Methodology to estimate the relative pressure field from noisy experimental velocity data

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Abstract. The determination of intravascular pressure fields is important to the characterization of cardiovascular pathology. We present a two-stage method that solves the inverse problem of estimating the relative pressure field from noisy velocity fields measured by phase contrast magnetic resonance imaging (PC-MRI) on an irregular domain with limited spatial resolution, and includes a filter for the experimental noise. For the pressure calculation, the Poisson pressure equation is solved by embedding the irregular flow domain into a regular domain. To lessen the propagation of the noise inherent to the velocity measurements, three filters – a median filter and two physics-based filters – are evaluated using a 2-D Couette flow. The two physics-based filters outperform the median filter for the estimation of the relative pressure field for realistic signal-to-noise ratios (SNR = 5 to 30). The most accurate pressure field results from a filter that applies in a least-squares sense three constraints simultaneously: consistency between measured and filtered velocity fields, divergence-free and additional smoothness conditions. This filter leads to a 5-fold gain in accuracy for the estimated relative pressure field compared to without noise filtering, in conditions consistent with PC-MRI of the carotid artery: SNR = 5, 20\times20 discretized flow domain (25\times25 computational domain).

1. Introduction

\textit{In vivo} intravascular pressure measurements is of importance to the characterization of cardiovascular pathology [1]. Catheterization allows for the direct albeit invasive measurement of pressure. Noninvasive imaging techniques such as computed tomography (CT) and magnetic resonance imaging (MRI) cannot provide direct pressure measurements. Unlike CT however, MRI does not use ionizing radiation. By using phase contrast (PC) MRI methods, it is possible to measure time-resolved 3-D velocity fields within the flow domain [2, 3]. Inverse problems can be posed that calculate relative pressure fields based on experimental velocity measurements. An approach relying on PC-MRI flow rate and vessel area data has been developed to calculate the 1-D relative pressure distribution along a vessel [4] and pressure-pulse waveforms [5]. Ebbers et al. [6] proposed to perform line-integrals of the pressure gradient measured by PC-MRI along streamlines. Iterative techniques have been developed to solve the Poisson pressure equation (PPE, see section 2.1) from MRI velocity maps and obtain a full-field pressure distribution [7]. These methods have been expanded to three dimensions [8], wherein a simple median filter was used to smooth the pressure gradient prior to integration. The PPE has also been used to determine pressure fields based on ultrafast CT images [9]. This method was expanded by
Ebbers et al. [10] to estimate time-varying 3-D pressure fields based on PC-MRI velocity data. Finally, velocity and geometry data based on PC-MRI have been used as boundary conditions for CFD simulations in order to generate relative pressure contours [11]. MRI acceleration measurements have also been used as a basis for pressure field estimations [12].

Experimental velocity measurements are inherently polluted by noise, which is amplified by the numerical differentiation required for most of the methods discussed above. One approach to mitigate these effects is to ignore terms with second-order derivatives, such as viscous terms in the Navier–Stokes equations (NSE), which can be justified in large vessels where the viscous layer is expected to be small based on the Womersley parameter [7]. If smaller vessels are to be investigated these high-order terms cannot be ignored. Tura et al. [13] proposed an iterative velocity field regularization technique based on a modified form of the NSE. Alternatively the experimental velocity field can be projected onto a space of divergence-free fields obtained either numerically [14, 15] or analytically [16], and both approaches have shown promising results.

Here, we propose a two-stage method that includes both a noise filtering algorithm for experimental PC-MRI velocity data and the estimation of the relative pressure field using the technique in [9]. The propagation of the experimental noise into three potential filtering algorithms and the subsequent estimation of the relative pressure field will be investigated using a mathematical phantom to establish the robustness and accuracy of our proposed method. Potential applications of the method include the estimation of the intravascular relative pressure field in the region of the interior-exterior bifurcation of the human carotid artery. Therefore, our validation will be carried out with realistic spatial resolutions and signal-to-noise ratios (SNR) for this particular application.

2. Methods
2.1. Theory
In a fluid flow, the relationship between the pressure field, $P$, and the fluid velocity field, $V$, is described by the NSE. For an incompressible Newtonian fluid in a steady laminar flow regime, the NSE can be written as:

$$\nabla P = -\rho (V \cdot \nabla V) + \mu \nabla^2 V + F$$  \hspace{1cm} (1)

where $\rho$ and $\mu$ are the density and dynamic viscosity of the fluid, respectively, and $F$ is the body force term [17]. In addition, the continuity equation needs to be satisfied:

$$\nabla \cdot V = 0.$$  \hspace{1cm} (2)

We will assume that $F$ can be ignored. If the velocity field is known, the pressure field can be solved for by integrating (1) along a path. However, because of the noise inherently present in experimental data, the resulting pressure field depends on the integration path.

In general, the method used to determine the pressure field from a velocity field is to solve the PPE which is derived by taking the divergence of (1)

$$\nabla^2 P = \nabla \cdot b$$  \hspace{1cm} (3)

where $b$ is the right-hand-side of (1). Neumann boundary conditions are prescribed for the PPE

$$\nabla P \cdot \hat{n} = b \cdot \hat{n}$$  \hspace{1cm} (4)

where $\hat{n}$ is the unit outward normal on the boundary [18]. The solution to (3) is unique up to an integration constant. The PPE has also be shown by Song et al. [9] to be a least-squares solution to (1) for cases where the system is inconsistent or overdetermined.
2.2. Numerical implementation

In the following, a 2-D domain is assumed for simplicity, and extension to 3-D is straightforward. On a rectangular domain $\Omega$, the PPE with the prescribed boundary conditions on $\partial\Omega$, can be solved directly in several ways [19, 20]. For most in vivo applications however, the flow domain $\Omega_F$ is irregular (e.g., nonconvex). Direct solutions for the Poisson equation on irregular domains exist by embedding $\Omega_F$ in an extended rectangular domain, $\Omega \supset \Omega_F$, on which the Poisson equation is solved [21]. Depending on the size of $\Omega_F$ and $\partial\Omega_F$, this technique may be inefficient as well as difficult to implement as it requires the careful accounting of points on $\partial\Omega$ for the proper application of the boundary conditions. We choose to use the alternative iterative approach proposed by Song et al. [9], which also uses embedding in an extended rectangular domain, but implements the boundary conditions in simpler manner. The value of $b$ outside the flow domain (i.e., on $\Omega \setminus \Omega_F$) is updated after each iteration (indexed by $k$) to the value of $\nabla P$ from the previous iteration:

\[
\nabla^2 P^{(k)} = \nabla \cdot b \quad \text{on } \Omega \\
\nabla P^{(k)} \cdot \hat{n} = b^{(k)} \cdot \hat{n} \quad \text{on } \partial\Omega
\]

where

\[
b^{(k+1)} = \begin{cases} 
    b_F & \text{on } \Omega \\
    \nabla P^{(k)} & \text{on } \Omega \setminus \Omega_F
\end{cases}
\]

The initial guess for the pressure field, $P^{(0)}$, is zero everywhere. In this method, the Neumann boundary conditions are applied on $\partial\Omega$. By choosing $\Omega$ to be rectangular, the calculation of $\hat{n}$ is trivial, which simplifies the application of the boundary conditions.

At each iteration, second-order finite difference schemes are used to approximate all the derivatives necessary to compute $b$ and its divergence. The pressure field in (5) is then solved for by using a direct method based on Fourier transforms [22]. The solution has converged when the absolute rate of change between iterations is less than $10^{-3}$, which typically takes about 80 iterations. Since pressure only appears as a gradient in equation (1), the converged solution is only accurate to an integration constant. Thus, the spatial average of the pressure field is subtracted from the pressure solution, yielding a relative pressure field, $P_{\text{num}}$ [9].

2.3. Mathematical phantom

To test the technique, a mathematical phantom of 2-D Couette flow between rotating cylinders is used. This flow provides a nontrivial flow with a analytical solution for both the pressure and velocity fields. It can be shown that the Cartesian velocity field $(u, v)$ in polar coordinates $(r, \theta)$ is given by:

\[
\begin{align*}
    u(r, \theta) &= -\left( A r + \frac{B}{r} \right) \sin \theta \\
    v(r, \theta) &= \left( A r + \frac{B}{r} \right) \cos \theta
\end{align*}
\]

where $A$ and $B$ are constants that depend on the angular velocity and radii of the two cylinders [17]. An analytical solution for the relative pressure field is similarly obtained:

\[
P(r) = \rho \left( \frac{A^2 r^2}{2} + 2 A B \ln r - \frac{B^2}{2r^2} + C \right)
\]

where $C$ is an integration constant, such that $P$ is the relative pressure field as defined earlier.
2.4. Proposed noise filters

In an effort to reduce the propagation of the noise present in the experimental velocity field into the computation of the relative pressure field, three filters are tested. The first choice is a median filter with radius 1 pixel, which has the benefits of being edge-preserving and easy to implement [23]. The velocity at each point is replaced by the median of its immediate neighbors, resulting in a new velocity field \( \mathbf{V}_{\text{med}} \). This reduces the effect of spurious noise peaks in the data at the cost of spatial resolution.

A physics-based approach is used for the development of two additional filters. At this stage, the continuity equation (2) has not been explicitly enforced. Our first proposed filter imposes (2) in a least-squares sense on \( \Omega_F \). An additional penalty function that minimizes the L_2 norm of the difference between the experimental (\( \mathbf{V}_{\text{exp}} \)) and filtered (\( \mathbf{V} \)) velocity fields is added, thus ensuring that the filtered field is still physically relevant. The function to be minimized is

\[
 f_1 := |\nabla \cdot \mathbf{V}| + \alpha_1 \| \mathbf{V} - \mathbf{V}_{\text{exp}} \|
\]

where \( \alpha_1 \) is a Lagrange multiplier to be calibrated (see section 3.2).

Our Filter 2 minimizes the value of the L_2 norm of the Laplacian in addition to the terms for Filter 1 in equation (9) in order to decrease the noise pollution on the computation of the second derivatives in the viscous terms of the NSE (1). Alternatively, a viscous energy norm can be used for regularization [16]. Thus, the cost function for Filter 2 is

\[
 f_2 := |\nabla \cdot \mathbf{V}| + \alpha_2 \| \mathbf{V} - \mathbf{V}_{\text{exp}} \| + \gamma_2 \| \nabla^2 \mathbf{V} \|
\]

where \( \alpha_2 \) and \( \gamma_2 \) are Lagrange multipliers to be calibrated (see section 3.2). The functions \( f_1 \) and \( f_2 \) for the proposed Filter 1 and 2 are minimized by using a built-in unconstrained nonlinear optimization algorithm in MatLab (The MathWorks, Natick, MA).

2.5. Numerical validation procedure

The relative pressure field estimation technique by [9] depends on spatial resolution and the strategy used to embed the flow domain \( \Omega_F \) into a regular domain \( \Omega \). Therefore, several numerical tests – not available in [9] or elsewhere to the authors’ knowledge – are necessary for validation, using noise-free velocity data as inputs. We define the embedding level \( \Gamma \) to be the percentage increase between the characteristic length scale of \( \Omega \) and that of \( \Omega_F \). First, discrete velocity fields are generated at several spatial resolutions to test the effect of spatial resolution on the estimation of the relative pressure field. For these tests, the velocity field is embedded in a square domain with \( \Gamma \) kept constant. Second, the effect of embedding on the accuracy of the estimated relative pressure field is assessed using the velocity field calculated at a single resolution and increasing \( \Gamma \). Third, the effect of off-center embedding is gauged using two cases. In case 1, \( \Omega_F \) is offset to one side of \( \Omega \) but centered in the other direction. In case 2, \( \Omega_F \) is offset towards one corner of \( \Omega \). The rationale for this last test is that in a 3-D blood vessel, the flow domain in one slice is connected geometrically with the flow domain in the slices above and below, and the extended regular domain \( \Omega \) is a parallelepiped that contains all of the flow domains for each slice along the blood vessel. Therefore, at each cross-section, the 2-D flow domain \( \Omega_F \) is not always be embedded in the center of the regular domain \( \Omega \).

The size of \( \Omega_F \) for the investigation of the experimental noise propagation is selected based on a typical in-plane spatial resolution used for PC-MRI (0.5 × 0.5 mm^2) of the human carotid artery (of diameter \( \approx 10 \) mm) resulting in a 20 × 20 flow domain. Gaussian noise is added with the SNR defined to be the ratio of the root-mean-square (RMS) of the difference between the analytical and noisy experimental velocity fields to the RMS of the analytical velocity field. We define the normalized RMS error, \( \bar{\varepsilon} \), of our numerical solution for the relative pressure field, \( P_{\text{num}}(x, y) \), to be the RMS of the difference between \( P_{\text{num}}(x, y) \) and the analytical relative pressure field,
\( P(x, y) \), normalized by the RMS of \( P(x, y) \), denoted by \( P_{\text{RMS}} \). The local normalized error is defined as 
\[ \varepsilon(x, y) := \frac{(P_{\text{num}} - P)}{P_{\text{RMS}}} \]
All RMS values are computed over the flow domain \( \Omega_F \).

The influence of the SNR of the experimental velocity data and the calibration and performance of the three proposed filters on the estimated relative pressure field are investigated.

### 3. Results

In this section, the effect of the parameters of the relative pressure field estimation and the influence of the experimental noise filters are investigated. The practical implementation of our proposed two-stage method is then discussed.

#### 3.1. Influence of spatial resolution and embedding strategy

The examination of the influence of the spatial resolution on the normalized RMS error, \( \bar{\varepsilon} \), reveals that the relative pressure field estimation is second-order accurate, as shown in figure 1(a). More importantly, \( \bar{\varepsilon} \approx 1\% \) for the lowest resolution expected for MRI studies of the carotid artery. The effect of embedding \( \Omega_F \) in increasingly larger \( \Omega \) is shown in figure 1(b). If \( \Omega_F \) is embedded in the center of \( \Omega \), \( \bar{\varepsilon} \) decays monotonically towards an asymptotic value of 0.8% with increasing embedding levels. In addition, figure 1(b) indicates that there is an error minimum \( (\varepsilon \approx 0.9\%) \) at an embedding level of approximately 45% when \( \Omega_F \) is located in the corner of \( \Omega \) and at about 110% for the case when \( \Omega_F \) is located off to one side of \( \Omega \). All in all, the effects of the embedding are relatively negligible \( (\varepsilon < 1\%) \) provided the embedding level is greater than 20%.

**Figure 1.** Plot of the normalized RMS error, \( \varepsilon \), of the estimated relative pressure field for Couette flow as a function of (a) the number of pixels \( n_{\text{pixels}} \) along one side of the computational domain, and (b) the embedding level \( \Gamma \) with a \( 20 \times 20 \) flow domain \( \Omega_F \).

#### 3.2. Calibration and performance of the proposed noise filters

The Lagrange multipliers used for Filter 1 \( (\alpha_1) \) and Filter 2 \( (\alpha_2 \text{ and } \gamma_2) \) are calibrated by optimizing the results for the worst noise conditions considered here \( (\text{SNR} = 5) \). The normalized RMS error for the estimated relative pressure field, \( \bar{\varepsilon} \), is plotted against \( \alpha_{1,2} := \alpha_1,2 \| \nabla \cdot V_{\text{exp}} \| / \| V_{\text{med}} - V_{\text{exp}} \| \) and \( \gamma_2 \) in figure 2(a,b). The optimal performance is obtained for \( \alpha_1 = 0.1 \) and \( \alpha_2 = \gamma_2 = 0.01 \), which are the conditions used henceforth.

The normalized RMS error, \( \bar{\varepsilon} \), obtained before and after filtering is plotted against SNR in figure 2(a) with a \( 20 \times 20 \) flow domain \( \Omega_F \) embedded in a \( 25 \times 25 \) computational domain \( \Omega \). The embedding level \( \Gamma = 25\% \) is deemed sufficient in view of the results presented in section 3.1. Without filtering, \( \bar{\varepsilon} \) exceeds the expected error \( (1/\text{SNR}) \) by less than 5%, so the PPE solver does not overly amplify the experimental noise. For all three filters, \( \bar{\varepsilon} \) decreases with increasing SNR,
as expected. Moreover, our two physics-based filters outperform the median filter over the full range of SNR values tested. For Filter 1, $\bar{\varepsilon}$ decreases from an average of 15.3% at SNR = 5 to a 5% at SNR = 30. For Filter 2, $\bar{\varepsilon}$ goes from 5.3% to 2% for the same SNR range. It is worth noting that the median filter becomes counterproductive for SNR > 10.

![Filter 1 calibration](image1)

(a) Filter 1 calibration

![Filter 2 calibration](image2)

(b) Filter 2 calibration

![Influence of SNR](image3)

(c) Influence of SNR

**Figure 2.** Plot of the normalized RMS error, $\bar{\varepsilon}$, of the estimated relative pressure field for Couette flow using a $20 \times 20$ flow domain $\Omega_F$, embedded in a $25 \times 25$ computational domain $\Omega$ (i.e., embedding level $\Gamma = 25\%$), as a function of: (a) $\alpha'_1$ for Filter 1; (b) $\alpha'_2$ and $\gamma_2$ for Filter 2; (c) SNR. The results shown in (a) and (b) indicate that optimal performance is obtained when $\alpha'_1 = 0.1$ for Filter 1, and when $\alpha'_2 = \gamma_2 = 0.01$ for Filter 2.

The dependence of estimated relative pressure field, $P_{\text{num}}(x, y)$, on the filtering of the noisy experimental velocity data with SNR = 5 is illustrated in figure 3. The analytical and noisy velocity vector fields are plotted on the upper half of $\Omega$ in figure 3(a). The analytical relative pressure field $P(x, y)$ is represented pixel-wise in figure 3(b). Contour plots of the normalized error distribution, $\varepsilon(x, y)$, are plotted when no filtering is applied in figure 4(c), and then for the median filter in figure 4(d), Filter 1 in figure 4(e), and Filter 2 in figure 4(f). The improved performance of Filter 2 is demonstrated by the clearly narrower range of values centered around zero for $\varepsilon(x, y)$ and the significantly lower relative RMS error ($\bar{\varepsilon}$) as compared to the results for the other filters (by factor of 2.7 w.r.t. Filter 1 and factor of 3.64 w.r.t. the median filter).

### 3.3. Discussion

Our investigation of the influence of the spatial resolution and embedding level $\Gamma$ on the estimation of the relative pressure field from noisy velocity measurements demonstrate that the PPE solution technique proposed by Song et al. [9] is capable of generating accurate relative pressure fields at spatial resolutions that are consistent with PC-MRI of small blood vessels. The accuracy of the solution is affected by the value for $\Gamma$, especially when $\Omega_F$ is not embedded at the center of $\Omega$, and there exists optimal embedding levels that provide more accurate solutions.
From a practical aspect however, these optimal embedding levels may require an excessively large computational domain $\Omega$, which then requires longer computational times. From a practical viewpoint, normalized RMS errors of the estimated relative pressure field on the order of 1% or less ($\bar{\varepsilon} \leq 1\%$) are achievable for realistic in-plane resolutions and embedding levels $\Gamma > 20\%$.

**Figure 3.** Numerical results for Couette flow: pixel-wise representation of the (a) analytical and noisy (SNR = 5) velocity fields for the upper half of $\Omega_F$; (b) analytical relative pressure field normalized by its RMS over $\Omega_F$, i.e. $P(x,y)/P_{RMS}$. The embedding level is set at $\Gamma = 25\%$. Panels (c–f) show the contour plots of the normalized error distribution, $\varepsilon(x,y)$, for the estimated relative pressure fields obtained using (d) no filter; (e) a median filter with radius 1 pixel; (f) Filter 1; and (g) Filter 2 on the noisy velocity field. The normalized RMS error, $\bar{\varepsilon}$, is specified for each case.

For PC-MRI, realistic RMS noise levels are about 10% to 20%, corresponding to SNR = 5 to 10. In figure 2(c), the curves for $\bar{\varepsilon}$ vs. SNR for no filter and the median filter cross at SNR = 10, which reveals that the median filter actually amplifies the propagation of the experimental noise into the estimated relative pressure field for SNR > 10. By implementing two physics-based filters on the experimental velocity field with a normalized RMS noise of 20%, the normalized RMS error in the estimated relative pressure field is reduced from $\bar{\varepsilon} = 23.6\%$ to 14.3% by Filter 1 and down to only 5% by Filter 2, which is quite encouraging. Filter 1 requires less computational time than Filter 2: 30 min compared to 75 to 90 min on a Dell Optiplex 755 with a 2.99 GHz
processor, 1.96 GB of RAM, cache speed of 3 GHz, and cache size of 4 MB (Dell Computers, Round Rock, TX). Thus, the significant gain in accuracy provided by Filter 2 vs. Filter 1 is inversely proportional to their relative computational time, which is an expected trade-off.

4. Conclusions
We propose a method to estimate the relative pressure field from noisy velocity field data acquired in an irregular flow domain with in-plane spatial resolutions consistent with PC-MRI protocols. Our two-stage method solves the inverse problem represented by the PPE by embedding the flow domain into a regular computational domain, and includes a noise filtering algorithm to alleviate the adverse effect of the inherent experimental noise on the estimation of the relative pressure field. The analysis of the effect of spatial resolution on the estimated relative pressure field for Couette flow indicates that accurate solutions (normalized RMS error $\bar{\varepsilon}$ less than 1%) can be generated for spatial resolutions and SNR that are realistically achievable with PC-MRI, with embedding levels above 20%. The two physics-based filters proposed here impose in a least-squares sense the following constraints: consistency with the velocity measurements (Filter 1 and 2), divergence-free velocity field (Filter 1 and 2) and additional smoothness (Filter 2). Both physics-based filters significantly dampen the propagation of the experimental noise present in the velocity measurements (e.g., 20%) into the estimation of the relative pressure field (e.g., $\bar{\varepsilon}$ reduced by a factor of 1.65 and 4.45, respectively) and appreciably outperform median filtering (e.g., $\bar{\varepsilon}$ reduced by a factor of 1.35 and 3.64, respectively). The relative performance of Filter 1 and Filter 2 scales inversely with their relative computational times. Further work includes the extension to 3-D and time-varying flows for an ongoing PC-MRI study of atherosclerosis in the human carotid artery conducted at the Biomedical Imaging Research Center at Michigan State University.

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