Analytical Dispersion Equations of a Lossy Coaxial Waveguide in the Microwave and Visible Spectra

Yong Heui Cho

Abstract

Analytical hybrid-mode dispersion relations of a lossy coaxial waveguide were rigorously analyzed using a mode-matching technique. In order to model a practical coaxial line with inevitable losses, we adopted an all-dielectric coaxial waveguide surrounded by the perfect electric conductor (PEC) boundary. The rigorous dispersion characteristics of the TM01, TE01, and EH11 modes were investigated for lossy coaxial waveguides filled with different electrical conductivities. Based on the exact solutions, approximate but accurate dispersion equations were proposed for the TM0p, TE0p, EHmp, and HEmp modes in order to estimate and compare the behaviors of complex propagation constants in the microwave and visible spectra.

Key Words: Coaxial Line, Complex Propagation Constant, Dispersion Relation, Guided Wave, Mode-Matching Technique.
scattering characteristics of canonical coaxial structures used in a coaxial calibration kit [5]. The quasi-TEM mode also yields a closed-form approximate solution for the TM01-mode complex propagation constant, which becomes identical to the propagation constant [5, 7] in the high electrical conductivity limit.

II. MODE-MATCHING ANALYSIS

A lossy coaxial waveguide can be modeled using a multilayered dielectric coaxial waveguide [1, 6–8] as shown in Fig. 1, where the medium constants—\( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) and \( \mu_1, \mu_2, \mu_3 \)—can be any complex number. We use and omit \( e^{i(\beta z - \omega t)} \) for the time convention, where \( \beta \) is a complex propagation constant composed of phase (\( \Re[\beta] \)) and attenuation (\( \Im[\beta] \)) constants. The interesting point of the geometry shown in Fig. 1 is that a dielectric coaxial waveguide [1, 6–8] shown in Fig. 2 is entirely surrounded by the PEC boundary. It is known that unwanted multiple leaky modes [11] are inevitably generated in open magnetodielectric waveguides including dielectric coaxial structures [1, 6] with an open boundary as shown in Fig. 2. The existence of leaky phenomena caused by the open boundary makes a dispersion analysis of the dielectric coaxial waveguide much more involved. For instance, the dielectric coaxial waveguide shown in Fig. 2 should satisfy a leaky dispersion relation in an open region as

\[

k_0^2 = k_\rho^2 + \beta^2,
\]

where \( k_0 = \omega/\sqrt{\mu_0 \varepsilon_0} \), and \( k_\rho \) is a leaky wavenumber for a radial direction \( \rho = \sqrt{x^2 + y^2} \). Equating the imaginary parts of the left and right sides of Eq. (1) yields a leaky-wave relation for \( k_\rho \) and \( \beta \) as

\[

\Re[k_\rho] = -\Re[\beta],
\]

where the phase term of a guided wave is \( e^{i(\kappa \rho + \beta z)} \). Eq. (2) indicates that the radial and longitudinal radiation conditions of \( \kappa_\rho \) and \( \beta \) cannot be satisfied at the same time, owing to the fact that the signs of the real or imaginary parts of \( \kappa_\rho \) and \( \beta \) are always different from each other. Contrary to the geometry shown in Fig. 2, the PEC boundary and lossy region (III) introduced in Fig. 1 enable us to block the leaky modes completely and model the conductor loss precisely. This is because the PEC boundary at \( \rho = b + d \) is imposed and \( \varepsilon_3 \) can be complex, thus confirming there are no leaky modes and the waves are evanescent in region (III). Therefore, the field representations for regions (I) through (III) as shown in Fig. 1 are formulated using magnetic and electric vector potentials [9]:

\[

A^I_m(r) = \sum_{m=-\infty}^{\infty} A_m J_m(k_\rho \rho) e^{im\phi},
\]

\[

A^I_m(r) = \sum_{m=-\infty}^{\infty} E^I_m(r) J_m(k_\rho \rho) + E^II_m(r) N_m(k_\rho \rho) e^{im\phi},
\]

\[

A^I_m(r) = \sum_{m=-\infty}^{\infty} G_m(r) J_m(k_\rho \rho),
\]

\[

F^I_m(r) = \sum_{m=-\infty}^{\infty} B_m N_m(k_\rho \rho),
\]

\[

F^I_m(r) = \sum_{m=-\infty}^{\infty} E^I_m(r) J_m(k_\rho \rho) + E^II_m(r) N_m(k_\rho \rho) e^{im\phi},
\]

\[

F^I_m(r) = \sum_{m=-\infty}^{\infty} H_m(r) N_m(k_\rho \rho) e^{im\phi},
\]

where \( A_m, B_m, E^I_m(r), E^II_m(r), G_m, \) and \( H_m \) are the unknown modal coefficients for the \( m \)th azimuthal mode, \( \rho = \sqrt{x^2 + y^2} \), \( \phi = \tan^{-1}(y/x) \), \( \kappa_n = \sqrt{k_n^2 - \beta^2} \), \( k_n = \omega/\sqrt{\mu_n \varepsilon_n} \),
and (·)' denotes differentiation for the entire argument; \( J_m(·) \) and \( N_m(·) \) are the \( m \)th order Bessel functions of the first and second kinds, respectively. Based on the standard mode-matching analysis \([9]\), we enforce \( E_{x^-}, E_{\phi^-}, H_{x^-}, \) and \( H_{\phi^-} \)-field continuities at \( \rho = a \) and \( b \) for the \( m \)th azimuthal mode. First, by multiplying the \( E_{x^-} \) and \( H_{x^-} \)-field continuities at \( \rho = a \) by \( e^{-i\phi} \) \((l = 0, \pm 1, \pm 2, \cdots)\) and integrating over \( 0 \leq \phi \leq 2\pi \) yields, respectively, we get:

\[
A_m J_m(u_n) = \frac{\mu_1 \varepsilon_k \kappa^2}{\mu_2 \varepsilon_k^2} e_m(u_n),
\]

(11)

\[
B_m J_m(u_n) = \frac{\mu_1 \varepsilon_k \kappa^2}{\mu_2 \varepsilon_k^2} \varphi_m(u_n),
\]

(12)

where \( u_n = \kappa_n a, \)

\[
\epsilon_m(u) = E_m^{(1)} J_m(u) + E_m^{(2)} N_m(u),
\]

(13)

\[
\varphi_m(u) = F_m^{(1)} J_m(u) + F_m^{(2)} N_m(u).
\]

Next, we apply and integrate the \( E_{\phi^-} \) and the \( H_{\phi^-} \)-field continuities at \( \rho = a \) with \( e^{-i\phi} \) to obtain the additional field-matching equations, respectively:

\[
\frac{im}{Z_{TE}^{(1)}} A_m J_m(u_1) - u_1 B_m J'_m(u_1) = \frac{\varepsilon_1}{\varepsilon_2} \frac{im}{Z_{TE}^{(2)}} e_m(u_2) - u_2 \varphi_m(u_2),
\]

(15)

\[
u_1 A_m J'_m(u_1) + \frac{im}{Y_{TM}^{(1)}} B_m J_m(u_1) = \frac{\mu_1}{\mu_2} \nu_2 e_m'(u_2) + \frac{im}{Y_{TM}^{(2)}} \varphi_m(u_2),
\]

(16)

where \( Z_{TE}^{(1)} = \omega \mu_n / \beta, \quad Y_{TM}^{(1)} = \omega \varepsilon_n / \beta, \quad e_m(u) = \frac{d}{du} e_m(u), \) \( \varphi_m(u) = \frac{d}{du} \varphi_m(u). \) Similar to Eq. (11), in Eqs. (12), (15), and (16), we utilize the tangential boundary conditions for the \( E_{x^-}, E_{\phi^-}, H_{x^-}, \) and \( H_{\phi^-} \)-fields at \( \rho = b. \) Then, we formulate the modal relations of \( E_m^{(1),(2)}, F_m^{(1),(2)}, G_m \) and \( H_m \) as

\[
G_m = \frac{\mu_3 \varepsilon_k \kappa^2}{\mu_2 \varepsilon_k^2} e_m(v_2),
\]

(17)

\[
H_m = \frac{\mu_3 \varepsilon_k \kappa^2}{\mu_2 \varepsilon_k^2} \varphi_m(v_2),
\]

(18)
Using Eq. (30) and the conductor condition ($\sigma$ with lossy conductors using $\epsilon$ with lossy dielectrics. Since Eq. (21) is applicable to a general coaxial waveguide of any $\mu_n$ and $\epsilon_n$, we can use Eq. (21) to analyze canonical waveguides, including the PEC circular and coaxial waveguides. For instance, when $\mu_n = \mu_0$ and $\epsilon_n = \epsilon_0$, the geometry shown in Fig. 1 becomes a PEC circular waveguide with a radius of $b + d$, and Eq. (21) is thus simplified to

$$\frac{\phi_{11}^{(1)} \times \phi_{32}^{(1)}}{TM \ mode \ TE \ mode} = 0.$$

Similarly, when $\mu_n = \mu_0$, $\epsilon_1 \rightarrow \infty$, and $\epsilon_3 = \epsilon_2$, the geometry in Fig. 1 is considered a PEC coaxial waveguide. Then, the dispersion equation in Eq. (21) reduces to that of a PEC coaxial waveguide as

$$\frac{\phi_{11}^{(1)} \times \phi_{32}^{(1)}}{TM \ mode \ TE \ mode} = 0.$$

III. NUMERICAL COMPUTATIONS

We used Davidenko’s method [10, 11] to search a complex root $\beta$ by equating $|\Phi_m(\beta)| = 0$ as shown in Eq. (21). Davidenko’s method in [11] uses the fourth-order Runge–Kutta method and the Newton–Raphson method in the complex domain, which is suitable for searching the complex propagation constant $\beta$. An iterative update equation for the next approximate solution $\beta_{n+1}$ is given by

$$\beta_{n+1} = \beta_n + \Delta \beta_{n+1},$$

where $\beta_n$ is the current approximate value for $\beta$ and $\Delta \beta_{n+1}$ is defined in [11, Eq. (10)]. The next differential step $\Delta \beta_{n+1}$ is computed using $\beta_n$, $\Delta \beta_n$, and $h$, where $\beta_0$ and $\Delta \beta_0$ are the initial value and step for root searching, respectively, and $h$ is a fixed step of the Runge–Kutta method. As $n$ increases very steeply, $\beta_{n+1}$ stably converges to $\beta$ [10, 11].

We set $h = 1$ and $\Delta \beta_0 = \beta_0 / 100$ for all root-searching computations.

The TM$_{01}$-mode dispersion results of the lossy coaxial waveguide are shown in Fig. 3 using $m = 0$ and $f = 100$ GHz, where the TM$_{01}$ mode ($m = 0$) is a dominant quasi-TEM mode. We assume that regions (I) and (III) are filled with lossy conductors using $\epsilon_1 = \epsilon_3 = \epsilon_0 \left(1 + i \frac{\sigma}{\omega \epsilon_0}\right)$, where $\sigma$ denotes the electrical conductivity of a lossy dielectric. Using Eq. (30) and the conductor condition ($|\epsilon_1| \gg 1$, $|\epsilon_3| \gg 1$), we obtain an approximate dispersion relation for the TM$_{0p}$ mode as

$$i U_0(v_2) - P_3 U'_0(v_2) + P_1 V_0(v_2) \approx 0,$$

where $P_n = \epsilon_2 K_n / \epsilon_n K_2$,

$$U_m(v) = f_m(u_2) N_m(v) - N_m(u_2) f_m(v),$$

$$V_m(v) = f'_m(u_2) N_m(v) - N'_m(u_2) f_m(v).$$

Note that setting $U_m(v_2)$ and $V_m(v_2)$ equal to zero yields...
the TMₘₙ⁻ and TEₘₙ⁺-mode dispersion relations of an ideal coaxial line, respectively, where the subscript mp means that m and p are the numbers of field variations in the φ- and ρ-directions, respectively. This property indicates that Eq. (34) is a perturbed but accurate solution for the lossy coaxial waveguide when P₁ and P₃ are very small owing to |ε₁| ≫ 1 and |ε₂| ≫ 1. Utilizing β ≈ k₂, |k₁| ≫ |β|, and |k₃| ≫ |β|, Eq. (34) can be further simplified to

\[ \kappa_2 \approx \frac{k_{01}(u_2) - k_{02}(v_2)}{u_0(v_2)} \approx \frac{(k_2^2 - \frac{\mu^2}{\omega^2}N_0(u_2) + \frac{\mu^4}{\omega^4}N_0(v_2)}{N_0(u_2) - N_0(v_2)}, \]

(37)

when u → 0, N₀(u) and N₀'(u) become \( \frac{2}{\pi} \ln \left( \frac{u}{2} \right) \) and \( \frac{2}{\pi u} \), respectively. Therefore, an approximate closed-form solution for Eq. (37) is finally obtained as

\[ \beta \approx k_2 \sqrt{1 + \frac{i}{\mu_2 \ln a} \left( \frac{\mu_1}{k_1 a} + \frac{\mu_3}{k_3 b} \right)}. \]

(38)

Note that Eq. (38) is identical to [7, Eq. (37)] even though the boundary conditions for the outer conductors are different from each other. When \( \mu_1 = \mu_0 \) and \( \sigma \gg 1 \), Eq. (38) becomes a complex propagation constant \( \gamma \) obtained by the transmission line theory and penetration depth \( \delta_p \). Fig. 3 clearly indicates that the thickness \( d \) of an outer conductor hardly affects complex dispersion relations for \( \sigma \geq 10^3 \) S/m and \( d \geq 0.2 \text{ mm} \approx 4\delta_p \), where \( \delta_p \approx 50 \text{ μm} \) and \( \sigma = 10^3 \) S/m. A comparison with [8] provides a favorable agreement when \( \sigma \geq 10^3 \) S/m. This is because [8] assumes an infinite outer conductor \( (d \to \infty) \) and the effects of the thick outer conductor are negligible for high electrical conductivity, where the outer conductor shown in Fig. 1 has finite thickness \( d \), which differs from [8]. As is also shown in Fig. 3, Eqs. (34) and (38) are practically good formulas in lieu of Eq. (21) for \( \sigma \geq 10^3 \) S/m. When \( \sigma \) approaches zero, Eq. (31) predicts that \( \beta \) becomes that of a PEC circular waveguide denoted as \( \bigcirc \) in Fig. 3. The insets in Fig. 3 illustrate the normalized real and imaginary parts of the \( E_\phi \)-field distributions when \( \sigma = 10^4 \) S/m and \( \beta = 2161.77 + 65.96i \) rad/m. The real and imaginary \( E_\phi \)-fields of the TM₀₁ mode are continuous across the boundaries at \( \rho = a \) and \( b \), thus verifying that Davidenko’s method in Eq. (33) is effective for searching the complex root \( \beta \) of a lossy coaxial waveguide. In addition, the magnitude of the \( E_\phi \)-fields is rapidly attenuated within lossy dielectrics in regions (I) and (III), and thus their field distributions illustrate the behaviors of the penetration depth very well.

Fig. 4 shows the dispersion behaviors of the TM₀₁, TE₀₁, and EH₁₁ modes in the lossy coaxial waveguide versus micro-wave frequency. The TM₀₁ mode has the lowest attenuation when compared with the TE₀₁ and EH₁₁ modes, thus indicating that the TM₀₁ mode is a dominant mode irrespective of \( \sigma \). Applying Eq. (30) and the conductor condition \( (|\epsilon_1| \gg 1, |\epsilon_2| \gg 1) \) yields an approximate dispersion equation for the TE₀₁ mode as

\[ \imath V_0'(v_2) + Q_3 V_0(v_2) - Q_1 U_0'(v_2) \approx 0, \]

(39)

where \( Q_n = \mu_n k_2 / \mu_2 k_n \). Similar to Eqs. (34) and (39), the dispersion relation Eq. (21) approximately reduces to

\[ \frac{[iU_m'(v_2) - P_3 U_m'(v_2) + P_1 V_m(v_2)]}{\text{HE}_m \text{ mode}} \times \frac{[iV_m'(v_2) + Q_3 V_m(v_2) - Q_1 U_m'(v_2)]}{\text{EH}_m \text{ mode}} \approx 0. \]

(40)
When \( \sigma \geq 10^7 \text{S/m} \), the magnitude of \( \varepsilon_3 \) and \( \varepsilon_5 \) becomes very high and thus the TE_{01}- and EH_{11}-mode \( \beta \) computed by Eqs. (39) and (40) agree well with more precise solutions obtained by Eqs. (30) and (21), respectively. In Fig. 4, the attenuation constant \( \Im(\beta) \) of the EH_{11} mode rapidly decreases above \( f_c \approx 135.9 \text{GHz} \). Therefore, the TM_{01} and EH_{11} modes coexist and propagate along the lossy coaxial waveguide above \( f_c \). Considering Eq. (40), the cutoff frequency \( f_c \) of the lossy EH_{11} mode can be approximately determined using \( V_1^c (v_2) \approx 0 \), which is related to \( f_c \) of the TE_{11} mode in an ideal coaxial line. The vertical dotted lines as shown in Fig. 4 denote \( f_c \) of the coaxial TE_{11} mode.

IV. CONCLUSION

Analytical hybrid-mode dispersion relations of the lossy coaxial waveguide are presented based on the mode-matching technique and the vector potential formulations. Precise phase and attenuation constants of the TM_{01}, TE_{01}, and EH_{11} modes have been numerically evaluated using the root-searching algorithm based on Davidenko’s method. Approximate but accurate dispersion equations for the TM_{0p}, TE_{0p}, EH_{np}, and HE_{np} modes are also proposed and agree well with more rigorous dispersion relations even for low electric conductivity or negative permittivity. Our dispersion solutions can be applied to the theoretical evaluation of a coaxial calibration kit or the analytical determination of light distribution within the optical coaxial waveguide.

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Yong Heui Cho

received his B.S. degree in electronics engineering from Kyungpook National University, Daegu, Korea in 1998, and his M.S. degree and Ph.D. in electrical engineering from the Korea Advanced Institute of Science and Technology (KAIST), Daejeon, Korea in 2000 and 2002, respectively. From 2002 to 2003, he was Senior Research Staff with the Electronics and Telecommunications Research Institute (ETRI), Daejeon, Korea. In 2003, he joined the School of Information and Communication Engineering, Mokwon University, Daejeon, Korea, where he is currently a professor. From 2011 to 2012, he was a visiting professor with the Department of Electrical and Computer Engineering, University of Massachusetts Amherst, Amherst, MA, USA. From 2019 to 2020, he was a visiting researcher with the Center for Electromagnetic Metrology, Korea Research Institute of Standards and Science (KRISS), Daejeon, Korea. His current research interests include electromagnetic wave theory and scattering, numerical analysis, design of reflectarrays, and dispersion characteristics of transmission lines.