1. Notations, Generalities, and Statements

Let $M$ be a real submanifold of $\mathbb{C}^N$ of codimension $l$ in a neighborhood of a point $p_0$. We assume all through the paper that $M$ is generic which means that its tangent plane $T_{p_0}M$ is not contained in any complex proper subspace of $\mathbb{C}^N$. A wedge $W$ in $\mathbb{C}^N$ is a domain which satisfies, for an open cone $\Gamma$ and a neighborhood $B$ of $p_0$,

\[
((M \cap B) + \Gamma) \cap B \subset V.
\]

The maximal cone for any whose proper subcone and for suitable $B$ (1.1) holds is invariant under $T_{p_0}M$ and is therefore identified to a cone $\Gamma$ in the normal space $(T_M\mathbb{C}^N)_{p_0}$, the so called “directional” cone of $W$ at $p_0$.

We will deal with the space CR$_M$ of the continuous CR functions on $M$ that is the solutions $f$ of the equation $\bar{\partial}_M f = 0$ where $\bar{\partial}_M f$ denotes the component of $\bar{\partial}f$ tangential to $M$. (When $f$ is not $C^1$ the equation $\bar{\partial}_M f = 0$ must be understood in the sense of currents.) A large class of CR functions is described as “topological” boundary values. Thus, if $F$ is a holomorphic function on a wedge $W$ with edge $M$, continuous up to $M$, then its boundary value $f = b(F)$ is a CR function on $M$ due to $\bar{\partial}_M f = b(\bar{\partial}F)(= 0)$. Note that by the Ajrapetyan-Henkin edge of the wedge theorem [1], there is a maximal directional cone $\Gamma$ for wedge extendibility of $f = b(F)$. In particular, if we denote by $\Gamma^*$ the polar of this maximal cone, we can meaningfully define the analytic wave front set of $b(F)$ by

\[
WF(b(F))_{p_0} = -\Gamma^*.
\]

The notion of wave front set for CR functions more general than just boundary values requires heavy microlocal machinery [5] and goes beyond the purpose of the present presentation. We write now complex coordinates as $(z, w) \in \mathbb{C}^l \times \mathbb{C}^n = \mathbb{C}^N$, $z = x + iy$, and suppose that $M$ is defined in a neighborhood of $p_0 = 0$ by a system of equations $y_j = h_j$ $j = 1, \ldots, l$ with $h(0) = 0$ and $\partial h(0) = 0$; we also write $r = (r_j)_j = (-y_j + h_j)_j$. We select one of the $w$-coordinates, say $w_1$, and define $\tilde{M} := M \cap (\mathbb{C}^l \times \mathbb{C}^1_{w_1})$. We decompose $l$ as $l = l_1 + \cdots + l_r$, write $I_1 = (1, \ldots, l_1), \ldots, I_r = (\sum_{j \leq r-1} l_j, \ldots, l)$ and decompose $z$ as $z = (z_{I_1}, \ldots, z_{I_r})$. For a set of integers $m_1 < \cdots < m_r$, where $m_r$ is possibly $+\infty$, we define the notions of “weighted” homogeneity and vanishing order. For a function $g = g(x_{I_1}, \ldots, x_{I_r}, w_1)$, with $j \leq r$, we say that $g$ is homogeneous of “weight” $m_j$ when
$h(t^{m_1}x_{I_1}, \ldots, t^{m_r}x_{I_r}, tw_1)$ is a homogeneous polynomial in $t$ of degree $m_j$. We say that $g$ is infinitesimal of weight $m_j$ and write $h = O^{m_j}$ when

$$g(t^{m_1}x_{I_1}, \ldots, t^{m_r}x_{I_r}, tw_1) = O(t^{m_j}).$$

A special definition is needed for $j = r$ and $m_r = +\infty$. In this case we say that $g$ is infinitesimal of weight $+\infty$ and write $g = O^{+\infty}$ when $g(t^{m_1}x_{I_1}, \ldots, t^{m_r}x_{I_r}, tw_1) = O(t^m)$ for any $m$. In other terms, $g$ is divisible by some monomial in $x_{I_r}$. We recall that equations for $\tilde{M}$ in “Bloom-Graham normal form” mean. Intrinsically associated to $\tilde{M}$ there are integers $m_1 < \cdots < m_r$, the so called “Hörmander numbers”, and $l_1, \ldots, l_r$ with $\sum_j l_j = l$, their respective “multiplicities”. For $m_r < +\infty$, in suitable coordinates at $p_0$, $\tilde{M}$ is described by equations

$$\begin{cases}
y_{I_1} = P_{I_1}(w_1) + O^{m_1+1}, \\
y_{I_2} = P_{I_2}(x_{I_1}, w_1) + O^{m_2+1}, \\
\vdots \\
y_{I_r} = P_{I_r}(x_{I_1}, \ldots, x_{I_{r-1}}, w_1) + O^{m_r+1},
\end{cases}$$

(1.2)

with each $P_{I_j}$ homogeneous of degree $m_j$ and such that for any $\xi^o \in \mathbb{R}^{l_j}$, $\langle \xi^o, P_{I_j} \rangle$ is not $\tilde{M}$-pluriharmonic. (A homogeneous polynomial $g$ of weight $m_j$ is said to be $\tilde{M}$-pluriharmonic of weight $m_j$ if there exists $F$ holomorphic in $\mathbb{C}^{l+1}$ such that $g = \text{Im} F|_{\tilde{M}} + O^{m_j+1}$.) When $m_r = +\infty$, then for any $m$, there are coordinates such that (1.2) holds with the last equation replaced by $y_{I_r} = O^m$. $\tilde{M}$ is said to be of “finite type” when $m_r < +\infty$. $\tilde{M}$ is said “semirigid” when each $P_{I_j}$ is a function of $w_1$ only. The similar notions of finite type and semirigidity for $M$ instead of $\tilde{M}$ apply when one deals with equations of type (1.2) involving all $w$-variables instead of $w_1$ only. We will see in §3 that finite type can be characterized by means of brackets instead of normal equations: iterated commutators of vector fields tangential to $M$, of (1, 0) and (0, 1) type, up to a certain finite number, the highest Hörmander number $m_r$, span the whole complexified tangent bundle $\mathbb{C} \otimes \mathbb{R} TM$. Let us recall that when $M$ is of finite type, then according to TUMANOV [12], CR functions $\tilde{M}$ are boundary values $f = b(F)$ of holomorphic functions $F$ on a wedge $W$; in particular, in this situation, the notion of wave front set applies to any $f$.

**Theorem 1.1.** Let $M$ be a generic manifold of $\mathbb{C}^N$ of finite type, and, for a choice of a complex tangent direction $w_1$, let (1.2) be a normal system of equations for $\tilde{M} = M \cap (\mathbb{C}_z^l \times \mathbb{C}_w^l)$. We assume that for some $j$, for $\xi^o \in \mathbb{R}^{l_j + \cdots + l_r}$ and with the notation $P := \langle \xi^o, P_{I_j} \rangle$ we have

$$\begin{cases}
P = P(w_1) & \text{for a homogeneous polynomial } P \text{ of degree } m_j, \\
P(w_1) \geq 0 & \text{for } w_1 \text{ in a sector } S \text{ of width } > \frac{\pi}{m_j}. 
\end{cases}$$

Then

$$\xi^o \notin WF(f) \quad \forall f \in CR_M.$$
The proof will follow in §2. The first of (1.3) is a sort of semirigidity in direction $w_1$ and codirection $\xi^o$. We will exhibit in §4 (Proposition 1.2 and Corollary 1.3) a large class of hypersurfaces $M$ for which when (1.3) is violated, we can find a “barrier” that is a holomorphic function $F$ with $M \subset \{ \text{Im} F < 0 \}$. In particular, for these $M$, there always exist CR functions $f \in CR_M$ such that $\xi^o \in WF(f)$ for $\xi^o = d(\text{Im} F)$. This shows that the statement in Theorem 1.1 is sharp.

**Remark 1.2.** When $j = 1$ the first of (1.3) is automatically fulfilled. Also, since we are assuming that $P_{I_1}$ is not $M$-harmonic, then it is divisible by $|w_1|^2$ and therefore it has at most $2m_1 - 2$ zeroes on the unit circle $|w_1| = 1$. In particular, for either of $\pm P$, the second of (1.3) is satisfied.

**Remark 1.3.** There is a sort of “hierarchy” between the Hörmander numbers $m_j$ whose geometric meaning will be fully clear from the proof in §2. According to it, (1.3) for $j > 1$, gives the control not of the whole $WF(f)$ but only of its section $WF(f) \cap (\{ 0 \} \times \cdots \times \{ 0 \} \times \mathbb{R}^{l_j + \cdots + l_r})$. In fact, the proof of the theorem will consist in proving CR extension in some extra direction $v$ close to the component normal to $M$ of the disc attached to $M$ over $S$, and (1.3) does not give informations for $v$ itself but for $v_{I_1, \ldots, I_r}$.

When $M$ is of finite type and semirigid (in the complex of its arguments $w$), then our proof provides an alternative proof of the extension of any $f$ to a wedge $W$. The first conclusion in this direction is due to [5] where a description of $W$ is also given. We improve this description by specifying the vanishing order in a precise direction $w_1$. Also, the semirigidity in the first of (1.3) can be released, as well the hypothesis that the equations are in canonical form as in (1.2). What is indeed essential is the weighted vanishing order; non-$M$-harmonicity in the homogeneous terms is not needed. Thus, suppose that $M$ is of finite type and that $M = M \cap (\mathbb{C}^l \times \mathbb{C})$ has equations in the (not necessarily normal) form $y_{I_j} = h_{I_j}$ with $h_{I_j} = O^{m_j}$, $j = 1, \ldots, r$.

**Theorem 1.4.** In the above situation suppose that for $j \leq r$ with $m_j < +\infty$ and for some $\xi^o \in \mathbb{R}^{l_j + \cdots + l_r}$, we have for a suitable constant $c$

\begin{equation}
\langle \xi^o, h_{I_j} \rangle > 0 \text{ for } w_1 \text{ in a sector } S \text{ of width } > \frac{\pi}{m_j} \text{ and for } |x_{I_j}| < c|w_1|^{m_j}.
\end{equation}

Then

$$\xi^o \notin WF(f) \quad \forall f \in CR_M.$$
Tumanov in [12] (cf. also Trepreau [11] in case $M$ is a hypersurface). He introduced the notion of “minimality” of $M$ as the absence of proper submanifolds $S \subset M$ with the same CR structure as $M$ that is $T^CS = T^CM|_S$. Note that when $M$ is of finite type then it is minimal. (First, finite type in the sense that $m_r < +\infty$ in a system of normal equations, is equivalent to “finite bracket type” according to the subsequent discussion of §3. But then the presence of $S$ as above would force all brackets to belong to $\mathbb{C} \otimes \mathbb{R} TS$.) He then proved that if $M$ is minimal, then there exist arbitrarily small discs of “defect” $0$ and hence endowed with infinitesimal deformations which span all normal directions to $M$. Collecting all these directions by the edge of the wedge theorem of [1] he got a common wedge $W$ to which all CR functions are forced to extend. Necessity of minimality for such an extension is on its hand a simpler result (and even trivial if $M$ is a hypersurface). However, a precise description of $W$ has not yet been found. Our paper aims at this attempt and deals with extension in directions produced by higher type commutators.

Let us briefly recall the related literature. The first theorems go back to Ajrapetyan-Henkin [1] and Boggess-Polking [9] and state extension in directions of the Levi cone. Next Boggess-Pitts [8] proved extension in directions shown by iterated brackets up to the first H"ormander number. More recently the authors obtained in collaboration with Zaitsev generalizations to the case of CR functions not defined on the whole $M$ but, instead, on a subwedge $V \subset M$. Let us point out the main novelties of the present paper.

In the equations (1.2) the weighted homogeneity degrees $m_1 < m_2 < \ldots$ are calculated with respect to $w_1$ and not to the complex of the variables. Also, $m_j$ is not the smallest among the $m_i$’s. On the contrary, most of other classical CR extension criteria concern the first H"ormander number: in all $w$ directions as in [8], or at least in one selected direction as in [3] Th. 11. (Let us point out that it seems that the method of [8] can be adapted to treat also this second situation though this is absent from their statements.) This paper [3], whose conclusions are the closest to ours, gives indeed extension also related to further H"ormander numbers, as e.g. in Theorem 8. But in this case its method founded on Fourier calculus, requires an assumption of semirigidity in the complex of the equations and of the arguments $w$. To explain the difference, let’s consider for instance in $\mathbb{C}^4$ with coordinates $(z_1, z_2, w_1, w_2)$ the manifold $M$ defined by

$$
\begin{align*}
y_1 &= |w_1|^2 + |w_2|^2 + f_1(x_1, x_2, w_1, w_2) \\
y_2 &= |w_2|^4 + x_1|w_1|^2 + f_2(x_1, x_2, w_1, w_2)
\end{align*}
$$

where $f_j = O^{2j+1}$. Extension in direction $v^1 \sim (1, 0)$ is clear according to all authors. For extension in directions with non-trivial $y_2$-component, we notice that the method by [8] fails because of the lack of semirigidity. (Also, [3] and [8] Th.11 cannot be applied because $w_2$ appears in the second equation in a higher homogeneity degree than in the first.) However our Theorem [11] applies for sectors in the $w_2$-plane, and yields extension in direction $v^2 = (1 + \sigma(\eta), \eta^2 + o(\eta^2))$ with $\sigma(\eta)$ infinitesimal with $\eta$. Our generalization goes also in another direction (though this was already achieved in [7]). We are able to
obtain extension in more general situations and to a larger set of directions. Thus, for instance, let $M$ be the manifold in $\mathbb{C}^3$ defined by $(y_1 = |w|^4 + a|w|^2 \Re w^2, y_2 = |w|^4)$. $\star$ gives extension for $a > 2$ in directions which satisfy $y_1 > -|y_2|(\frac{a}{2} - 1)$. On the other hand, by the proof of our Theorems 1.1 and 1.4, we have extension when $a > \sqrt{2}$ in directions satisfying $y_1 > -|y_2|(\frac{a}{\sqrt{2}} - 1)$. We refer to the subsequent §4 for a complete proof of these claims.

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2. Proof of Theorems 1.1, 1.4

(a) Preliminaries on $F^\alpha$ spaces. Let $0 < \alpha < 1$ and denote by $\tau = re^{i\theta}$ the variable in the standard disc $\Delta$. Let us recall from [13], [15] and [16] some basics about attaching analytic discs to $M$ in the subclasses $F^\alpha$ of the Hölder classes $C^\alpha$. These are the spaces of real continuous functions $\sigma(\theta)$, $\theta \in [-\pi, \pi]$ which are $C^{1,\alpha}$ out of $0$ and for which the following norm is finite

$$||\sigma||_{F^\alpha} := ||\sigma||_{C^\alpha} + ||\theta\sigma^{(1)}||_{C^\alpha}.$$  

(Here $\sigma^{(1)}$ denotes the first derivative.) We remark that for $\sigma \in F^\alpha$ we must have $\theta\sigma^{(1)}|_{\theta=0} = 0$ for otherwise $\theta\sigma^{(1)} \to c \neq 0$ which implies $|\sigma| \geq \log \frac{|c|}{2} + \log |\theta|$ which contradicts the boundedness of $\sigma$. This shows that $F^\alpha$ is continuously embedded into $C^\alpha$. It is easy to check that $F^\alpha$ is a Banach algebra. Also, if $\sigma_i \in F^{\alpha_i}$ $i = 1, 2$, then $\sigma_1 \cdot \sigma_2 \in F^{\alpha_1+\alpha_2}$ for $\mu_1 + \mu_2 < 1$, resp. $\sigma_1 \cdot \sigma_2 \in C^{1,\beta}$ with $\beta := (\mu_1 + \mu_2) - 1$ for $\mu_1 + \mu_2 > 1$. In both cases the multiplication is continuous with values in the respective spaces.

Let $T_1$ denote the Hilbert transform normalized by the condition $T_1(\cdot)(1) = 0$; it is easy to see that $T_1$ is a bounded operator in $F^\alpha$. We come back to our manifold $M$. We write coordinates in $\mathbb{C}^N \cong \mathbb{C} \times \mathbb{C}^{N-1}$ as $(z, w)$ with $z = x + iy$, choose a distinguished direction, say $w_1$, and describe $M := M \cap (\mathbb{C}_x \times \mathbb{C}_{w_1})$ by the system of equations $y_i = h_i(x, w)$ with $h_i = O^{m_i}$. (The Bloom-Graham normal form is not needed.) We will consider in $\mathbb{C}^N$ analytic discs $A(\tau) = (z(\tau), w(\tau)) \tau \in \Delta$ (the standard disc in $\mathbb{C}$), attached to $M$ that is satisfying $A(\partial \Delta) \subset M$. If we prescribe an analytic function $w_1(\tau) \tau \in \Delta$, the so called CR component, and a point $p = (z, w_1)$ with $y = h(x, w_1)$, and look for an analytic completion $z(\tau)$ for $A(\tau) = (z(\tau), w_1(\tau))$ with $A(1) = p$, we are lead to the Bishop’s equation

$$u(\tau) = -T_1h(u(\tau) + x, w_1(\tau)) \quad \tau \in \partial \Delta.$$  

(2.1)

In fact if $u(\tau)$ solves (2.1) then if set $z(\tau) = u(\tau) + iv(\tau) + z$, we obtain that $A(\tau) = (z(\tau), w_1(\tau))$ is holomorphic, $v(\tau) = h(u(\tau), w_1(\tau)) \forall \tau \in \partial \Delta$ and finally $A(1) = p$. We will consider equation (2.1) in the spaces $F^\alpha$, $F^{m,\alpha}$ and $C^{1,\beta}$ for which $T_1$ is bounded. We will also use the composition properties of $h_i$ for $i \geq j$ with functions in the above classes as stated in [7]. To take advantage of this composition we will assume $m_j\alpha > 1$ (and, to be sharp, $\alpha(m_j - 1) < 1$). Here is our main technical tool.
Proposition 2.1. Let $h_I$ be of class $C^{m_i+3}$, and satisfy $h_I = \Theta^{m_i}$. Then for any $\epsilon$ there is $\delta$ such that if $\|h_I\|_{C^{\alpha}_t} < \delta$, $\|w_i\|_{F_\alpha} < \delta$, $|x| < \delta$, then the equation \( (2.1) \) has an unique solution $u \in F_\alpha$ with $\|u\|_{F_\alpha} < \epsilon$. Moreover, $u_{I_1} \in F_{m_1}^{\alpha}$, ..., $u_{I_{j-1}} \in F_{m_{j-1}}^{\alpha}$ and $(u_{I_1}, \ldots, u_{I_j}) \in C^{1,\beta}$ for $\beta = m_j \alpha - 1$ and, if $w_i$ depends on some parameter $\lambda \in \mathbb{R}^d$ so that $\lambda \mapsto w_{1\lambda}$, $\mathbb{R}^d \rightarrow F_\alpha$ is $C^k$ for $k \leq m_i$, then also $\lambda, x \mapsto (u_{I})_{\lambda x}$, $\mathbb{R}^{d+l} \rightarrow C^{1,\beta}$ is $C^k$.

In particular there exist mixed derivatives in $\lambda, x$ and up to order $k$ and $1$ respectively, and they commute that is

\[
\partial_{\tau} \partial^{k'}_{x} u = \partial^{k'}_{x} \partial_{\tau} u \forall k' \leq k.
\]

Proof. One first solves Bishop’s equation \( (2.1) \) in the $F_\alpha$ - spaces by the aid of the implicit function theorem. To this end one considers the mapping $F: (\lambda, x, w, u) \mapsto u - T_1 h(u + x, w_1)$, $\mathbb{R}^d \times \mathbb{R}^l \times F_\alpha \times F_\alpha \rightarrow F_\alpha$. Then for the partial Jacobian $\partial_{\tau} F$ with respect to $u$, one has $\partial_{\tau} F : \dot{u} \mapsto \dot{u} - T_1 \partial_{x} \dot{h}$. In particular if we evaluate at $(\lambda, x, w, u) = (0, 0, 0, 0)$, then this is invertible since $\partial_{x} h|_{0} = 0$. The differentiability with respect to the parameters in the space $F_\alpha$ is also clear in view of \cite{L1} Prop. 11.

We show that the components $u_{I_i}$ $i \geq j$ of the solution to Bishop’s equation, as well as their harmonic conjugates $v_{I_i}$, are in fact in $F_{m_i}^{\alpha}$ for $i < j$ (resp. $C^{1,\beta}$ for $i \geq j$) with $\beta := m_j \alpha - 1$ and also prove differentiability with values in the space. The key point is that the composition $\varphi(1-\tau)^{\alpha}$, and in bigger generality $\varphi(w_1)$ for $w_1 \in F_\alpha$, $w_1(1) = 0$, with $\varphi = O^{m_i}$ belongs to $F_{m_i}^{\alpha}$ for $i < j$ (resp. $C^{1,\beta}$ for $i \geq j$). We put $z(\tau) = u(\tau) + iv(\tau) + z$ with $v = T_1 u$ and $z = x + ih(x, w)$ and also write $\tau = e^{i\theta}$ on $\partial \Delta$. We can check that if $z_{I_i}(\tau) \in F^{\kappa \alpha}$ for $k \leq m_i - 2$, then in fact $z(\tau) \in F^{(k+1)\alpha}$. In fact $v$ gains regularity at each step because $h_{I_i} = O^{m_i}$ together with the fact that if $\sigma(\theta) \in F^{\kappa \alpha}$ and $\sigma(0) = 0$, then $|\theta^{\alpha} \sigma(\theta)| \in F^{(k+1)\alpha}$ due to

\[
|(\theta^{\alpha} \sigma(\theta))^{(1)}| = |\theta^{\alpha-1} \sigma(\theta) + |\theta^{\alpha} \sigma^{(1)}(\theta)|
\leq c|\theta|^{(k+1)\alpha-1}.
\]

But the Hilbert transform interchanges the $F^{(k+1)\alpha}$ regularity from $v$ to $u$ and thus $z(e^{i\theta}) \in F^{(k+1)\alpha}$. This completes the proof when $i < j$. On the other hand, when $i \geq j$, in order to pass from $F^{(m_i-1)\alpha}$ to $C^{1,\beta}$, we have to prove that $(\theta^{\alpha} u)^{(1)} = \theta^{\alpha-1} u + \theta^{\alpha-1}(\theta u^{(1)})$ belongs to $C^{\beta}$. But in fact, since both $u$ and $\theta u^{(1)}$ are in $C^{(m_i-1)\alpha}$ and are 0 at $\theta = 0$, then their product by $\theta^{\alpha-1}$ is in $C^{\beta}$ as one can easily check by the Hardy-Littlewood principle. It follows that $(\theta^{\alpha} u)^{(1)} \in C^{\beta}$ and hence $\theta^{\alpha} u \in C^{1,\beta}$. Thus $u(e^{i\theta})$ and hence $z(e^{i\theta})$ itself is in $C^{1,\beta}$. As for the differentiability on $x$ and on the parameters, it is a variant of \cite{L15} Proposition 15 in \cite{L2} by the same feed back argument as above.

\[\square\]

We can think of the family of discs produced by the above statement as a deformation of the disc $A(\tau) \equiv 0$ which is a trivial solution to Bishop’s equation. By the next statement
we will show how it is possible to make infinitesimal deformations of discs which are no more small.

**Proposition 2.2.** Let \( h_{I_i} \in C^{m_i+3} \) satisfy \( h_{I_i} = O^{m_i} \), let \( \tilde{w}_i(\tau) \in C^{1,\beta} \), \( \tilde{w}_i(1) = 0 \) be small in \( F^\alpha \) (not necessarily in \( C^{1,\beta} \)), and let \( \tilde{u}(\tau) \in F^\alpha \) be a solution of Bishop’s equation \( \tilde{u} = -T_{\tau}h(\tilde{u},\tilde{w}) \); in particular \( \tilde{u}_{I_i} \in C^{1,\beta} \) for any \( i \geq j \) according to Proposition 2.1. Then for any \( w_i(\tau) \) with \( \|w_i - \tilde{w}_i\|_{C^{1,\beta}} < \delta \), \( |x| < \delta \) there is an unique solution \( w \in F^\alpha \) with \( w_i(\tau) \in C^{1,\beta} \) of Bishop’s equation with \( \|u_i - \tilde{u}_i\|_{C^{1,\beta}} < \epsilon \) \( \forall i \leq j \). Moreover, if \( \lambda \mapsto (w_1)_\lambda \) is \( C^k \), \( k \leq m_i \), then also \( \lambda, x \mapsto (u_1)_\lambda \) is \( C^k \).

**Proof.** In the present situation we define \( F: \mathbb{R}^d \times \mathbb{R}^l \times C^{1,\beta} \) similarly as in the proof of Proposition 2.1 and wish to prove that \( \partial_x F \) is still invertible. For this purpose it is enough to show that \( \partial_u h_{I_i}(\tilde{u},\tilde{w}) \) is small in \( C^{1,\beta} \) norm. But in fact recall that \( \|\partial_x h_{I_i}(u,w)\| = O(|w|^2) \) and therefore \( \|\partial_x h_{I_i}(\tilde{u},\tilde{w})(1)\|_{C^\beta} \leq c\|\tilde{w}_1\|_{C^\beta}\|\tilde{w}_1^{(1)}\|_{C^\beta} \leq \epsilon \).

(b) Construction of a singular disc attached to \( M \) with controlled normal component. Let us suppose that \( (\ref{eq:3}) \) is fulfilled. It is not restrictive to assume that the sector \( \mathbb{C}_{w_1} \) where \( g \geq 0 \) contains \((1 - \tau)^{\alpha}i\epsilon_{l+1}, \tau \in \Delta \). (Here \( \epsilon_{l+1} \) is the unit vector of the \( w_1 \)-plane.) Let \( \alpha \) satisfy \( \alpha m_j > 1, \alpha(m_j - 1) < 1 \). We define, for a small real parameter \( \eta > 0 \):

\[
(2.4) \quad w_1(\tau) = (w_1)_\eta(\tau) := \eta(1 - \tau)^{\alpha}i\epsilon_{l+1}.
\]

We attach to \( \tilde{M} \) a family of \( F^\alpha \) - discs \( A(\tau) = A_\eta(\tau) \) whose "\( w_1 \)-component" is \( w_1(\tau) \). We recall from (a) that for \( i \geq j \) we have \( \eta \mapsto (z_{I_i})_\eta(\tau), \mathbb{R} \to C^{1,\beta} \) is \( C^{m_i} \). We also write \( z_{I_i}(\tau) \) instead of \( (z_{I_i})_\eta(\tau) \), \( z_{I_i}(\tau) = u_{I_i}(\tau) + iT_{\tau}v_{I_i}(\tau) \), and finally \( A(\tau) = (z(\tau), w(\tau)) \). We note that we have

\[
(2.5) \quad \partial_\eta^s v_{I_i} |_{\eta=0} \equiv 0, \quad \partial_\eta^s u_{I_i} |_{\eta=0} \equiv 0 \quad \forall s \leq m_i - 1.
\]

This is clear for \( s = 0, 1 \). If it is true for any \( s \leq m_i - 2 \), then it is also true for \( s = m_i - 1 \) due to \( h_{I_i} = O^{m_i} \) by a “feed-back” procedure. If we then Taylor-expand \( \partial_\tau v_{I_i} \) at \( \eta = 0 \), we get

\[
(2.6) \quad \partial_\tau v_{I_i} = \frac{\partial_{\eta}^{m_i} \partial_\tau v_{I_i} |_{\eta=0}}{m_i!} + o(\eta^{m_i}).
\]

By a similar argument we can also prove that

\[
(2.7) \quad |v_{I_i}| \leq c|w_1|^{m_i}, \quad |u_{I_i}| \leq c|w_1|^{m_i}.
\]

In fact, in the classes \( F^{k\alpha} \) regularity and vanishing order are coincident: thus the equation \( v_{I_i} = h_{I_i} \) gives control of the vanishing order of \( v_{I_i} \) which is transferred as regularity to \( u_{I_i} \), through Hilbert transform, and again as vanishing order to \( v_{I_i} \). In this way we can prove that each \( v_{I_i} \) and \( u_{I_i} \) belongs to \( F^{m_i\alpha} \) (and also to \( C^{[m_i\alpha],[m_i\alpha]} \) where \([m_i\alpha]\), resp. \( \{m_i\alpha\}\), is the integer, resp. fractional, part of \( m_i\alpha \). Recall that if \( \xi_\alpha \) is, say, the unit vector in the
Hence (1.5) yields, through Hopf’s Lemma

By (2.6) we conclude

\[ \langle r, \partial_1 \rangle \geq 0 \forall \tau \in \bar{\Delta}. \]

Hence (1.4) yields, through Hopf’s Lemma

\[ \langle r, \partial_2 \rangle \geq 0 \forall \tau \in \bar{\Delta}. \]

By (2.6) we conclude \( \langle r, \partial_3 \rangle = -c_0 \eta^m \leq 0 \), for any \( \eta \) sufficiently small. We fix such a small \( \eta \) and, by rescaling, we even suppose \( \eta = 1 \) and define \( v_o = \partial_1 v |_{\tau=1} \). According to (2.9) we have \( \langle v_o, r \rangle < -\frac{c_0}{\bar{\Delta}} \).

(c) **Polynomial approximation of** \( (1-\tau)^\alpha \) **in** \( \mathcal{F}^\gamma(\bar{\Delta}) \) **for** \( \gamma < \alpha \). We have the Taylor expansion

\[ (1-\tau)^\alpha = 1 - \alpha\tau - \frac{\alpha(\alpha-1)}{2!} \tau^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{3!} \tau^3 + \ldots \]

\[ = 1 - \sum_{n=1}^{+\infty} \left(\begin{array}{c} n \\ n \end{array}\right) \alpha^n \tau^n. \]

We call \( S_N = S_N(\tau) \) its the partial sum of the series (2.10) for \( 1 \leq n \leq N \). Our goal is to prove next

**Theorem 2.3.** We have

\[ S_N(\tau) \to (1-\tau)^\alpha \text{ in } \mathcal{F}^\gamma(\bar{\Delta}) \text{ for any } \gamma < \alpha. \]

Before giving the proof of Theorem 2.3 let us recall that \( \|\sigma\|_{\mathcal{F}^\gamma} = \|\sigma\|_{C^0} + \|(1-\tau)\sigma\|_{C^\gamma} \). Hence we have to prove that

\[ S_N \to (1-\tau)^\alpha \text{ in } C^0(\bar{\Delta}), \quad (1-\tau)^\alpha \text{ in } C^\gamma(\bar{\Delta}). \]

To prove the first of (2.12) we note that since

\[ |S_N(\tau)| \leq \sum_{n=1}^{N} \left(\begin{array}{c} n \\ n \end{array}\right) \alpha^n n |\tau|^{n-1} \to \alpha(1-|\tau|)^{\alpha-1}, \]

then in particular the partial sums \( |S_N(\tau)| \) are bounded on \( \Delta \), uniformly over \( N \), by \( \alpha(1-||\tau|)^{\alpha-1} \). In particular the sequence of the \( S_N \)’s is uniformly continuous in \( \bar{\Delta} \), which yields at once the first of (2.12). As for the second of (2.12) we note that

\[ |S_N| \leq \sum_{n=1}^{N} \left(\begin{array}{c} n \\ n \end{array}\right) n(n-1) |\tau|^{n-2} \to \alpha |\alpha - 1| |1-|\tau||^{\alpha-2}. \]
It follows

\[(1 - \tau)S_N' \leq |S_N'| + |1 - \tau||S_N''| \leq \alpha(1 - |\tau|)^{\alpha - 1} + \alpha|\alpha - 1|(1 - |\tau|)^{\alpha - 1} = c(1 - |\tau|)^{\alpha - 1}.\]

To conclude the proof of Theorem 2.3 it suffices to use the following one real variable Lemma.

**Lemma 2.4.** Let \(\{f_N\}\) be a sequence of real functions such that

\[(2.16) \quad f_N \to 0 \text{ in } C^0([0, 1 - \epsilon]) \text{ for any } \epsilon,\]

and

\[(2.17) \quad |f_N'| \leq c(1 - t)^{\alpha - 1} \text{ in } [0, 1),\]

with \(c\) independent of \(\epsilon\). Then

\[(2.18) \quad f_N \to 0 \text{ in } C^\gamma([0, 1]) \text{ for any } \gamma < \alpha.\]

**Proof.** We have by integration

\[|f_N(x) - f_N(y)| \leq c|x - y|^{\alpha} \text{ (for a different } c).\]

It follows that for any \(\epsilon\) and for suitable \(\delta = \delta_\epsilon\) we have, when \(|x - y| < \delta\)

\[\frac{|f_N(x) - f_N(y)|}{|x - y|^{\alpha}} < \epsilon.\]

On the other hand, when \(|x - y| \geq \delta\), then

\[(2.19) \quad \frac{|f_N(x) - f_N(y)|}{|x - y|^{\gamma}} \leq \delta^{-\gamma}|f_N(x) - f_N(y)| \leq \delta^{-\gamma}(|f_N(x)| + |f_N(y)|).\]

Hence it suffices to prove that \(f_N \to 0\) in \(C^0([0, 1])\). By (2.17) \(\{f_N\}\) is equicontinuous.

Given \(\epsilon\) we thus have \(|f_N(x) - f_N(y)| \leq \epsilon\), uniformly on \(N\), for any \(x\) such that \(|x - y| \leq \delta\), in addition to \(\sup_{[0, 1 - \delta]} |f_N| < \epsilon\) for any \(N \geq N_\epsilon\). In conclusion given \(x\) we take \(\xi \in [0, 1 - \delta]\)

with \(|x - \xi| < \delta\), and then get, for any \(N \geq N_\epsilon\)

\[|f_N(x)| \leq |f_N(x) - f_N(\xi)| + |f_N(\xi)| < \epsilon.\]

This concludes the proof of the Lemma. The proof of Theorem 2.3 is also complete. \(\square\)

**(d) Construction of a smooth disc transversal to \(M\) and of its infinitesimal deformation.**

We put \(w_N(\tau) = S^*_N(\tau) - S_N(1)\), let \(u_N\) be the solution in \(F^\gamma\) to Bishop’s equation \(u_N = -T_1 h(u_N, w_N)\), and let \(z_N = u_N + iv_N\) for \(v_N = T_1 u_N\). Let \(u\) be the solution to \(u = -T_1 h(u, (1 - \tau)^\alpha)\), and set \(z = u + iv\) for \(v = T_1 u\). Since

\[w_N(\tau) \to i(1 - \tau)^\alpha \text{ in } F^\gamma(\Delta),\]
and since \(w_N(1) \equiv 0 \forall N\), then \((z_{I_j}, \ldots, t_r) N(\tau) \rightarrow z_{I_j}, \ldots, t_r(\tau)\) in \(C^{1,\beta'}(\Delta)\) by Proposition 2.1 (Clearly we are supposing \(\gamma\) close enough to \(\alpha\) so that \(\beta' := m_j \gamma - 1 > 0\).) In particular for any \(\epsilon\) and for large \(N\) the discs \(A_N = (z_N, w_N)\) are in \(C^{1,\beta'}\) and satisfy

\[
(2.20) \quad \partial_r (v_{I_j, \ldots, t_r}) N(1) = v'_o \text{ for } |v'_o - v_o| < \epsilon,
\]

uniformly in \(N\). We call \(\hat{A} = (\tilde{z}, \tilde{w})\) one of these discs. We are ready to construct a half-space \(M^+_1\) in a manifold \(M_1\) which contains \(M\) and gains one more direction by a deformation of the disc \(\hat{A}\) to which CR functions extend. For this we consider the Bishop’s equation

\[
(2.21) \quad u = -T_1 h(u + x, w + \tilde{w}),
\]

for \(x \in \mathbb{R}^l, w \in \mathbb{C}^n\) with \(|x| < \delta, |w| < \delta\). According to Proposition 3.1, for any \(\epsilon\) and for suitable \(\delta = \delta_\epsilon\) there is an unique solution \(u\) which satisfies \(\|u - \tilde{u}\|_{C^{1,\beta'}} < \epsilon\) for \(\beta' < \beta := k\alpha - 1\). We write \(p = x + ih(x, w), w\) with \(v = T_1 u\), and define \(A_p(\tau) = p + (u(\tau) + i\gamma(\tau), \tilde{w}(\tau))\). We also write \(I_p(A) = A_p|_{[1-\epsilon, 1]}\) and define

\[
(2.22) \quad M^+_1 = \bigcup_p I_p([1-\epsilon, 1]).
\]

Proposition 2.5. \(M^+_1\) is a half space in a manifold \(M_1\) of codimension \(l - 1\) with boundary \(M\) and inward conormal \(v'_o\) for \(v'_o\) close to \(v_o\).

Proof. We consider the mapping

\[
(2.23) \quad \Phi: \mathbb{C}^n \times \mathbb{R}^l \times [1-\epsilon, 1] \rightarrow V', \quad (w, x, r) \rightarrow I_p(r) \text{ for } p = (x + ih(x, w), w).
\]

By Proposition 2.1 \(\Phi\) is \(C^{1,\beta'}\) in the complex of its arguments \((w, x)\) and \(r\) up to \(r = 1\), and we have

\[
\Phi(0, 0, 1)(\mathbb{C}^n \times \mathbb{R}^l \times [1-\epsilon, 1]) = T_p M + \mathbb{R}^+ v'_o.
\]

In particular \(\Phi\) extends as a \(C^{1,\beta'}\) mapping to \(\mathbb{C}^n \times \mathbb{R}^l \times [1-\epsilon, 1 + \epsilon]\) whose image defines a manifold \(M_1 = \Phi(\mathbb{C}^n \times \mathbb{R}^l \times [1-\epsilon, 1 + \epsilon])\) which contains \(M^+_1\) and satisfies \(T_p M^+_1 = T_p M + \mathbb{R}^+ v'_o\).

(e) End of proof of Theorems 1.1, 1.3. First, we recall again that it suffices to prove Theorem 1.3. In fact, for \(h_{I_j} = P_{I_j} + O^{m_j+1}\) we have that \(\langle \xi^o, P_{I_j} \rangle > 0\) for \(w_1 \in S\) implies \(\langle \xi^o, h_{I_j} \rangle > 0\) for \(w_1 \in S\) and \(|x_{I_j}| \leq c |w_1|^{m_j}\). Hence (1.3) is a consequence of (1.3). Thus, let \(f\) be a CR function on \(M\). By the celebrated Baouendi-Treves approximation theorem of [4], \(f\) is the uniform limit of polynomials on compact subsets of \(M\). By the maximum principle it will extend to all analytic discs whose boundary is contained in this compact set. In particular it extends to the half-space \(M^+_1\) of (b) for this is defined as the union of discs attached to \(M\). On the other end it extends to a wedge \(W\) with edge \(M\) and directional cone, say \(\Gamma\) by [12] since we are assuming that \(M\) is of finite type. Thus by [1] it will extend to a larger wedge \(\hat{W}\) whose directional cone \(\hat{\Gamma}\) is the convex hull of \(\Gamma\) and
with \( \langle \xi^o, v'_o \rangle > 0 \). In particular, for any \( F \) holomorphic in \( \tilde{W} \), we have \( \xi^o \notin WF(b(F)) \).

This completes the proof of Theorem 1.4 and hence also Theorem 1.1.

We discuss some complements of our Theorems 1.1, 1.4. We keep our choice of the \( w_1 \)-direction, select an index \( i \), suppose \( P_{t_i} = P_{t_i}(w_1) \) and (1.3), or suppose (1.5), and define \( \Gamma_{w_1,i} = \text{convex hull}\{v'_o\} \) where \( v'_o \) ranges through the family of directions produced by Th. 1.1 or Th. 1.4 for different directions \( \xi_o \) and sectors \( S \). We use now the Ajrapetyan-Henkin edge of the wedge theorem. In our setting it allows to state that all different directions of extension produced by Theorems 1.1 or Theorem 1.4 and even those obtained as their convex combinations, can be collected to generate the directional cone of a wedge of extension. Precisely, for any \( \epsilon \) there is a wedge \( W' \) with edge \( M \) and directional cone \( \Gamma_{w_1} \), satisfying \( \Gamma_{w_1} \subset (\Gamma_{w_1})_\epsilon \), and \( \Gamma_{w_1} \subset (\Gamma_{w_1})_\epsilon \) such that CR functions extend from \( M \) to \( W' \). (Here \( \cdot_\epsilon \) denotes the \( \epsilon \) conical neighborhood of \( \cdot \). Also, in the above situation we will say that the cones \( \Gamma_{w_1} \) and \( \Gamma_{w_1}' \) are \( \epsilon \)-close.) We can also play with different directions of the \( w \)-plane, say \( w_k \). Thus if we have equations of type \( y_{w_k,l_i} = h_{w_k,l_i} \) with \( h_{w_k,l_i} = \mathcal{O}^{|m_i|} \), then through Theorems 1.1, 1.4 we get directions \( v'_{w_k,i} \) that we collect in a cone

\[
\Gamma := \sum_{k,i} \Gamma_{w_k,i}.
\]

For this cone \( \Gamma \) we have

**Proposition 2.6.** For any \( \epsilon \) there is a wedge \( W' \) with edge \( M \) and directional cone \( \Gamma' \) which is \( \epsilon \)-close to \( \Gamma \), such that CR functions extend from \( M \) to \( W' \).

As already mentioned the proof is an immediate consequence of Theorem 1.1 and 1.4 by the aid of the Ajrapetian-Henkin edge of the wedge theorem. We want to discuss now about the dimension of \( \Gamma_{w_k} \) and \( \Gamma \). Since we are dealing with various directions \( w_k \)'s, we will write \( m_{w_k}, l_{w_k} \) from now on. We have

**Proposition 2.7.** Assume that the equations \( y_{w_k,l_i} = h_{w_k,l_i} \) of \( \tilde{M} = \mathbb{C}^1 \times \mathbb{C}_{w_k}^1 \) satisfy \( h_{w_k,l_i} = \mathcal{O}^{|m_i|} \) and that \( h_{w_k,l_i} \) is not \( \tilde{M} \)-harmonic. Then

\[
\dim(\Gamma_{w_k}) = \sum_i l_{w_k,i}.
\]

**Proof.** We first prove that \( \dim(\Gamma_{w_k,1}) = l_{w_k,1} \). We write \( h_{w_k,1} = P_{w_k,1}(x, w_k) + \mathcal{O}^{|m_{w_k,1}| + 1} \) and know from the hypotheses that for any \( \xi_o \in \mathbb{R}^{|w_k,1|} \), \( \langle \xi_o, h_{w_k,1}(\tau w_k, x) \rangle \) is non-harmonic. In particular it is divisible by \( |\tau|^2 \) and hence, being of degree \( m_{w_k,1} \) it has at most \( 2(m_{w_k,1} - 2) \) zeroes on the unit circle \( |\tau| = 1 \). In particular there is a sector of width \( \geq \frac{\pi}{m_{w_k,1}} \) where it keeps constant sign and thus gives rise to a direction \( v^o \), such that \( \langle \xi_o, v^o \rangle \neq 0 \). If we play with all \( \xi_o \) and all corresponding sectors, we conclude that these directions \( v^o \) cannot be contained in any proper plane of \( \mathbb{R}^{|w_k,1|} \).
We prove now the statement in full generality. For any \(i\) we take a system of \(l_{w_k,i}\) independent vectors \(\xi \in \mathbb{R}^{w_k,i}\) and of corresponding sectors

\[ S_\xi = \eta_i e^{i \theta_\xi (1 - \tau)m_1 w_k}, \ \forall \tau \in \Delta, \] with \(\alpha_1\) satisfying \(\frac{1}{m_{w_k,i} - 1} > \alpha_1 > \frac{1}{m_{w_k,i}}\).

We assume \(\eta_1 << \eta_2 \cdots << 1\). This gives rise to a set of extension directions \(v' = v'_{w_k,i, \xi_\xi}\) of the type

\[ v' = (\eta_i^{m_1} v'_1, \eta_i^{m_2} v'_2, \ldots, \eta_i^{m_1} v'_1, \eta) \eta << \eta_i \ \forall i, \]

with the property that for each fixed \(i\):

\[ \dim(\text{Span}_{i, \xi_\xi} \{v'_{w_k,i, \xi_\xi}\}) = l_{w_k,i}. \] (2.25)

It is also clear, taking all \(i\) and playing with different \(\eta_i\), that

\[ \dim(\text{Span}_{i, \xi_\xi} v'_{i, \xi_\xi}) = \sum_i l_{w_k,i}. \] (2.26)

\[ \square \]

Again, if we play with different directions \(w_k\) we have the similar result as Proposition 2.7 that is

\[ \dim(\sum_{k,i} \Gamma_{w_k,i}) = \sum_i (\text{rank}\{v'_{w_k,i}\}_k). \] (2.27)

(In this context the assumption that \(M\) is of finite type that is \(m_r < +\infty\) for a system of equations in Bloom-Graham normal form for the whole \(M\), and not just for its \((l + 1)\)-dimensional sections \(\tilde{M}\), precisely means on account of Proposition 2.7 and (2.27) that \(\dim \Gamma = l\).

3. Hörmander’s numbers of submanifolds of \(\mathbb{C}^N\)

Let \(T^{1,0}M\) and \(T^{0,1}M\) denote the bundles of vector fields tangent to \(M\) which are holomorphic and antiholomorphic respectively. Let \(T^\mathbb{C} M = TM \cap iTM\) be the complex tangent bundle to \(M\); note that its complexification verifies \(\mathbb{C} \otimes_{\mathbb{R}} T^\mathbb{C} M = T^{1,0}M \oplus T^{0,1}M\). Note that \(\mathbb{C} \otimes_{\mathbb{R}} TM\) is integrable, that is closed under Lie brackets, but \(\mathbb{C} \otimes_{\mathbb{R}} T^\mathbb{C} M\) is not, in general. We introduce a finite interpolation between \(\mathbb{C} \otimes_{\mathbb{R}} T^\mathbb{C} M\) and \(\mathbb{C} \otimes_{\mathbb{R}} TM\). We set \(L^j = \mathbb{C} \otimes_{\mathbb{R}} T^\mathbb{C} M\) and denote by \(L^j\) the distribution of vector spaces spanned by Lie brackets of holomorphic and antiholomorphic vector fields of length \(\leq j\). Suppose that for an integer \(m_1 \geq 2\) we have

\[ L^j_{p_0} = T^{1,0}_{p_0} M \oplus T^{0,1}_{p_0} M \forall j \leq m_1 - 1, \quad L^{m_1}_{p_0} \supset T^{0,1}_{p_0} M \oplus T^{1,0}_{p_0} M. \] (3.1)
Let \( \dim \frac{c_m}{L_{p_o}} = l_1 \); in this situation it is usual to refer to \( m_1 \) as the first Hörmander number of \( M \) at \( p_o \), and to \( l_1 \) as its multiplicity. In case \( L^j = L^j \) for any \( j \), we set \( m_1 = +\infty \) with multiplicity \( l_1 = l \). Next, we look for \( m_2 > m_1 \) such that

\[
(3.2) \quad L^j_{p_o} = L^{m_1}_{p_o}, \quad \forall j < m_2, \quad L^{m_2}_{p_o} \neq L^{m_1}_{p_o},
\]

and set \( l_2 = \dim \left( L^{m_2}_{p_o} \right) \); again \( m_2 \) is possibly \( +\infty \). We continue the above processus. We will call \( M \) of finite type when commutators span the full \( \mathbb{C} \otimes \mathbb{R} T^C_{p_o} M \). Thus the above chain will end with a number \( m_r < +\infty \) or \( m_r = +\infty \) according to the case the type is finite or not. We want to discuss now in greater detail about the first Hörmander number. By the properties of linearity of commutators, one obtains easily the equivalence of (3.1) to

\[
(3.3) \quad [X_1, [X_2, \ldots, [X_{j-1}, X_j] \ldots] \in T^{1,0}M \oplus T^{0,1}M \quad \forall X_i \in T^{1,0}M \oplus T^{0,1}M, \quad \forall j \leq m_1 - 1
\]

\[
(3.4) \quad [X^{m_1}_o, \ldots, [X_o, X_0, \bar{X}_o] \ldots] \notin T^{1,0}M \oplus T^{0,1}M
\]

for some \( X_o \) and some choice of \( X^{m_1}_o = X_o \) or \( \bar{X}_o \).

One proves that commutators \([X_1[X_2, \ldots, [X_{j-1}, X_j] \ldots]_{p_o}, \mathbb{C} \otimes \mathbb{R} T^C_{p_o} M \) only depend on the initial values \( X_1(p_o), X_2(p_o) \ldots \) and not on the choice of the extended sections. This property is referred to as tensoriality of the iterated brackets of vector fields. We take a basis of equations \( y_j = h_j, j = 1, \ldots, l \) for \( M \) at \( z_0 = 0 \) with \( h(0) = 0 \) and \( \partial h(0) = 0 \) and also set \( r_j = -y_j + h_j \) and \( r = (r_j) \). We identify \( T_M^M \cong T^C M \) by the complex structure \( J \) and \( T_M \mathbb{C}^N \cong \mathbb{R}^l \) by the dual basis to \( \partial r_j \). We look closely to \( X_o \) in \((3.4)\), assume, say, \( X_o(p_o) = w_o \partial_{w_o} \), and denote by \( p'_o \) the projection of \( p_o \) on the plane of \((x, w)\). We denote by \( n - 1 \), resp. \( m - 1 \) the occurrences of \( X^{m_j} = X_o \) (resp. \( X^{m_j} = \bar{X}_o \)) in (3.4). We can prove that

\[
(3.5) \quad \begin{cases}
\frac{1}{2i} [X^{m_i}_o, \ldots, X^{m_j}_o, [X_o, \bar{X}_o], \ldots] (h)(p'_o) = 0 & \forall j < m_1 - 2 \\
\frac{1}{2i} [X^{m_i}_o, \ldots, X^{m_1 - 2}, [X_o, \bar{X}_o], \ldots] (h)(p'_o) = \partial^n_{w_o} \bar{\partial}^m_{w_o} h(p'_o).
\end{cases}
\]

This is a special case of subsequent Proposition 3.3. The above relation, together with the fact that harmonic terms can be removed by change of coordinates, makes (3.5) equivalent, in suitable coordinates, to

\[
(3.6) \quad \begin{cases}
\partial^n_{w_o} \bar{\partial}^m_{w_o} h(p'_o) = 0 & \forall |\alpha| + |\beta| \leq m_1 - 1 \\
\partial^n_{w_o} h(p'_o) = 0, \quad \bar{\partial}^m_{w_o} h(p'_o) = 0, \quad \forall |\alpha| \leq m_1 \\
\partial^n_{w_o} \bar{\partial}^m_{w_o} h(p'_o) \neq 0 \quad \text{for } X_o(p_o) = w_o \partial_w \text{ and for suitable } n + m = m_1.
\end{cases}
\]
We write also \( \partial_{w_0} \) instead of \( w_0 \partial_w \) and consider the homogeneous term of lowest degree in the Taylor expansion of \( h \) in the \( w_0 \)-plane:

\[
g(\tau w_0) = \sum_{m+n=k}^{m+n \geq 1} \partial_{w_0}^m \partial_{\bar{w}_0}^n h(p_0') \tau^m \bar{\tau}^n.
\]

The above polynomial is real homogeneous and has some non-null coefficient on account of the third of (3.6). Hence it has only a discrete set of zeroes for \( |\tau| = 1 \) that is, for all \( \theta \in [0, 2\pi] \) but a discrete set, we have \( \sum \partial_{w_0}^m \partial_{\bar{w}_0}^n h(p_0') e^{i(m-n)\theta} \neq 0 \). Sometimes we prefer to use the notation \( \bar{\omega}_0 = e^{i\theta} w_0 \) and then write in this notation

\[
\sum_{m+n=m_1}^{m+n \geq 1} \partial_{w_0}^m \partial_{\bar{w}_0}^n h(p_0') \neq 0.
\]

We also denote by \( \nu^o \) the vector in \( C_{w_0} \). We remark that if \( \xi_0 \in \mathbb{R}^t \) verifies \( \langle \xi_0, \nu^o \rangle \neq 0 \), then

\[
\langle \xi_0, g(w_0) \rangle \geq 0 \text{ in a sector of the plane } \mathbb{C}_{w_0} \text{ of width } \geq \frac{\pi}{m_1 - 2}.
\]

In fact each \( g_i(\tau w_0) \) is divisible by \( |\tau|^2 \) and hence \( |\tau|^{-2} \langle \xi_0, g(\tau w_0) \rangle \) has at most \( m_1 - 2 \) zeroes for \( |\tau| = 1 \). Hence (3.5) or its equivalent version (3.6) imply our condition (1.2).

To go further with our discussion, we need to fix better our notations. We fix numbers \( m_1 < ... < m_r \) (perhaps \( m_r = +\infty \)) and multiplicities \( l_i \) with \( \sum_i l_i = l \). We take multi-indices \( I_1 = (1, \ldots, l_1), \ldots, I_r = (\sum_{i < r} l_i, \ldots, l) \), give weight \( m_i \) to the \( x_{I_i} \) variables, and define the weighted vanishing order for a function \( f = f(...x_{I_i},..., w) \) by putting \( f = O^{+\infty} \) when \( m_r = +\infty \) and \( f \) contains some monomial in the \( x_{I_i} \)'s, and, otherwise, putting \( f = O^m \) when \( f(...x_{I_i}..., tw) = O(t^m) \). We then suppose that the equations of \( M \) are presented according to increasing vanishing orders

\[
\begin{align*}
y_{I_1} &= h_{I_1} \\
... \\
y_{I_r} &= h_{I_r},
\end{align*}
\]

with \( h_{I_i} = O^{m_i} \) for any \( i \). We point out that this is not necessarily the normal form in the BLOOM GRAHAM sense. In fact we are not assuming that each \( h_{I_i} \) is in the form \( h_{I_i} = P_{I_i}(x_{I_1}, ..., x_{I_{i-1}}, w) \) with \( \langle \xi, P_{I_i} \rangle \) non-\( M \)-pluriharmonic for any \( i \) and any \( \xi \in \mathbb{R}^t \). (In this situation, weighted homogeneity does not serve any purpose.) To carry on our discussion, we need a description of a basis \( \{X_j\} \) of vector fields for \( T^{1,0}M \). We put \( r_{I_i} = -y_{I_i} + h_{I_i}, r = (r_1, \ldots, r_l) \), define an \((N-l) \times l \) matrix \( A = (a_{jh}) \) by

\[
A = -t(\partial_w r)^t(\partial_z r)^{-1},
\]

and set \( X_j = \sum_{h=1}^{l} a_{jh} \partial_{z_h} + \partial_{w_j} \). We have

\[
\sum_{h} a_{jh} \partial_{z_h} (r_{I_i}) + \partial_{w_j} (r_{I_i}) = 0 \quad \forall i = 1, \ldots, r.
\]
Derivation of (3.10) yields
\[
\begin{align*}
\partial^\beta \partial^\alpha_{x_1} \cdots \partial^\alpha_{x_{i-1}} (a_{j,i}) &= 0 \text{ for } |\beta| + \sum_{j<i-1} m_j |a_j| \leq m_i - 2 \\
\sum_h \partial^\beta \partial^\alpha_{x_1} \cdots \partial^\alpha_{x_{i-1}} (a_{j,h}) &= -2i \partial_w \partial^\alpha_{x_1} \cdots \partial^\alpha_{x_{i-1}} \partial_w (r_i) \text{ for } |\beta| + \sum_{j<i-1} m_j |a_j| \leq m_i - 1.
\end{align*}
\]

Once the equations are ordered as in (3.8), we can introduce for any \( i \leq r \) a diagram
\[
\begin{array}{c}
\sum I \downarrow \downarrow \\
\sum \downarrow \downarrow \\
\sum \varphi_1 \varphi_2 \otimes I + \ldots + I_r,
\end{array}
\]
where \( \varphi_1 \) is defined by \( [v] \mapsto J(v)(\partial r) \) and \( \varphi_2 \) by \( [v] \mapsto (Jv)^t(\partial r_l, \ldots, \partial r_r) \). We have to show that \( \varphi_2 \) is well defined (in which case the diagram (3.11) is commutative). To see this, we preliminarily remark that, just by the vanishing condition in (3.8), we have \( \{ \partial r_l, \ldots, \partial r_r \} \subset \text{Span} \{ \partial w, \bar{\partial} w, \partial_{x_1}, \ldots, \partial_{x_{i-1}} \} \) (normal form being unessential for this conclusion). Thus our claim is a consequence of the following

**Proposition 3.1.** We have \( \mathcal{L}^{m_i-1} \subset \text{Span} \{ \partial w, \bar{\partial} w, \partial_{x_1}, \ldots, \partial_{x_{i-1}} \} \).

**Proof.** We have to show that
\[
[X^j_o, \ldots, [X^{m_i-3}_o, [X_o, \bar{X}_o] \ldots ](r_j)(p_o) = 0 \forall j \geq i \text{ and for any } \epsilon.
\]

We recall (3.9) and (3.10) and fix \( j = i \). We use the notation \([\cdot, \cdot]^k\) to denote brackets of \( X_o \) or \( \bar{X}_o \) performed \( k - 1 \) times. We assume, for instance, \( X^j_o = \sum_h a_{1h} \partial z_h + \partial w_1 \), and begin by remarking that
\[
\begin{align*}
[\cdot, \cdot]^{m_i-1}(r_i) &= \sum_h a_{1h} \partial z_h + \partial w_1, \quad [\cdot, \cdot]^{m_i-2}(r_i) \\
&= [\partial w_1, [\cdot, \cdot]^{m_i-2}(r_i),
\end{align*}
\]
due to \( a_{1h}(p_o) = 0 \) and \( [\cdot, \cdot]^{m_i-2}(a_{1h}) = 0 \). Continuing in this way we end up with
\[
\begin{align*}
[\partial^\beta \partial_{w_1 w_1}, \sum_h a_{1h} \partial z_h + \partial w_1, \sum_h \bar{a}_{1h} \bar{\partial} z_h + \bar{\partial} w_1]&(r_i) \\
= \partial^\beta \partial_{w_1 w_1} \left( \sum_h a_{1h} \partial z_h (\bar{a} 1 I_i) - \frac{i}{2} \partial_{w_1} (\bar{a} 1 I_i) + \sum_h a_{1h} \partial z_h \bar{\partial} w_1 (r_i) + \partial_{w_1} \bar{\partial} w_1 (r_i) \right) + \ldots
\end{align*}
\]
where \( \beta \) is a biindex of length \( |\beta| = m_i - 3 \) and the dots denote similar terms as the four in the right hand side of (3.13). Now:
\[
\begin{align*}
\partial^\beta \partial_{w_1 w_1} (\bar{a} 1 I_i) &= 0 \text{ (by (3.12))}, \\
\partial^\beta \partial_{w_1 w_1} \partial_{w_1} (r_i) &= \sum_{\gamma+\delta=\beta} \sum_h \partial^\gamma \partial^\delta \partial_{w_1 w_1} (a_{1h}) \partial_{w_1 w_1} \partial z_h \bar{\partial} w_1 (r_i).
\end{align*}
\]
Thus, if $h \in I_j$ for $j \geq i$, the above term is clearly 0. Otherwise, either $|\gamma| \leq m_h - 2$ and hence $\partial_{w_i}^\gamma (a_{1h}) = 0$, or else $|\delta| \leq m_i - 2 - m_h$ and hence $\partial_{w_i}^\delta \partial_{z_h} \partial_{w_1} (r_{1_i}) = 0$. By the same reason, we have for the remaining term in (3.13): $\partial_{w_i}^\delta \partial_{z_h} \partial_{w_1} (\sum_h a_{1h} \partial_{zh_1}) (a_{1i}) = 0$. Finally, $\partial_{w_i}^\delta \partial_{z_h} \partial_{w_1} (r_{1_i})$ is also 0 again by (3.12). The proof is complete.

\textbf{Remark 3.2.} Note that $\varphi_2$ is an isomorphism precisely when we have in fact equality in Proposition 3.1. But this is equivalent as to asking that the equations (3.8) are in normal form.

Let us choose a vector field $X_o \in T^{1,0}M$ with $X_o (p_o) = w_o \partial_x$; we will also use the notation $\partial_{w_0}$ instead of $w_o \partial_x$. We have

\textbf{Proposition 3.3.}

(3.16) \[ [X_o^{\epsilon_1}, ..., [X_o^{\epsilon_{m-2}}, [X_o, \bar{X}_o]...](r_{1_i}) (p_o) = 0 \forall j > i \text{ and any } \epsilon. \]

If moreover $\langle \xi_o, h_{1_i} \rangle$, restricted to $C_{w_0} \times \mathbb{R}^I$, is in the form $P + \mathcal{O}^{m+1}$ for $P = P(w_o)$ homogeneous of degree $m_i$ with $m_i < +\infty$, then

(3.17) \[ J[X_o^{\epsilon_1}, ..., [X_o^{\epsilon_{m-2}}, [X_o, \bar{X}_o]...]\langle \xi_o, r_{1_i} \rangle (p_o) = -2 \partial_{w_0} \partial_{w_0} (\langle \xi_o, h_{1_i} \rangle)(p_o). \]

\textbf{Proof.} The first statement is a variant of Proposition 3.1. As for the second, in the same way as in the proof of Proposition 3.1, we get for a suitable $|\beta| = m_i - 2$

(3.18) \[ [X_o^{\epsilon_1}, ..., [X_o^{\epsilon_{m-2}}, [X_o, \bar{X}_o]...](r_{1_i}) = [\partial_{w_0}^\beta, [\sum_h a_{1h} \partial_{zh} + \partial_{w_1}, \sum_h \bar{a}_{1h} \partial_{zh} + \partial_{w_1}]](r_{1_i}) \]

\[ = \partial_{w_0}^\beta \left( \sum_h a_{1h} \partial_{zh} (\bar{a}_{1i}) - \frac{i}{2} \partial_{w_1} (\bar{a}_{1i}) + \sum_h a_{1h} \partial_{zh} (r_{1_i}) + \partial_{w_1} \partial_{w_1} (r_{1_i}) \right) + \ldots, \]

where the dots denote similar terms. Now the fourth term disappears by elimination with the terms in the dots (where it appears with opposite sign). The first and third term are not 0, in general. However, they vanish if we apply vector fields not to the whole $r_{1_i}$ but just to $\langle \xi_o, r_{1_i} \rangle$ on account of the hypothesis of \textit{semirigidity} contained in the second statement of the proposition. Thus, for the third term, we have

\[ \partial_{w_i}^\beta \left( \sum_h a_{1h} \partial_{zh} \partial_{w_1} (r_{1_i}) = \sum_{\gamma + \delta = \beta} \partial_{w_i}^\gamma \partial_{w_1}^\delta \partial_{zh} \partial_{w_1} \langle \xi_o, r_{1_i} \rangle. \]

Again, if $|\gamma| \leq m_h - 2$ then $\partial_{w_i}^\gamma a_{1h} = 0$. If, instead, $|\delta| \leq m_i - 1 - m_h$ then $\partial_{w_i}^\delta \partial_{z_h} \partial_{w_1} (\xi, r_{1_i}) = \partial_{w_i}^\delta \partial_{z_h} \partial_{w_1} (P + \mathcal{O}^{m+1}) = 0$. In the same way one proves that the first term in the second line of (3.18) is 0. The only term which survives is the secon
(which also appears, with the same sign in the dots terms). We have thus got

\[ J[X_o^\epsilon_1, \ldots, X_o^{\epsilon_{m_1}-2}, [X_o, \bar{X}_o] \ldots] \langle \xi_o, r_{I_1} \rangle = -\frac{i}{2} (\bar{\partial}_{w_1\bar{w}_1} \partial_{w_1} \langle \xi_o, a_{I_1} \rangle + \ldots) \]

(3.19)

\[ = -\frac{i}{2} (\bar{\partial}_{w_1\bar{w}_1} \partial_{w_1} \bar{\partial}_{w_1} \langle \xi_o, r_{I_1} \rangle \frac{2}{i} + \ldots) \]

\[ = -2 \bar{\partial}_{w_1\bar{w}_1} \partial_{w_1} \bar{\partial}_{w_1} \langle \xi_o, r_{I_1} \rangle. \]

This completes the proof of the proposition.

We assume now that for some vector field \( X_o \) with \( X_o(p_o) = \partial_{w_o} \), for some \( \epsilon = (\epsilon_1, \ldots, \epsilon_{m_1}-2) \), and for some \( \xi_o \in \mathbb{R}^{l_i} \), we have

(3.20) \[ [X_o^{\epsilon_1}, \ldots, [X_o^{\epsilon_{m_1}-2}, [X_o, X_o] \ldots] \notin \text{Span}\{\partial_w, \partial_\bar{w}, \partial_{x_{I_1}}, \ldots, \partial_{x_{I_{l_i}-1}}\} \]

and

(3.21) \[ \langle \xi_o, h_{I_1} \rangle |_{\mathbb{C}_{w_o} \times \mathbb{R}^{l_i}} = P(w_o) + \mathcal{O}^{m_i+1}. \]

It follows that \( P(w_o) = |w_o|^2 Q(w_o) \) with \( Q \) real homogeneous of degree \( m_i - 2 \). Since \( Q \) has at most \( m_i - 2 \) zeroes on the circle \( |w_o| = 1 \), then

(3.22) \[ P \geq 0 \text{ for } w_o \text{ in a sector of width } > \frac{\pi}{m_i - 2}. \]

Hence we enter in the hypotheses of Theorem 1.3 and conclude that CR functions on \( M \) extend to a new direction \( v^o \) satisfying \( \langle \xi_o, v^o \rangle \geq 0 \). Note that in that Theorem normal equations as in (1.3) are not needed. What is really needed is, for equations as \( 3.20 \), to assume \( \langle \xi_o, h_{I_1} \rangle = P(w_o) + \mathcal{O}^{m_i+1} \) and \( P \geq 0 \) (or \( P \leq 0 \)) in a sector \( > \frac{\pi}{m_i} \).

Naturally, if the equations are normal, we have the significant simplification that \( \mathcal{L}^{m_i} = \text{Span}\{\partial_w, \partial_{\bar{w}}, \partial_{x_{I_1}}, \ldots, \partial_{x_{m_i-1}}\} \). Thus vector fields \( X_o \) which satisfy (3.20) do exist. If for one of them, with, say, \( X_o(p_o) = \partial_{w_o} \), and for some \( \xi_o \in \mathbb{R}^{l_i} \), (3.21) is also satisfied, then Proposition 2.3 yields CR extension to some \( v^o \) with \( \langle \xi_o, v^o \rangle \geq 0 \).

4. COMPARISON WITH BOGGESS-PITTS

Let \( M \) be a manifold of class \( C^{k+2} \) which satisfies (1.1) with \( g \) homogeneous of degree \( k \) and non-\( M \)-harmonic (in particular whose first Hörmander number is \( m_1 = k \)). Remember that in this situation (1.2) is also satisfied. Let \( v \) be the direction normal to \( M \) given by the formula

(4.1) \[ v = \sum_{\epsilon} C_\epsilon [X_o^{\epsilon_1}, X_o^{\epsilon_2}, \ldots, [X_o, \bar{X}_o] \ldots](r)(p_o) \text{ where } C_\epsilon := \frac{1}{\epsilon^{+!} \epsilon^{-!}} \]

with \( \epsilon^+ \) and \( \epsilon^- \) denoting the occurrences \( X_o^{\epsilon_1} = X_o \) and \( X_o^{\epsilon_1} = \bar{X}_o \) respectively. Note that the last two occurrences are fixed as \( X_o^{k-1} = X_o \) and \( X_o^{k} = \bar{X}_o \). Let \( X_o(p_o) = \partial_{w_o} \). By
tensoriality of brackets and by the combinatorial remark that the number of choices of \( \epsilon \)'s which give rise to the same pair of occurrences \( m, n \) is \( \binom{k-2}{m-1} \) one gets

\[
(4.2) \quad v = \sum_{m+n=k \atop m \geq 1, n \geq 1} \left( \frac{k-2}{m-1} \right) \frac{1}{m!n!} \partial_{w_0}^m \partial_{w_0}^n h(p'_o).
\]

Again, once the complex plane of \( X_o(p_o) \) is fixed, in our case the \( w_o \)-plane, there might be many vectors \( v = v_\varphi \) produced through (4.2) just by replacing \( w_o \) by \( e^{i\varphi}w_o \). The result by Boggess-Pitts [8] is that for each of these vectors \( v \), one obtains CR extension from \( M \) to \( M' \) where \( M' \) points to a direction \( v' \) close to \( v \). We first discuss this extension in case \( M \) is a hypersurface of \( \mathbb{C}^N \) defined, in coordinates \( (z, w) \in \mathbb{C}^1 \times \mathbb{C}^n \), \( w = (w_1, w') \), for a pair of even integers \( k \) and \( p \) with \( p \leq k - 2 \), for a choice of a coefficient \( a \geq 0 \), and with the notation \( w_o = (1, 0, \ldots) \), by an equation

\[
(4.3) \quad y_1 = |w_1|^k + a|w|^{k-p} \Re w_1^p + (O(|x_1|^2 + |w_1|^{k+1} + |x_1||w_1| + |w||w'|)).
\]

We denote by \( g = g(w_1) \) the homogeneous polynomial in the right side of (4.3). With \( p_o = 0 \) and \( X_o = \partial_{w_1} \) and with the notation \( k - 2 = p + 2q \), we have extension in directions \( v_\varphi = (ic_\varphi, 0, \ldots) \) for

\[
c_\varphi = \left( \frac{k-2}{p+q} \right) + a \cos(p\varphi) \left( \frac{k-2}{p+q} \right).
\]

In particular if we look for extension down, that is for \( v_\varphi \) with negative first component, we have to require \( k \geq 4 \), \( p \geq 2 \). Then \( v_\varphi < 0 \) will occur exactly for \( \varphi = \frac{\pi}{p} \) (which yields \( \cos(p\varphi) = -1 \)) and

\[
a \geq \frac{(p+q)!q!}{(\frac{p}{2} - 1)!(k-1-\frac{p}{2})!}.
\]

We compare the above condition with that which is given by sector property. We consider the restriction of \( g \) on the unit circle \( w_1 = e^{i\theta} \) given by \( g(e^{i\theta}) = 1 + a \cos(p\theta) \). It is clear that for any choice of \( a \) we have \( g \geq 0 \) in a sector of width bigger than \( \frac{\pi}{k} \) which is in turn bigger than \( \frac{\pi}{p} \). Hence by Theorem 1.1 we get holomorphic extension up.

If we search, instead, for extension down, we can use the following result which generalizes similar conclusions by Baouendi-Treves [17] concerning the case \( k = 4 \).

**Proposition 4.1.** We have

\[
(4.4) \quad g < 0 \text{ in a sector of width } > \frac{\pi}{k},
\]

if and only if

\[
(4.5) \quad a > \frac{1}{\cos \left( \frac{pa}{2\pi} \right)}.
\]
Proof. Let \( a > 0 \); it is clear that \( 1 + a \cos(p\theta) \) attains its minimum at \( \theta = \frac{\pi}{p} \). It is also clear that in order that the sector where \( g < 0 \) has angle bigger than \( \frac{\pi}{p} \) it is necessary and sufficient that
\[
a \cos(\pi + \frac{p\pi}{2k}) < -1,
\]
which is equivalent to the condition in the statement of the proposition.
\[ \square \]

We also have the following statement which shows necessity of sector property for holomorphic extendibility.

**Proposition 4.2.** Let \( p \) divide \( k \) and \( a \leq \cos(\frac{\pi}{p}) \). Then for \( b = \frac{k}{k} \tan(\frac{p\pi}{2k}) \), we have that the trigonometric polynomial \( g_1 = 1 - a \cos(p\theta) + b \cos(k\theta) \) verifies \( g_1 \geq 0 \ \forall \theta \). In particular, if in addition the plane of the \( w \) variables has dimension 1, by adding another harmonic term \( c \cos(k\theta) \) we can achieve \( g_1(w_1) \geq c_1 |w_1|^m \) for \( c_1 > 0 \).

**Proof.** Since \( a \leq \cos(\frac{\pi}{p}) \), then for \( g_1 \geq 0 \) it will suffice:
\[
(4.6) \quad \frac{b \cos(k\theta)}{\cos(\frac{p\pi}{2k})} \geq \frac{1}{\cos(\frac{p\pi}{2k})} \cos(p\theta) - 1.
\]

It is clear that it will suffice to take \( b \) such that
\[
(4.7) \quad (b \cos(k\theta))'|_{\frac{-\pi}{2k}} = \frac{1}{\cos(\frac{p\pi}{2k})} \cos(p\theta)'|_{\frac{-\pi}{2k}}.
\]

In fact this choice of \( b \) will imply that the derivative on the left of (4.7) dominates (respectively is dominated by) the one on the right in the interval \([-\frac{\pi}{2k}, 0]\) (respectively in \([-\frac{\pi}{k}, -\frac{\pi}{2k}]\)). Hence (4.6) holds in the interval \([-\frac{\pi}{2k}, 0]\) and also, by symmetry, in the whole \([-\frac{\pi}{k}, +\frac{\pi}{2k}]\). It also holds trivially in the remaining part of \([-\frac{\pi}{p}, +\frac{\pi}{p}]\). On the other hand this is a complete cycle of the trigonometric function \( 1 - a \cos(p\theta) + b \cos(k\theta) \) due to the assumption that \( p \) divides \( k \).
\[ \square \]

**Corollary 4.3.** Let \( M \) be a hypersurface in \( \mathbb{C}^N \) defined by (4.3) and assume that \( p \) divides \( k \). If \( a \leq \frac{1}{\cos(\frac{\pi}{p})} \), then there are CR functions \( f \in CR_M \) which do not extend down.

**Proof.** In new complex coordinates we can arrange that \( M \subset \{ y_1 > 0 \} \). Since \( \{ y_1 > 0 \} \) is pseudoconvex, the conclusion follows.
\[ \square \]

The comparison between the conditions related to \[\text{[8]}\] and to sector property is expressed by

**Lemma 4.4.** Let \( k - 2 = p + 2q \). Then
\[
(4.8) \quad \frac{(p+q)!q!}{(\frac{k}{2} - 1)!(\frac{k}{2} - 1)!} > \frac{1}{\cos(\frac{p\pi}{2k})}.
\]
Proof. The most delicate case is when $p = 2$. In this case (4.8) becomes
\[
\frac{k^2!(\frac{k}{2} - 2)!}{(k - 1)!(\frac{k}{2} - 1)!} > \frac{1}{\cos\left(\frac{\pi}{k}\right)},
\]
or else
\[
\frac{k}{k^2 - 1} > \frac{1}{\cos\left(\frac{\pi}{k}\right)}.
\]
\[\square\]

Hence the method of sectors is sharper than that of [5]. In particular it yields extension down for an extra range of values of $a$ that is for $\frac{1}{\cos\left(\frac{\pi}{k}\right)} \leq a < \frac{(p+q)!q!}{(\frac{k}{2}-1)!(\frac{k}{2}-1)!}$. The above conclusions are generalizations of former results by Baouendi-Treves [17].

We pass now to the higher codimensional case. We discuss CR-extension for $M \subset \mathbb{C}^3$ defined in coordinates $(z_1, z_2, w)$ by the system
\[
\begin{align*}
    y_1 &= |w|^k + a|w|^2\text{Re}w^p + O(|x|^2 + |w|^{k+1} + |x||w|) \\
    y_2 &= |w|^k + O(|x|^2 + |w|^{k+1} + |x||w|)
\end{align*}
\]
We also denote by $g = (g_j)_j$, $j = 1, 2$ the vector with polynomial entries on the right of (4.9) and, for $\xi \in \mathbb{R}^2$, we use the notation $g_\xi = \langle \xi, g \rangle$. We can express the extension directions $v_\varphi$ by [8] as
\[
(4.10) \quad v_\varphi = \left(\left(\frac{k-2}{k} - 1\right) + a \cos(\varphi) \left(\frac{k-2}{p+q}\right), \left(\frac{k-2}{k} - 1\right)^t\right),
\]
where $^t$ denotes transposition. Let us search for $v_\varphi$ whose first component is $< 0$. The first occurrence, which takes place for $\varphi = \pi$ is when $a > \frac{(p+q)!q!}{(\frac{k}{2}-1)!(\frac{k}{2}-1)!}$. In this case extension to directions arbitrarily close to $v = \left(\left(\frac{k-2}{k} - 1\right) - a \left(\frac{k-2}{p+q}\right), \left(\frac{k-2}{k} - 1\right)^t\right)$ holds according to [8]. If we look, instead, to our sector property and search for $v$ whose first component is $< 0$ and the second is, say, $> 0$, we are lead to the sector property of $g_\xi$ for suitable $\xi$ with $\xi_1 < 0$ and $\xi_2 < 0$. The condition reads in this case
\[
(4.11) \quad g_\xi(\theta) = \xi_1(1 + a \cos(p\theta)) + \xi_2 > 0 \text{ in a sector of width } > \frac{\pi}{k},
\]
that is
\[
1 + a \frac{\xi_1}{\xi_1 + \xi_2} \cos(p\theta) < 0 \text{ in a sector with angle } > \frac{\pi}{k}.
\]
We write $a_\xi = a \frac{\xi_1}{\xi_1 + \xi_2}$. Then the sector where $g_\xi > 0$ is centered at $\theta = \frac{\pi}{p}$ and its width is $> \frac{\pi}{k}$ if and only if
\[
a_\xi > \frac{1}{\cos\left(\frac{\pi}{2p}\right)}.
\]
Now the first such an occurrence is for $a > \frac{1}{\cos(\frac{\pi}{2k})}$ and for $\xi$ close to $(-1, 1 - \cos(\frac{\pi}{2k}))^i$. Hence we get extension to vectors $v$ with $\langle v, \xi \rangle > 0$ according to Theorem 1.1. Direct inspection of the second equation of $M$ shows that $v_2 > 0$. Also, extension to directions of a conic neighborhood of the diagonal is evident. In conclusion, using also the edge of the wedge theorem, we get extension to all intermediate directions, among whose some which is close to $v = (-\cos(\frac{\pi}{2k}) + 1, 1)^i$. We write now $a_1 = a \frac{(\frac{k}{2} - 1)!(k - 1 - \frac{k}{2})!}{(p + q)!q!}$ and $a_2 = \cos(\frac{\pi}{2k})$.

According to Lemma 4.4 we always have $a_1 < a_2$. (In the simplest cases we have $\frac{a_2}{a_1} = \frac{2}{\sqrt{2}}$ when $k = 4$, $p = 2$ and $\frac{a_2}{a_1} = \frac{3}{\sqrt{3}}$ for $k = 6$, $p = 4$.) Summarizing up we get (1) Extension for an extra range of values of $a$ that is $a_1 < a \leq a_2$ which were not taken care of by [8].

For this purpose the higher codimension is not really needed; the examples by Rea and Baouendi-Treves would suffice as well. (2) Extension to a wedge $V'$ with bigger directional cone $\Gamma'$ even for common values of $a > a_1$. (Here codimension $> 1$ is really essential.) In fact in [8] the cone is

$$\Gamma = \{ (y_1, y_2) : y_1 > -|y_2|/(a_1 - 1) \},$$

whereas in our case it is

$$\Gamma = \{ (y_1, y_2) : y_1 > -|y_2|/(a_2 - 1) \}.$$

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