Moving lemmas in mixed characteristic and applications

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Abstract

The present paper contains new geometric theorems proven in mixed characteristic case. We derive a bunch of cohomological consequences using these geometric theorems. Among them an isotropy result for quadratic spaces, a purity result for quadratic spaces, Grothendieck–Serre conjecture for groups $SL_{1,D}$, where $D$ is an Azumaya algebra. The Gersten conjecture for the functor $K_2$ is proved. Bloch-Ogus type result is obtained as well. Suslin's exact sequence is derived and its application to a finiteness result is given. A version of the Roitman theorem is proved. A very general result concerning the Cousin complex is obtained for any cohomology theory in the sense of Panin–Smirnov.

1 Introduction

Most of the result of this preprint are new. However some of them are well-known in the literature: [BW], [B-F/F/P], [BP], [C-T/S], [C-T/S 87], [C-T/H/K], [C], [Dru], [D-K-O] [G], [Gi1], [Gi2], [GiP], [Fe1], [Fe2], [J], [GL], [N], [Oj1], [Oj2], [Pa1], [Pa2], [SS]. There is a good hope that developing further methods of the present preprint one can get more applications. We decided to postpone these to a future. Also we postpone to another preprint proofs of three main geometric theorems formulated in Section 3. Point out that [C, Proposition 2.2.1] was the starting inspiring point of the departure.

2 Agreements

Through the paper

$A$ is a d.v.r., $m_A \subseteq A$ is its maximal ideal;

$\pi \in m_A$ is a generator of the maximal ideal;

$K$ is the field of fractions of $A$; $V = Spec(A)$ and $v \in V$ is its closed point;

$k(v)$ is the residue field $A/m_A$;

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$p > 0$ is the characteristic of the field $k(v)$;  
it is supposed in this notes that the field $K$ has characteritic zero;  
d $\geq 1$ is an integer;  
$X$ is an $A$-smooth $A$-scheme irreducible and equipped with a finite surjective $A$-morphism $X \to A^d_A$;  
Particularly, all open subschemes of $X$ are regular and all its local rings are regular.  
For an $A$-scheme $Y$ we write $Y_v$ for the closed fibre of the structure morphism $Y \to V$.  
If $x_1, x_2, \ldots, x_n$ are closed points in the scheme $X$, then write $\mathfrak{O}$ for the semi-local ring $\mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}}$, $\mathcal{K}$ for the fraction field of $\mathfrak{O}$, $U$ for $Spec(\mathfrak{O})$, $\eta$ for $Spec(\mathcal{K})$, $X' \subseteq X$ for an affine Zariski open which contains all generic points of the scheme $X_v$, but does not contain any irreducible component of the scheme $X_v$.  

Examples. Let $\bar{X}$ be an $A$-smooth projective irreducible $A$-scheme.  
Let $X_\infty \subseteq X$ be a divizor which does not contain any component of the closed fibre of $X$ over $V$. Then $X = X - X_\infty$ is an $A$-scheme of our interest in this paper.

More generally: let $\bar{X}$ be an $A$-projective irreducible $A$-scheme which is normal. Let $X_\infty \subseteq X$ be an effective Cartie divizor such that  
a) the $A$-scheme $X = X - X_\infty$ is $A$-smooth;  
b) the divisor $X_\infty$ does not contain any component of the closed fibre of $X$ over $V$.  
Then $X = X - X_\infty$ is an $A$-scheme of our interest in this paper.

all schemes are supposed to be separated  

$Sm/V$ is the category of $V$-smooth schemes of finite type over $V$ and $V$-morphisms;  

$Sm'/V$ is the category of essentially smooth $V$-schemes. Following [10], by an essentially smooth $V$-scheme we mean a Noetherian $V$-scheme $Y$ which is the inverse limit of a left filtering system $(Y_i)_{i \in I}$ with each transition morphism $Y_i \to Y_j$ being an étale affine $V$-morphism between smooth $V$-schemes of finite type;  

$Sm'Op/V$ is a category whose objects are pairs $(X, W)$, where $X \in Sm'/V$ and $W \subseteq X$ is an open subscheme. A morphism $f : (X, W) \to (X', W')$ in $Sm'Op/V$ is just a $V$-morphism $f : X \to X'$ such that $f(W) \subseteq W'$.  

$Sch/V$ is the category of $V$-schemes of finite type and $V$-morphisms;  

$Sch'/V$ is the category of $V$-schemes essentially of finite type. Recall that by a $V$-scheme essentially of finite type we mean a Noetherian $V$-scheme $S$ which is the inverse limit of a left filtering system $(S_j)_{j \in J}$ with each transition morphism $S_j \to S_{j'}$ being an affine $V$-morphism between $V$-schemes of finite type;
if \( l \) is a prime and \( B \) is an abelian group, then \( \{l\}B \) denotes the subgroup of \( l \)-primary elements in \( B \).

### 3 Geometric theorems

The main aim of the preprint is to state the following three geometric results and to get plenty of their applications. Let \( A \) be a d.v.r. and \( X \) be an \( A \)-scheme as in the section 2.

**Theorem 3.1 (geometric presentation).** Let \( x_1, x_2, \ldots, x_n \) be closed points in the scheme \( X \). Let \( X' \) be a neighborhood of points \( x_1, x_2, \ldots, x_n \) which contains all generic points of the scheme \( X_v \), but does not contain any irreducible component of the scheme \( X_v \). Then one can find inside \( X' \) an affine neighborhood \( X' \) of points \( x_1, x_2, \ldots, x_n \), an open affine subscheme \( S \subseteq \mathbb{P}^{d-1} \) and a smooth \( A \)-morphism

\[
q : X^o \to S
\]

equipped with a finite surjective \( S \)-morphism \( \Pi : X^o \to \mathbb{A}^1_S \).

Let \( Z \) be a closed subset in \( X' \) of codimension at least 2 in \( X' \) such that each its irreducible component contains at least one of the point \( x_i \)'s. Then one can choose the scheme \( X^o \) and the morphism \( q \) such that additionally \( Z^o/S \) is finite, where \( Z^o = Z \cap X^o \).

Let \( Y \) be a closed subset in \( X' \) of codimension 1 such that \( Y \) does not contain any irreducible component of the scheme \( X'_v \). Then one can choose the scheme \( X^o \) and the morphism \( q \) such that additionally \( Y^o/S \) is quasi-finite and \( Y^o_v \) is not-empty, where \( Y^o = Y \cap X^o \).

Here are two consequences of this Theorem.

**Corollary 3.2.** Let \( X, x_1, x_2, \ldots, x_n \in X, X' \subseteq X \) and \( Y \subseteq X' \) be as in Theorem 3.1. Then there is an affine Zariski open neighborhood \( U' \) of points \( x_1, x_2, \ldots, x_n \in X' \) and a diagram of \( A \)-schemes and \( A \)-morphisms of the form

\[
\mathbb{A}^1_{U'} \xleftarrow{\tau} X \xrightarrow{p_X} X'
\]

with an \( A \)-smooth irreducible scheme \( X \) and a finite surjective morphism \( \tau \) such that the canonical sheaf \( \omega_{X/Y} \) is isomorphic to the structure sheaf \( \mathcal{O}_X \). Moreover, if we write \( p : X \to U' \) for the composite map \( p|_{U'} \circ \tau \) and \( \mathcal{Y} \) for \( p_X^{-1}(Y) \), then these data enjoy the following properties:

1) there is a section \( \Delta : U' \to X \) of the morphism \( p \) such that \( \tau \circ \Delta = i_0 \) and \( p_X \circ \Delta = \text{can} \), where \( i_0 \) is the zero section of \( \mathbb{A}^1_{U'} \) and \( \text{can} : U' \hookrightarrow X' \) is the inclusion;
2) for \( \mathcal{D}_1 := \tau^{-1}(\{1\} \times U') \) one has \( \mathcal{D}_1 \cap \mathcal{Y} = \emptyset \);
3) for \( \mathcal{D}_0 := \tau^{-1}(\{0\} \times U') \) one has \( \mathcal{D}_0 = \Delta(U') \cup \mathcal{D}'_0 \) and \( \mathcal{D}_0 \cap \mathcal{Y} = \emptyset \);
Corollary 3.3. Let $X, x_1, x_2, \ldots, x_n \in X$, $X' \subseteq X$ and $Y \subset X'$ be as in Theorem 3.1. Suppose the field $k(v)$ is infinite. Then there is an affine Zariski open neighborhood $U'$ of points $x_1, x_2, \ldots, x_n \in X'$ and a diagram of $A$-schemes and $A$-morphisms of the form

$$A_{U'}^1 \xrightarrow{\tau} X \xrightarrow{p_X} X'$$

with an $A$-smooth irreducible scheme $X$ and a finite surjective morphism $\tau$ such that the canonical sheaf $\omega_{X/V}$ is isomorphic to the structure sheaf $\mathcal{O}_X$. Moreover, if we write $p : X \to U'$ for the composite map $pr_U \circ \tau$ and $\mathcal{Y}$ for $p_X^{-1}(Y)$, then these data enjoy the following properties:

0) the closed subset $\mathcal{Y}$ of $X$ is quasi-finite over $U'$;

1) there is a section $\Delta : U' \to X$ of the morphism $p$ such that $\tau \circ \Delta = i_0$ and $p_X \circ \Delta = \text{can}$, where $i_0$ is the zero section of $A_{U'}^1$, and $\text{can} : U' \to X'$ is the inclusion;

2) the morphism $\tau$ is étale as over $\{0\} \times U'$, so over $\{1\} \times U'$;

3) for $D_1 := \tau^{-1}(\{1\} \times U')$ one has $D_1 \cap \mathcal{Y} = \emptyset$;

4) for $D_0 := \tau^{-1}(\{0\} \times U')$ one has $D_0 = \Delta(U') \cup D_0'$ and $D_0' \cap \mathcal{Y} = \emptyset$;

5) let $(P', \varphi')$ be a quadratic space over $X'$ and $(P, \varphi) = \text{can}^*(P', \varphi')$, then one can construct the above diagram such that the properties (1) to (4) does hold and the quadratic spaces $p_{U'}^0(P, \varphi)$ and $p_X^0(P', \varphi')$ are isomorphic.

One more consequence of the Theorem 3.1 is the following moving lemma.

Theorem 3.4 (A moving lemma). Let $X, x_1, x_2, \ldots, x_n \in X$, $X' \subseteq X$ and $Z \subset X'$ be as in Theorem 3.1. Let $c \geq 2$ be an integer and suppose $Z$ has pure codimension $c$ in $X'$. Then there are an affine Zariski open neighborhood $U'$ of points $x_1, x_2, \ldots, x_n \in X'$ and a closed subset $Z^{\text{new}}$ in $X'$ containing $Z$ of pure codimension $c - 1$ and a morphism of pointed Nisnevich sheaves

$$\Phi_{t} : A^1 \times U'/(U' - Z^{\text{new}}) \to X'/(X' - Z)$$

in the category $\mathbf{Shv}_{\text{nis}}(Sm/V)$ such that for $\Phi_0 = \Phi \circ i_0$ and $\Phi_1 = \Phi \circ i_1$ one has:

1) $\Phi_0 : U'/(U' - Z^{\text{new}}) \to X'/(X' - Z)$ is the composite morphism $U'/(U' - Z^{\text{new}}) \xrightarrow{\text{can}} X'/(X' - Z^{\text{new}}) \xrightarrow{p_X} X'/(X' - Z)$;

2) $\Phi_1 : U'/(U' - Z^{\text{new}}) \to X'/(X' - Z)$ takes everything to the point $*$ in $X'/(X' - Z)$.

4 General results on Cousin complexes

Let $A, p > 0, d \geq 1, X, x_1, x_2, \ldots, x_n \in X$, $\emptyset$ and $U$ be as in Section 2.

Definition 4.1. One says that a presheaf $G$ on $Sm'/V$ is continuous if for each $Y \in Sm'/V$, each left filtering system $(Y_i)_{i \in I}$ as in Section 2 and each $V$-scheme isomorphism $Y \to \text{lim}_{i \in I} Y_i$ the map

$$\text{colim}_{i \in I} G(Y_i) \to G(Y)$$

is an isomorphism.
Definition 4.2. Let \( (Sm'Op/V)^{op} \to Ab \) together with
\[
(X,W) \mapsto [\partial_{X,W} : E(W) \to E(X,W)]
\]
be a cohomology theory on the category \( Sm'Op/V \) in the sense of [PW, Definition 2.1] (inspired by [PSm, Definition 2.1]). One says that \( E \) is \( \mathbb{Z} \)-graded, if for each pair \((X,W) \in Sm'Op/V\) the group \( E(X,W) \) is a graded abelian group \( \oplus E^n(X,W) \) and the differential \( \partial_{X,W} : E(W) \to E(X) \) is a graded abelian group homomorphism of degree +1.

It is convenient to suppose in this paper that all cohomology theories are \( \mathbb{Z} \)-graded.

Definition 4.3. One says that a \( \mathbb{Z} \)-graded cohomology theory \((E,\partial)\) is continuous if for each integer \( n \) the presheaf \( E^n \) on \( Sm'/V \) is continuous.

Agreement 4.4. Over this preprint each \( \mathbb{Z} \)-graded cohomology theory \((E,\partial)\) with the character "\( E \)" on the first place is supposed to be continuous.

Remark 4.5. It turns out that all specific \( \mathbb{Z} \)-graded cohomology theory regarded in this preprint are continuous.

Let \( z \in U \) be a point. We write in this preprint \( E^m_z(U) \) for \( E^m_z(\text{Spec } \mathcal{O}_{X,z}) = E^m_z(\text{Spec } \mathcal{O}_{U,z}) \). For each integers \( m \) and \( c \) with \( c \geq 0 \) and each codimension \( c \) point \( z \in U \) abelian groups \( E^{m,c}_z(U) \), \( E^{m,(c)}_z(U) \) and maps \( \partial : E^{m,(c)}_z(U) \to E^{m+1,(c)}_z(U) \) are defined in [Pan0, Section 9]. By the abuse of notation the same symbol \( \partial \) is used there to denote the composite map \( E^{m,(c)}_z(U) \xrightarrow{0} E^{m+1,(c)}_z(U) \to E^{m+1,(c+1)}_z(U) \). By the construction \( 0 = \partial \circ \partial : E^{m,(c)}_z(U) \to E^{m+2,(c+2)}_z(U) \). Combining all these together the complex of the form
\[
0 \to E^m(U) \xrightarrow{\eta} E^m(0) \xrightarrow{(1)} E^m(1) \xrightarrow{(2)} \cdots \xrightarrow{(d)} E^m(d+1) \to 0 \quad (1)
\]
is defined in [Pan0, Section 9, (8)]. Using the excision property of \((E,\partial)\) the equalities \( E^{m,(c)}_z(U) = \bigoplus_{z \in U(c)} E^m_z(U) \) are explained in [Pan0, Section 9, (8)]. In this way the complex
\[
0 \to E^m(U) \xrightarrow{\eta} E^m(\eta) \xrightarrow{\delta} \bigoplus_{y \in U^{(1)}} E^{m+1}_y(U) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bigoplus_{x \in U^{(d+1)}} E^{m+d+1}_x(U) \to 0 \quad (2)
\]
is constructed in [Pan0, Section 9, (8)].

Notation 4.6. Write \( \text{Cous}(E,U;m) \) for the complex \((2)\).

Theorem 4.7. Let \( m \) and \( c \) be an integers and \( c \geq 0 \). Then
(a) the complex \((2)\) is exact except possibly the terms \( E^m(U) \) and \( \bigoplus_{y \in U^{(1)}} E^{m+1}_y(U) \);
(b) particularly, the complex \((2)\) is exact at the term \( E^m(\eta) \);
(c) for each integer \( m \) and \( c \geq 1 \) the map \( E^{m,(c)}_{2,c+1}(U) \to E^{m,c}(U) \) vanishes;
(d) for each integer \( m \) and \( c \geq 1 \) the sequence \( 0 \to E^{m,(c)}_{2,c}(U) \to E^{m,(c)}_z(U) \xrightarrow{\partial} E^{m,(c+1)}_{2,c+1}(U) \to 0 \) is short exact.

Remark 4.8. The item (b) states the following. Let \( \alpha \in E^m(\eta) \) be an element such that for each codimension one point \( y \in U \) it can be lifted to an element \( \alpha_y \in E^m(\mathcal{O}_{X,y}) \). Then \( \alpha \) can be lifted to an element \( E^m(0) \). So, purity holds for the presheaf \( E^n \) on \( Sm/V \) and the ring \( \mathcal{O} \).
Corollary 4.14. Under the hypothesis of Theorem 4.13 the p-th cohomology group

\[ H^0[0 \to E^i(\eta) \xrightarrow{\partial} \bigoplus_{y \in X^{(1)}} E^i_y(U_y) \xrightarrow{\partial} \ldots \xrightarrow{\partial} \bigoplus_{x \in X^{(d+1)}} E^{i+d+1}(U_x) \to 0] \]  

(4)

coincides with the cohomology groups \( H^p_{\text{Zar}}(X, E^s) \), where \( E^s \) is the Zariski sheaf on \( X \) associated with the presheaf \( W \mapsto E^q(W) \).

If for each integer \( i \) and each closed point \( x \in X \) the map \( \eta^*: E^i(U_x) \to E^i(\eta) \) is injective, then there is a spectral sequence of the form \( H^p_{\text{Zar}}(X, E^0) \Rightarrow E^{p+q}(X) \).

5 Cousin complexes for certain theories are exact

The following result is a very weak form of [GiP] Theorem, page 2]. However the proof of Theorem 5.1 uses only the Balmer–Witt functor, whereas the main tool used in [GiP] is the coherent Witt-groups developed by S.Gille [Gi1], [Gi2], [Gi3], [Gi4].
Definition 5.2. Let $\text{Sm}$ be a contravariant functor on $\text{Sm}'/\text{Op}/V$. Then for each integer $m$, the Cousin complex $\text{Cous}(W, U, m)$ is exact. Particularly, the complex $\text{Cous}(W, U, 0)$ is exact. Moreover, for each $z \in \text{U}(\text{c})$, there is a non-canonical isomorphism $W_z^= W(z)$. So, a Gersten–Witt complex

$$0 \to W(U) \xrightarrow{\eta} W(\eta) \xrightarrow{\partial} \oplus_{y \in \text{U}(1)} W(y) \xrightarrow{\partial} \oplus_{x \in \text{U}(d+1)} W(x) \to 0$$

is exact. Particularly, the map $W(\emptyset) \to W(\emptyset)$ is injective and the purity holds for the functor $W$. Namely, if a class $[\varphi] \in W(\emptyset)$ can be lifted to the group $W(\emptyset)$ for each point $y$ in $\text{U}(1)$, then the class $[\varphi]$ can be lifted to the group $W(\emptyset)$.

Theorem 5.4. Let $\mathcal{F}$ be an étale sheaf on the Big étale site $(\text{Sch}'/V)_{\text{et}}$. Define a contravariant functor on $\text{Sm}'/\text{Op}/V$ by

$$(Y, Y - Z) \mapsto H^n_Z(Y, \mathcal{F}) := \text{Ext}^n(\mathbb{Z}(Y)/\mathbb{Z}(Y - Z), \mathcal{F}),$$

where the Ext-groups are taken in the category of sheaves on the small étale site $Y_{\text{et}}$. The boundary maps $\partial_{(Y, Y - Z)} : H^n(Y - Z, \mathcal{F}) \to H^{n+1}_Z(Y, \mathcal{F})$ is induced by the short exact sequence $0 \to \mathbb{Z}(Y - Z) \to \mathbb{Z}(Y) \to \mathbb{Z}(Y)/\mathbb{Z}(Y - Z) \to 0$ of representable sheaves on the small étale site $Y_{\text{et}}$.

Let $r > 1$ be an integer such that for any $Y \in \text{Sch}'/V$ one has $r : \mathcal{F}(Y) = 0$ and $r$ is coprime to $p$. Then for each $Y \in \text{Sch}'/V$ and each $n \geq 0$ and the projection $pr : Y \times \mathbb{A}^1 \to Y$ the map $pr^* : H^*_et(Y, \mathcal{F}) \to H^*_et(Y \times \mathbb{A}^1, \mathcal{F})$ is an isomorphism. Thus, the contravariant functor $(Y, Y - Z) \mapsto H^n_Z(Y, \mathcal{F})$ together with the boundary maps $\partial_{(Y, Y - Z)}$ form a cohomology theory on $\text{Sm}'/\text{Op}/V$ in the sense of [PW Definition 2.1] (inspired by [PSm Definition 2.1]).

Theorem 5.3. Suppose $r$ is coprime to the prime $p$ and $s$ be an integer. Let $(H^*_et(-, \mu_r^s), \partial)$ be the étale cohomology theory on $\text{Sm}'/\text{Op}/V$. Then for each integer $m$, the Cousin complex $\text{Cous}(H^*_et(-, \mu_r^s), U, m)$ is exact. Moreover, for each $z \in \text{U}(\text{c})$ there is a canonical isomorphism $H^{m+c}_{et, z}(U, \mu_r^{s-c}) = H^{m-c}_{et, z}(z, \mu_r^{s-c})$. So, the Bloch–Ogus type complex

$$0 \to H^m_{et}(U, \mu_r^{s-c}) \xrightarrow{\eta} H^m_{et}(\eta, \mu_r^{s-c}) \xrightarrow{\partial} \oplus_{x \in \text{U}(d+1)} H^{m-d-1}_{et}(x, \mu_r^{s-d-1}) \to 0$$

is exact.

Theorem 5.4. Suppose $r$ is coprime to the prime $p$. Let $\mathcal{F}$ be a locally constant $r$-torsion sheaf on the big étale site $(\text{Sch}'/V)_{\text{et}}$. Let $(H^*_et(-, \mathcal{F}), \partial)$ be the étale cohomology theory on $\text{Sm}'/\text{Op}/V$. Then for each integer $m$, the Cousin complex $\text{Cous}(H^*_et(-, \mathcal{F}), U, m)$ is exact. Moreover, for each $z \in \text{U}(\text{c})$ there is a canonical isomorphism $H^{m+c}_{et, z}(U, \mathcal{F}) = H^{m-c}_{et, z}(z, \mathcal{F})(-c)$. So, the Bloch–Ogus type complex

$$0 \to H^m_{et}(U, \mathcal{F}) \xrightarrow{\eta} H^m_{et}(\eta, \mathcal{F}) \xrightarrow{\partial} \oplus_{x \in \text{U}(d+1)} H^{m-d-1}_{et}(x, \mathcal{F}(-d-1)) \to 0$$

is exact.
Consider the Thomason-Throughbor $K$-groups with finite coefficients $\mathbb{Z}/r$. They form a cohomology theory on the category $Sm'/Op/V$ in the sense of $[PSm]$. Namely, for each integer $n$ and each pair $(X, X - Z)$ in $Sm'/Op/V$ put $K^{-n}(X, X - Z; \mathbb{Z}/r) = K_n(X\on X; \mathbb{Z}/r)$ (see $[TT]$). The definition of $\partial$ is contained in $[TT]$ Theorem 5.1] Repeating literally arguments of $[PSm]$ Example 2.1.8] we see that that this way we get a cohomology theory on $Sm'/Op/V$. Moreover, if $X$ is quasi-projective then $K^*_n(X\on X; \mathbb{Z}/r)$ coincides with the Quillens $K$-groups $K^Q_n(X; \mathbb{Z}/r)$ by $[TT]$ Theorems 3.9 and 3.10]. Write $K^{-n}_n(X; \mathbb{Z}/r)$ for $K^{-n}(X, X - Z; \mathbb{Z}/r)$.

**Theorem 5.5.** [Gillet, H., Levine M. $[GL]$] Suppose $r$ is coprime to the prime $p$. Let $(K^*(\cdot; \mathbb{Z}/r), \partial)$ be the Thomason-Throughbor $K$-theory with finite coefficients $\mathbb{Z}/r$. It is a cohomology theory on $Sm'/Op/V$ as explained just above. Then for each integer $m$ in the Cousin complex $\text{Cous}(K^*(\cdot; \mathbb{Z}/r), U, m)$ is exact. Moreover, for each $z \in U^{(c)}$ there is a canonical isomorphism $K^{m+c}_{\text{et}}(U; \mathbb{Z}/r) = K^{m+c}(z; \mathbb{Z}/r) = K_{-m-c}(z; \mathbb{Z}/r)$. So, the Gersten complex

$$0 \to K_m(U; \mathbb{Z}/r) \xrightarrow{\eta^*} K_m(\eta; \mathbb{Z}/r) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \oplus_{x \in U^{(d+1)}} K_{m-d-1}(x; \mathbb{Z}/r) \to 0 \quad (8)$$

is exact.

**Theorem 5.6.** Suppose $r$ is coprime to the prime $p$. The following Gersten type complexes are exact, where $K_*$ is the Quillen $K$-theory,

$$0 \to K_2(U) \xrightarrow{\eta^*} K_2(\eta) \xrightarrow{\partial} \oplus_{y \in U^{(1)}} K_1(k(y)) \xrightarrow{\partial} \oplus_{x \in U^{(2)}} K_0(k(x)) \to 0 \quad (9)$$

$$0 \to K_2(U)/r \xrightarrow{\eta^*} K_2(\eta)/r \xrightarrow{\partial} \oplus_{y \in U^{(1)}} K_1(k(y))/r \xrightarrow{\partial} \oplus_{x \in U^{(2)}} K_0(k(x))/r \to 0 \quad (10)$$

### 6 Suslin’s exact sequence in mixed characteristic

The following result gives a mixed characteristic version of the Suslin exact sequence.

**Theorem 6.1.** Suppose $r$ is coprime to the prime $p$. Then there is an exact sequence of the form

$$0 \to H^1_{\text{et}}(X, \mathbb{K}_2)/r \xrightarrow{\alpha} NH^3_{\text{et}}(X, \mu_r^{\otimes 2}) \xrightarrow{\beta} H^2_{\text{zar}}(X, \mathbb{K}_2) \to 0, \quad (11)$$

where $NH^3_{\text{et}}(X, \mu_r^{\otimes 2}) = \ker[H^3_{\text{et}}(X, \mu_r^{\otimes 2}) \xrightarrow{\eta_*} H^3_{\text{et}}(\eta, \mu_r^{\otimes 2})]$.

**Theorem 6.2.** Suppose additionally that the ring $A$ is henzelian, the residue field of $A$ is finite and let $l$ be a prime different of $p$. Suppose this time that $X$ is $A$-smooth projective irreducible and has relative dimension $d = 2$. Then

a) the map $\langle \eta \rangle H^2_{\text{zar}}(X, \mathbb{K}_2) \to \langle \eta \rangle CH^2(X, \mathbb{Q}_l)$ is an isomorphism;

b) the map $\beta : NH^3_{\text{et}}(X, \mathbb{Q}_l\otimes \mathbb{Z}_l(2)) \to \langle \eta \rangle H^2_{\text{zar}}(X, \mathbb{K}_2)$ is an isomorphism;

c) the map $NH^3_{\text{et}}(X, \mathbb{Q}_l\otimes \mathbb{Z}_l(2)) \to NH^3_{\text{et}}(X, \mathbb{Q}_l\otimes \mathbb{Z}_l(2))$ is an isomorphism;

d) the group $\langle \eta \rangle H^2_{\text{zar}}(X, \mathbb{K}_2)$ is finite.
Remark 6.3. Since $X$ is smooth projective over $V$ and irreducible and $A$ is henzelian it follows that the closed fibre $X_v$ is irreducible too. To prove this one has to use the Steiner decomposition of the morphism $p : X \to V$ as $X \xrightarrow{q} W \xrightarrow{\pi} V$, where $q$ is surjective with connected fibres and $\pi$ is finite surjective. Since $q$ is projective and surjective and $X$ is irreducible it follows that $W$ is irreducible. Since $V$ is local henzelian and $\pi$ is finite it follows that $W$ has a unique closed point, say $w$. Since the fibres of $q$ are connected it follows that $X_v = p^{-1}(v) = q^{-1}(w)$ is connected. Since $X_v$ is $v$-smooth, hence $X_v$ is irreducible.

The following colollary is a version of the Roitman theorem.

Corollary 6.4. Let $A,X,d,l$ be as in Theorem 6.3. Suppose the residue field of $A$ is an algebraic closure of the finite field $\mathbb{F}_p$. Then the natural map

$$\{l\} H^2_{\text{Zar}}(X, \mathbb{K}_2) \to \{l\} \text{Alb}(X/V)$$

is an isomorphism. There are equalities $NH^3_{\text{et}}(X_v, \mathbb{Q}_l/\mathbb{Z}_l(2)) = H^3_{\text{et}}(X_v, \mathbb{Q}_l/\mathbb{Z}_l(2))$ and $NH^3_{\text{et}}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) = H^3_{\text{et}}(X, \mathbb{Q}_l/\mathbb{Z}_l(2))$ and the map $\beta$ is an isomorphism

$$\beta : H^3_{\text{et}}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) = NH^3_{\text{et}}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to \{l\} H^2_{\text{Zar}}(X, \mathbb{K}_2).$$

Theorem 6.5. Let $A,X,l$ be as in Theorem 6.3 but this time $d = \dim_{\mathbb{K}} X$ is arbitrary. Suppose the residue field of $A$ is an algebraic extension of the finite field $\mathbb{F}_p$. Then

a) the map $\beta : NH^3_{\text{et}}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to \{l\} H^2_{\text{Zar}}(X, \mathbb{K}_2)$ is an isomorphism;

b) the map $NH^3_{\text{et}}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to NH^3_{\text{et}}(X_v, \mathbb{Q}_l/\mathbb{Z}_l(2))$ is injective;

c) the map $\{l\} H^2_{\text{Zar}}(X, \mathbb{K}_2) \to \{l\} CH^2(X_v)$ is injective;

d) if the residue field $A$ of $A$ is finite, then the group $\{l\} H^2_{\text{Zar}}(X, \mathbb{K}_2)$ is finite.

7 Rationally isotropic quadratic spaces

The results of this section extend to the mixed characteristic case isotropy and purity theorems for quadratic spaces proven in [P], [PP1], [PP2]. Following the method of [PP2] these results are derived in this section using Corollary 3.2 and Theorem 5.1. A proof of Theorem 5.1 is given in Section 10.

Let $A$, $p > 0$, $d \geq 1$, $X$, $x_1, x_2, \ldots, x_n \in X$, $\mathcal{O}$ and $U$ be as in Section 2. We suppose in this section that $1/2$ is in $A$. Write $\mathcal{K}$ for the fraction field of the ring $\mathcal{O}$.

We refer to [P] for definitions of a quadratic space and of an isotropic quadratic space. Here we just indicate the following: if $(Q, \psi)$ is a quadratic space over a Dedekind semi-local domain $R$ with a fraction field $L$ and $(Q, \psi) \otimes_R L$ is isotropic over $L$, then $(Q, \psi)$ is isotropic.

Theorem 7.1 (Rationally isotropic spaces are locally isotropic). (1) Suppose each irreducible component of $X_v$ contains at least one of the point $x_i$'s. Let $(P, \varphi)$ be a quadratic space over $\mathcal{O}$. If $(P, \varphi) \otimes_\mathcal{O} \mathcal{K}$ is isotropic over $\mathcal{K}$, then $(P, \varphi)$ is isotropic over $\mathcal{O}$, that is there exists a unimodular vector $v \in V$ with $\varphi(v) = 0$.

(2) If $X_v$ is irreducible, then the hypotheses of the item (1) hold automatically. So, each quadratic space over $\mathcal{O}$, which is isotropic over $\mathcal{K}$, is isotropic over $\mathcal{O}$. 

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To state the first corollary of the theorem we need to recall the notion of unramified spaces. Let $R$ be a Noetherian domain and $L$ be its fraction field. Recall that a quadratic space $(Q, \psi)$ over $L$ is unramified over $R$ if for every height one prime ideal $\mathfrak{p}$ of $R$ there exists a quadratic space $(Q_\mathfrak{p}, \psi_\mathfrak{p})$ over $R_\mathfrak{p}$ such that the spaces $(Q_\mathfrak{p}, \psi_\mathfrak{p}) \otimes_{R_\mathfrak{p}} L$ and $(Q, \psi)$ are isomorphic.

**Corollary 7.2 (Purity for quadratic spaces).** (1) Suppose each irreducible component of $X_v$ contains at least one of the point $x_i$’s. Let $(Q, \psi)$ be a quadratic space over $\mathcal{K}$ which is unramified over $\mathfrak{O}$. Then there exists a quadratic space $(P, \varphi)$ over $\mathfrak{O}$ extending the space $(Q, \psi)$, that is the spaces $(P, \varphi) \otimes_{\mathfrak{O}} \mathcal{K}$ and $(Q, \psi)$ are isomorphic.

(2) If $X_v$ is irreducible, then the hypotheses of the item (1) hold automatically. So, each quadratic space over $\mathcal{K}$, which is unramified over $\mathfrak{O}$, can be lifted to a quadratic space over $\mathfrak{O}$.

**Corollary 7.3.** (1) Suppose each connected component of $X_v$ contains at least one of the point $x_i$’s. Let $(P, \varphi)$ be a quadratic space over $\mathfrak{O}$ and let $u \in \mathfrak{O}^\times$ be a unit. Suppose the equation $\varphi = u$ has a solution over $\mathcal{K}$ then it has a solution over $\mathfrak{O}$, that is there exists a vector $v \in V$ with $\varphi(v) = u$.

(2) If $X_v$ is irreducible, then the hypotheses of the item (1) hold automatically. So, if the equation $\varphi = u$ has a solution over $\mathcal{K}$ then it has a solution over $\mathfrak{O}$.

**Notation 7.4.** Let $X_{v,j} \subseteq X_v$ ($j$ runs from 1 to l) be the irreducible components of the scheme $X_v$. Let $\eta_j$ be its generic point. Write $\mathcal{O}_{X,v}$ for $\mathcal{O}_{X_{v,\eta_1}, \ldots, \eta_l}$. If $X' \subset X$ is an open containing all the points $\eta_j$’s, then $\mathcal{O}_{X_{v,\eta_1}, \ldots, \eta_l} = \mathcal{O}_{X', \eta_1, \ldots, \eta_l}$. So, using short notation $\mathcal{O}_{X,v} = \mathcal{O}_{X',X_v}$. Put $S = \text{Spec} \mathcal{O}_{X,v} = \text{Spec} \mathcal{O}_{X',X_v} = S'$.

**Reducing Theorem 7.1 to Theorem 7.7 and Corollary 7.3.** First consider the case when the residue field $k(v)$ of $A$ is infinite. We may assume that there is an affine open neighborhood $X'$ of all the points $x_i$’s and a quadratic space $(P', \varphi')$ over $X'$ which extends to $X'$ the space $(P, \varphi)$. Moreover, this $X'$ can be chosen such that it contains all generic points of the scheme $X_v$, but it does not contain any irreducible component of the scheme $X_v$. By the hypothesis of the theorem the quadratic space $(P', \varphi')$ is isotropic over $\mathcal{K}$. Thus, it is isotropic over the semi-local Dedekind ring $\mathcal{O}_{X',X_v}$, where the ring $\mathcal{O}_{X',X_v}$ is defined in Notation 7.4. Hence there is an $f \in \Gamma(X', \mathcal{O}_{X'})$ such that the space $(P', \varphi')|_{X_v}$ is isotropic and the closed subset $Y := \{f = 0\}$ of $X'$ does not contain any irreducible component of the scheme $X_v$.

Now consider the diagram from Corollary 3.3. By the item (5) of the Corollary the quadratic spaces $p_X^*(P', \varphi')$ and $p_Y^*(P, \varphi)$ are isomorphic. Thus, by Corollary 3.3 the pull-backs of the quadratic space $(P, \varphi)$ as to $D_1$, so to $D'_0$ are both isotropic.

The schemes $X$ and $A_{1|1}$ are both regular and irreducible. The morphism $\tau$ is finite surjective. By Corollary 18.17 the $\mathfrak{O}[t]$-module $\Gamma(X, \mathcal{O}_X)$ is finitely generated projective (of constant rank). By the statements (1) and (4) of the Corollary 3.3 one has an equality $[D_1 : U] = 1 + [D'_0 : U]$. Hence one of the degree’s $[D_1 : U]$ or $[D'_0 : U]$ is an odd number. By the statement (3) of the Corollary 3.3 as $D_1$, so $D'_0$ are both finite étale.
over $U$. Applying now [PPI] Theorem 1.1 we conclude that the quadratic space $(P, q)$ is isotropic.

Now consider the case when the residue field $k(v)$ of $A$ is finite. Let $l$ be an odd prime number different of $p$. Consider a tower of finite étale extensions

$$A \subset A_1 \subset A_2 \subset A_3 \ldots \subset A_{\infty}$$

such that each $A_m$ is a d.v.r., each extension $A_m \subset A_{m+1}$ is finite étale of degree $l$, for each point $x_i$ the $k(x_i)$-algebra $k(x_i) \otimes_A A_m$ is a field. Replace $A$ with $A_{\infty}$, $X$ with $X_\infty = X \otimes_A A_{\infty}$, points $x_1, x_2, \ldots, x_n \in X$ with points $x_{1, \infty}, x_{2, \infty}, \ldots, x_{n, \infty} \in X_\infty$, $\mathcal{O}$ with $\mathcal{O}_\infty = \mathcal{O} \otimes_A A_{\infty}$ and $U$ with $U_\infty = U \otimes_A A_{\infty}$. Write $\mathcal{K}_\infty$ for $\mathcal{K} \otimes_A A_{\infty}$. Write $(P, \varphi)_{\infty}$ for the quadratic space $(P, \varphi) \otimes_A A_{\infty}$ over $\mathcal{O}_\infty$. Since $X$ is $A$-smooth it follows that $X_\infty$ is $A_{\infty}$-smooth. Thus, $X_\infty$ and $U_\infty$ are regular schemes. Since $U$ is irreducible, $U_\infty$ is regular and $x_{1, \infty}, x_{2, \infty}, \ldots, x_{n, \infty}$ are all its closed points it follows that $U_\infty$ is irreducible. Thus, $\mathcal{K}_\infty$ is a field (it is the fraction field of $\mathcal{O}_\infty$).

By the first part of the proof the the quadratic space $(P, \varphi) \otimes_A A_{\infty}$ is isotropic over $\mathcal{O}_\infty$. Thus, there exists an integer $m > 0$ such that the quadratic space $(P, \varphi) \otimes_A A_m$ is isotropic over $\mathcal{O}_m = \mathcal{O} \otimes_A A_m$. The extension $\mathcal{O} \subset \mathcal{O}_m$ is a finite étale extension of odd degree. Applying now [Sc] Theorem 5.1 we conclude that the quadratic space $(P, q)$ is isotropic.

Reducing Theorem 7.1 to Theorems 5.1 and 7.1. By Theorem 5.1 there exist a quadratic space $(V, \varphi)$ over $\mathcal{O}$ and an integer $n \geq 0$ such that $(P, \varphi) \otimes_\mathcal{O} \mathcal{K} \cong (Q, \psi) \perp \mathbb{H}_K^1$, where $\mathbb{H}_K$ is a hyperbolic plane. If $n > 0$ then the space $(V, \varphi) \otimes_\mathcal{O} \mathcal{K}$ is isotropic. By the Theorem 7.1 the space $(V, \varphi)$ is isotropic too. Thus $(V, \varphi) \cong (V', \varphi') \perp \mathbb{H}_0$ for a quadratic space $(V', \varphi')$ over $\mathcal{O}$. Now Witt’s Cancellation theorem over a field [La, Chap.I, Thm.4.2] shows that $(V', \varphi') \otimes_\mathcal{O} \mathcal{K} \cong (W, \psi) \perp \mathbb{H}_K^{n-1}$. Repeating this procedure several times we may assume that $n = 0$, which means that $(V, \varphi) \otimes_\mathcal{O} \mathcal{K} \cong (W, \psi)$.

Reducing Corollary 7.3 to Theorem 7.1. Let $(\mathcal{O}, -u)$ be a the rank one quadratic space over $\mathcal{O}$ corresponding the unit $-u$. The space $(V, \varphi)_\mathcal{K} \perp (K, -u)$ is isotropic. Thus, the space $(V, \varphi) \perp (\mathcal{O}, -u)$ is isotropic by Theorem 7.1. By the Claim below there exists a vector $v \in V$ with $\varphi(v) = u$. Clearly $v$ is unimodular.

Claim. Let $(W, \psi) = (V, \varphi) \perp (R, -u)$. The space $(W, \psi)$ is isotropic if and only if there exists a vector $v \in V$ with $\varphi(v) = u$.

This Claim is proved in [CT] the proof of Proposition 1.2.

8 Grothendieck–Serre conjecture for $SL_{1,D}$

Let $A$, $p > 0$, $d \geq 1$, $X$, $x_1, x_2, \ldots, x_n \in X$, $\mathcal{O}$ and $U$ be as in Section 2. Write $\mathcal{K}$ for the fraction field of the ring $\mathcal{O}$. 

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Theorem 8.1 (Grothendieck–Serre conjecture for $SL_{1,D}$). Let $D$ be an Azumaya $\mathcal{O}$-algebra and $\text{Nrd}_D : D^\times \to \mathcal{O}^\times$ be the reduced norm homomorphism. Let $a \in \mathcal{O}^\times$. If $a$ is a reduced norm for the central simple $\mathcal{K}$-algebra $D \otimes_{\mathcal{O}} \mathcal{K}$, then $a$ is a reduced norm for the algebra $D$.

The following result is the ”constant” case of Theorem 8.1. Its proof is easier than the one of Theorem 8.1. This is why we decided to include it in the paper.

Theorem 8.2 (Grothendieck–Serre conjecture for $SL_{1,A}$). Let $A$ be an Azumaya $A$-algebra. Let $a \in \mathcal{O}^\times$. If $a$ is a reduced norm for the central simple $\mathcal{K}$-algebra $A \otimes_A \mathcal{K}$, then $a$ is a reduced norm for the Azumaya $\mathcal{O}$-algebra $A \otimes_A \mathcal{O}$.

Proposition 8.3. Let $A$ be an Azumaya algebra over $A$. Let $K_*$ be the Quillen $K$-functor. Then for each integer $n \geq 0$ one has an exact sequence

$$K_n(A \otimes_A \mathcal{O}) \to K_n(A \otimes_A \mathcal{K}) \xrightarrow{\partial} \bigoplus_{y \in U(1)} K_{n-1}(A \otimes_A k(y)).$$

Particularly, this sequence is exact for $n = 1$.

Proposition 8.4. Let $D$ be an Azumaya $\mathcal{O}$-algebra. Let $K_*$ be the Quillen $K$-functor. Then for each integer $n \geq 0$ the sequence is exact

$$K_n(D) \to K_n(D \otimes_{\mathcal{O}} \mathcal{K}) \xrightarrow{\partial} \bigoplus_{y \in U(1)} K_{n-1}(D \otimes_{\mathcal{O}} k(y)).$$

Particularly, it is exact for $n = 1$.

Remark 8.5. Proposition 8.3 is a particular case of the item (b) of Theorem 4.7. This is the case since the Thomason-Throughbor $K$-groups with coefficients in $A$ form a cohomology theory on the category $\text{SmOp}/V$ in the sense of [PSm]. Details are given below in this section.

However, the Thomason-Throughbor $K$-groups with coefficients in $D$ does not form a cohomology theory on the category $\text{SmOp}/V$ in the sense of [PSm] (indeed, the algebra $D$ does not come from $V$). This show that a proof of Proposition 8.4 requires a different method (see below in this section).

Reducing Theorem 8.2 to Proposition 8.3. Consider a commutative diagram of groups

$$\begin{align*}
K_1(A \otimes_A \mathcal{O}) & \xrightarrow{\eta^*} K_1(A \otimes_A \mathcal{K}) \xrightarrow{\partial} \bigoplus_{y \in U(1)} K_0(A \otimes_A k(y)) \\
\mathcal{O}^\times & \xrightarrow{\eta^*} \mathcal{K}^\times \xrightarrow{\partial} \bigoplus_{y \in U(1)} K_0(k(y))
\end{align*}$$

By Proposition 8.3 the complex on the top is exact. The bottom map $\eta^*$ is injective. The right hand side vertical map $\text{Nrd}$ is injective. Thus, the map

$$\mathcal{O}^\times/\text{Nrd}(K_1(A \otimes_A \mathcal{O})) \to \mathcal{K}^\times/\text{Nrd}(K_1(A \otimes_A \mathcal{K}))$$
is injective. The image of the left vertical map coincides with \(\Nrd((A \otimes A \emptyset)^x)\) and the image of the middle vertical map coincides with \(\Nrd((A \otimes A \emptyset)^x)\). Thus, the map

\[
\emptyset^x / \Nrd((A \otimes A \emptyset)^x) \to \emptyset^x / \Nrd((A \otimes A \emptyset)^x)
\]

is injective. The derivation of Theorem 8.2 from Proposition 8.3 is completed.

**Proof of Proposition 8.3.** There are two ways of proving this Proposition. To use the Thomason-Throughbor \(K\)-groups with coefficients in \(A\) or to use Quillen’s \(K'\)-groups with coefficients in \(A\). We prefer to use the Thomason-Throughbor \(K\)-groups with coefficients in \(A\). They form a cohomology theory on the category \(\mathcal{SmOp}/V\) in the sense of [PSm]. Namely, for each integer \(n\) and each pair \((X, X - Z)\) in \(\mathcal{SmOp}/V\) put \(E^n(X, X - Z) = K_{-n}(X_{on Z}; A)\) (see [TT], Theorem 5.1). Repeating literally arguments of [PSm, Example 2.1.8] we see that this way we get a cohomology theory on \(\mathcal{SmOp}/V\). Moreover, If \(X\) is quasi-projective then \(K(X_{on X}; A)\) coincides with the Quillens \(K\)-groups \(K^n_q(X)\) by [TT] Theorems 3.9 and 3.10. Now our proposition follows from the item (b) of Theorem 4.7.

**Reducing Theorem 8.7 to Proposition 8.4.** Consider the commutative diagram of groups

\[
\begin{array}{ccc}
K_1(D) & \xrightarrow{\eta^*} & K_1(D \otimes \emptyset X) \\
\Nrd & \downarrow & \Nrd \\
\emptyset^x & \xrightarrow{\eta^*} & \emptyset^x
\end{array}
\]

\[\begin{array}{ccc}
& & \oplus_{y \in U(1)} K_0(D \otimes \emptyset k(y)) \\
& \xrightarrow{\partial} & \oplus_{y \in U(0)} K_0(k(y)) \\
& & \Nrd
\end{array}
\]

By Proposition 8.4 the complex on the top is exact. Repeat now the arguments from reducing Theorem 8.2 to Proposition 8.3. This will complete reducing the theorem to Proposition 8.3.

**Proof of Proposition 8.4.** To prove this proposition it sufficient to prove vanishing of the support extension map \(\text{ext}_{2,1} : K^n_1(U; D)_{\geq 2} \to K^n_1(U; D)_{\geq 1}\). Prove that \(\text{ext}_{2,1} = 0\). Take an \(a \in K^n_1(U; D)_{\geq 2}\). We may assume that \(a \in K^n_1(U; D)\) for a closed \(Z\) in \(U\) with \(\text{codim}_U(Z) \geq 2\). Enlarging \(Z\) we may assume that each its irreducible component contains at least one of the point \(x_i\)'s and still \(\text{codim}_U(Z) \geq 2\). Our aim is to find a closed subset \(Z_{\text{ext}}\) in \(U\) containing \(Z\) such that the element \(a\) vanishes in \(K^n_1(Z_{\text{ext}}; D)\) and \(\text{codim}_U(Z_{\text{ext}}) \geq 1\). We will follow the method of [PS].

One can find closed points \(x_{n+1}, \ldots, x_M\) in \(X\) such that

(a) each irreducible component of \(X_v\) contains at least one of the point \(x_j\)'s \((j \in \{1, ..., M\})");
(b) for the ring \(\emptyset' = \emptyset_{X,x_1,\ldots,x_M}\) and the scheme \(U' = \text{Spec} \emptyset'\) the set \(Z\) is closed in \(U'\).

Let \(X^*\) be a neighborhood of the points \(x_j\)'s \((j \in \{1, ..., M\})\) (containing all generic points of the scheme \(X_v\), but not containing any irreducible component of the scheme \(X_v\)). Let \(Z^*\) be the closure of \(Z\) in \(X^*\).

By Theorem 5.1 one can find inside \(X^*\) an affine neighborhood \(X^0\) of points \(x_1, x_2, \ldots, x_M\), an open affine subscheme \(S \subseteq \mathbb{P}_A^{d-1}\) and a flat \(A\)-morphism

\[
q : X^0 \to S
\]
such that \( q \) is smooth in a neighborhood of \( q^{-1}(S_\circ) \) and \( Z^\circ/S \) is finite, where \( Z^\circ = Z^* \cap X^\circ \).

Now take back to the points \( x_1, ..., x_n \), the scheme \( U \) and the Azumaya \( \mathcal{O} \)-algebra \( D \). Put \( s_i = q(x_i) \). Consider the semi-local ring \( \mathcal{O}_{S,s_1, ..., s_n} \), put \( B = \text{Spec} \mathcal{O}_{S,s_1, ..., s_n} \) and \( X_B = q^{-1}(B) \subset X^\circ \). Put \( Z_B = Z^\circ \cap X_B \). Write \( q_B \) for \( q|_{X_B} : X_B \to B \). Note that \( Z_B \) is finite over \( B \). Since \( B \) is semi-local, hence so is \( Z_B \). Also, \( Z_B \) contains all the points \( x_1, ..., x_n \). Hence any neighborhood of all the point \( x_1, ..., x_n \) in \( X_B \) contains \( Z_B \). These show that \( Z_B = Z \) and \( U \) is in \( X_B \). Since now on write \( Z \) for \( Z_B \).

Replacing \( X_B \) with an appropriate neighborhood of \( U \) in \( X_B \) we may assume that the Azumaya algebra \( D \) is an Azumaya algebra over \( X_B \). Since \( Z \subset U \) and \( Z \) is finite over \( S \) it is closed also in the chosen neighborhood of \( U \). We will still write \( X_B \) for the chosen neighborhood. Recall that we are given with the element \( a \in K'_n(Z; D) \).

Now put \( \mathcal{Z} = Z \times_B X_B \). Let \( \Pi : \mathcal{Z} \to X_B \) be the projection to \( X_B \) and \( p_Z : \mathcal{Z} \to Z \) be the projection to \( Z \). The closed embedding \( \text{in} : Z \to X_B \) defines a section \( \Delta = (id \times in) : Z \to \mathcal{Z} \) of the projection \( p_Z \). Also one has \( \text{in} = \Pi \circ \Delta \). Put \( D_Z = \text{in}^*(D) \) and write \( ZD_X \) for the Azumaya algebra \( \Pi^*(D) \otimes p_Z^*(D_Z^p) \) over \( \mathcal{Z} \). The \( \mathcal{O}_\mathcal{Z} \)-module \( \Delta_*(D_Z) \) has an obvious left \( ZD_X \)-module structure. And it is equipped with an obvious epimorphism \( \pi : ZD_X \to \Delta_*(D_Z) \) of the left \( ZD_X \)-modules. Following [PS] one can see that \( I = \text{Ker}(\pi) \) is a left projective \( ZD_X \)-module. Hence the left \( ZD_X \)-module \( \Delta_*(D_Z) \) defines an element \( [\Delta_*(D_Z)] = [ZD_X] - [I] \) in \( K_0(\mathcal{Z}; ZD_X) \). This element has rank zero. Hence by \( \Pi \) it vanishes semi-locally on \( \mathcal{Z} \).

Since \( Z/B \) is finite the morphism \( \Pi \) is finite. Hence any neighborhood of \( Z \times_B Z \) in \( \mathcal{Z} \) contains a neighborhood of the form \( Z \times_B W \), where \( W \subset X_B \) is an open containing \( Z \). Write \( \mathcal{Z}_W \) for \( Z \times_B W \). The set \( Z \) is finite over \( B \) and semi-local. Hence \( Z \times_B Z \) is finite over the first coordinate and semi-local. Hence there is an open \( W \) in \( X_B \) containing \( Z \) such that

\[
0 = [\Delta_*(D_Z)]|_{\mathcal{Z}_W} \in K_0(\mathcal{Z}_W; ZD_X).
\]

To simplify notation we will write below in this proof \( X_B \) for this \( W \), \( \mathcal{Z} \) for this \( \mathcal{Z}_W \), \( \Pi \) for \( \Pi|_{\mathcal{Z}_W} : \mathcal{Z}_W \to W \) and \( q_Z : \mathcal{Z} \to Z \) for the projection \( \mathcal{Z}_W \to Z \).

Since \( \Pi \) is a finite morphism. Hence the image \( \Pi(\mathcal{Z}_{\text{red}}) \) is closed in \( X_B \). Write \( Z_{\text{new}} \) for \( \Pi(\mathcal{Z}_{\text{red}}) \) in \( X_B \). Clearly, we have closed inclusions \( Z \subset Z_{\text{new}} \subset X_B \). The nearest aim is to show that for the inclusion \( i : Z \to Z_{\text{new}} \) one has \( i_*(a) = 0 \) in \( K'_n(Z_{\text{new}}; D) \). This will allow to put \( Z_{\text{ext}} = Z_{\text{new}} \cap U \) and to show that \( a \) vanishes in \( K'_n(Z_{\text{ext}}; D) \). Recall that the morphism \( q_B : X_B \to B \) is flat. Hence so is the morphism \( p_Z \).

The functor \( (P, M) \mapsto P \otimes p_Z^*(D_Z) \) induces a bilinear pairing

\[
\cup_{q_Z^*(D_Z)} : K_0(\mathcal{Z}; ZD_X) \times K_0(\mathcal{Z}; q_Z^*(D_Z))^p \to K_0(\mathcal{Z}; q_X^*(D))
\]

Each element \( \alpha \in K_0(\mathcal{Z}; ZD_X) \) defines a group homomorphism

\[
\alpha_* = \Pi_* \circ (\alpha \cup_{q_Z^*(D_Z)} -) \circ q_Z^p : K'_n(Z; D) = K'_n(\mathcal{Z}; ZD_Z) \to K'_n(Z_{\text{new}}; D)
\]

which takes an element \( b \in K'_n(Z; D_Z) \) to the one \( b \mapsto \Pi_*(\alpha \cup_{q_Z^*(D_Z)} q_Z^p(b)) \) in \( K'_n(Z_{\text{new}}; D) \).

Following [PS] we see that the map \( [\Delta_*(D_Z)]_* \) coincides with the map \( i_* : K'_n(Z; D) = K'_n(Z; D_Z) \to K'_n(Z_{\text{new}}; D). \) The equality \( 0 = [\Delta_*(D_Z)] \in K_0(\mathcal{Z}; ZD_X) \) proven just above shows that the map \( i_* \) vanishes.
Since $Z$ is in $U \cap Z_{\text{new}} = Z_{\text{ext}}$, hence we have a closed inclusion $i_n : Z \hookrightarrow Z_{\text{ext}}$. The inclusion $U \xrightarrow{\text{can}} X_B$ is a flat morphism. Hence the inclusion $i_n : Z_{\text{ext}} = Z_{\text{new}} \cap U \rightarrow Z_{\text{new}}$ is also a flat morphism. Thus the map $i_{n}^{*} : K_{n}(Z_{\text{new}}; D) \rightarrow K_{n}'(Z_{\text{ext}}; D)$ is well-defined. Moreover $i_{n}^{*} \circ i_{*} = i_{*}$. The map $i_{*}$ vanishes, hence the map $i_{n}^{*}$ vanishes also. Particularly, $i_{n}(a) = 0$. The proposition is proved.

\section{Proofs of Theorems 4.7, 4.9, 4.11, 4.13}

Reducing Theorem 4.7 to Theorem 3.4 The nearest aim is to prove the assertions (c) and (d). So, we must prove the vanishing of the support extension map

$$\text{ext}_{2,1} : E^{m}_{\geq 2}(U) \rightarrow E^{m}_{\geq 1}(U).$$

Prove that $\text{ext}_{2,1} = 0$. Take an $a \in E^{m}_{\geq 2}(U)$. We may assume that $a \in E^{m}_{Z}(U)$ for a closed $Z$ in $U$ with $\text{codim}_{U}(Z) \geq 2$. One can find closed points $x_{n+1}, \ldots, x_{M}$ in $X$ such that (a) each irreducible component of $X_{v}$ contains at least one of the point $x_{j}$’s ($j \in \{1, \ldots, M\}$); (b) for the ring $\mathcal{O}^{n} = \mathcal{O}_{X,x_{1},\ldots,x_{M}}$ and the scheme $U^{n} = \text{Spec} \mathcal{O}^{n}$ the set $Z$ is closed in $U^{n}$. By the excision property for the open embedding $i_{n} : U \hookrightarrow U^{n}$ the pull-back

$$i_{n}^{*} : E^{m}_{Z^{n}}(U^{n}) \rightarrow E^{m}_{Z}(U)$$

is an isomorphism. Let $a^{n} \in E^{m}_{Z}(U^{n})$ be a unique element such that $i_{n}^{*}(a^{n}) = a$. For an appropriate neighborhood $X^{n}$ of the points $x_{j}$’s ($j \in \{1, \ldots, M\}$) (containing all generic points of the scheme $X_{v}$, but not containing any irreducible component of the scheme $X_{v}$) and the closure $Z^{n}$ of $Z$ in $X^{n}$ one can find an element

$$\tilde{a}^{n} \in E^{m}_{Z^{n}}(X^{n})$$

such that $\tilde{a}^{n}|_{U^{n}} = a^{n} \in E^{m}_{Z^{n}}(U^{n})$.

Applying Theorem 3.4 to the scheme $X$, the points $x_{1}, x_{2}, \ldots, x_{M} \in X$, the open $X^{n} \subseteq X$, the closed subset $Z^{n} \subseteq X^{n}$, the ring $\mathcal{O}^{n}$ and the scheme $U^{n} = \text{Spec} \mathcal{O}^{n}$, we get a closed subset $Z_{\text{new}}^{n}$ in $X^{n}$ containing $Z^{n}$ with $\text{codim}_{X^{n}}(Z_{\text{new}}^{n}) \geq 1$ and a morphism of pointed Nisnevich sheaves

$$\Phi_{t} : A^{1} \times X/(U - Z_{\text{new}}^{n}) \rightarrow X^{n}/(X^{n} - Z^{n})$$

in the category $\text{Shv}_{\text{his}}(Sm/V)$. Moreover,

$$\Phi_{0} = [U/(U - Z_{\text{new}}^{n}) \xrightarrow{\text{can}} X/(X - Z_{\text{new}}^{n}) \xrightarrow{\mathcal{P}} X^{n}/(X^{n} - Z^{n})],$$

and $\Phi_{1}$ takes $U/(U - Z_{\text{new}}^{n})$ to the distinguished point $* \in X^{n}/(X^{n} - Z^{n})$.

The morphism $\Phi_{t}$ induces a map $\Phi_{t}^{*} : E^{m}_{Z^{n}}(X^{n}) \rightarrow E^{m}_{A^{1} \times Z_{\text{new}}^{n}}(A^{1} \times U^{n})$ such that (a) $i_{1}^{*} \circ \Phi_{t}^{*} : E^{m}_{Z^{n}}(X^{n}) \rightarrow E^{m}_{Z_{\text{new}}^{n}}(U^{n})$ is the zero map and

(b) $i_{0}^{*} \circ \Phi_{t}^{*} : E^{m}_{Z^{n}}(X^{n}) \rightarrow E^{m}_{Z_{\text{new}}(U^{n})}$ is the composite map of pull-backs $E^{m}_{Z^{n}}(X^{n}) \rightarrow E^{m}_{Z^{n}}(U^{n}) \rightarrow E^{m}_{Z_{\text{new}}(U^{n})}$.
Since the pull-back map $E^m_{Z_{new}}(U'') \to E^m_{A^1 \times Z_{new}}(A^1 \times U'')$ is an isomorphism one has an equality $i_1^* \circ \Phi_1^* = i_0^* \circ \Phi_0^*$. Since $0 = i_1^* \circ \Phi_1^*$ the map
\[
E^m_{\partial}(X'') \to E^m_Z(U'') \overset{ext}{\to} E^m_{Z_{new} \cap U''}(U'')
\]
vanishes. Since $\partial^m|_{U''} = a'' \in E^m_Z(U'')$, hence $ext(a'') = 0$ in $E^m_{Z_{new} \cap U''}(U'')$. Recall that $Z$ is closed in $U''$. Consider a commutative diagram
\[
\begin{array}{ccc}
a'' \in E^m_Z(U'') & \overset{ext}{\to} & E^m_{Z_{new} \cap U''}(U'') \\
\downarrow \quad in^* \downarrow & & \downarrow \quad in^* \\
a \in E^m_Z(U) & \overset{ext}{\to} & E^m_{Z_{new} \cap U}(U),
\end{array}
\]
in which the horizontal arrows are the support extension maps and the vertical ones are the pullback maps (any support extension map is a pullback map). Since $ext(a'') = 0$ and $a = in^*(a'')$, hence $ext(a) = 0$. We proved that $ext_{2,1} = 0$.

Let $c \geq 1$ and $m \in \mathbb{Z}$. Repeating literally the arguments above we prove the vanishing of the support extension map $ext_{c+1,1} : E^m_{\geq c+1}(U) \to E^m_{\geq c}(U)$. This gives for each integer $m$ a short exact sequence $0 \to E^m_{\geq c}(U) \to E^m_{(c)}(U) \overset{\partial}{\to} E^m_{\geq c+1}(U) \to 0$.

Prove now the assertion (b). The vanishing of the map $ext_{2,1}$ yields the injectivity of the map pull-back map $E^m_{\geq 1}(U) \to \oplus_{y \in U(1)} E^m_{y+1}(U)$. Consider the composite map
\[
\partial : E^m(\eta) \overset{\partial_{2,1}}{\to} E^m_{\geq 1}(U) \to \oplus_{y \in U(1)} E^m_{y+1}(U).
\]
Since the second arrow in this composition is injective, hence $\ker(\partial) = \ker(\partial_{2,1})$. But $\ker(\partial_{2,1}) = Im[E^m(U) \to E^m(\eta)]$. Thus, the sequence
\[
E^m(U) \to E^m(\eta) \overset{\partial}{\to} \oplus_{y \in U(1)} E^m_{y+1}(U)
\]
is exact. The purity for $E^m$ is proved. The assertion (a) is a direct consequence of statements (b), (c) and (d). The theorem is proved.

**Reducing Theorem 4.11 to Theorem 3.4.** Let $m \in \mathbb{Z}$ be as in Theorem 4.11. The hypotheses of the theorem yield exactness of the complex $0 \to E^m(\eta) \to E^m(U) \overset{\partial}{\to} E^m_{\geq 1}(U) \to 0$. This observation together with the item (d) of Theorem 4.7 complete the proof of the theorem.

**Reducing Theorem 4.11 to Theorem 3.4.** The nearest aim is to prove the assertions (c) and (d). So, take integers $m$ and $c$ with $c \geq 1$ and prove the vanishing of the support extension map
\[
\text{ext}_{c+1,1} : E^m_{\geq c+1}(U) \to E^m_{\geq c}(U).
\]
Take an $a \in E^m_{\geq c+1}(U)$. We may assume that $a \in E^m_Z(U)$ for a closed $Z$ in $U$ with $\text{codim}_U(Z) \geq c + 1$. For each closed point $x \in X$ which is in the closure $\{z\}$ of $\{z\}$ in $X$ one has $U_z \subset U_x$. Write $Z$ for the closure of $Z$ in $X$. The continuity of the
Theorem. Hence there exist a closed point $x$ of $X$ containing $\bar{z}$ and an element $a_x \in \text{Ext}^m_{Z_{\new}}(U_x, U_x)$. By the item (c) of Theorem 4.7 there exists a closed subset $Z_{\new}$ in $U_x$ containing $\bar{z}$ and of codimension $\geq c$ in $U_x$. Thus, $a$ vanishes in $\text{Ext}^m_{Z_{\new}}(U_x, U_x)$. The assertion (c) is proved. Since the assertion (c) yields the assertion (d) the assertion (d) is proved too.

The assertion (b) is an easy consequence of vanishing of the support extension map $\text{Ext}_{2,1} : \text{Ext}^m_{U_x} \to \text{Ext}^m_{U_x}$. Just repeat the arguments from the proof of the item (b) of Theorem 4.11.

The assertion (a) is a direct consequence of statements (b), (c) and (d). The theorem is proved.

Reducing Theorem 4.13 to Theorem 3.4. Let $m \in \mathbb{Z}$ be as in Theorem 4.13. The hypotheses of the theorem yield exactness of the complex $0 \to \text{Ext}^m(U_x) \to \text{Ext}^m(\eta) \to \text{Ext}_{m+1}^m(U_x) \to 0$. This observation together with the item (d) of Theorem 4.11 complete the proof of the first part of the theorem.

Now prove the second part. Let $i$ be an integer as in the second part. It is sufficient to show that the map $E^i(U_x) \to E^i(\eta)$ is injective for $i = m$ and $i = m + 1$. Let $\{\bar{z}\}$ be the closure of $\{z\}$. Then for each closed point $x \in \{\bar{z}\}$ one has the embedding $U_x \subseteq U_x$. Let $a \in E^i(U_x)$ be an element vanishing in $E^i(\eta)$. By the continuity of the presheaf $E^i$ there exist a closed point $x \in \{\bar{z}\}$ and an element $a_x \in E^i(U_x)$ such that $a_x|U_x = a$. Since $a|\eta = 0$ it follows that $a_x|\eta = 0$. Thus $a_x = 0$ by the hypotheses of the second part of the theorem. Hence $a = 0$. The second part is proved.

10 Proof of Theorem 5.1

Proof of Theorem 5.1. It is known [OP2] that for each integer $m$ which is not divisible by 4 the Balmer–Witt groups $W^m(\mathcal{O})$ and $W^m(\mathcal{X})$ vanish. Particularly, for these integers $m$ the map $\eta^* : W^m(\mathcal{O}) \to W^m(\mathcal{X})$ is injective. Thus, by Theorem 4.9 the injectivity of the map $\eta^* : W(\mathcal{O}) \to W(\mathcal{X})$ yields the exactness of the Cousin complex $\text{Cous}(W, U, m)$ for each integer $m$. To prove the injectivity of the map $\eta^* : W(\mathcal{O}) \to W(\mathcal{X})$ we closely follow ideas of [OP2].

Let $a \in W(\mathcal{O})$ which vanishes in $W(\mathcal{X})$. The Mayer-Vietories property and the continuity of the presheaf $W|_{S_{m}/V}$ on $S_{m}/V$ allows to find appropriate closed points $x_{n+1}, \ldots, x_{N}$ in $X_{\prime}$, a Zariski neighborhood $X_{\prime}$ of all the points $x_{i}$'s ($i = 1, 2, \ldots, N$) as in the section 2 and an element $\bar{a} \in W(X_{\prime})$ as in the section 2 and an element $\bar{a} \in W(X_{\prime})$. Recall that $X_{\prime}$ contains all generic points of the scheme $X_{\nu}$, but does not contain any irreducible component of the scheme $X_{\nu}$.

By the hypothesis of the theorem the element $\bar{a}$ vanishes at the generic point $\eta$ of the scheme $X_{\prime}$. Let $O_{X_{\nu}, X_{\nu}}$ be the ring defined in Notation 3.4. It is a semi-local Dedekind domain. Thus, the homomorphism $W(O_{X_{\nu}, X_{\nu}}) \to W(\mathcal{X})$ is injective. Hence $\bar{a}$ vanishes in $W(O_{X_{\nu}, X_{\nu}})$. So, there is a non-zero $f \in \Gamma(X_{\nu}, O_{X_{\nu}})$ such that the closed subset $Y := \{f = 0\}$ of $X_{\nu}$ does not contain any irreducible component of the scheme $X_{\nu}$, and the element $\bar{a}|_{X_{\nu}}$ vanishes.
Applying Corollary 3.22 to the scheme $X$, the points $x_1, x_2, \ldots, x_N \in X$, the open $X' \subseteq X$ the closed subset $Y \subseteq X'$ we get an affine Zariski open neighborhood $U'$ of points $x_1, x_2, \ldots, x_N \in X'$ and a diagram of $A$-schemes and $A$-morphisms of the form

$$A^1_{U'} \xleftarrow{\tau} X \xrightarrow{p_X} X'$$

with an $A$-smooth irreducible scheme $X$ and a finite surjective morphism $\tau_{new}$ such that the canonical sheaf $\omega_X/V$ is isomorphic to the structure sheaf $\mathcal{O}_X$. Moreover, if we write $p : X \to U'$ for the composite map $pr_U \circ \tau$ and $Y$ for $p_X^{-1}(Y)$, then these data enjoy the following properties:

(0) the closed subset $Y$ of $X$ is quasi-finite over $U'$;
(1) there is a section $\Delta : U' \to X$ of the morphism $p$ such that $\tau \circ \Delta = i_0$ and $p_X \circ \Delta = can$, where $i_0$ is the zero section of $A^1_{U'}$ and $can : U' \to X'$ is the inclusion;
(2) the morphism $\tau$ is étale as over $\{0\} \times U'$, so over $\{1\} \times U'$;
(3) for $D_1 := \tau^{-1}(\{1\} \times U')$ and $Y := p_X^{-1}(Y)$ one has $D_1 \cap Y = \emptyset$;
(4) for $D_0 := \tau^{-1}(\{0\} \times U')$ one has $D_0 = \Delta(U') \cup D_0'$ and $D_0 \cap Y = \emptyset$;

Consider a category $\text{Aff}$ of affine $A^1_{U'}$-schemes and $A^1_{U'}$-morphisms. For a scheme $T \in \text{Aff}$ write $\mathcal{F}$ for $T \times A^1_{U'} X$. There are two interesting presheaves on $\text{Aff}$:

$$T \mapsto W(T)$$

Choose an isomorphism $l : \mathcal{O}_X \to \omega_X/V$ of the $\mathcal{O}_X$-modules. Similarly to [OP2, Section 12] this isomorphism defines a functor transformation (an Euler trace)

$$Tr^E : T \mapsto [Tr^E_{\mathcal{O}_T} : W(\mathcal{F}) \to W(T)],$$

which enjoy properties similar to the properties (1) to (6) as in [OP2, Section 12]. Set $\alpha = p_X^*(\tilde{a}) \in W(X)$. Then following [OP2, Section 3] one gets an equality in $W(U')$

$$Tr^E_{\mathcal{O}_{U'}}(\alpha|_{D_1}) = Tr^E_{\mathcal{O}_{U'}}(\alpha|_{D_0}) + u \cdot \Delta^*(\alpha)$$

for a unit $u \in \Gamma(U', \mathcal{O}^\times)$. Our choice of the element $f$ and the properties (3),(4) show that $\alpha|_{D_1} = 0$ and $\alpha|_{D_0} = 0$. Hence $\Delta^*(\alpha) = 0$. By the property (1) one has $p_X \circ \Delta = can$. Thus, $\tilde{a}|_{U'} = \Delta^*(\alpha) = 0$. Hence $0 = (\tilde{a}|_{U'})|_U = a$ in $W(U)$. The injectivity of the map $W(0) \to W(X)$ is proved. Theorem 5.1 is proved.

\section{Presheaves with transfers}

Let $A, p > 0, V, Sm'/V, Sch'/V$ be as in Section 2.

\textbf{Definition 11.1.} A presheaf with transfers on $Sch'/V$ is an additive presheaf $\mathcal{F}$ such that for each finite flat morphism $\pi : Y \to S$ in $Sch'/V$ there is given a homomorphism $Tr_{\mathcal{F}/\mathcal{S}} : \mathcal{F}(Y) \to \mathcal{F}(S)$. Moreover these homomorphisms enjoy the following properties

(i) base change: if $f : S' \to S$ is a $V$-morphism, then for $Y' = S' \times_S Y$ and $F = pr_Y :
Y' \to Y$ one has an equality $f^* \circ \text{Tr}_{Y/S} = \text{Tr}_{Y'/S'} \circ F^*: \mathcal{F}(Y) \to \mathcal{F}(S)$;

(ii) additivity: if $Y = Y' \sqcup Y''$, then $\text{Tr}_{Y/S} = \text{Tr}_{Y'/S} + \text{Tr}_{Y''/S}$;

(iii) normalization: if $\pi = \text{id}_S: S \to S$, then $\text{Tr}_{Y/S} = \text{id}$.

An assignment $[\pi: Y \to S] \mapsto [\text{Tr}_{Y/S}: \mathcal{F}(Y) \to \mathcal{F}(S)]$ subjecting conditions (1)–(3) is called often a trace structure on the presheaf $\mathcal{F}$. So, a presheaf with transfers on $\text{Sch}'/V$ is an additive presheaf $\mathcal{F}$ equipped with a trace structure.

An étale sheaf with transfers on $\text{Sch}'/V$ is a presheaf with transfers which is an étale sheaf on $(\text{Sch}'/V)_{\text{et}}$.

**Example 11.2.** Let $\mathcal{F}$ be an étale sheaf with transfers on $\text{Sch}'/V$ and let $m \geq 0$ be an integer. Let $\pi: Y \to S$ in $\text{Sch}'/V$ be a finite flat morphism. Let $S_{et}$ be the small étale site of $S$. Each $S' \in S_{et}$ is in $\text{Sch}'/V$ and each morphism $S'' \to S'$ in $S_{et}$ is a morphism in $\text{Sch}'/V$. And similar observation is true for the small étale site $Y_{et}$. Then on the small étale site $Y_{et}$ there are two étale sheaves: $\mathcal{F}$ and $\pi_*(\mathcal{F})$. Since $\mathcal{F}$ is an étale sheaf with transfers on $\text{Sch}'/V$ it follows there is given a sheaf morphism $\text{Tr}_{Y/S}: \pi_*(\mathcal{F}) \to \mathcal{F}$ on $S_{et}$. This sheaf morphism induces a group homomorphism

$$H^m_{et}(Y, \mathcal{F}) = H^m_{et}(S, \pi_*(\mathcal{F})) \xrightarrow{\text{Tr}_{Y/S}^m} H^m_{et}(S, \mathcal{F}).$$

Clearly, the additive presheaf $H^m_{et}(-, \mathcal{F})$ on $\text{Sch}'/V$ together with the homomorphisms $\text{Tr}_{Y/S}^m$ is a presheaf with transfers on $\text{Sch}'/V$. The trace structure $Y/S \mapsto \text{Tr}_{Y/S}^m$ on the presheaf $H^m_{et}(-, \mathcal{F})$ is called the induced trace structure.

The following two examples show that constant and locally constant étale sheaves on $\text{Sch}'/V$ are naturally sheaves with transfers. And therefore their cohomology presheaves are equipped with distinguished trace structures.

**Example 11.3.** Let $A$ be an abelian group and let $d$ be an integer. Write $m_d: A \to A$ for the multiplication by $d$. Let $\underline{A}$ be the constant étale sheaf on $(\text{Sch}'/V)_{et}$. We show now that the sheaf $\underline{A}$ is naturally a sheaf with transfers on $\text{Sch}'/V$. Let $\pi: Y \to S$ be a surjective finite flat morphism in $\text{Sch}'/V$. Suppose schemes $S$ and $Y$ are connected. Then $\pi_*(\mathcal{O}_Y)$ is a locally free $\mathcal{O}_S$ module. Since $S$ is connected this module has a constant rank, say $[Y:S]$. In this case put $\text{Tr}_{Y/S} = m_d_{[Y:S]}: A = \underline{A}(Y) \to \underline{A}(S) = A$.

Suppose $S$ is connected and $Y = \sqcup_{i \in I} Y_i$, where each $Y_i$ is connected. Then each $Y_i$ is surjective finite flat over $S$. Put $\text{Tr}_{Y/S} = \sum_{i \in I} \text{Tr}_{Y_i/S} : \oplus_{i \in I} \underline{A}(Y_i) \to \underline{A}(S)$. Finally, let $S = \sqcup_{j \in J} S_j$, where each $S_j$ is connected. Put $Y_j = \pi^{-1}(S_j)$. Since $\pi$ is surjective it follows that $Y_j$ is not empty. Put

$$\text{Tr}_{Y/S} = \oplus_{j \in J} \text{Tr}_{Y_j/S_j} : \oplus_{j \in J} \underline{A}(Y_j) \to \oplus_{j \in J} \underline{A}(S_j).$$

It is easy to check that the assignment $[\pi: Y \to S] \mapsto [\text{Tr}_{Y/S}: \underline{A}(Y) \to \underline{A}(S)]$ defines a trace structure on the constant sheaf $\underline{A}$. This trace structure is called the standard trace structure on $\underline{A}$. By Example [11.2] for each integer $m$ the additive presheaf $H^m_{et}(-, \underline{A})$ on $\text{Sch}'/V$ is equipped with the trace structure induced the standard trace structure on $\underline{A}$. It is called the standard trace structure on $H^m_{et}(-, \underline{A})$. 

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Example 11.4. Recall that an étale sheaf $\mathcal{F}$ on $\text{Sch}'/V$ is called locally constant if there exists an abelian group $A$ such that the sheaf $\mathcal{F}$ is isomorphic to the constant sheaf $A$ locally for the étale topology on $\text{Sch}'/V$.

We prove now that the sheaf $\mathcal{F}$ is naturally a sheaf with transfers on $\text{Sch}'/V$. For each $S \in \text{Sch}'/V$ and each point $x \in S$ denote $\mathcal{F}_x$ the stalk of $\mathcal{F}$ at the point $x$. That is $\mathcal{F}_x = \mathcal{F}(S_x^{sh})$, where $S_x^{sh}$ is the strict henselization of $S$ at the point $x$. First note that for any surjective finite flat morphism $\pi : Y \to S$ and each point $x \in S$ and each point $y \in Y$ over $x$ the pull-back map $\mathcal{F}_x \to \mathcal{F}_y$ is an isomorphism. This yields that $\pi_* (\mathcal{F})_x = \oplus_{y/x} \mathcal{F}_y = \oplus_{y/x} \mathcal{F}_x$. Put $Tr_{\pi/x} = \sum_{y/x} m_{[Y_x^{sh}, S_x^{sh}]} : \oplus_{y/x} \mathcal{F}_x \to \mathcal{F}_x$.

The following claim we left to the reader: there exists a unique trace structure $\mathcal{F}$ such that for each $S \in \text{Sch}'/V$ and each point $x \in S$ one has an equality
\[(Tr_{\pi/Y})_x = Tr_{\pi/x} : \pi_* (\mathcal{F})_x \to \mathcal{F}_x.\]

This trace structure is called the standard trace structure on $\mathcal{F}$.

By Example 11.2 for each integer $m$ the additive presheaf $H^m_{et}(-, \mathcal{F})$ on $\text{Sch}'/V$ is equipped with the trace structure induced by the standard trace structure on $\mathcal{F}$. It is called the standard trace structure on $H^m_{et}(-, \mathcal{F})$.

Example 11.5. Let $m$ be an integer. Clearly, the Thomason-Throughbor $K_m$-groups is a presheaf with transfers on $\text{Sch}'/V$. The Thomason-Throughbor $K_m$-groups with finite coefficients $\mathbb{Z}/r$ is also a presheaf with transfers on $\text{Sch}'/V$. If $A$ is an Azumaya algebra over $A$, then the Thomason-Throughbor $K_m$-groups with coefficients in $A$ is also a presheaf with transfers on $\text{Sch}'/V$.

Definition 11.6. Let $\mathcal{F}$ be a presheaf of abelian groups on $\text{Sch}'/V$. One says that $\mathcal{F}|_{\text{Sm}'/V}$ satisfies dim 1 property, if for each data $d \geq 1$, $X$, $X'$, $\eta$ as in Section 2 and for $S = S'$ as in Notation 2.4 the map $\eta^* : \mathcal{F}(S) = \mathcal{F}(S') \to \mathcal{F}(\eta)$ is injective.

Definition 11.7. Let $\mathcal{F}$ be a presheaf on $\text{Sch}'/V$. One says that $\mathcal{F}|_{\text{Sm}'/V}$ is of cohomological type
a) if for each scheme $S \in \text{Sm}'/V$ the map $p_S^\#: \mathcal{F}(S) \to \mathcal{F}(\mathbb{A}^1_S)$ is an isomorphism;
b) if for each $S \in \text{Sm}'/V$ and each Zariski open cover $S_1 \cup S_2 = S$ the standard sequence is exact
\[\mathcal{F}(S) \to \mathcal{F}(S_1) \oplus \mathcal{F}(S_2) \to \mathcal{F}(S_1 \cap S_2)\]

Definition 11.8. (Continuity property) One says that a presheaf $G$ on $\text{Sch}'/V$ is continuous if for each $S \in \text{Sch}'/V$, each left filtering system $(S_j)_{j \in J}$ as in section 2 and each $V$-scheme isomorphism $S \to \lim_{j \in J} S_j$ the map
\[\text{colim}_{j \in J} G(S_j) \to G(S)\]
is an isomorphism. Clearly, if a presheaf $G$ on $\text{Sch}'/V$ is continuous, then $G|_{\text{Sm}'/V}$ is continuous.
Theorem 11.9. Let $\mathcal{F}$ be an additive continuous presheaf with transfers on $Sch'/V$. Suppose $\mathcal{F}|_{Sm'/V}$ satisfies the dim 1 property and is of cohomological type. Then for each data $d \geq 1$, $X, x_1, x_2, \ldots, x_n \in X$, $\emptyset, U, \eta$ as in Section 2 the map $\eta^* : \mathcal{F}(U) \to \mathcal{F}(\eta)$ is injective.

Proof of Theorem 11.9. Let $a \in \mathcal{F}(U)$ which vanishes in $\mathcal{F}(\eta)$. The Mayer-Vietories property and the continuity of the presheaf $\mathcal{F}|_{Sm'/V}$ on $Sm'/V$ allows to find appropriate closed points $x_{n+1}, \ldots, x_N$ in $X$, a Zariski neighborhood $X'$ of all the points $x_i$'s $(i = 1, 2, \ldots, N)$ as in the section 2 and an element $\tilde{a} \in \mathcal{F}(X')$ such that $a = \tilde{a}|_U$. Recall that $X'$ contains all generic points of the scheme $X_\eta$, but does not contain any irreducible component of the scheme $X_\eta$.

By the hypothesis of the theorem the element $\tilde{a}$ vanishes at the generic point $\eta$ of the scheme $X'$. Since $\mathcal{F}$ is continuous and $\mathcal{F}|_{Sm'/V}$ satisfies the dim 1 property it follows there is a non-zero $f \in \Gamma(X', O_{X'})$ such that the closed subset $Y := \{ f = 0 \}$ of $X'$ does not contain any irreducible component of the scheme $X'_\eta$ and the element $\tilde{a}|_{X'_\eta}$ vanishes.

Applying Corollary 3.2 to the scheme $X$, the points $x_1, x_2, \ldots, x_N \in X$, the open $X' \subseteq X$ and the closed subset $Y \subseteq X'$, we get an affine Zariski open neighborhood $U'$ of points $x_1, x_2, \ldots, x_N \in X'$ and a diagram of $A$-schemes and $A$-morphisms of the form

$$A_{U'} \leftarrow X \xrightarrow{p} X'$$

with an $A$-smooth irreducible schemes $X$ and a finite surjective morphism $\tau$. Moreover, if we write $p : X \to U'$ for the composite map $pr_{U'} \circ \tau$, then these data enjoy the following properties:

1. there is a section $\Delta : U' \to X$ of the morphism $p$ such that $\tau \circ \Delta = i_0$ and $p_X \circ \Delta = can$, where $i_0$ is the zero section of $A_{U'}^1$, and $can : U' \leftarrow X'$ is the canonical inclusion;
2. the morphism $\tau$ is étale as over $\{ 0 \} \times U'$, so over $\{ 1 \} \times U'$;
3. for $D_1 := \tau^{-1}(\{ 1 \} \times U')$ and $Y := p_X^{-1}(Y)$ one has $D_1 \cap Y = \emptyset$;
4. for $D_0 := \tau^{-1}(\{ 0 \} \times U')$ one has $D_0 = \Delta(U') \cup D_0'$ and $D_0' \cap Y = \emptyset$;

Consider a category $Aff$ of affine $A_{U'}$-schemes and $A_{U'}$-morphisms. For a scheme $T \in Aff$ write $\mathcal{F}$ for $T \times A_{U'}^1, X$. There are two interesting presheaves on Aff:

$$T \mapsto \mathcal{F}(T) \text{ and } T \mapsto \mathcal{F}(T).$$

The trace structure on the presheaf $\mathcal{F}$ defines a functor transformation

$$Tr : T \mapsto [Tr_{T/T} : \mathcal{F}(T) \to \mathcal{F}(T)],$$

which enjoys properties (ii) and (iii) as in Definition 11.1. Set $\alpha = p_X^*(\tilde{a}) \in \mathcal{F}(X)$. Using now the properties (2) and (3) and the homotopy invariance of the presheaf $\mathcal{F}|_{Sm'/V}$ one gets an equality in $\mathcal{F}(U')$

$$Tr_{D_1/U'}(\alpha|_{D_1}) = Tr_{D_0/U'}(\alpha|_{D_0}) + \Delta^*(\alpha).$$

Our choice of the element $f$ and the properties (3),(4) show that $\alpha|_{D_1} = 0$ and $\alpha|_{D_0} = 0$. Hence $\Delta^*(\alpha) = 0$. By the property (1) one has $p_X \circ \Delta = can$. Thus, $\tilde{a}|_U = \Delta^*(\alpha) = 0$. Hence $0 = (\tilde{a}|_{U'})|_U = \tilde{a}|_U = a$ in $\mathcal{F}(U')$. The injectivity of the map $\eta^* : \mathcal{F}(U) \to \mathcal{F}(\eta)$ is proved. Theorem 11.9 is proved. \qed
12 Proof of Theorems 5.5, 5.4, 5.3 and 5.6.

Here are some examples of presheaves on $\text{Sch}'/V$ which are continuous.

Example 12.1. For each integer $m$ the presheaves $K^m(-)$ and $K^m(-, \mathbb{Z}/r)$ on $\text{Sch}'/V$ satisfy the continuity property as in Definition 11.8.

Example 12.2. Let $\mathcal{F}$ be a locally constant $r$-torsion sheaf on the big étale site $(\text{Sch}'/V)_{\text{et}}$. Then for each integer $m$ the presheaf $H^m_{\text{et}}(-, \mathcal{F})$ on $\text{Sch}'/V$ satisfies the the continuity property.

Lemma 12.3. Let $S$ be semi-local irreducible Dedekind scheme $s_1, ..., s_l \in S$ be all its closed points. Suppose that all the residue fields $k(s_j)$ have characteristic $p$ and the general point $\eta := S - \{s_1, ..., s_l\}$ has characteristic zero. Let $r$ be coprime to $p$. Then for each integer $m$ the map $K_m(S, \mathbb{Z}/r) \to K_m(\eta, \mathbb{Z}/r)$ is injective.

Proof. (of the lemma) Let $i_j : w_j \hookrightarrow W$ is the closed embedding. It is sufficient to check that for each $j = 1, ..., l$ the direct image map $(i_j)_* : K_m(w_j, \mathbb{Z}/r) \to K_m(W, \mathbb{Z}/r)$ is zero. We may assume that $j = 1$ and write $w$ for $w_1$ and $i$ for $i_1$. So, it is sufficient to check that the map $i_* : K_m(w, \mathbb{Z}/r) \to K_m(W, \mathbb{Z}/r)$ is zero. Let $a \in K_m(w, \mathbb{Z}/r)$. We must check that $i_*(a) = 0$ in $K_m(W, \mathbb{Z}/r)$. By the Suslin rigidity theorem there is an étale neighbourhood $(W \leftarrow^{\tilde{W}^o} W^o, s : w \to W^o)$ of the point $w$ and an element $a^o \in K_m(W^o, \mathbb{Z}/r)$ such that $s^*(a^o) = a$. Write $\Pi^o$ as $\Pi \circ in$, where $in : W^o \to W$ is an open embedding, $\Pi : W \to W$ is a finite morphism and $W$ is a semi-local irreducible Dedekind scheme. Let $\tilde{\eta} \in \tilde{W}$ be the general point of $\tilde{W}$. Then $\tilde{\eta} = \tilde{W} - \Pi^{-1}(w)$, where $w = \{w_1, ..., w_l\}$. Clearly, $\tilde{\eta}$ is in $W^o$. Put $\tilde{a} := a^o|_{\tilde{\eta}}$. Let $\partial : K_{m+1}(\tilde{\eta}, \mathbb{Z}/r) \to K_m(\Pi^{-1}(w); \mathbb{Z}/r)$ be the boundary map.

Choose a function $f \in \Gamma(\tilde{W} - \{s(w)\}, \mathcal{O}^\infty)$ such that $f|_{\Pi^{-1}(w) - s(w)} \equiv 1$ and $\text{div}(s(w))(f) = 1$. In this case $\partial_{s(w)}(f \cup \tilde{a}) = a \in K_m(s(w), \mathbb{Z}/r)$ and for each $\tilde{w} \in \Pi^{-1}(w) - s(w)$ one has $\partial_{\tilde{w}}(f \cap \tilde{a}) = 0 \in K_m(\tilde{w}, \mathbb{Z}/r)$.

Let $\tilde{i} : \tilde{w} \hookrightarrow \tilde{W}$ and $\tilde{i} : \Pi^{-1}(w) \hookrightarrow \tilde{W}$ be the closed embeddings and $\pi = \Pi|_{\Pi^{-1}(w)} : \Pi^{-1}(w) \to w$. The above computation with the element $f \cup \tilde{a} \in K_{m+1}(\tilde{\eta}; \mathbb{Z}/r)$ shows that $(\tilde{i} \circ \pi)_*(\partial(\tilde{a})) = i_*(a)$. Since $\Pi \circ \tilde{i} = \tilde{i} \circ \pi : \Pi^{-1}(w) \to W$ and $\tilde{i} \circ \partial = 0$ it follows that $0 = i_*(a)$. This proves the lemma.

Proof of Theorem 5.3 Let $m$ be an integer. By Examples 11.5 and 12.1 the presheaf $K^m(-; \mathbb{Z}/r)$ is an additive continuous presheaf with transfers on $\text{Sch}'/V$. By Lemma 12.3 the presheaf $K^m(-; \mathbb{Z}/r)|_{\text{Sm}'/V}$ on $\text{Sm}'/V$ satisfies the dim 1 property. The Thomason-Adams-Throughbor $K$-theory $(K^*(-; \mathbb{Z}/r), \partial)$ with finite coefficients $\mathbb{Z}/r$ is a cohomology theory on $\text{Sm}'\text{Op}/V$ as explained just above Theorem 5.5. Thus, the presheaf $K^m(-; \mathbb{Z}/r)|_{\text{Sm}'/V}$ is of cohomological type in the sense of Definition 11.7. Thus, by Theorem 11.9 the map $\eta^* \colon K^m(U; \mathbb{Z}/r) \to K^m(\eta; \mathbb{Z}/r)$ is injective. Now Theorem 4.9 completes the proof.

Lemma 12.4. Let $\mathcal{F}$ be a locally constant $r$-torsion sheaf on the big étale site $\text{Ét}/V$. Let $S, s_1, ..., s_l \in S, \eta$ be such as in Lemma 12.3. Then for each integer $m$ the map $H^m_{\text{et}}(S, \mathcal{F}) \to H^m_{\text{et}}(\eta, \mathcal{F})$ is injective.
Proof of Theorem 5.4. Let $m$ be an integer. By Examples 11.4 and 12.2 the presheaf $H^0_m(-, \mathcal{F})$ is an additive continuous presheaf with transfers on $\text{Sch}'/V$. By Lemma 12.3 the presheaf $H^0_m(-, \mathcal{F})|_{\text{Sm}'/V}$ on $\text{Sm}'/V$ satisfies the dim 1 property. The étale cohomology theory $(H^0_m(-, \mathcal{F}), \partial)$ on $\text{Sm}'/V$ is a cohomology theory on $\text{Sm}'/V$. Thus, the presheaf $H^0_m(-, \mathcal{F})$ is of cohomological type in the sense of Definition 11.7. Thus, by Theorem 11.9 the map $\eta^*: H^m_{\text{et}}(U, \mathcal{F}) \to H^m_{\text{et}}(\eta, \mathcal{F})$ is injective. Theorem 4.9 completes the proof.

Proof of Theorem 5.3. This theorem is just a particular case of Theorem 5.4.

Lemma 12.5. Let $W$ be semi-local irreducible Dedekind scheme. Let $\eta \in W$ be its general point. Then the maps $K_2(W) \to K_2(\eta)$ is injective.

Proof. Let $w_1, \ldots, w_l \in W$ are all its closed points and $\underline{w} = \{w_1, \ldots, w_l\}$ be the closed subscheme of $W$ and $i: \underline{w} \hookrightarrow W$ be the closed embedding. It is sufficient to prove that the boundary map $\partial: K_3(\eta) \to K_2(\underline{w})$ is surjective.

Let $a \in K_2(\underline{w})$ be an element. It is sufficient to find an element $\alpha \in K_3(\eta)$ such that $\partial(\alpha) = a$. Recall that by Matzumoto theorem the pull-back map $i^*: K_2(W) \to K_2(\underline{w})$ is surjective. Thus, there exists an $\tilde{a} \in K_2(W)$ with $i^*(\tilde{a}) = a$. There is a rational function $f$ on $W$ such that for each point $w \in \underline{w}$ one has $\text{div}_w(f) = -1$. Take $\tilde{a} \cup (f) \in K_3(\eta)$ as $\alpha$. Then $\partial(\alpha) = a$.

Proof of Theorem 5.6. Let $m$ be an integer. By Example 11.5 and 12.1 the presheaf $K^{\leq 2}(-)$ is an additive continuous presheaf with transfers on $\text{Sch}'/V$. By Lemma 12.3 the presheaf $K^{\leq 2}(-)|_{\text{Sm}'/V}$ on $\text{Sm}'/V$ satisfies the dim 1 property. The Thomason-Thruholdor $K$-theory $(K^*(-), \partial)$ is a cohomology theory on $\text{Sm}'/V$ as explained just above Theorem 5.3. Thus, the presheaf $K^{\leq 2}(-)$ is of cohomological type in the sense of Definition 11.7. Thus, by Theorem 11.9 the map $\eta^*: K_2(U) \to K_2(\eta)$ is injective. The map $\eta^*: K_1(U) \to K_1(\eta)$ is injective also. Now Theorem 4.9 proves exactness of the complex (9).

It remains to prove the exactness of the complex (10). To do this recall that for each scheme $S \in \text{Sch}'/V$ and each integer $n$ there is the coefficient short exact sequence $0 \to K_n(S)/r \to K_n(S, \mathbb{Z}/r) \to rK_{n-1}(S) \to 0$. For an integer $n$ let $\mathfrak{S}_n$ be the complex (8) with $m = n$. Consider a complex

$$0 \to rK_1(U) \cong rK_1(\eta) \to 0 \to 0 \to 0 \quad (17)$$

and the coefficient short exact sequence $0 \to (\mathfrak{S})_2 \to (\mathfrak{S})_2 \to (\mathfrak{S})_2 \to 0$ of complexes. By Theorem 5.5 the complex $\mathfrak{S}_2$ is exact. Clearly, the complex $\mathfrak{S}_2$ is exact. Thus, the complex (10) is also exact. The theorem is proved.

13 Suslin’s exact sequence in mixed characteristic

Our next aim is to get a version of Suslin’s short exact sequence in mixed characteristic. Let $p > 0$, $d \geq 1$, $X, x_1, x_2, \ldots, x_n \in X$, $0$, $U$, $\mathcal{K}$ be as in Section 2. Let $r$ be an integer coprime to $p$. 

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Proof of Theorem 6.1. Consider a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & K_2(X) & \xrightarrow{\partial} & \oplus_{x \in X^{(1)}} H_1(k(x)) & \xrightarrow{\alpha} & 0 \\
& & \uparrow & & \uparrow & & \\
0 & \rightarrow & K_2(X) & \xrightarrow{\partial} & \oplus_{x \in X^{(1)}} K_1(k(x)) & \xrightarrow{\partial} & 0 \\
& & \times \gamma & & & & \\
0 & \rightarrow & K_2(X) & \xrightarrow{\partial} & \oplus_{x \in X^{(1)}} K_1(k(x)) & \xrightarrow{\partial} & 0 \\
& & \alpha & & \alpha & & \\
0 & \rightarrow & K_2(X)/r & \xrightarrow{\partial} & \oplus_{x \in X^{(1)}} K_1(k(x))/r & \xrightarrow{\partial} & 0 \quad (18)
\end{array}
\]

where \( \alpha \) are the reduction modulo \( r \) homomorphisms. Theorem 5.6 shows that the second, the third and the forth horizontal complexes compute Zariski cohomology \( H^*_{Zar}(X, \mathcal{K}_\alpha) \), \( H^*_{Zar}(X, \mathcal{K}_\beta) \) and \( H^*_{Zar}(X, \mathcal{K}_\gamma)/r \) respectively. The morphism \( \alpha \) between the third and the forth row of the diagram induces a homomorphism \( \alpha : H^1_{Zar}(X, \mathcal{K}_\gamma)/r \rightarrow H^1_{Zar}(X, \mathcal{K}_\beta)/r \).

Define a morphism \( \beta : H^1_{Zar}(X, \mathcal{K}_\beta)/r \rightarrow rH^2_{Zar}(X, \mathcal{K}_\beta) \) as follows: for an element \( a \in \oplus_{x \in X^{(1)}} K_1(k(x))/r \) with \( \partial(a) = 0 \) choose an element \( \tilde{a} \in \oplus_{x \in X^{(1)}} K_1(k(x)) \) such that \( \alpha(\tilde{a}) = a \). Then \( \alpha(\partial(\tilde{a})) = 0 \). Thus there exists a unique element \( b \in \oplus_{y \in X^{(2)}} K_0(k(y)) \).

Let \( \tilde{b} \) be the class of \( b \) modulo \( Im(\partial) \). Put \( \beta(a) = \tilde{b} \). Clearly, \( \tilde{b} \) is in \( rH^2_{Zar}(X, \mathcal{K}_\beta) \). Also, \( \beta \circ \alpha = 0 \). Thus, we get a complex of the form

\[
0 \rightarrow H^1_{Zar}(X, \mathcal{K}_\alpha)/r \xrightarrow{\alpha} H^1_{Zar}(X, \mathcal{K}_\beta)/r \xrightarrow{\beta} rH^2_{Zar}(X, \mathcal{K}_\beta) \rightarrow 0, \quad (19)
\]

Lemma 13.1. The sequence (19) is short exact.

Proof. Use a diagram chase in the diagram (18). □

Recall that \( NH^3_{et}(X, \mu^\otimes_n) = ker[H^3_{et}(X, \mu^\otimes_n) \xrightarrow{\eta^*} H^3_{et}(\eta, \mu^\otimes_n)] \). The nearest aim is to identify \( H^1_{Zar}(X, \mathcal{K}_\beta)/n \) with \( NH^3_{et}(X, \mu^\otimes_n) \). Recall that the norm residue homomorphisms identify the forth row of the diagram (18) with the complex

\[
0 \rightarrow H^2(X, \mu^\otimes_n) \xrightarrow{\partial} \oplus_{x \in X^{(1)}} H^1(k(x), \mu_n) \xrightarrow{\partial} \oplus_{y \in X^{(2)}} \mathbb{Z}/n \rightarrow 0 \quad (20)
\]

By Theorem 5.3 the first cohomology of the complex (20) equals to \( H^1_{Zar}(X, \mathcal{K}^2) \). Thus, \( H^1_{Zar}(X, \mathcal{K}_\beta)/n = H^1_{Zar}(X, \mathcal{K}^2) \). To identify \( H^2_{Zar}(X, \mathcal{K}_{\beta}/n) \) with \( NH^3_{et}(X, \mu^\otimes_n) \) it is remained to check that \( H^1_{Zar}(X, \mathcal{K}^2) = NH^3_{et}(X, \mu^\otimes_n) \).

Theorem 5.3 yields a spectral sequence of the form \( H^p_{Zar}(X, \mathcal{K}^q) \Rightarrow H^{p+q}_{et}(X, \mu^\otimes_n) \). The latter means that the group \( H^3_{et}(X, \mu^\otimes_n) \) relates to the groups \( H^0_{Zar}(X, \mathcal{K}^1) \), \( H^1_{Zar}(X, \mathcal{K}^2) \), \( H^2_{Zar}(X, \mathcal{K}^3) \) and \( H^3_{Zar}(X, \mathcal{K}^4) \). By Theorem 5.3 the latter two groups vanish. Thus, there is a short exact sequence \( 0 \rightarrow H^1_{Zar}(X, \mathcal{K}^2) \rightarrow H^3_{et}(X, \mu^\otimes_n) \rightarrow H^0_{Zar}(X, \mathcal{K}^3) \). By
Theorem 5.3 the map $H^0_{\text{Zar}}(X, \mathcal{F}^3) \xrightarrow{\varphi} H^0_{\text{Zar}}(\eta, \mathcal{F}^3) = H^3_{et}(\eta, \mu_n^{\otimes 2})$ is injective. Thus, the sequence $0 \to H^1_{\text{Zar}}(X, \mathcal{F}^2) \to H^3_{et}(X, \mu_n^{\otimes 2}) \xrightarrow{\psi} H^3_{et}(\eta, \mu_n^{\otimes 2})$ is short exact. This proves the equality $H^1_{\text{Zar}}(X, \mathcal{F}^2) = NH^3_{et}(X, \mu_n^{\otimes 2})$. Joining the latter equality with Lemma 13.1 we get the following mixed characteristic version of the Suslin exact sequence The following sequence is short exact

$$0 \to H^1_{\text{Zar}}(X, K_2) / n \to NH^3_{et}(X, \mu_n^{\otimes 2}) \xrightarrow{\beta} H^2_{Zar}(X) \to 0,$$

Theorem 6.1 is proved.

Let $A$ be a henselian d.v.r. Suppose the closed point $v$ of $V = \text{Spec}(A)$ is such that its residue field is finite. Let $l$ be a prime different of $p$. For an abelian group $B$ write $\varpi_1 B$ for the $l$-primary torsion subgroup of $B$.

**Remark 13.2.** Since $X$ is smooth projective over $V$ and irreducible and $A$ is henselian it follows that the closed fibre $X_v$ is irreducible too. To prove this one has to use the Steiner decomposition of the morphism $p : X \to V$ as $X \xrightarrow{\varphi} W \xrightarrow{\psi} V$, where $q$ is surjective with connected fibres and $\pi$ is finite surjective. Since $q$ is projective and surjective and $X$ is irreducible it follows that $W$ is irreducible. Since $V$ is local henselian and $\pi$ is finite it follows that $W$ has a unique closed point, say $w$. Since the fibres of $q$ are connected it follows that $X_v = p^{-1}(v) = q^{-1}(w)$ is connected. Since $X_v$ is $v$-smooth, hence $X_v$ is irreducible.

**Proof of Theorem 6.2.** Consider a commutative diagram of the form

$$
\begin{array}{lllllllll}
0 & \to & H^1_{\text{Zar}}(X, K_2) & \otimes & \mathbb{Q}_l / \mathbb{Z}_l & \xrightarrow{\alpha} & NH^3_{et}(X, \mathbb{Q}_l / \mathbb{Z}_l(2)) & \xrightarrow{\beta} & H^2_{Zar}(X, K_2) & \to 0 \\
\varphi & & & & & & \psi & & \rho \\
0 & \to & H^1_{\text{Zar}}(X_v, K_2) & \otimes & \mathbb{Q}_l / \mathbb{Z}_l & \xrightarrow{\alpha_v} & NH^3_{et}(X_v, \mathbb{Q}_l / \mathbb{Z}_l(2)) & \xrightarrow{\beta_v} & H^2_{Zar}(X_v, K_2) & \to 0
\end{array}
$$

The upper sequence is exact by Theorem 6.1. The bottom sequence is the original Suslin exact sequence. It is known [Pal Theorem 2.11] that $H^1_{\text{Zar}}(X_v, K_2) \otimes \mathbb{Q}_l / \mathbb{Z}_l = 0$ and $\beta_v$ is an isomorphism. The map $\psi$ is injective by the proper base change theorem. Thus, $H^1_{\text{Zar}}(X, K_2) \otimes \mathbb{Q}_l / \mathbb{Z}_l = 0$ and the map $\beta$ is an isomorphism. These yield the injectivity of $\rho$. One can show that the map $\rho$ is surjective. Hence $\rho$ is an isomorphism. So, the map $\psi$ is an isomorphism too. The assertion (d) will be proved below in the proof of Theorem 6.5. The theorem is proved.

**Proof of Corollary 6.4.** Since $A$ is algebraically closed and $d = 2$ it follows that for each integers $s$ and $n$ one has $H^3(A(X_v), \mu_n^{\otimes s}) = 0$. Thus, $NH^3_{et}(X_v, \mathbb{Q}_l / \mathbb{Z}_l(2)) = H^3_{et}(X_v, \mathbb{Q}_l / \mathbb{Z}_l(2))$. The proper base change theorem shows that

$$H^3_{et}(X, \mathbb{Q}_l / \mathbb{Z}_l(2)) = H^3_{et}(X_v, \mathbb{Q}_l / \mathbb{Z}_l(2)).$$

The item (c) of Theorem 6.2 shows now that $NH^3_{et}(X, \mathbb{Q}_l / \mathbb{Z}_l(2)) = H^3_{et}(X, \mathbb{Q}_l / \mathbb{Z}_l(2))$. Hence the map $\beta$ gives rise to an isomorphism $\beta : H^3_{et}(X, \mathbb{Q}_l / \mathbb{Z}_l(2)) \to \{\varpi_1 B \}$.
The second assertion of the corollary is proved. To prove the first one consider the commutative diagram

$$
\begin{array}{ccc}
\{l\}H^2_{zar}(X, K_2) & \xrightarrow{\beta} & \{l\}Alb(X/V) \\
\rho \downarrow & & \downarrow \\
\{l\}H^2_{zar}(X_v, K_2) & \xrightarrow{\beta_v} & \{l\}Alb(X_v).
\end{array}
$$

(23)

The map $\rho$ is an isomorphism by Theorem 6.2. The map $\epsilon$ is an isomorphism since $A$ is henzelian and $l$ is a prime different of $p$. The map $\beta_v$ is an isomorphism by the Roitman theorem (or due to [Pa, Thm. 2.11] and the equality $NH^3_{et}(X_v, \mathbb{Q}_l/\mathbb{Z}_l(2)) = H^3_{et}(X_v, \mathbb{Q}_l/\mathbb{Z}_l(2))$ proven just above). Thus, $\beta$ is an isomorphism.

**Proof of Theorem 6.3** To prove assertions (a), (b) and (c) repeat part of the arguments from the proof of Theorem 6.2.

To prove the assertion (d) it is sufficient to prove that the group $H^3_{et}(X, \mathbb{Q}_l/\mathbb{Z}(2)) = H^3_{et}(X_v, \mathbb{Q}_l/\mathbb{Z}(2))$ is finite. By the Weil conjecture [D74] the group $H^3_{et}(X_v, \mathbb{Q}_l/\mathbb{Z}(2))$ vanishes. It follows that the boundary map $\delta : H^3_{et}(X_v, \mathbb{Q}_l/\mathbb{Z}(2)) \to H^4_{et}(X_v, \mathbb{Z}(2))$ in the coefficient exact cohomology sequence identifies $H^3_{et}(X_v, \mathbb{Q}_l/\mathbb{Z}(2))$ with $\{l\}H^4_{et}(X_v, \mathbb{Z}(2))$. Let $F$ be the residue field of $A$ and $\Gamma = Gal(F_{sep}/F)$. Then there is a short exact sequence of the form $0 \to H^1(\Gamma, H^3_{et}(\bar{X}_v, \mathbb{Z}(2))) \to H^4_{et}(X_v, \mathbb{Z}(2)) \to H^0(\Gamma, H^4_{et}(\bar{X}_v, \mathbb{Z}(2))) \to 0$.

By [D] for each integer $i$ the $\mathbb{Z}_l$-module $H^i_{et}(\bar{X}_v, \mathbb{Z}(2))$ is finitely generated. Thus, the $\mathbb{Z}_l$-module $H^4_{et}(X_v, \mathbb{Z}(2))$ is also finitely generated. Hence its subgroup $\{l\}H^4_{et}(X_v, \mathbb{Z}(2))$ is finite and so is the group $H^3_{et}(X_v, \mathbb{Q}_l/\mathbb{Z}(2))$. The theorem is proved.

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