Research Article

Structural Stability for a Coupled System of Wave Plate Type

Jincheng Shi,1 Ze Wang,2 and Yuanlong Chen2

1College of Data Science, Huashang College Guangdong University of Finance & Economics, Huashang Road, Guangzhou 511300, China
2Department of Applied Mathematics, Guangdong University of Finance, Yingfu Road, Guangzhou 510521, China

Correspondence should be addressed to Yuanlong Chen; 26-022@gduf.edu.cn

Received 6 June 2020; Revised 2 July 2020; Accepted 8 July 2020; Published 4 August 2020

Academic Editor: Rigoberto Medina

Copyright ©2020 Jincheng Shi et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper focuses on the spatial properties of a coupled system of wave plate type in a two-dimensional pipe. Using the technique of differential inequality and the method of energy estimation, the effect of the coefficient λ is established.

1. Introduction

The research on the structural stability of various types of partial differential equations has always been the focus of current research and received long-term attention. One can refer to the book of Ames and Straughan [1] and the monograph of Straughan [2]. For more papers, one can also see [3–5] and the papers listed therein. The so-called structural stability is to study the continuous dependence and convergence of the model on the coefficients. In the process of model building, simplification, and numerical calculation, some errors will inevitably appear. And these errors will not be completely avoided with the progress of measurement methods like errors. Therefore, it is necessary to study the influence of these errors on the solution of the equation. Recently, there are many papers that study the spatial stability for the fluid flow in porous medium (one could see [6–9]).

In the field of differential equations, biharmonic equation represents a lot of physical models and has important applications in elastic mechanics and porous media. For the biharmonic equation on the two-dimensional pipe line, there are many articles studying the spatial properties of the solution and accumulating a lot of methods. For more specific results, readers can refer to Knowles [10, 11], Flavin [12], Flavin and Knops [13], Horgan [14], Payne and Scafeer [15], and Varlamov [16]. More references on structural stability can be found in literatures [17, 18]. Lin [19] and Knops and Lupoli [20] studied some time-dependent problems which involved the biharmonic operator. They transformed the biharmonic operator into a fourth-order differential equation and studied the spatial attenuation estimation along the semi-infinite pipeline. Lin’s study [19] was improved by Song [21, 22] who studied the time-dependent Stokes flow. For more articles on the spatial behavior, see [23–27]. For the viscoelasticity equations, there are some recent contributions [28–31].

For a review of new contributions about practical application of plates of waves systems, one could see [32–34].

The problems of the present paper will defined in an unbounded region which is denoted by Ω0:

\[ \Omega_0 := \{(x_1, x_2) \mid x_1 > 0, 0 < x_2 < h\}, \]

where \( h > 0 \) is a known constant. We also let

\[ L_z := \{(x_1, x_2) \mid x_1 = z \geq 0, 0 \leq x_2 \leq h\}. \]

The time interval of our problems is denoted as \([0, T]\), where \( T > 0 \).

We note that Tang et al. [35] considered the spatial behavior of the following equations:

\[ \rho_1 u_{tt} - \Delta u - \mu \Delta u_x + a \Delta v = 0, \]

\[ \rho_2 v_{tt} + \gamma \Delta^2 v + a \Delta u - \lambda \Delta v_x = 0. \]
The model is mainly used to describe the evolution of the system composed of elastic film and elastic plate. The system is subject to elastic force, which attracts the film to the plate with coefficient of $\alpha$, and is affected by the thermal effect (see [36, 37]). In (3) and (4), $u$ and $v$ denote the vertical deflections of the membrane and of the plate, respectively. The constants $\rho_1$, $\rho_2$, $\mu$, $\alpha$, $\gamma$, $m$, $r$, and $k$ are nonnegative.

In the boundary of $\Omega_0$ and the initial time, the solutions satisfy

$$u(x_1, 0, t) = v(x_1, 0, t) = v_2(x_1, 0, t) = 0, \quad x_1 > 0, t > 0,$$

(5)

$$u(x_1, h, t) = v(x_1, h, t) = v_2(x_1, h, t) = 0, \quad x_1 > 0, t > 0,$$

(6)

$$u(0, x_2, t) = g_1(x_2, t), \quad 0 \leq x_2 \leq h, t > 0,$$

(7)

$$v(0, x_2, t) = g_2(x_2, t), \quad 0 \leq x_2 \leq h, t > 0,$$

(8)

$$v_1(0, x_2, t) = g_3(x_2, t), \quad 0 \leq x_2 \leq h, t > 0,$$

(9)

$$u(x_1, x_2, 0) = v(x_1, x_2, 0) = u_{\xi}(x_1, x_2, 0) = \phi_\xi(x_1, x_2, 0) = 0, \quad 0 \leq x_2 \leq h, x_1 > 0,$$

(10)

where the symbol $\Delta$ is the harmonic operator and $\Delta^2$ is the biharmonic operator.

The present paper will consider the classical solutions to the problem (3)–(10). $g_1, g_2, g_3$ are known functions which satisfy the compatibility:

$$g_1(0, t) = g_1(h, t) = g_{12}(0, t) = g_{12}(h, t) = 0,$$

$$g_2(0, t) = g_2(h, t) = g_{22}(0, t) = g_{22}(h, t) = 0,$$

$$g_3(0, t) = g_3(h, t) = g_{32}(0, t) = g_{32}(h, t) = 0,$$

$$g_1(x_2, 0) = g_2(x_2, 0) = g_3(x_2, 0) = 0.$$

(11)

We shall frequently make use of the following Poincare inequality:

$$\int_0^h u^2 \, d\xi \leq \frac{h^2}{\pi^2} \int_0^h \left( \frac{du}{d\xi} \right)^2 \, d\xi,$$

(12)

where $u(\xi)$ is a smooth function which satisfies $u(0) = 0$(see [38]).

Throughout this paper, the Greek subscript is summed from 1 to 2. Comma is used to indicate partial differentiation, i.e., $u_{\xi j} = \partial u / \partial x_j$, $\phi_{\alpha\beta} = \sum_{\alpha=1}^2 (\partial^2 \phi / \partial x_\alpha^2)$, and $u_{\xi t}$ denotes $\partial u / \partial t$.

The paper is structured as follows. In Section 2, we define a weighted energy expression. Section 3 is devoted to seek the continuous dependence for the coefficient $\lambda$.

### 2. Weighted Energy Expressions

We define $(u, v)$ is the solution of (3) and (4) with $\lambda = \lambda_1$, $(u^*, v^*)$ is the solution of (3) and (4) with $\lambda = \lambda_2$, we now define the differences $\pi = u - u^*$, $\phi = v - v^*$; then, the difference $(\pi, \phi)$ satisfies

$$\rho_1 \pi_{tt} - \Delta \pi - \mu \Delta \pi_\xi + a \Delta \phi = 0,$$

(13)

$$\rho_2 \phi_{tt} + \gamma \Delta^2 \phi + a \Delta \pi - \lambda_2 \Delta \phi_j - \lambda \phi_j^* = 0,$$

(14)

and the initial boundary conditions are

$$\pi(x_1, 0, t) = \phi(x_1, 0, t) = \phi_2(x_1, 0, t) = 0, \quad x_1 > 0, t > 0,$$

(15)

$$\pi(x_1, h, t) = \phi(x_1, h, t) = \phi_2(x_1, h, t) = 0, \quad x_1 > 0, t > 0,$$

(16)

$$\pi(0, x_2, t) = 0, \quad 0 \leq x_2 \leq h, t > 0,$$

(17)

$$\phi(0, x_2, t) = 0, \quad 0 \leq x_2 \leq h, t > 0,$$

(18)

$$\phi_1(0, x_2, t) = 0, \quad 0 \leq x_2 \leq h, t > 0,$$

(19)

$$\pi(x_1, x_2, 0) = \phi(x_1, x_2, 0) = \phi_2(x_1, x_2, 0) = \phi_\xi(x_1, x_2, 0) = 0, \quad 0 \leq x_2 \leq h, x_1 > 0.$$  

(20)

We add some conditions on the solutions:

$$\pi_\pi, \pi_\phi, \pi_\alpha \longrightarrow 0 \text{ as } z \longrightarrow \infty,$$

(21)

$$\phi_\phi, \phi_\alpha \longrightarrow 0 \text{ as } z \longrightarrow \infty.$$  

(22)

From (13), we have

$$0 = \int_0^t \int_{L_1}^{\infty} \int_{L_1}^{L_2} \exp(-w) \pi_{\eta} \left( \rho_1 \pi_{\eta\eta} - \pi_{\alpha\alpha} - \mu \pi_{\alpha\alpha\eta} + a \phi_{\alpha\alpha\eta} \right) d\eta dA.$$  

(23)
\[
0 = \int_0^t \int_y^\infty \exp(-\eta y) \phi_y \left( \rho_2 \phi_y + \gamma \phi_{x\eta\eta} + a \pi_{xx} \right) d\Lambda y d\eta \\
- \lambda_2 \phi_{x\eta} - \gamma \phi_{x\eta}^* d\Lambda y d\eta \\
+ \frac{\rho_1 w}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta}^* d\Lambda y d\eta dA \\
+ \frac{\rho_2 w}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\omega t) \phi_{x\eta} dA \\
+ \frac{\rho_1 w}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta\eta} d\Lambda y d\eta dA \\
+ \frac{\gamma}{2} \int_y^\infty \int_{L_y} \exp(-\omega t) \phi_{x\eta\eta} dA d\eta \\
+ \frac{\lambda_2}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{xx} d\Lambda y d\eta dA \\
+ \frac{\lambda_2}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{xx} d\Lambda y d\eta dA \\
+ \lambda \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
+ \lambda \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
+ \lambda \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA.
\]

We now tackle the item
\[
a \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{xx} d\Lambda y d\eta dA \\
= -a \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{xx} d\Lambda y d\eta dA \\
- a \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
+ a \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
+ a \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
= -a \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
+ a \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
- a \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
+ a \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
- a \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA.
\]

We define a new function
\[
\psi(z,t) = \frac{\rho_1 w}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \pi_{x\eta} d\Lambda y d\eta dA \\
+ \frac{\rho_1 w}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\omega t) \pi_{x\eta} dA \\
+ \frac{\omega}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\omega t) \pi_{x\eta} dA d\eta \\
+ \frac{\mu}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \pi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
+ \frac{\rho_2 w}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\omega t) \phi_{x\eta} dA \\
+ \frac{\rho_2 w}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\omega t) \phi_{x\eta} dA d\eta \\
+ \frac{\gamma}{2} \int_y^\infty \int_{L_y} \exp(-\omega t) \phi_{x\eta} dA d\eta \\
+ \frac{\lambda_2}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\omega t) \phi_{x\eta} \pi_{xx} d\Lambda y d\eta dA \\
+ \frac{\lambda_2}{2} \int_0^t \int_y^\infty \int_{L_y} \exp(-\omega t) \phi_{x\eta} \pi_{xx} d\Lambda y d\eta dA \\
+ \lambda \int_0^t \int_y^\infty \int_{L_y} \exp(-\omega t) \phi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA.
\]

By combining (24)-(26), we can easily get
\[
\psi(z,t) = -a \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \pi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
- \mu \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \pi_{x\eta} \pi_{x\eta} d\Lambda y d\eta dA \\
- \gamma \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \phi_{x\eta} dA d\eta \\
+ \gamma \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \phi_{x\eta} dA d\eta dA \\
- \lambda_2 \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \phi_{x\eta} dA dA dA \\
+ \lambda \int_0^t \int_y^\infty \int_{L_y} \exp(-\eta y) \phi_{x\eta} \phi_{x\eta} dA dA dA.
\]
Using Cauchy’s inequality, we obtain

\[
\left| aw \int_0^t \int_{L_{l_1}} \exp(-\omega t) \phi_{\alpha \beta} \pi dA d\eta \right| \leq \frac{w^r}{8} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \phi_{\alpha \beta} \phi_{\alpha \beta} dA d\eta \\
+ \frac{2a^2 \omega h^2}{\gamma \pi^2} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \pi_{\alpha \beta} \pi_{\alpha \beta} dA d\eta,
\]

(28)

We choose suitable \( a \) such that

\[
a^2 h^2 \leq \frac{1}{8} \frac{\pi^2 \gamma}{
\]

By combining (26), (28), and (29), we have

\[
\psi(z, t) \geq \frac{\rho_1 w}{4} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \pi_{\eta} dA d\eta \\
+ \frac{\rho_1 w}{4} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \pi_{\eta} \pi_{\eta} dA d\eta \\
+ \frac{w}{4} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \pi_{\theta} dA d\eta \\
+ \frac{1}{4} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \pi_{\alpha \beta} dA d\eta \\
+ \frac{\mu}{2} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \pi_{\alpha \beta} \pi_{\alpha \beta} dA d\eta \\
+ \frac{\rho_2 w}{4} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \phi_{\alpha \beta} dA d\eta \\
+ \frac{w}{4} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \phi_{\alpha \beta} \phi_{\alpha \beta} dA d\eta \\
+ \frac{\gamma}{4} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \phi_{\alpha \beta} \phi_{\alpha \beta} dA d\eta.
\]

(31)

By combining (26) and (31), we also have

\[
\varphi(z, t) = \frac{\rho_1 w}{2} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \pi_{\eta} \pi_{\eta} dA d\eta \\
+ \frac{\rho_1}{2} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \pi_{\alpha \beta} \pi_{\alpha \beta} dA d\eta \\
+ \frac{w}{2} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \pi_{\alpha \beta} \pi_{\alpha \beta} dA d\eta \\
+ \frac{1}{2} \int_0^t \int_{L_{l_1}} \exp(-\omega t) \pi_{\alpha \beta} \pi_{\alpha \beta} dA d\eta.
\]

(32)
\[ \phi(z, t) \geq \frac{\rho_1 w}{4} \int_0^t \int_{L_i} (\xi - z) \exp(-\nu \eta) \pi_i \eta \pi_i d\eta dA \]

From (31), following Schwarz’s inequality, we can easily get

\[ \phi(z, t) \geq \frac{\rho_1 w}{4} \int_0^t \int_{L_i} (\xi - z) \exp(-\nu \eta) \pi_i \eta \pi_i d\eta dA \]

In this part, we will use the following lemmas to prove our result.

**Lemma 1.** We can get the following decay estimates:

\[ \int_0^t \int_{L_i} (\xi - z) \nu_{\eta \eta} \phi_{\eta \eta} d\eta dA \]

\[ \leq \frac{1}{\lambda_2} \left( 1 + \frac{1}{(k + \chi)} \right) c_1 (0, t) e^{-\frac{t}{2\chi}} , \]

with \( \bar{k} = (-\gamma_1 + \sqrt{\gamma_1^2 + 4\gamma_2})/2 \), \( \chi = \gamma_1 \) and \( c_1 (0, t) = [- (\partial F (0, t)/\partial z) + (k + \chi) F (0, t)] \).

**Proof.** From (54) of [35], we have

\[ \frac{\partial^2 F}{\partial z^2} - \gamma_1 \frac{\partial F}{\partial z} - \gamma_2 F \geq 0, \]

where \( \gamma_1 \) and \( \gamma_2 \) are positive constants.

We rewrite (38) as

\[ \left\{ e^{\frac{t}{2\chi}} \left[ F' - (\bar{k} + \chi) F \right]' \right\} \geq 0, \]
with $\tilde{k} = (-\gamma_1 + \sqrt{\gamma_1^2 + 4\gamma_2})/2$ and $\chi = \gamma_1$. Integrating (39) from 0 to $z$ gives

$$
- \frac{\partial F}{\partial z} + (\tilde{k} + \chi)F \leq \left[ - \frac{\partial F(0, t)}{\partial z} + (\tilde{k} + \chi)F(0, t) \right] e^{-\tilde{k}z}
$$

Choosing $\omega > \max\{4/\rho_2\gamma, 4\}$ and using Schwarz's inequality, (34) and (36), we obtain

$$
\gamma \int_0^t \int_z \int_{L_i} \exp(-\omega \eta)\phi_{,\eta}^s \phi_{,1\eta} dA d\eta \leq (-k_2) \frac{\partial \phi}{\partial z} + \frac{1}{2} \frac{\partial^2 \phi}{\partial z^2}.
$$

where $k_2$ is a positive constant.

Using the Schwarz inequality, we have

$$
\frac{\lambda}{4} \int_0^t \int_z \int_{L_i} \exp(-\omega \eta)(\xi - z)\phi_{,\eta}^s \phi_{,1\eta} dA d\eta
$$

$$
+ \lambda \int_0^t \int_z \int_{L_i} \exp(-\omega \eta)\phi_{,\eta}^s \phi_{,1\eta} dA d\eta
$$

$$
\leq \frac{\lambda^2}{4} \int_0^t \int_z \int_{L_i} \exp(-\omega \eta)(\xi - z)\phi_{,\eta}^s \phi_{,1\eta} dA d\eta
$$

A combination of (42) (44), (45), and (37) leads to the result

$$
\phi(z, t) \leq \frac{\phi(z, t)}{2} + (-k_1 - k_2) \frac{\partial \phi}{\partial z} + \frac{1}{2} \frac{\partial^2 \phi}{\partial z^2}
$$

$$
+ \frac{\lambda^2}{\lambda^2} \left( 1 + \frac{1}{(\tilde{k} + \chi)} \right) c_1(0, t) e^{-\tilde{k}z}.
$$

If we set $k_3 = 2(k_1 + k_2)$ and $k_4 = (2/\lambda^2)(1 + (1/(\tilde{k} + \chi)))$, we can get

$$
\phi(z, t) + k_4 \frac{\partial \phi}{\partial z} \leq \frac{\partial \phi}{\partial z} + \lambda^2 k_4 c_1(0, t) e^{-\tilde{k}z}.
$$

(47) can be rewritten as

$$
\gamma \int_0^t \int_z \int_{L_i} \exp(-\omega \eta)(\xi - z)\phi_{,\eta}^s \phi_{,1\eta} dA d\eta
$$

Inequality (48) is the result of Lemma 2.

**Lemma 3.** For the energy expression $\psi(z, t)$ defined in (26), we have the following estimates:

**Case 1:** if $a_0 - k_3 - \tilde{k} \neq 0$, we get

$$
\psi(z, t) \leq \left[ \Lambda(0, t) - \frac{1}{a_0 - k_3 - k_4\lambda^2 c_1(0, t)} \right] e^{-(a_0 - k_3)z}
$$

$$
+ \frac{1}{a_0 - k_3 - k_4\lambda^2 c_1(0, t)} e^{-\tilde{k}z}.
$$

(49)
Case 2: if \( \alpha_0 - k_3 - \bar{k} = 0 \), we get

\[
\psi(z, t) \leq \Lambda(0, t) e^{-\left(\alpha_0-k_z\right)z} + k_4 z \lambda^2 c_1(0, t) e^{-\left(\alpha_0-k_z\right)z},
\]

where \( \Lambda(0, t) \) is a positive function and \( \alpha_0 \) is a positive constant to be defined later.

Proof. Let

\[
\Lambda(z, t) = E(z, t) + \bar{\alpha} \int_0^z e^{k_1 (z - \xi)} E(\xi, t) d\xi,
\]

where \( E(z, t) \) is defined as

\[
E(z, t) = e^{-k_z} \psi(z, t),
\]

and \( \bar{\alpha} \) is an arbitrary constant to be chosen later.

(41) may be rewritten as

\[
\frac{\partial \Lambda(z, t)}{\partial z} + \bar{\alpha} \Lambda(z, t) \leq k_4 \lambda^2 c_1(0, t) e^{-k_z} e^{-\bar{k}z},
\]

if \( \bar{\alpha} \) satisfies the quadratic equation

\[
\bar{\alpha}^2 - k_3 \bar{\alpha} - 1 = 0.
\]

Solving (53), we have

\[
\bar{\alpha} = \alpha_0 = \frac{k_3 + \sqrt{k_3^2 + 4}}{2}.
\]

Choosing \( \bar{\alpha} = \alpha_0 \) and integrating (53), we have the following two cases.

Case 1: if \( \alpha_0 - k_3 - \bar{k} \neq 0 \), we get

\[
\Lambda(z, t) \leq \Lambda(0, t) - \frac{1}{\alpha_0 - k_3 - \bar{k}} k_4 \lambda^2 c_1(0, t) e^{-\left(\alpha_0-k_z\right)z} + \frac{1}{\alpha_0 - k_3 - \bar{k}} k_4 \lambda^2 c_1(0, t) e^{-\left(\alpha_0-k_z\right)z}.
\]

From (51), we may deduce that

\[
\psi(z, t) \leq \Lambda(0, t) - \frac{1}{\alpha_0 - k_3 - k} k_4 \lambda^2 c_1(0, t) e^{-\left(\alpha_0-k_z\right)z} + \frac{1}{\alpha_0 - k_3 - k} k_4 \lambda^2 c_1(0, t) e^{-\left(\alpha_0-k_z\right)z}.
\]

Case 2: if \( \alpha_0 - k_3 - \bar{k} = 0 \), we get

\[
\Lambda(z, t) \leq \Lambda(0, t) e^{-\alpha_0 z} + k_4 z \lambda^2 c_1(0, t) e^{-\alpha_0 z}.
\]

From (51), we can easily get that

\[
\psi(z, t) \leq \Lambda(0, t) e^{-\left(\alpha_0-k_z\right)z} + k_4 z \lambda^2 c_1(0, t) e^{-\left(\alpha_0-k_z\right)z}.
\]
From (68), we may deduce that

$$\psi(z,t) \leq \left[ \lambda^2 c_4(0,t) \right] e^{-\left(\alpha_0 - k_3 - k\right)z} + \frac{1}{\alpha_0 - k_3 - k} k_4 \lambda^2 c_1(0,t) e^{-\kappa z}. \quad (69)$$

Case 2: if $\alpha_0 - k_3 - k = 0$, we get

$$\Lambda(z,t) \leq \lambda^2 c_2(0,t)e^{-\alpha_0 z} + k_4 \lambda^2 c_1(0,t)e^{-\alpha_0 z}. \quad (70)$$

From (70), we can easily get that

$$\psi(z,t) \leq \lambda^2 c_2(0,t)e^{-\left(\alpha_0 - k_3\right)z} + k_4 \lambda^2 c_1(0,t)e^{-\left(\alpha_0 - k_3\right)z}. \quad (71)$$

Summarizing all the above discussions, we can establish the following theorem. □

**Theorem 1.** Let $(u,v)$ and $(u^*,v^*)$ be the classical solutions of equations (13)–(22) for different values of $\lambda = \lambda_1$ and $\alpha = \alpha_2$, respectively, and $(\psi, \phi)$ be the difference of $(u,v)$ and $(u^*,v^*)$; the estimates (68) and (71) are satisfied. Inequalities (68) and (71) exhibit not only exponential decay in $z$ but also show that the amplitude terms in (68) and (71) become small as $\lambda \to 0$.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors contributed equally to this study. All authors read and approved the final manuscript.

**Acknowledgments**

This study was supported by the National Natural Science Foundation of China (grant no. 61907010), the Foundation for Natural Science in Higher Education of Guangdong, China (grant no. 2018KZDXM048), the General Project of Science Research of Guangzhou (grant no. 201707010126), and the Science Foundation of Huashang College Guangdong University of Finance and Economics (grant no. 2019HSDS28).

**References**

[1] K. A. Ames and B. Straughan, "Non-standard and improperly posed problems," *Mathematics in Science and Engineering Series*, Vol. 194, Academic Press, San Diego, CA, USA, 1997.

[2] B. Straughan, *Stability and Wave Motion in Porous Media*, Springer, Berlin, Germany, 2008.

[3] R. Quintanilla, "Convergence and structural stability in thermoelasticity," *Applied Mathematics and Computation*, vol. 135, no. 2-3, pp. 287–300, 2003.

[4] R. Quintanilla, "Structural stability and continuous dependence of solutions of thermoelasticity of type III," *Discrete & Continuous Dynamical Systems-B*, vol. 1, no. 4, pp. 463–470, 2001.

[5] J. C. Song, "Phragmén-Lindelöf and continuous dependence type results in a Stokes flow," *Applied Mathematics and Mechanics*, vol. 31, no. 7, pp. 875–882, 2010.

[6] A. J. Harfash, "Structural stability for two convection models in a reacting fluid with magnetic field effect," *Annales Henri Poincaré*, vol. 15, no. 12, pp. 2441–2465, 2014.

[7] Y. Li and C. Lin, "Continuous dependence for the nonhomogeneous Brinkman-Forchheimer equations in a semi-infinite pipe," *Applied Mathematics and Computation*, vol. 244, pp. 201–208, 2014.

[8] Y. Liu, "Continuous dependence for a thermal convection model with temperature-dependent solubility," *Applied Mathematics and Computation*, vol. 308, pp. 18–30, 2017.

[9] Y. Liu, S. Xiao, and Y. Lin, "Continuous dependence for the Brinkman-Forchheimer fluid interface with a Darcy fluid in a bounded domain," *Mathematics and Computers in Simulation*, vol. 150, pp. 66–82, 2018.

[10] J. K. Knowles, "On Saint-Venant’s principle in the two-dimensional linear theory of elasticity," *Archive for Rational Mechanics and Analysis*, vol. 21, no. 1, pp. 1–22, 1966.

[11] J. K. Knowles, "An energy estimates for the biharmonic equation and its application to Saint-Venant’s principle in plane elastostatics," *Indian Journal of Pure and Applied Mathematics*, vol. 14, no. 7, pp. 791–805, 1983.

[12] J. N. Flavin, "On Knowles’ version of Saint-Venant’s principle in two-dimensional elastostatics," *Archive for Rational Mechanics and Analysis*, vol. 53, no. 4, pp. 366–375, 1974.

[13] J. N. Flavin and R. J. Knops, "Some convexity considerations for a two-dimensional traction problem," *ZAMP Zeitschrift für angewandte Mathematik und Physik*, vol. 39, no. 2, pp. 166–176, 1988.

[14] C. O. Horgan, "Decay estimates for the biharmonic equation with applications to Saint-Venant’s principle in plane elasticity and Stokes flows," *Quarterly of Applied Mathematics*, vol. 47, no. 1, pp. 147–157, 1989.

[15] L. E. Payne and P. W. Schaefer, "Some Phragmen-Lindelöf type results for the biharmonic equation," *ZAMP Zeitschrift für angewandte Mathematik und Physik*, vol. 45, no. 3, pp. 414–432, 1994.

[16] V. Varlamov, "Existence and uniqueness of a solution to the cauchy problem for the Damped Boussinesq Equation," *Mathematical Methods in the Applied Sciences*, vol. 19, no. 8, pp. 639–649, 1996.

[17] C. O. Horgan, "Recent developments concerning Saint-Venant’s principle: an update," *Applied Mechanics Reviews*, vol. 42, no. 11, pp. 295–303, 1989.

[18] C. O. Horgan and J. K. Knowles, "Recent developments concerning Saint-Venant’s principle," *Advances in Applied Mechanics*, vol. 23, pp. 179–269, 1983.

[19] C. Lin, "Spatial decay estimates and energy bounds for the Stokes flow equation," *Zeitschrift für angewandte Mathematik und Physik*, vol. 48, no. 6, pp. 905–920, 1997.
[21] J. C. Song, “Improved decay estimates in time-dependent Stokes flow,” *Journal of Mathematical Analysis and Applications*, vol. 288, no. 2, pp. 505–517, 2003.

[22] J. C. Song, “Improved spatial decay bounds in the plane Stokes flow,” *Applied Mathematics and Mechanics*, vol. 30, no. 7, pp. 833–838, 2009.

[23] C. D’Apice, “Convexity considerations and spatial behavior for the harmonic vibrations in thermoelastic plates,” *Journal of Mathematical Analysis and Applications*, vol. 312, no. 1, pp. 44–60, 2005.

[24] C. D’Apice, “On a generalized biharmonic equation in plane polar with applications to functionally graded material,” *Australian Journal of Mathematical Analysis and Applications*, vol. 3, no. 2, pp. 1–15, 2006.

[25] S. Chirita and C. D’Apice, “On spatial growth or decay of solutions to a non simple heat conduction problem in a semi-infinite strip,” *Analele Stiintifice ale Universitatii Al I Cuza din Iasi-Matematica*, vol. 48, no. 1, pp. 75–100, 2002.

[26] S. Chirita, M. Ciarletta, and M. Fabrizio, “Some spatial decay estimates in time-dependent Stokes slow flows,” *Applicable Analysis*, vol. 77, no. 3, pp. 211–231, 2001.

[27] M. Fabrizio and S. Chirita, “Some qualitative results on the dynamic viscoelasticity of the Reissner-Mindlin plate model,” *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 57, no. 1, pp. 59–78, 2004.

[28] A. Borrelli and M. C. Patria, “Energy bounds in dynamical problems for a semi-infinite magnetoelastic beam,” *ZAMP Zeitschrift fur angewandte Mathematik und Physik*, vol. 47, no. 6, pp. 880–893, 1996.

[29] J. I. Diaz and R. Quintanilla, “Spatial and continuous dependence estimates in linear viscoelasticity,” *Journal of Mathematical Analysis and Applications*, vol. 273, no. 1, pp. 1–16, 2002.

[30] R. Quintanilla, “A spatial decay estimate for the hyperbolic heat equation,” *Siam Journal on Mathematical Analysis*, vol. 27, no. 1, pp. 78–91, 1996.

[31] R. Quintanilla, “Phragmen-Lindelof alternative in nonlinear viscoelasticity,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 34, no. 1, pp. 7–16, 1998.

[32] O. Civalek, B. Uzun, MO. Yayli, and B. Akgoz, “Size-dependent transverse and longitudinal vibrations of embedded carbon and silica carbide nanotubes by nonlocal finite element method,” *European Physical Journal Plus*, vol. 135381 pages, 2020.

[33] Ç. Demir and Ö. Civalek, “Torsional and longitudinal frequency and wave response of microtubes based on the nonlocal continuum and nonlocal discrete models,” *Applied Mathematical Modelling*, vol. 37, no. 22, pp. 9355–9367, 2013.

[34] F. Ebrahimi, M. R. Barati, and Ö. Civalek, “Application of Chebyshev-Ritz method for static stability and vibration analysis of nonlocal microstructure-dependent nanostructures,” *Engineering with Computers*, vol. 36, no. 3, pp. 953–964, 2020.

[35] G. Tang, Y. Liu, and W. Liao, “Spatial behavior of a coupled system of wave-plate type,” *Abstract and Applied Analysis*, vol. 2014, pp. 1–13, 2014.

[36] A. E. H. Love, *Mathematical Theory of Elasticity*, Dover Publications, New York, NY, USA, Fourth edition, 1942.

[37] M. L. Santos and J. E. Munoz Rivera, “Analytic property of a coupled system of wave-plate type with thermal effect,” *Differential and Integral Equations*, vol. 24, no. 9, pp. 965–972, 2011.