IMBEDDINGS OF FREE ACTIONS ON HANDLEBODIES

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Abstract. Fix a free, orientation-preserving action of a finite group $G$ on a 3-dimensional handlebody $V$. Whenever $G$ acts freely preserving orientation on a connected 3-manifold $X$, there is a $G$-equivariant imbedding of $V$ into $X$. There are choices of $X$ closed and Seifert-fibered for which the image of $V$ is a handlebody of a Heegaard splitting of $X$. Provided that the genus of $V$ is at least 2, there are similar choices with $X$ closed and hyperbolic.

Introduction

Any finite group acts (smoothly and) freely preserving orientation on some 3-dimensional handlebody, and the number of inequivalent actions of a fixed $G$ on a fixed genus of handlebody can be arbitrarily large [6]. Consequently, the following imbedding property of such actions may appear surprising at first glance:

Theorem 1. Let $G$ be a finite group acting freely and preserving orientation on two handlebodies $V_1$ and $V_2$, not necessarily of the same genus. Then there is a $G$-equivariant imbedding of $V_1$ into $V_2$.

In fact, this result is almost a triviality, as is the following theorem of which it is a special case:

Theorem 2. Let $G$ be a finite group acting freely and preserving orientation on a handlebody $V$ and on a connected 3-manifold $X$. Then there is a $G$-equivariant imbedding of $V$ into $X$.

By a result of D. Cooper and D. D. Long [1], any finite group acts freely on some hyperbolic rational homology 3-sphere. So theorem 2 shows that a free $G$-action on a handlebody always has an extension to an action on such a 3-manifold. Also, by a result of S. Kojima [4], for any finite $G$ there is a closed hyperbolic 3-manifold whose full isometry group is $G$, and Kojima’s construction actually produces a free action. So there is an extension to a free action on a closed hyperbolic 3-manifold whose full isometry group is $G$.
One might ask for a more natural kind of extension, to a free $G$-action on a closed 3-manifold $M$, for which $V$ is one of the handlebodies in a $G$-invariant Heegaard splitting of $M$. Simply by forming the double of $V$ and taking an identical action on the second copy of $V$, one obtains such an extension with $M$ a connected sum of $S^2 \times S^1$'s. A better question is whether $V$ is an invariant Heegaard handlebody for a free action on an irreducible 3-manifold. Our main result answers this affirmatively.

**Theorem 3.** Let $G$ be a finite group acting freely and preserving orientation on a handlebody $V$. Then the action is the restriction of a free $G$-action on a closed irreducible 3-manifold $M$, which has a $G$-invariant Heegaard splitting with $V$ as one of the handlebodies. One may choose $M$ to be Seifert-fibered. Provided that $V$ has genus greater than 1, one may choose $M$ to be hyperbolic. In both cases, there are infinitely many choices of $M$.

We remark that any orientation-preserving action of a finite group on a closed 3-manifold, free or not, has an invariant Heegaard splitting. For the quotient is a closed orientable 3-orbifold with 1-dimensional (possibly empty) singular set. One may triangulate the quotient so that the singular set is a subcomplex of the 1-skeleton. Then, the preimage of a regular neighborhood of the 1-skeleton is invariant and is one of the handlebodies in a Heegaard splitting.

In the remaining sections of this paper, we prove theorems 2 and 3. In [6], a number of results about free $G$-actions on handlebodies are obtained using more algebraic methods.

1. **Proof of Theorem 2**

Recall that two $G$-actions on spaces $X$ and $Y$ are equivalent if there is a homeomorphism $j: X \to Y$ such that $h(x) = j^{-1}(h(j(x)))$ for all $x \in X$ and all $h \in G$. If $G$ acts properly discontinuously and freely on a path-connected space $X$, then the quotient map $X \to X/G$ is a regular covering map, so by the theory of covering spaces the action determines an extension

$$1 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X/G) \overset{\phi}{\longrightarrow} G \longrightarrow 1.$$ 

Since we have not specified basepoints, the homomorphism $\phi$ is well-defined only up to an inner automorphism of $G$.

Suppose now that $G$ is finite and acts freely and preserving orientation on a handlebody $V$. The quotient manifold $V/G$ is orientable and irreducible with nonempty boundary, so $\pi_1(V/G)$ is torsionfree. A torsionfree finite extension of a finitely generated free group is free (by [3] any finitely generated virtually free group is the fundamental group of a graph of groups with finite vertex groups, and if the group is torsionfree, the vertex groups must be trivial). So $\pi_1(V/G)$ is free, and theorem 5.2 of [2] shows that $V/G$ is a handlebody. In this context, we obtain a simple algebraic criterion for equivalence.

**Lemma 4.** Suppose that $G$ acts freely and preserving orientation on handlebodies $V_1$ and $V_2$, with quotient handlebodies $W_1$ and $W_2$, determining
homomorphisms $\phi_i: \pi_1(W_i) \to G$. The actions are equivalent if and only if there is an isomorphism $\Psi: \pi_1(W_1) \to \pi_1(W_2)$ for which $\phi_2 \circ \Psi = \phi_1$.

Proof. An equivalence of the actions $j: V_1 \to V_2$ induces a homeomorphism $\overline{j}: W_1 \to W_2$ for which $\phi_2 \circ \overline{j}_\# = \phi_1$. Conversely, suppose $\Psi$ exists. Since both $W_1$ and $W_2$ are orientable, there is a homeomorphism $f: W_1 \to W_2$. Using well-known constructions of homeomorphisms of $W_2$ (as for example in [5]), all of Nielsen’s generators of the automorphism group of the free group $\pi_1(W_2)$ can be induced by homeomorphisms, so $f$ may be selected to induce $\Psi$. The condition that $\phi_2 \circ \Psi = \phi_1$ then shows that $f$ lifts to a homeomorphism of covering spaces $j: V_1 \to V_2$, and moreover ensures that $h(x) = j^{-1}(h(j(x)))$.

Now we prove theorem 2. Let $W$ be the quotient handlebody of the action on $V$, let $Y = X/G$, and let $\phi: \pi_1(W) \to G$ and $\psi: \pi_1(Y) \to G$ be the homomorphisms determined by the actions.

There is an imbedding $k: W \to Y$ so that $\psi \circ k_\# = \phi$. For we can regard $W$ as a regular neighborhood of a 1-point union $K$ of circles, so that $\pi_1(K) = \pi_1(W)$, and construct a map $k_0$ of $K$ into $Y$ for which $\psi \circ (k_0)_\# = \phi$. Since $K$ is 1-dimensional, $k_0$ is homotopic to an imbedding, and since $W$ and $Y$ are orientable, this imbedding extends to an imbedding $k$ of $W$ into $Y$. Since $\psi \circ k_\# = \phi$, the preimage of $k(W)$ in $X$ is connected, and by lemma 4 the restricted $G$-action on it is equivalent to the original action on $V$.

Theorem 2 extends to the case when some elements of $G$ reverse the orientation. The equivariant imbedding exists if and only if the subgroups of elements of $G$ that reverse orientation on $V$ and on $X$ are identical. The proof is affected only at the step when the imbedding of $K$ into $Y$ is extended to an imbedding of $W$ into $Y$. The equality of the orientation-reversing subgroups is precisely the condition needed for the extension to exist.

2. PROOF OF THEOREM 3

If the genus of $V$ is 0, then $G$ is trivial and we take $M = S^3$. If the genus of $V$ is 1, so that $V = D^2 \times S^1$, then (using lemma 4) every free $G$-action is equivalent to a cyclic rotation in the $S^1$-factor. Regarding $V$ as a trivially fibered solid torus in the Hopf fibering of $S^3$, the action extends to a free action on $S^3$ with $V$ an invariant Heegaard splitting (it also extends to free actions on infinitely many lens spaces containing $V$ as a fibered Heegaard torus). So we may assume that the genus of $V$ is greater than 1. The quotient handlebody $W = V/G$ has genus at least 2 (since $V$ and consequently $W$ have negative Euler characteristic).

We first construct the Seifert-fibered extension. As in section 2, there is a homomorphism $\phi: \pi_1(W) \to G$ that determines the action. Let $g$ be the genus of $W$, and let $n$ be any positive integer divisible by the orders of all the elements of $G$. We consider a collection of simple closed curves $C_1, \ldots, C_g$ in the boundary $\partial W$, as shown in figure 1 for the case when $g = 3$ and
$n = 4$. Each $C_i$ winds $n$ times around one of the handles of $W$. Let $C'_i$ be the image of $C_i$ under the $n^{th}$ power of a Dehn twist of $\partial W$ about the curve $C$. The union of the $C_i$ does not separate $\partial W$, so neither does the union of the $C'_i$. So we can obtain a closed 3-manifold $Y$ with $W$ as a Heegaard handlebody by attaching 2-handles along the $C'_i$ and filling in the resulting 2-sphere boundary component with a 3-ball.

Let $x_1, \ldots, x_g$ be a standard set of generators of $\pi_1(W)$, where $x_i$ is represented by a loop that goes once around the $i^{th}$ handle. In $\pi_1(W)$, $C_i$ represents $x_i^n$ (up to conjugacy), and $C'_i$ represents $x_i^n (x_1 \cdots x_g)^{-n}$. Since every element of $G$ has order dividing $n$, it follows that $\phi$ carries each $C'_i$ to the trivial element of $G$, so induces a homomorphism $\psi: \pi_1(Y) \rightarrow G$. If $k: W \rightarrow Y$ is the inclusion, then $\psi \circ k\# = \phi$. The covering space $M$ of $Y$ has a free $G$-action and an invariant Heegaard splitting, one of whose handlebodies is the covering space of $W$ corresponding to the kernel of $\phi$, that is, $V$.

We will show that $Y$ is Seifert-fibered, from which it follows that $M$ is Seifert-fibered. Choose imbedded loops in the interior of $W$: $L$ near and parallel to $C$, and $L_1, \ldots, L_g$ near and parallel to loops $\ell_i$ in $\partial W$ with each $\ell_i$ going once around the $i^{th}$ handle, meeting $C_i$ in one point. The loop $\ell_1$ appears in figure 1. By a standard procedure, as explained for example on pp. 275-278 of [9], whose notation we follow, we may change the attaching curves for the discs by Dehn twists about $C$ and the $\ell_i$, at the expense of performing Dehn surgery on $L$ and the $L_i$. First we twist $n$ times along $C$, introducing a $1/n$ coefficient on $L$ and moving each $C'_i$ back to $C_i$. Then, $n-1$ twists along each $\ell_i$ move $C_i$ to a loop $C''_i$ in $\partial W$ that looks like $C_i$ except it goes only once around the handle. This creates surgery coefficients of $-1/(n-1)$ on the $L_i$. We may change the attaching homeomorphism of the Heegaard splitting by any homeomorphism of $\partial W$ that extends over $W$, without changing $Y$. In particular, we may perform left-hand twists in the meridional 2-discs of the 2-handles to move the $C''_i$ to the $\ell_i$. This subtracts 1 from the surgery coefficients of the $L_i$, and subtracts $g$ from the coefficient of $L$, yielding the diagram in figure 2. The $\ell_i$ are the attaching curves for
the discs of a Heegaard description of $S^3$, so $Y$ is obtained from $S^3$ by Dehn surgery using the diagram.

We can already see a Seifert fibering, but we will work out the exact Seifert invariants. The complement of a regular neighborhood of $L$ is a solid torus $T$, for which the $L_i$ are fibers of a product fibering. A cross-sectional surface in this fibering contained in a meridian disc of $T$ meets the boundary of a regular neighborhood of each $L_i$ in a meridian circle and meets the boundary of a regular neighborhood of $L$ in the negative of the longitude, while the fiber meets them in longitude circles and a meridian circle respectively.

A surgery coefficient $a/b$ means that a solid torus is filled in so that $a \cdot m + b \cdot \ell$ becomes contractible, where $m$ and $\ell$ are a meridian-longitude pair for a boundary torus of a regular neighborhood of the link component. A Seifert invariant $(\alpha, \beta)$ determines a filling in which $\alpha \cdot q + \beta \cdot t$ becomes contractible, where $q$ is the cross-section and $t$ is the fiber. So the surgery coefficients of our link produce one exceptional fiber with Seifert invariant $(n, 1-n)$ for each $L_i$, and one with Seifert invariants $(n, gn-1)$ for $L$. In the notation of [8], the unnormalized Seifert invariants of $Y$ are \{0; (o_1, 0); (n, 1-n), \ldots, (n, 1-n), (n, gn-1)\}, where there are $g+1$ exceptional orbits. The normalized invariants are \{-1; (o_1, 0); (n, 1), \ldots, (n, 1), (n, n-1)\}.

Now, we will construct the extension of the $G$-action on $V$ to a hyperbolic 3-manifold. As before, let $n$ be any positive integer divisible by the orders of all the elements of $G$. Assume for the time being that $g = 2$. We take the same curves $C_1$ and $C_2$ as in the Seifert-fibered construction, but for $C$ we take the image of the loop $C_0$ shown in figure 3 under the homeomorphism of $W$ which is a right-hand twist in a meridinal disc in each of the two 1-handles.
As before, we take the images of the $C_i$ under the $n^{th}$ power of a Dehn twist about $C$ as the attaching curves, form the closed manifold $Y$, and use the induced homomorphism $\psi$ to obtain the original $G$-action as an invariant Heegaard handlebody of a free $G$-action on a covering space $M$ of $Y$. Let $\ell_1$, $\ell_2$, $L_1$, $L_2$ and $L$ be as before. We choose the $L_i$ to lie closer to the boundary of $W$ than $L$. Again, change the attaching curves first by the inverse of the $n$ Dehn twists about $C$, introducing a surgery coefficient of $1/n$ on $L$, then by the $n-1$ twists about the $\ell_i$, introducing surgery coefficients $-1/(n-1)$ on the $L_i$. Applying left-hand twists in the meridian discs of the two 1-handles of $W$, we move the attaching curves to the $\ell_i$, obtaining the surgery description of $Y$ shown in figure 4. This time, the coefficient of $L$ is still $1/n$, because $L$ has algebraic intersection 0 with each of the meridian discs of the handles of $Y$ where the left-hand twists were performed. The complement of this link is a 2-fold covering of the complement of the Whitehead link, so is hyperbolic. By [10], Dehn surgery on the link produces a hyperbolic 3-manifold, provided that one avoids finitely many choices for the coefficients of each component. So all but finitely many choices for $n$ yield a hyperbolic 3-manifold for $Y$, and hence for $M$.

Finally, to adapt the construction to arbitrary genus, one simply adds more components to $C_0$ to obtain the $g-1$ circles shown in figure 5. In the surgery description for $Y$, the chain $L \cup L_1$ in figure 4 is replaced by a chain of length $2g-2$, in which $L_1, \ldots, L_{g-1}$ alternate with components from $C_0$, and the component $L_g$ links the chain as did $L_2$ in figure 4. The $L_i$ have surgery coefficients $-1-1/(n-1)$ as before, and the components coming from $C_0$ have coefficient $1/n$, since each had algebraic intersection 0 with the union of the meridian discs. The link complement is the $(2g-2)$-fold covering of the Whitehead link complement, so is hyperbolic, and the argument is completed as before.

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Figure 5. The loops $C_0$ for the general hyperbolic construction.

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