On the free rotations of rigid bodies with a liquid-filled gap

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Abstract

We consider the system constituted by a hollow rigid body whose cavity contains a homogeneous rigid ball, and let the gap between the solids be entirely filled by a viscous incompressible fluid. We investigate the free rotations of the whole system, i.e., motions driven only by the inertia of the fluid-solids system once an initial angular momentum is imparted on the whole system. We prove the existence of global weak solutions and local strong solutions to the equations of motion. In addition, we prove that the fluid velocity as well as the inner core angular velocity relative to the outer solid converge to zero as time approaches infinity.

Keywords: Fluid-solid interactions, Navier-Stokes equations, rigid body motion, Leray-Hopf weak solutions, global existence, critical spaces, strong solutions

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1. Introduction

Consider the system constituted by a hollow rigid body $B_1$ whose cavity contains a homogeneous rigid ball $B_2$. Assume further that $B_2$ is strictly contained in the cavity of $B_1$, and let the gap between $B_1$ and $B_2$ be entirely filled by a viscous incompressible fluid $L$ (simply called liquid). Let $G$ be the center of mass of the system $S_C$ constituted by the outer rigid body $B_1$ and the liquid. Suppose that $G$ is a fixed point in space and time with respect to an inertial frame of reference $I$, and it coincides with the (geometrical) center of the ball $B_2$. We are interested in the free rotations of the whole system of rigid bodies with a liquid-filled gap. This type of motion occurs when no external forces and torques are applied, and the system is constrained to rotate (without friction) around $G$ driven by only its inertia once an initial angular momentum is imparted, see Figure 1.

These type of fluid-solid interaction problems have been widely studied in connection to some geophysical problems related to the motion of the Earth’s inner (solid and liquid) core and its influence on the geodynamo (i.e., the mechanism responsible for the generation of Earth’s magnetic field and its maintenance against the Ohmic dissipation), see [19, 26, 3, 31, 32, 37]. From the mathematical point of view, there have been several contributions aimed at proving the existence of solutions to the relevant equations of motion and analyzing their stability properties. In the case where no rigid core is within the liquid-filled cavity, it was conjectured by Zhukovskii ([30]) and rigorously proved by the present author and

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1 The geometrical center of the ball is also its center of mass due to the homogeneity and geometrical symmetry of $B_2$. 

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collaborators that the liquid has a stabilizing effect on the motion of the solid (see [22, 23, 9, 25, 24]). In fact, there exists a finite interval of time (whose length depends on the liquid viscosity) where the motion of the system has a chaotic nature (as shown numerically in [9] and experimentally in [23]). After this interval of time, the system reaches (at an exponentially fast rate) a more orderly configuration, corresponding to a steady state in which the system moves as a whole rigid body with a constant angular velocity (see [25, 24] for a rigorous mathematical proof of this phenomena when the liquid is subject to no-slip and partial slip boundary conditions, respectively).

Concerning the motion of solids with fluid-filled gaps, known results mainly focus on the translational and rotational motions of rigid bodies in a liquid occupying a bounded domain with a prescribed motion of the liquid outer boundary. The works [31, 11, 32] provide the first results of existence of weak solutions à la Leray-Hopf to the Navier-Stokes equations in bounded regions with moving boundaries. For the fluid-solid interaction problems with a finite number of rigid bodies within a liquid, existence of weak solutions up to collisions are proved in [8, 4]. The work [17] deals with local strong solutions, whereas [30, 18, 10] provide the first results of existence of global weak solutions for both incompressible and compressible cases.

In this paper, we show that the problem of free rotations of rigid bodies with a liquid-filled gap admits global weak solutions à la Leray-Hopf. In addition, we determine the largest space of initial data for which the equations of motion are well-posed in the setting of maximal $L^p - L^q$ regularity and time-weighted $L^p$ spaces. It is worth emphasizing that for the problem at hand, possible translations of the solids are disregarded. This simplifying assumption has to be contrasted with the existing (cited above) literature in which the motion of the outer solid is instead prescribed. The novelty of the paper lies on the proof of the existence of weak solutions (Theorem 5.6) and important properties like a Serrin-type result for weak solutions that are “sufficiently regular” (Theorem 5.7). One of the main objective of this work is to show that, similarly to the case when no solid is within the liquid-filled cavity, the fluid has a stabilizing effect on the motion of both solids. In fact, it will be shown that the long-time dynamics of the whole system is completely characterized.
by the rest state for the liquid and solid cores relatively to the outer solid, and the system moving as a whole rigid body (see (51)). In addition, we prove a local well-posedness result (Theorem 6.1) in the functional setting of maximal \( L^p - L^q \) regularity in time-weighted \( L^p \) spaces. This result is the first of the kind for this class of fluid-solid interactions.

Here is the plan of the paper. After presenting the basic notation and recalling a well-known Gronwall-type lemma, we proceed with Section 2 containing the mathematical formulation of the problem as the coupled system of differential equations, given by the Navier-Stokes equations and the balances of angular momentums of \( B_1 \) and \( B_2 \), respectively. In Section 3, we introduce our functional setting. As equations (2) involve both differential and integral terms, in Section 4, we provide an equivalent formulation of the problem by replacing the (physical) equations of motion with those governing the motion of a rigid body with a cavity completely filled by a viscous impressible fluid with varying density. In Section 5, we prove the existence of weak solutions and related properties. In Section 6, we demonstrate the existence of strong solution in the \( L^p - L^q \) setting.

The notation used throughout this paper is quite standard. \( \mathbb{N} \) denotes the set of natural numbers. \( \mathbb{R} \) indicates the set of real numbers, and \( \mathbb{R}^3 \) the Euclidean three-dimensional space equipped with the canonical basis \( \{ e_1, e_2, e_3 \} \). The components of a vector \( \mathbf{v} \) with respect to the canonical base are indicated by \( (v_1, v_2, v_3) \), whereas \( |\mathbf{v}| \) represents the magnitude of \( \mathbf{v} \). We will use the Einstein convention for the summation of dummy indexes, and “:” will denote the tensor contraction. Moreover, \( B_R(G) \) denotes the ball in \( \mathbb{R}^3 \) with center at a point \( G \in \mathbb{R}^3 \) and radius \( R \). The ball centered at the origin of a coordinates system \( \{ e_1, e_2, e_3 \} \) will be simply denoted by \( B_R \).

If \( A \) is an open set of \( \mathbb{R}^3 \), \( L^p(A) \), \( W^{k,p}(A) \), \( W_0^{k,p}(A) \) denote the Lebesgue and Sobolev spaces, with norms \( \| \cdot \|_{L^p(A)} \) and \( \| \cdot \|_{W^{k,p}(A)} \), respectively.

For a bounded, Lipschitz domain \( A \), with outward unit normal \( \mathbf{n} \), we will often use the following well-known Helmholtz-Weyl decomposition (e.g., [12]):

\[
L^q(A) = H_q(A) \oplus G_q(A),
\]

where \( q \in (1, \infty) \), \( H_q(A) := \{ u \in L^q(A) : \text{div} u = 0 \text{ in } A, \text{ and } u \cdot \mathbf{n} = 0 \text{ on } \partial A \} \) (\( \text{div} u \) and \( u \cdot \mathbf{n} \) have to be understood in the sense of distributions), and \( G_q(A) := \{ w \in L^q(A) : w = \nabla \pi, \text{ for some } \pi \in W^{1,q}(A) \} \). In the case of \( q = 2 \), we will simply write \( H(A) \) and \( G(A) \), respectively.

If \( (X, \| \cdot \|_X) \) is a Banach space, for an interval \( I \) in \( \mathbb{R} \) and \( 1 \leq p < \infty \), \( L^p(I; X) \) (resp. \( W^{k,p}(I; X) \), \( k \in \mathbb{N} \)) will denote the space of functions \( f \) from \( I \) to \( X \) for which \( (\int_I \| f(t) \|_X^p \, dt)^{1/p} < \infty \) (resp. \( \sum_{k=0}^k J_I \| \partial_t^k f(t) \|_X^p \, dt)^{1/p} < \infty \)). Similarly, \( C^k(I; X) \) indicates the space of functions which are \( k \)-times differentiable with values in \( X \), and having \( \max_{t \in I} \| \partial_t^\ell f(t) \|_X < \infty \), for all \( \ell = 0, 1, ..., k \). Finally, \( C_w(I; X) \) is the space of functions \( f \) from \( I \) to \( X \) such that that the map \( t \in I \mapsto \phi(f(t)) \in \mathbb{R} \) is continuous for all bounded linear functionals \( \phi \) defined on \( X \).

We conclude this section by recalling the following two Gronwall-type lemmas that will be used in the paper. For its proof, we refer the interested reader to [28].

**Lemma 1.1.** Suppose that a function \( y \in L^\infty(0, \infty) \), \( y \geq 0 \), satisfies the following inequality

\[
\text{data with finite kinetic energy.}
\]

\(^3\)Unless confusion arises, we shall use the same symbol for spaces of scalar, vector and tensor functions.
for a. a. $s \geq 0$ and all $t \geq s$:

$$y(t) \leq y(s) - k \int_s^t y(\tau) \, d\tau + \int_s^t F(\tau) \, d\tau.$$  

Here, $k > 0$, and $F \in L^q(a, \infty) \cap L_1^2(0, \infty)$, for some $a > 0$ and $q \in [1, \infty)$, satisfies $F(t) \geq 0$ for a. a. $t \geq 0$. Then

$$\lim_{t \to \infty} y(t) = 0.$$  

If $F \equiv 0$, then

$$y(t) \leq y(s) e^{-k(t-s)}, \quad \text{for all } t \geq s.$$  

We are now ready to introduce the equations governing the motion of the system of rigid bodies with a liquid-filled gap.

2. A preliminary mathematical formulation of the problem

Consider $B_1 := \mathcal{V}_1 \setminus \mathcal{V}_3$, with $\mathcal{V}_1$ and $\mathcal{V}$ bounded domains in $\mathbb{R}^3$, $\overline{\mathcal{V}} \subset \mathcal{V}_1$, $\overline{B_R(G)} \subset \mathcal{V}$, and $B_2 := B_R(G)$. Let us denote $C := \partial \mathcal{V}$, $S := \partial B_2$ and $\mathcal{L} := \mathcal{V} \setminus \overline{B_R(G)}$ be the volume occupied by the liquid at each time. Throughout the paper, we will assume that $\mathcal{L}$ is of class $C^2$.

Let $\mathcal{F} \equiv \{G, e_1, e_2, e_3\}$ be the non-inertial reference frame with origin at $G$, and axes coinciding with the central axes of inertia of the coupled system $\mathcal{J}_C$; these axes are directed along the eigenvectors of the inertia tensor $I_C$ of $\mathcal{J}_C$ with respect to $G$, and with corresponding (positive and time-independent) eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (also called central moment of inertia). Let us denote by $I_B$ the inertia tensor of the rigid body $B_1$ with respect to $G$. Since $B_2$ is a homogeneous rigid ball with center at $G$, then any axis passing through its center is also a central axis of inertia. Thus, the inertial tensor of $B_2$ with respect to $G$ is simply $\lambda(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3)$ with $\lambda = 2/5 mR^2$, and $m$ the mass of the rigid ball. With respect to the reference frame $\mathcal{F}$, all the volumes considered above are time-independent.

The following system of differential equations describes the dynamics of the given system in the reference frame $\mathcal{F}$:

\[
\begin{aligned}
\rho \left( \frac{\partial u}{\partial t} + v \cdot \nabla u + \omega_1 \times u \right) &= \text{div} \, T(u, p) \quad \text{on } \mathcal{L} \times (0, \infty), \\
\text{div} \, u &= 0 \\
I_B \cdot \omega_1 + \omega_1 \times I_B \cdot \omega_1 &= - \int_{\mathcal{C}} x \times T(u, p) \cdot n \, d\sigma \quad \text{in } (0, \infty), \quad (2) \\
\lambda(\omega_2 + \omega_1 \times \omega_2) &= - \int_{S} x \times T(u, p) \cdot n \, d\sigma \quad \text{in } (0, \infty), \\
u &= \omega_1 \times x \quad \text{on } \mathcal{C}, \\
u &= \omega_2 \times x \quad \text{on } S.
\end{aligned}
\]
Moreover, \( T(u, p) \) denotes the Cauchy stress tensor for a viscous incompressible fluid

\[
T(u, p) := -p \mathbf{I} + 2\mu \mathbf{D}(u), \quad \text{where} \quad \mathbf{D}(u) := \frac{1}{2}(\nabla u + (\nabla u)^T).
\]

Finally, \( \omega_1 \) and \( \omega_2 \) are the angular velocities of \( B_1 \) and \( B_2 \), respectively. Equations (2) with (3) and (5) are the Navier-Stokes equations in the non-inertial reference frame \( \mathcal{F} \). These equations describe the dynamics of the liquid. Equations (2), (3), and (5) are the balances of angular momentum (with respect to \( G \)) of \( B_1 \) and \( B_2 \), respectively. In particular, the surface integrals in (2), (3), and (5) represent the total torque exerted by the liquid on the cavity surface \( \mathcal{C} \) and on the sphere \( \mathcal{S} \), respectively. The equations of motion are augmented with the no-slip boundary conditions (2), (3), and (5) at \( \mathcal{C} \) and \( \mathcal{S} \), respectively.

Equations (2) feature a combination of dissipative and conservative components. The dissipative role is played by the liquid variable through equations (2), (3), and (5). Whereas, the conservative feature comes from the coupling with the equations (2), (3), and (5) describing the dynamics of the solids. As a matter of fact, the energy dissipates only in the liquid variable (see equation (7) below), and the total angular momentum (with respect to \( G \)) of the whole system is conserved at all times (see equation (10) below). These properties are satisfied for “sufficiently regular” solutions.

**Lemma 2.1 (Energy Balance).** Consider \( t_0 \geq 0 \), and assume that the quadruple \((u, p, \omega_1, \omega_2)\) satisfies the following regularity properties for all \( T > 0 \):

\[
\begin{align*}
uin \quad & u \in C^0([t_0, t_0 + T]; W^{1,2}(\mathcal{L}) \cap H(\mathcal{L})) \cap L^2(t_0, t_0 + T; W^{2,2}(\mathcal{L})), \\
\frac{\partial u}{\partial t} \in L^2(t_0, t_0 + T; L^2(\mathcal{L})), \\
p \in L^2(t_0, t_0 + T; W^{1,2}(\mathcal{L})), \\
\omega_1, \omega_2 \in W^{1,\infty}(t_0, t_0 + T).
\end{align*}
\]

If \((u, p, \omega_1, \omega_2)\) satisfies (2) a.e. in \((t_0, \infty)\), then the following energy balance holds.

\[
\frac{1}{2} \frac{d}{dt} \left[ \rho \|u\|^2_{L^2(\mathcal{L})} + \omega_1 \cdot \mathbf{I} \cdot \omega_1 + \lambda \|\omega_2\|^2 \right] + 2\mu \|\mathbf{D}(u)\|^2_{L^2(\mathcal{L})} = 0.
\]

**Proof.** Let us take the \( L^2 \)-inner product of (2) with \( u \), we find that

\[
\frac{\rho}{2} \frac{d}{dt} \|u\|^2_{L^2(\mathcal{L})} + \int_{\mathcal{L}} (\nabla u) \cdot u \, dV - \int_{\mathcal{L}} u \cdot \nabla v \, dV = 0.
\]

Since \( \nabla v = \nabla u = 0 \) by (2), using (4) and Gauss' Theorem, we can infer the following

\[
\int_{\mathcal{L}} (v \cdot \nabla u) \cdot u \, dV = 0.
\]
By (4), and again by Gauss’ Theorem, we get
\[
\frac{d}{dt} \|\mathbf{u}\|_{L^2(\mathcal{L})}^2 = -\mathbf{\omega}_1 \cdot \int_{\mathcal{C}} \mathbf{x} \times T \cdot \mathbf{n} \, d\sigma - \mathbf{\omega}_2 \cdot \int_{\mathcal{S}} \mathbf{x} \times T \cdot \mathbf{n} \, d\sigma + 2\mu \|D(\mathbf{u})\|_{L^2(\mathcal{L})}^2 = 0.
\]
From the latter displayed equation, (7) immediately follows by using (2) dot-multiplied by \(\mathbf{\omega}_1\) and \(\mathbf{\omega}_2\), respectively.

With the same hypotheses of the previous lemma, we can show the following.

**Lemma 2.2** (Conservation of total angular momentum). If the quadruple \((u, p, \mathbf{\omega}_1, \mathbf{\omega}_2)\) satisfies (6) for some \(t_0 \geq 0\), and (2) a.e. in \((t_0, \infty)\), then
\[
\dot{A} + \mathbf{\omega}_1 \times A = 0,
\]
where
\[
A := \rho \int_{\mathcal{L}} \mathbf{x} \times \mathbf{u} \, dV + I_B \cdot \mathbf{\omega}_1 + \lambda \mathbf{\omega}_2
\]
is the total angular momentum of the whole system with respect to \(G\). In particular, equation (8) implies that
\[
|A(t)| = |A(t_0)|, \quad \text{all } t \geq t_0.
\]

**Proof.** From (2) and Gauss’ Theorem, we find that
\[
\dot{A} = \rho \int_{\mathcal{L}} \mathbf{x} \times \frac{\partial \mathbf{u}}{\partial t} \, dV + I_B \cdot \mathbf{\omega}_1 + \lambda \mathbf{\omega}_2
\]
\[
= \int_{\mathcal{L}} \mathbf{x} \times (\text{div} \, \mathbf{T}(u, p) - \rho \mathbf{v} \cdot \nabla \mathbf{u} - \rho \mathbf{\omega}_1 \times \mathbf{u}) \, dV - \mathbf{\omega}_1 \times I_B \cdot \mathbf{\omega}_1
\]
\[
- \int_{\mathcal{C}} \mathbf{x} \times T(u, p) \cdot \mathbf{n} \, d\sigma - \int_{\mathcal{S}} \mathbf{x} \times T(u, p) \cdot \mathbf{n} \, d\sigma.
\]
Since the Cauchy stress tensor is symmetric, by Gauss’ Theorem we get that
\[
\int_{\mathcal{L}} \mathbf{x} \times \text{div} \, T(u, p) \, dV - \int_{\mathcal{C}} \mathbf{x} \times T(u, p) \cdot \mathbf{n} \, d\sigma - \int_{\mathcal{S}} \mathbf{x} \times T(u, p) \cdot \mathbf{n} \, d\sigma = 0.
\]
Using again Gauss’ Theorem together with (4), we also find that
\[
-\rho \int_{\mathcal{L}} \mathbf{x} \times \left(\mathbf{v} \cdot \nabla \mathbf{u} + \mathbf{\omega}_1 \times \mathbf{u}\right) \, dV = \rho \int_{\mathcal{L}} \left[\mathbf{u} \times (\mathbf{\omega}_1 \times \mathbf{x}) + \mathbf{x} \times (\mathbf{u} \times \mathbf{\omega}_1)\right] \, dV
\]
\[
= -\rho \int_{\mathcal{L}} \mathbf{\omega}_1 \times (\mathbf{x} \times \mathbf{u}) \, dV.
\]
In the last equality, we have used the following property of the cross product in \(\mathbb{R}^3\):
\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}), \quad \text{all } \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3.
\]
Therefore, (11) becomes
\[
\dot{A} = -\mathbf{\omega}_1 \times \left(\rho \int_{\mathcal{L}} \mathbf{x} \times \mathbf{u} \, dV + I_B \cdot \mathbf{\omega}_1 + \lambda \mathbf{\omega}_2\right) = -\mathbf{\omega}_1 \times A.
\]
This shows (8), from which (10) immediately follows by taking the dot-product of (8) by \(A\).

In the next section, we will provide the functional setting in which we will study the existence of solutions to the equations of motions.
3. Functional spaces

Consider the spaces
\[ \mathcal{R}(\mathcal{V}) := \{ \mathbf{u} \in C^\infty(\mathcal{V}) : \mathbf{u} = \omega_u \times \mathbf{x} \text{ on } \mathcal{V}, \text{ for some } \omega_u \in \mathbb{R}^3 \}, \]
\[ C_R^\infty(\mathcal{V}) := \{ \mathbf{u} \in C^\infty(\mathcal{V}) : \mathbf{u} = \omega_u \times \mathbf{x} \text{ in a neighborhood of } B_2, \text{ for some } \omega_u \in \mathbb{R}^3 \}. \]

For every \( 1 \leq q < \infty \), let us consider the norm
\[ \| \mathbf{u} \|_q := \left( \int_\mathcal{V} \tilde{\rho} u^q \right)^{1/q} = \left( \rho \| \mathbf{u} \|_{L^q(\mathcal{L})}^q + \lambda |\omega_u|^q \right)^{1/q}, \]
for all \( \mathbf{u} \in C_R^\infty(\mathcal{V}) \). (12)

In the above equation,
\[ \tilde{\rho} := \begin{cases} \rho & \text{on } \mathcal{L} \\ \frac{15\lambda}{8\pi R^5} & \text{on } B_2. \end{cases} \] (13)

\( L^q_R(\mathcal{V}) \) indicates the completion of \( C_R^\infty(\mathcal{V}) \) in the norm \( \| \cdot \|_q \). In the particular case of \( q = 2 \), \( L^2_R(\mathcal{V}) \) is a Hilbert space endowed with the inner product
\[ (\mathbf{u}, \mathbf{v}) := \int_\mathcal{V} \tilde{\rho} u \cdot v = \int_\mathcal{L} \rho \mathbf{u} \cdot \mathbf{v} + \lambda \omega_u \cdot \omega_v. \] (14)

One can show that the following characterization holds for every \( 1 \leq q < \infty \) (see e.g. [36, Chapter 1, Section 1])
\[ L^q_R(\mathcal{V}) = \{ \mathbf{u} \in L^q(\mathcal{V}) : \mathbf{u} = \omega_u \times \mathbf{x} \text{ on } B_2 \text{ for some } \omega_u \in \mathbb{R}^3 \}. \]

Consider the spaces
\[ \mathcal{D}_R(\mathcal{V}) := \{ \mathbf{u} \in C_R^\infty(\mathcal{V}) \cap C_0^\infty(\mathcal{V}) : \text{div } \mathbf{u} = 0 \text{ on } \mathcal{V} \}, \]
and for \( T > 0 \)
\[ \mathcal{D}_R(\mathcal{V}_T) := \{ C_0^\infty(\mathcal{V} \times [0, T)) : \text{div } \mathbf{u} = 0 \text{ on } \mathcal{V} \times [0, T), \]
\[ \mathbf{u} = \omega_u \times \mathbf{x} \text{ in a neighborhood of } B_2, \text{ for some } \omega_u \in C_0^\infty([0, T)) \}. \]

In addition, \( \mathcal{H}_q(\mathcal{V}) \) denotes the completion of \( \mathcal{D}_R(\mathcal{V}) \) with respect to the norm \( \| \cdot \|_q \). In a similar fashion to the classical space of the hydrodynamics (see e.g. [12]), one can show that \( \mathcal{H}_q(\mathcal{V}) \) has the following representation
\[ \mathcal{H}_q(\mathcal{V}) = \{ \mathbf{u} \in L^q_R(\mathcal{V}) : \text{div } \mathbf{u} = 0 \text{ on } \mathcal{V}, \; \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \mathcal{C}, \]
\[ \mathbf{u} = \omega_u \times \mathbf{x} \text{ on } B_2 \text{ for some } \omega_u \in \mathbb{R}^3 \}. \]

Moreover, we can consider the orthogonal projection \( \mathcal{P}_q \) of \( L^q_R(\mathcal{V}) \) onto \( \mathcal{H}_q(\mathcal{V}) \). Let \( 1 < q < \infty \). The space \( \mathcal{H}^1_q(\mathcal{V}) \) denotes the completion of \( \mathcal{D}_R(\mathcal{V}) \) with respect to the norm
\[ \| \cdot \|_{1,q} := \left( \| \cdot \|_q^q + 2\mu \| D(\cdot) \|_{L^q(\mathcal{V})}^q \right)^{1/q}. \] (15)

The right-hand side of latter displayed equation defines indeed a norm due to the following Korn inequality ([16, Theorem 1]).
Lemma 3.1. For $1 < q < \infty$ the space $U_q := \{ u \in L^q(\mathcal{V}) : D(u) \in L^q(\mathcal{V}) \}$ is equal to $W^{1,q}(\mathcal{V})$. Moreover, there exist two constants $0 < c_1 < c_2$ such that

$$c_1 \| u \|_{W^{1,q}(\mathcal{V})} \leq \left( \| u \|_{L^q(\mathcal{V})}^q + \| D(u) \|_{L^q(\mathcal{V})}^q + \| \text{div}(u) \|_{L^q(\mathcal{V})}^q \right)^{1/q} \leq c_2 \| u \|_{W^{1,q}(\mathcal{V})},$$

for all $u \in U_q$.

The following characterization holds

$$H^1_q(\mathcal{V}) = \{ u \in W^{1,q}_0(\mathcal{V}) : \text{div} u = 0 \text{ on } \mathcal{V}, \ u = \omega_u \times x \text{ on } \mathcal{B}_2 \text{ for some } \omega_u \in \mathbb{R}^3 \}.$$

We notice that $D_R(\mathcal{V}) \subset H^1_q(\mathcal{V})$, so $H^1_q(\mathcal{V})$ is dense in $H_q(\mathcal{V})$. Moreover, since $W^{1,q}(\mathcal{V})$ is compactly embeded in $L^q(\mathcal{V})$ for all $1 \leq q < \infty$ (Theorem 6.3), we have the following lemma.

Lemma 3.2. If $1 \leq q < \infty$, then the embedding of $H^1_q(\mathcal{V})$ in $H_q(\mathcal{V})$ is compact.

We are now in position to state some inequalities that we will be used in the next sections. The proof are standards and will be omitted. We start with the following Korn-type equalities.

Lemma 3.3 (Korn’s equality in $H^1_2(\mathcal{V})$). For all $v, w \in H^1_2(\mathcal{V})$ the following equality holds

$$2 \int_{\mathcal{L}} D(v) : D(w) \ dV = \int_{\mathcal{V}} \nabla v : \nabla w \ dV.$$

In particular,

$$\| \nabla v \|_{L^2(\mathcal{V})} = \sqrt{2} \| D(v) \|_{L^2(\mathcal{L})}. \quad (16)$$

In a similar fashion as in [16, Proposition 3.], and applying Lemma 3.1 to (15) one can easily show the following Poincaré-Korn inequality.

Lemma 3.4 (Poincaré-Korn inequality in $H^1_q(\mathcal{V})$). Let $1 < q < \infty$. There exist two positive constants $k_1 < k_2$ such that

$$k_1 \| v \|_{W^{1,q}(\mathcal{V})} \leq \| v \|_{1,q} \leq k_2 \| D(v) \|_{W^{1,q}(\mathcal{L})}, \quad \text{for all } v \in H^1_q(\mathcal{V}). \quad (17)$$

Recall that $\mathcal{V} = \mathcal{L} \cup \overline{\mathcal{B}_2}$. Next lemma will also be useful.

Lemma 3.5. The following estimates hold for all $v \in H^1_2(\mathcal{V})$.

1. Let $\omega_v \in \mathbb{R}^3$ be such that $v = \omega_v \times x$ on $\mathcal{B}_2$, then

$$\| \nabla v \|_{L^2(\mathcal{L})}^2 + \frac{8}{3} \pi R^3 |\omega_v|^2 = 2 \| D(v) \|_{L^2(\mathcal{L})}^2. \quad (18)$$

2. There exists a positive constants $C_1$ depending only on $\mathcal{L}$ (and independent of $v$) such that

$$\| v \|_{L^2(\mathcal{L})} \leq C_1 \| D(v) \|_{L^2(\mathcal{L})}. \quad (19)$$

Since $D_R(\mathcal{V})$ is dense in $H^1_R(\mathcal{V})$, by Sobolev inequality together with (17), we can prove the following lemma.
Lemma 3.6. For all $s < 3$, there exists a positive constant $k$ depending only on $L$ (and independent of $v$) such that
\[ \|v\|_q \leq k\|D(v)\|_{L^s(L^2)}, \quad \text{for all } v \in H^1_q(V) \] (20)
if and only if $q = 6/(3 - s)$.

We conclude this section by introducing the space $H^k_q(V)$ as the completion of $D_R(V)$ with respect to the norm $\|\cdot\|_{W^k,q(V)}$ for all $1 \leq q < \infty$ and $k \geq 2$. $H^k_q(V)$ is a Banach space endowed with the norm
\[ \|\cdot\|_{2,q} := \left( \|\cdot\|_{L^q(V)}^q + \|D(\cdot)\|_{L^q(V)}^q + \|H(\cdot)\|_{L^q(V)}^q \right)^{1/q}, \] (21)
where $H$ denote the third order tensor of second order derivatives. Similarly to Lemma 3.2, the following embedding also holds.

Lemma 3.7. If $1 \leq q < \infty$ and $k \geq 1$, then the embedding of $H^k_q(V)$ in $H_q(V)$ is compact.

The previous results together with Lemma 2.1 and Lemma 2.2 allow us to present a new mathematical formulation of the problem. This new formulation is equivalent to (2), and it is more prone to reveal more features of the dynamics of the system.

4. An equivalent formulation

Let us introduce the new variable
\[ \omega := \omega_2 - \omega_1. \] (22)

Let $I := I_C + \lambda I_1 = (\lambda_1 + \lambda)e_1 \otimes e_1 + (\lambda_2 + \lambda)e_2 \otimes e_2 + (\lambda_3 + \lambda)e_3 \otimes e_3$ be the inertia tensor of the whole system with respect to $G$. Here, $I$ denotes the identity tensor in $\mathbb{R}^3 \times \mathbb{R}^3$. We note that $I_C = I_{\mathcal{L}} + I_B$, where
\[ b \cdot I_{\mathcal{L}} \cdot c = \rho \int_{\mathcal{L}} (x \times b) \cdot (x \times c) \, dV, \quad b, c \in \mathbb{R}^3. \]
The tensor $I$ is a symmetric and positive definite (thus, invertible). To simply the notation, let us introduce the vector field
\[ \omega_R := -I^{-1} \cdot \left[ \rho \int_{\mathcal{L}} x \times v + \lambda \omega \right]. \] (23)
The definition of the variable $\omega$ comes from the following heuristic reasoning. Due to the liquid viscosity (since also $D(u) = D(v)$), we expect the velocity of the liquid relative to $B_1$ (and also the one relative to $B_2$) to decay to zero as time approaches to infinity. If this happens, from the boundary conditions, also $\omega$ is expected to decay, and the system would then move as a whole rigid body. This kind of behavior has been rigorously proved to be true in other problems in liquid-solid interactions without the inner core for inertial motions and motions driven by gravity (see [4, 13, 14, 22, 23, 24, 25, 15]).
In terms of the variables \((v, p, \omega_1, \omega)\), and taking into account (5) together with (43), the equations of motion (2) can be equivalently reformulated as follows:

\[
\begin{aligned}
\rho \left( \frac{\partial v}{\partial t} + \dot{\omega}_1 \times x + v \cdot \nabla v + 2\omega_1 \times v \right) &= \frac{\rho}{2} \nabla |\omega_1 \times x|^2 + \text{div } T(v, p) \\
\text{div } v &= 0 \\
I \cdot (\dot{\omega}_1 - \dot{\omega}_R) + \omega_1 \times I \cdot (\omega_1 - \omega_R) &= 0 \\
\lambda (\dot{\omega} + \dot{\omega}_1 + \omega_1 \times \omega) &= -\int_S x \times T(v, p) \cdot n \, d\sigma \\
v &= 0 \\
v &= \omega \times x
\end{aligned}
\]

(24)

The proof of the equivalence between the formulations (2) and (24) goes along the one provided in the case when no rigid body is within the cavity of \(B_1\) (namely, if \(R \equiv 0\)). We refer the interested reader to [9, Appendix] and [23, Sections 2.1 and 2.2]. The energy balance (7) can be rewritten as follows

\[
\frac{1}{2} \frac{d}{dt} \left[ \rho \|v\|^2_{L^2(\mathcal{L})} + \lambda |\omega|^2 - \omega_R \cdot (\omega_1 - \omega_R) + I \cdot (\omega_1 - \omega_R) \right] + 2\mu \|D(v)\|^2_{L^2(\mathcal{L})} = 0.
\]

(25)

Consider the functionals

\[
b : w \in \mathcal{H}_q(\mathcal{V}) \mapsto b(w) := -I^{-1} \cdot \int_\mathcal{V} \tilde{\rho} x \times w
\]

\[
= -I^{-1} \cdot \left( \rho \int_\mathcal{L} x \times w + \lambda \omega w \right) \in \mathbb{R}^3,
\]

(26)

and taking \(q = 2\) in the previous definition, we define

\[
\mathcal{E} : w \in \mathcal{H}_2(\mathcal{V}) \mapsto \mathcal{E}(w) := \|w\|^2_2 - b(w) \cdot I \cdot b(w) \in \mathbb{R}.
\]

(27)

In particular, if we consider the field

\[
\tilde{v} := \begin{cases} v & \text{in } \mathcal{L}, \\ \omega \times x & \text{in } B_2,
\end{cases}
\]

(28)

and use (12) and (23), we find that \(b(\tilde{v}) = \omega_R\) and

\[
\mathcal{E}(\tilde{v}) = \rho \|v\|^2_{L^2(\mathcal{L})} + \lambda |\omega|^2 - \omega_R \cdot I \cdot \omega_R.
\]

(29)

The following lemma ensures that \(\mathcal{E}\) is a positive definite functional. Actually, it says a little more.

**Lemma 4.1.** There exists a constant \(c \in (0, 1)\) such that

\[
c \|w\|^2_2 \leq \mathcal{E}(w) \leq \|w\|^2_2.
\]

(30)

for all \(w \in \mathcal{H}_2(\mathcal{V})\). Moreover, for every \(w \in \mathcal{H}_2^2(\mathcal{V})\), there exists a positive constant \(C\) such that

\[
\mathcal{E}(w) \leq C \|D(w)\|^2_{L^2(\mathcal{L})}.
\]

(31)
In particular, since \( I \) admits a bounded inverse in \((\mathcal{V})\), thus, to complete the proof of (30), it is enough to show that the operator 

\[ (\mathbb{B}w, z) = -\rho \int_{\mathcal{F}} (b(w) \times x) \cdot z \, dV - \lambda \omega_z \cdot b(w) \]

\[ = -b(w) \cdot \left( \rho \int_{\mathcal{F}} x \times z \, dV + \lambda \omega_z \right) = b(w) \cdot I \cdot b(z) = (w, \mathbb{B}z). \]

In particular, since \( I \) is positive definite, \((\mathbb{B}w, w) = b(w) \cdot I \cdot b(w) \geq 0\). Moreover,

\[ ((1 - \mathbb{B})w, w) = \|w\|^2 - b(w) \cdot I \cdot b(w) = \mathcal{E}(w). \tag{33} \]

The inequality on the right-hand side of (30) follows immediately from the latter displayed equations. Thus, to complete the proof of (30), it is enough to show that the operator \( 1 - \mathbb{B} \) admits a bounded inverse in \((\mathcal{V}), \|\cdot\|_2\). First, we will show that \( 1 - \mathbb{B} \) is a nonnegative operator on \( \mathcal{V} \).

Using the above calculations, we have the following:

\[ ((1 - \mathbb{B})w, w) = \|w\|^2 - b(w) \cdot I \cdot b(w) \]

\[ = \|w + b(w) \times x\|^2 - \rho \|b(w) \times x\|^2 \]  

\[ - 2b(w) \cdot \left( \rho \int_{\mathcal{F}} x \times w \, dV + \lambda \omega_w \right) - \lambda \|b(w)\|^2 - b(w) \cdot I \cdot b(w) \]

\[ = \|w + b(w) \times x\|^2 - b(w) \cdot \left( I_\mathcal{F} + \lambda 1 \right) \cdot b(w) + 2b(w) \cdot I \cdot b(w) \]

\[ - b(w) \cdot I \cdot b(w) \]

\[ = \|w + b(w) \times x\|^2 - b(w) \cdot \left( I_\mathcal{F} + \lambda 1 \right) \cdot b(w) + b(w) \cdot I \cdot b(w) \]

\[ = \|w + b(w) \times x\|^2 + b(w) \cdot I_B \cdot b(w) \geq 0 \]

since \( I_B = I - I_\mathcal{F} - \lambda 1 \) is also a positive definite tensor. In addition to this, one can also show that \((1 - \mathbb{B})w, w) = 0 \) iff \( w = 0 \) on \( \mathcal{V} \). We need to show only that \((1 - \mathbb{B})w, w) = 0 \) implies that \( w = 0 \) (the converse implication is obvious). If \((1 - \mathbb{B})w, w) = 0 \), then \( b(w) \cdot I_B \cdot b(w) = 0 \). Since \( I_B \) is positive definite, then the previous statement implies that \( b(w) = 0 \), and also \((\mathbb{B}w, w) = 0 \). Thus,

\[ \|w\|_2 = (\mathbb{B}w, w) = 0, \]

implying that \( w = 0 \) in \( \mathcal{V} \). Summarizing, we have shown that \( \mathbb{B} \) is a finite dimensional, linear, nonnegative and self-adjoint operator for which \( \gamma = 1 \) is not an eigenvalue. Necessarily, \( \gamma = 1 \) is in the resolvent of \( \mathbb{B} \), implying that \( 1 - \mathbb{B} \) admits a bounded inverse in \( \mathcal{V} \) endowed with the norm defined in (12). This concludes the proof of (30).

The estimate (31) is an immediate consequence of (30) together with (20).

\[ \frac{d}{dt} \left[ \mathcal{E}(\mathbf{v}) + (\omega_1 - \omega_R) \cdot I \cdot (\omega_1 - \omega_R) \right] + 4\mu \|D(v)\|_{L^2(\mathcal{F})}^2 = 0, \tag{34} \]

Using (29) and (27) in (25), the balance of energy then reads as follows.
where \( \bar{v} \) has been defined in (28). From the physical point of view, \( \mathcal{E}(\bar{v}) + (\omega_1 - \omega_R) \cdot I \cdot (\omega_1 - \omega_R) \) represents the total kinetic energy of the whole system of rigid bodies with a liquid-filled gap.

Thanks to Lemma 4.1, we can introduce the inner product

\[
(v, w)_B := ((1 - B)v, w), \quad \text{for all} \ v, w \in H_2(V)
\]

with associated norm \( \| \cdot \|_B := \sqrt{((1 - B) \cdot \cdot)} = \sqrt{\mathcal{E}(\cdot)} \).

In addition to the energy balance, the conservation of the total angular momentum \( I \)
for the whole system can be rewritten in terms of the new variables

\[
|I \cdot (\omega_1(t) - \omega_R(t))| = |I \cdot (\omega_1(0) - \omega_R(0))| \quad \text{all} \ t \geq 0.
\]

One can also obtain (36) by taking the dot-product of (24) by \( I \cdot (\omega_1 - \omega_R) \).

5. Weak solutions and their properties

Our investigation on the inertial motion about a fixed point of the system of two rigid bodies with a liquid-filled gap is carried out in a considerably large class of solutions to (24) having finite kinetic energy.

A weak formulation for the problem (24), can be found by dot-multiplying both sides of (24) by \( \varphi \in H_2^1(V) \), integrating (by parts) the resulting equation over \( \mathcal{L} \times (0, t) \), and using (24) together with (A.1) and (A.2). This leads to the following of problem: find a solution \( (\bar{v}, \Omega) \) to the following system of equations

\[
\begin{align*}
(\bar{v}(t), \varphi)_B + 2\mu \int_0^t \int_V D(\bar{v}) : D(\varphi) \, dV \, dt
+ b(\varphi) \cdot \int_0^t [\Omega + b(\bar{v})] \times I \cdot \Omega \, dt \\
+ \int_0^t \int_V \rho [\bar{v} \cdot \nabla \bar{v} + 2(\Omega + b(\bar{v})) \times \bar{v}] \cdot \varphi \, dV \, dt
= (\bar{v}(0), \varphi)_B, \\
\text{for all} \ \varphi \in H_2^1(V), \ \text{and} \ \text{all} \ t \in [0, \infty).
\end{align*}
\]

\[
I \cdot \Omega(t) + \int_0^t [\Omega + b(\bar{v})] \times I \cdot \Omega \, dt = I \cdot \Omega(0), \quad \text{for all} \ t \in [0, \infty).
\]

Definition 5.1. The triple \((v, \omega_1, \omega)\) is a weak solution to (24) if the following requirements are met.

1. Consider the field \( \bar{v} \) in (28). Then,
\[
\bar{v} \in C_w([0, \infty); H_2(V)) \cap L^\infty(0, \infty; H_2(V)) \cap L^2(0, \infty; H_2^1(V)).
\]

2. The vector field \( \Omega = \omega_1 - b(\bar{v}) \in C^0([0, \infty)) \cap C^1((0, \infty)).
\]

3. \((\bar{v}, \Omega)\) satisfies (37).

4. The following strong energy inequality holds:
\[
\mathcal{E}(\bar{v}(t)) + \Omega(t) \cdot I \cdot \Omega(t) + 4\mu \int_s^t \|D(\bar{v}(\tau))\|_{L^2(\mathcal{L})}^2 \, d\tau \leq \mathcal{E}(\bar{v}(s)) + \Omega(s) \cdot I \cdot \Omega(s),
\]

\[
\text{for all} \ t \geq s \ \text{and a.a.} \ s \geq 0 \ \text{including} \ s = 0.
\]
From the previous definition, it immediately follows that the physical velocity fields $(v, \omega_1, \omega)$ enjoy the following properties

$$v \in C_\omega([0, \infty); H(\mathcal{L})) \cap L^\infty(0, \infty; H(\mathcal{L})) \cap L^2(0, \infty; H(\mathcal{L}) \cap W^{1,2}(\mathcal{L}))$$

$$\omega_1 \in C([0, \infty)) \cap L^\infty(0, \infty),$$

$$\omega \in C([0, \infty)) \cap L^\infty(0, \infty) \cap L^2(0, \infty),$$

$$v = 0 \quad \text{on} \quad \mathcal{C}, \quad \omega = \omega \times x \quad \text{on} \quad \mathcal{S} \quad \text{(in the trace sense)}.$$  

In particular, if $(v, \omega_1, \omega)$ is a weak solution, by (38) together with (30) and (14), it follows that there exists a constant $c_0 = c_0(v(0), \Omega(0), \omega(0))$ such that

$$\rho||v||_{L^2(\mathcal{L})}^2 + \lambda|\omega|^2 \leq c_0^2, \quad \text{for all} \quad t \geq 0.$$  

Furthermore, up to redefining the above constant $c_0$, we also have

$$|\omega_R(t)| \leq \rho \int_{\mathcal{S}} |x \times v| \, dV + \lambda|\omega| \leq c_0 \quad \text{for all} \quad t \geq 0,$$

$$\Omega(t) \cdot I \cdot \Omega(t) = \omega_1(t) \cdot I \cdot \omega_1(t) - 2\omega_R(t) \cdot I \cdot \omega_1(t) \leq c_0^2 \quad \text{for all} \quad t \geq 0.$$  

Thus, for every $\varepsilon > 0$,

$$\lambda_{\text{min}}|\omega_1(t)|^2 \leq c_0^2 + 2\omega_R(t) \cdot I \cdot \omega_1(t) \leq c_0^2 + 2\lambda_{\text{max}}|\omega_R(t)| |\omega_1(t)|$$

$$\leq c_0^2 + \frac{\lambda_{\text{max}}}{\varepsilon} |\omega_R(t)|^2 + \lambda_{\text{max}} \varepsilon |\omega_1(t)|^2.$$  

Here, $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ denote the minimum and maximum eigenvalue of $I$, respectively. Choosing $\varepsilon := \lambda_{\text{min}}/(2\lambda_{\text{max}})$, we can conclude that

$$\frac{1}{2} \lambda_{\text{min}} |\omega_1(t)|^2 \leq c_0^2 \quad \text{for all} \quad t \geq 0.$$  

**Remark 5.2.** Equations (37) together with (38) represent the “classical” weak formulation (á la Leray-Hopf) for the problem of a rigid body having a cavity $V$ completely filled by a viscous liquid with the varying viscosity $\tilde{\rho}$ defined in (13). However, setting $\omega_1 = \Omega + \omega_R$ and using (A.1) and (A.2), one can immediately observe that the system of equations in (37) is the appropriate weak formulation obtained by testing (24) with functions $\psi \in C^\infty(\mathcal{L})$, $\text{div} \psi = 0$ on $\mathcal{L}$ and satisfying the boundary conditions $\psi = 0$ on $\mathcal{C}$ and $\psi = \omega \times x$ on $\mathcal{S}$. In fact, for such test function and all $t \in (0, \infty)$,

$$\int_{\mathcal{L}} \rho[v(t) + \omega_1(t) \times x] \cdot \psi \, dV + \lambda \left[ \omega(t) + \omega_1(t) + \int_0^t \omega_1 \times \omega \, d\tau \right] \cdot \psi$$

$$+ 2\mu \int_0^t \int_{\mathcal{L}} D(v) : D(\psi) \, dV \, d\tau + \int_0^t \int_{\mathcal{L}} \rho[v \cdot \nabla v + 2\omega_1 \times v] \cdot \psi \, dV \, d\tau$$

$$\leq \int_{\mathcal{L}} \rho[v(0) + \omega_1(0) \times x] \cdot \psi \, dV + \lambda[\omega(0) + \omega_1(0)] \cdot \psi,$$

$$I \cdot (\omega_1(t) - \omega_R(t)) + \int_0^t \omega_1 \times I \cdot \Omega \, d\tau = I \cdot (\omega_1(0) - \omega_R(0)).$$

\footnote{Due to its regularity, we can extend $\psi$ by its boundary value on $\mathcal{S}$ and use it as test function in (37).}
Remark 5.3. Assume that $\tilde{v}$ possesses enough regularity to allow differentiation with respect to time and integration by parts in $(37)_1$. Then

$$\omega_1 = \Omega + b(\tilde{v}) = \Omega + \omega_R \in C^1(0, \infty),$$

and $(24)_3$ is satisfied for a.a. $t \in (0, \infty)$. Moreover, the fields $v$ and $\omega \times x$ in $(28)$ maintain the same regularity of $\tilde{v}$ on $\mathcal{L}$ and $\mathcal{B}_2$, respectively.

By $(37)_2$, we find that $\tilde{v}$ also satisfies

$$\frac{\partial \tilde{v}}{\partial t} + \omega_1 \times x + \tilde{v} \cdot \nabla \tilde{v} + 2\omega_1 \times \tilde{v}, \varphi) + 2\mu \int_{\mathcal{V}} D(\tilde{v}) : D(\varphi) = 0 \quad (41)$$

for all $\varphi \in H^1_{2}(\mathcal{V})$ and all $t \in (0, \infty)$. In particular,

$$\int_{\mathcal{L}} \rho \left( \frac{\partial v}{\partial t} + \omega_1 \times x + v \cdot \nabla v + 2\omega_1 \times v \right) - \mu \Delta v \right] \cdot \varphi = 0$$

for every $\varphi \in H(\mathcal{L}) \cap W^{1,2}_{0}(\mathcal{L})$. Thus, there exists $\tilde{p} \in L^2(0, \infty; W^{1,2}(\mathcal{L}))$ such that

$$\rho \left( \frac{\partial v}{\partial t} + \omega_1 \times x + v \cdot \nabla v + 2\omega_1 \times v \right) - \mu \Delta v = \nabla \tilde{p} \quad \text{a.e. in } \mathcal{L} \times (0, \infty).$$

Set

$$p := \tilde{p} - \rho \frac{1}{2} |\omega_1 \times x|^2 \quad \text{in } \mathcal{L},$$

then one immediately notices that equations $(24)_{1,2,5,6}$ are satisfied almost everywhere in space-time. Dot-multiplying $(24)_{1}$ by $\varphi \in H^1_{2}(\mathcal{V})$ such that $\omega_\varphi = e_i$, $i = 1, 2, 3$, and integrating the resulting equation over $\mathcal{L}$ we find

$$\int_{\mathcal{L}} \rho \left[ \frac{\partial v}{\partial t} + \omega_1 \times x + v \cdot \nabla v + 2\omega_1 \times v \right] \cdot \varphi = \int_{S} (x \times T \cdot n) \cdot e_i - 2\mu \int_{\mathcal{L}} D(v) : D(\varphi).$$

Using $(A.1)$ and $(A.2)$, the latter displayed equation is equivalent to the following one:

$$\frac{\partial \tilde{v}}{\partial t} + \omega_1 \times x + \tilde{v} \cdot \nabla \tilde{v} + 2\omega_1 \times \tilde{v}, \varphi) + 2\mu \int_{\mathcal{V}} D(\tilde{v}) : D(\varphi)$$

$$- \lambda (\tilde{\omega} + \tilde{\omega}_1 + \omega_1 \times \omega) \cdot e_i = \int_{S} (x \times T \cdot n) \cdot e_i.$$

By $(41)$, we can then conclude that

$$\lambda (\tilde{\omega} + \tilde{\omega}_1 + \omega_1 \times \omega) \cdot e_i = - \int_{S} (x \times T \cdot n) \cdot e_i,$$

for all $i = 1, 2, 3$, and this proves that also $(24)_{4}$ is satisfied.

The proof of the existence of weak solutions will be accomplished by using the Galerkin method together with a suitable approximation of the liquid velocity in $\mathcal{H}_2(\mathcal{V})$. To this aim, we will prove the existence of a special basis of $\mathcal{H}_2(\mathcal{V})$ and of a special basis of $\mathcal{H}_2^2(\mathcal{V})$. We start by noticing that, taking $(15)$ with $q = 2$, the norm $||\cdot||_{1,2}$ is induced by the following inner product

$$(v, w)_1 = (v, w) + 2\mu \int_{\mathcal{L}} D(v) : D(w) dV,$$  \quad (42)
and the latter makes $H_2^1(V)$ a Hilbert space.

Consider the bilinear form $a : H_2^1(V) \times H_2^1(V) \rightarrow \mathbb{R}$ defined as follows

$$a(v, w) := 2\mu \int_{\Omega} D(v) : D(w).$$

By (15) and (17) with $q = 2$, $a(\cdot, \cdot)$ is a continuous and coercive bilinear form in $H_2^1(V)$. Thus, by Lax-Milgram Theorem, for every $f \in H_2^1(V)$ there exists a unique solution $w \in H_2^1(V)$ to the variational problem

$$a(w, \varphi) = (f, \varphi), \quad \text{for all } \varphi \in H_2^1(V),$$

where the inner product $(\cdot, \cdot)$ has been defined in (14). In other words, $w$ is a generalized solution (with respect to the inner product (14)) to the problem

$$\left\{ \begin{array}{l}
-\frac{1}{\rho} \text{ div } T(\tilde{v}, p) = g \\
\text{ div } \tilde{v} = 0 \\
\tilde{v} = 0 
\end{array} \right\} \quad \text{in } V$$

where $g \in L^2(V)$ is such that $f = P g$. \footnote{We recall that $P$ is the orthogonal projection of $L^2(\Omega)$ onto $H_2(\Omega)$ with respect to the inner product $(\cdot, \cdot)$, defined in (13) (see Section 3).}

With an argument similar to the one that leads to the classical estimates for the Stokes problem (see [12, Theorem IV.6.1]), one can further show that $w \in H_2^1(V)$, and there exists a unique (up to a constant) pressure field $q \in W^{1,2}(V)$ such that equations (45) are satisfied almost everywhere on $V$. Moreover, $(w, q)$ satisfies the following estimates

$$\|w\|_{2,2} + \|q\|_{W^{1,2}(V)} \leq c\|g\|_2,$$

with $c = c(\mu, \rho, \lambda, R, V)$ a positive constant.

Consider the linear operator

$$A : u \in H_2^1(V) \mapsto Au := -\nu P(\Delta u) \in H_2(V),$$

where $\nu := \mu/\rho$ is the liquid coefficient of kinematic viscosity. An integration by parts implies that $a(u, w) = (Au, w)$ for all $u, w \in H_2^1(V)$. Thus, $A$ is a symmetric operator. Moreover, $A$ is invertible and closed. In fact, the inverse is defined by the operator

$$A^{-1} : f \in H_2(V) \mapsto A^{-1}f = \tilde{w} \in H_2^1(V),$$

the unique solution to (44), and $A^{-1}$ is bounded because of (16). Therefore, $A$ and $A^{-1}$ are self-adjoint. In addition, thanks to the estimate (16), we have the following lemma.

**Lemma 5.4.** There exists a positive constant $c$ such that

$$\|w\|_{2,2} \leq c\nu\|P(\Delta w)\|_2 \quad \text{for all } w \in H_2^1(V).$$

Let us consider the following inner product in $H_2^1(V)$

$$(u, w)_2 := (Au, Aw), \quad \text{for all } u, w \in H_2^1(V).$$

By (47), the associated norm is equivalent to $\|\cdot\|_{2,2}$. We are now ready to prove the existence of a special basis.
Theorem 5.5. The spectral problem

\[(u, \varphi)_2 = \lambda (u, \varphi)_B \quad \text{for all } \varphi \in \mathcal{H}_2^2(V)\]  

(49)

admits a denumerable number of positive eigenvalues \(\{\lambda_n\}_{n \in \mathbb{N}}\) clustering at \(+\infty\). The corresponding eigenfunctions \(\{w_n\}_{n \in \mathbb{N}}\) belong to \(\mathcal{H}_2^2(V)\) and form an orthonormal basis in \(\mathcal{H}_2^2(V)\) with respect to the inner product \((\cdot, \cdot)_B\) defined in (35).

Furthermore, \(\{w_n/\sqrt{\lambda_n}\}_{n \in \mathbb{N}}\) forms an orthonormal basis in \(\mathcal{H}_2^2(V)\) with respect to the inner product \((\cdot, \cdot)_2\) defined in (48).

Proof. By Lemma 4.1 and Lax-Milgram Theorem, for every \(f \in \mathcal{H}_2(V)\) there exists a unique solution to the problem

\[(u, \varphi)_2 = (f, \varphi)_B \quad \text{for all } \varphi \in \mathcal{H}_2^2(V).\]  

(50)

Consider the operator \(S_0 : f \in \mathcal{H}_2(V) \mapsto S_0 f := u \in \mathcal{H}_2^2(V)\) the unique solution to (50). By Lemma 3.7, the injection \(J : \mathcal{H}_2^2(V) \to \mathcal{H}_2(V)\) is compact. Thus, the operator \(S := J \circ S_0 : f \in \mathcal{H}_2(V) \mapsto S f \in \mathcal{H}_2(V)\) is also compact. Moreover, \(S\) is symmetric with respect to the inner product \((\cdot, \cdot)_B\) defined in (35). In fact, for every \(f_1, f_2 \in \mathcal{H}_2(V)\), we know that there exist unique \(u_1\) and \(u_2 \in \mathcal{H}_2^2(V)\) solutions to (50) with \(f\) replaced by \(f_1\) and \(f_2\), respectively. So, \(S f_1 = u_1\) and \(S f_2 = u_2\), and

\[(S f_1, f_2)_B = (u_1, f_2)_B = (f_2, u_1)_B = (u_2, u_1)_2 = (u_1, u_2)_2 = (f_1, u_2)_B = (f_1, S f_2)_B.\]

In addition, if \(f_1 = f_2 \equiv f\), then \(u_1 = u_2 \equiv u\) and \((S f, f)_B = (u, u)_2\). Thus, \(S\) is also a positive definite operator. Finally, \(S\) is self-adjoint. To prove the latter, we notice that \(S\) is a compact perturbation of the identity, and \(-1\) is not an eigenvalue of \(S\). Thus, \(\text{Range}(S) = \mathcal{H}_2(V)\) (38, Theorem 1, Section 5, Chapter X]). Since \(\text{Range}(S) = \mathcal{H}_2(V)\) and \(S\) is symmetric, then \(S\) is self-adjoint (38, Corollary to Theorem 1, Section 3, Chapter VII]). By the Hilbert-Schmidt Theorem, \((\mathcal{H}_2(V), (\cdot, \cdot)_B)\) admits an orthonormal basis of eigenfunctions \(\{w_n\}_{n \in \mathbb{N}}\) of \(S\) with corresponding positive eigenvalues \(\{\nu_n\}_{n \in \mathbb{N}}\) converging to 0 as \(n \to \infty\).

Let us denote \(\lambda_n := \nu_n^{-1} > 0\) for every \(n \in \mathbb{N}\). So, \(\{\lambda_n\}_{n \in \mathbb{N}}\) forms a sequence of eigenvalues of the problem (50) clustering at infinity as \(n \to \infty\) and with corresponding eigenfunctions \(\{w_n\}_{n \in \mathbb{N}}\). Indeed, by the definition of \(S\), we find that \(w_n \in \mathcal{H}_2^2(V)\) and

\[\nu_n(w_n, \varphi)_2 = (S w_n, \varphi)_2 = (w_n, \varphi)_B, \quad \text{for every } \varphi \in \mathcal{H}_2^2(V), \quad n \in \mathbb{N}.\]

Finally, \(\{w_n/\sqrt{\lambda_n}\}_{n \in \mathbb{N}}\) forms an orthonormal basis in \(\mathcal{H}_2^2(V)\) with respect to the inner product \((\cdot, \cdot)_2\) defined in (48). To see this, consider \(u \in \mathcal{H}_2^2(V)\) be such that \((w_n, u)_2 = 0\) for every \(n \in \mathbb{N}\). Then,

\[0 = (w_n, u)_2 = (S w_n, u)_2 = (w_n, u)_B\]

for every \(n \in \mathbb{N}\), and this implies that \(u = 0\) since \(\{w_n\}_{n \in \mathbb{N}}\) forms a basis in \(\mathcal{H}_2(V)\) endowed with the inner product \((\cdot, \cdot)_B\). Therefore, \(\{w_n/\sqrt{\lambda_n}\}_{n \in \mathbb{N}}\) is a basis of \(\mathcal{H}_2^2(V)\). Furthermore,

\[
\left(\frac{w_n}{\sqrt{\lambda_n}}, \frac{w_m}{\sqrt{\lambda_m}}\right)_2 = \frac{1}{\sqrt{\lambda_n}} \frac{1}{\sqrt{\lambda_m}} (w_n, w_m)_2 = \frac{\lambda_n}{\sqrt{\lambda_n} \sqrt{\lambda_m}} (S w_n, w_m)_2 = \frac{\lambda_n}{\sqrt{\lambda_n} \sqrt{\lambda_m}} \delta_{nm} \quad \text{for all } n, m \in \mathbb{N}.
\]
We are now in position to prove the following result about the existence of weak solutions to (24).

**Theorem 5.6.** For every \( v_0 \in H(\mathcal{L}), \omega_10, \omega_0 \in \mathbb{R}^3 \) such that \( v_0 = \omega_0 \times x \) on \( \mathcal{S} \). There exists at least one weak solution to (24) such that

1. \( \lim_{t \to 0^+} \| v(t) - v_0 \|_2 = \lim_{t \to 0^+} |\omega_1(t) - \omega_{10}| = \lim_{t \to 0^+} |\omega(t) - \omega_0| = 0 \).
2. The following decays hold

\[
\lim_{t \to +\infty} \| v \|_{L^2(\mathcal{L})} = 0 \quad \text{and} \quad \lim_{t \to +\infty} |\omega(t)| = 0.
\]

(51)

In particular, if \( \lambda_1 = \lambda_2 = \lambda_3 \), then the rate of the previous decays is exponential.

3. Equation (33) holds.

**Proof.** Consider the basis of \( \mathcal{H}_2(\mathcal{V}) \) constructed in Theorem 5.5. We look for “approximate” solutions

\[
\tilde{v}_n(x,t) = \sum_{p=1}^{n} c_{np}(t)w_p(x), \quad \Omega_n(t) = \sum_{i=1}^{3} \hat{c}_{ni}(t)e_i
\]

satisfying (37) with \( \varphi = w_r \), and (37)2. Set

\[
\tilde{v}_0 := \begin{cases} v_0 & \text{in} \mathcal{L}, \\ \omega_0 \times x & \text{in} \mathcal{B}_2. \end{cases}
\]

Then, \( \tilde{v}_0 \in \mathcal{H}_2(\mathcal{V}) \). Moreover, set \( \Omega_0 := \omega_{10} + b(\tilde{v}_0) \in \mathbb{R}^3 \).

Let \( \tilde{v}_{0n} \) denote the projection of \( \tilde{v}_0 \) on the \( \text{span} \{w_1, \ldots, w_n\} \). Replacing (52) in (37), we find that \( (c_{nr}, \hat{c}_{nk})_{r=1,\ldots,n, k=1,2,3} \) satisfy the following system of \((n + 3) \times (n + 3)\) first order initial value problem

\[
\begin{aligned}
\dot{c}_{nr}(t) &+ 2\mu \sum_{p=1}^{n} a_{pr} c_{np}(t) + \sum_{p=1}^{n} \sum_{q=1}^{n} b_{pqr} c_{np}(t) c_{nq}(t) \\
&+ \sum_{i=1}^{3} \sum_{k=1}^{n} d_{iqr} \hat{c}_{ni}(t) c_{nk}(t) + \sum_{i=1}^{3} \sum_{j=1}^{3} f_{ijr} \hat{c}_{ni}(t) \hat{c}_{nj}(t) = 0 && \text{for } r = 1, \ldots, n, \\
\hat{c}_{nr}(0) &\sim (\tilde{v}_{0n}, w_r)_B \\
\ell_k \hat{c}_{nk}(t) &+ \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ijk} \hat{c}_{ni}(t) \hat{c}_{nj}(t) + \sum_{p=1}^{n} \sum_{j=1}^{3} h_{pkj} c_{np}(t) \hat{c}_{nj}(t) = 0 && \text{for } k = 1, 2, 3, \\
\hat{c}_{nk}(0) &\sim \Omega_0 \cdot e_k
\end{aligned}
\]

(53)

where the (constant) coefficients are: \( \ell_k := e_k \cdot I \cdot e_k > 0 \),

\[
\begin{aligned}
a_{pr} &:= 2\mu \int_{\mathcal{V}} D(w_p) : D(w_r) \, dV, \\
b_{pqr} &:= \int_{\mathcal{V}} \tilde{\rho} \left[ w_p \cdot \nabla \tilde{w}_q - 2 \left( I^{-1} \cdot \int_{\mathcal{V}} \tilde{\rho} x \times w_p \, dV \right) \times \tilde{w}_q \right] \cdot w_r \, dV, \\
d_{iqr} &:= 2 \int_{\mathcal{V}} \tilde{\rho} (e_i \cdot w_p) \cdot w_r - e_i \cdot I \cdot \left[ \left( I^{-1} \cdot \int_{\mathcal{V}} \tilde{\rho} x \times w_p \, dV \right) \times \left( I^{-1} \cdot \int_{\mathcal{V}} \tilde{\rho} x \times w_r \, dV \right) \right] \, dV, \\
f_{ijr} &:= e_j \cdot \left[ \left( I^{-1} \cdot \int_{\mathcal{V}} \tilde{\rho} x \times w_r \, dV \right) \times I \cdot e_j \right], \\
g_{ijk} &:= e_k \cdot (e_i \times I \cdot e_j), \\
h_{pkj} &:= -e_k \cdot \left[ \left( I^{-1} \cdot \int_{\mathcal{V}} \tilde{\rho} x \times w_p \, dV \right) \times I \cdot e_j \right].
\end{aligned}
\]
By the classical theory of ordinary differential equations, the initial value problem (53) admits a unique solution \((c_{nr}, \tilde{c}_{nk})_{r=1,\ldots,n,k=1,2,3}\) defined in some interval \([0, T_n)\) with \(T_n > 0\). Actually, \(T_n = +\infty\) for all \(n \in \mathbb{N}\). In fact, the approximate solutions satisfy the following system of equations

\[
\begin{align*}
(d\tilde{v}_n/dt) , w_r)_B + 2\mu \int_V D(\tilde{v}_n) : D(w_r) \, dV + b(w_r) \cdot [\Omega_n + b(\tilde{v}_n) \times I \cdot \Omega_n] \\
+ \int_V \tilde{p}[\tilde{v}_n \cdot \nabla \tilde{v}_n + 2(\Omega_n + b(\tilde{v}_n)) \times \tilde{v}_n] \cdot w_r \, dV = 0, \quad \text{for all } r, n \in \mathbb{N}, \\
I \cdot \tilde{\Omega}_n + [\Omega_n + b(\tilde{v}_n)] \times I \cdot \Omega_n = 0, \quad \text{for all } n \in \mathbb{N},
\end{align*}
\]

and the energy equality

\[
\frac{1}{2} \frac{d}{dt} \left[\mathcal{E}(\tilde{v}_n) + \Omega_n \cdot I \cdot \Omega_n\right] + 2\mu \|D(\tilde{v}_n)\|_{L^2(\mathcal{X})}^2 = 0 \quad \text{in } (0, T_n), \quad \text{for all } n \in \mathbb{N}.
\]

The latter equality is obtained by multiplying (53) by \(c_{nr}\) and summing over \(r = 1, \ldots, n\), by multiplying (53) by \(\tilde{c}_{nk}\) and summing over \(k = 1, 2, 3\), and then adding the resulting equations. Integrating (55) in \([0, t], t < T_n\), and using (50), we find that

\[
c\|\tilde{v}_n(t)\|_2^2 + \Omega_n(t) \cdot I \cdot \Omega_n(t) + 2\mu \int_0^t \|D(\tilde{v}_n)\|_{L^2(\mathcal{X})}^2 \, d\tau \leq \|\tilde{v}_0\|_2^2 + \Omega_0 \cdot I \cdot \Omega_0,
\]

for all \(t \in [0, T_n)\). Since the right-hand side does not depend on \(n\), necessarily \(T_n = +\infty\). Moreover, the sequence \(\{(\tilde{v}_n, \Omega_n)\}_{n \in \mathbb{N}}\) enjoys the following properties.

(a) By (56), \(\{\tilde{v}_n\}_{n \in \mathbb{N}}\) is uniformly bounded in \(L^\infty(0, \infty; \mathcal{H}_2(\mathcal{V}))\).

(b) \(\{\tilde{v}_n\}_{n \in \mathbb{N}}\) is uniformly bounded also in \(L^2(0, \infty; \mathcal{H}_2^1(\mathcal{V}))\) by (56) and (20).

(c) \(\{\Omega_n\}_{n \in \mathbb{N}}\) is uniformly bounded in \(C^0((0, \infty)) \cap C^1(0, \infty)\), by (54) and (50).

(d) \(\{d\tilde{v}_n/dt\}_{n \in \mathbb{N}}\) is uniformly bounded in \(L^2(0, T; (\mathcal{H}_2^2(\mathcal{V})))\) for every \(T > 0\). To show this, let \(\mathbb{P}_n\) be the orthogonal projection of \(\mathcal{H}_2^2(\mathcal{V})\) onto \(\text{span}\{w_1/\sqrt{\lambda_1}, \ldots, w_n/\sqrt{\lambda_n}\}\).

By Theorem 5.5, for every \(w \in \mathcal{H}_2^2(\mathcal{V})\) one has

\[
w = \sum_{\ell=0}^{\infty} (w, w_\ell)_{2} w_\ell \quad \text{and} \quad \|\mathbb{P}_n w\|_{2,2} \leq \|w\|_{2,2}, \quad \text{for all } n \in \mathbb{N}.
\]

For every \(w \in \mathcal{H}_2^2(\mathcal{V})\),

\[
(d\tilde{v}_n/dt, w)_B = (d\tilde{v}_n/dt, \mathbb{P}_n w)_B = -2\mu \int_V D(\tilde{v}_n) : D(\mathbb{P}_n w) \, dV \\
- \int_V \tilde{p}[\tilde{v}_n \cdot \nabla \tilde{v}_n + 2(\Omega_n + b(\tilde{v}_n)) \times \tilde{v}_n] \cdot (\mathbb{P}_n w) \, dV \\
- b(\mathbb{P}_n w) \cdot [(\Omega_n + b(\tilde{v}_n)) \times I \cdot \Omega_n] \quad \text{for all } n \in \mathbb{N}.
\]

We recall the following classical estimates that can be obtained using an integration by parts together with Hölder inequality, (17) and (20). For every \(u_1, u_2 \in \mathcal{H}_2^1(\mathcal{V})\) and \(z \in \mathcal{H}_2^2(\mathcal{V})\) one has

\[
\left| \int_V \tilde{p}(u_1 \cdot \nabla u_2) \cdot z \, dV \right| = \left| \int_V \tilde{p}(u_1 \cdot \nabla z) \cdot u_2 \, dV \right| \\
\leq \|u_1\|_6 \|\nabla z\|_3 \|u_2\|_2 \leq c \|D(u_1)\|_{L^2(\mathcal{X})} \|z\|_{2,2} \|u_2\|_2.
\]
Using again Hölder inequality, and (59), we find that
\[
\left| \frac{d\tilde{\nu}_n}{dt}, w \right|_B = \left| \frac{d\tilde{\nu}_n}{dt}, \mathbb{P}_n w \right|_B \leq c_1 \|D(\tilde{\nu}_n)\|_{L^2(\mathcal{L}_2)} \|w\|_{2,2} + c_2 \|D(\tilde{\nu}_n)\|_{L^2(\mathcal{L}_2)} \|w\|_{2,2} + c_3 \|\tilde{\nu}_n\|_2 \|w\|_{2,2} + c_4 \|\Omega_n\| \|w\|_{2,2}.
\]

Since the previous estimates hold for every \( w \in \mathcal{H}_2(V) \) and \( \mathcal{H}_2(V) \hookrightarrow (\mathcal{H}_2(V))' \), by properties (a), (b) and (c), we can conclude that the sequence \( \{d\tilde{\nu}_n/dt\}_{n\in\mathbb{N}} \) belongs to a bounded set of \( L^2(0,T; (\mathcal{H}_2(V))') \) for every \( T > 0 \).

Properties (a) and (d) imply that the sequence \( \{\tilde{\nu}_n\}_{n\in\mathbb{N}} \) remains in a bounded set of the following space
\[
\{u \in L^2(0,T; \mathcal{H}_2(V)) : du/dt \in L^2(0,T; (\mathcal{H}_2(V))')\}.
\]

Moreover, \( \mathcal{H}_2(V) \hookrightarrow \mathcal{H}_2(V) \hookrightarrow (\mathcal{H}_2(V))' \), with the first embedding being compact (Lemma 3.2). Taking into account all these features and properties (a)-(d), we can claim the existence of functions
\[
\tilde{\nu} \in L^\infty(0,\infty; \mathcal{H}_2(V)) \cap L^2(0,\infty; \mathcal{H}_2^1(V)),
\]
\[
\Omega \in C^0([0,\infty)) \cap C^1(0,\infty),
\]
and subsequences, again denoted by \( \{\tilde{\nu}_n\}_{n\in\mathbb{N}} \) and \( \{\Omega_n\}_{n\in\mathbb{N}} \), such that
\[
\lim_{n \to \infty} \tilde{\nu}_n = \tilde{\nu} \quad \text{weakly-* in } L^\infty(0,\infty; \mathcal{H}_2(V)),
\]
\[
\lim_{n \to \infty} \tilde{\nu}_n = \tilde{\nu} \quad \text{weakly in } L^2(0,\infty; \mathcal{H}_2^1(V)),
\]
\[
\lim_{n \to \infty} \Omega_n = \Omega \quad \text{uniformly in every closed interval } J \subset [0,\infty),
\]
\[
\lim_{n \to \infty} \tilde{\nu}_n = \tilde{\nu} \quad \text{strongly in } L^2(0,T; \mathcal{H}_2(V)) \text{ for every } T > 0.
\]

The latter convergence is a consequence of properties (b) and (d), and of the Aubin-Lions compactness lemma (see [35, Theorem 2.3, Chapter III]).

To conclude the proof of the theorem, we need to show that the couple \((\tilde{\nu}, \Omega)\) satisfies (57). In other words, we need to pass to the limit as \( n \to \infty \) in the following equation obtained from (54), after an integration with respect to time:
\[
(\tilde{\nu}(t), \varphi)_B - (\tilde{\nu}(0), \varphi)_B + 2\mu \int_0^t \int_V D(\tilde{\nu}_n) : D(\varphi) \, dV d\tau
\]
\[
+ \int_0^t \int_V \tilde{\rho}[\tilde{\nu}_n n \cdot \nabla \tilde{\nu}_n + 2(\Omega_n + b(\tilde{\nu}_n)) \times \tilde{\nu}_n] \cdot \varphi \, dV d\tau
\]
\[
+ b(\varphi) \cdot \int_0^t [\Omega_n + b(\tilde{\nu}_n)] \times I \cdot \Omega_n \, d\tau = 0,
\]
\[
I \cdot \Omega_n(t) - I \cdot \Omega_n(0) + \int_0^t [\Omega_n + b(\tilde{\nu}_n)] \times I \cdot \Omega_n \, d\tau = 0, \text{ for all } t \in [0,\infty).
\]

Thanks to (59), the convergence of both linear and nonlinear terms in the above equations follows from standard arguments. We have then shown that, for every \( T > 0 \), the couple \((\tilde{\nu}, \Omega)\) satisfies (57) for every \( \varphi \in \mathcal{H}_2^1(V) \) and all \( t \in [0,T] \). Since \( \mathcal{H}_2^1(V) \) is dense in \( \mathcal{H}_2^1(V) \), (57) is also satisfied for every \( \varphi \in \mathcal{H}_2^1(V) \). Moreover, \( \tilde{\nu} \in C_w([0,T]; \mathcal{H}_2(V)) \) since
It satisfies \( B(\tilde{v}(t), \varphi) < \varepsilon \), for all \( \varphi \in H^1_2(\mathcal{V}) \).

By the density of \( H^1_2(\mathcal{V}) \) in \( H_2(\mathcal{V}) \), the latter properties continues to hold for every \( \varphi \in H_2(\mathcal{V}) \). In addition, taking the limit as \( n \to \infty \) in (56) and using (59) with \( \tilde{v} \in C_w([0,T); H_2(\mathcal{V})) \), we can conclude that \( (\tilde{v}, \Omega) \) satisfies the strong energy inequality (38).

Let us prove properties 1. to 3. in the statement. Let \( \omega_1 := \Omega + b(\tilde{v}) \) and recall that \( \tilde{v} \) has the following representation

\[
\tilde{v} = \begin{cases} 
    v & \text{in } \mathcal{L}, \\
    \omega \times x & \text{in } B_2.
\end{cases}
\]

Then, \( (v, \omega_1, \omega) \) satisfy (38).

Recall (29) and (30), thus property 1. immediately follows from the strong energy inequality (38) and the lower semicontinuity at zero of the map: \( t \mapsto \|v(t)\|_2^2 \).

For what concerns the decays stated in property 2., by (38) and (31), for all \( t \geq s \) and a.a. \( s \geq 0 \) including \( s = 0 \), we find that

\[
\mathcal{E}(\tilde{v}(t)) + C \mu \int_s^t \mathcal{E}(\tilde{v}(\tau)) \, d\tau \leq \mathcal{E}(\tilde{v}(s)) + G(t,s),
\]

where \( G(t,s) := \Omega(t) \cdot I \cdot \Omega(t) - \Omega(s) \cdot I \cdot \Omega(s) \). By (37) with \( s = 0 \) and Hölder inequality, we find that

\[
G(t,s) = 2 \int_s^t \Omega \cdot [b(\tilde{v}) \times I \cdot \Omega] \, d\tau \leq c_1 \int_s^t F(\tau) \, d\tau
\]

where \( c_1 \) is a positive constant (independent of time) and \( F(t) := \|\tilde{v}(t)\|_2 \). Hence, (51) follows by Lemma 1.1. In particular, if \( \lambda_1 = \lambda_2 = \lambda_3 \), then \( \Omega \cdot [b(\tilde{v}) \times I \cdot \Omega] = 0 \), and also the exponential decay follows.

Finally, we obtain (56) from (37)2 by dot-multiplying it by \( I \cdot \Omega \) and recalling that \( \Omega = \omega_1 - b(\tilde{v}) \).

Due to the coupling with the Navier-Stokes equations, also for the problem at hand, it is an open problem whether weak solutions constructed in Theorem 5.6 continuously depend upon the initial data, and are in particular unique. Nevertheless, such property holds for any weak solution possessing a further regularity, as for the classical Navier-Stokes case.

**Theorem 5.7.** Consider two weak solutions \((v, \omega_1, \omega)\) and \((v^*, \omega^*_1, \omega^*)\) to (24) corresponding to initial data \((v_0, \omega_{10}, \omega_0)\) and \((v^*_0, \omega^*_{10}, \omega^*_0)\), respectively. Suppose that there exists a time \( T > 0 \) such that

\[
v^* \in L^p(0,T; L^q(\mathcal{L})), \quad \frac{2}{p} + \frac{3}{q} = 1, \quad \text{for some } q > 3.
\]

Then, the following properties hold.
Lemma 5.8. Consider a weak solution standard, they are similar to the ones provided in [22, Chapter 3].

Lemma 5.9. Let \( \phi \) be a Hilbert space with the inner product \( \langle \cdot, \cdot \rangle \). Then, \( \int_I \phi \cdot \nabla v \, dV \) is a family of mollifiers. Then, the following lemma is an immediate consequence of \([2, Theorem 2.29] \) and \([3, Lemma 1.3.3. \& Remark 1.3.8 (b)]\).

Lemma 5.10. For every \( u, w \in C_w([0, T); L^2(\mathcal{V})) \cap L^2(0, T; L^2(\mathcal{V})) \)

\[
\lim_{h \to 0} \int_0^t \left( (u, \frac{\partial w_h}{\partial t})_B + \left( \frac{\partial u_h}{\partial t}, w \right)_B \right) \, d\tau = (u(t), w(t))_B - (u(0), w(0))_B
\]

\[ t \in [0, T). \]
Lemma 5.11. $D_R(V_T)$ is dense in $L^2(0,T;H^1_0(V))$. In particular, every $\mathbf{w} \in L^2(0,T;H^1_0(V))$ can be approximated in $L^2(0,T;H^1_2(V))$ by the family $\{\mathbf{w}_{n,h} : n \in \mathbb{N}, 0 < h < T\}$ of functions

$$\mathbf{w}_{n,h} := \sum_{k=1}^{n}(\mathbf{w}_h, \Psi_k)_1 \Psi_k,$$

where $\{\Psi_k\}_{k \in \mathbb{N}} \subset D_R(V)$ is a basis of $H_0^1(V)$. Moreover, the following convergences hold:

$$\lim_{n \to \infty} \|\mathbf{w}_{n,h} - \mathbf{w}_h\|_{1,2} = 0 \quad \text{for all } t \in [0,T] \text{ and } h < T,$n

$$\lim_{n \to \infty} \|\mathbf{w}_{n,h} - \mathbf{w}_h\|_{L^2(0,T;H^1(V))} = 0 \quad \text{for all } h < T,$n

$$\lim_{h \to 0} \left( \lim_{n \to \infty} \|\mathbf{w}_{n,h} - \mathbf{w}\|_{L^2(0,T;H^1_0(V))} \right) = 0.$$

We are now in position to prove Theorem 5.17.

Proof of Theorem 5.17. Consider the extensions $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{v}}^*$ of $\mathbf{v}$ and $\mathbf{v}^*$ (together with the corresponding initial conditions), defined in $V$, respectively. Set $\Omega = \omega_1 - b(\tilde{\mathbf{v}})$ and $\Omega^* = \omega_1^* - b(\tilde{\mathbf{v}}^*)$. Let $\{\tilde{\mathbf{v}}_{n,h} : n \in \mathbb{N}, 0 < h < T\}$ and $\{\tilde{\mathbf{v}}_{n,h}^* : n \in \mathbb{N}, 0 < h < T\}$ be the approximating families of $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{v}}^*$ in $L^2(0,T;H^1_0(V))$ given by Lemma 5.11 respectively. For every $n \in \mathbb{N}$ and $h \in (0,T)$, let us replace $\tilde{\mathbf{v}}_{n,h}$ and $\tilde{\mathbf{v}}_{n,h}^*$ in place of $\phi$ in (62) with $s = 0$, for $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{v}}^*$, respectively. The following equations hold:

$$- \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] \leq \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] d\tau + \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] d\tau$$

and

$$- \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] \leq \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] d\tau + \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] d\tau$$

Taking the limit as $n \to \infty$ in the preceding two equations, we find that

$$- \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] \leq \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] d\tau + \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] d\tau$$

and

$$- \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] \leq \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] d\tau + \int_0^t \left[ (\tilde{\mathbf{v}}^*, \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t})_{B} - b\left( \frac{\partial \tilde{\mathbf{v}}^*_{n,h}}{\partial t} \right) \cdot \mathbf{I} \cdot \Omega^* \right] d\tau$$

for $t \in [0,T]$ and $h < T$.
and
\[- \int_0^t \left[ (\tilde{\Omega}^*, \frac{\partial \tilde{\Omega}_h}{\partial t}) - b \left( \frac{\partial \tilde{\Omega}_h}{\partial t} \right) \cdot I \cdot \tilde{\Omega}^* \right] d\tau + (\tilde{\Omega}^*(t), \tilde{\Omega}_h(t))_B - (\tilde{\Omega}^*_0, \tilde{\Omega}_h(0))_B
\]
\[- b(\tilde{\Omega}_h(t)) \cdot I \cdot \tilde{\Omega}^*(t) + b(\tilde{\Omega}_h(0)) \cdot I \cdot \tilde{\Omega}^*_0 + 2\mu \int_0^t \int_V D(\tilde{\Omega}^*) : D(\tilde{\Omega}_h) \, dV \, d\tau \]
\[+ \int_0^t \int_V \rho(\tilde{\Omega}^* \cdot \nabla \tilde{\Omega}^* + 2(\tilde{\Omega}^* + b(\tilde{\Omega}^*)) \times \tilde{\Omega}^*) \cdot \tilde{\Omega}_h \, dV \, d\tau = 0. \tag{65}\]

In the previous limits, the convergence of the linear terms is standard thanks to Lemma \[5.11\]. For what concerns the nonlinear terms, the convergence follows from the following estimates, Lemma \[5.11\] and Lebesgue dominated convergence theorem. For every \( u_1, u_2 \in L^\infty(0,T; H(V)) \cap L^2(0,T; H^1_0(V)) \):
\[
\int_0^t \int_V \tilde{\rho}(u_1 \cdot \nabla u_1) \cdot [(u_2)_{n,h} - (u_2)_h] \, dV \, d\tau \leq \int_0^t \left\| u_1 \right\|_6 \left\| \nabla u_1 \right\|_2 \left\| (u_2)_{n,h} - (u_2)_h \right\|_3 \, d\tau
\]
\[\leq c_1 \int_0^t \left\| \nabla \tilde{u}_1 \right\|^3_2 \left\| (u_2)_{n,h} - (u_2)_h \right\|_{1,2} \, d\tau,\]

by Hölder inequality, \[20\], \[17\] and Sobolev embedding theorem. Moreover, for every \( a \in L^\infty(0,T) \), by Hölder inequality and \[17\]
\[
\int_0^t \int_V 2\tilde{\rho}(a + b(u_1) \times u_1) \cdot [(u_2)_{n,h} - (u_2)_h] \, dV \, d\tau \leq \int_0^t \left\| a \times u_1 \right\|_2 \left\| (u_2)_{n,h} - (u_2)_h \right\|_{1,2} \, d\tau
\]
\[\leq c_2 \int_0^t \left\| (u_2)_{n,h} - (u_2)_h \right\|^2_{1,2} \, d\tau, \tag{66}\]

where \( c_2 \) is a positive constant depending on \( \left\| u_1 \right\|_{L^2(0,T; H^1_0(V))} \) and \( \max_{t \in [0,T]} |a(t)| \).

From \[37\] for \( \Omega \) and \( \Omega^* \), we find that
\[
\int_0^t b \left( \frac{\partial \tilde{\Omega}_h}{\partial t} \right) \cdot I \cdot \tilde{\Omega} \, d\tau - b(\tilde{\Omega}_h(t)) \cdot I \cdot \tilde{\Omega}(t) + b(\tilde{\Omega}_h(0)) \cdot I \cdot \tilde{\Omega}_0 = \int_0^t b(\tilde{\Omega}_h^*) \cdot [(\Omega + b(\tilde{\Omega})) \times I \cdot \tilde{\Omega}] \, d\tau
\]
and
\[
\int_0^t b \left( \frac{\partial \tilde{\Omega}_h}{\partial t} \right) \cdot I \cdot \tilde{\Omega}^* \, d\tau - b(\tilde{\Omega}_h(t)) \cdot I \cdot \tilde{\Omega}^*(t) + b(\tilde{\Omega}_h(0)) \cdot I \cdot \tilde{\Omega}_0^* = \int_0^t b(\tilde{\Omega}_h) \cdot [(\Omega^* + b(\tilde{\Omega}^*)) \times I \cdot \tilde{\Omega}^*] \, d\tau.
\]
Hence, adding (64) and (65), we find that

\[-\int_0^t \left[ \left( \tilde{v}, \frac{\partial \tilde{v}_h^*}{\partial t} \right)_B + \left( \tilde{v}^*, \frac{\partial \tilde{v}_h^*}{\partial t} \right)_B \right] \, dt + \left( \tilde{v}(t), \tilde{v}_h^*(t) \right)_B - \left( \tilde{v}_0, \tilde{v}_h^*(0) \right)_B \]
\[+ \left( \tilde{v}^*(t), \tilde{v}_h(t) \right)_B - \left( \tilde{v}_0^*, \tilde{v}_h(0) \right)_B \]
\[+ \int_0^t b(\tilde{v}_h^*) \cdot [\Omega + b(\tilde{v})] \cdot I \cdot \Omega \, dt + \int_0^t b(\tilde{v}_h) \cdot [(\Omega^* + b(\tilde{v})) \times I \cdot \Omega^*] \, dt \]
\[+ 2\mu \int_0^t \int_{\mathcal{V}} \left[ D(\tilde{v}) : D(\tilde{v}_h^*) + D(\tilde{v}^*) : D(\tilde{v}_h^*) \right] \, dV \, dt \]
\[+ \int_0^t \int_{\mathcal{V}} \rho|\tilde{v} - \nabla \tilde{v} + 2(\Omega + b(\tilde{v})) \times \tilde{v}| \cdot \tilde{v}_h^* \, dV \, dt \]
\[+ \int_0^t \int_{\mathcal{V}} \rho|\tilde{v}^* - \nabla \tilde{v}^* + 2(\Omega^* + b(\tilde{v})) \times \tilde{v}^*| \cdot \tilde{v}_h \, dV \, dt = 0. \tag{67} \]

Next, we take the limit as $h \to 0$ in (67). Again, the convergence of the linear terms follows easily thanks to (63) and Lemma 5.9. For what concerns the nonlinear terms, we use (69) and the following classical inequality

\[ \left| \int_0^T \int_{\mathcal{V}} \rho(u_1 \cdot \nabla u_2) \cdot u_3 \, dV \, dt \right| \]
\[\leq c \left( \int_0^T \|\nabla u_1\|^2 \, dt \right)^{3/2q} \left( \int_0^T \|\nabla u_2\|^2 \, dt \right)^{1/2} \left( \int_0^T \|u_3\|^p \|u_1\|^2 \, dt \right)^{1/p} \tag{68} \]

which holds for every $u_1, u_2 \in L^\infty(0, T; \mathcal{H}(\mathcal{V})) \cap L^2(0, T; \mathcal{H}_2^1(\mathcal{V}))$ and $u_3 \in L^p(0, T; L^q(\mathcal{V}))$ with $p$ and $q$ satisfying (61) (see [34, Lemma 1.]). Moreover, from (67), we find that

\[ \Omega^*(t) \cdot I \cdot \Omega(t) - \Omega_0^* \cdot I \cdot \Omega_0 = -\int_0^t \Omega^* \cdot [(\Omega + b(\tilde{v})) \times I \cdot \Omega] \, dt - \int_0^t \Omega \cdot [(\Omega^* + b(\tilde{v}^*)) \times I \cdot \Omega^*] \, dt. \]

Hence, the couples $(\tilde{v}, \Omega)$ and $(\tilde{v}^*, \Omega^*)$ satisfy the following equality

\[ (\tilde{v}(t), \tilde{v}^*(t))_B - (\tilde{v}_0, \tilde{v}_0^*)_B + \int_0^t \left[ \Omega^* + b(\tilde{v}_h)^* \right] \cdot [(\Omega^* + b(\tilde{v})) \times I \cdot (\Omega - \Omega^*)] \, dt \]
\[+ \Omega^*(t) \cdot I \cdot \Omega(t) - \Omega_0^* \cdot I \cdot \Omega_0 + 4\mu \int_0^t \int_{\mathcal{V}} D(\tilde{v}) : D(\tilde{v}^*) \, dV \, dt \]
\[+ \int_0^t \int_{\mathcal{V}} \rho|\tilde{v} - \tilde{v}^*| \cdot \nabla \tilde{v} + 2(\Omega - \Omega^* + b(\tilde{v} - \tilde{v}^*)) \times \tilde{v} \cdot \tilde{v}_h^* \, dV \, dt = 0. \tag{69} \]

We recall that, by Definition 5.1, $(\tilde{v}, \Omega)$ and $(\tilde{v}^*, \Omega^*)$ satisfy the strong energy inequality (68) for all $t \in [0, T]$:

\[ E(\tilde{v}(t)) + \Omega(t) \cdot I \cdot \Omega(t) + 4\mu \int_s^t \| D(\tilde{v}(\tau)) \|^2_{L^2(\mathcal{V})} \, d\tau \leq E(\tilde{v}_0) + \Omega_0 \cdot I \cdot \Omega_0, \tag{70} \]

and

\[ E(\tilde{v}^*(t)) + \Omega^*(t) \cdot I \cdot \Omega^*(t) + 4\mu \int_s^t \| D(\tilde{v}^*(\tau)) \|^2_{L^2(\mathcal{V})} \, d\tau \leq E(\tilde{v}_0^*) + \Omega_0^* \cdot I \cdot \Omega_0^*. \tag{71} \]
Adding (70) and (71), and subtracting twice of (69), we find that the fields $w := \tilde{v} - \tilde{v}^*$ and $\xi := \Omega - \Omega^*$ must satisfy the following inequality

\[
E(w(t)) + \xi(t) \cdot I \cdot \xi(t) + 4\mu \int_s^t \|D(w(\tau))\|_{L^2(\mathcal{V})}^2 \, d\tau \\
\leq E(w_0) + \xi_0 \cdot I \cdot \xi_0 - 2 \int_0^t [\xi + b(\omega)] \cdot [(\Omega^* + b(\xi)) \times I \cdot \xi] \, d\tau \\
+ 2 \int_0^t \rho[w \cdot \nabla w + 2(\xi + b(\omega)) \times w] \cdot \tilde{v}^* \, dV \, d\tau,
\]

(72)

where $w_0 := \tilde{v}_0 - \tilde{v}_0^*$ and $\xi_0 := \Omega_0 - \Omega_0^*$. By Hölder inequality, (68) and Young’s inequality, we get the following estimates

\[
E(w(t)) + \xi(t) \cdot I \cdot \xi(t) + 2\mu \int_s^t \|D(w(\tau))\|_{L^2(\mathcal{V})}^2 \, d\tau \leq E(w_0) + \xi_0 \cdot I \cdot \xi_0 \\
+ c_3 \int_0^t \left[\|\tilde{v}^*(\tau)\|_{L_q(\mathcal{V})}^p + \|w(\tau)\|_{L^2(\mathcal{V})} + |\xi(\tau)|\|E(w(\tau)) + \xi(\tau) \cdot I \cdot \xi(\tau)\| \right] \, d\tau.
\]

Recalling (28) and (29) and using Gronwall’s Lemma together with (30), properties (a) and (b) of Theorem 5.7 immediately follow. \(\square\)

6. Existence of strong solution

In this section, we will prove the local in time existence and continuous dependence upon initial data of (local in time) strong solutions to (24) for a considerably “large” class of initial conditions. The approach is the one of maximal $\mathcal{X}$ upon initial data of (local in time) strong solutions to (24) for a considerably “large” class of initial conditions. The previous spaces are endowed with the norms

\[
\|u\|_{X_\mu} := \sqrt{\|\tilde{v}\|_{L^2(\mathcal{V})}^2 + |\omega_1|^2}, \quad \|u\|_{X_\alpha} := \sqrt{\|\tilde{v}\|_{W^{2,q}(\mathcal{V})}^2 + |\omega_1|^2}
\]

and similarly for the interpolation spaces. We recall the following characterization of Besov spaces $B_{qs}^s(\mathcal{V}) = (H_{qs_0}^s(\mathcal{V}), H_{qs_1}^s(\mathcal{V}))_{\theta,p}$ as real interpolation of Bessel potential spaces, and $H_{qs}^s(\mathcal{V}) = [H_{qs_0}^s(\mathcal{V}), H_{qs_1}^s(\mathcal{V})]_\theta$, with $[\cdot,\cdot]_\theta$ the complex interpolation, valid for $s_0 \neq s_1 \in \mathbb{R}$, $p, q \in [1, \infty)$, $\theta \in (0, 1)$, and where $s = (1-\theta)s_0 + \theta s_1$. We also recall that $B_{qq}^s(\mathcal{V}) = W_0^s(\mathcal{V})$ and $B_{22}^s(\mathcal{V}) = W_2^s(\mathcal{V}) = H_2^s(\mathcal{V})$. 25
Before stating our main result about existence and related properties of strong solutions to \([24]\), we need some preliminary observations. Let us consider the initial boundary value problem which describes the motion of a rigid body having a cavity \(\mathcal{V}\) completely filled by a viscous liquid with a varying density \(\tilde{\rho}\) defined in \([13]\).

\[
\begin{aligned}
&\frac{\partial \tilde{v}}{\partial t} + \tilde{\omega}_1 \times \tilde{x} + \tilde{v} \cdot \nabla \tilde{v} + 2\tilde{\omega}_1 \times \tilde{v} = \frac{\mu}{\tilde{\rho}} \Delta \tilde{v} - \frac{1}{\tilde{\rho}} \nabla \pi \\
&\text{div} \tilde{v} = 0 \\
&\tilde{\omega}_1 - b \left( \frac{\partial \tilde{v}}{\partial t} \right) + I^{-1} \cdot [\tilde{\omega}_1 \times I \cdot (\tilde{\omega}_1 - b(\tilde{v}))] = 0 \\
&\tilde{v}|_{t=0} = \tilde{v}_0, \quad \tilde{\omega}_1(0) = \omega_{10}
\end{aligned}
\]  \hspace{1cm} (74)

Assume that for some initial data \((\tilde{v}_0, \omega_{10})\) satisfying the condition

\[
\tilde{v}_0 = \begin{cases} v_0 & \text{on } \mathcal{L}, \\ \omega_0 \times x & \text{on } B_2, \end{cases}
\]

with \(v_0 = \omega_0 \times x\) on \(S\),

\hspace{1cm} (75)

there exists \((\tilde{v}, \omega_1)\) a strong solution to \((74)\) in the class \(E_{1,\mu}(0, T)\) with \(\mu = 1\), defined in \([78]\) below. Then there exist \(v \in H^1_p(0, t_1; H^2_q(\mathcal{L}')) \cap L^p(0, t_1; H^3_q(\mathcal{L}'))\) and \(\omega \in C^1((0, T]; \mathbb{R}^3)\) such that \(v = 0\) on \(\mathcal{C}\), \(v = \omega \times x\) on \(S\), and

\[
\tilde{v} = \begin{cases} v & \text{on } \mathcal{L}, \\ \omega \times x & \text{on } B_2. \end{cases}
\]

Using a duality argument (generalizing that in Remark 5.3), one can find that the triple \((v, \omega_1, \omega)\) is a strong solution to \([24]\). Moreover, \((v, \omega_1, \omega)\) satisfies the initial conditions thanks to \((75)\). Therefore, the goal of this section is to investigate the existence and related properties of strong solutions to \((74)\).

In the following we set

\[
0B^s_{qp,\sigma}(\mathcal{V}) := \begin{cases} u \in B^s_{qp}(\mathcal{V}) \cap H_q(\mathcal{V}) : u = 0 \text{ on } \mathcal{C}, & s > 1/q, \\ B^s_{qp}(\mathcal{V}) \cap H_q(\mathcal{V}), & s \in (0, 1/q). \end{cases}
\]

In view of the previous observations, next theorem turns out to be the main result of this section.

**Theorem 6.1.** Suppose

\[
p \in (1, \infty), \quad q \in (1, 3), \quad 2/p + 3/q \leq 3,
\]

and let (the time-weight) \(\mu\) satisfy

\[
\mu \in (1/p, 1], \quad \mu \geq \mu_{\text{crit}} = \frac{1}{p} + \frac{3}{2q} - \frac{1}{2}.
\]

**a)** Let \(u_0 = (\tilde{v}_0, \omega_{10}) \in 0B^{2-2/p}_{qp,\sigma}(\mathcal{V}) \times \mathbb{R}^3 = X_{\gamma, \mu}\) be given such that \((75)\) is satisfied. Then there are positive constants \(T = T(u_0)\) and \(\eta = \eta(u_0)\) such that \((74)\) admits a unique solution \(u(\cdot, u_0) = (\tilde{v}, \omega_1)\) in

\[
E_{1,\mu}(0, T) = H^1_{p,\mu}((0, T); X_0) \cap L^p_{\mu}((0, T); X_1).
\]

\hspace{1cm} (78)
(b) Suppose $p_j, q_j, \mu_j$ satisfy $([6], [11])$ and, in addition, $p_1 \leq p_2, q_1 \leq q_2$ as well as
\[ \mu_1 - \frac{3}{2q_1} \geq \mu_2 - \frac{3}{2q_2}. \]  
Then for each initial value $(\tilde{v}_0, \omega_1) \in 0P^{2u_{1\cdot 2/p_1}} \times R^3$ satisfying $[73]$, problem $[74]$ admits a unique solution $(\tilde{v}, \omega_1)$ in the class
\[ H^1_{p_1, \mu_1}((0, T); \mathcal{H}_{q_1}(V) \times R^3) \cap L^p_{p_2, \mu_2}((0, T); D(A_{q_2}) \times R^3) \]
\[ \cap H^1_{p_2, \mu_2}((0, T); \mathcal{H}_{q_2}(V) \times R^3) \cap L^p_{p_2, \mu_2}((0, T); D(A_{q_2}) \times R^3). \]
(c) Each solution exists on a maximal interval $[0, t_+) = [0, t_+(u_0))$, and enjoys the additional regularity property
\[ \tilde{v} \in C([0, t_+); 0B_{2q_2, \mu_1}^{2u_{2/p}}(V)) \cap C((0, t_+); 0B_{2q_1, \mu_1}^{2u_{2/p}}(V)), \quad \omega_1 \in C^1((0, t_+), R^3). \]
(d) The solution $u = (\tilde{v}, \omega_1)$ exists globally if $u([0, t_+)) \subset 0B_{2q_2, \mu_1}^{2u_{2/p}}(V) \times R^3$ is relatively compact.

Proof. The problem $[74]$ can be reformulated as a semilinear evolution equation for $u = (\tilde{v}, \omega_1):
E \cdot \frac{du}{dt} + Au = G(u, u), \quad u(0) = u_0,
where,
\[ E : (w, \xi) \in X_0 \mapsto E(w, \xi) := \begin{bmatrix} w + P_q(\xi \times x) \\ \xi - b(w) \end{bmatrix} \in X_0, \]
\[ A := \begin{bmatrix} A_q & 0 \\ 0 & 0 \end{bmatrix} : X_1 \to X_0, \quad A_q \text{ defined in } [73], \]
\[ G(u, u) := \begin{bmatrix} P_q(-v \cdot \nabla v - 2\omega_1 \times \tilde{v}) \\ -I^{-1} \cdot [\omega_1 \times I \cdot (\omega_1 - b(\tilde{v}))] \end{bmatrix}, \]
and the functional $b(\cdot)$ has been introduced in $[26]$. The operator $E$ is linear, bounded, invertible, and has a bounded inverse. The linearity and boundedness of $E$ is obvious from its definition. For what concerns its invertibility, we observe that $E = 1 + K$ with
\[ K := \begin{bmatrix} 0 & P_q(\cdot \times x) \\ -b(\cdot) & 0 \end{bmatrix} \]
a bounded operator with a finite dimensional range (see $[26]$). The null space of $K$ is given by \{$(e_i, P_q(e_i \times x))$ : $i = 1, 2, 3$\}). Thus, $K$ is a compact operator, and $E$ is a Fredholm operator of index zero (by $[21]$, Theorem 5.26, page 238). The invertibility of $E$ then follows if we prove that its null space reduces to $N(E) = \{0\}$. The latter immediately follows from Lemma 4.1 (actually, from its proof). In fact, $E$ is one-to-one when $q = 2$. In addition to the previous properties of $E$, we can also infer that $E^{-1} \equiv 1 + C$, where
\[ C := -K \cdot E^{-1} : X_0 \to R(V) \cap H_q(V) \times R^3 \]
is a bounded operator with a finite dimensional range, and then compact.

Let us consider the linear operator $L := E^{-1} \cdot A$ with domain $X_1$, and observe that
\[ L = (1 + C) \cdot A = \begin{bmatrix} A_q & 0 \\ 0 & 0 \end{bmatrix} + C \begin{bmatrix} A_q & 0 \\ 0 & 0 \end{bmatrix}, \]
(82)
and let us denote \( N(u) := E^{-1}G(u, u) \). Then, equation (80) (and thus (74)) can be equivalently rewritten as
\[
\frac{du}{dt} + Lu = N(u), \quad u(0) = u_0. \tag{83}
\]

Note that \( L \) has maximal \( L^p \)-regularity. In fact, the linearization of (74) reads as follows
\[
\begin{aligned}
\partial_t \tilde{v} + b \left( \frac{\partial \tilde{v}}{\partial t} \right) \times x &= \mu \Delta \tilde{v} - \frac{1}{\rho} \nabla \pi \\
\text{div} \tilde{v} &= 0 \\
\dot{\omega}_1 &= b \left( \frac{\partial \tilde{v}}{\partial t} \right) \quad \text{in } (0, \infty), \\
\tilde{v} &= 0 \quad \text{on } C, \\
\tilde{v}|_{t=0} &= \tilde{v}_0, \\
\omega_1(0) &= \omega_{10}. 
\end{aligned} \tag{84}
\]

The first two equations decouple from the third one. In other words, if we find a solution \( \tilde{v} \) to
\[
\begin{aligned}
\partial_t \tilde{v} + b \left( \frac{\partial \tilde{v}}{\partial t} \right) \times x &= \mu \Delta \tilde{v} - \frac{1}{\rho} \nabla \pi \\
\text{div} \tilde{v} &= 0 \\
\tilde{v} &= 0 \quad \text{on } C, \\
\tilde{v}|_{t=0} &= \tilde{v}_0,
\end{aligned} \tag{85}
\]
then the solution to (83)3 will be given by
\[
\omega_1(t) = \omega_{10} + b(\tilde{v}(t)) - b(\tilde{v}_0).
\]

Also (85) can be rewritten in a compact form as
\[
\frac{d\tilde{v}}{dt} + L_q \tilde{v} = 0, \quad u(0) = u_0, \tag{86}
\]
where the linear operator is defined by \( L_q := E_q^{-1}A_q \) with domain \( D(L) \equiv D(A_q) \), and for every \( u \in \mathcal{H}_q(V) \)
\[
E_q u := u + P_q (b(u) \times x) = u + P_q \left( x \times I^{-1} \cdot \int_V \hat{\rho} x \times u \, dV \right) \in \mathcal{H}_q(V). \tag{87}
\]

With an argument similar to the one done at the beginning of this proof (see also [24, 25]), it can be shown that the operator \( L_q : D(A_q) \to \mathcal{H}_q(V) \) is a compact perturbation of the operator \( A_q \) which is the Stokes operator with varying viscosity, and the latter has the property of \( L^p \)-maximal regularity as shown in [1] (see also [7]). In particular, \( A_q \in \mathcal{BIP}(X_0) \).

The statements in (a), (c) and (d) follow from [27, Theorem 1.2] provided that assumptions (S), (H1), (H2) and (H3) are satisfied. Hypotheses (H1), (H2) and (H3) can be verified in a similar fashion as in the proof of [24, Theorem 3.4]. It remains to show that the structural assumption (S) is also satisfied. By [29, Remark 1.1], it is enough to show that there exists an operator \( A_\# \in \mathcal{H}_\infty(X_0) \) with domain \( D(A_\#) = X_1 \) and \( \mathcal{H}_\infty \)-angle \( \phi_{A_\#}^\infty < \pi/2 \). As
our Stokes operator $A_q$ (defined in (63)) is just a particular case of the Stokes operator with variable viscosity studied in [1, Theorem 2], the proof of the latter can be adapted to show that $A_q \in \mathcal{H}^\infty(\mathcal{H}_q(V))$ with $\mathcal{H}^\infty$-angle $\phi_{A_q} = 0$. It follows that $A_q$ is sectorial with angle $\phi_{A_q} = 0$, while $CA_q$ is a relative compact perturbation. By [28, Lemma 3.1.7 and Corollary 3.1.6], it follows that there exists $\eta_0 > 0$ such that $\eta_0 + A_q + CA_q$ is invertible and sectorial with spectral angle less than $\pi/2$. Since $CA_q$ is a linear operator from $D(A_q)$ to $C^1(V)$, then there exists $s \in (0, 1/q)$ such that $CA_q : D(A_q) \to D(A_q)^s \equiv [\mathcal{H}_q(V), D(A_q)]_{s,\rho}$ is bounded. Hence, $\eta_0 + A_q + CA_q \in \mathcal{H}^\infty(\mathcal{H}_q(V))$ with $\mathcal{H}^\infty$-angle less than $\pi/2$ by [28, Corollary 3.3.15], and

\[
A_\# := \eta_0 + L = \begin{bmatrix} \eta_0 + A_q + CA_q & 0 \\ 0 & \eta_0 \end{bmatrix} \in \mathcal{H}^\infty(X_0), \text{ with } \phi_{A_\#} < \pi/2.
\]

The proof of part (b) follows from the fact that, under the stated hypotheses,

\[
B_{2\mu_1,2/\mu_1}^2(V) \times \mathbb{R}^3 \hookrightarrow B_{2\mu_2,2/\mu_2}^{2/\mu_2}(V) \times \mathbb{R}^3
\]

and for each fixed $j = 1, 2$, solutions $u_j \equiv (\tilde{v}_j, \omega_{1,j})$ to (83) in the class

\[
E_1(0, T) := H^1_{p_j,\mu_1}((0, T); \mathcal{H}_{q_j}(V) \times \mathbb{R}^3) \cap L^2_{p_j,\mu_1}((0, T); D(A_{q_j}) \times \mathbb{R}^3)
\]

are fixed points of the strict contraction

\[
T : \mathcal{M}_j \to \mathcal{M}_j, \quad Tu := e^{-tL}u_0 + e^{-tL}N(u),
\]

where $\mathcal{M}_j$ is a closed subset of $E_1(0, T)$. Since also $T : \mathcal{M}_1 \cap \mathcal{M}_2 \to \mathcal{M}_1 \cap \mathcal{M}_2$ is a strict contraction, then it admits a unique fixed point which is the unique solution $(\tilde{v}, \omega_1)$ in the class

\[
H^1_{p_1,\mu_1}((0, T); \mathcal{H}_{q_1}(V) \times \mathbb{R}^3) \cap L^2_{p_1,\mu_1}((0, T); D(A_{q_1}) \times \mathbb{R}^3)
\]

\[
\cap H^1_{p_2,\mu_2}((0, T); \mathcal{H}_{q_2}(V) \times \mathbb{R}^3) \cap L^2_{p_2,\mu_2}((0, T); D(A_{q_2}) \times \mathbb{R}^3).
\]

\[\square\]

**Remark 6.2.**

(a) In the case $p_1 = q_1 = 2$, we obtain $\mu_{\text{crit}} = 3/4$ and we find the largest space of initial data $X_{\text{crit}},$

\[
X_{\text{crit}} := (H(V) \times \mathbb{R}^3, H^2_0(V) \times \mathbb{R}^3)_{1/4,2} \subset H^{1/2}(V) \times \mathbb{R}^3,
\]

corresponding to which, there exists a unique solution to (83) in the class

\[
H^1_{2,3/4}((0, T); \mathcal{H}_1(V) \times \mathbb{R}^3) \cap L^2_{3/4}((0, T); 0H^2_{2,\sigma}(\Omega) \times \mathbb{R}^3)
\]

\[
\cap H_1^{p,\mu}((0, T); L^p_{q,\sigma}(\Omega) \times \mathbb{R}^3) \cap L^p_{p_2,\mu_2}((0, T); 0H^2_{q,\sigma}(\Omega) \times \mathbb{R}^3),
\]

for any $p \geq 2$, $q \in [2, 3]$, with $\mu = 1/p + 3/2q - 1/2$. In particular, we can conclude that $v \in C((0, t_+); B^2_{ap}((\tilde{v}, \omega_1)))$ for any $p \geq 2, q \in [2, 3]$.

(b) Theorem 6.1(b) asserts that problem (83) admits for each initial value $(\tilde{v}_0, \omega_{10}) \in \mathcal{H}_1^1(V) \times \mathbb{R}^3$ a unique solution in the class

\[
H^1((0, T); \mathcal{H}_1(V) \times \mathbb{R}^3) \cap L^2((0, T); \mathcal{H}_0^2(V) \times \mathbb{R}^3)
\]

\[
\cap H_1^{p,\mu}((0, T); \mathcal{H}_q(V) \times \mathbb{R}^3) \cap L^p_{p_2,\mu_2}((0, T); \mathcal{H}_q^2(V) \times \mathbb{R}^3),
\]

for any $p \geq 2$, $q \in [2, 3]$, with $\mu = 1/p + 3/2q - 1/4$. In particular, we can conclude that $v \in C((0, t_+); B^2_{ap}((\tilde{v}, \omega_1)))$ for any $p \geq 2, q \in [2, 3]$. 

29
Appendix A. Some useful integral equalities

We will recall some elementary integral equalities that have been widely used in the paper.

Let $B_R$ the open ball in $\mathbb{R}^3$ with radius $R$, centered at the origin of a coordinate system $\{O; e_1, e_2, e_3\}$. The following equalities hold:

1. $\int_{B_R} x \times (\omega \times x) = \frac{8\pi R^5}{15} \omega$ for all $\omega \in \mathbb{R}^3$. (A.1)

2. $\int_{B_R} (\omega \times x) \times (\xi \times x) = \frac{4\pi R^5}{15} \omega \times \xi$ for all $\omega, \xi \in \mathbb{R}^3$. (A.2)

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