SOLUTIONS OF QUASIANALYTIC EQUATIONS

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Abstract. The article develops techniques for solving equations $G(x, y) = 0$, where $G(x, y) = G(x_1, \ldots, x_n, y)$ is a function in a given quasianalytic class (for example, a quasianalytic Denjoy-Carleman class, or the class of $C^\infty$ functions definable in a polynomially-bounded o-minimal structure). We show that, if $G(x, y) = 0$ has a formal power series solution $y = H(x)$ at some point $a$, then $H$ is the Taylor expansion at $a$ of a quasianalytic solution $y = h(x)$, where $h(x)$ is allowed to have a certain controlled loss of regularity, depending on $G$. Several important questions on quasianalytic functions, concerning division, factorization, Weierstrass preparation, etc., fall into the framework of this problem (or are closely related), and are also discussed.

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1. INTRODUCTION

This article develops techniques for solving equations $G(x, y) = 0$, where $G(x, y) = G(x_1, \ldots, x_n, y)$ is a function in a given quasianalytic class (see Section 2). Assuming that $G(x, y) = 0$ has a formal power series solution $y = H(x)$ at some point $a$, we ask whether $H$ is the Taylor expansion at $a$ of a quasianalytic solution $y = h(x)$, where $h(x)$ is allowed to have a certain controlled loss of regularity, depending on $G$. Several important problems on quasianalytic functions, concerning division, factorization, Weierstrass preparation, etc., fall into the framework of this question, or are closely related, and they are also discussed in the paper.

There are two general categories of quasianalytic classes $\mathcal{Q}$ that are studied in the recent literature:

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(1) Quasianalytic Denjoy-Carleman classes $Q = Q_M$, going back to E. Borel [8] and characterized (following questions of Hadamard in studies of linear partial differential equations [16]) by the Denjoy-Carleman theorem [13, 9]. These are classes of $C^\infty$ functions whose partial derivatives have bounds on compact sets determined by a logarithmically convex sequence of positive real numbers $M = (M_j)_{j \in \mathbb{N}}$; several classical properties of $M$ guarantee that the functions of class $Q_M$ on an open subset $U$ of $\mathbb{R}^n$ form a ring $Q_M(U)$ that is closed under differentiation and (by the Denjoy-Carleman theorem) quasianalytic (i.e., the Taylor series homomorphism at a point of $U$ is injective, if $U$ is connected). See \[2.1\]

(2) Classes of $C^\infty$ functions that are definable in a given polynomially-bounded o-minimal structure. Such structures arise in model theory, and define quasianalytic classes $Q$ according to a result of C. Miller (see [22], [27] and Remark \[2.2(3)\]).

Loss of regularity will be expressed as follows. If the equation $G(x, y) = 0$ is of class $Q$, then a solution $y = h(x)$ will be allowed to belong to a (perhaps larger) quasianalytic class $Q'$. For classes $Q$ as in (2) above, we will always have $Q' = Q$. On the other hand, suppose that $Q$ is a quasianalytic Denjoy-Carleman class $Q_M$. Then we will find a positive integer $p$ depending on $G$, such that $h$ is of class $Q'$, where $Q_M \subseteq Q' \subseteq Q_M^{(p)}$ and $M^{(p)}$ denotes the sequence $M^{(p)}_j := M_{pj}$. More precisely, we can take $Q' = Q_M^{(p)} \cap C^\infty_M$ where $C^\infty_M(U)$ denotes the subring of $C^\infty(U)$ of functions $f$ such that, for every relatively compact definable open $V \subset U$, $f|_V$ is definable in the (polynomially bounded) o-minimal structure $\mathbb{R}Q_M$ generated by $Q_M$ (see [27] and \[2.2\]).

In the theorems following (proved in Sections 3 and 4 respectively) and in all results involving loss of regularity in Sections 4 and 7 $Q$ can be understood to mean a quasianalytic class in one of the two general categories above, and then $Q'$ will mean either $Q$, in the definable case (2), or $Q' \subseteq Q_M^{(p)}$, as above, in the case that $Q = Q_M$ (1), where $p$ depends on $G$. We fix this convention once and for all, and avoid repeating it in every result.

**Theorem 1.1.** Let $G(x, y)$ be a nonzero function of quasianalytic class $Q$, defined in a neighbourhood $U \times W$ of $(a, b) \in \mathbb{R}^n \times \mathbb{R}$. Then there is a (perhaps larger) quasianalytic class $Q' \supseteq Q$ such that, if the equation

$$G(x, y) = 0$$

admits a formal power series solution $y = H(x)$ at the point $a$, with $b = H(a)$, then there is a solution $y = h(x) \in Q'(V)$, where $V$ is a neighbourhood of $a$ in $U$, and $H$ is the formal Taylor expansion of $h$ at $a$.

Of course, it is enough to find $h \in Q'(V)$ with formal Taylor expansion $H$ at $a$, since it follows that $G(x, h(x)) = 0$, by quasianalyticity.

In the case that $G(x, y)$ is a monic polynomial in $y$ with quasianalytic coefficients, there is a result stronger than the above. Theorem 1.1 does not evidently reduce to the case of a monic polynomial equation because of the lack of a Weierstrass preparation theorem in quasianalytic classes; see Section 7 (cf. [11]).

**Theorem 1.2.** Let $Q$ denote a quasianalytic class. Let $U$ denote a (connected) neighbourhood of the origin in $\mathbb{R}^n$, with coordinates $x = (x_1, \ldots, x_n)$, and let

$$G(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x),$$

(1.1)
where the coefficients \(a_i \in \mathcal{Q}(U)\). Let

\[
G(x, y) = \prod_{j=1}^{k} \left( y^{d_j} + B_{j1}(x)y^{d_j-1} + \cdots + B_{jd_j}(x) \right)
\]

denote the irreducible factorization of \(G(x, y)\) as an element of \(\mathbb{R}[x][y]\). Then there is a (perhaps larger) quasianalytic class \(\mathcal{Q}' \supseteq \mathcal{Q}\) and a neighbourhood \(V\) of 0 in \(U\), such that each \(B_{ji}\) is the formal Taylor expansion \(\tilde{b}_{ji,0}\) at 0 of an element \(b_{ji} \in \mathcal{Q}'(V)\), and

\[
G(x, y) = \prod_{j=1}^{k} \left( y^{d_j} + b_{j1}(x)y^{d_j-1} + \cdots + b_{jd_j}(x) \right)
\]
in \(\mathcal{Q}'(V)[y]\).

These theorems and the other results in Sections 4–7 are not, however, particular to quasianalytic classes of types (1), (2) above. For \(G(x, y) = 0\) of any given quasianalytic class \(\mathcal{Q}\), the solutions \(y = h(x)\) will be \(C^\infty\) functions whose composites by a certain finite sequence \(\sigma\) of blowings-up and power substitutions (depending on \(G\)) belong to \(\mathcal{Q}\). Such functions satisfy the axiom of quasianalyticity (see Definitions 2.1). In the case (2) of functions definable in a given polynomially-bounded \(o\)-minimal structure, such \(C^\infty\) functions are evidently definable in the same structure, so we can take \(\mathcal{Q}' = \mathcal{Q}\). In Section 8 we will show that, if \(\mathcal{Q}\) is a quasianalytic Denjoy-Carleman class \(\mathcal{Q}_M\), then such \(C^\infty\) functions belong to \(\mathcal{Q}_M(p)\), for some \(p\) depending on the sequence \(\sigma\).

Theorems 1.1, 1.2 and the related results are proved using techniques of \emph{quasianalytic continuation} that are developed in Section 4. Quasianalyticity provides a generalization of the classical property of analytic continuation. We use the axiom of quasianalyticity to show that, if the formal Taylor expansion \(\tilde{f}_a\) of a quasianalytic function \(f\) at a given point \(a\) is the composite \(H \circ \tilde{\sigma}_a\) of a formal power series \(H\) with the formal expansion of a suitable quasianalytic mapping \(\sigma\), then this formal composition property extends to a neighbourhood of \(a\). The main problems are solved by reducing \(G(x, y)\) to a simpler form by composing with an appropriate sequence of blowings-up and power substitutions, finding a solution of the simpler problem, and using the quasianalytic continuation property to descend to a solution of the original equation.

2. QUASIANALYTIC CLASSES

We consider a class of functions \(\mathcal{Q}\) given by the association, to every open subset \(U \subset \mathbb{R}^n\), of a subalgebra \(\mathcal{Q}(U)\) of \(C^\infty(U)\) containing the restrictions to \(U\) of polynomial functions on \(\mathbb{R}^n\), and closed under composition with a \(\mathcal{Q}\)-mapping (i.e., a mapping whose components belong to \(\mathcal{Q}\)). We assume that \(\mathcal{Q}\) determines a sheaf of local \(\mathbb{R}\)-algebras of \(C^\infty\) functions on \(\mathbb{R}^n\), for each \(n\), which we also denote \(\mathcal{Q}\).

**Definition 2.1** (quasianalytic classes). We say that \(\mathcal{Q}\) is \emph{quasianalytic} if it satisfies the following three axioms:

1. **Closure under division by a coordinate.** If \(f \in \mathcal{Q}(U)\) and
   \[
   f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = 0,
   \]
   where \(a \in \mathbb{R}\), then \(f(x) = (x_i - a)h(x)\), where \(h \in \mathcal{Q}(U)\).
(2) **Closure under inverse.** Let \( \varphi : U \to V \) denote a \( Q \)-mapping between open subsets \( U, V \) of \( \mathbb{R}^n \). Let \( a \in U \) and suppose that the Jacobian matrix

\[
\frac{\partial \varphi}{\partial x}(a) := \frac{\partial(\varphi_1, \ldots, \varphi_n)}{\partial(x_1, \ldots, x_n)}(a)
\]

is invertible. Then there are neighbourhoods \( U' \) of \( a \) and \( V' \) of \( b := \varphi(a) \), and a \( Q \)-mapping \( \psi : V' \to U' \) such that \( \psi(b) = a \) and \( \psi \circ \varphi \) is the identity mapping of \( U' \).

(3) **Quasianalyticity.** If \( f \in Q(U) \) has Taylor expansion zero at \( a \in U \), then \( f \) is identically zero near \( a \).

**Remarks 2.2.**

(1) Axiom \( 2.1(1) \) implies that, if \( f \in Q(U) \), then all partial derivatives of \( f \) belong to \( Q(U) \).

(2) Axiom \( 2.1(2) \) is equivalent to the property that the implicit function theorem holds for functions of class \( Q \). It implies that the reciprocal of a nonvanishing function of class \( Q \) is also of class \( Q \).

(3) In the case of \( C^\infty \) functions definable in a given polynomially bounded \( o \)-minimal structure, we can define a quasianalytic class \( Q \) in the axiomatic framework above by taking \( Q(U) \) as the subring of \( C^\infty(U) \) of functions \( f \) such that \( f \) is definable in some neighbourhood of any point of \( U \) (or, equivalently, such that \( f|_V \) is definable, for every relatively compact definable open \( V \subset U \)).

The elements of a quasianalytic class \( Q \) will be called **quasianalytic functions**. A category of manifolds and mappings of class \( Q \) can be defined in a standard way. The category of \( Q \)-manifolds is closed under blowing up with centre a \( Q \)-submanifold [6].

Resolution of singularities holds in a quasianalytic class [5], [6]. Resolution of singularities of an ideal does not require that the ideal be finitely generated; see [7, Thm. 3.1]. Resolution of singularities of an ideal in a quasianalytic class is the main tool used in this article.

### 2.1. Quasianalytic Denjoy-Carleman classes

We use standard multiindex notation: Let \( \mathbb{N} \) denote the nonnegative integers. If \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), we write \( |\alpha| := \alpha_1 + \cdots + \alpha_n \), \( \alpha! := \alpha_1! \cdots \alpha_n! \), \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \), and \( \partial^{|\alpha|}/\partial x^\alpha := \partial^{\alpha_1} \cdots \partial^{\alpha_n}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \). We write \( \langle i \rangle \) for the multiindex with 1 in the \( i \)th place and 0 elsewhere.

**Definition 2.3** (Denjoy-Carleman classes). Let \( M = (M_k)_{k \in \mathbb{N}} \) denote a sequence of positive real numbers which is **logarithmically convex**; i.e., the sequence \( (M_{k+1}/M_k) \) is nondecreasing. A **Denjoy-Carleman class** \( Q = Q_M \) is a class of \( C^\infty \) functions determined by the following condition: A function \( f \in C^\infty(U) \) (where \( U \) is open in \( \mathbb{R}^n \)) is of class \( Q_M \) if, for every compact subset \( K \) of \( U \), there exist constants \( A, B > 0 \) such that

\[
|\frac{\partial^{\alpha}|f}{\partial x^\alpha}| \leq AB|\alpha|!M_{|\alpha|}
\]

on \( K \), for every \( \alpha \in \mathbb{N}^n \).

**Remarks 2.4.**

(1) The logarithmic convexity assumption implies that \( M_jM_k \leq M_0M_{j+k} \), for all \( j, k \), and that the sequence \( ((M_k/M_0)^{1/k}) \) is nondecreasing. The first of these conditions guarantees that \( Q_M(U) \) is a ring, and the second that
\( \mathcal{Q}_M(U) \) contains the ring \( \mathcal{O}(U) \) of real-analytic functions on \( U \), for every open \( U \subset \mathbb{R}^n \). (If \( M_k = 1 \), for all \( k \), then \( \mathcal{Q}_M = \mathcal{O} \).

(2) \( \mathcal{Q}_M \) can be defined equivalently using inequalities of the form \( |\partial^{\alpha} f/\partial x^\alpha| \leq AB^{\lceil \alpha \rceil} |\alpha|! M_{|\alpha|} \), instead of (2.1). This is true because, on the one hand, \( \alpha! \leq |\alpha|! \), and, on the other, \( |\alpha|! \leq n|\alpha| \alpha! \), since

\[
n^\alpha = (1 + \ldots + 1)^{\alpha_1 + \ldots + \alpha_n} = \sum (\alpha_1 + \ldots + \alpha_n)! \frac{\alpha! \cdots \alpha!}{\alpha_1! \cdots \alpha_n!},
\]

where the sum is over all partitions \( |\alpha| = \alpha_1 + \ldots + \alpha_n = \alpha \).

A Denjoy-Carleman class \( \mathcal{Q}_M \) is a quasianalytic class in the sense of Definition 2.1 if and only if the sequence \( M = (M_k)_{k \in \mathbb{N}} \) satisfies the following two assumptions in addition to those of Definition 2.3

(a) \( \sup \left( \frac{M_{k+1}}{M_k} \right)^{1/k} < \infty \).

(b) \( \sum_{k=0}^{\infty} \frac{M_k}{(k+1)M_{k+1}} = \infty \).

It is easy to see that the assumption (a) implies that \( \mathcal{Q}_M \) is closed under differentiation. The converse of this statement is due to S. Mandelbrojt [21]. In a Denjoy-Carleman class \( \mathcal{Q}_M \), closure under differentiation is equivalent to the axiom (2.2) 1 of closure under division by a coordinate—the converse of Remark 2.2(1) is a consequence of the fundamental theorem of calculus:

\[
f(x_1, \ldots, x_n) - f(x_1, \ldots, 0, \ldots, x_n) = x_i \int_0^1 \frac{\partial f}{\partial x_i}(x_1, \ldots, tx_1, \ldots, x_n) \, dt
\]

(where 0 in the left-hand side is in the \( i \)th place).

According to the Denjoy-Carleman theorem, the class \( \mathcal{Q}_M \) is quasianalytic (axiom (2.2) 3)) if and only if the assumption (b) holds [17, Thm. 1.3.8].

Closure of the class \( \mathcal{Q}_M \) under composition is due to Roumieu [28] and closure under inverse to Komatsu [18]; see [6] for simple proofs. The assumptions of Definition 2.3 and (a), (b) above thus guarantee that \( \mathcal{Q}_M \) is a quasianalytic class, in the sense of Definition 2.1.

If \( \mathcal{Q}_M, \mathcal{Q}_N \) are Denjoy-Carleman classes, then \( \mathcal{Q}_M(U) \subseteq \mathcal{Q}_N(U) \), for all \( U \), if and only if \( \sup (M_k/N_k)^{1/k} < \infty \) (see [29, §1.4]); in this case, we write \( \mathcal{Q}_M \subseteq \mathcal{Q}_N \).

2.2. Shifted Denjoy-Carleman classes. Given \( M = (M_j)_{j \in \mathbb{N}} \) and a positive integer \( p \), let \( M^{(p)} \) denote the sequence \( M^{(p)}_j := M_{pj} \).

If \( M \) is logarithmically convex, then \( M^{(p)} \) is logarithmically convex:

\[
\frac{M_{kp}}{M_{(k-1)p}} = \frac{M_{kp}}{M_{kp-1}} \cdots \frac{M_{kp-p+1}}{M_{kp-p}} \leq \frac{M_{kp+p}}{M_{kp+p-1}} \cdots \frac{M_{kp+1}}{M_{kp}} = \frac{M_{(k+1)p}}{M_{kp}}.
\]

Therefore, if \( \mathcal{Q}_M \) is a Denjoy-Carleman class, then so is \( \mathcal{Q}_{M^{(p)}} \). Clearly, \( \mathcal{Q}_M \subseteq \mathcal{Q}_{M^{(p)}} \). Moreover, the assumption (a) above for \( \mathcal{Q}_M \) immediately implies the same condition for \( \mathcal{Q}_{M^{(p)}} \). In general, however, it is not true that assumption (b) (i.e., the quasianalyticity axiom (3)) for \( \mathcal{Q}_M \) implies (b) for \( \mathcal{Q}_{M^{(p)}} \) [26, Example 6.6].

In particular, in general, \( \mathcal{Q}_{M^{(p)}} \nsubseteq \mathcal{Q}_M \). Moreover, \( \mathcal{Q}_{M^{(p)}} \) is the smallest Denjoy-Carleman class containing all \( g \in C^\infty(\mathbb{R}) \) such that \( g(t^2) \in \mathcal{Q}_M(\mathbb{R}) \) [26, Rmk. 6.2].
3. Regularity estimates

Let \( y = \sigma(x) \) denote a mapping of Denjoy-Carleman class \( Q_M \) (Definition 2.3). Given a \( C^\infty \) function \( g(y) \) such \( f(x) := g(\sigma(x)) \) is of class \( Q_M \), what can we say about the class of \( g \)? We will answer this question for mappings \( \sigma \) of two important kinds that will be needed for our main results: power substitutions and blowings-up.

3.1. Power substitutions. The following result is proved in [26, Thm. 6.1] for functions of a single variable, by an argument different from that below.

**Lemma 3.1.** Consider a Denjoy-Carleman class \( Q_M \). Let \( U = \prod_{i=1}^{n}(-r_i, r_i) \subset \mathbb{R}^n \), where each \( r_i > 0 \), and let \( \sigma : U \to V \) denote a power substitution

\[
(y_1, \ldots, y_n) = (x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}),
\]

where each \( k_i \) is a positive integer and \( V = \prod(-r_i^{k_i}, r_i^{k_i}) \). Let \( g \in C^\infty(V) \) and let \( f = g \circ \sigma \). If \( f \in Q_M(U) \), then \( g \in Q_M(\sigma(U)) \), where \( p = \max k_i \).

**Proof.** Let \( K \subset U \) denote the compact set \( \prod_{i=1}^{n}[-r_i/2, r_i/2] \). Since \( f \in Q_M(U) \), there are constants \( A > 0, B \geq 1 \) such that

\[
\frac{\partial^{[\alpha]} f}{\partial x^\alpha} \leq AB^{[\alpha]} |\alpha!| M_{[\alpha]}^{[\alpha]}
\]
on the compact set \( K \), for all \( \alpha \in \mathbb{N}^n \). We will show that

\[
\frac{\partial^{[\beta]} g}{\partial y^\beta} \leq AB^{[\beta]} |\beta!| M_{[\beta]}^{[\beta]}
\]
on \( \sigma(K) \), for all \( \beta \in \mathbb{N}^n \). In the following, we will not explicitly write “on \( K \)” or “on \( \sigma(K) \)”—all estimates will be understood to mean on these sets (and the left-hand side of (3.1) or (3.2) will sometimes be understood to mean the maximum on one of these sets, when the meaning is clear from the context). We will use the notation

\[
g^{(\beta)} := \frac{\partial^{[\beta]} g}{\partial y^\beta}.
\]

**Claim 3.2.** For each \( \beta \in \mathbb{N}^n \),

\[
\frac{\partial^{[\alpha]} (g^{(\beta)} \circ \sigma)}{\partial x^\alpha} \leq AB^{[\beta]}|\alpha + \beta!| |\alpha!| M_{[\alpha]}^{[\beta]} \Gamma(k, \alpha, \beta),
\]

for all \( \alpha \in \mathbb{N}^n \), where

\[
\Gamma(k, \alpha, \beta) := \frac{\alpha! \prod_{i=1}^{n} \beta_i!}{k^\beta (\alpha + \beta)!},
\]

and \( k := (k_1, \ldots, k_n), k^\beta := k_1^{\beta_1} \cdots k_n^{\beta_n} \).

Note that \( \Gamma(k, 0, \beta) = \left( \prod_{i=1}^{n} \beta_i! \right) / (k^\beta |\beta!|) = 1 \). In the case that \( \alpha = 0 \), therefore, (3.3) reduces to (3.2), so the lemma follows from the claim.

We will prove Claim 3.2 by induction on \( |\beta| \). Note that \( \Gamma(k, 0, 0) = 1 \). The claim is therefore true when \( \beta = 0 \), because in this case (3.3) reduces to (3.1).

Assume that (3.3) holds for a given multiindex \( \beta \). It is clearly then enough to prove (3.3) for \( \gamma := \beta + (1) \). The partial derivative \( \partial / \partial y_1 \) transforms by \( \sigma \) as follows:

\[
\frac{\partial}{\partial y_1} = \frac{1}{k_1 x_1^{k_1-1}} \frac{\partial}{\partial x_1} \quad \text{i.e.,} \quad \frac{\partial h}{\partial y_1} \circ \sigma = \frac{1}{k_1 x_1^{k_1-1}} \frac{\partial (h \circ \sigma)}{\partial x_1}, \quad h \in C^\infty(V).
\]
Therefore,
\[
\frac{\partial g^{(\beta)}}{\partial y_1} \circ \sigma = \frac{1}{k_1} \int_{[0,1]^{k_1-1}} \frac{\partial^{k_1} (g^{(\beta)} \circ \sigma)}{\partial x_1^{k_1}} \left(t_1 \cdots t_{k_1-1} x_1, x_2, \ldots, x_n\right) Q_0(t) dt,
\]
by (2.2) (applied \(k_1 - 1\) times), where \(t = (t_1, \ldots, t_{k_1-1})\) and \(Q_0(t)\) denotes the polynomial \(Q_0(t) := t_1^{k_1-2} t_2^{k_1-3} \cdots t_{k_1-2}\). It follows that, for all \(\alpha \in \mathbb{N}^n\),
\[
\left| \frac{\partial^{\alpha}(g^{(\gamma)} \circ \sigma)}{\partial x^\alpha} \right| = \left| \frac{\partial^{|\alpha|}(g^{(\beta)} \circ \sigma)}{\partial y_1} \right| \left| \frac{\partial^{\alpha}(g^{(\gamma)} \circ \sigma)}{\partial x^\alpha} \right|
\leq \frac{1}{k_1} \left| \frac{\partial^{\alpha+k_1}(g^{(\beta)} \circ \sigma)}{\partial x^{\alpha+k_1}(1)} \right| \cdot \int_{[0,1]^{k_1-1}} Q_0(t) dt,
\]
where \(Q_0(t) := Q_0(t)(t_1 \cdots t_{k_1-1}) \alpha_1 = t_1^{k_1-2} t_2^{k_1-3} \cdots t_{k_1-1-1}\), so that
\[
\int_{[0,1]^{k_1-1}} Q_0(t) dt = \frac{1}{(k_1 - 1 + \alpha_1)(k_1 - 2 + \alpha_1) \cdots (1 + \alpha_1)} = \frac{(k_1 + \alpha_1)!}{(k_1 + \alpha_1)!}.
\]
By the induction hypothesis,
\[
\left| \frac{\partial^{\alpha}(g^{(\gamma)} \circ \sigma)}{\partial x^\alpha} \right| \leq AB^{p|\beta|+|\alpha|+k_1}(\alpha + \gamma)! M_{p|\beta|+|\alpha|+k_1} \cdot \frac{1}{k_1} \left(\frac{(\alpha_1 + \beta_1 + k_1)!}{(\alpha_1 + \beta_1 + 1)!} \cdot \Gamma(k, \alpha + k_1(1), \beta) \cdot \frac{(k_1 + \alpha_1)\alpha_1!}{(k_1 + \alpha_1)!} \right).
\]
Since \(p|\beta| + |\alpha| + k_1 \leq p|\gamma| + |\alpha|\), we get
\[
\left| \frac{\partial^{\alpha}(g^{(\gamma)} \circ \sigma)}{\partial x^\alpha} \right| \leq AB^{p|\gamma|+|\alpha|}(\alpha + \gamma)! M_{p|\gamma|+|\alpha|} \Gamma(k, \alpha, \gamma),
\]
for all \(\alpha \in \mathbb{N}^n\), as required, using the following combinatorial identity, which can be easily checked:
\[
\frac{1}{k_1} \left(\frac{(\alpha_1 + \beta_1 + k_1)!}{(\alpha_1 + \beta_1 + 1)!} \cdot \frac{(k_1 + \alpha_1)\alpha_1!}{(k_1 + \alpha_1)!} \right) = \frac{(k_1 + \alpha_1)!}{(k_1 + \alpha_1)!} \Gamma(k, \alpha + k_1(1), \beta).
\]
\[\square\]

Remark 3.3. It follows from Lemma 3.1 that the same conclusion holds for a mapping \(\sigma\) of the form \((y_1, \ldots, y_n) = (\epsilon_1 x_1^{k_1}, \ldots, \epsilon_n x_n^{k_n})\), where each \(\epsilon_i \in \{-1,1\}\).

3.2. Blowing up.

Lemma 3.4. Let \(\mathcal{Q}_M\) denote a Denjoy-Carleman class. Let \(W\) be an open subset of \(\mathbb{R}^n\), and let \(\sigma : Z \to W\) denote a blowing-up with centre a \(\mathcal{Q}_M\)-submanifold of \(W\). Let \(g \in C^\infty(W)\) and let \(f = g \circ \sigma\). If \(f \in \mathcal{Q}_M(Z)\), then \(g \in \mathcal{Q}_M(\mathbb{R}^n)\).

In contrast to the case of a power substitution, we do not actually know whether the loss of regularity is necessary in Lemma 3.4; it is interesting to ask whether \(g \in \mathcal{Q}_M(\mathbb{R}^n)\).

Proof of Lemma 3.4. Any point of \(W\) has a coordinate neighbourhood \(V\), such that \(\sigma^{-1}(V)\) can be covered by finitely many coordinate charts \(U\), in each of which \(\sigma\) is given by a mapping of the form
\[
(y_1, \ldots, y_n) = (x_1, x_1 x_2, \ldots, x_1 x_s, x_{s+1}, \ldots, x_n),
\]
where \(2 \leq s \leq n\). In the following, we will use \(\sigma : U \to V\) to denote this mapping.
Let $K \subset U$ denote the compact set $\prod_{i=1}^{n} [-r_i, r_i]$, where each $r_i > 0$. Since $f \in \mathcal{Q}_M(U)$, there are constants $A > 0$, $B \geq 1$ such that
\begin{equation}
|\frac{\partial^{[\alpha]} f}{\partial x^\alpha}| \leq AB^{[\alpha]} \alpha! M_{[\alpha]}
\end{equation}
onumber
on the compact set $K$, for all $\alpha \in \mathbb{N}^n$. We will show that
\begin{equation}
|\frac{\partial^{[\beta]} g}{\partial y^\beta}| \leq A(4s^2 r B^{2})^{[\beta]} \beta! M_{2[\beta]}
\end{equation}
onumber
on $\sigma(K)$, for all $\beta \in \mathbb{N}^n$, where $r := \max\{1, r_i\}$. (Recall the conventions following (3.2) above.) The lemma then follows.

**Claim 3.5.** For each $\beta \in \mathbb{N}^n$,
\begin{equation}
|\frac{\partial^{[\alpha]} (g^{(\beta)} \circ \sigma)}{\partial x^\alpha}| \leq A(2s^2 r B^{2})^{\beta} B^{p_{\beta}(\alpha)} \Delta(\alpha, \beta) M_{p_{\beta}(\alpha)},
\end{equation}
for all $\alpha \in \mathbb{N}^n$, where $p_{\beta}(\alpha) := |\beta| + |\alpha| + \sum_{i=1}^{s} \beta_i$,
\begin{equation}
\Delta(\alpha, \beta) := \frac{\alpha! (\alpha + \beta + \gamma)!}{(\alpha + \xi)! (\beta_1 - \xi)!}.
\end{equation}
and
\[\gamma \text{ is chosen to maximize } (\alpha + \beta + \delta)! \text{ over the set } I(\beta) \text{ consisting of all } \delta = (0, \delta_2, \ldots, \delta_n, 0, \ldots, 0) \in \mathbb{N}^n \text{ such that } |\delta| = \beta_1, \]
\[\xi \text{ is chosen to minimize } (\alpha_1 + \eta)! (\beta_1 - \eta)! \text{ over the set } J(\eta) \text{ of all } \eta \in \mathbb{N} \text{ such that } 0 \leq \eta \leq \beta_1.\]

Note that $p_{\beta}(0) \leq 2|\beta|$, and that
\[\Delta(0, \beta) = \beta! \cdot \frac{(\beta)!}{(\xi)!(\beta_1 - \xi)!} \cdot \prod_{i=2}^{s} \frac{(\gamma_i + \beta_i)!}{\beta_i!} \cdot \prod_{i=2}^{s} \frac{\gamma_i!}{\beta_i!} \leq \beta! \cdot \frac{(\beta)!}{(\xi)!(\beta_1 - \xi)!} \cdot \prod_{i=2}^{s} \frac{(\gamma_i + \beta_i)!}{\beta_i!} \cdot \prod_{i=2}^{s} \frac{\gamma_i!}{\beta_i!} \leq 2\beta! \cdot \prod_{i=2}^{s} \frac{(\gamma_i + \beta_i)!}{\beta_i!} \cdot \prod_{i=2}^{s} \frac{\gamma_i!}{\beta_i!} \leq 2^{|\beta|} \prod_{i=2}^{s} \frac{\gamma_i + \beta_i}{\beta_i}.\]
(Recall the conventions following (3.2) above). Therefore, (3.7) in the case that $\alpha = 0$ implies (3.6); i.e., the lemma follows from Claim 3.5.

We will prove the claim by induction on $|\beta|$. Note that $\Delta(\alpha, 0) = \alpha!$. The claim is therefore true when $\beta = 0$, because in this case (3.7) reduces to (3.5). Fix a multiindex $\tilde{\beta}$, where $|\tilde{\beta}| > 0$. By induction, we assume the claim holds for all $\beta$ such that $|\beta| < |\tilde{\beta}|$. Now, the partial derivatives transform by $\sigma$ as follows (cf. (3.4)):
\begin{equation}
\frac{\partial}{\partial y_1} = \frac{\partial}{\partial x_1} - \sum_{j=2}^{s} \frac{x_j}{x_1} \frac{\partial}{\partial x_j},
\end{equation}
\begin{equation}
\frac{\partial}{\partial y_i} = \frac{\partial}{\partial x_i}, \quad i = 2, \ldots, s,
\end{equation}
\begin{equation}
\frac{\partial}{\partial y_i} = \frac{\partial}{\partial x_i}, \quad i = s + 1, \ldots, n.
\end{equation}

**Case 1.** $\tilde{\beta}_1 = 0$. Then there exists $\beta \in \mathbb{N}^n$ such that $\tilde{\beta} = \beta + (k)$, where $2 \leq k \leq n$.
Since $\beta_1 = 0$, (3.7) holds for all $\alpha \in \mathbb{N}^n$, with $\gamma = 0$ and $\xi = 0$ in $\Delta(\alpha, \beta)$. If $k > s$, then (3.7) for $\beta$ follows from (3.9) and the inductive assumption (3.7) for $\beta$. 

On the other hand, suppose that \(2 \leq k \leq s\). Then

\[
\frac{\partial g^{(\beta)}}{\partial y_k} \circ \sigma = \int_0^1 \frac{\partial^2 (g^{(\beta)} \circ \sigma)}{\partial x_1 \partial x_k} (tx_1, x_2, \ldots, x_n) dt.
\]

Given any \(\alpha \in \mathbb{N}^n\), let \(\delta := \alpha + (1) + (k)\). Then \(p_{\beta}(\delta) = p_{\beta}(\alpha)\) and

\[
\Delta(\delta, \beta) = (\delta + \beta)! = (\alpha + 1)(\alpha + \beta)! = (\alpha + 1)\Delta(\alpha, \beta).
\]

By (3.10) and the induction hypothesis,

\[
\left| \frac{\partial^{[\alpha]}(g^{(\tilde{\beta})} \circ \sigma)}{\partial x^{\alpha}} \right| = \left| \int_0^1 \frac{\partial^{[\alpha+1]}(g^{(\beta)} \circ \sigma)}{\partial x_1 \partial x_1 \partial x_k} (tx_1, x_2, \ldots, x_n) t^{\alpha_1} dt \right|
\]

\[
\leq AB^{p_{\beta}(\delta)} \Delta(\tilde{\delta}, \beta) M_{p_{\beta}(\delta)} \frac{1}{\alpha_1 + 1}
\]

\[
= AB^{p_{\beta}(\alpha)} \Delta(\alpha, \tilde{\beta}) M_{p_{\beta}(\alpha)},
\]

as required.

**Case 2.** \(\tilde{\beta}_1 > 0\). Then there exists \(\beta \in \mathbb{N}^n\) such that \(\tilde{\beta} = \beta + (1)\), and

\[
\frac{\partial g^{(\beta)}}{\partial y_1} \circ \sigma = \frac{\partial (g^{(\beta)} \circ \sigma)}{\partial x_1} - \sum_{j=2}^s \int_0^1 x_j \frac{\partial^2 (g^{(\beta)} \circ \sigma)}{\partial x_1 \partial x_j} (tx_1, x_2, \ldots, x_n) dt,
\]

by (3.9). Therefore, for all \(\alpha \in \mathbb{N}^n\),

\[
\left| \frac{\partial^{[\alpha]}(g^{(\tilde{\beta})} \circ \sigma)}{\partial x^{\alpha}} \right| \leq I + \sum_{j=2}^s II_j + \sum_{j=2}^s III_j,
\]

where

\[
I := \left| \frac{\partial^{[\alpha+1]}(g^{(\beta)} \circ \sigma)}{\partial x_1} \right|,
\]

\[
II_j := \int_0^1 x_j \left| \frac{\partial^{[\alpha+1]}(g^{(\beta)} \circ \sigma)}{\partial x_1} (tx_1, x_2, \ldots, x_n) \right| t^{\alpha_1} dt,
\]

\[
III_j := \int_0^1 \left| \frac{\partial^{[\alpha+2]}(g^{(\beta)} \circ \sigma)}{\partial x_1 \partial x_1} (tx_1, x_2, \ldots, x_n) \right| t^{\alpha_1} dt.
\]

We will use the inductive hypothesis to show that each term I, II, and III is bounded by

\[
A s (2s^2 r^{\tilde{\beta}_1 - 1} B^{p_{\beta}(\alpha)} \Delta(\alpha, \tilde{\beta}) M_{p_{\beta}(\alpha)}).
\]

The required estimate (3.7) for \(\tilde{\beta}\) follows since there are only \(2s - 1 \leq 2s\) such terms.

Consider the first term I. Set \(\delta := \alpha + (1)\). Then \(p_{\beta}(\delta) = p_{\beta}(\alpha) - 1\). Choose \(\gamma \in I(\beta), \xi \in J(\beta)\) to realize \(\Delta(\delta, \beta)\) according to the formula (3.8) (with \(\delta\) in place of \(\alpha\)). If \(\beta_1 = 1\), then \(\beta_1 = 0\), so that \(\gamma = 0, \xi = 0\); in this case, it is easy to see that

\[
\Delta(\delta, \beta) = (\delta + \beta)! = (\alpha + \beta)! \leq \Delta(\alpha, \tilde{\beta}).
\]

On the other hand, suppose that \(\tilde{\beta}_1 > 1\). Since \(|\gamma| = \tilde{\beta}_1 - 1\), there exists \(k \in \{2, \ldots, s\}\) such that \(\gamma_k \geq (\tilde{\beta}_1 - 1)/(s - 1) > 0\). Since \(\gamma_k \in \mathbb{N}\), it follows that,
in fact, $\gamma_k \geq \hat{\beta}_1/s$ (consider separately the cases that $\hat{\beta}_1 > s$ or $\hat{\beta}_1 \leq s$). Set $\gamma' := \gamma + (k)$. Then $\gamma' \in J(\hat{\beta})$ and

$$\Delta(\delta, \beta) = \frac{\alpha_1 + 1}{\alpha_k + \beta_k + \gamma_k + 1} \cdot \frac{\alpha_1!(\alpha + \hat{\beta} + \gamma')!}{(\alpha_1 + 1 + \xi)!} \cdot \frac{(\alpha_1 + 1 + \xi)!}{(\alpha_1 + 1 + \xi)!}.$$

Now,

$$\frac{\alpha_1 + 1}{\alpha_k + \beta_k + \gamma_k + 1} \leq \frac{s(\alpha_1 + 1)}{\beta_1} \leq \frac{s(\alpha_1 + 1 + \xi)}{\beta_1 - \xi}.$$

Since $\xi \in J(\hat{\beta})$, we get $\Delta(\delta, \beta) \leq s\Delta(\alpha, \hat{\beta})$.

In either case, by the induction hypothesis,

$$I \leq A(2s^2r)^{\hat{\beta}_1} B_{p_\beta}(\delta) \Delta(\delta, \beta) M_{p_\beta(\delta)}$$

$$\leq As(2s^2r)^{\hat{\beta}_1-1} B_{p_\beta(\alpha)} \Delta(\alpha, \hat{\beta}) M_{p_\beta(\alpha)}.$$

Secondly, consider a term $\Pi_j$. We can assume that $\alpha_j \neq 0$. Again set $\delta := \alpha + (1)$ (so that $p_\beta(\delta) = p_\beta(\alpha) - 1$), and choose $\gamma \in I(\beta), \xi \in J(\beta)$ to realize $\Delta(\delta, \beta)$. Set $\gamma' := \gamma + (j)$. Then $\gamma' \in I(\hat{\beta})$, and

$$\Delta(\delta, \beta) = \frac{\alpha_1 + 1}{\alpha_j + \beta_j + \gamma_j + 1} \cdot \frac{\alpha_1!(\alpha + \hat{\beta} + \gamma')!}{(\alpha_1 + 1 + \xi)!} \cdot \frac{(\alpha_1 + 1 + \xi)!}{(\alpha_1 + 1 + \xi)!}.$$

By the induction hypothesis,

$$\Pi_j \leq \alpha_j A(2s^2r)^{\hat{\beta}_1} B_{p_\beta(\delta)} \Delta(\delta, \beta) M_{p_\beta(\delta)} \cdot \frac{1}{\alpha_1 + 1}$$

$$\leq A(2s^2r)^{\hat{\beta}_1-1} B_{p_\beta(\alpha)} \Delta(\alpha, \hat{\beta}) M_{p_\beta(\alpha)}.$$

Finally, consider any of the terms $\Pi_j$. Set $\delta := \alpha + (1)+(j)$. Then $p_\beta(\delta) = p_\beta(\alpha)$. Choose $\gamma \in I(\beta), \xi \in J(\beta)$ to realize $\Delta(\delta, \beta)$. Set $\gamma' := \gamma + (j)$. Then $\gamma' \in I(\hat{\beta})$, and

$$\Delta(\delta, \beta) = \frac{\alpha_1 + 1}{\alpha_j + \beta_j + \gamma_j + 1} \cdot \frac{\alpha_1!(\alpha + \hat{\beta} + \gamma')!}{(\alpha_1 + 1 + \xi)!} \cdot \frac{(\alpha_1 + 1 + \xi)!}{(\alpha_1 + 1 + \xi)!}.$$

By the induction hypothesis,

$$\Pi_j \leq r A(2s^2r)^{\hat{\beta}_1} B_{p_\beta(\delta)} \Delta(\delta, \beta) M_{p_\beta(\delta)} \cdot \frac{1}{\alpha_1 + 1}$$

$$\leq Ar(2s^2r)^{\hat{\beta}_1-1} B_{p_\beta(\alpha)} \Delta(\alpha, \hat{\beta}) M_{p_\beta(\alpha)}.$$

Since $r, s \geq 1$, each of the terms $I, \Pi_j$ and $\Pi_j$ is bounded by $4.11$, and the proof is compete. \qed

4. Quasianalytic Continuation

Let $\mathcal{F}_a$ denote the ring of formal power series centred at a point $a \in \mathbb{R}^n$; thus $\mathcal{F}_a \cong \mathbb{R}[x_1, \ldots, x_n]$. If $U$ is open in $\mathbb{R}^n$ and $f \in C^\infty(U)$, then $\hat{f}_a \in \mathcal{F}_a$ denotes the formal Taylor expansion of $f$ at a point $a \in U$; i.e., $\hat{f}_a(x) = \sum_{\alpha \in \mathbb{N}^n} \partial^\alpha f(a)/\partial x^\alpha (a) x^\alpha /\alpha!$ (likewise for a $C^\infty$ mapping $U \to \mathbb{R}^m$).
Let $Q$ denote a quasianalytic class (Definition 2.1).

**Theorem 4.1.** Let $U, V$ denote open neighbourhoods of the origin in $\mathbb{R}^n$, with coordinate systems $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, respectively. (Assume $U$ is chosen so that every coordinate hyperplane $(x_i = 0)$ is connected). Let $\sigma : U \to V$ denote a $Q$-mapping such that the Jacobian determinant $\det(\partial \sigma / \partial x)$ is a monomial times an invertible factor in $Q(U)$. Let $f \in Q(U)$ and let $H \in F_0$ be a formal power series centred at $0 \in V$, such that $\hat{f}_0 = H \circ \hat{\sigma}_0$. Then, for all $\beta \in \mathbb{N}^n$, there exists $f_{\beta} \in Q(U)$ such that $f_{\beta,0} = f$ and

(1) for all $a \in U$, $\hat{f}_a = H_a \circ \hat{\sigma}_a$, where $H_a \in F_{\sigma(a)}$ denotes the formal power series

$$H_a := \sum_{\beta \in \mathbb{N}^n} \frac{f_{\beta}(a)}{\beta!} y^\beta;$$

(2) each $f_{\beta}$, $\beta \in \mathbb{N}^n$, and therefore also $H_a \in F_{\sigma(a)}$ (as a function of $a$) is constant on connected components of the fibres of $\sigma$.

**Proof.** (1) Write $\sigma = (\sigma_1, \ldots, \sigma_n)$ with respect to the coordinates of $V$. As formal expansions at $0 \in U$,

$$\sum_{j=1}^n \left( \frac{\partial H}{\partial y_j} \circ \sigma \right) \cdot \frac{\partial \sigma_j}{\partial x_i} = \frac{\partial f}{\partial x_i}, \quad i = 1, \ldots, n,$$

so that

$$\det \left( \frac{\partial \sigma}{\partial x} \right) \cdot \left( \frac{\partial H}{\partial y_j} \circ \sigma \right) = \left( \frac{\partial \sigma}{\partial x} \right)^* \left( \frac{\partial f}{\partial x_i} \right),$$

where $(\partial f / \partial x_i)$ denotes the column vector with components $\partial f / \partial x_i$, and $(\partial \sigma / \partial x)^*$ is the adjugate matrix of $\partial \sigma / \partial x$.

By axioms (2.1), (3), for each $j = 1, \ldots, n$, there is a quasianalytic function $f_{(j)} \in Q(U)$ such that

$$\hat{f}_{(j),0} = \frac{\partial H}{\partial y_j} \circ \hat{\sigma}_0$$

and

$$\det \left( \frac{\partial \sigma}{\partial x} \right) \cdot (f_{(j)}) = \left( \frac{\partial \sigma}{\partial x} \right)^* \left( \frac{\partial f}{\partial x_i} \right)$$

in $Q(U)$.

It follows by induction on the order of differentiation that, for each $\beta \in \mathbb{N}^n$, there is a quasianalytic function $f_{\beta} \in Q(U)$ such that

$$\hat{f}_{\beta,0} = \frac{\partial^{\mid \beta \mid} H}{\partial y^\beta} \circ \hat{\sigma}_0$$

and

$$\det \left( \frac{\partial \sigma}{\partial x} \right) \cdot (f_{\beta + (j)}) = \left( \frac{\partial \sigma}{\partial x} \right)^* \left( \frac{\partial f}{\partial x_i} \right).$$

Therefore, for all $a \in U$, $\hat{f}_a = H_a \circ \hat{\sigma}_a$, where $H_a$ is the formal power series centred at $\sigma(a) \in V$ given by (4.1). Likewise, for all $\beta \in \mathbb{N}^n$ and $a \in U$,

(4.2) $$\hat{f}_{\beta,a} = \frac{\partial^{\mid \beta \mid} H_a}{\partial y^\beta} \circ \hat{\sigma}_a$$
(2) It is enough to show that, for each $\beta \in \mathbb{R}^n$, $f_{\beta}$ is locally constant on every fibre of $\sigma$. This is immediate from Lemma 4.2 following applied at any given point $a \in U$ to the equation $\Omega$.

**Lemma 4.2.** Let $\sigma : U \to V$ denote a $Q$-mapping, where $U$, $V$ are open neighbourhoods of the origin in $\mathbb{R}^n$. Let $f \in Q(U)$ and let $H \in F_0$ be a formal power series centred at $0 \in V$, such that $f_0 = H \circ \sigma_0$. Then there is a neighbourhood $W$ of $0$ in $V$ such that $f$ is constant on the fibres of $\sigma$ in $W$.

**Proof.** The following argument is due to Nowak [25]. We can assume that $f(0) = 0$, $H(0) = 0$. Let

$$P := \{(\xi, \eta, \zeta) \in U \times U \times V : \sigma(\xi) = \zeta = \sigma(\eta), \ f(\xi) \neq f(\eta)\}.$$ 

Suppose the lemma is false. Then $(0, 0, 0) \in \overline{P}$. By the quasianalytic curve selection lemma (see [6] Thm. 6.2), there is a quasianalytic arc $(\alpha(t), \beta(t), \gamma(t)) \in U \times U \times V$ such that $(\alpha(0), \beta(0), \gamma(0)) = (0, 0, 0)$ and $(\alpha(t), \beta(t), \gamma(t)) \in P$ if $t \neq 0$. Then

$$(f \circ \alpha)^{\circ}_0 = f_0 \circ \alpha_0 = H \circ \sigma_0 \circ \alpha_0 = H \circ (\sigma \circ \alpha)^{\circ}_0 = H \circ \gamma_0.$$ 

Likewise, $(f \circ \beta)^{\circ}_0 = H \circ \gamma_0$, so that $(f \circ \alpha)^{\circ}_0 = (f \circ \beta)^{\circ}_0$. Since $f \circ \alpha, f \circ \beta$ are quasianalytic functions of $t$, $f \circ \alpha = f \circ \beta$; a contradiction. □

**Remark 4.3.** Theorem 4.1(2) also follows from Proposition 4.6 below, which is included here for completeness. Proposition 4.6 in the special case that $Q = O$ (the class of analytic functions) is proved in [4] Prop. 11.1, but the proof in the latter does not apply to quasianalytic classes, in general. We have chosen to prove Theorem 4.1(2) using Lemma 4.2 because the idea of its proof above will be needed again in our proof of Corollary 4.5.

**Lemma 4.4.** Let $\sigma : M \to V$ denote a proper $Q$-mapping, where $M$ is a $Q$-manifold of dimension $n$, and $V$ is an open neighbourhood of the origin in $\mathbb{R}^n$. Then, given any open covering $\{U\}$ of $\sigma^{-1}(0)$, there is a neighbourhood $W$ of $0$ in $V$ with the following properties:

1. $\sigma^{-1}(W) \subset \bigcup U$.
2. Let $H \in F_0$ be a power series centred at $0 \in V$, and suppose there exists $f_U \in Q(U)$, for each $U$, such that $f_{U,a} = \delta^*_a(H)$, for all $a \in \sigma^{-1}(0) \cap U$. Then there exists $f \in Q(\sigma^{-1}(W))$ such that $f_a = \delta^*_a(H)$, for all $a \in \sigma^{-1}(0)$.

**Proof.** There is a covering of the fibre $\sigma^{-1}(0)$ by finitely many open sets $\Omega_i$ with compact closure, such that, for each $i$, there exists $U$ such that $\overline{\Omega_i} \subset U$; write $f_U = f_i$ (of course, $U$ is not necessarily unique). We can assume that each $\Omega_i \cap \Omega_j$ has only finitely many connected components (e.g., take each $\Omega_i$ sub-quasianalytic).

For each $i$ and $j$, if $\Omega$ is a connected component of $\Omega_i \cap \Omega_j$ and its closure $\overline{\Omega}$ includes a point of $\sigma^{-1}(0)$, then $f_i = f_j$ in $\Omega$, by quasianalyticity. For each $i$, let $V_i$ denote the complement in $\Omega_i$ of the union of the $\overline{\Omega}$, for all connected components $\Omega$ of $\Omega_i \cap \Omega_j$, for every $j$, such that $\overline{\Omega} \cap \sigma^{-1}(0) = \emptyset$. Then $\{V_i\}$ is an open covering of $\sigma^{-1}(0)$. Let $W$ be any neighbourhood $0 \in V$ such that $\sigma^{-1}(W) \subset \bigcup V_i$. Then $f_i = f_j$ in $V_i \cap V_j \cap \sigma^{-1}(W)$, for all $i, j$, so the result follows. □

**Corollary 4.5.** Let $\sigma : M \to V$ denote a proper $Q$-mapping, where $M$ is a $Q$-manifold of dimension $n$, and $V$ is an open neighbourhood of the origin in $\mathbb{R}^n$. Let $H \in F_0$ be a power series centred at $0 \in V$. Suppose that, for all $a \in \sigma^{-1}(0)$, there
is a neighbourhood $U$ of $a$ with coordinates $(x_1,\ldots,x_n)$ such that $\sigma|_U$ satisfies the hypotheses of Theorem \ref{thm:gluing} and there exists $f_U \in \mathcal{Q}(U)$ such that $\hat{f}_{U,a} = \hat{\sigma}_a(H)$. Then:

1. There is a neighbourhood $W$ of $0$ in $V$ such that $\sigma^{-1}(W) \subset \bigcup U$, and a function $f_\beta \in \mathcal{Q}(\sigma^{-1}(W))$, for every $\beta \in \mathbb{N}^n$, with the following properties: each point of $\sigma^{-1}(0)$ has a neighbourhood $\Omega$ in $U \cap \sigma^{-1}(W)$, for some $U$, such that $f_\beta = f_{U,\beta}$ in $\Omega$, for all $\beta$ (where $f_{U,\beta}$ denotes the function associated to $f_U$ given by Theorem \ref{thm:gluing}).

2. (After perhaps shrinking $W$) $f = f_0$ is formally composite with $\sigma$; i.e., for all $b \in W$, there exists $H_b \in \mathcal{F}_b$ such that $\hat{f}_a = \hat{\sigma}_a(H_b)$, for all $a \in \sigma^{-1}(b)$.

3. In fact, there is a $C^\infty$ function $h \in C^\infty(W)$ such that $f = h \circ \sigma$.

Proof. (1) This gluing condition is immediate from Lemma \ref{lem:gluing}. 

(2) It is enough to show that, after shrinking $W$ if necessary, each $f_\beta$ is constant on the fibres of $\sigma$ over $W$. For every $k \in \mathbb{N}$, let

$$P_k := \{(\xi,\eta,\zeta) \in M \times M \times W : \sigma(\xi) = \zeta = \sigma(\eta), f_\beta(\xi) = f_\beta(\eta), |\beta| \leq k\}.$$ 

Then the decreasing sequence of closed quasianalytic sets $P_0 \supset P_1 \supset P_2 \supset \cdots$ stabilizes in some neighbourhood of the compact set $\sigma^{-1}(0) \times \sigma^{-1}(0) \times \{0\}$, by topological noetherianity \cite[Thm. 6.1]{topological}; say, $P_k = P_{k_0}$, $k \geq k_0$, in such a neighbourhood. It follows that, if $W$ is a sufficiently small neighbourhood of $0$ in $V$, and $f_\beta$ is constant on the fibres of $\sigma$ over $W$, for all $\beta \leq k_0$, then $f_\beta$ is constant on the fibres of $\sigma$ over $W$, for all $\beta$.

Therefore, it is enough to prove the following assertion: given $\beta \in \mathbb{N}^n$, there is an open neighbourhood $W$ of $0$ such that $f_\beta$ is constant on the fibres of $\sigma$ over $W$. We can now argue as in the proof of Lemma \ref{lem:gluing}. Define $P \subset M \times M \times W$ as in \ref{eq:gluing}. Suppose the assertion is false. Then there is a point $(a_1,a_2,0) \in \overline{P}$, and a quasianalytic arc $(\alpha(t),\beta(t),\gamma(t)) \in M \times M \times V$ such that $(\alpha(0),\beta(0),\gamma(0)) = (a_1,a_2,0)$ and $(\alpha(t),\beta(t),\gamma(t)) \in P$ if $t \neq 0$. We get a contradiction as before.

(3) The hypotheses on $\sigma$ imply that $\sigma$ is generically a submersion, so the assertion follows from (1) by a quasianalytic generalization \cite{generalization} of Glaeser’s composite function theorem \cite{glaeser} (cf. Corollary \ref{cor:generalization} if. below). \hfill \Box

**Proposition 4.6.** Let $\varphi : M \to \mathbb{R}^n$ denote a $\mathcal{Q}$-mapping, where $M$ is a $\mathcal{Q}$-manifold of dimension $m$. Let $f \in \mathcal{Q}(M)$ and let $H$ denote a formal power series at $b = 0 \in \mathbb{R}^n$. Then

$$S := \{a \in \varphi^{-1}(b) : \hat{f}_a = H \circ \hat{\varphi}_a\}$$

is open and closed in $\varphi^{-1}(b)$.

Proof. We work in a local coordinate chart of $M$ with coordinates $u = (u_1,\ldots,u_m)$ at a point $a = 0$ in $\varphi^{-1}(0)$. Write $H = \sum_{\beta \in \mathbb{N}^m} H_\beta v^\beta / \beta!$, where $v = (v_1,\ldots,v_n)$. For $x \in \varphi^{-1}(0)$,

$$\hat{f}_x(u) - (H \circ \hat{\varphi}_x)(u) = \sum_{\alpha \in \mathbb{N}^m} \frac{\partial^\alpha f(x)}{\alpha!} u^\alpha - \sum_{\beta \in \mathbb{N}^m} H_\beta \left( \sum_{|\alpha| > 0} \frac{\partial^\alpha \varphi(x)}{\alpha!} u^\alpha \right)^\beta$$

$$= \sum_{\alpha \in \mathbb{N}^m} \frac{1}{\alpha!} (\partial^\alpha f(x) - K_\alpha(x)) u^\alpha,$$
where each $K_\alpha(x)$ is a finite linear combination of products of derivatives of the components of $\varphi$ (defined in the coordinate neighbourhood). (We write $\partial^\alpha := \partial^{\alpha_1}/\partial x_1^{\alpha_1}$ in this proof, and use the same notation for the formal derivative of a power series, below.) If $x \in \varphi^{-1}(0)$, then $\hat{f}_x = H \circ \hat{\varphi}_x$ if and only if $\partial^\alpha f(x) - K_\alpha(x) = 0$, for all $\alpha$; i.e., $S$ is closed.

To show that $S$ is open, it is enough to prove that, if $a = 0 \in S$ (i.e., $\hat{f}_0 - H \circ \hat{\varphi}_0 = 0$), then $\partial^\alpha f - K_\alpha$ vanishes on a (common) neighbourhood of $a$ in $\varphi^{-1}(0)$, for all $\alpha$; i.e., $(\partial^\alpha f)(\gamma(t)) - K_\alpha(\gamma(t)) = 0$, for all $\alpha$, for any quasianalytic curve $\gamma(t)$ in $\varphi^{-1}(0)$, $\gamma(0) = 0$.

Consider such a curve $\gamma(t)$. Since $\hat{f}_0 - H \circ \hat{\varphi}_0 = 0$ and $\varphi \circ \gamma = 0$,

$$0 = \hat{f}_0(\gamma_0(t) + u) - H(\hat{\varphi}_0(\gamma_0(t) + u)) = \sum_\alpha \frac{\partial^\alpha \hat{f}_0(\gamma_0(t))}{\alpha!} u^\alpha - \sum_\alpha \frac{\hat{K}_\alpha(\gamma_0(t))}{\alpha!} u^\alpha;$$

i.e., for all $\alpha$, $(\partial^\alpha f \circ \gamma)_0(t) - (K_\alpha \circ \gamma)_0(t) = 0$; therefore, $(\partial^\alpha f)(\gamma(t)) - K_\alpha(\gamma(t)) = 0$, by quasianalyticity.

\textbf{Remarks 4.7.} (1) We will use the results above (apart from Proposition 4.6 in the case that $\sigma$ is a quasianalytic mapping of one of two kinds:

(a) $\sigma : M \rightarrow V$ is a blowing-up of $V$ with centre a closed submanifold of $V$ of class $Q$, or, more generally, $\sigma$ is a finite composite of admissible blowings-up, where a blowing-up is called admissible if its center is a $Q$-manifold that has only normal crossings with respect to the exceptional divisor.

(b) $\sigma : U \rightarrow V$ is a power substitution

$$(y_1, \ldots, y_n) = (x_1^{k_1}, \ldots, x_n^{k_n}),$$

where the exponents $k_i$ are positive integers, or, more generally,

$$\sigma : \coprod_{\epsilon \in \{-1, 1\}^n} U^\epsilon \rightarrow V,$$

where $\coprod$ means disjoint union, each $U^\epsilon$, $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$, is a copy of $U$, and $\sigma^\epsilon := \sigma|_{U^\epsilon}$ is given by

$$(y_1, \ldots, y_n) = (\epsilon_1 x_1^{k_1}, \ldots, \epsilon_n x_n^{k_n}).$$

(We assume that $V$ is of the form $\prod_{i=1}^n (-\delta_i, \delta_i)$, where each $\delta_i > 0$. The images $\sigma^\epsilon(U^\epsilon)$ are unions of closed quadrants, covering $V$.)

(2) Corollary 4.6 extends in an immediate way to the case that, instead of $\sigma : M \rightarrow V$, we have a locally finite covering $\{\sigma_j : M_j \rightarrow V\}$ of $V$ by quasianalytic mappings, where each $\sigma_j$ is a finite composite of admissible local blowings-up. This version of the corollary will be needed in the proof of Theorem 1.1.

A family of mappings $\{\sigma_j : M_j \rightarrow V\}$ is a locally finite covering of $V$ if (a) the images $\sigma_j(M_j)$ are subordinate to a locally finite covering of $V$ by open subsets; (b) if $K$ is a compact subset of $V$, then there is a compact subset $K_j$ of $M_j$, for each $j$, such that $K = \bigcup \sigma_j(K_j)$ (the union is finite, by (a)). A local blowing-up of $V$ is a blowing-up over an open subset of $V$. 

Corollary 4.8. Assume in Corollary 4.7 that \( \sigma \) is a mapping of either kind in Remarks 4.7(1). Then there is a (perhaps larger) quasianalytic class \( Q' \) depending only on \( Q \) and \( \sigma \) and there exists \( h \in Q'(W \cap \sigma(U)) \) such that \( f = h \circ \sigma \). Likewise for the version of Corollary 4.5 given in Remarks 4.7(2).

This is a consequence of Corollary 4.8 and Lemmas 3.1, 3.4 (see also Remark 4.9). Note that, in order to prove Corollary 4.8, we need to use only Glaeser’s original theorem [14] (rather than a quasianalytic version) in the proof of Corollary 4.5(3), because blowings-up or power substitutions are algebraic (polynomial) mappings (with respect to suitable quasianalytic coordinates).

We illustrate the use of the techniques above in two special cases of our main theorems:

**Proposition 4.9** (Membership in a principal ideal; cf. [25]). Let \( Q \) be a quasianalytic class and let \( g \in Q(V) \), where \( V \) is a neighbourhood of 0 in \( \mathbb{R}^n \). Then there is a quasianalytic class \( Q' \supseteq Q \) such that, given \( f \in Q(V) \) and a formal power series \( h \in F_0 \) such that \( f_0 = H \cdot g_0 \), there exists \( h \in Q'(W) \), where \( W \) is a neighbourhood of 0 in \( V \), such that \( f = h \cdot g \).

**Remarks 4.10.** (1) Thilliez has studied several cases of functions \( g \in Q(V) \), where \( Q \) is a quasianalytic Denjoy-Carleman class \( Q_M \), which satisfy the following property: if \( f \in Q(V) \) and \( \hat{f}_a \) is divisible by \( \hat{g}_a \), for all \( a \in V \), then \( f = h \cdot g \), where \( h \in Q(V) \) (see [29] Section 4 and Remark 3.3 below). By Proposition 4.9, the formal divisibility at a single point implies formal divisibility throughout a neighbourhood.

(2) If \( f \) is merely \( C^\infty \), the latter statement is not true, and it is necessary to assume that \( \hat{f}_0 \) is divisible by \( \hat{g}_0 \) throughout a neighbourhood \( W \) of 0 in order to guarantee that \( f = h \cdot g \), where \( h \in C^\infty(W) \) (quasianalytic version of the Łojasiewicz-Malgrange division theorem [6, Thm. 6.4]). The proof in [6] nevertheless works for Proposition 4.9 using the division axiom 2.1(1) and Corollary 4.8.

**Proof of Proposition 4.9** (Shrinking \( V \) if necessary) there is a mapping \( \sigma : M \to V \) given by a finite composite of blowings-up as in Remarks 4.7(1)(a), such that \( g \circ \sigma \) is a monomial times a non-normal factor (in suitable quasianalytic coordinates) in some neighbourhood of any point of \( \sigma^{-1}(0) \). By axiom 2.1(1), \( f \circ \sigma = \varphi \cdot g \circ \sigma \), where \( \varphi \in Q(M) \). Clearly, \( \hat{\varphi}_a = \hat{\sigma}_a'(H) \), for all \( a \in \sigma^{-1}(0) \), so the result follows from Corollary 4.8.

**Proposition 4.11** (\( k \)'th root of a quasianalytic function). Let \( Q \) be a quasianalytic class and let \( g \in Q(V) \), where \( V \) is a neighbourhood of 0 in \( \mathbb{R}^n \). Then there is a quasianalytic class \( Q' \supseteq Q \) such that, if \( k \) is a positive integer and \( g \) has a \( k \)'th root in formal power series at 0; i.e., \( \hat{g}_0 = H^k \), where \( H \in F_0 \), then there is a neighbourhood \( W \) of 0 in \( V \) and a quasianalytic function \( h \in Q'(W) \) such that \( g = h^k \).

**Proof.** (Shrinking \( V \) if necessary) there is a mapping \( \sigma : M \to V \) given by a finite composite of admissible blowings-up, such that \( g \circ \sigma \) is a monomial times an invertible factor (in suitable quasianalytic coordinates) in some neighbourhood \( U \) of any point of \( \sigma^{-1}(0) \). By the hypothesis, this monomial is a \( k \)'th power, and we can take \( h_U \in Q(U) \) such that \( g \circ \sigma|_U = f_U^k \) and \( \hat{f}_{U,a} = \hat{\sigma}_a'(H) \), for all \( a \in \sigma^{-1}(0) \cap U \). The result follows from Corollary 4.8.
5. Polynomial Equations with Quasianalytic Coefficients

Proof of Theorem 1.2. Let \( Q(U, \mathbb{C}) \) denote the ring of \( \mathbb{C} \)-valued functions of quasianalytic class \( Q \) on \( U \). We can consider \( G(x, y) \) as an element of \( Q(U, \mathbb{C})[y] \) and each \( B_{ji} \in \mathbb{C}[x][y] \), and it is enough to prove the result in the ring of polynomials with complex-valued quasianalytic coefficients. We break the proof into a number of lemmas.

Lemma 5.1. We can assume that \( a_1 = 0 \), and that there exists \( \alpha \in \mathbb{N}^n \setminus \{0\} \) such that
\[
a_i(x)^{d_{ji}} = x^{\alpha_i}a_i^*(x), \quad i = 2, \ldots, d,
\]
where each \( a_i^* \in Q(U, \mathbb{C}) \), and \( a_i^* \) is a unit, for some \( i \).

Proof. We can reduce to the case that \( a_1 = 0 \), by a coordinate change \( y' = y + a_1(x)/d \). Let \( I \) denote the ideal sheaf generated by the functions \( a_i^{d_{ji}}, i = 2, \ldots, d \). The theorem is trivial if \( I = (0) \). Otherwise, by resolution of singularities of \( I \), there is a finite composite of admissible blowings-up \( \sigma : M \to U \) (after shrinking \( U \) to a relatively compact neighbourhood of \( 0 \)) such that any point \( a \in \sigma^{-1}(0) \) admits a coordinate neighbourhood \( W \) (with coordinates \( z = (z_1, \ldots, z_n) \), say) in which the pullback of \( I \) is generated by a monomial \( x^\alpha, \alpha \in \mathbb{N}^n \); i.e.,
\[
a_i(\sigma(z))^{d_{ji}} = x^\alpha a_i^*(z), \quad i = 2, \ldots, d,
\]
where \( a_i^* \) is a unit in \( Q(W, \mathbb{C}) \), for some \( i \) [6, Thm. 5.9].

By Corollary 4.8, it is enough to find quasianalytic functions \( c_{ji}(z) \) such that \( \tilde{c}_{ji, \alpha} = \tilde{\sigma}_{ji}^*(B_{ji}) \), \( j = 1, \ldots, k \), \( i = 1, \ldots, d_j \), and
\[
G(\sigma(z), y) = \prod_{j=1}^k (y^{d_j} + c_{j1}(x)y^{d_j-1} + \cdots + c_{jd_j}(x))
\]
in \( Q'(V, \mathbb{C})[y] \). Therefore, we can replace \( G \) by \( G(\sigma(z), y) \) and each \( B_{ji} \) by \( \tilde{\sigma}_{ji}^*(B_{ji}) \) to get the lemma.

Notation 5.2. For any positive integer \( k \), we will write \( x^k \) to denote \( (x_1^k, \ldots, x_n^k) \).

For any \( \epsilon \in \{-1, 1\}^n \), we will write \( \tau_{\epsilon}^k \) to denote the substitution
\[
\tau_{\epsilon}^k(x) := (\epsilon_1 x_1^k, \ldots, \epsilon_n x_n^k).
\]

Lemma 5.3. We can assume, moreover, that
\[
a_i(x) = x^{\alpha_i}\tilde{a}_i(x), \quad i = 2, \ldots, d,
\]
where each \( \tilde{a}_i \in Q(U, \mathbb{C}) \), and \( \tilde{a}_i \) is a unit, for some \( i \).

Proof. For each \( i = 2, \ldots, d \), since \( a_i(x)^{d_{ji}} \) is divisible by \( x^{\alpha_i} \), it follows that \( a_i(x^{d_{ji}})^{d_{ji}} \) is divisible by \( x^{\alpha_i d_{ji}} \), and therefore that \( a_i(x^{d_{ji}}) \) is divisible by \( x^{\alpha_i} \) as a function of class \( Q \) (using unique factorization of formal power series and axioms 2.1(1), (3)); i.e., \( a_i(x^{d_{ji}}) = x^{\alpha_i} \tilde{a}_i(x), i = 2, \ldots, d \), where each \( \tilde{a}_i \) is of class \( Q \) (and \( \tilde{a}_i \) is a unit, for some \( i \)). Likewise for \( a_i(\tau_{\epsilon}^d(x)) \), for any \( \epsilon \in \{-1, 1\}^n \). The assertion now follows from Corollary 4.8, since, according to the latter, it is enough to prove the theorem after a power substitution \( \tau_{\epsilon}^d(x) \).

Lemma 5.4. Under the assumptions of Lemmas 5.1, 5.2, \( G(x, y) \) has a nontrivial factorization \( G = G_1G_2 \) in \( Q(U, \mathbb{C})[y] \), after perhaps shrinking \( U \), and each formal factor
\[
H_j(x, y) = y^{d_{ji}} + B_{j1}(x)y^{d_{ji}-1} + \cdots + B_{j, d_j}(x)
\]
splits as \( H_j = H_{j1}H_{j2} \), where \( H_{jl} \) is a formal factor of \( G_l \), \( l = 1, 2 \) (perhaps \( H_{j1} \) or \( H_{j2} = \text{constant} \)).

**Proof.** Since \( \tilde{a}_i(0) \neq 0 \), for some \( i \), we can write

\[
y^d + \tilde{a}_2(0)y^{d-2} + \cdots + \tilde{a}_d(0) \in \mathbb{C}[y]
\]
as a nontrivial product of two polynomials with no common factor. Therefore, there is also a nontrivial splitting

\[
y^d + \sum_{i=2}^d \tilde{a}_i(x)y^{d-i} = \left( y^k + \sum_{i=1}^k \xi_i(x)y^{k-i} \right) \cdot \left( y^l + \sum_{i=1}^l \eta_i(x)y^{l-i} \right),
\]
where \( k + l = d \) (see [3, Lemma 3.1] and Lemma 5.5 below), so that

\[
G(x,y) = \left( y^k + \sum_{i=1}^k \xi_i(x)y^{k-i} \right) \cdot \left( y^l + \sum_{i=1}^l \eta_i(x)y^{l-i} \right),
\]
where \( \xi_i(x) = x^{i\alpha}\tilde{\xi}_i(x) \), \( i = 1, \ldots, k \), and \( \eta_i(x) = x^{i\alpha}\tilde{\eta}_i(x) \), \( i = 1, \ldots, l \). The corresponding factorization of each \( H_j \) follows from unique factorization of formal power series.

Theorem 1.2 follows, by induction on the degree \( d \) of \( G \). \( \square \)

We recall [3, Lemma 3.1] and its proof, since this result is needed also in Section 7 below.

**Lemma 5.5.** Let \( P(x,y) = y^d + \sum_{i=1}^d A_i(x)y^{d-i} \), where the coefficients are functions in some class (e.g., formal power series in \( x = (x_1, \ldots, x_n) \), or \( C^\infty \) functions or functions of a quasianalytic class \( Q \) in a neighbourhood of \( 0 \in \mathbb{R}^n \)). Suppose that

\[
P(0,y) = y^d + \sum_{i=1}^d A_i(0)y^{d-i} = Q(\beta_0,y)R(\gamma_0,y),
\]
where

\[
Q(\beta_0,y) = y^k + \sum_{i=1}^k \beta_{0,i}y^{k-i}, \quad R(\gamma_0,y) = y^l + \sum_{i=1}^l \gamma_{0,i}y^{l-i}
\]
are polynomials in \( y \) with no common factor, \( k + l = d \). Then

\[
P(x,y) = \left( y^k + \sum_{i=1}^k B_i(x)y^{k-i} \right) \cdot \left( y^l + \sum_{i=1}^l C_i(x)y^{l-i} \right),
\]
where the coefficients \( B_i, C_j \) are functions of the given class.

**Proof.** Let

\[
Q(\beta,y) = y^k + \sum_{i=1}^k \beta_iy^{k-i}, \quad R(\gamma,y) = y^l + \sum_{i=1}^l \gamma_iy^{l-i},
\]
where \( \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^k \), \( \gamma = (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l \). Write

\[
Q(\beta,y)R(\gamma,y) = y^d + \sum_{i=1}^d \alpha_i(\beta,\gamma)y^{d-i}.
\]
Then the Jacobian determinant $\Delta(\beta, \gamma) := \det \partial \alpha(\beta, \gamma) / \partial (\beta, \gamma)$ is the resultant of $Q, R$ as polynomials in $y$. By the inverse function theorem, since $\Delta(\beta_0, \gamma_0) \neq 0$, we can write

$$y^d + \sum_{i=1}^d \alpha_i y^{d-i} = Q(\beta(\alpha), y)R(\gamma(\alpha), y),$$

where $\beta(A(0)) = \beta_0, \gamma(A(0)) = \gamma_0, A(x) = (A_1(x), \ldots, A_d(x))$. Then the assertion of the lemma holds with $B_i(x) = \beta_i(A(x)), C_j(x) = \gamma_j(A(x))$. \hfill \square

6. Quasianalytic Equations

In this section, we will prove Theorem 1.1 using Corollary 4.8. The latter allows us to follow the scheme of [7], in a simpler way.

**Lemma 6.1.** We can assume, without loss of generality, that, for some positive integer $d$, $G(x, y)$ is $y$-regular of order $d$ at $(a, b)$; i.e., $(\partial^d G / \partial y^d)(a, b) = 0$ if $j < d$, but $(\partial^d G / \partial y^d)(a, b) \neq 0$.

**Proof.** We can assume that $(a, b) = (0, 0)$, so that $G(0, 0) = 0$ and $H(0) = 0$. Let $\varphi_i := (\partial^i G / \partial y^i)(x, 0), i \in \mathbb{N}$. By resolution of singularities [7] Thm. 3.1, after shrinking $U$ to a relatively compact neighbourhood of $0$, there is a $Q$-mapping $\sigma : M \to U$ given by a finite composite of admissible blowings-up, such that any $a' \in \sigma^{-1}(0)$ admits a coordinate neighbourhood $W$ in which the ideal $J$ generated by the restrictions of $\varphi_i \circ \sigma, i \in \mathbb{N}$, is a principal ideal generated by a monomial $z^\alpha$, $\alpha \in \mathbb{N}^n$, where $z = (z_1, \ldots, z_n)$ (and $a'$ is the origin of the coordinate chart).

We claim that $G(\sigma(z), y)$ is divisible by $z^\alpha$; i.e., that $\tilde{G}(z, y) := z^{-\alpha} G(\sigma(z), y)$ is a quasianalytic function. It is enough to show that, for each $i = 1, \ldots, n$ such that $\alpha_i \neq 0, G(\sigma(z), y)$ is divisible by $z_i$, or (according to axiom 2.1(1)), that $G(\sigma(z), y)$ vanishes on the hyperplane $(z_i = 0)$. For fixed $z$ such that $z_i = 0, \varphi_i := G(\sigma(z), y)$ is of class $Q$ and

$$\frac{\partial^j \varphi_i}{\partial y^j}(0) = (\partial^i G / \partial y^i)(\sigma(z), 0) = 0, \quad j \in \mathbb{N}.$$

By axiom 2.1(3), $\varphi_i$ vanishes identically, as required. Thus $\tilde{G}(z, y)$ is of class $Q$, and $\tilde{G}(z, \sigma^*_\alpha(H)(z)) = 0$.

Since the ideal $J$ is generated by $z^\alpha$, there exists $d$ such that $\varphi_d \circ \sigma | W = z^\alpha$ times an invertible factor. Thus, $(\partial^d \tilde{G} / \partial y^d)(a', 0) \neq 0$. It follows from Corollary 4.8 that we can assume $G(x, H(x)) = 0$, where $G$ is $y$-regular of some order $d$ at 0. \hfill \square

We now prove the theorem by induction on $d$. The case $d = 1$ is a consequence of the implicit function theorem (axiom 2.1(2)). We can assume that $(a, b) = (0, 0)$, so that $G(0, 0) = 0$ and $H(0) = 0$.

**Lemma 6.2.** We can assume that $\frac{\partial^{d-1} G}{\partial y^{d-1}}(x, 0) = 0$.

**Proof.** Since $G$ is $y$-regular of order $d$ at $(0, 0)$, the function $(\partial^{d-1} G / \partial y^{d-1})(x, y)$ has nonvanishing derivative with respect to $y$ at $(0, 0)$. By the implicit function theorem (axiom 2.1(2)), there is a function $\varphi(x)$ of class $Q$ at 0 such that $\varphi(0) = 0$ and $(\partial^{d-1} G / \partial y^{d-1})(x, \varphi(x)) = 0$. We can replace $G(x, y)$ by $G(x, y + \varphi(x))$ and $H(x)$ by $H(x) - \varphi(y(x))$ to get the lemma. \hfill \square
Now, set
\[ c_i(x) := \frac{1}{(d-i)!} \partial_{d-i}^i G(x,0), \quad i = 2, \ldots, d; \]
thus \( c_1 = 0 \). Taking the Taylor expansion of \( G(x, y) \) with respect to \( y \), we can write
\[ G(x, y) = \rho(x, y)y^d + \sum_{i=2}^{d} c_i(x)y^{d-i}, \]
where \( \rho \) is \( C^\infty \) and thus of class \( Q \) (by axiom 2.1(1)), and \( \rho(0,0) \neq 0 \).

**Lemma 6.3.** We can assume there exists \( \alpha \in \mathbb{N}^n \setminus \{0\} \) such that
\[ c_i(x)^{d/i} = x^\alpha c_1^*(x), \quad i = 2, \ldots, d, \]
where each \( c_1^* \) is of class \( Q \) and \( c_i^* \) is a unit, for some \( i \).

**Proof.** By (6.1),
\[ \sum_{i=0}^{d-2} c_i(x)H(x)^i + \rho(x,H(x))H(x)^d = 0, \]
as a formal expansion at 0. Let \( \mathcal{I} \) denote the ideal sheaf generated by the functions \( c_i^{d/i}, i = 2, \ldots, d \). If \( \mathcal{I} = (0) \), then \( H = 0 \), by (6.2), so of course we can take \( h = 0 \) to solve our problem. Otherwise, we apply resolution of singularities to \( \mathcal{I} \), to obtain a finite composite of admissible blowings-up \( \sigma : M \rightarrow U \) (after shrinking \( U \) to a relatively compact neighbourhood of 0) such that any point \( a \in \sigma^{-1}(0) \) admits a coordinate neighbourhood \( W \) (with coordinates \( z = (z_1, \ldots, z_n) \), say) in which the pullback of \( \mathcal{I} \) is generated by a monomial \( z^\alpha, \alpha \in \mathbb{N} \setminus \{0\} \); i.e.,
\[ c_i(\sigma(z))^{d/i} = z^\alpha c_i^*(z), \quad i = 2, \ldots, d, \]
where \( c_i^* \) is a unit in \( Q(W) \), for some \( i \).

By Corollary 4.8, it is enough to find a quasianalytic function \( h(z) \) such that
\[ G(\sigma(z), h(z)) = 0 \quad \text{and} \quad \tilde{h}_a = \tilde{c}_a^*(H). \]
Therefore, we can replace \( G \) by \( G(\sigma(z), y) \) and \( H \) by \( \tilde{c}_a^*(H) \) to get the lemma. \( \square \)

As in the proof of Lemma 6.3 we can write \( c_i(x^{d/i}) = x^{i\alpha} \tilde{c}_i(x), i = 2, \ldots, d, \)
where each \( \tilde{c}_i \) is of class \( Q \) (and \( \tilde{c}_i \) is a unit, for some \( i \) ). (Recall Notation 5.2)

Consider
\[ G_1(x, y) := x^{-d\alpha} G(x^{d/i}, x^\alpha y) \]
\[ = \rho(x^{d/i}, x^\alpha y)y^d + \sum_{i=2}^{d} \tilde{c}_i(x)y^{d-i}, \]
\[ H_1(x) := x^{-\alpha} H(x^{d/i}). \]
Clearly, \( G_1(x, y) \) is a well-defined function of class \( Q \) in a neighbourhood of the \( y \)-axis. Since \( c_{d-1} = 0 \), \( G_1(x, y) \) is \( y \)-regular of order \( \leq d - 1 \) at any point \( (0, y_0) \). On the other hand, \( H_1(x) \) is a priori a Laurent series (with finitely many negative exponents). We have
\[ G_1(x, H_1(x)) = x^{-d\alpha} G(x^{d/i}, H(x^{d/i})), \]
Write
\[ H_1(x) = \sum_{i=0}^{n} x^{\beta_i_1} \cdots x^{\beta_n}, \]
(6.3)
where the exponents $\beta_j$ a priori may be negative.

**Lemma 6.4.** $H_1(x)$ is a formal power series; i.e., $H_1(x)$ has only nonnegative exponents $\beta_j$.

**Proof.** We first check that, for any formal curve $x(t) = (x_1(t), \ldots, x_n(t))$, $x(0) = 0$, the formal expansion $H_1(x(t))$ has nonnegative order; i.e., order $H(x(t)^d) \geq\deg x(t)^\alpha$.

Write $K(x) := H(x^d)$ to simplify the notation. Since $G_1(x, x^{-\alpha} K(x)) = 0$,

\begin{equation}
\rho(x(t)^d, K(x(t)))K(x(t))^d + \sum_{i=2}^{d} \tilde{c}_i(x(t))x(t)^i K(x(t))^{d-i} = 0.
\end{equation}

Suppose that order $K(x(t)) < \deg x(t)^\alpha$. Then, for each $i$, order $K(x(t))^i < \deg x(t)^i$, so that order $K(x(t))^d < \deg x(t)^i K(x(t))^{d-i}$, in contradiction to \((6.4)\).

Now suppose there is a negative exponent $\beta_j$ in \((6.3)\) (for some nonzero $\xi_{\beta}$). Let $b$ denote the smallest negative exponent that occurs; we can assume that $b = \beta_1$, for some $\beta = (\beta_1, \ldots, \beta_n)$. Let $a$ denote the smallest $\beta_1 > b$ that occurs in \((6.3)\), $A$ the smallest $\beta_2 + \cdots + \beta_B$ that occurs, and $B$ the smallest $\beta_2 + \cdots + \beta_n$ that occurs among those exponents with $\beta_1 = b$.

Choose $q \in \mathbb{N}$ such that $qb + B < 0$ and $qb + B < qa + A$. Let $I := \{\beta : \beta_1 = b, \beta_2 + \cdots + \beta_n = B\}$. Take $x(t) = (\lambda_1 t^\alpha, \lambda_2 t, \ldots, \lambda_n t)$, where $\lambda$ is chosen so that $\sum_{\beta \in I} \xi_{\beta} \lambda^B \neq 0$ ($\lambda$ exists because $\sum_{\beta \in I} \xi_{\beta} s^B$ is a nonzero polynomial). Then order $H_1(x(t)) < 0$; a contradiction. \hfill \Box

In the same way as above, for all $\epsilon \in \{-1, 1\}^n$, define

$G_1^\epsilon(x, y) := x^{-d \alpha} G(\tau_\epsilon^d(x), x^\alpha y)$,

$H_1^\epsilon(x) := x^{-\alpha} H(\tau_\epsilon^d(x))$

(cf. Notation \[6.2\]; then $H_1^\epsilon(x)$ is a formal power series, and $G_1^\epsilon(x, y)$ is a well-defined function of class $Q$ in a neighbourhood of the $y$-axis, which is $y$-regular of order $d - 1$ at any point $(0, y_0)$).

By Corollary \[1.3\] it is enough to show that there is a quasianalytic class $Q' \supseteq Q$ with the property that, for all $\epsilon \in \{-1, 1\}^n$, we can find a function $h^\epsilon$ quasianalytic of class $Q'$, such that $h^\epsilon_0 = H(\tau_\epsilon^d(x))$.

By induction on $d$, there exists $Q'$ with the property that, for each $\epsilon$, we can find $h^\epsilon$ of class $Q'$ such that $G_1(\tau_\epsilon^d(x), h^\epsilon_1(x)) = 0$ and $(h^\epsilon_1)_0 = H_1(\tau_\epsilon^d(x))$; then we can take $h^\epsilon(x) := x^\alpha h^\epsilon_1(x)$.

This completes the proof of Theorem \[1.1\] \hfill \Box

### 7. Remarks on Weierstrass preparation

Let $Q$ denote any subclass of $C^\infty$ functions which is closed under differentiation and taking the reciprocal of a nonvanishing function (we do not assume the axioms of Definition \[2.1\] to begin with). A **Weierstrass polynomial** in $y$ of degree $d$ at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$ means a function

$p(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x)$,

where the coefficients $a_i(x) = a_i(x_1, \ldots, x_n)$ are of class $Q$ and vanish at $0$. 

Definitions 7.1. (1) \( Q \) has the **Weierstrass preparation property** if, for every function \( g(x, y) \) of class \( Q \) that is \( y \)-regular of order \( d \) at \((0,0)\) (see Lemma 6.1), there exists a Weierstrass polynomial \( p(x, y) \) of degree \( d \) at \((0,0)\), such that \( g(x, y) = u(x, y)p(x, y) \) in a neighbourhood of \((0,0)\), where \( u \) is a unit of class \( Q \).

(2) \( Q \) has the **Weierstrass division property** if, given \( f(x, y), g(x, y) \) of class \( Q \), where \( g \) is \( y \)-regular of order \( d \) at \((0,0)\),

\[
f(x, y) = q(x, y)g(x, y) + \sum_{i=1}^{d} r_i(x)y^{d-i},
\]

where \( q \) and the \( r_i \) are of class \( Q \).

(3) \( Q \) has the **property of division by a Weierstrass polynomial** if (2) holds in the special case that \( g \) is a Weierstrass polynomial in \( y \) of degree \( d \).

The following lemma is classical, though it seems not so well known (see [12, §2], [15, Kap. I, §4. Supp. 3]).

**Lemma 7.2.** The Weierstrass preparation and division properties are equivalent. If \( Q \) satisfies the implicit function property and the property of division by a Weierstrass polynomial (axiom 7.1(2)), then all three properties of Definitions 7.1 are equivalent.

**Proof.** We first show that Weierstrass preparation in \( k+1 \) variables implies Weierstrass division in \( k \) variables. Suppose that \( g(x, y) = g(x_1, \ldots, x_n, y) \) is of class \( Q \) and \( y \)-regular of order \( d \) at \((0,0)\). Let \( f(x,y) \) be a function of class \( Q \). We want to divide \( f \) by \( g \). By Weierstrass preparation, \( g(x, y) = u(x, y)p(x, y) \) in class \( Q \), where \( u \) is a unit and \( p \) is a Weierstrass polynomial \( p(x, y) = g^d + \sum_{i=1}^{d} a_i(x)y^{d-i} \).

The function \( F(x,y,t) = p(x,y) + tf(x,y) \) is \( y \)-regular of order \( d \) at \((0,0,0)\). By Weierstrass preparation,

\[
(7.1) \quad p(x, y) + tf(x, y) = U(x, y, t)P(x, y, t),
\]

where \( U \) is a unit and \( P \) is a Weierstrass polynomial in \( y \) of degree \( d \). Clearly, \( U(x, y, 0) = 1 \) and \( P(x, y, 0) = p(x, y) \). Let

\[
r(x, y) = \left. \frac{\partial P(x, y, t)}{\partial t} \right|_{t=0};
\]

then \( r(x, y) \) is a polynomial of degree \(< d \) in \( y \). Apply \( \partial / \partial t \) to (7.1) and set \( t = 0 \); we get

\[
f(x, y) = q(x, y)g(x, y) + r(x, y),
\]

where

\[
q(x, y) = \left. \frac{\partial U(x, y, t)}{\partial t} \right|_{t=0} . u(x, y)^{-1},
\]

as required.

Clearly, Weierstrass division in \( k \) variables implies Weierstrass preparation in \( k \) variables. (Given \( g(x, y) \) regular of order \( d \) in \( y \), divide \( y^d \) by \( g \) and subtract the remainder term.)

Now assume that \( Q \) satisfies the implicit function property and the property of division by a Weierstrass polynomial. Let \( P(\lambda, y) \) denote the **generic polynomial** of degree \( d \),

\[
P(\lambda, y) := y^d + \sum_{i=1}^{d} \lambda_i y^{d-i}.
\]
Given $g(x, y)$, divide by $P(\lambda, y)$ as functions of $(x, \lambda, y)$:

(7.2) \[ g(x, y) = q(x, \lambda, y)P(\lambda, y) + \sum_{i=1}^{d} r_i(x, \lambda)y^{d-i}. \]

Suppose that $g(x, y)$ is $y$-regular of order $d$ at $(0, 0)$. Put $x = 0 = \lambda$ in (7.2); then $\text{unit} \cdot y^d = q(0, 0, y)y^d + \sum r_i(0, 0)y^{d-i}$. Clearly, $q(0, 0, 0) \neq 0$ and $r_i(0, 0) = 0$, for all $i$. It is easy to check that the Jacobian determinant $\det(\partial r_i/\partial \lambda_j)$ does not vanish at $(0, 0)$, so that the system of equations $r_i(x, \lambda) = 0$ has a solution $\lambda = \varphi(x)$, $\varphi(0) = 0$, and we get Weierstrass preparation $g(x, y) = q(x, \varphi(x), y)P(\varphi(x), y)$.

**Definition 7.3.** Let $g(x, y) = g(x_1, \ldots, x_n, y)$ denote a $C^\infty$ function that is $y$-regular of order $d$ at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$. We say that $g$ is $y$-hyperbolic at $(0, 0)$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for any given $x$ such that $|x| < \delta$ (where $|x| := (x_1^2 + \cdots + x_n^2)^{1/2}$), $g(x, y) = 0$ has $d$ real roots (counted with multiplicity) in the interval $(-\epsilon, \epsilon)$.

**Theorem 7.4.** Let $g(x, y)$ be a function of quasianalytic class $Q$ in a neighbourhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$. Assume that $g$ is regular of order $d$ and hyperbolic with respect to $y$ at $(0, 0)$. Then there is a (perhaps larger) quasianalytic class $Q' \supseteq Q$ such that $g(x, y) = u(x, y)p(x, y)$ near $(0, 0)$, where

\[ p(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x) \]

is a Weierstrass polynomial with coefficients $a_i(x)$ of class $Q'$, and $u$ is a unit of class $Q'$.

**Proof.** The proof is by induction on $d$. By the formal Weierstrass preparation theorem,

\[ \hat{g}(0,0)(x, y) = U(x, y)H(x, y), \]

where

\[ H(x, y) = y^d + \sum_{i=1}^{d} B_i(x)y^{d-i} \in \mathbb{R}[x][y] \]

and $U(x, y)$ is a unit in $\mathbb{R}[x, y]$.

We argue as in the proof of Theorem 1.1 (Section 6). Note that the hyperbolicity property of $g$ is preserved after pull-back to any point in the inverse image of the origin, by a blowing-up in the $x$-variables. We can assume that

\[ (\partial^{d-1}g/\partial y^{d-1})(0, 0) = 0 \]

(Lemma 6.2), so that

\[ g(x, y) = \rho(x, y)y^d + \sum_{i=2}^{d} c_i(x)y^{d-i}, \]

as in (6.11), where $\rho(0, 0) \neq 0$. We can also assume (as in Lemma 6.3 ff.) that there exists $\alpha \in \mathbb{N}^n \setminus \{0\}$ such that $c_i(x^{\alpha}) = x^{i\alpha}\tilde{c}_i(x)$, $i = 2, \ldots, d$, where each $\tilde{c}_i$ is of class $Q$ and $\tilde{c}_i$ is a unit, for some $i$.

Consider

\[ G_1(x, y) := x^{-d\alpha}g(x^{\alpha}, x^{\alpha}y) = \rho(x^{\alpha}, x^{\alpha}y)y^d + \sum_{i=2}^{d} \tilde{c}_i(x)y^{d-i}, \]

\[ H_1(x, y) := x^{-d\alpha}H(x^{\alpha}, x^{\alpha}y) = y^d + \sum_{i=1}^{d} x^{-i\alpha}B_i(x^{\alpha}y)y^{d-i}. \]
Then $G_1(x, y) = U(x^{d}, x^\alpha y)H_1(x, y)$ as formal expansions. Setting $y = 0$, we see that $\tilde{c}_i(x) = U(x^{d}, 0)x^{-\alpha -i}B_i(x^d)$, so that $B_d(x^{d})$ is divisible by $x^{-\alpha d}$; i.e., $x^{-\alpha}B_d(x^{d})$ is a formal power series $\tilde{B}_d(x)$. Successively taking $\partial^j/\partial y^j$, $j = 1, 2, \ldots$ and setting $y = 0$, we see that each $x^{-\alpha}B_i(x^{d})$ is a formal power series $\tilde{B}_i(x)$.

Setting $x = 0$, we have

$$\rho(0, 0)y^{d} + \sum_{i=2}^{d} \tilde{c}_i(0)y^{-i} = U(0, 0) \left( y^{d} + \sum_{i=1}^{d} \tilde{B}_i(0)y^{-i} \right).$$

Therefore, $\rho(0, 0) = U(0, 0), \tilde{B}_1(0) = 0,$ and $\tilde{c}_i(0) = U(0, 0)\tilde{B}_i(0), i = 2, \ldots, d$.

We claim that all roots of $y^{d} + \sum_{i=1}^{d} \tilde{B}_i(0)y^{-i} = 0$ are real. In fact, by the Malgrange preparation theorem [20, Ch. V], we can write $g(x, y) = u(x, y)h(x, y)$ in a neighbourhood of the origin, where $u(x, y)$ is a nonvanishing $C^\infty$ function, and

$$h(x, y) = y^{d} + \sum_{i=1}^{d} b_i(x)y^{-i},$$

where, for each $i$, $b_i(x)$ is $C^\infty$ and $b_i(0) = 0$. It follows that $h(x, y)$ is $y$-hyperbolic at 0, and $B_i(x)$ is the formal Taylor expansion at 0 of $b_i(x)$, for each $i$. Set $\tilde{b}_i(x) := b_i(x^{d})/x^{\alpha i}, i = 1, \ldots, d$. Then each $\tilde{b}_i(x)$ is a $C^\infty$ function in a neighbourhood of 0 (as can be seen by taking successive derivatives with respect to $y$ of the equation

$$G_1(x, y) = u(x^{d}, x^\alpha y)h_1(x, y),$$

where $h_1(x, y) := y^{d} + \sum_{i=1}^{d} \tilde{b}_i(x)y^{-i}$, and then setting $y = 0$). Moreover, $\tilde{b}_i(0) = \tilde{B}_i(0)$, for each $i$, and $y^{d} + \sum_{i=1}^{d} \tilde{B}_i(0)y^{-i} = 0$ has $d$ complex roots. So these are all real, by continuity of the roots of $h_1(x, y) = 0$ (for example, on a line $(x_1, \ldots, x_n) = (t, t, \ldots, t)$).

Moreover, since $\tilde{B}_1(0) = 0$ and $\tilde{B}_i(0)$ is a unit, for some $i$, we can write

$$y^{d} + \sum_{i=1}^{d} \tilde{B}_i(0)y^{-i} = \prod_{j=1}^{q} (y - \lambda_j)^{d_j},$$

where the $\lambda_j$ are distinct real numbers and $q \geq 2$; thus each $d_j < d$. It follows from Lemma 5.3 that

$$H_1(x, y) = \prod_{j=1}^{q} \left( y^{d_j} + \sum_{i=1}^{d_j} B_{ji}(x)y^{-i} \right),$$

where all $B_{ji}(x) \in \mathbb{R}[x]$.

For each $j$, $G(x, \lambda_j + y)$ is $y$-regular of order $d_j$ and hyperbolic at $(0, 0)$. By induction, there is a quasianalytic class $\mathcal{Q}' \supseteq \mathcal{Q}$, such that every coefficient $B_{ji}(x)$ is the formal Taylor expansion at 0 of a function $b_{ji}(x)$ of class $\mathcal{Q}'$.

The argument above applies equally to $G_1'(x, y)$ and $H_1'(x, y)$ (as defined in the proof of Theorem 1.1), for any $\epsilon \in \{-1, 1\}^{n}$, so the conclusion of Theorem 7.4 follows also as in the proof of Theorem 1.1. (Of course, $p(x, y)$ coincides with the function $h(x, y)$ above.)

Remarks 7.5. (1) Chaumat and Chollet proved that division by a hyperbolic Weierstrass polynomial of quasianalytic class $\mathcal{Q}$ (i.e., division according to property (3) of Definitions 7.1) holds with no loss of regularity in the quotient and remainder 10. It follows from 10 together with Theorem 7.4 that, if $g(x, y)$ is a function of class $\mathcal{Q}$ that is regular and hyperbolic with respect to $y$, then Weierstrass division by $g(x, y)$ (property (2) above) holds with loss of regularity given by Theorem 7.4. It
is not evident that this result follows directly either from \[10\] or from Theorem 7.4 (compare with the implications (1) \(\implies\) (2) and (3) \(\implies\) (1) in the proof of Lemma 7.2 above).

(2) Let \(Q\) be a quasianalytic Denjoy-Carleman class \(Q_M\), and let \(g(x,y)\) denote a hyperbolic Weierstrass polynomial of class \(Q_M\), of degree \(d\) in \(y\) (for \(x\) in a neighbourhood of 0 in \(\mathbb{R}^n\)). Let \(a_0 = (0,0) \in \mathbb{R}^n \times \mathbb{R}\). If \(f(x,y)\) is of class \(Q_M\) and \(f_{a_0} = H \cdot g_{a_0}\), where \(H \in \mathbb{R}[x,y]\), then there exists \(h(x,y)\) of class \(Q_M\) in a neighbourhood of \(a_0\), such that \(f = h \cdot g\). In fact, by Proposition 4.9 for each \(x\) near 0, all roots of \(g(x,y) = 0\) are roots of \(f(x,y) = 0\). By \[10\], \(f(x,y) = q(x,y)g(x,y) + r(x,y)\) near \(a_0\), where \(q, r\) are of class \(Q_M\), and \(r(x,y)\) is a polynomial in \(y\) of degree < \(d\); therefore, \(r = 0\).

(3) The loss of regularity in Theorem 7.4 depends on the function \(g(x,y)\). For any given quasianalytic Denjoy-Carleman class \(Q_M\) (except \(Q_M = \mathcal{O}\)), it is not true that there exists a (perhaps larger) Denjoy-Carleman class \(Q_M'\), such that, given \(g(x,y)\) \(\gamma\)-regular of class \(Q_M\) (or a Weierstrass polynomial of class \(Q_M\)), any function \(f(x,y)\) of class \(Q_M\) admits Weierstrass division by \(g(x,y)\) with quotient and remainder of class \(Q_M'\) \([11], [19], [30]\).

It is easy to see that, given \(Q_M\) and \(f \in Q_M([0,1])\) (say \(f(0) = 0, f'(0) > 0\)), the function \(g(x,y) := f(y^2) - x\) (which is \(\gamma\)-regular of order 2) satisfies the Weierstrass preparation property with \(u, p \in Q'\), for some quasianalytic class \(Q' \supseteq Q_M\), if and only if \(f\) extends to a function in \(Q'((-\delta,1))\), for some \(\delta > 0\). Nazarov, Sodin and Volberg \[23\] showed, in fact, that there is a quasianalytic Denjoy-Carleman class \(Q_M\), and a function \(f \in Q_M([0,1])\) which admits no extension to a function in \(Q_M'((-\delta,1))\), for any quasianalytic \(Q_M'\) and \(\delta > 0\).

It seems interesting to ask whether a Denjoy-Carleman class \(Q_M\) nevertheless does have the Weierstrass division property or the extension property as above, where \(Q'\) is some quasianalytic class that depends on the given functions.

(4) The point of view of this article seems relevant to the study of many algebraic properties of local rings of quasianalytic functions. For example, it is unknown (and unlikely) that such local rings are Noetherian, in general, although a topological version of Noetherianity follows from resolution of singularities \([9]\) Thm. 6.1; cf. the proof of Corollary 4.3 above. The local ring \(Q_n\) of germs of functions of quasianalytic class \(Q\) at the origin of \(\mathbb{R}^n\) is Noetherian if and only if, for all \(f,g_1, \ldots, g_p \in Q_n\), the equation \(f(x) = \sum_{i=1}^p y_i g_i(x)\) has a solution \(y_i = h_i(x), i = 1, \ldots, p\), of class \(Q\), provided there is a formal power series solution \(y_i = H_i(x)\). One can ask whether this formal condition implies rather the existence of a quasianalytic solution with loss of regularity depending on \(g_1, \ldots, g_p\) and perhaps \(f\).

REFERENCES

1. F. Acquistapace, F. Broglia, M. Bronshtein, A.Nicoara and N. Zobin, Failure of the Weierstrass preparation theorem in quasi-analytic Denjoy-Carleman rings, Adv. Math. 258 (2014), 397–413.

2. I. Biborski, On the geometric and differential properties of closed sets definable in quasianalytic structures, preprint, 2015, 32 pages, arXiv:1511.05671v1 [math.AC].

3. E. Bierstone and P.D. Milman, Arc-analytic functions, Invent. Math. 101 (1990), 411–424.

4. E. Bierstone and P.D. Milman, Relations among analytic functions, II, Ann. Inst. Fourier (Grenoble) 37:2 (1987), 49–77.

5. E. Bierstone and P.D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math. 128 (1997), 207–302.
6. E. Bierstone and P.D. Milman, Resolution of singularities in Denjoy-Carleman classes, Selecta Math. (N.S.) 10 (2004), 1–28.
7. E. Bierstone, P.D. Milman and G. Valette, Arc-quasianalytic functions, Proc. Amer. Math. Soc. 143 (2015), 3915–3925.
8. E. Borel, Sur la généralisation du prolongement analytique, C. R. Acad. Sci. Paris 130 (1900), 1115–1118.
9. T. Carleman, Les Fonctions Quasi-analytiques, Collection Borel, Gauthier-Villars, Paris, 1926.
10. J. Chaumat and A.-M. Chollet, Division par un polynôme hyperbolique, Can. J. Math. 56 (2004), 1121-1144.
11. C.L. Childress, Weierstrass division in quasianalytic local rings, Can. J. Math. 28 (1976), 938–953.
12. R. Cluckers and L. Lipshitz, Strictly convergent analytic structures, J. Euro. Math. Soc. 19 (2017), 107–149.
13. A. Denjoy, Sur les fonctions quasi-analytiques de variable réelle, C. R. Acad. Sci. Paris 173 (1921), 3120–1322.
14. G. Glaeser, Fonctions composées différentiables, Ann. of Math. 77 (1963), 193–209.
15. H. Grauert and R. Remmert, Analytische Stellenalgebren, Grundl. der mathematischen Wissenschaften 176, Springer-Verlag, Berlin-Heidelberg-New York, 1971.
16. J. Hadamard, Lectures on Cauchy’s Problem in Linear Partial Differential Equations, Yale Univ. Press, New Haven, 1923.
17. L. Hörmander, The Analysis of Linear Partial Differential Operators I, Springer-Verlag, Berlin-Heidelberg-New York, 1983.
18. H. Komatsu, The implicit function theorem for ultradifferentiable mappings, Proc. Japan Acad. 55 (1979), 69–72.
19. M. Langenbruch, Extension of ultradifferentiable functions, Manuscripta Math. 83 (1994), 123–143.
20. B. Malgrange, Ideals of Differentiable Functions, Tata Institute of Fundamental Research Studies in Math. 3, Oxford University Press, London 1966.
21. S. Mandelbrojt, Séries Adhérentes, Régularisation des Suites, Applications, Collection Borel, Gauthiers-Villars, Paris, 1952.
22. C. Miller, Infinite differentiability in polynomially bounded o-minimal structures, Proc. Amer. Math. Soc. 123 (1995), 2551–2555.
23. F. Nazarov, M. Sodin, A. Volberg, Lower bounds for quasianalytic functions. I. How to control smooth functions, Math. Scand. 95 (2004), 59–79.
24. K.J. Nowak, A note on Bierstone-Milman-Pawłucki’s paper “Composite differentiable functions”, Ann. Polon. Math. 102 (2011), 293–299.
25. K.J. Nowak, On division of quasianalytic function germs, Int. J. Math. 13 (2013), 1–5.
26. K.J. Nowak, Quantifier elimination in quasianalytic structures via non-standard analysis, Ann. Polon. Math. 114 (2015), 235–267.
27. J.-P. Rolin, P. Speissegger and A. J. Wilkie, Quasianalytic Denjoy-Carleman classes and o-minimality, J. Amer. Math. Soc. 16 (2003), 751–777.
28. C. Roumieu, Ultradistributions définies sur R^n et sur certaines classes de variétés différentiables, J. Analyse Math. 10 (1962–63), 153–192.
29. V. Thilliez, On quasianalytic local rings, Expo. Math. 26 (2008), 1–23.
30. V. Thilliez, On the non-extendability of quasianalytic germs, preprint, 2010, 4 pages, arXiv:1006:4171v1 [mathCA].

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