Twists of rational Cherednik algebras

Y. Bazlov, A. Berenstein, E. Jones-Healey and A. McGaw

Abstract

We show that braided Cherednik algebras introduced by the first two authors are cocycle twists of rational Cherednik algebras of the imprimitive complex reflection groups $G(m, p, n)$, when $m$ is even. This gives a new construction of mystic reflection groups which have Artin-Schelter regular rings of quantum polynomial invariants. As an application of this result, we show that a braided Cherednik algebra has a finite-dimensional representation if and only if its rational counterpart has one.

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1 Introduction

Cocycle twists of associative (and Lie) algebras have their origin in physics literature. Situations when a parametrised family of isomorphic groups of symmetry have a non-isomorphic group as a limit were formalised as “contractions” in Inonu and Wigner [13]: e.g., the Galilei group of classical mechanics is a limit of relativistic Lorentz groups. Moody and Patera [18] show that for graded Lie algebras, contractions are determined by 2-cocycles on the grading group. See also Vafa and Witten [20] where twists by cocycles of a finite abelian group are held as examples of mirror symmetry.

There are various applications of cocycle twists within mathematics, for example in non-commutative geometry (see Davies [8]) and colour Lie algebras (see Chen, Silvestrov and Oystaeyen [6]). Cocycle twists also find a generalisation in the language of Hopf algebras, leading to a twist originally due to
The Drinfeld twist has been well-studied, see Majid [17], and has also found applications in representation theory, see Giaquinto and Zhang [12] and Jordan [14]. To ascribe physical meaning to twists of an algebra, representations of the algebra also need to be twisted; but this proves to be more difficult, and there is no general approach to this so far.

We see the main result of this work, that rational and braided Cherednik algebras are related via a twist, as a stepping stone towards a better understanding of the representation theory of braided Cherednik algebras [2], for which very little is currently known. In this work we present one result in this direction, showing that finite-dimensional representations of one algebra exist if and only if they do for the twisted partner. We also give an example using one-dimensional representations of the Cherednik algebra, to show that twisting can non-trivially permute the characters of the underlying reflection group.

The contents of this paper are laid out as follows. Sections 2, 3, 4 introduce the main definitions and objects of the paper, with the only new result being Theorem 2.3, where we give a presentation of the mystic reflection groups \( \mu(G(m, p, n)) \). These groups arose independently in the work of Bazlov and Berenstein [2], as the groups over which the braided Cherednik algebras are defined, and of Kirkman, Kuzmanovich and Zhang [16] as a class of groups with Artin-Schelter regular rings of quantum polynomial invariants. Mystic reflection groups were comprehensively studied in [4]. The main result of the paper is found in Section 5, where the braided Cherednik algebra over \( \mu(G(m, p, n)) \) is shown to be the twist of the rational Cherednik algebra over the imprimitive complex reflection group \( G(m, p, n) \) by a cocycle (in fact, a quasitriangular structure) on a finite abelian group. A key step in the proof is to verify that the twist preserves the braid relations between the mystic reflection generators of \( \mu(G(m, p, n)) \) — this fact turns out to be related to the Clifford Braiding Theorem of Kauffman and Lomonaco [15]. Finally in Section 6 we use this twisting construction to obtain examples of non-trivial, and finite-dimensional, representations of a braided Cherednik algebra out of representations of a rational Cherednik algebra.

2 Reflection groups

2.1. Complex reflection groups. Let \( V \) be an \( n \)-dimensional \( \mathbb{C} \)-vector space, with dual space \( V^* \). If \( y \in V^*, x \in V \), we denote the evaluation of \( y \) on \( x \) by \( \langle y, x \rangle \). If \( G \) is a finite subgroup of \( \text{GL}(V) \), then for \( g \in G, x \in V \) we denote the action of \( g \) on \( x \) by \( g(x) \). Via the contragredient representation we have an action of \( G \) on \( V^* \): if \( g \in G, v \in V^* \), then \( \langle g(y), x \rangle = \langle y, g^{-1}(x) \rangle \) \( \forall x \in V \).

A complex reflection on \( V \) is an element \( s \in \text{GL}(V) \) that has finite order and satisfies \( \text{rank}(s - \text{id}) = 1 \). Equivalently, the characteristic polynomial for \( s \) is \( (t - 1)^{n-1}(t - \lambda) \) for some root of unity \( \lambda \neq 1 \). Note that in this case \( s \) acts on \( V^* \) also as a complex reflection with characteristic polynomial \( (t - 1)^{n-1}(t - \lambda^{-1}) \). A complex reflection group on \( V \) is a finite subgroup of \( \text{GL}(V) \) generated by complex reflections on \( V \).

We fix a basis \( \{x_1, \ldots, x_n\} \) for \( V \), and dual basis \( \{y_1, \ldots, y_n\} \) of \( V^* \). This allows us to identify \( \text{GL}(V) \) with the group \( \text{GL}_n(\mathbb{C}) \) of \( n \times n \)-invertible matrices. The groups of permutation matrices and diagonal
matrices are given respectively as
\[ S_n = \{ w \in \text{GL}(V) \mid \forall i \ w(x_i) \in \{ x_1, \ldots, x_n \} \} , \quad T_n = \{ t \in \text{GL}(V) \mid \forall i \ t(x_i) \in \mathbb{C} x_i \} . \]

We see that
\[ S_n \cap T_n = \{ \text{id} \} , \quad S_n \text{ normalises } T_n \text{ within } \text{GL}_n(\mathbb{C}) , \quad \text{so that } S_n \rtimes T_n \subseteq \text{GL}_n(\mathbb{C}) . \]

In particular, \( S_n \) acts on \( T_n \) by conjugation inside \( \text{GL}_n(\mathbb{C}) \); we will write this action as \( w(t) \) for \( w \in S_n \), \( t \in T_n \).

For parameters \( m, p \in \mathbb{N} \) with \( p \mid m \), let \( C_m^{\mathbb{Z}} \subseteq C_m \subset \mathbb{C}^\times \) be the finite multiplicative subgroups of \( \mathbb{C}^\times \) of \( \frac{m}{p} \)-th, respectively \( m \)-th, roots of unity. Besides the 34 exceptional cases, every irreducible complex reflection group belongs to the following family of imprimitive subgroups of \( S_n \rtimes T_n \),

\[ G(m, p, n) = S_n \rtimes T(m, p, n) , \]

where
\[ T(m, p, n) = \{ t \in T_n \mid t^m = \text{id}, \det(t) \in C_m^{\mathbb{Z}} \} \]

is the group of diagonal matrices with diagonal entries in \( C_m \) whose product is in \( C_m^{\mathbb{Z}} \). The complex reflections of \( G(m, p, n) \) are given by
\[ S = \{ s^{(e)}_{ij} \mid i, j \in [n], i \neq j , \ e \in C_m \} \cup \{ t^{(c)}_i \mid i \in [n], \ c \in C_m^{\mathbb{Z}} \setminus \{ 1 \} \} \]

where \([n] := \{ 1, 2, \ldots, n \} \), and for general \( e \in C_m^\times \),
\[ s^{(e)}_{ij}(x_k) := \begin{cases} x_k , & k \neq i, j , \\ e^{-1} x_j , & k = i , \\ e x_i , & k = j . \end{cases} \]
\[ t^{(c)}_i(x_k) = e^\delta_{ik} x_k . \]

Note that the groups \( G(1, 1, n) \), \( G(2, 1, n) \), \( G(2, 2, n) \) correspond to the Coxeter groups of type \( A_{n-1} \), \( B_n \), \( D_n \) respectively, whilst \( G(p, p, 2) \) corresponds to the dihedral group \( I_2(p) \).

It will be relevant to the construction of the rational Cherednik algebras in which conjugacy class each complex reflection lies in. The reflections \( s^{(e)}_{ij} \) with \( e \in C_m \) are involutions and form a single conjugacy class in \( G(m, p, n) \), unless \( n = 2 \) and \( p \) is even. Additionally for each \( c \in C_m^{\mathbb{Z}} \setminus \{ 1 \} \), the \( \{ t^{(c)}_i \mid i \in [n] \} \) forms a separate conjugacy class.

Recall also that complex reflection groups are characterised in terms of their polynomial invariants. If \( G \) is a finite subgroup of \( \text{GL}(V) \), the action of \( G \) on \( V \) extends naturally to algebra automorphisms of the symmetric algebra \( S(V) \). The invariant set \( S(V)^G \) forms a subalgebra of \( S(V) \).

**Theorem 2.1** (Chevalley-Shephard-Todd). For \( V \) and \( G \) as above, the invariant ring \( S(V)^G \) is a polynomial algebra if and only if \( G \) is a complex reflection group.

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2.2. Mystic reflection groups. Each of the complex reflection groups \(G(m, p, n)\), for \(m\) even, has a so called “mystic partner”, which is another subgroup of \(S_n \ltimes T_n\), defined as follows:

\[
\mu(G(m, p, n)) := \{ wt \in S_n \times T(m, 1, n) \mid \det(wt) \in C_\mathbb{F}_p \}.
\]

These are examples of mystic reflection groups, which are defined by generalising the Chevalley-Shephard-Todd characterisation of complex reflection groups (Theorem 2.1), see [16]. The same class of groups was obtained independently in [2], see also [4].

**Definition 2.2.** For a matrix \(q \in \text{Mat}_n(\mathbb{C})\) with \(q_{ij}q_{ji} = 1, q_{ii} = 1\), let \(S_q(V)\) be the algebra generated by \(V\) subject to relations \(x_i x_j = q_{ij} x_j x_i\) for \(i, j \in [n]\). A finite group \(G\) is a mystic reflection group if it has a faithful action by degree-preserving automorphisms on \(S_q(V)\) such that the invariant subalgebra \(S_q(V)^G\) is isomorphic to \(S_{q'}(V)\) for some \(q'\).

The groups \(G(m, p, n)\) are related to their mystic partners in the following ways:

\[
\mu(G(m, p, n)) = \begin{cases} 
G(m, p, n) & \text{if } \frac{m}{p} \text{ is even,} \\
\cong G(m, p, n) & \text{if } \frac{m}{p} \text{ and } n \text{ are odd,} \\
\not\cong G(m, p, n) & \text{if } \frac{m}{p} \text{ is odd and } n \text{ is even.}
\end{cases}
\] (1)

Even though the groups \(\mu(G(m, p, n))\) and \(G(m, p, n)\) in some cases coincide as a subgroup of \(S_n \ltimes T_n\), the generating set for \(\mu(G(m, p, n))\) relevant for what follows is the set

\[
\mathcal{S} := \{ \sigma_{ij}^{(r)} \mid i, j \in [n], i \neq j, \epsilon \in C_m \} \cup \{ t_i^{(\zeta)} \mid i \in [n], \zeta \in C_\mathbb{F}_p \setminus \{1\} \}
\] (2)

where

\[
\sigma_{ij}^{(r)}(x_k) = \begin{cases} 
x_k & k \neq i, j, \\
\epsilon^{-1} x_j & k = i, \\
-\epsilon x_i & k = j.
\end{cases}
\] (3)

We call the elements of \(\mathcal{S}\) mystic reflections. Notice \(\sigma_{ij}^{(r)} \in S_n \ltimes T_n\) are of order 4, with characteristic polynomial \((x - 1)^n - 2(x^2 + 1)\). Similarly to above, when \(n \geq 3\), the \(\sigma_{ij}^{(r)}\) form a single conjugacy class, whilst for each \(\zeta \in C_\mathbb{F}_p \setminus \{1\}\) the \(t_i^{(\zeta)}\) again form separate conjugacy classes.

2.3. A presentation of mystic reflection groups. It turns out that for fixed \(n\), each mystic reflection group \(\mu(G(m, p, n))\) contains the Tits group of type \(A_{n-1}\), introduced in [19, Section 4.6].

The Tits group is realised as \(\mu(G(2, 2, n))\), the mystic partner of the Coxeter group of type \(D_n\); it is the group of even elements in the Coxeter group \(G(2, 1, n)\) of type \(B_n\).

**Theorem 2.3** (a presentation of \(\mu(G(m, p, n))\)). For all even \(m\) and all divisors \(p\) of \(m\), the abstract group generated by symbols \(\sigma_1, \ldots, \sigma_{n-1}\) and the abelian group \(T(m, p, n)\), subject to the relations

\[
(i) \quad \sigma_i^2 = t_{i+1}^{(1)} t_{i+1}^{(-1)}, \\
(ii) \quad \text{the braid relations } \sigma_i \sigma_j = \sigma_j \sigma_i, i - j > 1, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, 1 \leq i \leq n - 2, \text{ and}
\]


(iii) \( \sigma_i \tau^{-1} = s_{i,i+1}(t) \) for all \( t \in T(m,p,n) \)

is isomorphic to \( \mu(G(m,p,n)) \) via the map \( \sigma_i \mapsto \sigma_{i,i+1}^{(1)} \) and the identity map on \( T(m,p,n) \).

**Proof.** Let \( W \) be the quotient of the free product of a free group on \( \{ \sigma_i \mid i \in [n-1] \} \) with \( T(m,p,n) \) by relations (i)–(iii). If \( m = p = 2 \), then by [1, Lemma 2.1], \( W \) is the Tits group \( T \subset SL_n(\mathbb{C}) \) of type \( A_{n-1} \), which surjects onto \( S_n \) with kernel \( T(2,2,n) \). Therefore, for \( m \) even, the subgroup of \( W \) generated by \( \sigma_1, \ldots, \sigma_{n-1} \) and \( T(2,2,n) \) is some quotient \( \overline{T} \) of \( T \). Moreover, by rearranging generators using relation (iii) we can write \( W \) as \( \overline{T} \cdot Q \) where \( Q \) is a transversal of \( T(2,2,n) \) in \( T(m,p,n) \). Hence

\[
|W| \leq |T| \cdot |Q| = |S_n||T(2,2,n)| \cdot |T(m,p,n)|/|T(2,2,n)| = |S_n||T(m,p,n)|.
\]

Now observe that relations (i)–(iii) hold in \( \mu(G(m,p,n)) \): one checks them using the factorisation \( \sigma_{i,i+1}^{(1)} = s_{i,i+1} \cdot t \) in \( S_n \times T_n \), the Coxeter relations for the \( s_{i,i+1} \) and the semidirect product relations. Hence the map \( W \to \mu(G(m,p,n)) \), given in the Theorem, is well-defined. We show that this map is surjective, i.e., \( \sigma_{i,i+1}^{(1)} \), \( i \in [n-1] \) and elements of \( T(m,p,n) \) generate the group \( \mu(G(m,p,n)) \). The composite homomorphism \( \mu(G(m,p,n)) \hookrightarrow S_n \times T_n \xrightarrow{\text{proj}} S_n \) carries the \( \sigma_{i,i+1}^{(1)} \) to the generators \( s_{i,i+1} \) of \( S_n \) so is surjective with kernel \( T(m,p,n) \). Hence the \( \sigma_{i,i+1}^{(1)} \) generate \( \mu(G(m,p,n)) \) modulo \( T(m,p,n) \), as required. We thus have

\[
|W| \leq |S_n||T(m,p,n)| = |\mu(G(m,p,n))|,
\]

so the surjective homomorphism \( W \to \mu(G(m,p,n)) \) is a bijection. \( \square \)

## 3 Rational and braided Cherednik algebras

### 3.1. Rational Cherednik algebras

For complex reflection group \( G \subset GL(V) \), let \( S \) denote the complex reflections in \( G \). Let \( c : S \to \mathbb{C}, s \mapsto c_s \) be such that \( c_{sg^{-1}} = c_s \forall g \in G, s \in S \). Using \( c \), we define a bilinear map

\[
\kappa_c : V^* \times V \to CG, \ (y, x) \mapsto \langle y, x \rangle 1 + \sum_{s \in S} c_s \langle y, (\text{id} - s)x \rangle s.
\]

The following algebras were introduced by Etingof and Ginzburg in [11]:

**Definition 3.1.** The **rational Cherednik algebra** \( H_c(G) \) is generated by \( V, CG, V^* \), subject to the relations: \( \forall x, x' \in V, y, y' \in V^*, g \in G, \)

- \( xx' - x'x = yy' - y'y = 0 \)
- \( gx = g(x)g, \ yg = g \cdot g^{-1}(y) \)
- \( yx - xy = \kappa_c(y, x) \)

Note that when \( c = 0 \), the algebra \( H_0(G) = A(V) \# CG \), the smash product of the Weyl algebra with the group algebra \( CG \). Whilst \( H_0(G) \) has no finite-dimensional modules, \( H_c(G) \) can have finite-dimensional modules for special values of \( c \).
Theorem 3.2 (The PBW theorem for rational Cherednik algebras, [11, Theorem 1.3]). Let \( x_1, \ldots, x_n \) be basis of \( V \), and \( y_1, \ldots, y_n \) a dual basis for \( V^* \). As a \( \mathbb{C} \)-vector space, \( H_c(G) \) has basis

\[
\{ x_1^{k_1} \cdots x_n^{k_n} g y_1^{l_1} \cdots y_n^{l_n} \mid g \in G, \ k_1, \ldots, k_n, l_1, \ldots, l_n \in \mathbb{Z}_{\geq 0} \}.
\]

In other words, as vector spaces, we have \( H_c(G) \cong \mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}G \otimes \mathbb{C}[y_1, \ldots, y_n] \).

In the following we restrict to the case \( n = 2 \) and \( p \) is odd and \( n = 2 \). This means the \( s_{ij}^{(e)} \) form a single conjugacy class, and we denote the value of \( c : S \to \mathbb{C} \) on this class as \( c_1 \). The groups with odd \( m \) are excluded because they have no mystic partner. If \( n = 2 \) and \( p \) is even, the algebra defined below is not the most general case of the rational Cherednik algebra because it only has a single parameter for the set of complex reflections of the form \( s_{ij}^{(e)} \), although this set is split into two conjugacy classes in \( G(m, p, 2) \).

Definition 3.3. The rational Cherednik algebra \( H_c(G(m, p, n)) \) is the algebra generated by \( x_1, \ldots, x_n \in V, g \in G(m, p, n), y_1, \ldots, y_n \in V^* \), subject to relations:

\[
\begin{align*}
    x_i x_j - x_j x_i &= 0 \\
    y_i y_j - y_j y_i &= 0 \\
    g x_i - g(x_i) g &= 0 \\
    g y_i - g(y_i) g &= 0 \\
    y_i x_j - x_j y_i &= c_1 \sum_{\epsilon \in C_m} \epsilon s_{ij}^{(e)} \\
    y_i x_i - x_i y_i &= 1 - c_1 \sum_{j \neq i \in C_m} s_{ij}^{(e)} - \sum_{\zeta \in C_{\frac{m}{p}} \setminus \{1\}} c_\zeta t_{ij}^{(c)}
\end{align*}
\]

where \( c_\zeta \) the value of \( c \) on conjugacy class of \( t_{ij}^{(c)} \), for each \( \zeta \in C_{\frac{m}{p}} \).

3.2. Braided Cherednik algebras. Consider the mystic reflection group \( \mu(G(m, p, n)) \), with mystic reflections \( S \) as in (2). We require \( m \) to be even in order for the mystic reflection group \( \mu(G(m, p, n)) \) to be defined; we also assume that \( n \geq 3 \) or \( p \) is odd and \( n = 2 \) as above, so that the mystic reflections \( d_{ij}^{(e)} \) form a single conjugacy class. Similarly to above, we consider a function \( c' : S \to \mathbb{C} \) that is invariant under conjugation in \( \mu(G(m, p, n)) \), and let \( x_1, \ldots, x_n \) be a basis of \( V \), with \( y_1, \ldots, y_n \), a dual basis for \( V^* \). The following algebras were introduced in [2]:

Definition 3.4. The negative braided Cherednik algebra \( H_{c'}(\mu(G(m, p, n))) \) is the algebra generated by \( x_1, \ldots, x_n \in V, g \in \mu(G(m, p, n)), y_1, \ldots, y_n \in V^* \), subject to relations:

\[
\begin{align*}
    x_i x_j + x_j x_i &= 0 \\
    y_i y_j + y_j y_i &= 0 \\
    g x_i - g(x_i) g &= 0 \\
    g y_i - g(y_i) g &= 0 \\
    y_i x_j + x_j y_i &= c'_1 \sum_{\epsilon \in C_m} \epsilon s_{ij}^{(e)} \\
    y_i x_i - x_i y_i &= 1 + c'_1 \sum_{j \neq i \in C_m} s_{ij}^{(e)} + \sum_{\zeta \in C_{\frac{m}{p}} \setminus \{1\}} c'_\zeta t_{ij}^{(c)}
\end{align*}
\]

The key property of \( H_{c'}(\mu(G(m, p, n))) \) is the PBW-type theorem [2, Theorem 0.2]. Below we obtain a new proof of this result, see Remark 5.10.

4 The cocycle twist

Although we will only use cocycles on a finite abelian group, we will work in a more general Hopf algebra setting as it provides the useful language of duality. Our notation generally follows [17]. If \( H \)
is a Hopf algebra over \( \mathbb{C} \), \( \triangle : H \to H \otimes H \) will denote the coproduct of \( H \) and \( \epsilon : H \to \mathbb{C} \), the counit. An example is the group algebra \( H = \mathbb{C} T \) of a group \( T \), with \( \triangle(t) = t \otimes t \) and \( \epsilon(t) = 1 \), extended from \( T \) to \( \mathbb{C} T \) by linearity. The action of \( h \in H \) on \( a \in A \) where \( A \) is an \( H \)-module will be written as \( h \triangleright a \). Recall that in an \( H \)-module algebra \( A \), the product map \( m : A \otimes A \to A \) is a morphism in the category \( H \)-Mod of \( H \)-modules, and \( h \triangleright 1_A = \epsilon(h)1_A \) for all \( h \in H \).

4.1. Quasitriangular structures and 2-cocycles. We begin with two well-known definitions, see [17, Definition 2.1.1 and Example 2.3.1].

**Definition 4.1.** A 2-cocycle on a Hopf algebra \( H \) is an invertible \( \chi \in H \otimes H \) such that \((\chi \otimes 1) \cdot (\Delta \otimes \text{id})(\chi) = (1 \otimes \chi) \cdot (\text{id} \otimes \Delta)(\chi) \) and \((\epsilon \otimes \text{id})(\chi) = 1 = (\text{id} \otimes \epsilon)(\chi)\).

**Definition 4.2.** A quasitriangular (QT-) structure on a Hopf algebra \( H \) is an invertible element \( R \in H \otimes H \) satisfying:

\[
\begin{align*}
\text{(QT1)} & \quad R \cdot (\Delta x) = (\Delta^{\text{op}}x) \cdot R \quad \forall x \in H, \\
\text{(QT2)} & \quad (\Delta \otimes \text{id})R = R_{13} \cdot R_{23}, \quad (\text{id} \otimes \Delta)R = R_{13} \cdot R_{12},
\end{align*}
\]

where \( R_{12} := R \otimes 1 \), \( R_{23} := 1 \otimes R \), and \( R_{13} \) similarly has 1 inserted in the middle leg.

QT-structures satisfy the quantum Yang-Baxter equation [17, Lemma 2.1.4]

\[
R_{12} \cdot R_{13} \cdot R_{23} = R_{23} \cdot R_{13} \cdot R_{12}. \tag{4}
\]

(QT2) and (4) imply that a QT-structure is a Hopf algebra 2-cocycle [17, Example 2.3.1].

4.2. Twists. It is natural to complement Majid’s description of a twisted \( H \)-module algebra [17, Section 2.3] by the observation that twisting by \( \chi \) is functorial:

**Proposition 4.3.** A 2-cocycle \( \chi \in H \otimes H \) for a Hopf algebra \( H \) gives rise to the functor

\[
(\chi) : \text{Alg}(H \text{-Mod}) \to \text{Alg}(H_{\chi} \text{-Mod})
\]

which takes an object \((A, m)\) to \((A, m_{\chi} = m(\chi^{-1} \triangleright -))\), and an arrow \((A, m) \xrightarrow{\phi} (B, m')\) to \((A, m_{\chi}) \xrightarrow{\phi_{\chi}} (B, m'_{\chi})\), where \( \phi_{\chi} = \phi \) as \( H \)-module morphisms. The twisted Hopf algebra \( H_{\chi} \) is defined as having the same algebra structure, and counit, as \( H \), but with coproduct \( \Delta_{\chi}(h) = \chi \cdot \Delta(h) \cdot \chi^{-1} \) where \( \Delta \) is the coproduct on \( H \).

**Proof.** That \((A, m_{\chi})\) is in \( \text{Alg}(H_{\chi} \text{-Mod}) \) is [17, Proposition 2.3.8], so we only need to check functoriality. Since \( A \otimes A \xrightarrow{\phi \otimes \phi} B \otimes B \) is an \( H \otimes H \)-module morphism, it commutes with the action of \( \chi^{-1} \in H \otimes H \), so \( \phi \circ m_{\chi} = m \circ (\phi \otimes \phi)(\chi^{-1} \triangleright -) = m(\chi^{-1} \triangleright -)(\phi \otimes \phi) \) shows that \( \phi_{\chi} \) is an algebra morphism if \( \phi \) is. Also, \( \phi_{\chi} \) is an \( H_{\chi} \)-module morphism because the actions of \( H \) and of \( H_{\chi} \) are the same. Therefore, \( \phi_{\chi} \) is indeed an arrow in \( \text{Alg}(H_{\chi} \text{-Mod}) \).

**Remark 4.4.** The functor given in Proposition 4.3 is essentially the restriction of the monoidal equivalence of categories \( H \text{-Mod} \to H_{\chi} \text{-Mod} \), see [10, Remark 5.14.3], to algebras.
4.3. The cocycle $F$. We will now define the cocycle $F$ which will be used for twisting in the rest of the paper. Let $T$ be the abelian group

$$T = \langle \gamma_1, \ldots, \gamma_n | \gamma_i^2 = 1, \gamma_i \gamma_j = \gamma_j \gamma_i, i, j \in [n] \rangle,$$

isomorphic to $T(2,1,n) \cong (C_2)^n$. Define

$$f_{ij} = \frac{1}{2} \left( 1 \otimes 1 + \gamma_i \otimes 1 + 1 \otimes \gamma_j - \gamma_i \otimes \gamma_j \right), \quad F = \prod_{1 \leq j < i \leq n} f_{ij}.$$

Let $a, b$ be elements of some associative algebra. It is easy to check that if $a, b$ are commuting involutions, then $\frac{1}{2}(1 + a + b - ab)$ is an involution, which implies that the $f_{ij}$ and $F$ are pairwise commuting involutions in $CT \otimes CT$. That they are cocycles follows from

**Lemma 4.5.** $F$ and $f_{ij}$ for all $i, j$ are quasitriangular structures on $CT$. 

**Proof.** (QT1) is vacuous as $CT$ is a commutative and cocommutative Hopf algebra. Rewriting $f_{ij}$ in the form $2^{-1} \sum_{a,b=0} (-1)^{ab} \gamma^a \otimes \gamma^b$, one checks (QT2) for $f_{ij}$ in the same way as in the case $n = 2$ of [17, Example 2.1.6]. Since (QT2) is multiplicative in $R$, (QT2) also holds for $F$. \hfill \square

4.4. $F$-twisted product of $T$-eigenvectors. Since $T$ is a finite abelian group, a $CT$-module is the same as a comodule for the dual Hopf algebra $\hat{C}T$, where the dual group $\hat{T}$ of $T$ is

$$\hat{T} = \{ \alpha : T \to \mathbb{C}^\times | \alpha \text{ is a group homomorphism} \} = \langle \alpha_1, \ldots, \alpha_n \rangle, \quad \alpha_i(\gamma_j) = (-1)^{\delta_{ij}}.$$

A coaction by the group algebra of $\hat{T}$ manifests itself as a $\hat{T}$-grading, so for a $CT$-module algebra $A$ we have $A = \bigoplus_{\alpha \in \hat{T}} A_\alpha$ where $A_\alpha := \{ a \in A | \forall \gamma \in T, \gamma \triangleright a = \alpha(\gamma) a \}$ is the $\alpha$-eigenspace of $T$. Denote by $*$ the twisted product on $A$ induced by the cocycle $F \in CT \otimes CT$ as in Proposition 4.3. Then

$$a * b = (\alpha \otimes \beta)(F^{-1}) a \cdot b \quad \text{for } a \in A_\alpha, b \in A_\beta. \quad (6)$$

Here $\alpha, \beta \in \hat{T}$ induce an algebra homomorphism $\alpha \otimes \beta : CT \otimes CT \to \mathbb{C}$. Observe that

$$(\alpha_i \otimes \alpha_j)(f_{ab}) = 1 \text{ if } (i, j) \neq (a, b), \quad (\alpha_i \otimes \alpha_j)(f_{ij}) = -1. \quad (7)$$

Every one of the $2^n$ elements of $\hat{T}$ is of the form

$$\alpha_I := \prod_{i \in I} \alpha_i \quad \text{where } I \subseteq [n],$$

so (6) and (7) imply the following

**Lemma 4.6.** Let $I, J$ be subsets of $[n]$. If $a \in A_{\alpha_I}, b \in A_{\alpha_J}$, then $a * b = (-1)^{d(I,J)ab}$ where $d(I,J) = |\{(i, j) : i \in I, j \in J, i > j\}|$. \hfill \square
**Remark 4.7.** In fact, Lemma 4.6 is a particular case of the construction in [3, Lemma 3.6] where a twist $A_F$ of a $G$-graded algebra $A$ by a 2-cocycle $F$ on the group $G$ is realised as the image of the coaction $A \to A \otimes CG$ viewed as a subalgebra of $A \otimes (CG)_F$. In the case $G = \hat{T}$ and the cocycle $F$ given above, the twisted group algebra $(\mathbb{C}\hat{T})_F$ is isomorphic to the complex Clifford algebra $Cl_n$ of a space with an orthonormal basis $\alpha_1, \ldots, \alpha_n$, [3, Example 1.7]. The calculations done in the next section can be interpreted as embedding the negative braided Cherednik algebra $\mathcal{H}_c(\mu(G(m,p,n)))$ in $H_c(G(m,p,n)) \otimes Cl_n$, although we do not explicitly write the Clifford algebra generators.

## 5 The Main Result

### 5.1. Statement of the main theorem.** To state Main Theorem 5.2 below, we need to define the action of the abelian group $T$ on the rational Cherednik algebra $H_c(G(m,p,n))$. This will allow us to twist the associative product in $H_c(G(m,p,n))$.

**Proposition 5.1.** Let $m$ be even and $n \geq 2$. The rational Cherednik algebra $H_c(G(m,p,n))$ is a $CT$-module algebra with respect to the action $\triangleright$ given by:

$$
\gamma_i \triangleright g = t_i^{(-1)} g t_i^{(-1)}, \quad \gamma_i \triangleright x_j = t_i^{(-1)}(x_j), \quad \gamma_i \triangleright y_j = t_i^{(-1)}(y_j)
$$

for all $i, j \in [n], \ g \in G(m,p,n)$.

**Proof.** The PBW theorem 3.2 and the defining relations in Definition 3.3 imply that $H_c(G(m,p,n))$ embeds as a subalgebra in $\tilde{H} = H_c(G(m,1,n))$, where $\tilde{c}$ is defined by $\tilde{c}_1 = c_1, \tilde{c}_\zeta = c_\zeta$ for $\zeta \in C_m \setminus \{1\}$ and $\tilde{c}_\zeta = 0$ whenever $\zeta \in C_m \setminus C_{m,1}$. This subalgebra is generated by $x_i, y_i$ for $i \in [n]$ and by the subgroup $G(m,p,n)$ of $\tilde{G} = G(m,1,n)$. Observe that $\tilde{H}$ is a $\tilde{C}G$-module algebra where $a \in \tilde{G}$ acts by conjugation, carrying $u \in \tilde{H}$ to $aua^{-1} \in \tilde{H}$. Since $G(m,p,n)$ is a normal subgroup of $\tilde{G}$, this action of $\tilde{G}$ preserves the subalgebra $H_c(G(m,p,n))$ of $\tilde{H}$. Now the embedding of $T$ in $\tilde{G}$ via $\gamma_i \mapsto t_i^{(-1)}$ defines the $T$-action on $H_c(G(m,p,n))$ given in the Proposition.

We now twist the rational Cherednik algebra by the cocycle $F \in \mathbb{C}T \otimes \mathbb{C}T$ from Section 4 and denote the result $(H_c(G(m,p,n)), \ast)$.

**Theorem 5.2.** There exists an isomorphism

$$
\phi : \mathcal{H}(\mu(G(m,p,n))) \to (H_c(G(m,p,n)), \ast)
$$

of associative algebras, where $\iota : S \to \mathbb{C}$ is defined by $\iota_1 := -c_1, \ i_\zeta := -c_\zeta \ \forall \zeta \in C_m \setminus \{1\}$, and

$$
\phi(\sigma_i) = s_i, \quad \phi(t) = t, \quad \phi(x_i) = x_j, \quad \phi(y_j) = y_j,
$$

for all $i \in [n-1], \ j \in [n], \ \sigma_i := \sigma_i^{(1)}, \ s_i := s_i^{(-1)}, \ t \in T(m,p,n)$.

### 5.2. Outline of the proof of the Theorem.** We fix the triple $(m,p,n)$ and denote $G = G(m,p,n), \ H_c = H_c(G), \ \mathcal{H}(\mu) = \mathcal{H}(\mu(G))$. The Theorem defines the algebra homomorphism $\phi$ on generators of $\mathcal{H}$, and so $\phi$, if it exists, is unique. We first extend $\phi$ from the generators $\sigma_i$ and $t$
of $\mu(G)$ to a homomorphism from $\mu(G)$ to a subgroup of the twisted group algebra $(CG, \ast)$. Then, by checking that the defining relations of $H_c$ from Definition 3.4 are satisfied by the elements $\phi(x)$, $\phi(y)$ and $\phi(g)$, $g \in \mu(G)$ of the algebra $(H_c, \ast)$, we show that $\phi$ extends from the generators to the whole algebra $H_c$. We then use the PBW theorem for $H_c$ to argue that $\phi$ is bijective.

5.3. $\ast$-multiplication by simple generators. We need several lemmas where we express the new, $F$-twisted associative product $\ast$ of certain elements of $H_c$ in terms of the usual product (written as $\cdot$ or omitted).

Lemma 5.3. For all $t \in T(m, p, n)$, $u \in H_c$, $t \ast u = tu$ and $u \ast t = ut$.

Proof. As $t$ is invariant under the action of $T$, i.e., is a $T$-eigenvector with eigencharacter $1 = \alpha_\emptyset$, by Lemma 4.6, $t \ast u = tu$ and $u \ast t = ut$ for all $T$-eigenvectors $u$, and so by linearity for all $u$ in $H_c$. \qed

We denote

$$s_i := s_{i,i+1} \in S_n, \quad r_{ij} := t_{ij} = t_{ij}^{-1} \in T(m, p, n), \quad p_i = \frac{1}{2}(1 + \gamma_i), \quad q_i = \frac{1}{2}(1 - \gamma_i) \in \mathbb{C}T$$

and let

$$s_i^+ := p_i \triangleright s_i = \frac{1}{2}(s_i + s_i), \quad s_i^- := q_i \triangleright s_i = \frac{1}{2}(s_i - s_i) \in \mathbb{C}G.$$

Observe that $\gamma_i \triangleright s_i = s_i$ if $\gamma_i \not\in \{i, i + 1\}$, so $s_i^+$ is $T$-invariant, and $s_i^- \in (H_c)_{\alpha_\emptyset, \alpha_{i+1}}$.

Lemma 5.4. For all $i \in [n - 1]$, $u \in H_c$,

$$s_i \ast u = s_i \cdot (p_i \triangleright u) + s_i \cdot (q_i \triangleright u) \quad \text{and} \quad u \ast s_i = (p_{i+1} \triangleright u)s_i + (q_{i+1} \triangleright u)s_i.$$

Proof. Since the expressions are linear in $u$, we may assume that $u$ is a $T$-eigenvector with eigencharacter $\alpha_J$, $J \subseteq [n]$. Apply Lemma 4.6 to $s_i^- \in (H_c)_{\alpha_J, \alpha_{i+1}}$ and $u$. In the string $i, i + 1, J$, the pair $i, i + 1$ forms zero or two inversions with every element of $J$ except possibly $i$, and forms exactly one inversion with $i$ if $i \in J$, so $s_i^- \ast u = s_i^- u$ if $i \not\in J$, and $s_i^- \ast u = -s_i^- u$ if $i \in J$. That is, $s_i^- \ast u = s_i^- \cdot (\gamma_i \triangleright u)$. Also by Lemma 4.6, $s_i^+ \ast u = s_i^+ u$. The formula for $s_i \ast u = (s_i^+ + s_i^-) \ast u$ follows. The proof for $u \ast s_i$ is similar. \qed

Lemma 5.5. The simple transpositions $s_i$, $i \in [n - 1]$, obey the relations $s_i \ast s_i = r_{i,i+1}$ and the braid relations with respect to the $\ast$-product on the group algebra of $G(m, p, n)$.

Proof. By Lemma 5.4, $s_i \ast s_i = s_i s_i^+ + s_i s_i^- = \frac{1}{2}(1 + r_{i,i+1}) + \frac{1}{2}r_{i,i+1}(1 - r_{i,i+1}) = r_{i,i+1}$.

If $|i - j| > 1$, then $s_j$ is $\gamma_j$-invariant, hence $q_i \triangleright s_j = 0$. Then by Lemma 5.4, $s_i \ast s_j = s_i s_j$. This is symmetric in $i, j$, so $s_i$ and $s_j$ $\ast$-commute.

To check the braid relation for $s_i$ and $s_{i+1}$, we can assume $i = 1$. We calculate the product

$$s_2 \ast s_{1,3} \ast s_1 = (s_2 \ast s_2 s_1 s_2) \ast s_1 = s_2 \ast (s_1 s_2 s_1 \ast s_1) \quad (8)$$

in two ways. First, since the transposition $s_{1,3} = s_2 s_1 s_2$ is $\gamma_2$-invariant, by Lemma 5.4 $s_2 \ast s_2 s_1 s_2 = s_2 s_2 s_1 s_2 = s_1 s_2$. This is the same as $s_1 \ast s_2$ because $s_1$ is $\gamma_3$-invariant, so (8) equals $s_1 \ast s_2 \ast s_1$. 

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On the other hand, by $\gamma_2$-invariance of $s_{1,3} = s_1s_2s_1$ and the second part of Lemma 5.4, $s_1s_2s_1 \star s_1 = s_1s_2s_1s_1 = s_1s_2 = s_1 \star s_2$, so (8) equals $s_2 \star s_1 \star s_2$. The $\star$-braid relations are proved.

**Remark 5.6.** If $(H_c, \star)$ is embedded in $H_c \otimes Cl_n$ as in Remark 4.7, the simple transposition $s_i$ becomes $s_i^+ \otimes 1 + s_i^- \otimes \alpha_i\alpha_{i+1}$. The calculation to prove Lemma 5.5 is then equivalent to verifying part of the Clifford Braiding Theorem of Kauffman and Lomonaco [15] which states that the $e_i := \frac{1+\alpha_i\alpha_{i+1}}{\sqrt{2}}$, $i \in [n-1]$, obey the braid relations in $Cl_n$. The Clifford Braiding Theorem goes further to assert the circular braid relations involving $e_n := 1 + \alpha_n\alpha_1$, but these do not arise from Lemma 5.5.

### 5.4. The extension of $\phi$ from the generators to the group algebra $\mathbb{C}\mu(G)$. To prove that the assignment $\phi(\sigma_i) = \bar{s}_i$, $\phi(t) = t$ extends to a homomorphism $\phi: \mathbb{C}\mu(G) \to (\mathbb{C}G, \star)$ of algebras, we check relations (i)-(iii) from the presentation of $\mu(G)$ given in Theorem 2.3.

(i) $\sigma_i\sigma_i = r_{i,i+1}$. We check that the relation $\phi(\sigma_i) \star \phi(\sigma_i) = \phi(r_{i,i+1})$ holds in $(H_c, \star)$. The left-hand side is $\bar{s}_i \bar{s}_i = (s_ir_{i,i+1}) \star (s_ir_{i,i+1})$, which by Lemma 5.3 is $s_i \star s_i$. By Lemma 5.5, this is $r_{i,i+1}$, the same as $\phi(r_{i,i+1})$.

(ii) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i - j > 1$. We need to check that $\phi(\sigma_i) = \bar{s}_i$ and $\phi(\sigma_j) = \bar{s}_j$ $\star$-commute. This follows from Lemma 5.3 and Lemma 5.5.

$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. By Lemma 5.5, $s_i \star s_{i+1} \star s_i = s_{i+1} \star s_i \star s_{i+1}$, and by Proposition 4.3, $(H_c, \star)$ is a $T$-module algebra. Acting by $\gamma_{i+1}$ on both sides gives the required relation $\bar{s}_i \star \bar{s}_{i+1} \star \bar{s}_i = \bar{s}_{i+1} \star \bar{s}_i \star \bar{s}_{i+1}$.

(iii) $\sigma_it\sigma_i^{-1} = s_i(t)$. We need to check $\bar{s}_i \star t \star \bar{s}_i = s_i(t)$. The left-hand side rewrites by Lemma 5.3 as $s_i \star s_i(t) = r_{i,i+1}(\bar{s}_i \star \bar{s}_i)s_i(t)$, which simplifies to $s_i(t)$ by (i).

We can now describe the map $\phi$ on the whole of $\mathbb{C}\mu(G)$ using a special basis of $\mathbb{C}\mu(G)$:

**Proposition 5.7.** There exist involutions $\theta_w \in CT(2, 1, n)$, indexed by $w \in S_n$, such that $\{ w\theta_w t : w \in S_n, t \in T(m, p, n) \}$ is a basis of $\mathbb{C}\mu(G)$. In this basis,

$$\phi(w\theta_w t) = wt. \quad (9)$$

**Proof.** First of all, we observe that, for each $i \in [n-1]$ and $w \in S_n$,

$$\bar{s}_i \star w = s_iw \cdot \eta_i(w)$$

for some involution $\eta_i(w) \in CT(2, 2, n)$. Indeed, denote $t_i^{(-1)} \cdot w^{-1}(t_i^{(-1)})$ by $r$ so that $\gamma_i \triangleright w = wr$. By Lemma 5.3 and Lemma 5.4,

$$\bar{s}_i \star w = s_i r_{i,i+1} : \frac{1}{2}(w + wr) + s_i : \frac{1}{2}(w - wr) = s_iw \cdot \frac{1}{2}(1 - r + r' + rr')$$

with $r' = w^{-1}(r_{i,i+1})$. Thus, $\eta_i(w) = \frac{1}{2}(1 + r + r' - rr')$, which is an involution by (5).

Factorise $w$ into simple transpositions as $w = s_{i_k} \ldots s_{i_2}s_{i_1}$, and let $s_{w} = \sigma_{i_k} \ldots \sigma_{i_2}\sigma_{i_1}$. Since $\sigma_i = s_i t_{i+1}$ and $\det s_i = 1$, in the group $S_n \times T(2, 1, n)$ one has $s_w = wt_w$ with $t_w \in T(2, 1, n)$ such that $\det(t_w) = \det(w)$. Therefore,

$$\phi(wt_w) = \bar{s}_{i_k} \cdots \bar{s}_{i_2} \bar{s}_{i_1} = s_{i_k} \ldots s_{i_2}s_{i_1} \cdot \eta_{i_k}(s_{i_1})\eta_{i_2}(s_{i_2}s_{i_1}) \ldots \eta_{i_k}(s_{i_{k-1}} \ldots s_{i_1}).$$
so (9) holds with $\theta_w = \eta_{i_2}(s_{i_2}) \ldots \eta_{i_k}(s_{i_k}) \ldots s_{i_1}t_w$. Note that $w t_w$ and $\eta_{i_j}(s)$, hence $w \theta_w$, lie in $\mathbb{C} G(2, 2, n)$ and so $\{w t_w : w \in S_n, t \in T(m, p, n)\}$ is a subset of $\mathbb{C} \mu(G)$. It is a basis of the space $\mathbb{C} \mu(G)$, because this set is carried by the linear map $\phi$ to the basis $\{w t\}$ of the space $\mathbb{C} G$ of the same dimension.

5.5. Commutation relations between the $x_i$ and between the $y_j$. We need to show that $\phi(x_i) \ast \phi(x_j) = -\phi(x_j) \ast \phi(x_i)$, $\phi(y_i) \ast \phi(y_j) = -\phi(y_j) \ast \phi(y_i)$ whenever $1 \leq i < j \leq n$. This is immediate by the following

**Corollary 5.8.** For all $i, j \in \{1, \ldots, n\}, i < j,$

$$x_i \ast x_i = x_i^2, \quad x_i \ast x_j = -x_j \ast x_i = x_i x_j, \quad x_i \ast y_i = x_i y_i, \quad x_i \ast y_j = x_i y_j, \quad y_j \ast x_i = -y_j x_i.$$  
The same holds where the letters $x$ and $y$ are swapped.

**Proof.** Since the $x_i$ and $y_i$ are $T$-eigenvectors with eigencharacter $\alpha_i$, this follows by Lemma 4.6.  

5.6. The main commutator relation between $x_i$ and $y_j$. We will now check the relation obtained by applying $\phi$ to both sides of the main commutator relation in $H_{x_i}$ for $i \neq j$:

$$\phi(y_i) \ast \phi(x_i) + \phi(x_i) \ast \phi(y_i) = \mathcal{C}_i \sum_{\epsilon \in C_m} \epsilon \phi(\sigma_{ij}^{(\epsilon)}).$$  

(10)

To calculate the right-hand side, we need

**Lemma 5.9.** $\phi(\sigma_{ij}^{(\epsilon)}) = s_{ij}^{(-\epsilon)}$ if $i < j$, and $\phi(\sigma_{ij}^{(\epsilon)}) = s_{ij}^{(\epsilon)}$ if $i > j$.

**Proof.** Since $\sigma_{ij}^{(\epsilon)} = \sigma_{ij}^{(1)} t_i^{(\epsilon)} t_j^{(-\epsilon)}$, $s_{ij}^{(\pm \epsilon)} = s_{ij}^{(\epsilon)} \ast t_i^{(\epsilon)} t_j^{(-\epsilon)}$ and $\phi(t_i^{(\epsilon)} t_j^{(-\epsilon)}) = t_i^{(\epsilon)} t_j^{(-\epsilon)}$, it is enough to prove the Lemma for $\epsilon = 1$.

The case $i < j$: if $j = i + 1$, the statement becomes $\phi(\sigma_i) = \bar{s}_i$ which is true by definition of $\phi$. To proceed by induction in $j$, we consider the identity $s_{ij} s_{i,j+1} = s_{i,j+1} s_{ij}$. Since $s_{i,j+1}$ is $\gamma_j$-invariant, and $s_{ij}$ is $\gamma_{j+1}$-invariant, this rewrites by Lemma 5.4 as $s_j s_{i,j+1} = s_{i,j+1} s_j$. Using $\bar{s}_j s_j = 1$,

$$s_{i,j+1} = \bar{s}_j s_{i,j+1} \ast s_j = \phi(\bar{s}_j) \phi(\sigma_{ij})^{-1} = \phi(\sigma_{i,j+1}^{(1)}),$$

so the inductive step is done, and the case $i < j$ follows.

The case $i > j$: $\phi(\sigma_{ij}^{(-\epsilon)}) = \phi(\sigma_{ji}^{(\epsilon)})$, which by the first part of the Lemma is $\bar{s}_{ij} = s_{ij}^{(-1)} = s_{ij}^{(\epsilon)}$.  

□

Now, by Corollary 5.8 and Lemma 5.9 relation (10) for $i < j$ rewrites as

$$y_i x_j - x_j y_i = \mathcal{C}_i \sum_{\epsilon \in C_m} \epsilon s_{ij}^{(-\epsilon)}.$$  

Substituting $\mathcal{C}_i = -c_1$ and recalling that $-1 \in C_m$, and so $-\epsilon \in C_m$ iff $\epsilon \in C_m$, transforms this equation into the relation between $y_i$ and $x_j$ in Definition 3.3. If $i > j$, (10) becomes $-(y_i x_j - x_j y_i) = -c_1 \sum_{\epsilon \in C_m} \epsilon s_{ij}^{(\epsilon)}$, which is again true by Definition 3.3. Thus, (10) is proved.
5.7. The main commutator relation between \( x \) and \( y \). We need to show that

\[
\phi(y) \star \phi(x) - \phi(x) \star \phi(y) = 1 + c_1 \sum_{j \neq 1, \epsilon \in C_m} \phi(\sigma_i^{(\epsilon)}) + \sum_{\zeta \in C_m \setminus \{1\}} C_\zeta \phi(t_i^{(\zeta)})
\]

where the left-hand side is \( y_i x_i - x_i y_i \) by Corollary 5.8. Apply Lemma 5.9, note that \( \sum_{\epsilon \in C_m} s_{ij}^{(-\epsilon)} \) is the same as \( \sum_{\epsilon \in C_m} t_{ij}^{(\epsilon)} \) because \(-1 \in C_m\), and substitute \( \phi(t_i^{(\zeta)}) = t_i^{(\zeta)} \), \( C_\zeta = -c_\zeta \) to rewrite the right-hand side of (11) as \( 1 - c_1 \sum_{j \neq 1} \sum_{\epsilon \in C_m} s_{ij}^{(\epsilon)} - \sum_{\zeta \in C_m \setminus \{1\}} c_\zeta t_i^{(\zeta)} \). This shows that (11) is true by the relation for \( y_i x_i - x_i y_i \) in Definition 3.3.

5.8. The semidirect product relations. We need to prove

\[
\phi(g) \star x_i = g(x_i) \star \phi(g), \quad \phi(g) \star y_i = g(y_i) \star \phi(g)
\]

for all \( g \in \mu(G) \) and all \( i \in [n] \). We will prove this only for \( x_i \), as the proof for \( y_i \) will be similar. Moreover, observe that if (12) holds for \( g = g_1 \) and for \( g = g_2 \), then it holds for \( g = g_1 g_2 \in \mu(G) \):

\[
\phi(g) \star x_i = \phi(g_1) \star \phi(g_2) x_i = \phi(g_1) \star (g_2(x_i)) \star \phi(g_2) = g_1(g_2(x_i)) \star \phi(g_1) \star \phi(g_2) = g(x_i) \star \phi(g).
\]

By Theorem 2.3, every element of \( \mu(G) \) is a product of some generators \( \sigma_j, j \in [n-1] \), and of some \( t \in T(m,p,n) \). If \( g = t \in T(m,p,n) \), one can omit \( \phi \) and \( \star \) in (12) by Lemma 5.3, and then (12) clearly holds by the semidirect product relations in \( H_c \). Hence it is enough to prove (12) when \( g = \sigma_j \):

\[
\phi(\sigma_j) \star x_i = \sigma_j(x_i) \star \phi(\sigma_j), \quad \text{equivalently } \bar{s}_j \star x_i = \sigma_j(x_i) \star \bar{s}_j.
\]

If \( i \notin \{j, j+1\} \), then \( x_i = \sigma_j(x_i) \) is both \( \gamma_j \) and \( \gamma_{j+1} \)-invariant, so by Lemma 5.4, (13) rewrites as \( \bar{s}_j x_i = x_i \bar{s}_j \).

If \( i = j + 1 \), \( x_{j+1} \) is \( \gamma_j \)-invariant and \( \sigma_j(x_{j+1}) = -x_j \) is \( \gamma_{j+1} \)-invariant, so by Lemma 5.4, (13) is \( \bar{s}_j x_{j+1} = x_j \bar{s}_j \).

If \( i = j \), \( x_j \) is a \( T \)-eigenvector with eigencharacter \( \alpha_j \) and \( \sigma_j(x_j) = x_{j+1} \), with \( \alpha_{j+1} \), so by Lemma 5.4 (13) is \( s_j x_j = x_{j+1} s_j \). In all three cases, (13) is true by the semidirect product relations in \( H_c \).

5.9. Bijectivity of \( \phi \). Hence all the relations are satisfied and \( \phi \) is a well-defined algebra homomorphism. We are left to prove that \( \phi \) is bijective. It is enough to construct a spanning set of \( H_c \) which is carried by \( \phi \) to a basis of \( H_c \).

Let \( w \in S_n \). Consider the coset \( wT(m,p,n) \) of \( T(m,p,n) \) inside \( G \) and let \( \langle wT(m,p,n) \rangle \) denote the span of this coset, a subspace of \( CG \). Observe that \( \langle wT(m,p,n) \rangle \) is a \( T \)-submodule of \( CG \), because, if \( i \in [n] \), \( t \in T(m,p,n) \), \( \gamma_i \triangleright (wt) = w(t_i^{(-1)} \cdot w^{-1}(t_i^{(-1)})) \in T(2,2,n) \subset T(m,p,n) \). Therefore, \( \langle wT(m,p,n) \rangle \) has \( T \)-eigenvectors \( wb_1(w), \ldots, wb_N(w) \) where \( N = |T(m,p,n)| \) and \( b_1(w), \ldots, b_N(w) \in CT(m,p,n) \). It follows that \( \{wb_m(w) : w \in S_n, m \in [N]\} \) is a basis of the group algebra \( CG = \oplus_{w \in S_n} \langle wT(m,p,n) \rangle \). The PBW-type tensor product factorisation of \( H_c \), see
Theorem 3.2, implies that
\[
\mathcal{B} = \{x_1^{k_1} \ldots x_n^{k_n} w b_m(w) y_1^{l_1} \ldots y_n^{l_n} : w \in \mathbb{S}_n, m \in [N], k_i, l_i \geq 0 \}
\] (14)
is a basis of $H_c$.

We replace the basis $\{w \theta_m t\}$ of $\mathbb{C} \mu(G)$, given by Proposition 5.7, by the following alternative basis: $\{w \theta_m b_m(w) : w \in \mathbb{S}_n, m \in [N]\}$. It is a basis of $\mathbb{C} \mu(G)$ because by Proposition 5.7, it is carried by $\phi$ to the basis $\{w b_m(w)\}$ of $G$, and dim $\mathbb{C} \mu(G) = \dim \mathbb{C} G$. It then follows from the defining relations in $H_c$ that the set
\[
\mathcal{B} = \{x_1^{k_1} \ldots x_n^{k_n} w \theta_m b_m(w) y_1^{l_1} \ldots y_n^{l_n} : w \in \mathbb{S}_n, m \in [N], k_i, l_i \geq 0 \}
\] (15)
spans $H_c$.

It is immediate from Corollary 5.8 that $\phi(x_1^{k_1} \ldots x_n^{k_n}) = x_1^{k_1} \ldots x_n^{k_n}$ and $\phi(y_1^{l_1} \ldots y_n^{l_n}) = y_1^{l_1} \ldots y_n^{l_n}$. We can now view how a general basis element of (15) is mapped under $\phi,
\[
\phi(x_1^{k_1} \ldots x_n^{k_n} w \theta_m b_m(w) y_1^{l_1} \ldots y_n^{l_n}) = x_1^{k_1} \ldots x_n^{k_n} \ast w b_m(w) \ast y_1^{l_1} \ldots y_n^{l_n} = \pm x_1^{k_1} \ldots x_n^{k_n} w b_m(w) y_1^{l_1} \ldots y_n^{l_n}
\]
where the second equality follows by Lemma 4.6, since $x_1^{k_1} \ldots x_n^{k_n}$, $wb_m(w)$ and $y_1^{l_1} \ldots y_n^{l_n}$ are $T$-eigenvectors. On noting that a spanning set of $H_c$ has been mapped (up to a scalar multiple of $\pm 1$) to the basis of $H_c$ given in (14), we conclude that $\mathcal{B}$ is a basis of $H_c$ and that $\phi$ is bijective, as required.

**Remark 5.10.** The above argument showing that $\mathcal{B}$ is a basis of $H_c$ is a new proof of the PBW-type theorem for negative braided Cherednik algebras, obtained earlier in [2].

### 6 Twists of representations

From Theorem 5.2 we know rational Cherednik algebras can be twisted, and the result is isomorphic to a negative braided Cherednik algebra. Next we show that representations of rational Cherednik algebras can also be twisted, generating a representation of the corresponding negative braided Cherednik algebra. A systematic approach to twists of representations, going beyond the examples considered in this section, will be explored in the upcoming paper [5].

#### 6.1 Twisting and finite-dimensional representations.

For the purposes of this section, we assume $\frac{m}{p}$ to be even. Recall that by Proposition 5.1 $H_c(G(m, p, n))$ is a $CT$-module algebra under the conjugation action of $T = (C_2)^n$. When $\frac{m}{p}$ is even, $T$ is embedded as the subgroup $T(2, 1, n)$ in $G(m, p, n)$, so a representation
\[
\rho : H_c(G(m, p, n)) \to \text{End}(V)
\]
of $H_c(G(m, p, n))$ induces a $T$-action on $\text{End}(V)$ via $t \triangleright f = \rho(t)f\rho(t)^{-1}$ for all $t \in T, f \in \text{End}(V)$. With this, $\rho$ becomes a $CT$-module algebra homomorphism. Denote by $\text{End}(V)_F$ the twist of the $CT$-module algebra $\text{End}(V)$ by the cocycle $F$ defined in Section 4.3.
Recall that the underlying vector space is unchanged by twisting, therefore \( \rho \) can be viewed as a linear map \( \rho : H_c(G(m, p, n))_\mathcal{F} \to \text{End}(V)_\mathcal{F} \).

**Proposition 6.1.** The linear map \( \rho : H_c(G(m, p, n))_\mathcal{F} \to \text{End}(V)_\mathcal{F} \) is an algebra homomorphism.

**Proof.** Let \( m \) denote the product on \( H_c(G(m, p, n)) \) and \( m' \) be the product on \( \text{End}(V) \), so that \( \rho \circ m = m' \circ (\rho \otimes \rho) \) because \( \rho \) is a homomorphism between the untwisted algebras. The twisted product maps are \( m_\mathcal{F} = m \circ (\mathcal{F}^{-1} \triangleright) \) and \( m'_\mathcal{F} = m' \circ (\mathcal{F}^{-1} \triangleright) \), and since \( \rho \) is a \( \mathbb{C}T \)-module algebra morphism, so that \( (\rho \otimes \rho)(\mathcal{F}^{-1} \triangleright) = (\mathcal{F}^{-1} \triangleright)(\rho \otimes \rho) \), we conclude that \( \rho \circ m_\mathcal{F} = m'_\mathcal{F} \circ (\rho \otimes \rho) \), as required. \( \square \)

We use this to deduce the following:

**Theorem 6.2.** For \( \frac{m}{p} \) even, if \( H_c(G(m, p, n)) \) has finite-dimensional representations, then so does the negative braided Cherednik algebra \( H\mathcal{C}_c(\mu(G(m, p, n))) \).

**Proof.** If \( \rho : H_c(G(m, p, n)) \to \text{End}(V) \) is a finite-dimensional representation, the algebra \( \text{End}(V)_\mathcal{F} \) is finite-dimensional, so it has finite-dimensional modules on which \( H\mathcal{C}_c(\mu(G(m, p, n))) \cong H_c(G(m, p, n))_\mathcal{F} \) acts via the algebra homomorphism \( \rho : H_c(G(m, p, n))_\mathcal{F} \to \text{End}(V)_\mathcal{F} \). \( \square \)

**6.2. Finite-dimensional representations: a general construction.** Let \( W \) be an irreducible complex reflection group with reflection representation \( V \), and \( H_c(W) \) be a rational Cherednik algebra over \( W \). We recall a general approach which produces finite-dimensional representations of \( H_c(W) \). Start with a simple \( CW \)-module \( \tau \) and extend \( \tau \) to an \( S(V)\#CW \)-module where \( V \) acts by zero. The standard \( H_c(W) \)-module \( M_c(\tau) \) is defined as

\[
M_c(\tau) = H_c(W) \otimes_{S(V)\#CW} \tau.
\]

The underlying vector space of \( M_c(\tau) \) is \( S(V^*) \otimes \tau \), hence these are infinite-dimensional representations of \( H_c(W) \). The standard module \( M_c(\mathbb{C}) \) given by \( \tau = \mathbb{C} \), the trivial \( CW \)-module, is the famous Dunkl (or polynomial) representation of \( H_c(W) \). Every \( M_c(\tau) \) has a unique simple quotient, denoted \( L_c(\tau) \).

For some \( \tau \) and some values of \( c \), \( L_c(\tau) \) are finite-dimensional.

If \( x_i \) is a basis of \( V \) and \( y_i \) the dual basis of \( V^* \), one has the following important element of \( H_c(W) \):

\[
h = \sum_i x_i y_i + \frac{n}{2} \sum_{s \in S} \frac{2c_s}{1 - \lambda_s} s
\]

where \( n = \dim(V) \), \( S \) is the set of complex reflections in \( W \), and \( \lambda_s \) is the non-trivial eigenvalue of \( s \) in the dual reflection representation. By [7, Section 2.1], \( h \) satisfies the commutator relations

\[
[h, x] = x \forall x \in V, \quad [h, y] = -y \forall y \in V^*.
\]

**6.3. Twisting one-dimensional representations of \( H_c(G(2, 1, n)) \).** We restrict the discussion above to the group \( G = G(2, 1, n) \cong S_n \rtimes (C_2)^n \), the Coxeter group of type \( B_n \), and consider the modules \( L_c(\tau) \) over \( H_c = H_c(G) \) which are one-dimensional. Such modules correspond to the four
linear characters \( \text{triv}, \kappa, \det \) and \( \kappa \det \) of \( G \); each character is determined by its values on \( s = s_i \) and \( t = t_i^{(-1)} \) as follows:

\[
\text{triv}: (s, t) \mapsto (1, 1), \quad \kappa: (s, t) \mapsto (1, -1), \quad \det: (s, t) \mapsto (-1, -1), \quad \kappa \det: (s, t) \mapsto (-1, 1). \tag{18}
\]

To find the parameters \( c \) where \( \dim L_c(\tau) = 1 \), we note that commutators must act on a one-dimensional module by zero, so (17) implies that the generators \( x_i \) and \( y_i, i \in [n] \), act by 0. Most relations in Definition 3.3 are satisfied by \( x_i = y_i = 0 \) automatically: the relation \( y_ix_j - x_jy_i = c_1 \sum_{\epsilon \in \{\pm 1\}} \epsilon s_{ij}^{(\epsilon)} \) holds because \( s_{ij} \) and \( s_{ij}^{(-1)} \) act on \( L_c(\tau) \) by the same scalar. The only constraint on the parameter \( c = (c_1, c_{-1}) \) arises from the last relation which reads

\[
0 = 1 - c_1 \cdot (n - 1) \cdot 2\tau(s_i) - c_{-1}\tau(t_i^{(-1)}). \tag{19}
\]

Let \( \tau \) be one of the four characters of \( G \) given in (18), and assume that \( c \) satisfies (19) so that \( L_c(\tau) \) is 1-dimensional. We apply the twisting procedure from Section 6.1 to the \( H_c \)-module \( L_c(\tau) \). The action of the group \( T \) on \( \text{End}(L_c(\tau)) \) is via conjugation, however, \( \text{End}(L_c(\tau)) \cong \mathbb{C} \) is commutative. Hence \( T \) acts trivially, and \( \text{End}(L_c(\tau))_x = \text{End}(L_c(\tau)) \).

We obtain the following 1-dimensional representation of the negative braided Cherednik algebra \( H_c = H_c(\mu(G)) \),

\[
H_c \xrightarrow{\phi} (H_c)_x \xrightarrow{\rho_c} \text{End}(L_c(\tau))_x \cong \mathbb{C}. \tag{20}
\]

Here \( \rho_c \) is the algebra homomorphism we arrive at from Proposition 6.1. Denote the 1-dimensional \( H_c \)-module given by (20) by \( L_c(\tau)_x \).

Recall from (1) that the group \( \mu(G) \) is the same as the group \( G \). To characters \( \tau \) of \( \mu(G) = G \) there correspond one-dimensional representations \( L_c(\tau) \) of the negative braided Cherednik algebra \( H_c(G) \) where the \( x_i \) and \( y_i \) act by 0 and elements of \( G \) act via \( \tau \). We can now identify the twists \( L_c(\tau)_x \) as certain representations \( L_c(\tau') \) of \( H_c(G) \), as follows:

\[
\tau'(\sigma_i) = \rho_c(\phi(\sigma_i)) = \rho_c(\hat{s}_i) = \rho_c(\sigma_i t_i) = \tau(\sigma_i)\tau(t_i), \quad \tau'(t_i) = \rho_c(\phi(t_i)) = \rho_c(t_i) = \tau(t_i).
\]

This means that twisting induces a non-trivial permutation of linear characters of the group \( G(2, 1, n) \), resulting in the following theorem which concludes the paper:

**Theorem 6.3** (Twists of one-dimensional representations of \( H_c(G(2, 1, n)) \)). The twisting procedure outlined above maps one-dimensional representations of \( H_c(G(2, 1, n)) \) to one-dimensional representations of \( H_c(G(2, 1, n)) \) as follows,

\[
L_c(\text{triv})_x = L_c(\text{triv}), \quad L_c(\kappa)_x = L_c(\kappa), \quad L_c(\det)_x = L_c(\det), \quad L_c(\kappa \det)_x = L_c(\kappa \det).
\]

7 References

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