QCD, Wick’s Theorem for KdV $\tau$-functions and the String Equation.

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Abstract

Two consistency conditions for partition functions established by Akemann and Damgaard in their studies of the fermionic mass dependence of the QCD partition function at low energy (a la Leutwiller-Smilga-Verbaarschot) are interpreted in terms of integrable hierarchies. Their algebraic relation is shown to be a consequence of Wick’s theorem for 2d fermionic correlators (Hirota identities) in the special case of the 2-reductions of the KP hierarchy (that is KdV/mKdV). The consistency condition involving derivatives is an incarnation of the string equation associated with the particular matrix model (the particular kind of the Kac-Schwarz operator).

1. Recently G.Akemann and P.Damgaard [1] established two consistency relations for the partition functions of matrix models which connect finite-volume partition functions with different fermion numbers. In the course of their paper the similarity of these relations with different bilinear relations for the $\tau$-functions of the KP hierarchy were noted and they conjectured they were in fact related. This letter provides a comment on the origin of these reduction formulae. We show they are direct corollaries of Wick’s theorem and the string equations of the $\tau$-function theory. Application of this theorem is possible because the partition functions of Cartanian matrix models are KP/Toda $\tau$-functions (see [3, 4] and references therein, especially [5] and [6], for the generic theory and [7, 2] for applications to unitary-matrix models). The algebraic relation (“Consistency Condition I” of ref.[1]) is true for any 2-reduction of KP $\tau$-function, which are essentially the KdV or mKdV reductions¹, of which the Brezin-Gross--Witten (BGW) model [8] under actual examination in [1] is an example. The differential relation (“Consistency Condition II”) is the string equation, depending on the specifics of the particular model. (The BGW model is studied from the perspective of integrability theory in ref.[2].)

These consistency conditions are imposed onto the QCD partition function as a function of fermionic masses $\mu_i$ at very low energies, which is of the general form

$$Z^{(N_f)}_\nu(\{\mu\}) = \det \Phi(\{\mu\})/\Delta(\{\mu^2\}).$$ (1)

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¹The KdV and mKdV hierarchies are basically the same reduction of the (m)KP hierarchy. One deals with the mKdV case when one additionally considers the discrete (Toda) time “evolution”. While generically the evolution of the KP $\tau$-function $\tau_n$ in this discrete time is governed by the Toda dynamics, in the case of the 2-reduction there exist only two different values of the zero time due to the periodic boundary conditions, $\tau_{n+2} = \tau_n$. The corresponding $\tau$-functions $\tau_0$ and $\tau_1$ give the mKdV hierarchy.
Here the matrix $\Phi$ is defined by

$$
\Phi(\{\mu\})_{ij} = \phi_i(\mu_j), \quad i, j = 1, \ldots, N_f,
$$

for some functions $\phi_i(\mu)$ yet to be specified. The denominator is given by the Vandermonde determinant of the squared masses

$$
\Delta(\{\mu^2\}) = \prod_{i>j}(\mu_i^2 - \mu_j^2) = \det(\mu_i^2)^{j-1}.
$$

Now the consistency conditions I and II of (4) acquire the forms respectively

$$
\det_{1 \leq a, b \leq k} \left[ \frac{Z^{(N_f+2k)}_\nu(\{\mu_i, \xi_a, \eta_b\})}{Z^{(N_f)}_\nu(\{\mu_i\})} \right] = \prod_{i<j}(\xi_i^2 - \xi_j^2)(\eta_i^2 - \eta_j^2) \frac{Z^{(N_f+2k)}_\nu(\{\mu_i, \xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k\})}{Z^{(N_f)}_\nu(\{\mu_i\})},
$$

and

$$
Z^{(N_f+2k)}_\nu(\{\mu_i, \xi, \eta\}) = \left[ \frac{1}{(\xi^2 - \eta^2)Z^{(N_f)}_\nu(\{\mu_i\})} \times \left[ \sum_{i=1}^{N_f} \mu_i \partial_{\xi_i} + \xi \partial_\xi \right] Z^{(N_f+1)}_\nu(\{\mu_i, \xi\}) \right] Z^{(N_f+1)}_\nu(\{\mu_i, \eta\}) - (\xi \leftrightarrow \eta).
$$

In what follows we discuss these equations, their origins and solutions.

2. We begin with the universal (in the world of 2-reductions of KP) algebraic relation (4). It states, in particular, that for certain sets of functions $\phi_i(\zeta)$, $i = 1, 2, \ldots$, and for any $k$

$$
\frac{\det_{(i,j)}^{(2k)}(\phi_i(\zeta_j))}{\Delta(\zeta^2)} = \frac{1}{\Delta(\zeta^2)\Delta(\eta^2)} \det_{(a,b)}^{(k)} \left( \begin{array}{c|c} \phi_1(\zeta_a) & \phi_1(\eta_b) \\ \hline \phi_2(\zeta_a) & \phi_2(\eta_b) \end{array} \right) \left( \begin{array}{c} \xi_a^2 - \eta_b^2 \end{array} \right).
$$

Here $i, j = 1, \ldots, 2k$, while $a, b = 1, \ldots, k$ and the set of $2k$ variables $\{\zeta_j\}$ is (arbitrarily) split into two subsets $\{\zeta_a\}$ and $\{\zeta_b\}$ of $k$ variables each (for example, $\zeta_a = \zeta_k$, $\zeta_{a+k} = \zeta_b$).

Obviously, relation (4) implies that all the functions $\phi_i(\zeta)$ are expressed in a certain way through the first two, $\phi_1(\zeta)$ and $\phi_2(\zeta)$. Sufficient conditions for (4) to be true are, for example, the recurrence relations:

$$
\phi_{i+2}(\zeta) = \zeta^2 \phi_i(\zeta) + \sum_{j=1}^{i+1} A_{ij} \phi_j(\zeta), \quad A_{ij} = \text{const.}
$$

In the $\tau$-function theory this recursion corresponds to the (m)KdV reduction. The derivation of (4) from (4) is an easy algebraic exercise (it can be done straightforwardly by induction in $k$ or one can expand $\phi_i(\zeta) = \zeta^{i-1} \sum_{k=0}^{\infty} \rho_{ik} \zeta^{-k}$ as a formal series in $\zeta^{-1}$ and express the determinants through the characters of $\text{SL}(N_f)$, see refs. (3) for examples).

Observe that, among other equations, the consistency condition I (4) and similarly (4) are invariant with respect to multiplication of all the $\phi_i(\mu)$ by an arbitrary function $f(\mu)$.
3. We shall now comment on how (6) and, more generally, (4), follow from the \( \tau \)-function theory. There are three ingredients needed from that theory:

1. Wick’s theorem and its realisation (the Hirota equation) in terms of the KP/Toda \( \tau \)-function in Miwa coordinates (involving positive and negative shifts of time-variables, see below);

2. Determinant formulae for \( \tau \) in Miwa coordinates (with positive shifts only);

3. The fact that \( \tau \) for a \( p \)-reduced hierarchy is essentially independent of the times \( t_{pk} \).

Specifically, the \((m)KdV\) \( \tau \)-function is independent of even times:

\[
\partial \tau_{KdV}/\partial t_{2n} = 0.
\]

Generally \[6, 4\] the \( \tau \)-function is only defined up to times linear in the exponential:

\[
\partial^2 \log \tau/\partial t_{pk}^2 = 0.
\]

This is equivalent to the freedom to multiply the functions \( \phi_i(\mu) \) by a function \( f(\mu) \) mentioned earlier. Here we are making the specific choice that yields (8).

This last property allows one to essentially identify the positive and negative Miwa shifts of time-variables for the \((m)KdV\) \( \tau \)-functions and combine Wick’s theorem and the determinant representation into a common identity (6) as we shall now demonstrate.

4. Wick’s theorem (in the restricted form which we need here) is an identity between correlators for any free-fermion theory with action of the form \( \int \psi(\xi)G(\xi, \eta)\tilde{\psi}(\eta) d^D \xi d^D \eta \). It says

\[
\langle \prod_{a=1}^{k} \psi(\xi_a)\tilde{\psi}(\eta_a) \rangle = \det^{(k)}_{(a,b)} \langle \psi(\xi_a)\tilde{\psi}(\eta_b) \rangle.
\]

For the KP/Toda \( \tau \)-functions, when \( D = 2 \) and \( G(\xi, \eta) \) is a first-order differential operator, \( G(\xi, \eta) = \delta(\xi, \eta)(\overline{\partial}_\xi + \overline{A}(\xi)) \), the correlators can be written as

\[
\langle \prod_{a=1}^{k} \psi(\xi_a)\tilde{\psi}(\eta_a) \rangle = \frac{\tau_{KP}\left(t + \sum_{a} ([\xi_a] - [\eta_a])\right)}{\tau_{KP}(t)} \cdot \frac{\Delta(\xi)\Delta(\eta)}{\prod_{a,b}(\xi_a - \eta_b)}.
\]

The dependence on times \( \{t\} \) on the left-hand-side of this expression is hidden in the definition of the brackets. Here the arguments of the \( \tau \)-function are Miwa-shifted time-variables

\[
t_n = \sum_{a=1}^{k} \left( \xi_a^{-n} - \eta_a^{-n} \right)
\]

and the Vandermonde determinants \( \Delta(\xi) = \prod_{a<b}(\xi_a - \xi_b) \) etc.) arise from the normal orderings. While

\[
\tau_{KP}\left(t + \sum_{a} ([\xi_a] - [\eta_a])\right) = \tau_{KP}(t)\langle \prod_{a=1}^{k} \psi(\xi_a)\tilde{\psi}(\eta_a) \rangle
\]

is non-singular at coincident points \( \xi \) and \( \eta \), the correlator in (10) vanishes when any two of \( \xi \) or any two of \( \eta \) collide and have a pole when any \( \xi \) approaches any \( \eta \). The Vandermonde determinants encode this singular behaviour and corresponds to the naive (“Riemann sphere”) normal ordering. One sometimes uses more sophisticated orderings associated with non-trivial Riemann surfaces, but this is not the case (for the matrix models) under consideration.
5. In Wick’s theorem, the numbers of $\xi$ and $\eta$ variables (i.e. of the $\psi$ and $\tilde{\psi}$ operators, or of the positive and negative Miwa shifts) are necessarily the same. However, one can take a limit when all the $\eta$ points collide, say at infinity. Then Wick’s theorem implies a peculiar determinant formula for the $\tau$-function in Miwa parametrisation with only positive Miwa shifts remaining:

\[
\tau_{\text{KP}}(t + \sum_a [\xi_a]) = \frac{\det(k^{a,b})\phi_a(\xi)}{\Delta(\xi)},
\]

From this perspective, the functions $\phi_i(\xi)$ arise as expansion coefficients in

\[
< \psi(\xi)\tilde{\psi}(\eta) > = \sum_{i=0}^{\infty} \phi_i(\xi)\eta^{-i}.
\]

Note that all the time-dependence is hidden in the shapes of the functions $\phi_i$. In particular, one can make a further Miwa transformation of the times $t$ in (12) enlarging the right-hand determinant of (12) and correspondingly changing the functions $\phi_i$, though they can of course be taken out to produce determinant formulae with additional sets of $N_f$ parameters $\mu_1, \ldots, \mu_{N_f}$.

At this stage nothing is gained by substituting the determinant formula (12) back into the Wick identity (9), from which it was derived: expansion of both sides of (9) in powers of $\eta^{-i}$ will produce nothing new.

6. However, in the case of (m)KdV $\tau$-functions there is an amusing way of introducing (another set of) $\eta$ parameters back into the game. Namely, because of property (8),

\[
\tau_{\text{KdV}}(t + [\xi] - [\eta]) = \tau_{\text{KdV}}(t + [\xi] + [\eta])
\]

since the difference between the arguments is seen (by (11)) only in the even times $t_{2n}$, on which $\tau_{\text{KdV}}$ does not depend. Now one can apply Wick’s theorem (9) with the substitution (10) to the left-hand-side of (13) and the determinant formula (12) to its right-hand-side. Straightforward calculation produces the relation (6).

Note now that for the ratio of determinants (12) to be a $\tau$-function, the functions $\phi_i(\mu)$ are to have prescribed asymptotics at large $\mu$, usually $\phi_i(\mu) \sim \mu^{i-1}$ (see, however, [6]). This ensures that the $\tau$-function is an $(N_f$-independent) function of the usual times $t_k = 1/k \sum_{i}^{N_f} \mu_i^{-k}$, and so allows an expansion as a formal series in the time variables $t_k$. These asymptotic properties remain unchanged when the functions $\phi_i(\mu)$ are multiplied by a function $f(\mu)$ that behaves like 1 for large $\mu$. In particular, if $\log f(\mu)$ expands into a Taylor series in $1/\mu$, such a redefined $\tau$-function is multiplied by a linear exponential of times $t_k$. Therefore, the property (8) holds for the (m)KdV $\tau$-function only for a very specific normalisation of the $\phi_i(\mu)$.

We have now shown how consistency condition I (4) can be understood in terms of Wick’s theorem for a particular (m)KdV $\tau$-function. It remains to discuss consistency condition II and the matrix theory origins.

7. While the algebraic relation (14) allows one to impose restrictions on $\phi_i(\mu)$ that lead to an (m)KdV $\tau$-function, i.e. express all $\phi_i(\mu)$’s through the first two, the differential relation (15) expresses all of the $\phi_i(\mu)$’s in terms of derivatives of the first. The combination of these two restrictions then fixes the $\phi_i(\mu)$ (up to a linear combination of rows in the determinant and a trivial prefactor). In terms of integrable hierarchies, this relation is called the string equation...
The role of this equation is to fix a unique solution of the integrable hierarchy (specified by the algebraic relation and the form of $Z^{(N_f)}$).

Let us start with the simplest case of $N_f = 0$. Then, implies that $\phi_2(\mu) \sim \mu \partial \phi_1(\mu)/\partial \mu$. This is, however, still not a restriction on $\phi_1(\mu)$ itself. In order to see how restricts all the $\phi_i(\mu)$, one needs to consider it at all values of $N_f$. In doing so, it is necessary to use the equation that is tautologically satisfied by the ratio of determinants:

$$\tau_{N_f+2}(\{\mu\}, \xi, \eta) = \frac{1}{(\xi - \eta)\tau_{N_f}(\{\mu\})} \left[ \tau_{N_f+1}(\{\mu\}, \xi) \tau_{N_f+1}(\{\mu\}, \eta) - (\xi \leftrightarrow \eta) \right]$$

(14)

where the hat over $\tau_N$ means that the last row $\phi_N(\{\mu\})$ is substituted with $\phi_{N+1}(\{\mu\})$. This equation is typically used for derivation of the string equation.

Now, since $\sum_i \mu_i \partial \mu_i \Delta_N(\mu^2) = \Delta_N(\mu^2)$, one can rewrite (3) as

$$\det^{(N_f+2)}(\{\mu\}, \xi, \eta) \cdot \det^{(N_f)}(\{\mu\})$$

$$= \left[ \left( \sum_i \mu_i \partial \mu_i + \xi \partial \xi \right) \det^{(N_f+1)}(\{\mu\}, \xi) \right] \det^{(N_f+1)}(\{\mu\}, \eta) - (\xi \leftrightarrow \eta).$$

Comparing with (14), we conclude that

$$\left[ \left( \sum_i \mu_i \partial \mu_i + \xi \partial \xi \right) \det^{(N_f+1)}(\{\mu\}, \xi) \right] \det^{(N_f+1)}(\{\mu\}, \eta) - (\xi \leftrightarrow \eta)$$

$$= \det^{(N_f+1)}(\{\mu\}, \xi) \det^{(N_f+1)}(\{\mu\}, \eta) - (\xi \leftrightarrow \eta).$$

(15)

with the evident notation $\hat{\Phi}$ denoting the matrix whose last row $\phi_{N+1}(\{\mu\})$ has been substituted with $\phi_{N+1}(\{\mu\})$. Induction on $N_f$ (we have already shown it true for $N_f = 0$ above) shows this equation is solved if

$$\det \hat{\Phi} = \left( \sum_i \mu_i \partial \mu_i \right) \det \Phi + N_f \alpha \det \Phi,$$

(16)

$\alpha$ being an arbitrary constant. In fact this relation is another version of the string equation (cf. with 3). The term proportional to $\det \Phi$ can always be absorbed by the simultaneous rescaling all $\phi_i(\mu)$ with $\mu^\alpha$ and yields in its turn that $\phi_i(\mu) \sim (\mu \partial \mu)^{\nu-i} \phi_1(\mu)$.

Therefore, the consistency condition II really restricts the functions $\phi_1(\mu)$. Combining the form $\phi_i(\mu) \sim (\mu \partial \mu)^{\nu-i} \phi_1(\mu)$ together with the reduction condition (consistency condition I), one arrives at the solution

$$\phi_i(\mu) \sim \mu^{\alpha+i-1} I_{\nu+i-1}(\mu)$$

(17)

where $I_k(\mu)$ are the modified Bessel functions, $\nu$ and $\alpha$ are arbitrary parameters (our solution slightly differs from that in 1 by the additional parameter $\alpha$).

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2 Up to a term proportional to $\phi_1(\mu)$; there is always a freedom to add to any $\phi_k(x)$ a linear combination of $\phi_i(\mu)$ with $i < k$ with constant coefficients, since it does not change the determinant in 1. Hereafter, we denote this freedom with the sign "°".

3 The authors of 1 correctly suggested its relevance to the problem. However, let us emphasise again that this equation does not select out any particular $\tau$-function, or any particular $\phi_i(\mu)$. This is just another form of formula 1.
The solution (17) obtained for $Z^{(Nf)}_\nu$ can also be presented in terms of the matrix integral

$$Z^{(Nf)}_\nu(\{\mu\}) \sim \int dX e^{Tr[\nu X^2/4-\nu X]} \log X + 1/X + (\alpha-\nu) \log M$$

where the integral is taken over Hermitian $N_f \times N_f$ matrix $X$ and $\mu_i$ are the eigenvalues of the matrix $M$. For the particular values $\alpha = \nu = 0$ this integral may be rewritten as an integral over unitary $N_f \times N_f$ matrix $U$,

$$Z^{(Nf)}_0(\{\mu\}) \sim \int dU e^{Tr[\nu X U^T U]} , \quad M^2 = JJ^\dagger.$$

The function $Z^{(Nf)}_\nu(\{\mu\})$ by itself is certainly not a $\tau$-function for the Miwa coordinates (11) we are using here (chosen because of the formulation of [1]). As we noted earlier, for the ratio of determinants (12) to be a $\phi$ over unitary $N$ matrix $M$ in $1/X$ we required the asymptotics of $\phi_i(\mu)$ at large $\mu$ to be normalised by $\mu^{N-1}$, which are not those of (17). An easy calculation shows that, in order to obtain a $\tau$-function, one needs to include several prefactors

$$\tau_\nu(\{t\}) = \prod_{i<j} (\mu_i + \mu_j) \prod_i \sqrt{\mu_i \mu_i^{-1}} e^{-\mu_i} Z^{(Nf)}_\nu(\{\mu\}).$$

Here the parameter $\nu$ plays role of the discrete time (in the standard notation of [2] it has the opposite sign). Such prefactors typically emerge in matrix model theory: to obtain a $\tau$-function, one should factor out a quasi-classical piece from the matrix integral. For the particular integral (18) this exactly gives (20), and in fact reference [2] also shows the independence of even times we have wanted here. Thus we return to the matrix model integrals that inspired the consistency conditions we have been discussing.

In concluding let us observe that these integrals can be further generalised yielding other $\tau$-functions. In particular, one can consider an arbitrary function $V(X)$ polynomial in $X$ or $1/X$. Then, the integral

$$C_V \int dX e^{Tr[X L - (\nu + N_f) \log X + V(X)]}$$

results in a two-dimensional Toda lattice $\tau$-function,

$$\tau_\nu(\{t\}) = \frac{\det_{ij} V_{ij}}{\Delta(\mu)} .$$

Here $C_V$ is the normalisation factor cancelling the quasi-classical part of the integral, and $\mu_i$ are related with the eigenvalues $l_i$ of the matrix $L$ via the saddle point condition $l + V'(\mu) = 0$ or $l + V'(1/\mu) = 0$ depending on whether $V(X)$ is a polynomial of $X$ or $1/X$ respectively. The time variables are again $t_k = 1/k \sum_i \mu_i^{-k}$ and

$$\psi_{i-\beta}(\mu) \equiv \frac{1}{\sqrt{V'(\mu)}} e^{\nu V'(\mu)} \int dx x^{(i,\beta)} e^{x=1+V(x)}$$

where $(i, \beta)$ is equal to $i - \beta - 1$ for $V(X)$ being a polynomial in $X$ and $\beta - i$ for a polynomial in $1/X$.

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