A GROUP ACTION ON HIGHER CHOW CYCLES ON A KUMMER SURFACE FAMILY

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Abstract. We construct higher Chow cycles on a 2-dimensional family $X^o \rightarrow T^o$ of Kummer surfaces and calculate their value under the transcendental regulator map. For the calculation, we use a finite group action on $X^o \rightarrow T^o$. As a result, we show that the rank of indecomposable cycles of $X_t$ is equal to or greater than 18 for very general $t \in T^o(\mathbb{C})$.

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1. Introduction

1.1. Contents of paper. In the celebrated paper [Blo86], Bloch defined higher Chow group $CH^p(X, q)$ for a variety $X$ over $k$. Higher Chow group is a natural generalization of Chow group. For a closed subvariety $Z \subset X$ of codimension $c$, the localization exact sequence of Chow group

$$\cdots \rightarrow CH^p(X, 1) \rightarrow CH^p(X - Z, 1) \rightarrow CH^{p-c}(Z) \rightarrow CH^p(X) \rightarrow CH^p(X - Z) \rightarrow 0$$

fits into the localization exact sequence of higher Chow group

$$\cdots \rightarrow CH^p(X, 1) \rightarrow CH^p(X - Z, 1) \rightarrow CH^{p-c}(Z) \rightarrow CH^p(X) \rightarrow CH^p(X - Z) \rightarrow 0$$

Thus higher Chow group is the analogy of the singular cohomology theory for algebraic varieties. Furthermore, there exists the canonical isomorphism

$$CH^p(X, q) \otimes \mathbb{Q} \cong H^{2p-q}_{\text{Mot}}(X, \mathbb{Q}(p))$$

where $H^{2p-q}_{\text{Mot}}(X, \mathbb{Q}(p))$ is the motivic cohomology of $X$. Motivic cohomology is an extension group in the triangulated category of mixed motives. Motivic cohomology and higher Chow groups appear in many aspects of algebraic geometry and number theory. However, its structure is still mysterious for many varieties.

For a surface $X$, elements in $CH^2(X, 1)$ are constructed geometrically (cf. [GL99], [Müll97], [CDKL16], [dAM02], [Col97] and [Asa16]). In this paper, we focus on the higher Chow group of a certain family of $K3$ surfaces, which are regarded as 2-dimensional analogue of elliptic curves. Higher Chow groups of general $K3$ surfaces are studied in [CDKL16]. In this paper, we will study the higher Chow group of a certain family of Kummer surfaces in detail.

We give an estimation of the rank of $CH^2(X, 1)$, which is one of basic problems. To study this, we construct elements in $CH^2(X, 1)$ explicitly, and consider their images under the following regulator map defined by Beilinson.

$$H^3_{\text{Mot}}(X, \mathbb{Q}(2)) \rightarrow H^3_D(X, \mathbb{Q}(2))$$

Here the target $H^3_D(X, \mathbb{Q}(2))$ is the Deligne cohomology. Among the cycles in $CH^2(X, 1)$, we are interested in indecomposable cycles. The space of indecomposable cycles is defined as the cokernel of the following group homomorphism

$$CH^1(X, 1) \otimes \mathbb{Q} \rightarrow CH^1(X) \rightarrow CH^2(X, 1)$$

induced by the intersection product (cf. Definition A.2). Since $CH^1(X) \simeq \text{Div}(X)$ and $CH^1(X, 1) \simeq \Gamma(X, O_X)$, the image of (4) can be written by the known invariants. Hence to study the structure of $CH^2(X, 1)$, it is important to study indecomposable cycles. In this paper, we focus on the indecomposable cycles.

In the articles [GL99], [Müll97], [CDKL16], [dAM02], [Col97] and [Asa16], they consider families of variety $\{X_t\}_{t \in T}$ (or deformation of the special variety) and construct a family of higher Chow cycles $\{\xi_t\}_{t \in T}$ (or deformation of the special cycles on the special variety). Then by studying the behavior of the value of the regulator map as a function of $t$, they show that $\xi_t$ does not vanish for very general $t \in T$. We follow this strategy.

In this paper, we consider a 2-dimensional family $\mathcal{X} \rightarrow T$ of Kummer surfaces associated with products of elliptic curves, which is constructed in subsection 3.2. We construct families of higher Chow cycle $\Xi \subset CH^2(\mathcal{X}, 1)$ and we compute their

---

1We use the word “very general” for the meaning that “outside of the countable union of proper(= not the whole space) analytic subsets”.

---
image under the following transcendental regulator map at fibers.

\[
\begin{align*}
    r : \text{CH}^2(X_t, 1) & \longrightarrow H^3_D(X_t^{an}, \mathbb{Z}(2)) \\
    & \longrightarrow (H^{2,0}(X_t^{an}))^\vee / H_2(X_t^{an}, \mathbb{Z}) \\
    \text{CH}^2(X_t, 1)_{\text{ind}} & \longrightarrow
\end{align*}
\]

The transcendental regulator map factors \(\text{CH}^2(X_t, 1)_{\text{ind}}\) which is the cokernel of the map \([4]\), namely, the space of indecomposable cycles. Thus we can use the transcendental regulator map for the rank estimate for indecomposable cycles. The main theorem of this paper is as follows.

**Theorem 1.1.** (Theorem 9.22) For a very general \(t \in T^\circ(\mathbb{C})\),

\[
    \text{rank } r(\Xi_t) = 18.
\]

where \(\Xi_t\) is the restriction of \(\Xi\) at the fiber \(X_t\). Especially, \(\text{rank } \text{CH}^2(X_t, 1)_{\text{ind}} \geq 18\).

By Proposition 1.1 in [Müll97], the image of the transcendental regulator map \(r\) is known to be at most countable for smooth projective surface \(X\) over \(\mathbb{C}\). Furthermore, since \(X^\circ \to T^\circ\) is a certain base change of the Kummer surface family treated in Section 6 of [CDKL16], \(\text{CH}^2(X_t, 1)_{\text{ind}} \neq 0\) was already known for very general \(t\). However, the explicit estimation for the rank of indecomposable cycles such as Theorem 1.1 is not known so far.

For the computation of the image of transcendental regulator, we use the formula obtained by Levine ([Lev88]) and construct topological chain on \(X_t^{an}\) explicitly (Section 8). By the Levine’s formula, we show that the following multivalued holomorphic function appears as an image of an element of \(\Xi\) under the transcendental regulator map (Theorem 8.16).

\[
    \int_{\Delta} \frac{dx dy}{\sqrt{x(1-x)(1-ax)\sqrt{y(1-y)(1-by)}}}
\]

(7)

Here \(\Delta = \{(x, y) \in \mathbb{R}^2 : 0 < y < x < 1\}\). This integral is similar to the representation of Appell’s hypergeometric function. The difference is that the boundary of the domain of integral is not necessarily contained in the branching locus of the integrand. In other words, (7) is a kind of incomplete integral.

The famous Beilinson’s conjecture predicts that if \(X\) is defined over number field, the value (in a suitable sense) of the regulator map (3) is related to the special value of \(L\)-function of motives of \(X\). Hence it is an important problem what kind of functions appear as the image of the regulator map.

Recently, in [AO18], Asakura and Otsubo gives examples of special varieties (which have hypergeometric fibrations) whose values of the regulator maps are represented by the value at \(z = 1\) of generalized hypergeometric function \(_3F_2\). Furthermore, by deforming such varieties, they give a 1-dimensional family of varieties such that the value of regulator map of members of such family is represented by generalized hypergeometric function \(_3F_2\) ([AO21]). Since our result is for 2-dimensional families, the function (7) seems to give a clue for the question “What higher dimensional generalized hypergeometric function should be?”

To compute the value of transcendental regulator for each element in \(\Xi\) is somewhat hard work. Hence we use automorphisms of the Kummer surface family. We consider the following automorphisms of a family of algebraic varieties.

**Definition 1.2.** Let \(\mathcal{X} \to T\) be a family of algebraic varieties over a field \(k\). An automorphism group \(\text{Aut}_k(\mathcal{X} \to T)\) of \(\mathcal{X} \to T\) consists of a pair \((g, g')\) with
g ∈ Aut_k(𝑋) and ̄g ∈ Aut_k(T) such that the following diagram commutes.

\[
\begin{array}{ccc}
𝑋 & \xrightarrow{g} & 𝑋 \\
\downarrow & & \downarrow \\
T & \xrightarrow{̄g} & T
\end{array}
\]  

(8)

In this paper, we construct the following finite group action on the Kummer surface family \( X^o \to T^o \).

**Proposition 1.3.** (Proposition 1.29) We have a finite subgroup \( ̂G \) of \( \text{Aut}_k(X^o \to T^o) \) which is isomorphic to a \( \mathbb{Z}/2\mathbb{Z} \) extension of \( G = H^2 = (H_0 \times \mathbb{Q}, H_1^2) \cong (\mathbb{S}_4 \times \mathbb{S}_3, \mathbb{S}_4) \).

Then we construct a standard subgroup \( Ξ_c \subset CH^2(X^o, 1) \) and show that \( Ξ \) is represented as the sum of \( ̂ρ, Ξ_c \) (\( ̂ρ \in ̂G \)). The higher Chow cycles in \( Ξ_c \) is constructed geometrically after Tomohide Terasoma’s idea. He also gave the author the idea of the geometric construction of higher Chow cycles in \( Ξ \).

We compute the image of the regulator map by using \( ̂G \)-action as follows: since \( Ξ \) is constructed as a family over \( T^o \), we consider “relative transcendental regulator map” \( R_ω \) (Definition 9.16)

\[ R_ω : Ξ \longrightarrow \mathcal{Q}_ω(T^o) \]

(9)

where \( \mathcal{Q}_ω \) is a sheaf on \((T^o)^\an\) such that restriction of \( \mathcal{Q}_ω \) at \( t \in T^o(\mathbb{C}) \) is isomorphic to \((H^2(\mathcal{X}^o_t))^{\text{V}}/H_2(\mathcal{X}^o_t, \mathbb{Q})\). The “relative transcendental regulator” means that the image of \( R_ω(Ξ) \) at \( t \in T^o(\mathbb{C}) \) coincides with that of transcendental regulator \( r(Ξ) \mod torsion \) part. This relative transcendental regulator map associates higher Chow cycle families to (a generalization of) normal functions. Though this kind of map can be defined in more general setting (cf. [San02]), we employ the ad hoc definition since we need only the explicit description for special cases. We define \( ̂G \)-action on \( \mathcal{Q}_ω \) so that \( R_ω \) is equivariant under this action. Hence we reduce the computation of \( r(Ξ) \) to that of \( R_ω(Ξ) \) with the \( ̂G \)-action on \( \mathcal{Q}_ω(T^o) \).

To study the image \( R_ω(Ξ) \) in the sheaf \( \mathcal{Q}_ω \), we consider the Picard-Fuchs differential operator

\[ \mathcal{D} : \mathcal{Q}_ω(T^o) \longrightarrow \mathcal{O}_{(T^o)^\an}(T^o)^{\mathbb{Q}/2} \]

(10)

as in the previous researches ([Mig97, JAM02] and [CDKL16]). We also define a \( ̂G \)-action on \( \mathcal{O}_{(T^o)^\an}(T^o)^{\mathbb{Q}/2} \) so that \( \mathcal{D} \) is \( ̂G \)-equivariant. Using a simple description of \( \mathcal{D} \circ R_ω(Ξ^{an}) \) (Proposition 9.21), we show that \( \mathcal{D} \circ R_ω(Ξ) \) has 18 \( \mathbb{Q} \)-linearly independent elements (Table 8), and get the Theorem 1.4.

Note that the order of \( ̂G \) is 18, 432 = 2^{11} \cdot 3^2 and \( r(Ξ_t^{an}) = 3 \) for very general \( t \in T^o(\mathbb{C}) \) (cf. Proposition 9.21). Hence the large subgroup of \( ̂G \) stabilizes the standard subgroup \( Ξ^{an} \) (not necessarily act trivially on \( Ξ^{an} \)). We will construct a subgroup \( ̂I \) which is contained in the stabilizer of \( Ξ^{an} \) (Section 6). We prove the following.

**Theorem 1.4.** (Definition 6.3, Theorem 6.10) The stabilizer of \( Ξ^{an} \subset CH^2(X^o, 1) \) contains a subgroup \( ̂I \subset ̂G \) which is isomorphic to the direct product of \( \mathbb{Z}/2\mathbb{Z} \) and \( I \simeq \mathbb{S}_4 \times \mathbb{S}_3, \mathbb{S}_4 \).

Furthermore, we have an explicit description of the \( ̂I \)-action on \( Ξ^{an} \) using a group cocycle \( δ \) defined in Proposition 6.9.

\( ̂I \) contains a subgroup \( I \) which is isomorphic to a Galois group of some field extension of the rational function field of the base scheme \( T^o \) (Remark 6.5). Since
I acts on the generic fiber of $\mathcal{X}^\circ \to T^\circ$. Theorem 1.4 can be regarded as the result about Galois action on (higher) algebraic cycles on the generic fiber of $\mathcal{X}^\circ \to T^\circ$.

1.2. **Outline of this paper.** This paper is divided into 3 parts.

Part 1 consists of Section 2, Section 3 and Section 4. The purpose of Part 1 is to fix the notation and to prove Proposition 1.3. In Section 2, we introduce a category $(\text{Sch}^{gp} / k)$, which is used to consider multiple finite group actions on multiple schemes simultaneously. In Section 3, we construct the Kummer surface family $\mathcal{X} \to T$. In Section 4, we prove Proposition 1.3 by using techniques in Section 2.

Part 2 consists of Section 5 and Section 6. The purpose of Part 2 is to explain the construction of $\Xi \subset \text{CH}^2(\mathcal{X}^\circ, 1)$ and considering $\tilde{G}$-action on $\Xi$. In Section 5, we construct basic higher Chow cycle families in $\Xi^{\text{can}} \subset \text{CH}^2(\mathcal{X}^\circ, 1)$ and define $\Xi$ as its image by $\tilde{G}$-action. In Section 6, we prove Theorem 1.4.

The purpose of Part 3 is to prove Theorem 1.1. Part 3 consists of Section 7, Section 8 and Section 9. In Section 7, we fix relative differential forms $\omega$ on $\mathcal{X} \to T$ and examine $\tilde{G}$-action on $\omega$. Furthermore, we find Picard-Fuchs differential operator $D$ which annihilates period functions. In Section 8, we calculate an element of $\Xi^{\text{can}}_t$ under the transcendental regulator map. In Section 9, we define relative transcendental regulator map $R_\omega$ in (9) and prove $\tilde{G}$-equivariance of $D$ and $R_\omega$. Finally, we prove Theorem 1.1.

In Appendix A, we recall the definition of decomposable part of higher Chow groups and how such cycles are represented by elements of the homology group of Gersten complex (cf. Proposition 5.1).

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1.4. **Conventions.**

1.4.1. **Conventions for algebraic geometry.**

1. For a field $k$, $(\text{Sch} / k)$ denotes the category of $k$-schemes. A **variety over $k$** is an integral separated scheme of finite type over $k$. For a variety $X$, the **function field** $R(X)$ is its residue field at the generic point.

2. A geometric point of scheme $X$ is a morphism from the spectrum of algebraically closed field to $X$. For geometric point $\overline{\pi}$ of $X$, $\kappa(\overline{\pi})$ denotes the algebraically closed field of the domain of $\overline{\pi}$. When $s$ is a point (resp. geometric point) in $S$, we usually denote the fiber (resp. geometric fiber) over $s$ by $X_s$.

3. For $S$-schemes $Y$ and $X$, $\text{Hom}_S(Y, X)$ denotes the set of $S$-morphisms. $\text{Aut}_S(X)$ denotes the group of $S$-automorphisms on $X$. For $\varphi \in \text{Hom}_S(Y, X)$, $\varphi^S$ denotes the morphism of sheaves of rings $\varphi^S : O_X \to \varphi^* O_X$. If $Y = \text{Spec} R$, elements in $\text{Hom}_S(Y, X)$ is called $R$-rational points. For any morphism $S' \to S$. We have a natural map $\text{Hom}_S(Y, X) \to \text{Hom}_S(Y \times_S S', X \times_S S')$. For a subset $\Sigma$ of $\text{Hom}_S(T, X)$, the image of $\Sigma$ under this map is called the **base change of $\Sigma$ by $S' \to S$**.
(4) For a finite disjoint family \( \{Y_\lambda\}_{\lambda \in \Lambda} \) of closed subschemes of \( X \), \( \bigsqcup_{\lambda \in \Lambda} Y_\lambda \) denotes the closed subscheme corresponding to the ideal sheaf \( \bigcap_{\lambda \in \Lambda} I_{Y_\lambda} \), where \( I_{Y_\lambda} \) is the ideal sheaf corresponding to \( Y_\lambda \). When \( \{Y_\lambda\}_{\lambda \in \Lambda} = \{Y_1, Y_2\} \), we often use the notation \( Y_1 \sqcup Y_2 \) for \( \bigsqcup_{\lambda \in \Lambda} Y_\lambda \).

1.4.2. Conventions for group theory.

(1) Let \( G \) be a group and \( M \) be an abelian group with a \( G \)-action. In this paper, we always consider a left group action. For a subgroup \( N \subset M \), the \( G \)-action of \( M \) stabilizes \( N \) if and only if for any \( g \in G \) and \( n \in N \), we have \( g \cdot n \in N \).

(2) For a set \( \Sigma \), \( S(\Sigma) \) denotes the symmetric group of \( \Sigma \). For \( n \in \mathbb{Z}_{\geq 1} \), \( S_n \) denotes the symmetric group of \( \{0, 1, \ldots, n-1\} \). For \( \sigma \in S(\Sigma) \), \( \text{sgn}(\sigma) \in \{\pm 1\} \) denotes its image by the sign character of \( S(\Sigma) \).

(3) For a set \( A \) and an abelian group \( M \), \( \text{Map}(A, M) \) or \( M^A \) denotes the set of maps from \( A \) to \( M \). \( \text{Map}(A, M) = M^A \) has a natural structure of an abelian group.

1.4.3. Others.

(1) For a set \( \Sigma \), \( |\Sigma| \) denotes the cardinality of \( \Sigma \).

(2) For a ring \( A \), the multiplicative group of \( A \) is denoted by \( A^\times \). If \( A \) is an integral domain, its fraction field is denoted by \( \text{Frac}(A) \).

(3) For \( n \in \mathbb{Z}_{>1} \) and a field \( k \), \( \mu_n(k) \) denotes the subgroup of \( k^\times \) consisting of \( n \)-th roots of unity.

(4) We use the symbol \( \sqcup \) for the fiber product as follows.

\[
\begin{array}{ccc}
X \times_S Y & \xrightarrow{pr_2} & Y \\
pr_1 \downarrow & & \downarrow \\
X & \xrightarrow{\_} & S
\end{array}
\]  

(11)

2. Generalities of Discrete Group Actions on Schemes

In this section, we introduce a category \( (\text{Sch}^{gp}/k) \) of schemes with group actions and prove some properties which we use in Section 4 to construct group actions on family of algebraic varieties.

All results in this section is more or less formal and proofs are often straightforward. Hence we give only sketches of proofs or completely omit proofs. We recommend readers only to check fundamental definitions (Definition 2.1, Definition 2.4, and Definition 2.6) and go back to rest propositions at the time when they are necessary.

For simplicity, throughout in this section, we fix a field \( k \) and all schemes and morphisms are defined over \( k \).

2.1. Schemes with group actions.

Definition 2.1. (The definition of \( (\text{Sch}^{gp}/k) \))

(1) A scheme with a group action \( (S, H, \varphi) \) is a triplet consist of a \( k \)-scheme \( S \), a group \( H \) and a group homomorphism \( \varphi : H \rightarrow \text{Aut}_k(S) \).

(2) If the group homomorphism is clear from the context, we usually omit \( \varphi \) from the notation and write \( (S, H) \). In that case, we use the same symbol for \( h \in H \) and its image in \( \text{Aut}_k(S) \).

(3) A pair \( (f, \psi) \) of a morphism of \( k \)-scheme \( f : T \rightarrow S \) and a group homomorphism \( \psi : G \rightarrow H \) is called a morphism of schemes with group actions from
(T, G) to (S, H) if the following diagram commutes for any \( g \in G \).

\[
\begin{array}{ccc}
T & \xrightarrow{f} & S \\
\downarrow{g} & & \downarrow{\psi(g)} \\
T & \xrightarrow{f} & S
\end{array}
\]

(12)

Then we have a category \((\text{Sch}_{\text{gp}}/k)\) of schemes with group actions by the natural composition of morphisms. We have a natural forgetful functor \((\text{Sch}_{\text{gp}}/k) \to (\text{Sch}/k)\).

(4) Let \((S, H) \in (\text{Sch}_{\text{gp}}/k)\). For a \(S\)-scheme \(X\), we define \(\text{Aut}(X; S, H)\) as a following group.

\[
\text{Aut}(X; S, H) = \left\{ (\mu, \nu) \in \text{Aut}_k(X) \times H : \begin{array}{c} \mu \quad \text{commutes.} \\ Y \rightarrow S \end{array} \right\}
\]

(13)

By the natural group homomorphism \(\text{Aut}(X; S, H) \to \text{Aut}_k(X)\) and \(\text{Aut}(X; S, H) \to H\), we have the following morphism and object in \((\text{Sch}_{\text{gp}}/k)\).

\(
(X, \text{Aut}(X; S, H)) \longrightarrow (S, H)
\)

(14)

(5) A scheme with a group action \((X, G)\) is called faithful if the group homomorphism \(G \to \text{Aut}_k(X)\) is injective.

First, we confirm the relation between morphisms \((X, G) \to (S, H)\) in \((\text{Sch}_{\text{gp}}/k)\) and subgroups of \(\text{Aut}_k(X \to S)\) (See Definition 1.2) as follows.

**Proposition 2.2.** Let \((f, \varphi) : (X, G) \to (S, H)\) be a morphism in \((\text{Sch}_{\text{gp}}/k)\). Then we have the group homomorphism

\[
G \longrightarrow \text{Aut}_k(X \to S); g \mapsto (g, \varphi(g)).
\]

(15)

If \((X, G)\) is a faithful, this group homomorphism is injective.

In this paper, we often use the following fiber product construction in \((\text{Sch}_{\text{gp}}/k)\). The proof is straightforward.

**Proposition 2.3.** Given the following diagram in \((\text{Sch}_{\text{gp}}/k)\).

\[
\begin{array}{ccc}
(S_1, H_1) & \xrightarrow{(f_1, \varphi_1)} & (S_2, H_2) \\
\downarrow{(f_2, \varphi_2)} & & \downarrow{(f_3, \varphi_3)} \\
(S_3, H_3)
\end{array}
\]

(16)

Then the fiber product of \((S_1, H_1) \times_{(S_2, H_2)} (S_2, H_2)\) exists and isomorphic to \((S_1 \times_{S_3} S_2, H_1 \times_{H_3} H_2)\). Here we have

\[
H_1 \times_{H_3} H_2 = \{(h_1, h_2) \in H_1 \times H_2 : \varphi_1(h_1) = \varphi_3(h_2)\}
\]

(17)

For a morphism \((X, G) \to (S, H)\) in \((\text{Sch}_{\text{gp}}/k)\) we introduce the concept of “compatibility” for a subset of sections.

**Definition 2.4.**

(1) Let \((X, G) \to (S, H)\) be a morphism in \((\text{Sch}_{\text{gp}}/k)\). For \(g \in G\), \(\bar{g}\) denotes its image in \(H\). A subset \(\Sigma\) of \(\text{Hom}_k(S, X)\) is compatible with respect to \((X, G) \to (S, H)\) if and only if for any \(\sigma \in \Sigma\) and \(g \in G\), \(g \circ \sigma \circ g^{-1} \in \Sigma\).

(2) If \(\Sigma\) is compatible with respect to \((X, G) \to (S, H)\), we have a \(G\)-action on \(\Sigma\) defined by

\[
G \times \Sigma \longrightarrow \Sigma \\
(g, \sigma) \longmapsto g \circ \sigma \circ g^{-1}
\]

(18)
We can keep track this action on $\Sigma$ after fiber product operations.

**Proposition 2.5.** We have the following properties.

1. Given the following diagram in $(\text{Sch}^{gp}/k)$.

   \[
   \begin{array}{ccc}
   (X_1, G_1) & \longrightarrow & (X_3, G_3) \\
   \downarrow & & \downarrow \\
   (S_1, H_1) & \longrightarrow & (S_3, H_3)
   \end{array}
   \]  
   \[
   \tag{19}
   \]

   Suppose $\Sigma_i \subset \text{Hom}_{S_i}(S_i, X_i)$ is compatible with respect to $(X_i, G_i) \rightarrow (S_i, H_i)$ for $i = 1, 2$. Put $(S, H) = (S_1, H_1) \times_{(S_3, H_3)} (S_2, H_2)$ and $(X, G) = (X_1, G_1) \times_{(S_3, H_3)} (X_2, G_2)$. Then the following morphism is induced.

   \[
   \begin{array}{ccc}
   (X, G) & \longrightarrow & (X_1, G_1) \\
   \downarrow & & \downarrow \\
   (S, H) & \longrightarrow & (S_1, H_1)
   \end{array}
   \]  
   \[
   \tag{20}
   \]

   Consider a map

   \[
   \text{Hom}_{S_1}(S_1, X_1) \times \text{Hom}_{S_2}(S_2, X_2) \longrightarrow \text{Hom}_S(S, X)
   \]  
   \[
   \tag{21}
   \]

   Let $\Sigma$ be the image of $\Sigma_1 \times \Sigma_2$ under this map. Then $\Sigma$ is compatible with respect to $(X, G) \rightarrow (S, H)$.

2. Let $f : \Sigma_1 \times \Sigma_2 \rightarrow \Sigma$ be the surjective map induced by (21). For any $(\mu_1, \mu_2) \in G_1 \times_{H_1} G_2$ and $(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$, we have the following.

   \[
   (\mu_1, \mu_2) \cdot f(\sigma_1, \sigma_2) = f(\mu_1 \cdot \sigma_1, \mu_2 \cdot \sigma_2)
   \]  
   \[
   \tag{22}
   \]

   Here the action of $G_1, G_2$ and $G_1 \times_{H_1} G_2$ on $\Sigma_1, \Sigma_2$ and $\Sigma$ is induced by \[18\] in Definition 2.4. In other words, $f : \Sigma_1 \times \Sigma_2 \rightarrow \Sigma$ is $G_1 \times_{H_1} G_2$-equivariant.

Especially, the base change of compatible section sets are compatible again. More precisely, as follows: Consider the following fiber product in $(\text{Sch}^{gp}/k)$.

\[
\begin{array}{ccc}
(X', G') & \longrightarrow & (X, G) \\
\downarrow & & \downarrow \\
(S', H') & \longrightarrow & (S, H)
\end{array}
\]  
\[
\tag{23}
\]

Assume $\Sigma \subset \text{Hom}_S(S, X)$ is compatible respect to $(X, G) \rightarrow (S, H)$. Then the base change $\Sigma' \subset \text{Hom}_S(S', X')$ of $\Sigma$ is compatible with respect to $(X', G') \rightarrow (S', H')$.

Furthermore, the natural map $f : \Sigma \rightarrow \Sigma'$ induced by the base change is $G'$-equivariant. i.e. For any $\sigma \in \Sigma$ and $(\mu, \nu) \in G \times_{H'} G'$, we have

\[
f(\mu \cdot \sigma) = (\mu, \nu) \cdot f(\sigma).
\]  
\[
\tag{24}
\]

### 2.2. Linearization of $\mathcal{O}_X$-modules.

We recall the definition of $G$-linearization of $\mathcal{O}_X$-module especially when $G$ is a discrete group scheme over $k$. In some references, $\mathcal{O}_X$-module with a $G$-linearization is called $G$-equivariant sheaf.
Definition 2.6. Let \((X, G) \in \text{Sch}^{\text{gp}}/k\) and \(\mathcal{L}\) be an \(\mathcal{O}_X\)-module. A \(G\)-linearization of \(\mathcal{L}\) is a collection of \(\mathcal{O}_X\)-module isomorphisms \(\{\Phi_g : g^* \mathcal{L} \sim \mathcal{L}\}_{g \in G}\) such that for any \(g, h \in G\), the following diagram commutes.

\[
\begin{array}{ccc}
(g \circ h)^* \mathcal{L} & \xrightarrow{c_{g,h}} & h^* g^* \mathcal{L} \\
\downarrow_{\phi_{g,h}} & & \downarrow^{h*(\phi_g)} \\
\mathcal{L} & \leftarrow & h^* \mathcal{L}
\end{array}
\]  

(25)

where \(c_{g,h}\) is the natural isomorphism. The commutativity of (25) is called cocycle condition.

Sheaves of relative differentials are the most fundamental example of linearized sheaves.

Proposition 2.7. Let \((f, \varphi) : (X, G) \to (S, H)\) be a morphism in \(\text{Sch}^{\text{gp}}/k\). We have a canonical \(G\)-linearization \(\{\Phi_g\}_{g \in G}\) of the sheaf of differentials \(\Omega^1_{X/S}\).

Proof. For \(g \in G\), we have the following diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
\downarrow^f & & \downarrow^f \\
S & \xrightarrow{\varphi(g)} & S
\end{array}
\]

(26)

Then by the universal property of the differential sheaf, we have the unique \(\mathcal{O}_X\)-module homomorphism \(\eta_g : g^! \Omega^1_{X/S} \to \Omega^1_{X/S}\). By the universality, this satisfies the cocycle condition. 

We list constructions of new linearized sheaves from other linearized sheaves. All proofs are formal, hence we omit the proofs.

Proposition 2.8. Let \((f, \varphi) : (X, G) \to (S, H)\) be a morphism of scheme with a group. Let \(\mathcal{L}\) be an \(\mathcal{O}_S\)-module and \(\{\Phi_h\}_{h \in H}\) be an \(H\)-linearization of \(\mathcal{L}\). For \(g \in G\), put

\[
f^* \Phi_{\varphi(g)} : g^*(f^* \mathcal{L}) \xrightarrow{c_{f,g}} (f \circ g)^* \mathcal{L} = (\varphi(g) \circ f)^* \mathcal{L} \xrightarrow{c_{\varphi(g),f}} f^* \varphi(g)^* \mathcal{L} \xrightarrow{f^* \Phi_{\varphi(g)}} f^* \mathcal{L}.
\]

Then \(\{f^* \Phi_{\varphi(g)}\}_{g \in G}\) is a \(G\)-linearization of \(f^* \mathcal{L}\).

Proposition 2.9. Let \((X, G)\) be a scheme with a group. Let \(\mathcal{L}, \mathcal{M}\) be \(\mathcal{O}_X\)-modules and \(\{\Phi_g\}_{g \in G}\) be a \(G\)-linearization of \(\mathcal{L}\) and \(\{\Psi_g\}_{g \in G}\) be a \(G\)-linearization of \(\mathcal{M}\).

(1) For \(g \in G\), put

\[
\Phi_g \otimes \Psi_g : g^*(\mathcal{L} \otimes \mathcal{O}_X \mathcal{M}) \simeq g^* \mathcal{L} \otimes g^* \mathcal{M} \xrightarrow{\Phi_g \otimes \Psi_g} \mathcal{L} \otimes \mathcal{O}_X \mathcal{M} \quad (28)
\]

Here, the first isomorphism is the canonical one. Then \(\{\Phi_g \otimes \Psi_g\}_{g \in G}\) is a \(G\)-linearization on \(\mathcal{L} \otimes \mathcal{O}_X \mathcal{M}\).

(2) Assume that \(\mathcal{L}\) is a locally free \(\mathcal{O}_X\)-module of finite rank. For \(g \in G\), put

\[
\Phi^\vee_g : g^* \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \simeq \text{Hom}_{\mathcal{O}_X}(g^* \mathcal{L}, \mathcal{O}_X) \xrightarrow{(\Phi_g)^\vee} \text{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \quad (29)
\]

The first isomorphism is the canonical one. Then \(\{\Phi^\vee_g\}_{g \in G}\) is a \(G\)-linearization of \(\mathcal{L}^\vee\). If \(\mathcal{L}\) is an invertible sheaf, we use the notation \(\{\Phi^\vee_g\}_{g \in G}\).

The group cocycles have close relations with sheaves with linearizations. In this paper, explicit cocycle calculations play a technically important role for the main result.
Definition 2.10. For a group $G$ and an abelian group $M$, an opposite $G$-action is a $G^{\text{op}}$-action\footnote{$G^{\text{op}}$ denotes the opposite group of $G$.} on $M$ which preserves the abelian group structure of $M$.

For an opposite $G$-action on $M$, an opposite 1-cocycle is a 1-cocycle of $G^{\text{op}}$ on $M$. In other words, for an opposite action $p : G^{\text{op}} \to \text{Aut}(M)$, opposite 1-cocycle is a map $\chi : G \to M$ which satisfies the following: For any $g, h \in G$,

$$\chi(gh) = \chi(h) + p(h)(\chi(g)).$$

(30)

Note that for any opposite 1-cocycle $\chi_1, \chi_2 : G \to M$, $\chi_1 + \chi_2$ and $-\chi_1$ are also opposite 1-cocycles.

Let $(X, G) \in (\text{Sch}^{\text{sp}}/k)$. We have a natural opposite $G$-action on the $k$-algebra $\Gamma(X, \mathcal{O}_X)$ defined by

$$G \times \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X); (g, a) \mapsto g^\ast(a).$$

(31)

We have an opposite $G$-action on the abelian group $\Gamma(X, \mathcal{O}_X^\times)$. If $X$ is an integral scheme, by the similar method, we have an opposite $G$-action on $R(X)^\times$.

We can get opposite 1-cocycles from linearizations of invertible sheaves and sections.

Proposition 2.11. Let $(X, G) \in (\text{Sch}^{\text{sp}}/k)$ where $X$ is an integral scheme. Let $\mathcal{L}$ be an invertible sheaf, $\{\Phi_g\}_{g \in G}$ be a $G$-linearization on $\mathcal{L}$ and $\eta$ be a non-zero rational section. For $g \in G$, we define $\phi(g) \in R(X)^\times$ by

$$\Phi_g(g^\ast(\eta)) = \phi(g)^{-1} \cdot \eta$$

(32)

Then $\phi : G \to R(X)^\times$ is an opposite $G$-cocycle. We say $\phi$ is the opposite 1-cocycle associated with $(\mathcal{L}, \{\Phi_g\}_{g \in G}, \eta)$.

Furthermore, if we take another rational section $\eta' = f\eta$ ($f \in R(X)^\times$), opposite 1-cocycle $\phi$ changes by the coboundary 1-cocycle associated with $f$.

Proof. By the cocycle condition of $\{\Phi_g\}_{g \in G}$, we see that $\phi$ in (32) defines an opposite 1-cocycle. If we construct another opposite 1-cocycle $\phi'$ from non-zero rational section $\eta' = f\eta$, we have

$$\phi'(g) \cdot \eta' = \Phi_g(g^\ast(\eta')) = \Phi_g(g^\ast(f^\ast(\eta))) = g^\ast(f)\phi(g^{-1}) \cdot \eta$$

(33)

Hence $\phi'(g)^{-1} = \phi(g) \cdot \frac{f}{g^\ast(f)}$, we have the latter statement. \hfill $\square$

Furthermore, if we are given an opposite 1-cocycle on $\Gamma(X, \mathcal{O}_X^\times)$, we get a new linearization as follows. The proof is straightforward.

Proposition 2.12. Let $(X, G) \in (\text{Sch}^{\text{sp}}/k)$. Let $\mathcal{L}$ be an invertible sheaf, $\{\Phi_g\}_{g \in G}$ be a $G$-linearization on $\mathcal{L}$. Let $\chi : G \to \Gamma(X, \mathcal{O}_X^\times)$ be an opposite 1-cocycle. Then we can define another $G$-linearization $\{X_g\}_{g \in G}$ on $\mathcal{L}$ by

$$X_g : g^\ast \mathcal{L} \xrightarrow{\Phi_g} \mathcal{L} \xrightarrow{\chi(g)^{-1}} \mathcal{L}$$

(34)

where the second morphism is induced by the multiplication of $\chi(g) \in \Gamma(X, \mathcal{O}_X^\times)$.

2.3. Lifting of group actions to cyclic coverings and blowing-ups. Finally, we prove the liftability of group actions to a cyclic covering and a blowing-up. We recall the construction of $m$-uple covering.

Definition 2.13. Let $X$ be a scheme and $m \in \mathbb{Z}_{\geq 1}$. Let $\mathcal{L}$ be an invertible sheaf on $X$ and $h \in \Gamma(X, \mathcal{L}^\otimes(-m))$. We have an $\mathcal{O}_X$-algebra structure on $\bigoplus_{i=0}^{m-1} \mathcal{L}$ by
the following rule: For an open subset $U \subset X$, $x \in \mathcal{L}^{\otimes i}(U)$ and $y \in \mathcal{L}^{\otimes j}(U)$ where $i, j \in \{0, 1, \ldots, m - 1\}$, we define

$$x \cdot y = \begin{cases} 
    x \otimes y \in \mathcal{L}^{\otimes(i+j)}(U) & (i + j < m) \\
    x \otimes y \otimes h|_U \in \mathcal{L}^{\otimes(i+j-m)}(U) & (i + j \geq m)
\end{cases}$$

(35)

We extend this multiplication rule $\sigma_X$-bilinearly. Note that commutativity and associativity follows from that $\mathcal{L}$ is an invertible sheaf. Then $m$-uple covering associated with $(\mathcal{L}, h)$ is defined by

$$\text{Spec } \bigoplus_{i=0}^{m-1} \mathcal{L} \longrightarrow X.$$  

(36)

**Proposition 2.14.** Let $(X, G) \in (\text{Sch}^{\text{gp}}/k)$. Let $\mathcal{L}$ be an invertible sheaf with $G$-linearization $\{\Phi_g\}_{g \in G}$. Let $\eta \in \Gamma(X, \mathcal{L}^{\otimes(-m)})$ be a global section and $\pi : Y \rightarrow X$ be a $m$-uple covering associated with $(\mathcal{L}, \eta)$.

Suppose that

$$\Phi_g^{(-m)}(g^*(\eta)) = \eta.$$  

(37)

Then we have a group homomorphism $G \rightarrow \text{Aut}_k(Y)$ such that $(\pi, \text{id}_G) : (Y, G) \rightarrow (X, G)$ is a morphism in $(\text{Sch}^{\text{gp}}/k)$.

**Proof.** For $g \in G$, we can construct $\tilde{g} : Y \rightarrow Y$ as follows.

1. Let $Y_1$ be the $m$-uple covering associated with $(g^*\mathcal{L}, g^*(\eta))$. Then $Y_1$ is a fiber product of $Y \rightarrow X$ and $X \overset{g}{\rightarrow} X$. Since $g$ is an isomorphism, $Y_1 \rightarrow Y$ is also isomorphism.
2. By the isomorphism $\Phi_g$, $(g^*\mathcal{L}, g^*(\eta))$ is isomorphic to $(\mathcal{L}, \eta)$. Hence we have an isomorphism $Y \overset{j}{\rightarrow} Y_1$ over $X$.

By composing these isomorphism, we get an automorphism $\tilde{g} \in \text{Aut}_k(X)$.

$$
\begin{array}{ccc}
Y & \overset{(2)}{\rightarrow} & Y_1 \\
\downarrow & & \downarrow j \\
X & \overset{\sim}{\rightarrow} & X
\end{array}
\quad
\begin{array}{ccc}
Y & \overset{(1)}{\rightarrow} & Y \\
\downarrow & & \downarrow \\
X & \overset{g}{\rightarrow} & X
\end{array}
$$

(38)

We can show that $G \rightarrow \text{Aut}_k(Y); g \mapsto \tilde{g}$ is a group homomorphism by the cocycle condition. Hence we can construct $(Y, G) \in (\text{Sch}^{\text{gp}}/k)$ and by construction, $(\pi, \text{id}_G) : (Y, G) \rightarrow (X, G)$ becomes a morphism in $(\text{Sch}^{\text{gp}}/k)$.

**Definition 2.15.** Let $(X, G) \in (\text{Sch}^{\text{gp}}/k)$. Let $Y$ be a subscheme of $X$ and $i : Y \hookrightarrow X$ be an inclusion. $Y$ is called $G$-stable if there exists a group homomorphism $G \rightarrow \text{Aut}_k(Y)$ such that $(Y, G) \in (\text{Sch}^{\text{gp}}/k)$ and $(i, \text{id}_G) : (Y, G) \rightarrow (X, G)$ is a morphism in $(\text{Sch}^{\text{gp}}/k)$.

Finally, we prove the liftable of group actions along blowing-ups.

**Proposition 2.16.** Let $(X, G) \in (\text{Sch}^{\text{gp}}/k)$ and $Y$ be a $G$-stable closed subscheme of $X$. Let $b : B_{\text{Bl}}YX \rightarrow X$ be a blowing up of $X$ along $Y$. Then we have a group homomorphism $G \rightarrow \text{Aut}_k(B_{\text{Bl}}YX)$ such that $(b, \text{id}_G) : (B_{\text{Bl}}YX, G) \rightarrow (X, G)$ is a morphism in $(\text{Sch}^{\text{gp}}/k)$.

**Proof.** Let $g \in G$. Since $Y$ is $G$-stable, we have the following commutative diagram on the left.

$$
\begin{array}{ccc}
Y & \overset{g}{\rightarrow} & Y \\
\downarrow & & \downarrow \\
X & \overset{b}{\rightarrow} & B_{\text{Bl}}YX
\end{array}
\quad
\begin{array}{ccc}
B_{\text{Bl}}YX & \overset{\tilde{g}}{\rightarrow} & B_{\text{Bl}}YX \\
\downarrow & & \downarrow h \\
X & \overset{h}{\rightarrow} & X
\end{array}
$$

(39)
By the universality of the blowing-up, we have the unique morphism \( \bar{g} : \text{Bl}_k Y \to \text{Bl}_k X \) which make the diagram on the right commute. The map \( G \to \text{Aut}_k (\text{Bl}_k X) ; g \mapsto \bar{g} \) becomes a group homomorphism by the uniqueness. By the construction, \((h, \text{id}_G) : (\text{Bl}_k X, G) \to (X, G)\) is a morphism in \((\text{Sch}^P / k)\). \(\square\)

3. Construction of a Kummer surface family

Hereafter we fix a field \( k \) whose characteristic is not 2 and containing \( \sqrt{-1} \). In this section, we explicitly construct the Kummer surface family \( X \to T \).

3.1. Construction of the Legendre Family of Elliptic Curves.

**Definition 3.1.** (1) We set \( A = k[c, \frac{1}{c(1-c)}] \), which is a localization of the polynomial ring of one variable \( k[c] \) and \( S = \text{Spec} \ A \). \( \mathbb{P}^1_S = \text{Proj} A[Z_0, Z_1] \) is the projective line over \( S \).

(2) We use the notations \( U_0 = D_+(Z_0) \subset \mathbb{P}^1_S \) and \( U_1 = D_+(Z_1) \subset \mathbb{P}^1_S \). We define the local coordinate \( z = Z_1/Z_0 \) on \( U_0 \).

(3) We define \( \tilde{h} = Z_0Z_1(Z_0 - Z_1)(Z_0 - czZ_1) \in \Gamma(\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(4)) \).

We construct the Legendre family of elliptic curves as the double covering of \( \mathbb{P}^1_S \).

**Definition-Proposition 3.2.** Let \( E \to \mathbb{P}^1_S \) be the double covering associated with \((\mathcal{O}_{\mathbb{P}^1_S}(-2), \tilde{h}) \). On the open subset \( U_0 \subset \mathbb{P}^1_S \), \( E \to \mathbb{P}^1_S \) can be described as the following morphism

\[
E_0 = \text{Spec} A[u, z]/(u^2 - h(z)) \longrightarrow \text{Spec} A[z] = U_0
\]

where \( h(z) = z(1 - z)(1 - cz) \in A[z] \).

**Definition 3.3.** (1) We define a set of \( A \)-rational points \( \Sigma(\mathbb{P}^1_S) \) on \( \mathbb{P}^1_S \) by

\[
\Sigma(\mathbb{P}^1_S) = \{ 0, 1, 1/c, \infty \} \subset \text{Hom}_S(S, \mathbb{P}^1_S)
\]

where \( 0, 1, 1/c, \infty \) denotes the \( A \)-rational points corresponding to \( z = 0, 1, 1/c, \infty \).

(2) Similarly, we define \( \Sigma(E) \) as the set of \( A \)-rational points on \( E \) corresponding to \( z = 0, 1, 1/c, \infty \) and \( u = 0 \).

(3) For \( \sigma \in \Sigma(\mathbb{P}^1_S) \), we can regard \( \sigma(S) \subset \mathbb{P}^2_S \) as Cartier divisor on \( \mathbb{P}^2_S \). Since \( S = \mathbb{P}^1 - \{ 0, 1, \infty \} \), \( \sigma \neq \sigma' \) in \( S \) implies \( \sigma(S) \cap \sigma'(S) = \emptyset \).

(4) For a morphism of scheme \( Z \to S \), \( \Sigma(E_Z) \) and \( \Sigma(\mathbb{P}^1_Z) \) denotes the base change of \( \Sigma(E_Z) \) and \( \Sigma(\mathbb{P}^1_Z) \). By the above property (2), for any morphism \( \emptyset \neq Z \to S \), the natural map \( \Sigma(\mathbb{P}^1_S) \to \Sigma(\mathbb{P}^1_Z) \) and \( \Sigma(E) \to \Sigma(E_Z) \) is bijective.

By the following, we have the involution \( \iota \) on \( E_Z \) associated with the structure of elliptic curves.

**Proposition 3.4.** Let \( \iota \) be an automorphism of \( E \) defined by the following ring homomorphism.

\[
A[u, z]/(u^2 - h(z)) \longrightarrow A[u, z]/(u^2 - h(z))
\]

Then for any geometric point \( S \) of \( S \), the induced morphism \( \iota_S : E_S \to E_S \) over \( \kappa(S) \) is the involution with respect to the elliptic curve structure \((E_S, O)\) with \( O \in \Sigma(E_S) \).

Since \( E_0 \) is written in Weierstrass form, if \( O = \infty \), we have the result. If \( O = 0, 1, 1/c \), we use the following lemma. The proof is standard.

\[\footnote{See Definition 2.13 for this notation.}\]
Lemma 3.5. Let $E$ be a smooth projective curve of genus 1 over an algebraically closed field $K$. $O$ and $O'$ are a $K$-rational point of $E$. $i$ and $i'$ are involutions on $E$ of taking inverses associated with the elliptic curve structure $(E, O)$ and $(E, O')$. Suppose $O'$ is a 2-torsion point for the elliptic curve structure $(E, O)$. Then $i = i'$.

3.2. Family of Kummer surfaces of product of Elliptic Curves.

Definition 3.6. We use the following notations.

1) Let $B$ denote $k$-algebra $A \otimes_k A$. We set $a = c \otimes 1, b = 1 \otimes c \in B$ and $T = \text{Spec } B$.

2) Let $\mathcal{Y} = \mathbb{P}^1_S \times_k \mathbb{P}^1_S$. By the direct product of the natural map $\mathbb{P}^1_S \to S$, we regard $\mathcal{Y}$ as a scheme over $T = S \times_k S$.

3) For $i, j \in \{0, 1\}, Y_{i,j} = U_i \times_k U_j$ are open subschemes of $\mathcal{Y}$.

4) Let $x, y$ denote local coordinates on $Y_{0,0}$ corresponding to $z \otimes 1$ and $1 \otimes z$ in $A[z] \otimes_k A[z]$, respectively. Using $x$ and $y$, we can write $Y_{0,0} = \text{Spec } B[x, y]$.

Definition 3.7. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{Y}$ corresponding to $pr_1^*O_{\mathbb{P}^1_S}(-2) \otimes_{O_\mathcal{Y}} pr_2^*O_{\mathbb{P}^1_S}(-2)$ where $pr_i : \mathcal{Y} \to \mathbb{P}^1_S$ denotes the $i$-th projection. Furthermore, we define a global section $\eta$ by

$$\eta = pr_1^*(\overline{h}) \otimes pr_2^*(\overline{h}) \in \Gamma(\mathcal{Y}, \mathcal{L} \otimes (-2)).$$

We define the following polynomial with coefficients in $B$.

$$f(x) = x(1-x)(1-ax)$$

$$g(y) = y(1-y)(1-by)$$

Definition-Proposition 3.8. We define $\widetilde{\mathcal{Y}} \to \mathcal{Y}$ as the double covering associated with $[\mathcal{L}, \eta]$. On $Y_{0,0} \subset \mathcal{Y}$, $\widetilde{\mathcal{Y}} \to \mathcal{Y}$ is described as

$$\widetilde{Y}_{0,0} = \text{Spec } B[u, x, y]/(u^2 - f(x)g(y)) \to \text{Spec } B[x, y] = Y_{0,0} \subset \mathcal{Y}$$

where $f(x), g(y) \in B[x, y]$ are the elements defined in (44). Then we have $\widetilde{Y}_{0,0} \subset \widetilde{\mathcal{Y}}$.

The double covering $\widetilde{\mathcal{Y}}$ and $E \times_k E$ are related as follows. Note that the coordinate ring of $E_0 \times_k E_0 \subset E \times_k E$ is described as follows.

$$A[u, z]/(u^2 - h(z)) \otimes_k A[u, z]/(u^2 - h(z)) \to B[u_1, u_2, x, y]/(u_1^2 - f(x), u_2^2 - g(y))$$

$$u \otimes 1, 1 \otimes u, z \otimes 1, 1 \otimes z \to u_1, u_2, x, y$$

Proposition 3.9. We have a morphism $E \times_k E \to \widetilde{\mathcal{Y}}$ over $T$ described as the following $B$-algebra homomorphism.

$$B[u, x, y]/(u^2 - f(x)g(y)) \quad \longrightarrow \quad B[u_1, u_2, x, y]/(u_1^2 - f(x), u_2^2 - g(y))$$

Then $E \times_k E \to \widetilde{\mathcal{Y}}$ corresponds to the universal categorical quotient of $E \times_k E$ under the $\mathbb{Z}/2\mathbb{Z}$-action induced by $i \times i'$.

Proof. By the description of $i$ in Proposition 3.4, $i \times i$ acts on $E_0 \times_k E_0$ as

$$B[u_1, u_2, x, y]/(u_1^2 - f(x), u_2^2 - g(y)) \quad \longrightarrow \quad B[u_1, u_2, x, y]/(u_1^2 - f(x), u_2^2 - g(y))$$

$$u_1, u_2 \quad \longrightarrow \quad -u_1, -u_2$$

Hence the image of (47) generates the ring of invariants. Since the map (47) is injective, we have the result. \[\square\]
Definition 3.10.  

(1) We define a set $\Sigma^2(\mathcal{Y})$ of $B$-rational points by

$$\Sigma^2(\mathcal{Y}) = \{ \sigma_1 \times \sigma_2 : \sigma_1, \sigma_2 \in \Sigma(\mathbb{P}^1_S) \}$$

(49)

where $\sigma_1 \times \sigma_2 : T = S \times_k S \to \mathbb{P}^1_S \times_k \mathbb{P}^1_S = \mathcal{Y}$ is the direct product of $\sigma_1$ and $\sigma_2$. More explicitly, $\Sigma^2(\mathcal{Y})$ is the $B$-rational point sets whose $x$-coordinates is in $\{0, 1, 1/a, \infty\}$ and $y$-coordinates in $\{0, 1, 1/b, \infty\}$. Especially, we often identify

$$\Sigma^2(\mathcal{Y}) = \{0, 1, 1/a, \infty\} \times \{0, 1, 1/b, \infty\}$$

(50)

and elements in $\Sigma^2(\mathcal{Y})$ is written like $(0, 0), (1, 1)$ and $(1/a, 1/b)$.

(2) Similarly, we define a set $\Sigma^2(E \times_k E)$ of $B$-rational points by

$$\Sigma^2(E \times_k E) = \{ \sigma_1 \times \sigma_2 : \sigma_1, \sigma_2 \in \Sigma(E) \}.$$

(51)

(3) We define a set $\Sigma^2(\tilde{\mathcal{Y}})$ of $B$-rational points by the image of $\Sigma^2(E \times_k E)$ under the morphism $E \times_k E \to \tilde{\mathcal{Y}}$ in Proposition 3.9. Since the following diagram commutes, the image of $\Sigma^2(\mathcal{Y})$ under $\mathcal{Y} \to \tilde{\mathcal{Y}}$ in Proposition 3.8 is $\Sigma^2(\tilde{\mathcal{Y}})$.

(52)

Hence the morphism $\tilde{\mathcal{Y}} \to \mathcal{Y}$ induces a bijection $\Sigma^2(\tilde{\mathcal{Y}}) \xrightarrow{\sim} \Sigma^2(\mathcal{Y})$. We have an identification

$$\Sigma^2(\tilde{\mathcal{Y}}) = \{0, 1, 1/a, \infty\} \times \{0, 1, 1/b, \infty\}.$$

(53)

(4) We sometimes simply write $\Sigma^2(\mathcal{Y}), \Sigma^2(\tilde{\mathcal{Y}})$ and $\Sigma^2(E \times_k E)$ as $\Sigma^2$ if the variety is clear from the context.

(5) For a morphism of schemes $Z \to T$, the base changes of $\Sigma^2(E \times_k E)$, $\Sigma^2(\tilde{\mathcal{Y}})$ and $\Sigma^2(\mathcal{Y})$ by $Z \to T$ are denoted by $\Sigma^2((E \times_k E)_Z)$, $\Sigma^2(\tilde{\mathcal{Y}}_Z)$ and $\Sigma^2(\mathcal{Y}_Z)$. If the variety is clear from the context, we denote these subset by $\Sigma^2$.

Definition 3.11. We define an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\tilde{\mathcal{Y}}}$ as the ideal sheaf corresponding to $\bigsqcup_{\sigma \in \Sigma^2} \sigma \subset \tilde{\mathcal{Y}}$. By the explicit equations of $f(x)$ and $g(y)$ given in (44), on $\tilde{Y}_{0,0}$, $\mathcal{I}$ is given by

$$\mathcal{I}|_{\tilde{Y}_{0,0}} = (f(x), g(y), u) \subset B[x, y, u]/(u^2 - f(x)g(y)).$$

(54)

Definition-Proposition 3.12. We define $X \to \tilde{\mathcal{Y}}$ by the blowing up of $\tilde{\mathcal{Y}}$ along $\bigsqcup_{\sigma \in \Sigma^2} \sigma$. Then $X \to \tilde{\mathcal{Y}}$ is described locally on $\tilde{Y}_{0,0}$ as follows.

$$V_{0,0} = \text{Spec } B[u, x, y]/(u^2f(x) - g(y)) \quad \xrightarrow{\sim} \quad \tilde{Y}_{0,0} = \text{Spec } B[u, x, y]/(u^2 - f(x)g(y))$$

$$W_{0,0} = \text{Spec } B[w, x, y]/(u^2g(y) - f(x))$$

(55)

where these morphisms are defined by $u \mapsto vf(x)$ and $u \mapsto wg(y)$. $v$ and $w$ and glued by the relations $v = \frac{1}{w}$. Then we have two open subsets $V_{0,0}$ and $W_{0,0}$ of $X$. 
Definition 3.13. For $\sigma \in \Sigma^2$, we define $Q_\sigma \subset X$ as the exceptional divisor over $\sigma \subset \tilde{Y}$. In other words, $Q_\sigma$ is defined by the following fiber product.

$$
\begin{array}{c}
Q_\sigma \\
\downarrow \\
T \to \tilde{Y}
\end{array}
\xrightarrow{j}
\begin{array}{c}
X \\
\downarrow \\
T \to \tilde{Y}
\end{array}
$$

(56)

See Figure 1 for the configurations of $Q_\sigma$ on $X$.

| $y = \infty$ | $Q_{(0,\infty)}$ | $Q_{(1,\infty)}$ | $Q_{(1/a,\infty)}$ | $Q_{(\infty,\infty)}$ |
| $y = \frac{1}{b}$ | $Q_{(0,1/b)}$ | $Q_{(1,1/b)}$ | $Q_{(1/a,1/b)}$ | $Q_{(\infty,1/b)}$ |
| $y = 1$ | $Q_{(0,1)}$ | $Q_{(1,1)}$ | $Q_{(1/a,1)}$ | $Q_{(\infty,1)}$ |
| $y = 0$ | $Q_{(0,0)}$ | $Q_{(1,0)}$ | $Q_{(1/a,0)}$ | $Q_{(\infty,0)}$ |

$X = \{ x = 0 \}$, $x = 1$, $x = \frac{1}{a}$, $x = \infty$

**Figure 1.** The exceptional divisors $Q_\sigma$ on $X$

We constructed the following $T$-schemes.

$$
\begin{array}{c}
\mathcal{E} \times_k \mathcal{E} \\
\downarrow \\
\mathcal{X} \cup \bigcup_{\sigma \in \Sigma^2} \mathcal{Y} \\
\downarrow \text{blowing-up along} \\
\mathcal{Y} \cup \bigcup \left( \mathcal{Y} \right) \\
\downarrow \text{double cover} \\
\mathcal{Y} = \mathbb{P}^1_S \times_k \mathbb{P}^1_S
\end{array}
\xrightarrow{\mathcal{Y}}
\begin{array}{c}
\mathcal{X} \\
\cup \\
\bigcup \\
\downarrow \\
\mathcal{Y} \\
\cup \\
\bigcup \\
\downarrow \\
\mathcal{Y}_{0,0}
\end{array}
\xrightarrow{\mathcal{Y}_{0,0}}
\begin{array}{c}
Y_{0,0} = U_0 \times_k U_0
\end{array}

(57)

We can check that these constructions are all stable under any base change of $T$.

**Proposition 3.14.** Let $Z$ be any scheme over $T$. $\mathcal{X}_Z, \tilde{\mathcal{Y}}_Z$, $(\mathcal{E} \times_k \mathcal{E})_Z$ and $\mathcal{Y}_Z$ denotes the base changes of $\mathcal{X}, \tilde{\mathcal{Y}}, \mathcal{E} \times_k \mathcal{E}$ and $\mathcal{Y}$ by $Z \to T$. Furthermore, $(\mathcal{L}_Z, \eta_Z)$ denotes the pull back of $(\mathcal{L}, \eta)$ by $\mathcal{Y}_Z \to \mathcal{Y}$, $(\iota \times \iota)_Z$ denotes the base change of $\iota \times \iota$. Then we have the following.
Let \((X, Y)\) of Legendre elliptic curves (Section 7) and for construction of topological 2-chains \(\Sigma\).

Proposition 3.16. \(\Sigma\) is not so obvious since the blowing-up is not stable under the base change. But in this case we see that \(O_{\tilde{Y}}/\mathbb{I}^n\) is flat over \(T\) for any \(n > 0\) by ring theoretic calculation, hence the statement holds.

Recall that \(E\) is the Legendre family of elliptic curves. By Proposition 3.14 the following holds.

Proposition 3.15. Let \(T\) be a geometric point of \(T\) and \(O \in \Sigma^2((E \times_k E)_T)\). Then the geometric fiber \((E \times_k E)_T\) is an abelian surface structure over \(\kappa(T)\) with the identity element \(O\). It satisfies the following properties.

1. \(\Sigma^2((E \times_k E)_T)\) is the set of 2-torsion sections of this abelian surface structure.
2. \((i \times i)_T\) is the involution of taking inverse with respect to this abelian surface structure.
3. Let \(a(T), b(T) \in \kappa(T)\) be the image of elements \(a, b \in O_T(T)\) by \(T\) : \(O_T(T) \to \kappa(T)\). Then \((E \times_k E)_T\) is isomorphic the direct product of the elliptic curves \(y^2 = x(1-x)(1-a(T)x)\) and \(y^2 = x(1-x)(1-b(T)x)\) over \(\kappa(T)\) with this abelian surface structure.

Finally, we prove that \(X \to T\) is a Kummer surface family.

Proposition 3.16. For each geometric point \(T\) of \(T\), the geometric fiber \(X_T\) is isomorphic to the Kummer surface associated with the Abelian surface \(((E \times_k E)_T, O)\) over \(\kappa(T)\) where \(O \in \Sigma^2((E \times_k E)_T)\).

Proof. By Proposition 3.15 \((i \times i)_T\) is the involution of taking inverses on the Abelian surface \(((E \times_k E)_T, O)\). By Proposition 3.14 (2), \((E \times_k E)_T \to \tilde{Y}_T\) corresponds to the quotient by \((i \times i)_T\). By the morphism \((E \times_k E)_T \to \tilde{Y}_T\), \(\kappa(T)\)-rational points in \(\Sigma^2((E \times_k E)_T)\) maps to \(\Sigma^2(\tilde{Y}_T)\) (cf. Definition 3.10). By Proposition 3.15 \(\Sigma^2((E \times_k E)_T)\) corresponds to 2-torsion points of the abelian surface \(((E \times_k E)_T, O)\) and \(\Sigma^2(\tilde{Y}_T)\) is 16 singular points on \(\tilde{Y}_T\). By Proposition 3.14 (3), \(X_T \to \tilde{Y}_T\) is the blowing-up of \(\tilde{Y}_T\) along \(\Sigma^2(\tilde{Y}_T)\). Hence \(X_T\) is isomorphic to the Kummer surface associated with \(((E \times_k E)_T, O)\).

3.3. Other constructions of \(X\). In this subsection, we explain other ways of construction of \(X\). These constructions are used to relate periods of \(X\) with periods of Legendre elliptic curves (Section 7) and for construction of topological 2-chains on \(X_t\) (Section 8).

Definition 3.17. Let \((E \times_k E)^-\) (resp. \(\bar{X}\)) be the blowing-up of \(E \times_k E\) (resp. \(Y\)) along \(\bigcup_{\gamma \in \Sigma^2} \sigma\). By the universal property of the blowing-up, we have the unique morphisms \((E \times_k E)^-\) \(\to X\) and \(X \to \bar{X}\) such that the following diagram commutes.
The morphisms appearing in the diagram above are described by the following $B$-algebra homomorphisms.

\[
\begin{align*}
B[u_1, v, x, y]/(u_1^2 - f(x), v^2 f(x) - g(y)) & \to B[u_2, w, x, y]/(u_2^2 - g(y), w^2 g(y) - f(x)) \\
B[u_1, x, y]/(u_1 - f(x)) & \to B[w, x, y]/(w - f(x)) \\
B[x, y]/(y - f(x)) & \to B[v, x, y]/(v - f(x)) \\
B[\tau, x, y]/(\tau^2 f(x) - g(y)) & \to B[\tau, x, y]/(\tau^2 f(x) - g(y)) \\
B[\tau, x, y]/(\tau f(x) - g(y)) & \to B[\tau, x, y]/(\tau f(x) - g(y)) \\
B[\tau, x, y]/(\tau^2 f(x) - g(y)) & \to B[\tau, x, y]/(\tau^2 f(x) - g(y))
\end{align*}
\]

The rings appearing in the first column are coordinate rings of affine open subsets in $(\mathcal{E} \times_k \mathcal{E})^\sim$ above $E_0 \times_k E_0 \subset \mathcal{E} \times_k \mathcal{E}$ and the rings appearing in the fifth column are coordinate rings of affine open subsets in $\overline{\mathcal{X}}$ above $Y_{0,0} \subset \mathcal{Y}$.

Finally, we name exceptional divisors on $\overline{\mathcal{X}}$. We use this notation in Section 8.

**Definition 3.18.** For $\sigma \in \Sigma^2$, we define the exceptional divisor $Q_\sigma \subset \overline{\mathcal{X}}$ by the following commutative diagram.

\[
\begin{array}{ccc}
\overline{Q}_\sigma & \longrightarrow & \overline{\mathcal{X}} \\
\downarrow & & \downarrow \\
T & \overset{\sigma}{\longrightarrow} & \mathcal{Y}
\end{array}
\]

The morphism $\mathcal{X} \to \overline{\mathcal{X}}$ induces the $2:1$ map $Q_\sigma \to \overline{Q}_\sigma$.

4. Construction of automorphisms of Kummer surface family

In this section, we will prove Proposition 1.3. As in Section 3, we fix a field $k$ whose characteristic is not 2 and contains $\sqrt{-1}$.

4.1. Summary. We will construct a group $\tilde{G}$ and its action to the scheme $\mathcal{X}'$, which is a base change of $\mathcal{X}$ in Definition 3.12. To construct $\tilde{G}$-action on $\mathcal{X}'$, we construct following objects in $(\text{Sch}^{pp}/k)$.

\[
\begin{array}{ccc}
(\mathcal{X}', \tilde{G}) & \longrightarrow & (\tilde{Y}', \tilde{G}) \\
\downarrow & & \downarrow \\
(\mathcal{Y}', G) & \longrightarrow & (\mathcal{Y}, G)
\end{array}
\]

We will construct them in the following order.

1. We start with $H_0 = \text{Aut}_k(S)$. Since $S = \mathbb{P}^1_k \setminus \{0,1,\infty\}$, we have $H_0 = \Theta(\{0,1,\infty\})$ (Definition 4.1). We define $H_0 \subset \text{Aut}(\mathbb{P}^1_S)$ so that $\Sigma(\mathbb{P}^1_S) \subset \text{Hom}_S(S, \mathbb{P}^1_S)$ is compatible (See Definition 2.4) with $(\mathbb{P}^1_S, H_0) \to (S, H_0)$.

By considering the $H_0$-action on $\Sigma(\mathbb{P}^1_S) = \{0,1,c,\infty\}$, we have $H_0 \simeq \cdots$
$$\mathfrak{S}((0,1,c,\infty))$$ (Proposition 1.3). Then we take a finite faithfully flat extension $\text{Spec} A' \to S$ (Definition 1.6). Then $H_0$-action on $S$ lifts to $\hat{H}$-action on $S'$ (Proposition 1.8). $\hat{H}$ is isomorphic to $\mathfrak{S}_1$ (Remark 1.9).

(2) We define the following objects in $(\text{Sch}^{p/k})$ (Definition 4.12)

$$\begin{align*}
(T, \hat{G}_0) &= (S \times_k S, \hat{H}_0 \times \hat{H}_0), \\
(Y, G_0) &= (\mathbb{P}^1_S \times \mathbb{P}^1_S, H_0 \times H_0).
\end{align*}$$

(3) To lift $G$-action on $Y$ to the double covering $\hat{Y}$, we use Proposition 2.14. Since $\hat{Y}$ is constructed from $(\mathcal{L}, \eta)$ in Definition 3.7 we will find linearization on $\mathcal{L}$ satisfying (37). The outline of the construction is as follows.

- Take a $\mathbb{Z}/2\mathbb{Z}$ extension $\hat{G}$ of $G$ (Definition 4.22).
- Find a good 1-cocycle $\hat{\chi} : \hat{G} \to (B')^\times$ (Definition 4.23).
- Modify the natural $G$-linearization $\{\Psi_{\rho}\}_{\rho \in G}$ (Definition 4.18) by $\hat{\chi}$ and get new $\hat{G}$-linearization $\{\chi_{\rho}\}_{\rho \in \hat{G}}$ (Proposition 4.24).

(4) Since $G$-action on $\hat{Y}$ stabilize the blowing-up locus of $\mathcal{X}' \to \hat{Y}$, we can lift $\hat{G}$-action on $\hat{Y}$ to $\mathcal{X}' = X \times_T T'$ (Proposition 4.26).

In addition, we calculate some opposite 1-cocycles in Subsection 4.5. The explicit cocycle representations are important for the description of the group action on the higher Chow subgroup $\Xi_{\text{can}}$ (Section 6), on the 2-form $\omega \in \Gamma(X, \Omega^2_{X'/T'})$ (Section 7) and on the sheaf $\mathcal{Q}_\omega$ (Section 9).

4.2. Automorphisms on $S$ and $\mathbb{P}^1_S$. In this subsection, we construct a morphism $(\mathbb{P}^1_S, H_0) \to (S, H_0)$ in $(\text{Sch}^{p/k})$.

**Definition 4.1.** We define $H_0 = \text{Aut}_k(S)$. If we regard $S = \mathbb{P}^1_k - \{0,1,\infty\}$, every $\tau_0 \in H_0$ extends to an automorphism on $\mathbb{P}^1_k$ which stabilizes the $k$-rational point set $\{0,1,\infty\}$. Hence we have the following group homomorphism.

$$\begin{array}{ccc}
\text{Aut}_k(S) & \text{via} & \mathfrak{S}((0,1,\infty)) \\
\downarrow \tau_0 & \Downarrow & \downarrow \tau_0 \\
\text{id} & \tau_0 & \tau_0(c)
\end{array}$$

As is well-known, this is an isomorphism. Hence we often identify $H_0$ with $\mathfrak{S}((0,1,\infty))$. The table of the correspondence of $H_0$ and $\mathfrak{S}((0,1,\infty))$ is given as follows. Note that the composition on $\mathfrak{S}((0,1,\infty))$ is defined as the usual order. For example, $(0 \infty)(0 1) = (0 1 \infty)$. $H_0$ induces opposite action on $A$.

| $\tau_0$ | $\tau_0^c(c)$ | $\tau_0$ | $\tau_0^c(c)$ |
|----------|--------------|----------|--------------|
| id       | $c$          | (0 1)    | 1 - $c$      |
| (1 $\infty$) | $c^{-1}$     | (0 1 $\infty$) | $1/c$ |
| (0 $\infty$) | $c$          | (0 $\infty$ 1) | $c^{-1}$ |

Next, we define the automorphism group $H_0$ on $\mathbb{P}^1_S$. Using the notation in Definition 2.1 we have a group

$$\text{Aut}(\mathbb{P}^1_S; H_0) = \left\{ (\tau_0, \tau_0^c) \in \text{Aut}_k(\mathbb{P}^1_S) : \begin{array}{ccc}
\mathbb{P}^1_S & \to & S \\
\downarrow \tau_0 & & \downarrow \tau_0 \\
\mathbb{P}^1_S & \to & S
\end{array} \text{commutes.} \right\}.$$
Since $\text{Aut}(\mathbb{P}^1_S; S, H_0) \to \text{Aut}_k(\mathbb{P}^1_S)$ is injective, we identify elements in $\text{Aut}(\mathbb{P}^1_S; S, H_0)$ with elements in $\text{Aut}_k(\mathbb{P}^1_S)$.

**Definition 4.2.**
We define $H_0$ as the following subgroup of $\text{Aut}(\mathbb{P}^1_S; S, H_0)$.

$$H_0 = \{ \tau_0 \in \text{Aut}(\mathbb{P}^1_S; S, H) : \text{For any } \sigma \in \Sigma(\mathbb{P}^1_S), \tau_0 \circ \sigma \circ \Sigma^{-1}_0 \in \Sigma \}$$

(65)

Then we have a natural morphism $\alpha : (\mathbb{P}^1_S, H_0) \to (S, H_0)$ in $(\text{Sch}^p/k)$. By the construction, $\Sigma(\mathbb{P}^1_S)$ is compatible with respect to $\alpha : (\mathbb{P}^1_S, H_0) \to (S, H_0)$. By Definition 2.4, $H_0$ has the following natural set-theoretic action on $\Sigma(\mathbb{P}^1_S)$.

$$\begin{array}{c}
H_0 \\
\downarrow \tau_0
\end{array} \xrightarrow{\psi} \begin{array}{c}
\Theta(\Sigma(\mathbb{P}^1_S)) \\
\downarrow \psi
\end{array} \xrightarrow{\Sigma(0, 1/c, \infty)} (\sigma \mapsto \tau_0 \circ \sigma \circ \Sigma^{-1}_0)$$

(66)

**Proposition 4.3.**
The group homomorphism (66) is an isomorphism.

**Proof.** Let $\tau_0 \in H_0$. We have the following diagram.

$$\begin{array}{c}
\mathbb{P}^1_S \\
\downarrow \tau_0
\end{array} \xrightarrow{\tau_0} \begin{array}{c}
\mathbb{P}^1_S \\
\downarrow \Sigma^{-1}
\end{array} \xrightarrow{\Sigma^{-1}} \begin{array}{c}
\mathbb{P}^1_S \\
\downarrow S \\
\downarrow S
\end{array}$$

(67)

where $\Sigma^{-1}_0 : \mathbb{P}^1_S \to \mathbb{P}^1_S$ is the morphism $\text{id}_{\mathbb{P}^1_k} \times \Sigma^{-1}_0 : \mathbb{P}^1_k \times S \to \mathbb{P}^1_k \times S = \mathbb{P}^1_S$.

Then $(\Sigma^{-1}_0 \circ \tau_0) : \mathbb{P}^1_S \to \mathbb{P}^1_S$ is morphism over $S$, and we can write $(\Sigma^{-1}_0 \circ \tau_0)^2(z) = \frac{pz + q}{pz + s}$ where $p, q, r, s \in \text{Frac}(A)$. Combining these results, we can write

$$\tau_0^2(z) = \frac{pz + q}{r z + s} \quad (p, q, r, s \in \text{Frac}(A))$$

(68)

First, we check (66) is injective. Suppose $\tau_0 \in H_0$ lies in the kernel of (66). Since $\tau_0$ acts trivially on $\{0, 1, 1/c, \infty\}$, $\tau_0(0) = 0$, $\tau_0(1) = 1$, $\tau_0(1/c) = 1/c$ and $\tau_0(\infty) = \infty$. Especially we have

$$\begin{align*}
p \cdot 0 + q \\
r \cdot 0 + s = 0
\end{align*}$$

(69)

Hence we see that $\tau_0^2(z) = z$ and $\tau_0^4(c) = c$, i.e. $\tau_0 = \text{id}_{H_0}$.

Next, we check that (66) is surjective. It is enough to find elements in $H_0$ corresponding to $(0, 1), (0, 1/1/c, \infty) \in \Theta(\{0, 1, 1/c, \infty\})$ since they are generators of $\Theta(\{0, 1, 1/c, \infty\})$. We can use the presentation in (68). For example, to find $\tau_0 \in H_0$ corresponds to $(0, 1/1/c, \infty)$, it is enough to find $p, q, r, s \in \text{Frac}(A)$ such that

$$\begin{align*}
p \cdot 0 + q \\
r \cdot 0 + s = 1
\end{align*}$$

(70)

Hence we have $\tau_0(z) = \frac{1}{r z}$ and $\tau_0^4(c) = 1 - c$. We can check that this automorphism is globally defined. Similarly, we can find elements in $H_0$ corresponding to $(0, 1)$. \(\square\)

**Remark 4.4.**
By Proposition 4.3, we often identify $H_0$ as $\Theta(\{0, 1, 1/c, \infty\})$. The explicit $H_0$ action on $\mathbb{P}^1_S$ is given in Table 2. We can find these correspondence by the same method we use in the proof of Proposition 4.3. In the table, for each
\[ \tau_0 \in H_0, \] the image of \( c \) under \( \mathbb{Z}_0^+ : A \to A \) and the image of the local coordinate \( z \) under \( \tau_0^+ : \mathcal{O}_{S_0}^+ \to (\tau_0)_* \mathcal{O}_{S_0}^+ \) are given.

**Table 2.** The table of \( H_0 \simeq \mathfrak{S}((0, 1, 1/c, \infty)) \) and \( H_0 \to H_0 \)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\tau_0 & \tau^0_0(c) & \tau^0_0(z) \\
\hline
\text{id} & c & z \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\tau_0 & \tau^0_0(c) & \tau^0_0(z_1) \\
\hline
(0 1) & \frac{1}{c-1} & 1 - z & (0 1/c) & 1 - c & \frac{1-c}{1-c} \\
\hline
(1/c \infty) & c & \frac{(1-c)x}{1-cz} & (1 \infty) & 1 - c & \frac{z}{z-1} \\
\hline
\end{array}
\]

**Remark 4.5.** We have a bijection

\[
\{\{0,1\}, \{\infty, 1/c\}\}, \{\{0, \infty\}, \{1, 1/c\}\}, \{\{0, 1/c\}, \{1, \infty\}\} \simeq \{0, 1, \infty\} \quad (71)
\]
defined by \( \{0, 1\}, \{\infty, 1/c\} \to 0, \{\{0, \infty\}, \{1, 1/c\}\} \to 1, \{\{0, 1/c\}, \{1, \infty\}\} \to \infty. \) Then we have a group homomorphism

\[
\mathfrak{S}((0, 1, 1/c, \infty)) \to \mathfrak{S}((0, 1, \infty)) \quad (72)
\]
The group homomorphism \( H_0 \to H_0 \) is identified with this group homomorphism under the identification in Definition 4.1 and Proposition 4.3.

### 4.3. The covering \( S' \to S \) and lifting of group actions

To get enough automorphisms of Kummer surface family, we have to enlarge the base scheme \( S. \) As we will see in Section 5, this base change is also necessary for constructing higher Chow cycles in \( \Xi^{\text{can}}. \)

**Definition 4.6.** We define \( A \)-algebra \( A' \) as \( A' = A[ \sqrt{c}, \sqrt{1 - c} ] \). \( S' \) denotes \( \text{Spec} A' \).

We have a natural morphism \( S' \to S \) induced by \( A \hookrightarrow A' \). Furthermore, we define \( H = \text{Aut}(S'; S, H_0) \), i.e.

\[
H = \left\{ (\tau, \tau_0) \in \text{Aut}_k(S') \times H_0 : \begin{array}{c}
S' \xrightarrow{\mu} S \\
\downarrow \quad \downarrow \\
S' \xrightarrow{\tau} S
\end{array} \text{ commutes} \right\} \quad (73)
\]
Then we have a natural morphism $\beta : (S', H) \to (S, H_0)$ in $\text{Sch}^{\text{sp}}/k$. Since the natural projection $H \to \text{Aut}_k(S')$ is injective, we regard $H$ as the subgroup of $\text{Aut}_k(S')$. The image of $\tau \in H$ under $H \to H_0$ is denoted by $\tau_0 \in H_0$.

**Proposition 4.7.** We have the following properties about $(S', H)$.

1. $S' \to S$ is a finite faithfully flat homomorphism.
2. We have the following isomorphism between $k$-algebras.

$$A' \xrightarrow{\cong} k\left[\gamma, \frac{1}{\gamma(\gamma - 1)}\right] ; \begin{cases} \sqrt{c} & \frac{\gamma + \frac{1}{2}}{2} \\ \sqrt{1 - c} & \frac{\gamma - \frac{1}{2}}{2\sqrt{-1}} \end{cases}$$  \quad (74)

Especially, $A'$ is an integral domain.

**Proposition 4.8.** The group $H$ has the following property.

1. The group homomorphism $H \to H_0$ is surjective.
2. The kernel of $H \to H_0$ is isomorphic to $\mu_2(k) \times \mu_2(k)$.

Especially, $H$ fits into the following exact sequence.

$$1 \to \mu_2(k) \times \mu_2(k) \to H \to H_0 \to 1 \quad (75)$$

Proof. To prove (1), we can check directly that following $\bar{\tau} : A' \to A'$ are the lifts of $\bar{\tau}_0 : A \to A$. (2) follows from the table below.

| $\bar{\tau}_0(c)$ | $\bar{\tau}^2(\gamma)$ | $\bar{\tau}_0(c)$ | $\bar{\tau}^2(\gamma)$ |
|-------------------|----------------------|-------------------|----------------------|
| $c$               | $\pm \gamma, \pm \frac{1}{\gamma}$ | $1 - c$           | $\pm \sqrt{-1} \gamma, \pm \sqrt{-1} \frac{1}{\gamma}$ |
| $\frac{c - 1}{c}$| $\pm \sqrt{\gamma - 1}, \pm \sqrt{\frac{1}{\gamma} - 1}$ | $\frac{1}{1 - c}$| $\pm \sqrt{-1} \sqrt{\gamma - 1}, \pm \sqrt{-1} \sqrt{\frac{1}{\gamma} - 1}$ |
| $\frac{c - 1}{c}$| $\pm \sqrt{\gamma - \frac{1}{\gamma}}, \pm \sqrt{\gamma + \frac{1}{\gamma}}$ | $\frac{1}{c}$     | $\pm \sqrt{-1} \sqrt{\gamma - \frac{1}{\gamma}}, \pm \sqrt{-1} \sqrt{\gamma + \frac{1}{\gamma}}$ |

\[\square\]

**Remark 4.9.** More strongly, we can show that $H$ is isomorphic to $\mathfrak{S}_4$ as follows. By the isomorphism (74) in Proposition 4.7, we can regard $S' = \mathbb{P}^1_k - \{1, \pm \sqrt{-1}, 0, \infty\}$. Let $N = \{\pm 1, \pm \sqrt{-1}, 0, \infty\} \subset \mathbb{P}^1(k)$. If we plot points of $N$ on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, $N$ forms an octahedron. We can check that $H$ acts on the set of pairs of opposite faces of this octahedron. Hence $H$ is naturally isomorphic to the octahedral group, which is isomorphic to $\mathfrak{S}_4$.

**Definition 4.10.** We define $(\mathbb{P}^1_S, H) \in (\text{Sch}^{\text{sp}}/k)$ as a fiber product of $(\mathbb{P}^1_S, H_0)$ and $(S', H)$ over $(S, H_0)$ in $\text{Sch}^{\text{sp}}/k$.

$$\begin{array}{ccc}
(\mathbb{P}^1_S, H) & \xrightarrow{\beta} & (S', H) \\
\downarrow & & \downarrow \\
(\mathbb{P}^1_S, H_0) & \xrightarrow{\alpha} & (S, H_0)
\end{array} \quad (76)$$

By Proposition 2.3, $H$ is equal to the fiber product

$$H_0 \times_{H_0} H = \{\tau = (\tau_0, \tau) \in H_0 \times H : \alpha(\tau_0) = \beta(\tau)\} \quad (77)$$

where $\alpha$ and $\beta$ are group homomorphisms corresponding to $\alpha : (\mathbb{P}^1_S, H_0) \to (S, H_0)$ and $\beta : (S', H) \to (S, H_0)$. Since $H_0 \simeq H \simeq \mathfrak{S}_4$ (Proposition 4.3 and Remark 4.9) and $H_0 \simeq \mathfrak{S}_3$ (Proposition 4.1), we have $H \simeq \mathfrak{S}_4 \times_{\mathfrak{S}_3} \mathfrak{S}_4$. 
By definition, we have the following natural group homomorphisms \( H \to H \) and \( H \to H_0 \). By Remark 4.5 and Proposition 4.8, they are surjective.

\[
\begin{array}{ccc}
H & \rightarrow & H \\
\downarrow & & \downarrow \\
H_0 & \rightarrow & H_0 \\
\tau & \mapsto & \tau \\
\downarrow & & \downarrow \\
\tau_0 & \mapsto & \tau_0
\end{array}
\]

(78)

The image of \( \tau \in H \) in \( H_0 \) and \( H \) are denoted by \( \tau_0 \in H_0 \) and \( \tau \in H \), respectively.

\( \Sigma(P_{S}^{1}) \) denotes the base change of \( \Sigma(P_{S}^{1}) \) by \( S' \to S \) (Definition 3.3). As we see in Definition 3.3, the natural surjection \( \Sigma(P_{S}^{1}) \to \Sigma(P_{S'}) \) induced by the base change is bijective.

**Corollary 4.11.** \( \Sigma(P_{S}^{1}) \) is compatible with respect to \( (P_{S}, H) \to (S', H') \). Furthermore, the bijection \( \Sigma(P_{S}^{1}) \to \Sigma(P_{S'}) \) is equivariant to \( H \)-action.

**Proof.** By Definition 4.12, \( \Sigma(P_{S}^{1}) \) is compatible with respect to \( (P_{S}, H_0) \to (S', H_0) \). Then by Proposition 2.5, we have the result. \( \square \)

By taking the direct product, we get the following automorphism groups.

**Definition 4.12.** We define the following objects in \( \text{Sch}^{op}/k \).

\[
\begin{align*}
(T, G_0) &= (S, H_0) \times (S, H_0) \\
(Y, G_0) &= (P_{S}^{1}, H_0) \times (P_{S}^{1}, H_0) \\
(T', G) &= (S', H) \times (S, H) \\
(Y', G) &= (P_{S'}^{1}, H) \times (P_{S}^{1}, H)
\end{align*}
\]

(79)

Since \( (P_{S}^{1}, H) \) is faithful, \( (Y', G) \) is also faithful. By Proposition 2.3, \( G_0, G_0, G \) and \( G \) coincide with \( H_0 \times H_0, H_0 \times H_0, H \times H, H \times H \). By considering homomorphisms in the morphisms of (82), we have the following diagram of groups.

\[
\begin{array}{ccc}
G & \rightarrow & G \\
\downarrow & & \downarrow \\
G_0 & \rightarrow & G_0 \\
\rho & \mapsto & \rho \\
\downarrow & & \downarrow \\
\rho_0 & \mapsto & \rho_0
\end{array}
\]

(80)

The images of \( \rho \in G \) in \( G, G_0 \) and \( G_0 \) by these maps are denoted as above. For \( \rho \in G \), its 1st and 2nd component are denoted by \( \rho^{(1)} \) and \( \rho^{(2)} \). i.e.

\[
G = \{(\rho^{(1)}, \rho^{(2)}); \rho^{(1)}, \rho^{(2)} \in H\}
\]

(81)

We define \( \rho^{(1)}, \rho^{(2)} \) similarly. By considering the direct products of morphisms in (76), we have the following morphisms in \( \text{Sch}^{op}/k \).

\[
\begin{array}{ccc}
(Y', G) & \rightarrow & (T', G) \\
\downarrow & & \downarrow \\
(Y, G_0) & \rightarrow & (T, G_0)
\end{array}
\]

(82)

By checking the universality, we see that the diagram (82) is the fiber product.

Especially, \( Y' \) is the fiber product of \( Y \) and \( T' \) over \( T \).

**Definition 4.13.** We define \( B' = A' \otimes_k A' \). By definition, \( T' = \text{Spec} B' \). For any schemes \( Z \) over \( T \), \( Z' \) denotes the base change of \( Z \) by \( T' \to T \). For example, \( Y' = Y \times_T T' \), \( X' = X \times_T T' \) and \( Q'_{\sigma} = Q_{\sigma} \times_T T' \). This notation is compatible with \( Y' = Y \times_T T' \).
**Proposition 4.14.** $\Sigma^2(Y')$ is compatible with respect to $(Y', G) \to (T', G)$. Especially, $G$ acts on the set $\Sigma^2(Y')$.

**Proof.** By Corollary 4.11, $\Sigma(\mathbb{P}^n_{Y'})$ is compatible with respect to $(\mathbb{P}^n_Y, H) \to (S', H)$. Hence by Proposition 2.5, we have the result. □

**Remark 4.15.** By Proposition 2.4. $G$-action on $\Sigma^2(Y')$ is described as follows: We identify $\Sigma^2 = \{0, 1, 1/a, \infty\} \times \{0, 1, 1/b, \infty\}$. Let $\rho \in G$. We identify $\rho_0^{(1)}, \rho_0^{(2)} \in H_0$ with

$$\begin{align*}
\rho_0^{(1)} & \in H_0 = \mathcal{S}((0, 1, 1/c, \infty)) \simeq \mathcal{S}((0, 1, 1/a, \infty)), \\
\rho_0^{(2)} & \in H_0 = \mathcal{S}((0, 1, 1/c, \infty)) \simeq \mathcal{S}((0, 1, 1/b, \infty))
\end{align*}$$

where the isomorphisms are induced by sending $1/c$ to $1/a$ and $1/b$. Then $\rho : \Sigma^2 \to \Sigma^2$ is the direct product of $\rho_0^{(1)} \times \rho_0^{(2)}$. For example, if $\rho \in G$ satisfies $\rho_0 = ((0, 1/\infty), (1, \infty))$, we have

$$\rho \cdot (0, 0) = (1/a, 0), \quad \rho \cdot (1/a, \infty) = (\infty, 1), \quad \rho \cdot (1, 1/b) = (1, 1/b).$$

**4.4. Linearizations and cocycles induced by group actions.** We define a linearization of $\mathcal{L}$ which give rise to a lifting of the $G$-action on $Y'$ to $Y'$.

Since $\mathcal{L} = pr_1^*O_{\mathbb{P}^1_{Y'}}(-2) \otimes O_{\mathbb{P}^1_{Y'}}(-2)$ and $O_{\mathbb{P}^1_{Y'}}(-2) \simeq \Omega^{1}_{\mathbb{P}^1_{Y'}}$, we have a $G$-linearization $\{\Psi_{\rho}\}_{\rho \in G}$ on $\mathcal{L}$. However, by this natural $G$-linearization, $\Psi^{(-2)}_{\rho}(\rho^*(\eta))$ and $\eta$ differs by

$$\Psi^{(-2)}_{\rho}(\rho^*(\eta)) = \chi_0(\rho) \cdot \eta.$$

where $\chi_0$ is an opposite 1-cocycle. The first aim is to get the explicit description of this $\chi_0$. Next, we will find an opposite 1-cocycle $\tilde{\chi}$ such that $\tilde{\chi}^2 = \chi_0$. For this purpose, we introduce an opposite coboundary of $1$-cocycles $\chi, \chi^{(1)}$ and $\chi^{(2)}$ and take a $\mathbb{Z}/2\mathbb{Z}$-extension $\tilde{G}$ of $G$. Finally, using $\tilde{\chi}$, we modify the linearization $\{\Psi_{\rho}\}_{\rho \in G}$ on $\mathcal{L}$ and get a new $\tilde{G}$-linearization $\{X_{\rho}\}_{\rho \in \tilde{G}}$ on $\mathcal{L}$ which satisfies the condition in Proposition 2.14.

The explicit presentation of these cocycles is useful not only for lifting group action, but also for describing group action by local coordinates. These local description enables us to describe explicit group actions on higher Chow subgroups $\Sigma^{\text{can}}$ in Section 6, on the distinguished relative $2$-form $\omega \in \Gamma(X', \Omega^{2}_{X'/T'})$ in Section 7 and on the image of higher Chow cycles by the regulator map in Section 9.

**Definition 4.16.** We define $H$-linearization $\Phi_{\tau} \in H \Omega^{1}_{\mathbb{P}^1_{Y'}}$ as the canonical one induced by Proposition 2.7. By definition, $\{\Phi_{\tau}\}_{\tau \in H}$ satisfies

$$\Phi_{\tau}((dz) \cdot \tau^2(z)) = \frac{\partial}{\partial z}((\tau^2(z)) \cdot dz).$$

By Proposition 2.9, $\{\Phi^{(-2)}_{\tau}\}_{\tau \in H}$ is the $H$-linearization of $\Omega^{1}_{\mathbb{P}^1_{Y'}}(\mathbb{P}^1_{Y'})$.

We define an opposite $1$-cocycle $\phi_{0} : H \to R(\mathbb{P}^1_{Y'})^\times$ as the opposite $1$-cocycle associated with

$$\left(\Omega^{1}_{\mathbb{P}^1_{Y'}}(\mathbb{P}^1_{Y'}), \Phi^{(-2)}_{\tau}\right)_{\tau \in H}, h(z)(dz)^{\otimes(-2)},$$

where $h(z) = z(1 - z)(1 - cz)$ in Definition 3.2. By definition, $\phi_{0}(\tau)$ can be computed as follows.

$$\phi_{0}(\tau) = \left(\frac{\partial}{\partial z}((\tau^2(z)))\right)^2 \frac{h(z)}{\tau^2(h(z))}.$$

**Proposition 4.17.** The opposite $1$-cocycle $\phi_{0}$ has the following properties.

---

5See Definition 3.10 for the definition of $B'$-rational point set $\Sigma^2(Y')$.

6See Definition 4.12 for the notation $\rho_0^{(1)}$.

7See Proposition 2.11.
(1) $\phi_0(\tau)$ is determined by the image of $\tau$ in $H \to H_0$.
(2) The explicit description of $\phi_0(\tau)$ is given by the following table.

| $\tau_0$ | $\tau_0^\flat(c)$ | $\phi_0(\tau_0)$ | $\tau_0^{-1}$ | $\tau_0^{-1}c$ | $\phi_0(\tau_0^{-1})$ |
|---|---|---|---|---|---|
| id | $c$ | 1 | (0 1) | $1 - c$ | $-1$ |
| $(1 \infty)$ | $\frac{c}{c - 1}$ | $1 - c$ | (0 1 $\infty$) | $\frac{1}{c - 1}$ | $c - 1$ |
| $(0 \infty)$ | $\frac{1}{c}$ | $c$ | (0 $\infty$) | $\frac{c - 1}{c}$ | $-c$ |

Especially, $\phi_0(\tau) \in A^\times$.

From these properties, we regard $\phi_0$ as the opposite 1-cocycle $H_0 \to A^\times$.

**Proof.** Since $\tau^i(z)$ depends only on $\tau_0 \in H_0$, by calculating $\phi_0(\tau)$ using the definition for each $\tau_0 \in H_0$ in Table 2, we have the result. □

We use the same symbol for $(\mathcal{L}, \eta)$ defined in Definition 3.7 and its pull-back by $\mathcal{Y} \to \mathcal{Y}$. We have an isomorphism

$$
\mathcal{L} = pr^*_1 \Omega_{\mathcal{Y}'}(-2) \otimes \mathcal{O}_{\mathcal{Y}'} \cong pr^*_2 \Omega_{\mathcal{Y}'}^{-2} \otimes \mathcal{O}_{\mathcal{Y}'} \cong \Omega_{\mathcal{Y}'/(S/)} \otimes \mathcal{O}_{\mathcal{Y}'} \cong \Omega_{\mathcal{Y}'/(S/)} \otimes \mathcal{O}_{\mathcal{Y}'} \quad (88)
$$

where the second isomorphism is induced by \( \mathcal{O}_{\mathcal{Y}'}(-2) \cong \text{Omega}_{\mathcal{Y}'/(S/)} \mathcal{O}_{\mathcal{Y}'} \), \( \frac{1}{\mathcal{Y}} \to dz \).

Then we define $G$-linearization \( \{ \Psi_\rho \}_{\rho \in \mathcal{G}} \) of $\mathcal{L}$ and important 3 opposite 1-cocycles $\chi_0^{(1)}$, $\chi_0^{(2)}$, and $\chi_0^0$.  

**Definition 4.18.** (Definition of $\chi_0^{(1)}$, $\chi_0^{(2)}$, $\chi_0^0$) For $i = 1, 2$, we have an $G$-linearization \( \{ pr^*_1 \Phi_{\rho,i} \}_{\rho \in \mathcal{G}} \) which is the pull-back (cf. Proposition 2.3) of $H$-linearization of $\Omega_{\mathcal{Y}'/(S/)}$ in Definition 4.16 by $pr_i : \mathcal{Y} \to \mathcal{Y}'$. If we regard $pr^*_1 \Omega_{\mathcal{Y}'}^{-1} \otimes \mathcal{O}_{\mathcal{Y}'}$ as a subsheaf of $\Omega_{\mathcal{Y}'/(S/)}$, this is the restriction of the natural $G$-linearization of $\Omega_{\mathcal{Y}'/(S/)}$.

Using isomorphism (88) above, we define $G$-linearization \( \{ \Psi_\rho \}_{\rho \in \mathcal{G}} \) of $\mathcal{L}$ as

$$
\Psi_\rho = pr^*_1 \Phi_{\rho,1} \otimes pr^*_2 \Phi_{\rho,2} \quad (89)
$$

For $i = 1, 2$, we define $\chi_0^{(i)}$ as the opposite 1-cocycles associated with

\( pr^*_1 \Omega_{\mathcal{Y}'}^{-1} \otimes \mathcal{O}_{\mathcal{Y}'} \) \{ $pr^*_1 \Phi_{\rho,i}^{(-2)}$ $pr^*_1 \Phi_{\rho,i}^{(-2)}$ $pr^*_1 \Phi_{\rho,i}^{(-2)}$ $pr^*_1 \Phi_{\rho,i}^{(-2)}$ $pr^*_1 \Phi_{\rho,i}^{(-2)}$ $pr^*_1 \Phi_{\rho,i}^{(-2)}$ \}. Since $pr^*_1 (h(z)(dz)^{(-2)}) = f(x)(dz)^{(-2)}$ and $pr^*_2 (h(z)(dz)^{(-2)}) = g(y)(dy)^{(-2)}$ where $f(x), g(y)$ are polynomials defined in (44) before Proposition 3.3 we have

$$
\begin{align*}
\chi_0^{(1)}(\rho) &= \left( \frac{\partial}{\partial x} (\rho^i(x)) \right)^2 \frac{f(x)}{\rho^i(f(x))} \\
\chi_0^{(2)}(\rho) &= \left( \frac{\partial}{\partial y} (\rho^i(y)) \right)^2 \frac{g(y)}{\rho^i(g(y))}
\end{align*}
$$

Since $\rho^i(x), \rho^i(y)$, $\rho^i(f(x))$ and $\rho^i(g(y))$ are determined by the image $\rho_0 \in \mathcal{G}_0$ of $\rho \in \mathcal{G}$, $\chi_0^{(i)}(\rho_0)$ is sometimes denoted by $\chi^{(i)}(\rho_0)$. Furthermore, we define $\chi_0$ as the opposite 1-cocycle associated with \( \mathcal{L}, \Omega_{\mathcal{Y}'/(S/)} \) \{ $\Psi_\rho^{(-2)}$ $\Psi_\rho^{(-2)}$ $\Psi_\rho^{(-2)}$ $\Psi_\rho^{(-2)}$ $\Psi_\rho^{(-2)}$ $\Psi_\rho^{(-2)}$ \}. By definition, we have

$$
\Psi_\rho^{(-2)}(\rho^i(\eta)) = \chi_0(\rho)^{-1} \cdot \eta.
$$

(91)

By the following Proposition 4.19, $\chi_0(\rho)$ is determined by $\rho_0$, $\chi_0(\rho)$ is also sometimes denoted by $\chi_0(\rho_0)$.

We can calculate $\chi_0^{(1)}, \chi_0^{(2)}$, and $\chi_0$ from Table 2 as follows.
Proposition 4.19. \( \chi_0^{(1)}, \chi_0^{(2)} \) and \( \chi_0 \) has the following properties.

1. For \( \rho \in G \), we have
\[
\chi_0^{(1)}(\rho) = pr_1^\sharp(\phi_0(\rho^{(1)})) \in B^\times
\]

2. For \( \rho \in G \), we have
\[
\chi_0(\rho) = \chi_0^{(1)}(\rho) \cdot \chi_0^{(2)}(\rho) \in B^\times.
\]

Proof. (1) follows from Proposition 1.17. Since \( \eta \) corresponds to \( pr_1^\sharp(h(z)(dz)^{\otimes(-2)}) \otimes pr_2^\sharp(h(z)(dz)^{\otimes(-2)}) \) under the isomorphism \( B ) \), (2) holds. \( \square \)

We will find an opposite 1-cocycle \( \tilde{\chi} \) such that \( \tilde{\chi}^2 = \chi_0 \). First, we will find an opposite coboundary 1-cocycle \( \phi \) of \( H \) whose square coincides with \( \phi_0 \) up to sign.

Definition-Proposition 4.20. For \( \tau \in H \), we define
\[
\phi(\tau) = \frac{1}{2}
\]

The explicit description of \( \phi \) is given in Table 9 in Section 9. By definition, this defines an opposite 1-cocycle of \( H \). \( \phi \) has the following properties.

1. For \( \tau \in H \), we have
\[
\phi_0(\tau) = \text{sgn}(\tau_0) \cdot \phi(\tau)^2.
\]

where \( \text{sgn} : \mathbb{H}_0 \simeq \mathbb{S}([0, 1, \infty]) \to \{\pm 1\} \) is the signature map.

2. \( \phi \) has the value in \((A')^\times\).

Proof. To prove (1), it is enough to calculate
\[
\phi(\tau)^2 = \frac{1}{2}
\]

Since the right hand side of the above equation depends only on \( \tau_0 \in \mathbb{H}_0 \) (which is the image of \( \tau \) by \( H \to \mathbb{H}_0 \)) and \( \phi_0(\tau) \) also depends only on \( \tau_0 \), it is enough to check (95) for each \( \tau_0 \in \mathbb{H}_0 \) by using Table 9 and Table 11. (2) follows from (1). \( \square \)

We get an opposite 1-cocycle \( \chi \) of \( G \) whose square coincides with \( \chi_0 \) up to sign.

Definition-Proposition 4.21. (Definition of \( \chi^{(1)}, \chi^{(2)} \) and \( \chi \)) For \( \rho \in G \), we define
\[
\chi^{(i)}(\rho) = pr_1^\sharp(\phi(\rho^{(i)})) \in (B')^\times \text{ for } i = 1, 2
\]
\[
\chi(\rho) = \chi^{(1)}(\rho) \cdot \chi^{(2)}(\rho) \in (B')^\times.
\]

Since \( G = H \times H \) acts on \( B' = A' \otimes_k A' \) component-wisely, we see that \( \chi : G \to (B')^\times \) defines an opposite 1-cocycle.

\( \chi \) satisfies the following equation for any \( \rho \in G \).
\[
\begin{aligned}
\chi^{0(1)}(\rho) & = \text{sgn}(\rho_0^{(1)}) \chi^{1(1)}(\rho)^2 \\
\chi^{0(2)}(\rho) & = \text{sgn}(\rho_0^{(2)}) \chi^{1(2)}(\rho)^2 \\
\chi^{0}(\rho) & = \text{sgn}(\rho_0^{(1)}) \text{sgn}(\rho_0^{(2)}) \cdot \chi(\rho)^2.
\end{aligned}
\]

where \( \rho_0 \in \mathbb{G} \) is the image of \( \rho \) in \( G \) by \( G \to \mathbb{G}_0 \) in Definition 4.13.

Proof. The equation (98) follows from Proposition 4.20. \( \square \)

By Proposition 4.21, to find an opposite 1-cocycle \( \tilde{\chi} \) such that \( \tilde{\chi}^2 = \chi_0 \), it is enough to find a square root of the map \( \mathbb{G} \to \{\pm 1\} : \rho \mapsto \text{sgn}(\rho_0^{(1)}) \text{sgn}(\rho_0^{(2)}) \). Hence we enlarge \( G \) as follows.
Definition 4.22. Let $\tilde{G}$ be the following fiber product of group homomorphisms.

$$
\begin{array}{ccc}
\tilde{G} & \xrightarrow{\text{sgn}} & \mu_4(k) \\
\downarrow & & \downarrow \\
G & \xrightarrow{\mu_2(k)} & \zeta^2 \\
\rho & \xrightarrow{\text{sgn}(\rho^{(1)})\text{sgn}(\rho^{(2)}) } & \\
\end{array}
$$

(99)

Then $\tilde{G}$ can be written as follows.

$$\tilde{G} = \{(\rho, \zeta) \in G \times \mu_4(k) : \text{sgn}(\rho^{(1)})\text{sgn}(\rho^{(2)}) = \zeta^2\}$$

(100)

We write elements in $\tilde{G}$ as $\tilde{\rho}$ or $(\rho, \zeta)$. We define $\text{sgn} : \tilde{G} \rightarrow \mu_4(k)$ as above.

Since $\sqrt{-1} \in k$, $\mu_4(k) \rightarrow \mu_2(k)$; $\zeta \mapsto \zeta^2$ is surjective and the kernel of this group homomorphism is $\mu_2(k) \subset \mu_4(k)$. Especially, we have the following group homomorphism.

$$1 \longrightarrow \mu_2(k) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

(101)

Finally, we get the desired cocycle $\tilde{\chi}$.

Definition-Proposition 4.23. For $\tilde{\rho} = (\rho, \zeta) \in \tilde{G}$, we define

$$\tilde{\chi}(\tilde{\rho}) = \text{sgn}(\tilde{\rho}) \cdot \chi(\rho) = \zeta \cdot \chi(\rho) \in (B')^\times.$$  

(102)

where $\rho \in G$ is the image of $\rho \in G$ under $G \rightarrow \tilde{G}$. Then $\tilde{\chi}$ defines an opposite 1-cocycle of $\tilde{G}$ on $(B')^\times$. Here $\tilde{G}$ acts oppositely on $(B')^\times$ by $\tilde{G} \rightarrow G \rightarrow \tilde{G}$.

Furthermore, $\tilde{\chi}$ satisfies the following equation for any $\tilde{\rho} = (\rho, \zeta) \in \tilde{G}$.

$$\tilde{\chi}(\tilde{\rho})^2 = \chi(\rho)$$

(103)

Proof. Since $\text{sgn}$ is the group homomorphism and each element in $\mu_4(k) \subset B'$ is stable under $\tilde{G}$-action, $\text{sgn}$ is an opposite 1-cocycle of $\tilde{G}$. Since $\chi$ is also an opposite 1-cocycle, their product $\tilde{\chi}$ is also an opposite 1-cocycle. Since $\text{sgn}$ satisfies $\text{sgn}(\rho)^2 = \text{sgn}(\rho^{(1)})\text{sgn}(\rho^{(2)})$, the equation (103) follows from (102) in Proposition 4.22. \(\square\)

4.5. A $\tilde{G}$-action on the Kummer surface family $X'$. Recall that $\tilde{Y}'$ is the base change of $Y$ by $T' \rightarrow T$ (Definition 4.13). Using the opposite 1-cocycle $\tilde{\chi}$ in Definition 4.23 we can lift $G$-action on $\tilde{Y}'$ to $\tilde{G}$-action on $\tilde{Y}'$.

Proposition 4.24. We have a $\tilde{G}$-action on $\tilde{Y}'$ such that $(\tilde{Y}', \tilde{G}) \rightarrow (Y', G)$ is a morphism in $(\text{Sch}_{\text{et}}/k)$. For $\tilde{\rho} = (\rho, \zeta) \in \tilde{G}$, $\tilde{\rho}^*: \mathcal{O}_{\tilde{Y}'} \rightarrow \rho_*\mathcal{O}_{\tilde{Y}'}$ is described as follows.

$$x \mapsto \rho^*(x), \quad y \mapsto \rho^*(y), \quad u \mapsto \tilde{\chi}(\tilde{\rho})^{-1} \frac{\partial}{\partial x}(\rho^*(x)) \frac{\partial}{\partial y}(\rho^*(y))u$$

(104)

where we use the notation in Proposition 3.8.

Proof. Recall the opposite 1-cocycle $\tilde{\chi}$ on $(B')^\times = \Gamma(Y', \mathcal{O}_{\tilde{Y}'})$ in Definition 4.23. For $\tilde{\rho} \in \tilde{G}$, consider the following $\mathcal{O}_{\tilde{Y}'}$-module isomorphism.

$$X_{\tilde{\rho}} : \rho^*\mathcal{L} \xrightarrow{\Phi_{\rho}} \mathcal{L} \xrightarrow{\tilde{\chi}(\tilde{\rho})^{-1}/\times} \mathcal{L}.$$  

(105)
where $\Psi_\rho$ is the $G$-linearization of $\mathcal{L}$ defined in Definition 4.18. By Proposition 2.12 \( \{X_\rho\}_{\rho \in \tilde{G}} \) defines $\tilde{G}$-linearization on $O_{\mathcal{Y}'}$. Then by Proposition 4.19 and Proposition 4.23 we have

\[
X_\rho^\otimes(-2)(\rho^*(\eta)) = \tilde{\chi}(\tilde{\rho})^2 \cdot \Psi_\rho^\otimes(-2)(\rho^*(\eta)) = \tilde{\chi}(\tilde{\rho})^2 \cdot \chi_0(\rho)^{-1} \cdot \eta = \eta. \tag{106}
\]

Especially \( \{X_\rho\}_{\rho \in \tilde{G}} \) satisfies (37) in Proposition 2.14. Since $\tilde{\psi}'$ is the double covering associated with $(\mathcal{L}, \eta)$ by Proposition 3.14 we have a $\tilde{G}$-action on $\tilde{\psi}'$ such that $(\tilde{\psi}', \tilde{G}) \to (\mathcal{Y}', G)$ is a morphism in $(\text{Sch}^{op}/k)$ by Proposition 2.14.

We can calculated the local description of $\tilde{G}$-action directly from the construction in Proposition 2.14. We can check that this action preserves the equation $u^2 - f(x)g(y) = 0$ as follows. For $\tilde{\rho} = (\rho, \zeta) \in \tilde{G}$, we have

\[
\tilde{\rho}^2(u^2 - f(x)g(y)) = \tilde{\rho}^2(u^2) - \rho^2(f(x))\rho^2(g(y))
\]

\[
= \tilde{\chi}(\tilde{\rho})^{-2} \left( \frac{\partial}{\partial x}(\rho^2(x)) \frac{\partial}{\partial y}(\rho^2(y)) \right)^2 u - \rho^2(f(x))\rho^2(g(y))
\]

\[
= \tilde{\chi}(\tilde{\rho})^{-2} \left( \frac{\partial}{\partial x}(\rho^2(x)) \frac{\partial}{\partial y}(\rho^2(y)) \right)^2 u
\]

\[
- (\chi_0^2(\rho))^{-1} \left( \frac{\partial}{\partial x}(\rho^2(x)) \frac{\partial}{\partial y}(\rho^2(y)) \right)^2 f(x)g(y)
\]

\[
\tag{107}
\]

\[\chi_0(\rho)^{-1} \left( \frac{\partial}{\partial x}(\rho^2(x)) \frac{\partial}{\partial y}(\rho^2(y)) \right)^2 (u^2 - f(x)g(y)) = 0.\]

Recall that $\mathcal{X}'$ is the base change of $\mathcal{X}$ by $T' \to T$ (Definition 4.13). We lift $\tilde{G}$-action on $\tilde{\psi}'$ to $\mathcal{X}'$. Since $\mathcal{X}' \to \mathcal{Y}'$ is blowing-up, it is enough to show blowing-up locus is stable under $\tilde{G}$-action by Proposition 2.16.

**Proposition 4.25.** The set $\Sigma^2(\tilde{\psi}')$ of $B'$-rational points is compatible with respect to $(\tilde{\psi}', \tilde{G}) \to (T', \tilde{G})$. Furthermore, the closed subscheme $\bigsqcup_{\sigma \in \Sigma^2} \sigma \to \tilde{\psi}'$ is $\tilde{G}$-stable.

**Proof.** By Proposition 4.14 $\Sigma^2(\mathcal{Y}')$ is compatible with respect to $(\mathcal{Y}', G) \to (T', \tilde{G})$. Since $\tilde{G}$-action on $\tilde{\psi}'$ is a lift of $G$-action on $\mathcal{Y}'$, we can check that $\Sigma^2(\tilde{\psi}')$ is compatible with respect to $(\tilde{\psi}', \tilde{G}) \to (T', \tilde{G})$. Especially $\tilde{G}$ acts on $\Sigma^2(\tilde{\psi}')$. \(\square\)

**Proposition 4.26.** There exists a $\tilde{G}$-action on $\mathcal{X}'$ such that $(\mathcal{X}', \tilde{G}) \to (\tilde{\psi}', \tilde{G})$ is a morphism in $(\text{Sch}^{op}/k)$.

**Proof.** By Proposition 3.14 $\mathcal{X}' \to \tilde{\psi}'$ is the blowing-up along $\bigsqcup_{\sigma \in \Sigma^2} \sigma$. By Proposition 4.23 this closed subscheme is $\tilde{G}$-stable. Hence by applying Proposition 2.16 we have the result. \(\square\)

Recall that for $\sigma \in \Sigma^2$, $Q_\sigma \subset \mathcal{X}$ denotes the exceptional divisor over $\sigma(T) \subset \tilde{\psi}'$ (Definition 3.13) and $Q'_\sigma \subset \mathcal{X}'$ denote the base change of $Q_\sigma$ by $T' \to T$ (Definition 4.13). This is the same as the inverse image of $\sigma \subset \tilde{\psi}'$ by $\mathcal{X}' \to \tilde{\psi}'$ where $\sigma \in \Sigma^2$. Hence we have the following.

**Proposition 4.27.** For $\tilde{\rho} = (\rho, \zeta) \in \tilde{G}$ and $\sigma \in \Sigma^2$, the following holds.

\[
\tilde{\rho}(Q'_\sigma) = Q'_{\rho \cdot \sigma} \tag{108}
\]

Finally, we describe $\tilde{G}$-action on $\mathcal{X}'$ locally.
Corollary 4.28. For \( \tilde{\rho} = (\rho, \zeta) \in \tilde{G} \), \( \tilde{\rho}^! : \mathcal{O}_{X'} \to \rho_* \mathcal{O}_{X'} \) can be described locally as

\[
x \mapsto \rho^t(x), \quad y \mapsto \rho^y(y), \quad v \mapsto \frac{\text{sgn}(\rho^{(1)}(\zeta) \chi^{(1)}(\rho) \frac{\partial}{\partial y}(\rho^y(y)))}{\chi^{(2)}(\rho) \frac{\partial}{\partial x}(\rho^t(x))} v
\]

where we use the notation in Proposition 3.12.

Proof. Using the relation \( u = v f(x) \) and Proposition 4.24, we can calculate \( \tilde{\rho}^!(v) \) as follows.

\[
\tilde{\rho}^!(v) = \frac{\tilde{\rho}^!(u)}{\rho^t(f(x))} = \frac{f(x)}{\rho^t(f(x))} \frac{\chi(\rho)^{-1} \frac{\partial}{\partial x}(\rho^t(x)) \frac{\partial}{\partial y}(\rho^y(y)) v}{\chi(\rho)}
\]

where the numbers under the \( = \) indicate the equations’ number in the transformation. \( \square \)

Finally, we can prove Proposition 1.3 as follows.

Proposition 4.29. \( \text{Aut}(X' \to T') \) has a finite subgroup \( \tilde{G} \) which is isomorphic to a \( \mathbb{Z}/2\mathbb{Z} \) extension of \( (\mathfrak{S}_1 \times \mathfrak{S}_1)^2 \).

Proof. It is enough to show the following.

(1) \( \tilde{G} \) is isomorphic to a subgroup of \( \text{Aut}_k(X \to T) \).

(2) \( \tilde{G} \) is isomorphic to a \( \mathbb{Z}/2\mathbb{Z} \) extension of \( (\mathfrak{S}_1 \times \mathfrak{S}_1)^2 \).

By Definition 4.12, Proposition 4.24 and Proposition 4.26, we have the following morphisms in \( \text{Sch}^{/k} \).

\[
(X', \tilde{G}) \to (\tilde{Y}', \tilde{G}) \to (Y', G) \to (T', G).
\]

By the explicit description in Corollary 4.28, \( \tilde{X}', \tilde{G} \) is faithful. Then by Proposition 2.2, we have (1).

By the exact sequence in Definition 4.22, \( \tilde{G} = \mu_2(k) \simeq \mathbb{Z}/2\mathbb{Z} \)-extension of \( G \). Furthermore, \( G = H \times H \) (Definition 4.12) and \( H = H_0 \times H_0 \) (Definition 4.10). We have isomorphisms \( H_0 \simeq \mathfrak{S}_3 \) (Proposition 4.1), \( H_0 \simeq \mathfrak{S}_4 \) (Proposition 4.3) and \( H \simeq \mathfrak{S}_4 \) (Remark 4.9). Hence, we have (2). \( \square \)

For later use, we check \( \tilde{G} \)-action on fibers of \( X' \to T' \) at \( k \)-rational points of \( T' \).

Definition 4.30. For a \( k \)-rational point \( t \in T'(k) \), let \( \mathcal{X}_t \) denote the fiber of \( X' \) over \( t \). In other words, \( \mathcal{X}_t \) is the fiber product of the following diagram.

\[
\begin{array}{ccc}
\mathcal{X}_t & \xrightarrow{i_t} & X' & \xrightarrow{j} & X \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spec } k & \xrightarrow{t} & T' & \xrightarrow{j} & T
\end{array}
\]

\( i_t : \mathcal{X}_t \to X' \) denotes the upper left horizontal map in the diagram above. We also denote the fibers of \( \tilde{Y} \) and \( Y \) at \( t \) by \( \tilde{Y}_t \) and \( Y_t \).

Definition-Proposition 4.31. For a \( k \)-rational point \( t \in T'(k) \) and \( \tilde{\rho} = (\rho, \zeta) \in \tilde{G} \), let \( \rho(t) \in T'(k) \) denote the \( k \)-rational point \( \rho \circ t \). We define \( \rho_t : \mathcal{X}_t \to \mathcal{X}_t^{\rho(t)} \) as
the unique isomorphism over $k$ which makes the following diagram commutes.

\[ \begin{array}{ccc}
X' & \xrightarrow{\iota} & T' \\
\downarrow{\rho} & & \downarrow{\iota} \\
\tilde{X} & \xleftarrow{\iota} & \Spec k
\end{array} \quad (113) \]

5. Construction of a Higher Chow Subgroup $\Xi^{\text{can}}$

As in Section 3, we fix a field $k$ whose characteristic is not 2 and contains $\sqrt{-1}$. In this section, we explain the construction of higher Chow subgroup $\Xi^{\text{can}} \subset \text{CH}^2(X^\circ, 1)$ where $X^\circ$ is an open subset of $X'$. For the construction, we use the following facts about higher Chow group.

**Proposition 5.1.** Let $X$ be a variety over $k$. The higher Chow group $\text{CH}^2(X, 1)$ is canonically isomorphic to the homology group of the following sequence.

\[ K^M_2(R(X)) \xrightarrow{T} \bigoplus_{Z \in X^{(1)}} R(Z)^{\times} \xrightarrow{\text{div}} \bigoplus_{p \in X^{(2)}} \Z \cdot p \quad (114) \]

where $X^{(1)}, X^{(2)}$ are the sets of integral closed subschemes of $X$ codimension 1 and 2, the map $\text{div}$ is the sum of the divisor map $\text{div}_Z$ for each $Z \in X^{(1)}$ and $T$ is the tame symbol map.

Hence our task is to find good subschemes and rational functions which lie in the kernel of $\text{div}$.

5.1. **Family of Curves $\mathcal{C}$ on $X^\circ$.** We construct a family of curves $\mathcal{C}$ on $X^\circ$. The subscheme $\mathcal{C}$ is the key for our construction of higher Chow cycles. First, we define an open subset $T^\circ \subset T'$. Hereafter we consider all things on this open subset.

**Definition 5.2.** Under the $G$-action on $B'$, the orbit of $a - b \in B'$ consists of the following 6 elements.

\[ a - b, a - (1 - b), a - b - 1, a - 1 - b, a - 1 - b, a - b - 1 \quad (115) \]

We define $k$-algebra $B^\circ$ by the localization of $B'$ by these 6 elements. We define $T^\circ = \text{Spec } B^\circ$, which is an open subscheme of $T'$. For a scheme $Z$ over $T'$, $Z^\circ$ denotes its base change by $T^\circ \hookrightarrow T'$. For example, $\mathcal{Y}^\circ = \mathcal{Y}' \times_{T'} T^\circ$, $\tilde{\mathcal{Y}}^\circ = \tilde{\mathcal{Y}}' \times_{T'} T^\circ$ and $X^\circ = X' \times_{T'} T^\circ$.

By the construction, $T^\circ \subset T'$ is invariant under $G$-action. Hence we have the following diagram in $(\text{Sch}^{\text{dp}}/k)$ from (111) whose vertical morphisms are open immersions.

\[ \begin{array}{ccc}
(A^\circ, \tilde{G}) & \longrightarrow & (\tilde{\mathcal{Y}}^\circ, \tilde{G}) \\
\downarrow & & \downarrow \\
(A'^\circ, \tilde{G}) & \longrightarrow & (\tilde{\mathcal{Y}}', \tilde{G})
\end{array} \quad (116) \]

**Definition 5.3.** We define a morphism $\Delta : \mathbb{P}^1_T \rightarrow \mathcal{Y}$ over $T$ as the following diagonal embedding.

\[ \Delta : \mathbb{P}^1_T = \mathbb{P}^1_{S'} \times_k S' \longrightarrow \mathbb{P}^1_{S'} \times_k S' = \mathcal{Y}' \\
\longrightarrow (z, z) \quad (117) \]
We use the symbol $\Delta$ for the $\Delta^\circ$. Then we define a closed subscheme $D \subset Y^\circ$ by the schematic image of $\Delta : \mathbb{P}^1_{T^\circ} \to Y^\circ$. By definition, the local equation of $D$ on $Y_{0,0}$ (which is the open subset of $Y$ defined in Definition 3.6) is $x = y$.

![Diagram](image)

**Figure 2.** The figure of $D$ on $Y^\circ$

**Definition-Proposition 5.4.** We define $\tilde{D} \hookrightarrow \tilde{Y}^\circ$ as the following fiber product.

$$
\begin{array}{c}
\tilde{D} \\
\downarrow \delta \\
D
\end{array} \hookrightarrow \begin{array}{c}
\tilde{Y}^\circ \\
\downarrow \\
Y^\circ
\end{array}
$$

(118)

The closed immersion $\tilde{D} \hookrightarrow \tilde{Y}^\circ$ is described locally on $\tilde{Y}_{0,0}^\circ \subset Y^\circ$ as follows.

$$
\text{Spec } B^\circ[u, z]/(u^2 - f(z)g(z)) \longrightarrow \text{Spec } B^\circ[u, x, y]/(u^2 - f(x)g(y)) = \tilde{Y}_{0,0}^\circ
$$

(119)

where $f(z), g(z)$ are polynomials in $[44]$ and the morphism is induced by sending $x, y$ to $z$.

**Definition-Proposition 5.5.** We define $C \hookrightarrow X^\circ$ as the strict transformation of $\tilde{D} \hookrightarrow \tilde{Y}^\circ$ by the blowing-up $X^\circ \to \tilde{Y}^\circ$. The closed immersion $C \hookrightarrow X^\circ$ is described locally on $V_{0,0}^\circ, W_{0,0}^\circ \subset X^\circ$ as follows.

$$
\begin{align*}
\text{Spec } B^\circ[v, z]/(v^2(1 - az) - (1 - bsz)) & \to \text{Spec } B^\circ[v, x, y]/(v^2f(x) - g(y)) = V_{0,0}^\circ \\
\text{Spec } B^\circ[w, z]/(w^2(1 - bz) - (1 - az)) & \to \text{Spec } B^\circ[w, x, y]/(w^2g(y) - f(x)) = W_{0,0}^\circ
\end{align*}
$$

(120)

where these morphisms are induced by sending $x, y$ to $z$.

By the description in Proposition 5.5, since $a - b$ is invertible on $T^\circ$, we see that $C$ is a conic bundle on $T^\circ$ with a section (e.g. $x = y = v - 1 = 0$). Hence we have the following corollary.

**Corollary 5.6.** $C$ is isomorphic to $\mathbb{P}^1_{T^\circ}$.
We have the following subschemes.

\[\xymatrix{ X^o \ar[r] & \breve{Y}^o \ar[r] & \breve{Y}^o \ar[u] & D \ar[u] \ar[r] & D \ar[u] }\]  \hfill (121)

5.2. Construction of Higher Chow Cycles $\Xi^{can}$ and restrictions to a fiber.

In this section, we will construct a family of higher Chow cycles $\Xi^{can} \subset CH^2(X^o, 1)$.

For the construction, we use Proposition 5.1 and closed subschemes $C$ in the previous section and exceptional divisors $Q^o_\sigma$ in Definition 3.13.

To define rational functions, we use the following local description of $Q^o_{(0,0)}$, $Q^o_{(1,1)}$ and $Q^o_{(\infty, \infty)}$. Since $Q^o_{(0,0)}$ and $Q^o_{(1,1)}$ are contained in $V_{0,0}$ and defined by the equation $x = y = 0$ and $x = y = 1$, we have the following description.

\[
\begin{align*}
V^o_{0,0} \cap Q^o_{(0,0)} &= \text{Spec } B^o[v, x, y]/(v^2 f(x) - g(y), x, y) \simeq \text{Spec } B^o[v] \\
V^o_{0,0} \cap Q^o_{(1,1)} &= \text{Spec } B^o[v, x, y]/(v^2 f(x) - g(y), x - 1, y - 1) \simeq \text{Spec } B^o[v]
\end{align*}
\]  \hfill (122)

To get the local description of $Q^o_{(\infty, \infty)}$, we consider the following affine open sub-scheme $V^o_{1,1}$ of $X^o$.

\[
V^o_{1,1} = \text{Spec } B^o[v', \xi, \eta]/((v')^2 \xi (\xi - 1)(\xi - a) - \eta(\eta - 1)(\eta - b))
\]  \hfill (123)

Here $\xi = \frac{1}{x}, \eta = \frac{1}{y}$ and $v' = \frac{a^2}{b^2}v$. Since $Q^o_{(\infty, \infty)}$ is defined by the equation $\xi = \eta = 0$, we have the following description.

\[
V^o_{1,1} \cap Q^o_{(\infty, \infty)} = \text{Spec } B^o[v', \xi, \eta]/(v'^2 \xi (\xi - 1)(\xi - a) - \eta(\eta - 1)(\eta - b), \xi, \eta) \simeq \text{Spec } B^o[v']
\]  \hfill (124)

Furthermore, the inclusion $C \hookrightarrow X^o$ on $V^o_{1,1}$ is corresponding to the $B^o$-algebra morphism. Here $\zeta = \frac{1}{2}$.

\[
B^o[v', \xi, \eta]/((v')^2 \xi (\xi - 1)(\xi - a) - \eta(\eta - 1)(\eta - b)) \twoheadrightarrow B^o[v, \zeta]/(v^2 (\zeta - a) - (\zeta - b))
\]  \hfill (125)

**Definition-Proposition 5.7.** We define six $B^o$-rational points $p^o_{\delta}(\bullet \in \{0, 1, \infty\}, \delta \in \{+, -\})$ on $X^o$ as follows.

1. $p^+_0$ and $p^-_0$ correspond to $B^o$-rational points on $V^o_{0,0} \subset X^o$ such that $(v, x, y) = (1, 0, 0)$ and $(v, x, y) = (-1, 0, 0)$.
2. $p^+_1$ and $p^-_1$ correspond to $B^o$-rational points on $V^o_{0,0} \subset X^o$ such that $(v, x, y) = \left(\frac{\sqrt{2}}{\sqrt{\sqrt{2} - 1}}, 1, 1\right)$ and $(v, x, y) = \left(-\frac{\sqrt{2}}{\sqrt{\sqrt{2} - 1}}, 1, 1\right)$.
3. $p^+_\infty$ and $p^-_\infty$ correspond to $B^o$-rational points on $V^o_{1,1} \subset X^o$ such that $(v', \xi, \eta) = \left(\frac{\sqrt{2}}{\sqrt{2} - 1}, 0, 0\right)$ and $(v, x, y) = \left(-\frac{\sqrt{2}}{\sqrt{2} - 1}, 0, 0\right)$.

Then we have the following relations.

\[
\begin{align*}
C \cap Q^o_{(0,0)} &= p^+_0 \cup p^-_0 \\
C \cap Q^o_{(1,1)} &= p^+_1 \cup p^-_1 \\
C \cap Q^o_{(\infty, \infty)} &= p^+_\infty \cup p^-_\infty
\end{align*}
\]  \hfill (126)

**Definition 5.8.** We define the following non-zero rational functions on $C$, $Q^o_{(0,0)}$, $Q^o_{(1,1)}$, and $Q^o_{(\infty, \infty)}$ using the local description in Proposition 5.7 and equation (122), (123).
A GROUP ACTION ON HIGHER CHOW CYCLES ON A KUMMER SURFACE FAMILY

Figure 3. The relation between $p_\delta$ and $C, Q_{(\bullet, \bullet)}$

\begin{align*}
(1) \quad & \psi_0 = (v + 1) \cdot (v - 1)^{-1}, \quad \psi_1 = \left( v + \frac{\sqrt{-1}}{\sqrt{1 - a}} \right) \cdot \left( v - \frac{\sqrt{-1}}{\sqrt{1 - a}} \right)^{-1}, \\
& \psi_\infty = \left( v + \frac{\sqrt{b}}{\sqrt{a}} \right) \cdot \left( v - \frac{\sqrt{b}}{\sqrt{a}} \right)^{-1} \in R(C)^\times \\
(2) \quad & \varphi_0 = (v - 1) \cdot (v + 1)^{-1} \in R\left( Q_{(0,0)}^\circ \right)^\times \\
(3) \quad & \varphi_1 = \left( v - \frac{\sqrt{-1}}{\sqrt{1 - a}} \right) \cdot \left( v + \frac{\sqrt{-1}}{\sqrt{1 - a}} \right)^{-1} \in R\left( Q_{(1,1)}^\circ \right)^\times \\
(4) \quad & \varphi_\infty = \left( v' - \frac{\sqrt{b}}{\sqrt{a}} \right) \cdot \left( v' + \frac{\sqrt{b}}{\sqrt{a}} \right)^{-1} \in R\left( Q_{(\infty, \infty)}^\circ \right)^\times \\
\end{align*}

By the explicit description of $\varphi_\bullet$ and $\psi_\bullet$, we have the following.

**Proposition 5.9.** The rational functions $\varphi_\bullet, \psi_\bullet$ satisfy the following relations.

\begin{align*}
(1) \quad & \text{div}_C(\psi_0) = p_0^- - p_0^+ = -\text{div}_{Q_{(0,0)}^\circ}(\varphi_0) \\
(2) \quad & \text{div}_C(\psi_1) = p_1^- - p_1^+ = -\text{div}_{Q_{(1,1)}^\circ}(\varphi_1) \\
(3) \quad & \text{div}_C(\psi_\infty) = p_\infty^- - p_\infty^+ = -\text{div}_{Q_{(\infty, \infty)}^\circ}(\varphi_\infty) \\
\end{align*}

Then we can construct at most rank 3 subgroup $\Xi_{\text{can}}$ of $\text{CH}^2(X^\circ, 1)$ as follows.

**Definition 5.10.** (Definition of $\Xi_{\text{can}}$) Consider the following elements of $\bigoplus_{Z \in (X^\circ)^{(1)}} Z \cdot p$.

\begin{align*}
\xi_0 = (C, \psi_0) + (Q_{(0,0)}^\circ, \varphi_0) \\
\xi_1 = (C, \psi_1) + (Q_{(1,1)}^\circ, \varphi_1) \\
\xi_\infty = (C, \psi_\infty) + (Q_{(\infty, \infty)}^\circ, \varphi_\infty)
\end{align*}

By Proposition 5.9, they are in $\text{Ker} \left( \bigoplus_{Z \in (X^\circ)^{(1)}} R(Z)^\times \xrightarrow{\text{div}} \bigoplus_{p \in (X^\circ)^{(1)}} \mathbb{Z} \cdot p \right)$. Hence these elements defines elements in $\text{CH}^2(X^\circ, 1)$ which are also denoted by $\xi_0, \xi_1, \xi_\infty$ respectively.

For $\epsilon \in \mathbb{Z}^{[0,1,\infty]}$, we define

\begin{equation}
\xi(\epsilon) = \epsilon(0)\xi_0 + \epsilon(1)\xi_1 + \epsilon(\infty)\xi_\infty.
\end{equation}

We define $\Xi_{\text{can}} \subset \text{CH}^2(X^\circ, 1)$ by a subgroup generated by these $\xi(\epsilon)$. By definition, we have the following surjective group homomorphism.

\begin{equation}
\mathbb{Z}^{[0,1,\infty]} \xrightarrow{\psi} \Xi_{\text{can}} \xrightarrow{\epsilon} \xi(\epsilon)
\end{equation}
Definition 5.11. Since \( T^o \) is regular scheme, for a \( k \)-rational point \( t \in T^o(k) \), \( i_t : X_t \to X^o \) in Definition 4.30 is an regular imbedding. Hence we have a pull-back map
\[
i_t^* : \text{CH}^2(X^o, 1) \to \text{CH}^2(X_t, 1).
\]
For each \( \xi \in \text{CH}^2(X^o, 1) \), we set \( \xi_t = i_t^* \xi \).

Since \( Q_\bullet, \bullet \) and \( C \) are smooth family on \( T^o \) and the zeros and poles of \( \varphi_\bullet, \psi_\bullet \) intersect properly with each fiber of \( X^o \to T^o \), we have the following.

Proposition 5.12. For a \( k \)-rational point \( t \in T^o(k) \) and \( \epsilon \in \mathbb{Z}^{(0,1,\infty)} \), \( \xi(\epsilon)_t \) is represented by the following element in \( \bigoplus_{Z \in \mathcal{X}(\mathcal{Y})(1)} R(Z)^\times \).
\[
\left( C_t, (\psi_\bullet)_t^{(0)}(\psi_1)_t^{(1)}(\psi_\infty)_t^{(\infty)}\right) + (Q_0(0,0)_t, (\varphi_\bullet)_t^{(0)} + (Q_1(1,1)_t, (\varphi_\bullet)_t^{(1)} + (Q_\infty(\infty,\infty)_t, (\varphi_\bullet)_t^{(\infty)})
\]
Here \( C_t, Q_\bullet, \bullet \) are the fibers of \( C \) and \( Q_\bullet, \bullet \) at \( t \) and \((\psi_\bullet)_t, (\varphi_\bullet)_t \) are the pull back of the rational function \( \psi_\bullet, \varphi_\bullet \) by \( C_t \to C \) and \( Q_\bullet, \bullet \to Q_\bullet, \bullet \).

5.3. Definition of a subgroup \( \Xi \) of higher Chow group. In this section, we define \( \Xi \subset \text{CH}^2(X^o, 1) \) and compute representatives in \( \bigoplus_{Z \in \mathcal{X}(\mathcal{Y})(1)} R(Z)^\times \).

Definition 5.13. We define a subgroup \( \Xi \) of \( \text{CH}^2(X^o, 1) \) as
\[
\Xi = \sum_{\rho \in \bar{G}} \rho_* \Xi^\text{can}
\]
where \( \Xi^\text{can} \subset \text{CH}^2(X^o, 1) \) is the subgroup of higher Chow group defined in Definition 5.10 and \( \rho_* : \text{CH}^2(X^o, 1) \to \text{CH}^2(X^o, 1) \) is the push-forward map induced by an automorphism \( \rho \in \bar{G} \) on \( X^o \). For a \( k \)-rational point \( t \in T^o(k) \), we define \( \Xi_t \subset \text{CH}^2(X_t, 1) \) as the pull-back of \( \Xi \) by \( i_t : X_t \to X^o \) in Definition 4.30.

Definition 5.14. For \( \rho \in G \), we define a closed subscheme \( D_\rho \subset \mathcal{Y}^o \) by the schematic image \( \rho(D) \). Note that \( D_\rho \) is determined under the image of \( \rho \in G \) by \( G \to G_0 \). The local equation of \( D_\rho \) is given by \( (\rho^{-1})^2(x - y) = 0 \). For example, if \( \rho \in G \) satisfies \( \rho_0 = ((0,1) / (1/c, 1 / c)) \in H_0 \times H_0 = G_0 \), the local equation of \( D_\rho \) is \( \frac{1}{1 - ax} = by = 0 \). (cf. Table 2).

We define \( \bar{D}_\rho \to \mathcal{Y}^o \) as the pull-back of \( D_\rho \) by \( \mathcal{Y}^o \to \mathcal{Y}^o \). Furthermore, we define \( \mathcal{C}_\rho \to \mathcal{X}^o \) as the strict transformation of \( \bar{D}_\rho \) by \( \mathcal{X}^o \to \mathcal{Y}^o \).

\[
\mathcal{X}^o \xrightarrow{\mathcal{C}_\rho} \mathcal{Y}^o \xrightarrow{\bar{D}_\rho} \mathcal{Y}^o \xrightarrow{\gamma} \mathcal{D}_\rho
\]
\[
\text{strict transform} \quad \text{double cover}
\]

Since \( \rho(D) = D_\rho \), for \( \rho \in \bar{G} \), we have \( \bar{\rho}(\bar{D}) = \bar{D}_\rho \) and \( \bar{\rho}(\mathcal{C}) = \mathcal{C}_\rho \).

Definition 5.15. (Definition of \( \sigma_\bullet \)) We define \( \sigma_0, \sigma_1, \sigma_\infty \in \Sigma^2 \) by
\[
\sigma_0 = \rho \cdot (0,0), \quad \sigma_1 = \rho \cdot (1,1), \quad \sigma_\infty = \rho \cdot (\infty, \infty).
\]
By Proposition 4.27, we have
\[
\bar{\rho}(Q_{(0,0)}^\rho) = Q_{\sigma_0}^\rho, \quad \bar{\rho}(Q_{(1,1)}^\rho) = Q_{\sigma_1}^\rho, \quad \bar{\rho}(Q_{(\infty, \infty)}^\rho) = Q_{\sigma_\infty}^\rho.
\]
As a consequence, the following elements in \( \bigoplus_{Z \in \mathcal{X}(\mathcal{Y})(1)} R(Z)^\times \) represents elements in \( \bar{\rho}_* \Xi^\text{can} \).
Proposition 5.16. Let $\tilde{\rho} \in \tilde{G}$. Then by the map $\tilde{\rho} : \mathcal{X}^o \to \mathcal{X}^o$, we have

\[
\tilde{\rho}(C) = C_{\rho}, \\
\tilde{\rho}(Q_{\bullet, \bullet}^o) = Q_{\bullet, \bullet}^o.
\]  

for $\bullet = 0, 1, \infty$. Let $\epsilon \in \mathbb{Z}^{(0,1,\infty)}$. Then $\rho_* \xi(\epsilon) \in \text{CH}^2(\mathcal{X}^o, 1)$ is represented by the following elements in $\bigoplus_{Z \in (\mathcal{X}^o)^{1,1}} R(Z)^{\times}$.

\[
\left( C_{\rho}, (\tilde{\rho}^{-1})^* (\psi_0^o (\varphi_0^{(0)}), \psi_1^o (\varphi_1^{(0)})) \right) \\
+ (Q_{\sigma_0, 1}^o, (\tilde{\rho}^{-1})^* (\varphi_0^{(0)})) + (Q_{\sigma_1, 1}^o, (\tilde{\rho}^{-1})^* (\varphi_1^{(1)})) + (Q_{\sigma_\infty, 1}^o, (\tilde{\rho}^{-1})^* (\varphi_\infty^{(\infty)}))
\]  

where $(\tilde{\rho}^{-1})^*$ are the field isomorphism $R(C) \to R(C_{\rho})$ and $R(Q_{\bullet, \bullet}^o) \to R(Q_{\bullet, \bullet}^o)$ induced by $\tilde{\rho}$.

Remark 5.17. As we stated in the introduction, elements in $\tilde{\rho}_* \Xi^{\text{can}}$ are at first constructed geometrically after Terasoma’s idea. The keys for construction are the following.

1. There exists the isomorphism $C_{\rho} \cong \mathbb{P}^1_T$ over $T^o$.
2. For $\bullet = 0, 1, \infty$, $C_{\rho} \cap Q_{\bullet, \bullet}^o$ decompose into the disjoint union of two $B^o$-rational points.

This gives another construction of $\tilde{\rho}_* \Xi^{\text{can}}$.

6. Subgroups $\tilde{I}$ and $\tilde{G}_{\text{fib}}$ of $\tilde{G}$

In this section, we construct subgroup $\tilde{I}$ of $\tilde{G}$ and prove that $\tilde{I}$ stabilize $\Xi^{\text{can}} \subset \text{CH}^2(\mathcal{X}^o, 1)$. Furthermore, we have the explicit description of $\tilde{I}$ action on $\Xi^{\text{can}}$ using the 1-cocycle $\delta$ (Theorem 6.10).

6.1. Definition of $\tilde{I}$ and stability of $\Xi^{\text{can}}$ under the $\tilde{I}$-action.

Definition 6.1. By Proposition 5.13, we have an identification $H_0$ with $\mathfrak{S}(\{0, 1, 1/c, \infty\})$. Let $\text{Stab}_{H_0}(1/c) \subset H_0$ be the stabilizer of $1/c \in \{0, 1, 1/c, \infty\}$. Then we define the subgroup $I_0$ of $G_0$ by the image of $\text{Stab}_{H_0}(1/c)$ under the following diagonal embedding.

\[
H_0 \xrightarrow{\Delta} H_0 \times H_0 \xrightarrow{\psi} G_0 \\
\tau_0 \xleftarrow{\psi^{-1}} (\tau_0, \tau_0)
\]  

Since this is injective, we identify $I_0 = \text{Stab}_{H_0}(1/c)$. We define $I_0 \subset G_0$ as the image of $I_0$ by $G_0 \to G_0$. Consider the following diagram.

\[
\begin{array}{cccc}
I_0 & \xleftarrow{\text{Stab}_{H_0}(1/c)} & \mathfrak{S}(\{0, 1, 1/c, \infty\}) & \xrightarrow{\Delta} H_0 \times H_0 \\
\downarrow \cong & & \downarrow \cong & \\
\text{Stab}_{H_0}(1/c) & \xrightarrow{\mathfrak{S}(\{0, 1, \infty\})} & H_0 & \xrightarrow{\Delta} H_0 \times H_0 \\
\end{array}
\]

(139)

By the description of $H_0 \to H_0$ in Remark 4.5, $\text{Stab}_{H_0}(1/c) \cong H_0 \to H_0$ is an isomorphism. Hence $I_0 \subset G_0$ coincides with the image of $H_0$ by the diagonal embedding and $I_0$ is an isomorphism.

Remark 6.2. Elements in $\text{Stab}_{H_0}(1/c)$ induce permutations on $\{0, 1, \infty\} \subset \{0, 1, 1/c, \infty\}$ and we have the following isomorphism.

\[
I_0 = \text{Stab}_{H_0}(1/c) \cong \mathfrak{S}(\{0, 1, \infty\})
\]  

(140)

By this map, we often identify $I_0$ with $\mathfrak{S}(\{0, 1, \infty\})$. 

\[

\]
Warning: This isomorphism is different from $I_0 \xrightarrow{\sim} L_0 \xrightarrow{\sim} H_0 = \mathcal{S}([0, 1, \infty])$ where the second isomorphism is induced by the diagonal embedding. However, these 2 morphisms coincide up to conjugate by $(0 \infty)$.

Under the identification $I_0 = \mathcal{S}([0, 1, \infty])$, the restriction of the action of $G_0$ on $\mathcal{Y}$ to its subgroup $I_0$ acts on $\mathcal{Y}$ as is follows.

**Table 5.** The action of $I_0$ on $\mathcal{Y}$

| $\rho_0$ | $(\rho_0^a(a), \rho_0^b(b))$ | $(\rho_0^c(x), \rho_0^d(y))$ | $\rho_0$ | $(\rho_0^a(a), \rho_0^b(b))$ | $(\rho_0^c(x), \rho_0^d(y))$ |
|---|---|---|---|---|---|
| id | $(a, b)$ | $(x_1, y_1)$ | $(0 \ 1)$ | $(\frac{a}{-1}, \frac{b}{-1})$ | $(1 - x, 1 - y)$ |
| $(1 \infty)$ | $(1 - a, 1 - b)$ | $(\frac{x_1}{1}, \frac{y_1}{1})$ | $(0 \ 1 \infty)$ | $(\frac{a - 1}{a}, \frac{b - 1}{b})$ | $(\frac{1 - x}{-1}, \frac{1 - y}{-1})$ |
| $(0 \infty)$ | $(\frac{1}{n}, \frac{1}{n})$ | $(\frac{x_1}{n}, \frac{y_1}{n})$ | $(0 \ 0 \infty)$ | $(\frac{1}{1 - a}, \frac{1}{1 - b})$ | $(\frac{1 - x}{-1}, \frac{y - 1}{-1})$ |

**Definition 6.3.** We define subgroups $I \subset \mathcal{G}$, $I \subset G$ and $\tilde{I} \subset \tilde{G}$ as follows.

\[
I = \{ \rho \in \mathcal{G} : \rho_0 \in I_0 \} \\
I = \{ \rho \in G : \rho_0 \in I_0 \} \\
\tilde{I} = \{ (\rho, \zeta) \in \tilde{G} : \rho \in I \}
\]

We have the following fiber product diagram for groups (cf. diagram (80)).

\[
\begin{array}{ccc}
I & \longrightarrow & \tilde{I} \\
\downarrow & & \downarrow \\
I_0 & \longrightarrow & \tilde{L}_0
\end{array}
\]

Since $I_0 \to L_0$ is an isomorphism by Definition 6.1, $I \to \tilde{I}$ is also an isomorphism.

Furthermore, since $\text{sgn}(\rho_0(1)) \cdot \text{sgn}(\rho_0(2)) = 1$ for $\rho \in I$ by Definition 6.1, we have the following splitting of $\tilde{I} \to I$.

\[
I \longrightarrow \tilde{I}; \rho \longmapsto (\rho, 1)
\]

By this splitting, we have the isomorphism $\tilde{I} \cong I \times \mathbb{Z}/2\mathbb{Z}$.

**Remark 6.4.** Since $G \to G_0$ is defined by $(\rho_0(1), \rho_0(2)) \mapsto (\rho_0(1), \rho_0(2))$ (See Definition 4.10 and Definition 4.12 for the notation) and $\tilde{L}_0 \subset G_0$ is the image of diagonal embedding (Definition 6.1), we have

\[
\tilde{I} = \{ (\rho_0(1), \rho_0(2)) \in \mathcal{G} : \rho_0(1) = \rho_0(2) \} = H \times_{\mathcal{L}_0} H
\]

Since $H \cong \mathcal{S}_4$ by Remark 4.9 and $H_0 \simeq \mathcal{S}_3$ by Proposition 4.4, $I$ is isomorphic to $\mathcal{S}_4 \times_{\mathcal{L}_0} \mathcal{S}_4$. Since $I \simeq L$, $\tilde{I}$ is also isomorphic to $\mathcal{S}_4 \times_{\mathcal{L}_0} \mathcal{S}_4$.

**Remark 6.5.** Let $T^{L_0}$ be the quotient of $T$ by $L_0$-action. Then degree of field extension $[R(T^\circ) : R(T^{L_0})]$ is 96. Since $L$ acts on $T^\circ$ faithfully, $L$ can be regarded as the Galois group of the field extension $R(T^\circ)/R(T^{L_0})$.

In this subsection, we show that $\tilde{I}$-action stabilize $\Xi_{\text{can}} \subset CH^2(X^\circ, 1)$ in Definition 5.10. In the following, we assume $\hat{\rho} = (\rho, \xi) \in \tilde{I}$.

**Proposition 6.6.**

1. We have $C_\rho = C$ where $C_\rho$ is the closed subscheme defined in Definition 5.14. Especially, we have $\hat{\rho}(C) = C$. 
(2) By definition of $\sigma_*$ in Definition 5.14 we have
\[ \sigma_0 = (\rho_0(0), \rho_0(0)), \quad \sigma_1 = (\rho_0(1), \rho_0(1)), \quad \sigma_\infty = (\rho_0(\infty), \rho_0(\infty)) \] (145)

Here $\rho_0$ is the image of $\rho$ by $I \to I_0 \xrightarrow{\sim} \mathcal{O}(\{0, 1, \infty\})$ where the last isomorphism is (140) in Remark 6.2. Especially, we have
\[ Q^\circ_{\sigma_*} = \tilde{\rho}(Q^\circ_{(\sigma_*)}(\bullet)) = Q^\circ_{(\rho_0)(\bullet), \rho_0(\bullet))} \] (146)

for $\bullet = 0, 1, \infty$.

Proof. (1) By Definition 5.14 it is enough to show $\mathcal{D} = \rho(\mathcal{D})$. By the description of $I_0$-action in Table 5 $I$-action on $\mathcal{Y}^0$ stabilize the local equation $x = y$ of $\mathcal{D}$ (Definition 5.3). Hence we have the result.

(2) The former part follows from the definition of $I_0$ (cf. Remark 4.15) and the way of identification of $I_0 = \mathcal{O}(\{0, 1, \infty\})$ in Remark 6.2. The latter part follows from Proposition 6.16.

By Proposition 6.6 to show that the subgroup of symbols in Proposition 5.16 is stable under $I$-action, it is enough to show that the sets of rational functions \{$\psi_{\pm1}$ : $\bullet = 0, 1, \infty$\} and \{$\psi_{\pm1}$ : $\bullet = 0, 1, \infty$\} are stable.

By Proposition 6.6 we have
\[ \tilde{\rho}(p^+_\bullet) \cup \tilde{\rho}(p^-_\bullet) = \tilde{\rho}(\mathcal{C} \cap Q^\circ_{\sigma_*}) = \mathcal{C} \cap Q^\circ_{\rho_0(\bullet)} = p^+_{\rho_0(\bullet)} \cup p^-_{\rho_0(\bullet)} \] (147)

for $\bullet = 0, 1, \infty$ where $p^+_{\rho_0}, p^-_{\rho_0}$ are $B^2$-rational points in Definition 5.7 and we use the relation (126).

Definition-Proposition 6.7. By comparing connected components in (147), we have either
\[ (A) \begin{cases} \tilde{\rho}(p^+_{\bullet}) = p^+_{\rho_0(\bullet)} \\ \tilde{\rho}(p^-_{\bullet}) = p^-_{\rho_0(\bullet)} \end{cases} \quad \text{or} \quad (B) \begin{cases} \tilde{\rho}(p^+_{\bullet}) = p^-_{\rho_0(\bullet)} \\ \tilde{\rho}(p^-_{\bullet}) = p^+_{\rho_0(\bullet)} \end{cases} \] (148)

for $\bullet = 0, 1, \infty$. Then we define $\delta(\tilde{\rho}) \in \{\pm 1\}^{\{0, 1, \infty\}}$ as follows.

(1) If the case (A) occurs for $\bullet = 0$, $\delta(\tilde{\rho})(\rho_0(0)) = 1$, else $\delta(\tilde{\rho})(\rho_0(0)) = -1$.
(2) If the case (A) occurs for $\bullet = 1$, $\delta(\tilde{\rho})(\rho_0(1)) = 1$, else $\delta(\tilde{\rho})(\rho_0(1)) = -1$.
(3) If the case (A) occurs for $\bullet = \infty$, $\delta(\tilde{\rho})(\rho_0(\infty)) = 1$, else $\delta(\tilde{\rho})(\rho_0(\infty)) = -1$.

Here $\rho_0 \in \mathcal{O}(\{0, 1, \infty\})$ is the image of $\rho$ under $I \to I_0 \in \mathcal{O}(\{0, 1, \infty\})$. Then the following equation holds for $\bullet = 0, 1, \infty$.
\[ (\tilde{\rho}^{-1})^2(\psi_{\bullet}) = \psi_{\rho_0(\bullet)}^{\delta(\tilde{\rho})(\rho_0(\bullet))}, \quad (\tilde{\rho}^{-1})^2(\varphi_{\bullet}) = \varphi_{\rho_0(\bullet)}^{\delta(\tilde{\rho})(\rho_0(\bullet))} \] (149)

To prove this proposition, we use the following lemma.

Lemma 6.8. Let $\varphi_1, \varphi_2 \in R(P^1_{T_\bullet})^\times$. Assume $\varphi_1$ satisfies the following condition.

If $p, q \in B^\circ$ satisfies $p + q \varphi_1 = 0$, then $p = q = 0$. (150)

Suppose that $\text{div}(\varphi_1) = \text{div}(\varphi_2)$ and $\text{div}(\varphi_1 + 1) = \text{div}(\varphi_2 + 1)$. Then we have
$\varphi_1 = \varphi_2$.

Proof. Since $P^1_{T_\bullet}$ is a Noetherian normal integral separated scheme, $\text{div}(\varphi_1) = \text{div}(\varphi_2)$ and $\text{div}(\varphi_1 + 1) = \text{div}(\varphi_2 + 1)$ implies that there exists $p, q \in \Gamma(P^1_{T_\bullet}, \mathcal{O}_{P^1_{T_\bullet}}^\times) = (B^\circ)^\times$ such that $\varphi_1 = p \varphi_2$ and $1 + \varphi_1 = q(1 + \varphi_2)$. This implies
\[ (1 - q) + (1 - p^{-1}q)\varphi_1 = 0 \] (151)

Then by the condition (150), we have $p = q = 1$. Hence we have the result. □
We define Proposition 6.9.\footnote{Note that $C$ and $Q^\circ_{\bullet, 0}$ are isomorphic to $\mathbb{P}^1_\mathbb{C}$ (Corollary 5.6). By the explicit presentation for $\varphi_{\bullet, \psi}$ in Definition 5.8 we see that $\varphi_{\bullet, \psi}^{\pm 1}$ satisfies the condition in Lemma 5.9 in Lemma 6.8. Hence to apply Lemma 6.8, we will compute divisors of these rational functions.}

By the definition of $\delta$, we have the following relations on divisors.

$$\text{div}_C((\tilde{\rho}^{-1})^\sharp(\psi_{\bullet})) = \text{div}_C(\psi_{\rho_0(\bullet)})$$

$$\text{div}_{Q^\circ_{\rho_0(\bullet), \rho_0(\bullet)}}((\tilde{\rho}^{-1})^\sharp(\varphi_{\bullet})) = \text{div}_{Q^\circ_{\rho_0(\bullet), \rho_0(\bullet)}}(\varphi_{\rho_0(\bullet)})$$

(152)

for $\bullet = 0, 1, \infty$. Here we use the relation in Proposition 5.9.

Next, we see the divisor associated with $1 + \psi_{\bullet}$ and $1 + \varphi_{\bullet}$. Consider a closed subscheme $Z \subset X^\circ$ defined by the local equation $v = 0$. Then we have $B^2$-rational points $q_0, q_0, q_\infty$ on $X^\circ$ such that

$$q_c = Z \cap C, \quad q_s = Z \cap Q^\circ_{\bullet, 0} \quad (\bullet = 0, 1, \infty).$$

(153)

This follows from the local descriptions in Proposition 5.5 and equation (122), (123). Using these $B^2$-rational points, we can describe the divisor of $1 + \psi_{\bullet}^{\pm 1}$ and $1 + \varphi_{\bullet}^{\pm 1}$ as follows.

$$\left\{ \begin{array}{l}
\text{div}_C(1 + \psi_{\bullet}) = q_c - p_{\bullet}^+

\text{div}_C(1 + \psi_{\bullet}^{-1}) = q_c - p_{\bullet}^-

\text{div}_{Q^\circ_{\rho_0(\bullet), \rho_0(\bullet)}}(1 + \psi_{\bullet}) = q_s - p_{\bullet}^+

\text{div}_{Q^\circ_{\rho_0(\bullet), \rho_0(\bullet)}}(1 + \psi_{\bullet}^{-1}) = q_s - p_{\bullet}^-
\end{array} \right.$$

(154)

where $\bullet = 0, 1, \infty$. This follows from the explicit description of Definition 5.8.

By the explicit description of $\tilde{G}$-action in Corollary 4.28, we see that the closed subscheme $Z$ is stable under $\tilde{I}$-action. Then we have

$$\tilde{\rho}(q_c) = \tilde{\rho}(Z \cap C) = Z \cap C = q_c$$

$$\tilde{\rho}(q_s) = \tilde{\rho}(Z \cap Q^\circ_{\bullet, 0}) = Z \cap Q^\circ_{\rho_0(\bullet), \rho_0(\bullet)} = q_{\rho_0(\bullet)}$$

(155)

By the definition of $\delta(\rho(\bullet))$, we have

$$\text{div}_C(1 + (\tilde{\rho}^{-1})^\sharp(\psi_{\bullet})) = q_c - \tilde{\rho}(p_{\bullet}^+) = \text{div}_C(1 + \psi_{\rho_0(\bullet)}(\rho(\bullet)))$$

$$\text{div}_{Q^\circ_{\rho_0(\bullet), \rho_0(\bullet)}}(1 + (\tilde{\rho}^{-1})^\sharp(\varphi_{\bullet})) = q_{\rho_0(\bullet)} - \tilde{\rho}(p_{\bullet}^+) = \text{div}_{Q^\circ_{\rho_0(\bullet), \rho_0(\bullet)}}(1 + \varphi_{\rho_0(\bullet)}(\rho(\bullet)))$$

(156)

for $\bullet = 0, 1, \infty$. Hence by (152) and (156), we have $(\tilde{\rho}^{-1})^\sharp(\psi_{\bullet}) = \psi_{\rho_0(\bullet)}(\rho(\bullet))$ and $(\tilde{\rho}^{-1})^\sharp(\varphi_{\bullet}) = \varphi_{\rho_0(\bullet)}(\rho(\bullet))$ by Lemma 6.8.

By the equality (149) in Definition 6.7, we have the following.

Proposition 6.9. We define $\tilde{I}$-action on $\{\pm 1\}^{1, 0, 1, \infty}$ by

$$\tilde{I} \times \{\pm 1\}^{1, 0, 1, \infty} \ni ((\rho, \zeta, \epsilon) \mapsto (\epsilon \circ \rho_0)^{-1}$$

(157)

Then the map $\delta : \tilde{I} \to \{\pm 1\}^{1, 0, 1, \infty} : \tilde{\rho} \mapsto \delta(\tilde{\rho})$ defines a 1-cocycle with respect to this $\tilde{I}$-action.

Theorem 6.10. We have $\tilde{\rho}_* (\Xi^{\text{can}}) = \Xi^{\text{can}}$. The $\tilde{I}$-action on $\Xi^{\text{can}}$ is given as follows:

$$\tilde{\rho}_* : \Xi^{\text{can}} \ni \xi(\epsilon) \mapsto \xi(\delta(\tilde{\rho}) \cdot (\epsilon \circ \rho_0)^{-1})$$

(158)
where \( \delta(\bar{\rho}) \in \{ \pm 1 \}^{(0,1,\infty)} \) is the map given Proposition 6.7 which acts on \( \mathbb{Z}^{(0,1,\infty)} \) by multiplication. Here \( \rho_0 \in \mathfrak{S}(\{0,1,\infty\}) \) is the image of \( \rho \) under \( I \to I_0 = \mathfrak{S}(\{0,1,\infty\}) \).

**Proof.** We have \( (C, \prod_{\rho \in \{0,1,\infty\}} (\bar{\rho}^{-1})^2(\psi_\alpha^{(\rho)})^{(\bullet)}) = \prod_{\rho \in \{0,1,\infty\}} (\bar{\rho}^{-1})^2(\psi_\alpha^{(\rho)})^{(\bullet)} \) and \( \prod_{\rho \in \{0,1,\infty\}} (\bar{\rho}^{-1})^2(\psi_\alpha^{(\rho)})^{(\bullet)} = (Q^\varphi_{\rho_0} (\bar{\rho}^{-1})^2(\psi_\alpha^{(\rho)})^{(\bullet)}) \) for \( \bullet \in \{ 0,1,\infty \} \).

(1) Let \( \rho^o \in I \) be the element satisfying that

\[
\rho^o \in \mathfrak{S}((0,1,\infty)) \quad \text{and} \quad \chi^{(2)}(\rho^o) = 1.
\]

Furthermore, \( \delta(\bar{\rho}) \) can be computed as

\[
\delta(\bar{\rho}^o)(0) = -1, \quad \delta(\bar{\rho}^o)(1) = 1, \quad \delta(\bar{\rho}^o)(\infty) = -1.
\]

(2) Let \( \rho^b \in I \) be the element satisfying that

\[
\rho^b \in \mathfrak{S}((0,1,\infty)) \quad \text{and} \quad \chi^{(1)}(\rho^b) = 1.
\]

Furthermore, \( \delta(\bar{\rho}) \) can be computed as

\[
\delta(\bar{\rho}^b)(1) = -1, \quad \delta(\bar{\rho}^b)(0) = 1, \quad \delta(\bar{\rho}^b)(\infty) = -1.
\]

### 6.2 Fiber-preserving subgroup \( \tilde{G}_{\text{fib}} \) of \( \tilde{G} \)

In this section, we define another subgroup \( \tilde{G}_{\text{fib}} \) of \( \tilde{G} \) and prove some group theoretic properties. As we will see in Corollary 9.19, \( \tilde{G}_{\text{fib}} \) acts on the image of \( \tilde{G} \) under the transcendental regulator map by \( \pm 1 \).

**Definition-Proposition 6.12.** We define a normal subgroup \( \tilde{G}_{\text{fib}} \subset \tilde{G} \) as

\[
\tilde{G}_{\text{fib}} = \text{Ker}(\tilde{G} \to \Omega).
\]

In other words, \( \tilde{G}_{\text{fib}} \) consists of elements in \( \tilde{G} \) which are automorphisms over \( T^o \). Then we have \( \tilde{G}_{\text{fib}} \cong (\mathbb{Z}/2\mathbb{Z})^5 \).

**Proof.** First we show the following homomorphisms are isomorphisms.

\[
\text{Ker}(H \to \Omega) \cong \text{Ker}(H_0 \to H_0) \cong (\mathbb{Z}/2\mathbb{Z})^2
\]
The first isomorphism is induced by the fact that fiber products preserves kernels (cf. Definition 4.10). The second isomorphism is from the Table 2. Then we have
\[ \text{Ker}(G \to \mathcal{G}) \simeq \text{Ker}(H \to H) \times \text{Ker}(H \to H) \simeq (\mathbb{Z}/22)^4. \quad (166) \]

By definition, we have the following cartesian diagrams.
\begin{align*}
\begin{array}{ccc}
\text{Ker}(\tilde{G} \to \mathcal{G}) & \xrightarrow{\rho} & \mu_4(k) \\
\downarrow & & \downarrow \\
\text{Ker}(G \to \mathcal{G}) & \xrightarrow{\rho} & \mu_2(k)
\end{array}
\end{align*}
(167)

Let \( \rho = (\rho^{(1)}, \rho^{(2)}) \in \text{Ker}(G \to \mathcal{G}) \). Then we have \( \rho^{(1)} = \rho^{(2)} = \text{id}_{\mathcal{G}} \). Hence the composition of lower horizontal group homomorphism in (167) is trivial. Hence the whole rectangle in (167) is just a direct product. Therefore, we have
\[ \text{Ker}(\tilde{G} \to \mathcal{G}) \simeq \text{Ker}(G \to \mathcal{G}) \times \text{Ker}(\mu_4(k) \xrightarrow{\rho^{(1)}} \mu_2(k)) \simeq (\mathbb{Z}/22)^5. \quad (168) \]

\[ \square \]

**Corollary 6.13.** \( \tilde{G}_{\text{fib}} \cap \tilde{I} = \{\text{id}_G, \pm 1\} \).

**Proof.** (\( \diamondsuit \)) follows from the definition. Let \( (\rho, \zeta) \in \tilde{G}_{\text{fib}} \cap \tilde{I} \). By Definition 6.12 we have \( \rho = \text{id}_G \). Since \( I \to \tilde{I} \) is an isomorphism, we have \( \rho = \text{id}_G \). Hence \( \zeta = \pm 1 \). \( \square \)

Since \( \tilde{G}_{\text{fib}} \) stabilize the image of \( \Xi \cap (\mathcal{G}) \) under the transcendental regulator (Corollary 9.19) and \( \tilde{I} \) stabilize \( \Xi \cap (\mathcal{G}) \) (Theorem 6.10). \( \tilde{G}_{\text{fib}} \tilde{I} \) stabilize the image of \( \Xi \cap (\mathcal{G}) \) under the transcendental regulator map. Hence \( \tilde{\rho}, \Xi \cap \mathcal{G} \) and \( \tilde{\rho}_I \Xi \cap \mathcal{G} \) have the same image under the transcendental regulator map if \( \tilde{\rho}, \tilde{\rho}_I \in \tilde{G} \) are in the same left cosets by \( \tilde{G}_{\text{fib}} \tilde{I} \) (For details, see Proposition 9.20). The following proposition is useful to determine whether \( \tilde{\rho}, \tilde{\rho}_I \in \tilde{G} \) are in the same left cosets or not.

**Proposition 6.14.** The group homomorphism \( \tilde{G} \to \mathcal{G}_0 \) induces the following bijection of sets.
\[ \tilde{G}/\tilde{G}_{\text{fib}} \tilde{I} \xrightarrow{\sim} \mathcal{G}_0/\mathcal{L}_0 \quad (169) \]

Especially, for \( \tilde{\rho} = (\rho, \zeta), \tilde{\rho}_I = (\rho_I, \zeta) \in \tilde{G} \), there exists a \( \tilde{\rho}_F \in \tilde{G}_{\text{fib}} \) and \( \tilde{\rho}_I \in \tilde{I} \) such that
\[ \tilde{\rho}_I = \tilde{\rho}_F \cdot \tilde{\rho}_I. \quad (170) \]

if and only if they satisfies \( \rho, \rho_I = \rho' \).

**Proof.** By the group homomorphism \( \tilde{G} \to \mathcal{G}_0 \), \( \tilde{G}_{\text{fib}} = \text{Ker}(\tilde{G} \to \mathcal{G}) \) maps to \( \{\text{id}_{\mathcal{G}_0}\} \) (Definition 6.12) and \( \tilde{I} \) maps to \( \mathcal{L}_0 \) (Definition 6.1 and Definition 6.3). Hence we see that \( \tilde{G} \to \mathcal{G}_0 / \mathcal{L}_0 \) induces a map (169). We will see this is bijective.

Since \( \tilde{G} \to \mathcal{G}_0 \) is surjective by Definition 4.12 and Proposition 4.23, the map (169) is also surjective. Hence it is enough to compare the cardinality of \( \tilde{G}/\tilde{G}_{\text{fib}} \tilde{I} \) with that of \( \mathcal{G}_0/\mathcal{L}_0 \). By Definition 6.1 \( |\mathcal{G}_0/\mathcal{L}_0| = 6 \). On the other hand, by Definition 6.3 and Remark 6.4 \(|\tilde{I}| = 192\). Hence by Proposition 6.12 and Corollary 6.13 we have
\[ |\tilde{G}_{\text{fib}} \tilde{I}| = \frac{|\tilde{G}_{\text{fib}}| \cdot |\tilde{I}|}{|\tilde{G}_{\text{fib}} \cap \tilde{I}|} = 3072 = 2^{10} \cdot 3 \quad (171) \]

By Proposition 4.29 we have \( |\tilde{G}| = 18432 = 2^{11} \cdot 3^2 \). Hence \( |\tilde{G}/\tilde{G}_{\text{fib}} \tilde{I}| = 6 \) and we confirm that (169) is bijective. \( \square \)
By this proposition and the above argument, if \( \tilde{\rho}, \tilde{\rho}'\) satisfies 
\( \tilde{\rho}_0 \in \mathcal{G}_0, \tilde{\rho}_0 \Xi^{\text{can}} \) and \( \tilde{\rho}' \Xi^{\text{can}} \) has the same image under the transcendental regulator map. Hence it is reasonable to define subgroups \( \Xi^\lambda \subset \Xi \) as follows.

**Definition 6.15.** For \( \lambda \in \mathcal{G}_0/L_0 \), we define \( \Xi^\lambda \subset CH^2(X^\circ, 1) \) by
\[
\Xi^\lambda = \sum_{\tilde{\rho} = (\rho, \zeta) \in \tilde{\mathcal{G}}_{\rho_0} \Xi^{\text{can}} \cap \lambda} \tilde{\rho}^* \Xi^{\text{can}}.
\]
(172)

By definition, the following holds.
\[
\Xi = \sum_{\lambda \in \mathcal{G}_0/L_0} \Xi^\lambda
\]
(173)

7. The Differential Form on \( X \) and Picard-Fuchs Differential Operator

Since \( X \) is a family of \( K3 \) surfaces, up to a multiplication of \( 
B \), we have a relative 2-form on each fiber which does not vanish everywhere. In the former part of this section, we specify such family \( \omega \in \Gamma(X', \Omega^2_{X'/T'}) \) and observe the \( \tilde{\mathcal{G}} \)-action on \( \omega \).

In the latter part, we assume \( k = \mathbb{C} \) and find a Picard-Fuchs differential operator with respect to \( \omega \). In other words, we find a differential operator on \( (T')^\text{an} \) which annihilate period functions associated with \( X' \to T' \) and relative 2-form \( \omega \in \Gamma(X', \Omega^2_{X'/T'}) \).

**7.1. The definition of the relative 2-form \( \omega \).** We define a relative 2-form \( \omega \) on \( X \) using a relative 2-form on \( E \times_k \mathcal{E} \). By Definition 3.17, we have the following morphisms over \( T \).
\[
\begin{array}{c}
X \\
\leftarrow (E \times_k \mathcal{E})^\sim \\
\rightarrow E \times_k \mathcal{E}
\end{array}
\]
(174)

**Definition 7.1.** We define a 1-form \( \theta \) on \( \mathcal{E} \) by
\[
\theta = \frac{dz}{u} \in \Gamma(\mathcal{E}, \Omega^1_{\mathcal{E}/S})
\]
(175)

where we use the local coordinates in Proposition 3.2. Then we have the following 2-form on \( E \times_k \mathcal{E} \) by
\[
pr_1^* (\theta) \wedge pr_2^* (\theta) = \frac{dx \wedge dy}{u_1 u_2} \in \Gamma(E \times_k \mathcal{E}, \Omega^2_{E \times_k \mathcal{E}/T'})
\]
(176)

where \( pr_i : E \times_k \mathcal{E} \to \mathcal{E} \) is the \( i \)-th projection and we use the local description of \( E \times_k \mathcal{E} \) in (46). Furthermore, we define the 2-form \( \tilde{\omega} \in \Gamma((E \times_k \mathcal{E})^\sim, \Omega^2_{(E \times_k \mathcal{E})^\sim/T'}) \) by the pull-back of \( pr_1^* (\theta) \wedge pr_2^* (\theta) \) by \( (E \times_k \mathcal{E})^\sim \to E \times_k \mathcal{E} \).

Finally, since \( (E \times_k \mathcal{E})^\sim \to X \) is separable, we have the unique element \( \omega \in \Gamma(X, \Omega^2_{X/T}) \) such that the pull back of \( \omega \) to \( (E \times_k \mathcal{E})^\sim \) coincides with \( \tilde{\omega} \). By Proposition 5.17, \( \omega \) is represented locally on \( V_{0,0} \) as
\[
\omega = \frac{dx \wedge dy}{vf(x)}.
\]
(177)

We use the same symbol \( \omega \) for its base change by \( X' \to X \) or \( X^\circ \to X \).

We describe the \( \tilde{\mathcal{G}} \)-action on \( \omega \).

**Proposition 7.2.** \( \bar{\rho} \in \tilde{\mathcal{G}} \) acts on \( \omega \) as follows.
\[
\bar{\rho}^* \omega = \tilde{\chi}(\bar{\rho}) \cdot \omega
\]
(178)

where \( \tilde{\chi}(\bar{\rho}) \) is the opposite 1-cocycle defined in Definition 4.23.
Proof. Since $\mathcal{X}'$ is smooth over $T'$, $\Omega^2_{\mathcal{X}'/T'}$ is locally free. Hence it is enough to show that the formula (178) on some non-empty open subset of $\mathcal{X}'$. We can show that

$$\rho^* \left( \frac{dx \wedge dy}{v f(x)} \right) = \frac{dp^2(x) \wedge dp^2(y)}{\rho^2(u)} = \frac{\partial}{\partial x} \left( \rho^2(x) \right) \frac{\partial}{\partial y} \left( \rho^2(y) \right) \frac{dx \wedge dy}{\rho^2(u)} = \chi(p) \frac{dx \wedge dy}{v f(x)}. \quad (179)$$

Here we use Proposition 7.2 and the relation $u = v f(x)$.

\[ \square \]

**Definition 7.3.** Let $t$ be a $k$-rational point on $T'$. Let $i_t : \mathcal{X}_t \hookrightarrow \mathcal{X}'$ be the morphism defined in Definition 4.3. We define $\omega_t \in \Gamma \left( \mathcal{X}_t, \Omega^2_{\mathcal{X}_t/k} \right)$ as the pull-back of $\omega$ in Definition 4.3.

For $\tilde{\rho} = (\rho, \zeta) \in \tilde{G}$, let $\tilde{\rho}_t : \mathcal{X}_t \sim \mathcal{X}_{\tilde{\rho}(t)}$ be the isomorphism defined in Proposition 4.3. Then by Proposition 7.2 we have

$$\tilde{\rho}_t^* \omega_{\tilde{\rho}(t)} = \chi(\tilde{\rho})(t) \cdot \omega_t \quad (180)$$

where $\chi(\tilde{\rho})(t) \in k$ is the image of $\chi(\tilde{\rho}) \in B'$ at $t$.

7.2. Calculation of periods of $\mathcal{X}_t$. Hereafter we assume $k = \mathbb{C}$. In this section, we calculate periods on $\mathcal{X}_t$ for $t \in T(\mathbb{C})$ with respect to relative 2-form $\omega_t$ in Definition 7.3. For notations about complex manifolds, see beginning of Section 9.

**Definition 7.4.** Let $X$ be a smooth projective surface over $\mathbb{C}$ and $\eta \in \Gamma(X, \Omega^2_{X/\mathbb{C}})$ be an algebraic 2-form on $X$. By considering the image of the natural morphism of sheaves $i^!_{\mathcal{X}} \Omega^2_{\mathcal{X}/\mathbb{C}} \to \Omega^2_{X_{an}}$, we can regard $\omega$ as a holomorphic 2-form on $X_{an}$. We define a subgroup $\mathcal{P}_\eta(X)$ of $\mathbb{C}$ by

$$\mathcal{P}_\eta(X) = \left\{ \int_G \eta \in \mathbb{C} : \Gamma \in Z_2(X_{an}) \right\}, \quad (181)$$

where $Z_2(X)$ denotes the group of topological closed 2-cycles on $X_{an}$. $\mathcal{P}_\eta(X)$ is a subgroup of periods of $X$ with respect to $\eta$.

We calculate $\mathcal{P}_{\omega_t}(\mathcal{X}_t)$ for $t \in T^o(\mathbb{C})$. First, we define elliptic curves $E^1_t$ and $E^2_t$ over $\mathbb{C}$ by

$$E^1_t : \quad y^2 = x(1-x)(1-a(t)x)$$

$$E^2_t : \quad y^2 = x(1-x)(1-b(t)x). \quad (182)$$

where $a(t), b(t)$ denotes the image of $a, b$ by $t^2 : B^o \to \mathbb{C}$. $E^1_t$ are geometric fibers of $\mathcal{E} \to S$ in Proposition 3.2 by the geometric point $Spec \mathbb{C} \xrightarrow{t} T' \to T \xrightarrow{\eta} S$. By Proposition 3.15 we see that $(\mathcal{E} \times_{\mathcal{E}} \mathcal{E})_t \simeq E^1_t \times_{\mathcal{X}} E^2_t$. By restricting the morphism (174) to fibers at $t \in T^o(\mathbb{C})$, we have the following diagram.

$$\begin{array}{ccc}
(\mathcal{E} \times_{\mathcal{E}} \mathcal{E})_t & \longrightarrow & (\mathcal{E} \times_{\mathcal{E}} \mathcal{E})_t \\
\downarrow & & \downarrow \\
\mathcal{X}_t & \sim & E^1_t \times_{\mathcal{X}} E^2_t
\end{array} \quad (183)$$

Let $p : (\mathcal{E} \times_{\mathcal{E}} \mathcal{E})_t \to E^1_t \times_{\mathcal{X}} E^2_t$ be the composition of the horizontal morphisms in (183) and $\pi : (\mathcal{E} \times_{\mathcal{E}} \mathcal{E})_t \to \mathcal{X}_t$ be the vertical morphism in (183). We have the following morphism $\phi$ of $\mathbb{Z}$-Hodge structures.

$$\phi : H^2(E^1_t \times_{\mathcal{X}} E^2_t) \xrightarrow{\rho^*} H^2 \left( (\mathcal{E} \times_{\mathcal{E}} \mathcal{E})_t \right) \xrightarrow{\pi^*} H^2(\mathcal{X}_t) \quad (184)$$

where $\rho^*$ is the pull-back by $\rho$ and $\pi^*$ is the Gysin morphism ([Voï02], p.178) induced by $\pi$. In other words, $\pi^*$ is the map

$$H^2((\mathcal{E} \times_{\mathcal{E}} \mathcal{E})_t) \xrightarrow{\sim} H_2((\mathcal{E} \times_{\mathcal{E}} \mathcal{E})_t) \xrightarrow{\pi^*} H_2(\mathcal{X}_t) \xrightarrow{\sim} H^2(\mathcal{X}_t) \quad (185)$$
where $\pi_*$ is the push-forward map induced on the homology group and the first map and the last map are Poincaré duality. We use the following fact about Kummer surfaces.

**Proposition 7.5.** ([BPV84], Chapter VIII, Proposition 5.1, Corollary 5.6) $\phi$ is injective. Furthermore, the image of the map $\phi : H^2(E_1^1 \times_c E_2^2, \mathbb{Z}) \to H^2(\mathcal{X}_t, \mathbb{Z})$ on singular cohomologies with coefficients in $\mathbb{Z}$ is primitive in $H^2(\mathcal{X}_t, \mathbb{Z})$.

Let $\theta'_t \in \Gamma(E^1_t, \Omega^1_{E^1_t/C})$ be the pull-back of $\theta$ in Definition 7.1 by Spec $\mathbb{C} \xrightarrow{\triangle} T' \to T \xrightarrow{p_t} S$.

**Proposition 7.6.** For $\phi$ in [184], the following relation holds in $H^2(\mathcal{X}_t, \mathbb{C})$.

$$
\phi([pr^*_1(\theta'_1) \land pr^*_2(\theta'_2)]) = 2\omega_t
$$

**Proof.** Under the isomorphism $(E \times_c E)_{\Gamma} \simeq E^1_1 \times_c E^2_2$, $pr^*_1(\theta'_1) \land pr^*_2(\theta'_2)$ coincides with the pull-back of $pr^*_1(\theta) \land pr^*_2(\theta)$ in Definition 7.1 at $t$. Let $\omega_t \in \Gamma((E \times_c E)_{\Gamma})$ be the pull-back of $\omega$ in Definition 7.1. Then we have

$$
\begin{align*}
\pi^* \omega_t &= \omega_t \\
\pi^* \omega_t &= \omega_t.
\end{align*}
$$

Since $\pi : (E \times_c E)_{\Gamma} \to \mathcal{X}$ is quotient by involution by Proposition 3.17, especially generically 2:1 map. Hence the mapping degree of $\pi$ is 2. By the definition of Gysin map in [185], $\pi_1 \circ \pi^* : H^2(\mathcal{X}) \to H^2(\mathcal{X}_t)$ equals to multiplication by 2 (cf. [Voi02], Remark 7.29). Then we have

$$
\phi([pr^*_1(\theta'_1) \land pr^*_2(\theta'_2)]) = \pi_1 \circ \pi^*[\pi^*([pr^*_1(\theta'_1) \land pr^*_2(\theta'_2)])] = \pi_1[\omega_t] = \pi_1[\pi^*\omega_t] = 2[\omega_t].
$$

By these propositions, we can reduce the computation of the periods of $\mathcal{X}_t$ to that of the period of the product of elliptic curves.

**Proposition 7.7.** The following equality holds for subgroups of $\mathbb{C}$.

$$
P_{\omega_t}(\mathcal{X}_t) = \frac{1}{2}P_{pr^*_1(\theta'_1) \land pr^*_2(\theta'_2)}(E^1_1 \times_c E^2_2)
$$

**Proof.** In the proof of this proposition, we omit the coefficient $\mathbb{Z}$ of singular cohomologies and homologies. Since $\mathcal{X}_t$ is $K3$ surface and $E^1_1 \times_c E^2_2$ is abelian surface, their singular cohomologies with coefficients in $\mathbb{Z}$ have no torsions and of finite rank as $\mathbb{Z}$-modules ([BPV84], Chapter VIII, Proposition 3.2). Hence $H_2(\mathcal{X}_t)$ and $H^2(E^1_1 \times_c E^2_2)$ are dual of $H^2(\mathcal{X}_t)$ and $H^2(E^1_1 \times_c E^2_2)$. Then the following morphism is the dual of $\phi$.

$$
\phi^\vee : H_2(\mathcal{X}_t) \xrightarrow{\pi^*} H^2(\mathcal{X} \times_c \mathcal{X}) \xrightarrow{pr} H_2(E^1_1 \times_c E^2_2)
$$

where $\pi^*$ is the following morphism.

$$
H_2(\mathcal{X}_t) \xrightarrow{\pi^*} H^2(\mathcal{X} \times_c \mathcal{X}) \xrightarrow{\pi^*} H^2((\mathcal{X} \times_c \mathcal{X})_{\Gamma})
$$

where the first and the last morphism is Poincaré duality. By Proposition 7.5 $\phi$ is a split injection. Hence its dual $\phi^\vee$ is surjective. By Proposition 7.6, we have the following equation for any $[\Gamma] \in H_2(\mathcal{X}_t)$.

$$
\int_{\Gamma} \omega_t = \langle [\omega_t], [\Gamma] \rangle = \frac{1}{2} \langle \phi([pr^*_1(\theta'_1) \land pr^*_2(\theta'_2)]), [\Gamma] \rangle
$$

$$
= \frac{1}{2} \langle [pr^*_1(\theta'_1) \land pr^*_2(\theta'_2)], \phi^\vee([\Gamma]) \rangle = \frac{1}{2} \langle \int_{\Gamma} pr^*_1(\theta'_1) \land pr^*_2(\theta'_2) \rangle
$$

\footnote{Its cokernel has no torsion.}
Proposition 7.10. The abelian subgroup $s$ is generated by $L$.

This follows from the condition (3).

Remark 7.9. If we take different path, we get another holomorphic function $s$ by $L$. Then we see that $s$ is the coordinate of $S^{an}$. By Proposition 3.2 we have the double covering $E_s \rightarrow \mathbb{P}_C^1$ induced by $(y, x) \mapsto x$. We introduce the following function.

Definition 7.8. We define local holomorphic function $P_1, P_2$ of $s \in S^{an} = \mathbb{C}$ as follows. For a while we fix $s \in S^{an}$. Let $\gamma, \delta$ be $C^\infty$ paths on $(\mathbb{P}_C^1)^{an}$ such that the following conditions holds.

1. $\gamma$ is a path from 0 to 1 and $\delta$ is a path from 1 to $\infty$.
2. $\gamma, \delta$ does not path through 0, 1, 1/c, $\infty$ unless edge points.
3. Let $\gamma_+, \gamma_-$ (resp. $\delta_+, \delta_-$) are liftings of $\gamma$ (resp. $\delta$) by $E^{an} \rightarrow (\mathbb{P}_C^1)^{an}$. Then $[\gamma_+] - [\gamma_-]$ and $[\delta_+] - [\delta_-]$ are generators of the singular homology group $H_1(E, \mathbb{Z})$.

If $c \in \mathbb{R}_{>0}$, the path $[0, 1]$ and $[1, \infty]$ along the real axis satisfies the conditions for $\gamma$ and $\delta$. Then we define

$$
P_1(c) = \int_{\gamma} \theta_s = \int_{\gamma} \frac{dx}{\sqrt{x(1-x)(1-cx)}}$$

$$
P_2(c) = \int_{\delta} \theta_s = \int_{\delta} \frac{dx}{\sqrt{x(1-x)(1-cx)}}.
$$

(193)

where $\theta_s$ be the pull-back of $\theta$ in Definition 7.1 by $E \hookrightarrow \mathcal{E}$. If $\gamma, \delta$ satisfies the condition (1) to (3) at $s \in S^{an}$, $\gamma, \delta$ satisfies the condition (1) to (3) at every point on the open neighborhood (in the classical topology) of $s$. Hence $P_1(c)$ and $P_2(c)$ defines the local holomorphic functions around $s$.

These functions form 2 linearly independent solution of the following hypergeometric differential equation.

$$
c(1-c) \frac{d^2P}{dc^2} + (1-2c) \frac{dP}{dc} - \frac{1}{4}P = 0
$$

(194)

We define a differential operator $L : \mathcal{O}_{S^{an}} \rightarrow \mathcal{O}_{S^{an}}$ of order 2 by

$$
L = c(1-c) \frac{d^2}{dc^2} + (1-2c) \frac{d}{dc} - \frac{1}{4}.
$$

(195)

Then we see that $L(P_1) = L(P_2) = 0$. If we emphasize the variable $c$, we also use the notation $L_c$.

Remark 7.9. If we take different path, we get another holomorphic function $P'_1$ and $P'_2$. However, there exists a $n_{ij} \in \mathbb{Z}$ such that $P'_i = n_{i1}P_1 + n_{i2}P_2$ for $i = 1, 2$. This follows from the condition (3).

Proposition 7.10. The abelian subgroup \( \mathcal{P}_{pr_1(\theta'_1) \wedge pr_2(\theta'_2)}(E_i^1 \times_C E_i^2) \) is generated by $4P_i(a(t))P_j(b(t))$ for $i, j \in \{1, 2\}$ where $a(t), b(t) \in \mathbb{C}$ are images of $a, b \in B'$ by the $t^i : B' \rightarrow C$.

Proof. Let $\gamma, \delta$ be paths on $(\mathbb{P}_C^1)^{an}$ which satisfies the conditions in Definition 7.8 for $E_i^1$ and $E_i^2$. Let $\gamma_1, \gamma_2, \gamma_1, \gamma_2$ be lift of $\gamma$ by $(E_i^1)^{an} \rightarrow (\mathbb{P}_C^1)^{an}$ and $\delta_1, \delta_2, \delta_1, \delta_2$ be lift of $\delta$ by $(E_i^2)^{an} \rightarrow (\mathbb{P}_C^1)^{an}$ for $i = 1, 2$. Then by K"unneth formula, $H_2(E_i^1 \times_C E_i^2, \mathbb{Z})$ is generated by $([\gamma_1] - [\gamma_2]) \times ([\gamma_2] - [\gamma_1])$, $([\gamma_1] - [\gamma_2]) \times ([\delta_2] - [\delta_2])$.
P
regulator (Theorem 205) using the local holomorphic function
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We define differential operators
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\[ \int \theta_1 \wedge \theta_2 \]
∈
Hereafter we use the following notations.
In this section, we calculate the image of the higher Chow cycle
P
Thus we have the following generators of
A GROUP ACTION ON HIGHER CHOW CYCLES ON A KUMMER SURFACE FAMILY 45
(\int \delta_1 \wedge \delta_2)
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pr
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\[ \int \delta_1 \wedge \delta_2 \]
\[ \int \delta_1 \wedge \delta_2 \]
\[ \int \delta_1 \wedge \delta_2 \]
\[ \int \delta_1 \wedge \delta_2 \]
By Definition 7.8, they coincide with 4P_1(a(t))P_1(b(t)), 4P_1(a(t))P_2(b(t)), 4P_2(a(t))P_1(b(t))
and 4P_2(a(t))P_2(b(t)).

Thus we have the following generators of \( \mathcal{P}_{\omega_i}(X_t) \).

**Definition-Proposition 7.11.** For \( i, j \in \{1, 2\} \), we define the germs of holomorphic function \( P_{ij} \) at \( t \in (T')^{an} \) by
\[ P_{ij} = 2P_1(a(t))P_2(b(t)) \tag{197} \]
where \( a(t), b(t) \in \mathbb{C} \) are images of \( a, b \in B' \) by \( t' : B' \rightarrow \mathbb{C} \) and \( P_1, P_2 \) are local holomorphic function defined in Definition 7.8. By Proposition 7.7 and Proposition 7.10, for each \( t \in T'(\mathbb{C}) \), the subgroup \( \mathcal{P}_{\omega_i}(X_t) \subset \mathbb{C} \) is generated by the value of \( P_{ij} \) at \( t \).

Finally, we get a system of differential operators \( \mathcal{D} \) which annihilate the period functions \( P_{ij} \).

**Definition 7.12.** We define differential operators \( \mathcal{D}_1, \mathcal{D}_2 : \mathcal{O}_{(T')^{an}} \rightarrow \mathcal{O}_{(T')^{an}} \) of order 2 by
\[ \mathcal{D}_1 = a(1 - a) \frac{\partial^2}{\partial a^2} + (1 - 2a) \frac{\partial}{\partial a} - \frac{1}{4} \tag{198} \]
\[ \mathcal{D}_2 = b(1 - b) \frac{\partial^2}{\partial b^2} + (1 - 2b) \frac{\partial}{\partial b} - \frac{1}{4} \]
Using these operators, we define a Picard-Fuchs differential operator \( \mathcal{D} \) by
\[ \mathcal{D} = \left( \mathcal{D}_1, \mathcal{D}_2 \right) : \mathcal{O}_{(T')^{an}} \rightarrow \mathcal{O}_{(T')^{an}}^{\mathbb{C}}. \tag{199} \]
These are \( \mathbb{C} \)-linear morphisms of sheaves. By Definition 7.8, the local holomorphic functions \( P_{ij} \) in Definition 7.11 are annihilated by the differential operator \( \mathcal{D} \).

8. BASIC CALCULATION OF THE REGULATOR MAP

In this section, we calculate the image of the higher Chow cycle \( \xi_{0,t} - \xi_{1,t} \in \text{CH}^2(X_t, 1) \) in Definition 5.10 via the regulator map using the Levine’s formula. For this purpose, we construct topological 2-chain \( K_+ \) and \( K_- \) on \( X_t^{an} \) explicitly (Proposition 8.12) and express the image of \( \xi_{0,t} - \xi_{1,t} \) under the transcendental regulator (Theorem 205) using the local holomorphic function \( \mathcal{L} \) (Definition 8.15).

Hereafter we use the following notations.

(1) For a smooth variety \( X \) over \( \mathbb{C} \), its analytification is denoted by \( X^{an} \). As a set, we have \( X^{an} = X(\mathbb{C}) \).
Theorem 8.3. ([Lev88], p.458 – 459) The following map is well-defined.

\[ \text{CH}^2(X,1) \quad \xrightarrow{\quad} \quad \frac{F^1H^2(X^{an},\mathbb{C})^\vee}{H_2(X^{an},\mathbb{Z})} \]

\[ [\sum_j(C_j,f_j)] \mapsto \left[ [\omega] \mapsto \int_{\gamma} \omega + \sum_j \frac{1}{2\pi \sqrt{-1}} \int_{D_j - \gamma_j} \log(f_j) \mu_j^* \omega \right] \mod H_2(X^{an},\mathbb{Z}) \]
Here \( \log(f_j) \) is the pull-back of the principal branch of the holomorphic function \( \log z \) on \((\mathbb{P}^1_{\mathbb{C}})^{an} - [0, \infty]\) by \( f_j \). By the isomorphism \( r \) is regarded as a map to \( H^2(X^{an}, \mathbb{Z}(2)) \). This map is called the regulator map.

**Remark 8.4.** This definition of the regulator map is different from the map defined \([Lev88]\) by the multiplication of \( 2 \pi \sqrt{-1} \).

In this paper, we do not treat whole Deligne cohomology. We only consider the image of the regulator in the space of functionals on holomorphic 2-forms. This map is often called “transcendental regulator” in the literature (cf. \([CDKL16]\)).

**Definition 8.5.** The transcendental regulator map is the composition of the following maps.

\[
\begin{align*}
&: \text{CH}^2(X, 1) \longrightarrow F^1H^2(X^{an}, \mathbb{C})^\vee \longrightarrow H^{2,0}(X^{an})^\vee \longrightarrow H_2(X^{an}, \mathbb{Z}) \\
\end{align*}
\]

where the first map is the regulator map in Theorem 8.3 and the second map is the projection induced by \( H^{2,0}(X^{an}) \hookrightarrow F^1H^2(X^{an}, \mathbb{C}) \).

The transcendental regulator map has the following simple formula.

**Proposition 8.6.** Let \( \xi \) be an element of \( \text{CH}^2(X, 1) \) and \( \Gamma \) be a 2-chain associated with \( \xi \). For a holomorphic 2-form \( \eta \) on \( X \), we have

\[
r(\xi)([\eta]) = \int_\Gamma \eta \mod \mathcal{P}_\eta(X).
\]

where \( \mathcal{P}_\eta(X) \subset \mathbb{C} \) is the subgroup defined in Definition 7.4.

**Proof.** Since we regard \( H_2(X^{an}, \mathbb{Z}) \) as a subgroup of \( F^1H^2(X^{an}, \mathbb{C})^\vee \) by integration, evaluation at \([\eta] \in H^{2,0}(X^{an})^\vee / H_2(X^{an}, \mathbb{Z})\) induces the following map.

\[
\begin{align*}
H^{2,0}(X^{an})^\vee / H_2(X^{an}, \mathbb{Z}) &\longrightarrow \mathbb{C} / \mathcal{P}_\eta(X) \\
\varphi &\longmapsto \varphi(\eta)
\end{align*}
\]

Hence \( r(\xi)([\eta]) \) should be an element of \( \mathbb{C} / \mathcal{P}_\omega(X) \).

The presentation of \( \mathcal{P}_\eta(X) \) follows from the formula in Theorem 8.3. Since \( \eta \) is the holomorphic 2-form and \( D_j \) is the complex manifold of dimension 1, we have \( \mu^*_j\eta = 0 \). Thus \( \int_{D_j} - \gamma_j \log(f_j) \mu^*_j\eta = 0 \) for all \( j \). Hence

\[
r(\xi)([\eta]) = \int_\Gamma \eta \mod \mathcal{P}_\eta(\eta).\]  

\( \square \)

Furthermore, we can detect indecomposable cycles by the transcendental regulator map by the following proposition.

**Proposition 8.7.** For a holomorphic 2-form \( \eta \) on \( X \) and a decomposable cycle \( \xi \in \text{CH}^2(X, 1)_{\text{dec}} \), we have

\[
r(\xi) = 0
\]

**Proof.** By the Proposition 8.3, \( \xi \) is represented as a sum of \((C, a)\) where \( a \in \Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^\times \). In this case, \( \gamma = 0 \) and we can take \( \Gamma = 0 \). Then results follows from Proposition 8.6. \( \square \)

When we compute the value of transcendental regulator, sometimes it is convenient to replace 1-cycle / 2-chain associated with \( \xi \) (Definition 8.2) by simple 1-cycle / 2-chain. Thus we define as follows.
Definition 8.8. Let $\xi$ be an element of $\mathrm{CH}^2(X, 1)$ and $\gamma$ be the 1-cycle associated with $\xi$. In this paper, $\gamma' \in Z_1(X^2)$ is called a 1-cycle associated with $\xi$ in a weak sense if there exists a family of smooth curves $\{D_\lambda \to X\}_\lambda$ such that $\gamma - \gamma' \in \sum_\lambda B_1(D^{an}_\lambda)$. Here we regard $B_1(D^{an}_\lambda)$ as a subgroup of $Z_1(X^2)$ by the natural inclusions.

Let $\Gamma \in S_2(X^2)$ be the 2-chain associated with $\xi$. $\Gamma' \in S_2(X^2)$ is called 2-chain associated with $\xi$ in a weak sense if there exists a family of smooth curves $\{D_\lambda \to X\}_\lambda$ such that $\Gamma - \Gamma' \in Z_2(X^2) + \sum_\lambda S_2(D^{an}_\lambda)$.

By the definition, if $\gamma'$ is a 1-cycle associated with $\xi$ in a weak sense and $\Gamma' \in S_2(X^2)$ satisfies $\partial \Gamma' = \gamma'$, $\Gamma$ is a 2-chain associated with $\xi$ in a weak sense.

The following lemma justifies this definition.

Lemma 8.9. Let $\{D_\lambda \to X\}_\lambda$ be a family of smooth curves on $X$. Suppose $\Gamma, \Gamma' \in S_2(X^2)$ satisfies that $\Gamma - \Gamma' \in \sum_\lambda S_2(D^{an}_\lambda)$. Then for a holomorphic 2-form $\eta$, we have

$$\int_\Gamma \eta = \int_{\Gamma'} \eta$$

(209)

Especially, if $\Gamma$ is a 2-chain associated with $\xi$ in a weak sense, we have

$$r(\xi)([\eta]) = \int_\Gamma \eta \mod P_\eta(X).$$

(210)

Proof. The former follows from that the restriction of $\eta$ on $D^{an}_\lambda$ is 0 since $D^{an}_\lambda$ are 1-dimensional complex manifolds. The latter follows from Proposition 8.6.

8.2. Construction of 2-chain associated with $\xi_{0,t} - \xi_{1,t}$ in a weak sense. By restricting the morphisms in Definition 3.17 to fibers at $t \in T^s(\mathbb{C})$, we have the following morphisms.

$$\mathcal{X}_t \longrightarrow \mathcal{X}^1_t \longrightarrow \mathcal{Y}_t$$

(211)

We construct a topological 2-chain $K_+ - K_-$ associated with $\xi_{0,t} - \xi_{1,t}$ from the topological 2-chains $\overline{\mathcal{X}}, K$ (Definition 8.10 and Definition 8.11).

$$\mathcal{X}^1_t \cup \overline{K}_+ \cup K_- \longrightarrow \mathcal{X}^1_t \cup \overline{K}_+ \cup K_- \longrightarrow \mathcal{Y}^1_t \cup \overline{K}_+ \cup K_-$$

(212)

First, we define a topological 2-chain $\overline{\mathcal{X}} \subset \mathcal{Y}^1_t$ and $K \subset \mathcal{X}^1_t$.

Definition 8.10. (Definition of $\overline{\mathcal{X}}$ and $K$) Let $t \in T^s(\mathbb{C})$. We use the same symbols $a, b, \sqrt{1-a}, \sqrt{1-b}$ for their image by $\beta : B^s \to \mathbb{C}$. We take the following $C^\infty$ path $\gamma$ on $(\mathbb{P}^1_t)^{an}$.

1. $\gamma(0) = 0$ and $\gamma(1) = 1$.
2. $\gamma(s) \neq 0, 1, \frac{2}{5}, \frac{3}{5}$ except $s = 0, 1$.
3. We can fix the branch of the functions $\sqrt{1-a\gamma}, \sqrt{1-b\gamma}$ along $\gamma$ so that $\sqrt{1-a\gamma(0)} = \sqrt{1-b\gamma(0)} = 1$ and $\sqrt{1-a\gamma(1)} = \sqrt{1-b\gamma(1)} = \frac{1}{a-b}$.
4. On a neighborhood of 0, we have $\gamma(s) = s^2$. Furthermore, we can fix the branch of $\sqrt{s}$ along $\gamma$ so that $\sqrt{\gamma(0)} = 1$ and $\sqrt{\gamma(s)} = s$ on a neighborhood of 0.
5. On a neighborhood of 1, we have $\gamma(s) = 1-(1-s)^2$. Furthermore, we can fix the branch of $\sqrt{1-s}$ along $\gamma$ so that $\sqrt{1-\gamma(0)} = 1$ and $\sqrt{1-\gamma(s)} = 1-s$ on a neighborhood of 1.
The condition (4) is necessary for $K_+$ and $K_-$ to be $C^\infty$ manifolds. If $\sqrt{1-a}, \sqrt{1-b} \in \mathbb{R}_{>1}$, the interval $[0,1]$ in real axis (with suitable parametrization) satisfies the conditions above. By the condition (3)(4)(5), we can fix the branch of the local holomorphic functions $\sqrt{z(1-z)(1-az)}$ and $\sqrt{z(1-z)(1-bz)}$ along $\gamma$. We define $\triangle \subset \mathcal{Y}_t^{an}$ by the image of the following map.

\[
\{(p, q) \in \mathbb{R}^2 : 0 < q < p < 1\} \rightarrow \mathcal{Y}_t^{an} \cup \mathcal{X}_t^{an}
\]

\[
(p, q) \rightarrow (x, y) = (\gamma(p), \gamma(q))
\]

We define $\overline{\triangle}$ as the closure (in the sense of classical topology) of $\triangle$ in $\mathcal{Y}_t^{an}$.

Since $\triangle \subset \mathcal{Y}_t^{an}$ does not intersect with the blowing-up locus of $\mathcal{X}_t \rightarrow \mathcal{Y}_t$, the inverse image of $\triangle$ by $\mathcal{X}_t \rightarrow \mathcal{Y}_t$ is homeomorphic to $\triangle$. We also denote the inverse image of $\triangle$ by $\triangle$. We define $K \subset \mathcal{Y}_t$ by the closure (in the sense of classical topology) of $\triangle \subset \mathcal{X}_t^{an}$.

\[
\mathcal{X}_t^{an} \supset \triangle \subset K
\]

\[
\mathcal{Y}_t^{an} \supset \triangle \subset \overline{\triangle}
\]

We name the paths appearing in the boundaries of $K$ as follows. We use the local coordinates $x, y, \tau$ and $x, y, \overline{\tau}$ on $\mathcal{X}_t$ in Definition 3.17. The location of these paths is described by Figure 4.

1. The path $\gamma_c$ is a path from $(x, y, \tau) = (0, 0, 1)$ to $(x, y, \overline{\tau}) = (1, 1, -\frac{1}{\tau})$ which is on the strict transformation of $\mathcal{D}_t$ in $\mathcal{X}_t$.
2. The path $\gamma_{11}$ is a path from $(x, y, \tau) = (1, 1, -\frac{1}{\tau})$ to $(x, y, \overline{\tau}) = (1, 1, 0)$ which is on the exceptional curve $\mathcal{Q}^{an}_{(1,1),t}$ (See Definition 3.18).
3. The path $\gamma_y$ is a path from $(x, y, \tau) = (1, 1, 0)$ to $(x, y, \overline{\tau}) = (1, 0, 0)$ which is contained in a submanifold defined by $x - 1 = \overline{\tau} = 0$.
4. The path $\gamma_{10}$ is a path from $(x, y, \tau) = (1, 0, 0)$ to $(x, y, \overline{\tau}) = (1, 0, 0)$ which is on the exceptional curve $\mathcal{Q}^{an}_{(1,0),t}$.
5. The path $\gamma_a$ is a path from $(x, y, \tau) = (1, 0, 0)$ to $(x, y, \overline{\tau}) = (0, 0, 0)$ which is contained in a submanifold defined by $y = \tau = 0$.
6. The path $\gamma_{00}$ is a path from $(x, y, \tau) = (0, 0, 0)$ to $(x, y, \overline{\tau}) = (0, 0, 1)$ which is on the exceptional curve $\mathcal{Q}^{an}_{(0,0),t}$.

Then we define $K_+$ and $K_-$ as follows.

**Definition 8.11.** Since $\triangle \subset \mathcal{X}_t^{an}$ does not intersect with the branching locus of $\mathcal{X}_t \rightarrow \mathcal{X}_t$, the inverse image of $\triangle$ by $\mathcal{X}_t \rightarrow \mathcal{X}_t^{an}$ decompose as the disjoint union of $\triangle_+$ and $\triangle_-$ where $\triangle_+$ and $\triangle_-$ are both homeomorphic to $\triangle \subset \mathcal{X}_t$ (Note that $\triangle$ is simply connected). We define $K_+$ and $K_-$ by the closure of $\triangle_+$ and $\triangle_-$. We choose $K_+$ and $K_-$ so that $K_+$ contains $(x, y, v) = (0, 0, 1)$ and $K_-$ contains $(x, y, v) = (0, 0, -1)$.

\[
\mathcal{X}_t^{an} \supset \triangle_+ \cup \triangle_- \subset K_+ \cup K_-
\]

\[
\mathcal{X}_t^{an} \supset \triangle \subset K
\]

Since $K_\pm$ are compact manifolds with corners and have the natural orientation induced by $\triangle_\pm$, we can regard them as 2-chains on $\mathcal{X}_t^{an}$. Furthermore, by the condition (4) in Definition 8.10 we can confirm that $K_+$ and $K_-$ are $C^\infty$ manifolds with corners (even at their boundary).
We name the paths appearing in the boundaries of $K$ as follows. The location of these paths is described by Figure 5.

1. The paths $\gamma_{c,+}$ and $\gamma_{c,-}$ are the lifts of $\gamma_c$. They are on the curve $C^\alpha_1$.
   \[ (x,y,v) = (0,0,1) \]
   Note that by the condition (3) in Definition 8.10, $\gamma_{c,+}$ is a path from $(x,y,v) = (0,0,1)$ to $(x,y,v) = \left(1,1,\frac{\sqrt{1-a}}{1-a}\right)$ and $\gamma_{c,-}$ is a path from $(x,y,v) = (0,0,1)$ to $(x,y,v) = \left(1,1,-\frac{\sqrt{1-a}}{1-a}\right)$.

2. The paths $\gamma_{11,+}$ and $\gamma_{11,-}$ are the lifts of $\gamma_{11}$. They are on the curve $Q^{an}_{11}(1,1,t)$.
   Note that since $\gamma_{11}(1)$ is located in the branching locus of $X_t \rightarrow \overline{X}_t$, $\gamma_{11,+}$ and $\gamma_{11,-}$ have the same terminal point.

3. Since $X \rightarrow \overline{X}$ is branches on $x = 1 = \overline{x} = 0$, we have the unique lift of $\gamma_{y}$ to $X$. we use the same symbol for its lift.

4. The paths $\gamma_{10,+}$ and $\gamma_{10,-}$ are the lifts of $\gamma_{10}$. They are on the curve $Q^{an}_{10}(1,0,t)$.
   Note that since $\gamma_{10}(0)$ and $\gamma_{10}(1)$ are located in the branching locus of $X_t \rightarrow \overline{X}_t$, $\gamma_{10,+}$ and $\gamma_{10,-}$ have the same initial point and terminal point.

5. Since $X \rightarrow \overline{X}$ is branches on $y = \overline{x} = 0$, we have the unique lift of $\gamma_{x}$ to $X$. we use the same symbol for its lift.

6. The paths $\gamma_{00,+}$ and $\gamma_{00,-}$ are the lifts of $\gamma_{00}$. They are on the curve $Q^{an}_{00}(0,0,t)$.
   Note that since $\gamma_{00}(0)$ is located on the branching locus of $X_t \rightarrow \overline{X}_t$, $\gamma_{00,+}$ and $\gamma_{00,-}$ have the same initial point.

**Proposition 8.12.** The 2-chain $K_+ - K_- \in Z_2(X^{\alpha}_t)$ is a 2-chain associated with $\xi_{0,t} - \xi_{1,t}$ in a weak sense (See Definition 8.8).

**Proof.** By Figure 5 we see that that
\[
\partial(K_+ - K_-) = (\gamma_{c,+} - \gamma_{c,-}) + (\gamma_{11,+} - \gamma_{11,-}) + (\gamma_{10,+} - \gamma_{10,-}) + (\gamma_{00,+} - \gamma_{00,-})
\]
(216)

Then it is enough to show that $\gamma_{c,+} - \gamma_{c,-} + (\gamma_{11,+} - \gamma_{11,-}) + (\gamma_{10,+} - \gamma_{10,-}) + (\gamma_{00,+} - \gamma_{00,-})$ is a 1-cycle associated with $\xi_{0,t} - \xi_{1,t}$ in a weak sense.

By Corollary 5.12, $\xi_{0,t} - \xi_{1,t}$ is represented by
\[
\left( C_t, \left(\frac{1}{v-1} \right) \frac{v+1}{\sqrt{1-a}} \right) + \left( Q^{(0,0),t}, \frac{v-1}{\sqrt{1-a}} \right) + \left( Q^{(1,1),t}, \frac{v+1}{\sqrt{1-a}} \right).
\]
Let $\gamma_{c, \text{tr}} \in S_1(C_{\text{an}})$, $\gamma_{00, tr} \in S_1(Q_{(0,0), t}^\text{an})$ and $\gamma_{11, tr} \in S_1(Q_{(1,1), t}^\text{an})$ and be the inverse images of the path $[0, \infty] \subset (\mathbb{P}_1^1)^{\text{an}}$ in the real axis by the rational functions $(v+1)(v-b \sqrt{1-a})/(v-1)(v-1)$ on $C_{\text{an}}$, $v^{-1}$ on $Q_{(0,0), t}^\text{an}$, and $v^{+1}$ on $Q_{(1,1), t}^\text{an}$. By using $(x, y, v)$-coordinate, we see that the following relations hold.

\[
\begin{align*}
\partial \gamma_{c, \text{tr}} &= (0, 0, 1) + \left(1, 1, -\frac{1-b}{\sqrt{1-a}}\right) - (0, 0, -1) - \left(1, 1, \frac{1-b}{\sqrt{1-a}}\right) = \partial \gamma_{c, -} - \partial \gamma_{c, +} \\
\partial \gamma_{11, \text{tr}} &= \left(1, 1, \frac{1-b}{\sqrt{1-a}}\right) - \left(1, 1, -\frac{1-b}{\sqrt{1-a}}\right) = \partial \gamma_{11, -} - \partial \gamma_{11, +} \\
\partial \gamma_{10, -} - \partial \gamma_{10, +} &= 0 \\
\partial \gamma_{00, \text{tr}} &= (0, 0, 1) - (0, 0, -1) = \partial \gamma_{00, -} - \partial \gamma_{00, +}
\end{align*}
\]

Since $Q_{(0,0), t}^\text{an}$, $Q_{(1,1), t}^\text{an}$, and $C_{t}$ are isomorphic to $\mathbb{P}_1^1$, and $B_1((\mathbb{P}_1^1)_{\text{an}}) = Z_1((\mathbb{P}_1^1)_{\text{an}})$, we can confirm that $(\gamma_{c, -} - \gamma_{c, +}) + (\gamma_{11, -} - \gamma_{11, +}) + (\gamma_{10, -} - \gamma_{10, +}) + (\gamma_{00, -} - \gamma_{00, +})$ is a 1-cycle associated with $\xi_{0, t} - \xi_{1, t}$ in a weak sense. Hence we have the result.

8.3. Calculation of the transcendental regulator at $t \in T^o(\mathbb{C})$. Since we construct a 2-chain associated with $\xi_{0, t} - \xi_{1, t}$, we can evaluate the image the transcendental regulator map by Lemma 8.9 at the 2-form $\omega_t$ on $X_t$.

Definition 8.13. Since $X_t$ is a $K3$ surface, we have $\dim_{\mathbb{C}} H^{2,0}(X_t^\text{an}) = 1$. Let $\omega_t$ be a holomorphic 2-form defined in Definition 7.3. Since $\omega_t$ is non-zero, the following map is an isomorphism of abelian groups.

\[
ev_t : H^{2,0}(X_t^\text{an})^+/H_2(X_t^\text{an}, \mathbb{Z}) \xrightarrow{\omega} \mathbb{C}/\mathcal{P}_{\omega_t}(X_t) \xrightarrow{\varphi} \mathbb{C}/\mathbb{P}(\omega_t) 
\]

Figure 5. The figure of $K_+, K_-$ and paths on their boundaries.
We denote this map by $\text{ev}_t$. Hereafter we concern periods of Kummer surfaces $X_t$, we simply write $P_{\omega_i}$ for $P_{\omega_i}(X_t)$. Furthermore, the image of $x \in C$ via the natural surjection $C \to \mathbb{C}/P_{\omega_i}$ is denoted by $[x] \in \mathbb{C}/P_{\omega_i}$.

**Proposition 8.14.** we have
\[
\int_{K_+} \omega_t = - \int_{K_-} \omega_t = \int_{\Delta} \frac{\gamma'(p)\gamma(q)dpdq}{\sqrt[4]{\gamma(p)(1-\gamma(p))(1-a\gamma(p))} \cdot \sqrt[4]{\gamma(q)(1-\gamma(q))(1-b\gamma(q))}}
\]
where the choice of the branch of the integrand is defined in Definition 8.10.

**Proof.** Since $\Delta_{\pm}$ are dense in $K_{\pm}$, integration of $\omega_t$ on $K_{\pm}$ is the same as integration on $\Delta_{\pm}$. By construction of $\Delta_{\pm}$, we see that $\Delta_{+}$ coincides with the following map.
\[
\{(p, q) \in \mathbb{R}^2 : 0 < q < p < 1\} \xrightarrow{\omega} \{x, y, v\} = \left(\gamma(p), \gamma(q), \frac{\sqrt[4]{\gamma(q)(1-\gamma(q))(1-b\gamma(q))}}{\sqrt[4]{\gamma(p)(1-\gamma(p))(1-a\gamma(p))}}\right)
\]
Hence we get the result. \hfill \square

**Definition 8.15.** Let $t \in T^\ell(C)$ and choose $\gamma$ satisfying the condition in Definition 8.10. We can take an open neighborhood $U$ of $t$ in $(T^\ell)^{an}$ in the classical topology such that $\gamma$ satisfies the condition in Definition 8.10 at every point on $U$. Then we can define a holomorphic function $L(t)$ on $U$ by
\[
L(t) = \int_{\Delta} \frac{\gamma'(p)\gamma(q)dpdq}{\sqrt[4]{\gamma(p)(1-\gamma(p))(1-a\gamma(p))} \cdot \sqrt[4]{\gamma(q)(1-\gamma(q))(1-b\gamma(q))}}
\]
This integral can be regarded as integration of a $C^\infty$ function on the compact $C^\infty$ manifolds with corners (Definition 8.11). The integrand is holomorphic with respect to $a, b$, which are local coordinates of $t \in (T^\ell)^{an}$. Thus we can check that $L$ is a local holomorphic function on $(T^\ell)^{an}$.

Using this function, we can describe the image of $\xi(0, -1, 1)_t$ by the transcendental regulator $r$ explicitly.

**Theorem 8.16.** The image of $\xi_{0,1} - \xi_{1,1}$ under the transcendental regulator is described as follows.
\[
\text{ev}_t(r(\xi_{0,1} - \xi_{1,1})) = 2[L(t)]
\]
For the notation $[x] \in \mathbb{C}/P_{\omega_i}$, see Definition 8.13.

**Proof.** This is the combination of Lemma 8.9, Proposition 8.14 and the definition of $L$ in Definition 8.15. \hfill \square

**Remark 8.17.** If we choose different $\gamma$ for the local holomorphic function $L$ in the Definition 8.15, we get new local holomorphic function $L'$. However by Theorem 8.16, the difference $L(t) - L'(t)$ should lie in $\frac{1}{2}P_{\omega_i}$.

At last, we calculate the image of $L$ under the Picard-Fuchs operator $\mathcal{D}$ in Definition 7.12. This calculation is used the rank estimation of the higher Chow group in Section 9. From this result, we have a system of differential equation satisfied by $L$. Furthermore, $L$ is linearly independent from period functions $P_{ij}$ in Definition 7.11.

**Theorem 8.18.** Let $L$ be the local holomorphic function defined in Definition 8.15. Then we have
\[
\mathcal{D}L = \frac{1}{a - b} \cdot \frac{\sqrt[4]{\frac{\sqrt[4]{\gamma(p)}}{\sqrt[4]{\gamma(q)}}} - 1}{\sqrt[4]{\gamma(p)}}
\]
(224)
Proof. A local holomorphic function \( H_c(z) = \frac{\sqrt{z(1 - z)}}{2(1 - cz)^\frac{3}{2}} \) satisfies
\[
L_c \left( \frac{1}{\sqrt{z(1 - z)(1 - cz)}} \right) = \frac{dH_c}{dz}.
\]
where \( L_c \) is the differential operator defined in Definition 7.8. Then we have
\[
\begin{align*}
\frac{d}{dp} H_a(\gamma(p)) &= L_a \left( \frac{\gamma'(p)}{\sqrt{\gamma(p)(1 - \gamma(p))(1 - a\gamma(p))}} \right), \\
\frac{d}{dq} H_b(\gamma(q)) &= L_b \left( \frac{\gamma'(q)}{\sqrt{\gamma(q)(1 - \gamma(q))(1 - b\gamma(q))}} \right).
\end{align*}
\]
By the definition of \( \mathcal{L} \) and Stokes theorem,
\[
\mathcal{D}_1(\mathcal{L}) = \mathcal{D}_1 \left( \int_{K_+} \frac{\gamma'(p)\gamma'(q)dqd\gamma}{\sqrt{\gamma(q)(1 - \gamma(q))(1 - a\gamma(p))(1 - b\gamma(q))}} \right) = \int_{\partial \Delta} \frac{H_a(\gamma(p))\gamma'(q)dq}{\sqrt{\gamma(q)(1 - \gamma(q))(1 - b\gamma(q))}}.
\]
Note that the condition on \( \gamma \) in Definition 8.10, \( \frac{H_a(\gamma(p))\gamma'(q)dq}{\sqrt{\gamma(q)(1 - \gamma(q))(1 - b\gamma(q))}} \) is well-defined 1-form at boundaries of \( K_+ \). Since this 1-form vanishes at \( \{q = 0\} \) and \( \{p = 1\} \), we have
\[
\mathcal{D}_1(\mathcal{L}) = \frac{1}{2} \int_0^1 \frac{\gamma'(q)dq}{(1 - b\gamma(q))(1 - a\gamma(q))}.
\]
where we use the coordinate transform \( u = \frac{\sqrt{1 - b\gamma(q)}}{\sqrt{1 - a\gamma(q)}} \). The computation of \( \mathcal{D}_2 \mathcal{L} \) is similar. \( \square \)

9. Estimation of the rank on \( \Xi \)

In this section, we prove the main result Theorem 9.22, which claims that the image of \( \Xi \) via the regulator map has rank 18. The basic strategy of the proof is as follows.

1. We construct a \( \mathbb{Q} \)-linear sheaf \( Q_\omega \) on \( T' \) such that at each point \( t \in (T')^{an} \), the fiber of \( Q_\omega \) is the space \( \mathbb{C}/\mathbb{Q}P_{\omega_t} \simeq H^{2,0}(X_t^{an})^\vee / H_2(X_t, \mathbb{Z}) \). We see that the Picard-Fuchs differential operator \( \mathcal{D} \) factors the sheaf \( Q_\omega \) (Definition 9.10).

2. The \( \mathbb{Q} \)-linear space \( Q_{\omega_t}(T^\circ) \) is the target of “relative transcendental regulator map” \( R_{\omega_t}: \Xi \to Q_{\omega_t}(T^\circ) \), (Definition 9.16). For each point \( t \in T^\circ(\mathbb{C}) \), the value of \( R_{\omega_t}(\xi) \) at fibers above \( t \) coincides with \( r(\xi) \) modulo torsion.

3. By the formula for \( G \)-action on \( \omega_t \) in Definition 7.3, we have the following commutative diagram (Proposition 9.6).
\[
\begin{array}{cccc}
\text{CH}^2(X_t, 1) & \xrightarrow{r} & H^{2,0}(X_t^{an})^\vee / H_2(X_t, \mathbb{Z}) & \xrightarrow{\text{ev}_{\omega_t}} & \mathbb{C}/\mathbb{Q}P_{\omega_t} \\
\downarrow \rho_t & & \downarrow \phi_{\omega_t}^\vee & & \downarrow \chi(\rho)(t) \\
\text{CH}^2(X_{\omega_t}(t), 1) & \xrightarrow{r} & H^{2,0}(X_{\omega_t}^{an})^\vee / H_2(X_t, \mathbb{Z}) & \xrightarrow{\text{ev}_{\omega_t}(t)} & \mathbb{C}/\mathbb{Q}P_{\omega_{\omega_t}(t)}
\end{array}
\]
(229)
(4) By the diagram above, we can define $G$-action $\{\rho_t\}_{t \in \mathcal{G}}$ on $\mathcal{Q}_\omega$ (Definition 9.7) so that the relative transcendental regulator $R_\omega$ is equivariant under $G$-actions (Proposition 9.17). Furthermore, we can also define $G$-action $\{\Theta_t\}_{t \in \mathcal{G}}$ on $\mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}} \oplus \mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}$ (Definition 9.11) so that $\mathcal{D}$ is equivariant to $G$-actions (Proposition 9.12).

(5) In conclusion, we have the following diagram for $\tilde{\rho} \in \tilde{G}$.

$$\begin{align*}
\Xi & \xrightarrow{R_\omega} \mathcal{Q}_\omega(\mathcal{T}^\circ) \xrightarrow{\mathcal{D}} \mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}(\mathcal{T}^\circ)^{\oplus 2} \\
\Xi & \xrightarrow{\tilde{\rho}^*} \mathcal{Q}_\omega(\mathcal{T}^\circ) \xrightarrow{\tilde{\rho}^*} \mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}(\mathcal{T}^\circ)^{\oplus 2}
\end{align*}$$

(230)

Hence we can compute the image $\mathcal{D} \circ R_\omega(\Xi)$ (Figure 8) and get the desired rank estimate (Theorem 9.22).

9.1. The definition of the sheaf $\mathcal{P}_\omega$ and $\mathcal{Q}_\omega$. In this section, we define the sheaf $\mathcal{P}_\omega$ and $\mathcal{Q}_\omega$ and prove some properties of them. The sheaf $\mathcal{Q}_\omega$ is a target of “relative transcendental regulator map” defined in Definition 9.15.

**Definition 9.1.** We regard the sheaf $\mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}$ as a $\mathbb{Q}$-linear sheaf. We define a subsheaf $\mathcal{P}_\omega \subset \mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}$ as the unique sheaf satisfying the following property:

For any open set $U \subset (\mathcal{T})^{an}$ in the classical topology such that $P_{ij}$ is defined, $\mathcal{P}_\omega|_U$ is the subsheaf generated (as $\mathbb{Q}$-linear sheaf) by $P_{ij}$ for $i, j \in \{1, 2\}$.

(231)

where $P_{ij}$ are the local holomorphic functions defined in Definition 7.11. Then we define a sheaf $\mathcal{Q}_\omega$ as the quotient sheaf $\mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}/\mathcal{P}_\omega$. For a local section $f \in \mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}$, $[f] \in \mathcal{Q}_\omega$ denotes the image of $f$ by $\mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}} \rightarrow \mathcal{Q}_\omega$.

The existence of the subsheaf $\mathcal{P}$ satisfying (231) is the consequence that $P_{ij}$ has the rational monodromy in Remark 7.9. The existence can also be confirmed by the following Remark.

**Remark 9.2.** Let $\pi : \mathcal{X} \rightarrow T'$ be the structure morphism. We define the following sheaves $\mathcal{P}, \mathcal{Q}$ on $(\mathcal{T})^{an}$.

$$\begin{align*}
\mathcal{P} &= \text{Im}(R^2\pi_*\mathcal{Q}(\mathcal{X})^{an}) \rightarrow \mathcal{H}om_{\mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}}(\pi_*\mathcal{O}^{\mathcal{T}}_{\mathcal{X}/\mathcal{T}'}, \mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}) \\
\mathcal{Q} &= \text{Coker}(R^2\pi_*\mathcal{Q}(\mathcal{X})^{an}) \rightarrow \mathcal{H}om_{\mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}}(\pi_*\mathcal{O}^{\mathcal{T}}_{\mathcal{X}/\mathcal{T}'}, \mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}})
\end{align*}$$

(232)

where $\mathcal{Q}(\mathcal{X})^{an}$ be the constant sheaf with coefficients in $\mathcal{Q}$ on $(\mathcal{X})^{an}$ and the morphism $R^2\pi_*\mathcal{Q}(\mathcal{X})^{an} \rightarrow \mathcal{H}om_{\mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}}(\pi_*\mathcal{O}^{\mathcal{T}}_{\mathcal{X}/\mathcal{T}'}, \mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}})$ is induced by the fiber integration.

Since $\mathcal{X}$ is a family of K3 surface, $\pi_*\mathcal{O}^{\mathcal{T}}_{\mathcal{X}/\mathcal{T}'}$ is a locally free $\mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}$-module of rank 1. By fixing $\omega \in \Gamma(\mathcal{X}, \pi_*\mathcal{O}^{\mathcal{T}}_{\mathcal{X}/\mathcal{T}'})$ in Definition 7.1 we have an isomorphism

$$\mathcal{H}om_{\mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}}(\pi_*\mathcal{O}^{\mathcal{T}}_{\mathcal{X}/\mathcal{T}'}, \mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}) \simeq \mathcal{O}^{\mathcal{T}}_{(\mathcal{T})_{\mathbb{R}}}$$

(233)

Under this isomorphism, $\mathcal{P} \simeq \mathcal{P}_\omega$ and $\mathcal{Q} \simeq \mathcal{Q}_\omega$. By Ehresmann’s fibration theorem, $\pi : \mathcal{X} \rightarrow T'$ is a topologically locally trivial fibration. Hence for sufficiently small open neighborhood in the classical topology, $P_{ij}$ ($i, j \in \{1, 2\}$) form a basis of $\mathcal{P}_\omega|_U \simeq \mathcal{P}|_U$. 


**Definition 9.3.** For each \( t \in T'(\mathbb{C}) \), \( \mathcal{O}_{(T')^n, t} \) denotes the stalk of \( \mathcal{O}_{(T')^n} \) at \( t \). We define the evaluation map \( m_t \) by

\[
m_t : \mathcal{O}_{(T')^n, t} \to \mathbb{C}
\]

\[
\varphi \quad \mapsto \varphi(t).
\]  

(234)

For an open neighborhood \( U \) of \( t \) in the classical topology, composition of \( m_t \) and the restriction map \( \mathcal{O}_{(T')^n}(U) \to \mathcal{O}_{(T')^n, t} \) is also denoted by \( m_t \). Furthermore, since \( \mathcal{P}_{\omega_i} \subset \mathbb{C} \) is generated by the values of \( P_{ij} \) at \( t \) by Definition 9.1, \( m_t \) induces the following map,

\[
\mathcal{O}_{(T')^n, t} \xrightarrow{m_t} \mathbb{C} \xrightarrow{\varphi} \varphi(t).
\]

(235)

where \( \mathbb{Q}\mathcal{P}_{\omega_i} \subset \mathbb{C} \) is a \( \mathbb{Q} \)-linear subspace of \( \mathbb{C} \) generated by \( \mathcal{P}_{\omega_i} \). We also denote this map by \( m_t \). Furthermore, the composition of \( m_t \) and the restriction map of \( \mathcal{Q}_\omega \) is also denoted by \( m_t \).

**Proposition 9.4.** Let \( U \) be an open subset of \( (T')^n \) in the classical topology and \( \varphi \in \mathcal{O}_{(T')^n}(U) \). If \( \varphi \not\in \mathcal{P}_{\omega}(U) \), then \( \varphi(t) \not\in \mathbb{Q}\mathcal{P}_{\omega_i} \) for very general \( t \in U \). Especially, if \( \varphi \in \mathcal{O}_{(T')^n} \) satisfies that \( \varphi(t) \in \mathbb{Q}\mathcal{P}_{\omega_i} \) holds for every \( t \in U \), then \( \varphi \) is a section of \( \mathcal{P}_{\omega}(U) \).

**Proof.** We will prove the former part of the proposition. We may assume \( U \) is so small that \( P_{ij} \) are defined on \( U \). For each quadruple \( \xi = (c_{ij}) \in \mathbb{Q}^{\mathbb{Z}_4} \), we define a holomorphic function \( F_\xi \) by

\[
F_\xi = \varphi - \sum_{i,j} c_{ij} P_{ij}
\]

(236)

Consider the countable family \( \{F_\xi\}_{\xi \in \mathbb{Q}^{\mathbb{Z}_4}} \) of holomorphic functions on \( U \). If \( \varphi \not\in \mathcal{P}_{\omega}(U) \), they are non-zero holomorphic functions. Especially, for very general \( t \in T \), \( F_\xi(t) \neq 0 \) holds for all \( \xi \in \mathbb{Q}^{\mathbb{Z}_4} \). Since \( \mathcal{P}_{\omega_i} \) is generated (as \( \mathbb{Q} \)-linear subspace of \( \mathbb{C} \)) by \( P_{ij}(t) \), we see that \( F_\xi(t) \neq 0 \) holds for all \( \xi \in \mathbb{Q}^{\mathbb{Z}_4} \) is equivalent to \( \varphi(t) \not\in \mathbb{Q}\mathcal{P}_{\omega_i} \).

To see the latter statement, note that the complement of countable union of proper analytic subsets are dense.

The sheaf \( \mathcal{Q}_\omega \) has the following property. This lemma enables us to the computation of \( \mathcal{Q}_\omega \) reduces to computation at each point on \( U \).

**Lemma 9.5.** For each open subset \( U \) in the classical topology and non-zero section \( x \in \mathcal{Q}_\omega(U) \), then the fiber \( m_t(x) \) is non-zero for very general \( t \in U \). Especially, the following map is injective,

\[
\begin{array}{ccc}
\mathcal{Q}_\omega(U) & \longrightarrow & \prod_{t \in U} \mathbb{C}/\mathbb{Q}\mathcal{P}_{\omega_i} \\
\varphi & \mapsto & (m_t(x))_t
\end{array}
\]

(237)

**Proof.** We can shrink \( U \) enough small so that \( x \in [\varphi] \) (for this notation, see Definition 9.1) for \( \varphi \in \mathcal{O}_{(T')^n}(U) \). Then the results follows from Proposition 9.4. \( \square \)

\( ^9 \)We use the word “very general” for the meaning of “outside of countable union of proper (not the whole space) analytic subsets”.

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9.2. \( \widetilde{G} \)-action on \( Q_\omega \). We will define the following group homomorphism for \( \tilde{\rho} \in \widetilde{G} \).

\[
\tau_{\tilde{\rho}} : Q_\omega(T^\circ) \longrightarrow Q_\omega(T^\circ)
\]  

(238)

This group action enables us to compute the image by transcendental regulator map of all elements in \( \Xi \) from Theorem 8.16. First, we see that how \( \widetilde{G} \) acts on fibers of \( Q_\omega \).

**Proposition 9.6.** Let \( t \in T^\circ(\mathbb{C}) \) and \( \tilde{\rho} = (\rho, \zeta) \in \widetilde{G} \). Let \( \tilde{\rho}_t : \mathcal{X}_t \overset{\sim}{\longrightarrow} \mathcal{X}_{an}(t) \) be the automorphism defined in Definition 4.31. Let \( \mathcal{X}_{an}(t) \). Then, we see that the whole rectangle in (240) commutes. Since \( \tilde{\rho}_t \)

\[
(1) \quad \mathcal{P}_{\omega_{an}(t)} = \bar{\chi}(\tilde{\rho})(t) \cdot \mathcal{P}_{\omega_t} \quad \text{as a subgroup of} \quad \mathcal{C}.
\]

Here \( \bar{\chi}(\tilde{\rho})(t) \in \mathcal{C} \) is the value of \( \bar{\chi}(\tilde{\rho}) \in B' \) in Definition 4.23 at \( t \).

(2) From (1), the following map is well-defined.

\[
\bar{\chi}(\tilde{\rho})(t) : \mathcal{C}/\mathcal{P}_{\omega_t} \longrightarrow \mathcal{C}/\mathcal{P}_{\omega_{an}(t)}
\]

(239)

\[
[x] \longmapsto [\bar{\chi}(\tilde{\rho})(t) \cdot x]
\]

See Definition 8.13 for the notation \([x]\).

(3) We have the following commutative diagram.

\[
\begin{array}{ccc}
\text{CH}^2(\mathcal{X}_t, 1) & \overset{r}{\longrightarrow} & H^{2,0}(\mathcal{X}_{an}(t))^*/H_2(\mathcal{X}_t, \mathbb{Z}) \\
\downarrow_{(\rho_t)_*} & & \downarrow_{(\rho_t)_*} \\
\text{CH}^2(\mathcal{X}_{an}(t), 1) & \overset{r}{\longrightarrow} & H^{2,0}(\mathcal{X}_{an}(t))^*/H_2(\mathcal{X}_{an}(t), \mathbb{Z})
\end{array}
\]

(240)

where the right vertical map is (239) above.

**Proof.** Note that the following equation holds for every 2-chain \( \Gamma \in S_2(\mathcal{X}_{an}(t)) \).

\[
\int_{(\rho_t)_*\Gamma} \omega_{\tilde{\rho}(t)} = \int_{\Gamma} (\rho_t)^* \omega_{\tilde{\rho}(t)} = \chi(\rho)(t) \cdot \int_{\Gamma} \omega_t
\]

(241)

For the 2nd equality, we use Proposition 7.3. If we take \( \Gamma \in Z_2(\mathcal{X}_t) \), computation shows (1) and the commutativity of the right square in (3). Note that if \( \Gamma \) is a 2-chain associated\(^{10}\) with \( \xi \in \text{CH}^2(\mathcal{X}_t, 1) \), \( (\rho_t)_*\Gamma \) is a 2-chain associated with \( (\rho_t)_*\xi \in \text{CH}^2(\mathcal{X}_{an}(t), 1) \). Hence if we take \( \Gamma \) as a 2-chain associated with \( \xi \) in the equation (241), we see that the whole rectangle in (240) commutes. Since \( ev_t, ev_{\tilde{\rho}(t)} \) are isomorphisms by Proposition 8.13, the left square in (240) commutes. \( \square \)

**Definition 9.7.** Let \( \tilde{\rho} = (\rho, \zeta) \in \widetilde{G} \). We define a morphism \( \Upsilon_{\tilde{\rho}} : \mathcal{O}_{(T^\circ)^{an}} \rightarrow (\rho^{-1})_*\mathcal{O}_{(T^\circ)^{an}} \) as follows. Let \( U \) be an open subset of \( (T^\circ)^{an} \) in the classical topology.

\[
\Upsilon_{\tilde{\rho}} : \mathcal{O}_{(T^\circ)^{an}}(U) \longrightarrow (\rho^{-1})_*\mathcal{O}_{(T^\circ)^{an}}(U) \\
\varphi \longmapsto (\rho^{-1})^\sharp (\bar{\chi}(\tilde{\rho}) \cdot \varphi) = \bar{\chi}(\tilde{\rho}^{-1})^{-1} \cdot (\rho^{-1})^\sharp (\varphi)
\]

(242)

Since \( \bar{\chi} \) is an opposite 1-cocycle, we have the equality \( (\rho^{-1})^\sharp (\bar{\chi}(\tilde{\rho})) = \bar{\chi}(\tilde{\rho}^{-1})^{-1} \). Furthermore, \( \{\Upsilon_{\tilde{\rho}}\}_{\tilde{\rho} \in \widetilde{G}} \) satisfies the following cocycle condition. i.e. the following

\(^{10}\)See Definition 8.2 for the definition.
diagram commutes for \( \tilde{\rho}, \tilde{\rho}' \in \tilde{G} \).

\[
\begin{array}{ccc}
\mathcal{O}(T^\circ)^{an} & \xrightarrow{\rho} & (\rho^{-1})_*\mathcal{O}(T^\circ)^{an} \\
\downarrow \tilde{\rho} \downarrow & & \downarrow \tilde{\rho}' \\
(\tilde{\rho}'\rho^{-1})*\mathcal{O}(T^\circ)^{an} & \xrightarrow{(\tilde{\rho}'\rho)^{-1}} & (\rho^{-1})_*\mathcal{O}(T^\circ)^{an}
\end{array}
\tag{243}
\]

**Proposition 9.8.** For \( \tilde{\rho} \in \tilde{G} \) and an open subset \( U \subset (T')^{an} \) in the classical topology, we have

\[
\Upsilon_{\tilde{\rho}}(P_{\omega}(U)) = P_{\omega}(\tilde{\rho}(U)).
\tag{244}
\]

**Proof.** By the cocycle condition \( (243) \), it is enough to show only \((\subset)\). Let \( \varphi \in P_{\omega}(U) \). Then for \( \rho(t) \in \tilde{\rho}(T) \), we have

\[
m_{\rho(t)}(\Upsilon_{\tilde{\rho}}(\varphi)) = m_{\rho(t)}((\rho^{-1})^\sharp (\chi(\rho) \cdot \varphi)) = \chi(\rho)(t) \cdot \varphi(t) \in \chi(\rho)(t) \cdot Q_{\omega t} = Q_{P_{\rho(t)}(U)}.
\tag{245}
\]

The last equality follows from Proposition 9.6. Then by Proposition 9.4, \( \Upsilon_{\tilde{\rho}}(\varphi) \in P_{\omega}(\tilde{\rho}(U)) \).

**Definition 9.9.** By Proposition 9.8 for \( \tilde{\rho} \in \tilde{G} \), \( \Upsilon_{\tilde{\rho}} : \mathcal{O}(T^\circ)^{an} \rightarrow (\rho^{-1})_*\mathcal{O}(T^\circ)^{an} \) induces the morphism \( Q \rightarrow (\rho^{-1})_*Q \). Since \( T^\circ \) is stable under \( \tilde{G} \)-action, we have a group homomorphism

\[
\Upsilon_{\tilde{\rho}} : Q_{\omega}(T^\circ) \longrightarrow Q_{\omega}(T^\circ)
\tag{246}
\]

for \( \tilde{\rho} \in \tilde{G} \). By the cocycle condition \( (243) \), \( \Upsilon_{\tilde{\rho}} \) defines \( \tilde{G} \)-action on the abelian group \( Q_{\omega}(T^\circ) \). By the definition of \( \Upsilon_{\rho} \) in Definition 9.7, the following diagram commutes for \( t \in T^\circ \).

\[
\begin{array}{ccc}
Q_{\omega}(T^\circ) & \xrightarrow{m_t} & C/Q_{\omega t} \\
\downarrow \Upsilon_{\rho} & & \downarrow \bar{\chi}(\rho)(t) \\
Q_{\omega}(T^\circ) & \xrightarrow{m_{\rho(t)}} & C/Q_{\omega_{\rho(t)}}
\end{array}
\tag{247}
\]

where the right vertical map is induced by \((2)\) of Proposition 9.6.

### 9.3. Differential Operator \( \mathscr{D} \) on \( Q_{\omega} \) and \( \tilde{G} \)-action.

Since it is difficult to study the rank of the image of \( \Xi \) in \( Q_{\omega} \), we consider the following differential operator \( \mathscr{D} \).

**Definition 9.10.** Since the local holomorphic functions \( P_{ij} \) are annihilated by the differential operator \( \mathscr{D} : \mathcal{O}(T^\circ)^{an} \rightarrow \mathcal{O}_{(T^\circ)^{an}}^{\otimes 2} \) in Definition 7.12, the following is 0-map.

\[
P_{\omega} \longrightarrow \mathcal{O}(T^\circ)^{an} \xrightarrow{\mathscr{D}} \mathcal{O}_{(T^\circ)^{an}}^{\otimes 2}
\tag{248}
\]

Hence the following morphism is induced. This morphism is also denoted by \( \mathscr{D} \).

\[
\begin{array}{ccc}
\mathcal{O}(T^\circ)^{an} & \xrightarrow{\mathscr{D}} & \mathcal{O}_{(T^\circ)^{an}}^{\otimes 2} \\
\downarrow & & \downarrow \\
Q_{\omega} & \xrightarrow{\mathscr{D}} & \mathcal{O}_{(T^\circ)^{an}}^{\otimes 2}
\end{array}
\tag{249}
\]

We will define a \( \tilde{G} \)-linearization on \( \mathcal{O}_{(T^\circ)^{an}}^{\otimes 2} \) such that \( \mathscr{D} : Q_{\omega} \rightarrow \mathcal{O}_{(T^\circ)^{an}}^{\otimes 2} \) becomes equivariant.
Definition 9.11. Let \( \tilde{\rho} = (\rho, \zeta) \in \tilde{G} \). Let \( U \) be an open subset of \((T')^\text{an}\) in the classical topology. We define a morphism \( \Theta_{\tilde{\rho}} : \mathcal{O}^\oplus_{(T')^\text{an}} \rightarrow (\tilde{\rho}^{-1})_* \mathcal{O}^\oplus_{(T')^\text{an}} \) as the following map.

\[
\Theta_{\tilde{\rho}} : \mathcal{O}^\oplus_{(T')^\text{an}}(U) \xrightarrow{\psi} \mathcal{O}^\oplus_{(T')^\text{an}}(\rho(U)) \xrightarrow{(\rho^{-1})_*} (\tilde{\rho}^{-1})_* \mathcal{O}^\oplus_{(T')^\text{an}}(U)
\]

\[
\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \mapsto \begin{pmatrix} (\rho^{-1})^2 (\tilde{\chi}(\tilde{\rho}) \chi^{(1)}(\rho)^2 \cdot \varphi_1) \\ (\rho^{-1})^2 (\tilde{\chi}(\tilde{\rho}) \chi^{(2)}(\rho)^2 \cdot \varphi_2) \end{pmatrix} = \begin{pmatrix} \tilde{\chi}(\rho^{-1})^{-1} \chi^{(1)}(\rho^{-1})^{-2} \cdot (\rho^{-1})^2 (\varphi_1) \\ \tilde{\chi}(\rho^{-1})^{-1} \chi^{(2)}(\rho^{-1})^{-2} \cdot (\rho^{-1})^2 (\varphi_2) \end{pmatrix}
\]

Here, \( \chi^{(1)}, \chi^{(2)} \) are the opposite 1-cocycles defined in Definition 4.21 and the last equality is obtained from the cocycle conditions of \( \tilde{\chi}, \chi^{(1)}, \chi^{(2)} \) are opposite 1-cocycles.

Then \( \{ \Upsilon_{\tilde{\rho}}, \rho \in \tilde{G} \} \) satisfies the following cocycle condition for \( \tilde{\rho}, \tilde{\rho}' \in \tilde{G} \).

\[
\begin{pmatrix} \mathcal{O}^\oplus_{(T')^\text{an}} \\ \Theta_{\tilde{\rho}} \end{pmatrix} \xrightarrow{\Theta_{\tilde{\rho}'}} \begin{pmatrix} \mathcal{O}^\oplus_{(T')^\text{an}} \\ \Theta_{\tilde{\rho}} \end{pmatrix} \xrightarrow{(\rho')^{-1}} \begin{pmatrix} \mathcal{O}^\oplus_{(T')^\text{an}} \\ \Theta_{\tilde{\rho}} \end{pmatrix} \xrightarrow{(\rho')^{-1}} \begin{pmatrix} \mathcal{O}^\oplus_{(T')^\text{an}} \\ \Theta_{\tilde{\rho}} \end{pmatrix}
\]

By the cocycle condition, \( \Theta_{\rho} : \mathcal{O}(T')^\text{an}(T^o)^\oplus \rightarrow \mathcal{O}(T')^\text{an}(T^o)^\oplus \) defines a \( \tilde{G} \)-action on \( \mathcal{O}(T')^\text{an}(T^o)^\oplus \).

The main purpose of this subsection is to prove the \( \tilde{G} \)-equivariance of \( \mathcal{D} \).

Proposition 9.12. For \( \rho \in \tilde{G} \), the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{Q}(T^o) & \xrightarrow{\mathcal{D}} & \mathcal{O}(T')^\text{an}(T^o)^\oplus \\
\downarrow \Upsilon_{\rho} & & \downarrow \Theta_{\rho} \\
\mathcal{Q}(T^o) & \xrightarrow{\mathcal{D}} & \mathcal{O}(T')^\text{an}(T^o)^\oplus
\end{array}
\]

(252)

We need some preparation for proving Proposition 9.12. First, we define some differential operators twisted by \( \tilde{G} \)-action.

Definition 9.13. For \( \rho \in \tilde{G} \), we define \( \mathcal{D}_i^\rho \) for \( i = 1, 2 \) as follows.

\[
\mathcal{D}_1^\rho = a'(1 - a') \frac{\partial^2}{(\partial a')^2} + (1 - 2a') \frac{\partial}{\partial a'} - \frac{1}{4}
\]

\[
\mathcal{D}_2^\rho = b'(1 - b') \frac{\partial^2}{(\partial b')^2} + (1 - 2b') \frac{\partial}{\partial b'} - \frac{1}{4}
\]

(253)

where \( a' = \rho^1(a) \) and \( b' = \rho^2(b) \). Furthermore, we define \( \mathcal{D}_3^\rho \) as follows.

\[
\begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \end{pmatrix} : \mathcal{O}(T')^\text{an} \rightarrow \mathcal{O}^\oplus_{(T')^\text{an}}
\]

(254)

By definition, for \( \rho \in \tilde{G} \) and local section \( \varphi \) of \( \mathcal{O}(T')^\text{an} \), we have

\[
\mathcal{D}_i^\rho(\rho^i(\varphi)) = \rho^i(\mathcal{D}_i \varphi).
\]

(255)
for $i = 1, 2$. Hence we have the following commutative diagram.

$$
\begin{array}{ccc}
\mathcal{O}(T') & \xrightarrow{\varphi^*} & \mathcal{O}(T') \\
\xrightarrow{(\rho^{-1})^*} & \xrightarrow{(\rho^{-1})^*} & \\
(\rho^{-1}) \cdot \mathcal{O}(T') & \xrightarrow{(\rho^{-1})^*} & \mathcal{O}(T')
\end{array}
$$

(256)

To prove Proposition 9.12 we need some transformation formula for $\varphi$ and cocycles $\chi^{(1)}, \chi^{(2)}, \tilde{\chi}$. Since $\varphi$ is the “pull-back” of the hypergeometric differential operator $L$ in Definition 7.8 by $p_r$, and cocycles $\chi^{(1)}, \chi^{(2)}$ are also pull-back of the cocycle $\phi$ in Definition 4.20, we have to see the transformation formula for the differential operator $L$ and the cocycle $\phi$.

**Proposition 9.14.** For $\tau \in H$ defined in Definition 4.6, we define $L^\tau : \mathcal{O}(S') \to \mathcal{O}(S')$ as follows.

$$
L^\tau = c'(1 - c') \frac{d^2}{dc'^2} + (1 - 2c') \frac{d}{dc'} - \frac{1}{4}
$$

(257)

where $c' = \tau^*(c)$. Then we have the following relation in the ring of differential operators on $(S')^n$.

$$
L^\tau \cdot \phi(\tau) = \phi(\tau)^3 \cdot L
$$

(258)

Here we regard $\phi(\tau) \in A' \subset \mathcal{O}(S')$ as a differential operator given by multiplication.

**Proof.** It is enough to prove the following.

$$
L^\tau = \phi(\tau)^3 \cdot L \cdot \phi(\tau)^{-1}
$$

(259)

To compute the right hand side of (259), we need the explicit description of $\phi(\tau)$ given in Table 6. Since we have the relation $\phi_0 = \text{sgn} \cdot \phi^2$ by Proposition 4.20 up to sign, $\phi(\tau)$ is determined from $\phi_0(\tau_0)$ which is given in Table 4 we have the following. Thus $\phi(\tau)$ is given by the following Table 6.

**Table 6. The Opposite 1-cocycle $\phi$**

| $\tau_0$ | $\tau^*(c)$ | $\phi(\tau)$ | $\tau_0$ | $\tau^*(c)$ | $\phi(\tau)$ |
|----------|-------------|--------------|----------|-------------|--------------|
| id       | $c$         | $\pm 1$      | (0 1)    | $1 - c$     | $\pm 1$      |
| $(1 \infty)$ | $\frac{1}{c-1}$ | $\pm \sqrt{-1} \sqrt{1 - c}$ | (0 1 $\infty$) | $\frac{1}{1 - c}$ | $\pm \sqrt{-1} \sqrt{1 - c}$ |
| $(0 \infty)$ | $\frac{1}{c}$  | $\pm \sqrt{-1} \sqrt{c}$   | (0 $\infty$ 1) | $\frac{c-1}{c}$ | $\pm \sqrt{-1} \sqrt{c}$ |

Hence to compute the right hand side of (259), we have to compute $L \cdot \frac{1}{\sqrt{c}}$ and $L \cdot \frac{1}{\sqrt{1 - c}}$. This can be computed easily by the formula $\frac{1}{\sqrt{c}} \cdot c^k = kc^{k-1} + c^k \cdot \frac{d}{dc}$ in the ring of differential operators. The result is as follows.

$$
L \cdot \frac{1}{\sqrt{c}} = \sqrt{c}(1 - c) \frac{d^2}{dc^2} - \sqrt{c} \frac{d}{dc} + \frac{1}{4c \sqrt{c}}
$$

$$
L \cdot \frac{1}{\sqrt{1 - c}} = c \frac{d^2}{dc^2} + \sqrt{1 - c} \frac{d}{dc} + \frac{1}{4(1 - c) \sqrt{1 - c}}
$$

(260)

We will compute the left hand side of (259). Note that $L \tau$ is determined determined by the image of $\tau$ in $H_0$, since $\tau^*(c)$ is determined by the image of $\tau$ in $H_0$. Hence it
is enough to check (259) for 6 elements in $H_0$. For example, we will check $z_0 = (1\infty)$ case. In this case, $c' = \frac{c}{c+1}$. Then

$$\frac{d}{dc'} = \frac{dc}{dc'} \cdot \frac{d}{dc} = -\frac{1}{(c' - 1)^2} \cdot \frac{d}{dc} = -(c - 1)^2 \cdot \frac{d}{dc}. \tag{261}$$

By using this, we have

$$\frac{d^2}{(dc')^2} = \left(-(c - 1)^2 \cdot \frac{d}{dc} \right)^2 = (c - 1)^4 \frac{d^2}{dc^2} + 2(c - 1)^3 \frac{d}{dc}. \tag{262}$$

Then by substituting $c', \frac{d}{dc'}, \frac{d^2}{(dc')^2}$ in (257) by the above differential operators, we get

$$L^\tau = -c(1 - c)^2 \frac{d}{dc^2} - (c - 1)^2 \frac{d}{dc} \cdot \frac{1}{4} \tag{263}$$

By the similar calculation, we get the following table and confirm (259).

**Table 7. The Differential Operator $L^\tau$**

| $z_0$       | $L^\tau$                  |
|-------------|---------------------------|
| id         | $c(1 - c) \frac{d^2}{dc^2} + (1 - 2c) \frac{d}{dc} - \frac{1}{4}$ |
| (0 1)      |                           |
| (1 $\infty$) | $-c(1 - c)^2 \frac{d^2}{dc^2} - (1 - c)^2 \frac{d}{dc} - \frac{1}{4}$ |
| (0 1 $\infty$) |                           |
| (0 $\infty$) | $-c^2(1 - c) \frac{d^2}{dc^2} + c^2 \frac{d}{dc} - \frac{1}{4}$ |
| (0 0 1)    |                           |

Then we get the transformation formulae for $\mathcal{D_i}$ and cocycles.

**Proposition 9.15.** For $\tilde{\rho} = (\rho, \zeta) \in \widetilde{G}$, we have the following relations in the ring of differential operators on $(T')^\text{an}$.

$$\mathcal{D}_1^\rho \cdot \tilde{x}(\tilde{\rho}) = \tilde{x}(\tilde{\rho}) \chi^{(1)}(\rho) \cdot \mathcal{D}_1$$

$$\mathcal{D}_2^\rho \cdot \tilde{x}(\tilde{\rho}) = \tilde{x}(\tilde{\rho}) \chi^{(2)}(\rho) \cdot \mathcal{D}_2 \tag{264}$$

where we regard $\tilde{x}(\tilde{\rho}), \chi^{(1)}(\rho), \chi^{(2)}(\rho) \in B' \subset \mathcal{O}(T')^\text{an}((T')^\text{an})$ as differential operators by multiplication.

**Proof.** By Definition 4.21 and Definition 4.23, we have

$$\tilde{x}(\tilde{\rho}) = \zeta \cdot pr_1^\rho(\phi(\rho^{(1)})) \cdot pr_2^\rho(\phi(\rho^{(2)})). \tag{265}$$

For any section $\varphi \in \mathcal{O}(T')^\text{an}$, $\frac{d}{d\rho}(pr_1^\rho(\varphi)) = 0$ by definition. Especially, we have the following relation in the ring of differential operators.

$$\mathcal{D}_1^\rho \cdot \zeta \cdot pr_1^\rho(\phi(\rho^{(1)})) = \zeta \cdot pr_2^\rho(\phi(\rho^{(2)})) \cdot \mathcal{D}_1^\rho \tag{266}$$

Furthermore, by Proposition 9.14, we have the following relation in the ring of differential operators.

$$\mathcal{D}_2^\rho \cdot pr_1^\rho(\phi(\rho^{(1)})) = pr_2^\rho(\phi(\rho^{(1)})) \cdot \mathcal{D}_1 \tag{267}$$

Using these relations, we have $\mathcal{D}_1^\rho \cdot \tilde{x}(\rho) = \tilde{x}(\rho) \chi^{(1)}(\rho)^2 \cdot \mathcal{D}_1$. Hence we have the $\mathcal{D}_1$ case. We can prove $\mathcal{D}_2$ case similarly. \qed
Finally, we can prove the equivariance of $\mathcal{D}$.

Proof. (Proposition 9.12) For $\tilde{\rho} = (\rho, \zeta) \in \tilde{G}$ and $i = 1, 2$, the following diagram commutes by Proposition 9.15 and (256) in Definition 9.13:

$$
\begin{array}{ccc}
\mathcal{O}_{(T')}^{an} & \xrightarrow{\mathcal{D}} & \mathcal{O}_{(T')}^{an} \\
\downarrow\tilde{\chi}(\rho) & & \downarrow\tilde{\chi}(\rho)^{i}(\zeta) \\
\mathcal{O}_{(T')}^{an} & \xrightarrow{\mathcal{D}_{T}} & \mathcal{O}_{(T')}^{an}
\end{array}
$$

(268)

By considering the definition of $\Upsilon_{\tilde{\rho}}$ in Definition 9.7 and that of $\Theta_{\tilde{\rho}}$ in Definition 9.11, we see that the upper face of the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{O}_{(T')}^{an} & \xrightarrow{\mathcal{D}} & \mathcal{O}_{(T')}^{an} \\
\downarrow\Upsilon_{\rho} & & \downarrow\Theta_{\rho} \\
(\rho^{-1})_{*}\mathcal{O}_{(T')}^{an} & \xrightarrow{\mathcal{D}_{T}} & (\rho^{-1})_{*}\mathcal{O}_{(T')}^{an}
\end{array}
$$

(269)

Since the front and back face commutes by Definition 9.10 the bottom diagram commutes since the vertical morphisms are epimorphisms. Hence we have the result.

9.4. Construction of the relative transcendental regulator map $R_\omega$.

Definition 9.16. For a subgroup $K \subset \text{CH}^2(X^o, 1)$, we say the relative transcendental regulator map exists for $K$ if and only if there exists a group homomorphism $R_\omega : K \longrightarrow \mathcal{Q}_\omega(T^o)$ such that for every $t \in T^o(\mathbb{C})$, the following diagram commutes.

$$
\begin{array}{ccc}
K & \xrightarrow{R_\omega} & \mathcal{Q}_\omega(T^o) \\
\downarrow i^*_t & & \downarrow m_t \\
\text{CH}^2(X_t, 1) & \xrightarrow{\text{ev}_t \circ r} & \mathbb{C}/\mathbb{P}_{\omega_t} \longrightarrow \mathbb{C}/\mathbb{Q}\mathbb{P}_{\omega_t}
\end{array}
$$

(271)

where $i^*_t$ is the pull-back map, $\text{ev}_t \circ r$ is the composition of transcendental regulator (Definition 8.5) and $\text{ev}_t$ in Definition 8.13, $m_t$ is the map defined in Definition 9.3 and $\mathbb{C}/\mathbb{P}_{\omega_t} \rightarrow \mathbb{C}/\mathbb{Q}\mathbb{P}_{\omega_t}$ is the natural projection. In this case, we say $R_\omega$ is the relative transcendental regulator map for $K$.

Proposition 9.17. The relative transcendental regulator map exists for the subgroup $\Xi \subset \text{CH}^2(X^o, 1)$. Furthermore, the image of $\xi_0 - \xi_1 \in \Xi$ under the relative transcendental regulator satisfies the following.

$$
R_\omega(\xi_0 - \xi_1) = 2[\mathcal{L}]
$$

(272)

Here $[\mathcal{L}] \in \mathcal{Q}_\omega(T^o)$ is the element satisfying $m_t([\mathcal{L}]) = [\mathcal{L}(t)]$ for $t \in T^o(\mathbb{C})$ where $\mathcal{L}(t) \in \mathbb{C}$ denotes the value of the local holomorphic function in Definition 8.13.
Proof. First we check the well-definedness of $[\mathcal{L}]$. Take an open cover $\{U_i\}_{i \in I}$ in the classical topology such that the branch of $\mathcal{L}$ is uniquely determined on each $U_i$. Let $\mathcal{L}_i \in \mathcal{O}(T_{T^\circ};U_i)$ denote the branch of $\mathcal{L}$ on $U_i$. It is enough to patch $[\mathcal{L}_i] \in \mathbb{Q}_\omega(U_i)$ (See Definition 8.11 for this notation). By Remark 8.17 for each $t \in U_i \cap U_j$, $\mathcal{L}_i(t) - \mathcal{L}_j(t) \in \mathcal{P}$, then we have $\mathcal{L}_i = \mathcal{L}_j \in \mathcal{P}_\omega(U_i \cap U_j)$ by Proposition 9.9. Hence we have

$$[\mathcal{L}_i]|_{U_i \cap U_j} - [\mathcal{L}_j]|_{U_i \cap U_j} = 0 \quad \text{in } \mathbb{Q}_\omega(U_i \cap U_j).$$

(273)

By gluing $\{[\mathcal{L}_i]\}_{i \in I}$, we get the element $[\mathcal{L}]$. Then the equality (272) follows from Theorem 8.16. Hence we see that the relative transcendental regulator exists for a subgroup of $\text{CH}^2(X^\circ_\xi, 1)$ generated by $\xi_0 - \xi_1$. Since every element $\xi \in \Xi$ is represented by the smooth curve family over $T^\circ$ (cf. Proposition 5.16), we can construct $C^\infty$-family $K_\xi$ of $2$-chain associated with $\xi_1$ as in Section 8. Then by the formula for transcendental regulator in Proposition 8.6, we see that $ev_{\xi}(r(\xi))$ is represented by the value of the local holomorphic function as in Theorem 8.16. Hence by the similar argument above, we can define $R_\omega(\xi) \in \mathbb{Q}_\omega(T^\circ)$. Especially, the relative transcendental regulator exists for $\Xi$. □

The following equivariance of $R_\omega$ is important for the rank estimate.

**Proposition 9.18.** For $\tilde{\rho} \in \tilde{G}$, the following diagram commutes.

$$
\begin{array}{ccc}
\Xi & \xrightarrow{R_\omega} & \mathbb{Q}_\omega(T^\circ) \\
\downarrow \rho & \quad & \downarrow \Upsilon_{\rho} \\
\Xi & \xrightarrow{R_\omega} & \mathbb{Q}_\omega(T^\circ)
\end{array}
$$

(274)

**Proof.** Consider the following diagram for $t \in T^\circ$.

$$
\begin{array}{ccc}
\Xi & \xrightarrow{R_\omega} & \mathbb{Q}_\omega(T^\circ) \\
\downarrow t & \quad & \downarrow \Upsilon_{t} \\
\Xi & \xrightarrow{R_\omega} & \mathbb{Q}_\omega(T^\circ)
\end{array}
$$

(275)

The left side face commutes by the associativity of pull-back map on higher Chow groups. The bottom face commutes by Proposition 9.9. and the right side face commutes by (277) in Definition 9.9. Hence the front face commutes for all $\tilde{\rho}(t) \in T^\circ(\mathbb{C})$. Then by Lemma 9.9, we see that the upper face commutes. □

As an application of $\tilde{G}$-equivariance, we have the following result.

**Corollary 9.19.** For $\tilde{\rho} \in \tilde{G}_{\text{fib}}$, we have $R_\omega(\tilde{\rho}_\ast \Xi_{\text{can}}) = \Xi_{\text{can}}$.

**Proof.** By $\tilde{G}$-equivariance of $R_\omega$, we have

$$R_\omega(\tilde{\rho}_\ast \Xi_{\text{can}}) = \Upsilon_{\tilde{\rho}}(R_\omega(\Xi_{\text{can}})).$$

(276)

By Definition 6.12 and definition of $\Upsilon_{\tilde{\rho}}$, $\Upsilon_{\tilde{\rho}}$ acts on $\mathbb{Q}_\omega(T^\circ)$ as $\pm 1$ for $\tilde{\rho} \in \tilde{G}_{\text{fib}}$. Hence we have the result. □

11Note that since $\tilde{\rho} \in \tilde{G}$ is an isomorphism, $\tilde{\rho}_\ast = (\tilde{\rho}^{-1})^\ast$.
9.5. The Proof of the Main Theorem. Finally, we prove the main theorem about the rank of the image of $\Xi$ by the transcendental regulator. First, we show the upper estimate.

**Proposition 9.20.** Let $R_\omega : \Xi \to \mathcal{Q}_\omega(T^\circ)$ be the relative transcendental regulator map defined in Definition 9.16. Then we have

$$\text{rank } R_\omega(\Xi^\lambda) \leq 3$$

(277)

where $\Xi^\lambda \subset \Xi^{\text{can}}$ is the subgroup defined in Definition 6.15. Especially,

$$\text{rank } R_\omega(\Xi) \leq 18.$$  

(278)

**Proof.** Since $|G_0/L_0| = 6$ by Definition 6.1 and $\Xi$ is sum of $\Xi^\lambda$ where $\lambda \in G_0/L_0$ by Definition 6.15, the latter statement follows from the former statement. Hence it is enough to show that rank $R_\omega(\Xi^\lambda) \leq 3$. Since $\Xi^\lambda$ is the sum of $\tilde{\rho}_\lambda, \Xi^{\text{can}}$ where $\tilde{\rho}_\lambda \in \lambda$ and rank $\Xi^{\text{can}} \leq 3$ by Definition 5.10, it is enough to show that if $\tilde{\rho}_\lambda, L_0 = \tilde{\rho}_3, L_0$, we have

$$R_\omega(\tilde{\rho}_\lambda, \Xi^{\text{can}}) = R_\omega(\tilde{\rho}_3, \Xi^{\text{can}}).$$

(279)

By Proposition 6.14, $\tilde{\rho}_\lambda, \tilde{\rho}_3$ are in the same left cosets by $\tilde{G}_{\text{fib}} \tilde{I}$. Hence there exist $\tilde{\rho}_F \in \tilde{G}_{\text{fib}}$ and $\tilde{\rho}_I \in \tilde{I}$ such that $\tilde{\rho}_3 = \tilde{\rho}_F \tilde{\rho}_I \tilde{\rho}_1$. By Theorem 6.10, we have $(\tilde{\rho}_I), \Xi^{\text{can}} = \Xi^{\text{can}}$. Furthermore, by Corollary 9.19, we have $R_\omega((\tilde{\rho}_F), \Xi^{\text{can}}) = R_\omega(\Xi^{\text{can}})$. Hence we have the result. \hfill \Box

To get lower estimate, we will consider the image under $\mathcal{D}$. First, we describe the image of $\mathcal{D} \circ R_\omega(\xi)$ for $\xi \in \Xi^{\text{can}}$.

**Proposition 9.21.** Let $R_\omega : \Xi^{\text{can}} \to \mathcal{Q}_\omega(T^\circ)$ be the relative transcendental regulator map in Definition 9.16. For $\xi_0, \xi_1, \xi_\infty \in \Xi^{\text{can}}$, we have

$$\mathcal{D} \circ R_\omega(\xi_0) = \frac{1}{b-a} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathcal{D} \circ R_\omega(\xi_1) = \frac{1}{b-a} \begin{pmatrix} \frac{\sqrt{1-a}}{\sqrt{1-a}} \\ -\frac{\sqrt{1-a}}{\sqrt{1-a}} \end{pmatrix}, \quad \mathcal{D} \circ R_\omega(\xi_\infty) = \frac{1}{b-a} \begin{pmatrix} \frac{\sqrt{a}}{\sqrt{\pi}} \\ -\frac{\sqrt{a}}{\sqrt{\pi}} \end{pmatrix}$$

(280)

Especially, we have rank $\mathcal{D} \circ R_\omega(\Xi^{\text{can}}) = 3$.

**Proof.** By Proposition 8.16, we have

$$\mathcal{D} \circ R_\omega(\xi_0 - \xi_1) = \mathcal{D}(2[\mathcal{L}]) = \frac{1}{a-b} \left( \frac{\sqrt{1-a}}{\sqrt{1-a}} - 1 \right)$$

(281)

where $[\mathcal{L}] \in \mathcal{Q}_\omega(T^\circ)$ is the element defined in Proposition 9.17. Let $\tilde{\rho}_a, \tilde{\rho}_b \in \tilde{I}$ be elements defined in Example 6.11. By Proposition 9.18, Proposition 9.12, and the computation of cocycles in Example 6.11, we have

$$\mathcal{D} \circ R_\omega(\xi_0 - \xi_\infty) = \mathcal{D} \circ R_\omega(\tilde{\rho}_a^\ast(\xi_0 - \xi_\infty)) = \Theta \Phi^a \left( \frac{1}{a-b} \left( \frac{\sqrt{1-a}}{\sqrt{1-a}} - 1 \right) \right) = \frac{1}{a-b} \left( \frac{1 - \sqrt{\frac{a}{\sqrt{\pi}}} \sqrt{\frac{a}{\sqrt{\pi}}}}{1 + \sqrt{\frac{a}{\sqrt{\pi}}} \sqrt{\frac{a}{\sqrt{\pi}}}} \right)$$

(282)

Here we use the equivariance in Proposition 9.18 and Proposition 9.12. Similarly, we have

$$\mathcal{D} \circ R_\omega(\xi_0 + \xi_\infty) = \mathcal{D} \circ R_\omega(\tilde{\rho}_b^\ast(\xi_0 - \xi_\infty)) = \Theta \Phi^b \left( \frac{1}{a-b} \left( \frac{\sqrt{1-a}}{\sqrt{1-a}} - 1 \right) \right) = \frac{1}{a-b} \left( \frac{1 - \sqrt{\frac{a}{\sqrt{\pi}}} \sqrt{\frac{a}{\sqrt{\pi}}}}{1 + \sqrt{\frac{a}{\sqrt{\pi}}} \sqrt{\frac{a}{\sqrt{\pi}}}} \right)$$

(283)

From (281), (282) and (283), we can deduce the result. \hfill \Box
Theorem 9.22. Let $\Xi \subset \text{CH}^2(X^\circ, 1)$ be the higher Chow subgroup defined in Definition 5.13 and $\Xi_t \subset \text{CH}^2(X_t, 1)$ be the restriction of $\Xi$ at the fiber over $t \in T^\circ(\mathbb{C})$.

(1) Let $R_\omega : \Xi \rightarrow Q_\omega(T^\circ)$ be the relative transcendental regulator defined in Definition 9.16. Then we have

$$\text{rank } R_\omega(\Xi) = 18. \quad (284)$$

(2) Let $r : \text{CH}^2(X_t, 1) \rightarrow H^{2,0}(X_t)^\vee/H_2(X_t, \mathbb{Z})$ be a transcendental regulator map. Then

$$\text{rank } r(\Xi_t) = 18 \quad (285)$$

for very general $t \in T^\circ(\mathbb{C})$. Especially, we have the following inequality for very general $t \in T^\circ(\mathbb{C})$.

$$\text{rank } \text{CH}^2(X_t, 1)^{\text{ind}} \geq 18 \quad (286)$$

Proof. (1) Since $\mathcal{D} : Q_\omega(T^\circ) \rightarrow \mathcal{O}_{(T^\circ)^{an}}(T^\circ)^{\otimes 2}$ is additive, if we can show that $\text{rank } \mathcal{D} \circ R_\omega(\Xi) \geq 18$, combining with Proposition 9.20, we can show the result. Since $\Xi$ is sum of $\tilde{\rho}_s \Xi^{\text{can}}$, by Proposition 9.18 and Proposition 9.12, $\mathcal{D} \circ R_\omega(\Xi)$ is generated by

$$\Theta(\mathcal{D} \circ R_\omega(\Xi^{\text{can}})) \quad (287)$$

By Proposition 9.20 and Proposition 9.12, we already know that $\Theta(\mathcal{D} \circ R_\omega(\Xi^{\text{can}})) = \Theta(\mathcal{D} \circ R_\omega(\Xi^{\text{can}}))$ if $\tilde{\rho} = \tilde{\rho}'$ is in the same left coset by $G_{\text{fib}} \tilde{I}$. Hence it is enough to calculate (287) for 6 representatives of $G/G_{\text{fib}} \tilde{I}$. By Proposition 6.11, if we take a lift of $(\text{id}, \text{id})$, $(\text{id}, (0, 1))$, $(\text{id}, (1, \infty))$, $(\text{id}, (0, 1))$, $(\text{id}, (0, \infty))$, $(\text{id}, (0, 1)) \in G_{\text{fib}}$ by $\tilde{G} \rightarrow G_0$, they become a complete system of representatives for $G/G_{\text{fib}} \tilde{I}$. Since $\Xi^{\text{can}}$ is generated by $\xi_0$, $\xi_1$ and $\xi_\infty$, we will calculate

$$\Theta(\mathcal{D} \circ R_\omega(\xi_0)), \quad \Theta(\mathcal{D} \circ R_\omega(\xi_1)), \quad \Theta(\mathcal{D} \circ R_\omega(\xi_\infty)) \quad (288)$$

for these representatives. Then we have the Table 8.

It is enough to show that the vectors in this table are linearly independent over $\mathbb{Q}$. To show this, it is enough to show that first component of these vectors are linearly independent over $\mathbb{C}$. Note that first component of these vectors are written in the form of

$$c \cdot F_1 \cdot F_2 \quad (289)$$

where $c \in \{ \pm 1, \pm \sqrt{-1} \}$ and $F_1$ is either

$$\frac{1}{a - b}, \frac{1}{a + b - 1}, \frac{1}{ab - a - b}, \frac{1}{ab - b + 1}, \frac{1}{ab - 1}, \frac{1}{a - ab - 1} \in \text{Frac}(B) \quad (290)$$

and $F_2$ is either

$$1, \frac{\sqrt{b}}{\sqrt{a}}, \frac{\sqrt{1 - b}}{\sqrt{1 - a}}, \frac{\sqrt{1 - b}}{\sqrt{1 - a}}, \frac{1}{\sqrt{a}}, \frac{1}{\sqrt{a}}, \frac{\sqrt{b}}{\sqrt{a}}, \frac{1}{\sqrt{a}}, \frac{\sqrt{b}}{\sqrt{a}} \in \text{Frac}(B') \quad (291)$$

Then it is enough to show that the elements in the form of $F_1 \cdot F_2$ where $F_1$ is in (290) and $F_2$ is in (291) are linearly independent over $\mathbb{C}$. Since elements in (290) are linearly independent over $\mathbb{C}$ and elements in (291) are linearly independent over $\text{Frac}(B)$, their products are linearly independent over $\mathbb{C}$. Hence we have the result.

(2) By Lemma 9.5, we have rank $m_t(R_\omega(\Xi)) = 18$ for very general $t \in T^\circ(\mathbb{C})$. By the definition of relative transcendental regulator map, we see that rank $ev_t \circ r(\Xi_t) = 18$. Since $ev_t$ is an isomorphism, we have rank $r(\Xi_t) = 18$. The statement about indecomposable part follows from Proposition 8.4. \qed

Remark 9.23. The simplified proof of the linearly independence of vectors in the Table 8 is by Tomohide Terasoma.
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Table 8. The basis of the image of $\Xi$ via $\mathcal{D} \circ R_\omega$

| $\lambda \in G_0/L_0$ | basis of $\mathcal{D} \circ R_\omega(\Xi^\lambda)$ |
|------------------------|----------------------------------|
| (id, id) $\cdot G_0/L_0$ | $\frac{1}{a - b} \left( -\frac{\sqrt{b}}{\sqrt{a}} \right) \cdot \frac{1}{a - b} \left( -\frac{\sqrt{1 - b}}{\sqrt{1 - a}} \right) \cdot \frac{1}{a - b} \left( -1 \right)$ |
| (id, (0 1)) $\cdot G_0/L_0$ | $\sqrt{\frac{1}{a + b - 1}} \left( -\frac{\sqrt{1 - b}}{\sqrt{a}} \right) \cdot \sqrt{\frac{1}{a + b - 1}} \left( -\frac{\sqrt{b}}{\sqrt{1 - a}} \right) \cdot \sqrt{\frac{1}{a + b - 1}} \left( -1 \right)$ |
| (id, (1 \(\infty\))) $\cdot G_0/L_0$ | $\sqrt{\frac{1}{a b - a - b}} \left( -\frac{\sqrt{b}}{\sqrt{a}} \right) \cdot \sqrt{\frac{1}{a b - a - b}} \left( -\frac{1}{\sqrt{a}} \right) \cdot \sqrt{\frac{1}{a b - a - b}} \left( 1 \right)$ |
| (id, (0 1 \(\infty\))) $\cdot G_0/L_0$ | $\sqrt{\frac{1}{a b - b + 1}} \left( -\frac{1}{\sqrt{a}} \right) \cdot \sqrt{\frac{1}{a b - b + 1}} \left( -\frac{1}{\sqrt{1 - a}} \right) \cdot \sqrt{\frac{1}{a b - b + 1}} \left( 1 \right)$ |
| (id, (0 \(\infty\))) $\cdot G_0/L_0$ | $\sqrt{\frac{1}{a b - 1}} \left( 1 \right) \cdot \sqrt{\frac{1}{a b - 1}} \left( 1 \right) \cdot \sqrt{\frac{1}{a b - 1}} \left( 1 \right)$ |
| (id, (0 \(\infty\))) $\cdot G_0/L_0$ | $\sqrt{\frac{1}{a b - 1}} \left( 1 \right) \cdot \sqrt{\frac{1}{a b - 1}} \left( 1 \right) \cdot \sqrt{\frac{1}{a b - 1}} \left( 1 \right)$ |

Appendix A. Decomposable cycles in higher Chow group

We assume that all scheme appearing in this section is a variety over a field $k$. We define a subgroup $\text{CH}^p(X, q)_{\text{dec}} \subset \text{CH}^p(X, q)$ called decomposable part.

Definition A.1. For $p, p', q, q' \geq 0$, there exists a bilinear map

$$\text{CH}^p(X, q) \times \text{CH}^{p'}(X, q') \rightarrow \text{CH}^{p+p'}(X \times_k Y, q + q')$$

(292)
called external product. If $X$ is smooth, the diagonal map $i : X \rightarrow X \times_k X$ is regular embedding. Then we can define a bilinear map

$$\text{CH}^p(X, q) \times \text{CH}^{p'}(X, q') \rightarrow \text{CH}^{p+p'}(X \times_k X, q + q') \xrightarrow{i^*} \text{CH}^{p+p'}(X, q + q')$$

(293)
called intersection product.

**Definition A.2.** For $p, q \geq 0$, decomposable part $\text{CH}^p(X, q)_{\text{dec}}$ of $\text{CH}^p(X, q)$ is the subgroup defined by

$$
\sum_{s, t} \text{Im} \left( \text{CH}^s(X, t) \otimes_{\mathbb{Z}} \text{CH}^{p-s}(X, q-t) \rightarrow \text{CH}^p(X, q) \right)
$$

(294)

where $(s, t)$ runs over $0 \leq s \leq p$, $0 \leq t \leq q$ and $(s, t) \neq (0, 0), (p, q)$ and the map $\text{CH}^s(X, t) \otimes_{\mathbb{Z}} \text{CH}^{p-s}(X, q-t) \rightarrow \text{CH}^p(X, q)$ is the intersection product. Elements in $\text{CH}^p(X, q)_{\text{dec}}$ are called decomposable cycles. We define

$$
\text{CH}^p(X, q)_{\text{ind}} = \frac{\text{CH}^p(X, q)}{\text{CH}^p(X, q)_{\text{dec}}}
$$

(295)

We describe decomposable part of $\text{CH}^2(X, 1)$. Recall that the elements of $\text{CH}^2(X, 1)$ is represented by the elements in $\text{Ker} \left( \bigoplus_{Z \in \text{X}^{(1)}} \text{R}(Z)^{\times} \rightarrow \bigoplus_{p \in \text{X}^{(2)}} \mathbb{Z} \cdot p \right)$.

**Proposition A.3.** The element of $\text{CH}^2(X, 1)_{\text{dec}}$ can be represented by $\sum_{\lambda} (Y_{\lambda}, c_{\lambda}) \in \bigoplus_{Z \in \text{X}^{(1)}} \text{R}(Z)^{\times}$ such that $c_{\lambda} \in \Gamma(X, \mathcal{O}_X^{\times})$.

*Proof.* Since $\text{CH}^0(X, 1) = 0, \text{CH}^2(X, 1)_{\text{dec}}$ is the image of the map

$$
\text{CH}^1(X, 1) \otimes_{\mathbb{Z}} \text{CH}^1(X, 0) \rightarrow \text{CH}^2(X, 1)
$$

(296)

By [GL01] Section 8, the external product $\text{CH}^1(X, 1) \times \text{CH}^1(Y) \rightarrow \text{CH}^2(X \times_k Y, 1)$ is induced by the following map. Here $Z^1(-, 1)$ denotes the subgroup of cycles on $- \times_k \Delta^1$ ($\Delta^1 = \text{Spec} \{ T_0, T_1 \}/(T_0 + T_1 - 1)$) which intersects properly with $- \times_k \{ T_0 = 0 \}$ and $- \times_k \{ T_1 = 0 \}$.

$$
Z^1(X, 1) \times Z^1(Y) \rightarrow Z^1(X \times_k Y, 1)
$$

(297)

\[ ([V], [W]) \longmapsto [V \times_k W] \]

where $V, W$ are integral subschemes. Recall that we regard elements in $\Gamma(X, \mathcal{O}_X^{\times})$ as cycles in $Z^1(X, 1)$ by considering its graph. Hence we see that the intersection product of graph of $c \in \Gamma(X, \mathcal{O}_X^{\times})$ and integral codimension 1-cycle $Y$ corresponds to the graph of $c$ on $Y$. Hence we have the result. \(\square\)

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