Cauchy problem of the generalized Zakharov type system in $\mathbb{R}^2$

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Abstract
In this paper, we consider the initial value problem for a two-dimensional generalized Zakharov system with quantum effects. We prove the existence and uniqueness of global smooth solutions to the initial value problem in the Sobolev space through making \textit{a priori} integral estimates and the Galerkin method.

Keywords: Zakharov system; Cauchy problem; existence; uniqueness

1 Introduction
In the recent years, special interest has been devoted to quantum corrections to the Zakharov equations for Langmuir waves in a plasma [1]. By use of a quantum fluid approach, the following modified Zakharov equations are obtained:

\begin{align}
    iE_t + E_{xx} - H^2 E_{xxxx} &= nE, \quad (1) \\
    n_{tt} - n_{xx} + H^2 n_{xxxx} &= |E|^2_{xx}, \quad (2)
\end{align}

where $H$ is the dimensionless quantum parameter given by the ratio of the ion plasmon and electron thermal energies. For $H = 0$, this system was derived by Zakharov in [2] to model a Langmuir wave in plasma. The Zakharov system attracted many scientists’ wide interest and attention [3–14].

In this paper, we deal with the following generalized Zakharov system:

\begin{align}
    iE_t + \Delta E - H^2 \Delta^2 E - nE &= 0, \quad (3) \\
    n_{tt} - \Delta n + H^2 \Delta^2 n - \Delta |E|^2 &= 0, \quad (4)
\end{align}

where $(E, n) : (x, t) \in \mathbb{R}^2 \times \mathbb{R}$ and the initial data are taken to be

\begin{align}
    E|_{t=0} = E_0(x), \quad n|_{t=0} = n_0(x), \quad n_t|_{t=0} = n_1(x). \quad (5)
\end{align}

To study a smooth solution of the generalized Zakharov system, we transform it into the following form:

\begin{align}
    iE_t + \Delta E - H^2 \Delta^2 E - nE &= 0, \quad (6)
\end{align}
\[ n_t - \Delta \varphi = 0, \]  
\[ \varphi_t - n + H^2 \Delta n - |E|^2 = 0, \]  
\[ \text{with initial data} \]
\[ E|_{t=0} = E_0(x), \quad n|_{t=0} = n_0(x), \quad \varphi|_{t=0} = \varphi_0(x). \]  

Now we state the main results of the paper.

**Theorem 1.1** Suppose that \( E_0(x) \in H^{l+4}(\mathbb{R}^2) \), \( n_0(x) \in H^{l+2}(\mathbb{R}^2) \), \( n_1(x) \in H^l(\mathbb{R}^2), l \geq 0. \) Then there exists a unique global smooth solution of the initial value problem (3)–(5).

\[ E(x, t) \in L^\infty(0, T; H^{l+4}(\mathbb{R}^2)), \quad E_t(x, t) \in L^\infty(0, T; H^l(\mathbb{R}^2)) \]
\[ n(x, t) \in L^\infty(0, T; H^{l+2}(\mathbb{R}^2)), \quad n_t(x, t) \in L^\infty(0, T; H^l(\mathbb{R}^2)) \]
\[ n_{tt}(x, t) \in L^\infty(0, T; H^{l-2}(\mathbb{R}^2)). \]

The obtained results may be useful for better understanding the nonlinear coupling between the ion-acoustic waves and the Langmuir waves in a two-dimensional space.

**2 A priori estimates**

**Lemma 2.1** Suppose that \( E_0(x) \in L^2(\mathbb{R}^2) \). Then, for the solution of problem (6)–(9), we have

\[ \|E\|^2_{L^2(\mathbb{R}^2)} = \|E_0(x)\|^2_{L^2(\mathbb{R}^2)}. \]

**Proof** Taking the inner product of (6) and \( E \), then taking the imaginary part, we have

\[ \text{Im}(AE - H^2 \Delta^2 E - nE, E) = 0. \]

Hence, we get

\[ \frac{d}{dt} \|E\|^2_{L^2} = 0. \]

We thus get Lemma 2.1.

**Lemma 2.2** (Sobolev’s estimations) Assume that \( u \in L^q(\mathbb{R}^n), D^m u \in L^r(\mathbb{R}^n), 1 \leq q, r \leq \infty, 0 \leq j \leq m \), we have the estimations

\[ \|D^j u\|_{L^q(\mathbb{R}^n)} \leq C \|D^m u\|_{L^r(\mathbb{R}^n)} \|u\|_{L^\infty(\mathbb{R}^n)}, \]

where \( C \) is a positive constant, \( 0 \leq \frac{j}{m} \leq \alpha \leq 1, \)

\[ \frac{1}{p} = \frac{j}{n} + \alpha \left( \frac{1}{r} - \frac{m}{n} \right) + (1 - \alpha) \frac{1}{q}. \]
Lemma 2.3 Suppose that $E_0(x) \in H^2(\mathbb{R}^2)$, $n_0(x) \in H^1(\mathbb{R}^2)$, $\varphi_0(x) \in H^1(\mathbb{R}^2)$. Then we have

\[ \mathcal{F}(t) = \mathcal{F}(0), \]

where

\[ \mathcal{F}(t) = \|\nabla E\|^2_{L^2} + H^2 \|\Delta E\|^2_{L^2} + \int_{\mathbb{R}^2} n|E|^2 \, dx + \frac{1}{2} \|\nabla \varphi\|^2_{L^2} + \frac{1}{2} \|n\|^2_{L^2} + \frac{H^2}{2} \|\n\|^2_{L^2}. \]

Proof Take the inner products of (6) and $-E_t$. Since

\[
\begin{align*}
\text{Re}(iE_t, -E_t) &= 0, \\
\text{Re}(\Delta E_t, -E_t) &= \frac{H}{2} \frac{d}{dt} \|\Delta E\|^2_{L^2}, \\
\text{Re}(-H^2 \Delta^2 E_t, -E_t) &= \frac{H}{2} \frac{d}{dt} \|\Delta E\|^2_{L^2}, \\
\text{Re}(-nE_t, -E_t) &= \frac{1}{2} \int_{\mathbb{R}^2} n|E|^2 \, dx \\
&= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n|E|^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n|E|^2 \, dx \\
&= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n|E|^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n|\varphi|^2 \, dx + \frac{1}{4} \frac{d}{dt} \|n\|^2_{L^2} + \frac{H}{4} \frac{d}{dt} \|\n\|^2_{L^2} \\
&= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n|E|^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \Delta \varphi \varphi \, dx + \frac{1}{4} \frac{d}{dt} \|\n\|^2_{L^2} + \frac{H}{4} \frac{d}{dt} \|\n\|^2_{L^2} \\
&= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n|E|^2 \, dx + \frac{1}{4} \frac{d}{dt} \|\n\|^2_{L^2} + \frac{1}{2} \frac{d}{dt} \|n\|^2_{L^2} + \frac{H}{4} \frac{d}{dt} \|\n\|^2_{L^2},
\end{align*}
\]

thus it follows that

\[
\frac{d}{dt} \left[ \|\nabla E\|^2_{L^2} + H^2 \|\Delta E\|^2_{L^2} + \int_{\mathbb{R}^2} n|E|^2 \, dx + \frac{1}{2} \|\nabla \varphi\|^2_{L^2} + \frac{1}{2} \|n\|^2_{L^2} + \frac{H^2}{2} \|\n\|^2_{L^2} \right] = 0. \tag{10}
\]

Letting

\[ \mathcal{F}(t) = \|\nabla E\|^2_{L^2} + H^2 \|\Delta E\|^2_{L^2} + \int_{\mathbb{R}^2} n|E|^2 \, dx + \frac{1}{2} \|\nabla \varphi\|^2_{L^2} + \frac{1}{2} \|n\|^2_{L^2} + \frac{H^2}{2} \|\n\|^2_{L^2}, \]

and noticing (10), we obtain

\[ \mathcal{F}(t) = \mathcal{F}(0). \]

Lemma 2.4 Suppose that $E_0(x) \in H^2(\mathbb{R}^2)$, $n_0(x) \in H^1(\mathbb{R}^2)$, $\varphi_0(x) \in H^1(\mathbb{R}^2)$. Then we have

\[
\sup_{0 \leq t \leq T} \left( \|E\|_{H^2} + \|n\|_{H^1} + \|\varphi\|_{H^1} \right) \leq C.
\]
Proof. By Hölder’s inequality, Young’s inequality and Lemma 2.2, it follows that
\[
\left| \int_{\mathbb{R}^2} n|E|^2 \, dx \right| \leq \|n\|_{L^2} \|E\|_{L^4}^2 \\
\leq \frac{1}{4} \|n\|_{L^2}^2 + \|E\|_{L^4}^4 \\
\leq \frac{1}{4} \|n\|_{L^2}^2 + C \|\Delta E\|_{L^2} \|E\|_{L^2}^3 \\
\leq \frac{1}{4} \|n\|_{L^2}^2 + \frac{H^2}{2} \|\Delta E\|_{L^2}^2 + C.
\]
From Lemma 2.3 we get
\[
\|\nabla E\|_{L^2}^2 + \frac{H^2}{2} \|\Delta E\|_{L^2}^2 + \frac{1}{2} \|\nabla \varphi\|_{L^2}^2 + \frac{1}{4} \|n\|_{L^2}^2 + \frac{H^2}{2} \|\nabla n\|_{L^2}^2 \leq \mathcal{F}(0) + C.
\]
Take the inner products of Eq. (8) and \(\varphi\). It follows that
\[
(\varphi_t - n + H^2 \Delta n - |E|^2, \varphi) = 0 \quad (11)
\]
since
\[
(\varphi, \varphi) = -\frac{1}{2} \frac{d}{dt} \|\varphi\|_{L^2}^2,
\]
\[
(n + |E|^2, \varphi) \leq (\|n\|_{L^2}^2 + \|E\|_{L^2}^2) \|\varphi\|_{L^2} \leq \frac{1}{2} \|\varphi\|_{L^2}^2 + C,
\]
where
\[
\|E\|_{L^4}^2 \leq C \|\nabla E\|_{L^2} \|E\|_{L^2} \leq C.
\]
\[
(H^2 \Delta n, \varphi) = H^2 (n, \Delta \varphi) = H^2 (n, n_t) = \frac{H^2}{2} \frac{d}{dt} \|n\|_{L^2}^2.
\]
Hence, from Eq. (11) we get
\[
\frac{d}{dt} \left( \|\varphi\|_{L^2}^2 + H^2 \|n\|_{L^2}^2 \right) \leq \|\varphi\|_{L^2}^2 + C.
\]
Using Gronwall’s inequality, we obtain that
\[
\sup_{0 \leq t \leq T} \left( \|\varphi\|_{L^2}^2 + H^2 \|n\|_{L^2}^2 \right) \leq C.
\]
We thus get Lemma 2.4.

Lemma 2.5 Suppose that \(E_0(x) \in H^4(\mathbb{R}^2)\), \(n_0(x) \in H^2(\mathbb{R}^2)\), \(\varphi_0(x) \in H^2(\mathbb{R}^2)\). Then we have
\[
\sup_{0 \leq t \leq T} \left( \|E\|_{H^4} + \|n\|_{H^2} + \|\varphi\|_{H^2} + \|E_t\|_{L^2} + \|n_t\|_{L^2} + \|\varphi_t\|_{L^2} \right) \leq C.
\]
Proof} Differentiating (6) with respect to $t$, then taking the inner products of the resulting equation and $E_t$, we have

$$
(iE_{tt} + \Delta E_t - H^2 \Delta^2 E_t - (nE_t)_t, E_t) = 0
$$

(12)

since

$$
\text{Im}(iE_{tt}, E_t) = \frac{1}{2} \frac{d}{dt} \|E_t\|_{L^2}^2,
\quad \text{Im}(\Delta E_t - H^2 \Delta^2 E_t - nE_t, E_t) = 0,
\quad |\text{Im}(-nE_t, E_t)| \leq C \|E\|_{L^\infty} \|n_t\|_{L^2} \|E_t\|_{L^2} \leq C (\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2).
$$

By Lemma 2.2, it follows that

$$
\|E\|_{L^\infty} \leq C \|\Delta E\|_{L^2} \|E\|_{L^2}^\frac{1}{2},
$$

thus from Eq. (12) we get

$$
\frac{d}{dt} \|E_t\|_{L^2}^2 \leq C (\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2).
$$

(13)

Differentiating Eq. (7) with respect to $t$, then taking the inner products of the resulting equation and $n_t$, we have

$$
(n_{tt} - \Delta \varphi, n_t) = 0
$$

(14)

since

$$
(n_{tt}, n_t) = \frac{1}{2} \frac{d}{dt} \|n_t\|_{L^2}^2,
\quad (-\Delta \varphi, n_t) = (-\Delta n + H^2 \Delta^2 n - \Delta |E|^2, n_t)
\quad = \frac{1}{2} \frac{d}{dt} \|\nabla n\|_{L^2}^2 + \frac{H^2}{2} \frac{d}{dt} \|\Delta n\|_{L^2}^2 - (\Delta |E|^2, n_t).
$$

Noting that

$$
|(\Delta |E|^2, n_t)| \leq C \|E\|_{L^\infty} \|\Delta E\|_{L^2} \|n_t\|_{L^2} \leq C (\|n_t\|_{L^2}^2 + 1),
$$

from Eq. (14) we get

$$
\frac{d}{dt} [\|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + H^2 \|\Delta n\|_{L^2}^2] \leq C (\|n_t\|_{L^2}^2 + 1).
$$

(15)

From Eq. (13) and (15) we get

$$
\frac{d}{dt} [\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + H^2 \|\Delta n\|_{L^2}^2] \leq C (\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2 + 1).
$$

By Gronwall’s inequality, it follows that

$$
\sup_{0 \leq t \leq T} [\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + H^2 \|\Delta n\|_{L^2}^2] \leq C.
$$

(16)
Take the inner products of Eq. (8) and \( \varphi_i \). It follows that

\[
(\varphi_i - n + H^2 \Delta n - |E|^2, \varphi_i) = 0
\]  

(17)

since

\[
(\varphi_i, \varphi_i) = \| \varphi_i \|_{L^2}^2,
\]

\[
(-n + H^2 \Delta n - |E|^2, \varphi_i) \leq \left( \| n \|_{L^2} + H^2 \| \Delta n \|_{L^2} + \| E \|_{L^4}^2 \right) \| \varphi_i \|_{L^2}
\]

\[
\leq C + \frac{1}{2} \| \varphi_i \|_{L^2}^2.
\]

From Eq. (17) we get

\[
\| \varphi_i \|_{L^2}^2 \leq C.
\]

Take the inner products of Eq. (6) and \( \Delta E \). It follows that

\[
(i E_i - nE, \Delta E) = 0
\]

(18)

since

\[
(i E_i - nE, \Delta E) \leq \left( \| E_i \|_{L^2} + \| E \|_{L^\infty} \| n \|_{L^2} \right) \| \Delta E \|_{L^2} \leq C,
\]

\[
(\Delta E - H^2 \Delta^2 E, \Delta E) = \| \Delta E \|_{L^2}^2 + H^2 \| \nabla^3 E \|_{L^2}^2.
\]

From Eq. (18) we get

\[
\| \nabla^3 E \|_{L^2}^2 \leq C.
\]

From (6) we obtain

\[
H^2 \| \Delta^2 E \|_{L^2} \leq \| E_i \|_{L^2} + \| \Delta E \|_{L^2} + \| nE \|_{L^2} \leq C,
\]

where

\[
\| nE \|_{L^2} \leq \| n \|_{L^4} \| E \|_{L^4} \leq C \| \nabla n \|_{L^2}^{1/2} \| n \|_{L^2}^{1/2} \| \nabla E \|_{L^2}^{1/2} \| E \|_{L^2}^{1/2} \leq C.
\]

From (7) we obtain

\[
\| \Delta \psi \|_{L^2} = \| n_i \|_{L^2} \leq C.
\]

We thus get Lemma 2.5. \( \square \)

**Lemma 2.6** Suppose that \( f_1, f_2 \in H^s(\Omega), \Omega \subseteq \mathbb{R}^n \). Then we have

\[
\| D'(f_1 \cdot f_2) \|_{L^2} \leq C_s \left( \| f_1 \|_{L^2} \| D'f_2 \|_{L^2} + \| D'f_1 \|_{L^2} \| f_2 \|_{L^2} \right),
\]

where the constant \( C_s \) is independent of \( f_1 \) and \( f_2 \).
Lemma 2.7 Suppose that $E_0(x) \in H^{m+2}(\mathbb{R}^2)$, $n_0(x) \in H^{m+2}(\mathbb{R}^2)$, $\varphi_0(x) \in H^{m+2}(\mathbb{R}^2)$, $m \geq 0$. Then we have

$$
\sup_{0 \leq t \leq T} \left( \|E(x,t)\|_{H^{m+4}} + \|n(x,t)\|_{H^{m+2}} + \|\varphi(x,t)\|_{H^{m+2}} \right) \leq C
$$

$$
\sup_{0 \leq t \leq T} \left( \|E_t(x,t)\|_{H^m} + \|n_t(x,t)\|_{H^m} + \|\varphi_t\|_{H^m} \right) \leq C.
$$

Proof Lemma 2.7 is true when $m = 0$ (Lemma 2.5). Suppose that Lemma 2.7 is true when $m = k$, $k \geq 0$. Take the inner products of (8) and $(-1)^{k+1} \Delta^{k+3} \varphi$. It follows that

$$
(\varphi_t - n + H^2 \Delta n - |E|^2, (-1)^{k+1} \Delta^{k+3} \varphi) = 0
$$

(19)

since

$$
(\varphi_t, (-1)^{k+1} \Delta^{k+3} \varphi) = \frac{1}{2} \frac{d}{dt} \|\nabla^{k+3} \varphi\|_{L^2}^2,
$$

$$
(-n, (-1)^{k+1} \Delta^{k+3} \varphi) = (-n, (-1)^{k+1} \Delta^{k+2} n_t) = \frac{1}{2} \frac{d}{dt} \|\nabla^{k+2} n\|_{L^2}^2,
$$

$$
(H^2 \Delta n, (-1)^{k+1} \Delta^{k+3} \varphi) = H^2 (\Delta n, (-1)^{k+1} \Delta^{k+2} n_t) = \frac{H^2}{2} \frac{d}{dt} \|\nabla^{k+3} n\|_{L^2}^2,
$$

$$
\left| (-|E|^2, (-1)^{k+1} \Delta^{k+3} \varphi) \right| \leq \left\| \left( \nabla^{k+3} |E|^2, \nabla^{k+3} \varphi \right) \right\| \leq \left\| \nabla^{k+3} |E|^2 \right\|_2 \left\| \nabla^{k+3} \varphi \right\|_2
$$

$$
\leq C \|E\|_2 \left\| \nabla^{k+3} E \right\|_2 \left\| \nabla^{k+3} \varphi \right\|_2
$$

$$
\leq C \left( \left\| \nabla^{k+3} \varphi \right\|_2^2 + 1 \right),
$$

thus from Eq. (19) it follows that

$$
\frac{d}{dt} \left( \|\nabla^{k+3} \varphi\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2 + H^2 \|\nabla^{k+3} n\|_{L^2}^2 \right) \leq C \left( \|\nabla^{k+3} \varphi\|_{L^2}^2 + 1 \right).
$$

(20)

By using Gronwall’s inequality, we have

$$
\sup_{0 \leq t \leq T} \left( \|\nabla^{k+3} \varphi\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2 \right) \leq C.
$$

From (7) and (8) we get

$$
\|\nabla^{k+1} n_t\|_{L^2} = \|\nabla^{k+3} \varphi\|_{L^2} \leq C,
$$

$$
\|\nabla^{k+1} \varphi_t\|_{L^2} \leq C \left( \|\nabla^{k+1} n_t\|_{L^2} + \|\nabla^{k+3} n\|_{L^2} + \|E\|_{L^2} \|\nabla^{k+1} E\|_{L^2} \right) \leq C.
$$

Differentiating (6) with respect to $t$, then taking the inner products of the resulting equation and $(-1)^{k+1} \Delta^{k+1} E_t$, we obtain

$$
(iE_{tt} + \Delta E_t - H^2 \Delta^2 E_t - (nE)_{tt} (-1)^{k+1} \Delta^{k+1} E_t) = 0.
$$

(21)

Since

$$
\text{Im}(iE_{tt}, (-1)^{k+1} \Delta^{k+1} E_t) = \frac{1}{2} \frac{d}{dt} \|\nabla^{k+1} E_t\|_{L^2}^2,
$$
thus from Eq. (21) we get

\[
\frac{d}{dt} \left\| \nabla^{k+1} E_t \right\|_{L^2}^2 \leq C \left( \left\| \nabla^{k+1} E_t \right\|_{L^2}^2 + 1 \right).
\]

By using Gronwall's inequality, we get

\[
\sup_{0 \leq t \leq T} \left\| \nabla^{k+1} E_t \right\|_{L^2}^2 \leq C.
\]

From (6) we obtain

\[
\left\| \nabla^{k+5} E \right\|_{L^2} \leq C \left( \left\| \nabla^{k+1} E_t \right\|_{L^2} + \left\| \nabla^{k+3} E \right\|_{L^2} + \left\| \nabla^{k+1} n \right\|_{L^2} \right) \leq C.
\]

Hence

\[
\sup_{0 \leq t \leq T} \left( \left\| E(x,t) \right\|_{H^{k+5}} + \left\| n(x,t) \right\|_{H^{k+3}} + \left\| \psi(x,t) \right\|_{H^{k+1}} \right) \leq C,
\]

\[
\sup_{0 \leq t \leq T} \left( \left\| E_t(x,t) \right\|_{H^{k+1}} + \left\| n_t(x,t) \right\|_{H^{k+1}} + \left\| \psi_t \right\|_{H^{k+1}} \right) \leq C.
\]

This means Lemma 2.7 is true when \( m = k + 1 \). Thus Lemma 2.7 is proved completely. \( \square \)

### 3 Existence and uniqueness of solution

Now, with these lemmas, we are able to prove Theorem 1.1. First we obtain the existence and uniqueness of the global generalized solution of problem (6)-(9).

**Definition 3.1** The functions \( E \in L^\infty(0,T;H^4) \cap W^{1,\infty}(0,T;L^2) \), \( n \in L^\infty(0,T;H^2) \) \( \cap \) \( W^{1,\infty}(0,T;L^2) \) and \( \psi \in L^\infty(0,T;H^2) \cap W^{1,\infty}(0,T;L^2) \) are called a generalized solution of problem (6)-(9) if for any \( \omega \in L^2 \) they satisfy the integral equality

\[
(i E_{\omega t}, \omega) + (\Delta E_j, \omega) = H^2 (\Delta^2 E_j, \omega) + (n E_j, \omega), \quad j = 1, 2, \ldots, N,
\]

\[
(n_t, \omega) = (\Delta \psi, \omega),
\]

\[
(\psi_t, \omega) + H^2 (\Delta n, \omega) = (n, \omega) + (|E|^2, \omega)
\]

with initial data

\[
E(x,0) = E_0(x), \quad n(x,0) = n_0(x), \quad \psi(x,0) = \psi_0(x).
\]
Now, one can estimate the following theorem.

**Theorem 3.1** Suppose that $E_0(x) \in H^{l+4}(\mathbb{R}^2)$, $n_0(x) \in H^{l+2}(\mathbb{R}^2)$, $\varphi_0(x) \in H^{l+2}(\mathbb{R}^2)$, $l \geq 0$. Then there exists a global smooth solution of the initial value problem (6)-(9).

$$
E(x,t) \in L^\infty (0, T; H^{l+4}(\mathbb{R}^2)), \quad E_t(x,t) \in L^\infty (0, T; H^l(\mathbb{R}^2)),
$$

$$
n(x,t) \in L^\infty (0, T; H^{l+2}(\mathbb{R}^2)), \quad n_t(x,t) \in L^\infty (0, T; H^l(\mathbb{R}^2)),
$$

$$
\varphi(x,t) \in L^\infty (0, T; H^{l+2}(\mathbb{R}^2)), \quad \varphi_t(x,t) \in L^\infty (0, T; H^l(\mathbb{R}^2)).
$$

**Proof** By using the Galerkin method, choose the basic periodic functions $\{\omega_k(x)\}$ as follows:

$$
-\Delta \omega_k(x) = \lambda_k \omega_k(x), \quad \omega_k(x) \in H^2(\Omega), k = 1, \ldots, l.
$$

The approximate solution of problem (6)-(9) can be written as

$$
E^j(x,t) = \sum_{k=1}^l \alpha^j_k(t) \omega_k(x), \quad n^j(x,t) = \sum_{k=1}^l \beta^j_k(t) \omega_k(x),
$$

$$
\varphi^j(x,t) = \sum_{k=1}^l \gamma^j_k(t) \omega_k(x),
$$

where

$$
E(x,t) = (E_1, E_2, \ldots, E_N), \quad \alpha^j_k(t) = (\alpha^j_{k1}, \alpha^j_{k2}, \ldots, \alpha^j_{kN}).
$$

$\Omega$ is a two-dimensional cube with $2D$ in each direction, that is, $\overline{\Omega} = \{x = (x_1,x_2)| |x| \leq 2D, j = 1, 2\}$. According to Galerkin’s method, these undetermined coefficients $\alpha^j_k(t)$, $\beta^j_k(t)$ and $\gamma^j_k(t)$ need to satisfy the following initial value problem of the system of ordinary differential equations:

$$
(iE_t^j, \omega_k) + (\Delta E^j_t, \omega_k) = H^2(\Delta E^j, \omega_k) + (n^j, \omega_k), \quad j = 1, 2, \ldots, N, \quad \text{(23)}
$$

$$
(n^j, \omega_k) = (\Delta \varphi^j, \omega_k), \quad \text{(24)}
$$

$$
(\varphi^j, \omega_k) + H^2(\Delta n^j, \omega_k) = (n^j, \omega_k) + (|E^j|_2^2, \omega_k) \quad \text{(25)}
$$

with initial data

$$
E^j(x,0) = E_0^j(x), \quad n^j(x,0) = n_0^j(x), \quad \varphi^j(x,0) = \varphi_0^j(x), \quad \text{(26)}
$$

where

$$
E^j_0(x) \xrightarrow{H^4} E_0(x), \quad n^j_0(x) \xrightarrow{H^2} n_0(x), \quad \varphi^j_0(x) \xrightarrow{H^2} \varphi_0(x), \quad l \to \infty.
$$

Similarly to the proof of Lemmas 2.1-2.5, for the solution $E^j(x,t)$, $n^j(x,t)$, $\varphi^j(x,t)$ of problem (23)-(26), we can establish the following estimates:

$$
\sup_{0 \leq t \leq T} \left( \|E^j\|_{H^4} + \|n^j\|_{H^2} + \|\varphi^j\|_{H^2} + \|E^j\|_{L^2} + \|n^j\|_{L^2} + \|\varphi^j\|_{L^2} \right) \leq C,
$$

where $C$ is a constant that depends only on $\|E_0^j\|_{H^4}$, $\|n_0^j\|_{H^2}$, and $\|\varphi_0^j\|_{H^2}$.
where the constants $C$ are independent of $l$ and $D$. By compact argument, some subsequence of $(E^l, n^l, \varphi^l)$, also labeled by $l$, has a weak limit $(E, n, \varphi)$. More precisely

\[
E^l(x, t) \to E(x, t) \quad \text{in } L^\infty(0, T; H^4) \text{ weakly star},
\]

\[
n^l(x, t) \to n(x, t) \quad \text{in } L^\infty(0, T; H^2) \text{ weakly star},
\]

\[
\varphi^l(x, t) \to \varphi(x, t) \quad \text{in } L^\infty(0, T; H^2) \text{ weakly star}
\]

and

\[
E^l_t \to E_t \quad \text{in } L^\infty(0, T; L^2) \text{ weakly star},
\]

\[
n^l_t \to n_t \quad \text{in } L^\infty(0, T; L^2) \text{ weakly star},
\]

\[
\varphi^l_t \to \varphi_t \quad \text{in } L^\infty(0, T; L^2) \text{ weakly star}.
\]

By using Guo and Shen’s method [15], one can prove the existence of a local solution for the periodic initial problem (6)-(9). Similarly to Zhou and Guo’s proof [16], letting $D \to \infty$, the existence of a local solution for the initial value problem (6)-(9) can be obtained. By the continuation extension principle, from the conditions of the theorem and a priori estimates in Section 2, we can get the existence of a global generalized solution for problem (6)-(9). By Lemma 2.7 and the Sobolev imbedding theorem, Theorem 3.1 is proved. □

Next, we prove the uniqueness of a solution for problem (6)-(9).

**Theorem 3.2** Suppose that $E_0(x) \in H^{l+4}(\mathbb{R}^2)$, $n_0(x) \in H^{l+2}(\mathbb{R}^2)$, $\varphi_0(x) \in H^{l+2}(\mathbb{R}^2)$, $l \geq 0$. Then the global solution of the initial value problem (6)-(9) is unique.

**Proof** Suppose that there are two solutions $E_1, n_1, \varphi_1$ and $E_2, n_2, \varphi_2$. Let

\[
E = E_1 - E_2, \quad n = n_1 - n_2, \quad \varphi = \varphi_1 - \varphi_2.
\]

From (6)-(9) we get

\[
iE_t + \Delta E - H^2 \Delta^2 E - n_1 E_1 + n_2 E_2 = 0, \tag{27}
\]

\[
n_t - \Delta \varphi = 0, \tag{28}
\]

\[
\varphi_t - n + H^2 \Delta n - |E_1|^2 + |E_2|^2 = 0, \tag{29}
\]

with initial data

\[
E|_{t=0} = 0, \quad n|_{t=0} = 0, \quad \varphi|_{t=0} = 0, \quad x \in \mathbb{R}^2. \tag{30}
\]

Take the inner product of (27) and $E$. Since

\[
\text{Im}(iE_t, E) = \frac{1}{2} \frac{d}{dt} \|E\|_{L^2}^2,
\]

\[
\text{Im}(\Delta E - H^2 \Delta^2 E, E) = 0,
\]

\[
\text{Im}(\Delta E - H^2 \Delta^2 E, E) = 0,
\]
\[ |\text{Im}(n_1E_1 - n_2E_2, E)| \leq |(nE_1 + n_2E, E)| \]
\[ \leq C(\|E_1\|_{L^\infty} \|r\|_{L^2} + \|n_2\|_{L^\infty} \|E\|_{L^2}) \|E\|_{L^2} \]
\[ \leq C(\|n\|_{L^2}^2 + \|E\|_{L^2}^2), \]
thus we obtain
\[
\frac{d}{dt}\|E\|_{L^2}^2 \leq C(\|n\|_{L^2}^2 + \|E\|_{L^2}^2). \tag{31}
\]
Take the inner product of (29) and \( \varphi \). Since
\[
(\varphi, \varphi) = \frac{1}{2} \frac{d}{dt}\|\varphi\|_{L^2}^2, \quad \langle -n, \varphi \rangle \leq C(\|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2),
\]
\[
(H^2 \Delta n, \varphi) = (H^2 n, \Delta \varphi) = (H^2 n, n_t) = \frac{H^2}{2} \frac{d}{dt}\|n\|_{L^2}^2,
\]
\[
|\langle \Delta E_1^2 + |E_1|^2, \varphi \rangle| = \left| \langle (E_1 - E_2)E_1 + E_2(E_1 - E_2), \varphi \rangle \right|
\]
\[
\leq C(\|E_1\|_{L^\infty} \|E\|_{L^2} + \|E_2\|_{L^\infty} \|E\|_{L^2}) \|\varphi\|_{L^2}
\]
\[
\leq C(\|E\|_{L^2}^2 + \|\varphi\|_{L^2}^2),
\]
thus we get
\[
\frac{d}{dt}(\|\varphi\|_{L^2}^2 + H^2\|n\|_{L^2}^2) \leq C(\|E\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2). \tag{32}
\]
Hence from (31) and (32) we get
\[
\frac{d}{dt}(\|E\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2) \leq C(\|E\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2). \tag{33}
\]
By using Gronwall's inequality and noticing (30), we arrive at
\[ E \equiv 0, \quad n \equiv 0, \quad \varphi \equiv 0. \]
Theorem 3.2 is proved. This completes the proof of Theorem 1.1. \( \square \)

4 Results and discussion
One can regard (3)-(4) as the Langmuir turbulence parameterized by \( H \) and study the asymptotic behavior of systems (3)-(4) when \( H \) goes to zero.

5 Conclusions
By a priori integral estimates and the Galerkin method, we have the following conclusion. Suppose that \( E_0(x) \in H^{l+4}(\mathbb{R}^2), n_0(x) \in H^{l+2}(\mathbb{R}^2), n_1(x) \in H^{l}(\mathbb{R}^2), \) \( l \geq 0. \) Then there exists a unique global smooth solution of the initial value problem (3)-(5).
\[
E(x, t) \in L^\infty(0, T; H^{l+4}(\mathbb{R}^2)), \quad E_1(x, t) \in L^\infty(0, T; H^{l}(\mathbb{R}^2))
\]
\[
n(x, t) \in L^\infty(0, T; H^{l+2}(\mathbb{R}^2)), \quad n_1(x, t) \in L^\infty(0, T; H^{l}(\mathbb{R}^2))
\]
\[
n_n(x, t) \in L^\infty(0, T; H^{l-2}(\mathbb{R}^2)).
\]
Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SY carried out the existence studies and drafted the manuscript. XN carried out the uniqueness of the solution and helped to draft the manuscript. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the National Natural Science Foundation of China (Grant No. 11501232), Research Foundation of Education Bureau of Hunan Province (Grant Nos. 15B185 and 16C1272) and Scientific Research Fund of Huaihua University (Grant No. HHUY2015-05) for the support.

Received: 29 September 2016 Accepted: 25 January 2017 Published online: 01 February 2017

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