We show that the density profile of a Fermi gas in rapidly rotating potential will develop prominent features reflecting the underlying Landau level like energy spectrum. Depending on the aspect ratio of the trap, these features can be a sequence of ellipsoidal volumes or a sequence of quantized steps.

Currently, there is an intense effort to cool trapped Fermi gases down to the degenerate limit. Recent experiments at JILA on $^{40}$K has reached one half of its Fermi temperature $\hbar \omega$. One of the motivations of cooling Fermions is to reveal their possible superfluid ground states, which can be quite novel in the case of multi-component Fermion systems. Current theoretical estimates, however, indicate that the interactions between different spin states of $^{40}$K are all positive, implying a normal instead of superfluid ground state. The absence of superfluid ground states, however, does not mean that the system can not have novel macroscopic quantum phenomena. Quantum Hall effect is an excellent example. In a strong magnetic field, the energy levels of a two dimensional electron system organize into highly degenerate Landau levels, leading to a whole host of dramatic effects. In this paper, we shall discuss the similarities and striking differences of rapidly rotating Fermi gases, leading to many macroscopic quantum phenomena.

For neutral atoms in rotating harmonic traps, we shall see that Landau level like energy spectrum will appear when the rotation frequency $\Omega$ approaches the transverse confining frequency $\omega$. This “fast rotating” limit might appear hard to achieve as the system is at the verge of flying apart due to centrifugal instability. Such instability, however, can be prevented by imposing an additional repulsive potential which dominates over the centrifugal force beyond certain radius. With centrifugal instability eliminated, the Landau-like levels will show up in many ways. We shall see that for cylindrical harmonic traps with $\omega_\perp << \omega$, where $\omega_\perp$ and $\omega_\parallel$ are the transverse and longitudinal trapping frequencies, the density is a sum of one-dimensional like density distributions residing in different “Landau volumes”. For traps with $\omega_\parallel$ comparable or smaller than $\omega_\perp$, the density consists of a set discs along $z$, each of which is made up of a sequence of density steps quantized in units of $M \omega_{\perp}^2 / (\pi \hbar)$.

2D case: We first consider 2D rotating Fermi gases in harmonic potentials since they illustrate the basic physics. The Hamiltonian in the rotating frame is

$$H - \Omega L_z = \frac{1}{2M} p_\perp^2 + \frac{1}{2} M \omega_\perp^2 r^2 - \Omega \hat{z} \cdot \mathbf{r} \times \mathbf{p}_\perp,$$

where $\mathbf{p}_\perp = (p_x, p_y)$ and $\mathbf{r} = (x, y)$. The eigenfunctions and eigenvalues of eq.(1) are

$$u_{n,m}(r, \theta) = \frac{e^{i|w|^2/2} \partial_x^{n} \partial_y^{m} e^{-|w|^2}}{\sqrt{\pi a_\perp^2 n!m!}}$$

$$\epsilon_{n,m} = \hbar (\omega_{\perp} + \Omega) n + \hbar (\omega_{\perp} - \Omega) m + \omega_{\perp},$$

where $n, m$ are non-negative integers, $0, 1, 2, \ldots$: $w \equiv (x + iy)/a_\perp$; $a_\perp = \sqrt{h/M \omega_{\perp}}$; and $\partial_{\pm} \equiv (a_\perp/2)(\partial_x \mp i \partial_y)$. To derive eqs. (2) and (3), we note that eq. (1) can be written as $\Pi^2/2M + (\omega_{\perp} - \Omega) L_z$ with $\Pi = p_{\perp} - M \omega_{\perp} \hat{z} \times \mathbf{r}$, which is precisely the canonical momentum $\Pi = p_{\perp} - \frac{\hbar}{M} \hat{z} \times \mathbf{r}$ of an electron in a magnetic field $B$ in the symmetric gauge, with $eB/Mc = 2\omega_{\perp}$. The eigenfunctions of $\Pi^2/2M$ are those in eq. (2), with eigenvalues $\epsilon_{n,m} = \hbar (2n + 1)$, where $n$ is the Landau level index, and $m$ is an “angular momentum” index labelling the degeneracy in each level. Since $L_z u_{n,m} = \hbar (n - m) u_{n,m}$, eq. (3) is also an eigenstate of eq. (1) with eigenvalues eq. (2). Note that the function $u_{n,m}$ in eq. (2) peaks at

$$r_{n,m} \equiv \langle r^2 \rangle_{n,m} = a_\perp^2 (n + m + 1),$$

and decays away as a Gaussian over a distance $a_\perp$.

Eq. (3) shows that the system is unbounded when $\Omega > \omega_{\perp}$ unless an additional repulsive potential $V_{\text{wall}}(r)$ (say, introduced by an additional optical trap) is present. We shall in particular consider potentials $V_{\text{wall}}(r)$ which are zero for $r < R$ but become strongly repulsive for $r > R$, with $R >> a_\perp$. The specific form of $V_{\text{wall}}$ is not important for the key features discussed below, as long as it is smooth over length scale $a_\perp$. The condition $R >> a_\perp$ however, allows us to fit many states inside $r < R$ and is a necessary feature for many effects discussed below.

Since $V_{\text{wall}}(r)$ is cylindrically symmetric, the eigenstates are still labeled by quantum numbers $(n, m)$. For states originally with $r_{n,m} < R$, eqs. (2) and (3) remain valid because $V_{\text{wall}} = 0$ for $r < R$. For states $(n, m)$ originally peaked beyond $R$, their energies increase rapidly because $V_{\text{wall}}$ is strongly repulsive. For $^{40}$K in a tight trap $\omega_{\perp} = 4000$Hz and $10^2$Hz, we have $a_\perp \approx 2.5 \times 10^{-6}$cm and $5 \times 10^{-6}$cm resp. The condition $R >> a_\perp$ is satisfied for $R > 5 \times 10^{-4}$cm.

Let us first consider the case $\Omega < \omega_{\perp}$ with $V_{\text{wall}} = 0$. The density in the ground state is $\rho(r) = \sum_{n,m} |\langle n,m| r \rangle|^2 \Theta(\mu - \epsilon_{n,m})$, where $\Theta(x)$ is 1 or 0 if $x > 0$ or $< 0$, $\mu$ is the chemical potential related to the particle number $N$ as $N = \sum_{n,m} \Theta(\mu - \epsilon_{n,m})$. We can write $\rho(r) = \sum_{n=0}^\infty \rho_n(r; m_n^*)$, where $\rho_n(r; L)$ is density contribution of the $n$-th Landau level with angular momentum states filled up to $m = L$;

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\[ \rho_n(r; L) = \sum_{m=0}^{L} |u_{n,m}(r)|^2 \Theta(\mu - \epsilon_{n,m}), \] (5)

\( m_n^* \) is the highest angular momentum state in the \( n \)-th Landau level with energy less than \( \mu \), and \( n^* \) is the highest Landau level below \( \mu \),

\[ m_n^* = \text{Int} \left[ \frac{\mu - \omega_L - (\omega_L + \Omega)n}{(\omega_L - \Omega)} \right], \] (6)

\( n = \int \left[ \frac{\mu - \omega_L}{\omega_L + \Omega} \right] \), where \( \text{Int}[x] \) denotes the integer part of \( x \), and \( x \) is understood to be positive. Since \( (n, m_n^*) \) is the state in \( \rho_n \) farthest from the origin, its peak location \( (r_n = r_{n,m_n^*} = a_\perp \sqrt{n + m_n^* + 1}) \) gives the size of \( \rho_n \).

When \( \Omega \) is very close to \( \omega_L \), we have \( m_n^* \gg 1 \) and

\[ r_n^2 = \frac{\mu - \hbar \Omega(2n + 1)}{M \omega_L(\omega_L - \Omega)}. \] (7)

Note that the difference in area between successive Landau discs is a constant

\[ \pi(r_{n-1}^2 - r_n^2) = (\pi a_\perp^2) \left( \frac{2\Omega}{\omega_L - \Omega} \right). \] (8)

Using eq.(4), it is straightforward to show that

\[ \rho_0(r; m_0^*) = \frac{1}{2\pi a_\perp^2} \left[ 1 - \text{erf} \left( \frac{s}{\sqrt{2m_0^*}} \right) (1 + [..]) \right]. \] (9)

where \( s = (r/a_\perp)^2 - m_0^* \), \( \text{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-z^2} \, dz \), and the term \([..]\) in eq.(4) is of order \((12m_0^*)^{-1}\) and smaller. The densities \( \rho_n(r) \) of the higher Landau levels \((n > 0)\) can be generated from \( \rho_0 \) as \( \rho_n(r; m_n^*) = \frac{1}{n!} \left( \frac{1}{\Omega} \hat{V}_{\omega_L} \right)^n \rho_0(r - a_\perp \hat{z}; m_0^*) \), . From the properties of \( \text{erf}(x) \), it is clear that \( \rho_n \) is a constant \( 1/(\pi a_\perp^2) \) within a disc of radius \( r_0 = \sqrt{m_0^* + 1} \), and has an edge of width \( \Delta_0 \approx \frac{1}{\sqrt{2} a_\perp} \). If \( m_0^* > 1 \), \( \Delta_0 \ll r_0 \), and \( \rho_0 \) can be approximated as a step function on scales larger than \( \Delta_0 \). Likewise, \( \rho_n \) can be approximated as a step function with somewhat larger width within the same approximation.

Thus, when \( m_n^* \gg 1 \), we have

\[ \rho_n(r) = (\pi a_\perp^2)^{-1} \Theta(r_n^2 - r^2). \] (10)

Eq.(10) and (9) then imply \( \rho(r) = \frac{M \omega_L}{\pi \hbar} I(r) \), with \( I(r) = \sum_n \Theta(\mu - M \omega_L(\omega_L - \Omega) r^2 - \hbar \Omega(2n + 1)) \). Using the identity \( \text{Int}[x + 1] = \sum_n \Theta(a(x - n)) \), for all \( x > 0 \) and \( a > 0 \), \( \rho(r) \) can be simplified to

\[ \rho(r) = \frac{M \omega_L}{\pi \hbar} \text{Int} \left[ \frac{\mu - M \omega_L(\omega_L - \Omega) r^2 + \hbar \Omega}{2\hbar \Omega} \right]. \] (11)

It is, however, instructive to re-derive eq.(11) in a way generalizable to arbitrary potentials. We rewrite eq.(4) as

\[ H - \Omega L_z = (\mathbf{p}_\perp - M \hbar \mathbf{z} \times \mathbf{r})^2 + \frac{1}{2} M (\omega_L^2 - \Omega^2) r^2. \] (12)

The first term gives a set of Landau levels with spacing \( 2\Omega \), each of which contributes \((\pi A^2)^{-1}\) to the density, where \( A^2 = \hbar/(M \Omega) \). If the second term in eq.(12) were absent, the density is given by \( \rho = I/(\pi A^2) \), where \( I \) is the total number of Landau levels below the chemical potential, \( I = \text{Int}[2\pi \hbar \Omega/\omega_L] \). When \( \Omega \) is close to \( \omega_L \), the second term in eq.(12) is slowly varying over the scale of \( A \approx a_\perp \) and can be absorbed in the chemical potential. The density profile within local density approximation (LDA) is then

\[ \rho(r) = \frac{\Omega}{\pi \hbar} I(r), \quad I(r) = \text{Int} \left[ \frac{\mu(r) + \hbar \Omega}{2\hbar \Omega} \right] \] (13)

where \( \mu(r) = \mu - \frac{1}{2} M (\omega_L^2 - \Omega^2) r^2 \). Clearly, eq.(13) is equivalent to eq.(11) up to correction \((1 - \Omega/\omega_L)<< 1\) as \( \Omega \to \omega_L \). Eq.(13), once established, is easily generalized to other potentials. In the presence of \( V_{\text{wall}} \), one simply replaces \( \mu(r) \) in eq.(13) by

\[ \mu(r) = \mu - \frac{1}{2} M (\omega_L^2 - \Omega^2) r^2 - V_{\text{wall}}(r). \] (2D) (14)

Eq.(13) and eq.(14) constitute the LDA solution for the 2D rotating Fermi gas for both \( \Omega < \omega_L \) and \( \Omega > \omega_L \). The schematics of LDA is shown in fig.1a and 1b.

To understand the validity of LDA (eq.(13)), we have calculated the density numerically using eq.(4). The result for a system of 2000 Fermions at \( \Omega/\omega_L = 0.996 \) is shown in figure 2a. The system exhibits a sequence of quantized steps at locations well described by LDA. The evolution of the density within the range \( 0.99 < \Omega/\omega_L < 1 \) of this Fermion system is shown in fig.2b. As \( \Omega \) decreases, more Landau levels are populated while the steps near the surfaces are closer together, (as expected from eq.(10) and (9)), yet the step structures remain discernable and correctly described by LDA, (the LDA construction is not displayed so as to keep the fig.2b readable). The behaviors of the densities at lower frequency \( 0.98 < \Omega/\omega_L < 1 \) are shown in fig.3a and 3b for a system of 1000 Fermions. At the lowest frequency displayed, (fig.3a), the step structure near the surface is completely smeared out by the spread of the edges. Nevertheless, the density of the innermost plateau remained quantized, with a size correctly described by LDA. This core of quantized density (or “quantized core” for short) is a clear evidence for Landau levels. Our studies show that for about 2000 particles, Landau levels will show up as a sequence of discernable steps only when \( 0.99 < \Omega/\omega_L < 1 \), which is achievable with the current capability to control frequencies, especially for large \( \omega_L \). For lower frequencies, the existence of Landau levels can only be revealed through the presence of a “quantized core”, which shrinks in size as \( \Omega \) decreases. On the other hand, the LDA in fig.1b shows that by introducing an additional potential \( V_{\text{wall}} \), Landau levels (in the form of a sequence of steps near
the center or a “quantized core”) can still be revealed at frequencies farther beyond \(\omega_L\), even though the steps near the surface are smeared out.

### 3D case

For a 3D harmonic trap, eq. (10) acquires terms \(p_z^2/2M + \frac{1}{2}M\omega_z^2z^2\), which give rise to harmonic oscillator eigenfunctions \(f_n(z)\) with eigenvalue \(\epsilon_n = \hbar\omega_z(n_x + \frac{1}{2})\). The density is

\[
\rho(r,z) = \sum_{n_x,n} \frac{|f_n(z)|^2}{\pi a_z^2} \Theta(\mu - \epsilon_{n_x,n})
\]

where \(\mu(r) = \mu - M\omega_z(\omega_L - \Omega)r^2 \sim \mu - \frac{M}{2}\omega_z^2r^2\).

The order of summation of the remaining integers \(n_x\) and \(n\) depends on the relative strength between \(\omega_z\) and \(\omega_L\). When \(\omega_L \ll \omega_z\), we first sum \(n_x\). To do that, we note that the density of a 1D Fermi gas is

\[
\rho_{1D} = \sqrt{2\mu/\pi \hbar}
\]

where \(\mu(r) = \mu - M\omega_z(\omega_L - \Omega)r^2 \sim \mu - \frac{M}{2}\omega_z^2r^2\). We have included \(V_{\text{wall}}\) in \(\rho_{1D}\) for the more general situation in the 3D case. Eq. (14) shows that \(\rho(r)\) is a sum of 1D densities (labelled by “\(n\)”) each of which distributed over in a “Landau volume” bounded by the “Landau surface” \(\mu(r,z) = \hbar\Omega(2n+1)\). When \(V_{\text{wall}} = 0\), \(\Omega \ll \omega_z\), the Landau surface are ellipsoidal surfaces. It is easy to verify that the surface areas \(A_n\) for successive ellipsoids differ by a constant \(A_{n-1} - A_n = \frac{h}{\Omega} \left[ \frac{16\pi \Omega^2}{e^2 - 4\pi} \right]\) when \(\Omega \ll \omega_z\), centrifugal instability against harmonic confinement sets in and stability can only be established by \(V_{\text{wall}}\).

To demonstrate the validity of the LDA eq. (14), we have evaluated the density numerically for a system of 2000 Fermions for \(\omega_z/\omega_L = 0.2\) at \(\Omega/\omega_L = 0.99\). The results are shown in figure 4. It shows that the LDA (dotted line) is a good approximation. The Landau volumes can be clearly identified by the change of slope in the density. The appearance of a plateau the center is because \(\omega_z/\omega_L\) is only 0.2, revealing the 2D feature of the \(n_z\) levels. For smaller ratios of \(\omega_z/\omega_L\), the plateau disappears and the LDA expression (eq. (14)) is achieved.

When \(\omega_L > \omega_z\), summation of \(n\) in eq. (14) gives

\[
\rho(r) = \frac{M\omega_L}{\pi \hbar} \sum_{n_z} |f_{n_z}(z)|^2 \text{Int} \left[ \frac{\mu(r) - \hbar\omega_L(n_z + \frac{1}{2}) + \hbar\Omega}{2\hbar\Omega} \right]
\]

where \(\mu(r) = \mu - \frac{1}{2}M(\omega_L^2 - \Omega^2)N - V_{\text{wall}}\). In this limit, the density consists of a sequence of discs (labelled by \(n_z\)) in the \(z\)-direction. Each disc \(|f_{n_z}(z)|^2\) consists of a sequence of density steps in the \(xy\)-plane reflecting the number of filled Landau levels. The behavior of the density within each disc in the \(xy\)-plane is identical to the 2D case discussed before. Finally, we note that as temperature increases, the Landau level structure near the surface will first melt away, and the melting will proceed toward the center. The temperature below which the Landau level effect begin to appear is \(T = 2\hbar\omega_L/k_B\), which is 3.8 x 10^{-7}K and 9.6 x 10^{-8}K for \(\omega_L = 4000\text{Hz}\) and 10^{2}\text{Hz\,resp., a\,temperature\ vary\ in\ current\ experiements}\. 5

So far, we have only discussed the effect of Landau levels on the density profiles of fast rotating Fermi gases. If the development of quantum Hall effect in the last decade is a guide, one expects many more novel phenomena in Fermi gases in the fast rotating regime. This work is completed during a workshop at the Lorentz Center of University of Leiden. We thank Professor Henk Stoof and the Lorentz Center for generous support. This work is supported by a Grant from NASA NAG8-1441, and by the NSF Grants DMR-9705295 and DMR-9807284.

1. DB. DeMarco and D.S. Jin, Science 285, 1703 (1999).
2. The pairing states in large spin Fermi gas are discussed in
3. J. Burke, private communication.
4. In the recent experiment of F. Chevy, K.W. Madison, and J. Dalibard, (cond-mat/000622) on rotating Bose gas, the highest rotational frequency used is quite closed to \(\omega_L\).
5. R.B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
6. The extent of erf(x) is about 1.5, we then have \(\frac{1}{2}\Delta(s/\sqrt{2m}) = 1.5\). Since \(\Delta(s) \approx 2\Delta(r)/a_z^2\), we have \(\sqrt{m^*\Delta(r)} \approx \frac{3}{\sqrt{2}} \sqrt{m^*a_z}\).
7. Note that \(\rho_0(r;L) = \sum_{m=0}^{L} e^{nw}/2\hbar\). Since \(u_{nm}\) is \(u_{nm,\lambda=0}\), where \(u_{nm,\lambda}\) is \((\pi\lambda!n!)^{-1/2}2\hbar e^{\lambda^2/2}\), which is \(-\lambda^2/2\). Explicitly, \(\rho_0(r;L) = (1 + \frac{1}{2}\lambda^2)\rho_0(r;M_0)\), where \(\rho_0(r;M_0) = (1 + \frac{1}{2}\lambda^2)\rho_0(r;M_1)\).
8. The studies of finite temperature effects will be presented elsewhere. At higher temperatures, the density profile is that of a Boltzman gas, \(e^{-M(\omega_L^2 - \Omega^2)r^2/2}/k_B\).
Caption
Fig.1a and 1b: Fig.1a and 1(b) corresponds to $\Omega < \omega_\perp$ and $\Omega > \omega_\perp$ resp. The rapid drop at large $r$ is due to the strongly repulsive potential $V_{\text{wall}}$. The LDA densities are indicated by the steps in thick lines. The integer value of $(\mu(r)+\hbar\Omega)/(2\hbar\Omega)$, (i.e. $I$), is related to the index $n$ of the intersected Landau level as $I = n + 1$. And the relation $(\mu(r)+\hbar\Omega)/(2\hbar\Omega) = I$ is equivalent to $\mu(r) = (2n + 1)\Omega$.

Figure 2a: LDA (dotted line) and numerical calculations (solid line) of the density for $N = 2000$ Fermions at $\Omega/\omega_\perp = 0.996$.

Figure 2b. Density profiles of $N = 2000$ Fermions at $\Omega/\omega_\perp = 0.993$ (crosses), 0.996 (dots), and 0.998 (solid line).

Figure 3a: LDA (dotted line) and numerical calculations (solid line) of the density for $N = 1000$ Fermions at $\Omega/\omega_\perp = 0.982$.

Figure 3b. Density profiles of $N = 2000$ Fermions at $\Omega/\omega_\perp = 0.982$ (crosses), 0.989 (dots), and 0.995 (solid line).

Figure 4. LDA (dotted line) and numerical calculations (solid line) of the density for $N = 2000$ Fermions at $\Omega/\omega_\perp = 0.99$ and $\omega_z/\omega_\perp = 0.2$. 
\[ (\mu(r) + \kappa \Omega) / 2 \kappa \Omega \]

(1a)

\[ \pi \rho a^2 \]

(2a)

\[ N = 2000 \]
\[ \Omega / \omega_\perp = 0.996 \]

(2b)

\[ (\pi a_i^2) \rho \]

(3a)

\[ N = 1000 \]
\[ \Omega / \omega_\perp = 0.982 \]

(3b)

\[ (\pi a_i^2) \rho \]

(1b)

\[ N = 2000 \]
\[ \Omega / \omega_\perp = 0.99 \]
\[ \omega / \omega_\perp = 0.20 \]

\[ (\pi a_i^3) \rho \]

\[ n=0 \]
\[ n=1 \]