The symplectic structure of rational Lax pair systems.

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December 1998

Abstract

We consider dynamical systems associated to Lax pairs depending rationally on a spectral parameter. We show that we can express the symplectic form in terms of algebro-geometric data provided that the symplectic structure on $L(\lambda)$ is of Kirillov type. In particular, in this case the dynamical system is integrable.
1 Introduction.

The algebro–geometrical approaches to integrable systems are the most powerful ways of solving the Lax equations of motion. These methods allow to introduce natural coordinates on phase space, namely action variables which characterize the spectral curve of genus $g$ and a divisor of $g$ points on this curve which is equivalent to a point on the Jacobian and can be related to angle variables. These coordinates are also known as the separated Sklyanin’s variables [3]. It is an important and natural problem to express the symplectic form in terms of these coordinates. The first conjecture was made by Veselov and Novikov in [1]. More recently the question has been reconsidered by Krichever and Phong [4] who proposed a general method to prove such a connection. Application of their technique to the spin Calogero model [2, 5, 6] met some difficulties and led to improvements and simplifications of the method. In particular the role of the branch points has been recognized in [6]. The spin Calogero model however is rather specific and we propose here to analyze the situation corresponding to a generic rational Lax pair. In this case the natural symplectic structure is a coadjoint orbit one, and we find that we have to use it in all details in order to complete the calculation of the symplectic form. This shows the nice interplay between the group theoretical and analytic approaches to integrable systems.

2 The setup.

Let us consider a $N \times N$ Lax matrix $L(\lambda)$, depending rationally on a spectral parameter $\lambda \in \mathbb{C}$, with poles at points $\lambda_k$

$$L(\lambda) = L_0 + \sum_k L_k$$

where $L_0 = \text{Diag}(a_1, \cdots, a_N)$ is independent of $\lambda$ and $L_k$ is the polar part of $L(\lambda)$ at $\lambda_k$, i.e. $L_k = \sum_{r=-n_k}^{-1} L_{k,r} (\lambda - \lambda_k)^r$. We assume that $L_k$ lives in a coadjoint orbit of the group of $N \times N$ matrix regular in the vicinity of $\lambda = \lambda_k$, i.e.

$$L_k = \left( g_k A_k g_k^{-1} \right)_-$$

where $A_k(\lambda)$ is a diagonal matrix with a pole of order $n_k$ at $\lambda = \lambda_k$, and $g_k$ has a regular expansion at $\lambda = \lambda_k$. The notation ($\cdot$)$_-$ means taking the singular part at $\lambda = \lambda_k$. This singular part only depends on the singular part $(A_k)_-$ and the first $n_k$ coefficients of the expansion of $g_k$ in powers of $(\lambda - \lambda_k)$. We see that $(A_k)_-$ specifies the coadjoint orbit, and that the physical degrees of freedom are contained in the first $n_k$ coefficients of $g_k(\lambda)$. Since $A_k$ commutes with diagonal matrices one has to take the quotient by $g_k \rightarrow g_k d_k$ where $d_k(\lambda)$ is a regular diagonal matrix, in order to correctly describe the dynamical variables on the orbit.

To interpret $L$ as living in a coadjoint orbit we consider the direct product of loop groups defined for each $\lambda_k$. Its Lie algebra is a direct sum of Lie algebras $\mathcal{G}_k$ where an element of $\mathcal{G}_k$ is a regular sum $X_k(\lambda) = \sum_{n \geq 0} X_{k,n}(\lambda - \lambda_k)^n$ and the Lie bracket is such that $[X_k(\lambda), X_l(\lambda)] = 0$ if $k \neq l$. The dual of this Lie algebra can be viewed as $N \times N$ matrix valued meromorphic functions $L(\lambda)$ with poles at the $\lambda_k$ and vanishing at $\infty$. The
duality is expressed by:

\[ < L(\lambda), \oplus_k X_k(\lambda) > = \sum_k \text{Res}_{\lambda_k} \text{Tr}(L(\lambda)X_k(\lambda)) \]

Note that \( L \) can be uniquely decomposed as \( \sum_k L_k \), where \( L_k \) is the polar part at \( \lambda_k \) and that only \( L_k \) appears in the residue \( \text{Res}_{\lambda_k} \). One can add an invariant character \( L_0 \) to \( L \).

The dimension of the orbit of \( L_k \) is \( N(N-1)n_k \) so that \( L(\lambda) \) depends on \( \sum_k N(N-1)n_k \) degrees of freedom. On this phase space we have a standard Kirillov symplectic form:

\[ \omega = \sum_k \text{Res}_{\lambda_k} \text{Tr}((A_k - g^{-1} \delta g_k \wedge g^{-1} \delta g_k) d\lambda) \]  \hspace{1cm} (2)

In the algebro–geometric approach we study the eigenvector equation:

\[ (L(\lambda) - \mu 1) \Psi(\lambda, \mu) = 0 \]  \hspace{1cm} (3)

and view the eigenvector \( \Psi(\lambda, \mu) \) with eigenvalue \( \mu \) as an analytic section of a natural line bundle on the Riemann surface defined by the characteristic equation:

\[ \Gamma : R(\lambda, \mu) \equiv \det(L(\lambda) - \mu 1) = 0 \]  \hspace{1cm} (4)

If \( N \) is the dimension of the Lax matrix, the equation of the curve is of the form:

\[ \Gamma : R(\lambda, \mu) \equiv (-\mu)^N + \sum_{q=0}^{N-1} r_q(\lambda)\mu^q = 0 \]  \hspace{1cm} (5)

The coefficients \( r_q(\lambda) \) are polynomials in the matrix elements of \( L(\lambda) \) and therefore have poles at \( \lambda_k \). Since the Lax equation \( \dot{L} = [L, M] \) is isospectral, these coefficients are time-independent and are related to the action variables.

From eq.(6), we see that the spectral curve appears as an \( N \)-sheeted covering of the Riemann sphere. To a given point \( \lambda \) on the Riemann sphere there correspond \( N \) points on the curve whose coordinates are \((\lambda, \mu_1), \cdots (\lambda, \mu_N)\) where the \( \mu_i \) are the solutions of the algebraic equation \( R(\lambda, \mu) = 0 \). By definition \( \mu_i \) are the eigenvalues of \( L(\lambda) \).

We assume for simplicity that all the \( a_i \)'s are different. Then on the spectral curve, we have \( N \) points \( Q_i \equiv (\lambda = \infty, \mu_i = a_i) \) above \( \lambda = \infty \). The analyticity properties of \( L(\lambda) \) are invariant under conjugation by constant matrices. To preserve the normalization at \( \infty \) these matrices have to be diagonal. Generically, these transformations form a group of dimension \( N - 1 \) and we will have to factor it out.

Before doing complex analysis on \( \Gamma \), one has to determine its genus. A general strategy is as follows. As we have seen, \( \Gamma \) is a \( N \)-sheeted covering of the Riemann sphere. There is a general formula expressing the genus \( g \) of an \( N \)-sheeted covering of a Riemann surface of genus \( g_0 \) (in our case \( g_0 = 0 \)). It is the Riemann-Hurwitz formula:

\[ 2g - 2 = N(2g_0 - 2) + \nu \]  \hspace{1cm} (6)

where \( \nu \) is the branching index of the covering. The branch points occur at the zeroes of \( \partial_\mu R \). The number of its zeroes is the same as the number of its poles, which are located above the points \( \lambda_k \). One gets:

\[ g = \frac{N(N-1)}{2} \sum_k n_k - N + 1 \]
For completeness of the method it is important to observe that the genus is equal to the number of action variables occurring as independent parameters in the eq.\(\text{[3]}\) which should also be half the dimension of phase space. This phase space \(\mathcal{M}\) is the above coadjoint orbit, quotiented by the action of constant diagonal matrices. The orbits of this action are of dimension \((N - 1)\), since the identity does not act. One has to perform a Hamiltonian reduction by this action. First one fixes the momentum, yielding \((N - 1)\) conditions, and then one takes the quotient by the stabilizer of the momentum which is here the whole group since it is Abelian. Hence the dimension of the phase space is reduced by \(2(N - 1)\), yielding:

\[
\dim \mathcal{M} = (N^2 - N) \sum_k n_k - 2(N - 1) = 2g
\]

Let us now count the number of independent coefficients in eq.\(\text{[3]}\). It is clear that \(r_j(\lambda)\) is a rational function of \(\lambda\). The value of \(r_j\) at \(\infty\) is known since \(\mu_j \to a_j\). Moreover \(r_j(\lambda)\) has a pole of order \(jn_k\) at \(\lambda = \lambda_k\). Hence it can be expressed on \(j \sum_k n_k\) parameters namely the coefficients of all these poles. So we have altogether \(\frac{1}{2}N(N + 1)\sum_k n_k\) parameters. They are not all independent however. Indeed above \(\lambda = \lambda_k\) the various branches can be written:

\[
\mu_j = \sum_{n=1}^{n_k} \frac{c_n^{(j)}}{(\lambda - \lambda_k)^n} + \text{regular}
\]

where all the coefficients \(c_1^{(j)}, \cdots, c_{n_k}^{(j)}\) are fixed and non–dynamical because they are the matrix elements of the diagonal matrices \((A_k)_-\), while the regular part is dynamical. This implies \(Nn_k\) constraints on the coefficients of \(r_j\) (note that \(r_j\) is the symmetrical function \(\sigma_j(\mu_1, \cdots, \mu_N)\) hence the \(n_k\) highest order terms in \(r_j\) are fixed). We are left with \(\frac{1}{2}N(N - 1)\sum_k n_k\) parameters, that is \(g + N - 1\) parameters.

It remains to take the quotient by the action of constant diagonal matrices. Consider the Hamiltonians \(H_n = (1/n) \text{res}_{\lambda=\infty} \text{Tr} (L^n(\lambda))d\lambda\), i.e. the term in \(1/\lambda\) in \(\text{Tr} (L^n(\lambda))\). These are functions of the \(r_j\) that we show to be the generators of the diagonal action. First we have:

\[
\text{Res}_{\lambda=\infty} \text{Tr} (L^n(\lambda))d\lambda = n \text{Res}_{\lambda=\infty} \text{Tr} (L_0^{n-1} \sum_k L_k(\lambda))d\lambda
\]

since all \(L_k\) are of order \(1/\lambda\) at \(\infty\). Computing with the above Kirillov bracket one obtains \(\{H_n, L(\mu)\} = -[L_0^{n-1}, L(\mu)]\) which is the coadjoint action of a diagonal matrix on \(L(\mu)\). Since \(L_0\) is generic the \(L_0^n\) generate the space of all diagonal matrices, so we get exactly \(N - 1\) generators \(H_1, \cdots, H_{N-1}\). In the Hamiltonian reduction procedure, the \(H_n\) are the moments of the group action and are to be set to fixed (non–dynamical) values. Hence when the system is properly reduced we are left with exactly \(g\) action variables.

The eigenvector line–bundle has Chern class \(-(g + N - 1)\). We get a non–vanishing section by requiring that the eigenvector \(\Psi(P)\) has its first component equal to 1. Then it has \((g + N - 1)\) poles on the spectral curve \(\text{[2]}\). Since \(L\) is diagonal at \(\infty\) the natural eigenvectors of \(L(\lambda)\) above \(\lambda = \infty\) are the canonical basis vectors. Since we impose that \(\Psi_1 = 1\) this implies that \(\Psi(P)\) has poles at \((N - 1)\) points above \(\lambda = \infty\), hence we are left with \(g\) dynamical poles. Note that this is the number of angle variables and we are led to describe the system in terms of the \(g\) action variables entering the equation of the spectral curve and \(g\) points on this curve, namely the dynamical poles of \(\Psi\).
3 Symplectic form.

We have seen that the Lax pair description of a dynamical system naturally provides coordinates on phase space, namely \( g \) independent action variables \( F_i \) which parameterize the spectral curve \( \Gamma \), and \( g \) points \( \nu_i = (\lambda_i, \mu_i) \) on the spectral curve, which we called the dynamical divisor. It is important to express the symplectic form in terms of these coordinates. The phase space appears as a fibered space whose base is the space of moduli of the spectral curve, explicitly described as coefficients of the equation \( R(\lambda, \mu) = 0 \) of the spectral curve, and the fiber at a given \( R \) is the Jacobian of the curve \( R = 0 \). On this space we introduce a differential \( \delta \) which varies the dynamical variables \( F_i, \lambda_i, \mu_i \) subjected to the constraint \( R(F_i)(\lambda_i, \mu_i) = 0 \).

We will need an auxiliary fiber bundle above the same base whose fiber is \( \Gamma \times \text{Jac}(\Gamma) \). We extend \( \delta \) to this space by keeping the previous definition on the \( \text{Jac}(\Gamma) \) part and on the \( \Gamma \) part, we differentiate any function of \( F_i, \lambda, \mu \) with \( R(F_i)(\lambda, \mu) = 0 \) keeping \( \lambda \) constant. Remark that \( \lambda \) is universally defined on the whole family of curves. For a function \( f(P; F_i) \) if we take \( \lambda \) as a local parameter \( \delta f = \sum_i \partial_{F_i} f \delta F_i \). At a branch point the local parameter is \( \mu \) and we have:

\[
\delta f = \partial_\mu f \delta \mu + \sum_i \partial_{F_i} f \delta F_i, \quad \text{with} \quad \delta \mu = -\frac{1}{\partial_\mu R(F_i)(\lambda, \mu)} \sum_i \partial_{F_i} R(F_i)(\lambda, \mu) \delta F_i \tag{7}
\]

Note that at a branch point \( \partial_\mu R(F_i)(\lambda, \mu) = 0 \), hence the differential \( \delta f \) acquires a pole even though \( f \) is regular. Remark however that if \( f \) depends rationally on \( \lambda \) and the \( F_i \), \( \delta f \) is regular at the branch points.

At each point \( P(\lambda, \mu) \) on \( \Gamma \) is defined a column eigenvector \( \Psi(P) \) of the Lax matrix up to normalization. This allows to define a matrix \( \hat{\Psi}(\lambda) \) whose columns are the \( N \) vectors \( \Psi(P_i) \) at the \( N \) points \( P_i \) above \( \lambda \) (assuming we have chosen locally some ordering of the sheets). We also consider its inverse matrix \( \hat{\Psi}^{-1}(\lambda) \) and denote its \( N \) lines by \( \hat{\Psi}^{-1}(P_i) \) (for the same ordering of the \( P_i \)). In particular we have \( < \hat{\Psi}^{-1}(P), \hat{\Psi}(P) >= 1 \). Note that \( \hat{\Psi}^{-1}(P) \) has poles at the branching points of the covering \( (\lambda, \mu) \to \lambda \) since the determinant of \( \hat{\Psi} \) vanishes here.

We define a three–form \( K \) on our extended fiber bundle, and we regard it as a one–form on \( \Gamma \) whose coefficients are two–forms on phase space.

\[
K = K_1 + K_2 + K_3 \tag{8}
K_1 = < \hat{\Psi}^{-1}(P) \delta L(\lambda) \land \delta \Psi(P) > d\lambda \\
K_2 = < \hat{\Psi}^{-1}(P) \delta \mu \land \delta \Psi(P) > d\lambda \\
K_3 = \delta (\log \partial_\mu R) \land \delta \mu d\lambda
\]

**Proposition.** Let us define the two–form on phase space: \( \omega = \sum_{k,i} \text{Res}_{P_{k,i}} K \) where \( P_{k,i} \) are the points above the poles \( \lambda_k \) of \( L(\lambda) \). Then we have:

\[
\omega = 2 \sum_{i=1}^g \delta \lambda_i \land \delta \mu_i
\]

hence \( \omega \) is a symplectic form on phase space.

**Proof.** The sum of the residues of \( K \) seen as a form on \( \Gamma \) vanishes. The poles of \( K \) are located at four different places, first the dynamical poles of \( \Psi \), then the poles at
the $P_{k,i}$ coming from $L$ and $\mu$, next the poles above $\lambda = \infty$ coming from $\Psi$ and $d\lambda$, and finally the poles at the branch points of the covering coming from the poles of $\Psi^{-1}$ and from eq.\([7]\).

Let us compute the residues at the dynamical poles $(\nu_1, \cdots, \nu_g)$. We write the coordinates of these points as: $\nu_i = (\lambda_i, \mu_i)$ for $i = 1, \cdots, g$. Near such a point we can choose $\lambda$ as a universal local parameter and $\Psi = 1/(\lambda - \lambda_i) \times \Psi_{\text{reg}}$ hence:

$$\delta \Psi = \frac{\delta \lambda_i}{\lambda - \lambda_i} (\Psi + O(1))$$

Since $(L - \mu)\Psi = 0$ and $\Psi^{-1}(L - \mu) = 0$, we have $(\delta L - \delta \mu)\Psi + (L - \mu)\delta \Psi = 0$. Multiplying by $\Psi^{(-1)}$ we get $\Psi^{(-1)}(\delta L\Psi = \delta \mu$, therefore:

$$\text{Res}_{\nu_i} K_1 = \delta \mu|_{\nu_i} \wedge \delta \lambda_i$$

Here $\delta \mu$ is to be seen as a meromorphic function on $\Gamma$ given by eq.(7). However varying $R(\lambda_i, \mu_i) = 0$ we obtain:

$$\delta \mu|_{\nu_i} = \delta \mu_i + \frac{\partial R}{\partial \mu} \bigg|_{\nu_i} \delta \lambda_i$$

and the second term does not contribute to the wedge product.

The contribution of $K_2$ is exactly the same. Finally, $K_3$ is regular at $\nu_i$ and does not contribute to the residue at this point. So we finally get:

$$\text{Res}_{\nu_i} K = 2\delta \mu_i \wedge \delta \lambda_i$$

We now show that there are no residues at the branch points due to the proper choice of $K_2$ and $K_3$. Let us look at the term $K_1$. At a branch point $b$, $\Psi^{(-1)}$ has a simple pole, $\delta L$ is regular, $\delta \Psi$ has a simple pole due to eq.(7) and the form $d\lambda$ has a simple zero, hence the considered expression has a simple pole at $b$. To compute its residue it is enough to keep the polar part in $\delta \Psi$, i.e. to replace $\delta \Psi$ by $\partial_\mu \Psi \delta \mu$ (recall that $\mu$ is a good local parameter around $b$). We get:

$$\text{Res}_b K_1 = \text{Res}_b < \Psi^{(-1)}(\delta L \partial_\mu \Psi > \wedge \delta \mu \frac{dL}{d\mu} \Psi$$

where in the last equation we have used the antisymmetry of the wedge product to replace $\delta L$ by $\delta L - \delta \mu$. Using again the eigenvector equation $(L - \mu)\Psi = 0$, and varying the point $(\lambda, \mu)$ on the curve around $b$ one gets

$$(L - \mu)\partial_\mu \Psi = \Psi - \frac{d\lambda}{d\mu} \frac{dL}{d\lambda} \Psi$$

(10)

where $d\lambda/d\mu$ vanishes at the branch point. We then differentiate with $\delta$ and get:

$$\text{Res}_b < \Psi^{(-1)}(\delta L - \delta \mu)\partial_\mu \Psi > \wedge \delta \mu \frac{dL}{d\mu} \Psi = \text{Res}_b < \Psi^{(-1)}(\delta \Psi > \wedge \delta \mu \frac{dL}{d\mu} \Psi$$

$$- \text{Res}_b < \Psi^{(-1)}(\delta \frac{d\lambda}{d\mu} \frac{dL}{d\lambda} \Psi > \wedge \delta \mu \frac{dL}{d\mu} \Psi$$
The first term exactly cancels the term $\text{Res}_b K_2$. The second term gives a non–vanishing contribution

$$\text{Res}_b \frac{\delta \mu_b}{\mu - \mu_b} \wedge \delta \mu \, d\lambda$$

Indeed let us call $\zeta = (d\lambda / d\mu)(dL / d\lambda) \Psi$ which vanishes at $b = (\lambda_b, \mu_b)$. Then $\zeta = (\mu - \mu_b)\zeta_1$ hence $\delta \zeta = -\delta \mu_b / (\mu - \mu_b) \zeta + \delta \mu \zeta_2 + \zeta_3$ with $\zeta_3$ regular. The second term does not contribute due to the antisymmetry of the wedge product and the third term has no residue. Using eq.(10) we have $< \Psi^{(-1)} \frac{d}{d\mu} \frac{d}{d\lambda} \Psi >= 1$ yielding the above formula. This contribution is exactly canceled by the contribution of $K_3$.

We now compute the residues above $\lambda = \infty$. Recall that we consider a reduced Hamiltonian system under the action of diagonal matrices. To fix this action on can normalize the eigenvectors at $\infty$ so that $\psi_i(Q_j) = \lambda \delta_{ij} + O(1)$ for $i, j = 2, \cdots, N$. Notice that $L = L_0 + O(1/\lambda)$ where $L_0$ is non–dynamical so $\delta L_0 = 0$, and that $\mu = a_i + O(1/\lambda)$ around $Q_i$ hence $\delta L = \delta \mu = O(1/\lambda)$. Moreover $\Psi^{(-1)}$ vanishes at $Q_i$ and $d\lambda$ has a double pole. Altogether $K_1$ and $K_2$ are regular at $Q_i$ since $(\delta \Psi)(Q_i) = O(1)$ due to the normalization condition. Finally $K_3$ is also regular since on the sheet $\mu = \mu_i(\lambda)$ one can write $\partial_\mu R = \prod_{j \neq i} (\mu_i - \mu_j)$ yielding $\delta \log \partial_\mu R = O(1/\lambda)$. All this shows that $K$ has no residues above $\lambda = \infty$.

**Proposition.** The symplectic form $\omega$ is given by:

$$\omega = 2 \sum_{i=1}^g \delta \lambda_i \wedge \delta \mu_i = 2 \sum_k \text{Res}_{\lambda_k} \text{Tr}((A_k)_- g_k^{-1} \delta g_k \wedge g_k^{-1} \delta g_k) \, d\lambda$$

where $(\lambda_i, \mu_i), i = 1, \cdots g,$ are the coordinates of the points of the dynamical divisor $D$.

**Proof.** Let us compute the residues at the poles $\lambda_k$ of $K_1$, where only $L_k$ contributes. Since locally we have $L_0 = g_k A_k g_k^{-1}$ around $\lambda = \lambda_k$, we may identify the matrix $\hat{\Psi}(\lambda)$ with $g_k$. More precisely we have $\hat{\Psi}(\lambda) = g_k d_k$ and $\hat{\Psi}^{-1}(\lambda) = d_k^{-1} g_k^{-1}$ with $d_k$ a diagonal matrix. The residues are obtained by integrating over small circles surrounding each of the $N$ points $P_{k,i}$ above $\lambda_k$. We can choose these small circles so that they project on the base $\lambda$ on a single small circle surrounding $\lambda_k$. Then we get

$$\sum_{i=1}^N \text{Res}_{P_{k,i}} K_1 = \sum_{i=1}^N \frac{1}{2i\pi} \oint_{C_{k,i}} < \Psi^{(-1)}(P_i) \delta L(\lambda) \wedge \delta \Psi(P_i) > \, d\lambda$$

$$\quad = \frac{1}{2i\pi} \oint_{C_k} \text{Tr}(\hat{\Psi}^{-1}(\lambda) \delta L(\lambda) \wedge \delta \hat{\Psi}(\lambda) \, d\lambda)$$

(12)

where we used the fact that $\hat{\Psi}^{-1}(\lambda)$ is equal to the matrix whose lines are the vectors $\Psi^{(-1)}(P_i)$. The trace has been reconstructed in eq.(12) because $\Psi(P_i), i = 1, \cdots N,$ form a basis of eigenvectors. Using the identification of $\Psi(\lambda)$ in terms of $g_k$ gives:

$$\text{Res}_{\lambda_k} K_1 = \text{Res}_{\lambda_k} \text{Tr}(d_k^{-1} g_k^{-1} (\delta g_k(A_k)_- g_k^{-1} - g_k(A_k)_- g_k^{-1} \delta g_k g_k^{-1}) \wedge \delta g_k d_k + g_k \delta d_k)) \, d\lambda$$

$$\quad = -2 \text{Res}_{\lambda_k} \text{Tr}((A_k)_- g_k^{-1} \delta g_k \wedge g_k^{-1} \delta g_k) \, d\lambda$$

$$\quad + \text{Res}_{\lambda_k} \text{Tr}(g_k^{-1} \delta g_k[(A_k)_-, \delta d_k d_k^{-1}]) \, d\lambda$$

(13)

The last term vanishes because it involves the commutator of two diagonal matrices. To compute the residues of $K_2$ at $\lambda_k$ we remark that $\delta \mu$ is regular on all sheets above $\lambda_k$. 

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This is because due to the form of $L(\lambda)$, we have $\dot{\mu} = (A_k)_- + \text{regular}$. Since $(A_k)_-$ characterizes the coadjoint orbit and is not dynamical, one has to take $\delta(A_k)_- = 0$. It follows immediately that $K_2$ is regular. To compute the residue of $K_3$ we note that if $\partial_\mu R$ has a pole of some order $m$ at $P_{k,i}$ and can be written $\partial_\mu R = c(\lambda)/(\lambda - \lambda_k)^m$ where $c(\lambda)$ is regular and non-vanishing, we get $\delta(\log \partial_\mu R) = \delta \log c(\lambda)$ which is regular, since $\delta \lambda = 0$ and $\delta \lambda_k = 0$. Hence $K_3$ has no residue again because $\delta \mu$ is regular.

This proposition means that the coordinates $(\lambda_i, \mu_i)$ of the point $\nu_i$ of the dynamical divisor are canonical coordinates. This type of result can be obtained in the $r$–matrix approach for specific models like the Toda chain [3].

4 Conclusion

We have shown under quite general but necessary assumptions that the natural symplectic structure on Lax pairs can be expressed in terms of algebro–geometric data. This result shows the nice interplay between the analytical and the group–theoretical approaches to integrable systems. We are able to show that $(\lambda_i, \mu_i)$ are canonical coordinates only using the fact that $L$ parametrizes a coadjoint orbit, specified by constant matrices $(A_k)_-$ and $L_0$.

Acknowledgements. We thank J. Avan, D. Bernard, I. Krichever and F. Smirnov for many interesting discussions on the subject of this paper.

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