Superconductivity from repulsion:  
Ginzburg – Landau phenomenology of cuprates

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We develop Ginzburg-Landau approach to the problem of superconducting pairing with large momentum under screened Coulomb repulsion ($\eta K$ - pairing). Two-component order parameter arising in this scheme can be associated with charge and orbital current degrees of freedom of the relative motion of $\eta K$ - pair corresponding to superconducting and orbital antiferromagnetic ordered states, respectively. All basic features of the phase diagram of cuprate superconductors result directly from the $\eta K$ - pairing concept.

Keywords: superconductivity, repulsive pairing, Ginzburg-Landau phenomenology

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1. Introduction

Since the discovery of superconductivity (H. Kamerling Onnes, 1911) a search of materials with high superconducting (SC) phase transition temperature $T_c$ has been at the focus of solid state physics. When the microscopic BCS theory of superconductivity was developed (J. Bardeen, L. N. Cooper, and J. R. Schrieffer, 1957; N. N. Bogoliubov, 1958), it became clear that there were some restrictions to SC transition temperature $T_c$ following from the BCS theory. According to BCS theory, the SC state arises when, due to phonon-mediated attraction (H. Fröhlich, 1950; J. Bardeen, 1950), electrons pair up into Cooper pairs (L. N. Cooper, 1956) which, at $T < T_c$, can form a coherent state (SC condensate) insensitive to crystal imperfections. One of the restrictions to $T_c$, following from the phonon scenario of SC pairing, is connected with the fact that Debye energy $\hbar \omega_D$ is much less than Fermi energy $E_F$.

It was proposed (W. A. Little, 1964; V. L. Ginzburg, 1964) that, in a low-dimensional electron system, $T_c$ might be considerably higher if pairing attraction were due to non-phonon origin. Such an exciton-mediated attraction, just as another ways of an increase of $T_c$, were theoretically studied in detail by Ginzburg’s group in Moscow [1]. A great many of the results obtained by this group, in particular, a possibility of superconductivity in three-dimensional insulating crystals (a coexistence of SC and insulating states in one-dimensional crystals was previously considered in [2]), turned out to be especially actual after the discovery of high temperature superconductivity (HTSC) of doped cuprate oxides (J. G. Bednorz and K. A. Müller, 1986).

Taking account of Coulomb repulsion in the framework of the phonon mechanism of the BCS theory (V. V. Tolmachev, 1958) leads to a conclusion that Cooper pairing occurs if effective coupling constant $V$ of phonon-mediated pairing attraction exceeds logarithmically suppressed average value of Coulomb repulsion energy $U$:

$$V > \frac{U}{1 + gU \ln (E_F/\hbar \omega_D)}.$$  \hspace{1cm} (1)

Here, $g$ is density of states on the Fermi level.

At the same time, there was investigated SC pairing under repulsive interaction itself. Such a pairing is possible if the expansion of repulsive pairing potential into the series over spherical harmonics contains at least one term with angular momentum $l \neq 0$ corresponding to negative scattering length (L. D. Landau, 1958). In a degenerate electron gas, this condition is satisfied (W. Kohn and J. M. Luttinger, 1964) due to Kohn singularity of screened Coulomb repulsion. In two-band superconductor, repulsion-induced SC pairing can be effective as well (V. A. Moskalenko, 1959; H. Suhl, B. T. Matthias, and L. R. Walker, 1959) when there is the Suhl inequality (inverse Cauchy-Bunyakovsky inequality),

$$U_{12}U_{21} - U_{11}U_{22} > 0,$$  \hspace{1cm} (2)

between interband ($U_{12} = U_{21}$) and intraband ($U_{11}, U_{22}$) matrix elements of the pairing interaction.

The discovery of HTSC in cuprates has resulted in not only a considerable increase of $T_c$, but put some fundamental problems in the theory of superconductivity. Anderson [3] has pointed out general statements of physics of cuprates: 1) these compounds are quasi-two-dimensional (2D) systems with considerably strong electron correlations in CuO$_2$ layers; 2) superconductivity arises under doping of parent spin antiferromagnetic (AF) insulator. To explain unconventional properties of cuprates, along with phonon-mediated attraction, AF magnon-mediated SC repulsion (N. F. Berk and J. R. Schrieffer, 1966) and direct Coulomb repulsion (most commonly, in the framework of Hubbard model or related models [4]) are considered as underlying pairing interactions.

A choice of the microscopic mechanism of SC pairing determines the symmetry and internal structure of SC order parameter which, in the cuprates, may be essentially...
other than simple one-component complex order parameter following from phonon scenario. It is quite natural to consider screened Coulomb repulsion as a reasonable underlying pairing interaction that determines both insulating and SC states of the cuprates. This mechanism results in the order parameter which has no less than two complex components that, under a description in the framework of Ginzburg-Landau phenomenology (developed previously to the microscopic BCS theory: V. L. Ginzburg and L. D. Landau, 1950), should be determined by an equation system that can admit of more than BCS-like solution only.

In this paper, using a generalized Ginzburg-Landau approach, we present the concept of Coulomb SC pairing with large momentum (next, pairing) and some of its applications to physics of cuprates.

2. Repulsion-induced superconducting pairing with large momentum

Angle-resolved photoemission spectroscopy (ARPES) study results in a conclusion that, in cuprates, Fermi contour (FC) is situated in an extended vicinity of saddle point of one-band 2D electron dispersion \( \varepsilon(p) \). At doping \( x \), a large FC bounds electron-filled region of the momentum space which is proportional to \( (1-x) \) and has a typical form of a square with rounded corners. Spin AF insulating gap of the corresponding parent compound survives under relatively low doping due to short-range AF order decreasing with an increase of \( x \). Therefore a part of hole filling proportional to \( x \) occupies small hole pockets in the lower electron subband that are centered at the points \( (\pm \pi/2, \pm \pi/2) \) on the boundary of the magnetic Brillouin zone of the parent compound. Relatively high spectral weight corresponds to the part of a pocket (main band) situated inside the first magnetic zone (this part belongs to the large FC) whereas the spectral weight corresponding to the other part (shadow band; A. Kampf and J.R. Schrieffer, 1990) turns out to be considerably suppressed and decreases with \( x \). Low energy quasiparticle excitations can arise only due to hole pockets resulting, after the transition into SC state, in low phase stiffness, \( \rho_{sf} \sim x \).

The FC in the form of hole pockets (Fig.1) possesses two special features. Firstly, it manifests perfect nesting feature, \( \varepsilon(K+p) + \varepsilon(p) = 2\mu \), (3) \( (\mu \) is the chemical potential) at the momentum \( K = (\pi, \pi) \) connecting the main and shadow parts of two antipodal pockets as it is shown in Fig.1. Nesting of the FC can promote a pairing in a particle-hole channel resulting in an insulating gap on the FC. Secondly, such a FC manifests perfect mirror nesting feature \( \varepsilon(K/2 + k) = \varepsilon(K/2 - k) \), (4) at the same momentum promoting pairing in a particle-particle channel that can result in a rise of SC condensate. In this case, the momenta \( K/2 \pm k \) of the particles composing a pair with total momentum \( K \) (next, pairing) belong to the main and shadow parts of the same pocket (\( k \) is a momentum of the relative motion of next, pairing).

In the case of Cooper pairing with \( K = 0 \), the mirror nesting condition (4) is satisfied trivially on the whole of any FC due to general feature of electron dispersion, \( \varepsilon(-p) = \varepsilon(p) \). If \( K \neq 0 \), the momenta \( k \) of the particles composing next, pairing belong to a domain of kinematic constraint (P. Fulde and R.A. Ferrel, 1964; A.I. Larkin and Yu.N. Ovchinnikov, 1964) being only a part of the crystal Brillouin zone. In the case of pocket-like FC, such a domain is a quarter of the Brillouin zone. Note that the spectral weight corresponding to a shadow band of the pocket decreases rapidly with an increase of a distance from the FC. There are four crystal equivalent domains of kinematic constraint \( \Xi_j \) that can be indexed by a label \( j = 1, 2, 3, 4 \).

Under mirror nesting condition, the mean-field self-consistency equation that determines SC energy gap parameter \( \Delta(k) \) can be written as

\[
\Delta(k) = -\frac{1}{2} \sum_{k'} \frac{U(k-k')\Delta'(k')}{E(k')} h(k'; T) \quad (5)
\]

where the summation has to be performed over \( k' \) inside one of the domains of kinematic constraint. Here, \( U(k-k') \) is pairing interaction matrix element,

\[
E(k) = \sqrt{\xi^2(k) + |\Delta(k)|^2} \quad (6)
\]
is quasiparticle energy,
\[ 2\xi(k) = \varepsilon(k_+) + \varepsilon(k_-) - 2\mu \]  
(7)
is kinetic energy of \( \eta_K \) - pair, and
\[ h(k; T) = \tanh[E(k)/2T]. \]  
(8)
One has to have in mind that the gap parameter in \( \Delta \) is defined inside the domain \( \Xi_j \) corresponding to the total momentum \( K_j \) of \( \eta_K \) - pair. Therefore, to take it into consideration, it is convenient to write this parameter in the explicit form, \( \Delta_j(k) \).

Repulsive interaction can result in SC pairing if and only if the kernel \( U(k-k') \) of the integral equation \( \psi \) has at least one negative eigenvalue \( \tilde{\gamma}_j \). Eigenfunctions \( \varphi_s(k) \) and eigenvalues \( \lambda_j \) of the pairing interaction operator are defined as the solutions to the equations
\[ \varphi_s(k) = \lambda_j \sum_{k'} U(k-k')\varphi_s(k'). \]  
(9)
One can consider complete orthonormal system of the functions \( \varphi_s(k) \) as generic basis convenient to solve the SC pairing problem \( \psi \).

A nontrivial solution \( \Delta(k) \) to the equation \( \psi \), if such a solution exists at all, must change its sign on a nodal line inside the domain of kinematic constraint \( \Xi \). As far as the arguments \( k \) and \( k' \) of the kernel \( U(k-k') \) are defined inside this domain, such a kernel, in the case of screened Coulomb repulsion, has to have one negative eigenvalue with necessity \( \tilde{\gamma}_j \). To obtain an approximate solution to the equation \( \psi \), it is convenient to reduce the true nondegenerate kernel to a degenerate one with a finite set of eigenfunctions. The simplest appropriate degenerate kernel \( \tilde{\gamma}_j \),
\[ U(k-k') = U_0[1-(k-k')^2r_0^2/2], \]  
(10)
where \( U_0 \) and \( r_0 \) have meaning of effective coupling constant and screening length, respectively, corresponds to two even and two odd (with respect to inversion transformation, \( k \rightarrow -k \)) eigenfunctions. Due to the fact that \( \Delta(-k) = \Delta(k) \), the gap parameter turns out to be a linear combination of even eigenfunctions only. Therefore, in the case of repulsion pairing \( \tilde{\gamma}_j \), one can conclude that there is a two-component SC order.

One can introduce the SC order parameter as a wave function of \( \eta_K \) - pair,
\[ \Psi(R, k) = \sum_j \gamma_j e^{iK_j R} \Psi_j(k), \]  
(11)
where \( R \) is center-of-mass radius vector, \( \Psi_j(k) \sim \Delta_j(k) \) has meaning of a wave function of the relative motion of \( \eta_K \) - pair with the total momentum \( K_j \). The coefficients \( \gamma_j \) determine a symmetry of the order parameter that is one of two one-dimensional irreducible representations of 2D crystal symmetry group: \( A_{1g} \) (extended s-wave) or \( B_{1g} \) (extended d-wave). A choice of \( \gamma_j \) is connected with the pairing interaction (for example, phonon-mediated or AF magnon-mediated) that mixes states of particles in different domains of kinematic constraint.

Eigenfunctions of pairing interaction operator defined in the region \( \Xi \) joining all domains of kinematic constraint can be written as
\[ \varphi_s(R, k) = \sum_j \gamma_j e^{iK_j R} \varphi_s(k). \]  
(12)

One can represent the order parameter as
\[ \Psi(R, k) = \sum_s \Psi_s(R) \varphi_s(R, k) \]  
(13)
that results in the components of the order parameter in the form of
\[ \Psi_s(R) = \sum_{k \in \Xi} \Psi(R, k) \varphi_s^*(R, k). \]  
(14)

As it follows from \( \eta_K \) - pairing concept, under mirror nesting condition, there are three peculiar nodal curves (Fig.2) belonging to each domain of kinematic constraint: 1) FC (or its finite piece) on which kinetic energy of \( \eta_K \) - pair equals zero, \( 2\xi(k) = 0 \), and quasiparticle charge changes its sign; 2) nodal curve on which \( \Delta_j(k) = 0 \) and SC order parameter changes its sign; 3) a curve on which \( \nabla_\xi \Delta_j(k) = 0 \) and quasiparticle group velocity changes its sign. These three curves do not coincide and, as a consequence, coherence factors manifest peculiar momentum dependence as it is shown schematically in Fig.2.
The mean-field SC transition temperature $T_{sc}$, corresponding to $\Delta_0(k) \to 0$, can be derived from the self-consistency equation \[15\]. It should be noted that pairing repulsion, in particular, the model potential \[11\], makes possible a rise of quasi-stationary states (QSS) of $\eta_K$-pair as solutions of corresponding Cooper problem \[11\]. Such QSS treated as developed fluctuations of noncoherent $\eta_K$-pairs creating above $T_{sc}$ can be associated \[11\] with strong pseudogap state observed in underdoped cuprates \[12\].

Complex two-component SC order parameter can be normalized in the following way:

$$\sum_{k \in \Xi} |\Psi(R, k)|^2 = \sum_s |\Psi_s(R)|^2 = n_s f(R)/2. \quad (15)$$

Here, $n_s f(R) \sim \rho_s f$ is the superfluid density. As it follows from \[15\], the components of the order parameter can be represented in the form

$$\Psi_1 = \psi_1 e^{i\phi}, \quad \Psi_2 = \psi_2 e^{i\beta \phi} \quad (16)$$

where $\psi_1$ and $\psi_2$ are the absolute values of the components, $\beta$ is the relative phase of these components and $\Phi$ is the phase relating to center-of-mass motion of $\eta_K$-pair. Due to the normalization condition \[15\], $\psi_1$ and $\psi_2$ are not independent and, to take into account the contribution of the relative motion of $\eta_K$-pair into SC order parameter, one needs only two variables, for example, $\psi_1$ and $\beta$. Further we restrict ourselves to the simplest case when $\psi_1 = \psi_2 \equiv \psi$.

3. Ginzburg-Landau approach

General expression of Ginzburg-Landau functional has the form

$$F = \int d^2 R \left[ f_1 + f_2 + f_m \right] \quad (17)$$

where, in the case of complex two-component order parameter, an expansion of free energy density in powers of the order parameter components should be written as

$$f_1 = \sum_{ss'} A_{ss'} \Psi_s^{*} \Psi_{s'} + \frac{1}{2} \sum_{ss'tt'} B_{ss'tt'} \Psi_s^{*} \Psi_{s'} \Psi_t^{*} \Psi_{t'} \quad (18)$$

A gradient contribution into free energy density can be presented as

$$f_g = \frac{\hbar^2}{4m} \sum_{ss'} [\nabla \Psi_s^{*}] [\nabla \Psi_{ss'}^{*}] \quad (19)$$

Here, $\mathbf{D} = -i\nabla - (2e/h c) \mathbf{A}$ is the operator of covariant differentiation, $\mathbf{A}$ is vector potential. The explicit form of the matrices $A_{ss'}$, $B_{ss'tt'}$, and $M_{ss'}$ following from $\eta_K$-pairing scheme is obtained in \[12\] within the framework of the original Gor’kov method (L.P.Gor’kov, 1959). Hermmitian $2 \times 2$ matrices $A_{ss}$ and $M_{ss}$ can be diagonalized simultaneously by a unitary transformation of the order parameter. Here, we suppose that such a transformation is already performed. Thus, each of the second order matrices $A_{ss'}$ and $M_{ss'}$ can be characterized by its two principal values whereas the fourth order term in $f_1$ should be described by five independent components of $B_{ss'tt'}$.

Magnetic field energy density,

$$f_m = (z_0/8\pi)(\text{rot} \mathbf{A})^2 \quad (20)$$

(here, $z_0$ is the separation between the neighbor cuprate planes) is determined by an average square of magnetic field strength $\mathbf{H} = \text{rot} \mathbf{A}$ which can be presented as a sum $\mathbf{H} = \mathbf{H}_c + \mathbf{H}_i$ where $\mathbf{H}_c$ and $\mathbf{H}_i$ are the contributions due to external sources and possible spontaneous magnetization of the system, respectively. Note that an average (over a macroscopic region of the real space) value of the internal field $\mathbf{H}_i$ equals zero (such a condition excludes any ferromagnetic order from the consideration) whereas the average value of $H_i^2$, of course, should not vanish.

The system of Ginzburg-Landau equations following from the functional \[17\] is derived in \[2\]. One can believe that this equation system, just as similar system of equations corresponding to a charged two-condensate Ginzburg-Landau model \[13\], may admit more various solutions as compared to the Ginzburg-Landau equations in the case of one-component order. To show this \[13\], let us consider here the simplest case of spatially homogeneous system without any magnetic fields when it is sufficient to remain only the contribution of $f_1$ into the functional \[17\]. One can rewrite \[15\] as

$$f_1 = a_1 \psi^2 + (B + 2C \cos \beta + D \cos \beta^2) \psi^4/2 \quad (21)$$

where $a_1 = A_1 + A_2$ is the trace of the matrix $A_{ss'}$, and the coefficients $B$, $C$, and $D$ can be expressed in terms of the independent elements of the matrix $B_{ss'tt'}$. Here $B$ and $D$ are positive by definition. For the sake of simplicity, we assume that $C > 0$ as well.

The coefficients $A_1$ and $A_2$ vanish at one and the same temperature $T_{sc}$ corresponding to the mean field approximation \[5\]. Therefore, at $|T - T_{sc}| \ll T_{sc}$, one can assume that $a_1 = -k_1 \tau_1$ where $k_1 > 0$ and $\tau_1 = (T_{sc} - T)/T_{sc}$. Besides, the coefficients $B$, $C$, and $D$ can be considered as taken at $T = T_{sc}$.

However, one should take into account that, aside from a dependence on temperature, the coefficients in \[21\] are dependent on doping level $x$. At $T < T_s$, a nontrivial solution minimizing the free energy density \[21\] corresponds to $\psi \neq 0$ and $\beta = \pi$ when $C \geq D$. It should be noted that the relative phase $\beta = \pi$ displays the fact that SC order parameter arising under repulsive pairing interaction should change its sign inside the domain of kinematic constraint. In the opposite case $C \leq D$ the relative phase of the components of the order parameter is determined by the equality $\cos \beta = -C/D$. The equation $C(x) = D(x)$ determines a doping level $x = x_0$ corresponding to a qualitative change of SC order.
4. Spontaneous orbital currents

One can naturally define the order parameter \( \alpha = \pi - \beta \) to distinguish two SC phases with \( \beta < \pi \) (\( \beta \)-phase) and \( \beta = \pi \) (\( \pi \)-phase). We assume that \( \pi \)-phase corresponding to \( \alpha = 0 \) may exist at \( x > x_0 \) whereas \( \beta \)-phase with nonzero \( \alpha \) occurs at \( x < x_0 \). In a small vicinity of the point \( x = x_0, T = T_{\text{opt}}(x_0) \), the free energy density can be represented in the form of an expansion in even powers of the order parameters \( \psi \) and \( \alpha \).

A rise of the relative phase \( \beta \neq \pi \) corresponding to complex coherence factors can be associated with an internal magnetic field-induced change in the phase of the destruction operator of an electron with spin \( \sigma \) on a lattice site with radius vector \( n \).

\[
\hat{c}_{n\sigma} \to \hat{c}_{n\sigma} \cdot \exp \left[ i(e/hc)A_{\beta}(n)n \right].
\] (22)

Therefore the phase of the order parameter in the real space representation can be written as

\[
\beta(n, n') = \pi - (e/hc)[A_{\beta}(n)n + A_{\alpha}(n)n'].
\] (23)

A contribution \( \Phi = (2e/hc)A_{\beta}(R)R \) into the phase (24) may be related to center-of-mass motion of the pair with radius vector \( R = (n + n')/2 \). One can assume that vector potential \( A_{\beta} \) of the internal magnetic field is due to orbital currents circulating inside a unit cell as a result of the relative motion of \( \eta_K \)-pair. When \( x \approx x_0 \), a small change in the relative phase can be represented as

\[
\alpha \approx \frac{e}{2hc} \frac{\partial A_{\beta}}{\partial x_k} x_k x_k.
\] (24)

where summation over repeated Cartesian indices \( i, k = 1, 2 \) is understood.

In the SC state, orbital antiferromagnetic (OAF) order \( \sigma \alpha \beta \) that can be associated with nonzero relative phase \( \alpha \) of the SC order parameter develops as AF correlated orbital current circulations \( \sigma \beta \). Such circulations may survive above \( T_{\text{opt}} \) in the form of either long-range \( \sigma \beta \) or short-range \( \sigma \alpha \) OAF order. In the case under consideration, real magnetic field due to orbital currents in \( \alpha \beta \) can be associated with a gauge field which links the charge and orbital current degrees of freedom (\( \psi \) and \( \alpha \), respectively). This field is similar to the gauge fields introduced into Ginzburg-Landau functional in boson version of spin-charge separation scheme \( \sigma \beta \).

In the framework of Ginzburg-Landau phenomenology, an order parameter should be understood as averaged over the relative motion of \( \eta_K \)-pair. Therefore, taking into account a checkerboard distribution of orbital currents, mean square value of OAF order parameter can be estimated as

\[
\alpha^2 \approx \left( \frac{ea}{4\hbar c} \right)^2 A_{\beta}^2.
\] (25)

where \( a \) is a separation between neighbor copper atoms in cuprate plane.

Spontaneous orbital currents in the SC state lead to a positive contribution of internal magnetic field energy into Ginzburg-Landau functional. In spatially homogeneous system without an external magnetic field, a rise of the relative phase change \( \alpha \) introduces another contribution into the free energy resulting from the gradient term of the Ginzburg-Landau functional \( \sigma \beta \). One can represent this contribution, that links two competing orders \( \psi \) and \( \alpha \), as \( F_{12} = b_{12} \psi^2 \alpha^2 \). Here we consider a positive constant \( b_{12}(x) \) as a phenomenological parameter.

These essentially positive contributions themselves exclude a possibility of a rise of a thermal stable state with \( \alpha \neq 0 \). Therefore, one has to realize a complete study of a competition between the SC and insulating OAF pairing channels.

Thus, Landau free energy density of the OAF insulating state,

\[
f_2 = a_2 \alpha^2 + b_2 \alpha^4/2,
\] (26)

should be added into the Ginzburg-Landau functional. Here, \( b_2 \) is a positive function of doping and the coefficient \( a_2 \) vanishing at the mean-field OAF transition temperature \( T_d(x) \), under the condition that \( |\tau_2| \ll 1 \), can be written as \( a_2 = -\kappa \tau_2 \) where \( \kappa_2 > 0 \). \( \tau_2 = (T_d - T)/T_d \). The magnetic field energy of circulating currents, being also proportional to \( \alpha^2 \), is presupposed to be included into the term \( a_2 \alpha^2 \). Without OAF ordering, the thermal stable SC \( \beta \)-phase turns out to be impossible and,
The free energy density of a homogeneous state, up to the terms of the fourth order, can be written as 

\[ f = a_1 \psi^2 + a_2 \alpha^2 + \frac{1}{2} b_1 \psi^4 + b_{12} \psi^2 \alpha^2 + \frac{1}{2} b_2 \alpha^4. \quad (27) \]

The conditions that \( a_1(T, x) = 0 \), \( a_2(T, x) = 0 \) determine the mean-field temperatures \( T_{sc}(x) \) and \( T_d(x) \), respectively. Doping suppresses both OAF and SC orders. At low doping, OAF order may dominate SC one. Therefore, one can assume that there is an intersection of the functions \( T_d(x) \) and \( T_s(x) \) at a point corresponding to a doping level \( x = x_0 \) as it is shown in Fig.3. It should be emphasized that the expression \( \beta \) represents the free energy inside relatively small region of the phase diagram where both \( |\tau_1| \ll 1 \) and \( |\tau_2| \ll 1 \).

5. Phase diagram

Free energy density expansion in powers of \( \psi \) and \( \alpha \), Eq. (27), is justified in a relatively small vicinity of the intersection point \( T_{tc}, x_0 \) of the functions \( T = T_d(x) \) and \( T = T_{sc}(x) \). At this point (tetracritical point \( c \) in Fig.3), four curves of second order phase transitions come together. In Fig.3, these curves (which, under the approximations we use, may terminate only on the coordinate axes of the \( T - x \) phase diagram) are prolonged outside of the tetracritical point vicinity, however, such an extension which does not directly follow from (27) should be considered as a very conditional one.

Sector \( N \) in Fig.3 is related to normal Fermi liquid state (\( N \)-phase corresponding to high values of both temperature and doping). Lowering of temperature or doping may result in second order transition from \( N \)-phase either into insulating \( \alpha \)-phase at \( T = T_d \) when \( T > T_{tc} \), \( x < x_0 \) or into SC \( \pi \)-phase at \( T = T_{sc} \) when \( T < T_{tc} \), \( x > x_0 \). A subsequent lowering of temperature leads to second order transition from \( \alpha \)-phase into SC \( \beta \)-phase at \( T_c(x) < T_{sc}(x) \). The curve \( T = T_{\beta \pi}(x) \) of second order transition between \( \beta \) and \( \pi \) phases inside the SC state is situated below \( T_d(x) \), starts from the tetracritical point and terminates at a point \( T = 0, x = x_0 \) on the \( T \)-axis. Thus, the point \( T = 0, x = x_b \) has meaning of a quantum critical point.

At \( x < x_0 \), the mean-field temperature \( T_{sc} \) is not a phase transition temperature, similar to the mean-field temperature \( T_d \) which cannot be a phase transition temperature at \( x > x_0 \). However, in the regions \( T_c < T < T_{sc}, x < x_0 \) of the insulating \( \alpha \)-phase and \( T_{\beta \pi} < T < T_d, x > x_0 \) of SC \( \pi \)-phase, there arise peculiar states corresponding to developed fluctuations of the absolute value \( \psi \) and relative phase \( \alpha \) of the order parameter, respectively.

In Fig.4, we present characteristic structure of isolines of the free energy density \( f(\psi, \alpha) \), Eq. (27), in different regions of the phase diagram shown in Fig.3. Different phases are separated by bold lines. Dashed lines bound the regions of developed fluctuations of the order parameter.
crossover temperature as an upper boundary of strong pseudogap state may be found considerably higher than \( T_{sc} \), because \( \eta_K \) - pairing admits QSS which may arise above the mean-field temperature \( T_{sc} \) (Fig.3).

The transition from \( \beta \)-phase into \( \pi \)-phase inside the SC state also occurs through a region \((T_{\beta \pi} < T < T_d, \quad x_0 < x < x_d)\) of developed fluctuations as it is seen from Fig.3. In this case, the relative phase \( \alpha \) turns out to be a fluctuating parameter and \( T_\beta(x) \) is a crossover temperature. One can imagine that QSS in the form of non-correlated circular orbital currents may exist inside the region of such fluctuations (\( \pi^* \), in Fig.3). At last, SC \( \beta \)-phase corresponds to a coexistence of superconductivity and orbital antiferromagnetism.

Phase transition in such quasi-two-dimensional systems as cuprates has to be similar to Berezinski - Kosterlitz - Thouless (BKT) transition (V.L. Berezinski, 1971; J.M. Kosterlitz and D.J. Thouless, 1973) which can be described by the thermal unbinding of vortex-antivortex pairs. In contrast to vortex lifetime in nonsuperconducting state of normal metal, BKT transition temperature \( T_{BKT} \) is weakly sensitive to vortex core energy. Therefore, in the case of transition from the SC state into normal metal state, unbinding vortices survive in a very narrow temperature interval above \( T_{BKT} \). We believe that such a case corresponds to the transition between \( N \) and \( \pi \)-pases.

One can think that, in underdoped and slightly overdoped cuprates, vortex core may be related not to a normal metal but to an insulating state competing with the SC state. It leads to highly low vortex core energy so that vortex excitations (as QSS of noncoherent \( \eta_K \) - pairs) may exist in rather wide temperature range which can be associated with the strong pseudogap state. It should be noted here that magnetic field-induced destruction of superconductivity in cuprates results in exactly insulating state.

In the framework of the BKT scenario, loss of SC phase coherence is due to a rise of Abrikosov vortices as topological defects of the phase \( \Phi \) relating to the center-of-mass motion of \( \eta_K \) - pair. One can think that, in this scenario, the transition \( \beta \rightarrow \pi \) is also accompanied by thermal excitation of vortices. As far as center-of-mass phase \( \Phi = \text{const} \) in the SC state, such vortices should be manifested in the relative motion of \( \eta_K \) - pair arising due to unbinding of bounded orbital current circulations of opposite sign. It should be emphasized especially that developed fluctuations arising in the mean-field scheme considered here emerge due to a competition of two ordered states and peculiar features of \( \eta_K \) - pairing under repulsive interaction.

5. Conclusion

The \( \eta_K \) - pairing concept presented here in the framework of a generalized Ginzburg-Landau approach seems quite adequate for the explanation of the phase diagram typical of SC cuprate compounds. Such a concept leads naturally to the relative phase of the two-component SC order parameter which can be associated with an insulating OAF (\textit{hidden}) order. Thus, a competition of the SC and OAF ordered states turns out to be a generic input principle of the macroscopic approach to the problem of superconductivity of cuprates.

All principal features of the phase diagram of the cuprates can be qualitatively described in the framework of the \( \eta_K \) - pairing concept: weak and strong pseudogaps, the form of the SC dome, the regions of developed fluctuations of the SC and OAF components of the order parameter. This concept predicts a possibility of the second order phase transition inside the SC dome: one of the two SC phases corresponds to a coexistence of superconductivity and AF ordered circulations of orbital currents whereas the other one is similar to conventional SC phase. An enhanced diamagnetic response and giant Nernst effect observed in underdoped cuprates in a broad temperature region above \( T_c \) can also be entered into the \( \eta_K \) - pairing scheme.

Screened Coulomb repulsion as an underlying pairing interaction of the \( \eta_K \) - pairing scheme allows to explain reasonably both the SC and insulating behavior of the cuprates. In particular, this scheme can qualitatively explain the observed evolution of the form of the FC with doping. Main and shadow parts of small hole pockets manifest perfect nesting and mirror nesting features of the FC at the same momentum \( \mathbf{K} = (\pi, \pi) \). These nesting features provide a possibility of a rise of both insulating and SC states at arbitrarily value of a coupling constant (asymptotic exact solution to the mean-field pairing problem).

The Drude-like behavior of the optical conductivity in the SC state of the cuprates up to extremely low temperature can be treated, in the framework of the \( \eta_K \) - pairing concept, as the fact that a considerable part of carriers does not belong to the SC condensate. In overdoped and nearly optimal doped cuprates, only a piece of the FC exhibits the mirror nesting feature. Therefore, the superfluid density is proportional to the length of this piece whereas the normal density remains finite up to the zero-temperature limit. In underdoped cuprates, the FC has the form of small hole pockets with perfect mirror nesting on the whole of each pocket. However, the considerable difference between the values of the spectral weight on the two opposite sides of the pocket (main and shadow bands) results in the fact that the SC pairing turns out to be governed by the shadow band with low spectral weight: only a small part of carriers in the main band can find the partners in the shadow band. Low-temperature heat capacity, \( C_V \sim \gamma T \) is also determined by such off-condensate carriers.

Nontrivial momentum dependence of the coherence factors resulting from the \( \eta_K \) - pairing scheme allows to explain observed tunnel conductance asymmetry, peak
and dip-hump structure of both tunnel and photoemission spectra \cite{29}, and typical of cuprates features \cite{30} of Andreev reflection \cite{21}.

The $\eta_K$ - pairing scheme, extended to account for phonon-mediated pairing, allows to explain \cite{32} qualitatively observed behavior of the isotope effect in different cuprates including both absence of this effect in some compounds \cite{33} and the so-called negative isotope effect \cite{34}.

A competition of screened Coulomb pairing repulsion and phonon-mediated attraction or AF magnon-mediated repulsion determines \cite{32} observed unconventional symmetry of the SC order parameter in cuprates \cite{35, 36}.

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