Potential Model for $\Sigma_u^-$ Hybrid Meson State

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Abstract

In this paper, lattice simulations are used to propose a potential model for gluonic excited $\Sigma_u^-$ states of bottomonium meson. This proposed model is used to calculate radial wave functions, masses and radii of $\Sigma_u^-$ bottomonium hybrid mesons. Here, gluonic field between a quark and an antiquark is treated as in the Born-Oppenheimer expansion, and Schrödinger equation is numerically solved employing shooting method. Results of calculated masses for $\Sigma_u^-$ state are in quite good agreement with the lattice simulations.

Keywords: Meson, gluonic excitations, Potential model, QCD

I. INTRODUCTION

Static quark potential models play important role in the understanding of Quantum chromodynamics. A hybrid static potential is defined as a potential of a static quark-antiquark pair with the gluonic field in the excited states. These hybrid static potentials for different states of mesons are computed in refs. [1][2][3][4][5]. Hybrid static potentials are characterized by quantum numbers, $\Lambda$, $\eta$, and $\epsilon$, where $\Lambda$ is the projection of the total angular momentum of gluons and for $\Lambda = 0, \pm 1, \pm 2, \pm 3, ..., $, meson states are represented as $\Sigma, \Pi, \Delta$ and so on [1]. $\eta$ is the combination of parity and charge and for $\eta = P\circ C = +, -$, states are labelled by sub-script $g, u$ [1]. $\epsilon$ is the eigen value corresponding to the operator $P$ and is equal to $+, -$. Parity and charge for hybrid static potentials are defined as [1]

$$P = \epsilon(-1)^{L+\Lambda+1}, C = \epsilon\eta(-1)^{L+\Lambda+S},$$

The low-lying static potential states are labelled as $\Sigma_g^+, \Sigma_g^-, \Sigma_u^+, \Sigma_u^-, \Pi_g, \Pi_u, \Delta_g, \Delta_u$ and so on [1]. $\Sigma_g^+$ is the low-lying potential state with ground state gluonic field and is approximated by a coulomb plus linear potential. The $\Pi_u$ and $\Sigma_u^-$ are the $Q\bar{Q}$ potential states with low lying gluonic excitations. Linear plus coulombic potential model is extended in [6] for $\Pi_u$ states by fitting the suggested ansatz with lattice data [5] and the extended model is tested by finding properties of mesons for a variety of $J^{PC}$ states in refs. [6][7][8][9]. In this Paper, linear plus coulombic potential model is extended for lowest excited hybrid state, $\Sigma_u^-$ by fitting the lattice data [1] with the suggested analytical expression (ansatz). The validity of suggested ansatz is tested by calculating the spectrum of $\Sigma_u^-$ states and comparing it with lattice results. For this purpose, Born-Oppenheimer formalism and adiabatic approximation is used. Relativistic corrections in the masses are incorporated through perturbation theory.

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Heavy hybrid mesons have been studied using theoretical approaches like lattice QCD \cite{1,2,3,5,10}, constituent gluon model \cite{11,12,13}, QCD sum rule \cite{14,15,16,17,18,19,20,21} and Bethe-Salpeter equation \cite{22}.

The paper is organised as: In the section II of this paper, Potential model for $\Sigma_+^g$ state is discussed while the proposed potential model for $\Sigma_u^-$ is defined in section III. The methodology to find radial wave functions, spectrum and radii is explained in section IV, while the discussion on results and concluding remarks are written in section V.

II. POTENTIAL MODEL FOR $\Sigma_+^g$ STATES

$\Sigma_+^g$ is the quarkonium state with ground state gluonic field and potential model for this state is defined as \cite{23}:

$$V(r) = -\frac{4\alpha_s}{3r} + br + \frac{32\pi\alpha_s}{9m_b m_g} \left(\frac{\sigma}{\sqrt{\pi}}\right)^3 e^{-\sigma^2 r^2} S_b S_g + \frac{4\alpha_s}{m_b^3} S_T + \frac{1}{m_b^2} \left(\frac{2\alpha_s}{r} - \frac{b}{2r}\right) L.S. \quad (2)$$

Here $\frac{-4\alpha_s}{3r}$ describes Coulomb-like interaction while linear term $br$ is due to linear confinement. The term with $S_b S_g$ is equal to

$$S_b S_g = \frac{S(S+1)}{2} - \frac{3}{4} \quad (3)$$

$L.S$ describes the spin orbit interactions defined as:

$$L.S = [J(J+1) - L(L+1) - S(S+1)]/2, \quad (4)$$

$S_T$ is the tensor operator defined in \cite{23} as:

$$<^3 L_J | S_T | ^3 L_J> = \begin{cases} -\frac{1}{6(2L+1)}, & J = L+1 \\ +\frac{1}{6}, & J = L \\ -\frac{L+1}{6(2L-1)}, & J = L-1 \end{cases} \quad (5)$$

Here, $L$ is the relative orbital angular momentum of the quark-antiquark and $S$ is the total spin angular momentum. Spin-orbit and colour tensor terms are equal to zero \cite{23} for $L = 0$. $m_b$ is the constituent mass of bottom quark.

III. POTENTIAL MODEL FOR $\Sigma_u^-$ STATES

Static potentials for different states of mesons (conventional and hybrid) are computed by lattice simulations. In Figure 3 of ref.\cite{5}, static potentials of various states are plotted with respect to quark-antiquark separation. In this paper, potential model (defined in eq. 2) is extended for $\Sigma_u^-$ state by adding the following ansatz:

$$V_\Sigma(r) = A' exp(-B'r^{P'}) + C', \quad (6)$$

whose parameters are found by fitting it with the lattice data \cite{5} obtained by taking difference between $\Sigma_+^g$ and $\Sigma_u^-$ states. The best fitted values of parameters are:

$$A' = 11.5917 \text{GeV}, \quad B' = 4.6119, \quad P' = 0.2810, \quad C' = 0.9589. \quad (7)$$

With these parameters, dimensionless $\chi^2$, defined as:

$$\chi^2 = \frac{\sum_{i=1}^{n} (\varepsilon_i - A exp[-B'r_i^{P'}])^2}{\sum_{i=1}^{n} \varepsilon_i^2}, \quad (8)$$

is found to be 0.0000296 for proposed model and lattice data\cite{5}. Here, $i = 1, 2, 3, ..., n$ is the number of data points. Proposed model ($V_\Sigma(r)$) and lattice data for difference between ($\Sigma_+^g$) and $\Sigma_u^-$ potential states are shown in Figure 1.
IV. CHARACTERISTICS OF $\Sigma_u^-$ HYBRID BOTTOMONIUM STATES

1. Radial wave function of $\Sigma_g^+$ and $\Sigma_u^-$ states

For $\Sigma_g^+$ state, radial Schrödinger equation is written as:

$$U''(r) + 2\mu \left( E - V(r) - \frac{L(L+1)}{2\mu r^2} \right) U(r) = 0,$$

(9)

where $V(r)$ is defined above in eq. (2). Here $U(r) = r R(r)$, where $R(r)$ is the radial wave function. To find numerical solutions of the Schrödinger equation for $\Sigma_g^+$ states, shooting method is used. At small distance ($r \to 0$), wave function becomes unstable due to very strong attractive potential. This problem is solved by applying smearing of position co-ordinates by using the method discussed in ref. [24]. To calculate the radial wave functions, parameters $\alpha_s = 0.36$, $b = 0.1340$ GeV$^2$, $\sigma = 1.34$ GeV, $m_b = 4.825$ GeV are taken from ref. [8].

For $\Sigma_u^-$ bottomonium hybrid states, radial Schrodinger equation can be modified as:

$$U''(r) + 2\mu \left( E - V(r) - A' \exp(-B' r P') - C' - \frac{L(L+1) - 2\Lambda^2 + \langle J^2_g \rangle}{2\mu r^2} \right) U(r) = 0,$$

(10)

Here, $\langle J^2_g \rangle$ is the square of gluon angular momentum and $\langle J^2_g \rangle = 2$ [1] for $\Sigma_u^-$ state. $\Lambda$ is the projection of gluon angular momentum and $\Lambda = 0$ [1] for $\Sigma_u^-$ state. Numerical solutions of the Schrödinger equation for $\Sigma_u^-$ states are found by the same method as discussed above and resultant radial wave functions with different $j^{PC}$ are shown in Figure. 2 and Figure 3. The quantum numbers ($L$ and $S$) for these states are given below in Table 1.

2. Spectrum of $\Sigma_u^-$ state

To check the validity of our model, masses of bottomonium mesons are calculated for $\Sigma_u^-$ states. To calculate the mass of a $b\bar{b}$ state, the constituent quark masses are added to the energy $E$, i.e;

$$m_{b\bar{b}} = 2m_b + E,$$

(11)
Figure 2: The radial wave functions for $\Sigma^+_g$ and $\Sigma^-_u$ states for $L=0$. Solid line curves indicate $\Sigma^-_u$ states and dashed curves are for $\Sigma^+_g$ states. Radial wave functions for $S=0$ and $S=1$ with $L=0$ are almost same in our numerical limits.

Figure 3: The radial wave functions for $\Sigma^+_g$ and $\Sigma^-_u$ states for $L=1$. Solid line curves indicate $\Sigma^-_u$ states and dashed curves are for $\Sigma^+_g$ states. Radial wave functions for $S=0$ and $S=1$ with $L=1$ are almost same in our numerical limits.
The lowest order relativistic correction in mass is incorporated by perturbation theory as adopted in refs. [7, 8] to calculate the spectrum of $\Sigma_+^+$ and $\Pi_u$ state. With relativistic correction, the expression to calculate mass becomes as:

$$m_{b\bar{b}} = 2m_b + E + \langle \Psi \mid \left( -\frac{1}{4m_b^2} \right) p^4 \mid \Psi \rangle,$$

(12)

The best fit values of parameters ($\alpha_s = 0.4, b = 0.11$ GeV$^2$, $\sigma = 1$ GeV, $m_b = 4.89$ GeV) with relativistic correction are taken from ref. [8]. Calculated masses for $\Sigma_u^-$ states with and without relativistic corrections are reported in Table 1.

3. Radii

The numerically calculated normalized wave functions are used to calculate the root mean square radii. To find the root mean square radii of the gluonic excited $\Sigma_u^-$ bottomonium states, following relation is used:

$$\sqrt{\langle r^2 \rangle} = \sqrt{\int U^* r^2 U dr}.$$  

(13)

Table 1

V. Discussion and conclusion

In this paper, potential model for lowest lying $\Sigma_u^-$ hybrid states is proposed whose parameters are found by fitting the model with lattice data [1, 5]. This model is used to calculate the numerical solutions of Schrodinger equation for $\Sigma_u^-$ states with different $J^{PC}$. In Figures (2-3), normalized radial wave functions of $\Sigma_g^+$ and $\Sigma_u^-$ states are plotted with respect to quark-antiquark separation $r$. Figures (2-3) show that peaks of radial wave functions are shifted away from origin for gluonic excited states ($\Sigma_u^-$) as compare to gluonic ground states. Figure 2 shows that shape of wave functions is different for $\Sigma_g^+$ and $\Sigma_u^-$ states.

The newly suggested model is used to calculate the masses and radii of the $\Sigma_u^-$ states and results are written in Table 1. Our calculated masses without relativistic corrections are close to the the results given in ref. [3] as shown in Table 1. In ref. [3], spectrum is calculated without including the spin, so the same mass is given for $\eta_b$ and $\Upsilon^0$. However, our proposed potential model gives distinguished results for $S = 0$ and $S = 1$. As observed from Table 1, the lowest calculated mass of the $\Sigma_u^-$ state is calculated to be 10.978 GeV with the incorporation of relativistic corrections in masses. In ref. [1, 5], the lowest mass of $\Sigma_u^-$ state is 11.1 GeV. This shows that our calculated masses with relativistic corrections are more closer to the masses calculated by lattice simulations [1, 5] than the nonrelativistic masses.

From Table 1, it is observed that masses and radii are increased by increasing the orbital quantum number (L). The similar behaviour is observed in ref. [8] while working on $\Sigma_g^+$ and $\Pi_u$ states of bottomonium meson. Spectrum of $\Sigma_g^+$ state bottomonium mesons is calculated in ref. [8] by shooting method and few of the results of ref. [8] are shown below in Table 2. The comparison of masses of $\Sigma_g^+$ and $\Sigma_u^-$ states shows that masses and radii of $\Sigma_u^-$ states are greater than $\Sigma_g^+$ states. Overall, we conclude that masses and radii increase towards higher gluonic excitations.

Table 2

Results of calculated radial wave functions, masses and radii can be used to find more properties like decay constant, decay widths and transition rates of gluonic excited $\Sigma_u^-$ states. Overall, we
conclude that our extended potential model can be used to study the gluonic excitations in a variety of meson sectors.

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Table 1: Our calculated masses of $b\bar{c}$ hybrid $\Sigma_u^-$ bottomonium mesons.

| Meson                       | $J^{PC}$ | calculated mass | mass | calculated $\sqrt{(r^2)}$ |
|-----------------------------|----------|-----------------|------|---------------------------|
|                             |          | Relativistic    | NR   | [3]                       |
|                             |          | GeV             | GeV  | fm                        |
| $\eta_b^q (1^1 S_0)$        | 0^{++}   | 10.9785         | 10.938 | 10.912(3) | 0.4634 |
| $\Upsilon_b^q (1^3 S_1)$    | 1^{++}   | 10.9809         | 10.94  | 0.4658                    |
| $\eta_b^q (2^1 S_0)$        | 0^{++}   | 11.2273         | 11.214 | 11.192(5) | 0.7358 |
| $\Upsilon_b^q (2^3 S_1)$    | 1^{++}   | 11.2292         | 11.2159 | 0.7368                    |
| $\eta_b^q (3^1 S_0)$        | 0^{++}   | 11.4278         | 11.4411 | 0.7369                    |
| $\Upsilon_b^q (3^3 S_1)$    | 1^{++}   | 11.4294         | 11.4429 | 0.9694                    |
| $\eta_b^q (4^1 S_0)$        | 0^{++}   | 11.6025         | 11.6413 | 1.1711                    |
| $\Upsilon_b^q (4^3 S_1)$    | 1^{++}   | 11.6039         | 11.643  | 1.1729                    |
| $\eta_b^q (5^1 S_0)$        | 0^{++}   | 11.7606         | 11.8238 | 1.3609                    |
| $\Upsilon_b^q (5^3 S_1)$    | 1^{++}   | 11.7618         | 11.8253 | 1.3625                    |
| $\eta_b^q (6^1 S_0)$        | 0^{++}   | 11.9066         | 11.9934 | 1.5384                    |
| $\Upsilon_b^q (6^3 S_1)$    | 1^{++}   | 11.9078         | 11.9948 | 1.5399                    |
| $h_b^q (1^1 P_1)$           | 1^{--}   | 11.0833         | 11.048  | 0.5507                    |
| $\chi_b^q (1^3 P_0)$        | 0^{++}   | 11.0684         | 11.0424 | 0.5481                    |
| $\chi_b^q (1^3 P_1)$        | 1^{++}   | 11.0763         | 11.0477 | 0.5511                    |
| $\chi_b^q (1^3 P_2)$        | 2^{++}   | 11.0872         | 11.0504 | 0.5529                    |
| $h_b^q (2^1 P_1)$           | 1^{--}   | 11.3052         | 11.2982 | 0.8078                    |
| $\chi_b^q (2^3 P_0)$        | 0^{++}   | 11.2926         | 11.2956 | 0.806                     |
| $\chi_b^q (2^3 P_1)$        | 1^{++}   | 11.3034         | 11.2986 | 0.8085                    |
| $\chi_b^q (2^3 P_2)$        | 2^{++}   | 11.3102         | 11.2998 | 0.8099                    |
| $h_b^q (3^1 P_1)$           | 1^{--}   | 11.4927         | 11.5123 | 1.0297                    |
| $\chi_b^q (3^3 P_0)$        | 0^{++}   | 11.4814         | 11.5109 | 1.0288                    |
| $\chi_b^q (3^3 P_1)$        | 1^{++}   | 11.4911         | 11.513  | 1.0306                    |
| $\chi_b^q (3^3 P_2)$        | 2^{++}   | 11.4975         | 11.5137 | 1.0316                    |
| $h_b^q (4^1 P_1)$           | 1^{--}   | 11.6596         | 11.7045 | 1.2303                    |
| $\chi_b^q (4^3 P_0)$        | 0^{++}   | 11.6492         | 11.7037 | 1.2298                    |
| $\chi_b^q (4^3 P_1)$        | 1^{++}   | 11.6582         | 11.7053 | 0.2313                    |
| $\chi_b^q (4^3 P_2)$        | 2^{++}   | 11.6641         | 11.7057 | 1.232                     |
| $h_b^q (5^1 P_1)$           | 1^{--}   | 11.8123         | 11.8814 | 1.4159                    |
| $\chi_b^q (5^3 P_0)$        | 0^{++}   | 11.8026         | 11.8809 | 1.4157                    |
| $\chi_b^q (5^3 P_1)$        | 1^{++}   | 11.8109         | 11.8822 | 1.4169                    |
| $\chi_b^q (5^3 P_2)$        | 2^{++}   | 11.8165         | 11.8825 | 1.4174                    |
| $h_b^q (6^1 P_1)$           | 1^{--}   | 11.9545         | 11.0468 | 1.5901                    |
| $\chi_b^q (6^3 P_0)$        | 0^{++}   | 11.9453         | 11.0466 | 1.5901                    |
| $\chi_b^q (6^3 P_1)$        | 1^{++}   | 11.9531         | 11.0477 | 1.5911                    |
| $\chi_b^q (6^3 P_2)$        | 2^{++}   | 11.9584         | 11.0479 | 1.5916                    |
| $\eta_b^q (1^3 D_2)$        | 2^{++}   | 11.206          | 11.1808 | 0.6629                    |
| $\Upsilon_b^q (1^3 D_1)$    | 1^{++}   | 11.2004         | 11.1759 | 0.6584                    |
| $\Upsilon_b^q (1^3 D_2)$    | 2^{++}   | 11.2054         | 11.1803 | 0.6623                    |
| $\Upsilon_b^q (1^3 D_3)$    | 3^{++}   | 11.2088         | 11.1835 | 0.6657                    |
| $\eta_b^q (2^3 D_2)$        | 2^{++}   | 11.4041         | 11.4069 | 0.9045                    |
| $\Upsilon_b^q (2^3 D_1)$    | 1^{++}   | 11.3978         | 11.4042 | 0.9017                    |
| $\Upsilon_b^q (2^3 D_2)$    | 2^{++}   | 11.4035         | 11.4068 | 9044                      |
| $\Upsilon_b^q (2^3 D_3)$    | 3^{++}   | 11.4076         | 11.4085 | 9066                      |
Table 2: Masses of $\Sigma_g^+$ states of bottomonium meson. These results are taken from our earlier work [8].

| Meson          | Relativistic mass GeV | NR mass GeV | $\sqrt{\langle r^2 \rangle}$ fm |
|----------------|-----------------------|-------------|----------------------------------|
| $\eta_b(1^1S_0)$ | 9.4926                | 9.5079      | 0.2265                           |
| $\Upsilon(1^3S_1)$ | 9.5098                | 9.5299      | 0.2328                           |
| $\eta_b(2^1S_0)$ | 10.0132               | 10.0041     | 0.5408                           |
| $\Upsilon(2^3S_1)$ | 10.0169               | 10.0101     | 0.5448                           |
| $h_b(1^1P_1)$   | 9.9672                | 9.9279      | 0.4347                           |
| $\chi_0(1^3P_0)$ | 9.8510                | 9.9232      | 0.4375                           |
| $\chi_1(1^3P_1)$ | 9.9612                | 9.9295      | 0.4379                           |
| $\chi_2(1^3P_2)$ | 9.9826                | 9.9326      | 0.4375                           |
| $\eta_b(1^1D_2)$ | 10.1661               | 10.1355     | 0.5933                           |
| $\Upsilon(1^3D_1)$ | 10.1548               | 10.1299     | 0.5930                           |
| $\Upsilon(1^3D_2)$ | 10.1649               | 10.1351     | 0.5939                           |
| $\Upsilon(1^3D_3)$ | 10.1772               | 10.1389     | 0.5942                           |