Brownian motion of a charged particle in electromagnetic fluctuations at finite temperature

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Abstract

The fluctuation-dissipation theorem is a central theorem in nonequilibrium statistical mechanics by which the evolution of velocity fluctuations of the Brownian particle under a fluctuating environment is intimately related to its dissipative behavior. This can be illuminated in particular by an example of Brownian motion in an ohmic environment where the dissipative effect can be accounted for by the first-order time derivative of the position. Here we explore the dynamics of the Brownian particle coupled to a supraohmic environment by considering the motion of a charged particle interacting with the electromagnetic fluctuations at finite temperature. We also derive particle’s equation of motion, the Langevin equation, by minimizing the corresponding stochastic effective action, which is obtained with the method of Feynman-Vernon influence functional. The fluctuation-dissipation theorem is established from first principles. The backreaction on the charge is known in terms of electromagnetic self-force given by a third-order time derivative of the position, leading to the supraohmic dynamics. This self-force can be argued to be insignificant throughout the evolution when the charge barely moves. The stochastic force arising from the supraohmic environment is found to have both positive and negative correlations, and it drives the charge into a fluctuating motion. Although positive force correlations give rise to the growth of the velocity dispersion initially, its growth slows down when correlation turns negative, and finally halts, thus leading to the saturation of the velocity dispersion. The saturation mechanism in a supraohmic environment is found to be distinctly different from that in an ohmic environment. The comparison is discussed.

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I. INTRODUCTION

One of the fundamental problems in statistical mechanics concerns the microscopic origin of dissipation and relaxation of a nonequilibrium system in its course toward equilibrium with the thermal bath. It is known that the particle moving in a randomly fluctuating medium undergoes Brownian motion. A phenomenological but rather successful approach to study the evolution of Brownin motion is by means of the Langevin equation. This Langevin equation is a classical equation of motion by phenomenologically adding the terms that account for the effects of dissipation and fluctuations under the influence of a fluctuating environment. These two effects are ultimately responsible for evolving into thermodynamic equilibrium of the Brownian particle with the bath, and are thus related by the fluctuation-dissipation theorem (FDT). The FDT is one of the cornerstones of statistical mechanics [1]. The static version of the FDT [2] relates the response of a system in equilibrium to action of a small external perturbation. The result of it gives a relation between noise correlations and susceptibility in frequency domain and the proportionality constant depends on temperature. The extension of the static FDT to the situation away from equilibrium can be realized in terms of the Langevin equation. The Einstein relation, one of the manifestations of the dynamic FDT, links the effects between friction and noise correlations in the Langevin equation, and thus plays an essential role in stabilizing the dynamics of the Brownian particle. The most striking feature of this relation is that it can determine the irreversible evolution of the particle from the fluctuations correlation of the heat bath. Thus, the origin of this irreversibility of the Brownian dynamics can be attributed to the fluctuating nature of the environment.

A very clear microscopic description, leading to the Langevin equation, within the context of one-particle quantum mechanics has been presented by Caldeira and Leggett. They considered a specific system-environment model that the particle interacts with an environment composed of a infinite number of harmonic oscillators by linear coupling of the oscillator and particle coordinates [3]. If the quantum states of harmonic oscillators are thermally distributed, the relative importance between quantum and thermal fluctuations depends on the temperature under consideration. The effects of environmental degrees of freedom on the particle can be investigated with the method of Feynman-Vernon influence functional by integrating out environment variables within the context of the closed-time-
path formalism \[3, 4, 5, 6, 7\]. The more complicated interaction by considering nonlinear couplings of the particle coordinate can also be studied perturbatively in \[8\]. Under the classical approximation where the intrinsic quantum fluctuations of the particle is ignored, the Langevin equation can be obtained by minimizing the corresponding effective stochastic action.

At low temperatures, the particle is affected mainly by quantum fluctuations of the environment, and it will lead to a new phenomenon, the so-called quantum Brownian motion. Many experimentally accessible systems (such as dissipation in quantum tunneling) \[3\], and the problems of quantum measurement theory (such as quantum decoherence of the system due to the interaction with its environment) \[10\] share the similar dynamics as quantum Brownian motion. At high temperatures in which the thermal fluctuations dominate, the problem of Brownian motion is described by the classical dynamics introduced in the previous paragraph. The relativistic Brownian motion has also been discussed in \[9\].

Here we would like to stress that under the approach of Feynman-Vernon influence functional, the corresponding FDT can be derived from first principles for a given microscopic model in terms of the Green’s functions of environment variables. The FDT is a central theorem in nonequilibrium statistical mechanics by which the evolution of velocity fluctuations of the particle under a fluctuating environment is intimately related to its dissipative behavior. Thus, in this paper, we wish to explore the dynamics of the Brownian particle in the supraohmic environment where backreaction dissipation is governed by the term with high-order time derivatives of the position than the one (a first-order time derivative) in an ohmic case. The known example is the motion of the charged particle under the electromagnetic fluctuations at finite temperature. The non-uniform motion of the charge will emit radiation that backreacts on itself through the electromagnetic self-force given by a third-order time derivative of the position. The stochastic noise, which encodes the influence of quantum/thermal statistics of the fields, drives the charge into a fluctuating motion \[11, 15\]. It is also known that, in nonrelativistic motion, the dissipation term is significant only on an extremely short time scale, which is the time for light to travel across the classical radius of the charged particle, \(\approx 10^{-23}\)s for an electron. It is then of interest to find out the mechanism by which the velocity fluctuations saturate, when backreaction dissipation is negligible during the evolution, that is, when the particle barely moves in a supraohmic environment.

The dynamics of a charged particle coupled with the electromagnetic fields has been stud-
ied quantum-mechanically in the system-plus-environment approach. We treat the particle as the system of interest, and the degrees of freedom of fields as the environment. The effects of fields on the particle is then obtained by integrating out field variables [3, 4, 5, 6, 7]. In this approach, the decoherence phenomena of the charged particle under the influence of electromagnetic vacuum fluctuations in the presence of the conducting plate has been studied in [11, 12]. The evolution of charge’s velocity dispersion affected by quantum electromagnetic fields near a conducting plate [15] and from the electromagnetic squeezed vacuum [16] are also studied. In the latter works, we investigate the possibility of reducing the velocity dispersion of the charged particle by tuning the parameters in these quantum states. In [17], the authors have shown that if the particle does not move significantly such that the electromagnetic self-force can be ignored, then the velocity dispersion still reaches a constant value at asymptotical times. Thus, constrained by the uncertainty principle, it is quite reasonable that the particle cannot extract energy indefinitely from the vacuum state of environment fields. It results from the fact that the integral of the correlation function of stochastic forces, which will be defined later, over the whole time domain vanishes. Similar results are also obtained in Ref. [18]. As will be seen later, it will also lead to the saturation of velocity fluctuations of a Brownian particle under a supraohmic environment at finite temperature. This work is a follow-up from our earlier investigations [15, 16], where more details on derivation can be found.

The Lorentz-Heaviside units with \( \hbar = c = 1 \) will be adopted unless otherwise noted. The metric is \( \eta^{\mu \nu} = \text{diag}(+1, -1, -1, -1) \).

II. LANGEVIN EQUATIONS

The dynamics of a nonrelativistic point particle of charge \( e \) interacting with the quantized electromagnetic fields can be described by the Lagrangian,

\[
L[\mathbf{q}, \mathbf{A}_T] = \frac{1}{2}m \dot{\mathbf{q}}^2 - \frac{1}{2} \int d^3x \int d^3y \ \varrho(x; \mathbf{q})G(x, y)\varrho(y; \mathbf{q}) + \int d^3x \left[ \frac{1}{2}(\partial_\mu \mathbf{A}_T)^2 + \mathbf{j} \cdot \mathbf{A}_T \right],
\]

in terms of the transverse components of the gauge potential \( \mathbf{A}_T \), and the position \( \mathbf{q} \) of the point charge in the Coulomb gauge. The instantaneous Coulomb Green’s function \( G(x, y) \) satisfies the Gauss’s law. The charge and the current densities take the form

\[
\varrho(x; \mathbf{q}(t)) = e \delta^{(3)}(\mathbf{x} - \mathbf{q}(t)), \quad \mathbf{j}(x; \mathbf{q}(t)) = e \dot{\mathbf{q}}(t) \delta^{(3)}(\mathbf{x} - \mathbf{q}(t)). \tag{1}
\]
The density matrix of the particle-field system $\hat{\rho}(t)$ evolves unitarily according to

$$\hat{\rho}(t_f) = U(t_f, t_i) \hat{\rho}(t_i) U^{-1}(t_f, t_i)$$

with $U(t_f, t_i)$ the time evolution operator. We assume that the state of the particle-field at an initial time $t_i$ is factorizable as $\hat{\rho}(t_i) = \hat{\rho}_e(t_i) \otimes \hat{\rho}_{AT}(t_i)$, and that the electromagnetic fields are initially in thermal equilibrium at temperature $T = 1/\beta$ so that its density operator takes the form

$$\hat{\rho}_{AT}(t_i) = e^{-\beta H_{AT}} / \text{Tr} \{ e^{-\beta H_{AT}} \},$$

where $H_{AT}$ is the Hamiltonian of the free fields.

After integrating out the degrees of freedom of the fields, the Langevin equation is obtained [15],

$$m \dddot{q} + e^2 \left( \delta^{il} \frac{d}{dt} - \dot{q}_l(t) \nabla_i \right) \int_{-\infty}^{t} dt' \, G^{ij}_{R}[q(t), q(t'); t - t'] \, \dot{q}_j(t') = f_s^i(t),$$

where

$$f_s^i(t) = -\hbar e \left( \delta^{il} \frac{d}{dt} - \dot{q}_l(t) \nabla_i \right) \xi^i(t),$$

with the noise-noise correlation functions,

$$\langle \xi^i(t) \rangle = 0, \quad \langle \xi^i(t) \xi^j(t') \rangle = \frac{1}{\hbar} G^{ij}_{H}[q(t), q(t'); t - t'],$$

and

$$\hbar G^{ij}_{H}(x - x') = i \theta(t - t') \left\langle \left[ A^i_T(x), A^j_T(x') \right] \right\rangle,$$

$$\hbar G^{ij}_{R}(x - x') = \frac{1}{2} \left\langle \left\{ A^i_T(x), A^j_T(x') \right\} \right\rangle,$$

are the Green’s function of the electromagnetic potentials at finite temperature. It is seen that the influence of the electromagnetic fields are expressed by an integral of the dissipation kernel $G^{ij}_{R}$ over the past history of charge’s motion, and by a stochastic noise $\xi$ that drives the charge into a fluctuating motion. As it stands, Eq. (4) is a nonlinear Langevin equation with non-Markovian backreaction, and the noise depends in a complicated way on the charge’s trajectory because the noise correlation function itself is a functional of the trajectory.

The fluctuation and dissipation effects of the electromagnetic fields on the motion of the charged particle are associated with the kernels $G^{ij}_{H}$ and $G^{ij}_{R}$ respectively. They in turn are
linked by the fluctuation-dissipation relation where the Fourier transform of the fluctuation kernel $G^{ij}_H$ is related to the imaginary part of the retarded kernel $G^{ij}_R$ as follows [15]

$$G^{ij}_H[q(t), q(t'); \omega] = \text{Im} \left\{ G^{ij}_R[q(t), q(t'); \omega] \right\} \coth \left[ \frac{\beta \hbar \omega}{2} \right]. \quad (9)$$

This relation is established from first principles. The explicit expression of the kernels $G^{ij}_H$ and $G^{ij}_R$ will be introduced when they are needed later.

III. VELOCITY FLUCTUATIONS

The nonlinear, non-Markovian Langevin equations are far too complicated to proceed further without any approximation. The appropriate approximation for nonrelativistic motion is the dipole approximation, which amounts to considering the backreaction solely from the electric fields. The Langevin equation under the dipole approximation reduces to

$$m\ddot{q}^i(t) + e^2 \int_0^t dt' \, \dot{g}^{ii}_R(t-t') \, \dot{q}^i(t') = f_s^i(t), \quad (10)$$

from Eq. (4) and

$$f_s^i(t) = -\hbar e \dot{\xi}^i(t). \quad (11)$$

The retarded Green’s function in the dipole approximation is denoted by $g_R^{ij}$, and can be expressed in terms of the spectral density $\rho$ as

$$g^{ij}_R(\tau) = -\theta(\tau) \int_0^\infty \frac{dk}{\pi} \, \rho^{ij}(k) \sin(k\tau). \quad (12)$$

In the isotropic thermal bath, the spectral density takes a simple form [15]

$$\rho^{ij}(k) = -\frac{k}{3\pi} \delta^{ij}. \quad (13)$$

The accompanying noise-noise correlation functions due to the fluctuations of the electric fields at finite temperature are given by

$$\langle f_s^i(t) \rangle = 0, \quad \langle f_s^i(t) f_s^j(t') \rangle = \hbar e^2 \frac{\partial^2}{\partial t \partial t'} g^{ij}_H(t - t'), \quad (14)$$

with

$$g^{ij}_H(\tau) = -\int_0^\infty \frac{dk}{2\pi} \, k^2 \rho^{ij}(k) \coth \left[ \frac{\beta \hbar k}{2} \right] \cos(k\tau). \quad (15)$$
Again the noise kernel $g^{ij}_{H}$ can be seen related to the dissipation kernel $g^{ij}_{R}$ via a fluctuation-dissipation relation under the dipole approximation, derived from Eq. (9). After carrying out the integration in Eq. (10) the Langevin equation becomes physically more transparent,

$$m_r \ddot{q}^i(t) - \frac{e^2}{6\pi} q^i(t) = f_s^i(t).$$

The non-uniform motion of the charge results in radiation that backreacts on the charge itself through the electromagnetic self-force. This backreaction occurs at the moment when radiation is emitted \[13, 14, 15\]. Thus, it may lead to short-distance divergence in the coincidence limit due to the assumption of a point-like particle. This ultraviolet divergence must be regularized to have a finite and unambiguous result. The divergent part is absorbed by particle mass renormalization, $m_r = m + e^2 \Lambda / 3\pi^2$, where $\Lambda$ is the energy cutoff scale related to the inverse of the charge’s wavepacket width. It essentially quantifies the intrinsic uncertainty of the charged particle. Then the finite backreaction effect is given by the well-known result, a third-order time derivative of the position \[20\].

For a nonrelativistic particle, the introduced cutoff wavelength should be much larger than its Compton wavelength, namely $\Lambda^{-1} \gg \lambda_C = \hbar / m_r c^2$, and in turn much greater than the classical radius of the charged particle, $r_e = e^2 / m_r c$. Thus, when the time scale associated with the cutoff frequency, $t \sim 2\pi / \Lambda$, is much longer than the characteristic time scale $\tau_e \sim r_e / c$, an extremely short time scale, the electromagnetic self-force can be safely ignored. If we assume that the charged particle starts off from the rest at $t = 0$, then the solution to the Langevin equation (16) has a very simple form

$$v^i(t) = \frac{1}{m_r} \int_0^t ds f_s^i(s),$$

and the corresponding velocity dispersion is given by

$$\langle \Delta v^2_i(t) \rangle = \frac{1}{m_r^2} \int_0^t ds \int_0^t ds' \langle f_s^i(s) f_s^i(s') \rangle.$$ (18)

We see that the evolution of velocity fluctuations is governed by the force-force correction function, whose finite temperature contribution $\langle f_s^i(\tau) f_s^i(0) \rangle_\beta$ is given by

$$C_{FF}(\tau) = \langle f_s^i(\tau) f_s^i(0) \rangle_\beta$$

$$= \frac{2 \hbar e^2}{\pi^2 \beta^4 \hbar^4} \text{Re} \left\{ \zeta(4, 1 + i \frac{\tau}{\beta \hbar}) \right\},$$ (19)
FIG. 1: The correlation function $C_{FF}(\tau)$ normalized with $\frac{2}{\pi^2} \frac{\hbar^2 e^2}{\beta^4}$ is drawn as a function of $\tau$ in units of $\frac{1}{\beta \hbar}$.

where its vacuum part has been subtracted. $\zeta(n, z)$ is the $n$th derivative of the zeta function. The correlation function $C_{FF}(\tau)$ is plotted against the time difference $\tau = t - t'$, scaled by $\beta \hbar$ in Fig. 1. It shows that the stochastic noise tends to have positive correlation for small $\tau$, and then $C_{FF}(\tau)$ turns negative at $\tau \sim O(\beta)$. It means that if we measure the electric field at certain time and find out the field points to one particular direction, then the moment $\tau$ later, within the time difference of order $O(\beta)$, if we perform a similar measurement again at the same location, there is high probability that the direction of the electric field remains the same. However, when the time difference between measurements is greater than $O(\beta)$, we probably have the result that the electric field points to the opposite direction instead of the same direction. Another way of understanding its significance is to compute the mean power $\overline{P}_s(t)$ done by the stochastic force $f_s(t)$,

$$\overline{P}_s(t) = \langle f_s(t)v(t) \rangle_\beta = \frac{1}{m_r} \int_0^t ds \langle f_s(t)f_s(s) \rangle_\beta = \frac{1}{m_r} \int_0^t ds C_{FF}(t - s) \quad (20)$$

$$= \frac{1}{3\pi^2} \frac{\hbar^2 e^2}{\beta^3 \hbar^3} \text{Im} \left\{ \psi^{(2)}(1 + i \frac{t}{\beta \hbar}) \right\} , \quad (21)$$

where $\psi^{(n)}(z)$ is the $n$th derivative of the digamma function $\psi(z)$. Again here only the finite temperature contribution is considered. We see that the mean power is positive at early times where the stochastic force does positive work to the charge such that its velocity dispersion or kinetic energy increases with time. The power reaches the maximum value at the time when correlation turns negative. Later on, the negative correlation makes the work
FIG. 2: The time evolution of $P_s(t)$ and $\langle \Delta v_i^2(t) \rangle_\beta$ is plotted where the time $t$ is in units of $\frac{1}{\beta \hbar}$.

These quantities are normalized with $\frac{1}{3\pi^2 m_r \beta^3 \hbar^2}$ and $\frac{2}{3\pi^2 m_r^2 \beta^2 \hbar}$ respectively

done by the force less positive and then the mean power $P_s(t)$ eventually approaches zero,

$$P_s(t) = \frac{1}{3\pi^2 m_r \beta^3 \hbar^3} \left\{ \begin{array}{ll}
\frac{\pi^4}{6} \frac{t}{\beta \hbar} + O\left(\frac{\beta \hbar}{t}\right)^3, & t \ll \beta \hbar, \\
\frac{\beta^3 \hbar^3}{t^5} + O\left(\frac{\beta \hbar}{t}\right)^5, & t \gg \beta \hbar.
\end{array} \right. $$

Then the finite temperature part of the velocity dispersion $\langle \Delta v_i^2(t) \rangle_\beta$ can be obtained by

$$\langle \Delta v_i^2(t) \rangle_\beta = \frac{2}{m_r} \int_0^t ds P_s(s)$$

$$= \frac{2}{3\pi^2 m_r^2 \beta^2 \hbar^2} \left[ \frac{\pi^2}{6} - \text{Re} \psi^{(1)}(1 + \frac{t}{\beta \hbar}) \right]$$

$$= \frac{2}{3\pi^2 m_r^2 \beta^2 \hbar^2} \left\{ \begin{array}{ll}
\frac{\pi^4}{30} \frac{t^2}{\beta^2 \hbar^2} + O\left(\frac{\beta \hbar}{t}\right)^4, & t \ll \beta \hbar, \\
\frac{\pi^2}{6} + O\left(\frac{\beta \hbar}{t}\right)^2, & t \gg \beta \hbar.
\end{array} \right. $$

In Fig. 2 we show the time evolution of $P_s(t)$ and $\langle \Delta v_i^2(t) \rangle_\beta$ respectively. It is seen that the averaged power $P_s(t)$ given by stochastic forces on the charged particle is positive for all times. This implies that the thermal bath keeps pumping energy to the particle during its evolution and increases the velocity dispersion. The presence of the negative correlation slows down the rate of energy transfer, and finally leads to the vanishing $P_s$ at asymptotic
times where velocity fluctuations are saturated. It implies that the integral of $C_{FF}(\tau)$ over the whole time domain vanishes, i.e.

$$\int_0^\infty d\tau \ C_{FF}(\tau) = 0 = \int_{-\infty}^\infty d\tau \ C_{FF}(\tau).$$

(22)

Thus, the contribution from the positive correlation for the small $\tau$ regime exactly cancels that from the negative correlation for the larger $\tau$ regime. It leads to the saturation of the velocity dispersion even though the self force is insignificant in this supraohmic case.

To understand Eq. (22) and its consequence from a different aspect, we may rewrite the correlation function in terms of its Fourier transform

$$C_{FF}(\tau) = -\int_0^\infty \frac{dk}{2\pi} \ k^2 \rho^{ii}(k) \left( \frac{1}{e^{\beta \hbar k} - 1} \right) e^{-ik\tau} + c.c..$$

Then the integration of the correlation function over the whole time domain becomes

$$\int_0^\infty d\tau \ C_{FF}(\tau) = -\int_0^\infty dk \ k^2 \rho^{ii}(k) \left( \frac{1}{e^{\beta \hbar k} - 1} \right) \delta(k) \bigg|_{k=0} = -\frac{1}{2\beta \hbar} \lim_{k \to 0} k \rho^{ii}(k).$$

(23)

If $k \rho^{ii}(k)$ behaves like $k^{l-1}$ with $l - 1 > 0$, then Eq. (22) holds. Thus, whether the integral of the correlation function over the whole time domain vanishes or not relies on the behavior of $k \rho^{ii}(k)$ in the zero momentum limit, which in turn depends on the spacetime dimension of the system, the dispersion relation of the environment field, and the coupling between the particle and environment.

In electrodynamics, the dynamics of a charged particle is governed by a local, gauge invariant interaction with the electromagnetic fields. Under the prescription of minimal coupling, the transverse component of the vector potential couples to the charged current density, which is proportional to the time derivative of particle’s position. This derivative coupling gives rise to the fact that the electromagnetic self-force should be given by a third-order time derivative of the position, and it is called supraohmic dynamics [3, 7]. From Eq.(13), we see that $l = 3$ and hence Eq. (22) holds. In this supraohmic case, the saturation of velocity dispersion can be achieved due to negative force-force correlations even when the backreaction effect of the self-force is found negligible in the evolution.

This is in striking contrast with the Brownian motion under an ohmic environment, characterized by the dissipative backreaction of the first-order time derivative of particle’s
position. It involves the coordinate coupling of the particle with the environment $B$ \cite{7}, and then $l = 1$. The positive force-force correlation \cite{1} has been shown to drive the growth of the velocity dispersion linearly in time within a time scale shorter than the relaxation time. It is also found \cite{21} that the integration on the force-force correlation function for the whole time regime then has a nonzero value with $l = 1$, so the energy transferred from the environment to the particle during the whole evolution of particle’s motion never slows down to halt. Then the dissipative backreaction must be taken into account to counterbalance the effect from the force fluctuations in order to finally stabilize the value of the velocity dispersion. Moreover, from the aspect of energy balancing between dissipative and fluctuation backreactions, the stronger dissipation is expected to occur in the subohmic case with $l < 1$, because the energy transfer rate increases even faster at late times \cite{7}. Thus, the mechanisms to stabilize the velocity dispersion of a particle in a fluctuating subohmic, ohmic, or supraohmic environment are rather different.

IV. SUMMARY AND CONCLUDING REMARKS

In this paper, the evolution of velocity fluctuations of the Brownian particle in a supraohmic environment is studied by considering the motion of the charged particle coupled to electromagnetic fluctuations at finite temperature. The Langevin equation of the particle incorporates the effects of fluctuation and dissipation backreaction in a self-consistent manner. In particular, the backreaction in a form of the electromagnetic self-force on the charge is a third-order time derivative of the position in this supraohmic environment. On the other hand, the thermal fluctuations of the electromagnetic fields are manifested as stochastic noise. Its correlation tends to be positive at shorter time difference $\tau$, and then turns negative at $\tau \sim O(\beta)$. We show that the integration of the force-force correlation function over the whole time regime vanishes. Throughout the evolution, the self-force can be argued to be insignificant for the charge having no significant motion. Then, the positive correlation contributes to the growth of the velocity dispersion initially, its growth slows down when correlation becomes negative, and finally halts where the velocity dispersion reaches a constant at asymptotical times. It is a rather different saturation mechanism for the velocity dispersion of the Brownian particle in an ohmic environment.

Additionally, the saturated value of the kinetic energy of the charge due to the thermal
electromagnetic fluctuations at temperature $T = \beta^{-1}$ can be estimated by
\begin{equation}
\frac{1}{2} m_r \langle \Delta v_i^2(\infty) \rangle_\beta \sim \alpha \frac{k_B T}{m_r c^2} k_B T \sim 10^{-5} \left( \frac{k_B T}{\text{keV}} \right)^2 \text{keV},
\end{equation}
with the fine structure constant $\alpha = e^2/\hbar c$. The $\alpha$ dependence implies that the saturated value of the velocity dispersion in this supraohmic case relies on the coupling between the system and the environment. The issue on how the velocity dispersion of the Brownian particle is stabilized in the subohmic environment deserves further investigation.

Thus, it comes to no surprise that the above saturated value of the velocity dispersion roughly becomes the value given by the equipartition theorem for a thermodynamic system as long as $k_B T \sim m_r c^2/\alpha \sim \hbar c \tau_e^{-1}$ because in that limit, the temperature is the highest energy scale in the system. However, in order to correctly describe the system in such a limit, a relativistic/field-theoretic formalism is needed for appropriately describing the dynamics of the charged particle. This is beyond the scope of the current investigation.

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