A Note on the Infinitesimal Baker-Campbell-Hausdorff Formula

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Abstract

We have studied the infinitesimal Baker-Campbell-Hausdorff formula up to $n = 4$ (Math. Appl. 2 (2013), 61-91). In this note we correct some errors in our calculation for $n = 4$ and presents the calculation for $n = 5$ by using Mathematica.

1 Introduction

We are going to work within synthetic differential geometry, in which a Lie group $G$ is a group and a microlinear space at the same time. For synthetic differential geometry, the reader is referred to \cite{1} and \cite{3}. Its Lie algebra (i.e., its tangent space $T_e G$ of $G$ at the identity $e \in G$), usually denoted by $\mathfrak{g}$, is endowed with a Lie bracket $[,]$ abiding by antisymmetry and the Jacobi identity. Each element $X \in \mathfrak{g}$ is a mapping $d \in D \mapsto X_d \in G$ with $X_0 = e$, where

$$D = \{d \in \mathbb{R} \mid d^2 = 0\}$$

We assume that the so-called exponential mapping $\exp : \mathfrak{g} \to G$ exists. The infinitesimal Baker-Campbell-Hausdorff formula expresses

$$\exp (d_1 + \ldots + d_n) X \cdot \exp (d_1 + \ldots + d_n) Y$$

as

$$\exp (a \text{ Lie polynomial of } X \text{ and } Y)$$

where $X, Y \in \mathfrak{g}$ and $d_1, \ldots, d_n \in D$. In \cite{4} we have calculated the infinitesimal Baker-Campbell-Hausdorff formula up to $n = 4$, but the second author \cite{5} found out some errors in the calculation for $n = 4$. 

This paper is based upon [5]. We correct some errors in the calculation of the infinitesimal Baker-Campbell-Hausdorff formula in case of \( n = 4 \) in our previous paper [4] and we present a calculation of the infinitesimal Baker-Campbell-Hausdorff formula in case of \( n = 5 \) newly. Both calculations were implemented by using Mathematica.

2 Preliminaries

The infinitesimal Baker-Campbell-Hausdorff formula for \( n = 3 \) goes as follows:

**Theorem 1** (cf. Theorem 7.5 and Theorem 8.3 of [4]) Given \( X, Y \in \mathfrak{g} \) and \( d_1, d_2, d_3 \in \mathbb{D} \), we have

\[
\exp (d_1 + d_2 + d_3) X \exp (d_1 + d_2 + d_3) Y = \exp (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \frac{1}{12} (d_1 + d_2 + d_3)^3 [X - Y, [X, Y]]
\]

The tangent space \( T_X \mathfrak{g} \) of \( \mathfrak{g} \) at \( X \in \mathfrak{g} \) is naturally identified with \( \mathfrak{g} \) itself. That is to say, each \( Y \in \mathfrak{g} \) gives rise to \( (d \in \mathbb{D} \mapsto X + dY) \in T_X \mathfrak{g} \), which yields a bijection between \( \mathfrak{g} \) and \( T_X \mathfrak{g} \). Its left logarithmic derivative \( \delta^{\text{left}} (\exp) \) and its right logarithmic derivative \( \delta^{\text{right}} (\exp) \) are characterized by the following formulas:

\[
\exp X + dY = \exp X \cdot (\delta^{\text{left}} (\exp) (X) (Y))_d
\]

and

\[
\exp X + dY = (\delta^{\text{right}} (\exp) (X) (Y))_d \cdot \exp X
\]

for any \( X, Y \in \mathfrak{g} \) and any \( d \in \mathbb{D} \). For logarithmic derivatives, the reader is referred to §5 of [4] and §38.1 of [2]. We have the following well-known formulas.

**Theorem 2** (cf. Theorem 5.3 and Theorem 5.8 of [4]) Given \( X \in \mathfrak{g} \) with \( (\text{ad} X)^{n+1} \) vanishing for some natural number \( n \), we have

\[
\delta^{\text{left}} (\exp) (X) = \sum_{p=0}^{n} \frac{(-1)^p}{(p+1)!} (\text{ad} X)^p
\]

and

\[
\delta^{\text{right}} (\exp) (X) = \sum_{p=0}^{n} \frac{1}{(p+1)!} (\text{ad} X)^p
\]

We note in passing that

**Proposition 3** (cf. Proposition 5.4 of [4]) For any \( X, Y \in \mathfrak{g} \) with \( [X, Y] \) vanishing, we have

\[
\exp X \cdot \exp Y = \exp X + Y
\]
The following simple proposition is very useful.

**Proposition 4** (cf. Proposition 4.9 of [4]) For any \( X \in \mathfrak{g} \) and any \( d \in D \), we have

\[
\exp dX = X_d
\]

### 3 The BCH Formula for n=4

**Theorem 5**

\[
\exp \left( (d_1 + d_2 + d_3 + d_4)X \right) \exp \left( (d_1 + d_2 + d_3 + d_4)Y \right) = \\
\exp \left( (d_1 + d_2 + d_3 + d_4)(X + Y) \right) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] + \\
\frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] + \\
\frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 [[X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y])
\]

**Proof.** We have

\[
\exp \left( (d_1 + d_2 + d_3 + d_4)X \right) \exp \left( (d_1 + d_2 + d_3 + d_4)Y \right) = \\
\exp \left( (d_1 + d_2 + d_3)X + d_4X \right) \exp \left( (d_1 + d_2 + d_3)Y + d_4Y \right) = \\
\exp d_4X \exp \left( (d_1 + d_2 + d_3)X \right) \exp \left( (d_1 + d_2 + d_3)Y \right) \exp d_4Y
\]

)By Proposition 3

= \exp d_4X.

\[
\exp \left( (d_1 + d_2 + d_3)(X + Y) \right) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \frac{1}{2} d_1d_2d_3 [[X, Y], Y - X]
\]

\[
\exp d_4Y
\]

)By Theorem 1

(3)
By the way, due to Theorem 2, we have

\[ \delta_{\text{left}} (\exp) \left( (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \frac{1}{2} d_1 d_2 d_3 [[X, Y], Y - X] \right) (Y) \]

\[ = -\frac{1}{2} d_1 [X, Y] - \frac{1}{2} d_2 [X, Y] + \frac{1}{3} d_1 d_2 [X, [X, Y]] + \frac{1}{3} d_1 d_2 [Y, [X, Y]] \]

\[ - \frac{1}{2} d_1 d_2 [[X, Y], Y] - \frac{1}{2} d_4 [X, Y] + \frac{1}{3} d_1 d_3 [X, [X, Y]] + \frac{1}{3} d_1 d_3 [Y, [X, Y]] \]

\[ - \frac{1}{2} d_2 d_3 [[X, Y], Y] + \frac{1}{4} d_1 d_2 d_3 [X, [X, Y]] - \frac{1}{4} d_1 d_2 d_3 [X, [X, Y]] \]

\[ + \frac{1}{4} d_1 d_2 d_3 [X, [X, Y], Y] + \frac{1}{4} d_1 d_2 d_3 [Y, [X, Y]] \]

\[ - \frac{1}{4} d_1 d_2 d_3 [Y, [X, Y]] + \frac{1}{2} d_1 d_2 d_3 [Y, [X, Y]] \]

\[ + \frac{1}{4} d_1 d_2 d_3 [[[X, Y], X], Y] - \frac{1}{4} d_1 d_2 d_3 [[[X, Y], Y], Y] + Y \] (4)

Letting \( n_{41} \) be the right-hand side of (4) with the last term \( Y \) deleted, we have

(5)

\[ = \exp d_4 X. \]

\[ \exp \left( (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \frac{1}{2} d_1 d_2 d_3 [[X, Y], Y - X] \right). \]

\[ \exp d_4 (n_{41} + Y). \exp -d_4 n_{41} \]

)By Proposition 3

\[ = \exp d_4 X. \]

\[ \exp \left( (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \frac{1}{2} d_1 d_2 d_3 [[X, Y], Y - X] \right). \]

\[ (n_{41} + Y)_{d_4}. \exp -d_4 n_{41} \]

)By Proposition 4

\[ = \exp d_4 X. \exp \left( (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] \right) . \exp -d_4 n_{41} \]

)By (4)
We let \( i_{41} \) be the result of \( n_{41} \) by deleting all the terms whose coefficients contain \( d_1d_2d_3 \). Then, due to Theorem \( \square \), we have

\[
\delta^{\text{left}}(\exp) \left( (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] + d_4Y \right)^{(i_{41})}
\]

\[
= \frac{1}{2}d_1[X, Y] + \frac{1}{2}d_2[X, Y] - \frac{5}{6}d_1d_2[X, [X, Y]] - \frac{5}{6}d_1d_2[Y, [X, Y]]
\]

\[
+ \frac{1}{2}d_1d_2[[X, Y], Y] + \frac{1}{2}d_3[X, Y] - \frac{5}{6}d_1d_3[X, [X, Y]] - \frac{5}{6}d_1d_3[Y, [X, Y]]
\]

\[
+ \frac{1}{2}d_1d_3[[X, Y], Y] - \frac{5}{6}d_2d_3[X, [X, Y]] - \frac{5}{6}d_2d_3[Y, [X, Y]]
\]

\[
+ \frac{1}{2}d_2d_3[[X, Y], Y] + d_1d_2d_3[X, [X, Y]]
\]

\[
+ d_1d_2d_3[X, [Y, [X, Y]]] - \frac{3}{4}d_1d_2d_3[X, [[X, Y], Y]] + d_1d_2d_3[Y, [X, [X, Y]]]
\]

\[
+ d_1d_2d_3[Y, [Y, [X, Y]]] - \frac{3}{4}d_1d_2d_3[Y, [[X, Y], Y]]
\]

(6)

Letting \( n_{42} \) be the result of \( (\cdot)^{(i_{41})} \), we have

\[
= \exp d_4X. \exp \left( (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] + d_4Y \right).
\]

\[
\exp d_4n_{42}. \exp -d_4n_{42} - d_4n_{41}
\]

\( \)By Proposition \( \square \)

\[
= \exp d_4X. \exp \left( (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] + d_4Y \right).
\]

\( (n_{42})_{d_4} \). \exp -d_4n_{42} - d_4n_{41}

\( \)By Proposition \( \square \)

\[
= \exp d_4X. \exp \left( (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X, Y] + \frac{1}{2}d_1d_2d_3[[X, Y], Y - X] + d_4Y + d_4i_{41} \right).
\]

\( \exp -d_4n_{42} - d_4n_{41} \)

\( \)By \( \square \)

(7)

5
We let $i_{42}$ be the result of $-n_{42} - n_{41}$ by deleting all the terms whose coefficients contain $d_1d_2d_3$. Then, thanks to Theorem 2, we have

$$\delta^{\text{left}} \left( \exp \left( (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X,Y] \right) + \frac{1}{2}d_1d_2d_3[[X,Y],Y - X] + d_4Y + d_{4i41} \right)(i_{42})$$

$$= \frac{1}{2}d_1d_2[X,[X,Y]] + \frac{1}{2}d_1d_2[Y,[X,Y]] + \frac{1}{2}d_1d_3[X,[X,Y]]$$

$$+ \frac{1}{2}d_1d_3[Y,[X,Y]] + \frac{1}{2}d_2d_3[X,[X,Y]] + \frac{1}{2}d_2d_3[Y,[X,Y]]$$

$$- \frac{3}{4}d_1d_2d_3[X,[X,Y]] - \frac{3}{4}d_1d_2d_3[X,[Y,[X,Y]]]$$

$$- \frac{3}{4}d_1d_2d_3[Y,[X,[X,Y]]] - \frac{3}{4}d_1d_2d_3[Y,[Y,[X,Y]]]$$

(8)

Letting $n_{43}$ be the right-hand side of (8), we have

$$= \exp d_4 X \exp \left( (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X,Y] \right).$$

$$= \exp d_4 X \exp \left( (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X,Y] \right).$$

(7)

$$= \exp d_4 X \exp \left( (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X,Y] \right).$$

$$= \exp d_4 X \exp \left( (d_1 + d_2 + d_3)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2[X,Y] \right).$$

(9)

Since the coefficient of every term in $-n_{43} - n_{42} - n_{41}$ contains $d_1d_2d_3$, we now turn our attention to the left $\exp d_4 X$. Now, thanks to Theorem 2, we
have
\[
\delta^{\text{right}}(\exp) \left( (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \frac{1}{12} d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y + d_4 i_{41} + d_4 i_{42} \right)(X)
\]
\[
= -\frac{1}{2} d_1 [X, Y] - \frac{1}{2} d_2 [X, Y] - \frac{1}{3} d_1 d_2 [X, [X, Y]] - \frac{1}{3} d_1 d_2 [Y, [X, Y]]
\]
\[
+ \frac{1}{2} d_1 d_2 [[X, Y], X] - \frac{1}{2} d_3 [X, Y] - \frac{1}{3} d_1 d_3 [X, [X, Y]] - \frac{1}{3} d_1 d_3 [Y, [X, Y]]
\]
\[
+ \frac{1}{2} d_1 d_3 [[X, Y], X] - \frac{1}{3} d_2 d_3 [X, [X, Y]] - \frac{1}{3} d_2 d_3 [Y, [X, Y]]
\]
\[
+ \frac{1}{2} d_2 d_3 [[X, Y], X] - \frac{1}{4} d_1 d_2 d_3 [X, [X, Y]] - \frac{1}{4} d_1 d_2 d_3 [Y, [X, Y]]
\]
\[
- \frac{1}{4} d_1 d_2 d_3 [Y, [X, Y]] + \frac{1}{2} d_1 d_2 d_3 [X, [X, Y], X]
\]
\[
- \frac{1}{2} d_1 d_2 d_3 [[X, Y], [X, Y]] - \frac{1}{4} d_1 d_2 d_3 [[[X, Y], X], X]
\]
\[
+ \frac{1}{4} d_1 d_2 d_3 [[[X, Y], Y], X] + X \quad (10)
\]

We let \( m_{41} \) be the right-hand side of (10) with the last term \( X \) deleted.

Then we have

\[
\text{(9)}
\]

\[
= \exp -d_4 m_{41} \cdot \exp d_4 X + d_4 m_{41}.
\]

\[
\exp \left( (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \frac{1}{12} d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y + d_4 i_{41} + d_4 i_{42} \right).
\]

\[
\exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41}
\]

)By Proposition 8

\[
= \exp -d_4 m_{41} \cdot (X + m_{41})_{d_4}.
\]

\[
\exp \left( (d_1 + d_2 + d_3) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \frac{1}{12} d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y + d_4 i_{41} + d_4 i_{42} \right).
\]

\[
\exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41}
\]

)By Proposition 4

\[
= \exp -d_4 m_{41}.
\]

\[
\exp \left( (d_1 + d_2 + d_3 + d_4) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] + \frac{1}{12} d_1 d_2 d_3 [[X, Y], Y - X] + d_4 Y + d_4 i_{41} + d_4 i_{42} \right).
\]

\[
\exp -d_4 n_{43} - d_4 n_{42} - d_4 n_{41}
\]

)By (10) (11)
We let $j_{41}$ be the result of $-m_{41}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3$. Then, thanks to Theorem 2 we have

$$
\delta^{\text{right \ (exp)}} \left( (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \frac{1}{2} d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 i_{42} \right) (j_{41})
$$

$$= \frac{1}{2} d_1 [X, Y] + \frac{1}{2} d_2 [X, Y] + \frac{5}{6} d_1 d_2 [X, [X, Y]] + \frac{5}{6} d_1 d_2 [Y, [X, Y]]$$

$$- \frac{1}{2} d_1 d_2 [[X, Y], X] + \frac{1}{2} d_3 [X, Y] + \frac{5}{6} d_1 d_3 [X, [X, Y]]$$

$$+ \frac{5}{6} d_1 d_3 [Y, [X, Y]] - \frac{1}{2} d_1 d_3 [[X, Y], X] + \frac{5}{6} d_2 d_3 [X, [X, Y]]$$

$$+ \frac{5}{6} d_2 d_3 [Y, [X, Y]] - \frac{1}{2} d_2 d_3 [[X, Y], X] + d_1 d_2 d_3 [X, [X, Y]]$$

$$+ d_1 d_2 d_3 [X, [Y, [X, Y]]] - \frac{3}{4} d_1 d_2 d_3 [X, [[X, Y], X]]$$

$$+ d_1 d_2 d_3 [Y, [X, [X, Y]]] + d_3 d_2 d_3 [Y, [Y, [X, Y]]]$$

$$- \frac{3}{4} d_1 d_2 d_3 [Y, [[X, Y], X]] + \frac{3}{4} d_1 d_2 d_3 [[X, Y], [X, Y]] \tag{12}$$

Letting $m_{42}$ be the right-hand side of (12), we have

$$\exp -d_4 m_{41} - d_4 m_{42} \cdot \exp \ d_4 m_{42}$$

By Proposition 2

$$\exp \left( (d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2} (d_1 + d_2 + d_3)^2 [X, Y] + \frac{1}{2} d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 i_{42} \right) \tag{13}$$
We let \( j_{42} \) be the result of \(-m_{41} - m_{42} \) by deleting all the terms whose coefficients contain \( d_1 d_2 d_3 \). Then, thanks to Theorem 2, we have

\[
\delta^{\text{right}}(\exp) \left( \frac{(d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y]}{+ \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 j_{41} + d_4 i_{42}} \right) (j_{42})
\]

\[
= -\frac{1}{2}d_1 d_2 [X, [X, Y]] - \frac{1}{2}d_1 d_2 [Y, [X, Y]] - \frac{1}{2}d_1 d_3 [X, [X, Y]]
\]

\[
- \frac{1}{2}d_1 d_3 [Y, [X, Y]] - \frac{1}{2}d_2 d_3 [X, [X, Y]] - \frac{1}{2}d_2 d_3 [Y, [X, Y]]
\]

\[
- \frac{3}{4}d_1 d_2 d_3 [X, [X, X]] - \frac{3}{4}d_1 d_2 d_3 [Y, [X, X]]
\]

\[
- \frac{3}{4}d_1 d_2 d_3 [Y, [X, X]] - \frac{3}{4}d_1 d_2 d_3 [Y, [Y, X]]
\]

(14)

Letting \( m_{43} \) be the right-hand side of (14), we have

\[
= \exp -d_4 m_{41} - d_4 m_{42} - d_4 m_{43} \cdot \exp d_4 m_{43}
\]

\[\exp \left( \frac{(d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y]}{+ \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 j_{41} + d_4 i_{42}} \right).\]

exp \(-d_4 m_{43} - d_4 n_{42} - d_4 n_{41}\)

\end{equation}

By Proposition 3

\[
= \exp -d_4 m_{41} - d_4 m_{42} - d_4 m_{43} \cdot (m_{43})_{d_4}
\]

\[
\exp \left( \frac{(d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y]}{+ \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 j_{41} + d_4 i_{42}} \right).\]

exp \(-d_4 m_{43} - d_4 n_{42} - d_4 n_{41}\)

\end{equation}

By Proposition 4

\[
= \exp -d_4 m_{41} - d_4 m_{42} - d_4 m_{43}\]

\[
\exp \left( \frac{(d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y]}{+ \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 j_{41} + d_4 i_{42} + d_4 j_{42}} \right).\]

exp \(-d_4 m_{43} - d_4 n_{42} - d_4 n_{41}\)

\end{equation}

By (14)

\[
(15)
\]

Since the coefficient of every term in \(-m_{41} - m_{42} - m_{43}\) contains \(d_1 d_2 d_3\), we are done, so that we have

\[
(15)
\]

\[\exp \left( \frac{(d_1 + d_2 + d_3 + d_4)(X + Y) + \frac{1}{2}(d_1 + d_2 + d_3)^2 [X, Y]}{+ \frac{1}{2}d_1 d_2 d_3 [[X, Y], Y - X] + d_4 i_{41} + d_4 j_{41} + d_4 i_{42} + d_4 j_{42}} \right).\]

\[
-d_4 m_{41} - d_4 m_{42} - d_4 m_{43} - d_4 n_{43} - d_4 n_{42} - d_4 n_{41}
\]

(16)
It is easy to see that
\[ j_{41} + i_{42} + j_{42} \]
\[ = d_1 [X, Y] + d_2 [X, Y] - \frac{1}{2} d_1 d_2 [[X, Y], X] + \frac{1}{2} d_1 d_2 [[X, Y], Y] \]
\[ - \frac{1}{2} d_2 d_3 [[X, Y], X] + \frac{1}{2} d_2 d_3 [[X, Y], Y] \]
whereas
\[ - m_{41} - m_{42} - m_{43} - n_{43} - n_{42} - n_{41} \]
\[ = \frac{1}{4} d_1 d_2 d_3 [X, [[X, Y], X]] + \frac{1}{4} d_1 d_2 d_3 [X, [[X, Y], Y]] \]
\[ + \frac{1}{4} d_1 d_2 d_3 [Y, [[X, Y], X]] + \frac{1}{4} d_1 d_2 d_3 [Y, [[X, Y], Y]] \]
\[ + \frac{1}{4} d_1 d_2 d_3 [[[X, Y], X], X] - \frac{1}{4} d_1 d_2 d_3 [[[X, Y], X], Y] \]
\[ - \frac{1}{4} d_1 d_2 d_3 [[[X, Y], Y], X] + \frac{1}{4} d_1 d_2 d_3 [[[X, Y], Y], Y] \]
Therefore we have the desired result. □

4 The BCH Formula for n=5

Theorem 6
\[ \exp (d_1 + d_2 + d_3 + d_4 + d_5) X \cdot \exp (d_1 + d_2 + d_3 + d_4 + d_5) Y \]
\[ = \exp (d_1 + d_2 + d_3 + d_4 + d_5) (X + Y) \]
\[ + \frac{1}{2} (d_1 + d_2 + d_3 + d_4 + d_5)^2 [X, Y] \]
\[ + \frac{1}{12} (d_1 + d_2 + d_3 + d_4 + d_5)^3 [[X, Y], Y - X] \]
\[ + \frac{1}{96} (d_1 + d_2 + d_3 + d_4 + d_5)^4 ([X + Y, [[X, Y], X + Y]] + [[X, Y], X - Y], X - Y) \]
\[ + \frac{1}{120} (d_1 + d_2 + d_3 + d_4 + d_5)^5 \left( \frac{5}{6} [X + Y, [X + Y, [[X, Y], X - Y]]] + \frac{1}{2} [X, Y], [[X, Y], X + Y] \right) \]
\[ + \frac{1}{8} [[X + Y, [[X, Y], X + Y]], Y - X] + \frac{1}{8} [[[X, Y], Y - X], X - Y] \]
Proof. We have

\[ \exp (d_1 + d_2 + d_3 + d_4 + d_5) X \cdot \exp (d_1 + d_2 + d_3 + d_4 + d_5) Y \]

\[ = \exp (d_1 + d_2 + d_3 + d_4) X + d_5 X \cdot \exp (d_1 + d_2 + d_3 + d_4) Y + d_5 Y \]

\[ = \exp d_5 X \cdot \exp (d_1 + d_2 + d_3 + d_4) X \cdot \exp (d_1 + d_2 + d_3 + d_4) Y \cdot \exp d_5 Y \]

By Proposition \(^3\)

\[ = \exp d_5 X \cdot \exp \left( (d_1 + d_2 + d_3 + d_4) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\
+ \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\
+ \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [X, Y], X + Y] + [[[X, Y], X - Y], X - Y]) \right) \cdot \exp d_5 Y \]

By Theorem \(^5\) \hspace{1cm} (17)
By the way, due to Theorem 2, we have

\[
\delta^{\text{left}} \left( \exp \left( \left( d_1 + d_2 + d_3 + d_4 \right) (X + Y) + \frac{1}{108} (d_1 + d_2 + d_3 + d_4)^3 [X, Y], Y - X \right) \right) \right) (Y)
\]

\[
= Y - \frac{1}{2} d_1 [X, Y] - \frac{1}{2} d_2 [X, Y] + \frac{1}{3} d_1 d_2 [X, [X, Y]] + \frac{1}{3} d_1 d_2 [Y, [X, Y]]
\]

\[- \frac{1}{2} d_1 d_2 [[X, Y], Y] - \frac{1}{2} d_3 [X, Y] + \frac{1}{3} d_1 d_3 [X, [X, Y]] + \frac{1}{3} d_1 d_3 [Y, [X, Y]]
\]

\[- \frac{1}{2} d_1 d_3 [[X, Y], Y] + \frac{1}{3} d_2 d_3 [X, [X, Y]] + \frac{1}{3} d_2 d_3 [Y, [X, Y]] - \frac{1}{2} d_2 d_3 [[X, Y], Y]
\]

\[- \frac{1}{4} d_1 d_2 d_3 [X, [X, Y]] - \frac{1}{4} d_2 d_3 [X, [X, Y]] + \frac{1}{2} d_1 d_2 d_3 [X, [[X, Y], Y]], Y\]

\[- \frac{1}{4} d_1 d_2 d_3 [Y, [X, Y]] + \frac{1}{4} d_1 d_2 d_3 [Y, [X, Y]] + \frac{1}{4} d_2 d_3 [X, [X, Y]] + \frac{1}{4} d_1 d_2 d_3 [Y, [X, Y]]
\]

\[- \frac{1}{2} d_2 d_3 [X, [X, Y]] - \frac{1}{4} d_1 d_2 d_3 [Y, [X, [X, Y]]] - \frac{1}{4} d_1 d_2 d_3 [X, [X, Y]]
\]

\[- \frac{1}{2} d_1 d_2 d_3 [Y, [X, Y]] + \frac{1}{4} d_1 d_2 d_3 [Y, [X, Y]] + \frac{1}{4} d_2 d_3 [X, [X, Y]] + \frac{1}{4} d_1 d_2 d_3 [Y, [X, Y]]
\]

\[- \frac{1}{4} d_1 d_3 d_4 [X, [X, Y]] + \frac{1}{2} d_1 d_3 d_4 [X, [[X, Y], Y]] - \frac{1}{4} d_1 d_3 d_4 [Y, [X, [X, Y]]]
\]

\[- \frac{1}{4} d_1 d_3 d_4 [Y, [X, Y]] + \frac{1}{2} d_1 d_3 d_4 [X, [[X, Y], Y]] - \frac{1}{4} d_1 d_3 d_4 [Y, [X, [X, Y]]]
\]

\[- \frac{1}{4} d_1 d_3 d_4 [[[X, Y], Y], Y] - \frac{1}{4} d_2 d_3 d_4 [X, [X, Y]] - \frac{1}{4} d_2 d_3 d_4 [X, [X, Y]]
\]

\[- \frac{1}{4} d_2 d_3 d_4 [X, [[X, Y], Y]] - \frac{1}{4} d_2 d_3 d_4 [Y, [X, Y]] - \frac{1}{4} d_2 d_3 d_4 [Y, [X, Y]]
\]

\[- \frac{1}{5} d_1 d_2 d_3 d_4 [X, [X, Y]] + \frac{1}{5} d_1 d_2 d_3 d_4 [X, [X, Y]] + \frac{1}{5} d_1 d_2 d_3 d_4 [X, [X, Y]]
\]

\[- \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, Y]] + \frac{1}{5} d_1 d_2 d_3 d_4 [X, [X, Y]]
\]

\[- \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, Y]] - \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, Y]]
\]

\[- \frac{1}{3} d_1 d_2 d_3 d_4 [X, [[X, Y], Y]] + \frac{1}{3} d_1 d_2 d_3 d_4 [X, [[X, Y], Y]]
\]
+ \frac{1}{5}d_1d_2d_3d_4 [Y, [X, [X, Y]]] + \frac{1}{5}d_1d_2d_3d_4 [Y, [X, [Y, [X, Y]]]]
- \frac{1}{2}d_1d_2d_3d_4 [Y, [X, [X, Y], Y]] + \frac{1}{5}d_1d_2d_3d_4 [Y, [X, [X, Y]]]
+ \frac{1}{5}d_1d_2d_3d_4 [Y, [Y, [X, Y]]] - \frac{1}{2}d_1d_2d_3d_4 [Y, [[X, Y], Y]]
- \frac{1}{3}d_1d_2d_3d_4 [Y, [[X, Y], X], Y] + \frac{1}{3}d_1d_2d_3d_4 [Y, [[[X, Y], Y], Y]]
- \frac{1}{3}d_1d_2d_3d_4 [[X, Y], [Y, [X, Y]]] - \frac{1}{3}d_1d_2d_3d_4 [[X, Y], [Y, [X, Y]]]
+ d_1d_2d_3d_4 [[X, Y], [[X, Y], Y]] - \frac{1}{8}d_1d_2d_3d_4 [[X, [[X, Y], X]], Y]
- \frac{1}{8}d_1d_2d_3d_4 [[X, [[X, Y], Y]], Y] - \frac{1}{8}d_1d_2d_3d_4 [[[X, Y], Y], X, Y]
+ \frac{1}{3}d_1d_2d_3d_4 [[[X, Y], Y], [X, Y]] - \frac{1}{3}d_1d_2d_3d_4 [[[X, Y], X], X, Y]
+ \frac{1}{3}d_1d_2d_3d_4 [[[X, Y], X], Y] + \frac{1}{3}d_1d_2d_3d_4 [[[X, Y], Y], X, Y]
- \frac{1}{3}d_1d_2d_3d_4 [[[X, Y], Y], Y](18)

Letting \( n_{51} \) be the right-hand side of (18) with the first term \( Y \) deleted, we have

(17)

\( = \exp d_5X \cdot \exp \left( \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \right) \cdot \exp \left( \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [X, Y], Y - X \right) \cdot \exp \left( \frac{1}{50} (d_1 + d_2 + d_3 + d_4)^4 (X + Y, [[X, Y], X + Y] + [[[X, Y], X - Y], X - Y]) \right) \)

\( \exp d_5X + d_5n_{51}, \exp -d_5n_{51} \)

) By Proposition 3

\( = \exp d_5X \cdot \exp \left( \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \right) \cdot \exp \left( \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [X, Y], Y - X \right) \cdot \exp \left( \frac{1}{50} (d_1 + d_2 + d_3 + d_4)^4 (X + Y, [[X, Y], X + Y] + [[[X, Y], X - Y], X - Y]) \right) \)

\( \exp -d_5n_{51} \)

) By (18)
We let $i_{n1}$ be the result of $-n_{51}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3 d_4$. Then, by dint of Theorem 1 we have

\[
\delta^{\text{left}}(\exp) \left( (d_1 + d_2 + d_3 + d_4) (X + Y) + d_5 Y + \frac{1}{4} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \right.
\]
\[
\left. + \frac{1}{27} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \right.
\]
\[
\left. + \frac{1}{81} (d_1 + d_2 + d_3 + d_4)^4 [X + Y, [[X, Y], X + Y]] \right.
\]
\[
\left. + \frac{1}{243} (d_1 + d_2 + d_3 + d_4)^4 [[[X, Y], X - Y], X - Y] \right) (i_{n1})
\]

\[
= \frac{1}{2} d_1 [X, Y] + \frac{1}{2} d_2 [X, Y] - \frac{5}{6} d_1 d_2 [X, [X, Y]] - \frac{5}{6} d_1 d_3 [Y, [X, Y]] + \frac{1}{2} d_1 d_2 [[X, Y], Y]
\]
\[
+ \frac{1}{2} d_3 [X, Y] - \frac{5}{6} d_1 d_3 [X, [X, Y]] - \frac{5}{6} d_1 d_3 [Y, [X, Y]] + \frac{1}{2} d_1 d_3 [[X, Y], Y]
\]
\[
- \frac{5}{6} d_2 d_3 [X, [X, Y]] - \frac{5}{6} d_2 d_3 [Y, [X, Y]] + \frac{1}{2} d_2 d_3 [[X, Y], Y] + \frac{5}{4} d_1 d_2 d_3 [X, [X, Y]]
\]
\[
+ \frac{5}{4} d_1 d_2 d_3 [Y, [X, Y]] - \frac{5}{4} d_1 d_2 d_3 [X, [[X, Y], Y]] + \frac{5}{4} d_1 d_2 d_3 [Y, [X, Y]]
\]
\[
+ \frac{1}{4} d_1 d_2 d_3 [[X, Y], Y], Y = \frac{1}{4} d_4 [X, Y] - \frac{5}{6} d_1 d_4 [X, [X, Y]] - \frac{5}{6} d_1 d_4 [Y, [X, Y]]
\]
\[
+ \frac{1}{2} d_1 d_4 [X, Y] - \frac{5}{6} d_2 d_4 [X, [X, Y]] - \frac{5}{6} d_2 d_4 [Y, [X, Y]] + \frac{1}{2} d_2 d_4 [[X, Y], Y]
\]
\[
+ \frac{5}{4} d_1 d_2 d_4 [X, [X, Y]] + \frac{5}{4} d_1 d_2 d_4 [Y, [X, Y]] - \frac{5}{4} d_1 d_2 d_4 [X, [[X, Y], Y]]
\]
\[
+ \frac{5}{4} d_1 d_2 d_4 [Y, [X, Y]] + \frac{5}{4} d_1 d_2 d_4 [Y, [X, Y]] - \frac{5}{4} d_1 d_2 d_4 [Y, [[X, Y], Y]]
\]
\[
- \frac{1}{4} d_4 d_2 d_4 [[X, Y], X], Y = \frac{1}{4} d_4 d_2 d_4 [[X, Y], Y], Y - \frac{5}{6} d_3 d_4 [X, [X, Y]]
\]
\[
- \frac{5}{6} d_3 d_4 [Y, [X, Y]] + \frac{1}{2} d_3 d_4 [[X, Y], X] + \frac{5}{4} d_1 d_3 d_4 [X, [X, Y]]
\]
\[
+ \frac{5}{4} d_1 d_3 d_4 [X, [X, Y]] - \frac{5}{4} d_1 d_3 d_4 [X, [[X, Y], Y]] + \frac{5}{4} d_1 d_3 d_4 [Y, [X, Y]]
\]
\[
+ \frac{5}{4} d_1 d_3 d_4 [Y, [X, Y]] - \frac{5}{4} d_1 d_3 d_4 [Y, [[X, Y], Y]] - \frac{1}{4} d_3 d_4 [[X, Y], X], Y
\]
\[
+ \frac{1}{4} d_3 d_4 [[X, Y], Y] + \frac{5}{4} d_2 d_3 d_4 [X, [X, Y]] + \frac{5}{4} d_2 d_3 d_4 [Y, [X, Y]]
\]
\[
- \frac{5}{4} d_2 d_3 d_4 [X, [[X, Y], Y]] + \frac{5}{4} d_2 d_3 d_4 [Y, [X, Y]] + \frac{5}{4} d_2 d_3 d_4 [Y, [X, Y]]
\]
\[
- \frac{5}{4} d_2 d_3 d_4 [Y, [[X, Y], Y]] - \frac{1}{4} d_2 d_3 d_4 [[[X, Y], X], Y] + \frac{1}{4} d_2 d_3 d_4 [[[X, Y], Y], Y]
\]
\[
- \frac{5}{3} d_1 d_2 d_3 d_4 [X, [X, [X, Y]]] - \frac{5}{3} d_1 d_2 d_3 d_4 [X, [X, [X, Y]]]
\]
\[
+ 2 d_1 d_2 d_3 d_4 [X, [X, [X, Y]]] - \frac{5}{3} d_1 d_2 d_3 d_4 [X, [X, [X, Y]]]
\]
\[
- \frac{5}{3} d_1 d_2 d_3 d_4 [X, [X, [Y, Y]]] + 2 d_1 d_2 d_3 d_4 [X, [Y, [X, Y]]]
\]
\[
+ \frac{1}{2} d_1 d_2 d_3 d_4 [X, [[[X, Y], X], Y]] - \frac{1}{2} d_1 d_2 d_3 d_4 [X, [[[X, Y], Y], Y]]
\]

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\[ -\frac{5}{3}d_1d_2d_3d_4 [Y, [X, [X, [X, Y]]]] - \frac{5}{3}d_1d_2d_3d_4 [Y, [X, [Y, [X, Y]]]] \\
+ 2d_1d_2d_3d_4 [Y, [[X, Y], [X, Y]]] - \frac{5}{3}d_1d_2d_3d_4 [Y, [Y, [X, [X, Y]]]] \\
- \frac{5}{3}d_1d_2d_3d_4 [Y, [Y, [X, [X, Y]]]] + 2d_1d_2d_3d_4 [Y, [Y, [[X, Y], [Y, X]]]]
\]

By Proposition (3)

\[ \frac{1}{2}d_1d_2d_3d_4 [Y, [[X, Y], [X, Y]]] - \frac{1}{2}d_1d_2d_3d_4 [Y, [[[X, Y], Y], Y]] \\
+ 2d_1d_2d_3d_4 [[X, Y], [X, [X, Y]]] + 2d_1d_2d_3d_4 [[X, Y], [Y, [X, Y]]]
\]

By Proposition (20)

\[ -\frac{3}{2}d_1d_2d_3d_4 [[X, Y], [[X, Y], [X, Y]]] + \frac{1}{2}d_1d_2d_3d_4 [[[X, Y], X], [X, Y]] \\
- \frac{1}{2}d_1d_2d_3d_4 [[[X, Y], [Y], [X, Y]]]
\]

(20)

Letting \( n_{52} \) be the right-hand side of (20), we have

[19]

\[ = \exp\ d_5 X.\]

\[ = \exp\ d_5 X.\]

\[ = \exp\ d_5 X.\]

\[ = \exp\ d_5 X.\]

\[ = \exp\ d_5 X.\]

(21)
We let \( i_{52} \) be the result of \(-n_{51} - n_{52}\) by deleting all the terms whose coefficients contain \(d_1d_2d_3d_4\). Then, thanks to Theorem 2, we have

\[
\delta_{\text{left}}^{\text{exp}} \left( \frac{(d_1 + d_2 + d_3 + d_4) (X + Y) + d_5 Y + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y]}{2} \right) (i_{52})
\]

\[
= \frac{1}{2} d_1d_2 [X, [X, Y]] + \frac{1}{2} d_1d_2 [Y, [X, Y]] + \frac{1}{2} d_1d_3 [X, [X, Y]] + \frac{1}{2} d_1d_3 [Y, [X, Y]]
\]

\[
+ \frac{1}{2} d_2d_3 [X, [X, Y]] + \frac{1}{2} d_2d_3 [Y, [X, Y]] - \frac{7}{4} d_1d_2d_3 [X, [X, Y]]
\]

\[
- \frac{7}{4} d_1d_2d_3 [X, [Y, [X, Y]]] + \frac{3}{4} d_1d_2d_3 [X, [[X, Y], Y]] - \frac{7}{4} d_1d_2d_3 [Y, [X, [X, Y]]]
\]

\[
- \frac{7}{4} d_1d_2d_3 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1d_2d_3 [Y, [[X, Y], Y]] + \frac{1}{2} d_1d_4 [X, [X, Y]]
\]

\[
+ \frac{1}{2} d_1d_4 [Y, [X, Y]] + \frac{1}{2} d_2d_4 [X, [X, Y]] + \frac{1}{2} d_2d_4 [Y, [X, Y]] - \frac{7}{4} d_1d_2d_4 [X, [X, [X, Y]]]
\]

\[
- \frac{7}{4} d_1d_2d_4 [X, [Y, [X, Y]]] + \frac{3}{4} d_1d_2d_4 [X, [[X, Y], Y]] - \frac{7}{4} d_1d_2d_4 [Y, [X, [X, Y]]]
\]

\[
- \frac{7}{4} d_1d_2d_4 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1d_2d_4 [Y, [[X, Y], Y]] + \frac{1}{2} d_3d_4 [X, [X, Y]]
\]

\[
+ \frac{1}{2} d_3d_4 [Y, [X, Y]] - \frac{7}{4} d_1d_3d_4 [X, [[X, X, Y]]] - \frac{7}{4} d_1d_3d_4 [X, [X, [X, Y]]]
\]

\[
+ \frac{3}{4} d_1d_3d_4 [X, [[X, Y], Y]] - \frac{7}{4} d_1d_3d_4 [Y, [X, [X, Y]]] - \frac{7}{4} d_1d_3d_4 [Y, [Y, [X, Y]]]
\]

\[
+ \frac{3}{4} d_1d_3d_4 [Y, [X, [X, Y]]] + \frac{7}{4} d_2d_3d_4 [X, [X, [X, Y]]] - \frac{7}{4} d_2d_3d_4 [Y, [X, [X, Y]]]
\]

\[
+ \frac{3}{4} d_2d_3d_4 [X, [[X, Y], Y]] - \frac{7}{4} d_2d_3d_4 [Y, [X, [X, Y]]] - \frac{7}{4} d_2d_3d_4 [Y, [Y, [X, Y]]]
\]

\[
+ \frac{3}{4} d_2d_3d_4 [Y, [X, [X, Y]]] + 3d_1d_2d_3d_4 [X, [X, [X, Y]]]
\]

\[
+ 3d_1d_2d_3d_4 [X, [X, [Y, [X, Y]]]] + \frac{3}{2} d_1d_2d_3d_4 [X, [X, [[X, Y], Y]]]
\]

\[
+ 3d_1d_2d_3d_4 [X, [Y, [X, [X, Y]]]] + 3d_1d_2d_3d_4 [X, [Y, [X, Y]]]
\]

\[
+ 3d_1d_2d_3d_4 [X, [Y, [X, [X, Y]]]] + 3d_1d_2d_3d_4 [Y, [X, [X, Y]]]
\]

\[
+ 3d_1d_2d_3d_4 [Y, [X, [X, [Y, Y]]]] - \frac{1}{2} d_1d_2d_3d_4 [Y, [X, [[X, Y], Y]]]
\]

\[
+ 3d_1d_2d_3d_4 [Y, [Y, [X, [X, Y]]]] + 3d_1d_2d_3d_4 [Y, [Y, [X, Y]]]
\]

\[
+ \frac{1}{2} d_1d_2d_3d_4 [Y, [Y, [[X, Y], Y]]] - \frac{3}{2} d_1d_2d_3d_4 [[X, Y], [X, [X, Y]]]
\]

\[
- \frac{3}{2} d_1d_2d_3d_4 [[X, Y], [Y, [X, Y]]]
\]
Letting $n_{53}$ be the right-hand side of (22), we have

\[ (21) = \exp d_5 X. \]

\[
\exp \left( (d_1 + d_2 + d_3 + d_4) (X + Y) + d_5 Y + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \right.
\]

\[
+ \frac{1}{72} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X]
\]

\[
+ \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 \left( [X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y] \right)
\]

\[ + d_5 i_{51} \]

\[ (n_{53})_{d_5}, \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} \]

By Proposition 3

\[ = \exp d_5 X. \]

\[ \exp \left( (d_1 + d_2 + d_3 + d_4) (X + Y) + d_5 Y + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \right. \]

\[
+ \frac{1}{72} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X]
\]

\[
+ \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 \left( [X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y] \right)
\]

\[ + d_5 i_{51} + d_5 i_{52} \]

\[ (n_{53})_{d_5}, \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} \]

By Proposition 4

\[ = \exp d_5 X. \]

\[ \exp \left( (d_1 + d_2 + d_3 + d_4) (X + Y) + d_5 Y + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \right. \]

\[
+ \frac{1}{72} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X]
\]

\[
+ \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 \left( [X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y] \right)
\]

\[ + d_5 i_{51} + d_5 i_{52} \]

\[ \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} \]

By (22)
We let $i_{53}$ be the result of $-n_{51} - n_{52} - n_{53}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3 d_4$. Then, due to Theorem 24 we have

$$\delta^{\text{left (exp)}} \left( \begin{array}{c} (d_1 + d_2 + d_3 + d_4) (X + Y) + d_5 Y + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X,Y] \\ + \frac{1}{4} (d_1 + d_2 + d_3 + d_4)^3 [[X,Y],Y - X] \\ + \frac{1}{24} (d_1 + d_2 + d_3 + d_4)^4 [X + Y, [[X,Y], X + Y]] \\ + \frac{1}{72} (d_1 + d_2 + d_3 + d_4)^4 [[[X,Y], X - Y], X - Y] \\ + d_5 i_{51} + d_5 i_{52} \end{array} \right) (i_{53})$$

$$= \frac{3}{4} d_1 d_2 d_3 [X, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [X, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [Y, [X, [X, Y]]]$$
$$+ \frac{3}{4} d_1 d_2 d_3 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [X, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [X, [X, [X, Y]]]$$
$$+ \frac{3}{4} d_1 d_2 d_4 [Y, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [Y, [X, [X, Y]]] + \frac{3}{4} d_1 d_3 d_4 [Y, [X, [X, Y]]]$$
$$+ \frac{3}{4} d_1 d_3 d_4 [X, [X, [X, Y]]] + \frac{3}{4} d_1 d_3 d_4 [X, [X, [X, Y]]] + \frac{3}{4} d_2 d_3 d_4 [Y, [X, [X, Y]]]$$
$$+ \frac{3}{4} d_2 d_3 d_4 [Y, [X, [X, Y]]] + \frac{3}{4} d_2 d_3 d_4 [Y, [X, [X, Y]]] + \frac{3}{4} d_2 d_3 d_4 [Y, [X, [X, Y]]]$$

$$- \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [X, Y]]] - \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [X, Y]]]$$
$$- \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [X, Y]]] - \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [X, [X, Y]]]$$
$$- \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [X, [X, Y]]] - \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [X, [X, Y]]]$$
$$- \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [Y, [X, Y]]]$$

(24)
Letting $n_{54}$ be the right-hand side of (24), we have

\[
\begin{align*}
\text{(23)} & = \exp d_5 X. \\
& = \exp d_5 X. \\
& = \exp \left( d_1 + d_2 + d_3 + d_4 \right) (X + Y) + d_5 Y + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\
& \quad + \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\
& \quad + \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [X, Y], X + Y] + [[[X, Y], X - Y], X - Y]) \\
& \quad + d_5 i_{51} + d_5 i_{52} \\
\end{align*}
\]

\[
\exp d_5 n_{54} \cdot \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54}
\]

By Proposition 3

\[
\begin{align*}
\text{(24)} & = \exp d_5 X. \\
& = \exp d_5 X. \\
& = \exp \left( d_1 + d_2 + d_3 + d_4 \right) (X + Y) + d_5 Y + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\
& \quad + \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\
& \quad + \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 ([X + Y, [X, Y], X + Y] + [[[X, Y], X - Y], X - Y]) \\
& \quad + d_5 i_{51} + d_5 i_{52} \\
\end{align*}
\]

\[
\exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54}
\]

)By (24)
Since the coefficient of every term in $-n_{51}-n_{52}-n_{53}-n_{54}$ contains $d_1 d_2 d_3 d_4$, we turn our attention to the left exp $d_5 X$. Now, thanks to Theorem 2, we have

$$\delta^{\text{right}} \left( \exp \left( \begin{array}{c} (d_1 + d_2 + d_3 + d_4) (X + Y) + d_5 Y + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\ + \frac{1}{3} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\ + \frac{1}{3} (d_1 + d_2 + d_3 + d_4)^4 [X + Y, [[X, Y], X + Y]] \\ + \frac{1}{3} (d_1 + d_2 + d_3 + d_4)^4 [[[X, Y], X - Y], X - Y] \\ + d_{5i1} + d_{5i32} + d_{5i33} \\ \end{array} \right) \right)(X)$$

$$= X - \frac{1}{2} d_1 [X, Y] - \frac{1}{2} d_2 [X, Y] - \frac{1}{3} d_1 d_2 [X, [X, Y]] - \frac{1}{3} d_1 d_2 [Y, [X, Y]]$$
$$+ \frac{1}{2} d_1 d_2 [[X, Y], X] - \frac{1}{3} d_3 [X, Y] - \frac{1}{3} d_1 d_3 [X, [X, Y]] - \frac{1}{3} d_1 d_3 [Y, [X, Y]]$$
$$+ \frac{1}{2} d_1 d_3 [[X, Y], X] - \frac{1}{3} d_2 d_3 [X, [X, Y]] - \frac{1}{3} d_2 d_3 [Y, [X, Y]] + \frac{1}{2} d_1 d_3 [X, [[X, Y], X]]$$
$$- \frac{1}{4} d_1 d_2 d_3 [X, [X, [X, Y]]] - \frac{1}{4} d_1 d_2 d_3 [X, [Y, [X, Y]]] + \frac{1}{2} d_1 d_2 d_3 [X, [[X, Y], X]]$$
$$- \frac{1}{4} d_1 d_2 d_3 [Y, [X, [X, Y]]] - \frac{1}{4} d_1 d_2 d_3 [Y, [Y, [X, Y]]] + \frac{1}{2} d_1 d_2 d_3 [Y, [[X, Y], X]]$$
$$- \frac{1}{4} d_1 d_2 d_3 [[[X, Y], X], X] + \frac{1}{4} d_1 d_3 d_4 [[[X, Y], Y], X] - \frac{1}{2} d_3 [X, Y] - \frac{1}{3} d_3 d_4 [X, [X, Y]]$$
$$- \frac{1}{3} d_1 d_3 [X, [X, Y]] - \frac{1}{3} d_3 d_4 [X, [X, Y]] + \frac{1}{2} d_3 d_4 [[X, Y], X] - \frac{1}{4} d_1 d_3 d_4 [X, [X, [X, Y]]]$$
$$- \frac{1}{4} d_1 d_3 [Y, [X, [X, Y]]] + \frac{1}{2} d_1 d_3 d_4 [X, [[X, Y], X]] - \frac{1}{4} d_1 d_3 d_4 [Y, [X, [X, Y]]]$$
$$+ \frac{1}{4} d_1 d_3 d_4 [[[X, Y], Y], X] - \frac{1}{4} d_2 d_3 d_4 [X, [X, [X, Y]]] - \frac{1}{4} d_2 d_3 d_4 [X, [Y, [X, Y]]]$$
$$+ \frac{1}{2} d_2 d_3 d_4 [X, [X, Y]] - \frac{1}{4} d_2 d_3 d_4 [[[X, Y], X], X] + \frac{1}{4} d_2 d_3 d_4 [[X, Y], Y], X$$
$$- \frac{1}{5} d_1 d_2 d_3 d_4 [X, [X, [X, Y]]] - \frac{1}{5} d_1 d_2 d_3 d_4 [X, [Y, [X, Y]]]$$
$$+ \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [X, Y]]] - \frac{1}{5} d_1 d_2 d_3 d_4 [X, [Y, [X, Y]]]$$
$$- \frac{1}{5} d_1 d_2 d_3 d_4 [X, [X, [X, Y]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [X, Y]]]$$
\begin{align}
- \frac{1}{3} & d_1 d_2 d_3 d_4 \left[ X, [[[X, Y], X], X] \right] + \frac{1}{3} d_1 d_2 d_3 d_4 \left[ X, [[[X, Y], Y], X] \right] \\
- \frac{1}{5} & d_1 d_2 d_3 d_4 \left[ Y, [X, [X, [X, Y]]] \right] - \frac{1}{5} d_1 d_2 d_3 d_4 \left[ Y, [X, [X, Y]] \right] \\
+ \frac{1}{2} & d_1 d_2 d_3 d_4 \left[ Y, [X, [[X, Y], X]] \right] - \frac{1}{5} d_1 d_2 d_3 d_4 \left[ Y, [X, [X, Y]] \right] \\
- \frac{1}{2} & d_1 d_2 d_3 d_4 \left[ Y, [Y, [X, [X, Y]]] \right] + \frac{1}{2} d_1 d_2 d_3 d_4 \left[ Y, [X, [Y, X]] \right] \\
- \frac{1}{3} & d_1 d_2 d_3 d_4 \left[ Y, [[[X, Y], X], X]] \right] + \frac{1}{3} d_1 d_2 d_3 d_4 \left[ Y, [[[X, Y], Y], X]] \right] \\
- \frac{1}{2} & d_1 d_2 d_3 d_4 \left[ [X, Y], [X, [X, Y]] \right] - \frac{1}{2} d_1 d_2 d_3 d_4 \left[ [X, Y], [Y, [X, Y]] \right] \\
+ \frac{1}{8} & d_1 d_2 d_3 d_4 \left[ [X, Y], [[X, Y], X]] \right] + \frac{1}{8} d_1 d_2 d_3 d_4 \left[ [X, Y], [X, X]] \right] \\
+ \frac{1}{8} & d_1 d_2 d_3 d_4 \left[ [Y, [X, [X, Y]]], X \right] + \frac{1}{8} d_1 d_2 d_3 d_4 \left[ [Y, [X, Y]], X \right] \\
+ \frac{1}{3} & d_1 d_2 d_3 d_4 \left[ [Y, [X, Y]], X \right] + \frac{1}{3} d_1 d_2 d_3 d_4 \left[ [Y, [X, Y]], [X, Y] \right] \\
- \frac{1}{3} & d_1 d_2 d_3 d_4 \left[ [[X, Y], Y]], [X, X] \right] + \frac{1}{8} d_1 d_2 d_3 d_4 \left[ [[[X, Y], X], X]], X \right] \\
- \frac{1}{8} & d_1 d_2 d_3 d_4 \left[ [[[X, Y], X], Y]], X \right] + \frac{1}{8} d_1 d_2 d_3 d_4 \left[ [[[X, Y], Y]], X \right] \\
+ \frac{1}{8} & d_1 d_2 d_3 d_4 \left[ [[[X, Y], Y]], X \right] 
\end{align}
Letting \( m_{51} \) be the right-hand side of (26) with the first term \( X \) deleted, we have

\[
(25) = \exp \left( -d_5 m_{51} \right).
\]

\[
\exp \left( (d_1 + d_2 + d_3 + d_4) (X + Y) + d_5 Y + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] + \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [X, Y], Y - X \right.
\]

\[
+ \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 \left( ([X + Y, [X, Y], X + Y] + [[[X, Y], X - Y], X - Y]) + d_{51} + d_{52} + d_{53}\right)
\]

\[
\exp \left( -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \right)\]

\[
\text{By Proposition 3,}
\]

\[
= \exp \left( -d_5 m_{51} \right) d_5.
\]

\[
\exp \left( (d_1 + d_2 + d_3 + d_4) (X + Y) + d_5 Y + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] + \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [X, Y], Y - X \right.
\]

\[
+ \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 \left( ([X + Y, [X, Y], X + Y] + [[[X, Y], X - Y], X - Y]) + d_{51} + d_{52} + d_{53}\right)
\]

\[
\exp \left( -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \right)\]

\[
\text{By Proposition 4,}
\]

\[
= \exp \left( -d_5 m_{51} \right).
\]

\[
\exp \left( (d_1 + d_2 + d_3 + d_4 + d_5) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4 + d_5)^2 [X, Y] + \frac{1}{12} (d_1 + d_2 + d_3 + d_4 + d_5)^3 [X, Y], Y - X \right.
\]

\[
+ \frac{1}{96} (d_1 + d_2 + d_3 + d_4 + d_5)^4 \left( ([X + Y, [X, Y], X + Y] + [[[X, Y], X - Y], X - Y]) + d_{51} + d_{52} + d_{53}\right)
\]

\[
\exp \left( -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \right)\]

By (26).
We let $j_{51}$ be the result of $-m_{51}$ by deleting all the terms whose coefficients contain $d_1d_2d_3d_4$. Then, due to Theorem 2 we have

$$
\delta^{\text{right}} (\exp \left( \left( d_1 + d_2 + d_3 + d_4 + d_5 \right) (X + Y) + \frac{1}{2} \left( d_1 + d_2 + d_3 + d_4 \right)^2 [X, Y] - \frac{1}{2} \left( d_1 + d_2 + d_3 + d_4 \right) Y - X + \frac{1}{2} \left( d_1 + d_2 + d_3 + d_4 \right)^2 [X + Y, [X, Y], X + Y] + \frac{1}{2} \left( d_1 + d_2 + d_3 + d_4 \right)^4 [X, Y, X - Y, X - Y] + d_5 s_{51} + d_5 s_{52} + d_5 s_{53} \right) \right) (j_{51})
$$

$$
= \frac{1}{2} d_1 [X, Y] + \frac{1}{2} d_2 [X, Y] + \frac{5}{6} d_1 d_2 [X, [X, Y]] + \frac{5}{6} d_1 d_2 [Y, [X, Y]] - \frac{1}{2} d_1 d_2 [[X, Y], X] + \frac{1}{2} d_3 [X, Y] + \frac{5}{6} d_1 d_3 [X, [X, Y]] + \frac{5}{6} d_1 d_3 [Y, [X, Y]] - \frac{1}{2} d_1 d_3 [[X, Y], X] + \frac{5}{4} d_1 d_2 d_3 [X, [X, Y], X] + \frac{5}{4} d_1 d_2 d_3 [Y, [X, Y], X] + \frac{1}{4} d_1 d_2 d_3 [[[X, Y], Y], X] + \frac{1}{2} d_4 [X, Y] + \frac{5}{6} d_1 d_4 [X, [X, Y]] + \frac{5}{6} d_1 d_4 [Y, [X, Y]] - \frac{1}{2} d_1 d_4 [[X, Y], X] + \frac{5}{4} d_1 d_2 d_4 [X, [X, Y]] + \frac{5}{4} d_1 d_2 d_4 [Y, [X, Y]] - \frac{5}{4} d_1 d_2 d_4 [[X, Y], Y, X] + \frac{5}{6} d_1 d_2 d_4 [[[X, Y], Y], X] - \frac{1}{4} d_1 d_2 d_4 [[[X, Y], Y], X] - \frac{1}{4} d_1 d_2 d_4 [[[X, Y], Y], X] + \frac{5}{6} d_1 d_2 d_4 [[[X, Y], Y], X] + \frac{5}{6} d_1 d_2 d_4 [[[X, Y], Y], X] - \frac{1}{2} d_3 d_4 [[X, Y], X] + \frac{5}{4} d_1 d_3 d_4 [X, [X, Y]] + \frac{5}{4} d_1 d_3 d_4 [X, [X, Y]] - \frac{5}{4} d_1 d_3 d_4 [[X, Y], X] + \frac{1}{4} d_1 d_3 d_4 [[[X, Y], Y], X] + \frac{5}{4} d_1 d_3 d_4 [X, [X, Y]] + \frac{5}{4} d_1 d_3 d_4 [X, [X, Y]] - \frac{5}{4} d_1 d_3 d_4 [[[X, Y], Y], X] + \frac{5}{4} d_1 d_3 d_4 [[[X, Y], Y], X] - \frac{1}{4} d_2 d_3 d_4 [[X, Y], X] + \frac{5}{4} d_2 d_3 d_4 [[[X, Y], Y], X] - \frac{1}{4} d_2 d_3 d_4 [[[X, Y], Y], X] + \frac{5}{3} d_1 d_2 d_3 d_4 [X, [X, Y]] + \frac{5}{3} d_1 d_2 d_3 d_4 [X, [X, Y]] - \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, Y]] + \frac{5}{3} d_1 d_2 d_3 d_4 [X, [X, Y]] - \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, Y]] + \frac{5}{3} d_1 d_2 d_3 d_4 [X, [X, Y]] - 2 d_1 d_2 d_3 d_4 [X, [X, Y]] + \frac{5}{3} d_1 d_2 d_3 d_4 [X, [X, Y]] + \frac{5}{3} d_1 d_2 d_3 d_4 [X, [X, Y]] - 2 d_1 d_2 d_3 d_4 [X, [X, Y]]$$
\[+ \frac{1}{2} d_1 d_2 d_3 d_4 [X, [[X, Y], X], X] - \frac{1}{2} d_1 d_2 d_3 d_4 [X, [[X, Y], Y], X] + \frac{5}{3} d_1 d_2 d_3 d_4 [Y, [X, [X, Y]]] + \frac{5}{3} d_1 d_2 d_3 d_4 [Y, [X, [X, Y]]] - 2 d_1 d_2 d_3 d_4 [Y, [X, [X, Y]], X] + \frac{5}{3} d_1 d_2 d_3 d_4 [Y, [Y, [X, Y]], X] + \frac{5}{3} d_1 d_2 d_3 d_4 [Y, [Y, [X, Y]]] - 2 d_1 d_2 d_3 d_4 [Y, [Y, [X, Y]], X]
\]

Letting \( m_{52} \) be the right-hand side of (28), we have

\[(27)\]

\[= \exp -d_5 m_{51} - d_5 m_{52} \exp d_5 m_{52} \exp \left( -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \right) \]

\,(28)\)

By Proposition 3

\[= \exp -d_5 m_{51} - d_5 m_{52} \cdot (m_{52})_{d_5} \cdot (m_{53})_{d_5} \cdot (m_{54})_{d_5} \cdot \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \]

By Proposition 3

\[= \exp -d_5 m_{51} - d_5 m_{52} \cdot \exp -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \]

By (28)
We let \( j_{52} \) be the result of \(-m_{51} - m_{52}\) by deleting all the terms whose coefficients contain \(d_1d_2d_3d_4\). Then, by dint of Theorem 2, we have

\[
\delta^{\text{right}} (\exp \begin{pmatrix}
(d_1 + d_2 + d_3 + d_4 + d_5) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\
+ \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\
+ \frac{1}{120} (d_1 + d_2 + d_3 + d_4)^4 [X + Y, [[X, Y], X + Y]] \\
+ \frac{1}{720} (d_1 + d_2 + d_3 + d_4)^5 [[[X, Y], X - Y], X - Y] \\
+ d_5 t_{51} + d_5 t_{52} + d_5 t_{53} + d_5 j_{51}
\end{pmatrix}) (j_{52})
\]

\[
= -\frac{1}{2} d_1 d_2 [X, [X, Y]] - \frac{1}{2} d_1 d_2 [Y, [X, Y]] - \frac{1}{2} d_1 d_3 [X, [X, Y]] - \frac{1}{2} d_1 d_3 [Y, [X, Y]] \\
- \frac{7}{4} d_1 d_2 d_3 [X, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [X, [[X, Y], X]] - \frac{7}{4} d_1 d_2 d_3 [Y, [X, [X, Y]]] \\
- \frac{7}{4} d_1 d_2 d_3 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [Y, [[X, Y], X]] - \frac{1}{2} d_1 d_4 [X, [X, Y]] \\
- \frac{7}{4} d_1 d_2 d_4 [X, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [X, [[X, Y], X]] - \frac{7}{4} d_1 d_2 d_4 [Y, [X, [X, Y]]] \\
- \frac{7}{4} d_1 d_2 d_4 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [Y, [[X, Y], X]] - \frac{1}{2} d_3 d_4 [X, [X, Y]] \\
- \frac{7}{4} d_1 d_3 d_4 [X, [X, [X, Y]]] - \frac{7}{4} d_1 d_3 d_4 [X, [Y, [X, Y]]] \\
+ \frac{3}{4} d_1 d_3 d_4 [X, [[X, Y], X]] - \frac{7}{4} d_1 d_3 d_4 [Y, [X, [X, Y]]] - \frac{7}{4} d_1 d_3 d_4 [Y, [Y, [X, Y]]] \\
+ \frac{3}{4} d_1 d_3 d_4 [Y, [[X, Y], X]] - \frac{7}{4} d_2 d_3 d_4 [X, [X, [X, Y]]] - \frac{7}{4} d_2 d_3 d_4 [Y, [X, [X, Y]]] \\
+ \frac{3}{4} d_2 d_3 d_4 [X, [[X, Y], X]] - \frac{7}{4} d_2 d_3 d_4 [Y, [X, [X, Y]]] - \frac{7}{4} d_2 d_3 d_4 [Y, [Y, [X, Y]]] \\
+ \frac{3}{4} d_2 d_3 d_4 [Y, [[X, Y], X]] - 3d_1 d_2 d_3 d_4 [X, [X, [X, Y]]]
\]

\[
- 3d_1 d_2 d_3 d_4 [X, [X, [Y, [X, Y]]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [[X, Y], X]]] \\
- 3d_1 d_2 d_3 d_4 [X, [Y, [X, [X, Y]]]] - 3d_1 d_2 d_3 d_4 [X, [Y, [Y, [X, Y]]]] \\
+ \frac{1}{2} d_1 d_2 d_3 d_4 [X, [[X, Y], X]] - 3d_1 d_2 d_3 d_4 [Y, [X, [X, Y]]] \\
- 3d_1 d_2 d_3 d_4 [Y, [X, [Y, [X, Y]]]] - 3d_1 d_2 d_3 d_4 [Y, [Y, [X, [X, Y]]]] \\
- 3d_1 d_2 d_3 d_4 [Y, [Y, [Y, [X, Y]]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [[X, Y], X]] \\
+ \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [[X, Y], X]] - \frac{3}{2} d_1 d_2 d_3 d_4 [[X, Y], [X, [X, Y]]] \\
- \frac{3}{2} d_1 d_2 d_3 d_4 [[X, Y], [Y, [X, Y]]]
\]
Letting $m_{53}$ be the right-hand side of (30), we have

\begin{align*}
= \exp & -d_5 m_{51} - d_5 m_{52} - d_5 m_{53}, \exp d_5 m_{53} \\
& - \left( (d_1 + d_2 + d_3 + d_4 + d_5) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\
& \quad + \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\
& \quad + \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 ( [[X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]]) \\
& \quad + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} + d_5 j_{51} \\
\exp & -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
\text{)} \text{By Proposition 3} \\
= \exp & -d_5 m_{51} - d_5 m_{52} - d_5 m_{53}. \exp m_{53}, d_5 \\
& - \left( (d_1 + d_2 + d_3 + d_4 + d_5) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\
& \quad + \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\
& \quad + \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 ( [[X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]]) \\
& \quad + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} + d_5 j_{51} \\
\exp & -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
\text{)} \text{By Proposition 4} \\
= \exp & -d_5 m_{51} - d_5 m_{52} - d_5 m_{53}. \\
& - \left( (d_1 + d_2 + d_3 + d_4 + d_5) (X + Y) + \frac{1}{2} (d_1 + d_2 + d_3 + d_4)^2 [X, Y] \\
& \quad + \frac{1}{12} (d_1 + d_2 + d_3 + d_4)^3 [[X, Y], Y - X] \\
& \quad + \frac{1}{96} (d_1 + d_2 + d_3 + d_4)^4 ( [[X + Y, [[X, Y], X + Y]] + [[[X, Y], X - Y], X - Y]]) \\
& \quad + d_5 i_{51} + d_5 i_{52} + d_5 i_{53} + d_5 j_{51} + d_5 j_{52} \\
\exp & -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54} \\
\text{)} \text{By (30)} (31)
\end{align*}
We let $j_{53}$ be the result of $-m_{51} - m_{52} - m_{53}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3 d_4$. Then, thanks to Theorem 2, we have

$$\delta_{\text{right}} \left( \exp \left( \frac{1}{2} \text{tr} \left( d_1 + d_2 + d_3 + d_4 \right) (X + Y) \right) + \frac{1}{3} \left( \frac{1}{2} \text{tr} \left( d_1 + d_2 + d_3 + d_4 \right)^3 \right) [X, Y] \right. $$

$$\left. + \frac{1}{12} \left( \frac{1}{2} \text{tr} \left( d_1 + d_2 + d_3 + d_4 \right)^4 \right) \left[[X, Y], Y - X\right] \right) + \frac{1}{180} \left( \frac{1}{2} \text{tr} \left( d_1 + d_2 + d_3 + d_4 \right)^4 \right) \left[[[X, Y], X + Y], X + Y\right) $$

$$\left. + \frac{1}{51} \left( \frac{1}{2} \text{tr} \left( d_1 + d_2 + d_3 + d_4 \right)^4 \right) \left[[[[X, Y], X - Y], X - Y], X - Y\right) $$

$$\left. + \frac{1}{51} \left( \frac{1}{2} \text{tr} \left( d_1 + d_2 + d_3 + d_4 \right)^4 \right) \left[[[X, Y], X - Y], X - Y\right) \right) $$

$$= \frac{3}{4} d_1 d_2 d_3 [X, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [X, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_3 [Y, [X, [X, Y]]]$$

$$+ \frac{3}{4} d_1 d_2 d_3 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [X, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [X, [Y, [X, Y]]]$$

$$+ \frac{3}{4} d_1 d_2 d_4 [Y, [X, [X, Y]]] + \frac{3}{4} d_1 d_2 d_4 [Y, [Y, [X, Y]]] + \frac{3}{4} d_1 d_3 d_4 [X, [X, [X, Y]]]$$

$$+ \frac{3}{4} d_1 d_3 d_4 [X, [Y, [X, Y]]] + \frac{3}{4} d_1 d_3 d_4 [Y, [X, [X, Y]]] + \frac{3}{4} d_1 d_3 d_4 [Y, [X, [X, Y]]]$$

$$+ \frac{3}{4} d_2 d_3 d_4 [X, [X, [X, Y]]] + \frac{3}{4} d_2 d_3 d_4 [X, [Y, [X, Y]]] + \frac{3}{4} d_2 d_3 d_4 [Y, [X, [X, Y]]]$$

$$+ \frac{3}{4} d_2 d_3 d_4 [Y, [Y, [X, Y]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [X, Y]]]$$

$$+ \frac{1}{2} d_1 d_2 d_3 d_4 [X, [X, [Y, [X, Y]]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [X, [Y, [X, [X, Y]]]]$$

$$+ \frac{1}{2} d_1 d_2 d_3 d_4 [X, [Y, [Y, [X, Y]]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [X, [X, [X, Y]]]]$$

$$+ \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [X, [Y, [X, Y]]]] + \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [Y, [X, [X, Y]]]]$$

$$+ \frac{1}{2} d_1 d_2 d_3 d_4 [Y, [Y, [Y, [X, Y]]]] \quad (32)$$
Letting $m_{54}$ be the right-hand side of (32), we have

$$
\begin{align*}
\exp & -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54}, \exp d_5 m_{54} \\
& = \exp -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54}, (m_{54})_{d_5} \\
& = \exp -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54}. (m_{54})_{d_5} \\
& = \exp -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54}.
\end{align*}
$$

By Proposition 31

$$
\begin{align*}
\exp & -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54}.
\end{align*}
$$

By Proposition 33

$$
\begin{align*}
\exp & -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54}.
\end{align*}
$$

By (32)
Since the coefficient of every term in \(-m_{51} - m_{52} - m_{53} - m_{54}\) contains \(d_1d_2d_3d_4\), we are done. We have

\[
i_{51} + i_{52} + j_{53} + j_{52} + j_{53} = d_1[X,Y] + d_2[X,Y] - \frac{1}{2}d_1d_2[[X,Y],X] + \frac{1}{2}d_1d_2[[X,Y],Y] + d_3[X,Y]
\]

\[
- \frac{1}{2}d_1d_3[[X,Y],X] + \frac{1}{2}d_1d_3[[X,Y],Y] - \frac{1}{2}d_2d_3[[X,Y],X]
\]

\[
+ \frac{1}{2}d_2d_3[[X,Y],Y] + \frac{1}{2}d_1d_2d_3[X,[[X,Y],X]] + \frac{1}{4}d_1d_2d_3[X,[[X,Y],Y]]
\]

\[
+ \frac{1}{4}d_1d_2d_3[Y,[[X,Y],X]] + \frac{1}{4}d_1d_2d_3[Y,[[X,Y],Y]] + \frac{1}{4}d_3d_4[[[X,Y],X] + \frac{1}{4}d_1d_2d_3[[[X,Y],Y],X]
\]

\[
- \frac{1}{4}d_1d_2d_3[[[X,Y],X],Y] - \frac{1}{4}d_1d_2d_3[[[X,Y],Y],X] + \frac{1}{4}d_1d_2d_3[[[X,Y],Y],Y]
\]

\[
d_4[X,Y] - \frac{1}{2}d_1d_4[[X,Y],X] + \frac{1}{2}d_1d_4[[X,Y],Y] - \frac{1}{2}d_2d_4[[X,Y],X]
\]

\[
+ \frac{1}{2}d_2d_4[[X,Y],Y] + \frac{1}{4}d_1d_2d_4[X,[[X,Y],X]] + \frac{1}{4}d_1d_2d_4[X,[[X,Y],Y]]
\]

\[
+ \frac{1}{4}d_1d_2d_4[Y,[[X,Y],X]] + \frac{1}{4}d_1d_2d_4[Y,[[X,Y],Y]] + \frac{1}{4}d_3d_4[[[X,Y],X] + \frac{1}{4}d_1d_2d_4[[[X,Y],Y],X]
\]

\[
- \frac{1}{4}d_1d_2d_4[[[X,Y],X],Y] - \frac{1}{4}d_1d_2d_4[[[X,Y],Y],X] + \frac{1}{4}d_3d_4[[[X,Y],Y],Y]
\]

\[
- \frac{1}{4}d_3d_4[[X,Y],X] + \frac{1}{2}d_3d_4[[X,Y],Y] + \frac{1}{4}d_1d_3d_4[X,[[X,Y],X]] + \frac{1}{4}d_1d_3d_4[X,[[X,Y],Y]]
\]

\[
+ \frac{1}{4}d_1d_3d_4[[[X,Y],X],X] - \frac{1}{4}d_1d_3d_4[[[X,Y],X],Y] - \frac{1}{4}d_1d_3d_4[[[X,Y],Y],X]
\]

\[
+ \frac{1}{4}d_1d_3d_4[[[X,Y],Y],Y] + \frac{1}{4}d_2d_3d_4[X,[[X,Y],X]] + \frac{1}{4}d_2d_3d_4[X,[[X,Y],Y]]
\]

\[
+ \frac{1}{4}d_2d_3d_4[Y,[[X,Y],X]] + \frac{1}{4}d_2d_3d_4[Y,[[X,Y],Y]] + \frac{1}{4}d_2d_3d_4[[[X,Y],X],X]
\]

\[
- \frac{1}{4}d_2d_3d_4[[[X,Y],X],Y] - \frac{1}{4}d_2d_3d_4[[[X,Y],Y],X] + \frac{1}{4}d_2d_3d_4[[[X,Y],Y],Y]
\]
on the one hand, and

\[-m_{51} - m_{52} - m_{53} - m_{54} - n_{51} - n_{52} - n_{53} - n_{54}\]

\[= d_1d_2d_3d_4 [X, [X, [[X, Y], X]], X] - d_1d_2d_3d_4 [X, [X, [[X, Y], Y]], X] + d_1d_2d_3d_4 [X, [Y, [[X, Y], X]], X] - d_1d_2d_3d_4 [X, [Y, [[X, Y], Y]], X] + d_1d_2d_3d_4 [Y, [X, [[X, Y], X]], X] - d_1d_2d_3d_4 [Y, [X, [[X, Y], Y]], X] + d_1d_2d_3d_4 [Y, [Y, [[X, Y], X]], X] - d_1d_2d_3d_4 [Y, [Y, [[X, Y], Y]], X] - \frac{1}{6}d_1d_2d_3d_4 [X, [[X, Y], X], X] - \frac{1}{6}d_1d_2d_3d_4 [X, [[X, Y], X], Y] + \frac{1}{6}d_1d_2d_3d_4 [X, [[X, Y], Y], X] + \frac{1}{6}d_1d_2d_3d_4 [X, [[X, Y], Y], Y] - \frac{1}{6}d_1d_2d_3d_4 [Y, [[X, Y], X], X] - \frac{1}{6}d_1d_2d_3d_4 [Y, [[X, Y], X], Y] + \frac{1}{6}d_1d_2d_3d_4 [Y, [[X, Y], Y], X] + \frac{1}{6}d_1d_2d_3d_4 [Y, [[X, Y], Y], Y] + \frac{1}{2}d_1d_2d_3d_4 [[X, Y], [[X, Y], X], X] + \frac{1}{2}d_1d_2d_3d_4 [[X, Y], [[X, Y], Y], X] - \frac{1}{8}d_1d_2d_3d_4 [[X, [X, Y], X], X] + \frac{1}{8}d_1d_2d_3d_4 [[X, [X, Y], X], Y] - \frac{1}{8}d_1d_2d_3d_4 [[X, [X, Y], Y], X] + \frac{1}{8}d_1d_2d_3d_4 [[X, [X, Y], Y], Y] - \frac{1}{8}d_1d_2d_3d_4 [[Y, [X, Y], X], X] + \frac{1}{8}d_1d_2d_3d_4 [[Y, [X, Y], X], Y] - \frac{1}{8}d_1d_2d_3d_4 [[Y, [X, Y], Y], X] + \frac{1}{8}d_1d_2d_3d_4 [[Y, [X, Y], Y], Y] - \frac{1}{8}d_1d_2d_3d_4 [[[X, Y], X], X], X] + \frac{1}{8}d_1d_2d_3d_4 [[[X, Y], X], X], Y] + \frac{1}{8}d_1d_2d_3d_4 [[[X, Y], X], Y], X] - \frac{1}{8}d_1d_2d_3d_4 [[[X, Y], X], Y], Y] + \frac{1}{8}d_1d_2d_3d_4 [[[X, Y], Y], X], X] - \frac{1}{8}d_1d_2d_3d_4 [[[X, Y], Y], X], Y] - \frac{1}{8}d_1d_2d_3d_4 [[[X, Y], Y], Y], X] + \frac{1}{8}d_1d_2d_3d_4 [[[X, Y], Y], Y], Y] \]

on the other. Therefore we have the desired result. ■

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