A Generalization of a Logarithmic Sobolev Inequality to the Hölder Class

H. Ibrahim

1 Mathematics Department, School of Arts and Sciences, Lebanese International University (LIU), Beirut Campus, Al-Mouseitbeh, P.O. Box 14-6404, Beirut, Lebanon
2 Mathematics Department, Faculty of Sciences, Lebanese University, Hadeth, Beirut, Lebanon

Correspondence should be addressed to H. Ibrahim, hassan.ibrahim@liu.edu.lb

Received 6 December 2010; Accepted 14 December 2010

1. Introduction and Main Results

In [1], a generalization of the Ogawa type inequality [2] to the parabolic framework has been shown. Ogawa inequality can be considered as a generalized version in the Lizorkin-Triebel spaces of the remarkable estimate of Brézis-Gallouët-Wainger [3, 4] that holds in a limiting case of the Sobolev embedding theorem. The inequality showed in [1, Theorem 1.1] provides an estimate of the $L^\infty$ norm of a function in terms of its parabolic BMO norm, with the aid of the square root of the logarithmic dependency of a higher order Sobolev norm. More precisely, for any vector-valued function $f = \nabla g \in W_2^{2m,m}(\mathbb{R}^{n+1})$, $g \in L^2(\mathbb{R}^{n+1})$ with $m, n \in \mathbb{N}^*, 2m > (n + 2)/2$, there exists a constant $C = C(m, n) > 0$ such that:

$$
\|f\|_{L^\infty(\mathbb{R}^{n+1})} \leq C \left(1 + \|f\|_{BMO(\mathbb{R}^{n+1})}\left(\log^+\left(\|f\|_{W_2^{2m,m}(\mathbb{R}^{n+1})} + \|g\|_{L^\infty(\mathbb{R}^{n+1})}\right)\right)^{1/2}\right),
$$

(1.1)

where $W_2^{2m,m}$ is the parabolic Sobolev space (we refer to [5] for the definition and further properties), and BMO is the parabolic bounded mean oscillation space (defined via parabolic
balls instead of Euclidean ones [1, Definition 2.1]). The above inequality reflects a limiting case of Sobolev embeddings in the parabolic framework (see [6, 7] for similar type inequalities, and [2–4, 8–11] for various elliptic versions). By considering functions \( f \in W^{2\gamma,m}_2(\Omega_T) \) defined on the bounded domain

\[
\Omega_T = (0,1)^n \times (0,T), \quad T > 0,
\]

we have the following estimate (see [1, Theorem 1.2]):

\[
\| f \|_{L^2(\Omega_T)} \leq C \left( 1 + \left( \| f \|_{BMO(\Omega_T)} + \| f \|_{L^1(\Omega_T)} \right) \left( \log^+ \| f \|_{W^{2\gamma,m}_2(\Omega_T)} \right)^{1/2} \right),
\]

The different norms of \( f \) appearing in inequalities (1.1) and (1.3) are finite since

\[
W^{2\gamma,m}_2 \hookrightarrow C^{\gamma/2} \hookrightarrow L^\infty \hookrightarrow \text{BMO}, \quad \text{for some } 0 < \gamma < 1,
\]

where \( C^{\gamma/2} \) is the parabolic Hölder space that will be defined later. Moreover, it is easy to check that \( g \) is bounded and continuous.

The purpose of this paper is to show that the condition \( f = \nabla g \in W^{2\gamma,m}_2 \) (vector-valued case), or \( f \in W^{2\gamma,m}_2 \) (scalar-valued case) can be relaxed. Indeed, inequalities (1.1) and (1.3) can be applied to a wider class of Hölder continuous functions \( f = \nabla g \in C^{\gamma/2}, 0 < \gamma < 1 \) (vector-valued case), or \( f \in C^{\gamma/2} \) (scalar-valued case). To be more precise, we now state the main results of this paper. Our first theorem is the following:

**Theorem 1.1 (Logarithmic Hölder inequality on \( \mathbb{R}^{n+1} \)).** Let \( 0 < \gamma < 1 \). For any \( f = \nabla g \in C^{\gamma/2}(\mathbb{R}^{n+1}) \cap L^2(\mathbb{R}^{n+1}) \) with \( g \in L^2(\mathbb{R}^{n+1}) \), there exists a constant \( C = C(\gamma, n) > 0 \) such that

\[
\| f \|_{L^2(\mathbb{R}^{n+1})} \leq C \left( 1 + \| f \|_{BMO(\mathbb{R}^{n+1})} \left( \log^+ \left( \| f \|_{C^{\gamma/2}(\mathbb{R}^{n+1})} + \| g \|_{L^2(\mathbb{R}^{n+1})} \right) \right)^{1/2} \right),
\]

The second theorem deals with functions defined on the bounded domain \( \Omega_T \).

**Theorem 1.2 (Logarithmic Hölder inequality on a bounded domain).** Let \( 0 < \gamma < 1 \). For any \( f \in C^{\gamma/2}(\Omega_T) \), there exists a constant \( C = C(\gamma, n, T) > 0 \) such that

\[
\| f \|_{L^2(\Omega_T)} \leq C \left( 1 + \left( \| f \|_{BMO(\Omega_T)} + \| f \|_{L^1(\Omega_T)} \right) \left( \log^+ \left( \| f \|_{C^{\gamma/2}(\Omega_T)} \right) \right)^{1/2} \right).
\]

We notice that inequalities (1.5) and (1.6) directly imply (with the aid of the embeddings (1.4)), (1.1), and (1.3).

**Remark 1.3.** The same inequality (1.5) still holds for scalar-valued functions \( f = \partial g / \partial x_i \in C^{\gamma/2}(\mathbb{R}^{n+1}) \cap L^2(\mathbb{R}^{n+1}), i = 1, \ldots, n + 1, \) with \( g \in L^\infty(\mathbb{R}^{n+1}) \).

This paper is organized as follows: in Section 2, we give the definitions of some basic functional spaces used throughout this paper. Section 3 is devoted to the proofs of the main results.
2. Definitions

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^{n+1}$. A generic element $z \in \mathbb{R}^{n+1}$ has the form $z = (x,t)$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We begin by defining parabolic H"older spaces $C^{\gamma/2}$.

Definition 2.1 (Parabolic H"older spaces). For $0 < \gamma < 1$, we define the parabolic space of H"older continuous functions of order $\gamma$ in the following way:

$$C^{\gamma/2}(\mathcal{O}) = \left\{ f \in C\left(\overline{\mathcal{O}}\right), \|f\|_{C^{\gamma/2}(\mathcal{O})} < \infty \right\},$$

(2.1)

where

$$\|f\|_{C^{\gamma/2}(\mathcal{O})} = \|f\|_{L^p(\mathcal{O})} + \langle f \rangle_{x,\mathcal{O}}^{(\gamma)} + \langle f \rangle_{t,\mathcal{O}}^{(\gamma/2)},$$

(2.2)

with

$$\langle f \rangle_{x,\mathcal{O}}^{(\gamma)} = \sup_{(x,t),(x',t) \in \mathcal{O}, x \neq x'} \frac{|f(x,t) - f(x',t)|}{|x - x'|^\gamma},$$

$$\langle f \rangle_{t,\mathcal{O}}^{(\gamma/2)} = \sup_{(x,t),(x',t) \in \mathcal{O}, t \neq t'} \frac{|f(x,t) - f(x,t')|}{|t - t'|^{\gamma/2}}.$$

For a detailed study of parabolic H"older spaces, we refer the reader to [5]. We now briefly recall some basic facts about Littlewood-Paley decomposition which are crucial in obtaining our logarithmic inequalities. Given the expansive $(n+1) \times (n+1)$ matrix $A = \text{diag}\{2, \ldots, 2, 2^2\}$ (parabolic anisotropy), the corresponding Littlewood-Paley decomposition asserts that any tempered distribution $f \in S'(\mathbb{R}^{n+1})$ can be decomposed as

$$f = \sum_{ j \in \mathbb{Z} } \varphi_j * f, \quad \text{where } \varphi_j(z) = |\det A| \varphi(A^j z),$$

(2.4)

with the convergence in $S'/\mathcal{P}$ (modulo polynomials). Here $\varphi \in S(\mathbb{R}^{n+1})$ is a test function such that supp $\varphi$ is compact and bounded away from the origin, and $\sum_{ j \in \mathbb{Z} } \varphi(A^j z) = 1$ for all $z \in \mathbb{R}^{n+1} \setminus \{0\}$, where $\varphi$ is the Fourier transform of $\varphi$. The sequence $(\varphi_j)_{j \in \mathbb{Z}}$ is mainly used to define parabolic homogeneous Lizorkin-Triebel, Hardy, and Besov spaces (see for instance [12, 13]). We only present here the spaces that are used throughout the analysis. For $1 \leq p \leq \infty$, we define the parabolic homogeneous Lizorkin-Triebel space $F^0_{p,2}$ as the space of functions $f \in S'(\mathbb{R}^{n+1})$ with finite quasinorms:

$$\|f\|_{F^0_{p,2}(\mathbb{R}^{n+1})} = \left\| \left( \sum_{ j \in \mathbb{Z} } |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{n+1})} < \infty.$$

(2.5)
The space $F^0_{p,2}$ can be identified with the parabolic Hardy space $H^p$, $1 \leq p < \infty$ having the following square function characterization stated informally as:

$$
H^p \left( \mathbb{R}^{n+1} \right) = \left\{ f \in S' \left( \mathbb{R}^{n+1} \right); \left( \sum_{|j|<\infty} |\varphi_j * f|^2 \right)^{1/2} \in L^p \right\}.
$$

(2.6)

This identification (see Bownik [14]) can be stated as follows: for all $1 \leq p < \infty$, we have

$$
F^0_{p,2} \left( \mathbb{R}^{n+1} \right) \cong H^p \left( \mathbb{R}^{n+1} \right).
$$

(2.7)

Now, for defining the inhomogeneous parabolic Besov space $B^I_{\infty,\infty}$ used later in obtaining our results, we use a slightly different sequence. Indeed, let $\theta \in C_0^\infty (\mathbb{R}^{n+1})$ be any cut-off function satisfying:

$$
\theta(z) = \begin{cases} 
1, & \text{if } |z|_p \leq 1, \\
0, & \text{if } |z|_p \geq 2,
\end{cases}
$$

(2.8)

where $|\cdot|_p$ is the parabolic quasinorm associated to the matrix $A$ (see [1]). Taking the new function (but keeping the same notation) $\varphi_0$ defined via the relation

$$
\hat{\varphi}_0 = \vartheta,
$$

(2.9)

we can give the definition of the Besov space $B^I_{\infty,\infty}$.

**Definition 2.2** (Parabolic inhomogeneous Besov spaces). Take the smoothness parameter $0 < \gamma < 1$. Let $(\varphi_j)_{j \in \mathbb{Z}}$ be the sequence such that $\varphi_0$ is given by (2.9), while $\varphi_j$ is given by (2.4) for all $j \geq 1$. We define the parabolic inhomogeneous Besov space $B^I_{\infty,\infty}$ as the space of all functions $f \in S'(\mathbb{R}^{n+1})$ with finite quasinorms

$$
\|f\|_{B^I_{\infty,\infty}}^p = \sup_{j \geq 0} 2^{ij} \|\varphi_j * f\|_{L^p(\mathbb{R}^{n+1})},
$$

(2.10)

**3. Proofs of Theorems**

The proof of Theorem 1.1 relies on the following two lemmas of different interests.

**Lemma 3.1.** Let $0 < \gamma < 1$ and let $N > 0$ be a positive integer. Then for any $f = \nabla g \in C^{1+\gamma/2}(\mathbb{R}^{n+1}) \cap L^2(\mathbb{R}^{n+1})$ with $g \in L^2(\mathbb{R}^{n+1})$, there exists a constant $C = C(\gamma, n) > 0$ such that

$$
\left\| \left( \sum_{j<-N} 2^{-2\gamma j} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^p} \leq C \|g\|_{L^p}.
$$

(3.1)
Proof. We provide a proof of (3.1) in the general case $N = 1$. We use the fact that $\partial_i g = f_i$ (for which we keep denoting it by $f$, i.e. $f = f_i$) for some $i = 1, \ldots, n + 1$, with $g \in L^\infty(\mathbb{R}^{n+1})$. For $z \in \mathbb{R}^{n+1}$, define

$$
\Phi(z) = (\partial_i \varphi)(z),
$$

(3.2)

$$
\Phi_j(z) = |\det A^j|\Phi(A^j z), \quad \forall j \leq -1.
$$

(3.3)

Using (2.4) we obtain:

$$
(\partial_i \varphi_j)(z) = \begin{cases}
2/\Phi_j(z) & \text{if } i = 1, \ldots, n, \\
2^{2j}/\Phi_j(z) & \text{if } i = n + 1.
\end{cases}
$$

(3.4)

We now compute (see (3.3) and (3.4)):

$$
\left\| \left( \sum_{j \leq -1} 2^{-2j/|\varphi_j * f|^2} \right)^{1/2} \right\|_{L^\infty} \leq C \sup_{j \leq -1} \|\Phi_j * g\|_{L^\infty},
$$

(3.5)

where the constant $C$ is given by:

$$
C^2 = \begin{cases}
\sum_{j \leq -1} 2^{2j(1-\gamma)}, & \text{if } i = 1, \ldots, n, \\
\sum_{j \leq -1} 2^{2j(2-\gamma)}, & \text{if } i = n + 1,
\end{cases}
$$

(3.6)

which is finite $0 < C < +\infty$ under the choice

$$
0 < \gamma < 1.
$$

(3.7)

In order to terminate the proof, it suffices to show that

$$
\|\Phi_j * g\|_{L^\infty} \leq C \|g\|_{L^\infty},
$$

(3.8)

which can be deduced, by translation and dilation invariance, from the following estimate:

$$
|\Phi * g(0)| \leq C \|g\|_{L^\infty}.
$$

(3.9)

Indeed, define the positive radial decreasing function $h(r) = h(\|z\|)$ as follows:

$$
h(r) = \sup_{\|z\| \geq r} |\Phi(z)|.
$$

(3.10)
From (3.2), we remark that the function $\Phi$ is the inverse Fourier transform of a compactly supported function. Hence, we have

$$h(0) = \|\Phi\|_{L^\infty} < +\infty,$$  

(3.11)

and the asymptotic behavior

$$h(r) \leq \frac{C}{r^{n+2}}, \quad \forall r \geq 1.$$  

(3.12)

We compute (taking $S^n_r$ as the $n$-dimensional sphere of radius $r$):

\[
\begin{aligned}
|((\Phi * g)(0)| &\leq \int_{\mathbb{R}^{n+1}} |\Phi(-z)||g(z)| dz \\
&\leq \int_0^\infty \left( \int_{S^n_r} |\Phi(-z)||g(z)| d\sigma(z) \right) dr \\
&\leq C \left( \int_0^\infty r^n h(r) dr \right) \|g\|_{L^\infty}.
\end{aligned}
\]

(3.13)

Using (3.11) and (3.12) we deduce that:

\[
\begin{aligned}
\int_0^\infty r^n h(r) dr &= \int_0^1 r^n h(r) dr + \int_1^\infty r^n h(r) dr \\
&\leq C \left( \int_0^1 h(0) dr + \int_1^\infty \frac{r^n}{r^{n+2}} dr \right) \\
&\leq C(\|\Phi\|_{L^\infty} + 1),
\end{aligned}
\]

(3.14)

which, together with (3.13), directly implies (3.9). As a conclusion, we obtain (see (3.5)):

\[
\left\| \left( \sum_{|j| \leq N} 2^{-2nj} |\varphi_j * f| \right)^{1/2} \right\|_{L^\infty} \leq C \|g\|_{L^\infty},
\]

(3.15)

and hence inequality (3.1) holds.

\[\square\]

**Lemma 3.2.** Let $N > 0$ be a positive integer. Then for any $f \in BMO(\mathbb{R}^{n+1})$ there exists a constant $C = C(n) > 0$ such that:

\[
\left\| \left( \sum_{|j| < N} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^\infty} \leq C \|f\|_{BMO}.
\]

(3.16)
Proof. The proof provides inequality (3.16) for all \( |j| < \infty \) by showing that \( F_{\infty,2}^{0} \approx \text{BMO} \) and then using (2.5). Before starting the proof, we remind the reader that:

\[
\|f\|_{\text{BMO}(\mathbb{R}^{n+1})} = \sup_{Q \subseteq \mathbb{R}^{n+1}} \inf_{c \in \mathbb{R}} \left( \frac{1}{|Q|} \int_{Q} |f - c| \right),
\]

(3.17)

where \( Q \) denotes any arbitrary parabolic cube. Using the result of Bownik [15, Theorem 1.2], we have the following duality argument (that can be viewed as the parabolic extension of the well-known isotropic result of Triebel [12], and Frazier and Jawerth [16]):

\[
\left( F_{1,2}^{0} \right)' \approx F_{\infty,2}^{0},
\]

(3.18)

where \( \left( F_{1,2}^{0} \right)' \) stands for the dual space of \( F_{1,2}^{0} \). Applying (2.7) with \( p = 1 \) we obtain:

\[
F_{1,2}^{0} \approx H^{1}.
\]

(3.19)

Using the description of the dual of parabolic Hardy spaces \( H^{p} \) for \( 0 < p \leq 1 \) (see Bownik [17, Theorem 8.3]), we get:

\[
(H^{p})' = C_{q,s}^{l}
\]

(3.20)

with the terms \( p, l, q, s \) chosen such that:

\[
l = \frac{1}{p} - 1, \quad 1 \leq \frac{q}{q-1} \leq \infty, \quad p < \frac{q}{q-1},
\]

(3.21)

\[s \in \mathbb{N}, \quad s \geq |l|, \quad |l| = \max\{n \in \mathbb{Z}; n \leq l\}.
\]

The function space \( C_{q,s}^{l} \), \( l \geq 0, 1 \leq q < \infty \) and \( s \in \mathbb{N} \) (called the Campanato space), is the space of all \( f \in L_{\text{loc}}^{q} (\mathbb{R}^{n+1}) \) (defined up to addition by \( P \in \mathcal{P}_{s} \), the set of all polynomials in \( n+1 \) variables of degree at most \( s \)) so that:

\[
\|f\|_{C_{q,s}^{l}(\mathbb{R}^{n+1})} = \sup_{Q \subseteq \mathbb{R}^{n+1}} \inf_{P \in \mathcal{P}_{s}} \left( \frac{1}{|Q|} \int_{Q} |f - P|^{q} \right)^{1/q} < \infty.
\]

(3.22)

Choosing \( p = 1, l = 0, q = 1 \) and \( s = 0 \), we can easily see that conditions (3.21) are all satisfied, and that (see (3.22) and (3.17)):

\[
C^{0}_{1,0} \approx \text{BMO}.
\]

(3.23)
This identification, together with (3.20), finally gives

$$
(H^1)' = \text{BMO}.
$$

(3.24)

The proof then directly follows from (3.18), (3.19), and (3.24).

**Proof of Theorem 1.1.** Let \( N \in \mathbb{N} \) be any arbitrary integer. Using (2.4), we estimate \( \| f \|_{L^\infty} \) in the following way:

$$
\| f \|_{L^\infty} \leq \left\| \sum_{j \leq -N} 2^{\gamma j/2} \left| \varphi_j * f \right| \right\|_{L^\infty} + \left\| \sum_{|j| \leq N} \left| \varphi_j * f \right| \right\|_{L^\infty} + \left\| \sum_{j > N} 2^{-\gamma j/2} \left| \varphi_j * f \right| \right\|_{L^\infty}
$$

\[ A_1 \]

$$
\leq C_1 2^{-\gamma N} \left\| \left( \sum_{j \leq -N} 2^{-\gamma j/2} \left| \varphi_j * f \right|^2 \right)^{1/2} \right\|_{L^\infty} + (2N + 1)^{1/2} \left\| \left( \sum_{|j| \leq N} \left| \varphi_j * f \right|^2 \right)^{1/2} \right\|_{L^\infty}
\]

\[ A_2 \]

$$
+ C'_1 2^{-\gamma N} \left( \sup_{j > N} \| \varphi_j * f \|_{L^\infty} \right),
$$

(3.25)

where

$$
C_1 = \left( \frac{1}{2^{2\gamma} - 1} \right)^{1/2}, \quad C'_1 = \frac{2^{-\gamma}}{1 - 2^{-\gamma}}.
$$

(3.26)

Using (3.1), we assert that:

$$
A_1 \leq C \| g \|_{L^\infty},
$$

(3.27)

while (3.16) gives:

$$
A_2 \leq C \| f \|_{\text{BMO}}.
$$

(3.28)

In order to estimate \( A_3 \), we proceed in the following way:

$$
A_3 \leq \sup_{j \geq 1} 2^{\gamma j} \| \varphi_j * f \|_{L^\infty} \leq \sup_{j \geq 1} 2^{\gamma j} \| \varphi_j * f \|_{L^\infty} + \| \varphi_0 * f \|_{L^\infty}, \quad \varphi_0 \text{ is given by (2.9),}
$$

(3.29)

hence (see Definition 2.2)

$$
A_3 \leq \| f \|_{\mathcal{B}_{\gamma, \infty}}.
$$

(3.30)
Using the well-known result (see, e.g., [18])

\[ B_{2,\infty}^\gamma = C\gamma^{1/2}, \] (3.31)

we finally obtain

\[ A_3 \leq \|f\|_{C\gamma^{1/2}}. \] (3.32)

Inequalities (3.25), (3.27), (3.28), and (3.32) imply

\[ \|f\|_{L^\infty} \leq C \left( (2N + 1)^{1/2} \|f\|_{\text{BMO}} + 2^{-\gamma N} (\|f\|_{C\gamma^{1/2}} + \|g\|_{L^\infty}) \right). \] (3.33)

We optimize (3.33) in \( N \) by setting:

\[ N = 1 \quad \text{if} \quad \|f\|_{C\gamma^{1/2}} + \|g\|_{L^\infty} \leq 2^\gamma \|f\|_{\text{BMO}}. \] (3.34)

Then it is easy to check (using (3.33)) that

\[ \|f\|_{L^\infty} \leq C \|f\|_{\text{BMO}} \left( 1 + \left( \log^+ \frac{\|f\|_{C\gamma^{1/2}} + \|g\|_{L^\infty}}{\|f\|_{\text{BMO}}} \right)^{1/2} \right)^{1/2}. \] (3.35)

In the case where \( \|f\|_{C\gamma^{1/2}} + \|g\|_{L^\infty} > 2^\gamma \|f\|_{\text{BMO}}, \) we take \( 1 \leq \beta < 2^\gamma \) such that

\[ N = N(\beta) = \log^+ \left( \beta \frac{\|f\|_{C\gamma^{1/2}} + \|g\|_{L^\infty}}{\|f\|_{\text{BMO}}} \right) - \frac{1}{2} \in \mathbb{N}. \] (3.36)

In fact this is valid since the function \( N(\beta) \) varies continuously from \( N(1) \) to \( N(2^\gamma) = 1 + N(1) \) on the interval \([1, 2^\gamma]\). Using (3.33) with the above choice of \( N \), we obtain:

\[ \|f\|_{L^\infty} \leq C \left[ 2^{1/2} \left( \log^+ \left( \beta \frac{\|f\|_{C\gamma^{1/2}} + \|g\|_{L^\infty}}{\|f\|_{\text{BMO}}} \right) \right)^{1/2} \|f\|_{\text{BMO}} + \frac{2^\gamma/2}{\beta} \|f\|_{\text{BMO}} \right], \] (3.37)

\[ \leq C \left[ \frac{2}{(\gamma \log 2)^{1/2}} \left( \log^+ \left( \frac{\|f\|_{C\gamma^{1/2}} + \|g\|_{L^\infty}}{\|f\|_{\text{BMO}}} \right) \right)^{1/2} \|f\|_{\text{BMO}} + \frac{2^\gamma/2}{\beta} \|f\|_{\text{BMO}} \right], \]

where for the second line we have used the fact that

\[ \log^+ \beta < \log^+ \frac{\|f\|_{C\gamma^{1/2}} + \|g\|_{L^\infty}}{\|f\|_{\text{BMO}}} \] (3.38)
The above computations again imply (3.35). By using the inequality:

\[ x \left( \log \left( e + \frac{y}{x} \right) \right)^{1/2} \leq \begin{cases} 
C \left( 1 + x \left( \log (e + y) \right)^{1/2} \right), & \text{for } 0 < x \leq 1, \\
Cx \left( \log (e + y) \right)^{1/2}, & \text{for } x > 1,
\end{cases} \tag{3.39} \]

in (3.35), we directly arrive to our result. \( \square \)

We now present the proof of Theorem 1.2 that involves finer estimates on the H"older norm.

**Proof of Theorem 1.2.** For the sake of simplifying the ideas of the proof, we only consider 1-spatial dimensions \( x = x_1 \). The general \( n \)-dimensional case can be easily deduced. Following the same notations of [1], we let \( \Omega_T = (-1,2) \times (-T,2T) \), \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \Omega_T \) such that

\[
\mathcal{L}_1 = \left\{ (x,t); -\frac{1}{4} < x < \frac{5}{4}, -\frac{T}{4} < t < \frac{5T}{4} \right\},
\]

\[
\mathcal{L}_2 = \left\{ (x,t); -\frac{3}{4} < x < \frac{7}{4}, -\frac{3T}{4} < t < \frac{7T}{4} \right\}. \tag{3.40}
\]

We also take the cut-off function \( \Psi \in C_0^\infty(\mathbb{R}^2), 0 \leq \Psi \leq 1 \) satisfying:

\[
\Psi(x,t) = \begin{cases} 
1, & \text{for } (x,t) \in \mathcal{L}_1, \\
0, & \text{for } (x,t) \in \mathbb{R}^2 \setminus \mathcal{L}_2.
\end{cases} \tag{3.41}
\]

The main idea of the proof consists in extending the function \( f \) to a suitable function of the form \( \Psi \tilde{f} \), where \( \tilde{f} \) is defined on \( \Omega_T \). We then apply inequality (1.5) (the scalar-valued version with \( n = 1 \)) to \( \Psi \tilde{f} \), and we estimate the different norms in order to get the result. However, away from the complicated extension (Sobolev extension) of the function \( \tilde{f} \) that was done in [1], we here consider a simpler symmetric extension. Indeed, we first take the spatial symmetry of the function \( f \):

\[
\tilde{f}(x,t) = \begin{cases} 
f(-x,t), & \text{for } -1 < x < 0, \ 0 \leq t \leq T, \\
f(2-x,t), & \text{for } 1 < x < 2, \ 0 \leq t \leq T,
\end{cases} \tag{3.42}
\]

and then the symmetry with respect to \( t \):

\[
\tilde{f}(x,t) = \begin{cases} 
f(x,-t), & \text{for } -1 < x < 2, \ -T < t \leq 0, \\
f(x,2T-t), & \text{for } -1 < x < 2, \ T \leq t < 2T.
\end{cases} \tag{3.43}
\]

We claim that \( \Psi \tilde{f} \in C^{\gamma/2}(\mathbb{R}^2) \) with

\[
\left\| \Psi \tilde{f} \right\|_{C^{\gamma/2}(\mathbb{R}^2)} \leq \left\| f \right\|_{C^{\gamma/2}(\Omega_T)}. \tag{3.44}
\]
In this case, we apply the scalar-valued version of inequality (1.5) (see Remark 1.3) to the function $\Psi \tilde{f}$ with $i = 1$ and $g(x, t) = \int_0^t \Psi(y, t) \tilde{f}(y, t) dy$. This, together with the fact that $\Psi = 1$ on $\Omega_T$, leads to the following estimate:

$$
\|f\|_{L^\infty(\Omega_T)} \leq \|\Psi \tilde{f}\|_{L^\infty(\mathbb{R}^2)} \leq C \left( 1 + \|\Psi \tilde{f}\|_{\text{BMO}(\mathbb{R}^2)} \left( \log^+ \left( \|\Psi \tilde{f}\|_{C^{1/2}(\mathbb{R}^2)} + \|g\|_{L^\infty(\mathbb{R}^2)} \right) \right)^{1/2} \right). \tag{3.45}
$$

It is worth noticing that choosing $i = 1$ above is somehow restrictive. In fact, we could also have used the inequality with $i = 2$ and $g(x, t) = \int_0^t \Psi(x, s) \tilde{f}(x, s) ds$.

In [7] it was shown that $\|\Psi \tilde{f}\|_{\text{BMO}(\mathbb{R}^2)} \leq C \|f\|_{L^1(\Omega_T)}$, while it is clear that $\|g\|_{L^\infty(\mathbb{R}^2)} \leq C \|\tilde{f}\|_{L^\infty(\Omega_T)}$. These arguments, along with (3.44) and (3.45), directly terminate the proof. The only point left is to show the claim (3.44). Recall the norm

$$
\|\Psi \tilde{f}\|_{C^{1/2}(\mathbb{R}^2)} = \|\tilde{f}\|_{L^2(\mathbb{R}^2)} + \left( \langle \Psi \tilde{f} \rangle_{x, \mathbb{R}^2}^{(1)} + \langle \Psi \tilde{f} \rangle_{x, \mathbb{R}^2}^{(1/2)} \right)_{L^2(\mathbb{R}^2)}. \tag{3.46}
$$

It is evident that

$$
\|\Psi \tilde{f}\|_{L^\infty(\mathbb{R}^2)} \leq C \|f\|_{L^\infty(\Omega_T)}, \tag{3.47}
$$

hence we only need to estimate the two terms $\langle \Psi \tilde{f} \rangle_{x, \mathbb{R}^2}^{(1)}$ and $\langle \Psi \tilde{f} \rangle_{x, \mathbb{R}^2}^{(1/2)}$. We only deal with $\langle \Psi \tilde{f} \rangle_{x, \mathbb{R}^2}^{(1)}$ since the second term can be treated similarly. We examine the different positions of $(x, t), (x', t) \in \mathbb{R}^2$. If $(x, t), (x', t) \in \mathbb{R}^2 \setminus \mathcal{Z}_2, x \neq x'$, then (since $\Psi = 0$ over $\mathbb{R}^2 \setminus \mathcal{Z}_2$):

$$
\left| \frac{(\Psi \tilde{f})(x, t) - (\Psi \tilde{f})(x', t)}{|x - x'|^p} \right| = 0. \tag{3.48}
$$

If both $(x, t), (x', t) \in \tilde{\Omega}_T, x \neq x'$, then the special extension (3.42) and (3.43) of the function $f$ guarantees the existence of

$$
\left( \tilde{x}, \tilde{t} \right), \left( \tilde{x}', \tilde{t} \right) \in \Omega_T, \tag{3.49}
$$

such that:

$$
\tilde{f}(x, t) = f \left( \tilde{x}, \tilde{t} \right), \quad \tilde{f}(x', t) = f \left( \tilde{x}', \tilde{t} \right). \tag{3.50}
$$

Two cases can be considered. Either $\tilde{x} = \tilde{x}'$ (see Figure 1), then we forcedly have

$$
\tilde{f}(x, t) = \tilde{f}(x', t), \tag{3.51}
$$
and therefore

\[
\left| \left( \Psi \tilde{f} \right)(x, t) - \left( \Psi \tilde{f} \right)(x', t) \right| \leq \langle \Psi \rangle_{x, \tilde{\Omega}_T} \left\| \tilde{f} \right\|_{L^\infty(\tilde{\Omega}_T)}
\]

\[
\leq C \left\| f \right\|_{L^\infty(\Omega_T)} \leq C \left\| f \right\|_{C^{\gamma/2}(\Omega_T)}
\]

or \( \bar{x} \neq \bar{x}' \), then we forcedly have (see Figure 2)

\[
|x - x'|^\gamma \geq |\bar{x} - \bar{x}'|^\gamma.
\]

In this case, we compute

\[
\left| \left( \Psi \tilde{f} \right)(x, t) - \left( \Psi \tilde{f} \right)(x', t) \right| \leq \frac{\tilde{f}(x, t) |\Psi(x, t) - \Psi(x', t)|}{|x - x'|^\gamma} + \frac{|\Psi(x', t)| |\tilde{f}(x, t) - \tilde{f}(x', t)|}{|x - x'|^\gamma}
\]

\[
\leq \left\| \tilde{f} \right\|_{L^\infty(\tilde{\Omega}_T)} \langle \Psi \rangle_{x, \tilde{\Omega}_T} + \frac{|\tilde{f}(x, t) - \tilde{f}(x', t)|}{|x - x'|^\gamma}.
\]

(3.54)
Figure 3: Case \((x,t) \in \mathcal{Z}_2\) and \((x',t) \in \mathbb{R}^2 \setminus \tilde{\Omega}_T\).

Using (3.50) and (3.53), we deduce that:

\[
\frac{\left| \tilde{f}(x,t) - \tilde{f}(x',t) \right|}{|x-x'|^\gamma} = \frac{\left| f(\tilde{x},\tilde{t}) - f(\tilde{x}',\tilde{t}) \right|}{|\tilde{x}-\tilde{x}'|^\gamma} \leq \frac{\left| f(\tilde{x},\tilde{t}) - f(\tilde{x}',\tilde{t}) \right|}{|\tilde{x}-\tilde{x}'|^\gamma} \leq \langle f \rangle_{x,\tilde{\Omega}_T},
\]

therefore, by (3.54), we obtain

\[
\left| \left( \Psi \tilde{f} \right)(x,t) - \left( \Psi \tilde{f} \right)(x',t) \right| \leq \| \tilde{f} \|_{L^\infty(\tilde{\Omega}_T)} \langle \Psi \rangle_{x,\tilde{\Omega}_T} + \langle f \rangle_{x,\tilde{\Omega}_T} \leq C \| f \|_{C^{1/2}(\Omega_T)}.
\]

The remaining case is when \((x,t) \in \mathcal{Z}_2\) and \((x',t) \in \mathbb{R}^2 \setminus \tilde{\Omega}_T\) (see Figure 3). In this case, we have \((\Psi \tilde{f})(x',t) = 0\) and

\[
|x-x'|^\gamma \geq \left( \frac{1}{4} \right)^{\frac{\gamma}{2}},
\]

hence

\[
\left| \left( \Psi \tilde{f} \right)(x,t) - \left( \Psi \tilde{f} \right)(x',t) \right| \leq 4^\gamma \| \tilde{f} \|_{L^\infty(\mathcal{Z}_2)} \leq C \| f \|_{C^{1/2}(\Omega_T)}.
\]

From (3.48), (3.52), (3.56), and (3.58), we finally deduce that

\[
\langle \Psi \tilde{f} \rangle_{x,\mathbb{R}^2}^{(1)} \leq C \| f \|_{C^{1/2}(\Omega_T)}.
\]

Arguing in exactly the same way as above, we also find that:

\[
\langle \Psi \tilde{f} \rangle_{t,\mathbb{R}^2}^{(1/2)} \leq C \| f \|_{C^{1/2}(\Omega_T)}.
\]
with a possibly different constant $C$ that depends on $T$. Indeed, the term $T$ enters in estimating $\langle \tilde{f} \rangle_{1,2}^{(\gamma/2)}$ since (3.57) is now replaced (see again Figure 3) by

$$|t - t'|^\gamma \geq \left( \frac{T}{4} \right)^\gamma.$$  \hspace{1cm} (3.61)

This shows the claim.

Remark 3.3. In the case of multispatial coordinates $x_i, i = 1, \ldots, n$, we simultaneously apply the extension (3.42) to each spatial coordinate while fixing all other coordinates including $t$. Finally, by fixing the spatial variables, we make the extension with respect to $t$ as in (3.43).

Acknowledgments

The author is highly indebted to Professor Lars-Erik Persson and to an anonymous referee for their useful comments on earlier versions of the paper.

References

[1] H. Ibrahim, “A critical parabolic Sobolev embedding via Littlewood-Paley decomposition,” preprint.
[2] T. Ogawa, “Sharp Sobolev inequality of logarithmic type and the limiting regularity condition to the harmonic heat flow,” SIAM Journal on Mathematical Analysis, vol. 34, no. 6, pp. 1318–1330, 2003.
[3] H. Brézis and T. Gallouet, “Nonlinear Schrödinger evolution equations,” Nonlinear Analysis, vol. 4, no. 4, pp. 677–681, 1980.
[4] H. Brézis and S. Wainger, “A note on limiting cases of Sobolev embeddings and convolution inequalities,” Communications in Partial Differential Equations, vol. 5, no. 7, pp. 773–789, 1980.
[5] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Uralceva, Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, RI, USA, 1967.
[6] H. Ibrahim, M. Jazar, and R. Monneau, “Global existence of solutions to a singular parabolic/Hamilton-Jacobi coupled system with Dirichlet conditions,” Comptes Rendus Mathématique. Académie des Sciences. Paris, vol. 346, no. 17-18, pp. 945–950, 2008.
[7] H. Ibrahim and R. Monneau, “On a parabolic logarithmic Sobolev inequality,” Journal of Functional Analysis, vol. 257, no. 3, pp. 903–930, 2009.
[8] H. Engler, “An alternative proof of the Brezis-Wainger inequality,” Communications in Partial Differential Equations, vol. 14, no. 4, pp. 541–544, 1989.
[9] H. Kozono, T. Ogawa, and Y. Taniuchi, “The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations,” Mathematische Zeitschrift, vol. 242, no. 2, pp. 251–278, 2002.
[10] H. Kozono, T. Ogawa, and Y. Taniuchi, “Navier-Stokes equations in the Besov space near $L^\infty$ and BMO,” Kyushu Journal of Mathematics, vol. 57, no. 2, pp. 303–324, 2003.
[11] H. Kozono and Y. Taniuchi, “Limiting case of the Sobolev inequality in BMO, with application to the Euler equations,” Communications in Mathematical Physics, vol. 214, no. 1, pp. 191–200, 2000.
[12] H. Triebel, Theory of Function Spaces, vol. 78 of Monographs in Mathematics, Birkhäuser, Basel, Switzerland, 1983.
[13] H. Triebel, Theory of Function Spaces. III, vol. 100 of Monographs in Mathematics, Birkhäuser, Basel, Switzerland, 2006.
[14] M. Bownik, “Anisotropic Triebel-Lizorkin spaces with doubling measures,” The Journal of Geometric Analysis, vol. 17, no. 3, pp. 387–424, 2007.
[15] M. Bownik, “Duality and interpolation of anisotropic Triebel-Lizorkin spaces,” Mathematische Zeitschrift, vol. 259, no. 1, pp. 131–169, 2008.
[16] M. Frazier and B. Jawerth, “A discrete transform and decompositions of distribution spaces,” *Journal of Functional Analysis*, vol. 93, no. 1, pp. 34–170, 1990.

[17] M. Bownik, “Anisotropic Hardy spaces and wavelets,” *Memoirs of the American Mathematical Society*, vol. 164, no. 781, p. vii+122, 2003.

[18] W. Farkas, J. Johnsen, and W. Sickel, “Traces of anisotropic Besov-Lizorkin-Triebel spaces—a complete treatment of the borderline cases,” *Mathematica Bohemica*, vol. 125, no. 1, pp. 1–37, 2000.
