A characterization of normal 3-pseudomanifolds with at most two singularities

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Abstract

Characterizing face-number-related invariants of a given class of simplicial complexes has been a central topic in combinatorial topology. In this regard, one of the well-known invariants is $g_2$. Let $K$ be a normal 3-pseudomanifold such that $g_2(K) \leq g_2(\text{lk}(v)) + 9$ for some vertex $v$ in $K$. Suppose either $K$ has only one singularity or $K$ has two singularities (at least) one of which is an $\mathbb{R}P^2$-singularity. We prove that $K$ is obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, edge expansions, vertex foldings, and edge foldings. In case $K$ has one singularity, $|K|$ is a handlebody with its boundary coned off. Further, we prove that the above upper bound is sharp for such normal 3-pseudomanifolds.

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1 Introduction

For a $d$-dimensional finite simplicial complex $K$, $g_2(K)$ is defined by $g_2 := f_1 - (d+1)f_0 + \binom{d+2}{2}$, where $f_0$ and $f_1$ denote the number of vertices and edges in $K$. The study on $g_2$ has been illuminated in a different prospect due to the lower bound conjecture for 3- and 4-manifolds by Walkup [16] in 1970. He proved that for any closed and connected triangulated 3-manifold $K$, $g_2(K) \geq 0$, and the equality occurs if and only if $K$ is a triangulation of a stacked sphere. Barnette [3, 4, 5] proved that if $K$ is the boundary complex of a simplicial $(d + 1)$-polytope or, more generally, a triangulation of a connected $d$-manifold, then $g_2(K) \geq 0$. In [10], Kalai proved that if $K$ is a normal pseudomanifold of dimension at least 3 with the 2-dimensional links as triangulated spheres, then $g_2(K) \geq g_2(\text{lk}(\sigma))$ for every face $\sigma$ of co-dimension at least 3. Fogelsanger’s results in [8, Chapter 8] made it possible to remove the restriction on 2-dimensional links. Therefore, $g_2(K) \geq 0$ for every normal $d$-pseudomanifold $K$. In [9], Gromov has similar work on the non-negativity of $g_2$.

Based on values of $g_2$, several classifications of combinatorial manifolds and normal pseudomanifolds are studied in the literature. In [14], Swartz proved that the number of combinatorial manifolds, up to PL-homeomorphism, of a given dimension $d$ with an upper bound on $g_2$, is finite. The combinatorial characterizations of normal $d$-pseudomanifolds are known due to Kalai [10] (for $g_2 = 0$), Nevo and Novinsky [11] (for $g_2 = 1$), and Zheng [17] (for $g_2 = 2$). In all

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three cases, the normal pseudomanifold is the boundary complex of a simplicial polytope. The classification of all triangulated pseudomanifolds of dimension \( d \) with at most \( d + 4 \) vertices can be found in [2]. For further developments in this direction, one may refer to [12, 13, 15].

In [7], Basak and Swartz introduced two new concepts, viz., vertex folding and edge folding. For a normal 3-pseudomanifold \( K \) with exactly one singularity, they proved that if \( g_2(K) = g_2(\text{lk}(v)) \) for some vertex \( v \) in \( K \), then \( |K| \) is a handlebody with its boundary coned off. This leads to a natural question: what will be the maximum value of \( g_2(K) \)? In this article, we answer this question. We prove that if \( g_2(K) \leq g_2(\text{lk}(t)) + 9 \), then \( |K| \) is a handlebody with its boundary coned off. Moreover, the above upper bound is sharp for such normal 3-pseudomanifolds (cf. Corollary 4.6). We give a combinatorial characterization of normal 3-pseudomanifolds with at most two singularities, where, in the case of two singularities, one is assumed to be an \( \mathbb{R}P^2 \)-singularity. If \( K \) has no singular vertices, then from [6, 16], we know that \( g_2(K) \leq g_2(\text{lk}(t)) + 9 \) implies \( K \) is a triangulated 3-sphere and is obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, and edge expansions. In this article, we extend this characterization to normal 3-pseudomanifolds with at most two singularities, that reads as follows:

**Theorem 1.1.** Suppose \( g_2(K) \leq g_2(\text{lk}(v)) + 9 \) for some vertex \( v \) of a normal 3-pseudomanifold \( K \). If either \( K \) has only one singularity or \( K \) has two singularities (at least) one of which is an \( \mathbb{R}P^2 \)-singularity, then \( K \) can be obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, edge expansions, vertex foldings, and edge foldings. Further, the above upper bound is sharp for such normal 3-pseudomanifolds.

2 Preliminaries

A simplicial complex \( K \) is a finite collection of simplices in \( \mathbb{R}^m \) for some \( m \in \mathbb{N} \), such that for any simplex \( \sigma \in K \), all of its faces are in \( K \), and for any two simplices \( \sigma, \tau \in K \), \( \sigma \cap \tau \) is either empty or a face of both. We assume that the empty set \( \emptyset \) (is a simplex of dimension \(-1\)) is a member of every simplicial complex. We define the dimension of a simplicial complex \( K \) to be the maximum of the dimension of simplices in \( K \). For a \( d \)-dimensional simplicial complex \( K \), the \( f \)-vector is defined as a \((d+2)\)-tuple \((f_{-1}, f_0, \ldots, f_d)\), where \( f_{-1} = 1 \) and for \( 0 \leq i \leq d \), \( f_i \) denotes the number of \( i \)-dimensional faces in \( K \). A maximal face in a simplicial complex \( K \) is called a facet, and if all the facets are of the same dimension, then \( K \) is said to be a pure simplicial complex. A subcomplex of \( K \) is a simplicial complex \( T \subseteq K \). Let \( S \subset V(K) \), where \( V(K) \) is the vertex set of \( K \). Then the subcomplex of \( K \) induced on the vertex set \( S \) is denoted by \( K[S] \). By \(|K|\) we mean the union of all simplices in \( K \) together with the subspace topology induced from \( \mathbb{R}^m \). A triangulation of a polyhedra \( X \) is a simplicial complex \( K \) together with a PL-homeomorphism between \(|K|\) and \( X \).

Two simplices \( \sigma = u_0u_1 \cdots u_k \) and \( \tau = v_0v_1 \cdots v_l \) in \( \mathbb{R}^n \) for some \( n \in \mathbb{N} \) are called skew if \( u_0, \ldots, u_k, v_0, \ldots, v_l \) are affinely independent. In that case \( u_0 \cdots u_k v_0 \cdots v_l \) is a \((k + l + 1)\)-simplex and is denoted by \( \sigma \star \tau \) or \( \sigma \tau \). Two simplicial complexes \( K \) and \( L \) in some \( \mathbb{R}^n \) are called skew if \( \sigma \) and \( \tau \) are skew for all \( \sigma \in K \) and \( \tau \in L \). If \( K \) and \( L \) are skew then we define \( K \star L = K \cup L \cup \{\sigma \tau : \sigma \in K, \tau \in L\} \). The simplicial complex \( K \star L \) is called the join of \( K \) and \( L \). If \( K \) and \( L \) are two simplicial complexes in \( \mathbb{R}^p \) and \( \mathbb{R}^q \) respectively, then we can define their join in a bigger space. More explicitly, let \( i_1 : \mathbb{R}^p \to \mathbb{R}^{p+q+1}, i_2 : \mathbb{R}^q \to \mathbb{R}^{p+q+1} \) be the inclusion maps.
given by \( i_1(x_1,\ldots,x_p) = (x_1,\ldots,x_p,0,\ldots,0) \) and \( i_2(x_1,\ldots,x_q) = (0,\ldots,0,x_1,\ldots,x_q,1) \). Let \( K' := \{ i_1(\sigma) : \sigma \in K \} \) and \( L' := \{ i_2(\tau : \tau \in L) \}. \) Then \( K \cong K' \), \( L \cong L' \) and \( K' \) and \( L' \) are skew in \( \mathbb{R}^{p+q+1} \). We define \( K \ast L = K' \ast L' \). For a simplex \( \sigma \) in \( \mathbb{R}^p \) and a simplicial complex \( K \) in \( \mathbb{R}^p \), by \( \sigma \ast K \) we mean \( \{ \alpha : \alpha \leq \sigma \} \ast K \) in \( \mathbb{R}^{p+q+1} \) for some \( p,q \in \mathbb{N} \). The link of a face \( \sigma \) in \( K \) is defined as \( \gamma \in K : \gamma \cap \sigma = \emptyset \) and \( \gamma \sigma \in K \), and it is denoted by \( \text{lk}(\sigma,K) \). The star of a face \( \sigma \) in \( K \) is defined as \( \{ \alpha : \alpha \leq \sigma \beta ; \beta \in \text{lk}(\sigma,K) \} \), and it is denoted by \( \text{st}(\sigma,K) \). If the underlying simplicial complex is specified, then we use the notations \( \text{lk}(\sigma) \) and \( \text{st}(\sigma) \) to refer to the link and star of the face \( \sigma \), respectively. For every face \( \sigma \) in \( K \), by \( d(\sigma,K) \) (or, \( d(\sigma) \) if \( K \) is specified) we mean the number of vertices in \( \text{lk}(\sigma) \). For two vertices \( x \) and \( y \), \( (x,y) \) denotes the semi-open and semi-closed edge \( xy \), where \( y \in (x,y) \) but \( x \notin (x,y) \). By \( (x,y) \), we denote the open edge \( xy \), where \( x,y \notin (x,y) \). By \( B_{x_1,\ldots,x_m}(p,q) \), we denote the bi-pyramid with \( m \) base vertices \( x_1,\ldots,x_m \) and apexes \( p \) and \( q \).

A normal \( d \)-pseudomanifold without boundary (respectively with boundary) is a connected pure simplicial complex in which every face of dimension \( (d-1) \) is contained in exactly two (respectively at most two) facets and the links of all the simplices of dimension \( \leq (d-2) \) are connected. For a normal \( d \)-pseudomanifold \( K \) with a connected boundary, its boundary \( \partial K \) is a normal \( (d-1) \)-pseudomanifold whose facets are \( (d-1) \)-dimensional faces of \( K \), each of which is contained in exactly one facet of \( K \). Throughout the article, by a normal \( d \)-pseudomanifold, we mean a normal \( d \)-pseudomanifold without boundary. For a simplex \( \sigma \), its boundary complex \( \partial(\sigma) \) is the collection of all of its proper faces. Let \( K \) be a normal \( d \)-pseudomanifold with boundary \( \partial K \). Let \( K' = K \cup (t \ast \partial K) \), where \( t \) is a new vertex. If \( \partial K \) is connected, then \( K' \) is a normal \( d \)-pseudomanifold without boundary. We say \( K' \) is obtained from \( K \) by coning off the boundary at \( t \), and the topological space \( |K'| \) is the topological space \( |K| \) with its boundary coned off. In a normal \( d \)-pseudomanifold \( K \), the vertices whose links are triangulated spheres are called non-singular vertices, and the remaining are called singular vertices. Let \( v \) be a singular vertex in a normal \( d \)-pseudomanifold \( K \) with \( |\text{lk}(v,K)| \cong S \). In this case, we say \( K \) has an \( S \)-singularity at \( v \).

**Definition 2.1.** Let \( K \) be a normal \( d \)-pseudomanifold and \( u,v \) be two vertices in \( K \) such that \( uv \in K \), and \( \text{lk}(u,K) \cap \text{lk}(v,K) = \text{lk}(uv,K) \). Consider \( K' = K \setminus \{ \{ \alpha \in K : u \leq \alpha \} \cup \{ \beta \in K : v \leq \beta \} \} \) and \( K_1 = K' \cup (w \ast \partial K') \), where \( w \) is a new vertex. Then, we say \( K_1 \) is obtained from \( K \) by an edge contraction at \( uv \).

Let \( L \) be a normal \( d \)-pseudomanifold and \( w \) be a vertex in \( L \). Let \( S \) be an induced normal \( (d-2) \)-pseudomanifold in \( \text{lk}(w) \) such that \( S \) separates \( \text{lk}(w) \) into two portions, say \( L_1 \) and \( L_2 \), where \( L_1 \) and \( L_2 \) are normal \( (d-1) \)-pseudomanifolds with the same boundary complex \( S \). Consider \( L' = (L \setminus \{ \alpha \in L : w \leq \alpha \}) \cup ((u \ast L_1) \cup (v \ast L_2) \cup (uv \ast S)) \), where \( u \) and \( v \) are two new vertices. We say \( L' \) is obtained from \( L \) by an edge expansion. Note that, \( L \) can be obtained from \( L' \) by contracting the edge \( uv \).

**Definition 2.2.** Let \( K \) be a normal \( 3 \)-pseudomanifold.

(A) Let \( uv \) be an edge in \( K \) such that \( \text{lk}(uv) = \partial(abc) \) and \( abc \notin K \). Consider \( K' = (K \setminus \{ \alpha \in K : uv \leq \alpha \}) \cup \{ abc, uabc, vabc \} \). Since \( abc \notin K \), we have \( |K'| \cong |K| \). Further, \( g_2(K') = g_2(K) - 1 \). We say \( K' \) is obtained from \( K \) by a bistellar 2-move. The reverse operation is called a bistellar 1-move.

(B) Let \( w \) be a non-singular vertex in \( K \) such that \( \partial(abc) \subset \text{lk}(w) \), where \( abc \notin K \). Then \( \partial(abc) \) separates \( \text{lk}(w) \) into two portions, say \( D_1 \) and \( D_2 \), where \( D_1 \) and \( D_2 \) are triangulated discs with the same boundary complex \( \partial(abc) \). Consider \( K' = (K \setminus \{ \alpha \in K : w \leq \alpha \}) \cup (u \ast D_1) \cup (v \ast D_2) \cup (abc \ast \partial(uv)) \), where \( u \) and \( v \) are two new vertices.
Then \( g_2(K') = g_2(K) - 1 \) and \( |K'| \cong |K| \). Note that this combinatorial operation is a combination of an edge expansion and a bistellar 2-move.

**Remark 2.3.** Let \( K \) be a normal 3-pseudomanifold. Let \( K' \) be obtained from \( K \) by one of the following combinatorial operations: (i) an edge contraction, (ii) a bistellar 2-move, (iii) the operation as in Definition 2.2 (B), or (iv) a combination of an edge expansion and an edge contraction. Then, \( K \) is obtained from \( K' \) by one of the following combinatorial operations: (i) an edge expansion, (ii) a bistellar 1-move, (iii) a combination of a bistellar 1-move and an edge contraction, or (iv) a combination of an edge expansion and an edge contraction.

**Lemma 2.4.** For \( d \geq 3 \), let \( K \) be a normal \( d \)-pseudomanifold. Let \( uv \) be an edge in \( K \) such that \( \text{lk}(u,K) \cap \text{lk}(v,K) = \text{lk}(uv,K) \) and \( |\text{lk}(v,K)| \cong S^{d-1} \). If \( K_1 \) is the normal pseudomanifold obtained from \( K \) by contracting the edge \( uv \), then \( |K| \cong |K_1| \).

**Proof.** Since \( \text{lk}(u,K) \cap \text{lk}(v,K) = \text{lk}(uv,K) \), the edge contraction is possible. Let \( w \) be the new vertex in \( K_1 \) obtained by identifying the vertices \( u \) and \( v \) in \( K \). Let \( K' = K \setminus \{ \{ \alpha \in K : u \leq \alpha \} \cup \{ \beta \in K : v \leq \beta \} \} \). Then \( K' \) is a normal \( d \)-pseudomanifold with boundary and \( \partial K' = \text{st}(u,K) \setminus \text{st}(v,K) \). Since \( \text{lk}(u,K) \cap \text{lk}(v,K) = \text{lk}(uv,K) \) and \( |\text{lk}(v,K)| \cong S^{d-1} \), \( \text{lk}(v,K) \setminus \{ \alpha \in \text{lk}(v,K) : u \leq \alpha \} \) is a triangulated \((d-1)\)-ball, say \( D \), with boundary \( \text{lk}(uv,K) \).

Further, \( K' \cap \text{st}(v,K) = D \). Since \( |\text{lk}(v,K)| \cong S^{d-1} \), \( |\text{st}(v,K)| \cong D^d \). Therefore, \( |K'| \) is PL-homeomorphic to \( |K' \cup \text{st}(v,K)| \). Let \( K'' := K' \cup \text{st}(v,K) \). Then \( K = K'' \cup (u \ast \partial K'') \) and \( K_1 = K' \cup (w \ast \partial K) \). Since \( |K''| \) and \( |K'| \) are PL-homeomorphic, \( |K| \) and \( |K_1| \) are also PL-homeomorphic.

From [16, Lemma 10.8], we have the following result (see [6, Section 4] for more details):

**Proposition 2.5 ([6, 16]).** If \( K \) is a triangulated 3-manifold with \( g_2(K) \leq 9 \), then \( K \) is a triangulated 3-sphere, and is obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, and edge expansions.

Let \( K \) be a normal \( d \)-pseudomanifold and \( f : V(K) \rightarrow \mathbb{R}^{d+1} \) be a function. A stress of \( f \) is a function \( \omega : E(K) \rightarrow \mathbb{R} \) such that for every vertex \( v \in V \), \( \sum_{vu \in E(K)} \omega(vu)(f(v) - f(u)) = 0 \), where \( E(K) \) denotes the set of edges in \( K \). The set of all stresses of \( f \) is an \( \mathbb{R} \)-vector space, which we denote by \( \mathcal{S}(K) \). From [10], we know that if \( K \) is a normal \( d \)-pseudomanifold (or a cone over a normal \((d-1)\)-pseudomanifold) and \( f \) is a generic map from \( V \) to \( \mathbb{R}^{d+1} \), then \( \dim \mathcal{S}(K_f) = g_2(K) \). Using the fact that a stress \( \omega \) on a subcomplex of \( K \) can be extended to \( K \) by setting \( \omega(uv) = 0 \) for any \( uv \) not in the subcomplex, we have the following result (due to Kalai [10]):

**Lemma 2.6 ([7, 10]).** Let \( K \) be a normal 3-pseudomanifold and \( v \) be a vertex in \( K \). Then \( g_2(K) \geq g_2(\text{st}(v,K)) = g_2(\text{lk}(v,K)) \). Moreover, if \( u \) and \( v \) are two vertices in \( K \) such that \( \text{st}(u,K) \cap \text{st}(v,K) = \emptyset \), then \( g_2(K) \geq g_2(\text{st}(u,K)) + g_2(\text{st}(v,K)) = g_2(\text{lk}(u,K)) + g_2(\text{lk}(v,K)) \).

Let \( K \) be a pure simplicial complex and \( \sigma_1, \sigma_2 \) be two facets of \( K \). A bijection \( \psi : \sigma_1 \rightarrow \sigma_2 \) is said to be admissible (cf. [11]) if for any vertex \( x \leq \sigma_1 \), length of every path between \( x \) and \( \psi(x) \) is at least 3. In this contest, any bijective map between two facets from different connected components of \( K \) is admissible. Now if \( \psi \) is an admissible bijection between \( \sigma_1 \) and \( \sigma_2 \), by identifying all the faces \( \rho_1 \leq \sigma_1 \) with \( \psi(\rho_1) \) and removing the identified facets, we get a new complex, say \( K^\psi \). If \( \sigma_1 \) and \( \sigma_2 \) are from the same component of \( K \), then we say \( K^\psi \) is formed via a handle addition (cf. [7]) to \( K \). If \( \sigma_1 \) and \( \sigma_2 \) are from different components of \( K \), then we say \( K^\psi \) is formed via a connected sum (cf. [7]), and we write it as \( K^\psi = K_1 \#_\psi K_2 \), where
A straightforward computation shows that if \( K \) is a similar spirit, \( \psi \) formed by identifying \( \sigma \) the links will remain the same.

A missing triangle of \( K \) is a triangle \( \sigma \) such that \( \sigma \notin K \) but \( \partial(\sigma) \subset K \). Similarly, a missing tetrahedron of \( K \) is a tetrahedron \( \tau \) such that \( \tau \notin K \) but \( \partial(\tau) \subset K \).

**Lemma 2.7 (\[7\]).** Let \( K \) be a normal 3-dimensional pseudomanifold, and suppose \( \tau \) is a missing tetrahedron in \( K \). If for every vertex \( x \leq \tau \), the missing triangle formed by the other three vertices separates the link of \( x \), then \( K \) is formed using either a handle addition or a connected sum.

A straightforward computation shows that for a \( d \)-dimensional complex \( K \), a handle addition and a connected sum satisfy the following:

\[
g_2(K^\psi) = g_2(K) + \binom{d+2}{2}, \tag{1}
\]
\[
g_2(K_1 \#_\psi K_2) = g_2(K_1) + g_2(K_2). \tag{2}
\]

**Lemma 2.8.** Let \( K \) be a normal 3-pseudomanifold such that \( g_2(K) \leq g_2(\text{lk}(v, K)) + 9 \) for some vertex \( v \). Let \( \sigma \) be a missing tetrahedron in \( K \) such that for every vertex \( x \leq \sigma \), the missing triangle formed by the other three vertices separates the link of \( x \). Then \( K \) is formed using a connected sum.

*Proof.* By Lemma 2.7, \( K \) is formed using either a handle addition or a connected sum. If possible, let \( K \) be formed using a handle addition from \( K' \) through the admissible bijection \( \psi : \sigma_1 \to \sigma_2 \). Then the identified simplex \( \sigma \) (obtained by identifying \( \psi(\sigma_1) \) with \( \sigma_2 \)) is a missing tetrahedron. If \( v \notin \sigma \), then \( g_2(\text{lk}(v, K)) \leq g_2(\text{lk}(v, K')) \leq g_2(K') = g_2(K) - 10 \). This is a contradiction. If possible, let \( v \leq \sigma \) be obtained by identifying \( v_1 \leq \sigma_1 \) and \( v_2 \leq \sigma_2 \) in \( K' \). Then \( g_2(\text{lk}(v, K)) = g_2(\text{lk}(v_1, K')) + g_2(\text{lk}(v_2, K')) \). Since \( \psi \) is admissible, \( \text{st}(v_1, K') \) and \( \text{st}(v_2, K') \) are disjoint. Then it follows from Lemma 2.7 that \( g_2(K') \geq g_2(\text{lk}(v_1, K')) + g_2(\text{lk}(v_2, K')) \). Therefore, \( g_2(K') \geq g_2(\text{lk}(v, K)) \) and \( g_2(K) = g_2(K') + 10 \). This implies, \( g_2(K) \geq g_2(\text{lk}(v, K)) + 10 \), which is a contradiction. Therefore, \( K \) is formed using a connected sum. \( \Box \)

Handle addition and connected sum are standard parts of combinatorial topology, but the operation of folding was recently introduced in \[7\].

**Definition 2.9 (Vertex folding \[7\]).** Let \( \sigma_1 \) and \( \sigma_2 \) be two facets of a simplicial complex \( K \), whose intersection is a single vertex \( x \). A bijection \( \psi : \sigma_1 \to \sigma_2 \) is vertex folding admissible if \( \psi(x) = x \) and for all other vertices \( y \) of \( \sigma_1 \), the only path of length two from \( y \) to \( \psi(y) \) is \( P(y, x, \psi(y)) \). For a vertex folding admissible map \( \psi \), we can form the complex \( K_x^\psi \) by identifying all faces \( \rho_1 \leq \sigma_1 \) and \( \rho_2 \leq \sigma_2 \), such that \( \psi(\rho_1) = \rho_2 \), and then removing the facet formed by identifying \( \sigma_1 \) and \( \sigma_2 \). In this case, we say that \( K_x^\psi \) is a vertex folding of \( K \) at \( x \). In a similar spirit, \( K \) is a vertex unfolding of \( K_x^\psi \).

A straightforward computation shows that if \( K_x^\psi \) is obtained from a \( d \)-dimensional simplicial complex \( K \) by a vertex folding at \( x \), then

\[
g_2(K_x^\psi) = g_2(K) + \binom{d+1}{2}, \tag{3}
\]

The definition of edge folding follows the same pattern as vertex folding.
Definition 2.10 (Edge folding [7]). Let $\sigma_1$ and $\sigma_2$ be two facets of a simplicial complex $K$, whose intersection is an edge $uv$. A bijection $\psi: \sigma_1 \to \sigma_2$ is edge folding admissible if $\psi(u) = u, \psi(v) = v$, and for all other vertices $y$ of $\sigma_1$, all paths of length two or less from $y$ to $\psi(y)$ pass through either $u$ or $v$. Identify all faces $\rho_1 \leq \sigma_1$ and $\rho_2 \leq \sigma_2$, such that $\psi: \rho_1 \to \rho_2$ is a bijection. The complex obtained by removing the facet resulting from identifying $\sigma_1$ and $\sigma_2$ is denoted by $K_{uv}^{\psi}$ and is called an edge unfolding of $K_{uv}$.

If $K$ is a normal $d$-pseudomanifold and $K_{uv}^{\psi}$ is obtained from $K$ by an edge folding at $uv$, then

$$g_2(K_{uv}^{\psi}) = g_2(K) + \binom{d}{2}. \tag{4}$$

Let $vabc$ be a missing tetrahedron in $K$. If $|\text{lk}(v,K)|$ is an orientable surface, then a small neighborhood of $|\partial(abc)|$ in $|\text{lk}(v,K)|$ is an annulus, and if $|\text{lk}(v,K)|$ is a non-orientable surface, then a small neighborhood of $|\partial(abc)|$ in $|\text{lk}(v,K)|$ is either an annulus or a Möbius strip.

Lemma 2.11 ([7]). Let $K$ be a normal 3-pseudomanifold. Let $abcd$ be a missing tetrahedron in $K$ such that (i) for $x \in \{b, c, d\}$, $\partial(K[a,b,c,d] \setminus \{x\})$ separates $\text{lk}(x,K)$, and (ii) $\partial(bcd)$ does not separate $\text{lk}(a,K)$. Then there exists $K'$, a normal 3-pseudomanifold such that $K = (K')_{uv}^{\psi}$, i.e., $K$ is obtained from a vertex folding at $a \in K'$, and $abcd$ is the image of the removed facet.

Lemma 2.12 ([7]). Let $K$ be a normal 3-pseudomanifold. Let $abuv$ be a missing tetrahedron in $K$ such that (i) for $x \in \{a, b\}$, $\partial(K[a,b,u,v] \setminus \{x\})$ separates $\text{lk}(x,K)$, and (ii) a small neighborhood of $|\partial(abu)|$ in $|\text{lk}(u,K)|$ is a Möbius strip. Then a small neighborhood of $|\partial(abu)|$ in $|\text{lk}(u,K)|$ is also a Möbius strip. Further, there exists $K'$ a normal 3-pseudomanifold such that $K = (K')_{uv}^{\psi}$, i.e., $K$ is obtained from an edge folding at $uv \in K'$, and $abuv$ is the removed facet.

### 3 Some lower bounds of $g_2$ for normal 3-pseudomanifolds with one or two singularities

In this section, we establish a few lower bounds of $g_2$ for a class of normal 3-pseudomanifolds with one or two singularities. Our general approaches are motivated by the idea used in [16].

**Definition of $\mathcal{R}$:** Let $\mathcal{R}$ be the class of all normal 3-pseudomanifolds $K$ such that $K$ has one or two singularities and $K$ satisfies the following two properties:

(i) If $K$ contains the boundary complex of a 3-simplex as a subcomplex, then $K$ contains the 3-simplex as well.

(ii) There is no normal 3-pseudomanifold $K'$ such that $K'$ is obtained from $K$ by a combinatorial operation mentioned in Remark 2.3 and $g_2(K') < g_2(K)$.

Now we state a few lemmas, the proofs of which follow from [16] using Lemma 2.4.

**Lemma 3.1** (Lemma 10.1, [16]). Let $K \in \mathcal{R}$, and $uv$ be an edge in $K$. Then $d(uv) \geq 4$, i.e., $\text{lk}(v,\text{lk}(u))$ has at least four vertices.

**Lemma 3.2** (Lemma 10.2, [16]). Let $K \in \mathcal{R}$, and $uv$ be an edge in $K$, where $v$ is a non-singular vertex. Then $\text{lk}(u) \cap \text{lk}(v) \setminus \text{lk}(uv) \neq \emptyset$. 
Lemma 3.3 (Lemma 10.4, [16]). Let $K \in \mathcal{R}$, and $u$ be a non-singular vertex in $K$. If $\text{lk}(u)$ contains the boundary complex of a 2-simplex $\sigma$ as a subcomplex, then $\text{lk}(u)$ contains the 2-simplex $\sigma$ as well. Thus, for every vertex $v \in \text{lk}(u)$, $\text{lk}(u) \setminus \{\alpha \in \text{lk}(u) : v \leq \alpha\}$ does not contain a diagonal edge.

Lemma 3.4 (Lemma 10.6, [16]). Let $K \in \mathcal{R}$, and $t$ be a singular vertex in $K$. Let $uv$ be an edge in $K$ such that $uv \notin \text{lk}(t)$. If $z \in \text{lk}(u) \cap \text{lk}(v) \setminus \text{lk}(uv)$, then $zw \notin \text{lk}(u) \cap \text{lk}(v)$ for every non-singular vertex $w \in \text{lk}(uv)$.

Lemma 3.5. Let $K \in \mathcal{R}$, and $uv$ be an edge in $K$, where $v$ is a non-singular vertex. Then $\text{lk}(u) \cap \text{lk}(v) \setminus \text{lk}(uv)$ contains some vertices.

Proof. It follows from Lemma 3.2 that $\text{lk}(u) \cap \text{lk}(v) \setminus \text{lk}(uv) \neq \emptyset$. If possible, let $\text{lk}(u) \cap \text{lk}(v) \setminus \text{lk}(uv)$ contain an open edge $(z, w)$, where $z, w \in \text{lk}(uv)$. Then $uvzw$ is a missing tetrahedron in $K$. This contradicts the fact that $K \in \mathcal{R}$. Thus the result follows.

By using Lemma 2.4 in the proof of Lemma 11.1 of [16], we have the following result:

Lemma 3.6. Let $K \in \mathcal{R}$, and $t$ be a singular vertex in $K$. Let $u$ be a non-singular vertex in $\text{lk}(t, K)$ such that $\text{lk}(t) \cap \text{lk}(u) \setminus \text{lk}(ut) = \{t_1, w\}$ or $\{w\}$, where $w \in \text{lk}(u) \cap \text{lk}(t) \setminus \text{lk}(ut)$ and $t_1 \in \text{lk}(ut)$ is a singular vertex. Then $d(tw), d(uw) \geq d(tu)$.

Lemma 3.7. If $K \in \mathcal{R}$, and $K$ contains exactly one singular vertex $t$, then $d(t) \geq 8$.

Proof. The proof follows from the hypothesis of Lemma 3.5 and the possible triangulations of $\text{lk}(t)$ for $d(t) \leq 7$.

Lemma 3.8. Let $K \in \mathcal{R}$, and $t, t_1$ be two singular vertices in $K$. If $d(t) = 7$, then $d(tt_1) = 6$ and $d(t_1) \geq 8$; otherwise $d(t), d(t_1) \geq 8$.

Proof. If either $d(t)$ or $d(t_1)$ is 6, then it contradicts the hypothesis of Lemma 3.5. Therefore, $d(t), d(t_1) \geq 7$. If $d(t) = 7$, then $f_1(\text{lk}(t)) \geq 18$. Thus $d(ut) = 6$ for some vertex $u \in \text{lk}(t)$. If $u$ is a non-singular vertex, then it contradicts the hypothesis of Lemma 3.5. If $d(tt_1) = 6$, then $\text{lk}(t) \cap \text{lk}(t_1) \setminus \text{lk}(tt_1)$ is either empty or an open edge. If it is an open edge, then $K$ contains a missing tetrahedron, which contradicts the fact that $K \in \mathcal{R}$. Therefore, $\text{lk}(t) \cap \text{lk}(t_1) \setminus \text{lk}(tt_1) = \emptyset$. If $d(t_1) = 7$, then the boundary complex of $\text{st}(t) \cup \text{st}(t_1)$ is a triangulated surface with 6 vertices with the first Betti number more than 1. This is not possible. Therefore, $d(t_1) \geq 8$.

Let $uv$ be an edge in $K$, where $u$ is a non-singular vertex. Define $D_vu := \text{lk}(u) \setminus \{\alpha \in \text{lk}(u) : v \leq \alpha\}$. We say $D_vu$ is of type $m(n)$ if $d(u) = m$ and $d(uv) = n$.

![Figure 1: All possible types of $D_vu$, where $u$ is a non-singular vertex in $K$ and $d(u) \leq 7$.](image)

Lemma 3.9. Let $K \in \mathcal{R}$, and $uv$ be an edge in $K$, where $u$ is a non-singular vertex. Suppose $\text{lk}(uv)$ contains at most one singular vertex.
(i) If $d(u) = 6$, then $d(v) \geq 9$.

(ii) If $d(u) = 7$ and $D_vu$ is of type 7(5), then $d(v) \geq 11$.

(iii) If $d(u) = 7$ and $D_vu$ is of type 7(4), then $d(v) \geq 8$.

Proof. (i) Let $d(u) = 6$. Then $D_vu$ is of type 6(4). Suppose $V(lk(uv)) = \{p_1, p_2, p_3, p_4\}$. It follows from Lemma 3.3 that $lk(u) \cap lk(v) \setminus lk(uv)$ contains exactly one vertex, say $w$. Then $uwp \in K$ for $1 \leq i \leq 4$. If some non-singular vertex $p_i \in lk(uv) \cap lk(vw)$, then $uwpt, uwp, vwp, w \in K$. This implies $\partial(uvw) \subset lk(p_i)$. It follows from Lemma 3.3 that $uvw \in K$. This is a contradiction. Therefore, $lk(uv) \cap lk(vw)$ does not contain any non-singular vertex. Since $lk(uv)$ contains at most one singular vertex, $lk(vw)$ contains at least three more vertices other than the vertices of $lk(uv)$. Thus, $d(v) \geq 9$.

(ii) Let $d(u) = 7$ and $D_vu$ be of type 7(5). Suppose $V(lk(uv)) = \{p_1, p_2, p_3, p_4, p_5\}$. Since $lk(u) \cap lk(v) \setminus lk(uv) \neq \emptyset$, it contains exactly one vertex, say $w$. Therefore $uwpt \in K$ for $1 \leq i \leq 5$. Since $w \in lk(v)$, it follows from Lemma 3.6 that $d(wv) \geq 5$. If possible, let there be a non-singular vertex $p_i \in lk(uv) \cap lk(vw)$. Then $\partial(uvw) \subset lk(p_i)$. It follows from Lemma 3.3 that $uvw \in K$, which is a contradiction. Therefore, $lk(uv) \cap lk(vw)$ does not contain any non-singular vertex. Since $lk(uv)$ contains at most one singular vertex, $lk(vw)$ contains at least four more vertices other than the vertices of $lk(uv)$. Thus, $d(v) \geq 11$.

(iii) Let $d(u) = 7$ and $D_vu$ be of type 7(4) for some vertex $v \in lk(u)$. Then Lemma 3.3 implies that $lk(u) \cap lk(v) \setminus lk(uv)$ contains a vertex, say $w$. It follows from Figure 1 that $d(wv) = 4$ and $w$ is connected to 3 vertices of $lk(uv)$. Let $V(lk(u)) = \{a, b, c, d, e, w, v\}$, $V(lk(uv)) = \{a, b, c, d\}$, and $w$ be connected to the vertices $a, b, c$ in $lk(uv)$. Since $lk(uv)$ contains at most one singular vertex, without loss of generality, let $a$ be the singular vertex (if any). It follows from Lemma 3.3 that $b, c \not\in lk(uw) \cap lk(vw)$. Therefore, $lk(vw)$ contains at least 2 vertices other than the vertices of $lk(uv)$. Thus, $d(v) \geq 8$. □

Lemma 3.10. Let $K \in \mathcal{R}$, and $uv$ be an edge in $K$, where $u$ is a non-singular vertex.

(i) If $d(u) = 6$, then $d(v) \geq 8$.

(ii) If $d(u) = 7$ and $D_vu$ is of type 7(5), then $d(v) \geq 10$.

Proof. The proof follows by the similar arguments as in the proof of Lemma 3.9. □

Definition 3.11. Let $K \in \mathcal{R}$. First, we fix a singular vertex $t$ such that $d(t) \geq 8$ (cf. Lemmas 3.7 and 3.8) and $g_2(lk(t, K)) \geq g_2(lk(v, K))$ for any other vertex $v$ in $K$. Let $u$ be a vertex in $K$. Then for every vertex $v \in lk(u, K)$, define the weight $\lambda(u, v)$ of the vertex $u$ with respect to $v$ as follows:

\[
\lambda(u, v) = \begin{cases} 
\frac{2}{3} & \text{if } d(u) = 6, \text{ and either } u \not\in st(t) \text{ or } v \not\in st(t), \\
\frac{4}{3} & \text{if } d(u) = 7, d(v, lk(u)) = 5, \text{ and either } u \not\in st(t) \text{ or } v \not\in st(t), \\
\frac{1}{2} & \text{if } d(u) = 7, d(v, lk(u)) = 4, \text{ and either } u \not\in st(t) \text{ or } v \not\in st(t), \\
\frac{1}{2} & \text{if } d(u) = 8, \text{ and either } u \not\in st(t) \text{ or } v \not\in st(t), \\
1 - \lambda(v, u) & \text{if } d(u) \geq 9, d(v) \leq 8, \text{ and either } u \not\in st(t) \text{ or } v \not\in st(t), \\
\frac{1}{2} & \text{otherwise.}
\end{cases}
\]
Then from Lemmas 3.8, 3.9 and 3.10 it follows that \( \lambda(u, v) + \lambda(v, u) = 1 \) for every edge \( uv \) of \( K \). For a vertex \( u \in K \), we define the weight of the vertex \( u \) as \( W_u := \sum_{v \in \text{lk}(u)} \lambda(u, v) \). For a vertex \( u \in \text{lk}(t) \), we define the outer weight of the vertex \( u \) as

\[
\mathcal{O}_u := \sum_{\substack{v \in \text{lk}(u) \\cap \text{lk}(t) \\setminus u \in \text{lk}(t) \}} \lambda(u, v).
\]

**Lemma 3.12.** Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11. If \( u \in K \setminus \text{st}(t, K) \), then \( W_u = \sum_{v \in \text{lk}(u)} \lambda(u, v) \geq 4 \).

**Proof.** If \( d(u) \leq 7 \), then it follows from Lemmas 3.7 and 3.8 that \( u \) is a non-singular vertex. If \( d(u) = 6 \), then for any vertex \( v \in \text{lk}(u) \), \( \lambda(u, v) = 2/3 \). Therefore, \( \sum_{v \in \text{lk}(u)} \lambda(u, v) = 6 \times 2/3 = 4 \). If \( d(u) = 7 \), then for any vertex \( v \in \text{lk}(u) \), \( D_v u \) is of type either \( 7(5) \) or \( 7(4) \). It follows from Figure 1 that in both cases, \( \text{lk}(u) \) contains five vertices whose \( D_v u \) is of type \( 7(4) \) and two vertices for which \( D_v u \) is of type \( 7(5) \). Therefore \( \sum_{v \in \text{lk}(u)} \lambda(u, v) = 5 \times 1/2 + 2 \times 3/4 = 4 \).

If \( d(u) = 8 \), then for every vertex \( v \in \text{lk}(u) \), \( \lambda(u, v) = 1/2 \). Therefore \( \sum_{v \in \text{lk}(u)} \lambda(u, v) = 4 \).

If \( d(u) = 9 \), then it follows from Lemma 3.9 that \( \text{lk}(u) \) contains no singular vertex \( v \) for which \( \lambda(u, v) = 1/4 \) holds. If possible, let \( v \) be the other singular vertex such that \( \lambda(u, v) = 1/4 \). Then \( d(v) = 7 \), \( d(uv) = 5 \) and \( t \in \text{lk}(v) \) such that \( d(ut) = 6 \). This implies \( t \in \text{lk}(u) \), a contradiction as \( u \notin \text{st}(t) \). If for every vertex \( v \in \text{lk}(u) \), \( \lambda(u, v) = 1/2 \), then we are done. Suppose there is a vertex \( v \in \text{lk}(u) \) such that \( \lambda(u, v) = 1/3 \), i.e., \( d(v) = 6 \). Let \( \text{lk}(v) = B_{u_1, u_2, u_3, u_4}(u, z) \). Then Lemma 3.5 implies that \( z \in \text{lk}(u) \). It follows from Lemma 3.10 that the five vertices \( u_1, u_2, u_3, u_4, z \in \text{lk}(u) \) have a degree of at least 8. Further, \( d(uz) \geq 4 \), and Lemma 3.4 suggests that one vertex from the set \( \{u_1, u_2, u_3, u_4\} \) must be a singular vertex; otherwise, \( d(uz) \geq 10 \). Let’s assume that \( u_1 \) is the singular vertex. Therefore, \( \text{lk}(u) \) forms a 4-cycle, denoted as \( C_4(u_1, z_1, z_2, z_3) \). As there are no two adjacent vertices in \( C_4(u_1, z_1, z_2, z_3) \) with a degree of 6, we conclude that at least one of the vertices \( z_1, z_2, \) or \( z_3 \) has a degree greater than or equal to 8. Hence, there are more than five vertices in \( \text{lk}(u) \) that contribute a value of \( 1/2 \) to \( \lambda \). Consequently, we deduce that \( \sum_{v \in \text{lk}(u)} \lambda(u, v) \geq 4 \).

If \( d(u) = 10 \), then by the same arguments as above, we have at least five vertices in \( \text{lk}(u) \) that contribute a value of \( 1/2 \) to \( \lambda \). Hence, \( \sum_{v \in \text{lk}(u)} \lambda(u, v) \geq 4 \).

Finally, consider \( d(u) \geq 11 \). If there exists a vertex \( v \in \text{lk}(u) \) such that \( \lambda(u, v) = 1/3 \) or \( 1/4 \), then \( D_v u \) must be of the type \( 6(4) \) or \( 7(5) \), respectively. Using similar reasoning as above, we conclude that there are at least five vertices in \( \text{lk}(u) \) that contribute a value of \( 1/2 \) to \( \lambda \). Therefore, \( \sum_{v \in \text{lk}(u)} \lambda(u, v) \geq 4 \). \( \square \)

**Lemma 3.13.** Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11. Then

\[
g_2(K) \geq g_2(\text{lk}(t)) + \sum_{u \in \text{lk}(t)} \mathcal{O}_u.
\]

**Proof.** We know that \( f_1(\text{st}(t)) = f_1(\text{lk}(t)) + f_0(\text{lk}(t)) = g_2(\text{lk}(t)) + 4f_0(\text{lk}(t)) - 6 = g_2(\text{lk}(t)) + 4f_0(\text{lk}(t)) - 6 \).
4f₀(st(t)) − 10. It follows from Lemma 3.12 that \( \sum_{u \in st(t)} \mathcal{W}_u \geq 4f₀(K \setminus st(t)) \). Thus

\[
\begin{align*}
f₁(K) &= f₁(st(t)) + f₁(K \setminus st(t)) \\
&= f₁(st(t)) + \sum_{uv \in (K \setminus st(t))} [λ(u, v) + λ(v, u)] \\
&= f₁(st(t)) + \sum_{u \in lk(t)} \sum_{v \in lk(u)} λ(u, v) + \sum_{u \notin st(t)} \sum_{v \in lk(u)} λ(u, v) \\
&= g₂(lk(t)) + 4f₀(st(t)) − 10 + \sum_{u \in lk(t)} \mathcal{O}_u + \sum_{u \notin st(t)} \mathcal{W}_u \\
&≥ g₂(lk(t)) + 4f₀(st(t)) − 10 + \sum_{u \in lk(t)} \mathcal{O}_u + 4f₀(K \setminus st(t)) \\
&= 4f₀(K) + g₂(lk(t)) − 10 + \sum_{u \in lk(t)} \mathcal{O}_u.
\end{align*}
\]

Therefore, \( g₂(K) \geq g₂(lk(t)) + \sum_{u \in lk(t)} \mathcal{O}_u \). This proves the result. \( \square \)

**Lemma 3.14.** Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11. Let \( u \in lk(t) \) be a non-singular vertex such that \( lk(u) = B_{u_1, \ldots, u_m}(t; z) \). If \( u_i \) is a non-singular vertex, for some \( i \in \{1, \ldots, m\} \), then \( zu_i \notin st(t) \).

**Proof.** If \( zu_i \in lk(t) \) for some non-singular vertex \( u_i \), then \( tzu_i \in K \). Further, \( zwu, u_i tu \in K \). Thus, \( \partial(tzu) \subset lk(u_i) \), and by Lemma 3.3, \( tzu \in K \). This is a contradiction as \( z \notin lk(tu) \). Therefore \( zu_i \notin lk(t) \). Since \( z, u_i \notin t \), we have \( zu_i \notin st(t) \). \( \square \)

**Lemma 3.15.** Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11. If \( u \) is a non-singular vertex in \( lk(t) \) with \( \mathcal{O}_u < 1 \), then \( lk(u) \cap lk(t) \setminus lk(ut) \) contains exactly one vertex, say \( z \). Moreover, if \( z \) is non-singular, then \( \mathcal{O}_u = 0.5 \) and \( lk(u) = B_{u_1, \ldots, u_m}(t; z) \).

**Proof.** It follows from Lemma 3.5 that there are some vertices in \( lk(u) \cap lk(t) \setminus lk(ut) \). Since \( \mathcal{O}_u < 1 \), there is only one such vertex, called \( z \).

Let \( lk(tu) = C_m(u_1, u_2, \ldots, u_m) \), for some \( u_1, \ldots, u_m \in lk(t) \). Then by Lemma 3.1, \( m \geq 4 \). Since \( lk(u) \cap lk(t) \setminus lk(tu) \) has only one vertex \( z \), \( \mathcal{O}_u \geq λ(u, z) = 0.5 \). If \( \mathcal{O}_u = 0.5 \), then \( lk(u) \setminus st(t, lk(u)) \) contains no vertex other than \( z \). Therefore, \( lk(u) = B_{u_1, \ldots, u_m}(t; z) \).

If \( 0.5 < \mathcal{O}_u < 1 \), then \( \mathcal{B}_u \setminus st(t, lk(u)) \) contains exactly two vertices \( z \) and \( w \) such that \( λ(u, z) = 0.5 \) and \( 0 < λ(u, w) < 0.5 \). This implies that \( w \notin st(t) \) and \( λ(u, w) = \frac{1}{2} \) or \( \frac{3}{4} \).

If \( λ(u, w) = \frac{1}{2} \), then \( lk(w) = B_{w_1, \ldots, w_q}(u; q) \) and if \( λ(u, w) = \frac{3}{4} \), then \( lk(w) = B_{w_1, \ldots, w_q}(u; q) \). Since \( lk(tu) = C_m(u_1, u_2, \ldots, u_m) \) and there are exactly two vertices \( z, w \in lk(u) \setminus st(t, lk(u)) \), we have \( d(uz) \leq m + 1 \).

Let \( z \) be a non-singular vertex and \( λ(u, w) = \frac{1}{2} \) while \( lk(w) = B_{w_1, w_2, \ldots, w_4}(u; q) \). It follows from Lemma 3.5 that \( q = u_k \) for some \( k \). Thus, \( u_kw \notin lk(u) \), and hence \( u_kz \in lk(u) \). Therefore, \( ukwz, uzw, u_kwz \in K \). Since \( \partial(uwu_k) \subset lk(z) \), by Lemma 3.3, \( uwu_k \in K \). But \( u_k \notin lk(uw) \). This is a contradiction. Thus, \( λ(u, w) \neq \frac{1}{2} \). By the same arguments, we can show that \( λ(u, w) \neq \frac{3}{4} \). Therefore, \( \mathcal{O}_u = 0.5 \) and \( lk(u) = B_{u_1, \ldots, u_m}(t; z) \), \( m \geq 4 \). \( \square \)

**Lemma 3.16.** Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11. Let \( u \in lk(t) \) be a non-singular vertex such that \( lk(u) = B_{u_1, \ldots, u_m}(t; z) \), where \( z \) is a non-singular vertex. If there is a vertex \( v \in lk(u, lk(t)) \) with \( lk(v) = B_{v_1, \ldots, v_k}(t; z_1) \), then \( z \neq z_1 \).
Proof. If \(|\text{lk}(z_1)| \not= S^2\), then clearly, \(z \neq z_1\). For \(|\text{lk}(z_1)| \cong S^2\), let \(z = z_1\), i.e., \(\text{lk}(u) = B_u, \ldots, u_m(t; z)\) and \(\text{lk}(v) = B_{v_1, \ldots, v_k(t; z)}\), where \(v \in \text{lk}(u, \text{lk}(t))\). Since \(vw\) is an edge in \(K\), it follows from Lemma 3.15 that \(\text{lk}(u) \cap \text{lk}(v) \setminus \text{lk}(uv)\) contains some vertices. Let \(w \in \text{lk}(u) \cap \text{lk}(v) \setminus \text{lk}(uv)\). Then \(uv \subseteq uvzw\), and \(uvz\) are in \(K\). This implies that \(\partial(uwv) \subset \text{lk}(z)\), but \(uwv \notin K\). This contradicts Lemma 3.3 and hence \(z \neq z_1\). □

Lemma 3.17. Let \(K \in \mathcal{R}\), and \(t\) be the singular vertex in \(K\) as in Definition 3.11. Then \(\sum_{v \in \text{lk}(t)} \mathcal{O}_v \geq f_0(\text{lk}(t)) - 1\). Moreover, if \(\text{lk}(t, K)\) does not contain the other singular vertex, then \(\sum_{v \in \text{lk}(t)} \mathcal{O}_v \geq f_0(\text{lk}(t))\).

Proof. If for all vertices \(v \in \text{lk}(t)\), \(\mathcal{O}_v \geq 1\) holds, then trivially \(\sum_{v \in \text{lk}(t)} \mathcal{O}_v \geq f_0(\text{lk}(t))\). We consider the case when some vertices have an outer weight of less than 1. Let \(p_1 \in \text{lk}(t)\) be a non-singular vertex such that \(\mathcal{O}_{p_1} < 1\). Then by Lemma 3.15, \(\text{lk}(p_1) \cap \text{lk}(t) \setminus \text{lk}(p_1 t)\) contains exactly one vertex, say \(z_1\). If \(z_1\) is non-singular, then \(\mathcal{O}_{p_1} = 0.5\) and \(\text{lk}(p_1) = B_{p_1, \ldots, p_1 m}(t; z_1)\).

Let \(S_1\) be the set of all non-singular vertices \(v \in \text{lk}(t)\) such that \(\mathcal{O}_v = 0.5\) and \(\text{lk}(v) = B_{v_1, \ldots, v_m(t; z_1)}\), where \(z_1\) is the non-singular vertex as above. Then it follows from Lemma 3.16 that \(p_1, \ldots, p_1 m \notin S_1\). Let \(S'_1 = \{p_1, p_1^2, p_1^3, p_1^4\}\). Then by Lemma 3.14, \(z_1 p_1^1 \notin \text{lk}(t)\) for at least three \(p_1^1\)’s of \(S'_1\). Therefore,

\[
\sum_{v \in S_1 \cup \{z_1\}} \mathcal{O}_v = \mathcal{O}_{z_1} + \sum_{v \in S_1} \mathcal{O}_v \\
\geq 1.5 + \sum_{v \in S_1} \lambda(z_1, v) + \sum_{v \in S_1} \mathcal{O}_v \\
= 1.5 + \text{card}(S_1) \quad \text{(since} \quad \lambda(z_1, v) \quad \text{and} \quad \mathcal{O}_v = 1)\).
\]

Suppose that there is another non-singular vertex \(p_2 \in \text{lk}(t) \setminus S_1\) such that \(\mathcal{O}_{p_2} = 0.5\) and \(\text{lk}(p_2) = B_{p_2^1, \ldots, p_2^m}(t; z_2)\), where \(z_2 \neq z_1\) is also a non-singular vertex. Let \(S_2\) be the set of all non-singular vertices \(v \in \text{lk}(t)\) such that \(\mathcal{O}_v = 0.5\) and \(\text{lk}(v) = B_{p_2, \ldots, p_2 m(t; z_2)}\), where \(z_2\) is the non-singular vertex as above. Then by Lemma 3.16, \(p_2^1, \ldots, p_2^m \notin S_2\). Let \(S'_2 = \{p_2^1, p_2^2, p_2^3, p_2^4\}\). From Lemma 3.14, \(z_2 p_2^1 \notin \text{lk}(t)\) for at least three \(p_2^1\)’s in \(S'_2\). By similar arguments as above, we have

\[
\sum_{v \in S_2 \cup \{z_2\}} \mathcal{O}_v \geq 1.5 + \text{card}(S_2).
\]

Further, by the assumptions on \(S_1\) and \(S_2\), we have \((S_1 \cup \{z_1\}) \cap (S_2 \cup \{z_2\}) = \emptyset\). Therefore, after a finite number of steps, say \(n\), we get a set \(\tilde{S} = (S_1 \cup \{z_1\}) \cup \cdots \cup (S_n \cup \{z_n\})\), where \(z_1, \ldots, z_n\) are non-singular vertices and \(\sum_{v \in \tilde{S}} \mathcal{O}_v \geq \text{card}(\tilde{S}) + n/2\).

Let \(t_1\) be an edge in \(K\), where \(t_1\) is the other singular vertex in \(K\). Then \(\tilde{S} \subset V(\text{lk}(t) \setminus t_1)\). Suppose that there is a non-singular vertex \(p_3 \in V(\text{lk}(t))\) such that \(0.5 \leq \mathcal{O}_{p_3} < 1\) and \(\text{lk}(p_3) \cap \text{lk}(t) \setminus \text{lk}(p_3 t)\) contains the vertex \(t_1\). Let \(P\) be the set of all non-singular vertices \(v \in \text{lk}(t)\) such that \(0.5 \leq \mathcal{O}_v < 1\) and \(\text{lk}(v) \cap \text{lk}(t) \setminus \text{lk}(vt)\) contains only \(t_1\). Then,

\[
\sum_{v \in P \cup \{t_1\}} \mathcal{O}_v = \mathcal{O}_{t_1} + \sum_{v \in P} \mathcal{O}_v \\
\geq \sum_{v \in P} \lambda(t_1, v) + \sum_{v \in P} \mathcal{O}_v \\
\geq \text{card}(P) \quad \text{(since} \quad \lambda(t_1, v) \quad \text{and} \quad \mathcal{O}_v \geq 1)\).
\]

From our constructions of \(\tilde{S}\) and \(P\), it is clear that \(\tilde{S} \cap (P \cup \{t_1\}) = \emptyset\). Further, \(v \notin P\).
Let \( \tilde{S} \cup (P \cup \{ t_1 \}) \) implies \( O_v \geq 1 \). Thus,

\[
\sum_{v \in \text{lk}(t)} O_v = \sum_{v \in \tilde{S}} O_v + \sum_{v \in P \cup \{ t_1 \}} O_v + \sum_{v \in V(\text{lk}(t)) \setminus (\tilde{S} \cup (P \cup \{ t_1 \}))} O_v \\
\geq \text{card}(\tilde{S}) + n/2 + \text{card}(P) + f_0(\text{lk}(t)) - \text{card}(\tilde{S} \cup (P \cup \{ t_1 \})) \\
= \text{card}(\tilde{S}) + n/2 + \text{card}(P) + f_0(\text{lk}(t)) - \text{card}(\tilde{S}) - \text{card}(P) - 1 \\
= f_0(\text{lk}(t)) + n/2 - 1 \\
\geq f_0(\text{lk}(t)) - 1.
\]

If \( tt_1 \) is not an edge in \( K \) then \( P \) becomes empty, and \( v \notin \tilde{S} \) implies \( O_v \geq 1 \). Thus,

\[
\sum_{v \in \text{lk}(t)} O_v = \sum_{v \in \tilde{S}} O_v + \sum_{v \in V(\text{lk}(t)) \setminus \tilde{S}} O_v \\
\geq \text{card}(\tilde{S}) + n/2 + f_0(\text{lk}(t)) - \text{card}(\tilde{S}) \\
\geq f_0(\text{lk}(t)).
\]

This proves the result. \( \square \)

**Lemma 3.18.** Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11. Let \( u \in \text{lk}(t,K) \) be a non-singular vertex such that \( O_u = 0.5 \) and \( \text{lk}(u, \text{lk}(t)) = C_m(u_1, \ldots, u_m) \), for some \( m \geq 4 \). Then \( f_0(\text{lk}(t)) \geq 2m + 1 \).

**Proof.** Since \( O_u = 0.5 \), \( \text{lk}(u) \setminus \text{st}(t, \text{lk}(u)) \) contains exactly one vertex, say \( z \). Then \( \text{lk}(u) \cap \text{lk}(t) \setminus \text{lk}(tu) = \{z\} \) or \( (u, z) \) for some singular vertex \( w \in \text{lk}(ut) \). It follows from Lemma 3.6 that \( d(tz), d(uz) \geq m \). Then \( utz \in \text{lk}(u) \), i.e., \( uu_iz \in K \) for \( 1 \leq i \leq m \). If \( u_i \in \text{lk}(tz) \cap \text{lk}(tu) \) is a non-singular vertex, then \( u_iz, uz \in K \). Since \( u_iz \mu \in K \), \( \partial(uz) \subset \text{lk}(ut) \). This implies \( u_iz \mu \in K \) and hence \( z \notin \text{lk}(tu) \), which is not possible. Thus, \( u_iz \notin \text{lk}(t) \) for each non-singular vertex \( u_i \). Therefore, \( f_0(\text{lk}(t,K)) \geq f_0(\text{lk}(tu)) - 1 + f_0(\text{lk}(tz)) + \text{card}(\{u\}) + \text{card}(\{z\}) = 2m + 1 \). \( \square \)

**Lemma 3.19.** Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11. Let \( u \in \text{lk}(t,K) \) be a non-singular vertex such that \( 4 \leq d(ut) \leq 5 \). Then either \( O_u = 0.5 \) or \( O_u \geq 1 \).

**Proof.** It follows from Lemma 3.5 that \( O_u \geq 0.5 \). If possible, let \( 0.5 < O_u < 1 \). Then \( \text{lk}(u) \cap \text{lk}(t) \setminus \text{lk}(ut) \) contains exactly one vertex, say \( z \), and \( \text{lk}(u, K) \setminus \text{st}(t, \text{lk}(u)) \) contains exactly one vertex, say \( w \notin \text{lk}(t,K) \), other than \( z \). Then \( d(ut) \leq 8 \) and \( \lambda(ut) \leq 0.5 \). Then \( \lambda(w,u) > 0.5 \), and Lemma 3.9 implies that \( d(u) \geq 9 \). This is a contradiction. Thus, the result follows. \( \square \)

**Lemma 3.20.** Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11 such that \( f_0(\text{lk}(t)) \leq 10 \). Let \( u \in \text{lk}(t,K) \) be a non-singular vertex such that \( 5 \leq d(ut) \leq 6 \). Then \( O_u \geq 1 \).

**Proof.** If \( \text{lk}(u) \cap \text{lk}(t) \setminus \text{lk}(ut) \) contains two or more vertices, then the result follows. Suppose \( \text{lk}(u) \cap \text{lk}(t) \setminus \text{lk}(ut) \) contains exactly one vertex, say \( z \). Then Lemma 3.6 implies that \( d(uz), d(tz) \geq d(ut) \). If \( d(ut) = 6 \), then \( \text{lk}(u) \) contains at least two vertices other than \( z \) and the vertices of \( \text{lk}(ut) \). Thus \( O_u \geq 1 \). If \( d(ut) = 5 \), then the result follows from Lemmas 3.18 and 3.19. \( \square \)

**Lemma 3.21.** Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11 such that \( f_0(\text{lk}(t)) \leq 10 \). Let \( u \in \text{lk}(t) \) be a non-singular vertex such that \( d(ut) = 4 \). Then either \( O_u \geq 1 \) or \( O_u = 0.5 \) and there exists a vertex \( z \in \text{lk}(t) \) such that \( O_z \geq 2 \).
Proof. If \( \text{lk} (u) \setminus \text{st} (t, \text{lk} (u)) \) contains more than two vertices, then the result follows. If \( \text{lk} (u) \setminus \text{st} (t, \text{lk} (u)) \) has two vertices, then \( D_1 u \) must be of type \( 7(4) \), and the outer weight is 0.5 for both the vertices of \( \text{lk} (u) \setminus \text{st} (t, \text{lk} (u)) \). Therefore, \( \mathcal{O}_u = 1 \). If \( \text{lk} (u) \setminus \text{st} (t, \text{lk} (u)) \) has exactly one vertex, say \( z \), then \( \mathcal{O}_u = 0.5 \) and from Lemma 3.18 \( f_0 (\text{lk} (t)) \geq 9 \). It follows from Lemma 3.22 that \( \text{lk} (uz) \cap \text{lk} (tz) \) does not contain any non-singular vertex. Thus, \( d (tz) = 4 \) and \( \text{lk} (z) \cap \text{lk} (t) \setminus \text{lk} (tz) \) contains at least 4 vertices, and therefore \( \mathcal{O}_z \geq 2 \).

Lemma 3.22. Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11. Let \( u \in \text{lk} (t) \) be a non-singular vertex such that \( \mathcal{O}_u = 1 \) and \( d (ut) = n \), where \( 4 \leq n \leq 6 \). Then there is a vertex \( z \in \text{lk} (t) \) such that \( \mathcal{O}_z \geq 1.5 \) for \( n = 4 \), and \( \mathcal{O}_z \geq 2 \) for \( n = 5, 6 \).

Proof. It follows from Lemma 3.15 that \( \text{lk} (u) \cap \text{lk} (t) \setminus \text{lk} (ut) \) contains some vertices. If \( \text{lk} (u) \setminus \text{st} (t, \text{lk} (u)) \) contains more than three vertices, then \( \mathcal{O}_u > 1 \), which is a contradiction. If \( \text{lk} (u) \setminus \text{st} (t, \text{lk} (u)) \) contains exactly three vertices, then \( \mathcal{O}_u = 1 \) implies that (i) \( \text{lk} (u) \cap \text{lk} (t) \setminus \text{lk}(ut) \) contains exactly one vertex, say \( z \), and (ii) other two vertices, say \( p, q \in \text{lk} (u) \) but \( p, q \notin \text{lk} (t) \), and \( \lambda (u, p) = \lambda (u, q) = 0.25 \). Further, \( d (u) \leq 10 \). However, \( \lambda (u, p) = 0.25 \), and \( \text{lk} (up) \) contains at most one singular vertex implying \( d (u) \geq 11 \) (cf. Lemma 3.9 (ii)), which is a contradiction.

Therefore, \( \text{lk} (u) \setminus \text{st} (t, \text{lk} (u)) \) contains exactly two vertices, say \( z \) and \( w \). It follows from Lemma 3.15 that \( \text{lk} (u) \cap \text{lk} (t) \setminus \text{lk}(ut) \) contains either both \( z \) and \( w \) or exactly one vertex, say \( z \).

Case 1: Let \( \text{lk} (u) \cap \text{lk} (t) \setminus \text{lk}(ut) \) contain only \( z \). It follows from Lemma 3.16 that \( d (uz) \geq d (tu) = n \). Therefore, \( z \) is connected with \( n - 1 \) vertices, say \( x_1, x_2, \ldots, x_{n-1} \), of \( \text{lk} (ut) \) in \( D_1 u \). Since \( \text{lk} (ut) \) can have at most one singular vertex, at least \( n - 2 \) vertices from \( x_1, x_2, \ldots, x_{n-1} \) are non-singular. Let \( x_1, x_2, \ldots, x_{n-2} \) be non-singular vertices. Then the edges \( zx_1, zx_2, \ldots, zx_{n-2}, zu \) are not in \( \text{lk} (t) \). Therefore, \( \mathcal{O}_z \geq \frac{n}{4} \).

Case 2: Let \( \text{lk} (u) \cap \text{lk} (t) \setminus \text{lk}(ut) \) contain both \( z \) and \( w \). For \( n = 4 \), both \( z \) and \( w \) are connected with precisely three vertices of \( \text{lk} (ut) \) in \( D_1 u \). By the same arguments as in Case 1, we get \( \mathcal{O}_z \geq 1.5 \). For \( n = 5, 6 \), one vertex, say \( z \), is connected with at least four vertices of \( \text{lk} (ut) \) in \( D_1 u \). Therefore, by similar arguments as in Case 1, we get \( \mathcal{O}_z \geq 2 \).

Lemma 3.23. Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11. Let \( u \in \text{lk} (t) \) be a non-singular vertex such that \( \text{lk} (ut) \) is a \( (n - 2) \)-cycle and \( f_0 (\text{lk} (t)) = n \). Then \( \mathcal{O}_u \geq \left\lceil \frac{n}{2} \right\rceil \times 0.5 + \left\lceil \frac{n}{2} \right\rceil \times 0.25 + 0.5 \).

Proof. It follows from Lemma 3.15 that \( \text{lk} (u) \cap \text{lk} (t) \setminus \text{lk}(ut) \) contains exactly one vertex, say \( z \). Then Lemma 3.16 implies that \( d (tz), d (uz) \geq d (tu) = n - 2 \). Therefore, \( d (tz) = n - 2 \). Let \( \text{lk} (uz) = C_m (z_1, z_2, \ldots, z_m) \), for some \( m \geq n - 2 \). It follows from Lemma 3.13 that \( \text{lk} (uz) \cap \text{lk} (tu) \) does not contain any non-singular vertices. Since \( K \) has at most two singularities, \( \text{lk} (uz) \) contains at most one singular vertex, say \( z_m \) (if it exists), and hence \( \mathcal{O}_u \geq \lambda (u, z) + \sum_{i=1}^{m-1} \lambda (u, z_i) \). If \( \lambda (u, z_i) = 0.25 \), then \( f_0 (\text{lk} (z_i)) = 7 \), and hence \( f_0 (\text{lk} (z_{i+1})) \) and \( f_0 (\text{lk} (z_{i+1})) \) must be bigger than 8. Therefore \( \lambda (u, z_{i-1}) = \lambda (u, z_{i+1}) = 0.5 \) (here the summation in subscripts is modulo \( m \)). Therefore \( \mathcal{O}_u \geq \left\lceil \frac{m}{2} \right\rceil \times 0.5 + \left\lceil \frac{m}{2} \right\rceil \times 0.25 + 0.5 \).

Lemma 3.24. Let \( K \in \mathcal{R} \), and \( t \) be the singular vertex in \( K \) as in Definition 3.11. Let \( u \in \text{lk} (t) \) be a non-singular vertex such that \( \text{lk} (ut) \) is a \( (n - 3) \)-cycle and \( f_0 (\text{lk} (t)) = n \), \( 8 \leq n \leq 10 \). Then \( \mathcal{O}_u \geq 1.33 \) for \( n = 8, 9 \) and \( \mathcal{O}_u \geq 1.25 \) for \( n = 10 \).

Proof. It follows from Lemma 3.15 that \( \text{lk} (u) \cap \text{lk} (t) \setminus \text{lk}(ut) \) contains at most two vertices.

Case 1: Let \( \text{lk} (u) \cap \text{lk} (t) \setminus \text{lk}(tu) \) contains \( \{ z \} \) or \( \{ y, z \} \) for some singular vertex \( y \in \text{lk} (tu) \). Then by Lemma 3.16 \( d (tz), d (uz) \geq d (tu) = n - 3 \). Since \( f_0 (\text{lk} (ut)) = f_0 (\text{lk} (t)) - 3 \), \( \text{lk} (t) \setminus \text{st} (u, \text{lk} (t)) \)
contains exactly two vertices, and one of them is \( z \). Therefore, at least \( n - 4 \) vertices of \( \text{lk}(ut) \) are joined with \( z \) in \( \text{lk}(t) \) and at least \( n - 5 \) of them are non-singular. If one of those \( n - 5 \) non-singular vertices is joined with \( z \) in \( \text{lk}(u) \), then this contradicts the hypothesis of the Lemma \[\ref{lem:4.3}\]. Therefore, \( \text{lk}(u) \) contains at least \( n - 5 \) vertices other than \( z \) and the vertices of \( \text{lk}(tu) \). Let \( z_1, z_2, \ldots, z_m \) be the vertices in \( \text{lk}(u) \), where \( m \geq n - 5 \). Therefore, \( O_u \geq \lambda(u, z) + \sum_{i=1}^{m} \lambda(u, z_i) \geq 0.5 + \sum_{i=1}^{n-5} \lambda(u, z_i) \geq 1.5, 1.75 \) for \( n = 9, 10 \), respectively.

For \( n = 8 \), \( f_0(\text{lk}(ut)) = 5 \) and \( \text{lk}(u) \) contains at least 3 vertices other than \( z \) and the vertices of \( \text{lk}(ut) \). Therefore, \( d(u) \geq 10 \). If \( d(u) = 10 \), then \( \text{lk}(u) \) contains exactly 3 vertices, say \( z_1, z_2, \) and \( z_3 \) other than vertices of \( \text{lk}(ut) \), where \( z_1, z_2, z_3 \notin \text{lk}(t) \). Then from Lemma \[\ref{lem:3.9}\] \( \lambda(u, z_i) \geq 1/3 \). Thus, \( O_u \geq 1.5 \). If \( d(u) \geq 11 \), then \( \text{lk}(u) \) contains at least 4 vertices other than \( z \) and the vertices of \( \text{lk}(ut) \), and hence \( O_u \geq 1.5 \).

**Case 2:** Let \( \text{lk}(u) \cap \text{lk}(t) \setminus \text{lk}(tu) \) contain exactly two vertices, say \( z \) and \( w \). We claim that \( \text{lk}(u) \setminus \text{st}(t, \text{lk}(u)) \) contains at least three vertices (i.e., one extra vertex other than \( z \) and \( w \)). If possible, let \( \text{lk}(u) \setminus \text{st}(t, \text{lk}(u)) \) have exactly two vertices \( z, w \). Since \( D_tu \) does not contain any diagonal (cf. Lemma \[\ref{lem:3.3}\]) and \( D_tu \) is a triangulated disc, \( zw \) must be an edge, and each vertex in \( \text{lk}(tu) \) is joined with either \( z \) or \( w \) (or both) in \( \text{lk}(u) \). Let \( \text{lk}(zw, \text{lk}(u)) = \{p, q\} \). Then \( p, q \in \text{lk}(tu) \) and \( p, q \) are joined with both \( z \) and \( w \).

If possible, let \( zw \notin \text{lk}(t) \). If \( p \) (respectively \( q \)) is non-singular, then it follows from Lemma \[\ref{lem:3.4}\] that \( p \) (respectively \( q \)) is not joined with \( z \) and \( w \) in \( \text{lk}(t) \). Further, Lemma \[\ref{lem:3.4}\] implies that a non-singular vertex in \( \text{lk}(tu) \), which is joined with \( z \) (respectively \( w \)) in \( \text{lk}(u) \), is not joined with \( z \) (respectively \( w \)) in \( \text{lk}(t) \). Therefore, a non-singular vertex \( v(\neq p, q) \in \text{lk}(tu) \) is joined with at most one of \( z \) and \( w \) in \( \text{lk}(t) \). Since \( V(\text{lk}(t)) = \{z, w, u\} \cup V(\text{lk}(tu)) \) and \( \text{lk}(t) \) contains at most one singular vertex, we have \( d(tz) + d(tw) \leq n - 3 \). If \( n \leq 10 \), then \( d(tz) + d(tw) \leq 7 \), which contradicts the hypothesis of Lemma \[\ref{lem:3.1}\]. Therefore, \( zw \) must be an edge in \( \text{lk}(t) \).

Let \( \text{lk}(zw, \text{lk}(t)) = \{r, s\} \). Then \( rz, rw, sz, \) and \( sw \) are edges in \( \text{lk}(t) \). Since \( V(\text{lk}(t)) = \{z, w, u\} \cup V(\text{lk}(tu)) \) and \( \text{lk}(t) \) contains at most one singular vertex, without loss of generality, we assume that \( r \in \text{lk}(tu) \) is a non-singular vertex. Since \( r \in \text{lk}(tu) \), \( r \) is joined with either \( z \) or \( w \) in \( \text{lk}(u) \). This contradicts the hypothesis of the Lemma \[\ref{lem:3.4}\].

Therefore, \( \text{lk}(u) \setminus \text{st}(t, \text{lk}(u)) \) has at least three vertices. Let \( x \) be the third vertex. If \( \text{lk}(u) \setminus \text{st}(t, \text{lk}(u)) \) has exactly three vertices, then \( d(u) = 1 + n - 3 + 3 = n + 1 \). Therefore, \( \lambda(u, x) \geq 0.33 \) for \( n = 8, 9 \), and \( \lambda(u, x) \geq 0.25 \) for \( n = 10 \). Thus, \( O_u \geq \lambda(u, z) + \lambda(u, w) + \lambda(u, x) \geq 1.33 \) for \( n = 8, 9 \), and \( O_u \geq 1.25 \) for \( n = 10 \). If \( \text{lk}(u) \setminus \text{st}(t, \text{lk}(u)) \) has more than three vertices, then \( O_u \geq 1.5 \).

## 4 Normal 3-pseudomanifolds with exactly one singularity

In this section we consider normal 3-pseudomanifolds with exactly one singularity. Let us denote \( \mathcal{R}_1 = \{K \in \mathcal{R} : K \text{ has exactly one singularity}\} \). Let \( K \in \mathcal{R}_1 \) and \( t \) be the singular vertex in \( K \). Then \( \text{lk}(t, K) \) is either a connected sum of tori or a connected sum of Klein bottles. In short, we say that \( |\text{lk}(t, K)| \) is a closed connected surface with \( h \) number of handles, for \( h \geq 1 \). For \( m \geq 4 \), let \( x_m \) be the number of vertices in \( \text{lk}(t, K) \) with degree \( m \) in \( \text{lk}(t, K) \).

**Lemma 4.1.** Let \( K \in \mathcal{R}_1 \), and \( t \) be the singular vertex in \( K \). Then \( \sum_{v \in \text{lk}(t)} O_v \geq 10 \).

**Proof.** It follows from Lemma \[\ref{lem:3.7}\] that \( f_0(\text{lk}(t, K)) \geq 8 \). First, let us assume \( f_0(\text{lk}(t, K)) = 8 \). It follows from Lemma \[\ref{lem:3.5}\] that \( x_m = 0 \) for \( m = 7 \). Let \( u \in \text{lk}(t, K) \) be any non-singular vertex such that \( d(ut) = 4 \). It follows from Lemmas \[\ref{lem:4.18}\] and \[\ref{lem:3.19}\] that \( O_u \geq 1 \). Further, Lemmas \[\ref{lem:3.23}\] and \[\ref{lem:3.24}\] imply \( \sum_{v \in \text{lk}(t)} O_v \geq x_4 + 1.33x_5 + 2.375x_6 \), where \( x_4 + x_5 + x_6 = 8 \) and \( 4x_4 + 5x_5 + 6x_6 = 48 \). Thus, by solving the L.P.P., \( \sum_{v \in \text{lk}(t)} O_v \geq 19 \).
Now, we assume that $f_0(\text{lk}(t, K)) = 9$. It follows from Lemma 3.2 that $x_m = 0$ for $m = 8$. Let $u \in \text{lk}(t, K)$ be any non-singular vertex such that $d(ut) = 4$. It follows from Lemma 3.10 that either $O_u = 0.5$ or $O_u \geq 1$. If $d(ut) = 4$ and $O_u = 0.5$, then by Lemma 3.5 \( \text{lk}(t) \cap \text{lk}(u) - \text{lk}(tu) \) contains exactly one vertex, say $z$. From Lemma 3.1 we get $d(tz), d(uz) \geq 4$. Since $\text{lk}(t, K)$ does not contain any singular vertex, by Lemma 3.4 \( \text{lk}(tz, K) \cap \text{lk}(tu, K) = \emptyset \). This implies, $f_0(\text{lk}(t, K)) \geq 10$. This is a contradiction. Therefore, $O_u \geq 1$. It follows from Lemmas 3.2, 3.3, and 3.4 that \( \sum_{v \in \text{lk}(t)} O_v \geq x_4 + x_5 + 1.33x_6 + 2.75x_7 \), where $x_4 + x_5 + x_6 + x_7 = 9$ and $4x_4 + 5x_5 + 6x_6 + 7x_7 = 54$. Thus, by solving the L.P.P., we get \( \sum_{v \in \text{lk}(t)} O_v \geq 11.97 \).

If $f_0(\text{lk}(t, K)) \geq 10$, then from Lemma 3.17 we have \( \sum_{v \in \text{lk}(t)} O_v \geq f_0(\text{lk}(t, K)) \geq 10 \). This proves the result. \( \square \)

**Remark 4.2.** Let $K \in \mathcal{R}_1$, and $t$ be the singular vertex in $K$. Then the lower bound for \( \sum_{v \in \text{lk}(t)} O_v \) can be easily improved from 10. However, we did not move in that direction, as the lower bound 10 serves all of our purposes.

**Theorem 4.3.** If $K \in \mathcal{R}_1$, then $g_2(K) \geq g_2(\text{lk}(v, K)) + 10$ for every vertex $v \in K$.

**Proof.** Let $t$ be the singular vertex in $K$. It follows from Lemma 4.1 that \( \sum_{v \in \text{lk}(t)} O_u \geq 10 \). Since $g_2(\text{lk}(t, K)) \geq g_2(\text{lk}(v, K))$ for any vertex $v$ in $K$, the result follows from Lemma 3.13. \( \square \)

**Theorem 4.4.** Let $K$ be a normal 3-pseudomanifold with exactly one singularity at $t$ such that \( |\text{lk}(t, K)| \) is a connected sum of $n$ copies of tori or Klein bottles for some positive integer $n$. Then $g_2(K) \leq 9 + 6n$ implies $K$ is obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, edge expansions, and vertex foldings. More precisely, the sequence of operations includes exactly $n$ vertex foldings and a finite number of remaining operations.

**Proof.** Let $\Delta$ be a normal 3-pseudomanifold with exactly one singularity at $t$ such that \( |\text{lk}(t, \Delta)| \) is a connected sum of $m$ copies of tori or Klein bottles for some positive integer $m$. Then $g_2(\text{lk}(t, K)) = 6m$. Let $g_2(\Delta) \leq 9 + 6m$. We have the following observation:

**Observation 1:** Let \( \Delta \) (may be $\Delta$ itself) be a normal 3-pseudomanifold obtained from $\Delta$ by the repeated applications of combinatorial operations mentioned in Remark 2.3 such that there is no normal 3-pseudomanifold $\Delta'$ that is obtained from $\Delta$ by a combinatorial operation mentioned in Remark 2.3 and $g_2(\Delta') < g_2(\Delta)$. If $\Delta$ has no missing tetrahedron, then $\Delta \in \mathcal{R}_1$ and hence by Theorem 4.3 $g_2(\Delta) \geq 6m + 10$. Thus, $g_2(\Delta) \geq g_2(\Delta) \geq 6m + 10$. This contradicts the given condition. Therefore, $\Delta$ must have a missing tetrahedron.

There can be four types of missing tetrahedra in $\Delta$:

**Type 1:** Let $\sigma$ be a missing tetrahedron in $\Delta$ such that $t$ is not a vertex of $\sigma$.

**Type 2:** Let $\sigma$ be a missing tetrahedron in $\Delta$ such that $t \leq \sigma$ and $\text{lk}(t, \Delta)$ is separated into two portions by the missing triangle formed by the other three vertices of $\sigma$, where one portion is a disc.

**Type 3:** Let $\sigma$ be a missing tetrahedron in $\Delta$ such that $t \leq \sigma$ and $\text{lk}(t, \Delta)$ is not separated into two portions by the missing triangle formed by the other three vertices of $\sigma$.

**Type 4:** Let $\sigma$ be a missing tetrahedron in $\Delta$ such that $t \leq \sigma$ and $\text{lk}(t, \Delta)$ is separated into two portions by the missing triangle formed by the other three vertices of $\sigma$, where no portions are triangulated discs.
Now, we are ready to prove our result. Let \( K \) be a normal 3-pseudomanifold with exactly one singularity at \( t \) such that \( |\text{lk} (t, K)| \) is a connected sum of \( n \) copies of tori or Klein bottles and \( g_2(K) \leq 9 + 6n \) for some positive integer \( n \). We use the principle of mathematical induction on \( n \). If a normal 3-pseudomanifold \( \Delta \) has no singular vertices and \( g_2(\Delta) \leq 9 \), then we can assume \( K = \Delta, n = 0, \) and \( t \) is any vertex of \( \Delta \). By Proposition 2.2, we can say that the result is true for \( n = 0 \). Let us assume that the result is true for \( 0, 1, \ldots, n - 1 \), and let \( \Delta \) be the normal 3-pseudomanifold that corresponds to \( n \).

**Step 1:** Let \( \Delta \) be a normal 3-pseudomanifold obtained from \( K \) by repeated applications of the combinatorial operations mentioned in Remark 2.3 such that there is no normal 3-pseudomanifold \( \Delta' \) that is obtained from \( \Delta \) by a combinatorial operation mentioned in Remark 2.3 and \( g_2(\Delta') < g_2(\Delta) \). Then by Observation 1, we get \( \Delta \) must have a missing tetrahedron.

**Step 2:** Let \( \Delta \) have a missing tetrahedron of Type 1 or 2. Then it follows from Lemma 2.5 that \( \Delta \) is formed using a connected sum of Type 1 and Type 2. Let \( t \in \Delta_1 \) (in case of Type 2, we take \( t \) as a vertex in \( \Delta_1 \) such that \( |\text{lk} (t, \Delta_1)| \) is a connected sum of \( n \) number of handles). Then \( g_2(\Delta_1) \geq g_2(\text{lk}(t, \Delta_1)) \geq 6n \). Therefore \( g_2(\Delta_2) = g_2(\Delta) - g_2(\Delta_1) \leq 9 \). Thus, after a finite number of steps, we have \( \Delta = \Delta_1 \# \Delta_2 \# \cdots \# \Delta_n \), where \( (i) \ t \in \Delta_1 \) and \( |\text{lk} (t, \Delta_1)| \) is a connected sum of \( n \) copies of tori or Klein bottles, \( (ii) \ \Delta_1 \) has no missing tetrahedron of Type 1 or 2, \( (iii) \ 6n \leq g_2(\Delta_1) \leq 6n + 9 \), and \( (iv) \ ) for \( 2 \leq i \leq n, \ \Delta_i \) has no singular vertices, and \( g_2(\Delta_i) \leq 9 \). Thus, by Proposition 2.5 we have, for \( 2 \leq i \leq n, \ \Delta_i \) is a triangulated 3-sphere and is obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, and edge expansions. If there is no normal 3-pseudomanifold \( \Delta'_1 \), which is obtained from \( \Delta_1 \) by a combinatorial operation mentioned in Remark 2.3 and \( g_2(\Delta'_1) < g_2(\Delta_1) \), then by Observation 1, \( \Delta_1 \) has a missing tetrahedron of Type 3 or 4, and we move to Step 3 or 4, respectively. Otherwise, we move to Step 1 and replace \( K \) with \( \Delta_1 \). Since \( K \) has a finite number of vertices and \( g_2(K) \) is also finite, after a finite number of steps, we must move to either Step 3 or Step 4.

**Step 3:** Let \( \Delta_1 \) have a missing tetrahedron of Type 3. It follows from Lemma 2.11 that \( \Delta_1 \) is formed using a vertex folding from a normal 3-pseudomanifold \( \Delta'_1 \) at \( t \in \Delta'_1 \) and \( g_2(\Delta'_1) = g_2(\Delta_1) - 6 \). Here \( \text{lk} (t, \Delta'_1) \) is a connected sum of \( n - 1 \) copies of tori or Klein bottle and \( g_2(\Delta'_1) \leq 9 + 6(n - 1) \). Now, the result follows by the induction hypothesis.

**Step 4:** Let \( \Delta_1 \) have a missing tetrahedron of Type 4. Then it follows from Lemma 2.8 that \( \Delta_1 \) is formed using a connected sum of \( \Delta'_1 \) and \( \Delta''_1 \). Let \( t_1 \in \Delta'_1 \) and \( t_2 \in \Delta''_1 \) be identified during the connected sum and produce \( t \in \Delta_1 \). Let \( \text{lk} (t_1, \Delta'_1) \) and \( \text{lk} (t_2, \Delta''_1) \) be the connected sum of \( n_1 \) and \( n_2 \) copies of tori or Klein bottles, respectively, where \( n_1 + n_2 = n \) for some positive integers \( n_1, n_2 \). Since \( n_1, n_2 > 0 \), both \( n_1, n_2 < n \). Further, \( g_2(\Delta'_1) \leq 9 + 6n_1 \) and \( g_2(\Delta''_1) \leq 9 + 6n_2 \). Then the result follows by the induction hypothesis.

From the construction, we can see that the sequence of operations includes exactly \( n \) number of vertex foldings and a finite number of remaining operations.

**Remark 4.5.** The upper bound in Theorem 4.4 is sharp. In other words, there exists a normal 3-pseudomanifold with exactly one singularity such that \( g_2(K) = 10 + 6n \) and \( K \) is not obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, edge expansions, and vertex foldings. Let us take
a connected sum of a finite number of the boundary complexes of the 4-simplices, and then apply a handle addition. Let $\Delta_0$ be the resulting 3-dimensional manifold. Then $g_2(\Delta_0) = 10$. Note that we can choose either an orientable or a non-orientable manifold according to our purpose. For $1 \leq i \leq n$, let $\Delta_i$ be a normal 3-pseudomanifold obtained from a connected sum of a finite number of the boundary complexes of the 4-simplices by applying a vertex folding at $v_i$, where $v_i \in \Delta_i$. Then $g_2(\Delta_i) = 6$ and $\text{lk}(v_i, \Delta_i)$ is a torus or Klein bottle (choose the surface according to the purpose). Let $\Delta$ be the connected sum $\Delta_0 \# \Delta_1 \# \cdots \# \Delta_n$, where the vertices $v_1, \ldots, v_n$ are identified to a single vertex $v$. Then $\Delta$ is a normal 3-pseudomanifold with exactly one singularity at $v$ such that $g_2(\Delta) = 10 + 6n$. This normal 3-pseudomanifold $\Delta$ will serve our purpose.

Let $K$ be formed using a vertex folding from a normal 3-pseudomanifold $K'$ at $t \in K'$ and $|K'|$ is a handlebody with its boundary coned off. Then $K$ is the following pseudomanifold: take $K'[V(K') \setminus \{t\}]$ (the induced subcomplex of $K'$ on the vertex set $V(K') \setminus \{t\}$), identify two triangles (with an admissible bijection between them) on the boundary, then coning off the boundary at $t$. Therefore, $|K|$ is a handlebody with its boundary coned off. Further, let $\text{lk}(t_1, \Delta_1)$ and $\text{lk}(t_2, \Delta_2)$ be the connected sum of $n_1$ and $n_2$ copies of tori or Klein bottles, respectively, where $n_1 + n_2 = n$ for some positive integers $n_1, n_2$. Let $\Delta_1$ and $\Delta_2$ be triangulated handlebodies with its boundary coned off at $t_1$ and $t_2$, respectively. Let $\Delta = \Delta_1 \# \Delta_2$, where $t_1 \in \Delta_1$ and $t_2 \in \Delta_2$ are identified to $t \in \Delta$ during the connected sum. Then $\Delta$ is the following pseudomanifold: take two triangulated handlebodies $\Delta_1[V(\Delta_1) \setminus \{t_1\}]$ and $\Delta_2[V(\Delta_2) \setminus \{t_2\}]$, identify two triangles from each of the boundaries, then coning off the boundary at $t$. Therefore, $|\Delta|$ is a handlebody with its boundary coned off. Therefore, from the proof of Theorem 4.4 and Remark 4.5 we have the following result:

**Corollary 4.6.** Let $K$ be a normal 3-pseudomanifold with exactly one singularity at $t$ such that $|\text{lk}(t, K)|$ is a connected sum of $n$ copies of tori or Klein bottles for some positive integer $n$. Then $g_2(K) \leq 9 + 6n$ implies $|K|$ is a handlebody with its boundary coned off. Moreover, there exists a normal 3-pseudomanifold with exactly one singularity such that $g_2(K) = 10 + 6n$ and $|K|$ is not a handlebody with its boundary coned off.

## 5 Normal 3-pseudomanifolds with exactly two singularities

In this section we will consider normal 3-pseudomanifolds with exactly two singularities. Let us denote $R_2 = \{K \in \mathcal{R} : K$ has exactly two singularities$\}$. Let $K \in R_2$ and $t$ be the singular vertex in $K$ as in Definition 3.11 i.e., $g_2(\text{lk}(t, K)) \geq g_2(\text{lk}(v, K))$ for any vertex $v$ in $K$. By Lemma 3.8 we can assume that $d(t) \geq 8$. Let $y_4$ be the number of non-singular vertices of degree 4 in $\text{lk}(t)$ whose outer weight is 0.5, and let $x_n$ be the number of non-singular vertices of degree $n$ in $\text{lk}(t)$ with outer weight greater than or equal to 1, for $4 \leq n \leq 9$.

**Lemma 5.1.** Let $K \in R_2$, and $t$ be the singular vertex in $K$ as in Definition 3.11. If $f_0(\text{lk}(t, K)) = 8$, then $\sum_{v \in \text{lk}(t)} O_v > 10$.

**Proof.** Since $f_0(\text{lk}(t, K)) = 8$, it follows from Lemma 3.2 that $x_7 = 0$. Let $u \in \text{lk}(t, K)$ be any non-singular vertex such that $d(ut) = 4$. It follows from Lemmas 3.18 and 3.19 that $O_u \geq 1$. We take the outer weight of any seven non-singular vertices of $K$ contained in $\text{lk}(t)$. It follows from Lemmas 3.23 and 3.24 that $\sum_{v \in \text{lk}(t)} O_v \geq x_4 + 1.33x_5 + 2.375x_6$. Further, we have $x_4 + x_5 + x_6 = 7$, and $4x_4 + 5x_5 + 6x_6 \geq 35$. Therefore, by solving the L.P.P., we get $\sum_{v \in \text{lk}(t)} O_v \geq 9.31$. 

\hspace{1cm} \square
Lemma 5.2. Let $K \in \mathcal{R}_2$, and $t$ be the singular vertex in $K$ as in Definition 3.11. If $f_0(\text{lk}(t, K)) = 9$, then $\sum_{v \in \text{lk}(t)} O_v > 9$.

Proof. Since $f_0(\text{lk}(t, K)) = 9$, Lemma 3.3 implies $x_8 = 0$. It follows from Lemma 3.19 that $O_u = 0.5$ or $\geq 1$, when $d(tu) = 4$. Further, Lemmas 3.20, 3.23 and 3.24 imply $O_u \geq 1$ when $d(tu) = 5$, $O_u \geq 1.33$ when $d(tu) = 6$, and $O_u \geq 2.75$ when $d(tu) = 7$. Let $t_1$ be the singular vertex in $K$ other than $t$. Now we have the following cases:

Case 1: Let there be a non-singular vertex $u \in \text{lk}(t, K)$ such that $O_u = 0.5$. Since $f_0(\text{lk}(t, K)) = 9$, it follows from Lemma 3.21 that there must be a vertex $z \in \text{lk}(t)$ such that (i) $O_z \geq 2$, (ii) $d(tu) = d(tz) = 4$, and (iii) $l(tu) \cap l(tz) = \{t_1\}$. Further, if $\text{lk}(t, K)$ contains at least two non-singular vertices, say $u_1$ and $u_2$, of $K$ such that $O_{u_1} = O_{u_2} = 0.5$, then we have two vertices $z_1 \neq z_2$ such that $O_{z_1}, O_{z_2} \geq 2$ and $d(tu_1) = d(tu_2) = d(tz_1) = d(tz_2) = 4$. Then $\sum_{v \in \text{lk}(t)} O_v \geq 0.5y_4 + x_4 + x_5 + 1.33x_6 + 2.75x_7 + 2z$, with one of the following conditions:

(i) $y_4 + x_4 + x_5 + x_6 + x_7 + z = 8$, $4y_4 + 4x_4 + 5x_5 + 6x_6 + 7x_7 + 4z \geq 40$, $y_4 = 1$, $z = 1$.

(ii) $y_4 + x_4 + x_5 + x_6 + x_7 + z = 8$, $4y_4 + 4x_4 + 5x_5 + 6x_6 + 7x_7 + 4z \geq 40$, $y_4 \geq 2$, $z \geq 2$.

Thus, by solving the L.P.P., we get $\sum_{v \in \text{lk}} O_v \geq 9.16$.

Case 2: Suppose that for all non-singular vertices $u \in \text{lk}(t, K)$, $O_u > 0.5$. It follows from Lemmas 3.19 and 3.20 that $O_u \geq 1$, when $d(tu) = 4$ and $5$. Further, Lemmas 3.23 and 3.24 imply $O_u \geq 1.33$ when $d(tu) = 6$, and $O_u \geq 2.75$ when $d(tu) = 7$.

Case 2a: Let there be a vertex $u \in \text{lk}(t)$ such that $d(u) = 5$ and $O_u = 1$. Then we must have a vertex $z \in \text{lk}(t)$ such that $O_z \geq 2$ and $d(tz) = 4$. Then $\sum_{v \in \text{lk}(t)} O_v \geq x_4 + x_5 + x_6 + 2z$, where $x_4 + x_5 + x_6 + x_7 + z \geq 8$, $4x_4 + 4x_5 + 4x_6 + 4x_7 + 4z \geq 40$, $x_4 \geq 1$, and $z \geq 1$. Thus, by solving the L.P.P., we get $\sum_{v \in \text{lk}(t)} O_v \geq 9.2$.

Case 2b: Suppose that $O_u > 1$ for all non-singular vertices $u \in \text{lk}(t, K)$, with $d(u) = 5$, i.e., $O_u > 1.16$. Then $\sum_{v \in \text{lk}(t)} O_v \geq x_4 + 1.16x_5 + 1.33x_6 + 2.75x_7$, where $x_4 + x_5 + x_6 + x_7 + z \geq 8$, and $x_4 + 4x_5 + 6x_6 + 7x_7 \geq 40$. Thus, by solving the L.P.P., we get $\sum_{v \in \text{lk}(t)} O_v \geq 9.28$.

Lemma 5.3. Let $K \in \mathcal{R}_2$, and $t \in K$ be the singular vertex as in Definition 3.11. If $f_0(\text{lk}(t)) = 10$, then $\sum_{v \in \text{lk}(t)} O_v > 9$.

Proof. If there is a non-singular vertex $u \in \text{lk}(t)$ such that $O_u = 0.5$, then from Lemma 3.18 we get $\sum_{v \in \text{lk}(t)} O_v \geq f_0(\text{lk}(t)) - 1/2$. Therefore, $\sum_{v \in \text{lk}(t)} O_v > 9$. Now, we assume that, for all non-singular vertex $u \in \text{lk}(t)$, $O_u > 0.5$. Then by Lemmas 3.20 and 3.21 we have $O_u \geq 1$, when $d(tu) = 4, 5$ and 6. Further, Lemmas 3.23 and 3.24 imply $O_u \geq 1.25$ when $d(tu) = 7$ and $O_u \geq 3.125$ when $d(tu) = 8$. Since $f_0(\text{lk}(t, K)) = 10$, it follows from Lemma 3.3 that $x_9 = 0$.

Case 1: If there is a non-singular vertex $u \in \text{lk}(t)$ such that $O_u = 1$ for $d(tu) = 4, 5$ or 6, then By Lemma 3.22 there is another vertex $z \in \text{lk}(t)$ such that $O_z \geq 1.5$. Then $\sum_{v \in \text{lk}(t)} O_v \geq x_4 + x_5 + x_6 + 1.25x_7 + 3.125x_8 + 1.5z$, where $z \geq 1$ and $x_4 + x_5 + x_6 + x_7 + x_8 + z \geq 9$. Therefore, $\sum_{v \in \text{lk}(t)} O_v \geq 9.5$.

Case 2: For all non-singular vertices $u \in \text{lk}(t)$, $O_u > 1$. Then $\sum_{v \in \text{lk}(t)} O_v > x_4 + x_5 + x_6 + x_7 + x_8 \geq 9$.

This proves the result.

Theorem 5.4. If $K \in \mathcal{R}_2$, then $g_2(K) \geq g_2(\text{lk}(v, K)) + 10$ for every vertex $v \in K$.

Proof. Let $t$ be the singular vertex in $K$ as in Definition 3.11 i.e., $g_2(\text{lk}(t, K)) \geq g_2(\text{lk}(v, K))$ for any vertex $v \in K$. If $8 \leq f_0(\text{lk}(t)) \leq 10$, then it follows from Lemmas 3.17, 5.2 and 5.3 that $\sum_{v \in \text{lk}(t)} O_u \geq 10$. If $f_0(\text{lk}(t)) \geq 11$, then it follows from Lemma 3.17 that $\sum_{v \in \text{lk}(t)} O_u \geq 10$. Now, the result follows from Lemma 3.13.
Let \( K \) be a normal 3-pseudomanifold with exactly two singularities at \( t \) and \( t_1 \) such that \( |\text{lk}(t_1)| \cong \mathbb{R}P^2 \). Then \( |\text{lk}(t)| \) is a connected sum of \((2m-1)\) copies of \( \mathbb{R}P^2 \) for some \( m \in \mathbb{N} \), and \( g_2(\text{lk}(t,K)) = 6m - 3 \). If \( K \in \mathcal{R}_2 \), then \( g_2(K) \geq 7 + 6m \).

**Theorem 5.5.** Let \( K \) be a normal 3-pseudomanifold with exactly two singularities at \( t \) and \( t_1 \) such that \( |\text{lk}(t)| \) is a connected sum of \((2m-1)\) copies of \( \mathbb{R}P^2 \) and \( |\text{lk}(t_1)| \cong \mathbb{R}P^2 \) for some positive integer \( m \). Then \( g_2(K) \leq 6 + 6m \) implies that \( K \) is obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, edge expansions, and edge foldings. More precisely, the sequence of operations includes exactly \((m - 1)\) vertex foldings, one edge folding and a finite number of remaining operations. Further, this upper bound is sharp for such normal 3-pseudomanifolds.

**Proof.** Let \( \Delta \) be a normal 3-pseudomanifold with exactly two singularities at \( t \) and \( t_1 \) such that \( |\text{lk}(t,\Delta)| \) is a connected sum of \((2k - 1)\) copies of \( \mathbb{R}P^2 \) and \( |\text{lk}(t_1,\Delta)| \cong \mathbb{R}P^2 \) for some positive integer \( k \). Then \( g_2(\text{lk}(t,K)) = 6k - 3 \). Let \( g_2(\Delta) \leq 6 + 6k \). We have the following observation:

**Observation 1:** Let \( \tilde{\Delta} \) (may be \( \Delta \) itself) be a normal 3-pseudomanifold obtained from \( \Delta \) by the repeated applications of the combinatorial operations mentioned in Remark 2.3 such that there is no normal 3-pseudomanifold \( \Delta' \) which is obtained from \( \Delta \) by a combinatorial operation mentioned in Remark 2.3 and \( g_2(\Delta') < g_2(\Delta) \). If \( \Delta \) has no missing tetrahedron, then \( \Delta \in \mathcal{R}_2 \) and hence by Theorem 3.4 \( g_2(\Delta) \geq (6k - 3) + 10 = 6k + 7 \). This contradicts the given condition. Therefore, \( \Delta \) must have a missing tetrahedron.

There can be five types of missing tetrahedra in \( \Delta \):

- **Type 1:** Let \( \sigma \) be a missing tetrahedron in \( \Delta \) such that \( t \) and \( t_1 \) are not vertices of \( \sigma \).

- **Type 2:** Let \( \sigma \) be a missing tetrahedron in \( \Delta \) such that \( \leq \sigma \) and \( \text{lk}(t,\Delta) \) is separated into two portions by the missing triangle formed by the other three vertices of \( \sigma \), where one portion is a disc. If \( t_1 \leq \sigma \), then \( \text{lk}(t_1,\Delta) \) is separated into two portions by the missing triangle formed by the other three vertices of \( \sigma \).

- **Type 3:** Let \( \sigma \) be a missing tetrahedron in \( \Delta \) such that \( t, t_1 \leq \sigma \) and \( \text{lk}(t_1,\Delta) \) is not separated into two portions by the missing triangle formed by the other three vertices. Then a small neighborhood of \( |\partial(\Delta[V(\sigma) \setminus \{t_1\}])| \) in \( |\text{lk}(t_1,\Delta)| \) is a Möbius strip, and it follows from Lemma 2.12 that a small neighborhood of \( |\partial(\Delta[V(\sigma) \setminus \{t\}])| \) in \( |\text{lk}(t,\Delta)| \) is also a Möbius strip. Further, there exists a normal 3-pseudomanifold \( \Delta' \) such that \( \Delta = (\Delta'_t)_{t_1} \) is obtained from an edge folding at \( t_1 \in \Delta' \). Therefore, \( \Delta' \) has exactly one singularity, say \( t \), such that \( (i) \ |\text{lk}(t,\Delta')| \) is a connected sum of \((k - 1)\) copies of tori or Klein bottles, and \( (ii) \ g_2(\Delta') \leq 6(k - 1) + 9 \). It follows from Theorem 4.4 that \( \Delta' \) is obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, edge expansions, and vertex foldings.

- **Type 4:** Let \( \sigma \) be a missing tetrahedron in \( \Delta \) such that \( \leq \sigma \) and \( \text{lk}(t,\Delta) \) is not separated into two portions by the missing triangle formed by the other three vertices of \( \sigma \). If \( t_1 \leq \sigma \), then \( \text{lk}(t_1,\Delta) \) is separated into two portions by the missing triangle formed by the other three vertices of \( \sigma \).

- **Type 5:** Let \( \sigma \) be a missing tetrahedron in \( \Delta \) such that \( \leq \sigma \) and \( \text{lk}(t,\Delta) \) is separated into two portions by the missing triangle formed by the other three vertices of \( \sigma \), where no portions are triangulated discs. If \( t_1 \leq \sigma \), then \( \text{lk}(t_1,\Delta) \) is separated into two portions by the missing triangle formed by the other three vertices of \( \sigma \).
Now, we are ready to prove our result. Let $K$ be a normal 3-pseudomanifold with exactly two singularities at $t$ and $t_1$ such that $|\text{lk}(t)|$ is a connected sum of $(2m - 1)$ copies of $\mathbb{RP}^2$, $|\text{lk}(t_1)| \cong \mathbb{RP}^2$ and $g_2(K) \leq 6 + 6m$ for some positive integer $m$.

Step 1: Let $\Delta$ be a normal 3-pseudomanifold obtained from $K$ by repeated applications of the combinatorial operations mentioned in Remark 2.3 such that there is no normal 3-pseudomanifold $\Delta'$ which is obtained from $\Delta$ by a combinatorial operation mentioned in Remark 2.3 and $g_2(\Delta') < g_2(\Delta)$. Then by Observation 1, $\Delta$ must have a missing tetrahedron.

Step 2: Let $\Delta$ have a missing tetrahedron of Type 1 or 2. Then it follows from Lemma 2.8 that $\Delta$ is formed using a connected sum of $\Delta_1$ and $\Delta_2$. Let $t \in \Delta_1$ (in case of Type 2, we can take $t$ as a vertex in $\Delta_1$ such that $|\text{lk}(t, \Delta_1)|$ is a connected sum of $(2m - 1)$ copies of $\mathbb{RP}^2$. Then $g_2(\Delta_1) \geq g_2(|\text{lk}(t, \Delta_1)|) \geq 3(2m - 1)$ and $t_1 \in \Delta_1$, where $|\text{lk}(t_1)| \cong \mathbb{RP}^2$. Therefore, $g_2(\Delta_2) = g_2(\Delta) - g_2(\Delta_1) \leq 9$. Thus, after a finite number of steps, we have $\Delta = \Delta_1 \# \Delta_2 \# \cdots \# \Delta_n$, where (i) $t, t_1 \in \Delta_1$ such that $|\text{lk}(t)|$ is a connected sum of $(2m - 1)$ copies of $\mathbb{RP}^2$ and $|\text{lk}(t_1)| \cong \mathbb{RP}^2$ (ii) $\Delta_1$ has no missing tetrahedron of Types 1 and 2, (iii) $6m - 3 \leq g_2(\Delta_1) \leq 6m + 6$, and (iv) for $2 \leq i \leq n$, $\Delta_i$ has no singular vertices and $g_2(\Delta_i) \leq 9$. Thus, by Proposition 2.5 we have, for $2 \leq i \leq n$, each $\Delta_i$ is a triangulated 3-sphere and is obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, and edge expansions. If there is no normal 3-pseudomanifold $\Delta_1'$, which is obtained from $\Delta_1$ by a combinatorial operation mentioned in Remark 2.3 and $g_2(\Delta_1') < g_2(\Delta_1)$, then by Observation 1, $\Delta_1$ has a missing tetrahedron of Type 3, 4 or 5, and we move to Steps 3, 4 or 5, respectively. Otherwise, we move to Step 1 and replace $K$ with $\Delta_1$. Since $K$ has a finite number of vertices and $g_2(K)$ is also finite, after a finite number of steps we must move to Steps 3, 4 or 5.

Step 3: Let $\Delta_1$ have a missing tetrahedron of Type 3. From the above arguments for Type 3, we get our result.

We use the principle of mathematical induction on $m$. First, we take $m = 1$. In this case, a missing tetrahedron will be of Type 1, 2 or 3 only. Therefore, from Step 2 we must move to Step 3 only. Thus, the result is true for $m = 1$. Let us assume that the result is true for $1, \ldots, m - 1$, and let $K$ be the normal 3-pseudomanifold that corresponds to $m$. Then we start from Step 1, and after a finite number of steps we must move to Steps 3, 4 or 5. If we move to Step 3, then we are done. We can use the induction hypothesis if we move to either Step 4 or Step 5.

Step 4: Let $\Delta_1$ have a missing tetrahedron of Type 4. Then it follows from Lemma 2.11 that $\Delta_1$ is formed using a vertex folding from a normal 3-pseudomanifold $\Delta_1'$ at $t \in \Delta_1'$ and $g_2(\Delta_1') = g_2(\Delta_1) - 6$. Here $|\text{lk}(t, \Delta_1')|$ is a connected sum of $2m - 3$ copies of $\mathbb{RP}^2$ and $g_2(\Delta_1') \leq 6 + 6(m - 1)$. Now, the result follows by the induction hypothesis.

Step 5: Let $\Delta_1$ have a missing tetrahedron $\sigma$ of Type 5. Then it follows from Lemma 2.8 that $\Delta_1$ is formed using a connected sum of $\Delta_1'$ and $\Delta_1''$. Let $t' \in \Delta_1'$ and $t'' \in \Delta_1''$ be identified during the connected sum and produce $t \in \Delta_1$. Without loss of generality, assume $t_1 \in \Delta_1'$ (if $t_1 \leq \sigma$, then we can take $t_1$ as a vertex in $\Delta_1'$ such that $|\text{lk}(t_1, \Delta_1')| \cong \mathbb{RP}^2$). Then $|\text{lk}(t', \Delta_1')|$ is a connected sum of $2n_1 - 1$ copies of $\mathbb{RP}^2$ and $|\text{lk}(t'', \Delta_1'')|$ is the connected sum of $n_2$ copies of tori or Klein bottles, where $n_1 + n_2 = m$ for some positive
integers \( n_1, n_2 \). Since \( n_1, n_2 > 0 \), both \( n_1, n_2 < m \). Further, \( g_2(\Delta'_1) \leq 6 + 6n_1 \) and \( g_2(\Delta''_1) \leq 9 + 6n_2 \). Now, the result follows by the induction hypothesis and Theorem 4.4.

From the construction, we can see that the sequence of operations includes exactly \( m - 1 \) number of vertex foldings, one edge folding, and a finite number of remaining operations. Further, the upper bound in Theorem 5.5 is sharp, i.e., there exists a normal 3-pseudomanifold with exactly two singularities \( \mathbb{R}P^2 \) and \( \#(2m-1)\mathbb{R}P^2 \) such that \( g_2(K) = 7 + 6m \) and \( K \) is not obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, edge expansions, vertex foldings, and edge foldings. We can construct a normal 3-pseudomanifold \( \Delta \) as in Remark 4.5, where \( |\text{lk}(t, \Delta)| \) is the connected sum of \( m - 1 \) copies of tori or Klein bottles, and then apply an edge folding at some edge \( ta \). Then the normal 3-pseudomanifold \( \Delta_\psi^a \) will serve our purpose.

Proof of Theorem 1.1. Let \( K \) have exactly one singularity at \( t \). Then \( |\text{lk}(t, K)| \) is a connected sum of \( n \) copies of tori or Klein bottles for some \( n \in \mathbb{N} \), and \( g_2(\text{lk}(t, K)) = 6n \). Since \( g_2(K) \leq g_2(\text{lk}(v)) + 9 \) for some vertex \( v \) in \( K \) and \( g_2(\text{lk}(v)) \leq g_2(\text{lk}(t)) \), we have \( g_2(K) \leq 6n + 9 \). Therefore, the result follows from Theorem 4.4.

Now consider, \( K \) has exactly two singularities at \( t \) and \( t_1 \) such that \( |\text{lk}(t_1, K)| \cong \mathbb{R}P^2 \). Then \( |\text{lk}(t, K)| \) is a connected sum of \( (2m - 1) \) copies of \( \mathbb{R}P^2 \) for some \( m \in \mathbb{N} \) and \( g_2(\text{lk}(t, K)) = 3(2m - 1) \). Since \( g_2(K) \leq g_2(\text{lk}(v)) + 9 \) for some vertex \( v \) in \( K \) and \( g_2(\text{lk}(v)) \leq g_2(\text{lk}(t)) \), we have \( g_2(K) \leq 6m + 6 \). Thus, the result follows from Theorem 5.5.

The sharpness of this bound follows from Remark 4.5 and Theorem 5.5.

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