A COMPUTER-ASSISTED STABILITY PROOF FOR A STATIONARY SOLUTION OF REACTION-DIFFUSION EQUATIONS

SHUTING CAI AND JING ZENG*

ABSTRACT. The main subject of this paper is a computer-assisted stability proof for a stationary solution of reaction-diffusion equations in one dimensional space. We use Nakao’s numerical verification method to enclose a stationary solution of reaction-diffusion equations. Considering the linearized stability of the solution, a method of excluding eigenvalues in a half plane is adopted. We first focus on the eigenvalues for an operator linearized at an approximate solution. An excluding theorem is presented such that we know under some condition, there is no eigenvalue in some disks. Some computable criteria is constructed to apply the theorem in a computer. And also the invertibility of some operator is proved theoretically in the paper. However, we need the information of the eigenvalues for the operator linearized at the exact solution. This can be obtained by combining with the verification results of the solution. Then we judge the stability of the solution from the domain where the eigenvalues are located. At last there are some verification results.

Keyword: Reaction-diffusion; Computer-assisted method; Eigenvalue excluding; Stability.

Mathematics Subject Classification (2010): 35K57; 65N25; 35B35.

1. INTRODUCTION

We are interested in the reaction-diffusion equations:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D_u \Delta u + f(u, v), \quad \text{in } \Omega \\
\frac{\partial v}{\partial t} &= D_v \Delta v + g(u, v), \quad \text{in } \Omega
\end{align*}
\]  

(1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n (n = 1, 2, 3) \) and \( D_u, D_v \) are positive constants, \( f, g \) are nonlinear functionals obtained from some model, such like Schnakenberg model, Predator-prey model and so on. In this paper, \( f \) and \( g \) are polynomial functionals with respect to \( u \) and \( v \). Equations (1.1) can be applied in biology and chemistry[7,8, 19]. According to Turing, the steady state of equations (1.1) could be stable to small perturbations without diffusion, but unstable to small spatial perturbations after diffusion is introduced into system[8, 19]. It has been attracted many researchers’ attention in Turing instability. For example, Dilao derived a necessary and sufficient condition for Turing instabilities to occur in two-component systems of reaction-diffusion equations with Neumann boundary conditions[3]. And Guo and Hwang in [5] considered the classical Turing instability in a reaction-diffusion system. Another one is that some quantified stability investigations for system (1.1) were done in [16] by Omarova. She had some research in Turing space, within which Turing instability could be observed. She is interested in the results on the stationary solutions’ existence, stability and - in the case of stability - in the size of their

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domain of attraction. In this paper, we are also interested in the stability of the stationary solution when Turing instability happens, but a different approach is considered.

This paper focuses on one-dimensional case. A stationary solution \((u^*, v^*)\) of system (1.1) with Neumann boundary satisfies

\[
\begin{align*}
    D_u \Delta u + f(u, v) &= 0 \quad \text{in } \Omega, \\
    D_v \Delta v + g(u, v) &= 0 \quad \text{in } \Omega, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \(\Omega \subset \mathbb{R}\). When we consider the Turning instability, the important part is to examine the behavior of the eigenvalues for the operator linearized at the stationary solution, that is, to investigate the eigenvalue problem

\[
\begin{align*}
    -D_u \Delta u - f_u(u^*, v^*)u - f_v(u^*, v^*)v &= \lambda u, \\
    -D_v \Delta v - g_u(u^*, v^*)u - g_v(u^*, v^*)v &= \lambda v.
\end{align*}
\]

For more details please refer to [4, 8]. Here, if all eigenvalues of problem (1.3) are positive, then we say the solution is stable. Thus, if the domain, where the eigenvalues are located, does not have intersection with the left half plane, that is, excluding the eigenvalues are in the left half plane, then it means the solution is stable. An eigenvalue excluding method is constructed here to achieve our aim. Our excluding method is inspired by [15] and [6]. In [15], the short note described a computer-assisted stability proof for the Orr-Sommerfeld problem with Poiseuille flow. And in [6], the authors had an improvement. They simplified the theory in verifying the invertibility of a linear elliptic operator. All those theories are application of a numerical verification method which was originated by Nakao [12] and then has been developed by him and his co-workers [9, 10, 11, 12, 13, 14]. We apply their excluding method in a different model and combine with the numerical verification method for the stationary solution. Moreover, in this paper, a different proof for the invertibility of a linear operator is given.

There are five sections in this paper. In Section 2, we define some notations. Also some imbedding constants and the constructive a priori estimation of the projection for the equations are described. Section 3 is about the stationary solution. We outline the method that how to get an approximate solution and prove the existence of the solution near the approximate solution. Meanwhile, the norm estimation for the residual part of the approximate solution is obtained. Then Section 4 is the eigenvalue excluding scheme. Our subject in this section is to obtain all eigenvalues of (1.3) are not in the left half plane. We consider the operator linearized at the approximate solution. An eigenvalue excluding theorem is given such that we know under some condition, there is no eigenvalue in some disks, which are not in the left half plane. Then we obtain those disks by the help of a computer, for which some computable criterion is constructed. Back to considering eigenvalues of problem (1.3), some verification results of the solution from Section 3 is used. And the last section presents some verification results.

2. SOME NOTATIONS AND PROJECTION ERROR ESTIMATION

The domain is \(\Omega = (0, l) \subset \mathbb{R}\). And we choose basis functions as

\[
\varphi_i(x) = \cos \left( \frac{i \pi x}{l} \right), \quad i = 0, 1, 2, \cdots
\]

The Sobolev space \(H^k(\Omega)(k \geq 0)\) is defined as

\[
H^k(\Omega) := \{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), \forall |\alpha| \leq k \}.
\]
The natural number \( k \) is called the order of the Sobolev space \( H^k(\Omega) \).

For \( z \in H^k(\Omega) (k \geq 0) \), define the usual norm as
\[
\| z \|_{H^k(\Omega)}^2 := \sum_{|\alpha| \leq k} \| D^\alpha z \|_{L^2(\Omega)}^2.
\]

Now we define functional space \( X^k (k \geq 0) \) by the closure in \( H^k(\Omega) \) of the linear hull of all basis functions \( \varphi_i (i = 1, 2, \cdots) \).

For a non-negative integer \( N \), let \( X^N \) denote a finite dimensional subspace as
\[
X^N := \left\{ v_N = \sum_{n=0}^{N} c_n \varphi_n \mid c_n \in \mathbb{R} \right\} \subset X^1.
\]

For \( z_1, z_2 \in X^1 \), define the usual inner product as
\[
(z_1, z_2)_{H^1(\Omega)} := (z_1, z_2)_{L^2(\Omega)} + (z'_1, z'_2)_{L^2(\Omega)},
\]
where \((\cdot, \cdot)_{L^2(\Omega)}\) means the inner product on \( L^2(\Omega) \).

And for \( z = \sum_{n=0}^{\infty} c_n \varphi_n \in X^1 \), let \( P_N : X^1 \rightarrow X_N \) denote the \( H^1 \)-projection defined by the truncation operator:
\[
P_N \left( \sum_{n=0}^{\infty} c_n \varphi_n \right) = \sum_{n=0}^{N} c_n \varphi_n,
\]
which satisfies
\[
(z - P_N z, z_N)_{H^1(\Omega)} = 0, \quad \text{for all } z_N \in X_N.
\]

Set \( P : X^1 \times X^1 \rightarrow X_N \times X_N \) as
\[
P(z_1, z_2) := (P_N z_1, P_N z_2), \quad z_1, z_2 \in X^1.
\]

For Hilbert spaces \( X \) and \( Y \), we define the inner product and the norm in \( X \times Y \) as
\[
\left( \begin{array}{c} x_1 \\ y_1 \\ x_2 \\ y_2 \end{array} \right)_{X \times Y} := (x_1, x_2)_X + (y_1, y_2)_Y
\]
and
\[
\| \left( \begin{array}{c} x \\ y \end{array} \right) \| := \sqrt{\| x \|^2_X + \| y \|^2_Y}.
\]

Defining the operator \( L : X^2 \rightarrow X^0 \) by \( L \psi := -\psi'' + \psi \), we get the following lemma.

**Lemma 2.1.** ([20], Lemma 1) For all \( \phi \in X^0 \), the linear equation
\[
L \psi = \phi \quad \text{in } \Omega
\]
has the unique solution \( \psi \in X^2 \).

Now we derive an estimation for the projection \( P_N \) and an imbedding constant from \( H^1(\Omega) \) to \( L^\infty(\Omega) \).

**Lemma 2.2.** ([20], Lemma 2) For all \( z \in X^2 \), we have
\[
\| z - P_N z \|_{H^1(\Omega)} \leq C(N) \| L z \|_{L^2(\Omega)},
\]
where \( C(N) = \sqrt{1 + (N+1)^2} \pi / \sqrt{2} \).

And
\[
\| z - P_N z \|_{L^2(\Omega)} \leq C(N) \| z - P_N z \|_{H^1(\Omega)}
\]
holds.
Lemma 2.3. (20, Lemma 3)(Imbedding Constants) \( \Omega = (a, b), a < b \). Then, for all \( u \in H^1(\Omega) \), we have
\[
\|u\|_{L^\infty(\Omega)} \leq K_\infty \|u\|_{H^1(\Omega)},
\]
where
\[
K_\infty = \sqrt{\max \left\{ \frac{2}{b-a}, 2(b-a) \right\}}.
\]

3. THE STATIONARY SOLUTION

We use Newton’s method to get an approximate solution \((\hat{u}_N, \hat{v}_N) \in X_N \times X_N\) of (1.2) and a verification method to prove the existence of the stationary solution \((u^*, v^*)\) of (1.2) near \((\hat{u}_N, \hat{v}_N)\). The method is similar to [1]. In [1], the authors considered the solution in two dimensional case. Here we prove the existence in one dimensional case, which is much more simple. Therefore, we omit the full process here. However, for the completeness of the paper, we outline the verification method here.

Set \( \tilde{u} := u^* - \hat{u}_N \) and \( \tilde{v} := v^* - \hat{v}_N \). Define a compact map from \( X^1 \times X^1 \) to \( X^1 \times X^1 \) as
\[
F(\tilde{u}, \tilde{v}) := \left( \frac{1}{D_u} L^{-1} \{ \gamma f(\hat{u}_N + \tilde{u}, \hat{v}_N + \tilde{v}) + D_u \hat{u}_N' + D_u \tilde{u} \} \right) + \left( \frac{1}{D_v} L^{-1} \{ \gamma g(\hat{u}_N + \tilde{u}, \hat{v}_N + \tilde{v}) + D_v \hat{v}_N' + D_v \tilde{v} \} \right),
\]
Then \( \tilde{w} = (\tilde{u}, \tilde{v}) \) becomes the solution of the fixed point equation
\[
\tilde{w} = F(\tilde{w}). \tag{3.1}
\]

Now if we enclose a fixed point of \( F \), then a solution of (1.2) can be enclosed by \( w^* = (u^*, v^*) \), \( u^* = \hat{u}_N + \tilde{u} \) and \( v^* = \hat{v}_N + \tilde{v} \).

Define the Newton-like operator
\[
N(\tilde{w}) := P\tilde{w} - [I - F'(0)]^{-1} (P\tilde{w} - PF(\tilde{w})),
\]
here, \( F'(0) \) is the Fréchet derivative of \( F \) at 0 and suppose that the restriction to \( X_N \times X_N \) of the operator \( P[I - F'(0)] : X^1 \times X^1 \to X_N \times X_N \) has an inverse
\[
[I - F'(0)]^{-1} : X_N \times X_N \to X_N \times X_N.
\]

Set a compact operator \( T : X^1 \times X^1 \to X^1 \times X^1 \) as
\[
T(\tilde{w}) := N(\tilde{w}) + (I - P) F(\tilde{w}).
\]
Then we have the equivalence relation
\[
\tilde{w} = F(\tilde{w}) \iff \tilde{w} = T(\tilde{w}).
\]
Thus, if there exists a non-empty, closed, convex and bounded set \( W \subset X^1 \times X^1 \) such that \( T(W) \subset W \), then by Schauder’s fixed point theorem, there exists a solution \( \tilde{w} \in W \) of \( \tilde{w} = T(\tilde{w}) \), i.e. \( \tilde{w} = F(\tilde{w}) \). We use a computer to find \( W \).

On a computer, we construct several sets:
\[
\begin{align*}
W &= U \times V, \\
U &= U_N + U_\perp, \\
V &= V_N + V_\perp, \tag{3.2}
\end{align*}
\]
with $U_N, U_\perp, V_N, V_\perp$ defined by

$$
U_N := \{ \phi_N \in X_N \mid \|\phi_N\|_{H^1(\Omega)} \leq \alpha_1 \}, \\
U_\perp := \{ \phi_\perp \in X_N^\perp \mid \|\phi_\perp\|_{H^1(\Omega)} \leq \alpha_2 \}, \\
V_N := \{ \phi_N \in X_N \mid \|\phi_N\|_{H^1(\Omega)} \leq \beta_1 \}, \\
V_\perp := \{ \phi_\perp \in X_N^\perp \mid \|\phi_\perp\|_{H^1(\Omega)} \leq \beta_2 \},
$$

for positive candidate constants $\alpha_1, \alpha_2, \beta_1, \beta_2$, where $X_N^\perp$ represents the orthogonal complement of $X_N$ in $X^1$.

We search some suitable constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$
\begin{align*}
N(W) &\subset PW, \\
(I-P)F(W) &\subset (I-P)W,
\end{align*}
$$

that is, $T(W) \subset W$.

Then we verify the existence of the solution in the set $[\{\hat{u}_N, \hat{v}_N\} + (\hat{u}, \hat{v})]$, where $(\hat{u}, \hat{v}) \in W$. And $\hat{u} = u_N + u_\perp, \hat{v} = v_N + v_\perp$, here $u_N \in U_N, u_\perp \in U_\perp, v_N \in V_N, v_\perp \in V_\perp$, that is, we have the norm estimation as $\|\hat{u}\|_{H^1(\Omega)} = \|u_N + u_\perp\|_{H^1(\Omega)} \leq \alpha_1 + \alpha_2$ and $\|\hat{v}\|_{H^1(\Omega)} = \|v_N + v_\perp\|_{H^1(\Omega)} \leq \beta_1 + \beta_2$.

## 4. Eigenvalue excluding scheme

After the preparation in those previous sections, we introduce a computer-assisted method that excluding the eigenvalues for the operator linearized at the verified solution in section 3. If the eigenvalues are not in the left half plane, then the verified solution is linearized stable. We will see in this section that our method only works for one dimensional case, since the lack of embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ in two dimensional case. But if we extend our theory to $X^2 \times X^2$ space, then it works for both one-dimensional and two dimensional case. However, it is more complicate and sometimes the results in $X^1 \times X^1$ are better than in $X^2 \times X^2$.

### 4.1. Eigenvalue excluding theorem

First we consider the eigenvalue problem for the operator linearized at the approximate solution

$$
\begin{align*}
-Du'' - f_u(\hat{u}_N, \hat{v}_N)u - f_v(\hat{u}_N, \hat{v}_N)v &= \lambda u, \\
-Dv'' - g_u(\hat{u}_N, \hat{v}_N)u - g_v(\hat{u}_N, \hat{v}_N)v &= \lambda v.
\end{align*}
$$

(4.1)

Let $\mu \in C$ be a given candidate excluding point which is suspected that no eigenvalue of Eq. (4.1) is close to $\mu$.

If $u, v \in X^1$ holds, then we observe that

$$
\begin{align*}
\| (Du + \mu)u + f_u(\hat{u}_N, \hat{v}_N)u + f_v(\hat{u}_N, \hat{v}_N)v \|_{H^1(\Omega)} \\
\leq &\| (Du + \mu)u \|_{H^1(\Omega)} + \| f_u(\hat{u}_N, \hat{v}_N)u \|_{L^\infty(\Omega)} + \| f_v(\hat{u}_N, \hat{v}_N)v \|_{L^\infty(\Omega)} \\
\| (Dv + \mu)v + g_u(\hat{u}_N, \hat{v}_N)u + g_v(\hat{u}_N, \hat{v}_N)v \|_{H^1(\Omega)} \\
\leq &\| g_u(\hat{u}_N, \hat{v}_N)u \|_{L^\infty(\Omega)} + \| g_v(\hat{u}_N, \hat{v}_N)v \|_{L^\infty(\Omega)} + \| (Dv + \mu)v \|_{H^1(\Omega)} + \| g_u(\hat{u}_N, \hat{v}_N)u \|_{L^\infty(\Omega)} + \| g_v(\hat{u}_N, \hat{v}_N)v \|_{L^\infty(\Omega)}
\end{align*}
$$

therefore, $f_1(u, v), f_2(u, v) \in X^1$.

Now we define a linear operator $\hat{L} : X^1 \times X^1 \to X^1 \times X^1$ as

$$
\hat{L}(u, v) := \begin{pmatrix}
Du + L^{-1}\{ -f_1(u, v) \} \\
Dv + L^{-1}\{ -f_2(u, v) \}
\end{pmatrix},
$$
Then the equations (4.1) can be rewritten as

\[ f_1(u, v) := (D_u + \mu)u + f_u(\tilde{u}_N, \tilde{v}_N)u + f_v(\tilde{u}_N, \tilde{v}_N)v, \]
\[ f_2(u, v) := (D_v + \mu)v + g_u(\tilde{u}_N, \tilde{v}_N)u + g_v(\tilde{u}_N, \tilde{v}_N)v. \]

Therefore, (4.3) becomes

\[ \hat{L}(u, v) = (\lambda - \mu) \left( \begin{array}{c} L^{-1}u \\ L^{-1}v \end{array} \right). \]

Next we have the eigenvalue excluding theorem.

**Theorem 4.1.** Suppose that \( \hat{L} \) has an inverse \( \hat{L}^{-1} : X^1 \times X^1 \to X^1 \times X^1 \) and there exists \( \hat{M}_\mu > 0 \) such that

\[ \| \hat{L}^{-1}(u, v) \|_{H^1(\Omega) \times H^1(\Omega)} \leq \hat{M}_\mu \| (u, v) \|_{H^1(\Omega) \times H^1(\Omega)}, \]

then there is no eigenvalue \( \hat{\lambda} \) of Eq. (4.1) in the disk given by \( |\hat{\lambda} - \mu| < \frac{1}{\hat{M}_\mu}. \)

**Proof.** For any eigenpair \( (u_1, v_1, \hat{\lambda})^T \in X^1 \times X^1 \times C \) of eq. (4.1) which satisfies

\[ \hat{L}(u_1, v_1) = (\lambda - \mu) \left( \begin{array}{c} L^{-1}u_1 \\ L^{-1}v_1 \end{array} \right), \]

where \( u_1, v_1 \neq 0 \), taking \( (u, v) \in X^1 \times X^1 \) as \( \hat{L}(u_1, v_1) \) in (4.2), we have

\[ \| (u_1, v_1) \|_{H^1(\Omega) \times H^1(\Omega)} \leq \hat{M}_\mu \| \hat{L}(u_1, v_1) \|_{H^1(\Omega) \times H^1(\Omega)} = \hat{M}_\mu |\hat{\lambda} - \mu| \cdot \| (L^{-1}u_1, L^{-1}v_1) \|_{H^1(\Omega) \times H^1(\Omega)}. \]

Since for every \( u, v \in X^1, \)

\[ (L^{-1}u, v)_{H^1(\Omega)} = ((L^{-1}u)', v')_{L^2(\Omega)} + (L^{-1}u, v)_{L^2(\Omega)} \]
\[ = (-L^{-1}u, v)_{L^2(\Omega)} + \int_{\partial\Omega} \frac{\partial L^{-1}u}{\partial \nu} v ds + (L^{-1}u, v)_{L^2(\Omega)} \]
\[ = (LL^{-1}u, v)_{L^2(\Omega)} = (u, v)_{L^2(\Omega)} \]

holds, we obtain, for all \( \phi \in X^0, \)

\[ \| L^{-1}\phi \|^2_{H^1(\Omega)} = \int_{\Omega} \phi L^{-1}\phi \leq \| \phi \|_{L^2(\Omega)} \| L^{-1}\phi \|_{L^2(\Omega)} \leq \| \phi \|_{L^2(\Omega)} \| L^{-1}\phi \|_{H^1(\Omega)}, \]

that is,

\[ \| L^{-1}\phi \|_{H^1(\Omega)} \leq \| \phi \|_{L^2(\Omega)}. \]

Therefore, (4.3) becomes

\[ \| (u_1, v_1) \|_{H^1(\Omega) \times H^1(\Omega)} \leq \hat{M}_\mu |\hat{\lambda} - \mu| \cdot \| (u_1, v_1) \|_{L^2(\Omega) \times L^2(\Omega)} \]
\[ \leq \hat{M}_\mu |\hat{\lambda} - \mu| \cdot \| (u_1, v_1) \|_{H^1(\Omega) \times H^1(\Omega)}. \]
4.2. Direct computation of upper bound for $\hat{L}^{-1}$. From Theorem 4.1 if we have the information of $\hat{M}_\mu$ for some candidate points $\mu$, that is, the upper bound for $\hat{L}^{-1}$, then eigenvalue excluding can be executed. This is our subject in this subsection.

For $0 \leq i, j \leq N$, let

$$A_{ij} := (\varphi_i, \varphi_j)_{L^2(\Omega)},$$

$$D_{ij} := (\varphi'_i, \varphi'_j)_{L^2(\Omega)} + (\varphi_i, \varphi_j)_{L^2(\Omega)},$$

$$G_{1ij} := \frac{1}{D_u}(D_u(\varphi'_i, \varphi'_j)_{L^2(\Omega)}) - (\varphi_i, \varphi_j)_{L^2(\Omega)} - (f_u(\hat{u}_N, \hat{v}_N)\varphi_i, \varphi_j)_{L^2(\Omega)},$$

$$G_{12j} := \frac{1}{D_u}(f_u(\hat{u}_N, \hat{v}_N)\varphi_i, \varphi_j)_{L^2(\Omega)},$$

$$G_{21j} := \frac{1}{D_v}(g_v(\hat{u}_N, \hat{v}_N)\varphi_i, \varphi_j)_{L^2(\Omega)},$$

$$G_{22j} := \frac{1}{D_v}(D_v(\varphi'_i, \varphi'_j)_{L^2(\Omega)}) - (\varphi_i, \varphi_j)_{L^2(\Omega)} - (g_v(\hat{u}_N, \hat{v}_N)\varphi_i, \varphi_j)_{L^2(\Omega)}.$$

It is clear that $D$ and $A$ are diagonal matrices, therefore, there exist diagonal matrices $D_{1/2}$ and $A_{1/2}$ such that $(D_{1/2})^2 = D$ and $(A_{1/2})^2 = A$. Then we set

$$\left(\begin{array}{cc}
\hat{G}_{11} & \hat{G}_{12} \\
\hat{G}_{21} & \hat{G}_{22}
\end{array}\right) := \left(\begin{array}{cc}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{array}\right)^{-1},$$

$$\hat{\rho}_1 := \|D_{1/2} \hat{G}_{11}A_{1/2}\|_E, \quad \hat{\rho}_2 := \|D_{1/2} \hat{G}_{12}A_{1/2}\|_E,$$

$$\hat{\rho}_3 := \|D_{1/2} \hat{G}_{21}A_{1/2}\|_E, \quad \hat{\rho}_4 := \|D_{1/2} \hat{G}_{22}A_{1/2}\|_E,$$

$$\rho_1 := \|D_{1/2} \hat{G}_{11}D_{1/2}\|_E, \quad \rho_2 := \|D_{1/2} \hat{G}_{12}D_{1/2}\|_E,$$

$$\rho_3 := \|D_{1/2} \hat{G}_{21}D_{1/2}\|_E, \quad \rho_4 := \|D_{1/2} \hat{G}_{22}D_{1/2}\|_E,$$

where $\| \cdot \|_E$ is the Euclidian norm for a matrix.

Next we make an assumption:

**Assumption 4.2.** Suppose there exist positive constants $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6$ satisfying

$$\frac{1}{D_u}\|P_N f_1(u, v)\|_{L^2(\Omega)} \leq \vartheta_1(\|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}),$$

$$\frac{1}{D_v}\|P_N f_2(u, v)\|_{L^2(\Omega)} \leq \vartheta_2(\|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}),$$

$$\frac{1}{D_u}\|(I - P_N) f_1(u, v)\|_{L^2(\Omega)} \leq \vartheta_3(\|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}),$$

$$\frac{1}{D_v}\|(I - P_N) f_2(u, v)\|_{L^2(\Omega)} \leq \vartheta_4(\|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}),$$

Then we obtain an estimation for $\hat{M}_\mu$.

**Theorem 4.3.** Under Assumption 4.2 if $\hat{L}$ is invertible and

$$\kappa_1 := C(N)(\vartheta_3 + \vartheta_5)(\hat{\rho}_1 \vartheta_1 + \hat{\rho}_2 \vartheta_2 + \hat{\rho}_3 \vartheta_3 + \hat{\rho}_4 \vartheta_4 + \vartheta_4 + \vartheta_6) < 1$$
holds, then $\tilde{M}_\mu > 0$ can be taken as

$$\tilde{M}_\mu = \sqrt{\hat{\rho}^2 + \left(\frac{\hat{\kappa}}{1 - \kappa_1}\right)^2},$$

here

$$\hat{\rho} = (\hat{\rho}_1 \hat{\vartheta}_1 + \hat{\rho}_2 \hat{\vartheta}_2 + \hat{\rho}_3 \hat{\vartheta}_1 + \hat{\rho}_4 \hat{\vartheta}_2) \frac{\hat{\kappa}}{1 - \kappa_1} + \max\left\{\frac{1}{D_u} (\rho_1 + \rho_3), \frac{1}{D_v} (\rho_2 + \rho_4)\right\},$$

$$\hat{\kappa} = \max\left\{\frac{1}{D_u} (C(N)(\vartheta_3 + \vartheta_5)(\rho_1 + \rho_3) + 1), \frac{1}{D_v} (C(N)(\vartheta_4 + \vartheta_5)(\rho_2 + \rho_4) + 1)\right\}.$$

**Proof.** Since $\hat{L}$ is invertible, for each $(u_1, v_1) \in X^1 \times X^1$, there exists $(u, v) \in X^1 \times X^1$, such that

$$\hat{L}(u, v) = \left(\begin{array}{c} u_1 \\ v_1 \end{array}\right) = \left(\begin{array}{c} D_u u + L^{-1}\{-f_1(u, v)\} \\ D_v v + L^{-1}\{-f_2(u, v)\} \end{array}\right) = \left(\begin{array}{c} u_1 \\ v_1 \end{array}\right).$$

(4.4) is equivalent to

$$\left(\begin{array}{c} u \\ v \end{array}\right) = \left(\begin{array}{c} \frac{1}{D_u} L^{-1}\{f_1(u, v) + Lu_1\} \\ \frac{1}{D_v} L^{-1}\{f_2(u, v) + Lv_1\} \end{array}\right) =: J(u, v) = Jw.$$

We rewrite $w = Jw$ as

$$\begin{cases} Pw = PJw, \\ (I - P)w = (I - P)Jw. \end{cases}$$

Then for the finite dimensional part, for all $\phi_{1, N}, \phi_{2, N} \in X_N$, we get

$$(u_N, \phi_{1, N})_{H^2(\Omega)} = \frac{1}{D_u} (f_1(u_N, v_N) + f_1(u_\perp, v_\perp) + Lu_1, \phi_{1, N})_{L^2(\Omega)},$$

$$(v_N, \phi_{2, N})_{H^2(\Omega)} = \frac{1}{D_v} (f_2(u_N, v_N) + f_2(u_\perp, v_\perp) + Lv_1, \phi_{2, N})_{L^2(\Omega)}.$$

So we obtain

$$\begin{cases} 
(u'_{N, \phi_{1, N}})_{L^2(\Omega)} + (u_{N, \phi_{1, N}})_{L^2(\Omega)} - \frac{1}{D_u} (f_1(u_N, v_N), \phi_{1, N})_{L^2(\Omega)} \\
= \frac{1}{D_u} (f_1(u_\perp, v_\perp), \phi_{1, N})_{L^2(\Omega)} + \frac{1}{D_u} (u'_{1, N}, \phi_{1, N})_{L^2(\Omega)} + \frac{1}{D_u} (u_{1, N}, \phi_{1, N})_{L^2(\Omega)}, \\
(v'_{N, \phi_{2, N}})_{L^2(\Omega)} + (v_{N, \phi_{2, N}})_{L^2(\Omega)} - \frac{1}{D_v} (f_2(u_N, v_N), \phi_{2, N})_{L^2(\Omega)} \\
= \frac{1}{D_v} (f_2(u_\perp, v_\perp), \phi_{2, N})_{L^2(\Omega)} + \frac{1}{D_v} (v'_{1, N}, \phi_{2, N})_{L^2(\Omega)} + \frac{1}{D_v} (v_{1, N}, \phi_{1, N})_{L^2(\Omega)}. 
\end{cases} \tag{4.5}$$
By setting
\[ u_N := \sum_{i=0}^{N} a_i \varphi_i, \quad v_N := \sum_{i=0}^{N} b_i \varphi_i, \]
\[ u_{1N} := \sum_{i=0}^{N} a_i^1 \varphi_i, \quad v_{1N} := \sum_{i=0}^{N} b_i^1 \varphi_i, \]
\[ a := (a_0, a_1, \ldots, a_N)^T, \quad b := (b_0, b_1, \ldots, b_N)^T, \]
\[ g_{11}(i) := \frac{1}{D_u} f_1(u_\perp, v_\perp, \varphi_i)_{L^2(\Omega)}, \quad g_{12}(i) := \frac{1}{D_u} (u_{1N}, \varphi_i)_{L^2(\Omega)} + \frac{1}{D_u} (u_{1N}', \varphi_i')_{L^2(\Omega)}, \]
\[ g_{21}(i) := \frac{1}{D_v} f_2(u_\perp, v_\perp, \varphi_i)_{L^2(\Omega)}, \quad g_{22}(i) := \frac{1}{D_v} (v_{1N}, \varphi_i)_{L^2(\Omega)} + \frac{1}{D_v} (v_{1N}', \varphi_i')_{L^2(\Omega)}, \]
\[ g_i(i) := g_{11}(i) + g_{12}(i), \quad g_2(i) := g_{21}(i) + g_{22}(i), \quad (0 \leq i \leq N) \]

Equation (4.5) can be written as
\[ \left( \begin{array}{cc} G^{11} & G^{12} \\ G^{21} & G^{22} \end{array} \right) \left( \begin{array}{c} \vec{a} \\ \vec{b} \end{array} \right) = \left( \begin{array}{c} \vec{g}_1 \\ \vec{g}_2 \end{array} \right), \]

therefore, we get
\[ \left( \begin{array}{c} \vec{a} \\ \vec{b} \end{array} \right) = \left( \frac{G^{11} \vec{g}_1 + G^{12} \vec{g}_2}{G^{21} \vec{g}_1 + G^{22} \vec{g}_2} \right). \]

Now, define the \( L^2 \)-projection \( P_0 : X^0 \to X_N \) as
\[ (s - P_0 s, s_N)_{L^2(\Omega)} = 0, \quad \forall s_N \in X_N. \]

It is easily seen that
\[ \frac{1}{D_u} \| P_0 f_1(u_\perp, v_\perp) \|_{L^2(\Omega)} = \left\| \frac{A^{-1/2}}{E} \vec{g}_{11} \right\|_E, \quad \frac{1}{D_u} \| P_0 u_1 \|_{H^1(\Omega)} = \left\| \frac{A^{-1/2}}{E} \vec{g}_{12} \right\|_E, \]
\[ \frac{1}{D_v} \| P_0 f_2(u_\perp, v_\perp) \|_{L^2(\Omega)} = \left\| \frac{A^{-1/2}}{E} \vec{g}_{21} \right\|_E, \quad \frac{1}{D_v} \| P_0 v_1 \|_{H^1(\Omega)} = \left\| \frac{A^{-1/2}}{E} \vec{g}_{22} \right\|_E. \]

Therefore, under Assumption 4.2, we have
\[ \| u_N \|_{H^1(\Omega)} = \left\| D^{1/2} \vec{a} \right\|_E = \left\| D^{1/2}(\vec{G}^{11} \vec{g}_1 + \vec{G}^{12} \vec{g}_2) \right\|_E \]
\[ \leq \left\| D^{1/2} \vec{G}^{11} \vec{g}_{11} \right\|_E + \left\| D^{1/2} \vec{G}^{11} \vec{g}_{12} \right\|_E + \left\| D^{1/2} \vec{G}^{21} \vec{g}_{21} \right\|_E + \left\| D^{1/2} \vec{G}^{22} \vec{g}_{22} \right\|_E \]
\[ \leq \left\| D^{1/2} \vec{G}^{11} A^{1/2} \right\|_E \| A^{-1/2} \vec{g}_{11} \|_E + \left\| D^{1/2} \vec{G}^{11} D^{1/2} \right\|_E \| D^{-1/2} \vec{g}_{12} \|_E \]
\[ + \left\| D^{1/2} \vec{G}^{21} A^{1/2} \right\|_E \| A^{-1/2} \vec{g}_{21} \|_E + \left\| D^{1/2} \vec{G}^{21} D^{1/2} \right\|_E \| D^{-1/2} \vec{g}_{22} \|_E \]
\[ \leq \frac{\rho_1}{D_u} \| P_0 (f_1(u_\perp, v_\perp)) \|_{L^2(\Omega)} + \frac{\rho_1}{D_u} \| P_0 u_1 \|_{H^1(\Omega)} \]
\[ + \frac{\rho_2}{D_v} \| P_0 (f_2(u_\perp, v_\perp)) \|_{L^2(\Omega)} + \frac{\rho_2}{D_v} \| P_0 v_1 \|_{H^1(\Omega)} \]
\[ \leq \frac{1}{D_u} \| D_u \delta_1 (\| u_\perp \|_{H^1(\Omega)} + \| v_\perp \|_{H^1(\Omega)}) + \rho_1 \| u_1 \|_{H^1(\Omega)} \]
\[ + \frac{1}{D_v} (D_v \delta_2 (\| u_\perp \|_{H^1(\Omega)} + \| v_\perp \|_{H^1(\Omega)}) + \rho_2 \| v_1 \|_{H^1(\Omega)} \]

(4.6)
and

\[
\frac{1}{D_u} (D_u \rho_3 \vartheta_1 (\| u_\perp \|_{H^1(\Omega)} + \| v_\perp \|_{H^1(\Omega)}) + \rho_3 \| u_1 \|_{H^1(\Omega)}) + \rho_3 \| u_1 \|_{H^1(\Omega)}) + \rho_3 \| u_1 \|_{H^1(\Omega)}) + \rho_3 \| u_1 \|_{H^1(\Omega)})
\]

(4.7)

And we know that

\[
\begin{align*}
u_\perp &= \frac{1}{D_u} \left( I - P_{N} \right) L^{-1} \{ f_1(u,v) + L u_1 \}, \\
u_\perp &= \frac{1}{D_v} \left( I - P_{N} \right) L^{-1} \{ f_2(u,v) + L v_1 \},
\end{align*}
\]

so we have

\[
\begin{align*}
\| u_\perp \|_{H^1(\Omega)} &\leq \frac{C(N)}{D_u} \| (I - P_{N}) f_1(u,v) \|_{L^2(\Omega)} + \frac{1}{D_u} \| u_1 \|_{H^1(\Omega)} \\
&\leq \frac{C(N)}{D_u} \| \vartheta_3 (\| u_N \|_{H^1(\Omega)} + \| v_N \|_{H^1(\Omega)}) + \vartheta_4 (\| u_\perp \|_{H^1(\Omega)} + \| v_\perp \|_{H^1(\Omega)}) + \frac{1}{D_u} \| u_1 \|_{H^1(\Omega)}, \\
\| v_\perp \|_{H^1(\Omega)} &\leq \frac{C(N)}{D_v} \| \vartheta_5 (\| u_N \|_{H^1(\Omega)} + \| v_N \|_{H^1(\Omega)}) + \vartheta_6 (\| u_\perp \|_{H^1(\Omega)} + \| v_\perp \|_{H^1(\Omega)}) + \frac{1}{D_v} \| v_1 \|_{H^1(\Omega)}
\end{align*}
\]

(4.8)

Substituting (4.6) and (4.7) into (4.8), we get

\[
\begin{align*}
\| u_\perp \|_{H^1(\Omega)} + \| v_\perp \|_{H^1(\Omega)} &\leq \frac{C(N) \rho_3 \vartheta_1 + \rho_3 \vartheta_2 + \rho_3 \vartheta_1 + \rho_3 \vartheta_2 + \rho_4 \vartheta_2 + \rho_4 \vartheta_4 + \vartheta_6 + \vartheta_6 + \vartheta_5 + \vartheta_5 + \vartheta_4 + \vartheta_4 + 1) \| u_1 \|_{H^1(\Omega)} + \frac{1}{D_u} \| u_1 \|_{H^1(\Omega)} + \frac{1}{D_v} \| v_1 \|_{H^1(\Omega)}
\end{align*}
\]

that is,

\[
\| u_\perp \|_{H^1(\Omega)} + \| v_\perp \|_{H^1(\Omega)} \leq \frac{\hat{\kappa}}{1 - \kappa_1} (\| u_1 \|_{H^1(\Omega)} + \| v_1 \|_{H^1(\Omega)}).
\]

(4.9)

And also substituting (4.9) into (4.6) and (4.7), we have

\[
\begin{align*}
\| u \|_{H^1(\Omega)} + \| v \|_{H^1(\Omega)} &\leq \begin{pmatrix} \hat{\rho} \vartheta_1 + \rho_2 \vartheta_2 + \rho_3 \vartheta_1 + \rho_4 \vartheta_2 + \rho_4 \vartheta_2 + \rho_4 \vartheta_4 \end{pmatrix} \| u \|_{H^1(\Omega)} + \| v \|_{H^1(\Omega)} + \frac{1}{D_u} \| u_1 \|_{H^1(\Omega)} + \frac{1}{D_v} \| v_1 \|_{H^1(\Omega)} \\
&\leq \begin{pmatrix} \hat{\rho} \vartheta_1 + \rho_2 \vartheta_2 + \rho_3 \vartheta_1 + \rho_4 \vartheta_2 + \rho_4 \vartheta_2 + \rho_4 \vartheta_4 \end{pmatrix} \| u_1 \|_{H^1(\Omega)} + \frac{1}{D_u} \| u_1 \|_{H^1(\Omega)} \\
&\leq \begin{pmatrix} \hat{\rho} \vartheta_1 + \rho_2 \vartheta_2 + \rho_3 \vartheta_1 + \rho_4 \vartheta_2 + \rho_4 \vartheta_2 + \rho_4 \vartheta_4 \end{pmatrix} \| u_1 \|_{H^1(\Omega)} + \frac{1}{D_v} \| v_1 \|_{H^1(\Omega)}.
\end{align*}
\]

Then, we obtain

\[
\| u \|_{H^1(\Omega)} + \| v \|_{H^1(\Omega)} \leq \hat{\rho} (\| u_1 \|_{H^1(\Omega)} + \| v_1 \|_{H^1(\Omega)}).
\]
Therefore,
\[
\|u\|_{H^1(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2 \\
\leq (\|u_N\|_{H^1(\Omega)}^2 + \|v_N\|_{H^1(\Omega)}^2)^2 + (\|u_1\|_{H^1(\Omega)}^2 + \|v_1\|_{H^1(\Omega)}^2)^2 \\
\leq \rho^2 (\|u_1\|_{H^1(\Omega)}^2 + \|v_1\|_{H^1(\Omega)}^2)^2 + \left(\frac{\kappa}{1 - \kappa_1}\right)^2 (\|u_1\|_{H^1(\Omega)}^2 + \|v_1\|_{H^1(\Omega)}^2)^2 \\
\leq \left(\rho^2 + \left(\frac{\kappa}{1 - \kappa_1}\right)^2\right) (\|u_1\|_{H^1(\Omega)}^2 + \|v_1\|_{H^2(\Omega)}^2)
\]
holds, that is,
\[
\|u\|_{H^1(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2 \leq \tilde{M}_\rho^2 (\|u_1\|_{H^1(\Omega)}^2) (\|v_1\|_{H^1(\Omega)}^2). \quad \square
\]

4.3. **Invertibility of \( \hat{L} \).** In Theorem 4.3 there is an assumption that \( \hat{L} \) is invertible. Similarly to [6], instead of giving some invertible criterion for \( \hat{L} \) like [15], we prove that when \( \kappa_1 < 1 \), \( \hat{L} \) is invertible. In [6], the authors prove the index of a Fredholm operator is 0. However, it needs to get the index of some compact operator is 0. Sometimes, it is not easy to approach. Here we prove the bijective of \( \hat{L} \) a little differently. We recall a Riesz lemma.

**Lemma 4.4.** ([13], Theorem 2.25) Suppose \( X \) is a normed vector space, \( X_0 \) is a closed linear subspace of \( X \) and \( X_0 \neq X \), then for \( 0 < \epsilon < 1 \), there exists \( y \in X \) such that \( \|y\| = 1 \) and \( \|y - x\| \geq 1 - \epsilon \) for all \( x \in X_0 \).

Now we show the connection between injection and surjection of \( \hat{L} \). Here some similar technique in [2] is used.

**Theorem 4.5.** If \( N(\hat{L}) = \{0\} \), then \( R(\hat{L}) = X^1 \times X^1 \).

**Proof.** Suppose that \( R(\hat{L}) \neq X^1 \times X^1 \) holds. Set \( X_0 = X^1 \times X^1 \) and \( X_k = \hat{L}(X_{k-1}) \) for \( k = 1, 2, \ldots \). Since \( N(\hat{L}) = \{0\} \), that is, \( \hat{L} \) is injective, and also \( R(\hat{L}) = X^1 \neq X_0 \), we have
\[
X_0 \supset X_1 \supset X_2 \supset \cdots, X_0 \neq X_1 \neq X_2 \neq \cdots.
\]
Then by Lemma 4.4 there exists \( y_k \in X_k, \|y_k\|_{H^1(\Omega)} = 1 \), such that
\[
\text{dist}(y_k, X_{k+1}) = \inf\{\|y_k - x_{k+1}\| : x_{k+1} \in X_{k+1}\} \geq \frac{1}{2} (k = 1, 2, \ldots).
\]
Set \( L_{-1} : X^1 \times X^1 \to X^1 \times X^1 \) as
\[
L_{-1}(u, v) := \left( \begin{array}{c}
L_{-1}^{-1}\{ -f_1(u, v) \} \\
L_{-1}^{-1}\{ -f_2(u, v) \}
\end{array} \right).
\]
Since \( L_{-1}^{-1} \) maps \( X^1 \) to \( X^3 \) and \( X^3 \) to \( X^1 \) holds, \( L_{-1} \) is compact.

It is easily seen that \( \hat{L} y_n - \hat{L} y_{n+p} + \left( \begin{array}{c}
D_u \\
D_v
\end{array} \right) y_{n+p} \in X_{n+1} \).

And \( \hat{L} = \left( \begin{array}{c}
D_u \\
D_v
\end{array} \right) + L_{-1} \) holds, then for \( \forall p, n \in \mathbb{N} \), we have
\[
\|L_{-1} y_n - L_{-1} y_{n+p}\|_{H^1(\Omega)} \times H^1(\Omega) \\
= \left\| \left( \begin{array}{c}
D_u \\
D_v
\end{array} \right) y_n + L_{-1} y_{n+p} + \left( \begin{array}{c}
D_u \\
D_v
\end{array} \right) y_{n+p} \right\|_{H^1(\Omega)} \times H^1(\Omega)} \\
\geq \frac{1}{2}.
\]
This is a contradiction with \( L_{-1} \) is compact. \( \square \)

Thus, whenever \( \hat{L} \) is injective, \( \hat{L} \) is surjective. In order to obtain \( \hat{L} \) is injective, an inequality is required.

**Lemma 4.6.** Under Assumption [4.1] if \( \kappa_1 < 1 \), then we obtain
\[
\| (u, v) \|_{H^1(\Omega) \times H^1(\Omega)} \leq \bar{M}_\mu \| \hat{L}(u, v) \|_{H^1(\Omega) \times H^1(\Omega)}, \forall (u, v) \in X^1 \times X^1.
\]

**Proof.** For \( \forall (u, v) \in X^1 \times X^1 \), set \((u_1, v_1) := \hat{L}(u, v)\). Then \((u, v)\) is the solution of the following equations,
\[
\hat{L}(u, v) = \left( \begin{array}{c}
D_u u + L^{-1}\{-f_1(u, v)\} \\
D_v v + L^{-1}\{-f_2(u, v)\}
\end{array} \right) = \left( \begin{array}{c}
u_1 \\
v_1
\end{array} \right).
\]

By the same calculation in Theorem 4.3, we have
\[
\| (u, v) \|_{H^1(\Omega) \times H^1(\Omega)} \leq \bar{M}_\mu \| \hat{L}(u, v) \|_{H^1(\Omega) \times H^1(\Omega)}.
\]

Now we show the invertibility of \( \hat{L} \).

**Theorem 4.7.** Under the same assumption in Lemma 4.6, the inverse of \( \hat{L} \) exists. Especially, \((\hat{L})^{-1} \in \mathcal{L}(X^1 \times X^1, X^1 \times X^1)\).

**Proof.** For \( \forall (u, v) \in N(\hat{L}) \subset X^1 \times X^1, \hat{L}(u, v) = 0 \) holds.

Set
\[
D_{M_1} := \max \left\{ \frac{1}{D_u}, \frac{1}{D_v} \right\},
\]
\[
D_{M_2} := \max \{|D_u + \mu|, |D_v + \mu|\},
\]
\[
D_{M_3} := \max \{\|f_u(\hat{u}_N, \hat{v}_N)\|_{L^\infty(\Omega)} + \|g_u(\hat{u}_N, \hat{v}_N)\|_{L^\infty(\Omega)},
\]
\[
\|f_v(\hat{u}_N, \hat{v}_N)\|_{L^\infty(\Omega)} + \|g_v(\hat{u}_N, \hat{v}_N)\|_{L^\infty(\Omega)} \}.
\]

From Lemma 4.6, we have
\[
\| (u, v) \|_{H^1(\Omega) \times H^1(\Omega)} \leq \left\| \left( \begin{array}{c}
1 \\
\frac{1}{D_u}
\end{array} \right) \hat{L}(u, v) - \left( \begin{array}{c}
L^{-1}\{-f_1(u, v)\} \\
L^{-1}\{-f_2(u, v)\}
\end{array} \right) \right\|_{H^1(\Omega) \times H^1(\Omega)} \leq D_{M_1} \| \hat{L}(u, v) \|_{H^1(\Omega) \times H^1(\Omega)} + D_{M_1} \| (u, v) \|_{L^\infty(\Omega) \times L^\infty(\Omega)} \leq D_{M_1} \| (u, v) \|_{H^1(\Omega) \times H^1(\Omega)} + D_{M_1} \| (u, v) \|_{H^1(\Omega) \times H^1(\Omega)} \leq D_{M_1} + D_{M_2}M_\mu + D_{M_1}D_{M_2}M_\mu \| (u, v) \|_{H^1(\Omega) \times H^1(\Omega)} = 0.
\]

Therefore, \( N(\hat{L}) = \{0\} \) is satisfied. Then from Theorem 4.5, we have \( R(\hat{L}) = X^1 \times X^1 \), that is, \( \hat{L} \) is bijective. The proof is complete. \( \square \)

**4.4. Eigenvalue problem of the linearized operator at the solution.** Now Back to the eigenvalue problem of the operator linearized at the verified solution, (1.3) as follows
\[
\begin{align*}
-D_u \Delta u - f_u(u^*, v^*)u - f_v(u^*, v^*)v &= \lambda u, \\
-D_v \Delta v - g_u(u^*, v^*)u - g_v(u^*, v^*)v &= \lambda v.
\end{align*}
\]

The eigenvalue \( \lambda \) of the equation (1.3) can be written as
\[ \lambda = (D_u(\nabla u, \nabla u)_{L^2(\Omega)} - (f_u(u^*, v^*)u, u)_{L^2(\Omega)} - (f_v(u^*, v^*)v, u)_{L^2(\Omega)} \\
+ D_v(\nabla v, \nabla v)_{L^2(\Omega)} - (g_u(u^*, v^*)u, v)_{L^2(\Omega)} - (g_v(u^*, v^*)v, v)_{L^2(\Omega)}) \\
/((u, u)_{L^2(\Omega)} + (v, v)_{L^2(\Omega)}). \]

Therefore, we have some estimation for the real part and the imaginary part of \( \lambda \) as follows,

\[ Re(\lambda) \geq -\|f_u(u^*, v^*)\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2 - \|f_v(u^*, v^*)\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} - \|g_u(u^*, v^*)\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)}^2 \]

\[ - \|g_v(u^*, v^*)\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \]

\[ + (\|u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2) \]

\[ \geq -\|f_u(u^*, v^*)\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2 - 1/2(\|f_v(u^*, v^*)\|_{L^\infty(\Omega)} + \|g_u(u^*, v^*)\|_{L^\infty(\Omega)}) \]

\[ \cdot (\|u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2) \]

\[ - \|g_v(u^*, v^*)\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)}^2 /((\|u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2)) \]

Set \( C_\lambda := \max \{\|f_u(u^*, v^*)\|_{L^\infty(\Omega)}, 1/2(\|f_v(u^*, v^*)\|_{L^\infty(\Omega)} + \|g_u(u^*, v^*)\|_{L^\infty(\Omega)}), \|g_v(u^*, v^*)\|_{L^\infty(\Omega)}\} \), then we obtain

\[ Re(\lambda) \geq -C_\lambda. \]

And also we get

\[ Im(\lambda) \geq -C_\lambda. \]

It is clear that if \( \lambda \) is an eigenvalue of problem (1.3), then \( \bar{\lambda} \) (the conjugate number of \( \lambda \)) is also an eigenvalue of problem (1.3), therefore, from (4.10), we have

\[ Im(\lambda) \leq C_\lambda. \]

And thus, the eigenvalues of problem (1.3) are in the domain

\[ \{x + iy | x \geq -C_\lambda, |y| \leq C_\lambda\}. \]

In application, we do eigenvalue excluding in the domain

\[ \{x + iy | x \geq -C_\lambda, 0 \leq y \leq C_\lambda\}. \]

**Remark 4.8.** Recall that \( f \) and \( g \) here are polynomial functionals with respect to \( u \) and \( v \). Thus, by using the imbedding constant from \( L^\infty(\Omega) \) to \( H^1(\Omega) \) and the verification results, we can fix \( C_\lambda \) here. We will explain how to do that in Section 5.

Set

\[ L_\mu(u, v) := \left\{ \begin{array}{ll} D_u u + L^{-1}( - (D_u + \mu)u - f_u(u^*, v^*)u - f_v(u^*, v^*)v) \\
D_v v + L^{-1}( - (D_v + \mu)v - g_u(u^*, v^*)u - g_v(u^*, v^*)v) \end{array} \right\}. \]

Same as the proof of Theorem 4.1, we have the following theorem.

**Theorem 4.9.** Suppose that \( L_\mu \) has an inverse \( L_\mu^{-1} : X^1 \times X^1 \rightarrow X^1 \times X^1 \) and there exists \( M_\mu > 0 \) such that

\[ ||L_\mu^{-1}(u, v)||_{H^1(\Omega) \times H^1(\Omega)} \leq M_\mu \|(u, v)||_{H^1(\Omega) \times H^1(\Omega)}, \]

then there is no eigenvalue \( \tilde{\lambda} \) of Eq. (1.3) in the disk given by \( |\tilde{\lambda} - \mu| < \frac{1}{M_\mu} \).
Now we discuss how to get $M_{\mu}$ from $\hat{M}_{\mu}$ in Theorem \[4.1\] Note that

\[
\|L_\mu(u, v) - \hat{L}(u, v)\|^2_{H^1(\Omega) \times H^1(\Omega)} \\
= \|f_u(u^*, v^*)u - f_u(\hat{u}_N, \hat{v}_N)u + f_v(u^*, v^*)v - f_v(\hat{u}_N, \hat{v}_N)v\|^2_{L^2(\Omega)} \\
+ \|g_u(u^*, v^*)u - g_u(\hat{u}_N, \hat{v}_N)u + g_v(u^*, v^*)v - g_v(\hat{u}_N, \hat{v}_N)v\|^2_{L^2(\Omega)} \\
\leq 2(\|f_u(u^*, v^*) - f_u(\hat{u}_N, \hat{v}_N)\|_{L^\infty(\Omega)}^2 + \|g_u(u^*, v^*) - g_u(\hat{u}_N, \hat{v}_N)\|_{L^\infty(\Omega)}^2)\|u\|_{H^1(\Omega)}^2 \\
+ 2(\|g_u(u^*, v^*) - g_u(\hat{u}_N, \hat{v}_N)\|_{L^\infty(\Omega)}^2 + \|g_v(u^*, v^*) - g_v(\hat{u}_N, \hat{v}_N)\|_{L^\infty(\Omega)}^2)\|v\|_{H^1(\Omega)}^2.
\]

Hence, if we set $\varsigma := \max\{\|f_u(u^*, v^*) - f_u(\hat{u}_N, \hat{v}_N)\|_{L^\infty(\Omega)}, \|g_u(u^*, v^*) - g_u(\hat{u}_N, \hat{v}_N)\|_{L^\infty(\Omega)}, \|g_v(u^*, v^*) - g_v(\hat{u}_N, \hat{v}_N)\|_{L^\infty(\Omega)}\}$, then we have

\[
\|L_\mu(u, v) - \hat{L}(u, v)\|_{H^1(\Omega) \times H^1(\Omega)} \leq \sqrt{2}\varsigma\|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} \\
\leq \sqrt{2}\varsigma\sqrt{2(\|u\|_{H^1(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2)} = 2\sqrt{\varsigma}(\|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}).
\]

Remark 4.10. Same reason as Remark 4.9, $\varsigma$ above can be fixed. We will also explain it in Section 5.

And therefore, we obtain

\[
\|(u, v)\|_{H^1(\Omega) \times H^1(\Omega)} \leq \hat{M}_{\mu}\|L_\mu(u, v)\|_{H^1(\Omega) \times H^1(\Omega)} \\
\leq \hat{M}_{\mu}\|L_\mu(u, v)\|_{H^1(\Omega) \times H^1(\Omega)} + \|L_\mu(u, v) - \hat{L}(u, v)\|_{H^1(\Omega) \times H^1(\Omega)} \\
\leq \hat{M}_{\mu}\|L_\mu(u, v)\|_{H^1(\Omega) \times H^1(\Omega)} + 2\sqrt{\varsigma}(\|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}).
\]

If $1 - 2\sqrt{\varsigma}\hat{M}_{\mu} > 0$ holds, we have

\[
\|(u, v)\|_{H^1(\Omega) \times H^1(\Omega)} \leq \hat{M}_{\mu}\|L_\mu(u, v)\|_{H^1(\Omega) \times H^1(\Omega)},
\]

where $\hat{M}_{\mu} := \frac{\hat{M}_{\mu}}{1 - 2\sqrt{\varsigma}\hat{M}_{\mu}}$.

Thus, in a computer, we construct some disks given by $\lambda \in \mathbb{R} | |\lambda - \mu| < \frac{1}{\hat{M}_{\mu}}$ for some candidate points $\mu$ in the domain \[4.1\], then there is no eigenvalue in an area which is formed by those disks. If the parts in the left half plane of the domain \[4.1\] are contained in the area, then the verified solution is stable.

5. NUMERICAL RESULTS

In this section we apply our method to Schnakenberg model:

\[
\begin{aligned}
-\Delta u + \gamma(a - u + u^2v) &= 0, & \text{in} & & (0, l), \\
-\Delta v + \gamma(b - u^2v) &= 0, & \text{in} & & (0, l), \\
\frac{\partial u(0)}{\partial \nu} &= \frac{\partial u(l)}{\partial \nu} = \frac{\partial v(0)}{\partial \nu} &= \frac{\partial v(l)}{\partial \nu} = 0,
\end{aligned}
\]

where $d, \gamma, a, b > 0$, that is, $f(u, v) = \gamma(a - u + u^2v), g(u, v) = \gamma(b - u^2v), D_u = 1$, $D_v = d$.

Then the steady stationary solution of \[1.1\] becomes $u = a + b$ and $v = \frac{b}{(a + b)^2}$. According to (2.35) in [8], Turing instability could be observed only if $a, b, d$ satisfy the
following conditions:

\[
0 < b - a < (a + b)^3, \\
(a + b)^2 > 0, \\
d(b - a) > (a + b)^3, \\
d(b - a) - (a + b)^3 > 4d(a + b)^4.
\]

(5.1)

Now we prove \( f, g \) satisfy Assumption 4.2. Recall that

\[
f_1(u, v) = (1 + \mu)u + f_u(\hat{u}_N, \hat{v}_N)u + f_v(\hat{u}_N, \hat{v}_N)v,
\]

\[
f_2(u, v) = (d + \mu)v + g_u(\hat{u}_N, \hat{v}_N)u + g_v(\hat{u}_N, \hat{v}_N)v,
\]

then

\[
\|P_N(f_1(u_\perp, v_\perp))\|_{L^2(\Omega)} = \|P_N((1 + \mu)u_\perp + \gamma(-1 + 2\hat{u}_N\hat{v}_N)u_\perp + \gamma\hat{u}_N^2v_\perp)\|_{L^2(\Omega)} \\
\leq C(N)(2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_\perp\|_{H^1(\Omega)} + \gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}\|v_\perp\|_{H^1(\Omega)}),
\]

\[
\frac{1}{d}\|P_N(f_2(u_\perp, v_\perp))\|_{L^2(\Omega)} = \frac{1}{d}\|P_N(-2\gamma\hat{u}_N\hat{v}_Nu_\perp + (-\gamma\hat{u}^2_N + d + \mu)v_\perp)\|_{L^2(\Omega)} \\
\leq \frac{1}{d}C(N)(\gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}\|v_\perp\|_{H^1(\Omega)} + 2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_\perp\|_{H^1(\Omega)}),
\]

\[
\|P_N(f_1(u, v))\|_{L^2(\Omega)} = \|P_N((1 + \mu)u + \gamma(-1 + 2\hat{u}_N\hat{v}_N)u + \gamma\hat{u}_N^2v)\|_{L^2(\Omega)} \\
\leq |1 + \mu - \gamma|C(N)\|u_\perp\|_{H^1(\Omega)} + 2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_\perp\|_{L^2(\Omega)} + C(N)\|u_\perp\|_{H^1(\Omega)} \\
+ \gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}\|v_\perp\|_{L^2(\Omega)} + C(N)\|v_\perp\|_{H^1(\Omega)} \\
\leq 2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_\perp\|_{H^1(\Omega)} + \gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}\|v_\perp\|_{H^1(\Omega)} \\
+ C(N)\|1 + \mu - \gamma| + 2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_\perp\|_{H^1(\Omega)} + \gamma C(N)\|\hat{u}_N^2\|_{L^\infty(\Omega)}\|v_\perp\|_{H^1(\Omega)}
\]

and

\[
\frac{1}{d}\|P_N(f_2(u, v))\|_{L^2(\Omega)} = \frac{1}{d}\|P_N((d + \mu)v + \gamma\hat{u}_N\hat{v}_N(u_N + u_\perp) + \gamma\hat{u}_N^2(v_N + v_\perp))\|_{L^2(\Omega)} \\
\leq \frac{1}{d}(2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_N\|_{H^1(\Omega)} + \gamma\|\hat{u}_N\|_{L^\infty(\Omega)}\|v_N\|_{H^1(\Omega)} \\
+ 2C(N)\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}\|u_\perp\|_{H^1(\Omega)} + \gamma C(N)\|\hat{u}_N^2\|_{L^\infty(\Omega)} + |d + \mu|)\|v_\perp\|_{H^1(\Omega)}
\]

hold. Thus, if we set

\[
\nu_1 := C(N)\max\{2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}, \gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}\},
\]

\[
\nu_2 := \frac{\nu_1}{d},
\]

\[
\nu_3 := \max\{2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}, \gamma\|\hat{u}_N\|_{L^\infty(\Omega)}\},
\]

\[
\nu_4 := C(N)\max\{1 + \mu - \gamma, + 2\gamma\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}, \gamma\|\hat{u}_N^2\|_{L^\infty(\Omega)}\},
\]

\[
\nu_5 := \frac{\gamma}{d}\max\{2\|\hat{u}_N\hat{v}_N\|_{L^\infty(\Omega)}, \|\hat{u}_N^2\|_{L^\infty(\Omega)}\},
\]

\[
\nu_6 := \frac{\nu_1}{d}. \quad \text{and}\quad \nu_7 := \frac{\nu_3}{d}.
\]
\[ \nu_6 := \frac{C(N)}{d} \max \{2\gamma \| \hat{u}_N \hat{v}_N \|_{L^\infty(\Omega)}, |d + \mu| + \gamma \| \hat{u}_N^2 \|_{L^\infty(\Omega)} \}, \]

then \( f, g \) satisfy Assumption 4.2.

In the following, we explain how to get \( C_\lambda \) in (4.12) and \( \zeta \) in (4.14). Recall that the solution \( u^* = (u^*, v^*) \) of (1.2) can be enclosed as \( u^* = \hat{u}_N + \tilde{u} \) and \( v^* = \hat{v}_N + \tilde{v} \), where \( \hat{u}, \hat{v} \) are the residual part of \( \hat{u}_N \) and \( \hat{v}_N \) respectively. From the method in Section 3, we get the norm estimation \( \| \tilde{u} \|_{H^1(\Omega)} \leq \alpha_1 + \alpha_2 \) and \( \| \tilde{v} \|_{H^1(\Omega)} \leq \beta_1 + \beta_2 \).

Notice that

\[ \| f_u(u^*, v^*) \|_{L^\infty(\Omega)} = \| f_u(\hat{u}_N + \tilde{u}, \hat{v}_N + \tilde{v}) \|_{L^\infty(\Omega)} \]

\[ \leq \gamma \| \hat{u}_N \hat{v}_N - 1 \|_{L^\infty(\Omega)} + 2 \| \hat{u}_N \|_{L^\infty(\Omega)} \| \tilde{v} \|_{L^\infty(\Omega)} + 2 \| \hat{v}_N \|_{L^\infty(\Omega)} \| \tilde{u} \|_{L^\infty(\Omega)} \]

\[ \leq \gamma \| \hat{u}_N \hat{v}_N - 1 \|_{L^\infty(\Omega)} + 2K_\infty \| \hat{u}_N \|_{L^\infty(\Omega)} (\beta_1 + \beta_2) \]

\[ + 2K_\infty (\alpha_1 + \alpha_2) \| \tilde{v} \|_{L^\infty(\Omega)} + K_\infty^2 (\alpha_1 + \alpha_2)^2) =: C_\lambda_1, \]

\[ \frac{1}{2} \gamma \| f_u(u^*, v^*) \|_{L^\infty(\Omega)} + \| g_u(u^*, v^*) \|_{L^\infty(\Omega)} \]

\[ \leq \frac{3}{2} \gamma \| \hat{u}_N \|_{L^\infty(\Omega)} + 2 \| \hat{u}_N \|_{L^\infty(\Omega)} \lambda \infty(\alpha_1 + \alpha_2) + K_\infty^2 (\alpha_1 + \alpha_2)^2) =: C_\lambda_2, \]

\[ \| g_u(u^*, v^*) \|_{L^\infty(\Omega)} = \gamma \| \hat{u}_N + \tilde{u} \|^2_{L^\infty(\Omega)} \]

\[ \leq \gamma \| \hat{u}_N \|_{L^\infty(\Omega)} + 2K_\infty \| \hat{u}_N \|_{L^\infty(\Omega)} (\alpha_1 + \alpha_2) + K_\infty^2 (\alpha_1 + \alpha_2)^2) =: C_\lambda_3, \]

therefore, \( C_\lambda := \max \{C_\lambda_1, C_\lambda_2, C_\lambda_3 \}. \)

And

\[ \| f_u(u^*, v^*) - f_u(\hat{u}_N, \hat{v}_N) \|_{L^\infty(\Omega)} \]

\[ = \| f_u(\hat{u}_N + \tilde{u}, \hat{v}_N + \tilde{v}) - f_u(\hat{u}_N, \hat{v}_N) \|_{L^\infty(\Omega)} \]

\[ + \| g_u(\hat{u}_N + \tilde{u}, \hat{v}_N + \tilde{v}) - g_u(\hat{u}_N, \hat{v}_N) \|_{L^\infty(\Omega)} \]

\[ = \gamma^2 \| (-1 + 2(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v})) - (-1 + 2\hat{u}_N \hat{v}_N) \|_{L^\infty(\Omega)} \]

\[ + \| 2(\hat{u}_N + \tilde{u})(\hat{v}_N + \tilde{v}) - 2\hat{u}_N \hat{v}_N \|_{L^\infty(\Omega)} \]

\[ = 2 \gamma \| 2\hat{u}_N \hat{v}_N + 2\hat{u}_N \tilde{v} + 2\tilde{u} \hat{v}_N \|_{L^\infty(\Omega)} \]

\[ \leq 8 \gamma^2 (K_\infty^2 \| \hat{u}_N \|_{L^\infty(\Omega)} (\beta_1 + \beta_2)^2 + K_\infty^2 \| \hat{v}_N \|_{L^\infty(\Omega)} (\alpha_1 + \alpha_2)^2 \]

\[ + K_\infty^4 (\alpha_1 + \alpha_2)^2 (\beta_1 + \beta_2)^2) =: \zeta, \]

\[ \| f_u(u^*, v^*) - f_u(\hat{u}_N, \hat{v}_N) \|_{L^\infty(\Omega)} \]

\[ + \| g_u(u^*, v^*) - g_u(\hat{u}_N, \hat{v}_N) \|_{L^\infty(\Omega)} \]

\[ = \gamma^2 \| (\hat{u}_N + \tilde{u})^2 - \hat{u}^2 \|_{L^\infty(\Omega)} + \| (\hat{v}_N + \tilde{v})^2 - \hat{v}^2 \|_{L^\infty(\Omega)} \]

\[ \leq 2 \gamma \| (\hat{u}_N + \tilde{u})^2 - \hat{u}^2 \|_{L^\infty(\Omega)} \]

\[ + \| (\hat{v}_N + \tilde{v})^2 - \hat{v}^2 \|_{L^\infty(\Omega)} \]

hold, thus, \( \zeta := \max \{\zeta_1, \zeta_2 \}. \)

Then we made use of an interval arithmetic based on the interval library (17) to avoid the effects of rounding errors in the floating-point computations. The computations were carried out on a SONY VPCZ11AFJ(Intel(R) Core(TM) i5 M520 2.40GHz) using Matlab(Ver.7.5.0).
We apply our method to an example with some parameters satisfying (5.1) and get some numerical results. Here, \( a = 0.1, b = 0.9, d = 30, \gamma = 20, \Omega = (0, 1) \) and \( N = 200 \) are adopted. Then the steady stationary solution becomes \( u = 1 \) and \( v = 0.9 \). Now we choose initial value as \( u = 1 - \cos(\pi x) \) and \( v = 0.9 + \cos(\pi x) \). By using Newton’s method, we have an approximate solution in Figure 1.

![Figure 1](image)

**Figure 1.** The left one is \( \hat{u}_N \) and the right one is \( \hat{v}_N \).

From the verification method, we get the verification results for the stationary solution in Table 1.

| \( \alpha_1 \) | \( \beta_1 \) | \( \alpha_2 \) | \( \beta_2 \) |
|----------------|----------------|----------------|----------------|
| 4.555E-14      | 8.810E-15      | 2.771E-15      | 9.234E-17      |

**Table 1.** Verification results

For the eigenvalue problem (1.3), we have the following eigenvalue excluding results (Fig. 2) in the domain (4.12). The points are the approximate eigenvalues for problem (4.1) and the green domain is the domain (4.11), where \( C_\lambda = 39.2355 \). The red circles denote there is no eigenvalue inside them.

![Figure 2](image)

**Figure 2.** The eigenvalue excluding results. The right one is a zoom in of the left one.
From above excluding results, we see the eigenvalues of problem (1.3) are not in the left half plane, therefore, the verified stationary solution is stable.

**Remark 5.1.** From Figure 2, those red circles are a little far away from the approximate eigenvalues of (4.1). If we want, then we can obtain some better results. But the results now are sufficient to prove the stability of the solution.

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