THE CLASS OF A FIBRE IN NONCOMMUTATIVE GEOMETRY

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Abstract. This article addresses the K-homology of a crossed product of a discrete group acting smoothly on a manifold, with a better understanding of the noncommutative geometry of the crossed-product as the primary goal, and the Baum-Connes apparatus as the main tool. Examples suggest that the correct notion of the ‘Dirac class’ of such a noncommutative space is the image under the equivalence determined by Baum-Connes of the fibre of the fibration of the Borel space associated to the action and a smooth model for the classifying space of the group. We give a systematic study of such fibre, or ‘Dirac classes,’ with applications to the construction of interesting spectral triples, and computation of their K-theory functionals, and we prove in particular that both the well-known deformation of the Doleault operator on the noncommutative torus, and the class of the boundary extension of a hyperbolic group, are both Dirac classes in this sense and therefore can be treated topologically in the same way.

1. Introduction

The purpose of this article is to use the Baum-Connes apparatus to shed some light on the noncommutative geometry of some examples of C*-algebras that probably deserve to be thought of as ‘noncommutative manifolds,’ since they are canonically KK-equivalent to classical manifolds.

We do this by fixing a definition of a class in the K-homology of a crossed-product $C_0(X) \rtimes \Gamma$ of a smooth action by a discrete group which we call the Dirac class of the action, and which is determined by the K-homology class of a fibre in the natural fibration $p: E\Gamma \rtimes \Gamma X \to B\Gamma$ and the Dirac map involved in the Baum-Connes apparatus. This makes the Dirac class dependent on not only the action, but on aspects of the group itself including, in a certain sense, its coarse geometry. Our class differs from the transverse Dirac classes studied by A. Connes and others, which is, roughly speaking, invariant under the whole diffeomorphism group of the manifold, and doesn’t really involve the group as such.

We use this set-up to prove that the boundary extension of a classical hyperbolic group acting on its sphere at infinity, and the deformed Dolbeault spectral triple over the irrational rotation algebra of Connes, are, at the level of K-homology, instances of the same same construction: they are each Dirac classes for the respective actions.

We also deduce an index theorem for Dirac classes, which computes the K-theory functional determined by a Dirac class, in terms of topological data (the intersection index of a Baum-Douglas cycle with a fibre.) When specialized to either the boundary extension of a hyperbolic group, where it computes the boundary map on K-theory, or the irrational rotation situation, the resulting index formulas seem quite promising.

We now explain of all of this in more detail.

The Baum-Connes conjecture seeks to reduce the analytic problem of computing the K-theory groups of a crossed-product $C_0(X) \rtimes \Gamma$ to topology (by and large we work with the max crossed-product in this paper, as it is functorial, and since most of the actions we consider specifically...
are amenable.) In the work of Meyer and Nest, following a tradition initiated by Kasparov, Lusztig, Higson and others, it is shown that to \( \Gamma \) one can associate a proper \( \Gamma \)-C*-algebra \( \mathcal{P} \) and a Kasparov morphism \( D \in KK^\Gamma_*(\mathcal{P}, \mathbb{C}) \) (the ‘Dirac morphism’) with the property that the forgetful map \( KK^\Gamma \to KK^\mathcal{P} \) maps \( D \) to an equivalence, for any finite subgroup \( H \) of \( \Gamma \). This condition determines \( D \). External product in \( KK^\Gamma \) gives a map

\[
(1.1) \quad KK^\Gamma_*(A, B) \to KK^\Gamma_*(\mathcal{P} \otimes A, B)
\]

for any \( \Gamma \)-C*-algebras \( A, B \), and the co-domain of this map is of a purely topological nature because it is KK-dual to \( KK^\mathcal{P}_*(\mathcal{P}; A, B) \).

The C*-algebra \( \mathcal{P} \) and the morphism \( D \) are not always easy to represent concretely. In this paper, we assume that \( \mathcal{P} = C_0(Z) \) and \( D = [Z] \in KK^{\mathcal{T}}_{-d}(C_0(Z), \mathbb{C}) \) the class of the \( \Gamma \)-equivariant Dirac operator on \( Z \). The assumptions may sound strong, but they cover nearly all fundamental groups \( \Gamma = \pi_1(M) \) of compact, oriented, aspherical manifolds, as ‘most’ such manifolds are also spin* and the Dirac spinor bundles etc on \( M \) can be lifted to \( \Gamma \)-equivariant bundles and so on, on \( Z = \hat{M} \), providing a \( \Gamma \)-equivariant K-orientation in our sense.

The geometric content of the Dirac map starts to appear if one puts \( A = B = \mathbb{C} \) and \( \Gamma = \mathbb{Z}^d \). The domain of the Dirac map is \( KK^{\mathbb{Z}^d}_*((\mathbb{C}, \mathbb{C}) \cong KK_*((\mathbb{C}, \mathbb{C}) \cong KK_*(C_0((\mathbb{R}^d), \mathbb{C}) \cong KK_*(C_0((\mathbb{R}/\mathbb{Z}^d), \mathbb{C}) = KK_*(C(T^d), \mathbb{C}) \) where \( T^d := \mathbb{R}^d/\mathbb{Z}^d \) and \( \mathbb{T}^d \) is by definition \( \mathbb{Z}^d \), the ‘dual’ torus. The Dirac map therefore is a map

\[
(1.2) \quad K_*(\mathbb{T}^d) \to K_{*+d}(T^d),
\]

and, it is not that difficult to compute that it is precisely the well-known Fourier-Muñoz transform, implemented by composing cohomology cycles with the smooth correspondence

\[
T^d \leftarrow (T^d \times (\mathbb{T}^d, \beta)) \to \mathbb{T}^d,
\]

from \( T^d \) to the dual torus \( \mathbb{T}^d \), where \( \beta \) is the Mischenko-Poincaré element, and the maps are the projection maps. Furthermore, as we show, it has an interesting effect on Baum-Douglas K-homology, for it interchanges (the K-homology class of) a \( j \)-dimensional subtorus in \( \mathbb{T}^d \) to (the class of a) certain canonical \( d-j \)-dimensional ‘dual’ torus in \( T^d \). It was this observation that first made the author want to study the Dirac map more closely, and especially for actions.

If \( A = C_0(X) \), for a \( \Gamma \)-space \( X \), and \( B = \mathbb{C} \), the Dirac map looks like

\[
[Z] \otimes \mathbb{C} : KK^\Gamma_*(C_0(X), \mathbb{C}) \to KK^\Gamma_{*-d}(C_0(Z \times \Gamma X), \mathbb{C})
\]

and the domain is the K-homology of the crossed-product \( C_0(X) \rtimes \Gamma \), while the co-domain, if \( \Gamma \) is torsion-free, is naturally isomorphic to \( K_*((Z \times \Gamma X) \rtimes \Gamma) \) – the K-homology of the mapping cylinder \( Z \times \Gamma X \), which fibres over \( \Gamma \backslash Z \cong B^j \), with fibre \( X \). The Dirac map shifts degrees by \( -d \).

We define a Dirac class for the action to be any class in \( K^{d-j}(C_0(X) \rtimes \Gamma) \) mapped by the Dirac map to the Baum-Douglas K-homology class of the fibre \( X \): a spin* manifold mapping (properly) to \( Z \times \Gamma X \), by including it as a fibre.

This definition does not guarantee that a Dirac class exists, nor that it is unique, because the Dirac map is neither onto nor 1-1 in general. However, if \( \Gamma \) has a dual-Dirac morphism, then the Dirac map can be split, yielding a existence result about Dirac classes (although still not uniqueness). It is this method which, when applied to isometric actions of nice discrete groups \( \Gamma \) (like \( \mathbb{Z}^d \)), leads to spectral triple representations of the Dirac class by spectral triples over \( C_0(X) \rtimes \Gamma \), whose general format, of the Schrödinger kind, \( D + \delta \), with \( \delta \) an operator on the group \( \Gamma \), \( D \) the Dirac on \( X \), have appeared in the work d e.g. in [27]. When \( \mathbb{Z} \) acts by irrational
rotation on the circle, we get the famous spectral triple (the deformed Dolbeault operator $\overline{\partial}_\theta$ over $A_\theta$ first defined by A. Connes).

Actions of discrete (co-compact) groups of Möbius transformations $\Gamma \subset \text{SL}_2(\mathbb{R})$ on the circle $\mathbb{T}$ are smooth actions preserving a $K$-orientation; they are special cases of broader classes of hyperbolic groups acting on their boundaries. These examples cannot be treated like isometric actions as in the previous paragraph, by $\mathbb{Z}^d$: one cannot form an external product of the type $D + \delta$ as in the previous paragraph because there is no $\Gamma$-invariant Dirac operator $D$ on the circle (because there is no $\Gamma$-invariant probability measure) with which one can take external product with. We show how one can construct a spectral triple representative, by ideas of Connes and Moscovici, but it is not very satisfactory, as it is not finitely summable. But in fact it turns out that the Dirac class of such an action is represented by an (odd) Fredholm module with finite summability, and which depends only on an ergodic-theoretic property of the action, and, furthermore, represents the class in $\text{KK}^1(C(\partial \Gamma) \rtimes \Gamma, C)$ of the boundary extension

$$0 \to C_0(\Gamma) \rtimes \Gamma \to C(\overline{\Gamma}) \rtimes \Gamma \to C(\partial \Gamma) \rtimes \Gamma \to 0 (1.3)$$

with $\overline{\Gamma}$ the compactification of $\Gamma$ obtained by mapping it in as an orbit in the disk $\mathbb{D}$, and compactifying in the closed disk $\overline{\mathbb{D}}$. (What is done here is to prove that the Dirac class is the class of the boundary extension. The Fredholm representation was carried out in [EM]).

What the Dirac class detects, topologically, is a certain intersection number. If one has a Baum-Douglas cycle (or coycle) for $\mathbb{Z} \times_\Gamma X$, a higher index construction produces a K-theory class for $C_0(X) \rtimes \Gamma$ which pairs with the Dirac class to give a certain analytic index. The index theorem is that this analytic index is the topological intersection number of the Baum-Douglas cycle (or coycle) with the fibre $X \subset \mathbb{Z} \times_\Gamma X$. The irrational rotation algebra is already an interesting example. In Connes’ work it is shown that $\overline{\partial}_\theta$ can be extended, by constructing a connection, and so on, to act on sections of various ‘noncommutative vector bundles’ over $A_\theta$ – that is, f.g.p. modules $\mathcal{E}_{p,q}$. These bundles are parameterized by pairs of relatively prime integers and are higher indices of the 1-dimensional Baum-Douglas cocycles for the ordinary torus given by loops $L_{p,q}$; the content of our index theorem here is that the index of the operator $D_\theta$ extended to act on $L^2(\mathcal{E}_{p,q})$ is the topological intersection number of the loop with the standard meridian loop of the torus (that is, $q$, in this parameterization.)

In the case of the boundary extension class of a hyperbolic group, which due to work of the author and Ralf Meyer, is torsion of order $\chi(\Gamma)$ if $\Gamma$ is torsion-free, the intersection index computes the boundary map

$$\delta: K_1(C(\partial \Gamma) \rtimes \Gamma) \to K_0(C_0(\Gamma) \rtimes \Gamma) = K_0(C) = \mathbb{Z},$$

and one of the author’s initial interests was in finding $K$-theory classes in $K_1(C(\partial \Gamma) \rtimes \Gamma)$ in the case of zero Euler characteristic (e.g. for Kleinian groups, rather than surface groups) with nonzero pairing with the boundary extension. We give in fact a direct geometric construction of such a K-theory class, based on a non-vanishing vector field on $\mathbb{Z} \setminus \Gamma$, using the intersection index formula to compute it’s image under $\delta$.

I would like to thank Paul Baum, for his relentless enthusiasm for the subject of Dirac operators and K-homology, and all that I have learned from him over the years in our numerous conversations, and Nigel Higson, for answering, with his usual élan, a number of my questions on the matters herein.

2. **Transverse Dirac classes and the Dirac map**

If $\Gamma$ is a locally compact group acting smoothly on a smooth Riemannian manifold $X$, then a $\Gamma$-equivariant $K$-orientation on $X$ consists of

a) A $\Gamma$-invariant Riemannian metric on $X$.  

b) A $\Gamma$-equivariant complex vector bundle $S \to X$ (the spinor bundle), equipped with a $\Gamma$-invariant Hermitian metric, and, if $n$ is even, a $\Gamma$-invariant $\mathbb{Z}/2$-grading on $S$.

c) A $\Gamma$-equivariant fibrewise irreducible representation of the Clifford algebra bundle of $X$, on $S$ compatible with the $\mathbb{Z}/2$-gradings if relevant.

Assuming that the action is proper, one can then construct a $G$-invariant connection on $X$ compatible with the Levi-Civita connection on $TX$, and corresponding $G$-equivariant Dirac operator on $X$, providing a cycle and corresponding class in $\text{KK}^G_{\mathbb{Z}/2}(C_0(X), \mathbb{C})$.

This definition can be adapted without great trouble (see [23]) to the case where one has a bundle $p: X \to G^0$ of smooth manifolds over the unit space of a proper groupoid $G$ (so that $p$ is the momentum map). With a suitable ($G$-equivariant) $K$-orientation assumption on $X$ one manufactures a class which we denote by $p_{an!} \in \text{KK}^G_{\mathbb{Z}/2}(C_0(X), C_0(G^0))$, represented by a $G$-equivariant bundle of Dirac operators over $G^0$.

In [21] we develop the theory of equivariant, normally non-singular maps $f: X \to Y$. Normally non-singular maps admit, by definition, factorizations into three types of maps: zero sections of vector bundles, open embeddings, and vector bundle projections. When the vector bundles are required to be $K$-oriented, all three lead to $KK$-morphisms, and hence, when combined, to a topologically defined morphism $f! \in \text{KK}_* (C_0(X), C_0(Y))$.

Equivalence classes of $K$-oriented normally non-singular maps can be composed and form a category. Coupling a $K$-oriented normally non-singular map with further data gives a correspondence. If $X$ and $Y$ are smooth manifolds, a (smooth) correspondences from $X$ to $Y$ is a pair of maps and a $K$-theory class, usually depicted by a diagram

\[
\begin{array}{c}
\begin{array}{c}
X \\ b
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xleftarrow{(M, \xi)} \xrightarrow{f} Y
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xleftarrow{M'} \xrightarrow{f'} Z
\end{array}
\end{array}
\]

where $f$ is a smooth $K$-oriented map; $b$ is an ordinary smooth map (not necessarily proper), and the class $\xi$ lies in the representable $K$-theory of $M$ with $b$ compact support. If $b$ is proper, this is just the ordinary $K$-theory of $M$.

The equivalence relation is generated by Thom modification and bordism. One can compose two correspondences

\[
X \leftarrow b \xrightarrow{M} \xrightarrow{f} Y \xleftarrow{b'} \xrightarrow{M'} \xrightarrow{f'} Z
\]

using transversality, provided that the maps $f$ and $b'$ are transverse. The composition, with this assumption, and neglecting any $K$-theory data, is represented by

\[
X \leftarrow M \times_Y M' \to Z,
\]

with $M \times_Y M'$ having its canonical smooth manifold structure, and where the map $M \times_Y M' \to Z$ (the composition of the projection $M \times_Y M \to M'$ and the map $f'; M' \to Z$) carries a certain $K$-orientation induced by the $K$-orientations on $f$ and $f'$. The left map $M \times_Y M' \to X$ is similarly the composition of the first coordinate projection and the map $b$.

This efficient recipe of composing correspondences (KK-elements) will be used prolifically in this paper.

The dimension of the correspondence (2.1) is $\dim Y - \dim M + \deg \xi$. If $\xi$ is the class of the trivial line bundle, which is common in the most geometric examples, then the dimension is $\dim Y - n$. With this notion of dimension, correspondences composed by transversality do so additively with respect to dimension.

**Example 2.2.** Here is a beautiful example of a correspondence, whose corresponding analytic cycle can also be described in an interesting way. We call it the Fourier-Mukai correspondence, here. Let $T$ be the circle understood as $\mathbb{R}/\mathbb{Z}$. And $\hat{T}$ means $\mathbb{Z}$. The correspondence is from $T$ to $\hat{T}$ and is given by

\[
T \leftarrow (T \times \hat{T}, \beta) \to \hat{T}
\]
where the two maps are the coordinate projections, K-oriented in the evident way, and \( \beta \) is the Mischenko-Poincaré line bundle over \( \mathbb{T} \times \hat{\mathbb{T}} \).

The corresponding analytic cycle in \( \text{KK}_d(C(\mathbb{T}), C(\hat{\mathbb{T}})) \) may be described in the following way. It is given by a bundle of Dirac operators on \( \mathbb{T} \), parameterized by the points of \( \hat{\mathbb{T}} \). An element of \( \hat{\mathbb{T}} \) is by definition a group character \( \chi : \mathbb{Z} \to \mathbb{T} \). We form the flat induced vector bundle \( L_\chi := \mathbb{R} \times_{\mathbb{Z}, \chi} \mathbb{C} \) and twist the Dirac operator on \( \mathbb{T} \) by this flat bundle. The flat connection means this can be done canonically, and continuously. This canonical bundle of Dirac operators determines an unbounded cycle in \( \text{KK}_d(C(\mathbb{T}), C(\hat{\mathbb{T}})) \).

Correspondences modulo an appropriate geometric equivalence relation gives a category equivalent to KK when the arguments are suitably restricted (to smooth manifolds, for example.) The theory is developed equivariantly, for proper groupoids \( \mathcal{G} \) in the papers cited above, acting on \( \mathcal{G} \)-spaces.

This general, equivariant version of the correspondence framework will be used extensively in this article. This has nothing to do with seeking greater generality than group actions; it is only group actions we are really interested in. The Baum-Connes methodology, however, involves forming a very simple transformation groupoid \( \mathcal{G}_\Gamma := \Gamma \ltimes \mathcal{E} \Gamma \) from a discrete group \( \Gamma \) and the equivariant correspondence theory applies to this category. This allows us to make correspondence arguments in \( \text{KK}_\Gamma \) by making them in \( \text{KK}^\mathcal{G}_\Gamma \) instead and invoking Baum-Connes.

If \( \mathcal{G} \) is a proper groupoid, a smooth \( \mathcal{G} \)-manifold \( X \) is a bundle of smooth manifolds over the base \( \mathcal{G}^0 \) of \( \Gamma \), and morphisms in \( \mathcal{G} \) act by diffeomorphisms between fibres. There is a natural notion of smooth \( \mathcal{G} \)-equivariant map, etc., between two \( \mathcal{G} \)-manifolds, meaning that it is fibrewise smooth (and equivariant). This leads naturally to a definition of \( \mathcal{G} \)-equivariant smooth correspondence between two \( \mathcal{G} \)-manifolds, by making the non-equivariant version fibrewise.

However, to define a topological wrong-way element \( f! \in \text{KK}^\mathcal{G}_{\dim Y - \dim (X)}(C_0(X), C_0(Y)) \), for any smooth K-oriented \( \mathcal{G} \)-map between \( \mathcal{G} \)-manifolds \( X \) and \( Y \), we need additional hypotheses that ensure that \( X \) embeds equivariantly into the total space of a \( \mathcal{G} \)-equivariant vector bundle over \( \mathcal{G}^0 \). The hypothesis is ensured if \( \mathcal{G} \) has a full vector bundle over its base, and this is the case if \( \mathcal{G} = \Gamma \ltimes \mathcal{E} \Gamma \) for a discrete group \( \Gamma \) with co-compact \( \mathcal{E} \Gamma \), the classifying space for proper actions, by a result of Lück and Oliver. The basic index theorem of Kasparov theory, shown (pp. 36-37) in [21], that the topologically defined element \( f! \) agrees with the analytic one \( f_{an}! \), discussed above.

More generally, it is proved in [22] that with the hypothesis, that \( Z \) has a full equivariant vector bundle over its base, all three theories: of \( \mathcal{G} \)-equivariant correspondences using normally non-singular maps, smooth \( \mathcal{G} \)-equivariant correspondences (without requiring normal data) , and the analytically defined theory \( \text{KK}^\mathcal{G} \) of Kasparov, all give the same theory, for smooth \( \mathcal{G} \)-manifolds.

In order to avoid any pathologies, then, we will put a blanket assumption on the discrete groups \( \Gamma \) studied in this paper, that \( \Gamma \backslash \mathcal{E} \Gamma \) is compact. As above, then, the transformation groupoid \( \mathcal{G}_\Gamma := \Gamma \ltimes \mathcal{E} \Gamma \) is proper and has a full vector bundle on its base, and the above results apply.

**Definition 2.3.** Let \( X \) a smooth manifold with a smooth action of the discrete group \( \Gamma \) with \( \Gamma \)-compact \( \mathcal{E} \Gamma \). Let \( \mathcal{G}_\Gamma := \mathcal{E} \Gamma \ltimes \Gamma \) the corresponding (proper) transformation groupoid.

We will say that \( X \) is \( \Gamma \)-equivariantly orientable if \( \mathcal{G}_\Gamma \) preserves a K-orientation on the \( \mathcal{G}_\Gamma \)-equivariant bundle of smooth manifolds \( \mathcal{E} \Gamma \ltimes X \) over \( \mathcal{E} \Gamma \).

Let \( p_{X, \mathcal{E} \Gamma} : X \times \mathcal{E} \Gamma \to \mathcal{E} \Gamma \) the momentum map, then \( p_{X, \mathcal{E} \Gamma} \) defines a smooth wrong-way element

\[
\text{pr}_{X, \mathcal{E} \Gamma}! \in \text{KK}^\mathcal{G}_n(C_0(X \times \mathcal{E} \Gamma), C_0(\mathcal{E} \Gamma)),
\]

since \( p_{X, \mathcal{E} \Gamma} \) gets a K-orientation from the hypothesis.
The groupoid-equivariant KK-group $\text{KK}^\xi_\gamma(C_0(\mathcal{E}\Gamma) \otimes A, C_0(\mathcal{E}\Gamma) \otimes B)$ featuring here is identical to Kasparov’s $\text{RKK}^\xi_\gamma(\mathcal{E}\Gamma; A, B)$. The homomorphism of groupoids $\gamma : \mathcal{E}\Gamma \to \Gamma$ by mapping $\mathcal{E}\Gamma$ to a point induces natural maps

\[(2.4) \quad \text{inflate} : \text{KK}^\xi_\gamma(A, B) \to \text{KK}^\xi_\gamma(C_0(\mathcal{E}\Gamma) \otimes A, C_0(\mathcal{E}\Gamma) \otimes B)\]

which we call the inflation map. Isomorphism of the inflation map for a given $\Gamma$ and arbitrary $A, B$ is implied by the $\gamma = 1$ version of the Baum-Connes conjecture (see [38]). More generally, if $\Gamma$ acts amenably on $A$, then (2.4) is an isomorphism by the Higson-Kasparov-Tu theorem ([28], [39].)

**Remark 2.5.** For compact groups, $\mathcal{E}\Gamma$ can be taken to be a point, and Definition 2.3 reduces to the standard definition of equivariant (K-)orientation on a smooth $\Gamma$-manifold $X$.

The definition of equivariant K-orientation used in [30] is stronger than ours; we do not prove this here, however, as it is not essential to us.

**Definition 2.6.** Let $\Gamma$ be a discrete group and $X$ be a $\Gamma$-equivariantly K-orientable $n$-dimensional manifold $X$ in the sense of Definition 2.3.

A transverse Dirac class for a $K$-orientation-preserving action (in the sense of Definition 2.3) of the locally compact group $\Gamma$ on the smooth manifold $X$, is a class

$$[X] \in \text{KK}^\Gamma_{-n}(C_0(X), \mathbb{C})$$

such that

$$\text{inflate}([X]) = \text{pr}_{X,\mathcal{E}\Gamma} \in \text{KK}^\xi_\gamma_n(C_0(X \times \mathcal{E}\Gamma), C_0(\mathcal{E}\Gamma)),$$

where $\text{pr}_{X,\mathcal{E}\Gamma} \in \text{KK}^\xi_\gamma_n(C_0(X \times \mathcal{E}\Gamma), C_0(\mathcal{E}\Gamma))$ is the class of the $\mathcal{G}$-equivariant fibrewise smooth and $\mathcal{G}$-equivariantly $K$-oriented non-singular map $\text{pr}_{\mathcal{E}\Gamma} : X \times \mathcal{E}\Gamma \to \mathcal{E}\Gamma$ discussed above.

**Example 2.7.** If $\Gamma$ is the trivial group, then the class $[X] \in \text{KK}_{-n}(C_0(X), \mathbb{C}) = \text{K}_n(X)$ of the Dirac operator on $X$ is a transverse Dirac class.

If $X$ is a point, $\Gamma$ an arbitrary locally compact group, then the the class $1 \in \text{KK}^\Gamma_n(\mathbb{C}, \mathbb{C})$ of the trivial representation of $\Gamma$ is a transverse Dirac class for the action of $\Gamma$ on a point.

**Example 2.8.** If $\Gamma$ is compact, so $\mathcal{E}\Gamma$ is a point, then our notion of a smooth, equivariantly orientable action agrees with the usual one and the associated equivariant Dirac operator on $X$ determines a Kasparov cycle and transverse Dirac class $[X] \in \text{K}^{-n}(C(X) \rtimes \Gamma)$. Of course in this case it is unique.

More generally, if $\Gamma$ acts properly on $X$, there is a well-known (direct) construction of a $\Gamma$-equivariant Dirac operator and class $[X] \in \text{KK}^\xi_{-n}(C_0(X), \mathbb{C})$, as discussed in the first paragraph of this section, and this class can be easily checked to be a transverse Dirac class.

Note that in this case, the inflation map

$$\text{inflate} : \text{KK}^\xi_\gamma(C_0(X), \mathbb{C}) \to \text{KK}^\xi_\gamma(C_0(X \times \mathcal{E}\Gamma), C_0(\mathcal{E}\Gamma)) = \text{RKK}^\xi_\gamma(\mathcal{E}\Gamma; C_0(X), \mathbb{C})$$

is an isomorphism, so this alone implies both existence and uniqueness of a transverse Dirac class.

The work of Hilsum and Skandalis shows existence of transverse Dirac classes in essentially complete generality (see [30]). Their construction uses hypoelliptic operator theory and a frame bundle construction.

**Theorem 2.9.** (Hilsum and Skandalis). If $X$ is a smooth $\Gamma$-equivariantly K-oriented manifold, then $X$ has a transverse Dirac class $[X] \in \text{KK}^\xi_{-n}(C_0(X), \mathbb{C})$.

Moreover, $X$ has a unique transverse Dirac class if $\Gamma$ acts amenably on $X$. 

If $\Gamma$ has property $T$, the class in $KK^\Gamma(\mathbb{C}, \mathbb{C})$ of the $\gamma$-element maps to $1 \in RKK^\Gamma(\mathcal{E}\Gamma; \mathbb{C}, \mathbb{C})$ and hence is a transverse Dirac class for $\Gamma$ acting on a point; however, it is not equal to the class $[\epsilon] \in KK^0(\mathbb{C}, \mathbb{C})$ of the class of the trivial representation of $\Gamma$. Thus there is more than one transverse Dirac class for a property $T$ group $\Gamma$ acting on a point.

Our definition of Dirac class (distinct from transverse Dirac class) for a smooth action of $\Gamma$ on $X$, will rely on assuming that the group $\Gamma$, as a discrete group, has some nice geometric features. We will then build these features into a different definition, which will then encode things both about $X$, the action, and the group $\Gamma$ itself.

**Definition 2.10.** A K-orientation on the discrete group $\Gamma$ will refer to a smooth, proper $\Gamma$-equivariantly K-oriented co-compact $\Gamma$-compact manifold $Z$ which is $H$-equivariantly contractible for every compact subgroup $H$ of $\Gamma$. We refer to the pair $(\Gamma, Z)$ as a smooth K-oriented group.

The contractibility assumption means that $Z$ is a model for the classifying space $\mathcal{E}\Gamma$ for proper actions of $\Gamma$.

**Example 2.11.** Every compact group admits a smooth K-orientation with $Z$ a point.

The group $\mathbb{Z}^d$ admits a smooth K-orientation using $Z := \mathbb{R}^d$ with the smooth action of $\mathbb{R}^d$ by translation; since $\mathbb{Z}^d$ is a closed subgroup of $\mathbb{R}^d$, $(\mathbb{Z}^d, \mathbb{R}^d)$ is a smooth K-oriented group.

Suppose that $M$ is a compact Riemann surface (a compact two-dimensional manifold equipped with a complex structure). Then it admits a canonical orientation. The universal cover $\tilde{Z} := \tilde{M}$ has a free and proper action of $\Gamma := \pi_1(M)$, and can be equipped with a $\Gamma$-invariant metric and orientation (lifted from $M$, i.e., a complex structure) and metric of constant negative curvature, making it contractible, and more generally, $H$-equivariantly contractible for any compact group of isometries of $Z$.

The Riemannian manifold $Z$ can of course be identified with the hyperbolic plane $\mathbb{H}^2$ with an appropriate proper, isometric action of $\Gamma$.

Thus, $(\Gamma, \mathbb{H}^2)$ is a smooth, oriented group.

Similarly, any orientation-preserving discrete group of hyperbolic isometries of $\mathbb{H}^3$ admits a smooth structure, since any compact oriented 3-manifold also carries a K-orientation.

More generally, if $\Gamma$ is a torsion-free lattice in a semi-simple Lie group with associated symmetric space $Z = G/K$, then K-orientability of $\Gamma\backslash X$ implies $\Gamma$-equivariant K-orientability of $Z$, and hence $(\Gamma, Z)$ admits a canonical structure of a smooth oriented group in this case, from a K-orientation on $\Gamma\backslash Z$.

This amounts to the well-known procedure of ‘lifting’ a K-orientation (Clifford and spinor bundles, indeed the Dirac operator itself, etc) under a covering map $\pi: \tilde{X} \to X$.

Associated to a smooth oriented group $(\Gamma, Z)$ of dimension $d$ is the associated $\Gamma$-equivariant Dirac operator on $Z$ and corresponding class $[Z] \in KK^\Gamma(C_0(Z), \mathbb{C})$. Equivalently, $[Z]$ is the transverse Dirac class for the $\Gamma$-action on the smooth manifold $Z$, in the sense of Definition 2.6.

The element $[Z]$ is also a Dirac morphism for $\Gamma$ in the sense of Meyer and Nest [38].

**Definition 2.12.** Let $(\Gamma, Z)$ be a smooth oriented group. The associated Dirac map is the map

$$\text{Dirac} : KK^\Gamma(A, B) \xrightarrow{\otimes[\gamma]} KK^\Gamma(C_0(Z) \otimes A, B)$$

induced by external product in $KK^\Gamma$, with the class $[Z]$ of the transverse Dirac class for $\Gamma$ acting on $Z$.

**Remark 2.13.** If $B$ is a trivial $\Gamma$-$C^*$-algebra, and $A$ is an arbitrary $\Gamma$-$C^*$-algebra, recalling that $\Gamma$ is discrete, there is a completely canonical isomorphism

$$KK^\Gamma(A, B) \cong KK\Gamma_*(A \rtimes \Gamma, B).$$
Indeed, the two groups involved here have exactly the same cycles. Taking this into account, the Dirac map, in the case of (smooth, oriented..) discrete groups, can be interpreted as a map (2.14)
\[ \text{K}^\Gamma(A \times \Gamma) := \text{KK}_s(A \times \Gamma, \mathbb{C}) \cong \text{KK}^\Gamma(A, \mathbb{C}) \to \text{KK}^\Gamma_{-d}(C_0(Z) \times A, \mathbb{C}) \cong \text{K}^{* - n}(C_0(Z, A) \times \Gamma, \mathbb{C}) \]
for any $\Gamma$-\text{C}*-algebra $A$. If $\Gamma$ is torsion-free, $A = C_0(X)$, some $\Gamma$-space $X$, then the target of the
the Dirac map is $\text{KK}_s(C_0(X \times_\Gamma Z), \mathbb{C}) = K_{-*}(Z \times_\Gamma X)$, by a standard Morita equivalence, in
other words, with the Kasparov K-homology of the Borel space, or ‘mapping cylinder’, $X \times_\Gamma Z$, which fibres over $\Gamma \setminus Z$ under the second projection map, with fibre $X$.

For a compact group, the Dirac map is the identity map.

The Dirac map can be difficult to compute in general, because its definition involves an external product in (equivariant) analytic Kasparov theory. This external product can be computed explicitly sometimes at the level of cycles on classes represented by $\Gamma$-equivariant spectral triples, but finding such representatives is not always easy.

However, the Dirac map can be shown to factor in a simple way through what, in terms of the analysis involved in cycles, essentially behaves like a forgetful map, and a map which is purely topologically defined. We will use this factorization to compute it in certain cases.

If $(\Gamma, Z)$ is a smooth oriented group, then there is a Poincaré duality isomorphism
\[ \text{K}^\Gamma(C_0(Z) \otimes A, B) \cong \text{KK}^\Gamma_{+d}(C_0(Z) \otimes A, C_0(Z) \otimes B) = \text{RKK}^\Gamma(Z; A, B) \]
shifting degrees by $d$, which is explained in [23].

The Poincaré duality isomorphism uses as one of its ingredients $[Z] \in \text{KK}^\Gamma_{-d}(C_0(Z), \mathbb{C})$ of the
group.

Proposition 2.16. Let $(\Gamma, Z)$ be a smooth $\text{K}$-oriented $d$-dimensional group. Then the Dirac map factors as
\[ \text{K}^\Gamma(A, B) \xrightarrow{\text{infl}} \text{RKK}^\Gamma(Z; A, B) \xrightarrow{\text{PD}} \text{K}^\Gamma_{+d}(C_0(Z) \otimes A, B) \]
where PD is Poincaré duality.

For the proof, see Theorem 4.34 of [23].

Since PD is always an isomorphism, the inflation map and the Dirac map are equivalent; thus one is an isomorphism if and only if the other is.

The following shows how the transverse Dirac class is related to the Dirac map.

Proposition 2.18. If $\Gamma$ is discrete, $(\Gamma, Z)$ is a smooth, oriented group, $X$ is a compact smooth
equivariantly orientable $\Gamma$-manifold, $[X] \in \text{KK}^\Gamma_{-n}(C_0(X), \mathbb{C})$ a transverse Dirac class for $X$ and $[X \times Z]$ the transverse Dirac class for the $\Gamma$-manifold $X \times Z$, then
\[ \text{Dirac}([X]) = [X \times Z] \in \text{K}^\Gamma_{-n-d}(C_0(X \times Z), \mathbb{C}). \]
where Dirac is the Dirac map.

Moreover, if $\Gamma$ is torsion-free, then $\text{KK}^\Gamma_{-d}(C_0(X \times Z), \mathbb{C}) \cong K_*(X \times_\Gamma Z)$ and with this identification
\[ \text{Dirac}([X]) = [X \times_\Gamma Z] \in K_{n+d}(X \times_\Gamma Z), \]
where $[X \times_\Gamma Z]$ is the class of the Dirac operator on $X \times_\Gamma Z$ (the transverse Dirac class of the trivial group acting on $X \times_\Gamma Z$).
Since Dirac([X]) = [X] \otimes \mathbb{C} [Z], the external product in KK^\Gamma, the Proposition is the multiplicity statement for transverse Dirac classes

\[ [X] \otimes \mathbb{C} [Z] = [X \times Z] \]

when \( X \) has a transverse Dirac class.

There is a quite easy proof of Proposition 2.18, using the fact that the inflation map

\[ \text{inflate}: K^\Gamma_\ast(C_0(Z \times W), \mathbb{C}) \to \text{RKK}^\Gamma_\ast(\mathbb{C} \times W, \mathbb{C}) = \text{KK}^\Gamma_\ast(C_0(\mathbb{C} \times Z \times W), C_0(\mathbb{C})) \]

is an isomorphism for any \( \mathbb{C} \times W \). But instead we give a proof independent of this fact below, in order to be explicit with the machinery being introduced. In any case, the proofs end up being similar.

**Proof.** In view of the commutative diagram of Proposition 2.16, we need to prove that

\[ \text{PD}(\text{pr}_{X,Z}) = [Z \times X] \in \text{KK}^\Gamma_{d-n}(C_0(Z \times X), \mathbb{C}). \]

Poincaré duality

\[ \text{PD} : \text{RKK}^\Gamma_\ast(\mathbb{C} \times A, B) \to \text{KK}^\Gamma_{d-n}(C_0(Z) \otimes A, B) \]

is the composition of the map

\[ \text{RKK}^\Gamma_{\ast}(\mathbb{C} \times A, B) \to \text{KK}^\Gamma_{\ast}(C_0(\mathbb{C}) \otimes A, C_0(\mathbb{C}) \otimes B) \]

which forgets the \( \mathbb{C} \times \mathbb{C} \) equivariance on a cycle, remembering only \( \mathbb{C} \)-equivariance, which we denote by \( f \mapsto \overline{f} \), and the map

\[ \text{inflate}_{\ast}(C_0(Z) \otimes A, C_0(Z) \otimes B) \overset{\otimes_{C_0(Z)[Z]}}{\longrightarrow} \text{KK}^\Gamma_{d-n}(C_0(Z) \otimes A, B). \]

of composition with the Dirac class \([Z] \in \text{KK}^\Gamma_{d-n}(C_0(Z), \mathbb{C})\). Here we have identified \( \mathbb{C} \times \mathbb{C} \) with \( Z \), for the geometric step in the map. A general fact, easily checked, is that the composition

\[ \text{ inflate}_{\ast}(Z; A, B) \overset{f \mapsto \overline{f}}{\longrightarrow} \text{RKK}^\Gamma_{\ast}(C_0(Z) \otimes A, C_0(Z) \otimes B) \overset{\text{inflate}}{\longrightarrow} \text{RKK}^\Gamma_{\ast}(Z; C_0(Z) \otimes A, C_0(Z) \otimes B) \]

is external product in \( \text{RKK}^\Gamma_{\ast}(Z) \) with \( 1_{C_0(Z)} \).

We are interested in the case \( A = C_0(X), B = \mathbb{C} \). We can thus write the equation we want to prove as

\[ [X] \otimes \mathbb{C} [Z] = \text{pr}_{X,Z} \ast \otimes_{C_0(Z)[Z]} [Z] \in \text{KK}^\Gamma_{d-n}(C_0(X \times Z), \mathbb{C}). \]

Now the inflation map

\[ \text{inflate} : \text{pr}_{X,Z} \ast \otimes_{C_0(Z)} \text{inflate}( [Z] ) \in \text{RKK}^\Gamma_{d-n}(\mathbb{C} \times Z, \mathbb{C}). \]

By the discussion above, \( \text{inflate}(\text{pr}_{X,Z}) = \text{pr}_{X,Z} \ast \otimes_{C_0(Z)} [C_0(Z)] \), the external product in \( \text{RKK}^\Gamma_{\ast}(Z) \).

Hence, the two sides of the equation we want to prove \( \text{inflate} \) can be represented naturally by \( \mathbb{C} \)-equivariant correspondences. Both are constant bundles over \( Z \). We just describe them fibrewise. The left hand side of \( \text{inflate} \) is fibrewise - represented by the composition of correspondences

\[ Z \times X \xleftarrow{\text{id}} Z \times X \xrightarrow{\text{pr}_{Z}} Z \xleftarrow{\text{id}} Z \to \cdot. \]

The right hand side is also a bundle, with fibres the correspondence

\[ Z \times X \xleftarrow{\text{id}} Z \times X \to \cdot. \]
The equality is immediate by computing the composition \((2.26)\) by (fibrewise) transversality. 

**Corollary 2.27.** A class \([X] \in \text{KK}^\Gamma_{-n}(C_0(X), \mathbb{C})\) is a transverse Dirac class if and only if

\[
[X] \otimes_C [Z] = [X \times Z] \in \text{KK}^\Gamma_{-n-d}(C_0(X \times Z), \mathbb{C}),
\]

where \([X \times Z]\) is the transverse Dirac class of the proper space \(X \times Z\) (and the left-hand side is the external product in \(\text{KK}^\Gamma\).)

The corollary gives an alternative, and attractive way of trying to verify that a given construction has produced, at the level of K-homology, the transverse Dirac class, especially if one has at hand an *unbounded* representative, for then the external product \([X] \otimes_C [Z]\) can be described in a concrete and simple manner, by a recipe due to Baaj and Julg.

**Example 2.28.** Let \(\Gamma\) be a countable group of diffeomorphisms of the circle \(\mathbb{T}\). Then \(\mathbb{T}\) is \(\Gamma\)-equivariantly K-orientable, in the sense explained at the beginning of this paper, as soon as \(\Gamma\) acts by orientation-preserving diffeomorphisms. One can represent the wrong-way element \(p_\Gamma^! \in \text{RRK}^\Gamma_1(\mathbb{E}\Gamma; C(\mathbb{T}), \mathbb{C})\) in a more analytic way, in terms of an equivariant family of Dirac operators on the circle, parameterized by the points of \(\mathbb{E}\Gamma\).

Assume for simplicity that \(\mathbb{E}\Gamma\) can be triangulated, with vertex set a single \(\Gamma\)-orbit. At vertex \(v_g\) we define the measure (equivalently, the Riemannian metric) \(\mu_g := g_*(\mu)\), \(\mu\) Lebesgue. If \(v_{g_1}, \ldots, v_{g_n}\) span a simplex of \(\mathbb{E}\Gamma\) and \(p\) is a point of the simplex with barycentric coordinates \((t_1, \ldots, t_n)\) set \(\mu_p := \sum t_i \mu_{v_{g_i}}\). The resulting map from \(\mathbb{E}\Gamma\) into measures on \(\mathbb{T}\) is clearly equivariant.

The field of Hilbert spaces \(L^2(\mathbb{T}, \mu_p)\) for \(p \in \mathbb{E}\Gamma\) carries a fibrewise unitary action of \(\Gamma\) by \(U_g \xi := \xi \circ g\). Conjugating \(D := -i \frac{d}{dx}\) on \(\mathbb{T}\) (with Lebesgue measure) by \(U_g\) produces the operator on \(L^2(\mathbb{T}, \mu_g)\)

\[
D_g = e^{h(g \cdot \cdot)} D, \ h(g, x) = -\log((g^{-1})'(x)).
\]

We have thus constructed a family \(\{(L^2(\mathbb{T}, \mu_p), D_p)\}_{p \in \mathbb{E}\Gamma}\) of Dirac operators on \(\mathbb{T}\), parameterized by \(\mathbb{E}\Gamma\), which is exactly \(\Gamma\)-equivariant. The operator \(D_h\) is clearly homotopic to \(D\), so in each fibre of this bundle we have constructed a perturbation of \(D\), so that the whole family is equivariant.

This gives the two usual descriptions of \(p_\Gamma^!\), as a topological morphism in the category of equivariant correspondences, and as a (groupoid-equivariant) analytic cycle.

Now, if the group \(\Gamma\) is both orientation preserving and metric preserving, *e.g.* in the case of irrational rotation, then already \((L^2(\mathbb{T}, \mu), D)\) is a \(\Gamma\)-equivariant spectral triple, and the inflation map

\[
\text{inflate}: \text{KK}^\Gamma_1(C(\mathbb{T}), \mathbb{C}) \rightarrow \text{RRK}^\Gamma_{-1}(\mathbb{E}\Gamma; C(\mathbb{T}), \mathbb{C})
\]

maps it to the class of the corresponding constant family of cycles over \(\mathbb{E}\Gamma\). In this case, in reference to the general description of \(p_\Gamma^!\) above, the function \(h\) is zero, so that the two cycles match up exactly.

Thus:

**Proposition 2.29.** The transverse Dirac class for an isometric, orientation-preserving action of \(\Gamma\) on \(\mathbb{T}\) is represented by the (odd) spectral triple \((L^2(\mathbb{T}), \pi, D = -i \frac{d}{dy})\), where \(\pi: C(\mathbb{T}) \times \Gamma \rightarrow \mathcal{L}(L^2(\mathbb{T}))\) is induced by the covariant pair \(\pi(f) \xi := f \xi, \pi(g) \xi := \xi \circ g^{-1}\).

**Remark 2.30.** If \(\Gamma\) is a smooth oriented group, then it has a transverse Dirac class \([Z] \in \text{KK}^\Gamma_{-d}(C_0(\mathbb{T}), \mathbb{C})\), and representing it by a spectral triple in the usual way, we can prove that \((L^2(\mathbb{T}), D = -i \frac{d}{dy})\) represents \([\mathbb{T}]\) in an alternative way by arguing that the external product of the class of \((L^2(\mathbb{T}), D = -i \frac{d}{dy})\) and the class \([Z]\) equals \([\mathbb{T} \times \mathbb{T}]\), and invoking Corollary 2.27.
3. The Dirac map for free abelian groups

The Dirac map for $\mathbb{Z}^d$ acting on a point is the Fourier-Mukai transform, and has some very interesting geometric features, as we show next.

Note firstly that, in general, if $\Gamma$ is torsion-free, $(\Gamma, \mathbb{Z})$ a smooth K-oriented group, then $\Gamma$ acts freely on $\mathbb{Z}$, and the commutative diagram (2.17) becomes

$$(3.1) \quad K^*(C^*\Gamma) \xrightarrow{\text{inflate}} K^*(\Gamma/\mathbb{Z}) \xrightarrow{\text{PD}} K_{*-d}(\Gamma/\mathbb{Z})$$

with PD Poicaré duality for the K-oriented manifold $\Gamma/\mathbb{Z}$.

In particular if $\Gamma = \mathbb{Z}^d$, $T^d$ the torus $T^d : = \mathbb{R}^d/\mathbb{Z}^d$, and $\widetilde{T}^d : = \mathbb{R}^d$ the dual torus then based on the elementary isomorphisms

$$(3.2) \quad \text{KK}^*_{\mathbb{Z}^d}(\mathbb{C}, \mathbb{C}) \cong \text{KK}^*(C^*(\mathbb{Z}^d), \mathbb{C}) \cong \text{KK}^*(C(\widetilde{T}^d), \mathbb{C}),$$

$$\text{KK}^*_{\mathbb{Z}^d}(C_0(\mathbb{R}^d), \mathbb{C}) \cong \text{KK}^*(C(\mathbb{R}^d/\mathbb{Z}^d), \mathbb{C}) = \text{KK}^*(C(T^d), \mathbb{C}),$$

we may interpret the Dirac map as a map

$$(3.3) \quad K_*\big(\widetilde{T}^d\big) \to K_{*-d}(T^d)$$

shifting degrees by $+d \mod 2$. We insist the need to introduce Bott Periodicity at this point (i.e. to work mod 2 in discussion of degrees,) because the Fourier transform has the effect in the first isomorphism to introduce a non-standard real structure on the first dual torus. Thus, (3.3) as stated does not quite correctly describe the Dirac map in real KK-theory, one has to introduce a non-standard real structure on $C(T^d)$ to make it accurate.

With this understanding, both groups $K_*(T^d)$ and $K_*(\widetilde{T}^d)$ can be identified, by the Künneth Theorem, with the graded tensor product

$$K_*(T) \hat{\otimes}_\mathbb{Z} \cdots \hat{\otimes}_\mathbb{Z} K_*(T).$$

As for $K_*(T)$, it has two generating K-homology classes: the $K_0(T)$-class $[\cdot]$ of a point $p$ in $T$, and the $K_1(T)$-class $[T]$ of the Dirac operator on the circle. Now for an $r$-tuple $k = k_1 < \cdots < k_r$ of integers from $\{1, \ldots, n\}$ we associate the $l$-dimensional correspondence (or Baum-Douglas cycle)

$$(3.4) \quad T^d \leftarrow T^l \to \cdot,$$

with $T^j$ carrying its standard (product) K-orientation, and the left map $T^l \to T^d$ sending the $j$th coordinate circle of $T^d$ into the $k_j$th coordinate of $T^d$, and putting the point $p$ into the other coordinates. We call this a standard coordinate embedding of an $l$-torus in $T^d$. Let $[k] \in K_r(T^d)$ be it’s Baum-Douglas class.

Obiously, every standard coordinate embedding $T^l \to T^d$ comes along with a ‘dual’ coordinate embedding $T^{d-l} \to T^d$, mapping circles into the complementary coordinates, and putting $p$’s in the other coordinate spots. Let $[k]^+ \in K_{r+d}(T^d)$ be its class.

If $k$ is given by $k_1 < \cdots < k_r$ and the complementary embedding $k^+$ by $k_{r+1} < \cdots < k_n$, set $\epsilon = \pm 1$ depending on whether $k_1 < k_2 < \cdots < k_n$ is an even or odd permutation.

**Theorem 3.5.** If $[k]$ is the class of a standard coordinate embedding,

$$\text{Dirac}([k]) = \epsilon(k) \cdot [k]^+,\quad$$

with $[k]^+$ the class of the of the dual embedding, the sign $\epsilon(k)$ defined above.
In particular, the Dirac map, that is, composition with (3.3), interchanges the K-homology classes of a point in $\widehat{T}^d$, and the class of the Dirac operator on $T^d$.

Proof. Given the Künneth theorem, and taking products, it suffices to verify the assertion for $\mathbb{Z}$, since the K-homology classes we are considering are all external products of K-homology classes for $\mathbb{T}$.

The Dirac map is thus

$$\text{KK}_\ast(C(\widehat{\mathbb{Z}}), \mathbb{C}) \cong \text{KK}_\ast(C^\ast(\mathbb{Z}), \mathbb{C}) = \text{KK}_\ast^\mathbb{Z}(\mathbb{C}, \mathbb{C}) \xrightarrow{[\mathbb{R}] \otimes} \text{KK}_{-1}^\mathbb{Z}(C_0(\mathbb{R}), \mathbb{C}) \cong \text{KK}_{-1}(C(\mathbb{R}/\mathbb{Z}), \mathbb{C}).$$

The K-homology class of a point in $\mathbb{T}$ is the class of the f.g.p. module over $\mathbb{Z}$, and we can easily verify that restricting $\beta$ to a the slice $T^d \times \{\chi\}$ is understood as $\mathbb{T}$-periodic in the first variable.

Under the identification $K_0(\mathbb{T}) \cong \text{KK}_0^\mathbb{R}(C_0(\mathbb{R}), \mathbb{C})$, the point homology class for $\mathbb{T}$ corresponds to the class $[ev] \in \text{KK}_0^\mathbb{R}(C_0(\mathbb{R}), \mathbb{C})$ of the Z-equivariant representation $C_0(\mathbb{R}) \to C_0(\mathbb{Z}) \subset K^\mathbb{Z}(\mathbb{Z})$ due to the inclusion $\mathbb{Z} \to \mathbb{R}$. We need to prove that $[\mathbb{T}] \otimes_C [\mathbb{R}] = [ev] \in \text{KK}_0^\mathbb{Z}(C_0(\mathbb{R}), \mathbb{C})$. To do so we employ the fact that $[Z] \in \text{KK}_0^\mathbb{Z}(C_0(\mathbb{R}), \mathbb{C})$ is invertible in $\text{KK}_0^\mathbb{Z}(\mathbb{C}, \mathbb{C})$. Let $\eta \in \text{KK}_1^\mathbb{Z}(C_0(\mathbb{R}))$ be the class of the self-adjoint unbounded multiplier $\delta(x) = x$ of $C_0(\mathbb{R})$. Then

$$\eta \otimes_{C_0(\mathbb{R})} [\mathbb{R}] = [ev] \in \text{KK}_0^\mathbb{Z}(\mathbb{C}, \mathbb{C}), \quad [\mathbb{R}] \otimes \eta = 1 \in \text{KK}_0^\mathbb{R}(\mathbb{C}, \mathbb{C})$$

is the content of the Dirac-dual-Dirac method for $\mathbb{Z}$.

Therefore the equation $[\mathbb{T}] \otimes_C [\mathbb{R}] = [ev] \in \text{KK}_0^\mathbb{Z}(C_0(\mathbb{R}), \mathbb{C})$ we want to prove is equivalent to the equation

$$\eta \otimes_{C_0(\mathbb{R})} [ev] = [\mathbb{T}] \in \text{KK}_0^\mathbb{Z}(\mathbb{C}, \mathbb{C}).$$

But the composition on the left is clearly represented by the spectral triple with Hilbert space $l^2(\mathbb{Z})$, and operator the densely defined operator of multiplication by $n$. It’s Fourier transform is therefore the Dirac operator on the circle $\mathbb{T}$. This completes the proof.

Now, let $\beta \in K^0(T^d \times \widehat{T}^d)$ be the Miscenko (or ‘Poincaré) K-theory element, defined in the following way. It is the class of the f.g.p. module over $C(T^d \times \widehat{T}^d)$ consisting of all continuous functions $f$ on $\mathbb{R}^d \times \widehat{T}^d$ such that

$$f(x + v, \chi) = \chi(v)f(x), \quad x \in \mathbb{R}^d, \quad \chi \in \widehat{\mathcal{D}} := \mathbb{Z}^d.$$

The bimodule structure over $C(T^d \times \widehat{T}^d)$ is given by

$$(f \cdot h)(x, \chi) = f(x, \chi)h(x, \chi),$$

where $T^d$ is understood as $\mathbb{R}^d/\mathbb{Z}^d$ so $h$ in this formula is to be interpreted as a continuous function on $\mathbb{R}^d \times \widehat{T}^d$ which is $\mathbb{Z}^d$-periodic in the first variable.

If $\chi: \mathbb{Z}^d \to \mathbb{T}$ is a character, it induces a complex line bundle $L_\chi$ over $\mathbb{R}^d/\mathbb{Z}^d = T^d$. Then the reader can easily verify that restricting $\beta$ to a the slice $T^d \times \{\chi\}$ is the induced vector bundle $L_\chi$. This is the essential point about $\beta$.

Give now $T^d$ and $\widehat{T}^d$ their standard K-orientations, then the two coordinate projections $T^d \times \widehat{T}^d \to T^d$ and $T^d \times \widehat{T}^d \to \mathbb{T}$ are each K-oriented maps, and

$$T^d \xrightarrow{pr_1} (T^d \times \widehat{T}^d, \beta) \xrightarrow{pr_2} \widehat{T}^d,$$
is a (smooth) correspondence, whose analytic class in $\text{KK}_d(C(T^d), C(\hat{T}^d))$ may described in the following way. To each $\chi \in \hat{T}^d$, we associate the flat Hermitian vector bundle $L_\chi := \mathbb{R}^d \times \mathbb{Z} \mathbb{C}$, as above. The Dirac operator $D$ on $T^d$ can be twisted by $L_\chi$ giving an elliptic self-adjoint operator on smooth sections of $L_\chi$. The twisting is accomplished at the level of cycles since $L_\chi$ has a flat connection. The ensemble $\{D_\chi\}_{\chi \in \hat{T}^d}$ makes up a bundle of elliptic operators along the fibres of the coordinate projection $T^d \times \hat{T}^d \to \hat{T}^d$, and each fibre is torus, so we obtain a canonical cycle for $\text{KK}_d(C(T^d), (\hat{T}^d, \hat{T}^d))$.

**Theorem 3.7.** The Dirac map $K_*((\hat{T}^d)) \to K_{*-d}(T^d)$ for $\mathbb{Z}^d$ acting on a point is the Fourier-Mukai transform, given by the map on Baum-Douglas cycles of composition with the smooth correspondence

\[(3.8) \quad T^d \xrightarrow{\text{pr}_1} (T^d \times \hat{T}^d, \beta) \xrightarrow{\text{pr}_2} \hat{T}^d,\]

where $\beta$ is the Miscenko line bundle.

**Proof.** For any discrete group, the standard identification (at the level of cycles) $\text{KK}_{*}^\Gamma (A, \mathbb{C}) \cong \text{KK}_{*}(A \times \Gamma, \mathbb{C})$ factors through the descent map

\[j_* : \text{KK}_{*}^\Gamma (A, \mathbb{C}) \to \text{KK}_{*}(A \times \Gamma, \mathbb{C}(\Gamma))\]

and the map $\epsilon_* : \text{KK}_{*}(A \times \Gamma, \mathbb{C}(\Gamma)) \to \text{KK}_{*}(A \times \Gamma, \mathbb{C})$ induced by the trivial representation $\mathbb{C}(\Gamma) \to \mathbb{C}$. It follows from a quick calculation that if $(\Gamma, Z)$ is $K$-oriented, then the Dirac map interpreted as a map

\[\text{KK}_{*}(\mathbb{C}(\Gamma), \mathbb{C}) \to \text{KK}_{*-d}(C_0(Z) \rtimes \Gamma, \mathbb{C})\]

is given by Kasparov composition with the image $j_*(Z) \in \text{KK}_{*-d}(C_0(Z) \rtimes \Gamma, \mathbb{C}(\Gamma))$ of the Dirac class $[Z] \in \text{KK}_{*-d}(C_0(Z), \mathbb{C})$ under descent.

I claim that under descent, the Dirac class maps to the Fourier-Mukai correspondence $\mathbb{A}_Z$. As this could be viewed as a fairly standard exercise in manipulating unbounded cycles in KK-theory, we sketch the proof only, and do it for $d = 1$.

In this case the Hilbert space $L^2(\mathbb{R})$ on which the Dirac operator $-i \frac{d}{dx}$ acts, to give a cycle for $\text{KK}_{-1}^\Gamma (C_0(\mathbb{R}), \mathbb{C})$, is replaced a the right $C^*(\mathbb{Z})$-module $L^2(\mathbb{R}) \rtimes \mathbb{Z}$, on which $-i \frac{d}{dx}$ now acts in the $L^2(\mathbb{R})$-variable as a right $C^*(\mathbb{Z})$-module map. This gives a cycle for $\text{KK}_{-1}(C_0(\mathbb{R}) \rtimes \mathbb{Z}, C^*(\mathbb{Z}))$.

To complete the picture we compose this $\text{KK}_{-1}(C_0(\mathbb{R}) \rtimes \mathbb{Z}, C^*(\mathbb{Z}))$ element with the class in $\text{KK}_{0}(C(\mathbb{R}/\mathbb{Z}), C_0(\mathbb{R}) \rtimes \mathbb{Z})$ of the standard Morita equivalence $C(\mathbb{R}/\mathbb{Z})-C_0(\mathbb{R}) \rtimes \mathbb{Z}$ bimodule $\mathcal{E}$, which is the completion of $C_0(\mathbb{R})$ with respect to the right $C_0(\mathbb{R}) \rtimes \mathbb{Z}$-valued inner product

\[(f_1, f_2) (x, n) = \overline{f_1 (x)} f_2 (x + n),\]

carrying the right $C_0(\mathbb{R}) \rtimes \mathbb{Z}$-module structure

\[(f \cdot h)(x) := f(x) h(x), \quad (f \cdot n)(x) = f(x + n),\]

$C(\mathbb{R}/\mathbb{Z})$ acts by periodic functions on this module, by module operators.

We obtain, therefore, the $C(\mathbb{R}/\mathbb{Z}) \rtimes C^*(\mathbb{Z})$-bimodule

\[\mathcal{E} \otimes_{C_0(\mathbb{R}) \rtimes \mathbb{Z}} L^2(\mathbb{R}) \rtimes \mathbb{Z},\]

which, with some manipulations, one shows is a the right $C^*(\mathbb{Z})$-module of continuous $f : \mathbb{R} \to C^*(\mathbb{Z})$ such that $f(x + n) = f(x) [n] \in C^*(\mathbb{Z})$. The operator $-i \frac{d}{dx}$ acts as a densely defined unbounded Hilbert module map, since differentiation is translation invariant.

The result now follows from the description of the correspondence $\mathbb{A}_Z$ in analytic KK given just before the statement of Theorem 3.7 by taking Fourier transform. 

\[\square\]
Naturally, one can try to prove that the Dirac map for \( \mathbb{Z} \) interchanges the point class and the Dirac classes for \( T \) and \( \hat{T} \), using the correspondence \((3.13)\) rather directly.

Composition of the correspondences
\[
T \leftarrow (T \times \hat{T}, \beta) \rightarrow \hat{T} \leftarrow \cdot \rightarrow \cdot
\]
gives by transversality
\[
\hat{T} \leftarrow (T, \beta|_T) \rightarrow \cdot
\]
where we understand \( T \) as a subset of \( T \times \hat{T} \) by \( T \times \{p\} \), the point we have picked. Taking \( p = 0 \), the zero character of \( \mathbb{Z} \), is convenient. The inclusion \( T \rightarrow T \times T \) of the slice pulls back the f.g.p. \( C(T \times T) \)-module defining \( \beta \) to the module of continuous functions on \( \mathbb{R} \times \{0\} \) such that \( f(x + n, 0) = f(x, 0) \), i.e. periodic functions, and hence the restricted module is precisely \( C(T) \). Hence the restriction of \( \beta \) to \( \hat{T} \) is the class of the trivial line bundle, proving that the Dirac map sends the point K-homology class to the correspondence
\[
T \overset{\text{id}}{\rightarrow} T \rightarrow \cdot
\]
which describes the Dirac class \([T]\).

Things do not run so smoothly if one starts with the class \([T]\). The resulting correspondence involves a manifold of dimension 2 which then must be Thom, or Bott un-modified, to a manifold of dimension zero (a point), and one has to take care of the bundle as well. It is easier to use the dual-Dirac method to do the calculation, as we did in the proof of Theorem 3.5, but we will prove it more geometrically in a forthcoming note with D. Hudson.

The following gives a more geometric description of the Dirac (or Fourier-Mukai) map.

Suppose \( T \subset T^d \) is a \( j \)-dimensional torus subgroup. It lifts to a \( j \)-dimensional linear subspace \( L \subset \mathbb{R}^d \).

Let \( L^\perp \) be all characters of \( \mathbb{R}^d \) which vanish on \( L \), \( (\mathbb{Z}^d)^\perp \) all \( \chi \in \mathbb{R}^d \) that vanish on \( \mathbb{Z}^d \) and \( T^\perp \) the projection to \( \mathbb{R}^d / (\mathbb{Z}^d)^\perp \cong \mathbb{Z}^d = T^\perp \) of \( L^\perp \). It is a \( d - j \)-dimensional subtorus of \( \hat{T}^d \).

**Theorem 3.11.** Let \( T \subset T^d \) be a \( j \)-dimensional linear torus, \( T^\perp \subset T^\hat{d} \) its \( d - j \)-dimensional dual torus. Then the Dirac-Fourier-Mukai transform
\[
K_{j}(T^d) \rightarrow K_{j+d}((\hat{T}^d)
\]
maps the cycle \([T]\) to the cycle \([T^\perp]\).

With some labour, the theorem can be proved using Theorem 3.5, but we will prove it more geometrically in a forthcoming note with D. Hudson.

Finally, we note that just as application of descent and the Fourier transform to the Dirac class of \((\mathbb{R}^d, \mathbb{Z}^d)\) produced the Fourier-Mukai correspondence
\[
\mathcal{F}_d \in \text{KK}_d(C(T^d), C(\hat{T}^d)),
\]
we obtain the following when we do it to the dual-Dirac element
\[
\eta \in \text{KK}_d^Z(\mathcal{C}, C_0(\mathbb{R}^d))
\]
Let \( \mathcal{F}'_d \) be the class in \( \text{KK}_d(\hat{C}(T^d), C(\hat{T}^d)) \) of the reversed Fourier-Mukai correspondence
\[
\hat{T}^d \overset{\text{pr}}{\rightarrow} (\hat{T}^d \times T^d, \beta) \overset{\text{pr}_2}{\rightarrow} T^d,
\]

**Corollary 3.13.** Application of the composition of functors
\[
\text{KK}_d^Z(\mathcal{C}, C_0(\mathbb{R}^d)) \rightarrow \text{KK}_d(\mathcal{C}^*(\mathbb{Z}^d), C_0(\mathbb{R}^d / \mathbb{Z}^d)) \cong \text{KK}_d(C(\hat{T}^d), C(T^d))
\]
to the dual-Dirac element \( \eta \) for \( \mathbb{R}^d \) gives the class of the dual Fourier-Mukai correspondence \( \mathcal{F}'_d \) of \((3.12)\).
In particular,
\[ F_d' \circ F_d = \text{id}, \quad F_d \circ F_d' = \text{id}, \]
as morphisms in the category KK.

It would appear to be a very interesting project to work out precisely happens in real K-theory, but we are not going to pursue this here.

We conclude this section by noting that the fact that the Dirac map descends to the Fourier-Mukai transform has been known to at least Nigel Higson and the author (in that order) since around 2000. However, as far as I have been able to determine, the connection has not been explored or written down anywhere since. The paper \[7\] contains some related material, though.

4. The Dirac class of a discrete group

In the previous section we defined the Γ-equivariant transverse Dirac class on a smooth and equivariantly oriented Γ-manifold \(X\). This contained the case where Γ is trivial, and reduces to the usual notion of a Dirac class on a compact oriented manifold. We also computed the Dirac map for \(\mathbb{Z}^d\) acting on a point.

Motivated especially by the latter calculations, we now proceed to a notion of the Dirac class on the ‘noncommutative manifold’ underlying the C*-algebra \(C^*(\Gamma)\) for more general discrete groups Γ. We will continue to assume that Γ admit a smooth orientation \((\Gamma, Z)\). The group C*-algebra in this case may be considered – at least at the level of homology – as a ‘noncommutative compact smooth oriented manifold’ of dimension \(d\) with the ‘Dirac class’ \([\hat{\Gamma}]\) defined below, playing the role of the class of the associated Dirac operator on that smooth compact (oriented) manifold.

Choose a point \(x_0 \in Z\). The group Γ acts on \(l^2 \Gamma\) by the regular representation. The composition
\[
\text{ev}: C_0(Z) \xrightarrow{\text{restr}} C_0(\Gamma x_0) \to C_0(\Gamma) \xrightarrow{\text{mult}} K(l^2 \Gamma)
\]
is a Γ-equivariant *-homomorphism and determines a class
\[
[\text{ev}] \in KK^0_0(C_0(Z), \mathbb{C}).
\]
It is an ‘equivariant point homology class’ for Z. Since Z is a classifying space, it is path-connected and [ev] is independent of the choice of point.

**Definition 4.3.** Let \((\Gamma, Z)\) be a smooth oriented group of dimension \(d\). A **Dirac class** for Γ is any class \([\hat{\Gamma}] \in K^d(C^*(\Gamma)) = KK^d_{\mathbb{C}}(\mathbb{C}, \mathbb{C})\) such that
\[
\text{Dirac}(\hat{\Gamma}) = [\text{ev}] \in KK^d_0(C_0(Z), \mathbb{C}).
\]

**Remark 4.4.** One could make a variant of the definition involving the **reduced** C*-algebra of the group. The projection \(C^*(\Gamma) \to C^*_r(\Gamma)\) induces a pullback map \(K^*(C^*_r(\Gamma)) \to K^*(C^*(\Gamma))\) so there is an associated (reduced) Dirac map
\[
K^*(C^*_r(\Gamma)) \to K^*(C^*(\Gamma)) \cong KK^d_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \otimes_{\mathbb{C}} [Z_\Gamma] \xrightarrow{\text{ev}} KK^d_{*+d}(C_0(Z), \mathbb{C}).
\]

Occasionally we will call a class mapping to \([\text{ev}] \in KK^d_0(C_0(Z), \mathbb{C})\) a **reduced Dirac class**; note that the projection \(C^*(\Gamma) \to C^*_r(\Gamma)\) pulls a reduced Dirac class back to a Dirac class, so the existence of a reduced class is stronger.
Example 4.6. If $\Gamma$ is finite, so $Z$ is a point, the Dirac map is the identity map and the Dirac class for $\Gamma$ is represented by the homomorphism

$$\lambda: C^*(\Gamma) \rightarrow K(l^2\Gamma),$$

induced by the regular representation.

In particular, if $\Gamma$ is finite abelian, we recover the usual notion of the Dirac class on the finite set $\hat{\Gamma}$ as the sum of the Dirac classes on the points (a Dirac class on a point is a specification of a generator $\pm 1$ of $KK_0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$.)

Remark 4.7. As a noncommutative space, $C^*(\Gamma)$ is the space of irreducible representations of the discrete group $\Gamma$, appropriately topologized. The class of the trivial representation may be thought of as a homology class for this space: it is the class associated to the corresponding point of $\hat{\Gamma}$.

The *transverse* Dirac class for a discrete group (acting on a point) is the class of the trivial representation in $KK_0(\mathbb{C}, \mathbb{C}) \cong K^0(C^*(\Gamma))$, and so can be thought of as a point homology class in the noncommutative space $\hat{\Gamma}$. But the Dirac class is the sum of all the points of $\hat{\Gamma}$, with the usual multiplicities involved in the Peter-Weyl Theorem.

**Proposition 4.8.** The Dirac class $[\hat{\mathbb{Z}^d}] \in KK^d_\mathbb{C}(\mathbb{C}, \mathbb{C}) \cong K^d(C^*(\mathbb{Z}^d))$ of the smooth oriented group $(\mathbb{Z}^d, \mathbb{R}^d)$ is the Fourier transform of the transverse Dirac class $[\hat{\mathbb{Z}}^d]$ of its dual torus (equipped with a non-standard real structure).

The notation $[\hat{\mathbb{Z}^d}]$ is thus consistent, but only in complex $K$-theory: it is off by a sign in real $K$-theory since the Dirac operator on the spin manifold $\hat{\mathbb{Z}}^d$ lives in $KK_{-d}(C(\hat{\mathbb{Z}}^d), \mathbb{C})$.

Proposition 4.8 follows from Theorem 3.5.

The following proves that $[\hat{\Gamma}]$ is always nonzero and non-torsion in $K^d(C^*(\Gamma))$. The result will be refined later.

**Proposition 4.9.** Let $(\Gamma, Z)$ be a smooth, oriented group, and $[\hat{\Gamma}] \in KK^d(C^*(\Gamma))$ a Dirac class. Let

$$\mu: KK^d_\Gamma(C_0(Z), \mathbb{C}) \rightarrow K_*(C^*(\Gamma))$$

be the Baum-Connes assembly map. Then

$$(\mu([\hat{\Gamma}]), [\Gamma]) := \mu([Z]) \otimes_{C^*(\Gamma)} [\hat{\Gamma}] = 1 \in KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}.$$  

In particular, the Dirac class $[\hat{\Gamma}]$ of any smooth orientable group induces a surjection $K_*(C^*(\Gamma)) \rightarrow \mathbb{Z}$, and is never zero in $K$-homology, nor torsion.

The analogous statement holds for a reduced Dirac class.

**Proof.** Denote by $f \mapsto \overline{f}$ the descent map $KK^d_\Gamma(A, B) \rightarrow KK_*(A \times \Gamma, B \times \Gamma)$. If $(\Gamma, Z)$ is a smooth oriented group, then $\delta\Gamma = Z$ and the Baum-Connes assembly map with coefficients $B$ is, by definition, $\mu(f) = [P_\varphi] \otimes C_0(\Gamma) \times \Gamma \overline{f}$ for any $f \in KK^d_\Gamma(C_0(Z), B)$, and any cut-off function $\varphi \in C_c(\mathbb{Z})$, $0 \leq \varphi \leq 1$ and $\sum_{g \in \Gamma} g(\varphi) = 1$. The space of cut-off functions is convex and nonempty, and $P_\varphi := \sum_{g \in \Gamma} g(\varphi) |g| \in C_0(\mathbb{Z}) \times \Gamma$ is a projection whose homotopy-class does not depend on the choice of cut-off function. For the element $[ev] \in KK^d_\Gamma(C_0(Z), \mathbb{C})$ and the element $[P_\varphi] \in K_0(C_0(\mathbb{Z}) \rtimes \Gamma)$, observe that the map $C_0(Z) \rtimes \Gamma \rightarrow C_0(\Gamma) \rtimes \Gamma \cong C_0(\Gamma) \rtimes \Gamma = K(l^2\Gamma)$ maps $P_\varphi$ to a projection homotopic through projections to a rank-one projection. Hence

$$(\mu([ev]), [P_\varphi]) = 1.$$  

On the other hand, descent maps the Dirac class $[Z] \in KK^d_{-d}(C_0(Z), \mathbb{C})$ to a class

$$[\mathbb{Z}] \in KK_{-d}(C_0(Z) \rtimes \Gamma, C^*(\Gamma)),$$
and identifying $\text{KK}^\Gamma_d(C_0(Z), \mathbb{C}) \cong \text{KK}^\Gamma_d(C_0(Z) \rtimes \Gamma, \mathbb{C})$, the Dirac map can be identified with the map

$$\text{KK}_s(C^*(\Gamma), \mathbb{C}) \to \text{KK}^\Gamma_{\tau-d}(C_0(Z) \rtimes \Gamma, \mathbb{C})$$

of composition with $[Z]$. Hence

$$\langle \mu([Z]), \tilde{[\Gamma]} \rangle = [P_\phi] \otimes_{C_0(Z) \rtimes \Gamma} [Z] \otimes_{C^*(\Gamma)} [\tilde{\Gamma}] = [P_\phi] \otimes_{C_0(Z) \rtimes \Gamma} \text{Dirac}(\tilde{\Gamma}) = [P_\phi] \otimes_{C_0(Z) \rtimes \Gamma} [\text{ev}] = 1 \in Z \cong \text{KK}_0(\mathbb{C}, \mathbb{C})$$

as claimed. 

If $(\Gamma, Z)$ is a smooth, oriented group, then a dual-Dirac class for $\Gamma$ is a class

$$\eta \in \text{KK}^\Gamma_d(\mathbb{C}, C_0(Z))$$

such that $D_Z \otimes C \eta = 1 = C_0(Z)(C_0(Z), C_0(Z))$. The existence or non-existence question of such $\eta$ is determined by a certain coarse co-assembly map (see [24]), and so is a question about the large-scale geometry of $\Gamma$. The fact that the two coordinate projections $Z \times Z \to Z$ are $\Gamma$-equivariantly homotopy implies by an easy exercise that $[Z] \otimes \eta = 1$ as well.

If $Z$ admits a $\Gamma$-invariant Riemannian metric of nonpositive curvature, then a dual-Dirac class exists and is described in detail below. More generally, $\eta$ exists if $\Gamma$ admits a uniform embedding in Hilbert space, e.g. if $\Gamma$ is linear or word hyperbolic.

**Proposition 4.12.** If $(\Gamma, Z)$ is a smooth oriented, group with a dual-Dirac morphism $\eta \in \text{KK}^\Gamma_d(\mathbb{C}, C_0(Z))$, then

$$\eta \otimes_{C_0(Z)} [\text{ev}] \in \text{KK}^\Gamma_d(\mathbb{C}, \mathbb{C}) \cong K^d(C^*(\Gamma))$$

is a Dirac class for $\Gamma$.

Hence if $\Gamma$ is torsion-free, then isomorphism of the coarse co-assembly map of [24] coarse co-assembly map

$$\tilde{K}_s(c(\Gamma)) \to K^*(\Gamma \backslash Z)$$

for the coarse space $|\Gamma|$ underlyng $\Gamma$, implies existence of a Dirac class for $\tilde{\Gamma}$.

**Proof.** $[Z] \otimes_C (\eta \otimes_{C_0(Z)} [\text{ev}]) = ([Z] \otimes_C \eta) \otimes_{C_0(Z)} [\text{ev}] = [\text{ev}]$ since $[Z] \otimes_C \eta = 1 = C_0(Z)$.

The $C^*$-algebra $c(\Gamma)$ figuring in (4.13) is the stable Higson corona of the coarse space $|\Gamma|$ underlyng $\Gamma$. $\tilde{K}_s(c(\Gamma))$ is its reduced $K$-theory.

Let $\Gamma$ be torsion-free, so $K_s(C_0(Z) \rtimes \Gamma) \cong K^*(\Gamma \backslash Z) \cong K^*(B\Gamma)$ and a class $a$ in this ring yields a higher signature, associating to any compact manifold $M$ with fundamental group $\Gamma$ the index $(D_M^{\text{alg}} \cdot \chi^* a)$ of the signature operator on $M$ twisted by the $K$-theory class $\chi^* a$, where $\chi: M \to B\Gamma$ is the classifying map for $M$.

By a theorem of Kaminker and Miller [31], building on work of many others, such classes are homotopy-invariant when they are in the range of the inflation map

$$\text{infl}: \text{KK}^\Gamma_s(\mathbb{C}, \mathbb{C}) \to \text{RKK}^\Gamma_s(Z; \mathbb{C}, \mathbb{C}) \cong K^*(\Gamma \backslash Z) \cong K^*(B\Gamma).$$

Since the composition of the inflation map and Poincaré duality

$$\text{KK}^\Gamma_s(\mathbb{C}, \mathbb{C}) \xrightarrow{\text{infl}} \text{RKK}^\Gamma_s(Z; \mathbb{C}, \mathbb{C}) \cong K^*(\Gamma \backslash Z) \xrightarrow{PD} \text{ KK}^\Gamma_{\tau-d}(C_0(Z), \mathbb{C}) \cong K_{\tau-d}(\Gamma \backslash Z) = K_{\tau-d}(B\Gamma)$$

is exactly the Dirac map, it follows that the higher signature associated to the Poincaré dual of any class in the range of the Dirac map, is homotopy-invariant.
This applies in particular to the point class \([\gamma] \in K_0(\Gamma\backslash Z)\) (or \([\text{ev}] \in KK_0(C_0(Z), \mathbb{C})\), as we have been denoting it above.)

**Corollary 4.15.** Let \((\Gamma, Z)\) be a smooth oriented, torsion-free group. Then a Dirac class for \(\Gamma\) is a pre-image in \(KK_0^r(\mathbb{C}, \mathbb{C})\) of the Poincaré dual of a point \(\in K^{-d}(\Gamma\backslash Z) \cong K^{-d}(BG)\), under the inflation map.

In particular, the higher signature associated to the Poincaré dual of a point in \(\Gamma\backslash Z \cong BG\) is homotopy-invariant as soon as a Dirac class exists for \(\Gamma\).

The Poincaré dual of a point, and the question of the homotopy-invariance of the higher signature associated to it, is studied in \[13\] by Connes, Gromov and Moscovici. In their paper, they analyze geometric conditions one can put on a group under which one can write down a 'Dirac class', thus guaranteeing homotopy-invariance (for a single cohomology class). The procedure they use to build such 'Dirac cycles' (they build more general ones as well), is based on the non-positive curvature idea we exploit below.

**Remark 4.16.** We note as well that A. Connes has described Dirac cycles for discrete groups – representatives, in our terminology, of the Dirac class \([\hat{\Gamma}]\) – in his book \[11\], and again, these are the same cycles we describe below.

Assume that \((\Gamma, Z)\) is a smooth oriented group such that \(Z\) admits a \(\Gamma\)-invariant metric of nonpositive curvature. For example, \(\Gamma\) could be a discrete subgroup of a connected and semisimple Lie group \(G\); if \(K \subset G\) is the maximal compact subgroup, then \(G/K\) is a symmetric space of nonpositive curvature admitting a \(G\)-invariant (and hence \(\Gamma\)-invariant) orientation.

The hypotheses say that there is a \(\Gamma\)-equivariant spinor bundle \(S\) on \(Z\), and a \(\Gamma\)-equivariant bundle map

\[
\text{Cliff}(TZ) \otimes S \to S
\]

specifying the Clifford multiplication on \(S\). We denote Clifford multiplication by a tangent vector \(\xi\) by \(c(\xi)\), so if \(z \in Z\) and \(\xi \in T_zZ\) is a unit tangent vector then \(c(\xi)\) is a certain self-adjoint endomorphism of \(S_z\) with square 1.

Fix a point \(z_0 \in Z\), and let \(\exp : T_{z_0}Z \to Z\) be the exponential map, a diffeomorphism, and \(\log : Z \to T_{z_0}Z\) its inverse. Then nonpositive curvature implies that \(\log\) is a Lipschitz map, that is, \(|\log(x) - \log(y)| \leq d(x, y)\), where \(d\) is the metric on \(Z\) induced by the Riemannian metric on \(Z\).

Note that the fibre \(S_{z_0}\) of the spinor bundle at \(z_0\) is a finite-dimensional Hilbert space, \(Z/2\)-graded if \(\dim(Z)\) is even, and trivially-graded otherwise. We define a map \(\delta : \Gamma \to B(V)\) by \(\delta(\gamma) := c(\log \gamma z_0)\). On the Hilbert space \(L^2(\Gamma, V)\) with the induced \(\Gamma\)-action, \(\delta\) determines an unbounded, self-adjoint multiplication operator which we denote by the same letter; \(\delta \in B(L^2(\Gamma, V))\); it commutes modulo bounded operators with the \(\Gamma\)-action because of the Lipschitz condition.

It defines a spectral triple \((L^2(\Gamma, V), \pi, \delta)\) for \(C^*(\Gamma)\) (not finitely summable in general, see Remark 4.18).

**Theorem 4.17.** If \((\Gamma, Z)\) is a smooth oriented group and \(Z\) is equipped with a \(\Gamma\)-invariant metric of nonpositive curvature, then the spectral triple \((L^2(\Gamma, V), \pi, \delta)\) defined above represents a Dirac class \([\Gamma]\) for \(\Gamma\).

Furthermore, it is defined over the reduced \(C^*\)-algebra of \(\Gamma\) and hence there is a reduced Dirac class for \(\Gamma\) in this case, given by the same cycle but regarded as defined over \(C^*_r(\Gamma)\).

**Remark 4.18.** The Fredholm module (defined over \(C^*_r(\Gamma)\), its class is a reduced Dirac class) just described is finitely summable if \(\Gamma\) has polynomial growth. By a result of Connes, the existence of a finitely summable spectral triple representing a nonzero class in \(KK_0^r(\mathbb{C}, \mathbb{C})\) implies amenability of \(\Gamma\) (see \[11\]).
5. Dirac classes for smooth actions of smooth K-oriented groups

In this section we simultaneously generalize transverse Dirac classes for manifolds and Dirac classes for discrete groups, to define Dirac classes for actions.

Definition 5.1. Let \( (\Gamma, Z) \) be a smooth oriented group, and let \( \Gamma \) act smoothly and on the smooth compact oriented manifold \( X \) preserving a K-orientation in the sense of Definition 2.3.

Let \([X] \in KK_{\Gamma 0}^r(C(X), \mathbb{C})\) be a transverse Dirac class for the action. \([X]\) exists by Theorem 2.9.

A Dirac class for \( \Gamma \ltimes X \) is any class
\[
[X \ltimes \Gamma] \in KK_{\Gamma 0}^r(C_0(X) \ltimes \Gamma, \mathbb{C})
\]
such that
\[
\text{Dirac}([X \ltimes \Gamma]) = [ev] \otimes [X] \in KK_{\Gamma 0}^r(C_0(Z \times X), \mathbb{C}).
\]

A reduced Dirac class is a class \([X \ltimes \Gamma] \in K^{d-n}(C(X) \ltimes \Gamma, \Gamma)\) which pulls back to a Dirac class under the map on K-homology induced by the projection \(C(X) \ltimes \Gamma \to C(X) \ltimes G\).

If \( \Gamma \) is torsion-free, \( KK_{\Gamma}^r(C_0(Z \times X), \mathbb{C}) \cong K_n(Z \ltimes \Gamma X)\), where \( Z \ltimes \Gamma X \) denotes the quotient of \( Z \times X \) by the diagonal action of \( \Gamma \). The smooth manifold \( Z \ltimes \Gamma X \) is foliated into the images of the slices \( Z \times \{x\} \), for \( x \in X \), and, \( Z \times \Gamma X \) is also a bundle of compact manifolds over \( \Gamma \backslash Z \) under the coordinate projection
\[
p: Z \times \Gamma X \to \Gamma \backslash Z.
\]
The submanifold \( p^{-1}(\Gamma z_0) \) is a closed transversal \( X_{z_0} \) for the foliation, naturally diffeomorphic to \( X \), for any \( z_0 \in Z \).

Let \( e_{z_0, X}: X \to Z \times \Gamma X \) the corresponding embedding. The class
\[
[ev] \otimes [X] \in KK_{\Gamma 0}^r(C_0(Z \times X), \mathbb{C}) \cong K_n(Z \times \Gamma X)
\]
may be then simply described as the class of the Baum-Douglas cycle
\[
(5.2)
\]
from \( Z \times \Gamma X \) to a point, obtained by mapping the K-oriented compact manifold \( X \) into \( Z \times \Gamma X \) as a transversal to the foliation described above, or, even more explicitly,

Proposition 5.3. If \( \Gamma \) is torsion-free, and \( X \) is smooth and \( \Gamma \)-equivariantly K-orientable, then a Dirac class for \( \Gamma \ltimes X \) is any class mapping to
\[
e_{z_0, X}^r \otimes C(X) [X] \in KK_{\Gamma 0}^r(C_0(Z \times \Gamma X), \mathbb{C}) = K_n(Z \times \Gamma X),
\]
under the Dirac map, where \( e_{z_0, X}^r \in KK_{0}(C(Z \times \Gamma X), C(X)) \) is the class of the \(*\)-homomorphism induced by mapping \( X \) into \( Z \times \Gamma X \) as a fibre, and \([X] \in K_{-n}(X)\) is the transverse Dirac class of \( X \).

The analogue of proposition 4.1 holds for groupoids as well.

Proposition 5.4. If \( (\Gamma, Z) \) is a smooth oriented group, \( X \) an equivariantly oriented \( \Gamma \)-manifold, then a Dirac class \([\Gamma \ltimes X] \) exists as soon as the groupoid \( \Gamma \ltimes X \) has a dual-Dirac morphism \( \eta \in KK_{d}^{r \times X}(C_0(X), C_0(Z \times X)) \).

The Dirac class for \( \Gamma \ltimes X \) is unique as soon as \( \gamma_{r \times X} = 1_{r \times X} \in KK_{0}^{r \times X}(C_0(X), C_0(X)) \), where \( \gamma_{r \times X} \) is the corresponding \( \gamma \)-element for the groupoid \( \Gamma \ltimes X \).

The hypotheses are met if \( X \) is a proper, or more generally, an amenable \( \Gamma \)-space; they are also met if \( \Gamma \) itself has a \( \gamma \)-element.

In the latter case, if \( X \) is also compact, then \( \Gamma \) itself has a \( \gamma \)-element (and not just the groupoid \( \Gamma \ltimes X \)). More generally, if \( (\Gamma, Z) \) is a smooth, oriented group, \( X \) a smooth oriented

THE CLASS OF A FIBRE IN NONCOMMUTATIVE GEOMETRY 19
Γ-manifold, and if η ∈ KR_{d}^{Γ}(C, C_0(Z)) is a dual-Dirac morphism for Γ, then the groupoid Γ × X also has a dual-Dirac morphism and corresponding Dirac class, given by the formula

\[ [Γ × X] = (η ⊗ C 1_{C(X)}) ⊗ C_0(Z × X) ([ev] ⊗ C [X]). \]

More generally, existence of a Dirac class for Γ implies one for any action.

**Proposition 5.6.** If (Γ, Z) is a smooth oriented group, \([\hat{Γ}] \in KR_{d}^{Γ}(C, C)\) is a Dirac class for Γ, and if \([X] \in KK_{r}^{Γ-n}(C_0(X), C)\) is a transverse Dirac class then

\[ [\hat{Γ}] ⊗ C [X] \in KK_{d-n}^{Γ}(C_0(X), C) \]

is a Dirac class for Γ × X, where the Kasparov product is the external product in KK\(^{Γ}\).

**Proof.** The proof is a trivial consequence of associativity of the Kasparov product. □

Suppose (Γ, Z) is a smooth K-oriented group with Z carrying a Γ-invariant metric of non-positive curvature, and suppose that Γ acts by Riemannian isometries of X preserving a K-orientation. The Dirac class of Γ is represented by the spectral triple \((L^2(Γ, \pi, δ))\) of Theorem 4.17. It is finitely summable if Γ has polynomial group (not otherwise).

Since the Γ-action is assumed isometric on X, the transverse Dirac class is also represented by a spectral triple, of the type \((L^2(S), \pi, D_X)\), where \(D_X\) is the Γ-equivariant Dirac operator associated to the equivariant metric and K-orientation.

**Corollary 5.7.** If (Γ, Z) is a smooth K-oriented group with Z carrying a metric of nonpositive curvature, and if Γ acts isometrically and preserving a Γ-orientation on a Riemannian manifold X, then, in the notation above, the Dirac class for Γ × X is represented by the spectral triple \((L^2(S) ⊗ L^2(Γ, \pi, δ), D_X ⊗ 1 + 1 ⊗ δ)\).

In particular, if Γ has polynomial growth \(\sim n^d\) then the (reduced) Dirac class is represented by a \(dim(X) + d\)-summable spectral triple over \(C_0(X) \times _r Γ\).

**Proof.** The proof is an immediate consequence of Proposition 5.6 and the standard recipe for taking external products of unbounded cycles in KK\(^{Γ}\). □

**Example 5.8.** The Dirac class of a K-orientation preserving action of a finite group Γ on X is represented by the spectral triple \((L^2(S) ⊗ L^2(Γ, D_X ⊗ 1), D_X, GNS Hilbert space associated to τ)\). The algebra \(A_θ\) is represented on \(L^2(A_θ)\) by left multiplication, and the two densely defined derivations

\[ \delta_1(\sum_{n∈Z} f_n[n]) := \sum_{n∈Z} f'_n[n], \quad \delta_2(\sum_{n∈Z} f_n[n]) := \sum_{n∈Z} nf_n[n], \]

using group-algebra notation. They assemble to a densely defined self-adjoint operator \(\overline{θ}_θ := \begin{bmatrix} 0 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & 0 \end{bmatrix}\),

**Dirac classes for actions of \(Z^d\)**

We start with computing the Dirac class of an irrational rotation.

**Example 5.9.** The irrational rotation algebra.

The Dirac class of the irrational rotation algebra \(A_θ := C(\mathbb{T}) ×_θ Z\) is represented by a spectral triple first described by A. Connes: let τ : \(A_θ → C\) be the standard trace, \(L^2(A_θ)\) the GNS Hilbert space associated to τ. The algebra \(A_θ\) is represented on \(L^2(A_θ)\) by left multiplication, and the two densely defined derivations

\[ \delta_1(\sum_{n∈Z} f_n[n]) := \sum_{n∈Z} f'_n[n], \quad \delta_2(\sum_{n∈Z} f_n[n]) := \sum_{n∈Z} nf_n[n], \]

using group-algebra notation. They assemble to a densely defined self-adjoint operator \(\overline{θ}_θ := \begin{bmatrix} 0 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & 0 \end{bmatrix}\),
on \( L^2(A_\theta) \oplus L^2(A_\theta) \), a deformation of the Dolbeault operator \( \overline{\partial}_{T^2} \) on \( T^2 \).

Note that, as a \textit{(unbounded) operator on a Hilbert space}, \( \overline{\partial}_\theta \) is absolutely identical to the ordinary Dolbeault operator \( \overline{\partial}_{T^2} \) operating with its usual initial domain of smooth functions in the graded Hilbert space \( L^2(T^2) \oplus L^2(T^2) \). Thus as far as the \textit{operator} is concerned, we are dealing with a classical operator. However, the dynamics is encoded by the representation of \( A_\theta \), so that the noncommutative aspect of this spectral triple lies entirely in the representation.

From this point of view, thus, the spectral triple involves the Hilbert space \( L^2(T^2) \), the Dolbeault operator on \( T^2 \), and the representation of \( \pi : A_\theta = C(T) \rtimes \mathbb{Z} \to L(L^2(T^2) \oplus L^2(T^2)) \), is

\[
\pi(f)\xi(x,y) = f(x)\xi(x,y), \quad (n \cdot \xi)(x,y) = e^{2\pi i n y} \xi(R_{\theta}^{-n}(x),y).
\]

More generally, suppose that \( \mathbb{Z} \) acts on a complete Riemannian manifold \( X \), isometrically, and preserving a K-orientation in the sense that there is a Hermitian \( \mathbb{Z} \)-equivariant spinor bundle \( S_X \) over \( X \), graded or ungraded or \( p \)-multigraded (if one is working over the reals), and a \( \mathbb{Z} \)-equivariant connection on \( S_X \) and associated \( \mathbb{Z} \)-equivariant Dirac operator

\[
D_X : L^2(S_X) \to L^2(S_X)
\]
determining the transverse Dirac class \( [X] \in KK_{\mathbb{Z}}(C_0(X), \mathbb{C}) \).

Following the recipe of Corollary 6.7, we form an external product and get the following explicit representative of \( [Z \rtimes X] \in KK_{\mathbb{Z}}(C_0(X) \rtimes \mathbb{Z}, \mathbb{C}) \). We assume for simplicity that \( \dim X \) is odd, and that \( S_X \) is ungraded, \( D_X \) self-adjoint. Then the Hilbert space of our spectral triple consists of two copies of \( L^2(S_X) \otimes \ell^2(\mathbb{Z}) \) with standard even grading, and the operator

\[
\begin{bmatrix}
0 & D_X \otimes 1 + i(1 \otimes \delta) \\
D_X \otimes 1 - i(1 \otimes \delta) & 0
\end{bmatrix}
\]

where \( \delta \) is the number operator on \( \ell^2(\mathbb{Z}) \). The representation of \( C(X) \rtimes \mathbb{Z} \) on this Hilbert space is by

\[ f(\xi \otimes e_k) := f\xi \otimes e_k, \quad m(\xi \otimes e_k) := m(\xi) \otimes e_{k+n}. \]

This of course directly generalizes the irrational rotation example.

We next consider an action of \( \mathbb{Z}^2 \) preserving a K-orientation on the compact manifold \( X \) of dimension \( n \).

If \( n \) is odd, the spinor bundle \( S_X \) for \( X \) is ungraded; let \( D_X \) be the corresponding Dirac operator. Let \( \overline{\partial} \) be the Dolbeault operator on the \( \mathbb{Z}/2 \)-graded Hilbert space \( L^2(T^2) \oplus L^2(T^2) \). Then the Dirac class of the action is represented by the following odd-dimensional spectral triple over \( C(X) \rtimes \mathbb{Z}^2 \). The Hilbert space is \( L^2(T^2) \otimes L^2(S_X) \oplus L^2(T^2) \otimes L^2(S_X) \) with \textit{no} grading, and the operator with respect to this decomposition is

\[
\begin{bmatrix}
0 & \overline{\partial}_{T^2} \otimes 1 + i (1 \otimes D_X) \\
\overline{\partial}_{T^2} \otimes 1 - i (1 \otimes D_X) & 0
\end{bmatrix}. \tag{5.10}
\]

The action of \( C(X) \) is by letting \( f \in C(X) \) act by a multiplication operator in the \( L^2(S_X) \) factor. The group \( \mathbb{Z}^2 \) acts diagonally, with the implicitly assumed unitary action on sections of the spinor bundle \( S_X \), and the action of \( \mathbb{Z}^2 \) on \( L^2(T^2) \) given by the Fourier transform

\[ (g \cdot \xi)(x) := \chi(g)\xi(x), \quad \chi \in \hat{\mathbb{Z}} \cong \mathbb{T}^2. \]

This spectral triple is clearly \( 2 + n \)-summable.

If \( n \) is even, \( S_X \) graded into \( S_X^0 \oplus S_X^1 \), then the Dirac class is represented by the odd operator

\[
\begin{bmatrix}
0 & \overline{\partial}_{T^2} \otimes 1 + 1 \otimes D_X^0 \\
\overline{\partial}_{T^2} \otimes 1 + 1 \otimes D_X^1 & 0
\end{bmatrix}. \tag{5.11}
\]
on the $\mathbb{Z}/2$-graded Hilbert space with even part
\[ L^2(\mathbb{T}^2) \otimes L^2(S^0_X) \oplus L^2(\mathbb{T}^2) \otimes L^2(S^1_X) \]
and odd part
\[ L^2(\mathbb{T}^2) \otimes L^2(S^1_X) \oplus L^2(\mathbb{T}^2) \otimes L^2(S^0_X). \]
Here $D^0_X := D_X|_{L^2(S^0_X)} : L^2(S^0_X) \to L^2(S^1_X)$, and $D^1_X := D_X|_{L^2(S^1_X)} : L^2(S^1_X) \to L^2(S^0_X)$.

The action of $C(X) \times \mathbb{Z}^2$ is as before.
This Dirac spectral triple is $n + 2$-summable, of course.

6. Further properties of Dirac classes: Dirac classes for proper actions

We first describe the image of a Dirac class under the inflation map.
Choose $z_0 \in Z$, let $o_{z_0} : \Gamma \to Z$ be the orbit map at $z_0$. It is a smooth, \( \Gamma \)-equivariantly $\mathbb{K}$-oriented embedding. The normal bundle is $\cong o^{*}_{z_0}(TZ)$, using the usual tubular neighbourhood embedding of the form
\[ \varphi(g, \xi) := \exp_{g z_0}(\xi), \]
where $\xi \mapsto \xi'$ is an appropriate re-scaling of tangent vectors (e.g. by $\xi' := \frac{\xi}{(1 + |\xi|^2)^{1/2}}$) into an open disk sub bundle of the tangent bundle, on which the Riemannian exponential map is a diffeomorphism.

The $\Gamma$-equivariant map $o_{z_0} : \Gamma \to Z$ gives $\Gamma$ the structure of a $G$-space, where $G := \Gamma \times Z$ as before, and
\[ (6.1) \]
\[ Z \xleftarrow{o_{z_0}} \Gamma \xrightarrow{o_{z_0}} Z. \]
is a $G$-equivariant correspondence from $Z$ to $Z$, with class
\[ o^{*}_{z_0} \otimes C_0(\Gamma) \circ o_{z_0} \in KK^G_d(C_0(Z), C_0(Z)) = RKK^G_d(Z; \mathbb{C}, \mathbb{C}) \]
where $o^{*}_{z_0} \in KK^G_d(C_0(Z), C_0(Z))$ is the class of the induced $^*$-homomorphism.

Similarly, if $X$ is a smooth, compact, orientable $\Gamma$-manifold, then
\[ (6.2) \]
\[ Z \times X \xleftarrow{o_{z_0} \times \text{id}_X} \Gamma \times X \xrightarrow{o_{z_0} \circ \text{pr}_G} Z \]
is a $G$-equivariant correspondence from $Z \times X$ to $Z$, with class
\[ (o_{z_0} \times \text{id}_X)^* \otimes C_0(Z) \circ (o_{z_0} \circ \text{pr}_G)! \in KK^G_d(C_0(Z \times X), C_0(Z)) = RKK^G_d(Z; C_0(X), \mathbb{C}). \]

**Proposition 6.3.** If $\Gamma \times Z$ is a smooth oriented group, $X$ an equivariantly oriented $\Gamma$-manifold, $[\tilde{X} \times \Gamma]$ a Dirac class for the action, then
\[ \text{infl}([\tilde{X} \times \Gamma]) = (o_{z_0} \times \text{id}_X)^* \otimes C_0(Z) \circ (o_{z_0} \circ \text{pr}_G)! \in RKK^G_d(Z; C_0(X), \mathbb{C}). \]

**Proof.** We need to show that $PD^{-1}([\text{ev}] \otimes C[X])(o_{z_0} \times \text{id}_X)^* \otimes C_0(Z) \circ (o_{z_0} \circ \text{pr}_G)$, where
\[ PD^{-1} : KK^\Gamma_d(C_0(Z \times X), \mathbb{C}) \xrightarrow{\cong} \text{KK}^{G \times d}_+ (C_0(Z \times X), C_0(Z)) \cong RKK^{G \times d}_+ (Z; C_0(X), \mathbb{C}) \]
is Poincaré duality.

The map $PD^{-1}$ is the composition of inflation
\[ \text{infl} : \text{KK}^{\Gamma}_d(C_0(Z) \otimes A, B) \to RKK^{G\times d}_+ (Z; C_0(Z) \otimes A, B), \]
with Kasparov product in $RKK^{\Gamma}_d(\mathbb{Z})$ with a class $\Theta \in RKK^G_d(Z; \mathbb{C}, C_0(Z)).$ The class $\Theta$ is the class of the $G$-equivariant correspondence
\[ Z \xleftarrow{id} Z \xrightarrow{\delta} Z \times Z, \]
where the momentum map for the $G$-space $Z \times Z$ is in the first variable, and $\delta_Z: Z \to Z \times Z$ be the diagonal map.

By definition, inflate([X]) $\in RKK^\Gamma_n(Z; C(X), C)$ is represented by the $G$-equivariant correspondence

$$Z \times X \xleftarrow{id} Z \times X \xrightarrow{pr_Z} Z$$

and since inflate is compatible with external products,

$$\text{inflate}([\text{ev}] \otimes_C [X]) = \text{inflate}([\text{ev}]) \otimes_{Z,C} \text{inflate}([X]) \in RKK^\Gamma_n(Z; C(X), C)$$

is represented by the $\Gamma \times Z$-equivariant correspondence

$$(6.4) \quad Z \times Z \times X \xleftarrow{id \times o \times id_X} Z \times Z \times X \xrightarrow{id \times o \times id_X, Z \times \Gamma \times X} pr_Z \times Z,$$

where the momentum map for $Z \times Z \times X$ is projection to the first coordinate. We want to compose this with the class $\Theta$, thus, compute the composition of correspondences

$$Z \times X \xleftarrow{id} Z \times X \xrightarrow{\delta \times id_X} Z \times Z \times X \xrightarrow{id \times o \times id_X, Z \times \Gamma \times X} pr_Z \times Z$$

but the maps $\delta \times id_X$ and $id_Z \times o \times id_X$ are not transverse, due to the orbit $\Gamma \cdot 0$.

We can rectify this by a standard argument. The diagonal map $\delta: Z \to Z \times Z$ has a $\Gamma$-equivariant normal bundle $\nu$ and associated Thom class $\xi_\nu \in KK^\Gamma_d(TZ, C_0(Z), C_0(\nu))$ and an open $\Gamma \times Z$-equivariant open embedding $\varphi: \nu \to Z \times Z$.

We can identify $\nu$ with the tangent bundle $TZ$, which is, by assumption, a equivariantly oriented real $\Gamma$-vector bundle over $Z$, and the open embedding $\varphi: TZ \to Z \times Z$ may be taken to be $\varphi(z, \xi) = (z, \exp_z(\xi'))$, as discussed above, with $\xi \mapsto \xi'$ a re-scaling of the tangent space into an open neighbourhood $D_\epsilon := \{(z, \xi) \in TZ \mid |\xi| < \epsilon\}$ on which the Riemannian exponential map is a diffeomorphism.

Since the support of the Thom class may be shrunk arbitrarily close to the diagonal, we can take it supported in $D_\epsilon$. A standard bordism-Thom modification argument shows that the class $\Theta$ is also represented by the $\Gamma \times Z$-equivariant correspondence

$$(6.5) \quad Z \xleftarrow{\varphi} (D_\epsilon, \xi_{TZ}) \xrightarrow{\pi} Z \times Z$$

with $\varphi(z, \xi) := \exp_z(\xi')$ as above and $\pi$ the bundle projection to $Z$.

We now can compose this with (6.4) (after multiplying it by $id_X$) by transversality, since $\varphi$ is a bundle of submersions. We obtain the coincidence manifold

$$W_\epsilon \times X = \{(z, \xi, g) \in D_\epsilon \times \Gamma \mid \exp_z(\xi') = g \cdot 0\}$$

as the middle space of the resulting correspondence. $W_\epsilon$ is a $\Gamma$-invariant open submanifold of $Z \times \Gamma$; let

$$\pi: W_\epsilon \to Z, \pi(z, \xi, g) := z,$$

then $\pi$ gives the required bundle structure over $Z$, i.e. the momentum map, and a $G_\Gamma$-manifold.

Let $pr_{D_\epsilon} : W_\epsilon \subset D_\epsilon \times \Gamma \to D_\epsilon$ be the bundle projection, and let $\zeta$ denote the pull-back $pr_{D_\epsilon}^* (\xi_{TZ})$ of the Thom class $\xi_{TZ} \in KK^\Gamma_d(Z, C_0(D_\epsilon)) \cong KK^\Gamma_d(Z, C_0(TZ))$ to an equivariant K-theory class for $W_\epsilon \times X$.

Then the $\Gamma \times Z$-equivariant correspondence

$$Z \xleftarrow{pr_{W_\epsilon}} (W_\epsilon \times X, \zeta) \xrightarrow{\pi} Z,$$

represents the Poincaré dual of $[\text{ev}] \otimes_C [X]$.

Note that $W_\epsilon$ is diffeomorphic to the total space of the vector bundle $o_{g\cdot 0}^* (TZ)$ over $\Gamma$: indeed, $W_\epsilon \cong \bigsqcup_{g \in \Gamma} W_{\epsilon, g \cdot 0}$, with $W_{\epsilon, g \cdot 0} := \{(z, \xi) \in D_\epsilon \mid \exp_z(\xi) = g \cdot 0\}$, and the map $T_{g \cdot 0} Z \to W_{\epsilon, g \cdot 0}$, $\xi \mapsto \exp_{g \cdot 0}(\xi')$ is a diffeomorphism, for each $g$. Taking the product of these maps gives the required $\Gamma \times Z$-equivariant diffeomorphism.
Under the identification given by this diffeomorphism, the $Z \ltimes \Gamma$-structure on $\alpha_{z_0}^*(TZ)$ is using the bundle map $\rho_z(\xi, g) := \exp_{g z_0}(\xi')$. Denoting this map by $\rho$ we get the slightly modified description
\[ Z \times X \xrightarrow{\rho_z \times \text{id}_X} (\alpha_{z_0}^*(TZ), \alpha_{z_0}^*(\xi_{TZ})) \xrightarrow{\rho} Z \]
from $Z \times X$ to $Z$, representing the Poincaré dual of $[\text{ev}] \otimes_{\mathbb{C}} [X]$.

Now by an obvious bordism of $\Gamma \ltimes Z$-manifolds, we can replace this correspondence by simply
\[ Z \xrightarrow{\alpha_{z_0} \circ \pi} (\alpha_{z_0}^*(TZ), \alpha_{z_0}^*(\xi_{TZ})) \xrightarrow{\alpha_{z_0} \circ \pi} Z \]
where $\pi: \alpha_{z_0}^*(TZ) \to \Gamma$ is the vector bundle projection, which is the Thom modification of the correspondence
\[ Z \times X \xleftarrow{\alpha_{z_0} \times \text{id}_X} \Gamma \times X \xrightarrow{\alpha_{z_0} \circ \text{pr}_X} Z, \]
i.e. (6.2).

This proves the statement.

The case where $\Gamma$ is torsion free and hence acts freely on $Z$ can be expressed and proved more simply.

Since $Z \times X$ is a $\mathcal{G}$-equivariantly $K$-oriented bundle of smooth manifolds, by hypothesis, $G_\Gamma$-equivariant Poincaré duality holds and gives an isomorphism
\[ (6.6) \quad \text{RK}_\Gamma^0(Z; C_0(X), \mathbb{C}) \cong \text{RK}_{0+n}^\Gamma(Z; \mathbb{C}, C_0(X)) \]
and composing this with the generalized Green-Julg isomorphism and a standard Morita equivalence identifies $\text{RK}_{0+n}^\Gamma(Z; \mathbb{C}, C_0(X))$ with $K^{-n-n}(Z \times \Gamma X)$, the $K$-theory of the mapping cylinder. A routine manipulation gives the following; it also follows from Proposition 6.3 and the fact, easily verified, that the composition
\[ \text{KK}_\Gamma^0(C_0(Z \times X), \mathbb{C}) \xrightarrow{\text{PD}} \text{RK}_\Gamma^0(Z; C_0(X), \mathbb{C}) \cong \text{RK}_{0+n}^\Gamma(Z; \mathbb{C}, C_0(X)) \cong K^{-n-n}(Z \times \Gamma X) \]
is ordinary Poincaré duality for $Z \times \Gamma X$.

**Corollary 6.7.** If $\Gamma$ is torsion-free, then $\Gamma \ltimes Z$-equivariant Poincaré duality for $X$
\[ \text{RK}_\Gamma^0(Z; C_0(X), \mathbb{C}) \to K^{-n-n}(Z \times \Gamma X) \]
maps inflate($[\Gamma \ltimes X]$) to the class in $K^{-n-d}(X)$ of the correspondence
\[ \cdot \leftarrow X \xrightarrow{e_{z_0}X} Z \times \Gamma X, \]
where $e_{z_0}X: X \to Z \times \Gamma X$ is the inclusion of the fibre $X$ at $\Gamma z_0 \in \Gamma \backslash Z$.

**Dirac classes for proper actions**

We now discuss Dirac classes for proper actions. Suppose that $X$ is a smooth proper $\Gamma$-manifold, $\chi \to Z \cong \mathcal{E}\Gamma$ a smooth classifying map for the proper action of $\Gamma$ on $X$. By Sard’s theorem, $\chi$ has a regular value $z_0$, so $F := \chi^{-1}(z_0)$ is a smooth submanifold of $X$ of dimension $n - d$, carrying a smooth action of the finite group $\text{Stab}_\Gamma(z_0)$. If $\Gamma \backslash X$ is compact, then $F$ is compact.

The fibres $F_g := \chi^{-1}(g z_0)$ as $g$ ranges over $\Gamma$ are isomorphic copies, and come with actions of the corresponding conjugate isotropy groups. We set
\[ (6.8) \quad F_\Gamma := \{(x, g) \in X \times \Gamma \mid \chi(x) = g z_0\}. \]
which is a bundle over $\Gamma z_0 \subset X$ with fibre $F_g \times \text{Stab}_\Gamma(g z_0)$ over $g z_0$. A $\Gamma$-equivariant $K$-orientation on $X$ induces a canonical $\Gamma$-equivariant $K$-orientation on $F_\Gamma$. 
Set \([F_\Gamma] \in \text{KK}^\Gamma_{d-n}(\mathbb{C}, C_0(F_\Gamma), \mathbb{C})\) the transverse Dirac class for \(\Gamma\) acting on \(F_\Gamma\). Let \(i: F_\Gamma \to X\) be the projection to the first factor.

**Theorem 6.9.** Let \((\Gamma, Z)\) be a smooth oriented group, and \(X\) is a \(\Gamma\)-equivariantly \(\mathbb{K}\)-oriented proper \(\Gamma\)-manifold, \(\chi: X \to Z\) be a smooth \(\Gamma\)-map, \(z_0 \in Z\) a regular orbit, and \(F_\Gamma, i: F_\Gamma \to X\) etc as in the discussion above. Then the Dirac class for \(\Gamma \times X\) is given by the class of the fibre of \(\chi\):

\[
\left[\Gamma \times X\right] = i_*([F_\Gamma]) \in \text{KK}^\Gamma_{d-n}(C_0(X), \mathbb{C}).
\]

In particular, if \(n < d\) then the Dirac class vanishes, and otherwise, the Dirac class is represented by a \(n - d\)-dimensional spectral triple over \(C_0(X) \rtimes \Gamma\).

**Proof.** We show that

\[
(6.10)\quad \text{infl}e (i_*([F_\Gamma])) = \text{PD}^{-1}([ev] \otimes \mathbb{C} [X]) \in \text{KK}^\Gamma_{d-n}(Z; C_0(Z \times X), \mathbb{C}).
\]

The result will follow from Proposition 2.10. As in the proof of Proposition 6.3 PD\(^{-1}\) involves composition with the class \(\Theta \in \text{RKK}^\Gamma (Z; \mathbb{C}, C_0(Z))\), where \(\Theta = \delta!\) where \(\delta: Z \to Z \times Z\) is the diagonal map, canonically \(\mathbb{K}\)-oriented, and \(G_\Gamma\)-equivariant.

Let \(\chi: X \to Z\) be the smooth classifying map with regular value \(z_0\) discussed above. Set

\[
G_\chi: X \to Z \times X
\]

the graph of \(\chi\): \(G_\chi(x) := (z, \chi(x))\).

**Lemma 6.11.** The equality

\[
(\delta \otimes \mathbb{C} \text{id}_X)! = (\text{id}_Z \otimes \mathbb{C} G_\chi)!
\]

holds in

\[
\text{KK}^\Gamma_{d} (Z; C_0(Z \times X), C_0(Z \times Z \times X)) = \text{RKK}^\Gamma (Z; C_0(X), C_0(Z \times X)).
\]

**Proof.** By the universal property of \(\mathcal{E} \Gamma \cong Z\), the coordinate projections \(Z \times Z \to Z\) are \(\Gamma\)-equivariantly homotopic. Fix a \(\Gamma\)-equivariant smooth homotopy

\[
F: Z \times Z \times [0, 1] \to Z
\]

between the two coordinate maps and pull it back in one coordinate using the map \(\chi\) to get

\[
\tilde{F}: Z \times X \times [0, 1] \to Z, \quad \tilde{F}(z, x, t) := F(z, \chi(x), t).
\]

Then, as is easily checked, \(\text{id}_Z \times \tilde{F}\) gives a smooth \(G_\Gamma\)-equivariant homotopy between the smooth \(\mathbb{K}\)-oriented \(G_\Gamma\)-equivariant maps \(\delta \times \text{id}_X\) and \(\text{id}_Z \times G_\chi\), as claimed.

To complete the proof, we need to evaluate the composition of \(G_\Gamma\)-equivariant correspondences

\[
(6.12)\quad Z \times X \leftarrow Z \times X \xrightarrow{\text{id}_Z \times G_\chi} Z \times Z \times X \xrightarrow{\text{id}_X \times \epsilon_{z_0} \times \text{id}_X} Z \times \Gamma \times X \to Z
\]

The maps \(G_\chi: X \to Z \times X\) and \(\epsilon_{z_0} \times \text{id}_X: \Gamma \times X \to Z \times X\) are transverse since \(z_0\) is a regular value (and hence so is \(g \ast z_0\) for all \(g \in \Gamma\)) and the associated coincidence manifold

\[
\{(x, g, y) \in X \times \Gamma \times X \mid G_\chi(x) = (g \ast z_0, y)\}
\]

is the smooth \(\mathbb{K}\)-oriented manifold \(F_\Gamma\) described above in (6.8). Taking the product of everything with \(\text{id}_Z\) gives the identity (6.10) as required.

**Corollary 6.13.** \([\Gamma \times Z] = [ev] \in \text{KK}^\Gamma_{0}(\mathbb{C}, \mathbb{C})\).
This is the case of Theorem 6.9 where $Z = X$, $\chi$ the identity map.

Finally, we note that when $\Gamma$ is finite, acting on $X$, arbitrary, our description $i_\ast([F_\Gamma])$ of the Dirac class for $C_0(X) \rtimes \Gamma$ just given matches that given in Example 5.8 since then $Z$ becomes a point, and $[ev]$ the class of the regular representation of the finite group.

Theorem 6.9 is quite satisfying, from a certain point of view, as it gives a case where the homological subtraction involved in forming a Dirac class, which lies in dimension $d - n$, matches precisely the geometric dimension: there is a spectral triple representative of summability dimension $n - d$.

7. DIRAC CLASSES FOR BOUNDARY ACTIONS OF NEGATIVE CURVED GROUPS

Along with the Dirac class for the irrational rotation algebra, and it’s spectral triple representative, an important example in noncommutative geometry is the action of a co-compact discrete subgroup of $\text{SL}_2(\mathbb{R})$ acting on the circle by Möbius transformations. This is a special case of a Gromov hyperbolic group $\Gamma$ on it’s boundary $\partial \Gamma$.

If $\Gamma$ is a Gromov hyperbolic group (see [25] for an exposition) hyperbolicity leads to a compact, metrizable, $\Gamma$-space $\overline{\Gamma}$ containing $\Gamma$ as a dense open $\Gamma$-invariant open subset, and complement $\partial \overline{\Gamma}$. This produces an exact sequence

\[ 0 \to C_0(\Gamma) \rtimes \Gamma \to C(\overline{\Gamma}) \rtimes \Gamma \to C(\partial \overline{\Gamma}) \rtimes r, \Gamma \to 0 \]

and, using the canonical isomorphism $C_0(\Gamma) \rtimes \Gamma \cong K(\ell^2 \Gamma)$, and amenability of the action, we obtain a KK-class $[\partial \Gamma] \in KK^{1}(C(\partial \Gamma), C) = K^{-1}(C(\partial \Gamma) \rtimes r, \Gamma)$. Alternatively, in [18] a $\Gamma$-equivariant completely positive splitting of (7.1) is provided, which implies the extension determines a $KK_1$-class.

The main result of this section is that when $\Gamma$ also fits into the previous framework, the boundary extension class is the Dirac class of the action, with its boundary $K$-orientation.

Some fine points about the statement are discussed following the Theorem.

**Theorem 7.2.** Let $(\Gamma, Z)$ be a smooth $K$-oriented group with $Z$ negatively curved. Then $\partial Z$ is canonically $\Gamma$-equivariantly $K$-oriented manifold in the sense of Definition 2.3, and the Dirac class is given by

\[ \widehat{[\Gamma \rtimes \partial Z]} = [\partial \Gamma] \in K^{-1}(C(\partial Z) \rtimes \Gamma) \cong K^{-1}(C(\partial \Gamma) \rtimes r, \Gamma). \]

where $[\partial \Gamma]$ is the boundary extension class.

The boundary extension class is represented by a cycle which doesn’t involve any smooth structures, but only, in a sense, on the asymptotic geometric of the group $\Gamma$, and its action on it.

The specific example we for the most part have in mind in this article is the case where $\Gamma = \pi_1(M)$ for a negatively curved, compact, $d$-dimensional, $K$-oriented manifold, on which, therefore, we can assemble a Dirac operator $D$ acting on the spinor bundle $S_X$.

The metric, $K$-orientation, etc on $M$ lifts to $\Gamma$-equivariant data: spinor bundles on $\tilde{Z} := \tilde{M}$, connection, and so on, and one assembles the ‘lifted,’ $\Gamma$-equivariant Dirac operator representing $[Z] \in KK^{1}_{\ell^2}(C_0(Z), \mathbb{C})$.

The boundary sphere $\partial Z$ of the negatively curved space $Z$ is the boundary of the usual geodesic compactification of $Z$, and agrees with the Gromov boundary $\partial \Gamma$ (the latter of course only depends on the group, up to homeomorphism, but here we have modelled it topologically by the smooth sphere $S^{d-1}$.)

The isometric group action of $\Gamma$ on $Z$ extends to an action of $\Gamma$ on $\partial Z$. The action when the curvature of $M$ is constant is by Möbius transformations and is smooth. It preserves, in the weak sense we are using here, the boundary $K$-orientation on the sphere, because by assumption it preserves the $K$-orientation on $Z$. 

Due to the precise situation, we can give a more precise geometric description of the Dirac map in this case. We use the same notation as above, with \( M = \Gamma \setminus Z \), a negatively curved manifold. Let \( SM \) be the sphere bundle of its tangent bundle. Then
\[
Z \times_\Gamma \partial Z \cong SM,
\]
in the following canonical way. Given \( z \in Z \), and \( z, \xi \) a unit tangent vector at \( z \),
\[
\exp_{z_0}: S_{z_0}Z \to Z
\]
the Riemannian exponential map determines a geodesic ray beginning at \( z_0 \) and ending at a boundary point in \( \partial Z \). This construction is clearly equivariant and determines the required homeomorphism.

In this notation, the Dirac map can be written
\[
K^*(C(\partial Z) \rtimes \Gamma) \to K_{-\ast+d}(SM)
\]
and a Dirac class is one which maps to the class of a single fibre of the bundle projection \( \pi: SM \to M \), K-oriented as a sphere, and regarded as a Baum-Douglas cycle for \( SM \).

Remark 7.3. The \( \Gamma \)-action on \( \partial Z \) in general (for variable curvature) is by \( C^1+\epsilon \)-diffeomorphisms: even compact surfaces with variable negative curvature can produce non-smooth actions on the boundary sphere.

There is a natural way to amend our definition of Dirac class, amend, in order to admit at least some non-smooth actions of \( \Gamma \) on \( X \), however, and we will point this out and leave the issue alone, since it seems of little importance. Indeed, the Dirac map itself (or the inflation map) does not depend on smoothness of the \( \Gamma \)-action on \( X \), and the target of the inflation map is \( \text{RKK}^\Gamma(Z; C(\partial Z), \mathcal{C}) \), and the Dirac class is defined as one hitting a certain target there defined by a smooth \( Z \times \Gamma \)-equivariant correspondence, so that it is only needed that the action of the proper groupoid \( Z \times \Gamma \) is smooth, on the bundle of smooth manifolds \( Z \times \partial Z \), in order to apply the smooth correspondence framework. That this can be achieved, without changing the topology, we leave as an exercise.

In any case, in these examples, it appears that the Dirac class forgets any issues related to smoothness whatsoever – in stark contrast to the irrational rotation algebra \( A_\theta \) and other \( Z^d \)-actions.

Proceeding to the proof of Theorem 7.2, we begin with a discussion of (known) constructions relating to extensions and KK-theory.

Suppose \( \mathcal{G} \) is any locally compact groupoid with Haar system and
\[
(7.4) \quad 0 \to J \to B \to A \to 0
\]
is an exact sequence of \( \mathcal{G} \)-C*-algebras. The Busby invariant of the equivariant extension (7.4) is the *-homomorphism \( \tau(a) := \pi(\tilde{a}) \in Q(J) := \mathcal{M}(J)/J \), where \( \pi: \mathcal{M}(B) \to \mathcal{M}(B)/B \) is the quotient map, and where \( \tilde{a} \) denotes a lift of \( a \) under \( \beta \), regarded as a multiplier of \( B \). (If \( J \) is any ideal in \( B \) then \( J \) maps canonically to the multiplier algebra of \( J \), we have suppressed this map in the notation.)

The Busby invariant is uniquely associated to the strong isomorphism class of the extension: if \( \tau: A \to Q(J) \) is any equivariant *-homomorphism, then \( B := \left\{ (a, m) \in A \oplus \mathcal{M}(J) \mid \pi(m) = \tau(a) \right\} \) determines a \( \mathcal{G} \)-equivariant extension with of \( A \) by \( J \) with the given Busby invariant, and if \( \tau \) comes from (7.4) then this procedure determines a strongly isomorphic extension. Thus, equivariant strong isomorphism classes of equivariant extensions are in 1-1 correspondence with equivariant *-homomorphisms \( A \to Q(J) \).

Similarly, if \( \mathcal{M}^s(B) \) denotes multipliers of \( B \otimes \mathcal{K} \) and \( Q^s(J) := \mathcal{M}^s(J)/J \otimes \mathcal{K} \) then we obtain a bijective correspondence between strong isomorphism classes of equivariant extensions of \( A \) by \( J \otimes \mathcal{K} \) and \( \mathcal{G} \)-equivariant *-homomorphisms \( A \to Q^s(J) \).
We generally work with Busby invariants rather than extensions themselves.

**Example 7.5.** Suppose that \( \rho: A \to \mathcal{L}(J \otimes I^2) \) is a representation, and that \( P \in \mathcal{L}(J \otimes I^2) \) such that
\[
[P, \rho(a)], \ \rho(a)(P^2 - P) \in \mathcal{K}(J \otimes I^2)
\]
for all \( a \in A \). Then
\[
\tau: A \to \mathbb{Q}^+(J), \ \tau(a) := P\rho(a)P \mod J \otimes \mathcal{K}
\]
is an equivariant \( * \)-homomorphism and hence is the Busby invariant of some strong isomorphism class of equivariant extension of \( A \) by \( J \otimes \mathcal{K} \). It is the extension associated to the Kasparov cycle \((J \otimes I^2, \rho, P)\) for \( KK^G_1(A, J) \).

Say that a Busby invariant \( \tau: A \to \mathbb{Q}^+(J) \) is **equivariantly dilatable** if there is a \( G \)-equivariant completely positive map \( s: A \to \mathcal{M}^+(J) \) such that \( \pi \circ s = \tau \). We say that \( s \) is a splitting of \( \tau \). It determines a completely positive splitting of the corresponding extension. The **Stinespring construction** realizes any \( G \)-equivariant, dilatable Busby invariant as one of the form \( \tau(a) = P\rho(a)P \mod J \otimes \mathcal{K} \), i.e. as one of the form of Example 7.5 where \( P \) is an actual projection (rather than just an almost-projection).

In fact, in Kasparov theory, almost-projections can be replaced by actual projections at the expense of adding a degenerate to the cycle: suppose that \( \mathcal{E} = J \otimes I^2 \) is the standard Hilbert \( J \)-module and \( (\mathcal{E}, \pi, P) \) is an odd cycle for \( KK^G_1(A, J) \), where \( P \) is a self-adjoint almost-projection in the sense of Definition \( 7.5 \) such that \( \|P\| \leq 1 \), then the triple \( (\mathcal{E}, 0, 1 - P) \) is evidently a triple as well, where \( 0 \) denotes the zero representation of the algebra on \( \mathcal{E} \), but it is degenerate. Hence the cycles \( (\mathcal{E}, \pi, P) \) and \( (\mathcal{E}, \pi \oplus 0, P \oplus 1 - P) \) are equivalent in \( KK \). Let
\[
\hat{\mathcal{P}} := \begin{bmatrix} P & (P - P^2)^{\frac{1}{2}} \\ (P - P^2)^{\frac{1}{2}} & 1 - P \end{bmatrix},
\]
then \( \hat{\mathcal{P}}^2 = \hat{\mathcal{P}}, \) that is, \( \hat{\mathcal{P}} \) is an actual projection. The operator homotopy with
\[
\hat{\mathcal{P}}_t := \begin{bmatrix} P & (tP - t^2P^2)^{\frac{1}{2}} \\ (tP - t^2P^2)^{\frac{1}{2}} & 1 - P \end{bmatrix}
\]
gives a homotopy between the Kasparov cycles \( (\mathcal{E}, \pi \oplus 0, P \oplus 1 - P) \) and \( \mathcal{E} \oplus \mathcal{E}, \rho \oplus 0, \hat{\mathcal{P}} \), so \( (\mathcal{E}, \pi, P) \) and \( (\mathcal{E} \oplus \mathcal{E}, \rho \oplus 0, \hat{\mathcal{P}}) \) determine the same class: \( P \) has been replaced by an actual projection. This leads to a completely positive map
\[
\hat{s}(a) := \hat{\mathcal{P}}\hat{\rho}(a)\hat{\mathcal{P}},
\]
for the Busby invariant
\[
\hat{\tau}(a) := \hat{\mathcal{P}}\hat{\rho}(a)\hat{\mathcal{P}} \mod J \otimes \mathcal{K}.
\]
Computing, this equals
\[
\left[ \begin{array}{cc} P\rho(a)P & P\rho(a)\sqrt{P - P^2} \\ P\rho(a)\sqrt{P - P^2} & P\rho(a)\sqrt{P - P^2} \end{array} \right] \mod M_2(J)
\]
and as all entries except for the top left corner are zero mod \( M_2(J) \), this equals \( P\rho(a)P = \tau(a) \) after stabilization by the compacts. That is, \( \hat{\tau} = \tau \) (up to stabilization).

We are primarily interested in the following example of a geometric source.

Let \( \tilde{M} \) be a compact manifold-with-boundary \( \partial M \) and interior \( M \). Then we have an extension
\[
0 \to C_0(M) \to C(\tilde{M}) \to C(\partial M) \to 0.
\]
With the identification \( \mathcal{M}(C_0(M)) \cong C_0(M) \) we can describe the Busby invariant as the \( * \)-homomorphism
\[
\tau: C(\partial M) \to C_0(M), \ \tau(f) = \tilde{f} \mod C_0(M)
\]
where \( \tilde{f} \) is any extension of \( f \) to a continuous function on \( \tilde{M} \).
It will be convenient to work with another extension rather than (7.6). Let \( \nu \subset \overline{M} \) be a collar of the boundary, thus \( \nu \cong \partial M \times [0,1) \) with the boundary identifying as \( M \times \{0\} \). Let \( \chi \in C_b(\nu) \) with \( 0 \leq \chi \leq 1 \), \( \chi = 1 \) on \( \partial M \) and \( \chi \) has compact support in \( \nu \). We may take \( \chi(x,t) = 1 - t \) for example, defined on \( U \cong \partial M \times [0,1) \) and extended to zero outside \( U \), where we have used the identification \( \nu \cong \partial M \times [0,1) \). Let

\[
\nu := \nu \cap M
\]

an open subset of \( M \) homomorphic (via the collaring) to \( \partial M \times (0,1) \). Let \( r: \nu \to \partial M \) be the projection, then dualizing \( r \) gives a \(^*\)-homomorphism

\[
\hat{r}: C(\partial M) \to C_b(\nu) \subset C_b(\nu),
\]

and since \( \chi^2 - 1 \in C_0(\nu) \), we get a Kasparov cycle \((C_0(\nu), \hat{r}, \chi)\) for KK\(_1(C(\partial M), C_0(\nu))\). On the other hand, the extension

(7.7)

\[
0 \to C_0(\nu) \to C_0(\nu) \to C(\partial M) \to 0
\]

admits the following completely positive splitting:

\[
s: C(\partial M) \to C_0(\nu), s(f) := \chi \cdot \hat{r}(f) \cdot \chi,
\]

since \( \chi \cdot \hat{r}(f) \cdot \chi = f \) on \( \partial M \).

Now, carrying out the general procedure discussed above, with \( A := C(\partial M), B := C(\nu), J := C_0(\nu) \) and \( P := \chi \), and we realize the Busby invariant of the extension (7.7) as the upper left corner of the representation

\[
\hat{r}(f) := \begin{bmatrix}
\chi \cdot \hat{r}(f) \cdot \chi & \chi \cdot \hat{r}(f) \cdot \sqrt{\chi^2} \\
\chi \cdot \hat{r}(f) \cdot \sqrt{\chi^2} & \sqrt{\chi^2} \cdot \chi \cdot \hat{r}(f) \cdot \sqrt{\chi^2}
\end{bmatrix} \mod M_2(C_0(\nu)),
\]

This proves that the Kasparov cycle associated to the extension (7.7) is the triple \((C_0(\nu), \hat{r}, \chi)\) for KK\(_1(C(\partial M), C_0(\nu))\).

Let \( \iota: \partial M \to M \) the smooth embedding \( \iota(x) = \xi(x, \frac{1}{\sqrt{2}}) \), where \( \xi: \partial M \times [0,1) \to M \) is the collaring \( \xi \) is a diffeomorphism onto the open collar \( \nu \). Then \( \iota \) has an evidently trivial normal bundle with total space \( \nu \) the same as above, which therefore carries a canonical K-orientation and Thom class \( t_\nu \in KK_1(C(\partial M), C_0(\nu)) \).

**Lemma 7.8.** 1) The class \([\partial \nu]\) \( \in KK_1(C(\partial M), C_0(\nu)) \) of the extension (7.7) is equal to the Thom class \( t_\nu \) of the normal bundle to the embedding \( \iota \). 2) Let \([\partial \nu]\) \( \in KK_1(C(\partial M), C_0(M)) \) be the class of the extension (7.6). Then

\[
(\varphi^\dagger)_*([\partial \nu]) = [\partial \nu] \in KK_1(C(\partial M), C_0(M))
\]

holds, where \( \varphi^\dagger: C_0(\nu) \to C_0(M) \) is the \(^*\)-homomorphism induced by the open embedding \( \varphi: \nu \to M \).

In the notation of correspondences, this says that

\[
\iota! = [\partial \nu] \in KK_1(C(\partial M), C_0(M)).
\]

In the equivariant case, the same arguments go through.

**Lemma 7.9.** Let \( G \) be a proper groupoid acting smoothly on a bundle of smooth manifolds-with-boundary \( \overline{M} \) with boundary \( \partial M \) and interior \( M \). Assume there is a \( G \)-equivariant collaring

\[
c: \partial M \times [0,1) \to \overline{M},
\]

i.e. \( c \) is a \( G \)-equivariant fibrewise diffeomorphism onto an open neighbourhood of \( \partial M \) in \( \overline{M} \) whose restriction to \( \partial M \times \{0\} \) is the inclusion \( \partial M \to \overline{M} \). Then the \( G \)-equivariant extension

\[
0 \to C_0(M) \to C(\overline{M}) \to C(\partial M) \to 0
\]
admits a $G$-equivariant completely positive splitting, the smooth embedding $\iota: \partial M \to M$, $\iota(x) = c(x, \frac{1}{2})$ admits a canonical $G$-equivariant orientation, and the equation
\[ \iota! = [\partial M] \in KK^G_1(C(\partial M), C_0(M)) \]
holds.

We now return to hyperbolic groups.

Let $(Z, \Gamma)$ be a smooth $K$-oriented group with $Z$ negatively curved, so that it is Gromov hyperbolic with Gromov compactification $\overline{Z}$ and boundary $\partial Z \cong \partial \Gamma$ (noting that $\Gamma$ acts co-compactly and isometrically on $Z$.)

The boundary $\partial Z$ can be identified with $S^{n-1}$, if $\dim(Z) = n$, and the group acts by $C^{1+\varepsilon}$-diffeomorphisms of this sphere. We have a $\Gamma$-equivariant exact sequence
\[ (7.10) \quad 0 \to C_0(Z) \to C(Z) \to C(\partial Z) \to 0. \]
Clearly the boundary does not admit a $\Gamma$-equivariant collaring. For if it did, the image $\iota(\partial Z) \subset Z$ would be $\Gamma$-invariant, which is impossible, since $\partial Z$ is compact and $\Gamma$ acts properly.

However, after inflating this whole situation over $Z \cong \mathcal{E} \Gamma$ we obtain a $\mathcal{G}$-equivariant bundle of manifolds-with-boundary and corresponding extensions, and now it is possible to find an equivariant collaring $c: Z \times \partial Z \times [0, 1) \to Z \times \overline{Z}$ quite explicitly, for example as follows. For a point $(x, a) \in Z \times \partial Z$, the geodesic ray emanating from $x$ and pointing towards the boundary point $a$ determines a canonical (geometrically defined) map $r_{x,a}: [0, \infty) \to Z$ (and endpoint $a$ in $\overline{Z}$), so that
\[ r_{g_{x,a}}(t) = gr_{x,a}(t) \]
for any isometry of $Z$. Fix a re-scaling $[0, 1) \xrightarrow{\sim} [0, \infty)$ mapping 1 to $\infty$. Then combining the re-scaling with the map $r$ just defined gives the required collaring
\[ c: Z \times \partial Z \times [0, 1) \to Z \times \overline{Z}, \]
the explicit formula may be taken to be
\[ c(x, a, t) = (x, r_{x,a}(\sqrt{1 - \frac{t}{t}})), \text{ if } t \neq 0, \text{ and } c(x, a, 0) := a. \]

Lemma 7.9 now applies. Let
\[ \iota: Z \times \partial Z \to Z \times Z, \quad \iota(x, a) := c(x, a, \frac{1}{2}). \]
This is a $\mathcal{G}$-equivariant oriented fibre wise embedding, i.e. a bundle of smooth embeddings $\partial Z \to Z$ parameterized by the points of $Z$, and equivariant, as a bundle of maps, under the $\Gamma$-action. It yields a class
\[ (7.11) \quad \iota! \in KK^G_1(C_0(Z \times \partial Z), C_0(Z \times Z)) \cong RKK^G_1(Z; C(\partial Z), C_0(Z)) \]
which, by Lemma 7.9, is the same as the corresponding extension class, that is, the class obtained via the $\mathcal{G}_\Gamma$-equivariant Stinespring construction from the $\mathcal{G}_\Gamma$-equivariant bundle of extensions
\[ (7.12) \quad 0 \to C_0(Z \times Z) \to C_0(Z \times \overline{Z}) \to C_0(\partial Z) \to 0, \]
over $Z$.

For convenience in the argument we are using, we point out that the $\Gamma$-equivariant extension $(7.10)$, though it doesn’t admit a $\Gamma$-equivariant collaring, it does admit a $\Gamma$-equivariant completely positive splitting of another kind. If $x \in Z$, the exponential map determines a map
exp_z : S_x(Z) → ∂Z, and pushing forward the volume element on S_xZ determined by the metric, we obtain a measure µ_x ∈ Prob(∂Z). The formula

\[(Pf)(z) := \int_{\partial Z} f(\xi) d\mu_x(\xi)\]

provides a Poisson-transform in this situation: Pf extends continuously to Z and restricts to f on the boundary. The construction is clearly equivariant.

Therefore the extension

\[0 \to C_0(Z) \to C(\overline{Z}) \to C(\partial Z) \to 0\]
determines a class \([\partial Z] \in \text{KK}^1(Z, C(\partial Z), C_0(Z))\).

See [12] for a treatment of measures on the boundaries of hyperbolic spaces.

It is clear from computing with the cycles that \([\partial Z] \otimes C_0(Z) [\text{ev}] = [\partial \Gamma] \in \text{KK}^1(Z(\partial \Gamma), C)\), where \([\partial \Gamma] \) is the boundary extension class (the Dirac class, as we aim to prove).

**Lemma 7.13.** The inflation of the boundary extension class \([\partial \Gamma] \in \text{KK}^1(C(\partial \Gamma) \times \Gamma)\) factors as

\[\text{inflate}([\partial \Gamma]) = ! \otimes C_0(Z \times Z) \text{inflate}(\text{[ev]}) \in \text{RKK}^1(Z; C(\partial Z), C)\]

with \(!\) as in (7.11).

**Proof.** The class \(\text{inflate}([\partial Z])\) is that of an extension (7.12) with two natural \(G_\Gamma\)-equivariant completely positive splittings: one using the product of the identity map on Z and the Poisson splitting of (7.10) explained above, and the other using the \(G_\Gamma\)-equivariant collaring. The space of \(G_\Gamma\)-equivariant completely positive maps is convex, and so the two corresponding cycles are homotopic. We obtain

\[! = \text{inflate}([\partial Z]) \in \text{RKK}^1(Z; C(\partial Z), C_0(Z))\]

by Lemma 7.9. We get

\[! \otimes C_0(Z \times Z) \text{inflate}(\text{[ev]}) = \text{inflate}([\partial Z]) \otimes C_0(Z \times Z) \text{inflate}(\text{[ev]}) = \text{inflate}([\partial Z] \otimes C_0(Z) [\text{ev}] = \text{inflate}([\partial \Gamma])\]

We now complete the proof of Theorem 7.2.

We have shown that

\[\text{inflate}([\partial \Gamma]) = ! \otimes C_0(Z \times Z) \otimes C_0(Z \times Z) [1_{C_0(Z)} \otimes C [\text{ev}]]\]

where \(\iota : Z \times \partial Z \to Z \times Z\) is

\[\iota(x, a) = (x, c(x, a, \frac{1}{2}))\]

This equality holds in \(\text{RKK}^1(Z; C(\partial Z), C)\).

To complete the proof, we show that

\[\text{PD}(! \otimes C_0(Z \times Z) \otimes C_0(Z \times Z) [1_{C_0(Z)} \otimes C [\text{ev}]]) = [\text{ev}] \otimes C [\partial Z]\]

Using the definition of PD, the left hand side of this equation is

\[\overline{t} ! \otimes C_0(Z \times Z) (1_{C_0(Z)} \otimes [\text{ev}]) \otimes C_0(Z) [Z] = \overline{t} ! \otimes C_0(Z \times Z) ([\text{ev}] \otimes C [Z])\]

By commutativity of the external product \(\otimes C\) and the fact that the two coordinate projections \(Z \times Z \to Z\) are \(\Gamma\)-equivariantly homotopic, this is the same as

\[\overline{t} ! \otimes C_0(Z \times Z) ([Z] \otimes C [\text{ev}]) = \overline{t} ! \otimes C_0(Z \times Z) ([Z] \otimes C 1_{C_0(Z)}) \otimes C_0(Z) [\text{ev}]\]
Now we claim that
\[(7.20) \quad \tau ! \otimes_{C_0(Z \times Z)} ([Z] \otimes C_0(Z)) = 1_{C_0(Z)} \otimes C[\partial Z]\]
with $[\partial Z]$ the transverse Dirac class on $\partial Z$.

The class $\tau ! \in KK^\Gamma(C_0(Z \times \partial Z), C_0(Z \times Z))$ is represented by the wrong-warp map
\[r: Z \times \partial Z \to Z, r(x, a) := (x, c(x, a, \frac{1}{2})),\]
or, more precisely, by the smooth correspondence
\[Z \times \partial Z \xleftarrow{\text{id}} Z \times \partial Z \xrightarrow{r} Z \times Z.
\]
And $[Z] \otimes C_0(Z)$ is of course represented by the correspondence
\[\tau \xleftarrow{\text{id}} Z \times Z \xrightarrow{pr_2} Z.
\]
These correspondences are transverse (the left map of the second is a submersion) so can be composed using transversality and coincidence spaces and the outcome is easily computed to be the correspondence
\[Z \times \partial Z \xleftarrow{\text{id}} Z \times \partial Z \xrightarrow{r'} Z,
\]
with $r'(z, a) := c(z, a, \frac{1}{2})$.

But moving $c(z, a, \frac{1}{2})$ along the ray $[z, a)$ from $c(z, a, \frac{1}{2})$ to $z = \lim_{t \to 1} c(z, a, t)$ gives a homotopy between $r'$ and the projection $pr_2: Z \times \partial Z \to Z$.

Now given the claim (7.20), plug it into (7.19), to give that the left hand side of (7.17) equals
\[\left(1_{C_0(Z)} \otimes C_0(Z)\right) \otimes C_0(Z) \left[\text{ev}\right] = [\partial Z] \otimes C_0(Z) \left[\text{ev}\right],\]
as required.

Remark 7.21. Conceptually, what is going on in the proof just given is as follows. It is a well-known fact, proved in Proposition 11.2.15 of [29], that if $\partial Z$ is a K-oriented manifold-with-boundary $\partial Z$, $[M] \in KK_{-n}(C_0(M), \mathbb{C})$ the class of the Dirac on $M$, $[\partial Z] \in KK_{-n+1}(C(\partial Z), \mathbb{C})$ the class of the Dirac operator on the boundary (with the boundary K-orientation) then the boundary map
\[\delta: K_n(Z) \to K_{n-1}(\partial Z)\]
associated to the exact sequence
\[(7.22) \quad 0 \to C_0(Z) \to C(\overline{Z}) \to C(\partial Z) \to 0\]
maps $[Z]$ to $[\partial Z]$. Let us assume that these statements all hold equivariantly with respect to $\Gamma$ acting on the whole arrangement. The functoriality of Dirac classes with respect to boundaries just discussed would amount to the identity
\[(7.23) \quad [\partial Z] \otimes_{C_0(Z)} [Z] = [\partial Z] \in KK^\Gamma_{-n+1}(C(\partial Z), \mathbb{C}).\]
Now if $Z$ is also a universal proper $\Gamma$-space, then the two coordinate maps $Z \times Z \to Z$ are $\Gamma$-homotopic, and hence
\[(7.24) \quad [\text{ev}] \otimes C_0(Z) = [Z] \otimes C_0(Z) \left[\text{ev}\right] \in KK^\Gamma_{-n}(C_0(Z \times Z), \mathbb{C})\]
holds. Putting (7.23) and (7.24) together gives
\[[\partial Z] \otimes C_0(Z) \left[\text{ev}\right] = [\partial Z] \otimes C_0(Z) \otimes C_0(Z) \left[\text{ev}\right] = [\partial Z] \otimes C_0(Z) \left[\text{ev}\right] \otimes C_0(Z) = [\partial Z] \otimes C_0(Z) \left[\text{ev}\right] = [\partial Z] \otimes C_0(Z) \left[\text{ev}\right] \in KK_{-n+1}(C(\partial Z), \mathbb{C})\]
because $[\overline{Z}] \otimes C_0(Z) \left[\text{ev}\right]$ is (clearly) equal to the boundary extension class $[\partial Z]$. This shows that the latter is the Dirac class, by the defining property of Dirac classes.

The main difficulty with this argument is showing that the equivariant version (7.23) of the Higson-Roe result holds. In fact, one can do so, by using the inflation map, and the fact that
it is an isomorphism for proper $\Gamma$-spaces in the first variable, but we have taken the relatively more direct route, instead.

8. The intersection index formula

The procedure followed by Connes, Gromov and Moscovici say in their paper [13], amounts to, as they put it, a sort of reverse index theorem. It builds an analytic object (a KK-class) from to a given topological one (group cohomology class). In their case, the purpose was to prove homotopy-invariance of the topological object (the higher signature determined by the cohomology class).

We are doing something similar. The class of a fibre in $Z \times \Gamma X \to \Gamma \backslash X$ may or may not admit an analytic representative (as a cycle representing the Dirac class). But any time this is done, an automatic topological formula for its induced K-theory pairing is determined.

Let $(Z, \Gamma)$ be a smooth, $d$-dimensional, K-oriented group, and let $X$ be a smooth equivariantly K-oriented $\Gamma$-manifold of dimension $n$. Assume that a Dirac class $\overset{\sim}{[\Gamma \times X]} \in \text{KK}_{d-n}(C_0(X) \times \Gamma, \mathbb{C})$ exists, then it determines a pairing and corresponding map

$$K_{n-d}(C_0(X) \times \Gamma) \to \mathbb{Z}.$$  

We are going to compute this map geometrically on the range of the Baum-Connes assembly map

$$\mu: \text{KK}_{n-d}^\Gamma(C_0(Z), C_0(X)) \to K_{n-d}(C_0(X) \times \Gamma).$$

This is possible due to the topological definition of Dirac class.

To simplify matters, we will assume that $\Gamma$ is torsion-free in the rest of this section. As before we let $e_{z_0, X}: X \to Z \times \Gamma X$ the inclusion of $X$ as a fibre in $p: Z \times \Gamma X \to \Gamma \backslash Z$.

Using inverse of the Poincaré duality from Proposition 2.10

$$\text{PD}^{-1}: \text{KK}_{d}^\Gamma(Z, C_0(X)) \cong \text{RKK}_{d}^\Gamma(Z; \mathbb{C}, C_0(X))$$

and the generalized Green-Julg Theorem,

$$\text{RKK}_{d}^\Gamma(Z; \mathbb{C}, C_0(X)) \cong \text{KK}_{\ast}(\mathbb{C}, C_0(Z \times X \times \Gamma)) \cong K^{-\ast}(Z \times \Gamma X),$$

we may re-act the Baum-Connes assembly map as a map

$$(8.1) \quad \hat{\mu}: K^{-\ast}(Z \times \Gamma X) \to K_{n-d}(C_0(X) \times \Gamma),$$

shifting degrees by $-d$. The relevant dimension for purposes of pairing with the Dirac class, is then $\ast = n$. The domain of $\hat{\mu}$ may be described in terms of geometric equivalence classes of smooth correspondences

$$(8.2) \quad \cdot \leftarrow (M, \xi) \xrightarrow{f} Z \times \Gamma X.$$  

with $\xi$ a compactly supported K-theory class on $M$, $f$ K-oriented. Since $Z \times \Gamma X$ is already endowed with a fixed K-orientation, by the 2-out-of-3 result for K-oriented vector bundles, K-orientations on $f$, that is, on the vector bundle $f^\ast(T(Z \times \Gamma X)) \oplus TM$, are in 1-1 correspondence with K-orientations on $M$, so that we may regard the data of a geometric cycle for the domain of $\hat{\mu}$ as being a smooth, K-oriented manifold $M$, a smooth map $f: M \to Z \times \Gamma X$, and a compactly supported K-theory class $\xi \in K^{-i}(M)$ on $M$.

The dimension of the corresponding class in $K^\ast(Z \times \Gamma X)$ is $-i - n - d + \dim M$, and after Poincaré dualizing it we obtain thus a class in $\text{KK}_{i+n-d}^\Gamma(C_0(Z), C_0(X))$, so that in order to get a $n - d$-dimensional K-theory class, the right dimension to pair with the Dirac class, for $C_0(X) \times \Gamma$, we need $i = \dim M - d$.

In particular, if $M$ is compact, and $\xi = 1$ is the class of the trivial line bundle, the case of most immediate geometric interest, and the one we will focus on, then we need $\dim M = d$, the dimension of $Z$. 

THE CLASS OF A FIBRE IN NONCOMMUTATIVE GEOMETRY 33
**Definition 8.3.** A $d$-dimensional geometric cocycle for the $\Gamma$ action on $X$ is a pair consisting of a compact, $d$-dimensional $K$-oriented manifold $M$ and a smooth map $f : M \rightarrow Z \times_\Gamma X$.

Its class in $K^{-d}(Z \times \Gamma X)$ is denoted $\text{Index}(f)$. 

**Example 8.4.** If the integers $\mathbb{Z}$ acts on a compact manifold $X$ of dimension $n$, then $d = 1$ and geometric 1-cocycles correspond roughly to (homotopy classes of) loops in the mapping cylinder $\mathbb{R} \times Z X$.

**Example 8.5.** Let $\mathbb{Z}^2$ act on a torus $\mathbb{T}^n$ by a pair of group translations, so that $\mathbb{R}^2 \times Z \mathbb{T}^n \cong \mathbb{T}^{2+n}$. Let $\hat{L}$ be a plane in $\mathbb{R}^{2+n}$ specified by a set of $n$ equations

$$a_i x + b_i y + u_{i1} t_1 + \cdots + u_{in} t_n = 0, \quad i = 1, \ldots, n$$

with integer coefficients with the $n$-by-$n$ matrix $U := (u_{ij})$ invertible over $\mathbb{Q}$.

Then $\hat{L}/\mathbb{Z}^{n+2}$ is a 2-dimensional torus which maps canonically to $\mathbb{T}^{2+n}$. We get a 2-dimensional geometric cocycle

$$p : \mathbb{T}^2 \rightarrow \mathbb{R}^2 \times Z \mathbb{T}^n.$$ 

**Example 8.6.** Let $M$ be a negatively curved compact $d$-dimensional $K$-oriented manifold and $\Gamma = \pi_1(M)$ acting on the universal cover $\tilde{Z} := \tilde{M}$. Then

$$Z \times_\Gamma \partial Z \cong SM$$

the sphere bundle of the tangent bundle of $M$. If the Euler characteristic $\chi(M)$ is zero, then there is a non-vanishing vector field on $M$ and hence a smooth map

$$\xi : M \rightarrow SM,$$

and, $K$-orienting $\xi$ by the $K$-orientations on its domain and range, we get a $d$-dimensional geometric cocycle $\text{Index}(\xi)$ for $\Gamma$ acting on $\partial Z$.

Let $(M, f)$ be a (slightly inapted named) $d$-dimensional geometric cycle, as it determines a smooth $n$-dimensional correspondence as in [8.2] with class $[M, f] \in K_{K}(\mathbb{C}, C(Z \times_\Gamma X))$. The dualized version of the assembly map of [8.1] shifts degrees by $-d$ and so

$$\hat{\mu}(\{M, f\}) \in K_{K-n-d}(\mathbb{C}, C_0(X) \times_\Gamma \mathbb{C}) = K_{K-n-d}(C_0(X) \times_\Gamma \mathbb{C})$$

can be paired with the Dirac class $[\Gamma \times_\Gamma X] \in K_{K-n-d}(C_0(X) \times_\Gamma \mathbb{C})$ to give an integer. We call this integer the *Dirac index* of the cocycle. It is an analytic invariant.

We now define a (topological invariant of) a cocycle $f : M \rightarrow Z \times_\Gamma X$. Consider the inclusion $e_{z_0, X} : X \rightarrow Z \times_\Gamma X$ of the fibre $X_{z_0}$. By perturbing $f$ through a homotopy if necessary, we may assume that $f$ and $e_{z_0, X}$ are transverse. Therefore we can compose the correspondences

$$\cdot \leftarrow M \rightarrow Z \times_\Gamma X \xleftarrow{e_{z_0, X}} X \rightarrow \cdot$$

by transversality, yielding the $K$-oriented smooth, 0-dimensional manifold $f^{-1}(X_{z_0})$, where $X_{z_0}$ is the fibre. This inverse image is is a finite set of points, suitably $K$-oriented.

We call the algebraic sum of these $K$-oriented (e.g. signed) points the *intersection index* of the cocycle.

**Example 8.7.** In the case of integer actions as in Example 8.4, the intersection index of a loop in $\mathbb{R} \times_\mathbb{Z} X$ is the algebraic number of times the loop crosses the hypersurface $X \cong F \subset \mathbb{R} \times_\mathbb{Z} X$.

**Example 8.8.** The intersection index of the 2-dimensional geometric cocycles for a $\mathbb{Z}^2$-action on $\mathbb{T}^n$ as in Example 8.5 is given by the cardinality of the finite group $U(\mathbb{Z}^n)/\mathbb{Z}^n$ with $U \in M_n(\mathbb{Z}) \cap \text{GL}_n(\mathbb{Q})$ the integer matrix used to define the plane.
Remark 8.9. Given that $\Gamma$ is torsion-free, the long exact sequence of the fibration $p : Z \times \Gamma X \to \Gamma \backslash (Z \times \Gamma X)$ gives that the inclusion of the fibre $X$ in $Z \times \Gamma X$ induces an isomorphism $\pi_d(X) \to \pi_d(Z \times \Gamma X)$ on homotopy groups as long as $d \geq 2$, since $\Gamma \backslash Z$ is aspherical.

This shows that for $d > 1$, one cannot achieve geometric cocycles with nonzero intersection indices by mapping spheres $S^d \to Z \times \Gamma X$, since they factor through the fibre inclusion, and up to obvious homotopy one can alter one of any two fibre inclusions to make them have disjoint range.

The main point of Dirac classes for actions is the following result, which can be considered a kind of ‘black box’ index theorem. It applies automatically every time one constructs a representative of the Dirac class. The result below is a special case of a more general one, Theorem 8.15.

**Theorem 8.10.** If $f : M \to Z \times \Gamma X$ is a $d$-dimensional geometric cocycle for $\Gamma$ acting on $X$, then it’s Dirac (analytic) index equals its (topological) intersection index.

This result is a formal consequence of functorial properties of the Dirac method and is discussed following the examples below.

**Integer actions**

In the case of integer actions, say of $\mathbb{Z}$ acting on $X$ smoothly by a diffeomorphism $\varphi : X \to X$, the Dirac class $[\mathbb{Z} \ltimes X] \in \mathrm{KK}_{1-n}(C(X) \times \mathbb{Z}, \mathbb{C})$ always is non-vanishing and non-torsion in K-homology. This follows from Theorem 8.10 and the following construction of a 1-cocycle.

Choose a point $x_0$ of $X$ and let $\gamma : [0, 1] \to X$ be a smooth path from $x_0$ to $\varphi(x_0)$ with $\gamma'(0) \neq 0$. Let $f : [0, 1] \to \mathbb{R} \times \mathbb{Z} X$ be the loop $f(t) := [(t, \gamma(t))]$. It’s intersection index is clearly $+1$.

If the action is isometric, then we may represent the Dirac class by the spectral triple(s) described in Example 5.9.

Suppose $X = T$ with $\mathbb{Z}$ acting by irrational rotation. The Dirac class is represented by the deformed Dolbeault operator $D_\theta$ acting on $L^2(A_\theta)$, while $\mathbb{R} \times \mathbb{Z} T \cong T^2$ can be identified with the 2-torus. If $L_{p,q} \subset \mathbb{R}^2$ is a line through the origin of rational slope $\frac{p}{q}$, then it projects to a loop

$$f_{p,q} : T \to T^2,$$

that is, a geometric 1-cocycle, whose intersection index is $= +q$.

The analytic counterpart of the index theorem is as follows.

Let $L_\theta$ be a line in $T^2$ of slope $\theta$. The real line acts by Kronecker flow on $T^2$ and the loops $f_{p,q}$ are transverse to the flow. Restricting the groupoid $\mathbb{R} \times \mathbb{R}^2$ to the transversals $f_{p,q}$ and identifying the transversals with $T$ yields, as one computes, the groupoid $\mathbb{Z} \ltimes_{\gamma'} T$ of irrational rotation by $\theta' := \frac{p \theta + q}{r \theta + s}$. Forgetting the left action of $A_{\theta'}$ on these modules gives (since $A_{\theta'}$ is unital and acts by compact operators) a family of $\mathcal{E}_{p,q}$ of finitely generated projective modules over $A_{\theta'}$ (studied by Rieffel.)

The Dirac index computes the Kasparov pairing

$$\langle [\mathcal{E}_{p,q}], [D_\theta] \rangle.$$

Intuitively, this is the index of the Dirac operator on $T^2_{\theta}$ ‘twisted by’ the ‘bundle’ $\mathcal{E}_{p,q}$, and by choosing a suitable connection one can represent it quite concretely as a deformed Doloault operator, acting on sections of the relevant bundle. The Dirac index (the Fredholm index of this operator) is thus $+q$. 
Vanishing of the Fredholm index

The intersection index formula allows us to dispose rapidly of the problem of computing the ordinary Fredholm index of a Dirac class, for an action of $\Gamma$ on $X$ compact, that is, the pairing

$$\langle [1], [\hat{\Gamma} \rtimes X] \rangle \in \text{KK}_{d-n}(\mathbb{C}, \mathbb{C}) = \mathbb{Z},$$

which, of course, is only potentially nonzero when $d - n = 0 \mod 2$.

**Theorem 8.11.** The Fredholm index of the Dirac class $[\hat{\Gamma} \rtimes X]$ of any action is zero.

For example, the ordinary Fredholm index of the deformed Dolbeault operator $D_\theta$ on $T^2_\theta$ is zero.

The proof is based on the following simple Lemma, whose proof already follows from the discussion in the proof of Proposition 4.9.

**Lemma 8.12.** $\mu([\text{ev}]) = [1_{C^*\Gamma}] \in \text{K}_0(C^*\Gamma)$ where $1_{C^*\Gamma}$ is the unit in $C^*\Gamma$.

**Proof.** (of Theorem 8.11). We lift the class $[1] \in \text{K}_0(C_0(Z) \rtimes \Gamma \times \Gamma)$ under $\hat{\mu}$ to a geometric coycle with zero intersection index.

Let $u: C \to C(X)$ be the $\Gamma$-equivariant inclusion, $u \times \Gamma: C^*(\Gamma) \to C(X) \rtimes \Gamma$ the induced map. The diagram

$$\begin{align*}
\text{KK}_0^\Gamma(C_0(Z), C) & \xrightarrow{\mu} \text{K}_0(C^*\Gamma) \\
u_* & \downarrow \\
\text{KK}_0^\Gamma(C_0(Z), C(X)) & \xrightarrow{\mu_X} \text{K}_0(C(X) \rtimes \Gamma)
\end{align*}$$

and thus $\mu_X(u_*([\text{ev}]) \in \text{KK}_0^\Gamma(C_0(Z), C(X)) = [1_{C(X) \rtimes \Gamma}]$ by the Lemma. To show this has zero pairing with the Dirac class, it suffices to compute the Poincaré dual of $u_*([\text{ev}]) \in \text{RKK}_0^\Gamma(Z ; C, C(X)) = \text{K}^{-d}(Z \times \Gamma X)$, and show that its intersection index is zero. As the proof is similar to that of Proposition 6.3, we merely sketch it. The class $u_*([\text{ev}]$ is represented (analytically) by the $\Gamma$-equivariant correspondence obtained by composing

$$Z \xleftarrow{\alpha_{z_0}} \Gamma \to \cdot \xleftarrow{\text{id}} X \xrightarrow{X} X,$$

where $\alpha_{z_0}$ is the orbit map at $z_0$.

The composition gives

$$Z \xleftarrow{\alpha_{z_0} \circ \text{pr}_\Gamma} \Gamma \times X \xrightarrow{\text{pr}_X} X$$

from $Z$ to $X$. Poincaré dualizing as in the proof of Proposition 6.3 and taking $\Gamma$-invariants gives the smooth correspondence

$$\cdot \xleftarrow{e_{z_0,X}} Z \times \Gamma X.$$

Finally, replacing the point $\Gamma z_0$ to any different point $\Gamma z_1$, we obtain the equivalent (because the two points can be connected by a path) correspondence

$$\cdot \xleftarrow{e_{z_1,X}} Z \times \Gamma X$$

and the map $e_{z_1,X}$ now has disjoint image from the image of $e_{z_0,X}$. Hence the intersection index is zero as claimed. \qed

**Boundary actions of hyperbolic groups**
Corollary 8.13. Let $\Gamma$ be the fundamental group of a negatively curved compact $d$-dimensional manifold $M$ with universal cover $\mathbb{Z}$ and Gromov boundary $\partial \mathbb{Z} \cong \partial \Gamma$. Then the boundary extension class

$$[\partial \Gamma] \in K^1(C(\partial \Gamma) \rtimes \Gamma)$$

is a non-torsion, nonzero class in K-homology, and the Dirac index of the $d$-dimensional geometric cocycle of Example 8.6 determined by a non-vanishing vector field on $M$, is $+1$.

Remark 8.14. We have implicitly K-oriented $\xi: M \to SM$ by the separate K-orientations on $M$ and on $SM$ given to us. One could also K-orient it differently, by switching the K-orientation on $M$. This would result in an intersection index of $-1$.

In the case of isometry groups of classical (say, real), where $SM$ identifies with $G/\Gamma$, if $\Gamma' \subset \Gamma$ is a lattice of finite index, and $\xi: M' \to SM' \cong G/\Gamma'$ is a non-vanishing vector field, then

$$\cdot \leftarrow M' \xrightarrow{\xi} G/\Gamma' \to G/\Gamma$$

also gives a geometric $d$-cocycle with intersection index $[\Gamma; \Gamma']$, as the reader can easily check.

Of course the existence of a non-vanishing vector field $\xi$ in order to make a positive index is equivalent to vanishing of the Euler characteristic $\chi(M)$. For surface groups $\Gamma = \pi_1(M^g)$, this is not the case, and in fact the boundary extension class, that is, the Dirac class, is torsion of order $\chi(M^g) = 2 - 2g$. We show how to prove this using our interpretations, and the theory of $\mathbb{Z}/k$-manifolds.

We conclude with the proof of the intersection index formula, which is essentially a formal consequence of functoriality results in equivariant KK-theory.

Theorem 8.15. If $\cdot \leftarrow (M, \xi) \xrightarrow{\Gamma} Z \times_\Gamma X$ is a smooth correspondence from a point to $Z \times_\Gamma X$, Index$(f, \xi) \in KK_*((C, C_0(Z) \times_\Gamma X))$ its class, $[\text{ev}] \otimes_C [X] \in KK_*(-(Z \times_\Gamma X, C))$ the K-homology class of a fibre of $Z \times_\Gamma X \to \Gamma \backslash X$ then

$$\langle \hat{\mu}(\text{Index}(f, \xi)), [\hat{\Gamma} \times X] \rangle = \langle \text{Index}(f, \xi), [\text{ev}] \otimes_C [X] \rangle,$$

with in both cases the pairings between Kasparov $K$-theory and K-homology.

Proof. The work of Meyer and Nest, part of which was extended in [23], on formalizing and abstracting the Dirac method, show that in a rather more general context, the Baum-Connes assembly map

$$(8.16) \quad KK_*^\Gamma(C_0(\mathcal{E}^\Gamma)), B \xrightarrow{\mu} KK_*^\Gamma(C, B \rtimes \Gamma)$$

agrees with the following map, supposing that one has a suitable dual (see [23] Theorem 6.9 and environmental discussion). The dual involves various data, including a proper $\Gamma$-$C^*$-algebra $\mathcal{P}$ which in the present case of interest is $C_0(Z)$. Duality identifies the domain of $[8.10]$ with the group

$$\mathcal{R}KK_*^{\Gamma^*}(\mathcal{E}^\Gamma; C_0(\mathcal{E}^\Gamma), \mathcal{P} \otimes C_0(X)).$$

The generalized Green-Julg Theorem identifies this group with

$$KK_*^{\Gamma^*}(\mathcal{E}^\Gamma; C_0(\mathcal{E}^\Gamma), \mathcal{P} \otimes C_0(X)) \times \Gamma).$$

Hence assembly is equivalent to a map

$$(8.17) \quad KK(C, \mathcal{P} \otimes C_0(X) \rtimes \Gamma) \to KK_*^{\Gamma^*}(\mathcal{E}^\Gamma; C_0(X) \rtimes \Gamma).$$

The map in question is induced by Kasparov product with with the Dirac morphism $D \in KK_*^{\Gamma^*}(\mathcal{P}, C)$.

Translating this into the present context, where $\mathcal{P} = C_0(Z)$, $D = [Z] \in KK_*^{\Gamma^*}(C_0(Z), C)$, gives the following Lemma, from which the Theorem follows immediately from putting $\psi := [\hat{\Gamma} \times X]$ to be the Dirac class in equivariant theory.
Lemma 8.18. Let \( \varphi \in KK^\Gamma_*(C_0(Z), C_0(X)) \) and \( \check{\varphi} \in K^{-*+}(Z \times_\Gamma X) \) its Poincaré dual. Let \( \psi \in KK^\Gamma_*(C_0(X), C_0(Z)) \) and \( \text{Dirac}(\psi) \in KK_{-*+}(C_0(Z \times_\Gamma X), C_0(Z)) \) its image under the Dirac map. Then

\[
\langle \mu(\varphi), \psi \rangle = \langle \check{\varphi}, \text{Dirac}(\psi) \rangle \in \mathbb{Z},
\]

where the pairing is that between \( K \)-theory and \( K \)-homology of \( Z \times_\Gamma X \).

This concludes the proof of the intersection index formula. \( \square \)

Finally, we note that the more general statement of the Intersection Index Formula specializes to a topological formula for computing the boundary map of the boundary extension of a hyperbolic group. We record it here, by way of conclusion.

Corollary 8.20. The boundary map

\[
\delta : K_1(C(\partial Z) \rtimes \Gamma) \to \mathbb{Z}
\]

associated to the boundary extension admits the following topological description. Let

\[
\hat{\mu} : K^{-*}(SM) \to K_{*+d}(C(\partial Z) \rtimes \Gamma)
\]

be the Baum-Connes assembly map. Then if \( \cdot \leftrightarrow (W, \xi) \overset{f}{\rightarrow} SM \) is a Baum-Douglas cocycle for \( SM \), with class \( \text{Index}(f!, \xi) \), and \( f : W \to SM \) transverse to the fibre \( S_{z_0}M \), then

\[
(\delta \circ \hat{\mu})(\text{Index}(f!, \xi)) = \text{Index}(f^{-1}(S_{z_0}M) \cdot \xi)
\]

the index of the Dirac operator on the spin\(^c\)-manifold \( f^{-1}(S_{z_0}M) \), twisted by \( \xi \).

References

[1] S. Adams: Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups, Topology 33 (1994), no. 4, 765–783
[2] C. Anantharaman-Delaroche: Purely infinite \( C^* \)-algebras arising from dynamical systems, Bull. Soc. Math. France 125 (1997), no. 2, 199–225
[3] C. Anantharaman-Delaroche, J. Renault: Amenable groupoids, Monographie de L’Enseignement Mathématique 36, 2000
[4] C. Anantharaman-Delaroche: Amenability and exactness for dynamical systems and their \( C^* \)-algebras, Trans. Amer. Math. Soc. 354 (2002), no. 10, 4153–4178
[5] M.-D. Choi, E.G. Effros: The completely positive lifting problem for \( C^* \)-algebras, Ann. of Math. (2) 104 (1976), no. 3, 585–609
[6] P. Baum, A. Connes, N. Higson: Classifying space for propositioner actions and \( K \)-theory of group \( C^* \)-algebras, from \( C^* \)-Algebras: 1943–1993, San Antonio, TX, Contemp. Math. 167 (1994), Amer. Math. Soc., Providence, RI, 240–291.
[7] O. Ben-Bassat, J. Block, T. Pantev: Noncommutative tori and Fourier-Mukai duality, Compositio Math. 143 (2007) 423?475 doi:10.1112/S0010437X06002636
[8] M. Bonk, T. Foertsch: Asymptotic upper curvature bounds in coarse geometry, Math. Z. 253 (2006), no. 4, 753–785
[9] A. Connes: Cyclic cohomology and the transverse fundamental class of a foliation, Geometric methods in operator algebras (Kyoto, 1983), Pitman Res. Notes Math. Ser. 123 (1986), pp. 52–144
[10] A. Connes: Compact metric spaces, Fredholm modules, and hyperfiniteness, Ergodic Theory Dynam. Systems 9 (1989), no. 2, 207–220
[11] A. Connes: Noncommutative Geometry, Academic Press 1994
[12] M. Coornaert: Mesures de Patterson-Sullivan sur le bord d’un espace hyperbolique au sens de Gromov, Pacific J. Math. 159 (1993), no. 2, 241–270
[13] A. Connes, M. Gromov, H. Moscovici: Group cohomology with Lipschitz control and higher signatures, Geom. Funct. Anal. 3 (1993) no. 1, 1–78.
[14] A. Connes, G. Skandalis: The longitudinal index theorem for foliations, Publ. Res. Inst. Math. Sci. 20 (1984), no. 6, 1139–1183
[15] M. Coornaert, T. Delzant, A. Papadopoulos: Géométrie et théorie des groupes. Les groupes hyperboliques de Gromov, Lecture Notes in Mathematics 1441, Springer 1990
[16] J. Cuntz, W. Krieger: A class of $C^*$-algebras and topological Markov chains, Invent. Math. 56 (1980), no. 3, 251–268
[17] H. Emerson: Noncommutative Poincaré duality for boundary actions of hyperbolic groups, J. Reine Angew. Math. 564 (2003), 1–33
[18] H. Emerson, B. Nica: $K$-homological finiteness and hyperbolic groups, J. Reine Angew. Math., published online J. 2016-04-07 — DOI: https://doi.org/10.1515/crelle-2015-0115.
[19] H. Emerson, R. Meyer: Euler characteristics and Gysin sequences for group actions on boundaries, Math. Ann. 334 (2006), no. 4, 853–904
[20] H. Emerson, R. Meyer: Equivariant representable $K$-theory, Topology. (2009), no. 2 123–156.
[21] H. Emerson, R. Meyer: Equivariant embedding theorems and topological index maps, Adv. Math. 225 (2010), 2840–2882
[22] H. Emerson, R. Meyer: Bivariant $K$-theory via correspondences, Adv. Math. 225 (2010), 2883–2919
[23] H. Emerson, R. Meyer: Dualities in equivariant Kasparov theory, New York J. Math.16 (2010), 245–313
[24] H. Emerson, R. Meyer: A descent principle for the dual-Dirac method, Topology 46, no. 2 (2007), 185-209.
[25] E. Ghys, P. de la Harpe (eds.), Sur les groupes hyperboliques d’après Mikhael Gromov, Progress in Mathematics 83, Birkhäuser (1990)
[26] M. Gromov: Hyperbolic groups, in Essays in group theory (Publ. MSRI 8, Springer 1987), 75–263
[27] A. Hawkins, A. Skalski, S. White and J. Zacharias, On spectral triples on crossed products arising from equicontinuous actions, Math. Scand. 113 (2013), 262-291.
[28] N. Higson, G. G. Kasparov: $E$-theory and $KK$-theory for groups which act properly and isometrically on Hilbert space, Invent. Math. 144 (2001), no. 1, 23–74
[29] N. Higson, J. Roe: Analytic $K$-homology, Oxford Science Publications (2000).
[30] M. Hilsum, G. Skandalis: Morphismes $K$-orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov, Ann. scient. Éc. Norm. Sup., 4e série, t. 20 (1987), 325–390.
[31] J. Kaminker, J. Miller: A comment on the Novikov conjecture, Proc. Amer. Math. Soc. 83 (1981), 656–658.
[32] I. Kapovich, N. Benakli: Boundaries of hyperbolic groups, in Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001), 39–93, Contemp. Math. 296, Amer. Math. Soc. 2002
[33] M. Kapovich, B. Kleiner: Hyperbolic groups with low-dimensional boundary, Ann. Sci. École Norm. Sup. (4) 33 (2000), no. 5, 647–669
[34] B. Kleiner: The asymptotic geometry of negatively curved spaces: uniformization, geometrization and rigidity, International Congress of Mathematicians. Vol. II, 743–768, Eur. Math. Soc. 2006
[35] E. Kirchberg: The classification of purely infinite $C^*$-algebras using Kasparov’s theory, 3rd draft
[36] M. Koubi: Croissance uniforme dans les groupes hyperboliques, Ann. Inst. Fourier (Grenoble) 48 (1998), no. 5, 1441–1453
[37] M. Laca, J. Spielberg: Purely infinite $C^*$-algebras from boundary actions of discrete groups, J. Reine Angew. Math. 480 (1996), 125–139
[38] Meyer, Ralf; Nest, Ryszard: The Baum-Connes conjecture via localisation of categories, Topology 45 (2006), no. 2, 209–259.
[39] J.-L. Tu: La conjecture de Baum-Connes pour les feuilletages moyennables, K-Theory 17 (1999), no. 3, 215–264
[40] N.C. Phillips: A classification theorem for nuclear purely infinite simple $C^*$-algebras, Doc. Math. 5 (2000), 49–114
[41] J. Rosenberg, C. Schochet: The Künneth theorem and the universal coefficient theorem for Kasparov’s generalized $K$-functor, Duke Math. J. 55 (1987), no. 2, 431–474
[42] D. Sullivan: The density at infinity of a discrete group of hyperbolic motions, Publ. IHES 50 (1979), 171–202
[43] J.-L. Tu: La conjecture de Baum-Connes pour les feuilletages moyennables, K-Theory 17 (1999), no. 3, 215–264
[44] J. Väisälä: Gromov hyperbolic spaces, Expo. Math. 23 (2005), no. 3, 187–231

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