Almost global solutions to two classes of 1-d Hamiltonian Derivative Nonlinear Schrödinger equations

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Abstract

Consider two kinds of 1-d Hamiltonian Derivative Nonlinear Schrödinger (DNLS) equations with respect to different symplectic forms under periodic boundary conditions. The nonlinearities of these equations depend not only on $(x, \psi, \bar{\psi})$ but also on $(\psi_x, \bar{\psi}_x)$, which means the nonlinearities of these equations are unbounded. Suppose that the nonlinearities depend on the space-variable $x$ periodically. Under some assumptions, for most potentials of these two kinds of Hamiltonian DNLS equations, if the initial value is smaller than $\varepsilon \ll 1$ in $p$-Sobolev norm, then the corresponding solution to these equations is also smaller than $2\varepsilon$ during a time interval $(-c\varepsilon^{-r^*}, c\varepsilon^{-r^*})$ (for any given positive $r^*$). The main methods are constructing Birkhoff normal forms to two kinds of Hamiltonian systems which have unbounded nonlinearities and using the special symmetry of the unbounded nonlinearities of Hamiltonian functions to obtain a long time estimate of the solution in $p$-Sobolev norm.

Keyword. Derivative Nonlinear Schrödinger (DNLS) equations, Hamiltonian systems, unbounded, long time stability, momentum, Birkhoff normal form

AMS subject classifications. 37K55, 37J40

1 Introduction

It is very interesting to research the behavior of the solution in high-index Sobolev norm to nonlinear evolution equations with derivative in nonlinearities during a long time interval.

Consider a nonlinear Schrödinger equation

$$i\psi_t = \partial_{xx}\psi + F(x, \psi, \bar{\psi}, \partial_x\psi, \partial_x\bar{\psi}), \ x \in [0, 2\pi]$$

(1.1)

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under periodic boundary condition
\[ \psi(t, x) = \psi(t, x + 2\pi). \]

Suppose that \( F \) satisfies
\[ F(x + 2\pi, \psi, \bar{\psi}, \partial_x \psi, \partial_x \bar{\psi}) = F(x, \psi, \bar{\psi}, \partial_x \psi, \partial_x \bar{\psi}) \quad \text{and} \quad F(x, 0, 0, 0, 0) = 0. \]

Under this assumption \( \psi = 0 \) is an equilibrium solution to equation (1.1). I am interested in the behavior of solutions around \( \psi = 0 \) during a long time interval.

If \( F \) only depends on \((x, \psi, \bar{\psi})\) and vanishes at order \( n+1 \) about \((\psi, \bar{\psi})\) at the origin \((n \text{ is a positive integer)}\), local existence theory implies that when initial value \( \|\psi(x, 0)\|_{H^p} \leq \varepsilon \ll 1 \) the corresponding solution \( \psi(x,t) \) exists at least over an interval \((-c\varepsilon^{-n}, c\varepsilon^{-n})\) and \( \|\psi(x,t)\|_{H^p} \) stays bounded on such an interval. The problem that I am interested in is that construct almost global solutions when \( F \) depends on \((\psi_x, \bar{\psi}_x)\). An almost global solution means that for any given \( k > 0 \), when the initial value \( \psi(x,0) \) is smaller than \( 0 \ll \varepsilon \ll 1 \), solution \( \psi(x,t) \) is also small in a high index Sobolev norm for any \( t \in (-c\varepsilon^{-k}, c\varepsilon^{-k}) \) (refer [31]).

When investigation concerns equation (1.1) on a compact manifold, no dispersion is available. Nevertheless, two ways may be used to obtain solutions, defined on time-intervals larger than the one given by local existence theory. The first one is using KAM theory to get small amplitude periodic or quasi-periodic (hence global) solutions. A lot of work have been devoted to these questions and readers refer [5, 6, 8, 9, 11, 13, 21, 39, 24, 28, 29, 30, 32, 33, 34, 35].

The second approach concerns the construction of almost global \( H^p \)-small solutions for (1.1) on compact manifold. Use Birkhoff normal form method to improve the order of normal form and then exploit integral principle to get almost global solutions. When nonlinearity \( F \) to equation (1.1) depends only on \((\psi, \bar{\psi})\), small initial data give rise to global solutions and keep uniform control of the \( p \)-Sobolev norm of solutions \((p \text{ large enough})\), over time-intervals of length \( \varepsilon^{-k} \), for any given positive \( k \). This has been initiated by Bourgain [10], who stated results of almost global existence and uniform control to equation
\[ i\psi_t - \psi_{xx} + V(x)\psi + \frac{\partial H}{\partial \psi}(\psi, \bar{\psi}) = 0, \quad (1.2) \]
for any typical \((\text{with large probability})\) \( V(x) \). Bourgain in [11] stated that for any small typical initial value to equation
\[ i\psi_t - \psi_{xx} + v|\psi|^2\psi + \frac{\partial H}{\partial \psi}(\psi, \bar{\psi}) = 0, \quad (1.3) \]
the solution \( \psi \) will satisfy
\[ \|\psi(t)\|_{H^p} < C\varepsilon, \quad \text{for any } |t| \leq \varepsilon^{-B}. \]

More results may be found in the book of Bourgain [12]. Almost global solutions for Hamiltonian Semi-linear Klein-Gordon Equations \((\text{without derivative in nonlinearity})\) on spheres and Zoll manifolds have been obtained by Bambusi, Delort, Grêbert and Szeftel in [3]. Berti and Delort in [7] give almost global existence of solutions for capillarity-gravity water waves equations with periodic spatial boundary conditions.

Bambusi and Grêbert [4] (see also Bambusi [1] and Grêbert [25]) prove an abstract Birkhoff normal form theorem for Hamiltonian partial differential equations and apply
this theorem to semi-linear equations: nonlinear wave equation, nonlinear Schrödinger equation on the d-dimensional (d \geq 1) torus with nonlinearities satisfying a property-tame modulus. In a non-resonant case they deduce that any small amplitude solution remains very close to a torus for a very long time. In \[2\] Bambusi researches the NLW tame modulus. In a non-resonant case they deduce that any small amplitude solution of differential systems and prove that if the \( (0 < \varepsilon \text{ small enough}) \) then the solution is bounded by \( 2\varepsilon \) during time of order \( \varepsilon^{-r} \) with \( r \) arbitrary. This theorem applies to a class of reversible semi-linear PDEs including nonlinear Schrödinger equation on the d-dimensional torus and a class of coupled NLS equations which is reversible but not Hamiltonian. Feola and Iandoli in \[23\] give the long time existence for a large class of fully nonlinear, reversible and parity preserving Schrödinger equations on the one dimensional torus.

Delort and Szeftel in \[17, 18\], Delort in \[19, 20\] research semi-linear Klein-Gordon equation on tori and \( S^1 \), and obtain that when the initial value is small than \( \varepsilon > 0 \), the corresponding solution exists when \( |t| \leq \varepsilon^{-r} \).

Given a DNLS equation
\[
\mathbf{i} \psi_t = \partial_{xx} \psi + V \ast \psi + \left( \frac{\partial f(\psi, \bar{\psi})}{\partial \psi} \right)_x, \quad x \in [0, 2\pi],
\]
Yuan and Zhang in \[37\] obtain that for most \( V \) the solution to \[(1.4)\] is still smaller than \( 2\varepsilon \) among time \( |t| \leq \varepsilon^{-r} \) (for any given positive \( r \)), if the initial value is smaller than \( \varepsilon \ll 1 \). The nonlinearity in \[(1.4)\] does not directly depend on space variable \( x \). In \[38\] Yuan and Zhang research the long time behavior of the solution to the perturbed KdV equation the nonlinearity of which is trigonometric polynomial about \( x \).

In this paper, I focus on the behavior of solutions during a long time interval to two types of Hamiltonian Derivative Nonlinear Schrödinger (DNLS) equations which depend on \( x \) periodically. One type is of the following form
\[
\mathbf{i} \psi_t = \partial_{xx} \psi + V \ast \psi + \frac{1}{2} \partial_x \partial_{\psi} F(x, \psi, \bar{\psi}) + \partial_{\psi} \bar{\psi} F(x, \psi, \bar{\psi}) \psi_x,
\]
where \( V \) belongs to
\[
\Theta^0_m := \left\{ V(x) \in L^2([0, 2\pi], \mathbb{R}) \mid \hat{V}_j \cdot \max\{1, |j|^m\} \in [-\frac{1}{2}, \frac{1}{2}], \hat{V}_j = \hat{V}_{-j}, \forall j \in \mathbb{Z} \right\}
\]
and the other type is as follow
\[
\mathbf{i} \psi_t = \partial_{xx} \psi + V \ast \psi + \mathbf{i} \partial_x \left( \partial_{\psi} F(x, \psi, \bar{\psi}) \right),
\]
where \( V \) belongs to
\[
\Theta^1_m := \left\{ V \in L^2([0, 2\pi], \mathbb{C}) \mid \hat{V}_j \cdot \max\{1, |j|^m\} \in [-\frac{1}{2}, \frac{1}{2}], \forall j \in \mathbb{Z} \setminus \{0\}, \hat{V}_0 = 0 \right\}.
\]
Under some assumptions, \[(1.5)\] becomes into a Hamiltonian equation with respect to a symplectic form \( w^0 := J_0 d\psi \wedge d\bar{\psi} \), \( J_0^{-1} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) and \[(1.6)\] is Hamiltonian under a symplectic form \( w^1 := J_1 d\psi \wedge d\bar{\psi} \), \( (J_1)^{-1} := \begin{pmatrix} 0 & \frac{\partial_x}{\partial_{\psi}} \\ \frac{\partial_x}{\partial_{\psi}} & 0 \end{pmatrix} \).
When $\partial_\psi F(x,0,0) = 0$ and $\partial^2_{\psi\psi} F(x,0,0) = 0$, $\psi = 0$ is an equilibrium point of equations (1.5) and (1.6). In order to get the almost global solution around the origin to (1.5) and (1.6), it is required to research the behavior of solutions around $\psi = 0$ during a long time interval.

The result in [37] holds ture for Hamiltonian DNLS equation with nonlinearity independent of $x$. In other words the momentum of the corresponding Hamilton function equals to zero. This property is important in proving the long time stability result. In [14] one researches an unbounded perturbed KdV equation the nonlinearity of which is a trigonometric polynomial about $\sin kx$ and $\cos kx$ ($|k| \leq M$), i.e., the momentum of the corresponding Hamilton function are bounded. But generally, the sets of the momentum of Hamiltonian functions to equation (1.5) and (1.6) under Fourier transformation may be unbounded. Even if assume that for any $(\psi, \bar{\psi})$ around origin $F \in H^\delta([0,2\pi], \mathbb{R})$ ($\beta$ big enough), the corresponding nonlinear vector field of equations (1.5) and (1.6) are still unbounded. Denote the part of $F$, the momentum of which is bigger than $\delta > 0$, as $F_1$. Even if $\delta$ is very large, the Hamiltonian vector field of $F_1$ in equations (1.5) and (1.6) are still unbounded. The results and methods in [37] and [38] do not work to equations (1.5) and (1.6), directly. In [14] one consider quasi-linear Klein-Gordon equation on $S^1$. The nonlinearities are polynomials and smooth depend on $x$. Their methods are not suitable to DNLS equations (1.5) and (1.6). In [23] they consider the reversible and parity preserving Schrödinger equation. It is necessary to construct a long time stability theory to solutions of Hamiltonian DNLS equations (1.5) and (1.6) around the origin.

Under Fourier transformation, equations (1.5) and (1.6) are transformed into two types of Hamiltonian systems $\theta \in \{0, 1\}$

$$
\begin{align*}
\dot{u}_j &= -\text{sgn}(j) \cdot \partial_u H^{w_\theta}(u, \bar{u}), \\
\dot{\bar{u}}_j &= \text{sgn}(j) \cdot \partial_{\bar{u}} H^{w_\theta}(u, \bar{u}),
\end{align*}
$$

when $\theta \in \{0, 1\}$ and

$$
H^{w_\theta}(u, \bar{u}) = H_0^{w_\theta} + P^{w_\theta}(u, \bar{u}), \quad (u, \bar{u}) \in H^p, \quad \theta \in \{0, 1\}
$$

under symplectic form

$$
w_\theta := \begin{cases}
\sum_{j \in \mathbb{Z}^+} d u_j \wedge d \bar{u}_j & \text{when } \theta = 0, \\
\sum_{j \in \mathbb{Z}^+} \text{sgn}(j) d u_j \wedge d \bar{u}_j & \text{when } \theta = 1,
\end{cases}
$$

with Hamiltonian function $H_0^{w_\theta} = \sum_{j \in \mathbb{Z}^+} |u_j|^2$ and $\omega_j^{w_\theta} := \sum_{j \in \mathbb{Z}^+} \text{sgn}(j)(-j^2 + V_j) \theta = 0$. $P^{w_\theta}(u, \bar{u})$ is a power series having $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C_\theta > 0$ (refer definitions 3.3 and 3.4 in section 3). Note that the coefficients of $P^{w_\theta}(u, \bar{u})$ are not bounded. This leads to the Hamiltonian vector field of $P^{w_\theta}(u, \bar{u})$ being unbounded. See Proposition 4.1 in section 4.

The problem of finding almost global solutions around the origin to equations (1.5) and (1.6) is changed into considering a long time stability of solutions around equilibrium point $(u, \bar{u}) = 0$ of (1.7).

In section 3 Theorem 3 states that under some assumptions, the solution to the two type of Hamiltonian systems which have $(\beta, \theta)$-type symmetric coefficients ($\theta \in \{0, 1\}$) is
still smaller than $2\varepsilon$ during a time interval $(-\varepsilon^{-r^*}, \varepsilon^{-r^*})$, if its initial value is smaller than $\varepsilon \ll 1$.

The idea of proving Theorem 3 is to combine Birkhoff normal form method with the property of $(\beta, \theta)$-type symmetric coefficients used to obtain energy inequalities.

Let us introduce the important steps in proving Theorem 3.

First step: construct a coordination transformation $\mathcal{T}_{w_0}^{(r)}$ under which the Hamiltonian function $H_{w_0}(u, \bar{u})$ in (1.8) can be transformed into a new Hamiltonian function

$$H^{(r,w_0)} = H_{w_0} \circ \mathcal{T}_{w_0}^{(r)} = H_{0}^{w_0} + \mathcal{R}^{N(w_0)} + \mathcal{R}^{T(r,w_0)}$$

with a high degree ($\theta, \gamma, \alpha, N$)-normal form $Z^{(r,w_0)}$ (see definition 5.1). Because the system (1.7) is in an infinite dimension, one can only get a partial normal form. $\mathcal{R}^{N(r,w_0)}$ is at least 3 order about $(u_j, \bar{u}_j)_{|j|>N}$ ($N$ is large enough) and $\mathcal{R}^{T(r,w_0)}$ has a zero of high order about $(u, \bar{u})$. The Hamiltonian vector field of $P^{(r,w_0)}$ is still unbounded. The construction of $\mathcal{T}_{w_0}^{(r)}$ is from solving Homological equation (refer Lemma 5.2). Because the perturbation in equation (1.7) is unbounded, a strong non resonant condition to frequencies $\{\omega_j^{w_0}(V)\}$ is needed to keep the transformation $\mathcal{T}_{w_0}^{(r)}$ bounded. This condition will estimate the sets of potential $V(x)$ and the expression of $(\theta, \gamma, \alpha, N)$-normal form. Moreover, if $P^{w_0}(u, \bar{u})$ has $(\beta, \theta)$-type symmetric coefficients, then $P^{(r,w_0)}(u, \bar{u})$ is still of $(\beta, \theta)$-type symmetric coefficients.

Second step: The solution to the new Hamiltonian system satisfies

$$\frac{d\|u\|^2}{dt} = \{\|u\|^2, H^{(r,w_0)}(u, \bar{u})\}_{w_0}.$$  \hspace{1cm} (1.10)

From above equation, it is obvious that estimating $\{\|u\|^2, H^{(r,w_0)}(u, \bar{u})\}_{w_0}$ is the key to get a long time behavior of the solution. For a general function $f(u, \bar{u})$ with unbounded coefficients, $\{\|u\|^2, f(u, \bar{u})\}_{w_0}$ is not bounded even if $\|u\|_p$ is small enough. Fortunately, $P^{(r,w_0)}(u, \bar{u})$ has $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C_\theta > 0$. Studying the Poisson bracket of $\|u\|^2_p$ and $P^{(r,w_0)}(u, \bar{u})$ with $(\beta, \theta)$-type symmetric coefficients is an important problem in this paper. Proposition 4.2 in section 4 and Lemma 5.1 in section 5 state that

$$\{|\{\|u\|^2_p, \mathcal{R}^{N(w_0)}(u, \bar{u})\}_{w_0} + \mathcal{R}^{T(r,w_0)}(u, \bar{u})\} \sim R^{r+1}$$ \hspace{1cm} (1.11)

and

$$\{|\{\|u\|^2_p, Z^{(r,w_0)}(u, \bar{u})\}_{w_0} \sim R^{r+1} \hspace{1cm} (1.12)$$

hold true for any $\|u\|_p \leq R \ll 1$ and large enough $N$. With the help of (1.11), (1.12) and (1.10), the long time behavior of solution to the new Hamiltonian system can be obtained.

Since the two Hamiltonian DNLS equations have some difference, there still are some differences in the results of existence of almost global solution. The main difference is the sets of the potentials. From Lemma 7.1 there exist positive measure subsets $\Theta^0_m \subset \Theta^0_m$ ($\theta \in \{0, 1\}$) such that when $V \in \Theta^0_m$, frequencies $\{\omega_j^{w_0}(V)\}$ are $(\theta, \gamma, \alpha, N)$-non resonant (see definition 5.1). When $V \in \Theta^0_m$, its Fourier coefficients satisfy $\hat{V}_j = \hat{V}_{-j} \in \mathbb{R}$, for any $j \in \mathbb{Z}$, which makes the corresponding frequencies satisfying $\omega_j^{w_0} = \omega_{-j}^{w_0}$, while $V \in \Theta^1_m$, $\hat{V}_j$ does not always equal to $\hat{V}_{-j}$. Thus $\omega_j^{w_1}$ is not related to $\omega_j^{w_1}$ for any $j \in \mathbb{N}$. The potential sets to equations (1.5) and (1.6) are different, because (1.5) and (1.6) have different.
symplectic forms and nonlinearities. To be specific, from the definitions of symplectic structures, the following equations hold true for any $j \in \mathbb{Z} \setminus \{0\}$

$$\{u_j \bar{u}_{-j} + \bar{u}_j u_{-j}, \|u\|^2\}_{w_0} = 0$$

(1.13)

and

$$\{u_j \bar{u}_{-j} + \bar{u}_j u_{-j}, \|u\|^2\}_{w_1} \neq 0.$$  

(1.14)

In order to make $|\{H^{w_0}(u, \bar{u}), \|u\|^2\}_{w_0}|$ being high order small as $\|u\|_p$ is small, when $\theta = 0$, from (1.13) the terms depending on $(u_j \bar{u}_{-j} + \bar{u}_j u_{-j})$ in $H^{w_0}(u, \bar{u})$ will not need to be eliminated by symplectic transformations; when $\theta = 1$, from (1.14) it needs to eliminate the terms depending on $(u_j \bar{u}_{-j} + \bar{u}_j u_{-j})$ in $H^{w_0}(u, \bar{u})$. Therefore, it needs more parameters in the case $\theta = 1$ than in the case $\theta = 0$ and the sets of potential $V(x)$ are different to equations (1.5) and (1.6).

The paper is organized as follows: The section 2 of this paper is devoted to introduction of two types of Hamiltonian DNLS equations with respect to different symplectic forms. There are many differences between these two types of equations (see Remark 2.1). Then I give the main results in this paper, the existence of global solutions with small initial values to these two types of DNLS equations (See Theorem 1 and Theorem 2).

In the third section I present a definition of $(\beta, \theta)$-type symmetric coefficients with respect to different symplectic forms. To be specific, from the definitions of symplectic forms and nonlinearities, the following equations hold true for any $j \in \mathbb{Z} \setminus \{0\}$

$$\{f_j \bar{f}_{-j} + \bar{f}_j f_{-j}, \|f\|^2\}_{\theta} = 0$$

(4.1)

It is easy to found that the Hamiltonian vector field of $f^{w_0}(u, \bar{u})$ with $(\beta, \theta)$-type symmetric coefficients under symplectic form $w_0 (\theta \in \{0, 1\})$. This estimate is given in Proposition 4.2.

Proposition 4.2 states that $|\{f^{w_0}(u, \bar{u}), \|u\|^2\}_{w_0}|$ is small when $\|u\|_p$ is small enough and $f^{w_0}(u, \bar{u})$ has $(\beta, \theta)$-type symmetric coefficients. Even if the set of momentum of $f^{w_0}(u, \bar{u})$ is unbounded, the result still holds true. The property of having $(\beta, \theta)$-type symmetric coefficients is invariant under some operators, such as truncated operators $\Gamma^N_{\leq 2}$ and $\Gamma^N_{> 2}$ defined in (4.2) and (4.3). See Corollary 4.1.

In the fifth section, in order to improve the order of Birkhoff normal forms of Hamiltonian systems under two different symplectic forms, I will find suitable bounded symplectic transformations (See Theorem 5.1). These transformations are constructed by solving Homological equations. Since the nonlinear vector fields of Hamiltonian systems are unbounded (see Proposition 4.1), a stronger non-resonant condition (see definition 5.1) is needed. Under these transformations, new Hamiltonian systems are obtained. The nonlinearities of the new Hamiltonian functions still have $(\beta, \theta)$-type symmetric coefficients (See Lemma 5.3). Although the high order normal forms $Z^{(r, w_0)}(u, \bar{u}) (\theta \in \{0, 1\})$ in the new Hamiltonian functions are not standard Birkhoff normal forms, from Lemma 5.1 $\{|\|u\|^2, Z^{(r, w_0)}(u, \bar{u})\}_{w_0}$ is high order small when $\|u\|_p$ is small enough. The detail of the proof of Theorem 5.1 is listed in Appendix.

In the sixth section Theorem 5.1 is proved by applying Theorem 1, Proposition 4.2, Corollary 1 and Lemma 5.1.

In the seventh section the proofs of Theorem 1 and Theorem 2 are given. Using Lemma 7.1 there exists a positive measure subset $\hat{\Theta}_m^\theta \subset \Theta_m^\theta (\theta \in \{0, 1\})$ such that when $V \in \hat{\Theta}_m^\theta$ the eigenvalues of linear operator $\partial_{xx} + V(x)*$ are stronger non resonant.
2 Hamiltonian DNLS equations and main results

2.1 Hamiltonian DNLS equations

Let

\[ H^p([0, 2\pi], \mathbb{C}) := \left\{ \psi \in L^2([0, 2\pi], \mathbb{C}) \mid \frac{\partial^r \psi}{\partial x^r} \in L^2([0, 2\pi], \mathbb{C}), \forall 0 \leq r \leq p \right\} \]

be a p-Sobolev space. The inner product of the space \( L^2([0, 2\pi], \mathbb{C}) \) is defined as

\[ \langle \zeta, \eta \rangle := \text{Re} \int_0^{2\pi} \zeta \cdot \bar{\eta} \, dx, \quad \text{for any } \zeta, \eta \in L^2([0, 2\pi], \mathbb{C}). \]

The important definition of Hamiltonian PDEs is introduced in [32]. I list it as following. Consider an evolution equation

\[ \dot{\xi} = A\xi + f(\xi) \tag{2.1} \]

defined in symplectic Hilbert scales \( \{H^p([0, 2\pi], \mathbb{C}) \times H^p([0, 2\pi], \mathbb{C})\}, \alpha \), where \( \alpha \) is a non-degenerate closed 2-form. Equation (2.1) is called a Hamiltonian equation, if there exists a Hamiltonian function \( H(\xi) \) defined in a domain \( O_p \subset H^p([0, 2\pi], \mathbb{C}) \times H^p([0, 2\pi], \mathbb{C}) \) making

\[ \alpha(A\xi + f(\xi), \eta) = -\langle dH(\xi), \eta \rangle \quad \text{for any } \xi \in O_p, \eta \in TO_p (TO_p \text{ is the tangent space of } O_p). \]

The dual space and the tangent space of \( H^p([0, 2\pi], \mathbb{C}) \times H^p([0, 2\pi], \mathbb{C}) \) are isometry to \( H^p([0, 2\pi], \mathbb{C}) \times H^p([0, 2\pi], \mathbb{C}) \), without confusion I denote them in the same signal in the following content.

Denote \( d_A \) as the order of the linear operator

\[ A : H^p([0, 2\pi], \mathbb{C}) \times H^p([0, 2\pi], \mathbb{C}) \to H^{p-d_A}([0, 2\pi], \mathbb{C}) \times H^{p-d_A}([0, 2\pi], \mathbb{C}) \]

and \( d_f \) as the order of the mapping

\[ f : H^p([0, 2\pi], \mathbb{C}) \times H^p([0, 2\pi], \mathbb{C}) \to H^{p-d_f}([0, 2\pi], \mathbb{C}) \times H^{p-d_f}([0, 2\pi], \mathbb{C}). \]

When the nonlinearity of a partial differential equation includes derivative, the corresponding order of the nonlinear vector field is positive. Otherwise, the order is non-positive. The following notations “bounded” and “unbounded” are given by the signs of the order of the vector field, and readers can refer [28], [37], [38]. For the sake of reference I list the definitions again.

**Definition 2.1.** If \( d_f \leq 0 \), call \( f \) in (2.1) **bounded**; If \( d_f > 0 \), \( f \) is called **unbounded**. Moreover, If \( d_A - 1 = d_f > 0 \), call \( f \) **critical unbounded**.

In this paper, I focus on two kinds of Hamiltonian Derivative Nonlinear Schrödinger (DNLS) equations.

**Type I**–DNLS equation has the following form

\[ i\psi_t = \partial_{xx} \psi + V \ast \psi + i f(x, \psi, \psi_x, \psi_{xx}), \quad \psi \in H^p([0, 2\pi], \mathbb{C}) \tag{2.2} \]

under periodic boundary condition

\[ \psi(x, t) = \psi(x + 2\pi, t), \tag{2.3} \]
where $V$ belongs to

$$
\Theta_m^0 := \left\{ V(x) = \sum_{j \in \mathbb{Z}} \hat{V}_j e^{ijx} \in L^2([0,2\pi], \mathbb{R}) \bigg| v_j^{w_0} := \hat{V}_j(j)^m \in [-\frac{1}{2}, \frac{1}{2}], v_j^{w_0} = v_j^{\bar{w}_0}, \forall j \in \mathbb{Z} \right\} \tag{2.4}
$$

with $m > 1/2$, $\langle j \rangle := \max\{1, |j|\}$ and $\bar{\psi}$ is the complex conjugate of $\psi$.

Suppose that there exists a function $F(x, \psi, \bar{\psi})$ such that

$$
f(x, \psi, \bar{\psi}, \psi_x, \bar{\psi}_x) = \frac{1}{2} \left| \partial_x \psi \right|^2 + (V * \psi) \bar{\psi} + \frac{1}{2} \partial_x F(x, \psi, \bar{\psi}) + \partial_{\psi} \bar{\psi} F(x, \psi, \bar{\psi}) \psi_x. \tag{2.5}
$$

Moreover, $F$ satisfies assumptions as follows.

**A$_1$:** $F(x, \xi, \eta)$ is analytic about $(\xi, \eta)$ in a neighborhood of the origin and satisfies

$$
F(x, \psi, \bar{\psi}) = F(x, \psi, \bar{\psi}) \tag{2.6}
$$

and $F(x, \psi, \bar{\psi})$ vanishes at least at order 2 in $(\psi, \bar{\psi})$ at the origin.

**A$_2$:** For any fixed $(\psi, \bar{\psi})$ a neighborhood of the origin, $F \in H^{\beta+1}([0,2\pi], \mathbb{C})$ ($\beta$ is a big enough positive real number) satisfies

$$
F(x + 2\pi, \psi, \bar{\psi}) = F(x, \psi, \bar{\psi}).
$$

Then (2.2) becomes a Hamiltonian PDE with a real value Hamiltonian function

$$
H_{(2.2)}(\psi, \bar{\psi}) = \int_0^{2\pi} -|\partial_x \psi|^2 + (V * \psi) \bar{\psi} + \frac{1}{2} \partial_x F(x, \psi, \bar{\psi}) + i \partial_{\psi} \bar{\psi} F(x, \psi, \bar{\psi}) \psi_x dx. \tag{2.8}
$$

on symplectic space $(H^p([0,2\pi], \mathbb{C}) \times H^p([0,2\pi], \mathbb{C}), w^0)$, where

$$
w^0 = J_0 d\psi \wedge d\bar{\psi}, \quad J_0^{-1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.9}
$$

The corresponding Hamiltonian vector of $H_{(2.2)}(\psi, \bar{\psi})$ under symplectic form $w^0$ is

$$
X^w_{(2.2)} := \begin{pmatrix} -i \partial_\psi H_{(2.2)} \\ i \partial_{\psi} H_{(2.2)} \end{pmatrix}^T
$$

*Since $F(x, \psi, \bar{\psi})$ satisfies assumptions A$_1$-A$_2$, the following equation holds true for any $\psi \in H^p([0,2\pi], \mathbb{C})$ fulfilling $\psi(x + 2\pi, t) = \psi(x, t)$

$$
0 = \int_0^{2\pi} \frac{d}{dt} F(x, \psi, \bar{\psi}) dx = \int_0^{2\pi} \partial_x F(x, \psi, \bar{\psi}) + \partial_{\psi} F(x, \psi, \bar{\psi}) \psi_x + \partial_{\bar{\psi}} F(x, \psi, \bar{\psi}) \bar{\psi}_x dx,
$$

i.e.,

$$
\int_0^{2\pi} \frac{1}{2} \partial_x F(x, \psi, \bar{\psi}) + \partial_{\psi} F(x, \psi, \bar{\psi}) \psi_x dx = - \int_0^{2\pi} \frac{1}{2} \partial_x F(x, \psi, \bar{\psi}) - \partial_{\bar{\psi}} F(x, \psi, \bar{\psi}) \bar{\psi}_x dx. \tag{2.7}
$$

From (2.7) and assumptions A$_1$-A$_2$, it follows

$$
\int_0^{2\pi} \frac{1}{2} \partial_x F(x, \psi, \bar{\psi}) + i \partial_{\psi} F(x, \psi, \bar{\psi}) \psi_x dx = \int_0^{2\pi} (-i) \frac{1}{2} \partial_x F(x, \psi, \bar{\psi}) - i \partial_{\bar{\psi}} F(x, \psi, \bar{\psi}) \bar{\psi}_x dx = \int_0^{2\pi} \frac{1}{2} \partial_x F(x, \psi, \bar{\psi}) + i \partial_{\psi} F(x, \psi, \bar{\psi}) \psi_x dx,
$$

which means that the Hamiltonian function $H_{(2.2)}(\psi, \bar{\psi})$ is real.
and the equation (2.2) can be written as

\[
\begin{align*}
\dot{\psi} &= -i \frac{\partial}{\partial \bar{\psi}} H_{2.2} (\psi, \bar{\psi}), \\
\dot{\bar{\psi}} &= i \frac{\partial}{\partial \psi} H_{2.2} (\psi, \bar{\psi}).
\end{align*}
\] (2.10)

**Type II–DNLS equation** has the form as following

\[
i \psi_t = \partial_{xx} \psi + V * \psi + i \partial_x \left( \frac{\partial F(x, \psi, \bar{\psi})}{\partial \psi} \right)
\] (2.11)

defined on

\[
H_0^p([0, 2\pi], \mathbb{C}) := \left\{ \psi \in H^p([0, 2\pi], \mathbb{C}) \mid \int_0^{2\pi} \psi(x, t) dx = 0 \right\}
\] (2.12)

under periodic boundary condition

\[
\psi(x, t) = \psi(x + 2\pi, t).
\] (2.13)

The potential \( V \) belongs to

\[
\Theta_m^1 := \left\{ V \in L^2([0, 2\pi], \mathbb{C}) \mid v_j^{m_1} := \hat{V}_j^m (j) \in \left[ -\frac{1}{2}, \frac{1}{2} \right], \forall j \in \mathbb{Z} \setminus \{0\}, \ v_0^{m_1} = \hat{V}_0 = 0 \right\}
\] (2.14)

with \( m > 1/2 \).

If equation (2.11) satisfies the following assumptions:

\( B_1 \) F\( (x, \xi, \eta) \) is analytic at the origin about \((\xi, \eta) \in H_0^p([0, 2\pi], \mathbb{C}) \times H_0^p([0, 2\pi], \mathbb{C}) \) and vanishes at least at order 2 in \((\psi, \bar{\psi})\) at origin. For any \( \psi \in H_0^p([0, 2\pi], \mathbb{C}) \), it holds

\[
F(x, \psi, \bar{\psi}) = F(x, \psi, \bar{\psi}).
\]

\( B_2 \) For any fixed \((\psi, \bar{\psi})\) in a neighborhood of the origin, \( F \in H^{\beta+1}([0, 2\pi], \mathbb{C}) \) (\( \beta \) is a big enough positive real number) satisfies

\[
F(x + 2\pi, \psi, \bar{\psi}) = F(x, \psi, \bar{\psi});
\]

then equation (2.11) becomes into a Hamiltonian PDE with a real Hamiltonian

\[
H_{2.11} (\psi, \bar{\psi}) = \int_0^{2\pi} -i \partial_x \psi \bar{\psi} - i (\partial_x)^{-1} (V(x) * \psi) \cdot \bar{\psi} + F(x, \psi, \bar{\psi}) dx
\]

under symplectic space \((H_0^p([0, 2\pi], \mathbb{C}) \times H_0^p([0, 2\pi], \mathbb{C}), w^1))\), where

\[
w^1 := J_1 d\psi \wedge d\bar{\psi}, \quad (J_1)^{-1} := \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}
\] (2.15)

is a symplectic from \((w^1)\) is a non-degenerate closed two form in space \(H_0^p([0, 2\pi], \mathbb{C}) \times H_0^p([0, 2\pi], \mathbb{C}))\).
The Hamiltonian vector $X_{H_{2.11}}^{w_1}$ of $H_{2.11}^{2.11}(\psi, \bar{\psi})$ equals to

$$\left( \partial_x(\partial_{\psi}H_{2.11}^{2.11}), \partial_x(\partial_{\bar{\psi}}H_{2.11}^{2.11}) \right)^T.$$  

Equation (2.11) can be written as follow

$$\left\{ \begin{array}{l}
\dot{\psi} = \partial_x \partial_{\bar{\psi}}H_{2.11}^{2.11}(\psi, \bar{\psi}), \\
\dot{\bar{\psi}} = \partial_x \partial_{\psi}H_{2.11}^{2.11}(\psi, \bar{\psi}).
\end{array} \right.$$  

The DNLS equation researched in [37] is a special case of equation (2.11), i.e., $F(x, \psi, \bar{\psi})$ is independent of $x$.

**Remark 2.1.** There are some differences between type I-DNLS equations and type II-DNLS equations:

- Nonlinearities of these two kinds of DNLS equations are different. It is an essential difference.
- Symplectic spaces are different. Type I-DNLS equation is defined in $(H^p([0, 2\pi], \mathbb{C}) \times H^p([0, 2\pi], \mathbb{C}), w^1)$ and type II-DNLS equation is defined in $(H^p_0([0, 2\pi], \mathbb{C}) \times H^0_0([0, 2\pi], \mathbb{C}), w^0)$. $w^1$ and $w^0$ are also different.
- The Potential $V$ in type I-DNLS equation belongs to $\Theta_0^m$, and the one in type II-DNLS equation belongs to $\Theta^1_m$. $\Theta^1_m$ is different from $\Theta_0^m$. When $V \in \Theta_0^m$ it fulfills $\hat{V}_j = \hat{V}_{-j} \in \mathbb{R}$ which means $V(x) = V(x)$; while $V \in \Theta^1_m$, $V$ is a complex valued potential. The potential $V$ will directly determine the eigenvalues of the linear operator $-\partial_{xx} + V(x)$. It is clear that when $V \in \Theta_0^m$ the corresponding eigenvalues $\omega_j$ and $\omega_{-j}$ of the linear operator $-\partial_{xx} + V(x)$ are resonant, when $V \in \Theta^1_m$, they are independent. The measures of $\Theta_0^m$ and $\Theta^1_m$ are defined as follows

$$\text{meas}(\Theta_0^m) := \prod_{j \in \mathbb{N} \cup \{0\}} \text{meas}\{ v_j \in [-\frac{1}{2}, \frac{1}{2}] \mid V \in \Theta_0^m, \hat{V}_j(j)^m = v_j \}$$

and

$$\text{meas}(\Theta^1_m) := \prod_{j \in \mathbb{Z} \setminus \{0\}} \text{meas}\{ v_j \in [-\frac{1}{2}, \frac{1}{2}] \mid V \in \Theta_0^m, \hat{V}_j(j)^m = v_j \},$$

where “meas” means the Lebesgue measure.

**Remark 2.2.**

- If type II-DNLS equation (2.11) is defined in space $H^p([0, 2\pi], \mathbb{C})$, it is easy to verify that for any solution $\psi(x,t)$ to equation (2.11) the quantity $\int_0^{2\pi} \psi(x,t)dx$ is a constant for any $t \in \mathbb{R}$. Set

$$\phi := \psi - c, \quad c := \int_0^{2\pi} \psi(x,0)dx = \int_0^{2\pi} \psi(x,0)dx.$$  

If $\psi \in H^p([0, 2\pi], \mathbb{C})$, then $\phi \in H^p_0([0, 2\pi], \mathbb{C})$. When $G(x, \phi, \bar{\phi}) := F(x, \phi + c, \phi + c)$ satisfies $B_1$ and $B_2$, then equation (2.11) becomes into a Hamiltonian equation under symplectic form $w^1$ about $(\phi, \bar{\phi})$.

- From Proposition 4.1 in section 4, the nonlinearities of type I-DNLS equation (2.2) and type II-DNLS equation (2.11) are unbounded.
2.2 Main result

The long time behavior of the solutions around equilibrium point to type I and type II-DNLS Hamiltonian equations are given in this subsection.

**Theorem 1.** Suppose that the equation (2.2) satisfies assumptions $A_1$-$A_2$. For any integer $r_\ast > 1$, there exist an almost full measure set $\tilde{\Theta}_0^m \subset \Theta_0^m$ and $p_\ast > 0$ such that for any fixed $V \in \tilde{\Theta}_0^m$ and any $p$ fulfilling $(\beta - 4)/2 > p > p_\ast$, if the initial data of the solution to (2.2) satisfies

$$\|\psi(x, 0)\|_{H^p([0, 2\pi], C)} \leq \varepsilon < \varepsilon_\ast,$$

then one has

$$\|\psi(x, t)\|_{H^p([0, 2\pi], C)} < 2\varepsilon, \quad \forall \ |t| < \varepsilon - r_\ast - 1.$$

**Theorem 2.** Suppose that equation (2.11) fulfills assumptions $B_1$-$B_2$. For any integer $r_\ast > 1$, there exist a positive $p_\ast$ and an almost full measure set $\tilde{\Theta}_1^m \subset \Theta_1^m$ such that for any fixed $V \in \tilde{\Theta}_1^m$ and any $p$ fulfilling $(\beta - 4)/2 > p > p_\ast$, the solution to (2.11) satisfies

$$\|\psi(x, t)\|_{H^{p+1/2}_0([0, 2\pi], C)} < 2\varepsilon, \quad \text{for any } \ |t| < \varepsilon - r_\ast - 1,$$

if the initial value fulfills

$$\|\psi(x, 0)\|_{H^{p+1/2}_0([0, 2\pi], C)} < \varepsilon \ll 1.$$

**Remark 2.3.** As type I and type II-DNLS equations have many differences (Readers can refer Remark 2.1), the proofs of Theorem 2 and Theorem 2 still have some differences.

**Remark 2.4.** The DNLS equations researched in our paper are not always invariant under gauge transformation. For example, take

$$F(\psi, \bar{\psi}) = \psi^3 \bar{\psi} + \psi \bar{\psi}^3 \quad (2.16)$$

in equation (2.11), which fulfills assumption $B_1$. It is easy to check that equation (2.11) with $F$ in (2.16) is not invariant under the transformation $\phi = e^{i\theta} \psi$, $\theta \in \mathbb{R}$.

3 Long time stability result to infinite dimension Hamiltonian systems with $(\beta, \theta)$-type symmetric coefficients $(\theta \in \{0, 1\})$

3.1 $(\beta, \theta)$-type symmetric coefficients $(\theta \in \{0, 1\})$

Under Fourier transformation, Hamiltonian DNLS equations with respect to periodic boundary condition can be transformed into two classes of infinite dimension Hamiltonian systems with “unbounded” nonlinearities. In this section, I will introduce long time stability results to two classes of infinite dimension Hamiltonian systems with “unbounded” nonlinearities. First, give some notations and annotations. In this paper, $\mathbb{Z}^\ast$ means $\mathbb{Z}$ or $\mathbb{Z} \setminus \{0\}$. Denote weighted Hilbert spaces

$$\ell^2_p(\mathbb{Z}^\ast, \mathbb{C}) := \left\{ u \in \ell^2(\mathbb{Z}^\ast, \mathbb{C}) \mid \|u\|_p^2 := \sum_{j \in \mathbb{Z}^\ast} \langle j \rangle^{2p} \cdot |u_j|^2 < +\infty, \langle j \rangle := \max\{|j|, 1\} \right\},$$
and \( \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C}) := \{(u, \bar{u}) \in \ell_2(\mathbb{Z}^*, \mathbb{C}) \times \ell_2(\mathbb{Z}^*, \mathbb{C}) \mid v = \bar{u}\} \) with norm \\
\[ \|(u, \bar{u})\|_p := \sqrt{\|u\|_p^2 + \|ar{u}\|_p^2}. \]

Let the neighborhood of the origin with a radius \( R \) be noted by \\
\[ B_p(R) := \{(u, \bar{u}) \in \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C}) \mid \|(u, \bar{u})\|_p < R\}. \]

**Definition 3.1.** For any fixed \( l, k \in \mathbb{N}^{\mathbb{Z}^*} \), call the integer \\
\[ \sum_{j \in \mathbb{Z}^*} j(l_j - k_j) \]
be the momentum of the ordered vector \((l, k)\) and denote it as \( \mathcal{M}(l, k) \).

Readers can refer this definition in \[37\] and \[38\].

**Remark 3.1.** If \( \mathcal{M}(l, k) = i \), from the definition of momentum, it holds that \\
\[ \mathcal{M}(k, l) = -i. \]

**Definition 3.2.** Call a power series \\
\[ f(u, \bar{u}) = \sum_{t \geq 3} \sum_{|l + k| = t, l, k \in \mathbb{N}^{\mathbb{Z}^*}} f_{l, k}^i u^l \bar{u}^k, \quad (u, \bar{u}) \in \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C}) \]
have symmetric coefficients, if for any \( l, k \) fulfilling \( |l + k| = t \) and \( \mathcal{M}(l, k) = i \), the coefficient holds \\
\[ f_{l, k}^i = f_{k, l}^{-i}. \]

Moreover, fixed \( \beta > 0 \), call \( f(u, \bar{u}) \) has \( \beta \)-bounded coefficients bounded by \( C > 0 \), if \\
\[ |f_{l, k}^i| \leq C \frac{t^2 - 2}{(i)^\beta}, \]
for any \( l, k \) satisfying \( |l + k| = t \) and \( \mathcal{M}(l, k) = i \).

**Remark 3.2.** A power series \( f : \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C}) \to \mathbb{C} \) is of symmetric coefficients, if and only if \( f \) satisfies \\
\[ f(u, \bar{u}) = f(u, \bar{u}), \quad \text{for any } (u, \bar{u}) \in \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C}). \]
Hence, a real-value Hamiltonian function has symmetric coefficients.

Now define two kinds of power series with “unbounded” special symmetric coefficients.

**Definition 3.3.** Given \( \beta > 0 \) and \( C_f > 0 \), call a power series \\
\[ f(u, \bar{u}) = \sum_{t \geq 3} \sum_{|l + k| = t, l, k \in \mathbb{N}^{\mathbb{Z}^*}} f_{l, k}^i u^l \bar{u}^k, \quad (u, \bar{u}) \in \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C}) \]
have \( (\beta, 0) \)-type symmetric coefficients, if its coefficients have the following form \\
\[ f_{l, k}^i := \sum_{(i^0, k^0, t^0) \in \mathcal{A}_{l, k}^i} f_{l, k}^{(i^0, k^0, t^0)} (\mathcal{M}(t^0, k^0) - \frac{t^0}{2}), \]
where
\[ A_{f_{l,ik}} \subset \{(\tilde{l}, \tilde{k}, \tilde{i}) \mid 0 \leq \tilde{l} \leq l, 0 \leq \tilde{k} \leq k, \text{ for any } j \in \mathbb{Z}^*, (\tilde{l}, \tilde{k}, \tilde{i}) \in \mathbb{N}_{+}^2 \times \mathbb{N}_{+}^2, \tilde{i} \in \mathbb{Z}\}, \]
and for any \((l^0, k^0, i^0) \in A_{f_{l,ik}}\), the followings hold true
\[ (k - k^0, l - l^0, i^0 - 2i) \in A_{f_{l,ik}^{-1}}, \quad f_{l,ik}^{-i(l^0, k^0, i^0)} = f_{l,ik}^{-i(k - k^0, l - l^0, i^0 - 2i)}. \]
Moreover, call \(f(u, \bar{u})\) have \((\beta, 0)\)-type symmetric coefficients semi-bounded by \(C_f\), if \(f(u, \bar{u})\) have \((\beta, 0)\)-type symmetric coefficients and there exists a constant \(C_f > 0\) such that for any \(l, k \in \mathbb{N}_{+}^2\) with \(|l + k| = t\) and \(M(l, k) = i\), the following inequality holds true
\[ \sum_{(p, k^0, i^0) \in A_{f_{l,ik}}} |f_{p, k^0, i^0}| \cdot \max\{\langle t^0, \rangle, \langle t^0 - 2i \rangle\} \leq \frac{C_f^{t-2}}{(i)\beta}. \quad (3.1) \]

Suppose that Type I-DNLS equation satisfies assumption \(A_1-A_2\). Under Fourier transformation, there exists a constant \(C > 0\) such that the Hamiltonian function of Type I-DNLS equation have \((\beta, 0)\)-type symmetric coefficients semi-bounded by \(C\). See section 7 for details. This symmetric property is invariant under a symplectic transformation. Refer Lemma 5.3 in section 5.

**Definition 3.4.** Given \(\beta > 0\) and \(C_g > 0\), call a power series
\[ g(u, \bar{u}) = \sum_{l \geq 3} \sum_{|k + l| = t, l, k \in \mathbb{N}_{+}^2, M(l, k) = i, M_g \subset \mathbb{Z}} g_{l,ik} u^l \bar{u}^k \]
have \((\beta, 1)\)-type symmetric coefficients, if for any \(l, k \in \mathbb{N}_{+}^2\) with \(|l + k| = t\) and \(M(l, k) = i\), its coefficient has the following form
\[ g_{l,ik} = g_{l,ik}^t \prod_{j \in \mathbb{Z}^*} \langle j \rangle^{\frac{1}{2} (l_j + k_j)} \]
and satisfies
\[ g_{l,ik} = g_{l,ik}^{-i}. \]
Moreover, call \(g(u, \bar{u})\) have \((\beta, 1)\)-type symmetric coefficients semi-bounded by \(C_g\), if \(g(u, \bar{u})\) have \((\beta, 1)\)-type symmetric coefficients and there exists a constant \(C_g > 0\) such that
\[ |g_{l,ik}| \leq \frac{C_g^{t-2}}{(i)\beta}. \quad (3.2) \]
for any \(l, k\) fulfilling \(|l + k| = t\) and \(M(l, k) = i\).

**Remark 3.3.** Suppose a power series \(f^{\omega}(u, \bar{u})\) is of \((\beta, \theta)\)-type symmetric coefficients semi-bounded by \(C_\theta > 0\) (\(\theta \in \{0, 1\}\)). Thence,
- the “semi-bounded” does not mean the coefficients of \(f^{\omega}(u, \bar{u})\) are bounded, even if \(f^{\omega}(u, \bar{u})\) is an \(r\)-degree polynomial;
\[ \mathcal{M}(l, 0) - \frac{l_0}{2} = \mathcal{M}(k - k_0, l - l_0) - \left( \frac{l_0}{2} - i \right) \]

and \((k - k_0, l - l_0, i_0 - 2i) \in \mathcal{A}_{f, i} \); when \(\theta = 1\) for any \(l, k \in \mathbb{N}^{\mathbb{Z}^*}\) and any \(i \in \mathbb{Z}\),

\[ \frac{(f^{wu})^i_{t, kl}}{(f^{wu})^0_{t, kl}} = (f^{wu})^i_{t, kl} \prod_{j \in \mathbb{Z}^*} (j)^{\frac{1}{2}(j_i + k_j)} = (f^{wu})^i_{t, kl} \prod_{j \in \mathbb{Z}^*} (j)^{\frac{1}{2}(j_i + k_j)} = (f^{wu})^{-i}_{t, kl}. \quad (3.4) \]

Hence, the coefficients of \(f^{wu}(u, \tilde{u})\) are symmetric.

Let \((\mathcal{H}^p(\mathbb{Z}^*, \mathbb{C}), w_\theta) (\theta \in \{0, 1\})\) be a symplectic space endowed with symplectic form

\[ w_\theta := \begin{cases} 
  \i \sum_{j \in \mathbb{Z}^*} du_j \land d\tilde{u}_j & \theta = 0, \\
  \i \sum_{j \in \mathbb{Z}^*} \text{sgn}(j) du_j \land d\tilde{u}_j & \theta = 1.
\end{cases} \quad (3.5) \]

When \(\theta = 0\), \(\mathbb{Z}^*\) can be either \(\mathbb{Z}\) or \(\mathbb{Z} \setminus \{0\}\). When \(\theta = 1\), \(\mathbb{Z}^*\) means \(\mathbb{Z} \setminus \{0\}\) only.

The poisson bracket of differential functions \(f_1\) and \(f_2\) defined in the domain of \(\mathcal{H}^p(\mathbb{Z}^*, \mathbb{C})\) under the symplectic form \(w_\theta (\theta \in \{0, 1\})\) has the following form

\[ \{f_1, f_2\}_{w_\theta} = w_\theta(\nabla f_1, \nabla f_2). \quad (3.6) \]

Given a differential function \(f\), its corresponding Hamiltonian vector field under the symplectic form \(w_\theta\) is defined as

\[ X^{w_\theta}_f := J_\theta \nabla f, \quad J_\theta := \begin{pmatrix} 0 & -I_0 \\ I_0 & 0 \end{pmatrix}, \quad \theta \in \{0, 1\}, \quad (3.7) \]

where \(I_0\) is an identity operator on space \(l^2_p(\mathbb{Z}^*, \mathbb{C})\), and for any \(u = (u_j)_{j \in \mathbb{Z}^*} \in l^2_p(\mathbb{Z}^*, \mathbb{C})\),

\[ I_1 u = \left( (I_1 u)_j \right)_{j \in \mathbb{Z}^*}, \quad (I_1 u)_j := \text{sgn}(j) \cdot u_j, \quad j \in \mathbb{Z}^*. \]

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3.2 Result of Hamiltonian system with $(\beta, \theta)$-Type symmetric coefficients

In order to use an uniformly formula to describe two kinds of Hamiltonian equations, in this paper, denote $0^0 = 1$.

Let $\theta \in \{0, 1\}$. Consider Hamiltonian systems defined in $\mathcal{H}^p(Z^*, \mathbb{C}, w_{\theta})$, for any $j \in Z^*$

\[
\begin{align*}
\dot{u}_j &= -\text{sgn}(j) \cdot \partial_{u_j} H^{w_{\theta}}(u, \bar{u}), \\
\dot{u}_j &= \text{sgn}(j) \cdot \partial_{u_j} H^{w_{\theta}}(u, \bar{u}),
\end{align*}
\]

with a Hamiltonian function

\[ H^{w_{\theta}}(u, \bar{u}) = H^{w_{\theta}}_0 + P^{w_{\theta}}(u, \bar{u}), \]

where $H^{w_{\theta}}_0 := \sum_{j \in Z^*} \omega_j^{w_{\theta}} |u_j|^2$.

**Theorem 3.** Suppose that equation (3.8) satisfies the following assumptions:

$\mathcal{A}_\theta : \omega^{w_{\theta}} := (\omega_j^{w_{\theta}})_{j \in Z^*}$, $\omega_j^{w_{\theta}} \in \mathbb{R}$. $\omega^{w_{\theta}}$ satisfies strong non resonant condition.$^1$

$\mathcal{B}_\theta : P^{w_{\theta}}(u, \bar{u})$ is a power series beginning with at least at order 2 in $(u, \bar{u})$ and has $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C_\theta > 0$ ($\beta$ is big enough positive number).

Given integer $r_*> 1$, there exists an integer $p_{r_*} > 0$, for any $p$ fulfilling $(\beta - 4)/2 > p > p_{r_*}$, there exists $\varepsilon_{r_*, p} > 0$ such that the solution to (3.8) satisfies

\[ \| (u(t), \bar{u}(t)) \|_p < 2\varepsilon, \quad \text{for any} \quad |t| \leq \varepsilon^{-r_*-1}, \]

if the initial data fulfills

\[ \| (u(0), \bar{u}(0)) \|_p < \varepsilon < \varepsilon_{r_*, p}. \]

Let us give the basic procedure of proving Theorem 3 which consists of the following steps.

The first step is to construct a bounded symplectic transformation around the origin under which the nonlinearity of Hamiltonian function (3.9) becomes into the sum of the following three parts: one is a high order ($\theta, \gamma, \alpha, N$)-normal form $Z^{(r_*, w_{\theta})}(u, \bar{u})$; one of the others is $R^{T(r_*, w_{\theta})}(u, \bar{u})$ which vanishes at $r_* + 3$ order of $(u, \bar{u})$ at origin and the last one denoted as $R^{N(r_*, w_{\theta})}(u, \bar{u})$ has zero at least order 3 about high index variable $(u_j, \bar{u}_j)_{|j| > N}$ ($N$ is big enough). Moreover, when $P^{w_{\theta}}(u, \bar{u})$ in (3.9) has $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C_\theta > 0$, the new Hamiltonian function is still of $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C(\theta, r) > 0$ ($C(\theta, r)$ is defined in Theorem 4). In order to guarantee the boundedness of the symplectic transformation, a strong non-resonant condition is presented in Definition 5.1. See Theorem 4 in section 5 for details.

Since (3.8) is a Hamiltonian system, the following equation holds true

\[ \| u(t) \|_p^2 - \| u(0) \|_p^2 = \int_0^t \frac{d}{d\tau} \| u(\tau) \|_p^2 d\tau = \int_0^t \{ \| u \|_p^2, H^{w_{\theta}}(u, \bar{u}) \}_w \|_w d\tau. \]

$^1$See Definition 5.1 in section 5.

$^2$See Definition 5.1.
Researching the Possion bracket of Hamiltonian function $H^w(u, \bar{u})$ and $\|u\|_p^2$ under the corresponding symplectic form $w_\theta$ is important. The second step is to estimate the Possion bracket of function $f_w^\theta(u, \bar{u})$ and $\|u\|_p^2$. Suppose that $f_w^\theta(u, \bar{u})$ has $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C_{\theta} > 0$. If the momentum of $f_w^\theta(u, \bar{u})$ are bounded, partial result can be found in [37] and [38]. When the set of the momen-
tric coefficients semi-bounded by $C$ Hamiltonian function $H$ is small under $H^{p-1}$ norm but not $H^p$ norm (see Proposition 4.1 and Remark 4.1). In order to deal with it, I will make use of the Hamiltonian structure and $(\beta, \theta$)-type symmetric coefficients semi-bounded by $C_{\theta} > 0$ to get the estimate of Possion bracket of Hamiltonian function $H^w(u, \bar{u})$ and $\|u\|_p^2$ under the corresponding symplectic form $w_\theta$. See Proposition 4.1 and Corollary 4.1 for details. By Proposition 4.1 and Corollary 4.1 $\{|\mathcal{R}(r, w_p)(u, \bar{u}) + \mathcal{R}^N(r, w_p)(u, \bar{u}), \|u\|_{p, w_\theta}^2 \prec R^{r+1} \}$ holds true for $\|(u, \bar{u})\|_p \leq R$. From Remark 4.1 $(\theta, \gamma, \alpha, N)$-normal form $Z(r, w_p)(u, \bar{u})$ is not a standard Birkhoff normal form. By Lemma 4.1 when $Z(r, w_p)(u, \bar{u})$ has $(\beta, \theta$)-type symmetric coefficients semi-bounded by $C(\theta, r_\alpha)$, for any $\|(u, \bar{u})\|_p < R \ll 1$ and $N$ satisfying [4.28], $|\{Z(r, w_p)(u, \bar{u}), \|u\|_{p}^2\}| \prec R^{r+1}$.

4 Estimate $\{f_r^{w_\theta}(u, \bar{u}), \|u\|_{p}^2\}_{w_\theta}$ and Hamiltonian vector field $X_{f_r^{w_\theta}}(f_r^{w_\theta}(u, \bar{u})$ has $(\beta, \theta$)-type symmetric coefficients $\theta \in \{0, 1\}$

Suppose that an $r$-degree homogeneous power series $f_r^{w_\theta}(u, \bar{u})$ defined on $\mathcal{H}^p(\mathbb{Z}^*, \mathbb{C})$ is of $(\beta, \theta$)-type symmetric coefficients semi-bounded by $C_{\theta} > 0$.

First of all I present that the Hamiltonian vector field of $f_r^{w_\theta}(u, \bar{u})$ under symplectic form $w_\theta$ is unbounded with order 1. See Proposition 4.1.

Next, the estimate of the possion bracket of the power series $f_r^{w_\theta}(u, \bar{u})$ and $\|u\|_p^2$ is given in Proposition 4.1.

Last but not least, I introduce truncated operators $\Gamma^N_{<2}$ and $\Gamma^N_{>2}$, and estimate the Hamiltonian vector fields of the functions $\Gamma^N_{<2}f_r^{w_\theta}(u, \bar{u}), \Gamma^N_{>2}f_r^{w_\theta}(u, \bar{u})$ in $\mathcal{H}^{p-1}(\mathbb{Z}^*, \mathbb{C})$-norm, $\{|\Gamma^N_{<2}f_r^{w_\theta}(u, \bar{u}), \|u\|_p^2\}$ and $\{|\Gamma^N_{>2}f_r^{w_\theta}(u, \bar{u}), \|u\|_p^2\}$ for any $(u, \bar{u}) \in \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C})$ in Corollary 4.1.

These results will be used in proving Theorem 3.

Proposition 4.1. Suppose that an $r$-degree $(r \geq 3)$ homogeneous polynomial $f_r^{w_\theta}(u, \bar{u})$ $(\theta \in \{0, 1\})$ defined on $\mathcal{H}^p(\mathbb{Z}^*, \mathbb{C})$ has $(\beta, \theta$)-type symmetric coefficients semi-bounded by $C_{f_r^{w_\theta}} > 0$, and $\beta - p \geq 2$. Then for any $(u, \bar{u}) \in \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C})$ and any $\theta \in \{0, 1\}$,

$$\|X_{f_r^{w_\theta}}(u, \bar{u})\|_{p-1} \leq 16C_{f_r^{w_\theta}}^{-2}r^{p+1}c^{-1}\|u\|_{2}^{-2}\|u\|_{p}.$$  \hspace{1cm} (4.1)

If an $r$-degree homogeneous polynomial $f_r(u, \bar{u})$ has $\beta$-bounded symmetric coefficients bounded by $C_f > 0$, then the following inequality holds true for any $(u, \bar{u}) \in \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C})$ and any $\theta \in \{0, 1\}$

$$\|X_{f_r^{w_\theta}}(u, \bar{u})\|_{p} \leq 16C_{f_r}^{-2}r^{p+1}c^{-1}\|u\|_{2}^{-2}\|u\|_{p}.$$ \hspace{1cm} (4.2)

Remark 4.1. If an $r$-degree homogeneous polynomial $f_r^{w_\theta}(u, \bar{u}) : B_\theta(R_\star) \to \mathbb{C}$ $(\theta \in \{0, 1\}, R_\star > 0)$ has $(\beta, \theta$)-type symmetric coefficients semi-bounded by $C_{f_r^{w_\theta}} > 0$ $(\theta \in \{0, 1\})$, from Proposition 4.1 it holds that
• The Hamiltonian vector field $X^{u_0}_{f_{u_0}}$ is from $B_p(R_\ast)$ to $\mathcal{H}^{p-1}(Z^*, \mathbb{C})$, but not to $\mathcal{H}^p(Z^*, \mathbb{C})$. It means that $X^{u_0}_{f_{u_0}}$ is unbounded with order 1.

• There exists a constant $\delta \in (0, 1)$ such that

$$|f_{r}^{u_0}(u, \bar{u})| = |\langle \nabla_{(u_0, 0)} f_{r}^{u_0}(\delta u, \delta \bar{u}), (u, \bar{u}) \rangle|.$$  \hspace{1cm} (4.3)

Together with Cauchy estimate and (4.2), one has that the function $f_{r}^{u_0}(u, \bar{u})$ ($\theta \in \{0, 1\}$) is analytic about $(u, \bar{u})$ on some $B_p(R) \subset \mathcal{H}^p(Z^*, \mathbb{C})$ ($p > 1$). (In this paper, when I mention the “analyticity” of functions or vector fields, I take $u$ and $\bar{u}$ as independent variables).

**Proposition 4.2.** Suppose that an $r$-degree ($r \geq 3$) homogeneous polynomial $f_{r}^{u_0}(u, \bar{u})$ ($\theta \in \{0, 1\}$) has $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C_{f_{r}^{u_0}} > 0$. Then the following inequality holds true for any $(u, \bar{u}) \in \mathcal{H}^p(Z^*, \mathbb{C})$ ($p \geq 2$, $\beta - p \geq 2$)

$$|\{f_{r}^{u_0}(u, \bar{u}), \|u\|_{\mathcal{H}^p}^2\}| \leq C_{f_{r}^{u_0}}^{r-2}2^{r+1}p^{p-1}e^{-1}\|u\|_{\mathcal{H}^p}^2\|u\|_{\mathcal{H}^p}^2.$$  \hspace{1cm} (4.4)

Given an integer $N > 0$, two projection operators $\Gamma_{\geq N}$ and $\Gamma_{\leq N}$ on $\ell_2^Z(Z^*, \mathbb{C})$ are defined as follows. For any $l = (l_j)_{j \in \mathbb{Z}} \in \ell_2^Z$, $\Gamma_{\geq N} l$ and $\Gamma_{\leq N} l \in \ell_2^Z$ with

$$(\Gamma_{\geq N} l)_j := \begin{cases} l_j, & |j| > N, \\ 0, & |j| \leq N, \end{cases}$$  \hspace{1cm} (4.4)

For any $u = (u_j)_{j \in \mathbb{Z}} \in \ell_2^Z(Z^*, \mathbb{C})$, $\Gamma_{\geq N} u$, $\Gamma_{\leq N} u \in \ell_2^Z(Z^*, \mathbb{C})$ with

$$(\Gamma_{\geq N} u)_j := \begin{cases} u_j, & |j| > N, \\ 0, & |j| \leq N. \end{cases}$$

Now I will introduce two truncated operators $\Gamma_{\leq 2}$ and $\Gamma_{\geq 2}$ defined as follows. For any power series

$$f(u, \bar{u}) := \sum_{r \geq 3} \sum_{|l+k|=r, M(l+k)=i} f_{r,l,k}^i u^l \bar{u}^k$$

denote

$$\Gamma_{\leq 2}^N f(u, \bar{u}) := \sum_{r \geq 3} \sum_{|l+k|=r, M(l+k)=i, |\Gamma_{\geq N}(l+k)| \leq 2, |i| \leq N} f_{r,l,k}^i u^l \bar{u}^k,$$  \hspace{1cm} (4.4)

$$\Gamma_{\geq 2}^N f(u, \bar{u}) := f(u, \bar{u}) - \Gamma_{\leq 2}^N f(u, \bar{u}).$$  \hspace{1cm} (4.5)

**Remark 4.2.** Fix a positive integer $N$. Suppose that a power series $f_{r}^{u_0}(u, \bar{u})$ has $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C_{f_{r}^{u_0}} > 0$ ($\theta \in \{0, 1\}$). Then $\Gamma_{\leq 2}^N f_{r}^{u_0}(u, \bar{u})$, $\Gamma_{\geq 2}^N f_{r}^{u_0}(u, \bar{u})$ also have $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C_{f_{r}^{u_0}} > 0$.

**Corollary 1.** Suppose that an $r$-degree ($r \geq 3$) homogeneous polynomials $f_{r}^{u_0}(u, \bar{u})$ ($\theta \in \{0, 1\}$) defined on $\mathcal{H}^p(Z^*, \mathbb{C})$ has $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C_{f_{r}^{u_0}} > 0$, and $\beta - p \geq 2$. Given an integer $N > 0$, then

$$\|X_{f_{r}^{u_0}}^{u_0}(u, \bar{u})\|_{\mathcal{H}^{p-1}} \leq 16C_{f_{r}^{u_0}}^{r-2}r^{p+1}e^{-1}\|u\|_{\mathcal{H}^p}^2\|u\|_{\mathcal{H}^p}^2,$$
\[ \| X_{t\geq N}^{f_{w^0}}(u, \bar{u}) \|_{p-1} \leq 16C_{f_{w^0}}^{r-2}r^{p+1}c^{r-1-\|u\|_2^2} + 16N^{-(\beta-\theta-2)}, \]
\[ \| \Gamma_N^{f_{w^0}}(u, \bar{u}), \| u \|_{p}^2 \omega \| \leq C_{f_{w^0}}^{-2}p^{p+1}c^{r-1} \| u \|_{p}^2, \]
\[ \| \{\Gamma_N^{f_{w^0}}(u, \bar{u}), \| u \|_{p}^2 \omega \} \leq C_{f_{w^0}}^{-2}p^{p+1}c^{r-1} \| u \|_{p}^2. \] (4.6)

The proof of Corollary 4.5 is similar with Proposition 4.1-4.2 and I omit it. In order to give the proof of Proposition 4.1-4.2 the following Lemmas are needed.

**Lemma 4.1.** If power series \( f_{w^0}(u, \bar{u}) \) and \( g_{w^0}(u, \bar{u}) \) have \((\beta, \theta)\)-type symmetric coefficients \((\theta \in \{0, 1\})\) semi-bounded by \( C_{f_{w^0}} > 0 \) and \( C_{g_{w^0}} > 0 \), respectively, then for any \( a, b \in \mathbb{R} \), \((af_{w^0} + bg_{w^0})(u, \bar{u})\) also has \((\beta, \theta)\)-type symmetric coefficients semi-bounded by

\[ C_{af_{w^0}+bg_{w^0}} := \max\{|a|C_{f_{w^0}} + |b|C_{g_{w^0}}, \quad |a|C_{f_{w^0}} + C_{g_{w^0}}, \quad C_{f_{w^0}} + |b|C_{g_{w^0}}, \quad C_{f_{w^0}} + C_{g_{w^0}}\} > 0. \] (4.7)

**Proof.** I only give the proof in the case \( \theta = 0 \), while in the case \( \theta = 1 \) the proof is similar to the case \( \theta = 0 \). For any \( a, b \in \mathbb{R} \),

\[ (af_{w^0} + bg_{w^0})(u, \bar{u}) = \sum_{t \geq 3} \sum_{M(l, k) = i \in M(af_{w^0} + bg_{w^0})} (af_{w^0} + bg_{w^0})_{t,lk}^i u^l \bar{u}^k, \] (4.8)

where \( M(af_{w^0} + bg_{w^0}) := M_{f_{w^0}} \cup M_{g_{w^0}} \). For any \( l, k \in \mathbb{N}^+ \) with \( M(l, k) = i \in M(af_{w^0} + bg_{w^0})_t \), the corresponding coefficient of \( af_{w^0} + bg_{w^0} \) has the following form

\[ (af_{w^0} + bg_{w^0})_{t,lk}^i = \sum_{(l^0, k^0, i^0) \in A_{(af_{w^0} + bg_{w^0})_{t,lk}}} (af_{w^0} + bg_{w^0})_{l^0, k^0, i^0}^i (\mathcal{M}(l^0, k^0) - i^0), \]

where

\[ A_{(af_{w^0} + bg_{w^0})_{t,lk}} := A_{(af_{w^0})_{l,tk}} \cup A_{(bg_{w^0})_{l,tk}}, \]

and

\[ (af_{w^0} + bg_{w^0})_{l^0, k^0, i^0}^i := \begin{cases} af_{w^0}^i(l^0, k^0, i^0), & (l^0, k^0, i^0) \in A_{(af_{w^0})_{t,lk}} \cap A_{(g_{w^0})_{t,lk}}; \\
bg_{w^0}^i(l^0, k^0, i^0), & (l^0, k^0, i^0) \in A_{(f_{w^0})_{t,lk}} \cap A_{(g_{w^0})_{t,lk}}; \\
af_{w^0}^i(l^0, k^0, i^0) + bg_{w^0}^i(l^0, k^0, i^0), & (l^0, k^0, i^0) \in A_{(f_{w^0})_{t,lk}} \cap A_{(g_{w^0})_{t,lk}}. \end{cases} \]

It is easy to check that

\[ (af_{w^0} + bg_{w^0})_{l,tk}^i = (af_{w^0} + bg_{w^0})_{l,tk}^{i - (k - l) - l^0, i^0 - 2i}, \]

and

\[ \sum_{(l^0, k^0, i^0) \in A_{(af_{w^0} + bg_{w^0})_{t,lk}}} |(af_{w^0} + bg_{w^0})_{l^0, k^0, i^0}^i| \cdot \max \{i^0, \ i^0 - 2i\} \]

\[ \leq \sum_{(l^0, k^0, i^0) \in A_{(f_{w^0})_{t,lk}}} |a(f_{w^0})_{l^0, k^0, i^0}^i| \cdot \max \{i^0, \ i^0 - 2i\} \]

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Given real numbers $a$ and $b$, where $a, b > 0$, then

\[ \sum_{(l, k, m) \in A} |b(g_{l, k, m})| \leq \frac{(C_{a} + b)g_{0}^{\frac{t-2}{t}}}{{(t)^{\beta}}} \]

where $C_{a}g_{0}^{t}$ is defined in (4.7).

**Lemma 4.2.** Given real numbers $q_{i} \leq p$ ($1 \leq i \leq r$), suppose that $F = (F_{j})_{j \in \mathbb{Z}^{*}}$ is an $r$-multiple linear vector field defined as following

\[ F_{j}(u^{(1)}, \ldots, u^{(r)}) := \sum_{j=j_{(1)} \pm j_{(2)} \pm \cdots \pm j_{(r)}} F_{j_{(1)}j_{(2)} \cdots j_{(r)}} u^{(1)}_{j_{(1)}} \cdots u^{(r)}_{j_{(r)}}, \quad j \in \mathbb{Z}^{*} \]

where $u^{(1)} := (u^{(1)})_{j_{(1)} \in \mathbb{Z}^{*}}$, $\ldots$, $u^{(r)} := (u^{(r)})_{j_{(r)} \in \mathbb{Z}^{*}} \in \ell_{p}^{r}(\mathbb{Z}^{*}, \mathbb{C})$. If there exist a positive constant $C$ and an integer $n \in \{1, \ldots, r\}$ such that

\[ |F_{j_{(1)} \cdots j_{(r)}}| \leq C \langle j_{(n)} \rangle^{q_{n}} \cdot \prod_{t=1, t \neq n}^{r} \langle j_{(t)} \rangle^{q_{t}-1}, \quad (4.9) \]

then

\[ \|F(u^{(1)}, \ldots, u^{(r)})\|_{\ell^{2}(\mathbb{Z}^{*}, \mathbb{C})} \leq C c^{r-1}\|u^{(n)}\|_{q_{n}} \cdot \prod_{i=1, i \neq n}^{r} \|u^{(i)}\|_{q_{i}}, \]

where $c := \sqrt{\sum_{j \in \mathbb{Z}} \langle j \rangle^{-2}}$.

**Proof.** Using Young’s inequality the following inequality holds true for any $a \in \ell^{2}(\mathbb{Z}^{*}, \mathbb{C})$ and $b \in \ell^{1}(\mathbb{Z}^{*}, \mathbb{C})$

\[ \left\| \left( \sum_{k \in \mathbb{Z}^{*}} a_{j} b_{k} \right)_{j \in \mathbb{Z}^{*}} \right\|_{\ell^{2}(\mathbb{Z}^{*}, \mathbb{C})} \leq \|a\|_{\ell^{2}(\mathbb{Z}^{*}, \mathbb{C})} \cdot \|b\|_{\ell^{1}(\mathbb{Z}^{*}, \mathbb{C})}. \quad (4.10) \]

Together with (4.9), using (4.10) repeatedly, one has

\[ \|F(u^{(1)}, \ldots, u^{(r)})\|_{\ell^{2}(\mathbb{Z}^{*}, \mathbb{C})} \leq C \left( \sum_{j_{(1)} \pm j_{(2)} \pm \cdots \pm j_{(r)}} \langle j_{(1)} \rangle^{q_{1}} \cdots \langle j_{(n-1)} \rangle^{q_{n-1}} \langle j_{(n)} \rangle^{q_{n}} \langle j_{(n+1)} \rangle^{q_{n+1}} \cdots \langle j_{(r)} \rangle^{q_{r}} \right)^{\frac{1}{q_{n}}} \cdot \|u^{(n)}\|_{q_{n}} \cdot \prod_{1 \leq t \leq r, t \neq n} \|u^{(t)}\|_{q_{t}} \leq C \|u^{(n)}\|_{q_{n}} \cdot \prod_{1 \leq t \leq r, t \neq n} \|u^{(t)}\|_{q_{t}} \|u^{(n)}\|_{q_{n}} \cdot \prod_{1 \leq t \leq r, t \neq n} \|u^{(t)}\|_{q_{t}}. \quad (4.11) \]

Since $q_{i} \leq p$, the following inequality holds true for any $u := (u_{j})_{j \in \mathbb{Z}^{*}} \in \ell_{p}^{r}(\mathbb{Z}^{*}, \mathbb{C})$

\[ \|u\|_{q_{n}} = \sum_{j \in \mathbb{Z}^{*}} \langle j \rangle^{q_{n}} |u_{j}| \cdot \frac{1}{\langle j \rangle} \leq \sum_{j \in \mathbb{Z}} \langle j \rangle^{q_{n}} \cdot |u_{j}| \leq c \|u\|_{q_{n}}. \quad (4.12) \]

\[ ^{5} \text{Suppose } a \in \ell^{p}(\mathbb{Z}^{*}, \mathbb{C}), b \in \ell^{q}(\mathbb{Z}^{*}, \mathbb{C}) \text{ and } \frac{1}{p} + \frac{1}{q} = \frac{1}{2} + 1, \text{ with } 1 \leq p, q, r \leq \infty. \text{ Then } \|f \ast g\|_{\ell^{r}(\mathbb{Z}^{*}, \mathbb{C})} \leq \|f\|_{\ell^{p}(\mathbb{Z}^{*}, \mathbb{C})} \cdot \|g\|_{\ell^{q}(\mathbb{Z}^{*}, \mathbb{C})}. \]
In view of (4.12) and (4.11), one has
\[ \|F(u^{(1)}, \ldots, u^{(r)})\|_{\ell^2(\mathbb{Z}^*, \mathbb{C})} \leq C c^{r-1} \|u^{(n)}\|_{q_n} \cdot \prod_{i=1, i \neq n}^{r} \|u^{(i)}\|_{q_i}. \]

\square

**Corollary 2.** Given integers \( p > q \geq 1 \) and real number \( \rho \geq 2 \), suppose that there exists a positive number \( C_f > 0 \) such that the coefficients of an \( r \)-degree homogeneous polynomial
\[ f(u, \bar{u}) = \sum_{l, k \in \mathbb{N}^*} f_{r, kl} u^l \bar{u}^k \]
satisfy that for any \( l, k \in \mathbb{N}^* \) with \( \mathcal{M}(l, k) = i \in M_f, \)
\[ |f_{r, kl}^i| \leq \frac{C_f}{(i)^p} (l + e_j)(p-q+1) \sum_{t \in \mathbb{Z}^*} (l + k - e_j) t (p-q+1) \prod_{m \in \mathbb{Z}^*} \langle m \rangle^{(q-1)(l_m + k_m)}. \quad (4.13) \]
Then for any \((u, \bar{u}) \in \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C})\), it satisfies that
\[ |f(u, \bar{u})| \leq C_f c^{r-1} r \|u\|_p^2 \|u\|_q^{r-2}. \]

**Remark 4.3.** The result of Corollary 2 still holds true for \( \rho > 1 \). To simplify the process of proof, assume \( \rho \geq 2 \).

**Proof.** By Cauchy estimate,
\[ |f(u, \bar{u})| \leq |\langle F, G \rangle| \leq \|F\|_{\ell^2} \cdot \|G\|_{\ell^2}, \quad (4.14) \]
where \( G := ((j)^p |u_j|)_{j \in \mathbb{Z}^*} \) and \( F(u, \bar{u}) = (F_j(u, \bar{u}))_{j \in \mathbb{Z}^*} \)
\[ F_j(u, \bar{u}) := \sum_{|l-e_j+k|=r-1, \atop \mathcal{M}(l, k)=i \in M_f} f_{r, kl}^i \langle j \rangle^p (l - e_j) \bar{u}_j^k. \]
For any \( n \in \{0, 1, \ldots, r-1\} \), there exist an \((r-1)\)-multiple linear vector field
\[ \tilde{F}^n(u^{(1)}, \ldots, u^{(r-1)}) := \left( \sum_{|l-e_j+n|=r-1-n, \atop i \in M_f} f_{r, kl}^i \langle j \rangle^p u^{(1)} \cdots u^{(n)} \bar{u}^{(n+1)} \cdots \bar{u}^{(r-1)} \right)_{j \in \mathbb{Z}^*} \]
such that
\[ F_j(u, \bar{u}) = \sum_{n=0}^{r-1} \tilde{F}^n_j(u, \ldots, u, \bar{u}, \ldots, \bar{u}), \quad \text{for any } j \in \mathbb{Z}^*. \]
By condition (4.13), the coefficients of each \((r-1)\)-multiple linear vector fields \( \tilde{F}^n \) satisfy the condition (4.9) of Lemma 4.2 with \( q_n = p \) and \( q_i = q \leq p \) \((i \neq n)\). Hence, by Lemma 4.2 for any \((u, \bar{u}) \in \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C})\), one has
\[ \|F(u, \bar{u})\|_{\ell^2} \leq \sum_{n=0}^{r-1} \|\tilde{F}^n(u, \ldots, u, \bar{u}, \ldots, \bar{u})\|_{\ell^2(\mathbb{Z}^*, \mathbb{C})}. \]
\[
\begin{align*}
&\leq C_f c^{-2} r \|u\|_p \cdot \|u\|_{q}^{r-2} \sum_{i \in M, r \subseteq Z} \langle i \rangle^{-\rho} \\
&\leq C_f c^{-1} r \|u\|_p \cdot \|u\|_{q}^{r-2}.
\end{align*}
\]

In view of (4.14) and (4.15), the following inequality holds true

\[
|f(u, \bar{u})| \leq \|F(u, \bar{u})\|_{\ell^2} \cdot \|G(u, \bar{u})\|_{\ell^2} \\
\leq \|F(u, \bar{u})\|_{\ell^2} \cdot \|u\|_p \leq C_f c^{-1} r \|u\|_p \cdot \|u\|_{q}^{r-2}.
\]

\[
\square
\]

Now give the proof of Proposition 4.1.

**Proof.** In the case \(\theta = 0\),

\[
\|X^{w_0}_{r, t}(u, \bar{u})\|_{\ell^1} = \sqrt{\|\nabla f^{w_0}_{r, t}(u, \bar{u})\|_{p-1}^2 + \|\nabla f^{w_0}_{r, t}(u, \bar{u})\|_{q-1}^2}.
\]

Note that \(\|\nabla f^{w_0}_{r, t}(u, \bar{u})\|_{p-1}\) equals to equals to \(\ell^2(\mathbb{Z}^*, \mathbb{C})\) norm of the following vector field

\[
\left( \sum_{j = M(l,k,e_j)-i} \sum_{l, k \in \mathbb{N}^Z, i \in M_{f, w_0} \subseteq Z} k_j(j)^{p-1} \cdot (f^{w_0}_{r, t})(j, k) \cdot (M(l, k, e_j)) \right) \cdot \mathcal{M}(l, k, e_j). \quad (4.17)
\]

Accordingly, \(\|\nabla f^{w_0}_{r, t}(u, \bar{u})\|_{p-1}\) equals to equals to \(\ell^2(\mathbb{Z}^*, \mathbb{C})\) norm of the following vector field

\[
\left( \sum_{j = M(l,k,e_j)-i} \sum_{l, k \in \mathbb{N}^Z, i \in M_{f, w_0} \subseteq Z} l_j(j)^{p-1} \cdot (f^{w_0}_{r, t})(j, k) \cdot (M(l, k, e_j)) \right) \cdot \mathcal{M}(l, k, e_j). \quad (4.18)
\]

In order to use Lemma 4.2 to give the \(\ell^2\)-norm of (4.17) and (4.18), it is required to estimate the coefficients of vector field (4.17) and (4.18). Note that for any fixed \(l, k \in \mathbb{N}^Z\) and \(j \in \mathbb{Z}^*\) with \((f^{w_0}_{r, t})(j, k) \neq 0\) and \(k_j \neq 0\), the indices satisfy

\[
\text{sgn}(k_j) \cdot j = M(l, k, e_j) - i. \quad (4.19)
\]

Since \(p > 2\), it follows

\[
\begin{align*}
|j|^{p-1} &\leq r^{p-1} \left( \sum_{t \in \mathbb{Z}^*} |l_t|^p + \sum_{t \in \mathbb{Z}^*, t \neq j} |k_t|^p + (k_j - 1) \cdot |j|^{p-1} + |i|^{p-1} \right) \\
&\leq 2r^{p-1} \left( \sum_{t \in \mathbb{Z}^*} |l_t|^p + \sum_{t \in \mathbb{Z}^*, t \neq j} |k_t|^p + (k_j - 1) \cdot |j|^{p-1} \cdot (i)^{p-1} \right), \quad (4.20)
\end{align*}
\]

the last inequality holds true by the fact that for any integer \(a, b \geq 1\),

\[
a + b \leq ab + 1 \leq 2ab. \quad (4.21)
\]

Furthermore, for any \((l, k, 0, 0) \in A_{(f^{w_0}_{r, t})}, i \), it holds that

\[
0 \leq l_j^{(l, k, 0, 0)} \leq l_j, \quad 0 \leq k_j^{(l, k, 0, 0)} \leq k_j, \quad \text{for any } j \in \mathbb{Z}^* \quad (4.22)
\]
and the momentum of \((t^0, k^0)\) equals to

\[
\mathcal{M}(t^0, k^0) = \begin{cases} 
\mathcal{M}(t^0, k^0 - \text{sgn}(k^0) \epsilon_j) - \text{sgn}(k^0) \cdot j, & k^0_j \neq 0, \\
\mathcal{M}(t^0, k^0), & k^0_j = 0.
\end{cases}
\tag{4.23}
\]

Together with (4.19), (4.21), (4.22) and (4.23), one has

\[
|\mathcal{M}(t^0, k^0) - \frac{t^0_j}{2}| \leq 4 \left( \sum_{n \in \mathbb{Z}^*} l_n|n| + \sum_{n \in \mathbb{Z}^* \atop n \neq j} k_n|n| + (k_j - 1)|j| \right) \cdot \max\{\langle i^0 \rangle, \langle i - i^0 \rangle \}. \tag{4.24}
\]

In view of (4.19), (4.20), (4.24) and \(f^{u_0}(u, \bar{u})\) having the \((\beta, 0)\)-type symmetric coefficients semi-bounded by \(C_{f^{u_0}} > 0\), one has

\[
\langle j \rangle^{p-1} \cdot \sum_{(l^0, k^0, i^0) \in \mathcal{A}(f^{u_0})_{r, l, k}} k_j (f^{u_0})_{l, j, k} \left( \mathcal{M}(t^0, k^0) - \frac{t^0_j}{2} \right) \leq \frac{8C_{r-2}^p}{\langle j \rangle^{d-p+1}} \left( \sum_{m \in \mathbb{Z}^*} \langle m \rangle \text{sgn}(l_m + (k - \epsilon_j)m) \prod_{t \in \mathbb{Z}^*} \langle t \rangle^{(p-1)\text{sgn}(l_t + (k - \epsilon_j)t)} \right). \tag{4.25}
\]

Therefore, using (4.25) and Lemma 4.2 by taking \(q_n = p\) and \(q_t = 2\) \((i \neq n)\), the following inequality holds true

\[
\left\| \nabla_u f^{u_0}_r (u, \bar{u}) \right\|_{p-1} \leq 8r^{p+1} C_{f^{u_0}} \left\| u \right\|_2^{p-2} \left\| u \right\|_p.
\]

Similarly, one has

\[
\left\| \nabla_{u} f^{u_0}_r (u, \bar{u}) \right\|_{p-1} \leq 8r^{p+1} C_{f^{u_0}} \left\| u \right\|_2^{p-2} \left\| u \right\|_p.
\]

Thus,

\[
\left\| X^{u_0}_r (u, \bar{u}) \right\|_{p-1} \leq 16r^{p+1} C_{f^{u_0}} \left\| u \right\|_2^{p-2} \left\| u \right\|_p.
\]

In the case \(\theta = 1\),

\[
\left\| X^{w_1}_r (u, \bar{u}) \right\|_{p-1} = \sqrt{\left\| \nabla_u f^{w_1}_r (u, \bar{u}) \right\|_{p-1}^2 + \left\| \nabla_u f^{w_1}_r (u, \bar{u}) \right\|_{p-1}^2}. \tag{4.26}
\]

Similarly, \(\left\| \nabla_u f^{w_1}_r (u, \bar{u}) \right\|_{p-1}\) and \(\left\| \nabla_u f^{w_1}_r (u, \bar{u}) \right\|_{p-1}\) equal to \(\ell^2(\mathbb{Z}^*, \mathbb{C})\)-norm of the following vector fields respectively

\[
(\langle k_j \cdot \langle j \rangle^{p-1} \langle j \rangle \rangle)^{\frac{1}{2}} \sum_{j = M(l, k - \epsilon_j - -i) \atop l, k \in \mathbb{N}^*} \langle l \rangle^{\frac{1}{2}} \sum_{n \in \mathbb{Z}^* \atop n \neq j} \langle n \rangle^{\frac{1}{2}} u^j_n u^{k-\epsilon_j} \langle j \rangle \in \mathbb{Z}^*, \tag{4.27}
\]

and

\[
(\langle l_j \cdot \langle j \rangle^{p-1} \langle j \rangle \rangle)^{\frac{1}{2}} \sum_{j = 1 - M(l, \epsilon_j, k) \atop l, k \in \mathbb{N}^*} \langle l \rangle^{\frac{1}{2}} \sum_{n \in \mathbb{Z}^* \atop n \neq j} \langle n \rangle^{\frac{1}{2}} u^{l-\epsilon_j} u^{k} \langle j \rangle \in \mathbb{Z}^*. \tag{4.28}
\]
For any fixed \( l, k \in \mathbb{N}^* \) and \( j \in \mathbb{Z}^* \) with \((f^{u_1})^i_{r,lk} \neq 0\). \( \mathcal{M}(l, k) = i \) and \( k_j \neq 0 \), (4.19) still holds true. By (4.19) and (4.20), the coefficients of the vector field in (4.27) are bounded by

\[
\langle j \rangle^{p-\frac{1}{2}} \cdot |k_j(f^{u_1})^i_{r,lk}| \prod_{n \in \mathbb{Z}^*} \langle n \rangle^{\frac{1}{2}l_n} \prod_{n \in \mathbb{Z}^*} \langle n \rangle^{\frac{1}{2}k_n}
\]

\[
\leq 2C^{r-2}_{r-1}p^1 \sum_{m \in \mathbb{Z}^*} l_m(m)^{p-\frac{1}{2}} + \sum_{m \in \mathbb{Z}^*} k_m(m)^{p-\frac{1}{2}} + (k_j - 1) \langle j \rangle^{p-\frac{1}{2}} \prod_{t \in \mathbb{Z}^*} \langle t \rangle^{\frac{1}{2}(l_i + (k-e_j))}.
\]

(4.29)

Using Lemma (4.2) and (4.29), one has

\[
\| \nabla_{\tilde{u}} f^{u_1}_r (u, \tilde{u}) \|_{p-1} \leq 8C^{r-2}_{r-1}p^1c^{-1-\theta} \|u\|_{2}^{r-2} \|u\|_{p}.
\]

Similarly,

\[
\| \nabla_{u} f^{u_1}_r (u, \tilde{u}) \|_{p-1} \leq 8C^{r-2}_{r-1}p^1c^{-1} \|u\|_{2}^{r-2} \|u\|_{p}.
\]

Thus,

\[
\| X^{u_1}_r (u, \tilde{u}) \|_{p-1} \leq 16r^{1+1}c^{-1}C^{r-2}_{r-1} \|u\|_{2}^{r-2} \|u\|_{p}.
\]

By the same approach, when \( f \) has \( \beta \)-bounded symmetric coefficients bounded by \( C_f > 0 \), one has

\[
\| X^{u_2}_r \|_{p} \leq 16C^{r-2}_{r-1}p^1c^{-1} \|u\|_{2}^{r-2} \|u\|_{p}.
\]

Next the proof of Proposition 4.2 is given.

**Proof.** Step 1: (delete unbounded part)

In the case \( \theta = 0 \), since \( f^{u_0}_r (u, \tilde{u}) \) has \( (\beta, 0) \)-type symmetric coefficients, assume that \( f^{u_0}_r (u, \tilde{u}) \) has the following form

\[
f^{u_0}_r (u, \tilde{u}) = \sum_{\mathcal{M}(l, k)=i \in M} \sum_{f^{u_0} \subseteq \mathbb{Z}^*} (f^{u_0})^{i}_{r,lk} \left( \mathcal{M} (l, k) - \frac{i^0}{2} \right) u^l \tilde{u}^k.
\]

Under the definition of Possion bracket, it holds that

\[
\{ f^{u_0}_r (u, \tilde{u}), u \}_p^2 \sum_{j \in \mathbb{Z}^*} \frac{\partial f^{u_0}_r}{\partial u_j} i \frac{\partial u}{\partial u_j} - \sum_{j \in \mathbb{Z}^*} \frac{\partial f^{u_0}_r}{\partial u_j} i \frac{\partial u}{\partial u_j} = i \sum_{\mathcal{M}(l, k)=i \in M} \sum_{f^{u_0} \subseteq \mathbb{Z}^*} (f^{u_0})^{i}_{r,lk} \left( \mathcal{M} (l, k) - \frac{i^0}{2} \right) u^l \tilde{u}^k.
\]

For the sake of convenience, rewrite \( \{ f^{u_0}_r (u, \tilde{u}), u \}_p^2 \sum_{j \in \mathbb{Z}^*} \frac{\partial f^{u_0}_r}{\partial u_j} i \frac{\partial u}{\partial u_j} \) as the sum of the following two parts

\[
O^+ (u, \tilde{u}) := i \sum_{\mathcal{M}(l, k)=i \in M} \sum_{f^{u_0} \subseteq \mathbb{Z}^*} \sum_{(l^0, k^0, \beta^0) \in A(f^{u_0})^{i}_{r,lk}} (f^{u_0})^{i}_{r,lk} \left( \mathcal{M} (l, k) - \frac{i^0}{2} \right) u^l \tilde{u}^k,
\]

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\[ O^-(u, \bar{u}) := \sum_{\substack{l+k=r \in i \in M_{\ell r} \subseteq Z}} (l_0^j - k_0^j)(j)^{2p} \sum_{(l^0, k^0, \ell^0) \in A_{\ell r}^{(l^0, k^0)}} (f^{w_0})_{r, \ell k}^{(l^0, k^0, \ell^0)} (M(l^0, k^0) - \frac{\ell^0}{2}) u^l \bar{u}^k. \]

Since the coefficients of \( f_r^{w_0}(u, \bar{u}) \) are \((\beta, 0)-\)type symmetric, take complex conjugation to \( O^-(u, \bar{u}) \) and obtain

\[
\begin{align*}
O^-(u, \bar{u}) &= i \sum_{\substack{l+k=r \in i \in M_{\ell r} \subseteq Z}} (l_0^j - k_0^j)(j)^{2p} \sum_{(l^0, k^0, \ell^0) \in A_{\ell r}^{(l^0, k^0)}} (f^{w_0})_{r, \ell k}^{(l^0, k^0, \ell^0)} (M(l^0, k^0) - \frac{\ell^0}{2}) u^l \bar{u}^k \\
&= i \sum_{\substack{-l+k=r \in i \in M_{\ell r} \subseteq Z}} (-l_0^j - k_0^j)(j)^{2p} \sum_{(l^0, k^0, \ell^0) \in A_{\ell r}^{(l^0, k^0)}} (f^{w_0})_{r, \ell k}^{(l^0, k^0, \ell^0)} (M(k - l^0, l - \ell^0, \ell^0 - 2\beta))(l^0 - k^0, l - l^0, \ell^0 - 2\beta)) u^l \bar{u}^k \\
&= O^+(u, \bar{u}).
\end{align*}
\]

Together with \((4.30)\), rewrite \( \{ f_r^{w_0}(u, \bar{u}), \|u\|^2 \}_w \) as the following

\[
\begin{align*}
\{ f_r^{w_0}(u, \bar{u}), \|u\|^2 \}_w &= 2\text{Re}O^+(u, \bar{u}) \\
&= A^0(u, \bar{u}) + A^1(u, \bar{u}) + A^2(u, \bar{u}),
\end{align*}
\]

where

\[
\begin{align*}
A^0(u, \bar{u}) := & -\text{Re} \sum_{\substack{l+k=r \in i \in M_{\ell r} \subseteq Z}} \text{i}(l_0^j - k_0^j, k_0^j)(j)^{2p} \sum_{(l^0, k^0, \ell^0) \in A_{\ell r}^{(l^0, k^0)}} (f^{w_0})_{r, \ell k}^{(l^0, k^0, \ell^0)} u^l \bar{u}^k; \quad (4.31) \\
A^1(u, \bar{u}) : &= 2\text{Re} \sum_{\substack{l+k=r \in i \in M_{\ell r} \subseteq Z}} \text{i}(l_0^j - k_0^j, k_0^j)(j)^p \\
&\quad \cdot \sum_{(l^0, k^0, \ell^0) \in A_{\ell r}^{(l^0, k^0)}} (f^{w_0})_{r, \ell k}^{(l^0, k^0, \ell^0)} \sum_{t \in \mathbb{Z}^*} (l_0^t - k_0^t)(j)^p - \langle t \rangle^p u^l \bar{u}^k \quad (4.32) \\
A^2(u, \bar{u}) : &= 2\text{Re} \sum_{\substack{l+k=r \in i \in M_{\ell r} \subseteq Z}} \text{i}(l_0^j - k_0^j, k_0^j)(j)^p \\
&\quad \cdot \sum_{(l^0, k^0, \ell^0) \in A_{\ell r}^{(l^0, k^0)}} (f^{w_0})_{r, \ell k}^{(l^0, k^0, \ell^0)} \sum_{t \in \mathbb{Z}^*} \cdot \langle t \rangle^p u^l \bar{u}^k. \quad (4.33)
\end{align*}
\]

The estimate of \( \{ f_r^{w_0}(u, \bar{u}), \|u\|^2 \}_w \) follows the estimates of \( A^0(u, \bar{u}), A^1(u, \bar{u}) \) and \( A^2(u, \bar{u}) \). In fact, I cannot estimate \( A^2(u, \bar{u}) \) by Corollary \(2\) directly, because the coefficients of \( A^2(u, \bar{u}) \) are not satisfy condition \((4.13)\). Fortunately the bad unbounded part (not satisfy the condition \((4.13)\)) in \( A^2(u, \bar{u}) \) can be handled by \((\beta, 0)\)-type symmetric property of
Together with (4.35) and (4.36), it follows
\[ A^2(u, \bar{u}) \] is transformed into a new form, the coefficients of which satisfy (4.34). Thus, the estimate of \( A^2(u, \bar{u}) \) can be obtained by Corollary 2.

Now the details of deleting the unbounded terms in \( A^2(u, \bar{u}) \) are given in the follows. For any \((l^0, k^0, l^0) \in A_{(f_{wo})_{l,k}},\) it holds that

\[
(l^0 - k^0) \cdot t = \sum_{n \neq t} (l^0 - k^0) \cdot n + \sum_{n \neq t} (l^0 - k^0) \cdot n \]

Using (4.34)

\[
A^2(u, \bar{u}) = 2 \text{Re} \sum_{i \in M_{f_{wo}} \subseteq Z, [l+k] = i, \Lambda(l,k) = i} \sum_{(f_{wo})_{r,l,k}} \left( i \left( l^0 - k^0 - j \right) \langle j \rangle^p \right)
\]

\[
\cdot \sum_{(l^0, k^0, l^0) \in A_{(f_{wo})_{l,k}}} (f_{wo})_{r,l,k} \sum_{t \in \mathbb{Z}^*} R^{wo}(l, k, t, j, i) \langle t \rangle^p u^l \bar{u}^k + 2 \text{Re} \sum_{i \in M_{f_{wo}} \subseteq Z, [l+k] = i, \Lambda(l,k) = i} \sum_{(l^0, k^0, l^0) \in A_{(f_{wo})_{l,k}}} \left( i \left( l^0 - k^0 \right) \langle t \rangle^p \right)
\]

\[
\cdot \sum_{(l^0, k^0, l^0) \in A_{(f_{wo})_{l,k}}} (f_{wo})_{r,l,k} \sum_{t \in \mathbb{Z}^*} \left( (l^0 - k^0) \cdot t \right)^p u^l \bar{u}^k.
\]

Since the two parts in the right side of (4.35) are real value functions, they are invariant under complex conjugation. Taking complex conjugation to the second part of the right side of (4.35) and using that fact that \( f_{wo}(u, \bar{u}) \) has \((\beta, 0)\)-type symmetric coefficients, it leads to

\[
2 \text{Re} \sum_{i \in M_{f_{wo}} \subseteq Z, [l+k] = i, \Lambda(l,k) = i} \sum_{(f_{wo})_{r,l,k}} \left( i \left( l^0 - k^0 - j \right) \langle j \rangle^p \right)
\]

\[
\cdot \sum_{(l^0, k^0, l^0) \in A_{(f_{wo})_{l,k}}} (f_{wo})_{r,l,k} \sum_{t \in \mathbb{Z}^*} \left( (l^0 - k^0 - j) \cdot t \right)^p u^l \bar{u}^k
\]

\[ = -2 \text{Re} \sum_{i \in M_{f_{wo}} \subseteq Z, [l+k] = i, \Lambda(l,k) = i} \sum_{(f_{wo})_{r,l,k}} \left( i \left( l^0 - k^0 \right) \langle t \rangle^p \right)
\]

\[
\cdot \sum_{(l^0, k^0, l^0) \in A_{(f_{wo})_{l,k}}} (f_{wo})_{r,l,k} \sum_{t \in \mathbb{Z}^*} \left( (l^0 - k^0 - l) \cdot t \right)^p u^l \bar{u}^k
\]

\[ = -A^2(u, \bar{u}).
\]

Together with (4.35) and (4.36), it follows

\[
A^2(u, \bar{u}) = \text{Re} \sum_{i \in M_{f_{wo}} \subseteq Z, [l+k] = i, \Lambda(l,k) = i} \sum_{(f_{wo})_{r,l,k}} \left( i \left( l^0 - k^0 - l + k^0 \right) \langle j \rangle^p \right)
\]

\[
\cdot \sum_{(l^0, k^0, l^0) \in A_{(f_{wo})_{l,k}}} (f_{wo})_{r,l,k} \sum_{t \in \mathbb{Z}^*} R^{wo}(l, k, t, j, i) \langle t \rangle^p u^l \bar{u}^k.
\]

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In the case $\theta = 1$,
\[
\{ f_r^{w_1}(u, \bar{u}), \|u\|_p^2 \}_{w_1} = Q^+(u, \bar{u}) + Q^-(u, \bar{u}),
\]  
where
\[
Q^+(u, \bar{u}) := \sum_{l+k=r, \mathcal{M}(l, k) = i \in \mathcal{M}} \text{sgn}(j) l_j \cdot \langle j \rangle^{2p} \prod_{n \in \mathbb{Z}^+} \langle n \rangle^\frac{1}{2}(l_n + k_n) (\tilde{f}^{w_1})_{r, l, k} u^l \bar{u}^k,
\]
\[
Q^-(u, \bar{u}) := -\sum_{l+k=r, \mathcal{M}(l, k) \neq i \in \mathcal{M}} \text{sgn}(j) k_j \cdot \langle j \rangle^{2p} \prod_{n \in \mathbb{Z}^+} \langle n \rangle^\frac{1}{2}(l_n + k_n) (\tilde{f}^{w_1})_{r, l, k} u^l \bar{u}^k.
\]

Since the coefficients of $f_r^{w_1}(u, \bar{u})$ are $(\beta, 1)$-type symmetric, it holds that
\[
\overline{Q^-(u, \bar{u})} = Q^+(u, \bar{u}).
\]  

From (4.38) and (4.39), one has
\[
\begin{align*}
\{ f_r^{w_1}, \|u\|_p^2 \}_{w_1} &= 2\text{Re}Q^+(u, \bar{u}) \\
&= 2\text{Re} \sum_{i \in \mathcal{M}} \sum_{l+k=r, \mathcal{M}(l, k) = i} \text{sgn}(j) l_j \cdot \langle j \rangle^{2p} \prod_{n \in \mathbb{Z}^+} \langle n \rangle^\frac{1}{2}(l_n + k_n) (\tilde{f}^{w_1})_{r, l, k} u^l \bar{u}^k \\
&= 2\text{Re} \sum_{i \in \mathcal{M}} i \sum_{l+k=r, \mathcal{M}(l, k) = i} (\tilde{f}^{w_1})_{r, l, k} \langle j \rangle^{2p-1} l_j \cdot j \prod_{n \in \mathbb{Z}^+} \langle n \rangle^\frac{1}{2}(l_n + k_n) u^l \bar{u}^k,
\end{align*}
\]  
the last equation is obtained by $\text{sgn}(j) |j| = j$. For any $l, k \in \mathbb{N}\mathbb{Z}^+$ and any $j \in \mathbb{Z}^+$ with $\mathcal{M}(l, k) = i \in \mathcal{M}$, and $l_j \neq 0$, one has that
\[
l_j \cdot j = i - \mathcal{M}(l - l_j e_j, k) = i - \sum_{n \in \mathbb{Z}^+} l_n \cdot n + \sum_{n \in \mathbb{Z}^+} k_n \cdot n.
\]  
Together with (4.40) and (4.41),
\[
\begin{align*}
\{ f_r^{w_1}(u, \bar{u}), \|u\|_p^2 \}_{w_1} &= E(u, \bar{u}) + B(u, \bar{u}) + D(u, \bar{u}),
\end{align*}
\]  
where
\[
E(u, \bar{u}) := 2\text{Re} \sum_{l+k=r, \mathcal{M}(l, k) = i \in \mathcal{M}} i (\tilde{f}^{w_1})_{r, l, k} \text{sgn}(l_j) \langle j \rangle^{2p-1} (\sum_{n \in \mathbb{Z}^+} k_n \cdot n) \prod_{m \in \mathbb{Z}^+} \langle m \rangle^\frac{1}{2}(l_m + k_m) u^l \bar{u}^k,
\]
\[
B(u, \bar{u}) := -2\text{Re} \sum_{l+k=r, \mathcal{M}(l, k) \neq i \in \mathcal{M}} i (\tilde{f}^{w_1})_{r, l, k} \text{sgn}(l_j) \langle j \rangle^{2p-1} (\sum_{n \in \mathbb{Z}^+} l_n \cdot n) \prod_{m \in \mathbb{Z}^+} \langle m \rangle^\frac{1}{2}(l_m + k_m) u^l \bar{u}^k,
\]
\[
D(u, \bar{u}) := 2\text{Re} \sum_{l+k=r, \mathcal{M}(l, k) = i \in \mathcal{M}} i (\tilde{f}^{w_1})_{r, l, k} \text{sgn}(l_j) \langle j \rangle^{2p-1} \prod_{m \in \mathbb{Z}^+} \langle m \rangle^\frac{1}{2}(l_m + k_m) u^l \bar{u}^k,
\]
In order to estimate of $E(u, \bar{u})$ and $B(u, \bar{u})$, $E(u, \bar{u})$ is rewritten as the sum of $E^1(u, \bar{u})$ and $E^2(u, \bar{u})$, where
\[
E^1(u, \bar{u}) := 2\text{Re} \sum_{l+k=r, \mathcal{M}(l, k) = i \in \mathcal{M}} i (\tilde{f}^{w_1})_{r, l, k} \text{sgn}(l_j) \langle j \rangle^{p-\frac{1}{2}} ((j)^{p-\frac{1}{2}} - \langle n \rangle^{p-\frac{1}{2}}) \sum_n nk_n \prod_{m \in \mathbb{Z}^+} \langle m \rangle^\frac{1}{2}(l_m + k_m) u^l \bar{u}^k,
\]  
(4.46)
\[ E^2(u, \bar{u}) := 2 \text{Re} \sum_{|\{l,k\}|=r, \mathcal{M}(l,k) = i \in \mathcal{M}_{F_r^{w_1}}} i (\hat{f}^{w_1})^i_{r,l,k} \text{sgn}(l,j)(n)^{p-\frac{1}{2}} \left( \sum_n k_n \cdot n(n)^{p-\frac{1}{2}} \right) \prod_{m \in \mathbb{Z}^*} \langle m \rangle^{\frac{1}{2}} (l_n + k_m) u^l \bar{u}^k \] (4.47)

and \( B(u, \bar{u}) \) is rewritten as the sum of \( B^1(u, \bar{u}) \) and \( B^2(u, \bar{u}) \), where

\[ B^1(u, \bar{u}) := -2 \text{Re} \sum_{|\{l,k\}|=r, \mathcal{M}(l,k) = i \in \mathcal{M}_{F_r^{w_1}}} i (\hat{f}^{w_1})^i_{r,l,k} \text{sgn}(l,j)(n)^{p-\frac{1}{2}} \left( \sum_n l_n \cdot n(n)^{p-\frac{1}{2}} - (n)^{p-\frac{1}{2}} \right) \prod_{m \in \mathbb{Z}^*} \langle m \rangle^{\frac{1}{2}} (l_n + k_m) u^l \bar{u}^k \] (4.48)

\[ B^2(u, \bar{u}) := -2 \text{Re} \sum_{|\{l,k\}|=r, \mathcal{M}(l,k) = i \in \mathcal{M}_{F_r^{w_1}}} i (\hat{f}^{w_1})^i_{r,l,k} \text{sgn}(l,j)(n)^{p-\frac{1}{2}} \left( \sum_n l_n \cdot n(n)^{p-\frac{1}{2}} \right) \prod_{m \in \mathbb{Z}^*} \langle m \rangle^{\frac{1}{2}} (l_n + k_m) u^l \bar{u}^k \] (4.49)

Using the \((\beta,1)\)-type symmetric property of \( f^{w_1}_r(u, \bar{u}) \), delete the bad unbounded parts (not satisfy the condition \([1.13]\)) in \( B^2(u, \bar{u}) \) and \( E^2(u, \bar{u}) \) in the followings. For any \( l, k \in \mathbb{N}^* \) and any \( j, n \in \mathbb{Z}^* \) with \( k_n \neq 0 \), \( l_j \neq 0 \) and \( \mathcal{M}(l,k) = i \), the following equation holds true

\[ k_n \cdot n = i + \sum_{t \in \mathbb{Z}^*, t \neq j} l_t \cdot t - \sum_{t \in \mathbb{Z}^*, t \neq n} k_t \cdot t + l_j \cdot j. \] (4.50)

Take (4.50) into \( E^2(u, \bar{u}) \), one obtains that

\[ E^2(u, \bar{u}) = -2 \text{Re} \sum_{|\{l,k\}|=r, \mathcal{M}(l,k) = i \in \mathcal{M}_{F_r^{w_1}}} i (\hat{f}^{w_1})^i_{r,l,k} \text{sgn}(l,j)(n)^{p-\frac{1}{2}} \left( \sum_n R^{w_1}_+(l, k, n, j, i) \langle n \rangle^{p-\frac{1}{2}} \prod_{m \in \mathbb{Z}^*} \langle m \rangle^{\frac{1}{2}} (l_n + k_m) u^l \bar{u}^k \right) \] (4.51)

Take complex conjugation to the second part of the right side of (4.51) and get

\[ E^2(u, \bar{u}) = -2 \text{Re} \sum_{|\{l,k\}|=r, \mathcal{M}(l,k) = i \in \mathcal{M}_{F_r^{w_1}}} i (\hat{f}^{w_1})^i_{r,l,k} \text{sgn}(l,j)(n)^{p-\frac{1}{2}} \left( \sum_n R^{w_1}_-(l, k, n, j, i) \langle n \rangle^{p-\frac{1}{2}} \prod_{m \in \mathbb{Z}^*} \langle m \rangle^{\frac{1}{2}} (l_n + k_m) u^l \bar{u}^k \right) \] (4.52)
the last equation is obtained by the coefficients of \( f^{w_3}_r(u, \bar{u}) \) being \((\beta, 1)\)-type symmetric. Equation (4.52) leads to

\[
E^2(u, \bar{u}) = -\text{Re} \sum_{|k| = r \in M} i(\tilde{f}^{w_1})_r,l,k(j)^{p-1} \sum_{n \in \mathbb{Z}^*} R^{w_1}_r(l, k, n, j, i) \langle n \rangle^{p-1} \prod_{m \in \mathbb{Z}^*} \langle m \rangle^{1/2} (l_m + k_m) u^l \bar{u}^k. \tag{4.53}
\]

Similarly, by the fact

\[
l_n \cdot n = i + \sum_{t \in \mathbb{Z}^*} k_t \cdot t - \sum_{t \neq j, n} l_t \cdot t - l_j \cdot j,
\]

the following equation holds true

\[
B^2(u, \bar{u}) = -\text{Re} \sum_{|k| = r \in M} i(\tilde{f}^{w_1})_r,l,k(j)^{p-1} \sum_{n \neq j, n \neq j, n \neq j} R^{w_1}_r(l, k, n, j, i) \langle n \rangle^{p-1} \prod_{m \in \mathbb{Z}^*} \langle m \rangle^{1/2} (l_m + k_m) u^l \bar{u}^k. \tag{4.54}
\]

Summarize this step, it satisfies that

\[
\{ f^{w_0}_r, \|u\|_p \}^{w_0} \downarrow \quad \text{Re} O^+ (u, \bar{u}) = \begin{cases} A^0(u, \bar{u}) \text{ defined in (4.31)} + A^1(u, \bar{u}) \text{ defined in (4.32)} + A^2(u, \bar{u}) \text{ in (4.33) } \text{\((\beta, 0)\)-type symmetrical} \end{cases} \rightarrow A^2(u, \bar{u}) \text{ in (4.37)}.
\]

and

\[
\{ f^{w_1}_r, \|u\|_p \}^{w_1} \downarrow \quad \text{Re} Q^+ (u, \bar{u}) = \begin{cases} B(u, \bar{u}) \text{ in (4.41)} = \begin{cases} B^1(u, \bar{u}) \text{ in (4.48)} \end{cases} + \begin{cases} B^2(u, \bar{u}) \text{ in (4.49) } \text{\((\beta, 1)\) } B^2(u, \bar{u}) \text{ in (4.54)} \end{cases} \end{cases} \text{ in (4.44)}.
\]

+ \begin{cases} E(u, \bar{u}) \text{ in (4.43)} = \begin{cases} E^1(u, \bar{u}) \text{ in (4.46)} \end{cases} + \begin{cases} E^2(u, \bar{u}) \text{ in (4.47) } \text{\((\beta, 1)\) } E^2(u, \bar{u}) \text{ in (4.53)} \end{cases} \end{cases} \text{ in (4.45)}.
\]

Step 2: Estimate \(A^0(u, \bar{u}) \cdot A^2(u, \bar{u}), B^1(u, \bar{u}), B^2(u, \bar{u}), E^1(u, \bar{u}), E^2(u, \bar{u})\) and \(D(u, \bar{u})\). It is clear that \(A^0(u, \bar{u})\) can be written as an inner product of the vector fields \(G := (j)^p \bar{u}^j\) and \(F := (F_j)_j \in \mathbb{Z}\), where

\[
F_j(u, \bar{u}) := \sum_{|k + e_j| = r - 1} \sum_{m \in \mathbb{Z}^*} F^{i}_{r,l,k-e_j} u^l \bar{u}^{k-e_j}, \tag{4.55}
\]

and

\[
F^{i}_{r,l,k-e_j} := (l_j - l^0_j - k_j + k^0_j) \langle j \rangle^p \sum_{m \in \mathbb{Z}^*} \langle m \rangle^{1/2} (l_m + k_m) u^l \bar{u}^k. \tag{4.56}
\]
Noting the fact the momentum of \((f_{w_0})_{l,l,k}^{i} u_l u_k\) being \(i\) and 0 \(\leq l_j^0 \leq l_j\), \(0 \leq k_j^0 \leq k_j\), the following equation holds true
\[
(l_j - l_j^0 - k_j + k_j^0) \cdot j = i - (l_j^0 - k_j^0) \cdot j - \sum_{l \neq j} (l_t - k_t) \cdot t. \tag{4.57}
\]

Using the fact that \(|x|^p (p \geq 2)\) is convex function and (4.21), it follows that
\[
| i - (l_j^0 - k_j^0) \cdot j - \sum_{l \neq j} (l_t - k_t) \cdot t |^p \leq 2r^{p-1} \left( \sum_{l \neq j} (l_t + k_t) |t|^p + (l_j^0 + k_j^0) |j|^p \right) \cdot \langle i \rangle^p. \tag{4.58}
\]

In view of (4.57) and (4.58), it holds that
\[
|l_j - l_j^0 - k_j + k_j^0|^p \leq 2r^{p-1} \left( \sum_{l \neq j} (l_t + k_t) |t|^p + (l_j^0 + k_j^0) |j|^p \right) \cdot \langle i \rangle^p. \tag{4.59}
\]

Since \(f_{r}^{w_0}\) has \((\beta, 0)\)-type symmetric coefficients semi-bounded by \(C_{f_{w_0}}\), together with (4.59), the coefficients of vector field \(F\) in (4.55) are bounded by the following
\[
|F_{j,l,k-\epsilon,j}^{i}| \leq 2r^{p-1} C_{f_{w_0}}^{-2} \left( \sum_{l \neq j} (l_t + k_t) |t|^p + (l_j^0 + k_j^0) |j|^p \right) \cdot \frac{1}{\langle j \rangle^{\alpha}}. \tag{4.60}
\]

Then by Corollary 2 the following inequality holds true
\[
|A^0(u, \bar{u})| \leq |\langle F, G \rangle| \leq \|F\|_{\mathcal{E}} \cdot \|G\|_{\mathcal{E}} = \|F\|_{\mathcal{E}} \cdot \|u\|^2_p \leq 2C_{f_{w_0}}^{-2} r^{p-1} \|u\|^2_p \|u\|^2_r. \tag{4.61}
\]

Using the same method, one has
\[
|D(u, \bar{u})| \leq 2C_{f_{w_0}}^{-2} r^{p-1} \|u\|^2_p \|u\|^2_r. \tag{4.62}
\]

In order to estimate \(A^1(u, \bar{u})\), the following inequality is given for any \(j, \ m \in \mathbb{Z}^*\)
\[
| |j|^a - |m|^a | \leq \int_0^1 d|m + \theta (j - m)|^a d\theta
\leq \int_0^1 a |j - m| \cdot \left| m + \theta (j - m) \right|^{a-1} d\theta
\leq 2^{a-2} a \left( |m|^{a-1} |j - m| + |j - m|^a \right), \tag{4.63}
\]
with \(a \geq 2\).

Take \(a = p\) into (4.62). Given \(l, k \in \mathbb{N}^{\mathbb{Z}}\) fulfilling \(\mathcal{M}(l, k) = i\) with \(k_j \neq 0\) and \(l_m^0 \neq 0\), together with (4.21), it holds that
\[
| |j|^p - |m|^p | \leq 2^{p-1} \left( |m|^p-1 \left( \sum_{n \neq j} k_n |n| + \sum_{n \neq m} l_n |n| \right) \cdot \langle i \rangle \right.
\left. + (r - 2) \sum_{n \neq j} |k_n|^p + \sum_{n \neq m} |l_n|^p \cdot \langle i \rangle^p \right). \tag{4.64}
\]

The similar inequality holds in the case \(k_j \neq 0, k_m^0 \neq 0 (m \neq j)\).

Since
\[
|A^1(u, \bar{u})| \leq \sum_{l \in \mathcal{M}_{l,l,k}^{i} \forall_{i \in \mathcal{M}(l, k)}} \sum_{j \neq l, k-\epsilon} \left( l_j - l_j^0 - k_j + k_j^0 \right) \cdot j |j|^p \sum_{m \in \mathbb{Z}^*} \left( f_{w_0}^{i(\bar{l}, \bar{k}, \bar{u}, \omega)} \right)_{r,i,l,k}.
\]
\[
\cdot (t^0, m(\langle j \rangle^p - \langle m \rangle^p) - k^0 m(\langle j \rangle^p - \langle m \rangle^p)) u^k u^j,
\]
(4.64)

take the right side of (4.64) as an inner product of vectors \( F \) and \( G \), where
\[
F := (2\langle j \rangle^p u_j)_{j \in \mathbb{Z}^*}, \quad G := \left( \sum_{i \in M_{r_0}} \sum_{j \in M_{l_0}} (G_j)_{i - 1,lk - ej} u^l u^k u^j \right)_{j \in \mathbb{Z}^*}
\]

and
\[
(G_j)_{i - 1,lk - ej} := (l_j - l^0_j - k_j + k^0_j) \sum_{(l^0, k^0, n) \in A_{\langle j^0 \rangle}} (f_{r_0}^{\langle j^0, k^0, n \rangle}) (t^0_m m(\langle j \rangle^p - \langle m \rangle^p) - k^0 m(\langle j \rangle^p - \langle m \rangle^p)).
\]

By (4.63) and \( f_{r_0}^{\langle j^0 \rangle} (u, \bar{u}) \) having \((\beta, 0)\)-type symmetric coefficients semi-bounded by \( C_{f^0} \), the coefficients of \( G_j \) are bounded by
\[
\frac{|(G_j)_{i - 1,lk - ej}|}{(\langle j \rangle^p)^{p-1} p(r - 2)^{p-1}} \leq \frac{C_{f^0}^{r-2}}{(\langle j \rangle^p)^{p-1}} (r - 2)^{p-1} \|u\|_p^2 \|u\|_2^{p-2}.
\]

Using Corollary 2 one has
\[
|A^1(u, \bar{u})| \leq \|G\|_{\ell^2} \cdot \|F\|_{\ell^2} \leq C_{f^0}^{r-2} (r - 2)^{p-1} \|u\|_p^2 \|u\|_2^{p-2}.
\]

By the same method, \( E^1(u, \bar{u}) \) and \( B^1(u, \bar{u}) \) satisfy the following inequalities
\[
|B^1(u, \bar{u})| \leq C_{f^0}^{r-2} (r - 2)^{p-1} \|u\|_p^2 \|u\|_2^{p-2}
\]

and
\[
|E^1(u, \bar{u})| \leq C_{f^0}^{r-2} (r - 2)^{p-1} \|u\|_p^2 \|u\|_2^{p-2}.
\]

Since \( A^2(u, \bar{u}), B^2(u, \bar{u}) \) and \( E^2(u, \bar{u}) \) can be estimated by the same method, I only give the details of estimate of \( A^2(u, \bar{u}) \) in (4.37).
\[
|A^2(u, \bar{u})| \leq \|F\|_{\ell^2} \cdot \|G\|_{\ell^2},
\]

where \( F = (F_j)_{j \in \mathbb{Z}^*} \) and \( G = (G_j)_{j \in \mathbb{Z}^*} \) with \( F_j = \langle j \rangle^p u_j \) and
\[
G_j(u, \bar{u}) := \sum_{i \in M_{r_0}} \sum_{j \in M_{l_0}} i(\langle l - l^0 \rangle_j - \langle k - k^0 \rangle_j)
\]
\[
\cdot \sum_{(l^0, k^0, n) \in A_{\langle j^0 \rangle}} (f_{r_0}^{\langle j^0, k^0, n \rangle}) \sum_{t \in \mathbb{Z}} R_{\langle j^0 \rangle}^{\langle l^0, k^0, n \rangle} (l, k, t, j, i) \langle t \rangle^p u^l u^k u^j.
\]

From (4.21) and (4.50), one has
\[
|R_{\langle j^0 \rangle}^{\langle l^0, k^0, n \rangle}(l, k, t, j, i)| \leq 2 \left( \sum_{n \neq j} |(k - k^0)_n - (l - l^0)_n| \cdot \langle n \rangle + \sum_{n \neq l} |l^0_n - k^0_n| \cdot \langle n \rangle \right) \cdot \langle i \rangle.
\]
Thus, the following inequalities hold true

\[ |(G_j)_{r,l,k-e_j}^t| \leq 2 \frac{C_{f_p}^{n-2}}{(t)^{d-1}} \left( \sum_{n \neq j} |(k - k^0)_n - (l - l^0)_n| \cdot \langle n \rangle + \sum_{n \neq l} |l_n^0 - k_n^0| \cdot \langle n \rangle \right) \langle t \rangle.

Using Corollary 2 it holds that

\[ |A^2(u, \bar{u})| \leq 2C_{f_p}^{r-2} e^{r-1} r \|u\|_p \|u\|_2^{r-1}.

Similarly,

\[ |B^2(u, \bar{u})|, |E^2(u, \bar{u})| \leq 2C_{f_p}^{r-2} e^{r-1} r \|u\|_p \|u\|_2^{r-1}.

Thus, the following inequalities hold true

\[ |\{ f_w^r(u, \bar{u}) \}, \|u\|^2 \rangle_{w_0} \leq |A^0(u, \bar{u})| + |A^1(u, \bar{u})| + |A^2(u, \bar{u})| \leq C_{f_p}^{r-2} 2^{p+1} p^{p-1} e^{r-1} \|u\|_p \|u\|_2^{r-2}

and

\[ |\{ f_w^r(u, \bar{u}) \}, \|u\|^2 \rangle_{w_1} \leq |B^1(u, \bar{u})| + |B^2(u, \bar{u})| + |D(u, \bar{u})| + |E^1(u, \bar{u})| + |E^2(u, \bar{u})| \leq C_{f_p}^{r-2} 2^{p+1} p^{p-1} e^{r-1} \|u\|_p \|u\|_2^{r-2}.

\]

**5 Birkhoff Normal form and non resonant condition**

**5.1 (θ, γ, α, N)-normal form**

In order to guarantee the boundedness of the symplectic transformation, it is required a strong non resonant condition. Given integers \( N > 0 \) and \( r \geq 3 \), let

\[ E_{r,N} := \{ (l, k) \mid l, k \in \mathbb{N}^Z, \ 3 \leq |l + k| = t \leq r, \ |\Gamma_{>N}(l + k)| \leq 2 \}.

**Definition 5.1.** Given \( \gamma > 0, \alpha > 0, \theta \in \{0, 1\} \) and \( N, r \in \mathbb{N} \), frequencies \( \omega = (\omega_j)_{j \in \mathbb{Z}} \) is said to be \( r \)-degree \((\theta, \gamma, \alpha, N)\)-non resonant, if for any \( (l, k) \) belongs to

\[ O_{r,N}^{\omega_{\theta}} := \left\{ (l, k) \in E_{r,N} \right\} \begin{cases} \text{when} \quad |\Gamma_{>N}(l + k)| < 2, \\ \theta \sum_{j \in \mathbb{Z}} |l_j - k_j| + (1 - \theta) \sum_{j \in \mathbb{Z}} |l_j + l_{-j} - k_j - k_{-j}| \neq 0 \end{cases},

\[ \text{when} \quad |\Gamma_{>N}(l + k)| = 2, \\ \sum_{|j| > N} |l_j + l_{-j} - k_j - k_{-j}| \neq 0 \}

it satisfies

\[ |\langle \omega, I_\theta(l - k) \rangle| > \gamma \frac{M_{l,k}}{N^\alpha}, \]

where

\[ M_{l,k} := \max \{ |j| \mid k_j \neq 0 \text{ or } l_j \neq 0, \ k, \ l \in \mathbb{N}^Z \} \cup \{N\} \]
With the $r$-degree $(\theta, \gamma, \alpha, N)$-non resonant condition, a symplectic transformation will be obtained. Under this transformation the Hamiltonian function $H(u, \bar{u})$ are transformed into the sum of an $r$-degree normal form and a remainder term. However this $r$-degree normal form is not a standard Hamiltonian Birkhoff normal form (A standard $2r$-degree Hamiltonian Birkhoff normal form in variables $(u, \bar{u})$ is an $r$-degree polynomial which only depends on variables $([|u_j|^2])_{j \in \mathbb{Z}^r}$). Now I introduce a definition to describe this normal form.

**Definition 5.2.** Given $\gamma > 0$, $\alpha > 0$ and an integer $N > 0$, call an $r$-degree polynomial

$$f(u, \bar{u}) = \sum_{r \geq t \geq 3} \sum_{|i|+|k|=t} \sum_{M(l,k)=i \in M_f} f_{t,ik}^i u^l \bar{u}^k$$

$(\theta, \gamma, \alpha, N)$-normal form with respect to $\omega \in \mathbb{R}^{\mathbb{Z}^r}$, if for any $(l,k) \in E_{r,N}$ ($E_{r,N}$ is defined in (3.7)) with $M(l,k) = i \in M_f$ and $f_{t,ik}^i \neq 0$, it satisfies that

$$|\langle \omega, I_\theta(l-k) \rangle| \leq \frac{\gamma M_{k,l}}{N^\alpha},$$

where $\langle , \rangle$ is the inner product of space $\ell^2(\mathbb{Z}^r, \mathbb{C})$, $I_\theta$ and $M_{l,k}$ are defined in (3.7) and (5.7).

**Remark 5.1.** Let $f(u, \bar{u})$ be an $r$-degree polynomial. For any given $\gamma > 0$, $\alpha > 0$, $\theta \in \{0, 1\}$, integers $N > 0$ and $\omega \in \mathbb{R}^{\mathbb{Z}^r}$, denote

$$\Gamma_{\omega, \theta, \gamma, \alpha, N} f(u, \bar{u}) := \sum_{r \geq t \geq 3} \sum_{|i|+|k|=t} \sum_{M(l,k)=i \in M_f} (\Gamma_{\omega, \theta, \gamma, \alpha, N} f)^i_{t,ik} u^l \bar{u}^k$$

with

$$(\Gamma_{\omega, \theta, \gamma, \alpha, N} f)^i_{t,ik} := \begin{cases} f_{t,ik}^i, & \text{if } l, k \text{ fulfills } |\langle \omega, I_\theta(l-k) \rangle| \leq \frac{\gamma M_{k,l}}{N^\alpha} \\ 0, & \text{the other cases} \end{cases}$$

as $(\theta, \gamma, \alpha, N)$-normal form of $f(u, \bar{u})$ with respect to $\omega$. Moreover, suppose that $f^{\omega_0}(u, \bar{u})$ has $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C_\theta > 0$ ($\theta \in \{0, 1\}$). So does $\Gamma_{\omega, \theta, \gamma, \alpha, N} f^{\omega_0}(u, \bar{u})$.

**Remark 5.2.** Assume that $\omega = (\omega_j)_{j \in \mathbb{Z}^r}$ is an $r_*$-degree $(\theta, \gamma, \alpha, N)$-non resonant frequencies and $f^{\omega_0}(u, \bar{u})$ is an $r_*$-degree $(\theta, \gamma, \alpha, N)$-normal form with respect to $\omega$. Then $f^{\omega_0}(u, \bar{u})$ has the following form

$$f^{\omega_0}(u, \bar{u}) := A^{\omega_0}(u, \bar{u}) + B^{\omega_0}(u, \bar{u}) + C^{\omega_0}(u, \bar{u}),$$

where

$$A^{\omega_0}(u, \bar{u}) := \sum_{3 \leq r \leq r_*} \sum_{(l,k) \in \Pi_{\omega_0}^N, i=M(l,k)} (f^{\omega_0})_{r,ik}^i u^l \bar{u}^k,$$

$$B^{\omega_0}(u, \bar{u}) := \sum_{3 \leq r \leq r_*} \sum_{(l,k) \in \Pi_{\omega_0}^N, i=M(l,k)} (f^{\omega_0})_{r,ik}^i u^l \bar{u}^k,$$

$$C^{\omega_0}(u, \bar{u}) := \sum_{3 \leq r \leq r_*} \sum_{(l,k) \in \Pi_{\omega_0}^N, i=M(l,k)} (f^{\omega_0})_{r,ik}^i u^l \bar{u}^k,$$
\[ C^w(u, \bar{u}) := \sum_{3 \leq r \leq \tau^*} \sum_{(l,k) \in \Omega_{\epsilon}\epsilon N(i) \in \mathcal{M}(l,k)} (f^w)_{r,ik}^i u^l \bar{u}^k, \quad (5.6) \]

and

\[ \Omega_{\epsilon}\epsilon N := \begin{cases} \{ (l,k) \in E_{\epsilon}\epsilon N \mid |\Gamma_{\epsilon}\epsilon N(l+k)| < 2, \sum_{j \in \mathbb{Z}^*} |l_j + l_{-j} - k_j - k_{-j}| = 0 \}, & \theta = 0 \\ \{ (l,k) \in E_{\epsilon}\epsilon N \mid |\Gamma_{\epsilon}\epsilon N(l+k)| = 2, \sum_{j \in \mathbb{Z}^*} |l_j - k_j| = 0 \}, & \theta = 1 \end{cases} \]

\[ \Omega_{\epsilon}\epsilon N := \begin{cases} \{ (l,k) \in E_{\epsilon}\epsilon N \mid |\Gamma_{\epsilon}\epsilon N(l+k)| = 2, \text{ there exists } |j_0| > N \text{ such that } l_{j_0} = k_{j_0} = 1 \} \}
\]

\[ \Omega_{\epsilon}\epsilon N := \begin{cases} \{ (l,k) \in E_{\epsilon}\epsilon N \mid |\Gamma_{\epsilon}\epsilon N(l+k)| = 2, \text{ there exists } |j_0| > N \text{ such that } l_{j_0} = k_{j_0} = 1 \} \].

**Lemma 5.1.** Let \( \omega = (\omega_j)_{j \in \mathbb{Z}^*} \) be an r-degree \((\theta, \gamma, \alpha, N)\) non-resonant frequency. Suppose that \( f^w_r(u, \bar{u}) \) is an r-degree \((r \geq 3)\) homogeneous \((\theta, \gamma, \alpha, N)\)-normal form with respect to \( \omega \) and has \((\beta, \theta)\)-type symmetric coefficients semi-bounded by \( \mathcal{C}^r_{\epsilon\epsilon^w} > 0 \). Then for any \((u, \bar{u}) \in \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C}) \) it has

\[ \| f^w_r(u, \bar{u}), \| u \|_p^2 \| \leq 20r^{r+1}c^{r-1}c^{r-2}N \| \Gamma_{\epsilon\epsilon N}u \| \| u \|_2 \cdot \| u \|_2^{-3} \cdot \| u \|_p \quad (5.7) \]

**Remark 5.3.** Although \( f^w_r(u, \bar{u}) \) in Lemma 5.1 is at most 2 degree about \((\Gamma_{\epsilon\epsilon N}u, \Gamma_{\epsilon\epsilon N}\bar{u})\), it still satisfies an inequality similar to (4.6) in Corollary 1. But for the general polynomials being at most degree about \((\Gamma_{\epsilon\epsilon N}u, \Gamma_{\epsilon\epsilon N}\bar{u})\), this inequality does not always hold.

**Proof of Lemma 5.1.** From Remark 5.2

\[ f^w_r(u, \bar{u}) = A^w_r(u, \bar{u}) + B^w_r(u, \bar{u}) + C^w_r(u, \bar{u}), \quad (5.8) \]

where \( A^w_r(u, \bar{u}), B^w_r(u, \bar{u}) \) and \( C^w_r(u, \bar{u}) \) are defined in (5.4)-(5.6) in Remark 5.2.

**Step 1:** Calculate \( \{ A^w_r(u, \bar{u}), \| u \|_p^2 \}_{w_0} \).

It is easy to verify that

\[ \{ A^w_r(u, \bar{u}), \| u \|_p^2 \}_{w_0} = \sum_{j \in \mathbb{Z}^*} \sum_{(l,k) \in \Omega_{\epsilon}\epsilon N} \mathbf{i}(l_j + l_{-j} - k_j - k_{-j})(j)^{2p}(f^w)_{r,ik}^i u^l \bar{u}^k = 0; \quad (5.9) \]

and

\[ \{ A^w_r(u, \bar{u}), \| u \|_p^2 \}_{w_1} = \sum_{j \in \mathbb{Z}^*} \sum_{(l,k) \in \Omega_{\epsilon}\epsilon N} \mathbf{i} \text{ sgn}(j) \cdot (l_j - k_j)(j)^{2p}(f^w)_{r,ik}^i u^l \bar{u}^k = 0. \quad (5.10) \]

**Step 2:** Estimate \( \{ B^w_r(u, \bar{u}), \| u \|_p^2 \}_{w_y} \).

Since the function \( B^w_r(u, \bar{u}) \) depends on \((u_j, \bar{u}_j)_{|j| \leq N}\) and \( \| u_j \|_{|j| > N}^2 \); the following equation holds true

\[ \{ B^w_r(u, \bar{u}), \sum_{|j| > N} (j)^{2p} |u_j|^2 \}_{w_y} = 0. \]

Thus,

\[ \{ B^w_r(u, \bar{u}), \| u \|_p^2 \}_{w_y} = \{ B^w_r(u, \bar{u}), \sum_{|j| \leq N} (j)^{2p} |u_j|^2 \}_{w_y}. \quad (5.11) \]
From the definition of \( \{, \}_{w_0} \) and the structure of \( \mathcal{B}_{f_{w_0}}^{u}(u, \bar{u}), \{ \mathcal{B}_{f_{w_0}}^{u}(u, \bar{u}), \sum_{|j| \leq N} \langle j \rangle^{2p}|u_j|^2 \}_{w_0} \) still depends on \( (|u_j|^2)_{|j|>N} \) and \( (\Gamma_{\leq N} u, \Gamma_{\leq N} \bar{u}) \). To be more specific,

\[
\{ \mathcal{B}_{f_{w_0}}^{u}(u, \bar{u}), \sum_{|j| \leq N} \langle j \rangle^{2p}|u_j|^2 \}_{w_0} = 2\Re \sum_{(l,k) \in \Omega_{r,l}^p, |j| \leq N} \text{sgn}(\theta)\langle j \rangle^{2p}(f_{w_0})^l_{r,l,k} u^l \bar{u}^k. \tag{5.12}
\]

For any non-zero term \( u^l \bar{u}^k \) of the right side of (5.12), its index \((l, k)\) satisfies

\[
\mathcal{M}(l, k) = \mathcal{M}(\Gamma_{\leq N} l, \Gamma_{\leq N} k) = \sum_{|j| \leq N} (l_j - k_j) \cdot j = i. \tag{5.13}
\]

From (5.13), for any \(|j| \leq N\) with \( l_j \neq 0\), it satisfies that

\[
j = \sum_{|t| \leq N, t \neq j} (l_t - k_t) \cdot t + (l_j - 1 - k_j) \cdot j = i. \tag{5.14}
\]

Moreover, given \( p \geq 2\), by (1.21) and (5.14), the following inequalities hold true

\[
|j|^p \leq 2r^p \left( \sum_{|t| \leq N, t \neq j} (l_t + k_t) \cdot |t|^p + (l_j - 1 + k_j) \cdot |j|^p \right) \cdot \langle i \rangle^p \tag{5.15}
\]

and

\[
|j|^{2p} \leq |j| \cdot |j|^{p-\frac{1}{2}} \cdot |j|^{p-\frac{1}{2}} \leq 2N r^{p-\frac{1}{2}} |j|^{p-\frac{1}{2}} \left( \sum_{|t| \leq N, t \neq j} (l_t + k_t) \cdot |t|^{p-\frac{1}{2}} + (l_j - 1 + k_j) \cdot |j|^{p-\frac{1}{2}} \right) \cdot \langle i \rangle^{p-\frac{1}{2}}. \tag{5.16}
\]

I will estimate the coefficients of \( \{ \mathcal{B}_{f_{w_0}}^{u}(u, \bar{u}), \|u\|_{p}^2 \}_{w_0} \). When \( \theta = 0 \), for any \((l^0, k^0, i^0) \in \mathcal{A}_{(f_{w_0})}^{r, l, k}\) with \((l, k) \in \Omega_{r,l}^p\), it holds that

\[
|\mathcal{M}(l^0, k^0) - \frac{i^0}{2}| \leq \sum_{|j| \leq N} (l_j^0 + k_j^0) \cdot |j| \right. \left. + \sum_{|j_0| > N} |\text{sgn}(l_{j_0} + k_{j_0})j_{0}| + \frac{|i^0|}{2} \right.

\leq (r - 2)N + \sum_{|j_0| > N} |\text{sgn}(l_{j_0} + k_{j_0})j_{0}| + |i^0|

\leq 4(r - 2)N \cdot \left( \sum_{|j_0| > N} |\text{sgn}(l_{j_0} + k_{j_0})j_{0}| \right) \cdot \langle i^0 \rangle \tag{5.17}
\]

the last inequality hold by (1.21). From (5.15) and (5.17), the coefficients of \( \{ \mathcal{B}_{f_{w_0}}^{u}(u, \bar{u}), \|u\|_{p}^2 \}_{w_0} \) are bounded by

\[
8 \frac{C_{r+2}}{(i)^{3-p}} \sum_{|j| \leq N, t \neq j} (l_t + k_t) \cdot |t|^{p-\frac{1}{2}} + (l_j - 1 + k_j) \cdot |j|^{p-\frac{1}{2}} \sum_{|j_0| > N} |\text{sgn}(l_{j_0} + k_{j_0})j_{0}| \cdot i^0. \tag{5.18}
\]

Similarly, in the case \( \theta = 1 \), using (5.16), the coefficients of \( \{ \mathcal{B}_{f_{w_1}}^{u}(u, \bar{u}), \|u\|_{p}^2 \}_{w_1} \) are bounded by

\[
8 \frac{C_{r+2}}{(i)^{3-p+\frac{1}{2}}} \sum_{|j| \leq N, t \neq j} (l_t + k_t) \cdot |t|^{p-\frac{1}{2}} + (l_j - 1 + k_j) \cdot |j|^{p-\frac{1}{2}} \prod_{t \in \mathbb{Z}^*} \|t|^{\frac{k+1}{2}}. \tag{5.19}
\]

By Corollary 2 it holds that

\[
\|\{ \mathcal{B}_{f_{w_0}}^{u}(u, \bar{u}), \|u\|_{p}^2 \}_{w_0} \| \leq 8r^{p+1} Nc^{r-1} C_{f_{w_0}}^{r-2} \|\Gamma_{> N} u\|_{\frac{3}{2}} \|u\|_{p}^{-2}. \tag{5.20}
\]
Step 3: Estimate \( \{C^w_r(u, \bar{u}), \|u\|_p^2\}\)w.

When \( \theta = 0 \),
\[
|\{C^w_r(u, \bar{u}), \|u\|_p^2\}\w| = \{C^w_r(u, \bar{u}), \sum_{|t| \leq N} |u_t|^2 \langle t \rangle^{2p}\}_w + \{C^w_r(u, \bar{u}), \sum_{|t| > N} |u_t|^2 \langle t \rangle^{2p}\}_w \\
= \{C^w_r(u, \bar{u}), \sum_{|t| \leq N} |u_t|^2 \langle t \rangle^{2p}\}_w = 0,
\]
the last equation holds by the fact that
\[
\{u_j \bar{u}_{-j} \mid j \}^{2p} (|u_j|^2 + |u_{-j}|^2) \w = 0.
\]

It is easy to verify that \( \{C^w_r(u, \bar{u}), \sum_{|t| \leq N} |u_t|^2 \langle t \rangle^{2p}\}_w \) is still dependent on \( \{u_{j_0} \bar{u}_{-j_0} \mid j_0 \} > N \) and \( (\Gamma \leq N, \Gamma \leq N \bar{u}) \). Using the method of estimate \( \{B^w_r(u, \bar{u}), \sum_{|t| \leq N} |u_t|^2 \langle t \rangle^{2p}\}_w \), the estimate of \( \{C^w_r(u, \bar{u}), \sum_{|t| \leq N} |u_t|^2 \langle t \rangle^{2p}\}_w \) is obtained.

When \( \theta = 1 \),
\[
|\{C^w_r(u, \bar{u}), \|u\|_p^2\}\w| = \{C^w_r(u, \bar{u}), \sum_{|t| \leq N} |u_t|^2 \langle t \rangle^{2p}\}_1 + \{C^w_r(u, \bar{u}), \sum_{|t| > N} |u_t|^2 \langle t \rangle^{2p}\}_1.
\]

Using the method of estimate \( \{B^w_r(u, \bar{u}), \sum_{|t| \leq N} |u_t|^2 \langle t \rangle^{2p}\}_w \) in step 2, the estimate of \( \{C^w_r(u, \bar{u}), \sum_{|t| \leq N} |u_t|^2 \langle t \rangle^{2p}\}_1 \) can be obtained. The estimate of \( \{C^w_r(u, \bar{u}), \sum_{|t| > N} |u_t|^2 \langle t \rangle^{2p}\}_1 \) will be obtained by the following. For any nonzero term of \( C^w_r(u, \bar{u}) \) with index \( (l, k) \), there exists \( |j_0| > N \) with \( l_{j_0} = 1, k_{-j_0} = 1 \) (or \( l_{-j_0} = 1, k_{j_0} = 1 \)) such that
\[
2|j_0| = \sum_{|t| \leq N} (l_t - k_t) \cdot t - i,
\]
which follows from \( M(l, k) = i \). From the relation \([5.22]\), using \([4.21]\) it holds that
\[
|j_0| \leq \frac{1}{2} \left( \sum_{|t| \leq N} (l_t + k_t) \cdot |t| + |i| \right) \leq \langle i \rangle \left( \sum_{|t| \leq N} (l_t + k_t) \cdot |t| \right) \leq rN \langle i \rangle
\]
and
\[
|j_0|^{p-\frac{1}{2}} \leq 2r^{p-\frac{1}{2}} \left( \sum_{|t| \leq N} (l_t + k_t) \cdot \langle t \rangle^{p-\frac{1}{2}} \right) \cdot \langle i \rangle^{p-\frac{1}{2}}.
\]
By \([5.23]\) and \([5.24]\), the coefficients of
\[
|\{C^w_r(u, \bar{u}), \sum_{|t| > N} |u_t|^2 \langle t \rangle^{2p}\}_1| = |\Re \sum_{(l, k) \in \Omega^1_N} \sum_{t \in \mathbb{Z}^r} |i(j_0)^{2p}(\tilde{f}^{w_1})_{r,l,k} \prod_{t \in \mathbb{Z}^r} |t|^{l_{j_0} + k_t} u_t \bar{u}_t|^{2p}\]
are smaller than
\[
2|j_0|^{p-\frac{1}{2}} r^{p-\frac{1}{2}} N^{C^r-2 \frac{|\langle i \rangle^{\beta - p + \frac{1}{2}|}{|n| \leq N} (l_n + k_n) \langle n \rangle^{p-\frac{1}{2}} \prod_{t \in \mathbb{Z}^r} |t|^{l_{j_0} + k_t}}.
\]
Using Corollary \([2]\) it holds that
\[
|\{C^w_r(u, \bar{u}), \|u\|_p^2\}_w| \leq 12r^{p+1} e^{r-1} N^{C^r-2 \frac{|\Gamma > N u\|_2 \cdot \|u\|_p^{-3} \cdot \|u\|_p}.\]

Summing \([5.9]\), \([5.10]\), \([5.21]\) and \([5.27]\), inequality \([5.7]\) is obtained.

\]
5.2 Birkhoff normal form theorem

In this subsection, construct a coordinate transformation under which the Hamiltonian system \((3.8)\) will have an \(r_\ast + 3\) degree \((\theta, \gamma, \alpha, N)\)-normal form, for any given positive \(r_\ast\).

**Theorem 4** (Birkhoff normal form theorem). Suppose that system \((3.8)\) satisfies assumptions \(A_0-B_0\) and \(P^{w_\theta}(u, \bar{u})\) in \(H^{w_\theta}(u, \bar{u})\) defined in \((3.24)\) has \((\beta, \theta)\)-type symmetric coefficients semi-bounded by \(C_\theta > 0\) (\(\beta\) is big enough). Given \(\alpha > 1\), \(0 < \gamma < 1\) and integer \(r_\ast > 0\), take \(p\) satisfying \((\beta - 4)/2 > p > 2 + 4\alpha(r_\ast + 1)r_\ast^2\). There exist a positive real number \(\bar{R} > 0\) and a Lie-transformation \(T^{(r_\ast)}_{w_\theta} : B_p(\bar{R}/3) \to B_p(\bar{R})\) such that:

For any \(R < \bar{R}\) and any integer \(N\) fulfilling

\[
(R^{r_\ast - 2\gamma r_\ast + 1})^{\frac{1}{p - 2 - 2\gamma(r_\ast + 1)}} \leq N \leq (\gamma R)^{\frac{1}{(r_\ast + 1)(r_\ast + 2)}} \theta^2,
\]

the transformation \(T^{(r_\ast)}_{w_\theta}\) puts Hamiltonian \(H^{w_\theta}\) into

\[
H^{(r_\ast, w_\theta)} := H^{w_\theta} \circ T^{(r_\ast, w_\theta)} = H^{w_\theta} + Z^{(r_\ast, w_\theta)} + R^{N(r_\ast, w_\theta)} + R^{T(r_\ast, w_\theta)}
\]

which satisfies that

1) Both \(Z^{(r_\ast, w_\theta)}\) and \(R^{N(r_\ast, w_\theta)}\) are \((r_\ast + 3)\)-degree polynomials and \(R^{T(r_\ast, w_\theta)}\) is a power series which starts with \(r_\ast + 4\) degree polynomial. All of them have \((\beta, \theta)\)-type symmetric coefficients semi-bounded by \(C(\theta, r_\ast) := C_\theta \left(\frac{(r_\ast + 2)^{r_\ast + 1}}{\gamma} \right)\).

2) The polynomial \(Z^{(r_\ast, w_\theta)}(u, \bar{u})\) is \(r_\ast + 3\)-degree \((\theta, \gamma, \alpha, N)\)-normal form with respect to \(\omega^{w_\theta}\).

3) The polynomial \(R^{N(r_\ast, w_\theta)}(u, \bar{u}) := \sum^{r_\ast + 3}_{r=3} \Gamma^N_{r+4} g^{r, w_\theta}_{r+4}\) is an \((r + 4)\)-degree homogeneous polynomial with \((\beta, \theta)\)-type symmetric coefficients semi-bounded by \(C(\theta, r)\):

4) The canonical Lie-transformation \(T^{(r_\ast)}_{w_\theta}\) satisfies

\[
\sup_{(u, \bar{u}) \in B_p(\bar{R}/3)} \|T^{(r_\ast)}_{w_\theta}(u, \bar{u}) - (u, \bar{u})\|_p \leq C(\theta, p, r_\ast) R^{2 - \frac{1}{2(\gamma + 1)r_\ast^2}}, \tag{5.29}
\]

where \(C(\theta, p, r_\ast)\) is a constant dependent on \(\theta, p\) and \(r_\ast\).

5.3 Important Lemmas in the Proof of Theorem 4

In order to prove Theorem 4 it need not only to construct a bounded canonical transformation under which the Hamiltonian \(H^{w_\theta}\) in \((3.39)\) has an \(r_\ast + 3\) degree \((\theta, \gamma, \alpha, N)\)-normal form, but also to show that the new Hamiltonian function has \((\beta, \theta)\)-type symmetric coefficients semi-bounded by \(C(\theta, r_\ast)\). First, let us review the definition of canonical transformation.

**Definition 5.3.** Call a map \(\phi : \mathcal{H}(\mathbb{Z}^*, \mathbb{C}) \ni (u, \bar{u}) \to (\xi, \bar{\xi}) \in \mathcal{H}(\mathbb{Z}^*, \mathbb{C})\) canonical transformation under a symplectic form \(w_\theta\) (or a symplectic change of coordinates), if \(\phi\) is a diffeomorphism and preserves the Poisson bracket, i.e. \(\{f, g\}_{w_\theta} \circ \phi = \{f \circ \phi, g \circ \phi\}_{w_\theta}\).
A convenient way of constructing canonical transformations is as follows. Let \( \Phi^t_{S^u} \) be the flow generated by a regular function \( S^{u\theta}(u, \bar{u}) \) defined in \( \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C}) \) with respect to the symplectic structure \( u_\theta \). \( \Phi^t_{S^u} \mid_{t=0} = \text{id} \). If \( \Phi^t_{S^u} \) is well defined up to \( t = 1 \), then the map \( \Phi^t_{S^u} \mid_{t=1} \) is called a Lie transformation associated to \( S^{u\theta}(u, \bar{u}) \) under symplectic form \( u_\theta \). \( \Phi^t_{S^u} \) is canonical.

Given a regular function \( g \), the new function \( g \circ \Phi^t_{S^u} \) satisfies
\[
\frac{d^n}{dt^n} \left( g \circ \Phi^t_{S^u} \right) = \left\{ \left( g, S^{u\theta} \right)_{u_\theta}, \ldots \right\}_{u_\theta} \circ \Phi^t_{S^u}.
\]
Thus the Taylor expansion of \( g \circ \Phi^t_{S^u} \) in the variable \( t \) is
\[
g \circ \Phi^t_{S^u} = \sum_{\nu=0}^{\infty} g(\nu, S^{u\theta}) \circ \Phi^t_{S^u} \mid_{t=0} \nu = \sum_{\nu=0}^{\infty} g(\nu, S^{u\theta}) v^\nu,
\]
where
\[
g(0, S^{u\theta}) := g, \quad g(\nu, S^{u\theta}) := \frac{1}{\nu} \left( g(\nu-1, S^{u\theta}), S^{u\theta} \right)_{u_\theta}, \quad \nu \geq 1.
\] (5.30)

Take \( t = 1 \) and it follows that
\[
g \circ \Phi^t_{S^u} \mid_{t=1} = \sum_{\nu=0}^{\infty} g(\nu, S^{u\theta}).
\]

In this paper, denote \( \prec \) as \( \leq \prec \), where \( \prec \) is independent of \( R \). To improve the order of the \((\theta, \gamma, \alpha, N)\)-normal form of \( H^{u\theta} \), it needs to solve a linear equation to find a suitable generated function \( S^{u\theta} \) under symplectic form \( u_\theta \). The following lemma is to do this with respect to \( u_\theta \)-Poisson bracket.

**Lemma 5.2. (Homological Equation)** Given an integer \( N > 0 \), real numbers \( \gamma > 0 \) and \( \alpha > 0 \), suppose that an \( r \)-degree homogeneous polynomial \( g^{u\theta}(u, \bar{u}) \) has \((\beta, \theta)\)-type symmetric coefficients semi-bounded by \( C_{g^{u\theta}} > 0 \) \((\theta \in \{0, 1\})\). Then there exists an unique \( S^{u\theta}(u, \bar{u}) \) such that
\[
\{ H_0^{u\theta}, S^{u\theta}(u, \bar{u}) \}_{u_\theta} + \Gamma_{(\theta, \gamma, \alpha, N)}^{\omega^{u\theta}} \Gamma_{\leq 2}^{N} g^{u\theta}(u, \bar{u}) = \Gamma_{\leq 2}^{N} g^{u\theta}(u, \bar{u}),
\] (5.31)

where \( H_0^{u\theta} := \sum_{j \in \mathbb{Z}} \omega^{u\theta}_j |u_j|^2 \) with \( \omega^{u\theta}_j \in \mathbb{R} \). Moreover for any \((u, \bar{u}) \in \mathcal{H}^p(\mathbb{Z}^*, \mathbb{C})\) the Hamiltonian vector of \( S^{u\theta}(u, \bar{u}) \) holds
\[
\| X_{S^{u\theta}}(u, \bar{u}) \|_p \leq 4r^{p+1} C_{g^{u\theta}}^{-2} c^{-1} \gamma^{N} \| u \|_p \| u \|_2^{r-2}.
\]

**Proof.** By the definition of Poisson bracket \( \{ \; , \; \}_{u_\theta} \), the solution \( S^{u\theta}(u, \bar{u}) \) of (5.31) is still an \( r \)-degree homogeneous polynomial and has the following form
\[
S^{u\theta}(u, \bar{u}) = \sum_{i \in M^{u\theta}, |i| + k |r, M| l(k) = i} (S^{u\theta})^{i}_{r,l,k} u^i \bar{u}^k
\] (5.32)

with undetermined coefficients. Since \( g^{u\theta}(u, \bar{u}) \) has \((\beta, \theta)\)-type symmetric coefficients semi-bounded by \( C_{g^{u\theta}} \), by Remark 1.2 and Remark 5.1 \( \Gamma_{(\theta, \gamma, \alpha, N)}^{\omega^{u\theta}} \Gamma_{\leq 2}^{N} g^{u\theta}(u, \bar{u}) \) is an \( r \)-degree \((\theta, \gamma, \alpha, N)\)-normal form of \( \Gamma_{\leq 2}^{N} g^{u\theta}(u, \bar{u}) \) with \((\beta, \theta)\)-type symmetric coefficients semi-bounded by \( C_{g^{u\theta}} \), and its coefficients have the following form
\[
(\Gamma_{(\theta, \gamma, \alpha, N)}^{\omega^{u\theta}})_{r,l,k} := \begin{cases} 
(\Gamma_{\leq 2}^{N} g^{u\theta})_{r,l,k}, & \text{if } |\langle \omega^{u\theta}, I_\theta(l-k) \rangle| \leq \frac{C_{g^{u\theta}}}{N^{\alpha}}, \\
0, & \text{if } |\langle \omega^{u\theta}, I_\theta(l-k) \rangle| > \frac{C_{g^{u\theta}}}{N^{\alpha}}. 
\end{cases}
\] (5.33)
where $M_{l,k}$ is defined in Definition 5.1. Take (5.32) into equation (5.31) and get that for any $i \in M_{s,w}$ and any $l, k \in \mathbb{N}^\ast$ with $|l + k| = r$ and $\mathcal{M}(l, k) = i$,

$$-i\langle \omega_{w^g}, I_\theta(l - k) \rangle (S_{w^g})^i_{r,lk} + (\Gamma_{(\gamma, \alpha, N)} \Gamma_{\leq 2g_{w^g}}^N \Gamma_{\leq 2g_{w^g}}^N)^i_{r,lk} = (\Gamma_{\leq 2g_{w^g}}^N)^i_{r,lk},$$

(5.34)

which means that the coefficients of $S_{w^g}(u, \bar{u})$ has the following form

$$(S_{w^g})^i_{r,lk} = \begin{cases} -\frac{(\Gamma_{\leq 2g_{w^g}}^N)^i_{r,lk}}{i\langle \omega_{w^g}, I_\theta(l - k) \rangle}, & \text{when } |\langle \omega_{w^g}, I_\theta(l - k) \rangle| > \frac{\gamma M_{l,k}}{N^\ast} \\ 0, & \text{when } |\langle \omega_{w^g}, I_\theta(l - k) \rangle| \leq \frac{\gamma M_{l,k}}{N^\ast} \end{cases}$$

(5.35)

and satisfy that

$$\frac{(S_{w^g})^i_{r,lk}}{i\langle \omega_{w^g}, I_\theta(l - k) \rangle} = -\frac{(\Gamma_{\leq 2g_{w^g}}^N)^i_{r,lk}}{i\langle \omega_{w^g}, I_\theta(k - l) \rangle} = (S_{w^g})^{-i}_{r,lk},$$

(5.36)

the second equality holds by $\Gamma_{\leq 2g_{w^g}}^N(u, \bar{u})$ having symmetric coefficients from Remark 4.12 and $\omega_{j,w^g} \in \mathbb{R} \ (j \in \mathbb{Z}^\ast)$.

The norm of Hamiltonian vector field $X_{w^g}$

$$\| X_{w^g}^\ast (u, \bar{u}) \|_p = \sqrt{\| \nabla_u S_{w^g}^\ast (u, \bar{u}) \|^2_2 + \| \nabla_u S_{w^g}^\ast (u, \bar{u}) \|^2_2}$$

$$\leq \| (\sum_{j = \mathcal{M}(l, k - e_j) - i}^{\mathcal{M}(l, k - e_j) - i} j) \|_p + \| (\sum_{j = -\mathcal{M}(l, e_j - k) - i}^{\mathcal{M}(l, e_j - k) - i} j) \|_p$$

equals to the $\ell^2$ norm of the vector fields

$$Q_{1,w^g} := \left[ \sum_{i \in M_{s,w}} \sum_{j = \mathcal{M}(l, k - e_j) - i}^{\mathcal{M}(l, k - e_j) - i} (j) \rho_j k_j \cdot (S_{w^g})^i_{r,lk} u_l \bar{u}_k \right]_{j \in \mathbb{Z}^\ast}$$

and

$$Q_{2,w^g} := \left[ \sum_{i \in M_{s,w}} \sum_{j = -\mathcal{M}(l, e_j - k) - i}^{\mathcal{M}(l, e_j - k) - i} (j) \rho_j l_j \cdot (S_{w^g})^i_{r,lk} u_l \bar{u}_k \right]_{j \in \mathbb{Z}^\ast}.$$

When $\theta = 0$, for any $l, k \in \mathbb{N}^\ast$ with $\mathcal{M}(l, k) = i$ and $k_j \neq 0$, by (4.21), it holds

$$|j|^p \leq 2r^{p-1} (\sum_{t \in \mathbb{Z}^\ast} l_t |t|^p + \sum_{t \in \mathbb{Z}^\ast\neq j} k_t |t|^p + (k_j - 1)|j|^p) \cdot \langle i \rangle^p.$$  

(5.37)

For any $(l^0, k^0, l^0) \subset \mathcal{A}_{r,lk}$, by (4.21) the following inequality holds

$$|\mathcal{M}(l^0, k^0) - \frac{j^0}{2}| \leq 2r M_{l,k} \cdot \langle i_0 \rangle.$$  

(5.38)

By (5.35) and (5.38), the coefficients of $S_{w^g}$ satisfy that

$$\sum_{(l^0, k^0, l^0) \in \mathcal{A}_{(w^g)}(l^0, k^0, l^0)}^i \left| (\Gamma_{\leq 2g_{w^g}}^N)^i_{r,lk} (\mathcal{M}(l^0, k^0) - \frac{j^0}{2}) \right| \leq \frac{\gamma M_{l,k}}{N^\ast}.$$  

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Moreover, its coefficients satisfy some inequalities. By Corollary 2 and (5.42)-(5.43), the following estimate is obtained

\[ \langle j \rangle^p \cdot |k_j(S^{u_0})_{r,lk}|, \quad \langle j \rangle^p \cdot |l_j(S^{u_0})_{r,lk}| \leq 2r^{p+1} \sum_{t \in \mathbb{Z}^*} l_t(t)^p + \sum_{t \notin j} k_t(t)^p + (k_j - 1) \langle j \rangle^p \frac{N^\alpha C^\gamma_{g^{u_0}}}{\gamma(\gamma(\beta - p))}. \]  

(5.40)

By (5.40), using Corollary 2 it holds that

\[ \|X^{w_0}_{S^{u_0}}(u, \bar{u})\|_p \leq \|Q^{w_0}_1\|_\theta + \|Q^{w_0}_2\|_\theta \leq 4r^{p+1} C^{-1} \frac{N^\alpha C^\gamma_{g^{u_0}}}{\gamma} \|u\|_p \|u\|_2^{-2}. \]  

(5.41)

When \( \theta = 1 \), in order to estimate the \( \ell^2 \)-norm of \( Q^{w_1}_1(u, \bar{u}) \) and \( Q^{w_1}_2(u, \bar{u}) \), let us consider the coefficients of \( Q^{w_1}_1(u, \bar{u}) \) and \( Q^{w_1}_2(u, \bar{u}) \) firstly. For any \( i \in M_{w_1} \) and any \( l, k \in \mathbb{N}^\mathbb{Z} \) satisfying \( |l + k| = r \), \( M(l, k) = i \) and \( k_j \neq 0 \) (or \( l_j \neq 0 \)), using (5.35), the coefficients of \( u^l \bar{u}^k \) in \( Q^{w_1}_1 \) are bounded by the following

\[ \frac{2r^{p+1} C^\gamma_{g^{u_0}}}{\gamma(\beta - p + 2)} \langle \sum_{t \in \mathbb{Z}^*} \sum_{t \notin j} \sum_{t \notin j} k_t(t)^p \prod_{t \notin j} \frac{1}{t^{(l - e_j)t}} \rangle \]  

(5.42)

and the coefficients of \( u^l \bar{u}^k \) in \( Q^{w_1}_2(u, \bar{u}) \) are bounded by

\[ \frac{2r^{p+1} C^\gamma_{g^{u_0}}}{\gamma(\beta - p + 2)} \langle \sum_{t \notin j} \sum_{t \notin j} k_t(t)^p \prod_{t \notin j} \frac{1}{t^{(l - e_j)t + k_t}} \rangle. \]  

(5.43)

By Corollary 2 and (5.42), (5.43), the following estimate is obtained

\[ \|X^{w_1}_{S^{u_0}}(u, \bar{u})\|_p \leq \|Q^{w_1}_1\|_\theta + \|Q^{w_1}_2\|_\theta \leq 4r^{p+1} C^{-1} \frac{N^\alpha C^\gamma_{g^{u_0}}}{\gamma} \|u\|_p \|u\|_2^{-2}. \]

The following Lemma shows that the Poisson bracket of an \( \tilde{r} \)-degree homogeneous polynomial \( f^{w_0}(u, \bar{u}) \) with \( (\beta, \theta) \)-type symmetric coefficients semi-bounded by \( C_{f^{w_0}} > 0 \) and the solution \( S^{w_0}(u, \bar{u}) \) to equation (5.31) is still of \( (\beta, \theta) \)-type symmetric coefficients. Moreover, its coefficients satisfy some inequalities.

**Lemma 5.3.** Let an \( \tilde{r} \)-degree homogeneous polynomial \( f^{w_0}(u, \bar{u}) (\theta \in \{0, 1\}) \) have \( (\beta, \theta) \)-type symmetric coefficients semi-bounded by \( C_{f^{w_0}} > 0 \). Then the poisson bracket of \( f^{w_0}(u, \bar{u}) \) and the solution \( S^{w_0}(u, \bar{u}) \) to equation (5.31) under the symplectic form \( w_0 \) is an \( (\tilde{r} + r - 2) \)-degree homogeneous polynomial with \( (\beta, \theta) \)-type symmetric coefficients and it holds that

- when \( \theta = 0 \), for any \( l', k' \), \( i \) fulfilling \( |l' + k'| = \tilde{r} + r - 2 \) and \( M(l', k') = i \), it holds that

\[ \sum_{(l^0, k^0, \theta) \in A_{(1, S^{u_0})}^{(l^0, k^0, \theta)}} |(f_{1}^{w_0})^i(l^0, k^0, \theta_{i})| \cdot \max\{i^0, |i^0 - 2i\} \leq 2^{\beta + 2} r^{2} C_{f^{w_0}} \frac{C^\gamma_{g^{u_0}}}{\gamma} \langle \tilde{r} \rangle \beta \cdot (2N + 1)^{-2} N_{a+1}. \]  

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• when \( \theta = 1 \) the following inequality holds true

\[
|f_{\nu}^{w_1}(v_{\nu}(S_{\nu}^1))_{r+(\nu - 2)v}^{|r|} \leq 2^{\beta + 1} c r(\bar{r} + 1) \frac{C_{\nu}^{\nu - 2} C_{\nu}^{\nu - 2}}{\gamma(i)^\beta} \prod_{t \in \mathbb{Z}^*} |t|^{\frac{r+k}{\bar{r}+1}}.
\]

**Remark 5.4.** Under the same assumptions of Lemma 5.3, for any integer \( \nu \geq 1 \), \( f_{\nu}^{w_0}(v_{\nu}(S_{\nu}^0)) \) is an \((\bar{r} + \nu(r - 2))\)-degree homogeneous polynomial with \((\beta, \theta)\)-type symmetric coefficients.

- When \( \theta = 0 \), the following inequality holds

\[
\sum_{(l', k', i') \in A(l', k', i')} |f_{\nu}^{w_0}(v_{\nu}(S_{\nu}^0))_{l+(\nu - 2)v}^{|l'|} \cdot \max\{\langle i'^0\rangle, \langle i'^0 - 2i\rangle\} \leq C_{\nu}^{\nu - 2}(2^{\beta + 2} r^2 c(2N + 1)^{(r-2)} \frac{N^\alpha}{\gamma C_{\nu}^{\nu - 2}} \prod_{n=0}^{\nu-1} (\bar{r} + n(r - 2) + 1) \prod_{t \in \mathbb{Z}^*} |t|^{\frac{r+k}{\bar{r}+1}}.
\]

- When \( \theta = 1 \), it holds that

\[
|f_{\nu}^{w_1}(v_{\nu}(S_{\nu}^1))_{l+(\nu - 2)v}^{|l|} \leq C_{\nu}^{\nu - 2}(2^{\beta + 1} c(2N)^{(r-2)} \frac{N^\alpha}{\gamma C_{\nu}^{\nu - 2}} \prod_{n=0}^{\nu-1} (\bar{r} + n(r - 2) + 1) \prod_{t \in \mathbb{Z}^*} |t|^{\frac{r+k}{\bar{r}+1}}.
\]

Before proving Lemma 5.3 I denote a set of indexes and give a Lemma to count the number of this set. This Lemma is used to prove Lemma 5.3. For any \((l', k') \in \mathbb{N}^{2^*} \times \mathbb{N}^{2^*}\) and any \(i' \in \mathbb{Z}\), let

\[
\Omega(l', k', i') := \left\{ (l, k, i_1), (L, K, i_2), j) \mid l, k, L, K \in \mathbb{N}^{2^*}; i_1, i_2 \in \mathbb{Z}; j \in \mathbb{Z}^*; \right. \left. \text{satisfying A, B, D1 or A, B, D2} \right\},
\]

where

- **A:** \(|l + k| = \bar{r}, |L + K| = r, |\Gamma > N(L + K)| \leq 2;\)
- **B:** \(\mathcal{M}(l, k) = i_1, \mathcal{M}(L, K) = i_2, i' = i_1 + i_2;\)
- **D1:** \((l - e_j) + L = l', k + (K - e_j) = k', \) with \(l_j > 0 \) and \(K_j > 0;\)
- **D2:** \((l - e_j) + L = l', (k - e_j) + K = k', \) with \(L > 0 \) and \(k_j > 0;\)

From the definition of set \(\Omega(l', k', i')\), if element \((l, k, i_1), (L, K, i_2), j) \in \Omega(l', k', i')\), then \((k, l - i_1), (K, L - i_2), j) \in \Omega(k', l', -i').\)

**Lemma 5.4.** Fix \(\beta \geq 2\). For any given \(l', k' \in \mathbb{N}^{2^*}\) with \(|l' + k'| = r + \bar{r} - 2\) and \(\mathcal{M}(l', k') = i'\), it holds

\[
\sum_{((l,k,i_1),(L,K,i_2),j)\in\Omega(l',k',i')} \frac{K_j l_j}{(i_2 - i')^\beta \cdot (\bar{i}_2)^\beta} + \sum_{((l,k,i_1),(L,K,i_2),j)\in\Omega(l',k',i')} \frac{k_j L_i}{(i_2 - i')^\beta \cdot (\bar{i}_2)^\beta} \leq \frac{2^{\beta + 1}}{(\bar{i}_2)^\beta} c r(\bar{r} + 1)(2N + 1)^{r-2}.
\]
Proof. Consider the non-zero components of vectors $k'$ and $l'$. For example, $e_j$ has only one non-zero component with index $j$, being 1; Taking multiplicity into account, regard that $ke_j(k$ is a positive integer) has $k$ non-zero components whose values are 1 and their indexes are $j$. So $(l', k')$ with $|l' + k'| = r + \tilde{r} - 2$ have $r + \tilde{r} - 2$ non-zero components whose values are 1. Denote

$$\Omega_{j,i_2}(l', k', i') := \left\{ ((l, k, i_1), (L, K, \tilde{i}), t) \in \Omega(l', k', i') \mid t = j \text{ and } \tilde{i} = i_2 \right\}.$$  

It follows

$$\Omega(l', k', i') = \bigcup_{j \in \mathbb{Z}^*} \bigcup_{i_2 \in \mathbb{Z}} \Omega_{j,i_2}(l', k', i').$$

The element in $\Omega_{j,i_2}(l', k', i')$ is unique determined, if $(l - e_j, k)$ is fixed. The estimate of $\Omega_{j,i_2}(l', k', i')$ is obtained as follows. In the case $|j| \leq N$, since $|\Gamma_{\leq N}(L + K)| \leq 2$, there are at least $r - 3$ non-zero components of $(l', k')$ coming from $(L, K)$ with the indexes being bounded by $N$ and the choices of that is smaller than $(2N + 1)^{r-3}$. As for the remaining three components of $(L, K)$ whose values are 1, one of their positions is $j$ with $|j| \leq N$; One position among the other two can be selected from the rest non-zero components of $(l', k')$ and the choices is $\tilde{r} + 1$; The last one may be determined by the fact that $\mathcal{M}(L, K) = i_2$. It holds

$$\sum_{(l, k, i_1), (L, K, i_2), j \in U_{|j| \leq N} \bigcup_{i_2 \in \mathbb{Z}} \Omega_{j,i_2}(l', k', i'),} \frac{K_j l_j}{\langle i_1 \rangle^\beta \cdot \langle i_2 \rangle^\beta} \leq \frac{r}{\langle i' \rangle^\beta} \sum_{|j| \leq N} \sum_{i_2 \in \mathbb{Z}} \frac{l_j (i')^\beta}{\langle i_1 \rangle^\beta \cdot \langle i_2 \rangle^\beta} \sum_{i_2 \in \mathbb{Z}} \frac{(i_1)^\beta + (i_2)^\beta}{\langle i_1 \rangle^\beta \cdot \langle i_2 \rangle^\beta} \leq \frac{rc2^\beta}{\langle i' \rangle^\beta} (\tilde{r} + 1)(2N + 1)^{r-2}. \quad (5.44)$$

In the case $|j| > N$, there are at least $r - 2$ value-1 components of $(l', k')$ coming from $(L, K)$ whose indexes are bounded by $N$, and there are at most $(2N + 1)^{r-2}$ choices; One position of the last two value-1 components of $(L, K)$ is chosen from the rest $\tilde{r}$ non-zero components of $(l', k')$ and the choices is $\tilde{r}$; The position of the last component of $(L, K)$ is determined by the momentum of $(L, K)$ being $i_2$. It holds

$$\sum_{(l, k, i_1), (L, K, i_2), j \in U_{|j| > N} \bigcup_{i_2 \in \mathbb{Z}} \Omega_{j,i_2}(l', k', i'),} \frac{K_j l_j}{\langle i_1 \rangle^\beta \cdot \langle i_2 \rangle^\beta} \leq \frac{r}{\langle i' \rangle^\beta} \sum_{|j| > N} \sum_{i_2 \in \mathbb{Z}} \frac{l_j (i')^\beta}{\langle i_2 \rangle^\beta \cdot \langle i_1 \rangle^\beta} \sum_{i_2 \in \mathbb{Z}} \frac{(i_1)^\beta + (i_2)^\beta}{\langle i_1 \rangle^\beta \cdot \langle i_2 \rangle^\beta} \leq \frac{rc2^\beta}{\langle i' \rangle^\beta} \tilde{r}(2N + 1)^{r-2}. \quad (5.45)$$

The result is obtained from (5.44) and (5.45). \qed

In the following, the proof of the Lemma 5.3 is given.
Proof. By the definition of \( \{ \cdot, \cdot \}_w \), the following equation holds
\[
\left( f^{(u_0)}_{(1,S^{w_0})} \right)^{\nu}_{r + r - 2, l', k'} = \sum_{i \in \mathbb{Z}} \frac{\sum_{i \in \mathbb{Z}} (f^{(u_0)}_{(1,S^{w_0})})^{\nu}_{r + r - 2, l', k'} u^{\nu}_l u^{\nu}_{l'}
\]
with \( M_{(1,s^{w_0})} := \{ i = i_1 + i_2 \mid i_1 \in M_{f^{u_0}} \subset \mathbb{Z}, \ i_2 \in M_{S^{w_0}} \subset \mathbb{Z} \} \) and
\[
(f^{u_0}_{(1,S^{w_0})})^{\nu}_{r + r - 2, l', k'} = \sum_{j \in \mathbb{Z}^+} \frac{\sum_{j \in \mathbb{Z}^+} (l_j K_j (f^{u_0}_{(1,S^{w_0})})^{\nu}_{r, l_k} (S^{w_0})^{\nu}_{r, L, K})}{i_j K_j (f^{u_0}_{(1,S^{w_0})})^{\nu}_{r, l_k} (S^{w_0})^{\nu}_{r, L, K}}
\]
In the case \( \theta = 0 \), I will give the exact definition of \( A^{(u_0)}_{(1,s^{w_0})} \) and \( (f^{(u_0)}_{(1,S^{w_0})})^{\nu}_{r + r - 2, l', k'} \) and prove that the coefficients \( (f^{(u_0)}_{(1,S^{w_0})})^{\nu}_{r + r - 2, l', k'} \) can be rewritten as the following form
\[
(f^{(u_0)}_{(1,S^{w_0})})^{\nu}_{r + r - 2, l', k'} := \sum_{(l_0, k_0, i_0) \in A^{(u_0)}_{(1,s^{w_0})}} \frac{(f^{(u_0)}_{(1,S^{w_0})})^{\nu}_{r + r - 2, l', k'}}{(f^{(u_0)}_{(1,S^{w_0})})^{\nu}_{r + r - 2, l', k'}}
\]
and satisfy
\[
(f^{(u_0)}_{(1,S^{w_0})})^{\nu}_{r + r - 2, l', k'} = (f^{(u_0)}_{(1,S^{w_0})})^{\nu}_{r + r - 2, l', k'}
\]
In order to describe the set \( A^{(u_0)}_{(1,s^{w_0})} \) clearly, for any fixed \( (l, k, i_0), (L, k, i_0), j \) \( \in \Omega^{u_0}(l', k', i') \), define a mapping \( D \) on set \( A^{(u_0)}_{(1,s^{w_0})} \), for any \( (l_0, k_0, i_0) \in A^{(u_0)}_{(1,s^{w_0})} \),
\[
D^{(l_0, k_0, i_0)} := \begin{cases} 
(l_0, k_0, i_0) & \text{when } (l_0, k_0) = ((l - e_j) + (l - e_j) + (K - e_j)) \\
(l_0 - e_j + L + k_0 - e_j, i_0 + 2i_2) & \text{when } (l_0, k_0) = ((l - e_j) + (l - e_j) + (K - e_j)) \\
(l_0, k_0, i_0) & \text{when } (l_0, k_0) = ((l - e_j) + (l - e_j) + (K - e_j)) \\
(l_0 - e_j + L + k_0 - e_j, i_0 + 2i_2) & \text{when } (l_0, k_0) = ((l - e_j) + (l - e_j) + (K - e_j))
\end{cases}
\]
Base on the set \( \Omega^{u_0}(l', k', i') \) and the map \( D \), denote
\[
A^{(u_0)}_{(1,s^{w_0})} := \bigcup_{(l, k, j_0), (L, k, j_2), j} \bigcup_{n \in \Omega^{u_0}(l', k', i')} \bigcup_{(l, k, j_0), (L, k, j_2), j} \bigcup_{n \in \Omega^{u_0}(l', k', i')}
\]
and
\[
A^{(u_0)}_{(1,s^{w_0})} := \bigcup_{(l, k, j_0), (L, k, j_2), j} \bigcup_{n \in \Omega^{u_0}(l', k', i')} \bigcup_{(l, k, j_0), (L, k, j_2), j} \bigcup_{n \in \Omega^{u_0}(l', k', i')}
\]
where
\[
D^{(l_0, k_0, i_0)} := \{ D(l_0, k_0, i_0) \mid (l_0, k_0, i_0) \in A^{(u_0)}_{(1,s^{w_0})} \}
\]
It is easy to check that $D$ is not an inverse mapping from $\mathcal{A}_{(f^{\mathrm{wo}})_{r,ik}}$ to $\mathcal{D}\mathcal{A}_{(f^{\mathrm{wo}})_{r,ik}}$. Denote

\[
(f^{\mathrm{wo}})_{(1,S^{\mathrm{wo}})}(\ell, k^0, i^0) := \sum_{(l, k^0, i^0) \in D^{-1}(l^0, k^0, i^0)} i(l_j K_j - L_j k_j) (f^{\mathrm{wo}})_{l,ik} S^{\mathrm{wo}}_{l,ik}^2, \tag{5.47}
\]

where $D^{-1}(l^0, k^0, i^0) := \{(l^0, k^0, i^0) \in \mathcal{A}_{(f^{\mathrm{wo}})_{r,ik}} \mid D(l^0, k^0, i^0) = (l^0, k^0, i^0)\}$. For any $(l^0, k^0, i^0) \in \mathcal{A}_{(f^{\mathrm{wo}})_{r,ik}}$, it is easy to verify that $(l' - l^0, k' - k^0, i' - i^0) \in \mathcal{A}_{(f^{\mathrm{wo}})^{-1}_{(1,S^{\mathrm{wo}})}(\ell, k^0, i^0)}$. Moreover, by (5.47) and the facts that $f^{\mathrm{wo}}$ having $(\beta, 0)$-type symmetric coefficients and $S^{\mathrm{wo}}$ having symmetric coefficients, it holds

\[
\left\langle \frac{f^{\mathrm{wo}}}{(1,S^{\mathrm{wo}})} \right\rangle_{\ell(\ell')}^{\prime(\ell^0, k^0, i^0)} = i \sum_{(l, k^0, i^0) \in D^{-1}(l^0, k^0, i^0)} (l_j K_j - k_j L_j) (f^{\mathrm{wo}})_{l,ik} S^{\mathrm{wo}}_{l,ik}^2,
\]

\[
= i \sum_{(k-k^0, l-l^0, i-i^0) \in D^{-1}(k-k^0, l-l^0, i-i^0)} (l_j K_j - k_j L_j) (f^{\mathrm{wo}})_{l,ik} S^{\mathrm{wo}}_{l,ik}^2 - 2i,
\]

\[
= (f^{\mathrm{wo}})_{(1,S^{\mathrm{wo}})}(\ell, k^0, i^0) - \frac{i^0}{2} = \mathcal{M}(l' - l^0, k' - k^0, i' - i^0) - \frac{i^0}{2} - i'.
\]

So $f^{\mathrm{wo}}$ has $(\beta, 0)$-type symmetric coefficients semi-bounded by $C_{g^r c^{\mathrm{wo}}}$. By (5.47), (5.39) in Lemma 5.2, it holds

\[
\sum_{(l^0, k^0, i^0) \in \mathcal{A}_{(f^{\mathrm{wo}})_{r,ik}}} |(f^{\mathrm{wo}})_{(1,S^{\mathrm{wo}})}(\ell, k^0, i^0)| \cdot \max \{\langle i^0 \rangle, \langle i^0 - 2i' \rangle\} \leq \sum_{(l^0, k^0, i^0) \in \mathcal{A}_{(f^{\mathrm{wo}})_{r,ik}}} \sum_{(l^0, k^0, i^0) \in D^{-1}(l^0, k^0, i^0)} |(k_j L_j - K_j l_j) (f^{\mathrm{wo}})_{l,ik} S^{\mathrm{wo}}_{l,ik}^2| \cdot \max \{\langle i^0 \rangle, \langle i^0 - 2i' \rangle\} \leq 2r \cdot \sum_{(l^0, k^0, i^0) \in \mathcal{A}_{(f^{\mathrm{wo}})_{r,ik}}} \left\langle \frac{k_j L_j - K_j l_j}{(i_2)^2} \right\rangle^{(i_2)^2} \gamma \cdot \sum_{(l, k^0, i^0) \in D^{-1}(l^0, k^0, i^0)} \sum_{(l^0, k^0, i^0) \in \mathcal{A}_{(f^{\mathrm{wo}})_{r,ik}}} \left\langle \frac{k_j L_j - K_j l_j}{(i_2)^2} \right\rangle^{(i_2)^2} \cdot \gamma^{(i_2)^2}. \tag{5.48}
\]

By (5.48) and Lemma 5.4, it follows that

\[
\sum_{(l^0, k^0, i^0) \in \mathcal{A}_{(f^{\mathrm{wo}})_{r,ik}}} |(f^{\mathrm{wo}})_{(1,S^{\mathrm{wo}})}(\ell, k^0, i^0)| \cdot \max \{\langle i^0 \rangle, \langle i^0 - 2i' \rangle\} \leq 2^{\beta+2} r^2 c(\hat{r} + 1) N^{\alpha+1} C_{g^r c^{\mathrm{wo}}} \gamma \cdot \frac{(2N + 1) r^2}{\gamma^{(i_2)^2}}. \tag{5.49}
\]
When $\theta = 1$, the coefficients of $f_{(1,S^{w_1})}^{w_1}$ have the following form

$$\left(f_{(1,S^{w_1})}^{w_1}\right)^{i}_{\tilde{f} + r - 2, k'} := (\tilde{f}_{(1,S^{w_1})}^{w_1})^{i}_{\tilde{f}, r - 2, k'} \prod_{t \in \mathbb{Z}^*} |t|^{k_t' - k_t},$$

(5.50)

where

$$\left(f_{(1,S^{w_1})}^{w_1}\right)^{i}_{\tilde{f} + r - 2, k'} := \sum_{(l,k,i), (L,K,i), (r,lk)} \text{sgn}(j)(l_j K_j - L_j k_j)(\tilde{f}_{(1,S^{w_1})}^{w_1})^{i}_{r, l k} \frac{|j| (\tilde{g}_{(1,S^{w_1})}^{w_1})^{i}_{r, L K}}{|\omega^{w_1}, I_1(l - k)|}. \quad (5.51)$$

Since $f_{w_1}^{w_1}(u, \bar{u})$ and $g_{w_1}^{w_1}(u, \bar{u})$ have $(\beta, 1)$-type symmetric coefficients ($g_{w_1}^{w_1}(u, \bar{u})$ is given in equation (5.34)), from (5.51) the coefficients of $f_{(1,S^{w_1})}^{w_1}$ satisfy that

$$\left(f_{(1,S^{w_1})}^{w_1}\right)^{i}_{\tilde{f} + r - 2, k'} = \sum_{(l,k,i), (L,K,i), (r,lk)} \text{sgn}(j)(l_j K_j - L_j k_j)(\tilde{f}_{(1,S^{w_1})}^{w_1})^{i}_{r, l k} \frac{|j| (\tilde{g}_{(1,S^{w_1})}^{w_1})^{i}_{r, L K}}{|\omega^{w_1}, I_1(l - k)|}$$

$$= \sum_{(l,k,i), (L,K,i), (r,lk)} \text{sgn}(j)(L_j k_j - l_j K_j)(\tilde{f}_{(1,S^{w_1})}^{w_1})^{i}_{r, l k} \frac{|j| (\tilde{g}_{(1,S^{w_1})}^{w_1})^{i}_{r, L K}}{|\omega^{w_1}, I_1(k - l)|}$$

From (5.35), for any $(l, k)$ with nonzero $(S^{w_1})^{i}_{r, l k}$,

$$|\langle \omega^{w_1}, I_1(l - k) \rangle| > \frac{\gamma M_{l k}}{N^\alpha},$$

which implies that

$$|\langle \tilde{f}_{(1,S^{w_1})}^{w_1} \rangle^{i}_{\tilde{f} + r - 2, k'}|$$

$$\leq \sum_{(l,k,i), (L,K,i), (r,lk)} |(l_j K_j - L_j k_j) \cdot (\tilde{f}_{(1,S^{w_1})}^{w_1})^{i}_{r, l k} \frac{j (\tilde{g}_{(1,S^{w_1})}^{w_1})^{i}_{r, L K}}{|\omega^{w_1}, I_1(l - k)|}|$$

$$\leq \frac{N^\alpha C^{\alpha - 2} C^{\alpha - 2}_{\omega^{w_1}}}{\gamma} \sum_{(l,k,i), (L,K,i), (r,lk)} \frac{|l_j K_j - L_j k_j|}{(i_1)^{\beta}(i_2)^{\beta}}$$

$$\leq (\tilde{r} + 1)cr(2N)^{\tau - 2} \frac{N^{\alpha \beta + 1} C^{\alpha - 2} C^{\alpha - 2}_{\omega^{w_1}}}{\gamma (i)^{\beta}},$$

the last inequality holds by Lemma (5.4)

The proof of Theorem 4 is a purely technical matter and is relegated to Appendix.

6 Proof of Theorem 3

For any given integer $r_\ast \geq 0$, using Theorem 4, there exists a transformation $T_{\omega_0}^{(r_\ast)}$ changing the system (3.8) into

$$\begin{cases}
\dot{\tilde{u}}_j = -\text{sgn}(j) \cdot \partial_{\tilde{u}_j} H^{(r_\ast, \omega_0)}(\tilde{u}, \bar{u}), \\
\dot{\tilde{u}}_j = \text{sgn}(j) \cdot \partial_{\tilde{u}_j} H^{(r_\ast, \omega_0)}(\tilde{u}, \bar{u}),
\end{cases}$$

(6.1)
with Hamiltonian

$$H^{(r_*, w_0)}(\tilde{u}, \tilde{u}) = H_0^{w_0} + Z^{(r_*, w_0)}(\tilde{u}, \tilde{u}) + \mathcal{R}^{N(r_*, w_0)}(\tilde{u}, \tilde{u}) + \mathcal{R}^{T(r_*, w_0)}(\tilde{u}, \tilde{u}). \quad (6.2)$$

The solution $(\tilde{u}, \tilde{u})$ to (6.1) satisfies

$$\begin{align*}
\frac{d}{dt} \|\tilde{u}(t)\|_{p}^{2} &= \{\|\tilde{u}\|_{p}^{2}, H^{(r_*, w_0)}(\tilde{u}, \tilde{u})\}_{w_0} \\
&= \{\|\tilde{u}\|_{p}^{2}, H_0^{w_0} + Z^{(r_*, w_0)}(\tilde{u}, \tilde{u}) + \mathcal{R}^{N(r_*, w_0)}(\tilde{u}, \tilde{u}) + \mathcal{R}^{T(r_*, w_0)}(\tilde{u}, \tilde{u})\}_{w_0}. \quad (6.3)
\end{align*}$$

It is easy to get that

$$\{\|\tilde{u}\|_{p}^{2}, H_0^{w_0}(\tilde{u}, \tilde{u})\}_{w_0} = 0. \quad (6.4)$$

Using Theorem 4, Proposition 4.2 and Corollary 1, when $N$ satisfies (5.28), it holds that

$$\sup_{(\tilde{u}, \tilde{u}) \in B_p(R/4)} \{\|\tilde{u}\|_{p}^{2}, \mathcal{R}^{N(r_*, w_0)}(\tilde{u}, \tilde{u}) + \mathcal{R}^{T(r_*, w_0)}(\tilde{u}, \tilde{u})\}_{w_0} \leq \frac{1}{2} C(\theta, p, r_*) R^{r_*+1}, \quad (6.5)$$

the inequality is holding by the fact that for any $(\tilde{u}, \tilde{u}) \in B_p(R)$

$$\|\Gamma_N \tilde{u}\|_p \leq \frac{\|\Gamma_N \tilde{u}\|_p}{N^{p-3}} \quad \text{and} \quad \sum_{|i| > N} \frac{1}{|i|^\beta} \leq \frac{1}{N^p} \left( \sum_{|i| > N} \frac{1}{|i|^2} \right), \quad \text{as} \quad \beta > 2p + 4. \quad (6.6)$$

By Lemma 5.1 and (6.6), when $N$ satisfies (5.28), it follows that

$$\sup_{(\tilde{u}, \tilde{u}) \in B_p(R/4)} \{\|\tilde{u}\|_{p}^{2}, Z^{(r_*, w_0)}(\tilde{u}, \tilde{u})\}_{w_0} \leq \frac{1}{2} C(\theta, p, r_*) R^{r_*+1}. \quad (6.7)$$

Suppose that the initial value to (3.8) satisfies $(u(0), \tilde{u}(0)) \in B_p(R/6)$. If $R$ is small enough, the initial value $(u(0), \tilde{u}(0)) \in B_p(R/6)$ is transformed into

$$(\tilde{u}(0), \tilde{u}(0)) \in B_p(R/4). \quad (6.8)$$

Together with (6.3)-(6.5) and (6.7)-(6.8), the following inequality holds true

$$\|\tilde{u}(t)\|_{p}^{2} - \|\tilde{u}(0)\|_{p}^{2} \leq \int_{0}^{T} \|\tilde{u}(\tau)\|_{p}^{2} d\tau \leq C(\theta, p, r_*) R^{r_*+1} T, \quad (6.9)$$

where $T := \min\{ |t| \mid \|\tilde{u}(t), \tilde{u}(t)\|_{p} = R/2 \}$, which means that for any $|t| \leq T := \frac{1}{144 C(\theta, p, r_*) R^{r_*+1}}$,

$$\|\tilde{u}(t)\|_{p} \leq R/4, \quad \|\tilde{u}(t), \tilde{u}(t)\|_{p} \leq R/2. \quad (6.10)$$

From Theorem 4 when $R \ll 1$, $T_{w_0}(r_*)$ is an inverse transformation from $B_p(R/2)$ to $B_p(R)$. Then by (6.10), the solution $(u(t), \tilde{u}(t))$ to systems (3.8) with $(u(0), \tilde{u}(0)) \in B_p(R/6)$ satisfies

$$\|(u(t), \tilde{u}(t))\|_{p} \leq R, \quad \text{for any} \quad |t| < \frac{1}{R^{r_*-1}}.$$

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7 Proof of Theorem 1 and Theorem 2

7.1 Proof of Theorem 1

It is common knowledge that \((g^2)_{j \in \mathbb{Z}}\) are the eigenvalues of \(-\partial_{xx}\) under periodic boundary condition \(\psi(x, t) = \psi(x + 2\pi, t)\) with the corresponding eigenfunctions \(\{\phi_j(x) := \frac{\sqrt{\pi}}{\sqrt{2\pi}} e^{jix}\}_{j \in \mathbb{Z}}\). Take

\[
\psi(x, t) = \sum_{j \in \mathbb{Z}} u_j(t) \phi_j(x), \quad u_j := \int_0^{2\pi} \psi(x, t) \phi_{-j}(x) dx \tag{7.1}
\]

into equation (2.2) and obtain a Hamiltonian system,

\[
\begin{align*}
\dot{u}_j &= -\frac{i}{\partial u_j} H^{w_0}(u, \bar{u}), \\
\ddot{u}_j &= \frac{i}{\partial u_j} H^{w_0}(u, \bar{u}),
\end{align*}
\tag{7.2}
\]

for any \(j \in \mathbb{Z}\), with respect to 2-form \(w_0\) in (3.5), and the Hamiltonian function has the form

\[
H^{w_0}(u, \bar{u}) := H_0^{w_0} + P^{w_0}(u, \bar{u}),
\tag{7.3}
\]

where

\[
H_0^{w_0} := \sum_{j \in \mathbb{Z}} \omega_j^0 |u_j|^2, \quad \omega_j^0 := -j^2 + \tilde{V}_j = -j^2 + \frac{v_j^{w_0}}{(j)^m}.
\tag{7.4}
\]

Under assumptions \(A_1\) and \(A_2\) in section 2.1, the power series \(P^{w_0}(u, \bar{u})\) has the following form

\[
P^{w_0}(u, \bar{u}) = \sum_{r \geq 3} \sum_{M(l, k) = r \in M_{P^{w_0}}} \sum_{(l', k', -i_1) \in A(P^{w_0})_{r, l,k}} (P^{w_0})_{r, l, k}^{i(l', k', -i_1)} \cdot (\mathcal{M}(l', k') - \frac{i_1}{2}) u^{l'}\bar{u}^{k'},
\]

where \(A(P^{w_0})_{r, l,k} := \{(l, 0, -i) \mid i \in M_{P^{w_1}} \subset \mathbb{Z}\}, M_{P^{w_1}}\) is a symmetric set and

\[
(P^{w_0})_{r, l, k}^{i(l, 0, -i)} := \frac{\partial^{|[l]|}_{\phi_{[l]}} \partial^{[k]}_{\tilde{G}^{[k]}_{\tilde{u}^{[k]}}} \tilde{F}_{(0, 0)}(-i)}{(-2\pi)^{\frac{r}{2}!l!k!}}.
\]

Moreover, the following equation holds true for any \(l, k\) with \(|l + k| = r\) and \(\mathcal{M}(l, k) = i\)

\[
(P^{w_0})_{r, l, k}^{i(l, 0, -i)} = \frac{\partial^{|[l]|}_{\phi_{[l]}} \partial^{[k]}_{\tilde{G}^{[k]}_{\tilde{u}^{[k]}}} \tilde{F}_{(0, 0)}(-i)}{(-2\pi)^{\frac{r}{2}!l!k!}} = \frac{\partial^{[k]}_{\phi_{l}} \partial^{[i]}_{\tilde{G}^{[i]}_{\tilde{u}^{[i]}}} \tilde{F}_{(0, 0)}(i)}{(-2\pi)^{\frac{r}{2}!l!k!}} = (P^{w_0})_{r, l, k}^{i(k, 0, -i)}
\]

and there exists a constant \(C_1 > 0\) such that

\[
\sum_{(l', k', -i_1) \in A(P^{w_0})_{r, l,k}} \max\{|i_1|, (2i - i_1)|\} |(P^{w_0})_{r, l, k}^{i(l', k', -i_1)}| \leq C_1^r \frac{2^{-r}}{|i|^\beta},
\]

which means that \(P^{w_0}(u, \bar{u})\) has \((\beta, 0)\)-type symmetric coefficients semi-bounded by \(C_1 > 0\).
Lemma 7.1. For any given integers \( r_* \), \( N > 0 \) and real numbers \( \alpha \geq 4m + r_* + 8 \), \( 1 \gg \gamma > 0 \), there exists an open subset \( \Theta^\theta_m \subset \Theta^\theta_\alpha \) (\( \Theta^\theta_m \) defined in (2.4) and (2.14)), respectively) such that for any \( V \in \Theta^\theta_m \) and any \( (l, k) \) belongs to \( O_{r_* + 3, N}^{w_{\theta}} \) defined in (5.2), it satisfies
\[
|\langle \omega^{w_\theta}(V), I_\theta(l - k) \rangle| > \frac{\gamma M_{l,k}}{N^\alpha}.
\]

where
\[
\omega^{w_\theta}(V) = (\omega^{w_\theta}_j)_{j \in \mathbb{Z}^n}, \quad \omega^{w_\theta}_j := \text{sgn}^\theta(j) \cdot \left( -j^2 + \frac{v^{w_\theta}_j}{(j)^m} \right), \quad v^{w_\theta}_j \in [-1/2, 1/2].
\]

Moreover,
\[
\text{meas} \left( \frac{\Theta^\theta_m}{\Theta^\theta_\alpha} \right) \leq \frac{4r_* + 4 + m + 3}{N^\alpha - 2m - r_* - 3} \gamma.
\]

Remark 7.1. If \( \gamma > 0 \) is small enough, the set \( \Theta^\theta_m \) will have a positive measure. In particular, if \( \gamma \) approaches to 0, then the measure of \( \Theta^\theta_m \) will approach to the measure of \( \Theta^\theta_\alpha \).

Now give the proof of Lemma 7.1.

Proof. Denote
\[
\Theta^\theta_m/\Theta^\theta_\alpha := \bigcup_{3 \leq r \leq r_* + 3} \bigcup_{(l, k) \in O_{r_* + 3, N, 0}^{w_{\theta}} \cup O_{r_* + 3, N, 1}^{w_{\theta}} \cup O_{r_* + 3, N, 2}^{w_{\theta}}} \Theta^\theta_{r,l,k},
\]
where
\[
\Theta^\theta_{r,l,k} := \left\{ V \in \Theta^\theta_m \mid \right|\langle \omega^{w_\theta}(V), I_\theta(l - k) \rangle \right| \leq \frac{\gamma M_{l,k}}{N^\alpha} \}
\]
and
\[
O_{r_* + 3, N, n}^{w_{\theta}} := \left\{ (l, k) \in O_{r_* + 3, N}^{w_{\theta}} \mid |\Gamma_{>N}(l + k)| = n, \ |l + k| = r \right\},
\]
for \( n \in \{0, 1, 2\} \) and \( 1 \leq r \leq r_* + 3 \).

I only give the estimate of the measure of \( \Theta^\theta_{r,l,k} \) in the case \( (l, k) \in O_{r_* + 3, N, 2}^{w_{\theta}} \), which is more complex than the case \( (l, k) \in O_{r_* + 3, N, 0}^{w_{\theta}} \cup O_{r_* + 3, N, 0}^{w_{\theta}} \).

When the multi-index \( (l, k) \in O_{r_* + 3, N, 2}^{w_{\theta}} \), estimate the measure of \( \Theta^\theta_{r,l,k} \) in two cases.

(1) The first case
\[
(l, k) \in O_{r_* + 3, N, 2}^{w_{\theta}} := \{ (l, k) \in O_{r_* + 3, N, 2}^{w_{\theta}} \mid |\Gamma_{>N}| = 2 \text{ or } |\Gamma_{>N}k| = 2 \}.
\]
In this case, there exists \( |j_1|, |j_0| > N \) such that \( l_{j_1} = l_{j_0} = 1 \) or \( k_{j_1} = k_{j_0} = 1 \).

Without loss of generality, assume \( |j_0| \geq |j_1| > N \) with \( l_{j_0} = l_{j_1} = 1 \). So \( M_{l,k} = |j_0| \) and \( |\omega^{w_\theta}_j(V)| \geq j_0^2 - 1 \), \( |\omega^{w_\theta}_j(V)| \geq j_1^2 - \frac{1}{2} \). The other frequencies \( \omega^{w_\theta}_j(V)(|j| \leq N) \) are bounded by \( |\omega^{w_\theta}_j| \leq N^2 + 1 \).

If \( |j_0| > 4\sqrt{N} \), it follows that
\[
|\langle \omega^{w_\theta}(V), I_\theta(l - k) \rangle| = |l_{j_0} \omega^{w_\theta}_j + l_{j_1} \omega^{w_\theta}_j + \sum_{|j| \leq N} \omega^{w_\theta}_j(l_j - k_j)|
\]
\[
\geq |\omega^{w_\theta}_{j_0} + \omega^{w_\theta}_{j_1}| - \sum_{j \neq j_0, j_1, |j| \leq N} (l_j - k_j) \omega^{w_\theta}_j | \geq j_0^2 - \frac{1}{2} - (N^2 + 1)(r - 1)
\]
> \frac{\gamma |j_0|}{N^\alpha} = \frac{\gamma M_{l,k}}{N^\alpha}.

That means when \(|j_0| > 4\sqrt{r} N\), the set \(\Theta^\theta_{r,l,k}\) is empty. So it is only need to calculate the measure of \(\Theta^\theta_{r,l,k}\) whose multi-index \((l, k)\) being in the following set,

\[
O^{\nu_{u,g}}_{r+3,N,2a} := \{ (l, k) \in O^{\nu_{u,g}}_{r+3,N,2a} | N \leq M_{l,k} \leq 4\sqrt{r} N \} \subset O^{\nu_{u,g}}_{r+3,N,2a},
\]

(7.6)

the number of which are bounded by

\[
\# O^{\nu_{u,g}}_{r+3,N,2a} \leq 4\sqrt{r}(4N)^r.
\]

(7.7)

For any fixed \((l, k) \in O^{\nu_{u,g}}_{r+3,N,2a}\), there exists \(4\sqrt{r} N \geq |j_0| > N\) fulfilling

\[
\begin{cases}
    l_{j_0} + l_{-j_0} - k_{j_0} - k_{-j_0} \neq 0, & \theta = 0, \\
    l_{j_0} - k_{j_0} \neq 0 \text{ or } l_{-j_0} - k_{-j_0} \neq 0, & \theta = 1.
\end{cases}
\]

such that

\[
\begin{align*}
\left| \frac{\partial g_{u,g}}{\partial \nu^u \nu_j} \right| &= \frac{|l_{j_0} - l_{-j_0} - k_{j_0} - k_{-j_0}|}{|j_0|^m} \geq \frac{1}{|j_0|^m} \neq 0, & \theta = 0, \\
\left| \frac{\partial g_{u,g}}{\partial \nu^u \nu_{-j_0}} \right| &= \frac{|l_{j_0} - k_{j_0}|}{|j_0|^m} \geq \frac{1}{|j_0|^m} \neq 0, \text{ or } \left| \frac{\partial g_{u,g}}{\partial \nu^{-u} \nu_{-j_0}} \right| = \frac{|l_{-j_0} - k_{-j_0}|}{|j_0|^m} \geq \frac{1}{|j_0|^m} \neq 0, & \theta = 1.
\end{align*}
\]

(7.8)

The measure of \(\Theta^\theta_{r,l,k}\) has the following estimate by (7.8)

\[
\text{meas}(\Theta^\theta_{r,l,k}) \leq \frac{M_{l,k} \gamma}{N^\alpha} \left( \frac{\partial g_{u,g}}{\partial v_{-j_0}^{u,g}} \right)^{-1} \leq \frac{\gamma |j_0|^{m+1}}{N^\alpha} \leq \frac{\gamma (4\sqrt{r})^{m+1}}{N^\alpha - m - 1}.
\]

(7.9)

From (7.7) and (7.9), it holds that

\[
\text{meas}(\bigcup_{(l,k) \in O^{\nu_{u,g}}_{r+3,N,2a}} \Theta^\theta_{r,l,k}) = \text{meas}(\bigcup_{(l,k) \in O^{\nu_{u,g}}_{r+3,N,2a}} \Theta^\theta_{r,l,k}) \leq \frac{2^{m+2r} + \sqrt{r}^{m+2} \gamma}{N^\alpha - m - r - 1}.
\]

(7.10)

(2) The second case

\[(l,k) \in O^{\nu_{u,g}}_{r+3,N,2b} := \{ (l, k) \in O^{\nu_{u,g}}_{r+3,N,2a} | \text{there exist } |j_0| \neq |j_1| > N, \ l_{j_0} = k_{j_1} = 1 \text{ or } l_{j_1} = k_{j_0} = 1 \} \]

Without loss of generality, assume \(|j_1| > |j_0|\) and \(M_{l,k} = |j_1|\). By (7.5), it holds \(|\omega_{j_1}^{u,g}| \geq j_1^2 - \frac{1}{2}, \ |\omega_{j_0}^{u,g}| \leq j_0^2 + \frac{1}{2}\) and \(\omega_{j}^{u,g} \leq N^2 + 1, (|j| \leq N)\). If \(|j_1| > 4rN^2\), the following inequality holds

\[
\begin{align*}
|\langle (\omega_{j}^{u,g}, I_\theta(l - k)) \rangle| &\geq |\omega_{j_0}^{u,g} - \omega_{j_1}^{u,g}| - | \sum_{j \neq j_1, j_0} \omega_{j}^{u,g} (l_j - k_j) | \\
&\geq (|j_1|^2 - (|j_0|^2 + 1)) - (N^2 + 1)(r - 2) \\
&\geq (|j_1| + |j_0|)(|j_1| - |j_0|) - (N^2 + 1)(r - 2) - 1 \\
&\geq \gamma \frac{|j_1|}{N^\alpha} = \frac{\gamma M_{l,k}}{N^\alpha}.
\end{align*}
\]

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That means when $M_{t,k} > 4rN^2$, the set $\Theta_{r,l,k}^{\theta}$ is empty. It only needs to calculate the sum of the set $\Theta_{r,l,k}^{\theta}$ with $(l,k)$ being in the following set

$$O_{r+3,N,2b}^{w_{a,r}} = \{(l,k) \in O_{r+3,N,2b}^{w_{a,r}} \mid M_{l,k} \leq 4rN^2\}$$

which is bounded by

$$\sharp O_{r+3,N,2b}^{w_{a,r}} \leq 4r^2(4N)^{r+2}$$

(7.11)

There exists $j$ with $4rN^2 \geq |j|$ such that $l_j + l_{-j} - k_j - k_{-j} \neq 0$ and

$$\left| \frac{\partial g^{w_a}(v)}{\partial v_j} \right| > \frac{1}{|j|^m}.\quad (7.12)$$

Denote a set

$$\hat{\Theta}_{r,l,k}^{\theta} := \left\{ V(x) \in \Theta_{m}^{\theta} \mid |\langle \omega^{w_a}(V), I_\theta(l-k) \rangle| \leq \frac{4r\gamma}{N^{\alpha-2}} \right\}.$$

When $(l,k) \in O_{r+3,N,2b}^{w_{a,r}}$, using the fact $\Theta_{r,l,k}^{\theta} \subset \hat{\Theta}_{r,l,k}^{\theta}$ and (7.12), it implies that

$$\text{meas} \Theta_{r,l,k}^{\theta} \leq \text{meas} \hat{\Theta}_{r,l,k}^{\theta} \leq \left( \frac{\partial g^{w_a}(v)}{\partial v_j} \right)^{-1} \frac{4r\gamma}{N^{\alpha-2}} \leq \frac{2(4r)^{m+1}\gamma}{N^{\alpha-2m-2}}.\quad (7.13)$$

From (7.11) and (7.13), it holds that

$$\text{meas} \left( \bigcup_{(l,k) \in O_{r+3,N,2b}^{w_{a,r}}} \Theta_{r,l,k}^{\theta} \right) = \text{meas} \left( \bigcup_{(l,k) \in O_{r+3,N,2b}^{w_{a,r}}} \Theta_{r,l,k}^{\theta} \right) \leq \frac{4r^{m+4}\gamma^{m+3}}{N^{\alpha-2m-4}}.\quad (7.14)$$

Using the same method, the following inequality holds true

$$\text{meas} \left( \bigcup_{(l,k) \in O_{r+3,N,2b}^{w_{a,r}} \cup O_{r+3,N,1}^{w_{a,r}}} \Theta_{r,l,k}^{\theta} \right) \leq \frac{2^{m+2r+3}}{N^{\alpha-m-1}}.\quad (7.15)$$

In view of (7.15) and (7.14), one has

$$\text{meas}(\Theta_{m}^{\theta} \setminus \hat{\Theta}_{m,N}^{\theta}) \leq \sum_{r=3}^{r+3} \sum_{(l,k) \in (O_{r+3,N,0}^{w_{a,r}} \cup O_{r+3,N,1}^{w_{a,r}} \cup O_{r+3,N,2}^{w_{a,r}})} \text{meas} \Theta_{r,l,k}^{\theta} < \frac{4r^{+4+m}\gamma^{m+3}}{N^{\alpha-2m-4}}.$$

Now Theorem 1 is obtained by Theorem 3 and Lemma 7.1. The transformation $\psi(x,t) = \sum_{j \in Z} u_j(t) \phi_j(x)$ is from $L^2_p$ to $H^p([0,2\pi], \mathbb{C})$ and satisfies

$$\|u(t)\|_p \leq \|\psi(x,t)\|_{H^p([0,2\pi], \mathbb{C})} = \sup_{0 \leq |n| \leq p} \|D^a_x \psi(x,t)\|_{L^2} \leq (2\pi)^p \|u(t)\|_p.$$
7.2 Proof of Theorem \[2\]

The following statements deal with the solution to equation (2.11). It is obvious that \(j^2 (j \in \mathbb{Z}^\ast)\) is the eigenvalue of \((-\partial_{xx})\) under periodic boundary condition and \(\phi_j(x) := \frac{1}{\sqrt{2\pi}} e^{ij\pi x}\) is the corresponding eigenfunction. Precisely,

\[
(-\partial_{xx})\phi_j(x) = j^2 \phi_j(x) \quad \forall \ j \in \mathbb{Z}^\ast.
\]

For any \(\psi \in H_0^{p+1/2}([0, 2\pi], \mathbb{C})\),

\[
\psi(x, t) = \sum_{j \in \mathbb{Z}^\ast} \hat{\psi}_j(t) \phi_j(x),
\]

where \(\hat{\psi}_j(t) := \int_0^{2\pi} \overline{\psi(x, t)} \phi_{-j}(x) dx\). In order to transform equation (2.11) into an infinite dimensional Hamiltonian system under a standard symplectic form, I will use a tool given in [30]

\[
u(t) = (u_j(t))_{j \in \mathbb{Z}^\ast}, \quad u_j(t) := \overline{\hat{\psi}_j(t)} \quad \text{for any} \ j \in \mathbb{Z}^\ast. \tag{7.16}
\]

It is easy to get that if \(\psi \in H_0^{p+1/2}([0, 2\pi], \mathbb{C})\) then the corresponding Fourier coefficients vector \((\hat{\psi}_j)_{j \in \mathbb{Z}^\ast} \in \ell^p_{p+1/2}(\mathbb{Z}^\ast, \mathbb{C})\) and \(u \in \ell^p(\mathbb{Z}^\ast, \mathbb{C})\). Moreover, there exist constants \(\tilde{C}_1, \tilde{C}_2 > 0\) such that

\[
\tilde{C}_1 \|u\|_p \leq ||\hat{\psi}||_{H_0^{p+1/2}([0,2\pi],\mathbb{C})} \leq \tilde{C}_2 \|u\|_p. \tag{7.17}
\]

Under transformation (7.16), equation (2.11) therefore can be written into the following Hamiltonian system with respect to symplectic from \(w_1\) defined in (3.5), for any \(j \in \mathbb{Z}^\ast\),

\[
\begin{cases}
\dot{u}_j = -\text{isgn}(j) \frac{\partial H^{w_1}(u, \bar{u})}{\partial u_j} \\
\dot{\bar{u}}_j = \text{isgn}(j) \frac{\partial H^{w_1}(u, \bar{u})}{\partial u_j}
\end{cases} \tag{7.18}
\]

with the Hamiltonian

\[
H^{w_1}(u, \bar{u}) := H_0^{w_1} + P^{w_1}(u, \bar{u}), \tag{7.19}
\]

where

\[
H_0^{w_1} := \sum_{j \in \mathbb{Z}^\ast} \omega^1_j |u_j|^2, \quad \omega^1_j := \text{sgn}(j)(-j^2 + \hat{\nu}_j) = \text{sgn}(j)(-j^2 + \frac{\nu_j^{w_1}}{|j|^m}) \in \mathbb{R}. \tag{7.20}
\]

By assumptions B1-B2 in Theorem \[2\] \(P^{w_1}(u, \bar{u})\) has a zero at origin at last order 3 with the following form

\[
P^{w_1}(u, \bar{u}) := \sum_{r=3}^{+\infty} \sum_{\sum |m| = r} \left(\hat{P}^{w_1}\right)_{r,l,k} \prod_{l \in \mathbb{Z}^\ast} |t^{\frac{l+k}{2}} u^l \bar{u}^{-k},
\]

where \(M_P^{w_1} \subseteq \mathbb{Z}\) is a symmetric set

\[
\left(\hat{P}^{w_1}\right)_{r,l,k} := \frac{1}{(2\pi)^{r-1} l! k!} \left. \frac{\partial^{l+k} F(u)}{\partial u^l \bar{u}^k} \right|_{(0,0)} (-i)^l, \quad l! = \prod_{j \in \mathbb{Z}^\ast} l_j!.
\]
And there exists a constant $C_2 > 0$ such that for any $r \geq 3$ and any $i \in M_{P^{(w)}} \subseteq \mathbb{Z}$,

$$\left| (P^{(w)})^i_{r,kl} \right| \leq \frac{C_2^{r-2}}{(i)^3}$$

and

$$\left( P^{(w)} \right)^{i}_{r,kl} = \frac{1}{(2\pi)^{\frac{r+1}{2}lkl!}} \left| \frac{\partial^{i+k} F}{\partial \psi^i \partial \psi^k} \right|_{(0,0)} (-i) = \frac{1}{(2\pi)^{\frac{r+1}{2}lkl!}} \left| \frac{\partial^{i+k} F}{\partial \psi^i \partial \psi^k} \right|_{(0,0)} (i) = (\tilde{P}^{(w)})^{-i}_{r,kl}$$

which means that the power series $P^{(w)}(u, \bar{u})$ has $(\beta, 1)$-type symmetric coefficients semi-bounded by $C_2$.

From (7.20), the origin is the elliptic equilibrium point of the equation (7.18). Using Theorem 3, Lemma 7.3, and (7.17), for any $V \in \Theta^1_{m}$, there exists $\tilde{\varepsilon} \ll 1$ such that for any $0 < \varepsilon < \tilde{\varepsilon}$, if

$$\|\tilde{\psi}(0, x)\|_{H^{5+1/2}_0([0, 2\pi], \mathbb{C})} < \varepsilon < \tilde{\varepsilon},$$

then it satisfies

$$\|\tilde{\psi}(t, x)\|_{H^{5+1/2}_0([0, 2\pi], \mathbb{C})} < 2\varepsilon, \quad \text{for any } |t| < \varepsilon^{-r+1}.$$

### 8 Appendix

Now the proof of Theorem 3 is given in this section.

**Proof.** For any $\theta \in \{0, 1\}$, denote

$$g^{(-1, w_0)}(u, \bar{u}) := \sum_{r=3}^{r+s+3} P^{(w)}_r(u, \bar{u}) , \ R^{N(-1, w_0)}(u, \bar{u}) := 0, \ R^T(-1, w_0)(u, \bar{u}) := \sum_{r=r+s+4}^{\infty} P^{(w)}_r(u, \bar{u}),$$

where $P^{(w)}_r(u, \bar{u})$ is an $r$-degree homogeneous polynomial of $P^{(w)}(u, \bar{u})$. Thus (3.9) can be rewritten as

$$H^{(-1, w_0)} = H^{w_0}_0 + g^{(-1, w_0)} + R^{N(-1, w_0)} + R^T(-1, w_0), \quad \text{defined in } B_p(R_3). \quad (8.1)$$

To start with, the results hold at rank $r = 0$. For any $R < R_3$ and any $N$ satisfying (5.28), I will look for a bounded Lie-transformation $\mathcal{T}^{w_0}_0$ to eliminate the non-normalized monomials of $\Gamma^{N}_{\leq 2g^3_3}(-1, w_0)$. The Lie-transformation $\mathcal{T}^{w_0}_0$ is constructed from 1-time flow $\Phi^{l}_{S^{(0)}_0}$ of the following equations,

$$\begin{cases}
\dot{u}_j = -i \text{sgn}^\theta(j) \nabla_{u_j} S^{(0)}_0(u, \bar{u}), \\
\dot{\bar{u}}_j = i \text{sgn}^\theta(j) \nabla_{\bar{u}_j} S^{(0)}_0(u, \bar{u}),
\end{cases} \quad j \in \mathbb{Z}^*, \ \theta \in \{0, 1\},$$

where $S^{(0)}_0$ is undetermined. Under transformation $\mathcal{T}^{w_0}_0$ the new Hamiltonian $H^{(0, w_0)}$ has the following form,

$$H^{(0, w_0)} = H^{(-1, w_0)} \circ \mathcal{T}^{w_0}_0 = (H^{w_0}_0 + g^{(-1, w_0)} + R^T(-1, w_0)) \circ \Phi^{l}_{S^{(0)}_0}$$

$$= H^{w_0}_0 + \{H^{w_0}_0, S^{(0)}_0 \}_{w_0} + g^{(-1, w_0)} \quad (8.2)$$

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The Lie-transformation \( T \) by solving the following homological equation

\[
\frac{\partial}{\partial t} \Gamma_{\nu, S_{w}^{(0)}} = (H^{w}_{0})_{(\nu, S_{w}^{(0)})} + \sum_{r+4}^{\infty} (\gamma_{r})(\nu, S_{w}^{(0)}) + \sum_{\nu+4}^{\infty} (\gamma_{r})(\nu, S_{w}^{(0)}) + \sum_{p=0}^{\infty} (\gamma_{r})(\nu, S_{w}^{(0)}), \tag{8.3}
\]

where \((\cdot)_{(\nu, S_{w}^{(0)})}\) is defined in (5.31). The auxiliary Hamiltonian function \( S_{w}^{(0)} \) are obtained by solving the following homological equation

\[
\{ H^{w}_{0}, S_{w}^{(0)} \}_{w} + \Gamma_{\nu, S_{w}^{(0)}}^{N} P_{3}^{w} = Z_{3}^{w}. \tag{8.4}
\]

Using Remark 4.2, \( \Gamma_{\nu, S_{w}^{(0)}}^{N} \) and \( \Gamma_{\nu, S_{w}^{(0)}}^{N} P_{3}^{w} \) are still having \((\beta, \theta)\)-type symmetric coefficients semi-bounded by \( C(\theta, -1) = C_{\theta} > 0 \). From Lemma 5.2, \( Z_{3}^{w} \) is \((\theta, \gamma, \alpha, N)\)-normal form of \( \Gamma_{\nu, S_{w}^{(0)}}^{N} P_{3}^{w} \) and the Hamiltonian vector field of \( S_{w}^{(0)} \) satisfies

\[
\| X_{S_{w}^{(0)}}^{w} \|_{p} \leq 8c^{2} C_{\theta}^{3p+1} \frac{N^{\alpha}}{\gamma} \| u \|_{p} \| u \|_{2} \quad \text{for any} \quad (u, \bar{u}) \in B_{p}(2R). \tag{8.5}
\]

From (8.4), the following holds true

\[
(8.2) = Z_{3}^{w}(u, \bar{u}) + \Gamma_{\nu, S_{w}^{(0)}}^{N} P_{3}^{w}. \tag{8.6}
\]

The Lie-transformation \( T_{0}^{w} \) satisfies

\[
\sup_{(u, \bar{u}) \in B_{p}(R)} \| T_{0}^{w}(u, \bar{u}) - (u, \bar{u}) \|_{p} = \sup_{(u, \bar{u}) \in B_{p}(R)} \| \Phi_{S_{w}^{(0)}}^{w}(u, \bar{u}) - (u, \bar{u}) \|_{p} = \sup_{(u, \bar{u}) \in B_{p}(R)} \| \int_{t=0}^{1} X_{S_{w}^{(0)}}^{w}(\tau) d\tau \|_{p}. \tag{8.7}
\]

Use the bootstrap method to estimate \( T_{0}^{w} \). First, assume that

\[
\Phi_{S_{w}^{(0)}}^{w} : B_{p}(R) \to B_{p}(2R), \quad \text{for any} \quad t \in [0, 1]. \tag{8.8}
\]

By (8.5), (8.7), the following inequality holds true

\[
\sup_{(u, \bar{u}) \in B_{p}(R)} \| T_{0}^{w}(u, \bar{u}) - (u, \bar{u}) \|_{p} \leq \sup_{(u, \bar{u}) \in B_{p}(2R)} \| \int_{t=0}^{1} X_{S_{w}^{(0)}}^{w}(\tau) d\tau \|_{p} \leq 8C_{\theta}^{2} \frac{N^{\alpha}}{\gamma} 3^{p+1}(2R)^{2}. \tag{8.9}
\]

Since \( R \) is small enough, from (5.28) and (8.8), the transformation \( T_{0}^{w} \) satisfies

\[
\sup_{(u, \bar{u}) \in B_{p}(R)} \| T_{0}^{w}(u, \bar{u}) - (u, \bar{u}) \|_{p} \leq R,
\]

which means

\[
T_{0}^{w} : B_{p}(R) \to B_{p}(2R). \tag{8.9}
\]

Denote \( T_{w}^{(0)} := T_{0}^{w} \). By (5.28), (8.6) and (8.8), it is verified that (5.29) holds for rank \( r = 0 \):

\[
\sup_{(u, \bar{u}) \in B_{p}(R)} \| T_{w}^{(0)}(u, \bar{u}) - (u, \bar{u}) \|_{p} \leq C(\theta, p, r_{s}) R^{2^{r_{s}+1}}. \tag{8.10}
\]

Set

\[
Z^{(0,w)} := Z^{(-1,w)} + Z_{3}^{w}, \quad R^{N(0,w)} := R^{N(-1,w)} + \Gamma_{\nu, S_{w}^{(0)}}^{N}(-1,w). \tag{8.10}
\]

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Since $Z^w_3$ and $g_3^{(-1, w_g)}$ having $(\beta, \theta)$-type symmetric coefficients, then $Z^{(0, w_g)}$ and $R^{N(0, w_g)}$ are still having $(\beta, \theta)$-type symmetric coefficients. Denote the $r_\ast + 3$-degree polynomial of power series $g^{(0, w_g)}$ and the remainder as $R^{T(0, w_g)}$, i.e.,

$$g^{(0, w_g)} := \sum_{t=1}^{r_\ast+3} g_t^{(0, w_g)}, \quad R^{T(0, w_g)} := \sum_{t>r_\ast+3} R_t^{T(0, w_g)},$$

where for any $4 \leq t \leq r_\ast + 3$,

$$g_t^{(0, w_g)} := g_t^{(-1, w_g)} + (H^{w_g}_0)_{t-2, S^{(0)}_{w_g}} + \sum_{n'=1}^{t-3} (g_{t-n'}^{(-1, w_g)})_{(n', S^{(0)}_{w_g})}$$

and for any $t > r_\ast + 3$

$$R_t^{T(0, w_g)} := (H^{w_g}_0)_{t-2, S^{(0)}_{w_g}} + \sum_{n'=1}^{t-3} (g_{t-n'}^{(-1, w_g)})_{(n', S^{(0)}_{w_g})} + \sum_{n'=1}^{t-3} (R_{t-n'}^{T(1, w_g)})_{(n', S^{(0)}_{w_g})}.$$

From Remark 5.4 and Lemma 4.3, $g^{(0, w_g)}$ and $R^{T(0, w_g)}$ have $(\beta, \theta)$-type symmetric coefficients. In order to estimate them, one needs to estimate the coefficients of functions $(H^{w_g}_0)_{(t-2, S^{(0)}_{w_g})}$, $\sum_{n'=1}^{t-3} (g_{t-n'}^{(-1, w_g)})_{(n', S^{(0)}_{w_g})}$ and $\sum_{n'=1}^{t-3} (R_{t-n'}^{T(1, w_g)})_{(n', S^{(0)}_{w_g})}$. By Remark 5.4, when $\theta = 0$, for any $|l + k| = t \geq 4$, any $1 \leq n' \leq t - 3$ and any $i \in M(g^{(-1, w_g)})_{(n', S^{(0)}_{w_g})}$, it holds

$$\sum_{(p, k, n, i) \in A} \left| \langle g_{t-n'}^{(-1, w_g)} \rangle_{(n', S^{(0)}_{w_g})} (p, k, n, i) \right| \leq \frac{C_4^{t-2}}{\langle i \rangle^2} \left( 144 N^{\alpha + 2} \gamma 2^\beta \right) \frac{1}{n'} \sum_{n=0}^{n'-1} \prod_{n=0}^{t-1} (t - n' + n + 1),$$

and when $\theta = 1$, it holds

$$\sum_{(p, k, n, i) \in A} \left| \langle Z^{w_0}_3 \rangle_{(t-3, S^{(0)}_{w_g})} (p, k, n, i) \right| \leq \frac{C_4^{t-2}}{\langle i \rangle^2} \left( 72 c \gamma \right) \frac{1}{n'} \sum_{n=0}^{n'-1} \prod_{n=0}^{t-1} (t - n' + n + 1).$$

By equation 8.4, when $\theta = 0$ it follows

$$\sum_{(p, k, n, i) \in A} \left| \langle Z^{w_0}_3 \rangle_{(t-3, S^{(0)}_{w_g})} (p, k, n, i) \right| \leq \frac{1}{\langle i \rangle^2} \left( 144 N^{\alpha + 2} \gamma 2^\beta \right) \frac{1}{t-3} \prod_{n=0}^{t-4} (4 + n);$$

when $\theta = 1$ it follows

$$\sum_{(p, k, n, i) \in A} \left| \langle Z^{w_0}_3 \rangle_{(t-3, S^{(0)}_{w_g})} (p, k, n, i) \right| \leq \frac{1}{\langle i \rangle^2} \left( 72 c \gamma \right) \frac{1}{t-3} \prod_{n=0}^{t-4} (4 + n).$$
When $N$ satisfies (6.28), using (8.12)-(8.14), in the case $\theta = 0$ it holds that
\[
\sum_{(\ell, k_0, \delta) \in A_r} |(g^{(0, u_0)})_t|_{t, l_k} \cdot \max \{ (i^0), (2i - i^0) \}
\leq \sum_{(\ell, k_0, \delta) \in A_r} |(g^{(-1, u_0)})_t|_{t, l_k} \cdot \max \{ (i^0), (2i - i^0) \}
\]
\[
+ \sum_{(\ell, k_0, \delta) \in A_r} (Z^{(1)})_t \cdot \max \{ (i^0), (2i - i^0) \}
\]
\[
\leq \frac{1}{(i^0)^{-2}} (C(0, 0))^{t^{-2}}.
\]
When $\theta = 1$ it follows
\[
|\tilde{\gamma}^{(0, u_1)}_t|_{t, l_k}
\leq |(\gamma^{(-1, u_1)}_t)_{t, l_k}| + |(Z^{(W_1)})_t \cdot \max \{ (i^0), (2i - i^0) \}
\]
\[
\leq \frac{1}{(i^0)^{-2}} (C(1, 0))^{t^{-2}}.
\]
Similarly, $R^{T_{(u, v)}}(u, \bar{u})$ has still $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C(\theta, 0)$. Now assume that the results hold for rank $r < r_*$ and a Lie-transformation which changes Hamiltonian (8.1) into the following form
\[
H^{(r, u_0)} = H_0^{u_0} + Z^{(r, u_0)} + N^{(r, u_0)} + g^{(r, u_0)} + R^{T_{(r, u_0)}},
\]
which is defined in $B_r(R_r)$ ($R < \tilde{R} < R_*$), where $R_r := \frac{2r - r_0}{2r} R$. One should construct a bounded Lie-transformation $T^{r_{(u, v)}}$ to eliminate the non-normalized monomials of $\Gamma_{\leq 2r_+, 4}^{(r, u_0)}$. Because $g^{(r, u_0)}$ have $(\beta, \theta)$-type symmetric coefficients, by Remark 4.24 the coefficients of $\Gamma_{\leq 2r_+, 4}^{(r, u_0)}$ and $\Gamma_{> 2r_+, 4}^{(r, u_0)}$ are $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C(\theta, r)$. Make use of the 1-time flow of the following equation, for any $j \in \mathbb{Z}^*$
\[
\begin{cases}
\dot{u}_j = \text{isgn}^0 (j) \partial_{u_j} S^{(r_{(u, v)})}(u, \bar{u}), \\
\ddot{u}_j = -\text{isgn}^0 (j) \partial_{u_j} S^{(r_{(u, v)})}(u, \bar{u}),
\end{cases}
\]
to define a Lie-transformation $T^{r_{(u, v)}}$, under which the new Hamiltonian has the following form formally,
\[
\begin{align}
H^{(r+1, u_0)} := H^{(r, u_0)} \circ T^{r_{(u, v)}} = H_0^{u_0} + Z^{(r+1, u_0)} + N^{(r+1, u_0)} + g^{(r+1, u_0)} + R^{T_{(r+1, u_0)}} \\
+ \sum_{l=0}^{r+3} T^{(r+1, u_0)} + \sum_{\nu \geq 2} (H_0^{(r, u_0)})_{(\nu, S^{(r_{(u, v)})})} + \sum_{\nu \geq 1} (Z^{(r, u_0)} + g^{(r, u_0)} + R^{(r, u_0)})_{(\nu, S^{(r_{(u, v)})})} + \sum_{\nu \geq 0} (R^{T_{(r, u_0)})}_{(\nu, S^{(r_{(u, v)})})},
\end{align}
\]
The auxiliary Hamiltonian $S_{w^g}^{(r)}$ can be obtained by solving the following homological equation
\[
\{H_{w^g}^{(r)}, S_{w^g}^{(r)}\} + \Gamma_{\leq 2g + 4}^{N} = Z_{r+4}.
\] (8.17)

From Lemma 5.2, $Z_{r+4}$ is $(\theta, \gamma, \alpha, N)$-normal form of $\Gamma_{\leq 2g + 4}^{N}$ and
\[
Z_{r+4} = Z_{r+4} + \Gamma_{> 2g + 4}^{N}.
\]  

The Hamiltonian vector field $X_{S_{w^g}^{(r)}}^{w^g}$ satisfies
\[
\sup_{(u, \bar{u}) \in B_p(R_r)} \|X_{S_{w^g}^{(r)}}^{w^g}(u, \bar{u})\|_p \leq 8(C(\theta, r))^{r+2}(r + 4)p^{r+3}N^{\alpha/\gamma}R^{r+3}. 
\] (8.18)

Using (8.13) and bootstrap method, suppose that
\[
\Phi_{S_{w^g}^{(r)}}^{(r)} : B_p(R_{r+1}) \to B_p(R_r),
\] (8.19)
for any $t \in [0, 1]$.

\[
\left\| T_{r}^{w^g}(u, \bar{u}) - (u, \bar{u}) \right\|_p = \sup_{(u, \bar{u}) \in B_p(R_r)} \| \Phi_{S_{w^g}^{(r)}}^{(r)}(u, \bar{u}) - (u, \bar{u}) \|_p
\]
\[
\leq 16(C(\theta, r))^{r+2}(r + 4)p^{r+3}N^{\alpha/\gamma}R^{r+3}. 
\] (8.20)

By (5.28) and (8.20), the transformation $T_{r}$ satisfies
\[
\sup_{(u, \bar{u}) \in B_p(R_r)} \| T_{r}^{w^g}(u, \bar{u}) - (u, \bar{u}) \|_p \leq \delta/2 = (R_r - R_{r+1})/2,
\]
which verifies (8.19). Denote $T_{w^g(r+1)} := T_{r}^{w^g} \circ T_{r}^{w^g}$. By (8.20) and (8.19), noting that $R < R_r < R_{r+1} < 1$, it holds
\[
\sup_{(u, \bar{u}) \in B_p((R_r + R_{r+1})/2)} \| T_{w^g(r+1)}(u, \bar{u}) - (u, \bar{u}) \|_p
\]
\[
\leq \sup_{(u, \bar{u}) \in B_p((R_r + R_{r+1})/2)} \left(\| T_{w^g(r)} \circ T_{r}^{w^g}(u, \bar{u}) - T_{r}^{w^g}(u, \bar{u}) \|_p + \| T_{r}^{w^g}(u, \bar{u}) - (u, \bar{u}) \|_p \right)
\]
\[
\leq \sup_{(u, \bar{u}) \in B_p(R_r)} \| T_{r}(u, \bar{u}) - (u, \bar{u}) \|_p + 16(C(\theta, r))^{r+2}(r + 4)p^{r+3}N^{\alpha/\gamma}R^{r+3}. 
\] (8.21)

Because $C(\theta, t) \leq C(\theta, t + 1)$ for any positive integer $t$, from (8.21), one has that
\[
\sup_{(u, \bar{u}) \in B_p((R_r + R_{r+1})/2)} \| T_{w^g(r+1)}(u, \bar{u}) - (u, \bar{u}) \|_p
\]
\[
\leq 16 \sum_{i=3}^{\frac{r+3}{2}} \frac{N^{\alpha}}{\gamma} (C(\theta, t - 3))^{r-2}p^{r+1}c^{r-1}R^{r-1} + 16 \frac{N^{\alpha}}{\gamma} (C(\theta, r + 1))^{r+2}(r + 4)p^{r+1}c^{r+3}R^{r+3}
\]
\[
\leq 16 \frac{N^{\alpha}}{\gamma} \sum_{i=3}^{r+4} (C(\theta, t - 3))^{r-2}p^{r+1}c^{r-1}R^{r-1} \leq R^{2-\frac{1}{p-1}}.
\]
Denote
\[ Z^{(r+1,u)} := Z^{(r,u)} + Z^{r+4}, \quad R^{N(r+1,u)} := R^{N(r,u)} + \Gamma^{N(r+1,u)} \]
(8.22)
By Remark 5.4 and Lemma 4.1, \( Z^{(r+1,u)} \) and \( R^{N(r+1,u)} \) have \((\beta, \theta)\)-type symmetric coefficients. Denote
\[
g^{(r+1,u)} = \sum_{t=r+3}^{r_x+3} g^{(r+1,u)}_t, \quad R^{T(r+1,u)} = \sum_{t>r+3} R^{T(r+1,u)}_t,
\]
where
\[
g^{(r+1,u)}_t := \left\{ \begin{array}{l}
g^{(r,u)} + (Z^{r+4} - \Gamma^{N(r+1,u)}_{\leq 2}G^{r+4}_t)\left(\frac{r}{r+2}, S^{(r)}_{u}\right) + \sum_{n'=0}^{\frac{t-r}{r+2}} (R^{T(r,u)}_{t-n'}(r+2)) (n', S^{(r)}_{u}) \\
+ \sum_{n'=0}^{\frac{t-r}{r+2}} (g^{(r,u)}_{t-n'}(r+2)) (n', S^{(r)}_{u}) + \sum_{n'=2}^{\frac{t-r}{r+2}} (Z_t^{r=0}) (n', S^{(r)}_{u}) \quad \text{when } (r + 2) | (t - 2);
\end{array} \right.
\]
and
\[
R^{T(r+1,u)}_t := \left\{ \begin{array}{l}
(Z_{r+4} - \Gamma^{N(r+1,u)}_{\leq 2}G^{r+4}_t)\left(\frac{r}{r+2}, S^{(r)}_{u}\right) + \sum_{n'=1}^{\frac{t-r}{r+2}} (R^{T(r,u)}_{t-n'}(r+2)) (n', S^{(r)}_{u}) \\
+ \sum_{n'=1}^{\frac{t-r}{r+2}} (g^{(r,u)}_{t-n'}(r+2)) (n', S^{(r)}_{u}) + \sum_{n'=1}^{\frac{t-r}{r+2}} (Z_t^{r=0}) (n', S^{(r)}_{u}) \\
+ \sum_{n'=0}^{\frac{t-r}{r+2}} (R^{T(r,u)}_{t-n'}(r+2)) (n', S^{(r)}_{u}) \quad \text{when } (r + 2) | (t - 2);
\end{array} \right.
\]
where \([a]\) denotes the integer part of the real number \(a\). Using Lemma 4.1 and Remark 5.4, from the fact that \(g^{(r,u)}\), \(R^{T(r,u)}\), \(R^{N(r,u)}\) and \(Z^{(r,u)}\) have \((\beta, \theta)\)-type symmetric coefficients semi-bounded by \(C(\theta, r)\), then \(g^{(r+1, u)}\) and \(R^{T(r+1,u)}\) also have \((\beta, \theta)\)-type symmetric coefficients.
When \(\theta = 0\), using Remark 5.4, the followings estimates hold: for any \(|l + k| = t\) with \(M(t, k) = i \in M^{r+4}_{(r+4)}\),
\[
\sum_{(l, k, v) \in A(z_{r+4} - \Gamma^N_{\leq 2}G^r_t)} \left| (Z_{r+4} - \Gamma^N_{\leq 2}G^r_t) \left(\frac{r}{r+2}, S^{(r)}_{u}\right) \right|_{l, i, k} \\
\cdot \max\{(\gamma^0), (2i - 0)\} \\
\leq \frac{(C(\theta, r))^t + 2}{(t+1)} (2^{r+2} (r+4)^2 (C(\theta, r))^t + 2) \left(\frac{N^{\alpha+1}}{\gamma^{t-r-4}(r+2)!} \prod_{n=1}^{t-r-4} (t+1-n(r+2)) \right);
\]
(8.23)
for any $|l + k| = t$ with $\mathcal{M}(l, k) = i \in M_{g_{l-n'(r+2)}^{(r,w_0)}((r,w_0))}$:

$$
\sum_{(l', k', i') \in A} |((g_{l-n'(r+2)}^{(r,w_0)}(n', S_{w_0}))_{l', k'})| \cdot \max\{\langle i', 2i - i' \rangle\} 
\leq \frac{(C(\theta, r))^t}{(i)\beta} (2^{\beta+2}(r + 4)^2(C(\theta, r))^2 + 2\frac{N^{\alpha+1}}{\gamma} n' \frac{(2N)^{r+2}n'}{n'} \prod_{n=1}^{t+1-n(r+2)} (t+1-n(r+2)) ;
$$

(8.24)

for any $|l + k| = t$ with $\mathcal{M}(l, k) = i \in M_{z_{l-n'(r+2)}^{(r,w_0)}((r,w_0))}$:

$$
\sum_{(l', k', i') \in A} |((z_{l-n'(r+2)}^{(r,w_0)}(n', S_{w_0}))_{l', k'})| \cdot \max\{\langle i', 2i - i' \rangle\} 
\leq \frac{(C(\theta, r))^t}{(i)\beta} (2^{\beta+2}(r + 4)^2(C(\theta, r))^2 + 2\frac{N^{\alpha+1}}{\gamma} n' \frac{(2N)^{r+2}n'}{n'} \prod_{n=1}^{t+1-n(r+2)} (t+1-n(r+2)) ,
$$

(8.25)

and for any $|l + k| = t$ with $\mathcal{M}(l, k) = i \in M_{\mathcal{R}^{(r,w_0)}^{T_{l-n'(r+2)}^{(r,w_0)}(n', S_{w_0})}}$:

$$
\sum_{(l', k', i') \in A} |((\mathcal{R}_{l-n'(r+2)}^{T_{l-n'(r+2)}^{(r,w_0)}}(n', S_{w_0}))_{l', k'})| \cdot \max\{\langle i', 2i - i' \rangle\} 
\leq \frac{(C(\theta, r))^t}{(i)\beta} (2^{\beta+2}(r + 4)^2(C(\theta, r))^2 + 2\frac{N^{\alpha+1}}{\gamma} n' \frac{(2N)^{r+2}n'}{n'} \prod_{n=1}^{t+1-n(r+2)} (t+1-n(r+2)) .
$$

(8.26)

By [8.23]-[8.27] and assumption [5.27], for any $r + 5 \leq t \leq r_4 + 3$, $|l + k| = t$ and $i \in M_{g_{l-n'(r+2)}^{(r,w_0)}}$, the following estimate holds

$$
\sum_{(l', k', i') \in A} |((g_{l-n'(r+2)}^{(r+1,w_0)}(n', S_{w_0}))_{l', k'})| \cdot \max\{\langle i', 2i - i' \rangle\} \leq \frac{(C(\theta, r + 1))^t}{(i)\beta} ,
$$

(8.28)

which means that $g_{l-n'(r+2)}^{(r+1,w_0)}(u, \bar{u})$ has $(\beta, 0)$-type symmetric coefficients semi-bounded by $C(\theta, r + 1) > 0$.

Similarly, $\mathcal{R}_{l-n'(r+2)}^{T_{l-n'(r+2)}^{(r,w_0)}}(u, \bar{u})$ and $\mathcal{R}_{l-n'(r+2)}^{N_{l-n'(r+2)}^{(r,w_0)}}(u, \bar{u})$ are also of $(\beta, \theta)$-type symmetric coefficients semi-bounded by $C(\theta, r + 1) > 0$. 

□
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