HERMITE-HADAMARD INEQUALITIES FOR UNIFORMLY CONVEX FUNCTIONS AND ITS APPLICATIONS IN MEANS

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Abstract. In this paper, we prove Hermite-Hadamard inequality for uniformly convex, uniformly s-convex functions. Also, we obtain Hermite Hadamard inequality for fractional integral by using these functions. Finally, some applications of these inequalities are given.

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1. INTRODUCTION AND PRELIMINARIES

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}.$$ 

The above inequality is well known in the literature as the Hermite-Hadamard inequality. Recently, the generalizations, improvements, variations and applications for convexity and the Hermite-Hadamard inequality have attracted the attention of many researchers, see [4–8, 11] and the references therein.

The following definitions can be found in [2, 12] and [1].

Definition 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $f$ is called uniformly convex with modulus $\psi : [0, +\infty) \to [0, +\infty]$ if $\psi$ is increasing, $\psi$ vanishes only at 0, and

$$f(tx+(1-t)y) + t(1-t)\psi(|x-y|) \leq tf(x) + (1-t)f(y), \quad (1.1)$$

for each $x, y \in \mathbb{R}$ and $t \in (0, 1)$.

If (1.1) holds with $\psi = \frac{\beta}{2}||x-y||^2$ for some $\beta > 0$, then $f$ is called strongly convex with constant $\beta$.

In the following we give a simple example of a uniformly convex function (see [2], Corollary 2.14).

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Example 1. In view of the following equality,
\[(\alpha x + (1 - \alpha) y)^2 + \alpha(1 - \alpha)(x - y)^2 = \alpha x^2 + (1 - \alpha) y^2,\]
for all \(\alpha \in (0, 1)\) and \(x, y \in \mathbb{R}\), the function \(f(t) = t^2\) for \(t \in \mathbb{R}\) is uniformly convex with modulus \(\psi(t) = t^2\) for all \(t \geq 0\).

In the following proposition, the relation between convex functions and strongly convex functions is expressed. For more details about uniformly and strongly convex functions see [2].

Proposition 1. Let \(f : \mathbb{R} \to \mathbb{R}\) be a function and \(\beta > 0\). Then \(f\) is a strongly convex function with constant \(\beta\) if and only if \(f - \frac{\beta}{2} |.|^2\) is a convex function.

Clearly, strong convexity implies uniformly convexity, uniformly convexity implies strict convexity, and strict convexity implies convexity.

We can define the concept of uniformly \(s\)-convexity as follows:

Definition 2. Let \(f : \mathbb{R} \to \mathbb{R}\) be a function. Then \(f\) is called \(s\)-uniformly convex function with modulus \(\psi : [0, +\infty) \to [0, +\infty]\) if \(\psi\) is increasing, \(\psi\) vanishes only at 0, and
\[f(tx + (1 - t)y) + t^s(1 - t)\psi(|x - y|) \leq t^s f(x) + (1 - t)^s f(y),\]
for each \(x, y \in \mathbb{R}\), \(t \in (0, 1)\) and \(s \in (0, 1)\).

If Definition (1.2) holds with \(\psi = \frac{\beta}{2} |.|^2\) for some \(\beta > 0\), then \(f\) is called strongly \(s\)-convex with constant \(\beta\).

Definition 3. Let \(f \in L[a, b]\). The left-sided and right-sided Riemann-Liouville fractional integrals \(J^a_\alpha f\) and \(J^b_\alpha f\) of order \(\alpha > 0\) with \(a \geq 0\) are defined by
\[J^a_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) \, dt \quad \text{with} \quad x > a\]
\[J^b_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t) \, dt \quad \text{with} \quad x < b\]
respectively, where \(\Gamma(\alpha)\) is the Gamma function and its definition is
\[\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} \, dt.\]

It is to be noted that \(J^a_0 f(x) = J^0_b f(x) = f(x)\). In the case of \(\alpha = 1\), the fractional integral reduces to the classical integral.

In [12], M. Z. Sarikaya et al. presented the following Hermite-Hadamard’s inequalities for fractional integrals.
**Theorem 1** ([12]). Let $f : I \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a,b]$. If $f$ is a convex function on $[a,b]$, then the following inequality for fractional integrals holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_a^\alpha f(b) + J_b^\alpha f(a)\right] \leq \frac{f(a) + f(b)}{2}.$$

**2. MAIN RESULTS**

In this section, we shall state our main results. At the first, we obtain Hermite-Hadamard type inequalities for the class of uniformly convex, uniformly $s$-convex and strongly convex functions.

**Theorem 2.** Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly convex function. Then, the following inequality holds:

$$f\left(\frac{a+b}{2}\right) + \frac{1}{8(b-a)} \int_{a-b}^{b-a} \psi(|t|)dt \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2} - \frac{1}{6} \psi(|a-b|).$$

**Proof.** In (1.1), set $t = \frac{1}{2}$, then one has

$$f\left(\frac{x+y}{2}\right) + \frac{1}{4} \psi(|x-y|) \leq \frac{f(x) + f(y)}{2}. \quad (2.1)$$

Now in (2.1), set $x = ta + (1-t)b$ and $y = (1-t)a + tb$, and integrate inequality (2.1) on $[0,1]$ with respect to $t$. We conclude

$$f\left(\frac{a+b}{2}\right) + \frac{1}{4} \int_0^1 \psi(|(2t-1)(a-b)|)dt$$

$$\leq \frac{1}{2} \int_0^1 f(ta + (1-t)b)dt + \frac{1}{2} \int_0^1 f((1-t)a + tb)dt.$$

Also, the following equalities hold

$$\frac{1}{4} \int_0^1 \psi(|(2t-1)(a-b)|)dt = \frac{1}{4} \int_{b-a}^{a-b} \psi(|u|) \frac{du}{2(a-b)} = \frac{1}{8(b-a)} \int_{a-b}^{b-a} \psi(|t|)dt$$

and

$$\int_0^1 f((1-t)a + tb)dt = \int_0^1 f(ta + (1-t)b)dt = \frac{1}{b-a} \int_a^b f(t)dt.$$

Therefore,

$$f\left(\frac{a+b}{2}\right) + \frac{1}{8(b-a)} \int_{a-b}^{b-a} \psi(|t|)dt \leq \frac{1}{b-a} \int_a^b f(t)dt.$$
On the other hand, in (1.1) put \( x = a, y = b \) and integrate on \([0,1]\) with respect to \( t \). Hence
\[
\int_0^1 f(ta + (1-t)b)dt + \int_0^1 t(1-t)\psi(|a-b|)dt \leq \int_0^1 \frac{f(a) + f(b)}{2} dt,
\]
and so
\[
\frac{1}{b-a} \int_a^b f(t)dt + \psi(|a-b|) \frac{\Gamma(2)^2}{\Gamma(4)} \leq \frac{f(a) + f(b)}{2}.
\]
Therefore,
\[
\frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2} - \frac{1}{6} \psi(|a-b|),
\]
which completes the proof. It is worth noting that we used the following fact:
\[
\int_0^1 t(1-t)dt = B(2,2) = \frac{\Gamma(2)^2}{\Gamma(4)} = \frac{1}{6},
\]
where
\[
B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > 0, \ y > 0, \\
B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\]

In order to prove the main theorems, we need the following lemma that has been proved in [3].

**Lemma 1.** Let \( f : I^o \rightarrow \mathbb{R} \) be a differentiable function on \( I^o \), \( a, b \in I^o \) with \( a < b \). If \( f' \in L[a,b] \), then the following equality holds:
\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt = \frac{b-a}{2} \int_0^1 (1-2t)f'(ta + (1-t)b)dt.
\]

**Theorem 3.** Let \( f : I^o \rightarrow \mathbb{R} \) be a differentiable function on \( I^o \), \( a, b \in I^o \) with \( a < b \). If \( |f'| \) is uniformly convex function on \( I^o \), then the following inequality holds:
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|) - \frac{b-a}{32} \psi(|a-b|).
\]

**Proof.** In view of Lemma 1 and uniformly convexity of \( |f'| \), one has
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{2} \int_0^1 |(1-2t)||f'(ta + (1-t)b)|dt
\]
\[
\leq \frac{b-a}{2} \int_0^1 |1-2t||f'(a)| + (1-t)|f'(b)| + t(1-t)\psi(|a-b|)dt
\]
\[
\leq \frac{b-a}{2} \int_0^1 |1-2t||f'(a)|dt + \int_0^1 |1-2t||(1-t)|f'(b)|dt
\]
\[ + \int_0^1 |1 - 2t(r(t - 1))\psi(|a - b|)|dt \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|) - \frac{b-a}{32} \psi(|a - b|), \]

which completes the proof. Also, note that
\[ \int_0^1 |1 - 2t(r(t - 1))\psi(|a - b|)|dt = 1 - 4, \]
\[ \int_0^1 |1 - 2t(r(t - 1))\psi(|a - b|)|dt = -\frac{1}{16} \psi(|a - b|). \]

\[ \square \]

**Theorem 4.** Let \( f : I' \to \mathbb{R} \) be a differentiable mapping on \( I' \), \( a, b \in I' \) with \( a < b \) and \( p > 1 \). If \( |f'|^q \) is uniformly convex on \( I' \), then the following inequality holds:
\[ \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{2(p+1)^\frac{1}{p}} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} - \frac{1}{6} \psi(|a - b|) \right)^\frac{1}{q}, \]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** By Lemma 1 and Hölder’s inequality, we conclude
\[ \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{2} \int_0^1 |(1 - 2t)||f'(ta + (1 - t)b)|dt \]
\[ \leq \frac{b-a}{2} \left( \int_0^1 |1 - 2t|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(ta + (1 - t)b)|^q dt \right)^{\frac{1}{q}} \]
\[ \leq \frac{b-a}{2} \left( \frac{1}{(p+1)^{\frac{1}{p}}} \right) \left( \frac{|f(a)|^q}{2} \int_0^1 t dt + |f'(b)|^q \int_0^1 (1 - t) dt + \psi(|a - b|) \int_0^1 |t(t - 1)| dt \right)^{\frac{1}{q}} \]
\[ \leq \frac{b-a}{2} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} - \frac{1}{6} \psi(|a - b|) \right)^{\frac{1}{q}}. \]

Hence, the proof is complete. \( \square \)

**Theorem 5.** Let \( f : \mathbb{R} \to \mathbb{R} \) be strongly convex function. Then
\[ f\left( \frac{a+b}{2} \right) + \frac{\beta}{24} (b-a)^2 \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2} - \frac{\beta}{12} (b-a)^2. \]

**Proof.** From Hermite-Hadamard inequality for convex functions, we have
\[ f\left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}. \] (2.2)

Since from Proposition 1 \( f \) is a strongly convex function, we have \( f - \frac{\beta}{2} \cdot \cdot ^2 \) is convex. Hence in (2.2) replace \( f \) by \( f - \frac{\beta}{2} \cdot \cdot ^2 \) and after some calculations the result is obtained. \( \square \)
Theorem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly s-convex function. Then

$$2^{s-1}f\left(\frac{a+b}{2}\right) + \frac{1}{8(b-a)} \int_{a-b}^{b-a} \psi(|t|)dt \leq \frac{1}{b-a} \int_{a}^{b} f(t)dt$$

$$\leq \frac{f(a) + f(b)}{s+1} - \frac{1}{(s+1)(s+2)} \psi(|a-b|).$$

Proof. In (1.2), set $t = \frac{1}{2}$, then we have

$$f\left(\frac{x+y}{2}\right) + \frac{1}{2^{s+1}} \int_{0}^{1} \psi(|(2t-1)(a-b)|)dt \leq \frac{f(x) + f(y)}{2^{s}}. \quad (2.3)$$

Now, set $x = ta + (1-t)b$ and $y = (1-t)a + tb$ in (2.5) and integrate on $[0, 1]$ with respect to $t$. We get

$$f\left(\frac{a+b}{2}\right) + \frac{1}{2^{s+1}} \int_{0}^{1} \psi(|(2t-1)(a-b)|)dt$$

$$\leq \frac{1}{2^{s}} \int_{0}^{1} f(ta+(1-t)b)dt + \frac{1}{2^{s}} \int_{0}^{1} f((1-t)a+tb)dt.$$ 

Now,

$$\frac{1}{2^{s+1}} \int_{0}^{1} \psi(|(2t-1)(a-b)|)dt = \frac{1}{2^{s+1}} \int_{b-a}^{a-b} \psi(|u|) \frac{du}{2(a-b)}$$

$$= \frac{1}{2^{s+2}(b-a)} \int_{a-b}^{b-a} \psi(|t|)dt.$$ 

Also, we have $\int_{0}^{1} f((1-t)a+tb)dt = \frac{1}{b-a} \int_{a}^{b} f(t)dt$. Therefore

$$f\left(\frac{a+b}{2}\right) + \frac{1}{2^{s+2}(b-a)} \int_{a-b}^{b-a} \psi(|t|)dt \leq \frac{1}{2^{s+1}(b-a)} \int_{a}^{b} f(t)dt.$$ 

On the other hand, in (1.1) put $x = a$, $y = b$ and integrate on $[0, 1]$ with respect to $t$. Then we obtain

$$\int_{0}^{1} f(ta+(1-t)b)dt + \int_{0}^{1} t^{s}(1-t)\psi(|a-b|)dt \leq \int_{0}^{1} t^{s} f(a) + (1-t)^{s} f(b)dt$$

so,

$$\frac{1}{b-a} \int_{a}^{b} f(t)dt + \psi(|a-b|) \Gamma(s+1)\Gamma(2) \Gamma(s+3) \leq \frac{f(a) + f(b)}{s+1},$$

finally,

$$\frac{1}{b-a} \int_{a}^{b} f(t)dt \leq \frac{f(a) + f(b)}{s+1} - \frac{1}{(s+1)(s+2)} \psi(|a-b|),$$

which completes the proof. \qed
Theorem 7. Let \( p \in [2, +\infty) \), then the following inequality holds:
\[
\frac{|a+b|}{2}p + \frac{1}{8(b-a)}2^{1-p} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} \int_{a-b}^{b-a} |t|^p dt \leq \frac{1}{b-a} \int_a^b |t|^p dt
\]
\[
\leq \frac{|a|^p + |b|^p}{2} - \frac{1}{6} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} |a-b|^p.
\]

Proof. According to ([2], Proposition 10.13), since \( |\cdot|^2 \) is uniformly convex with modules of convexity \(|\cdot|^2\). Hence for \( p \in [2, +\infty) \) is uniformly convex with modules of convexity \( \psi \) such that \( \psi \) satisfying
\[
\psi \geq 2^{1-p} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}|\cdot|^p,
\]
Hence, in view of Theorem 2 for function \( f(t) = |t|^p \) and (2.4), one has
\[
\frac{|a+b|}{2}p + \frac{1}{8(b-a)}2^{1-p} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} \int_{a-b}^{b-a} |t|^p dt \\
\leq \frac{|a|^p + |b|^p}{2} - \frac{1}{6} \psi(|a-b|)
\]
\[
\leq \frac{|a|^p + |b|^p}{2} - \frac{1}{6} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} |a-b|^p.
\]
\( \square \)

Proposition 2. Let \( p \) be an even number and let \( a, b \in \mathbb{R} \) with \( 0 < a < b \), then the following inequality holds:
\[
(p+1)(\frac{a+b}{2})^p + \frac{(b-a)^{p+1}}{2^{p+1}(b-a)} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}
\]
\[
\leq \frac{b^{p+1} - a^{p+1}}{b-a}
\]
\[
\leq \left( \frac{a^p + b^p}{2} - \frac{(b-a)^p}{6} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} \right) (p+1).
\]

Proof. The proof is immediate consequence of Theorem 7.\( \square \)

2.1. Hermite-Hadamard’s inequalities for fractional integrals

Theorem 8. Let \( f : [a, b] \to \mathbb{R} \) be a uniformly convex function. Then, for \( \alpha > 0 \) the following inequality for fractional integrals holds:
\[
f\left(\frac{a+b}{2}\right) + \frac{\Gamma(\alpha+1)}{2^{\alpha+2}(b-a)^\alpha \Gamma(a-b)} \psi(|a-b|) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[ J_a^\alpha f(b) + J_b^\alpha f(a) \right]
\]
we obtain (2.5) by \( t \alpha \) and then integrating the resulting inequality with respect to \( t \) over \([0,1]\), we obtain
\[
\int_0^1 t^{\alpha-1}f(a+b)dt + \frac{1}{4} \int_0^1 t^{\alpha-1}\psi((2t-1)(a-b))dt \\
\leq \frac{1}{2} \int_0^1 t^{\alpha-1}f(ta + (1-t)b)dt + \frac{1}{2} \int_0^1 t^{\alpha-1}f((1-t)a + tb)dt.
\]
Let \( ta + (1-t)b = r \), \((1-t)a + tb = s \) and \((2t-1)(a-b) = x \), then
\[
f\left(\frac{a+b}{2}\right) + \frac{1}{2} \int_{b-a}^{a-b} \left(\frac{x}{2(b-a)}\right)^{\alpha-1}\psi(|x|) \frac{dx}{2(a-b)} \\
\leq \frac{1}{2} \int_b^a \left(\frac{b-r}{b-a}\right)^{\alpha-1}f(r) \frac{dr}{a-b} + \frac{1}{2} \int_a^b \left(\frac{s-a}{b-a}\right)^{\alpha-1}f(s) \frac{ds}{b-a}.
\]
So, we have
\[
\frac{f\left(\frac{a+b}{2}\right)}{\alpha} + \frac{1}{2\alpha+2(b-a)\alpha} \int_{a-b}^{a-b} \frac{1}{b-a}\alpha \psi(|a-b|) \leq \frac{\Gamma(\alpha)}{2(b-a)\alpha} \left[j_{\alpha}^a f(b) + j_{\alpha}^a f(a)\right].
\]
Conversely, since \( f \) is uniformly convex one has
\[
f(tx + (1-t)y) + t(1-t)\psi(|x-y|) \leq tf(x) + (1-t)f(y). \tag{2.6}
\]
Now, replacing \( x \) by \( y \) we have
\[
f(ty + (1-t)x) + t(1-t)\psi(|x-y|) \leq tf(y) + (1-t)f(x). \tag{2.7}
\]
Adding the two equations (2.6) and (2.7) we obtain
\[
f(tx + (1-t)y) + f((1-t)x+ty) + 2t(1-t)\psi(|x-y|) \leq f(x) + f(y). \tag{2.8}
\]
Set \( x = a \) and \( y = b \) in (2.8) and also multiplying both sides of (2.8) by \( t^{\alpha-1} \) and then integrating the resulting inequality with respect to \( t \) over \([0,1] \), we obtain
\[
\int_0^1 t^{\alpha-1}f(ta + (1-t)b)dt + \int_0^1 t^{\alpha-1}f((1-t)a + tb)dt + \int_0^1 2\alpha(1-t)\psi(|a-b|)dt \\
\leq \int_0^1 t^{\alpha-1}f(a)dt + \int_0^1 t^{\alpha-1}f(b)dt.
\]
So,
\[
\frac{\Gamma(\alpha)}{2(b-a)\alpha} \left[j_{\alpha}^a f(b) + j_{\alpha}^a f(a)\right] \leq \frac{f(a) + f(b)}{2\alpha} - \beta(\alpha + 1,2)\psi(|a-b|),
\]
which completes the proof. □

3. APPLICATIONS TO SPECIAL MEANS

Consider the following special means for two nonnegative real numbers \( \alpha, \beta \) with \( \alpha \neq \beta \) as follows (see [3, 5, 9, 10]):

(1) The arithmetic mean:

\[
A = A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R},
\]

with \( \alpha, \beta > 0 \).

(2) The logarithmic mean:

\[
L = L(\alpha, \beta) = \frac{\beta - \alpha}{\ln\beta - \ln\alpha}, \quad \alpha \neq \beta, \alpha, \beta \in \mathbb{R},
\]

with \( \alpha, \beta > 0 \).

(3) The generalized logarithmic mean:

\[
L_n(\alpha, \beta) = \left[ \frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)} \right]^\frac{1}{n}, \quad n \in \mathbb{R} \setminus \{-1, 0\}, \alpha \neq \beta, \alpha, \beta \in \mathbb{R},
\]

with \( \alpha, \beta > 0 \).

Proposition 3. Let \( a, b \in \mathbb{R} \) with \( 0 < a < b \) and let \( p \) be an even number. Then the following inequality holds:

\[
\left( \left( \frac{a+b}{2} \right)^p + \frac{(b-a)^{p+1}}{2p+2(p+1)(b-a)} \min\{p2^{-\frac{p}{2}}, 1 - 2^{-\frac{p}{2}}\} \right)^{\frac{1}{p}} \leq L_p(a, b) \leq \left( \left( \frac{a+b}{2} \right)^p - \frac{(b-a)^p}{6} \min\{p2^{-\frac{p}{2}}, 1 - 2^{-\frac{p}{2}}\} \right)^{\frac{1}{p}}.
\]

Proof. Since the function \( g(t) = t^\frac{1}{p} \) is increasing for \( t \geq 0 \) and \( p > 0 \), in view of Proposition 2, the proof is complete. □

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