On solvability for inverse problem of compact support source determination for the heat equation

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Abstract. An inverse problem of reconstructing the source for the heat equations on a plane is considered. As an “overdetermination” (additional information about the solution of the direct problem) a trace of it’s solution is given on a line segment inside of a bounded region. We give sufficient conditions for uniqueness of the solution of the task at hand, prove Fredholm alternative and sufficient conditions for existence and uniqueness of solution of the task. The studying of the problem is performed in the spaces of functions satisfying Hölder condition.

1. The problem statement
Let $T > 0$, $0 < \alpha < 1$ be a fixed number, and there is a bounded domain $\Omega$ with a smooth boundary of class $C^{2, \alpha}$ on the plane with real coordinates of points $x = (x_1, x_2)$. In the space of points $(x, t)$ we define a cylinder $\Omega_T = \Omega \times (0, T]$ with side boundary $\Gamma_T = \partial \Omega \times [0, T]$. In the closed cylinder $\overline{\Omega}_T = \overline{\Omega} \times [0, T]$ we consider the first boundary value problem for the heat equation, that is, the problem of determining the function $u : \overline{\Omega}_T \to \mathbb{R}$ from the following conditions:

\begin{align}
(Lu)(x, t) &= u_t(x, t) - (\Delta u)(x, t) = f(x_1)h(x, t) + g(x, t), \quad (x, t) \in \Omega_T, \\
u(x, t) &= \mu(x, t), \quad (x, t) \in \Gamma_T, \\
u(x, 0) &= \varphi(x), \quad x \in \overline{\Omega}.
\end{align}

In equation (1), $g, h : \overline{\Omega}_T \to \mathbb{R}$, $\mu : \Gamma_T \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ are given functions. The function $f$ is assumed to have compact support, more precisely, there exist numbers $a < b$ and this function is nonzero only on the interval $[a, b]$. In addition, we assume that a number $c$ is given such that for the interval $\omega = (a, b) \times c$ the inclusion is valid:

$\omega = [a, b] \times c \subset \Omega$.

For convenience of the following constructions we define the sets:

$P = \{(x_1, x_2) : a < x_1 < b, x_2 \in \mathbb{R}\}$,

$P = \{(x_1, x_2) : a \leq x_1 \leq b, x_2 \in \mathbb{R}\},$

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$$P_T = P \times (0, T], \quad \overline{P}_T = \overline{P} \times [0, T].$$

We define the following space of functions that satisfy the Hölder condition (for the definition of all standard Hölder functions used here, see [1], p. 16):

$$U(\Omega_T) = C(\overline{\Omega}_T) \cap C^{2+\alpha,1+\frac{\alpha}{2}} (\overline{P}_T \cap \Omega_T) \cap C^{2+\alpha,1+\frac{\alpha}{2}} (\Omega_T \setminus \overline{P}_T) \cap C^{1,0}_{x,t}(\Omega_T).$$

In the future, when formulating the inverse problem for equation (1), we will assume that the solution of the direct problem (1), (2) lies in the class of functions $U(\Omega_T)$. Since the class of functions $U(\Omega_T)$ is not generally accepted for solutions of equation (1), we present here the existence and uniqueness theorem for the solution of problem (1) - (2) in this class.

**Theorem 1.** Let the inclusions are valid: $\varphi \in C(\overline{\Omega})$, $\mu \in C(\Gamma_T)$, $h, g \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)$, $f \in C^\alpha[a, b]$, the matching conditions $\mu(x, 0) = \varphi(x)$, $x \in \partial\Omega$ hold. Then problem (1) - (2) has a unique solution in the class of functions $U(\Omega_T)$.

Theorem follows from the results of the monograph [1], c. 256.

**Remark** An important property of the class of functions $U(\Omega_T)$, which will be used in the formulation of the inverse problem for equation (1), is that if we denote $u(x_1, c, T) = \chi(x_1)$, then for the function $\chi(x_1)$ the inclusion is valid $\chi(x_1) \in C^{2+\alpha}[a, b]$.

In order to formulate the inverse problem for equation (1), we assume that in addition to the function $u(x, t)$, the function $f(x_1)$ is also unknown, but there is an additional information on the function $u(x, t)$. We assume that in addition to the condition (2), the trace of the function $u(x, t)$ on the interval $\overline{\omega}$ is also known (this kind of information in the formulation of the inverse problem is usually called an overdetermination). In this way, we start to study the inverse problem of determining the pair of functions $(u, f) \in U(\Omega_T) \times C^\alpha[a, b]$ from the conditions:

$$(Lu)(x, t) = f(x_1)h(x, t) + g(x, t), \quad (x, t) \in \Omega_T, \quad (3)$$

$$(x, t) \in \Gamma_T, \quad u(x, 0) = \varphi(x), \quad x \in \overline{\Omega}, \quad (4)$$

$$u(x_1, c, T) = \chi(x_1), \quad x_1 \in [a, b]. \quad (5)$$

The inverse problem for the heat equation with compact support sources was considered in [2]. In that paper, the question of the uniqueness of its solution with another overdetermination in an unbounded domain was studied.

Other statements of the inverse problems of determining the sources for the heat equation, which are close to problem (3) - (5), and also the history of the study of such problems, see [3] - [6]. In this paper, for the inverse problem (3) - (5) sufficient conditions for unique solvability and the formulation of the Fredholm alternative are presented.

**2. Main results**

We consider an important matter for any inverse problem, that is the sufficient conditions for the uniqueness of its solution. For problem (3) - (5) the following uniqueness theorem holds.

**Theorem 2.** Suppose that the conditions $h, h_t, h_{x_2} \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)$, $h(x, 0) = 0$, $x \in \partial\Omega$, $h(a, x_2, 0) = h(b, x_2, 0) = 0$, $(a, x_2), (b, x_2) \in \Omega$, $|h(x_1, 0, T)| \geq h_T > 0$, where $h_T$ is a positive constant, are satisfied for the function $h(x, t)$ Then there exists a number $v_0 > 0$ such that if the condition $b - a < v_0$ is satisfied, then the inverse problem (3) - (5) can not have two different solutions.

The proof of Theorem 2 is based on the representation of the solution of problem (3) - (5) as a sum of heat potentials.
The nature of the solvability of the inverse problem (3) - (5) is closely related to the solvability of the homogeneous inverse problem corresponding to this one. We call a homogeneous inverse problem corresponding to the inverse problem (3) - (5) the similar problem (3) - (5) where \( \varphi = 0, \mu = 0, g = 0, \chi = 0 \), that is, the problem of defining a pair of functions \( (u, f) \in U(\Omega_T) \times C^\alpha[a, b] \) from the following conditions:

\[
(Lu)(x, t) = f(x_1)h(x, t), \quad (x, t) \in \Omega_T,
\]

\[
u(x, t) = 0, \quad (x, t) \in \Gamma_T, \quad u(x), 0) = 0, \quad x \in \Omega,
\]

\[
u(x_1, c, T) = 0, \quad x_1 \in [a, b].
\]

The relation between problems (3) - (5) and (6) - (8) is established in the following statement.

**Theorem 3.** (Fredholm Alternative) Suppose that the conditions \( h, h_1, h_2 \in C^\alpha, \overline{\Omega}(\Omega_T) \), \( h(x, 0) = 0, \ x \in \partial \Omega, \ h(a, x_2, 0) = h(b, x_2, 0) = 0, \ (a, x_2), (b, x_2) \in \partial \Omega, \ |h(x_1, c, T)| \geq h_T > 0 \) are satisfied for the function \( h(x, t) \). Then there exists a number \( \nu_0 > 0 \) such that if the condition \( b - a < \nu_0 \) is fulfilled, then for the inverse problem (3) - (5) one of the following statements (alternative) holds:

(i) The homogeneous inverse problem (6) - (8) has a finite number of linearly independent solutions (in other words, the homogeneous problem has a nontrivial solution)

(ii) For any functions \( \varphi \in C(\Omega), \mu \in C(\Gamma_T) \) satisfying the matching conditions \( \mu(x, 0) = \varphi(x), \ x \in \partial \Omega \) and any functions \( \chi \in C^2, \alpha[a, b] \) the inhomogeneous problem (3) - (5) has an unique solution.

The proof of Theorem 3 is based on reducing the problem of the solvability of the inverse problem (3) - (5) to the question of the solvability of a linear equation of the second kind with a compact operator in the Banach space \( C^\alpha[a, b] \) and using of the Riesz-Schauder theorem for this equation.

As a corollary of Theorem 2 and Theorem 3, we obtain a theorem that gives a sufficient condition for the existence of a unique solution of the inverse problem (3) - (5).

**Theorem 4.** Suppose that the conditions \( h, h_1, h_2 \in C^\alpha, \overline{\Omega}(\Omega_T) \), \( h(x, 0) = 0, \ x \in \partial \Omega, \ h(a, x_2, 0) = h(b, x_2, 0) = 0, \ (a, x_2), (b, x_2) \in \partial \Omega, \ |h(x_1, c, T)| \geq h_T > 0, \ x_1 \in [a, b] \) are satisfied for the function \( h(x, t) \). Then there exists a number \( \nu_0 > 0 \) such that for any segment \( \overline{\omega} \in \Omega \) such that \( b - a < \nu_0 \) and any functions \( \varphi \in C(\Omega), \mu \in C(\Gamma_T) \) satisfying the matching conditions \( \mu(x, 0) = \varphi(x), \ x \in \partial \Omega \) and any functions \( g \in C^\alpha, \overline{\Omega}(\Omega_T), \chi \in C^2, \alpha[a, b] \) the inverse problem (3) - (5) has a unique solution.

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