Towards a Theory of Molecular Forces between Deformed Media

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(Dated: March 22, 2022)

A macroscopic theory for the molecular or Casimir interaction of dielectric materials with arbitrarily shaped surfaces is developed. The interaction is generated by the quantum and thermal fluctuations of the electromagnetic field which depend on the dielectric function of the materials. Using a path integral approach for the electromagnetic gauge field, we derive an effective Gaussian action which can be used to compute the force between the objects. No assumptions about the independence of the shape and material dependent contributions to the interaction are made. In the limiting case of flat surfaces our approach yields a simple and compact derivation of the Lifshitz theory for molecular forces \[ F \]. For ideal metals with arbitrarily deformed surfaces the effective action can be calculated explicitly. For the general case of deformed dielectric materials the applicability of perturbation theory and numerical techniques to the evaluation of the force from the effective action is discussed.

PACS numbers: 12.20.-m, 33.90.+h, 42.50.Ct

I. INTRODUCTION

The last years have witnessed a resurgence of theoretical and experimental research on Casimir interactions between macroscopic and mesoscopic objects \[ 2, 3, 4, 5, 6, 7 \]. The most mentionable recent achievements include on the experimental side the high precision measurements of the attractive Casimir force between metallic surfaces \[ 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 \] and the simultaneous study of both the critical Casimir force due to (thermal) order parameter fluctuations and the electrodynamic Casimir force in superfluid Helium films near the critical point \[ 18, 19, 20 \]. On the theoretical side there is an ongoing attempt to describe the Casimir interaction between non-ideal metals with finite conductivity, including simultaneously the effect of finite temperature, see \[ 21 \] for a recent summary of the used and partly controversial approaches. Another direction of great interest is the study of the strong geometry dependence of the Casimir force which is inevitably linked to the non-additivity of fluctuation induced forces \[ 22, 23, 24, 25 \]. Even a potential change from attractive to repulsive forces due to either the material properties \[ 26, 27, 28 \] or the geometry of the objects \[ 29 \] has been discussed. As to the comparison between experiment and theory, Casimir’s prediction \[ 30 \]

\[
\frac{F_0}{A} = -\frac{\pi^2}{240} \frac{\hbar c}{H^4}
\]

for the force \( F_0 \) between ideal metallic and parallel flat surfaces of area \( A \) and distance \( H \) at zero temperature has been confirmed by the recent experiments within a few percent of accuracy. However, there is broad agreement that the experiments have shown that the inclusion of the material properties of the surfaces, of the roughness and geometry of the surfaces, and of finite temperature effects is indispensable. A solid theoretical account of these effects is necessary in view of the importance of the experimental results to the test of unified gauge theories of fundamental interactions \[ 31 \] and the design of nanotechnological devices \[ 32, 33 \].

Each of the modifications to the ideal prediction of Eq. \( 1 \) introduces at least one corresponding length scale in the interaction energy. If the interacting materials are not ideal metals, the electromagnetic fluctuations are not reflected perfectly at all wavelengths by the surfaces. Then one expects Eq. \( 1 \) to hold only for distances \( H \) much larger than the plasma wavelength \( \lambda_p \) of the metal. In the recent experiments \( \lambda_p \) is of the order of 0.1μm. At smaller distances the force will be reduced to \( F = \eta_m F_0 \) by a material dependent factor \( \eta_m < 1 \) compared to the ideal force. Since the experiments are usually performed at room temperature, thermal fluctuations of the electromagnetic field tend to increase the force for separations \( H \) which are larger than the de Broglie wavelength of photons, \( \lambda_T = \hbar c/(k_B T) \) (≈ 7μm at 300°K), leading to \( F = \eta_T F_0 \) with the temperature dependent factor \( \eta_T > 1 \). Finally the presumably most difficult to calculate change of the force \( F_0 \) comes from the surface geometry. The geometry modifications can be divided into two different types. The first is a general unwanted stochastic surface roughness, the second is an intentionally designed surface structure like a corrugation. The latter has been studied in a recent experiment to study the geometry dependence of the normal and lateral Casimir force \[ 34 \]. Both types can be characterized by the deformation amplitude \( a \) and the roughness correlation or corrugation length \( \lambda \). Recent theoretical work for ideal metals at zero temperature has shown that the force is generally increased by deviations from the flat surface geometry, at least for uni-axial structures \[ 22, 23, 24 \]. Therefore we can write \( F = \eta_g F_0 \) with a factor \( \eta_g \) accounting for geometry dependent changes of the force.
So far, we have discussed the different modifications of the ideal force \( F_0 \) independently. One can expect that this is indeed justified if the characteristic length scales of the modifications are widely different from each other. However, this is by no means always the case in the recent experiments, especially at the point of closest approach of the interacting surfaces, and in nanotechnological devices. In view of this, it is of importance to consider correlations between the force modifications. That means that one has to assume a more general form of the force

\[
F = \eta_m \eta_T \eta_g (1 + \Delta_{\text{corr}}) F_0
\]

with a new term \( \Delta_{\text{corr}} \) accounting for correlations between different modifications. The present state of the techniques available in literature, however, does not allow to determine \( \Delta_{\text{corr}} \) in general. A commonly used but uncontrolled approximation is to set \( \Delta_{\text{corr}} = 0 \). Most of the recent theoretical studies of correlation effects have been devoted to the simultaneous effect of finite conductivity of metallic surfaces and finite temperature, see [21] for a recent overview and [34] for a proposed experiment to measure these effects. Common to almost all of the existing theoretical work on the correlations between conductivity and temperature corrections is that it starts from the so-called Lifshitz theory for molecular interactions between macroscopic objects [1]. The Lifshitz theory provides a formula for the force in the rather general case of dielectric bodies at arbitrary temperature but with flat and parallel surfaces. The key difference in the approaches to correlation effects is the treatment of the zero Matsubara frequency term of the Lifshitz formula when it is applied to different models (Drude, ideal metal or free plasma model) for the dielectric function of the metals. In addition, there are a few approaches which are not based on the Lifshitz theory but employ a surface impedance boundary condition in order to account for the coupling between electromagnetic fluctuations and a metallic surface. To date, however, it appears that there is no broad agreement on a correct prescription for the evaluation of the Lifshitz formula in the case of non-ideal metallic surfaces at finite temperatures.

However, this is by no means always the case in the recent experiments, especially at the point of closest approach of the interacting surfaces, and in nanotechnological devices. In view of this, it is of importance to consider correlations between the force modifications. That means that one has to assume a more general form of the force modifications. The present state of the techniques available in literature, however, does not allow to determine \( \Delta_{\text{corr}} \) in general. A commonly used but uncontrolled approximation is to set \( \Delta_{\text{corr}} = 0 \). Most of the recent theoretical studies of correlation effects have been devoted to the simultaneous effect of finite conductivity of metallic surfaces and finite temperature, see [21] for a recent overview and [34] for a proposed experiment to measure these effects. Common to almost all of the existing theoretical work on the correlations between conductivity and temperature corrections is that it starts from the so-called Lifshitz theory for molecular interactions between macroscopic objects [1]. The Lifshitz theory provides a formula for the force in the rather general case of dielectric bodies at arbitrary temperature but with flat and parallel surfaces. The key difference in the approaches to correlation effects is the treatment of the zero Matsubara frequency term of the Lifshitz formula when it is applied to different models (Drude, ideal metal or free plasma model) for the dielectric function of the metals. In addition, there are a few approaches which are not based on the Lifshitz theory but employ a surface impedance boundary condition in order to account for the coupling between electromagnetic fluctuations and a metallic surface. To date, however, it appears that there is no broad agreement on a correct prescription for the evaluation of the Lifshitz formula in the case of non-ideal metallic surfaces at finite temperatures.

The situation becomes even worse when one considers deviations from the parallel flat plate geometry and correlations between geometry, conductivity and temperature modifications of the Casimir force. To our knowledge there is no complete theory for correlations between geometry induced modifications of the force and the above discussed corrections available in literature. In fact, only recently an exact description for the sole geometry dependence of the force, making no additivity assumptions of two-body forces, has been given [22, 23, 24]. So far, surface roughness in combination with finite conductivity has been studied only by neglecting correlations between both effects [35]. However, the characteristic length scales of surface roughness or designed surface corrugations in nanotechnological devices can be close to the relevant length scales of the material as, e.g., the plasma wavelength. Therefore it would be very useful to have an analog of the Lifshitz theory for more general geometries. Such a general theory should yield the Casimir interaction depending on the dielectric function of the material, temperature and a height profile describing the surface geometry as input parameters. The development of such a theory is the purpose of the present work.

In this paper we will introduce a novel macroscopic approach to molecular forces between dielectric media. It will be based on a path integral technique for fluctuation induced forces which was previously developed for ideal metals at zero temperature [7, 22, 36]. Non-local boundary conditions for the electromagnetic gauge field are employed to treat the interaction between electromagnetic fluctuations and matter. The boundary conditions can be viewed as a reformulation of the so-called extinction theorem of classical electrodynamics [37, 38, 39]. The important new property of our approach is that surface deformations can be included without any assumption about the correlations between contributions to the force from geometrical and material properties. We derive an effective Gaussian action which is a functional of the frequency dependent dielectric function of the material and the height profile of the surfaces. The effective action is a possible starting point for future detailed analyses of the effect of correlation on the force as described by the \( \Delta_{\text{corr}} \) in Eq. (4). We demonstrate the efficiency of our approach by looking at two particular limits of interest. First, we consider flat surfaces of dielectric media. In this case we obtain, as a byproduct of our theory, a compact and concise derivation of the Lifshitz formula for molecular forces in the language of quantum statistical mechanics. An even simpler derivation of this formula is found by a scalar field approach, cf. the Appendix, which should be compared to other derivations [10, 11] of the original result of Lifshitz. As second limiting case, we consider ideal metals with arbitrary deformations. Then the effective action assumes a simple form which can be determined explicitly. For general deformed dielectric media, the effective action can be used as a basis for perturbative [22, 24] or numerical [21] computations of the correlation term \( \Delta_{\text{corr}} \).

The outline of the rest of the paper is as follows. In the following section we develop the path integral approach for deformed surfaces of materials which are characterized by a general frequency dependent dielectric function. We derive the non-local boundary conditions which describe the reflection and refraction properties of the interacting bodies. By integrating out the electromagnetic gauge field, an effective action for the interaction between the bodies is obtained. In Sec. [11] we apply our theory to calculate the force between two flat surfaces of dielectric media. In this case, the known Lifshitz formula for molecular forces is found. The effective action for deformed surfaces of ideal metals is computed explicitly in section [16]. Section [17] provides a discussion of the relevance of our results to perturbative and numerical analyses of the Casimir or molecular interaction between macroscopic objects. A rather
short and concise derivation of the standard Lifshitz theory in terms of a scalar field is left to the Appendix.

II. PATH-INTEGRAL FORMULATION OF MOLECULAR FORCES

We will develop a macroscopic theory which allows to calculate the interaction between materials of rather general shape. Instead of considering directly the field emitted by the fluctuating dipoles in the material we view the interaction as occurring through the modifications of the quantum (and thermal) fluctuations of the electromagnetic field between the materials. No direct reference is made to the electromagnetic field fluctuations in the interior of the materials. The effect of the dipoles induced by the external fluctuating field will be described by material dependent boundary conditions which are defined on the surface of the material. Our method is based on a path integral quantization of the electromagnetic gauge field. This approach has full generality in the sense that it can be applied to any body, characterized by its dielectric function, with any surface profile, described by a height field, at any temperature.

The common approaches for computing the force between materials is to first determine the solution of Maxwell’s equations both inside and outside the materials, and than to evaluate the force either from the stress tensor or from the zero-point energy of the modes using the so-called argument theorem, see, e.g., Ref. [3]. The problem with these approaches is that they are not suited to treat non-flat surfaces since deformations in general lead to a complicated modification of the mode structure and make the solution to Maxwell’s equations a hard task. In the following, we will formulate the interaction between deformed materials within the language of quantum statistical mechanics. Since this formulation makes no explicit use of the individual eigenfrequencies of the modes it is better targeted for the treatment of deformations.

We consider the two interacting media as filling half spaces which are bounded by deformed surfaces \( S_\alpha, \alpha = 1, 2 \). The deformations from a plan-parallel geometry of mean surface distance \( H \) are described by the height functions \( h_\alpha(x_\parallel) \) with \( x_\parallel \) the lateral surface coordinates, see Fig. 1. The media are characterized by their complex dielectric functions \( \epsilon_\alpha(\omega) \), respectively. The gap between the media is assumed to be vacuum, \( \epsilon(\omega) = 1 \). The free energy \( F \) of the photon gas in the gap between the two surfaces can be calculated from the imaginary time path integral for the electromagnetic gauge field \( A \). In the absence of media, the vacuum partition function \( Z_0 \) is given by

\[
Z_0^2 = \int \mathcal{D}(A^* A) \, e^{-S_E(A^*, A)},
\]

where we have introduced a complex valued gauge field which leads to a double counting of each degree of freedom. The reason for this will become clear below when we discuss the boundary conditions at the surfaces. The Euclidean action \( S_E(A^*, A) \) is the imaginary time version of the action \( S(A^*, A) \) of the electromagnetic field in the Minkowski

\[\text{FIG. 1: Two deformed surfaces } S_1 \text{ and } S_2 \text{ of dielectric media with dielectric functions } \epsilon_1(\omega) \text{ and } \epsilon_2(\omega), \text{ respectively, separated by a gap of mean size } H \text{ along the } x_3 \text{-direction. The meaning of the auxiliary surfaces } R_1 \text{ and } R_2 \text{ is explained in the text.}\]
space time with coordinates \( X = (t, \mathbf{x}) = (t, x_x, x_3) \),
\[
S(A^*, A) = -\frac{1}{2} \int_X \left( F_{\mu
u}^* F^{\mu
u} \right)(X) - \frac{1}{\xi} \int_X \left( \partial_\mu A^\mu \right) \left( \partial_\nu A^\nu \right)(X)
\]
where the first term comes from the Lagrangian of the electromagnetic field \( F_{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu \) and the second term results from the Fadeev-Popov gauge fixing procedure which assures that each physical field configuration is counted only once in the path integral over the gauge field. The parameter \( \xi \) allows to switch between different gauges: all gauge invariant quantities calculated from this action like, e.g., the Casimir force, do not depend on \( \xi \). In the following we will use the Feynman gauge corresponding to \( \xi = 1 \). The coefficients in the action of Eq. (4) differ by a factor of 1/2 from the conventional definition of the action for a real valued gauge field in order to obtain the correct photon propagator which in Feynman gauge reads \( G_{\mu\nu} = g_{\mu\nu}/K^2 \) with momentum \( K = (\omega, \mathbf{k}) \), \( K^2 = K^\mu K^\mu = \omega^2 - k^2 \) and Minkowski metric tensor \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \). The Euclidean action \( S_E(A^*, A) \) is obtained from Eq. (4) by applying a Wick rotation to imaginary time which amounts to the transformations \( t \to -i\tau \), \( \omega \to i\zeta \) and \( A^0 \to iA^0 \), \( A^{*0} \to iA^{*0} \), while the remaining components \( A^\mu \) remain unchanged [12, 13]. Since this transformation corresponds to the change \( g_{\mu\nu} \to -\delta_{\mu\nu} \) for the metric tensor, the Euclidean action in momentum space becomes
\[
S_E(A^*, A) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int_k A^{*\mu}(\zeta_n, k) G_{\mu\nu}^{-1}(\zeta_n, k) A^{\nu}(\zeta_n, k)
\]
where we allowed for a finite temperature \( T \) by introducing bosonic Matsubara frequencies \( \zeta_n = 2\pi n/\beta \) with \( \beta = 1/T \). The Euclidean Green function is given by \( G_{\mu\nu}(\zeta_n, k) = \delta_{\mu\nu} \bar{G}(\zeta_n, k) \) with \( \bar{G}(\zeta_n, k) = (\zeta^2 + k^2)^{-1} \).

In the presence of the two media the free energy of the photon gas in the gap between the media is obtained from a restricted partition function. The restrictions are due to the boundary conditions for the gauge field which are imposed by the dielectric properties of the media. It will be shown below that there are three boundary conditions on each surface \( S_\alpha \) which we number by \( \alpha = 1, 2, 3 \). Each of these conditions implies the vanishing of a non-local linear combination of derivatives of the components of the gauge field. Thus the mean distance \( H \)-dependent restricted partition function \( Z(H) \) can be written as
\[
2^2(H) = \frac{1}{2\theta} \int \mathcal{D}(A^* A) \prod_{\alpha,j} \prod_{\zeta_n} \prod_{x \in R_\alpha} \delta \left[ \int_{x^\prime \in S_\alpha} \mathcal{L}_{j\mu}^\alpha(\zeta_n; x, x') A^{\mu}(\zeta_n, x') \right] e^{-S_E(A^*, A)}
\]
where we enforced the boundary conditions by inserting delta-functions for all positions \( x \) on (flat) auxiliary surfaces \( R_\alpha \) which are placed at \( x_3 = \pm L \) with sufficiently large \( L \) so that the surfaces \( S_\alpha \) are located between them, see Fig. 1. The final result for the force between the media will be of course independent of \( L \). The differential operators \( \mathcal{L}_{j\mu}^\alpha \) depend via both the dielectric function \( \epsilon_\alpha \) and the normal vector \( \mathbf{n}_\alpha \) on the surface index \( \alpha \). Their actual form will be computed below. The interaction (Casimir) free energy per unit area of the two surfaces \( S_\alpha \) is then given by
\[
F(H) = -\frac{1}{A\beta} \ln \left[ Z(H) Z_{\infty}^{-1}(H) \right],
\]
where \( A \) is the surface area and \( \beta = 1/T \) the inverse temperature. \( Z_{\infty} \) is the asymptotic limit of \( Z \) for large \( H \) so that \( F \) is measured relative to two surfaces which are infinitely apart from each other. The force per unit area between the surfaces is then given by \( F = -\partial F/\partial H \).

### A. Boundary conditions

In this section we will derive the boundary conditions at the surfaces of the dielectric media. The boundary conditions are based on the optical extinction theorem of Ewald [37] and Oseen [38], see also [39]. This theorem states that part of the electromagnetic field produced by the molecular dipoles inside a medium exactly cancels the incident field, while the remainder propagates according to Maxwell’s equations in continuous media. Ewald and Oseen proved the theorem for crystalline media and amorphous, isotropic dielectrics, respectively, from a point of view of classical molecular optics. Later, Born and Wolf extended the theorem to more general classes of materials [38]. A relationship between the extinction theorem and the Lifshitz theory of dispersion forces for continuous media has been pointed out by Milonni and Lerner [14]. They use the fact that the extinction theorem permits a reduction of the multiple-scattering problem for the molecular dipoles to the solution of the wave equation for the gauge field \( A \) with appropriate boundary conditions. From this they conclude that the extinction theorem shows that the macroscopic
Lifshitz theory for continuous media correctly accounts for all multiple-scattering nonadditive contributions to the force between flat surfaces. We will demonstrate that these concepts are useful to describe the interaction of even deformed surfaces.

In the following we use an (equivalent) reformulation of the extinction theorem as a non-local boundary condition which enforces the laws of reflection and refraction at the surfaces of the interacting media. Our derivation of the boundary conditions follows closely the approach outlined in [12]. Let us start with the Helmholtz wave equation for the magnetic field $\mathbf{B}$ inside a dielectric medium which occupies the volume $V$ with surface $S = \partial V$,

$$\nabla^2 + \epsilon(\omega)\omega^2 \mathbf{B}(\omega, \mathbf{x}) = 0.$$  

The propagation of the field inside the medium is described by the material Green function satisfying

$$\left[\nabla'^2 + \epsilon(\omega)\omega^2\right] G'(\omega; \mathbf{x}, \mathbf{x}') = \delta^{(3)}(\mathbf{x} - \mathbf{x'}),$$  

where we allow for a general frequency dependent complex dielectric function $\epsilon(\omega)$. Applying Green’s theorem to the components of $\mathbf{B}$ and to $G'$, one easily obtains, using (8) and (9),

$$\int_{x' \in S} \left[ G'(\omega; \mathbf{x}, \mathbf{x}')(\mathbf{n}' \cdot \nabla')(\mathbf{B}(\omega, \mathbf{x}') - \mathbf{B}(\omega, \mathbf{x}')) + (\mathbf{n}' \cdot \nabla(\omega; \mathbf{x}')) \nabla' + (\mathbf{n}' \times \mathbf{B}(\omega, \mathbf{x}')) \times \nabla' \right] G'(\omega; \mathbf{x}, \mathbf{x}') = 0.$$  

The usual continuity conditions for the electric and magnetic field at dielectric surfaces [16] without external surface charges and currents show that the three terms in Eq. (11) have to be continuous across the surface. Therefore we can use the vanishing of the integral as a boundary condition for the electromagnetic field on the vacuum side of the surface. As a side remark we note that if we had started with the wave equation for the electric field instead of the magnetic field we had obtained a similar expression as that in Eq. (11). However, due to the discontinuity of the dielectric function across the surface the condition that the integral vanishes had not translated to the field on the vacuum side. Finally, in the case of an ideal metal, $\epsilon(\omega) \rightarrow \infty$, and the integral in Eq. (11) is dominated by the first term. In this limit the integration can be carried out, leading to the well-known condition $\mathbf{n} \times \mathbf{E} = 0$ for ideal metals.

The condition of Eq. (11) can now be used to determine the differential operators $\mathcal{L}_{\mu\nu}^E(\zeta; \mathbf{x}, \mathbf{x}')$ appearing in Eq. (6). We express the electric and magnetic field in terms of the gauge field. After a Wick rotation to imaginary time, the corresponding relations read $E_j = -\zeta A^j - i\partial_\nu A^\nu$ and $B_j = \varepsilon_{j\mu\nu} \partial_\nu A^\mu$ in Euclidean space. Multiplying Eq. (11) with $(\zeta\epsilon_\alpha)^{-1}$ and decomposing $\mathcal{L}_{\mu\nu}^E(\zeta; \mathbf{x}, \mathbf{x}') = \hat{n}_\mu^\alpha(\mathbf{x}') \mathcal{L}_{\mu\nu}^E(\zeta; \mathbf{x}, \mathbf{x}')$ with respect to the components $\hat{n}_\mu^\alpha$ of the normal vector by using the standard summation convention for $\hat{k}$, one gets after some algebra the explicit result

$$\mathcal{L}_1^\alpha(\zeta; \mathbf{x}, \mathbf{x}') = \begin{pmatrix} 0 & -\frac{1}{\zeta\epsilon_\alpha} [\partial_3 \partial_3 - \partial_2 \partial_1] & \frac{1}{\zeta\epsilon_\alpha} [\partial_1 \partial_3 + \partial_3 \partial_1] & -\frac{1}{\zeta\epsilon_\alpha} [\partial_1 \partial_2 + \partial_2 \partial_1] \\ i\partial_3 & -\frac{1}{\zeta\epsilon_\alpha} \partial_1 \partial_3 & -\frac{1}{\zeta\epsilon_\alpha} \partial_2 \partial_3 & \zeta - \frac{1}{\zeta\epsilon_\alpha} [\partial_2 \partial_3 - \partial_3 \partial_2] \\ -i\partial_2 & \frac{1}{\zeta\epsilon_\alpha} \partial_1 \partial_2 & -\zeta + \frac{1}{\zeta\epsilon_\alpha} [\partial_3 \partial_3 - \partial_1 \partial_1] & -\frac{1}{\zeta\epsilon_\alpha} \partial_2 \partial_1 \\ -i\partial_1 & \frac{1}{\zeta\epsilon_\alpha} \partial_2 \partial_1 & \zeta - \frac{1}{\zeta\epsilon_\alpha} [\partial_3 \partial_3 - \partial_1 \partial_1] & 0 \end{pmatrix} G_F^E(\zeta; \mathbf{x} - \mathbf{x'})$$  

$$\mathcal{L}_2^\alpha(\zeta; \mathbf{x}, \mathbf{x}') = \begin{pmatrix} -i\partial_3 & -\frac{1}{\zeta\epsilon_\alpha} \partial_1 \partial_3 & \frac{1}{\zeta\epsilon_\alpha} [\partial_2 \partial_3 - \partial_3 \partial_2] & -\zeta + \frac{1}{\zeta\epsilon_\alpha} [\partial_3 \partial_3 - \partial_1 \partial_1] \\ 0 & -\frac{1}{\zeta\epsilon_\alpha} [\partial_3 \partial_3 - \partial_2 \partial_2] & -\frac{1}{\zeta\epsilon_\alpha} [\partial_2 \partial_3 - \partial_3 \partial_2] & \frac{1}{\zeta\epsilon_\alpha} [\partial_2 \partial_3 + \partial_3 \partial_2] \\ i\partial_1 & \zeta - \frac{1}{\zeta\epsilon_\alpha} [\partial_3 \partial_3 - \partial_1 \partial_1] & 0 & 1 \zeta \frac{1}{\zeta\epsilon_\alpha} \partial_3 \partial_1 \\ i\partial_2 & \frac{1}{\zeta\epsilon_\alpha} \partial_1 \partial_2 & -\frac{1}{\zeta\epsilon_\alpha} [\partial_2 \partial_3 - \partial_3 \partial_2] & 0 \end{pmatrix} G_F^E(\zeta; \mathbf{x} - \mathbf{x'})$$  

$$\mathcal{L}_3^\alpha(\zeta; \mathbf{x}, \mathbf{x}') = \begin{pmatrix} i\partial_2 & \frac{1}{\zeta\epsilon_\alpha} \partial_1 \partial_2 & \zeta - \frac{1}{\zeta\epsilon_\alpha} [\partial_2 \partial_3 - \partial_3 \partial_2] & -\frac{1}{\zeta\epsilon_\alpha} \partial_3 \partial_2 \\ -i\partial_1 & -\zeta + \frac{1}{\zeta\epsilon_\alpha} [\partial_2 \partial_3 - \partial_3 \partial_2] & 0 & \frac{1}{\zeta\epsilon_\alpha} \partial_2 \partial_1 \\ 0 & -\frac{1}{\zeta\epsilon_\alpha} [\partial_3 \partial_3 - \partial_2 \partial_2] & 1 \zeta \frac{1}{\zeta\epsilon_\alpha} \partial_3 \partial_1 & \frac{1}{\zeta\epsilon_\alpha} [\partial_3 \partial_3 - \partial_1 \partial_1] \\ -i\partial_3 & \frac{1}{\zeta\epsilon_\alpha} \partial_1 \partial_3 & -\frac{1}{\zeta\epsilon_\alpha} \partial_2 \partial_3 & 0 \end{pmatrix} G_F^E(\zeta; \mathbf{x} - \mathbf{x'}).$$  

where $G_F^E$ is the Euclidean version of the Green function inside the medium which is defined by Eq. (9). The partial differential operators $\partial_j$ are acting on the spatial argument of $G_F^E$, whereas the “free” operators $\partial_j$ are acting on the gauge field to which $\mathcal{L}_{\mu\nu}$ is applied. For non-deformed surfaces as described by the conventional Lifshitz theory, i.e., $\mathbf{n} = (0, 0, \pm 1)$, only the last matrix is relevant.
Now we are in the position to calculate the partition function defined by Eq. (6) and by the operators in Eqs. (12-14). Similar to the approach of Refs. [36, 47] we introduce auxiliary fields in order to treat the delta-function constraints. However, here we will use complex valued auxiliary fields since the arguments of the delta-functions are complex in our problem. Moreover, the fields will be not defined on the original surfaces \( S_\alpha \) itself but on the flat auxiliary surfaces \( R_\alpha \) since these are the regions on which the “external” positions \( x \) of the boundary conditions are located, cf. Eq. (6). Introducing on each of the two surfaces \( R_\alpha \) at \( x_3 = L_\alpha = (-1)^{\alpha-1}L \) with lateral coordinates \( x_l \) the three fields \( \psi_{\alpha j}(\zeta, x_l) \), \( j = 1, 2, 3 \), the delta-functions for fixed \( \alpha \) and \( j \) can be written as

\[
\prod_\zeta \prod_{x \in R_\alpha} \delta \left[ \int_{x' \in S_\alpha} \mathcal{L}^\alpha_{j\mu}(\zeta; x, x') A^\mu(\zeta, x') \right] = \int \mathcal{D} [\psi_{\alpha j}^* \psi_{\alpha j}] \exp \left[ i \sum_n \int_{x' \in S_\alpha} \left\{ \psi_{\alpha j}^*(\zeta, x_l) \mathcal{L}^\alpha_{j\mu}(\zeta; x_l, L_\alpha), x') A^\mu(\zeta, x') + \text{c.c.} \right\} \right].
\]

Inserting this representation in the partition function of Eq. (6), the complex gauge field \( A^\mu \) can be integrated out, using the free action \( S_E(A^+, A) \) of Eq. (5). The partition function can then be expressed in terms of an effective quadratic action for the auxiliary fields,

\[
\mathcal{Z}^2(H) = \int \prod_\alpha \mathcal{D} [\psi_{\alpha j}^* \psi_{\alpha j}] e^{-S_{\text{eff}}[\psi_{\alpha j}^*, \psi_{\alpha j}]} \quad (16)
\]

with

\[
S_{\text{eff}}[\psi_{\alpha j}^*, \psi_{\alpha j}] = \sum_{n, n'} \int_x \int_{x'} \psi_{\alpha j}^*(\zeta, x_l) \mathcal{M}^{\alpha\beta, jl}(\zeta, x_l; \zeta', x'_l) \psi_{\beta j}(\zeta', x'_l),
\]

where the usual summation convention applies to all indices. Since in Feynman gauge the propagator of \( A^\mu \) is diagonal in \( \mu \), the resulting matrix kernel can be simply written as

\[
\mathcal{M}^{\alpha\beta, jl}(\zeta, x_l; \zeta', x'_l) = 2 \pi \delta(\zeta - \zeta') \int_y \int_y' \mathcal{L}^\alpha_{j\mu}(\zeta; x_l, \mu, y) \mathcal{L}^{\mu\beta}(\zeta'; x'_l, \beta, y') \mathcal{G}(\zeta; y - y'),
\]

where a summation over \( \mu \) is implicit and \( \mathcal{G}(\zeta, y) \) is the free photon propagator with Fourier transform \( \mathcal{G}(\zeta, k) = (\zeta^2 + k^2)^{-1} \). To simplify this result, and to prove the independence of the free energy on the choice of \( L \), it is useful to rewrite the operators of Eqs. (12-14) as \( \mathcal{L}^{\alpha\beta}(\zeta; x, y) \equiv \mathcal{L}^{\alpha\beta}(\zeta) \mathcal{G}_E(\zeta; x - y) \) in order to make their proportionality to the material Green function explicit. It is important to keep in mind that the differential operators \( \mathcal{L}^{\alpha\beta} \) act on the spatial arguments of \( \mathcal{G}_E \) as well as on those of the free propagator \( \mathcal{G} \). Now the kernel in Eq. (18) acquires the form

\[
\mathcal{M}^{\alpha\beta, jl}(\zeta, x_l; \zeta', x'_l) = 2 \pi \delta(\zeta - \zeta') \int_y \int_y' \mathcal{L}^{\mu\beta}_{j\mu}(\zeta; x_l, \mu, y) \mathcal{L}^{\mu\beta}_{j\mu}(\zeta'; x'_l, \beta, y') \mathcal{G}(\zeta; y - y'),
\]

where \( \mathcal{L}^{\alpha\beta}(\zeta; x, y) \) acts on the primed coordinates and a summation over \( k \) and \( s \) is implicit. In momentum space, using \( \mathcal{G}_E(\zeta, k) = (\zeta^2 + k^2)^{-1} \), the partially Fourier transformed material Green function can be written as

\[
\mathcal{G}_E^{\alpha}(\zeta, k_l, z) = \frac{e^{-p_\alpha(\zeta, k_l)|z|}}{2p_\alpha(\zeta, k_l)}
\]

with \( p_\alpha(\zeta, k_l) = \sqrt{e_\alpha(\zeta)\zeta^2 + k_l^2} \). With this representation the kernel becomes

\[
\mathcal{M}^{\alpha\beta, jl}(\zeta, k_l; \zeta', k'_l) = 2 \pi \delta(\zeta - \zeta') \int_y \int_y' \frac{e^{-ik_l y_l + ik_l y'_l}}{2p_\alpha(\zeta, k_l)} \frac{e^{-p_\alpha(\zeta, k_l)|L_\alpha - y_l|}}{2p_\alpha(\zeta, k_l)} \frac{e^{-p_\beta(\zeta', k'_l)|L_\beta - y'_l|}}{2p_\beta(\zeta', k'_l)} \mathcal{G}(\zeta; y - y'),
\]

for \( l = 1, 2, 3 \).
where the differential operators $\hat{L}^{\kappa\alpha}(\zeta, \mathbf{k}_\parallel)$ are obtained from the $L^{\kappa\alpha}$ of Eqs. (12-14) by the replacements $\vec{\nabla}_\parallel \equiv (\partial_1, \partial_2) \rightarrow i\hat{k}_\parallel$, $\partial_3 \rightarrow (-1)^{\alpha}p_\alpha$. Thus the operators $\hat{L}^{\kappa\alpha}(\zeta, \mathbf{k}_\parallel)$ are acting via the remaining derivatives $\partial_3$ only on the spatial coordinates of the vacuum Green function $G(\zeta; y)$. At this stage it will become obvious that the free energy or force is independent of the positions $x_3 = \pm L$ of the auxiliary surfaces $R_{\alpha}$. Due to the construction of the surfaces $R_{\alpha}$, we have $|L_{\alpha} - y_3| = (-1)^{\alpha-1}(L_{\alpha} - y_3)$. The resulting kernel can be factorized into

$$M^{\alpha\beta, jl}(\zeta, \mathbf{k}_\parallel; \zeta', \mathbf{k}_\parallel') = \eta_\alpha(\zeta, \mathbf{k}_\parallel) \hat{M}^{\alpha\beta, jl}(\zeta, \mathbf{k}_\parallel; \zeta', \mathbf{k}_\parallel') \eta_\beta(\zeta', \mathbf{k}_\parallel')$$  \hspace{2cm} (22)

with the functions $\eta_\alpha(\zeta, \mathbf{k}_\parallel) = \exp(-p_\alpha(\zeta, \mathbf{k}_\parallel)L)/2p_\alpha(\zeta, \mathbf{k}_\parallel)$ and the simplified $L$-independent kernel

$$\hat{M}^{\alpha\beta, jl}(\zeta, \mathbf{k}_\parallel; \zeta', \mathbf{k}_\parallel') = 2\pi \delta(\zeta - \zeta') \int_{y \in S_{\alpha}} \int_{y' \in S_{\beta}} e^{-i\hat{k}_\parallel y_1 + i\hat{k}_\parallel' y_1'} e^{-(1)^{\alpha}p_\alpha(\zeta, \mathbf{k}_\parallel) y_3 + (1)^{\beta}p_\beta(\zeta', \mathbf{k}_\parallel') y_3'} \]

$$\times \hat{n}_\parallel^\alpha \hat{n}_\parallel'^\beta \left[ L^{\kappa\alpha}(\zeta, \mathbf{k}_\parallel); L^{\kappa\beta}(\zeta', \mathbf{k}_\parallel') \right]_{jl} G(\zeta; y - y').$$  \hspace{2cm} (23)

From Eq. (16) follows that the partition function $Z(H) = \det^{-1/2}M$ with the determinant taken with respect to both the continuous $(\zeta, \mathbf{k}_\parallel)$ and the discrete $(\alpha, j)$ arguments. Due to the structure of Eq. (22) one has $\det M \propto \det \hat{M}$. Since the functions $\eta_\alpha(\zeta, \mathbf{k}_\parallel)$ are independent of the mean surface distance $H$, the proportionality constant of the two determinants is independent of $H$, too. Therefore, this constant, and the $L$-dependence, will drop out of the free energy of Eq. (17) which can now be written as

$$F(H) = \frac{1}{2A\beta} \ln \det \left( \hat{M}M^{-1}_\infty \right),$$  \hspace{2cm} (24)

where $\hat{M}_\infty$ denotes $\hat{M}$ in the limit of asymptotically large $H$. The force per unit area between the two surfaces can be directly obtained from the kernel $\hat{M}$ by

$$F = -\frac{1}{2A\beta} \text{Tr} \left( \hat{M}^{-1} \partial_H \hat{M} \right)$$  \hspace{2cm} (25)

without the need to subtract the asymptotic expansion for large $H$. Here the trace has to be taken with respect to the Matsubara frequencies $\zeta_n$, the lateral momenta $\mathbf{k}_\parallel$, and the discrete $(\alpha, j)$ arguments. Eqs. (24) and (25) together with Eq. (22) represent the main result of our general approach. We will apply this formula below to specific model situations. Before proceeding along these lines, it might be interesting to discuss the above result. During the derivation of the result we worked within the Feynman gauge. This, however, poses no problem since the restricted partition function $Z(H)$ can be considered as the expectation value of the boundary condition enforcing delta-functions with respect to the free action of the gauge field. The arguments of the delta-functions are composed of the electromagnetic field components, and are thus manifestly gauge invariant. Therefore, the kernel $\hat{M}$ must be gauge invariant.

Let us first discuss the most simple situation where the kernel $\hat{M}$ is diagonal in momentum space so that the force can be calculated exactly. This will be the case when the geometry has translational symmetry in the lateral directions, i.e., for flat surfaces. Then the integrals in Eq. (23) can be easily computed, and the resulting kernel provides a compact account of Lifshitz theory as we will show in the next Section. Even if the surfaces are deformed the kernel can be obtained explicitly if one considers the limit of ideal metals, i.e., a diverging dielectric function $\epsilon(i\zeta)$. In this particular limit both $p_\alpha$ and the operators $\hat{L}^{\kappa\alpha}$ become independent of the lateral momentum $\mathbf{k}_\parallel$. Therefore, after parameterizing the surfaces so that $y_3, y'_3$ are replaced by functions of the lateral coordinates $y_\parallel, y'_\parallel$, respectively, the integrals in Eq. (23) correspond to Fourier transformations with respect to the lateral coordinates, and the kernel assumes a simple form in position space as we will demonstrate explicitly below. However, any kind of deviation from flat surfaces (even for ideal metals) renders $\hat{M}$ non-diagonal and makes the evaluation of Eq. (23) a hard problem. There are basically two approaches to tackle this problem. First, one can consider the amplitude of the surface deformations as small compared to both the mean surface distance and other characteristic lateral length scales as, e.g., the roughness correlation length. Then one can apply perturbation theory to obtain the force in powers of the deformation profiles $h_\alpha(x_\parallel)$. This program has been carried out in detail for ideal metals in Refs. (22, 23). Second, one can try to compute the force exactly by a numerical algorithm. For periodically deformed (corrugated) surfaces of ideal metals it has been demonstrated recently that the corresponding kernel can be transformed to a form which is particularly suited for an efficient numerical evaluation of the force (24). We expect that these techniques can be applied to the general case of deformed surfaces of dielectric media using our approach of expressing the force in terms of a kernel [Eq. (23)] which contains all information about material and geometrical properties of the surfaces. As for ideal metals, the kernel is proportional to the vacuum Green function which, however, is now dressed by the operators $\hat{L}^{\kappa\alpha}$ which contain the reflection and refraction properties of the material.
III. FLAT SURFACES

As a simple application of our approach, we consider in this section the case of flat surfaces of general dielectric media. In this particular limit the force between the surfaces is well-known from more conventional approaches. The corresponding result is known as the so-called Lifshitz theory for molecular forces [1]. In the following we will show that our path-integral approach provides a compact derivation of the Lifshitz result without the need to solve Maxwell’s equations with a random source explicitly and to calculate the expectation value of the stress tensor. In the flat surface limit, the surfaces $S_{\alpha}$ are parameterized by $(y_{\|}, H_{\alpha})$ with $H_{\alpha} = 0, H$ for $\alpha = 1, 2$, respectively. Due to the translational symmetry of the problem, it is convenient to work in momentum space. Using the representation

$$\mathcal{G}(\zeta, y) = \int_{q_{\parallel}} e^{i q_{\parallel} y} \frac{e^{-p(\zeta, q_{\parallel})|y_{\parallel}|}}{2p(\zeta, q_{\parallel})}$$

(26)

of the vacuum Green function in Eq. (20) with $p(\zeta, q_{\parallel}) = \sqrt{\zeta^2 + q_{\parallel}^2}$ yields

$$\mathcal{M}^{\alpha\beta\lambda\mu}(\zeta, k_{\parallel}; \zeta', k_{\parallel}') = 2\pi \delta(\zeta - \zeta') \int_{q_{\parallel}} \int_{y_{\|}} \int_{y'_{\|}} \int_{y''_{\|}} e^{ik_{\parallel}y + ik_{\parallel}'y'} e^{-[(1)^\alpha p_\alpha(\zeta, k_{\parallel})H_{\alpha} + (1)^\beta p_\beta(\zeta', k_{\parallel}')H_{\beta}]} (-1)^{\alpha + \beta}$$

$$\times \left[ \mathcal{L}^{3\alpha}(\zeta, k_{\parallel}) \cdot \mathcal{L}^{3\beta}(\zeta', k_{\parallel}') \right] \left[ \mathcal{L}_{\mu}(\zeta) - \mathcal{L}_{\mu}(\zeta', k_{\parallel}') \right] e^{-p(\zeta, q_{\parallel})|y_{\parallel} - y'_{\parallel}|} \frac{1}{2p(\zeta, q_{\parallel})}$$

(27)

where we made use of the surface normal vectors $\hat{n}_\alpha = (0, 0, (-1)^{\alpha-1})$ for flat surfaces. The differential operators $\mathcal{L}^{3\alpha}$ and $\mathcal{L}^{3\beta}$ can now be expressed in momentum space with the replacements $\nabla_{\parallel} \rightarrow i k_{\parallel}$ and $\nabla'_{\parallel} \rightarrow i k'_{\parallel}$ yielding

$$\mathcal{M}^{\alpha\beta\lambda\mu}(\zeta, k_{\parallel}; \zeta', k_{\parallel}') = (2\pi)^3 \delta(\zeta - \zeta') \delta(2)(k_{\parallel} - k_{\parallel}') \eta^{\alpha\beta} \left[ \mathcal{L}^{3\alpha}(\zeta, k_{\parallel}) \cdot \mathcal{L}^{3\beta}(\zeta', k_{\parallel}') \right] e^{-p(\zeta, q_{\parallel})|y_{\parallel} - y'_{\parallel}|} \frac{1}{2p(\zeta, q_{\parallel})}$$

(28)

where we separated the factor $\eta^{\alpha\beta} = (-1)^{\alpha + \beta} e^{-[(1)^\alpha p_\alpha(\zeta, k_{\parallel})H_{\alpha} + (1)^\beta p_\beta(\zeta', k_{\parallel}')H_{\beta}]}$ which we will discuss below. The differential operator acquires now the form

$$\mathcal{L}^{3\alpha}(\zeta, k_{\parallel}) = \begin{pmatrix} -k_2 - \frac{k_1 k_2}{\zeta_{\alpha}} & \zeta + \frac{1}{\zeta_{\alpha}} \left[ k_1^2 + (1)^\alpha p_\alpha \partial_{\alpha} \right] & \frac{k_1 k_2}{\zeta_{\alpha}} \frac{(-1)^\alpha p_\alpha}{\zeta_{\alpha}} k_2 \\ k_1 & -\zeta - \frac{1}{\zeta_{\alpha}} \left[ k_1^2 + (1)^\alpha p_\alpha \partial_{\alpha} \right] & \frac{k_1 k_2}{\zeta_{\alpha}} \frac{(-1)^\alpha p_\alpha}{\zeta_{\alpha}} k_1 \\ 0 & \frac{k_1 k_2}{\zeta_{\alpha}} \left[ (1)^\alpha p_\alpha + \partial_{\alpha} \right] & 0 \end{pmatrix},$$

(29)

and the primed adjoint operator acts via $\partial_{\alpha}$ on $y'_{\parallel}$. Before we calculate from this expression the free energy and force between the surfaces, it is instructive to examine the structure of the above matrix. It is easily checked that the third row of the matrix–operator $\mathcal{L}^{3\alpha}$ can be expressed in terms of the other two rows via $(-1)^\alpha p_\alpha \mathcal{L}_{3\alpha} = k_1 \mathcal{L}_{1\alpha} + k_2 \mathcal{L}_{2\mu}$. The physical reason for this can be easily understood. The relation between the rows reflects the fact that there exist only two independent boundary conditions for each surface. Since the surfaces are flat here, any field configurations can be considered as a superposition of transversal electric (TE) and magnetic (TM) modes. Each mode type is characterized by a scalar field which has to satisfy only one boundary condition at each surface. This will be demonstrated more explicitly in the Appendix where the problem is formulated from the outset in terms of two scalar fields representing TE and TM modes. However, for deformed surfaces this reduction to separated mode types is generally no longer expected to hold since the modes will mix under the scattering at deformations.

For flat surfaces we are thus led to introduce the reduced matrix–operator

$$\Omega^\alpha(\zeta, k_{\parallel}; \partial_{\alpha}) = \begin{pmatrix} -k_2 - \frac{k_1 k_2}{\zeta_{\alpha}} & \zeta + \frac{1}{\zeta_{\alpha}} \left[ k_1^2 + (1)^\alpha p_\alpha \partial_{\alpha} \right] & \frac{k_1 k_2}{\zeta_{\alpha}} \frac{(-1)^\alpha p_\alpha}{\zeta_{\alpha}} k_2 \\ k_1 & -\zeta - \frac{1}{\zeta_{\alpha}} \left[ k_1^2 + (1)^\alpha p_\alpha \partial_{\alpha} \right] & \frac{k_1 k_2}{\zeta_{\alpha}} \frac{(-1)^\alpha p_\alpha}{\zeta_{\alpha}} k_1 \\ 0 & \frac{k_1 k_2}{\zeta_{\alpha}} \left[ (1)^\alpha p_\alpha + \partial_{\alpha} \right] & 0 \end{pmatrix},$$

(30)

which consists of two linear independent rows only. Defining $\Omega^\alpha(\zeta, k_{\parallel}) = \Omega^\alpha(\zeta, k_{\parallel}; \pm \partial_{\alpha})$ the kernel can be written as

$$\mathcal{M}^{\alpha\beta\lambda\mu}(\zeta, k_{\parallel}; \zeta', k_{\parallel}') = (2\pi)^3 \delta(\zeta - \zeta') \delta(2)(k_{\parallel} - k_{\parallel}') \eta^{\alpha\beta} \left[ (\Omega^\alpha(\zeta, k_{\parallel}) \cdot \Omega^\beta(\zeta', k_{\parallel}')) \right] e^{-p(\zeta, q_{\parallel})|z - H_{\alpha} - H_{\beta}|} \frac{1}{2p(\zeta, q_{\parallel})}$$

(31)
The entries of this diagonal matrix consist of the $4 \times 4$ matrices which are given by the expression in the square brackets. Inserting now Eq. (30) into Eq. (31) we obtain for the expression in the square brackets the $4 \times 4$ matrix

$$\tilde{\mathcal{M}}(\zeta, k) = \begin{pmatrix} A_1 & \frac{e^{-p_H}}{e^{z_2}z_4} B_1 \\ \frac{e^{-p_H}}{e^{z_2}z_4} B_2 & A_2 \end{pmatrix}$$

(32)

in terms of the symmetric $2 \times 2$ matrices

$$A_\alpha = \begin{pmatrix} \frac{\epsilon_{\alpha}-1}{\epsilon_{\alpha}} \zeta^2 + \frac{\epsilon_{\alpha}-1}{\epsilon_{\alpha}} k_1^2 + \frac{\epsilon_{\alpha}-1}{\epsilon_{\alpha}} k_2^2 \\
-\frac{\epsilon_{\alpha}-1}{\epsilon_{\alpha}} k_1 k_2 
\end{pmatrix},
B = \begin{pmatrix} (p_1^2 - k_2^2 - pp_1)(p_2^2 - k_2^2 - pp_2) + k_2^2(\epsilon_1 \epsilon_2 \zeta^2 + k_1^2 - p_1 p_2) \\
k_1 k_2 \left( (\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2) \zeta^2 + k_1^2 - p(p_1 + p_2) + p_1 p_2 \right) \\
(\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2) \zeta^2 + k_1^2 - p(p_1 + p_2) + p_1 p_2 \end{pmatrix}
$$

(33)

using $p_\alpha = \sqrt{\epsilon_{\alpha} \zeta^2 + k_2^2}$, $p = \sqrt{\zeta^2 + k_2^2}$. The Casimir free energy per unit area can now be obtained from Eq. (24). In the limit $H \to \infty$ the off-diagonal elements of $\tilde{\mathcal{M}}$ in Eq. (32) vanish, so that we have to compute the determinant of the matrix

$$\tilde{\mathcal{M}}(\zeta, k) \tilde{\mathcal{M}}^{-1}(\zeta, k) = \begin{pmatrix} I & \frac{e^{-p_H}}{e^{z_2}z_4} BA^{-1} \\ \frac{e^{-p_H}}{e^{z_2}z_4} BA^{-1} & I \end{pmatrix},
$$

(35)

where $I$ is the $2 \times 2$ identity matrix. In the above matrix we have neglected the factor $\eta_{\alpha \beta}$ appearing in Eq. (21). This factor will have no effect on the free energy as will show at the end of this section. The determinant of the matrix of Eq. (35) can be calculated using the relation

$$\det(Y) = 1 - \text{Tr}(X_1 X_2) + \det(X_1 X_2)$$

(36)

for a general $4 \times 4$ matrix of the form

$$Y = \begin{pmatrix} I & X_1 \\ X_2 & I \end{pmatrix}.$$

(37)

Thus the free energy can be obtained by calculating the determinant of just a $2 \times 2$ matrix. Using Eq. (24), the logarithm of the product of all the determinants for different $\zeta_n$ and $k_\parallel$ becomes a corresponding sum and integral, respectively,

$$\mathcal{F}(H) = \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int_{k_\parallel}^{\infty} \frac{k dk}{2\pi} \ln |\tilde{\mathcal{M}}(\zeta_n, k_\parallel) \tilde{\mathcal{M}}^{-1}(\zeta_n, k_\parallel)|,$$

(38)

where $| \ldots |$ denotes the determinant of the $4 \times 4$ matrix at fixed $\zeta$ and $k_\parallel$. Calculating explicitly the determinant with the aid of Eq. (30), we obtain the final result for the Casimir or interaction free energy per unit area of the surfaces,

$$\mathcal{F}(H) = \frac{1}{\beta} \sum_{n=0}^{\infty} \int_{0}^{\infty} \frac{k dk}{2\pi} \ln \left[ \left(1 - e^{-2p_n H} \frac{p_{n1} - p_n}{p_{n1} + p_n} \frac{p_{n2} - p_n}{p_{n2} + p_n} \right) \left(1 - e^{-2p_n H} \frac{p_{n1} - \epsilon_1 p_n}{p_{n1} + \epsilon_1 p_n} \frac{p_{n2} - \epsilon_2 p_n}{p_{n2} + \epsilon_2 p_n} \right) \right].$$

(39)

with $k = |k_\parallel|$. The corresponding force per unit area is given by

$$F(H) = \frac{1}{\beta} \sum_{n=0}^{\infty} \int_{0}^{\infty} \frac{k dk}{\pi} p_n \left( \frac{p_{n1} + p_n}{p_{n1} - p_n} + \frac{p_{n2} + p_n}{p_{n2} - p_n} e^{2p_n H} - 1 \right)^{-1} + \left( \frac{p_{n1} + \epsilon_1 p_n}{p_{n1} - \epsilon_1 p_n} + \frac{p_{n2} + \epsilon_2 p_n}{p_{n2} - \epsilon_2 p_n} e^{2p_n H} - 1 \right)^{-1},$$

(40)

where we defined $p_{n\alpha} = \sqrt{\epsilon_{\alpha} \zeta^2 + k_\parallel^2}$ and $p_n = \sqrt{\zeta^2 + k_\parallel^2}$. The primed sum indicates that the term for $n = 0$ is to be multiplied by $1/2$. It is important to note that the dielectric function in the above expressions is evaluated along the
imaginary axis only, since \( \epsilon_\alpha = \epsilon_\alpha(i\zeta) \) due to the initial Wick rotation to the Euclidean field theory. Since \( \epsilon_\alpha(i\zeta) \) is completely determined by the imaginary part of the dielectric function for real frequencies \( \omega \), the force depends only the dissipative properties of the media, as expected from the fluctuations-dissipation theorem. Our result of Eqs. [39, 40] is in perfect agreement with the original result by Lifshitz [1, 2].

Next we consider now the zero temperature limit. This limit is obtained by the replacements \( \zeta_n \to \zeta \) and \( 1/\beta \sum_{n>0} \to \int_0^\infty d\zeta/2\pi \) in Eqs. [39] and [40]. Following Lifshitz, we change the integration variable to \( q = \sqrt{1 + k^2/\zeta^2} \) and define \( s_\alpha \equiv \sqrt{q^2 - 1 + \epsilon_\alpha(i\zeta)} \), yielding

\[
F(H) = \int_0^\infty \frac{\zeta^2 d\zeta}{2\pi} \int_1^\infty \frac{dq}{2\pi} \ln \left( \left[ 1 - e^{-2\zeta q H} \frac{s_1 + q + s_2 - q}{s_1 + q + s_2} \right] \left[ 1 - e^{-2\zeta q H} \frac{s_1 - \epsilon_1 q + s_2 - \epsilon_2 q}{s_1 + \epsilon_1 q + s_2 + \epsilon_2 q} \right] \right)
\]

for the free energy, and

\[
F(H) = -\frac{1}{2\pi^2} \int_0^\infty \frac{\zeta^3 d\zeta}{2\pi} \int_1^\infty q^2 dq \left( \left[ \frac{s_1 + q + s_2 - q}{s_1 - q + s_2 - q} e^{2\zeta q H} - 1 \right]^{-1} + \left[ \frac{s_1 + \epsilon_1 q + s_2 + \epsilon_2 q}{s_1 - \epsilon_1 q + s_2 - \epsilon_2 q} e^{2\zeta q H} - 1 \right]^{-1} \right)
\]

for the force. The above result agrees again with the original Lifshitz theory, see Eq. (2.9) in [1].

Finally, we come back to the omitted factor \( \eta_{\alpha\beta} \). The effect of taking into account this factor is that in the matrix-kernel of Eq. [29] the matrix \( A_2 \) is multiplied by \( e^{-2p_2 H} \), \( B \) is multiplied by \( -e^{-p_2 H} \) while \( A_1 \) remains unchanged. For the matrix in Eq. [38] this means that the off-diagonal matrix \( A_2^{-1} \) gets multiplied by the factor \( -e^{p_2 H} \) while the matrix \( A_1^{-1} \) gets multiplied by the inverse factor \( e^{-p_2 H} \). Due to Eq. [39] the determinant depends only on the product of the two off-diagonal matrices so that the factors coming from the \( \eta_{\alpha\beta} \) drop out in the determinant of \( \mathcal{M}_\infty \).

### IV. DEFORMED SURFACES OF IDEAL METALS

In the previous section we saw that our general approach reproduces the Lifshitz theory for flat surfaces of dielectric media. In this section we will apply our theory to deformed surfaces. As an example we consider ideal metals with infinite dielectric functions \( \epsilon_\alpha \). This is a reasonable approximation for surface separations which are large compared to the plasma wavelength of the material. However, our general result for the kernel of Eq. [29] contains all information which is necessary to treat deformed surfaces of non-ideal metals or general dielectric media as well. In the latter case the kernel \( \mathcal{M} \) assumes in general no particular simple form and has to be computed numerically in order to obtain the force. For ideal metals the kernel can be calculated explicitly and the result provides another interesting limit which has not been obtained previously. In previous works only special deformations of ideal metals have been studied by a path integral approach. If the surface deformations are translational invariant in one direction as for, e.g. uni-axial corrugations, the electromagnetic field can be separated into TE and TM modes. This property has been used in [22, 23, 24] to describe the surface interaction by a scalar field theory. In contrast, here we will allow for general deformations so that no separation into TE and TM modes is possible anymore.

Our starting point is the general result for the kernel of Eq. [29]. After taking the limit \( \epsilon_\alpha \to \infty \) both \( p_\alpha(\zeta, \mathbf{k}_\parallel) \) and the operators \( \hat{\mathcal{L}}_{\alpha} \) become independent of the lateral momentum \( \mathbf{k}_\parallel \). The kernel can then be written as

\[
\hat{\mathcal{M}}^{\alpha\beta;jl}(\zeta, \mathbf{k}_\parallel; \zeta', \mathbf{k}_\parallel') = 2\pi \delta(\zeta - \zeta') \int_{y \in S_\alpha} \int_{y' \in S_\beta} e^{-i\mathbf{k}_\parallel \cdot \mathbf{y}} e^{-i\mathbf{k}_\parallel' \cdot \mathbf{y}'} e^{-[\zeta(1 - \sqrt{\zeta}) y_3 + (1 - \zeta)^2 \sqrt{\zeta} y_3']}
\]

\[
\times \hat{\mathcal{L}}^{\alpha\beta;jl}_\parallel \left[ \hat{\mathcal{L}}^{\alpha\beta}(\zeta) \cdot \hat{\mathcal{L}}^{\gamma\delta}(\zeta') \right]_{jl} \mathcal{G}(\zeta; \mathbf{y} - \mathbf{y}')
\]

with the differential operators

\[
\hat{\mathcal{L}}^{1\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ i\partial_3 & 0 & 0 & \zeta \\ -i\partial_2 & 0 & -\zeta & 0 \end{pmatrix}, \quad \hat{\mathcal{L}}^{2\alpha} = \begin{pmatrix} -i\partial_3 & 0 & 0 & -\zeta \\ 0 & 0 & 0 & 0 \\ i\partial_1 & \zeta & 0 & 0 \end{pmatrix}, \quad \hat{\mathcal{L}}^{3\alpha} = \begin{pmatrix} i\partial_2 & 0 & \zeta & 0 \\ -i\partial_1 & -\zeta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Due to the simple exponential dependence of the integrand of Eq. [30] on \( \mathbf{k}_\parallel \) it is more convenient to transform the kernel to position space. When we insert the height profile of the surfaces with

\[
y_3 = H_\alpha + h_\alpha(\mathbf{y}), \quad y_3' = H_\beta + h_\beta(\mathbf{y}_\parallel'),
\]
\[ H_\alpha = 0, \quad H \text{ for } \alpha = 1, 2, \] the position space form of the kernel can be read off from Eq. 13:

\[ \mathcal{M}^{\alpha\beta,jl}(\zeta, y_\parallel; \zeta', y'_\parallel) = 2\pi \delta(\zeta - \zeta') e^{-|\zeta|\sqrt{\hbar_\alpha + h_{\alpha,1}}(y_1) + |\zeta'|\sqrt{\hbar_\beta + h_{\beta,1}}(y'_1)} \times \begin{pmatrix} h_{\alpha,1} \\ h_{\alpha,2} \\ -1 \end{pmatrix} \]

Before proceeding, it is useful to discuss the two exponential factors depending on \( \sqrt{\alpha} \). Let us start with the second one which depends on the height profiles \( h_{\alpha,i}(y_\parallel) \) but is independent of the mean surface distance \( H \). Defining

\[ \eta_\alpha(\zeta, y_\parallel) = e^{-|\zeta|\sqrt{\hbar_\alpha + h_{\alpha,1}}(y_1)}, \]

due to the arguments given below Eq. 22 but with \( k_\parallel \) replaced by \( y_\parallel \). The first exponential factor in Eq. 45 this argument does not apply since the factor depends on \( H \). However, we can make use of the fact that the factor does not depend on the lateral coordinates \( y_\parallel \). The effect of this exponential factor is that every \( 2 \times 2 \) sub-matrix of \( \mathcal{M} \) resulting from keeping \( (j, l) \) or \( (\zeta, y_\parallel) \) or \( (\zeta', y'_\parallel) \) fixed is multiplied by the same only \( \zeta \) dependent factors. The two diagonal elements are multiplied by 1 and \( e^{-|\zeta|\sqrt{\hbar_\alpha}H} \), respectively, while the off-diagonal elements are multiplied by \( e^{-|\zeta'|\sqrt{\hbar_\beta}H} \). It is easy to check that this leads to the simple factor \( e^{-N|\zeta|\sqrt{\hbar_\beta}H} \) for the determinant of \( \mathcal{M} \) if \( N \) denotes the dimension of the matrix \( \mathcal{M} \). However, this factor will drop out when taking the determinant of the ratio \( \mathcal{M}_1^\infty \). Therefore, the first exponential factor in Eq. (45) can be omitted as well.

Now the kernel assumes a simple form. Expressing the surface normal vectors in terms of the height profile by

\[ \hat{n}_\alpha = \begin{pmatrix} -1 \alpha \\ \sqrt{\alpha} \end{pmatrix} \begin{pmatrix} h_{\alpha,1} \\ h_{\alpha,2} \\ -1 \end{pmatrix} \]

with \( h_{\alpha,j} = \partial_j h_\alpha \) and \( g_\alpha = 1 + (\nabla_\perp h_\alpha)^2 \) the kernel can be written as a functional of the height profile. For the same reason as the factors \( \eta_\alpha(\zeta, y_\parallel) \) could be omitted above, we can neglect the normalization factor \( (-1)^\alpha/\sqrt{g_\alpha} \) of the normal vector. Thus we obtain for the differential operator

\[ \left[ \hat{n}_k^\alpha \hat{L}^{k\alpha} \right](\zeta, y_\parallel) = \begin{pmatrix} -ih_{\alpha,2}\partial_3 - i\partial_2 & 0 & -\zeta & -h_{\alpha,2}\zeta \\ ih_{\alpha,1}\partial_3 + i\partial_1 & \zeta & 0 & h_{\alpha,1}\zeta \\ -ih_{\alpha,1}\partial_2 + ih_{\alpha,2}\partial_1 & h_{\alpha,2}\zeta & -h_{\alpha,1}\zeta & 0 \end{pmatrix} = \hat{\Omega}^\alpha. \]

We observe that the third row of the matrix in (47) is linearly dependent since \( h_{\alpha,1}\hat{\Omega}^{\alpha}_{3\beta} + h_{\alpha,2}\hat{\Omega}^{\alpha}_{3\mu} = \Omega^{\alpha}_{3\mu} \). Therefore, as in the Lifshitz theory limit discussed earlier, the matrix has to be reduced to its first two rows. The linear dependence of rows reflects the fact that for ideal metals there are, in fact, only two independent boundary conditions for each surface. As mentioned earlier, for general deformations a reduction to TE and TM modes as it appears in Lifshitz theory is not possible. However, for ideal metals the boundary conditions can be simply written as

\[ [\hat{n}_\alpha(y) \times E(\zeta, y)]_{y \in S_\alpha} = 0. \]

This boundary condition requires the two tangential components of the electric field at the surface to vanish locally. Saying it differently, the limit of infinite conductivity converts the three originally non-local boundary condition into two local conditions. The final result for the matrix kernel is now given by a \( 4 \times 4 \)–matrix with \( j, l = 1, 2, \)

\[ \mathcal{M}^{\alpha\beta,jl}(\zeta, y_\parallel; \zeta', y'_\parallel) = 2\pi \delta(\zeta - \zeta') \left[ \hat{\Omega}^\alpha \cdot \hat{\Omega}^{\beta\parallel} \right]_{jl} G(\zeta; y - y') \]

with the deformation dependent differential operator

\[ \left[ \hat{\Omega}^\alpha \cdot \hat{\Omega}^{\beta\parallel} \right]_{jl} = (-1)^{j+l} \left( \zeta^2 \left[ \delta_{jl} + h_{\alpha,3-j}h_{\beta,3-l} \right] + \left[ h_{\alpha,3-j}\partial_3 + \partial_{3-j} \right] \left[ h_{\beta,3-l}\partial_3 + \partial_{3-l} \right] \right) \]

acting on the vacuum Green function. Here the prime on \( h \) indicates the dependence on the primed variable \( y'_\parallel \). This kernel together with the formula of Eq. 24 yields the exact free energy of the interacting surfaces. Generally it is not possible to give a closed analytical expression for the determinant of \( \mathcal{M} \) of (19). However, either perturbative [22, 23] or numerical [24] techniques can be used to evaluate the free energy and force from the kernel using Eq. (24) and Eq. (25).
V. DISCUSSION AND OUTLOOK

We derived from a path-integral quantization of the electromagnetic gauge field with non-local boundary conditions an effective action for the molecular forces between dielectric media with deformed surfaces. From this effective action we derived the Lifshitz formula for flat surfaces and an explicit expression for the matrix kernel which determines the interaction between ideal metals with arbitrary deformations. We believe that our approach will be a useful starting point for the calculation of correlations between modifications of the ideal Casimir force of Eq. (1) due to material properties (finite conductivity) and geometry. Our theory is certainly not a final answer to the problem of correlation effects. Rather it is intended for a generalization of the Lifshitz formula to deformed surfaces. As for the Lifshitz formula, for an explicit calculation of the force one, of course, has to specify a suitable dielectric function and, in addition, a height profile describing the geometry. An precise result for the force can presumably only be obtained from a numerical evaluation of the effective action. However, certain limiting cases should be accessible to a perturbative analysis. For example, it would be interesting to consider the cases of small and large distances \( H \) between the surfaces separately. For small \( H \), the interaction is dominated by high frequencies and is sensitive to the model from which the frequency dependence of the dielectric function \( \epsilon(\omega) \) is obtained. The correct choice of \( \epsilon(\omega) \) for real metals is currently being discussed but this is not the matter of the present discussion. For large \( H \) a useful approximation is to assume \( \epsilon \) to be constant. In a large \( \epsilon \) expansion a strong geometry dependence and non-additivity of Casimir forces between ideal metals should be found. In the opposite limit of rarefied media the non-additivity effects and thus the sensitivity to geometry should decrease if \( \epsilon \) approaches one. Another point of recent dispute is the choice of boundary conditions for real metals. Although this is not the subject of the present paper, we note that our path integral approach is sufficiently general to include any type of boundary conditions as, e.g., the so-called impedance boundary condition.

APPENDIX A: SCALAR FIELD THEORY FOR FLAT SURFACES

The scope of application of scalar field theories is more restricted than that of the gauge field formalism, but calculations become simpler in many cases. An interesting example is the case of flat surfaces of dielectric media. In Section III we re-derived the Lifshitz theory for dielectric media. There we argued that the field can be decomposed into TE and TM modes each of which have to obey only one boundary condition on each surface, leading to a reduced matrix kernel. In this Appendix we show explicitly that the Lifshitz theory can be obtained directly from a path integral quantization of a scalar field which fulfills a suitable boundary condition. Therefore, no gauge fixing procedure is needed in this special situation.

In order to decompose the electromagnetic field into transversal electric (TE) and magnetic (TM) modes, we have to specify a preferred spatial direction. Due to the rotational symmetry in the lateral plane of the flat surface geometry, we can choose without any restrictions the \( y_2 \) direction. Actually, it turns out that it is useful to adapt the choice of the direction to the lateral momentum \( k_\parallel \) of the field mode. Note that we can do this since the modes for different \( k_\parallel \) are decoupled for flat surfaces. Therefore, in the following we will choose for a given \( k_\parallel \) the lateral coordinates so that \( k_2 = 0 \), and \( y_2 \) defines the longitudinal direction. Following the parameterization of the TE and TM modes for waveguides, the longitudinal components of the electric (TM modes) and magnetic (TE modes) field define a (real-valued) scalar field \( \Phi \),

\[
E_2 = \Phi \quad B_2 = 0 \quad \text{for TM modes}
\]

\[
B_2 = \Phi \quad E_2 = 0 \quad \text{for TE modes}.
\]

The transversal components \( E_j \equiv E_j(\zeta, k_\parallel; z) \) and \( B_j \equiv B_j(\zeta, k_\parallel; z) \) of the electromagnetic field are then given by

\[
E_1 = \frac{k_1 k_2}{\zeta + k_2^2} \Phi \quad B_1 = \frac{\zeta}{\zeta + k_2^2} \partial_z \Phi
\]

\[
E_3 = -\frac{ik_2}{\zeta + k_2^2} \partial_z \Phi \quad B_3 = \frac{-i\zeta k_1}{\zeta + k_2^2} \Phi
\]

for TM modes, and

\[
E_1 = \frac{\zeta}{\zeta + k_2^2} \partial_z \Phi \quad B_1 = \frac{k_1 k_2}{\zeta + k_2^2} \Phi
\]

\[
E_3 = \frac{ik_2}{\zeta + k_2^2} \Phi \quad B_3 = \frac{ik_2^2}{\zeta + k_2^2} \partial_z \Phi
\]

for TE modes, where we performed already a Wick rotation to imaginary frequency, \( \omega \to i\zeta \). Using Maxwell equations it can be shown that the dynamics of the scalar field \( \Phi \) are governed by the usual wave equation, corresponding to...
the Euclidean action

\[ S_E[\Phi] = \frac{1}{2} \int_X (\nabla \Phi)^2 \]  \hspace{1cm} (A4)

with the partition function

\[ Z_0 = \int \mathcal{D}\Phi \, e^{-S_E[\Phi]}. \]  \hspace{1cm} (A5)

In this section we consider only the zero temperature case. Finite temperatures can be treated in analogy to the gauge field approach in section II B by introducing Matsubara frequencies. The boundary condition for \( \Phi \) can be derived from the general condition we found in section II A. From Eq. (11) we obtain, using the Fourier representation of the material Green function \( g^\ast \), in Euclidean space the conditions

\[ -\zeta \epsilon_\alpha (i\zeta) E_2 - ik_1 B_3 - (-1)^\alpha p_\alpha B_1 = 0 \]  \hspace{1cm} (A6)

\[ \zeta \epsilon_\alpha (i\zeta) E_1 - ik_2 B_3 - (-1)^\alpha p_\alpha B_2 = 0 \]  \hspace{1cm} (A7)

\[ ik_\parallel \cdot B_\parallel - (-1)^\alpha p_\alpha B_3 = 0 \]  \hspace{1cm} (A8)

for the flat surface \( S_a \) for \( \alpha = 1, 2 \), and with \( p_\alpha = \sqrt{\epsilon_\alpha (i\zeta) \zeta^2 + k_\parallel^2} \). Now we make use of the fact that we can constrain the analysis to the case \( k_2 = 0 \) by a suitable choice of the lateral coordinates. After inserting the electromagnetic field components as given by Eqs. (A1), (A2) and (A3), the above conditions collapse to a single boundary condition for the scalar field \( \Phi \). Depending on the type of mode, we obtain the condition

\[ \left[ 1 - \Gamma_\alpha \partial_{n_\alpha} \right] \Phi|_{z = H_a} = 0 \]  \hspace{1cm} (A9)

with

\[ \Gamma_\alpha = \frac{1}{p_\alpha} \]  \hspace{1cm} (A10)

for TM modes, \( \Gamma_\alpha = \frac{\epsilon_\alpha}{p_\alpha} \) for TE modes, where \( \partial_{n_\alpha} = (-1)^{\alpha-1} \partial_z \) denotes the normal derivatives of the surfaces. Due to the \( k_\parallel \)-dependence, the condition (A9) is non-local in position space. In the limit of ideal metals, \( \epsilon_\alpha \to \infty \), the above condition reduces to the well known Dirichlet and Neumann boundary conditions for TM and TE modes, respectively. We note that \( \Gamma_\alpha \) is real, since the same is valid for \( \epsilon_\alpha \) on the imaginary frequency axis. Therefore we need to consider only a real-valued field \( \Phi \). The restricted partition function for this field reads for both types of modes

\[ Z = Z_0^{-1} \int \mathcal{D}\Phi \, e^{-S_E[\Phi]} \prod_\alpha \prod_{\zeta, k_\parallel} \delta \left[ \left( 1 - \Gamma_\alpha \partial_{n_\alpha} \right) \Phi \right]_{z = H_a}, \]  \hspace{1cm} (A11)

where we implemented the boundary constraints again by delta-functions. Now we proceed in analogy to the treatment of the gauge field path integral in section II B. We introduce two auxiliary field \( \psi_\alpha \), one for each surface, in order to replace the delta-function. After integrating out \( \Phi \), we obtain for the partition function

\[ Z = \int \mathcal{D}\psi_\alpha \, e^{-S_{\text{eff}}[\psi_\alpha]} \]  \hspace{1cm} (A12)

with the effective action

\[ S_{\text{eff}}[\psi_\alpha] = \frac{1}{2} \int \int_{\zeta, k_\parallel} \int_{\zeta', k'_\parallel} \psi_\alpha(\zeta, k_\parallel) M^{\alpha\beta}(\zeta, k_\parallel; \zeta', k'_\parallel) \psi_\beta(\zeta', k'_\parallel) \]  \hspace{1cm} (A13)

and the \( 2 \times 2 \) matrix kernel

\[ M^{\alpha\beta}(\zeta, k_\parallel; \zeta', k'_\parallel) = (2\pi)^3 \delta(\zeta + \zeta') \delta(k_\parallel + k'_\parallel) \frac{1}{2p} \left( \begin{array}{cc} 1 - p^2 \Gamma_1^2 & (1 - p \Gamma_1)(1 - p \Gamma_2) e^{-pH} \\ (1 - p \Gamma_1)(1 - p \Gamma_2) e^{-pH} & 1 - p^2 \Gamma_2^2 \end{array} \right). \]  \hspace{1cm} (A14)

with \( k = |k_\parallel|, p = \sqrt{\zeta^2 + k^2} \) and a summation over \( \alpha \) and \( \beta \) is implicit. From this kernel, the Casimir energy can be quite easily calculated. Using Eq. (24) in the zero temperature limit, we obtain for the Casimir energy per unit surface area

\[ \mathcal{F}(H) = \frac{1}{2} \int_{\zeta, k_\parallel} \ln |M M^{-1}| = \int_0^\infty \frac{d\zeta}{2\pi} \int_0^\infty \frac{dk}{2\pi} \ln \left[ 1 - e^{-2pH} \frac{1}{1 + p \Gamma_1} \frac{1}{1 + p \Gamma_2} \right], \]  \hspace{1cm} (A15)
where $|\ldots|$ denotes the determinant of the $2 \times 2$ matrix at fixed $\zeta$ and $k_\parallel$. The matrix $M_\infty$ is diagonal since for $H \to \infty$ the off-diagonal elements in Eq. (A14) vanish. Therefore, the determinant of the matrix product can be easily obtained, yielding the last expression in Eq. (A15). If we substitute $\Gamma_\alpha$ according to Eq. (A10) in Eq. (A15) by $1/\rho_\alpha$ or by $\epsilon_\alpha/\rho_\alpha$ we obtain the contribution of the TM or TE modes, respectively, to the energy. It is easily seen that the sum of the energies from both modes reproduces the Lifshitz theory result we obtained before in section III from the gauge field approach, cf. Eq. (39).

ACKNOWLEDGMENTS

This work was supported by the Deutsche Forschungsgemeinschaft through the Emmy Noether grant No. EM70/2-2.

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