Integrability of spin-$\frac{1}{2}$ fermions with charge pairing and Hubbard interaction.

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**Abstract**

In this paper we study the exact solution of a one-dimensional model of spin-$\frac{1}{2}$ electrons composed by a nearest-neighbor triplet pairing term and the on-site Hubbard interaction. We argue that this model admits a Bethe ansatz solution through a mapping to a Hubbard chain with imaginary kinetic hopping terms. The Bethe equations are similar to that found by Lieb and Wu \[3\] but with additional twist phases which are dependent on the ring size. We have studied the spectrum of the model with repulsive interaction by exact diagonalization and through the Bethe equations for large lattice sizes. One feature of the model is that it is possible to define the charge gap for even and odd lattice sites and both converge to the same value in the infinite size limit. We analyze the finite-size corrections to the low-lying spin excitations and argue that they are equivalent to that of the spin-$\frac{1}{2}$ isotropic Heisenberg model with a boundary twist depending on the lattice parity. We present the classical statistical mechanics model whose transfer matrix commutes with the model Hamiltonian. To this end we have used the construction employed by Shastry \[6,7\] for the Hubbard model. In our case, however, the building block is a free-fermion eight-vertex model with a particular null weight.

**Keywords:** Lattice fermions model, Integrability, Bethe ansatz, Yang-Baxter.

June 2020
1 The Model Hamiltonian

In general, correlations among fermions in one-dimension give rise to complex phase diagram with charge and spin ordering. One of the simplest lattice system model that describes the effect of such correlations is the Hubbard model \[1,2\]. This model encodes the basics physics concerning the competition between electron kinetic energy and the on-site Coulomb interaction denoted here by \( U \). The Hamiltonian of this model on a ring of size \( L \) with a electron-hole symmetric interaction is given by,

\[
H = - \sum_{j=1}^{L} \sum_{\alpha=\uparrow,\downarrow} [c_{\alpha}^\dagger(j)c_{\alpha}(j+1) + c_{\alpha}^\dagger(j+1)c_{\alpha}(j)] + U \sum_{j=1}^{L} (n_{\uparrow}(j) - \frac{1}{2})(n_{\downarrow}(j) - \frac{1}{2}),
\]

(1)

where \( c_{\alpha}^\dagger(j) \) and \( c_{\alpha}(j) \) creates and annihilates fermions on site \( j \) with spin \( \alpha \) and \( n_{\alpha}(j) = c_{\alpha}^\dagger(j)c_{\alpha}(j) \) is the occupation number operator. Here we apply periodic boundary conditions by identifying the sites \( L + 1 \equiv 1 \).

In 1968 Lieb and Wu showed that the Hamiltonian (1) can be diagonalized by an extension of the Bethe ansatz technique \[3\]. They used this solution to argue that the Hubbard model at half-filling is an insulator for positive values of \( U \) and undergoes a Mott transition at \( U = 0 \). The literature exploring the solution by Lieb and Wu is nowadays vast and for a collection of reprints and an extensive review on this subject see for instance \[4, 5\]. In the context of this paper we mention the progresses made by Shastry towards the understanding of the algebraic structure associated to the integrability of the one-dimensional Hubbard model \[6,7\]. In particular, this author discovered a two-dimensional vertex model of classical statistical mechanics whose transfer matrix commutes among themselves and with the Hubbard Hamiltonian.

The purpose of this work is to introduce a variant of the Hubbard model and to discuss its solution by the Bethe ansatz as well as to uncover the underlying covering vertex model. The model is defined replacing the hopping term of the Hubbard chain by a nearest-neighbor charge pairing potential. The corresponding model Hamiltonian is,

\[
H_c = \sum_{j=1}^{L} \sum_{\alpha=\uparrow,\downarrow} [c_{\alpha}(j)c_{\alpha}(j+1) + c_{\alpha}^\dagger(j+1)c_{\alpha}^\dagger(j)] + U \sum_{j=1}^{L} (n_{\uparrow}(j) - \frac{1}{2})(n_{\downarrow}(j) - \frac{1}{2}),
\]

(2)
where periodic boundary conditions is assumed.

We observe that the first term of Hamiltonian (2) causes charges to be created or annihilated in pairs being similar to a triplet pairing in the p-wave theory of superconductivity. Here we are considering the situation in which the pairing energy is the same for both spin up and down channels. The interaction term is the same as that of the Hubbard model which is taken symmetric under the electron-hole transformation $c_\alpha(j) \leftrightarrow c_\alpha^\dagger(j)$.

The charge pair model (2) enjoys of translation invariance, the symmetry under spin flips and the invariance under two $\mathbb{Z}_2$ symmetries represented by the unitary transformation $V_\alpha H_c V_\alpha^\dagger$ with $V_\alpha = e^{i\pi \sum_{j=1}^L n_\alpha(j)}$. Besides that we have other global invariance with respect to specific rotations associated to the spin space. In order to describe that we first recall the structure of the isomorphic $SU(2)$ algebras which can be constructed out of the possible six non-vanishing on-site combinations of spin-$\frac{1}{2}$ fermionic operators. The on-site generators of the standard spin $SU(2)$ algebra is known to be given by,

$$S_x^z = \frac{1}{2} [c_\uparrow^\dagger(j)c_\downarrow(j) + c_\downarrow^\dagger(j)c_\uparrow(j)], \quad S_y^z = \frac{i}{2} [c_\uparrow^\dagger(j)c_\uparrow(j) - c_\downarrow^\dagger(j)c_\downarrow(j)], \quad S_z^z = \frac{1}{2} [n_\uparrow(j) - n_\downarrow(j)]. \quad (3)$$

Yet another basis can be obtained by applying for instance the electron-hole transformation on the spin down fermionic operators defined by (3). This gives rises to the so-called pseudo-spin or charge $SU(2)$ algebra whose on-site generators are,

$$R_x^z = \frac{1}{2} [c_\uparrow^\dagger(j)c_\downarrow^\dagger(j) + c_\downarrow^\dagger(j)c_\uparrow(j)], \quad R_y^z = \frac{i}{2} [c_\uparrow^\dagger(j)c_\uparrow(j) - c_\downarrow^\dagger(j)c_\downarrow(j)], \quad R_z^z = \frac{1}{2} [n_\uparrow(j) + n_\downarrow(j) - 1]. \quad (4)$$

For arbitrary values of $L$ the Hubbard model (1) is invariant by the full non-Abelian spin $SU(2)$ symmetry (3). However, this is not the case of the Hamiltonian (2) which is invariant only when this symmetry is broken down to rotations around the $y$ axis. The same observation applies to the pseudo-spin $SU(2)$ algebra (4) since the charge pair model is invariant by such symmetry when it is restricted to rotations around the $x$ axis. More precisely, for arbitrary values of $L$ we have the following conservation laws,

$$[H_c, \sum_{j=1}^L S_j^y] = [H_c, \sum_{j=1}^L R_j^z] = 0. \quad (5)$$

\textsuperscript{1}For $L$ even we will see that such rotations around specific axes are enlarged to $SU(2)$ symmetries.
In next section we shall explore the commutations (5) in order to determine the eigenspectrum of the model by the coordinate nested Bethe ansatz method. These charges can be made equivalent to the conservation of particle numbers by means of a redefinition of the original electrons operators. In this new fermionic basis, the Hamiltonian (2) is mapped to the form of the Hubbard model but with pure imaginary and asymmetric hopping terms. We find that the corresponding Bethe equations for $L$ even are distinct from that with $L$ odd through suitable boundary twists. In section 3 we investigate the properties of the spectrum of the model for repulsive interactions. We argue that for both $L$ even and odd we can define a lattice charge gap with respect to the ground state which in the thermodynamic limit converges to the value computed by Lieb and Wu for the Hubbard model at half-filling [3]. We have used the Bethe equations to study the finite-size corrections associated to the excitations due to the spin degrees of freedom. We find that they are equivalent to that of the isotropic spin-$\frac{1}{2}$ Heisenberg model with periodic boundary for $L$ even and with a twisted toroidal boundary when $L$ is odd. In section 4 we describe the lattice vertex model whose transfer matrix commutes with the Hamiltonian (2). This is done by using a construction due to Shastry devised to couple two symmetric six-vertex models satisfying the free-fermion condition [6,7]. However, in our case the building block has the form of an eight-vertex model in which one of the weights is zero. The fact that Shastry’s formulation also works for such special $Z_2$ invariant vertex model seems to have been unnoticed in the literature. Our concluding remarks are given in section 5 and in Appendix A we present the technical details on the underlying Yang-Baxter algebra.

2 The Energy Hamiltonian Spectrum

The space of states of spin-$\frac{1}{2}$ fermions associated with every lattice is four-dimensional and they can be represented as,

$$|0\rangle, \quad c_\uparrow(j)|0\rangle, \quad c_\downarrow(j)|0\rangle, \quad c_\uparrow(j)c_\downarrow(j)|0\rangle,$$  \hspace{1cm} (6)

where $|0\rangle$ denotes the vacuum state defined by the condition $c_\alpha(j)|0\rangle = 0$. 

3
In the above canonical basis the local conserved charges $S_y^j$ and $L_x^j$ are viewed as anti-diagonal matrices. However, they can be both diagonalized by on-site unitary transformation with the following similarity matrix,

$$V_j = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & -1 & i & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$  

(7)

We can use this transformation to define new on-site fermionic operators,

$$d_\alpha(j) = V_j c_\alpha(j) V_j^\dagger, \quad d_\alpha^\dagger(j) = V_j c_\alpha^\dagger(j) V_j^\dagger,$$

(8)

and their explicit expressions in terms of the electrons operators are,

$$d_\uparrow(j) = \frac{i}{2} c_\uparrow(j) + \frac{1}{2} c_\downarrow(j) - \frac{1}{2} c_\downarrow(j), \quad d_\downarrow^\dagger(j) = -\frac{i}{2} c_\downarrow^\dagger(j) + \frac{1}{2} c_\uparrow(j) - \frac{1}{2} c_\uparrow(j),$$

$$d_\downarrow(j) = \frac{1}{2} c_\uparrow(j) + \frac{i}{2} c_\downarrow(j) - \frac{i}{2} c_\downarrow(j) + \frac{1}{2} c_\uparrow(j), \quad d_\uparrow^\dagger(j) = \frac{1}{2} c_\downarrow^\dagger(j) - \frac{i}{2} c_\uparrow(j) + \frac{i}{2} c_\downarrow(j) + \frac{1}{2} c_\uparrow(j).$$

(9)

By transforming back the above relations we can represent the conserved charges in terms of the new fields $d_\alpha(j)$ and $d_\alpha^\dagger(j)$. The expression of the spin algebra charge component is,

$$\sum_{j=1}^L S_y^j = \frac{1}{2} \sum_{j=1}^L \left[ d_\uparrow^\dagger(j)d_\uparrow(j) - d_\downarrow^\dagger(j)d_\downarrow(j) \right],$$

(10)

while the one associated to the pseudo-spin algebra is,

$$\sum_{j=1}^L R_x^j = \frac{1}{2} \sum_{j=1}^L \left[ d_\uparrow^\dagger(j)d_\uparrow(j) + d_\downarrow^\dagger(j)d_\downarrow(j) - 1 \right].$$

(11)

By the same token the Hamiltonian (2) of the charge pair model can be expressed as follows,

$$\tilde{H}_c = \sum_{j=1}^L \left[ e^{i\pi/2} d_\uparrow^\dagger(j)d_\uparrow(j + 1) + e^{-i\pi/2} d_\downarrow^\dagger(j)d_\downarrow(j + 1) + e^{i\pi/2} d_\downarrow^\dagger(j)d_\downarrow(j + 1) + e^{-i\pi/2} d_\uparrow^\dagger(j)d_\uparrow(j) \right]$$

$$+ U \sum_{j=1}^L (d_\uparrow^\dagger(j)d_\uparrow(j) - \frac{1}{2})(d_\downarrow^\dagger(j)d_\downarrow(j) - \frac{1}{2}),$$

(12)
which has the typical form of the Hubbard Hamiltonian however with imaginary and asymmetric hopping terms.

The conserved charges (10,11) imply that the Hilbert space of the Hamiltonian (12) can be separated into block disjoint sectors labeled by the total number $N_{\alpha} = \sum_{j=1}^{L} d_{\alpha}^\dagger(j)d_{\alpha}(j)$ of fermions of spin $\alpha$. This means that eigenvalue problem can be formally written as,

$$\tilde{H}_c \left| N_{\uparrow}, N_{\downarrow} \right\rangle = E(N_{\uparrow}, N_{\downarrow}, U) \left| N_{\uparrow}, N_{\downarrow} \right\rangle. \hspace{1cm} (13)$$

The range of the quantum numbers can be constrained observing that Hamiltonian (12) is invariant under the particle-hole symmetry $d_{\alpha}(j) \leftrightarrow d_{\alpha}^\dagger(j)$. In fact, taking into account this invariance we obtain the spectral identity,

$$E(N_{\uparrow}, N_{\downarrow}, U) = E(L - N_{\uparrow}, L - N_{\downarrow}, U). \hspace{1cm} (14)$$

Here we remark that such spectral relation is valid for arbitrary $L$ in the case of the transformed charge pair model Hamiltonian (12). Therefore, unlike the Hubbard model no restriction to bipartite lattices is necessary in order to relate the energies of different sectors with the same coupling $U$. The spectral identity (14) together with spin flip invariance tell us that we may restrict our considerations to states,

$$N_{\uparrow} + N_{\downarrow} \leq L \hspace{0.5cm} \text{and} \hspace{0.5cm} N_{\uparrow} \geq N_{\downarrow}, \hspace{1cm} (15)$$

for even and odd values of $L$. We emphasize that in the case of the Hubbard model (1) the spectral relation (14) and constraint (15) only works when $L$ is even.

We now can determine the eigenspectrum of the Hamiltonian (12) by adapting the nested Bethe ansatz approach employed Lieb and Wu in the presence of hopping phases. In a given sector with total number of fermions $N = N_{\uparrow} + N_{\downarrow}$ the wave function may be represented as linear combination of $N$-particle states,

$$\left| N_{\uparrow}, N_{\downarrow} \right\rangle = \sum_{x_1, x_2, \ldots, x_N} \psi_{\alpha_1, \alpha_2, \ldots, \alpha_N}(x_1, x_2, \ldots, x_N) \prod_{j=1}^{N} e^{i\phi_{\alpha_j}(1-x_j)} d_{\alpha_j}^\dagger(x_j) \left| \tilde{0} \right\rangle, \hspace{1cm} (16)$$
where the reference state $|\tilde{0}\rangle$ is taken such that $d_\alpha(j) |\tilde{0}\rangle = 0$ for any site $j$ and spin $\alpha$. The exponentials terms in (16) are able to pull the hopping bulk phases up to the boundary terms. In our case this happens when we choose the twists to be fixed as $\phi_\uparrow = \frac{\pi}{2}$ and $\phi_\downarrow = -\frac{\pi}{2}$.

In the Bethe ansatz approach one assumes that the $N$-particle amplitudes have a plane wave form 

$$\psi_{\alpha_1,\alpha_2,\ldots,\alpha_N}(x_1, x_2, \ldots, x_N) = \sum_{P} A(Q|P) \exp[i k_{P_1} x_{Q_1} + i k_{P_2} x_{Q_2} + \cdots + i k_{P_N} x_{Q_N}],$$

where it is assumed the ordering $x_{Q_1} \leq x_{Q_2} \leq \cdots \leq x_{Q_N}$. The partition $Q = \{Q_1, Q_2, \ldots, Q_N\}$ denotes the $N!$ permutations of fermions with positions $x_{Q_1}, x_{Q_2}, \ldots, x_{Q_N}$ and spins $\alpha_1, \alpha_2, \ldots, \alpha_N$ while $P = \{P_1, P_2, \ldots, P_N\}$ refers to similar permutations on the fermions momenta $k_{P_1}, k_{P_2}, \ldots, k_{P_N}$. The coefficients associated to these permutation are denoted by $A(Q|P)$.

In this formulation the hopping phases are all removed excepted those associated to fermions hopping among the boundary sites $j = 1$ and $j = L$. At this point the situation becomes equivalent to that of generalized diagonal boundary conditions discussed for the Hubbard model in [8,9]. This fact is taken into account by requiring that $A(Q, P)$ satisfy the condition,

$$\exp(ik_{P_N}L) A(Q|P) = \left[ \exp(i \frac{\pi L}{2}) \delta_{Q,N} + \exp(-i \frac{\pi L}{2}) (1 - \delta_{Q,N}) \right] A(\bar{Q} | \bar{P}),$$

where $\bar{Q} = \{Q_n, Q_1, \ldots, Q_{N-1}\}$ and $\bar{P} = \{P_n, P_1, \ldots, P_{N-1}\}$ are cyclic permutations of the partitions $Q$ and $P$, respectively.

From now on the procedure is analog to that already exposed by Lieb and Wu [3] and we shall present only the main results. The spectrum of the Hamiltonian (12) is parametrized in terms of a set of variables $\{k_j, \mu_j\}$ which fulfill the following nested Bethe equations,

$$e^{ik_j L} = e^{i \frac{\pi L}{2}} \prod_{l=1}^{N_\uparrow} \frac{\sin(k_j) - \mu_l - \frac{i U}{4}}{\sin(k_j) - \mu_l + \frac{i U}{4}}, \quad j = 1, 2, \ldots, N_\uparrow + N_\downarrow,$$

$$\prod_{j=1}^{N_\uparrow + N_\downarrow} \frac{\sin(k_j) - \mu_l + \frac{i U}{4}}{\sin(k_j) - \mu_l - \frac{i U}{4}} = e^{i \pi L} \prod_{k=1}^{N_\uparrow} \frac{\mu_k - \mu_l - \frac{i U}{2}}{\mu_k - \mu_l + \frac{i U}{2}}, \quad l = 1, \ldots, N_\downarrow,$$

while the eigenvalue of the transformed Hamiltonian (12) associated with the state specified by
the rapidities \( \{k_j, \mu_j\} \) is given by,

\[
E(N_\uparrow, N_\downarrow, U) = -2 \sum_{j=1}^{N_\uparrow + N_\downarrow} \cos(k_j) + \frac{U}{2} \left( \frac{L}{2} - N_\uparrow - N_\downarrow \right).
\] (21)

We would like to close this section with the following comments. We first observe that the fermionic chain for \( L = 2 \) is somehow special since the charge pairing terms are canceled and the Hamiltonian (2) becomes a diagonal operator. From the Bethe solution point of view this peculiarity is associated with the presence of the minus sign factor in the first level Bethe equation (19). We have checked this fact by solving the two sites Bethe equations (19,20) for roots configurations satisfying the restriction (15). These solutions indeed reproduce the expected Hamiltonian energies and our findings have been summarized in Table 1.

| \((N_\uparrow, N_\downarrow)\) | \(E(N_\uparrow, N_\downarrow, U)\) | Bethe roots |
|---|---|---|
| (0, 0) | \(\frac{U}{2}\) | empty set |
| (1, 0) | 0 | \(k_1 = \pm \frac{\pi}{2}\) |
| (2, 0) | \(-\frac{U}{2}\) | \(k_1 = \frac{\pi}{2}, k_2 = -\frac{\pi}{2}\) |
| (1, 1) | \(\frac{U}{2}\) | \(e^{ik_1} = \frac{-U - \sqrt{U^2 - 16}}{4}, e^{ik_2} = \frac{-U + \sqrt{U^2 - 16}}{4}, \mu_1 = 0\) |
| (1, 1) | \(-\frac{U}{2}\) | \(k_1 = 0, k_2 = \pi, \mu_1 = 0\) |

Table 1: The spectrum of Hamiltonians (2,12) for \( L = 2 \) where the sectors \((N_\uparrow,N_\downarrow)\) satisfy (15). The eigenvalues are obtained substituting the Bethe roots into the relation (21).

We next note that for arbitrary \( L \) the Bethe equations (19,20) are similar to that of the Hubbard model and the main difference are the presence of certain phase factors depending on the lattice parity. In particular, when \( L \) is multiple of four such phase factors are unity resulting in the same Bethe equations of the Hubbard model. We conclude that for \( L = 4, 8, 12, \ldots \) the spectrum of the Hubbard Hamiltonian (1) and the charge pair model (2) should be exactly the same. This fact has been verified for \( L = 4, 8 \) by comparing all the energy levels of both Hamiltonians using exact diagonalization. However, the structure of the wave-function on the electron basis is expected to be rather different because of the canonical transformation (9). We can see that considering
examples of simple states whose energy per site is independent of the size $L$. From the Hubbard model perspective we already know that there exists two such eigenvalues associated to the trivial ferromagnetic and anti-ferromagnetic states. We find out that these energies also belong to the spectrum of the charge pair model Hamiltonian (2) but with distinct wave-function structure. The form of the wave-function on the canonical basis can be uncovered with the help of the transformation (9). The final results for such states have been summarized on Table 2.

| Eigenvalue | Hubbard Eigenvector | Charge Pair Eigenvector |
|------------|---------------------|-------------------------|
| $+\frac{LU}{4}$ | $|0\rangle, \prod_{j=1}^{L} c^\dagger_{\uparrow}(j)c^\dagger_{\downarrow}(j)|0\rangle$ | $\prod_{j=1}^{L} \left(1 \pm c^\dagger_{\uparrow}(j)c^\dagger_{\downarrow}(j)\right)|0\rangle$ |
| $-\frac{LU}{4}$ | $\prod_{j=1}^{L} c^\dagger_{\uparrow}(j)|0\rangle, \prod_{j=1}^{L} c^\dagger_{\downarrow}(j)|0\rangle$ | $\prod_{j=1}^{L} \left(c^\dagger_{\uparrow}(j) \pm ic^\dagger_{\downarrow}(j)\right)|0\rangle$ |

Table 2: Example of eigenstates the Hubbard model (1) and the charge pair model (2) with common eigenvalue for arbitrary $L$.

Finally, we remark that for a bipartite lattice the rotation invariance (5) of the charge pair Hamiltonian around specific axes are enlarged to the invariance under two $SU(2)$ symmetries. This is similar to the case of the Hubbard model (1) which for $L$ even has besides the spin $SU(2)$ symmetry (3) another distinct $SU(2)$ invariance named the $\eta$-pairing symmetry [10, 11].

In fact, for an even number of lattice sites the invariance of the Hamiltonian (2) under the rotation around the $y$-axis of the spin algebra (3) extends to a full “spin” $SU(2)$ symmetry, namely

$$[H_c, \sum_{j=1}^{L} \tilde{S}_{j}^x] = [H_c, \sum_{j=1}^{L} S_{j}^y] = [H_c, \sum_{j=1}^{L} \tilde{S}_{j}^z] = 0$$ (22)

where now the extra on-site generators $\tilde{S}_{j}^x$ and $\tilde{S}_{j}^z$ alternate among the lattice sites,

$$\tilde{S}_{j}^x = \frac{1}{2}(-1)^j[c^\dagger_{\uparrow}(j)c_{\downarrow}(j) + c^\dagger_{\downarrow}(j)c_{\uparrow}(j)], \quad \tilde{S}_{j}^z = \frac{1}{2}(-1)^j[n_{\uparrow}(j) - n_{\downarrow}(j)].$$ (23)

The same happens to the rotation around the $x$-axis of the charge algebra (4). For $L$ even it
is enlarged to the following “charge” $SU(2)$ symmetry,

$$[H_c, \sum_{j=1}^L R^x_j] = [H_c, \sum_{j=1}^L \tilde{R}^y_j] = [H_c, \sum_{j=1}^L \tilde{R}^z_j] = 0 \quad (24)$$

where the expression for the additional staggered on-site generators $\tilde{R}^y_j$ and $\tilde{R}^z_j$ are given by,

$$\tilde{R}^y_j = \frac{i}{2}(-1)^j[c_\downarrow(j)c_\uparrow(j) - c_\uparrow(j)c_\downarrow(j)], \quad \tilde{R}^z_j = \frac{1}{2}(-1)^j[n_\uparrow(j) + n_\downarrow(j) - 1]. \quad (25)$$

3 The Spectrum Properties for $U > 0$

As far as the energy spectrum is concerned the difference among the charge pair model (2) and the Hubbard chain (1) is the presence of size dependent twists in the Bethe equations. However, these fluxes are not expected to affect the value of ground state energy per site in the thermodynamic limit. The value should be same as that of the Hubbard model in the half-filled case determined long ago by Lieb and Wu [3]. Denoting this energy by $e_\infty$ we have,

$$e_\infty = -4 \int_0^\infty \frac{J_0(x)J_1(x)}{x[\exp(Ux/2) + 1]}dx - \frac{U}{4} \quad (26)$$

where $J_0(x)$ and $J_1(x)$ are Bessel functions.

The other basic feature of the half-filled Hubbard model is the presence of energy gap in the charge excitation sector. For $L$ even this mass gap was defined by Lieb and Wu [3] as the energy $\Delta(L)$ of a particle or a hole excitation with respect to the half-filled state. In the thermodynamic limit its value was computed to be [3],

$$\Delta(\infty) = 4 \int_0^\infty \frac{J_1(x)}{x[\exp(Ux/2) + 1]}dx + \frac{U}{2} - 2 \quad (27)$$

We shall argue that for the charge pair model (2) it is possible to define the energy gap for both even and odd lattice sites. It turns out that the phase factors for $L$ odd compensate the effects of frustration due to the lattice parity and the energy gap of either a hole or a particle excitation over the double degenerated ground state is the same. We shall present numerical evidences that the value of the gap for even and odd sites converges in the thermodynamic limit to the result (27).
The other known feature of the Hubbard model at half-filling is that the spin excitations are gapless in the repulsive regime. The phases twists for the charge pair model will not change this behaviour but the conformal data will be dependent on the parity of the lattice size. In what follows we will also study the finite-size effects for some of the gapless states of the charge pair Hamiltonian [2].

3.1 Finite-size effects for L even

From the Bethe solution we concluded that the energy spectrum of the charge pair model and the Hubbard model coincides for \( \frac{L}{2} \) even. However, when \( \frac{L}{2} \) is odd the energy spectrum of these two models are not the same due to the presence of a minus sign in the first Bethe equation [19]. In what follows we shall therefore restrict our analysis of the spectrum for lattice sites not multiple of four.

From the exact diagonalization of the Hamiltonian (2) we conclude that the ground state is a singlet and lies in the sector \((\frac{L}{2}, \frac{L}{2})\). The situation is similar to that of the Hubbard model at half-filling. In Figure 1 we exhibit the low-lying energies per site for \( L = 6 \) in which the states are label using the quantum numbers associated with the Bethe ansatz solution of the transformed Hamiltonian [12].

We find out that the ground states as well as many of the low-lying excitations can be described by real roots of the Bethe ansatz equations [19,20]. Based on this observation we can take the logarithm of the Bethe equations to obtain,

\[
Lk_j = 2\pi Q_j^{(1)} - 2 \sum_{l=1}^{N} \arctan \left( \frac{\sin(k_j) - \mu_l}{U/4} \right), \quad j = 1, \ldots, N_\uparrow + N_\downarrow,
\]

\[
2 \sum_{l=1}^{N_\uparrow+N_\downarrow} \arctan \left( \frac{\mu_j - \sin(k_l)}{U/4} \right) = 2\pi Q_j^{(2)} + 2 \sum_{l=1}^{N_\downarrow} \arctan \left( \frac{\mu_j - \mu_l}{U/2} \right), \quad j = 1, \ldots, N_\downarrow \tag{28}
\]

where the numbers \( Q_j^{(1,2)} \) define the many possible branches of the logarithm.

In Table (3) we give the numbers \( Q_j^{(1,2)} \) for the ground state and lowest states associated to the charge and spin sectors. We remark that such sequence of numbers are not the same as that of
Figure 1: The low-lying energies of Hamiltonian (2) for \( L = 6 \) as function of the interaction parameter \( U \). The energies are labeled using the quantum numbers \((N_\uparrow, N_\downarrow)\) of the Bethe equations (19,20).

The energy gap of one hole excitation over the singlet ground state can then be defined as,

\[
\Delta_{ev}(L) = E_0\left(\frac{L}{2}, \frac{L}{2} - 1, U\right) - E_0\left(\frac{L}{2}, \frac{L}{2}, U\right)
\]

(29)

We now start to report on the numerical analysis about the eigenstates described in Table (3). In what follows we shall denote by \( E_j(N_\uparrow, N_\downarrow, U) \) the \( j \)-th energy level in a given sector \((N_\uparrow, N_\downarrow)\). The energy gap of one hole excitation over the singlet ground state can then be defined as,

\[
\Delta_{ev}(L) = E_0\left(\frac{L}{2}, \frac{L}{2} - 1, U\right) - E_0\left(\frac{L}{2}, \frac{L}{2}, U\right)
\]

(29)

For the Hubbard models such Bethe numbers depend on whether \( L/2 \) is even or odd see for instance [12].
Table 3: The Bethe numbers $Q_j^{(1,2)}$ for the ground state and the lowest charge and spin excitations.

| $(N_1, N_2)$ | $Q_j^{(1)}$ | $Q_j^{(2)}$ | state          |
|--------------|-------------|-------------|----------------|
| $(\frac{L}{2}, \frac{L}{2})$ | $\frac{L}{2} - j + 1$ | $\frac{(L-2)}{4} + j - 1$ | ground state   |
| $(\frac{L}{2}, \frac{L}{2} - 1)$ | $\frac{L-1}{2} - j + 1$ | $\frac{(L-2)}{4} + j - 1$ | charge excitation |
| $(\frac{L}{2} + 1, \frac{L}{2} - 1)$ | $\frac{L-1}{2} - j + 1$ | $\frac{(L-4)}{4} + j - 1$ | spin excitation |

In order to verify the behaviour of the gap in the thermodynamic limit we have solved the Bethe equations (28) for lattice sizes up to $L = 1038$. In this solution we have used the corresponding configurations of the numbers $Q_j^{(1,2)}$ of Table (3). The numerical data for mass gap is presented in Table (4) together with the extrapolation for large lattice sizes. The extrapolated data is in accordance with the Lieb and Wu’s result (27).

| $\Delta_{ev}(L)$ | $U = 2$ | $U = 3$ | $U = 4$ |
|------------------|--------|--------|--------|
| 62               | 0.1397049178 | 0.3583388520 | 0.6783598211 |
| 142              | 0.1081504685 | 0.3327705853 | 0.6577213650 |
| 222              | 0.0995719373 | 0.3263342354 | 0.6523806267 |
| 302              | 0.0957870201 | 0.3234174755 | 0.6499337362 |
| 382              | 0.0936871136 | 0.3217548135 | 0.6485307420 |
| 462              | 0.0923434712 | 0.3206809062 | 0.6476212362 |
| 638              | 0.0906289820 | 0.3192820048 | 0.6464324386 |
| 1038             | 0.0889500759 | 0.317852239 | 0.6452407134 |
| Extrap.          | 0.08645(1)   | 0.31566(1)   | 0.64335(2)   |
| $\Delta(\infty)$| 0.0863890951 | 0.3156956889 | 0.6433635110 |

Table 4: The finite-size sequence (29) of the charge gap for $U = 2, 3, 4$ and the respective extrapolated value. The exact value is obtained from expression (27).

We now turn to the analysis of the finite-size corrections to the spin degrees of freedom. As
remarked these excitations should be gapless since the spectrum of the charge pair model (2) and
the Hubbard model (1) is the same for lattice sizes multiple of four. For the Hubbard model such
spin excitations is known to show the same critical behaviour of the isotropic spin-$\frac{1}{2}$ Heisenberg
model with periodic boundary conditions [13]. It is therefore expected similar critical behaviour
for charge pair model (2) in the case of even number of lattice sizes. In particular, the finite-size
dependence of the ground state energy should be governed by a conformal theory with central
charge $c = 1$. More precisely, following the results obtained by Woynarovich and Eckle [13] one
expects,

$$E_0 \left( \frac{L}{2}, \frac{L}{2}, U \right) - e_\infty L = -\frac{\pi \xi}{6L} \left( 1 + \mathcal{O}(1/\ln[L I_0(2\pi/U)])^3 \right)$$

(30)

where $\xi = 2I_1(2\pi/U)/I_0(2\pi/U)$ is the sound velocity of the spin excitation. The functions $I_0(x)$
and $I_1(x)$ are modified Bessel functions.

We have checked the above result by numerically computing the estimators,

$$C(L) = \frac{6L}{\pi \xi} \left[ E_0 \left( \frac{L}{2}, \frac{L}{2}, U \right) - e_\infty L \right]$$

(31)

for lattice sizes up to $L = 1038$. In Table (5) we have presented these estimates and we observe
the rapid converge to the expected value $c = 1$.

From the above informations we conclude that for $L$ even the leading behaviour of the finite-
size corrections of charge pair model should be same of that discussed Woynarovich and Eckle [13]
for the Hubbard model. As far as finite-size effects are concerned the difference among these two
models for $L/2$ odd appears to be associated with the subleading corrections. The amplitudes
of subleading terms are probably affected by presence of distinct phase factors in the first level
Bethe equation.

3.2 Finite-size effects for L odd

We have performed numerical diagonalization of the charge pair Hamiltonian (2) for small
values of odd lattice sites. In Figure (2) we present the low-lying energies in the spectrum of
the charge pair model for $L = 7$. Considering this analysis we conclude that the ground state
sits in the sectors $(\frac{L-1}{2}, \frac{L-1}{2})$ and $(\frac{L+1}{2}, \frac{L+1}{2})$. We find that these states have zero momenta and

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Table 5: The finite-size sequence (31) for $U = 2, 3, 4$ together with the respective extrapolation for large systems. The predicted value for the central charge is $c = 1$.

![Table 5](image)

consequently the energy of the ground state is double degenerared. For sake of comparison we note that these same states for the Hubbard model carries non-zero momenta and the respective energy is therefore four-fold degenerated.

We find that the low-lying states are well described by real Bethe roots and as before we can take the the logarithm of the Bethe equations (19,20). Considering the presence of the phase factors we obtain,

$$Lk_j = 2\pi \left[ \tilde{Q}_j^{(1)} - \frac{1}{4} \right] - 2 \sum_{l=1}^{N_j} \arctan \left[ \frac{\sin(k_j) - \mu_l}{U/4} \right], \quad j = 1, \ldots, N_\uparrow + N_\downarrow,$$

$$2 \sum_{l=1}^{N_\uparrow + N_\downarrow} \arctan \left[ \frac{\mu_j - \sin(k_l)}{U/4} \right] = 2\pi \left[ \tilde{Q}_j^{(2)} + \frac{1}{2} \right] + 2 \sum_{l=1}^{N_\downarrow} \arctan \left[ \frac{\mu_j - \mu_l}{U/2} \right], \quad j = 1, \ldots, N_\downarrow,$$

where in Table (6) we exhibit the numbers $\tilde{Q}_j^{(1,2)}$ of selected low-lying states which we shall discuss here.

A distinguished feature of the charge pair model is that it permits us to define a mass gap for
Figure 2: The low-lying energies of Hamiltonian (2) for \( L = 7 \) as function of the interaction \( U \). The energies are labeled by the quantum numbers \((N^\uparrow, N^\downarrow)\) of the Bethe equations (19, 20).

odd number of sites in analogy to what has been done for \( L \) even. In fact, from Figure 2 we observe that either a hole or a particle excitation has the same energy with respect to the ground state. This leads us to define the following charge gap for odd number of sites,

\[
\Delta_{od}(L) = E_0 \left( \frac{L-1}{2}, \frac{L-1}{2}, U \right) - E_0 \left( \frac{L+1}{2}, \frac{L-1}{2}, U \right)
\]  

We have computed the gap estimates (33) by numerically solving the Bethe equations (32) for the respective energies up to \( L = 1025 \). The results are exhibited in Table (7) and we see that the extrapolated estimators are very close to the exact values (27).

Let us now discuss the behaviour of the finite-size corrections to the ground state energy. The
computation of these corrections can be done within the root density formalism \cite{14, 15} since the Bethe equations are solved by real roots. At this point we recall that this approach has already been applied to the Hubbard model with even number of sites \cite{13}. By adapting the computations of \cite{13} to tackle the Bethe equations (32) we find that the leading behaviour of the finite-size corrections for the ground state is,

$$E_0 \left( \frac{L+1}{2}, \frac{L-1}{2}, U \right) - e_{\infty} L = \frac{2\pi \xi}{L} \left( \frac{1}{8} - \frac{1}{12} + O(1/\ln[L/\theta_0(2\pi/U)]) \right)$$  \hspace{1cm} (34)

and from the predictions of conformal field theory \cite{16} we conclude that this state has conformal dimension $X_0 = \frac{1}{8}$.

To support the above result for the scaling dimension we compute the following finite size two-step estimators for large sizes,

$$X_j(L) = \frac{L}{2\pi \xi} \left[ E_j \left( \frac{L+1}{2}, \frac{L-1}{2}, U \right) - e_{\infty} L \right] + \frac{1}{12} + \frac{A_j(L)}{\ln[L/\theta_0(2\pi/U)]}$$  \hspace{1cm} (35)

The strong logarithmic correction in the finite-size estimators is considered as follows. For each two consecutive values of lattice sites we eliminate the logarithmic amplitude $A_0(L)$ and calculate the respective scaling dimension $X_0(L)$. In Table (8) we present the results for $X_0(L)$ together with the extrapolated value for large $L$. We observe that the data approach the value predicted by the root density method $X_0 = \frac{1}{8}$ with reasonable precision.

We observe that in analogy to the Hubbard model with $L$ even the scaling dimension lacks of dependence on the coupling $U$ \cite{13}. This fact suggests that further insights about the finite-size corrections may be easily obtained by exploring the strong coupling limit of the Bethe equations \cite{19, 20}. In what follows we will pursue this analysis for the sector with total number of fermions
Table 7: The finite-size sequence (33) of the charge gap for $U = 2, 3, 4$ and the respective extrapolated value. The exact value is obtained from expression (27).

\[
\begin{array}{|c|c|c|c|}
\hline
\Delta_{\text{od}}(L) & U = 2 & U = 3 & U = 4 \\
\hline
65 & 0.0908120137 & 0.3180826815 & 0.6455305736 \\
145 & 0.0874329662 & 0.3166431214 & 0.6442188253 \\
225 & 0.0869930499 & 0.3162724101 & 0.6438819345 \\
305 & 0.0868183283 & 0.3161057094 & 0.6437310541 \\
385 & 0.0867207267 & 0.3160118618 & 0.6436463224 \\
465 & 0.0886658335 & 0.3159520064 & 0.6435923748 \\
625 & 0.0865835133 & 0.3158804564 & 0.6435279994 \\
1025 & 0.0865021020 & 0.3158029708 & 0.6434584567 \\
\text{Extrap.} & 0.08635(2) & 0.31567(1) & 0.64336(2) \\
\Delta(\infty) & 0.0863890951 & 0.3156965889 & 0.6433635110 \\
\hline
\end{array}
\]

$N_\uparrow + N_\downarrow = L$ and spin $N_\uparrow - N_\downarrow = 2n$ where $n$ takes values on half-integers for $L$ odd. Formally, this limit may be performed by scaling the spin rapidities as $\mu_j = \frac{U}{2} \lambda_j$ and afterwards taking the limit $U \rightarrow \infty$. For real momenta $\sin(k_j)$ is always bounded and through lowest order in $1/U$ the two-level Bethe equations for the momenta and spin variables decouple. The first Bethe equation (19) turn into a momenta condition for free-fermions while the second level one (20) becomes equivalent to that of the isotropic spin-$\frac{1}{2}$ model with twisted boundary condition. More precisely, the equation for the renormalized spin rapidities becomes,

\[
\left( \frac{\lambda_l + \frac{i}{2}}{\lambda_l - \frac{i}{2}} \right)^L = e^{i\pi \frac{L}{2} - n} \prod_{k=1, k\neq l}^{L-1} \frac{\lambda_l - \lambda_k + i}{\lambda_l - \lambda_k - i}, \quad l = 1, \ldots, \frac{L}{2} - n. \tag{36}
\]

The critical exponents associated to the spin degrees of freedom can therefore be inferred from previous analytical and numerical works for the spin-$\frac{1}{2}$ Heisenberg, see for instance [17,19]. Here we have to combine the frustrated character of the ground state of the Heisenberg chain with the presence of boundary twist. Following [19] and performing the adaptation to our situation we find
Table 8: The ground state $j = 0$ finite-size sequence (35) for $U = 2, 3, 4$ and the respective extrapolation for large systems. The predicted value for the exponent is $X_0 = 0.125$.

that such conformal dimensions are

$$
\tilde{X}(n, m) = \frac{n^2}{2} + \frac{(m - \frac{1}{2})^2}{2}, \quad n = \frac{1}{2}, \frac{3}{2}, \ldots; \quad m = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots,
$$

(37)

where the number $m$ indicates the vorticity of the state.

We note that the ground state scaling dimension $X_0 = \frac{1}{8}$ coincides with the lowest dimension $\tilde{X}(1/2, 1/2)$ of the twisted Heisenberg chain. Thus it is plausible to believe that the conformal dimensions (37) should be present in the finite-size corrections of the charge pair model for $L$ odd.

In order to give further support to this conjecture we now consider the first excitation in the sector $(L + \frac{1}{2}, L - \frac{1}{2})$ of the charge pair model. This state has momenta being double degenerated and the respective logarithmic branch numbers are given in the third line of Table (6). By applying the root density method to this state we obtain,

$$
E_1 \left( \frac{L + 1}{2}, \frac{L - 1}{2}, U \right) - e_{\infty}L = \frac{2\pi \xi}{L} \left( \frac{5}{8} - \frac{1}{12} + \mathcal{O}(1/\ln[L J_0(2\pi/U)]) \right)
$$

(38)

3Note that in [19] the vorticity are integer numbers while in our situation they take values on half-integers.

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whose corresponding conformal dimension is $X_1 = \frac{5}{8}$. In Table (9) we provide numerical support for the above analytical computation. The extrapolated value is in reasonable accordance with the analytical prediction.

| $X_1(L)$ | $U = 2$ | $U = 3$ | $U = 4$ |
|----------|---------|---------|---------|
| 65       | 0.6260129719 | 0.6335332494 | 0.6345026359 |
| 145      | 0.6291377527  | 0.6301830740  | 0.6307714643  |
| 225      | 0.6284896233  | 0.6293039768  | 0.6297667556  |
| 305      | 0.6281555616  | 0.6288656639  | 0.6292626234  |
| 385      | 0.6279480092  | 0.6285905082  | 0.6289457396  |
| 465      | 0.6278021291  | 0.6283961414  | 0.6287220234  |
| 625      | 0.6276205639  | 0.6281536039  | 0.6284433391  |
| 1025     | 0.6274273472  | 0.6278947826  | 0.6281470300  |
| Extrapol.| 0.62535(1)    | 0.62542(1)    | 0.62545(2)    |

Table 9: The first-excitation $j = 1$ finite-size sequence (35) for $U = 2, 3, 4$ and the respective extrapolation for large systems. The predicted value for the exponent is $X_1 = 0.625$.

Once again we note the dimension $X_1 = \frac{5}{8}$ can be obtained either from $\tilde{X}(1/2, -1/2)$ or $\tilde{X}(1/2, 3/2)$ in agreement with fact we are dealing with a momenta state. We think that the above arguments strongly suggests that the contributions of the spin degrees of freedom to the finite-size corrections of the charge pair model with $L$ odd are indeed governed by the conformal dimensions (37).

We conclude with the following comments. We expect that the finite-size behaviour of the Hubbard model for $L$ odd will be different from that described above for the charge pair model. First we remark that the gap definition (33) does not apply for the Hubbard model because its eigenvalues do not satisfy the spectral property (14). Besides that exact diagonalization of the Hubbard Hamiltonian (1) reveal us that there exits level crossing among the first two lowest energies states for some finite value of $U$. For $L = 5$ we find that the level crossing is among the
ground states lying in the sectors (3, 2) and (3, 3). For \( L = 7 \) the crossing occurs for states in the sectors (3, 4) and (3, 3) rather than among the energies in sectors (3, 4) and (4, 4). Therefore, the nature of such crossings seems to depend on the parity of the number \((L - 1)/2\) and the understanding of the large \( L \) behaviour of the low-lying states requires further investigation. We plan to expand on this preliminary analysis and present it elsewhere since most of finite-size results for the Hubbard model appears to be concentrated on even number of sites.

4 The Covering Vertex Model

Here we argue that the fermionic Hamiltonian (2) can be derived in the context of the commuting transfer matrix approach [20]. We start by recalling the Boltzmann weights structure of the symmetric free-fermion eight-vertex model. The model has four weights \( a, b, c, d \) and its Lax operator can be represented as,

\[
\mathcal{L}_{0j} = \begin{bmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{bmatrix}_j,
\] (39)

where the indices 0 and \( j \) refer to the horizontal and vertical spaces of states of the vertex model, respectively. It is assumed the free-fermion condition among the weights,

\[
a^2 + b^2 - c^2 - d^2 = 0.
\] (40)

For \( d=0 \) Shastry devised a way to couple two free-fermion six-vertex models by a particular diagonal vertex interaction. As a result was obtained a new integrable vertex model of statistical mechanics with non-additive \( R \)-matrix [6,7]. In addition, Shastry showed that the transfer matrix of such model commutes with the Hamiltonian of an equivalent spin chain derived from that of the Hubbard model by means of the Jordan-Wigner transformation,

\[
c_{\uparrow}(j) = \prod_{k=1}^{j-1} \sigma_k^z \sigma_j^-,
\quad
c_{\downarrow}(j) = \prod_{k=1}^{j-1} \sigma_k^z \prod_{k=1}^{L} \tau_k^z \tau_j^-.
\] (41)
where $\{\sigma^+_j, \sigma^-_j\}$ and $\{\tau^+_j, \tau^-_j\}$ are two commuting sets of Pauli matrices acting the $j$-th lattice site.

In what follows we shall point out that Shastry’s approach also works in the subspace of weights with $b = 0$. Before that we recall that Shastry’s construction has been shown to be applicable when we couple certain special vertex models invariant under the $gl(n|m)$ superalgebra \cite{21,23}. We emphasize such generalizations lead to Hamiltonian models with higher number of states per site than that of the charge pair model \cite{2} introduced here. The fact that Shastry’s method also works using an eight-vertex model with $b = 0$ appears to have been overlooked so far. For $b = 0$ free-fermion condition \cite{40} is a circle in the affine plane and it can be parametrized as,

$$a = 1, \quad c = \cos(\lambda), \quad d = \sin(\lambda)$$

(42)

where $\lambda$ is the spectral parameter. Note that at $\lambda = 0$ the Lax operator \cite{39} becomes a two-dimensional permutator.

The construction of coupled vertex models for $b = 0$ is fairly parallel to that devised by Shastry and in what follows we shall summarize only the main results. The Lax operator of the coupled model has the standard Shastry’s form,

$$L_{0j}(\lambda) = \exp \left[ \frac{h(\lambda)}{2} (\sigma^-_0 \tau^+_0 + I_0) \right] I_j \left[ L^{(\sigma)}_{0j}(\lambda) L^{(\tau)}_{0j}(\lambda) \right] \exp \left[ \frac{h(\lambda)}{2} (\sigma^-_0 \tau^+_0 + I_0) \right] I_j,$$

(43)

where $I$ denotes the four-dimensional identity matrix and $h(\lambda)$ characterizes the strength of the coupling.

In our case, however, the operators $L^{(\sigma)}_{0j}(\lambda)$ and $L^{(\tau)}_{0j}(\lambda)$ are two copies of the free-fermion eight-vertex model with $b = 0$. These Lax operators can be expressed in terms of Pauli matrices as,

$$L^{(\sigma)}_{0j}(\lambda) = \frac{1}{2} [I_0 I_j + \sigma^-_0 \sigma^+_j] + \cos(\lambda) [\sigma^+_0 \sigma^-_j + \sigma^-_0 \sigma^+_j] + \sin(\lambda) [\sigma^+_0 \sigma^-_j + \sigma^-_0 \sigma^+_j],$$

$$L^{(\tau)}_{0j}(\lambda) = \frac{1}{2} [I_0 I_j + \tau^-_0 \tau^+_j] + \cos(\lambda) [\tau^+_0 \tau^-_j + \tau^-_0 \tau^+_j] + \sin(\lambda) [\tau^+_0 \tau^-_j + \tau^-_0 \tau^+_j]$$

(44)

As usual the transfer matrix of the respective vertex model on the square lattice can be written as the trace of an ordered product of Lax operators \cite{43} on the horizontal space,

$$T(\lambda) = Tr_0[L_{01}(\lambda)L_{02}(\lambda) \ldots L_{0L}(\lambda)]$$

(45)
which gives rise to a family of commuting of transfer matrices provide the coupling \( h(\lambda) \) satisfies the Shastry’s spectral constraint,

\[
\sinh [2h(\lambda)] = \frac{U}{4} \sin(2\lambda)
\]  

(46)

At this point we remark that the condition (40) with \( b = 0 \) and the constraint (46) can be translated into a single algebraic relation after a suitable definition of the ring variables. Indeed, following [24] it is possible to define new affine variables,

\[
x = c \exp [h(\lambda)], \quad y = d \exp [h(\lambda)]
\]  

(47)

such that the spectral curve assuring the integrability of the model is the following genus one quartic curve,

\[
(x^2 + y^2)^2 - Uxy - 1 = 0
\]  

(48)

Now the spin Hamiltonian \( H_s \) associated with this vertex model is obtained by expanding the logarithm of the transfer matrix (45) around the regular point \( \lambda = 0 \). Apart from an additive constant its expression is given by,

\[
H_s = \sum_{j=1}^{L} \left[ \sigma_j^- \sigma_{j+1}^- + \sigma_j^+ \sigma_{j+1}^+ + \tau_j^- \tau_{j+1}^- + \tau_j^+ \tau_{j+1}^+ \right] + \frac{U}{4} \sum_{j=1}^{L} \sigma_j^z \tau_j^z,
\]  

(49)

with periodic boundary conditions imposed.

With the help of the Jordan-Wigner transformation (41) the fermionic Hamiltonian (2) can be rewritten in terms of Pauli operators. It turns out that this transformation is able to reproduce only bulk part of the coupled spin chain (49),

\[
H_c = \sum_{j=1}^{L-1} \left[ \sigma_j^- \sigma_{j+1}^- + \sigma_j^+ \sigma_{j+1}^+ + \tau_j^- \tau_{j+1}^- + \tau_j^+ \tau_{j+1}^+ \right] + \frac{U}{4} \sum_{j=1}^{L} \sigma_j^z \tau_j^z
\]  

\[
- (\sigma_L^- \sigma_1^- \prod_{k=1}^{L-1} \sigma_k^z - (\tau_L^- \tau_1^- \prod_{k=1}^{L-1} \tau_k^z) \prod_{k=1}^{L-1} \tau_k^z
\]  

(50)

since the boundary terms are clearly distinct from that of the coupled spin model (49).

In order to match the boundary term we can exploit the fact that integrability is still preserved by performing certain suitable twist transformations on the Lax operators [25]. Besides that we
have to consider that the local states of the fermionic Hamiltonian (2) are constituted by a graded space with two bosonic and two fermionic degrees of freedom. We expect that the respective Lax operator at the regular point should be proportional to the graded permutation operator,

$$P(g) = \sum_{j,k=1}^{4} (-1)^{p_j p_k} e_{j k} \otimes e_{k j}$$

(51)

where $e_{j k}$ are the standard Weyl matrices. We choose the Grassmann parities $p_j$ according to the basis ordering (6) and therefore we set $p_1 = 0, p_2 = 1, p_3 = 1, p_4 = 0$.

Combining the procedures mentioned above we find that the suitable fermionic Lax operator is obtained by the following twist transformation,

$$L_{0 j}^{(g)}(\lambda) = M L_{0 j}(\lambda) \overline{M}$$

(52)

where the twists $M$ and $\overline{M}$ are the following diagonal matrices,

$$M = \text{diag}(1, 1, -1, -1|1, 1, -1, -1|1, -1, 1, -1, 1, -1)$$

$$\overline{M} = \text{diag}(1, 1, 1|1, -1, -1|1, -1, 1, -1, -1, -1)$$

(53)

It turns out that the explicit matrix representation of the Lax operator (52) in terms of the spectral variables $x$ and $y$ is given by,

$$L_{0 j}^{(g)}(x, y) = \begin{pmatrix}
\omega_1 & 0 & 0 & 0 & 0 & -y & 0 & 0 & 0 & -y^2 \\
0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -xy \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

(54)
where the weights $w_1, \ldots, w_4$ dependence on the spectral variables are,

\[ w_1 = x^2 + y^2, \quad w_2 = \frac{xy}{x^2 + y^2}, \quad w_3 = \frac{-y^2}{x^2 + y^2}, \quad w_4 = \frac{-x^2}{x^2 + y^2}. \tag{55} \]

We now show that this fermionic Lax operator is able to produce the two-body part of the charge pair Hamiltonian \[ (2) \] through its expansion around the regular point $x = 1$ and $y = 0$. It turns out that the first order expansion of the spectral variables constrained by the curve \[ (48) \] is given by,

\[ x = 1 + \frac{U}{4} \epsilon + \mathcal{O}(\epsilon^2), \quad y = \epsilon + \mathcal{O}(\epsilon^2) \tag{56} \]

where $\epsilon$ is the expansion parameter.

Now considering the expansion of the Lax operator \[ (54) \] we obtain,

\[ L_{j+1}^{(g)}(x,y) = P^{(g)}(1 + \epsilon H_{j,j+1}) \tag{57} \]

where the operator $H_{j,j+1}$ is given by

\[ H_{j,j+1} = c^-_\sigma(j)c^+_\sigma(j+1) + c^+_\sigma(j+1)c^-_\sigma(j) + \frac{U}{2}(n^\uparrow_\sigma(j) - \frac{1}{2})(n^\downarrow_\sigma(j) - \frac{1}{2}) \]

\[ + \frac{U}{2}(n^\uparrow_\sigma(j+1) - \frac{1}{2})(n^\downarrow_\sigma(j+1) - \frac{1}{2}) + \frac{U}{4} I_j \otimes I_{j+1} \tag{58} \]

which coincides with the the two-body term of Hamiltonian \[ (2) \] apart from a trivial additive factor.

We close this section mentioning that both Lax operators \[ (43,54) \] fulfill the Yang-Baxter relation. This factorization condition together with corresponding $R$-matrices has been summarized in Appendix A.

5 Concluding Remarks

In this paper we have introduced a variant of the Hubbard model whose next-neighbor term plays the role of a triplet charge pair potential. For arbitrary lattice sizes the model has two conserved charges which can be added to the Hamiltonian without affecting its integrability. Besides that gauge fluxes can be attached to both the pair potential and the conserved charges...
and an extended charge pair Hamiltonian can be written,

\[ H_c = \sum_{j=1}^{L} \sum_{\alpha=\uparrow,\downarrow} \left[ e^{i\theta_{\alpha}} c_{\alpha}(j)c_{\alpha}(j+1) + e^{-i\theta_{\alpha}} c_{\alpha}^\dagger(j+1)c_{\alpha}^\dagger(j) \right] + U \sum_{j=1}^{L} (n_{\uparrow}(j) - \frac{1}{2})(n_{\uparrow}(j) - \frac{1}{2}) \]

\[ + \; i h_1 \sum_{j=1}^{L} \left[ e^{i\left(\frac{\theta_{\uparrow}-\theta_{\downarrow}}{2}\right)} c_{\uparrow}^\dagger(j)c_{\uparrow}(j) - e^{-i\left(\frac{\theta_{\uparrow}-\theta_{\downarrow}}{2}\right)} c_{\downarrow}^\dagger(j)c_{\downarrow}(j) \right] \]

\[ + \; h_2 \sum_{j=1}^{L} \left[ e^{-i\left(\frac{\theta_{\uparrow}+\theta_{\downarrow}}{2}\right)} c_{\uparrow}^\dagger(j)c_{\downarrow}^\dagger(j) + e^{i\left(\frac{\theta_{\uparrow}+\theta_{\downarrow}}{2}\right)} c_{\downarrow}(j)c_{\uparrow}(j) \right] \]

(59)

where \( \theta_{\uparrow}, \theta_{\downarrow} \) are flux phases and \( h_1, h_2 \) are the chemical potentials associated to the conserved charges.

The fluxes can be removed from the Hamiltonian (59) by means of the canonical transformation \( c_{\alpha}(j) \rightarrow e^{-i\theta_{\alpha}} c_{\alpha}(j) \) and \( c_{\alpha}^\dagger(j) \rightarrow e^{i\theta_{\alpha}} c_{\alpha}^\dagger(j) \). The Bethe ansatz solution for extended Hamiltonian (59) follows that given in section 3 and the respective Bethe equations are given by the same relations (19,20). The basic change is in the expression for the eigenenergies which now is,

\[ E(N_{\uparrow}, N_{\downarrow}, U) = -2 \sum_{j=1}^{N_{\uparrow}+N_{\downarrow}} \cos(k_j) + U \left( \frac{L}{2} - N_{\uparrow} - N_{\downarrow} \right) + h_1(N_{\uparrow} - N_{\downarrow}) + h_2(N_{\uparrow} + N_{\downarrow} - L) \]

(60)

We have argued that the exact integrability of the charge pair model (2) can be established by using a construction devised by Shastry for the Hubbard model [6,7]. This procedure gives rise to an equivalent spin chain (49) which can be seen as two coupled special XY models. We now show that such spin chain can be mapped into two coupled XX models where the boundary conditions depend on if we have an even or odd number of sites. To this end we define the following transformation acting on the even sites of the lattice,

\[ \sigma_{j}^{+} \rightarrow \sigma_{j}^{-}, \quad \sigma_{j}^{-} \rightarrow -\sigma_{j}^{+}, \quad \tau_{j}^{+} \rightarrow \tau_{j}^{-}, \quad \tau_{j}^{-} \rightarrow -\tau_{j}^{+}, \quad \text{for} \quad j = 2, 4, 6, \ldots \]

(61)

For \( L \) even the form of the transformed Hamiltonian (49) is,

\[ \tilde{H}_s = \sum_{j=1}^{L-1} \left[ \sigma_{j}^{-} \sigma_{j+1}^{+} + \sigma_{j+1}^{+} \sigma_{j}^{-} + \tau_{j}^{-} \tau_{j+1}^{+} + \tau_{j+1}^{+} \tau_{j}^{-} \right] + U \sum_{j=1}^{L} \sigma_{j}^{z} \tau_{j}^{z} + \sigma_{L}^{+} \sigma_{1}^{-} + \sigma_{1}^{+} \sigma_{L}^{-} + \tau_{L}^{+} \tau_{1}^{-} + \tau_{1}^{+} \tau_{L}^{-}, \quad L = 2, 4, 6, \ldots \]

(62)
which is exactly the same spin chain associated to integrability of the Hubbard model \[6,7\]. The corresponding Bethe equations have been discussed before \[27,28\] and for sake of completeness we also present them here,

$$e^{ik_j L} = (-1)^{N_\uparrow} \prod_{t=1}^{N_\uparrow} \frac{\sin(k_{j_t}) - \mu_l - \frac{iU}{4}}{\sin(k_{j_t}) - \mu_l + \frac{iU}{4}}, \quad j = 1, 2, \ldots, N_\uparrow + N_\downarrow,$$

$$\prod_{j=1}^{N_\uparrow + N_\downarrow} \frac{\sin(k_{j_l}) - \mu_l + \frac{iU}{4}}{\sin(k_{j_l}) - \mu_l - \frac{iU}{4}} = (-1)^{N_\downarrow + N_\uparrow} \prod_{k=1}^{N_\downarrow} \frac{\mu_l - \mu_k - \frac{iU}{2}}{\mu_l - \mu_k + \frac{iU}{2}}, \quad l = 1, \ldots, N_\downarrow,$$

where now the phase factors depend on the combined parities of the quantum numbers of the model. The eigenvalues are once again determined by the expression (21).

On the other hand when \(L\) is odd the transformed Hamiltonian (49) is given by,

$$\tilde{H}_s = L - 1 \sum_{j=1}^{L-1} \left[ \sigma_j^- \sigma_{j+1}^+ + \sigma_{j+1}^- \sigma_j^+ + \tau_j^- \tau_{j+1}^+ + \tau_{j+1}^- \tau_j^+ \right] + \frac{U}{4} \sum_{j=1}^{L} \sigma_j^z \tau_j^z + \sigma_L^- \sigma_1^+ + \sigma_1^- \sigma_L^+ + \tau_L^- \tau_1^+ + \tau_1^- \tau_L^+, \quad L = 3, 5, 7, \ldots,$$

Now we see that the boundary term in (64) breaks explicitly the two \(U(1)\) symmetries present in the bulk part of the Hamiltonian. Despite of this fact we found out that the transformed model (64) still preserves the property of having factorized reference states associated with the exact eigenvalues \(E = \pm \frac{L(U)}{4}\). The situation is similar to what we have found for the charge pair model as shown in Table (2). The structure of such eigenstates for the spin model (64) are however a bit different since it contain alternating phases in the tensor product. The form of these reference states are summarized in Table (10) where \(e^{(l)}_j\) denote the four dimensional orthogonal vectors acting on the \(j\)-th site of the lattice,

$$e^{(1)}_j = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_j, \quad e^{(2)}_j = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_j, \quad e^{(3)}_j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_j, \quad e^{(4)}_j = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_j.$$

In addition to that we have been able to built few low-lying states on top of the reference state given in Table (10). Carrying on the Bethe ansatz analysis for such states we end up with the
Table 10: Factorized eigenvectors of the transformed coupled spin chain (64). The explicit form of the vectors $e_j^{(l)}$ are given in (65).

same Bethe equations of the charge pair model, see equations (19, 20). This strongly suggests that the eigenenergies of the charge pair model (2) and the coupled spin chain (49) are the same for $L$ odd. We have indeed confirmed this fact by comparing the spectrum of these models with the help of exact diagonalization for $L = 3, 5, 7$ sites. The eigenfunctions structure of such two models should be related but a more concrete relationship among them has eluded us so far.

Lastly, one characteristic of the one-dimensional charge pair model is that the thermodynamic limit properties do not depend on fact that the lattice is bipartite. It was argued that the charge gap can be defined for even and odd number of sites both converging to the same value in the infinite size limit. This should be contrasted to the case of the Hubbard model in which the lattice bipartiteness plays important role to establish certain exact results for the repulsive interaction in any lattice dimension [29]. It seems interesting to investigate whether or not the methods used to obtain significant informations for the Hubbard model in all dimensions can also be adapted to the case of the charge pair model. In particular, if one can state concrete informations for the charge pair model in higher dimension without the need of a bipartite lattice assumption.

Acknowledgments

This work was supported in part by the Brazilian Research Council CNPq through the grants 304758/2017-7 and 401694/2016-0.
Appendix A: The Yang-Baxter algebra.

A sufficient condition for exact integrability of vertex model is that its Lax operator satisfies the Yang-Baxter equation for some invertible $R$-matrix [20]. In the case of the Lax operator (43) this relation can be stated as,

$$R_{12}(\lambda_1, \lambda_2) L_{13}(\lambda_1) L_{23}(\lambda_2) = L_{23}(\lambda_2) L_{13}(\lambda_1) R_{12}(\lambda_1, \lambda_2),$$  \hfill (A.1)

where the $R$-matrix has the same structure of that proposed by Shastry for the Hubbard model [7],

$$R_{12}(\lambda_1, \lambda_2) = \cos(\lambda_1 + \lambda_2) \cosh [h(\lambda_1) - h(\lambda_2)] L_{12}^{(\sigma)}(\lambda_1 - \lambda_2) L_{12}^{(\tau)}(\lambda_1 - \lambda_2)$$

$$+ \cos(\lambda_1 - \lambda_2) \sinh [h(\lambda_1) - h(\lambda_2)] L_{12}^{(\sigma)}(\lambda_1 + \lambda_2) L_{12}^{(\tau)}(\lambda_1 + \lambda_2) \sigma_1^z \tau_1^z$$  \hfill (A.2)

except by the fact that the building block operators $L_{12}^{(\sigma)}(\lambda)$ and $L_{12}^{(\tau)}(\lambda)$ are given by the special eight-vertex models (44).

The Yang-Baxter algebra for the fermionic Lax operator (54) has similar form,

$$R_{12}^{(g)}(x_1, y_1, x_2, y_2) L_{13}^{(g)}(x_1, y_1) L_{23}^{(g)}(x_2, y_2) = L_{23}^{(g)}(x_2, y_2) L_{13}^{(g)}(x_1, y_1) R_{12}^{(g)}(x_1, y_1, x_2, y_2),$$  \hfill (A.3)

but now the tensor products in (A.3) have to consider the gradation of the three subspaces.

The explicit form of the $R$-matrix turns out to be,

$$R_{12}^{(g)}(x_1, y_1, x_2, y_2) = 
\begin{pmatrix}
  h & 0 & 0 & 0 & 0 & -d & 0 & 0 & 0 & 0 & a & -h \\
  0 & d & 0 & 0 & 0 & q & 0 & 0 & 0 & g & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$  \hfill (A.4)
where the expressions of the matrix entries are,

\[
\begin{align*}
\mathbf{a} &= \frac{y_1y_2}{x_1^2 + y_1^2} + \frac{x_1x_2}{x_2^2 + y_2^2}, & \mathbf{b} &= -\frac{x_1y_2}{x_1^2 + y_1^2} + \frac{y_1x_2}{x_2^2 + y_2^2}, & \mathbf{b} &= -\frac{y_1x_2}{x_1^2 + y_1^2} - \frac{x_1y_2}{x_2^2 + y_2^2}, \\
\mathbf{d} &= \frac{x_1y_1 - x_2y_2}{x_1^2x_2^2 - y_1^2y_2^2}, & \mathbf{g} &= -\frac{x_1x_2}{x_1^2 + y_1^2} - \frac{y_1y_2}{x_2^2 + y_2^2}, & \mathbf{h} &= \frac{x_1x_2(x_1^2 + y_1^2) - y_1y_2(x_2^2 + y_2^2)}{x_1^2x_2^2 - y_1^2y_2^2}, \\
\mathbf{q} &= \frac{y_1y_2(x_1^2 + y_1^2) - x_1x_2(x_2^2 + y_2^2)}{x_1^2x_2^2 - y_1^2y_2^2}
\end{align*}
\]

such that \(\{x_1, y_1\}\) and \(\{x_2, y_2\}\) denote two arbitrary points on the quartic curve \((48)\).

We finally recall that it is possible to rewrite the Yang-Baxter relation in an alternative form which is insensitive to the grading of the spaces [26]. With the help of the graded permutation one can define new operators \(\tilde{S} = P^{(g)}S\) and the algebraic relation (A.3) becomes,

\[
\tilde{R}^{(g)}_{23}(x_1, y_1, x_2, y_2)\tilde{L}^{(g)}_{12}(x_1, y_1)\tilde{L}^{(g)}_{23}(x_2, y_2) = \tilde{L}^{(g)}_{12}(x_2, y_2)\tilde{L}^{(g)}_{23}(x_1, y_1)\tilde{R}^{(g)}_{12}(x_1, y_1, x_2, y_2).
\]  

(A.6)

References

[1] J. Hubbard, *Proc.Roy.Soc.* London Ser. A 276 (1963) 238; ibid. 277 (1963).

[2] M.C. Gutzwiller, *Phys.Rev.Lett.* 10 (1963) 159.

[3] E.H. Lieb and F.Y. Wu, *Phys.Rev.Lett.* 20 (1988) 1445;

[4] A. Montorsi, *The Hubbard Model*, World Scientific, Singapore, 1992.

[5] F.H.L Essler, H. Frahm, F. Göhmann, A. Klümper and V. Korepin,*The One-Dimensional Hubbard Model*, Cambridge University Press, Cambridge, 2005.

[6] B.S. Shastry, *Phys.Rev.Lett.* 56 (1986) 2453; ibid 56 (1986) 2453.

[7] B.S. Shastry, *J.Stat.Phys.* 30 (1988) 57.

[8] B.S. Shastry and B. Sutherland, *Phys.Rev.Lett.* 65 (1990) 243.
[9] M.J. Martins and R.M. Fye, *J.Stat.Phys.* 64 (1991) 271.

[10] C.N. Yang, *Phys.Rev.Lett.* 63 (1989) 2144; C.N. Yang and S.C. Zhang, *Mod.Phys.Lett. B* 4 (1990) 759.

[11] M. Pernici, *Europhys.Lett.* 12 (1990) 75.

[12] M. Takahashi, *Thermodynamics of one-dimensional Solvable Models*, Cambridge University Press, New York, 1999.

[13] F. Woynarovich and H.-P. Eckle, *J.Phys.A:Math.Gen.* 20 (1987) L443.

[14] H.J. de Vega and F. Woynarovich, *Nucl.Phys.B.* 251 (1985) 439.

[15] H.J. de Vega, *J.Phys.A:Math.Gen.* 21 (1988) L1089; J. Suzuki, *J.Phys.A:Math.Gen.* 21 (1988) L1175.

[16] J.L.Cardy, *Nucl.Phys.B.* 270 (1986) 186.

[17] F. Woynarovich and H.-P. Eckle, *J.Phys.A:Math.Gen.* 20 (1987) L97.

[18] C.J. Hamer, G.R.W. Quispel and M.T. Batchelor, *J.Phys.A* 20 (1987) 567.

[19] F.C. Alcaraz, M.N. Barber and M.T. Batchelor, *Ann.Phys.NY* 182 (1988) 280.

[20] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1992.

[21] Z.Maassarani, *Phys.Lett.A* 239 (1998) 187; *Int.J.Mod.Phys.B* 12 (1988) 1893.

[22] M.J. Martins, *Phys.Lett.A* 247 (1998) 218.

[23] J. Drummond, G. Feverati, L. Frappat and E. Racoucy, *JHEP* 05 (2007) 05008; G. Feverati, L. Frappat and E. Racoucy, *STAT* 04 (2009) P04014.

[24] M.J. Martins, *Nucl.Phys.B* 907 (2016) 479.

[25] A. Kundu, *Nucl.Phys.B* 618 (2001) 500.
[26] P.P. Kulish, *J.Sov.Math.* 35 (1986) 2648.

[27] R. Yue and T. Deguchi, *J.Phys.A:Math.Gen.* 30 (1997) 849.

[28] M.J. Martins and P.B. Ramos, *Nucl.Phys.B* 522 (1998) 413.

[29] E.H. Lieb, *Phys.Rev.Lett.* 62 (1989) 1201.