EXISTENCE OF THE MAXIMIZING PAIR FOR THE DISCRETE HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

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Abstract. In this paper, we study the best constant of the following discrete Hardy-Littlewood-Sobolev inequality,

\[ \sum_{i,j \neq j} \frac{f_i g_j}{|i-j|^{n-\alpha}} \leq C_{r,s,\alpha} |f|_r |g|_s, \]

where \( i,j \in \mathbb{Z}^n, r,s > 1, 0 < \alpha < n, \) and \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} \geq 2. \) Indeed, we can prove that the best constant is attainable in the supercritical case \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} > 2. \)

1. Introduction

In the present paper, we investigate the attainability of the best constant of the following discrete Hardy-Littlewood-Sobolev(DHLS for abbreviation) inequality

\[ \sum_{i,j \neq j} \frac{f_i g_j}{|i-j|^{n-\alpha}} \leq C_{r,s,\alpha} |f|_r |g|_s, \]

where \( i,j \in \mathbb{Z}^n, r,s > 1, 0 < \alpha < n, \) and \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} \geq 2. \) In fact, DHLS inequality is directly related to the classical Hardy-Littlewood-Sobolev(HLS) inequality

\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{n-\alpha}} \, dx \, dy \leq C'_{r,s,\alpha} \|f\|_r \|g\|_s \]

for any \( f \in L^r(\mathbb{R}^n) \) and \( g \in L^s(\mathbb{R}^n) \) provided that

\[ 0 < \alpha < n, 1 < r, s < \infty \]

with

\[ \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} = 2 \]

\( C'_{r,s,\alpha} \) is the best constant for (1.2).

We now provide a proof from (1.2) to get (1.1). One may consider the special case of (1.2) for

\[ f(x) \equiv f_i, g(x) \equiv g_i, \text{ in } |x-i| < \frac{1}{n}, \forall i \in \mathbb{Z}^n, \text{ otherwise } f(x) = g(x) = 0. \]

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Obviously, we have
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)g(y)|}{|x-y|^{n-\alpha}} \, dx \, dy = \sum_{i,j \in \mathbb{Z}^n} \int_{B_{\frac{n}{\alpha}}(i)} \int_{B_{\frac{n}{\alpha}}(j)} \frac{|f_i||g_j|}{|x-y|^{n-\alpha}} \, dx \, dy > \sum_{i,j,i \neq j} \int_{B_{\frac{n}{\alpha}}(0)} \int_{B_{\frac{n}{\alpha}}(0)} \frac{1}{|i-j|^n} \, dx \, dy \geq \sum_{i,j,i \neq j} \int_{B_{\frac{n}{\alpha}}(0)} \int_{B_{\frac{n}{\alpha}}(0)} \frac{1}{|i-j|^n} \, dx \, dy \geq c_n \sum_{i \neq j} \frac{|f_i||g_j|}{|i-j|^{n-\alpha}}
\]
(1.3)

Then by (1.2), we get (1.1) immediately for \(\frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} = 2\). For the supercritical situation, we will present a simple lemma in Section 2 to illustrate it.

It is well-known that (1.2) was studied by a remarkable paper of Lieb [14]. In [14], Lieb proved the existence of the maximizing pair \((f,g)\), i.e. the attainability of the best constant of (1.2). In particular, Lieb also gave the explicit \((f,g)\) and \(C'_{r,s,\alpha}\) in the case \(p = q\). The method Lieb used was to examine the Euler-Lagrange equation that the maximizing pair \((f,g)\) satisfies. Also, we will analysis the Euler-Lagrange equation corresponding to (1.1). After Lieb [14], Stein and Weiss first completed Lieb’s work for weighted HLS inequality. There are also many other works concerning the Euler-Lagrange equations corresponding to HLS inequality, see [3]-[10].

Now we turn to the discrete situation. For \(n=1\), (1.1) is just the Hardy-Littlewood-Pólya (HLP) inequality [12]. In [13], the authors considered (1.1) in a finite form under the assumptions that \(r = s = 2\), \(\alpha = 0\),

\[
\sum_{i,j=1,i \neq j}^{N} \frac{f_i g_j}{|i-j|} \leq \lambda_N |f|_2 |g|_2.
\]
(1.4)

As this is a finite summation, (1.4) always holds by Hölder inequality for some constant \(\lambda_N\) depending on \(N\). From (1.1), one can see that (1.4) fails for a uniform bound as \(N \to \infty\). They proved that

\[
\lambda_N = 2 \ln N + O(1).
\]

Recently, Cheng-Li [11] generalized this result to high dimension for \(r = s = 2\), \(\alpha = 0\). They pointed out that the best constant \(\lambda_N\) satisfied

\[
\lambda_N = |S^{n-1}| \ln N + o(\ln N),
\]

here \(|S^{n-1}|\) represents the Lebesgue measure of the \(n - 1\) dimensional unit sphere. The regularities of the maximizing pair \((f,g)\) are also important in analysis. Chen-Li-Zhen [2] use the regularity lifting theorem obtained in [3] to get the optimal summation interval of the solution of the Euler-Lagrange equation of (1.1). They also get some non-existence results.
In our paper, we have the following theorem.

**Theorem 1.1.** If \( r, s > 1, \alpha \in (0, n) \), \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} > 2 \), then the best constant \( C_{r,s,\alpha} \) for DHLS inequality (1.1) is attainable.

**Remark 1.1.** In fact, the assumptions of DHLS inequality (1.1) derived from HLS inequality (1.2) should be \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} = 2 \). Later, we will give a simple lemma to verify that (1.1) still holds for \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} \geq 2 \).

**Remark 1.2.** In the above theorem, we only proved the existence of the maximizing pair \((f, g)\) in the supercritical case \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} > 2 \). But, we believe it is also valid for the critical case \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} = 2 \).

The main idea to prove Theorem 1.1 is to consider a sequence of DHLS inequalities with finite elements as follows,

\[
\sum_{|i| \leq N} \sum_{|j| \leq N, i \neq j} \frac{f_i g_j}{|i-j|^{n-\alpha}} \leq C_{r,s,\alpha,N}|f|_r |g|_s, \tag{1.5}
\]

here \( f = (f_i)_{|i| \leq N}, r, s > 1, \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} > 2 \). It is easy to see that (1.5) is the restriction of (1.1) on \( f, g \) with \( f_i, g_i \equiv 0 \) for \(|i| > N\). For later use, we denote \( J(f,g) \) by

\[
J(f,g) = \sum_{i \in \mathbb{Z}^n} \sum_{j \in \mathbb{Z}^n, i \neq j} \frac{f_i g_j}{|i-j|^{n-\alpha}}.
\]

Also we take \( f^N, g^N \) with \(|f^N|_r = |g^N|_s = 1\) satisfy that

\[
J(f^N, g^N) = C_{r,s,\alpha,N}.
\]

We want to prove \( f^N, g^N \to f, g \) strongly in \( l^r, l^s \) respectively. If it is right, we have proved Theorem 1.1. Unfortunately, this is always false as we can see DHLS inequality (1.1) is invariant under translation. We should use the Concentration Compactness ideas introduced by P.L. Lions. The following theorem is important for using Concentration Compactness ideas.

**Theorem 1.2.** If \( r, s > 1, \alpha \in (0, n) \), \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} > 2 \), then

\[
\max_{|i| \leq N} f^N_i, \max_{|i| \leq N} g^N_i \geq c_{r,s,\alpha} > 0,
\]

here \( c_{r,s,\alpha} \) is a uniform constant independent of \( N \).

Theorem 1.2 tells us that after a translation \( \tilde{f}_i^N = f_{i-i_0}^N \), we will have \( \tilde{f}_0^N = \max_{|i| \leq N} f^N = f_{i_0}^N \). This excludes the case \( \tilde{f}^N, g^N \to 0 \). We will have after translation,

**Theorem 1.3.** Let \( \tilde{f}^N, \tilde{g}^N \) be the translation of \( f^N, g^N \), then \( J(\tilde{f}^N, \tilde{g}^N) \to C_{r,s,\alpha} \) and \( \tilde{f}^N, \tilde{g}^N \to f, g \) strongly respectively in \( l^r, l^s \) as \( N \to \infty \).

The present paper is organized as follows. In Section 2, we will prove Theorem 1.2 and the first part of Theorem 1.3. This is the main part of this paper and the Concentration Compactness ideas is used in this section. We will prove Theorem 1.1 in Section 3 and the last part of Theorem 1.3 in Section 4.
2. Concentration Compactness Property

This section is devoted to prove Theorem 1.2. First we shall illustrate (1.1) with the following lemma.

**Lemma 2.1.** Suppose \( a \in l^p(\mathbb{Z}^n) \), then \( |a|_q \leq |a|_p \) for all \( q \geq p \).

**Proof.** For simplicity, we may assume \( |a|_p = 1 \) which means \( |a_i| \leq 1, \ i \in \mathbb{Z}^n \). This implies that

\[
\sum_{i \in \mathbb{Z}^n} |a_i|^q \leq \sum_{i \in \mathbb{Z}^n} |a_i|^p = 1.
\]

This ends the proof of the present lemma. □

By Lemma 2.1, one can directly get (1.1) from the critical case \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} = 2 \).

A directly computation easily yields the Euler-Lagrange equation for (1.5):

\[
\begin{cases}
  r(f_i^N)^{r-1} = \lambda \sum_{j \neq i} \frac{g_j^N}{|i-j|^{n-\alpha}} \\
  s(g_i^N)^{s-1} = \mu \sum_{j \neq i} \frac{f_j^N}{|i-j|^{n-\alpha}}
\end{cases}
\]

If we multiply the first equation of (2.1) by \( f_i^N \), the second equation by \( g_i^N \) and sum up both sides, we can find out that \( \frac{r}{\lambda} = \frac{s}{\mu} = C_{r,s,\alpha,N} \). From the definition of \( C_{r,s,\alpha,N} \), it is easy to see that \( C_{r,s,\alpha,N} > 0 \) is non-decreasing with respect to \( N \). Moreover, we have the following lemma which corresponds to the first part of Theorem 1.3.

**Lemma 2.2.** Let \( C_{r,s,\alpha} \) and \( C_{r,s,\alpha,N} \) be defined as in (1.1) and (1.5) respectively. We have

\[
\lim_{N \to \infty} C_{r,s,\alpha,N} = C_{r,s,\alpha}
\]

**Proof.** It is obvious that

\[
\lim_{N \to \infty} C_{r,s,\alpha,N} \leq C_{r,s,\alpha}.
\]

Now we choose a maximizing sequence \( f^{(m)}, g^{(m)} > 0 \) with \( |f^{(m)}|_r = |g^{(m)}|_s = 1 \) such that

\[
\sum_{i \in \mathbb{Z}^n} \sum_{j \in \mathbb{Z}^n, i \neq j} \frac{f^{(m)}_i g^{(m)}_j}{|i-j|^{n-\alpha}} \geq C_{r,s,\alpha}(1 - \frac{1}{m}).
\]

Then we can choose \( N_m \) large enough depending on \( m \) such that

\[
\sum_{i \in \mathbb{Z}^n} \sum_{j \in \mathbb{Z}^n, i \neq j} \frac{f^{(m),N_m}_i g^{(m),N_m}_j}{|i-j|^{n-\alpha}} \geq C_{r,s,\alpha}(1 - \frac{1}{m})^2.
\]

Here \( f^{(m),N_m}_i \) means that

\[
f^{(m),N_m}_i = \begin{cases} f^{(m)}_i, & \text{for } |i| \leq N_m \\ 0, & \text{for } |i| > N_m. \end{cases}
\]
From the cut-off above, we have
\[ |f^{(m)}_i, N_m|_{L^r} \leq 1, C_{r,s,\alpha,N_m} \geq C_{r,s,\alpha}(1 - \frac{1}{m})^2. \]

Passing \( m \to \infty \), we get the desired result. \( \square \)

By Lemma 2.2, it is true that
\[ 0 < c_0 \leq C_{r,s,\alpha,N} \leq C_0 < \infty \]
for some uniform constants \( c_0, C_0 \). Therefore without loss of generality, we may assume \( C_{r,s,\alpha,N} = 1 \) in the proof of Theorem 1.2, since we only use the uniform up bound and lower bound of \( C_{r,s,\alpha,N} \).

The proof for Theorem 1.2: Taking the equation of \( f^N \) for instance, by (2.1) we have
\[ (2.2) \quad (f^N_i)^{r-1} = \sum_{|j| \leq N, j \neq i} \frac{g_j^N}{|i - j|^{n-\alpha}} \leq \max_{|j| \leq N} (g_j^N)^\epsilon \sum_{|j| \leq N, j \neq i} \frac{(g_j^N)^{1-\epsilon}}{|i - j|^{n-\alpha}}. \]

Here \( 0 < \epsilon < 1 \) is a parameter to be determined later. This means that
\[ (2.3) \quad 1 = \sum_{|i| \leq N} (f^N_i)^r \leq \max_{|k| \leq N} (g_k^N)^\epsilon \sum_{|i| \leq N} \left( \sum_{|j| \leq N, j \neq i} \frac{(g_j^N)^{1-\epsilon}}{|i - j|^{n-\alpha}} \right)^{\frac{r}{r-1}}. \]

Now we define an operator \( T \) satisfying:
\[ (Tf)_i = \sum_{j \in \mathbb{Z}^N, j \neq i} \frac{f_i}{|j - i|^{n-\alpha}}. \]

Then by DHLS inequality, we have
\[ |Tf|_p \leq C|f|_q \]
for \( \frac{1}{q} + \frac{n-\alpha}{n} = 1 + \frac{1}{p} \). We take \( p = \frac{r}{r-1}, q = \frac{s}{1-\epsilon} \), then we can get the righthand side of (2.3),
\[ (2.4) \quad |T((g^N)^{1-\epsilon})|_{L^{\frac{r}{r-1}}} \leq C|(g^N)^{1-\epsilon}|_{L^{\frac{s}{1-\epsilon}}} = C. \]

To guarantee (2.4), we need
\[ \frac{r - 1}{r} + 1 = \frac{1 - \epsilon}{s} + \frac{n - \alpha}{n}, \quad \text{i.e.,} \quad 2 + \frac{\epsilon}{s} = \frac{1}{r} + \frac{n - \alpha}{n}. \]

By the assumption of Theorem 1.2, we see \( \epsilon > 0 \). Also, as \( \frac{1}{r} + \frac{n - \alpha}{n} < 2 \), we must have \( \epsilon < 1 \). From (2.4), one can get
\[ \max_{|k| \leq N} (g_k^N)^\epsilon \leq C^{\frac{r}{r-1}} \geq 1. \]

Or we have
\[ \max_{|k| \leq N} (g_k^N) \geq c_{r,s,\alpha}. \]

The proof for the lower bound of \( \max_{|k| \leq N} (f^N_k) \) is just the same, we omit the details here. \( \square \)
Lemma 3.1. \( \forall \) \( f \) and \( g \) choose a subsequence still denoted by \( f \) with max \( f_i \). We only need to show the first part of (3.1) is right. As Proof. Here \( \Omega \) is finite summation. We can pass the limit in the left-hand side and the first part of right-hand side of (3.2) and \( f^{(N)} \rightarrow f, g^{(N)} \rightarrow g, \) weakly in \( l^r, l^s \) respectively. It is easy to see that \( f_0, g_0 \geq c > 0 \) and \( |f|_{l^r}, |g|_{l^s} \leq 1 \). Now we can have the following lemma.

Lemma 3.1. \( \forall i, j \in \mathbb{Z}^n, f_i > 0, g_j > 0, \) we have

\[
\begin{align*}
C_{r,s,\alpha} f_i^{r-1} &= \sum_{k, k \neq i} \frac{g_k}{|k - i|^{n-\alpha}} \\
C_{r,s,\alpha} g_j^{s-1} &= \sum_{k, k \neq i} \frac{f_k}{|k - j|^{n-\alpha}}.
\end{align*}
\]

Proof. We only need to show the first part of (3.1) is right. As \( f_i > 0 \), we can see \( f_i^{(N)} > 0 \) for \( N \) large, then for any fixed \( M \),

\[
(3.2) \quad C_{r,s,\alpha,N} \left( f_i^{(N)} \right)^{r-1} = \sum_{k, k \neq i} \frac{g_k^{(N)}}{|k - i|^{n-\alpha}} = \sum_{|k| \leq M, k \neq i} \frac{g_k^{(N)}}{|k - i|^{n-\alpha}} + \sum_{|k| > M, k \neq i} \frac{g_k^{(N)}}{|k - i|^{n-\alpha}}
\]

We can pass the limit in the left-hand side and the first part of right-hand side of (3.2) since it is finite summation.

\[
\sum_{|k| > M, k \neq i} \frac{g_k^{(N)}}{|k - i|^{n-\alpha}} \leq \left( \sum_{|k| > M, k \neq i} \left( g_k^{(N)} \right)^s \right)^{\frac{1}{s}} \left( \sum_{|k| > M, k \neq i} \left| k - i \right|^{-(n-\alpha)s} \right)^{\frac{s-1}{s}}
\]

(3.3) \( \leq CM^{n-\frac{(n-\alpha)s}{s}} \rightarrow 0, \) as \( M \rightarrow \infty \).

In getting the last inequality, we have used \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} > 2 \) which means \( \frac{1}{s} > \frac{\alpha}{n}. \) \( \square \)

Lemma 3.2.

\( |f|_{l^r} = |g|_{l^s} = 1. \)
Proof. If it’s not true, by Lemma 3.1, we can easily see that $0 < |f|^r_r = |g|^s_s < 1$. Set

$$
\bar{f}_i = \frac{f_i}{|f|^r_r}, \quad \bar{g}_i = \frac{g_i}{|g|^s_s}.
$$

Then

$$
J(\bar{f}, \bar{g}) = \sum_i \sum_{j, j \neq i} \frac{f_i g_j}{|i - j|^n - \alpha} |f|^{r-1}_r |g|^{s-1}_s
$$

(3.4)

which is a contradiction to the definition of best constant. The last inequality follows from

$$
\frac{1}{r} + \frac{1}{s} > \frac{\alpha}{n} + 1 > 1.
$$

□

In fact, Lemma 3.2 implies Theorem 1.1 with $(f, g)$ as the maximizing pair. Although in passing the limit to get the maximizing pair $(f, g)$, we may only have $f_i, g_i > 0$ for $i \in \Omega \subset \mathbb{Z}^n$. But in fact, as we know $(f, g)$ are maximizing pair, they should satisfy (3.1) for all $i \in \mathbb{Z}^n$ which means $f, g > 0$.

4. THE STRONG CONVERGENCE OF $\bar{f}^N, \bar{g}^N$

This section is denoted to prove the second part of Theorem 1.3. The following lemma is a special case of Theorem 2 in [1]. We provide a simple proof here.

**Lemma 4.1.** Suppose $f^N \in l^r(\mathbb{Z}^n)$ and $f^N \to f \in l^r(\mathbb{Z}^n)$ pointwise in $\mathbb{Z}^n$ as $N \to \infty$. Then we will have

$$
\lim_{N \to \infty} |f^N - f|^r_r = 0,
$$

provided $\lim_{N \to \infty} |f^N|^r_r = |f|^r_r$.

**Proof.** For any fixed $\epsilon > 0$, we can choose $M$ large enough such

$$
\sum_{|i| \leq M} |f_i|^r \geq |f|^r_r(1 - \epsilon).
$$

(4.1)

For such fixed $M$, by the pointwise convergence of $f^N$, we can choose $N_M$ large enough so that

$$
\sum_{|i| \leq M} |f_i^N - f_i|^r \leq \epsilon |f|^r_r, \quad \forall N \geq N_M.
$$

(4.2)

Combining (4.1) and (4.2), we have $(1 - 2\epsilon)|f|^r_r \leq \sum_{|i| \leq M} |f_i^N|^r \leq |f|^r_r$. From $\lim_{N \to \infty} |f^N|^r_r = |f|^r_r$, we have another $N_\epsilon$

$$
(1 - \epsilon)|f|^r_r \leq |f^N|^r_r \leq (1 + \epsilon)|f|^r_r, \quad \forall N \geq N_\epsilon.
$$
Hence for $N \geq \max(N_M, N_\epsilon) = N_0$,

$$\sum_{|i| > M} |f_i|^r_N \leq 3\epsilon |f|^r.$$

Now we have

$$|\tilde{f}^N - f|^r = \sum_{|i| \leq M} |f_i^N - f_i|^r + \sum_{|i| > M} |f_i^N - f_i|^r \leq \epsilon |f|^r + C \sum_{|i| > M} (|f_i^N|^r + |f_i|^r) \leq 5\epsilon |f|^r$$

for $N \geq N_0$. Passing $\epsilon \rightarrow 0$, we have finished the proof of the present lemma. \qed

The second part of Theorem 1.3 is the direct conclusion of Lemma 3.2 and Lemma 4.1.

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