MEAN VALUES OF LONG DIRICHLET POLYNOMIALS WITH HIGHER DIVISOR COEFFICIENTS

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Abstract. In this article, we prove an asymptotic formula for mean values of long Dirichlet polynomials with higher order shifted divisor functions, assuming a smoothed additive divisor conjecture for higher order shifted divisor functions. As a consequence of this work, we prove special cases of conjectures of Conrey-Keating [8] on mean values of long Dirichlet polynomials with higher order shifted divisor functions as coefficients.

1. Introduction

An important field of research with a long history in analytic number theory is the study of the $2k$-th moments of the Riemann zeta function, $\zeta(s)$. These moments are given by

$$I_k(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt \text{ where } k > 0 \text{ and } T \geq 1.$$

(1.1)

A driving force in this field is the conjectural asymptotic

$$I_k(T) \sim \frac{c_k g_k}{(k^2)!} T(\log T)^{k^2},$$

(1.2)

where

$$c_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{\alpha=0}^{\infty} \frac{\tau_k(p^{\alpha})^2}{p^{\alpha}} \quad \text{and} \quad g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

(1.3)

This asymptotic is only known to be true in the cases $k = 1$ and $k = 2$ as established by Hardy-Littlewood [20] and Ingham [24] respectively. The conjecture in the form (1.2) with $c_k$ given in (1.3) is folklore. For a long time, the values of $g_k$ were unknown until Keating and Snaith [30] famously computed $g_k$ via a random matrix model and then announced their result at a conference in Vienna in 1998. At the same conference Conrey and Gonek announced the conjecture $g_4 = 24024$, based on mean values of long Dirichlet polynomials. Previously, Conrey and Ghosh [4] conjectured $g_3 = 42$ by studying various mean value formulae for $\zeta(s)$ and Dirichlet polynomials.

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The Keating-Snaith conjecture (1.2) is intimately related to the size of the Riemann zeta function and the Riemann hypothesis. Littlewood showed that the Riemann hypothesis implies that

$$|\zeta\left(\frac{1}{2} + it\right)| \ll \exp\left(\frac{C \log t}{\log \log t}\right),$$

(1.4)

for some positive constant $C$. Note that the Lindelöf hypothesis (LH) is the assertion

for all $\varepsilon > 0$, $|\zeta\left(\frac{1}{2} + it\right)| \ll \varepsilon |t|^\varepsilon$.

(1.5)

From Littlewood’s bound (1.4) it follows that the Riemann hypothesis implies LH. The first non-trivial subconvexity bound was established by Hardy-Littlewood, using a method of Weyl (see [37, Chapter 5]):

$$|\zeta\left(\frac{1}{2} + it\right)| \ll \varepsilon |t|^\vartheta$$

(1.6)

where $\vartheta = \frac{1}{6} = 0.1666\ldots$. The current record due to Bourgain is (1.6) with $\vartheta = \frac{13}{84} = 0.1547\ldots$. Hardy and Littlewood introduced the moments $I_k(T)$ in an attempt to solve LH. This is since it is known that the bound $I_k(T) \ll_{k, \varepsilon} T^{1+\varepsilon}$ for all $k \geq 1$ implies (1.5). Therefore, the Keating-Snaith conjecture (1.2) implies LH. The Riemann hypothesis has a number of arithmetic consequences. In some instances these arithmetic consequences can be proved only assuming LH. For instance, Ingham [24] showed that LH implies that the gaps between consecutive primes satisfies $p_{n+1} - p_n \ll p_n^{1+\varepsilon}$. One application of a uniform version of the Keating-Snaith conjecture is to the maximal size of the Riemann zeta function on the critical line. Farmer, Gonek, and Hughes [16] (see also [22]) conjectured that

$$\max_{0 \leq t \leq T} |\zeta\left(\frac{1}{2} + it\right)| = \exp\left((1 + o(1))\sqrt{\frac{1}{2} \log T \log \log T}\right).$$

(1.7)

Another reason for studying the moments (1.1) is that the techniques, tools, and ideas used in evaluating them can often be useful in the context of moment problems for other families of L-functions. In the last thirty years, there has been a flurry of activity in the study of the distribution and moments of L-functions and the distribution of values of L-functions. For a comprehensive overview of many of the recent advances in the theory see [36].

In studying the moments (1.1) it is useful to consider smoothed and shifted versions of them given by

$$\int_{-\infty}^{\infty} \omega(t) \left(\prod_{j=1}^{k} \zeta\left(\frac{1}{2} + a_j + it\right) \prod_{j=1}^{\ell} \zeta\left(\frac{1}{2} + b_j - it\right)\right) dt,$$

(1.8)

where $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_\ell\}$ are multisets of complex numbers and $\omega$ is a suitable real or complex valued function. The idea of using complex shifting parameters was introduced by Ingham [21], and the idea of introducing smoothing weights has long been known and was used by Atkinson [2] and Titchmarsh [37]. In [19] a smooth weight $\omega(t)$ was removed and replaced by an indicator function. Generalized moments such as (1.8) are known to reveal the combinatorial structure of the moments (1.1) (see [11]).

The mean values (1.1) and (1.8) can be modelled by mean values of long Dirichlet polynomials.
This approach had previously been introduced by Conrey-Gonek [5] and Ivić [26]. We set

\[ A_{a,\varphi}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \varphi\left(\frac{n}{K}\right), \quad B_{b,\varphi}(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \varphi\left(\frac{n}{K}\right), \]

(1.9)

where \( K = T^\theta \), \( \{a(n)\} \) and \( \{b(n)\} \) are arbitrary sequences, and \( \varphi \) is a real valued function. Attached to these polynomials is the mean value

\[ D_{a,b}\omega(K) = \int_{\mathbb{R}} \omega(t)A_{a,\varphi}\left(\frac{1}{2} + it\right)B_{b,\varphi}\left(\frac{1}{2} - it\right) dt. \]

(1.10)

Such mean values are simple to evaluate when \( 0 < \theta \leq 1 \). In the case that \( \theta > 1 \), they become harder to evaluate and they are called mean values of a long Dirichlet polynomial. Goldston and Gonek [17] considered such mean values and provided certain formulae for \( D_{a,b}\omega(K) \) based on correlation sum estimates for \( a(n) \) and \( b(n) \). In this article we shall evaluate \( D_{a,b}\omega(K) \) in the cases that \( a(n) \) and \( b(n) \) are generalized divisor functions, \( \theta \in (1,2) \), and \( \omega \) and \( \varphi \) are suitably chosen smooth functions (see Section 2 for details). Throughout this article \( k \) and \( \ell \) denote natural numbers and \( I \) and \( J \) denote multisets of complex numbers given by

\[ I = \{a_1, \ldots, a_k\} \quad \text{and} \quad J = \{b_1, \ldots, b_\ell\}. \]

(1.11)

It will be convenient to use the notation

\[ K = \{1, \ldots, k\} \quad \text{and} \quad L = \{1, \ldots, \ell\}. \]

(1.11)

We shall assume throughout the article that we have the following size condition: there exists a positive absolute constant \( \delta \) such that

\[ |a_i|, |b_j| \leq \delta \quad \text{for} \quad i \in K \quad \text{and} \quad j \in L. \]

(1.12)

At times we shall require the more restrictive size conditions

\[ |a_i|, |b_j| \ll \frac{1}{\log T} \quad \text{for} \quad i \in K \quad \text{and} \quad j \in L, \]

(1.13)

and

\[ |a_{i_1} - a_{i_2}| \gg \frac{1}{\log T} \quad \text{and} \quad |b_{j_1} - b_{j_2}| \gg \frac{1}{\log T} \quad \text{for} \quad i_1 \neq i_2 \in K \quad \text{and} \quad j_1 \neq j_2 \in L, \]

(1.14)

where \( T \geq 2 \) is a parameter. Note that if \( T \) is taken sufficiently large, and if \( I \) and \( J \) satisfy (1.13), then they will also satisfy (1.12). For \( k \in \mathbb{N} \), we define the \( k \)-th divisor function to be

\[ \tau_k(n) = \#\{(n_1, \ldots, n_k) \in \mathbb{N}^k \mid n_1 \cdots n_k = n\} \quad \text{for} \quad n \in \mathbb{N} \]

and for a multiset \( J = \{a_1, \ldots, a_k\} \subset \mathbb{C} \), the shifted divisor function\(^1\) is given by

\[ \sigma_J(n) = \sum_{d_1 \cdots d_k = n} d_1^{-a_1} \cdots d_k^{-a_k}. \]

(1.15)

\(^1\)In the articles [6], [7], [8], [9], and [10], the authors use the notation \( \tau_J(n) \) instead of our \( \sigma_J(n) \).
Observe that if \( J = \{0, \ldots, 0\} \), then \( \sigma_J(n) = \tau_k(n) \) where \( k = |J| \). We shall evaluate \( \mathcal{D}_{a,b,\omega}(K) \) in the cases
\[
a(n) = \tau_k(n) \quad \text{and} \quad b(n) = \tau_\ell(n),
\]
and
\[
a(n) = \sigma_J(n) \quad \text{and} \quad b(n) = \sigma_\delta(n).
\]
We shall use the short hand notation
\[
\mathcal{D}_{k,\ell,\omega}(K) := \mathcal{D}_{\tau_k,\tau_\ell,\omega}(K) \quad \text{and} \quad \mathcal{D}_{\sigma_J,\sigma_\delta,\omega}(K) := \mathcal{D}_{\sigma_J,\sigma_\delta,\omega}(K).
\]
In \cite{5}, Conrey and Gonek gave heuristic arguments which showed how to model the sixth and eighth moments in terms of \( \mathcal{D}_{k,k,\omega}(K) \) for \( k = 3, 4 \). We have the following conjecture for \( \mathcal{D}_{k,\ell,\omega}(K) \). In the case \( k = \ell \) this is due to Conrey-Gonek \cite{5} Conjecture 4, p.583].

**Conjecture 1.** Let \( k, \ell \in \mathbb{N} \). Let \( T \) be sufficiently large, \( K = T^{1+\eta} \) where \( \eta \in (0, 1) \). Then we have
\[
\mathcal{D}_{k,\ell,\omega}(K) \sim \frac{a_{k,\ell}}{\Gamma(k\ell + 1)} w_{k,\ell}(1 + \eta) T (\log T)^{k\ell},
\]
where
\[
a_{k,\ell} = \prod_p \left( 1 - \frac{1}{p} \right) k\ell \sum_{\alpha=0}^{\infty} \frac{\tau_k(p^\alpha)\tau_\ell(p^\alpha)}{p^\alpha},
\]
\[
w_{k,\ell}(x) = x^{k\ell} \left( 1 - \sum_{n=0}^{k\ell-1} \binom{k\ell}{n+1} \gamma_{k,\ell}(n)(-1)^{n+\ell+k}(1-x^{-n-1}) \right),
\]
\[
\gamma_{k,\ell}(n) = \sum_{i=1}^{\ell} \sum_{j=1}^{k} \binom{\ell}{i} \binom{k}{j} \binom{n-1}{i+j-2} \binom{i+k-\ell-1}{i+k-\ell-1},
\]
for \( n \in \mathbb{N} \), and
\[
\gamma_{k,\ell}(0) = \sum_{i=1}^{\ell} \sum_{j=1}^{k} (-1)^{k+i+j} \binom{\ell}{i} \binom{k}{j} \binom{i+j-2}{i+k-\ell-1}.
\]

Let \( \eta \in (0, 1) \) and \( \omega = \mathbb{1}_{[T,2T]} \). In \cite{5} an argument with the classical approximate functional equation for \( \zeta^k(s) \) is given which suggests the asymptotics
\[
\int_T^{2T} |\zeta\left(\frac{1}{2} + it\right)|^6 \, dt \sim \mathcal{D}_{3,3,\omega}(T^{1+\eta}) + \mathcal{D}_{3,3,\omega}(T^{2-\eta}) \sim (w_{3,3}(1+\eta) + w_{3,3}(2-\eta)) \frac{C_3}{9!} T (\log T)^9
\]
and
\[
\int_T^{2T} |\zeta\left(\frac{1}{2} + it\right)|^8 \, dt \sim 2 \mathcal{D}_{4,4,\omega}(T^2) \sim 2w_{4,4}(2) \frac{C_4}{16!} T (\log T)^{16}.
\]

In fact, one can verify by straightforward computations that
\[
w_{3,3}(1+\eta) + w_{3,3}(2-\eta) = 42 = g_3 \quad \text{and} \quad w_{4,4}(2) = 24024 = g_4. \tag{2}
\]

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Note that \( w_{3,3}(x) = -2x^9 + 27x^8 - 324x^7 + 2268x^6 - 8694x^5 + 19278x^4 - 25452x^3 + 19764x^2 - 8343x + 1479 \) and \( w_{4,4}(x) = -3x^{16} + 16x^{15} - 1320x^{14} + 14000x^{13} - 78260x^{12} + 179088x^{11} - 152152x^{10} + 11440x^9 + 12870x^8 + 11440x^7 + 8008x^6 + 4368x^5 + 1820x^4 + 560x^3 + 120x^2 + 16x + 1 \).
where $g_k$ is as defined in (1.3). Thus (1.18) and (1.19) agree with the Keating-Snaith conjecture. This was one of the main motivations for evaluating mean values of long Dirichlet polynomials with divisor coefficients. In [32], the heuristic (1.18) is made precise assuming the 3-3 additive divisor conjecture (see Conjecture 4 below) with error term $O(P^C X^{\frac{1}{2} - \delta})$, for some $\delta > 0$, uniformly for $|r| \leq \sqrt{X}$. In [33], (1.19) is made precise in the same way where the 4-4 additive divisor conjecture is required with an error term $O(P^C X^{1 - \varepsilon_1} + \varepsilon_1)$, uniformly for $|r| \leq X^{1 - \varepsilon_2}$ where $\varepsilon_1, \varepsilon_2$ are arbitrarily small positive constants. Note that the argument of Conrey-Gonek using the additive divisor sums (1.33) does not seem to extend to the moments $I_k(T)$ with $k > 4$. In order to further understand $I_k(T)$, Conrey and Keating undertook an extensive study [6], [7], [8], [9], [10] of $D_{I,J,\omega}(K)$, the mean values of Dirichlet polynomials with shifted divisor functions. This work has led to the consideration of more complicated additive divisor sums. Furthermore, they have formulated a conjecture on the asymptotic size of $D_{I,J,\omega}(K)$.

In order to state their conjecture we must introduce some notation and definitions.

**Definition 1.** Let $I, J$ be finite multisets of complex numbers. We define $B(I, J)$ as the series

$$B(I, J) = \sum_{n=1}^{\infty} \frac{\sigma_I(n) \sigma_J(n)}{n},$$

if the series converges (for example, when $\Re(a), \Re(b) > 0$ for all $a \in I$ and $b \in J$), and by analytic continuation otherwise.

Observe that when the series (1.20) converges, we use the multiplicativity of $\sigma_I \sigma_J$ to write

$$B(I, J) = \prod_p \sum_{u=0}^{\infty} \frac{\sigma_I(p^u) \sigma_J(p^u)}{p^u}.$$  

Upon factoring out $\prod_p \prod_{i \in \mathcal{X}_I, j \in \mathcal{E}_J} (1 - p^{-1-a_i-b_j})^{-1}$ from the right hand side of (1.21), we obtain

$$B(I, J) = \prod_p \prod_{i \in \mathcal{X}_I, j \in \mathcal{E}_J} (1 - p^{-1-a_i-b_j})^{-1} \prod_p \prod_{i \in \mathcal{X}_I, j \in \mathcal{E}_J} (1 - p^{-1-a_i-b_j}) \sum_{u=0}^{\infty} \sigma_I(p^u) \sigma_J(p^u) \frac{p^u}{p^u}.$$  

**Definition 2.** For $p$ prime and $s \in \mathbb{C}$, we set $z_p(s) = (1 - p^{-s})^{-1}$. Attached to the local factors $z_p(s)$, we define

$$\mathcal{Z}(I, J) = \prod_p \prod_{i \in \mathcal{X}_I, j \in \mathcal{E}_J} z_p(1 + a_i + b_j),$$

$$\mathcal{A}(I, J) = \prod_p \prod_{i \in \mathcal{X}_I, j \in \mathcal{E}_J} z_p^{-1}(1 + a_i + b_j) \sum_{u=0}^{\infty} \sigma_I(p^u) \sigma_J(p^u) \frac{p^u}{p^u}.$$  

Observe that we have

$$\mathcal{Z}(I, J) = \prod_{i \in \mathcal{X}_I, j \in \mathcal{E}_J} \zeta(1 + a_i + b_j),$$

where $\zeta$ is the Riemann zeta function.
Definition 3. Given a multiset \( U = \{\alpha_1, \ldots, \alpha_n\} \) and \( w \in \mathbb{C} \), we define \( U_w := U + \{w\} = \{\alpha_1 + w, \ldots, \alpha_n + w\} \). We also set \( -U = \{-\alpha_1, \ldots, -\alpha_n\} \). With this notation, observe that we have the identity

\[
\sigma_{U_w}(n) = n^{-w}\sigma_U(n).
\]

We can now state the Conrey-Keating conjectures for the mean values \( \mathcal{D}_{\mathfrak{J}, \mathfrak{J}, \omega}(K) \) (see [8] pages 739-740).

Conjecture 2 (Conrey-Keating). Let \( K = T^\theta \) with \( \theta > 0 \). Then for \( T \) sufficiently large

\[
\mathcal{D}_{\mathfrak{J}, \mathfrak{J}, \omega}(K) = \int_0^\infty \omega(t) \frac{1}{(2\pi i)^2} \int_{(c_1)} \int_{(c_2)} \Phi(s_1)\Phi(s_2)K^{s_1 + s_2} \times \\
\sum_{U \subset \mathfrak{J}, V \subset \mathfrak{J}, |U| = |V|} \left( \frac{t}{2\pi} \right)^{-\sum_{x \in U, y \in V} (x + s_1 + y + s_2)} \mathcal{B}((\mathfrak{J}_x \setminus U_x) \cup (-V_x), (\mathfrak{J}_y \setminus V_y) \cup (-U_y)) \text{ds}_1\text{ds}_2\text{dt} + o(T),
\]

where \( c_1, c_2 > 0 \) and \( \Phi \) is the Mellin transform of \( \varphi \).

In the above conjecture, we used the terminology of [8]; for example, by \( (\mathfrak{J}_x \setminus U_x) \cup (-V_x) \) we mean the following: remove the elements of \( U_x \) from \( \mathfrak{J}_x \) and then include the negatives of the elements of \( V_x \). The notation \( (\mathfrak{J}_y \setminus V_y) \cup (-U_y) \) can be explained similarly. Since the subsets \( U \) and \( V \) have equal cardinalities, this process is referred to as swapping equal numbers of elements between \( \mathfrak{J}_x \) and \( \mathfrak{J}_y \). The cardinality \( |U| \) is referred to as the number of swaps in the associated term.

To give more insight on the terms \( \mathcal{B}((\mathfrak{J}_x \setminus U_x) \cup (-V_x), (\mathfrak{J}_y \setminus V_y) \cup (-U_y)) \) appearing in (1.27), we give precise formulae when \( |\mathfrak{J}| = |\mathfrak{J}| = 2 \). Suppose that \( \mathfrak{J} = \{a_1, a_2\} \) and \( \mathfrak{J} = \{b_1, b_2\} \). The term \( \mathcal{B}((\mathfrak{J}_x \setminus U_x) \cup (-V_x), (\mathfrak{J}_y \setminus V_y) \cup (-U_y)) \) corresponding to \( |U| = |V| = 0 \) simplifies to

\[
\mathcal{B}\{\{a_1 + s_2, a_2 + s_2\}, \{b_1 + s_1, b_2 + s_1\}\}
\]

\[
= \frac{\zeta(1 + s_1 + s_2 + a_1 + b_1)\zeta(1 + s_1 + s_2 + a_1 + b_2)\zeta(1 + s_1 + s_2 + a_2 + b_1)\zeta(1 + s_1 + s_2 + a_2 + b_2)}{\zeta(2 + 2s_1 + 2s_2 + a_1 + a_2 + b_1 + b_2)}.
\]

(1.28)

which can be derived from a formula of Ramanujan (see [37] Eq 1.3.3). To describe the terms corresponding to \( |U| = |V| = 1 \), we consider the case \( U = \{a_1\} \) and \( V = \{b_1\} \) as an example. In this case, the term \( \mathcal{B}((\mathfrak{J}_x \setminus U_x) \cup (-V_x), (\mathfrak{J}_y \setminus V_y) \cup (-U_y)) \) simplifies to

\[
\mathcal{B}\{\{a_2 + s_2, -b_1 - s_1\}, \{-a_1 - s_2, b_2 + s_1\}\}
\]

\[
= \frac{\zeta(1 + a_2 - a_1)\zeta(1 + b_2 - b_1)\zeta(1 + s_1 + s_2 + a_2 + b_2)\zeta(1 - s_1 - s_2 - a_1 - b_1)}{\zeta(2 + a_2 - a_1 + b_2 - b_1)}.
\]

(1.29)
The remaining three terms corresponding to \(|U| = |V| = 1| can be computed similarly. When \(|U| = |V| = 2|, we see that the term \(B(\mathcal{J}_{s_2} \setminus U_{s_2}) \cup (-V_{s_1}), (\mathcal{J}_{s_1} \setminus V_{s_1}) \cup (-U_{s_2})\) simplifies to

\[
B(\{-b_1 - s_1, -b_2 - s_1\}, \{-a_1 - s_2, -a_2 - s_2\})
\]

\[
= \zeta(1 - s_1 - s_2 - a_1 - b_1)\zeta(1 - s_2 - s_2 - a_2 - b_1)\zeta(1 - s_1 - s_2 - a_2 - b_2)
\]

\[
\zeta(2 - 2s_1 - 2s_2 - a_1 - a_1 - b_1 - b_2)
\]

(1.30)

Thus, the size of \(K\) determines which swaps contribute to our main term. In the special case when \(K = o(T^2)\), we only get a contribution from \(U, V\) such that \(|U| = |V| \leq 1|, In particular, if \(K = o(T^2)\) and \(|U| = |V| \geq 2|, then these terms do not contribute to the main term. The contribution of the terms with \(|U| = |V| = 0\) to the integral is

\[
M_{0, \mathcal{J}; \omega}(K) = \int_0^\infty \omega(t) \frac{1}{(2\pi i)^2} \int_{(c_1)} \int_{(c_2)} \Phi(s_1) \Phi(s_2) K^{s_1 + s_2} B(\mathcal{J}_{s_2}, \mathcal{J}_{s_1}) ds_1 ds_2 dt
\]

\[
= \frac{\hat{\omega}(0)}{2\pi i} \int_{(c_1 + c_2)} K^s \Phi_2(s) B(\mathcal{J}, \mathcal{J}) ds,
\]

(1.31)

where \(\Phi_2\) is defined in (2.14) in Section 2. The second equality in (1.31) is obtained by applying the change of variables \(s = s_2 + s_1\) and observing that \(B(\mathcal{J}_{s_2}, \mathcal{J}_{s_1}) = B(\mathcal{J}_{s_1 + s_2}, \mathcal{J})\), which follows from two applications of (1.26). The contribution of the terms with \(|U| = |V| = 1\) is

\[
M_{1, \mathcal{J}; \omega}(K) := \int_0^\infty \omega(t) \frac{1}{(2\pi i)^2} \int_{(c_1)} \int_{(c_2)} \Phi(s_1) \Phi(s_2) K^{s_1 + s_2} \sum_{i \in \mathcal{J}, j \in \mathcal{L}} \left(\frac{2\pi}{t}\right)^{a_i + b_j + s_1 + s_2}
\]

\[
\times B((\mathcal{J}_{s_2} \setminus \{a_i + s_2\}) \cup \{b_j - s_1\}, (\mathcal{J}_{s_1} \setminus \{b_j + s_1\}) \cup \{-a_i - s_2\}) ds_1 ds_2 dt.
\]

Observe that (see [8, page 740])

\[
\mathcal{Z}(\mathcal{J}_{s_2} \setminus \{a_i + s_2\}) \cup \{b_j - s_1\}, (\mathcal{J}_{s_1} \setminus \{b_j + s_1\}) \cup \{-a_i - s_2\})
\]

\[
= \mathcal{Z}(\mathcal{J} \setminus \{a_i\}, \{-a_i\}) \mathcal{Z}(\{b_j\} \setminus \{b_j\}) \mathcal{Z}(\mathcal{J} \setminus \{a_i\}) + s, (\mathcal{J} \setminus \{b_j\}) \mathcal{Z}(1 - a_i - b_j - s).
\]

By (1.25) and the change of variables \(s = s_1 + s_2\), we get

\[
M_{1, \mathcal{J}; \omega}(K) = \int_0^\infty \omega(t) \sum_{i \in \mathcal{J}, j \in \mathcal{L}} \left(\frac{t}{2\pi}\right)^{-a_i - b_j} \mathcal{Z}(\mathcal{J} \setminus \{a_i\}, \{-a_i\}) \mathcal{Z}(\{b_j\} \setminus \{b_j\}) ds dt
\]

\[
\times \left(\frac{2\pi K}{t}\right)^s \mathcal{Z}(\mathcal{J} \setminus \{a_i\}) + s, (\mathcal{J} \setminus \{b_j\}) \mathcal{Z}(1 - a_i - b_j - s) \]

\[
\times \mathcal{A}(\mathcal{J} \setminus \{a_i\}) \cup \{-b_j - s\}, ((\mathcal{J} \setminus \{b_j\}) + s) \cup \{-a_i\}) ds dt.
\]

(1.32)

Based on these observations, Conjecture 2 simplifies as follows in the case \(K = o(T^2)\).

Conjecture 3 (Conrey-Keating). If \(T \ll K = o(T^2)\), then

\[
\mathcal{B}_{\mathcal{J}; \omega}(K) = M_{0, \mathcal{J}; \omega}(K) + M_{1, \mathcal{J}; \omega}(K) + o(T)
\]
where $M_{0,\ell,\omega}(K)$ and $M_{1,\ell,\omega}(K)$ are given in (1.31) and (1.32).

A key goal of this article is to establish Conjecture 3 under the assumption of an averaged additive divisor conjecture which provides an asymptotic formula for certain smoothed additive divisor sums (see Theorem 1.1). Let us now introduce these additive divisor sums. We put

$$D_f; I, J(r) = \sum_{m,n \geq 1} \sigma_1(m)\sigma_3(n)f(m,n).$$

Moreover, the partial derivatives of $f$ satisfy growth conditions. That is, there exist $X, Y,$ and $P \geq 1$ such that

$$\text{support}(f) \subset [X, 2X] \times [Y, 2Y]$$

and

$$x^m y^n f(m,n)(x,y) \ll m,n P^{m+n}.$$

Before we state a conjectural asymptotic formula for the shifted convolution sum $D_f; I, J(r),$ we need to introduce the following definition.

**Definition 4.** Let $A = \{a_1, \cdots, a_m\}$ be a finite multiset of complex numbers and $s \in \mathbb{C}$. We define two multiplicative functions $n \mapsto g_A(s, n)$ and $n \mapsto G_A(s, n)$ by

$$g_A(s, n) = \prod_{p \mid n} \frac{\sum_{j=0}^{\infty} \sigma_A(p^{j+e})}{p^s}$$

and

$$G_A(s, n) = \sum_{d \mid n} \frac{\mu(d)d^s}{\phi(d)} \sum_{e \mid d} \frac{\mu(e)}{e^s} g_A(s, ne/d).$$

Notice that, for $n \in \mathbb{N}$ we have $\sum_{j=1}^{\infty} \frac{\sigma_A(jn)}{j^s} = g_A(s, n) \prod_{a \in A} \zeta(s+a)$.

Simpler expressions for $G_A(s, n)$ can be derived from Lemmas 6.1, Lemma 6.2, and Lemma 6.3 below.

We are now prepared to state the averaged additive divisor conjecture.

**Conjecture 4.** (k-\ell Additive divisor conjecture) Let $k, \ell \in \mathbb{N}$. There exists a triple $(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell}) \in [\frac{1}{2}, 1) \times [0, \infty) \times (0, 1]$ for which the following (henceforth to be referred to as $AD_{k,\ell}(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell})$, or the ‘additive divisor hypothesis’) holds. Let $\varepsilon$ be a positive absolute constant. Let $P > 1, H \geq 1,$ and let $X, Y > \frac{1}{2}$ satisfy $Y \asymp X$. For each integer $r$ with $1 \leq |r| \leq H$, let $f_r$ be a smooth function satisfying (1.34) and (1.35), and suppose $I = \{a_1, a_2, \ldots, a_k\}$ and $J = \{b_1, \ldots, b_\ell\}$ are sets of distinct complex numbers satisfying $|a_i|, |b_j| \ll (\log X)^{-1}$ where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \ell\}$ (where the implicit constants are absolute). Then, in those cases where $X$ is sufficiently large (in
absolute terms), one has
\[
D_{f,\vartheta,\beta}(r) = \sum_{i_1=1}^{k} \sum_{j_2=1}^{\ell} \prod_{i_1 \neq i_1} \zeta(1-a_{i_1}+a_{j_1}) \prod_{j_2 \neq i_2} \zeta(1-b_{i_2}+b_{j_2})
\times \sum_{q=1}^{\infty} c_q(r) G_1(1-a_{i_1}, q) G_2(1-b_{i_2}, q) \int_{\max(0,r)}^{\infty} f_r(x,x-r)x^{-a_{i_1}}(x-r)^{-b_{i_2}} dx + \Delta_{f,\vartheta,\beta}(r),
\]
where
\[
\sum_{1 \leq |r| \leq H} \Delta_{f,\vartheta,\beta}(r) = O \left( HP^{C_{k,\ell}} X^{\beta_{k,\ell}+\varepsilon} \right)
\]
uniformly for $1 \leq H \ll X^{\beta_{k,\ell}}$.

Remarks.

1. Note that $c_q(r) = \sum_{a=1}^{q} e(\frac{r a}{q})$ is Ramanujan’s sum where $e(\theta) := e^{2\pi i \theta}$.

2. It should be observed that from Lemma 6.3 below, we can see that
\[
G_1(1-a_{i_1}, q) \approx \sigma_{\{a_{i_1}\}}(q) \text{ and } G_2(1-b_{i_2}, q) \approx \sigma_{\{b_{i_2}\}}(q).
\]

3. The main term in the above conjecture can be derived using Duke, Friedlander, and Iwaniec’s $\delta$-method [14]. The conjecture provides a bound for the error term on average over $r$ which is sufficient for our purposes. The reader is referred to [27] and [28] among other references for a treatment of the additive divisor conjecture on average.

4. In the case $|\vartheta| = |\beta| = 2$, Hughes and Young [23, page 218] have proven that this holds with
\[
\vartheta_{2,2} = \frac{3}{4}, C_{2,2} = \frac{5}{4}, \text{ and } \beta_{2,2} = 1 \text{ (even without the averaging over } r). \]
The main term in the result of Hughes and Young can be established by using Duke, Friedlander, and Iwaniec’s [14] $\delta$-method.

5. Using work of Aryan [1] and Topacogullari [35], it may be possible to establish $\vartheta_{2,2} = \frac{1}{2} + \vartheta_0$, where $\vartheta_0$ is the current best bound for the Ramanujan conjecture.

6. Topacogullari [35, 36] and Drappeau [13] have recently established asymptotic formula for the additive divisor sums $D_{k,\ell}(x,r) = \sum_{n=1}^{\infty} \tau_k(n) \tau_{\ell}(n+r)$ in the cases $k \geq 3$, $\ell = 2$ where $0 \neq r \in \mathbb{Z}$. It is likely that their work will lead to the additive divisor conjecture in the case $\ell = 2$ with some $\vartheta_{k,2} < 1$ and $C_{k,2} > 0$.

7. It has been conjectured by Conrey and Keating [8, p.740] that $k-\ell$ additive divisor conjecture in the unsmoothed case holds with $\vartheta_{k,\ell} = \frac{1}{2}$ and $\beta_{k,\ell} = 1 - \varepsilon_0$, for sufficiently small $\varepsilon_0$. It is thus reasonable to expect that $AD_{k,\ell}(\frac{1}{2}, C_{k,\ell}, 1 - \varepsilon_0)$ holds for some $C_{k,\ell} > 0$ and $\varepsilon_0 > 0$.

The main goal of this paper is to prove that Conjecture 4 implies Conjecture 3. More precisely, we establish the following theorem.

**Theorem 1.1.** Let $|\vartheta| = k$ and $|\beta| = \ell$ with $k, \ell \geq 2$, and suppose that $\vartheta$ and $\beta$ satisfy (1.13) and (1.14). Assume that $AD_{k,\ell}(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell})$ holds for some triple $(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell}) \in [\frac{1}{2}, 1) \times [0, \infty) \times (0, 1]$. Let $K = T^{1+\eta}$ with $\eta > 0$, and let $\omega$ satisfy (2.1), (2.2), and (2.3) with $b > \frac{1-\beta_{k,\ell}(1+\eta)}{1-\epsilon}$ and
\[ 0 < \epsilon < 1. \text{ Then we have for any } \varepsilon > 0, \]
\[ D_{\beta, \omega}(K) = M_{0, \beta, \omega}(K) + M_{1, \beta, \omega}(K) + O\left(K^{\theta_{k, \ell} + \varepsilon} \left(\frac{T}{T_0}\right)^{1+C_{k, \ell}}\right). \]

Remarks.

(1) This result provides a rigorous proof of some of the arguments in \[8\]. The main difference is that in \[8\] the authors focus on the main terms without providing bounds for any error terms. Another key difference is that in \[8\] Perron’s formula is applied twice whereas we make use of Mellin inversion.

(2) In order for this result to be non-trivial the error term needs to be \( o(T) \). This is the case if \( \omega \) satisfies (2.3) below with \( b > \frac{C_{k, \ell} + (\theta_{k, \ell} + \varepsilon)(\eta + 1)}{1 + C_{k, \ell}} \). Observe that since we require \( b \leq 1 \), this condition implies \( \eta < \frac{1}{\theta_{k, \ell}} - 1 \). If the additive divisor conjecture is true with \( \theta_{k, \ell} = \frac{1}{2} \), then this theorem allows one to take Dirichlet polynomials with length \( K = T^c \), for any \( c < 2 \).

(3) In the case \( k = \ell = 2 \), this is an unconditional theorem, due to the work of Hughes-Young \[23\] as they have established \( AD_{2,2}^2 \left(\frac{3}{4}, \frac{5}{4}, 1\right) \). In this case, we have an asymptotic formula for \( 0 < \eta < \frac{1}{2} \). Using ideas from [1], it may be possible to increase the range to \( \eta \in (0, 0.61) \). Furthermore, if the Ramanujan conjecture on the size of the Fourier coefficients of Maass forms is true, then the range can be increased to \( \eta \in (0, 1) \).

(4) We expect to establish this theorem unconditionally in the cases \( k \geq 3 \) and \( \ell = 2 \), by using techniques from \[38\], \[39\], and \[13\] and this is current work in progress.

As a consequence of the work in this article, we deduce in an accompanying article \[21\] the special case \( k = \ell = 2 \) and \( \eta \in (0, \frac{1}{3}) \) of Conjecture \[1\] with all lower order terms and a power savings in the error term. Moreover, we expect to deduce a version of Conjecture \[1\] with the full main term and a power savings error term for all \( k \geq 2 \) and \( \ell = 2 \) for some range of \( \eta \). We have

**Theorem 1.2.** Let \( K = T^{1+\eta} \) with \( 0 < \eta < \frac{1}{3} \). Let \( \omega \) satisfy (2.1), (2.2) and (2.3) with \( b > \frac{5+3(\eta+1)}{9} \). Then we have

\[ D_{2, \omega}(K) = \int_{-\infty}^{\infty} \omega(t) \left( \sum_{i=0}^{4} Q_i(\log K, \log \frac{t}{T_0}) \right) dt + O \left( T^{\frac{1}{4}(1+\eta)+\varepsilon} \left( \frac{T}{T_0} \right)^\frac{2}{7} + T^{1-\frac{2}{7}} \right), \]

where the \( Q_i(x, y) \in \mathbb{R}[x, y] \) are polynomials of degree \( i \) and the leading term is given by

\[ Q_4(x, y) = \frac{1}{\zeta(2)} \cdot \frac{1}{4!} (-x^4 + 8x^3y - 24x^2y^2 + 32xy^3 - 14y^4). \]

Precise formulae for the \( Q_i \) are given in \[21\].

We note here that \( Q_4(x, y) = \frac{1}{\zeta(2)} \cdot \frac{1}{4!} y^4 w_{2,2}(x/y) \), where \( w_{2,2} \) is the polynomial given in (1.16) with \( k = \ell = 2 \).

We now explain how our results relate to the previous literature on mean values of Dirichlet polynomials. Setting \( \varphi_0 = 1_{[0,1]} \) in (1.9) we get the following Dirichlet polynomials associated to the
real sequences \( \{a(n)\} \) and \( \{b(n)\} \):
\[
A(s) := A_{a,\varphi_0}(s) = \sum_{n \leq K} \frac{a(n)}{n^s} \quad \text{and} \quad B(s) := B_{b,\varphi_0}(s) = \sum_{n \leq K} \frac{b(n)}{n^s}.
\]
Set \( \omega(t) \) to be \( \mathbf{1}_{[0,T]}(t) \) in \((1.10)\). A standard tool in analytic number theory is the mean value estimate
\[
D_{a,a;\omega_0} = \int_R \omega_0(t) A(\tfrac{1}{2} + it)^2 dt = \int_0^T |A(\tfrac{1}{2} + it)|^2 dt = \sum_{n \leq K} a(n)^2 \frac{1}{n} (T + O(n)),
\]
which follows from the work of Montgomery and Vaughan \[31\] Corollary 3. When \( K = o(T) \) this implies \( D_{a,a;\omega_0} \sim T \sum_{n \leq K} \frac{a(n)^2}{n} \), and this is referred to as the ‘diagonal contribution’. In the case that \( K \gg T \), this is not always the correct asymptotic. Goldston and Gonek \[17\] studied \( D_{a,b;\omega} \) when \( K \gg T \) for a certain smooth weight \( \omega \) supported in \( [c_1 T, c_2 T] \) with \( 0 < c_1 < c_2 \). They showed that \( D_{a,b;\omega} \) is intimately related to the correlation sums \( C_{a,b}(x, r) = \sum_{n \leq x} a(n)b(n + r) \). They assume uniform formulae of the type \( C_{a,b}(x, r) = \mathcal{M}_{a,b}(x, r) + \epsilon_{a,b}(x, r) \) where \( \mathcal{M}_{a,b}(x, r) \) is a main term and \( \epsilon_{a,b}(x, r) \) is an error term. They show that when \( K \gg T \) the main term of \( D_{a,b;\omega} \) also contains an additional off-diagonal contribution which arises from certain averages of \( \mathcal{M}_{a,b}(x, r) \) and that the error term is related to averages of \( \epsilon_{a,b}(x, r) \) (see \[17\] Theorems 1, 2, Corollary 1).

In this article we only consider \( D_{a,b;\omega}(K) \) in the case \( a = \sigma_2 \) and \( b = \sigma_3 \). A key difference in our approach as opposed to \[17\] is that we make use of the smoothed additive sums \( D_{f,j;\omega}(r) \) rather than unsmoothed sums \( C_{a,b}(x, r) \). We handle the off-diagonals by using smooth partitions of unity so that we can deal with additive divisor sums of the type \( D_{f,j;\omega}(r) \). Goldston and Gonek \[17\] treat the off-diagonals via partial summation and the additive divisor sums \( C_{a,b}(x, r) \). A disadvantage of using the unsmoothed correlation sums is that most proofs of asymptotic formula of \( C_{a,b}(x, r) \) begin with considering a smoothed version of such sums. At the end of such arguments the smooth function is removed and this increases the size of the error term \( \epsilon_{a,b}(x, r) \).

It is natural to wonder if the mean values \( D_{a,b;\omega}(K) \) can be computed for other sequences. The main fact we have used is that generalized divisor functions are coefficients of \( L \)-functions (in the sense of Selberg). Here they correspond to the non-primitive \( L \)-functions \( \zeta(s + a_1) \cdots \zeta(s + a_k) \) and \( \zeta(s + b_1) \cdots \zeta(s + b_{\ell}) \). A natural generalization would be to consider sequences \( a(n), b(n) \) which are the Dirichlet series coefficients of \( L(s + a_1, \pi_1) \cdots L(s + a_k, \pi_k) \) and \( L(s + b_1, \pi_1') \cdots L(s + b_{\ell}, \pi_{\ell}') \) where the \( L(s, \pi_i) \) and \( L(s, \pi_j') \) are automorphic \( L \)-functions. It seems likely that the approach in this article would lead to an asymptotic evaluation of \( D_{a,b;\omega}(K) \), subject to a suitable additive divisor conjecture. In some cases, the additive divisor conjecture is known to be true. For instance, when \( a(n) = b(n) = r_2(n) \) with \( r_2(n) \) being the sum of two squares function, the conjecture holds (see \[20\] page 174).

1. Conventions and Notation. Given two functions \( f(x) \) and \( g(x) \), we shall interchangeably use the notation \( f(x) = O(g(x)) \), \( f(x) \ll g(x) \), and \( g(x) \gg f(x) \) to mean there exists \( M > 0 \) such

\[31 \mathbf{1}_B(t) \] denotes the indicator function corresponding to \( B \subset \mathbb{R} \).
that $|f(x)| \leq M|g(x)|$ for all sufficiently large $x$. We write $f(x) \asymp g(x)$ to mean that the estimates $f(x) \ll g(x)$ and $g(x) \ll f(x)$ simultaneously hold.

In this article we shall use the convention that $\varepsilon$ denotes an arbitrarily small positive constant which may vary from instance to instance. In addition, $B$ shall denote a positive constant, which may be taken arbitrarily large and which may change from line to line. The letter $p$ will always be used to denote a prime number. For a function $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$, $\varphi^{(m,n)}(x,y) = \frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial y^n} \varphi(x,y)$. The integral notation $\int_c f(s)ds$ for a complex function $f(s)$ and $c \in \mathbb{R}$ will be used frequently and is defined by the following contour integral

$$\int_c f(s)ds = \int_{c-i\infty}^{c+i\infty} f(s)ds.$$  \hfill (1.39)

In this article we shall consider $s \in \mathbb{C}$ and usually we shall write its real part as $\sigma = \Re(s)$. Throughout this article we often use the fact that $\omega(t)$ has support in $[c_1T, c_2T]$ so that $t \asymp T$. For a polynomial $P(X_1, \ldots, X_m) \in \mathbb{C}[X_1, \ldots, X_m]$ we write $\deg(P)$ to denote its degree.

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### 2. Preliminaries

In this section we describe the smooth weights $\varphi$ which occur in (1.9) and $\omega$ which occur in (1.10). We should think of $\varphi$ as a smooth approximation to $1_{[0,K]}$ and $\omega$ as a smooth approximation to $1_{[T,2T]}$. The function $\varphi$ is used so that we can work with the Mellin transform $\Phi(s)$ of $\varphi$ instead of Perron’s formula (see [15] and [31] for other such uses of the Mellin transform in analytic number theory). The smoothness of $\varphi$ allows integrals in the $s$-variable to be absolutely convergent. Similarly the weight $\omega$ is used so that its Fourier transform $\hat{\omega}(u)$ is small if $u$ satisfies $|u| \gg T_0^{1-\varepsilon}$, where $T_0$ is a parameter satisfying of $T^b \ll T_0 \ll T$ for some $b > 0$. In certain situations it is standard to remove the weight $\omega$ from the integral and replace it with a sharp cutoff function (for instance, if the integrand is positive). Previously, the use of functions such as $\omega$ can be found in the works [3] and [19].

**1. Properties of $\omega$.** Let $b$ be a positive absolute constant, and let $\omega$ be a function from $\mathbb{R}$ to $\mathbb{C}$ that satisfies the following:

- $\omega$ is smooth, \hfill (2.1)
- the support of $\omega$ lies in $[c_1T, c_2T]$ where $0 < c_1 < c_2$, \hfill (2.2)
- there exists $T_0 \geq T^b$ such that $T_0 \ll T$ and $\omega^{(j)}(t) \ll T_0^{-j}$.

The Fourier transform of $\omega$ is

$$\hat{\omega}(u) = \int_{\mathbb{R}} \omega(t)e^{-2\pi i ut}dt.$$  \hfill (2.4)
Integrating by parts $j$ times and using the second part of (2.3) we see that
\[
\hat{\omega}(u) = \frac{1}{(2\pi i)^j} \int_{\mathbb{R}} \omega^{(j)}(t)e^{-2\pi i ut} dt \ll \frac{T}{(T_0 u)^j}.
\] (2.5)

Thus,
\[
\text{if } |u| \gg T_0^{-1+\varepsilon}, \text{ then } |\hat{\omega}(u)| \ll T^{-A} \text{ for any } A > 0 \quad (2.6)
\]
by choosing $j \geq \frac{A+1}{\varepsilon_0}$.

2. Properties of $\varphi$. Let $\varphi \in (0, \frac{1}{2})$. Let $\varphi$ be a smooth, non-negative function defined on $\mathbb{R}_{\geq 0}$ such that it equals one on $[0, 1]$ and zero on $[1 + \varphi, \infty)$ and for all $j \geq 0$
\[
\varphi^{(j)}(t) \ll \varphi^{-j}. \quad (2.7)
\]

Its Mellin transform is
\[
\Phi(s) = \int_{0}^{\infty} \varphi(t)t^{s-1}dt, \quad (2.8)
\]
which converges absolutely for $\Re(s) > 0$. By Mellin inversion,
\[
\varphi(t) = \frac{1}{2\pi i} \int_{(c)} \Phi(s)t^{-s}ds, \quad (2.9)
\]
for $c > 0$. We now study $\Phi(s)$ further. Integrating by parts,
\[
\Phi(s) = \frac{1}{s} \Psi(s) \text{ where } \Psi(s) = -\int_{0}^{\infty} \varphi'(t)t^{s}dt. \quad (2.10)
\]
This is valid for $\Re(s) > 0$. It may be shown that $\Psi(s)$ is entire. Thus $\Phi(s)$ is holomorphic on $\mathbb{C}$ with the exception of a simple pole at $s = 0$. We have the Laurent expansion
\[
\Phi(s) = \frac{\Psi(0)}{s} + \Psi'(0) + \frac{\Psi''(0)}{2}s + \cdots. \quad (2.11)
\]

Note that
\[
\Psi(0) = -\int_{0}^{\infty} \varphi'(t)dt = \varphi(0) - \varphi(1 + \varphi) = 1.
\]

Next we provide useful bounds for $\Phi$. Integrating (2.8) by parts $m$ times, we find that
\[
\Phi(s) = \frac{(-1)^m}{s(s+1)\cdots(s+m-1)} \int_{0}^{\infty} \varphi^{(m)}(t)t^{s+m-1}dt,
\]
which is valid for all $s \in \mathbb{C} \setminus \{0\}$. Note that for $m \geq 2$ the integrand has simple zeros at $s = -1, \ldots, -(m-1)$. Thus, for $m \geq 1$ and $s \in \mathbb{C} \setminus \{0, -1, \ldots, -(m-1)\}$,
\[
|\Phi(s)| \leq \frac{1}{|s(s+1)\cdots(s+m-1)|} \int_{1}^{1+\varepsilon} |\varphi^{(m)}(t)|t^{\sigma+m-1}dt \ll_{m} \frac{\varphi^{1-m}(1+\varphi)^{\sigma+m-1}}{|s(s+1)\cdots(s+m-1)|}. \quad (2.12)
\]

Observe that this implies
\[
|\Phi(s)| \ll_{\sigma, \varphi, m} |\Im(s)|^{-m} \text{ for } |\Im(s)| \geq 1. \quad (2.13)
\]

Let $c > 0$. For $\Re(s) > c$, we define
\[
\Phi_2(s) = \frac{1}{2\pi i} \int_{(c)} \Phi(s_1)\Phi(s-s_1)ds_1. \quad (2.14)
\]
We shall encounter this function frequently. Observe that by a version of the convolution formula (see [35] eq. (3.1.14), p. 83)

$$
\Phi_2(s) = \int_0^\infty \varphi(t)^2 t^{s-1} dt \quad \text{and} \quad \varphi(t)^2 = \frac{1}{2\pi i} \int_c \Phi_2(s)t^{-s} ds \quad \text{for} \quad c > 0.
$$

(2.15)

Note that

$$
\Phi_2(s) - \Phi(s) = \int_1^{1+\rho} \varphi(t)(\varphi(t) - 1) t^{s-1} dt.
$$

By [12] Ch. 5, p.108 it follows that the right hand side is an entire function. Thus we see that \( \Phi_2(s) \) has a simple pole at \( s = 0 \) and \( \Phi_2(s) = \Phi(s) + R(s) \) with \( R(s) \) an entire function.

3. The Dirichlet series \( Z_{\mathcal{J},\mathcal{J}}(s) \). In this article we shall encounter the Dirichlet series

$$
Z_{\mathcal{J},\mathcal{J}}(s) = \sum_{m=1}^{\infty} \frac{\sigma_2(m)\sigma_2(m)}{m^{1+s}}.
$$

(2.16)

From (1.20) and (1.25), we have the alternate expression

$$
Z_{\mathcal{J},\mathcal{J}}(s) = B(\mathcal{J},\mathcal{J}) = Z(\mathcal{J},\mathcal{J})A(\mathcal{J},\mathcal{J}),
$$

(2.17)

where the first equality follows from (1.26) and \( Z \) and \( A \) are defined in Definition 2. For example, when \( \mathcal{J} = \{a_1, a_2\} \) and \( \mathcal{J} = \{b_1, b_2\} \), it follows from a formula of Ramanujan (see [37] Eq 1.3.3) that

$$
Z_{\mathcal{J},\mathcal{J}}(s) = \frac{\zeta(1+s+a_1+b_1)\zeta(1+s+a_1+b_2)\zeta(1+s+a_2+b_1)\zeta(1+s+a_2+b_2)}{\zeta(2+2s+a_1+a_2+b_1+b_2)},
$$

and so

$$
A(\mathcal{J},\mathcal{J}) = \frac{1}{\zeta(2+2s+a_1+a_2+b_1+b_2)}.
$$

(2.18)

The next lemma gives an analytic continuation of \( Z_{\mathcal{J},\mathcal{J}}(s) \) and demonstrates that it has simple poles at

$$
s = -a_i - b_j \quad \text{for} \quad i \in \mathcal{K}, j \in \mathcal{L},
$$

(2.19)

as long as the elements are distinct.

**Lemma 2.1.** Let \( \delta \in (0, \frac{1}{2}) \). Assume that \( \mathcal{J} \) and \( \mathcal{J} \) satisfy (1.12). For \( \Re(s) > 2\delta \), we have that

$$
Z_{\mathcal{J},\mathcal{J}}(s) = \left( \prod_{i \in \mathcal{K}} \zeta(1+s+a_i+b_i) \right) A_{\mathcal{J},\mathcal{J}}(s),
$$

(2.20)

where \( A_{\mathcal{J},\mathcal{J}}(s) = A(\mathcal{J},\mathcal{J}) \) is holomorphic in \( \Re(s) > -\frac{1}{2} + 2\delta \).

**Proof.** We let \( \sigma = \Re(s) \). By multiplicativity we have

$$
Z_{\mathcal{J},\mathcal{J}}(s) = \prod_p \sum_{u=0}^{\infty} \frac{\sigma_2(p^u)\sigma_2(p^u)}{p^{u(1+s)}} = \prod_p \left( 1 + \sum_{i \in \mathcal{K}} \sum_{j \in \mathcal{L}} p^{-(1-s)-a_i-b_j} + \sum_{u=2}^{\infty} \frac{\sigma_2(p^u)\sigma_2(p^u)}{p^{u(1+s)}} \right).
$$
We now factor out \( \prod_{i \in \mathcal{X}} \prod_{j \in \mathcal{L}} p^{(1-p^{-1-s-a_i-b_j})-1} \) to obtain

\[
Z_{j,\beta}(s) = \prod_{i \in \mathcal{X}} \prod_{j \in \mathcal{L}} \zeta(1 + s + a_i + b_j) A_{j,\beta}(s),
\]

where \( A_{j,\beta}(s) = \prod_{i \in \mathcal{X}} A_{p,j,\beta}(s) \) and

\[
A_{p,j,\beta}(s) = \prod_{i \in \mathcal{X}, j \in \mathcal{L}} (1 - p^{-1-s-a_i-b_j}) \left(1 + \sum_{i \in \mathcal{X}, j \in \mathcal{L}} p^{-1-s-a_i-b_j} + \sum_{u=2}^{\infty} \frac{\sigma_j(p^u) \sigma_j(p^u)}{p^u(1+s)}\right).
\]

Expanding out and using (1.12) the first factor is

\[
\prod_{i \in \mathcal{X}, j \in \mathcal{L}} (1 - p^{-1-s-a_i-b_j}) = 1 - \sum_{i \in \mathcal{X}, j \in \mathcal{L}} p^{-1-s-a_i-b_j} + O\left(\sum_{u=1}^{k\ell} p^{-2u-2u\sigma+4j\delta}\right) = 1 - \sum_{i \in \mathcal{X}, j \in \mathcal{L}} p^{-1-s-a_i-b_j} + O\left(p^{-2-2\sigma+4\delta}\right)
\]

if \( \sigma > 2\delta - 1 \). Using (1.12) and the fact that \( \sigma_j(p^u) \ll_{k,\ell} p^{u\delta} \), we find that

\[
\sum_{u=2}^{\infty} \frac{\sigma_j(p^u) \sigma_j(p^u)}{p^u(1+s)} \ll \sum_{u=2}^{\infty} \frac{p^{2u\delta}}{p^u(1+\sigma)} \ll p^{-2-2\sigma+4\delta}.
\]

It follows that

\[
A_{p,j,\beta}(s) = \left(1 - \sum_{i \in \mathcal{X}, j \in \mathcal{L}} p^{-1-s-a_i-b_j} + O\left(p^{-2-2\sigma+4\delta}\right)\right) \left(1 + \sum_{i \in \mathcal{X}, j \in \mathcal{L}} p^{-1-s-a_i-b_j} + O\left(p^{-2-2\sigma+4\delta}\right)\right)
\]

\[
= 1 + O_{k,\ell}\left(p^{-2-2\sigma+4\delta}\right).
\]

Hence, \( A_{j,\beta}(s) = \prod_{p} A_{p,j,\beta}(s) \) converges absolutely if \( \sigma > -\frac{1}{2} + 2\delta \).

\[
\square
\]

3. Setting up the evaluation of \( \mathcal{D}_{j,\beta\omega}(K) \)

Let us begin our evaluation of \( \mathcal{D}_{j,\beta\omega}(K) \). By splitting into diagonal terms \( m = n \) and off-diagonal terms \( m \neq n \) we have

\[
\mathcal{D}_{j,\beta\omega}(K) = \sum_{m,n=1}^{\infty} \frac{\sigma_j(m) \sigma_j(n) \varphi(m) \varphi(n)}{\sqrt{mn}} \hat{o}\left(\frac{1}{\pi^2} \log(\frac{m}{n})\right) = \mathcal{D}_{j,\beta\omega}^{\text{diag}}(K) + \mathcal{D}_{j,\beta\omega}^{\text{off}}(K)
\]

where

\[
\mathcal{D}_{j,\beta\omega}^{\text{diag}}(K) = \hat{o}(0) \sum_{m=1}^{\infty} \frac{\sigma_j(m) \sigma_j(m) \varphi(m)^2}{m} \frac{\varphi(m)}{\pi^2} \hat{o}\left(\frac{1}{\pi^2} \log(\frac{m}{n})\right),
\]

\[
\mathcal{D}_{j,\beta\omega}^{\text{off}}(K) = \sum_{m \neq n} \frac{\sigma_j(m) \sigma_j(n) \varphi(m) \varphi(n)}{\sqrt{mn}} \hat{o}\left(\frac{1}{\pi^2} \log(\frac{m}{n})\right).
\]

\[
(3.2)
\]
First we examine the contribution of \( \mathcal{D}_{2,\beta,\omega}^{\text{diag}}(K) \). Using (2.14) and (2.15), we have for \( c > 0 \)
\[
\mathcal{D}_{2,\beta,\omega}^{\text{diag}}(K) = \hat{\omega}(0) \sum_{m=1}^{\infty} \frac{\sigma_2(m) \sigma_3(m) \varphi(m)}{m} = \frac{\hat{\omega}(0)}{2\pi i} \sum_{m=1}^{\infty} \frac{\sigma_1(m) \sigma_3(m)}{m} \int_{(2c)} \Phi_2(s) \left( \frac{m}{K} \right)^{-s} ds.
\]
Swapping summation and integration order we obtain
\[
\mathcal{D}_{2,\beta,\omega}^{\text{diag}}(K) = \frac{\hat{\omega}(0)}{2\pi i} \int_{(2c)} K^s \Phi_2(s) Z_{J,\beta}(s) ds = \mathcal{M}_{0,\beta,\omega}(K),
\]
where \( Z_{J,\beta}(s) \) is defined in (2.16) and \( \mathcal{M}_{0,\beta,\omega}(K) \) is defined in (1.31). Observe that these two expressions are equal since by (2.17) we have \( Z_{J,\beta}(s) = B(J, \beta) \).

The key part of the calculation of \( \mathcal{D}_{J,\beta,\omega}(K) \) is that of the off-diagonal term \( \mathcal{D}_{2,\beta,\omega}^{\text{off}}(K) \). We establish the following result.

**Proposition 3.1.** Let \( |J| = k \) and \( |\beta| = \ell \) with \( k, \ell \geq 2 \), and suppose that \( J \) and \( \beta \) satisfy (1.13) and (1.14). Assume that \( AD_{k,\ell}(\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell}) \) holds for some triple \( (\vartheta_{k,\ell}, C_{k,\ell}, \beta_{k,\ell}) \in \left[ \frac{1}{2}, 1 \right] \times [0, \infty) \times (0, 1] \). Let \( K = T^{1+\eta} \) with \( \eta > 0 \), and let \( \omega \) satisfy (2.11), (2.2), and (2.3) with \( b > \frac{(1-\beta_{k,\ell})(1+\eta)}{1-\epsilon} \) and \( 0 < \epsilon < 1 \). Then we have for any \( \varepsilon > 0 \)
\[
\mathcal{D}_{2,\beta,\omega}^{\text{off}}(K) = \mathcal{M}_{1,\beta,\omega}(K) + O \left( K^{2+\varepsilon} \left( \frac{T}{T_0} \right)^{1+C_{k,\ell}} \right),
\]
where \( \mathcal{M}_1 \) is given in (1.32).

This result builds on previous work of Hughes-Young [23] on the twisted fourth moment of the Riemann zeta function and work of the second author [32] on the sixth moment of the Riemann zeta function. Those results are directly related to the cases \( (k, \ell) \in \{(2, 2), (3, 3)\} \) of this theorem.

Combining (3.1), (3.3), and Proposition 3.1 yields Theorem 1.1. The proof of Proposition 3.1 will be given in Sections 4 and 5.

### 4. Off-Diagonal Terms

In this section we evaluate the off-diagonal terms \( \mathcal{D}_{2,\beta,\omega}^{\text{off}}(K) \). In order to evaluate this we must impose some initial size conditions on \( J \) and \( \beta \). We require that the coefficients satisfy (1.14). These conditions are imposed since some error terms involve factors of the form \( \zeta(1-a_{11}+a_{12}) \) and \( \zeta(1-b_{11}+b_{12}) \) which are unbounded unless (1.14) is imposed. These conditions can be removed via use of an argument with Cauchy’s integral formula (see the argument in [32 pp.20-21]). Recall that
\[
\mathcal{D}_{2,\beta,\omega}^{\text{off}}(K) = \sum_{m \neq n} \frac{\sigma_1(m) \sigma_3(n) \varphi(m/n) \varphi(n)}{\sqrt{mn}} \hat{\omega}(\frac{1}{2\pi} \log(\frac{m}{n})).
\]

1. **Smooth partition of unity.** First, we apply a dyadic partition of unity to the sums over \( m \) and \( n \). To do this, we consider a smooth non-negative function \( W_0 \) on \((0, \infty)\) whose support lies in
[1, 2], and which satisfies \( \sum_{k \in \mathbb{Z}} W_0(x/2^k) = 1 \), for all \( x > 0 \). This implies that
\[
\sum_{\substack{M=2^k \geq 1 \\text{k} \geq -1}} W_0 \left( \frac{x}{M} \right) = 1 \text{ for } x \geq 1.
\] (4.2)

An example of such a function is given in [18, Section 5]. Given two integers \( m, n \geq 1 \), we have
\[
\sum_{M} W_0 \left( \frac{m}{M} \right) \sum_{N} W_0 \left( \frac{n}{N} \right) = 1,
\] (4.3)
where \( M, N \in \{2^k \mid k \in \mathbb{Z} \text{ and } k \geq -1\} \). We shall often use the fact that \( \# \{M \mid M \leq X \} \ll \log X \).

Upon inserting the identity (4.3) in (4.1), we get
\[
\sum_{M} W_0 \left( \frac{m}{M} \right) \sum_{N} W_0 \left( \frac{n}{N} \right) = 1,
\]
where \( K' = (1 + \varrho) K \) and \( W(x) = x^{-\frac{1}{2}} W_0(x) \). By setting
\[
I_{M,N} = \sum_{\substack{m \neq n \\text{M}\text{n}, M \leq m \leq 2M \\text{N}\text{n}, N \leq n \leq 2N}} \sigma_3(m) \sigma_3(n) W \left( \frac{m}{M} \right) W \left( \frac{n}{N} \right) \varphi \left( \frac{m}{K} \right) \varphi \left( \frac{n}{K} \right) \frac{\varpi \left( \frac{1}{2\pi} \log \left( \frac{m}{n} \right) \right)}{T},
\] (4.4)
we can write
\[
\mathcal{G}_{2,3,\omega}(K) = \sum_{M,N \leq K'} \frac{T}{\sqrt{MN}} I_{M,N}.
\] (4.5)

2. Restricting \( M \) and \( N \). First, we observe that if \( M < \frac{N}{3} \) or \( M > 3N \), then the variables \( m \) and \( n \) satisfy \( |\log \left( \frac{m}{n} \right)| > \log \left( \frac{3}{2} \right) \). In this case, we know from (2.6) that for any \( A > 0 \), we have \( \varpi \left( \frac{1}{2\pi} \log \left( \frac{m}{n} \right) \right) \ll T^{-A} \). Hence, we see that \( I_{M,N} \) can be made very small. Throughout the rest of this article we write \( M \asymp N \) to mean \( \frac{N}{3} \leq M \leq 3N \). Furthermore, we may restrict the sum in (4.3) to integers \( m, n \) such that
\[
|\log \left( \frac{m}{n} \right)| \ll T_0^{-1+\varepsilon}.
\] (4.6)

In fact, by (2.6) we know that if \( |\log \left( \frac{m}{n} \right)| \gg T_0^{-1+\varepsilon} \), then for any \( A > 0 \), we have \( \varpi \left( \frac{1}{2\pi} \log \left( \frac{m}{n} \right) \right) \ll T^{-A} \). Therefore, the integers \( m, n \) satisfying \( |\log \left( \frac{m}{n} \right)| \gg T_0^{-1+\varepsilon} \) lead to an error term of the form \( O(T^{O(1)-A}) \) for any \( A > 0 \). Without loss of generality, we may assume that \( m > n \) and \( m = n + r \) with \( r \geq 1 \). Thus, condition (4.6) becomes \( |\log \left( 1 + \frac{r}{n} \right)| \ll T_0^{-1+\varepsilon} \), and so \( |r| \ll nT_0^{-1+\varepsilon} \ll MT_0^{-1+\varepsilon} \) since \( N \asymp M \). Hence,
\[
I_{M,N} = \sum_{1 \leq |r| \ll nT_0^{-1+\varepsilon}} \sum_{m, n \geq 1} \sigma_3(m) \sigma_3(n) W \left( \frac{m}{M} \right) W \left( \frac{n}{N} \right) \varphi \left( \frac{m}{K} \right) \varphi \left( \frac{n}{K} \right) \frac{\varpi \left( \frac{1}{2\pi} \log \left( \frac{m}{n} \right) \right)}{T} + O(T^{-A}).
\]
Finally, observe that the condition on $r$ can be replaced by $0 < |r| \ll (MN)^{1/2} T_0^{-1+\epsilon}$ since $M \asymp N$.

To summarize, we state the following proposition.

**Proposition 4.1.** If $\frac{N}{3} \leq M \leq 3N$, then for any $A > 0$ we have

$$I_{M,N} = \sum_{0 < |r| \ll \sqrt{MN} T_0^{-1+A}} \sum_{m,n \geq 1} \sigma_3(m) \sigma_3(n) f^*(m,n) + O(T^{-A}),$$

where

$$f^*(m,n) = W \left( \frac{m}{M} \right) W \left( \frac{n}{N} \right) \varphi \left( \frac{m}{K} \right) \varphi \left( \frac{n}{K} \right) \frac{\tilde{\omega}(\frac{1}{2\pi} \log(\frac{M}{n}))}{T}. \tag{4.7}$$

If $M < \frac{N}{3}$ or $M > 3N$, then for any $B > 0$ we have $I_{M,N} = O(T^{-B}).$

Observe that if $m - n = r$, we have

$$f^*(m,n) = W \left( \frac{m}{M} \right) W \left( \frac{n}{N} \right) \varphi \left( \frac{m}{K} \right) \varphi \left( \frac{n}{K} \right) \frac{\tilde{\omega}(\frac{1}{2\pi} \log(1 + \frac{r}{y}))}{T}. \tag{4.8}$$

3. **Applying the Additive Divisor Conjecture.** By combining (4.5), Proposition 4.1 and (4.8), we obtain

$$\mathcal{D}_{\beta,\omega}(K) = \sum_{M,N \leq K} \frac{T}{\sqrt{MN}} \sum_{0 < |r| \ll \frac{\sqrt{MN}}{T_0 - A}} D_{f^*,\beta}(r) + O(T^{-A}), \tag{4.9}$$

where for $0 \neq r \in \mathbb{Z}$ we define

$$f_r(x,y) := f_{r,M,N}(x,y) = W \left( \frac{x}{M} \right) W \left( \frac{y}{N} \right) \varphi \left( \frac{x}{K} \right) \varphi \left( \frac{y}{K} \right) \frac{\tilde{\omega}(\frac{1}{2\pi} \log(1 + \frac{r}{y}))}{T}. \tag{4.10}$$

when $x, y > 0$ (and put $f_r(x,y) = 0$ otherwise) and $D_{f^*,\beta}(r)$ is the additive divisor sum associated to $f_r, \beta, \gamma, \delta$, and $r$ as defined in (1.33). Observe that in (4.9), for each fixed $0 \neq |r| \ll \frac{\sqrt{MN}}{T_0}$, the inner sum contains the additive divisor sum $D_{f^*,\beta}(r)$ associated to distinct functions $f_r$.

In order to apply the additive divisor conjecture, Conjecture [1] we use the following lemma which shows that the smooth function $f_r$ and its partial derivatives of all orders satisfy certain bounds.

The proof of this technical lemma is given in Section [4].

**Lemma 4.2.** Let $M \asymp N$ and $(x,y) \in [M,2M] \times [N,2N]$ with $1 \leq |r| \ll \frac{\sqrt{MN}}{T_0} T^\epsilon$. Then

$$x^m y^n f_r^{(m,n)}(x,y) \ll P^n, \text{ where } P = \left( \frac{T}{T_0} \right) T^\epsilon.$$

We must also ensure that the length of the sum over $r$ in (4.9) is within the range $H \ll M^{\beta_k,\ell}$ as required by Conjecture [4]. We have

$$\frac{\sqrt{MN}}{T_0^{1-\epsilon}} \ll MT^{-b(1-\epsilon)} \leq M^{1-\frac{b+1}{1+\eta}}.$$

If we assume that $b > \frac{(1-\beta_k,\ell)(1+\eta)}{1-\epsilon}$ for $0 < \epsilon < 1$, we get that $\frac{\sqrt{MN}}{T_0} \ll M^{\beta_k,\ell}$ as desired. We are now ready to apply the additive divisor conjecture to compute the main terms in $\mathcal{D}_{\beta,\omega}(K)$. To
simplify notation, it is convenient to set
\[ c_{i_1,i_2} = \prod_{j_1 \in \mathcal{K} \setminus \{i_1\}} \zeta(1 - a_{i_1} + a_{j_1}) \prod_{j_2 \in \mathcal{L} \setminus \{i_2\}} \zeta(1 - b_{i_2} + b_{j_2}), \tag{4.11} \]
where \( \mathcal{K} \) and \( \mathcal{L} \) are as in (1.11). We also set
\[ N_{f;i_1,i_2}(r) = c_{i_1,i_2} \sum_{q=1}^{\infty} \frac{c_q(r)G_2(1 - a_{i_1}, q)G_3(1 - b_{i_2}, q)}{q^{a_{i_1} - b_{i_2}}} \int_{\max(0,r)}^{\infty} f(x, x - r)x^{-a_{i_1}}(x - r)^{-b_{i_2}} \, dx, \tag{4.12} \]
and
\[ u_{q,r;i_1,i_2} = c_q(r)G_2(1 - a_{i_1}, q)G_3(1 - b_{i_2}, q)q^{-2 + a_{i_1} + b_{i_2}}. \tag{4.13} \]
It follows from the definition of \( D_{f;\beta}(r) \) in (1.33), Proposition (4.1), and an application of Conjecture \( H \) with \( X = M, \ Y = N, \ M \asymp N \) and \( P = (T_0)T^\varepsilon \) that
\[ I_{M,N} = \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} \sum_{0 < |r| \ll \sqrt{MN}/T_0^\varepsilon} N_{f;i_1,i_2}(r) + \mathcal{E}_{M,N} + O \left( T^{-A} \right), \tag{4.14} \]
where
\[ \mathcal{E}_{M,N} = \sum_{0 < |r| \ll \sqrt{MN}/T_0^\varepsilon} \Delta_{f;\beta}(r) = O \left( \left( \frac{T}{T_0} \right)^{C_{k,\ell}} \frac{T^\varepsilon}{T_0^\varepsilon} M^{g_{k,\ell} + 1 + \varepsilon} \right). \tag{4.15} \]
We now estimate the contribution of the error terms \( \mathcal{E}_{M,N} \) when substituted in (4.5).

**Lemma 4.3.** Let \( M \asymp N \). We have
\[ \sum_{M \leq N \leq K'} \sum_{M \asymp N} \frac{T}{\sqrt{MN}} \mathcal{E}_{M,N} \ll T^\varepsilon \left( \frac{T}{T_0} \right)^{1 + C_{k,\ell}} K^{g_{k,\ell}}. \]

**Proof.** We have
\[ \frac{T}{M} \mathcal{E}_{M,N} \ll T^\varepsilon \left( \frac{T}{T_0} \right)^{1 + C_{k,\ell}} M^{g_{k,\ell} + \varepsilon}. \]
It follows that
\[ \sum_{M \leq N \leq K'} \sum_{M \asymp N} \frac{T}{\sqrt{MN}} \mathcal{E}_{M,N} \ll T^\varepsilon \left( \frac{T}{T_0} \right)^{1 + C_{k,\ell}} \sum_{M \leq K'} \sum_{N \asymp M} M^{g_{k,\ell} + \varepsilon} \ll K^{g_{k,\ell} + \varepsilon} \left( \frac{T}{T_0} \right)^{1 + C_{k,\ell}}. \]

\[ \square \]
Hence, by (1.5), (4.14), and Lemma 4.3, we have
\[ \mathcal{D}_{\beta;\omega}(K) = \sum_{M \leq N \leq K'} \frac{T}{\sqrt{MN}} \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} \sum_{0 < |r| \ll \sqrt{MN}/T_0^\varepsilon} N_{f;\beta}(r) + O \left( K^{g_{k,\ell}} T^\varepsilon \left( \frac{T}{T_0} \right)^{1 + C_{k,\ell}} \right). \tag{4.16} \]
The next step is to extend the summation over \( r \) to \( |r| \leq R_0 \), where we set \( R_0 = T^A \) for some large enough fixed \( A \). This will be useful later when we extend the summation over \( r \) to all integers.
Recalling the definition of $f$ in (4.7), the integral in (4.12) is

$$i = \int_{\max(0, r)}^{\infty} W \left( \frac{x}{M} \right) W \left( \frac{x - r}{N} \right) \varphi \left( \frac{x}{K} \right) \varphi \left( \frac{x - r}{K} \right) \frac{\tilde{\omega} \left( \frac{1}{2\pi} \log \left( \frac{x - r}{T} \right) \right)}{T} x^{-a_1} (x - r)^{-b_2} \, dx.$$  

First, observe that the integrand is zero unless $x \in [M, 2M]$ and $x \in [N + r, 2N + r]$, since otherwise $W \left( \frac{x}{M} \right) W \left( \frac{x - r}{N} \right) = 0$. Suppose without loss of generality that $r \geq 1$. In order for these conditions to hold, the intervals $[M, 2M]$ and $[N + r, 2N + r]$ must intersect. In particular, we must have $N + r \leq 2M$ and thus $r \leq 2M - N \leq 2M - M = \frac{5M}{3}$, since $\frac{M}{3} \leq N \leq 3N$.

Notice that $x - r \geq N \geq 2^{-\frac{1}{2}}$, and so $x \geq r + 2^{-\frac{1}{2}}$. By (2.5) we have

$$\frac{\tilde{\omega} \left( \frac{1}{2\pi} \log \left( \frac{x - r}{T} \right) \right)}{T} \ll \frac{1}{|\log \left( \frac{x - r}{T} \right)|^j} \leq \frac{1}{|\log \left( \frac{M - r}{T} \right)|^j} \text{ since } x \in [r + 2^{-\frac{1}{2}}, 2M]$$

(4.17)

where for the last inequality we used $|\log \left( \frac{1}{2\pi} \right)| > \frac{r}{2M}$ since $0 < \frac{r}{2M} \leq \frac{5}{6}$. If $\frac{r}{2M} \gg \frac{c}{T^0}$, then (4.7) and (4.17) give

$$f(x, x - r) \ll T^{-j} \ll T^{-B} \text{ for any } B > 0$$

(4.18)

by choosing $j$ sufficiently large. We note that (4.18) still holds for $\frac{r}{2M} > \frac{5}{6}$, since then $f(x, x - r) = 0$. It follows that $i \ll MT^{-B}$ and that for $|r| \leq R_0$ all of these terms contribute

$$\mathcal{E}^{\mathcal{T}} := \frac{T}{\sqrt{MN}} \sum_{\substack{\mathbf{x} \leq |r| \leq R_0 \\mathbf{x} \leq 1 \mathbf{i} \leq 1 \mathbf{j}_1 = 1 \mathbf{j}_2 = 1}} \sum_{q = 1}^{\infty} \sum_{u_{q, r, i_1, i_2}} |u_{q, r, i_1, i_2}| MT^{-B},$$

where $c_{i_1, i_2}$ and $u_{q, r, i_1, i_2}$ are defined in (4.11) and (4.13) respectively. Observe that by (4.14) it follows that

$$c_{i_1, i_2} = O((\log T)^{(k - 1)(\ell - 1)})$$

(4.19)

and

$$\sum_{q \geq 1} |u_{q, r, i_1, i_2}| \ll \sum_{q \geq 1} \gcd(q, r) q^{-2 + 4\delta} = \sum_{g | r} g^{-1 + 4\delta} \sum_{\substack{q' \geq 1 \\gcd(q', r/g) = 1}} (q')^{-2 + 4\delta} \ll \tau_2(r),$$

(4.20)

since $|c_q(r)| \leq \gcd(q, r)$ and

$$|G_2(1 - a_{i_1}, q)G_3(1 - b_{i_2}, q)| \ll C^{\omega(q)} q^{2\delta}$$

(4.21)

for some $C > 0$. This follows from the the multiplicativity of $G_2(s, n)$ and the bound (6.24) established in Lemma 6.3 below. Hence it follows that

$$\mathcal{E}^{\mathcal{T}} \ll T^{1 - B} (\log T)^{(k - 1)(\ell - 1)} \left( \sum_{1 \leq |r| \leq R_0} \tau_2(r) \right) \ll R_0 (\log T)^{(k - 1)(\ell - 1) + 1} T^{1 - B} \ll T^{-A},$$

(4.22)

by choosing $B = 1 + 2A + \varepsilon$. This shows that we can extend the summation over $r$ in (4.16) to $|r| \leq R_0$ with a negligible error as desired.
We now extend the sum in (4.16) to all \( M, N \). If \( M \neq N \), then we have \( N > 3M \) or \( N < M/3 \). Suppose without loss of generality that \( N > 3M \). Since \( M \leq x \leq 2M \) and \( N \leq x - r \leq 2N \), it follows that \( |\log(\frac{x}{r})| = \log(\frac{2M}{x}) \geq \log(\frac{N}{3M}) \geq \log(\frac{1}{2}) \). Therefore, \( i \ll \max(2M, 2N + r)T^{-B} \) for any \( B > 0 \). It follows that we can add in the condition \( M \neq N \) into (4.16) with an error of size \( O(T^{-A}) \), by arguing as we did in the previous paragraph. Finally, we note that the terms with \( M > K' \) or \( N > K' \) vanish since \( f(x, x - r) = 0 \) then. Hence, we have shown that we can extend the summations in (4.16) to \( |r| \leq R_0 \) and all \( M, N \in \mathbb{N} \) with negligible error, thus proving the following proposition.

**Proposition 4.4.**

\[
\mathcal{G}_{3,\beta,\omega}^{\text{off}}(K) = \sum_{M,N} \frac{T}{\sqrt{MN}} \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} \sum_{1 \leq |r| \leq R_0} N_{f;i_1,i_2}(r) + O\left(K^{\theta_{k,\ell}} T^{\frac{1}{1+c_{k,\ell}}} \right),
\]

(4.23)

where by (4.7) and (4.12) we have

\[
N_{f;i_1,i_2}(r) = c_{i_1,i_2} \sum_{q=1}^{\infty} \frac{c_q(r)G_j(1-a_{i_1},q)G_j(1-b_{i_2},q)}{q^{2-a_{i_1}-b_{i_2}}} \times \int_{\max(0,r)}^{\infty} x^{-a_{i_1}} (x-r)^{-b_{i_2}} W\left(\frac{a}{\ell}\right) W\left(\frac{b}{\ell}\right) \varphi\left(\frac{x}{\ell}\right) \varphi\left(\frac{x}{2}\right) \frac{\hat{\omega}(\frac{1}{2\pi} \log(\frac{x}{x-r}))}{T} \, dx.
\]

Our goal is to evaluate \( \mathcal{G}_{3,\beta,\omega}^{\text{off}}(K) \) asymptotically. More precisely, we want to prove that

\[
\sum_{M,N} \frac{T}{\sqrt{MN}} \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} \sum_{1 \leq |r| \leq R_0} N_{f;i_1,i_2}(r) = M_{1,\beta,\omega}(K) + O(T^{8\delta} \log T),
\]

(4.24)

where \( M_{1,\beta,\omega}(K) \) is the double integral given by (1.32). Towards proving (4.24), we shall establish the following proposition in the remaining part of this section.

**Proposition 4.5.**

\[
\sum_{M,N} \frac{T}{\sqrt{MN}} \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} \sum_{1 \leq |r| \leq R_0} N_{f;i_1,i_2}(r) = \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} \frac{c_{i_1,i_2}}{(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int_{(1)} \int_{(1)} \Phi(s_1) \Phi(s_2) K^{s_1+s_2} H_{3,\beta;i_1,i_2}(s_1+s_2) \times \Gamma(a_{i_1}+b_{i_2}+s_1+s_2) \left( \frac{\Gamma\left(\frac{1}{2} - b_{i_2} - s_2 + it\right)}{\Gamma\left(\frac{1}{2} + a_{i_1} + s_1 + it\right)} + \frac{\Gamma\left(\frac{1}{2} - a_{i_1} - s_1 + it\right)}{\Gamma\left(\frac{1}{2} + b_{i_2} + s_2 + it\right)} \right) \, ds_1 \, ds_2 \, dt + O(1),
\]

where

\[
H_{3,\beta;i_1,i_2}(s) = \sum_{r=1}^{\infty} \sum_{q=1}^{\infty} \frac{c_q(r)G_j(1-a_{i_1},q)G_j(1-b_{i_2},q)}{q^{2-a_{i_1}-b_{i_2},a_{i_1}+b_{i_2}+s}}.
\]

(4.25)

There are quite a few steps involved in proving this proposition which we now outline:

1. The sum over \( M, N \) is executed and the smooth partition of unity (4.2) is applied again. This removes the functions \( W_0 \).
We often use the following weak form of Stirling’s formula
\[ \frac{1}{2\pi i} \int_{(c)} \Phi(s)u^{-s} \, du. \]

This creates a double integral in the variables \( s_1, s_2 \).

(3) The sum over \( r \) is extended to all integers which introduce the Dirichlet series \( H_{\beta}^{\mathbb{R}}(a_1), (b_2) (s) \).

In establishing Proposition 4.5 we require a number of technical results related to Stirling’s formula. We often use the following weak form of Stirling’s formula
\[ |\Gamma(\sigma + iu)| = \sqrt{2\pi}|u|^\sigma e^{-\frac{1}{2}\pi|u|}(1 + O(u^{-1})) \]for \( 0 < \sigma < 1 \) and \( |u| \geq 1 \), in addition to the trivial bound
\[ |\Gamma(\sigma + iu)| \ll \eta_0, A 1 \text{ for } |u| \leq 1, \sigma \in [-1 + \eta_0, -\eta_0] \cup [\eta_0, A], \]
where \( \eta_0 \in (0, \frac{1}{2}) \) and \( A > 0 \) are fixed constants. It follows that for \( \Re(s_1) = \Re(s_2) = 1 \), we have
\[
\Gamma(a_1 + b_2 + s_1 + s_2) \ll \begin{cases} 
|\Im(s_1 + s_2)|^{\frac{3}{2} + 2\delta} \exp\left(-\frac{\pi}{2} |\Im(s_1 + s_2)| \right) & \text{if } |\Im(s_1 + s_2)| \geq 1 \\
1 & \text{if } |\Im(s_1 + s_2)| \leq 1.
\end{cases} \quad (4.26)
\]

We also need the following lemma, the proof of which is given in Section 7.

**Lemma 4.6.** Suppose that \( a_1 \) and \( b_2 \) satisfy (1.12). Let \( \eta_0 \in (0, \frac{1}{2}) \) and \( A > 0 \). Assume that \( \Re(a_1 + s_1) \in [0, A] \) and \( \Re(b_1 + s_2) \in [0, \frac{1}{2} - \eta_0] \cup [\frac{3}{2} + \eta_0, \frac{3}{2} - \eta_0] \).

(i) Assume that \( \Re(s_1 + s_2 + a_1 + b_2) \leq 1 \). When \( |\Im(s_1)| \leq t + 1 \) and \( |\Im(s_2)| \leq t + 1 \), we have
\[
\frac{\Gamma(\frac{1}{2} - b_2 - s_2 + it)}{\Gamma(\frac{1}{2} + a_1 + s_1 + it)} = t^{-(s_1 + s_2 + a_1 + b_2)} \exp \left( \mp \frac{\pi}{2} (s_1 + s_2 + a_1 + b_2) \right) \quad (4.27)
\]
\[
\times \left( 1 + O \left( \frac{1 + |s_1|^2 + |s_2|^2}{t} \right) \right).
\]

(ii) When \( |\Im(s_1)| \geq t + 1 \) or \( |\Im(s_2)| \geq t + 1 \), we have the bound
\[
\frac{\Gamma(\frac{1}{2} - b_2 - s_2 + it)}{\Gamma(\frac{1}{2} + a_1 + s_1 + it)} \ll \frac{\Im(s_1)^2 + \Im(s_2)^2}{t^2} e^{\pm \frac{\pi}{2} |\Im(s_1 + s_2)|}, \quad (4.28)
\]

We remark that if \( \Re(s_1 + s_2 + a_1 + b_2) \geq 1 \), \( \Re(a_1 + s_1) \in [0, A] \) and \( \Re(b_1 + s_2) \in [0, \frac{1}{2} - \eta_0] \cup [\frac{3}{2} + \eta_0, \frac{3}{2} - \eta_0] \), then the proof of Lemma 4.6 (i) yields the asymptotic bound
\[
\frac{\Gamma(\frac{1}{2} - b_2 - s_2 + it)}{\Gamma(\frac{1}{2} + a_1 + s_1 + it)} \ll \frac{\Im(s_1)^2 + \Im(s_2)^2}{t^2} e^{\pm \frac{\pi}{2} |\Im(s_1 + s_2)|}, \quad (4.29)
\]
for \( |\Im(s_1)| \leq t + 1 \) and \( |\Im(s_2)| \leq t + 1 \).
Moreover, by applying (4.27) twice we get the asymptotic formula:

\[
\frac{\Gamma\left(\frac{1}{2} - b_{12} - s_2 + it\right)}{\Gamma\left(\frac{1}{2} + a_{11} + s_1 + it\right)} + \frac{\Gamma\left(\frac{1}{2} - a_{11} - s_1 - it\right)}{\Gamma\left(\frac{1}{2} + b_{12} + s_2 - it\right)} = 2t^{-a_{11} - b_{12} - s_1 - s_2} \cos\left(\frac{\pi}{2} (a_{11} + b_{12} + s_1 + s_2)\right) + O\left(e^{\frac{\pi}{2} |\Im(s_1 + s_2)|} \left(\frac{1 + |s_1|^2 + |s_2|^2}{\ell^{1-2\delta + R(s_1 + s_2)}}\right)\right)
\]

for \(|\Im(s_1)| \leq t + 1\) and \(|\Im(s_2)| \leq t + 1\) under the assumptions \(\Re(a_{11} + s_1), \Re(b_{12} + s_2) \in (0, \frac{1}{2} - \eta_0] \cup [\frac{1}{2} + \eta_0, 1]\), and \(\Re(s_1 + s_2 + a_{11} + b_{12}) \leq 1\).

**Proof of Prop 4.5** Recall that

\[
N_{f;i_1,i_2}(r) = c_{i_1,i_2} \sum_{q=1}^{\infty} u_{q;r;i_1,i_2} \int_{\max(0,r)}^{\infty} f(x, x - r)x^{-a_{11}}(x - r)^{-b_{12}} \, dx,
\]

where

\[
u_{q;r;i_1,i_2} = c_q(r)G_2(1 - a_{11}, q)G_3(1 - b_{12}, q)q^{-2 + a_{11} + b_{12}}.
\]

and

\[
f(x, x - r) = W\left(\frac{x}{M}\right)W\left(\frac{x - r}{N}\right) \varphi\left(\frac{x}{K}\right) \varphi\left(\frac{x - r}{K}\right) \hat{\omega}\left(\frac{1}{2\pi} \log(\frac{x}{x - r})\right).
\]

Using the fact that \(W(x) = x^{-\frac{1}{2}}W_0(x)\), we see that \(N_{f;i_1,i_2}(r)\) can be written as

\[
N_{f;i_1,i_2}(r) = \frac{\sqrt{MN}}{T}c_{i_1,i_2} \sum_{q=1}^{\infty} u_{q;r;i_1,i_2} \int_{\max(0,r)}^{\infty} W_0\left(\frac{x}{M}\right)W_0\left(\frac{x - r}{N}\right) x^{-a_{11} - \frac{1}{2}}(x - r)^{-b_{12} - \frac{1}{2}}
\]

\[
\varphi\left(\frac{x}{K}\right) \varphi\left(\frac{x - r}{K}\right) \hat{\omega}\left(\frac{1}{2\pi} \log(\frac{x}{x - r})\right) \, dx.
\]

Hence,

\[
\sum_{M,N} \frac{T}{\sqrt{MN}} \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} \sum_{1 \leq |r| \leq R_0} N_{f;i_1,i_2}(r)
\]

\[= \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} c_{i_1,i_2} \sum_{1 \leq |r| \leq R_0} \left(\sum_{M,N} W_0\left(\frac{x}{M}\right)W_0\left(\frac{x - r}{N}\right)\right) x^{-a_{11} - \frac{1}{2}}(x - r)^{-b_{12} - \frac{1}{2}}
\]

\[
\varphi\left(\frac{x}{K}\right) \varphi\left(\frac{x - r}{K}\right) \hat{\omega}\left(\frac{1}{2\pi} \log(\frac{x}{x - r})\right) \, dx.
\]

Using (4.2) we observe that, for \(0 \neq r \in \mathbb{Z}\), we have

\[
\sum_{M,N} W_0\left(\frac{x}{M}\right)W_0\left(\frac{x - r}{N}\right) = \begin{cases} 1 & \text{if } x > \max(0, r) + 1, \\ W_0\left(\frac{x - \max(0, r)}{2^{-\frac{1}{2}}}\right) & \text{if } \max(0, r) + 2^{-\frac{1}{2}} < x \leq \max(0, r) + 1, \\ 0 & \text{if } \max(0, r) \leq x \leq \max(0, r) + 2^{-\frac{1}{2}}. \end{cases}
\]
Substituting this expression, we see that we may replace \( \sum_{M,N} W_0(\frac{x}{M}) W_0(\frac{x-r}{N}) \) by 1 with an error of size \( O(1) \). This is very similar to an analogous calculation in [32] pp.28-29]. Therefore,

\[
\sum_{M,N} \frac{T}{\sqrt{MN}} \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} \sum_{1 \leq |r| \leq R_0} \sum_{q=1}^{\infty} N_f;i_1,i_2(r) \\
= \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} c_{i_1,i_2} \sum_{1 \leq |r| \leq R_0} \sum_{q=1}^{\infty} c_q(r) G_{\ell}(1 - a_{i_1} - q) G_{\ell}(1 - b_{i_2} - q) q^{-2+a_{i_1}+b_{i_2}} \\
\times \int_{max(0,r)}^{\infty} x^{-a_{i_1}} \frac{1}{2} (x-r)^{-b_{i_2} - \frac{1}{2}} \varphi \left( \frac{x}{K} \right) \varphi \left( \frac{x-r}{K} \right) \omega \left( \frac{1}{2\pi} \log \left( \frac{x-r}{x-r} \right) \right) \, dx + O(1). 
\]

(4.32)

Applying (2.4) and (2.9) we get

\[
\sum_{M,N} \frac{T}{\sqrt{MN}} \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} \sum_{1 \leq |r| \leq R_0} \sum_{q=1}^{\infty} N_f;i_1,i_2(r) \\
= \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} \sum_{1 \leq |r| \leq R_0} \sum_{q=1}^{\infty} c_q(r) G_{\ell}(1 - a_{i_1} - q) G_{\ell}(1 - b_{i_2} - q) q^{-2+a_{i_1}+b_{i_2}} \\
\times \int_{max(0,r)}^{\infty} x^{-a_{i_1}} \frac{1}{2} (x-r)^{-b_{i_2} - \frac{1}{2}} \phi \left( \frac{K}{x-r} \right) \phi \left( \frac{K}{x} \right) \omega \left( \frac{1}{2\pi} \log \left( \frac{x-r}{x-r} \right) \right) \, dx + O(1) 
\]

(4.33)

\[
= \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} (I^+_{i_1,i_2} + I^-_{i_1,i_2}) + O(1), 
\]

where

\[
I^+_{i_1,i_2} = c_{i_1,i_2} \sum_{r=1}^{R_0} \sum_{q=1}^{\infty} c_q(r) G_{\ell}(1 - a_{i_1} - q) G_{\ell}(1 - b_{i_2} - q) q^{-2+a_{i_1}+b_{i_2}} \\
\times \int_{max(0,r)}^{\infty} x^{-a_{i_1}} \frac{1}{2} (x-r)^{-b_{i_2} - \frac{1}{2}} \phi \left( \frac{K}{x-r} \right) \phi \left( \frac{K}{x} \right) \omega \left( \frac{1}{2\pi} \log \left( \frac{x-r}{x-r} \right) \right) \, dx 
\]

and

\[
I^-_{i_1,i_2} = c_{i_1,i_2} \sum_{r=-R_0}^{-1} \sum_{q=1}^{\infty} c_q(r) G_{\ell}(1 - a_{i_1} - q) G_{\ell}(1 - b_{i_2} - q) q^{-2+a_{i_1}+b_{i_2}} \\
\times \int_{max(0,r)}^{\infty} x^{-a_{i_1}} \frac{1}{2} (x-r)^{-b_{i_2} - \frac{1}{2}} \phi \left( \frac{K}{x-r} \right) \phi \left( \frac{K}{x} \right) \omega \left( \frac{1}{2\pi} \log \left( \frac{x-r}{x-r} \right) \right) \, dx 
\]

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Note that in view of (4.23) we have
\begin{equation}
G_{\text{off}}(K) = \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} (I_{i_1,i_2}^+ + I_{i_1,i_2}^-) + O \left( K^{qk,t} T \left( \frac{T}{T_0} \right)^{1+C_{k,t}} + 1 \right). \tag{4.34}
\end{equation}

First, we carry out computations for $I_{i_1,i_2}^+$. The change of variable $x \mapsto rx + r$ gives
\begin{align*}
I_{i_1,i_2}^+ &= c_{i_1,i_2} \sum_{r=1}^{R_0} \sum_{q=1}^{\infty} \frac{c_q(r)G_q(1-a_{i_1}, q)G_q(1-b_{i_2}, q)}{q^{2-a_{i_1}-b_{i_2}}a_{i_1}+b_{i_2}} \int_0^\infty (1+x)^{-a_{i_1}-\frac{1}{2}+b_{i_2}-\frac{1}{2}} \\
&\quad \times \frac{1}{(2\pi i)^2} \int_{(c_1)} \int_{(c_2)} \Phi(s_1)\Phi(s_2) \frac{K^{s_1+s_2}}{r^{s_1+s_2}(1+x)^{s_1+s_2}} ds_1 ds_2 \int_{-\infty}^\infty \omega(t)x^{it}(x+1)^{-it} dt dx.
\end{align*}

We rearrange the orders of integration to get
\begin{align*}
I_{i_1,i_2}^+ &= c_{i_1,i_2} \sum_{r=1}^{R_0} \sum_{q=1}^{\infty} \frac{c_q(r)G_q(1-a_{i_1}, q)G_q(1-b_{i_2}, q)}{q^{2-a_{i_1}-b_{i_2}}a_{i_1}+b_{i_2}(2\pi i)^2} \int_{-\infty}^\infty \omega(t) \int_{(c_1)} \int_{(c_2)} \Phi(s_1)\Phi(s_2) \frac{K^{s_1+s_2}}{r^{s_1+s_2}} ds_1 ds_2 dt.
\end{align*}

We now simplify the inner integral with the following useful identities. The first gives an integral representation for the Beta function and the second relates it to Gamma functions:
\begin{equation}
\int_0^\infty \frac{x^{u-1}}{(1+x)^v} dx = B(u,v-u) = \frac{\Gamma(u)\Gamma(v-u)}{\Gamma(v)}, \quad \Re(u), \Re(v) > 0. \tag{4.36}
\end{equation}

Applying (4.36) with $u = \frac{1}{2} - b_{i_2} - s_2 + it$ and $v = \frac{1}{2} + a_{i_1} + s_1 + it$ to the innermost integral in (4.35), we obtain
\begin{align*}
I_{i_1,i_2}^+ &= c_{i_1,i_2} \sum_{r=1}^{R_0} \sum_{q=1}^{\infty} \frac{c_q(r)G_q(1-a_{i_1}, q)G_q(1-b_{i_2}, q)}{q^{2-a_{i_1}-b_{i_2}}a_{i_1}+b_{i_2}(2\pi i)^2} \int_{-\infty}^\infty \omega(t) \int_{(c_1)} \int_{(c_2)} \Phi(s_1)\Phi(s_2) \frac{K^{s_1+s_2}}{r^{s_1+s_2}} \\
&\quad \times \frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)\Gamma(a_{i_1} + b_{i_2} + s_1 + s_2)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} ds_1 ds_2 dt.
\end{align*}

The computations for $I_{i_1,i_2}^-$ are similar. We make the change of variable $x \mapsto rx$ and interchange the orders of integration to get
\begin{align*}
I_{i_1,i_2}^- &= c_{i_1,i_2} \sum_{r=1}^{R_0} \sum_{q=1}^{\infty} \frac{c_q(r)G_q(1-a_{i_1}, q)G_q(1-b_{i_2}, q)}{q^{2-a_{i_1}-b_{i_2}}a_{i_1}+b_{i_2}(2\pi i)^2} \int_{-\infty}^\infty \omega(t) \int_{(c_1)} \int_{(c_2)} \Phi(s_1)\Phi(s_2) \frac{K^{s_1+s_2}}{r^{s_1+s_2}} \\
&\quad \times \int_0^\infty (1+x)^{-\frac{1}{2}-b_{i_2}-s_2+it} x^{-\frac{1}{2}-a_{i_1}-s_1-it} dx ds_1 ds_2 dt. \tag{4.37}
\end{align*}
Applying \((4.36)\) in equation \((4.37)\) gives

\[
I_{i_1, i_2}^- = c_{i_1, i_2} \sum_{r=1}^{R_0} \sum_{q=1}^{\infty} \frac{c_q(r)G_1(1-a_{i_1}, q)G_3(1-b_{i_2}, q)}{q^2-a_{i_1}-b_{i_2}+t_{a_{i_1}+b_{i_2}}(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int_{(c_1)} \Phi(s_1) \Phi(s_2) \frac{K^{s_1+s_2}}{r^{s_1+s_2}} ds_1 ds_2 \ dt.
\]

We combine \(I_{i_1, i_2}^-\) and \(I_{i_1, i_2}^+\) to obtain

\[
I_{i_1, i_2}^+ + I_{i_1, i_2}^- = c_{i_1, i_2} \sum_{r=1}^{R_0} \sum_{q=1}^{\infty} \frac{c_q(r)G_1(1-a_{i_1}, q)G_3(1-b_{i_2}, q)}{q^2-a_{i_1}-b_{i_2}+t_{a_{i_1}+b_{i_2}}(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int_{(c_1)} \Phi(s_1) \Phi(s_2) \frac{K^{s_1+s_2}}{r^{s_1+s_2}} ds_1 ds_2 \ dt.
\]

We move the \(s_2\) contour right to \(\Re(s_2) = 1\) and pass a pole at \(s_2 = P_2\), where \(P_2 = \frac{1}{2} - b_{i_2} + it\). Therefore,

\[
\frac{1}{2\pi i} \int_{(c_2)} f(s_1, s_2) ds_2 = \frac{1}{2\pi i} \int_{(1)} f(s_1, s_2) ds_2 - \text{Res}_{s_2=P_2} f(s_1, s_2),
\]

where

\[
f(s_1, s_2) = \Phi(s_2) \frac{K^{s_1+s_2}}{r^{s_1+s_2}} \Gamma(a_{i_1} + b_{i_2} + s_1 + s_2) \left( \frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_2} + s_2 - it)} \right).
\]

Since \(\Gamma(z) \sim \frac{1}{z}\) as \(z \to 0\), it follows that

\[
\text{Res}_{s_2=P_2} f(s_1, s_2) = -\Phi(P_2) \left( \frac{K}{r} \right)^{s_1+P_2} = -\Phi \left( \frac{1}{2} - b_{i_2} + it \right) \left( \frac{K}{r} \right)^{s_1+P_2} \ll T^{-B} \left( \frac{K}{r} \right)^{c_1+\frac{1}{2}+\delta},
\]

for any \(B > 0\). By \((4.19), (4.12),\) and \((4.21)\), the residue’s contribution to \((4.38)\) is

\[
\log T^{(k-1)(\ell-1)} \sum_{r=1}^{R_0} \sum_{q=1}^{\infty} \frac{|c_q(r)|^2 \delta C_{\omega(q)}}{q^{2-4\delta}} \int_{-\infty}^{\infty} \omega(t) \int_{(c_1)} |\Phi(s_1)| \left( \frac{K}{r} \right)^{c_1+\frac{1}{2}+\delta} T^{-B}|ds_1| dt \ll T^{-B+O(1)}.
\]

Thus, we have

\[
I_{i_1, i_2}^+ + I_{i_1, i_2}^- = c_{i_1, i_2} \sum_{r=1}^{R_0} \sum_{q=1}^{\infty} \frac{c_q(r)G_1(1-a_{i_1}, q)G_3(1-b_{i_2}, q)}{q^2-a_{i_1}-b_{i_2}+t_{a_{i_1}+b_{i_2}}(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int_{(c_1)} \Phi(s_1) \Phi(s_2) \frac{K^{s_1+s_2}}{r^{s_1+s_2}} ds_1 ds_2 \ dt + O(T^{-A}),
\]

for any \(A > 0\). Next, we move the \(s_1\) contour right to \(\Re(s_1) = 1\) and pass a pole at \(s_1 = P_1\), where \(P_1 = \frac{1}{2} - a_{i_1} - it\). We have

\[
\text{Res}_{s_1=P_1} \Phi(s_1) \frac{K^{s_1+s_2}}{r^{s_1+s_2}} \Gamma(a_{i_1} + b_{i_2} + s_1 + s_2) \left( \frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_2} + s_2 - it)} \right) \ll T^{-B} \left( \frac{K}{r} \right)^{\frac{1}{2}+\delta},
\]

\[
= -\Phi(P_1) \left( \frac{K}{r} \right)^{s_2+P_1} = -\Phi \left( \frac{1}{2} - a_{i_1} - it \right) \left( \frac{K}{r} \right)^{s_2+P_1} \ll T^{-B} \left( \frac{K}{r} \right)^{\frac{1}{2}+\delta}.
\]
for any \( B > 0 \). By (4.19), (4.12), and (4.21), the residue’s contribution to (4.39) is

\[
(\log T)^{(k-1)(\ell-1)} \sum_{q=1}^{\infty} \sum_{r=1}^{R_0} c_q(r) \frac{\zeta(\gamma q)}{q^2 - \delta} \int_{-\infty}^{\infty} \omega(t) \int_{1}^{\infty} |\Phi(s_2)| \left( K \right)^{\frac{3}{4} + \delta} |T^{-B}ds_2|dt \ll T^{-B + O(1)}.
\]

Thus, we have

\[
I_{i_1,i_2}^+ + I_{i_1,i_2}^- = c_{i_1,i_2} \sum_{q=1}^{\infty} \sum_{r=1}^{R_0} c_q(r) G_2(1 - a_{i_1}, q) G_3(1 - b_{i_2}, q) \frac{\zeta(\gamma q)}{q^2 - \delta} \int_{-\infty}^{\infty} \omega(t) \int_{1}^{\infty} |\Phi(s_2)| \left( K \right)^{\frac{3}{4} + \delta} |T^{-B}ds_2|dt \ll T^{-B + O(1)}.
\]

\[
\times \Gamma(a_{i_1} + b_{i_2} + s_1 + s_2) \left( \frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_2} + s_2 - it)} \right) ds_1 ds_2 dt + O(T^{-A}),
\]

(4.40)

for any \( A > 0 \).

By (4.28) and (4.29), we have

\[
\frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_2} + s_2 - it)} \ll t^{-2} (\Re(s_1)^2 + \Re(s_2)^2) e^{\frac{\pi}{2}(\Re(s_1+s_2))},
\]

(4.41)

for all \( s_1, s_2 \) with \( \Re(s_1) = \Re(s_2) = 1 \).

Using (2.12), (4.26) and (4.41) we see that the absolute value of the \( s_1, s_2 \) integrals in (4.40) is

\[
\ll \frac{K^2}{r^{2\ell^2}} \int_{1}^{\infty} |\Phi(s_1)| |\Phi(s_2)| |\Gamma(a_{i_1} + b_{i_2} + s_1 + s_2)| |(\Re(s_1)^2 + \Re(s_2)^2) e^{\frac{\pi}{2}(\Re(s_1+s_2))}||ds_1||ds_2|
\]

\[
\ll \frac{K^2}{r^{2\ell^2}}.
\]

Therefore,

\[
c_{i_1,i_2} \sum_{r=R_0}^{\infty} \sum_{q=1}^{\infty} c_q(r) G_2(1 - a_{i_1}, q) G_3(1 - b_{i_2}, q) \frac{\zeta(\gamma q)}{q^2 - \delta} \int_{-\infty}^{\infty} \omega(t) \int_{1}^{\infty} |\Phi(s_2)| \left( K \right)^{\frac{3}{4} + \delta} |T^{-B}ds_2|dt \ll \frac{K^2}{r^{2\ell^2}}.
\]

\[
\times \Gamma(a_{i_1} + b_{i_2} + s_1 + s_2) \left( \frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_2} + s_2 - it)} \right) ds_1 ds_2 dt
\]

\[
\ll K^2 (\log T)^{(k-1)(\ell-1)} \sum_{r=R_0}^{\infty} \tau_2(r) r^{-2\delta} \int_{1}^{\infty} |\Phi(s_2)| \left( K \right)^{\frac{3}{4} + \delta} |T^{-B}ds_2|dt \ll K^2 T^{-1} R_0^{-1 + 2\delta} (\log T)^{(k-1)(\ell-1)} \ll 1,
\]

where the last bound is obtained by choosing \( R_0 = T^A \) for a sufficiently large value of \( A \) and assuming that \( \delta \) is small enough. Therefore, at the cost of increasing the error term in (4.40) to \( O(1) \), we may extend the range of summation for \( r \) to include all positive integers. By absolute convergence, we may swap the order of summation and integration to get

\[
I_{i_1,i_2}^+ + I_{i_1,i_2}^- = J_{i_1,i_2} + O(1),
\]

(4.42)
Combining Proposition 4.4 and Proposition 4.5 gives
\[ J_{1,i_2} = \frac{c_{i_1,i_2}}{(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int (1) \int (1) \Phi(s_1)\Phi(s_2)K^{s_1+s_2}H_{\beta,\beta;\{a_1\},\{b_2\}}(s_1+s_2) \times \Gamma(a_{i_1}+b_{i_2}+s_1+s_2) \left( \frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_2} + s_2 - it)} \right) ds_1 ds_2 dt, \] (4.43)
and \( H_{\beta,\beta;\{a_1\},\{b_2\}}(s) \) is the Dirichlet series defined in (4.25).

Proposition 4.5 relates the off-diagonal terms to a certain double integral of the Dirichlet series \( H_{\beta,\beta;\{a_1\},\{b_2\}}(s) \). By Proposition 5.2 below we are able to meromorphically extend this series to \( \Re(s) > -\frac{1}{2} + 2\delta \). Thus a main term arising from the off-diagonals can be obtained by moving the contours to the left.

5. Completing the proof of Proposition 3.1

Combining Proposition 4.4 and Proposition 4.5 gives
\[ \mathcal{D}_{\beta,\beta;\omega}(K) = \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} J_{i_1,i_2} + O \left( K^{\vartheta_{k,\ell}T^e} \left( \frac{T}{T_0} \right)^{1+C_{k,\ell}} + 1 \right), \] (5.1)
where
\[ J_{i_1,i_2} = \frac{c_{i_1,i_2}}{(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int (1) \int (1) \Phi(s_1)\Phi(s_2)K^{s_1+s_2}H_{\beta,\beta;\{a_1\},\{b_2\}}(s_1+s_2) \times \Gamma(a_{i_1}+b_{i_2}+s_1+s_2) \left( \frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_2} + s_2 - it)} \right) ds_1 ds_2 dt. \]

In order to complete the proof of Proposition 3.1 we require the following proposition.

**Proposition 5.1.** We have
\[ \sum_{i_1=1}^{k} \sum_{i_2=1}^{\ell} J_{i_1,i_2} = M_{1,\beta,\beta;\omega}(K) + O(T^{8\delta} \log T). \]

With this proposition in hand, we are able to establish Proposition 3.1.

**Proof of Proposition 3.1.** From (5.1) and Proposition 5.1 we get
\[ \mathcal{D}_{\beta,\beta;\omega}(K) = M_{1,\beta,\beta;\omega}(K) + O \left( K^{\vartheta_{k,\ell}T^e} \left( \frac{T}{T_0} \right)^{1+C_{k,\ell}} + T^{8\delta} \log T \right). \]
If we assume that \( \delta \) is small enough (e.g. \( 8\delta < (1+\eta)\vartheta_{k,\ell} \)), we find that the second error term is smaller than the first, i.e.
\[ T^{8\delta} \log T = O \left( K^{\vartheta_{k,\ell}T^e} \left( \frac{T}{T_0} \right)^{1+C_{k,\ell}} \right) \]
as \( T \to \infty. \)
The rest of this section is dedicated to establishing Proposition 5.1. Note that we require the following proposition which gives a meromorphic continuation of \( H_{3,3;\{a_i\},\{b_j\}}(s) \).

**Proposition 5.2.** Let \( \delta \in (0, \frac{1}{12}) \), \( \varepsilon \in (0, \frac{1}{2}) \), and assume \( |a_i|, |b_j| \leq \delta \) for \( i \in \mathcal{K}, j \in \mathcal{L}. \)

Fix \( i_1 \in \mathcal{K} \) and \( i_2 \in \mathcal{L} \). For \( \Re(s) > 2\delta \), we have

\[
H_{3,3;\{a_i\},\{b_j\}}(s) = \zeta(a_{i_1} + b_{i_2} + s) \prod_{k_1 \in \mathcal{K} \setminus \{i_1\}} \prod_{k_2 \in \mathcal{L} \setminus \{i_2\}} (1 + a_{k_1} + b_{k_2} + s) \mathcal{C}_{3,3;\{a_i\},\{b_j\}}(s),
\]

where

\[
\mathcal{C}_{3,3;\{a_i\},\{b_j\}}(s) = \prod_p \mathcal{C}_{3,3;\{a_i\},\{b_j\}}(p; s).
\]

For \( \Re(s) = \sigma \geq -1 + 2\delta + \varepsilon \), we have

\[
\mathcal{C}_{3,3;\{a_i\},\{b_j\}}(p; s) = 1 - \left( \sum_{\substack{\mathcal{S} \subset \mathcal{P} \setminus \{a_i\} \\mathcal{T} \subset \mathcal{P} \setminus \{b_j\} \\mathcal{V} = |\mathcal{S}| = |\mathcal{T}| = 2}} p^{-|\mathcal{S}| - |\mathcal{T}|} \right) p^{-2 - 2s} + O(\varepsilon p^{8\delta + \vartheta(\sigma)}),
\]

where

\[
\vartheta(\sigma) = \begin{cases} -2 & \text{if } \sigma \geq 0; \\ -2 - \sigma & \text{if } 0 > \sigma \geq -\frac{1}{2}; \\ -3 - 3\sigma & \text{if } -\frac{1}{2} > \sigma, \end{cases}
\]

and we use the following notation: if \( \mathcal{S} \subset \mathcal{J} = \{a_1, \ldots, a_k\} \) and \( \mathcal{S} = \{a_{i_1}, \ldots, a_{i_s}\} \) and if \( \mathcal{T} \subset \mathcal{J} = \{b_1, \ldots, b_\ell\} \) and \( \mathcal{T} = \{b_{j_1}, \ldots, b_{j_r}\} \)

then

\[
p^{-\mathcal{S}} := p^{-a_{i_1} - a_{i_2} - \cdots - a_{i_s}} \text{ and } p^{-\mathcal{T}} := p^{-b_{j_1} - b_{j_2} - \cdots - b_{j_r}}.
\]

It follows that \( \mathcal{C}_{3,3;\{a_i\},\{b_j\}}(s) \) is absolutely convergent and holomorphic for \( \Re(s) > -\frac{1}{2} + 2\delta \). Hence, \( H_{3,3;\{a_i\},\{b_j\}}(s) \) has an analytic continuation to \( \Re(s) > -\frac{1}{2} + 2\delta \) with the exception of simple poles at

\[
1 - (a_{i_1} + b_{i_2}) \text{ and } - (a_{k_1} + b_{k_2}) \text{ for } k_1 \neq i_1, k_2 \neq i_2.
\]

The proof of Proposition 5.2 is given in Section 6. We note that special cases of this result were previously proven in [32, Corollary 6.2, p.223] for the case \( |\mathcal{J}| = |\mathcal{J}| = 2 \) (by setting \( h = k = 1 \) in their article) and in [32, Proposition 5.1] for the case \( |\mathcal{J}| = |\mathcal{J}| = 3 \). Furthermore, our proof simplifies the proof of Proposition 5.1 in [32], as we do not use Maple. We also require the following bounds on \( H_{3,3;\{a_i\},\{b_j\}}(s) \) which follow from (5.2) and the well known bounds for \( \zeta(x + iy) \) when \( 0 \leq x < 1 \).

**Lemma 5.3.** Let \( \delta \in (0, \frac{1}{12}) \) and suppose that \( |a_{i_1}|, |b_{i_2}| < \delta \). When \( \Re(s) = 4\delta \), we have

\[
H_{3,3;\{a_i\},\{b_j\}}(s) \ll \delta (1 + |\Im(s)|)^{\frac{1}{2}}.
\]

When \( |\Im(s) - 1| \leq 4\delta \) and \( \Im(s) \times T \), we have

\[
H_{3,3;\{a_i\},\{b_j\}}(s) \ll T^{4\delta}.
\]
Note that the implied constant in  (5.7) goes to infinity as \( \delta \to 0 \).

**Proof.** By  (5.2) we have for \( s = 4 \delta + iu \)

\[
H_{5,3;(a_1),\{b_2\}}(s) \ll |\zeta(a_1 + b_2 + 4 \delta + iu)| \prod_{k_1 \in \mathcal{X}\{i_1\}} |\zeta(1 + a_{k_1} + b_{k_2} + 4 \delta + iu)|
\]

\[
\ll |\zeta(a_1 + b_2 + 4 \delta + iu)|,
\]

since \( \Re(1 + a_{k_1} + b_{k_2} + 4 \delta + iu) > 1 + 2 \delta \). By \cite{25} Theorem 1.9], we know that

\[
\zeta(a_1 + b_2 + 4 \delta + iu) \ll (1 + |u|)^{\frac{1-2i}{2}} \log(2 + |u|),
\]

since \( 0 < 2 \delta \leq \Re(a_1 + b_2 + 4 \delta \leq 6 \delta < 1 \) and \( u - 2 \delta \leq \Im(a_1 + b_2) + u \leq u + 2 \delta \). Hence,

\[
H_{5,3;(a_1),\{b_2\}}(s) \ll (1 + |u|)^{\frac{1}{2}}.
\]

Let us now assume that \( 1 - 4 \delta \leq \Re(s) \leq 1 + 4 \delta \), then \( \Re(1 + a_1 + b_2 + s) \geq 2 - 6 \delta > 1 \). It follows that \( \prod_{k_1 \in \mathcal{X}\{i_1\}} |\zeta(1 + a_{k_1} + b_{k_2} + s)| \ll 1 \). If we further assume that \( \Im(s) \asymp T \), then

\[
\zeta(a_1 + b_2 + s) \ll T^{\frac{1}{1-6 \delta}} \log(T).
\]

Hence,

\[
H_{5,3;(a_1),\{b_2\}}(s) \ll T^{\frac{1}{2}} \log T \ll T^{4 \delta},
\]

when \( |\Re(s) - 1| \leq 4 \delta \) and \( \Im(s) \asymp T \). \( \square \)

We also require an identity relating \( C_{5,3;(a_1),\{b_2\}}(s) \) to \( A(\bar{I}, \bar{J}) \) for certain sets \( \bar{I} \) and \( \bar{J} \).

**Proposition 5.4.** Let \( \delta > 0 \), and let \( \mathcal{I} \) and \( \mathcal{J} \) satisfy the size restrictions  (5.12]. Let \( C > 0 \), and suppose that \( |\Re(s)| \leq C \delta \). If \( \delta < \frac{1}{2(C+1)} \), then we have

\[
C_{5,3;(a_1),\{b_2\}}(s) = A((\mathcal{J} \setminus \{a_1\}) \cup \{-b_2 - s\}, ((\mathcal{J} \setminus \{b_2\}) + s) \cup \{-a_1\}). \tag{5.8}
\]

The proof of the identity (5.8) is given in \cite{8} Section 4]. Special cases of this result were proven in \cite{23} Lemmas 6.10-6.12] and \cite{32} Proposition 6.2] corresponding to \( |\mathcal{I}| = |\mathcal{J}| = 2 \) and \( |\mathcal{I}| = |\mathcal{J}| = 3 \) respectively.

**Proof of Proposition 5.1.** By Proposition 5.2, \( H_{5,3;(a_1),\{b_2\}}(s) \) has a meromorphic continuation to \( \Re(s) > -\frac{1}{2} + 2 \delta \) with simple poles at \( -a_{i_1} - b_{i_2} \) and \( \{-a_{j_1} - b_{j_2}\}_{j_1 \neq i_1, j_2 \neq i_2} \). Going back to  (4.33], we move the line of integration in \( s_1 \) to \( \Re(s_1) = \varepsilon_1 \) with \( \varepsilon_1 = 2 \delta \). In doing so, we pass a pole of

\[
\frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_1} + s_2 - it)} \quad \text{at } s_1 = \frac{1}{2} - a_{i_1} - it
\]

which contributes an error term of \( o(1) \) as \( T \to \infty \). Next, we move the line of integration in \( s_2 \) to \( \Re(s_2) = \varepsilon_1 \), passing a pole of

\[
\frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_1} + s_2 - it)} \quad \text{at } s_2 = \frac{1}{2} - b_{i_2} + it
\]

which also contributes an error term of \( o(1) \) as \( T \to \infty \). Notice that the pole of \( H_{5,3;(a_1),\{b_2\}}(s_1 + s_2) \) at \( s_2 = 1 - a_{i_1} - b_{i_2} - s_1 \) cancels with a corresponding zero of

\[
\frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_1} + s_2 - it)},
\]

and the other poles at \( s_2 = -a_{j_1} - b_{j_2} - s_1 \)
for $j_1 \neq i_1$, $j_2 \neq i_2$ are avoided by our choice of $\epsilon_1$. Now we have

$$J_{i_1,i_2} = \frac{c_{i_1,i_2}}{(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int_{(\epsilon_1)} \Phi(s_1) \Phi(s_2) K^{s_1 + s_2} H_{\beta,\beta,j_1,i_1,b_{i_2}}(s_1 + s_2) \times \Gamma(a_{i_1} + b_{i_2} + s_1 + s_2) \left( \frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_2} + s_2 - it)} \right) \, ds_1 \, ds_2 \, dt + O(1).$$

We consider the portions of the $s_1$, $s_2$ integrals with $|\Im(s_1)| \geq t + 1$ or $|\Im(s_2)| \geq t + 1$. We have

$$\int_{\Re(s_2) = \epsilon_1} \int_{|\Im(s_2)| \geq t + 1} \Phi(s_1) \Phi(s_2) K^{s_1 + s_2} H_{\beta,\beta,j_1,i_1,b_{i_2}}(s_1 + s_2) \times \Gamma(a_{i_1} + b_{i_2} + s_1 + s_2) \left( \frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_2} + s_2 - it)} \right) \, ds_1 \, ds_2 \leq K^{2\epsilon_1} \int_{\Re(s_2) = \epsilon_1} \int_{|\Im(s_2)| \geq t + 1} \Phi(s_1) |\Phi(s_2)| |H_{\beta,\beta,j_1,i_1,b_{i_2}}(s_1 + s_2)| \times |\Gamma(a_{i_1} + b_{i_2} + s_1 + s_2)| \left( \frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_2} + s_2 - it)} \right) \, |ds_1| \, |ds_2| \quad (5.9)$$

Using (2.13), (4.26), Lemma 4.6(ii) and Lemma 5.3, we see that (5.9) is

$$\leq K^{2\epsilon_1} t^{-2} \int_{\Re(s_2) = \epsilon_1} \int_{|\Im(s_2)| \geq t + 1} |\Im(s_1)|^{-m} |\Im(s_2)|^{-m} |H_{\beta,\beta,j_1,i_1,b_{i_2}}(s_1 + s_2)| \times |\Gamma(a_{i_1} + b_{i_2} + s_1 + s_2)| (\Im(s_1)^2 + \Im(s_2)^2) \exp \left( \frac{\pi}{2} (|\Im(s_1 + s_2)|) \right) \, |ds_1| \, |ds_2| \quad (5.10)$$

$$\leq K^{2\epsilon_1} t^{-2}. \quad (5.11)$$

It follows that the contribution to (5.9) arising from $|\Im(s_1)|, |\Im(s_2)| \geq t + 1$ is $\ll T^{2\epsilon_1(1+\eta)} T^{-1} = O(1)$. The contribution to (5.9) arising from the other portions is likewise $O(1)$. Therefore, we have

$$J_{i_1,i_2} = \frac{c_{i_1,i_2}}{(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int_{\epsilon_1 - it}^{\epsilon_1 + it(1+1)} \Phi(s_1) \Phi(s_2) K^{s_1 + s_2} H_{\beta,\beta,j_1,i_1,b_{i_2}}(s_1 + s_2) \times \Gamma(a_{i_1} + b_{i_2} + s_1 + s_2) \left( \frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)} + \frac{\Gamma(\frac{1}{2} - a_{i_1} - s_1 - it)}{\Gamma(\frac{1}{2} + b_{i_2} + s_2 - it)} \right) \, ds_1 \, ds_2 \, dt + O(1)$$

$$= \frac{c_{i_1,i_2}}{(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int_{\epsilon_1 - it(1+1)}^{\epsilon_1 + it(1+1)} \Phi(s_1) \Phi(s_2) K^{s_1 + s_2} H_{\beta,\beta,j_1,i_1,b_{i_2}}(s_1 + s_2) \times \Gamma(a_{i_1} + b_{i_2} + s_1 + s_2) 2t^{-a_{i_1} - b_{i_2} - s_1 - s_2} \cos \left( \frac{\pi}{2} (a_{i_1} + b_{i_2} + s_1 + s_2) \right) \, ds_1 \, ds_2 \, dt$$

$$+ \frac{c_{i_1,i_2}}{(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int_{\epsilon_1 - it(1+1)}^{\epsilon_1 + it(1+1)} \Phi(s_1) \Phi(s_2) K^{s_1 + s_2} H_{\beta,\beta,j_1,i_1,b_{i_2}}(s_1 + s_2) \times \Gamma(a_{i_1} + b_{i_2} + s_1 + s_2) O \left( \exp \left( \frac{\pi}{2} |\Im(s_1 + s_2)| \right) \left( 1 + |s_1|^2 + |s_2|^2 \right) \right) \, ds_1 \, ds_2 \, dt + O(1), \quad (5.12)$$

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where the last equality follows from (4.30).

We use (2.12), (4.26) and Lemma 5.3 to get that the contribution of the second term to (5.12) is bounded by \( T^{3\delta} \log T \). Therefore,

\[
J_{1,i_2} = \frac{c_{i_1,i_2}}{(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int_{\epsilon_1-i(t+1)}^{\epsilon_1+i(t+1)} \Phi(s_1)\Phi(s_2)K^{s_1+s_2}H_{\beta;\{a_1\},\{b_2\}}(s_1+s_2) \times \Gamma(a_1 + b_2 + s_1 + s_2)2t^{-a_1-b_2-s_1-s_2} \cos\left(\frac{\pi}{2}(a_1 + b_2 + s_1 + s_2)\right) \, ds_1 \, ds_2 \, dt + O(T^{3\delta} \log T).
\]

Since

\[
\Gamma(a_1 + b_2 + s_1 + s_2) \cos\left(\frac{\pi}{2}(a_1 + b_2 + s_1 + s_2)\right) \ll |\Re(s_1 + s_2)|^{\Re(s_1 + s_2) + 2\delta - \frac{1}{2}},
\]

the part of the integral where \(|\Re(s)| \geq t + 1\) or \(|\Re(s_2)| \geq t + 1\) is \( O(1) \) and can be added to the error term. Hence, we may extend the bounds of integration in \( s_1 \) and \( s_2 \) to all of \( \Re(s_1) = \epsilon_1 \) and \( \Re(s_2) = \epsilon_1 \). This yields,

\[
J_{1,i_2} = \frac{c_{i_1,i_2}}{(2\pi i)^2} \int_{-\infty}^{\infty} \omega(t) \int_{\epsilon_1-i(1)}^{\epsilon_1+i(1)} \Phi(s_1)\Phi(s_2)K^{s_1+s_2}H_{\beta;\{a_1\},\{b_2\}}(s_1+s_2) \times 2t^{-a_1-b_2-s_1-s_2} \cos\left(\frac{\pi}{2}(a_1 + b_2 + s_1 + s_2)\right) \, ds_1 \, ds_2 \, dt + O(T^{3\delta} \log T).
\]

Observe that the inner double integral is of the shape \( \int_{\epsilon_1}^{\epsilon_1} \int_{\epsilon_1}^{\epsilon_1} \Phi(s_1)\Phi(s_2)f(s_1+s_2)ds_2ds_1 \), where \( f(z) = K^2H_{\beta;\{a_1\},\{b_2\}}(z)\Gamma(a_1 + b_2 + z)2t^{-a_1-b_2-s} \cos\left(\frac{\pi}{2}(a_1 + b_2 + s)\right) \). In the inner integral, we make the variable change \( s = s_1 + s_2 \) and then change order of integration to find

\[
\frac{1}{2\pi i} \int_{\epsilon_1}^{\epsilon_1} \int_{\epsilon_1}^{\epsilon_1} \Phi(s_1)\Phi(s_2)f(s_1+s_2)ds_2ds_1 = \int_{\epsilon_1}^{\epsilon_1} \Phi(s_2)f(s)ds,
\]

where we recall that \( \Phi_2 \) is defined in (2.14). It follows that

\[
J_{1,i_2} = \frac{c_{i_1,i_2}}{2\pi i} \int_{-\infty}^{\infty} \omega(t) \int_{\Re(s) = \epsilon_1} \Phi_2(s)K^sH_{\beta;\{a_1\},\{b_2\}}(s)\Gamma(a_1 + b_2 + s) \times 2t^{-a_1-b_2-s} \cos\left(\frac{\pi}{2}(a_1 + b_2 + s)\right) \, ds \, dt + O(T^{3\delta} \log T).
\]

By the functional equation \( \zeta(1 - z) = 2^{-1-z} \pi^{-z} \cos\left(\frac{\pi}{2}z\right)\Gamma(z)\zeta(z) \) and (5.2) it follows that

\[
2H_{\beta;\{a_1\},\{b_2\}}(s)\Gamma(a_1 + b_2 + s) \cos\left(\frac{\pi}{2}(a_1 + b_2 + s)\right)
= 2\left(\zeta(a_1 + b_2 + s) \prod_{a_j \in \mathcal{A}_{\beta\{a_1\}}} \zeta(1 + a_j + b_j + s)\right)\mathcal{C}_{\beta,a_1,b_2}(s)
\times \Gamma(a_1 + b_2 + s) \cos\left(\frac{\pi}{2}(a_1 + b_2 + s)\right)
= \prod_{a_j \in \mathcal{A}_{\beta\{a_1\}}} \zeta(1 + a_j + b_j + s)\mathcal{C}_{\beta,a_1,b_2}(s) \zeta(1 - a_1 - b_2 - s)(2\pi)^{a_1+b_2+s}.
\]
From these identities, it follows from (5.13) and (1.32) that

\[
J_{i_1, i_2} = \frac{c_{i_1, i_2}}{2\pi i} \int_{-\infty}^{\infty} \omega(t) \int_{\mathbb{R}(s)=2\pi} \Phi_2(s) K^s \left( \frac{2\pi}{t} \right)^{a_{i_1} + b_{i_2} + s} \prod_{a_{j_1} \in J \setminus \{a_{i_1}\}} \zeta(1 + a_{j_1} + b_{j_2} + s) \\
\times \prod_{b_{j_2} \in \mathcal{J} \setminus \{b_{i_2}\}} \zeta(1 - a_{i_1} - b_{i_2} - s) \ ds \ dt + O(T^{8\delta} \log T).
\]

By an application of (5.8) we obtain

\[
J_{i_1, i_2} = \frac{c_{i_1, i_2}}{2\pi i} \int_{-\infty}^{\infty} \omega(t) \int_{\mathbb{R}(s)=2\pi} \Phi_2(s) K^s \left( \frac{2\pi}{t} \right)^{a_{i_1} + b_{i_2} + s} \zeta(1 - a_{i_1} - b_{i_2} - s) \\
\times \prod_{a_{j_1} \in J \setminus \{a_{i_1}\}} \zeta(1 + a_{j_1} + b_{j_2} + s) A((J \setminus \{a_{i_1}\}) \cup \{-b_{i_2} - s\}, ((J \setminus \{b_{i_2}\}) + s) \cup \{-a_{i_1}\}) \ ds \ dt \\
+ O(T^{8\delta} \log T).
\]

Hence,

\[
\sum_{i_1 = 1}^{k} \sum_{i_2 = 1}^{\ell} J_{i_1, i_2} = \sum_{i_1 = 1}^{k} \sum_{i_2 = 1}^{\ell} \frac{c_{i_1, i_2}}{2\pi i} \int_{-\infty}^{\infty} \omega(t) \int_{\mathbb{R}(s)=2\pi} \Phi_2(s) K^s \left( \frac{2\pi}{t} \right)^{a_{i_1} + b_{i_2} + s} \zeta(1 - a_{i_1} - b_{i_2} - s) \\
\times \prod_{a_{j_1} \in J \setminus \{a_{i_1}\}} \zeta(1 + a_{j_1} + b_{j_2} + s) A((J \setminus \{a_{i_1}\}) \cup \{-b_{i_2} - s\}, ((J \setminus \{b_{i_2}\}) + s) \cup \{-a_{i_1}\}) \ ds \ dt \\
+ O(T^{8\delta} \log T).
\]

We now remark that from definitions (4.11) and (1.24) we have

\[
c_{i_1, i_2} = \mathcal{Z}(J \setminus \{a_{i_1}\}, \{a_{i_1}\}) \mathcal{Z}(\{a_{i_2}\}, J \setminus \{a_{i_2}\}),
\]

and similarly from (1.24) we have

\[
\prod_{a_{j_1} \in J \setminus \{a_{i_1}\}} \zeta(1 + a_{j_1} + b_{j_2} + s) = \mathcal{Z}(J \setminus \{a_{i_1}\} + \{s\}, J \setminus \{b_{i_2}\}).
\]

From these identities, it follows from (5.13) and (1.32) that

\[
\sum_{i_1 = 1}^{k} \sum_{i_2 = 1}^{\ell} J_{i_1, i_2} = \mathcal{M}_{1, J, \beta, \omega}(K) + O(T^{8\delta} \log T).
\]

\[\square\]

6. The function \(H_{J, \beta, \{a_{i_1}\}, \{b_{i_2}\}}(s)\)

In this section, we study the behaviour of the Dirichlet series \(H_{J, \beta, \{a_{i_1}\}, \{b_{i_2}\}}(s)\) that was introduced in Section 4.
Lemma 6.1. Let $k \in \mathbb{N}$, $I = \{1, \ldots, k\}$, and $X = \{x_1, x_2, \ldots, x_k\}$, where the $x_i$’s are distinct complex numbers. For $\Re(s) > -\min_{i=1,\ldots,k} \Re(x_i)$, $p$ prime and $j \geq 1$, we have

$$G_X(s, p^j) = (1 - p^{-s-x_1}) \cdots (1 - p^{-s-x_k}) \frac{1}{p-1} \sum_{i=1}^{k} \frac{p^{1-x_i,j} - p^{s-x_i(j-1)}}{1 - p^{-x_i-s}} \prod_{\ell \in I \setminus \{i\}} (1 - p^{x_i-x_{i\ell}})^{-1}. \quad (6.1)$$

We now further simplify the multiplicative functions $G_X(s, p^j)$. We shall express $G_X(s, p^j)$ in terms of the rational function

$$F_a(Y_1, \ldots, Y_m; Z) := \sum_{i=1}^{m} Y_i^a \prod_{\ell \in M \setminus \{i\}} \frac{(1 - ZY_i)}{(1 - Y_i/Y_\ell)}, \quad (6.2)$$

where $Y_1, \ldots, Y_m, Z$ are variables and $M = \{1, 2, \ldots, m\}$. A key point will be to demonstrate that $F_a$ is a polynomial and this shall be established in the combinatorial result, Lemma 6.2, which follows. This lemma was proven by the authors in the case $a = 1$. In the case $a = 2$, Gabriel Verret conjectured the formula (6.7), where $q_{2,j}$ is given by (6.9). Based on this, Dave Morris extended the lemma to the case $a > 1$. In order to describe the polynomials $q_{a,j}$ which appear in the lemma, we require the elementary symmetric polynomials. Associated to variables $Y_1, \ldots, Y_m$, we let

$$e_1 := e_1(Y_1, \ldots, Y_m) = Y_1 + \cdots + Y_m, \quad (6.3)$$
$$e_2 := e_2(Y_1, \ldots, Y_m) = Y_1Y_2 + \cdots + Y_{m-1}Y_m = \sum_{1 \leq i_1 < i_2 \leq m} Y_{i_1}Y_{i_2}, \quad (6.4)$$

$$\vdots$$
$$e_j := e_j(Y_1, \ldots, Y_m) = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq m} Y_{i_1}Y_{i_2} \cdots Y_{i_j}, \quad (6.5)$$
$$\vdots$$
$$e_m := e_m(Y_1, \ldots, Y_m) = Y_1 \cdots Y_m. \quad (6.6)$$

It is convenient to set $e_0 := e_0(Y_1, \ldots, Y_m) = 1$ and $e_{m+1} := e_{m+1}(Y_1, \ldots, Y_m) = 0$. With this notation in hand, we may now state the lemma.

Lemma 6.2. Let $a, m \in \mathbb{N}$. Then

$$F_a(Y_1, \ldots, Y_m; Z) = \sum_{j=0}^{m-1} q_{a,j}(Y_1, \ldots, Y_m) Z^j, \quad (6.7)$$
where \( q_{a,j}(Y_1, \ldots, Y_m) \) is a polynomial in \( \mathbb{Q}[Y_1, \ldots, Y_m] \) of degree \( j + a \). We have

\[
q_{1,j} := q_{1,j}(Y_1, \ldots, Y_m) = (-1)^j e_{j+1},
\]

\[
q_{2,j} := q_{2,j}(Y_1, \ldots, Y_m) = (-1)^j e_{j+2} - e_1 e_{j+1},
\]

and in general,

\[
q_{a,j} := q_{a,j}(Y_1, \ldots, Y_m) \in \mathbb{Z}[e_1, \ldots, e_m].
\]

A direct consequence of this lemma, is the following bound. Let \( Z \in \mathbb{C} \) and \( X \geq 1 \). If \( |Y_j| \leq X \) for each \( 1 \leq j \leq m \), then

\[
|F_a(Y_1, \ldots, Y_m; Z)| \ll X^{j+a} |Z|^j.
\]

**Proof of Lemma 6.2** In order to simplify notation we set \( F = F_a(Y_1, \ldots, Y_m; Z) \) and write

\[
P = P(Y_1, \ldots, Y_m; Z) := \prod_{i \in M} (1 - Z Y_i) = \sum_{j=0}^{m} (-1)^j e_j Z^j.
\]

Note that since \( P(Z) = \sum_{j=0}^{m} (P(j)(0)/j!) Z^j \), it follows that

\[
P(j)(0) = (-1)^j j! e_j \quad \text{for } 0 \leq j \leq m.
\]

Let us consider the partial fraction decomposition of \( 1/(Z^a P) \). This is of the form

\[
\frac{1}{Z^a P} = \sum_{j=1}^{a} \frac{c_j}{Z^j} + \sum_{i=1}^{m} \frac{Y_i^a}{\prod_{\ell \in M \setminus \{i\}} (1 - Y_\ell / Y_i)} \cdot \frac{1}{(1 - Z Y_i)},
\]

for certain polynomials \( c_j \in \mathbb{R}[Y_1, \ldots, Y_m] \). Letting

\[
R(Z) := \sum_{i=0}^{a-1} c_{a-i} Z^i,
\]

we can rewrite this as

\[
\frac{1}{Z^a P} = \sum_{j=1}^{a} \frac{c_j}{Z^j} + \frac{F(Z)}{P(Z)} = \frac{R(Z)}{Z^a} + \frac{F(Z)}{P(Z)}
\]

which follows from (6.2) and (6.12). Note that

\[
R^{(j)}(0) = j! c_{a-j} \quad \text{for } 0 \leq j \leq a - 1.
\]

We now compute the coefficients \( c_j \). Rearranging (6.14) gives

\[
1 = P(Z) R(Z) + F(Z) Z^a.
\]

Letting \( Z = 0 \) we obtain

\[
1 = P(0) R(0) = c_a.
\]
Then we differentiate (6.17) \( i \) times where \( 1 \leq i \leq a - 1 \). Observe that \( \frac{d^i}{dz^i}(F(Z)Z^a) \bigg|_{Z=0} = 0 \), and thus we obtain by the generalized product rule, (6.13), and (6.16)

\[
0 = \sum_{u+v=i} \binom{i}{u} P^{(u)}(0) R^{(v)}(0) = \sum_{u+v=i} \binom{i}{u} (-1)^u u! e_u v! c_{a-v}. \tag{6.19}
\]

Simplifying yields the condition

\[
0 = \sum_{u+v=i} (-1)^u e_u c_{a-v} \text{ for } 1 \leq i \leq a - 1.
\]

Note that if \( i = 1 \) then this simplifies to \( c_{a-1} = c_a \cdot e_1 = e_1 \) (since \( e_0 = 1 \)), and we have used (6.18). Since \( e_0 = 1 \), we have

\[
c_{a-i} = - \sum_{u+v=i} (-1)^u e_u c_{a-v} \text{ for } 1 \leq i \leq a - 1.
\]

It follows from this that \( c_{a-i} \in \mathbb{Z}[e_1, \ldots, e_i] \) and \( \text{degree}(c_{a-i}) = i \). From (6.17) and the definitions of \( P(Z) \) and \( R(Z) \) in (6.12) and (6.15), we get

\[
F = \frac{1 - P(Z)R(Z)}{Z^a} = Z^{-a} \left( 1 - \sum_{i=0}^{m+a-1} \theta_i Z^i \right),
\]

where \( \theta_i = \sum_{u+v=i} (-1)^u e_u c_{a-v} \).

From the definition of \( \theta_i \) and (6.18), we observe that \( \theta_0 = 1 \). In addition, we have \( \theta_i = 0 \) for \( 1 \leq i \leq a - 1 \) since \( \frac{d^i}{dz^i} (P(Z)R(Z)) \bigg|_{Z=0} = 0 \). This follows from (6.17) and the observation that \( \frac{d^i}{dz^i}(F(Z)Z^a) \bigg|_{Z=0} = 0 \). Thus, we have

\[
F = - \sum_{i=0}^{m+a-1} \theta_i Z^{i-a} = - \sum_{j=0}^{m-1} \theta_{a+j} Z^j,
\]

where we note that the degree of \( \theta_{a+j} \) is \( a + j \). Setting \( q_{a,j} := q_{a,j}(Y_1, \ldots, Y_m) = -\theta_{a+j} \) gives the desired result.

\[
\square
\]

**Lemma 6.3.** Let \( k \geq 2 \), let \( J = \{a_1, a_2, a_3, \ldots, a_k\} \subset \mathbb{C} \), \( p \) a prime, and \( j \geq 1 \). Then

\[
G_j(1 - a_1, p^n) = \sum_{j=0}^{k-2} q_{n,j}(X_2, \ldots, X_k)(X_1^{-1})^j p^{-j}, \tag{6.20}
\]

where \( q_{n,j} \) is defined in (6.10) and

\[
X_i = p^{-a_i} \text{ for } 1 \leq i \leq k. \tag{6.21}
\]

In particular, we obtain

\[
G_j(1 - a_1, p) = \sum_{j=0}^{k-2} (-1)^j e_{j+1}(X_2, \ldots, X_k)(X_1^{-1})^j p^{-j}, \tag{6.22}
\]
and
\[ G_3(1 - a_1, p^2) = \sum_{j=0}^{k-2} (-1)^{j-1} (e_{j+2}(X_2, \ldots, X_k) - e_{j+1}(X_2, \ldots, X_k)e_1(X_2, \ldots, X_k))(X_1^{-1})^j p^{-j}. \] (6.23)

For any \( n \geq 1 \) and \( \delta \in (0, \frac{1}{2}) \) satisfying (6.12), we have the bound
\[ |G_3(1 - a_1, p^n)| \ll p^{n \delta}. \] (6.24)

**Proof.** We apply Lemma 6.1 with \( s = 1 - a_1 \) and \( x_i = a_i \) for \( 1 \leq i \leq k \) to obtain
\[ G_3(1 - a_1, p^j) = \left( \prod_{i=2}^{k} (1 - p^{-1+a_i-a_{i-1}}) \right) \sum_{i=2}^{k} p^{-a_i-1}(1 - p^{-1+a_i-a_{i-1}}) \prod_{\ell \in I \setminus \{i\}} (1 - p^{a_i-a_\ell})^{-1} \]
\[ = \frac{1}{p^j} \left( \prod_{r=2}^{k} (1 - p^{-1+a_r}) \right) \sum_{i=1}^{k} \frac{p^{-a_i-1}(1 - p^{a_i-a_1})}{1 - p^{-1+a_i-a_1}} \prod_{\ell \in I \setminus \{i\}} (1 - p^{a_i-a_\ell})^{-1}. \]

Observe that the \( i = 1 \) term vanishes and further simplification yields
\[ G_3(1 - a_1, p^j) = \left( \prod_{r=2}^{k} (1 - p^{-1+a_r}) \right) \sum_{i=2}^{k} \frac{p^{-a_i-1}(1 - p^{a_i-a_1})}{1 - p^{-1+a_i-a_1}} \prod_{\ell \in I \setminus \{i\}} (1 - p^{a_i-a_\ell})^{-1} \]
\[ = \sum_{i=2}^{k} \frac{p^{-a_i}}{1 - p^{-1+a_i}} \left( \prod_{r \in I \setminus \{i, 1\}} (1 - p^{-1+a_r}) \right) \prod_{\ell \in I \setminus \{1, i\}} (1 - p^{a_i-a_\ell})^{-1} \]
\[ = \sum_{i=2}^{k} X_i \left( \prod_{r \in I \setminus \{i, 1\}} (1 - X_r Z) \right) \prod_{\ell \in I \setminus \{1, i\}} (1 - X_\ell/X_i)^{-1}, \]
where \( X_i = p^{-a_i} \) and \( Z = p^{-1+a_1} \). We now let \( m = k - 1 \) and set \( Y_1 = X_2, \ldots, Y_m = X_k \) to get
\[ G_3(1 - a_1, p^j) = F_n(X_2, \ldots, X_k; Z), \]
where \( F_n \) is defined in (6.2). By Lemma 6.2 we immediately obtain (6.22), (6.23), and (6.24). Finally, we note that by (6.11) it follows that
\[ |G_3(1 - a_1, p^n)| \ll \sum_{j=0}^{k-2} (p^{\delta})^j n^{(p^{\delta}-1)/j} \ll p^{n \delta} \sum_{j=0}^{k-2} (p^{2\delta-1})^j \ll p^{n \delta}, \]
since \( \delta < \frac{1}{2} \).

With these two lemmas in hand we are ready to prove Proposition 5.2.

**Proof of Proposition 5.2.** We shall prove the Proposition in the case \( i_1 = 1 \) and \( i_2 = 1 \). The general case follows by a permutation of the variables. Throughout this proof we let \( \sigma = \Re(s) \).

Using the bound \( |c_\sigma(r)| \leq (q, r) \) and Lemma 6.3 we can check that the series for \( H_{j,\sigma}(a_1, \{b_i\}_1(s)), \)
defined in (4.25), is absolutely convergent for \(\Re(s) > 1 + 2\delta\) and \(\delta < \frac{1}{4}\). Furthermore, since \(c_q(r) = \sum_{d|q,r} d\mu\left(\frac{q}{d}\right)\) we have

\[
H_{j,q;\{a_1,\ldots,a_l\},\{b_1\}}(s) = \sum_{r=1}^{\infty} \sum_{q=1}^{\infty} \frac{G_2(1-a_1,q)G_2(1-b_1,q)}{q^{2-a_1-b_1+s}a_1+a_1+b_1+s} \sum_{d|q,r} d\mu\left(\frac{q}{d}\right) = \sum_{q=1}^{\infty} \alpha_q \sum_{r=1}^{\infty} \frac{1}{r^c} \sum_{d|q,d|r} d\mu\left(\frac{q}{d}\right),
\]

where \(\alpha_q = \frac{G_j(1-a_1,q)G_2(1-b_1,q)}{q^{2-a_1-b_1}}\) and \(c = a_1 + b_1 + s\). Thus,

\[
H_{j,q;\{a_1,\ldots,a_l\},\{b_1\}}(s) = \sum_{q=1}^{\infty} \alpha_q \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sum_{r \geq 1, d|r} \frac{1}{r^c} = \sum_{q=1}^{\infty} \alpha_q \sum_{d|q} \frac{d\mu\left(\frac{q}{d}\right)}{d^{1-c}} \zeta(c) = \zeta(c) \sum_{q=1}^{\infty} \alpha_q \sum_{d|q} d^{1-c} \mu\left(\frac{q}{d}\right) = \zeta(c) \sum_{q=1}^{\infty} \alpha_q q^{1-c} \sum_{d|q} \frac{\mu(d)}{d^{1-c}}.
\]

For prime powers \(p^j\) we have \(\sum_{d|p^j} \frac{\mu(d)}{d^{1-c}} = 1 - \frac{1}{p^{1-c}}\). By multiplicativity, we have

\[
\sum_{q=1}^{\infty} \alpha_q q^{1-c} \sum_{d|q} \frac{\mu(d)}{d^{1-c}} = \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{G_j(1-a_1,p^j)G_j(1-b_1,p^j)}{(p^j)^{2-a_1-b_1}} \frac{(p^j)^{1-a_1-b_1-s} \left(1 - p^{a_1+b_1+s-1}\right)}{(p^j)^{1-a_1-b_1-s}} \right)
\]

\[
= \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{G_j(1-a_1,p^j)G_j(1-b_1,p^j)}{(p^j)^{1-a_1-b_1-s}} \frac{1 - p^{a_1+b_1+s-1}}{(p^j)^{1+s}} \right)
\]

\[
= \prod_p \left(1 + \sum_{j=1}^{\infty} T_j f_j \right)
\]

where we have set

\[
T_j = G_j(1-a_1,p^j)G_j(1-b_1,p^j) \text{ and } f_j = \frac{1 - p^{a_1+b_1+s-1}}{(p^j)^{1+s}} \text{ for } j \in \mathbb{Z}_{\geq 0}.
\]

We now aim to simplify the last expression within the brackets in (6.26). At this point it will be convenient to introduce some notation. Let

\[
U = p^{-1}, \quad V = p^{-1-s}, \quad X_i = p^{-a_i}, \quad Y_i = p^{-b_i}, \text{ for } i \in \mathbb{N}.
\]

Observe that we have the bounds

\[
|V| \leq p^{-1-s}, \quad |X_i^{\pm 1}|, |Y_i^{\pm 1}| \leq p^\delta \text{ for } i = 1, 2, \ldots.
\]

Notice that

\[
f_j = V^j - X_1^{-1}Y_1^{-1}U^2V^{j-1} \text{ for } j \in \mathbb{N}.
\]

Given \(I = \{i_1, \ldots, i_s\} \subset \mathcal{K} = \{1, \ldots, k\}, \text{ and } J = \{j_1, \ldots, j_t\} \subset \mathcal{L} = \{1, \ldots, \ell\}, \text{ we put}

\[
X_I := X_{i_1}X_{i_2} \cdots X_{i_s} \text{ and } Y_J := Y_{j_1}Y_{j_2} \cdots Y_{j_t}.
\]
We also define the polynomial rings
\[
R_1 = \mathbb{Z}[X_1^{-1}, X_2, \ldots, X_k], \quad R_2 = \mathbb{Z}[Y_1^{-1}, Y_2, \ldots, Y_\ell],
\]
and \( R = \mathbb{Z}[X_1^{-1}, X_2, \ldots, X_k, Y_1^{-1}, Y_2, \ldots, Y_\ell] \).

By (6.20) it follows that
\[
T_n = G_3(1 - a_1, p^n)G_3(1 - b_1, p^n) = \left( \sum_{j=0}^{k-2} \alpha_{n,j} U^n \right) \left( \sum_{j'=0}^{\ell-2} \beta_{n,j'} U^{n-1} \right) = \sum_{i=0}^{k+\ell-4} A_{n,i} U^i,
\]
where by Lemma 6.2
\[
\alpha_{n,j} = q_{n,j}(X_2, \ldots, X_k)(X_1^{-1})^j \in R_1,
\]
\[
\beta_{n,j'} = q_{n,j'}(Y_2, \ldots, Y_\ell)(Y_1^{-1})^{j'} \in R_2,
\]
and thus
\[
A_{n,i} = \sum_{j+j' = i} \alpha_{n,j} \beta_{n,j'} \in R.
\]

In particular, by (6.8) we have
\[
A_{1,0} := \alpha_{1,0} \beta_{1,0} = e_1(X_2, \ldots, X_k) \cdot e_1(Y_2, \ldots, Y_\ell) = \sum_{I \subseteq K_1, \, |I| = 1} X_I Y_J,
\]
where
\[
K_1 = K \setminus \{1\} \quad \text{and} \quad L_1 = L \setminus \{1\}.
\]

By (6.9) we have
\[
A_{2,0} = \alpha_{2,0} \beta_{2,0} = (e_1(X_2, \ldots, X_k)^2 - e_2(X_2, \ldots, X_k))(e_1(Y_2, \ldots, Y_\ell)^2 - e_2(Y_2, \ldots, Y_\ell))
\]
\[
= \left( \sum_{I \subseteq K_1, \, |I| = 1} X_I + \sum_{I \subseteq K_1, \, |I| = 2} X_{I}^2 \right) \cdot \left( \sum_{J \subseteq L_1, \, |J| = 1} Y_J + \sum_{J \subseteq L_1, \, |J| = 2} Y_J^2 \right)
\]
\[
= \sum_{I \subseteq K_1, \, |I| = 1} X_I Y_J + \sum_{I \subseteq K_1, \, |I| = 2} X_{I} Y_J^2 + \sum_{J \subseteq L_1, \, |J| = 1} X_I Y_J + \sum_{J \subseteq L_1, \, |J| = 2} X_I Y_J^2.
\]

Observe that \( \deg(\alpha_{n,j}) = 2j + n \) since \( q_{n,j} \) has degree \( j + n \) and \( \deg(\beta_{n,j'}) = 2j' + n \) since \( q_{n,j'} \) has degree \( j' + n \). It follows from (6.35) that \( \deg(A_{n,i}) = 2(i + n) \). Using this fact, with the bounds (6.30) we find that
\[
|A_{n,i}| \ll p^{2(n+i)\delta}.
\]

Therefore by (6.28), (6.29), and (6.30), we also have
\[
A_{n,i} U^{i} V^{n} \ll p^{-(1-2\delta)i-(1+\sigma-2\delta)n},
\]
and
\[
A_{n,i} X_1^{-1} Y_1^{-1} U^{i+2} V^{n-1} \ll p^{-(1-2\delta)i-(1+\sigma-2\delta)n+2\delta-1+\sigma},
\]
for \( n \geq 1 \) and \( i \geq 0 \). These bounds will be employed frequently in the sequel. By (6.31) and (6.33) we have

\[
T_{1} f_{1} = \left( \sum_{i=0}^{k+\ell-4} A_{1,i} U^{i} \right) \left( V - X_{1}^{-1} Y_{1}^{-1} U^{2} \right) = A_{1,0} V + \sum_{i=1}^{k+\ell-4} A_{1,i} U^{i} V - \sum_{i=0}^{k+\ell-4} A_{1,i} X_{1}^{-1} Y_{1}^{-1} U^{i+2}.
\]

(6.41)

It follows from (6.39) and (6.40) that

\[
1 + T_{1} f_{1} = 1 + A_{1,0} V + O \left( \sum_{i=1}^{k+\ell-4} p^{-(1-2\delta)i-(1+\sigma-2\delta)} + \sum_{i=0}^{k+\ell-4} p^{-(1-2\delta)(i+2)} \right)
\]

\[
= 1 + \left( \sum_{I \subseteq \mathcal{K}_{1}, \, |I| \neq |J| = 1} X_{I} Y_{J} \right) V + O \left( (p^{-2-\sigma} + p^{-2}) p^{3\delta} \right).
\]

(6.42)

Next, observe that by (6.27) and (6.24)

\[
|T_{j}| \ll p^{2j\delta}
\]

(6.43)

and by (6.27), (6.28), (6.29), and (6.30)

\[
|f_{j}| \ll |V|^{j} + |X_{1}^{-1} Y_{1}^{-1} p^{-2} V|^{j-1} \ll (p^{-1-\sigma})^{j} + (p^{-1-\sigma})^{j-1} p^{2\delta-2}.
\]

(6.44)

Combining these bounds gives

\[
\sum_{j=2}^{\infty} T_{j} f_{j} \ll \sum_{j=2}^{\infty} p^{2j\delta} ((p^{-1-\sigma})^{j} + (p^{-1-\sigma})^{j-1} p^{2\delta-2}) = \sum_{j=2}^{\infty} (p^{2\delta-1-\sigma})^{j} (1 + p^{\sigma+2\delta-1}) \ll 1
\]

(6.45)

for \( \Re(s) \geq 0 \). Therefore, from (6.41) and (6.45) we deduce, that for \( \Re(s) > 2\delta \), the sum over \( q \) in (6.26) equals

\[
\prod_{(i,j) \in \mathcal{K}_{1} \times \mathcal{L}_{1}} \zeta(1 + a_{i} + b_{j} + s) c_{s,j;\{a_{1}\},\{b_{1}\}}(s),
\]

where

\[
c_{s,j;\{a_{1}\},\{b_{1}\}}(s) = \prod_{p} c_{s,j;\{a_{1}\},\{b_{1}\}}(p; s),
\]

and

\[
c_{s,j;\{a_{1}\},\{b_{1}\}}(p; s) = \left( 1 + \sum_{j=1}^{\infty} \frac{G_{j}(1 - a_{1} + p^{j}) G_{j}(1 - b_{1} + p^{j})}{(p^{j})^{1+s}} (1 - p^{a_{1}+b_{1}+s-1}) \right)
\]

\[
\times \prod_{(i,j) \in \mathcal{K}_{1} \times \mathcal{L}_{1}} \left( 1 - \frac{1}{p^{1+a_{i}+b_{j}+s}} \right).
\]

(6.46)

Hence,

\[
H_{s,j;\{a_{1}\},\{b_{1}\}}(s) = \zeta(a_{1} + b_{1} + s) \prod_{(i,j) \in \mathcal{K}_{1} \times \mathcal{L}_{1}} \zeta(1 + a_{i} + b_{j} + s) c_{s,j;\{a_{1}\},\{b_{1}\}}(s).
\]

We now establish (5.4). It is convenient to set

\[
\Pi = \sum_{i=0}^{k+\ell-2} g_{i} V^{i},
\]

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where \( g_i \in R \), \( \deg(g_i) = 2i \), and \( g_0 = 1 \). We find that
\[
g_1 = - \sum_{I \subseteq \mathcal{X}_2, \|I\| = \|J\| = 1} X_I Y_J = -e_1(X_2, \ldots, X_k)e_1(Y_2, \ldots, Y_\ell) = -A_{1,0},
\]
by \((6.36)\), and
\[
g_2 = \sum_{I \subseteq \mathcal{X}_2, \|I\| = \|J\| = 2} X_I^2 Y_J + \sum_{I \subseteq \mathcal{X}_2, \|I\| = \|J\| = 1} X_I Y_J^2 + 2 \sum_{I \subseteq \mathcal{X}_2} X_I Y_J. \tag{6.47}
\]

From \((6.46)\) it follows that
\[
\mathcal{C}_{j,3;(a_1),(b_1)}(p,s) = \left(1 + \sum_{j=1}^{\infty} T_j f_j\right) \Pi. \tag{6.48}
\]

We now consider \( T_2 f_2 \). By \((6.31)\) and \((6.33)\) we have
\[
T_2 f_2 = \left( \sum_{i=0}^{k+\ell-4} A_{2,i} U^i \right) \left( V^2 - X_1^{-1} Y_1^{-1} U^2 V \right)
= A_{2,0} V^2 + \sum_{i=1}^{k+\ell-4} A_{2,i} U^i V^2 - \sum_{i=0}^{k+\ell-4} A_{2,i} X_1^{-1} Y_1^{-1} U^{i+2} V.
\]

Combining \((6.41)\) and \((6.49)\) yields
\[
1 + T_1 f_1 + T_2 f_2 = 1 + A_{1,0} V + \sum_{i=1}^{k+\ell-4} A_{1,i} U^i V - \sum_{i=0}^{k+\ell-4} A_{1,i} X_1^{-1} Y_1^{-1} U^{i+2}
+ A_{2,0} V^2 + \sum_{i=1}^{k+\ell-4} A_{2,i} U^i V^2 - \sum_{i=0}^{k+\ell-4} A_{2,i} X_1^{-1} Y_1^{-1} U^{i+2} V.
\]

Thus,
\[
(1 + T_1 f_1 + T_2 f_2) \Pi =
\left( 1 + A_{1,0} V + A_{2,0} V^2 + \sum_{i=1}^{k+\ell-4} A_{1,i} U^i V - \sum_{i=0}^{k+\ell-4} A_{1,i} X_1^{-1} Y_1^{-1} U^{i+2}
+ \sum_{i=1}^{k+\ell-4} A_{2,i} U^i V^2 - \sum_{i=0}^{k+\ell-4} A_{2,i} X_1^{-1} Y_1^{-1} U^{i+2} V \right) \left( 1 - A_{1,0} V + g_2 V^2 + \sum_{i=3}^{k+\ell-2} g_i U^i \right). \tag{6.50}
\]

Observe that
\[
(1 + A_{1,0} V + A_{2,0} V^2)(1 - A_{1,0} V + g_2 V^2) = 1 + (g_2 + A_{2,0} - A_{1,0}^2) V^2 + (A_{1,0} g_2 - A_{1,0} A_{2,0}) V^3 + g_2 A_{2,0} V^4.
\]

By expanding out \((6.50)\) and using \((6.51)\) we find that
\[
(1 + T_1 f_1 + T_2 f_2) \Pi = 1 + c_{02} V^2 + \sum_{u \geq 2} c_{u0} U^u + \sum_{u \geq 1} c_{u1} U^u V + \sum_{v \geq 2, (u,v) \in S} c_{uv} U^u V^v , \tag{6.52}
\]
where \( S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \) is finite, \( c_{uv} \in R, \) \( \deg(c_{uv}) = 2(u + v), \) and \( c_{02} = g_2 + A_{2,0} - A_{1,0}^2. \) Here we have made use of the fact that \( \deg(A_{n,i}) = 2(i + n) \) and \( \deg(g_i) = 2i. \) Furthermore, we have

\[
c_{uv}u^v v^w \ll p^{2(u+v)\delta - u - (1+\sigma)v} = p^{-(1-\delta)u - (1+\sigma)v} \text{ for } u, v \geq 0.
\]

(6.53)

In particular, if \( v \geq 2 \) and \( (u, v) \neq (0, 2), \) we have

\[
c_{uv}u^v v^w \ll p^{-(u+v)\min(1-\delta,\sigma, 1-\delta)} \leq p^{-(1-2\delta+\min(\sigma,0))}.
\]

(6.54)

Given that \( S \) is finite, that \( \sigma > -1 + 2\delta, \) and that we certainly have \( \delta < \frac{1}{2}, \) it follows from (6.28), (6.32), (6.53) and (6.54) that

\[
(1 + T_1f_1 + T_2f_2)\Pi = 1 + c_{02}V^2 + O\left(p^{-2(1-\delta)} + p^{-(1-\delta) - (1+\sigma - 2\delta) + p^{-3(1-2\delta+\min(\sigma,0))}}\right)
\]

\[
= 1 + c_{02}V^2 + O\left(p^{6\delta+\max(-2, -2-\sigma, -3-3\min(\sigma,0))}\right)
\]

\[
= 1 + c_{02}V^2 + O\left(p^{6\delta+\vartheta(\sigma)}\right),
\]

(6.55)

where \( \vartheta(\sigma) \) is defined by (5.5). Finally, we bound the contribution from \( j \geq 3 \) to (6.48). We make use of (6.43), (6.44) and \( \Pi \leq (1 + p^{(1-\sigma - 2\delta)(k-1)(\ell-1)}) < 2^{(k-1)(\ell-1)} \) (since \( \sigma > -1 + 2\delta \)) to obtain

\[
\left(\sum_{j=3}^{\infty} T_j f_j\right)\Pi \ll \sum_{j=3}^{\infty} p^{2j\delta}\left(\frac{1}{p^{1+\sigma}}\right)^j + p^{2\delta}\left(\frac{1}{p^{1+\sigma}}\right)^{j-1}
\]

\[
\ll p^{6\delta}\left(\frac{1}{p^{1+\sigma}}\right)^3 + p^{2\delta}\left(\frac{1}{p^{1+\sigma}}\right)^2 \leq p^{6\delta}\left(\frac{1}{p^{3+3\sigma}} + \frac{1}{p^{4+2\sigma}}\right) \ll p^{8\delta}p^{\vartheta(\sigma)}
\]

(6.56)

for \( \sigma > -1 + 2\delta + \varepsilon. \)

Next, we simplify the formula for \( c_{02} \) in (6.55). By (6.36) we have

\[
A_{1,0}^2 = \sum_{I, I' \subset X_1, \ |I| = |I'|} X_I X_{I'} Y_J Y_{J'}.
\]

We proceed as follows in 4 cases: (i) \( I = I', J = J', \) (ii) \( I \neq I', J = J', \) (iii) \( I = I', J \neq J', \) and (iv) \( I \neq I', J \neq J' \) to obtain

\[
A_{1,0}^2 = \sum_{I \subset X_1, \ |I| = |I'|} X_I Y_J^2 + 2 \sum_{I \subset X_1, \ |I| = |I'|} X_I Y_J^2 + 2 \sum_{I \subset X_1, \ |I| = |I'|} X_I Y_J^2 + 4 \sum_{I \subset X_1, \ |I| = |I'|} X_I Y_J.
\]

From (6.37) and (6.47) we see that

\[
g_2 + A_{2,0} = 3 \sum_{I \subset X_1, \ |I| = |I'|} X_I Y_J + 2 \sum_{I \subset X_1, \ |I| = |I'|} X_I Y_J^2 + 2 \sum_{I \subset X_1, \ |I| = |I'|} X_I Y_J^2 + 2 \sum_{I \subset X_1, \ |I| = |I'|} X_I Y_J^2 + \sum_{I \subset X_1, \ |I| = |I'|} X_I Y_J^2
\]

and it follows that

\[
c_{02} = g_2 + A_{2,0} - A_{1,0}^2 = - \sum_{I \subset X_1, \ |I| = |I'|} X_I Y_J.
\]

(6.57)
Combining (6.48), (6.55), (6.56), and (6.57) we establish (5.4) along with (5.5). Note that from (5.5), it follows that if $\sigma = \Re(s) = -\frac{1}{2} + 2\delta + \varepsilon$ with $\varepsilon > 0$, then
\[
C_{I,\mathcal{J};\{a_1\},\{b_1\}}(p; s) = 1 + O(p^{-1-2\varepsilon}).
\]
Therefore, $C_{I,\mathcal{J};\{a_1\},\{b_1\}}(s)$ is holomorphic and absolutely convergent for $\Re(s) > -\frac{1}{2} + 2\delta$. Furthermore, we see that the poles listed in (5.6) arise from the zeta factors in (5.2). \hfill \Box

7. Technical Lemmas

In this section we establish Lemma 4.2 and Lemma 4.6.

Proof of Lemma 4.2. We have
\[
f_{r,M,N}(x, y) = W \left( \frac{x}{M} \right) W \left( \frac{y}{N} \right) \varphi \left( \frac{x}{K} \right) \varphi \left( \frac{y}{K} \right) a_r(y)
\]
where
\[
a(y) = a_r(y) = \frac{\omega \left( \frac{1}{r} \log(1 + \frac{r}{y}) \right)}{T} = \frac{1}{T} \int_{-\infty}^{\infty} \omega(t) \left( 1 + \frac{r}{y} \right)^{-it} dt,
\]
and $1 \leq |r| \ll \frac{M}{T_0} T^\varepsilon$. First we shall show
\[
y^v \frac{d^v}{dy^v} a(y) \ll P^v \text{ for } v \geq 0 \text{ where } P = \left( \frac{T}{T_0} \right)^{T_0}
\]
and then deduce the lemma from this bound. The case $v = 0$ is trivial. Observe that for $v \geq 1$
\[
\frac{d^v}{dy^v} a(y) = \frac{1}{T} \int_{-\infty}^{\infty} \omega(t) \frac{d^v}{dy^v} \left( 1 + \frac{r}{y} \right)^{-it} dt.
\]
It is shown in [32, Equation 8.18] that for $v \geq 1$, one has
\[
\left| \frac{d^v}{dy^v} \left( 1 + \frac{r}{y} \right)^{-it} \right| \ll \left( \frac{P}{y} \right)^v
\]
when $y \asymp N \asymp M$, $t \asymp T$, and $1 \leq |r| \ll \frac{M}{T_0} T^\varepsilon = o(M)$. Inserting this last bound in (7.3) establishes (7.4).

We now deduce the lemma. Observe that for $m \geq 0$, we have
\[
\frac{d^m}{dx^m} W \left( \frac{x}{M} \right) \varphi \left( \frac{x}{K} \right) = \sum_{i+j=m} \binom{m}{i} \frac{d^i}{dx^i} W \left( \frac{x}{M} \right) \frac{d^j}{dx^j} \varphi \left( \frac{x}{K} \right) \ll \sum_{i+j=m} \binom{m}{i} M^{-i} K^{-j}
\]
\[
= \left( \frac{1}{M} + \frac{1}{K} \right)^m \ll M^{-m},
\]
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since \( M \leq K \). Similarly, \( \frac{d^n}{dz^n} W \left( \frac{y}{K} \right) \varphi \left( \frac{y}{K} \right) \ll N^{-u} \) for \( u \geq 0 \). By the generalized product rule in conjunction with the last two derivative bounds and \((7.2)\)

\[
f^{(m,n)}(x,y) = \frac{d^m}{dx^m} \left( W \left( \frac{x}{M} \right) \varphi \left( \frac{x}{K} \right) \right) \frac{d^n}{dy^n} \left( W \left( \frac{y}{N} \right) \varphi \left( \frac{y}{K} \right) a(y) \right)
\]

\[
\ll M^{-m} \sum_{u+v=n} \left( \frac{n}{u} \right) d^n dy^n \left( W \left( \frac{y}{N} \right) \varphi \left( \frac{y}{K} \right) a^{(v)}(y) \right) = M^{-m} \sum_{u+v=n} \left( \frac{n}{u} \right) N^{-u} y^{-v} P^v
\]

\[
\ll M^{-m} N^{-n} \sum_{u+v=n} \left( \frac{n}{u} \right) P^v = M^{-m} N^{-n} (1 + P)^n.
\]

Therefore \( x^m y^n f^{(m,n)}(x,y) \ll P^n \) since \( P \geq 1 \), \( x \asymp M \), and \( y \asymp N \). \( \square \)

We now prove Lemma 4.6 which makes extensive use of Stirling’s formula.

**Proof of Lemma 4.6 (i).** Let \( i_1 \in \mathcal{K} \) and \( i_2 \in \mathcal{L} \). We write

\[
a_{i_1} + s_1 = \sigma_1 + iu_1 \quad \text{and} \quad b_{i_2} + s_2 = \sigma_2 + iu_2.
\]

We begin by assuming that \( |u_i| \leq \sqrt{7} \) for \( i = 1, 2 \). Let \( \log z \) be the principal branch of the logarithm, so that \(-\pi < 3 \log z < \pi \) for \( z \in \mathbb{C} \setminus (-\infty,0) \). For \( \epsilon > 0 \) and \( 0 < a \leq 1 \), we have

\[
\log \Gamma(z + a) = \left( z + a - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log(2\pi) + O\left( |z|^{-1} \right)
\]

in the sector \( |\arg(z)| \leq \pi - \epsilon \) (see [40, Section 13.6]). We have

\[
\log \Gamma \left( \frac{1}{2} - b_{i_2} - s_2 + it \right) = -\sigma_2 \ln(t - u_2) - \frac{\pi}{2}(t - u_2) + i \left( (t - u_2) \ln(t - u_2) - \sigma_2 \frac{\pi}{2} \right) + O \left( \frac{1}{t - u_2} \right),
\]

and

\[
\log \Gamma \left( \frac{1}{2} + a_{i_1} + s_1 + it \right) = \sigma_1 \ln(t + u_1) - \frac{\pi}{2}(t + u_1) + i \left( (t + u_1) \ln(t + u_1) + \sigma_1 \frac{\pi}{2} \right) + O \left( \frac{1}{t + u_1} \right).
\]

Hence,

\[
\log \Gamma \left( \frac{1}{2} - b_{i_2} - s_2 + it \right) - \log \Gamma \left( \frac{1}{2} + a_{i_1} + s_1 + it \right)
\]

\[
= (-s_2 - b_{i_2}) \ln(t - u_2) - (s_1 + a_{i_1}) \ln(t + u_1) + it(\ln(t - u_2) - \ln(t + u_1))
\]

\[
+ i(u_1 + u_2) - i \frac{\pi}{2}(s_1 + s_2 + a_{i_1} + b_{i_2}) + O \left( \frac{1}{t - u_2} \right) + O \left( \frac{1}{t + u_1} \right).
\]

Notice that

\[
\ln(t - u_2) = \ln t + \ln \left( 1 - \frac{u_2}{t} \right) \quad \text{(7.6)}
\]

and

\[
\ln(t + u_1) = \ln t + \ln \left( 1 + \frac{u_1}{t} \right). \quad \text{(7.7)}
\]
Using this observation, we get
\[
\log \Gamma \left( \frac{1}{2} - b_{i_2} - s_2 + it \right) - \log \Gamma \left( \frac{1}{2} + a_{i_1} + s_1 + it \right) \\
= -(s_1 + s_2 + a_{i_1} + b_{i_2}) \ln t - (s_2 + b_{i_2}) \ln \left( 1 - \frac{u_2}{t} \right) - (s_1 + a_{i_1}) \ln \left( 1 + \frac{u_1}{t} \right) + it(\ln(t - u_2) - \ln(t + u_1)) \\
+ i(u_1 + u_2) - \frac{\pi}{2} s_1 + s_2 + a_{i_1} + b_{i_2} + O \left( \frac{1}{t - u_2} \right) + O \left( \frac{1}{t + u_1} \right).
\]

Exponentiating both sides of the above equation yields
\[
\frac{\Gamma \left( \frac{1}{2} - b_{i_2} - s_2 + it \right)}{\Gamma \left( \frac{1}{2} + a_{i_1} + s_1 + it \right)} = t^{-(s_1 + s_2 + a_{i_1} + b_{i_2})} \exp \left( -i \frac{\pi}{2} (s_1 + s_2 + a_{i_1} + b_{i_2}) \right) \exp \left( it(\ln(t - u_2) - \ln(t + u_1)) \right) \\
\times \exp \left( i(u_1 + u_2) \right) \exp \left( -(s_2 + b_{i_2}) \ln \left( 1 - \frac{u_2}{t} \right) \right) \exp \left( -(s_1 + a_{i_1}) \ln \left( 1 + \frac{u_1}{t} \right) \right) \\
\times \exp \left( O \left( \frac{1}{t - u_2} \right) + O \left( \frac{1}{t + u_1} \right) \right).
\]

Notice that (7.6) and (7.7) imply
\[
\ln(t - u_2) - \ln(t + u_1) + i(u_1 + u_2) = O \left( \frac{u_1^2 + u_2^2}{t} \right).
\]

Moreover, we have
\[
-(s_2 + b_{i_2}) \ln \left( 1 - \frac{u_2}{t} \right) - (s_1 + a_{i_1}) \ln \left( 1 + \frac{u_1}{t} \right) = -(s_2 + b_{i_2})O \left( \frac{u_2}{t} \right) - (s_1 + a_{i_1})O \left( \frac{u_1}{t} \right) = O \left( \frac{u_1^2 + u_2^2}{t} \right),
\]
and
\[
\exp \left( O \left( \frac{u_1^2 + u_2^2}{t} + \frac{1}{t} \right) \right) = 1 + O \left( \frac{u_1^2 + u_2^2 + 1}{t} \right) = 1 + O \left( \frac{|s_1|^2 + |s_2|^2 + 1}{t} \right).
\]

Therefore, when \(|s_1|^2 + |s_2|^2 \ll t\), we have
\[
\frac{\Gamma \left( \frac{1}{2} - b_{i_2} - s_2 + it \right)}{\Gamma \left( \frac{1}{2} + a_{i_1} + s_1 + it \right)} = t^{-(s_1 + s_2 + a_{i_1} + b_{i_2})} \exp \left( -i \frac{\pi}{2} (s_1 + s_2 + a_{i_1} + b_{i_2}) \right) \left( 1 + O \left( \frac{|s_1|^2 + |s_2|^2 + 1}{t} \right) \right).
\]

When \(|s_1|^2 + |s_2|^2 \gg t\), the \(O\) term in (4.27) becomes larger than 1, and so the dominant term becomes
\[
\frac{|s_1|^2 + |s_2|^2}{t} t^{-s_1 - s_2 - a_{i_1} - b_{i_2}} \exp \left( i \frac{\pi}{2} (-s_1 - s_2 - a_{i_1} - b_{i_2}) \right).
\]

Hence, it suffices to establish for \(|s_1|^2 + |s_2|^2 \gg t\) that
\[
\left| \frac{\Gamma \left( \frac{1}{2} - b_{i_2} - s_2 + it \right)}{\Gamma \left( \frac{1}{2} + a_{i_1} + s_1 + it \right)} \right| \ll \frac{|s_1|^2 + |s_2|^2}{t^{1 + \Re(s_1 + s_2 + a_{i_1} + b_{i_2})}} e^{\frac{\pi}{2} (3(s_1 + s_2 + a_{i_1} + b_{i_2}))},
\]
assuming
\[
\sigma_1 \in (0, A - \frac{1}{2}), \sigma_2 \in (0, \frac{1}{2} - \eta_0) \cup \left[ \frac{1}{2} + \eta_0, \frac{3}{2} - \eta_0 \right], \text{ and } \sigma_1 + \sigma_2 \leq 1. \tag{7.8}
\]

Here \(\eta_0 \in (0, \frac{1}{2})\) and \(A > 0\) are fixed constants. Since we assume that \(\sigma_1\) and \(\sigma_2\) are in bounded intervals, then \(|s_1| \asymp |u_1|\) and \(|s_2| \asymp |u_2|\). We proceed to prove the asymptotic bound
\[
\left| \frac{\Gamma \left( \frac{1}{2} - b_{i_2} - s_2 + it \right)}{\Gamma \left( \frac{1}{2} + a_{i_1} + s_1 + it \right)} \right| \ll \frac{u_1^2 + u_2^2}{t^{1 + |\sigma_1| + |\sigma_2|}} e^{\frac{\pi}{2} (u_1 + u_2)} \text{ where } \sqrt{t} \leq |u_1|, |u_2| \leq t + 1. \tag{7.9}
\]
We write the interval \([\sqrt{t}, t + 1] = I_1 \cup I_2 \cup I_3\) where
\[
I_1 = [\sqrt{t}, \frac{t}{2}], \quad I_2 = [\frac{t}{2}, t - 1], \quad I_3 = [t - 1, t + 1].
\] (7.10)

There are nine cases according to \(|u_1| \in I_i, |u_2| \in I_j\) with
\[
(i, j) \in \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 2)\}.
\]
Recall the Stirling estimate
\[
|\Gamma(\sigma + iu)| = \sqrt{2\pi}|u|^{\sigma - 1/2}e^{-\frac{1}{2}\pi|u|}(1 + O(u^{-1})) \quad \text{for } 0 < \sigma < 1 \text{ and } |u| \geq 1. \tag{7.11}
\]
Note that we also have the bounds
\[
|\Gamma(\sigma + iu)| \ll_{\eta_0, A} 1 \quad \text{for } |u| \leq 1, \sigma \in [-1 + \eta_0, -\eta_0] \cup [\eta_0, A], \tag{7.12}
\]
and
\[
|\Gamma(\sigma + iu)|^{-1} \ll_{A} 1 \quad \text{for } -A < \sigma < A, |u| \leq 1, \tag{7.13}
\]
where \(\eta_0 \in (0, \frac{1}{2})\) and \(A > 0\) are fixed constants. The last two bounds follow from the facts that \(\Gamma(s)\) is holomorphic on and within the given region and \(\Gamma(s)^{-1}\) is entire. Thus, we obtain
\[
|\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)| \ll \begin{cases} |t - u_2|^{-\sigma_2}e^{-\frac{\pi}{2}|t-u_2|} & \text{if } |t - u_2| \geq 1, \\ 1 & \text{if } |t - u_2| \leq 1, \sigma_2 \in [\frac{1}{2} - A, \frac{1}{2} - \eta_0] \cup [\frac{1}{2} + \eta_0, \frac{3}{2} - \eta_0] \end{cases}\tag{7.14}
\]
and
\[
|\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)|^{-1} \ll \begin{cases} |t + u_1|^{-\sigma_1}e^{\frac{\pi}{2}|t+u_1|} & \text{if } |t + u_1| \geq 1, \\ 1 & \text{if } |t + u_1| \ll 1, \sigma_1 \in [-A - \frac{1}{2}, A - \frac{1}{2}] \end{cases}. \tag{7.15}
\]
We provide full details for Cases (i), (v), (vi), and (ix) and note that the remaining cases are treated similarly.

Case (i): \((i, j) = (1, 1)\). It may be checked that the conditions \(\sqrt{t} \leq |u_1| \leq \frac{t}{2}\) and \(\sqrt{t} \leq |u_2| \leq \frac{t}{2}\) imply \(t + u_1, t - u_2 \in [t - \sqrt{t}, \frac{3t}{2}]\). Thus
\[
\left|\frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)}\right| \ll \frac{1}{(t - u_2)^{\sigma_2}(t + u_1)^{\sigma_1}} e^{-\frac{\pi}{2}(t-u_2)} = \frac{e^{\frac{\pi}{2}(u_1+u_2)}}{(t - u_2)^{\sigma_2}(t + u_1)^{\sigma_1}} \ll \frac{(u_1^2 + u_2^2)e^{\frac{\pi}{2}(u_1+u_2)}}{t^{1+\sigma_1+\sigma_2}}.
\]

Case (v): \((i, j) = (2, 2)\). In this case, we have \(\frac{t}{2} \leq |u_1|, |u_2| \leq t - 1\). This implies that \(t + u_1 \geq 1\) and \(t - u_2 \geq 1\). Thus it follows that
\[
\left|\frac{\Gamma(\frac{1}{2} - b_{i_2} - s_2 + it)}{\Gamma(\frac{1}{2} + a_{i_1} + s_1 + it)}\right| \ll \frac{1}{(t - u_2)^{\sigma_2}(t + u_1)^{\sigma_1}} e^{-\frac{\pi}{2}(t-u_2)} = \frac{e^{\frac{\pi}{2}(u_1+u_2)}}{(t - u_2)^{\sigma_2}(t + u_1)^{\sigma_1}} \ll \frac{(u_1^2 + u_2^2)e^{\frac{\pi}{2}(u_1+u_2)}}{t^2}.
\]
since \( u_1^2 + u_2^2 \times t^2 \). By considering cases, we find that

\[
\frac{1}{(t - u_2)^{\sigma_2}(t + u_1)^{\sigma_1}} \ll \begin{cases} 
\frac{t^{-\sigma_1}}{\sigma_1} & \text{if } u_1, u_2 > 0, \\
\frac{t^{-\sigma_1 - \sigma_2}}{\sigma_1 \sigma_2} & \text{if } u_1 > 0, u_2 < 0, \\
1 & \text{if } u_1 < 0, u_2 > 0, \\
\frac{t^{-\sigma_2}}{\sigma_2} & \text{if } u_1, u_2 < 0,
\end{cases}
\]

and thus

\[
\left| \frac{\Gamma\left( \frac{1}{2} - b_{i_2} - s_2 + it \right)}{\Gamma\left( \frac{1}{2} + a_{i_1} + s_1 + it \right)} \right| \ll \begin{cases} 
\frac{(u_1^2 + u_2^2)^{\frac{1}{2}}}{t^{\sigma_1 + \sigma_2}}\frac{\Gamma\left( u_1 + u_2 \right)}{\Gamma\left( t + u_1 \right)^{\sigma_1}} & \text{if } u_1, u_2 > 0, \\
\frac{(u_1^2 + u_2^2)^{\frac{1}{2}}}{t^{\sigma_1 + \sigma_2}}\frac{\Gamma\left( u_1 + u_2 \right)}{\Gamma\left( t + u_1 \right)^{\sigma_1}} & \text{if } u_1 > 0, u_2 < 0, \\
\frac{(u_1^2 + u_2^2)^{\frac{1}{2}}}{t^{\sigma_1 + \sigma_2}}\frac{\Gamma\left( u_1 + u_2 \right)}{\Gamma\left( t + u_1 \right)^{\sigma_1}} & \text{if } u_1 < 0, u_2 > 0, \\
\frac{(u_1^2 + u_2^2)^{\frac{1}{2}}}{t^{\sigma_1 + \sigma_2}}\frac{\Gamma\left( u_1 + u_2 \right)}{\Gamma\left( t + u_1 \right)^{\sigma_1}} & \text{if } u_1, u_2 < 0,
\end{cases}
\]

Observe that the assumptions \( \sigma_1 + \sigma_2 \leq 1 \) and \( \sigma_i \geq 0 \) imply \( t^{-2-\sigma_i} \), \( t^{-2} \leq t^{-1-\sigma_1-\sigma_2} \) for \( i = 1, 2 \). Inserting these bounds implies (7.9).

Case (vi): \((i,j) = (2,3)\). In this case, \( \frac{1}{2} \leq |u_1| \leq t - 1 \) and \( t - 1 \leq |u_2| \leq t + 1 \). We demonstrate the proof in the case \( u_1 < 0 \) and \( u_2 > 0 \) in which case \( 1 < t + u_1 < \frac{1}{2} \) and \( -1 < t + u_2 < 1 \). It follows from (7.14) and (7.15) that

\[
\left| \frac{\Gamma\left( \frac{1}{2} - b_{i_2} - s_2 + it \right)}{\Gamma\left( \frac{1}{2} + a_{i_1} + s_1 + it \right)} \right| \ll \frac{e^{\pi t(t + u_1)}}{(t + u_1)^{\sigma_1}} \ll \frac{(u_1^2 + u_2^2)e^{\pi t(u_1 + u_2)}}{t^{\sigma_1 + \sigma_2}},
\]

where the last step holds provided that \( \sigma_1 + \sigma_2 \leq 1 \). The other cases are similar.

Case (ix): \((i,j) = (3,3)\). In this case, we have \( t - 1 \leq |u_1|, |u_2| \leq t + 1 \). Splitting into cases, depending on the sign of \( u_i \) we find that

\[
\begin{align*}
\text{if } u_1, u_2 > 0, & \text{ then } t + u_1 \in [2t - 1, 2t + 1], t - u_2 \in [-1, 1]; \\
\text{if } u_1 < 0, u_2 > 0, & \text{ then } t + u_1, t - u_2 \in [-1, 1]; \\
\text{if } u_1 > 0, u_2 < 0, & \text{ then } t + u_1, t - u_2 \in [2t - 1, 2t + 1]; \\
\text{if } u_1, u_2 < 0, & \text{ then } t + u_1 \in [-1, 1], t - u_2 \in [2t - 1, 2t + 1].
\end{align*}
\]

It follows from the above cases (7.16) and (7.14) and (7.15) that

\[
\left| \frac{\Gamma\left( \frac{1}{2} - b_{i_2} - s_2 + it \right)}{\Gamma\left( \frac{1}{2} + a_{i_1} + s_1 + it \right)} \right| \ll \begin{cases} 
\frac{e^{\pi t(t + u_1)}}{(t + u_1)^{\sigma_1}} & \text{if } u_1, u_2 > 0, \\
1 & \text{if } u_1 < 0, u_2 > 0, \\
\frac{e^{\pi t(u_1 + u_2)}}{(t - u_2)^{\sigma_2}} & \text{if } u_1 > 0, u_2 < 0, \\
\frac{e^{\pi t(u_1 + u_2)}}{(t - u_2)^{\sigma_2}} & \text{if } u_1, u_2 < 0.
\end{cases}
\]

In this case we have \( u_1^2 + u_2^2 \times t^2 \). Using this along with the size conditions (7.16) on \( t + u_1 \) and \( t - u_2 \) in the various cases, and the assumptions \( \sigma_1 + \sigma_2 \leq 1, \sigma_1 \geq 0, \) and \( \sigma_2 \geq 0 \), it follows that (7.9) holds.

This completes the proof of (7.9). To complete the proof of lemma we must establish (4.27) in the case that \( \sqrt{t} \leq |u_1| \leq t + 1 \) and \( |u_2| \leq \sqrt{t} \) or \( \sqrt{t} \leq |u_2| \leq t + 1 \) and \( |u_1| \leq \sqrt{t} \). The first case is dealt
with by considering three subcases, depending on whether \(|u_1|\) lies in the interval \([\sqrt{t}, \frac{t}{2}],[\frac{t}{2}, t-1]\), or \([t-1,t+1]\). The second case, \(\sqrt{t} \leq |u_2| \leq t+1\) and \(|u_1| \leq \sqrt{t}\) is treated in the same way and in each case we obtain the desired bound

\[
\left| \frac{\Gamma\left(\frac{1}{2} - b_2 - s_2 + it\right)}{\Gamma\left(\frac{1}{2} + a_1 + s_1 + it\right)} \right| \leq \frac{u_1^2 + u_2^2}{t^1+\sigma_1+\sigma_2} e^{\frac{\pi}{2}(u_1+u_2)}.
\]

This completes the proof of part (i) of the lemma. \(\square\)

**Proof of Lemma 4.6 (ii).** We use the notation from the previous part and consider the following cases: (i) \(|u_1| \geq t+1\) and \(|u_2| \geq t+1\), (ii) \(|u_1| \geq t+1\) and \(|u_2| \leq t+1\), and (iii) \(|u_1| \leq t+1\) and \(|u_2| \geq t+1\). We provide a proof in the first case and just observe that cases (ii) and (iii) can be treated similarly. Note that we have the following inequalities

\[
\begin{align*}
\text{if } u_1, u_2 & > 0, \text{ then } t + u_1 \geq 2t + 1 \text{ and } t - u_2 \leq -1; \\
\text{if } u_1 & < 0, u_2 > 0, \text{ then } t + u_1 \leq -1 \text{ and } t - u_2 \leq -1; \\
\text{if } u_1 & > 0, u_2 < 0, \text{ then } t + u_1 \geq 2t + 1 \text{ and } t - u_2 \geq 2t + 1; \\
\text{if } u_1, u_2 & < 0, \text{ then } t + u_1 \leq -1 \text{ and } t - u_2 \geq 2t + 1.
\end{align*}
\]

Using these facts, it follows from (7.14) and (7.15) that

\[
\left| \frac{\Gamma\left(\frac{1}{2} - b_2 - s_2 + it\right)}{\Gamma\left(\frac{1}{2} + a_1 + s_1 + it\right)} \right| \leq \begin{cases} 
\frac{e^{\frac{\pi}{2}(2t+u_1-u_2)}}{|t-u_2|^{|\frac{1}{2}|2|t+u_1|^{|\frac{1}{2}}|}} & \text{if } u_1, u_2 > 0, \\
\frac{e^{-\frac{\pi}{2}(u_1+u_2)}}{|t-u_2|^{|\frac{1}{2}|2|t+u_1|^{|\frac{1}{2}}|}} & \text{if } u_1 < 0, u_2 > 0, \\
\frac{\exp\left(-2t+u_2-u_1\right)}{|t-u_2|^{|\frac{1}{2}|2|t+u_1|^{|\frac{1}{2}}|}} & \text{if } u_1 > 0, u_2 < 0, \\
\frac{\exp\left(-2t+u_1-u_2\right)}{|t-u_2|^{|\frac{1}{2}|2|t+u_1|^{|\frac{1}{2}}|}} & \text{if } u_1, u_2 < 0.
\end{cases}
\]

By the conditions (7.17) it may be checked that each numerator on the right hand side of (7.18) is bounded by \(e^{\frac{\pi}{2}|u_1+u_2|}\). Since \(|u_1|^2 + |u_2|^2 \geq t^2\) and \(|t+u_1|, |t-u_2| \geq 1\) in all cases, we get

\[
\left| \frac{\Gamma\left(\frac{1}{2} - b_2 - s_2 + it\right)}{\Gamma\left(\frac{1}{2} + a_1 + s_1 + it\right)} \right| \leq \frac{|u_1|^2 + |u_2|^2}{t^2} e^{\frac{\pi}{2}|u_1+u_2|},
\]

as desired. \(\square\)

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