RESOLVENT CONVERGENCE OF STURM-LIOUVILLE OPERATORS
WITH SINGULAR POTENTIALS

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Abstract. In this paper we consider the Sturm-Liouville operator in the Hilbert space $L_2$ with the singular complex potential of $W^{-1/2}$ and two-point boundary conditions. For this operator we give sufficient conditions for norm resolvent approximation by the operators of the same class.

1. Main result

Let on a compact interval $[a, b]$ the formal differential expression

$$l(y) = -y''(t) + q(t)y(t), \quad q(\cdot) \in L_2([a, b], \mathbb{C}) =: L_2,$$

be given.

This expression can be defined as the Shin-Zettl [1] quasi-differential expression with following quasi-derivatives [2]:

$$D^{[0]}y = y, \quad D^{[1]}y = y' - qy, \quad D^{[2]}y = -(D^{[1]}y)' - qD^{[1]}y - q^2y.$$

In this paper we consider the set of quasi-differential expressions $l_\varepsilon(\cdot)$ of the form (1) with potentials $q_\varepsilon(\cdot) \in L_2, \varepsilon \in [0, \varepsilon_0]$. In the Hilbert space $L_2$ with norm $\| \cdot \|_2$ each of these expressions generates a dense closed quasi-differential operator

$$L_\varepsilon y := l_\varepsilon(y),$$

$$Dom(L_\varepsilon) := \{ y \in L_2 : \exists D^{[2]}_\varepsilon y \in L_2; \quad \alpha(\varepsilon)\mathcal{Y}_a(\varepsilon) + \beta(\varepsilon)\mathcal{Y}_b(\varepsilon) = 0 \},$$

where matrices $\alpha(\varepsilon), \beta(\varepsilon) \in \mathbb{C}^{2 \times 2}$, and vectors

$$\mathcal{Y}_a(\varepsilon) := \{ y(a), D^{[1]}_\varepsilon y(a) \}, \quad \mathcal{Y}_b(\varepsilon) := \{ y(b), D^{[1]}_\varepsilon y(b) \} \in \mathbb{C}^2.$$

Recall that operators $L_\varepsilon$ converge to $L_0$ in the sense of norm resolvent convergence, $L_\varepsilon \xrightarrow{R} L_0$, if there exists a number $\mu \in \mathbb{C}$ such that $\mu \in \rho(L_0)$ and $\mu \in \rho(L_\varepsilon)$ (for all sufficiently small $\varepsilon$) and

$$\|(L_\varepsilon - \mu)^{-1} - (L_0 - \mu)^{-1}\| \to 0, \quad \varepsilon \to 0.$$\

This definition does not depend on the point $\mu \in \rho(L_0)$ [3].

For the case where matrices $\alpha(\varepsilon), \beta(\varepsilon)$ do not depend on $\varepsilon$, paper [2] gives following

Theorem 1. Suppose $\|q_\varepsilon - q_0\|_2 \to 0$ for $\varepsilon \to +0$ and the resolvent set of the operator $L_0$ is not empty. Then $L_\varepsilon \xrightarrow{R} L_0$.

Our goal is to generalize Theorem [1] onto the case of boundary conditions depending on $\varepsilon$ and to weaken conditions on potentials applying results of papers [4, 5].

Denote by $c'(t) := \int_a^t c(x)dx$ and by $\| \cdot \|_C$ the sup-norm.

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Theorem 2. Suppose the resolvent set of the operator \( L_0 \) is not empty and for \( \varepsilon \to 0 \):

1) \( \| q_\varepsilon \|_2 = O(1) \);
2) \( \| (q_\varepsilon - q_0) \|_C \to 0 \);
3) \( \| (q_\varepsilon^2 - q_0^2) \|_C \to 0 \);
4) \( \alpha(\varepsilon) \to \alpha(0), \quad \beta(\varepsilon) \to \beta(0) \).

Then \( L_\varepsilon \to L_0 \).

Note that condition 3) is not additive.

Condition 1) (taking into account 2), 3)) may be weakened in several directions.

Actually we will prove a stronger statement on the considered operators’ Green functions’ convergence with respect to the norm \( \| \cdot \|_\infty \) of the space \( L_\infty \) on the square \([a, b] \times [a, b] \).

2. Comparison of Theorems 1 and 2

We are going to show that if \( \| q_\varepsilon - q_0 \|_2 \to 0, \varepsilon \to +0 \), then conditions 1), 2), 3) of Theorem 2 are true.

Indeed, \( \| q_\varepsilon \|_2 \leq \| q_\varepsilon - q_0 \|_2 + \| q_0 \|_2 = O(1) \).

Also

\[
| \int_a^t (q_\varepsilon - q_0) ds \| \leq \int_a^b |q_\varepsilon - q_0| ds \leq (\int_a^b |q_\varepsilon - q_0|^2 ds)^{1/2} (b - a)^{1/2} \to 0, \varepsilon \to +0.
\]

\[
| \int_a^t (q_\varepsilon^2 - q_0^2) ds \| \leq \int_a^b |q_\varepsilon^2 - q_0^2| ds \leq \int_a^b |q_\varepsilon - q_0| |q_\varepsilon + q_0| ds \leq (\int_a^b |q_\varepsilon - q_0|^2 ds)^{1/2} (\int_a^b |q_\varepsilon + q_0|^2 ds)^{1/2} \to 0, \varepsilon \to +0.
\]

Following example proves Theorem 2 to be stronger than Theorem 1.

Example 1. Suppose \( q_0(t) \equiv 0 \), \( q_\varepsilon(t) = e^{it/\varepsilon}, t \in [0, 1] \).

The set of operators \( L_\varepsilon \) defined by these potentials does not satisfy assumptions of Theorem 1 because

\[
\| q_\varepsilon - q_0 \|^2_2 = \| q_\varepsilon \|^2_2 = \int_0^1 |q_\varepsilon|^2 ds \equiv 1.
\]

It is evident that functions \( q_\varepsilon(\cdot) \) do not converge to 0 even with respect to the Lebesgue measure. However, they satisfy conditions 1), 2), 3) of Theorem 2. Indeed, \( \| q_\varepsilon \|_2 \leq 1 \). Moreover,

\[
\| q_\varepsilon' \|_C = \| \int_0^t e^{-is/\varepsilon} ds \|_C \leq 2\varepsilon \to 0, \varepsilon \to +0.
\]

\[
\| (q_\varepsilon^2)' \|_C = \| \int_0^t (e^{-is/\varepsilon})^2 ds \|_C \leq \varepsilon \to 0, \varepsilon \to +0.
\]

3. Preliminary result

Consider a boundary-value problem

\[
y'(t; \varepsilon) = A(t; \varepsilon)y(t; \varepsilon) + f(t; \varepsilon), \quad t \in [a, b], \quad \varepsilon \in [0, \varepsilon_0] \tag{3.1_\varepsilon}
\]

\[
U_\varepsilon y(\cdot; \varepsilon) = 0, \tag{3.2_\varepsilon}
\]

where matrix functions \( A(\cdot, \varepsilon) \in L_1^{m \times m} \), vector-functions \( f(\cdot, \varepsilon) \in L_1^m \), and linear continuous operators \( U_\varepsilon : C([a, b] ; \mathbb{C}^m) \to \mathbb{C}^m \).

We recall from [4, 5]
**Definition** Denote by $\mathcal{M}^m[a,b] := \mathcal{M}^m$, $m \in \mathbb{N}$ the class of matrix functions $R(\cdot; \varepsilon) : [0, \varepsilon_0] \to L_1^{m \times m}$, such that the solution of the Cauchy problem

$$Z'(t; \varepsilon) = R(t; \varepsilon)Z(t; \varepsilon), \quad Z(a; \varepsilon) = I_m$$

satisfies the limit condition

$$\lim_{\varepsilon \to +0} \|Z(\cdot; \varepsilon) - I_m\|_C = 0.$$

Sufficient conditions for $R(\cdot; \varepsilon) \in \mathcal{M}^m$ derive from [6]. To prove Theorem 2 we apply the simplest of them

$$\|R(\cdot; \varepsilon]\|_1 = O(1), \quad \|R(\cdot; \varepsilon]\|_C \to 0,$$

where $\| \cdot \|_1$ is the norm in $L_1^{m \times m}$.

Paper [5] gives the following general

**Theorem 3.** Suppose

1) the homogeneous limit boundary-value problem (3.1$_0$), (3.2$_0$) with $f(\cdot; 0) \equiv 0$

   has only zero solution;

2) $A(\cdot; \varepsilon) - A(\cdot; 0) \in \mathcal{M}^m$;

3) $\|U_\varepsilon - U_0\| \to 0, \quad \varepsilon \to +0$.

Then for sufficiently small $\varepsilon$ Green matrices $G(t, s; \varepsilon)$ of problems (3.1$_\varepsilon$), (3.2$_\varepsilon$) exist and on the square $[a, b] \times [a, b]$

$$\|G(\cdot, \cdot; \varepsilon) - G(\cdot, \cdot; 0)\|_\infty \to 0, \quad \varepsilon \to +0.$$

Condition 3) of Theorem 3 cannot be replaced by a weaker condition of the strong convergence of the operators $U_\varepsilon \to U_0$ [5]. However, one may easily see that for multi-point ”boundary” operators

$$U_\varepsilon y := \sum_{k=1}^n B_k(\varepsilon)y(t_k), \quad \{t_k\} \subset [a,b], \quad B_k(\varepsilon) \in \mathbb{C}^{m \times m}, \quad n \in \mathbb{N},$$

both conditions of strong and norm convergence are equivalent to

$$\|B_k(\varepsilon) - B_k(0)\| \to 0, \quad \varepsilon \to +0, \quad k \in \{1, \ldots, n\}.$$

**4. Proof of Theorem 2**

We give two lemmas to apply Theorem 3 to proof of Theorem 2.

**Lemma 1.** Function $y(t)$ is a solution of a boundary-value problem

$$D_\varepsilon^{[2]}y(t) = f(t; \varepsilon) \in L_2, \quad \varepsilon \in [0, \varepsilon_0],$$

$$\alpha(\varepsilon)\mathcal{Y}_a(\varepsilon) + \beta(\varepsilon)\mathcal{Y}_b(\varepsilon) = 0.$$

if and only if vector-function $w(t) = (y(t), D_\varepsilon^{[1]}y(t))$ is a solution of a boundary-value problem

$$w'(t) = A(t; \varepsilon)w(t) + \varphi(t; \varepsilon),$$

$$\alpha(\varepsilon)w(a) + \beta(\varepsilon)w(b) = 0,$$

where matrix function

$$A(\cdot; \varepsilon) := \begin{pmatrix} q_\varepsilon & 1 \\ -q_\varepsilon^2 & -q_\varepsilon \end{pmatrix} \in L_1^{2 \times 2},$$
and \( \varphi(\cdot; \varepsilon) := (0, -f(\cdot; \varepsilon)) \).

**Proof.** Consider the system of equations
\[
\begin{cases}
(D_\varepsilon^{[0]} y(t))' = q_\varepsilon(t) D_\varepsilon^{[0]} y(t) + D_\varepsilon^{[1]} y(t) \\
(D_\varepsilon^{[1]} y(t))' = -q_\varepsilon^2(t) D_\varepsilon^{[0]} y(t) - q_\varepsilon(t) D_\varepsilon^{[1]} y(t) - f(t; \varepsilon)
\end{cases}
\]

If \( y(\cdot) \) is a solution of equation (7), then definition of quasi-derivatives derives that \( y(\cdot) \) is a solution of this system. On the other hand with this system may be rewritten in the form of equation (7).

As \( Y_a(\varepsilon) = w(a), \ Y_b(\varepsilon) = w(b) \) then it is evident that boundary conditions (6) are equivalent to boundary conditions (8).

**Lemma 2.** Let the assumption
\((\mathcal{E})\) Homogeneous boundary-value problem \( D_0^{[2]} y(t) = 0, \ a(0) Y_a(0) + \beta(0) Y_b(0) = 0 \) has only zero solution
be fulfilled. Then for sufficiently small \( \varepsilon \) Green function \( \Gamma(t, s; \varepsilon) \) of the semi-homogeneous boundary problem (3), (6) exists and
\[
\Gamma(t, s; \varepsilon) = -g_{12}(t, s; \varepsilon) \quad \text{a.e.,}
\]
where \( g_{12}(t, s; \varepsilon) \) is the corresponding element of the Green's matrix
\[
G(t, s; \varepsilon) = (g_{ij}(t, s; \varepsilon))_{i,j=1}^2
\]
of two-point vector boundary-value problem (7), (8).

**Proof.** Taking into account Theorem (3) and Lemma (1) assumption (\( \mathcal{E} \)) derives that homogeneous boundary-value problem
\[
w'(t) = A(t; \varepsilon) w(t), \quad a(\varepsilon) w(a) + \beta(\varepsilon) w(b) = 0
\]
for sufficiently small \( \varepsilon \) has only zero solution.

Then for problem (7), (8) Green matrix
\[
G(t, s, \varepsilon) = (g_{ij}(t, s))_{i,j=1}^2 \in L_{\infty}^{2 \times 2}
\]
exists and the unique solution of (7), (8) is written in the form
\[
w_\varepsilon(t) = \int_a^b G(t, s; \varepsilon) \varphi(s; \varepsilon) ds, \quad t \in [a, b], \quad \varphi(\cdot; \varepsilon) \in L_2.
\]

The last equality can be written in the form
\[
\begin{cases}
D_\varepsilon^{[0]} y_\varepsilon(t) = \int_a^b g_{12}(t, s; \varepsilon)(-\varphi(s; \varepsilon)) ds \\
D_\varepsilon^{[1]} y_\varepsilon(t) = \int_a^b g_{22}(t, s; \varepsilon)(-\varphi(s; \varepsilon)) ds,
\end{cases}
\]
where \( y_\varepsilon(\cdot) \) is the unique solution of problem (5), (8). This implies the assertion of Lemma 2.

Now, passing to the proof of Theorem 2, we note that since
\[
(q_\varepsilon + \mu)^2 - (q_0 + \mu)^2 = (q_\varepsilon^2 - q_0^2) + 2\mu(q_\varepsilon - q_0),
\]

in view of conditions 2), 3) we can assume without loss of generality that \(0 \in \rho(L_0)\). Let’s prove that

\[
\sup_{\|f\|_2=1} \|L_\varepsilon^{-1}f - L_0^{-1}f\| \to 0, \quad \varepsilon \to +0.
\]

Equation \(L_\varepsilon^{-1}f = y_\varepsilon\) is equivalent to the relation \(L_\varepsilon y_\varepsilon = f\), that is \(y_\varepsilon\) is the solution of the problem (5), (6) and due to inclusion \(0 \in \rho(L_0)\) the assumption (E) of Lemma 2 holds. Conditions 1) – 3) of Theorem 2 imply that \(A(\cdot; \varepsilon) - A(\cdot; 0) \in \mathcal{M}^2\), where \(A(\cdot; \varepsilon)\) is given by (9). Therefore assumption of Theorem 2 derives that assumption of Theorem 3 for problem (7), (8) is fulfilled. This means that Green matrices \(G(t, s; \varepsilon)\) of the problems (7), (8) exist and limit relation (4) holds. Taking into account Lemma 2, this implies the limit equality

\[
\|\Gamma(\cdot, \cdot; \varepsilon) - \Gamma(\cdot, \cdot; 0)\|_\infty \to 0, \quad \varepsilon \to +0.
\]

Then

\[
\|L_\varepsilon^{-1} - L_0^{-1}\| = \sup_{\|f\|_2=1} \|f\|_2 \to 0,
\]

which implies the assertion of Theorem 2.

5. THREE EXTENSIONS OF THEOREM 2

As was already noted, the assumptions of Theorem 2 may be weakened. Let

\[
R(\cdot; \varepsilon) := A(\cdot; \varepsilon) - A(\cdot; 0)
\]

where \(A(\cdot; \varepsilon)\) is given by (9).

**Theorem 4.** In the statement of Theorem 2, condition 1) can be replaced by any one of the following three more general (in view of 2) and 3)) asymptotic conditions as \(\varepsilon \to +0\):

- (I) \(\|R(\cdot; \varepsilon)R'(\cdot; \varepsilon)\|_1 \to 0\);
- (II) \(\|R'(\cdot; \varepsilon)R(\cdot; \varepsilon)\|_1 \to 0\);
- (III) \(\|R(\cdot; \varepsilon)R'(\cdot; \varepsilon) - R'(\cdot; \varepsilon)R(\cdot; \varepsilon)\|_1 \to 0\).

**Proof.** The proof of Theorem 4 is similar to the proof of Theorem 2 with following remark to be made. Condition 2) of Theorem 3 holds if (see [6]) \(\|R'(\cdot; \varepsilon)\|_C \to 0\) and either the condition \(R(\cdot; \varepsilon)\|_1 = O(1)\) (as in Theorem 2), or any of three conditions (I), (II), (III) of Theorem 4 holds.

Following example shows each part of Theorem 4 to be stronger than Theorem 2

**Example 2.** Let \(q_0(t) \equiv 0\), \(q_\varepsilon(t) = \rho(\varepsilon)e^{it/\varepsilon}\), \(t \in [0, 1]\).

One may easily calculate that conditions

\[
\rho(\varepsilon) \uparrow \infty, \quad \varepsilon^3 \rho(\varepsilon) \to 0, \quad \varepsilon \to +0,
\]

imply assumptions 2), 3) of Theorem 2 and any one of assumptions (I), (II), (III) of Theorem 4 But assumption 1) of Theorem 2 does not hold because \(\|q_\varepsilon - q_0\|_2 \uparrow \infty\).

For Schrödinger operators of the form (11) on \(\mathbb{R}\) with real-valued periodic potential \(q'\), where \(q \in L^{2loc}\), self-adjointness and sufficient conditions for norm resolvent convergence were established in [7]. For other problems related to those studied in [2], see also [8], [9].
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