A semigroup theoretic approach
to the Whitehead asphericity problem

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Abstract

Related to a group presentation \( P = \mathcal{G}P(x, r) \) we consider two monoids. The first one is the monoid \( \Upsilon \) defined by the monoid presentation \( \mathcal{M}P(Y \cup Y^{-1}, P) \) where \( Y^\varepsilon (\varepsilon = \pm 1) \) is the set of symbols \( (r^u)^\varepsilon \) \( (\varepsilon = \pm 1) \) with \( r \in r \) and \( u \in FG(x) \) and \( P \) consists of all crossed commutations \( (ab, ba^\theta b) \) and \( (ab, b^\theta a^{-1}a) \) where \( a, b \in Y \cup Y^{-1} \). The second one is the universal enveloping group \( \mathcal{G}(\Upsilon) \) of \( \Upsilon \) given by the group presentation \( \hat{\mathcal{G}}P(Y \cup Y^{-1}, \hat{P}) \) where \( \hat{P} \) is the set of all words \( ab\iota(b) \) and \( ab\iota(b^\theta a^{-1}) \) with \( \iota(c) \) standing for the inverse of \( c \) in the free group over \( Y \cup Y^{-1} \). In terms of these monoids we prove that if \( d = (a_1, ..., a_n) \) is an identity \( Y \)-sequence over \( P \), then \( d \) is Peiffer equivalent to the empty sequence if and only if, the image of \( d \) in \( \mathcal{G}(\Upsilon) \) belongs to the subgroup \( \hat{\Upsilon} \) of \( \mathcal{G}(\Upsilon) \) generated by the images of \( aa^{-1} \) with \( a \in Y \cup Y^{-1} \). We use this to prove a necessary and sufficient condition under which a subpresentation of an aspherical group presentation is aspherical.

1 Introduction

The Whitehead asphericity conjecture, raised as a problem in [12], asks whether any subcomplex of an aspherical 2-complex is also aspherical. In group theoretic terms it can be rephrased as follows: given an aspherical presentation \( P = \mathcal{G}P(x, r) \) of a group \( G \), is it true that every subpresentation \( P' = \mathcal{G}P(x', r') \) of the first is also aspherical? The aim of this paper is to give a necessary and sufficient condition under which a subpresentation of an aspherical group presentation is aspherical. To achieve this we use among other things, some results from the theory of monoid acts. In this section we give a rough idea of how monoid acts come into play. First, we recall that a group presentation \( P = \mathcal{G}P(x, r) \) is aspherical if its geometric realisation \( K(\mathcal{P}) \) is an aspherical 2-complex, that is \( \pi_2(K(\mathcal{P})) = 0 \). In [2]
Brown and Huebschmann have proved several key results about aspherical group presentation one of which is their proposition 14 that gives sufficient and necessary conditions under which a group presentation $\mathcal{P} = \mathcal{GP}(x, r)$ is aspherical. As we use two of them in particular, we will state them here and explain their meanings. One of these conditions states that the relation module $\mathcal{N}(\mathcal{P})$ is a free $Z$-$G$ module. We give below the definition of $\mathcal{N}(\mathcal{P})$ and afterwards introduce its basis when $\mathcal{P}$ is aspherical. If $\mathcal{P} = \mathcal{GP}(x, r)$ is a presentation for a group $G$, we denote by $FG(x)$ the free group on $x$ and let $\alpha : FG(x) \to G$ and $\beta : N \to N/[N, N]$ be the canonical homomorphisms where $N$ is the normal closure of $r$ in $FG(x)$ and $[N, N]$ its commutator subgroup. There is a well defined $G$-action on $\mathcal{N}(\mathcal{P}) = N/[N, N]$ given by $$w^\alpha \cdot s^\beta = (w^{-1}sw)^\beta$$ for every $w \in FG(x)$ and $s \in N$. This action extends to an action of $Z$-$G$ over $\mathcal{N}(\mathcal{P})$ by setting $$(w^1_1 \pm w^0_1) \cdot s^\beta = (w^{-1}_1sw_1w_2^{-1}s^{\pm1}w_2)^\beta.$$  

Now the bases of $\mathcal{N}(\mathcal{P})$ as a free $Z$-$G$ module is the set of elements $r^\beta$ with $r \in r$. The other condition of proposition 14 states that any identity $Y$-sequence for $\mathcal{P}$ is Peiffer equivalent to the empty sequence. Related to the given data, it is denoted by $H$ the free group on the set $Y$ of symbols $r^u$ where $r \in r$ and $u \in FG(x)$. The group homomorphism $\theta : H \to FG(x)$ defined by $\theta(r^u) = u^{-1}ru$ has kernel $E$ the set of identities among the relations for $\mathcal{P}$. Besides $H$ it is considered the free monoid on the set $Y \cup Y^{-1}$ consisting of strings $(a_1, ..., a_n)$ where $n \geq 0$ and each $a_i \in Y \cup Y^{-1}$. The elements of this monoid are usually called $Y$-sequences and a string $(a_1, ..., a_n)$ for which $\theta(a_1) \cdots \theta(a_n) = 1$ in $FG(x)$ is called an identity $Y$-sequences for $\mathcal{P}$. Of a particular importance is the concept of Peiffer operations on $Y$-sequences.

(i) An elementary Peiffer exchange replaces an adjacent pair $(a, b)$ in a $Y$-sequence by either $(b, a^\theta b)$ or $(b^\theta a^{-1}, a)$. Note here that the second type of Peiffer exchange is the inverse of the first.

(ii) A Peiffer deletion deletes an adjacent pair $(a, a^{-1})$ in a $Y$-sequence.

(iii) A Peiffer insertion is the inverse of the Peiffer deletion.

The equivalence relation on the set of $Y$-sequences generated by the above operations is called Peiffer equivalence. In the next section we will see that, when it comes for the study of aspherical group presentations, Peiffer operations on $Y$-sequences can be better understood within the framework of the theory of monoid actions. For the benefit of the reader not familiar with monoid actions we will list below some basic notions and results that are used in the paper. For further results on the subject the reader may consult the monograph [7].

If $S$ is a monoid with identity element 1 and $X$ a nonempty set, we say that $X$ is a left $S$-system if there is an action $(s, x) \mapsto sx$ from $S \times X$ into $X$ with the properties

$$(st)x = s(tx) \text{ for all } s, t \in S \text{ and } x \in X,$$

$$1x = x \text{ for all } x \in X.$$
Right $S$-systems are defined dually in the obvious way. If $S$ and $T$ are (not necessarily different) monoids, we say that $X$ is an $(S,T)$-bisystem if it is a left $S$-system, a right $T$-system, and if

$$(sx)t = s(xt)$$

for all $s \in S$, $t \in T$ and $x \in X$.

If $X$ and $Y$ are both left $S$-systems, then an $S$-morphism or $S$-map is a map $\phi : X \to Y$ such that

$\phi(sx) = s\phi(x)$

for all $s \in S$ and $x \in X$.

Morphisms of right $S$-systems and of $(S,T)$-bisystems are defined in an analogue way. If we are given a left $T$-system $X$ and a right $S$-system $Y$, then we can give the cartesian product $X \times Y$ the structure of an $(T,S)$-bisystem by setting

$t(x,y) = (tx,y)$ and $(x,y)s = (x,ys)$.

Let now $A$ be an $(T,U)$-bisystem, $B$ an $(U,S)$-bisystem and $C$ an $(T,S)$-bisystem. As explained above, we can give to $A \times B$ the structure of an $(T,S)$-bisystem. With this in mind we say that a $(T,S)$-map $\beta : A \times B \to C$ is a bimap if

$\beta(au,b) = \beta(a,ub)$

for all $a \in A$, $b \in B$ and $u \in U$.

A pair $(A \otimes_U B, \psi)$ consisting of a $(T,S)$-bisystem $A \otimes_U B$ and a bimap $\psi : A \times B \to A \otimes_U B$ will be called a tensor product of $A$ and $B$ over $U$ if for every $(T,S)$-bisystem $C$ and every bimap $\beta : A \times B \to C$, there exists a unique $(T,S)$-map $\bar{\beta} : A \otimes_U B \to C$ such that the diagram

$$A \times B \xrightarrow{\psi} A \otimes_U B$$

$$\downarrow \beta$$

$$C \xleftarrow{\beta}$$

commutes. It is proved that $A \otimes_U B$ exists and is unique up to isomorphism. The existence theorem reveals that $A \otimes_U B = (A \times B)/\tau$ where $\tau$ is the equivalence on $A \times B$ generated by the relation

$$T = \{(au,b), (a,ub) : a \in A, b \in B, u \in U\}.$$

The equivalence class of a pair $(a,b)$ is usually denoted by $a \otimes_U b$. To us is of interest the situation when $A = S = B$ where $S$ is a monoid and $U$ is a submonoid of $S$. Here $A$ is clearly regarded as an $(S,U)$-bisystem with $U$ acting on the right of $A$ by multiplication, and $B$ as an $(U,S)$-bisystem where $U$ acts on the left of $B$ by multiplication. Another concept that is important to our approach is that of the dominion. If $U$ is a submonoid of a monoid $S$, then we say that $a$ is in the dominion of $U$ in $S$, written as $a \in \text{Dom}_S(U)$, if for all monoids $T$ and all monoid homomorphisms $f, g : S \to T$ that agree on $U$, we have that $f(a) = g(a)$. Related to dominions there is the well know zigzag theorem of Isbell. We will present here the Stenstrom version of it which reads. Let $U$ be a submonoid of a monoid $S$ and let $d \in S$. Then, $d \in \text{Dom}_S(U)$ if and only if $d \otimes_U 1 = 1 \otimes_U d$ in the tensor product $A = S \otimes_U S$. We mention here that this result holds true if $S$ turns out to be a group and $U$.
a subgroup, both regarded as monoids. A key result that is used to prove our main theorem in the next section is the fact that any inverse semigroup \( U \) is absolutely closed in the sense that for every semigroup \( S \) containing \( U \) as a subsemigroup, \( \text{Dom}_S(U) = U \). It is obvious that groups are absolutely closed as special cases of inverse monoids (see [3]).

The monoids involved in our approach are the following. The first one is the monoid \( \Upsilon \) defined by the monoid presentation \( \mathcal{MP}(Y \cup Y^{-1}, P) \) where \( Y^{-1} \) is the set of group inverses of the elements of \( Y \) and \( P \) consists of all pairs \((ab, ba)\) and \((ab, b^{a^{-1}}a)\) where \( a, b \in Y \cup Y^{-1} \).

The second one is the group \( \mathcal{G}(\Upsilon) \) given by the group presentation \( \mathcal{GP}(Y \cup Y^{-1}, \hat{P}) \) where \( \hat{P} \) is the set of all words \( ab(a^{b}) \iota(b) \) and \( abu(a)\iota(b^{a^{-1}}) \) where by \( \iota(c) \) we denote the inverse of \( c \) in the free group over \( Y \cup Y^{-1} \). Before we introduce the next two monoids and the respective monoid actions, we stop to explain that \( \Upsilon \) and \( \mathcal{G}(\Upsilon) \) are special cases of a more general situation.

If a monoid \( S \) is given by the monoid presentation \( \mathcal{MP}(X, R) \), then its universal enveloping group \( \mathcal{G}(S) \) (see [1] and [4]) is defined to be the group given by the group presentation \( \mathcal{GP}(X, \hat{R}) \) where \( \hat{R} \) consists of all words \( u \iota(v) \) whenever \((u, v) \in R \) where \( \iota(v) \) is the inverse of \( v \) in the free group over \( X \). We let for future use \( \sigma : FM(X) \rightarrow S \) and \( \hat{\sigma} : FG(X) \rightarrow \mathcal{G}(S) \) be the respective canonical homomorphisms where \( FM(X) \) and \( FG(X) \) are the free monoid and the free group on \( X \). It is easy to see that there is a monoid homomorphism \( \mu_S : S \rightarrow \mathcal{G}(S) \) which maps each generator \( \sigma(x) \) of \( S \) to \( \hat{\sigma}(x) \) and satisfies the following universal property. For every group \( G \) and monoid homomorphism \( f : S \rightarrow G \), there is a unique group homomorphism \( \hat{f} : \mathcal{G}(S) \rightarrow G \) such that \( \hat{f} \mu_S = f \). This universal property is indication of an adjoint situation. Specifically, the functor \( \mathcal{G} : \text{Mon} \rightarrow \text{Grp} \) which maps every monoid to its universal group, is a left adjoint to the forgetful functor \( U : \text{Grp} \rightarrow \text{Mon} \). This ensures that \( \mathcal{G}(S) \) is an invariant of the presentation of \( S \).

The third monoid we consider is the submonoid \( \mathfrak{U} \) of \( \Upsilon \), having the same unit as \( \Upsilon \), and is generated from all the elements of the form \( \sigma(a)\sigma(a^{-1}) \) with \( a \in Y \cup Y^{-1} \). This monoid, acts on the left and on the right of \( \Upsilon \) by the multiplication in \( \Upsilon \). The last monoid considered is the subgroup \( \mathfrak{U} \) of \( \mathcal{G}(\Upsilon) \) generated by \( \mu(\mathfrak{U}) \). Similarly to above, \( \mathfrak{U} \) acts on \( \mathcal{G}(\Upsilon) \) by multiplication.

In the next section we will see that an identity \( Y \)-sequence \((a_1, ..., a_n)\) is Peiffer equivalent to the empty sequence if and only if for the element \( a = \mu(\sigma(a_1) ... \sigma(a_n)) \) of \( \mathcal{G}(\Upsilon) \) we have \( a \otimes_{\mathfrak{U}} 1 = 1 \otimes_{\mathfrak{U}} a \) in the tensor product \( \mathcal{G}(\Upsilon) \otimes_{\mathfrak{U}} \mathcal{G}(\Upsilon) \). From the zigzag theorem of Isbell the last equality is equivalent to assuming that \( a \in \text{Dom}_{\mathcal{G}(\Upsilon)}(\hat{\mathfrak{U}}) \), where \( \text{Dom}_{\mathcal{G}(\Upsilon)}(\hat{\mathfrak{U}}) \) is the dominion of \( \hat{\mathfrak{U}} \) in \( \mathcal{G}(\Upsilon) \). Recalling that the group \( \hat{\mathfrak{U}} \) is absolutely closed we infer that an identity \( Y \)-sequence \((a_1, ..., a_n)\) is Peiffer equivalent to the empty sequence if and only if \( a = \mu(\sigma(a_1) ... \sigma(a_n)) \in \mathfrak{U} \). Having proved this it is not to difficult to prove our theorem [2,7] which gives a necessary and sufficient condition under which a subpresentation of an aspherical group presentation is itself aspherical. We also recall a result of Ivanov from [9] which states that if the Whitehead conjecture is false, then there is an aspherical presentation \( E = \langle A, R \cup z \rangle \) of the trivial group \( E \), where the alphabet \( A \) is finite or countably infinite and \( z \in A \), such that its subpresentation \( \langle A, R \rangle \) is not aspherical. In virtue of this, we see that the conjecture is true if and only if it is true for subpresentations that differs from the given aspherical presentation by a single defining relation.
Finally we mention that results related to ours can be found in [3], [5], [6] and [11]. Also a good account on the theory of identity sequences over group presentations can be found in [10].

2 Peiffer operations and monoid actions

If \( \alpha = (a_1, \ldots, a_n) \) is any \( Y \)-sequence over the group presentation \( P = \langle x, r \rangle \), then performing an elementary Peiffer operation on \( \alpha \) can be interpreted in a simple way in terms of monoids \( \Upsilon \) and \( \mathcal{U} \) defined in the introduction. In what follows we will denote by \( \sigma(\alpha) \) the element \( \sigma(a_1) \cdots \sigma(a_n) \in \Upsilon \). If \( \beta = (b_1, \ldots, b_n) \) is obtained from \( \alpha = (a_1, \ldots, a_n) \) by performing an elementary Peiffer exchange, then from the definition of \( \Upsilon \), \( \sigma(\alpha) = \sigma(\beta) \), therefore an elementary Peiffer exchange or a finite sequence of such has no effect on the element \( \sigma(a_1) \cdots \cdot \cdots \sigma(a_n) \in \Upsilon \). Before we see the effect that a Peiffer insertion in \( \alpha \) has on \( \sigma(\alpha) \) we need the first claim of the following.

Lemma 2.1. The elements of \( \mathcal{U} \) are central in \( \Upsilon \) and those of \( \hat{\mathcal{U}} \) are central in \( G(\Upsilon) \).

Proof. We see that for every \( a \) and \( b \in Y \cup Y^{-1} \), \( \sigma(a) \sigma(a^{-1}) \sigma(b) = \sigma(b) \sigma(a) \sigma(a^{-1}) \). Indeed,

\[
\sigma(b) \sigma(a) \sigma(a^{-1}) = \sigma(baa^{-1}) = \\
= \sigma(ab^\theta_\alpha a^{-1}) = \sigma(aa^{-1}(b^\theta_\alpha)^\theta a^{-1}) \\
= \sigma(a) \sigma(a^{-1}) \sigma(b).
\]

Since elements \( \sigma(b) \) and \( \sigma(a) \sigma(a^{-1}) \) are generators of \( \Upsilon \) and \( \mathcal{U} \) respectively, then the first claim holds true. The second claim follows easily. \( \Box \)

If we insert \( (a, a^{-1}) \) at some point in \( \alpha = (a_1, \ldots, a_n) \) to obtain \( \alpha' = (a_1, \ldots, a, a^{-1}, \ldots, a_n) \), then from lemma 2.1

\[
\sigma(\alpha) = \sigma(\alpha) \cdot (\sigma(a) \sigma(a^{-1})),
\]

which means that inserting \( (a, a^{-1}) \) inside a \( Y \)-sequence \( \alpha \) has the same effect as multiplying the corresponding \( \sigma(\alpha) \) in \( \Upsilon \) by the element \( \sigma(a) \sigma(a^{-1}) \) of \( \mathcal{U} \) and conversely. Of course the deletion has the obvious interpretation in our semigroup theoretic terms as the inverse of the above process. We retain the same names for our semigroup operations, that is insertion for multiplication by \( \sigma(a) \sigma(a^{-1}) \) and deletion for its inverse. Related these operations on the elements of \( \Upsilon \) we make the following definition.

Definition 2.2. We denote by \( \sim_\mathcal{U} \) the equivalence relation in \( \Upsilon \) generated by all pairs \( (\sigma(\alpha), \sigma(\alpha) \cdot \sigma(a) \sigma(a^{-1})) \) where \( \alpha \in FM(Y \cup Y^{-1}) \) and \( a \in Y \cup Y^{-1} \). We say that two elements \( \sigma(a_1) \cdots \sigma(a_n) \) and \( \sigma(b_1) \cdots \sigma(b_m) \) where \( m, n \geq 0 \) are Peiffer equivalent in \( \Upsilon \) if they fall in the same \( \sim_\mathcal{U} \)-class.

It is obvious that two \( Y \)-sequences \( \alpha \) and \( \beta \) are Peiffer equivalent in the usual sense if and only if \( \sigma(\alpha) \sim_\mathcal{U} \sigma(\beta) \), but it should be mentioned that the study of \( \sim_\mathcal{U} \) might be as hard
as the study of Peiffer operations on \( Y \)-sequences, and at this point it seems we have not made any progress at all. In fact this definition will become useful latter in this section and yet we have to prove a few more things before we utilize it.

The process of inserting and deleting generators of \( \mathcal{U} \) in an element of \( \Upsilon \) is related to the following new concept. If in general \( U \) is a submonoid of a monoid \( S \) and \( d \in S \), then we say that \( d \) belongs to the weak dominion of \( U \), shortly written as \( d \in W\text{Dom}_S(U) \), if for every group \( G \) and every monoid homomorphisms \( f, g : S \to G \) such that \( f(u) = g(u) \) for every \( u \in U \), then \( f(d) = g(d) \). An analogue of Stenström version of Isbell theorem (theorem 8.3.3 of [7]) for weak dominion holds true. The proof of the if part of its analogue is similar to that of Isbell theorem apart from some minor differences that reflect the fact that we are working with \( W\text{Dom} \) rather than \( \text{Dom} \) and that will become clear along the proof, while the converse relies on the universal property of \( \mu : S \to \mathcal{G}(S) \).

**Proposition 2.3.** Let \( S \) be a monoid, \( U \) a submonoid and let \( \hat{U} \) be the subgroup of \( \mathcal{G}(S) \) generated by elements \( \mu(u) \) with \( u \in U \). Then \( d \in W\text{Dom}_S(U) \) if and only if \( \mu(d) \in \hat{U} \).

**Proof.** The set \( \hat{A} = \mathcal{G}(S) \otimes_{\hat{U}} \mathcal{G}(S) \) has an obvious \((\mathcal{G}(S), \mathcal{G}(S))\)-bisystem structure. The free abelian group \( \mathbb{Z}\hat{A} \) on \( \hat{A} \) inherits a \((\mathcal{G}(S), \mathcal{G}(S))\)-bisystem structure if we define

\[
\begin{equation}
\sum\limits_{i} z_i(g_i \otimes_{\hat{U}} h_i) = \sum\limits_{i} z_i(g_i g_i \otimes_{\hat{U}} h_i) \quad \text{and} \quad \left( \sum\limits_{i} z_i(g_i \otimes_{\hat{U}} h_i) \right) \cdot g = \sum\limits_{i} z_i(g_i \otimes_{\hat{U}} h_i g).
\end{equation}
\]

The set \( \mathcal{G}(S) \times \mathbb{Z}\hat{A} \) becomes a group by defining

\[
(g, \sum\limits_{i} z_i g_i \otimes_{\hat{U}} h_i) \cdot (g', \sum\limits_{i} z'_i g'_i \otimes_{\hat{U}} h'_i) = (g g', \sum\limits_{i} z_i g_i \otimes_{\hat{U}} h_i g + \sum\limits_{i} z'_i g'_i \otimes_{\hat{U}} h'_i).
\]

The associativity is proved easily. The unit element is \((1, 0)\) and for every \((g, \sum\limits_{i} z_i g_i \otimes_{\hat{U}} h_i)\) its inverse is the element \((g^{-1}, -\sum\limits_{i} z_i g^{-1} g_i \otimes_{\hat{U}} h_i g^{-1})\). Let now define

\[
\beta : S \to \mathcal{G}(S) \times \mathbb{Z}\hat{A} \text{ by } s \mapsto (\mu(s), 0),
\]

which is clearly a monoid homomorphism, and

\[
\gamma : S \to \mathcal{G}(S) \times \mathbb{Z}\hat{A} \text{ by } s \mapsto (\mu(s), \mu(s) \otimes_{\hat{U}} 1 - 1 \otimes_{\hat{U}} \mu(s)),
\]

which is again seen to be a monoid homomorphism. These two coincide on \( U \) since for every \( u \in U \)

\[
\gamma(u) = (\mu(u), \mu(u) \otimes_{\hat{U}} 1 - 1 \otimes_{\hat{U}} \mu(u)) = (\mu(u), 0) = \beta(u).
\]

The last equality and the assumption that \( d \in W\text{Dom}_S(U) \) imply that \( \beta(d) = \gamma(d) \), therefore

\[
(\mu(d), 0) = (\mu(d), \mu(d) \otimes_{\hat{U}} 1 - 1 \otimes_{\hat{U}} \mu(d)),
\]

which shows that \( \mu(d) \otimes_{\hat{U}} 1 = 1 \otimes_{\hat{U}} \mu(d) \) in the tensor product \( \mathcal{G}(S) \otimes_{\hat{U}} \mathcal{G}(S) \) and therefore theorem 8.3.3, [7], applied for monoids \( \mathcal{G}(S) \) and \( \hat{U} \), implies that \( \mu(d) \in \text{Dom}_{\mathcal{G}(S)}(\hat{U}) \). But
Dom_{G(S)}(\hat{U}) = \hat{U}$ as from theorem 8.3.6, every inverse semigroup is absolutely closed, whence $\mu(d) \in \hat{U}$.

Conversely, suppose that $\mu(d) \in \hat{U}$ and want to show that $d \in WDom_S(U)$. Let $G$ be a group and $f, g : S \to G$ two monoid homomorphisms that coincide in $U$, therefore the group homomorphisms $\hat{f}, \hat{g} : G(S) \to G$ of the universal property of $\mu$ coincide in $\hat{U}$ which, from our assumption, implies that $\hat{f}(\mu(d)) = \hat{g}(\mu(d))$, and then $f(d) = g(d)$ proving that $d \in WDom_S(U)$. \hfill \Box

Before we reveal the connection between Peiffer deletions (insertions) and weak dominion, we need a few more technical result. Let

$$\Psi : FG(Y \cup Y^{-1}) \to \Upsilon$$

be the map defined as follows.

$$\Psi(u) = \sigma(u)$$ if the reduced word $u$ does not contain any $\iota(a)$ with $a \in Y \cup Y^{-1}$,

otherwise if $u$ has occurrences of $\iota(a)$ with $a \in Y \cup Y^{-1}$, then

$$\Psi(u) = \sigma(u')$$ where $u'$ is obtained from $u$ by replacing any $\iota(a)$ by $a^{-1}$.

Let $u, v, x \in FG(Y \cup Y^{-1})$ be irreducibles such that $u = u_1x$, $v = \iota(x)v_1$ where $\iota(x)$ is the inverse of $x$ and $u_1, v_1, u_1v_1$ are irreducibles. It is easy to see that

$$\Psi(uv) = \Psi(u_1v_1) = \Psi(u_1)\Psi(v_1),$$

and that

$$\Psi(u)\Psi(v) = \Psi(u_1)\Psi(x)\Psi(\iota(x))\Psi(v_1)$$

$$= \Psi(u_1)\Psi(v_1)\Psi(x)\Psi(\iota(x))$$ since $\Psi(x)\Psi(\iota(x)) \in \Upsilon$

$$= \Psi(uv)[u, v]$$ where $[u, v]$ is $\Psi(x)\Psi(\iota(x))$ for short.

In this way we have proved that for any irreducibles $u, v \in FG(Y \cup Y^{-1})$, there is $[u, v] \in \Upsilon$ such that $\Psi(uv)[u, v] = \Psi(u)\Psi(v)$.

**Lemma 2.4.** Let $\rho$ be any defining relation of $G(\Upsilon)$ or its inverse and $\xi \rho \iota(\xi)$ any conjugate of $\rho$ in $FG(Y \cup Y^{-1})$. Then there is $u \in \Upsilon$ such that $\Psi(\xi \rho \iota(\xi)) \sim_{\Upsilon} u$.

**Proof.** First we see that for any defining relation $\rho$ of $G(\Upsilon)$ we have that $\Psi(\rho) \in \Upsilon$. Indeed, if $\rho = ab(a^{\theta b})\iota(b)$, then

$$\Psi(ab(a^{\theta b})\iota(b)) = \sigma(a)\sigma(b)\sigma((a^{\theta b})^{-1})\sigma(b^{-1})$$

$$= \sigma(b)\sigma(a^{\theta b})\sigma((a^{\theta b})^{-1})\sigma(b^{-1})$$

$$= \sigma(b)\sigma(b^{-1})\sigma(a^{\theta b})\sigma((a^{\theta b})^{-1}) \in \Upsilon.$$
The proof for the second type of relations is similar. In the same way one can show that for every defining relation \( \rho \), \( \Psi(\tau(\rho)) \in \mathfrak{U} \). Finally, if \( \xi_\rho(\xi) \) is a conjugate of a defining relation or its inverse, then \( \Psi(\xi_\rho(\xi)) \) is Peiffer equivalent in \( \mathfrak{Y} \) to an element \( \mathfrak{U} \). Indeed,

\[
\Psi(\xi)(\Psi(\rho)(\Psi(\xi(\rho)))\varepsilon = \Psi(\xi)(\Psi(\xi(\rho)))\varepsilon = [\xi, \xi(\rho)]\Psi(\xi(\rho))\varepsilon \]

On the other hand,

\[
\Psi(\xi)(\Psi(\rho)(\Psi(\xi(\rho))) = [\xi, \rho][\xi, \xi(\rho)]\Psi(\xi(\rho)).
\]

Since \([\xi, \rho][\xi, \xi(\rho)] \in \mathfrak{U} \), and from above \( \Psi(\xi)(\Psi(\rho)(\Psi(\xi(\rho))) \in \mathfrak{U} \), then we have that \( \Psi(\xi_\rho(\xi)) \sim_\mathfrak{U} u \) where \( u \in \mathfrak{U} \).

The reason why we had to define the map \( \Psi \) will become apparent shortly. It is obvious that when \( A \in FM(Y \cup Y^{-1}) \), then \( \hat{\sigma}(A) \) is nothing but \( \sigma(A) \). The following lemma shows that if two words which contain letters from \( Y \cup Y^{-1} \) but not inverses in \( FG(Y \cup Y^{-1}) \) represent the same element in \( \mathcal{G}(\mathfrak{Y}) \), then seen as elements of \( \mathfrak{Y} \), they are \( \sim_\mathfrak{U} \) equivalent.

**Lemma 2.5.** If \( A, B \in FM(Y \cup Y^{-1}) \) such that \( \hat{\sigma}(A) = \hat{\sigma}(B) \) in \( \mathcal{G}(\mathfrak{Y}) \), then \( \sigma(A) \sim_\mathfrak{U} \sigma(B) \).

**Proof.** Suppose that \( A = (\xi_1\rho_1t(\xi_1)) \cdots (\xi_n\rho_nt(\xi_n))B \) and want to prove that \( \sigma(A) \sim_\mathfrak{U} \sigma(B) \). For every \( 1 \leq i \leq n - 1 \) make the following notations

\[
\varepsilon_i = [\xi_i\rho_1t(\xi_i), (\xi_{i+1}\rho_{i+1}t(\xi_{i+1})), \cdots (\xi_n\rho_nt(\xi_n)) \cdot B].
\]

Also set

\[
\varepsilon_n = [\xi_n\rho_nt(\xi_n), B].
\]

The following hold true

\[
\Psi(A) \cdot \varepsilon_1 = \Psi((\xi_1\rho_1t(\xi_1)) \cdot \Psi((\xi_2\rho_2t(\xi_2)) \cdots (\xi_n\rho_nt(\xi_n)) \cdot B)
\]

\[
\Psi(A) \cdot \varepsilon_1 \cdot \varepsilon_2 = \Psi((\xi_1\rho_1t(\xi_1)) \cdot \Psi((\xi_2\rho_2t(\xi_2)) \cdots (\xi_n\rho_nt(\xi_n)) \cdot B)
\]

\[
\cdots
\]

\[
\Psi(A) \cdot \varepsilon_1 \cdots \varepsilon_{n-1} = \Psi((\xi_1\rho_1t(\xi_1)) \cdots (\xi_n\rho_nt(\xi_n)) \cdot (\xi_1\rho_1t(\xi_1)) \cdots (\xi_n\rho_nt(\xi_n)) \cdot B)
\]

\[
\Psi(A) \cdot \varepsilon_1 \cdots \varepsilon_{n-1} \cdot \varepsilon_n = \Psi((\xi_1\rho_1t(\xi_1)) \cdots (\xi_n\rho_nt(\xi_n)) \cdot B)
\]

Since from the proceeding lemma, each \( \Psi((\xi_i\rho_1t(\xi_i)) \sim_\mathfrak{U} u_i \) with \( u_i \in \mathfrak{U} \) and since every \( \varepsilon_i \in \mathfrak{U} \), one can easily see that \( \Psi(A) \sim_\mathfrak{U} \Psi(B) \), hence \( \sigma(A) \sim_\mathfrak{U} \sigma(B) \).

The relation between insertion (deletion) and the weak dominion is now revealed from the following.

**Theorem 2.6.** Let \( d \in \mathfrak{Y} \), then \( d \sim_\mathfrak{U} 1 \) if and only if \( d \in WDom_\mathfrak{Y}(\mathfrak{U}) \).
Proof. Let $G$ be any group and $f, g : \Upsilon \to G$ two monoid homomorphisms that coincide in $\mathcal{U}$ and want to show that $f(d) = g(d)$. The proof will be done by induction on the minimal number $h(d)$ of insertions and deletions needed to transform $d = \sigma(a_1) \cdots \sigma(a_n)$ to 1. If $h(d) = 1$, then $d \in \mathcal{U}$ and $f(d) = g(d)$. Suppose that $h(d) = n > 1$ and let $\tau$ be the first operation performed on $d$ in a series of operations of minimal length. After $\tau$ is performed on $d$, it is obtained an element $d'$ with $h(d') = n - 1$. By induction hypothesis, $f(d') = g(d')$ and want to prove that $f(d) = g(d)$. There are two possible cases for $\tau$. First, $\tau$ is an insertion and let $u = \sigma(a)\sigma(a^{-1}) \in \mathcal{U}$ be the element inserted. It follows that $f(d') = f(d)f(u)$ and $g(d') = g(d)g(u)$, but $f(u) = g(u)$, therefore from cancellation law in the group $G$ we get $f(d) = g(d)$. Second, $\tau$ is a deletion and let $u = \sigma(a)\sigma(a^{-1}) \in \mathcal{U}$ be the element deleted, that is $d = d' u$. It follows immediately from the assumptions that $f(d) = g(d)$.

Conversely, assume that $d \in \text{WDom}_\Upsilon(\mathcal{U})$ and want to prove that $d \sim_\mathcal{U} 1$. From proposition 2.3 $\mu(d) \in \mathcal{U}$ and let $u_1, ..., u_n$ be group generators from $\mathcal{U}$ such that $\mu(d) = u_1 \cdots u_n$. For $i = 1, ..., n$ define

$$\omega_{u_i} = \begin{cases} \{ (\sigma(a)\sigma(a^{-1}))^2 & \text{if } u_i = i(\mu\sigma(a)\mu\sigma(a^{-1})) \\ 1 & \text{if } u_i \text{ is not an inverse} \end{cases}$$

We may now write

$$\mu(\omega_{u_1} \cdots \omega_{u_n}d) = \mu(\omega_{u_1})u_1 \cdots \mu(\omega_{u_n})u_n,$$

where the right hand side belongs to $\mu(\mathcal{U})$ and let $u \in \mathcal{U}$ be such that

$$\mu(\omega_{u_1} \cdots \omega_{u_n}d) = \mu(u).$$

Lemma 2.3 implies that $\omega_{u_1} \cdots \omega_{u_n}d \sim_\mathcal{U} u$. Since each $\omega_{u_i}$ is either 1 or square of a generator from $\mathcal{U}$ and since $u \sim_\mathcal{U} 1$, we infer that $d \sim_\mathcal{U} 1$ concluding the proof. \qed

Let $\mathcal{P} = \mathcal{GP}(x, r)$ be an aspherical group presentation and $\mathcal{P}_1 = \mathcal{GP}(x, r_1)$ a subpresentation of the first where $r_1 = r \setminus \{r_0\}$ and $r_0 \in r$ is a fixed relation. We denote by $\Upsilon_1$, $\mathcal{U}_1$ monoids associated with $\mathcal{P}_1$ and by $\mathcal{G}(\Upsilon_1)$ and $\mathcal{U}_1$ their respective groups. Also we consider $\hat{\mathcal{A}}_1$ the subgroup of $\hat{\mathcal{U}}_1$ generated by all $\mu_1\sigma_1(bb^{-1})$ where $b \in Y_1 \cup Y_1^{-1}$. Finally note that the monomorphism $f : \Upsilon_1 \to \Upsilon$ induced by the map $\sigma_1(a) \to \sigma(a)$ induces a homomorphism $\hat{f} : \mathcal{G}(\Upsilon_1) \to \mathcal{G}(\Upsilon)$. With the above notation we have the following.

Theorem 2.7. The subpresentation $\mathcal{P}_1 = \mathcal{GP}(x, r_1)$ is aspherical if and only if $\hat{f}^{-1}(\hat{\mathcal{A}}_1) = \hat{\mathcal{U}}_1$.

Proof. Suppose that $(a_1, ..., a_n)$ is an identity $Y_1$-sequence. Since it is also an identity $Y$-sequence and $\mathcal{P} = \mathcal{GP}(x, r)$ is aspherical, then from [2] $(a_1, ..., a_n)$ is Peiffer equivalent in $\mathcal{P}$ to the empty sequence. The latter is equivalent to assuming that $d = (\sigma(a_1) \cdots \sigma(a_n)) \sim_\mathcal{U} 1$, and then theorem 2.6 and proposition 2.3 imply that $\mu(d) \in \mathcal{U}$. We claim that $\mu(d) \in \mathcal{U}_1$. To see this we first let

$$\mu(d) = (\mu\sigma(b_1b_1^{-1}) \cdots \mu\sigma(b_kb_k^{-1})) \cdot (\nu(\mu\sigma(b_{s+1}b_{s+1}^{-1})) \cdots \nu(\mu\sigma(b_r b_r^{-1})))$$

$$(\mu\sigma(c_1c_1^{-1}) \cdots \mu\sigma(c_tc_t^{-1})) \cdot (\nu(\mu\sigma(d_1d_1^{-1})) \cdots \nu(\mu\sigma(d_kd_k^{-1}))).$$
where the first half involves elements from \( Y_1 \cup Y_1^{-1} \) and the second one is 

\[
\mu \sigma(C) \iota(\mu \sigma(D))
\]

with

\[
C = c_1 c_1^{-1} \cdots c_i c_i^{-1} \quad \text{and} \quad D = d_1 d_1^{-1} \cdots d_k d_k^{-1},
\]

where \( C \) and \( D \) involve only elements of the form \((r_0^\varepsilon)\) with \( \varepsilon = \pm 1 \). Define

\[
\psi : FM(Y \cup Y^{-1}) \to \mathcal{N}(\mathcal{P})
\]
on free generators as follows

\[
(r^u)^\varepsilon \mapsto (u^{-1} ru)^\beta.
\]

It is easy to see that \( \psi \) is compatible with the defining relations of \( \mathcal{Y} \), hence there is \( g : \mathcal{Y} \to \mathcal{N}(\mathcal{P}) \) and then the universal property of \( \mu \) implies the existence of \( \hat{g} : \mathcal{G}(\mathcal{Y}) \to \mathcal{N}(\mathcal{P}) \) such that \( \hat{g} \mu = g \). Recalling from above that in \( \mathcal{G}(\mathcal{Y}) \) we have

\[
\mu \sigma((a_1 \cdots a_n) \cdot ((b_{s+1} b_{s+1}^{-1}) \cdots (b_r b_r^{-1})) \cdot ((d_1 d_1^{-1}) \cdots (d_k d_k^{-1}))) = \mu \sigma(((b_1 b_1^{-1}) \cdots (b_s b_s^{-1})) \cdot ((c_1 c_1^{-1}) \cdots (c_t c_t^{-1}))).
\]

we can apply \( \hat{g} \) on both sides and get

\[
g \sigma((a_1 \cdots a_n) \cdot ((b_{s+1} b_{s+1}^{-1}) \cdots (b_r b_r^{-1})) \cdot ((d_1 d_1^{-1}) \cdots (d_k d_k^{-1}))) = g \sigma(((b_1 b_1^{-1}) \cdots (b_s b_s^{-1})) \cdot ((c_1 c_1^{-1}) \cdots (c_t c_t^{-1}))).
\]

If we now write each \( c_i = (r_0^\varepsilon) \) and each \( d_j = (r_0^\delta) \) where \( \varepsilon_i \) and \( \delta_j = \pm 1 \), while we write each \( a_\ell = (r_\ell^w) \) and each \( b_p = (\rho_p^\eta)^\rho \) where all \( r_\ell \) and \( \rho_p \) belong to \( r_1 \) and \( r_\ell, \rho_p = \pm 1 \), then the definition of \( g \) yields

\[
(w_1^\alpha \cdot r_1^\beta + \cdots + w_n^\alpha \cdot r_n^\beta) + (2 \eta_{s+1}^\alpha \cdot \rho_{s+1}^\beta + \cdots + 2 \eta^\alpha \cdot \rho^\beta) + (2 v_1^\alpha + \cdots + 2 v_k^\alpha) \cdot r_0^\delta = (2 \eta_1^\alpha \cdot \rho_1^\beta + \cdots + 2 \eta^\alpha \cdot \rho^\beta) + (2 u_1^\alpha + \cdots + 2 u_t^\alpha) \cdot r_0^\delta
\]

The freeness of \( \mathcal{N}(\mathcal{P}) \) on the set of elements \( r_\ell^\beta \) implies in particular that

\[
(2 v_1^\alpha + \cdots + 2 v_k^\alpha) \cdot r_0^\delta = (2 u_1^\alpha + \cdots + 2 u_t^\alpha) \cdot r_0^\delta
\]

from which we see that \( k = t \), and after a rearrangement of terms \( u_i^\alpha = v_i^\alpha \) for \( i = 1, \ldots, k \).

One can see that in general if \( v = u \cdot \prod_{i=1}^k w_i^{-1} r_i^{\lambda_i} w_i \) in \( FG(\mathbf{x}) \) where \( \lambda_i = \pm 1 \) and \( r_i \in r \), then in \( \mathcal{G}(\mathcal{Y}) \) we have

\[
\mu \sigma((r_0^\varepsilon)^\lambda) = \iota \left( \prod_{i=1}^k \mu \sigma(r_i^\lambda) \right) \cdot \mu \sigma((r_0^\varepsilon)^\lambda) \cdot \left( \prod_{i=1}^k \mu \sigma(r_i^\lambda) \right).
\]

Using this it is easy to see that

\[
\mu \sigma((r_0^\varepsilon)^\delta(r_0^\varepsilon)^{-\delta}) = \mu \sigma((r_0^\varepsilon)^\delta(r_0^\varepsilon)^{-\delta}).
\]
The easily verified fact that in $G(\Upsilon)$, $\mu\sigma(aa^{-1}) = \mu\sigma(a^{-1}a)$, implies
\[ \mu\sigma((r_0^u)^\delta(r_0^u)^{-\delta}) = \mu\sigma((r_0^u)^\varepsilon(r_0^u)^{-\varepsilon}). \]

If we apply the latter to pairs $(c_i, d_i)$ for which $v_i^\alpha = v_i^\beta$, we get that $\mu\sigma(C)(\mu\sigma(D)) = 1$ which shows that $\mu\sigma(a_1 \cdots a_n) \in \hat{\Lambda}_1$. If we are now given that $\phi^{-1}(\hat{\Lambda}_1) = \hat{U}_1$, then $\mu_1\sigma_1(a_1 \cdots a_n) \in \hat{U}_1$. Proposition 2.3 and theorem 2.6 imply that $\sigma_1(a_1 \cdots a_n) \sim_{\mu_1} 1$ proving that $\mathcal{P}_1$ is aspherical. For the converse, assume that $\hat{U}_1 \neq \phi^{-1}(\hat{\Lambda}_1)$. It follows that there is an identity $Y_1$-sequence $(a_1, ..., a_n)$ such that $\mu_1\sigma_1(a_1 \cdots a_n) \in \phi^{-1}(\hat{\Lambda}_1) \setminus \hat{U}_1$ contrary to the assumption of the asphericity for $\mathcal{P}_1$. \qed

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