Negative mode problem in false vacuum decay with gravity *

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There is a single negative mode in the spectrum of small perturbations about the tunneling solutions describing a metastable vacuum decay in flat spacetime. This mode is needed for consistent description of decay processes. When gravity is included the situation is more complicated. There are different answers in different reduction schemes. An approach based on elimination of scalar field perturbations shows no negative mode, whereas the recent approach based on elimination of gravitational perturbations indicates presence of a negative mode. In this contribution we analyse and compare the present approaches to the negative mode problem in false vacuum decay with gravity.

1. INTRODUCTION

The first order phase transitions might play an important role in the Early Universe [1]. They proceed via nucleation of the bubbles of true vacuum in metastable (false) one and subsequent growth of the bubbles. The false vacuum decay is usually discussed in frame of self-interacting scalar field theory. This process is described by the $O(4)$-symmetric bounce solution [2,3] of the Euclidean equations of motion. Value of the Euclidean action at the bounce gives leading exponential factor in decay rate. The perturbations about the bounce solution define the one loop corrections to the bubble nucleation rate and determine quantum state of materialized bubble [4]. It is remarkable that in the spectrum of small perturbations about the bounce in flat spacetime there is exactly one negative mode [5]. This mode is responsible for making correction to the ground-state energy imaginary, thus justifying decay interpretation.

When gravity is included the model contains gauge degrees of freedom and to extract information about the spectrum one should find proper reduction to physical variables. In the Euclidean gravity the presence of the gauge degrees of freedom manifests itself in the well known conformal factor problem. In pure gravity it is possible to cure this problem either at the level of prescription [6], or via careful gauge fixing [7]. For scalar perturbations in the theory of a scalar field coupled to gravity it is the only spatially homogeneous modes, which suffer from the conformal factor problem (see e.g. [8]). On the other hand the negative mode is expected exactly in this problematic sector of scalar homogeneous perturbations, which consists of coupled scalar field and metric perturbations. The task is to reduce this coupled system to a system with a single physical variable. Note that there is no problem in elimination of unphysical variables in perturbations about the flat Friedmann-Robertson-Walker (FRW) type model with scalar field [9]. Problem arises for perturbations in the theory of scalar field in closed FRW type Universe.

There are two possible ways of reduction of coupled system of perturbations: either to eliminate metric perturbations or perturbations of scalar field. The approach based on elimination of scalar field perturbations shows no negative mode in...
the spectrum of remaining variable $[9]$. On the other hand, the recent investigation [10] indicates presence of a single negative mode in the approach based on elimination of gravitational perturbations. This indicates on lack of full understanding of the role of gauge-fixing procedure in description of false vacuum decay with gravity and need for further investigation of this subject.

The aim of present contribution is to discuss and compare existing approaches to the negative mode problem. The rest of the article is organized as follows. In the next section we recall main formulas relevant to false vacuum decay in flat spacetime. In Sec. 3 we discuss inclusion of gravity, remind some important classical solutions of Euclidean equations of motion and derive quadratic action for scalar perturbations about $O(4)$-symmetric background solutions. We will restrict ourselves to the most problematic sector of $O(4)$-symmetric perturbations. In Sec. 4 we analyse three different reduction schemes. Sec. 5 contains the concluding remarks.

Figure 1. Schematic view of a scalar field potential $V(\phi)$ with a metastable vacuum $\phi = \phi_+$, true vacuum $\phi = \phi_-$ and local maximum $\phi = \phi_{\text{top}}$.

2. FALSE VACUUM DECAY IN FLAT SPACETIME

Let’s consider self-interacting scalar field $\phi$ in flat spacetime and assume that the scalar field potential $V(\phi)$ has shape shown in Fig.1, with local minimum (false vacuum) at $\phi = \phi_+$ and absolute minimum (true vacuum) at $\phi = \phi_-$. The false vacuum decay is described by the Euclidean path integral

$$< \phi_- | e^{-HT} | \phi_+ > = \int e^{\int \left( i \pi \phi \frac{\partial}{\partial \phi} - H \right) d\phi d\pi_\phi} , \quad (1)$$

where $T$ is a large positive number and $H$ is the scalar field Hamiltonian

$$H(\pi, \phi) = \int \left( \frac{\pi(x)^2}{2} + \frac{1}{2} \partial_{\mu} \phi(x) \partial^{\mu} \phi(x) + V(\phi) \right) d^3 x, \quad n = 1, 2, 3. \quad (2)$$

Integrating Eq.(1) over $\pi_\phi$ one gets

$$< \phi_- | e^{-HT} | \phi_+ > = \int e^{-S_E} d\phi , \quad (3)$$

where $S_E$ is the usual Euclidean action

$$S_E = \int \left( \frac{1}{2} \partial_\mu \phi(x) \partial^{\mu} \phi(x) + V(\phi) \right) d^3 x d\tau , \quad (4)$$

with $\mu = 0, 1, 2, 3$. In the limit $T \to \infty$ the l.h.s. of Eq.(3) contains information about the lowest energy eigenvalue and wave function. The r.h.s. can be calculated in the WKB approximation. The energy $E_0$ of the lowest state gets correction due to tunneling phenomenon

$$E = E_0 - K e^{-B} , \quad (5)$$

with

$$B = S_E(\phi_{\text{bounce}}) - S_E(\phi_+), \quad (6)$$

where $\phi_{\text{bounce}}$ is the $O(4)$-symmetric euclidean bounce solution with the lowest action, which interpolates between true and false vacua. The factor $K$ is obtained by the Gaussian integration of exponent of quadratic action of small perturbations about the bounce

$$K = \frac{B^2}{4 \pi^2} \left( \frac{\det'[-\partial^2 + V''(\phi)]}{\det[-\partial^2 + V''(\phi_+)]} \right)^{-1/2}. \quad (7)$$
The operator \(-\partial^2 + V''(\phi)\) has zero modes and \(det'\) means that the determinant computed with the zero eigenvalues omitted. It turns out that there is exactly one mode with the negative eigenvalue in the spectrum of small perturbations about the bounce solution. This makes \(K\) and hence the energy shift in the Eq.\((\ref{en})\) purely imaginary and supports the false vacuum decay interpretation. Decay rate per unit volume \(\mathcal{V}\), per unit time is given by

\[
\Gamma = |K|e^{-\mathcal{B}}. \tag{8}
\]

Note that the functional integral Eq.\((\ref{eq1})\) is written for unconstrained (physical) degrees of freedom. If one has \(m\) first class constraints

\[
C_\alpha(\phi, \pi) \approx 0 , \quad \alpha = 1, \ldots, m , \tag{9}
\]

then according to general recipy \([\ref{2.2}]) one should choose some gauge fixing conditions

\[
\chi_\beta(\phi, \pi) = 0 , \quad \beta = 1, \ldots, m , \tag{10}
\]

and modify the functional integral, Eq.\((\ref{eq1})\), as follows

\[
\int e^{i\int d\phi d\pi \chi_{\beta} \partial_{\beta} \Delta(C_{\alpha}) \delta(\chi_{\beta}) d\phi d\pi} , \tag{11}
\]

with the Faddeev-Popov determinant

\[
\Delta \equiv \text{det}\{C_{\alpha}, \chi_{\beta}\} \neq 0 . \tag{12}
\]

### 3. INCLUSION OF GRAVITY

The Euclidean action of system composed of scalar field minimally coupled to gravity is

\[
S = \int d^4x \sqrt{g} \left[ -\frac{1}{2\kappa} R + \frac{1}{2} \nabla^\mu \phi \nabla^\nu \phi + V(\phi) \right] , \tag{13}
\]

where \(\kappa = 8\pi G\) is the reduced Newton’s constant.

Since we are interested in solutions having \(O(4)-\)symmetry, the metric is parametrised as follows

\[
ds^2 = ds^2 + a^2(\sigma) \gamma_{ij} dx^i dx^j , \quad \phi = \phi(\sigma) , \tag{14}
\]

where \(\gamma_{ij}\) is the three-dimensional metric on the constant curvature space sections. The field equations are

\[
a^2 - \mathcal{K} - \frac{k a^2}{3} \left( \frac{\dot{\phi}^2}{2} - V(\phi) \right) = 0 , \tag{15}
\]

\[\dot{\phi}^2 + 3 \frac{\dot{a}}{a} \dot{\phi} - \frac{\delta V}{\delta \phi} = 0 , \tag{16}\]

where a dot denotes a derivative with respect to proper time \(\sigma\) and \(\mathcal{K}\) is the curvature parameter, which has the values 1, 0, -1 for closed, flat and open universes respectively. In what follows we are interested in closed case, but we do not specify the value of \(\mathcal{K}\) now in order to see how it enters in the final expressions.

#### 3.1. Euclidean solutions

There are few celebrated solutions of these equations, which play an important role in Euclidean quantum gravity.

1. The Hawking-Moss solution is a 4-sphere corresponding to scalar field sitting on the top of the potential barrier

\[
\phi(\sigma) = \phi_{top}, \quad a(\sigma) = H_{top}^{-1} \sin(H_{top} \sigma) , \tag{18}
\]

with \(H_{top} = \sqrt{\kappa V(\phi_{top})/3}\). Note that the Euclidean time \(\sigma\) varies in finite interval \(\sigma = (0, \sigma_f)\).

2. The Coleman-De Luccia bounce is a deformed 4-sphere. It starts with some \(\phi = \phi_0\) at \(\sigma = 0\) close to \(\phi_-\), stops at \(\sigma = \sigma_f\) close to \(\phi_+\) and obeys the regularity conditions

\[
a(0) = \dot{\phi}(0) = 0 , \quad a(\sigma_f) = \dot{\phi}(\sigma_f) = 0 . \tag{19}
\]

3. Relaxing the regularity conditions at the second zero of \(a\) one gets the Hawking-Turok singular instanton. It was introduced to describe creation of an open inflationary Universe.

Our main goal is to investigate spectrum of small perturbations about the Euclidean solutions and, in particular, to determine presence of a negative mode.

#### 3.2. Quadratic action of scalar \(O(4)-\) symmetric perturbations

Investigation of perturbations would be convenient to perform in conformal frame. We expand the metric and the scalar field over a \(O(4)-\)symmetric background

\[
ds^2 = a(\tau)^2 \left[ (1 + 2A(\tau)) d\tau^2 + \gamma_{ij} (1 - 2\Psi(\tau)) dx^i dx^j \right] ,
\]

\[
\phi = \varphi(\tau) + \Phi(\tau) , \tag{20}
\]
where $\tau$ is the conformal time, $a$ and $\varphi$ are the background field values and $A, \Psi$ and $\Phi$ are small perturbations.

The background field equations in the conformal time are

$$\mathcal{H}^2 - \mathcal{H}' - \mathcal{K} = \frac{\kappa}{2} \varphi'^2, \quad (21)$$

$$2 \mathcal{H}' + \mathcal{H}^2 - \mathcal{K} = \frac{\kappa}{2}(\varphi'^2 + 2a^2V(\varphi)), \quad (22)$$

$$\varphi'' + 2H\varphi' - a^2 \frac{\delta V}{\delta \varphi} = 0, \quad (23)$$

where a prime denotes a derivative with respect to $\tau$ and $\mathcal{H} := a'/a$.

Expanding the total action, keeping terms of the second order in perturbations and using the background equations, we find

$$S = S^{(0)} + S^{(2)}, \quad (24)$$

where $S^{(0)}$ is the action for the background solution and $S^{(2)}$ is quadratic in perturbations with the Lagrangian for scalar $O(4)$–symmetric perturbations

$$\mathcal{L}^{(s)} = \frac{1}{2\kappa}a^2 \sqrt{\gamma} \left[-6\Psi'^2 + 6K\Psi^2 + \kappa(\Psi'^2 + a^2 \frac{\delta^2 V}{\delta \varphi^2} \Phi^2 + 6\varphi' \Psi' \Phi) - 2(\kappa \varphi' \Phi' - 2\kappa a^2 \frac{\delta V}{\delta \varphi} \Phi + 12H \Psi' + 12K \Psi)A - 2(\mathcal{H}' + 2\mathcal{H}^2 + \mathcal{K})A^2 \right], \quad (25)$$

Note that the variation with respect to $A$ gives the first order (constraint) equation

$$2\kappa \varphi' \Phi' - 2\kappa a^2 \frac{\delta V}{\delta \varphi} \Phi + 12H \Psi' + 12K \Psi + 4(\mathcal{H}' + 2\mathcal{H}^2 + \mathcal{K})A = 0. \quad (26)$$

4. **THREE APPROACHES TO THE REDUCTION**

To obtain the unconstrained system corresponding to the degenerate Lagrangian $\mathcal{L}^{(s)}$ we will follow the conventional Dirac formulation of generalized Hamiltonian dynamics [17,18]. Calculating the canonical momenta

$$\Pi_{\Phi} := \frac{i\delta^{(s)} \mathcal{L}}{\delta \Phi'} = ia^2 \sqrt{\gamma}(\Phi' - \varphi' A),$$

$$\Pi_{\Psi} := \frac{i\delta^{(s)} \mathcal{L}}{\delta \Psi'} = i\frac{6a^2}{\kappa} \sqrt{\gamma} (-\Psi' + \frac{\kappa}{2} \varphi' \Phi - \mathcal{H} A),$$

we find the primary constraint $C_1 := \Pi_A = 0$.

Thus the evolution is governed by the total Hamiltonian

$$H_T = H_C + u_1(\tau)C_1, \quad (28)$$

with arbitrary function $u_1(\tau)$ and canonical Hamiltonian

$$H_C = -\frac{\kappa}{12a^2 \sqrt{\gamma}} \Pi_{\Psi}^2 + \frac{1}{2a^2 \sqrt{\gamma}} \Pi_{\Phi}^2 + i\frac{\kappa \varphi'}{2} \Pi_{\Phi} \Phi + a^2 \sqrt{\gamma} \left[\frac{3\kappa}{\kappa} \Psi^2 + \frac{1}{2} \left(a^2 \frac{\delta^2 V}{\delta \varphi^2} + \frac{3}{2} \kappa \varphi'^2\right) \Phi^2 \right] + A[i\varphi' \Pi_{\Phi} - iH \Pi_{\Psi} + a^2 \sqrt{\gamma} \left(a^2 \frac{\delta V}{\delta \varphi} - 3\varphi' \mathcal{H} \Phi - \frac{6\kappa}{\kappa} \Psi\right)]. \quad (29)$$

Conservation of primary constraint gives the secondary constraint

$$C_2 := i\varphi' \Pi_{\Psi} - iH \Pi_{\Phi} + a^2 \sqrt{\gamma} \left(a^2 \frac{\delta V}{\delta \varphi} - 3\varphi' \mathcal{H} \Phi - \frac{6\kappa}{\kappa} \Psi\right). \quad (30)$$

The primary and secondary constraints are first class one and there are no ternary constraints.

Under the infinitesimal shift $\tau \rightarrow \tau + \lambda$ these constraints generate the gauge transformations

$$\delta \Psi = -H \lambda, \quad \delta \Pi_{\Psi} = -i\frac{6a^2}{\kappa} \sqrt{\gamma} \kappa \lambda,$$

$$\delta \Phi = \varphi' \lambda, \quad \delta \Pi_{\Phi} = ia^2 \sqrt{\gamma}(\varphi'' - \varphi' \mathcal{H}) \lambda,$$

$$\delta A = \lambda' + H \lambda. \quad (31)$$

The existence of constraints in the system as usually means the presence of unphysical degrees of freedom and that the whole system has no unique dynamics. To find the physical variables with unique evolution one should fix the gauge and solve the constraints. There are two possible general strategies: either to eliminate the gravitational degrees of freedom ($\Pi_{\Psi}$ and $\Psi$) or perturbations of scalar field ($\Pi_{\Phi}$ and $\Phi$).
4.1. The I approach

In the first investigation on this subject the Lagrangian approach was used \cite{19}. The gauge was fixed by the condition

\[ \Psi = 0. \] (32)

Using this gauge condition and eliminating \( A \) with the help of the constraint equation Eq.\,(26) we obtain the unconstrained quadratic action in the form

\[ S_{LRT}^{(2)} = \int \frac{a^4}{2Q_{LRT}} \left( \frac{d^2}{d\tau^2} - \frac{a^2}{3} \frac{\kappa}{\psi} \Phi \right) d\tau d^3x, \] (33)

with

\[ Q_{LRT} := \left( \mathcal{H}^2 + 2\mathcal{H}^2 + 2\mathcal{H} \right) / 3 = \mathcal{H}^2 - \frac{\kappa a^2}{6}. \] (34)

We see that the action Eq.\,(33) has correct overall sign if \( Q_{LRT} > 0 \). As it was shown in \cite{19} under certain choice of parameters the bounce can develop a region with negative \( Q_{LRT} \). The main conclusion in \cite{19} was that the region with \( Q_{LRT} < 0 \) leads to catastrophic particles creation and is pathological.

However the gauge fixing Eq.\,(32) has problems where \( a' = 0 \). This can be seen in the Hamiltonian formalism, while the Faddeev-Popov determinant is proportional to \( a' \).

4.2. The II approach

Eliminating \( \Pi_{\Psi} \) using the constraint Eq.\,(30) and working in the gauge invariant variables

\[ \Psi = \Phi + \frac{\mathcal{H}}{\psi} \Phi, \]
\[ \Pi_{\Psi} = \frac{6i\kappa a^2}{a^2} \sqrt{\psi} \Phi, \] (35)

one obtains the reduced Hamiltonian

\[ H^* = -\frac{\kappa}{12a^2\sqrt{\psi}} \Pi_{\Psi}^2 + \frac{a^2}{2a^2} \sqrt{\frac{3\kappa}{\psi}} \Psi^2 - \frac{2a^2}{\psi^2} \left( \frac{3\kappa}{\kappa} \Phi + \frac{i\mathcal{H}}{2a^2\sqrt{\psi}} \Pi_{\Psi} \right)^2. \] (36)

Performing the canonical transformation

\[ \Psi = \frac{i\kappa \psi'}{4} \bar{q} - \frac{\mathcal{H}}{3ka^2\sqrt{\psi \psi'}} \bar{p}, \] (37)

and solving for the momenta \( \bar{p} \) one obtains the quadratic part of the Euclidean action

\[ S_{(2)} = \frac{(1 - 4\kappa)}{2} \int \left[ \left( \frac{dq}{d\tau} \right)^2 + Uq^2 \right] \sqrt{\gamma} d^3x d\tau, \] (38)

with a potential \( U \) depending on the background fields

\[ U = \frac{\kappa}{2} \phi^2 + \phi' \left( \frac{1}{\psi} \right)'' + 1 - 4\kappa. \] (39)

We see that quadratic action for the homogeneous harmonic has “wrong” overall sign. To overcome this problem it was suggested that analytic continuation

\[ q \rightarrow -iq \] (40)

is performed while integrating over this mode (compare \cite{23}).

The equation for the mode functions, which diagonalize the action \cite{8}, has form of the Schrödinger equation

\[ -\frac{d^2}{d\tau^2} q + Uq = Eq. \] (41)

Note that the singularities \( \frac{1}{\psi'} \), which appear in this approach do not allow one to investigate straightforwardly the perturbations about the Hawking-Moss solution (with \( \phi = \phi_{\text{top}} = \text{const} \)).

The spectrum about the Coleman-De Luccia bounce was investigated and no negative mode theorem was proven \cite{11,12}. The Hawking-Turok instanton was investigated \cite{6} and it was found that for monotonous scalar field potentials there is no negative mode in the spectrum. On the other hand it was found that the negative mode is present in “exotic” cases when the Hawking-Turok instanton “overshoots” the Coleman-De Luccia bounce. This means that in this case the Hawking-Turok instanton is unstable and has larger action then the corresponding Coleman-De Luccia bounce.

Note that the formulation in terms of the gauge invariant variables Eq.\,(33) corresponds to the
gauge choice $\chi_{GMST} := \Phi = 0$ (compare [11]).
Hence the Faddeev-Popov determinant is $\varphi'$ and this approach is singular for certain configurations.

4.3. The III approach

Having in mind that i) spherically symmetric gravity has no propagating degrees of freedom and ii) in the limit $\kappa \to 0$ we should recover scalar field theory in flat space, we prefer an approach based on elimination of gravitational perturbations [13]. At the same time from the Eq.(30) we see that solving constraint we divide either to $\varphi'$ or to $\mathcal{H}$ or to $a^2$. Since $\varphi'$ and $a'$ might have zero(s) by solving constraints we might introduce extra singularities in quadratic action. For closed universe the scale factor $a$ is vanishing for some $\tau = \tau_{\pm}$. Since we are interested in perturbations which vanish for $\tau \to \tau_{\pm}$, the coordinate singularity $a(\tau_{\pm}) = 0$ looks most harmless.

We use the following gauge fixing conditions
\[ \chi_1 := A = 0, \quad \chi_2 := \frac{\kappa}{6\kappa a^2 \sqrt{\gamma}} \Pi_\varphi = 0, \quad (42) \]
which obey the following canonical algebra:
\[ \{ C_i, C_j \} = \{ \chi_i, \chi_j \} = 0, \quad \{ \chi_i, C_j \} = \delta_{ij}. \quad (43) \]
As a next step we consider all constraints in a strong sense and define the physical Hamiltonian as
\[ H^* := H_C \big|_{\chi_i = 0, C_i = 0}. \quad (44) \]
For the physical Hamiltonian we find
\[ H^* = \frac{1}{2 a^2 \sqrt{\gamma}} \left( 1 - \frac{\kappa}{6\kappa} \varphi'^2 \right) \Pi_\varphi^2 + \frac{i\kappa \varphi'}{6\kappa} \left( a^2 \frac{\delta V}{\delta \varphi} - 3\varphi' \mathcal{H} \right) \Pi_\varphi \Phi + \frac{1}{2} a^2 \sqrt{\gamma} \left( \frac{\kappa}{6\kappa} \left( a^2 \frac{\delta V}{\delta \varphi} - 3\varphi' \mathcal{H} \right) \Phi^2 + \left( a^2 \frac{\delta^2 V}{\delta \varphi^2} + \frac{3}{2} \kappa \varphi'^2 \right) \right) \Phi^2, \quad (45) \]
which corresponds to the following unconstrained quadratic action for one physical dynamical degree of freedom
\[ S^{(2)} = \int \frac{a^2 \sqrt{\gamma}}{2Q} \left[ \Phi^2 - \frac{\kappa \varphi'}{3\kappa} \left( a^2 \frac{\delta V}{\delta \varphi} - 3\varphi' \mathcal{H} \right) \Phi \Phi' + \left( \frac{\kappa}{6\kappa} \left( a^2 \frac{\delta V}{\delta \varphi} - 3\varphi' \mathcal{H} \right) \Phi \Phi' + Q \left( a^2 \frac{\delta^2 V}{\delta \varphi^2} + \frac{3}{2} \kappa \varphi'^2 \right) \right) \Phi \right] d\tau d^3x, \quad (46) \]
with
\[ Q := \left( 1 - \frac{\kappa}{6\kappa} \varphi'^2 \right). \quad (47) \]
In what follows we will consider the background configurations for which the factor $Q$ is positive definite.

Introducing a new variable $q = a/\sqrt{Q}$ $\Phi$ and integrating by parts we obtain the unconstrained quadratic action in the form
\[ S^{(2)} = \frac{1}{2} \int \left( q^2 + W[a(\tau), \varphi(\tau)] q^2 \right) d\tau \sqrt{\gamma} d^3x, \quad (48) \]
with frequency $W$ whose time dependence is determined by the background solutions
\[ W[a(\tau), \varphi(\tau)] = \frac{a^2 \delta^2 V}{Q \delta \varphi \delta \varphi} + \frac{\kappa a^4}{2KQ^2} \frac{\delta V}{\delta \varphi} - \frac{2\kappa \varphi' a^2 \mathcal{H} \delta V}{KQ^2} \frac{\delta \varphi}{\delta \varphi} - \frac{10H^2}{Q} + \frac{12\mathcal{H}^2}{Q^2} + \frac{8}{Q} - 6\kappa - 3\kappa Q. \quad (49) \]
Note that the obtained action allows us to consider the slowly varying scalar field, $\varphi' \to 0$, and vanishing gravity, $\kappa \to 0$, limits.

The equation for the mode functions, which diagonalize the action Eq.(18), has form of the Schrödinger equation
\[ -\frac{d^2}{d\tau^2} q + W[a(\tau), \varphi(\tau)] q = Eq, \quad (50) \]
with the potential Eq.(13). Having this equation, one can investigate the negative mode problem for concrete cases numerically. We investigated the case with the scalar field potential
\[ V(\varphi) = \frac{m^2}{2} (\varphi^2 (\varphi - v)^2 + B \varphi^4) . \quad (51) \]
This potential for $m^2 = 2, B = 0.12$ and $v = 0.5$ has local maximum at $\varphi_{\text{top}} = 0.31250$, and local minimum (false vacuum) at $\varphi_{\text{false}} = 0.3571428$. In $2\kappa = 1$ units are used.
For this potential there exists the Coleman-De Luccia bounce solution \cite{21} with \( \varphi_0 = 0.1123579 \). The quantum mechanical Schrödinger equation Eq.\,(50) was solved numerically for this potential and exactly one bound state was found. Note that the factor \( Q \) was positive definite for this case.

Within this approach it is problematic to investigate the Hawking-Turok instanton, while scalar field \( \varphi \) runs away and eventually the factor \( Q \) becomes negative.

We found that the Schrödinger equation Eq.\,(50) for the Hawking-Moss background solution has six boundstates. This number is in perfect agreement with the analytic formula for the eigenvalues \( \lambda \):

\[
\lambda_n = n(n + 3)\mathcal{H}_{\text{top}}^2 + V''(\varphi_{\text{top}}), \quad n = 0, 1, 2, \ldots (52)
\]

while for our example

\[
\frac{-V''(\varphi_{\text{top}})}{\mathcal{H}_{\text{top}}^2} = 40.96 . \tag{53}
\]

Presence of many negative modes about the Hawking-Moss solution means that the corresponding Coleman-De Luccia bounce for the given potential and set of parameters gives dominant contribution to tunneling \cite{23}.

Note that this approach based on gauge fixing Eq.\,(42) can be equally well formulated in the gauge invariant variables

\[
\Phi = \Phi - \frac{i\kappa\varphi'}{6a^2\sqrt{\gamma\kappa}}\Pi_\Phi ,
\]

\[
\Pi_\Phi = \Pi_\Phi + \frac{\kappa}{6\sqrt{\kappa}}(\varphi'' - \varphi'\mathcal{H})\Pi_\Phi . \tag{54}
\]

5. CONCLUDING REMARKS

There are two important types of the Euclidean solutions in flat spacetime: instantons and bounces. The instanton describes quantum-mechanical mixing between the equal energy states, whereas the bounce describes decay of a metastable state. There are no negative modes in the spectrum of perturbations about the instantons (only zero modes). The bounce has a single negative mode. When gravity is included plenty of new euclidean solutions are found. Presence or absence of a negative mode might be a good indicator for understanding to which type each solution belongs: instanton (mixing) or bounce (decay).

The approach based on elimination of gravitational degrees of freedom shows no negative mode in the spectrum. It was suggested \cite{9} that the \( i \) which comes from the conformal rotation Eq.\,(40) plays the same role as a negative mode. This argument has a weak point. The conformal rotation Eq.\,(40) is the same for all euclidean background solutions, while the imaginary shift in energy is peculiarity of an unstable state. Moreover, for the Giddings-Strominger wormhole a negative mode was found on top of conformal rotation \cite{20}.

The approach based on elimination of gravitational degrees of freedom indicates the presence of a negative mode. It has smooth \( \kappa \to 0 \) limit and can be straightforwardly used for the Hawking-Moss solution (with \( \varphi = 0 \)). At the same time there are still many open questions. A detailed numerical study of different cases is needed. Another issue which needs further investigation is an understanding of the role of configurations with the negative factor \( Q \): is it real physical instability or it is the breakdown of the reduction scheme.

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Note added: After this contribution was essentially completed, further progress in investigation of negative mode problem was made. The results are summarized in the revised version of \cite{13}.

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