The geometry of $\mathcal{W}_3$ algebra: a twofold way for the rebirth of a symmetry

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Abstract

The purpose of this note is to show that $\mathcal{W}_3$ algebras originate from an unusual interplay between the breakings of the reparametrization invariance under the diffeomorphism action on the cotangent bundle of a Riemann surface. It is recalled how a set of smooth changes of local complex coordinates on the base space are collectively related to a background within a symplectic framework. The power of the method allows to calculate explicitly some primary fields whose OPEs generate the algebra as explicit functions in the coordinates: this is achieved only if well defined conditions are satisfied, and new symmetries emerge from the construction. Moreover, when primary fields are introduced outside of a coordinate description the $\mathcal{W}_3$ symmetry byproducts acquire a good geometrical definition with respect to holomorphic changes of charts.

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1 Introduction

It is hard to decide if it is more relevant, from the physical point of view, either the birth or the decease of a symmetry.

The origin of a symmetry clarifies its deep purest form of manifestation and discloses its harmonies, while the symmetry breaking mechanisms, even if they display the physical world faults, shed some new light on the dynamical realization of the symmetry. Indeed the breaking consistency conditions [1] prevent a wild discharge of the symmetry constraints. Moreover if we are in presence of several symmetry violation mechanisms, a mutual liking of them could generate a fascinating guide-line leading to a new symmetry.

The purpose of this note is to show that one of the possible origins of $W_3$ [2, 3, 4, 5, 6, 7, 8, 9] algebras comes from an unusual interplay between diffeomorphism symmetry breakings between the base space (with local complex coordinates $(z, \bar{z})$) and a well defined chain of spaces (with local complex coordinates $(Z^{(n)}(z, \bar{z}), \bar{Z}^{(n)}(z, \bar{z}))$) whose construction preserves the symplectic diffeomorphism invariance.

Indeed the $W$ algebras saga [10] comes from many sources: they were originally discovered within an OPE construction by Zamolodchikov [11, 12] as a natural extension of the Virasoro algebra; later on they were derived by Drinfeld and Sokolov [13] through a reduction procedure, which is in fact (classically) simply a Poisson reduction in infinitely many dimensions, taking as Poisson structure the Kirillov-Poisson structure naturally associated to any Lie algebra [14, 15].

The same algebras were derived within the Korteweg de Vries hierarchy of equations an approach which generalized Toda system formalism [16] giving fundamental insights in the field of the 2D matter physics [17], integrable models [18], topological 2D field theory [19], 2D conformal field theory [20], 2D quantum gravity [21] as well as matrix models [15].

So, having a lot of possible sources, one may suppose that a common general trend could link the widespread variety of these physical and mathematical fields of research.

The two dimensional space exclusive feature is that it admits an infinite dimensional diffeomorphism group of transformations [22, 23] for which higher spin extensions of the group of representations are allowed.

This fact was already sensed within the $W$ field of interest from the very beginning: indeed in the instance of $W$ chiral sector, the transformation laws of the BRS ghosts, $C^{(r)}(z, \bar{z})$, was found to be

$$SC^{(r)}(z, \bar{z}) = \sum_s s C^{(s)}(z, \bar{z}) \partial C^{(r-s+1)}(z, \bar{z}) + \text{(stuff)}. \quad (1.1)$$

This gives a slight indication that this symmetry is involved with coordinate transformation laws (the so-called $W$ diffeomorphisms) [21, 24, 25, 26, 27, 28].

This ansatz addresses both the questions to which the $(Z^{(r)}(z, \bar{z}), \bar{Z}^{(r)}(z, \bar{z}))$ spaces are locally diffeomorphic to the $(z, \bar{z})$ base space and what could be the general construction law which originates them.

Already Witten [29] pointed out, with argument of algebraic topology, that the use of Poisson brackets induces a kind of symplectic geometry, and gave the conjecture that $W$ algebras could be related to symplectomorphisms. In a recent paper [30] we have taken inspiration from this idea in an attempt to provide an alternative approach to these algebras.

In our scheme a sequence of local complex coordinates $(Z^{(n)}(z, \bar{z}), \bar{Z}^{(n)}(z, \bar{z}))$ will be found. The latter are built up in such a way to preserve a symplectic form, and turn out to be the
images under canonical diffeomorphisms of the \((z, \bar{z})\) base. Under the diffeomorphism action on the cotangent bundle, these coordinate transformations give rise to BRS variations similar to that of Eq (1.1) and introduce an alternative attempt to a \(\mathcal{W}\) algebra realization. For example, some questions will find an easy answer: first of all a Lagrangian formalism can be naturally embedded and the role of complex structure [24, 31] (along the lines of [22]) can be explained [32]. However not all the \(\mathcal{W}\) algebras that can be found in the literature are naturally explained in our formalism, but the \(W_3\) algebras in Refs [2] and [3] acquire a particular role in our approach: on the one hand, it does not maintain the reparametrization invariance previously introduced, but on the other hand, the breaking mechanisms are fully under control in our treatment by imposing a reparametrization invariance in a \((z, \bar{z}, \theta)\) space, where \(\theta\) is a constant Grassmann variable which allows to manage in an algebraic way the symmetry breakings with respect to the \((z, \bar{z})\) background.

This problem is the subject of the paper, and it will be shown that in the most general case the there are two distinct mechanisms for the symmetry breakings, whose accordance produces a liking mechanism which generates a symmetry criterion.

On the other side, in the particular case of the \(W_3\) of a surviving reparametrization invariance maintains its validity.

As a byproduct of our approach we shall try to use the symmetry as a firepower to construct primary fields (whose OPE gave historically the origin to \(\mathcal{W}\) algebras) as explicit functions of the \((Z^{(n)}(z, \bar{z}), \bar{Z}^{(n)}(z, \bar{z}))\) coordinates. This construction will be possible, and in that case an explicit reparametrization invariance will survive under the diffeomorphism action as in the situation of [3].

Otherwise primary fields have to be introduced as independent fields depending on the background \((z, \bar{z})\) space. This point of view enhances for a good geometrical definition of the theory, in particular a well defined anomaly under holomorphic changes of charts will be found: this feature in not naively shown in the first approach.

Our paper is organized as follows:

In Section 2 our approach to \(\mathcal{W}\) algebras from symplectomorphisms is reviewed and the \(W_3\) algebras are derived from a general construction.

In Section 3 the \(W_3\) algebra is studied as a breaking of the symmetry under reparametrizations on the base space in a well defined geometric scenario. In particular, we discuss the different aspects which emerge if an explicit coordinate description of the algebra is required.

## 2 \(\mathcal{W}\) algebras from symplectomorphisms: the chiral \(\mathcal{W}_{n}\)-gravity

In this Section we sum up our approach to \(\mathcal{W}\) algebras in term of symplectomorphisms we have performed in Refs [20, 22] to which we refer for more details.

The canonical 1-form \(\Theta\) on the cotangent bundle \(T^*\) writes in a local chart frame \(U_{(z,y)}\)

\[
\Theta|_{U_{(z,y)}} = \left[ y_z dz + \bar{y}_{\bar{z}} d\bar{z} \right]
\]

(2.1)

In a frame \(U_{(Z,Y)}\), \(\Theta\) will take the form

\[
\Theta|_{U_{(Z,Y)}} = \left[ Y_Z dZ + \bar{Y}_{\bar{Z}} d\bar{Z} \right]
\]

(2.2)
The $(2,0) - (0,2)$ form $\Omega \equiv d\Theta$ is globally defined both in $U(z,y)$ and in $U(Z,Y)$

$$\Omega|_{U(z,y)} \equiv \left[ dy_z \wedge dz + d\overline{Y}_{Z} \wedge d\overline{Z} \right] \quad (2.3)$$

$$\Omega|_{U(Z,Y)} = \left[ dY_Z \wedge dZ + d\overline{Y}_{Z} \wedge d\overline{Z} \right] \quad (2.4)$$

Recall that a change of charts is canonical if in $U(z,y) \cap U(Z,Y)$

$$\Omega|_{U(z,y)} = \Omega|_{U(Z,Y)} \quad (2.5)$$

which implies

$$\Theta|_{U(z,y)} - \Theta|_{U(Z,Y)} = dF \quad (2.6)$$

$F$ is a function on $U(z,y) \cap U(Z,Y)$. In the $(z, Y)$ plane, an extra coordinate $\theta$ is now introduced which will turn out to be fundamental in the sequel. A generating function $\Phi(z, Y, \theta)$ is thus defined as:

$$d\Phi(z, Y, \theta) \equiv d\left( F(z, Y, \theta) + (Y_Z Z(z, Y, \theta) + \overline{Y}_{Z} \overline{Z}(z, Y, \theta) \right)$$

$$= \left( y_z(z, Y, \theta)dz + d\overline{Y}_{Z}(z, Y, \theta)d\overline{Z} \right) + \left( dY_Z Z(z, Y, \theta) + d\overline{Y}_{Z} \overline{Z}(z, Y, \theta) \right) \quad (2.7)$$

showing that the mappings:

$$y_z(z, Y, \theta) = \partial \Phi(z, Y, \theta), \quad Z(z, Y, \theta) = \frac{\partial}{\partial Y_Z} \Phi(z, Y, \theta) \quad (2.8)$$

are canonical.

So that in the $(z, Y, \theta)$ chart, $\Omega|_{U(z,y,\theta)}$ takes the elementary form:

$$\Omega|_{U(z,y,\theta)} = dY_Z \wedge dz + d\overline{Y}_{Z} \wedge d\overline{Z}(z, Y, \theta)$$

$$= dY_Z y_z(z, Y, \theta) \wedge dz + d\overline{Y}_{Z} \overline{Z}(z, Y, \theta) \wedge d\overline{Z}$$

$$= d_z dY \Phi(z, Y, \theta) \quad (2.9)$$

where $dY$ and $dz$ are the differentials operating in the $(Y_Z, \overline{Y}_{Z})$ and $(z, \overline{z})$ respectively.

At this point a crucial remark is in order: in Eq (2.3) the terms $dY_Z Z \wedge dY_Z + dY_Z \wedge d\overline{Y}_{Z}$ and $d_z y_z \wedge dz + d\overline{Y}_{Z} \wedge d\overline{Z}$ will identically vanish in $\Omega$. So an infinitesimal variation of $Z(z, Y, \theta)$ in $Y_Z$ does not modify, for fixed $(z, \overline{z}, \theta)$, the 2-form $\Omega$.

From this comment originates the important Theorem [30].

**Theorem 2.1** On the smooth trivial bundle $\Sigma \times \mathbb{R}^2$, the vertical holomorphic change of local coordinates,

$$Z((z, \overline{z}, \theta) \rightarrow Z((z, \overline{z}, \theta), F(Y_Z), \overline{Y}_{Z}), \quad (2.10)$$

where $F$ is a holomorphic function in $Y_Z$, and the horizontal holomorphic change of local coordinates,

$$y_z(z, \overline{z}, \theta(Y_Z, \overline{Y}_{Z})) \rightarrow y_z(f(z), \overline{z}, \theta(Y_Z, \overline{Y}_{Z})), \quad (2.11)$$

where $f$ is a holomorphic function in $z$, are both canonical transformations.
So the local changes of complex coordinates $z \rightarrow Z((z, \overline{z}), Y_Z)$ and $z \rightarrow Z((z, \overline{z}), Y_Z + dY_Z)$, will be related to the same two form $\Omega$.

Expanding around, say, $Y_Z = 0, \overline{Y} = 0$ the generating function $\Phi$ will be written as a formal power series in $Y$:

$$\Phi(z, Y, \theta) = \sum_{n=1}^{n_{\text{max}}} \left[ Y^n Z^{(n)}(z, \overline{z}, \theta) \right] + \sum_{n=1}^{n_{\text{max}}} \left[ \overline{Y}^n Z^{(n)}(z, \overline{z}, \theta) \right]$$

(2.12)

The extra coordinate $\theta$ previously introduced is now specified as a constant Grassmann variable with Faddeev-Popov charge equal to $-1$. One has the splitting

$$\Phi(z, Y, \theta) = \Phi_0(z, Y) + \theta \Phi_{\theta}(z, Y)$$

(2.13)

where the $Z^{(r)}(z, \overline{z}, \theta), 1 \leq r \leq n$ are local coordinates defined by

$$Z^{(r)}(z, \overline{z}, \theta) = \frac{1}{r!} \left( \frac{\partial}{\partial Y_{Z}} \right)^r \Phi(z, Y, \theta)\bigg|_{Y_{Z}=0} = Z_{0}^{(r)}(z, \overline{z}) + \theta Z_{0}^{(r)}(z, \overline{z})$$

(2.14)

We introduce:

$$\lambda^{(r)}(z, \overline{z}, \theta) \equiv \partial Z^{(r)}(z, \overline{z}, \theta) = \frac{1}{r!} \left[ \frac{\partial}{\partial Y_{Z}} \right]^r \Phi(z, Y, \theta)$$

$$\lambda^{(r)}(z, \overline{z}, \theta) \mu(r, (z, \overline{z}, \theta)) \equiv \partial Z^{(r)}(z, \overline{z}, \theta) = \frac{1}{r!} \left[ \frac{\partial}{\partial Y_{Z}} \right]^r \Phi(z, Y, \theta)$$

(2.15)

and we shall denote $\lambda^{(1)}(z, Y, \theta) =: \lambda(z, Y, \theta), \mu(1, (z, Y, \theta)) =: \mu(z, Y, \theta)$

Note that for a given level $r$ the coordinate $Z^{(r)}(z, \overline{z}, \theta)$ is related to other ones with index less than $r$ by

$$Z^{(r)}(z, \overline{z}, \theta) = \frac{1}{r!} \left( \frac{\partial}{\partial Y_{Z}} \right)^r \left( \Phi(z, Y, \theta)\bigg|_{Y_{Z}=0} \right) = \lambda^{(r)}(z, \overline{z}, \theta) \mu(r, (z, \overline{z}, \theta))$$

for all $0 \leq r \leq n$ and where the functions $M^{(j)}(z, \overline{z}, \theta)$ are given by:

$$M^{(j)}(z, \overline{z}, \theta) \equiv \frac{1}{j!} \left( \frac{1}{\lambda(z, Y, \theta) \partial Y_{Z}} \right)^j \Phi(z, Y, \theta)\bigg|_{Y_{Z}=0, Y_{\overline{Z}}=0} = M^{(j)}_{0}(z, \overline{z}) + \theta M^{(j)}_{\theta}(z, \overline{z})$$

(2.17)

The differential operator $\frac{1}{\lambda(z, Y, \theta) \partial Y_{Z}}$ can be formally defined using the usual Grassmaniann inverse procedure
\[
\frac{1}{\lambda(z, Y, \theta)} \frac{\partial}{\partial Y} = \left( \frac{1}{\lambda_0(z, Y)} - \theta \frac{\lambda_0(z, Y)}{(\lambda_0(z, Y))^2} \right) \frac{\partial}{\partial Y} \equiv D_0(z, Y) + \theta D_0(z, Y) \tag{2.18}
\]
so that for \(1 \leq r \leq n\)

\[
M_0^{(r)}(z, \varpi) \equiv \frac{1}{r!} \left[ D_0(z, Y) \right]^r \Phi_0(z, Y) \big|_{Y \bar{Z} = 0}
\]

\[
M_\theta^{(r)}(z, \varpi) \equiv \frac{1}{r!} \left( \left[ D_0(z, Y) \right]^r \Phi_\theta(z, Y) + \sum_{j=0}^{r-1} D_0^{(n-j)}(z, Y) D_\theta(z, Y) D_0^j(z, Y) \Phi_0(z, Y) \right) \big|_{Y \bar{Z} = 0} \tag{2.19}
\]

The previous geometrical structure allows us to derive now a \(W\)-symmetry in terms of the algebra of diffeomorphisms combining both the diffeomorphism action and the canonical transformations via the BRS machinery. Calling \(S\) the BRS differential operator for the diffeomorphism action on the cotangent bundle \([35, 30]\), according to the decomposition (2.13), one has

\[
S \Phi(z, Y, \theta) = \Lambda(z, Y, \theta) \equiv \Lambda_0(z, Y) + \theta \Lambda_\theta(z, Y) \tag{2.20}
\]
where \(\theta\) is a constant field with Faddeev-Popov charge equals to \(-1\), and we shall fix its BRS variation as \(S \theta = -1\). One identifies

\[
\Lambda_0(z, Y) = S \Phi_0(z, Y) - \Phi_\theta(z, Y)
\]

\[
\Lambda_\theta(z, Y) = S \Phi_\theta(z, Y) \tag{2.21}
\]
and the nilpotency condition on the generating function, \(S^2 \Phi(z, Y) = 0\) means:

\[
SA_0(z, Y) = \Lambda_1(z, Y)
\]

\[
SA_\theta(z, Y) = 0. \tag{2.22}
\]

So for each \(n\) we define the diffeomorphism action on the local complex coordinate \(Z^{(r)}\) by,

\[
S Z^{(r)}(z, \varpi, \theta) \equiv \Upsilon^{(r)}(z, \varpi, \theta) = \frac{1}{r!} \left[ \frac{\partial}{\partial Y_{\bar{Z}}} \right]^r \left[ \Lambda(z, Y, \theta) \right] \big|_{Y \bar{Z} = 0}
\]

\[\text{, } 1 \leq r \leq n \tag{2.23}\]

which, once decomposed according to its \(\theta\) content, gives:

\[
S Z^{(r)}_0(z, \varpi) - Z^{(r)}_\theta(z, \varpi) = \frac{1}{r!} \left( \frac{\partial}{\partial Y_{\bar{Z}}} \right)^r \Lambda_0(z, Y) \big|_{Y \bar{Z} = 0}
\]

\[
S Z^{(r)}_\theta(z, \varpi) = -\frac{1}{r!} \left( \frac{\partial}{\partial Y_{\bar{Z}}} \right)^r \Lambda_\theta(z, Y) \big|_{Y \bar{Z} = 0}. \tag{2.24}
\]

One can easily verify, that \(S^2 Z^{(r)}(z, \varpi, \theta) = 0\) in the \((z, \varpi, \theta)\) space (as in Eq(2.23)), but note that \(S^2 Z_0(z, \varpi) \neq 0\) in the \((z, \varpi)\) space (as in Eq (2.24)) which leads to a diffeomorphism symmetry breaking through the local smooth changes of complex coordinates \((z, \varpi) \rightarrow \ldots\)
(Z^{(r)}(z, \bar{z}), \overline{Z}^{(r)}(z, \bar{z}))$, while the one for $(z, \bar{z}) \rightarrow (Z^{(r)}(z, \bar{z}, \theta), \overline{Z}^{(r)}(z, \bar{z}, \theta))$ holds its full validity.

This shows the important role played by the $\theta$ field, and how the $\theta$ decomposition can be managed in the present treatment: in order to fix all the algebraic relations, calculations must be performed in the $(z, \bar{z}, \theta)$ space, and only after the $\theta$ decomposition is performed. To sum up the construction:

**Statement 2.1** The introduction of the $\theta$ coordinate in the $(z, \bar{z})$ space gives rise to a supers-election sector where the BRS breaking terms of the diffeomorphism transformations $(z, \bar{z}) \rightarrow (Z^{(r)}(z, \bar{z}), \overline{Z}^{(r)}(z, \bar{z}))$ lie.

Moreover the consistency conditions of these breakings and the B.R.S behaviour of the $\theta$ field preserves the diffeomorphism symmetry $(z, \bar{z}) \rightarrow (Z^{(r)}(z, \bar{z}, \theta)\overline{Z}^{(r)}(z, \bar{z}, \theta))$.

The diffeomorphism ghosts in the $(z, \bar{z}, \theta)$ space, are introduced as usual [30] and can also be written with respect to the $\theta$ decomposition,

$$K^{(r)}(z, \bar{z}, \theta) = \frac{\partial \Upsilon^{(r)}(z, \bar{z}, \theta)}{\partial Z^{(r)}(z, \bar{z}, \theta)} \equiv \kappa^{(r)}_0(z, \bar{z}) + \theta \kappa^{(r)}_\theta(z, \bar{z})$$

(2.25)

with the following transformation laws

$$SK^{(r)}(z, \bar{z}, \theta) = K^{(r)}(z, \bar{z}, \theta)\partial K^{(r)}(z, \bar{z}, \theta)$$

(2.26)

and

$$SK^{(r)}_0(z, \bar{z}) = \kappa^{(r)}_0(z, \bar{z})\partial \kappa^{(r)}(z, \bar{z}) + \kappa^{(r)}_\theta(z, \bar{z})$$

(2.27)

$$SK^{(r)}_\theta = \kappa^{(r)}_0(z, \bar{z})\partial \kappa^{(r)}_\theta(z, \bar{z}) - \kappa^{(r)}_\theta(z, \bar{z})\partial \kappa^{(r)}_0(z, \bar{z})$$

(2.28)

respectively.

The nesting of the $(Z^{(r)}(z, \bar{z}), \overline{Z}^{(r)}(z, \bar{z}))$ spaces brings to a decomposition [31] of the ghost $\Upsilon^{(r)}(z, \bar{z}, \theta)$ into the spaces of order lower than $r$,

$$\Upsilon^{(r)}(z, \bar{z}, \theta) = \sum_{j=1}^r j! \prod_{i=1}^m m_j \left[ \frac{\lambda^m(z, \bar{z}, \theta)^{a_i}}{a_i!} \right] \left\{ \sum_{a_i = j, p_i > \sum_{a_i, a_j} = r} \right\}$$

(2.29)

where it has been defined for $1 \leq r \leq n$

$$J^{(r)}(z, \bar{z}, \theta) = \left( C^{(r)}(z, \bar{z}) + \theta \chi^{(r)}(z, \bar{z}) \right)$$

$$C^{(r)}(z, \bar{z}) = \frac{1}{r!} \left[ D_0(z, Y) \right]^r \Lambda_0(z, Y)\|Y_2=0$$

$$\chi^{(r)}(z, \bar{z}) = \left( \frac{1}{r!} \left[ D_0(z, Y) \right]^r \Lambda_1(z, Y) + \Sigma_0(z, Y) \Lambda_0(z, Y) \right)\|Y_2=0$$

(2.30)
With some tedious but straightforward algebraic manipulations the B.R.S variations of the previous fields can be computed by using:

$$[S, D(z, \bar{z})] = -\left( C(z, \bar{z}) \partial \ln \lambda_0(z, \bar{z}) + \partial C(z, \bar{z}) + \frac{\lambda_0(z, \bar{z})}{\lambda_0(z, \bar{z})} \right) D(z, \bar{z}) \quad (2.31)$$

(from now on the summation procedure is explicitly expressed: repeated indices do not mean summation). One obtains as said in the introduction variations of the type of Eq(1.1),

$$SC^{(r)}(z, \bar{z}) = \sum_{s=1}^{r} sC^{(s)}(z, \bar{z}) \partial C^{(r-s+1)}(z, \bar{z}) + \chi^{(r)}(z, \bar{z}), \quad r < n \quad (2.32)$$

and, to insure nilpotency

$$S\chi^{(r)}(z, \bar{z}) = \sum_{s=1}^{r} \left( sC^{(s)}(z, \bar{z}) \partial \chi^{(r-s+1)}(z, \bar{z}) \right) - s\chi^{(s)}(z, \bar{z}) \partial C^{(r-s+1)}(z, \bar{z}) \quad (2.33)$$

$$= - \sum_{s=1}^{r} \left( (r-s+1)\chi^{(r-s+1)}(z, \bar{z}) \partial C^{(s)}(z, \bar{z}) - sC^{(s)}(z, \bar{z}) \partial \chi^{(r-s+1)}(z, \bar{z}) \right)$$

The $\chi^{(r)}(z, \bar{z})$ fields are still undetermined objects: our aim is to fix a recipe in order to identify them as functions of the $C^{(r)}(z, \bar{z})$ ghost fields; we now show that if we impose by hand

$$\chi^{(n)}(z, \bar{z}) = 0 \quad (2.34)$$

then all the $\chi^{(r)}(z, \bar{z}), \quad r < n$ fulfill our requirements. Indeed from the fact that the ghost of maximum order $C^{(n)}(z, \bar{z})$ has no breaking term, that is:

$$SC^{(n)}(z, \bar{z}) = \sum_{s=1}^{n} sC^{(s)}(z, \bar{z}) \partial C^{(n-s+1)}(z, \bar{z}) \quad (2.35)$$

so we can derive:

$$S^2C^{(n)}(z, \bar{z}) = -\theta \left[ \sum_{s=1}^{n-1} s\chi^{(s)}(z, \bar{z}) \partial C^{(n-s+1)}(z, \bar{z}) - \sum_{s=2}^{n} sC^{(s)}(z, \bar{z}) \partial \chi^{(n-s+1)}(z, \bar{z}) \right] = 0 \quad (2.36)$$

and try the more general solution as a differential polynomial in the ghost fields

$$\chi^{(n-r+1)}(z, \bar{z}) = \sum_{r=2}^{n} \sum_{l=0}^{l' \geq 0} \partial^{r'} \partial \chi^{(l)}(z, \bar{z}) \partial^{m'} \partial \chi^{(l)}(z, \bar{z}) T^{(n-s+1-l-t+m+r),(r'+m')}(z, \bar{z})$$

we obtain (there is no summation):

$$\chi^{(n-r+1)}(z, \bar{z}) = C^{(r)}(z, \bar{z}) \partial C^{(r)}(z, \bar{z}) T^{(n-3r+2)}(z, \bar{z})$$

$$+ \alpha^{(n-r+1)} \left( \partial C^{(r)}(z, \bar{z}) \partial^2 C^{(r)}(z, \bar{z}) - \frac{r}{n+1} C^{(r)} \partial^3 C^{(r)}(z, \bar{z}) \right) \delta^{(2r+3)}_{(n-r+1)} \quad (2.37)$$

So the existence of $\alpha$ terms is possible only if the condition $n - 3r + 4 = 0$ is solved by $n$ and $r$ integers. This is achieved for example for $n = r = 2$ which turns out to be the $\mathcal{W}(3)$ case. Rewriting the above solution for $1 \leq r < n$ as
\[ \mathcal{X}^{(r)}(z, \bar{z}) = C^{(n-r+1)}(z, \bar{z}) \partial C^{(n-r+1)}(z, \bar{z}) \mathcal{T}_{(2n-3r+1)}(z, \bar{z}) \]

\[ + \alpha(r) \left( \partial C^{(n-r+1)}(z, \bar{z}) \partial^2 C^{(n-r+1)}(z, \bar{z}) - \frac{n-r+1}{n+1} C^{(n-r+1)} \partial^2 C^{(n-r+1)}(z, \bar{z}) \right) \delta^{2(n-r+1)-3}(r) \]

we then substitute into Eq(2.32) in order to get the properties of coefficients \( \mathcal{T} \).

The terms of the type \( C^{(n-r)}(z, \bar{z})(\text{stuff}) \) fix the B.R.S variations of \( \mathcal{T}_{(2n-3r+1)}(z, \bar{z}) \) the other terms must cancel. Let us first consider the monomials \( C^{(n-r)}(z, \bar{z}) \partial^2 C^{(n-r-1)}(z, \bar{z}) \). The terms:

1) \[ \sum_{s=1}^{r} \left[ sC^{(s)}(z, \bar{z}) C^{(n-(r-s+1)+1)}(z, \bar{z}) \partial^2 C^{(n-(r-s+1)+1)}(z, \bar{z}) \mathcal{T}_{(2n-3s+1-1)}(z, \bar{z}) \right] \]

2) \[ C^{(n-r+1)} \sum_{s=1}^{n-r+1} \left[ sC^{(s)} \partial^2 C^{(n-r+s+2)} \mathcal{T}_{(2n-3r+1)} \right] \]

cancel each other only for \( s = 1 \) while for \( s > 1 \) they give inconsistently; so a priori we have to put in the term 1) \( r = 1 \). In this case this term collapses into

\[ C^{(1)}(z, \bar{z}) C^{(n)}(z, \bar{z}) \partial^2 C^{(n)}(z, \bar{z}) \mathcal{T}_{(2n-2)}(z, \bar{z}) \]

which is the first term (s=1) of the series 2). The remaining terms in the latter can be dropped out giving prescription for the parameter \( n \).

The “a priori logical choice” would be \( n = 1 \) which reproduces the previous term, and trivializes the \( \mathcal{W} \) content. But it has to be noted that if \( n = 2 \) the top term of (2.40) is zero for obvious Faddeev-Popov reason and the cancellation mechanism is then achieved.

For the highest value of \( n \) the cancellation mechanism does not hold unless all the ghosts \( C^{(j)}(z, \bar{z}); \ j = 2 \cdots n - 1 \) are put to be zero and we next repeat the same Faddeev-Popov trick for the \( n \)-th term of the sum in order to get

\[ SC(z, \bar{z}) = C(z, \bar{z}) \partial C(z, \bar{z}) + C^{(n)}(z, \bar{z}) \partial C^{(n)}(z, \bar{z}) \mathcal{T}_{(2n-2)}(z, \bar{z}) \]

\[ + \alpha \left( \partial C^{(n)}(z, \bar{z}) \partial^2 C^{(n)}(z, \bar{z}) - \frac{n}{n+1} C^{(n)} \partial^2 C^{(n)}(z, \bar{z}) \right) \delta^{(n=2)} \]

\[ SC^{(n)}(z, \bar{z}) = C(z, \bar{z}) \partial C^{(n)}(z, \bar{z}) + nC^{(n)}(z, \bar{z}) \partial C(z, \bar{z}) \]

for \( n \neq 2 \) this algebra was found in [37]. So we conclude:

**Statement 2.2** The most general algebra of the type Eqs(2.32)- (2.33) with constraints Eq(2.33)- (2.34) is given in the Equations (2.41)-(2.44). In particular the \( \mathcal{W}_3 \) algebra is the only one which contain an \( \alpha \) dependent term.

### 2.1 The chiral \( \mathcal{W}_3 \)-gravity algebra

We specialize here to the case \( n = 2 \) (\( \mathcal{W}_3 \)) whose peculiarity among all the other situations is stated just above: it is the only case which admits \( \alpha \neq 0 \). We shall see that the presence (or the absence) of a term proportional to \( \alpha \) is fundamental for the geometrical setting of the problem.
For different values of \( n \) the general discussion can be easily performed along the lines here for \( n = 2 \).

From (2.35), let us recall that the highest order \( X \) term has been put to zero in order to fix all the lowest order terms; here for \( n = 2 \) we have to impose:

\[
X^{(2)}(z, \bar{z}) \equiv \frac{1}{2} \left[ \partial^2_0 \Lambda_1(z, Y) - \frac{\lambda_0(z, Y)}{\lambda_0(z, Y)} \partial^2_0 \Lambda_0(z, Y) - \partial_0 \left( \frac{\lambda_0(z, Y)}{\lambda_0(z, Y)} \partial_0 \Lambda(z, Y) \right) \right]|_{Y_2 = 0} = 0 \quad (2.42)
\]

This constraint has a twofold face. The first one concerns the implications on the \( C \)'s algebra, as in the previous Section. The second involves the role that \( X^{(2)}(z, \bar{z}) \) is supposed to play in the symplectic formalism by exploiting all the geometrical aspects emerging from the vanishing of the previous formula. Only the former will be discussed here, the latter being postponed to the next Section.

First of all, the condition \( X^{(2)}(z, \bar{z}) = 0 \) implies:

\[
SC^{(2)}(z, \bar{z}) = C(z, \bar{z}) \partial C^{(2)}(z, \bar{z}) + 2C^{(2)}(z, \bar{z}) \partial C(z, \bar{z}) \quad (2.43)
\]

and setting \( X := X^{(1)} \), Eq (2.36) yields

\[
X(z, \bar{z}) \partial C^{(2)}(z, \bar{z}) = 2C^{(2)}(z, \bar{z}) \partial X(z, \bar{z}) \quad (2.44)
\]

which is solved by

\[
X(z, \bar{z}) = -C^{(2)}(z, \bar{z}) \partial C^{(2)}(z, \bar{z}) \frac{16}{3} T(z, \bar{z})
+ \alpha \left( \partial C^{(2)}(z, \bar{z}) \partial^2 C^{(2)}(z, \bar{z}) - \frac{2}{3} C^{(2)}(z, \bar{z}) \partial^3 C^{(2)}(z, \bar{z}) \right) \quad (2.45)
\]

where we have been forced to introduce a spin \((2,0)\)-conformal field \( T(z, \bar{z}) \). Summing up, we find the algebra:

\[
SC(z, \bar{z}) = C(z, \bar{z}) \partial C(z, \bar{z}) - \frac{16}{3} T(z, \bar{z}) C^{(2)}(z, \bar{z}) \partial C^{(2)}(z, \bar{z})
+ \alpha \left( \partial C^{(2)}(z, \bar{z}) \partial^2 C^{(2)}(z, \bar{z}) - \frac{2}{3} C^{(2)}(z, \bar{z}) \partial^3 C^{(2)}(z, \bar{z}) \right) \quad (2.46)
\]

\[
SC^{(2)}(z, \bar{z}) = C(z, \bar{z}) \partial C^{(2)}(z, \bar{z}) + 2C^{(2)}(z, \bar{z}) \partial C(z, \bar{z})
\]

Note that the \( X(z, \bar{z}) \), as in our primary purpose in Eq (2.30), expresses the breaking at the level of the diffeomorphism algebra. It depends only on the \( C^{(2)}(z, \bar{z}) \) ghost and its derivative; this moves us to make the following Remark:

**Remark 2.1** In the limit of \( C^{(2)}(z, \bar{z}) \) going to zero, we find the diffeomorphism algebra describing the ordinary \((z, \bar{z}) \rightarrow (Z(z, \bar{z}), \overline{Z}(z, \bar{z}))\) reparametrization.

So the ghost \( C^{(2)}(z, \bar{z}) \) parametrizes the breaking of this symmetry at the level of BRS algebra.

The BRS variation of \( T(z, \bar{z}) \) is obtained from Eq (2.41) leading to:
\[ ST(z, \overline{z}) = C(z, \overline{z}) \partial T(z, \overline{z}) + 2T(z, \overline{z}) \partial C(z, \overline{z}) - W(z, \overline{z}) \partial C^{(2)}(z, \overline{z}) - \frac{2}{3} C^{(2)}(z, \overline{z}) \partial W(z, \overline{z}) + \alpha \partial^3 C(z, \overline{z}), \] (2.47)

in terms of a spin \((3, 0)\)-conformal field \(W\) whose BRS behaviour can be calculated from the nilpotency condition applied to (2.47),

\[ SW(z, \overline{z}) = C(z, \overline{z}) \partial W(z, \overline{z}) + 3 \partial C(z, \overline{z}) W(z, \overline{z}) + 16 T(z, \overline{z}) \partial \left( C^{(2)}(z, \overline{z}) T(z, \overline{z}) \right) + \alpha \left( \partial^5 C^{(2)}(z, \overline{z}) + 2 C^{(2)}(z, \overline{z}) \partial^3 T(z, \overline{z}) + 10 T(z, \overline{z}) \partial^3 C^{(2)}(z, \overline{z}) \right) + 15 \partial T(z, \overline{z}) \partial^2 C^{(2)}(z, \overline{z}) + 9 \partial^2 T(z, \overline{z}) \partial C^{(2)}(z, \overline{z}) \right) \] (2.48)

The algebra here closes, since the nilpotency condition holds whatever \(\alpha\). In order to realize the role played by the parameter \(\alpha\), the stability under holomorphic changes of charts \(z \rightarrow w(z)\) of the algebra defined by the Eqs (2.47) is now discussed. One has

\[ C^w = w' C^z, \quad C^{ww} = (w')^2 C^{zz}, \quad \partial_w = \frac{1}{w'} \partial_z, \] (2.49)

and the glueing rule of \(T\) will be also worked out. Writing first (2.47) in the \(w\) system of complex coordinates and then going back to the \(z\) one, we have

\[ C^w \partial_w C^w = w' C^z \frac{1}{w'} \partial_z (w' C^z) = w' C^z \partial_z C^z \]

\[ - \frac{16}{3} T_{ww} C^{ww} \partial_w C^{ww} = - \frac{16}{3} (w')^3 T_{ww} C^{zz} \partial_z C^{zz} \]

while the \(\alpha\)-term is more involved

\[ \alpha \left( \frac{1}{w'} \partial_z ((w')^2 C^{zz}) \frac{1}{w'} \partial_z [\frac{1}{w'} \partial_z ((w')^2 C^{zz})] - \frac{2}{3} (w')^2 C^{zz} \frac{1}{w'} \partial_z [\frac{1}{w'} \partial_z (\frac{1}{w'} \partial_z ((w')^2 C^{zz}))] \right) \]

\[ = \alpha w' \left( \partial_z C^{zz} \partial_z^2 C^{zz} - \frac{2}{3} C^{zz} \partial_z^3 C^{zz} - \frac{16}{3} \{w, z\} C^{zz} \partial_z C^{zz} \right) \]

where \(\{w, z\}\) denotes the Schwarzian derivative. Covariance requires that

\[ \left( (w')^3 T_{ww} + w' \alpha \{w, z\} \right) C^{zz} \partial_z C^{zz} = w' T_{zz} C^{zz} \partial_z C^{zz} \] (2.50)

so that

\[ (w')^2 T_{ww} + \alpha \{w, z\} = T_{zz} \] (2.51)

showing that \(T_{zz}\) is a projective connection. On the other hand if \(\alpha = 0\), \(T\) is a tensor. It is also easy to recover from Eqs (2.47) (2.48):

\[ (w')^3 W_{www} = W_{zzz} \] (2.52)
showing that $W$ behaves as a true tensor of order three.

The solution can be found if we notice that the parameter $\alpha$ can be reabsorbed in the B.R.S. operator by rescaling:

$$
\mathcal{C}(2)(z, \bar{z}) = \alpha^{-\frac{1}{2}} \mathcal{C}(2)(z, \bar{z}) \\
\mathcal{T}(z, \bar{z}) = \alpha \mathcal{T}(z, \bar{z}) \\
\mathcal{W}(z, \bar{z}) = \alpha^{\frac{3}{2}} \mathcal{W}(z, \bar{z})
$$

(2.53)

So if $\alpha \neq 0$ we can fix it to be equal to one without any trouble. In the case $\alpha = 1$ the B.R.S. transformations of $\mathcal{T}(z, \bar{z})$ and $\mathcal{W}(z, \bar{z})$ can be rewritten in terms of Bol derivatives [18].

3 $\mathcal{W}_3$ algebra and coordinate transformations: two different approaches for the same symmetry

The purpose of this Section is to discuss the algebra in Eq (2.47) (2.47) (2.48) found previously in terms of coordinate transformations. Indeed, we have introduced the coordinates $Z^{(r)}(z, \bar{z}, \theta) = Z_0^{(r)}(z, \bar{z}) + \theta Z_\theta^{(r)}(z, \bar{z}), r = 1, 2$ in Eq (2.14) and the reparametrizations:

$$(z, \bar{z}) \rightarrow (Z(z, \bar{z}, \theta), Z(z, \bar{z}, \theta))
$$

$$(z, \bar{z}) \rightarrow (Z^{(2)}(z, \bar{z}, \theta), Z^{(2)}(z, \bar{z}, \theta))
$$

(3.1)

whose algebra under symplectomorphisms reads:

$$
\mathcal{S}Z(z, \bar{z}, \theta) = \mathcal{K}^{(1)}(z, \bar{z}, \theta) \partial Z(z, \bar{z}, \theta)
$$

$$
\mathcal{S}Z^{(2)}(z, \bar{z}, \theta) = \mathcal{K}^{(2)}(z, \bar{z}, \theta) \partial Z^{(2)}(z, \bar{z}, \theta)
$$

(3.2)

with ghosts:

$$
\mathcal{K}^{(1)}(z, \bar{z}, \theta) = \mathcal{C}(z, \bar{z}) + \theta \mathcal{X}^{(1)}(z, \bar{z})
$$

$$
\mathcal{K}^{(2)}(z, \bar{z}, \theta) = \mathcal{C}(z, \bar{z}) + \left(\frac{\partial Z_0(z, \bar{z})}{\partial Z_0^{(2)}(z, \bar{z})}\right)^2 \mathcal{C}(2)(z, \bar{z})
$$

$$
+ \theta \left[ \mathcal{X}^{(1)}(z, \bar{z}) + \left(\frac{\partial Z_0(z, \bar{z})}{\partial Z_0^{(2)}(z, \bar{z})}\right)^2 \mathcal{X}(2)(z, \bar{z}) \right] + \theta \left[ 2 \frac{\partial Z_\theta(z, \bar{z}) \partial Z_0(z, \bar{z})}{\partial Z_0^{(2)}(z, \bar{z})} \mathcal{C}(2)(z, \bar{z}) \right] (3.3)
$$

vanishing in $\mathcal{W}_3$

(3.4)

The previous symmetry is broken at the $(z, \bar{z})$ level, such that the mappings:

$$(z, \bar{z}) \rightarrow (Z_0(z, \bar{z}), Z_0(z, \bar{z}))
$$

$$(z, \bar{z}) \rightarrow (Z_0^{(2)}(z, \bar{z}), Z_0^{(2)}(z, \bar{z}))
$$

(3.5)
do not yield any reparametrization.

However a closed algebra can be written in terms of the BRS operation acting on the previous coordinates:

\[
SZ_0(z, \bar{z}) = C(z, \bar{z})\partial Z_0(z, \bar{z}) + Z_0(z, \bar{z}) \tag{3.6}
\]

\[
SZ_0(z, \bar{z}) = C(z, \bar{z})\partial Z_0(z, \bar{z}) - \partial Z_0(z, \bar{z})A^{(1)}(z, \bar{z}) \tag{3.7}
\]

\[
SZ_0^{(2)}(z, \bar{z}) = C(z, \bar{z})\partial Z_0^{(2)}(z, \bar{z}) + C^{(2)}(z, \bar{z})(\partial Z_0(z, \bar{z}))^2 + Z_0^{(2)}(z, \bar{z}) \tag{3.8}
\]

\[
SZ_0^{(2)}(z, \bar{z}) = C(z, \bar{z})\partial Z^{(2)}(z, \bar{z}) - A^{(1)}(z, \bar{z})\partial Z^{(2)}(z, \bar{z})
- 2\partial Z_0(z, \bar{z})\partial Z_0(z, \bar{z})C^{(2)}(z, \bar{z}) - \left(\frac{\mathcal{A}^{(2)}(z, \bar{z})(\partial Z_0(z, \bar{z}))^2}{\text{vanishing in } \mathcal{W}_3}\right) \tag{3.9}
\]

where the 0 label identifies good coordinate frames and the \(\theta\) ones the breakings of the symmetry under reparametrizations.

So this algebra contains anomalies \(\mathcal{A}(z, \bar{z})\) and \(\mathcal{A}^{(2)}(z, \bar{z})\) which in the same formalism can be expressed in terms of coordinates as:

\[
\mathcal{A}(z, \bar{z}) := \mathcal{A}^{(1)}(z, \bar{z}) \equiv \frac{\tilde{\delta}Z_0(z, \bar{z})}{\partial Z_0(z, \bar{z})} \tag{3.10}
\]

\[
\mathcal{A}^{(2)}(z, \bar{z}) = \frac{1}{(\partial Z_0(z, \bar{z}))^2} \left[ \tilde{\delta}Z_0^{(2)}(z, \bar{z}) - \tilde{\delta}Z_0(z, \bar{z})\frac{\partial Z_0^{(2)}(z, \bar{z})}{\partial Z_0(z, \bar{z})} \right]
- 2\frac{\partial Z_0(z, \bar{z})}{\partial Z_0(z, \bar{z})}(\tilde{\delta}Z_0^{(2)}(z, \bar{z}) - Z_0^{(2)}(z, \bar{z})) \tag{3.11}
\]

where we have introduced the operator associated to \(\delta_{[B.R.S.]} := S\)

\[
\tilde{\delta}(z, \bar{z}) := \left[ \delta_{[B.R.S.]} - C(z, \bar{z})\partial \right]. \tag{3.12}
\]

These parameters are related to the breaking consistency conditions \(\tilde{\delta}Z_0(z, \bar{z})\) and \(\tilde{\delta}Z_0^{(2)}(z, \bar{z})\): since the realization of the \(\mathcal{W}_3\) algebra requires, as shown before, the constraint \(\mathcal{A}^{(2)}(z, \bar{z}) = 0\), we have to investigate in which manner this condition could affect these transformations. In the symplectic framework and from Eqs (2.30)(2.42), the previous condition will imply

\[
\tilde{\delta}Z_0^{(2)}(z, \bar{z}) = \tilde{\delta}Z_0(z, \bar{z})\frac{\partial Z_0^{(2)}(z, \bar{z})}{\partial Z_0(z, \bar{z})} + 2C^{(2)}(z, \bar{z})\partial Z_0(z, \bar{z})\partial Z_0(z, \bar{z}) \tag{3.13}
\]

when written in terms of coordinates. This formula encodes the geometrical content of the \(\mathcal{W}_3\) symmetry.

So this interplay between the breaking consistency conditions \(\tilde{\delta}Z_0^{(2)}(z, \bar{z}), \tilde{\delta}Z_0(z, \bar{z})\) and coordinate transformations induces the symmetry criterion which gives the origin to \(\mathcal{W}_3\) algebra through the condition \(\mathcal{A}^{(2)}(z, \bar{z}) = 0\). Hence we can state:

**Statement 3.1** The condition \(\mathcal{A}^{(2)}(z, \bar{z}) = 0\) not only fixes the breaking terms of the diffe algebra, but also generates a mutual conspiracy between the breaking consistency conditions: the relationship between these violation mechanisms generates the coordinate counterpart symmetry of the \(\mathcal{W}_3\) algebra.
### 3.1 Primary fields as explicit functions of coordinates

This part of our paper wants to investigate an intriguing aspect of our problem to which our approach gives a full meaning.

The conclusion reached in the previous statement \([\text{(3.1)}]\) transforms our algebra with the help of a differential complex for the infinitesimal transformations written in terms of the coordinates \(Z_0(z, \bar{z})\) and \(Z_0^{(2)}(z, \bar{z})\) in presence of breaking terms \(Z_0(z, \bar{z})\) and \(Z_0^{(2)}(z, \bar{z})\). From the physical (and even the mathematical) point of view a coordinate system represents a tool which is hard to do without.

Our approach embeds them in a \((z, \bar{z})\) background whose presence gives a meaning to our model; nevertheless the explicit presence of coordinates amounts to investigating the possibility of writing \(Z_0(z, \bar{z})\) and \(Z_0^{(2)}(z, \bar{z})\) as explicit expression of both \(Z_0(z, \bar{z})\) and \(Z_0^{(2)}(z, \bar{z})\).

We have seen before in the Ansatz \([\text{(2.1)}]\) that the Faddeev-Popov content of the breaking terms can only be carried by the \(C^{(2)}(z, \bar{z})\) so we argue as follows. Let us consider in a matrix-like notation the following ansatz.

**Ansatz 3.1**

\[
\begin{align*}
\left\{ \begin{array}{l}
Z_{\theta}(z, \bar{z}) \\
Z_{\theta}^{(2)}(z, \bar{z})
\end{array} \right\} &= \sum_{r \geq 0} \sum_{j=1}^{3} \left( \begin{array}{l}
r \\
0
\end{array} \right) \partial^j C^{(2)}(z, \bar{z}) \partial^{r-j+1} C^{(2)}(z, \bar{z}) \left\{ \begin{array}{l}
P_{(2-r)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z})) \\
R_{(2-r)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z}))
\end{array} \right\} \\
-2 \sum_{r \geq 0} \sum_{j=0}^{r} \left( \begin{array}{l}
r \\
j
\end{array} \right) \partial^{j+1} C(z, \bar{z}) \partial^{r-j} C^{(2)}(z, \bar{z}) \left\{ \begin{array}{l}
P_{(2-r)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z})) \\
R_{(2-r)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z}))
\end{array} \right\} \\
- \sum_{r \geq 0} \left( \partial^r C^{(2)}(z, \bar{z}) \right) \sum_{n \geq 0} \left( \partial(\partial^n Z_0(z, \bar{z})) \right) \left\{ \begin{array}{l}
P_{(2-r)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z})) \\
R_{(2-r)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z}))
\end{array} \right\} \\
&+ \frac{\partial}{\partial(\partial^n Z_0^{(2)}(z, \bar{z}))} \left\{ \begin{array}{l}
P_{(2-r)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z})) \\
R_{(2-r)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z}))
\end{array} \right\} \partial^n \left( C(z, \bar{z}) \partial Z_0^{(2)}(z, \bar{z}) \right) \\
&+ C^{(2)}(z, \bar{z}) \partial Z_0^{(2)}(z, \bar{z})^2 + \sum_{s \geq 0} \partial^s C^{(2)}(z, \bar{z}) \left\{ \begin{array}{l}
P_{(2-s)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z})) \\
R_{(2-s)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z}))
\end{array} \right\} \\
&- C(z, \bar{z}) \partial \left\{ \begin{array}{l}
P_{(2-r)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z})) \\
R_{(2-r)}^{Z}(Z_0(z, \bar{z}), Z_0^{(2)}(z, \bar{z}))
\end{array} \right\}
\end{align*}
\]

\((3.14)\)
The previous $\tilde{\delta}$ variations have Faddeev-Popov charge equal to two and contain both $\partial^r C(z,\bar{z})\partial^s C(\partial^2)(z,\bar{z})$ and $\partial^r C(z,\bar{z})\partial^m C(\partial^2)(z,\bar{z})$ ghost monomials. In order to implement Eqs\((3.7)\) and \((3.9)\), recall that in the solutions \((2.13)\) for $X(z,\bar{z})$, all the mixed terms $\partial^r C(z,\bar{z})\partial^s C(\partial^2)(z,\bar{z})$ have to cancel out. This gives rise to the following set of constraints on both the unknowns $P$ and $R$. It will be useful for the sequel to proceed as follows; we have two distinct systems of conditions. First the vanishing of the coefficient of the monomials $C(\partial^r)^{r+1}C$, $r \geq 0$, yields

\[
\begin{align*}
\left\{ \begin{array}{l}
P^Z_{(2-r)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z})) \\
R^Z_{(2-r)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))
\end{array} \right\} &= \frac{1}{2} \mathcal{V}^{(r)} \left\{ \begin{array}{l}
P^Z_{(2)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z})) \\
R^Z_{(2)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))
\end{array} \right\}, \quad r \geq 0, \\
\end{align*}
\]

and the one for the monomials $\partial^r C(\partial^s)^{s+1}C$, $r \geq 1$, $s \geq 0$ gives rise to

\[
\begin{align*}
\left\{ \begin{array}{l}
P^Z_{(2-(r+s))}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z})) \\
R^Z_{(2-(r+s))}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))
\end{array} \right\} &= \frac{r!(s+1)!}{(s+r)!(2s+1-r)} \mathcal{V}^{(s)} \left\{ \begin{array}{l}
P^Z_{(2-r)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z})) \\
R^Z_{(2-r)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))
\end{array} \right\}, \\
\end{align*}
\]

(3.17)

for $r \geq 1$, $s \geq 0$, and $r \neq 2(s+1)$,

where we have set for $r \geq 0$:

\[
\mathcal{V}^{(r)} = \sum_{n \geq r+1} \left( \begin{array}{c}
n \\
r+1
\end{array} \right) \left[ \frac{\partial^{n-r} Z_0(z,\bar{z})}{\partial(\partial^{n} Z_0(z,\bar{z}))} + \frac{\partial^{n-r} Z_0^{(2)}(z,\bar{z})}{\partial(\partial^{n} Z_0^{(2)}(z,\bar{z}))} \right]
\]

(3.18)

It is worthwhile to remark that $\mathcal{V}^{(0)}$ is the counting operator with respect to the little indices $z$. Note however that the conditions \((3.16)\) is contained in \((3.17)\) by allowing for $r = 0$. This decomposition of the constraints leads advantageously to the following

**Statement 3.2** Owing to Eq\((3.14)\), for $r, s \geq 0$, the functions $P^Z_{(2-(r+s))}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$ and $R^Z_{(2-(r+s))}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$ can be completely fixed in terms of $P^Z_{(2-r)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$ and $R^Z_{(2-r)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$, respectively, in an independent way.

The operator $\mathcal{V}^{(r)}$ defined in Eq\((3.18)\) decreases the order of all derivatives by $r$. Thus the construction of $P^Z_{(2-r)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$ in terms of $P^Z_{(2)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$ means that, if $\tilde{m}$ is the highest order of derivative terms of $Z(z,\bar{z})$ or $Z^{(2)}(z,\bar{z})$ contained in $P^Z_{(2)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$, then the functions $P^Z_{(2-(\tilde{m}-r))}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$, $r > 1$ will be zero.

In particular, any $\mathcal{V}^{(r)}$ for $r \geq 1$ gives no action on $P^Z_{(2)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$ involving zero and first order derivative terms. The same argument holds for $R^Z_{(2)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$ in relation to $R^Z_{(2)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$.

An important remark is in order. By acting with $\mathcal{V}^{(r)}$ on the lower states $P^Z_{(2)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$ and $R^Z_{(2)}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$ all the upper states can be constructed. But due to \((3.17)\) the states $P^Z_{(2-(r+s))}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$ and $R^Z_{(2-(r+s))}(Z_0(z,\bar{z}), Z_0^{(2)}(z,\bar{z}))$ are obtained from...
\( \mathcal{P}^{Z}_{(2-r)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \) and \( \mathcal{R}^{Z(2)}_{(2-r)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \) acting with \( \mathcal{V}^{(s)} \) only if \( r \neq 2(s + 1) \). The latter limits the cancellation mechanism of the \( \partial^s \mathcal{C}(2) \partial^{s+1} \mathcal{C} \) monomials in Eq. (3.15) only to selected states (since the sum \( r + s \) can be reached in a non unique way without spoiling the inequality \( r \neq 2(s+1) \)) in the decomposition (3.14).

The previous cancellation mechanisms could hide an \textit{a priori} inconsistency: this is not the case. In fact the local operators \( \mathcal{V}^{(r)} \) defined in (3.18) verify the algebra

\[
\left[ \mathcal{V}^{(r)}, \mathcal{V}^{(s)} \right] = (r - s) \frac{(r + s + 1)!}{(r+1)!(s+1)!} \mathcal{V}^{(r+s)},
\]

so that Eqs (3.16) and (3.17) can be related. Indeed, combining both Eqs (3.16) and (3.19) one obtains

\[
\left\{ \begin{array}{l}
\mathcal{P}^{Z}_{(2-(r+s))}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \\
\mathcal{R}^{Z(2)}_{(2-(r+s))}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z}))
\end{array} \right\}
= \frac{1}{2} \left\{ \begin{array}{l}
\mathcal{V}^{(r+s)} \mathcal{P}^{Z}_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \\
\mathcal{V}^{(r+s)} \mathcal{R}^{Z(2)}_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z}))
\end{array} \right\}
\]

Using Eq. (3.17) we get an identity which holds for \( r \neq 2(s + 1) \) and \( s \neq 2(r+1) \). The latter formula shows moreover that \( \mathcal{P}^{Z}_{(2-(r+s))} \) and/or \( \mathcal{R}^{Z(2)}_{(2-(r+s))} \) can be obtained without any restriction. Hence (3.13) reduces to

\[
\tilde{\delta} \left\{ Z_0(z, \overline{z}) \right\} = \sum_{r,s \geq 0} \partial^s \mathcal{C}(2)(z, \overline{z}) \partial_r \mathcal{C}(2)(z, \overline{z}) \mathcal{L}_{(2-s)}(z, \overline{z}) \mathcal{V}^{(r)} \frac{1}{2} \left\{ \begin{array}{l}
\mathcal{P}^{Z}_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \\
\mathcal{R}^{Z(2)}_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z}))
\end{array} \right\}
\]

where the differential operator \( \mathcal{L}_{(2-s)} \) is defined by

\[
\mathcal{L}_{(2-s)}(z, \overline{z}) := \sum_{k,l \geq 0} \sum_{n \geq k} \left( \begin{array}{c}
\frac{n}{k}
\end{array} \right) \left( \begin{array}{c}
\partial^n \mathcal{V}^{(l)} \left( \begin{array}{c}
\frac{l}{2}
\end{array} \right) \mathcal{P}^{Z}_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z}))
\end{array} \right) \frac{\partial}{\partial \partial^n Z_0(z, \overline{z})}
\]

\[
+ \left( \begin{array}{c}
\partial^n \mathcal{V}^{(l)} \left( \begin{array}{c}
\frac{l}{2}
\end{array} \right) \mathcal{R}^{Z(2)}_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z}))
\end{array} \right) \frac{\partial}{\partial \partial^n Z_0^{(2)}(z, \overline{z})}
\]

\[
+ \sum_{n \geq s} \left( \begin{array}{c}
\frac{n}{s}
\end{array} \right) \left( \begin{array}{c}
\partial^n \partial Z_0(z, \overline{z})
\end{array} \right) \frac{\partial}{\partial \partial^n Z_0^{(2)}(z, \overline{z})}
\]

(3.21)

The \( \mathcal{L}_{(2-s)}(z, \overline{z}) \) operator looks like a coordinate tranformation operator wich maps \( Z_0(z, \overline{z}) \) and its derivatives into particular derivatives of \( \mathcal{P}_{(2-l)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \) and similarly for \( Z_0^{(2)}(z, \overline{z}) \) with respect \( \mathcal{R}^{Z(2)}_{(2-l)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \) plus an inhomogeneous term depending on \( Z_0(z, \overline{z}) \).

But \( \tilde{\delta} Z_0(z, \overline{z}) \) has a well defined Faddeev-Popov ghost content due to the condition (2.45). Only the \( \mathcal{C}(2)(z, \overline{z}) \partial \mathcal{C}(2)(z, \overline{z}) \), \( \mathcal{C}(2)(z, \overline{z}) \partial^2 \mathcal{C}(2)(z, \overline{z}) \), and \( \partial \mathcal{C}(2)(z, \overline{z}) \partial^2 \mathcal{C}(2)(z, \overline{z}) \) ghost monomials
are present in the expansion \( (3.20) \): the first term will provide a \( Z_0(z, \overline{z}) \), \( Z_0^{(2)}(z, \overline{z}) \) for the \( T \) function; while the other ones give the \( \alpha \) term in Eq \( (2.45) \). All the other terms must be zero.

These conditions provide a system of equations for \( \mathcal{P} \) and \( \mathcal{R} \).

A capital role is played by the equations for \( \alpha \) terms, since they provide a coordinate representation for the constant \( \alpha \) which has to be true for each \( Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z}) \) change of charts. This covariance requirement, combined with the algebraic structure of the two local operators \( \mathcal{V}^{(s)} \) and \( \mathcal{L}_{(2-s)} \) show that only the value \( \alpha = 0 \) is consistent for this approach.

\[
\mathcal{L}_{(2-s)} \mathcal{P}_{(2-r)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) = \mathcal{L}_{(2-r)} \mathcal{P}_{(2-s)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z}))
\]

\((r + s) \neq 1 \quad (3.22)\)

This equation can be treated as an integrability condition and contains both a linear and a bilinear part in the \( \mathcal{P}_{(2-r)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \) and the \( \mathcal{R}_{(2-r)}^{Z_0^{(2)}}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \) functions.

Filtering with the counting operators of the previous functions we select the linear part of Eq \( (3.22) \). So we have an infinite set of conditions on \( \mathcal{P}_{(2-r)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \)

For \( r = 0 \), \( s > 1 \) we get:

\[
\mathcal{L}_{(2-s)}^{lin} \mathcal{P}_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) = \mathcal{L}_{(2-s)}^{lin} \mathcal{P}_{(2-s)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z}))
\]

\((3.23)\)

where:

\[
\mathcal{L}_{(2-s)}^{lin} = \sum_{n \geq s} \binom{n}{s} \frac{\partial^{n-s} (\partial Z_0)^2(z, \overline{z})}{\partial (\partial^n Z_0^{(2)}(z, \overline{z}))}
\]

\((3.24)\)

This condition must be compatible with the \( \mathcal{V}^{(r)} \) algebra stated in the equation \( (3.16) \) for \( \mathcal{P}_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \).

The form of the operator \( \mathcal{L}_{(2-s)}^{lin} \) forbids this possibility; the only way to escape is to prevent the dependence of these function on the derivatives of \( Z_0^{(2)}(z, \overline{z}) \) with order greater or equal than two. Going on with the same filtration mechanism we forbid the dependence on the derivatives of \( Z_0(z, \overline{z}) \) of the same order.

This shows that \( Z_0(z, \overline{z}) \) depends only on the derivatives of \( Z_0(z, \overline{z}) \) and \( Z_0^{(2)}(z, \overline{z}) \) of order less than two: now the same argument can be repeated for \( \tilde{Z}_0^{(2)}(z, \overline{z}) \) in the same Faddeev-Popov and derivatives sectors as before; so we obtain an analogous equation as Eq \( (3.22) \) for the \( \mathcal{R}_{(2-r)}^{Z_0^{(2)}}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \) functions obtaining the same results as before for the \( \mathcal{R}_{(2)}^{Z_0^{(2)}}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \) functions.

So we get:

**Statement 3.3** The only non zero functions in Eq \( (3.14) \) will be \( \mathcal{P}_{(2)}^{Z_0}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \), and \( \mathcal{R}_{(2)}^{Z_0}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) \).

The other functions will be zero if, as stated by Eq \( (2.14) \), the previous functions depend only on \( Z_0(z, \overline{z}) \) and \( Z_0^{(2)}(z, \overline{z}) \) and their first order derivatives.
This means that $\mathcal{P}^Z_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z}))$, $\mathcal{R}^Z_{(2)}(Z_0^{(2)}(z, \overline{z}), Z_0^{(2)}(z, \overline{z}))$ and $\mathcal{T}(z, \overline{z})$ are well-defined tensors under holomorphic change of charts.

Assuming the general expressions:

$$
\mathcal{P}^Z_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) = \sum_i A_i \prod_j \left( \frac{\partial^{m_j} Z_0(z, \overline{z})}{\partial Z_0^{(2)}(z, \overline{z})} \right)^{\beta_j} \left( \frac{\partial^{n_j} Z_0^{(2)}(z, \overline{z})}{\partial Z_0^{(2)}(z, \overline{z})} \right)^{\sigma_j} \left\{ \sum_{\nu_i, j} \beta_j \nu_i^{m_j} + \sigma_j \nu_i^{n_j} = 1 \right\} \sum_{\nu_i, j} \beta_j \nu_i^{m_j} + \sigma_j \nu_i^{n_j} = 2 \right\}
$$

$$
f_i \left( \frac{Z_0^{(2)}(z, \overline{z})}{(Z_0(z, \overline{z}))^2}, \frac{\partial Z_0^{(2)}(z, \overline{z})}{\partial (Z_0(z, \overline{z}))^2} \right) \right)
$$

$$
\mathcal{R}^Z_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) = \sum_i B_i \prod_j \left( \frac{\partial^{l_j} Z_0(z, \overline{z})}{\partial Z_0^{(2)}(z, \overline{z})} \right)^{\rho_j} \left( \frac{\partial^{h_j} Z_0^{(2)}(z, \overline{z})}{\partial Z_0^{(2)}(z, \overline{z})} \right)^{\psi_j} \left\{ \sum_{\nu_i, j} \rho_j \nu_i^{l_j} + \psi_j \nu_i^{h_j} = 2 \right\} \sum_{\nu_i, j} \rho_j \nu_i^{l_j} + \psi_j \nu_i^{h_j} = 2 \right\}
$$

where $f_i, g_i$ are arbitrary scalar functions.

with the constraints imposed in Eq (3.25), (3.28) by the Statement (3.3):

$$
m_j^i, n_j^i, l_i^j, h_j^i = 0, 1
$$

At this stage the role of Eq (3.13) appears as the condition which links together the functions $\mathcal{P}^Z_{(2-r)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z}))$ and $\mathcal{R}^Z_{(2-r)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z}))$ since:

$$
\partial Z_0(z, \overline{z}) \mathcal{L}_{(1)}(z, \overline{z}) \mathcal{R}^Z_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) = \partial Z_0^{(2)}(z, \overline{z}) \mathcal{L}_{(1)}(z, \overline{z}) \mathcal{P}^Z_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) + 2(\partial Z_0(z, \overline{z}))^2 (\mathcal{P}^Z_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})))
$$

we get

$$
\mathcal{P}^Z_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) = A(\partial Z_0(z, \overline{z}))^3
$$

$$
\mathcal{R}^Z_{(2)}(Z_0(z, \overline{z}), Z_0^{(2)}(z, \overline{z})) = B(\partial Z_0(z, \overline{z}))^2
$$

so:

$$
\mathcal{S} Z_0(z, \overline{z}) = \left( C(z, \overline{z}) + AC^{(2)}(z, \overline{z}) \right) \frac{(\partial Z_0(z, \overline{z}))^2}{\partial Z_0^{(2)}(z, \overline{z})} \partial Z_0(z, \overline{z})
$$

$$
\mathcal{S} Z_0^{(2)}(z, \overline{z}) = \left( C(z, \overline{z}) + AC^{(2)}(z, \overline{z}) \right) \frac{(\partial Z_0(z, \overline{z}))^2}{\partial Z_0^{(2)}(z, \overline{z})} \partial Z_0^{(2)}(z, \overline{z})
$$
so that for a rescaling \( C^{(2)}(z, \overline{z}) \rightarrow \frac{1}{\lambda} C^{(2)}(z, \overline{z}) \) we find a perfect covariance of \( Z_0(z, \overline{z}) \) under the ghost \( K_2(z, \overline{z}) = \left( C(z, \overline{z}) + C^{(2)}(z, \overline{z}) \frac{\partial Z_0(z, \overline{z})}{\partial Z_0^{(2)}(z, \overline{z})} \right) \).

Finally we find:

\[
\mathcal{T}(z, \overline{z}) = \frac{3}{8} \left( \frac{\partial Z_0(z, \overline{z})}{\partial Z_0^{(2)}(z, \overline{z})} \right)^2
\]

\[
\mathcal{W}(z, \overline{z}) = \frac{3}{4} \left( \frac{\partial Z_0(z, \overline{z})}{\partial Z_0^{(2)}(z, \overline{z})} \right)^3
\]

Indeed it is easy to verify that the Eq (3.28) is fully equivalent to say that the \( \theta \) part of Eq(3.4) becomes zero.

The explicit dependence on the coordinates reconstructs the diffeomorphism symmetry.

In the case of the algebra Eq(2.41) if we put:

\[
Z_\theta = A C^{(n)}(z, \overline{z}) \left( \frac{\partial Z_0(z, \overline{z})}{\partial Z_0^{(n)}(z, \overline{z})} \right)^{(n+1)}
\]

\[
Z_\theta^{(n)}(z, \overline{z}) = B (\partial Z_0(z, \overline{z}))^{(n)}, \text{ with } B = A - 1
\]

we find the covariance

\[
S Z_0(z, \overline{z}) = \left( C(z, \overline{z}) + C^{(n)}(z, \overline{z}) \frac{\partial Z_0(z, \overline{z})}{\partial Z_0^{(n)}(z, \overline{z})} \right) \partial Z_0(z, \overline{z})
\]

\[
S Z_0^{(n)}(z, \overline{z}) = \left( C(z, \overline{z}) + C^{(n)}(z, \overline{z}) \frac{\partial Z_0(z, \overline{z})}{\partial Z_0^{(n)}(z, \overline{z})} \right) \partial Z_0^{(n)}(z, \overline{z})
\]

under the ghost field

\[
K^{(n)}(z, \overline{z}) = \left( C(z, \overline{z}) + C^{(n)}(z, \overline{z}) \frac{\partial Z_0(z, \overline{z})}{\partial Z_0^{(n)}(z, \overline{z})} \right)
\]

3.2 Advantages and drawbacks of the different approaches

In this, we will point out the worths and the faults of choosing the primary fields as explicit functions of the coordinates or not.

First of all the coordinate description honors its scoreboard with the explicit residual reparametrization symmetry \((z, \overline{z}) \rightarrow (Z^{(2)}(z, \overline{z}), \overline{Z}^{(2)}(z, \overline{z}))\). The price to pay for it is to lose in the way the \( \mathcal{T}(z, \overline{z}) \) fies as a projective connection. These geometrical objects are welcome in conformal “covariant” theories, when the holomorphic derivative operators are to be defined in a covariant way.

It is a well-known problem for a right definition of the diffeomorphism anomaly [18]. Its solution is quite involved and its origin is not a clearly identified.

We want to show that the anomaly of our algebra is, in the explicit coordinate approach \((\alpha = 0)\) not naively well defined, so that complicated surgery methods have to be applied, while if \( \mathcal{T}(z, \overline{z}) \) can be considered as a projective connection \((\alpha \neq 0)\), the anomaly becomes well defined under change of charts.
We define the anomaly in terms of Gel'fand-Fuchs cocycles as in reference \textsuperscript{34}. The holomorphic cocycles $\Delta^{(a,n)}(z,\overline{z})$ of Faddeev-Popov charge equal to $n$ is defined as:

$$\mathcal{S}\Delta^{(a,n)}(z,\overline{z}) = 0$$ \hspace{1cm} (3.39)

So if we decompose:

$$\Delta^{(a,n)}(z,\overline{z}) = C(z,\overline{z})\Delta_1^{(n-1)}(z,\overline{z}) + \Delta_0^{n}(z,\overline{z})$$ \hspace{1cm} (3.40)

Eq. (3.39) implies:

$$\delta(z,\overline{z})\Delta_1^{(n-1)}(z,\overline{z}) - \partial C(z,\overline{z})\Delta_1^{(n-1)}(z,\overline{z}) - \partial \Delta_0^{n}(z,\overline{z}) = 0$$
$$\delta(z,\overline{z})\Delta_0^{n}(z,\overline{z}) + \mathcal{X}(z,\overline{z})\Delta_1^{(n-1)}(z,\overline{z}) = 0$$ \hspace{1cm} (3.41)

The Feigin-Fuchs cocycle is the Faddeev-Popov charge three well defined element of the cohomology space.

Somewhat elaborated calculations will give:

$$\Delta_0^{3}(z,\overline{z}) = (\mathcal{S}C(z,\overline{z}))\partial^2 C(z,\overline{z}) + \partial(C(z,\overline{z})\partial C(z,\overline{z}) - \mathcal{S}C(z,\overline{z})\partial C(z,\overline{z})$$

$$-\frac{2}{3}\mathcal{W}(z,\overline{z})C^{(2)}(z,\overline{z})\partial C^{(2)}(z,\overline{z})\partial^2 C^{(2)}(z,\overline{z})$$

$$+ \frac{1}{3}(\alpha(\partial^2 C^{(2)}(z,\overline{z})\partial^3 C^{(2)}(z,\overline{z}) - \partial C^{(2)}(z,\overline{z})\partial^4 C^{(2)}(z,\overline{z}))$$

$$+ 2\partial^2 \mathcal{T}(z,\overline{z})C^{(2)}(z,\overline{z})\partial C^{(2)}(z,\overline{z}) - 10\mathcal{T}(z,\overline{z})\partial C^{(2)}(z,\overline{z})\partial^2 C^{(2)}(z,\overline{z})$$

$$- 2\partial\mathcal{T}(z,\overline{z})C^{(2)}(z,\overline{z})\partial^2 C^{(2)}(z,\overline{z}) C(z,\overline{z})$$ \hspace{1cm} (3.42)

Going at $\alpha = 0$ and substituting the explicit expression of $\mathcal{T}(z,\overline{z})$ and $\mathcal{W}(z,\overline{z})$ as in Eqs (3.32)(3.33) we recover the explicit coordinate description, so Eq(3.42) can be written in term of the $\mathcal{K}^{(2)}(z,\overline{z})$ as:

$$\Delta_0^{3}(z,\overline{z}) = \mathcal{K}^{(2)}(z,\overline{z})\partial \mathcal{K}^{(2)}(z,\overline{z})\partial^2 \mathcal{K}^{(2)}(z,\overline{z})$$ \hspace{1cm} (3.43)

The anomaly can be calculated along the lines found in Ref \textsuperscript{33, 34}:

$$\Delta_2^{1}(z,\overline{z})dz \wedge d\overline{z} = \frac{\partial}{\partial c(z,\overline{z})} \frac{\partial}{\partial \overline{c}(z,\overline{z})} \Delta_0^{3}(z,\overline{z})dz \wedge d\overline{z}$$

$$= \partial \mu(z,\overline{z})\partial^2 C(z,\overline{z}) - \partial C(z,\overline{z})\partial^2 \mu(z,\overline{z})$$

$$+ \frac{1}{3}(\alpha(\partial^2 \mu^{(2)}(z,\overline{z})\partial^3 C^{(2)}(z,\overline{z}) - \partial^2 C^{(2)}(z,\overline{z})\partial^3 \mu^{(2)}(z,\overline{z})$$

$$- \partial \mu^{(2)}(z,\overline{z})\partial^4 C^{(2)}(z,\overline{z}) + \partial C^{(2)}(z,\overline{z})\partial^4 \mu^{(2)}(z,\overline{z}))$$

$$+ 2\partial^2 \mathcal{T}(z,\overline{z})(\mu^{(2)}(z,\overline{z})\partial C^{(2)}(z,\overline{z}) - C^{(2)}(z,\overline{z})\partial \mu^{(2)}(z,\overline{z})$$

$$- 10\mathcal{T}(z,\overline{z})(\partial \mu^{(2)}(z,\overline{z})\partial^2 C^{(2)}(z,\overline{z}) - \partial C^{(2)}(z,\overline{z})\partial^2 \mu^{(2)}(z,\overline{z})$$

$$- 2\partial\mathcal{T}(z,\overline{z})(\mu^{(2)}(z,\overline{z})\partial^2 C^{(2)}(z,\overline{z}) - C^{(2)}(z,\overline{z})\partial^2 \mu^{(2)}(z,\overline{z})$$ \hspace{1cm} (3.44)
where the (extended) Beltrami multipliers are:

\[ \mu(z, \bar{z}) = \frac{\partial C(z, \bar{z})}{\partial c(z, \bar{z})} \quad \mu^{(2)}(z, \bar{z}) = \frac{\partial C^{(2)}(z, \bar{z})}{\partial c(z, \bar{z})} \] (3.45)

for \( \alpha = 0 \) we get the usual anomaly

\[ \Delta^{1/2}(z, \bar{z})dz \wedge d\bar{z} = K^{(2)}(z, \bar{z})\partial^3 \left( \mu(z, \bar{z}) + \frac{(\partial Z_0(z, \bar{z}))^2}{\partial Z_0^{(2)}(z, \bar{z})} \mu^{(2)}(z, \bar{z}) \right) \] (3.46)

since \( \mu(z, \bar{z}) + \frac{(\partial Z_0(z, \bar{z}))^2}{\partial Z_0^{(2)}(z, \bar{z})} \mu^{(2)}(z, \bar{z}) \) is the Beltrami multiplier of \( Z_0^{(2)}(z, \bar{z}) \) with respect the \( (z, \bar{z}) \) background.

This object is not well defined under holomorphic changes of charts, and an intricate game, necessary to introduce a projective connection (useful for geometrical purposes but unessential for the dynamical ones) is needed.

On the other hand for \( \alpha \neq 0 \), so we can fix \( \alpha = 1 \) due to the rescaling property Eq (2.53) a projective connection \( \mathcal{T}(z, \bar{z}) \) is at our disposal. So, adding total derivatives and cocycles to Eq (3.44) (unessential from the cohomological but essential from the geometrical point of view) we can rewrite:

\[
\begin{aligned}
\Delta^{1/2}(z, \bar{z})dz \wedge d\bar{z} &= \left\{ C(z, \bar{z})L_3 \mu^{(2)}(z, \bar{z}) - \mu^{2}(z, \bar{z})L^3 C(z, \bar{z}) \right. \\
&- \left. \frac{1}{3} \left( C^{(2)}(z, \bar{z})L_5 \mu^{(2)}(z, \bar{z}) - \mu^{(2)}(z, \bar{z})L^5 C^{(2)}(z, \bar{z}) \right) \right. \\
&- \left. 8 \left( C(z, \bar{z})\mu^{(2)}(z, \bar{z}) - \mu^{(2)}(z, \bar{z})C^{(2)}(z, \bar{z}) \right) \mathcal{W}(z, \bar{z}) \right. \\
&- \left. 24 \mathcal{W}(z, \bar{z}) \left( C(z, \bar{z})\partial \mu^{(2)}(z, \bar{z}) - \mu^{(2)}(z, \bar{z})\partial C(z, \bar{z}) \right) \right. \\
&+ \left. C^{(2)}(z, \bar{z})\partial \mu^{2}(z, \bar{z}) - \mu^{2}(z, \bar{z})\partial C^{(2)}(z, \bar{z}) \right) \right\} dz \wedge d\bar{z}
\end{aligned}
\] (3.47)

where the Bolt derivatives are recalled to be:

\[
\begin{align*}
L_3 &= \partial^3 + 2T(z, \bar{z})\partial + (\partial T(z, \bar{z})) \\
L_5 &= \partial^5 + 10T(z, \bar{z})\partial^3 + 15(\partial T(z, \bar{z}))\partial^2 \\
&+ \left[ 9(\partial T(z, \bar{z})) + 16T^2(z, \bar{z}) \right] \partial + 2 \left[ (\partial^3 T(z, \bar{z}) + 8T(z, \bar{z})(\partial T(z, \bar{z})) \right] \end{align*}
\] (3.48)

We see that the anomaly is well defined under holomorphic change of charts, due to the fact that \( \mathcal{T}(z, \bar{z}) \) is a projective connection.

4 Conclusions

We have shown in this paper the geometrical origin and the strengths and weaknesses of \( \mathcal{W}_3 \) and the other algebras related to the same construction.

The symplectic approach to this problem fits in a coordinate scenario the O.P.E method from which, historically speaking, the \( \mathcal{W} \) algebras were derived. There are more questions
that the reader can ask. What deeper insights will lock the symplectic geometry? Is there a
further relation between O.P.E. and the symmetry constraints? The answers could reveal the
connections between Physics and Geometry, but unfortunately they are not at our hand.

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