LOCAL STRUCTURES IN POLYHEDRAL MAPS ON SURFACES, 
AND PATH TRANSFERABILITY OF GRAPHS

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ABSTRACT. We extend Jendrol’ and Skupień’s results about the local structure 
of maps on the 2-sphere: In this paper we show that if a polyhedral map \( G \) on 
a surface \( M \) of Euler characteristic \( \chi(M) \leq 0 \) has more than \( 126|\chi(M)| \) vertices, 
then \( G \) has a vertex with "nearly" non-negative combinatorial curvature. As 
a corollary of this, we can deduce that path transferability of such graphs are 
at most 12.

keyword 1. Polyhedral maps, Embedding, Light vertex, Combinatorial curvature, 
Path transferability

1. introduction

In this paper we use standard terminology and notation of graph theory. The 
graphs discussed here are finite, simple, undirected, and connected. A orientable 
surface \( S_g \) of genus \( g \) is obtained from the sphere by adding \( g \) handles. A non-
orientable surface \( N_q \) of genus \( q \) is obtained from the sphere by adding \( q \) crosscaps. 
The Euler characteristic is defined by 
\[
\chi(S_g) = 2 - 2g, \quad \chi(N_q) = 2 - q.
\]

If a graph \( G \) is embedded in a surface \( M \) then the connected components of \( M - G \) 
are called the faces of \( G \). If each face is an open disc then the embedding is called 
a 2-cell embedding. If \( G \) is a 2-cell embedding in a surface \( M \) and each vertex has 
degree at least three, then \( G \) is called a map on \( M \). If in addition, \( G \) is 3-connected 
and the embedding has representativity at least three, then \( G \) is called polyhedral 
map in \( M \) (see e.g. [11]). Let us recall that the representativity (or face-width) of
a 2-cell embedded graph $G$ in a surface $M$ is equal to the smallest number $k$ such that $M$ contains a non-contractible closed curve that intersects the graph $G$ in $k$ points.

The facial walk of a face $\alpha$ in a 2-cell embedding is the shortest closed walk that follows the edges in order around the boundary of the face $\alpha$. The degree of a face $\alpha$ is the length of its facial walk. Vertices and faces of degree $i$ are called $i$-vertices and $i$-faces, respectively. A vertex $v$ is said to be an $(a_1, a_2, \ldots, a_n)$-vertex if the faces incident with $v$ have degree $a_1, a_2, \ldots, a_n$. An edge $e$ is said to be an $(i, j)$-edge if two vertices incident with $e$ have degree $i, j$.

If each facial walk of a 2-cell embedding consists of distinct vertices, then the embedding is called closed 2-cell embedding. If $G$ is a closed 2-cell embedding and the subgraph of $G$ bounding the faces incident with any vertex is a wheel with $\geq 3$ spokes and a possibly subdivided rim, the embedding is called a wheel-neighborhood embedding. The following Proposition is due to Negami and Vitray (see [9], [10], [11], [16]).

**Proposition 1.** An embedding of a graph is a polyhedral map if and only if it is a wheel-neighborhood embedding.

By Euler polyhedral formula, a simple planar graph has a vertex of degree $\leq 5$. Local structures of planar graphs are further studied by Jendrol’ and Skupień [5]. Local structures of graphs on general surfaces is investigated by several researchers, see [6], [7], [8], [15] etc. In this paper we will show the following by using the Discharging method that is slightly changed from the ones of [5], [6]:

**Theorem 2.** Let $G$ be a simple polyhedral map on a surface $M$ of Euler characteristic $\chi(M) \leq 0$. If $G$ has more than $126|\chi(M)|$ vertices, then $G$ contains an $(a_1, a_2, \ldots, a_n)$-vertex, where $n = 3, 4, 5, 6$ and $(a_1, a_2, \ldots, a_n)$ satisfies one of the lists in Table [7].
Table 1. Local structure in polyhedral map with sufficient large vertices

Such $i$-vertices, $i = 3, 4, 5, 6$, are called light vertices in $G$. After Section 4, these vertices are also called the ones with nearly non-negative curvature.

On the other hand, path transferability is introduced in [13]: We consider a path as an ordered sequence of distinct vertices with a head and a tail. Given a path, a transfer-move is to remove the tail and add a vertex at the head. A graph is $n$-path-transferable if any path with length $n$ can be transformed into any other such path by a sequence of transfer-moves. The maximum number $n$ for which $G$ is $n$-path-transferable is called the path transferability of $G$. The author in [14] showed the following result for planar graphs.

**Theorem 3** ([14]). Path transferability of a simple planar graph with minimum degree $\geq 3$ is at most 10.

For graphs on general surfaces, we will show the following result.
Main Theorem. If a polyhedral map $G$ on a surface $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$ has more than $126|\chi(\mathbb{M})|$ vertices, then path transferability of $G$ is at most 12.

2. Proof of Theorem 2

Let $G$ be a counterexample with $n$ vertices. We consider only 2-cell embeddings of graphs. Hence Euler's formula implies:

$$\sum_{v \in V(G)} (2\deg(v) - 6) + \sum_{\alpha \in F(G)} (\deg(\alpha) - 6) = 6|\chi(\mathbb{M})|$$

because $\chi(\mathbb{M}) \leq 0$.

In the following we use the Discharging method. We assign to each vertex $v$ the charge $c(v) = 2\deg(v) - 6$ and to each face $\alpha$ the charge $c(\alpha) = \deg(\alpha) - 6$. These charges of the vertices and faces will be locally redistributed to charges $c^*(v)$ and $c^*(\alpha)$, respectively, by the following rules. We first apply Rule A1 − 4, and next apply Rule B. An edge $e$ is called weak or semi-weak if two or exactly one of its endvertices are of degree 3, respectively. A face of $G$ is called minor if its degree is at most 5, and is called major if its degree is at least 7.

Rule A1.

Suppose that $\alpha$ is a face of $G$ incident with a vertex $v$, and that $\deg(\alpha) \leq 5$, $\deg(v) \geq 4$. Then $v$ sends to $\alpha$ the following charge:

- 1 if $\deg(\alpha) = 3$,
- $\frac{1}{2}$ if $\deg(\alpha) = 4$,
- $\frac{1}{6}$ if $\deg(\alpha) = 5$.

Rule A2.

Every $k$-vertex, $k \geq 4$, which has at least one 3-face and at least two 6-faces sends additional charge $\frac{1}{10}$ to each 3-face after Rule A1.

Rule A3.

Suppose that $e$ is a common edge of adjacent faces $\alpha$ and $\alpha'$ of $G$, and that
deg(\(\alpha\)) \leq 5, \deg(\alpha') \geq 7. If \(e\) is an weak edge, then \(\alpha'\) sends to \(\alpha\) the following charge:

| \(\deg(\alpha')\) | \(\deg(\alpha) = 3\) | \(4\) | \(5\) |
|---------------------|-----------------|-----|-----|
| 7, 8                | \(\frac{1}{5}\) | \(\frac{1}{5}\) | \(\frac{1}{5}\) |
| 9, 10, 11, 12      | \(\frac{1}{2}\) | \(\frac{1}{2}\) | \(\frac{1}{5}\) |
| 13, \ldots, 2518   | 1               | \(\frac{1}{2}\) | \(\frac{1}{5}\) |
| \(\geq 2519\)      | \(\frac{19}{10}\) | 1   | \(\frac{1}{5}\) |

If \(e\) is a semi-weak edge, then \(\alpha'\) sends to \(\alpha\) the following charge.

| \(\deg(\alpha')\) | \(\deg(\alpha) = 3\) | \(4\) | \(5\) |
|---------------------|-----------------|-----|-----|
| 7, 8                | \(\frac{1}{10}\) | \(\frac{1}{10}\) | \(\frac{1}{10}\) |
| 9, 10, 11, 12      | \(\frac{1}{2}\) | \(\frac{1}{2}\) | \(\frac{1}{10}\) |
| 13, \ldots, 2518   | \(\frac{1}{7}\) | \(\frac{1}{7}\) | \(\frac{1}{10}\) |
| \(\geq 2519\)      | 1               | \(\frac{1}{2}\) | \(\frac{1}{5}\) |

**Rule A4.**

Every \(k\)-face, \(k \geq 2519\), supplies the charge \(\frac{1}{2}, \frac{1}{5}\) to each incident \((3, 3, 4, k)\)-, \((3, 3, 5, k)\)-vertices, respectively.

**Lemma 4.** After applying Rule A1 – 4 to each vertex \(v\) and each face \(\alpha\), the new charge \(c^*(v)\) and \(c^*(\alpha)\) are

\[
c^*(v) \geq 0,
\]

\[
c^*(\alpha) \geq \begin{cases} 
0 & \text{if } d = 3, 4, 5, 6, \\
\frac{2}{5}, \frac{6}{5}, 1, \frac{3}{2}, \frac{5}{2}, 3 & \text{if } d = 7, 8, 9, 10, 11, 12, \\
\frac{1}{d} & \text{if } d \geq 13,
\end{cases}
\]

Here \(d = \deg(\alpha)\).
Proof. We consider several cases.

Case 1. Let \( v \) be a \( k \)-vertex, \( k \geq 3 \). Then \( c(v) = 2k - 6 \).

1-1: \( k = 3 \). Then \( c^*(v) = c(v) = 0 \).

1-2: \( k = 4 \). This 4-vertex \( v \) corresponds to one of the following types: \((3, 3, 4, \geq 2519), (3, 3, 5, \geq 2519), (3, 3, \geq 6, \geq 7), (3, 4, 4, \geq 7), (3, \geq 4, \geq 5, \geq 5), (\geq 4, \geq 4, \geq 4, \geq 5)\) since \( G \) is a counterexample. For the first type \((3, 3, 4, \geq 2519), c^*(v) = c(v) - (1 \times 2 + \frac{1}{2}) + \frac{1}{7} = 0\) by Rule A4. For the second type \( c^*(v) = 0 \) similarly. If this vertex has the type \((3, 6, 6, i), i \geq 4\), then \( c^*(v) \geq c(v) - (1 + \frac{1}{2} + \frac{1}{16}) = \frac{2}{5} \geq 0\) by Rule A2. For the other cases we can deduce that \( c^*(v) \geq 0 \).

1-3: \( k = 5 \). Then \( v \) corresponds to one of the following types: \((3, 3, 3, \geq 7), (3, 3, \geq 4, \geq 5), (\geq 3, \geq 3, \geq 4, \geq 4, \geq 4)\). For the first type \( c^*(v) = c(v) - 1 \times 4 = 0\), and for the second type \( c^*(v) \geq c(v) - (1 + 1 + 1 + \frac{1}{2} + \frac{1}{7}) = \frac{3}{10} \geq 0\). For the third type \( c^*(v) = c(v) - (1 + 1 + \frac{1}{2} + \frac{3}{2} + \frac{1}{5}) = \frac{7}{10} \geq 0\).

1-4: \( k = 6 \). Then \( v \) has the type \((\geq 3, \geq 3, \geq 3, \geq 3, \geq 3, \geq 4)\), and \( c^*(v) \geq c(v) - (1 \times 5 + \frac{1}{2}) = \frac{1}{2} \geq 0 \).

1-5: \( k \geq 7 \). The charge transferred from \( v \) is maximized when all incident faces of \( v \) are of degree 3 or when all incident faces except two 6-faces are of degree 3. Therefore \( c^*(v) \geq c(v) - \max\{1 \times k; (1 + \frac{1}{2}) \times (k - 2)\} = \min\{k - 6; \frac{1}{10}(9k - 38)\} \geq 0\).

Case 2. Let \( \alpha \) be a \( k \)-face, \( k \geq 3 \). Then \( c(\alpha) = k - 6 \).

2-1: \( k = 3 \). Let \( x_1, x_2, x_3 \) be the three vertices of \( \alpha \), and \( \beta_1, \beta_2, \beta_3 \) the three adjacent faces such that they have \( x_1x_2, x_2x_3, x_3x_1 \) in common with \( \alpha \), respectively.

[a] We first assume that all of \( x_1, x_2, x_3 \) are of degree 3. One of the followings holds: [1] all of \( \beta_1, \beta_2, \beta_3 \) are of degree \( \geq 13 \); or [2] two of \( \beta_1, \beta_2, \beta_3 \) are of degree \( \geq 2519 \). Anyway \( c^*(\alpha) \geq c(\alpha) + \min\{1 \times 3; \frac{10}{10} \times 2\} \geq 0 \).
[b] We next assume that exactly one of $x_1, x_2, x_3$ has degree $\geq 4$. Without loss of generality, let $\deg(x_1) = 4$ and $\deg(x_2) = \deg(x_3) = 3$. We further assume that $\deg(\beta_1) = 6$. Then $\deg(\beta_2) \geq 2519$ because $x_2$ is not $(3, 6, \leq 2518)$-vertex.

If $\deg(\beta_3) = 6$, then $c^*(\alpha) = c(\alpha) + (1 + \frac{1}{10}) + \frac{19\alpha}{10} = 0$ by Rule A2, therefore we set $\deg(\beta_3) \geq 7$. The face $\beta_3$ sends to $\alpha$ the charge $\geq \frac{1}{10}$, hence $c^*(\alpha) \geq c(\alpha) + 1 + (\frac{19}{10} + \frac{1}{10}) = 0$. We thus assume that $\deg(\beta_1) \geq 7$. We similarly deduce that $\deg(\beta_4) \geq 7$ by its symmetry. If one of $\beta_1, \beta_3$ has degree $\leq 12$, then $\deg(\beta_2) \geq 2519$, and then $c^*(\alpha) \geq c(\alpha) + 1 + (\frac{19}{10} \times 2 + \frac{4}{10}) \geq 0$. Therefore both $\beta_1$ and $\beta_3$ are of degree $\geq 13$. If $\beta_2$ has degree $\leq 12$, then $\beta_1$ and $\beta_3$ are of degree $\geq 2519$, and then $c^*(\alpha) \geq c(\alpha) + 1 + 1 \times 2 = 0$. Hence $\beta_2$ has degree $\geq 13$, and then $c^*(\alpha) \geq c(\alpha) + 1 + (\frac{1}{2} \times 2 + 1) = 0$.

c] We assume that exactly two of $x_1, x_2, x_3$ have degree $\geq 4$. Let $\deg(x_1) = 4$, $\deg(x_2) \geq 4$, and $\deg(x_3) = 3$. If one of the face $\beta_2, \beta_3$ has degree $\leq 12$, then the other face has degree $\geq 2519$, and then $c^*(\alpha) \geq c(\alpha) + 1 \times 2 + 1 = 0$. Therefore both $\beta_2$ and $\beta_3$ are of degree $\geq 13$, and $c^*(\alpha) \geq c(\alpha) + 1 \times 2 + (\frac{1}{2} \times 2) = 0$.

d] We finally assume that $x_1, x_2, x_3$ have degree $\geq 4$. Then $c^*(\alpha) \geq c(\alpha) + 1 \times 3 = 0$.

2-2: $k = 4$. Let $x_1, x_2, x_3, x_4$ be the four vertices of $\alpha$ in a natural circular ordering, and $\beta_1, \beta_2, \beta_3, \beta_4$ the adjacent faces such that they have $x_1, x_2, x_3, x_4, x_4, x_1$ in common with $\alpha$, respectively.

[a] We assume that all of $x_1, x_2, x_3, x_4$ are of degree $3$. Then [1] all of $\beta_1, \beta_2, \beta_3, \beta_4$ are of degree $\geq 9$; or [2] at least two of the four faces are of degree $\geq 2519$. Therefore $c^*(\alpha) \geq c(\alpha) + \min\{1 \times 2; \frac{1}{2} \times 4\} = 0$.

[b] We next assume that exactly one of $x_1, x_2, x_3, x_4$ are of degree $\geq 4$. Let $\deg(x_1) \geq 4$ and $\deg(x_2) = \deg(x_3) = \deg(x_4) = 3$. If $\deg(\beta_2) \leq 8$, then $\beta_1$ and $\beta_3$ are of degree $\geq 2519$, and then $c^*(\alpha) \geq c(\alpha) + \frac{1}{2} + (\frac{1}{2} + 1) = 0$. Therefore $\deg(\beta_2) \geq 9$, and similarly $\deg(\beta_4) \geq 9$ by its symmetry. If $\deg(\beta_1) \leq 8$, then $\beta_2$ has degree $\geq 2519$, and then $c^*(\alpha) \geq c(\alpha) + \frac{1}{2} + (1 + \frac{1}{2}) = 0$. We thus conclude
Therefore the sum of the charge sent from these two faces is at least 9, or one of them has degree $\geq \beta$ following holds; both of faces $\beta$ have degree $\geq c$. Let $\deg(\beta_1) \geq 4$, $\deg(\beta_4) \geq 9$, and $c^*(\alpha) \geq c(\alpha) + \left(\frac{1}{2} \times 2 + \frac{1}{2} \times 2\right) = 0$.

[c] We assume that consecutive two vertices of $x_1, x_2, x_3, x_4$ are of degree $\geq 4$. Let $\deg(x_1) \geq 4$, $\deg(x_2) \geq 4$, and $\deg(x_3) = \deg(x_4) = 3$. If $\deg(\beta_3) \leq 8$, then $\beta_2$ and $\beta_4$ are of degree $\geq 2519$, and then $c^*(\alpha) \geq c(\alpha) + \left(\frac{1}{2} \times 2 + \frac{1}{2} \times 2\right) = 0$. Therefore $\deg(\beta_2) \geq 9$. If $\deg(\beta_2) \leq 8$, then $\beta_3$ has degree $\geq 2519$, and then $c^*(\alpha) \geq c(\alpha) + \left(\frac{1}{2} \times 2 + 1\right) = 0$. Hence $\deg(\beta_2) \geq 9$, and similarly $\deg(\beta_4) \geq 9$. Then $c^*(\alpha) \geq c(\alpha) + \left(\frac{1}{2} \times 2 + \frac{1}{2} \times 2\right) = 0$.

[d] We assume that opposite two vertices of $x_1, x_2, x_3, x_4$ are of degree $\geq 4$. Let $\deg(x_1) \geq 4$, $\deg(x_3) \geq 4$, and $\deg(x_2) = \deg(x_4) = 3$. Then one of the following holds: [1] both of $\beta_1, \beta_2$ have degree $\geq 9$; [2] one of $\beta_1, \beta_2$ has degree $\geq 2519$. Therefore the sum of the charge sent from these two faces is at least $\frac{1}{2}$. The two faces $\beta_3, \beta_4$ similarly send to $\alpha$ the charge at least $\frac{1}{2}$. Hence $c^*(\alpha) \geq c(\alpha) + \left(\frac{1}{2} \times 2\right) = 0$.

[e] We assume that three vertices of $x_1, x_2, x_3, x_4$ are of degree $\geq 4$. Let $\deg(x_1) \geq 4$, $\deg(x_2) \geq 4$, $\deg(x_3) \geq 4$, and $\deg(x_4) = 3$. Then [1] both of $\beta_3, \beta_4$ have degree $\geq 9$, or [2] one of them has degree $\geq 2519$. Therefore $c^*(\alpha) \geq c(\alpha) + \left(\frac{1}{2} \times 3 + \frac{1}{2}\right) = 0$.

[f] We finally assume that all of $x_1, x_2, x_3, x_4$ are of degree $\geq 4$. Then $c^*(\alpha) \geq c(\alpha) + \frac{1}{2} \times 4 = 0$.

2. Assume $k = 5$. Let $x_1, \ldots, x_5$ be the vertices of $\alpha$, and $\beta_1, \ldots, \beta_5$ the faces, similarly as in the previous cases.

[a] We assume that all of $x_1, \ldots, x_5$ are of degree 3. Then at most two of $\beta_1, \ldots, \beta_5$ are 5-, 6-faces. If two of them, say $\beta_1, \beta_3$, are 5-, 6-faces, then the other three faces have degree $\geq 2519$, and then $c^*(\alpha) \geq c(\alpha) + \frac{2}{5} \times 3 \geq 0$. If exactly one of them, say $\beta_1$, is 5-, 6-faces, then its neighboring faces $\beta_2, \beta_5$ are of degree $\geq 2519$ and the other two faces $\beta_3, \beta_4$ are of degree $\geq 7$, and then $c^*(\alpha) \geq c(\alpha) + \frac{2}{5} \times 2 + \frac{1}{2} \times 2 \geq 0$. Hence all five faces are of degree $\geq 7$, and then $c^*(\alpha) \geq c(\alpha) + \frac{1}{2} \times 5 = 0$.  

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[b] We assume that one of the five vertices, say $x_1$, has degree $\geq 4$. If $\deg(\beta_3) = 5, 6$, then its neighboring faces $\beta_2, \beta_4$ have degree $\geq 2519$, and then $c^*(\alpha) \geq c(\alpha) + \frac{1}{5} + \frac{2}{5} \times 2 = 0$. Thus $\deg(\beta_3) \geq 7$. One of the followings holds; [1] both of $\beta_4, \beta_5$ have degree $\geq 7$; or [2] one of $\beta_4, \beta_5$ has degree $\geq 2519$. The sum of the charge sent from these two faces is at least $\frac{1}{5}$ in either case. If $\deg(\beta_2) = 5, 6$, then $\beta_1$ and $\beta_3$ have degree $\geq 2519$, and then $c^*(\alpha) \geq c(\alpha) + \frac{1}{5} + (\frac{1}{5} + \frac{2}{5} + \frac{1}{5}) = 0$. Thus $\deg(\beta_2) \geq 7$. If $\deg(\beta_1) = 5, 6$, then $\beta_2$ has degree $\geq 2519$, and then $c^*(\alpha) \geq c(\alpha) + \frac{1}{5} + (\frac{1}{5} \times 3 + \frac{1}{10} \times 2) = 0$. Thus $\deg(\beta_1) \geq 7$. We can similarly deduce that $\deg(\beta_1) \geq 7, \deg(\beta_5) \geq 7$, therefore $c^*(\alpha) \geq c(\alpha) + \frac{1}{5} + (\frac{1}{5} \times 3 + \frac{1}{10} \times 2) = 0$.

[c] We assume that consecutive two vertices of $x_1, \ldots, x_5$ are of degree $\geq 4$. Let $\deg(x_1) \geq 4, \deg(x_2) \geq 4$, and $\deg(x_3) = \deg(x_4) = \deg(x_5) = 3$. If $\deg(\beta_3) = 5, 6$, then $\beta_2$ and $\beta_4$ are of degree $\geq 2519$, and then $c^*(\alpha) \geq 0$. Therefore $\deg(\beta_3) \geq 7$, and similarly $\deg(\beta_4) \geq 7$. If $\deg(\beta_2) = 5, 6$, then $\beta_3$ is of degree $\geq 2519$, and then $c^*(\alpha) \geq 0$. Therefore $\deg(\beta_2) \geq 7$ and $\deg(\beta_5) \geq 7$ are similarly deduced. And then $c^*(\alpha) \geq c(\alpha) + \frac{1}{5} \times 2 + (\frac{1}{5} \times 2 + \frac{1}{10} \times 2) = 0$. The other case that non-consecutive two vertices of $x_1, \ldots, x_5$ are of degree $\geq 4$ is similar.

[d] We can treat the remaining cases that three, four or five vertices of $x_1, \ldots, x_5$ are of degree $\geq 4$ as well as above cases, and can deduce that $c^*(\alpha) \geq 0$.

2-4: $k = 6$. Then $c^*(\alpha) = c(\alpha) = 0$.

2-5: $k = 7, 8$. The transfer from $\alpha$ is possible along $(3, k)$-,$(4, k)$-,$(5, k)$-edges which are weak or semi-weak. Since there are no consecutive two such edges which are weak, and since there are no consecutive three such edges which are semi-weak, $c^*(\alpha) \geq c(\alpha) - \frac{1}{3} \times 3 = \frac{2}{3}$ if $k = 7$, and $c^*(\alpha) \geq c(\alpha) - \frac{1}{5} \times 4 = \frac{6}{5}$ if $k = 8$.

2-6: $9 \leq k \leq 12$. we can similarly deduce that $c^*(\alpha) \geq c(\alpha) - \frac{1}{2} \times \lfloor \frac{k}{2} \rfloor$. This value is $1, \frac{3}{2}, \frac{5}{2}, 3$ for $k = 9, 10, 11, 12$, respectively.
2-7: $13 \leq k \leq 2518$. Then $c^*(\alpha) \geq c(\alpha) - 1 \times \lceil \frac{k}{2} \rceil = (k - 6) - \lfloor \frac{k}{2} \rfloor$, and this value is not less than $\frac{1}{21}k$ when $13 \leq k \leq 2518$.

2-8: $k \geq 2519$. The transfer from $\alpha$ is possible along $(3,k)$-, $(4,k)$-, $(5,k)$-edges which are weak or semi-weak. We notice that there are no consecutive two (resp. consecutive three) $(3,k)$-edges which are weak (resp. semi-weak), and that there are no consecutive $(4,k)$-edges which are weak. Therefore the transfer from $\alpha$ is maximized when [1] every other edges on $\alpha$ are weak $(3,k)$-edges and $k$ is even; [2] except two consecutive $(3,k)$-edges which are incident with a $(3,3,4,k)$-vertex on $\alpha$, every other edges on $\alpha$ are weak $(3,k)$-edges and $k$ is odd. If $k$ is even, $c^*(\alpha) \geq c(\alpha) - \frac{19}{10} \times \frac{k}{2} = \frac{1}{20}k - 6 \geq \frac{1}{21}k$ because $k \geq 2520$. If $k$ is odd, $c^*(\alpha) \geq c(\alpha) - \left(\frac{19}{10} \times \frac{k-3}{2} + 2 + \frac{1}{2}\right) = \frac{1}{20}k - 6 + \frac{7}{20} > \frac{1}{21}k$ because $k \geq 2519$.

As a consequence, we establish Lemma 4. $\square$

Rule B.

Each major faces $\alpha$ sends its charge $c^*(\alpha)$ equally to its incident vertices.

We notice that each major face sends at least $\frac{1}{21}$ charge to its incident vertices because $\min\{\frac{2}{3} \times \frac{1}{3}; \frac{5}{6} \times \frac{1}{3}; \frac{1}{4} \times \frac{1}{3}; \frac{5}{6} \times \frac{1}{4}; \frac{3}{4} \times \frac{1}{3}; \frac{1}{3}k \times \frac{1}{3}\} = \frac{1}{21}$.

Lemma 5. After applying Rule B, we have the new charge $c^{**}(v)$:

$$c^{**}(v) \geq \frac{1}{21} \text{ for all vertices } v \in V(G).$$

Proof. Let $v$ be a $k$-vertex, $k \geq 3$.

We first assume that $k = 3$. Since $G$ is a counterexample, this vertex $v$ corresponds to one of the following types: $(3,i, \geq 2519)$; $i = 6, \ldots, 12$, $(3, \geq 13, \geq 13)$, $(4,i, \geq 2519)$; $j = 5, \ldots, 8$, $(4, \geq 9, \geq 9)$, $(5,5, \geq 2519)$, $(5,6, \geq 2519)$, $(5,7, \geq 7)$, $(\geq 6, \geq 6, \geq 7)$. In each case, $v$ is incident with at least one major face, therefore
\(c^{**}(v) \geq \frac{1}{21}\).

We next assume that \(k = 4\). This vertex is one of the following types: \((3, 3, 4, \geq 2519)\); \((3, 3, 5, \geq 2519)\); \((3, 3, \geq 6, \geq 7)\); \((3, 4, 4, \geq 7)\); \((3, \geq 4, \geq 5, \geq 5)\); \((\geq 4, \geq 4, \geq 4, \geq 5)\). If \(v\) has the type \((3, i, 6, 6)\) as in Rule A2, then \(c^{**}(v) \geq c(v) - (1 + \frac{1}{10} + \frac{1}{2}) = \frac{2}{5} \geq \frac{1}{21}\). If \(v\) has the type \((3, \geq 4, \geq 5, \geq 5)\), then the charge of \(v\) is still remained, i.e., \(c^{**}(v) \geq c^*(v) \geq c(v) - (1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10}) = \frac{1}{10} \geq \frac{1}{21}\). If \(v\) has the type \((\geq 4, \geq 4, \geq 4, \geq 5)\), then similarly \(c^{**}(v) \geq c(v) - (\frac{1}{2} + \frac{1}{5} + \frac{1}{10}) = \frac{1}{5} \geq \frac{1}{21}\). For the other cases, \(v\) is incident with at least one major face, and \(c^{**}(v) \geq \frac{1}{21}\).

We set \(k = 5\). This vertex is one of the following types: \((3, 3, 3, 3, \geq 7)\); \((3, 3, \geq 4, \geq 5)\); \((3, \geq 3, \geq 4, \geq 4, \geq 4)\). If \(v\) is a \((3, 3, 3, 3, \geq 7)\)-vertex, then \(c^{**}(v) \geq c^*(v) + \frac{1}{21} \times 1 = \frac{1}{21}\). If \(v\) is a \((3, 3, 3, 6, 6)\)-vertex, then \(c^{**}(v) = c(v) - (1 + \frac{1}{10}) \times 3 = 4 - \frac{33}{10} \geq \frac{1}{21}\). For the other cases, we observe that the charge of \(v\) is still remained, and \(c^{**}(v) \geq \frac{1}{21}\).

We set \(k \geq 6\). This vertex \(v\) has the type \((\geq 3, \geq 3, \geq 3, \geq 3, \geq 3, \geq 4)\) if \(k = 6\). In any case, the charge of \(v\) is still remained after applying Rule A1-4, and we can deduce that \(c^{**}(v) \geq \frac{1}{21}\). \(\square\)

Euler’s formula together with Lemma 5 and the hypothesis \(n > 126|\chi(M)|\) yields

\[
6|\chi(M)| = \sum_{v \in V(G)} c(v) + \sum_{\alpha \in \mathcal{F}(G)} c(\alpha) = \sum_{v \in V(G)} c^{**}(v) + \sum_{\alpha \in \mathcal{F}(G)} c^{**}(\alpha)
\geq \sum_{v \in V(G)} c^{**}(v) \geq \frac{1}{21}n > 6|\chi(M)|,
\]

a contradiction. This completes the proof of Theorem 2.
3. Path Transferability of Graphs on Surfaces

In this section we treat path transferability of graphs on general surfaces. We first prepare several notations: A path consists of distinct vertices \( v_0, v_1, \ldots, v_n \) and edges \( v_0v_1, v_1v_2, \ldots, v_{n-1}v_n \). Through this paper we assume that each path has a direction. The reverse path of \( P \) is denoted by \( P^{-1} \). The number of edges in a path \( P \) is called its length, and a path of length \( n \) is called an \( n \)-path. The last (resp. first) vertex of a path \( P \) in its direction is called the head (resp. tail) of \( P \) and is denoted by \( h(P) \) (resp. \( t(P) \)); for \( P = \langle v_0v_1 \cdots v_{n-1}v_n \rangle \), we set \( h(P) = v_n \) and \( t(P) = v_0 \). The set of all inner vertices of \( P \), the vertices that are neither the head nor the tail, is denoted by \( \text{Inn}(P) \).

We are interested in the movement of a path along a graph, which seems as a "train" moving on the graph: Let \( P \) be an \( n \)-path. If \( h(P) \) has a neighboring vertex \( v \notin \text{Inn}(P) \), then we have a new \( n \)-path \( P' \) by removing the vertex \( t(P) \) from \( P \) and adding \( v \) to \( P \) as its new head. We say that \( P \) take a step to \( v \), and denote it by \( P \xrightarrow{v} P' \) (or briefly \( P \to P' \)). If there is a sequence of \( n \)-paths \( P \to \cdots \to Q \), then we say that \( P \) can transfer (or move) to \( Q \), and denote it by \( P \rightsquigarrow Q \). A graph \( G \) is called \( n \)-path-transferable or \( n \)-transferable if \( G \) has at least one \( n \)-path and if \( P \rightsquigarrow Q \) for any pair of directed \( n \)-paths \( P, Q \) in \( G \). The maximum number \( n \) for which \( G \) is \( n \)-path-transferable is called the path transferability of \( G \). The following result says that for a fixed surface the number of polyhedral maps whose path transferability are more than 12 is finite:

**Main Theorem.** If a polyhedral map \( G \) on a surface \( \mathcal{M} \) of Euler characteristic \( \chi(\mathcal{M}) \leq 0 \) has more than \( 126|\chi(\mathcal{M})| \) vertices, then path transferability of \( G \) is at most 12.

**Proof.** Let \( G \) be a graph as above. By Theorem 2, \( G \) contains one of the light vertices in Table 1. We assume that such a vertex has the type \((3, 12, \leq 2518)\). Then we can find a path of length 13 which cannot move any longer (see Fig.1), therefore path transferability of \( G \) is at most 12. For the other types, we can
similarly find clogged paths of length $\leq 12$ around the light vertices. Hence path transferability of $G$ is at most 12. \hfill \Box

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{Figure1.png}
\caption{A path clogged around a light vertex (3, 12, $\leq 2518$).}
\end{figure}

Let $G$ be a polyhedral map on a surface whose faces are of degree 6, and $G^\Delta$ the truncated graph of $G$. This graph $G^\Delta$ has path transferability 12, thus the value 12 in this theorem is best possible (see Fig.2).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{Figure2.png}
\caption{Paths of length twelve can move from one to another in this graph.}
\end{figure}

4. Combinatorial Curvature

By considering the graphs in Fig.3, we can see that the several types in Table 1 is in some sense tight; we cannot remove the types $(a_1, a_2, \ldots, a_n)$ with $\dagger$ mark from the list. On the other hand, Higuchi [11] studied the combinatorial curvature, introduced by Gromov [3], that is defined as

$$\Phi(v) = 1 - \frac{\deg(v)}{2} + \sum_{\alpha \in F(v)} \frac{1}{\deg(\alpha)}.$$
where $F(v)$ is the set of faces incident with a vertex $v$. Higuchi conjectured the following:

**Conjecture 1 (Higuchi).** Let $G$ be a finite or infinite planar graph. If $\Phi(v) > 0$ for all $v \in V(G)$, then $G$ has finite number of vertices.

This conjecture was partly confirmed by Higuchi himself [4] for some special cases, and by Sun and Yu [12] for the case of 3-regular graphs. The conjecture is fully solved by Devos and Mohar [2] by establishing a Gauss-Bonnet inequality on polygonal surface. B. Chen and G. Chen [1] further investigated this study.

We will expect the following for a polyhedral map on a fix surface:
Conjecture 2. Let \( G \) be a simple polyhedral map on a surface \( M \) of Euler characteristic \( \chi(M) \leq 0 \). There exists a constant number \( c_M \) for each \( M \) such that \( G \) contains a vertex \( v \) with \( \Phi(v) \geq 0 \) if \( |V(G)| > c_M|\chi(M)| \).

This means that there exists a vertex whose surrounding area looks convex or flat if a polyhedral map has sufficiently large number of vertices. From the aspect of the combinatorial curvature, the upper bound 2518 in Table 1 will be expected to improve as \((3, 7, \leq 42), (3, 8, \leq 24), (3, 9, \leq 18), (3, 10, \leq 15), (3, 11, \leq 13), (3, 12, 12), (4, 5, \leq 20), (4, 6, \leq 12), (4, 7, \leq 9), (4, 8, 8), (5, 5, \leq 10), (5, 6, \leq 7), (3, 3, 4, \leq 12), (3, 3, 5, \leq 7)\), respectively, for some number \( c_M \).

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