INTERVAL MV-ALGEBRAS AND GENERALIZATIONS

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ABSTRACT. For any MV-algebra $A$ we equip the set $I(A)$ of intervals in $A$ with pointwise Łukasiewicz negation $\neg x = \{-\alpha \mid \alpha \in x\}$, (truncated) Minkowski sum, $x \oplus y = \{\alpha \ominus \beta \mid \alpha \in x, \beta \in y\}$, pointwise Łukasiewicz conjunction $x \otimes y = \neg (\neg x \oplus \neg y)$, the operators $\Delta x = [\min x, \min x]$, $\nabla x = [\max x, \max x]$, and distinguished constants $0 = [0, 0], 1 = [1, 1], i = A$. We list a few equations satisfied by the algebra $I(A) = (I(A), 0, 1, i, \neg, \Delta, \nabla, \oplus, \otimes)$, call IMV-algebra every model of these equations, and show that, conversely, every IMV-algebra is isomorphic to the IMV-algebra $I(B)$ of all intervals in some MV-algebra $B$. We show that IMV-algebras are categorically equivalent to MV-algebras, and give a representation of free IMV-algebras. We construct Łukasiewicz interval logic, with its coNP-complete consequence relation, which we prove to be complete for $I([0,1])$-valuations. For any class $Q$ of partially ordered algebras with operations that are monotone or antimonotone in each variable, we consider the generalization $I_Q$ of the MV-algebraic functor $I$, and give necessary and sufficient conditions for $I_Q$ to be a categorical equivalence. These conditions are satisfied, e.g., by all subquasivarieties of residuated lattices.

1. Foreword

As shown in [28, §1.6], truth-values in Łukasiewicz logic may be thought of as arising from normalized measurements of bounded physical observables, just as boolean truth-values arise from \{yes, no\}-observables. Łukasiewicz implication is uniquely characterized among all binary operations on $[0,1]$ by the Smets-Magrez theorem, [1, 31], as the only $[0,1]$-valued continuous map on $[0,1]^2$ satisfying natural monotonicity conditions with respect to the natural order of $[0,1]$. These conditions yield the classical Łukasiewicz axioms of infinite-valued logic $L_\infty$, which, via Modus Ponens, determine the consequence relation of $L_\infty$. Closing a circle of ideas about truth-values as real numbers, we recover the intended meaning of $L_\infty$-formulas, by a well known completeness theorem [30], [11], [13, §2.5] to the effect that $L_\infty$-tautologies (i.e., $L_\infty$-consequences of the Łukasiewicz axioms) coincide with formulas taking value 1 for every $[0,1]$-valuation.

To achieve greater adherence to actual physical measurements—and more generally, to formally handle the imprecise estimations/evaluations of everyday life—one might envisage logics whose truth-values are the closed intervals in $[0,1]$. This paper is devoted to developing the algebraic and categorical tools for the construction of such logics, using Łukasiewicz logic as a template. Mimicking the approach to Łukasiewicz logic via MV-algebras, the set $I(A)$ of intervals in any MV-algebra $A$ is equipped with pointwise Łukasiewicz negation $\neg x = \{-\alpha \mid \alpha \in x\}$, (truncated) Minkowski sum, $x \oplus y = \{\alpha \ominus \beta \mid \alpha \in x, \beta \in y\}$, pointwise Łukasiewicz conjunction $x \otimes y = \neg (\neg x \oplus \neg y)$, the operators $\Delta x = [\min x, \min x]$, $\nabla x = [\max x, \max x]$, and distinguished constants $0 = [0, 0], 1 = [1, 1], i = A$.

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We list nineteen simple equations (9)-(27) satisfied by the algebra
\[ \mathcal{I}(A) = (I(A), 0, 1, \neg, \Delta, \nabla, \oplus, \odot), \]
and call IMV-algebra every model of these equations. The adequacy of these equations is shown in
Theorem 3.4 stating that, conversely, every IMV-algebra is isomorphic to the IMV-algebra \( \mathcal{I}(B) \) of
all intervals in some MV-algebra \( B \). While no IMV-algebra reduct is an MV-algebra, the functor
\( \mathcal{I} \) establishes a categorical equivalence between MV-algebras and IMV-algebras (Theorem 4.4).

In Section 5 we obtain the following equational completeness theorem (Theorem 5.1): an equation is satisfied by all IMV-algebras iff it is satisfied by the IMV algebra \( \mathcal{I}([0, 1]) \) of all intervals in
the standard MV-algebra \( [0, 1] \) iff it is derivable from the IMV-axioms (9)-(27) by the familiar
rules of replacing equals by equals according to Birkhoff equational logic. This result has a deeper
algebraic counterpart in the representation Theorem 5.2 of free IMV-algebras.

Every IMV-algebra \( J \) is equipped with two types of partial orders: the product order and the
inclusion order. The first endows \( J \) with a distributive lattice structure \( (J, \sqcup, \sqcap) \); the second yields
an upper semilattice structure \( (J, \cup) \). In the final part of Section 5 it is proved that each operation
\( \sqcap, \sqcup, \cup \) is definable from the IMV-structure. With reference to our initial remarks on truth-values
as intervals, IMV-algebras can express the fundamental inclusion order \( u \subseteq v \) between actual
measurements/estimations \( u, v \), meaning that \( u \) is more precise than \( v \). Within the MV-algebraic
framework, the inclusion order has no meaning, despite MV-algebras are categorically equivalent
to IMV-algebras.

Closing another circle of ideas, about truth-values as intervals, in Section 7 we will introduce
Lukasiewicz interval logic, with its consequence relation based on the only rule of Modus Ponens,
and prove a completeness theorem for \( \mathcal{I}([0, 1]) \)-valuations. The consequence problem in this logic,
just like the equational theory of IMV-algebras turns out to be coNP-complete (Theorem 6.1,
Corollary 7.2). Thus, the increased expressive power of Lukasiewicz interval logic with respect to
Lukasiewicz logic does not entail greater complexity of the consequence problem.

In the all-important Section 8, for any class \( Q \) of partially ordered algebras with operations
that are monotone or antimonotone in each variable, we consider the generalization \( \mathcal{I}_Q \) of
the MV-algebraic functor \( \mathcal{I} \), and give necessary and sufficient conditions for \( \mathcal{I}_Q \) to be a categorical
equivalence. As shown in Corollary 8.12, these conditions are satisfied, e.g., by every quasivariety
\( Q \) having a lattice reduct—including many classes of ordered algebras related with logical systems
of general interest, such as BL-algebras, Heyting algebras, Gödel algebras, MTL-algebras, and
more generally every subquasivariety of residuated lattices (Corollary 8.13).

Remarkably enough, the pervasiveness of the categorical equivalence \( \mathcal{I}_Q \) has gone virtually
unnoticed in the literature on interval and triangle algebras and their logics, interval constructors
and triangularizations. The final Section 9 is devoted to relating our results to the extensive
literature on this subject, [2, 3, 4, 14, 15, 32, 33].

The only prerequisite for this paper is some acquaintance with MV-algebras [13], Birkhoff-style
universal algebra [8], and the rudiments of category theory [24].

2. The equational class of IMV-algebras

Let \( A = (A, 0, 1, \neg, \oplus, \odot) \) be an MV-algebra. By an interval of \( A \) we mean a subset \( x \) of \( A \) of
the form \( x = [\alpha, \beta] = \{ \xi \in A \mid \alpha \leq \xi \leq \beta \} \), where \( \alpha, \beta \in A \) and \( \alpha \leq \beta \). In case \( \alpha = \beta \) we say that \( x \) is
degenerate. We let \( I(A) \) denote the set of intervals in \( A \), and \( D(A) \subseteq I(A) \) the set of degenerate
intervals. We record here a first result, to the effect that “sums of intervals are intervals”:

**Proposition 2.1.** For any MV-algebra \( A \) and intervals \( H, K \in I(A) \), let \( H \oplus K \) denote the
(truncated) Minkowski sum of \( H \) and \( K \),
\[ H \oplus K = \{ \alpha \in A \mid \alpha = \mu \oplus \nu \text{ for some } \mu \in H \text{ and } \nu \in K \}. \]

Then \( H \oplus K \) is an interval in \( A \). Similarly, the set
\[ H \odot K = \{ \alpha \in A \mid \alpha = \mu \odot \nu \text{ for some } \mu \in H \text{ and } \nu \in K \} \]
is an interval in $A$.

Proof. We first prove the following special case:

$$[0, \alpha] \oplus [0, \beta] = [0, \alpha \oplus \beta] \quad \text{for all } \alpha, \beta \in A.$$  \hfill (1)

We will make use of the derived lattice operations $\wedge, \vee$ and the natural order $\leq$ of $A$, [13, §1.1].

Given $x \in [0, \alpha \oplus \beta]$ let $\beta' = \beta \wedge x$ and $\alpha' = x \ominus \beta'$. Then

$$\alpha' \oplus \beta' = (x \ominus \beta') \oplus \beta' = \neg (\neg x \ominus \beta') \oplus \beta' = x \vee \beta' = x.$$  

Further, $\alpha' = x \ominus \beta' = x \ominus (x \ominus \beta) = x \ominus (\neg x \lor \neg \beta) = (x \ominus \neg x) \lor (x \ominus \neg \beta) = x \ominus \neg \beta$.

Now from $\alpha' \ominus \neg \alpha = (x \ominus \neg \beta) \ominus \neg \alpha = x \ominus \neg (\beta \ominus \alpha) \leq x \ominus \neg x = 0$ we get $\alpha' \leq \alpha$. Since $\beta' \leq \beta$, (1) is settled. One now easily proves $[\alpha, \alpha] \oplus [0, \beta] = [\alpha, \alpha \oplus \beta]$, and more generally, $[\alpha, \alpha] \oplus [\delta, \beta] = [\alpha \oplus \delta, \alpha \oplus \beta]$. A final verification using all these preliminary results yields the desired conclusion $[\delta, \alpha] \oplus [\theta, \beta] = [\delta \oplus \theta, \alpha \oplus \beta]$. Since in every MV-algebra $\alpha \ominus \beta = \neg (\neg \alpha \ominus \neg \beta)$, one immediately verifies that $H \ominus K$ is an interval in $A$. (Readers familiar with the categorical equivalence $\Gamma$ between MV-algebras and unital abelian $\ell$-groups ([27, §3]) will observe that (1) is a special case of the Riesz decomposition property ([5, Lemma 1, page 310, and Theorem 49, p. 328]) of the unital abelian $\ell$-group $(G, u)$ defined by $\Gamma(G, u) = A$.) \hfill \Box

In view of the foregoing result, for any MV-algebra $A$, the set $I(A)$ is made into an algebra $I(A) = (I(A), 0, 1, i, \ominus, \Delta, \nabla, \ominus, \odot)$ of type $(0, 0, 0, 1, 1, 1, 2, 2)$, by equipping it with the distinguished constants

$$0 = [0, 0], \; 1 = [1, 1], \; i = A,$$  \hfill (2)

the operations

$$\neg x = \{ \neg \alpha \mid \alpha \in x \}, \quad \text{pointwise negation}$$  \hfill (3)

$$x \oplus y = \{ \alpha \oplus \beta \mid \alpha \in x, \beta \in y \}, \quad \text{(truncated) Minkowski sum}$$  \hfill (4)

$$x \odot y = \{ \alpha \odot \beta \mid \alpha \in x, \beta \in y \}, \quad \text{pointwise Lukasiewicz conjunction},$$  \hfill (5)

and the lower/upper collapse operations

$$\Delta x = [\min x, \min x],$$  \hfill (6)

$$\nabla x = [\max x, \max x],$$  \hfill (7)

where the min (resp., the max) of an interval $x = [\alpha, \beta] \in I(A)$ equals $\alpha$ (resp., equals $\beta$), according to the natural order $\leq$ of $A$. From the context it will always be clear whether the constants $0, 1$ and the operations $\neg, \ominus, \odot$ are those of $I(A)$ or those of $A$.

Form the definition of $I(A)$, the set $D(A) \subseteq I(A)$ is the universe of a subalgebra $D(A)$ of the MV-algebra reduct of $I(A)$. Clearly, the map $\iota_A : \alpha \in A \mapsto [\alpha, \alpha] \in D(A)$ becomes an isomorphism of $A$ onto $D(A)$, in symbols,

$$\iota_A : A \cong D(A).$$  \hfill (8)
Proposition 2.2. The algebra $\mathcal{I}(A)$ satisfies the following equations:

\[
\begin{align*}
x \oplus (y \oplus z) &= (x \oplus y) \oplus z \quad (9) \\
x \oplus y &= y \oplus x \quad (10) \\
x \oplus 0 &= x \quad (11) \\
x \oplus -0 &= -0 \quad (12) \\
\neg x &= x \quad (13) \\
\neg(\Delta x \oplus \Delta y) \oplus \Delta y &= \neg(\Delta y \oplus \Delta x) \oplus \Delta x \quad (14) \\
x \odot y &= \neg(\neg x \oplus -y) \quad (15) \\
1 &= \neg 0 \quad (16) \\
\nabla x &= \neg \Delta \neg x \quad (17) \\
\neg i &= i \quad (18) \\
\Delta 0 &= 0 \quad (19) \\
\Delta 1 &= 1 \quad (20) \\
\Delta i &= 0 \quad (21) \\
\Delta \Delta x &= \Delta x \quad (22) \\
\nabla \Delta x &= \nabla x \quad (23) \\
\nabla \Delta (x \odot y) &= \Delta \nabla x \quad (24) \\
\nabla \Delta (x \odot y) &= \Delta \nabla \Delta x \quad (25) \\
\Delta x \odot \neg \nabla x &= 0 \quad (26) \\
\Delta x \oplus (i \odot \nabla x \odot \neg \Delta x) &= x. \quad (27)
\end{align*}
\]

Proof. (9)-(13) are easily verified properties of pointwise negation and Minkowski (truncated) sum (4). They are inherited by $\mathcal{I}(A)$ from the corresponding properties of negation and truncated sum in $A$, [13, Definition 1.1.1].

To prove (14), we first observe that both terms in this equation depend on $x$ and $y$ only via $\Delta x$ and $\Delta y$. Since $\Delta x$ and $\Delta y$ are degenerate intervals of $\mathcal{I}(A)$, we can write $\Delta x = [\alpha, \alpha] = \iota_A(\alpha)$ and $\Delta y = [\beta, \beta] = \iota_A(\beta)$ for some $\alpha, \beta \in A$. Since $A$ is an MV-algebra, it satisfies the identity $\neg(\neg \alpha \oplus \beta) \oplus \beta = \neg(\neg \beta \oplus \alpha) \oplus \alpha$. Thus by (8), $\mathcal{I}(A)$ satisfies (14).

For the proof of (15) one uses (5) and the well known definability, [13, §1], of $\odot$ from $\neg$ and $\oplus$ in $A$, as well as in its isomorphic copy $D(A)$.

Equations (16)-(23) are immediate consequences of (2) and definitions (6)–(7).

For the verification of (24) one combines (4) with the translation invariance of the natural order of $D(A)$, i.e., the distributivity of $\oplus$ and $\odot$ over $\lor$ and $\land$ [13, Proposition 1.1.6].

Equation (25) easily follows by (6), (15) and (24).

As a preliminary step for the proof of (26), one notes that the left hand term therein depends on $x$ only via $\Delta x$ and $\nabla x$. Now, $\Delta x, \nabla x \in D(A)$ are degenerate intervals in $A$, acted upon by the $\iota_A$-images of the MV-algebraic operations of $A$. Equation (26) states that for every interval $x = [\alpha, \beta] \in \mathcal{I}(A)$, $\Delta x \leq \nabla x$, i.e., $[\min x, \min x] \leq [\max x, \max x]$, i.e., $[\alpha, \alpha] \leq [\beta, \beta]$ in the natural order of $D(A)$ inherited from the natural order of $A$ via the isomorphism $\iota_A$. Going back via $\iota_A^{-1}$, (26) amounts to the inequality $\alpha \leq \beta$, which holds in $A$ by definition of interval. Thus $\mathcal{I}(A)$ satisfies (26).

Finally, for the verification of (27) letting $[\alpha, \beta] \in \mathcal{I}(A)$ we can write

\[
\Delta [\alpha, \beta] \oplus ([0, 1] \odot \nabla [\alpha, \beta] \odot \neg \Delta [\alpha, \beta]) = [\alpha, \alpha] \oplus ([0, 1] \odot [\beta, \beta] \odot \neg [\alpha, \alpha])
\]

\[
= [\alpha, \alpha] \oplus [0, \beta \odot \neg \alpha] = [\alpha, \alpha \odot (\beta \odot \neg \alpha)]
\]

\[
= [\alpha, \beta]. \quad \square
\]
3. Representation of IMV-algebras

**Definition 3.1.** An IMV-algebra is a structure $J = (J, 0, 1, i, \neg, \Delta, \nabla, \oplus, \odot)$ of type $(0, 0, 0, 1, 1, 2, 2)$ satisfying equations (9)-(27). The center $C(J)$ of $J$ is the set of all elements $x \in J$ such that $\Delta x = \nabla x$.

From Proposition 2.2 we immediately obtain:

**Proposition 3.2.** For any MV-algebra $A$ let $\mathcal{I}(A) = (I(A), 0, 1, i, \neg, \Delta, \nabla, \oplus, \odot)$ be the algebra of intervals in $A$, as defined by (2)–(5) in view of Proposition 2.1. Then

(i) $\mathcal{I}(A)$ is an IMV-algebra.

(ii) Every IMV-algebra satisfies the following quasiequation:

$$\text{If } \Delta x = \Delta y \text{ and } \nabla x = \nabla y \text{ then } x = y.$$ (28)

(iii) For any IMV-algebra $J$, its $(0, 1, \neg, \oplus, \odot)$-reduct is not an MV-algebra. Indeed, the equation $x \oplus \neg x = 1$ fails in $\mathcal{I}([0, 1])$ (already with $x = i$). The MV-axiom $\neg(\neg \alpha \oplus \beta) \oplus \beta = \neg((\neg \beta \oplus \alpha) \oplus \alpha)$ fails in $\mathcal{I}([0, 1])$ for $x = i, y = 1$.

(iv) IMV-algebras are not term-equivalent ([23, §4]) to MV-algebras: indeed, there is no two-element IMV-algebra.

As a converse of (i) above, in Theorem 3.4 we will see that, up to isomorphism, algebras of the form $\mathcal{I}(A)$ exhaust all possible IMV-algebras.

The quasiequation (28) in is reminiscent of Moisil’s “determination principle”, [6, p.106].

**Proposition 3.3.** Let $J$ be an IMV-algebra.

(i) For all $x \in J$, $\Delta x = \nabla x$ iff $x = \nabla x$ iff $x = \Delta x$ iff $x = \Delta y$ for some $y \in J$ iff $x = \nabla z$ for some $z \in J$.

(ii) The center $C(J)$ is closed under the operations $\neg, \oplus, \odot$ of $J$ and contains the elements $0, 1$ of $J$. The resulting subreduct $C(J) = (C(J), 0, 1, \neg, \oplus, \odot)$ of $J$ is an MV-algebra, called the central MV-algebra of $J$.

(iii) For any IMV-algebra $K$ and homomorphism $\theta: J \to K$, the restriction $\theta|C(J)$ of $\theta$ to $C(J)$ is a homomorphism of $C(J)$ into $C(K)$.

(iv) Let us define the map $\gamma_J: J \to \mathcal{I}(C(J))$ by the following stipulation:

$$\gamma_J(x) = [\Delta x, \nabla x], \text{ for all } x \in J.$$ (29)

Then $\gamma_J$ maps $J$ one-one into $\mathcal{I}(C(J))$.

Proof. (i) easily follows from (22), (23) and (28).

(ii) The closure of $C(J)$ under $\neg, \oplus, \odot$ follows from (i) and (17), (24), (25). Thus, $C(J) = (C(J), 0, 1, \neg, \oplus, \odot)$ is a $(0, 1, \neg, \oplus, \odot)$-subreduct of $J$ and, as such, it necessarily satisfies equations (9)–(13). Further, any two elements $\alpha, \beta \in C(J)$ satisfy the characteristic MV-algebraic equation $\neg(\neg \alpha \oplus \beta) \oplus \beta = \neg((\neg \beta \oplus \alpha) \oplus \alpha)$. This is so because by Proposition 3.3(i) we can write $\alpha = \Delta x$, $\beta = \Delta y$ for suitable $x, y \in J$, and $J$ satisfies (14). Finally, since the constant 1 and the operation $\odot$ of $C(J)$ are definable via (16) and (15), then $C(J)$ is an MV-algebra.

(iii) Easy.

(iv) As already noted, every element $x \in J$ satisfies equation (26) stating that $(\Delta x, \nabla x)$ is a monotone pair of $C(J)$. Equation (28) ensures that different elements of $J$ are mapped by $\gamma_J$ into different elements.

The following converse of Proposition 3.2(i) shows that $\gamma_J$ is an isomorphism of $J$ onto the IMV-algebra of all intervals in $C(J)$. \qed
Lemma 4.2. Both proved that where the homomorphism $\gamma$ is evidently, then $x \in J \mapsto [\Delta x, \nabla x] \in \mathcal{I}(C(J))$ of Proposition 3.3(iv) is an isomorphism of $J$ onto $\mathcal{I}(C(J))$.

Proof. Evidently, $\gamma_J(0) = [0,0]$, $\gamma_J(1) = [1,1]$, $\gamma_J(i) = C(J)$. By (24) and Proposition 2.1, $\gamma_J(x \oplus y) = [\Delta x \oplus \Delta y, \nabla(x \oplus y)] = [\Delta x, \nabla x] \oplus [\Delta y, \nabla y] = \gamma_J(x) \oplus \gamma_J(y)$. By (13) and (17), $\gamma_J(-x) = -\gamma_J(x)$. By (25), $\gamma_J(x \otimes y) = \gamma_J(x) \otimes \gamma_J(y)$. By (22), (23) and (17) for all $x \in J$ we have $\gamma_J(\Delta x) = [\Delta \Delta x, \nabla \Delta x] = [\Delta x, \Delta x] = \Delta [\Delta x, \nabla x] = \Delta \gamma_J(x)$. Similarly, $\nabla \gamma_J(x) = \nabla \gamma_J(x)$. Let $[\alpha, \beta] \in \mathcal{I}(C(J))$, $\gamma_J(\alpha \oplus i) = [\alpha, \alpha] \oplus [0,1] = [\alpha, 1]$ and $\gamma_J(\beta \otimes i) = [0, \beta]$. Then $\gamma_J((\alpha \oplus i) \circ (\beta \otimes i)) = [\alpha, \beta]$. Thus every interval in $C(J)$ is in $\gamma_J(J)$.

4. IMV-algebras are categorically equivalent to MV-algebras

In (8) we observed that the map $\iota_A: A \to D(A) = C(I(A))$ is an isomorphism of MV-algebras. Also, in Theorem 3.4 we proved that the map $\gamma_J: J \to \mathcal{I}(C(J))$ defined by $\gamma_J(x) = [\Delta(x), \nabla(x)]$ is an isomorphism of IMV-algebras. In this section we prove that the category $\text{MV}$ of MV-algebras with homomorphisms and the category $\text{IMV}$ of IMV-algebras with homomorphisms are equivalent, and that $\iota$ and $\gamma$ are the natural isomorphisms determining the equivalence.

For all unexplained notions in category theory used in this paper, we refer to [24].

Lemma 4.1. Let $\mathcal{I}: \text{MV} \to \text{IMV}$ be the assignment defined by:

- **objects:** $A \mapsto \mathcal{I}(A)$
- **morphisms:** $h: A \to B \mapsto \mathcal{I}(h): \mathcal{I}(A) \to \mathcal{I}(B)$,

where the homomorphism $\mathcal{I}(h)$ is given by $(\mathcal{I}(h))(\alpha, \beta) = [h(\alpha), h(\beta)]$, for any interval $[\alpha, \beta] \in \mathcal{I}(A)$. Conversely, let $\mathcal{C}: \text{IMV} \to \text{MV}$ be the assignment defined by:

- **objects:** $J \mapsto C(J)$
- **morphisms:** $f: J \to K \mapsto C(f) = f \circ C(J)$.

Both $\mathcal{I}$ and $\mathcal{C}$ are well-defined functors, respectively called interval and central functor.

Lemma 4.2. The assignment $\iota: A \to \iota_A$ is a natural isomorphism from the identity functor $\text{IMV}: \text{MV} \to \text{MV}$ to the composite functor $\mathcal{I} \circ \mathcal{C}$.

Proof. By (8), for each MV-algebra $A$, the map $\iota_A$ is an IMV-isomorphism. There remains to be proved that $\iota$ is a natural transformation, that is, for every MV-algebras $A, B$ and homomorphism $h: A \to B$, the diagram of Figure 1 commutes.

![Figure 1](image)

Indeed, for each $\alpha \in A$ we can write $\mathcal{C}(\mathcal{I}(h))(\iota_A(\alpha)) = \mathcal{C}(\mathcal{I}(h))(\alpha, \alpha) = \mathcal{I}(h)(\alpha, \alpha) = [h(\alpha), h(\alpha)] = \iota_B(h(\alpha))$. 

Lemma 4.3. The assignment $\gamma: J \to \gamma_J$ is a natural isomorphism from the identity functor $\text{IMV}: \text{IMV} \to \text{IMV}$ to the composite functor $\mathcal{I} \circ \mathcal{C}$.
Proof. By Theorem 3.4, for each IMV-algebra \( J \) the map \( \gamma_J : J \to \mathcal{I}(\mathcal{C}(J)) \) is an isomorphism. To prove that \( \gamma \) is a natural transformation, let \( J, K \) be IMV-algebras and \( f : J \to K \) an IMV-homomorphism. Then the diagram of Figure 2 commutes.

\[
\begin{array}{ccc}
J & \xrightarrow{f} & K \\
\xdownarrow{\gamma_J} & & \xdownarrow{\gamma_K} \\
\mathcal{I}(\mathcal{C}(J)) & \xrightarrow{\mathcal{I}(f)} & \mathcal{I}(\mathcal{C}(K))
\end{array}
\]

Indeed, since \( f \) commutes with \( \Delta \) and \( \nabla \), for each \( x \in J \) we can write

\[
\mathcal{I}(\mathcal{C}(f))(\gamma_J(x)) = \mathcal{I}(\mathcal{C}(f))(\Delta x, \nabla x) = [\mathcal{C}(f)(\Delta x), \mathcal{C}(f)(\nabla x)] = [f(\Delta x), f(\nabla x)]
\]

\[
= [\Delta f(x), \nabla f(x)] = \gamma_K(f(x)).
\]

\( \square \)

From Lemmas 4.2 and 4.3, we obtain:

**Theorem 4.4 (Categorical equivalence).** The functors \( \mathcal{I} \) and \( \mathcal{C} \) and the natural isomorphisms \( \iota \) and \( \gamma \) determine a categorical equivalence between MV-algebras and IMV-algebras.

5. Completeness, free IMV-algebras, product and inclusion order

As a first application of the categorical equivalence \( \mathcal{I} \) between MV-algebras and IMV-algebras, we give a complete description of free IMV-algebras. Since categorical equivalence does not necessarily preserve freeness, the description of free IMV-algebras cannot be directly derived from a description of free MV-algebras.

We refer to [13] for all unexplained MV-algebraic notions used here. Fix an integer \( n > 0 \). Then a **McNaughton function** on \([0,1]^n\) is a continuous piecewise linear map \( \xi : [0,1]^n \to [0,1] \) such that each linear piece of \( \xi \) has integer coefficients. More generally, for any arbitrary (possibly infinite) set \( X \neq \emptyset \), a function \( \eta : [0,1]^X \to [0,1] \) is said to be a **McNaughton map** if for some finite \( Y \subseteq X \), and McNaughton function \( \xi : [0,1]^Y \to [0,1] \) we have

\[
\eta = \xi \circ \pi_Y^X,
\]

where \( \pi_Y^X : [0,1]^X \to [0,1]^Y \) is the projection map.

For each closed set \( S \subseteq [0,1]^X \) we let \( \mathcal{M}(S) \) denote the set of restrictions to \( S \) of the McNaughton functions on the cube \([0,1]^X\), in symbols,

\[
\mathcal{M}(S) = \{ f \mid S \cap [0,1]^X \to [0,1] \text{ a McNaughton map} \},
\]

equipped with the pointwise operations of the standard MV-algebra \([0,1]\). In this paper \( S \) will always be nonempty, so that \( \mathcal{M}(S) \) is a nontrivial MV-algebra.

We let the triangle \( \Theta \subseteq [0,1]^2 \) be defined by

\[
\Theta = \{ (\alpha, \beta) \in [0,1]^2 \mid \alpha \leq \beta \}.
\]

For each \( i = 1, 2 \) we also let \( \pi_i : \Theta \to [0,1] \) be the projection maps \( \pi_i(\alpha_1, \alpha_2) = \alpha_i \).

**Theorem 5.1 (Equational Completeness).** The operations of the IMV-algebra

\[
\Theta = (\Theta, (0,0), (1,1), (0,1), \neg, \Delta, \nabla, \oplus, \odot)
\]

are defined for each \( (a,b), (c,d) \in \Theta \) as follows:

\[
\neg(a,b) = (\neg b, \neg a), \; \Delta(a,b) = (a,a), \; \nabla(a,b) = (b,b),
\]

\( i = 1, 2 \).
(a, b) ⊕ (c, d) = (a ⊕ c, b ⊕ d) and (a, b) ⊙ (c, d) = (a ◦ c, b ◦ d).

Let the map \( \omega: \mathcal{I}([0,1]) \to \Theta \) be defined by \([a, b] \mapsto (a, b)\). Then \( \omega \) is an isomorphism between the IMV-algebras \( \mathcal{I}([0,1]) \) and \( \Theta \), in symbols,

\[ \omega: \mathcal{I}([0,1]) \cong \Theta. \tag{31} \]

Further, with the notation of \([8, \text{Defintion 9.1}]\),

\[ \text{IMV} = \text{HSP}(\mathcal{I}([0,1])) = \text{HSP}(\Theta). \tag{32} \]

Thus an equation holds in all IMV-algebras iff it holds in \( \mathcal{I}([0,1]) \).

**Proof.** (31) is immediately verified in the light of Proposition 2.1. As a categorical equivalence between two varieties (Theorem 4.4) the interval functor \( \mathcal{I} \) preserves products, subalgebras and homomorphic images (the later are preserved since homomorphic images in every variety are codomains of regular epi-morphisms), [24]. Using Chang completeness theorem [13, Theorem 2.5.3] in combination with \([8, \text{Theorem 9.5}]\), we can write \( \text{MV} = \text{HSP}([0,1]) \). Then (32) immediately follows from Theorem 4.4 and (31). The rest now follows from Birkhoff completeness theorem for equational logic, \([8, \text{Theorem 14.19}]\). \( \square \)

**Theorem 5.2 (Free IMV-algebras).** For any cardinal \( \kappa \geq 1 \) the free \( \kappa \)-generator IMV-algebra is the algebra \( \mathcal{I}(\mathcal{M}(\Theta^\kappa)) \). A free generating set for this algebra is given by the intervals \( \langle \pi_1 \circ \pi_\alpha \circ \pi_\beta \circ \pi_\alpha \rangle \) where \( \pi_\alpha: \Theta^\kappa \to \Theta \) is the projection map into the \( \alpha \)-th coordinate, for each ordinal \( \alpha < \kappa \).

**Proof.** Let \( X \) be a set of cardinality \( \kappa \). As a consequence of (32), the free IMV-algebra on \( \kappa \) generators is isomorphic to the IMV-subalgebra \( \mathbf{F}_{\text{IMV}}(X) \) of \( \Theta^{\Theta^X} \) generated by the projection maps.

**Claim 1:** Let \( f: \Theta^X \to \Theta \) be an element of \( \mathbf{F}_{\text{IMV}}(X) \). Then both functions \( \pi_1 \circ f \) and \( \pi_2 \circ f \) are members of \( \mathcal{M}(\Theta^X) \).

The proof is by induction on the number of applications of the IMV-operations needed to obtain \( f \) from the projection maps \( \pi_x \).

**Basis Step:** If for some \( x \in X \), \( f = \pi_x \), then the map \( \pi_1 \circ f = \pi_1 \circ \pi_x : \Theta^X \to [0,1] \) coincides over \( \Theta^X \) with the map \( \pi_{x,1} : ([0,1]^2)^X \to [0,1] \) sending any \( v \in ([0,1]^2)^X \) into \( \pi_1(v_x) \). Since \( \pi_{x,1} \) is continuous piecewise linear with only one piece, and its unique linear piece has integer coefficients, then \( \pi_{x,1} \) is a member of \( \mathcal{M}(([0,1]^2)^X) \), and hence \( f \in \mathcal{M}(\Theta^X) \). Similarly, \( \pi_2 \circ \pi_x \in \mathcal{M}(\Theta^X) \).

**Induction Step:** By (18) and (16) it is enough to argue only for \( \Delta, \neg \) and \( \oplus \). Let \( f, g: \Theta^X \to \Theta \) be members of \( \mathcal{M}(\Theta^X) \). For each \( v \in \Theta^X \) we can write

\[
(\pi_1 \circ (\Delta_{\text{inj}}(x))f)(v) = \pi_1(\Delta f(v)) = \pi_1(f(v)) = (\pi_1 \circ f)(v);

(\pi_2 \circ (\Delta_{\text{inj}}(x))f)(v) = \pi_2(\Delta f(v)) = \pi_2(f(v)) = (\pi_2 \circ f)(v);

(\pi_2 \circ (\neg_{\text{inj}}(x))f)(v) = \pi_2(\neg f(v)) = \neg(\pi_2(f(v))) = (\neg_{\mathcal{M}(\Theta^X)}(\pi_2 \circ f))(v);

(\pi_1 \circ (f \oplus_{\text{inj}}(x))g)(v) = \pi_1(f(v) \oplus g(v)) = \pi_1(f(v)) \oplus \pi_1(g(v)) = (\pi_1 \circ f) \oplus_{\mathcal{M}(\Theta^X)}(\pi_1 \circ g)(c);

(\pi_2 \circ (f \oplus_{\text{inj}}(x))g)(v) = \pi_2(f(v) \oplus g(v)) = \pi_2(f(v)) \oplus \pi_2(g(v)) = (\pi_2 \circ f) \oplus_{\mathcal{M}(\Theta^X)}(\pi_2 \circ g)(c).
\]

Thus whenever \( f \) and \( g \) satisfy \( \pi_1 \circ f, \pi_2 \circ f, \pi_1 \circ g, \pi_2 \circ g \in \mathcal{M}(\Theta^X) \), all the maps \( \pi_1 \circ (\Delta_{\text{inj}}(x))f \), \( \pi_2 \circ (\Delta_{\text{inj}}(x))f \), \( \pi_1 \circ (\neg_{\text{inj}}(x))f \), \( \pi_2 \circ (\neg_{\text{inj}}(x))f \), \( \pi_1 \circ (f \oplus_{\text{inj}}(x))g \), \( \pi_2 \circ (f \oplus_{\text{inj}}(x))g \) belong to \( \mathcal{M}(\Theta^X) \).

**Claim 2:** \( \mathcal{C}(\mathbf{F}_{\text{IMV}}(X)) \cong \mathcal{M}(\Theta^X) \).

Let \( \eta: \mathcal{C}(\mathbf{F}_{\text{IMV}}(X)) \to \mathcal{M}(\Theta^X) \) be defined by \( \eta(f) = \pi_1 \circ f \) for any map \( f: \Theta^X \to \Theta \) in \( \mathcal{C}(\mathbf{F}_{\text{IMV}}(X)) \). By Claim 1, \( \eta(f) = \pi_1 \circ f \in \mathcal{M}(\Theta^X) \) for each \( f \in \mathcal{C}(\mathbf{F}_{\text{IMV}}(X)) \), thus proving that \( \eta \)
is well defined. To see that \( \eta \) is an MV-homomorphism, first observe that 0 is the constant map on \( F_{\text{IMV}}(X) \) for each \( x \in \Theta^X \). Thus \( \eta(F_{\text{IMV}}(X))(x) = 0 \) for each \( x \in \Theta^X \). Since \( f = \langle f_1, f_2 \rangle \in C(F_{\text{IMV}}(X)) \), then

\[
f_1 = \pi_1 \circ f = \eta(f) = \eta(\nabla f) = \pi_1 \circ (\pi_2 \circ f, \pi_2 \circ f) = \pi_2 \circ f = f_2,
\]

and for every \( x \in \Theta^X \),

\[
\eta(\neg F_{\text{IMV}}(X))f(x) = \eta(\neg F_{\text{IMV}}(X))f(x) = \pi_1(\neg F_{\text{IMV}}(X)f(x)) = \pi_1(\neg f(x)) = \pi_1(\neg f_1(x), \neg f_2(x)) = \neg f_1(x) = (\neg \eta(f))(x).
\]

Now let \( f = \langle f_1, f_2 \rangle \), \( g = \langle g_1, g_2 \rangle \in C(F_{\text{IMV}}(X)) \). It follows that

\[
(\eta(f \oplus F_{\text{IMV}}(X))g)(x) = \eta(f(x) \oplus g(x)) = \eta(f_1(x) \oplus g_1(x), f_2(x) \oplus g_2(x)) = f_1(x) \oplus g_1(x) = (\eta(f) \oplus \eta(g))(x).
\]

It is easy to see that \( \eta \) is one-to-one. Indeed, \( \eta(f) = \eta(g) \), implies \( \pi_1 \circ f = \pi_1 \circ g \) and \( \pi_2 \circ f = \pi_2 \circ g \), that is, \( f = g \). To see that \( \eta \) is onto, for each \( x \in X \) let \( \pi_{1,x}, \pi_{2,x} : (0,1]^2 \rightarrow [0,1]^2 \) denote the projection maps \( \pi_{1,x}(v) = (\pi_1(v), \pi_2(v)) \) and \( \pi_{2,x}(v) = (\pi_2(v), \pi_1(v)) \). Since the MV-algebra \( \mathcal{M}((0,1]^2)^X \) is generated by these projection maps, then \( \mathcal{M}(\Theta^X) \) is generated by the restriction of the projection maps \( \pi_{1,x} \) and \( \pi_{2,x} \) to \( \Theta^X \). Now for each \( x \in X \) we have the identities \( \pi_{1,x} \mid \Theta^X = \pi_1 \circ (\Delta F_{\text{IMV}}(X) \pi_x) \) and \( \pi_{2,x} \mid \Theta^X = \pi_1 \circ (\nabla F_{\text{IMV}}(X) \pi_x) \). Further, \( \Delta F_{\text{IMV}}(X) \pi_x, \nabla F_{\text{IMV}}(X) \pi_x \in C(F_{\text{IMV}}(X)) \). As a consequence, \( \eta \) is onto \( \mathcal{M}(\Theta^X) \).

From Claim 2 and Theorem 3.4, it follows that

\[
F_{\text{IMV}}(X) \cong \mathcal{I}(C(F_{\text{IMV}}(X))) \cong \mathcal{I}(\mathcal{M}(\Theta^X)),
\]

whence we can write \( \mu : F_{\text{IMV}}(X) \cong \mathcal{I}(\mathcal{M}(\Theta^X)) \) for some one-one onto map. For every \( x \in X \),

\[
\mu(\pi_x) = [\eta(\Delta F_{\text{IMV}}(X) \pi_x), \eta(\nabla F_{\text{IMV}}(X) \pi_x)] = [\pi_1 \circ \pi_x, \pi_2 \circ \pi_x]. \quad \Box
\]

**The product order in IMV-algebras.** Just as every MV-algebra \( A \) has an underlying lattice structure that is definable from the monoidal structure of \( A \), also every IMV-algebra has the following (product) lattice order.

**Proposition 5.3.** Let \( J \) be an IMV-algebra, identified with \( \mathcal{I}(C(J)) \) via the isomorphism \( \gamma_J \) of Theorem 3.4. For any two intervals \( x, y \in J \) let us write without fear of ambiguity \( x = [\alpha, \beta], y = [\gamma, \delta] \) for uniquely determined elements \( \alpha, \beta, \gamma, \delta \) of the MV-algebra \( C(J) \). Let us define

\[
u \wedge v = \neg(\neg u \circ v) \circ v \quad \text{and} \quad u \vee v = \neg(u \circ v) \oplus v, \quad \text{for all } u, v \in J. \tag{33}
\]

We then have:

(i) Over the MV-algebra \( C(J) \) the operations \( \wedge, \vee \) coincide with the lattice operations \( \wedge, \vee \) of the MV-algebra \( C(J) \).

(ii) Upon writing \( x \cap y = [\alpha \wedge \gamma, \beta \wedge \delta] \) and \( x \cup y = [\alpha \vee \gamma, \beta \vee \delta] \), the algebra \n
\[ J^* = (J, 0, 1, i, \neg, \Delta, \nabla, \oplus, \odot, \sqcup, \sqcap) \tag{34} \]

has a distributive lattice reduct, \( (J, 0, 1, \sqcup, \sqcap) \) with largest element 1 and smallest 0.

(iii) Denoting by \( \sqsubseteq \) the resulting partial order on \( J^* \), it follows that \( \oplus \) and \( \otimes \) are monotone in both arguments, \( \neg \) is order-reversing, and \( \Delta x \sqsubseteq x \sqsubseteq \nabla x \).

(iv) Generalizing the definition of the left-hand term of (27), let the binary IMV-term \( \zeta(u, v) \) be defined by

\[
\zeta(u, v) = \Delta u \oplus (i \circ \nabla u \circ \neg \Delta u), \quad \text{for all } u, v \in J. \tag{35}
\]

Suppose \( u = [\alpha, \alpha] \) and \( v = [\beta, \beta] \) belong to \( C(J) \), and \( u \sqsubseteq v \), which in the present case is equivalent to \( \alpha \leq \beta \) in the MV-algebra \( C(J) \). Then \( \Delta \zeta(u, v) = u \) and \( \nabla \zeta(u, v) = v \), that is,

\[
\zeta(u, v) = [\alpha, \beta].
\]
(v) The lattice operation \( \sqcap \) of \( J^* \) is definable from the operations of \( J \), by
\[
x \sqcap y = \zeta(\Delta x \land \Delta y, \nabla x \land \nabla y).
\]

(vi) The lattice operation \( \sqcup \) of \( J^* \) is definable from the operations of \( J \), by
\[
x \sqcup y = \zeta(\Delta x \lor \Delta y, \nabla x \lor \nabla y).
\]

(vii) For all \( x, y \in J \) we have \( x \subseteq y \iff x \sqcup y = y \iff (\neg \Delta x \oplus \Delta y) \odot (\neg \nabla x \oplus \nabla y) = 1 \).

Proof. Routine.

\[
\square
\]

Remark 5.4. In the IMV-algebra \( J \) the operations \( \land \) defined in (33) and \( \sqcap \) defined in (v) above do not coincide in general. Similarly, \( \lor \) need not coincide with \( \sqcup \).

In [20] the authors study the algebra (denoted \( I^{[2]} \)) given by the \((0,1,\Delta,\nabla,\sqcup,\sqcap)\)-reduct of \((I([0,1]))^* \) as defined in (34). Thus in \( I^{[2]} \), \([\alpha,\beta] \subseteq [\alpha',\beta'] \) means \( \alpha \leq \alpha' \) and \( \beta \leq \beta' \). A commutative associative operation \( \odot : I([0,1]) \times I([0,1]) \to I([0,1]) \) is said to be a \( t \)-norm if it satisfies the following conditions:

(a) \( C(I([0,1])) \odot C(I([0,1])) \subseteq C(I([0,1])) \),

(b) \( \odot \) distributes over the lattice operations \( \sqcup, \sqcap \) of \( I^{[2]} \),

(c) for all \( x \in I([0,1]) \), \( 1 \odot x = x \), \( i \odot [\alpha,\beta] = [0,\beta] \).

In addition, \( \odot \) is said to be convex if whenever \( x \in I([0,1]) \) satisfies \( \alpha \odot \alpha' \subseteq x \sqcup \beta \odot \beta' \), it follows that \( x = \epsilon \odot \delta \) for some \( \alpha \leq \epsilon \leq \beta \) and \( \alpha' \leq \delta \leq \beta' \).

Theorem 5.5. The \( \odot \) operation of \( I([0,1]) \) equips the lattice \( I^{[2]} \) with a convex \( t \)-norm in the sense of [20].

Proof. Use Propositions 2.1 and 5.3.

\[
\square
\]

The inclusion order in IMV-algebras. Beyond the lattice order \( \sqcap, \sqcup \), every IMV-algebra \( J \) is equipped with a partial order relation \( \subseteq \), given by the inclusion relation between intervals of \( J = I(C(J)) \). Notwithstanding the categorical equivalence between IMV-algebras and MV-algebras, the inclusion order has no role in MV-algebras.

Proposition 5.6. Adopt the hypotheses and notation of Proposition 5.3. We then have:

(i) Upon writing \( x \sqcup y = [\alpha \land \gamma, \beta \lor \delta] \), the algebra
\[
J^{**} = (J,0,1,i,\neg,\Delta,\nabla,\oplus,\odot,\sqcup)
\]
becomes a sup-semilattice with maximum element \( i \), whose set of minimal elements coincides with the center of \( J \).

(ii) Denoting by \( \subseteq \) the resulting partial order on \( J^{**} \), it follows that \( \oplus \) and \( \odot \) are monotone in both arguments, \( \neg \) is monotone, and \( \Delta x \subseteq x \sqsupseteq \nabla x \).

(iv) The sup-semilattice operation \( \sqcup \) of \( J^{**} \) is definable from the operations of \( J \), by \( x \sqcup y = \zeta(\Delta x \land \Delta y, \nabla x \lor \nabla y) = [\Delta x \land \Delta y, \nabla x \lor \nabla y] \).

(vi) For all \( x, y \in J \) we have \( x \subseteq y \iff x \sqcup y = y \iff (\Delta y \leq \Delta x) \) and \( (\nabla x \leq \nabla y) \) in the MV-algebra \( C(J) \) iff \( (\neg \Delta y \oplus \Delta x) \odot (\neg \nabla x \oplus \nabla y) = 1 \) in \( J \).

Proof. From Proposition 5.3.

\[
\square
\]

Remark 5.7. Going back to the outset of this paper, it is interesting to note that IMV-algebras, while categorically equivalent to MV-algebras, can express the fundamental inclusion order \( u \subseteq v \) between actual measurements/estimations \( u, v \), meaning that \( u \) is more precise than \( v \). Within the MV-algebraic framework, the inclusion order has no meaning, because truth-values in Lukasiewicz logic are real numbers, corresponding to error-free normalized measurements.
6. The equational theory of IMV-algebras is coNP-complete

We refer to [19] for algorithmic complexity theory.

Let $\mathcal{X}$ be a countable set, whose elements are called variables. For any finite subset $X = \{X_1, \ldots, X_n\}$ of $X$ we denote by $\text{MV}(X)$ (resp., $\text{IMV}(X)$) the set of MV-terms (resp., IMV-terms) $\omega$ such that all variables occurring in $\omega$ belong to $X$. Letting $X$ range over all finite subsets of $\mathcal{X}$ we obtain the set $\text{MV}(\mathcal{X})$ (resp., $\text{IMV}(\mathcal{X})$) of MV-terms (resp., IMV-terms) over $\mathcal{X}$. The equational theory $\text{Id}_{\text{MV}(\mathcal{X})}$ of MV-algebras (resp., $\text{Id}_{\text{IMV}(\mathcal{X})}$ of IMV-algebras) is the set of all equations over $\mathcal{X}$ satisfied by every MV-algebra (resp., every IMV-algebra). In the jargon of algorithmic complexity theory, the set $\text{Id}_{\text{MV}(\mathcal{X})}$ (resp., $\text{Id}_{\text{IMV}(\mathcal{X})}$) amounts to the following “problem”:

\textbf{INSTANCE} : Two MV-terms (resp., two IMV-terms) $\sigma_1$ and $\sigma_2$ in the variables $X_1, \ldots, X_n$, ($n = 1, 2, \ldots$).

\textbf{QUESTION} : Does the equation $\sigma_1 = \sigma_2$ hold in every MV-algebra (resp., in every IMV-algebra)?

The equational theory of MV-algebras has the same algorithmic complexity as the set of tautologies (in the variables of $\mathcal{X}$) of infinite-valued Lukasiewicz logic, whence it is coNP-complete by [13, Theorem 9.3.8]. A similar result holds for IMV-algebras.

\textbf{Theorem 6.1.} The equational theory $\text{Id}_{\text{IMV}(\mathcal{X})}$ is coNP-complete.

\textbf{Proof.} We first describe a polytime reduction $\psi : (\sigma_1, \sigma_2) \mapsto (\sigma'_1, \sigma'_2)$ of the equational theory $\text{Id}_{\text{MV}(\mathcal{X})}$ of MV-algebras to the equational theory $\text{Id}_{\text{IMV}(\mathcal{X})}$. Using Chang’s distance function, [13, §1], it is no loss of generality to assume $\sigma_2 = 0$. So $\psi$ will transform any given MV-term $\sigma_1 = \sigma(X_1, \ldots, X_n)$ into an IMV-term $\sigma'$ such that $\text{MV} \models \sigma = 0$ iff $\text{IMV} \models \sigma' = 0$. Here, as usual, the symbol $\models$ stands for the tarskian satisfaction relation of first-order logic, [8, pp. 71 and 195]. Let $[0, 1]$ be the standard MV-algebra. We have the following equivalences

\[
\text{MV} \models \sigma(X_1, \ldots, X_n) = 0 \\
\iff [0, 1] \models \sigma(X_1, \ldots, X_n) = 0, \text{ by [13, Theorem 2.5.3]} \\
\iff \mathcal{I}([0, 1]) \models \forall X_1 \ldots \forall X_n \in C(\mathcal{I}([0, 1])) \sigma(X_1, \ldots, X_n) = 0, \text{ by Theorem 4.4} \\
\iff \mathcal{I}([0, 1]) \models \forall X_1 \ldots \forall X_n \in \mathcal{I}([0, 1]) \sigma(\Delta X_1, \ldots, \Delta X_n) = 0, \text{ by Prop. 3.3(i)} \\
\iff \text{IMV} \models \sigma(\Delta X_1, \ldots, \Delta X_n) = 0, \text{ because IMV = HSP} \text{(}\mathcal{I}([0, 1]))\text{, by (32).}
\]

Letting $\sigma' = \sigma(\Delta X_1, \ldots, \Delta X_n)$, a direct inspection shows that the map $\sigma \mapsto \sigma'$ can be computed in deterministic polynomial time. This provides the desired reduction $\psi$ of $\text{Id}_{\text{MV}(\mathcal{X})}$ to $\text{Id}_{\text{IMV}(\mathcal{X})}$. As already noted, the equational theory of MV-algebras is coNP-complete, whence the equational theory of IMV-algebras is coNP-hard.

In order to prove that $\text{Id}_{\text{IMV}(\mathcal{X})}$ is in coNP we will construct a polytime reduction $\chi$ of $\text{Id}_{\text{IMV}(\mathcal{X})}$ to the consequence problem in Lukasiewicz logic, [28, §18]. Given an equation $\omega = \sigma$, for IMV-terms $\omega, \sigma$ in the variables $X_1, \ldots, X_n$, ($n = 1, 2, \ldots$), we prepare $2n$ new variables $Y_1, \ldots, Y_n, Z_1, \ldots, Z_n$, and write down the following equivalences:

\[
\text{IMV} \models \omega = \sigma \iff \mathcal{I}([0, 1]) \models \omega = \sigma, \text{ by (32)} \\
\iff \mathcal{I}([0, 1]) \models \Delta \omega = \Delta \sigma \text{ and } \mathcal{I}([0, 1]) \models \nabla \omega = \nabla \sigma, \text{ by (28).}
\]

We now focus on the first leg

\[
\mathcal{I}([0, 1]) \models \Delta \omega = \Delta \sigma \tag{36}
\]

of the last equivalence.

In every IMV-algebra, repeated application of the equations (9)–(27) shows that the IMV-term $\Delta \omega(X_1, \ldots, X_n)$ can be transformed in polytime into an equivalent “normal form” IMV-term $\omega'(\Delta X_1, \ldots, \Delta X_n, \nabla X_1, \ldots, \nabla X_n)$, where the constant $i$ does not appear, the $\Delta, \nabla$ symbols only occur immediately before variables, and the total number of symbols in $\omega'$ is proportional to that of $\omega$. Replacing in $\omega'$ every occurrence of $\Delta X_i$ by the variable $Y_i$ and every occurrence of $\nabla X_i$ by $Z_i$, we obtain the IMV-term

\[
\omega^\Delta(Y_1, \ldots, Y_n, Z_1, \ldots, Z_n) \tag{37}
\]
which is also readable as an MV-term. Similarly there is a polytime transformation
\[ \sigma(X_1, \ldots, X_n) \mapsto \sigma^\Delta(Y_1, \ldots, Y_n, Z_1, \ldots, Z_n). \]  
(38)

By construction,
\[ \mathcal{I}([0,1]) \models \Delta \omega = \Delta \sigma \iff \mathcal{I}([0,1]) \models \omega = \sigma' \text{ for all } X_1, \ldots, X_n \]
\[ \iff [0,1] \models \omega^\Delta = \sigma^\Delta \text{ whenever } Y_1 \leq Z_1, \ldots, Y_n \leq Z_n \]
\[ \iff Y_1 \leq Z_1, \ldots, Y_n \leq Z_n \vdash L_\infty \omega^\Delta = \sigma^\Delta, \]

where \( \vdash L_\infty \) denotes consequence in Łukasiewicz logic.

Turning to the second leg \( \mathcal{I}([0,1]) \models \nabla \omega = \nabla \sigma \), we similarly reduce it to an equivalent instance \( \vdash L_\infty \omega^\nabla = \sigma^\nabla \) of the consequence problem in Łukasiewicz logic, for suitable polytime computable MV-terms \( \omega^\nabla \) and \( \sigma^\nabla \).

In conclusion, the map \( \chi : (\omega, \sigma) \mapsto (Y_1, Z_1, \ldots, Y_n, Z_n, \omega^\Delta, \sigma^\Delta, \omega^\nabla, \sigma^\nabla) \) determines a polytime reduction of \( \text{Id}_{\text{IMV}(X)} \) to the consequence problem in Łukasiewicz logic. A refinement of the proof of [28, Theorem 18.3] in the light of [13, Theorem 9.3.4] shows that this latter problem is in \( \text{coNP} \), whence so is the problem \( \text{Id}_{\text{IMV}(X)} \).

7. Tautologies and consequence in Łukasiewicz interval logic

The proof of Theorem 5.1 routinely yields a transformation of IMV-equational logic into a deductive algorithmic machinery on IMV-terms, which we will call “Łukasiewicz interval logic”. One first says that an IMV-term \( \sigma(X_1, \ldots, X_n) \) is an IMV-tautology if the equation \( \sigma = 1 \) is satisfied by all IMV-algebras. We call \( \sigma \) an \( \mathcal{I}([0,1]) \)-tautology if the IMV-algebra \( \mathcal{I}([0,1]) \) satisfies equation \( \sigma = 1 \). By Theorem 5.1, IMV-tautologies coincide with \( \mathcal{I}([0,1]) \)-tautologies, and will be called tautologies without fear of confusion.

Let \( \text{FORM}(X_1, \ldots, X_n) \) be the (absolutely free) algebra of IMV-terms in the variables \( X_1, \ldots, X_n \). Given \( \theta_1, \ldots, \theta_m, \psi \in \text{FORM}(X_1, \ldots, X_n) \) we say that \( \psi \) is a consequence of \( \theta_1, \ldots, \theta_m \), in symbols,
\[ \theta_1, \ldots, \theta_m \vdash_{\text{IMV}} \psi \]
if every homomorphism (valuation, evaluation, truth-value assignment, interpretation, model) \( \eta : \text{FORM}(X_1, \ldots, X_n) \to \mathcal{I}([0,1]) \) that evaluates to 1 each \( \theta_i \), also satisfies \( \eta(\psi) = 1 \). More generally, when \( \Phi \) is an infinite set of IMV-terms, we write \( \Phi \vdash_{\text{IMV}} \psi \) if \( \Phi' \vdash_{\text{IMV}} \psi \) for some finite \( \Phi' \subseteq \Phi \). In particular, the notation
\[ \emptyset \vdash_{\text{IMV}} \psi \]  
(39)
precisely means that \( \psi \) is a tautology.

In Łukasiewicz interval logic one can reduce the consequence problem to the tautology problem, because of the following counterpart of the “local deduction theorem” of Łukasiewicz logic,

**Theorem 7.1.** For any \( \theta_1, \ldots, \theta_m, \psi \in \text{FORM}(X_1, \ldots, X_n) \) the following conditions are equivalent:

(i) \( \theta_1, \ldots, \theta_m \vdash_{\text{IMV}} \psi. \)

(ii) For some integer \( k > 0 \) we have a tautology
\[ \neg (\Delta \theta_1 \odot \cdots \odot \Delta \theta_m) \oplus \cdots \oplus \neg (\Delta \theta_1 \odot \cdots \odot \Delta \theta_m) \oplus \Delta \psi. \]

(k times)

(iii) For some integer \( k > 0 \) we have a tautology
\[ (\Delta \theta_1 \odot \cdots \odot \Delta \theta_m) \oplus \cdots \oplus (\Delta \theta_1 \odot \cdots \odot \Delta \theta_m) \Rightarrow \Delta \psi, \]

where the implication interval connective \( \Rightarrow \) is defined by \( u \Rightarrow v = -u \oplus v. \)

(iv) \( \Delta \psi \) is obtainable via Modus Ponens from \( \Delta \theta_1 \odot \cdots \odot \Delta \theta_m \) and the tautologies.
Proof. One first notes that a homomorphism $\eta$ of $\text{FORM}(X_1, \ldots, X_n)$ into $I([0, 1])$ evaluates to 1 an IMV term $\sigma$ iff $\eta(\Delta \sigma) = 1$. So without loss of generality the consequence relation $\vdash_{\text{IMV}}$ deals with interpretations of IMV-terms in the center of $I([0, 1])$—which we can safely identify with $[0, 1]$, in the light of Proposition 3.2(ii). Letting $\vdash_{L_\infty}$ denote consequence in Lukasiewicz logic, [28, §1], we can write:

\[
\theta_1, \ldots, \theta_m \vdash_{\text{IMV}} \psi
\]

\[
\iff \Delta \theta_1, \ldots, \Delta \theta_m \vdash_{\text{IMV}} \Delta \psi
\]

\[
\iff \theta_1, \ldots, \theta_m \vdash_{L_\infty} \psi^\Delta
\]

where the IMV-terms $\theta_1^\Delta$ and $\psi^\Delta$ are readable

as $L_\infty$-formulas via the polytime transformation $\sigma \mapsto \sigma^\Delta$ of (37)-(38)

\[
\iff \emptyset \vdash_{L_\infty} \neg \left( (\theta_1^\Delta \odot \cdots \odot \theta_m^\Delta) \oplus \cdots \oplus \neg (\theta_1^\Delta \odot \cdots \odot \theta_m^\Delta) \right) \Rightarrow \Delta \psi
\]

for some positive integer $k$. This follows from the local deduction theorem in Lukasiewicz logic, [28, 1.7]. To conclude the proof, let us note that the IMV-term

\[
\neg \left( (\theta_1^\Delta \odot \cdots \odot \theta_m^\Delta) \oplus \cdots \oplus \neg (\theta_1^\Delta \odot \cdots \odot \theta_m^\Delta) \right) \Rightarrow \Delta \psi
\]

can be equivalently rewritten as

\[
(\theta_1^\Delta \odot \cdots \odot \theta_m^\Delta) \odot \cdots \odot (\theta_1^\Delta \odot \cdots \odot \theta_m^\Delta) \Rightarrow \psi^\Delta
\]

A final application of [28, Definition 1.7] yields the desired conclusion. 

As an immediate consequence of Theorem 6.1 we have

**Corollary 7.2.** The tautology problem in Lukasiewicz interval logic is coNP-complete, and so is the consequence problem $\theta_1, \ldots, \theta_m \vdash_{\text{IMV}} \psi$

**Remark 7.3.** Proof-theoretically oriented readers may envisage other approaches to Lukasiewicz interval logic, beyond Theorem 7.1. For instance, one can obtain from Theorem 5.1 a proof system (with axioms and rules) originating from the equational logic of IMV-algebras. To this purpose, one first notes that the IMV-term $\delta(x, y) = (x \odot \neg y) \oplus (y \odot \neg x)$ coincides with Chang distance [13, Definition 1.2.4] when interpreted in the center $[0, 1]$ of $I([0, 1])$. Secondly, one may transform each IMV-axiom $\omega = \sigma$ of the list (9)-(27) into the tautology $\omega = \sigma$ given by the IMV-term $\neg \delta(\Delta \omega, \Delta \sigma) \odot \neg \delta(\nabla \omega, \nabla \sigma)$. These nineteen tautologies are the axioms of Lukasiewicz interval logic. Finally, whenever a rule

\[
\omega_1 = \sigma_1, \ldots, \omega_m = \sigma_m
\]

of equational logic is applied to obtain a new equation $\omega = \sigma$ from old equations $\omega_1 = \sigma_1, \ldots, \omega_m = \sigma_m$, the corresponding rule of Lukasiewicz interval logic

\[
\omega_1 = \sigma_1, \ldots, \omega_m = \sigma_m
\]

is applied to derive $\omega = \sigma$ from $\omega_1 = \sigma_1, \ldots, \omega_m = \sigma_m$. Since equational logic has finitely many rules (essentially: instantiation and congruence), then so does the resulting proof system for Lukasiewicz interval logic. A main reason of interest in the proof system of Theorem 7.1 is that Modus Ponens is its only rule.

As of today, whatever proof system $S$ one may choose for Lukasiewicz interval logic, $S$ will unavoidably require exponential space. This is a consequence of Corollary 7.2—in want of an answer to the P/NP problem.
A term-equivalent implicative reformulation of IMV-algebras. Just as MV-algebras have a term-equivalent variant [13, §4] where the monoidal operations $\oplus, \odot$ are replaced by Łukasiewicz implication $x \Rightarrow y = \neg x \oplus y$, also IMV-algebras have a term-equivalent counterpart based on the operation $\Rightarrow$ of Theorem 7.1(iii). The main interest in this reformulation is in Theorem 7.1(iv), stating that consequences in Łukasiewicz interval logic can be computed using Modus Ponens as the only rule, once formulas are written using $\Rightarrow$ instead of the monoidal connectives $\oplus, \odot$.

**Proposition 7.4.** For any IMV-algebra $J$ let $(J, \Rightarrow)$ denote $J$ enriched with the $\Rightarrow$ operation.

(i) Then $(J, \Rightarrow)$ obeys the following equations:

\[
x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z) \quad (40)
\]

\[
x \Rightarrow y = \neg y \Rightarrow \neg x \quad (41)
\]

\[
\neg x = x \Rightarrow \Delta i \quad (42)
\]

\[
\neg \Delta i = \Delta i \Rightarrow x \quad (43)
\]

\[
\neg \neg x = x \quad (44)
\]

\[
(\Delta x \Rightarrow \Delta y) \Rightarrow \Delta y = (\Delta y \Rightarrow \Delta x) \Rightarrow \Delta x \quad (45)
\]

\[
\neg i = i \quad (46)
\]

\[
\Delta \Delta x = \Delta x \quad (47)
\]

\[
\Delta \neg \Delta x = \neg \Delta x 
\]

\[
\Delta (\neg x \Rightarrow y) = \neg \Delta x \Rightarrow \Delta y \quad (49)
\]

\[
\neg \Delta (x \Rightarrow y) = \Delta x \Rightarrow \neg \Delta \neg y \quad (50)
\]

\[
\neg \Delta x \Rightarrow \neg \Delta x = \neg \Delta i \quad (51)
\]

\[
(i \Rightarrow (\neg \neg \neg x \Rightarrow \Delta x)) \Rightarrow \Delta x = x. 
\] (52)

(ii) Conversely, suppose an algebra $L = (L, i, \neg, \Delta, \Rightarrow)$ of type $(0, 1, 1, 2)$ satisfies equations (40)–(52). Define $0 = \Delta i$, $1 = \neg \Delta i$, $\nabla x = \neg \Delta \neg x$, $x \oplus y = \neg \Delta x \Rightarrow y$, $x \odot y = \neg (x \Rightarrow \neg y)$. Then the algebra $(L, 0, 1, i, \neg, \Delta, \nabla, \oplus, \odot)$ is an IMV-algebra.

(iii) IMV-algebras are term-equivalent to the algebras satisfying the equations (40)–(52).

(iv) If an equation is satisfied the algebra $([0, 1], i, \neg, \Delta, \Rightarrow)$ then it is satisfied by all algebras satisfying the equations (40)–(52).

(v) Thus an equation is obtainable from equations (40)–(52) in equational logic iff it is satisfied by the algebra $([0, 1], i, \neg, \Delta, \Rightarrow)$.

**Proof.** A routine transcription. \hfill \Box

8. The interval functor for general classes of ordered algebras

Using MV-algebras and IMV-algebras as a template, in this section we will extend the definition of “interval algebra” $I(A)$ to all algebras $A$ in very general classes $C$ of ordered structures. We will introduce the associated class $I(C)$ of interval algebras, and provide a necessary and sufficient condition for $I$ to be a categorical equivalence between $C$ and $I(C)$.

In our approach to IMV-algebras the underlying order of MV-algebras has been overshadowed by the monoidal operations $\oplus$ and $\odot$. By contrast, the partial order of any algebra $A$ considered in this section will play a fundamental role from the outset in defining its associated interval algebra: indeed, any other operation of $A$ will be required to be monotone or antimonotone in each input argument. As another dissimilarity from IMV-algebras, the operations on the intervals of $A$ are no longer defined in terms of Minkowski pointwise operations—because the counterpart of Proposition 2.1 need no longer hold for $A$ (see Example 8.4).

A suitable framework for our generalized approach to interval algebras is provided by quasivarieties of “$\rho$-partially ordered algebras”, where $\rho$ is a polarity as defined by Pigozzi in [29].
As usual, for any set \( \Sigma \) of constant and functions symbols, and \( \Sigma \)-algebra \( A \), we write \( c^A \) and \( f^A \) for the interpretation in \( A \) of the constant symbol \( c \) (resp., the function symbol \( f \)) of \( \Sigma \). More generally, for each \( \Sigma \)-term \( t \), we let \( t^A \) denote the interpretation of \( t \) in \( A \).

**Definition 8.1** ([29, Definition 2.1]). A polarity for an algebraic language \( \Sigma \) is a map \( \rho \) defined on each symbol in \( \Sigma \) such that whenever \( f \in \Sigma \) is an \( n \)-place function symbol, \( \rho(f) \in \{+, -\}^n \).

Given a \( \Sigma \)-algebra \( A \) and a partial order \( \leq \) on \( A \), we say that \( \leq \) is a \( \rho \)-partial order of \( A \) if for each \( f \in \Sigma \) of arity \( n \), \((a_1, \ldots, a_n) \in A^n \), \( k \leq n \), and \( a, b \in A \) with \( a \leq b \), the following conditions hold:

1. If \( \rho(f)_k = + \) then \( f^A(a_1, \ldots, a_k, a, a_{k+1}, \ldots, a_n) \leq f^A(a_1, \ldots, a_k, b, a_{k+1}, \ldots, a_n) \);

2. If \( \rho(f)_k = - \) then \( f^A(a_1, \ldots, a_k, b, a_{k+1}, \ldots, a_n) \leq f^A(a_1, \ldots, a_k, a, a_{k+1}, \ldots, a_n) \).

The pair \((A, \leq)\) is called a \( \rho \)-partially ordered \( \Sigma \)-algebra, (for short, \( \rho \)-poalgebra when \( \Sigma \) is clear from the context). We further say that \((A, \leq)\) is a bounded \( \rho \)-poalgebra if there exist constant symbols \( 0, 1 \in \Sigma \) such that \( 0^A \leq a \leq 1^A \) for each \( a \in A \).

**Definition 8.2.** Given a polarity \( \rho \) for an algebraic language \( \Sigma \) and a bounded \( \rho \)-poalgebra \((A, \leq)\) we let \( \mathcal{I}(A) \) be the \((\Sigma \cup \{\Delta, \vee, \wedge\})\)-algebra whose universe is the set \( I(A) = \{[a, b] \mid a, b \in A \land a \leq b\} \) of intervals in \( A \), and whose operations are defined as follows:

1. For each \( n \)-ary symbol \( f \in \Sigma \) and intervals \([a_1, b_1], \ldots, [a_n, b_n] \) in \( I(A) \),
\[
 f^\mathcal{I}(A)\left([a_1, b_1], \ldots, [a_n, b_n]\right) = [f^A(c_1, \ldots, c_n), f^A(d_1, \ldots, d_n)],
\]
where \( c_k = a_k \) and \( d_k = b_k \) if \( \rho(f)_k = + \), and \( c_k = b_k \) and \( d_k = a_k \) if \( \rho(f)_k = - \).

2. For each \([a, b] \in I(A)\), \( \Delta^\mathcal{I}(A)\left([a, b]\right) = [a, b] \) and \( \vee^\mathcal{I}(A)\left([a, b]\right) = [b, b] \).

3. \( [0, 1]^A \).

**Remark 8.3.** By Proposition 2.1, for any MV-algebra \( A \) (equipped with its natural order \( \leq \)), the algebra \( \mathcal{I}(A) \) given by Definition 8.1 coincides with the algebra defined in Section 2 in terms of Minkowski operations. It turns out that Proposition 2.1 cannot be extended to arbitrary classes of \( \rho \)-poalgebras. For each \( \rho \)-poalgebra \( A \), \( n \)-ary function symbol \( f \in \Sigma \), and intervals \([a_1, b_1], \ldots, [a_n, b_n] \) in \( I(A) \) we have the inclusion
\[
\{f^A(e_1, \ldots, e_n) \mid (e_1, \ldots, e_n) \in [a_1, b_1] \times \cdots \times [a_n, b_n]\} \subseteq f^\mathcal{I}(A)\left([a_1, b_1], \ldots, [a_n, b_n]\right).
\]
The converse inclusion need not hold in general, because the set
\[
 f^A([a_1, b_1] \times \cdots \times [a_n, b_n]) = \{f^A(e_1, \ldots, e_n) \mid (e_1, \ldots, e_n) \in [a_1, b_1] \times \cdots \times [a_n, b_n]\}
\]
need not be an interval. However, \( f^\mathcal{I}(A)\left([a_1, b_1] \times \cdots \times [a_n, b_n]\right) \) has a smallest and a largest element. Thus \( f^\mathcal{I}(A)\left([a_1, b_1], \ldots, [a_n, b_n]\right) \) coincides with the smallest interval of \( A \) containing the set \( f([a_1, b_1] \times \cdots \times [a_n, b_n]) \).

**Example 8.4.** Following [8, p. 44], by a Heyting algebra \((A, \wedge, \vee, \rightarrow, 0, 1)\) we mean a bounded distributive lattice such that \( a \wedge b \leq c \) iff \( a \leq b \rightarrow c \). The underlying order \( \leq \) of \( A \) is given by the stipulation: \( a \leq b \) iff \( a \wedge b = a \). As is well known, the class \( H \) of Heyting algebras is a variety. Let us denote by \([0, 1]\) the uniquely determined Heyting algebra whose lattice reduct is \((\{0, 1\}, \min, \max, 0, 1)\). Then \( a \rightarrow b = 1 \) if \( a \leq b \), and \( a \rightarrow b = b \) otherwise. For each \([a, b], [c, d] \in \mathcal{I}([0, 1])\) we can write
\[
[a, b] \wedge^\mathcal{I}(A) [c, d] = \min\{a, c\}, \min\{b, d\} = \min\{a, b\} \times [c, d];
\]
\[
[a, b] \vee^\mathcal{I}(A) [c, d] = \max\{a, c\}, \max\{b, d\} = \max\{a, b\} \times [c, d].
\]
However, when \( a \neq b \) we have \([a, b] \rightarrow^\mathcal{I}(A) [a, a] = [b \rightarrow a, a \rightarrow a] = [a, 1] \), but \( \{e \rightarrow f \mid (e, f) \in [a, b] \times [a, a]\} = \{a, 1\} \).
Therefore, I

Theorem 8.7. The following conditions hold:

As an example, for every MV-algebra $A$, the single equation $\neg x \oplus y = 1$ determines the order of $A$. In any class of lattices, the order is determined by the single equation $x \land y = x$.

Notational convention. Throughout the rest of this section, $\Sigma$ will denote a set of constant and function symbols, and $Q$ will denote a $\Sigma$-quasivariety of $\rho$-poalgebras having a set $E(x,y)$ of $\Sigma$-equations such that the order of each algebra $A \in Q$ is determined by $E(x,y)$. We also write

$$IQ = Q(I(Q))$$

for the quasivariety of algebras generated by $I(Q)$. We regard $Q$ and $IQ$ as categories whose morphisms are $Q$-homomorphisms and $IQ$-homomorphisms, respectively.

Theorem 8.6. For any $Q$ we have:

(i) Every $Q$-homomorphism $h: A \rightarrow B$ is order-preserving.

(ii) Let the assignment $I: Q \rightarrow IQ$ be defined by:

objects: \quad $A \mapsto I(A)$

morphisms: \quad $h: A \rightarrow B \mapsto I(h): I(A) \rightarrow I(B)$,

where the homomorphism $I(h)$ is given by $(I(h))([a,b]) = [h(a), h(b)]$, for any interval $[a,b] \in I(A)$. Then $I$ is a well-defined functor.

Proof. (i) follows because the same equations $E(x,y)$ define the order in $A$ and in $B$.

(ii) By (i), $I(h)$ is a well-defined map from $I(A)$ to $I(B)$. To see that $I(h)$ is a homomorphism, first consider $f \in \Sigma$ and $[a_1, b_1], \ldots, [a_n, b_n] \in I(A)$. Then

$$I(h)(f^{(A)}([a_1, b_1], \ldots, [a_n, b_n])) = I(h)([f^{A}(c_1, \ldots, c_n), f^{A}(d_1, \ldots, d_n)])$$

$$= [h(f^{A}(c_1, \ldots, c_n)), h(f^{A}(d_1, \ldots, d_n))],$$

where $c_k = a_k$ and $d_k = b_k$ if $\rho(f,k) = +$; and $c_k = b_k$ and $d_k = a_k$ if $\rho(f,k) = -$. Since $h$ commutes with $f$,

$$I(h)(f^{(A)}([a_1, b_1], \ldots, [a_n, b_n])) = [f^{B}(h(c_1), \ldots, h(c_n)), f^{B}(h(d_1), \ldots, h(d_n))]$$

$$= f^{(B)}([h(a_1), h(b_1)], \ldots, [h(a_n), h(b_n)]).$$

For every $[a, b] \in I(A)$,

$$I(h)(\Delta^{(A)}[a,b]) = I(h)([a,a]) = [h(a), h(a)] = \Delta^{(B)}[h(a), h(b)]$$

$$= \Delta^{(B)}(I(h)([a,b]))$$

$$I(h)(\nabla^{(A)}[a,b]) = I(h)([b,b]) = [h(b), h(b)] = \nabla^{(A)}[h(a), h(b)]$$

$$= \nabla^{(A)}(I(h)([a,b]))$$

$$I(h)(i^{(A)}) = I(h)([0^{A}, 1^{A}]) = [h(0^{A}), h(1^{A})] = [0^{B}, 1^{B}] = i^{(B)}.$$

Therefore, $I$ is a functor. \hfill \Box

For each quasivariety of $\rho$-poalgebras $Q$ the functor $I: Q \rightarrow IQ$ defined in Theorem 8.6 is called the interval functor of $Q$.

Next we will prove that $I$ has always a left adjoint, and $Q$ is isomorphic to a retractive subcategory of $IQ$.

Theorem 8.7. The following conditions hold:
(i) For each \( J \in \mathcal{I} \mathcal{Q} \) and \( f \in \Sigma \), the set \( C(J) = \{ x \in J \mid \nabla x = x = \Delta x \} \) is closed under the operation \( f^J \).

(ii) Let the assignment \( C : \mathcal{I} \mathcal{Q} \rightarrow \mathcal{Q} \) be defined by:

\[
\begin{align*}
\text{objects:} & \quad J & \rightarrow & \quad C(J) \\
\text{morphisms:} & \quad \rho : J \rightarrow K & \rightarrow & \quad C(\rho) = \rho \upharpoonright C(J),
\end{align*}
\]

where \( C(J) \) is the \( \Sigma \)-algebra whose universe is \( C(J) \) and for each \( f \in \Sigma \), \( f^C(J) = f^J \upharpoonright C(J) \).

Then \( C \) is a well-defined faithful functor.

(iii) For each \( A \in \mathcal{Q} \) the map \( \iota_A : A \rightarrow C(\mathcal{I}(A)) \) defined by \( \iota_A(a) = [a, a] \) is an isomorphism; further, the map \( A \mapsto \iota_A \) is a natural isomorphism from the identity functor \( \mathcal{I} \mathcal{Q} : \mathcal{Q} \rightarrow \mathcal{Q} \) to the composite functor \( C \upharpoonright \mathcal{I} \).

(iv) For each \( J \in \mathcal{Q}(\mathcal{I}(\mathcal{Q})) \), the map \( \gamma_J : J \rightarrow \mathcal{I}(C(J)) \) defined by \( \gamma_J(a) = [\Delta a, \nabla a] \) is a one-to-one homomorphism; the map \( J \mapsto \gamma_J \) is a natural transformation from the identity functor \( \mathcal{I} \mathcal{Q} : \mathcal{Q} \rightarrow \mathcal{Q} \) to the composite functor \( \mathcal{I} \upharpoonright C \).

Proof. (i) Let \( A \in \mathcal{Q} \) and \( [a, b] \in \mathcal{I}(A) \). Observe that \( \Delta[a, b] = [a, b] \) implies \( b = a \). Therefore, for any \( n \)-ary function symbol \( f \in \Sigma \) and \( [a_1, b_1], \ldots, [a_n, b_n] \in \mathcal{I}(A) \), if \( \Delta[a_i, b_i] = [a_i, b_i] \) for each \( i \), then

\[
f^J([a_1, b_1], \ldots, [a_n, b_n]) = f^J([a_1, a_1], \ldots, [a_n, a_n]) = f(a_1, \ldots, a_n), \quad f^J([a_1, b_1], \ldots, [a_n, b_n]) = \Delta^J([a_1, b_1], \ldots, [a_n, b_n]).
\]

As a consequence, the quasiequation

\[
\Delta x_1 = x_1, \ldots, \Delta x_n = x_n \Rightarrow \Delta f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)
\]

(53)
is satisfied by each algebra \( \mathcal{I}(\mathcal{Q}) \). Since each \( J \in \mathcal{I} \mathcal{Q} \) satisfies (53), \( C(J) \) is closed under \( f^J \).

(ii) Let \( s_1 = t_1, \ldots, s_n = t_n \Rightarrow s = t \) be a quasiequation in the variables \( x_1, \ldots, x_m \) satisfied by all algebras in \( \mathcal{Q} \). The same argument used in (i) shows that the quasiequation

\[
\Delta x_1 = x_1, \ldots, \Delta x_m = x_m, \quad s_1 = t_1, \ldots, s_n = t_n \Rightarrow s = t
\]

(54)
is satisfied by every algebra in \( \mathcal{I}(\mathcal{Q}) \). Since \( C(J) \) satisfies every quasiequation satisfied by \( \mathcal{Q} \), then \( C(J) \in \mathcal{Q} \). Every homomorphism \( h : J \rightarrow K \) commutes with \( f^J \) (for each \( f \in \Sigma \)), whence \( C(h) \) is a homomorphism.

To prove that \( C \) is faithful, let \( J, K \in \mathcal{I} \mathcal{Q} \) and \( g, h : J \rightarrow K \) be homomorphisms such that \( C(g) = C(h) \). The quasiequation

\[
\Delta x = \Delta y, \quad \nabla x = \nabla y \Rightarrow x = y
\]

(55)
is satisfied by each algebra in \( \mathcal{I}(\mathcal{Q}) \), whence it is satisfied by \( J \). Now for each \( a \in J \) we can write \( \Delta g(a) = g(\Delta a) = h(\Delta a) = \Delta h(a) \) and \( \nabla g(a) = g(\nabla a) = h(\nabla a) = \nabla h(a) \). Since \( K \) satisfies (55), it follows that \( g(a) = h(a) \) for each \( a \in K \), that is, \( g = h \).

(iii) It is easy to check that the map \( \iota_A : A \rightarrow C(\mathcal{I}(A)) \) is an isomorphism, which is natural in \( \mathcal{Q} \).

(iv) To see that \( \gamma_J \) is well defined observe that

\[
t(\Delta x, \nabla x) = s(\Delta x, \nabla x)
\]

(56)
is satisfied by all algebras of \( \mathcal{I}(\mathcal{Q}) \), for each \( s = t \in E(x, y) \). Then for each \( a \in J \), \( \Delta a \leq \nabla a \) in \( C(J) \). The fact that \( \gamma_J \) is one-to-one follows directly from (55). For each \( A \in \mathcal{Q} \) and \( f \in \Sigma \), let

\[
t_{f,i}(x_i) = \Delta x_i \quad \text{and} \quad s_{f,i}(x_i) = \nabla x_i \quad \text{if} \quad \rho(f)_i = +, \\
t_{f,i}(x_i) = \nabla x_i \quad \text{and} \quad s_{f,i}(x_i) = \Delta x_i \quad \text{if} \quad \rho(f)_i = -.
\]
By definition of $f^I(A)$, the equations
\begin{align*}
\Delta(f(x_1, \ldots, x_n)) &= f(t_f(x_1), \ldots, t_f(x_n)) \\
\nabla(f(x_1, \ldots, x_n)) &= f(s_f(x_1), \ldots, s_f(x_n))
\end{align*}
are satisfied by every algebra in $I(Q)$. Since $J$ satisfies these equations, $\gamma_J$ preserves $f^J$ for each $f \in \Sigma$. Finally, a direct inspection shows that every algebra in $I(Q)$ satisfies the following equations:
\begin{align*}
\Delta i &= 0, \quad \nabla i = 1, \quad \Delta \Delta x = \Delta x, \quad \nabla \Delta x = \Delta x, \quad \nabla \nabla x = \nabla x, \quad \Delta \nabla x = \nabla x.
\end{align*}
We have shown that $\gamma_J$ also preserves $i^J$, $\Delta^J$ and $\nabla^J$, as required to complete the proof. \hfill \square

The proof of Theorem 8.7 yields a method which, having in input a set of quasiequations axiomatizing $Q$ outputs a set of quasiequations axiomatizing $IQ$:

**Corollary 8.8.** Suppose $Q$ is axiomatized by a finite set $M$ of $\Sigma$-quasiequations. Then $IQ$ is axiomatized by the following quasiequations, for every function symbol $f \in \Sigma$, quasiequation $s_1 = t_1, \ldots, s_n = t_n \Rightarrow s = t \in M$ and equation $s = t \in E(x, y)$:
\begin{align*}
\Delta x_1 = x_1, \ldots, \Delta x_m = x_m \Rightarrow \Delta f(x_1, \ldots, x_n) &= f(x_1, \ldots, x_n) \\
\Delta x_1 = x_1, \ldots, \Delta x_m = x_m, s_1 = t_1, \ldots, s_n = t_n \Rightarrow s = t \\
\Delta x &= \Delta y, \quad \nabla x = \nabla y \Rightarrow x = y
\end{align*}

Together with the equations
\begin{align*}
t(\Delta x, \nabla x) &= s(\Delta x, \nabla x) \\
\Delta(f(x_1, \ldots, x_n)) &= f(t_f(x_1), \ldots, t_f(x_n)) \\
\nabla(f(x_1, \ldots, x_n)) &= f(s_f(x_1), \ldots, s_f(x_n)),
\end{align*}
\begin{align*}
\Delta i &= 0, \quad \nabla i = 1, \quad \Delta \Delta x = \Delta x, \quad \nabla \Delta x = \Delta x, \quad \nabla \nabla x = \nabla x, \quad \Delta \nabla x = \nabla x.
\end{align*}

**Proof.** From the proof of Theorem 8.7 it follows that each algebra in $IQ$ satisfies (53)-(59). Conversely, let $J$ be an algebra satisfying (53)-(59). Mimicking the proof of Theorem 8.7(iv), from (53) it follows that the set $C(J) = \{a \in J \mid \Delta a = a = \nabla a\}$ is closed under $f^J$ for each $f \in \Sigma$. By (54), the algebra $C(J)$ equipped with all restrictions $f^J | C(J)$ belongs to $Q$. Further, from (56) we get $\Delta a \leq \nabla a$ in $C(J)$, and hence the map $\gamma_J(a) = [\Delta a, \nabla a] : J \to \mathbb{I}(C(J))$ is well defined. By (55), $\gamma_J$ is one-to-one; by (57)-(59), $\gamma_J$ is a homomorphism. In conclusion, $J \in \mathbb{I}(I(Q)) = IQ$. \hfill \square

The following theorem gives necessary and sufficient conditions on a quasivariety $Q$ for its interval functor $I$ to be a categorical equivalence.

**Theorem 8.9.** The following conditions are equivalent:
\begin{enumerate}
\item The interval functor $I : Q \to IQ$ is a categorical equivalence.
\item $IQ = \mathbb{I}(I(Q)) = \text{the class of isomorphic copies of algebras of } I(Q)$.
\item For each $A \in Q$ the set $C(I(A)) = \{[a, a] \mid a \in A\}$ generates $I(A)$.
\item For some $(\Sigma \cup \{\Delta, \nabla, i\})$-term $t(y, z)$, the equation $t(\Delta(x), \nabla(x)) = x$ is satisfied by every algebra of $IQ$.
\end{enumerate}

In particular, if $Q$ is a variety, these conditions (i)-(iv) are also equivalent to
\begin{enumerate}
\item $\forall(I(Q)) = \mathbb{I}(I(Q))$, where $\forall(I(Q))$ is the variety generated by $I(Q)$.
\end{enumerate}

**Proof.** (i)⇒(ii). Trivially, if $I$ is an equivalence every $J \in IQ$ belongs to $\mathbb{I}(I(Q))$.

(ii)⇒(iii). Let $A \in Q$ and $J \subseteq I(A)$ be the subalgebra generated by $\{[a, a] \mid a \in A\}$. Then $C(J) = \{[a, a] \mid a \in A\}$ and $C(J) \cong A$. Let $g : J \to I(A)$ be the inclusion map. By (ii), there exists $B \in Q$, and an isomorphism $f : J \to I(B)$. By Theorem 8.7(iii), $B \cong C(I(B)) = f(C(J)) \cong$
$C(J) \cong A$. The map $h: A \to B$ defined by $h(a) = b$ whenever $f([a, a]) = [b, b]$ is an isomorphism. Therefore, $k = f \circ \mathcal{I}(h^{-1}): J \to \mathcal{I}(A)$ is an isomorphism onto $\mathcal{I}(A)$, and

\[ k([a, a]) = f \circ \mathcal{I}(h^{-1})([a, a]) = f([h^{-1}(a), h^{-1}(a)]) = [a, a]. \]

Having thus proved $k = g$, we conclude that $J = g(J) = k(J) = \mathcal{I}(A)$.

(iii) $\Rightarrow$ (iv). Let $F(x) \in \mathcal{IQ}$ be the free algebra with one free generator. By Theorem 8.7(iv), $F(x)$ is isomorphic to a subalgebra of $\mathcal{I}(C(F(x)))$. By (iii), $\mathcal{I}(C(F(x)))$ is generated by $C(\mathcal{I}(C(F(x))))$, $C(F(x))$ generates $F(x)$ and there exists a $\Sigma\{\Delta, \nabla, i\}$-term $s(x_1, \ldots, x_n)$ together with elements $a_1, \ldots, a_n$ in $C(F(x))$ such that $s(F(x))(a_1, \ldots, a_n) = x$. Since $F(x)$ is generated by $x$, $C(F(x))$ is generated by $\Delta x$ and $\nabla x$. Thus for each $i \in \{1, \ldots, n\}$ there is a term $t_i(y, z)$ such that $a_i = t_i^{F(x)}(\Delta x, \nabla x)$. Therefore, the term $t(y, z) = s(t_1(y, z), \ldots, t_n(y, z))$ satisfies $t^{F(x)}(\Delta x, \nabla x) = x$. Since $F(x)$ is the free algebra in $\mathcal{IQ}$ with free generator $x$, the equation $t(\Delta x, \nabla x) = x$ is satisfied by every algebra of $\mathcal{IQ}$.

(iv) $\Rightarrow$ (i). Since, by Theorem 8.7(iii) $t_A$ is an isomorphism for each $A \in Q$, it is enough to check that $\gamma_J$ is an isomorphism for each $J \in \mathcal{IQ}$. By Theorem 8.7(iv), $\gamma_J$ is one-to-one. Letting $[a, b] \in \mathcal{I}(C(J))$ we can write

\[ \gamma_J(t_J(a, b)) = t^{\mathcal{I}(C(J))}(\gamma_J(a), \gamma_J(b)) = t^{\mathcal{I}(C(J))}(\Delta^J a, \nabla^J a, \Delta^J b, \nabla^J b) = t^{\mathcal{I}(C(J))}([a, b], [a, b]) = [a, b], \]

and $\gamma_J$ is onto $\mathcal{I}(C(J))$.

Trivially (v) implies (ii) even if $Q$ is not a variety. Now assume $Q$ is a variety and (i)-(iv) holds. Let $K \in \mathcal{IQ}$ and a homomorphism $h: J \to K$ onto $K$. Since $C(K) = \{x \in K \mid \Delta x = \nabla x = x\} = h(C(J))$, then $C(K)$ is closed under $f^K$ for each $f \in \Sigma$. Let $C(K)$ be the $\Sigma$-algebra whose universe is $C(K)$ and whose operations are given by restricting to $C(K)$ the operations of $K$. Thus the map $h|C(J): C(J) \to C(K)$ is a homomorphism onto $C(K)$. Since $Q$ is a variety, $C(K) \in Q$. Since $J$ satisfies (57)–(58) then so does $K$. As a consequence, the map $\gamma_K: K \to \mathcal{I}(C(K))$ defined by $\gamma_K(a) = [\Delta a, \nabla a]$ is a homomorphism. Finally, let $a, b \in K$ be such that $\gamma_K(a) = \gamma_K(b)$. Equivalently, $\Delta a = \Delta b$ and $\nabla a = \nabla b$. Since $K \in \mathcal{IQ}(A)$, recalling (iv) we can write

\[ a = t^K(\Delta^K a, \nabla^K a) = t^K(\Delta^K b, \nabla^K b) = b, \]

which shows that $K \in \mathcal{IQ}(Q)$. By (iii), $\mathcal{IQ} = \mathcal{IQ}(Q) = \mathcal{IQ}(Q)$. We have proved that condition (v) follows from (i)-(iv), as desired.

\[ \square \]

**Corollary 8.10.** Let $R \subseteq Q$ be a subquasivariety of $Q$ and $\mathcal{IR} = \mathcal{IQ}(\mathcal{IR})$. If the functor $\mathcal{I}: Q \to \mathcal{IQ}$ is a categorical equivalence, then its restriction $R$ is a categorical equivalence between $R$ and $\mathcal{IR}$.

**Corollary 8.11.** Every quasivariety of MV-algebras is categorically equivalent to the quasivariety of its interval algebras.

The next two corollaries show that $\mathcal{I}$ is a categorical equivalence for several classes of lattice ordered algebras having an important role in (many-valued) logic.

**Corollary 8.12.** Suppose the quasivariety $Q$ has a lattice reduct. In other words, there are binary operation symbols $\land, \lor \in \Sigma$ such that for each algebra $A \in Q$ the $(\land, \lor)$-reduct of $A$ is a lattice. Then the interval functor $\mathcal{I}: Q \to \mathcal{IQ}$ is an equivalence.

**Proof.** Directly from Theorem 8.9, using the IQ-term $t(y, z) = (i \lor y) \land z$. \[ \square \]

**Corollary 8.13.** For each subquasivariety of the following varieties, the functor $\mathcal{I}$ is an equivalence:

- Heyting algebras [8, p. 44], BL-algebras [21], MTL-algebras [18], and, more generally, residedated lattices [18].
- Modal algebras [9].
Remark 8.14. Corollary 8.12 does not apply directly to MV-algebras, since the lattice structure of an MV-algebra is not a reduct of the MV-structure, but, rather, it is term-definable from the basic operations \(-, \odot\). However, by (27), the term \(t(y, z) = y \odot (i \odot z \odot \neg y)\) does satisfy condition (iv) of Theorem 8.9. In this way, Theorem 4.4 can be seen as a consequence of Theorem 8.9(iv) \(\rightarrow\)(i).

The IMV-term \(t(y, z) = (i \lor y) \land z\), (where \(\lor\) and \(\land\) are as defined in (33)) satisfies condition (iv) of Theorem 8.9. This is a very special feature of MV-algebras, depending on the actual definition of \(\lor, \land\), as well as on the identity \(-i = i\) being satisfied by all IMV-algebras.

To find an example of a quasivariety for which \(\mathcal{I}\) does not determine a categorical equivalence, we need to leave the domain of lattice ordered structures. The following example exhibits a class of algebras whose underlying order is determined by the equation \(a \rightarrow b = 1\), but where the interval functor \(\mathcal{I}\) is not a categorical equivalence:

Example 8.15. Following [16], a Hilbert algebra is an algebra \((A, \rightarrow, 1)\) satisfying the equations

\[ a \rightarrow (b \rightarrow a) = 1, \quad (a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1 \]

and the quaequation

\[ a \rightarrow b = 1, \quad b \rightarrow a \Rightarrow a = b. \]

Hilbert algebras are the algebraic equivalent semantics of the implicational fragment of intuitionistic logic. A bounded Hilbert algebra is a structure \((A, \rightarrow, 0, 1)\) such that \((A, \rightarrow, 1)\) is a Hilbert algebra and \(0 \rightarrow a = 1\) for each \(a \in A\).

Let \(\mathcal{BH}\) denote the variety of bounded Hilbert algebras.

Every Hilbert algebra \(A\) admits a natural order defined by \(a \leq b\) if \(a \rightarrow b = 1\). With this order, \(A\) satisfies \(0 \leq a \leq 1\). Since \(\rightarrow\) reverses the order in the first coordinate and preserves the order in the second coordinate, upon defining \(\rho(\rightarrow) = (-, +)\), each bounded Hilbert algebra becomes a bounded \(\rho\)-poalgebra. Let \(B = \{\{0, 1\}, \rightarrow, 0, 1\}\) be the unique 3-element bounded Hilbert algebra. It is easy to prove that \(D = \{[0, 1], [0, 0], [a, a], [1, 1], [a, 1]\}\) is the universe of a subalgebra of \(\mathcal{I}(B)\). By Theorem 8.9(iii), the functor \(\mathcal{I}: \mathcal{BH} \rightarrow \mathcal{IBH}\) is not a categorical equivalence.

9. Related work

Corollary 8.12 shows that the interval functor of most quasivarieties \(\mathcal{Q}\) of partially ordered algebras existing in the literature is in fact a categorical equivalence. Thus, intuitively, \(\mathcal{Q}\) and the quasivariety of its interval algebras \(\mathcal{I}(\mathcal{Q})\) stand in the same relation as MV-algebras and IMV-algebras: the functor \(\mathcal{I}\) preserves subalgebras, homomorphic images, products, coproducts, projectives, injectives.

Remarkably enough, as the following brief survey will show, the pervasiveness of this categorical equivalence has gone virtually unnoticed in the vast literature on interval algebras, triangle algebras, interval constructors and triangularizations of algebras whose underlying order is a lattice.

Interval analysis, Minkowski sums. From the very outset, the basic operations in interval analysis include Minkowski sum, [26, (2.15)]. By contrast, in the literature on interval algebras, interval constructors, triangularizations, interval t-norms, [4, 14, 15, 20, 32, 33, 34], every monotone binary operation \(*\) on the set of intervals of a partially ordered algebra \(A\), possibly equipped with lattice and/or t-norm operations, is usually defined by \([\alpha, \beta] \star [\gamma, \delta] = [\alpha \star \gamma, \beta \star \delta]\), as we have done in Section 8. This is because the set \(\{\xi \star \chi \mid \xi \in [\alpha, \beta], \chi \in [\gamma, \delta]\}\) need not be an interval of \(A\) — whenever the counterpart of Proposition 2.1 does not hold for \(A\). Proposition 2.1 holds for any MV-algebra \(A\) because by [27, §3], (up to isomorphism), \(A\) is the unit interval of a unique unitary \(\ell\)-group \((G, u)\), and \(G\), like the ordered group of real numbers considered in interval analysis, has the Riesz decomposition property, [5, Lemma 1, page 310, and Theorem 49, p. 328].

Other basic operations in interval analysis include \(\overline{x}, \overline{\overline{x}}\), which are also found in interval algebra theory, and correspond to our \(\Delta x, \nabla x\), to \(\nu x, \mu x\) of [33], to \(l, r\), or \(\pi_1, \pi_2\) of [4]. An important property of any IMV-algebra \(J\) is Proposition 3.2(ii), stating that

\[ \text{by equation (27), any } x \in J \text{ is uniquely determined by } \Delta x \text{ and } \nabla x. \]

Mutatis mutandis, for certain classes \(K\) of algebras, (e.g., the “triangle algebras” of [33, Definition 3]), the property above turns out to be definable by equations, thus allowing the equational definability of the interval algebras of \(K\). For every quasivariety \(\mathcal{Q}\) considered in Corollary 8.12, the existence of a term \(t(\Delta x, \nabla x)\) equal to \(x\) yields a categorical equivalence between \(\mathcal{Q}\) and its associated category \(\mathcal{I}(\mathcal{Q})\) of interval algebras.
Monadic, modal, Girard algebras. The operations $\Delta, \nabla$, sometimes written as $\exists, \forall$, or $\diamondsuit, \square$, also occur in the realm of monadic algebras, to express some properties of quantifiers. Thus for instance, the paper [17] represents every monadic MV-algebra as an algebra of suitable (generally non-monotone) pairs of MV-algebras, equipped with unary operators that are reminiscent of our $\Delta$ and $\nabla$. However, monadic MV-algebras are not (term-equivalent to) IMV-algebras. A particular case of monadic MV-algebras is given by monadic boolean algebras: a celebrated theorem by A. Monteiro [25] shows that each monadic boolean algebra $A$ yields a three-valued Łukasiewicz algebra (i.e., an $MV_3$-algebra, as defined by Grigolia, see [13, §8.5] and [12]) $L_A$, in such a way that, up to isomorphism, each three-valued Łukasiewicz algebra arises has the form $L_A$ for some monadic boolean algebra $A$. And again, for any boolean algebra $B$, $I(B)$ will not be an $MV_3$ algebra, because of Proposition 3.2(iv).

The operations $\Delta, \nabla$ are also found in the theory of modal algebras to express the notions of “necessity” and “possibility”. In the specific domain of MV-algebras, modal operators are studied in [22] as a special case of general modal operators on residuated lattices [7]. However, the IMV-operations $\Delta, \nabla$ do not satisfy the same equations of the modal operators of [22] and [7]. Further examples are mentioned in [33, Section 3.3].

In the paper [34], the set of intervals in an MV-algebra $A$ is equipped with the structure of a “Girard algebra” $G(A)$. The main aim of the paper is to represent conditionals in many-valued logic. By [34, Theorem 2.2], and Proposition 3.2(iv), no interval algebra arising from the constructions of [34] can be an IMV-algebra. The paper [10] deals with “interval MV-algebras”, having no relations to IMV-algebras. The main result of [10] is that every interval $[\alpha, \beta]$ in an MV-algebra $A$ can be equipped with the structure of an MV-algebra. Again by Proposition 3.2(iv), no “interval MV-algebra” in the sense of [10] can be the subreduct of an IMV-algebra.

Interval t-norms on lattices, orders, logics. The book chapter in [14] entitled “Interval-Valued Algebras and Fuzzy Logics” investigates “triangularization” as an operator from bounded lattices into bounded lattices. Also see [33]. The action of triangularization on bounded lattices homomorphisms is not considered. In the same paper, triangularization is applied to classes of residuated lattices. Triangularization operators of general lattice-ordered structures are also considered in [2], using a different terminology. “Interval functors” from t-normed bounded lattices into t-norm bounded lattices are explicitly considered, e.g., in [2, Subsection 6.1]. In [3] the authors define a category whose objects are t-norms on arbitrary bounded lattices and whose morphisms are suitable generalizations of automorphisms. We refer to [3, 4, 15, 21], and to the bibliography of [14] for an account of interval t-norms on classes of bounded lattices. Similarly as in IMV-algebras, the product and the inclusion order are naturally definable in most interval algebras arising from triangularizations, [20, 15, 34, 4]. One can find in the literature various interval-valued logics, [4, 14, 33]. For instance, the semantics of the logic considered in [14, Section 4.2] is in terms of “triangle algebras”, and satisfies a soundness and completeness theorem. For any of these logic systems one can naturally ask about the algorithmic complexity of the tautology/consequence problem.

References

[1] M. Baczyński, B. Jayaram, (S,N)- and R-implications: A state-of-the-art survey, Fuzzy Sets and Systems, 159 (2008) 1836–1859.
[2] R. Callejas-Bedregal, B.C. Bedregal, Acióly-Scott Interval Categories, Electronic Notes in Theoretical Computer Science, 95 (2004) 169-187.
[3] B.C. Bedregal, R. Callejas-Bedregal, H.S. Santos, Bounded Lattice T-Norms as an Interval Category. In: D. Leivant and R. de Queiroz (Eds.):WoLLIC 2007, Lecture Notes in Computer Science, 4576 (2007) 26–37.
[4] B.C. Bedregal, R.H. Nunes Santiago, Interval representations, Łukasiewicz implicators and Smets–Magrez axioms, Information Sciences, 221 (2013) 192–200.
[5] G. Birkhoff, Lattice-Ordered Groups, Annals of Mathematics, Second Series, Vol. 43, No. 2 (1942), pp. 298-331.
[6] V. Boicescu, A. Filipoiu, G. Georgescu, S. Rudeanu, Łukasiewicz–Moisil Algebras, North-Holland, Amsterdam, 1991.
[7] F. Bou, F. Esteva, L.Godo, On the Minimum Many-Valued Modal Logic over a Finite Residuated Lattice, Journal of Logic and Computation, Published online 7 October 2009 doi:10.1093/logcom/exp062
[8] S. Burris, H. P. Sankappanavar, A course in universal algebra, Graduate Texts in Mathematics, Vol. 78, Springer, 1981.
[9] A. Chagrov and M. Zakharyaschev, Modal Logic, Oxford Logic Guides vol. 35, Oxford University Press, 1997.
[10] I. Chajda, J. Kühr, A note on interval MV-algebras, Mathematica Slovaca, 56.1 (2006) 47–52.
[11] C.C. Chang, A new proof of the completeness of the Lukasiewicz axioms, Transactions of the American Mathematical Society, 93 (1959) 74–80.
[12] R. Cignoli, Coproducts in the categories of Kleene and three-valued Lukasiewicz algebras, Studia Logica, 38 (1979) 237–245.
[13] R. Cignoli, I.M.L. D’Ottaviano, D. Mundici, Algebraic Foundations of many-valued Reasoning, Trends in Logic, Vol. 7, Kluwer Academic Publishers, Dordrecht, 2000.
[14] C. Cornelis, G. Deschrijver, M. Nachtegaele, S. Schockaert, Yun Shi (Eds.), 35 Years of Fuzzy Set Theory, Studies in Fuzziness and Soft Computing, Vol. 261, Springer, Berlin, 2010.
[15] Dechao Li, Yongming Li, Algebraic structures of interval-valued fuzzy (S,N)-implications, International Journal of Approximate Reasoning, 53 (2012) 892–900.
[16] A. Diego, Sur les algèbres de Hilbert, Colléction de Logique Mathématique Serie A, vol. 21, Gauthiers-Villars, Paris, 1966.
[17] A. Di Nola, R. Grigolia, On monadic MV-algebras, Annals of Pure and Applied Logic, 128 (2004) 125–139.
[18] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, Residuated Lattices. An Algebraic Glimpse at Substructural Logics, Studies in Logic and the Foundations of Mathematics, Vol.151, Elsevier, 2007.
[19] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness. W.H. Freeman, New York, 1979.
[20] M. Gehrke, C. Walker, E. Walker, Some Comments on Interval Valued Fuzzy Sets, International Journal of Intelligent Systems, 11 (1996) 751–759.
[21] P. Hájek, Metamathematics of Fuzzy Logic, Trends in Logic Vol. 4, Kluwer Academic Publishers, 1998.
[22] D. Mundici, Interpretation of AF C*-algebras in Lukasiewicz sentential calculus, Journal of Functional Analysis, 65 (1986) 15–63.
[23] D. Mundici, Advanced Lukasiewicz calculus and MV-algebras, Trends in Logic, Vol. 35, Springer, NY, 2011.
[24] A. Rose, J.B. Rosser, Fragments of many-valued statement calculi, Transactions of the American Mathematical Society, 87 (1958) 1–53.
[25] P. Smets, P. Magrez, Implication in fuzzy logic, International Journal of Approximate Reasoning, 1 (1987) 327–347.

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