4}' + a_{v_9}).
\]

The reason why we consider the specific linear combinations of analogs of Frohman-Sikora coordinates as in eq\.[3.2] and eq\.[3.4], as well as the word \textit{tropical}, is related to the coordinate change formula

\[d_t(\ell) = \frac{1}{3} \sum_{i=1}^{3} c_{out, e}(\ell) - \frac{1}{3} \sum_{i=1}^{3} c_{in, e}(\ell).
\]

\[d_t(\ell) \geq 0, \quad d_t(\ell) \leq 0.
\]

The numbers \(a_v(\ell)\) are called \textit{tropical coordinates} for \(\ell\).

**Remark 3.28.** The degree \(d_t(\ell)\) is a generalization of a corresponding concept defined for \(A_2\)-webs in [FS20 §12], which detects the degree of the pyramid in triangles of \(\Delta\).

It is easy to see that the degree for a triangle \(t\) can be expressed using the edge coordinates for the sides \(e_1, e_2, e_3\) of that triangle:
\[
d_t(\ell) = \sum_{\alpha=1}^{3} a_{v_{\alpha-1}}(\ell) - \sum_{\alpha=1}^{3} a_{v_{\alpha}}(\ell).
\]

The reason why we consider the specific linear combinations of analogs of Frohman-Sikora coordinates as in eq\.[3.2] and eq\.[3.4], as well as the word \textit{tropical}, is related to the coordinate change formula
under flips of triangulations which we will soon discuss, as seen for Douglas-Sun coordinates of $A_2$-webs. Indeed, one can verify that our coordinates agree with Douglas-Sun’s on reduced non-elliptic $A_2$-webs, which can naturally be viewed as $A_2$-laminations (with all weights being $1$).

**Lemma 3.29.** Let $\ell$ be represented by a reduced non-elliptic $A_2$-web $W$ with weight $1$, in a triangulable generalized marked surface $\mathcal{G}$. For an ideal triangulation $\Delta$ and for each node $v$ of $Q_\Delta$, our coordinate $a_v(\ell)$ coincides with Douglas-Sun’s coordinate of $W$ at $v$.

This lemma may be useful already, as Douglas-Sun’s coordinates are defined in [DS20] like in Def 3.24 but explicit formulas for them in terms of Frohman-Sikora coordinates are not given. We postpone a proof of this lemma until a little bit later.

Meanwhile, here is the first major assertion about our coordinates for $A_2$-laminations.

**Proposition 3.30.** The coordinates of Def.3.27 provide a well-defined map

$$a_\Delta : \{A_2\text{-laminations in }\mathcal{G}\} \longrightarrow B_\Delta \subset (\mathbb{Z}/3)^{V(Q_\Delta)}$$

where $B_\Delta$ is the set of all balanced elements of $(\mathbb{Z}/3)^{Q_\Delta}$, where an element $(a_v)_v \in (\mathbb{Z}/3)^{Q_\Delta}$ is said to be balanced if for each triangle $t$ of $\Delta$, if we denote by $e_1, e_2, e_3$ the sides of $t$, then the number $d_t := \sum_{\alpha=1}^3 a_{v_{e_\alpha}} - \sum_{\alpha=1}^3 a_{v_{e_\alpha+1}}$ belongs to $\mathbb{Z}$.

**Proof.** Well-definedness of the above coordinates follows from that of Frohman-Sikora coordinates for (weakly) reduced non-elliptic $A_2$-webs. Next, we should check whether the coordinates have values in $\mathbb{Z}/3$; this is clear for the edge coordinates. Let’s show that the triangle coordinates also have values in $\mathbb{Z}/3$. Let $\hat{\ell}$ be a triangle of $\hat{\Delta}$ corresponding to a triangle $t$ of $\Delta$, and let $e_1, e_2, e_3$ be the sides of $\hat{\ell}$ appearing clockwise in this order along $\partial \hat{\ell}$. Represent $\ell$ as a weighted reduced non-elliptic $A_2$-web $W(\ell)$ that is canonical with respect to $\hat{\Delta}$. For $\alpha, \beta \in \{1, 2, 3\}$, let $c_{\alpha, \beta} = c_{\alpha, \beta}(\ell)$ be the sum of weights of corner arcs of the $A_2$-web $W(\ell) \cap \hat{\ell}$ in $\hat{\ell}$ going from edge $e_\alpha$ to edge $e_\beta$. By Def.3.17(CW1) and Def.3.12 it follows that $W(\ell) \cap \hat{\ell}$ is a union of a single pyramid $H_{d_t}$ of some degree $d_t \in \mathbb{Z}$ and corner arcs. By Def.3.9(L1), $H_{d_t}$ has weight $1$. So we have

$$e_{\text{out}, e_\alpha}(\ell) = c_{\alpha+1, \alpha} + c_{\alpha-1, \alpha} + [d_t]_+, \quad e_{\text{in}, e_\alpha}(\ell) = c_{\alpha, \alpha+1} + c_{\alpha, \alpha-1} + [-d_t]_+ \quad \alpha = 1, 2, 3,$$

where the subscript indices in $c_{\alpha, \beta}$ are considered modulo $3$ (e.g. $c_{12} = c_{12}$), and $[a]_+$ is as in eq.(2.1). In particular, note $[a]_+ = (a + |a|)/2$, and hence $[a]_+ - [-a]_+ = a$. Thus we observe

$$d_t(\ell) = \frac{1}{3} \sum_{\alpha=1}^3 e_{\text{out}, e_\alpha}(\ell) - \frac{1}{3} \sum_{\alpha=1}^3 e_{\text{in}, e_\alpha}(\ell) = [d_t]_+ - [-d_t]_+ = d_t.$$

When $d_t(\ell) = d_t \geq 0$, note

$$r_t(\ell) + 3 \sum_{\alpha=1}^3 a_{v_{e_\alpha}} \ell(\ell) = \left(\sum_{\alpha=1}^3 c_{\alpha, \alpha+1} - \sum_{\alpha=1}^3 c_{\alpha+1, \alpha}\right) + \left(2 \sum_{\alpha=1}^3 e_{\text{out}, e_\alpha} + \sum_{\alpha=1}^3 e_{\text{in}, e_\alpha}\right)$$

$$= 4 \sum_{\alpha=1}^3 c_{\alpha, \alpha+1} + 2 \sum_{\alpha=1}^3 c_{\alpha+1, \alpha} + 2[d_t]_+ + [-d_t]_+ \in 2\mathbb{Z}$$

so $a_{v_\ell}(\ell) = \frac{1}{6}(r_t(\ell) + 3 \sum_{\alpha=1}^3 a_{v_{e_\alpha}} \ell(\ell)) \in \mathbb{Z}/3$. Similarly, when $d_t(\ell) = d_t \leq 0$, one observes $a_{v_\ell}(\ell) = \frac{1}{6}(r_t(\ell) + 3 \sum_{\alpha=1}^3 a_{v_{e_\alpha}} \ell(\ell)) = \frac{1}{6}(4 \sum_{\alpha=1}^3 c_{\alpha, \alpha+1} + 2 \sum_{\alpha=1}^3 c_{\alpha+1, \alpha} + [d_t]_+ + [d_t]_+ \in 2\mathbb{Z}$. So indeed, all coordinate values lie in $\mathbb{Z}/3$. Also, since $d_t(\ell) = d_t \in \mathbb{Z}$, it follows that each image $(a_v(\ell))_v$ of the coordinate map is balanced.

We shall prove that the image of the coordinate map coincides with $B_\Delta$. We first establish one useful lemma, which is straightforward to see.

**Definition 3.31.** We say that $A_2$-laminations $\ell_1, \ldots, \ell_n$ in a generalized marked surface are disjoint if they can be represented by weighted reduced non-elliptic $A_2$-webs that are mutually disjoint. We denote by $\ell_1 \cup \cdots \cup \ell_n$ the $A_2$-lamination obtained by taking the union of them.

**Lemma 3.32 (additivity of coordinates under disjoint union).** Suppose that $\ell_1, \cdots, \ell_n$ are disjoint $A_2$-laminations in a triangulable generalized marked surface $\mathcal{G}$. For any triangulation $\Delta$ of $\mathcal{G}$, we have

$$a_\Delta(\ell_1 \cup \cdots \cup \ell_n) = a_\Delta(\ell_1) + \cdots + a_\Delta(\ell_n),$$

i.e. $a_v(\ell_1 \cup \cdots \cup \ell_n) = a_v(\ell_1) + \cdots + a_v(\ell_n)$ holds for every node $v$ of $Q_\Delta$. 

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Proof. It suffices to prove the assertion when $\mathcal{S}$ is a triangle, and when each $\ell_i$ can be represented by a single-component weakly reduced non-elliptic $A_2$-web in a triangle. One can see that at most one of $\ell_1, \ldots, \ell_n$ can contain an internal vertex, i.e. can be a pyramid $H_d$ with $d \neq 0$, and others are all corner arcs. If $d > 0$ or there is no pyramid, then $d_1(\ell_i) \geq 0$ for all $i$ and $d_1(\ell_1 \cup \cdots \cup \ell_n) \geq 0$, hence $a_{v_0}(\ell_i)$ as well as $a_{v_0}(\ell_1 \cup \cdots \cup \ell_n)$ is given by the first line formula of eq. (3.4). So, all of $a_{v_0}(\cdot), a_{v_0}(\cdot), r_1(\cdot)$, and hence also $a_{v_0}(\cdot)$ are additive for $\ell_1, \ldots, \ell_n$. Likewise, if $d < 0$ or there is no pyramid, the second line formula of eq. (3.4) applies to all $\ell_1, \ldots, \ell_n$ and $\ell_1 \cup \cdots \cup \ell_n$, so the coordinates are additive. 

Before proceeding further, we use this lemma to prove the promised easy lemma, Lem. 3.29

Proof of Lem. 3.29 It suffices to check this for each node $v$ living in a triangle $t$ of $\Delta$. Let $\hat{t}$ be the triangle of the split ideal triangulation $\hat{\Delta}$. When $W$ is canonical with respect to $\hat{\Delta}$, note that $W \cap \hat{t}$ is a (weakly) reduced $A_2$-web in $t$, and the tropical coordinates of $W \cap \hat{t}$ coincide with the tropical coordinates of $W$ for the nodes of $Q_\Delta$ living in $t$. Since our coordinates are additive (Lem. 3.32) and so are Douglas-Sun’s by construction, it suffices to show the equality for each component of $W \cap \hat{t}$, which is a corner arc or a pyramid. For these elementary cases, the Douglas-Sun coordinates are as in Fig. 5 which we verify to be same as ours as follows.

Let the side names $e_{\alpha}$, node names $v_{e_0,1}, v_{e_0,2}, v_i$ be as in Def. 3.27. Let $W_{\alpha,\alpha+1}$ be a left turn corner arc $A_2$-web in $t$, going from side $e_{\alpha}$ to $e_{\alpha+1}$ (where $e_4 = e_1$). The Frohman-Sikora coordinates are $0 = a_{v_{e_0,1}}(W_{\alpha,\alpha+1}) = a_{\alpha}(W_{\alpha,\alpha+1}) = a_{v_{e_0,2}}(W_{\alpha,\alpha+1}) = a_{v_{e_0,1}}(W_{\alpha,\alpha+1}), 1 = a_{v_{e_0,1}}(W_{\alpha,\alpha+1}) = a_{v_{e_0,2}}(W_{\alpha,\alpha+1}), r_1(W_{\alpha,\alpha+1}) = 1$, and $d_1(W_{\alpha,\alpha+1}) = 0$, so

$$
\begin{align*}
(3.8) \quad & a_{v_{e_0,1}}(W_{\alpha,\alpha+1}) = \frac{2}{3} = a_{v_{e_0,2}}(W_{\alpha,\alpha+1}), \quad a_{\alpha}(W_{\alpha,\alpha+1}) = \frac{1}{3} = a_{v_{e_0,2}}(W_{\alpha,\alpha+1}), \\
& a_{v_{e_0,1}}(W_{\alpha,\alpha+1}) = a_{v_{e_0,2}}(W_{\alpha,\alpha+1}) = 0, \quad a_{\alpha}(W_{\alpha,\alpha+1}) = \frac{1}{3}(1 + 3) = \frac{2}{3},
\end{align*}
$$

which matches Fig 5. Let $W_{\alpha+1,\alpha}$ be a right turn corner arc $A_2$-web in $t$, going from side $e_{\alpha+1}$ to $e_{\alpha}$. The Frohman-Sikora coordinates are $0 = a_{v_{e_0,1}}(W_{\alpha+1,\alpha}) = a_{\alpha}(W_{\alpha+1,\alpha}) = a_{v_{e_0,2}}(W_{\alpha+1,\alpha}) = a_{v_{e_0,1}}(W_{\alpha+1,\alpha}), 1 = a_{v_{e_0,1}}(W_{\alpha+1,\alpha}) = a_{v_{e_0,2}}(W_{\alpha+1,\alpha}), r_1(W_{\alpha+1,\alpha}) = -1$, and $d_1(W_{\alpha+1,\alpha}) = 0$, so

$$
(3.9) \quad \begin{align*}
& a_{v_{e_0,1}}(W_{\alpha+1,\alpha}) = \frac{\frac{1}{3} = a_{v_{e_0,2}}(W_{\alpha+1,\alpha}), \quad a_{\alpha}(W_{\alpha+1,\alpha}) = \frac{2}{3} = a_{v_{e_0,2}}(W_{\alpha+1,\alpha}), \\
& a_{v_{e_0,1}}(W_{\alpha+1,\alpha}) = a_{v_{e_0,2}}(W_{\alpha+1,\alpha}) = 0, \quad a_{\alpha}(W_{\alpha+1,\alpha}) = \frac{1}{6}(1 + 3) = \frac{2}{3},
\end{align*}
$$

which matches Fig 5. Now let $H_d$ be a pyramid with $d > 0$. The Frohman-Sikora coordinates are $0 = a_{v_{e_0,1}}(H_d), d = a_{\alpha}(H_d), \alpha = 1, 2, 3, r_1(H_d) = 0$, and $d_1(H_d) = d > 0$, so

$$
(3.10) \quad \begin{align*}
& a_{v_{e_0,1}}(H_d) = \frac{d}{3}, \quad a_{v_{e_0,2}}(H_d) = \frac{2d}{3}, \quad \alpha = 1, 2, 3, \quad a_{\alpha}(H_d) = \frac{1}{6}(0 + 6d) = d.
\end{align*}
$$

which matches Fig 5. Finally, let $H_d$ be a pyramid with $d < 0$. The Frohman-Sikora coordinates are $-d = a_{v_{e_0,1}}(H_d), 0 = a_{v_{e_0,2}}(H_d), \alpha = 1, 2, 3, r_1(H_d) = 0$, and $d_1(H_d) = d < 0$, so

$$
\begin{align*}
& a_{v_{e_0,1}}(H_d) = -\frac{d}{3}, \quad a_{v_{e_0,2}}(H_d) = -\frac{2d}{3}, \quad \alpha = 1, 2, 3, \quad a_{\alpha}(H_d) = \frac{1}{6}(0 - 6d) = -d.
\end{align*}
$$

which matches Fig 5. 

What we will use right away is another easy observation.

Lemma 3.33. For each marked point $p$ of a triangulable generalized marked surface $\mathcal{S}$, let $\ell_0$ be a $A_2$-lamination in $\mathcal{S}$ represented by a weighted $A_2$-web whose only components are oriented peripheral curves. Then for any $A_2$-lamination $\ell$ in $\mathcal{S}$,

1. $\ell_0$ is disjoint from $\ell$;
2. If we denote by $-\ell_0$ the $A_2$-lamination obtained from $\ell_0$ by multiplying the weight on each constituent peripheral curve by $-1$, then $\ell \cup \ell_0 \cup (-\ell_0) = \ell$ as $A_2$-laminations. 

We now prove:

Proposition 3.34. The coordinate map in Prop. 3.30 is a bijection onto $B_\Delta$.

Proof. We construct an inverse map to the coordinate map. Let $\hat{a} = (a_{v})_{v \in V(Q_\Delta)}$ be any balanced element of $(\frac{1}{2}\mathbb{Z})^{Q_\Delta}$. We will construct an $A_2$-lamination $\ell$ having these as its coordinates. We shall construct a weighted $A_2$-web in each triangle of $\hat{t}$, ‘fill in’ the biangles, then move the boundary 4-gons, to construct a sought-for $A_2$-lamination $\ell$ in $\mathcal{S}$. Let $t$ be a triangle of $\Delta$, and let $\hat{t}$ be the corresponding
Consider an $A_2$-lamination $\ell_t$ in $\hat{t}$ represented by a canonical $A_2$-web in $\hat{t}$ (Def 3.12). Let $e_1, e_2, e_3$ be the sides of $t$ appearing clockwise, and for each $\alpha, \beta \in \{1, 2, 3\}$ let $c_{\alpha, \beta}$ be the sum of weights of corner arcs of $\ell_t$ going from edge $e_\alpha$ to edge $e_\beta$. Then $\ell_t$ is a union of a single pyramid $H_{d_t}$ for some $d_t \in \mathbb{Z}$ and corner arcs. The lamination $\ell_t$ is completely determined by these numbers $c_{\alpha, \beta}$ and $d_t$. Let’s show that there exists a unique $\ell_t$ whose tropical coordinates coincide with those assigned by the element $\tilde{a}$. By definition of the coordinates, we must have eq. (3.6) and eq. (3.7), with $\ell$ replaced by $\ell_t$. In view of eq. (3.5) and eq. (3.7), we have $d_t = d_t(\ell_t) = \sum_{\alpha=1}^{3}a_{\alpha e_{\alpha}} - \sum_{\alpha=1}^{3}a_{\alpha e_{\alpha-1}}$; so, the degree $d_t$ of the pyramid for the sought-for $A_2$-lamination $\ell_t$ is determined by $\tilde{a}$, and it is an integer, by the balancedness condition of $\tilde{a}$. From eq. (3.2) it follows that for each $\alpha = 1, 2, 3$,

\[
e_{\text{out}, e_\alpha}(\ell_t) = 2a_{\alpha e_{\alpha}} - a_{\alpha e_{\alpha+1}} - a_{\alpha e_{\alpha-1}},
\]

so the intersection weights of $\ell_t$ are determined by $\tilde{a}$. In turn, from eq. (3.4) it follows that the rotation weight $r_t(\ell_t)$ is also determined by $\tilde{a}$.

In [FS20 Lem.23] it is observed that the intersection coordinates and the rotation numbers of an $A_2$-web completely determine the number of each kind of corner arcs. Likewise, we show that the weights $c_{\alpha, \beta}$ of corner arcs are completely determined by the intersection weights and the rotation weight; we will give explicit reconstruction formulas. For convenience, let $\ell_t$ be the $A_2$-lamination in $\hat{t}$ obtained from $\ell_t$ by removing the pyramid $H_{d_t}$. Then $\ell_t$ and $\ell_t$ have same corner weights $c_{\alpha, \beta}$, hence the same rotation weight $r_t$. The intersection weights of $\ell_t$, denoted by $\tilde{e}_{\text{out}, e_\alpha}$ and $\tilde{e}_{\text{in}, e_\alpha}$, are obtained from those of $\ell_t$ by suitably subtracting $[d_t]_+$ and $[-d_t]_+$:

\[
\tilde{e}_{\text{out}, e_\alpha} = e_{\text{out}, e_\alpha}(\ell_t) - [d_t]_+ = c_{\alpha+1, \alpha} + c_{\alpha-1, \alpha}, \quad \tilde{e}_{\text{in}, e_\alpha} = e_{\text{in}, e_\alpha}(\ell_t) - [-d_t]_+ = c_{\alpha, \alpha+1} + c_{\alpha, \alpha-1}.
\]

Note $r_t = \sum_{\alpha=1}^{3}c_{\alpha, \alpha+1} - \sum_{\alpha=1}^{3}c_{\alpha, \alpha-1}$. Let the left and the right rotation weights be

\[
r_{\text{left}} = \sum_{\alpha=1}^{3}c_{\alpha, \alpha+1}, \quad r_{\text{right}} = \sum_{\alpha=1}^{3}c_{\alpha, \alpha+1}.
\]

Note $\sum_{\alpha=1}^{3}\tilde{e}_{\text{in}, e_\alpha} = \sum_{\alpha=1}^{3}\tilde{e}_{\text{out}, e_\alpha} = r_{\text{left}} + r_{\text{right}}$, while $r_t = r_{\text{left}} - r_{\text{right}}$. Thus we can express $r_{\text{left}}$ and $r_{\text{right}}$ in terms of the intersection weights and the rotation weight as

\[
r_{\text{left}} = \frac{1}{2}(\sum_{\alpha=1}^{3}\tilde{e}_{\text{out}, e_\alpha}) + \frac{1}{2}r_t, \quad r_{\text{right}} = \frac{1}{2}(\sum_{\alpha=1}^{3}\tilde{e}_{\text{out}, e_\alpha}) - \frac{1}{2}r_t.
\]

Observe now

\[
r_{\text{left}} + \tilde{e}_{\text{in}, e_1} + \tilde{e}_{\text{out}, e_2} = \tilde{e}_{\text{out}, e_3} + \tilde{e}_{\text{in}, e_3}
\]

\[
= (c_{1, 2} + c_{2, 3} + c_{1, 3}) + (c_{1, 2} + c_{1, 3}) + (c_{3, 2} + c_{1, 2}) - (c_{1, 3} + c_{2, 3}) - (c_{1, 1} + c_{2, 3}) = 3c_{1, 2}.
\]

Exchanging $r_{\text{left}}$ with $r_{\text{right}}$, and each $\tilde{e}_{\text{in}, e_\alpha}$ with $\tilde{e}_{\text{out}, e_\alpha}$ and vice versa results in exchanging the order of subscripts of $c_{\alpha, \beta}$, so we obtain $r_{\text{right}} = \tilde{e}_{\text{out}, e_1} + \tilde{e}_{\text{in}, e_2} - \tilde{e}_{\text{out}, e_3} - \tilde{e}_{\text{in}, e_3} = 3c_{2, 1}$. By the cyclicity of the subscript indices 1, 2, 3, we thus get

\[
c_{\alpha, \alpha+1} = \frac{1}{4}(r_{\text{left}} + \tilde{e}_{\text{in}, e_\alpha} + \tilde{e}_{\text{out}, e_{\alpha+1}} - \tilde{e}_{\text{out}, e_{\alpha-1}} - \tilde{e}_{\text{in}, e_{\alpha-1}}),
\]

\[
c_{\alpha+1, \alpha} = \frac{1}{4}(r_{\text{right}} + \tilde{e}_{\text{out}, e_\alpha} + \tilde{e}_{\text{in}, e_{\alpha+1}} - \tilde{e}_{\text{in}, e_{\alpha-1}} - \tilde{e}_{\text{out}, e_{\alpha-1}}).
\]

We expressed all corner weights $c_{\alpha, \beta}$ in terms of the intersection weights and the rotation weight, as desired. So, for each triangle $\hat{t}$ of $\hat{\Delta}$ we constructed unique canonical $A_2$-lamination $\ell_t$ in $\hat{t}$ whose tropical coordinates equal to those assigned by $\tilde{a}$.

We now modify $\tilde{a}$ before proceeding to biangles. Let $k \in (\mathbb{Z} \times \mathbb{Z})^P$ be the choice of two integers $k_{C\!W,P}$ and $k_{C\!C\!W,P}$ for each marked point $p \in P$ of $\hat{S}$. Let $\ell_k$ be the $A_2$-lamination in $\hat{S}$ consisting of two peripheral curves per marked point $p \in P$ surrounding $p$ (so having $2|P|$ components), where one of them has clockwise orientation around $p$ and has weight $k_{C\!W,P}$, and the other is counterclockwise with weight $k_{C\!C\!W,P}$. Let $\tilde{a}_k := a_{\Delta}(\ell_k)$ be the element of $(1/3)^{V(Q\Delta)}$ for the tropical coordinates of $\ell_k$. Let

\[
\tilde{a}'' := \tilde{a} + \tilde{a}_k \in (1/3)^{V(Q\Delta)}.
\]

Now, repeat the previous process for this new element $\tilde{a}''$, to get $A_2$-lamination $\ell_\ell'$ in each triangle $\hat{t}$. We claim that we can choose $\tilde{a}$ so that all corner weights of $\ell_\ell'$ for all triangles $\hat{t}$ are non-negative. For example, for a fixed positive integer $k$, let $\tilde{a}$ be such that $k_{C\!W,P} = k_{C\!C\!W,P} = k$ for all $p \in P$. Each corner of a triangle $t$ of $\hat{\Delta}$ is attached at some unique marked point of $P$, and hence in this corner there are $k$ corner arcs of $\ell_k \cap \hat{\ell}$ in one direction and $k$ corner arcs of $\ell_k \cap \hat{\ell}$ in the opposite direction.
Meanwhile, the degree $d_t(\ell_\Delta')$ is zero for all triangles $t$. Hence, each corner weight $c_{a,\beta}'$ of $\ell_\Delta'$ constructed from $\vec{a}'$ equals $c_{a,\beta} + k$, while the degree $d_t(\ell_\Delta)$ equals $d_t(\ell) = d_t$. So, for a sufficiently large $k$, we see that all corner weights of $\ell_\Delta'$ for each triangle $t$ are non-negative.

Now, represent $\ell_\Delta'$ by a weighted $A_2$-web $W'_\Delta$ such that all weights are 1. So, for each ordered pair $(a, \beta)$ of distinct indices in $\{1, 2, 3\}$, we draw $c_{a,\beta}'$ number of corner arcs going from side $e_a$ to side $e_\beta$, and a pyramid $H_{\Delta'}$, so that these are all disjoint. This way, in each side $e_i$ of the triangle $\hat{t}$, there are some $e_{\text{out},e_i}'$ number of source external vertices, and some $e_{\text{out},e_i}'$ number of sink external vertices.

Pick one side $e_\alpha$ of $\hat{t}$. Let $f_\beta$ be the edge of $\Delta'$ parallel to $e_\alpha$ hence forming a biangle with $e_\alpha$, where $f_1, f_2, f_3$ are edges of the triangle $\hat{t}$ of $\Delta$ (corresponding to triangle $r$ of $\Delta$ adjacent to $t$), where the $A_2$-lamination $\ell_r'$ in $\hat{t}$ is drawn as a weighted $A_2$-web $W_r'$ with all weights being 1. So, on $f_\beta$, there are $e_{\text{in},f_\beta}'$ source external vertices of $\ell_\Delta'$ and $e_{\text{out},f_\beta}'$ sink external vertices of $\ell_\Delta'$. By construction of the $A_2$-laminations $\ell_r'$ and $\ell_\Delta'$, one has the compatibility $e_{\text{in},e_\alpha}' = e_{\text{out},f_\beta}'$ and $e_{\text{out},e_\alpha}' = e_{\text{in},f_\beta}'$ at the common edge of $t$ and $r$. Then, by Lem. 3.33(MC3), we can fill in the biangle formed by $e_\alpha$ and $f_\beta$ by a (uniquely determined) minimal crossbar web.

Gluing the canonical $A_2$-webs $W'_\Delta$ in $\hat{t}$ for all triangles $\hat{t}$ of $\Delta'$ and the minimal crossbar webs for all biangles of $\Delta'$, we obtain a crossingless $A_2$-web $W'$, without boundary 1-gon or 2-gon. Replace each

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
\end{tikzpicture}
\end{center}

by \begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
\end{tikzpicture}
\end{center}

and each \begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
\end{tikzpicture}
\end{center}

by \begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- (0,0);
\end{tikzpicture}
\end{center}

, and do the same for the cases with all orientations reversed. Repeat until no more replacing is possible, so that the resulting $A_2$-web $W''$ is a reduced non-elliptic $A_2$-web in $\mathfrak{G}$. One can observe that the internal 3-valent vertices appearing in this process must be in biangles, so the process removes some crossbars. Thus, in each triangle $\hat{t}$ of $\Delta$, each step of such a process results only in exchanging positions of some corner arcs, hence the part in $\hat{t}$ is still canonical. For each biangle $B$ of $\Delta'$, one such process removes an ‘outermost’ crossbar, and one can observe that the resulting picture is still a non-elliptic crossbar $A_2$-web in $\mathfrak{G}$, and hence is a minimal crossbar $A_2$-web. Thus, the final $A_2$-web $W''$ is non-elliptic, reduced, and is canonical with respect to $\Delta$, and for each triangle $\hat{t}$ of $\Delta$, the $A_2$-webs $W'' \cap \hat{t}$ and $W' \cap \hat{t} = W'_\Delta$ in $\hat{t}$ have the same numbers of each kind of corner arcs and the same degree of pyramid. Thus the tropical coordinates of $W''$ form the vector $\vec{a}'$. Let $\mu''$ be the $A_2$-lamination represented by the reduced non-elliptic $A_2$-web $W''$ with weight 1. Now let $\ell$ be the $A_2$-lamination defined as $\ell := \mu'' \cup \ell_{-\vec{k}}$, which makes sense by Lem. 3.33(1). Note that the tropical coordinates of $\ell$ form the vector $\vec{a}' - \vec{a}'_k = \vec{a}$, as desired. This shows the surjectivity of the coordinate map.

Let’s now show the injectivity of the coordinate map. Let $\ell_1$ and $\ell_2$ be $A_2$-laminations in $\mathfrak{G}$ having same tropical coordinates. As discussed above, we can find some $\vec{k}$ such that the $A_2$-laminations $\ell_1' := \ell_1 \cup \ell_{-\vec{k}}$ and $\ell_2' := \ell_2 \cup \ell_{-\vec{k}}$ have non-negative corner weights, so that they can be represented as weighted $A_2$-webs with all weights being 1; they can be viewed as reduced non-elliptic $A_2$-webs. The intersection weights and the rotation weights, which can be easily seen to be determined by the tropical coordinates, then coincide with Frohman-Sikora’s intersection coordinates and rotation numbers. Thus from Prop. 3.23(FS2) it follows that $\ell_1'$ and $\ell_2'$ are equivalent as $A_2$-webs. Hence $\ell_1' = \ell_2'$ as $A_2$-laminations. Thus by Lem. 3.33(2) we get $\ell_1 = \ell_1' \cup \ell_{-\vec{k}} = \ell_2' \cup \ell_{-\vec{k}} = \ell_2$ as $A_2$-lamimations, finishing the proof of injectivity.

As mentioned already, one of the favorable properties of our coordinates is the compatibility formulas under change of ideal triangulations.

**Proposition 3.35** (coordinate change formula for tropical coordinates). Let $\Delta$ and $\Delta'$ be ideal triangulations of a triangulable punctured surface $\mathfrak{G}$ related to each other by a flip at an edge. Let $\ell$ be an $A_2$-lamination in $\mathfrak{G}$. The tropical coordinates $(a_v(\ell))_v \in (\mathbb{Z}_{\geq 0})^{V(\mathfrak{G}_\Delta)}$ and $(a'_v(\ell))_v \in (\mathbb{Z}_{\geq 0})^{V(\mathfrak{G}_{\Delta'})}$ of $W$ with respect to $\Delta$ and $\Delta'$ are related by the sequence of tropical $\alpha'$-mutations with respect to the sequence of mutations associated to a flip, i.e. by the same coordinate change formulas as described in Prop. 3.26.

**Proof.** Core of a proof of this proposition is just the corresponding statement Douglas-Sun coordinates of $A_2$-webs, i.e. Prop. 3.26. Indeed, by additivity (Lem. 3.32), it suffices to show the statement for $A_2$-laminations $\ell$ that be represented as a single-component $A_2$-web $W$, with some weight. Let $W$
be a single-component non-elliptic $A_2$-web, and for each integer $k$, define $k\ell$ be the $A_2$-lamination represented by $W$ with weight $k$, whenever it can be defined. Then, in view of the definition of the tropical coordinates, it is easy to observe $a_v(k\ell) = ka_v(\ell)$ for all $v \in \mathcal{V}(Q_\Delta)$ and $a'_v(k\ell) = ka'_v(\ell)$ for all $a' \in \mathcal{V}(Q_\Delta')$. Note $a_v(\ell)$ and $a'_v(\ell)$ coincide with Douglas-Sun coordinates of the $A_2$-web $W$ (Lem 3.29), and they transform as asserted, by Prop 3.20. The transformation formulas as presented in Prop 3.20 are equivariant under the multiplication action by $\mathbb{Z}$, hence $a_v(k\ell)$’s and $a'_v(k\ell)$’s also transform as wanted. 

**Remark 3.36.** This compatibility with tropical $\mathscr{A}$-mutation formulas is how we found the definition of our coordinates, up to scalar. Namely, we verified that those particular linear combinations of Frohman-Sikora coordinates enjoy these coordinate change formulas, at least for some simple cases.

We expect that the proof of Prop 3.20 is ‘local’ in the sense that it would involve only the triangles having the flipped edges as one of the side, and therefore would apply also for triangulable generalized marked surfaces.

**Conjecture 3.37.** Prop 3.20 and Prop 3.35 hold also for triangulable generalized marked surfaces. Note

$$\mathbb{Z}^{Q_\Delta} \subset B_\Delta \subset (\frac{1}{4}\mathbb{Z})^{Q_\Delta}.$$  

What will eventually play a major role are the $A_2$-laminations whose tropical coordinates lie in $\mathbb{Z}^{Q_\Delta}$. One consequence of Prop 3.35 is that, if all tropical coordinates of an $A_2$-lamination $\ell$ with respect to some ideal triangulation $\Delta$ are integers, then so are those of $\ell$ with respect to any ideal triangulation.

**Definition 3.38.** An $A_2$-lamination in a triangulable generalized marked surface $\mathcal{G}$ is said to be congruent if for some, hence for every, ideal triangulation $\Delta$ of $\mathcal{G}$, its tropical coordinates are all integers.

So we have a bijection

$$a_\Delta : \{\text{congruent }A_2\text{-laminations in }\mathcal{G}\} \to \mathbb{Z}^{Q_\Delta} \cong \mathcal{A}_{Q_\Delta}(\mathbb{Z})^t,$$

which is compatible under the tropical $\mathscr{A}$-mutations; see §2.3 for $\mathcal{A}_{Q_\Delta}(\mathbb{Z})^t$. Hence the set of all congruent $A_2$-laminations in $\mathcal{G}$ works as a geometric model of $\mathcal{A}_{SL_3,\mathcal{G}}(\mathbb{Z})^t$, the set of tropical integer points of the moduli space $\mathcal{A}_{SL_3,\mathcal{G}}$, or that of the corresponding cluster $\mathscr{A}$-variety.

**Theorem 3.39.** For a triangulable punctured surface $\mathcal{G}$, we have a geometric model of $\mathcal{A}_{SL_3,\mathcal{G}}(\mathbb{Z})^t$, the set of tropical integer points of the moduli space $\mathcal{A}_{SL_3,\mathcal{G}}$, or that of the corresponding cluster $\mathscr{A}$-variety:

\[(3.11) \quad \mathcal{A}_{SL_3,\mathcal{G}}(\mathbb{Z})^t \leftrightarrow \{\text{congruent }A_2\text{-laminations in }\mathcal{G}\}.\]

This theorem would also hold for generalized marked surface, provided that Conjecture 3.37 holds.

We suggest the readers to compare our model with previously proposed models of Le [L16] (‘higher’ laminations) and Goncharov-Shen [GS15] (top-dimensional components of ‘surface affine Grassmannian’ stack).

Note that the assertion that our congruent $A_2$-laminations indeed provides a model of $\mathcal{A}_{SL_3,\mathcal{G}}(\mathbb{Z})^t$ depends on Prop 3.35 which in turn heavily depends on Prop 3.20 which is a result of an upcoming work [DS2] of Douglas and Sun. So, one might say that we haven’t fully justified this assertion yet. However, as a corollary of our main result (which is algebraic) whose proof does not depend on the validity of Prop 3.35 or Prop 3.20, we will provide a proof of a weaker version of Thm 3.39.

**Definition 3.40.** Let $\mathcal{G}$ be a triangulable punctured surface. For an ideal triangulation $\Delta$ of $\mathcal{G}$, we say that an $A_2$-lamination $\ell \in \mathcal{A}_t(\mathcal{G};\mathbb{Z})$ in $\mathcal{G}$ is $\Delta$-congruent if all tropical coordinates of $\ell$ for $\Delta$ are integers, i.e. $a_v(\ell) \in \mathbb{Z}$, $\forall v \in \mathcal{V}(Q_\Delta)$. Let

$$\mathcal{A}_\Delta(\mathbb{Z})^t := \{\Delta\text{-congruent }A_2\text{-laminations in }\mathcal{G}\} \subset \mathcal{A}_t(\mathcal{G};\mathbb{Z}).$$

**Proposition 3.41** (congruence condition is independent on triangulation). For any ideal triangulations $\Delta$ and $\Delta'$, we have $\mathcal{A}_\Delta(\mathbb{Z})^t = \mathcal{A}_{\Delta'}(\mathbb{Z})^t$.

In particular, this would justify Def 3.38 and also eq (3.11) of Thm 3.39. In the next section, Prop 3.41 will be proved only at the end, so until then, we will use the set $\mathcal{A}_\Delta(\mathbb{Z})^t$ instead of $\mathcal{A}_{SL_3,\mathcal{G}}(\mathbb{Z})^t$. 


4. Regular functions on moduli spaces

One of the original Fock-Goncharov’s duality conjectures [FG06] is on the existence of a basis of the ring $L(X_{PGL_3, \mathfrak{S}})$ (Def. 4.1) enumerated by $\mathcal{A}_{SL_3, \mathfrak{S}}(Z)$). We will construct a map $\mathcal{A}_{SL_3, \mathfrak{S}}(Z^1) \rightarrow L(X_{PGL_3, \mathfrak{S}})$. By mimicking Fock-Goncharov’s argument [FG06] for $SL_2$ and $PGL_2$, we show that the image of this map is a basis of $\mathcal{O}(X_{PGL_3, \mathfrak{S}})$. To do that, we investigate the ring $\mathcal{O}(X_{PGL_3, \mathfrak{S}})$, and its relationship with $\mathcal{O}(X_{PGL_2, \mathfrak{S}})$. We will observe that $\mathcal{O}(X_{PGL_2, \mathfrak{S}})$ coincides with $\mathcal{O}_3(X_{PGL_2, \mathfrak{S}}) = \mathcal{O}(X_{SL_3, \mathfrak{S}})$. We also investigate favorable properties of our duality map. Throughout this section, $\mathfrak{S} = \Sigma \setminus \mathcal{P}$ is a triangulable punctured surface.

For a stack or a scheme $\mathcal{X}$, we denote by $\mathcal{O}(\mathcal{X})$ the ring of all its regular functions.

4.1. Bases of rings of regular functions on $SL_3$-moduli spaces. We begin with $\mathcal{O}(X_{SL_3, \mathfrak{S}})$, the ring of regular functions on the moduli space $X_{SL_3, \mathfrak{S}}$ of $SL_3$-local systems on the punctured surface $\mathfrak{S}$. Here is a standard element.

Definition 4.1 (trace of monodromy on $X_{SL_3, \mathfrak{S}}$). Let $\gamma$ be an oriented loop in $\mathfrak{S}$. Denote by $f_\gamma$ the function on $X_{SL_3, \mathfrak{S}}$ given by the trace of monodromy along $\gamma$. That is, for an $SL_3$-local system $\mathcal{L}$ on $\mathfrak{S}$, if $\rho: \pi_1(\mathfrak{S}) \rightarrow SL_3$ is the monodromy representation of $\mathcal{L}$ defined up to conjugation, define

$$f_\gamma(\mathcal{L}) := \text{tr}(\rho(\gamma)).$$

It is easy to see that $f_\gamma$ is a well-defined regular function on $X_{SL_3, \mathfrak{S}}$. It is known from Procesi [P76] that $\mathcal{O}(X_{SL_3, \mathfrak{S}})$ is generated by these trace-of-monodromy functions along loops. Sikora [S01] found a complete set of relations among the trace-of-monodromy functions, and thus obtained an algebra isomorphism between the $A_2$-skein algebra $S(\mathfrak{S}; \mathbb{Z})$ (Def. 3.2) and $\mathcal{O}(X_{SL_3, \mathfrak{S}})$. Note that a single-component $A_2$-web $W$ in $\mathfrak{S}$ with no internal or external vertices (Def. 4.1) is an oriented loop $\gamma$ in $\mathfrak{S}$; let’s denote this $W$ by $W_\gamma$.

Proposition 4.2 (S01, P76). There is a unique isomorphism of rings

$$\Phi: S(\mathfrak{S}; \mathbb{Z}) \rightarrow \mathcal{O}(X_{SL_3, \mathfrak{S}})$$

that sends each $A_2$-skein $[W_\gamma]$ consisting of one oriented loop $\gamma$ to the trace-of-monodromy function $f_\gamma$.

Recall from Prop. 3.5 the result of Sikora and Westbury [SW07] saying that the non-elliptic $A_2$-webs form a basis of $S(\mathfrak{S}; \mathbb{Z})$, and recall that the set of all non-elliptic $A_2$-webs is in bijection with the set $A_L^0(\mathfrak{S}; \mathbb{Z})$ of all integral $A_2$-laminations with non-negative weights (Lem. 3.10); recall also Rmk. 3.4.

Corollary 4.3 ($A_2$-bangle basis of $\mathcal{O}(X_{SL_3, \mathfrak{S}})$). The above construction yields an injective map

$$\mathbb{I}_{SL_3}^0: A_L^0(\mathfrak{S}; \mathbb{Z}) \rightarrow \mathcal{O}(X_{SL_3, \mathfrak{S}})$$

whose image set forms a basis of $\mathcal{O}(X_{SL_3, \mathfrak{S}})$, which we call the $A_2$-bangle basis of $\mathcal{O}(X_{SL_3, \mathfrak{S}})$.

As mentioned earlier for the $A_2$-skein algebra $S(\mathfrak{S}; \mathbb{Z})$, we may think of this basis of $\mathcal{O}(X_{SL_3, \mathfrak{S}})$ as an $A_2$ version of a bangle basis for the well known $A_1$ theory. We will also discuss the $A_2$ version of the so-called bracelet basis. We recall the meaning of bangle and bracelet.

Definition 4.4. Let $\gamma$ be an oriented simple loop in $\mathfrak{S}$, hence forming a single-component non-elliptic $A_2$-web $W_\gamma$, and therefore a single-component $A_2$-lamination with weight 1. Let $k \in \mathbb{Z}_{>0}$.

1. Define a $k$-bangle $W_\gamma^k$ of $W_\gamma$ as a non-elliptic $A_2$-web consisting of $k$ copies of mutually disjoint oriented loops isotopic to $\gamma$.

2. Define a $k$-bracelet $W_\gamma^{(k)}$ as a single-component $A_2$-web obtained from the loop $\gamma^k = \gamma \cdot \cdots \cdot \gamma$ by deforming it by a homotopy so that its self-intersections are transverse double.

When $W_\gamma$ is given as an element of the skein algebra $S(\mathfrak{S}; \mathbb{Z})$, note that $W_\gamma^k$ and $W_\gamma^{(k)}$ yield well-defined elements $[W_\gamma^k]$ and $[W_\gamma^{(k)}]$ of $S(\mathfrak{S}; \mathbb{Z})$. The notation for $k$-bangle is instructive, since $[W_\gamma^k] = [W_\gamma]^k$, with respect to the product structure of $S(\mathfrak{S}; \mathbb{Z})$. By construction, (for an oriented simple loop $\gamma$) the $k$-bangle $W_\gamma^k$ can be viewed as an $A_2$-lamination, and we have

$$\mathbb{I}_{SL_3}^0(W_\gamma^k) = (\mathbb{I}_{SL_3}^0(W_\gamma))^k = (f_\gamma)^k,$$

where $f_\gamma$ is the trace-of-monodromy function along $\gamma$ (Def. 4.1); note that the $A_2$-lamination $W_\gamma^k$ can be represented by one component $W_\gamma$ with weight $k$. We will define the bangle version in a later subsection.
On the other hand, from the defining relations and the product structure of \( S(\mathfrak{g}; \mathbb{Z}) \), we immediately get the following useful result.

**Corollary 4.5.** The structure constants of the basis \( \mathbb{Z}_0(\mathfrak{g}_L(\mathfrak{s}; \mathbb{Z})) \) of \( \mathcal{O}(\mathcal{L}_{\text{SL}_3, \mathfrak{s}}) \) are integers. That is, for any \( \ell, \ell' \in \mathfrak{s}_0 \), we have
\[
\mathbb{Z}_0(\mathfrak{g}(\ell)) \mathbb{Z}_0(\mathfrak{g}(\ell')) = \sum_{\ell'' \in \mathfrak{s}_0} c^0_{\mathfrak{g}_3}(\ell, \ell'; \ell'') \mathbb{Z}_0(\mathfrak{g}(\ell''))
\]
where \( c^0_{\mathfrak{g}_3}(\ell, \ell'; \ell'') \in \mathbb{Z} \) are zero for all but at most finitely many \( \ell'' \).

Next step is to consider \( \mathcal{O}(\mathcal{Z}_{\text{SL}_3, \mathfrak{s}}) \). We recall the result from [FG06] §12.5 showing that it is a free module over \( \mathcal{O}(\mathcal{L}_{\text{SL}_3, \mathfrak{s}}) \). By abuse of notation, let \( B \) be a Borel subgroup of \( \text{SL}_3 \), say the subgroup of all upper triangular matrices in \( \text{SL}_3 \), and let \( U := [B, B] \) be the corresponding maximal unipotent subgroup, which would be the subgroup of all upper triangular matrices with diagonal entries being all 1. Define the Cartan group of \( \text{SL}_3 \) as \( H := B/U \), which for our case is canonically isomorphic to the subgroup of \( \text{SL}_3 \) of diagonal matrices, and the quotient map \( B \to B/U \) just reads the diagonal entries. Let \( W \) be the corresponding Weyl group. For each puncture \( p \in \mathcal{P} \), there is a canonical map
\[
\pi_p : \mathcal{Z}_{\text{SL}_3, \mathfrak{s}} \to H
\]
provided by the framing and the semi-simple part of the monodromy along a peripheral loop surrounding \( \mathcal{L} \). In fact, we need to choose the orientation of the loop carefully.

**Definition 4.6.** A peripheral loop in \( \mathfrak{s} \) (Def. 3.9) around a puncture \( p \in \mathcal{P} \) is positively oriented if it is isotopic to the hole of \( \tilde{\mathfrak{s}} \) corresponding to \( p \), given the boundary-orientation (Def. 2.8). We say it is negatively oriented otherwise.

Now, given a framed \( \text{SL}_3 \)-local system \( (\mathcal{L}, \beta) \) on \( \mathfrak{s} \), for each puncture \( p \in \mathcal{P} \), consider the monodromy along a positively oriented peripheral loop \( \gamma \) around \( p \). This monodromy is defined only up to conjugation in \( \text{SL}_3 \), and lives in some Borel subgroup of \( \text{SL}_3 \), hence can be thought of as living in our fixed choice \( B \). The semi-simple part of this element of \( B \) can be obtained as the image of the quotient map \( B \to B/U = H \). As said in [FG06], the semi-simple part of the monodromy alone yields an element of \( H/W \), giving a map
\[
\mathcal{L}_{\text{SL}_3, \mathfrak{s}} \to H/W,
\]
and together with the framing data we get the map \( \pi_p : \mathcal{Z}_{\text{SL}_3, \mathfrak{s}} \to H \). Let us give a more precise explanation of \( \pi_p \) as it is important in the present paper, but is not described in [FG06] in detail.

Consider \( p \) as a point of the hole of \( \tilde{\mathfrak{s}} \), hence in particular \( p \) can be thought of as a point of \( \mathfrak{s} \), as in Def. 2.8 and the discussion after that. So the framing \( \beta \) yields a distinguished point of the fiber \( (\mathcal{L}_B)_p \) of the associated flag bundle \( \mathcal{L}_B \); recall \( (\mathcal{L}_B)_p = \mathcal{L}_p \times_G G/B = \{(v, gB) | v \in \mathcal{L}_p, g \in G\} \), where \( [v, gB] = [v', g'B] \) iff \( v' = vgb \) for some \( b \in B \). In particular, \( [v, gB] = [vg, B] = [vgb, B] \). Hence we can write the distinguished element of \( (\mathcal{L}_B)_p \) assigned by \( \beta \) as \( \beta(p) = [v_0, B] \) for some \( v_0 \in \mathcal{L}_p \) that is uniquely determined up to right action of \( B \). The parallel transport map of \( \mathcal{L} \) along \( \gamma \) gives the monodromy map \( \Pi_\gamma : \mathcal{L}_p \to \mathcal{L}_p \) that is equivariant under the right \( G \)-actions. The induced monodromy \( (\Pi_\gamma)_* : (\mathcal{L}_B)_p \to (\mathcal{L}_B)_p \) for \( \mathcal{L}_B \) then sends \( [v, gB] \) to \( [\Pi_\gamma(v), gB] \), but since \( \beta \) is a flat section, we have \( [v_0, B] = [\Pi_\gamma(v_0), B] \). This means \( \Pi_\gamma(v_0) = v_0b_0 \) for some \( b_0 \in B \) which is unique determined by \( v_0 \). If \( \beta(p) = [v_0, B] \), then \( v_0 = v_0b \) for some \( b \in B \), then \( \Pi_\gamma(v_0) = \Pi_\gamma(v_0b) = \Pi_\gamma(v_0)b = v_0b^{-1}b_0b = v_0b^{-1}b_0b \), so \( b_0 = b^{-1}b_0b \) makes \( \Pi_\gamma(v_0) = v_0b_0 \). This means that out of the monodromy of \( (\mathcal{L}, \beta) \) along \( \gamma \) we get an element of \( B \) uniquely determined up to conjugation by an element of \( B \). But \( b_0 \) and \( b^{-1}b_0b \) have same semi-simple parts (i.e. they have same diagonal entries), i.e. we get a well-defined element of \( B/U = H \). Thus we get the well-defined map \( \pi_p : \mathcal{Z}_{\text{SL}_3, \mathfrak{s}} \to H \).

The maps \( \pi_p \) for all \( p \in \mathcal{P} \) constitute the map
\[
\pi : \mathcal{Z}_{\text{SL}_3, \mathfrak{s}} \to H^P
\]
where \( \mathcal{P} \) is the set of all punctures of \( \mathfrak{s} \); here \( H^P \) may be understood as \( H^{[\mathcal{P}]} \). Likewise, the maps \( \mathcal{L}_{\text{SL}_3, \mathfrak{s}} \to H/W \) for punctures \( p \in \mathcal{P} \) constitute the map \( \mathcal{L}_{\text{SL}_3, \mathfrak{s}} \to (H/W)^\mathcal{P} \). Fock and Goncharov...
**Proposition 4.8.**

For a triangulable punctured surface \( \mathcal{S} \), define a map

\[
\mathbb{I}_{\mathcal{SL}_3} : \mathcal{A}_L(\mathcal{S}; \mathbb{Z}) \to \mathcal{O}(\mathcal{X}_{\mathcal{SL}_3, \mathcal{S}})
\]

as follows. Let \( \ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z}) \). Represent \( \ell \) as disjoint union \( \ell = \ell_1 \cup \ell_2 \cup \cdots \cup \ell_n \) (Def. 3.31) of single-component \( A_2 \)-laminations \( \ell_1, \ldots, \ell_n \), whose underlying non-elliptic \( A_2 \)-webs are mutually non-isotopic. Define \( \mathbb{I}_{\mathcal{SL}_3}(\ell_i) \) as:

(CB1) If \( \ell_i \) consists of a peripheral loop \( \gamma_i \) with weight \( k_i \in \mathbb{Z} \), surrounding a puncture \( p \in \mathcal{P} \),

(CB1-1) if \( \gamma_i \) is positively oriented (Def. 4.6), then

\[
\mathbb{I}_{\mathcal{SL}_3}(\ell_i) := ((\pi_p)_i)^{k_i};
\]

(CB1-2) if \( \gamma_i \) is negatively oriented (Def. 4.6), then

\[
\mathbb{I}_{\mathcal{SL}_3}(\ell_i) := ((\pi_p)_i)^{-k_i};
\]

(CB2) Otherwise, define

\[
\mathbb{I}_{\mathcal{SL}_3}(\ell_i) = F^* \mathbb{I}_{\mathcal{SL}_3}(\ell_i).
\]

Define

\[
\mathbb{I}_{\mathcal{SL}_3}(\ell) := \mathbb{I}_{\mathcal{SL}_3}(\ell_1) \mathbb{I}_{\mathcal{SL}_3}(\ell_2) \cdots \mathbb{I}_{\mathcal{SL}_3}(\ell_n).
\]

By convention, we set \( \mathbb{I}_{\mathcal{SL}_3}(\emptyset) := 1 \).

The image set \( \mathbb{I}_{\mathcal{SL}_3}(\mathcal{A}_L(\mathcal{S}; \mathbb{Z})) \) is called the \( A_2 \)-bangle basis of \( \mathcal{O}(\mathcal{X}_{\mathcal{SL}_3, \mathcal{S}}) \), by a slight abuse of notation.

**Proposition 4.8.** For a triangulable punctured surface \( \mathcal{S} \), one has:

1. The map \( \mathbb{I}_{\mathcal{SL}_3} \) is injective, and the image set of \( \mathbb{I}_{\mathcal{SL}_3} \) is indeed a basis of \( \mathcal{O}(\mathcal{X}_{\mathcal{SL}_3, \mathcal{S}}) \).
2. The structure constants of this \( A_2 \)-bangle basis of \( \mathcal{O}(\mathcal{X}_{\mathcal{SL}_3, \mathcal{S}}) \) are integers. That is, for any \( \ell, \ell' \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z}) \), we have

\[
\mathbb{I}_{\mathcal{SL}_3}(\ell) \mathbb{I}_{\mathcal{SL}_3}(\ell') = \sum_{\ell'' \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})} c_{\mathcal{SL}_3}(\ell, \ell'; \ell'') \mathbb{I}_{\mathcal{SL}_3}(\ell'')
\]

where \( c_{\mathcal{SL}_3}(\ell, \ell'; \ell'') \in \mathbb{Z} \) and \( c_{\mathcal{SL}_3}(\ell, \ell'; \ell'') \) are zero for all but at most finitely many \( \ell'' \).
Proof. (1) We first consider the restriction of \( \mathcal{L}_3 \) to the subset \( \mathcal{L}_3^0(\mathbb{G};\mathbb{Z}) \) of \( \mathcal{L}_3(\mathbb{G};\mathbb{Z}) \) consisting of \( \mathcal{A}_2 \)- laminations with non-negative weights. This restricted map doesn’t exactly equal \( F'^* \circ \mathcal{L}_3^0 \) because of the peripheral loops. Let \( \ell_i \) be a single-component \( \mathcal{A}_2 \)-lamination consisting of a single peripheral (simple) loop \( \gamma_i \) around a puncture \( p \in \mathcal{P} \), with weight \( k_i \in \mathbb{Z} \). One may view \( \gamma_i \) as the hole of \( \mathcal{S} \) corresponding to \( p \); assume that the orientation matches the boundary-orientation of the hole (Def. 2.8).

As discussed before, the monodromy around \( \gamma_i \) can be thought of as living in \( \mathbb{B} \), i.e. being an upper triangular matrix. Recall that the map \( \pi_p : \mathcal{A}_3(\mathbb{G};\mathbb{Z}) \to \mathbb{H} \) reads the diagonal part. Hence it follows that

\[
F'^* \mathcal{L}_3^0(\ell_i) = f_{\gamma_i} \ell_i = ((\pi_{p_1})^{k_1} + ((\pi_{p_2})^{k_2} + ((\pi_{p_3})^{k_3}.
\]

By definition, one notes \( \mathcal{L}_3^0(\ell_i) = (\pi_{p_i})^{k_i} \). As in Lem. 3.33 - \( \ell_i \) would denote a single-component \( \mathcal{A}_2 \)- lamination consisting of \( \gamma_i \) with weight \(-k_i\); then \( \mathcal{L}_3^0(-\ell_i) = ((\pi_{p_i})^{k_i} \). On the other hand, denote by \( \bar{\ell}_i \) the single-component \( \mathcal{A}_2 \)-lamination consisting of the peripheral loop \( \gamma_i \) with weight \( k_i \), where \( \bar{\ell}_i \) is same as \( \gamma_i \) with the orientation reversed. So, by definition, \( \mathcal{L}_3^0(\ell_i) = (\pi_{p_i})^{k_i} \), and \( \mathcal{L}_3^0(-\ell_i) = (\pi_{p_i})^{k_i} \). From \( \pi_p(\pi_p) = 1 \) it follows that \((\pi_{p_i})^{k_i} = ((\pi_{p_i})^{k_i} - k_i) = \mathcal{L}_3^0(-\ell_i) \mathcal{L}_3^0(\ell_i) \), which, in turn, by Lem. 3.33 equals \( \mathcal{L}_3^0(-\ell_i) \cup \mathcal{L}_3^0(\ell_i) \).

To summarize,

\[
(4.6) \quad F'^* \mathcal{L}_3^0(\ell_i) = \mathcal{L}_3^0(\ell_i) + \mathcal{L}_3^0((-\ell_i) \cup \bar{\ell}_i) + \mathcal{L}_3^0(-\ell_i)
\]

when \( \ell_i \) is a single-component \( \mathcal{A}_2 \)-lamination of a peripheral loop, oriented according to the boundary-orientation along the corresponding hole of \( \mathcal{S} \). Now suppose that \( \ell_i \) is a single peripheral loop \( \gamma_i \) with weight \( k_i \), but \( \gamma_i \) is negatively oriented (Def. 4.6). Then, by definition of \( \pi_p \), we have \( F'^* \mathcal{L}_3^0(\ell_i) = f_{\gamma_i} \ell_i = (((\pi_{p_1})^{k_1}) + ((\pi_{p_2})^{k_2}) + ((\pi_{p_3})^{k_3}). \) This time, we can observe that \( \mathcal{L}_3^0(\ell_i) = ((\pi_{p_i})^{k_i} \) and \( \mathcal{L}_3^0(-\ell_i) = ((\pi_{p_i})^{k_i} \) hence eq. (4.6) still holds.

On the other hand, if \( \ell \in \mathcal{L}_3(\mathbb{G};\mathbb{Z}) \) does not contain any peripheral loop, then \( \ell \in \mathcal{L}_3^0(\mathbb{G};\mathbb{Z}) \), and

\[
(4.7) \quad F'^* \mathcal{L}_3^0(\ell) = \mathcal{L}_3^0(\ell).
\]

Note that \( \mathcal{L}_3^0(\mathcal{L}_3(\mathbb{G};\mathbb{Z})) \) spans \( \mathcal{O}(\mathcal{L}_3(\mathbb{G};\mathbb{Z})) \) (Cor. 4.3). We just saw that the set \( F'^* (\mathcal{L}_3^0(\mathcal{L}_3(\mathbb{G};\mathbb{Z}))) \subseteq F'^* (\mathcal{O}(\mathcal{L}_3(\mathbb{G};\mathbb{Z}))) \subseteq \mathcal{O}(\mathcal{L}_3(\mathbb{G};\mathbb{Z})) \) lies in the span of \( \mathcal{L}_3^0(\mathcal{L}_3(\mathbb{G};\mathbb{Z})) \). In view of eq. (4.3), elements of \( F'^* (\mathcal{O}(\mathcal{L}_3(\mathbb{G};\mathbb{Z}))) \) spanned with elements of \( \mathcal{O}(\mathcal{H}) \) give \( \mathcal{O}(\mathcal{H}) \) span in \( \mathcal{L}_3(\mathbb{G};\mathbb{Z}) \). One copy of \( \mathcal{O}(\mathcal{H}) \) \( \equiv \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]/(abc - 1) \) (where \( a, b, c \) are coordinate functions of \( H \) for the diagonal entries) is associated to each puncture \( p \in \mathcal{P} \), and by definition of \( \pi_p \) and \( (\pi_{p_i}) \), one can observe that the functions \( \pi_{p_i} k_i \), \( i = 1, 2, 3 \), \( k \in \mathbb{Z} \), span this copy of \( \mathcal{O}(\mathcal{H}) \). By (CB1-1)-(CB1-2), \( (\pi_{p_i} k_i \) and \( (\pi_{p_i}) k_i \) (for each \( k \in \mathbb{Z} \) follow to \( \mathcal{L}_3(\mathcal{L}_3(\mathbb{G};\mathbb{Z})) \) and we saw above that \( (\pi_{p_i} k_i \) also belongs to \( \mathcal{L}_3^0(\mathcal{L}_3(\mathbb{G};\mathbb{Z})) \). This shows that \( \mathcal{L}_3^0(\mathcal{L}_3(\mathbb{G};\mathbb{Z})) \) spans \( \mathcal{L}_3(\mathcal{L}_3(\mathbb{G};\mathbb{Z})) \). We only sketch a proof for the linear independence of this set and the injectivity of \( \mathcal{L}_3^0 \), as we will not really use these facts; but we will definitely be using the spanning property. From the injectivity of \( F^* \), the injectivity of \( F'^* \) and the linear independence of the set \( F'^* (\mathcal{L}_3^0(\mathcal{L}_3(\mathbb{G};\mathbb{Z}))) \) follow. One can explicitly write down this much result in terms of \( \mathcal{L}_3 \). The remaining is essentially the investigation of a basis of \( \mathcal{L}_3 \) \( \subseteq \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]/(abc - 1) \) (a non-redundant set of all possible) Laurent monomials in \( a, b, c \). This basis is in the set \( \mathcal{L}_3^0(\mathcal{L}_3(\mathbb{G};\mathbb{Z})) \);

As the empty \( \mathcal{A}_2 \)- lamination (Lemma 3.33(2)).

(2) Let’s first establish a lemma, which is easily proved (with the help of Lem. 3.33):

**Lemma 4.9.** Let \( \ell, \ell' \in \mathcal{L}_3(\mathbb{G};\mathbb{Z}) \). If \( \ell \) and \( \ell' \) are disjoint (Def. 3.31), so that \( \ell \cup \ell' \) makes sense as an \( \mathcal{A}_2 \)- lamination, then

\[
\mathcal{L}_3(\ell \cup \ell') = \mathcal{L}_3(\ell) \mathcal{L}_3(\ell').
\]

If furthermore \( \ell, \ell' \in \mathcal{L}_3^0(\mathbb{G};\mathbb{Z}) \), then

\[
\mathcal{L}_3^0(\ell \cup \ell') = \mathcal{L}_3^0(\ell) \mathcal{L}_3^0(\ell').
\]

For example, if \( \ell \) or \( \ell' \) (both belonging to \( \mathcal{L}_3(\mathbb{G};\mathbb{Z}) \) or to \( \mathcal{L}_3^0(\mathbb{G};\mathbb{Z}) \), respectively) consists only of peripheral loops, then \( \ell \) and \( \ell' \) are disjoint, and the above holds.

Now let \( \ell, \ell' \in \mathcal{L}_3(\mathbb{G};\mathbb{Z}) \). We can decompose them into disjoint unions as \( \ell = \ell_1 \cup \ell_2 \) and \( \ell' = \ell'_1 \cup \ell'_2 \), where each of \( \ell_2 \) and \( \ell'_2 \) is either empty or consists only of peripheral loops, while each of \( \ell_1 \) and
\(\ell'_1\) is either empty or does not contain any peripheral loop. In particular, \(\ell_1, \ell'_1 \in A^0_L(\mathbb{G}; \mathbb{Z})\), and \(F^*\mathbb{P}_L(\ell_1) = \mathbb{P}_L(\ell_1), F^*\mathbb{P}_L(\ell'_1) = \mathbb{P}_L(\ell'_1)\). So
\[
\mathbb{P}_L(\ell) \mathbb{P}_L(\ell') = F^* (\mathbb{P}_L(\ell_1) \mathbb{P}_L(\ell'_1)) \mathbb{P}_L(\ell_2) \mathbb{P}_L(\ell'_2)
= F^* \left( \sum_{\ell'' \in A^0_L(\mathbb{G}; \mathbb{Z})} \mathbb{P}_L(\ell_1, \ell'_1; \ell'') \mathbb{P}_L(\ell''') \right) \mathbb{P}_L(\ell_2) \mathbb{P}_L(\ell'_2)
= \sum_{\ell'' \in A^0_L(\mathbb{G}; \mathbb{Z})} \mathbb{P}_L(\ell_1, \ell'_1; \ell'') (F^* \mathbb{P}_L(\ell''')) \mathbb{P}_L(\ell_2) \mathbb{P}_L(\ell'_2).
\]

Decompose \(\ell'' \in A^0_L(\mathbb{G}; \mathbb{Z})\) into disjoint union \(\ell''_1 \cup \ell''_2\), where \(\ell''_2\) consists only of peripheral loops and \(\ell''_1\) has no peripheral loop. Then, as seen,
\[
F^* \mathbb{P}_L(\ell'') = F^* \mathbb{P}_L(\ell''_1) F^* \mathbb{P}_L(\ell''_2) \quad \text{(: Lem. 4.9)}
= \mathbb{P}_L(\ell''_1) \left( \mathbb{P}_L(\ell''_1') + \mathbb{P}_L\left((-\ell''_2) \cup \mathbb{P}_L\right) + \mathbb{P}_L\left(-\mathbb{P}_L\right) \right) \quad \text{(: eq. (4.6) - (4.7))}
= \mathbb{P}_L(\ell''_1) \cup \mathbb{P}_L(\ell''_2) + \mathbb{P}_L(\ell''_1' \cup (-\ell''_2) \cup \mathbb{P}_L) + \mathbb{P}_L(\ell''_1 \cup (-\mathbb{P}_L)) \quad \text{(: Lem. 4.9)}.
\]

Putting into eq. (4.8) and using Lem. 4.9 we obtain the desired statement for item (2). 

4.2. **Lifting** \(\mathbb{P}_L\) **monodromy to** \(\mathbb{P}_L\). Consider the natural regular map
\[
P : \mathcal{R}_{\mathbb{P}_L, \mathbb{G}} \to \mathcal{R}_{\mathbb{P}_L, \mathbb{G}}
\]
induced by the natural quotient \(\mathbb{P}_L \to \mathbb{P}_L\), yielding a map
\[
P^* : \mathcal{O}(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}) \to \mathcal{O}(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}})
\]
In the previous subsection, we obtained a basis of \(\mathcal{O}(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}})\). Now we have to figure out which elements of \(\mathcal{O}(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}})\) belong to the image of \(P^*\). Or, going in the other direction, given an element of \(\mathcal{O}(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}})\), the image of it under \(P^*\) would be an element of \(\mathcal{O}(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}})\), hence can be written as linear combination of elements of the \(A_2\)-bangle basis \(\mathbb{P}_L(A_2(\mathbb{G}; \mathbb{Z}))\) we obtained. Each \(A_2\)-bangle basis vector is a product of trace-of-monodromy functions along loops and certain functions associated to punctures. Now, for example, what kind of function on \(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}\) should correspond to the trace-of-monodromy function on \(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}\)? The monodromy for a point of \(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}\) gives only a homomorphism \(\pi_1(\mathbb{G}) \to \mathbb{P}_L\) (defined up to conjugation), hence the naive trace-of-monodromy along a loop is not well-defined (or, its value is defined in \(\mathbb{A}^1\) only up to \(\mathbb{G}_m\), which is not useful).

As an auxiliary device, we will make use of the set of positive real points, i.e.
\[
\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}^+ := \mathcal{R}_{\mathbb{P}_L, \mathbb{G}}(\mathbb{R}_{>0})
\]
which was studied by Fock and Goncharov [FG03] and called a higher Teichmüller space. It is topologized e.g., as a subspace of \(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}(\mathbb{R})\). Or, one could think of it as being obtained by gluing \((\mathbb{R}_{>0})^{|Q_\Delta|}\) associated to each cluster \(\mathcal{R}\)-chart (not just the cluster charts for ideal triangulations \(\Delta\)), along the mutation gluing maps. In this case, the gluing maps are diffeomorphisms, and so \(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}^+\) is a smooth manifold diffeomorphic to \((\mathbb{R}_{>0})^{|Q_\Delta|}\). Given a regular function on \(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}\), i.e. an element of \(\mathcal{O}(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}})\), for the cluster \(\mathcal{R}\)-chart associated to any ideal triangulation \(\Delta\), this function can be written as a Laurent polynomial in the coordinate functions \(X_v\)'s, \(v \in V(\Delta)\). This Laurent polynomial expression can be thought of as a smooth function on the manifold \(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}^+\). In particular, each \(X_v\) is a positive real valued smooth function on \(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}^+\).

We will observe that there is an embedding
\[
\Psi : \mathcal{R}_{\mathbb{P}_L, \mathbb{G}}^+ \to \mathcal{R}_{\mathbb{P}_L, \mathbb{G}}(\mathbb{R}),
\]
whose inverse map on the image coincides with the map \(P\). Then we use this to translate the functions \(\mathbb{P}_L(\ell) \in \mathcal{O}(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}})\) (for \(\ell \in A_L(\mathbb{G}; \mathbb{Z})\)) to functions on the manifold \(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}^+\).

**Definition 4.10** (translation of \(\mathbb{P}_L\) regular functions to \(\mathbb{P}_L\)). For each \(\ell \in A_L(\mathbb{G}; \mathbb{Z})\), denote by \(\mathbb{P}_L(\ell)\) the function on \(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}^+\) obtained as the pullback under the map eq. (4.10) of the function \(\mathbb{P}_L(\ell) \in \mathcal{O}(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}})\) (evaluated at \(\mathbb{R}\)). Call this
\[
\mathbb{P}_L(\ell) := \Psi^*(\mathbb{P}_L(\ell))(\mathbb{R}) \in C^\infty(\mathcal{R}_{\mathbb{P}_L, \mathbb{G}}^+).
\]
a basic semi-regular function on $\mathcal{X}_{\mathrm{PGL}_3,\mathcal{E}}^+$. They can be viewed as forming a map

$$I_{\mathcal{PGL}_3}^+ : \mathcal{L}(\mathcal{E}; \mathbb{Z}) \to C^\infty(\mathcal{X}_{\mathrm{PGL}_3,\mathcal{E}}^+)$$

To construct the map $\Psi$ in eq.(4.10), we partially recall Fock-Goncharov’s [FG06] reconstruction of a framed $\mathrm{PGL}_3$-local system on $\mathcal{E}$ out of the cluster $\mathcal{X}$-coordinates; see also [D20]. Let $\Delta$ be an ideal triangulation of a punctured surface $\mathcal{E}$. Let $\gamma$ be an oriented loop in $\mathcal{E}$, not necessarily simple. Deform $\gamma$ by an isotopy if necessary, so that $\gamma$ meets $\Delta$ transversally, at finitely many points. We call the elements of $\gamma \cap \Delta$ the $\Delta$-junctures of $\gamma$. For each juncture of $\gamma$, choose a small neighborhood of it in $\gamma$, which is an oriented path meeting $\Delta$ exactly once; call this a juncture segment of $\gamma$ corresponding to this $\Delta$-juncture. Each connected component of the complement in $\gamma$ of the union of all juncture segments is called a triangle segment of $\gamma$. A segment of $\gamma$ is either a juncture segment or a triangle segment. Then, by choosing a starting segment of $\gamma$, one can express $\gamma$ as a concatenation (or, path product) of a sequence of segments

$$(4.11) \quad \gamma = \gamma_1 \cdot \gamma_2 \cdots \cdot \gamma_N;$$

here $\gamma_1$ is the initial segment, and as one travels on $\gamma$ along its orientation, one then meets $\gamma_2$, and then $\gamma_3$, etc. Notice that this sequence alternates between juncture segments and triangle segments, and that $N$ is even. So, if $\gamma_1$ is a triangle segment, then $\gamma_2$ is a juncture segment, $\gamma_3$ is a triangle segment, and so on, and the last $\gamma_N$ is a juncture segment. Note that a triangle segment is exactly one of a left turn, a right turn, or a U-turn. Examples are shown below.

To each segment $\gamma_i$, we assign a monodromy matrix $M_{\gamma_i} \in \mathrm{SL}_3(\mathbb{Z}[\{X_v^{\pm \frac{1}{2}} \mid v \in \mathcal{V}(Q_\Delta)\}])$

as follows, where $X_v$’s are the Fock-Goncharov $\mathcal{X}$-coordinates of the space $\mathcal{X}_{\mathrm{PGL}_3,\mathcal{E}}$ associated to nodes $v$ of the quiver $Q_\Delta$, i.e. the coordinates for the cluster $\mathcal{X}$-chart for $\Delta$. One can view the symbol $X_v^{\frac{1}{2}}$ as a generator of a formally defined Laurent polynomial ring $\mathbb{Z}[\{X_v^{\pm \frac{1}{2}} \mid v \in \mathcal{V}(Q_\Delta)\}]$, in which $\mathbb{Z}[\{X_v^{\pm 1} \mid v \in \mathcal{V}(Q_\Delta)\}]$ embeds into as $X_v \mapsto (X_v^{\frac{1}{2}})^3$. Or, as in [FG06], we can also view $X_v^{\frac{1}{2}}$ as functions on a covering space $\mathcal{X}_{\mathrm{PGL}_3,\mathcal{E}}^+$ of $\mathcal{X}_{\mathrm{PGL}_3,\mathcal{E}}$. Our approach here will be to view each $X_v$ as a positive-real valued smooth function on the manifold $\mathcal{X}_{\mathrm{PGL}_3,\mathcal{E}}^+$; then $X_v^{\pm \frac{1}{2}}$ is well defined as a positive real valued smooth function on $\mathcal{X}_{\mathrm{PGL}_3,\mathcal{E}}^+$.

**Figure 7.** Juncture segment $\gamma_i$, intersecting an edge of $\Delta$

(MM1) (edge matrix) If $\gamma_i$ is a juncture segment that meets the edge $e$ of $T$, then if the Fock-Goncharov $\mathcal{X}$-coordinates at the two nodes of $Q_\Delta$ lying in $e$ are $X_1$ and $X_2$, where $X_1$ is located at the
along γ (MM4) (U-turn matrix) If γ is a triangle segment in a triangle t and turns to left, then if X is the Fock-Goncharov \( \mathcal{X} \)-coordinate at the node of \( Q_\Delta \) lying in the interior of t, we have
\[
\mathbf{M}_{\gamma} = \begin{pmatrix}
X^{1/3} & X^{2/3} & 0 \\
0 & X^{-1/3} & 0 \\
0 & 0 & X^{-2/3}
\end{pmatrix};
\]

(MM2) (left turn matrix) If γi is a triangle segment in a triangle t and turns to left, then if X is the Fock-Goncharov \( \mathcal{X} \)-coordinate at the node of \( Q_\Delta \) lying in the interior of t, we have
\[
\mathbf{M}_{\gamma} = \begin{pmatrix}
X^{1/3} & 0 & 0 \\
0 & X^{1/3} & 0 \\
0 & 0 & X^{2/3}
\end{pmatrix};
\]

(MM3) (right turn matrix) If γi is a triangle segment in a triangle t and turns to right, then if X is the Fock-Goncharov \( \mathcal{X} \)-coordinate at the node of \( Q_\Delta \) lying in the interior of t, we have
\[
\mathbf{M}_{\gamma} = \begin{pmatrix}
0 & 0 & 1 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

An important example is a peripheral loop surrounding a puncture. An easy observation:

An important example is a peripheral loop surrounding a puncture. An easy observation:

**Lemma 4.11.** An oriented loop γ is a peripheral loop if and only if it can be isotoped so that the triangle segments are either all left turns, or all right turns.

If γ is a peripheral loop surrounding a puncture, then \( \mathbf{M}_\gamma \) (hence also \( \tilde{\mathbf{M}}_\gamma \)) is either an upper triangular matrix or a lower triangular matrix. It is upper triangular if and only if γ matches the boundary-orientation of the corresponding hole of \( \tilde{\mathcal{S}} \) (Def \[2.3\], the triangle segments are all left turns. Otherwise, all right turns.

Therefore we get:

**Lemma 4.12.** If γ is a peripheral loop surrounding a puncture, then \( \mathbf{M}_\gamma \) (hence also \( \tilde{\mathbf{M}}_\gamma \)) is either an upper triangular matrix or a lower triangular matrix. It is upper triangular if and only if γ matches the boundary-orientation of the corresponding hole of \( \tilde{\mathcal{S}} \).

So, given a tuple of (nonzero) coordinates \( (X_v)_{v \in \mathcal{V}(Q_\Delta)} \), i.e. given a point of \( (\mathbb{G}_3)_{\mathcal{V}(Q_\Delta)} \), one can reconstruct a monodromy representation \( \pi_1(\mathcal{S}) \to \text{PGL}_3 \) defined up to conjugation, by setting the image of \( [\gamma] \) for a loop γ under the sought-for map \( \pi_1(\mathcal{S}) \to \text{PGL}_3 \) to be the image of the matrix \( \mathbf{M}_\gamma = \mathbf{M}_\gamma(\mathcal{X}_v)_{v \in \mathcal{V}(Q_\Delta)}) \in \text{GL}_3 \) under the projection \( \text{GL}_3 \to \text{PGL}_3 \). More precisely:

**Proposition 4.13.** Let γ be an oriented simple loop in \( \mathcal{S} \) and Δ an ideal triangulation of \( \mathcal{S} \). Then, for a framed \( \text{PGL}_3 \)-local system on \( \mathcal{S} \), the image under the underlying monodromy \( \pi_1(\mathcal{S}) \to \text{PGL}_3 \) of the equivalence class \( [\gamma] \in \pi_1(\mathcal{S}) \) of γ coincides with the image under the projection \( \text{GL}_3 \to \text{PGL}_3 \) of the matrix \( \mathbf{M}_\gamma = \mathbf{M}_\gamma(\mathcal{X}_v)_{v \in \mathcal{V}(Q_\Delta)}) \in \text{GL}_3 \) constructed above.

Define then the **monodromy matrix** along γ to be the product

\[
\mathbf{M}_\gamma := \mathbf{M}_{\gamma_1} \mathbf{M}_{\gamma_2} \cdots \mathbf{M}_{\gamma_N} \in \text{SL}_3(\mathbb{Z}[\{X_v^{\pm \frac{1}{3}} \mid v \in \mathcal{V}(Q_\Delta)\}]).
\]

One can always isotope γ so that there is no U-turn, so the entries of \( \mathbf{M}_\gamma \) actually lies in \( \mathbb{Z}_{\geq 0}[\{X_v^{\pm \frac{1}{3}} \mid v \in \mathcal{V}(Q_\Delta)\}] \). Hence its trace \( \text{tr}(\mathbf{M}_\gamma) \) is an element of \( \mathbb{Z}_{\geq 0}[\{X_v^{\pm \frac{1}{3}} \mid v \in \mathcal{V}(Q_\Delta)\}] \):

\[
\text{tr}(\mathbf{M}_\gamma) \in \mathbb{Z}_{\geq 0}[\{X_v^{\pm \frac{1}{3}} \mid v \in \mathcal{V}(Q_\Delta)\}].
\]

Observe that for each segment \( \gamma_i \), the monodromy matrix \( \mathbf{M}_{\gamma_i} \) times a monomial \( \prod v_k X_v^k/3 \) for some integer \( k_v \)’s lies in \( \text{GL}_3(\mathbb{Z}[\{X_v^{\pm 1} \mid v \in \mathcal{V}(Q_\Delta)\}] \). Let \( \tilde{\mathbf{M}}_{\gamma} = (\prod v_k X_v^k/3) \mathbf{M}_{\gamma} \) for each \( i \), and

\[
\tilde{\mathbf{M}}_{\gamma} := \tilde{\mathbf{M}}_{\gamma_1} \cdots \tilde{\mathbf{M}}_{\gamma_N} \in \text{GL}_3(\mathbb{Z}[\{X_v^{\pm 1} \mid v \in \mathcal{V}(Q_\Delta)\}]).
\]

Since the determinant of each \( \tilde{\mathbf{M}}_{\gamma_i} \) is a Laurent monomial in \( X_v \)’s, so is that of \( \tilde{\mathbf{M}}_{\gamma} \). However, note that such a normalization for \( \tilde{\mathbf{M}}_{\gamma} \) is not unique, and it is defined only up to a Laurent monomial in \( X_v \)’s. Later, we will use a specific choice of normalization (which makes the \( (1,1) \)-th entry to be 1).

The image of \( \gamma \) for a loop γ under the sought-for map \( \pi_1(\mathcal{S}) \to \text{PGL}_3 \) of \( \mathcal{S} \) coincides with the image under the projection \( \text{GL}_3 \to \text{PGL}_3 \) of the matrix \( \mathbf{M}_\gamma = \mathbf{M}_\gamma(\mathcal{X}_v)_{v \in \mathcal{V}(Q_\Delta)}) \in \text{GL}_3 \) constructed above.
In fact, the monodromy matrices $M_{p\ell}$ let us to completely reconstruct a point of $\mathcal{X}_{PGL_3,\mathfrak{S}}$; namely, Fock Goncharov \cite[§9]{FG06} considered a certain graph on $\mathfrak{S}$ and assigned these matrices to its graph, and constructed a $PGL_3$-local system explicitly (not just its monodromy), together with a framing.

Coming back to our strategy, let us construct the promised map eq.(4.10). Given a point of the domain $\mathcal{X}_{PGL_3,\mathfrak{S}}^+$, we can record it by its positive real coordinates $X_v$'s, for $v \in \mathcal{V}(\Delta)$, for any chosen ideal triangulation $\Delta$. Consider the monodromy $\rho : \pi_1(\mathfrak{S}) \to PGL_3(\mathbb{R})$ for this point, defined up to conjugation. Above, we saw explicitly how $\rho([\gamma]) \in PGL_3(\mathbb{R})$ is given in terms of the coordinates $X_v$'s, for each $\gamma \in \pi_1(\mathfrak{S})$. In fact, we can lift it to a $SL_3(\mathbb{R})$ monodromy $\tilde{\rho} : \pi_1(\mathfrak{S}) \to SL_3(\mathbb{R})$. Pick any basepoint $x \in \mathfrak{S}$, and let $\gamma$ be a loop based at $x$. Define $\tilde{\rho}([\gamma]) := M_{p\ell} \in SL_3(\mathbb{R})$. It is proved in \cite[see the above Prop.4.13]{FG06} that this map $\tilde{\rho} : \pi_1(\mathfrak{S}) \to SL_3(\mathbb{R})$ is a projective representation; that is, $\tilde{\rho}([\gamma_1])\tilde{\rho}([\gamma_2]) = c_{[\gamma_1],[\gamma_2]}\tilde{\rho}([\gamma_1][\gamma_2])$ holds for some constants $c_{[\gamma_1],[\gamma_2]}$. Since both sides of this equation belong to $SL_3$, by taking determinants we see that $c_{[\gamma_1],[\gamma_2]}$ is a 3rd root of unity. But since the entries of matrices are real now, this constant $c_{[\gamma_1],[\gamma_2]}$ must be real, hence $c_{[\gamma_1],[\gamma_2]} = 1$. This means that $\tilde{\rho}$ is a genuine group homomorphism. As mentioned above, the monodromy matrices $M_{p\ell}$ also yield the specific choice of the framing data at punctures, so that one indeed obtains a point of $\mathcal{X}_{SL_3,\mathfrak{S}}(\mathbb{R})$. For us, what we need to know about this framing data are the functions $\pi_p : \mathcal{X}_{SL_3,\mathfrak{S}}(\mathbb{R}) \to H(\mathbb{R})$ at punctures $p$ (eq.(4.3)), and corresponding regular functions $(\pi_p)_i$, (eq.(4.3)). As observed above, the $SL_3(\mathbb{R})$ monodromy $M_{p\ell}$ along a peripheral loop $\gamma_p$ surrounding a puncture $p$ along the orientation matching the boundary-orientation of the corresponding hole of $\mathfrak{S}$ is upper triangular, hence belongs to our choice of the Borel subgroup $B(\mathbb{R})$ of $G(\mathbb{R}) = SL_3(\mathbb{R})$. Composing with the quotient map $B(\mathbb{R}) \to B(\mathbb{R})/U(\mathbb{R}) = H(\mathbb{R})$ which extracts the semi-simple, i.e. the diagonal, part, we obtain an element of $H(\mathbb{R})$, yielding the value of the function $\pi_p : \mathcal{X}_{SL_3,\mathfrak{S}}(\mathbb{R}) \to H(\mathbb{R})$ at this point of $\mathcal{X}_{SL_3,\mathfrak{S}}(\mathbb{R})$. Namely, for this $\gamma_p$ we have:

\begin{equation}
(\pi_p)_i = \text{the } i\text{-th diagonal entry of the upper triangular monodromy matrix } M_{p\ell}. \tag{4.13}
\end{equation}

Now, using the map just constructed, we can apply Def.4.10 and get the basic semi-regular functions $\mathbb{I}_{PGL_3}(\ell)$ on $\mathcal{X}_{PGL_3,\mathfrak{S}}^+$ by pulling back the $A_2$-bangle basis regular functions $\mathbb{L}_{SL_3}(\ell) \in \mathcal{O}(\mathcal{X}_{SL_3,\mathfrak{S}})$ for $\ell \in \mathcal{A}_L(\mathfrak{S};\mathbb{Z})$. A basic example is when $\ell$ is a single oriented simple loop. More generally, we consider the pullback of trace-of-monodromy function along any oriented loop; we refer to it still as trace-of-monodromy.

**Definition 4.14.** For an oriented loop $\gamma$ in $\mathfrak{S}$, we define the trace-of-monodromy function $f_\gamma^+$ on $\mathcal{X}_{PGL_3,\mathfrak{S}}^+$ as

\[ f_\gamma^+ := \text{tr}(M_{p\ell}) \]

which is a smooth function on the manifold $\mathcal{X}_{PGL_3,\mathfrak{S}}^+$.

For any oriented loop $\gamma$, for each triangulation $\Delta$, this function $f_\gamma^+$ can be written as a Laurent polynomial in $\{X_v^{1/3} \mid v \in \mathcal{V}(\Delta)\}$, with non-negative integer coefficients. Changing the basepoint $x$ results in a new matrix $M_{p\ell}$ related to the previous one by a conjugation, hence the trace doesn’t change.

If $\ell$ is a single-component $A_2$-lamination consisting of an oriented simple loop $\gamma$ with weight 1, then

\begin{equation}
\mathbb{I}_{PGL_3}^+(\ell) = f_\gamma^+. \tag{4.14}
\end{equation}

We will be dealing with more general $A_2$-laminations in the coming subsections.

4.3. Functions for punctures. In order to investigate $\mathbb{I}_{PGL_3}^+(\ell)$ in case $\ell$ consists only of a peripheral loop, we study the trace-of-monodromy function $f_\gamma^+ = \text{tr}(M_{p\ell})$ on $\mathcal{X}_{PGL_3,\mathfrak{S}}^+$ for a peripheral loop $\gamma$.

**Proposition 4.15** (peripheral monodromy). Let $\gamma$ be an oriented peripheral loop. Denote by $\ell$ the $A_2$-lamination consisting just of $\gamma$ with weight 1, and by $\bar{\ell}$ the $A_2$-lamination consisting of just of the orientation-reversed loop $\bar{\gamma}$ with weight 1.
If $\gamma$ is positively oriented (Def. 4.6), then for any choice of basepoint of $\gamma$, the monodromy matrix $M_{\gamma}$ is given by the upper triangular matrix

$$
M_{\gamma} = \begin{pmatrix}
\prod_{v} X_{v}^{a_{v}(\ell)} & 0 & \prod_{v} X_{v}^{-a_{v}(\ell) + a_{v}(\ell)} & 0 \\
0 & 0 & \prod_{v} X_{v}^{-a_{v}(\ell)} & 0
\end{pmatrix}
$$

in terms of the coordinate functions of any triangulation $\Delta$, where the products $\prod_{v}$ are taken over all $v$ in $V(Q_{\Delta})$. Hence the trace-of-monodromy function $f_{\gamma}^{+}$ on $\mathcal{PGL}_{\mathbb{R}}$ is

$$
f_{\gamma}^{+} = \prod_{v \in V(Q_{\Delta})} X_{v}^{a_{v}(\ell)} + \prod_{v \in V(Q_{\Delta})} X_{v}^{-a_{v}(\ell) + a_{v}(\ell)} + \prod_{v \in V(Q_{\Delta})} X_{v}^{-a_{v}(\ell)}
$$

where $a_{v}(\ell), a_{v}(\ell) \in \frac{1}{2} \mathbb{Z}$ are the tropical coordinates of the $A_{2}$-laminations $\ell$ and $\ell$ at the node $v$ of the quiver $Q_{\Delta}$; in particular, $a_{v}(\ell)$ and $a_{v}(\ell)$ are non-negative. The trace formula eq. (4.16) holds also when $\gamma$ is negatively oriented.

**Proof.** Pick any basepoint of $\gamma$, and express $\gamma$ as a concatenation of segments as in eq. (4.11). The triangle segments can be assumed to be all right turns, or all left turns (Lem. 4.11). We assume that $\gamma$ is upper triangular, it follows that $M_{\gamma} = M_{\gamma} \cdots M_{\gamma}$ is also upper triangular, and cyclic shift of the product order yields an upper triangular matrix $M_{\gamma}'$ with same diagonal entries as $M_{\gamma}$.

Let $p$ be the puncture that $\gamma$ is surrounding. For an edge $e$ of $\Delta$, $\gamma$ meets $e$ once if only one of the two endpoints of $e$ is $p$, twice if both endpoints of $e$ are $p$, and does not meet $e$ if none of the endpoints of $e$ is $p$. When $\gamma$ meets $e$ twice, they meet in different configuration of orientations as follows; given an arbitrary orientation on $e$, at each of the two intersection points $x$ of $\gamma$ and $e$, the velocity vectors of $\gamma$ and of $e$ (in this order) form a positively oriented basis of $T_{x}S$ (according to the orientation of the surface $S$) at one $x$ and a negatively oriented basis for the other $x$. Now, let $t$ be any ideal triangle of $\Delta$, in which there is at least one triangle segment of $\gamma$. The triangle segments of $\gamma$ in $t$ are all left turns, and by the above discussion, each corner of $t$ can have at most one such triangle segment; if there were two, then $\gamma$ would meet some edge of $\Delta$ twice with same configuration of orientations.

To investigate the tropical coordinates of $\ell = \gamma$, consider a split ideal triangulation $\hat{\Delta}$ for $\Delta$. For convenience, one can isotope so that the intersection points of $\gamma$ with $\hat{\Delta}$ are exactly the breaking points of the concatenation decomposition of $\gamma$ as in eq. (4.11). That is, the intersection points $\gamma \cap \hat{\Delta}$ divide $\gamma$ into the pieces, where a piece in a biangle is a juncture segment, and a piece in a triangle is a triangle segment. In particular, now a triangle segment is what we called a corner arc before. Observe that as of now, $\ell = \gamma$ is an $A_{2}$-lamination that is canonical with respect to $\Delta$ (Def. 3.17), so we can read the tropical coordinates as in Def. 3.27.

Let $t$ be an ideal triangle of $\Delta$, and $\hat{t}$ be the corresponding triangle of $\hat{\Delta}$. Let $e_{1}, e_{2}, e_{3}$ be the sides of $\hat{t}$, appearing clockwise in this order along $\partial \hat{t}$. On each $e_{a}$, there are two nodes $v_{e_{a},1}$ and $v_{e_{a},2}$ of $Q_{\Delta}$ so that the direction $v_{e_{a},1} \rightarrow v_{e_{a},2}$ matches the clockwise orientation of $\partial \hat{t}$; in fact, these nodes should be viewed as living on an edge of $\Delta$, but now we are focusing on only one triangle, so we can be ambiguous. Let $v_{t}$ be the node of $Q_{\Delta}$ lying in the interior of $\hat{t}$. So, in total, we are considering seven nodes of $Q_{\Delta}$ in $\hat{t}$ (or in $t$). Let $\gamma_{j}$ be a triangle segment of $\gamma$ in $\hat{t}$, which is a left turn segment and hence a left turn corner arc in $\hat{t}$. Say, the initial endpoint of $\gamma_{j}$ lies in the side $e_{a}$; then the terminal endpoint of $\gamma_{j}$ lies in $e_{a}^{+1}$ (where $e_{1} = e_{1}$). The tropical coordinates of this $\gamma_{j}$ are given as in eq. (3.8) with $W_{\alpha, \alpha + 1} = \gamma_{j}$. Denoting by $\gamma_{j}$ the triangle segment of the orientation-reversed loop $\gamma$ corresponding to $\gamma_{j}$, by viewing it as an $A_{2}$-lamination in $\hat{t}$ that is a right turn corner arc in $\hat{t}$, its tropical coordinates are as given in eq. (3.9) with $W_{\alpha, \alpha + 1} = \gamma_{j}$.

On the other hand, let's now consider the monodromy matrix contribution, from the three segments $\gamma_{j-1}, \gamma_{j}, \gamma_{j+1}$. We claim that, for a fixed triangle $t$, the basepoint of $\gamma$ could have been chosen in the beginning such that for each triangle segment $\gamma_{j}$ in $\hat{t}$ we have $1 < j < N$. Indeed, since there are at least two triangles meeting $\gamma$, one could choose the basepoint of $\gamma$ such that the initial segment $\gamma_{1}$ is a triangle segment not living in $t$; thus $1 < j$ for any triangle segment $\gamma_{j}$ living in $t$. Meanwhile, the
concatenation sequence $\gamma_1, \ldots, \gamma_N$ must end with a juncture segment, hence it follows that $j < N$ for any triangle segment $\gamma_j$ living in $t$, as desired.

Note that the triples $(\gamma_{j-1}, \gamma_j, \gamma_{j+1})$ associated to different triangle segments $\gamma_j$ living in $\hat{t}$ (or $t$) are disjoint with each other. From (MM1)–(MM2), it follows that the corresponding product of monodromy matrices $M_{\gamma_{j-1}}M_{\gamma_j}M_{\gamma_{j+1}}$ equals

$$
\begin{pmatrix}
X_{\gamma_{j-1},\gamma_j}^{1/3}X_{\gamma_j,\gamma_{j+1}}^{2/3} & 0 & 0 \\
0 & X_{\gamma_{j-1},\gamma_j}^{1/3} & 0 \\
0 & 0 & X_{\gamma_{j-1},\gamma_j}^{1/3}
\end{pmatrix}
= 
\begin{pmatrix}
\prod_v X_v^{a_v(\gamma_j)} & 0 & 0 \\
0 & \prod_v X_v^{a_v(\gamma_j)+a_v(\gamma_{j+1})} & 0 \\
0 & 0 & \prod_v X_v^{a_v(\gamma_{j+1})}
\end{pmatrix}
$$

with the last equality holding in view of the tropical coordinate values as in eq. (3.8) with $W_{\alpha,\alpha+1} = \gamma_j$ and eq. (3.9) with $W_{\alpha+1,\alpha} = \gamma_j$, where $\prod_v$ is taken over seven nodes of $\Delta$ living in $\hat{t}$ (or $t$).

Note $M_{\gamma_j} = M_{\gamma_1}M_{\gamma_2} \cdots M_{\gamma_N}$, where each factor $M_{\gamma_j}$ is upper triangular with diagonal entries being Laurent monomials in $X_v^{1/3}$, $v \in \mathcal{V}(\Delta)$. For each of the three diagonal entries of $M_{\gamma_j}$, we need to know the power of $X_v^{1/3}$ for each $v \in \mathcal{V}(\Delta)$. Let’s read the powers of $X_v^{1/3}$ for nodes $v$ living in $t$ (or $\hat{t}$). Note that for each $\gamma_j$ that is not part of a triangle $(\gamma_{j-1}, \gamma_j, \gamma_{j+1})$ for a triangle segment $\gamma_j$ living in $t$, the monodromy matrix $M_{\gamma_j}$ does not involve any $X_v^{1/3}$ for nodes $v$ living in $t$. So we should focus on the product of $M_{\gamma_{j-1}}M_{\gamma_j}M_{\gamma_{j+1}}$ over all triples $(\gamma_{j-1}, \gamma_j, \gamma_{j+1})$ associated to triangle segments $\gamma_j$ living in $t$. The diagonal entries of this product are $\prod_v X_v^{a_v(\gamma_j)}$, $\prod_v X_v^{a_v(\gamma_j)+a_v(\gamma_{j+1})}$, and $\prod_v X_v^{a_v(\gamma_{j+1})}$, in this order, where $\prod_v$ is over all nodes $v$ living in $t$, and the sum $\sum_j$ is over all $j$’s such that $\gamma_j$ is a triangle segment in $t$. By Lemmas 3.33 we have $\sum_j a_v(\gamma_j) = a_v(\gamma_{j-1} \gamma_j) = a_v(\gamma_{j+1})$, and $\sum_j a_v(\gamma_{j+1}) = a_v(\gamma_{j+1}) = a_v(\gamma_{j+1})$, in view of definition of the tropical coordinates, we can see that $a_v(\gamma_{j+1}) = a_v(\gamma_{j+1})$ and $a_v(\gamma_{j+1}) = a_v(\gamma_{j+1})$ for these $v$’s. Thus, we showed that, for each node $v$ of $\Delta$ living in each triangle $t$ of $\Delta$, hence for each node $v$ in $\Delta$, the powers of $X_v^{1/3}$ in the monomials appearing as the three diagonal entries of $M_{\gamma_j}$ are $a_v(\gamma_{j-1} \gamma_j) = a_v(\gamma_{j+1})$, and $a_v(\gamma_{j+1})$, in this order, as desired in eq. (4.15). We showed this statement for any chosen triangle $t$ of $\Delta$. For any other triangle $t'$, one might have to choose a different basepoint of $\gamma$ for the above arguments to work, so that in the new resulting monodromy matrix $M'_{\gamma_j}$, the diagonal entries have correct powers for $X_v^{1/3}$ for all nodes $v$ living in $t'$. As mentioned in the beginning of the proof, $M_{\gamma_j}$ and $M'_{\gamma_j}$ have same diagonal entries. This finishes the proof for the case when $\gamma$ is a positively oriented peripheral loop.

When $\gamma$ is a negatively oriented peripheral loop surrounding $p$, the proof goes similarly, using the triples $(\gamma_{j-1}, \gamma_j, \gamma_{j+1})$ for triangle segments $\gamma_j$ living in $\hat{t}$. Now $\gamma_j$ is a right turn, so we can assume it goes from the side $e_{\alpha+1}$ to $e_\alpha$ of $\hat{t}$. By (MM1) and (MM3), the product $M_{\gamma_{j-1}}M_{\gamma_j}M_{\gamma_{j+1}}$ now looks

$$
\begin{pmatrix}
X_{\gamma_{j-1},\gamma_j}^{1/3}X_{\gamma_j,\gamma_{j+1}}^{2/3} & 0 & 0 \\
0 & X_{\gamma_{j-1},\gamma_j}^{1/3} & 0 \\
0 & 0 & X_{\gamma_{j-1},\gamma_j}^{1/3}
\end{pmatrix}
= 
\begin{pmatrix}
\prod_v X_v^{a_v(\gamma_j)} & 0 & 0 \\
0 & \prod_v X_v^{a_v(\gamma_j)+a_v(\gamma_{j+1})} & 0 \\
0 & 0 & \prod_v X_v^{a_v(\gamma_{j+1})}
\end{pmatrix}
$$

with the last equality holding in view of the tropical coordinate values as in eq. (3.9) with $W_{\alpha+1,\alpha} = \gamma_j$ and eq. (3.8) with $W_{\alpha,\alpha+1} = \gamma_j$, where $\prod_v$ is taken over seven nodes of $\Delta$ living in the triangle $\hat{t}$ (or $t$). The rest of the arguments goes the same.

As seen in eq. (4.13), the three diagonal entries of $M_{\gamma}$ for a peripheral loop $\gamma$ are the sought-for puncture functions on $\mathcal{B}_{PGL_3, \mathbb{E}}$, corresponding to the regular functions $(\pi_p)_t$ on $\mathcal{B}_{SL_3, \mathbb{E}}$; for a single-component $A_2$-lamination $\ell_p$ (resp. $\ell_p$) consisting of a positively oriented (resp. negatively oriented)
Peripheral loop surrounding $p$ with weight 1, we let
\[
(p_{\pi})_1^+ := \prod_{v \in \mathcal{V}(Q_{\Delta})} X_v^{a_v(\ell_p)} , \quad (p_{\pi})_2^+ := \prod_{v \in \mathcal{V}(Q_{\Delta})} X_v^{-a_v(\ell_p) + a_u(\ell_p)}, \quad (p_{\pi})_3^+ := \prod_{v \in \mathcal{V}(Q_{\Delta})} X_v^{-a_u(\ell_p)},
\]
defined as smooth functions on the smooth manifold $\mathcal{F}_{PGL_3, \mathcal{E}}$. The following statement is not trivial, but is immediate from definitions.

**Lemma 4.16.** Each of these functions $(p_{\pi})_i^+$ on $\mathcal{F}_{PGL_3, \mathcal{E}}^+$ does not depend on the choice of an ideal triangulation $\Delta$.

For example, if $\Delta'$ is any other ideal triangulation, then $\prod_{v \in \mathcal{V}(Q_{\Delta})} X_v^{a_v(\ell_p)} = \prod_{v \in \mathcal{V}(Q_{\Delta'})} X_v^{a'_v(\ell_p)}$.

Recall the $A_2$-bundle basis of $\mathcal{O}(\mathcal{F}_{SL_3, \mathcal{E}})$ constructed by the map $\mathbb{I}_{SL_3} : A_1(\mathcal{G}; \mathbb{Z}) \rightarrow \mathcal{O}(\mathcal{F}_{SL_3, \mathcal{E}})$ in Def 4.7. For each $\ell \in A_1(\mathcal{G}; \mathbb{Z})$, the function $\mathbb{I}_{SL_3}(\ell)$ is constructed by products (and powers) and $\mathbb{Z}$-linear combinations of the trace-of-monodromy functions along loops $\gamma$ and the puncture functions $(\pi_p)_i$. Hence, now this function can be translated as a smooth function on the manifold $\mathcal{F}_{PGL_3, \mathcal{E}}$ using $f^+_\gamma$ and $(\pi_p)_i^+$, which is given for each ideal triangulation $\Delta$ as a Laurent polynomial in $\{X_v^{1/3} \mid v \in \mathcal{V}(Q_{\Delta})\}$ with integer coefficients. Denote this function by
\[
\mathbb{I}_{PGL_3}(\ell) \subset C^\infty(\mathcal{F}_{PGL_3, \mathcal{E}}).
\]
In particular, if $\ell$ consists of only peripheral loops with arbitrary integer weights, then we have
\[
(4.17) \quad \mathbb{I}_{PGL_3}(\ell) = \prod_{v \in \mathcal{V}(Q_{\Delta})} X_v^{a_v(\ell)}.
\]

### 4.4. Basis of ring of regular functions on $\mathcal{F}_{PGL_3, \mathcal{E}}$: main theorem.

We go back to the strategy set out in §4.2. Let $f \in \mathcal{O}(\mathcal{F}_{PGL_3, \mathcal{E}})$. By eq.(4.19) we get $P^* f \in \mathcal{O}(\mathcal{F}_{SL_3, \mathcal{E}})$. By Prop. 4.8(1), we have
\[
(4.18) \quad P^* f = \sum_{\ell \in A_1(\mathcal{G}; \mathbb{Z})} c_\ell(f) \mathbb{I}_{SL_3}(\ell)
\]
for some $c_\ell(f) \in \mathbb{Z}$, which are zero for all but finitely many $\ell \in A_1(\mathcal{G}; \mathbb{Z})$. Evaluating at the field $\mathbb{R}$, we view $P^* f$ and each $\mathbb{I}_{SL_3}(\ell)$ as functions on $\mathcal{F}_{SL_3, \mathcal{E}}(\mathbb{R})$. Pulling back by the map in eq.(4.10), these can be viewed as functions on $\mathcal{F}_{PGL_3, \mathcal{E}}$. The pullback of $P^* f$ on $\mathcal{F}_{PGL_3, \mathcal{E}}$ is just $f$ evaluated at the semi-field $\mathbb{R}_{>0}$, and the pullback of each $\mathbb{I}_{SL_3}(\ell)$ is what we denoted by $\mathbb{I}_{PGL_3}(\ell)$. For any ideal triangulation $\Delta$, since $f$ is regular on the cluster $\mathcal{F}_{PGL_3, \mathcal{E}}$ for $\Delta$, it can be written as a Laurent polynomial in the variables $\{X_v \mid v \in \mathcal{V}(Q_{\Delta})\}$ with integer coefficients. By evaluating at $\mathbb{R}_{>0}$, this Laurent polynomial expression can be viewed as a function on $\mathcal{F}_{PGL_3, \mathcal{E}}$. On the other hand, this Laurent polynomial function on $\mathcal{F}_{PGL_3, \mathcal{E}}$ must equal the function $\sum \ell c_\ell(f) \mathbb{I}_{PGL_3}(\ell)$, which is a priori a Laurent polynomial in $\{X_v^{1/3} \mid v \in \mathcal{V}(Q_{\Delta})\}$ with integer coefficients. In our investigation of when this becomes a Laurent polynomial in $\{X_v \mid v \in \mathcal{V}(Q_{\Delta})\}$, what play crucial roles are the highest term of each basic semi-regular function $\mathbb{I}_{PGL_3}(\ell)$, and the congruence property of all (cube root) Laurent monomial terms for $\mathbb{I}_{PGL_3}(\ell)$.

**Definition 4.17** (partial ordering and congruence on Laurent monomials). Let $\Delta$ be an ideal triangulation of a punctured surface $\mathcal{E}$.

- On the set of all Laurent monomials in $\{X_v^{1/3} \mid v \in \mathcal{V}(Q_{\Delta})\}$, define the partial ordering as follows: for $(a_v)_{v \in \mathcal{V}(Q_{\Delta})}, (b_v)_{v \in \mathcal{V}(Q_{\Delta})} \in (\frac{1}{3}\mathbb{Z})^{\mathcal{V}(Q_{\Delta})}$,
\[
\prod_v X_v^{a_v} \triangleright \prod_v X_v^{b_v} \quad \text{def.} \quad a_v \geq b_v, \quad \forall v \in \mathcal{V}(Q_{\Delta}).
\]

By convention, the zero monomial is set to be of the lowest ordering, i.e. $\prod_v Z_v^{a_v} > 0$.

- For $(a_v)_{v \in \mathcal{V}(Q_{\Delta})}, (b_v)_{v \in \mathcal{V}(Q_{\Delta})} \in (\frac{1}{3}\mathbb{Z})^{\mathcal{V}(Q_{\Delta})}$, we say
\[
\prod_v X_v^{a_v} \text{ and } \prod_v X_v^{b_v} \text{ are congruent to each other } \quad \text{def.} \quad a_v - b_v \in \mathbb{Z}, \quad \forall v \in \mathcal{V}(Q_{\Delta}).
\]

**Proposition 4.18** (highest term of basic semi-regular function). Let $\Delta$ be an ideal triangulation of a punctured surface $\mathcal{E}$. For each $\ell \in A_1(\mathcal{G}; \mathbb{Z})$, the basic semi-regular function $\mathbb{I}_{PGL_3}(\ell) \subset C^\infty(\mathcal{F}_{PGL_3, \mathcal{E}})$ can be written as a Laurent polynomial in $\{X_v^{1/3} \mid v \in \mathcal{V}(Q_{\Delta})\}$ with integer coefficients such that the monomial $\prod_{v \in \mathcal{V}(Q_{\Delta})} X_v^{a_v(\ell)}$ appears with coefficient 1 and is the unique Laurent monomial having the highest partial order among all Laurent monomials appearing in this expression.
Proposition 4.19 (congruence of terms of basic regular function). Let $\mathcal{S}$ be a triangulable punctured surface. For each $\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$, the basic semi-regular function $\mathbb{I}^+_{PGL_3}(\ell)$ on $\mathcal{X}^+_{PGL_3,\mathcal{S}}$ satisfies the following, for each ideal triangulation $\Delta$ of $\mathcal{S}$:

$$\mathbb{I}^+_{PGL_3}(\ell) \in (\prod_{v \in V(\Delta)} X_v^{a_v(\ell)}) \cdot \mathbb{Z}[\{X_v^{\pm1} \mid v \in V(\Delta)\}]$$

That is, $\mathbb{I}^+_{PGL_3}(\ell)$ can be written as a Laurent polynomial in $\{X_v^{1/3} \mid v \in V(\Delta)\}$ such that all Laurent monomials appearing are congruent to each other.

In fact, proofs of these two propositions are much more involved than it might look at the first glance, so we postpone them until the next section. In the present section, let’s assume them.

Corollary 4.20 (congruence and integrality of powers). Let $\mathcal{S}$ be a punctured surface. Let $\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$, where $c_\ell$’s are zero for all but finitely many $\ell$’s. For any ideal triangulation $\Delta$ of $\mathcal{S}$,

$$\sum_{\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})} c_\ell \mathbb{I}^+_{PGL_3}(\ell) \in \mathbb{Z}[\{X_v^{1/3} \mid v \in V(\Delta)\}]$$

belongs to $\mathbb{Z}[\{X_v^{1/3} \mid v \in V(\Delta)\}]$ if and only if $c_\ell = 0$ for all $\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$ not belonging to $\mathcal{A}_L(\mathcal{S}; \mathbb{Z})$ (Def. 3.4), i.e. $c_\ell = 0$ for all $\ell$ such that $a_\ell(\ell) \neq 0$ does not belong to $\mathcal{A}_L(\mathcal{S}; \mathbb{Z})$ for at least one $v \in V(\Delta)$.

Proof of Cor. 4.20 Let $f^+ := \sum_{\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})} c_\ell \mathbb{I}^+_{PGL_3}(\ell)$ be a function on $\mathcal{X}^+_{PGL_3,\mathcal{S}}$, with $c_\ell \in \mathbb{Z}$, which are zero for all but finitely many $\ell$’s. One direction is easy. Suppose $c_\ell = 0$ whenever $\ell$ is not in $\mathcal{A}_L(\mathcal{S}; \mathbb{Z})$, so we can write $f^+ := \sum_{\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})} c_\ell \mathbb{I}^+_{PGL_3}(\ell)$ by Prop 4.19. $\mathbb{I}^+_{PGL_3}(\ell)$ belongs to $\mathbb{Z}[\{X_v^{1/3} \mid v \in V(\Delta)\}]$ for each $\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$. Hence $f^+ \in \mathbb{Z}[\{X_v^{1/3} \mid v \in V(\Delta)\}]$.

Now, for the converse, suppose there exists a map $f^+ \in \mathbb{Z}[\{X_v^{1/3} \mid v \in V(\Delta)\}]$. Recall the partial ordering on the set of all Laurent monomials in $\{X_v^{1/3} \mid v \in V(\Delta)\}$. Choose any ordering on the set $V(\Delta)$, and consider the induced lexicographic total ordering on the set of all Laurent monomials in $\{X_v^{1/3} \mid v \in V(\Delta)\}$, which is compatible with the previous partial ordering. We expressed each $\mathbb{I}^+_{PGL_3}(\ell)$ so that it has the unique Laurent monomial term of highest partial order (Prop 4.18). Among all these highest Laurent monomials appearing in the sum $f^+ = \sum_{\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})} c_\ell \mathbb{I}^+_{PGL_3}(\ell)$, there must be one with the highest lexicographic ordering; in view of Prop 4.18 it is $\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$ (which is the highest term of $\mathbb{I}^+_{PGL_3}(\ell_0)$) for some $\ell_0$ contributing to the sum. This is in fact the unique term of highest lexicographic order, because of the injectivity of the coordinate-system map $a_\ell : \ell \mapsto (a_\ell(\ell)) \in V(\Delta)$ (Prop 3.34). Therefore, in order for $f^+$ to be a function that can be written as a Laurent polynomial in $\{X_v^{1/3} \mid v \in V(\Delta)\}$, it follows that the term $\prod_v X_v^{a_\ell(\ell)}$ of the highest lexicographic order must be a Laurent monomial in $\{X_v \mid v \in V(\Delta)\}$, so $a_\ell(\ell) \in \mathbb{Z}$ for all $v \in V(\Delta)$, or equivalently, $\ell_0 \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$. By Prop 4.19 we know $\mathbb{I}^+_{PGL_3}(\ell_0) \in \mathbb{Z}[\{X_v^{1/3} \mid v \in V(\Delta)\}]$. Now $f^+ - c_{\ell_0} \mathbb{I}^+_{PGL_3}(\ell_0)$ equals $\sum_{\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z}) \setminus \{\ell_0\}} c_\ell \mathbb{I}^+_{PGL_3}(\ell)$, and therefore it has fewer summands than $f^+$ (i.e. fewer $\ell$’s contributing to the sum) and it belongs to $\mathbb{Z}[\{X_v^{1/3} \mid v \in V(\Delta)\}]$ again. By induction, we get that all $\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$ contributing to the sum $\sum_{\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})} c_\ell \mathbb{I}^+_{PGL_3}(\ell)$ must belong to $\mathcal{A}_L(\mathcal{S}; \mathbb{Z})$.

Corollary 4.21 (congruent $A_2$-laminations gives genuinely regular function). Let $\mathcal{S}$ be a triangulable punctured surface. Let $\Delta$ be any ideal triangulation of $\mathcal{S}$. For $\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$, the function $\mathbb{I}^+_{PGL_3}(\ell)$ on $\mathcal{X}^+_{PGL_3,\mathcal{S}}$ can be written as a Laurent polynomial in $\{X_v \mid v \in V(\Delta)\}$ with integer coefficients if and only if $\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$.

So, for $\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$, $\mathbb{I}^+_{PGL_3}(\ell)$ comes from a rational function on $\mathcal{X}^+_{PGL_3,\mathcal{S}}$ that is regular on the cluster $\mathcal{S}$-chart associated to each ideal triangulation $\Delta$. In fact, this rational function on $\mathcal{X}^+_{PGL_3,\mathcal{S}}$ is a regular function on the entire moduli space $\mathcal{X}^+_{PGL_3,\mathcal{S}}$.

Proposition 4.22. Let $\Delta$ be an ideal triangulation of a punctured surface $\mathcal{S}$. For $\ell \in \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$, the basic semi-regular function $\mathbb{I}^+_{PGL_3}(\ell)$ on $\mathcal{X}^+_{PGL_3,\mathcal{S}}$ comes from a regular function on $\mathcal{X}^+_{PGL_3,\mathcal{S}}$.

Prop 4.22 will be proved in the next subsection through several steps. For now, let’s assume it.

Combining the results so far, we arrive at the main theorem of the paper.

Theorem 4.23 (main theorem; $A_2$-bangle basis of $\mathcal{O}(\mathcal{X}^+_{PGL_3,\mathcal{S}})$). Let $\mathcal{S}$ be a triangulable punctured surface. Then the sets $\mathcal{A}_L(\mathcal{S}; \mathbb{Z}) \subseteq \mathcal{A}_L(\mathcal{S}; \mathbb{Z})$ (Def 3.4) for all ideal triangulations $\Delta$ of $\mathcal{S}$ coincide with each other (i.e. Prop 3.41 holds); denote any one of them by $\mathcal{A}_{SL_3,\mathcal{S}}(\mathcal{S}; \mathbb{Z})$. Then, there exists a map

$$\mathbb{I} : \mathcal{A}_{SL_3,\mathcal{S}}(\mathcal{S}; \mathbb{Z}) \to \mathcal{O}(\mathcal{X}^+_{PGL_3,\mathcal{S}})$$
such that

1. $\Pi$ is injective and the image set $\pi(\mathcal{A}_{\text{SL}_3}(\mathbb{Z}'))$ forms a basis of $\mathcal{O}(\mathcal{X}_{\text{PGL}_3})$, which we call an $A_2$-bangle basis of $\mathcal{O}(\mathcal{X}_{\text{PGL}_3})$.

2. For $\ell \in A_{\text{SL}_3}(\mathbb{Z}')$, for any ideal triangulation $\Delta$ of $\mathcal{S}$, $\pi(\ell)$ can be written as a Laurent polynomial in $\{X_v \mid v \in V(\Delta)\}$ with integer coefficients, with the unique highest Laurent monomial being $\prod_{v \in V(\Delta)} X_v^{a_v(\ell)}$, with coefficient 1.

3. If $\ell \in A_{\text{SL}_3}(\mathbb{Z}')$ consists only of peripheral loops, then for each ideal triangulation $\Delta$, we have $\pi(\ell) = \prod_{v \in V(\Delta)} X_v^{a_v(\ell)}$.

4. The structure constants of this $A_2$-bangle basis of $\mathcal{O}(\mathcal{X}_{\text{PGL}_3})$ are integers. That is, for any $\ell, \ell' \in A_{\text{SL}_3}(\mathbb{Z}')$, we have

$$\Pi(\ell) \Pi(\ell') = \sum_{\ell'' \in \mathcal{A}_{\text{SL}_3}(\mathbb{Z}')} c(\ell, \ell'; \ell'') \Pi(\ell''),$$

where $c(\ell, \ell'; \ell'') \in \mathbb{Z}$ and $c(\ell, \ell'; \ell'')$ are zero for all but at most finitely many $\ell''$.

The rest of this section is devoted to proof of Prop. 4.22 and Thm. 4.23 as said, Prop. 4.18 and Prop. 4.19 will be proved in the next section.

4.5. Mutation of basic regular functions. In this subsection we prove Prop. 4.22. First, recall from Def. 2.17 the notion $\mathcal{O}_{\text{cl}}(\mathcal{X}_{\text{PGL}_3})$, the ring of all rational functions on $\mathcal{X}_{\text{PGL}_3}$ that are regular on all cluster $\mathcal{X}$-charts. Any element of $\mathcal{O}_{\text{cl}}(\mathcal{X}_{\text{PGL}_3})$ is universally Laurent for all cluster $\mathcal{X}$-charts, hence in particular is universally Laurent in the Fock-Goncharov’s weaker sense that it is a Laurent polynomial in the cluster $\mathcal{X}$-chart associated to every ideal triangulation $\Delta$, i.e. belongs to $L(\mathcal{X}_{\text{PGL}_3})$ (Def. 1.3).

We recall the result of Shen:

**Proposition 4.24** ([S20, Thm.1.1]). $\mathcal{O}_{\text{cl}}(\mathcal{X}_{\text{PGL}_3}) = \mathcal{O}(\mathcal{X}_{\text{PGL}_3})$.

Shen’s result is written in terms of a slightly different moduli space $\mathcal{M}_{G,S}$ for a generalized marked surface $S$. Putting $G = \text{PGL}_3$ and when $S$ is a punctured surface $\mathcal{S}$, this moduli space is same as our $\mathcal{X}_{\text{PGL}_3}$.

Next, we need the following statement, which follows from Gross-Hacking-Keel [GHK15]. It tells us that, to check the universally Laurent condition, it suffices to check it for one cluster chart and for all charts obtained by applying a single mutation to this chart.

**Proposition 4.25** ([GHK15, Thm.3.9], [S20, Lem.2.2]). Let $f$ be a rational function on $\mathcal{X}_{\text{PGL}_3}$.

Let $\Delta$ be an ideal triangulation of a punctured surface $\mathcal{S}$, and suppose that $f$ is regular on the cluster $\mathcal{X}$-chart for $\Delta$; that is, $f$ is a Laurent polynomial in the cluster $\mathcal{X}$-variables for this chart. If, for every node $v$ of $Q_{\Delta}$, $f$ is regular on the cluster $\mathcal{X}$-chart obtained from the cluster $\mathcal{X}$-chart for $\Delta$ by applying the mutation at node $v$, then $f$ belongs to $\mathcal{O}_{\text{cl}}(\mathcal{X}_{\text{PGL}_3})$.

Our strategy to prove Prop. 4.22 is as follows. For $\ell \in A_{\mathcal{S}}(\mathbb{Z}')$, we know that $\Pi_{\mathcal{X}_{\text{PGL}_3}}(\ell)$ comes from a rational function on $\mathcal{X}_{\text{PGL}_3}$, say $\Pi_{\mathcal{X}}(\ell)$, that is regular on the cluster $\mathcal{X}$-chart for each ideal triangulation $\Delta$. We fix any triangulation $\Delta$, and will show that if we mutate at any node of $Q_{\Delta}$, the result is still a Laurent polynomial in the new cluster $\mathcal{X}$-variables. Then by Prop. 4.22, it follows that $\Pi_{\mathcal{X}}(\ell)$ is regular in the cluster $\mathcal{X}$-chart for $\Delta$ by applying the mutation at node $v$, hence in $\mathcal{O}_{\text{cl}}(\mathcal{X}_{\text{PGL}_3})$, as desired in Prop. 4.22.

In order to study the effect of mutation, we study the basic semi-regular functions $\Pi_{\mathcal{X}_{\text{PGL}_3}}(\ell)$ for $\ell \in \mathcal{A}_{\mathcal{S}}(\mathbb{Z}')$, which are functions on the manifold $\mathcal{X}_{\text{PGL}_3}$ that can be written as Laurent polynomials in the cube roots of (positive real evaluations of) cluster $\mathcal{X}$-coordinate functions. We investigate the effect of mutation for these functions explicitly.

**Proposition 4.26** (mutation of basic semi-regular function at interior node of triangle). Let $\Delta$ be an ideal triangulation of a triangulable punctured surface $\mathcal{S}$. Consider the cluster $\mathcal{X}$-chart associated to $\Delta$, and mutate it at a node of $Q_{\Delta}$ lying in the interior of some triangle of $\Delta$. Denote the resulting quiver by $Q'$, while we naturally identify the sets of nodes $V(Q_{\Delta})$ and $V(Q')$. Denote by $X'_v$ the $\mathcal{X}$-coordinate for the node $v$ of $Q'$ for this new chart obtained as the result of mutation. Then for any $\ell \in \mathcal{A}_{\mathcal{S}}(\mathbb{Z}')$, we have

$$\Pi_{\mathcal{X}_{\text{PGL}_3}}(\ell) \in \prod_{v \in V(Q')} X'_v \cdot \mathbb{Z}^{\{X'_v^{\pm 1} \mid v \in V(Q')\}}.$$
Proof. Consider mutation at \( v_t \) of some triangle \( t \) of \( \Delta \). Let \( e_1, e_2, e_3 \) be the sides of \( t \) appearing in this order clockwise along \( \partial t \). For each \( e_\alpha \), let \( v_{e_\alpha,1} \) and \( v_{e_\alpha,2} \) be the nodes of \( Q_\Delta \) so that \( v_{e_\alpha,1} \to v_{e_\alpha,2} \) matches the clockwise orientation of \( \partial t \); see the triangle on the left of Fig. 8. The cluster \( \mathcal{X} \)-variables change under the mutation at \( v_t \), by the formulas (eq. (2.2))

\[
X'_v = X^{−1}_v, \quad X'_{v_{e_\alpha,1}} = X'_{v_{e_\alpha,1}}(1 + X_v), \quad X'_{v_{e_\alpha,2}} = X'_{v_{e_\alpha,2}}(1 + X_{v_1}^{−1})^{-1},
\]

for \( \alpha = 1, 2, 3 \), with \( X'_v = X_v \) for all \( v \in V(Q') = V(Q_\Delta) \) not appearing in \( t \); so, seven variables change. Writing the old variables in new variables:

\[
(4.20) \quad X_{v_1} = X'_{v_1}, \quad X_{v_{e_\alpha,1}} = X'_{v_{e_\alpha,1}}X_{v_1}(1 + X_{v_1})^{-1}, \quad X_{v_{e_\alpha,2}} = X'_{v_{e_\alpha,2}}(1 + X_{v_1'}), \quad \text{for} \quad \alpha = 1, 2, 3, \quad X_v = X'_v, \quad \text{for all} \quad v \in V(Q_\Delta) \text{not appearing in} \ t.
\]

Let \( \gamma \) be an oriented loop, decomposed into concatenation \( \gamma_1 \gamma_2 \cdots \gamma_N \) of triangle segments and juncture segments, as in eq. (4.11). We study the monodromy matrix \( M_{\gamma} = M_{\gamma_1} \cdots M_{\gamma_N} \). A triangle segment in \( t \), going from edge \( e_{\alpha_1} \) to \( e_{\alpha_2} \) is denoted by \( \gamma_{\alpha_1 \alpha_2} \). A juncture segment at the side \( e_\alpha \) of \( t \) coming out of this triangle \( t \) is denoted by \( \gamma_{\alpha, \text{out}} \), and that going into \( t \) by \( \gamma_{\alpha, \text{in}} \). See the left triangle \( t \) of Fig. [7]

We should consider all possibilities of concatenations of segments in \( t \) forming a ‘complete’ concatenation in this quadrilateral:

\[
(4.21) \quad \gamma_{\alpha_1, \text{in}} \gamma_{\alpha_1 \alpha_2} \gamma_{\alpha_2, \text{out}}, \quad \alpha_1, \alpha_2 \in \{1, 2, 3\}, \quad \alpha_1 \neq \alpha_2.
\]

For each case of a complete concatenation, we should compute the effect of mutation on the product of corresponding monodromy matrices \( M \) defined in (MM1)–(MM3) of §4.2. We use the normalized matrices \( \tilde{M} \), defined as follows. For a juncture segment \( \gamma_t \) as in Fig. [7] let

\[
(4.22) \quad \tilde{M}_{\gamma_t} := \text{diag}(1, X_1^{-1}, X_1^{-1}X_2^{-1}).
\]

Define the left and the right turn matrices for triangle \( t \) as

\[
(4.23) \quad \tilde{M}^{\text{left}}_t = \begin{pmatrix} 1 & 1 + X_1^{-1} & X_1^{-1} \\ 0 & X_1^{-1} & X_1^{-1} \\ 0 & 0 & X_1^{-1} \end{pmatrix}, \quad \tilde{M}^{\text{right}}_t = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 + X_1^{-1} & X_1^{-1} \\ 0 & 0 & X_1^{-1} \end{pmatrix},
\]

So these \( \tilde{M} \) matrices are obtained by dividing by the (1, 1)-th entry of the corresponding original matrix \( M \). A good way to keep track of the (1, 1)-th entries of the original monodromy matrices is using \( A_2 \)-webs and their tropical coordinates. Let \( W_{\alpha_1 \alpha_2; t} \) be an \( A_2 \)-web in \( t \) consisting just of one corner arc, from edge \( e_{\alpha_1} \) to \( e_{\alpha_2} \). In terms of the segments of \( \gamma_t \), this can be viewed as a concatenation of (part of) \( \gamma_{\alpha_1, \text{in}} \), then \( \gamma_{\alpha_1 \alpha_2} \), then (part of) \( \gamma_{\alpha_2, \text{out}} \). When writing the corresponding product of matrices \( \tilde{M} \) as \( \tilde{M}_{W_{\alpha_1 \alpha_2; t}} := M_{\gamma_{\alpha_1, \text{in}}} M_{\gamma_{\alpha_1 \alpha_2}} M_{\gamma_{\alpha_2, \text{out}}} \) as the product of normalized matrices \( \tilde{M}_{W_{\alpha_1 \alpha_2; t}} := \tilde{M}_{\gamma_{\alpha_1, \text{in}}} \tilde{M}_{\gamma_{\alpha_1 \alpha_2}} \tilde{M}_{\gamma_{\alpha_2, \text{out}}} \) times some factor, this factor is the product of (1, 1)-th entries of \( M_{\gamma_{\alpha_1, \text{in}}} \), \( M_{\gamma_{\alpha_1 \alpha_2}} \), \( M_{\gamma_{\alpha_2, \text{out}}} \), and one can observe that the power of each generator \( X_v \) in this factor equals the tropical coordinate \( a_v(W_{\alpha_1 \alpha_2; t}) \) of the \( A_2 \)-web \( W_{\alpha_1 \alpha_2; t} \) (see Fig. 5), i.e. this factor equals \( \prod_{v \in V(Q_\Delta) \cap t} a_v(W_{\alpha_1 \alpha_2; t}) \); this was already seen in the proof of Prop. 4.15.

Now, we will investigate the effect of mutation on the (1,1)-entry-factor and on the (products of) normalized matrices \( \tilde{M}_{W_{\alpha_1 \alpha_2; t}} \). Note that, using this language of \( A_2 \)-webs in \( t \) and \( r \), the cases to be
checked are $W_{\alpha_1 \alpha_2 \ell}$ with $\alpha_1, \alpha_2 \in \{1, 2, 3\}$, $\alpha_1 \neq \alpha_2$. For convenience when studying the effect of mutation, we let

$$X_t := 1 + X_{vt}^t,$$

By eq. (4.20), the effect of mutation on a monomial $\prod_{v \in V(Q_\Delta)} X_{\ell}^k$, for $(k_v) \in \mathbb{Z}^{V(Q_\Delta)}$, is

$$\prod_{v \in V(Q_\Delta)} X_{\ell}^{k_v} = X_{\ell t}^{(k_{vt} + \sum_{\alpha=1}^{3} k_{v_{\alpha t}})/3} X_{\ell t}^{\sum_{\alpha=1}^{3} (-k_{v_{\alpha t}} + k_{v_{\alpha t}} - 2)/3} \prod_{v \in V(Q') \setminus \{vt\}} X_{\ell}^{k_v^{vt}/3}.$$ 

For all the cases of $A_2$-webs $W = W_{\alpha_1 \alpha_2 \ell}$ to be checked, we let $k_v = 3a_v(W)$ for nodes $v$ of $Q_\Delta$ living in $t$, let $k_t = 0$ for other $v \in V(Q_\Delta)$. Note from eq. (3.5) that $\sum_{\alpha=1}^{3} (k_{v_{\alpha t}} + k_{v_{\alpha t}} - 2) = 0$.

In fact, by cyclic symmetry, it suffices to check only two cases $W_{12; t}$ and $W_{13; t}$. As can be seen in eq. (3.8) and eq. (3.9), we have $d_t(W) = 0$ for these $A_2$-webs, as well as $\sum_{\alpha=1}^{3} k_{v_{\alpha t}}/3 = \sum_{\alpha=1}^{3} a_{v_{\alpha t}}(W) \in \mathbb{Z}$ (which appears in the power of $X_{vt}^t$).

We now investigate the effect of mutation on the normalized monodromy matrices. By eq. (4.20), the left and the right turn matrices mutate as:

$$\tilde{M}_t^{\text{left}} = \begin{pmatrix} 1 & 1 & X_{vt}^t & X_{vt}^t \\ 0 & X_{vt}^t & X_{vt}^t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{M}_t^{\text{right}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & X_{vt}^t & X_{vt}^t \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$ 

The edge matrices mutate as:

$$\tilde{M}_{\gamma_{a, in}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X_{v_{a, t}}^{-1} & 0 & 0 \\ 0 & 0 & X_{v_{a, t}}^{-1} & 0 \end{pmatrix}, \quad \tilde{M}_{\gamma_{a, out}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X_{v_{a, t}}^{-1} & 0 & 0 \\ 0 & 0 & X_{v_{a, t}}^{-1} & 0 \end{pmatrix}.$$ 

What we would like to check is, for each $A_2$-web $W = W_{12; t}$ and $W_{13; t}$, that the corresponding product of normalized monodromy matrices lives in $GL_3(\mathbb{Z}[\{X_{v, \ell}^{\pm 1} \mid v \in V(Q')\}])$, i.e., the entries are $X_{\ell}^{\text{Laurent}},$ i.e. Laurent polynomials in $\{X_{v, \ell}^{\pm} \mid v \in V(Q')\}$ with integer coefficients. The point is to make sure that there is no negative powers of $X_t$. For $W_{12; t}$, the corresponding product of normalized matrices is

$$\tilde{M}_{\gamma_{1, in}}^{\text{left}} \tilde{M}_{\gamma_{2, out}}^{\text{right}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X_{v_{1, t}}^{-1} X_{v_{2, t}}^{-1} X_{v_{3, t}}^{-1} & 0 & 0 \\ 0 & 0 & X_{v_{1, t}}^{-1} X_{v_{2, t}}^{-1} X_{v_{3, t}}^{-1} & 0 \end{pmatrix},$$

and when we multiply these matrices, it is easy to see that in each entry there is no $X_t^{-1}$ left, so that it is $X_{\ell}^{\text{Laurent}}$. For $W_{13; t}$, the corresponding product is

$$\tilde{M}_{\gamma_{1, in}}^{\text{left}} \tilde{M}_{t}^{\text{right}} \tilde{M}_{\gamma_{3, out}}^{\text{right}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X_{v_{1, t}}^{-1} X_{v_{2, t}}^{-1} X_{v_{3, t}}^{-1} & 0 & 0 \\ 0 & 0 & X_{v_{1, t}}^{-1} X_{v_{2, t}}^{-1} X_{v_{3, t}}^{-1} & 0 \end{pmatrix},$$

and again, when we multiply these matrices, we see that $X_t^{-1}$ is cancelled, so that the entries are $X_{\ell}^{\text{Laurent}}$.

Let’s summarize the results so far. Writing the trace-of-monomody $f_\gamma^+ = \text{tr}(M_{\gamma_1} \cdots M_{\gamma_N})$ along oriented simple loop $\gamma$ as a Laurent polynomial in the (cube-root) old variables $\{X_v^{1/3} \mid v \in V(Q_\Delta)\}$, by Prop 4.19 we know

$$f_\gamma^+ \in \prod_{v \in V(Q_\Delta)} X_v^{a_v(\ell)} \cdot \mathbb{Z}[X_v^{\pm 1} \mid v \in V(Q_\Delta)],$$

where $\ell = \gamma$. We investigated the monodromy matrices $M_{\gamma_{\ell}}$ in terms of new variables $\{X_v^{1/3} \mid v \in V(Q')\}$, and found out that for the entries of the product matrix $M_{\gamma_1} \cdots M_{\gamma_N}$, the discrepancy between the power of the old variable $X_v$ and the corresponding new variable $X_v'$ (via natural identification $V(Q_\Delta) \leftrightarrow V(Q')$), considered up to integers, occurs only for the node $v_t$ which we are mutating at, where the previous power of $X_v$ is $a_v(\ell)$ while the new power of $X_v'$ is $-a_v(\ell)$ (modulo $\mathbb{Z}$). So

$$f_\gamma^+ \in X_v^{-a_v(\ell)} \prod_{v \in V(Q') \setminus \{vt\}} X_v^{a_v(\ell)} \cdot \mathbb{Z}[\{X_v^{\pm 1} \mid v \in V(Q')\}].$$

In fact, when we apply the above investigation of monodromy matrices to a peripheral loop $\gamma$ around a puncture $p$, by looking at the diagonal entries, we obtain the following: if $\ell$ is a single-component
A2-lamination consisting only of this peripheral loop with an integer weight, we already know from eq.(4.17) that
\[
I^+_{\text{PGL}_3}(\ell) = X_v^{a_v}(\ell) \left( \prod_{e \in \mathcal{V}(Q_\Delta) \setminus \{v_1\}} X_e^{a_e}(\ell) \right),
\]
we now know
\[
II^+_{\text{PGL}_3}(\ell) = X_v^{t-a_v}(\ell) \left( \prod_{e \in \mathcal{V}(Q') \setminus \{v_1\}} X_e^{t_a-e}(\ell) \right).
\]

Now let \( \ell \) be any A2-lamination that can be represented by an A2-web (with weights). Let \( \ell = \ell_1 \cap \ell_2 \), where \( \ell_1 \) consists only of peripheral loops and \( \ell_2 \) does not contain any peripheral loop. By additivity of tropical coordinates (Lem.3.32) it suffices to show the statement separately for \( \ell_1 \) and \( \ell_2 \). The statement for \( \ell_1 \) follows from above observation.

So, now assume that \( \ell \) is an A2-lamination without peripheral loops. Then \( \ell \) can be represented by an A2-web \( W \) with all weights being 1. Note from Def.4.10 that \( \text{II}^+_{\text{PGL}_3}(\ell) = \Psi^*(\text{II}^+_{\text{SL}_3}(\ell)(\mathbb{R})) \), and by eq.(4.17) we have \( \text{II}^+_{\text{SL}_3}(\ell) = F_*^+\text{II}^+_{\text{SL}_3}(\ell) \). In view of Prop.4.2 and Cor.4.3 we have \( \text{II}^+_{\text{SL}_3}(\ell) = \Phi(W) \). Hence by using A2-skein relations in Fig.2 one can remove (or ‘unresolve!’) all internal 3-valent vertices of \( W \), so that the element \( W \) of the skein algebra \( \mathcal{S}(\mathcal{G}; \mathbb{Z}) \) is expressed as \( \mathbb{Z} \)-linear combination of products of oriented simple non-contractible loops \( \gamma \). The sought-for statement for \( \text{II}^+_{\text{PGL}_3}(\ell) \) follows from that for the trace-of-monomodromy function \( f_\gamma^+ \) for each \( \gamma \), the facts that \( \Psi^*, F^* \), \( \Phi \) are ring homomorphisms, and the additivity of tropical coordinates (Lem.3.32).

**Proposition 4.27** (mutation of basic semi-regular function at edge node of triangle). Let \( \Delta \) be any ideal triangulation of a punctured surface \( \mathcal{G} \). Consider the cluster \( \mathcal{X} \)-chart associated to \( \Delta \), and mutate it at a node \( v_0 \) of \( Q_\Delta \) lying in an edge of \( \Delta \). Denote the resulting quiver by \( Q'' \), naturally identifying \( \mathcal{V}(Q_\Delta) \) and \( \mathcal{V}(Q'') \). Denote by \( X_v'' \) the \( \mathcal{X} \)-coordinate for the node \( v \) of \( Q'' \) for this new chart after mutation. Then for any \( \ell \in \mathcal{A}_\Delta(\mathcal{G}) \) we have
\[
II^+_{\text{PGL}_3}(\ell) = \Psi^*(F^*(\Phi(W))(\mathbb{R})�).\]

**Proof.** We use same notations as in the proof of Prop.4.26 for \( t, e_1, e_2, e_3 \) and for nodes \( v_t, v_{e_1}, v_{e_2}, v_{e_3} \) (\( a = 1, 2, 3 \)) of \( Q_\Delta \) appearing in \( t \). It suffices to investigate the mutation at the node \( v_1 \); we do not lose generality. Let \( r \) be the other triangle of \( \Delta \) sharing \( e_1 \) as a side. Label the sides of \( r \) as \( e_4, e_5, e_6 \) clockwise in \( \partial r \), so that \( e_4 \) coincides with \( e_1 \). For each \( \beta = 4, 5, 6 \), let \( e_{\beta,1} \) and \( e_{\beta,2} \) be the nodes of \( Q_\Delta \) lying in \( e_\beta \). In particular, we have \( v_{e_1,1} = v_{e_4,2}, v_{e_1,2} = v_{e_4,1} \); see Fig.8 Under the mutation at the node \( v_{e_1,1} \), the cluster \( \mathcal{X} \)-variables change as (eq.(22))
\[
X_{v_{e_1,1}}'' = X_{v_{e_1,1}}^{-1}, \quad X_{v_{e_2,2}}'' = X_{v_{e_2,2}}(1 + X_{v_{e_1,1}}), \quad X_{v_{e_3,1}}'' = X_{v_{e_3,1}}(1 + X_{v_{e_1,1}}), \quad X_{v_{e_1}}'' = X_{v_{e_1}}(1 + X_{v_{e_1,1}})^{-1},
\]
and \( X_v'' = X_v \) for all other nodes \( v \) of \( Q_\Delta \). Writing the old variables as new ones,
\[
\left\{
\begin{array}{l}
X_{v_{e_1,1}} = X_{v_{e_1,1}}^{-1}, \quad X_{v_{e_2,2}} = X_{v_{e_2,2}}(1 + X_{v_{e_1,1}})^{-1} = X_{v_{e_3,2}}X_{v_{e_1}}(1 + X_{v_{e_1,1}})^{-1},
X_{v_{e_1,2}} = X_{v_{e_1,2}}(1 + X_{v_{e_1,1}})^{-1}, \quad X_{v_{e_3,1}} = X_{v_{e_3,1}}(1 + X_{v_{e_1,1}})^{-1},
X_{v_{e_5}} = X_{v_{e_5}}(1 + X_{v_{e_3,1}}), \quad X_{v_{e_6}} = X_{v_{e_6}}(1 + X_{v_{e_1,1}}),
X_{v} = X_v'' \end{array}\right.
\]
(4.25)
We then proceed as in the proof of Prop.4.26 to study the monodromy matrices of triangle and puncture segments. Triangle segments in \( t \) are denoted by \( \gamma_{a,2} \) and puncture segments in \( t \) by \( \gamma_{a,3} \), as in the proof of Prop.4.26. Define triangle segments \( \gamma_{\beta,1,2} \) and puncture segments \( \gamma_{\beta,1,3} \) and \( \gamma_{\beta,2,3} \) for triangle \( r \) analogously. In particular, \( \gamma_{1,1} = \gamma_{4,3} \) and \( \gamma_{1,3} = \gamma_{4,1} \) under this notation; see Fig.9

Using a similar argument used near the end of the proof of Prop.4.26 it suffices to verify the sought-for statement only in the case when \( \ell \) is a single oriented simple loop \( \gamma \), where we replace \( II^+_{\text{PGL}_3}(\ell) \) in the statement by the trace-of-monomodromy function \( f_\gamma^+ \). We express \( \gamma \) as concatenation \( \gamma = \gamma_1\gamma_2 \cdots \gamma_N \) of triangle segments and puncture segments, and make use of \( f_\gamma^+ = \text{tr}(M_{\gamma_1} \cdots M_{\gamma_N}) \). This time we are considering
two adjacent triangles forming a quadrilateral. So we should consider all possibilities of concatenations of segments in Fig. 9 forming a ‘complete’ concatenation in this quadrilateral:

\[ \gamma_{1,\text{in}}, \gamma_{1,\alpha}, \gamma_{1,\beta}, \gamma_{2,\text{out}} \quad \{\alpha, \beta\} = \{2, 3\}, \quad \{\gamma_{i}, \beta\} = \{5, 6\}, \]

\[ \gamma_{i,\alpha}, \gamma_{i,\beta} \quad \alpha \in \{2, 3\}, \quad \beta \in \{5, 6\}, \]

\[ \gamma_{i,\alpha}, \gamma_{i,\beta} \quad \alpha \in \{2, 3\}, \quad \beta \in \{5, 6\}. \]

For each case of a complete concatenation, we should compute the effect of mutation on the product of corresponding monodromy matrices \( \tilde{M} \) defined in (MM1)–(MM3) of §4.2. We use the normalized matrices \( \tilde{M} \) as in the proof of Prop. 4.26. For a juncture segment \( \gamma_{i,\text{in}} \) as in Fig. 9 let \( \tilde{M}_{\gamma_{i,\text{in}}} \) be as in eq. (4.22). For triangle \( t \), the left and the right turn matrices \( \tilde{M}_{\ell}^{\text{left}} \) and \( \tilde{M}_{\ell}^{\text{right}} \) are as in eq. (4.23); define the corresponding matrices \( \tilde{M}_{\ell}^{\text{left}} \) and \( \tilde{M}_{\ell}^{\text{right}} \) for triangle \( r \) by replacing each \( X_{v}^{-1} \) by \( X_{v}^{-1} \). So these \( \tilde{M} \) matrices are obtained by dividing by the \((1,1)\)-th entry of the corresponding original matrix \( M \).

Let \( W_{\alpha,\beta,\gamma} \) be an \( A_2 \)-web in triangle \( t \) consisting just of one corner arc, from edge \( e_{\alpha} \) to \( e_{\beta} \). In terms of the segments of \( \gamma \), this can be viewed as concatenation of (part of) \( \gamma_{\alpha,\text{in}} \), then \( \gamma_{\alpha,\beta} \), then (part of) \( \gamma_{\alpha,\text{out}} \). We saw in the proof of Prop. 4.26 that the corresponding product of original monodromy matrices \( M_{W_{\alpha,\beta,\gamma}} := M_{\gamma_{\alpha,\text{in}}} M_{\gamma_{\alpha,\beta}} M_{\gamma_{\alpha,\text{out}}} \) equals the product of normalized matrices \( \tilde{M}_{W_{\alpha,\beta,\gamma}} := \tilde{M}_{\gamma_{\alpha,\text{in}}} \tilde{M}_{\gamma_{\alpha,\beta}} \tilde{M}_{\gamma_{\alpha,\text{out}}} \) times \( \prod_{v \in V(\Delta)} X_{v}^{a_{v}(W_{\alpha,\beta,\gamma})} \). Likewise for a corner arc \( A_2 \)-web \( W_{\beta,\gamma,\delta} \) in triangle \( r \). Also, for an \( A_2 \)-web in the union \( t \cup r \) of two triangles given as union of a corner arc in \( t \) and a corner arc in \( r \), similar statement holds. Such web can be either in the form \( W_{\alpha,\beta,\gamma} := W_{\alpha,\beta,\gamma} \cup W_{\gamma,\delta,\epsilon} \) or \( W_{\alpha,\beta,\gamma} := W_{\alpha,\beta,\gamma} \cap W_{\gamma,\delta,\epsilon} \). In the former case \( W_{\alpha,\beta,\gamma} \), the corresponding product of monodromy matrices is \( M_{W_{\alpha,\beta,\gamma}} := M_{\gamma_{\alpha,\text{in}}} M_{\gamma_{\alpha,\beta}} M_{\gamma_{\alpha,\text{out}}} M_{\gamma_{\beta,\gamma}} M_{\gamma_{\beta,\delta}} M_{\gamma_{\beta,\epsilon}} \) (note \( M_{\gamma_{\alpha,\text{in}}} = M_{\gamma_{\alpha,\text{out}}} \) and its \((1,1)\)-th entry can be seen to be \( \prod_{v \in V(\Delta)} X_{v}^{a_{v}(W_{\alpha,\beta,\gamma})} \)). In the latter case \( W_{\alpha,\beta,\gamma} \), the corresponding product of matrices is \( M_{W_{\alpha,\beta,\gamma}} := M_{\gamma_{\alpha,\text{in}}} M_{\gamma_{\beta,\gamma}} M_{\gamma_{\beta,\delta}} M_{\gamma_{\beta,\epsilon}} M_{\gamma_{\beta,\delta}} M_{\gamma_{\beta,\epsilon}} \) (note \( M_{\gamma_{\alpha,\text{in}}} = M_{\gamma_{\alpha,\text{out}}} \) and its \((1,1)\)-th entry can be seen to be \( \prod_{v \in V(\Delta)} X_{v}^{a_{v}(W_{\alpha,\beta,\gamma})} \)). Now, we will investigate the effect of mutation on the \((1,1)\)-entry-factor and on the \((1,1)\)-th entry of \( (\gamma_{\alpha,\beta,\gamma}) \) on the \((1,1)\)-th entry of \( \tilde{M}_{W_{\alpha,\beta,\gamma}} \) and \( \tilde{M}_{W_{\alpha,\beta,\gamma}} \). Note that, using this language of \( A_2 \)-webs in \( t \) and \( r \), the cases to be checked are

\[
\begin{cases}
W_{\alpha,\beta,\gamma} & \text{for } (\alpha, \beta) = (2, 3), \quad (\gamma_{i}, \beta) = (5, 6), \\
W_{\alpha,\beta,\gamma} & \text{for } (\alpha, \beta) = (2, 3), \quad (\gamma_{i}, \beta) = (5, 6), \\
W_{\alpha,\beta,\gamma} & \text{for } (\alpha, \beta) = (2, 3), \quad (\gamma_{i}, \beta) = (5, 6).
\end{cases}
\]

For convenience when studying the effect of mutation, we let

\[
X_{1} := 1 + X_{v_{1,1}}^{2}.
\]

By eq. (4.25), the effect of mutation on a monomial \( \prod_{v \in V(\Delta)} X_{v}^{k_{v}/3} \) is

\[
\prod_{v \in V(\Delta)} X_{v}^{k_{v}/3} = X_{v_{1,1}}^{(k_{v_{1,1}} + k_{v_{3,2}} + k_{v_{3}})/3} X_{1}^{(k_{v_{3,2}} - k_{v_{3}} + k_{v_{0,1}} + k_{v_{3}})/3} \prod_{v \in V(\Delta) \backslash \{v_{1,1}\}} X_{v}^{k_{v}/3}.
\]

For all the cases of \( A_2 \)-webs \( W \) in eq. (4.26), to be checked, we let \( k_{v} = 3a_{v}(W) \) for nodes \( v \) of \( Q_{\Delta} \) living in triangles containing part of \( W \) and let \( k_{v} = 0 \) for other \( v \in V(\Delta) \), and let \( k(W) \) be \( k_{v_{3,2}} - k_{v_{3}} + k_{v_{0,1}} + k_{v_{3}} \).
\[ k_{v_{e,1}} + k_v \] for this \( W \), i.e.
\[ k(W) := 3(-a_{v_{e,2}}(W) - a_v(W) + a_{v_{e,1}}(W) + a_v(W)). \]

To compute this \( k(W) \) for each of our \( A_2 \)-web \( W \), we first compute the following numbers for \( A_2 \)-webs \( W_{\alpha_1\alpha_2} \) and \( W_{\beta_1\beta_2} \) living in one triangle:
\[ k_t(W_{\alpha_1\alpha_2}) := 3(-a_{v_{e,2}}(W_{\alpha_1\alpha_2}) + a_v(W_{\alpha_1\alpha_2})), \quad k_t(W_{\beta_1\beta_2}) := 3(-a_v(W_{\beta_1\beta_2}) + a_{v_{e,1}}(W_{\beta_1\beta_2})). \]

We list the results, which is easily verified from eq. (3.8) and eq. (3.9):
\[ k_t(W_{12}) = 2, \quad k_t(W_{21}) = 1, \quad k_t(W_{13}) = -1, \quad k_t(W_{23}) = 0, \quad k_t(W_{31}) = 0, \quad k_t(W_{32}) = 0. \]
\[ k_t(W_{54}) = -1, \quad k_t(W_{55}) = 1, \quad k_t(W_{46}) = -1, \quad k_t(W_{44}) = -2, \quad k_t(W_{64}) = 0, \quad k_t(W_{65}) = 0. \]

Then, for \( \alpha, \alpha_1, \alpha_2 \in \{2, 3\} \) and \( \beta, \beta_1, \beta_2 \in \{5, 6\} \) one can compute the \( k(W) \) using
\[ k(W_{\alpha_1\alpha_2}) = k_t(W_{\alpha_1\alpha_2}), \quad k(W_{\beta_1\beta_2}) = k_t(W_{\beta_1\beta_2}), \quad k(W_{\alpha_1\beta_2}) = k_t(W_{\alpha_1\beta_2}) + k_t(W_{\beta_4}). \]

Now, we study the effect of mutation on the normalized monodromy matrices \( \tilde{M} \) using eq. (4.25). The edge matrices for edges \( e_2 \) and \( e_6 \) are easy.
\[ \tilde{M}_{74,in} = \text{diag}(1, X^{-1}_{v_{e,2}}, X^{-1}_{v_{e,1}}), \quad \tilde{M}_{74,out} = \text{diag}(1, X^{-1}_{v_{e,2}}, X^{-1}_{v_{e,1}}), \quad \tilde{M}_{76,in} = \text{diag}(1, X^{-1}_{v_{e,1}}, X^{-1}_{v_{e,2}}), \quad \tilde{M}_{76,out} = \text{diag}(1, X^{-1}_{v_{e,1}}, X^{-1}_{v_{e,2}}). \]

The remaining edge matrices are
\[ \tilde{M}_{73,in} = \text{diag}(1, X^{-1}_{v_{e,3}}), \quad \tilde{M}_{37,out} = \text{diag}(1, X^{-1}_{v_{e,3}}), \quad \tilde{M}_{75,in} = \text{diag}(1, X^{-1}_{v_{e,4}}), \quad \tilde{M}_{57,out} = \text{diag}(1, X^{-1}_{v_{e,4}}). \]
\[ \tilde{M}_{71,in} = \text{diag}(1, X^{-1}_{v_{e,1}}), \quad \tilde{M}_{71,out} = \text{diag}(1, X^{-1}_{v_{e,1}}). \]

The left and the right turn matrices are:
\[ \tilde{M}_l = \begin{pmatrix}
1 + X^{-1}_{v_{e}} & 1 + X^{-1}_{v_{e}} & 1 + X^{-1}_{v_{e}} \\
0 & X^{-1}_{v_{e}} & X^{-1}_{v_{e}} \\
0 & 0 & X^{-1}_{v_{e}}
\end{pmatrix}, \quad \tilde{M}_r = \begin{pmatrix}
1 & 0 & 0 \\
1 + X^{-1}_{v_{e}} & 1 & X^{-1}_{v_{e}} \\
1 + X^{-1}_{v_{e}} & 1 + X^{-1}_{v_{e}} & 1 + X^{-1}_{v_{e}}
\end{pmatrix}. \]

We should check, for each \( A_2 \)-web \( W \) in eq. (4.26), that \( X_1^{k(W)/3} \) times the corresponding product of normalized monodromy matrices lives in \( GL_3(\mathbb{Z}[\{X_1^{\pm}\}, v \in V(Q^0)]) \), i.e. the entries are \( X^n \)-\textit{Laurent}, i.e. Laurent polynomials in \( \{X_1^{v} | v \in V(Q^0)\} \) with integer coefficients. There are 12 cases to check in total. The point is to check that in the entries of the final matrices, we see no negative powers of \( X_1 \). Note that the only basic monodromy matrices that are not \( X^n \)-Laurent are \( \tilde{M}_{35,*}, \tilde{M}_l^*, \tilde{M}_r^* \), because they involve \( X_1^{-1} \), so we should keep an eye on them; on the other hand, keep track of \( \tilde{M}_{74,*}, \tilde{M}_l^*, \tilde{M}_r^* \) as they involve \( X_1 \), We cannot assume much symmetry, so we deal with all 12 cases explicitly. Case 1 is \( W_{23,t} \), where we have \( k(W_{23,t}) = 0 \), and the product of normalized matrices is
\[ \tilde{M}_{W_{23,t}} = \tilde{M}_{74,in} \tilde{M}_l^* \tilde{M}_{73,out} \] which is manifestly \( X^n \)-Laurent because each factor already is. In Case 2 we have \( k(W_{32,t}) = 0 \), and
\[ \tilde{M}_{W_{32,t}} = \tilde{M}_{74,in} \tilde{M}_l^* \tilde{M}_{72,out} = \begin{pmatrix}
1 & 0 & 0 \\
0 & X^{-1}_{v_{e,1}} & X^{-1}_{v_{e,2}} \\
0 & 0 & X^{-1}_{v_{e,1}} X^{-1}_{v_{e,2}}
\end{pmatrix} \tilde{M}_{72,out} \]
when the first two matrices are multiplied, $X_1^{-1}$ are cancelled, so the resulting entries are $X''$-Laurent. In Case 3 we have $k(W_{56, tr}) = 0$, and

$$
\tilde{M}_{W_{56, tr}} = \tilde{M}_{75, in} \tilde{M}_{r}^{\text{left}} \tilde{M}_{76, out} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & X_1^{-1} X_{v_{1,1}} X_{v_{1,2}} X_{v_{1,3}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X_1^{-1} X_{v_{1,2}} X_{v_{1,3}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & X_1^{-1} X_{v_{1,3}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X_1^{-1} X_{v_{1,1}} X_{v_{1,2}} X_{v_{1,3}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & X_1^{-1} X_{v_{1,1}} X_{v_{1,2}} X_{v_{1,3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_1^{-1} X_{v_{1,1}} X_{v_{1,2}} X_{v_{1,3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & X_1^{-1} X_{v_{1,1}} X_{v_{1,2}} X_{v_{1,3}}
\end{pmatrix} \tilde{M}_{76, out};
$$

when the first two matrices are multiplied, $X_1^{-1}$’s are cancelled. In Case 4, $k(W_{65, tr}) = 0$, and

$$
\tilde{M}_{W_{65, tr}} = \tilde{M}_{76, in} \tilde{M}_{r}^{\text{right}} \tilde{M}_{75, out} = \tilde{M}_{76, in} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & X_{v_{1,1}} X_{v_{1,2}} & X_{v_{1,3}} & 0 & 0 & 0 & 0 \\
0 & 0 & X_{v_{1,1}} X_{v_{1,2}} X_{v_{1,3}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & X_{v_{1,1}} X_{v_{1,2}} X_{v_{1,3}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X_{v_{1,1}} X_{v_{1,2}} & X_{v_{1,3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & X_{v_{1,1}} X_{v_{1,2}} & X_{v_{1,3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_{v_{1,1}} X_{v_{1,2}} & X_{v_{1,3}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{v_{1,1}} X_{v_{1,2}} X_{v_{1,3}}
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & X_{v_{1,1}} X_{v_{1,2}} & X_{v_{1,3}} & 0 & 0 & 0 & 0 \\
0 & 0 & X_{v_{1,1}} X_{v_{1,2}} X_{v_{1,3}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & X_{v_{1,1}} X_{v_{1,2}} X_{v_{1,3}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X_{v_{1,1}} X_{v_{1,2}} & X_{v_{1,3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & X_{v_{1,1}} X_{v_{1,2}} & X_{v_{1,3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_{v_{1,1}} X_{v_{1,2}} & X_{v_{1,3}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{v_{1,1}} X_{v_{1,2}} X_{v_{1,3}}
\end{pmatrix}
$$

when the latter two matrices are multiplied, $X_1^{-1}$ is cancelled. In Case 5, $k(W_{25, tr}) = 1 - 1 = 0,

$$
\tilde{M}_{W_{25, tr}} = \tilde{M}_{72, in} \tilde{M}_{r}^{\text{right}} \tilde{M}_{71, out} \tilde{M}_{r}^{\text{left}} \tilde{M}_{75, out};
$$

When taking product of the latter two matrices, the only entry that is not manifestly $X''$-Laurent is the $(3, 3)$-th entry, which equals $X''_{v_{1,1}} X''_{v_{1,2}} X''_{v_{1,3}}$ times

$$
(4.27) \quad 1 + (1 + X_1^{-1} X_{v_{1,1}}) X''_{v_{1,2}} + X_1^{-1} X_{v_{1,1}} X''_{v_{1,2}} X''_{v_{1,3}} = 1 + X''_{v_{1,2}} + (1 + X''_{v_{1,1}}) X''_{v_{1,2}} X_{v_{1,1}}
$$

hence is $X''$-Laurent. In Case 6, $k(W_{26, tr}) = 1 - 1 = 0,

$$
\tilde{M}_{W_{26, tr}} = \tilde{M}_{72, in} \tilde{M}_{r}^{\text{right}} \tilde{M}_{71, out} \tilde{M}_{r}^{\text{right}} \tilde{M}_{76, out};
$$

When taking the product of the middle two matrices, the only entries that are not manifestly $X''$-Laurent are $(3, 1)$-th and $(3, 2)$-th entries. The $(3, 1)$-th entry is $X''$-Laurent due to the computation in eq.[4.27]. The $(3, 2)$-th entry is

$$
(4.28) \quad (1 + X_1^{-1} X_{v_{1,1}}) X''_{v_{1,2}} + X_1^{-1} X_{v_{1,1}} X''_{v_{1,2}} X''_{v_{1,3}} (1 + X''_{v_{1,2}} X_1 X''_{v_{1,1}}) = (\text{underlined part in eq.}(4.27)) + X_1^{-1} X_{v_{1,2}} X''_{v_{1,1}} X''_{v_{1,3}} X_1 X''_{v_{1,1}}
$$

hence is $X''$-Laurent. In Case 7, we have $k(W_{35, tr}) = 1 - 1 = 0,

$$
\tilde{M}_{W_{35, tr}} = \tilde{M}_{73, in} \tilde{M}_{r}^{\text{left}} \tilde{M}_{74, out} \tilde{M}_{r}^{\text{left}} \tilde{M}_{76, out};
$$

When taking the product of latter two matrices, the only entries that are not manifestly $X''$-Laurent are $(1, 3)$-th, $(2, 3)$-th, and $(3, 3)$-th entries. The $(1, 3)$-th entry is $X''$-Laurent due to eq.[4.27], and the $(3, 3)$-th entry become $X''$-Laurent when we also multiply the matrix $\tilde{M}_{73, in}$ from left because its $(3, 3)$-th entry is divisible by $X_1$. The $(2, 3)$-th entry is $X''_{v_{1,1}} X''_{v_{1,2}} X''_{v_{1,3}}$ times

$$
(4.29) \quad X_1^{-1} X_{v_{1,2}} X''_{v_{1,1}} + X_1^{-1} X_{v_{1,2}} X''_{v_{1,1}} X''_{v_{1,3}} X_{v_{1,1}} = (1 + X''_{v_{1,1}}) X_1^{-1} X_{v_{1,2}} X''_{v_{1,1}}
$$
which is $X''$-Laurent. In Case 8, we have $k(W_{36;tr}) = 1 - 1 = 0$, and

$$\tilde{M}_{W_{36;tr}} = \tilde{M}_{73,in} \tilde{M}_{\tilde{M}} \tilde{M}_{\text{out}} \tilde{M}_{\text{right}} \tilde{M}_{76;out}$$

$$= \tilde{M}_{73,in} \left( \begin{array}{ccc}
1 & 1 & 0 \\
0 & x_{1}^{-1} & x_{1}^{-1} \\
0 & 0 & 0
\end{array} \right) \left( \begin{array}{ccc}
1 & 1 & 0 \\
0 & x_{1}^{-1} & x_{1}^{-1} \\
0 & 0 & 0
\end{array} \right) \right) \tilde{M}_{76;out}.$$ 

In the product of middle two matrices, we consider the following entries not manifestly $X''$-Laurent. The (1,1)-th, the (1,2)-th and the (2,1)-th entries are $X''$-Laurent due to eq.(4.27), eq.(4.28) and eq.(4.29) respectively. So is (2,2)-th entry essentially due to eq.(4.29). The (3,1)-th, the (3,2)-th and the (3,3)-th become $X''$-Laurent when multiplied by $\tilde{M}_{73,in}$ from the left, cancelling the $X_{1}$ factor. 

In Case 9, we have $k(W_{52;tr}) = 1 + 2 = 3$, and

$$\tilde{M}_{W_{52;tr}} = \tilde{M}_{75,in} \tilde{M}_{\tilde{M}} \tilde{M}_{\text{out}} \tilde{M}_{\text{left}} \tilde{M}_{72;out}$$

$$= \tilde{M}_{75,in} \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \right) \tilde{M}_{72;out}.$$ 

When we take the product of the middle two matrices, the (3,3)-th entry is $X_{1}^{-1}X_{v_{1}}^{-1}$ times

$$X_{1} = 1 + (1 + X_{v_{1}}^{-1}X_{1}X_{v_{1}}^{-1})X_{v_{1}}^{-1} + X_{1}X_{v_{1}}^{-1}X_{v_{1}}^{-1}$$

$$= 1 + X_{v_{1}}^{-1} + X_{1}X_{v_{1}}^{-1}(1 + X_{v_{1}}^{-1}) = X_{1}(1 + X_{v_{1}}^{-1}X_{v_{1}}^{-1}),$$

the (3,2)-th entry equals $1 + \xi_{1}^{-1}X_{v_{1}}^{-1}X_{v_{1}}^{-1}X_{v_{1}}^{-1}X_{v_{1}}^{-1}X_{v_{1}}^{-1}X_{v_{1}}^{-1}X_{v_{1}}^{-1}X_{v_{1}}^{-1}X_{v_{1}}^{-1}$. 

\[ \tilde{M}_{W_{52;tr}} = \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \tilde{M}_{72;out} \]

If we multiply the first two matrices, in the entries we see some $X_{1}^{-1}$ (but not higher powers of $X_{1}^{-1}$); hence, multiplying $\tilde{M}_{W_{52;tr}}$ by $X_{1}$ yields $X'$-Laurent matrix. In Case10, we have $k(W_{53;tr}) = 1 - 1 = 0$, and

$$\tilde{M}_{W_{53;tr}} = \tilde{M}_{75,in} \tilde{M}_{\tilde{M}} \tilde{M}_{\text{out}} \tilde{M}_{\text{left}} \tilde{M}_{73;out}$$

$$= \tilde{M}_{75,in} \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \tilde{M}_{73;out}.$$ 

In the product of the middle two matrices, the (3,1)-th entry is as in eq.(4.30), the (3,2)-th entry is

$$\tilde{M}_{W_{53;tr}} = \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \tilde{M}_{73;out}.$$ 

(4.31)

So $\tilde{M}_{W_{53;tr}}$ equals

$$\left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \tilde{M}_{73;out}.$$ 

Concentrating on the $X_{1}$'s and $X_{1}^{-1}$'s, one sees that when one multiplies these three matrices, there is no $X_{1}^{-1}$ left, hence the entries are $X'$-Laurent. In Case 11, we have $k(W_{62;tr}) = -2 + 2 = 0$, and

$$\tilde{M}_{W_{62;tr}} = \tilde{M}_{76,in} \tilde{M}_{\tilde{M}} \tilde{M}_{\text{out}} \tilde{M}_{\text{left}} \tilde{M}_{72;out}$$

$$= \tilde{M}_{76,in} \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \tilde{M}_{72;out}.$$
In the product of the middle two matrices, there are two entries that are not manifestly $X''$-Laurent: 

$$1 + X_{v_1}^{-1} X_{v_1}^{-1} + (1 + X_{v_1}^{-1} X_{v_1}^{-1}) X_{v_1}^{-1} X_{v_1}^{-1} = 1 + X_{v_1}^{-1} X_{v_1}^{-1} (1 + X_{v_1}^{-1} + X_{v_1}^{-1} X_{v_1}^{-1})$$

which is $X''$-Laurent and the (1,3)-th entry which is $X_{v_1}^{-1} X_{v_1}^{-1}$ times eq.\(4.30\), hence is $X''$-Laurent. In Case 12, we have $k(W_{63;rt}) = -2 - 1 = -3$, and 

$$\hat{M}_{W_{63;rt}} = M_{66;in} M_{66;out} M_{11;in} M_{11;out} = M_{66;out} = 1 + (1 + X_{v_1}^{-1} X_{v_1}^{-1}) X_{v_1}^{-1} X_{v_1}^{-1} X_{v_1}^{-1} X_{v_1}^{-1} + 1 + X_{v_1}^{-1} X_{v_1}^{-1}$$

In the product of the middle two matrices, the (1,1)-th entry is eq.\(4.30\), the (1,2)-th entry is eq.\(4.31\); hence all entries are $X''$-Laurent, and the entries of the first column are divisible by $X_1$. Multiplying by $\hat{M}_{66;out}$, one observes that all entries are $X''$-Laurent and are divisible by $X_1$. Therefore $X_{1} k(W_{63;rt}) \hat{M}_{W_{63;rt}} = X_{1}^{-1} M_{W_{63;rt}}$ is a $X''$-Laurent matrix.

Let’s summarize the results so far. Writing the trace-of-monodromy $f^{+}_\gamma = \text{tr}(M_{\gamma_1} \cdots M_{\gamma_N})$ along oriented simple loop $\gamma$ as a Laurent polynomial in (cube-root) old variables $\{X_{v}^{1/3} \mid v \in \mathcal{V}(Q)\}$, by Prop.\(4.19\), we know $f^{+}_\gamma \in \left(\prod_{v \in \mathcal{V}(Q)} X_v^{-1}\right) \cdot \mathbb{Z}[X_v^\pm 1 \mid v \in \mathcal{V}(Q)]$ where $\ell = \gamma$. We investigated the monodromy matrices $M_{\gamma_1}$ in terms of new variables $\{X_v^{1/3} \mid v \in \mathcal{V}(Q)\}$, and found out that for the entries of the product matrix $M_{\gamma_1} \cdots M_{\gamma_N}$, the discrepancy between the power of an old variable $X_v$ and the corresponding new variable $X_v''$, considered up to integers, occurs only for the node $v_{v_1}$ which we are mutating at, where the previous power of $X_v = a_{v_{v_1}}(\ell)$ while the new power of $X_v''$ is $-a_{v_{v_1}}(\ell) + a_{v_{v_2}}(\ell) + a_{v_1}(\ell)$, which can be written as $-a_{v_{v_1}}(\ell) + \sum_{v \in \mathcal{V}(Q)} a_{v}(\ell)$.

The rest of the argument goes as in the proof of Prop.\(4.26\) as already mentioned. ■

Apply Propositions\(4.26\) and \(4.27\) to $\gamma \in \mathcal{A}(S; \mathbb{Z})$ lying in $\mathcal{A}(\mathbb{Z})$, i.e. when $a_\gamma(\ell) \in \mathbb{Z}$ for all nodes $v$ of $Q$.

**Corollary 4.28.** Let $\Delta$ be any ideal triangulation. Consider the cluster $\mathcal{X}$-chart associated to $\Delta$, and mutate it at a node of $Q$. Denote the resulting quiver by $Q'$, naturally identifying $\mathcal{V}(Q_{\Delta})$ and $\mathcal{V}(Q')$. Denoting by $X'_{v}$ the $\mathcal{X}$-coordinate for the node $v$ of $Q'$ for this new chart after mutation. Then for any $\ell \in \mathcal{A}(\mathbb{Z})$, we have 

$$\hat{I}_{PGL_3}^{\mathcal{X}}(\ell) \in \mathbb{Z}[\{X_v^{1/3} \mid v \in \mathcal{V}(Q')\}]$$

Thus the argument we gave right after Prop.\(4.25\) works, and this proves Prop.\(4.22\) as promised.

**4.6 Proof of the main theorem.** We prove the main theorem, Thm.\(4.23\). First, choose any ideal triangulation $\Delta$ of $\mathcal{S}$. Define the map 

$$I_{\Delta} : \mathcal{A}(\mathbb{Z}) \to \mathcal{O}(\mathcal{X}_{PGL_3;\mathcal{S}})$$

as follows: for each $\ell \in \mathcal{A}(\mathbb{Z})$, let $\hat{I}_{\Delta}(\ell) \in \mathcal{O}(\mathcal{X}_{PGL_3;\mathcal{S}})$ be the regular function on $\mathcal{X}_{PGL_3;\mathcal{S}}$ yielding $I_{PGL_3}^{\mathcal{X}}(\ell)$ when evaluated at the semi-field $\mathbb{R}_{\geq 0}$; such $I_{\Delta}(\ell)$ exists by Prop.\(4.22\), which we have explicitly constructed during the proof. We recall the arguments in \(4.4\) to prove items Thm.\(4.23\)\(1)–\(4)\ for $I_{\Delta}$.

(1) To show that $I_{\Delta}(\mathcal{A}(\mathbb{Z}))$ spans $\mathcal{O}(\mathcal{X}_{PGL_3;\mathcal{S}})$, start from any $f \in \mathcal{O}(\mathcal{X}_{PGL_3;\mathcal{S}})$. Bring it to $P^{+} f = \sum_{\ell \in \mathcal{A}(S;\mathbb{Z})} c_{\ell}(f) I_{\Delta}(\ell) \in \mathcal{O}(\mathcal{X}_{PGL_3;\mathcal{S}})$ as in \(4.18\), then pullback by \(4.10\) to the function $f^{+} := \sum_{\ell \in \mathcal{A}(S;\mathbb{Z})} c_{\ell}(f) I_{PGL_3}^{\mathcal{X}}(\ell)$ on $\mathcal{X}_{PGL_3;\mathcal{S}}$, which should coincide with the evaluation of $f$ at the semi-field $\mathbb{R}_{\geq 0}$. Since $f$ is regular on $\mathcal{X}_{PGL_3;\mathcal{S}}$, it must be Laurent polynomial in $\{X_v \mid v \in \mathcal{V}(Q_\Delta)\}$ for every ideal triangulation $\Delta$. So, $f^{+}$ must equal to a Laurent polynomial in $\{X_v \mid v \in \mathcal{V}(Q_\Delta)\}$, as a function on $\mathcal{X}_{PGL_3;\mathcal{S}}$. By Cor.\(4.20\) we see that each $\ell \in \mathcal{A}(S;\mathbb{Z})$ contributing to the sum all belongs
to $\mathcal{A}_\Delta(Z')$. For each $\ell \in \mathcal{A}_\Delta(Z')$, recall that $I_\Delta(\ell)$ is the element of $\mathcal{O}(X_{PGL_3, \mathfrak{c}})$ yielding $I^+_{PGL_3}(\ell)$ by evaluation at the semi-field $\mathbb{R}_{>0}$. So we have the equality

$$f = \sum_{\ell \in \mathcal{A}_\Delta(Z')} c_\ell(f) I_\Delta(\ell)$$

of elements of $\mathcal{O}(X_{PGL_3, \mathfrak{c}})$, where the right hand side is a finite sum. This proves that $I_\Delta(\mathcal{A}_\Delta(Z'))$ spans $\mathcal{O}(X_{PGL_3, \mathfrak{c}})$. To show the linear independence, suppose the right hand side is zero, as a regular function on $X_{PGL_3, \mathfrak{c}}$. Evaluate at the semi-field $\mathbb{R}_{>0}$, and consider the corresponding sum $\sum_{\ell \in \mathcal{A}_\Delta(Z')} c_\ell(f) I^+_{PGL_3}(\ell)$. By using a similar argument as in the proof of Corollary 4.20 employing the lexicographic total ordering on the set of all Laurent monomials, one can show by induction that the coefficients $c_\ell(f)$ must be all zero. Hence the linear independence. This shows that $I_\Delta(\mathcal{A}_\Delta(Z'))$ is a basis of $\mathcal{O}(X_{PGL_3, \mathfrak{c}})$, and also shows the injectivity of the map $I_\Delta$.

(2) This is immediate from the definition of $I_\Delta$ and Proposition 4.18.

(3) This is immediate from the definition of $I_\Delta$ and eq. (4.17).

(4) Let $\ell, \ell' \in \mathcal{A}_\Delta(Z') \subset \mathcal{A}_L(\mathfrak{G}; \mathbb{Z})$. By Proposition 4.8(2) we get the product-to-sum decomposition as in eq. (4.32) for some $c_{SL_3}(\ell, \ell'; \ell'') \in \mathbb{Z}$; this is an equality of elements of $\mathcal{O}(X_{SL_3, \mathfrak{c}})$. Pulling back by the map eq. (4.10), we get

$$I^+_{PGL_3}(\ell) I^+_{PGL_3}(\ell') = \sum_{\ell'' \in \mathcal{A}_L(\mathfrak{G}; \mathbb{Z})} c_{SL_3}(\ell, \ell'; \ell'') I^+_{PGL_3}(\ell'')$$

Now, since $\ell, \ell' \in \mathcal{A}_\Delta(Z')$, both $I^+_{PGL_3}(\ell)$ and $I^+_{PGL_3}(\ell')$, hence also their product, belong to $\mathbb{Z}[\{X^\pm v \mid v \in V(\mathfrak{G})\}]$. So the right hand side of eq. (4.32) belongs to $\mathbb{Z}[\{X^\pm v \mid v \in V(\mathfrak{G})\}]$. By Corollary 4.20 all $\ell'' \in \mathcal{A}_L(\mathfrak{G}; \mathbb{Z})$ contributing to the sum in the right hand side belong to $\mathcal{A}_\Delta(Z')$. Then one can recognize that the resulting eq. is the evaluation at the semi-field $\mathbb{R}_{>0}$ of an equality

$$I_\Delta(\ell) I_\Delta(\ell') = \sum_{\ell'' \in \mathcal{A}_\Delta(Z')} c_{SL_3}(\ell, \ell'; \ell'') I_\Delta(\ell''),$$

which is the desired statement in item (4).

So, for each chosen ideal triangulation $\Delta$, the items (1) and (4) of Theorem 4.23 hold for $I_\Delta$, with $\mathcal{A}_{SL_3, \mathfrak{c}}(Z')$ in the statements replaced by $\mathcal{A}_\Delta(Z')$, while the items (2) and (3) for $I_\Delta$ hold only for this particular $\Delta$ at the moment.

Let $\Delta'$ be any other ideal triangulation. Let $\ell \in \mathcal{A}_\Delta(Z') \subset \mathcal{A}_L(\mathfrak{G}; \mathbb{Z})$. Then $I_\Delta(\ell) \in \mathcal{O}(X_{PGL_3, \mathfrak{c}})$, and since $I_\Delta(\mathcal{A}_\Delta(Z'))$ is a basis of $\mathcal{O}(X_{PGL_3, \mathfrak{c}})$ (by item (1) for $I_\Delta$), we have

$$I_\Delta(\ell) = \sum_{\ell' \in \mathcal{A}_\Delta(Z')} c(\ell') I_\Delta(\ell')$$

for some $c(\ell') \in \mathbb{Z}$ which are zero for all but finitely many $\ell' \in \mathcal{A}_\Delta(Z') \subset \mathcal{A}_L(\mathfrak{G}; \mathbb{Z})$. Evaluating at $\mathbb{R}_{>0}$ we obtain

$$I^+_{PGL_3}(\ell) = \sum_{\ell' \in \mathcal{A}_\Delta(Z')} c(\ell') I^+_{PGL_3}(\ell').$$

Now, view all functions in eq. (4.33) as Laurent polynomials in $\{X^v \mid v \in V(\mathfrak{G})\}$, for $\Delta$. Since the left hand side $I^+_{PGL_3}(\ell)$ belongs to $\mathbb{Z}[\{X^v \mid v \in V(\mathfrak{G})\}]$ (because $\ell \in \mathcal{A}_\Delta(Z')$, and by item (2) for $I_\Delta$), from Corollary 4.20 for $\Delta$ we deduce that all $\ell'$’s contributing to the sum belongs to $\mathcal{A}_\Delta(Z')$, hence for these $\ell'$ the function $I^+_{PGL_3}(\ell')$ comes from $I_\Delta(\ell') \in \mathcal{O}(X_{PGL_3, \mathfrak{c}})$. Thus, from eq. (4.33) we get

$$I_\Delta(\ell) = \sum_{\ell' \in \mathcal{A}_\Delta(Z')} c(\ell') I_\Delta(\ell').$$

However, since $I_\Delta$ is injective and $I_\Delta(\mathcal{A}_\Delta(Z'))$ is a basis (by item (1) for $I_\Delta$), it follows that the only contributing $\ell'$ in the right hand side is $\ell' = \ell$ itself, with $c(\ell') = 1$. This yields:

**Proposition 4.29.** For each $\ell \in \mathcal{A}_\Delta(Z')$, we have $\ell \in \mathcal{A}_\Delta(Z')$ and $I_\Delta(\ell) = I_\Delta(\ell)$.

As a corollary, this proves Proposition 4.11 i.e. the sets $\mathcal{A}_\Delta(Z') \subset \mathcal{A}_L(\mathfrak{G}; \mathbb{Z})$ for all ideal triangulations $\Delta$ coincide with each other, as promised, and the main theorem Theorem 4.23 holds as is written, with $\mathcal{A}_{SL_3, \mathfrak{c}}(Z')$ being understood as $\mathcal{A}_\Delta(Z')$ for any ideal triangulation $\Delta$. 
5. SL$_3$ classical trace map and state-sum formula

In the present section we prove Prop. 4.18 and Prop. 4.19 as promised. In order to do so, we develop a classical SL$_3$ version of Bonahon-Wong's SL$_2$ quantum trace map [BW11]. The SL$_3$ classical trace map that we construct in the present section is interesting in its own right, for explicit computation, and for providing basic framework for quantization. Notice that, in this section, $\mathcal{S}$ may be a generalized marked surface having boundary, not even necessarily triangulable.

5.1. SL$_3$ classical trace for stated $A_2$-skein algebra. The goal is to study the properties of the map $I^{+}_{\text{PGL}} : \mathcal{A}_2(\mathbb{Z} ; \mathbb{Z}) \to \mathcal{C}^\infty(\mathcal{X}^\pm_{\text{PGL}}, \mathbb{Z})$ defined in Def. 4.10. Crucial is the restriction to $A_2^0(\mathbb{Z} ; \mathbb{Z})$, which embeds to the $A_2$-skein algebra $\mathcal{S}(\mathbb{Z} ; \mathbb{Z})$; the image under $I^{+}_{\text{PGL}}$ of each element of $\mathcal{S}(\mathbb{Z} ; \mathbb{Z})$ is a Laurent polynomial in $\{ X_v^{1/3} \mid v \in \mathcal{V}(\Delta) \}$ per each chosen $\Delta$, and we would like to investigate this Laurent polynomial. Although the main interest of the present paper is on the case when $\mathcal{S}$ is a punctured surface, i.e. without boundary, a full treatment of the SL$_3$ classical trace map requires us to consider the case when $\mathcal{S}$ has boundary. In fact, the domain of the sought-for trace map is a ‘stated’ version of the $A_2$-skein algebra.

**Definition 5.1** (stated $A_2$-skein algebra). Let $\mathcal{S}$ be a generalized marked surface, not necessarily triangulable.

- A state of an $A_2$-web $W$ in $\mathcal{S}$ is a map $s : \partial W \to \{1, 2, 3\}$ which assigns a number in $\{1, 2, 3\}$ to each endpoint of $W$, i.e. to each external vertex of $W$. A stated $A_2$-web in $\mathcal{S}$ is a pair $(W, s)$ of an $A_2$-web $W$ in $\mathcal{S}$ and a state $s$ of $W$.

- An isotopy of stated $A_2$-webs in $\mathcal{S}$ is an isotopy within the class of stated $A_2$-webs in $\mathcal{S}$. Two stated $A_2$-webs in $\mathcal{S}$ are said to be equivalent if they are related to each other by a sequence of isotopies and the stated versions of the moves (M1), (M2) and (M3) in Def. 3.3. The states should be compatible under isotopy and moves, and in particular, the move (M3) should be understood as $\quad \leftrightarrow \quad $ so that the endpoints of same labels must carry same state values.

- Let $\mathcal{R}$ be a commutative ring with unity 1. The (commutative) stated $A_2$-skein algebra $\mathcal{S}_s(\mathcal{S} ; \mathcal{R})$ is the free $\mathcal{R}$-module with the set of all equivalence classes of stated $A_2$-webs in $\mathcal{S}$ as a free basis, mod out by the $A_2$-skein relations (S1), (S2), (S3) and (S4) in Fig 2.

- For an equivalence class of a stated $A_2$-web $(W, s)$ in $\mathcal{S}$, the corresponding element of the stated $A_2$-skein algebra $\mathcal{S}_s(\mathcal{S} ; \mathcal{R})$ is denoted by $[W, s]$ and is called a stated $A_2$-skein.

Note that we do not impose any boundary relations, other than the boundary exchange move (M3), hence $\mathcal{S}_s(\mathcal{S} ; \mathcal{R})$ is different from (the commutative version of) the version used and studied by Higgins [H20]; we will invoke his version in the next subsection.

We now define the target ring of the trace map.

**Definition 5.2** (cube-root Fock-Goncharov algebra). Let $\Delta$ be an ideal triangulation of a triangulable generalized marked surface $\mathcal{S}$.

For each ideal triangle $t$ of $\Delta$, let $e_1, e_2, e_3$ be the sides of $t$ appearing clockwise in $\partial t$ in this order. On each side $e_i$, let $v_{e_{i,1}}, v_{e_{i,2}}$ be the nodes of $Q_{\Delta}$ on $e_i$ such that the direction $v_{e_{i,1}} \to v_{e_{i,2}}$ matches the clockwise orientation of $\partial t$. Let $v_t$ be the node of $Q_{\Delta}$ in the interior of $t$ (see the left triangle of Fig 3). Define the (classical) cube-root Fock-Goncharov triangle algebra (or just triangle algebra in short) $\mathcal{Z}_t$ as the commutative free associative $\mathbb{Z}$-algebra generated by (the seven) $Z_{t,v}$’s and their inverses for the nodes $v$ of $Q_{\Delta}$ appearing in $t$.

Consider tensor product algebra $\bigotimes_{t \in \mathcal{F}(\Delta)} \mathcal{Z}_t$, where each $\mathcal{Z}_t$ naturally embeds into, where $\mathcal{F}(\Delta)$ is the set of all ideal triangles of $\Delta$.

For each node $v$ of $Q_{\Delta}$, define the element $Z_v$ of the tensor product algebra $\bigotimes_{t \in \mathcal{F}(\Delta)} \mathcal{Z}_t$ as follows:

1. If $v$ is an interior node $v_t$ of some triangle $t$, then $Z_v := Z_{t,v_t}$.
2. If $v$ is a node $v_{e_{i,1}}$ lying in a boundary arc of $\mathcal{S}$, and if this node lies in triangle $t$, then $Z_v := Z_{t,v_{e_{i,1}}}$.
(3) If $v$ is a node lying in an interior edge of $\Delta$ so that it equals $v_{\alpha,1}$ of a triangle $t$ and $v_{\alpha,2}$ of a triangle $r$, then $Z_v := Z_{t,v_{\alpha,1}} Z_{r,v_{\alpha,2}}$.

Let $Z_\Delta = Z_{\Delta;\mathcal{S}}$ be the (classical) cube-root Fock-Goncharov algebra for $\Delta$.

This is a classical counterpart of what was called the Fock-Goncharov algebra in [D20]. In particular, the Fock-Goncharov algebra $Z_\Delta$ is just the Laurent polynomial ring over $\mathbb{Z}$ with generators $\{Z_v | v \in \mathcal{V}(Q_\Delta)\}$. More explanation and justification of this construction will be given soon. The generators $Z_v$ will play the role of $X_v^{1/3}$. In the end, we actually want to have Laurent polynomials in the variables $X_v$.

**Definition 5.3.** In Def. 5.2, for each node $v$ of $Q_\Delta$, let

$$X_v := Z_v^3 \in Z_\Delta = Z_{\Delta;\mathcal{S}}.$$ 

Define the (classical) Fock-Goncharov algebra $\mathcal{X}_\Delta = \mathcal{X}_{\Delta;\mathcal{S}}$ as the $\mathbb{Z}$-subalgebra of $Z_\Delta$ generated by $\{X_v^{\pm 1} | v \in \mathcal{V}(Q_\Delta)\}$.

One more technical preliminary is the following.

**Lemma 5.4** (cutting process). Let $(\Sigma, \mathcal{P})$ be a generalized marked surface, with $\mathcal{S} = \Sigma \setminus \mathcal{P}$. Let $e$ be a $\mathcal{P}$-arc in $\Sigma$ (or an ideal arc in $\mathcal{S}$) whose interior lies in the interior of $\Sigma$. Cutting $(\Sigma, \mathcal{P})$ along $e$ yields a possibly-disconnected generalized marked surface $(\Sigma', \mathcal{P}')$, uniquely determined up to diffeomorphism. Denote by $g : \Sigma' \to \Sigma$ a corresponding gluing map.

Let $(W, s)$ be a stated $A_2$-web in $\mathcal{S}$ that is transverse to $e$. Then $W' := g^{-1}(W)$ is an $A_2$-web in $\mathcal{S}' = \Sigma' \setminus \mathcal{P}'$, and we say that $W'$ is obtained from $W$ by cutting along $e$. A state $s' : \partial W' \to \{1, 2, 3\}$ of $W'$ is said to be compatible with $s$ if $s'(x) = s(g(x))$ for each $x \in \partial W' \cap g^{-1}(\partial W)$ and $s'(x_1) = s'(x_2)$ for all $x_1, x_2 \in \partial W' \cap g^{-1}(e)$ such that $g(x_1) = g(x_2)$.

If $\mathcal{S}$ is triangulable, with a chosen ideal triangulation $\Delta$, then $\Delta' := g^{-1}(\Delta)$ is an ideal triangulation of $\mathcal{S}'$, which is said to be obtained from $\Delta$ by cutting along $e$. Triangles of $\Delta$ are naturally in bijection with $\Delta'$, where for each triangle $t$ of $\Delta$ there is a canonical bijection from the sides of $t$ to the sides of the corresponding triangle $t'$ of $\Delta'$. The induced isomorphism $Z_t \to Z_{t'}$ between triangle algebras naturally induces the injection

$$i_{\Delta, \Delta'} : Z_\Delta \to Z_{\Delta'}.$$  

For convenience, we define:

**Definition 5.5.** For a generalized marked surface $\mathcal{S}$, an $A_2$-web $W$ in $\mathcal{S}$ is called a 3-way web in $\mathcal{S}$ if it has three external vertices, one internal vertex, no crossing, and has only one component, which consists of three edges, all meeting the internal vertex.

The main object of study of the present section is the following $SL_3$ classical trace map.

**Proposition 5.6** ($SL_3$ classical trace map). There exists a unique family of ring homomorphisms

$$\text{Tr}_\Delta = \text{Tr}_{\Delta;\mathcal{S}} : S_3(\mathcal{S}; \mathbb{Z}) \to Z_\Delta$$

defined for each triangle $\Delta$ of a generalized marked surface $\mathcal{S}$ and each ideal triangulation $\Delta$, such that:

(CT1) **(cutting/gluing property)** Let $(W, s)$ be a stated $A_2$-web in $\mathcal{S}$, and $e$ be a constituent arc of $\Delta$ that is not a boundary arc of $\mathcal{S}$. Let $\mathcal{S}'$ be the generalized marked surface obtained from $\mathcal{S}$ by cutting along $e$, $\Delta'$ be the triangulation of $\mathcal{S}'$ obtained from $\Delta$ by cutting along $e$, and $W'$ be the $A_2$-web in $\mathcal{S}'$ obtained from $W$ by cutting along $e$ (Lem. 5.4). Then

$$i_{\Delta, \Delta'} \circ \text{Tr}_{\Delta;\mathcal{S}'}([W, s]) = \sum_{s'} \text{Tr}_{\Delta';\mathcal{S}'}([W', s']),$$

where the sum is over all states $s'$ of $W'$ that are compatible with $s$ in the sense as in Lem. 5.4, and $i_{\Delta, \Delta'}$ is as in eq. (5.1).

(CT2) **(values at triangle)** Let $(W, s)$ be a stated $A_2$-web in a triangle $t$, viewed as a generalized marked surface with a unique ideal triangulation $\Delta$. Denote the sides of $t$ by $e_1, e_2, e_3$, and the nodes
of $Q_\Delta$ lying in $t$ by $v_{\epsilon,1}$, $v_{\epsilon,2}$, $v_{\epsilon}$ (for $\alpha = 1, 2, 3$) as in Def. 5.2. For each $\alpha = 1, 2, 3$, define the following matrices with entries in the triangle $Z_t$:

- $M^\text{out}_{t,\alpha}$ is the edge matrix $M_{t,\alpha}$ in (MM1) of $\Delta$ with $X_{1,2}^{1/3}$ replaced by $Z_t, v_{\epsilon,1}$, $Z_t, v_{\epsilon,2}$;
- $M^\text{in}_{t,\alpha}$ is the edge matrix $M_{t,\alpha}$ in (MM1) of $\Delta$ with $X_{1,2}^{1/3}$ replaced by $Z_t, v_{\epsilon,2}$, $Z_t, v_{\epsilon,1}$;
- $M^\text{left}_{t,\alpha}$ is left turn matrix $M_{t,\alpha}$ in (MM2) of $\Delta$ with $X_{1,2}^{1/3}$ replaced by $Z_{v_{\epsilon}}$;
- $M^\text{right}_{t,\alpha}$ is right turn matrix $M_{t,\alpha}$ in (MM3) of $\Delta$ with $X_{1,2}^{1/3}$ replaced by $Z_{v_{\epsilon}}$;
- $M^U_t$ is the $U$-turn matrix $M_{t,\alpha}$ in (MM) of $\Delta$.

They are defined as elements of $\text{SL}_3(\mathbb{Z})$. Now we define the outgoing 3-way matrix $M_t^\text{out}$ as the following $3 \times 3$ matrix with entries in $Z_t$. For each $\epsilon_1, \epsilon_2, \epsilon_3 \in \{1, 2, 3\}$, the $(\epsilon_1, \epsilon_2, \epsilon_3)$-th entry $(M_t^\text{out})_{\epsilon_1, \epsilon_2, \epsilon_3} \in Z_t$ is given by

\begin{equation}
(M_t^\text{out})_{\epsilon_1, \epsilon_2, \epsilon_3} = \det \begin{pmatrix}
(M_t^\text{left})_{\epsilon_1, \epsilon_2, \epsilon_3} & (M_t^\text{right})_{\epsilon_1, \epsilon_2, \epsilon_3} \\
(M_t^\text{left})_{\epsilon_2, \epsilon_3, \epsilon_1} & (M_t^\text{right})_{\epsilon_2, \epsilon_3, \epsilon_1} \\
(M_t^\text{left})_{\epsilon_3, \epsilon_1, \epsilon_2} & (M_t^\text{right})_{\epsilon_3, \epsilon_1, \epsilon_2}
\end{pmatrix}
\end{equation}

where the index-inversion $(r_1(\epsilon), r_2(\epsilon))$ for $\epsilon \in \{1, 2, 3\}$ is given by

\begin{equation}
(r_1(1), r_2(1)) = (1, 2), \quad (r_1(2), r_2(2)) = (1, 3), \quad (r_1(3), r_2(3)) = (2, 3).
\end{equation}

Similarly, define the incoming 3-way matrix $M_t^\text{in}$ as the $3 \times 3$ matrix with entries

\begin{equation}
(M_t^\text{in})_{\epsilon_1, \epsilon_2, \epsilon_3} := \det \begin{pmatrix}
(M_t^\text{right})_{\epsilon_1, \epsilon_2, \epsilon_3} & (M_t^\text{left})_{\epsilon_1, \epsilon_2, \epsilon_3} \\
(M_t^\text{right})_{\epsilon_2, \epsilon_3, \epsilon_1} & (M_t^\text{left})_{\epsilon_2, \epsilon_3, \epsilon_1} \\
(M_t^\text{right})_{\epsilon_3, \epsilon_1, \epsilon_2} & (M_t^\text{left})_{\epsilon_3, \epsilon_1, \epsilon_2}
\end{pmatrix}
\end{equation}

(CT2-1) If $W$ consists of a single left turn corner arc in $t$, with its initial point $x$ in the side $e_{\alpha}$ and the terminal point $y$ in the side $e_{\alpha+1}$, then

\[ \text{Tr}_{\Delta, t}([W, s]) = (s(x), s(y)) \text{-th entry of } \begin{pmatrix} M_{t,\alpha}^\text{in} & M_{t,\alpha}^\text{left} & M_{t,\alpha}^\text{out} \end{pmatrix} \] 

(CT2-2) If $W$ consists of a single right turn corner arc in $t$, with its initial point $x$ in the side $e_{\alpha+1}$ and the terminal point $y$ in the side $e_{\alpha}$, then

\[ \text{Tr}_{\Delta, t}([W, s]) := (s(x), s(y)) \text{-th entry of } \begin{pmatrix} M_{t,\alpha+1}^\text{in} & M_{t,\alpha+1}^\text{right} & M_{t,\alpha+1}^\text{out} \end{pmatrix} \]

(CT2-3) If $W$ consists of a single 3-way arc in $t$ with endpoints $x_1, x_2, x_3$ on sides $e_1$, $e_2$, $e_3$ respectively, if we let $\epsilon_{\alpha} := s(x_{\alpha})$ for each $\alpha = 1, 2, 3$, then

\[ \text{Tr}_{\Delta, t}([W, s]) := \begin{cases} \prod_{\alpha=1}^{3}(M_{t,\alpha}^\text{out})_{\epsilon_{\alpha}, \epsilon_{\alpha}}, \frac{1}{\prod_{\alpha=1}^{3}(M_{t,\alpha}^\text{in})_{\epsilon_{\alpha}, \epsilon_{\alpha}}} & \text{if } W \text{ is outgoing 3-way,} \\
\prod_{\alpha=1}^{3}(M_{t,\alpha}^\text{in})_{\epsilon_{\alpha}, \epsilon_{\alpha}}, \frac{1}{\prod_{\alpha=1}^{3}(M_{t,\alpha}^\text{out})_{\epsilon_{\alpha}, \epsilon_{\alpha}}} & \text{if } W \text{ is incoming 3-way.}
\end{cases} \]

(CT2-4) Suppose $W$ consists of a single 3-way arc in $t$ with endpoints $x_1, x_2, x_3$, where $x_1, x_2$ on a different side $e_\beta$, where $x_1 \to x_2$ matches the clockwise orientation of $\partial t$, and let $\epsilon_\beta := s(x_{\beta})$ for each $\beta = 1, 2, 3$. If $\epsilon_1 = \epsilon_2$ then $\text{Tr}_{\Delta, t}([W, s]) = 0$. If $\epsilon_1 \neq \epsilon_2$, let $\epsilon$ be the unique element of $\{1, 2, 3\}$ such that $\{r_1(\epsilon), r_2(\epsilon)\} = \{\epsilon_1, \epsilon_2\}$. Let $W'$ be an $A_2$-web in $t$ consisting of a single corner arc connecting sides $e_\alpha$ and $e_\beta$, where the endpoint on $e_\beta$ is a sink if and only if $x_3$ is a sink of $W$. Let $s'$ be the state of $W'$ that assigns $\epsilon'$ to the endpoint in $e_\alpha$ and $\epsilon_3$ to the endpoint in $e_\beta$. Then

\[ \text{Tr}_{\Delta, t}([W, s]) = \text{sgn}(\epsilon_2 - \epsilon_1) \text{Tr}_{\Delta, t}([W', s']), \]

where $\text{Tr}_{\Delta, t}([W', s'])$ is given by (CT2-1) or (CT2-2).

(CT2-5) If $W$ consists of a single 3-way arc in $t$ with endpoints $x_1, x_2, x_3$ all lying in one side $e_\alpha$, where $x_1, x_2, x_3$ appear in this order on $e_\alpha$ such that $x_1 \to x_2 \to x_3$ matches the clockwise orientation on $\partial t$, if we let $\epsilon_i := s(x_i)$ for each $i = 1, 2, 3$, then

\[ \text{Tr}_{\Delta, t}([W, s]) = \begin{cases} 1 & \text{if } (\epsilon_1, \epsilon_2, \epsilon_3) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}, \\
-1 & \text{if } (\epsilon_1, \epsilon_2, \epsilon_3) \in \{(1, 3, 2), (3, 2, 1), (2, 1, 3)\}, \\
0 & \text{otherwise.}
\end{cases} \]
If $W$ consists of a single $U$-turn arc in $t$, i.e. $W$ consists of an oriented edge with no crossing whose two endpoints lie on one side $e_x$, then
\[\text{Tr}_{\Delta,t}(|W,s|) = (s(x),s(y))-\text{th entry of } M^U.\]

A priori, the triangle factor involving 3-way webs (i.e. (CT2-3)) may look to be depending on the labeling of sides of $t$, i.e. on the choice of which side of $t$ to call $e_1$. In fact, it is independent of such a choice, due to this lemma:

Lemma 5.7 (cyclic symmetry of 3-way matrices). $M^\text{out}_t$ and $M^\text{in}_t$ have cyclic symmetries, i.e.
\[
(M^\text{out}_t)_{e_1,e_2,1} = (M^\text{out}_t)_{e_2,e_3,1} = (M^\text{out}_t)_{e_3,e_1,2},
\]
\[
(M^\text{in}_t)_{e_1,e_2,1} = (M^\text{in}_t)_{e_2,e_3,1} = (M^\text{in}_t)_{e_3,e_1,2}.
\]

Proof. From (MM2)–(MM3) of §4.2 we have
\[
(M^\text{left}_t)_{i,j} = 0 \text{ if } i > j, \quad (M^\text{right}_t)_{i,j} = 0 \text{ if } i < j,
\]
\[
\begin{align*}
(M^\text{left}_t)_{1,1,1} &= Z_{v_1}^2, \quad (M^\text{left}_t)_{1,2,2} = Z_{v_1}^2 + Z_{v_2}^{-1}, \quad (M^\text{left}_t)_{2,2,2} = Z_{v_2}^{-1}, \quad (M^\text{left}_t)_{i,3,2} = Z_{v_i}^{-1}, \quad i = 1, 2, 3, \\
(M^\text{right}_t)_{1,1,1} &= Z_{v_1}, \quad (M^\text{right}_t)_{1,2,2} = Z_{v_1}, \quad (M^\text{right}_t)_{2,2,2} = Z_{v_1} + Z_{v_2}^{-2}, \quad (M^\text{right}_t)_{3,3,2} = Z_{v_2}^{-2},
\end{align*}
\]

if $Z_{v_i}$ denotes the generator of $Z_\Delta$ for the node of $Q_\Delta$ in the interior of $t$. From eq.(5.3), we compute all values of $(M^\text{out}_t)_{*,*,*}$, which is also useful for later use.

Investigate the case $e_3 = 1$: $(M^\text{out}_t)_{e_1,e_2,1} = (M^\text{left}_t)_{e_1,e_1} (M^\text{right}_t)_{2,2,2} - (M^\text{left}_t)_{2,2,2} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 1$: $(M^\text{out}_t)_{1,1,2} = (M^\text{left}_t)_{1,1} (M^\text{right}_t)_{2,2,2} - (M^\text{left}_t)_{2,1,1} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 2$: $(M^\text{out}_t)_{2,1,2} = (M^\text{left}_t)_{2,1} (M^\text{right}_t)_{2,2,2} - (M^\text{left}_t)_{2,2,2} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 3$: $(M^\text{out}_t)_{3,1,2} = (M^\text{left}_t)_{3,1} (M^\text{right}_t)_{2,2,2} - (M^\text{left}_t)_{2,3} (M^\text{right}_t)_{1,1,1}.
\]

Investigate the case $e_3 = 2$: $(M^\text{out}_t)_{e_1,e_2,2} = (M^\text{left}_t)_{e_1,e_1} (M^\text{right}_t)_{3,3,2} - (M^\text{left}_t)_{3,3,2} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 1$: $(M^\text{out}_t)_{1,1,2} = (M^\text{left}_t)_{1,1} (M^\text{right}_t)_{3,3,2} - (M^\text{left}_t)_{3,1,1} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 2$: $(M^\text{out}_t)_{2,1,2} = (M^\text{left}_t)_{2,1} (M^\text{right}_t)_{3,3,2} - (M^\text{left}_t)_{3,2,1} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 3$: $(M^\text{out}_t)_{3,1,2} = (M^\text{left}_t)_{3,1} (M^\text{right}_t)_{3,3,2} - (M^\text{left}_t)_{3,3,1} (M^\text{right}_t)_{1,1,1}.
\]

Investigate the case $e_3 = 3$: $(M^\text{out}_t)_{e_1,e_2,3} = (M^\text{left}_t)_{e_1,e_1} (M^\text{right}_t)_{3,3,2} - (M^\text{left}_t)_{3,3,2} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 1$: $(M^\text{out}_t)_{1,1,3} = (M^\text{left}_t)_{1,1} (M^\text{right}_t)_{3,3,2} - (M^\text{left}_t)_{3,1,1} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 2$: $(M^\text{out}_t)_{2,1,3} = (M^\text{left}_t)_{2,1} (M^\text{right}_t)_{3,3,2} - (M^\text{left}_t)_{3,2,1} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 3$: $(M^\text{out}_t)_{3,1,3} = (M^\text{left}_t)_{3,1} (M^\text{right}_t)_{3,3,2} - (M^\text{left}_t)_{3,3,1} (M^\text{right}_t)_{1,1,1}.
\]

By inspection on all 27 values of $(M^\text{out}_t)_{e_1,e_2,e_3}$, one indeed observes the cyclicity.

Proof of cyclicity of $(M^\text{in}_t)_{e_1,e_2,e_3}$ goes similarly; we compute them by eq.(5.5).

Investigate the case $e_3 = 1$: $(M^\text{in}_t)_{e_1,e_2,1} = (M^\text{left}_t)_{e_1,e_1} (M^\text{right}_t)_{2,2,2} - (M^\text{left}_t)_{2,2,2} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 1$: $(M^\text{in}_t)_{1,1,2} = (M^\text{left}_t)_{1,1} (M^\text{right}_t)_{2,2,2} - (M^\text{left}_t)_{2,1,1} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 2$: $(M^\text{in}_t)_{2,1,2} = (M^\text{left}_t)_{2,1} (M^\text{right}_t)_{2,2,2} - (M^\text{left}_t)_{2,2,2} (M^\text{right}_t)_{1,1,1}.
\]

Put $e_1 = 3$: $(M^\text{in}_t)_{3,1,2} = (M^\text{left}_t)_{3,1} (M^\text{right}_t)_{2,2,2} - (M^\text{left}_t)_{3,3,1} (M^\text{right}_t)_{1,1,1}.
\]

so $(M^\text{in}_t)_{2,1,2} = (M^\text{left}_t)_{2,2,2} - (M^\text{left}_t)_{2,2,2} (M^\text{right}_t)_{1,1,1}.$
\[\varepsilon_1 = 3: (M_{t}^{\text{int}})_{3,v_1,1} = (M_{t}^{\text{right}})_{3,1}(M_{t}^{\text{left}})_{c,2} - (M_{t}^{\text{right}})_{2,2}(M_{t}^{\text{left}})_{c,1} = Z_{v_1}(M_{t}^{\text{right}})_{c,2} - (Z_{v_1} + Z_{v_1}^2)(M_{t}^{\text{left}})_{c,1},\]

so \( (M_{t}^{\text{int}})_{3,1,1} = (Z_{v_1}^3 + 1) - (Z_{v_1}^2 + 1) = 0, (M_{t}^{\text{int}})_{3,2,1} = 1, (M_{t}^{\text{int}})_{3,3,1} = 0.\)

Investigate the case \(\varepsilon_2 = 2: (M_{t}^{\text{int}})_{1,v_1,2} = (M_{t}^{\text{right}})_{1,1}(M_{t}^{\text{left}})_{c,3} - (M_{t}^{\text{right}})_{1,3}(M_{t}^{\text{left}})_{c,1} = Z_{v_1}(M_{t}^{\text{left}})_{c,2}.\)

So \((M_{t}^{\text{int}})_{1,2,2} = 1, (M_{t}^{\text{int}})_{1,3,2} = 1.\)

\[\varepsilon_1 = 1: (M_{t}^{\text{int}})_{1,v_1,2} = (M_{t}^{\text{right}})_{1,1}(M_{t}^{\text{left}})_{c,3} - (M_{t}^{\text{right}})_{1,3}(M_{t}^{\text{left}})_{c,1} = Z_{v_1}(M_{t}^{\text{left}})_{c,2}.\]

\((M_{t}^{\text{int}})_{1,2,2} = 1, (M_{t}^{\text{int}})_{1,3,2} = 1.\)

Investigate the case \(\varepsilon_3 = 3: (M_{t}^{\text{int}})_{1,v_1,2} = (M_{t}^{\text{right}})_{1,2}(M_{t}^{\text{left}})_{c,3} - (M_{t}^{\text{right}})_{1,3}(M_{t}^{\text{left}})_{c,2} = 0.\)

So \((M_{t}^{\text{int}})_{1,1,3} = 0, (M_{t}^{\text{int}})_{1,2,3} = 0, (M_{t}^{\text{int}})_{1,3,3} = 0.\)

By inspection on all 27 values of \((M_{t}^{\text{int}})_{1,v_1,2}\), one indeed observes the cyclicity.

A nice consequence of the above computation is:

**Lemma 5.8** (positivity of 3-way matrices). Entries of \(M_{t}^{\text{int}}, M_{t}^{\text{out}}\) are among 0, 1, \(Z_{v_1}^{\pm 1}, 1 + Z_{v_1}^{\pm 1}\), hence are Laurent polynomials in \(Z_{v_1}^3 = X_{v_1}\) with non-negative integer coefficients.

Anyhow, the definitions of \(M_{t}^{\text{out}}\) and \(M_{t}^{\text{int}}\) in eq. (5.3) and eq. (5.5) may still seem ad hoc at the moment. We will later justify them.

Throughout the present section, we shall prove Prop. 5.6. Uniqueness is easy to see. By Property (CT1), the values of the SL_3 classical trace maps are completely determined by the values of the SL_4 classical trace map for a single triangle \(t\). Since Tr_{Aelta} must be a ring homomorphism, the values of Tr_{Aelta} are determined by the values of stated A_2-webs in \(t\) with single component. Suppose \((W, s)\) is a stated A_2-web with a single component. If the number of 3-valent internal vertices is at least 2, then pick an edge connecting two 3-valent internal vertices, and apply the A_2-skein relation Fig. 2 to unresolved it into a sum of stated A_2-webs with less numbers of 3-valent vertices. By induction, one observes that the value of \((W, s)\) under Tr_{Aelta} is determined by the values under Tr_{Aelta} of single-component stated A_2-webs in \(t\) with the number of 3-valent vertices being 0 or 1, which are determined by (CT2.1)–(CT2.6).

In the upcoming subsections, we shall prove the existence of this SL_3 classical trace maps, study the properties of the values, and relate to the map \(\text{Tr}_{\text{PGL}_3}\) of our original interest.

### 5.2. Biangle SL_3 Trace

Our proof of the existence part of Prop. 5.6 follows the style of Bonahon-Wong [11]. In particular, we first study the SL_3 classical trace map for a biangle \(B\) (Def. 2.2), i.e., a generalized marked surface diffeomorphic to a closed disc, with two marked points on the boundary, with no punctures. According to later developments by Costantino-Lec [19] and Higgins [20], it is wise to consider that for a monogon \(M\), i.e., a generalized marked surface diffeomorphic to a closed disc with one marked point on the boundary. Note that \(B\) and \(M\) are not triangulable, so they don't really fit into the setting of Prop. 5.6, hence the SL_3 classical trace for them must be dealt with separately. The following is a biangle analog of Prop. 5.6.

**Proposition 5.9** (biangle SL_3 trace). There exists a unique family of ring homomorphisms

\[\text{Tr}_B : S_3(B; \mathbb{Z}) \rightarrow \mathbb{Z}\]

defined for biangles \(B\), such that

(BT1) (cutting/gluing property) Let \((W, s)\) be a stated A_2-web in \(B\), and \(e\) is an ideal arc in \(B\) whose interior lies in the interior of \(B\). Let \(B'\) be the generalized marked surface obtained from \(B\) by
cutting along $e$ (Lem.5.4), so that $B'$ is disjoint union of two biangles $B_1$ and $B_2$. Let $W'$ be the $A_2$-web in $B'$ obtained from $W$ by cutting along $e$, and let $W_1$ and $W_2$ be the $A_2$-webs in $B_1$ and $B_2$ such that $W' = W_1 \cup W_2$. Then
\[
\text{Tr}_B([W, s]) = \sum_{s_1, s_2} \text{Tr}_{B_1}([W_1, s_1]) \cdot \text{Tr}_{B_2}([W_2, s_2]),
\]
where the sum is over all states $s_1$ and $s_2$ of $W_1$ and $W_2$ such that the state $s' := s_1 \cup s_2$ of $W' = W_1 \cup W_2$ is compatible with $s$ in the sense as in Lem.5.4.

(BT2) (values at elementary single-component stated $A_2$-webs with at most one 3-valent vertex)

(BT2-1) If the $A_2$-web $W$ in $B$ consists of a single edge with no crossing connecting two distinct sides of $B$, and if $\varepsilon, \varepsilon'$ are the values of a state $s$ of $W$ for its two endpoints, then
\[
\text{Tr}_B([W, s]) = (\varepsilon, \varepsilon')\text{-th entry of the } 3 \times 3 \text{ identity matrix}.
\]

(BT2-2) If the $A_2$-web $W$ in $B$ consists of a single edge with no crossing with the two endpoints lying in a single side of $B$ (i.e. is a U-turn arc), and if $\varepsilon, \varepsilon'$ are the values of a state $s$ of $W$ for its two endpoints, then
\[
\text{Tr}_B([W, s]) = (\varepsilon, \varepsilon')\text{-th entry of the U-turn matrix } M^U.
\]

(BT2-3) If the $A_2$-web $W$ in $B$ is a 3-way web with endpoints $x_1, x_2, x_3$, with $x_1, x_2$ lying in one side of $B$ while $x_3$ on the other side, where $x_1 \to x_2$ matches the clockwise orientation of the boundary $\partial B$, and if $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the values of a state $s$ of $W$ for the endpoints $x_1, x_2, x_3$, then
\[
\text{Tr}_B([W, s]) = \begin{cases} 
1 & \text{if } (r_1(\varepsilon_3), r_2(\varepsilon_3)) = (\varepsilon_1, \varepsilon_2), \\
-1 & \text{if } (r_1(\varepsilon_3), r_2(\varepsilon_3)) = (\varepsilon_2, \varepsilon_1), \\
0 & \text{otherwise}.
\end{cases}
\]

(BT2-4) If the $A_2$-web $W$ in $B$ is a 3-way web with endpoints $x_1, x_2, x_3$ lying in one side of $B$, where $x_1, x_2, x_3$ appear in this order in that side, with $x_1 \to x_2 \to x_3$ matching the clockwise orientation of the boundary $\partial B$, and if $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the values of a state $s$ of $W$ for the endpoints $x_1, x_2, x_3$, then $\text{Tr}_B([W, s])$ is given by the right-hand side of eq.5.6.

There are several strategies to prove this proposition. We take a possibly shortest one, by mimicking the idea of Costantino and Lê for the $\text{SL}_2$ case [CL18, CL19], applied to the $\text{SL}_3$ case with the help of the results of Higgins on stated $A_2$-skein algebras [H20]. The situation for us is easier because we only deal with the classical setting, i.e. we put $q = 1$ (or $q^{1/3} = 1$) in these works. We first recall the version of stated skein algebras studied by Higgins; our state values $1, 2, 3$ correspond to $-0, +$ of [H20].

**Definition 5.10** ([H20]). Let $\mathcal{G}$ be a generalized marked surface, not necessarily triangulable. For a commutative ring $\mathcal{R}$ with unity $1$, define the (commutative) reduced stated $A_2$-skein algebra $\mathcal{S}^\text{red}_s(\mathcal{G}; \mathcal{R})$ as the quotient of Def.5.1 by the boundary relations in Fig.10.

![Diagram](image)

**Figure 10.** Boundary relations for stated $A_2$-skeins (horizontal blue line is boundary)

We begin with the monogon.

**Proposition 5.11** ([H20 Prop.3]). Let $M$ be a monogon. Then $\mathcal{S}^\text{red}_s(M; \mathbb{Z})$ is spanned by the empty diagram, so
\[
\mathcal{S}^\text{red}_s(M; \mathbb{Z}) \cong \mathbb{Z}.
\]

In [H20] it is shown that the non-commutative reduced stated $A_2$-skein algebra for a biangle $B$ is a Hopf algebra and is isomorphic to the quantum group $\mathcal{O}_q(\text{SL}_3)$. We put $q^{1/3} = 1$ there, and just use the fact that $\mathcal{S}^\text{red}_s(B; \mathbb{Z})$ is a bialgebra. We already know its algebra structure, i.e. the product. To describe the remaining structures, a priori we should choose preferred orientations on boundary arcs of $M$ and $B$.
Definition 5.12. A direction of a biangle $B$ is the choice of a distinguished marked point of $B$, denoted by $\text{dir}$. The distinguished marked point is called the top marked point, while the other marked point the bottom marked point. The pair $(B, \text{dir})$ is called a directed biangle and is denoted by $\vec{B}$. For a directed biangle $\vec{B}$, the induced orientation on the boundary arcs of $B$ are the orientations pointing toward the distinguished marked point. The boundary arc of $B$ whose induced orientation matches the clockwise orientation (coming from the surface orientation of $B$) is called the left side of $\vec{B}$, and the other boundary arc the right side.

For a monogon $M$, the induced orientation on the boundary arc of $M$ is the clockwise orientation (coming from the surface orientation of $M$).

We will use the notation $S_\text{dir}(\vec{B};R)$ and $S_{\text{dir}}^\text{red}(\vec{B};R)$ to mean the algebras $S_\text{dir}(B;R)$ and $S_{\text{dir}}^\text{red}(B;R)$ together with the information of direction on $B$.

Definition 5.13 (coproduct for biangle $A_2$-skein algebra; [H20]). Let $\vec{B} = (B, \text{dir})$ be a directed biangle, and $e$ be an ideal arc in $B$ whose interior lies in the interior of $B$. Cutting $B$ along $e$ yields disjoint union of two biangles $B_1$ and $B_2$ as in Prop. 5.9(B1). The direction naturally inherits to $B_1$ and $B_2$, making them directed biangles $\vec{B}_1$ and $\vec{B}_2$; assume that the left side of $\vec{B}_1$ corresponds to the left side of $\vec{B}$, and the right side of $\vec{B}_2$ to the right side of $\vec{B}$. Let $(W, s)$ be a stated $A_2$-web in $B$, and let $W_1$ and $W_2$ the $A_2$-webs in $B_1$ and $B_2$ obtained by cutting $W$ along $e$, as in Prop 5.9(B1). Define the map

$$\Delta_{\vec{B},e} : S_{\text{dir}}^\text{red}(\vec{B};\mathbb{Z}) \rightarrow S_{\text{dir}}^\text{red}(\vec{B}_1;\mathbb{Z}) \otimes_{\mathbb{Z}} S_{\text{dir}}^\text{red}(\vec{B}_2;\mathbb{Z})$$

as

$$\Delta_{\vec{B},e}([W,s]) := \sum_{s_1,s_2} [W_1,s_1] \otimes [W_2,s_2],$$

where the sum is over all states $s_1$ and $s_2$ of $W_1$ and $W_2$ such that the state $s' := s_1 \cup s_2$ of $W' = W_1 \cup W_2$ is compatible with $s$ in the sense as in Lem. 5.2. Composing with the canonical isomorphisms $S_{\text{dir}}^\text{red}(\vec{B}_1;\mathbb{Z}) \cong S_{\text{dir}}^\text{red}(\vec{B};\mathbb{Z})$ define the map

$$\Delta : S_{\text{dir}}^\text{red}(\vec{B};\mathbb{Z}) \rightarrow S_{\text{dir}}^\text{red}(\vec{B};\mathbb{Z}) \otimes_{\mathbb{Z}} S_{\text{dir}}^\text{red}(\vec{B};\mathbb{Z}).$$

Definition 5.14 (counit for biangle $A_2$-skein algebra; [H20]). Let $\vec{B} = (B, \text{dir})$ be a directed biangle. For a stated $A_2$-web $(W, s)$ in $\vec{B}$, let $\partial_{\text{left}} W$ (resp. $\partial_{\text{right}} W$) be the set of all endpoints of $W$ lying in the left side of $\vec{B}$ (resp. right side of $\vec{B}$). Let $s' : \partial W \rightarrow \{1,2,3\}$ be the state of $W$ defined as

$$s'(x) := \begin{cases} s(x) & \text{if } x \in \partial_{\text{left}} W, \\ 4 - s(x) & \text{if } x \in \partial_{\text{right}} W. \end{cases}$$

Define the inversion map

$$\text{inv} : S_{\text{dir}}^\text{red}(\vec{B};\mathbb{Z}) \rightarrow S_{\text{dir}}^\text{red}(\vec{B};\mathbb{Z})$$

as

$$\text{inv}([W,s]) := (-1)^{\partial_{\text{right}} W}[W,s'] \in S_{\text{dir}}^\text{red}(\vec{B};\mathbb{Z}).$$

Consider an orientation-preserving embedding

$$\iota : \vec{B} \rightarrow M$$

that sends the bottom marked point of $\vec{B}$ to the marked point of $M$. Define the map

$$\varepsilon : S_{\text{dir}}^\text{red}(\vec{B};\mathbb{Z}) \rightarrow \mathbb{Z}$$

as the composition

$$S_{\text{dir}}^\text{red}(\vec{B};\mathbb{Z}) \xrightarrow{\text{inv}} S_{\text{dir}}^\text{red}(\vec{B};\mathbb{Z}) \xrightarrow{\iota_*} S_{\text{dir}}^\text{red}(M;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}.$$
Lemma 5.16. $\varepsilon \circ \pi : \mathcal{S}_n(\bar{B}; \mathbb{Z}) \to \mathbb{Z}$ is a ring homomorphism that satisfies (BT1) of Prop. 5.9.

Indeed, since $\varepsilon$ is a counit of a bialgebra, it respects the product structure and hence is a ring homomorphism, and also satisfies $(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} : \mathcal{S}_n^{\text{red}}(\bar{B}; \mathbb{Z}) \to \mathcal{S}_n^{\text{red}}(\bar{B}; \mathbb{Z})$, thus $(\varepsilon \otimes \varepsilon) \circ \Delta = \varepsilon: \mathcal{S}_n^{\text{red}}(\bar{B}; \mathbb{Z}) \to \mathbb{Z}$ holds, from which one can deduce the property (BT1) for $\varepsilon \circ \pi$.

Now we twist $\varepsilon \circ \pi$ to define

$$\varepsilon' : \mathcal{S}_n(\bar{B}; \mathbb{Z}) \to \mathbb{Z} \quad \text{as}$$

$$\varepsilon'([W, s]) = (-1)^{|\partial_1 W|}(\varepsilon \circ \pi)([W, s]), \quad \text{where} \quad \partial_1 W := \{ x \in \partial W \mid s(x) = 1 \}.$$ 

It is easy to observe that

Lemma 5.17. $\varepsilon' : \mathcal{S}_n(\bar{B}; \mathbb{Z}) \to \mathbb{Z}$ is a ring homomorphism that satisfies (BT1) of Prop. 5.9.

Indeed, the twist by $(-1)^{|\partial_1 W|}$ is multiplicative, hence $\varepsilon'$ is a ring homomorphism, and for (BT1), these twists for $B_1$ and $B_2$ are cancelled on the side of $B_1$ and that of $B_2$ that came from the cut (internal) arc $e$ of $B$, for compatible states $s_1$ and $s_2$, hence (BT1) still holds.

What remains is to check (BT2), i.e. the values of $\varepsilon'$ at single-component stated $A_2$-webs with at most one 3-valent vertex, which is a straightforward exercise; we show it here for completeness.

It is convenient to make use of Prop. 1 of [H20], which lists some consequences of the defining relations of stated $A_2$-skine algebras. Among them, particularly useful to us are:

Lemma 5.18 (H20 Prop. 1). In the reduced $A_2$-skine algebra $\mathcal{S}_n^{\text{red}}(\mathcal{S}; \mathcal{R})$ for a generalized marked surface $\mathcal{S}$ and a commutative ring $\mathcal{R}$ with unity, one has the following, each of which also holds when orientations of all edges are reversed: (blue horizontal line is boundary)

\begin{equation}
(\varepsilon_1 \varepsilon_2, 4) = -\delta_{\varepsilon_1, \varepsilon_2, 4} \cdot 
\end{equation}

\begin{equation}
(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\text{right-hand-side of eq.} \,(5.6)). 
\end{equation}

We will be denoting the image under $\pi$ of $[W, s] \in \mathcal{S}_n(\bar{B}; \mathbb{Z})$ as $[W, s]$ too, by slight abuse of notation.

Let $(W, s)$ be as in (BT2-1); let $x$ be the endpoint of $W$ in the left side of $\bar{B}$ and $y$ be the endpoint of $W$ in the right side of $\bar{B}$. Then $\text{inv}[W, s] = \varepsilon([W, s])$ is as in $\mathcal{S}_n^{\text{red}}(\bar{B}; \mathbb{Z})$ with $s'(x) = s(x)$ and $s'(y) = 4 - s(y)$. Note $\iota_* [W, s']$ falls into eq. (5.7), and equals $-1$ times identity if $\varepsilon(s(x)) \leq \varepsilon(s(y))$, i.e. if $s(x), s(y) \in \{(1, 1), (1, 2), (2, 2), (3, 3)\}$, and equals 0 otherwise. So $\varepsilon \circ \pi$ satisfies (BT2-1), and therefore so does $\varepsilon'$.

Let $(W, s)$ be as in (BT2-2). The two endpoints $x, y$ of $W$ either all lie on the left side of $\bar{B}$, or on the right side of $\bar{B}$. In either case, again by applying inv and then using eq. (5.7), it follows that $\varepsilon([W, s])$ equals $-1$ if $(s(x), s(y)) \in \{(3, 2), (2, 3), (1, 1)\}$ and equals 0 otherwise. Hence $\varepsilon'$ satisfies (BT2-2).

Let $(W, s)$ be as in (BT2-3). First, suppose that $x_1, x_2$ lie in the left side of $\bar{B}$. Then $\text{inv}[W, s] = \varepsilon([W, s])$ is as in $\mathcal{S}_n^{\text{red}}(\bar{B}; \mathbb{Z})$ with $s'(x_1) = s(x_1) = \varepsilon_1$, $s'(x_2) = s(x_2) = \varepsilon_2$ and $s'(x_3) = s(x_3) = 4 - s(x_3) = 4 - \varepsilon_3$. Note $\iota_* [W, s']$ falls into eq. (5.8), and therefore equals $-1$ if $(s'(x_1), s'(x_2), s'(x_3)) \in \{(1, 2, 3), (3, 2, 1), (3, 1, 2)\}$, equals 1 if $(s'(x_1), s'(x_2), s'(x_3)) \in \{(1, 3, 2), (3, 2, 1), (2, 1, 3)\}$, and equals 0 otherwise.

Now suppose that $x_1, x_2$ lie in the right side of $\bar{B}$. Then $\text{inv}[W, s] = \varepsilon([W, s])$ is as in $\mathcal{S}_n^{\text{red}}(\bar{B}; \mathbb{Z})$ with $s'(x_1) = 4 - s(x_1) = 4 - \varepsilon_1$, $s'(x_2) = 4 - s(x_2) = 4 - \varepsilon_2$ and $s'(x_3) = s(x_3) = \varepsilon_3$. Note $\iota_* [W, s']$ falls into eq. (5.8), and therefore equals $-1$ if $(s'(x_1), s'(x_2), s'(x_3)) \in \{(1, 2, 3), (2, 3, 1), (1, 2, 3)\}$, equals 1 if $(s'(x_1), s'(x_2), s'(x_3)) \in \{(1, 3, 2), (2, 3, 1), (2, 1, 3)\}$, and equals 0 otherwise. So $\varepsilon([W, s])$ equals $-1$ if $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{(3, 2, 3), (2, 3, 1), (3, 1, 2)\}$, and equals 0 otherwise, hence $\varepsilon'$ satisfies (BT2-3) in this case. Now suppose that $x_1, x_2$ lie in the right side of $\bar{B}$. Then $\text{inv}[W, s] = \varepsilon([W, s])$ is as in $\mathcal{S}_n^{\text{red}}(\bar{B}; \mathbb{Z})$ with $s'(x_1) = 4 - s(x_1) = 4 - \varepsilon_1$, $s'(x_2) = 4 - s(x_2) = 4 - \varepsilon_2$ and $s'(x_3) = s(x_3) = \varepsilon_3$. Note $\iota_* [W, s']$ falls into eq. (5.8), and therefore equals $-1$ if $(s'(x_1), s'(x_2), s'(x_3)) \in \{(1, 2, 3), (2, 3, 1), (1, 2, 3)\}$, equals 1 if $(s'(x_1), s'(x_2), s'(x_3)) \in \{(1, 3, 2), (2, 3, 1), (2, 1, 3)\}$, and equals 0 otherwise. So $\varepsilon([W, s])$ equals $-1$ if $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{(3, 2, 3), (2, 3, 1), (3, 1, 2)\}$, and equals 0 otherwise, hence $\varepsilon'$ satisfies (BT2-3) in this case.
Let \((W, s)\) be as in (BT2-4). First, suppose that \(x_1, x_2, x_3\) lie in the left side of \(\tilde{B}\). Then \(\text{inv}[W, s] = [W, s] \in S^s_{\text{red}}(\tilde{B}; \mathbb{Z})\). Note \(\epsilon_s(W, s')\) falls into eq. (5.8), hence equals \(-1\) if \((\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}\), equals \(1\) if \((\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{(1, 3, 2), (3, 2, 1), (2, 1, 3)\}\), and equals \(0\) otherwise. Hence \(\epsilon(W, s)\) equals \(1\) if \((\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}\), equals \(-1\) if \((\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{(1, 3, 2), (3, 2, 1), (2, 1, 3)\}\), and equals \(0\) otherwise, and thus \(\epsilon'\) satisfies (BT2-4) in this case. Now suppose \(x_1, x_2, x_3\) lie in the right side of \(\tilde{B}\). Then \(\text{inv}[W, s] = -[W, s'] \in S^s_{\text{red}}(\tilde{B}; \mathbb{Z})\), with \(s'(x_i) = 4 - s(x_i) = 4 - \varepsilon_i\) for all \(i = 1, 2, 3\). Note \(\epsilon_s(W, s')\) falls into eq. (5.8), and therefore equals \(-1\) if \((s'(x_1), s'(x_2), s'(x_3)) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}\), equals \(1\) if \((s'(x_1), s'(x_2), s'(x_3)) \in \{(1, 3, 2), (3, 2, 1), (2, 1, 3)\}\), and equals \(0\) otherwise. Thus \(\epsilon(W, s)\) equals \(1\) if \((\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}\), equals \(-1\) if \((\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{(1, 3, 2), (3, 2, 1), (2, 1, 3)\}\), and equals \(0\) otherwise. Therefore, \(\epsilon(W, s)\) equals \(1\) if \((\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}\), equals \(-1\) if \((\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{(1, 3, 2), (3, 2, 1), (2, 1, 3)\}\), and equals \(0\) otherwise, and hence \(\epsilon'\) satisfies (BT2-4) in this case.

To finish the proof, observe that the values at the single-component stated \(A_2\)-webs with at most one 3-valent vertex do not depend on the choice of the direction on \(B\). Hence the final result may be written in terms of \(B\), instead of \(\tilde{B}\).

5.3. State-sum construction. In order to prove the existence part of Prop. 5.6, we provide an explicit formula for computation of the value \(\text{Tr}_{\Delta, \Theta}([W, s])\) of the sought-for trace map \(\text{Tr}_{\Delta, \Theta}\) for a triangulated surface. Like in Bonahon-Wong’s argument \([\text{BW}11]\) for the \(SL_2\) quantum trace map, we consider the split ideal triangulation \(\hat{\Delta}\) for \(\Delta\) (Def. 3.13), put the complicated parts of \(W\) into the biangles of \(\hat{\Delta}\) by isotopy, and use the biangle \(SL_2\) classical trace to deal with these parts. We begin by defining the \(A_2\) analog of Bonahon-Wong’s good position (\([\text{BW}11]\)) for skeins.

**Definition 5.19** (good position and gool position). Let \(\mathcal{S}\) be a triangulable generalized marked surface, \(\Delta\) an ideal triangulation of \(\mathcal{S}\), and \(\hat{\Delta}\) a split ideal triangulation for \(\Delta\) (Def. 3.13). An \(A_2\)-web \(W\) in \(\mathcal{S}\) is said to be in a good position with respect to \(\hat{\Delta}\) if it satisfies the following conditions:

(GP1) Each intersection point of \(W\) and \(\hat{\Delta}\) is a transverse double intersection, and in particular is not an internal 3-valent vertex of \(W\);

(GP2) For each triangle \(\hat{t}\) of \(\hat{\Delta}\) (viewed as a generalized marked surface on its own), each component of the \(A_2\)-web of \(W \cap \hat{t}\) in \(\hat{t}\) has at most one internal 3-valent vertex.

If furthermore the following also holds, we say \(W\) is in a gool position with respect to \(\hat{\Delta}\):

(GP3) For each triangle \(\hat{t}\) of \(\hat{\Delta}\), each component of the \(A_2\)-web \(W \cap \hat{t}\) in \(\hat{t}\) is a corner arc (Def. 3.12).

By isotopy, one can push all (or almost) of the 3-valent vertices into the biangles.

**Lemma 5.20.** An \(A_2\)-web \(W\) in \(\mathcal{S}\) is isotopic to an \(A_2\)-web \(W'\) in a gool position with respect to \(\hat{\Delta}\).

**Definition 5.21** (state-sum trace for gool position). Let \(\mathcal{S}, \Delta, \hat{\Delta}\) be as in Def. 5.19. Let \((W, s)\) be a stated \(A_2\)-web in \(\mathcal{S}\) in a gool position with respect to \(\hat{\Delta}\).

The points of \(W \cap \hat{\Delta}\) are called \(\hat{\Delta}\)-junctures of \(W\), and a \(\hat{\Delta}\)-juncture-state of \(W\) is a map

\[ J : W \cap \hat{\Delta} \to \{1, 2, 3\}. \]

For each ideal triangle \(\hat{t}\) of \(\hat{\Delta}\) corresponding to a triangle \(t\) of \(\Delta\), view \(W \cap \hat{t}\) as an \(A_2\)-web in \(\hat{t}\), where \(\hat{t}\) is viewed as a generalized marked surface on its own. Let \(W_{i_1}, \ldots, W_{i_n}\) be components of this \(A_2\)-web \(W \cap \hat{t}\). A \(\hat{\Delta}\)-juncture-state \(J\) of \(W\) restricts to a state \(J_t : \partial(W \cap \hat{t}) \to \{1, 2, 3\}\) for the external vertices of \(W \cap \hat{t}\), and to a state \(J_{i,j} : \partial W_{i,j} \to \{1, 2, 3\}\) for the external vertices of \(W_{i,j}\). To each stated component \((W_{i,j}, J_{i,j})\), i.e. a pair of a component \(W_{i,j}\) and a state for its external vertices, define the element

\[ \hat{\text{Tr}}(W_{i,j}, J_{i,j}) \in \mathbb{Z}_4 \]

as in Prop. 5.6 (CT2-1)–(CT2-2). Define the triangle factor of \(W\) for the triangle \(t\) (or \(\hat{t}\)) with respect to \(J\) as

\[ \hat{\text{Tr}}_t(W \cap \hat{t}, J_t) := \prod_{i=1}^n \hat{\text{Tr}}(W_{i,j}, J_{i,j}) \in \mathbb{Z}_4. \]

\(^1\)“Gool” sounds like honey (or oyster!) in Korean.
For each biangle $B$ of $\hat{\Delta}$, view $W \cap B$ as an $A_2$-web in $B$. The $\hat{\Delta}$-juncture-state $J$ of $W$ restricts to a state $J_B : \partial(W \cap B) \to \{1, 2, 3\}$. Let the biangle factor of $W$ for the biangle $B$ with respect to $J$ be
\[ \Tr_B([W \cap B, J_B]) \in \mathbb{Z}, \]
as given by Prop. 5.4.

Define the state-sum trace of the stated $A_2$-web $(W, s)$ in a goal position with respect to $\hat{\Delta}$ as
\[ \mathcal{Tr}_\Delta(W, s) := \sum_j (\prod_B \mathcal{Tr}_B([W \cap B, J_B]) \otimes \mathcal{Tr}_t(W \cap \hat{J}, J_t)) \in \otimes_{t \in \mathcal{F}(\Delta)} \mathbb{Z}_t \]
where the sum $\sum_j$ is over all $\hat{\Delta}$-juncture-states $J$ for $W$ that restrict to $s$ at $\partial W$, and the products $\prod_B$ and $\prod_t$ are over all biangles of $\hat{\Delta}$ and triangles $t$ of $\Delta$.

To use the state-sum trace as the sought-for $\text{SL}_3$ classical trace, one must show that the value $\mathcal{Tr}_\Delta(W, s)$ lies in the subalgebra $\mathcal{Z}_\Delta$ of $\otimes_{t \in \mathcal{F}(\Delta)} \mathbb{Z}_t$ (Def. 5.2).

**Proposition 5.22** (balancedness of state-sum trace). Let $\mathcal{S}$, $\Delta$, and $\hat{\Delta}$ be as in Def. 5.19. For a stated $A_2$-web $(W, s)$ in a goal position with respect to $\hat{\Delta}$,
\[ \mathcal{Tr}_\Delta(W, s) \in \mathcal{Z}_\Delta \subset \otimes_{t \in \mathcal{F}(\Delta)} \mathbb{Z}_t. \]

To prove this, we first establish the following lemma, which is interesting in its own right, and is an $\text{SL}_3$ analog of the corresponding statement for $\text{SL}_2$.

**Lemma 5.23** (charge conservation property of biangle $\text{SL}_3$ trace). Let $(W, s)$ be a stated $A_2$-web in a directed biangle $\hat{B}$. Let $b_{\text{left}}, b_{\text{right}}$ be the left and the right sides of $\hat{B}$ (Def. 5.12). For $\varepsilon \in \{1, 2, 3\}$, let $n^{\left.\text{left}, \varepsilon\right.}$ be the number of endpoints $x$ of $W$ lying in $b_{\text{left}}$ with $s(x) = \varepsilon$ such that $x$ is a source (resp. sink) of $W$ near $x$ is going toward (resp. away from) the interior of $B$, or equivalently, going from left to right (resp. right to left). For $\varepsilon \in \{1, 2, 3\}$, let $n^{\left.\text{right}, \varepsilon\right.}$ be the number of endpoints $x$ of $W$ lying in $b_{\text{right}}$ with $s(x) = \varepsilon$ such that $x$ is a sink (resp. source) of $W$, i.e. the orientation of $W$ near $x$ is going from left to right (resp. right to left).

For $h \in \{\text{left}, \text{right}\}$, the first charge of $(W, s)$ at the boundary arc $b_h$ is defined as
\[ C^{(1)}_h(W, s) = -n_{h,1}^+ + n_{h,3}^− - n_{h,1}^− + n_{h,3}^+ \in \mathbb{Z}. \]
and the second charge of $(W, s)$ at boundary arc $b_h$ as
\[ C^{(2)}_h(W, s) = n_{h,2}^+ - 2n_{h,2}^− + n_{h,3}^− - n_{h,1}^− + 2n_{h,2}^− - n_{h,3}^+ \in \mathbb{Z}. \]

If $\Tr_B([W, s]) \neq 0$, then
\[ C^{(1)}_{\text{left}}(W, s) = C^{(1)}_{\text{right}}(W, s) \quad \text{and} \quad C^{(2)}_{\text{left}}(W, s) = C^{(2)}_{\text{right}}(W, s). \]

**Corollary 5.24.** Define
\[ C^{(3)}_h(W, s) := -\frac{3}{2}C^{(1)}_h(W, s) - \frac{1}{2}C^{(2)}_h(W, s) = n_{h,1}^− + n_{h,2}^+ - 2n_{h,2}^− + 2n_{h,1}^+ - n_{h,2}^− - n_{h,3}^+, \]
\[ C^{(4)}_h(W, s) := -\frac{3}{2}C^{(1)}_h(W, s) + \frac{1}{2}C^{(2)}_h(W, s) = 2n_{h,1}^+ - n_{h,2}^+ - n_{h,3}^− + n_{h,1}^− + n_{h,2}^− - 2n_{h,3}^+. \]
If $\Tr_B([W, s]) \neq 0$, then
\[ C^{(3)}_{\text{left}}(W, s) = C^{(3)}_{\text{right}}(W, s) \quad \text{and} \quad C^{(4)}_{\text{left}}(W, s) = C^{(4)}_{\text{right}}(W, s). \]

The first charge can be understood as
\[ C^{(1)}_h(W, s) = \sum_{x \in (\partial W) \cap b_h} (s(x) - 2), \]
i.e. the sum of signs for the values of the state $s$ at the endpoints of $W$ lying in $b_h$, where the sign of the state value $\varepsilon \in \{1, 2, 3\}$ is defined as $\varepsilon = 2 \in \{-1, 0, +1\}$; this matches the convention of Higgins [H20] who use the symbols $\{-, 0, +\}$ as the values of states.

**Proof of Lem. 5.23.** Let $(W, s)$ be a stated $A_2$-web in $\hat{B}$, such that $\Tr_B([W, s]) \neq 0$. One can observe that $W$ can be decomposed as ‘composition’ of ‘elementary’ $A_2$-webs in biangles. More precisely, there exists a finite collection of ideal arcs $e_1, \ldots, e_n$ of $B$ connecting its two marked points, dividing $\hat{B}$ into
directed biangles $\vec{B}_1, \ldots, \vec{B}_{n+1}$, appearing in this order from the left side of $\vec{B}$ toward the right side of $\vec{B}$, so that for each $i = 1, \ldots, n + 1$, the $A_2$-web $W_i := W \cap \vec{B}_i$ in $\vec{B}_i$ is union of single-component $A_2$-webs in $\vec{B}_i$ with at most one 3-valent vertex. One strong way of doing so is to require that each $W_i$ has at most one 3-valent vertex, which is possible.

Denote the left and the right sides of $\vec{B}_i$ as $b_{\text{left},i}$ and $b_{\text{right},i}$. Then $b_{\text{left},i} = e_{i-1}$ and $b_{\text{right},i} = e_i$ for each $i = 1, \ldots, n + 1$, where we denote the left and the right sides $b_{\text{left}}$ and $b_{\text{right}}$ of $\vec{B}$ by $e_0$ and $e_{n+1}$, respectively. Let $J : W \cap (\Omega B \cup e_1 \cup \cdots \cup e_n) \to \{1, 2, 3\}$ be a juncture-state for this decomposition of $B$. By Prop 5.9 (BT1), we have

$$\text{Tr}_B([W, s]) = \sum_j \Pi^{n+1}_i \text{Tr}_B,([W, J|_{\partial W_i}])$$

where the sum is over all juncture-states $J$ restricting to $s$ at $\partial W = W \cap \partial B$. Since $\text{Tr}_B([W, s]) \neq 0$, there exists a juncture-state $J_0$ restricting to $s$ such that the corresponding summand $\Pi^{n+1}_i \text{Tr}_B,([W, J_0|_{\partial W_i}])$ is nonzero. For this $J_0$, we therefore have $\text{Tr}_B,([W, J_0|_{\partial W_i}]) \neq 0$ for all $i = 1, \ldots, n + 1$.

For each $i = 1, \ldots, n + 1$, let $W_{i,1}, \ldots, W_{i,l_i}$ be the components of $W_i$; so $\text{Tr}_B,([W_i, J_0|_{\partial W_i}]) = \prod_{j=1}^{l_i} \text{Tr}_B,([W_{ij}, J_0|_{\partial W_{ij}}])$, hence $\text{Tr}_B,([W_{ij}, J_0|_{\partial W_{ij}}]) \neq 0$ for all $j = 1, \ldots, l_i$. Note that $(W_{ij}, J_0|_{\partial W_{ij}})$ is a single-component stated $A_2$-web in the biangle $\vec{B}_i$ with at most one 3-valent vertex, hence falls into one of the cases in (BT2-1)-(BT2-4) of Prop 5.5. In case (BT2-1), $\text{Tr}_B,([W_{ij}, J_0|_{\partial W_{ij}}]) \neq 0$ iff $J_0|_{\partial W_{ij}}$ assigns same state value in $\{1, 2, 3\}$ to the two endpoints of $W_{ij}$, while the two endpoints lie in distinct sides of $B_i$ (i.e. $e_1-1$ and $e_i$), hence $n_{e_1-1,e_i} = n_{e_1,e_i}$ and $n_{e_1-1,e} = n_{e_1,e}$ hold for all $e \in \{1, 2, 3\}$, so in view of eq.(5.12) and eq.(5.13) we can observe

$$C^{(1)}_{\text{left}}(W_{ij}, J_0|_{\partial W_{ij}}) = C^{(1)}_{\text{right}}(W_{ij}, J_0|_{\partial W_{ij}}),$$

$$C^{(2)}_{\text{left}}(W_{ij}, J_0|_{\partial W_{ij}}) = C^{(2)}_{\text{right}}(W_{ij}, J_0|_{\partial W_{ij}}).$$

In case (BT2-2), note $\text{Tr}_B,([W_{ij}, J_0|_{\partial W_{ij}}]) \neq 0$ iff the pair of values of $J_0|_{\partial W_{ij}}$ at the two endpoints of $W_{ij}$ is one of $(1,3), (2,3), (3,1)$, while the two endpoints lie in a single side of $B_i$ (i.e. either $e_1-1$ or $e_i$, one being a source and the other a sink. So, for one $h \in \{\text{left}, \text{right}\}$ for which $W_{ij}$ has no endpoints on $b_{h,i}$, manifestly $C^{(1)}_h(W_{ij}, J_0|_{\partial W_{ij}}) = C^{(2)}_h(W_{ij}, J_0|_{\partial W_{ij}}) = 0$. For the other $h$, one easily observes

$$C^{(1)}_h(W_{ij}, J_0|_{\partial W_{ij}}) = C^{(2)}_h(W_{ij}, J_0|_{\partial W_{ij}}) = 0 \text{ too, hence eq.}(5.12) \text{ and eq.}(5.13) \text{ hold. In case (BT2-3)}.$$

where one side of $B_i$ has two endpoints $x_1, x_2$ of $W_{ij}$, and the other side of $B_i$ has one endpoint $x_3$ of $W_{ij}$, where the three endpoints are either all sinks or all sources, note $\text{Tr}_B,([W_{ij}, J_0|_{\partial W_{ij}}]) \neq 0$ iff $\{J_0(x_1), J_0(x_2)\} = \{r_1(J_0(x_3)), r_2(J_0(x_3))\}$. Suppose $x_1, x_2, x_3$ are sources, and $x_3$ is at $e_1-1 = b_{\text{left},i}$. If $J_0(x_3) = 1$, then $n_{e_1-1}^+, n_{e_1-1}^- = n_{e_1}^-, n_{e_1}^+$, while the remaining $n_{e_i}^+$ are all zero; so for $(W_{ij}, J_0|_{\partial W_{ij}})$ we have $C^{(1)}_{\text{left}} = -1$, $C^{(2)}_{\text{right}} = -1$, $C^{(2)}_{\text{left}} = 1$, $C^{(2)}_{\text{right}} = -1 + 2 = 1$. If $J_0(x_3) = 2$, then $n_{e_1-1}^+, n_{e_1-1}^- = n_{e_1}^-, n_{e_1}^+$, with other $n_{e_i}^+$ being zero, so $C^{(1)}_{\text{left}} = 0$, $C^{(1)}_{\text{right}} = -1 + 1 = 0$, $C^{(2)}_{\text{left}} = -2$, $C^{(2)}_{\text{right}} = -1 - 1 = -2$. If $J_0(x_3) = 3$, then $n_{e_1}^+, n_{e_1}^- = n_{e_1}^-, n_{e_1}^+$, with other $n_{e_i}^+$ being zero, so $C^{(1)}_{\text{left}} = 1$, $C^{(1)}_{\text{right}} = 1$, $C^{(2)}_{\text{left}} = 1$, $C^{(2)}_{\text{right}} = 2 - 1 = 1$. In any case, eq.(5.15) and eq.(5.16) hold. Proof of eq.(5.15) and eq.(5.16) for the cases when $x_1, x_2, x_3$ may be sinks and $x_3$ may be at $e_i = b_{\text{right},i}$ follows, due to the symmetry and skew-symmetry of the definition of the charges as in eq.(5.12) and eq.(5.13) under the exchange $+ \leftrightarrow -$ of the superscripts, and the symmetry of the sought-for eq.(5.15) and eq.(5.16) under $e_i-1 \leftrightarrow e_i$ (i.e. left$\leftrightarrow$right). In case (BT2-4), where one side of $B_i$ has three endpoints $x_1, x_2, x_3$ of $W_{ij}$, and the other side of $B_i$ has no endpoint of $W_{ij}$, where the three endpoints are either all sinks or all sources, note $\text{Tr}_B,([W_{ij}, J_0|_{\partial W_{ij}}]) \neq 0$ iff $\{J_0(x_1), J_0(x_2), J_0(x_3)\} = \{1, 2, 3\}$. Suppose $x_1, x_2, x_3$ are sources lying in $e_{i-1} = b_{\text{left},i}$. Then $n_{e_i}^+, n_{e_i}^- = 1$ for all $e_i \in \{1, 2, 3\}$, while the remaining $n_{e_i}^+$ are all zero. So for $(W_{ij}, J_0|_{\partial W_{ij}})$ one observes $C^{(1)}_{\text{left}} = -1 + 1 = 0$ and $C^{(2)}_{\text{left}} = 1 - 2 + 1 = 0$, while $C^{(1)}_{\text{right}}$ and $C^{(2)}_{\text{right}}$ are manifestly zero. Hence eq.(5.15) and eq.(5.16) hold. Proof of eq.(5.15) and eq.(5.16) for the cases when $x_1, x_2, x_3$ may be sinks and they may be at $e_i$ follow from skew-symmetry and symmetry.

Now, summing eq.(5.15) and eq.(5.16) over $j = 1, \ldots, l_i$, we get

$$C^{(1)}_{\text{left}}(W_i, J_0|_{\partial W}) = C^{(1)}_{\text{right}}(W_i, J_0|_{\partial W}) \text{ and } C^{(2)}_{\text{left}}(W_i, J_0|_{\partial W}) = C^{(2)}_{\text{right}}(W_i, J_0|_{\partial W}).$$
For each $k = 1, 2$, observe from definition that $c^{(k)}_{\text{right}}(W_i, J_0|\partial W_i) = c^{(k)}_{\text{left}}(W_{i+1}, J_0|\partial W_{i+1})$ holds for each $i = 1, \ldots, n$, and $c^{(k)}_{\text{left}}(W_i, J_0|\partial W_i) = c^{(k)}_{\text{left}}(W, s)$ and $c^{(k)}_{\text{right}}(W_{n+1}, J_0|\partial W_{n+1}) = c^{(k)}_{\text{right}}(W, s)$. So, by using above equalities repeatedly, one obtains $C^{(k)}_{\text{left}}(W, s) = C^{(k)}_{\text{right}}(W, s)$, as desired. 

**Proof of Prop. 5.22.** Let $\mathcal{S}, \Delta$, and $\hat{\Delta}$ be as in Def. 5.19, and $(W, s)$ be a stated $A_2$-web $(W, s)$ in a gool position with respect to $\hat{\Delta}$. Recall the state-sum formula for $\hat{\text{Tr}}_\Delta(W, s)$ as in eq. (5.11) of Def. 5.21.

Let $J$ be a $\hat{\Delta}$-juncture-state for $W$ restricting to $s$ such that the corresponding summand in eq. (5.11) is nonzero; then the biangle factor $\text{Tr}_B([W \cap B, J_B])$ for each biangle $B$ of $\hat{\Delta}$ is nonzero. Pick any internal (i.e., non-boundary) arc $e$ of $\Delta$, and let $B$ be the corresponding biangle of $\hat{\Delta}$. Let $t, r$ be the ideal triangles of $\Delta$ having $e$ as a side, and let $\hat{t}, \hat{r}$ be the corresponding triangles of $\hat{\Delta}$; say, $e \in \hat{\Delta}$ is a side of $\hat{t}$ and $e' \in \hat{\Delta}$ is a side of $\hat{r}$. Note that quiver $Q_\Delta$ has two nodes on the arc $e$, say $v_1$ and $v_2$, such that the direction $v_1 \rightarrow v_2$ matches the clockwise orientation on $\partial t$ and counterclockwise orientation on $\partial r$. We investigate the generators of $Z_{t, v_1}, Z_{t, v_2}, Z_{r, v_1}, Z_{r, v_2}$ in the (tensor) product of the triangle factors $\hat{\text{Tr}}_t(W \cap \hat{t}, J_t) \otimes \hat{\text{Tr}}_r(W \cap \hat{r}, J_r)$; these generators do not appear in the triangle factors for triangles other than $t, r$.

For convenience, choose a direction of $B$ to make it a directed biangle $\hat{B}$, so that the left side $b_{\text{left}}$ is a side of $\hat{t}$ and the right side $b_{\text{right}}$ belongs to $\hat{r}$. For $h \in \{\text{left, right}\}$, $\epsilon \in \{+, -\}$ and $\varepsilon \in \{1, 2, 3\}$, define the numbers $n_{h, \epsilon, \varepsilon}$ as in Lem. 5.25. Investigating the triangle factor $\hat{\text{Tr}}_t(W \cap \hat{t}, J_t)$ as in eq. (5.10), since $W$ is in a gool position, each factor as in eq. (5.9) falls into (CT2-1) or (CT2-2) of Prop. 5.6. Looking at (CT2-1) and (CT2-2), $Z_{t, v_1}$ or $Z_{t, v_2}$ may appear in the entries of the matrices $M^{\text{left}}_{t, \alpha}$, $M^{\text{left}}_{t, J}$, $M^{\text{left}}_{t, \alpha+1}$, $M^{\text{right}}_{t, \alpha}$, which are all diagonal matrices, but not in the matrices $M^{\text{left}}_{t, \alpha}$ and $M^{\text{right}}_{t, \alpha}$. In view of the edge matrix $M_{v_1}$ in (MM1) of §4.2, the triangle factor $\hat{\text{Tr}}_t(W \cap \hat{t}, J_t)$ equals

$$(Z_{t, v_1} Z_{t, v_2})^{n_{\text{left, 1}}} (Z_{t, v_1} Z_{t, v_2})^{-1} n_{\text{left, 2}} (Z_{t, v_1} Z_{t, v_2})^{n_{\text{left, 3}}} (Z_{t, v_1} Z_{t, v_2})^{n_{\text{left, 4}}} (Z_{t, v_1} Z_{t, v_2})^{n_{\text{left, 5}}} = Z_{t, v_1}^{n_{\text{left, 1}}} Z_{t, v_2}^{n_{\text{left, 2}}} Z_{t, v_1}^{n_{\text{left, 3}}} Z_{t, v_2}^{n_{\text{left, 4}}} Z_{t, v_1}^{n_{\text{left, 5}}}$$

times a Laurent polynomial in the generators of the triangle algebra $Z_t$ not involving the nodes $v_1$ or $v_2$. Similarly, the triangle factor $\hat{\text{Tr}}_r(W \cap \hat{r}, J_r)$ equals

$$(Z_{r, v_1} Z_{r, v_2})^{n_{\text{right, 1}}} (Z_{r, v_1} Z_{r, v_2})^{n_{\text{right, 2}}} (Z_{r, v_1} Z_{r, v_2})^{n_{\text{right, 3}}} (Z_{r, v_1} Z_{r, v_2})^{n_{\text{right, 4}}} (Z_{r, v_1} Z_{r, v_2})^{n_{\text{right, 5}}} = Z_{r, v_1}^{n_{\text{right, 1}}} Z_{r, v_2}^{n_{\text{right, 2}}} Z_{r, v_1}^{n_{\text{right, 3}}} Z_{r, v_2}^{n_{\text{right, 4}}} Z_{r, v_1}^{n_{\text{right, 5}}}$$

times a Laurent polynomial in the generators of the triangle algebra $Z_r$ not involving the nodes $v_1$ or $v_2$. Since $\hat{\text{Tr}}_B([W \cap B, J_B]) \neq 0$, from Cor. 5.24 we observe that the power of $Z_{t, v_1}$ matches the power of $Z_{r, v_1}$, and that the power of $Z_{t, v_2}$ matches the power of $Z_{r, v_2}$, hence establishing the balancedness as being asserted in the present Proposition 5.22.

Crucial thing to show is the isotopy invariance of the state-sum trace formulated as follows, which we prove in the next subsection.

**Proposition 5.25** (isotopy invariance of state-sum trace for gool positions). Let $\mathcal{S}, \Delta$, and $\hat{\Delta}$ be as in Def. 5.19. If $(W, s)$ and $(W', s')$ are isotopic stated $A_2$-webs in $\mathcal{S}$ in gool positions with respect to $\hat{\Delta}$, then $\text{Tr}_\Delta(W, s) = \hat{\text{Tr}}_\Delta(W', s')$.

**Proof of the sought-for Prop. 5.6 assuming Prop. 5.22.** Let $\mathcal{S}, \Delta$, and $\hat{\Delta}$ be as in Def. 5.19. We will construct a map $\text{Tr}_\Delta : \mathcal{S}(\mathcal{S}; Z) \rightarrow \hat{\mathcal{Z}}$ which is well-defined, one must show that the defining relations of the stated $A_2$-skein algebra $\mathcal{S}(\mathcal{S}; Z)$ are satisfied. By pushing all the relations to biangles, one observes that this is the case. For example, take the $A_2$-skein relation (S4) of Fig. 2 so that we have $[W, s] = [W_1, s_1] + [W_2, s_2]$ in $\mathcal{S}(\mathcal{S}; Z)$, where the stated $A_2$-webs $(W, s)$, $(W_1, s_1)$, $(W_2, s_2)$ in $\mathcal{S}$ are
identical except at a small disk as shown in the three figures appearing in (S4). By applying same isotopies to these three stated $A_2$-webs, one can push this disk to the interior of a biangle $B$ of $\Delta$. Note that the $\Delta$-juncture-states for these three stated $A_2$-webs are naturally in bijection. For each such $\Delta$-juncture-state $J$, in eq. (5.11) the only difference among the three is the biangle factor for $B$, where $\hat{\text{Tr}}_B([W \cap B, J_B]) = \text{Tr}_B([W_1 \cap B, J_B]) + \text{Tr}_B([W_2 \cap B, J_B])$ holds because $\text{Tr}_B$ is a well-defined map on the stated $A_2$-skein algebra $\mathcal{S}_\text{t}(B; \mathbb{Z})$ (Prop. 5.9). Hence it follows that $\hat{\text{Tr}}_{\Delta}(W, s) = \hat{\text{Tr}}_{\Delta}(W_1, s_1) + \hat{\text{Tr}}_{\Delta}(W_2, s_2)$, as desired. The facts that $\text{Tr}_{\Delta} s$ constructed this way is a ring homomorphism and that it satisfies the properties (CT1) and (CT2-1)–(CT2-2) of Prop 5.6 are built in from the very construction of the state-sum trace $\hat{\text{Tr}}_{\Delta}$. The properties (CT2-3)–(CT2-6) would follow from the following:

**Proposition 5.26** (state-sum trace for good positions). Let $\mathfrak{S}$, $\Delta$, and $\hat{\Delta}$ be as in Def 5.19. Let $(W, s)$ be a stated $A_2$-web in $\mathfrak{S}$ in a good position with respect to $\Delta$. Define the state-sum trace

$$\hat{\text{Tr}}_{\Delta}(W, s) \in \bigotimes_{t \in T(\Delta)} \mathbb{Z}_t$$

precisely as in eq. (5.11) of Def 5.21 where the value $\hat{\text{Tr}}_t(W_{t,j}, J_{j,t}) \in \mathbb{Z}_t$ (replacing eq. (5.9)) of each stated component $(W_{t,j}, J_{j,t})$ for a triangle $t$ of $\Delta$ is now defined using Prop 5.6(C)-((CT2-6)).

If $(W', s')$ is a stated $A_2$-web in $\mathfrak{S}$ in a good position and $(W, s)$ is isotopic to $(W', s')$, then

$$\hat{\text{Tr}}_{\Delta}(W, s) = \hat{\text{Tr}}_{\Delta}(W', s').$$

### 5.4. Isotopy invariance of state-sum formula.

In order to complete our proof for the existence part of Prop 5.6, it remains to show Prop 5.25 and Prop 5.26. For both propositions, it helps to establish the following statement first.

**Proposition 5.27** (elementary isotopy invariance). Let $t$ be a triangle, viewed as a generalized marked surface. Let $\hat{\Delta}$ be the collection of four arcs in $t$, three of them being the boundary arcs of $t$, and the remaining one an arc connecting two marked points of $t$ whose interior lies in the interior of $t$; so $\hat{\Delta}$ divides $t$ into one triangle $\hat{t}$ and one biangle $B$. Consider the state-sum trace for stated $A_2$-webs in $t$ in a good position with respect to $\hat{\Delta}$ as defined in Prop 5.26, which we denote by $\hat{\text{Tr}}_{\hat{\Delta}}$. Then $\hat{\text{Tr}}_{\hat{\Delta}}$ satisfies the following elementary isotopy invariance: if $(W, s)$ and $(W', s')$ are stated $A_2$-webs in $t$ in good positions with respect to $\hat{\Delta}$ and are related to each other by one of the moves in Fig 11, 12, 13, possibly with different possible orientations on the components, they have same values under $\hat{\text{Tr}}_{\hat{\Delta}}$.

**Proof.** Label the arcs of $\hat{\Delta}$ by $e_1, e_2, e_3, e_3'$, and the names of endpoints and junctures as $x, y, z, w, w', w''$ as in the pictures. Denoting by $\Delta$ the unique triangulation of $t$, label the seven nodes of the quiver $Q_{\Delta}$.
Lemma 5.28. For states \(s\) and \(s'\), we consider
\[
\bar{\text{Tr}}_\Delta(W, s) = \sum_{s_1, s_2} \text{Tr}_B([W \cap B, s_1]) \bar{\text{Tr}}_r(W \cap \hat{t}, s_2),
\]
where the first sum is over all states \(s_1, s_2\) of \(W \cap B\) and \(W \cap \hat{t}\) that are compatible with \(s\) and similarly for the second sum.

Consider the case of Fig. 11. Let \(s_1, s_2\) be states of \(W \cap B\) and \(W \cup \hat{t}\) compatible with \(s\) and whose corresponding summand in eq. (5.17) is nonzero. Note \(W \cap B\) has just one component, which is an edge connecting two sides of \(B\), so by (BT2-1) of Prop. 5.9 \(\text{Tr}_B([W \cap B, s_1]) \neq 0\) implies \(s_1(w) = s_1(z)\), in which case \(\text{Tr}_B([W \cap B, s_1]) = 1\). By compatibility, \(s_2(w) = s_1(w)\) and \(s_2(z) = s_1(z)\), \(s_2(x) = s(x)\), \(s_2(y) = s(y)\). So there is only one pair of \(s_1, s_2\) contributing to the sum, and hence \(\bar{\text{Tr}}_\Delta(W, s) = \bar{\text{Tr}}_r(W \cap \hat{t}, s_2) = \bar{\text{Tr}}_{\Delta, t}([W, s])\). Now, let \(s'_1, s'_2\) be states of \(W' \cap B\) and \(W' \cap t\) compatible with \(s'\). In particular, \(s'_1(z) = s'(z), s'_2(x) = s'(x), s'_2(y) = s'(y)\). As \(W' \cap B\) has just one component, which is a 3-way web with two endpoints on a same arc and the remaining on the other, by (BT2-3) Prop. 5.9 it follows that \(\text{Tr}_B([W' \cap B, s'_1]) \neq 0\) equals 1 if \((s'_1(w'), s'_1(w'')) = (r_1(s'(z)), r_2(s'(z)))\), equals \(-1\) if \((s'_1(w'), s'_1(w'')) = (r_2(s'(z)), r_1(s'(z)))\), and equals zero otherwise. By compatibility, \(s'_2(w') = s'_1(w')\) and \(s'_2(w') = s'_1(w')\). Denoting \(s'_2\) by the ordered quadruple of values \((s'_2(x), s'_2(y), s'_2(w'), s'_2(w''))\) it follows
\[
\bar{\text{Tr}}_\Delta(W', s') = \bar{\text{Tr}}_r(W' \cap t, (s'(x), s'(y), r_1(s'(z)), r_2(s'(z)))) - \bar{\text{Tr}}_r(W' \cap \hat{t}, (s'(x), s'(y), r_1(s'(z))))
\]
For convenience, denote \(\varepsilon_1 = s'(x), \varepsilon_2 = s'(y), \varepsilon_3 = s'(z)\). Note \(W' \cap \hat{t}\) is union (or, product) of a left turn arc from \(w'\) (in \(e_1\)) to \(x\) (in \(e_1\)) and a right turn arc from \(w''\) (in \(e_3\)) to \(y\) (in \(e_2\)). Hence, using notations as in Prop 5.6 we have
\[
\bar{\text{Tr}}_\Delta(W', s') = (M_{t,3}^{\text{in}})_{r_3(e), r_1(e)}(M_{t,1}^{\text{out}})_{r_3(e), r_1(e)} (M_{t,2}^{\text{right}})_{r_2(e), r_1(e)} (M_{t,4}^{\text{out}})_{r_2(e), r_1(e)}
\]
Observe that the underlined part equals \((M_{t,3}^{\text{out}})_{e_1, e_2, e_3}\) defined in eq. (5.5) of Prop 5.6 (CT2), justifying why it was defined like it there: in fact, this is how we came up with the definition of \(M_{t,3}^{\text{out}}\) as written there. For the \(M_{t,1}^{\text{left}}\) parts, we need a small lemma:

Lemma 5.28 (edge matrix inversion formula). For each \(\varepsilon \in \{1, 2, 3\}\) and \(\alpha \in \{1, 2, 3\}\), one has
\[
(M_{t,3}^{\text{in}})_{r_3(e), r_1(e)}(M_{t,1}^{\text{in}})_{r_2(e), r_1(e)} = (M_{t,3}^{\text{out}})_{r_3(e), e_2} (M_{t,1}^{\text{out}})_{r_2(e), r_1(e)} = (M_{t,3}^{\text{out}})_{r_3(e), e_2}.
\]
Proof of Lem. 5.28. Writing the three diagonal entries of \(M_{t,3}^{\text{out}}\) as \(Z_1Z_2^2, Z_2Z_1^{-1}, Z_1^{-1}Z_2\) in this order, those of \(M_{t,1}^{\text{out}}\) are \(Z_1Z_2^2, Z_2Z_1^{-1}, Z_1^{-1}Z_1^{-1}\) in this order. In view of eq. (5.4), for the first equality we
check \((Z_2Z_1^2)(Z_2Z_1^{-1}) = Z_1Z_2^2\), \((Z_2Z_1^3)(Z_2^{-1}Z_1^{-1}) = Z_1Z_2^{-1}\), and \((Z_2Z_1^{-1})(Z_2^{-1}Z_1) = Z_1^{-2}Z_2\). The second equality follows from the three equations with the subscripts 1 and 2 exchanged.

Hence we showed \(\widetilde{\text{Tr}}_\Delta(W', s')\) equals \(\prod_{i=1}^{3}(M_{i,\ell}^{\text{out}})_{\epsilon_1,\epsilon_2}(M_{i}^{\text{out}})_{\epsilon_1,\epsilon_2,\epsilon_3}\), which equals \(\text{Tr}_\Delta([W, s])\) in view of Prop.5.6 CT2-3, if \(s\) assigns \(\epsilon_1, \epsilon_2, \epsilon_3\) to \(x, y, z\) respectively. This finishes the proof for the case of Fig.11.

For the case as in Fig.11 with the reverse orientation, the proof goes similarly. In particular, the arguments goes almost verbatim for \(\widetilde{\text{Tr}}_\Delta(W, s)\), while in our investigation of \(\widetilde{\text{Tr}}_\Delta(W', s')\), we should exchange in and out, left and right, and change the orders of two index subscripts of each \((M_{i,\ell}^{\text{left}})_{\epsilon_1,\epsilon_2}\) and \((M_{i,\ell}^{\text{right}})_{\epsilon_1,\epsilon_2}\). In particular, the underlined part now becomes \((M_{i,\ell}^{\text{right}})_{\epsilon_1,\epsilon_2,\epsilon_3}(M_{i}^{\text{left}})_{\epsilon_1,\epsilon_2,\epsilon_3} - (M_{i,\ell}^{\text{left}})_{\epsilon_1,\epsilon_2,\epsilon_3}(M_{i}^{\text{right}})_{\epsilon_1,\epsilon_2,\epsilon_3}\), which equals \((M_{i}^{\text{left}})_{\epsilon_1,\epsilon_2,\epsilon_3}\) as defined in eq.5.5 of Prop.5.6. The rest arguments work similarly.

Now take the case as in Fig.12 possibly with all orientations reversed. Denote by \(\epsilon_1, \epsilon_2, \epsilon_3\) the state values of \(s\) and \(s'\) at endpoints \(z', z, x\), respectively. Look at \(W'\) first, which is on the right (i.e. the second or the fourth picture from the left in Fig.12). Note \(W' \cap B\) consists of two edges connecting distinct sides, so in the sum in eq.5.18, by Prop.5.9 BT2-1 the bangle factor ‘goes away’, and we just have \(\widetilde{\text{Tr}}_\Delta(W', s') = \text{Tr}_\Delta(W' \cap \hat{t}, (\epsilon_1, \epsilon_2, \epsilon_3))\), where \(\epsilon_1, \epsilon_2, \epsilon_3\) denotes \(s_3'\) that assigns \(\epsilon_1, \epsilon_2, \epsilon_3\) to \(w, \epsilon, x\). On the other hand, consider \(W\), which is on the left (i.e. the first or the third picture from the left in Fig.12). Note \(W \cap B\) consists of one 3-way web component. If the state \(s_1\) of \(W \cap B\) assigns \(\epsilon, \epsilon_1, \epsilon_2\) to \(w, \epsilon, x\), then by Prop.5.9 BT2-3 we see that \(\text{Tr}_B([W \cap B, s_1])\) equals 1 if \((r_1(\epsilon), r_2(\epsilon)) = (\epsilon_1, \epsilon_2)\), equals \(-1\) if \((r_1(\epsilon), r_2(\epsilon)) = (\epsilon_2, \epsilon_1)\), and equals 0 otherwise. So if \(\epsilon_1 = \epsilon_2\) then \(\widetilde{\text{Tr}}_\Delta(W, s) = 0\), and if \(\epsilon_1 \neq \epsilon_2\), we see that

\[
\widetilde{\text{Tr}}_\Delta(W, s) = \text{sgn}(\epsilon_2 - \epsilon_1)\text{Tr}_\Delta([W \cap \hat{t}, (\epsilon \mapsto \epsilon, x \mapsto \epsilon_3)]),
\]

where \(\epsilon\) is the unique element of \(\{1, 2, 3\}\) with \(\{r_1(\epsilon), r_2(\epsilon)\} = \{\epsilon_1, \epsilon_2\}\). In view of Prop.5.6 CT2-4, this matches \(\widetilde{\text{Tr}}_\Delta(W', s')\); in fact, this is how we came up with (CT2-4).

Take the left case of Fig.13 possibly with all orientations reversed. Denote by \(\epsilon_1, \epsilon_2\) the state values of \(s\) and \(s'\) at \(z_1, z_2\) respectively. Look at \(W\) on the left (i.e. the first picture from the left in Fig.13). There is nothing in the triangle, so we have \(\widetilde{\text{Tr}}_\Delta(W, s) = \text{Tr}_\Delta([W \cap B, (z_1 \mapsto \epsilon_1, z_2 \mapsto \epsilon_2)])\). Look at \(W'\), which is on the right (i.e. the second picture from the left in Fig.13). Note \(W' \cap B\) consists of two edges connecting distinct sides, so in the sum in eq.5.18, by Prop.5.9 BT2-1 the bangle factor ‘goes away’, and we just have \(\widetilde{\text{Tr}}_\Delta(W', s') = \text{Tr}_\Delta(W' \cap \hat{t}, (w_1 \mapsto \epsilon_1, w_2 \mapsto \epsilon_2))\). In view of Prop.5.6 CT2-6) and Prop.5.9 BT2-2), we have \(\widetilde{\text{Tr}}_\Delta(W, s) = \widetilde{\text{Tr}}_\Delta(W', s')\).

Take the right case of Fig.13. Denote by \(\epsilon_1, \epsilon_2\) the state values of \(s\) and \(s'\) at \(x, y\) respectively. For \(W'\) (i.e. the third picture from the left of Fig.13), there is nothing in the bangle, so \(\widetilde{\text{Tr}}_\Delta(W, s) = \text{Tr}_\Delta([W \cap \hat{t}, (x \mapsto \epsilon_1, y \mapsto \epsilon_2)])\), which equals

\[
(M_{i,\ell}^{\text{in}}M_{\ell,\ell}^{\text{left}}M_{\ell,\ell}^{\text{out}})_{\epsilon_2,\epsilon_1}.
\]

For \(W'\) (i.e. the fourth picture from the left of Fig.13), \(W' \cap B\) consists of one U-turn component, hence the value under \(\text{Tr}_B\) is governed by Prop.5.9 BT2-2). Meanwhile, \(W' \cap \hat{t}\) consists of two left turn corner arcs, so the values under \(\text{Tr}_\Delta\) are given by Prop.5.6 CT2-1). Taking into account that these values are given as matrix entries of \(3 \times 3\) matrices, one observes that the final value \(\widetilde{\text{Tr}}_\Delta(W', s')\) is one matrix entry of the corresponding product of matrices multiplied in the order of appearance along the curve \(W'\). (see our proof of Prop.5.39 for a similar interpretation of state-sum value as a matrix entry of product of monodromy matrices taken while traveling along a curve), so equals

\[
\left(\begin{array}{ccc}
M_{k,\ell}^{\text{out}} & M_{\ell,\ell}^{\text{left}}M_{\ell,\ell}^{\text{out}} & M_{1,\ell}^{\text{left}}M_{1,\ell}^{\text{out}} \\
\text{left turn in } & \text{U-turn in } & \text{left turn in } \\
\hat{t} & B & \hat{t}
\end{array}\right)_{\epsilon_2,\epsilon_1}.
\]

So the problem boils down to proving the matrix identity:

\[
(M_{1,2}^{\text{in}}M_{2,1}^{\text{right}}M_{1,1}^{\text{out}}) = M_{1,2}^{\text{in}}M_{1,2}^{\text{left}}M_{1,3}^{\text{out}} \cdot M_{1,3}^{\text{left}}M_{1,1}^{\text{out}}.
\]

Before showing this, note that since all remaining cases of the present proposition are for cases when \(W\) and \(W'\) are oriented curves, the problem for each of them also boils down to checking an identity of
two products of monodromy matrices for segments. We first collect all such matrix identities to check, and show them altogether. First, still for the right case of Fig.13 but with opposite orientations, we should show

\[ M_{\text{out}}^{t_{1,2}}M_{\text{left}}^{t_{1,3}}M_{\text{right}}^{t_{2,3}}M_{\text{out}}^{t_{3,1}} = M_{\text{in}}^{t_{1}}M_{\text{right}}^{t_{1,2}}M_{\text{out}}^{t_{3,2}} \cdot M_{\text{U}} \cdot M_{\text{in}}^{t_{3}}M_{\text{right}}^{t_{2,3}}, \]  

(5.20)

For the two cases in Fig.14, we should show

\[ M_{\text{U}} = M_{\text{in}}^{t_{1}}M_{\text{right}}^{t_{1,2}}M_{\text{out}}^{t_{3,1}} \cdot M_{\text{in}}^{t_{3}}M_{\text{right}}^{t_{2,3}}, \]  

(5.21)

\[ M_{\text{U}} = M_{\text{in}}^{t_{2}}M_{\text{left}}^{t_{1,3}}M_{\text{out}}^{t_{3,1}} \cdot M_{\text{U}} \cdot M_{\text{right}}^{t_{2,3}} \cdot M_{\text{left}}^{t_{1}}. \]  

(5.22)

The orientation reversed versions of these two yield identical matrix identities as themselves. To prove these four matrix identities, we observe:

**Lemma 5.29** (compatibility relations among monodromy matrices; [FG06]). One has

\[ M_{\text{out}}^{t_{1,2}}M_{\text{left}}^{t_{1,3}}M_{\text{right}}^{t_{2,3}}M_{\text{out}}^{t_{3,1}} = M_{\text{U}} = M_{\text{in}}^{t_{1}}M_{\text{right}}^{t_{1,2}}M_{\text{out}}^{t_{3,2}} \cdot M_{\text{U}} \cdot M_{\text{in}}^{t_{3}}M_{\text{right}}^{t_{2,3}}. \]  

(5.23)

\[ M_{\text{left}}^{t_{1}}M_{\text{U}}M_{\text{left}}^{t_{2}} = M_{\text{right}}^{t_{1}}, \quad M_{\text{right}}^{t_{2}}M_{\text{U}}M_{\text{right}}^{t_{1}} = M_{\text{left}}. \]  

(5.24)

\[ M_{\text{left}}^{t_{1}}M_{\text{U}}M_{\text{right}}^{t_{2}} = M_{\text{U}} = M_{\text{right}}^{t_{1}}M_{\text{U}}M_{\text{left}}^{t_{2}}. \]  

(5.25)

See [FG06, Thm.9.2] for a proof; one can easily check by hand. Using this lemma, it is easy to see that the remaining sought-for equalities eq.(5.19),(5.20),(5.21) and (5.22) are satisfied.

We now observe the following topological lemma, whose proof can be obtained in the style of Lem.24 of [BW11].

**Lemma 5.30** (moves between gool positions). Let \( \mathcal{S}, \Delta, \) and \( \tilde{\Delta} \) be as in Def.5.19. Let \( W \) and \( W' \) be \( A_2 \)-webs in \( \mathcal{S} \) in gool positions with respect to \( \tilde{\Delta} \), such that \( W \) is isotopic to \( W' \) as \( A_2 \)-webs in \( \mathcal{S} \). Then \( W \) can be connected to \( W' \) by a sequence of \( A_2 \)-webs \( W = W_1, W_2, \ldots, W_n = W' \) in gool positions with respect to \( \tilde{\Delta} \), such that for each \( i = 1, \ldots, n - 1 \), \( W_i \) is related to \( W_{i+1} \) either by an isotopy within the class of \( A_2 \)-webs in gool positions with respect to \( \tilde{\Delta} \) or by one of the moves in Fig.15, 16 and 17, possibly with different possible orientations on the components.

In fact, the statement of Lem.5.30 should be more refined. Namely, in the definition of the above moves, each biangle in the pictures may not precisely be one entire biangle in the split ideal triangulation \( \tilde{\Delta} \). Before applying the move as depicted in the picture, one may have to divide a biangle of \( \tilde{\Delta} \) into several biangles, by introducing some ideal arcs in this biangle connecting the two marked points. This will yield a generalized version \( \tilde{\Delta}' \) of split ideal triangulation, which may have more than one biangles per
each edge of \( \Delta \). We require that we draw the new arcs so that the \( A_2 \)-web in question is still transverse to \( \hat{\Delta}' \). Then apply the moves as in the above pictures, for the part of \( A_2 \)-web living in the union of a triangle of \( \hat{\Delta}' \) and its three neighboring biangles. For example, a move like

is redundant, as one can show that this can be obtained as composition of the above moves, applied in the sense just described.

Proof of Prop. 5.25. In view of Lem 5.30 it suffices to show that \( \hat{\Tr}(W,s) = \hat{\Tr}(W',s') \) in case \( (W,s) \) and \( (W',s') \) are related by one of those moves, in the sense just described above, using generalized split ideal triangulation \( \hat{\Delta}' \). In each union of one triangle and three neighboring (thin) biangles, one observes that these moves can be obtained as compositions of the moves dealt with in Lem 5.27. Now, write \( \hat{\Tr}(W,s) \) and \( \hat{\Tr}(W',s') \) using a state-sum formula adapted to \( \hat{\Delta}' \), instead of \( \hat{\Delta} \); so we may have more numbers of junctures, and more numbers of biangle factors. By Prop 5.9 (BT1), the new state-sum formulas give same answers as before which used \( \hat{\Delta} \). Now, in these new state-sum expressions, the parts involving the above mentioned union of a triangle and three biangles have equal values for \( (W,s) \) and \( (W',s') \), due to Prop 5.27. Thus \( \hat{\Tr}(W,s) = \hat{\Tr}(W',s') \).

Proof of Prop 5.26. Let \( \mathcal{S}, \Delta, \) and \( \hat{\Delta} \) be as in Def 5.19. Let \( (W,s) \) be a stated \( A_2 \)-web in \( \mathcal{S} \) in a good position with respect to \( \hat{\Delta} \). Let \( t \) be a triangle of \( \Delta \), and \( \hat{t} \) be the corresponding triangle of \( \hat{\Delta} \), such that the \( A_2 \)-web \( W \cap \hat{t} \) contains a component that is either a U-turn arc or a 3-way arc. One can push the whole of such U-turn arc or the 3-valent vertex to a neighboring biangle \( B \), by an isotopy. Before pushing, one can divide \( B \) into two biangles \( B_1 \) and \( B_2 \) by considering an ideal arc in \( B \), such that \( B_1 \) is adjacent to \( \hat{t} \) and \( W \cap B_1 \) consists only of the components of the form in as in Prop 5.9 (BT2-1). Then, push the U-turn arc or the 3-valent vertex living in \( \hat{t} \) into the biangle \( B_1 \). Then, by Prop 5.27, the value of \( \hat{\Tr}(W \cap (\hat{t} \cup B_1), s \mid_{W \cap (\hat{t} \cup B_1)}) \) under the state-sum trace \( \hat{\Tr}_{\hat{t} \cup B_1} \) is unchanged by such a pushing. Meanwhile, one can observe that the state-sum trace \( \hat{\Tr}_{\hat{\Delta}} \) as defined in Prop 5.26 using the split ideal triangulation \( \hat{\Delta} \) equals the new state-sum trace defined also as in Prop 5.26 (i.e. as in eq 5.11 of Def 5.21) but this time for the finer decomposition \( \hat{\Delta} \cup \{e\} \) with one more number of biangles, using Prop 5.9 (BT1). This shows that the value under \( \hat{\Tr}_{\hat{\Delta}} \) does not change after such a pushing. By a finite number of such pushing moves, one can put \( (W,s) \) into an \( A_2 \)-web \( (W',s') \) in a good position with respect to \( \hat{\Delta} \), so \( \hat{\Tr}_{\hat{\Delta}}(W,s) = \hat{\Tr}_{\hat{\Delta}}(W',s') \). Meanwhile, we have \( \hat{\Tr}_{\hat{\Delta}}(W',s') = \hat{\Tr}_{\hat{\Delta}}(W',s') \) by construction. Hence \( \hat{\Tr}_{\hat{\Delta}}(W,s) = \hat{\Tr}_{\hat{\Delta}}(W',s') \).

Consequently, Prop 5.6 is finally proved, via the argument at the end of §5.3.

A computationally useful corollary:

Corollary 5.31. Let \( \mathcal{S}, \Delta, \) and \( \hat{\Delta} \) be as in Def 5.19. For any stated \( A_2 \)-web \( (W,s) \) in \( \mathcal{S} \) in a good position with respect to \( \hat{\Delta} \), one has

\[
\Tr_{\Delta}(\lbrack W,s \rbrack) = \hat{\Tr}_{\hat{\Delta}}(W,s).
\]

5.5. Congruence of terms, and highest term. Recall that the major motivation for our study of the \( SL_3 \) classical trace map was to prove Prop 4.18 and Prop 4.19 which are on the highest term and the congruence of terms of the basic semi-regular function \( \int_{PGL_3}(\ell) \in C^\infty(\mathbb{Z}_{PGL_3}) \) for each \( \ell \in \mathcal{A}_k(\mathcal{S}; \mathbb{Z}) \).

In the present subsection we first establish the counterparts for our \( SL_3 \) classical trace.

Proposition 5.32 (congruence of terms of \( SL_3 \) classical trace). Let \( \Delta \) be an ideal triangulation of a triangulable generalized marked surface \( \mathcal{S} \), and let \( (W,s) \) be a stated \( A_2 \)-web in \( \mathcal{S} \). Then the value of the \( SL_3 \) classical trace map \( \Tr_{\Delta}(\lbrack W,s \rbrack) \in \mathbb{Z}_{\Delta} \) can be written as a Laurent polynomial in the generators \( \{Z_v \mid v \in \mathcal{V}(Q_{\Delta})\} \) of \( Z_{\Delta} \) with integer coefficients so that all appearing Laurent monomials are congruent to each other in the following sense: for any two Laurent monomials \( \prod_v Z_v^{\alpha_v} \) and \( \prod_v Z_v^{\beta_v} \), appearing in this Laurent polynomial (with \( (\alpha_v, \beta_v) \in \mathcal{V}(Q_{\Delta}) \)), we have \( \alpha_v - \beta_v \in 3\mathbb{Z} \) for all \( v \in \mathcal{V}(Q_{\Delta}) \).

Proof. We may assume that \( (W,s) \) is in a good position with respect to a split ideal triangulation \( \hat{\Delta} \) for \( \Delta \), by applying an isotopy. Let’s use the state-sum formula for \( \Tr_{\Delta}(\lbrack W,s \rbrack) \) as in eq 5.11 in Def 5.21. Note that the biangle factors \( \Tr_B(\lbrack W \cap B, J_B \rbrack) \) are integers. The triangle factor \( \hat{\Tr}_t(\lbrack W \cap \hat{t}, J_t \rbrack) \) is a
Proposition 5.33 (highest term of SL₃ classical trace). Let $\Delta$ be an ideal triangulation of a triangulable generalized marked surface $\mathcal{S}$, and $\hat{\Delta}$ be a split ideal triangulation for $\Delta$. Let $W$ be a reduced non-elliptic $A_2$-web in $\mathcal{S}$ in a canonical position with respect to $\hat{\Delta}$ (Def.3.14). View $W$ as an $A_2$-lamination by giving the weight $1$; let $a_v(W) \in \mathbb{Q}$, $v \in V(\mathcal{Q}_\Delta)$, be the tropical coordinates defined in Def.3.27. Then $\mathcal{Q}_\Delta$ can be written as a Laurent polynomial in $\{Z_v | v \in V(\mathcal{Q}_\Delta)\}$ with integer coefficients so that $\prod_v Z_v^{a_v(W)}$ is the unique Laurent monomial among all appearing Laurent monomials that has higher partial ordering than any other appearing Laurent monomials, and this highest Laurent monomial has coefficient $1$.

This very important proposition is proved in several steps. The core lies in the following treatment of single-component canonical $A_2$-webs in a triangle (Def.3.12).

Proposition 5.34 (highest term of classical SL₃ trace for a triangle). Let $t$ be a triangle, viewed as a generalized marked surface. Let $W$ be a canonical $A_2$-web in $t$ (Def.3.12). Let $\Delta$ be the unique triangulation of $t$, so that $Q_\Delta$ has seven nodes. For each $v \in V(\mathcal{Q}_\Delta)$, let $a_v(W) \in \mathbb{Q}$ be the tropical coordinate of $W$ as defined in Def.3.27 when $W$ is viewed as an $A_2$-lamination in $t$ with weight $1$. Denote by $1_W$ the state of $W$ assigning $1 \in \{1, 2, 3\}$ to all endpoints of $W$. Then

1. $\mathcal{T}_\Delta([1_W]) = \prod_{v \in V(\mathcal{Q}_\Delta)} Z_v^{a_v(W)} = \prod_{v \in V(\mathcal{Q}_\Delta)} Z_v^{a_v(W)} \in Z_\Delta = Z_t$.
2. For any other state $s$ of $W$, $\mathcal{T}_\Delta([W, s]) \in Z_\Delta = Z_t$ can be written as a Laurent polynomial in $\{Z_v | v \in V(\mathcal{Q}_\Delta)\}$ so that each appearing Laurent monomial has strictly lower partial ordering than $\prod_v Z_v^{a_v(W)}$.

The hard case is when $W$ involves a pyramid (Def.3.12). To compute $\mathcal{T}_\Delta([W, s])$, we will push all $3$-valent vertices into the biangle for one of the sides of the triangle. Thus we find it convenient to build some lemmas for $A_2$-webs in a biangle.

Lemma 5.35 (SL₃ biangle trace for pyramid $P_d$ in biangle). Let $d \in \mathbb{Z}$, $d \neq 0$, and let $P_d$ be the $A_2$-web in a biangle $B$ as in the left picture of Fig.18 (there, an example is drawn for $d = 3$), called a degree $d$ pyramid in a biangle $B$. One way of constructing $P_d$ is from a degree $d$ pyramid $H_d$ in a triangle as in Def.3.12 by removing (i.e. forgetting, or ‘filling in’) one marked point of the triangle to turn it into a biangle. Label the endpoints of $P_d$ as $x_1, x_2, \ldots, x_{|d|}$, $y_1, y_2, \ldots, y_{|d|}$, $z_1, z_2, z_3$, appearing clockwise in this order on $\partial B$, so that $x_1, y_s$ lie on one side and $z_s$ lie on the other side. Suppose $s$ is a state of $P_d$ assigning $1$ to all $x_i$’s and $z_i$’s. Then $\mathcal{T}_B([P_d, s]) \neq 0$ if and only if $s$ assigns $2$ to all $y_i$’s, and for that $s$ we have $\mathcal{T}_B([P_d, s]) = 1$.

Proof of Lemma 5.35. We will show the statement for $d > 0$. The proof for $d < 0$ is completely symmetric. We use induction on $d$. The base case is $d = 1$. One notes that $P_1$ is a $3$-way web in $B$ falling into the case of (BT2-3) of Prop.5.9 and it is easy to verify the desired statement. Now, let $d \geq 2$, and suppose we showed the statement holds for $P_{d-1}$. Observe that $P_d$ ‘contains’ $P_{d-1}$ in its lower right corner (which is easier to see for $H_d$ and $H_{d-1}$ in a triangle), so that there exists an ideal arc $e$ in $B$ (drawn as a dotted line in the left picture of Fig.18) connecting the two marked points of $B$, cutting $B$ into two biangles $B_1$ and $B_2$, such that the non-elliptic $A_2$-web $P_d \cap B_1$ in $B_1$ consists of one edge
connecting the two sides of $B_1$ (having a red dot as one endpoint in the right picture of Fig.18) and a degree $d-1$ pyramid $P_{d-1}$ in $B_1$ (having blue dots as some endpoints in the right picture of Fig.18). Let's label the junctures of $P_d$ at $e$, i.e. the elements of $P_d \cap e$, as $w_1, w_2, \ldots, w_d, u_2, \ldots, u_d$, as in the right picture of Fig.18. By the cutting property (BT1) of Prop.5.9 one has
\begin{equation}
\text{Tr}_B([P_d, s]) = \sum_{s_1, s_2} \text{Tr}_{B_1}([P_d \cap B_1, s_1]) \text{Tr}_{B_2}([P_d \cap B_2, s_2])
\end{equation}
where the sum is over all states $s_1, s_2$ of the $A_2$-webs $P_d \cap B_1$ in $B_1$ and $P_d \cap B_2$ in $B_2$ compatible with $s$, in the sense as in Prop.5.9 (BT1). In particular, any such $s_1$ assigns 1 to all $z_1, z_2, \ldots, z_d$ and any such $s_2$ assigns 1 to all $x_1, \ldots, x_d$. If $\text{Tr}_{B_1}([P_d \cap B_1, s_1]) \neq 0$, then the value under $\text{Tr}_{B_1}$ of each of the two components of $(P_d \cap B_1, s_1)$ must be nonzero, by multiplicativity of $\text{Tr}_{B_1}$. The edge component, which connects the endpoints $z_1$ and $w_1$, falls into the case Prop.5.9 (BT2-1), hence it has nonzero $\text{Tr}_{B_1}$ value iff $s_1$ assigns same value to $z_1$ and $w_1$, so $s_1(w_1) = 1$.

We now investigate the $A_2$-web $P_d \cap B_2$ in $B_2$. It consists of $d$ components, where $d - 1$ of them are edges connecting $u_i$ in $e$ and $y_i$ in the other side of $B_2$ (with $i = 2, 3, \ldots, d$); see the right picture of Fig.18. Denote the remaining component as the $A_2$-web $K_d$ in a biangle $B_2$. Its endpoints on one side are $x_1, x_2, \ldots, x_d, y_1$ appearing in this order along clockwise orientation on $\partial B_2$, and the endpoints on the other side are $w_d, w_{d-1}, \ldots, w_2, y_1$ appearing in this order along clockwise orientation on $\partial B_2$.

Note $x_1, \ldots, x_d, y_1, w_1$ are sinks, $w_2, \ldots, w_d$ are sources, and there are $2d - 1$ internal 3-valent vertices. We prove:

**Lemma 5.36.** Suppose $s_2$ is a state for $K_d$ assigning 1 to all $x_1, \ldots, x_d$ and $w_1$. Then $\text{Tr}_{B_2}([K_d, s_2]) \neq 0$ if and only if $s_2$ assigns 1 to all $w_2, \ldots, w_d$ and 2 to $y_1$. For this $s_2$, we have $\text{Tr}_{B_2}([K_d, s_2]) = 1$.

**Figure 19.** $A_2$-web $K_d$ in a biangle $B_2$ and its decomposition (for $d = 3$)

**Proof of Lem.5.36.** We use induction on $d$. For the base case $K_1$, note that the endpoints on one side of $B_2$ are $x_1, y_1$, while there is only one endpoint $z_1$ in the other side. There is only one internal 3-valent vertex, so $K_1$ is just a 3-way web falling into (BT2-3) of Prop.5.9 so the statement of Lem.5.36 holds. Let $d \geq 2$, and suppose Lem.5.36 holds for $K_{d-1}$. Observe that $K_d$ ‘contains’ $K_{d-1}$ in its lower left corner, so that there exists an ideal arc $e'$ in $B_2$ (drawn as a dotted line in the left picture of Fig.19) cutting $B_2$ into biangles $B_3$ and $B_4$, such that the $A_2$-web $K_d \cap B_3$ in $B_3$ consists of $K_{d-1}$ in $B_3$ (having red dots and the blue dot as vertices in the right picture of Fig.19) and an edge connecting the two sides of $B_3$ (having purple dot as a vertex in the right picture of Fig.19). Label the junctures of $K_d$ at $e'$ as $r_1, \ldots, r_{d-1}, r, r'$ as in Fig.19. By the cutting property (BT1) of Prop.5.9 one has
\begin{equation}
\text{Tr}_{B_2}([K_d, s_2]) = \sum_{s_3, s_4} \text{Tr}_{B_3}([K_d \cap B_3, s_3]) \text{Tr}_{B_4}([K_d \cap B_4, s_4])
\end{equation}
where the sum is over all states $s_3, s_4$ of the $A_2$-webs $K_d \cap B_3$ in $B_3$ and $K_d \cap B_4$ in $B_4$ compatible with $s_2$, in the sense as in Prop.5.9 (BT1). In particular, any such $s_4$ assigns 1 to all $x_1, \ldots, x_d$, and any such $s_3$ assigns 1 to $w_1$.

Note $K_d \cap B_4$ has $d$ components, where $d - 1$ of them are edges connecting $r_1$ in $e'$ and $x_i$ in the other side of $B_4$ (with $i = 1, \ldots, d - 1$). The remaining component can be called an $I$-web, denoted by $I$. If $\text{Tr}_{B_4}([K_d \cap B_4, s_4]) \neq 0$, then the value under $\text{Tr}_{B_4}$ of each of the $d$ components of $(K_d \cap B_4, s_4)$ must be nonzero, by multiplicativity of $\text{Tr}_{B_4}$. The edge component, which connects the endpoints $r_2$ and $x_i$, falls into the case Prop.5.9 (BT2-1), hence it has nonzero $\text{Tr}_{B_4}$ value iff $s_4$ assigns same value to $r_2$ and $x_i$, so $s_4(r_i) = 1$ for all $i = 1, \ldots, d - 1$. For the remaining $I$-web $I$, we further break it into composition
of two 3-way webs in biangles $B_5$ and $B_6$ as in Fig. 20 in particular, the juncture of $I$ at the common arc of $B_5$ and $B_6$ is $r''$.

By (BT1) of Prop. 5.9 one has

$$\text{Tr}_{B_4}([I, s_4|_{∂I}]) = \sum_{s_5, s_6} \text{Tr}_{B_5}([I \cap B_5, s_5]) \text{Tr}_{B_6}([I \cap B_6, s_6])$$

where the sum is over all states $s_5, s_6$ of the $A_2$-webs $I \cap B_5$ in $B_5$ and $I \cap B_6$ in $B_6$ compatible with $s_4|_{∂I}$, in the sense as in Prop. 5.9(BT1). In particular, any such $s_6$ assigns 1 to $x_d$. Note $I \cap B_5$ is a 3-way web in $B_6$, falling into (BT2-3) of Prop. 5.9, so $\text{Tr}_{B_5}([I \cap B_5, s_5]) \neq 0$ iff $s_5(y_1) = 2$ and $s_3(r'') = 1$.

Now, since $I \cap B_6$ is a 3-way web falling into (BT2-3) of Prop. 5.9 it follows that, in case $s_6(r'') = 1$, we have $\text{Tr}_{B_6}([I \cap B_6, s_6]) \neq 0$ iff $(s_4(r), s_6(r''))$ is either $(1, 2)$ or $(2, 1)$. It follows, that under the condition $s_4(x_d) = 1$, we have $\text{Tr}_{B_4}([I, s_4|_{∂I}]) \neq 0$ if $s_1(y_1) = 2$, $s_4(r) = 2$, $s_4(r') = 1$ hold or $s_4(y_1) = 2$, $s_4(r) = 1$, $s_4(r') = 2$ hold. In the former case the value of $\text{Tr}_{B_4}([I, s_4|_{∂I}])$ is 1, while in the latter case this value is $-1$.

Let $s_3$ be as above, and let $s_3$ be some compatible state of $K_d \cap B_3$ such that $\text{Tr}_{B_3}([K_d \cap B_3, s_3]) \neq 0$. That is, so far we are requiring that $s_3, s_4$ be compatible with $s_2$, and that $\text{Tr}_{B_4}([K_d \cap B_4, s_4]) \neq 0$. By multiplicativity of $\text{Tr}_{B_3}$, the value under $\text{Tr}_{B_3}$ of each of the $d$ components of $(K_d \cap B_3, s_3)$ must be nonzero. By compatibility, we have $s_3(r_i) = s_4(r_i) = 1$ for all $i = 1, \ldots, d - 1$; we also had $s_3(w_1) = 1$. Hence, the induction hypothesis applies for the $K_{d-1}$ component of $(K_d \cap B_3, s_3)$; so the value under $\text{Tr}_{B_3}$ of this $K_{d-1}$ component is nonzero iff $s_3$ assigns 1 to all $w_2, \ldots, w_{d-1}$ and 2 to $r$, and in this case, the value is 1. So $s_3(r) = 2$. By compatibility, $s_3(r) = s_4(r) = 2$, hence by the above observation on the I-web $I$, it must be $s_4(r') = 1$, and $\text{Tr}_{B_4}([I, s_4|_{∂I}]) = 1$. Again by compatibility, $s_4(r'') = s_4(r') = 1$. The edge component of $(K_d \cap B_3, s_3)$, which connects $r'$ and $w_d$, falls into Prop. 5.9(BT2-1), hence it has nonzero $\text{Tr}_{B_5}$ value iff $s_3$ assigns same value to $r'$ and $w_d$, hence it follows $s_3(w_d) = 1$.

To summarize, the unique pair of states $s_3, s_4$ whose corresponding summand in eq. (5.27) is nonzero assign the values 1 to $r_1, \ldots, r_{d-1}, r'$, $w_2, \ldots, w_{d}$, and the value 2 to $y_1, r$. For this choice of states, the summand is 1. This finishes proof of Lem. 5.36.

We go back to proof of Lem. 5.35, investigating the sum in eq. (5.26). Let $s_1, s_2$ be states of $P_d \cap B_1$ and $P_d \cap B_2$ compatible with $s$, and whose corresponding summand of eq. (5.26) is nonzero. Recall that we already know $s_1$ assigns value 1 to $z_1, z_2, \ldots, z_d, w_1$, and $s_2$ assigns 1 to $x_1, \ldots, x_d$. By multiplicativity of $\text{Tr}_{B_2}$, it follows that the value of $\text{Tr}_{B_2}$ at the $K_d$ component is nonzero. Since $s_2$ assigns 1 to $x_1, \ldots, x_d, w_1$, Lem. 5.36 that we just showed applies, and so the value of $\text{Tr}_{B_2}$ at this component is nonzero iff $s_2$ assigns 1 to all $w_2, \ldots, w_d$ and 2 to $y_1$, in which case the value is 1. By compatibility, $s_1$ assigns 1 to all $w_2, \ldots, w_d$. Since $s_1$ also assigns 1 to $z_2, \ldots, z_d$, the induction hypothesis (of our proof of Lem. 5.35) applies to the $P_{d-1}$ component of $P_d \cap B_1$, hence the value of $\text{Tr}_{B_1}$ at this component is nonzero iff $s_1$ assigns 2 to all $u_2, \ldots, u_d$, in which case the value is 1. By compatibility, $s_2$ assigns 2 to all $u_2, \ldots, u_d$. Each edge component of $(P_d \cap B_2, s_2)$, connecting $u_i$ and $y_i$, falls into Prop. 5.9(BT2-1), hence it has nonzero $\text{Tr}_{B_2}$ value iff $s_2$ assigns same value to $u_i$ and $y_i$, so $s_2(y_i) = 2$ for all $i = 2, \ldots, d$. To summarize, there is only one pair of $s_1, s_2$ contributing to the sum in eq. (5.26), which assign 2 to all $y_1, \ldots, y_d$, and the corresponding summand in the sum in eq. (5.26) is 1. This finishes the proof of Lem. 5.35.

We now prove Prop. 5.34.

**Proof of Prop. 5.34.** By the fact that $\text{Tr}_{A}$ is a ring homomorphism and from the additivity of the tropical coordinates $a_v$ (Lem. 3.32), it suffices to prove the statement for canonical $A_2$-web $W$ having a
single component. Denote by $e_1, e_2, e_3$ the sides of the triangle $\hat{t}$ appearing clockwise this order along $\partial t$. Denote the nodes of $Q_\Delta$ by $v_{e_1,1}$, $v_{e_2,2}$, $v_{e_3,3}$ as in Def 5.2.

Suppose that $W$ is a left turn corner arc in $t$, as in Prop 5.6 (CT2-1); so

$$
\text{Tr}_\Delta([W, s]) = (M^\text{in}_{t,\alpha}(x, s(x)); M^\text{left}_{t}(x, s(x)) (M^\text{out}_{t,\alpha+1}(y), s(y)),
$$
where $\alpha, x, y$ are as in Prop 5.6 (CT2-1). We used the fact that $M^\text{in}_{t,\alpha}$ and $M^\text{out}_{t,\alpha+1}$ are diagonal matrices. So, $M^\text{in}_{t,\alpha}$ involves variables $Z_{v_{e_1,1}}, Z_{v_{e_2,1}}$, but no others, $M^\text{left}_{t}$ involves $Z_{v_{e_3,3}}$ but no others, and $M^\text{out}_{t,\alpha+1}$ involves variables $Z_{v_{e_4,4}}, Z_{v_{e_5,5}}$, but no others. In view of (MM1), (MM2) of 4.2 note

$$
\text{Tr}_\Delta([W, 1]) = Z_{v_{e_1,1}} Z_{v_{e_2,1}} Z_{v_{e_3,1}} Z_{v_{e_4,1}} Z_{v_{e_5,1}}.
$$

In view of eq. (3.8), item (HT1) is satisfied. By inspection of the monodromy matrices in (MM1), (MM2), it follows this Laurent monomial indeed has higher or equal partial ordering than any other Laurent monomials appearing in $\text{Tr}_\Delta([W, s])$. Also, if $s(x) \neq 1$, then $(M^\text{in}_{t,\alpha}(x, s(x))$ has strictly lower partial ordering than $(M^\text{in}_{t,\alpha+1}(x, s(x))$. If $s(x) \neq 1$, then $(M^\text{out}_{t,\alpha+1}(y, s(y))$ has strictly lower partial ordering than $(M^\text{out}_{t,\alpha+1}(y, s(y))$. Thus (HT2) is satisfied.

When $W$ is a right turn corner arc as in Prop 5.6 (CT2-2), the proof goes completely parallel. We just have to check (HT1) precisely. Indeed,

$$
\text{Tr}_\Delta([W, 1]) = Z_{v_{e_1,1}} Z_{v_{e_2,1}} Z_{v_{e_3,1}} Z_{v_{e_4,1}} Z_{v_{e_5,1}},
$$

hence (HT1) is satisfied, in view of eq. (3.9).

Now suppose that $W$ is a degree $d$ pyramid $H_d$ for some nonzero $d \in \mathbb{Z}$. We only present how to deal with $d > 0$; the case $d < 0$ is completely parallel. To compute $\text{Tr}_\Delta([W, s])$, we decompose $t$ into one triangle $\hat{t}$ and one biangle $B$, as done in Lem 4.27. Let's say that the biangle is attached at the side $e_3$ of $t$. Let $e_1, e_2, e_3$ be sides of $\hat{t}$, and let $e'_3$ be the other side of $B$. Push all 3-valent vertices of $W$ to the biangle $B$ to form a stated $A_2$-web $(W', s')$ in $t$ as shown in Fig 21 and apply the state-sum formula in eq. (4.11) in Def 5.21 to define $\tilde{\text{Tr}}_\Delta(W', s') \in Z_\Delta$. By the isotopy invariance of the state-sum formula that we proved, we know $\text{Tr}_\Delta([W, s]) = \tilde{\text{Tr}}_\Delta(W', s')$ (if one wants to write down a proof explicitly, one may want to consider a genuine split ideal triangulation of $\Delta$, which has three biangles).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure21.png}
\caption{Pushing a pyramid $H_d$ from triangle to biangle (for $d = 3$)}
\end{figure}

Denote the endpoints of $W$ by $x_1, \ldots, x_d, y_1, \ldots, y_d, z_1, \ldots, z_d$, appearing clockwise in this order along $\partial t$, where $x_i$'s are in $e_1$, $y_i$'s in $e_2$, and $z_i$'s in $e_3$; see Fig 21. Inherit these labelings to $W'$, so that $z_1, \ldots, z_d$ lie in the outermost side $e'_3$ of the biangle $B$. Denote the junctures of $W'$ at the common arc $e_3$ of $\hat{t}$ and $B$ as $w_1, \ldots, w_d, u_1, \ldots, u_d$ as in Fig 21. Then

$$
\text{Tr}_\Delta([W, s]) = \sum_{s_1, s_2} \text{Tr}_{B'}([W' \cap B, s_1]) \text{Tr}_{\hat{t}}([W' \cap \hat{t}, s_2])
$$

where the sum is over all states $s_1, s_2$ of the $A_2$-webs $W' \cap B$ in $B$ and $W' \cap \hat{t}$ in $\hat{t}$ compatible with $s$, in the sense as in Prop 5.9 (BT1). Note that $W' \cap \hat{t}$ has 2$d$ components, which are corner arcs. Denote
the component connecting $w_i$ and $x_i$ by $W_{i}^{\text{left}}$, and the component connecting $u_i$ to $y_i$ by $W_{i}^{\text{right}}$, for $i = 1, \ldots, d$. So

\begin{equation}
\text{Tr}_{\hat{t}}([W' \cap \hat{t}, s_2]) = \prod_{i=1}^{d} \text{Tr}_{\hat{t}}([W_{i}^{\text{left}}, s_2]) \text{Tr}_{\hat{t}}([W_{i}^{\text{right}}, s_2]),
\end{equation}

where the $s_2$'s appearing in the right hand side mean appropriate restrictions. By Prop\,5.6 (CT2-1) and (CT2-2) we have

\begin{align}
\text{Tr}_{\hat{t}}([W_{i}^{\text{left}}, s_2]) &= (M_{i}^{\text{in}}_{\ell}, s_{2}(w_{i}), s_{2}(x_{i})) (M_{i}^{\text{left}})_{s_{2}(w_{i}), s_{2}(x_{i})} (M_{i}^{\text{out}})_{s_{2}(x_{i}), s_{2}(x_{i})}, \\
\text{Tr}_{\hat{t}}([W_{i}^{\text{right}}, s_2]) &= (M_{i}^{\text{in}}_{\ell}, s_{2}(u_{i}), s_{2}(y_{i})) (M_{i}^{\text{right}})_ {s_{2}(u_{i}), s_{2}(y_{i})} (M_{i}^{\text{out}})_{s_{2}(y_{i}), s_{2}(y_{i})}.
\end{align}

We first compute $\text{Tr}_{\Delta}([W, 1_{W}])$. Let $s_1, s_2$ be states of $W' \cap B$ and $W' \cap \hat{t}$ compatible with $1_{W}$ and such that the corresponding summand in eq\,(5.28) is nonzero. In particular, $s_1$ assigns 1 to $z_1, \ldots, z_d$, and $s_2$ assigns 1 to $x_1, \ldots, x_d, y_1, \ldots, y_d$. By multiplicativity of $\text{Tr}_{\hat{t}}$, the value under $\text{Tr}_{\hat{t}}$ of each component of $(W' \cap \hat{t}, s_2)$, i.e. eq\,(5.30) and eq\,(5.31), must be nonzero. Since $s_2(x_i) = 1$ and $M_{i}^{\text{left}}$ is upper triangular, it follows $s_2(w_i) = 1$, for all $i = 1, \ldots, d$. By compatibility, $s_1(1_i) = 1$ for all $i = 1, \ldots, d$. Since $W' \cap B$ is the $A_2$-web $P_{d}$, where $s_1$ assigns 1 to $w_1, w_d, z_1, \ldots, z_d$, Lem\,[5.35] applies. So $\text{Tr}_{B}([W' \cap B, s_1]) \neq 0$ implies $s_1(1_i) = 2$ for all $i = 1, \ldots, d$, in which case $\text{Tr}_{B}([W' \cap B, s_1]) = 1$. By compatibility, $s_2(1_i) = 2$ for all $i = 1, \ldots, d$. So there is a unique such pair of states $s_1, s_2$. For this pair of states, we have

\begin{equation}
\text{Tr}_{\hat{t}}([W_{i}^{\text{left}}, s_2]) = (Z_{e_{v_{3,2}}} \cdot Z_{e_{v_{3,3}}} \cdot Z_{e_{v_{3,1}}} \cdot Z_{e_{v_{2,2}}}) \text{Tr}_{\hat{t}}([W_{i}^{\text{right}}, s_2]) = (Z_{e_{v_{3,2}}} \cdot Z_{e_{v_{3,1}}} \cdot Z_{e_{v_{2,2}}})
\end{equation}

in view of (MM1), (MM2), (MM3) of \ref{4.2}. Thus

\begin{equation}
\text{Tr}_{\Delta}([W, 1_{W}]) = 1 \cdot (Z_{e_{v_{3,2}}} \cdot Z_{e_{v_{3,1}}} \cdot Z_{e_{v_{2,2}}}) = Z_{e_{v_{3,2}}}^{3d} \prod_{v=1}^{2} Z_{e_{v_{3,2}}}^{2d} Z_{e_{v_{2,2}}}^{2d}.
\end{equation}

One can easily verify that these powers are indeed 3 times the tropical coordinates of the degree $d$ pyramid $W = H_d$ (eq\.,\,[5.10]), hence (HT1) is satisfied.

Now let\,'s prove (HT2). Let $s$ be any state of $W$, and consider eq\.,\,[5.28]. Let $s_1, s_2$ be any pairs of states of $W' \cap B$ and $W' \cap \hat{t}$ compatible with $s$. The biangle factor $\text{Tr}_{B}([W' \cap B, s_1])$ is an integer, so it does not involve any generator of $Z_{e}$. The triangle factor is given by eq\.,\,[5.29], \,[5.30] and \,[5.31]. The only places where the variable $Z_{e_{3}}$ appears are $M_{i}^{\text{in}}$ of eq\.,\,[5.30] and $M_{i}^{\text{right}}$ of eq\.,\,[5.31]. In view of (MM2) and (MM3) of \ref{4.2} the highest power of $Z_{e_{3}}$ in $M_{i}^{\text{left}}$ is $Z_{e_{3}}^{2d}$ and that in $M_{i}^{\text{right}}$ is $Z_{e_{3}}$. Hence, the maximum possible power of $Z_{e_{3}}$ that can appear in a summand of the sum expression for $\text{Tr}_{\Delta}([W, s])$ in eq\.,\,[5.28] is $(Z_{e_{3}}^{2d}(Z_{e_{3}}^{d})^{d} = Z_{e_{3}}^{3d}$, which is the power of $Z_{e_{3}}$ that does appear in $\text{Tr}_{\Delta}([W, 1_{W}])$. Now, among eq\.,\,[5.30] and \,(5.31), the only place where the variables $Z_{e_{v_{3,1}}}$ or $Z_{e_{v_{2,2}}}$ is $M_{i}^{\text{out}}$ of eq\.,\,[5.30]. In view of (MM1) of \ref{4.2} the highest monomial in $Z_{e_{v_{3,1}}}$ and $Z_{e_{v_{2,2}}}$ appearing in $M_{i}^{\text{out}}$ is the $(1,1)$-entry ($M_{i}^{\text{out}})_{1,1} = Z_{e_{v_{3,1}}}, Z_{e_{v_{2,2}}}$, So the highest possible Laurent monomial in $Z_{e_{v_{3,1}}}$ and $Z_{e_{v_{2,2}}}$ that can appear in the sum expression for $\text{Tr}_{\Delta}([W, s])$ is $(Z_{e_{v_{3,2}}}, Z_{e_{v_{2,2}}})^{d}$, which may happen when $s$ assigns values 1 to all $x_1, \ldots, x_d$, e.g. in $\text{Tr}_{\Delta}([W, 1_{W}])$. If $s$ does not assign 1 to some $x_i$, then one notes that $(M_{i}^{\text{in}})_{s_{2}(x_{i}), s_{2}(x_{i})} = (M_{i}^{\text{out}})_{s_{2}(x_{i}), s_{2}(x_{i})}$ in eq\.,\,[5.30] is either $Z_{e_{v_{3,1}}}, Z_{e_{v_{2,2}}}$ or $Z_{e_{v_{3,1}}}, Z_{e_{v_{2,2}}}$, hence is of strictly lower partial order than $(M_{i}^{\text{in}})_{1,1} = Z_{e_{v_{3,1}}}, Z_{e_{v_{2,2}}}$.

Now, go back to the beginning, before we split $t$ into the triangle $\hat{t}$ and a biangle $B$ at the side $e_3$. This time, decompose $t$ into a triangle and a biangle where the biangle is at a different side than $e_3$. Apply the same arguments as we have seen so far, which is possible because the $A_2$-web $W = H_d$ has cyclic symmetry. Then we obtain similar results about the variables $Z_{e_{v_{3,1}}}$ and $Z_{e_{v_{2,2}}}$ lying in the side $e_3$, and also for the variables $Z_{e_{v_{3,1}}}$ and $Z_{e_{v_{3,2}}}$. So, for each $i = 1, 2, 3$, the highest possible Laurent monomial in $Z_{e_{v_{3,1}}}$ and $Z_{e_{v_{2,2}}}$ that can appear in the sum expression for $\text{Tr}_{\Delta}([W, s])$ is $(Z_{e_{v_{3,2}}}, Z_{e_{v_{2,2}}})^{d}$, which may happen when $s$ assigns values 1 to all endpoints of $W$ lying in $e_3$. If $s$ does not assign 1 to some endpoint in $e_3$, then one notes that the Laurent monomials in $Z_{e_{v_{3,1}}}$ and $Z_{e_{v_{2,2}}}$ appearing in the summands of eq\.,\,[5.28] have strictly lower partial order than $(Z_{e_{v_{3,1}}}, Z_{e_{v_{2,2}}})^{d}$. This completes the proof of (HT2).
Before proceeding to a proof of the general highest-term statement, Prop \[5.33\] we need one more easy lemma about biangles.

**Lemma 5.37.** Let \( W \) be a crossbar \( A_2 \)-web in a biangle \( B \) (Def \[5.14\]), and let \( 1_W \) be the state of \( W \) assigning the value \( 1 \in \{1, 2, 3\} \) to all the endpoints of \( W \). Then \( \text{Tr}_B([W, 1_W]) = 1 \).

**Proof of Lem. 5.37.** Note that \( W \) can be decomposed as composition of crossbar webs having exactly one crossbar, i.e. contains exactly two internal 3-valent vertices. That is, there is a finite collection of \( \text{Tr}_{B_i} \)'s, \( s_i \) of one edge component and a 3-way component. Let the junctures of \( W \) be made, \( \text{Tr}_{B_i} \) of \( B_i \) is a crossbar web in \( B_i \) with two internal 3-valent vertices. We have the state-sum formula eq. \[5.14\] with \( s = 1_W \). Denote the endpoints of \( W \) lying in one side of \( B_i \) by \( x_1, \ldots, x_n \), and the endpoints of \( W \) lying on the other side by \( y_1, \ldots, y_n \), so that \( x_j \) is connected to \( y_j \) by an edge of \( W \) for all \( j = 1, \ldots, n \) except for some two adjacent \( j \)’s.

Assume that \( s_i \) is a state of \( W \) assigning 1 to all \( x_1, \ldots, x_n \). Assume \( \text{Tr}_{B_i}([W, s_i]) \neq 0 \). By multiplicativity of \( \text{Tr}_{B_i} \) the value under \( \text{Tr}_{B_i} \) of each component of \( (W, s_i) \) must be nonzero. The edge component, connecting \( x_i \) and \( y_i \), falls into Prop \[5.9\] BT2-1, hence it follows \( s_i(y_i) = 1 \). For the single-crossbar component \( (L, s_i|_{\partial L}) \), connecting \( x_j, x_{j+1} \) and \( y_j, y_{j+1} \) as in the left picture of Fig. 22 we decompose \( B_i \) into biangles \( B_i' \) and \( B_i'' \) as shown in Fig. 22 so that each of \( L \cap B_i' \) and \( L \cap B_i'' \) consists of one edge component and a 3-way component. Let the junctures of \( L \) at the common side of \( B_i' \) and \( B_i'' \) be \( z_j, z_j, z_{j+1} \) as in Fig. 22.

![Figure 22. Single-crossbar A_2-web L in a biangle B_i and its decomposition](image)

By (BT1) of Prop \[5.9\] one has
\[ \text{Tr}_{B_i}([L, s_i|_{\partial L}]) = \sum_{s_i', s_i''} \text{Tr}_{B_i'}([L \cap B_i', s_i']) \text{Tr}_{B_i''}([L \cap B_i'', s_i'']) \]

where the sum is over all states \( s_i', s_i'' \) of the \( A_2 \)-webs \( L \cap B_i' \) in \( B_i' \) and \( L \cap B_i'' \) in \( B_i'' \) compatible with \( s_i|_{\partial L} \), in the sense as in Prop \[5.9\] BT1. Let \( s_i', s_i'' \) be a pair of states compatible with \( s_i|_{\partial L} \) and whose corresponding summand in the above sum is nonzero. In particular, \( s_i' \) assigns 1 to \( x_j \) and \( x_{j+1} \). Since \( \text{Tr}_{B_i'}([L \cap B_i', s_i']) \neq 0 \), by multiplicativity of \( \text{Tr}_{B_i'} \) it follows that values of \( \text{Tr}_{B_i'} \) of the components of \( (L \cap B_i', s_i') \) are nonzero. The edge component, connecting \( x_j \) and \( z_j \), falls into Prop \[5.9\] BT2-1, hence it follows \( s_i'(z_j) = s_i'(x_j) = 1 \). By similar arguments applied to \( B_i'' \), the \( \text{Tr}_{B_i''} \) values for the components of \( (L \cap B_i'', s_i'') \) are nonzero, and from the edge component we get \( s_i''(y_{j+1}) = s_i''(z_{j+1}) \) by compatibility at \( z_j \), we have \( s_i''(z_j) = s_i'(z_j) = 1 \). Since the 3-way component of \( L \cap B_i'' \) should have nonzero \( \text{Tr}_{B_i''} \) value, from Prop \[5.9\] BT2-3, it follows that \( s_i''(y_j) = 1, s_i''(z_j) = 2 \), in which case the \( \text{Tr}_{B_i''} \) value at this component is \( -1 \). By compatibility at \( z_j \), we have \( s_i''(z_j) = s_i''(z_{j+1}) = 2 \). Since the 3-way component of \( L \cap B_i'' \) has nonzero \( \text{Tr}_{B_i''} \) value, from Prop \[5.9\] BT2-3 it follows \( s_i''(z_{j+1}) = 1 \), in which case the \( \text{Tr}_{B_i''} \) value of this component is \( -1 \). By compatibility at \( z_{j+1} \), \( s_i''(z_{j+1}) = s_i''(z_{j+1}) = 1, \) and \( s_i''(y_{j+1}) = s_i''(z_{j+1}) = 1 \). So there is only one pair of \( s_i', s_i'' \) that has a nonzero contribution to the sum in eq. (5.32), in this case assigns 1 to \( y_j, x_{j+1} \), and the corresponding unique nonzero summand, hence the value \( \text{Tr}_{B_i}([L, s_i|_{\partial L}]) \), equals \(( -1 \cdot (-1) = 1) \).

We go back to the state-sum formula eq. \[5.14\] with \( s = 1_W \). Let \( J \) be any juncture-state of \( W \) compatible with \( s = 1_W \) whose corresponding summand in the sum in eq. \[5.14\] is nonzero. Then \( \text{Tr}_{B_i}([W_i, J|_{\partial W_i}]) \neq 0 \) for all \( i = 1, \ldots, n + 1 \). Look at the first biangle \( B_1 \) whose one side equals one side of \( B_i \); so \( J|_{\partial W_i} \) assigns 1 to all endpoints lying on this side of \( B_i \). Applying the above observation we made, \( \text{Tr}_{B_i}([W_i, J|_{\partial W_i}]) \neq 0 \) if \( J|_{\partial W_i} \) assigns 1 to all endpoints of \( W_i \), in which case \( \text{Tr}_{B_i}([W_i, J|_{\partial W_i}]) = 1 \). Then we go to biangle \( B_2 \), where now we know \( J|_{\partial W_i} \) assigns 1 to all endpoints lying in one side. Apply the above observation. Repeating this till the end, we deduce that \( J \) must assign \( 1 \) to all junctures, in which case the corresponding summand in eq. \[5.14\] is \( 1 \). So \( \text{Tr}_B([W, 1_W]) = 1 \) as desired. \( \blacksquare \)
We finally can provide a proof of Prop. \ref{5.33}.

**Proof of Prop. \ref{5.33}** Let $W$ be a reduced non-elliptic $A_2$-web in a generalized marked surface $\mathcal{S}$ in a canonical position with respect to a split ideal triangulation $\Delta$ of $\mathcal{S}$. We use the state-sum formula eq. (5.11) of Def. \ref{5.24}. Let $J_0$ be the $\Delta$-juncture-state of $W$ that assigns the value $1 \in \{1, 2, 3\}$ to all junctures. Since each $W \cap B$ is a crossbar $A_2$-web in a biangle $B$, by Len. \ref{5.37} the biangle factor $\text{Tr}_B([W \cap B, (J_0)_B])$ equals 1. For each triangle $t$ of $\Delta$, by Prop. \ref{5.34}(HT1) the triangle factor $\hat{\text{Tr}}_t(W \cap \hat{\ell}, (J_0)_t)$ equals $\prod_{v \in V(Q_\Delta) \cap t} Z_{t,v}^{3a_v(W)}(W^{\hat{\ell} t}) \in \mathbb{Z}_t$. For the node $v_i$ of $Q_\Delta$ lying in the interior of $t$, we have $Z_{t,v_i}^{3a_v(W)}(W^{\hat{\ell} t}) = \hat{Z}_{t,v_i}^{3a_v(W)}(W) \in \mathbb{Z}_t$. Now let $v$ be a node of $Q_\Delta$ lying in an internal arc of $\Delta$, say a common side of triangles $t$ and $v$. Let $B$ be the biangle in between the triangles $\hat{\ell}$ and $\hat{\gamma}$. By the well-definedness of the tropical coordinates at arcs of $\Delta$, note that $a_v(W \cap \hat{\ell}) = a_v(W \cap \hat{\gamma}) = a_v(W)$. Then note $Z_{t,v}^{3a_v(W^{\hat{\ell} \gamma})} Z_{v,\hat{\gamma}}^{3a_v(W^{\hat{\ell} \gamma})} = \hat{Z}_{t,v}^{3a_v(W)} \in \mathbb{Z}_t$. Therefore, the summand of eq. (5.11) corresponding to $J_0$ exactly equals $\prod_{v \in V(Q_\Delta)} \hat{Z}_{t,v}^{3a_v(W)}$. Now, let $J$ be any $\Delta$-juncture-state of $W$ different from $J_0$. Note $\text{Tr}_B([W \cap B, J_B])$ is an integer, hence does not involve any generator of $Z_\Delta$. By Prop. \ref{5.34}(HT2), for each triangle $t$ of $\Delta$, the triangle factor $\hat{\text{Tr}}_t(W \cap \hat{\ell}, (J_0)_t)$ in $\mathbb{Z}_t$ only involves Laurent monomials $Z_t$ having lower or equal order than $\hat{\text{Tr}}_t(W \cap \hat{\ell}, (J_0)_t)$ which is a single monomial in $Z_t$. Also by Prop. \ref{5.34}(HT2), there exists a triangle $t$ such that the triangle factor $\hat{\text{Tr}}_t(W \cap \hat{\ell}, (J_0)_t)$ in $\mathbb{Z}_t$ only involves Laurent monomials of $Z_t$ having strictly lower order than $\hat{\text{Tr}}_t(W \cap \hat{\ell}, (J_0)_t)$. This finishes the proof of Prop. \ref{5.33}. \hfill \blacksquare

### 5.6. Relationship with basic semi-regular function

In order to prove Prop. \ref{4.18} and Prop. \ref{4.19} we should translate the results from the previous subsection about the $\text{SL}_3$ classical (state-sum) trace into those of basic semi-regular functions $\mathbb{H}_\Delta^\pm$ into $\mathcal{S}$. Then $\mathbb{H}_\Delta^\pm(\ell) \in C^\infty(\mathcal{X}_\mathbb{H}^\pm, \mathcal{S})$.

**Definition 5.38.** For each ideal triangulation $\Delta$ define

$$\iota_\Delta : \mathbb{Z}_\Delta \to C^\infty(\mathcal{X}_\mathbb{H}^\pm, \mathcal{S})$$

as the unique ring homomorphism sending $\mathbb{Z}_\Delta$ to $C^\infty(\mathcal{X}_\mathbb{H}^\pm, \mathcal{S})$, $\forall \nu \in V(Q_\Delta)$.

**Proposition 5.39** (SL$_3$ classical trace and basic semi-regular function). Let $\mathcal{S}$ be a triangulable generalized marked surface, $\Delta$ an ideal triangulation of $\mathcal{S}$, and $\hat{\Delta}$ a split ideal triangulation of $\Delta$. Let $\ell \in \mathcal{S}_0^\Delta(\mathcal{S}; \mathbb{Z})$ be an $A_2$-lamination in $\mathcal{S}$ that can be represented as a reduced non-elliptic $A_2$-web $W$ in $\mathcal{S}$ such that

- (E1) $W$ has no external vertices,
- (E2) $W$ contains no peripheral loops,
- (E3) All weights (of components of $W$) are 1.

Then

$$\mathbb{H}_\Delta^\pm(\ell) = \iota_\Delta \text{Tr}_\Delta; \mathbb{S}([W; \mathbb{O}])$$

**Proof of Prop. \ref{5.39}** First, not precisely being fit to the current situation, assume that $W$ is an $A_2$-web consisting of a single oriented simple loop, say $\gamma$, which is not necessarily simple. By applying an isotopy, we may assume that $\gamma$ meets $\Delta$ transversally in a minimal possible number of points. We apply the construction in §4.3 of the monodromy matrix for $\gamma$. The $\Delta$-junctures of $\gamma$, i.e. the points of $\Delta \cap \gamma$, divide $\gamma$ into segments $\gamma_1, \ldots, \gamma_N$, so that $\gamma$ is the concatenation $\gamma = \gamma_1 \gamma_2 \cdots \gamma_N$. The $\gamma_i$ in a triangle of $\Delta$ work as a triangle segment, and $\gamma_i$ in a biangle of $\Delta$ work as a juncture segment, so that $f_\gamma^+ = \text{tr}(M_{\gamma_1} \cdots M_{\gamma_N})$. Also, the sequence $\gamma_1, \ldots, \gamma_N$ alternates between triangle segments and juncture segments. Let $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \{1, 2, 3\}^N$. Denote by $\bar{M}_{\gamma_i}$ the matrix defined by the same formula as $M_{\gamma_i}$, where the entries are thought of as elements of $Z_\Delta$, so that $M_{\gamma_i} = \iota_\Delta \bar{M}_{\gamma_i}$, where $\iota_\Delta$ applied to a matrix means $\iota_\Delta$ applied to each entry. Denoting by $(\bar{M}_{\gamma_i})_{\alpha_i, \alpha_{i+1}}$ the $(\alpha_i, \alpha_{i+1})$-th entry of $\bar{M}_{\gamma_i}$, as usual, we have

$$f_\gamma^+ = \iota_\Delta \text{tr}^{\alpha_{N+1} = \alpha_1} \iota_\Delta \sum_{\alpha_1, \ldots, \alpha_N}^\{1, 2, 3\} \iota_\Delta \prod_{i=1}^N \bar{M}_{\gamma_i}^{\alpha_i, \alpha_{i+1}}$$

where $\alpha_{N+1} := \alpha_1$. View $\alpha_i$ as being associated to the $\Delta$-juncture of $\gamma$ that is the initial point of $\gamma_i$ (which is the terminal point of $\gamma_{i-1}$; let $\gamma_0 := \gamma_N$). So $\bar{\alpha}$ can be viewed as a $\Delta$-juncture-state $J = J^{\bar{\alpha}}$.
of $\gamma$, and the above sum is over all $\hat{\Delta}$-juncture-states $J$. For each juncture segment $\gamma_i$, note $\hat{M}_{\gamma_i}$ is diagonal, so $(\hat{M}_{\gamma_i})_{\alpha_i,\alpha_{i+1}} = 0$ unless $\alpha_i = \alpha_{i+1}$. Hence only the $\hat{\Delta}$-juncture-states $J^{\hat{\Delta}}$ that are biangle-coherent may contribute to the above sum, where we say that $\hat{\Delta}$-juncture-state is biangle-coherent if it assigns the same value to the two endpoints of each segment of $\gamma$ living in a biangle. For each juncture segment $\gamma_i$ as in Fig. 7 we have $\hat{M}_{\gamma_i} = \text{diag}(Z_1 Z_2^2, Z_2 Z_1^{-1}, Z_1^{-2} Z_2^{-1})$. Suppose the initial and terminal points of $\gamma_i$ lives in triangles $\hat{t}$ and $\bar{t}$ of $\hat{\Delta}$ corresponding to triangles $t$ and $r$ of $\Delta$. Define

$$\hat{M}_{\gamma_i}^{\text{ini}} := \text{diag}(Z_1 Z_2^2, Z_2^{-1} Z_1^{-2}, Z_1^{-2} Z_2^{-1}), \quad \hat{M}_{\gamma_i}^{\text{ter}} := \text{diag}(Z_{r,1} Z_{r,2}^2, Z_{r,2}^{-1} Z_{r,1}^{-2}, Z_{r,1}^{-2} Z_{r,2}^{-1}),$$

so that $\hat{M}_{\gamma_i} = \hat{M}_{\gamma_i}^{\text{ini}} \hat{M}_{\gamma_i}^{\text{ter}}$ and $(\hat{M}_{\gamma_i})_{\alpha_i,\alpha_i} = (\hat{M}_{\gamma_i}^{\text{ini}})_{\alpha_i,\alpha_i} (\hat{M}_{\gamma_i}^{\text{ter}})_{\alpha_i,\alpha_i}$. Meanwhile, for each triangle segment $\gamma_j$, living in triangle $\hat{t}$ (or $t$), one observes from Def. 5.21 (eq.(5.9)) that

$$\hat{\text{Tr}}_i(\gamma_j, (J^{\hat{\Delta}})|_{\partial \gamma_j}) = (\hat{M}_{\gamma_j}^{\text{ter}} \hat{M}_{\gamma_{j+1}}^{\text{ini}})_{\alpha_j,\alpha_{j+1}} \in Z_r.$$

Now, assuming that $\gamma_i$ is a triangle segment (so that $\gamma_N$ is a juncture segment), for each biangle-coherent $\hat{\Delta}$-juncture-state $J^{\hat{\Delta}}$ (so that $\alpha_2 = \alpha_3, \alpha_4 = \alpha_5, \ldots, \alpha_{N-2} = \alpha_{N-1}, \alpha_N = \alpha_1$), observe

$$\prod_{i=1}^N (\hat{M}_{\gamma_i})_{\alpha_i,\alpha_{i+1}} = (\hat{M}_{\gamma_1})_{\alpha_1,\alpha_2} (\hat{M}_{\gamma_2})_{\alpha_2,\alpha_3} \cdots (\hat{M}_{\gamma_N})_{\alpha_N,\alpha_1} = (\hat{M}_{\gamma_1})_{\alpha_1,\alpha_2} (\hat{M}_{\gamma_2}^{\text{ter}})_{\alpha_2,\alpha_3} \cdots (\hat{M}_{\gamma_N}^{\text{ini}})_{\alpha_N,\alpha_1}.$$

Meanwhile, consider the state-sum formula in eq.(5.11), which we can apply because $W = \gamma$ is in a good position with respect to $\hat{\Delta}$ (Def.5.19). The $A_2$-web $W \cap B$ in each biangle $B$ consists of edge components of type as in Prop 5.21 BT2-1, hence it follows that each $\hat{\Delta}$-juncture-state $J$ whose corresponding summand in the sum in eq.(5.11) is nonzero is biangle-coherent, in the sense as defined above, and for such $J$’s the biangle factors $\hat{\text{Tr}}_B([W \cap B, J_B])$ are 1. So it follows that

$$f_j^{\hat{\Delta}} = \iota_\Delta \sum_{\alpha} \prod_i \hat{\text{Tr}}_i(\gamma \cap \hat{t}, (J^{\hat{\Delta}})|_{\partial (\gamma \cap \hat{t})}) = \iota_\Delta \sum_{\alpha} \prod_i \hat{\text{Tr}}_i(W \cap \hat{t}, J_i) = \iota_\Delta \text{Tr}_{\hat{\Delta},\hat{\Sigma}}([W, \hat{\Sigma}]),$$

where the middle equality holds because both are sums over biangle-coherent juncture-states.

Now, coming back the original situation of the problem, let $W$ be a reduced non-elliptic $A_2$-web in $\hat{\Sigma}$ representing $\ell \in \mathcal{A}_L(\hat{\Sigma}; \hat{\mathbb{Z}})$, satisfying the conditions (E1), (E2) and (E3) as in the statement of the present proposition. Since $\ell$ has no peripheral loop component, by eq.(4.21) we have $\Gamma_{\text{PGL}_3}(\ell) = \Psi^*((F^*\Phi(W))(\mathbb{R}))$, where $W$ is viewed as an element of the $A_2$-skew algebra $\mathcal{S}(\hat{\Sigma}; \hat{\mathbb{Z}})$, which is naturally a subalgebra of the stated $A_2$-skew algebra $\mathcal{S}(\hat{\Sigma}; \hat{\mathbb{Z}})$. Thus we should prove the equality

$$\Psi^*((F^*\Phi(W))(\mathbb{R})) = \iota_\Delta \text{Tr}_{\Delta,\Sigma}([W, \hat{\Sigma}]),$$

for all $A_2$-webs $W$ satisfying conditions (E1), (E2) and (E3). Note that all maps $\Psi^*, F^*, \Phi$, evaluation at $\mathbb{R}$, $\iota_\Delta$, and $\text{Tr}_{\Delta,\Sigma}$ are ring homomorphisms, and that the map $\Phi$ defined on $\mathcal{S}(\hat{\Sigma}; \hat{\mathbb{Z}})$ and the map $\text{Tr}_{\Delta,\Sigma}$ defined on $\mathcal{S}(\hat{\Sigma}; \hat{\mathbb{Z}})$ respect the defining $A_2$-skew relations. It is known that $\mathcal{S}(\hat{\Sigma}; \hat{\mathbb{Z}})$ is generated by oriented loops, so it suffices show the above equality when $W$ is an oriented loop $\gamma$. In this case, the left-hand-side $\Psi^*((F^*\Phi(W))(\mathbb{R}))$ equals the trace-of-monodromy $f_\gamma^+ (\text{Def.4.14})$, by construction. And we showed $f_\gamma^+ = \iota_\Delta \text{Tr}_{\Delta,\Sigma}([W, \hat{\Sigma}])$ above.

Before proceeding, we state one immediate but non-trivial consequence of the proof of Prop 5.39

**Corollary 5.40** (SL$_3$ classical trace is independent on triangulation). Let $W$ be an $A_2$-web in a triangulable generalized marked surface $\hat{\Sigma}$, without external vertices. Let $\Delta, \Delta'$ be ideal triangulations of $\hat{\Sigma}$. Then

$$\iota_\Delta \text{Tr}_\Delta([W, \hat{\Sigma}]) = \iota_{\Delta'} \text{Tr}_{\Delta'}([W, \hat{\Sigma}]).$$
Remark 5.41. In order to relate $\text{Tr}_\Delta(W; \mathcal{O}) \in \mathcal{Z}_\Delta$ and $\text{Tr}_\Delta(W; \mathcal{O}) \in \mathcal{Z}_{\Delta'}$ directly, one first needs to come up with a coordinate change isomorphism between (fields of fractions of) some ‘balanced’ subalgebras of the algebras $\mathcal{Z}_\Delta$ and $\mathcal{Z}_{\Delta'}$; compare with the $\text{SL}_2$ case studied in [BW11] [H10]. We do not undertake that task in the present paper. We also conjecture that Corollary 5.40 holds also for stated $A_2$-webs $(W, s)$ (in particular, with $\partial W \neq \emptyset$).

After a long journey, we finally prove the following.

Proof of Proposition 4.18 and Proposition 4.19. Let $\mathcal{S}$ be a punctured surface, $\Delta$ be an ideal triangulation of $\mathcal{S}$, and let $\ell \in A_1(\mathcal{S}; \mathbb{Z})$. One can write $\ell = \ell_1 \cup \ell_2$ as disjoint union, where $\ell_1$ consists only of peripheral loops, and $\ell_2$ has no peripheral loop. Recall eq. (4.17), which says $\Pi^+_{\text{PGL}_3}(\ell_1) = \prod_{v \in \mathcal{V}(Q_\Delta)} X_v^{a_v(\ell_1)}$.

Meanwhile, $\ell_2$ can be represented as an $A_2$-web $W_2$ satisfying (E1), (E2) and (E3) of Proposition 5.39, hence eq. (5.33) holds for $\ell_2$: $\Pi^+_{\text{PGL}_3}(\ell_2) = \iota_\Delta \text{Tr}_{\Delta, \mathcal{S}}([W_2; \emptyset])$.

Since $\text{Tr}_{\Delta, \mathcal{S}}([W_2; \mathcal{O}]) \in \mathcal{Z}_\Delta$ (Proposition 5.22), and in view of Definition 5.38 it follows that $\Pi^+_{\text{PGL}_3}(\ell_2)$ can be written as a Laurent polynomial in $\{X_v^{1/3} | v \in \mathcal{V}(Q_\Delta)\}$ with integer coefficients. By Proposition 5.33, such a Laurent polynomial expression can be chosen so that there is a unique highest order Laurent monomial, which is $\prod_{v \in \mathcal{V}(Q_\Delta)} X_v^{a_v(\ell_2)}$ and is of coefficient $1$. And by Proposition 5.32, such a Laurent polynomial expression can be chosen so that other Laurent monomials appearing in this expression are $\prod_{v \in \mathcal{V}(Q_\Delta)} X_v^{a_v(\ell_2)}$ times some integer powers of $X_v$’s. By (partial) multiplicativity of $\Pi^+_{\text{PGL}_3}$ (see Lemma 4.9), we have $\Pi^+_{\text{PGL}_3}(\ell) = \Pi^+_{\text{PGL}_3}(\ell_1) \Pi^+_{\text{PGL}_3}(\ell_2)$, hence it follows that $\Pi^+_{\text{PGL}_3}(\ell)$ can be written as a Laurent polynomial $\{X_v^{1/3} | v \in \mathcal{V}(Q_\Delta)\}$ with integer coefficients, so that $\prod_v X_v^{a_v(\ell_1)} \prod_v X_v^{a_v(\ell_2)} = \prod_v X_v^{a_v(\ell)}$ (cf. Lemma 3.32) is the unique highest order term with coefficient $1$, while the other terms are $\prod_v X_v^{a_v(\ell)}$ times some integer powers of $X_v$’s.

At last, this justifies our proof of the main theorem, Theorem 4.23 given in the previous section.

6. Conjectures

6.1. List of conjectures. We state some naturally arising conjectures and questions, besides those which already appeared in the text.

Question 6.1. If a rational function $f$ on $\mathcal{X}_{\text{PGL}_3, \mathcal{S}}$ is regular on all the cluster $\mathcal{X}$-charts for ideal triangulations of $\mathcal{S}$, then is it regular on all cluster $\mathcal{X}$-charts, hence is a regular function on $\mathcal{X}_{\text{PGL}_3}$? That is, does universality Laurent for triangulations imply universally Laurent for all cluster $\mathcal{X}$-charts?

It might be convenient to have an affirmative answer to the above, but maybe it is more natural for us to consider more general class of ideal triangulations, called tagged ideal triangulations (in particular incorporating the self-folded triangles). For these, the construction of Fock-Goncharov $\mathcal{X}$-coordinates must be modified; see [FG06] §10.7 for a discussion, and [AB20] for SL2 case. Next, we consider:

Conjecture 6.2 (Laurent coefficient positivity). For each $\ell \in A_1(\mathcal{S}; \mathbb{Z})$, the basic regular function $\mathbb{I}(\ell) \in \mathcal{O}(\mathcal{X}_{\text{PGL}_3, \mathcal{S}})$ can be written, for any cluster $\mathcal{X}$-chart, as a Laurent polynomial in the generators with non-negative integer coefficients.

We have a partial result, due to our state-sum formula:

Proposition 6.3. Conjecture 6.2 holds for a cluster $\mathcal{X}$-chart for an ideal triangulation $\Delta$ of a punctured surface $\mathcal{S}$, for each $\ell$ that can be represented by a non-elliptic $A_2$-web in a canonical position with respect to $\hat{\Delta}$ such that there is at most one internal 3-valent vertex in each triangle of $\hat{\Delta}$ (i.e. degree of the pyramid in each triangle is 1).

To prove the full version for cluster $\mathcal{X}$-chart for all ideal triangulations, one must show that the values of the $\text{SL}_3$ classical trace for all pyramids $H_2$ in a triangle are Laurent polynomials with non-negative coefficients; for that, one should use results established in [FG06]. Once Conjecture 6.2 is settled, then one can try to check whether $\mathbb{I}(\ell)$ are extremal, i.e. cannot be a sum of two universally positive Laurent functions (as predicted in Conjecture 12.3 of [FG06]).
Another perhaps more important kind of positivity is the following:

**Question 6.4** (structure constant positivity). Does our $A_2$-bangle basis of $\mathcal{O}(\mathcal{X}_{SL_3,\mathfrak{g}})$ have non-negative structure constants? Namely, are the structure constants $c(\ell, \ell'; \ell'')$ in eq. (4.19) of Thm 4.23 are non-negative?

One can ask similar question for the basis of $\mathcal{O}(\mathcal{X}_{SL_3,\mathfrak{g}})$ in Def 4.7 or the basis of $\mathcal{O}(\mathcal{X}_{SL_3,\mathfrak{g}})$ as in Cor 4.3 and Cor 4.5. Then the question is related to a similar question for the basis of the $A_2$-skein algebra $\mathcal{S}(\mathfrak{g}; \mathbb{Z})$, consisting of non-elliptic $A_2$-webs (SW07). Note that such a positivity holds true for Fock-Goncharov’s basis of $\mathcal{O}(\mathcal{X}_{PGL_3,\mathfrak{g}})$, and a core part of the proof relies on the corresponding positivity of a certain basis of the $A_1$-skein algebra (i.e. the usual Kauffman bracket skein algebra), proved by Dylan Thurston [T14]. Notice that one important aspect of this statement and proof for $A_1$ (or SL$_2$) is that the positive basis is not a bangle basis, but is a bracelet basis (see Def 4.4). So we propose a new basis that is an $A_2$-analogue of the bracelet basis of Kauffman bracket skein algebra.

**Definition 6.5.** Define a map

$$\mathcal{I}_{SL_3}^b : \mathcal{A}_L(\mathfrak{g}; \mathbb{Z}) \to \mathcal{O}(\mathcal{X}_{SL_3,\mathfrak{g}})$$

as in Def 4.7 with the following modification. Let $\ell \in \mathcal{A}_L(\mathfrak{g}; \mathbb{Z})$, and let $\ell = \ell_1 \cup \cdots \cup \ell_n$ be the disjoint union of single-component $A_2$-webs that are mutually non-isotopic. Define $\mathcal{I}_{SL_3}^b(\ell_1)$ as $\mathcal{I}_{SL_3}(\ell_1)$ unless $\ell_1$ consists of a non-peripheral loop $\gamma_1$, say with weight $k_1 \in \mathbb{Z}_{>0}$, in which case we define $\mathcal{I}_{SL_3}(\ell_1) := F^*\Phi([W^{(k_1)}_{\gamma_1}]),$

where $W^{(k_1)}_{\gamma_1}$ is as in Def 4.4, that is, as the trace of monodromy along $\gamma_1^{k_1}$, the $k_1$-time-going-along-$\gamma_1$. Define $\mathcal{I}_{SL_3}(\ell) := \mathcal{I}_{SL_3}(\ell_1) \cdots \mathcal{I}_{SL_3}(\ell_n)$.

**Conjecture 6.6.** $\mathcal{I}_{SL_3}^b$ is injective, and its image forms a basis of $\mathcal{O}(\mathcal{X}_{SL_3,\mathfrak{g}})$.

One should be able to prove this by showing an analogous statement for the $A_2$-skein algebra, by observing that the ‘base change’ transformation between $\mathcal{I}_{SL_3}$ and $\mathcal{I}_{SL_3}^b$ is “upper triangular”. Let’s denote the resulting basis $\mathcal{I}_{SL_3}^b(\mathcal{A}_L(\mathfrak{g}; \mathbb{Z}))$ of $\mathcal{O}(\mathcal{X}_{SL_3,\mathfrak{g}})$ by $A_2$-bracket basis of $\mathcal{O}(\mathcal{X}_{SL_3,\mathfrak{g}})$.

**Conjecture 6.7.** By mimicking the present paper’s construction $\mathcal{I}_{SL_3} \sim \mathcal{I}_{PGL_3}^b \sim 1$ to $\mathcal{I}_{SL_3}^b$, one can obtain a map

$$\mathcal{I}_{SL_3}^b : \mathcal{A}_{SL_3,\mathfrak{g}}(\mathbb{Z}^t) \to \mathcal{O}(\mathcal{X}_{PGL_3,\mathfrak{g}}),$$

which is injective and whose image forms a basis of $\mathcal{O}(\mathcal{X}_{PGL_3,\mathfrak{g}})$.

We call this conjectural basis $\mathcal{I}_{SL_3}^b(\mathcal{A}_{SL_3,\mathfrak{g}}(\mathbb{Z}^t))$ the $A_2$-bracket basis of $\mathcal{O}(\mathcal{X}_{PGL_3,\mathfrak{g}})$, by a slight abuse of notation. We then formulate the positivity conjecture.

**Conjecture 6.8.** $\mathcal{I}_{SL_3}^b$ satisfies the Laurent coefficient positivity as in Conjecture 6.2.

**Conjecture 6.9.** $\mathcal{I}_{SL_3}^b$ satisfies the structure constant positivity as in Question 6.4.

Meanwhile, the work of Gross-Hacking-Keel-Kontsevich [GHKK18] yields a duality map

$$\mathcal{I}_{GHKK} : \mathcal{A}_{Q_\Delta}(\mathbb{Z}^t) \to \mathcal{O}(\mathcal{X}_{Q_\Delta}),$$

where $\mathcal{A}_{Q_\Delta}$ and $\mathcal{X}_{Q_\Delta}$ denote the cluster $\mathcal{A}$- and $\mathcal{X}$- varieties for the quiver mutation equivalence class $Q_\Delta$ of the exchange matrix of the 3-triangulation quiver $Q_\Delta$ for a triangulation $\Delta$ of a generalized marked surface $\mathfrak{g}$. The definition is quite combinatorial and computationally much involved (and abstract), and the existence is guaranteed by the result of Goncharov-Shen [GS18]. Of course, a natural question is whether their duality map equals ours, which is much more geometric and intuitive.

**Conjecture 6.10.** $\mathcal{I}_{GHKK}$ coincides with $\mathcal{I}_{SL_3}^b$ (or with $\mathcal{I}$).

In fact, even for SL$_2$-PGL$_2$ (or $A_1$), Gross-Hacking-Keel-Kontsevich’s duality map is not known to coincide with Fock-Goncharov’s [FG06], except for couple of small surfaces. For SL$_3$-PGL$_3$ (or $A_2$), one may get some evidence by computing some examples.

Another natural direction of research is the quantization of our duality map $\mathcal{I}$, in the flavor of [AKL17] which quantized Fock-Goncharov’s SL$_2$-PGL$_2$ duality map.

**Conjecture 6.11.** There is a quantum version $\mathcal{I}^q$ of $\mathcal{I}_{SL_3}^b$ and/or $\mathcal{I}$. 
More precise formulation would require us to recall the notion of (non-commutative) Fock-Goncharov algebra. In fact, this task of quantization was partially carried out by Douglas in his thesis \cite{D20}; there, functions for oriented loops are quantized. The puncture function $\prod_{\ell} X^{m}_{\ell}$ (when $\ell$ is an $A_2$-lamination consisting of a single peripheral loop surrounding a puncture) was not known there, but can be canonically quantized by its Weyl-ordered product, since it has only one term. More involved is figuring out how general $\mathbb{I}(\ell)$ or $\mathbb{I}^{br}(\ell)$ should be quantized, i.e. what non-commutative Laurent polynomials should deform these functions. Like in \cite{AK17}, one of the core aspect of quantization should be captured as an SL$_3$ quantum trace map, which should be a map defined on stated A$_2$-skein algebras and which should be a quantum version of the SL$_3$ classical trace studied in the present paper. We believe the ideas developed in \cite{FG06} should be helpful. In particular, the SL$_3$ biangle quantum trace should be obtained from Higgins’ work \cite{H20} (which was on the quantum non-commutative stated A$_2$-skein algebras) by taking the counit and applying a twist; it seems that finding an appropriate quantum twist is one of the first challenges to encounter. The values for left, right and U-turn arcs in a triangle should be as already studied in \cite{D20}. The rest of the arguments and flows of logic can be mimicked from ours given in \cite{MR20}.

Conjecture 6.12.

1. Davisdon-Mandel’s quantum duality map \cite{DM19} is a quantum version of our $\mathbb{I}^{qu}$ (or $\mathbb{I}$).
2. Lê’s SL$_n$ quantum trace map $\mathbb{I}$ is related to a quantum version of our $\mathbb{I}^{br}$ (or $\mathbb{I}$), when $n = 3$.

Another possible further topic is on the $A_2$-laminations of tropical $\mathcal{V}$-type. Note that for $A_1$ theory, or SL$_2$-PGL$_2$ theory, there are dual notions of $\mathcal{V}$-laminations and $\mathcal{V}$-laminations \cite{FG06}. Our $A_2$-lamination is an $A_2$-analog of Fock-Goncharov’s $\mathcal{V}$-laminations, so there should also be the $A_2$-counterpart for Fock-Goncharov’s $\mathcal{V}$-laminations. We expect that they should be similar to the $A_2$-laminations studied in the present paper but different, and the tropical $\mathcal{V}$-coordinates should be defined in some other interesting way.

Lastly, an obvious way to explore is the higher dimensional generalization to SL$_m$-PGL$_m$, which is dealt with in the following subsection.

6.2. Generalization to SL$_m$-PGL$_m$. There are two possible approaches to $A_{m-1}$-webs. One is Sikora’s approach \cite{S01} using graphs with $n$-valent sinks or sources, to define $A_{m-1}$-web and $A_{m-1}$-skein algebra; there is analog of Prop 4.2. If one finds a basis of this $A_{m-1}$-skein algebra, then together with the puncture functions $(\pi_p)_i : \mathcal{K}_{SL_m, s} \to \mathbb{G}_m$ as in eq. (4.3), one would obtain a basis of $\mathcal{O}(\mathcal{K}_{SL_m, s})$ as in Def 4.7 and Prop 4.8. To proceed to $\mathcal{O}(\mathcal{K}_{SL_m, s})$ one needs tropical coordinates for basic $A_{m-1}$-skeins. For such a task, we propose to use the other approach to $A_{m-1}$-webs by Murakami-Ohtsuki-Yamada \cite{MOY98} and others \cite{CKM14}, in terms of graphs with 3-valent graphs but with edges equipped with an orientation and a label in $\{1, \ldots, m\}$, so that the signed sum at each 3-valent vertex is divisible by $m$. We suggest the following definition.

**Definition 6.13 (A$_{m-1}$-web and A$_{m-1}$-lamination).** Let $\mathcal{S}$ be a generalized marked surface. Let $m \in \mathbb{Z}_{\geq 2}$.

- Define the set of $m$-labels

$$m = \{1, 2, \ldots, m\},$$

and for each $i \in m$ define the inverted $m$-label

$$\tilde{i} := m - i.$$
• For a (smooth) (closed or non-closed) curve $\gamma$ in $\mathcal{G}$, an oriented $m$-label on $\gamma$ is the choice of an equivalence class of a pair $(o, i)$ of an orientation $o$ of $\gamma$ and an $m$-label $i \in m$, where $(o, i)$ and $(o', i')$ is equivalent iff $\gamma' = \overline{\gamma}$ (the opposite orientation of $o$) and $i' = i$.

• An $A_{m-1}$-web $W$ in $\mathcal{G}$ is a finite union of its components, where a component is one of

(mW1) a loop (i.e. a closed curve) in $\mathcal{G}$ with an oriented $m$-label,
(mW2) a connected graph in $\mathcal{G}$ such that each vertex is either a 1-valent vertex lying in $\mathcal{G}$ or a 3-valent vertex lying in the interior of $\mathcal{G}$, with each edge equipped with an oriented $m$-label, such that each edge connects two distinct vertices, and at each 3-valent vertex, if the oriented $m$-labels of the edges of $W$ meeting this vertex are represented by pairs $(o, i)$ so that $o$ is going toward this vertex, then the sum of the $m$-labels $i$ for these three edges is divisible by $m$,

such that

(mW3) every self-intersection of $W$ is a transverse intersection lying in the interior of $\mathcal{G}$, and is a double intersection (called a crossing) unless it is a 3-valent vertex of a constituent graph.

• An $A_{m-1}$-web $W$ is non-elliptic if

(mNE1) $W$ has no crossings;
(mNE2) $W$ has no contractible loop;
(mNE3) none of the components of the complement in $\mathcal{G}$ of the union of all edges of $W$ is a contractible region bounded by either two or four edges of $W$, i.e. there is no $\circlearrowleft$ or $\circlearrowright$ (with all possible oriented labels)

• A non-elliptic $A_m$-web $W$ is weakly reduced if it contains none of $\circlearrowleft$ and $\circlearrowright$, and is reduced if furthermore it contains none of $\circlearrowright$; in these pictures, the blue line is boundary, the edges can be given all possible oriented labels, and the boundary 2-gon, 3-gon and 4-gon are contractible.

• An $A_{m-1}$-lamination $\ell$ in $\mathcal{G}$ is a reduced non-elliptic $A_m$-web $W(\ell)$ in $\mathcal{G}$ with each component equipped with an integer weight, subject to the following conditions and equivalence relations:

(mL1) the weight for each component of $W(\ell)$ containing a 3-valent (i.e. internal) vertex is 1;
(mL2) the weight for each component of $W(\ell)$ that is not a peripheral curve (Def.3.6) is non-negative;
(mL3) an $A_{m-1}$-lamination containing a component of weight zero is equivalent to the $A_{m-1}$-lamination with this component removed;
(mL4) an $A_{m-1}$-lamination with two of its components being homotopic with weights $a$ and $b$ is equivalent to the $A_{m-1}$-lamination with one of these components removed and the other having weight $a + b$.

Let $\mathcal{A}^{(m)}(\mathcal{G}; \mathbb{Z})$ be the set of all $A_{m-1}$-laminations in $\mathcal{G}$.

Note that when $m = 2$, there cannot be a 3-valent vertex, and the orientations of loops and edges are irrelevant, and so $A_1$-laminations are precisely Fock-Goncharov’s integral $A$-laminations $\mathcal{A}$ [FG06, FG07], while for $m = 3$, the $A_2$-laminations are as already defined in the present paper.

We go on to propose $A_{m-1}$ analogs of some constructions about $A_2$-webs, to define tropical coordinates.

**Definition 6.14.** An $A_{m-1}$-web $W$ in a triangle is canonical if it is a disjoint union of corner arcs (Def.3.12) and pyramids, where an $A_{m}$-web is called a pyramid if, when the oriented labels are forgotten, it is isotopic to a pyramid $H_3$ in Def.3.12 with orientations forgotten.

• An $A_{m-1}$-web $W$ in a triangle is a crossbar $A_{m-1}$-web if, when the oriented labels are forgotten, it is a crossbar graph in Def.3.14, and is called a minimal crossbar $A_{m-1}$-web if it is furthermore non-elliptic.

• An $A_{m-1}$-web $W$ in a triangulable generalized marked surface $\mathcal{G}$ is canonical with respect to a split ideal triangulation $\tilde{\Delta}$ for an ideal triangulation $\Delta$ of $\mathcal{G}$ if, for each triangle $\ell$ of $\Delta$, the part $W \cap \hat{\ell}$ in the corresponding triangle $\hat{\ell}$ of $\tilde{\Delta}$ is a canonical $A_{m-1}$-web in $\hat{\ell}$, and for each biangle $B$ of $\Delta$, the part $W \cap B$ is a minimal crossbar $A_{m-1}$-web in $B$.

**Definition 6.15** (FG06). An $m$-triangulation of a triangle is a quiver as follows. Realize the triangle (by applying a diffeomorphism) in the oriented plane $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, m)$, $(m, m)$;
then the nodes of the $m$-triangulation for this triangle are the integer points $(a, b)$ (with $a, b \in \mathbb{Z}$) not coinciding with a vertex of the triangle. Draw the arrows as $(a, b) \rightarrow (a+1, b)$, $(a, b) \rightarrow (a, b+1)$, and $(a+1, b+1) \rightarrow (a, b)$, whenever possible.

- An $m$-triangulation of an ideal triangulation $\Delta$ of a generalized marked surface $S$ is the quiver $Q^{(m)}_\Delta$ obtained by gluing the $m$-triangulations for the triangles of $\Delta$ (in particular, having $m-1$ nodes in each constituent arc of $\Delta$, and $\frac{(m-1)(m+1)}{2}$ nodes in the interior of each triangle of $\Delta$).

Xie [X13] suggested tropical coordinates for some basic $A_{m-1}$-webs in a triangle. We use some modified version.

**Definition 6.16** (tropical coordinates of basic $A_{m-1}$-webs in triangle; [X13]). Let $t$ be a triangle, with unique ideal triangulation $\Delta$. For a weakly reduced single-component $A_{m-1}$-web in $t$ with at most one $3$-valent vertex, define the tropical coordinates $a_{(a,b)}^{(m)}(W) \in \frac{1}{m}\mathbb{Z}$ of $W$ at each node $v$ of $Q^{(m)}_\Delta$ as follows.

Consider a quadrilateral in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,m)$, $(m,m)$, $(m,0)$, divided into two triangles $t$ and $r$ by the diagonal connecting $(0,0)$ and $(m,m)$, where $t$ is the upper one. View each of these triangles, as well as the quadrilateral, as generalized marked surface, so that the integer points (not being vertices of $t$ or $r$) in each of them are the nodes of the $m$-triangulation.

- Consider an $A_{m-1}$-web $W_{i}^{\text{up}}$ in this quadrilateral consisting of one oriented arc going vertically upward, say from $(x,0)$ to $(x,m)$ for some $x \in \mathbb{R}$ with $0 < x < m$, with label $i \in m$. Define the tropical coordinates of $W_{i}^{\text{up}}$ at each node $(a,b)$ of the above $m$-triangulation(s) as follows:

$$a_{(a,b)}^{(m)}(W_{i}^{\text{up}}) = \begin{cases} \frac{1}{m} \cdot a \cdot i & \text{if } 0 \leq a \leq i, \\ \frac{1}{m} \cdot \overline{a} \cdot \overline{i} & \text{if } i \leq a \leq m. \end{cases}$$

That is, the nodes $(a,b)$ having some $a$ get same coordinates, where $(0, \cdot)$ gets zero coordinate, and the coordinate value for $(a, \cdot)$ increases by $\frac{1}{m}$ as a increases by 1 until $a = i$ (value $\overline{i}$ there), then after $a = i$ the coordinate for $(a, \cdot)$ decreases by $\frac{1}{m}$ as a increases by 1.

- Note that $W_{i}^{\text{up}} \cap t$ is a right turn corner arc in $t$ with label $i$, and $W_{i}^{\text{up}} \cap r$ is a left turn corner arc in $r$ with label $i$. Now, for any left turn corner arc or a right turn corner in any triangle (with some label $\in m$), define the tropical coordinates according to this recipe.

- In the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,m)$, $(m,m)$, for $i,j \in m$ consider a single-component $A_{m-1}$-web $W_{i,j}^{\text{up, left}}$ having precisely one internal $3$-valent vertex and whose endpoints lie in distinct sides of the triangle, such that the oriented labels of the edges can be represented so that the edge meeting the top side of the triangle is going upward with label $i$, and the edge meeting the left side of the triangle is going toward left label $j$ (then the oriented label for the remaining edge is determined). Let $W_{i}^{\text{up}}$ be the one already considered. Let $W_{j}^{\text{left}}$ be the oriented arc in this triangle going horizontally to the left, say from $(y,y)$ to $(0,y)$ for some $y \in \mathbb{R}$ with $0 < y < m$, with label $j \in m$. Then the tropical coordinates $a_{(a,b)}^{(m)}(W_{i}^{\text{up}})$ and $a_{(a,b)}^{(m)}(W_{j}^{\text{left}})$ for nodes $(a,b)$ of the $m$-triangulation of the triangle are defined using the above recipe. Define the tropical coordinates of $W_{i,j}^{\text{up, left}}$ at the nodes $(a,b)$ of the $m$-triangulation of this triangle as follows:

$$a_{(a,b)}^{(m)}(W_{i,j}^{\text{up, left}}) = \begin{cases} a_{(a,b)}^{(m)}(W_{i}^{\text{up}}) + a_{(a,b)}^{(m)}(W_{j}^{\text{left}}) - 1, & \text{if } a \geq i \text{ and } b \leq j, \\ a_{(a,b)}^{(m)}(W_{i}^{\text{up}}) + a_{(a,b)}^{(m)}(W_{j}^{\text{left}}), & \text{otherwise}. \end{cases}$$

This gives recipe for tropical coordinates of any 3-way $A_{m-1}$-web in any triangle, with three endpoints lying in distinct sides.

In particular, one can check that the above tropical coordinates are well-defined, i.e. if an oriented label $(o,i)$ of an edge is represented by $\overline{(i,m)}$, the coordinates do not change.

**Question 6.17.** How do we define tropical coordinates for general pyramid $A_{m-1}$-webs in a triangle?

Let’s assume for now that Question 6.17 is answered.

**Definition 6.18** (conjectural tropical coordinates for $A_{m-1}$-laminations). Let $W$ be a weakly reduced non-elliptic $A_{m-1}$-web in a triangulable generalized marked surface $S$, with a chosen split ideal triangulation $\hat{\Delta}$ (for $\Delta$). Put $W$ into a canonical position with respect to $\hat{\Delta}$ by isotopy. For each triangle
t of \(\Delta\), for the nodes \(v\) of the \(m\)-triangulation quiver \(Q^{(m)}_\Delta\) living in \(t\), define the tropical coordinate \(a_v^{(m)}(W)\) of \(W\) at \(v\) as the sum of coordinates at \(v\) of the components of \(W \cap t\), as defined in Def. 6.16 and Question 6.17.

For an \(A_{m-1}\)-lamination \(\ell\) in \(\mathcal{S}\), represented by a weighted reduced non-elliptic \(A_{m-1}\)-web \(W(\ell)\), for each triangle \(t\) of \(\Delta\) define the coordinate \(a_v^{(m)}(\ell)\) of \(\ell\) at node \(v\) in \(Q^{(m)}_\Delta \cap t\) as the weighted sum of the tropical coordinates at \(v\) of the components of \(W\).

**Conjecture 6.19.** Question 6.17 can be answered so that Def. 6.18 yields a well-defined injective map

\[
\mathcal{A}_L^{(m)}(\mathcal{S}; \mathbb{Z}) \rightarrow (\frac{1}{m} \mathbb{Z})^{\mathcal{V}(Q^{(m)}_\Delta)}
\]

\[
\ell \mapsto ((a_v^{(m)}(\ell))_{v \in \mathcal{V}(Q^{(m)}_\Delta)})
\]

**Conjecture 6.20.** These tropical coordinates transform under change of ideal triangulations \(\Delta \sim \Delta'\) according to the composition of the tropicalized versions of the sequence of cluster \(\mathcal{A}\)-mutations that relate Fock-Goncharov’s cluster \(\mathcal{A}\)-charts of the moduli space \(\mathcal{A}_{SL_m, \mathcal{S}}\) associated to \(\Delta, \Delta'\) [FG06].

**Definition 6.21.** For an ideal triangulation \(\Delta\) of \(\mathcal{S}\), we say \(\ell \in \mathcal{A}_L^{(m)}(\mathcal{S}; \mathbb{Z})\) is \(\Delta\)-congruent if \(a_v^{(m)}(\ell) \in \mathbb{Z}\) for all \(v \in \mathcal{V}(Q^{(m)}_\Delta)\).

**Conjecture 6.22.** For two ideal triangulations \(\Delta, \Delta', \ell \in \mathcal{A}_L^{(m)}(\mathcal{S}; \mathbb{Z})\) is \(\Delta\)-congruent iff \(\Delta'\)-congruent.

If one of Conjectures 6.20 and 6.22 holds, then we may assert the identification

\[
\mathcal{A}_{SL_m, \mathcal{S}}(\mathcal{S}') \leftrightarrow \{\text{congruent } A_{m-1}\text{- laminations in } \mathcal{S}\},
\]

providing a geometric model of the tropical integer points of the moduli space \(\mathcal{A}_{SL_m, \mathcal{S}}\).

To relate to the regular functions on \(\mathcal{A}_{PGL_m, \mathcal{S}}\), we note from [X13] that the label \(i\) of an edge equals the \(i\)-th fundamental representation \(\omega_i\) of \(SL_m/GL_m/PGL_m\), where \(i = 1\) means the tautological standard \(m\)-dimensional representation. We apply this representation before taking trace of monodromy along loops (as in [FG06 §9.11]).

**Conjecture 6.23.** For a label \(i \in m\), let \(\omega_i: G \rightarrow GL(V)\) be the \(i\)-th fundamental representation of \(G = SL_m\). For an oriented loop \(\gamma\) in a punctured surface \(\mathcal{S}\), the trace-of-monodromy function \(f_{\gamma, i}\) on the moduli space \(\mathcal{A}_{SL_m, \mathcal{S}}\) along \(\gamma\) with respect to \(\omega_i\)

\[
f_{\gamma, i}(\mathcal{L}, \beta) := \text{tr}(\omega_i(\rho_\mathcal{L}))
\]

is a regular function on \(\mathcal{A}_{SL_m, \mathcal{S}}\), that yields, via similar arguments in the present paper, the function \(f_{\gamma, i}^+\) on \(\mathcal{A}_{PGL_m, \mathcal{S}} = \mathcal{A}_{PGL_m, \mathcal{S}}(\mathbb{R}_{>0})\) that is, for each ideal triangulation \(\Delta\), can be written as a Laurent polynomial in the \(m\)-th roots \(X_v^{1/m}\) of (positive real evaluations of) the cluster \(\mathcal{X}\)-coordinates \(X_v\), \(v \in \mathcal{V}(Q^{(m)}_\Delta)\).

When \(\gamma\) is simple, and if \(\ell\) denotes the \(A_{m-1}\)-lamination consisting of this oriented simple loop \(\gamma\) with label \(i\), then for each ideal triangulation \(\Delta\), the unique highest Laurent monomial term of \(f_{\gamma, i}^+\) equals

\[
\prod_{v \in \mathcal{V}(Q^{(m)}_\Delta)} X_v^{a_v^{(m)}(\ell)}
\]

with coefficient 1.

Even if we do not have control over full \(A_{m-1}\)-laminations at hand, the above conjecture can be tried; some were done in [X13], and this is how the coordinates of [X13] were found. What would be particularly doable is when \(i = 1\), for which the basic monodromy matrices \(M\) are given in [FG06].

The easiest to investigate would be the peripheral loops.

**Conjecture 6.24.** For a peripheral loop \(\gamma\) around a puncture \(p\), analog of Prop. 4.15 holds.

- There exists a constant \(m \times m\) matrix \(\sigma = (\sigma_{ij})\) of rank \(m - 1\), whose first row is \((10 \cdots 0)\) and the last row is \((0 \cdots 0 - 1)\), satisfying the following. Let \(\gamma_i\) be a positively oriented peripheral loop (Def. 4.6) around \(p\) with label \(i \in m\), which may be viewed as an \(A_{m-1}\)-lamination. For the \(i\)-th puncture function \((3\pi) : \mathcal{A}_{SL_m, \mathcal{S}} \rightarrow \mathbb{G}_m\), the corresponding \((3\pi)_{\gamma_i}^+ \in C^\infty(\mathcal{A}_{PGL_m, \mathcal{S}})\) equals \(\prod_{k=1}^m (\prod_{v} X_v^{a_v^{(m)}(\gamma_i)})^{\sigma_{ij}}\).

In particular, the above conjecture holds for \(m = 3\); for general \(m\) it should be a simple check.

Although there were already lots of gaps to fill in so far, another major step to be done is the following.
Conjecture 6.25. There is a version of (stated and non-stated) $A_{m-1}$-webs defined in Def. 6.13 such that it is isomorphic to Sikora’s $A_{m-1}$-skein algebra $\mathcal{S}_m$, and that under the composition of this isomorphism and the isomorphism from Sikora’s $A_{m-1}$-skein algebra to $\mathcal{O}(\mathbf{Z}_{\mathbf{S}_m})$ (analog of Prop. 4.2.1), the oriented loop $\gamma$ with label $i$ goes to the $i$-th trace-of-monodromy function $f_{\gamma,i}$, defined as $f_{\gamma,i}(L) = \text{tr}(\omega_i(\rho_L))$.

We will also need:

Conjecture 6.26. Reduced non-elliptic $A_{m-1}$-webs form a basis of this $A_{m-1}$-skein algebra.

For these conjectures, one may want to consult the results in [CKM14].

Finally, following the rest of the arguments used in the present paper, if one can check the $A_{m-1}$-analog of all statements, one would get:

Conjecture 6.27. The rest of the arguments of the present paper yields a duality map

$$I^{(m)} : \mathcal{A}_{\mathbf{S}_m, \mathcal{E}}(\mathbf{Z}^l) \to \mathcal{O}(\mathbf{X}_{\mathbf{PGL}_m, \mathcal{E}}).$$

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