A $q$-analogue of some binomial coefficient identities of Y. Sun

Victor J. W. Guo and Dan-Mei Yang

Department of Mathematics, East China Normal University
Shanghai 200062, People’s Republic of China

1 jwguo@math.ecnu.edu.cn, 2 plain_dan2004@126.com

Submitted: Dec 1, 2010; Accepted: Mar 24, 2011; Published: Mar 31, 2011

Mathematics Subject Classifications: 05A10, 05A17

Abstract

We give a $q$-analogue of some binomial coefficient identities of Y. Sun [Electron. J. Combin. 17 (2010), #N20] as follows:

$\lfloor \frac{n}{2} \rfloor \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m+k}{k} q^{(n-2k)/2} \binom{m+1}{n-2k} q^{2k} = \binom{m+n}{n},$

$\lfloor \frac{n}{4} \rfloor \sum_{k=0}^{\lfloor n/4 \rfloor} \binom{m+k}{k} q^{(n-4k)/4} \binom{m+1}{n-4k} q^{2k} = \lfloor \frac{n/2}{2} \rfloor \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{m+k}{k} q^{2k} \binom{m+n-2k}{n-2k},$

where $\binom{n}{k}_q$ stands for the $q$-binomial coefficient. We provide two proofs, one of which is combinatorial via partitions.

1 Introduction

Using the Lagrange inversion formula, Mansour and Sun [2] obtained the following two binomial coefficient identities:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2k+1} \binom{3k}{k} \binom{n+k}{3k} = \frac{1}{n+1} \binom{2n}{n},$$

$$\sum_{k=0}^{(n-1)/2} \frac{1}{2k+1} \binom{3k+1}{k+1} \binom{n+k}{3k+1} = \frac{1}{n+1} \binom{2n}{n} \quad (n \geq 1).$$

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In the same way, Sun [3] derived the following binomial coefficient identities

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{3k+a} \binom{3k+a}{k} \binom{n+a+k-1}{n-2k} = \frac{1}{2n+a} \binom{2n+a}{n}, \quad (1.3)
\]

\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \frac{1}{4k+1} \binom{5k}{k} \binom{n+k}{5k} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n+k}{k} \binom{2n-2k}{n}, \quad (1.4)
\]

\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \frac{n+a+1}{4k+a+1} \binom{5k+a}{k} \binom{n+a+k}{5k+a} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n+a+k}{k} \binom{2n+a-2k}{n+a}, \quad (1.5)
\]

It is not hard to see that both (1.1) and (1.2) are special cases of (1.3), and (1.4) is the \(a=0\) case of (1.5). A bijective proof of (1.1) and (1.3) using binary trees and colored ternary trees has been given by Sun [3] himself. Using the same model, Yan [4] presented an involutive proof of (1.4) and (1.5), answering a question of Sun.

Multiplying both sides of (1.3) by \(n+a\) and letting \(m=n+a\), we may write it as

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m+k}{k} \binom{m+1}{n-2k} = \binom{m+n}{n}, \quad (1.6)
\]

while letting \(m=n+a\), we may write (1.5) as

\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \binom{m+k}{k} \binom{m+1}{n-4k} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{m+k}{k} \binom{m+n-2k}{n}, \quad (1.7)
\]

The purpose of this paper is to give a \(q\)-analogue of (1.6) and (1.7) as follows:

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left[ \begin{array}{c} m+k \\ k \end{array} \right]_q \left[ \begin{array}{c} m+1 \\ n-2k \end{array} \right]_q q^{n-2k} = \left[ \begin{array}{c} m+n \\ n \end{array} \right]_q, \quad (1.8)
\]

\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \left[ \begin{array}{c} m+k \\ k \end{array} \right]_q \left[ \begin{array}{c} m+1 \\ n-4k \end{array} \right]_q q^{n-4k} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left[ \begin{array}{c} m+k \\ k \end{array} \right]_q \left[ \begin{array}{c} m+n-2k \\ n-2k \end{array} \right]_q, \quad (1.9)
\]

where the \(q\)-binomial coefficient \(\left[ \begin{array}{c} x \\ k \end{array} \right]_q\) is defined by

\[
\left[ \begin{array}{c} x \\ k \end{array} \right]_q = \begin{cases} \prod_{i=1}^{k} \frac{1-q^{x-i+1}}{1-q^i}, & \text{if } k \geq 0, \\ 0, & \text{if } k < 0. \end{cases}
\]

We shall give two proofs of (1.8) and (1.9). One is combinatorial and the other algebraic.
2 Bijective proof of (1.8)

Recall that a partition $\lambda$ is defined as a finite sequence of nonnegative integers $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. A nonzero $\lambda_i$ is called a part of $\lambda$. The number of parts of $\lambda$, denoted by $\ell(\lambda)$, is called the length of $\lambda$. Write $|\lambda| = \sum_{i=1}^{r} \lambda_i$, called the weight of $\lambda$. The sets of all partitions and partitions into distinct parts are denoted by $\mathcal{P}$ and $\mathcal{D}$ respectively. For two partitions $\lambda$ and $\mu$, let $\lambda \cup \mu$ be the partition obtained by putting all parts of $\lambda$ and $\mu$ together in decreasing order.

It is well known that (see, for example, [1, Theorem 3.1])

$$
\sum_{\lambda \in \mathcal{P}, \ell(\lambda) = n} q^{\lambda} = q^n \left[ \begin{array}{c} m+n \cr n \end{array} \right]_q,
$$

$$
\sum_{\lambda \in \mathcal{D}, \ell(\lambda) = n} q^{\lambda} = \left[ \begin{array}{c} m+1 \cr n \end{array} \right]_q \left( \frac{n+1}{2} \right)_q.
$$

Therefore,

$$
\sum_{\lambda \in \mathcal{A}, |\lambda| = m+1} q^{2|\lambda| + |\mu|} = q^n \sum_{k=0}^{n/2} \left[ \begin{array}{c} m+k \cr k \end{array} \right] q^k \left[ \begin{array}{c} m+1 \cr n-2k \end{array} \right] \left( \frac{n-2k}{2} \right)_q,
$$

where $k = \ell(\lambda)$. Let

$$
\mathcal{A} = \{ \lambda \in \mathcal{P} : \lambda_1 \leq m+1 \text{ and } \ell(\lambda) = n \},
$$

$$
\mathcal{B} = \{ (\lambda, \mu) \in \mathcal{P} \times \mathcal{D} : \lambda_1, \mu_1 \leq m+1 \text{ and } 2\ell(\lambda) + \ell(\mu) = n \}.
$$

We shall construct a weight-preserving bijection $\phi$ from $\mathcal{A}$ to $\mathcal{B}$. For any $\lambda \in \mathcal{A}$, we associate it with a pair $(\overline{\lambda}, \mu)$ as follows: If $\lambda_i$ appears $r$ times in $\lambda$, then we let $\lambda_i$ appear $\lfloor r/2 \rfloor$ times in $\overline{\lambda}$ and $r - 2\lfloor r/2 \rfloor$ times in $\mu$. For example, if $\lambda = (7, 5, 5, 4, 4, 4, 2, 2, 1)$, then $\overline{\lambda} = (5, 4, 4, 2)$ and $\mu = (7, 2, 1)$. Clearly, $(\overline{\lambda}, \mu) \in \mathcal{B}$ and $|\lambda| = 2|\overline{\lambda}| + |\mu|$. It is easy to see that $\phi : \lambda \mapsto (\overline{\lambda}, \mu)$ is a bijection. This proves that

$$
\sum_{\lambda \in \mathcal{A}} q^{\lambda} = \sum_{(\lambda, \mu) \in \mathcal{B}} q^{2|\lambda| + |\mu|}.
$$

Namely, the identity (1.8) holds.
3 Involution proof of (1.9)

It is easy to see that

\[
q^n \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{m+k}{k} q^2 \binom{m+n-2k}{n-2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \sum_{\Delta_1 \leq m+1} q^{2|\lambda|} \sum_{\Delta_1 \leq m+1} q^{\mu} \sum_{\ell(\lambda) = k} \sum_{\ell(\mu) = m-2k} (-1) \ell(\lambda) q^{2|\lambda|+|\mu|}, \tag{3.1}
\]

and

\[
q^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m+k}{k} q^4 \binom{m+1}{4k} q^{(n-4k)} = \sum_{\lambda \in \mathcal{D}} q^{4|\lambda|+|\mu|}. \tag{3.2}
\]

Let

\[
\mathcal{U} = \{ (\lambda, \mu) \in \mathcal{D} : \lambda_1, \mu_1 \leq m+1 \text{ and } 2\ell(\lambda) + \ell(\mu) = n \},
\]

\[
\mathcal{V} = \{ (\lambda, \mu) \in \mathcal{U} : \text{ each } \lambda_i \text{ appears an even number of times and } \mu \in \mathcal{D} \}.
\]

We shall construct an involution \( \theta \) on the set \( \mathcal{U} \setminus \mathcal{V} \) with the properties that \( \theta \) preserves \( 2|\lambda| + |\mu| \) and reverses the sign \( (-1)^{\ell(\lambda)} \).

For any \( (\lambda, \mu) \in \mathcal{U} \setminus \mathcal{V} \), notice that either some \( \lambda_i \) appears an odd number of times in \( \lambda \), or some \( \mu_j \) is repeated in \( \mu \), or both are true. Choose the largest such \( \lambda_i \) and \( \mu_j \) if they exist, denoted by \( \lambda_{i_0} \) and \( \mu_{j_0} \) respectively. Define

\[
\theta((\lambda, \mu)) = \begin{cases} ((\lambda \setminus \lambda_{i_0}), \mu \cup (\lambda_{i_0}, \lambda_{i_0})), & \text{if } \lambda_{i_0} \geq \mu_{j_0} \text{ or } \mu \in \mathcal{D}, \\ ((\lambda \cup \mu_{j_0}), \mu \setminus (\mu_{j_0}, \mu_{j_0})), & \text{if } \lambda_{i_0} < \mu_{j_0} \text{ or } \lambda_{i_0} \text{ does not exist.} \end{cases}
\]

For example, if \( \lambda = (5, 5, 4, 4, 3, 3, 1, 1) \) and \( \mu = (5, 3, 2, 2, 1) \), then

\[
\theta(\lambda, \mu) = ((5, 5, 4, 4, 3, 3, 1, 1), (5, 4, 4, 3, 2, 2, 1)).
\]

It is easy to see that \( \theta \) is an involution on \( \mathcal{U} \setminus \mathcal{V} \) with the desired properties. This proves that

\[
\sum_{(\lambda, \mu) \in \mathcal{U}} (-1)^{\ell(\lambda)} q^{2|\lambda|+|\mu|} = \sum_{(\lambda, \mu) \in \mathcal{V}} (-1)^{\ell(\lambda)} q^{2|\lambda|+|\mu|} = \sum_{\mu \in \mathcal{D}} q^{4|\tau|+|\mu|}, \tag{3.3}
\]

where \( \lambda = \tau \cup \tau \). Combining (3.1)–(3.3), we complete the proof of (1.9).
4 Generating function proof of (1.8) and (1.9)

Recall that the $q$-shifted factorial is defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \ldots.$$

Then we have

$$\frac{1}{(z^2; q^2)_{m+1}}(-z; q)_m = \frac{1}{(z; q)_{m+1}}$$

and

$$\frac{1}{(z^4; q^4)_{m+1}}(-z; q)_m = \frac{1}{(z; q)_{m+1} (-z^2; q^2)_{m+1}}.$$  \hfill (4.2)

By the $q$-binomial theorem (see, for example, [1, Theorem 3.3]), we may expand (4.1) and (4.2) respectively as follows:

$$\left( \sum_{k=0}^{\infty} \binom{m+k}{k} q^{2k} z^{2k} \right) \left( \sum_{k=0}^{m+1} \binom{m+1}{k} q^{(k)} z^k \right) = \sum_{k=0}^{\infty} \binom{m+k}{k} q^{k} z^k,$$  \hfill (4.3)

and

$$\left( \sum_{k=0}^{\infty} \binom{m+k}{k} q^{4k} z^{4k} \right) \left( \sum_{k=0}^{m+1} \binom{m+1}{k} q^{(k)} z^k \right) = \left( \sum_{k=0}^{\infty} \binom{m+k}{k} q^{2} (-1)^k z^{2k} \right).$$  \hfill (4.4)

Comparing the coefficients of $z^n$ in both sides of (4.3) and (4.4), we obtain (1.8) and (1.9) respectively.

Finally, we give the following special cases of (1.8):

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+k}{k} q^{2k} \left[ \frac{n+1}{2k+1} \right] q^{\left( \frac{n-2k}{2} \right)} = \left[ \frac{2n}{n} \right]$$

and

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+k}{k} q^{2k} \left[ \frac{n}{2k+1} \right] q^{\left( \frac{n-2k-1}{2} \right)} = \left[ \frac{2n}{n-1} \right].$$  \hfill (4.6)

When $q = 1$, the identities (4.5) and (4.6) reduce to (1.1) and (1.2) respectively.

Acknowledgments. This work was partially supported by the Fundamental Research Funds for the Central Universities, Shanghai Rising-Star Program (#09QA1401700), Shanghai Leading Academic Discipline Project (#B407), and the National Science Foundation of China (#10801054).
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