Fermi-Frenet coordinates for space-like curves

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Abstract. We generalize Fermi coordinates, which correspond to an adapted set of coordinates describing the vicinity of an observer’s worldline, to the worldsheet of an arbitrary spatial curve in a static spacetime. The spatial coordinate axes are fixed using a covariant Frenet triad so that the metric can be expressed using the curvature and torsion of the spatial curve. As an application of Fermi-Frenet coordinates, we show that they allow covariant inertial forces to be expressed in a simple and physically intuitive way.

PACS numbers: 04.20.-q, 04.20.Cv, 02.40.-k

1. Introduction

Finding a set of coordinates that is adapted to a particular physical situation is a very useful tool that helps to simplify the analysis and interpretation of physical phenomena. In particular, Fermi coordinates [1, 2, 3, 4] allow one to describe the vicinity of an observer’s worldline by using only geometrically defined quantities: the time coordinate is the observer’s proper time, and direction and modulus of the spatial coordinates are constructed using a tetrad on the worldline and the length of a geodesic starting on the worldline. The metric tensor expressed in this coordinate system is locally flat, i.e., it is of Minkowskian type on every point on the worldline. In the vicinity of the worldline the metric can be expressed as a Taylor series in the geodesic distance from the worldline. The original formulation of Fermi coordinates has later been extended to include non-inertial observers [5, 6] and explicitly constructed for weak gravitational fields [7]. Attempts to define Fermi coordinates away from the worldline are subtle [8] and lead to paradoxical phenomena even in very simple situations [9].

Fermi coordinates are particularly useful to describe situations that require a spatially extended analysis around a localized object. This applies in particular to the coupling between point particles and waves, where the wave dynamics in the vicinity of the point particles has to be taken into account. For instance, Fermi coordinates have been used to study macroscopic electrodynamics in rotating reference frames [10], Dirac fields in non-inertial frames [11], gravitational corrections to the spectrum of hydrogen atoms [12], and gravitationally induced phase shifts in atom interferometers [13].
In this paper we generalize Fermi coordinates to the case of the ‘worldsheet’ associated with a spatial curve in a static spacetime with metric $\tilde{g}_{\alpha\beta}$ in which $\tilde{g}_{00}$ is constant. Given a static metric a one-parameter family of spacelike hypersurfaces $\Sigma_\tau$ foliates spacetime and furnishes a natural time coordinate $\tau$ \[14\]. If on each of these hypersurfaces the same (time independent) spatial curve exists, then the union of these curves over all $\tau$ is a two-dimensional subspace, the ‘worldsheet’ of the curve. We will construct a set of coordinates that are locally flat on the worldsheet and determine the second-order expansion of the metric about the worldsheet. This is different from previous efforts in that the expansion is about a two-dimensional surface rather than a one-dimensional worldline, meaning that a greater volume of the background spacetime is covered by the expansion. Any number of observers constrained to otherwise arbitrary motion on the worldsheet can agree on a single set of coordinates, and any spacetime event sufficiently near the worldsheet can be expressed in them. The only requirements on the curve are that it is smooth. For simplicity we also assume that it is not a geodesic, although this requirement can be lifted.

We make use of metrics with signature $(+−−−)$, and use Greek indices to run over $0$ to $3$. For reasons that will become clear, we make non-standard use of Latin indices by having them take on only the values $2$ and $3$. The summation convention is employed throughout.

\[\{\tilde{\alpha}\}_\mu\] denotes an arbitrary set of spacetime coordinates and $\tau$ is the natural time coordinate. Points on the spatial curve that we will study are parametrized by $f(\ell)$, with $\ell$ the arclength parameter. The worldsheet of this curve then corresponds to the set of events $\Sigma := \bigcup_{\tau,\ell} (\tau, f(\ell))$.

2. Definition of Fermi-Frenet coordinates

Our goal is to establish a construction principle for the adapted set of coordinates and to express the metric $g_{\alpha\beta}$ as a Taylor series around the worldsheet. The metric in Fermi-Frenet coordinates takes the form $g_{\alpha\beta}|_{\Sigma} = \eta_{\alpha\beta}$ for all values of $\tau$ and $\ell$. This is accomplished by determining a tetrad $\{e_\alpha\}$ of orthonormal vectors defined everywhere on the worldsheet. Underlined indices run from $0$ to $3$ and label the four tetrad vectors.

For a fixed $\tau$ let $\tilde{z} = (\tau, f(\ell)) \subset \Sigma_{\tau}$ be a (spacelike) curve. We want to define an orthonormal tetrad $\{e_\alpha\}$ along $\tilde{z}$ in an analogous way to how the Frenet frame is defined along a curve $\alpha(s) \subset \mathbb{R}^3$ \[15\]. The normalized timelike vector and the tangent vector to the spatial curve are given by

\[
\tilde{e}_0 := \tilde{m} := \left( \frac{1}{\sqrt{\tilde{g}_{00}}}, 0, 0, 0 \right), \quad \tilde{e}_1 := \tilde{t} := \left( 0, \frac{df(\ell)}{d\ell} \right),
\]

with $\tilde{t} \cdot \tilde{m} = -1$ because $\ell$ is the arc length for the curve. With the covariant derivative of a 4-vector $\tilde{\alpha}$ along the curve, $\nabla_\ell \tilde{\alpha} = \tilde{t} \alpha^0 \partial_\alpha \tilde{\alpha} + \tilde{\Gamma}^{\mu}_{\alpha\beta} \tilde{m} \cdot \tilde{\alpha} \tilde{\alpha}^\beta$, we find that $\nabla_\ell (\tilde{t} \cdot \tilde{m}) = 0$ implies that $\nabla_\ell \tilde{m} =: \tilde{N}$ is orthogonal to $\tilde{t}$.

In three-dimensional Euclidean space the Frenet frame is only unambiguously defined if the curvature $\kappa$ is non-zero, because of the definition of the normal vector through $\kappa''(\ell) = \kappa(\ell) n$, with $\kappa^2 = \alpha^\alpha \cdot \alpha^\alpha$. Analogously we define $K(\ell) := (\tilde{N} \cdot \tilde{N})^{1/2}$. 
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While it is possible to proceed with the construction even when $K$ vanishes for some range on $\dot{z}^\mu(\ell)$, we assume for simplicity that the curve is not a geodesic, i.e. $\nabla_\ell \dot{t}^\mu \neq 0$. We then can introduce the definition of the normal vector to the curve,

$$\dot{e}_2^\mu := \dot{n}^\mu := \frac{\dot{N}^\mu}{K}. \quad (2)$$

The last tetrad vector corresponds to Frenet’s binormal vector which is normalized and orthogonal to each of the other vectors. This can be achieved by defining

$$\dot{e}_3^\mu := \dot{b}^\mu := \delta_{\alpha\beta\gamma} \dot{m}_\alpha \dot{t}_\beta \dot{n}_\gamma, \quad (3)$$

where $\delta_{\alpha\beta\delta}$ is the natural volume element on the background spacetime.

It can be shown that $\nabla_\ell \dot{b}^\mu = -T(\ell) \dot{n}^\mu$, where $T(\ell)$ corresponds to the torsion of the spatial curve (which is not related to a torsion of spacetime). Furthermore one has $\dot{n}^\mu = \delta_{\mu\beta\gamma} \dot{m}_\alpha \dot{b}_\beta \dot{t}_\gamma$ so that the change of the tetrad along the spatial curve can be expressed as

$$\nabla_\ell \dot{e}_0^\mu = 0, \quad (4a)$$

$$\nabla_\ell \dot{e}_1^\mu = K \dot{e}_2^\mu, \quad (4b)$$

$$\nabla_\ell \dot{e}_2^\mu = -K \dot{e}_1^\mu + T \dot{e}_3^\mu, \quad (4c)$$

$$\nabla_\ell \dot{e}_3^\mu = -T \dot{e}_2^\mu, \quad (4d)$$

where $(4a)$ follows from the fact that the covariant derivative of the metric vanishes.

Having constructed the tetrad on the worldsheet we can introduce Fermi-Frenet coordinates in the vicinity of the worldsheet. Consider some event $\dot{x}^\mu$ sufficiently near the worldsheet. It can be uniquely parametrized by the following set of quantities: (i) a point $\dot{z}(\tau, \ell)$ on the worldsheet, (ii) a geodesic $\dot{y}(s)$ that connects $\dot{x}$ and $\dot{z}(\tau, \ell)$, and (iii) the geodesic arc length $s_0$ between the two events along $\dot{y}$. We can therefore take $\dot{y}(0) = \dot{z}(\tau, \ell)$ and $\dot{y}(s_0) = \dot{x}$. The point $\dot{z}(\tau, \ell)$ and the geodesic are fixed by requiring that the tangent of the geodesic on the worldsheet is orthogonal to it, i.e., $\nabla_s \dot{y}^\mu|_{s=0} = \cos \theta \dot{n}^\mu + \sin \theta \dot{b}^\mu$ for some angle $\theta$. The event $\dot{x}$ can then be labeled in a new coordinate system by the four Fermi-Frenet coordinates $x^\mu = (\tau, \ell, s_0 \cos \theta, s_0 \sin \theta)$.

3. Expansion of the metric around the worldsheet

For events $x^\mu$ that are sufficiently close to the worldsheet we can expand the metric in Fermi-Frenet coordinates to second order in the transverse coordinates $x^2, x^3$ as

$$g_{\alpha\beta}|_{x^\mu} \approx g_{\alpha\beta}|_{\mathcal{O}} + x^i g_{\alpha\beta,i}|_{\mathcal{O}} + \frac{1}{2} x^i x^j g_{\alpha\beta,ij}|_{\mathcal{O}}, \quad (5)$$

with $\mathcal{O} = (\tau, \ell, 0, 0) \in \Sigma$ and summation on latin indices running over 2 and 3. To find the derivatives of the metric on the worldsheet we first deduce the Christoffel symbols from various propagation equations. The covariant derivative of a tetrad vector along the curve is given by $\nabla_\ell \dot{e}_2^\mu = \nabla_\ell e_2^\mu = \Gamma^\mu_{\alpha\beta} e_2^\mu$. Comparing this with $(\mathbb{H})$ yields the nonzero components $\Gamma^2_{11} = -\Gamma^1_{12} = K$ and $\Gamma^3_{12} = -\Gamma^2_{13} = T$. To show that all other components
do vanish we start with the geodesic equation for \( y^\mu(s) = (\tau, \ell, s \cos \theta, s \sin \theta) \), the curve that defines the coordinates of an event near the worldsheet,

\[
0 = \frac{d^2 y^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dy^\beta}{ds} \frac{dy^\gamma}{ds} = \Gamma^\alpha_{22} \cos^2 \theta + 2 \Gamma^\alpha_{23} \cos \theta \sin \theta + \Gamma^\alpha_{33} \sin^2 \theta .
\]

(6)

Since this must hold for any event near \( \Sigma \), and hence \( \forall \theta \), each of the Christoffel symbols involved must vanish independently. Finally, since the background spacetime is static, we have \( 0 = \nabla_{\ell} e^\mu_\alpha = \Gamma^\beta_{0\alpha} e^\mu_\beta \) so that \( \Gamma^\beta_{0\alpha} = 0 \).

The derivatives of the metric can be found by exploiting the fact that the covariant derivative of the metric vanishes. Using \( g_{\alpha\beta} = \eta_{\alpha\beta} \) on the worldsheet one easily finds

\[
g_{11,2} = 2 K; \quad g_{12,3} = T; \quad g_{13,2} = -T .
\]

(7)

To find the second derivatives of the metric on \( \Sigma \) we look to the first derivatives of the Christoffel symbols for use in

\[
g_{\alpha\beta,\gamma\delta} = g_{\beta\mu,\delta} \Gamma^\mu_{\alpha\gamma} + g_{\beta\mu} \Gamma^\mu_{\alpha\gamma,\delta} + g_{\alpha\mu,\delta} \Gamma^\mu_{\beta\gamma} + g_{\alpha\mu} \Gamma^\mu_{\beta\gamma,\delta} .
\]

(8)

As (5) requires derivatives only with respect to the final two coordinates it only remains to derive the quantities \( \Gamma^\mu_{\beta_i j} \). This can be done by using their relation to the the Riemann curvature tensor through

\[
R_{\alpha\beta\gamma\delta}^\mu = \Gamma^\mu_{\alpha\gamma,\beta} - \Gamma^\mu_{\beta\gamma,\alpha} + \Gamma^\nu_{\alpha\gamma} \Gamma^\mu_{\beta\nu} - \Gamma^\nu_{\beta\gamma} \Gamma^\mu_{\alpha\nu} .
\]

(9)

Its components on the worldsheet are given by \( R_{\rho\sigma\mu\nu} = \tilde{R}_{\rho\sigma\gamma\delta} e^\alpha_\rho e^\beta_\sigma e^\gamma_\mu e^\delta_\nu \). Because the metric is static one quickly finds \( \Gamma^\mu_{0i j} = R_{0ji}^\mu \).

To find the other derivatives we can make use of the equation of geodesic deviation [4]. For a family of geodesics with tangent vectors \( Y^\mu \) and deviation vectors \( X^\mu \) this is given by

\[
0 = \frac{d^2 X^\mu}{ds^2} + 2 \frac{dX^\alpha}{ds} Y^\beta \Gamma^\mu_{\alpha\beta} + Y^\alpha Y^\beta X^\gamma \left( R_{\beta\gamma\alpha\mu} + \Gamma^\mu_{\alpha\gamma} + \Gamma^\lambda_{\gamma\alpha} \Gamma^\mu_{\lambda\beta} - \Gamma^\mu_{\gamma\lambda} \Gamma^\lambda_{\alpha\beta} \right) .
\]

(10)

In our case we consider \( Y^\mu = x^i \delta^\mu_i \) and we will look first at \( X^\mu = \delta^\mu_1 \). That is, the tangents to the geodesics lie in the plane spanned by \( n^\mu \) and \( t^\mu \), and their deviation along the \( x^1 \)-direction is being examined. This then yields

\[
0 = Y^i Y^j \left( R_{ji\mu} + \Gamma^\mu_{1i} + \Gamma^\lambda_{1i} \Gamma^\mu_{\lambda j} - \Gamma^\mu_{1\lambda} \Gamma^\lambda_{ij} \right) .
\]

(11)

The symmetric part (over \( i \) and \( j \)) of the parenthetical term must vanish,

\[
0 = R_{ji\mu} + R_{i1\mu} + \Gamma^\mu_{1i} + \Gamma^\lambda_{1i} \Gamma^\mu_{\lambda j} + \Gamma^\lambda_{ij} \Gamma^\mu_{\lambda i} - 2 \Gamma^\mu_{1\lambda} \Gamma^\lambda_{ij} .
\]

(12)

Equation (12) and the analogue of (9) for \( R_{i1j}^\mu \) can be solved respectively for the sum and difference of \( \Gamma^\mu_{1i,j} \) and \( \Gamma^\mu_{i1,j} \). Combined, these give us

\[
\Gamma^\mu_{i1,j} = R_{i1j}^\mu - \Gamma^\lambda_{1i} \Gamma^\mu_{\lambda j} .
\]

(13)

Finally we utilize the geodesic deviation equation with deviation vector \( X^\mu = \delta^\mu_0 \), i.e. deviation in the direction of the time coordinate. For the two coordinate directions
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defined by geodesics (i.e. $x^2$ and $x^3$) this gives the same result as obtained in Ref. [4] for the three spatial coordinates,

$$\Gamma_{ij,k}^\mu = \frac{1}{3} (R_{jki}^\mu + R_{ikj}^\mu) . \quad (14)$$

With a little algebra the results derived above allow us to determine the second-order derivatives of the metric on the worldsheet. Defining the quantity $\Delta_{\alpha\beta}(x^\mu) = \frac{1}{3} x^i x^j (R_{\alpha ij \beta})|_\Sigma$ we can express the Taylor expansion (5) of the metric at $x^\mu$ to second order in the geodesic distance from the worldsheet as

$$g_{00} \approx 1 + 3\Delta_{00} , \quad (15a)$$
$$g_{01} \approx 3\Delta_{01} , \quad (15b)$$
$$g_{0i} \approx 2\Delta_{0i} , \quad (15c)$$
$$g_{11} \approx -1 + 2K x^2 - (K^2 + T^2) (x^2)^2 - T^2 (x^3)^2 + 3\Delta_{11} , \quad (15d)$$
$$g_{ii} \approx T (\delta_{i2} x^3 - \delta_{i3} x^2) + 2\Delta_{ii} , \quad (15e)$$
$$g_{ij} \approx -\delta_{ij} + \Delta_{ij} . \quad (15f)$$

This is the main result of the paper.

4. Application of Fermi-Frenet coordinates: Identification of inertial forces

Fermi-Frenet coordinates allow one to describe physical situations in which a spatially extended description in the vicinity of a given spatial curve is needed. As in the case of Fermi coordinates, this is especially interesting in the context of wave dynamics. One situation where Fermi-Frenet coordinates would be favorable is the electromagnetic interaction between charged particles that are constrained to move on a given spatial curve. A second example would be the propagation of extended light pulses inside an optical fibre. Because of the well-known equivalence between Maxwell’s equations in curved space and Maxwell’s equations in a dielectric medium [16], Fermi-Frenet coordinates could also be used to describe the propagation of electromagnetic waves in inhomogeneous dielectric media.

To demonstrate the use of Fermi-Frenet coordinates we here discuss the observer-independent definition of inertial forces in a general relativistic setting, a problem that has been well studied (see, for example, Refs. [17] [18] [19]).

Consider a particle of mass $m$ constrained to motion on the worldsheet $\Sigma$. In the Fermi-Frenet coordinates we have derived, the 4-momentum of the particle is given by

$$p^\mu = (p^0, p^1, 0, 0) . \quad (16)$$

Following Abramowicz, et al. [17] we consider this trajectory to be an integral curve of some vector field extrapolated from the curve, such that in the following expression for the 4-force $f^\mu$ experienced by the particle the derivatives are well-defined,

$$mf_\mu = p^\nu \nabla_\nu p_\mu = p^\nu \partial_\nu p_\mu - \frac{1}{2} p^\nu p^\rho \partial_\mu g_{\nu \rho} . \quad (17)$$
The components of the 4-force can be found to be given by

\[ mf_0 = p^\nu \partial_\nu p^0 \, , \quad (18a) \]
\[ mf_1 = -p^\nu \partial_\nu p^1 \, , \quad (18b) \]
\[ mf_2 = K (p^1)^2 \, , \quad (18c) \]
\[ mf_3 = 0 \, . \quad (18d) \]

We see then that the particle feels an inertial force dependent upon the covariant curvature \( K \) as the mechanism constraining it to the worldsheet. Equations (18) have simple and intuitive physical interpretations. We note that

\[ p^\mu \partial_\mu = m \partial_t \, , \quad (19) \]

where \( t \) is the proper time experienced by the particle. With the energy and kinetic 3-momentum of the particle given by \( E = p^0 \) and \( p = mv = p^1 \), respectively, (18) can be written

\[ f_0 = \partial_t E \, , \quad (20a) \]
\[ f_1 = -\partial_t p \, , \quad (20b) \]
\[ f_2 = Kv^2 \, , \quad (20c) \]
\[ f_3 = 0 \, . \quad (20d) \]

We see that the 0-component of the 4-force experienced by the particle is given by the change in its energy, while the force felt in the direction of motion, (20b), corresponds to Newton’s second law. The fact that \( f_3 \) vanishes is due to the coordinate system we have constructed, and demonstrates that an appropriate choice of coordinates helps to simplify the equations of motion.

This leaves us with only (20c) to interpret. Since \( K \) is the curvature of the spatial curve it is equal to the inverse of the radius of its curvature. This means that \( f_2 \) can be viewed as the centripetal force constraining the particle to the worldsheet, corresponding to the classical force \( mv^2/r \).

It is clear then that examination of the forces felt by a particle on the worldsheet will yield information about the curvature \( K \). We might also expect the torsion \( T \) to play a role, and indeed by examining the parallel transport of an arbitrary 4-vector \( s^\mu = (s^0, s^1, s^2, s^3) \) carried by the particle we see exactly that. We require

\[ p^\nu \nabla_\nu s^\mu = 0 \, , \quad (21) \]

which yields the equations

\[ \partial_t s^0 = 0 \, , \quad (22a) \]
\[ \partial_t s^1 = Kv^2 \, , \quad (22b) \]
\[ \partial_t s^2 = -Kv^2 + Tv^3 \, , \quad (22c) \]
\[ \partial_t s^3 = -Tv^2 \, , \quad (22d) \]

showing the effect that \( T \) has on the transverse components of \( s^\mu \).
5. Concluding remarks

In conclusion, we have derived an adapted set of coordinates that allows for a simple and physically intuitive description of particles and fields that are constrained to move on, or in the vicinity of, a given spatial curve in a static spacetime. In a neighbourhood of the curve the metric can be expressed in terms of purely geometric properties: the curvature and torsion of the curve, and the Riemann tensor evaluated on the curve. This form allows an easy identification of inertial forces and is of practical use when considering extended objects or wave phenomena in the vicinity of the curve.

Acknowledgments

We thank David Hobill for helpful discussions. This work was supported by iCORE and NSERC.

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