LOCAL THETA CORRESPONDENCES BETWEEN EPIPELAGIC SUPERCUSPIDAL REPRESENTATIONS

HUNG YEAN LOKE, JIA-JUN MA, AND GORDAN SAVIN

Abstract. In this paper we study the local theta correspondences between epipelagic supercuspidal representations of a type I classical dual pair \((\tilde{G}, \tilde{G}')\) over \(p\)-adic fields. We show that an epipelagic supercuspidal representation \(\pi\) of \(\tilde{G}\) lifts to an epipelagic supercuspidal representation \(\pi'\) of \(\tilde{G}'\) if and only if the epipelagic data of \(\pi\) and \(\pi'\) are related by the moment maps.

1. Introduction

1.1. Let \(k\) be \(p\)-adic field with ring of integers \(\mathcal{O}_k\), prime ideal \(\mathfrak{p}_k = (\varpi)\) and residue field \(\mathcal{O}_k / \mathfrak{p}_k\) of odd characteristic \(p\) where \(\varpi\) is a fixed uniformizer. Let \(\nu : k \rightarrow \mathbb{Z} \cup \{ \infty \}\) denote the valuation map where \(\nu(\varpi) = 1\). We will denote an algebraic variety by a boldface letter, say \(H\). We will denote its \(E\)-points by \(H(E)\) where \(E\) is an extension field of \(k\) and we will denote its \(k\) points by the corresponding normal letter \(H\).

If \(G\) is an algebraic group, then we let \(\mathfrak{g}\) be its Lie algebra or the \(k\) points of the Lie algebra, depending on the context. If \(G = G(k)\) acts on a set \(X\), then \(g \cdot x\) will denote \(g \cdot x\) for \(g \in G\) and \(x \in X\). In order to simplify the situation, we always assume that \(p \neq 2\) and sufficiently large compare to the rank of \(G\). Let \(k_{ur}\) denote the maximal unramified extension of \(k\). For a reductive algebraic group, we will fix a maximally \(k_{ur}\)-split torus \(S\), a maximally \(k_{ur}\)-split torus \(T\) which is defined over \(k\) and contains \(S\). Since \(G(k_{ur})\) is always quasi-split, we set \(Y := Z_G(T)\) to be the fixed Cartan subgroup of \(G\).

1.2. We recall the classification of irreducible type I reductive dual pairs. Let \(D\) be a \(k\)-division algebra with a fixed involution, which is either (i) \(k\), (ii) a quadratic field extension of \(k\), or (iii) the quaternionic division \(k\)-algebra. We continue to use \(\nu\) to denote the unique extension of the valuation \(\nu\) from \(k\) to \(D\). Let \(\mathfrak{d}\) be the ring of integers of \(D\). Let \(V\) be a right \(D\)-module with \(\epsilon\)-Hermitian form \(\langle \cdot, \cdot \rangle_V\) and \(G = G(k) = U(V, \langle \cdot, \cdot \rangle_V)\) be the unitary groups preserving \(\langle \cdot, \cdot \rangle_V\). Similar notation applies to \(G'\) and \(\epsilon\epsilon' = -1\). The \(k\)-vector space \(W = V \otimes_D V'\) has a natural symplectic form and \((G, G')\) is an irreducible type I reductive dual pair in \(Sp := Sp(W)\).

Let \(\mathbb{C}^1\) denote the norm 1 complex numbers. We will let \(\tilde{E}\) be the inverse image of a subgroup \(E\) in the metaplectic \(\mathbb{C}^1\)-cover \(Mp\) of \(Sp\). We fix a unitary non-trivial character \(\psi : k \rightarrow \mathbb{C}^\times\) with conductor \(\mathfrak{p}_k\) and consider local theta correspondence \(\theta\) arising from the oscillator representation of \(Mp\) with central character \(\psi\). For general information on theta correspondences, see [6, 9, 24]. In this paper, we will investigate the theta correspondence between epipelagic supercuspidal representations of \(\tilde{G}\) and \(\tilde{G}'\). In many situations, the roles of \(G\) and \(\tilde{G}'\) are interchangeable. In such cases, we only discuss the situations for \(G\) and the extend all notations to \(G'\) implicitly by adding ‘primes’.

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1.3. We briefly review some facts about epipelagic supercuspidal representations. See Section 2 and [20] for more details. Let \( \mathcal{B}(G, k) \) be the (extended) building of \( G \). In the setting of type I dual pairs, \( G \) and \( G' \) have compact centers. Therefore \( Z(G) \subseteq G_x \) and \( G_x = G_{[x]} \) for any \( x \in \mathcal{B}(G, k) \) where \( [x] \) is the image of \( x \) in the reduced building. Following Reeder-Yu [20], we construct a tamely ramified irreducible supercuspidal representation \( \pi_{\Sigma} \) of \( \tilde{G} \) in Section 3.4 from the data \( \Sigma = (x, \lambda, \chi) \) where

\[ (SD1) \quad x \in \mathcal{B}(G, k) \text{ is an epipelagic point of order } m \text{ where } p \nmid m \text{ (c.f. Definition 2.2.1)}, \]

\[ (SD2) \quad \lambda \text{ is a stable vector in } g_x \cdot \frac{1}{m} / g_x \cdot \frac{1}{m} + \text{ and} \]

\[ (SD3) \quad \chi \text{ is a character of the stabilizer } S_\lambda \text{ of } \lambda \text{ in } G_x, k. \]

We call \( \Sigma = (x, \lambda, \chi) \) an epipelagic data of order \( m \) for \( G \). The supercuspidal representation \( \pi_{\Sigma} \) has depth \( \frac{1}{m} \) and it is called an epipelagic supercuspidal representation of \( \tilde{G} \). Since the cover \( \tilde{G} \) depends on the dual pair \((G, G')\), the notation \( \pi_{\Sigma} \) only makes sense relative to the dual pair \((G, G')\).

1.4. Let \( \Sigma = (x, \lambda, \chi) \) and \( \Sigma' = (x', \lambda', \chi') \) be epipelagic data of \( G \) and \( G' \) of order \( m \) and \( m' \) respectively. Suppose \( \theta(\pi_{\Sigma}) = \pi_{\Sigma'}' \). By [18, 19], \( \pi_{\Sigma} \) and \( \pi_{\Sigma}' \) have the same depth \( \frac{1}{m} = \frac{1}{m'} \), i.e. \( m = m' \). It turned out that the data \((x, \lambda, \chi)\) and \((x', \lambda', \chi')\) are related by a geometric picture which we now briefly explain.

The points \( x \) and \( x' \) correspond to self dual \( \mathfrak{g}_D \)-lattice functions \( L \) and \( L' \) in \( V \) and \( V' \) respectively (c.f. [3, 5, 7, 12]). The tensor product \( \mathcal{B} = L \otimes L' \) is a self dual \( \mathfrak{g}_k \)-lattice function in \( W \). The quotient \( X = \mathcal{B} \cdot \frac{1}{m} / \mathcal{B} \cdot \frac{1}{m} + \) is an \( \mathfrak{j} \)-vector space. In [6] we define the moment maps \( M = \frac{1}{2m} \) and \( M' = \frac{1}{2m} \):

\[ \mathfrak{g}_x, \frac{1}{m} \overset{M}{\longrightarrow} X \overset{M'}{\longrightarrow} \mathfrak{g}'_x, \frac{1}{m}. \]

Our first result is a refinement of special cases of [19].

**Proposition 1.4.1** (Proposition 7.1.1). Suppose \( \theta(\pi_{\Sigma}) = \pi_{\Sigma}' \). Then

\[ (M) \quad \text{there exists a } \omega \in X \text{ such that } \lambda = M \cdot \frac{1}{2m}(\omega) \text{ and } \lambda' = -M' \cdot \frac{1}{2m}(\omega). \]

Condition \((M)\) imposes severe restriction to the ranks of \( G \) and \( G' \):

**Proposition 1.4.2** (Proposition 7.3.2). Suppose \( \theta(\pi_{\Sigma}) = \pi_{\Sigma}' \). Then \((G, G')\) or \((G', G)\) is one of the following types:

(i) \((D_n, C_n)\), (ii) \((C_n, D_{n+1})\), (iii) \((C_n, B_n)\), (iv) \((A_n, A_n)\) or (v) \((A_n, A_{n+1})\).

Let \((x, \lambda)\) and \((x', \lambda')\) be parts of data for \( G \) and \( G' \) of order \( m \) respectively satisfying (SD1) and (SD2). It turns out that Condition \((M)\), in all but one exceptional case (see Case \( \mathbf{I} \) in Section 8), is a sufficient condition for the epipelagic supercuspidal representation \( \pi_{\Sigma} \) to lift to an epipelagic supercuspidal representation of \( \tilde{G}' \). For the case of explaining in this introduction, we will omit the exceptional case. Using Condition \((M)\), we will construct a group homomorphism \( \alpha : S_{\lambda'} \rightarrow S_{\lambda} \) in Lemma 8.1.1. We can now state part of the main 8.1.2.

**Theorem 1.4.3.** Let \((G, G')\) be an irreducible type I reductive dual pair such that \((G, G')\) has the form \((i)-(v)\) in Proposition 1.4.4. Let \((x, \lambda)\) and \((x', \lambda')\) be data of \( G \) and \( G' \) respectively of order \( m \) satisfying (SD1), (SD2) and Condition \((M)\). Suppose we are not in the exceptional Case \( \mathbf{I} \). Then for every character \( \chi \) of \( S_{\lambda} \),

\[ \theta(\pi_{\Sigma}) = \pi_{\Sigma}', \]

where \( \Sigma = (x, \lambda, \chi) \), \( \Sigma' = (x', \lambda', \chi' \circ \alpha) \) and \( \chi' \) is the contragredient of \( \chi \). In particular the theta lift is nonzero.
2. Epipelagic representations

In this section we review Reeder-Yu’s construction of epipelagic supercuspidal representations (c.f. [20]).

2.1. Let \( k \) be a \( p \)-adic field as in Section 1.4. Let \( k^\ur \) be the maximal unramified extension of \( k \) with residue field \( \overline{f} \). We let \( \Fr \) denote the Frobenius element such that \( \Gal(k^\ur/k) = \Gal(\overline{f}/f) = \langle \Fr \rangle \).

2.2. Epipelagic points. Let \( G \) be an algebraic group defined over \( k \). Let \( E \) be a tamely ramified extension of \( k \). For \( x \in B(G, k) \subseteq B(G, E) \), we set up some notation which will be used in the rest of the paper. Let

- \( G(E)_{x,r} \) and \( g(E)_{x,r} \) be the Moy-Prasad filtrations corresponding to \( x \) for \( r \geq 0 \);
- \( G(E)_x = \text{Stab}_{G(E)}(x) \) and \( G(E)_{[x]} = \text{Stab}_{G(E)}([x]) \) where \([x]\) is the image of \( x \) in the reduced building;
- \( G_x(E) = G(E)_x/G(E)_{x,0^+}; \)
- \( G(E)_{x,r} := G(E)_{x,r}/G(E)_{x,r^+} := G(E)_{x,r}/G(E)_{x,r^+} \), and
- \( g(E)_{x,r} := g(E)_{x,r}/g(E)_{x,r^+} \).

In order to abbreviate the notations of objects correspond to \( G = G(k) \), let

- \( G_{x,r} := G(k)_{x,r} \cap g(k) \) and
- \( g_{x,r} := g(k)_{x,r} = G(k^\ur)_{x,r} \cap g(k) \).

In order to abbreviate the notations of objects correspond to \( G(k^\ur) \), we let

- \( G_x := G_x(k^\ur), G_{x,r} := G(k^\ur)_{x,r} \) and \( g_{x,r} := g(k^\ur)_{x,r} \).

The quotient space \( g_{x,r} \) is an \( \overline{f} \)-vector space and we denote its dual space \( \text{Hom}(g_{x,r}, \overline{f}) \) by \( \overline{g}_{x,r} \). We have assumed in the introduction that \( p \) is large compared to the rank of \( G \). Then by [2 Prop. 4.1], \( \overline{g}_{x,r} \) could be identified with \( \overline{g}_{x,-r} \) via an invariant bilinear form on \( g \). Since we are only treating classical groups and \( p \neq 2 \), we will use the trace form defined in Definition 6.1 in this paper. The group \( G_x \) acts on \( \overline{g}_{x,r} \). A vector \( \lambda \in \overline{g}_{x,r} \) is called a stable vector if the \( G_x \)-orbit of \( \lambda \) is a closed subset of \( \overline{g}_{x,r} \) and the stabilizer of \( \lambda \) in \( G_x \) modulo \( Z(G)_{x,0^+} \) is a finite group.

Let \( \Psi_K \) be the set of affine \( T(k^\ur) \)-roots. Suppose \( x \in A(S, k) = A(T, K)^\Fr \), i.e. \( x \) is in the apartment defined by \( S \). Let \( r(x) \) be the smallest positive value in \( \{ \psi(x) : \psi \in \Psi_K \} \). Then \( G(k^\ur)_{x,0^+} = G(k^\ur)_{x,r(x)} \).

**Definition 2.2.1.** Let \( m \) be an integer where \( p \nmid m \). A point \( x \) in the apartment \( A(S, k) \) is called an epipelagic point of order \( m \) if \( r(x) = \frac{1}{m} \) and \( \overline{g}_{x,\frac{1}{m}} \) contains a stable vector.

By [8], \( m \) is an even integer except when \( G(k^\ur) \) is split of type \( A_{m-1} \). In particular \( m \geq 2 \).
2.3. Epipelagic supercuspidal representations. Let $x$ be an epipelagic point of order $m$ so that $r(x) = \frac{1}{m}$. We fix an isomorphism of abelian groups $c : G_{x,r(x);r(x)^+} \rightarrow \mathfrak{g}_{x,r(x);r(x)^+}$. For the classical group, we choose the isomorphism to be the one induced by the Cayley transform $c(y) = (1 + g/2)(1 - g/2)^{-1}$.

Let $\lambda \in \mathfrak{g}_{x,\frac{1}{m}}$ be an $\mathfrak{f}$-rational functional on $\mathfrak{g}_{x,\frac{1}{m}}$. Then we get a character
\[
\psi : \psi \circ \lambda : G_{x,\frac{1}{m}+} = g(k)_{x,\frac{1}{m}} \cong \mathfrak{g}_{x,\frac{1}{m}} \rightarrow f \circ \psi \rightarrow \mathbb{C}^\times.
\]
The inflation of $\psi$ to $G_{x,\frac{1}{m}}$ will also be denoted by $\psi$.

Let
\[
H_{x,\lambda} := \text{Stab}_{G_x} (\lambda), \quad S_{x,\lambda} := H_{x,\lambda}/G_{x,\frac{1}{m}}.
\]
Then $H_{x,\lambda}$ is the stabilizer of $\psi$ in $G_x$.

We will assume that $\lambda$ is $G_x$-stable in $\mathfrak{g}_{x,\frac{1}{m}}$. In all the cases that we will consider in this paper, the finite group $S_{x,\lambda}$ is abelian. Note that the order of $S_{x,\lambda}$ is prime to $p$ and $G_{x,\frac{1}{m}}$ is pro-$p$. This implies that $H^2(S_{x,\lambda}, G_{x,\frac{1}{m}}) = 0$ which in turn implies that $S_{x,\lambda}$ splits in $H_{x,\lambda}$, i.e. $H_{x,\lambda} = S_{x,\lambda} \ltimes G_{x,\frac{1}{m}}$. We extend $\psi$ to a character of $H_{x,\lambda}$ by setting $\psi$ to be trivial on $S_{x,\lambda}$. By $H^1(S_{x,\lambda}, G_{x,\frac{1}{m}}) = 0$, we know that all splittings are conjugate up to $G_{x,\frac{1}{m}}$-conjugation. Hence the extension $\psi$ is unique and therefore canonical.

Let $\chi$ be a character of $S_{x,\lambda}$ and let $\pi_x(\lambda, \chi) := \text{ind}_{H_{x,\lambda}}^{G_x} \psi \otimes \chi$. By [20] Prop. 5.2, $\pi_x(\lambda, \chi)$ is an irreducible supercuspidal representation of $G$. We will call $(x, \lambda, \chi)$ an epipelagic data of order $m$ and we call $\pi_x(\lambda, \chi)$ an (irreducible) epipelagic supercuspidal representation attached to the data. It contains a minimal $K$-type represented by a coset
\[
\lambda = [\Gamma] = \Gamma + \mathfrak{g}_{x,-\frac{1}{m}+} + \mathfrak{g}_{x,-\frac{1}{m}} + \mathfrak{g}_{x,-\frac{1}{m}+}
\]
where $K = G_{x,\frac{1}{m}}$.

Proposition 2.3.1. (i) All unrefined minimal $K$-types of $\pi_x(\lambda, \chi)$ are $G$-conjugate to $\Gamma + \mathfrak{g}_{x,-\frac{1}{m}+}$.

(ii) If $\pi_x(\lambda, \chi)$ and $\pi_x(\lambda', \chi')$ are isomorphic $G$-modules, then $\lambda$ and $\lambda'$ are in the same $G_x$-orbit. In addition if $\lambda = \lambda'$ then $\chi = \chi'$.

Proof. (i) Let $\Gamma + \mathfrak{g}_{x,-\frac{1}{m}+}$ represent the unrefined minimal $K$-type of $\pi$ as above. We will see in Lemma 2.3.1 later that $\Gamma \in \mathfrak{g}_{x,-\frac{1}{m}}$ is a good element and $T = Z_G(\Gamma)$ is a torus.

Let $T = T(k)$ and let $t = Z_G(\Gamma)$ be its Lie algebra. Since $(x, \Gamma, \rho)$ is a tamely ramified supercuspidal pair, then $T/Z(G)$ is anisotropic (or see for example, [10] Proposition 14.5). We are considering type I classical dual pairs so $Z(G)$ is anisotropic. Therefore $T$ is $k$-anisotropic. Hence $B(T, k) = \{x\}$.

Suppose $\Gamma_y + \mathfrak{g}_{y,-\frac{1}{m}+}$ is another unrefined minimal $K$-type of $\pi$ for some $y \in B(G, k)$.

Now show that we can move $y$ to $x$. Since $\Gamma + \mathfrak{g}_{x,-\frac{1}{m}+}$ and $\Gamma_y + \mathfrak{g}_{y,-\frac{1}{m}+}$ are minimal $K$-types of $\pi$, they are associates [16], i.e. there exists a $g \in G$ such that
\[
g(\Gamma_y + \mathfrak{g}_{y,-\frac{1}{m}+}) \cap (\Gamma + \mathfrak{g}_{x,-\frac{1}{m}+}) = (\Gamma_y + \mathfrak{g}_{y,-\frac{1}{m}+}) \cap (\Gamma + \mathfrak{g}_{x,-\frac{1}{m}+}).
\]
is nonempty. Here $\Gamma_{gy} = g\Gamma_y$.

Now we apply the argument in the proof of [11] Corollary 2.4.8. By (2), there are $X'' \in \mathfrak{g}_{x,-\frac{1}{m}+}$ and $Y'' \in \mathfrak{g}_{y,-\frac{1}{m}+}$ such that $\Gamma_{gy} + Y'' = \Gamma + X''$. By [11] Corollary 2.3.5, there exists $h \in G_{x,0^+}$ such that $b(\Gamma + X'') = \Gamma + X' \in (\Gamma + \mathfrak{g}_{x,-\frac{1}{m}+}) \cap t \cap \mathfrak{g}_{hyy,-\frac{1}{m}}$ where $X' \in t, \mathfrak{g}_{x,-\frac{1}{m}+}$. By [11] Lemma 2.4.7 $hgy \in B(T, k) = \{x\}$, i.e. $hgy = x$. We consider the isomorphism $\text{Ad}(hg^{-1}) : \mathfrak{g}_{y,-\frac{1}{m}+} / \mathfrak{g}_{y,-\frac{1}{m}+} \rightarrow \mathfrak{g}_{x,-\frac{1}{m}+} / \mathfrak{g}_{x,-\frac{1}{m}+}$. It is now clear that the coset
\[
h\gamma \Gamma_y = [\Gamma] \lambda.\]
This proves (i).
(ii) The last assertion of (ii) is \cite{20} Lemma 2.2. Now we prove the first assertion. Note that \( \lambda = \Gamma + g \mathbf{x} \) and \( \lambda' = \Gamma' + g \mathbf{x} \) represent unrefined minimal \( K \)-types of \( \pi_x(\lambda, \chi) \) and \( \pi_x(\lambda', \chi') \) respectively. Since \( \pi_x(\lambda, \chi) \simeq \pi_x(\lambda', \chi') \), the two minimal \( K \)-types are associates. By the proof in (i) where \( y = x \) and \( \Gamma_y = \Gamma' \), we conclude that there exists \( g \in G \) such that \( gx = x \) and \( g \lambda' = \lambda \). In particular \( g \in G_x \). Hence \( \lambda \) and \( \lambda' \) are in the same \( G_x \)-orbit in \( g(k)_{x, -\frac{1}{m}} \). \( \square \)

3. Classical reductive dual pairs and local theta correspondence

3.1. Classical groups. In this section, we will define the classical groups which appear in the irreducible dual pairs.

Let \( D \) be a division algebra over \( k \) with an involution \( \tau \) in one of the following cases:

1. \( D = k; \tau = \mathrm{id} \) and \( \varpi_D = \infty \).
2. \( D \) is a quadratic extension of \( k \), \( \tau \) is the nontrivial Galois element in \( \mathrm{Gal}(D/k) \), \( \varpi_D \) is a uniformizer of \( D \) so that \( \varpi_D = \infty \) if \( D/k \) is unramified or \( \tau(\varpi_D) = -\varpi_D \) if it is ramified.
3. \( D \) is the quaternionic division algebra over \( k \), \( \tau \) is the usual involution on \( D \), \( \varpi_D \) is a uniformizer of \( D \) such that \( \varpi_D^2 = \infty \).

Let \( \mathfrak{o}_D \) denote the ring of integers of \( D \), \( \mathfrak{p}_D = \varpi_D \mathfrak{o}_D \) denote its maximal prime ideal and \( f_D = \mathfrak{o}_D/\mathfrak{p}_D \) denote its residue field. We set \( \nu_D = \nu(\varpi_D) \).

Let \( V \) be a right \( D \)-vector space. Let \( \mathrm{End}_D(V) \) denote the space of \( D \)-linear endomorphisms of \( V \) which acts on the left. For \( \epsilon = \pm 1 \), let \( \langle , \rangle_v : V \times V \to D \) be an \( \epsilon \)-Hermitian sesquilinear form, i.e.

\[
\langle v_1, v_2 \rangle = \epsilon \langle v_2, v_1 \rangle^\tau \quad \text{and} \quad \langle v_1 a_1, v_2 a_2 \rangle = a_1^* \langle v_1, v_2 \rangle a_2
\]

for all \( v_1, v_2 \in V \) and \( a_1, a_2 \in D \). The \( \epsilon \)-Hermitian form induces a conjugation \( * : \mathrm{End}_D(V) \to \mathrm{End}_D(V) \) such that \( \langle gv_1, v_2 \rangle_v = \langle v_1, g^* v_2 \rangle_v \) for all \( v_1, v_2 \in V \) and \( g \in \mathrm{End}_D(V) \). Then

\[
G = U(V) = U(V, \langle , \rangle_V) := \{ g \in \mathrm{End}_D(V) : gg^* = \mathrm{Id} \} \quad \text{and} \quad \mathfrak{g} = \mathfrak{u}(V) = \mathfrak{u}(V, \langle , \rangle_V) := \{ X \in \mathrm{End}_D(V) : X + X^* = 0 \}
\]

is a classical group and its Lie algebra defined over \( k \).

3.2. Irreducible reductive dual pairs of type I. Let \( V \) be a right \( D \)-vector space equipped with an \( \epsilon \)-Hermitian sesquilinear form \( \langle , \rangle_V \) and let \( V' \) be a right \( D \)-vector space equipped with an \( (-\epsilon) \)-Hermitian sesquilinear form \( \langle , \rangle_{V'} \). Let \( G \) and \( G' \) be the classical groups defined by \( (V, \langle , \rangle_V) \) and \( (V', \langle , \rangle_{V'}) \) respectively.

We view \( V' \) as a left \( D \)-module by \( av = va^\tau \) for all \( a \in D \) and \( v \in V' \). Let \( W = V \otimes_D V' \). It is a symplectic \( k \)-vector space with symplectic form \( \langle , \rangle_W \) given by

\[
\langle v_1 \otimes v'_1, v_2 \otimes v'_2 \rangle_W = \mathrm{tr}_{D/k}(\langle v_1, v_2 \rangle_V \langle v'_1, v'_2 \rangle_{V'}^\tau).
\]

Then \( G \) and \( G' \) are mutual centralizers of each other in the symplectic group \( \mathrm{Sp}(W) \). We call \( (G, G') \) an irreducible reductive dual pair of type I.

3.3. Lattice model. We recall that \( \psi : k \to \mathbb{C}^\times \) is an additive unitary character of conductor \( \mathfrak{p}_k \). Let \( A \) be a self dual lattice in \( W \) (c.f. Definition \[4.3.1\]). The lattice model with respect to \( A \) of the oscillator representation \( \omega \) with central character \( \psi \) is defined by

\[
\mathcal{S}(A) = \left\{ f : W \to \mathbb{C} \mid f(a + w) = \psi(\frac{1}{2} \langle w, a \rangle_W f(w) \forall a \in A \right. \right. \left. \left. \text{f locally constant, compactly supported} \right\}
\]

Let \( \mathbb{C}^1 \) be the group of norm 1 complex numbers. Let \( \mathrm{Mp}(W) \) be the metaplectic \( \mathbb{C}^1 \)-covering of \( \mathrm{Sp}(W) \) which acts on the oscillator representation naturally by its definition. The lattice model with respect to \( A \) gives a section \( \omega_A : \mathrm{Sp}(W) \to \mathrm{Mp}(W) \) of
the natural projection $\text{Mp}(W) \to \text{Sp}(W)$ (c.f. [14, 24]). Let $\text{Sp}_A := \text{Stab}_\text{Sp}(W)(A) = \{ g \in \text{Sp}(W) \mid gA \subseteq A \}$. We only describe $\omega_A(g)$ for $g \in \text{Sp}_A$:

$$
(\omega_A(g)f)(w) = f(g^{-1}w) \quad \forall g \in \text{Sp}_A, f \in \mathcal{F}(A) \text{ and } w \in W.
$$

The splitting $\omega_A$ does not depend on the choice of the self-dual lattice $A$. More precisely, we have the following proposition whose proof is given in Appendix C.1.

**Proposition 3.3.1.** There is a section

$$
\omega_0 : \bigcup_{A \text{ is self-dual in } W} \text{Sp}_A \to \text{Mp}(W)
$$

such that $\omega_0|_{\text{Sp}_A} = \omega_A$ for every self-dual lattice $A$.

### 3.4. Epipelagic supercuspidal representations of covering groups.

Let $\Sigma = (x, \lambda, \chi)$ be an epipelagic datum of order $m$. We retain the notation for the subgroup $H_{x,\lambda} = S_{x,\lambda} \ltimes G_{x,\frac{\lambda}{m}}$ and its character $\psi_\lambda \otimes \chi$ in Section 2.3. Since $G$ is a member of a type I dual pair, we recall that $G_x = G[\omega]$ and $H_{x,\lambda}$ is a subgroup of $G_x$. We will show in [7.1.2] later that $G_x$ stabilizes a self-dual lattice $A$ in $W$. Then Proposition 3.3.1 gives a splitting

$$
\omega_0|_{G_x} : G_x \to \tilde{G}_x
$$

of $\tilde{G}_x \to G_x$. We will identify $H_{x,\lambda}$ and $G_x$ as subgroups of $\tilde{G}_x$ via $\omega_0$. Let $\chi_1 : \mathbb{C}^1 \to \mathbb{C}^\times$ be the inclusion map. Under this splitting, $\tilde{G}_x = G_x \times \mathbb{C}^1$ with $\mathbb{C}^1$ acting on the oscillator representation $\mathcal{F}$ via $\chi_1$. Now

$$
\pi_\Sigma := \text{ind}^G_{H_{x,\lambda} \rtimes \mathbb{C}^1} (\psi_\lambda \otimes \chi \otimes \chi_1)
$$

is an irreducible supercuspidal representation of $\tilde{G}$ which is also denoted by $\pi_\Sigma^G$ or $\pi_\Sigma^G(\lambda, \chi)$. We will also call $\pi_\Sigma$ an epipelagic supercuspidal representation attached to the epipelagic data $\Sigma$.

By Proposition 3.3.1 the splitting of $G_{x,0^+}$ is canonically defined for any $x \in \mathcal{B}(G, k)$. In particular, it still makes sense to talk about positive depth minimal $K$-types. In addition Proposition 3.3.1 holds if we replace $\pi_X(\lambda, \chi)$ with $\pi_X^G$ without any modification.

### 4. BRUHAT-TITS BUILDING AND MOY-PRASAD GROUPS OF CLASSICAL GROUPS

In this section we recall some known facts about the Bruhat-Tits buildings of classical groups. Our references are [3, 5].

#### 4.1. Lattice functions.

A (right) $\mathfrak{o}_D$-lattice $L$ in a right $D$-vector space $V$ is a right $\mathfrak{o}_D$-submodule such that $L \otimes_{\mathfrak{o}_D} D = V$.

**Definition 4.1.1.**

1. Let $\mathcal{L}_s$ be the set of $\mathfrak{o}_D$-lattice valued functions $s \mapsto L_s$ on $\mathbb{R} \cup \mathbb{R}^+$ such that (i) $L_s \supseteq L_t$ if $s < t$, (ii) $L_{s+t_D} = L_s \otimes_D L_t$ and, (iii) $L_{s^+} = \bigcup_{t>s} L_t$.

2. We set $\text{Jump}(\mathcal{L}) = \{ r \in \mathbb{R} : \mathcal{L}_r \supseteq \mathcal{L}_{r^+} \}$.

3. Given any lattice function $\mathcal{L}$, we define

$$
\mathfrak{gl}(V)_{\mathcal{L}, r} := \{ X \in \mathfrak{gl}(V) \mid X L_s \subseteq L_{s+r}, \forall s \in \mathbb{R} \} \quad \forall r \in \mathbb{R} \cup \mathbb{R}^+,
$$

$$
\text{GL}(V)_{\mathcal{L}, r} := \{ g \in \text{GL}(V) \mid (g - 1)L_s \subseteq L_{s+r}, \forall s \in \mathbb{R} \} \quad \forall r > 0,
$$

$$
\text{GL}(V)_{\mathcal{L}} := \{ g \in \text{GL}(V) \mid g L_s \subseteq L_s \}.
$$

4. For $r < s$, we denote $\mathcal{L}_{s-r} = \mathcal{L}_r / \mathcal{L}_s$.

**Definition 4.1.2.**

1. A $D$-norm of $V$ is a function $l : V \to \mathbb{R} \cup \{ \infty \}$ such that for all $x, y \in V$ and $d \in D$, (i) $l(xd) = l(x) + \nu(d)$, (ii) $l(x + y) \geq l(x) + l(y)$ and (iii) $l(x) = \infty$ if and only if $x = 0$. 

2. The norm \( l \) is called **splittable** if there is a \( D \)-basis \( \{ e_i : i \in I \} \) of \( V \) such that 
\[
l(\sum_{i \in I} e_i d_i) = \inf_{i \in I} (l(e_i) + \nu(d_i)) \]
Let \( \mathcal{SN}(V) \) denote the splittable norms on \( V \). In this paper, all norms refer to splittable \( D \)-norms.

There is a natural bijection between \( \mathcal{SN}(V) \) and \( \text{Latt}_V \) given by \( l \mapsto (L_r = l^{-1}([r, +\infty))) \).
Then \( \text{Jump}(L_r) \) is the image of \( l \).

The following theorem is well known and follows directly from the definition of Moy-Prasad filtration [16].

**Theorem 4.1.3.** The (extended) building \( B(\text{GL}(V)) \) could be identified with \( \text{Latt}_V \) as \( \text{GL}(V) \)-sets. This identification is unique up to translation (c.f. [4, Theorem 2.11]). Suppose \( x \in B(\text{GL}(V)) \) corresponds to the lattice function \( \mathcal{L} \in \text{Latt}_V \). Then

(a) \( \text{gl}(V)_{x,r} = \text{gl}(V)_{\mathcal{L},r} \) for \( r \in \mathbb{R} \),
(b) \( \text{GL}(V)_{x,r} = \text{GL}(V)_{\mathcal{L},r} \) for \( r > 0 \) and
(c) \( \text{GL}(V)_{\mathcal{L}} = \text{GL}(V)_{x} \).

For the rest of this paper, we will freely interchange the notion of points in a building, \( D \)-norms and lattice functions.

4.2. **Tensor products.** Suppose \( l \) and \( l' \) are two norms on \( D \)-modules \( V \) and \( V' \). Then there is an induced norm on \( W := V \otimes_D V' \) such that \( (l \otimes l')(v \otimes v') = l(v) + l'(v') \) (c.f. [3, § 1.11]). Let \( \mathcal{L} \) and \( \mathcal{L}' \) be the corresponding lattice functions. We denote by \( \mathcal{L} \otimes \mathcal{L}' \) the corresponding \( u_k \)-lattice function on \( V \otimes_D V' \) where

\[
(\mathcal{L} \otimes \mathcal{L}')_l = \sum_{r + r' = l} \mathcal{L}_r \otimes_{\mathcal{O}_D} \mathcal{L}'_{r'}.
\]

It is easy to see that

\[
\text{Jump}(\mathcal{L} \otimes \mathcal{L}') = \text{Jump}(\mathcal{L}) + \text{Jump}(\mathcal{L}').
\]

The norms \( l \) and \( l' \) also induce a natural norm \( \text{Hom}(l, l') \) on \( \text{Hom}_D(V, V') \) whose corresponding lattice function is

\[
(\text{Hom}(\mathcal{L}, \mathcal{L}'))_r := \{ w \in \text{Hom}_D(V, V') \mid w(\mathcal{L}_s) \subseteq \mathcal{L}_{s+r}, \forall s \in \mathbb{R} \}.
\]

In particular, every norm \( l \) on \( V \) defines a dual norm \( l^* := \text{Hom}(l, \nu) \) on \( V^* := \text{Hom}_D(V, D) \).

Under the isomorphism \( \text{Hom}_D(V, V') \simeq V^* \otimes_D V'^* \), the norms \( \text{Hom}(l, l') \) and \( l' \otimes l^* \) coincide. If \( V = V' \), then the Moy-Prasad lattice function \( r \mapsto \text{gl}(V)_r \), defined in [11, 11] is the tensor product lattice function \( \mathcal{L} \otimes \mathcal{L}^* \) on \( \text{gl}(V) = \text{End}_D(V) \).

4.3. **Self-dual lattice functions.** Let \( V \) be a space with a non-degenerate sesquilinear form \( \langle , \rangle_V \).

**Definition 4.3.1.** 1. For a lattice \( L \) in \( V \), we set

\[
L^\perp = \{ v \in V \mid \langle v, v' \rangle \in \mathcal{P}_D, \forall v' \in L \}.
\]

A lattice \( L \) is called **self-dual** when \( L = L^\perp \).

2. For a lattice function \( \mathcal{L} \) we define its dual lattice function \( \mathcal{L}^\perp \) by \( (\mathcal{L}^\perp)_s = (\mathcal{L}_{-s})^\perp \).

If \( l \) is the norm corresponding to \( \mathcal{L} \), then we denote the norm corresponding to \( \mathcal{L}^\perp \) by \( l^\perp \).

If we identify \( V \) with \( V^* \) using the form \( \langle , \rangle_v \), then the norm \( l^* \) on \( V^* \) translates to the norm \( l' \) on \( V \).

3. A lattice function \( \mathcal{L} \) is called **self-dual** if and only if \( \mathcal{L} = \mathcal{L}^\perp \). In terms of norm, it is equivalent to \( l^\perp = l \) (c.f. [3, Prop. 3.3]) and we say that \( l \) is self-dual. Let \( \text{Latt}^\perp_V \) be the set of self-dual lattice functions. Clearly \( \text{Latt}^\perp_V \) is the \( \sharp \)-fixed point set of \( \text{Latt}_V \).

4. When \( \mathcal{L} \) is self-dual, we define \( \mathcal{g}_{\mathcal{L},r} = \mathcal{g} \cap \text{gl}(V)_{\mathcal{L},r} \), \( G_{\mathcal{L},r} = G \cap \text{GL}(V)_{\mathcal{L},r} \), \( G_{\mathcal{L}} = G \cap \text{GL}(V)_{\mathcal{L}} \), \( G_{\mathcal{L}} = G_{\mathcal{L}}/G_{\mathcal{L},0}^\perp \) and \( \mathcal{g}_{\mathcal{L},r,s} = \mathcal{g}_{\mathcal{L},r}/\mathcal{g}_{\mathcal{L},s} \).
If we identify $V^* \otimes_D V''$ as $(V \otimes_D V')^*$, then by a calculation on a splitting basis, we have $(l \otimes l')^* = l^* \otimes l'^*$ (c.f. [4] (18),(21), Sect. 1.12). In particular, suppose that $V$ and $V'$ are formed spaces, and $\mathcal{L}$ and $\mathcal{L}'$ are self-dual lattice functions. It is easy to see that $(l \otimes l')^2 = l \otimes l'$, i.e. it is self-dual. Hence $\mathcal{L} \otimes \mathcal{L}'$ is a self-dual lattice on $V \otimes_D V'$. We will use this fact freely from now on.

4.4. We recall that $k$ is a $p$-adic field with $p \neq 2$. For a classical group $G$ defined over $k$, $\mathcal{B}(G, k)$ could be identified canonically with the set of (splittable) self-dual norms on $V$ (c.f. [5, 7]). The following theorem is the cumulation of [5, 3, 12] and [7]. The following theorem is the cumulation of [5, 3, 12] and [7].

**Theorem 4.4.1.**  (i) There is a unique $G$-equivariant bijection between $\mathcal{B}(G, k)$ and $\text{Latt}_V^D$.

(ii) Suppose $x \in \mathcal{B}(G, k)$ corresponds to $\mathcal{L} \in \text{Latt}_V^D$. Then

(a) $g_{\mathcal{L}, r} = g_{x, r}$ for $r \in \mathbb{R}$,

(b) $G_{\mathcal{L}, r} = G_{x, r}$ for $r > 0$ and

(c) $G_{\mathcal{L}, x} = G_x$.

4.5. Let $r \in \text{Jump}(\mathcal{L})$ so that $L_r := \mathcal{L}_{x, r}^+ = \mathcal{L}_r / \mathcal{L}_{x, r}$ which is nonzero. The sesquilinear form $\langle \cdot, \cdot \rangle_V$ induces a nonzero pairing $L_r \times L_{x, r}^+ \to \mathfrak{o}_D$ and a non-degenerate pairing over $\mathfrak{f}_D$:

$$L_r \times L_{x, r} \to \mathfrak{f}_D.$$ 

In particular we have

$$\text{Jump}(\mathcal{L}) = -\text{Jump}(\mathcal{L}).$$

The structure of $g_{\mathcal{L}, x}$ is described in the following lemma. It is well known so we omit its proof. Also see Appendix $\mathcal{A}$.

**Lemma 4.5.1.** Let $v_D = \nu(\mathcal{w}_D)$. Then

$$G_{\mathcal{L}, x} \cong G_0 \times G_{\mathcal{w}_D} \times \prod_{r \in \text{Jump}(\mathcal{L}) \cap (0, \frac{1}{2} v_D)} \text{GL}(L_r),$$

where $G_0 = U(L_0)$ and $G_{\mathcal{w}_D} \cong U(L_{\mathcal{w}_D})$ where $L_{\mathcal{w}_D}$ is equipped with the form $\langle [v_1], [v_2] \rangle = \langle v_1, v_2 \mathcal{w}_D^{-1} \rangle_V \pmod{\mathfrak{p}_D}$.

5. **Tame base changes and epipelagic points**

5.1. Let $D$ be the division $k$-algebra as in Section 3.1. Let $G = U(V, \langle \cdot, \cdot \rangle_V)$ be a classical group defined over a $D$-module $V$ with an $\epsilon$-Hermitian sesquilinear form $\langle \cdot, \cdot \rangle_V$. Suppose $E$ is a tamely ramified finite extension of $k$ or $K$ such that $G$ splits. In this section we study the relations between buildings under tamely ramified field extensions.

In all cases, it is standard to construct a $E$-vector space $\tilde{V}$ obtained by certain base change of $V$ so that $G(k) \subseteq G(E)$ are subgroups of $\text{GL}_E(\tilde{V})$. By [7] there is a canonical bijection $\mathcal{B}(G, k) \to \mathcal{B}(G, E)^{\text{Gal}(E/k)}$. The next proposition describes this bijection in terms of split norm lattices on $V$ and $\tilde{V}$.

**Proposition 5.1.1.**  (i) Suppose $D = k$. Let $\tilde{V} = V \otimes_k E$ and let $i_V : V \to \tilde{V}$ be given by $v \mapsto v \otimes 1$. Let $\langle \cdot, \cdot \rangle_{\tilde{V}}$ be the $E$-linear extension of $\langle \cdot, \cdot \rangle_V$ and let $G(E) = U(\tilde{V}, \langle \cdot, \cdot \rangle_{\tilde{V}})$.

The bijection $\mathcal{B}(G, k) \to \mathcal{B}(G, E)^{\text{Gal}(E/k)}$ is given by $l \mapsto l_V \otimes_k l_E$ and its inverse map is $l_{\tilde{V}} \mapsto l_V \otimes i_V$.

(ii) Suppose $D$ is a quadratic extension of $k$. We fix a field embedding $i \in \text{Hom}_k(D, E)$ and view $D$ as an subfield of $E$. Let $\tilde{V} = V \otimes_D E$ and let $i_V : V \to \tilde{V}$ be given by $v \mapsto v \otimes 1$. Then $G(E) = \text{GL}_E(\tilde{V})$. The bijection $\mathcal{B}(G, k) \to \mathcal{B}(G, E)^{\text{Gal}(E/k)}$ is given by $l_V \mapsto l_V \otimes_D l_E$ and its inverse map is $l_{\tilde{V}} \mapsto l_V \otimes i_V$. 


(iii) Suppose $D$ is the quaternion division $k$-algebra. We fix a degree two subfield $L$ of $E$, identify $L$ with a subfield of $D$ and fix a $d \in D$ such that $d^2 \in k^\times$, $d^* = -d$ and $\text{Ad}(d)$ acts on $L$ by the non-trivial Galois action. Let $\text{pr} : D \to L$ be the projection of $D = L \oplus Ld$. Then $Q(v_1, v_2) := \text{pr}((v_1, d, v_2)E)$ defines an $L$-bilinear from on $V$. Let $\tilde{V} = V \otimes_L E$ and let $\langle , \rangle_{\tilde{V}}$ be the $(\epsilon)$-symmetric $E$-linear extension of $Q$. The bijection $\mathcal{B}(G, k) \cong \mathcal{B}(G, E)^{\text{Gal}(E/k)}$ is given by $l_V \mapsto l_{\tilde{V}} \otimes l_E$ and its inverse map is $l_{\tilde{V}} \mapsto l_V \circ i_{\tilde{V}}$.

Before we give the proof of Proposition 5.1.1, we first recall the uniqueness observation stated in [22] §2.1] and in [4] §1.2.

Lemma 5.1.2. Let $\mathcal{B}$ and $\mathcal{B}'$ be two sets satisfy the axioms of building of $G$ over $k$. Let $j : \mathcal{B} \to \mathcal{B}'$ be a bijection such that

(i) $j$ is $G$-equivariant, i.e. $j(g \cdot x) = gj(x)$ for all $g \in G$, and

(ii) its restriction to an apartment $\mathcal{A}$ is affine.

Then $j$ is unique up to the translation by an element in $X_*(Z(G)^\circ) \otimes R$.

In our cases, $Z(G)$ is anisotropic so the map $j$ is unique.

Proof of Proposition 5.1.1. In the proof below we identify buildings with the corresponding set of norms.

(i) In this case $G(E) = U(\tilde{V}, \langle , \rangle_{\tilde{V}})$ and $G(k) = G(E)^{\text{Gal}(E/k)}$. We consider the following diagram.

$$
\mathcal{B}(G, k) \xrightarrow{=} \mathcal{B}(G, E)^{\text{Gal}(E/k)}
$$

The top row is viewed as the fixed point sets of the involution $\tilde{\tau}$ on the bottom row.

For any $l_V \in \mathcal{B}(GL(V))$, we set $l_{\tilde{V}} = l_V \otimes v_E$. By direct computation or applying the main theorem of [4], one sees that $l_V \mapsto l_{\tilde{V}}$ is the unique $GL(V)$-equivariant map for the bottom row which maps self-dual norms to self-dual norms.

By Lemma 5.1.2, we conclude that $l_V \mapsto l_{\tilde{V}}$ realizes the unique bijection $\mathcal{B}(G, k) \cong \mathcal{B}(G, E)^{\text{Gal}(E/k)}$. This proves (i).

(ii) We refer to the computation in [4] §1.13. Let $\nu_2$ be the embedding of $D$ different from $\nu$. Let $V_2 = V \otimes_{D,\nu_2} E$. Then $\text{Res}_{D/k}(GL_D(V))(E) \cong GL_E(\tilde{V}) \times GL_E(\tilde{V}_2)$ Part (ii) follows by applying a similar argument as in (i) to the following diagram.

$$
\mathcal{B}(G, k) \xrightarrow{=} \mathcal{B}(\text{GL}_D(V))^\text{Gal}(E/k)
$$

(iii) Suppose $D$ is a quaternion division $k$-algebra. Note that $H(v_1, v_2) = \text{pr}((v_1, v_2))$ defines a Hermitian form on $V$. Moreover, $G(k) = U(V, Q) \cap U(V, H)$. Part (iii) follows by applying a similar argument as in (ii) to the following diagram.

$$
\mathcal{B}(G, k) \xrightarrow{=} \mathcal{B}(U(V, H))^\text{Gal}(E/k)
$$
For an $\mathfrak{o}_D$-lattice function $L$ corresponding to $x \in B(G, k)$, we will denote by $L^E$ the $\mathfrak{o}_E$-lattice function in $V$ corresponding to $x \in B(G, E)^{\text{Gal}(E/k)}$ in the above proposition. We need the following application of Lemma 5.1.2 in our study of the epipelagic points.

**Lemma 5.1.3.** Let $L$ be the self-dual lattice function corresponding to a point $x$ in $B(G, k)$ of order $m$. Suppose $G$ splits under a tamely ramified extension $E$ with ramification index $m$. Then $\text{Jump}(L)$ is contained in either $\frac{1}{m}Z$ or $\frac{1}{2m} + \frac{1}{m}Z$.

**Proof.** We have $\nu(E) = \frac{1}{m}Z$. Let $L^E$ be the $\mathfrak{o}_E$-lattice function. Let $J = \text{Jump}(L)$ and $J^E = \text{Jump}(L^E)$. By [20] §4.2, $L^E$ corresponds to a hyperspecial point in $G(E)$. Hence $J^E = J + 1 \frac{1}{m}Z$ for some $j_0 \in [0, \frac{1}{m}]$.

By the Proposition 5.1.1, we have $J \subseteq J^E = J + \frac{1}{m}Z$. Since $L$ is self-dual, $J = -J$. Hence $-j_0 + \frac{1}{m}Z = \frac{1}{m}Z \subseteq J + \frac{1}{m}Z \subseteq J^E + \frac{1}{m}Z = j_0 + \frac{1}{m}Z$. Therefore $j_0 = 0$ or $\frac{1}{2m}$. The lemma follows. \hfill \Box

### 5.2. Epipelagic points. 

We recall that $k^{ur}$ is the maximal unramified extension of $k$. Let $E$ be the totally ramified extension of $k^{ur}$ of degree $m$. We fix a uniformizer $\varpi_E$ such that $\varpi_E^m = 1$. Let $\text{Gal}(E/k^{ur}) = \langle \sigma \rangle$ where $\sigma(\varpi_E) = \varpi_E$ and $\varpi_E$ is a primitive $m$-th root of unity. Let $F_r \in \text{Gal}(E/k)$ denote the lift the Frobenius automorphism in $\text{Gal}(k^{ur}/k)$ such that $F_r(\varpi_E) = \varpi_E$. Now $\text{Gal}(E/k) = \langle F_r, \sigma \rangle$.

The group $G$ splits over $E$ [20] §4.1. We recall that $T$ is a maximally $k^{ur}$-split torus in $G$ containing $S$ and defined over $k$. Let $Y = Z_G(T)$ be a Cartan subgroup of $G$. Then $x$ is a hyperspecial point in $A(Y, E)$. We have $G(E)^{x}_r = G(k^{ur})_x$ where $r > 0$, and $\mathfrak{g}^d(E)_{x, r} = 0$. The building $B(G, k^{ur})$ embeds into $B(G, E)$ as the $\sigma$-invariant set and $B(G, k) = B(G, k^{ur})^{Fr} = (B(G, E)^{\sigma})^{Fr}$.

We set $D(E) = D \otimes_k E$. We equip $D(E)$ with the tensor product norm of valuations of $D$ and $E$. Let $\mathcal{S}$ be the lattice functions defined by their valuations. Then $\mathcal{S} \otimes \mathcal{S}$ is the corresponding lattice function on $D(E)$. Let $\mathfrak{o}_{D(E)} = (\mathcal{S} \otimes \mathcal{S})_0$ which is a hereditary order, $\mathfrak{p}_{D(E)} = (\mathcal{S} \otimes \mathcal{S})_{0+}$ and $\mathfrak{f}_{D(E)} = \mathfrak{o}_{D(E)}/\mathfrak{p}_{D(E)}$ which is a semisimple algebra over $\mathfrak{f}_E$.

Let $L$ be the self-dual lattice function in $V$ corresponding to $x \in B(G, k)$. Let $L_E := L \otimes \mathcal{S}$ be the self-dual $\mathfrak{o}_k$-lattice function in $V(E) := V \otimes_k E$. In fact it is an $\mathfrak{o}_{D(E)}$-lattice function. We have following situations.

(i) If $D = k$, then $L_E = L^E$.

(ii) If $D$ is a quadratic extension of $k$, then $D(E) \cong E \times E, V(E) = \tilde{V}^{11} \oplus \tilde{V}^{12}, L_E = L^{\varphi_{11},E} \oplus L^{\varphi_{12},E}$ where $\tilde{V}^{11}$ and $\tilde{V}^{12}$, $L^{\varphi_{11},E}$ and $L^{\varphi_{12},E}$ corresponds to the two different $k$-embeddings of $D$ into $E$.

(iii) Suppose $D$ is a quaternion algebra. We fix a quadratic extension $L$ of $k$ in $E$. Then $D(E) \cong \text{Mat}_2(E), V(E) = \tilde{V}^{11} \oplus \tilde{V}^{12}$ and $L_E = L^{\varphi_{11},E} \oplus L^{\varphi_{12},E}$ where $L^{\varphi_{11},E}$ and $L^{\varphi_{12},E}$ are $\mathfrak{o}_E$-lattice functions corresponding to the two different $k$-embeddings of $L$ into $E$.

Clearly, $\text{Jump}(L_E) = \text{Jump}(L^E) = \text{Jump}(L) + \frac{1}{m}Z$. The Galois group $\text{Gal}(E/k)$ acts on $V(E)$ by $\sigma(v \otimes x) = v \otimes \sigma(x)$ for $\sigma \in \text{Gal}(E/k), v \in V$ and $x \in E$. For $g \in G(E)$ and $\sigma \in \text{Gal}(E/k)$, we have $\sigma(g) = \sigma \circ g \circ \sigma^{-1}$ as $D(E)$-linear automorphism on $V(E)$. In Cases (ii) and (iii), under the decomposition, $G(E)$ acts diagonally on $V(E) = \tilde{V}^{11} \oplus \tilde{V}^{12}$.

Extending the notation in Definition 4.1.1, we have $G(E)^x = G(E)^{\mathfrak{g}_E}, G(E)_{x, r} = G(E)_{x, r}$ and $g(E)_{x, r} = g(E)_{x, r}$ by Proposition 5.1.1.

Following [20] §4, we pick a hyperspecial point $x_0 \in \mathcal{A}(T, K)$ such that $\text{Jump}(\mathcal{L}_E^{x_0}) = \text{Jump}(\mathcal{L}_E)$ where $\mathcal{L}_E$ is the lattice function in $V(E)$ corresponding to $x_0$. Let $\bar{\eta} \in X^0$ such that $x = x_0 + \frac{1}{m} \bar{\eta}$. By the classification of hyperspecial points for classical groups, $\bar{\eta}$ is
in the lattice generated by coroots. Let \( t := \tilde{\eta}(\varpi_E) \in G(E) \). The \( \mathfrak{a}_D(E) \)-lattice function corresponding to \( x_0 \) is \( \mathcal{L}_E^0 = t^{-1} \mathcal{L}_E \).

As in [20, §4], there are isomorphisms

\[
G(E)_{x_0} \xrightarrow{\text{conj}(t)} G(E)_x \quad \text{and} \quad g(E)_{x_0,0} \xrightarrow{\text{Ad}(t)} g(E)_{x,0} \xrightarrow{\varpi^*_{\theta,\zeta}} g(E)_{x,\frac{1}{\theta}}.
\]

Let \( \hat{G} = G(E)_{x_0}/G(E)_{x_0,0}^+ \), \( \hat{g} = g(E)_{x_0,0}/g(E)_{x_0,0}^+ \), \( \hat{Y} = Y(E)_{x_0,0,0}^+ \) and \( \hat{y} = y(E)_{x_0,0,0}^+ \). Let \( \vartheta \) be the algebraic automorphisms (resp. Frobenius automorphisms) on \( \hat{G} \) and \( \hat{g} \) induced by the \( \sigma \) action on \( G(E)_{x_0,0} \) and \( g(E)_{x_0,0} \) respectively. Let \( \theta := \text{Ad}(t^{-1}) \circ \sigma \circ \text{Ad}(t) \) (resp. \( \tilde{\text{Fr}} := \text{Ad}(t^{-1}) \circ \tilde{\text{Fr}} \circ \text{Ad}(t) \)) be the automorphisms on \( \hat{G} \) and on \( \hat{g} \) induced by the \( \sigma \) (resp. \( \tilde{\text{Fr}} \)) action on \( G(E)_x \) and \( g(E)_{x,0} \). Let \( \hat{g}^{\theta,\zeta} \) be the \( \zeta^{-1}\)-eigenspace of \( \theta \) on \( \hat{g} \). Then

(a) \( \theta = \text{Ad}(\hat{t}) \vartheta \) where \( \hat{t} = t^{-1} t^\sigma = \tilde{\eta}(\zeta) \mod G(E)_{x_0,0}^+ \);
(b) \( \text{Ad}(t) : \hat{G}^0 \xrightarrow{\sim} \hat{G}(k_{ur})_x := G(k_{ur})_x/G(k_{ur})_{x,0}^+ \) is an isomorphism and
(c) \( \varpi^*_{\theta,\zeta} \text{Ad}(t) : \hat{g}^{\theta,\zeta} \xrightarrow{\sim} \hat{g}(k_{ur})_{x,\frac{1}{m}} = g(k_{ur})_{x,\frac{1}{m}} \) is \( \hat{G}^0 \)-equivariant with \( \hat{G}^0 \) acting on the right hand side via (b).

By putting \( j = -1 \) in (c), we define \( \iota_g := \varpi_{-1}^{-1} \text{Ad}(t) : \hat{g}^{\theta,\zeta} \to g(k_{ur})_{x,\frac{1}{m}} \). Then \( \iota_g \) is a bijection between the set of stable vectors for the \( \hat{G}^0 \) action on \( \hat{g}^{\theta,\zeta} \) and the set of stable vectors for the action \( G(k_{ur})_x \) on \( g(k_{ur})_{x,\frac{1}{m}} \). The former was studied by Vinberg [23] and Levy [13].

6. Moment maps

6.1. Let \( W = V \otimes_D V' \). Using the sesquilinear forms, we identify \( \Psi : W \rightarrow \text{Hom}_D(V, V') \) and \( \Psi' : W \rightarrow \text{Hom}_D(V', V) \) by

\[
\Psi(v \otimes v')(v_1) = v' \langle v, v_1 \rangle_V \quad \text{and} \quad \Psi'(v \otimes v')(v'_1) = v \langle v', v'_1 \rangle_{V'}
\]

for all \( v, v_1 \in V \) and \( v', v'_1 \in V' \). Now \( g \in G \) and \( g' \in G'(k) \) acts on \( \text{Hom}_D(V', V) \) by the formula \( (g, g') \cdot w = g'gw^{-1} \).

**Definition 6.1.1.** 1. We define a non-degenerate \( G \)-invariant symmetric \( k \)-bilinear form\(^1\) \( B_g : g \times g \to k \) by \( B_g(X_1, X_2) = \frac{1}{2} \text{tr}_{D/k} \langle X_1^2 X_2 \rangle \).
2. We define an operator \( \ast : \text{Hom}_D(V, V') \to \text{Hom}_D(V', V) \) by

\[
\langle w \cdot v, v' \rangle_{V'} = \langle v, w^* \cdot v' \rangle_V \quad \forall w \in \text{Hom}_D(V, V'), v \in V, v' \in V'.
\]

We note that if \( x \in W \) and \( \Psi(x) = w \), then \( \Psi'(x) = w^* \).
3. We define the moment map \( M : W \cong \text{Hom}_D(V, V') \to g \) and \( M' : W \to g' \) by

\[
M(w) = w^* w \quad \text{and} \quad M'(w) = ww^*.
\]

By definition \( M \) and \( M' \) are \( G \times G' \)-equivariant.

**Lemma 6.1.2.** We have

(a) \( \langle u_1, w_2 \rangle_W = \text{tr}_{D/k} \langle w_2^* w_1 \rangle \) and
(b) \( \langle X \cdot w, w \rangle_W = 2B_g(M(w), X) \) and \( \langle X' \cdot w, w \rangle_W = 2B_g(-M'(w), X') \).

The proof is a straightforward computation using [3] and the definition of \( \ast \). We will leave it to the reader.
6.2. **Gradings.** Let $\mathcal{L}$ and $\mathcal{L}'$ be two self-dual lattice functions on $V$ and $V'$ respectively. Let $\mathcal{B} = \mathcal{L} \otimes \mathcal{L}'$ on $W = V \otimes_D V'$.

**Lemma 6.2.1.** (i) We have $\text{Jump}(\mathcal{B}) = \text{Jump}(\mathcal{L}) + \text{Jump}(\mathcal{L}')$.
(ii) The lattice function $\mathcal{B}_r$ is self-dual in $W$, i.e. $\mathcal{B}_r^2 = \mathcal{B}_r$. 
(iii) Under the isomorphism $\Psi : W \rightarrow \text{Hom}_D(V,V')$,
$$\Psi(\mathcal{B}_r) = \left\{ w \in \text{Hom}_D(V,V') \mid w, \mathcal{L}_s \subseteq \mathcal{L}_{s+r} \forall s \in \mathbb{R} \right\}.$$  
(iv) We have $(\Psi(\mathcal{B}_r))^* = \Psi'(\mathcal{B}_r)$.
(v) We have $M(\mathcal{B}_r) \subseteq \mathfrak{g}_{\mathcal{L,2s}}$ and $M'(\mathcal{B}_r) \subseteq \mathfrak{g}'_{\mathcal{L',2s}}$.

**Proof.** Part (i) is [5]. Parts (ii) and (iii) are explained in Section 4.3 and Section 4.2 respectively. If $x \in W$ and $\Psi(x) = w$, then $\Psi'(x) = w^*$. This proves (iv). Part (v) follows easily from (iii) and (iv).

6.3. We could view $\mathcal{B}_s$, $\mathfrak{g}_{\mathcal{L,2s}}$ and $\mathfrak{g}'_{\mathcal{L',2s}}$ as schemes over $\mathfrak{o}_k$. Since $\ast$ is $\mathfrak{o}_k$-linear, the moment maps defined over the generic fiber as in Section 6.1 extends to morphisms between these $\mathfrak{o}_k$-schemes. The $\mathfrak{o}_k$-group scheme $G_{\mathcal{L}} \times G'_{\mathcal{L'}}$ acts on all these objects and the moment maps are equivariant maps.

Let $W_s = \mathcal{B}_s / \mathcal{B}_s^+$, $\mathfrak{g}_{\mathcal{L,2s}} = \mathfrak{g}_{\mathcal{L,2s}} / \mathfrak{g}_{\mathcal{L,2s}}^+$ and $\mathfrak{g}'_{\mathcal{L',2s}} = \mathfrak{g}'_{\mathcal{L',2s}} / \mathfrak{g}'_{\mathcal{L',2s}}^+$. We get morphisms, as certain quotients of the moment maps over the special fiber,
$$M : W_s \rightarrow \mathfrak{g}_{\mathcal{L,2s}} \quad \text{and} \quad M' : W_s \rightarrow \mathfrak{g}'_{\mathcal{L',2s}}.$$  
The action of $G_{\mathcal{L}} \times G'_{\mathcal{L'}}$ reduces to $G_{\mathcal{L}} \times G'_{\mathcal{L'}}$, actions on $W_s$, $\mathfrak{g}_{\mathcal{L,2s}}$ and $\mathfrak{g}'_{\mathcal{L',2s}}$. These are the moment maps over the residual field $\mathfrak{f}$ which we will study later.

7. **Theta correspondences I**

7.1. In this section we let $(G, G')$ be a reductive dual pair in $\text{Sp}(W)$ as in Section 3.2.

Let $\Sigma = (x, \lambda, \chi)$ and $\Sigma' = (x', \lambda', \chi')$ be epipelagic supercuspidal data for $G$ and $G'$ respectively. Let $\pi_\Sigma = \pi_{\Sigma'}^G(\chi, \lambda')$ (resp. $\pi'_\Sigma = \pi_{\Sigma'}^G(x', \chi')$) be an epipelagic representation of $\tilde{G}$ (resp. $\tilde{G}'$). From now on, we assume $\theta(\pi) = \pi'$. As discussed in Section 1.4, $x, x'$ are both epipelagic points of order $m$ for some $m \geq 2$.

Let $\mathcal{L}$ and $\mathcal{L}'$ be self dual lattice functions corresponding to $x$ and $x'$ respectively. Let $\mathcal{B} = \mathcal{L} \otimes \mathcal{L}'$. Let $X = W_m \otimes \mathcal{B} \otimes \mathcal{B}'$, $\mathfrak{m}, \mathfrak{m}'$ denote the moment maps defined in $[13]$ by $M : X \rightarrow \frac{1}{m} \mathfrak{g}_{\mathcal{L,2s}}$ and $M' : X \rightarrow \frac{1}{m} \mathfrak{g}'_{\mathcal{L',2s}}$.

**Proposition 7.1.1.** Suppose that $\theta(\pi_\Sigma) = \pi'_\Sigma$. Then there exists a $\vec{w} \in X$ such that $\lambda = M(\vec{w})$ and $\lambda' = -M'(\vec{w})$.

**Proof.** By [19] there exists $(y, y') \in \mathcal{B}(G,k) \times \mathcal{B}(G', k)$, $\mathcal{B} = \mathcal{L}_y \otimes \mathcal{L}'_{y'}$ and $w \in \mathcal{B}' \otimes \mathcal{B}'$ such that $M(w) + \mathfrak{g}_{\mathcal{L,2s}} - \frac{1}{m}$ is an unrefined minimal $K$-type of $\pi$, and $-M'(w) + \mathfrak{g}'_{\mathcal{L',2s}} - \frac{1}{m}$ is an unrefined minimal $K'$-type of $\pi'$. The moment maps $M$ and $M'$ commute with the action of $G \times G'$-conjugation. By Proposition 2.3.3(i) and conjugating by $G \times G'$, we may assume that $y = x$, $y' = x'$, $w \in \mathcal{B}' \otimes \mathcal{B}'$, $M(w) + \mathfrak{g}_{\mathcal{L,2s}} - \frac{1}{m} = M(\vec{w}) = \lambda$ and $-M'(w) + \mathfrak{g}'_{\mathcal{L',2s}} - \frac{1}{m} = -M'(\vec{w}) = \lambda'$ where $\vec{w} = w + \mathcal{B}' \otimes \mathcal{B}' \subseteq X$.

**Corollary 7.1.2.** We have $\text{Jump}(\mathcal{B}) \subseteq \frac{1}{2m} + \frac{1}{m} \mathbb{Z}$. Moreover either
(i) $\text{Jump}(\mathcal{L}) \subseteq \frac{1}{2m} \mathbb{Z}$ and $\text{Jump}(\mathcal{L}') \subseteq \frac{1}{2m} + \frac{1}{m} \mathbb{Z}$ or
(ii) $\text{Jump}(\mathcal{L}) \subseteq \frac{1}{2m} + \frac{1}{m} \mathbb{Z}$ and $\text{Jump}(\mathcal{L}') \subseteq \frac{1}{m} \mathbb{Z}$.
Proof. By Lemma 5.1.3, Jump(L) (resp. Jump(L')) is a subset of \( \frac{1}{m} \mathbb{Z} \) or \( \frac{1}{2m} + \frac{1}{m} \mathbb{Z} \). From the proof of the last proposition, \( \hat{w} \) is a nonzero element in \( \mathcal{B} \setminus \frac{1}{m} \mathbb{Z} \). In particular \( \frac{1}{2m} \notin \text{Jump}(\mathcal{B}) = \text{Jump}(L) + \text{Jump}(L') \). The corollary follows. \( \square \)

7.2. In [20] §4, Reeder and Yu connect the Moy-Prasad filtration at an epipelagic point with the Kac-Vinberg grading of Lie algebras over the residue fields. Now we relate this with the moment maps.

Let \( L_0 = t^{-1} L_0 \) and \( L'_0 = t^{-1} L'_0 \) denote the \( \mathfrak{g}_P(E) \)-lattice functions corresponding to \( x_0 \) and \( x'_0 \) respectively as in Section 5.2. By 7.1.3 we are in one of the following two cases.

(a) We have Jump(\( L_0^0 \)) = \( \frac{1}{m} \mathbb{Z} \) and Jump(\( L'_0^0 \)) = \( \frac{1}{2m} + \frac{1}{m} \mathbb{Z} \). In this case we set \( V = L_0^0 \) and \( V' = L'_0^0, \frac{1}{2m} + \frac{1}{m} \mathbb{Z} \).

(b) We have Jump(\( L_0^0 \)) = \( \frac{1}{2m} + \frac{1}{m} \mathbb{Z} \) and Jump(\( L'_0^0 \)) = \( \frac{1}{m} \mathbb{Z} \). In this case we set \( V = L_0^0, \frac{1}{2m} + \frac{1}{m} \mathbb{Z} \) and \( V' = L'_0^0, \frac{1}{m} \mathbb{Z} \).

Both \( V \) and \( V' \) are \( f_{D(E)} \)-modules. In Case (a), we assign a non-degenerate \( f_{D(E)} \)-bilinear forms \( \langle \cdot, \cdot \rangle \) (mod \( \mathfrak{p}_{D(E)} \)) and a non-degenerate \( f_{D(E)} \)-bilinear forms on \( \langle \cdot, \cdot \rangle = \omega \langle \cdot, \cdot \rangle \) (mod \( \mathfrak{p}_{D(E)} \)). In the Case (b), the bilinear forms are defined similarly. Thus \( G = U(V) \) (resp. \( G = U(V') \)) as \( f_{D(E)} \)-linear transformations on \( V \) (resp. \( V' \)) preserving the form.

Let \( L^0(E)_r = L_0^0, r \times r \). The actions \( \sigma \) and \( t^{-1} \circ \sigma \circ t \) on \( L_0^0 \) induce actions on \( L^0(E)_r \), which we denote by \( \vartheta \) and \( \theta \) respectively. It is compatible with the \( \vartheta \) and \( \theta \) actions on \( G \) in the sense that for \( g \in G \) we have \( \vartheta(g) = \vartheta \circ \sigma \circ \vartheta^{-1} \) and \( \theta(g) = \theta \circ \sigma \circ \theta^{-1} \) as linear transformations on \( L_0^0 \).

Let \( \mathcal{B}(E) = \mathcal{B} \otimes_{\mathcal{O}_S} E \) and \( \mathcal{B}^0(E) = (t^{-1}, t^{-1}) \mathcal{B}(E) = L_0^0 \otimes_{\mathcal{O}_{D(E)}} L_0^0 \). Let \( W_{m} = \mathcal{B}^0(E), \frac{1}{m} + \mathcal{B}^0(E), \frac{1}{m} \cong W := V \otimes_{f_{D(E)}} V' \cong \text{Hom}_{f_{D(E)}}(V, V') \). We observe the following diagram:

\[
\begin{array}{cccc}
W & \overset{\sim}{\rightarrow} & W_{m} & \overset{\sim}{\rightarrow} & \mathcal{B}^0(E), \frac{1}{2m} & \overset{(t,t')}{\rightarrow} & \mathcal{B}(E), \frac{1}{2m} \\
\downarrow M & & \downarrow M & & \downarrow M & & \downarrow M \\
g & \overset{\omega}{\rightarrow} & g^0, \frac{1}{m} & \overset{\omega}{\rightarrow} & g(E), x_0, \frac{1}{m} & \overset{\text{Ad}(t)}{\rightarrow} & g(E), x, \frac{1}{m} \\
\end{array}
\]

In the above diagram, the \( G(E)_x \times G'(E)_x \)-equivariant map \( M \) on the far right translates to the \( G \times G \)-equivariant map \( M \) on the far left. The explicit formula for the map \( M \) is given by exactly the same as that for \( k \) with respect to the forms \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \).

We remark that \( W_{m} \) is not equipped with any sesquilinear form. On the other hand \( W \) has isomorphic vector space structure as \( W_{m} \) but equipped with a tensor product form.

7.3. The ranks of \( G \) and \( G' \). Let \( r \) be the rank of \( g \) and let \( P = P(X) \) be the coefficient of \( z^r \) in \( \det(z + \text{ad}X) \). Then \( P \) is an \( \text{Ad}(G) \)-invariant homogenous rational function on \( g \) defined over \( k \) such that the set of regular semisimple elements in \( g \) is \( P^{-1}(\mathbb{A}^1 \setminus \{ 0 \}) \).

The following lemma is a consequence of [8, Lemma 13].

**Lemma 7.3.1.** Let \( \lambda \) be a stable vector in \( g_{x, -\frac{1}{m}}, -\frac{1}{m} \) and \( \gamma \in g_{x, -\frac{1}{m}} \) be a lifting of \( \lambda \). Then \( \gamma \) is a regular semisimple element. Let \( T = \mathbb{Z}_G(\gamma) \). Then \( \gamma \) is a good element of depth \( -\frac{1}{2m} \) with respect to \( T \), i.e. for every root \( \alpha \) of \( G(K) \) with respect to \( T(K) \), \( \alpha(\gamma) \) is nonzero and \( \nu(\alpha(\gamma)) = -\frac{1}{m} \).
Proof. Let \( f(X) = \varphi_X \text{Ad}(t)(X) \) and let \( \overrightarrow{f} \) be the induced map in the following diagram,

\[
\begin{array}{cccc}
\gamma & \in & g_{\frac{e}{m}, \frac{1}{m}} & \subset g(E)_{\frac{e}{m}, \frac{1}{m}} \xrightarrow{f} g(E)_{x_0, 0} \\
\lambda & \in & g_{\frac{e}{m}, \frac{1}{m}} & \subset g(E)_{\frac{e}{m}, \frac{1}{m}} \xrightarrow{\gamma} g(E)_{x_0} = \hat{g}.
\end{array}
\]

Since \( \lambda \) is a stable vector, \( \overrightarrow{f}(\lambda) \) is a regular semisimple element in \( \hat{g} \) and \( [P(f(\gamma))] \neq 0 \in \mathfrak{f}_E \). Therefore, \( P(\gamma) \neq 0 \) and \( \gamma \) is a regular semisimple element in \( g \).

Next we prove that \( \gamma \) is good. Let \( R(G(\overline{k}), T(\overline{k})) \) be the set of roots. Let \( T'(\overline{k}) = \text{Ad}(t)T(\overline{k}) \). Then \( \alpha \mapsto \alpha' := \alpha \circ \text{Ad}(t^{-1}) \) gives a map \( R(G(\overline{k}), T(\overline{k})) \rightarrow R(G(\overline{k}), T'(\overline{k})) \).

Moreover \( \alpha' \) reduces to a root \( \overrightarrow{\alpha'} \in \hat{g} \). For any \( \alpha \in R(G(\overline{k}), T(\overline{k})) \), \( d\alpha'([f(\gamma)]) \neq 0 \) since \( [f(\gamma)] = \overrightarrow{f}(\lambda) \) is regular semisimple in \( \hat{g} \). This implies that \( \nu(d\alpha(\gamma)) = -\frac{1}{m} \) which proves the lemma.

\[\square\]

**Proposition 7.3.2.** Suppose \( \theta(\pi_X) = \pi_{Y'} \). Then \( (G, G') \) or \( (G', G) \) is one of the following types: (i) \( (D_n, C_n) \), (ii) \( (C_n, D_{n+1}) \), (iii) \( (C_n, B_n) \), (iv) \( (A_n, A_n) \), (v) \( (A_n, A_{n+1}) \).

**Proof.** By Proposition 7.1.1 there exist \( w \in \mathcal{B}_{\frac{e}{m}} \subset W \) such that \( [M(w)] = M(\overline{w}) \in g_{\frac{e}{m}, \frac{1}{m}} \) and \( [M'(w)] = M'(\overline{w}) \in g'_{\frac{e}{m}, \frac{1}{m}} \) are stable vectors. By Lemma 7.3.1 both \( M(w) \) and \( M'(w) \) are regular semisimple elements.

Now we show that (i) to (iv) list all possible cases which satisfy the following condition.

\[(8) \text{ There is a } w \in W \text{ such that } M(w) \text{ and } M'(w) \text{ are both regular semisimple.} \]

We may base change to the algebraic closure \( \overline{k} \) so that \( G \) and \( G' \) are split groups. We fix a maximal (split) torus \( Y \) and identify its Lie algebra \( \mathfrak{a} \) with \( \mathfrak{h}^{\text{dim}g} \) where \( \mathfrak{a} \) is the affine line. We list \( P|_{\mathfrak{a}} = \prod_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{y})} d\alpha \) explicitly when \( G = \text{one the following groups.} \)

- **GL(\( n \))**: \( \mathfrak{h} = \mathbb{A}^n \), \( P(a_1, \cdots, a_n) = \prod_{i \neq j} (a_i - a_j) \).
- **Sp(\( 2n \))**: \( \mathfrak{h} = \mathbb{A}^n \), \( P(a_1, \cdots, a_n) = \prod_{i \neq j} (a_i^2 - a_j^2) \prod_j (-4a_i^2) \).
- **O(\( 2n \))**: \( \mathfrak{h} = \mathbb{A}^n \), \( P(a_1, \cdots, a_n) = \prod_{i \neq j} (a_i^2 - a_j^2) \prod_j (-a_i^2) \).
- **O(\( 2n + 1 \))**: \( \mathfrak{h} = \mathbb{A}^n \), \( P(a_1, \cdots, a_n) = \prod_{i \neq j} (a_i^2 - a_j^2) \prod_j (-a_j^2) \).

By the classification of dual pairs, we only have to consider one of the following reductive dual pairs:

(a) \( (O(2n), \text{Sp}(2n')) \),
(b) \( (\text{Sp}(2n), O(2n' + 1)) \),
(c) \( (\text{GL}(n), \text{GL}(n')) \).

We may assume that \( \text{rank } G(\overline{k}) \leq \text{rank } G'(\overline{k}) \). Let \( \mathfrak{w} \) and \( \mathfrak{w}' \) be the Weyl groups of \( G \) and \( G' \) with respect to \( Y \) and \( Y' \) respectively. By classical invariant theory, we get following diagram:

\[
\begin{array}{cccc}
\mathfrak{h}' & \xrightarrow{\mathfrak{g}} & W & \xrightarrow{M'} \mathfrak{g}' \\
\mathfrak{h}/\mathfrak{w} & \xrightarrow{\cong} & \mathfrak{g}/\mathfrak{G} & \xrightarrow{\cong} \mathfrak{w}/\mathfrak{G}'.
\end{array}
\]

We remind the readers that in the lemma below all the vector spaces and algebraic groups are defined over \( \overline{k} \).

**Lemma 7.3.3.** Suppose \( \text{rank } G(\overline{k}) \leq \text{rank } G'(\overline{k}) \). Then there is a vector subspace \( \mathcal{A} \) of \( W \) which is stable under the action of \( Y \times Y' \) and such that the following diagram below...
Lemma 8.1.1. Dimensional characters.

By (C1) and the classification of epipelagic points in [8, 20], this is exactly in the following situation:

(C2) Let \( \bar{w} \in W \times G \times G' \) be an inclusion map \( \Upsilon \). Using the bottom row in (9), we have an inclusion \( \Upsilon / \bar{w} \hookrightarrow \Upsilon / \bar{w}' \) induced by \( \Upsilon_\eta \). Let \( P \) and \( P' \) be the invariant polynomials for \( G \) and \( G' \) defined before Lemma 7.3.1. Then (9) is equivalent to the following statement:

There is an \( X \in \eta \) such that \( P|_\eta(X) \neq 0 \) and \( P'|_\eta(\Upsilon_\eta(X)) \neq 0 \).

It follows by inspection that (i) to (v) are all the possible cases. \( \Box \)

8. Theta correspondences II

Suppose \( \theta(\pi_V) = \pi_{V'} \). By Proposition 7.3.2, Proposition 7.1.1 and 7.1.2, the following statements hold:

(C1) Let \((G, G')\) be a dual pair of one the following types: (i) \((D_n, C_n)\), (ii) \((C_n, D_{n+1})\), (iii) \((C_n, B_n)\), (iv) \((A_n, A_n)\), (v) \((A_n, A_{n+1})\).

(C2) Let \( x \in \mathcal{B}(G, k) \) and \( x' \in \mathcal{B}(G', k) \) be epipelagic points of order \( m \). In particular, \( m \geq 2 \). Let \( \mathcal{L} \) and \( \mathcal{L}' \) denote the corresponding \( \mathcal{O}_D \)-lattice functions.

(C3) We set \( \mathcal{B} = \mathcal{L} \otimes \mathcal{L}' \). Then Jump\((\mathcal{B}) \subseteq 1 + \frac{1}{m} \mathbb{Z} \).

(C4) There exists a \( \bar{w} \in X = \mathcal{B} \mathcal{L}' / \mathcal{L}' \) such that \( M(\bar{w}) = \lambda \in g_{x,-\frac{1}{m}} \) and \(-M'(\bar{w}) = \lambda' \in g_{x',-\frac{1}{m}} \) are stable vectors.

8.1. We now study the geometry of \( X \) and the moment maps which will eventually determine the local theta correspondences.

By (C4) \( \lambda \) is a stable vector so \( \hat{\lambda} := t_\bar{g}^{-1} \lambda \) is a regular semisimple element in \( \bar{g} \) (c.f. Section 5.2). When \( \hat{\lambda} \in \bar{g} \subseteq \text{End}_{\mathcal{L}(\mathcal{V})} (V) \) is not of full rank, it requires a special treatment. By (C1) and the classification of epipelagic points in [8, 20], this is exactly in the following situation:

(E) \( D \) is a ramified quadratic extension of \( k \), the dual pair \((G, G')\) is a pair of unitary groups of the same rank \( n \), and \( \hat{\lambda} \in \bar{g} \) has rank \( n - 1 \).

Define

\[
X_{\lambda,\lambda'} := M^{-1}(\lambda) \cap M'^{-1}(-\lambda') \\
S_{\bar{w}} := \text{Stab}_{S_{\lambda} \times S'_{\lambda'}}(\bar{w})
\]

We note that \( S_{\lambda} \) is abelian in all our cases so all its irreducible representations are one dimensional characters.

Lemma 8.1.1. (i) The set \( X_{\lambda,\lambda'} \) is the \( S_{\lambda} \)-orbit of \( \bar{w} \) in \( X \).

(ii) There is a group homomorphism \( \alpha : S'_{\lambda'} \rightarrow S_{\lambda} \) such that \( \Delta_{\alpha}(S'_{\lambda'}) := \{ (\alpha(g'), g') : g' \in S'_{\lambda'} \} \) is a subgroup of \( S_{\bar{w}} \).

(iii) If we are not in Case (E), then \( S_{\lambda} \) acts freely on \( X_{\lambda,\lambda'} \) and \( S_{\bar{w}} = \Delta_{\alpha}(S'_{\lambda'}) \).
In Case (E), let \( \mathcal{S}_\omega = \text{Stab}_{\mathcal{S}_\lambda}(\bar{w}) \) and \( \mathcal{S}_\lambda := \{ g \in \mathcal{S}_\lambda : g \circ \lambda = \lambda \} \). Then \( \mathcal{S}_\omega = \mathcal{S}_\lambda \) and so the character \( \chi \) of \( \mathcal{S}_\lambda \) occurs in \( \mathbb{C}[X_{\lambda, \lambda'}] \) if and only if \( \chi|_{\mathcal{S}_\lambda} \) is trivial.

The proof is given in Appendix B.2.

**Remark:** (1) The homomorphism \( \alpha \) induces a map \( \alpha^* : \hat{\mathcal{S}}_\lambda \to \hat{\mathcal{S}}_{\lambda'} \), given by \( \alpha^*(\chi) = \chi \circ \alpha \). The definition depends on the choice of \( \bar{w} \in X_{\lambda, \lambda'} \). On the other hand, it is well-defined up to conjugation by the Grothendieck groups induced by \( \alpha^* \) is independent of the choice of \( \bar{w} \).

(2) In the exceptional Case (E), (iv) will lead to the fact that not all epipelagic representations can occur in this local theta correspondence. The extreme case is the well known fact [15] that not all characters of \( \text{U}(1) \) occur in the oscillator representation of \( \text{Mp}(2) \).

**Theorem 8.1.2.** Suppose (C1) to (C4) hold. For any character \( \chi \) of \( \mathcal{S}_\lambda \), let \( \Sigma = (x, \lambda, \chi) \) and \( \Sigma' = (x', \lambda', \chi') \) where \( \chi' = \chi^* \circ \alpha \) and \( \chi^* \) is the dual representation of \( \chi \).

(i) Suppose we are not in the exceptional case (E). Then

\[
\theta(\pi_{\Sigma}) = \pi_{\Sigma'}.
\]

In particular the theta lift is nonzero.

(ii) Suppose we are in the exceptional case (E). Then (10) holds for \( \chi \in \hat{\mathcal{S}}_\lambda \) such that \( \chi|_{\mathcal{S}_\lambda} \) is trivial.

The rest of this section is devoted to the proof of the above theorem.

### 8.2

We set \( B = \mathcal{B}_{-\frac{1}{2m}} \) and \( A = \mathcal{B}_{\frac{1}{2m}} \). By (C2), \( A^\sharp = A \) and \( B^\sharp = \mathcal{B}_{\frac{1}{2m}} = \mathcal{B}_{\frac{1}{2m}} \). Since \( m \geq 2 \), \( \mathcal{B}_{-\frac{1}{2m}} \subseteq \mathcal{B}_{\frac{1}{2m}} \) and \( B \) is a good lattice. We define following \( f \)-vector spaces, \( W := \mathcal{B}_{-\frac{1}{2m}} \subseteq \mathcal{B}_{\frac{1}{2m}} \) and \( B \) is a maximal isotropic subspace in \( W \). Let \( \mathcal{P} = (G^X_{\mathcal{X}}/G^X_{\mathcal{X}}) \times (G^X_{\mathcal{X}}/G^X_{\mathcal{X}}) \) and \( J = (G^X_{\mathcal{X}}/G^X_{\mathcal{X}}) \times (G^X_{\mathcal{X}}/G^X_{\mathcal{X}}) \). We have an exact sequence

\[
0 \rightarrow Y \rightarrow W \rightarrow X \rightarrow 0
\]

of \( \mathcal{P} \)-modules. The proof of following lemma is given in Appendix A.

**Lemma 8.2.1.** The natural quotient \( \mathcal{P} \rightarrow J \) has a splitting such that the exact sequence (11) splits as \( J \)-modules. We denote the splitting by \( W = Y \oplus X \).

### 8.3

**Proof of 8.1.2.** We recall the lattice model \( \mathcal{S}(A) \) in Section 3.3. Let \( \mathcal{S}(A)_B \) be the subspace of functions in \( \mathcal{S}(A) \) with support in \( B \). For \( f \in \mathcal{S}(A)_B \), \( w \in B \) and \( b' \in B^\sharp \), \( f(w + b') = \psi(\frac{1}{2}(w, b'))f(w) = f(w) \). Therefore, we could view \( \mathcal{S}(A)_B \) as a subspace in \( \mathbb{C}[W] \). We fix the splitting \( W = Y \oplus X \) in Lemma 8.2.1. Let \( J \) act on \( \mathbb{C}[X] \) by translation. Since \( f(w + a) = \psi(\frac{1}{2}(w, a))f(w) \) for all \( a \in A \), the restriction map \( R_X \) from \( W \) to \( X \) induces a \( J \)-module isomorphism

\[
R_X : \mathcal{S}(A)_B \rightarrow \mathbb{C}[X]
\]

whose inverse map \( R_X^{-1} \) is given by

\[
(R_X^{-1}F)(w) = \psi\left(\frac{1}{2}(x, y)\right)F(x)
\]

---

2 Let \( h \in G_x \) and \( \gamma \in g_{-\frac{1}{2m}} \) be any lifts of \( g \) and \( \lambda \) respectively. We consider \( h, \gamma \in \text{Hom}_{\mathcal{A}}(V, V) \). Then \( g \circ \lambda := h \circ \gamma + g_{-\frac{1}{2m}} \in g_{-\frac{1}{2m}} \) is well defined.

3 A lattice \( B \) is called **good** if \( B^\sharp \mathcal{P}_D \subseteq B \subseteq B^\sharp \). Good lattices correspond to vertices in the building.
for all \( w \in B \) such that \( w \equiv x + y \pmod{B^3} \) with \( x + B^2 \in X \) and \( y + B^3 \in Y \).

For \( f \in \mathcal{S}(A)_B, w \in B \) and \( g \in G_{\mathcal{L}, \frac{1}{m}} \), we have \((g^{-1} - 1)w \in A\). Applying Lemma 6.1.2, we have\(^4\)

\[
\omega_A(g)f(w) = f((g^{-1} - 1)w + w) = \psi \left( \frac{1}{2} (w, (g^{-1} - 1)w) \right) f(w) = \psi(\mathcal{B}_g(M(w), C(g))) f(w) = \psi_M(\bar{w}) g f(w)
\]

where \( \bar{w} \) is the image of \( w \) in \( X = B/A \). Similarly \( \omega_A(g')f(w) = \psi^{-1}(\bar{w}) (g') f(w) \) for all \( g' \in G_{\mathcal{L}, \frac{1}{m}} \).

Let \( \mathcal{S}(A)^\lambda_B \) be the subspace of functions in \( \mathcal{S}(A)_B \) such that \( G_{\mathcal{L}, \frac{1}{m}} \) acts by \( \psi_\lambda \). Then it follows from (13) and (12) that

\[
\mathcal{S}(A)^\lambda_B = R^{-1}(\mathcal{C}[M^{-1}(\lambda)]).
\]

A similar consideration applies to \( \lambda' \in \mathfrak{g}'_{\mathcal{L}, \frac{1}{m}} \) too. Let \( \mathcal{S}^{\lambda, \lambda'} := \mathcal{S}(A)^\lambda_B \cap \mathcal{S}(A)^{\lambda'}_B \). Then

\[
\mathcal{S}^{\lambda, \lambda'} = R^{-1}(\mathcal{C}[\lambda, \lambda'])
\]

We fix a \( \varpi \) in \( X_{\lambda, \lambda'} \). By Lemma 8.1.1, \( S_\lambda \rightarrow X_{\lambda, \lambda'} \) given by \( s \mapsto s \cdot \varpi \) is a surjection of \( S_\lambda \times S_{\lambda'} \)-set. Here \((s, s') \in S_\lambda \times S_{\lambda'} \) acts on \( S_\lambda \) by \((s, s') \cdot s_0 = ss_0s'(s')^{-1} \) for all \( s_0 \in S_\lambda \).

By the decomposition of regular representation of \( S_\lambda \), we have

\[
\mathcal{C}[X_{\lambda, \lambda'}] \subseteq \mathcal{C}[S_\lambda] = \bigoplus_{\chi \in \hat{S}_\lambda} \mathcal{C}_\chi \otimes \mathcal{C}_{\chi^* \circ \alpha}
\]

as an \( S_\lambda \times S_{\lambda'} \)-module. In (i), (14) is an equality. In (ii) the summand \( \mathcal{C}_\chi \otimes \mathcal{C}_{\chi^* \circ \alpha} \) occurs in \( \mathcal{C}[X_{\lambda, \lambda'}] \) if and only if \( \chi \mid s_\lambda \) is trivial by Lemma 8.1.1.

Fix any \( S_\lambda \)-character \( \chi \) which occurs in \( \mathcal{C}[X_{\lambda, \lambda'}] \). It is clear that the \( H_\lambda \) and \( H_{\lambda'} \) in (11) act on the space \( R^{-1}(\mathcal{C}_\chi \otimes \mathcal{C}_{\chi^* \circ \alpha}) \) by the characters \( \psi_\lambda \otimes \chi \) and \( \psi_{\lambda'} \otimes \chi^* \circ \alpha \) respectively. By Frobenius reciprocity, the function \( R^{-1}(\mathcal{C}_\chi \otimes \mathcal{C}_{\chi^* \circ \alpha}) \subset \mathcal{S}(A) \) induces a non-zero intertwining map

\[
\pi^G_\lambda(\lambda, \chi) \otimes \pi^{G'}_{\lambda'}(\lambda', \chi^* \circ \alpha) \rightarrow \mathcal{S}(A).
\]

Since the left hand side is irreducible and supercuspidal, the above map is an injection. By the smoothness of \( \mathcal{S}(A) \) we conclude that the left hand side is a discrete component in \( \mathcal{S}(A) \) and we have a projection map from \( \mathcal{S}(A) \) to the left hand side. This completes the proof of 8.1.2. \(\square\)

**Appendix A. Proofs of Lemma 8.2.1**

We recall \( \nu_D = \nu(\varpi_D) \). Let \( [r] \) denote the largest integer not greater than \( r \in \mathbb{R} \). We recall the explicit description of an apartment in \( \mathcal{B}(G, k) \) (c.f. [5, §2.9] and [3, §2-4]). Let \( n^+ \) be the dimension of a maximally isotropic subspace in \( V \). Let \( I := I^+ \sqcup I^- \sqcup I^0 \) where \( I^+ = \{ 1, \ldots, n^+ \} \), \( I^- = -I^+ \) and \( I^0 \) is any index set with \( \dim_D V - 2n^+ \) elements. Then there is a maximally split \( D \)-basis \( \{ e_i : i \in I \} \) of \( V \) which splits \( \mathcal{L} \), i.e.

(a) \( e_i \) is isotropic for \( i \in I^+ \sqcup I^- \) and \( \langle e_i, e_j \rangle = \delta_{i, j} \) for \( i \in I^+ \);
(b) \( e_i \) is anisotropic and \( \langle e_i, e_j \rangle = 0 \) for \( i \in I^0 \) and \( i \neq j \in I \) and
(c) there are \( a_i \in \mathbb{R} \) such that \( \mathcal{L}_r = \bigoplus_{i \in I} e_i \mathcal{P}_D^{[(r+a_i)/\nu_D]} \).

---

\(^4\)See also the proof of [19, Theorem 5.5]
For \( r \in \text{Jump} (L) \), define \( V^{[r]} = \text{span}\{ c_i \mid a_i \equiv r \pmod{\nu_D} \} \) and \( V^r = V^{[r]} \cap L_r \). For \( r \equiv 0, \frac{1}{2}\nu_D \pmod{\nu_D} \), \( V^{[r]} \) is non-degenerate and \( V^r \) is totally isotropic if otherwise. Moreover \( V^{[r]} \) is in perfect pairing with \( V^{-[r]} \). Let \( \mathcal{M} = G_L \cap (\prod_{r \in \text{Jump} (L)} (1 - \frac{1}{2\nu_D}, \frac{1}{\nu_D}) \text{GL}(V^{[r]})) \) be the \( \mathfrak{p}_k \)-subgroup scheme of \( G_L \) preserving \( \{ V^r : r \in \text{Jump} (L) \} \). The filtration \( G_{L,s} \) induces a filtration \( \mathcal{M}_s \). Then the embedding \( \mathcal{M} \to G_{L,s} \) (resp. \( \mathcal{V}^r \to L_s \)) reduces to an isomorphism \( \mathcal{M} / \mathcal{M}_{PD} \cong G_L / G_{L,0} \) (resp. \( \mathcal{V}^r / \mathcal{V}^r_{PD} \to L_s / L_{s+} \)). This is a more precise statement for Lemma 4.5.1.

We define similar notations for \( V^\prime \). We recall \( \mathcal{B} = L \otimes L^\prime \). For \( \mu \in \mathbb{R} \), we define \( \mathcal{X}_\mu = \sum_{\nu \in \mathcal{B}} \phi \otimes \phi \). For any \( w \), \( M \) induces a filtration \( \mathcal{O} \). Then \( \mathcal{X}_\mu \to \mathcal{B} \) induces an embedding \( \mathcal{X}_\mu \to \mathcal{W} = \mathcal{B} / \mathcal{O}_\mu + \mathcal{B}_{\mu+} \). The embedding \( \mathcal{M} \times \mathcal{M}^\prime \to \mathcal{G}_L \times \mathcal{G}_L^\prime \) induces an embedding \( J = G_L \times G_{L,0} \to \mathcal{G}_L^\prime \), \( \mathcal{G}_L^\prime \) J-module isomorphism via the splitting. We get a decomposition \( \mathcal{W} = X \oplus Y \) as J-modules under the above chosen splitting \( J \to \mathcal{P} \).

**Appendix B. Matrix calculations**

**B.1.** We will construct an \( \mathfrak{A} \) defined over an algebraically closed field \( \overline{k} \) which satisfies Lemma 7.3.3. The lemma and the proof below is valid for any field extension provided \((G, G')\) is an irreducible dual pair such that \( G \) and \( G' \) are both split.

There are only several cases.

1. \((G, G') = (\text{GL}(n), \text{GL}(n'))\) with \( n \leq n' \). We can identify (a) \( W = \text{Mat}_{n,n'} \oplus \text{Mat}_{n,n''} \), (b) \((x, y)^o = (y, -x)\) for \((x, y) \in W\), (c) \( M(x, y) = xy^\dagger \in \text{GL}(n) \) and (d) \( M(x, y) = y^\dagger x \in \text{GL}(n') \). We set

\[
\mathfrak{A} = \left\{ w = (\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}) \mid a = \text{diag}(a_1, \ldots, a_n), \ b = \text{diag}(b_1, \ldots, b_n), \text{ with } a_i, b_i \in \overline{k} \right\}.
\]

For any \( w \in \mathfrak{A}, M(w) = ab \) and \( M'(w) = \begin{pmatrix} ab & 0 \\ 0 & 0 \end{pmatrix} \). Hence \( \mathfrak{A} \) satisfies Lemma 7.3.3.

2. \((G, G') = (\text{Sp}(2n), \text{O}(2n' + 1))\) with \( n \leq n' \). We can choose suitable basis so that \( V = k^{2n} \) and \( V' = k^{2n'+1} \) such that \( \langle v_1, v_2 \rangle_V = v_1^\dagger J v_2 \) and \( \langle v'_1, v'_2 \rangle_{V'} = v'_1^\dagger J' v'_2 \) where

\[
J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad \text{and} \quad J' = \begin{pmatrix} 0 & 0 & I_n \\ 0 & I_{2n'+1} & 0 \\ I_n & 0 & 0 \end{pmatrix}.
\]

Now we can identify (a) \( W = M_{2n'+1,2n} \), (b) \( w^* = J^{-1} w^\dagger J' \), (c) \( M(w) = w^* w \) and (d) \( M'(w) = w w^* \). We consider

\[
\mathfrak{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a = \text{diag}(a_1, \ldots, a_n), \ b = \text{diag}(b_1, \ldots, b_n), \text{ with } a_i, b_i \in \overline{k} \right\}.
\]

For any \( w \in \mathfrak{A}, M(w) = \begin{pmatrix} ab & 0 \\ 0 & -ab \end{pmatrix} \) and \( M'(w) = \begin{pmatrix} ab & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \).

3. We leave the all other cases where \((G, G') = (\text{Sp}(2n), \text{O}(2n' + 2)), (\text{O}(2n), \text{Sp}(2n'))\) or \((\text{O}(2n+1), \text{Sp}(2n'))\) where \( n \leq n' \) to the reader. The formulas are similar to 2.
B.2. **Proof of Lemma 8.1.1.** We translate everything to the left hand side of (7) and denote the images of \( \hat{w}, \lambda, \lambda', \chi_{\lambda, \lambda'}, S_{\lambda}, \cdots, \) by \( \hat{\lambda}, \lambda', \tilde{X}_{\lambda, \lambda'}, \tilde{S}_{\lambda}, \cdots \) respectively. We also transport implicitly the Galois actions. Then \( \hat{\lambda} \) and \( \lambda' \) are regular semisimple elements. It is enough to prove the statements for \( \hat{\lambda} \) and \( \lambda' \).

(i) First we assume that \( \hat{\lambda} \) is a full rank matrix. In this case \( \hat{w} \) is a full rank matrix. By Witt’s theorem, \( M^{t-1}(\lambda') \) is a single free \( \tilde{G} \)-orbit. Let \( \hat{w}' \in \tilde{X}_{\lambda', \lambda'} \). Then there is a unique \( g \in \tilde{G} \) such that \( \hat{w}' = g \cdot \hat{w} \). Clearly \( g \in \text{Stab}_{\tilde{G}}(\hat{\lambda}) \). For every \( \sigma \in \text{Gal}(E/k) \),

\[
g \cdot \hat{w} = \hat{w}' = \sigma(\hat{w}') = \sigma(g) \cdot \sigma(\hat{w}) = \sigma(g) \cdot \hat{w}.
\]

Since \( \tilde{G} \) acts freely, we have \( g = \sigma(g) \). Hence \( g \in \tilde{S}_{\lambda} = (\text{Stab}_{\tilde{G}}(\hat{\lambda}))^{\text{Gal}(E/k)} \). This proves (i) in these cases.

Now we suppose that \( \hat{\lambda} \) is not full rank i.e. Case (E). This only occurs in the unitary group case. We refer to the Appendix B.1 for the notation. In this case, \( \tilde{W} = M_{nn}(\tilde{f}) \oplus M_{n}(\tilde{f}) \) are two copies of \( n \) by \( n \) matrices and \( \lambda' \) is of rank \( n - 1 \). There is an element in \( \text{Gal}(E/k) \times \tilde{G} \) exchanging the two components of \( \hat{w} = (A, B) \), hence \( A \) and \( B \) have the same rank \( n - 1 \). The group \( \tilde{G} \) is a general linear group. Let \( SL \) be the special linear subgroup of \( \tilde{G} \). Let \( \hat{w}' \in \tilde{X}_{\lambda', \lambda'} \). Let \( \tilde{S}_{\lambda} := \text{Stab}_{\tilde{SL}}(\hat{\lambda}) \). It is straightforward to check that \( \hat{w}' \) and \( \hat{w} \) are in the same \( SL_{\lambda} \)-orbit on which \( SL_{\lambda} \) acts freely. Let \( g \in SL_{\lambda} \) such that \( \hat{w}' = g \cdot \hat{w} \).

Again by (15), \( g \) is Galois invariant, i.e. \( g \in \tilde{S}_{\lambda} \). This proves (i).

(ii) Let \( \hat{w} \in \tilde{B}_{\tilde{\chi}_{\lambda, \lambda'}} \) be a lift of \( \hat{w} \). Without loss of generality, we may assume that \( \hat{w} \) is of full rank. Then \( \Gamma = M(\hat{w}) \) and \( \Gamma' = M'(\hat{w}) \) are lifts of \( \lambda \) and \( \lambda' \) respectively. Let \( S_{\Gamma} \) (resp. \( S'_{\Gamma} \)) be the stabilizer of \( \Gamma \) (resp. \( \Gamma' \)) in \( G \) (resp. \( G' \)). We recall that \( S_{\Gamma} \) is anisotropic so \( \tilde{B}(S_{\Gamma}) = \{ x \} \) as shown in the proof of Proposition 2.3.1. This implies that \( S_{\Gamma} \subseteq G_{\tilde{\chi}} \) and \( S'_{\Gamma} \subseteq G'_{\tilde{\chi}} \). Using the same argument and Witt’s theorem as in (i), for every \( g' \in S'_{\Gamma} \), there is a unique \( g \in S_{\Gamma} \) such that \( g'w = g^{-1}w \). The map \( \tilde{\alpha} : S'_{\Gamma} \rightarrow S_{\Gamma} \) given by \( g' \mapsto g \) is a (surjective) homomorphism. Note that \( S_{\Gamma} \) (resp. \( S'_{\Gamma} \)) surjects onto \( S_{\lambda} \) (resp. \( S'_{\lambda} \)) since \( \Gamma \) (resp. \( \Gamma' \)) is a good element (c.f. [11, Corollary 2.3.5 and Lemma 2.3.6]). Then \( \tilde{\alpha} \) induces a homomorphism \( \alpha : S'_{\lambda} \rightarrow S_{\lambda} \).

(iii) This follows from the proofs of (i) and (ii).

(iv) Note that \( \text{Im } \hat{w}^* = \text{Im } \lambda \subseteq V \). Therefore \( g \in \tilde{S}_{\hat{w}} \Leftrightarrow g|_{\text{Im } \hat{w}^*} = id \Leftrightarrow g|_{\text{Im } \lambda} = id \Leftrightarrow g \in \tilde{S}_{\lambda} \), i.e. \( \hat{w} \in \tilde{S}_{\lambda} \). The last assertion is clear. \( \square \)

**APPENDIX C. Lattice model and splitting**

C.1. **Independence of lattice models.** Let \( \text{Sp}(W) \) be a symplectic group of a symplectic space \( W \). Proposition 3.3.1 follows from Lemma C.1.1 below. One may compare the lemma with [21, §4.1] and [17].

**Lemma C.1.1.** Let \( A_1 \) and \( A_2 \) be two self-dual lattices in \( W \). For \( i = 1, 2 \), let \( \text{Sp}_{A_i} = \text{Stab}_{\text{Sp}(W)}(A_i) \) be the maximal compact subgroup in \( G \) stabilizing \( A_i \) and let \( \omega_{A_i} : \text{Sp}(W) \rightarrow \text{Mp}(W) \) be the section defined by the lattice model \( \mathcal{J}(A_i) \) as in (4). Then

\[
\omega_{A_1}(g) = \omega_{A_2}(g) \quad \forall g \in \text{Sp}_{A_1} \cap \text{Sp}_{A_2}.
\]

**Proof.** We have an intertwining operator \( \Xi : \mathcal{J}(A_1) \rightarrow \mathcal{J}(A_2) \) given by \( (\Xi f)(w) = \int_{A_2} \psi_{\frac{1}{2}}(a, w))f(w + a)da \) between the two lattice models. This intertwining operator is unique up to scalar.
Let \( g \in \text{Sp}_{A_1} \cap \text{Sp}_{A_2} \). Since \( g : A_2 \to A_2 \) is measure preserving,

\[
((\omega_{A_2}(g) \circ \Xi f)(w) = \int_{A_2} \psi\left(\frac{1}{2} \langle a, g^{-1}w \rangle\right) f(g^{-1}w + a) da \\
= \int_{A_2} \psi\left(\frac{1}{2} \langle ga, w \rangle\right) f(g^{-1}w + a) da \\
= \int_{A_2} \psi\left(\frac{1}{2} \langle a, w \rangle\right) f(g^{-1}w + g^{-1}a) da \\
= (\Xi \circ \omega_{A_1}(g) f)(w).
\]

This proves the lemma. \( \square \)

Let \( x \in B(G, k) \). We pick any \( x' \in B(G', k) \) and let \( \mathcal{L} \) and \( \mathcal{L}' \) be the lattice functions corresponding to \( x \) and \( x' \) respectively. Let \( \mathcal{B} = \mathcal{L} \otimes \mathcal{L}' \) be the tensor product lattice function and \( A \) be any self-dual lattice such that \( \mathcal{B}_{0+} \subseteq A \subseteq \mathcal{B}_0 \). We have \( G_{x,0+} \) stabilizes \( A \), i.e. \( G_{x,0+} \subseteq \text{Sp}_A \) (see also \cite{GPR} §3.3.2). As a corollary of Lemma \cite{GPR} and Proposition \cite{GPR}, the lattice models give a canonical splitting

\[
\omega_0 : \bigcup_{x \in B(G, k)} G_{x,0+} \to \tilde{G}.
\]

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HUNG YEAN LOKE, DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 2 SCIENCE DRIVE 2, SINGAPORE 117543

E-mail address: matlhy@nus.edu.sg

JIA-JUN MA, UNIT 408, ACADEMIC BUILDING NO.1, THE INSTITUTE OF MATHEMATICAL SCIENCES, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, N.T., HONG KONG

E-mail address: jjma@ims.cuhk.edu.hk

GORDAN SAVIN, DEPARTMENT OF MATHEMATICS UNIVERSITY OF UTAH SALT LAKE CITY, UT 84112

E-mail address: savin@math.utah.edu