Derivation of the 1d NLS equation from the 3d quantum many-body dynamics of strongly confined bosons

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Abstract
We consider the dynamics of $N$ interacting bosons initially exhibiting Bose–Einstein condensation. Due to an external trapping potential, the bosons are strongly confined in two spatial directions, with the transverse extension of the trap being of order $\varepsilon$. The non-negative interaction potential is scaled such that its scattering length is positive and of order $(N/\varepsilon^2)^{-1}$, the range of the interaction scales as $(N/\varepsilon^2)^{-\beta}$ for $\beta \in (0,1)$. We prove that in the simultaneous limit $N \to \infty$ and $\varepsilon \to 0$, the condensation is preserved by the dynamics and the time evolution is asymptotically described by a cubic defocusing nonlinear Schrödinger equation in one dimension, where the strength of the nonlinearity depends on the interaction and on the confining potential. This is the first derivation of a lower-dimensional effective evolution equation for singular potentials scaling with $\beta \geq \frac{1}{2}$ and lays the foundations for the derivation of the physically relevant one-dimensional Gross–Pitaevskii equation ($\beta = 1$) in [4]. For our analysis, we adapt an approach by Pickl [28] to the problem with strong confinement.

1 Introduction
We consider a system of $N$ identical bosons in $\mathbb{R}^3$ interacting among each other through repulsive pair interactions. The bosons are trapped within a cigar-shaped trap, which effectively confines the particles in two spatial directions to a region of order $\varepsilon$. To describe this mathematically, let us first introduce the coordinates

$$z = (x, y) \in \mathbb{R}^{1+2}.$$ 

The cigar-shaped confinement is given by the scaled potential $\frac{1}{\varepsilon^2} V_\perp \left( \frac{y}{\varepsilon} \right)$ for some $0 < \varepsilon \ll 1$ and $V_\perp : \mathbb{R}^2 \to \mathbb{R}$. The Hamiltonian of this system is

$$H_\beta(t) = \sum_{j=1}^{N} \left( -\Delta_j + \frac{1}{\varepsilon^2} V_\perp \left( \frac{y_j}{\varepsilon} \right) + V_\parallel(t, z_j) \right) + \sum_{1 \leq i < j \leq N} w_\beta(z_i - z_j),$$

where $\Delta$ denotes the Laplace operator on $\mathbb{R}^3$ and $V_\parallel$ is a possibly time-dependent additional external potential. The units are chosen such that $\hbar = 1$ and $m = \frac{1}{2}$. In the limit $\varepsilon \to 0$, the system becomes effectively one-dimensional, in the sense that excitations in the transverse direction are energetically strongly suppressed.

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The interaction between the particles is described by the potential \( w_\beta \) with scaling parameter \( \beta \in (0, 1) \). For the sake of this introduction, let us for the moment assume that

\[
w_\beta(z) = \left( \frac{N}{\varepsilon} \right)^{1-3\beta} w \left( \frac{N}{\varepsilon} \beta z \right)
\]

for some compactly supported, spherically symmetric, non-negative, bounded potential \( w \).\(^1\) This scaling describes a dilute gas, where the scaling parameter \( \beta \) interpolates between Hartree \( (\beta = 0) \) and Gross–Pitaevskii \( (\beta = 1) \) regime. The proof of the physically relevant Gross–Pitaevskii \( (\beta = 1) \) regime relies essentially on the result for \( \beta \in (0, 1) \) and is given in [4]. An important parameter characterising the interaction \( w_\beta \) is its effective range,

\[
\mu := \left( \frac{N}{\varepsilon} \right)^{-\beta}.
\]

We study the dynamics of the system in the simultaneous limit \( (N, \varepsilon) \to (\infty, 0) \). The state \( \psi_{N,\varepsilon}(t) \) of the system at time \( t \) is determined by the \( N \)-body Schrödinger equation

\[
i \frac{d}{dt} \psi_{N,\varepsilon}(t) = H_\beta(t) \psi_{N,\varepsilon}(t)
\]

with initial data \( \psi_{N,\varepsilon}(0) = \psi^N_{0,\varepsilon} \in L^2_\varepsilon(\mathbb{R}^3 N) := \otimes_{sym}^N L^2(\mathbb{R}^3) \). We assume that the system initially exhibits Bose–Einstein condensation, i.e. that the one-particle reduced density matrix \( \gamma_{\psi_{0,\varepsilon}}^{(1)} \) of \( \psi_{0,\varepsilon} \),

\[
\gamma_{\psi_{0,\varepsilon}}^{(k)} := \text{Tr}_{k+1,\ldots,N} |\psi_{0,\varepsilon}^N \rangle \langle \psi_{0,\varepsilon}^N |
\]

for \( k = 1 \), is asymptotically close to the projection onto a one-body state \( \varphi_0^\varepsilon \). At low energies, the state factorises as a consequence of the strong confinement and is of the form \( \varphi_0^\varepsilon(z) = \Phi_0(x) \chi^\varepsilon(y) \in L^2(\mathbb{R}^3) \) (see Remark 1e). Here, \( \Phi_0 \) denotes the wavefunction along the \( x \)-axis and \( \chi^\varepsilon \) is the normalised ground state of \(-\Delta_y + \frac{1}{\varepsilon^2} V^\perp(y)\) in the confined directions. Due to the rescaling by \( \varepsilon \), \( \chi^\varepsilon \) is given by

\[
\chi^\varepsilon(y) = \frac{1}{\varepsilon} \chi(\frac{y}{\varepsilon}),
\]

where \( \chi \) is the normalised ground state of \(-\Delta_y + V^\perp(y)\).

In Theorem 1, we show that if the state of the system is initially such a factorised Bose–Einstein condensate with condensate wavefunction \( \varphi_0^\varepsilon = \Phi_0 \chi^\varepsilon \), i.e. if

\[
\lim_{(N,\varepsilon) \to (\infty, 0)} \text{Tr}_{L^2(\mathbb{R}^3)} \left| \gamma_{\psi_{0,\varepsilon}}^{(1)} - |\varphi_0^\varepsilon \rangle \langle \varphi_0^\varepsilon | \right| = 0,
\]

where the limit \( (N, \varepsilon) \to (\infty, 0) \) is taken in an appropriate way, then the condensation of the system into a factorised state is preserved by the dynamics, i.e. for all \( t \in \mathbb{R} \) and \( k \in \mathbb{N} 

\[
\lim_{(N,\varepsilon) \to (\infty, 0)} \text{Tr}_{L^2(\mathbb{R}^{3k})} \left| \gamma_{\psi_{0,\varepsilon}(t)}^{(k)} - |\varphi_0^\varepsilon(t) \rangle \langle \varphi_0^\varepsilon(t) | \otimes k \right| = 0.
\]

The condensate wavefunction at time \( t \) is given by \( \varphi^\varepsilon(t) = \Phi(t) \chi^\varepsilon \), where \( \Phi(t) \) is the solution of the one-dimensional nonlinear Schrödinger (NLS) equation

\[
i \frac{\partial}{\partial t} \Phi(t, x) = \left( -\frac{\partial^2}{\partial x^2} + V(t, (x, 0)) + b_\beta |\Phi(t, x)|^2 \right) \Phi(t, x) =: h(t) \Phi(t, x)
\]

with \( \Phi(0) = \Phi_0 \) and coupling parameter \( b_\beta = \|w\|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}^2} |\chi(y)|^4 \, dy \).

\(^1\)In our main result, the interaction is of a more generic form.
To our knowledge, Theorem 1 is the first rigorous derivation of an effectively lower-dimensional evolution equation directly from the three-dimensional $N$-body dynamics for $\beta \geq 1/2$. In [18], von Keler and Teufel consider a similar problem for $\beta \in (0, 1/3)$ and in [6] and [8], Chen and Holmer study interactions for values of $\beta$ in subsets of the interval $(0, 1/2)$. The extension to $\beta \in (0, 1)$ requires a non-trivial adaptation of methods used for the fully three-dimensional problem without strong confinement [32] to handle the additional limit $\varepsilon \to 0$ and the associated dimensional reduction. Not only is this an interesting mathematical problem on its own but it lays the foundations for the derivation of the physically relevant effectively one-dimensional Gross–Pitaevskii equation corresponding to the scaling $\beta = 1$ [4].

In fact, the main idea of the proof in [4] is to approximate the interaction $w_{\beta=1}$ by a softer scaling interaction which is covered by our Theorem 1, and to show that the remainders from this substitution vanish in the limit. The dimensional reduction occurs in the approximated interaction, hence the result for $\beta = 1$ relies essentially on the tools and results proven here.

Let us give a brief motivation of the effective equation (5). The $N$-body problem is interacting, hence the effective evolution is nonlinear and the strength of the linearity depends on the two-body scattering process. This process is to leading order described by the scattering length $a_{\beta}$ of $w_{\beta}$, which scales as $(N\varepsilon^{2})^{-1}$ for $\beta \in (0, 1)$ [9, Lemma A.1]. This implies that, for $\beta \in (0, 1)$, the length scale of the inter-particle correlations is small compared to the range $\mu = (N\varepsilon^{2})^{-\beta}$ of $w_{\beta}$. Hence, the correlations are negligible in the limit and the two-body scattering process is described by the first order Born approximation to the scattering length, $8\pi a_{\beta} \approx \int w_{\beta}(z) \, dz$. The additional factor $\int_{R^{2}} |\chi(y)|^{4} \, dy$ in the coupling parameter arises from integrating out the transverse degrees of freedom in the course of the dimensional reduction.

Quasi one-dimensional Bose gases in highly elongated traps have been studied experimentally [12, 14] and the dynamical behaviour of such systems is of great physical interest [11, 19, 27]. The first rigorous derivation of an NLS evolution for three-dimensional bosons was by Erdős, Schlein and Yau [9]. The main tool of their proof is the convergence of the BBGKY hierarchy, a system of coupled equations determining the time evolution of all $k$-particle density matrices. Later, the authors adapted their proof to handle the Gross–Pitaevskii scaling of the interaction [10]. A different approach providing rates for the convergence of the reduced density matrices was proposed by Pickl [28, 31], who derived effective evolution equations for NLS and Gross–Pitaevskii scaling of the interaction, including time-dependent external potentials [32] as well as non-positive [30, 17] and singular interactions [21]. A third method for the Gross–Pitaevskii case, based on Bogoliubov transformations and coherent states on Fock space, was developed by Benedikter, De Oliveira and Schlein [3], and a presumably optimal rate of convergence was recently proven by Brennecke and Schlein [5]. Further results concern bosons in one [1, 7] and two [20, 15, 16] spatial dimensions.

Some authors have considered the problem of dimensional reduction for the NLS equation. In [26], Méhats and Raymond study the cubic NLS equation in a two-dimensional quantum waveguide, i.e. within a tube of width $\varepsilon$ around a curve in $\mathbb{R}^{2}$. They show that in the limit $\varepsilon \to 0$, the nonlinear evolution is well approximated by a one-dimensional cubic NLS equation with an additional potential term due to the curvature. Ben Abdallah, Méhats, Schmeiser and Weishäupl consider in [2] an $(n + d)$-dimensional NLS equation subject to a strong confinement in $d$ directions and derive an effective $n$-dimensional NLS equation with a modified nonlinearity.

As mentioned above, there are few results concerning the derivation of lower-dimensional NLS equations from the underlying three-dimensional $N$-body dynamics. Chen and Holmer consider three-dimensional bosons with pair interactions in a harmonic potential that is
strongly confining in one [6] or two [8] directions. For a repulsive interaction scaling with \( \beta \in (0, \frac{2}{3}) \) in case of the disc-shaped and for an attractive interaction with \( \beta \in (0, \frac{2}{3}) \) in case of the cigar-shaped confinement, they prove that the dynamics are effectively described by a two- or respectively one-dimensional NLS equation. In [18], von Keler and Teufel study a Bose gas confined to a quantum waveguide with non-trivial geometry for scaling parameters \( \beta \in (0, \frac{1}{2}) \). They prove that the evolution is well captured by a one-dimensional NLS equation with additional potential terms arising from the twisting and bending of the waveguide.

The remainder of this paper is structured as follows: in Section 2, we specify our assumptions and present the result. Our proof follows an approach by Pickl, which is outlined in Section 3. This section also contains the proof of our main Theorem 1, relying essentially on two propositions. Finally, these propositions are proven in Section 4.

## 2 Main Result

To study the effective behaviour of the many-body system in the simultaneous limit \((N, \varepsilon) \to (\infty, 0)\), let us consider families of initial data \(\psi_0^{N,\varepsilon}\) along sequences \((N_n, \varepsilon_n) \to (\infty, 0)\).

**Definition 2.1.** A sequence \((N_n, \varepsilon_n)\) in \(\mathbb{N} \times (0, 1)\) is called *admissible* if

\[
\lim_{n \to \infty} (N_n, \varepsilon_n) = (\infty, 0) \quad \text{and} \quad \lim_{n \to \infty} \frac{\varepsilon_n^2}{\mu_n} = 0 \quad \text{for} \quad \mu_n := \left(\frac{N_n}{\varepsilon_n^2}\right)^{-\beta}.
\]

It is called *moderately confining* if

\[
\lim_{n \to \infty} \frac{\mu_n}{\varepsilon_n} = 0.
\]

Moderate confinement means that the extension \(\varepsilon\) of the confining potential shrinks to zero but is still large compared to the range of the interaction \(\mu\). This prevents the interaction from being supported mainly in a region that is quasi inaccessible to the particles due to the strong confinement. As \(\mu/\varepsilon = N^{-\beta} \varepsilon^{2\beta - 1}\), this condition is a restriction only for \(\beta < \frac{1}{2}\).

The admissibility condition ensures that \(\varepsilon\) shrinks sufficiently fast compared to \(\mu\) that the system becomes effectively one-dimensional. Note that for \(\delta > 0\), \(\varepsilon^\delta/\mu = N^{-\beta} \varepsilon^{\delta - 2\beta}\), hence \(\delta = 2\) is the smallest exponent for which \(\varepsilon^\delta/\mu \to 0\) is possible for all \(\beta \in (0, 1)\). Both conditions are comparable to the assumptions in [8] for an attractive interaction scaling with \(\beta \in (0, \frac{3}{7})\).

We will use the notation \(A \lesssim B\) to indicate that there exists a constant \(C > 0\) independent of \(\varepsilon, N, t, \psi_0^{N,\varepsilon}, \Phi_0\) such that \(A \lesssim CB\). The constant may depend on the quantities fixed by the model, such as \(V^\perp, \chi\) and \(V^\parallel\).

We consider interactions of the following type:

**Definition 2.2.** Let \(\beta \in (0, 1)\) and \(\eta > 0\). Define the set \(W_{\beta, \eta}\) as the set containing all families

\[
w_{\beta} : \mathbb{N} \times (0, 1) \to L^\infty(\mathbb{R}^3, \mathbb{R}), \quad (N, \varepsilon) \mapsto w_{\beta}((N, \varepsilon)),
\]

\(^2\)In our notation, the assumptions in [8] are \(N^{\nu_1(\beta)} \lesssim \varepsilon^{-\delta} \lesssim N^{\nu_2(\beta)}\), where \(\nu_1\) and \(\nu_2\) are given by \(\nu_1(\beta) = \frac{1}{1 - \beta}\) and \(\nu_2 = \min\left\{\frac{1 - \beta}{2 - \beta}, \frac{2 - \beta}{\beta} + 1 + \infty \cdot \frac{1}{\beta} < 1 + \frac{2\beta}{1 - 2\beta}, \frac{2 - \beta}{\beta}\right\}\). Note that \(N^{\nu_1(\beta)} \varepsilon^2 = \left(\frac{\nu_1(\beta)}{\beta}\right)^{\frac{4}{1 - \beta}}\) and \(N^{\nu_2(\beta)} \varepsilon^2 \leq \left(\frac{1}{\beta}\right)^{\frac{4}{1 - \beta}}\) as \(\nu_2(\beta) \leq \frac{2\beta}{1 - 2\beta}\), hence these conditions are comparable to our assumptions.
such that for any \((N, \varepsilon) \in \mathbb{N} \times (0, 1)\)

\[
\begin{cases}
(a) & \|w_\beta((N, \varepsilon))\|_{L^\infty(\mathbb{R}^3)} \leq \left(\frac{N}{\varepsilon^2}\right)^{-1+3\beta}, \\
(b) & w_\beta((N, \varepsilon)) \text{ is non-negative and spherically symmetric}, \\
(c) & \text{supp} w_\beta((N, \varepsilon)) \subseteq \left\{ z \in \mathbb{R}^3 : |z| \leq \left(\frac{N}{\varepsilon^2}\right)^{-\beta}\right\}, \\
(d) & \lim_{(N, \varepsilon) \to (\infty, 0)} \left(\frac{N}{\varepsilon^2}\right)^2 |b_{N, \varepsilon}((N, \varepsilon), w_\beta) - b_\beta(w_\beta)| = 0,
\end{cases}
\]

where

\[
b_{N, \varepsilon}((N, \varepsilon), w_\beta) := N \int_{\mathbb{R}^3} w_\beta((N, \varepsilon), z) \, dz \int_{\mathbb{R}^3} |\chi(y)|^4 \, dy = N \int_{\mathbb{R}^2} w_\beta((N, \varepsilon), z) \, dz \int_{\mathbb{R}^2} |\chi(y)|^4 \, dy,
\]

\[
b_\beta(w_\beta) := \lim_{(N, \varepsilon) \to (\infty, 0)} b_{N, \varepsilon}((N, \varepsilon), w_\beta).
\]

We will in the following abbreviate \(w_\beta((N, \varepsilon)) \equiv w_\beta, b_{N, \varepsilon}((N, \varepsilon), w_\beta) \equiv b_{N, \varepsilon}\) and \(b_\beta(w_\beta) \equiv b_\beta\).

Condition (d) ensures that the \((N, \varepsilon)\)-dependent parameter \(b_{N, \varepsilon}\) converges sufficiently fast to its limit \(b_\beta\). Clearly, the interaction \(\left(\frac{N}{\varepsilon^2}\right)^{-1+3\beta} w((\frac{N}{\varepsilon^2})^\beta z)\) from the introduction is contained in this set. In this case, \(b_{N, \varepsilon} = \|w\|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}^2} |\chi(y)|^4 \, dy = b_\beta\), hence (d) is true for any \(\eta > 0\).

In order to formulate our main theorem, we will need two different notions of one-particle energies:

- The “renormalised” energy per particle: for \(\psi \in \mathcal{D}(H_\beta(t))\),
  \[
  E^\psi(t) := \frac{1}{N} \langle \psi, H_\beta(t) \psi \rangle_{L^2(\mathbb{R}^3N)} - \frac{E_0}{\varepsilon^2},
  \]
  where \(E_0\) denotes the lowest eigenvalue of \(-\Delta_y + V^\perp(y)\). By rescaling, \(\frac{E_0}{\varepsilon^2}\) is the lowest eigenvalue of \(-\Delta_y + \frac{1}{\varepsilon^2} V^\perp(\frac{N}{\varepsilon^2})\).

- The effective energy per particle: for \(\Phi \in H^2(\mathbb{R})\),
  \[
  E^\Phi(t) := \left\langle \Phi, \left(-\frac{\partial^2}{\partial x^2} + V(t, (x, 0)) + \frac{b_\beta}{\varepsilon^2} |\Phi|^2\right) \Phi \right\rangle_{L^2(\mathbb{R})}.
  \]

Further, we define the function \(\varepsilon : \mathbb{R} \to [1, \infty)\) by

\[
\varepsilon^2(t) := 1 + |E^{\psi_0,N,\varepsilon}(0)| + |E^{\Phi_0}(0)| + \int_0^t \|V^{\perp}(s)\|_{L^\infty(\mathbb{R}^3)} \, ds + \sup_{i,j \in \{0,1\}, \ k \in \{1,2\}} \|\partial_x^i \partial_y^j \psi^{\perp}(t)\|_{L^\infty(\mathbb{R}^3)}.
\]

This function will be of use because, by the fundamental theorem of calculus,

\[
|E^{\psi,N,\varepsilon}(t)(t)| \leq \varepsilon^2(t) - 1 \quad \text{and} \quad |\varepsilon^{\Phi(t)}(t)| \leq \varepsilon^2(t) - 1
\]

for any time \(t \in \mathbb{R}\). Note that if the external field \(V^{\perp}\) is time-independent, \(\varepsilon^2(t) \lesssim 1\) for any \(t\), hence in this case, \(E^{\psi,N,\varepsilon}(t)(t)\) and \(\varepsilon^{\Phi(t)}(t)\) are bounded uniformly in time.

Let us now state our assumptions:
A1 Interaction. Let the interaction \( w_\beta \in W_{\beta, \eta} \) for some \( \eta > 0 \).

A2 Confining potential. Let \( V^\perp : \mathbb{R}^2 \to \mathbb{R} \) such that \( -\Delta_y + V^\perp \) is self-adjoint and has a non-degenerate ground state \( \chi \) with energy \( E_0 < \inf \sigma_{\text{ess}}(-\Delta_y + V^\perp) \). Assume that the negative part of \( V^\perp - E_0 \) is bounded, i.e. that \( (V^\perp - E_0)^- \in L^\infty(\mathbb{R}^2) \), and that \( \chi \in C^1_0(\mathbb{R}^2) \), i.e. \( \chi \) is bounded and continuously differentiable with bounded derivative. We choose \( \chi \) normalised and real.

A3 External field. Let \( V^\parallel : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \) such that for fixed \( z \in \mathbb{R}^3 \), \( V^\parallel(\cdot, z) \in C^1(\mathbb{R}) \). Further, assume that for each fixed \( t \in \mathbb{R} \), \( V^\parallel(t, (\cdot, 0)) \in H^2(\mathbb{R}), V^\parallel(t, \cdot), \dot{V}^\parallel(t, \cdot) \in L^\infty(\mathbb{R}^3) \cap C^1(\mathbb{R}^3) \) and \( \nabla_y V^\parallel(t, \cdot), \nabla_y \dot{V}^\parallel(t, \cdot) \in L^\infty(\mathbb{R}^3) \).

A4 Initial data. Assume that the family of initial data, \( \psi_0^{N, \varepsilon} \in \mathcal{D}(H_\beta(0)) \cap L^2(\mathbb{R}^3) \) with \( \|\psi_0^{N, \varepsilon}\|^2 = 1 \), is close to a condensate with condensate wavefunction \( \varphi_0^\varepsilon = \Phi_0 \chi^\varepsilon \) for some normalised \( \Phi_0 \in H^2(\mathbb{R}) \) in the following sense: for some admissible, moderately confining sequence \( (N, \varepsilon) \), it holds that

\[
\lim_{(N, \varepsilon) \to (\infty, 0)} \| \gamma^{(1)}_{\psi_0^{N, \varepsilon}} - |\Phi_0 \chi^\varepsilon\rangle \langle \Phi_0 \chi^\varepsilon| \| = 0 \quad (10)
\]

and

\[
\lim_{(N, \varepsilon) \to (\infty, 0)} \| E^{\psi_0^{N, \varepsilon}}(0) - \mathcal{E}^{\Phi_0}(0) \| = 0. \quad (11)
\]

Remark 1. (a) Assumption A1 includes the interaction \( w_\beta(z) = \left( \frac{N}{\varepsilon^2} \right)^{-1+3\beta} w \left( \left( \frac{N}{\varepsilon^2} \right)^\beta z \right) \) for \( w : \mathbb{R}^3 \to \mathbb{R} \) spherically symmetric, non-negative and with \( \text{supp } w \subseteq B_1(0) \).

(b) Assumption A2 is, for instance, fulfilled by a harmonic potential or by any bounded smooth potential with a bound state below the essential spectrum. Note that it is not necessary that the potential increases as \( |y| \to \infty \). The confining effect of the potential is due to the rescaling by \( \varepsilon \) because the ground state of \( -\Delta_y + V^\perp \) is exponentially localised [13, Theorem 1].

(c) The regularity condition on \( V^\parallel(t, (\cdot, 0)) \) in A3 ensures the global existence of \( H^2 \)-solutions of the NLS equation (5) (see Appendix A and Lemma 4.8). The further requirements for \( V^\parallel, \nabla_y V^\parallel \) and \( \nabla_y \dot{V}^\parallel \) are needed to control the one-particle energies and the interactions of bosons with the external field \( V^\parallel \).

(d) Due to assumptions A1 – A3, \( H_\beta(t) \) is self-adjoint on \( \mathcal{D}(H_\beta(t)) = \mathcal{D}(H_\beta) \). As \( t \to V^\parallel(t) \in \mathcal{L}(L^2(\mathbb{R}^3)) \) is continuous, \( H_\beta(t) \) generates a strongly continuous unitary evolution on \( \mathcal{D}(H_\beta) \).

(e) We assume in A4 that the system is initially given by a Bose–Einstein condensate with factorised condensate wavefunction. Both parts (10) and (11) of the assumption are standard when deriving effective evolution equations. For the scaling parameter \( \beta = 1 \), it is shown in [25] that the ground state of the corresponding system satisfies assumption A4. For related results without strong confinement, we refer to the review [24] for \( \beta = 1 \) and to [22] for \( \beta < 1 \).

Theorem 1. Let \( \beta \in (0, 1) \) and assume that \( w_\beta, V^\perp \) and \( V^\parallel \) satisfy A1 – A3. Let \( \psi_0^{N, \varepsilon} \) be a family of initial data satisfying A4, let \( \psi^{N, \varepsilon}(t) \) denote the solution of the \( N \)-body Schrödinger equation with initial data \( \psi_0^{N, \varepsilon} \). Then...
equation (2) with initial datum \( \psi_{N,\varepsilon}^{(0)}(0) = \psi_0^{N,\varepsilon} \) and let \( \gamma_{\psi_{N,\varepsilon}}^{(k)} \) denote its \( k \)-particle reduced density matrix as in (3). Then for any \( t \in \mathbb{R} \) and \( k \in \mathbb{N} \),

\[
\lim_{(N,\varepsilon) \to (\infty,0)} \text{Tr}_{L^2(\mathbb{R}^3)} \left| \gamma_{\psi_{N,\varepsilon}}^{(k)}(t) - \langle \Phi(t) \chi^\varepsilon \rangle \langle \Phi(t) \chi^\varepsilon \rangle^\otimes k \right| = 0 \tag{12}
\]

and

\[
\lim_{(N,\varepsilon) \to (\infty,0)} \left| \mathcal{E}_{\psi_{N,\varepsilon}}(t) - \mathcal{E}_\Phi(t) \right| = 0, \tag{13}
\]

where the limits are taken along the sequence from A4. \( \Phi(t) \) is the solution of the NLS equation (5) with initial datum \( \Phi(0) = \Phi_0 \) from A4, where the strength of the nonlinearity in (5) is given by \( b_\beta \) from Definition 2.2, namely

\[
b_\beta = \lim_{(N,\varepsilon) \to (\infty,0)} b_{N,\varepsilon} = \lim_{(N,\varepsilon) \to (\infty,0)} \frac{N}{\varepsilon} \int_{\mathbb{R}^3} \frac{\chi(y)}{\varepsilon^2} \int_{\mathbb{R}^2} \left| \chi(y) \right|^4 \, dy. \tag{14}
\]

Remark 2.  
(a) For the specific choice \( w_\beta(z) = \left( \frac{N}{\varepsilon^2} \right)^{-1+3\beta} = \left( \frac{N}{\varepsilon^2} \right)^{-1} \), we obtain the coupling parameter \( b_\beta = \| w \|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}^2} \left| \chi(y) \right|^4 \, dy \).

(b) For any fixed \( t \in \mathbb{R} \), our proof yields an estimate of the rate of the convergence of (12), which is explicitly stated in Corollary 3.6. The rate is not uniform in time but depends on it in terms of a double exponential. Note, however, that times of order one already correspond to long times on the microscopic scale.

(c) Let us comment on the difference of our work to the result of von Keler and Teufel [18], who consider \( \beta \in (0, \frac{1}{3}) \). The extension to \( \beta \in (0,1) \) means a physically relevant improvement of the result: for \( \beta < \frac{1}{3} \), the problem can still be considered as a mean-field problem since the mean inter-particle distance \( \theta^{-\frac{1}{2}} \sim \left( \frac{N}{\varepsilon^2} \right)^{-\frac{1}{3}} \) is small compared to the range of the interaction \( \mu = \left( \frac{N}{2} \right)^{-\beta} \). For \( \beta > \frac{1}{3} \), the mean-field description breaks down and one must handle interactions which are too singular to be covered by the approach of [18]. We solve this by an integration by parts of the interaction, which comes at the price that one must control the kinetic energy of the \( N \)-particle wavefunction (Lemma 4.10 and Lemma 4.17). Also, note that our admissibility condition is weaker than the respective condition \( \varepsilon^2/\mu \to 0 \) in [18], which cannot be satisfied for \( \beta > \frac{3}{7} \).

In [18], the bosons are trapped within a quantum waveguide with non-trivial geometry. The confinement is realised by means of Dirichlet boundary conditions, which restrict the system to a tube of width \( \varepsilon \) around some curve in \( \mathbb{R}^3 \). In our model, the confinement is by potentials. However, our result can be easily modified to a confinement via Dirichlet boundary conditions, corresponding to a straight and untwisted quantum waveguide. The main difference in the proof is the estimate of \( \gamma_b^{(1)} \) (Section 4.4.2): one divides the expression (46) into an integral over those \( y \) sufficiently distant from the boundary that \( \text{supp} w_\beta((x,y) - \cdot) \) is completely contained in the waveguide, and into an integral over the rest, which is easily estimated.

In addition to moderately confining sequences, the authors of [18] consider sequences \( (N,\varepsilon) \to (\infty,0) \) with \( \varepsilon/\mu \to 0 \). This is possible for \( \beta \in (0, \frac{1}{2}) \) and leads to \( b_\beta = 0 \) in the effective equation because an essential part of the interaction is cut off such that the limiting effective equation becomes linear. We conjecture that the same effect occurs in our setup.
(d) In [8], Chen and Holmer study attractive interactions, i.e. \( \int_{\mathbb{R}^3} w_\beta(z) \, dz \leq 0 \). In distinction from that work, we exclusively consider repulsive interactions with \( w_\beta \geq 0 \). However, as the condition \( w_\beta \geq 0 \) seems to be crucial only to the proofs of Lemma 4.10 and Lemma 4.17, it is likely that our result can be extended to include repulsive interactions with a certain negative part.

### 3 Proof of the main theorem

To prove Theorem 1, we need to show that the expressions in (12) and (13) vanish in the limit \((N,\varepsilon) \to (\infty,0)\), given suitable initial data. Instead of estimating these differences directly, we adhere to the idea by Pickl [28, 29, 30, 31, 32] to define a functional \( \alpha_\xi(\psi^{N,\varepsilon}(t),\varphi^\varepsilon(t)) \) which provides a measure of the part of the \( N \)-particle wavefunction \( \psi^{N,\varepsilon} \) that has not condensed into the single-particle orbital \( \varphi^\varepsilon \). The functional is chosen such that \( \alpha_\xi(\psi^{N,\varepsilon}(t),\varphi^\varepsilon(t)) \to 0 \) is equivalent to (12) and (13). We follow in general [32]. However, the strongly asymmetric confinement requires a nontrivial modification of the formalism to treat the dimensional reduction and the more singular scaling of the interaction. For the construction of \( \alpha_\xi \), we need the following projections:

**Definition 3.1.** Let \( \varphi^\varepsilon(t) = \Phi(t)\chi^\varepsilon \), where \( \Phi(t) \) is the solution of the NLS equation (5) with initial datum \( \Phi_0 \) from A4 and with \( \chi^\varepsilon \) as in (4). Let
\[
p := |\varphi^\varepsilon(t)\rangle \langle \varphi^\varepsilon(t)|,
\]
where we have dropped the time dependence of \( p \) in the notation. For \( i \in \{1, \ldots, N\} \), define the projection operators on \( L^2(\mathbb{R}^{3N}) \)
\[
p_j := \bigotimes_{j=1}^{j-1} 1 \otimes p \otimes 1 \otimes \cdots \otimes 1 \quad \text{and} \quad q_j := 1 - p_j.
\]
Further, define the orthogonal projections on \( L^2(\mathbb{R}^3) \)
\[
p^\Phi := |\Phi(t)\rangle \langle \Phi(t)| \otimes 1_{L^2(\mathbb{R}^2)},
q^\Phi := 1_{L^2(\mathbb{R}^3)} - p^\Phi,
p^\chi^\varepsilon := 1_{L^2(\mathbb{R}^3)} \otimes |\chi^\varepsilon\rangle \langle \chi^\varepsilon|,
q^\chi^\varepsilon := 1_{L^2(\mathbb{R}^3)} - p^\chi^\varepsilon,
\]
and define \( p^\Phi_j, q^\Phi_j, p^\chi^\varepsilon_j \) and \( q^\chi^\varepsilon_j \) on \( L^2(\mathbb{R}^{3N}) \) analogously to \( p_j \) and \( q_j \). Finally, for \( 0 \leq k \leq N \), define the many-body projections
\[
P_k = (q_1 \cdots q_k p_{k+1} \cdots p_N)_{\text{sym}} := \sum_{J \subseteq \{1, \ldots, N\}} \prod_{J \neq J} q_j \prod_{l \notin J} p_l
\]
and \( P_k = 0 \) for \( k < 0 \) and \( k > N \).

We will write \( p_j = |\varphi^\varepsilon(t, z_j)\rangle \langle \varphi^\varepsilon(t, z_j)| \), \( p^\Phi_j = |\Phi(t, x_j)\rangle \langle \Phi(t, x_j)| \), and \( p^\chi^\varepsilon_j = |\chi^\varepsilon(y_j)\rangle \langle \chi^\varepsilon(y_j)| \).

Some useful identities of the projections are listed in the following corollary:

**Corollary 3.1.** For \( 0 \leq k \leq N \) and \( 1 \leq j \leq N \), it holds that
\[
(a) \quad \sum_{k=0}^{N} P_k = 1, \quad \sum_{j=1}^{N} q_j P_k = k P_k,
(b) \quad p_j = p^\Phi_j p^\chi^\varepsilon_j, \quad p^\Phi_j p^\chi^\varepsilon_j p_j = p_j, \quad q^\Phi_j q_j = q^\Phi_j q^\chi^\varepsilon_j = q^\Phi_j q^\chi^\varepsilon_j = q^\Phi_j q^\chi^\varepsilon_j p_j = 0,
\]
\]

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(c) \( q_j = q_j^p p_j^{\chi} + p_j^q q_j^{\chi} + q_j^p q_j^{\chi} = q_j^{\chi} + q_j^p p_j^{\chi} = q_j^p + p_j^q q_j^{\chi} \).

**Proof.** The first identity in (a) is due to the relation \( p_j + q_j = 1 \). The second identity follows from the fact that
\[
\sum_{j=1}^{N} q_j = \sum_{j=1}^{N} q_j \sum_{k=0}^{N} P_k = \sum_{k=0}^{N} \sum_{j=1}^{N} q_j P_k = \sum_{k=0}^{N} k P_k
\]

together with \( P_k P_{k'} = \delta_{k,k'} P_k \). While part (b) is an immediate consequence of Definition 3.1, part (c) is implied by \( q = 1 - p = (p^p + q^q)(p^{\chi} + q^{\chi}) - p^p p^{\chi} = p^p q^{\chi} + q^q p^{\chi} + q^q q^{\chi} \).

**Definition 3.2.** For any function \( f : \mathbb{N}_0 \to \mathbb{R}^+_0 \), define the operator \( \hat{f} \in \mathcal{L}(L^2(\mathbb{R}^{3N})) \) by
\[
\hat{f} := \sum_{k=0}^{N} f(k) P_k
\]

and, for any \( d \in \mathbb{Z} \), the shifted operator \( \tilde{f}_d \in \mathcal{L}(L^2(\mathbb{R}^{3N})) \) by
\[
\tilde{f}_d := \sum_{j=-d}^{N-d} f(j + d) P_j.
\]

We will in particular need the weight function \( n \) defined by \( n(k) := \sqrt{\frac{k}{N}} \).

We will exclusively use the symbol \( \tilde{\cdot} \) to denote such weighted many-body operators. Besides, we will in the following use the abbreviations
\[
\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^{3N})}, \quad \| \cdot \| := \| \cdot \|_{L^2(\mathbb{R}^{3N})} \quad \text{and} \quad \| \cdot \|_{op} := \| \cdot \|_{\mathcal{L}(L^2(\mathbb{R}^{3N}))}.
\]

**Definition 3.3.** Define the functional \( \alpha_f : L^2(\mathbb{R}^{3N}) \times L^2(\mathbb{R}^3) \to \mathbb{R} \) by
\[
\alpha_f(\psi, \varphi^s(t)) := \left\langle \psi, \tilde{f} \psi \right\rangle = \sum_{k=0}^{N} f(k) \left\langle \psi, P_k \psi \right\rangle.
\]

The \( \varphi^s \)-dependence of \( \alpha_f \) is due to the \( \varphi^s \)-dependence of the projectors \( P_k \). As the operators \( P_k \) project onto states with exactly \( k \) particles outside the condensate, \( \alpha_f \) is a measure of the relative number of such particles in the state \( \psi \). We choose the weight \( f \) increasing and \( f(0) \approx 0 \), hence those parts of \( \psi \) with a larger “distance” to the condensate contribute more to \( \alpha_f(\psi, \varphi^s) \). On the other hand, \( P_0 \psi \) — the state where all particles are condensed into \( \varphi^s \) — contributes hardly anything. The weight \( \tilde{n} \) is in particular distinguished because for any symmetric wavefunction \( \psi \in L^2_+(\mathbb{R}^{3N}) \),
\[
\alpha_{\tilde{n}}(\psi, \varphi^s(t)) = \sum_{k=0}^{N} \frac{k}{N} \left\langle \psi, P_k \psi \right\rangle = \sum_{k=0}^{N} \sum_{j=1}^{N} \frac{1}{N} \left\langle \psi, q_j P_k \psi \right\rangle = \| q_1 \psi \|^2
\]

by Corollary 3.1a.

**Lemma 3.2.** Let \( \psi^N \in L^2_+(\mathbb{R}^{3N}) \) be a sequence of normalised \( N \)-particle wavefunctions and let \( \gamma_N^{(k)} \) be the sequence of corresponding \( k \)-particle reduced density matrices for some fixed \( k \in \mathbb{N} \). Let \( t \in \mathbb{R} \). Then the following statements are equivalent:

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(a) \( \lim_{N \to \infty} \alpha_n^a(\psi^N, \varphi^\varepsilon(t)) = 0 \) for some \( a > 0 \),

(b) \( \lim_{N \to \infty} \alpha_n^a(\psi^N, \varphi^\varepsilon(t)) = 0 \) for any \( a > 0 \),

(c) \( \lim_{N \to \infty} \| \gamma^{(k)}_N - |\varphi^\varepsilon(t)\rangle \langle \varphi^\varepsilon(t)| \|_{L^2} = 0 \) for all \( k \in \mathbb{N} \),

(d) \( \lim_{N \to \infty} \text{Tr}_{L^2(\mathbb{R}^3)} \left( \gamma^{(k)}_N - |\varphi^\varepsilon(t)\rangle \langle \varphi^\varepsilon(t)| \right) = 0 \) for all \( k \in \mathbb{N} \),

(e) \( \lim_{N \to \infty} \text{Tr}_{L^2(\mathbb{R}^3)} \left( \gamma^{(1)}_N - |\varphi^\varepsilon(t)\rangle \langle \varphi^\varepsilon(t)| \right) = 0 \).

For the proof of this lemma, we refer to [18, Lemma 3.1] and to corresponding results in [21, 31]. We will in the following choose the weight function \( m : \mathbb{N}_0 \to \mathbb{R}^+ \) with

\[
m(k) := \begin{cases} n(k) & \text{for } k \geq N^{1-2\varepsilon}, \\ \frac{1}{2} (N^{-1+\xi}k + N^{-\xi}) & \text{else} \end{cases}
\]

for some \( \xi \in (0, \frac{1}{2}) \), i.e. \( m \) equals \( n \) with a smooth cut-off to soften the singularity of \( \frac{dn}{dk} \) for small \( k \). Clearly, \( n(k) \leq m(k) \leq n(k) + \frac{1}{2} N^{-\xi} \) for all \( k \geq 0 \) and \( \xi \in (0, \frac{1}{2}) \), hence \( \alpha_m(\psi, \varphi^\varepsilon(t)) \to 0 \) is equivalent to \( \alpha_n(\psi, \varphi^\varepsilon(t)) \to 0 \) and thus to all cases in Lemma 3.2 for any choice of the parameter \( \xi \). For the actual proof, we will consider a modified version of this functional, namely

\[
\alpha_\varepsilon(t) := \alpha_m(\psi^{N,\varepsilon}(t), \varphi^\varepsilon(t)) + \left| E^{\psi^{N,\varepsilon}(t)}(t) \right| E^{\Phi}(t) \left| t \right|.
\]

The convergence of \( \alpha_\varepsilon(t) \) to zero is equivalent to (12) and (13). Conversely, (10) and (11) imply \( \alpha_\varepsilon(0) \to 0 \) as \( (N, \varepsilon) \to (\infty, 0) \). The main idea of the proof is therefore to derive a bound for \( \frac{d}{dt} \alpha_\varepsilon(t) \) (Propositions 3.4 and 3.5), from which one obtains an estimate for \( \alpha_\varepsilon(t) \) by Grönwall’s inequality. The propositions will be proven in Sections 4.3 and 4.4. The estimate of the rate of the convergence of \( \alpha_\varepsilon(t) \) gained from this procedure translates to a rate for the reduced density matrices:

**Lemma 3.3.** For \( \alpha_\varepsilon(t) \) as in (15), it holds that

\[
\alpha_\varepsilon(t) \leq \frac{1}{2} N^{-\xi}.
\]

**Proof.** Let us abbreviate \( \psi^{N,\varepsilon}(t) \equiv \psi \) and drop all time dependencies. [21, Lemma 2.3] implies

\[
\langle \psi, \hat{N}^2 \psi \rangle \leq \text{Tr} \left| \gamma^{(1)}_\psi - |\varphi^\varepsilon\rangle \langle \varphi^\varepsilon| \right| \leq \sqrt{8 \langle \psi, \hat{N}^2 \psi \rangle}.
\]

The first inequality is thus immediately clear as \( n(k)^2 \leq n(k) \leq m(k) \). For the second inequality, recall that \( m(k) \leq n(k) + \frac{1}{2} N^{-\xi} \), hence

\[
\langle \psi, \hat{n} \psi \rangle \leq \left| \langle \psi, \hat{n} \psi \rangle \right| \leq \sqrt{8 \langle \psi, \hat{n}^2 \psi \rangle} + \frac{1}{2} N^{-\xi} \leq \sqrt{8 \langle \psi, \hat{n}^2 \psi \rangle} + \frac{1}{2} N^{-\xi} \leq \left| \gamma^{(1)}_\psi - |\varphi^\varepsilon\rangle \langle \varphi^\varepsilon| \right| + \frac{1}{2} N^{-\xi}.
\]

\( \square \)
\begin{align}
\| \frac{d}{dt} \alpha(t) \| & \leq |\gamma_a(t)| + |\gamma_b(t)| \\
& \leq |\gamma_a(t)| + |\gamma_b^{(1)}(t)| + |\gamma_b^{(2)}(t)| + |\gamma_b^{(3)}(t)|
\end{align}

for almost every \( t \in \mathbb{R} \), where

\begin{align}
\gamma_a(t) & := \left\| \left\langle \psi^{N,\varepsilon}(t), \hat{V}(t, z_1) \psi^{N,\varepsilon}(t) \right\rangle - \left\langle \Phi(t), \hat{V}(t, (x, 0)) \Phi(t) \right\rangle \right\|_{L^2(\mathbb{R})} \\
& - 2N\Im \left\langle \left\langle \psi^{N,\varepsilon}(t), q_1 \hat{m}^{a}_{-1}(V || (t, z_1) - V || (t, (x_1, 0)))p_1 \psi^{N,\varepsilon}(t) \right\rangle \right\rangle;
\end{align}

\begin{align}
\gamma_b(t) & := -N(N-1)\Im \left\langle \psi^{N,\varepsilon}(t), \bar{Z}_{\beta}^{(12)} \hat{m} \psi^{N,\varepsilon}(t) \right\rangle, \\
\gamma_b^{(1)}(t) & := -2N(N-1)\Im \left\langle \psi^{N,\varepsilon}(t), q_1^{\Phi} \hat{m}^{a}_{-1} p_1^{\chi} p_2 \bar{Z}_{\beta}^{(12)} p_1 p_2 \psi^{N,\varepsilon}(t) \right\rangle, \\
\gamma_b^{(2)}(t) & := -N(N-1)\Im \left\langle \psi^{N,\varepsilon}(t), q_1^{\chi} \psi^{N,\varepsilon}(t), \left( 2p_2 \hat{m}^{a}_{-1} q_2 + q_2 (1 + p_2^{\varepsilon}) \hat{m}^{b}_{-2} \right) w_{\beta}^{(12)} p_1 p_2 \psi^{N,\varepsilon}(t) \right\rangle \\
& - 2N(N-1)\Im \left\langle \psi^{N,\varepsilon}(t), (q_1^{\xi} q_2 + q_1^{\Phi} q_2^{\chi}) \hat{m}^{a}_{-1} w_{\beta}^{(12)} p_1 q_2 \psi^{N,\varepsilon}(t) \right\rangle, \\
& - 2N(N-1)\Im \left\langle \psi^{N,\varepsilon}(t), q_1^{\Phi} q_2^{\chi} \hat{m}^{a}_{-1} q_1^{\chi} q_2^{\chi} w_{\beta}^{(12)} p_1 q_2 \psi^{N,\varepsilon}(t) \right\rangle,
\end{align}

\begin{align}
\gamma_b^{(3)}(t) & := -N(N-1)\Im \left\langle \psi^{N,\varepsilon}(t), q_1^{\Phi} q_2^{\Phi} \hat{m}^{a}_{-1} p_1^{\chi} p_2^{\chi} w_{\beta}^{(12)} p_1 p_2 \psi^{N,\varepsilon}(t) \right\rangle \\
& - 2N(N-1)\Im \left\langle \psi^{N,\varepsilon}(t), q_1^{\Phi} q_2^{\Phi} \hat{m}^{a}_{-1} p_1^{\chi} p_2^{\chi} w_{\beta}^{(12)} q_1 p_2 q_2 \psi^{N,\varepsilon}(t) \right\rangle, \\
& + 2Nb_{\beta} \Im \left\langle \psi^{N,\varepsilon}(t), q_1 q_2 \hat{m}^{a}_{-1} |\Phi(t, x_1)|^2 p_1 q_2 \psi^{N,\varepsilon}(t) \right\rangle.
\end{align}

Here,

\[ w_{\beta}^{(12)} := w_{\beta}(z_1 - z_2) \quad \text{and} \quad Z_{\beta}^{(12)} := w_{\beta}^{(12)} - b_{\beta} \left( |\Phi(t, x_1)|^2 + |\Phi(t, x_2)|^2 \right) \]

and \( \hat{m}^{a} \), \( \hat{m}^{b} \) denote the many-body operators corresponding to the weight functions

\[ m^{a}(k) := m(k) - m(k+1) \quad \text{and} \quad m^{b}(k) := m(k) - m(k+2). \]

The first term, \( \gamma_a \), merely contains one-body contributions, i.e. interactions between the bosons and the external field \( V^{\parallel} \), and is therefore the easiest to estimate. Note that (16) is small only if the system is in a state \( \psi^{N,\varepsilon} \) close to the condensate with condensate wavefunction \( \varphi^{\varepsilon} = \Phi^{\varepsilon} \) (see Lemma 4.6). The term \( \gamma_b \) handles the two-body contributions, i.e. interactions among bosons. The expressions \( \gamma_b^{(1)} \) and \( \gamma_b^{(3)} \) contain the quasi one-dimensional interaction \( \varpi(x_1 - x_2) \) defined by \( p_1^{\varepsilon} p_2^{\varepsilon} w_{\beta}(z_1 - z_2)p_1^{\chi} p_2^{\chi} =: \varpi(x_1 - x_2)p_1^{\varepsilon} p_2^{\varepsilon} \) (see Definition 4.4), where the transverse degrees of freedom are integrated out. These terms are comparable to the corresponding three-dimensional terms in [32]. \( \gamma_b^{(2)} \) has no equivalent in the situation without strong confinement as it collects the remainders that arise upon approximating the three-dimensional interaction \( w_{\beta} \) with the quasi one-dimensional interaction \( \varpi \).

\( \gamma_b^{(1)} \) is physically most relevant because it depends on the difference between the quasi one-dimensional interaction \( \varpi \) and the one-dimensional effective potential \( b_{\beta}|\Phi(t)|^2 \). In other
words, this term is small if and only if (5) is the right effective equation, in particular with the
 correct coupling parameter $b_\beta$. Note that for this term it is crucial that the sequence $(N, \varepsilon)$ is
 moderately confining, i.e. that $\mu/\varepsilon \to 0$.

For $\gamma_b^{(2)}$ to be small, we require in particular the admissibility of the sequence $(N, \varepsilon)$,
 i.e. that $\varepsilon^2/\mu \to 0$. The other key tool for the estimate is the observation that due to the
 strong confinement, it is unlikely that a particle is excited in the transverse directions. This
 implies in particular that $||q_N^N \psi^{N, \varepsilon}(t)|| = O(\varepsilon)$ (Lemma 4.10).

The estimate of $\gamma_b^{(3)}$ relies on a bound for the kinetic energy of the part of $\psi^{N, \varepsilon}(t)$ with
 at least one particle orthogonal to $\Phi(t)$, i.e. a bound for $||\partial_{x_1} q_1^N \psi^{N, \varepsilon}(t)||$ (Lemma 4.17). The proof
 of this bound again involves the splitting of the interaction $w_\beta$ into a quasi one-dimensional
 part $\overline{w}$ and remainders. Hence for $\gamma_b^{(3)}$ to be small, we require both moderate confinement
 and the admissibility of the sequence $(N, \varepsilon)$. The last line (25) is a remainder which is easily
 controlled.

**Proposition 3.5.** Let $\mu$ be sufficiently small. Under assumptions A1 – A4, $\gamma_a$ to $\gamma_b^{(3)}$ from
 Proposition 3.4 are bounded by

$$
|\gamma_a(t)| \lesssim (\alpha \xi(t) + \varepsilon) \epsilon^3(t),
$$

$$
|\gamma_b^{(1)}(t)| \lesssim \left( \frac{\mu}{\varepsilon} + N^{-1} + (\frac{N}{\varepsilon^2})^{-\eta} \right) \epsilon^2(t),
$$

$$
|\gamma_b^{(2)}(t)| \lesssim \left( \frac{\epsilon^2}{\mu} \right)^{\frac{1}{2}} \epsilon^3(t),
$$

$$
|\gamma_b^{(3)}(t)| \lesssim \left( \alpha \xi(t) + \frac{\mu}{\varepsilon} + \left( \frac{\epsilon^2}{\mu} \right)^{\frac{1}{2}} + N^{-\frac{\beta_1}{2}} + N^{-1+\beta_1+\xi} + (\frac{N}{\varepsilon^2})^{-\eta} \right) \epsilon(t) \exp \left\{ \epsilon^2(t) + \int_0^t \epsilon^2(s) \, ds \right\}
$$

for any $\xi \in (0, \frac{\beta_1}{4}]$, any $\beta_1 \in (0, \beta]$ and with $\eta$ from Definition 2.2 and $\epsilon(t)$ as in (8).

The estimate of $\gamma_b^{(1)}$ is essentially the same as in the case $\beta \in (0, \frac{1}{3})$ in [18]. $\gamma_a$ must be treated
 in a different way because the confinement is by a potential and not via Dirichlet boundary
 conditions. For the terms $\gamma_b^{(2)}$ and $\gamma_b^{(3)}$, the argument from [18] does not work because the
 interaction becomes too singular for $\beta > \frac{1}{3}$. To cope with this, we follow an idea from [32]:
 we write the interaction as $w_\beta = \Delta h_\varepsilon$ for some function $h_\varepsilon$ and integrate by parts. $\nabla h_\varepsilon$ is
 less singular, and the expressions resulting from $\nabla$ acting on $\psi^{N, \varepsilon}(t)$ can be controlled with
 Lemma 4.10 (or the refined version, Lemma 4.17).

Our strategy differs from [32] in a relevant point: in [32], the interaction $w_\beta$ is approximated
 by a potential $U_{\beta_1}$ with a softer scaling behaviour ($\beta_1 < \frac{1}{3}$). The author first proves bounds
 for $\beta < \frac{1}{3}$, the second step it to estimate the contribution from the difference $w_\beta - U_{\beta_1}$ using
 integration by parts. Instead of these two steps, we define $h_\varepsilon$ as the solution of $\Delta h_\varepsilon = w_\beta$
 on a ball with Dirichlet boundary conditions and integrate by parts on the ball. To prevent
 the emergence of boundary terms, we use smooth step functions whose derivatives can be
 controlled. This mathematical trick enables us to avoid the separate estimate for $\beta < \frac{1}{3}$.

The control of the kinetic energy (Lemma 4.17) required for the integration by parts in $\gamma_b^{(3)}$
 is also different from the corresponding Lemma 5.2 in [32]. Instead of following that path, we
 extend ideas from [18, Lemma 4.7] and [29, Lemma 4.6] and estimate the part of the kinetic
 energy in the free direction. Besides, we use with Lemma 4.7a a slightly sharpened version of
 [32, Lemma 4.3].
Proof of Theorem 1. From Propositions 3.4 and 3.5, we gather that for sufficiently small $\mu$,

$$\left| \frac{d}{dt} \alpha(t) \right| \lesssim C(t) \left( \alpha(t) + R_{\xi, \beta_1, \eta}(N, \varepsilon) \right)$$

for almost every $t \in \mathbb{R}$, where

$$C(t) := \epsilon(t) \exp \left\{ \epsilon^2(t) + \int_0^t \epsilon^2(s) \, ds \right\},$$

(26)

$$R_{\xi, \beta_1, \eta}(N, \varepsilon) := \frac{\mu}{\varepsilon} + \left( \frac{\varepsilon^2}{\mu} \right)^\frac{1}{2} + N^{-\frac{\alpha}{4}} + N^{-1+\beta_1+\xi} + \left( \frac{N}{\varepsilon^2} \right)^{-\eta}.$$

The differential version of Grönwall’s inequality yields

$$\alpha(t) \leq \left( \alpha(0) + R_{\xi, \beta_1, \eta}(N, \varepsilon) \right) \exp \left\{ 2 \int_0^t C(s) \, ds \right\}$$

for all $t \in \mathbb{R}$. Due to assumption A4 and by Lemma 3.2, $\lim_{(N, \varepsilon) \to (\infty, 0)} \alpha(0) = 0$ and $R_{\xi, \beta_1, \eta}(N, \varepsilon)$ vanishes in the limit $(N, \varepsilon) \to (\infty, 0)$ for $\beta_1 \in (0, \beta]$ and $\xi \in (0, \frac{\beta_1}{4}]$, $\xi < 1 - \beta_1$, because the sequence $(N, \varepsilon)$ is by assumption A4 admissible and moderately confining. Again by Lemma 3.2, this implies (12) and (13) for any $t \in \mathbb{R}$. \( \square \)

Corollary 3.6. Let $t \in \mathbb{R}$. Then

$$\text{Tr} \left| \chi_{N, \varepsilon}^{(1)}(\gamma_{N, \varepsilon}^{(1)}(t) - \frac{\varphi^c(t)}{\varphi^c(t)}) \right| \lesssim \left( A(0) + \frac{\mu}{\varepsilon} + \left( \frac{\varepsilon^2}{\mu} \right)^\frac{1}{2} + N^{-\frac{\beta_1}{4}} + \left( \frac{N}{\varepsilon^2} \right)^{-\eta} \right)^\frac{1}{2} \exp \left\{ \int_0^t C(s) \, ds \right\}$$

for $C(t)$ as in (26) and where

$$A(0) := \left| \mathcal{E}_{N, \varepsilon}^{(1)}(0) - \mathcal{E}_{(0, 0)}^{(1)} \right| + \sqrt{\text{Tr} \left| \chi_{N, \varepsilon}^{(1)}(\gamma_{N, \varepsilon}^{(1)}(0) - \frac{\varphi^c(0)}{\varphi^c(0)}) \right|}.$$

Proof. This follows from Lemma 3.3 after optimisation over $\xi$ and $\beta_1$. \( \square \)

4 Proofs of the propositions

4.1 Preliminaries

In this section, we prove several lemmata which are needed for the proofs of the propositions. The first ones establish several properties of the weighted operators $\hat{f}$, Lemma 4.6 and Lemma 4.7 contain some useful estimates for scalar products, and the remainder of the section covers properties of the condensate wavefunction $\varphi^c(t)$. In the following, we will always assume that assumptions A1 – A4 are satisfied.

Lemma 4.1. Let $f : \mathbb{N}_0 \to \mathbb{R}^+_0$, $d \in \mathbb{Z}$ and define

$$\hat{l} := N \max\{ \hat{m}^a_{-1}, \hat{m}^b_{-2} \},$$

where the max is to be understood in the sense of inequalities between operators, i.e. $\hat{l} = N \hat{m}^a_{-1}$ if $\hat{m}^a_{-1} - \hat{m}^b_{-2}$ is a positive operator and vice versa. Then

(a) $\| \hat{f} \|_{op} = \| \hat{f}_d \|_{op} = \| \hat{f} \|_{op}^2 = \sup_{0 \leq k \leq N} f(k)$,
(b) \( \| l \hat{m} \|_{op} \leq 1, \quad \| \hat{l} \|_{op} \leq N^\xi. \)

**Proof.** Part (a) is obvious. For part (b), note that

\[
\hat{m}^a = \sum_{k=1}^N (m(k) - m(k)) n(k) P_k, \quad \hat{m}^b = \sum_{k=2}^N (m(k) - m(k)) n(k) P_k.
\]

The derivative of \( m \) with respect to \( k \) is given by

\[
m'(k) \equiv \frac{d}{dk} m(k) = \begin{cases} \frac{1}{2^{kN}} = \frac{1}{2} N^{-1} n(k)^{-1} & \text{for } k \geq N^{1-2\xi}, \\ \frac{1}{2} N^{-1+\xi} & \text{else.} \end{cases}
\]

By the mean value theorem, \( |m(k) - m(k-j)| = j|m'(k)| \) for \( j \in \{1, 2\} \) and \( \kappa = (k-j,k) \). For \( \kappa \geq N^{1-2\xi}, |m'(\kappa)| = \frac{1}{2} N^{-1} n(k)^{-1}. \) For \( \kappa < N^{1-2\xi}, \) we obtain \( |m'(\kappa)| = \frac{1}{2} N^{-1+\xi} < \frac{1}{2^\epsilon} = \frac{1}{2} N^{-1} n(k)^{-1}. \) Consequently,

\[
\sum_{k=j}^N |m(k) - m(k)| n(k) P_k \leq \frac{1}{2} N^{-1} j \sum_{k=j}^N \frac{k}{\kappa} P_k \lesssim N^{-1} 1
\]

in the sense of operators. This proves the first part of (b). For the second identity, observe that \( |m'(k)| \leq \frac{1}{2} N^{-1+\xi} \) uniformly in \( k \geq 0. \)

**Lemma 4.2.** Let \( f, g : \mathbb{N}_0 \to \mathbb{R}_0^+ \) be any weights and \( i, j \in \{1, \ldots, N\} \).

(a) For \( k \in \{0, \ldots, N\} \),

\[
\hat{f} \hat{g} = \hat{f} \hat{g} = \hat{g} \hat{f}, \quad \hat{f} p_j = p_j \hat{f}, \quad \hat{f} q_j = q_j \hat{f}, \quad \hat{f} P_k = P_k \hat{f}.
\]

(b) Define \( Q_0 := p_j, \quad Q_1 := q_j, \quad \tilde{Q}_0 := p_i p_j, \quad \tilde{Q}_1 := q_i q_j \) and \( \tilde{Q}_2 := q_i q_j \). Let \( S_j \) be an operator acting only on factor \( j \) in the tensor product and \( T_{ij} \) acting only on \( i \) and \( j \).

Then for \( \mu, \nu \in \{0, 1, 2\} \)

\[
Q_\mu \hat{f} S_j Q_\nu = Q_\mu S_j \hat{f}_{\mu-i} Q_\nu \quad \text{and} \quad \tilde{Q}_\mu \hat{f} T_{ij} \tilde{Q}_\nu = \tilde{Q}_\mu T_{ij} \tilde{f}_{\mu-i} \tilde{Q}_\nu.
\]

(c) Let \( S_{x_j} \) be an operator acting only on the \( x \)-component of factor \( j \). Then

\[
q_j^\Phi \hat{f} S_{x_j} p_j^\Phi = q_j^\Phi S_{x_j} (\hat{f} q_j^\chi + \hat{f}_1 p_j^\chi) p_j^\Phi \quad \text{and} \quad q_j^\Phi \hat{f} S_{x_j} q_j^\Phi = q_j^\Phi S_{x_j} \hat{f} q_j^\Phi.
\]

(d)

\[
[T_{12}, \hat{f}] = [(T_{12}, p_1 p_2 (\hat{f} - \hat{f}_2) + (p_1 q_2 + q_1 p_2) (\hat{f} - \hat{f}_1)].
\]

We will apply parts (b) and (c) to unbounded operators, for instance to \( S_j \equiv \nabla_j \) and \( S_{x_j} \equiv \partial_{x_j} \). In this case, the respective equality holds on the intersection of the domains of the operators on both sides of the equation.
Finally, observe that

\[ Q_\mu P_k S_1 Q_\nu = Q_\mu \left( \sum_{J \subseteq \{2, \ldots, N\}} \prod_{j \in J} q_j \prod_{l \not\in J} p_l \right) S_1 Q_\nu = Q_\mu S_1 \left( \sum_{J \subseteq \{2, \ldots, N\}} \prod_{j \in J} q_j \prod_{l \not\in J} p_l \right) Q_\nu = Q_\mu S_1 P_{k-\mu+\nu} Q_\nu, \]

hence

\[ Q_\mu \hat{f} S_1 Q_\nu = Q_\mu S_1 \left( \sum_{k=-(\mu-\nu)}^{N-(\mu-\nu)} f(k+\mu-\nu) P_k \right) Q_\nu = Q_\mu S_1 \hat{f}_{\mu-\nu} Q_\nu. \]

Assertion (c) is a consequence of part (b) and Corollary 3.1b, for example

\[ q_j^\Phi \hat{f} S_x \hat{p}_j^\Phi = q_j^\Phi \left( q_j^\Phi \hat{f} S_x (p_j + q_j) \right) \hat{p}_j^\Phi = q_j^\Phi \hat{f} S_x (\hat{p}_j^\Phi + \hat{f} q_j^\Phi) \hat{p}_j^\Phi. \]

Finally, observe that

\[ [T_{12}, p_1 p_2 (\hat{f} - \hat{f}_2) + (p_1 q_2 + q_1 p_2) (\hat{f} - \hat{f}_1)] = [T_{12}, \hat{f}] - [T_{12}, q_1 q_2 \hat{f} + (p_1 q_2 + q_1 p_2) \hat{f}_1 + p_1 p_2 \hat{f}_2]. \]

The second commutator equals zero, which can be seen by inserting \( 1 = p_1 p_2 + (p_1 q_2 + q_1 p_2) + q_1 q_2 \) in front of the commutator and applying part (d).

**Lemma 4.3.** Let \( f : \mathbb{N}_0 \to \mathbb{R}_0^+ \). Then

(a) \( P_k, \hat{f} \in C^1(\mathbb{R}, \mathcal{L}(L^2(\mathbb{R}^3))) \) for \( 0 \leq k \leq N \),

(b) \( -\Delta_{y_j} + \frac{1}{\varepsilon^2} V^\perp(\frac{y_j}{\varepsilon}), \hat{f} \) = 0 for \( 1 \leq j \leq N \),

(c) \( \frac{d}{dt} \hat{f} = i \left[ \hat{f}, \sum_{j=1}^{N} h_j(t) \right] \),

where \( h_j(t) \) denotes the one-particle operator corresponding to \( h(t) \) from (5) acting on the \( j \)-th factor in \( L^2(\mathbb{R}^3) \).

**Proof.** Part (a) is clear as \( \varphi^\varepsilon \in C^1(\mathbb{R}, L^2(\mathbb{R}^3)) \). Assertion (b) is due to the fact that \( -\Delta_{y_j} \frac{1}{\varepsilon^2} V^\perp(\frac{y_j}{\varepsilon}) \) commutes with its spectral projection \( p_j^\varepsilon \). For the last part, note that

\[ \frac{d}{dt} p = \frac{d}{dt} |\Phi(t)\chi^\varepsilon \rangle \langle \Phi(t)\chi^\varepsilon | = i \left[ |\Phi(t)\chi^\varepsilon \rangle \langle \Phi(t)\chi^\varepsilon |, h(t) \right] = i[q, h(t)] \quad \text{and} \quad \frac{d}{dt} q = i[q, h(t)] \]

as \( \Phi(t) \) is a solution of (5). \( \square \)

We will consider functions which are symmetric only in the variables of a subset of \( \{1, \ldots, N\} \), for instance the expressions \( q_1 \psi \) and \( w_{12}^{(12)} \psi \) for \( \psi \in L^2_+(\mathbb{R}^3) \).

**Definition 4.1.** Let \( \mathcal{M} \subseteq \{1, \ldots, N\} \). Define \( \mathcal{H}_\mathcal{M} \subseteq L^2(\mathbb{R}^3) \) as the space of functions which are symmetric in all variables in \( \mathcal{M} \), i.e. for \( \psi \in \mathcal{H}_\mathcal{M} \),

\[ \psi(z_1, \ldots, z_j, \ldots, z_k, \ldots, z_N) = \psi(z_1, \ldots, z_k, \ldots, z_j, \ldots, z_N) \quad \forall j, k \in \mathcal{M}. \]
**Lemma 4.4.** Let \( f : \mathbb{N}_0 \to \mathbb{R}_0^+ \) and \( M_1, M_{1,2} \subseteq \{1, 2, \ldots, N\} \) with \( 1 \in M_1 \) and \( 1, 2 \in M_{1,2} \).

(a) \( \hat{n}^2 = \frac{1}{N} \sum_{j=1}^{N} q_j \),

(b) \( \| \hat{f}_1 q_1 \| \leq \frac{N}{|M_1|} \| \hat{f} \hat{n} \psi \|^2 \quad \forall \psi \in \mathcal{H}_{M_1} \),

(c) \( \| \hat{f}_1 q_2 \| \leq \frac{N^2}{|M_{1,2}|(|M_{1,2}| - 1)} \| \hat{f} \hat{n}^2 \psi \|^2 \quad \forall \psi \in \mathcal{H}_{M_{1,2}} \).

**Proof.** Part (a) follows immediately from Corollary 3.1a. Consequently, for \( \psi \in \mathcal{H}_{M_1} \),

\[
\| \hat{f} \hat{n} \psi \|^2 = \frac{1}{N} \sum_{j=1}^{N} \langle \psi, \hat{f}^2 q_j \psi \rangle \geq \frac{1}{N} \sum_{j \in M_1} \langle \psi, \hat{f}^2 q_j \psi \rangle = \frac{|M_1|}{N} \| \hat{f}_1 q_1 \|^2
\]

and analogously for \( \psi \in \mathcal{H}_{M_{1,2}} \),

\[
\| \hat{f} \hat{n}^2 \psi \|^2 \geq \frac{1}{N^2} \sum_{j,k \in M_{1,2}} \langle \psi, \hat{f}^2 q_j q_k \psi \rangle \geq \frac{|M_{1,2}|(|M_{1,2}| - 1)}{N^2} \| \hat{f}_1 q_2 \|^2.
\]

\[\square\]

**Corollary 4.5.** Let \( f : \mathbb{N}_0 \to \mathbb{R}_0^+ \) and \( \mathcal{H}_{M_1}, \mathcal{H}_{M_{1,2}} \) as in Lemma 4.4.

(a) For \( \psi \in \mathcal{H}_{M_1} \),

\[
\| \nabla_1 \hat{f}_1 q_1 \psi \| \lesssim \| \hat{f} \|_{\text{op}} \| \nabla_1 q_1 \psi \| \quad \text{and} \quad \| \partial_{x_1} \hat{f}_1 q_1 \psi \| \lesssim \| \hat{f} \|_{\text{op}} \| \partial_{x_1} q_1 \psi \|.
\]

(b) For \( \psi \in \mathcal{H}_{M_{1,2}} \),

\[
\| \nabla_1 \hat{f}_1 q_2 \psi \| \lesssim \| \hat{f} \hat{n} \|_{\text{op}} \| \nabla_1 q_1 \psi \| \quad \text{and} \quad \| \partial_{x_1} \hat{f}_1 q_2 \psi \| \lesssim \| \hat{f} \hat{n} \|_{\text{op}} \| \partial_{x_1} q_2 \psi \|.
\]

**Proof.** Insertion of \( 1 = p_1 + q_1 \) in front of \( \nabla_1 \psi \) yields with Lemma 4.2b

\[
\| \nabla_1 \hat{f}_1 q_1 \psi \| \leq (\| \hat{f} \|_{\text{op}} + \| \hat{f}_1 \|_{\text{op}}) \| \nabla_1 q_1 \psi \| \lesssim \| \hat{f} \|_{\text{op}} \| \nabla_1 q_1 \psi \|
\]

and

\[
\| \nabla_1 \hat{f}_1 q_2 \psi \| \leq \| \hat{f}_1 q_2 \| \| \nabla_1 q_1 \psi \| + \| \hat{f}_2 \| \| \nabla_1 q_1 \psi \| \lesssim \left( \| \hat{f}_1 \hat{n} \|_{\text{op}} + \| \hat{f} \|_{\text{op}} \right) \| \nabla_1 q_1 \psi \|
\]

by Lemma 4.4b as \( \nabla_1 q_1 \psi \in \mathcal{H}_{\{2, \ldots, N\}} \). As \( n(k) \leq n(k+1) \), \( \| \hat{f}_1 \hat{n} \|_{\text{op}} \leq \| \hat{f} \hat{n} \|_{\text{op}} = \| \hat{f} \hat{n} \|_{\text{op}} \) by Lemma 4.1a. The respective second identities are shown analogously, using that \( q_1 \Phi = q_2 \Phi \)

and that \( \partial_{x_1} q_1 \psi \in \mathcal{H}_{\{2, \ldots, N\}} \).

The next lemma provides an estimate of the difference between expectation values with respect to a symmetric \( N \)-body wavefunction \( \psi \) and with respect to \( \Phi(t) \).

**Lemma 4.6.** Let \( \psi \in L_2^{2N}(\mathbb{R}^{3N}) \) be normalised and \( f \in L_\infty(\mathbb{R}) \). Then

\[
\| \langle \psi, f(x_1) \psi \rangle - \langle \Phi(t), f \Phi(t) \rangle_{L^2(\mathbb{R})} \| \lesssim \| f \|_{L_\infty(\mathbb{R})} \| \langle \psi, \hat{n} \psi \rangle \|.
\]
We drop the time dependence of $\Phi$. Inserting $1 = p_1 + q_1$ on both sides of $f(x_1)$ yields
\[
\|\langle \psi, f(x_1)\rangle \| - \langle \Phi, f\rangle_{L^2(\mathbb{R})} \leq \|\langle \psi, p_1 f(x_1)p_1\rangle \| - \langle \Phi, f\rangle_{L^2(\mathbb{R})}
+ \|\langle q_1 \psi, f(x_1)q_1\rangle\| + 2\|\langle \psi, p_1 f(x_1)\rangle\|.
\]
We estimate the first term as
\[
\|\langle \psi, p_1^{\chi} \rangle \langle \Phi(x_1) \rangle f(x_1) \langle \Phi(x_1) \rangle p_1^{\chi} \rangle \langle \Phi, f\rangle_{L^2(\mathbb{R})} \| \leq \|\langle \Phi, f\rangle_{L^2(\mathbb{R})}\| \|\langle \psi, q_1\rangle\| \leq \|f\|_{L^\infty(\mathbb{R})} \|\langle \psi, q_1\rangle\|
\]
by Lemma 4.4a and as $\tilde{n}^2 \leq \hat{n}$. The second term is bounded by
\[
\|\langle q_1 \psi, f(x_1)q_1\rangle\| \leq \|f\|_{L^\infty(\mathbb{R})} \|q_1\| \|q_1\| \leq \|f\|_{L^\infty(\mathbb{R})} \|\langle \psi, q_1\rangle\|.
\]
For the third term, we compute
\[
\|\langle \psi, p_1^{\chi} \rangle \langle \Phi(x_1) \rangle f(x_1) \langle \Phi(x_1) \rangle p_1^{\chi} \rangle \langle \Phi, f\rangle_{L^2(\mathbb{R})} \| \leq \|f\|_{L^\infty(\mathbb{R})} \|\langle \psi, q_1\rangle\| \leq \|f\|_{L^\infty(\mathbb{R})} \|\langle \psi, q_1\rangle\|,
\]
where we have used that $\sqrt{k+1} \leq \sqrt{k} + 1$, hence $n_1(k) \leq n(k) + N^{-\frac{1}{2}} \leq 2n(k) \leq n(k)$. □

In the following lemma, we estimate two particular scalar products.

**Lemma 4.7.** Let $O_{j,k}$ be an operator that acts nontrivially only on the $j$th and $k$th coordinate and let $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ for $d \in \mathbb{N}$.

(a) Let $\Gamma, \Lambda \in \mathcal{H}_{\mathcal{M}}$ for some $\mathcal{M}$ such that $j \not\in \mathcal{M}$ and $k, l \in \mathcal{M}$. Then
\[
\|\langle \Gamma, O_{j,k} \rangle\| \leq \|\Gamma\| \left|\|O_{j,k} \Lambda, O_{j,l} \Lambda\| + \|\mathcal{M}\|^{-1}\|O_{j,k} \Lambda\|^2\right|^\frac{1}{2}.
\]

(b) Let $r_k, s_k$ and $t_j$ denote operators acting only on the factors $j$ and $k$ of the tensor product, respectively. Then for $j \neq k \neq l \neq j$,
\[
\|\langle r_k F(z_j, z_k) s_k t_j \Gamma, r_l F(z_j, z_l) s_l t_j \Gamma\| \leq \|s_k F(z_j, z_k) r_k t_j \Gamma\|^2.
\]

**Proof.** Using the symmetry of $\Gamma, \Lambda$ in all coordinates contained in $\mathcal{M}$, we find
\[
\|\langle \Gamma, O_{j,k} \rangle\| \leq \|\Gamma\| \left|\sum_{m \in \mathcal{M}} O_{j,m} \Lambda\right|^\frac{1}{2}
\]
\[
\leq \|\Gamma\| \left|\sum_{m \in \mathcal{M}} \sum_{n, m \in \mathcal{M}, n \neq m} \|O_{j,m} \Lambda\|^2\right|^\frac{1}{2}
\]
\[
= \|\Gamma\|^2 \left|\sum_{m \in \mathcal{M}} \|O_{j,m} \Lambda\|\right|^\frac{1}{2}.
\]
For part (b), we use that, for instance, \( r_t \) and \( F(z_j, z_k) \) commute, hence
\[
\| \langle t_j^\Gamma, s_k F(z_j, z_k) r_k r_t F(z_j, z_t) s_t t_j^\Gamma \rangle \| = \| \langle r_t s_k F(z_j, z_t) F(z_j, z_k) s_t r_k t_j^\Gamma \rangle \| \\
= \| \langle r_t s_k F(z_j, z_t) s_t s_k F(z_j, z_k) r_k t_j^\Gamma \rangle \| \\
\leq \| s_k F(z_j, z_k) r_k r_t t_j^\Gamma \| ^2.
\]

The next lemma collects estimates for the time evolved condensate wavefunction.

**Lemma 4.8.** \( H^2(\mathbb{R}) \) solutions of the NLS equation (5) exist globally.

(a) For any fixed time \( t \in \mathbb{R} \),
\[
\| \Phi(t) \| _{L^2(\mathbb{R})} = 1, \quad \| \Phi(t) \| _{H^1(\mathbb{R})} \leq c(t), \\
\| \Phi(t) \| _{L^\infty(\mathbb{R})} \leq c(t), \quad \| \Phi(t) \| _{H^2(\mathbb{R})} \lesssim \exp \left\{ c^2(t) + \int_0^t c^2(s) \, ds \right\}.
\]

(b) For sufficiently small \( \varepsilon \) and fixed \( t \in \mathbb{R} \),
\[
\| \chi^\varepsilon \| _{L^\infty(\mathbb{R}^2)} \lesssim \varepsilon ^{-1}, \quad \| \nabla \chi^\varepsilon \| _{L^\infty(\mathbb{R}^2)} \lesssim \varepsilon ^{-2}, \\
\| \varphi^\varepsilon (t) \| _{L^\infty(\mathbb{R}^3)} \lesssim c(t) \varepsilon ^{-1}, \quad \| \nabla \varphi^\varepsilon (t) \| _{L^\infty(\mathbb{R}^3)} \lesssim c(t) \varepsilon ^{-2}, \\
\| \nabla \varphi^\varepsilon (t) \| ^2 _{L^2(\mathbb{R}^3)} \lesssim c(t) \varepsilon ^{-2}.
\]

**Proof.** For \( \frac{1}{2} < r \leq 4 \) and \( \Phi_0 \in H^r(\mathbb{R}) \), (5) has a unique strong \( H^r(\mathbb{R}) \)-solution \( \Phi \in C(\mathbb{R}; H^r(\mathbb{R})) \) depending continuously on the initial data. The proof of this is sketched in Appendix A. By assumption A4, \( \Phi_0 \in H^2(\mathbb{R}) \) and consequently \( \Phi(t) \in H^2(\mathbb{R}) \). This implies \( \frac{d}{dt} \| \Phi(t) \| _{L^2(\mathbb{R})}^2 = 0 \) and by definition of \( \mathcal{E} \Phi(t) \) and \( c(t) \),
\[
\| \Phi(t) \| _{H^1(\mathbb{R})}^2 \leq \mathcal{E} \Phi(t)(t) + \| V \| (t, \cdot) \| _{L^\infty(\mathbb{R}^3)} \leq c^2(t).
\]

Besides, \( \Phi(t) \in H^2(\mathbb{R}) \subset C^1(\mathbb{R}) \), hence
\[
| \Phi(t, x) | ^2 = \int_{-\infty}^{x} \left( \Phi'(t, \zeta) \Phi(t, \zeta) + \bar{\Phi}(t, \zeta) \Phi'(t, \zeta) \right) d\zeta \\
\leq \int_{-\infty}^{x} \left( | \Phi'(t, \zeta) | ^2 + | \Phi(t, \zeta) | ^2 \right) d\zeta = \| \Phi(t) \| _{H^1(\mathbb{R})}^2 \leq c^2(t),
\]
\[
\| \frac{\partial}{\partial x} \Phi(t) \| _{L^2(\mathbb{R})}^2 \leq 4 \int_{\mathbb{R}} | \Phi'(t, x) | ^2 | \Phi(t, x) | ^2 dx \leq 4 \| \Phi(t) \| _{L^\infty(\mathbb{R})} \| \Phi(t) \| _{H^1(\mathbb{R})} \| \Phi(t) \| _{L^2(\mathbb{R})}^2 \lesssim c^4(t).
\]

For \( \Phi(t) \in H^4(\mathbb{R}) \), we obtain
\[
\frac{d}{dt} \left( 1 + \| \Phi(t) \| _{L^2(\mathbb{R})}^2 \right) = -2 \Im \left\langle \hat{V}(t, \Phi(t), \Phi(t)) \hat{\Phi}(t) , \hat{\Phi}(t) \right\rangle _{L^2(\mathbb{R})} - 2b \| \Phi(t) \| _{L^2(\mathbb{R})}^2 \lesssim \| \Phi(t) \| _{L^\infty(\mathbb{R})} \| \Phi(t) \| _{L^2(\mathbb{R})} \| \Phi(t) \| _{L^2(\mathbb{R})},
\]

hence by Grönwall’s inequality and as \( \| \Phi(t) \| _{L^\infty(\mathbb{R})} \leq c(t) \),
\[
\| \Phi(t) \| _{L^2(\mathbb{R})}^2 \leq \left( 1 + \| \Phi(0) \| _{L^2(\mathbb{R})}^2 \right) \exp \left\{ 2 \int_0^t \left( \| \hat{V}(s, \cdot) \| _{L^\infty(\mathbb{R}^3)} + b \| \Phi(s) \| _{L^2(\mathbb{R})}^2 \right) ds \right\}
\]

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\[ \exp \left\{ 2 \epsilon^2(t) + 2 \int_0^t \epsilon^2(s) \, ds \right\}. \]

This implies a bound for \( \|\Phi(t)\|_{H^2(\mathbb{R})} \) because
\[ \|\dot{\Phi}(t)\|_{L^2(\mathbb{R})} \geq \|\Phi''(t)\|_{L^2(\mathbb{R})} - b_\beta \|\Phi(t)\|_{L^2(\mathbb{R})}^2 - \|V(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \geq \|\Phi''(t)\|_{L^2(\mathbb{R})} - \epsilon^2(t) \]
and consequently
\[ \|\Phi(t)\|_{H^2(\mathbb{R})} \leq \|\Phi''(t)\|_{L^2(\mathbb{R})} + 2 \|\Phi(t)\|_{H^1(\mathbb{R})} \]
\[ \lesssim \epsilon^2(t) + \exp \left\{ \epsilon^2(t) + \int_0^t \epsilon^2(s) \, ds \right\} \lesssim \exp \left\{ \epsilon^2(t) + \int_0^t \epsilon^2(s) \, ds \right\}. \]

By continuity of the solution map, this bound extends to \( \Phi(t) \in H^2(\mathbb{R}) \). If the solution \( \Phi(t) \in H^3(\mathbb{R}) \subset C^2(\mathbb{R}) \), we find further
\[ |\Phi'(x)|^2 = \int_{-\infty}^x \left( \Phi''(\zeta) \Phi'((\zeta) + \Phi''(\zeta) \Phi'(\zeta) \right) \, d\zeta \leq \|\Phi\|_{H^2(\mathbb{R})}^2, \]
which extends to \( \Phi(t) \in H^2(\mathbb{R}) \) by continuity of the solution map. For part (b), recall that \( \chi^\epsilon(y) = \frac{1}{\epsilon} \chi\left(\frac{y}{\epsilon}\right) \), hence \( \|\chi^\epsilon\|_{L^\infty(\mathbb{R}^2)} = \frac{1}{\epsilon} \|\chi\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{\epsilon} \) and analogously \( \|\nabla \chi^\epsilon\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{\epsilon^2} \).

Together with (a), this implies the bounds for \( \|\varphi^\epsilon(t)\|_{L^\infty(\mathbb{R}^3)} \) and \( \|\nabla \varphi^\epsilon(t)\|_{L^\infty(\mathbb{R}^3)} \) as
\[ |\nabla \varphi^\epsilon(t, z)|^2 \leq |\Phi'(t, x)|^2 |\chi^\epsilon(y)|^2 + |\Phi(t, x)|^2 |\nabla \chi^\epsilon(y)|^2 \lesssim \|\Phi(t)\|_{H^2(\mathbb{R})}^2 \epsilon^{-2} + \epsilon^2(t) \epsilon^{-4} \]
for any fixed time \( t \) and \( \epsilon \) small enough. Finally,
\[ \|\nabla \varphi^\epsilon(t)\|_{L^2(\mathbb{R}^3)} = \|\frac{\partial}{\partial x} \Phi(t)\|_{L^2(\mathbb{R}^3)}^2 \int_{\mathbb{R}^2} |\chi^\epsilon(y)|^4 \, dy + \int_{\mathbb{R}} |\Phi(t, x)|^4 \, dx \int_{\mathbb{R}^2} |\nabla_y |\chi^\epsilon(y)|^2 \|^2 \, dy \]
\[ \lesssim \epsilon^4(t) \epsilon^{-2} + 4 \epsilon^2(t) \int_{\mathbb{R}^2} |\nabla_y |\chi^\epsilon(y)|^2 \|^2 \epsilon^2(y) \, dy \lesssim \epsilon^2(t) \epsilon^{-4}. \]

Now we prove some elementary facts enabling us to estimate one- and two-body potentials.

**Lemma 4.9.** Let \( t \in \mathbb{R} \) be fixed and let \( j, k \in \{1, \ldots, N\} \). Let \( g \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \) be a measurable function such that \( |g(z_j, z_k)| \leq G(z_k - z_j) \) almost everywhere for some \( G \colon \mathbb{R}^3 \to \mathbb{R} \).

(a) For \( G \in L^1(\mathbb{R}^3) \),
\[ \|p_j g(z_j, z_k)p_j\|_{op} \lesssim \epsilon^2(t) \epsilon^{-2} \|G\|_{L^1(\mathbb{R}^3)}. \]

(b) For \( G \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \),
\[ \|p_j g(z_j, z_k)p_j\|_{op} = \|p_j g(z_j, z_k)p_j\|_{op} \lesssim \epsilon(t) \epsilon^{-1} \|G\|_{L^2(\mathbb{R}^3)}. \]

(c) For \( G \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \),
\[ \|g(z_j, z_k) \nabla_j p_j\|_{op} = \|\varphi^\epsilon(t, z_j) \langle \nabla \varphi^\epsilon(t, z_j) | g(z_j, z_k) \rangle\|_{op} \lesssim \epsilon(t) \epsilon^{-2} \|G\|_{L^2(\mathbb{R}^3)}. \]
(d) Now let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that $|g(x_j, x_k)| \leq G(x_k - x_j)$ almost everywhere for some $G \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then

\[
\|g(x_j, x_k)p_j^\Phi\|_{op} = \|p_j^\Phi g(x_j, x_k)\|_{op} \leq \epsilon(t)\|G\|_{L^2(\mathbb{R})},
\]

\[
\|g(x_j, x_k)\partial_x p_j^\Phi\|_{op} = \|\langle \Phi(t, x_j)\rangle g(x_j, x_k)\|_{op} \leq \|\Phi\|_{H^2(\mathbb{R})}\|G\|_{L^2(\mathbb{R})}.
\]

Proof. Let $\psi \in L^2(\mathbb{R}^{3N})$ and drop the time dependence of $\varphi^\varepsilon$ and $\Phi$ in the notation. Then

\[
\|p_j g(z_j, z_k)\|_{op} = \|\varphi^\varepsilon(z_j) \langle \varphi^\varepsilon(z_j)\rangle g(z_j, z_k) \varphi^\varepsilon(z_j) \psi\|
\]

\[
\leq \int_{\mathbb{R}^3} |\varphi^\varepsilon(z_j)|^2 |g(z_j, z_k)|\,dz_j \|p_j\|_{op}
\]

\[
\leq \|\varphi^\varepsilon\|^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}^3} |G(z_j - z_k)|\,dz_j \|\psi\|.
\]

The multiplication operators corresponding to $G$ and $g$ as well as $p_j$, $\nabla_j p_j$ and $\partial_x p_j^\Phi$ are bounded. This implies the first equalities in (b) to (d). The second equalities follow from

\[
\|g(z_j, z_k)p_j\|^2_{op} = \sup_{\|\psi\|_1 = 1} \left\langle \psi, p_j g(z_j, z_k)^2 p_j \psi \right\rangle \leq \|p_j g(z_j, z_k)^2 p_j\|_{op} \lesssim \|G\|^2_{L^2(\mathbb{R})} \epsilon^2(t) \varepsilon^{-2},
\]

\[
\|G(x_j)p_j^\Phi\|^2_{op} \leq \|p_j^\Phi G(x_j)^2 p_j^\Phi\|_{op} \lesssim \|G\|^2_{L^2(\mathbb{R})}\|\Phi\|^2_{L^\infty(\mathbb{R})},
\]

\[
\|g(z_j, z_k)\nabla_j p_j\|^2_{op} = \sup_{\|\psi\|_1 = 1} \left\langle \psi, |\varphi^\varepsilon(z_j)| \langle \nabla_j \varphi^\varepsilon(z_j)\rangle g(z_j, z_k)^2 |\nabla_j \varphi^\varepsilon(z_j)\rangle \varphi^\varepsilon(z_j) \psi\right\rangle
\]

\[
\leq \int_{\mathbb{R}^3} |\nabla \varphi^\varepsilon(z_j)|^2 G(z_j - z_j)\,dz_j \|p_j\|^2_{op} \lesssim \|\nabla \varphi^\varepsilon\|^2_{L^\infty(\mathbb{R})}\|G\|^2_{L^2(\mathbb{R})}
\]

and analogously for the second part of (d).

\[
\Box
\]

### 4.2 A priori estimate of the kinetic energy

In this section, we prove estimates for the kinetic energy $\|\nabla_j \psi^{N,\varepsilon}(t)\|$ and related quantities, which follow from the fact that the renormalised energy per particle $E^{N,\varepsilon}(t)$ is bounded by $\epsilon(t)$. Particularly meaningful is assertion (a) of the following lemma: it states that the part of the wavefunction with one particle excited in the confined directions is of order $\varepsilon$. The lemma provides a sufficient estimate for most of the terms in Proposition 3.4. To bound (24), we require a better estimate (see Section 4.5).

**Lemma 4.10.** Let $\varepsilon$ be small enough and $t \in \mathbb{R}$ be fixed. Then

(a) $\|q_1^\varepsilon \psi^{N,\varepsilon}(t)\| \leq \epsilon(t)\varepsilon,$ $\|\hat{q}_1^\varepsilon \psi^{N,\varepsilon}(t)\| \leq \epsilon(t)N^\varepsilon\varepsilon,$

(b) $\|\partial_{x_1} p_1^\Phi\|_{op} \leq \epsilon(t),$ $\|\partial_{x_1}^2 p_1^\Phi\|_{op} \leq \|\Phi(t)\|_{H^2(\mathbb{R})}$ $\|\nabla_{x_1} p_1^\Phi\|_{op} \lesssim \varepsilon^{-1},$ $\|\nabla_{x_1} p_1\|_{op} \lesssim \varepsilon^{-1},$

(c) $\|\partial_{x_1} \hat{q}_1^\Phi \psi^{N,\varepsilon}(t)\| \lesssim \epsilon(t),$ $\|\nabla_{x_1} \hat{q}_1^\Phi \psi^{N,\varepsilon}(t)\| \lesssim \epsilon(t),$ $\|\nabla_{x_1} \hat{q}_1^\Phi \psi^{N,\varepsilon}(t)\| \lesssim N^\varepsilon\varepsilon(t),$ $\|\nabla_{x_1} \hat{q}_1^\Phi \psi^{N,\varepsilon}(t)\| \lesssim \varepsilon^{-1},$

(d) $\|\partial_{x_1} \psi^{N,\varepsilon}(t)\| \leq \epsilon(t),$ $\|\nabla_{x_1} \psi^{N,\varepsilon}(t)\| \lesssim \varepsilon^{-1},$ $\|\nabla_{x_1} \psi^{N,\varepsilon}(t)\| \lesssim \varepsilon^{-1},$

(e) $\|\nabla_{x_1} \hat{q}_1^\Phi \psi^{N,\varepsilon}(t)\| \lesssim \varepsilon^{-1},$ $\|\nabla_{x_1} \hat{q}_1^\Phi \psi^{N,\varepsilon}(t)\| \lesssim \varepsilon(t).$
Proof. Abbreviating $\psi^{N,\varepsilon}(t) \equiv \psi$, we compute

$$E^\psi(t) = \frac{1}{N} \langle \psi, H_\beta(t) \psi \rangle - \frac{E_0}{\varepsilon^2}$$

$$= \left\langle \psi, \frac{1}{N} \left( \sum_{j=1}^{N} \left( -\partial^2_{x_j} + (-\Delta_{y_j} + \frac{1}{\varepsilon^2} V^\perp(\frac{y_j}{\varepsilon}) - \frac{E_0}{\varepsilon^2}) + V(t, z_j) \right) + \sum_{i<j} w_\beta(z_i - z_j) \right) \psi \right\rangle$$

$$\geq \|\partial_x \psi\|^2 + \left\langle q_1^{\varepsilon} \psi, \left( -\Delta_{y_1} + \frac{1}{\varepsilon^2} V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2} \right) q_1^{\varepsilon} \psi \right\rangle - \|V(t)\|_{L^\infty(\mathbb{R}^3)}$$

since $w_\beta \in \mathcal{W}_{\beta,\eta}$ is non-negative and $(-\Delta_{y_1} + \frac{1}{\varepsilon^2} V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2}) \chi^\varepsilon(y_1) = 0$. $\frac{E_0}{\varepsilon^2}$ is the smallest eigenvalue of $-\Delta_{y_1} + \frac{1}{\varepsilon^2} V^\perp(\frac{y_1}{\varepsilon})$ and as a consequence of the rescaling by $\varepsilon$, the spectral gap to the next eigenvalue is of order $\varepsilon^{-2}$. Hence

$$\left\langle q_1^{\varepsilon} \psi, \left( -\Delta_{y_1} + \frac{1}{\varepsilon^2} V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2} \right) q_1^{\varepsilon} \psi \right\rangle \geq \frac{1}{\varepsilon^2} \left\langle \psi, q_1^{\varepsilon} \psi \right\rangle,$$

which implies

$$\|\partial_x \psi\|^2 + \frac{1}{\varepsilon^2} \|q_1^{\varepsilon} \psi\|^2 \leq \|V(t)\|_{L^\infty(\mathbb{R}^3)} + |E^\psi(t)| \leq \varepsilon^2(t). \quad (28)$$

Besides, by assumption A2, $\|(V^\perp - E_0)\|_{L^\infty(\mathbb{R}^3)} \lesssim 1$, hence

$$\varepsilon^2(t) \geq \left\langle q_1^{\varepsilon} \psi, \left( -\Delta_{y_1} + \frac{1}{\varepsilon^2} V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2} \right) q_1^{\varepsilon} \psi \right\rangle$$

$$= \|\nabla_y q_1^{\varepsilon} \psi\|^2 + \frac{1}{\varepsilon^2} \left\langle q_1^{\varepsilon} \psi, \left( V^\perp(\frac{y_1}{\varepsilon}) - E_0 \right) q_1^{\varepsilon} \psi \right\rangle - \frac{1}{\varepsilon^2} \left\langle q_1^{\varepsilon} \psi, \left( V^\perp(\frac{y_1}{\varepsilon}) - E_0 \right) q_1^{\varepsilon} \psi \right\rangle$$

$$\geq \|\nabla_y q_1^{\varepsilon} \psi\|^2 - \frac{1}{\varepsilon^2} \|\nabla \chi\|_{L^\infty(\mathbb{R}^3)} \|q_1^{\varepsilon} \psi\|^2 \geq \|\nabla_y q_1^{\varepsilon} \psi\|^2 - \varepsilon^2(t)$$

and consequently $\|\nabla_y q_1^{\varepsilon} \psi\|^2 \lesssim \varepsilon^2(t)$. The remaining inequalities of (a) to (d) follow by Lemma 4.1b, Lemma 4.2b, by using that $q_1^{(0)} = 1 - p_1^{(0)}$ and from $\|\partial_x p_1\|_{op} \leq \|\partial_x p_1\|_{op} \leq \|\Phi(t)\|_{L^2(\mathbb{R})}$ and $\|\nabla y p_1^{(\varepsilon)}\|_{op} \leq \|\nabla \chi\|_{L^2(\mathbb{R}^2)} \lesssim \varepsilon^{-1}$. For the second part of (d), note that

$$\|\nabla_y \psi\| \leq \|\nabla_y q_1^{\varepsilon} \psi\| + \|\nabla_y p_1^{\varepsilon} \psi\| \lesssim \varepsilon(t) + \varepsilon^{-1} \lesssim \varepsilon^{-1}$$

for sufficiently small $\varepsilon$ and fixed $t \in \mathbb{R}$. Assertion (c) is a consequence of parts (a) to (d) and Corollary 4.5, Lemma 4.1 and Lemma 4.4:

$$\|\nabla^2 \Delta_1 p_1^{\varepsilon} q_1^{\Phi} q_2^{\Phi} \psi\|^2 \leq \|\partial_x \nabla q_1^{(\Phi)} q_2^{\Phi} \psi\|^2 + \|\nabla^2 \Delta_1 p_1^{\varepsilon}\|_{op}^2 \|\nabla q_1^{(\Phi)} q_2^{\Phi} \psi\|^2 \lesssim \varepsilon^2(t) + \varepsilon^{-2} \|\nabla \psi\|^2,$$

$$\|\nabla^2 \Delta_1 p_1^{(\Phi)} q_1^{\Phi} q_2^{\Phi} \psi\|^2 \leq \|\partial_x q_1^{\Phi} \psi\|^2 + \|\nabla^2 \Delta_1 q_1^{\Phi} \psi\|_{op}^2 \|q_2^{\Phi} \psi\|^2 \lesssim \varepsilon^2(t).$$

For the last lemma in this section, we make use of Lemma 4.10a to prove an estimate which is crucial for the control of $\gamma_\alpha(t)$.

**Lemma 4.11.** Let $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that $f(t) \in C^1(\mathbb{R}^3)$ and $\nabla y f(t) \in L^\infty(\mathbb{R}^3)$ for any $t \in \mathbb{R}$. Then

(a) $\|f(t, z_1) - f(t, (x_1, 0)) p_1^{\varepsilon} \psi^{N,\varepsilon}(t)\| \leq \varepsilon \|\nabla y f(t)\|_{L^\infty(\mathbb{R}^3)}$,

(b) $\|f(t, z_1) - f(t, (x_1, 0)) \psi^N(t)\| \leq \varepsilon (\varepsilon(t) \|f(t)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla y f(t)\|_{L^\infty(\mathbb{R}^3)})$.  

\[\boxed{}\]
Proof. For the first part, we expand \( f(t, (x_1, \cdot)) \) around \( y = 0 \), which yields

\[
||f(t, z_1) - f(t, (x_1, 0)) p_1^x \psi^{N, \varepsilon}(t)||^2 = \left\| p_1^x \psi^{N, \varepsilon}(t) \right\| ^2 \int_{\mathbb{R}^2} dy_1 |\chi_\varepsilon(y_1)|^2 (f(t, z_1) - f(t, (x_1, 0)))^2 \\
\leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} dy_1 |\chi(\frac{y_1}{\varepsilon})|^2 \left( \int_0^1 ds \nabla_y f(x_1, s y_1) \cdot y_1 \right)^2 \\
\leq \varepsilon^2 \int_{\mathbb{R}^2} dy_1 |y|^2 |\nabla_y f(t)|^2 \|L_{\infty}(\mathbb{R}^3)\| \lesssim \varepsilon^2 \|\nabla_y f(t)||_{L_{\infty}(\mathbb{R}^3)}^2.
\]

The last step follows because \( \chi \) decays exponentially by \([13, \text{Theorem 1}]\) since \( E_0 < \sigma_{\text{ess}}(\Delta_y + V^{-1}) \) (A2). To prove the second part, we insert \( 1 = q_1^x + p_1^x \) and apply Lemma 4.10a. □

4.3 Proof of Proposition 3.4

Let us from now on drop the time dependence of \( \Phi, \varphi^\varepsilon \) and \( \psi^{N, \varepsilon} \) in the notation and further abbreviate \( \psi^{N, \varepsilon} \equiv \psi \). The time derivative of \( \alpha_\varepsilon(t) \) is bounded by

\[
\frac{d}{dt} \alpha_\varepsilon(t) \leq \left| \frac{d}{dt} \langle \psi, \hat{m} \psi \rangle \right| + \left| \frac{d}{dt} \left| E^\psi(t) - E^\Phi(t) \right| \right|. \tag{29}
\]

For the second term in (29), we compute first

\[
\left| \frac{d}{dt} \left( E^\psi(t) - E^\Phi(t) \right) \right| = \left| \left( \psi, \hat{V} \| (t, z_1) \psi \right) - \left( \Phi, \hat{V} \| (t, (x, 0)) \Phi \right) \right|_{L^2(\mathbb{R})}. \tag{30}
\]

By \([23, \text{Theorem 6.17}]\), \( \left| \frac{d}{dt} \left( E^\psi(t) - E^\Phi(t) \right) \right| = \left| \frac{d}{dt} \left( E^\psi(t) - E^\Phi(t) \right) \right| \) for almost every \( t \in \mathbb{R} \) because \( t \mapsto \frac{d}{dt} \left( E^\psi(t) - E^\Phi(t) \right) \) is continuous due to assumption A3. The first term in (29) yields

\[
\frac{d}{dt} \langle \psi, \hat{m} \psi \rangle \stackrel{4.3c}{=} i \left| \left( \psi, \left[ H_\beta(t) - \sum_{j=1}^N h_j(t), \hat{m} \right] \psi \right) \right| \]

\[
\stackrel{4.3b}{=} i N \left\langle \psi, \left( V \| (t, z_1) - V \| (t, (x, 0)), \hat{m} \right) \psi \right\rangle + i \frac{N(N-1)}{2} \left\langle \psi, \left[ Z^{(12)}_\beta, \hat{m} \right] \psi \right\rangle \]

\[
+ i \frac{N(N-1)}{2} \left\langle \psi, \left[ Z^{(12)}_\beta, Q_0(\hat{m} - \hat{m}_1) + Q_1(\hat{m} - \hat{m}_1) \right] \psi \right\rangle, \tag{31}
\]

where \( Q_0 := p_1 p_2, Q_1 := p_1 q_2 + q_1 p_2 \) and \( Q_2 := q_1 q_2 \). To expand (32), we write the commutator explicitly and insert \( 1 = Q_0 + Q_1 + Q_2 \) appropriately before or after \( Z^{(12)}_\beta \). Terms with the same \( Q_\mu \) on both sides cancel as a consequence of Lemma 4.2b. Hence

\[
\frac{\left(32\right)}{N(N-1)} = i \left( \left( Q_0 + Q_2 \right) Z^{(12)}_\beta (\hat{m} - \hat{m}_2) Q_0 - Q_0(\hat{m} - \hat{m}_2) Z^{(12)}_\beta (Q_0 + Q_2) \right) \right| \psi \right\rangle
\]

\[
+ \frac{i}{2} \left\langle \psi, \left( \left( Q_0 + Q_2 \right) Z^{(12)}_\beta (\hat{m} - \hat{m}_1) Q_1 - Q_1(\hat{m} - \hat{m}_1) Z^{(12)}_\beta (Q_0 + Q_2) \right) \right| \psi \right\rangle
\]

\[
= \frac{i}{2} \left\langle \psi, \left( Q_1(\hat{m} - \hat{m}_1) Z^{(12)}_\beta Q_0 + Q_2(\hat{m}_2 - \hat{m}) Z^{(12)}_\beta Q_0 \right) \right| \psi \right\rangle
\]

\[
- \frac{i}{2} \left\langle \psi, \left( Q_0 Z^{(12)}_\beta (\hat{m} - \hat{m}_1) Q_1 + Q_0 Z^{(12)}_\beta (\hat{m}_2 - \hat{m}) Q_2 \right) \right| \psi \right\rangle \]
\[
\begin{align*}
&+ \frac{i}{2} \left\langle \psi, \left( Q_0 Z^{(12)}_\beta (\hat{m} - \hat{m}_1) Q_1 + Q_2 (\hat{m}_{-1} - \hat{m}) Z^{(12)}_\beta Q_1 \right) \psi \rightangle \\
&- \frac{i}{2} \left\langle \psi, \left( Q_1 (\hat{m} - \hat{m}_1) Z^{(12)}_\beta Q_0 + Q_1 Z^{(12)}_\beta (\hat{m}_{-1} - \hat{m}) Q_2 \right) \psi \rightangle \\
&= \Im \left\langle \psi, Q_1 (\hat{m} - \hat{m}_{-1}) Z^{(12)}_\beta Q_0 \psi \rightangle + \Im \left\langle \psi, Q_2 (\hat{m} - \hat{m}_{-2}) Z^{(12)}_\beta Q_0 \psi \rightangle \\
&+ \Im \left\langle \psi, Q_2 (\hat{m} - \hat{m}_{-1}) Z^{(12)}_\beta Q_1 \psi \rightangle .
\end{align*}
\]

To simplify this expression, note that
\[
\hat{m} - \hat{m}_{-1} = \sum_{k=0}^{N} m(k) P_k - \sum_{k=1}^{N} m(k - 1) P_k = \sum_{k=1}^{N} (m(k) - m(k - 1)) P_k + m(0) P_0
\]
and analogously
\[
\hat{m} - \hat{m}_{-2} = -\hat{m}_{-2} + m(0) P_0 + m(1) P_1.
\]
Using that \( Q_1 P_0 = Q_2 P_0 = Q_2 P_1 = 0 \), we obtain consequently
\[
\frac{(32)}{N(N - 1)} = -2 \Im \left\langle \psi, q_1 p_2 \hat{m}^{a}_{-1} Z^{(12)}_{\beta} p_1 p_2 \psi \right\rangle
\]
where we have in (33) and (35) exploited the symmetry of \( \psi \) in coordinates 1 and 2. According to Corollary 3.1c, \( q = q^\epsilon + q^\phi p^\epsilon \), hence
\[
(33) = -2 \Im \left\langle q_1^\epsilon \psi, q_2 p_2 \hat{m}^{a}_{-1} w^{(12)}_{\beta} p_1 p_2 \psi \right\rangle
\]
\[
(34) = - \Im \left\langle q_1^\epsilon \psi, q_2 \hat{m}^{b}_{-2} w^{(12)}_{\beta} p_1 p_2 \psi \right\rangle
\]
and
\[
(35) = - \Im \left\langle q_1^\epsilon \psi, q_2 \hat{m}^{a}_{-1} w^{(12)}_{\beta} p_1 p_2 \psi \right\rangle
\]
In (34), we have used that the contribution of \( |\Phi(x_1)|^2 + |\Phi(x_2)|^2 \) vanishes as \( q_1^\epsilon |\Phi(x_1)|^2 p_1^\epsilon = q_1^\epsilon |\Phi(x_2)|^2 p_1^\epsilon = 0 \). Similarly, we expand (34) and (35) into terms containing \( q^\epsilon \) and terms containing \( p_1^\epsilon p_2^\epsilon w^{(12)}_{\beta} p_1^\epsilon p_2^\epsilon \):
\[
(34) = - \Im \left\langle q_1^\epsilon \psi, q_2 \hat{m}^{b}_{-2} w^{(12)}_{\beta} p_1 p_2 \psi \right\rangle - \Im \left\langle q_2^\epsilon \psi, q_1 p_1^\epsilon \hat{m}^{b}_{-2} w^{(12)}_{\beta} p_1 p_2 \psi \right\rangle
\]
\[
(35) = - \Im \left\langle q_1^\epsilon \psi, q_2 (1 + p_2^\epsilon) \hat{m}^{b}_{-2} w^{(12)}_{\beta} p_1 p_2 \psi \right\rangle
\]
\[
(36) = - \Im \left\langle q_1^\epsilon \psi, q_2 p_2 \hat{m}^{a}_{-1} w^{(12)}_{\beta} p_1 p_2 \psi \right\rangle
\]
and
\[
(37) = - \Im \left\langle q_1^\epsilon \psi, q_2 \hat{m}^{a}_{-1} w^{(12)}_{\beta} p_1 p_2 \psi \right\rangle
\]
Finally, we insert $1 = p_1 + q_1$ on both sides of the commutator in (31) and apply Lemma 4.2b. Analogously to above, we obtain

\begin{align*}
(31) = & \imath N \left< \psi, (p_1 + q_1)(V^\parallel(t, z_1) - V^\parallel(t, (x_1, 0))) \hat{m}(p_1 + q_1)\psi \right> \\
& - \imath N \left< \psi, (p_1 + q_1)\hat{m}(V^\parallel(t, z_1) - V^\parallel(t, (x_1, 0)))(p_1 + q_1)\psi \right> \\
& = - 2N \Im \left< \psi, q_1 \hat{m}_{a-1}^\ast V^\parallel(t, z_1) - V^\parallel(t, (x_1, 0))p_1\psi \right>.
\end{align*}

Collecting and regrouping all terms arising from (29) yields $\gamma_a = (30) + (45), \gamma_b = (32), \gamma_b^{(1)} = N(N - 1)(37), \gamma_b^{(2)} = N(N - 1)\left[\left((36) + (38)\right) + \left((40) + (41)\right) + (42)\right]$ and $\gamma_b^{(3)} = N(N - 1)(39) + (43) + (44)$.

\[
\Gamma(x_1) := N \int_{\mathbb{R}^2} |\chi^\varepsilon(y_1')|^2 dy_1' \left( \int_{\mathbb{R}^3} |\varphi^\varepsilon(z_1'' - z)|^2 w_\beta(z) dz - |\varphi^\varepsilon(z_1'')|^2 \|w_\beta\|_{L^1(\mathbb{R}^3)} \right). \tag{46}
\]
Let us first consider an analogous expression where \(|\varphi^\varepsilon|^2\) is replaced by some \(g \in C_0^\infty(\mathbb{R}^3)\). Expanding \(g(z_1'' - \cdot)\) around \(z_1''\) yields
\[
\int_{\mathbb{R}^3} g(z_1'' - z)w_\beta(z) \, dz = g(z_1'')\|w_\beta\|_{L^1(\mathbb{R}^3)} - \int_{\mathbb{R}^3} dz \int_0^1 \nabla g(z_1'' - sz) \cdot zw_\beta(z) \, ds
\]
\[= g(z_1'')\|w_\beta\|_{L^1(\mathbb{R}^3)} + R(z_1''),\]
where
\[|R(z_1'')| \lesssim \sup_{s \in [0,1]} |\nabla g(z_1'' - sz)| \int_{\mathbb{R}^3} dz |zw_\beta(z)|.
\]
Hence
\[\|R\|_{L^2(\mathbb{R}^3)}^2 \lesssim \varepsilon^4 N^{-2}\mu^2\|\nabla g\|_{L^2(\mathbb{R}^3)}^2
\]
because \(|z| \leq \mu\) for \(z \in \text{supp} \ w_\beta\) and as \(w_\beta \in W_{\beta, \eta}\) implies
\[
\int_{\mathbb{R}^3} w_\beta(z) \, dz \lesssim \varepsilon^2 N^{-1} b_{N,\varepsilon} = \varepsilon^2 N^{-1}(b_{N,\varepsilon} - b_\beta) + \varepsilon^2 N^{-1} b_\beta \lesssim \varepsilon^2 N^{-1}.
\]
Consequently,
\[
\left\| N \int_{\mathbb{R}^2} |x_1^\varepsilon(y_1')|^2 \, dy_1' \left( \int_{\mathbb{R}^3} g(z_1'' - z)w_\beta(z) \, dz - g(z_1'')\|w_\beta\|_{L^1(\mathbb{R}^3)} \right) \right\|_{L^2(\mathbb{R}^3)}^2
\]
\[\leq N^2 \int_{\mathbb{R}} dx_1 \left| \int_{\mathbb{R}^2} dy_1' |x_1^\varepsilon(y_1')|^2 R(z_1'') \right|^2 \leq N^2 \|\varphi^\varepsilon\|_{L^2(\mathbb{R}^2)}^2 \|R\|_{L^2(\mathbb{R}^3)}^2 \lesssim \mu^2 \varepsilon^2 \|\nabla g\|_{L^2(\mathbb{R}^3)}^2,
\]
where we have in the second step used Hölder’s inequality. By density, this bound extends to \(g \in H^1(\mathbb{R}^3)\) and in particular to \(g \equiv |\varphi^\varepsilon|^2\), hence
\[
\|\Gamma\|_{L^2(\mathbb{R})} \lesssim \mu \varepsilon \|\nabla |\varphi^\varepsilon|^2\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^4 \|\varepsilon\|_{C^0(\mathbb{R})}^4
\]
and
\[
|\gamma_{b}^{(1)}| \leq \|\tilde{\eta}^\Phi \varphi\|_{\text{op}} + \left( N^{-1} + \left( \frac{N}{\varepsilon^2} \right)^{-\eta} \right) \varepsilon^2(t) \lesssim \left( \frac{\mu}{\varepsilon} + \left( \frac{N}{\varepsilon^2} \right)^{-\eta} + N^{-1} \right) \varepsilon^2(t).
\]

\[\]

\[4.4.3\ \text{Proof of the bound for } \gamma_{b}^{(2)}(t)\]

Let us first define the functions needed for the integration by parts of the interaction.

\textbf{Definition 4.2.} Define \(h_{\varepsilon}: \mathbb{R}^3 \to \mathbb{R}\) by
\[
h_{\varepsilon}(z) := \begin{cases} \frac{1}{4\pi} \left( \int_{\mathbb{R}^3} \frac{w_\beta(\zeta)}{|z - \zeta|} \, d\zeta - \int_{\mathbb{R}^3} \frac{\varepsilon w_\beta(\zeta)}{|z - \zeta|^{\varepsilon^2 + 1}} \, d\zeta \right) & \text{for } |z| < \varepsilon, \\ 0 & \text{else} \end{cases}
\]
where
\[\zeta^* := \frac{\varepsilon^2}{|z|}\zeta.
\]
We will abbreviate
\[h_{\varepsilon}^{(ij)} := h_{\varepsilon}(z_i - z_j).
\]
Lemma 4.12. Let $\mu \ll \varepsilon$. Then

(a) $h_\varepsilon$ solves the boundary value problem

$$\begin{cases}
\Delta h_\varepsilon(z) = w_\beta(z) & \text{for } z \in B_\varepsilon(0), \\
h_\varepsilon(z) = 0 & \text{for } z \in \partial B_\varepsilon(0),
\end{cases} \quad (49)$$

where $B_\varepsilon(0) := \{ z \in \mathbb{R}^3 : |z| < \varepsilon \}$.

(b) $\|\nabla h_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \lesssim N^{-1}\mu^{-2}\varepsilon^2$, \quad $\|\nabla h_\varepsilon\|_{L^2(\mathbb{R}^3)} \lesssim N^{-1}\mu^{-\frac{3}{2}}\varepsilon^2$.

Proof. Green’s function for the problem (49) is $G(z, \zeta) = \frac{1}{4\pi} \left( \frac{1}{|z-\zeta|} - \frac{\varepsilon}{|z-\zeta|^2} \right)$, hence $h_\varepsilon|_{B_\varepsilon(0)}$ is the unique solution of (49). For part (b), define

$$h^{(1)}(z) := \begin{cases}
\frac{w_\beta(\zeta)}{|z-\zeta|} & \text{for } |z| < \varepsilon, \\
0 & \text{else},
\end{cases} \quad \text{and} \quad h^{(2)}(z) := \begin{cases}
\frac{\varepsilon w_\beta(\zeta)}{|\zeta|} & \text{for } |z| < \varepsilon, \\
0 & \text{else},
\end{cases}$$

hence $h_\varepsilon(z) = \frac{1}{4\pi} \left( h^{(1)}(z) + h^{(2)}(z) \right)$. We estimate $h^{(1)}$ and $h^{(2)}$ separately.

Estimate of $|\nabla h^{(1)}|$. Let $|z| \leq 2\mu$ and substitute $\zeta \mapsto \zeta' := \zeta - z$. As $\text{supp} w_\beta \subseteq B_\mu(0)$, we conclude that $|\zeta'| \leq |\zeta| + |z| \leq 3\mu$ for $\zeta \in \text{supp} w_\beta$ and consequently

$$|\nabla h^{(1)}(z)| \leq \|w_\beta\|_{L^\infty(\mathbb{R}^3)} \int_{|\zeta| \leq \mu} \frac{1}{|z-\zeta|^2} d\zeta \lesssim \left( \frac{\varepsilon}{N} \right)^{-1+3\beta} \int_{|\zeta'| \leq 3\mu} \frac{1}{|\zeta'|^2} d\zeta' \lesssim N^{-1}\varepsilon^2\mu^{-2}.$$  

For $2\mu \leq |z| < \varepsilon$, note that $\zeta \in \text{supp} w_\beta$ implies $|\zeta| \leq \mu \leq \frac{1}{2}|z|$, hence $|z-\zeta| \geq |z| - |\zeta| \geq \frac{1}{2}|z|$ and consequently

$$|\nabla h^{(1)}(z)| \leq \frac{4}{|z|^2} \int_{\mathbb{R}^3} w_\beta(\zeta) d\zeta \lesssim N^{-1}\varepsilon^2|z|^{-2} \lesssim N^{-1}\varepsilon^2\mu^{-2}$$

due to (47). Hence

$$\int_{\mathbb{R}^3} |\nabla h^{(1)}(z)|^2 dz \lesssim \int_{|z| \leq 2\mu} N^{-2}\varepsilon^4\mu^{-4} dz + \int_{2\mu \leq |z| \leq \varepsilon} N^{-2}\varepsilon^4 \frac{1}{|z|^4} dz \lesssim N^{-2}\varepsilon^4\mu^{-1}.$$

Estimate of $|\nabla h^{(2)}|$. $\zeta \in \text{supp} w_\beta$ implies $|\zeta| \leq \mu$, hence $|\zeta^*| = \frac{\varepsilon^2}{|\zeta|} \geq \frac{\varepsilon^2}{\mu}$. For $\mu$ sufficiently small that $\frac{\varepsilon}{\mu} > 2$, we observe $|z| < \varepsilon < \frac{1}{2}\frac{\varepsilon^2}{\mu} \leq \frac{1}{2}|\zeta^*|$ and consequently $|\zeta^*-z| \geq |\zeta^*|-|z| > \frac{1}{2}|\zeta^*| = \frac{1}{2} \frac{\varepsilon^2}{|\zeta|}$. This yields

$$|\nabla h^{(2)}(z)| = \int_{\mathbb{R}^3} \frac{\varepsilon w_\beta(\zeta)}{|\zeta| |\zeta^*-z|^2} d\zeta \lesssim \varepsilon^{-3}\|w_\beta\|_{L^\infty(\mathbb{R}^3)} \int_{|\zeta| \leq \mu} |\zeta| d\zeta \lesssim N^{-1}\varepsilon^{-1}\mu < N^{-1}\varepsilon^2\mu^{-2}$$

and consequently $\int_{\mathbb{R}^3} |\nabla h^{(2)}(z)|^2 dz \lesssim N^{-2}\mu^2\varepsilon < N^{-2}\varepsilon^4\mu^{-1}$. \qed

Besides, we need a smooth step function to prevent contributions from the boundary when integrating by parts over the ball $B_\varepsilon(0)$.  

26
Definition 4.3. Define $\Theta_\varepsilon : \mathbb{R}^3 \to [0, 1]$ by

$$
\Theta_\varepsilon(z) = \begin{cases} 
1 & \text{for } |z| \leq \mu, \\
\theta_\varepsilon(|z|) & \text{for } \mu < |z| < \varepsilon, \\
0 & \text{for } |z| \geq \varepsilon,
\end{cases}
$$

where $\theta_\varepsilon : [\mu, \varepsilon] \to [0, 1]$ is given by

$$
\theta_\varepsilon(x) := \frac{\exp\left(-\frac{x-\mu}{\varepsilon-x}\right)}{\exp\left(-\frac{x-\mu}{\varepsilon-x}\right) + \exp\left(-\frac{x-\mu}{\varepsilon-x}\right)}.
$$

Clearly, $\theta_\varepsilon$ is a smooth, decreasing function with $\theta_\varepsilon(\mu) = 1$ and $\theta_\varepsilon(\varepsilon) = 0$. We will write

$$
\Theta_\varepsilon^{(ij)} := \Theta_\varepsilon(z_i - z_j).
$$

Lemma 4.13. Let $\mu \ll \varepsilon$. Then

(a) $\|\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1$, \hspace{1em} $\|\Theta_\varepsilon\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^{\frac{3}{2}}$,

(b) $\|\nabla \Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \lesssim \varepsilon^{-1}$, \hspace{1em} $\|\nabla \Theta_\varepsilon\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^{\frac{3}{2}}$.

Proof. Part (a) follows immediately from the definition of $\Theta_\varepsilon$. For part (b), observe that $\frac{\partial}{\partial z_\varepsilon} \Theta_\varepsilon(x) \lesssim 2(\varepsilon - \varepsilon^{-1}) = 2\varepsilon^{-1}(1 - \frac{\mu}{\varepsilon^2}) \lesssim -\varepsilon$. \hfill \Box

Corollary 4.14. Let $\mu \ll \varepsilon$ and $j \in \{1, 2\}$. Then

(a) $\|p_j(\nabla_1 h_\varepsilon^{(12)})\|_{\text{op}} = \|(\nabla_1 h_\varepsilon^{(12)})p_j\|_{\text{op}} \lesssim \varepsilon(t)N^{-1}\mu^{-\frac{1}{2}}\varepsilon$,

(b) $\|p_j\Theta_\varepsilon^{(12)}\|_{\text{op}} = \|\Theta_\varepsilon^{(12)}p_j\|_{\text{op}} \lesssim \varepsilon(t)\varepsilon^{\frac{3}{2}}$,

(b) $\|p_j(\nabla_1 \Theta_\varepsilon^{(12)})\|_{\text{op}} = \|(\nabla_1 \Theta_\varepsilon^{(12)})p_j\|_{\text{op}} \lesssim \varepsilon(t)\varepsilon^{-\frac{1}{2}}$,

(b) $\|\Theta_\varepsilon^{(12)}\nabla_j p_j\|_{\text{op}} = \|\nabla \varepsilon(z_j)\|_{\Theta_\varepsilon^{(12)}} \lesssim \varepsilon(t)\varepsilon^{-\frac{1}{2}}$.

Proof. This follows immediately from Lemma 4.9, Lemma 4.12 and Lemma 4.13. \hfill \Box

Estimate of (20). Define $t_2 := 2p_2 + q_2(1 + p_2^{(i)})$. Then we obtain with $\tilde{t}$ from Lemma 4.1

$$
|\langle (20) \rangle| \lesssim \mathcal{N} \left\| \left\langle \tilde{t}q_1^{(i)} \psi, w_\beta^{(12)}p_1p_2 \psi \right\rangle \right\| = \mathcal{N} \left\| \left\langle \left(\tilde{t}q_1^{(i)} \psi, \Theta_\varepsilon^{(12)}w_\beta^{(12)}p_1p_2 \psi \right) \right\| \right.
$$

$$
= \mathcal{N} \int_{\mathbb{R}^{(N-1)}} dz_1 \int_B (\tilde{t}q_1^{(i)} \psi)(z_1, \ldots, z_N) \Theta_\varepsilon(z_1 - z_2) w_\beta(z_1 - z_2)(p_2p_1 \psi)(z_1, \ldots, z_N)
$$

as $\Theta_\varepsilon(z_1 - z_2) = 1$ for $z_1 - z_2 \in \text{supp } w_\beta$ and $\supp \Theta_\varepsilon = B_\varepsilon(0)$. Thus $w_\beta(z_1 - z_2) = \Delta_1 h_\varepsilon(z_1 - z_2)$ on the whole domain of integration in the $dz_1$-integral. Integration by parts in $z_1$ yields

$$
|\langle (20) \rangle| \lesssim \mathcal{N} \left\| \left\langle \tilde{t}q_1^{(i)} \psi, t_2(\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)}) \cdot \nabla_1 p_1p_2 \psi) \right\rangle \right\|
$$

$$
+ \mathcal{N} \left\| \left\langle \tilde{t}q_1^{(i)} \psi, t_2(\nabla_1 \Theta_\varepsilon^{(12)}) \cdot (\nabla_1 h_\varepsilon^{(12)})p_1p_2 \psi \right\rangle \right\|
$$

(51)

(52)
where the boundary terms vanish because $\Theta_\varepsilon(|z|) = 0$ for $|z| = \varepsilon$. We estimate these expressions by application of Lemma 4.7. To this end, we write each term as $\langle \Gamma, O_{1,2} \Lambda \rangle$, where $\Gamma$ and $\Lambda$ are symmetric in the coordinates $\{2, \ldots, N\}$. Hence

$$\left| (51) \right| \lesssim N \left\| \hat{t}_1 q_1^\varepsilon \psi \right\| \left( \left\| t_2 \Theta_\varepsilon^{(12)} (\nabla \chi_\varepsilon^{(12)}) p_2 \cdot \nabla \chi_\varepsilon \right\|^2 + N^{-1} \left\| t_2 \Theta_\varepsilon^{(12)} (\nabla \chi_\varepsilon^{(12)}) p_2 \cdot \nabla \chi_\varepsilon \right\|^2 \right)^{\frac{1}{2}}$$

by Lemma 4.10, Lemma 4.13 and Corollary 4.14. Analogously,

$$\left| (52) \right| \lesssim N \left\| \hat{t}_1 q_1^\varepsilon \psi \right\| \left( \left\| t_2 \Theta_\varepsilon^{(12)} (\nabla \chi_\varepsilon^{(12)}) p_1 \right\|^2 + N^{-1} \left\| t_2 \Theta_\varepsilon^{(12)} (\nabla \chi_\varepsilon^{(12)}) p_1 \right\|^2 \right)^{\frac{1}{2}} \lesssim \varepsilon^3(t) \left( \frac{\varepsilon^2}{\mu} \right)^{\frac{1}{2}} \left( \varepsilon + N^{-1} \right)^{\frac{1}{2}} N^\xi$$

$$\left| (53) \right| \lesssim N \left\| \hat{t}_1 q_1^\varepsilon \psi \right\| \left( \left\| t_2 \Theta_\varepsilon^{(12)} (\nabla \chi_\varepsilon^{(12)}) p_1 \right\|^2 + N^{-1} \left\| t_2 \Theta_\varepsilon^{(12)} (\nabla \chi_\varepsilon^{(12)}) p_1 \right\|^2 \right)^{\frac{1}{2}} \lesssim \varepsilon^3(t) \left( \frac{\varepsilon^2}{\mu} \right)^{\frac{1}{2}} \left( \varepsilon + N^{-1} \right)^{\frac{1}{2}} N^\xi.$$
\[ = N \left| \left( t_{12} \psi, \Theta_{\varepsilon}^{(12)}(\nabla_1 h_{\varepsilon}^{(12)}) \cdot (\widehat{t}_1 p_1 + \widehat{t}_1 q_1) \nabla_1 q_2 \psi \right) \right| \]
\[ = N \left| \left( t_{12} \psi, \Theta_{\varepsilon}^{(12)}(\nabla_1 h_{\varepsilon}^{(12)}) \cdot \nabla_1 p_1 \widehat{t}_1 q_2 \psi \right) \right| \]
\[ \leq N \| t_{12} \psi \| \| \Theta_{\varepsilon} \|_{L^\infty(\mathbb{R}^3)} \| (\nabla_1 h_{\varepsilon}^{(12)}) \cdot \nabla_1 p_1 \|_{\text{op}} \| \widehat{t}_1 q_2 \psi \| \leq \varepsilon^2(t) \left( \frac{\varepsilon^2}{\mu} \right)^{\frac{1}{2}} \]
and analogously
\[ (55) = N \left| \left( t_{12} \psi, (\nabla_1 \Theta_{\varepsilon}^{(12)}) \cdot (\nabla_1 h_{\varepsilon}^{(12)}) p_1 \widehat{t}_1 q_2 \psi \right) \right| \]
\[ \leq N \| t_{12} \psi \| \| \nabla \Theta_{\varepsilon} \|_{L^\infty(\mathbb{R}^3)} \| (\nabla_1 h_{\varepsilon}^{(12)}) p_1 \|_{\text{op}} \| \widehat{t}_1 q_2 \psi \| \leq \varepsilon^2(t) \left( \frac{\varepsilon^2}{\mu} \right)^{\frac{1}{2}}, \]
\[ (56) = N \left| \left( \nabla_1 t_{12} \psi, \Theta_{\varepsilon}^{(12)}(\nabla_1 h_{\varepsilon}^{(12)}) p_1 \widehat{t}_1 q_2 \psi \right) \right| \]
\[ \leq N \left( \| \nabla_1 q_1 \psi p_1 \|_{L^1(\mathbb{R}^3)} \| q_2 \nabla_1 q_1 \psi \| \right) \| \Theta_{\varepsilon} \|_{L^\infty(\mathbb{R}^3)} \| (\nabla_1 h_{\varepsilon}^{(12)}) p_1 \|_{\text{op}} \| \widehat{t}_1 q_2 \psi \| \leq \varepsilon^2(t) \left( \frac{\varepsilon^2}{\mu} \right)^{\frac{1}{2}} \]
by Lemma 4.13, Corollary 4.14a and Lemma 4.10.

**Estimate of (22).** Analogously to before,
\[ \| (22) \| \leq N \left( \| \widehat{t}_1 q_1 \psi, p_1 \psi x^\varepsilon_{\beta} p_2 \psi (12) p_1 q_2 \psi \| \right) \]
\[ \leq N \left( \| \widehat{t}_1 q_1 \psi, p_1 \psi x^\varepsilon_{\beta} (12) \psi \| \right) \]
\[ + N \left( \| \widehat{t}_1 q_1 \psi, (12) p_1 q_2 \psi \| \right) \]
\[ + N \left( \| \nabla_1 \widehat{t}_1 q_1 \psi, p_1 q_2 \psi \| \right) \]
\[ \leq N \left( \| \widehat{t}_1 q_1 \psi, \psi \| \right) \]
\[ + N \| \nabla_1 \widehat{t}_1 q_1 \psi \| \| p_1 \|_{\text{op}} \| \nabla_1 \psi \| \]
\[ \| q_2 \psi \| \leq \varepsilon^2(t) \left( \frac{\varepsilon^2}{\mu} \right)^{\frac{1}{2}} \]
by Lemma 4.10, Lemma 4.13, Corollary 4.14 and Lemma 4.4.

**Proof of the bound for \( \gamma_b^{(3)}(t) \)**

We estimate (25) as
\[ \| (25) \| \leq \left( \| \widehat{t}_1 q_2 \psi, |\Phi(x)| p_1 \psi \| \right) \]
\[ \leq \| \Phi \|_{L^\infty(\mathbb{R}^3)} \| \widehat{t}_1 q_2 \psi \| \| q_2 \psi \| \leq \varepsilon^2(t) \| \psi, \widehat{\psi} \| \]
by Lemma 4.8 and Lemma 4.4c. For (23) and (24), we proceed similarly as in Section 4.4.3 for the quasi one-dimensional interaction \( \overline{w} \) instead of the three-dimensional interaction \( w_\beta \).

**Definition 4.4.** Define
\[ \overline{w}(x) := \int_{\mathbb{R}^2} dy_1 |\chi^\varepsilon(y_1)|^2 \int_{\mathbb{R}^2} dy_2 |\chi^\varepsilon(y_2)|^2 w_\beta(x, y_1 - y_2). \quad (57) \]

Further, for \( \beta_1 \in [0, 1] \), define \( \overline{h}_{\beta_1} : \mathbb{R} \to \mathbb{R} \) by
\[ \overline{h}_{\beta_1}(x) := \begin{cases} \int_{-N^{-\beta_1}}^{N^{-\beta_1}} G(x', x') \overline{w}(x') \, dx' & \text{for } |x| \leq N^{-\beta_1}, \\ 0 & \text{else}, \end{cases} \quad (58) \]
where
\[ G(x', x) := \frac{1}{2} N^{\beta_1} \begin{cases} (x' + N^{-\beta_1}) (x - N^{-\beta_1}) & \text{for } x' < x, \\ (x' - N^{-\beta_1}) (x + N^{-\beta_1}) & \text{for } x' > x. \end{cases} \] (59)

Besides, define
\[ \Theta_{\beta_1}(x) := \begin{cases} 1 & \text{for } |x| \leq \mu, \\ \theta_{\beta_1}(|x|) & \text{for } \mu < |x| < N^{-\beta_1}, \\ 0 & \text{for } |x| \geq N^{-\beta_1}, \end{cases} \] (60)

where \( \theta_{\beta_1} : [\mu, N^{-\beta_1}] \to [0, 1] \) is a smooth decreasing function with \( \theta_{\beta_1}(\mu) = 1, \theta_{\beta_1}(N^{-\beta_1}) = 0 \) analogously to (50). As before, we will write
\[ \omega^{(ij)} := \omega(x_i - x_j), \quad \Theta_{\beta_1}^{(ij)} := \Theta_{\beta_1}(x_i - x_j), \quad \Theta_{\beta_1}^{(ij)} := \Theta_{\beta_1}(x_i - x_j). \]

**Lemma 4.15.** (a) \( \Theta_{\beta_1} \) solves the boundary-value problem
\[ \begin{cases} \frac{\partial^2}{\partial x^2} \Theta_{\beta_1} = \omega & \text{for } x \in [-N^{-\beta_1}, N^{-\beta_1}], \\ \Theta_{\beta_1} = 0 & \text{for } |x| = N^{-\beta_1}. \end{cases} \] (61)

(b) \( \| \frac{\partial}{\partial x} \Theta_{\beta_1} \|_{L^\infty(\mathbb{R})} \lesssim N^{-1} \), \( \| \frac{\partial}{\partial x} \Theta_{\beta_1} \|_{L^2(\mathbb{R})} \lesssim N^{-1 - \frac{\beta_1}{2}} \).

(c) \( \| \Theta_{\beta_1} \|_{L^\infty(\mathbb{R})} \lesssim 1 \), \( \| \Theta_{\beta_1} \|_{L^2(\mathbb{R})} \lesssim N^{-\frac{\beta_1}{2}} \), \( \| \frac{\partial}{\partial x} \Theta_{\beta_1} \|_{L^\infty(\mathbb{R})} \lesssim N^{\beta_1} \), \( \| \frac{\partial}{\partial x} \Theta_{\beta_1} \|_{L^2(\mathbb{R})} \lesssim N^{\frac{\beta_1}{2}} \).

**Proof.** Part (a) is evident as \( G(x', x) \) is Green’s function for the problem (61). For part (b), we compute for \( x \in [-N^{-\beta_1}, N^{-\beta_1}] \)
\[ \left| \frac{\partial}{\partial x} \Theta_{\beta_1}(x) \right| = \frac{N^{\beta_1}}{2} \int_{-N^{-\beta_1}}^{x} (x' + N^{-\beta_1}) \omega(x') \, dx' + \int_{x}^{N^{-\beta_1}} (x' - N^{-\beta_1}) \omega(x') \, dx' \lesssim \| \omega \|_{L^1(\mathbb{R})} \lesssim N^{-1} \]

since
\[ \| \omega \|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}^2} dy_1 |\xi(y_1)|^2 \int_{\mathbb{R}^2} dy_2 |\xi(y_2)|^2 w_\beta(x, y_1 - y_2) \]
\[ \leq \| \xi \|_{L^\infty(\mathbb{R}^2)}^2 \int_{\mathbb{R}^2} dy_1 |\xi(y_1)|^2 \| w_\beta \|_{L^1(\mathbb{R}^3)} \lesssim N^{-1} \]
(62)
by (47). The second inequality in (b) follows from this as \( \text{supp } \Theta_{\beta_1} = [-N^{-\beta_1}, N^{-\beta_1}] \). Part (c) is shown analogously to Lemma 4.13.

**Corollary 4.16.** Let \( j \in \{0, 1\} \). Then

(a) \( \| p_j^{\phi} \left( \frac{\partial}{\partial x} \left( \Theta_{\beta_1}^{(12)} \right) \right) \|_{op} \lesssim \varepsilon(t) N^{-1 - \frac{\beta_1}{2}} \), \( \| \left( \frac{\partial}{\partial x} \Theta_{\beta_1}^{(12)} \right) (\partial_{x_j} p_j^{\phi}) \|_{op} \lesssim \| \Phi(t) \|_{H^2(\mathbb{R})} N^{-1 - \frac{\beta_1}{2}} \).

(b) \( \| p_j^{\phi} \left( \frac{\partial}{\partial x} \Theta_{\beta_1}^{(12)} \right) \|_{op} \lesssim \varepsilon(t) N^{\frac{\beta_1}{2}} \).

**Proof.** This follows immediately from Lemma 4.9d and Lemma 4.15.
Estimate of (23). Observing that $p_1^x p_2^x w_{12} (12) p_1^x p_2^x = \bar{w}_{12} (12) p_1^x p_2^x$, we obtain analogously to the estimate of (20)

$$\left| (23) \right| \lessapprox N \left\| \left( \hat{q}_1^{\Phi} q_2^{\Phi} \psi, \bar{w}_{12} (12) p_1 p_2 \psi \right) \right\| = N \left\| \left( \hat{q}_1^{\Phi} q_2^{\Phi} \psi, \bar{\Theta}_{12} \left( \frac{d^2}{dx_1^2} \bar{h}_{12} \right) \right) \right\|$$

$$\leq N \left\| \left( \hat{q}_1^{\Phi} q_2^{\Phi} \psi, \bar{\Theta}_{12} \left( \frac{d}{dx_1} \bar{h}_{12} \right) \right) \right\| + N \left\| \left( \hat{q}_1^{\Phi} q_2^{\Phi} \psi, \bar{\Theta}_{12} \left( \frac{d}{dx_1} \bar{h}_{12} \right) \right) \right\|$$

$$\leq N \left\| \left( \hat{q}_1^{\Phi} q_2^{\Phi} \psi, \bar{\Theta}_{12} \left( \frac{d}{dx_1} \bar{h}_{12} \right) \right) \right\| + N \left\| \left( \hat{q}_1^{\Phi} q_2^{\Phi} \psi, \bar{\Theta}_{12} \left( \frac{d}{dx_1} \bar{h}_{12} \right) \right) \right\|.$$

(63)

The boundary terms upon integration by parts vanish as $\bar{\Theta}_{12} (\pm N^{1-\beta_1}) = 0$. With Lemmata 4.1b, 4.4c, 4.10, 4.15 and Corollary 4.16, we conclude

$$\left(63\right) \leq N \left\| \hat{q}_1^{\Phi} q_2^{\Phi} \psi \right\| \left\| \Theta_{12} \right\| (\Psi) \left\| \left( \frac{d}{dx_1} \bar{h}_{12} \right) p_2 \right\| \left\| \partial_x p_2 \right\| \lesssim c^2 (t) \left\| \psi, \bar{n} \psi \right\|^\frac{1}{2} N^{-\frac{\beta_1}{2}},$$

(64)

$$\left(64\right) \leq N \left\| \hat{q}_1^{\Phi} q_2^{\Phi} \psi \right\| \left\| \Theta_{12} \right\| (\Psi) \left\| \left( \frac{d}{dx_1} \bar{h}_{12} \right) p_2 \right\| \left\| \partial_x p_2 \right\| \lesssim c^2 (t) \left\| \psi, \bar{n} \psi \right\|^\frac{1}{2} N^{-\frac{\beta_1}{2}}.$$

(65) Hence

$$\left| (23) \right| \lesssim c^2 (t) \left\| \psi, \bar{n} \psi \right\|^\frac{1}{2} N^{-\frac{\beta_1}{2}} + N^{-1+\beta_1+\epsilon}.$$

Estimate of (24). For this term, we choose $\beta_1 = 0$. Analogously to the estimate of (23),

$$\left| (24) \right| \lesssim N \left\| \left( \hat{q}_1^{\Phi} q_2^{\Phi} \psi, \Theta_{00} \right) \left( \frac{d^2}{dx_1^2} \bar{h}_{00} \right) \right\|$$

$$\leq N \left\| \left( \hat{q}_1^{\Phi} q_2^{\Phi} \psi, \Theta_{00} \right) \left( \frac{d}{dx_1} \bar{h}_{00} \right) \right\| + N \left\| \left( \hat{q}_1^{\Phi} q_2^{\Phi} \psi, \Theta_{00} \right) \left( \frac{d}{dx_1} \bar{h}_{00} \right) \right\|$$

$$\leq N \left\| \left( \hat{q}_1^{\Phi} q_2^{\Phi} \psi, \Theta_{00} \right) \left( \frac{d}{dx_1} \bar{h}_{00} \right) \right\| + N \left\| \left( \hat{q}_1^{\Phi} q_2^{\Phi} \psi, \Theta_{00} \right) \left( \frac{d}{dx_1} \bar{h}_{00} \right) \right\|.$$
The estimate \( \| \partial_x q_1^\Phi \psi \| \lesssim \epsilon(t) \) (Lemma 4.10c) is not sharp enough to see that this expression is small. We need a better control of the kinetic energy, which is established in the following refined energy lemma:

**Lemma 4.17.** Let \( \beta \in (0, 1) \). Then

\[
\| \partial_x q_1^\Phi \psi^{N,\epsilon} (t) \| \lesssim \exp \left\{ c^2(t) + \int_0^t \epsilon^2(s) \, ds \right\} \left( \alpha_\epsilon(t) + \frac{\mu}{\epsilon} + \left( \frac{\epsilon}{\mu} \right)^{\frac{1}{2}} + N^{-\beta} + \left( \frac{N}{\epsilon^2} \right)^{-\eta} \right)^{\frac{1}{2}}.
\]

The proof is given in the next section. As a consequence,

\[
| (24) | \lesssim \epsilon(t) \exp \left\{ c^2(t) + \int_0^t \epsilon^2(s) \, ds \right\} \left( \alpha_\epsilon(t) + \frac{\mu}{\epsilon} + \left( \frac{\epsilon}{\mu} \right)^{\frac{1}{2}} + N^{-\beta} + \left( \frac{N}{\epsilon^2} \right)^{-\eta} \right).
\]

\[ \square \]

### 4.5 Proof of Lemma 4.17.

We prove a refined bound for the kinetic energy. The basic idea of the proof is comparable to Lemma 4.10. However, we estimate the single terms in terms of \( \alpha_\epsilon(t) \) instead of using \( \epsilon^2(t) \). Abbreviating \( \psi^{N,\epsilon}(t) \equiv \psi \) and \( \Phi(t) \equiv \Phi \), we obtain

\[
\alpha_\epsilon(t) \geq E_\psi(t) - E_\Phi(t)
\]

\[
= \| \partial_x \psi \|^2 - \| \Phi \|^2_{L^2(\mathbb{R})} + \left\langle \psi, \left( -\Delta y_1 + \frac{1}{2} V'(\frac{y_1}{\epsilon}) - \frac{E_\Phi}{\epsilon^2} \right) \psi \right\rangle
\]

\[
+ \frac{N-1}{2} \left\langle \psi, w_\beta^{(12)} \psi \right\rangle - \frac{b_\beta}{2} \left\{ \psi, |\Phi(x_1)|^2 \psi \right\}
\]

\[
+ \frac{b_\beta}{2} \left\{ \psi, |\Phi(x_1)|^2 \psi \right\} - \left\langle \psi, \left( (N-1)w_\beta^{(12)} - b_\beta |\Phi(x_1)|^2 \right) \psi \right\} - \left\langle \psi, V(t, (x_1)) \psi \right\rangle - \alpha_\epsilon(t) \geq \frac{\mu}{\epsilon} + \left( \frac{\epsilon}{\mu} \right)^{\frac{1}{2}} + N^{-\beta} + \left( \frac{N}{\epsilon^2} \right)^{-\eta}
\]

as \( \left\langle \psi, \left( -\Delta y_1 + \frac{1}{2} V'(\frac{y_1}{\epsilon}) - \frac{E_\Phi}{\epsilon^2} \right) \psi \right\rangle \geq 0 \). The last step follows by Lemma 4.6, Lemma 4.8 and Lemma 4.11, analogously to Section 4.4.1. Further, using that \( \| \partial_x p_1^\Phi \psi \|^2 = \| \Phi' \|^2_{L^2(\mathbb{R})} \| p_1^\Phi \|^2 = \| \Phi' \|^2_{L^2(\mathbb{R})} (1 - q_{1}^{\Phi} \psi)^2 \), we obtain

\[
\| \partial_x \psi \|^2 = \| \partial_x q_1^\Phi \psi \|^2 + \| \partial_x p_1^\Phi \psi \|^2 + \left\{ \left( \| \partial_x q_1^\Phi \psi, \partial_x p_1^\Phi \psi \right) + c.c. \right\}
\]

\[
\geq \| \partial_x q_1^\Phi \psi \|^2 + \| \Phi' \|^2_{L^2(\mathbb{R})} (1 - q_{1}^{\Phi} \psi)^2 - 2 \left\langle \left( \tilde{n} q_{1}^{\Phi} \psi, \partial_x p_1^\Phi \tilde{n} q_{1}^{\Phi} \psi \right) \right\rangle + \| \Phi \|^2_{L^2(\mathbb{R})},
\]

where we have used that \( \tilde{n} = \tilde{n} \) and Lemma 4.4b. (66) and (67) yield

\[
\| \partial_x q_1^\Phi \psi \|^2 \lesssim \| \Phi \|^2_{H^2(\mathbb{R})} \alpha_\epsilon(t) + \left\langle \left( \psi, \left( b_\beta |\Phi(x_1)|^2 - (N-1)w_\beta^{(12)} \right) \psi \right) \right\rangle + \epsilon^3(t) \epsilon.
\]
We estimate the second term of (68) by inserting $1 = p_1 p_2 + 1 - p_1 p_2$ into both slots of the scalar product:

$$
\left\langle \psi, (p_1 p_2 + 1 - p_1 p_2) \left( b_\beta | \Phi(x_1) |^2 - (N - 1) w^{(12)}_\beta \right) (p_1 p_2 + 1 - p_1 p_2) \psi \right\rangle 
$$

$$
= \left\langle \psi, p_1 p_2 \left( b_\beta | \Phi(x_1) |^2 - N w^{(12)}_\beta \right) p_1 p_2 \psi \right\rangle + \| w^{(12)}_\beta p_1 p_2 \psi \|^2
$$

$$
+ \left\langle \psi, (1 - p_1 p_2) b_\beta | \Phi(x_1) |^2 (1 - p_1 p_2) \psi \right\rangle - (N - 1) \| w^{(12)}_\beta (1 - p_1 p_2) \psi \|^2
$$

$$
+ \left\langle \left\langle \psi, p_1 p_2 b_\beta | \Phi(x_1) |^2 (1 - p_1 p_2) \psi \right\rangle + c.c. \right\rangle
$$

$$
- (N - 1) \left( \left\langle \left\langle \psi, p_1 p_2 w^{(12)}_\beta (1 - p_1 p_2) \psi \right\rangle + c.c. \right\rangle \right).
$$

Making use of $\Gamma(x_1)$ from (46), the first term can be estimated as

$$
(69) = \left\langle \psi, p_1 \Gamma(x_1) p_2 \psi \right\rangle + \left\langle \psi, p_1 p_2 (b_{N, \epsilon} - b_\beta) | \Phi(x_1) |^2 p_1 p_2 \psi \right\rangle + \| w^{(12)}_\beta p_1 \|_{op}^2
$$

$$
\leq c^2(t) \left( \frac{m}{\epsilon} + N^{-1} + \left( \frac{N}{\epsilon} \right)^{-\eta} \right)
$$

by (48) and (47) with $\eta$ from Definition 2.2. Note that at this point, it is crucial that $\beta < 1$.

For the second and third term, note that $1 - p_1 p_2 = q_2 + q_1 p_2$ and $\| w^{(12)}_\beta (1 - p_1 p_2) \|^2 \geq 0$.

Hence

$$
(70) \leq \left\langle \psi, q_2 b_\beta | \Phi(x_1) |^2 q_2 \psi \right\rangle + \left\langle \psi, q_1 p_2 b_\beta | \Phi(x_1) |^2 q_1 p_2 \psi \right\rangle \leq \left\langle \psi, \tilde{n} \psi \right\rangle c^2(t),
$$

$$
(71) \leq 2 \left| \left| \tilde{n}^{\frac{1}{2}} \psi, p_1 p_2 b_\beta | \Phi(x_1) |^2 p_2 q_1 \tilde{n}^{-\frac{1}{2}} \psi \right| \right| \leq c^2(t) \left\langle \psi, \tilde{n} \psi \right\rangle
$$

by Lemma 4.4a and Lemma 4.8. For the last term, observe that $1 - p_1 p_2 = p_1 q_2 + q_1 p_2 + q_1 q_2$, hence, by symmetry of $\psi$,

$$
(72) \leq 2N \left| \left| \left\langle \psi, p_1 q_2 w^{(12)}_\beta p_1 p_2 \psi \right\rangle \right| + N \left| \left| \left\langle \psi, q_1 q_2 w^{(12)}_\beta p_1 p_2 \psi \right\rangle \right| \right|
$$

$$
\leq N \left| \left| \left\langle \tilde{n}^{-\frac{1}{2}} q_2 \psi, p_1 w^{(12)}_\beta p_1 p_2 \tilde{n}^{\frac{1}{2}} \psi \right\rangle \right| \right|
$$

$$
+ N \left| \left| \left\langle q_1 \tilde{\psi}, q_2 (1 + p_2^\epsilon) w^{(12)}_\beta p_1 \psi \right\rangle \right| \right|
$$

$$
+ N \left| \left| \left\langle \psi, q_1 \tilde{\psi}, q_2 \tilde{\psi}, p_1 p_2 w^{(12)}_\beta \psi \right\rangle \right| \right|
$$

(73)

analogously to the decomposition of (35). Using (47), (73) is easily estimated as

$$
(73) \leq c^2(t) \left\langle \psi, \tilde{n} \psi \right\rangle.
$$

For (74), we obtain with $t_2 := q_2 (1 + p_2^\epsilon)$, similarly to the estimate of (20),

$$
(74) \leq N \left| \left| q_1^\epsilon \tilde{\psi}, t_2 \Theta^{(12)}_\epsilon (\nabla_1 h^{(12)}_\epsilon) \cdot \nabla_1 p_1 p_2 \psi \right| \right|
$$

$$
+ N \left| \left| q_1^\epsilon \tilde{\psi}, t_2 (\nabla_1 \Theta^{(12)}_\epsilon \cdot (\nabla_1 h^{(12)}_\epsilon)) p_1 p_2 \psi \right| \right|
$$

$$
\leq N \left| q_1^\epsilon \tilde{\psi} \right| \left( \left| \left| \Theta_\epsilon \right| \right|_{L^\infty(R^3)} \left| \left| \nabla_1 h^{(12)}_\epsilon \right| \right|_{op} + \left| \left| p_1 \left| \left| \nabla_1 h^{(12)}_\epsilon \right| \right|_{op} \left| \left| \nabla \Theta_\epsilon \right| \right|_{L^\infty(R^3)} \right|
$$

$$
+ N \left| \left| \nabla_1 q_1^\epsilon \tilde{\psi} \right| \right| \left| \left| \Theta_\epsilon \right| \right|_{L^\infty(R^3)} \left| \left| p_1 \left| \left| \nabla_1 h^{(12)}_\epsilon \right| \right|_{op} \right| \right| \leq c^2(t) \left( \frac{\epsilon}{\nu} \right)^{\frac{3}{2}}.
$$
(75) is of the same structure as (23). Choosing \( \beta_1 = \beta \), one computes analogously to (63) to (65)

\[
(75) = N \left\| \left( \begin{array}{c}
\n \frac{-1}{2} q_1 \phi_{\psi}, \phi_{\psi}, \phi_{\psi}^{(12)} \left( \frac{d^2}{dx^2} T_{\beta}^{(12)} \right) p_1 p_1 p_2 \phi_{\psi} 
\end{array} \right) \right\|
\]

\[
\leq N \left\| \left( \begin{array}{c}
\n \n \frac{-1}{2} q_1 \phi_{\psi}, \phi_{\psi}, \phi_{\psi}^{(12)} \left( \frac{d^2}{dx^2} T_{\beta}^{(12)} \right) p_1 p_1 p_2 \phi_{\psi} 
\end{array} \right) \right\|
\]

\[
+ N \left\| \left( \begin{array}{c}
\n \n \frac{-1}{2} q_1 \phi_{\psi}, \phi_{\psi}, \phi_{\psi}^{(12)} \left( \frac{d^2}{dx^2} T_{\beta}^{(12)} \right) p_1 p_1 p_2 \phi_{\psi} 
\end{array} \right) \right\|
\]

\[
\leq \epsilon^2 (t) \left( \langle \psi, \hat{\n} \psi \rangle + N^{-\beta} \right),
\]

since \( n_2(k) \lesssim n(k) \) and by Corollary 4.5b and Lemma 4.10c. Besides, we have used that \( N^{-1+\beta} < 1 \) and \( \| \Phi \|_{H^2(\mathbb{R})} N^{-\beta} \lesssim \epsilon^2 (t) \) for sufficiently large \( N \) at fixed time \( t \). Thus,

\[
(72) \lesssim \epsilon^2 (t) \left( \left( \frac{\epsilon^2}{\mu} \right)^{\frac{1}{2}} + N^{-\beta} + \langle \psi, \hat{\n} \psi \rangle \right). \tag{76}
\]

Finally, inserting the bounds for (69) to (72) into (68) yields

\[
\| \partial x_{1} q_1 \phi_{\psi} \|^{2} \lesssim \| \Phi \|_{H^2(\mathbb{R})} \alpha (t) + \epsilon^2 (t) \left( \left( \frac{\epsilon^2}{\mu} \right)^{\frac{1}{2}} + \frac{\mu}{\epsilon} + N^{-\beta} + \frac{1}{N^{\frac{1}{10}}} \right) \exp \left\{ 2 \epsilon^2 (t) + 2 \int_{0}^{t} \epsilon^2 (s) \, ds \right\}
\]

\[
\lesssim \left( \alpha (t) + \frac{\mu}{\epsilon} + \left( \frac{\epsilon^2}{\mu} \right)^{\frac{1}{2}} + N^{-\beta} + \frac{1}{N^{\frac{1}{10}}} \right) \exp \left\{ 2 \epsilon^2 (t) + 2 \int_{0}^{t} \epsilon^2 (s) \, ds \right\}
\]

since \( \epsilon \leq \left( \frac{\epsilon^2}{\mu} \right)^{\frac{1}{2}} \) and \( \epsilon^2 (t) \leq \exp \{ 2 \epsilon^2 (t) \}. \]

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**A Well-posedness of the effective equation**

Let \( \frac{1}{2} < r \leq 4 \) and let the initial datum \( \phi_0 \in H^r (\mathbb{R}) \). Local existence of \( H^r \)-solutions of (5) on the maximal time interval \( t \in [0, T_r] \) follows from the usual contraction argument on the subset \( K := \{ u \in X : \| u \|_X \leq 2R \} \) of the Banach space \( X := C ([0, T]; H^r (\mathbb{R})) \) for some \( R > 0 \) and \( T < T_r \), where one uses that the map \( f : u \mapsto b_\beta |u|^2 u + V (t, \cdot) u \) is locally Lipschitz continuous.

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on $H^r(\mathbb{R})$. To prove global existence, one shows first that $T_s = T_r$ for all $\frac{1}{2} < r, s \leq 4$ and concludes from an estimate of $\|\Phi(t)\|_{H^1(\mathbb{R})}$ that no blow-up can occur [33]. Let $\frac{1}{2} < r < s \leq 4$ and $\Phi_0 \in H^s(\mathbb{R})$. Clearly, $T_s \leq T_r$. Assume now $T_s < T_r$. Then $C_{T_s} := \sup_{t \in [0,T_s]} \|\Phi(t)\|_{H^r(\mathbb{R})} < \infty$. Applying twice the inequality
\[
\|uv\|_{H^s(\mathbb{R})} \leq C (\|u\|_{H^r(\mathbb{R})}\|v\|_{H^s(\mathbb{R})} + \|u\|_{H^r(\mathbb{R})}\|v\|_{H^s(\mathbb{R})})
\]
and using the fact that $H^s(\mathbb{R})$ is an algebra, one concludes that for $t \in [0, T_s]$
\[
\|\Phi(t)\|_{H^s(\mathbb{R})} \leq \|\Phi_0\|_{H^s(\mathbb{R})} + \int_0^t \|f(\Phi(s))\|_{H^s(\mathbb{R})} \, ds
\]
\[
\leq \|\Phi_0\|_{H^s(\mathbb{R})} + C \int_0^t \left( C_{T_s}^2 + \|V(s, \cdot)\|_{H^s(\mathbb{R})} \right) \|\Phi(s)\|_{H^r(\mathbb{R})} \, ds.
\]
Grönwall’s inequality implies that $\|\Phi(t)\|_{H^s(\mathbb{R})}$ cannot blow up at $t = T_s$, which contradicts $[0,T_s)$ being the maximal time interval where $H^s$-solutions exist. Therefore $T_s = T_r =: T_{\text{max}}$. Hence for $\Phi_0 \in H^2(\mathbb{R})$, $\Phi(t) \in H^2(\mathbb{R})$ for $t \in [0, T_{\text{max}})$. Consequently, (27) implies that $\lim_{t \to T_{\text{max}}} \|\Phi(t)\|_{H^1(\mathbb{R})} < \infty$, hence $T_1 = T_{\text{max}} = \infty$.

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