Temperature patches for the subcritical Boussinesq–Navier–Stokes System with no diffusion

Calvin Khor* and Xiaojing Xu

Abstract

In this paper, we prove that temperature patch solutions to the subcritical Boussinesq–Navier–Stokes System with no diffusion preserve the Hölder regularity of their boundary for all time, which generalises the previously known result by F. Gancedo and E. García-Juárez [Annals of PDE, 3(2):14, 2017] to the full range of subcritical viscosity.

Keywords Boussinesq–Navier–Stokes System, temperature patch, subcritical dissipation, global existence.

MSC Classification 35R05 (Primary) 35Q35, 35F25, 35A01, 35A02 (Secondary)

1 Introduction

This paper studies temperature patch solutions of the following initial-value problem for the subcritical Boussinesq–Navier–Stokes equations, $\alpha \in \left(\frac{1}{2},1\right)$:

$$
\begin{cases}
    u_t + u \cdot \nabla u + \Lambda^{2\alpha} u &= \nabla p + \theta e_2, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
    \nabla \cdot u &= 0, \\
    \theta_t + u \cdot \nabla \theta &= 0, \\
    u|_{t=0} &= u_0, \\
    \theta|_{t=0} &= \theta_0.
\end{cases}
$$

Here, $\Lambda^{2\alpha} = (-\Delta)^\alpha$ is a fractional Laplacian on $\mathbb{R}^2$ defined initially for Schwartz functions as a Fourier multiplier $\hat{\Lambda^{2\alpha} f}(\xi) = |2\pi \xi|^{2\alpha} \hat{f}(\xi)$. Also, $e_2 := (0,1)^T \in \mathbb{R}^2$, $u = u(t,x) = (u_1(t,x), u_2(t,x))^T \in \mathbb{R}^2$ is the velocity of the fluid, $p = p(t,x) \in \mathbb{R}$ is the pressure of the fluid, $\theta = \theta(t,x) \in \mathbb{R}$ is the temperature of the fluid, and $u_0, \theta_0$ are the initial data of the system.

This active scalar transport system for $\theta$ arises as a natural generalisation of the Boussinesq–Navier–Stokes system, where the dissipation is the full Laplacian (corresponding to $\alpha = 1$). Details of the physics described by the Boussinesq–Navier–Stokes system can be found in [44]. Actually,
is a special case of a two parameter family of equations where there is also a diffusion term $\Lambda^{2\alpha}\theta$ in the temperature equation; see for instance the papers [54], [43], [50], [53], and citations within. Only local results for the zero viscosity and zero diffusion case are known, such as those in [6].

In addition, some authors consider the opposite situation, where $\alpha = 0$ and $\beta > 0$ (called the Euler–Boussinesq system) such as [30], [29], and [52], but to the best of the authors’ knowledge, there are no other papers currently available studying temperature patches for (1.1) with $\alpha \in (1/2, 1)$.

We consider solutions in the following sense:

**Definition 1.1.** We say that $(u, \theta) \in L^2_{\text{loc}}([0, \infty) \times \mathbb{R}^2; \mathbb{R}^2) \times L^2_{\text{loc}}([0, \infty) \times \mathbb{R}^2)$ with $\nabla \cdot u = 0$ is a solution to (1.1) if for every divergence-free $\phi \in C^\infty_c((0, T) \times \mathbb{R}^2; \mathbb{R}^2)$, and for every $\psi \in C^\infty_c((0, T) \times \mathbb{R}^2)$,

\[
\int_0^\infty \int_{\mathbb{R}^2} u \cdot \phi_t + u \otimes u : \nabla \phi - u \cdot \Lambda^{2\alpha} \phi - \theta \phi \cdot e_2 \, dx \, dt = 0,
\]

\[
\int_0^\infty \int_{\mathbb{R}^2} \nabla \psi_t + \theta u \cdot \nabla \psi \, dx \, dt = 0.
\]

By a ‘temperature patch solution’, we mean a solution of (1.1) where $\theta$ is an indicator function for each time $t \geq 0$. These are similar to the sharp front solutions of the SQG equation and their variants which have been and are extensively studied [47], [46], [20], [19], [18], [21], [16], [8], [38], [34], [35], [32], [33], and the more classical theory of vortex patches for the 2D Euler equation as laid out in [41], and also [10], [11], and [37]. The Boussinesq system also supports initial data of ‘Yudovich’ type. We mention a recent paper [42] working on local-in-time Yudovich solutions to the full inviscid equation. The case of critical diffusion was studied in [55].

Global unique classical solutions to (1.1) with $\alpha = 1$ were shown to exist in $H^s$ spaces in [4] and [31]; global (weak) solutions in $(L^2 \cap B^{1-1}_{\infty,1}) \times B^0_{2,1}$ were studied in [2], and global (weak) solutions in $L^2 \times L^2$ were proven to exist in [27] with uniqueness proven in [14]. In the case of temperature patches, Abidi and Hmidi [1] proved the persistence of $C^1$ regularity of the boundary, and Dauchin and Zhang [15] improved this result to $C^{1+\varepsilon}$ regularity. In Gancedo and García–Juárez [22], a second proof of Dauchin and Zhang’s result was given, and they improved the result to $C^{2+\varepsilon}$ persistence, in particular implying that the curvature of the patch remains bounded.

In [28], the critical equation $\alpha = 1/2$ has been studied in a low regularity scenario. However, the well-posedness result there requires initial data $\theta_0 \in B^{0}_{1,1} \hookrightarrow C^0_0$ which falls short of allowing $L^\infty$ data, of which temperature patches are a special case. Our work shows that this seems to be a special feature of the critical $\alpha = 1/2$ case, as $L^\infty$ data is allowed for all $\alpha \in (1/2, 1]$. This is a curious property, as the the transport evolution of $\theta$ gives that weak solutions cannot increase $L^p$ norms of $\theta$. At the same time, there was some foreshadowing, as the method of Gancedo and
García-Juárez which relies on the explicit formula for the heat kernel can only be replicated for the critical case $\alpha = 1/2$, and $e^{-t\Lambda} \nabla f$ has roughly the same regularity as $f$ (as opposed to $e^{t\Delta} \nabla f$ being better behaved than $f$; see [22] for details.)

Our main technical result is the following existence and uniqueness result:

**Theorem 1.2.** Suppose that $\alpha \in (\frac{1}{2}, 1)$, $u_0 \in H^1 \cap W^{1,p}$ for some $p \in (2, \infty)$, $\nabla \cdot u_0 = 0$, and $\theta_0 \in L^1 \cap L^\infty$. Then there is a unique global solution $(u, \theta)$ in the sense of Definition 1.1 to (1.1) such that for each $\rho \in [1, \min(2, p/2))$ and for each $\alpha' \in [0, \alpha)$,

$$u \in L^\infty_t W^{1,p} \cap L^\rho_t C^{2\alpha'}, \quad \text{and} \quad \theta \in L^\infty_t L^1 \cap L^\infty.$$

Using Theorem 1.2, for any $\alpha' < \alpha$, we can control the $C^{2\alpha'}$ boundary regularity of temperature patch solutions to (1.1):

**Theorem 1.3.** Suppose that $\alpha \in (\frac{1}{2}, 1)$, $u_0 \in H^1 \cap W^{1,p}$ for some $p \in (2, \infty]$, $\nabla \cdot u_0 = 0$, and $\theta_0 = 1_{D_0}$ be an indicator function of a simply connected set $D_0$ with $D_0 \subset C^{2\alpha'}$ for some $\alpha' < \alpha$. Then the global solution $\theta$ to (1.1) remains an indicator, $\theta(t) = 1_{D(t)}$ where $\partial D(t) \subset L^\infty_t C^{2\alpha'}$ for all $t \geq 0$.

The main method of proof is the use of the special structure of the equation by the study of $\Gamma := \omega - R^{2\alpha} \theta$, where $R^{2\alpha}$ is a smoothing operator. It turns out that it is easier to study this combination of terms than the vorticity $\omega = \nabla \perp \cdot u$ by itself. This is the method used in [24], but since $\alpha > 1/2$ the proof is more streamlined.

Theorem 1.2 raises the following interesting questions: (a) Is it possible to control the curvature for $\alpha \in (1/2, 1)$, as it is possible in $\alpha = 1$? (b) can the critical equation $\alpha = 1$ support unique temperature patch solutions, and what regularity of their boundary is preserved?

The remainder of the paper is organised as follows. In Section 2 we list the notation that we use for function spaces and inequalities. In Section 3 we explain how introducing the term $\Gamma$ leads to better estimates. In Section 4 we give some easy a priori estimates from the equation obtained by classical means. In Section 5 we use the equation for $\Gamma$ to derive better a priori estimates for the vorticity (and hence the velocity). In Section 6 we use the Osgood Lemma to prove uniqueness of solutions. In Section 7 we use the quantitative bound from the Osgood Lemma to show existence of solutions, and also present the proof for conservation of Hölder regularity of temperature patch boundaries.

## 2 Preliminaries

We follow the notation of [24] as follows.

- We write $\phi_k(t)$ to denote any function of the form

$$\phi_k(t) = C_0 \exp \exp \ldots \exp (C_1 t).$$

$k$ times
• We write $A \lesssim B$ to mean that $|A| \leq CB$ for some constant $C$ that does not depend on $A$, $B$, or any other variable under consideration. We write $A \lesssim_{\phi_1, \ldots, \phi_N} B$ to emphasise that the implicit constant $C$ depends on the $N$ quantities $\phi_1, \ldots, \phi_N$.

• When $(X, \| \cdot \|_X)$ is a Banach space, we will write for $p \in [1, \infty]$

$$L^p_t X$$

to denote the Bochner space $L^p(0, t; X)$; a function $f = f(t)$ with values in $X$ is in $L^p_t X$ if

$$\|f\|_{L^p_t X} := \|\int_0^t |f(t)|_X \, dt\|_{L^p} < \infty.$$

In a norm, we also abusively adopt the notation that a subscripted variable $q$, like $\ell^q$ below, means that the norm is to be taken with respect to $q$. This should not cause confusion with the above convention for time integrals. See for instance [17] or [48] for details on Bochner spaces.

2.1 Littlewood–Paley Decomposition and Function spaces

We fix our notation for Littlewood–Paley decompositions here; see [3], [8], [45], [51], or [49] for more details about Littlewood–Paley, Besov spaces, and Tilde spaces.

There exist two radial non-negative functions $\chi \in D(\mathbb{R}^d)$ and $\varphi \in D(\mathbb{R}^d \setminus \{0\})$ such that

(i) $\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1; \forall q \geq 1$, supp $\chi \cap$ supp $\varphi(2^{-q}) = \emptyset$,

(ii) supp $\varphi(2^{-j}\bullet) \cap$ supp $\varphi(2^{-k}\bullet) = \emptyset$, if $|j - k| \geq 2$.

For every $v \in S'(\mathbb{R}^d)$ we define Fourier multipliers $\Delta_q, S_q$ by

$$\Delta^{-1}v = \chi(D)v; \forall q \in \mathbb{N}, \Delta_q v = \varphi(2^{-q}D)v; \text{ and } S_q v = \sum_{-1 \leq p \leq q - 1} \Delta_p v.$$

Lemma 2.1 (Bernstein Inequalities). There exists a constant $C$ such that for $q, k \in \mathbb{N}, 1 \leq a \leq b$ and for $f \in L^a(\mathbb{R}^d)$

$$\sup_{|\alpha| = k} \|\partial^\alpha S_q f\|_{L^b} \leq C^k 2^{k(\frac{d}{a} - \frac{d}{b})} \|S_q f\|_{L^a}, \text{ and }$$

$$C^{-k}2^{dk} \|\Delta_q f\|_{L^a} \leq \sup_{|\alpha| = k} \|\partial^\alpha \Delta_q f\|_{L^a} \leq C^k 2^{dk} \|\Delta_q f\|_{L^a}.$$

2.1.1 Besov and Tilde Spaces

The Besov space $B^{p,q}_{s}(\mathbb{R}^d)$ for $p, q \in [1, \infty]$, $s \in \mathbb{R}$ is the space of distributions $u$ such that

$$\|u\|_{B^{p,q}_{s}} := \|2^{qs} \|\Delta_q u\|_{L^p} \|_{L^q} < \infty.$$

In a norm, a subscripted variable $q$ like $\ell^q$ above means that the norm is to be taken with respect to $q$. Since we will only consider functions on
\(\mathbb{R}^2\), we will only write \(B^s_{p,q}\) (and similarly \(L^s_{p,q}\) instead of \(L^s_{p,q} (\mathbb{R}^2)\)). Also write \(H^s := B^s_{2,2}\). The Besov spaces trivially satisfy

\[s_1 \leq s_2 \implies B^s_{p,r} \hookrightarrow B^{s_2}_{p,r}, \quad \forall p, r, \tilde{r} \in [1, \infty],\]

\[r_1 \leq r_2 \implies B^s_{p,r_1} \hookrightarrow B^s_{p,r_2}, \quad \forall s \in \mathbb{R}, p \in [1, \infty],\]

and there is also the analogue of Sobolev embedding in 2D:

\[B^{s_0 + \epsilon}_{p,r} \hookrightarrow B^{s_0}_{p,r}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{s}{2}, \quad \forall s_0 \in \mathbb{R}, p, r \in [1, \infty].\]

In addition, we have the embeddings for any \(p \in [1, \infty],\)

\[B^0_{p,r} \hookrightarrow L^p \hookrightarrow B^0_{p,\infty}, \quad k \in \mathbb{N}_0 \implies B^k_{p,1} \hookrightarrow W^{k,p} \hookrightarrow B^k_{p,\infty}.\]

We say that a function \(u = u(x,t)\) belongs to the ‘Tilde space’ \(\tilde{L}^p_{t} B^s_{p,q}\) if

\[\|u\|_{\tilde{L}^p_{t} B^s_{p,q}} := \left\|2^q \left\|\Delta_q u(x,t)\right\|_{L^p_t}\right\|_{L^q_t}.\]

In comparison with

\[\|u\|_{L^p_t B^s_{p,q}} = \left\|\|u\|_{(B^s_{p,q})_x}\right\|_{L^p_t} = \left\|2^q \left\|\Delta_q u(x,t)\right\|_{L^p_t}\right\|_{L^q_t},\]

From the generalised Minkowski inequality \(\left\|f\right\|_{L^q_t} \leq \left\|f\right\|_{L^p_t} \left\|L^1_t\right\|_{L^q_t}\) if \(q \geq p\) and embeddings between Besov spaces, we have the following relations:

\[L^p_t B^{s+} \hookrightarrow \tilde{L}^p_{t} B^s_{p,q} \hookrightarrow L^p_t B^{s-}_p, \quad \text{if } r \geq p,\]

\[L^p_t B^{s+} \hookrightarrow \tilde{L}^p_{t} B^s_{p,q} \hookrightarrow L^p_t B^s_{p,r}, \quad \text{if } \rho \geq r,\]

where \(\epsilon > 0\) is arbitrarily small. In particular \(\tilde{L}^p_{t} B^s_{p,q} = L^p_t B^s_{p,q}.\)

### 3 Study of \(\mathcal{R}_{2\alpha}\) and some commutators

#### 3.1 Introduction of the \(\Gamma = \omega - \mathcal{R}_{2\alpha} \theta\) term

We use the technique of [28], [52], [43], and other papers of introducing an auxiliary quantity \(\Gamma\) determined from the equation, that has better regularity properties than the original functions under consideration. The \(\omega := \nabla^+ \cdot u\) equation is obtained by applying the \(\nabla^+ \cdot := (\partial_\theta^\alpha)\) operator to the first equation of (1.1). By writing \(\partial_\theta^\alpha \Theta = \Lambda^{2\alpha} \Theta, \text{ i.e. } \Theta \in \mathcal{R}_{2\alpha} \Theta := \Lambda^{1-2\alpha} \mathcal{R}_1 \Theta,\) the \(\omega\) equation takes the form

\[(\partial_t + u \cdot \nabla + \Lambda^{2\alpha})\omega - \Lambda^{2\alpha} \Theta = 0.\]

By adding \(- (\partial_t + u \cdot \nabla) \Theta\) to both sides of the equation, we obtain an equation for \(\Gamma = \omega - \Theta\) which has a commutator:

\[(\partial_t + u \cdot \nabla + \Lambda^{2\alpha})\Gamma = [\mathcal{R}_{2\alpha}, u \cdot \nabla] \theta.\]

This structure will be key in deriving the a priori estimates for \(u\). An important lemma for the study of the critical dissipation case is Lemma 3.3 in Hmidi–Keraani–Rousset’s paper [28]. The generalisation of Lemma 3.3 (ii) is the following result of Wu–Xue [52]:

5
Proposition 3.1 (Proposition 4.2 of [52]). Let $\beta \in [1, 2)$, $(p, r) \in [2, \infty] \times [1, \infty)$, $u$ be a smooth divergence-free vector field of $\mathbb{R}^n$ ($n \geq 2$) with vorticity $\omega$ and $\theta$ be a smooth scalar function. Then we have that for every $s \in (\beta - 2, \beta)$,

$$\|[\mathcal{R}_\beta, u \cdot \nabla] \theta\|_{B^{s, \beta}_{p, r}} \lesssim_s \|\nabla u\|_{L^p} \left(\|\theta\|_{B^{s+1-\beta}_{\infty, r}} + \|\theta\|_{L^2}\right).$$

Besides, if $p = \infty$, we also have

$$\|[\mathcal{R}_\beta, u \cdot \nabla] \theta\|_{B^{s, \beta}_{\infty, r}} \lesssim_s (\|\omega\|_{L^\infty} + \|u\|_{L^2}) \|\theta\|_{B^{s+1-\beta/2}_{\infty, r}} + \|u\|_{L^2} \|\theta\|_{L^2}.$$

We will similarly generalise Lemma 3.3 (i) of [28] by using Bony’s decomposition, as follows.

Theorem 3.2. For any $s \in (0, 2\alpha)$ and any smooth $v, \theta$ with $\nabla \cdot v = 0$,

$$\|\mathcal{R}_{2\alpha, v} \theta\|_{H^s} \lesssim_s \|\nabla v\|_{L^2} \|\theta\|_{B^{-2\alpha}_{\infty, 2}} + \|v\|_{L^2} \|\theta\|_{L^2}.$$

We prove this result using the following lemma and the equivalent of Hmidi–Keraani–Rousset’s Proposition 3.1 (Wu–Xue’s Proposition 4.1). This lemma is also quoted in [29]. We give its short proof here, which is a variant of Lemma 2.97 of [5].

Lemma 3.3 (Lemma 3.2 of [28]). Let $p \in [1, \infty]$, and let $f, g, h$ be three functions such that $\nabla f \in L^p$, $g \in L^\infty$, and $xh \in L^1$. Then

$$\|h \ast (fg) - f(h \ast g)\|_{L^p} \leq \|xh(x)\|_{L^1} \|\nabla f\|_{L^p} |g|_{L^\infty}.$$

Proof. By a direct computation and the fundamental theorem of calculus, setting $F := h \ast (fg) - f(h \ast g)$,

$$F(x) = \int_{\mathbb{R}^n} h(x-y)(f(y) - f(x))g(y) \, dy$$

$$= \int_0^1 \int_{\mathbb{R}^n} h(x-y) \nabla f((1-\tau)x + \tau y) \cdot (y-x)g(y) \, dy \, d\tau$$

$$= -\int_0^1 \int_{\mathbb{R}^n} h(z) \nabla f(x - \tau z) \cdot zg(x-z) \, dz \, d\tau.$$

Therefore, Hölder’s inequality and translation invariance of the Lebesgue measure gives

$$\|F\|_{L^p} \leq \int_0^1 \int_{\mathbb{R}^n} |zh(z)||\nabla f(x - \tau z)g(x-z)| \, dz \, d\tau$$

$$\leq \int_{\mathbb{R}^n} |zh(z)| \int_0^1 \|\nabla f(x - \tau z)\|_{L^p} \, d\tau \|g(x-z)\|_{L^2} \, dz$$

$$\leq \|zh(z)\|_{L^p} \|\nabla f\|_{L^p} |g|_{L^\infty},$$

for any $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$. In particular, setting $q = p$, $r = \infty$, we obtain the claimed result. \qed

Proposition 3.4 (Proposition 4.1 of [52]). Let $j \in \mathbb{N}$, $\alpha \in (\frac{1}{2}, 1)$, and $\mathcal{R}_{2\alpha} = \Lambda^{1-2\alpha} \mathcal{R}$. Then:
1. For every $p \in (1, \infty)$ and $q > p$ defined by $\frac{1}{q} = \frac{1}{p} - \frac{2\alpha - 1}{n}$, $R_{2\alpha}$ is a bounded map $L^p \to L^q$.

2. Let $\chi \in \mathcal{D}(\mathbb{R}^n)$. Then for each $(p, s) \in [1, \infty] \times (2\alpha - 1, \infty)$ and $f \in L^p(\mathbb{R}^n)$,
   \[ \||\nabla|^s \chi(2^{-j} \nabla)|| R_{2\alpha} f \|_{L^p} \leq 2^{j(s+1-2\alpha)} \|f\|_{L^p}. \]
   Moreover, $|\nabla|^s \chi(2^{-j} \nabla)$ is a convolution operator with kernel $K$ satisfying
   \[ |K(x)| \lesssim \frac{1}{(1+|x|)^{n+s+1-\beta}}. \]

3. Let $\mathcal{O}$ be an annulus centered at the origin. Then there exists $\phi \in \mathcal{S}(\mathbb{R}^n)$ with spectrum supported away from 0 such that for every $f$ with spectrum in $2\mathcal{O}$,
   \[ R_{2\alpha} f = 2^{(n+1-2\alpha)} \phi(2^j \bullet) \ast f. \]

Proof of Theorem 3.2. We use Bony’s decomposition,
\[
[R_{2\alpha}, v] \theta = I + II + III,
\]
\[
I = \sum_{q \in \mathbb{N}} [R_{2\alpha}, S_{q-1} v] \Delta_q \theta = \sum_{q \in \mathbb{N}} I_q,
\]
\[
II = \sum_{q \in \mathbb{N}} [R_{2\alpha}, \Delta_{q-1} v] S_{q-1} \theta = \sum_{q \in \mathbb{N}} II_q,
\]
\[
III = \sum_{q \geq -1} [R_{2\alpha}, \Delta_q \theta] \tilde{\Delta}_{q-1} \theta = \sum_{q \geq -1} III_q,
\]
where $\tilde{\Delta}_q := \Delta_{q-1} + \Delta_q + \Delta_{q+1}$ (and $\Delta_{-2} := 0$). The terms $I$ and $II$ are low-high and high-low interactions; the term $III$ are the interactions at similar frequencies (low-low and high-high).

Estimation of $I$

We have
\[
I_q = 2^{(1-2\alpha)q} (h_q \ast (S_{q-1} \Delta_q \theta) - (S_{q-1} v \Delta_q \theta) (h_q \ast \Delta_q \theta)),
\]
for some $h_q = 2^{d/q} h(2^q x)$ coming from the third part of Proposition 3.3.

By Lemma 3.3
\[
\|I_q\|_{L^2} \lesssim 2^{(1-2\alpha)q} \|x h_q\|_{L^1} \|\nabla S_{q-1} u\|_{L^2} \|\Delta_q \theta\|_{L^\infty}
\lesssim 2^{-(2\alpha-1)q} 2^{-q} \|\nabla u\|_{L^2} \|\Delta_q \theta\|_{L^\infty}
\lesssim 2^{-2\alpha q} \|\nabla u\|_{L^2} \|\Delta_q \theta\|_{L^\infty}.
\]

Therefore, we have (there are no low frequency terms in $I$)
\[
\|I\|^2_{L^2} \lesssim \sum_{q \geq 0} 2^{2\alpha q} \|I_q\|^2_{L^2}
\lesssim \|\nabla u\|^2_{L^2} \sum_{q \geq 0} 2^{2\alpha q} \|\Delta_q \theta\|^2_{L^\infty}
\lesssim \|\nabla u\|^2_{L^2} \|\theta\|^2_{B^{s-2\alpha}_{\infty,2}}.
\]
Estimation of II

Similarly to I, we write

$$\Pi_q = 2^{(1-2\alpha)q} \left( h_q \ast (\Delta_q u S_q-\theta) - (\Delta_q v)(h_q \ast S_q-\theta) \right)$$

Lemma 3.3 this time gives

$$\|\Pi_q\|_{L^2} \lesssim 2^{(1-2\alpha)q} \|xh_q\|_{L^1} \|\nabla \Delta_q u\|_{L^2} \|S_q-\theta\|_{L^\infty}$$

$$\lesssim 2^{2\alpha q} \|\nabla u\|_{L^2} \sum_{j \leq q-2} \|\Delta_j \theta\|_{L^\infty}.$$

We want to use $\|\Pi\|_{H^s} \sim \sum_{q \geq 0} 2^q \|\Pi_q\|_{L^2}$ again; this time we will use the discrete Young’s inequality. Multiplying by $2^q$, we note that

$$2^q \|\Pi_q\|_{L^2} = 2^{-(2\alpha-s)q} \|\nabla u\|_{L^2} \sum_{j \leq q-2} \|\Delta_j \theta\|_{L^\infty}$$

$$\lesssim \|\nabla u\|_{L^2} \left( 2^{-(2\alpha-s)} \ast 2^{-(2\alpha-s)} \ast (\Delta_s \theta) \|L^\infty \right)(q),$$

where $\ast$ denotes the discrete convolution on $\mathbb{Z}_{\geq -1}$,

$$a \ast b(q) = \sum_{q_1+q_2=q} a_{q_1} b_{q_2}.$$  

Applying the discrete Young’s inequality (note that $2^{-(2\alpha-s)} \ast \in \ell^1 \iff s < 2\alpha$) with parameters $1 + \frac{1}{s} = \frac{3}{4} + \frac{1}{2}$,

$$\|\Pi\|_{H^s} \lesssim \|\nabla u\|_{L^2} \left( 2^{-(2\alpha-s)} \ast 2^{-(2\alpha-s)} \ast (\Delta_s \theta) \|L^\infty \right)(q) \|_{L^2(dq)}$$

$$\lesssim 2^{\alpha-s} \|\nabla u\|_{L^2} \|\theta\|_{B^{-\alpha,s}_\infty}.$$  

Estimation of III

We further split III into high-high and low-low interactions,

$$III = J_1 + J_2,$$

$$J_1 = \sum_{q \geq 1} |R_{2\alpha}, \Delta_q v| \Delta_q \theta,$$

$$J_2 = \sum_{q \leq 0} |R_{2\alpha}, \Delta_q v| \Delta_q \theta.$$  

$J_1$ with no low frequency terms is dealt with as before, giving

$$q \geq 1 \implies \|R_{2\alpha}, \Delta_q v| \Delta_q \theta\|_{L^2} \lesssim 2^{-2\alpha q} \|\nabla v\|_{L^2} \|\Delta_q \theta\|_{L^\infty},$$

so that

$$\|\Delta_j J_1\|_{L^2} \lesssim \|\nabla v\|_{L^2} \sum_{q, q \geq j-4} 2^{-2\alpha q} \|\Delta_q \theta\|_{L^\infty},$$

$$2^{2j} \|\Delta_j J_1\|_{L^2} \lesssim \|\nabla v\|_{L^2} \sum_{q \geq j-4} 2^{3j} 2^{-2\alpha q} \|\Delta_q \theta\|_{L^\infty}$$

$$= \|\nabla v\|_{L^2} \sum_{q \geq j-4} 2^{(j-q)\alpha} 2^{-(2\alpha-s)q} \|\Delta_q \theta\|_{L^\infty}.$$
Under the assumption that $s > 0$, we have by the discrete Young’s inequality again,
\[ \|J_1\|_{H^s} \lesssim \|\nabla v\|_{L^2} \|\theta\|_{H^{-(2\alpha - s)}}. \]

For $J_2$, the lowest frequency terms don’t have a corresponding annulus for us to apply Lemma 3.3, so the Riesz transform has to be dealt with in a different way. We proceed without using the commutator structure, instead relying on Bernstein inequalities and $L^2 \rightarrow L^p$ boundedness of $\mathcal{R}_{2\alpha}$ for $p \in (2, \infty)$ defined by $\frac{1}{p} = \frac{1}{2} - \frac{d}{2\alpha + 1}$ (as in Proposition 3.4):
\[ \|\left[\mathcal{R}_{2\alpha}, \Delta_q\right]\theta\|_{L^2} \lesssim \|\Delta_q \theta\|_{L^2} (\|\Delta_q \theta\|_{L^\infty} + \|\mathcal{R}_{2\alpha} \Delta_q \theta\|_{L^\infty}) \leq \|\theta\|_{L^2} (\|\theta\|_{L^\infty} + \|\mathcal{R}_{2\alpha} \theta\|_{L^p}) \lesssim \|\theta\|_{L^2} \|\theta\|_{L^2}. \]

(Note that all constants from Bernstein inequalities are left implicit because $J_2$ is a finite sum.) This completes the estimation of III and the Theorem is proved.

4 Basic A Priori Estimates

Here we collect some estimates for smooth solutions of (1.1). Since $\theta$ solves a transport equation with no diffusion, and by testing the $u$ equation with itself, we obtain
\[ \|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}, \quad p \in [1, \infty], \]
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \|\Lambda^\alpha u(t)\|_{L^2}^2 \leq \|\theta_0\|_{L^2} \|u\|_{L^2}. \]

These imply
\[ \frac{d}{dt} \|u(t)\|_{L^2} \leq \|\theta_0\|_{L^2}, \]
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \|\Lambda^\alpha u(t)\|_{L^2}^2 \leq \|\theta_0\|_{L^2} (\|u_0\|_{L^2} + \|\theta_0\|_{L^2} t), \]

and therefore:

**Proposition 4.1.** Suppose $(u, \theta)$ are smooth solutions of (1.1) with initial data $u_0 \in L^2$ and $\theta_0 \in L^2 \cap L^p_\alpha$. Then
\[ \|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}, \]
\[ \|u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha u(s)\|_{L^2}^2 \, ds \lesssim u_0, \theta_0, 1 + t^2. \]

**Proposition 4.2.** Suppose $(u, \theta)$ are smooth solutions of (1.1) with initial data $u_0 \in H^1$ and $\theta \in L^2 \cap L^\infty$. Then for $\omega = \nabla^\perp \cdot u$,
\[ \|\omega(t)\|_{L^2}^2 + \int_0^t \|\omega(s) - \mathcal{R}_{2\alpha} \theta(s)\|_{H^\infty}^2 \, ds \leq \Phi_1(t). \]
Proof. As mentioned, we set \(\Gamma = \omega - \mathcal{R}_{2\alpha} \theta\). It solves the equation

\[
(\partial_t + u \cdot \nabla + \Lambda^{2\alpha} \Gamma) = [\mathcal{R}_{2\alpha}, u \cdot \nabla] \theta
\]

Taking the \(L^2\) inner product with \(\Gamma\) and using that \(\nabla \cdot u = 0\) in the form of the identity \([\mathcal{R}_{2\alpha}, u \cdot \nabla] \theta = \nabla \cdot ([\mathcal{R}_{2\alpha}, v] \theta)\),

\[
\frac{1}{2} \frac{d}{dt} \|\Gamma\|_{L^2}^2 + \|\Gamma\|_{H^\alpha}^2 = \int_{\mathbb{R}^2} \nabla \cdot ([\mathcal{R}_{2\alpha}, v] \theta) \Gamma \\
\leq \|\mathcal{R}_{2\alpha}, v\|_{B^{1-\alpha}_\infty} \|\Gamma\|_{H^\alpha}.
\]

By Theorem 3.2, Proposition 4.1, and Proposition 3.4,

\[
\|\mathcal{R}_{2\alpha}, u\|_{B^{1-\alpha}_\infty} \lesssim \|\nabla u\|_{L^p} \|\theta\|_{B^{1-2\alpha}_\infty} + \|u\|_{L^2} \|\theta\|_{L^2} \\
\lesssim \|\nabla u\|_{L^p} + t \\
\lesssim \|\Gamma\|_{L^2} + \|\mathcal{R}_{2\alpha} \theta\|_{L^2} + 1 + t \\
\lesssim \|\Gamma\|_{L^2} + 1 + t.
\]

Therefore,

\[
\frac{1}{2} \frac{d}{dt} \|\Gamma\|_{L^2}^2 + \|\Gamma\|_{H^\alpha}^2 = \int_{\mathbb{R}^2} \nabla \cdot ([\mathcal{R}_{2\alpha}, v] \theta) \Gamma \lesssim \|\Gamma\|_{L^2}^2 + 1 + t^2.
\]

Young’s inequality for products gives

\[
\frac{d}{dt} \|\Gamma(t)\|_{L^2}^2 + \|\Gamma\|_{H^\alpha}^2 \lesssim \|\Gamma\|_{L^2}^2 + 1 + t^2.
\]

An application of Gronwall’s inequality finishes the proof.

5 A Priori Estimates for the Vorticity

Proposition 5.1. Let \((u, \theta)\) be a smooth solution of (1.1), and let \(u_0 \in H^1 \cap W^{1,p}\), with \(p \in (2, \infty)\), \(\nabla \cdot u_0 = 0\), and \(\theta_0 \in L^2 \cap L^\infty\). Then for \(\omega = \nabla \cdot u\),

\[\|\omega(t)\|_{L^p} \leq \Phi_1(t).\]

Proof. We will again use the equation for \(\Gamma\),

\[
\partial_t \Gamma + u \cdot \nabla \Gamma + \Lambda^{2\alpha} \Gamma = [\mathcal{R}_{2\alpha}, u \cdot \nabla] \theta
\]

From the \(L^p\) estimates in Corollary 3.6 of Cordoba–Cordoba (they prove the estimate for the homogeneous equation \(\Gamma_i + u \cdot \nabla \Gamma + \Lambda^{2\alpha} \Gamma = 0\); what we need follows by Duhamel’s principle),

\[
\|\Gamma(t)\|_{L^p} \leq \|\Gamma_0\|_{L^p} + \int_0^t \|([\mathcal{R}_{2\alpha}, u \cdot \nabla] \theta)\|_{L^p} \, ds.
\]

By Proposition 3.1,

\[
\|([\mathcal{R}_{2\alpha}, u \cdot \nabla] \theta)\|_{B^{1-\alpha}_\infty} \lesssim \|\nabla u\|_{L^p} \left(\|\theta\|_{B^{1-2\alpha}_\infty} + \|\theta\|_{L^2}\right) \lesssim \|\omega\|_{L^p} (\|\theta_0\|_{L^2} + \|\theta_0\|_{L^\infty}).
\]
Therefore, using $\Gamma = \omega - \mathcal{R}_{2a} \theta$ and the boundedness of $\mathcal{R}_{2a}$,

$$
\|\omega(t)\|_{L^P} \lesssim_{\theta_0, \omega_0} 1 + \int_0^t \|\omega(s)\|_{L^P} \, ds,
$$

and Gronwall’s inequality completes the proof.

The following result is the analogue of Theorem 6.3 in [28].

**Theorem 5.2.** Let $(u, \theta)$ solve (1.1) with initial data $u_0 \in H^1 \cap \dot{W}^{1,p}$, $\nabla \cdot u = 0$, and $\theta_0 \in L^2 \cap L^\infty$ with $p \in (2, \infty)$. Then for $r \in [1, p/2]$ and $\rho \in [0, 1/2)$,

$$
\|\omega - \mathcal{R}_{2a} \theta\|_{L_t^r B^0_{\rho, r}} \leq \Phi_1(t).
$$

**Proof.** For $q \in \mathbb{N}$ we set $\Gamma_q = \Delta_q \Gamma$. Then, we localise in frequencies the equation (5.1) for $\Gamma$ to get

$$
\partial_t \Gamma_q + u \cdot \nabla \Gamma_q + \Lambda^{2q} \Gamma_q = -[\Delta_q, u \cdot \nabla] \Gamma + \Delta_q([\mathcal{R}_{2a}, u \cdot \nabla] \theta) := f_q.
$$

Multiplying the above equation by $|\Gamma_q|^{-2} \Gamma_q$ and integrating in the space variable we find

$$
\frac{1}{r} \frac{d}{dt} \|\Gamma_q(t)\|_{L^r}^r + \int_{\mathbb{R}^d} (\Lambda^{2q} \Gamma_q) |\Gamma_q|^{-2} \Gamma_q \, dx \leq \|\Gamma_q(t)\|_{L^r}^{r-1} \|f_q(t)\|_{L^r}.
$$

From [30] or [40], we have the following generalised Bernstein inequality,

$$
\int_{\mathbb{R}^d} (\Lambda^{2q} \Gamma_q) |\Gamma_q|^{-2} \Gamma_q \, dx \geq c \alpha^{2q} \|\Gamma_q\|_{L^r}^r,
$$

valid for some $c > 0$ independent of $q$, and hence we find

$$
\frac{1}{r} \frac{d}{dt} \|\Gamma_q(t)\|_{L^r}^r + c \alpha^{2q} \|\Gamma_q(t)\|_{L^r} \leq \|\Gamma_q(t)\|_{L^r}^{r-1} \|f_q(t)\|_{L^r}.
$$

This yields (writing $\Gamma_0 := \omega_0 - \mathcal{R}_{2a} \theta_0$ and $(\Gamma_0)_q := \Delta_q \Gamma_0$)

$$
\|\Gamma_q(t)\|_{L^r} \leq e^{-ct \alpha^{2q}} \|\Gamma_0\|_{L^r} + \int_0^t e^{-c(t-\tau) \alpha^{2q}} \|f_q(\tau)\|_{L^r} \, d\tau.
$$

By taking the $L^r[0, t]$ norm and by using convolution inequalities, we find (since $f_q = -[\Delta_q, u \cdot \nabla] \Gamma + \Delta_q([\mathcal{R}_{2a}, u \cdot \nabla] \theta)$, and $\rho \in [1, p/2]$)

$$
2^{2q} \gamma \|\Gamma_q\|_{L_t^r L^r} \lesssim \|\Gamma_0\|_{L^r} + 2^{2q} \gamma \int_0^t \|\Delta_q([\mathcal{R}_{2a}, u \cdot \nabla] \theta)(\tau)\|_{L^r} \, d\tau
$$

$$
+ 2^{2q} \gamma \int_0^t \|\Delta_q([\mathcal{R}_{2a}, u \cdot \nabla] \theta)(\tau)\|_{L_t^r L^r} \, d\tau.
$$

To estimate the second integral of the RHS of (5.1), we use Proposition [53] This gives uniformly in $q \geq 0$,

$$
\|\Delta_q([\mathcal{R}_{2a}, u \cdot \nabla] \theta)\|_{L_t^r L^r} \leq \|\mathcal{R}_{2a}, u \cdot \nabla\|_{B^{\rho, \infty}_{1, \infty}} \leq \|\nabla u\|_{L_t^r L^\infty(\mathbb{R}^1_{t}, B^{\rho-2, \infty}_{1, \infty})} + \|\theta\|_{L^2} \leq \Phi_1(t).
$$

For the first integral of the RHS of (5.1) we use the following lemma:
Lemma 5.3 (Lemma 6.4 of [28]). Let \( v \) be a smooth divergence-free vector field and \( f \) be a smooth scalar function. Then, for all \( a \in [1, \infty] \) and \( q \geq -1 \),

\[
\| \|\Delta_q v \cdot \nabla \| f \| \|_{L^a} \lesssim \| \nabla v \|_{L^r} \| f \|_{B^{a/2}_q}.
\]

Using Lemma 5.3, the \( L^2 \) bound (Proposition 1.2), and the \( L^p \) bound on \( \omega \) (Proposition 5.1), we can interpolate (recall \( r \in [2, p] \)) to bound \( \|\nabla v\|_{L^r} \leq \Phi_1(t) \), giving

\[
\int_0^t \| \Delta_q u \cdot \nabla \| \| f \|_{L^r} \, d\tau \leq \| \nabla u \|_{L^r} \|\Gamma\|_{B^{2/r}_{\infty,1}} \leq \Phi_1(t) \|\Gamma\|_{L^1 B^{2/r}_{\infty,1}}.
\]

Now we show how to estimate the sum in \( q \geq -1 \) of (5.1). For high frequencies, since \( r > 2p \) i.e. \( \frac{2}{r} - \frac{1}{2} < 0 \), we have

\[
\sum_{q \geq N} 2^{2aq} \left[ \right] \|\Gamma\|_{L^r_q L^r} \leq \|\Gamma_0\|_{L^r_q L^r} 2^{-\alpha N} \|\Gamma\|_{L^1 B^{2/r}_{\infty,1}} \frac{1}{2^{2\alpha N}} \leq \Phi_1(t).
\]

(The 1 and \( \|\Gamma\|_{L^1 B^{2/r}_{\infty,1}} \) is from the first and second integral terms, respectively.) For low frequencies of \( \Gamma \), we just use

\[
\sum_{q < N} 2^{2aq} \|\Gamma\|_{L^r_q L^r} \leq 2^{2\alpha N} \|\Gamma\|_{L^r_q L^r} \leq 2^{\alpha N} \|\Gamma\|_{L^r_q L^r} \frac{1}{2^{2\alpha N}} \Phi_1(t).
\]

Replacing \( \|\Gamma\|_{B^{2/r}_{\infty,1}} \) on the right by the worse term \( \|\Gamma\|_{B^{3\alpha/2}_{\infty,1}} \), we see that

\[
\|\Gamma\|_{L^p B^{3\alpha/2}_{\infty,1}} \leq 2^{-\alpha N} \|\Gamma\|_{L^1 B^{4\alpha/3}_{\infty,1}} 2^{\alpha N} \|\Gamma\|_{L^p B^{4\alpha/3}_{\infty,1}} \frac{1}{2^{2\alpha N}} \Phi_1(t) \]

\[
\leq 2^{-\alpha N} \|\Gamma\|_{L^1 B^{4\alpha/3}_{\infty,1}} 2^{\alpha N} \|\Gamma\|_{L^p B^{4\alpha/3}_{\infty,1}} \frac{1}{2^{2\alpha N}} \Phi_1(t).
\]

Taking \( N \gg 1 \), we obtain \( \|\Gamma\|_{L^p B^{4\alpha/3}_{\infty,1}} \leq \Phi_1(t) \) as claimed. \( \square \)

Proposition 5.4. Let \((u, \theta)\) be a smooth solution of (1.1), and let \( u_0 \in H^1 \cap W^{1,p} \), with \( p \in (2, \infty) \), \( \nabla \cdot u_0 = 0 \), and \( \theta_0 \in L^2 \cap L^\infty \). Then we have

\[
\|u\|_{L^r_p B^{(2\alpha-1)_+}_{\infty,1}} \leq \Phi_1(t),
\]

for every \( \rho \in [1, 2 \wedge \frac{3}{2}] \) and \( r > 2 \).
Proof. The result is stronger for \( r \) closer to 2, so without loss of generality, make \( r \) smaller so that \( r < p \) (this is for the application of Theorem 5.2).

By triangle inequality and \( \omega = \Gamma + R_{2\alpha} \theta \),

\[
\|\omega\|_{L^p B_{\infty,1}^2} \lesssim \|\Gamma\|_{L^p B_{\infty,1}^2} + \|R_{2\alpha} \theta\|_{L^p B_{\infty,1}^2}.
\]

The first term is controlled by some \( \Phi_1 \) by the embedding \( \tilde{L}^p B_{r,1}^{4\alpha/r} \hookrightarrow \tilde{L}^p B_{\infty,1}^2 \) and the previous result, Proposition 5.3. Recalling that \( \mathcal{R}_{2\alpha} = \Lambda^{1-2\alpha} \mathcal{R} \), we can use Bernstein inequalities to see that

\[
\|\mathcal{R}_{2\alpha} \theta\|_{L^p B_{\infty,1}^2} \lesssim \sum_{q \geq 1} 2^q \|\Delta_q \mathcal{R}_{2\alpha} \theta\|_{L^p L^\infty} + \sum_{q \geq 0} 2^q \|\theta\|_{L^p L^\infty},
\]

where we have used that \( \|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}. \) Since \( r > 2 \),

\[
\sum_{q \geq 0} 2^q \|\theta\|_{L^p} < \infty.
\]

Hence, \( \|\mathcal{R}_{2\alpha} \theta\|_{L^p B_{\infty,1}^2} \lesssim \theta_0 \) and therefore \( \|\omega\|_{L^p B_{\infty,1}^{2(2\alpha-1)}} \leq \Phi_1(t). \)

Then we obtain (using Bernstein’s inequality for the low frequency term)

\[
\|u\|_{L^p B_{\infty,1}^{2(2\alpha-1)+1}} \lesssim \|\Delta^{-1} u\|_{L^p L^\infty} + \|\omega\|_{L^p B_{\infty,1}^{2(2\alpha-1)}} \lesssim 1 + \|\omega\|_{L^p B_{\infty,1}^{2(2\alpha-1)}} \leq \Phi_1(t).
\]

\[\square\]

6 Uniqueness of solutions

In order to prove the uniqueness, we will rely on the following Osgood lemma, which can be found as Theorem 5.2.1 in [3], or in [10]:

**Lemma 6.1 (Osgood Lemma).** Let \( \gamma \in L^\infty_0 (\mathbb{R}_+; \mathbb{R}_+) \), \( a \) a continuous non-decreasing function, \( \alpha \in \mathbb{R}_+ \) and \( \alpha \) a measurable function satisfying

\[
0 \leq \alpha(t) \leq a + \int_0^t \gamma(\tau) a(\alpha(\tau)) \, d\tau, \quad \forall t \in \mathbb{R}_+.
\]

If we assume that \( a > 0 \) then

\[
-M(\alpha(t)) + M(\alpha) \leq \int_0^t \gamma(\tau) \, d\tau \quad \text{with} \quad M(x) := \int_x^{t_1} \frac{dr}{\mu(r)}.
\]

If we assume \( a = 0 \) and \( \lim_{x \to 0+} M(x) = +\infty \), then \( \alpha(t) = 0, \forall t \in \mathbb{R}_+ \).
Remark 6.2. In the case $\mu(x) = x(1 - \log x)$, an explicit bound is also given in [8]. We will use the similar function $\mu(x) := x \log(e + 1/x)$, for which in the case $a < 1/e$, we have the estimate

$$\alpha(t) \leq \exp \left[ -\exp \left( e - 1 + M(a) - \int_0^t \gamma \right) \right].$$

(6.1)

This estimate follows elementarily from $\mu(r) \geq -r \log r$ for $0 < r < 1/e$. In particular, note that $\alpha(t) \to 0$ as $a \to 0$.

We now recall some results (with mild modifications) from Section 4 of Hmidi–Keraani–Rousset’s paper [28].

6.1 Estimates for transport-diffusion models

First, we have a result for transported scalars, which we will apply to the temperature $\theta$ of our system.

Proposition 6.3. Let $v$ be a smooth divergence-free vector field. Then every scalar solution $\psi$ of the equation

$$\partial_t \psi + v \cdot \nabla \psi = f, \quad \psi|_{t=0} = \psi_0,$$

satisfies for every $p \in [1, \infty]$,

$$\|\psi(t)\|_{B^{-1}_{p,\infty}} \leq C e^{C \int_0^t \|v(r)\|_{B^{-1}_{p,\infty}} \, dr} \left( \|\psi_0\|_{B^{-1}_{p,\infty}} + \int_0^t \|f(r)\|_{B^{-1}_{p,\infty}} \, d\tau \right).$$

Secondly, we have the following proposition for the linearised velocity equation. The proof is similar to the proof in Hmidi–Keraani–Rousset, so we omit the details. We will only apply this proposition with $s = 0$.

Proposition 6.4. Let $v$ be a smooth divergence-free vector field, $s \in (-1, 1)$, and $\rho \in [1, \infty]$. Let $u$ be a smooth solution of the linear system

$$\partial_t u + v \cdot \nabla u + \Lambda^{2\alpha} u + \nabla p = f, \quad \nabla \cdot u = 0.$$

Then we have for each $t \in [0, \infty)$, with $s' := s - 2\alpha(1 - \frac{1}{p})$,

$$\|u\|_{L^\infty_t B^{-s'}_{2,\infty}} \leq C e^{C \int_0^t \|v(r)\|_{L^\infty} \, dr} \left( \|u_0\|_{B^{-s'}_{2,\infty}} + (1 + t^{1-1/\rho}) \|f\|_{L^2_t B^{-s}_{2,\infty}} \right).$$

We also need the following estimates adapted from the Appendix of [28]; essentially the same proofs can prove these lemmas, so we will omit them.

Lemma 6.5. For every $s \in [-1, 0]$, if $v$ is a smooth divergence-free vector field, and $f$ is a smooth function, then

$$\|v \cdot \nabla f\|_{B^{-s}_{2,\infty}} \lesssim \|f\|_{B^{1+s}_{2,\infty}} \|v\|_{B^{0}_{2,1}}.$$ 

Lemma 6.6. If $v \in H^1$, then $\|v\|_{B^0_{2,1}} \lesssim \|v\|_{B^0_{2,\infty}} \log \left( e + \frac{\|v\|_{H^1}}{\|v\|_{B^0_{2,\infty}}} \right).$

Now we prove the uniqueness of solutions.
Theorem 6.7. Let $\alpha \in (\frac{1}{2}, 1)$. Then the equation (1.1) can have at most one solution pair $(u, \theta)$ (in the sense of Definition 1.1) in the space $\mathcal{X}_T := (L^1_t B^1_{2,1} \cap L^\infty_t H^1) \times (L^\infty_t B^{-1}_{2,1} \cap L^1_t L^\infty).

Proof. Let $(u^1, \theta^1)$ and $(u^2, \theta^2)$ be two solutions to (1.1) in the space $\mathcal{X}_T$. Set $u = u^1 - u^2$ and $\theta = \theta^1 - \theta^2$. Then, they solve

$$
\begin{align*}
\partial_t u + u^2 \cdot \nabla u + \Lambda^{2\alpha} u + \nabla p &= -u \cdot \nabla u^1 + \theta \varepsilon_2, \\
\partial_t \theta + u^2 \cdot \nabla \theta &= -u \cdot \nabla \theta^1,
\end{align*}
$$

with zero initial data. For the velocity $u$, we write $u = V_1 + V_2$ where $V_i$ solve the equations

$$
\begin{align*}
\partial_t V_1 + u^2 \cdot \nabla V_1 + \Lambda^{2\alpha} V_1 + \nabla p_i &= -u \cdot \nabla u^1, \\
\partial_t V_2 + u^2 \cdot \nabla V_2 + \Lambda^{2\alpha} V_2 + \nabla p_i &= \theta \varepsilon_2,
\end{align*}
$$

Proposition 6.4 first with $\rho = 1$ and then with $\rho = \infty$, gives

$$
\begin{align*}
\|V_1\|_{L^\infty_t B^0_{2,\infty}} &\leq C e^{C \int_0^t \|\nabla u^2(r)\|_{L^\infty} \, dr} \left(\|u_0\|_{B^0_{2,\infty}} + \|u \cdot \nabla u^1\|_{L^1_t B^0_{2,\infty}}\right), \\
\|V_2\|_{L^\infty_t B^0_{2,\infty}} &\leq C e^{C \int_0^t \|\nabla u^2(r)\|_{L^\infty} \, dr} \left(\|u_0\|_{B^0_{2,\infty}} + (1 + t) \|\theta\|_{L^\infty_t B^{-1}_{2,\infty}}\right),
\end{align*}
$$

since $L^\infty_t B^{-1}_{2,\infty} \hookrightarrow L^\infty_t B^{-2\alpha}_{2,\infty} = L^\infty_t B^{-2\alpha}_{2,\infty}$. Together, we obtain

$$
\|u\|_{L^\infty_t B^0_{2,\infty}} \leq C e^{C \int_0^t \|\nabla u^2(r)\|_{L^\infty} \, dr} \left(\|u_0\|_{B^0_{2,\infty}} + \|u \cdot \nabla u^1\|_{L^1_t B^0_{2,\infty}} + (1 + t) \|\theta\|_{L^\infty_t B^{-1}_{2,\infty}}\right) \tag{6.2}
$$

By Lemma 6.3 with $s = 0$, we have

$$
\|u \cdot \nabla u^1\|_{B^0_{2,\infty}} \lesssim \|u\|_{B^1_{2,\infty}} \|u\|_{B^0_{2,1}} \tag{6.3}
$$

By using the logarithmic interpolation inequality of Lemma 6.5, we obtain

$$
\|u\|_{B^0_{2,1}} \lesssim \|u\|_{B^0_{2,\infty}} \log \left(e + \frac{1}{\|u\|_{B^0_{2,\infty}}} \right) \log \left(e + \|u\|_{H^1}\right). \tag{6.4}
$$

We have used $\log(e + ab) \leq \log(e + a) \log(e + b)$ which can be proven for instance by using Bernoulli’s inequality for $a, b \geq 0$ in the form $(1 + (1 + a)b) \leq (1 + a)^{1+b}$. Writing $\mu(a) := a \log(e + 1/a)$, the above inequality can be rewritten as

$$
\|u\|_{B^0_{2,1}} \lesssim \mu(\|u\|_{B^0_{2,\infty}}) \log \left(e + \|u\|_{H^1}\right). \tag{6.4}
$$

For the temperature, Proposition 6.3 with $p = 2$ gives

$$
\|\theta\|_{L^\infty_t B^{-1}_{2,\infty}} \leq C e^{C \|u\|_{B^1_{2,1}} \int_0^t \|u \cdot \nabla \theta^1\|_{B^{-1}_{2,\infty}} \, dt} \tag{6.5}
$$

By Lemma 6.5 with $s = -1$,

$$
\|u \cdot \nabla \theta^1\|_{B^{-1}_{2,\infty}} \leq \|\theta^1\|_{B^0_{2,\infty}} \|u\|_{B^1_{2,1}} \leq \|\theta_0\|_{L^\infty} \|u\|_{B^1_{2,1}} \tag{6.6}
$$

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By combining (6.2) with (6.3), (6.4), (6.5), (6.6), we obtain

\[ \|u\|_{L^\infty_t B^{1}_{2,\infty}} + \|\theta\|_{L^\infty_t B^{-1}_{2,\infty}} \leq Ce^{C\|u^2\|_{L^1_t B^1_{\infty,1}}} \left( \|\theta_0\|_{B^{-1}_{2,\infty}} + \|\theta_0\|_{L^\infty} \int_0^t \mu(\|u\|_{B^0_{2,\infty}}) \log(e + \|u\|_{H^1}) \, d\tau \right) + C e^{C\|u^2\|_{L^1_t B^1_{\infty,1}}} \left( \|u_0\|_{B^0_{2,\infty}} + \int_0^t \|u^1\|_{B^1_{\infty,\infty}} \mu(\|u\|_{B^0_{2,\infty}}) \log(e + \|u\|_{H^1}) \, d\tau \right) \]

+ \left[ C(1 + t) e^{C\|u^2\|_{L^1_t B^1_{\infty,1}}} \left( \|\theta_0\|_{B^{-1}_{2,\infty}} + \|\theta_0\|_{L^\infty} \int_0^t \mu(\|u\|_{B^0_{2,\infty}}) \log(e + \|u\|_{H^1}) \, d\tau \right) \right] \leq C(1 + t) e^{C\|u^2\|_{L^1_t B^1_{\infty,1}}} \left( \|u_0\|_{B^0_{2,\infty}} + \|\theta_0\|_{B^{-1}_{2,\infty}} + \log(e + \|u\|_{L^\infty}) (1 + \|\theta_0\|_{L^\infty}) \right) \times \int_0^t (1 + \|u^1(\tau)\|_{B^1_{\infty,\infty}}) \mu(\|u\|_{L^\infty} \|\theta\|_{B^0_{2,\infty}} + \|\theta\|_{L^\infty}) \, d\tau \right).

Setting

X(t) := \|u\|_{L^\infty_t B^0_{2,\infty}} + \|\theta\|_{L^\infty_t B^{-1}_{2,\infty}},

f(t) := C(1 + t) e^{C\|u^2\|_{L^1_t B^1_{\infty,1}}} + \log(e + \|u\|_{L^\infty}) (1 + \|\theta_0\|_{L^\infty}),

g(t) := 1 + \|u^1(\tau)\|_{B^1_{\infty,\infty}}.

We obtain the following integral inequality for X:

\[ X(t) \leq f(t) \left( X(0) + \int_0^t g(\tau) \mu(X(\tau)) \, d\tau \right). \]

Since X(0) = 0, by Lemma 6.1, X(t) = 0 for every t, which proves the uniqueness of solutions.

7 Existence of solutions

We now use the above results to prove our main technical result.

Proof of Theorem 6.3. The uniqueness is guaranteed by Theorem 6.7, so we focus on the existence. Following [4] (or Chapter 3 of [41]), if the initial data are smooth, we can obtain smooth solutions in $H^m$ to (1.1). By the a priori estimate of Proposition 5.4 and the Beale–Kato–Majda type blow-up criterion [4], the solution is globally defined.

Now let $u_0 \in H^1 \cap W^{1,p}$ with $\nabla \cdot u_0 = 0$, for some $p > 2$, and $\theta_0 \in L^1 \cap L^\infty$. Consider the sequence of initial data $u_{0,n}, \theta_{0,n}$ defined by Littlewood–Paley projections,

\[ u_{0,n} := S_n u_0, \quad \theta_{0,n} := S_n \theta_0. \]
These functions are smooth, so they define smooth solutions \( u_n, \theta_n \). Moreover, we have a priori estimates uniform in \( n \) that imply for any \( \varepsilon > 0 \) and \( \rho \in [1, 2 \wedge \frac{4}{n}) \),

\[
\theta_n \in L_t^\infty (L^1 \cap L^\infty), \\
u_n \in L_t^\infty W^{1, \rho} \cap L_t^\infty B^{2a - \varepsilon}_{\infty, 1}.
\]

The spaces \( B^{2a - \varepsilon}_{\infty, 1} \) and \( C^{2a - \varepsilon} = B^{2a - \varepsilon}_{\infty, 1} \) are essentially equivalent by standard embeddings of Besov spaces due to the small loss in \( \varepsilon \), so we can replace one with the other at any point.

Up to subsequences, we have that \( \theta_n \) and \( u_n \) converge weakly to functions \( \theta \) and \( u \). By taking \( (S_n u_0 - S_m u_0, S_n \theta_0 - S_m \theta_0) \) as initial data, the proof of Theorem 6.7 and Remark 6.2 gives that as soon as

\[
\|S_n u_0 - S_m u_0\|_{B^0_{\infty, \infty}} + \|S_n \theta_0 - S_m \theta_0\|_{B^{-1}_{\infty, \infty}} \leq 1/\varepsilon,
\]

we obtain

\[
\|u_n - u_m\|_{L_t^\infty B^0_{2, \infty}} + \|\theta_n - \theta_m\|_{L_t^\infty B^{-1}_{2, \infty}} \\
\leq F(t, \|S_n u_0 - S_m u_0\|_{B^0_{2, \infty}} + \|S_n \theta_0 - S_m \theta_0\|_{B^{-1}_{2, \infty}}),
\]

for an explicit function \( F \) given by (6.1) with \( F(t, s) \to 0 \) as \( s \to 0 \). Thus \( u_m \) is Cauchy and hence strongly convergent to \( u \) in \( L_t^\infty B^0_{2, \infty} \). Interpolation yields strong convergence of \( u_m \) to \( u \) in \( L^2([0, t] \times \mathbb{R}^2) \). This implies \( u_n \odot u_m \to u \odot u \) strongly in \( L^1([0, t] \times \mathbb{R}^2) \). Also, since \( \theta_n \to \theta \) in \( L^2 \), the product \( u_n \theta_n \to u \theta \) in \( L^1 \). Thus, all terms in Definition 1.1 make sense, and \( (u, \theta) \) is a solution to (1.1).

Finally, for completeness, we give a standard argument that shows the boundary regularity for the temperature patches is preserved.

**Proof of Theorem 1.3.** Let \( X = X(t, \xi) \) denote the flow of the vector field \( u \in L_t^1 C^{2a'}, \) i.e. the solution of

\[
\begin{align*}
X_1(t, \xi) &= u(X(t, \xi), t), \\
X(0, \xi) &= \xi.
\end{align*}
\]

Taking the gradient in \( \xi \) gives

\[
(\nabla_{\xi} X)_t = \nabla u(X, t) \nabla_{\xi} X,
\]

so that

\[
\|\nabla_{\xi} X(t, \xi)\|_{L^\infty_x} \leq \|\nabla u(X(0, \xi))\|_{L^\infty_x} e^\int_0^t \|\nabla u(s)\|_{L^\infty_x} ds,
\]

and similarly,

\[
[\nabla_{\xi} X(t, \xi)]_{C^{2a' - 1}} \leq \|\nabla_{\xi} X(0, \xi)\|_{C^{2a' - 1}} e^\int_0^t \|\nabla u(s)\|_{C^{2a'}} ds.
\]

This shows that the flow remains in \( C^{2a' - 1} \). To apply this to the boundary of the patch, suppose that at time \( t = 0, \partial D_0 \) is parameterised by the curve \( \gamma_0 \in C^{2a'}([0, 1]; \mathbb{R}^2) \). Then at time \( t \), the flow transports it to the curve \( \gamma(t, s) := X(t, \gamma_0(s)) \).
Then $\partial_s \gamma = \nabla \xi X(t, \gamma_0(s)) \gamma_0'$, and the Hölder seminorm of $\partial_s \gamma$ is controlled:

$$[\partial_s \gamma]_{C^{2\alpha'-1}} \leq [\nabla \xi X(t, \gamma_0)]_{C^{2\alpha'-1}} \|\gamma_0'\|_{L^\infty} + [\nabla \xi X]_{C^{2\alpha'-1}} \|\gamma_0'\|_{L^\infty}.$$ 

This proves that the Hölder regularity is preserved for all time. \qed

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C. Khor  
Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People’s Republic of China.  
C.Khor@bnu.edu.cn

X. Xu  
Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People’s Republic of China.  
xjxu@bnu.edu.cn