Field Theory of the Random Flux Model

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Abstract

The random flux model (defined here as a model of lattice fermions hopping under the influence of maximally random link disorder) is analysed field theoretically. It is shown that the long range physics of the model is described by the supersymmetric version of a field theory that has been derived earlier in connection with lattice fermions subject to weak random hopping. More precisely, the field theory relevant for the behaviour of $n$-point correlation functions is of non-linear $\sigma$ model type, where the group $\text{GL}(n|n)$ is the global invariant manifold. It is argued that the model universally describes the long range physics of random phase fermions and provides further evidence in favour of the existence of delocalised states in the middle of the band in two dimensions. The same formalism is applied to the study of non-Abelian generalisations of the random flux model, i.e. $N$-component fermions whose hopping is mediated by random $\text{U}(N)$ matrices. We discuss some physical applications of these models and argue that, for sufficiently large $N$, the existence of long range correlations in the band center (equivalent to metallic behaviour in the Abelian case) can be safely deduced from the RG analysis of the model.

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I. INTRODUCTION

Quantum disordered systems are typically described in terms of Hamiltonians of the form

\[ \hat{H} = \hat{H}_0 + \hat{V}, \quad (1) \]

where \( \hat{H}_0 \) models the underlying clean system and disorder is introduced via the randomly distributed Hermitian operator \( \hat{V} \). Sometimes, however, it is preferable to describe disordered systems differently, viz. by Hamiltonians where disorder is implemented in terms of unitary stochastic operators. A simple prototype example is the system of lattice electrons subject to a (perpendicular) strong random magnetic flux (see fig. 1). An electron hopping from one site to the next acquires a field dependent phase which, if the field is strong enough, becomes completely random. Under these conditions the system is described by the Hamiltonian,

\[ \hat{H} = - \sum_{\langle i,j \rangle} c_i^\dagger e^{i\phi_{ij}} c_j, \quad (2) \]

where \( \sum_{\langle i,j \rangle} \) is a summation over all nearest neighbour sites of a \( d \)-dimensional bipartite lattice, \( c_i \) is an electron field operator at site \( i \), \( \phi_{ij} \) a phase living on the link \( i \rightarrow j \), and maximum stochasticity implies that the phase \( \phi_{ij} \) is drawn randomly from the full interval \([0, 2\pi]\) subject only to the Hermiticity requirement \( \phi_{ij} = -\phi_{ji} \). Notice that the definition (2) implicitly assumes that all energies are measured in units of the modulus of the hopping matrix elements.

It has been suggested that the model (2) describes the physics of the fractional quantum Hall effect close to half filling (where the problem is expected to map onto a system of composite fermions subject to a strong random field with zero mean [1]). Furthermore, it is believed to be of some relevance in the gauge theory of high-\( T_c \) superconductivity [2]. (However, it is important to recognise that the denotation ‘random flux model’ is, to some extent, misleading in the sense that the Hamiltonian (2) does not actually model a short range correlated random magnetic field but rather the lattice version of a random vector potential. The difference can be of importance since a short range correlated field would imply the presence of long range correlations in the vector potential [3,4]. Keeping this difference in mind we nevertheless continue to refer to (2) as a random flux Hamiltonian.)

The random phase Hamiltonian (2), describing the lattice dynamics of scalar fermions, can straightforwardly be generalized so as to accommodate fermions with an internal structure: Let us assume that a) the lattice fermions carry an internal quantum number, \( c_i \rightarrow c_i^\alpha, \alpha = 1, \ldots, N \) such as, e.g., spin and b) that the fundamental hopping mechanism leads to a full randomisation of the internal states. Under these circumstances, the Hamiltonian (2) generalises to (c.f. fig. 2)

\[ H = - \sum_{\langle i,j \rangle} c_i^\dagger U_{ij} c_j, \quad (3) \]

More precisely, maximum stochasticity requires fields stronger than one flux-quantum per lattice cell.
where \( U_{ij} \equiv \{ U_{ij}^{\alpha,\beta} \} \in \text{U}(N) \) are \( N \)-dimensional unitary matrices sitting on the nearest neighbour links of the lattice, and \( c_i \equiv \{ c_i^\alpha_{a} \} \), \( \alpha = 1, \ldots, N \), \( a = 1, \ldots, n \) are \( N \times n \)-component field operators. The lower index \( a \), which has been introduced for future reference, plays a passive role in the sense that it is summed over and does not couple to the \( U \)'s. ‘Full randomisation’ upon each hopping process translates to the \( U_{ij} \) being drawn democratically from the Haar measure of the group (subject to the Hermiticity requirement \( U_{ij} = U_{ji}^\dagger \)). Following a convention introduced in Ref. \[5\] we refer to the index \( \alpha \) as a ‘colour’ index, whilst \( a \) will be denoted as a ‘flavour’ index (for the motivation of this terminology, see below). The original RF-Hamiltonian fits into the general framework in the sense that it is the \( N = 1 \) version of (3). For the sake of simplicity we will also refer to (3) as a ‘random flux’ model.

The generalized RF-model has applications in both condensed matter and high energy physics:

**Physics of manganese oxides:** Suggestions have been put forward to explain the effect of giant magnetoresistance, shown by transition metal oxide compounds, in terms of effective models consisting of lattice electrons hopping in a matrix of localised \( S = 3/2 \) core spins \[6\]. The hopping process is essentially influenced by the Hund’s rule coupling between the spin of the itinerant electrons and the localised spins. In the limit where the coupling strength becomes infinite, the hopping matrix element along any given bond \( i \to j \) depends solely on the orientation of the spins at the sites \( i \) and \( j \) \[7\], i.e. the hopping is mediated by an \( \text{SU}(2) \) matrix which acts in the \( S = 1/2 \) space of the electron spins and depends on the orientation of the two adjacent core spins. Assuming that a) the spin orientations are random, b) that the effect of the randomness can be mimicked by drawing the link matrices from a random distribution, and c) that, with regard to universal properties, the difference between random \( \text{SU}(2) \) hopping and random \( \text{U}(2) \) hopping is of no concern, we arrive at the \( (N = 2, n = 1) \) version of (3).

**Lattice gauge theory:** The simplest lattice action of a matter field with \( N \) ‘colour’ indices subject to a non-Abelian local gauge symmetry is given by

\[
S = - \sum_{(i,j)} c_i^a U_{ij} c_j^a - \frac{1}{2g} \sum_\square \left[ \left( 1 - \frac{1}{N} \text{tr}_\square(U^4) \right) \right] + \text{h.c.}. \tag{4}
\]

Here, \( a = 1, \ldots, n \) has the physical significance of a fermionic flavour index, \( \sum_\square \) is a sum over all plaquettes of the lattice, and \( \text{tr}_\square(U^4) \equiv \text{tr} (U_{ij} U_{jk} U_{kl} U_{li}) \) represents the trace over all \( U \)'s sitting on the links of the plaquette. Physically, the plaquette action represents the discrete version of a continuum field strength tensor (c.f., for example, \[8\]). In the strong coupling limit, \( g \to \infty \) (i.e. the long distance-confinement phase of the theory) the gauge fields become free and (4) is governed by a structure like (3).

\[2\] A more realistic lattice representation of the QCD Lagrangian would involve Dirac \( \gamma \)-matrices acting in the Dirac-spinor space of the fermion matter field. Although the embedding of this additional structure into the formalism discussed below is straightforward, we prefer to keep the discussion simple and not to include the Dirac-spinor structure.
What can be said, in general terms, about the physical behaviour of the scalar random flux Hamiltonian and its non-abelian generalizations:

- At first sight it looks like the random flux Hamiltonian falls into the general category of Anderson Hamiltonians for disordered systems. More precisely, the model \( (2) \) bears similarity with a lattice Anderson Hamiltonian with strong disorder in the hopping matrix elements. Being Hermitian but not real, the Hamiltonian \( (2) \) is expected to fall into the general class of models with unitary symmetry. In particular, all states might be expected to be Anderson localised. As will be discussed below, this picture is correct as long as one stays away from zero energy:

- **The middle of the tight binding band, \( \epsilon = 0 \).** In the band center, the Hamiltonian \( (2) \) possesses a discrete symmetry which, as will be explored in detail below, heavily influences the physical properties of the model. The nature of the symmetry can easily be understood by inspection of fig. [1]: The essential point is that the Hamiltonian of the RF problem is of pure nearest neighbour type. (There are no matrix elements connecting any one site with itself.) To connect this statement with the existence of a symmetry, one may subdivide the lattice into two sublattices (the \( \circ \)'s and the \( \times \)'s in fig. [1]). When represented in a \( \circ/\times \) block decomposition the Hamiltonian becomes purely off-diagonal,

\[
\hat{H} = \begin{pmatrix}
\hat{H}_{\circ/\times}
\end{pmatrix}.
\]  

(5)

Formally, the block ‘off-diagonality’ of the Hamiltonian can be expressed as

\[
\sigma_3 \hat{H} \sigma_3 = -\hat{H} \Leftrightarrow [\hat{H}, \sigma_3]_+ = 0,
\]

(6)

where \([ \ , \ ]_+ \) is the anticommutator and \( \sigma_3 \) a Pauli matrix in \( \circ/\times \) space. Symmetries of the type (6) are commonly referred to as chiral symmetries.

The existence of a chiral symmetry has far reaching consequences for the physics of the model at the band-center. Before turning to a more comprehensive discussion of this point, let us summarise a few elementary facts that follow straightforwardly from the definition of the model. Firstly, we note that the Hamiltonian is always chiral, i.e. not just in the band center. One may then ask why, for finite energies, the chiral symmetry is ineffective. The reason is that the information about both thermodynamic and transport properties of the system is encoded in the single-particle Green function \( G^\pm(\epsilon) \equiv (\epsilon^\pm - \hat{H})^{-1} \) rather than in the Hamiltonian itself (here \( \epsilon^\pm \equiv \epsilon \pm i0 \)). Now,

\[
G^\pm(\epsilon) = -\sigma_3 G^{\mp}(\epsilon) \sigma_3,
\]

i.e. the inversion property under adjunction with \( \sigma_3 \) is broken by finite \( \epsilon \). Some physical implications of this symmetry breaking will be discussed in detail below.

Secondly, we note that stochastic Hamiltonians with a chiral structure have previously been studied in the random matrix literature \([9–14]\). It has been shown that they
belong to symmetry classes that are fundamentally different from the standard Wigner Dyson classes. In particular, the band center RF model belongs to the universality class of the so-called ‘chiral GUE’, ChGUE \cite{15,9–12,14}. For finite $\epsilon$, the model undergoes a symmetry crossover from ChGUE to ordinary GUE symmetry.

To the best of our knowledge the full extent of the consequences of the chirality of the RF model was first noticed in a seminal paper by Miller and Wang \cite{16}.

- Beginning with Lee and Fisher \cite{17}, there has been intensive research into the Anderson localization behaviour of the model. So far the question whether or not all states are localised has not been conclusively settled: Despite extensive numerical and analytic studies, opinion is divided between those who conclude that all states are localised \cite{18,19,20–22}, and those who believe that delocalised states persist at the center of the band \cite{23–28,16,29–31}. Below we will provide further evidence for the existence of a metallic phase in the middle of the band. Needless to say, the phenomenon is induced by the unconventional symmetry of the model. By contrast, for finite $\epsilon$, the model can be shown to map onto the standard Anderson model of unitary symmetry which is known to be localised in dimensions two and less. For weak random magnetic fields, this connection has already been established microscopically in Ref. \cite{4}.

Besides the localization behaviour, several other physical properties of the RF system are affected by the chiral symmetry. In particular the presence of the symmetry gives rise to non-trivial quantum-interference corrections to average single-particle properties such as, e.g. the density of states. Some of these features will be discussed below.

The subject of this paper is the construction of an effective field theory describing the low energy/long range properties of the generalized random flux model (3). Some emphasis will be placed on exploring the consequences of the chiral symmetry of these systems. Specifically, it will be shown that the random flux model maps onto a field theory of a structure similar to that constructed earlier by Gade \cite{32} in connection with weakly disordered sublattice models introduced by Oppermann and Wegner \cite{33}. Both the symmetries and the structure of Gade’s field theory are known to be essentially influenced by chirality. In particular, renormalization group (RG) flow of the theory is drastically different from the one observed in conventional low dimensional disordered systems. Indeed, the fact that the RF models (3) and the Wegner/Oppermann sublattice systems of broken time reversal invariance, respectively, map onto the same field theory might have been anticipated from the outset, the reason being that the latter represents the maximally disordered counterpart of the former. On the field theoretical side, the main difference between the models lies in the value of the coupling constants. The consequences of this difference for the large scale analysis of the RF model will be discussed below, after the field theory has been microscopically derived. A second difference to Gade’s analysis is that the present construction will be based on the supersymmetry formalism (as opposed to Gade’s boson replicas). As will be discussed below, the supersymmetric formulation enables one to explicitly demonstrate that the ergodic limit of the model is described by chiral random matrix theory.

The paper is organized as follows: We start out from a qualitative explanation for the importance of the chiral symmetry in section II. The central sections III and IV contain the derivation of the long range field theory of the model. In section V we discuss what kind
of physical predictions follow from the field theoretical modelling. To clarify our discussion, much of the technical content of the analysis has been moved to appendices.

II. THE CHIRAL SYMMETRY: QUALITATIVE CONSIDERATIONS

Prior to embarking on the quantitative analysis of the RF model, it is instructive to develop some intuition regarding the consequences of the presence of the chiral symmetry. Perhaps the most apparent manifestation of the symmetry is the non-triviality of ensemble averaged single-particle properties such as the DoS: As shown in Ref. [32] within a perturbative renormalization group scheme, the DoS of the two-dimensional (weakly disordered) sublattice model diverges upon approaching the middle of the band. The same type of behaviour has been predicted for one-dimensional chiral systems [34]. That these results can, in fact, not hold true all the way down to $\epsilon = 0$ follows from exact random matrix theory [9].

The random matrix analysis which, by general symmetry arguments, is expected to effectively describe the ergodic limit of chiral problems (i.e. the asymptotic vicinity of the band center) yields a vanishing DOS at $\epsilon = 0$. Since the breakdown of the DoS takes place on an energy scale set by the level spacing, it is generally difficult to detect within perturbative approaches. In fact it is known, for example from the conceptually related analysis of the level-spacing gap in the DoS of superconductor/normal metal systems without proximity effect [35], that a non-perturbative level spacing gap may signal its presence in terms of spurious divergences appearing in perturbative analyses. Thus, judging from perturbation theory alone, it may be difficult to tell what part of the band center DoS divergence is ‘real’ and what is merely an unphysical repercussion of a tiny non-perturbative effect.

As will be discussed in more detail below, the complex behaviour of the DoS finds its origin in the fact that, in chiral models, mechanisms of large scale quantum interference — which are commonly represented by diffusion-type modes like ‘diffusons’ and ‘Cooperons’ — already operate on the level of single-particle properties. This should be contrasted with the situation in conventional disordered models where the average Green function picks up a largely structureless imaginary self energy but otherwise is not affected by the disorder.

To understand why the DoS of the RF model behaves unusually in the band center, it is not even necessary to invoke the concept of diffusion modes. A simple path counting argument analogous to that employed by Brinkman and Rice [36] is sufficient: Consider the average DoS per unit volume, $\rho$, represented in terms of the single-particle Green function,

$$\rho(\epsilon) = \frac{1}{\pi} \text{Im} \sum_i \langle \langle i | \frac{1}{\epsilon - \hat{H}} | i \rangle \rangle,$$

where the outer brackets represent the phase averaging

$$\langle \ldots \rangle \equiv \prod_{\langle i,j \rangle} \frac{1}{2\pi} \int_0^{2\pi} d\phi_{ij} \langle \ldots \rangle.$$

Expanding in powers of $\hat{H}$ and inserting lattice resolutions of unity, $\sum_j |j\rangle \langle j|$, we obtain
\[
\rho(\varepsilon) = -\frac{1}{\pi} \operatorname{Im} \sum_{n=0,i,i_1,\ldots,i_n}^{\infty} (\varepsilon^-)^{-(n+1)} \left. \left\langle \varepsilon^i \sum_{m=0}^{n-1} \phi_{i_m,i_{m+1}} \right\rangle \right|_{i_0=i_n=i} \\
= \frac{1}{\pi} \left( \frac{L}{a} \right)^d \operatorname{Im} \sum_{n \text{ even}}^{\infty} \sum_{i_1,\ldots,i_n} (\varepsilon^-)^{-(n+1)} \cdot 1 \\
= \frac{1}{\pi} \left( \frac{L}{a} \right)^d \operatorname{Im} \sum_{n \text{ even}} (\varepsilon^-)^{-(n+1)} (2d)^{n/2},
\]

where \( L \) is the linear extension of the lattice and \( a \) the lattice spacing. The expression \( \sum_{m=0}^{n-1} \phi_{i_m,i_{m+1}} \) has the meaning of the total phase acquired during the propagation along the path \( i \rightarrow i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i \) (c.f. fig. [3]). The second equality in (9) states that the contribution of all paths with a non-vanishing total phase averages to zero. The surviving 0-phase paths are self retracing — indicated by the symbol ‘s.r.’. In the third equality we have used the fact that the total number of self retracing paths of length \( n \) is \( (2d)^{n/2} \). Evidently, the summation over \( n \) in the last line diverges for \( \varepsilon < \sqrt{2d} \), indicating a breakdown of perturbation theory in the vicinity of the band center. Notice that the presence of 0-phase paths is intimately connected to the sublattice structure of the system. No analogous contribution to the mean DoS of conventional (on-site) disordered systems exists.

A second and more formal way of demonstrating the non-triviality of single-particle properties is based on a few algebraic manipulations of the single-particle Green function. Introducing a system of Pauli matrices in the two-component \( o/x \)-partition, the Hamiltonian takes the form

\[
\hat{H} = \begin{pmatrix}
0 & \hat{H}_1 - i\hat{H}_2 \\
\hat{H}_1 + i\hat{H}_2 & 0
\end{pmatrix} = U^\dagger \begin{pmatrix}
\hat{H}_1 & -i\hat{H}_2 \\
i\hat{H}_2 & -\hat{H}_1
\end{pmatrix} U,
\]

where \( U = \exp(i\frac{\pi}{4}\sigma_2) \) and both \( \hat{H}_1 \) and \( \hat{H}_2 \) are Hermitian. Similarly, the Green function can be represented as

\[
G^+(\varepsilon) = U^\dagger \begin{pmatrix}
\varepsilon^+ - \hat{H}_1 & +i\hat{H}_2 \\
-i\hat{H}_2 & \varepsilon^+ + \hat{H}_1
\end{pmatrix}^{-1} U = U^\dagger \begin{pmatrix}
\varepsilon^+ - \hat{H}_1 & i\hat{H}_2 \\
i\hat{H}_2 & -\varepsilon^+ - \hat{H}_1
\end{pmatrix}^{-1} \sigma_3 U.
\]

The point now is that the central matrix can be interpreted as a retarded and an advanced stochastic Green function \( g^+(\varepsilon) \equiv (\varepsilon^+ - \hat{H}_1)^{-1} \) and \( g^-(\varepsilon) \equiv (-\varepsilon^+ - \hat{H}_1)^{-1} \), coupled by the operator \( i\hat{H}_2 \). Since the the manipulations with the matrices \( U \) and \( \sigma_3 \) do not affect the structure of the singularities of the Green function, we conclude that the single-particle \( G^+(\varepsilon) \) has the effective complexity of two coupled stochastic Green functions. In particular, we can anticipate that the average \( G^+(\varepsilon) \) will be affected by mechanisms of quantum interference between \( g^+(\varepsilon) \) and \( g^-(\varepsilon) \). Notice that both large energy arguments \( \varepsilon \) and on-site matrix elements of the Hamiltonian (corresponding to diagonal blocks in the \( o/x \)-partition) would lead to a destruction of this two-particle coherence.

After these general considerations, we proceed to the quantitative study of the generalized RF-problem. Our general strategy parallels that employed by Zirnbauer [37] in a loosely related study of the mapping of the Chalker-Coddington network model of the Integer Quantum Hall Effect onto the effective field theory proposed by Pruisken.
III. MAPPING THE RANDOM FLUX MODEL TO A FIELD THEORY: COLOUR-FLAVOUR TRANSFORMATION

As a first step towards the construction of a field theory of the RF problem, we need to introduce some degrees of freedom capable of describing the long range features of the model (3). A mathematical tool for mapping model systems governed by unitary stochastic operators onto functional integrals containing a ‘low energy’ sector describing the large distance physics has recently been introduced and applied by Zirnbauer [5]. For reasons that will become clear below, the transformation has been christened the ‘colour-flavour transformation’. Our application of the colour-flavour transformation to the U(N) RF-model closely parallels the analysis of Ref. [5] of the spectral properties of single U(N) matrices. For this reason the presentation of the initial stages of the construction of the theory will be comparatively concise but nevertheless self contained. (For a deeper discussion of the mathematics of the colour-flavour transformation we refer the reader to the original reference.)

As a first step towards the transformed theory, we consider the Gaussian functional integral

\[
Z[\hat{J}] = \left\langle \int \mathcal{D}(\bar{\psi}, \psi) e^{i \sum_{i \in A, j \in B} \left\{ \bar{\psi}_i s^{1/2} U_{ij} s^{1/2} \psi_j + \bar{\psi}_j s^{1/2} U^\dagger_{ij} s^{1/2} \psi_i \right\} + i \sum_{k,l} \bar{\psi}_k s^{1/2} (\hat{\varepsilon}_{kli} + \hat{J}_{kli}) s^{1/2} \psi_l} \right\rangle,
\]

(10)

where

\[
\langle \ldots \rangle \equiv \prod_{(i,j)} \int dU_{ij}(\ldots)
\]

is the non-Abelian generalisation of the phase average (8), \(dU_{ij}\) denotes the Haar measure on U(N) and \(A (B)\) is the sublattice of \(\circ\)-sites (\(\times\)-sites). The fields

\[
\bar{\psi} = \{ \bar{\psi}_{i,a} \}, \quad \psi = \{ \psi^\alpha_{i,a} \}
\]

carry three types of indices: A lattice index \(i\), a ‘colour’ index \(\alpha\) (coupling to the internal group indices of the \(U\)’s), and a two component ‘flavour’ index \(a \equiv (\sigma, m)\). Here \(m = 1, \ldots, n\) is a discrete counting variable whose significance will become clear shortly, whereas \(\sigma = 1, 2\) introduces a boson/fermion grading into the theory: Field components \(\psi^\alpha_{i,\sigma=1,m} \equiv S^\alpha_{i,m}\) are complex commuting whilst \(\psi^\alpha_{i,\sigma=2,m} \equiv \chi^\alpha_{i,m}\) are odd elements of a Grassmann algebra (‘anticommuting variables’). Convergence of the integral (10) requires that the components \(S^\alpha_{i,m}\) be the complex conjugates of \(S^\alpha_{i,m}\), while the variables \(\chi^\alpha_{i,m}\) and \(\bar{\chi}^\alpha_{i,m}\) are independent of each other. In cases where no confusion can arise we will employ the more explicit notation, \(\sigma = B, F\) instead of \(\sigma = 1, 2\).

The elements of the diagonal matrix

\[
\hat{\varepsilon} = \text{diag}(z_1, \ldots, z_n) \otimes \mathbb{1}_{2N},
\]

where (see below) \(z_m \in \mathcal{C}\), have the significance of energy arguments of single-particle Green functions. The presence of the matrix
\[ s = \text{diag}(\text{sgn Im } z_1, \ldots, \text{sgn Im } z_n) \otimes \mathbb{I}_{2N} \]

ensures convergence of the Gaussian integral. Finally, the ‘source field’ \( \hat{J} \) is block diagonal in flavour space,

\[ \hat{J} = \text{bdiag} (\hat{J}_1, \ldots, \hat{J}_n) \otimes \sigma_3^{bf}, \]

where \( \hat{J}_m = \{ (\hat{J}_m)_{ij} \}, m = 1, \ldots, n \) are arbitrary matrices on the lattice, and \( \sigma_3^{bf} \) is the Pauli matrix in the space of grading or \( \sigma \)-indices.

The significance of the functional integral (10) lies in the fact that averaged products of single-particle Green functions can be obtained from it by differentiation with respect to matrix elements of the source field \( \hat{J} \): As with the application of supersymmetric field integrals to the analysis of Green functions of conventional disordered systems, we have the identity

\[ \frac{\delta^n Z[\hat{J}]}{\delta (J_1)_{i_1j_1} \cdots \delta (J_n)_{i_nj_n}} = 2^n \left\langle \prod_{m=1}^{n} G_{j_m,i_m}(z_m) \right\rangle. \]  

(11)

In the following it is important to notice the existence of a symmetry of the functional (10) which is directly inherited from the chirality of the Hamiltonian: Consider a linear transformation of the integration variables,

\[ \psi_i \rightarrow T \psi_i, \quad \bar{\psi}_i \rightarrow \bar{\psi}_i T, \quad i \in A, \]
\[ \psi_j \rightarrow V \psi_j, \quad \bar{\psi}_j \rightarrow \bar{\psi}_j V, \quad j \in B, \]

(12)

where \( T, \bar{T}, V, \bar{V} \in \text{Gl}(n|n) \) are arbitrary invertible supermatrices. In the limit \( \hat{z} \rightarrow 0 \), the (source free, \( \hat{J} = 0 \)) action remains invariant provided that the transformation matrices are related to each other by

\[ \bar{T}_s V = s, \quad \bar{V}_s T = s. \]

(13)

In other words, the \( \hat{z} = 0 \) functional integral possesses a global \( \text{Gl}(n|n) \times \text{GL}(n|n) \) symmetry. Notice that for a general non-chiral Hamiltonian (i.e. a Hamiltonian including diagonal blocks in the representation (5)), the symmetry of the functional integral collapses to \( \text{GL}(n|n) \).

What is missing in the above argument is that the matrices \( T, \bar{T}, V \) and \( \bar{V} \) can in fact not be chosen independently: The relation \( \bar{\psi}_B = \psi_B^\dagger \) implies that \( \bar{V}_{BB} = V_{BB}^\dagger \) and \( \bar{T}_{BB} = T_{BB}^\dagger \). Consequently, in the BB-sector of the theory, the global symmetry collapses from \( \text{GL}(n) \times \text{GL}(n) \) to just \( \text{GL}(n) \). Later on we will see that the symmetry manifold is even smaller than that. We will come back to discussing the consequences of such reduction mechanisms after the theory has been subjected to the colour-flavour transformation.

Having identified the basic symmetries of the functional integral, the next conceptual step in the construction of the field theory is the average over the bond matrices \( U_{ij} \). At this stage the ‘colour-flavour transformation’, mentioned above, is applied. It effectively trades the group integrals over the \( U_{ij} \) for integrations over a set of auxiliary field variables \( Z_{ij} \): Applying the identity Eq. (3) in Ref. [5], i.e. the transformation identity for a single \( U(N) \) integration, to all links of the lattice we obtain
\[
Z[\hat{J}] = \int \mathcal{D}(Z, \tilde{Z}) \mathcal{D}(\psi, \overline{\psi}) e^{N \sum_{i \in A, j \in B} \text{str} \ln (1 - Z_{ij} \tilde{Z}_{ij})} \times \\
i \sum_{i \in A} \overline{\psi}_i s^{1/2} (\hat{\xi} + \sum_{j \in N_i} Z_{ij}) s^{1/2} \psi_i + i \sum_{j \in B} \overline{\psi}_j s^{1/2} (\hat{\xi} + \sum_{i \in N_j} \tilde{Z}_{ij}) s^{1/2} \psi_j + i \sum_{k,l} \overline{\psi}_k s^{1/2} j_{kl} s^{1/2} \psi_l.
\]

(14)

Here
\[
Z_{ij} = \{(Z_{ij})_{aa'}\} \in \text{GL}(n|n), \quad \tilde{Z}_{ij} = \{(\tilde{Z}_{ij})_{aa'}\} \in \text{GL}(n|n)
\]

are $2n \times 2n$-dimensional supermatrices living on the directed link from the $A$ site $i$ to the nearest neighbour $B$-site $j$. (By ‘directed’ we mean that there is no $Z_{ji}$.) The symbol $\sum_{j \in N_i}$ stands for a summation over all $j$’s that are connected with $i$ and ‘str’ is the standard supertrace \cite{33}.

In \cite{14}, the integration $\int \mathcal{D}(Z, \tilde{Z})$ over all pairs $(Z_{ij}, \tilde{Z}_{ij})$ is subject to a number of constraints: The fermion-fermion blocks of the matrices $Z_{ij}$ and $\tilde{Z}_{ij}$ are anti-Hermitian adjoint to each other,
\[
Z_{ij,FF} = -\tilde{Z}_{ij,FF}^\dagger.
\]

(16)

In the boson-boson block, a similar but more subtle constraint needs to be imposed. In the original formulation of the colour-flavour transformation \cite{33} the BB-blocks of $Z_{ij}$ and $\tilde{Z}_{ij}$ were chosen to be Hermitian adjoint, $Z_{ij,BB,orig} = \tilde{Z}_{ij,BB,orig}^\dagger$. As will become clear later on, this convention is not suitable for the analysis of the RF-problem. We rather need to chose a different integration manifold, where
\[
\tilde{Z}_{ij,BB} \equiv Z_{ij,BB}^\dagger \neq Z_{ij,BB}^\dagger.
\]

(17)

In order to define the meaning of the symbol ‘$\dagger$’, we first note that $Z_{BB} \in \text{GL}(n)$. Furthermore, due to $\text{GL}(n) = \text{Gl}(n)/\hat{U}(n) \times U(n)$, any matrix $Z_{BB}$ can be decomposed as
\[
Z_{BB} = e^{iH'} e^H,
\]

(18)

where both $H$ and $H'$ are Hermitian (implying that the factor $e^H \in U(n) \backslash \text{GL}(n) \simeq \text{GL}(n)/U(n)$ whilst $e^{iH'} \in U(n)$). The $\dagger$-operation is then defined by
\[
Z^\dagger = (e^{iH'} e^H)^\dagger \equiv e^{-H} e^{iH'},
\]

(19)

i.e. it changes the signs of the exponents in a manner that is opposite from the standard Hermitian adjunction ‘$\dagger$’. As detailed in Appendix \ref{app}, the new integration domain can be reached from the original one by analytic continuation. Hence, switching between the two conventions does not affect the result of the integral. Finally, the Grassmann valued components, $Z_{ij,BF}, Z_{ij,FB}, \tilde{Z}_{ij,BF}, \tilde{Z}_{ij,FB}$ are independent $n \times n$-matrices.

Conceptually, the transformation \cite{14} has a status similar to the Hubbard-Stratonovich transformation employed in the standard construction of field theories for systems with Hermitian disorder \cite{39}: The motivation for introducing the $Z$-fields is that they couple to the fields $\psi$ in a way different from the coupling $\overline{\psi} U \psi$ between the $\psi$’s and the unitary disorder (see fig. \ref{fig:4}). Qualitatively, the $Z$’s connect pairs of fields that, by construction,
represent segments of ‘paired paths’. As discussed in section II, segments of this structure are stabilised by mechanisms of quantum interference, implying that the \( Z \)'s connect to a sector of the theory which contains information on long range behaviour. In addition, the \( Z \)'s couple non-trivially to the ‘flavour’ space thereby potentially allowing for the description of quantum interference between different flavour species (\( \sim \) different Green functions). The subsequent analysis of the field theory will essentially amount to extracting the low energy interference stabilised sector contained in the \( Z \)-functional.

The first step towards an effective long range model is to integrate out the \( \psi \)-fields. At this stage one runs into an unexpected problem, viz. the Gaussian integration over the bosonic component of the \( \psi \)'s does not converge for general complex energy arguments \( z_m \).

In other words, it looks like the orders of integrations over \( Z \) and \( \psi \) cannot be exchanged which would imply that the colour-flavour transformation is useless.

The problem can be surmounted by analytically continuing the arguments \( z_m \) from their natural definition range \( z_m \equiv \epsilon_m \pm i\delta \), \( \delta \) positive infinitesimal, \( \epsilon_m \) a real ‘energy’ argument, deeply into the complex plane: \( \epsilon_m \pm i\delta \rightarrow \epsilon \pm 2i\delta \). As shown in Appendix B, this manipulation cures the problems associated with the \( \psi \)-integration. Moreover, we will see shortly that the remaining integration over \( Z \) is not plagued by convergence problems (as long as the \( z_m \)'s stay away from the real axis.) As a consequence the shift into the complex plane can be redone immediately after the \( \psi \)'s have been integrated out.

Integrating the partition function (14) over the field \( \psi \) (the manipulation of the energy arguments mentioned above being understood), we obtain the functional integral

\[
Z[\hat{J}] = \int \mathcal{D}(Z, \tilde{Z}) e^{-S[Z,\tilde{Z}]} X[\hat{J}],
\]

\[
S[Z, \tilde{Z}] = -N \sum_{(i \in A, j \in B)} \text{str} \ln (1 - Z_{ij} \tilde{Z}_{ij}) + N \sum_{i \in A} \text{str} \ln \left( \hat{z} + \sum_{j \in N_i} Z_{ij} \right) + N \sum_{j \in B} \text{str} \ln \left( \hat{\tilde{z}} + \sum_{i \in N_j} \tilde{Z}_{ij} \right),
\]

where \( X[\hat{J}] \) represents certain pre-exponential contributions resulting from an expansion in powers of the source field \( \hat{J} \) to \( n \)th order. Since, in this study, the focus will be on an analysis of the general structure of the theory rather than on the computation of specific correlation functions, we will henceforth ignore this contribution. (To actually compute correlation functions one may proceed in a standard manner, i.e. keep track of \( X[\hat{J}] \), differentiate with respect to \( \hat{J} \) at any convenient stage, and eventually carry out the \( Z \)-integral weighted with the effective action derived below.)

**IV. EFFECTIVE FIELD THEORY**

So far all manipulations have been exact. In order to make further progress towards an effective long range theory, we next subject the functional (20) to a standard saddle point analysis, that is we seek for solutions of the equations

\[
\left. \frac{\delta S[Z, \tilde{Z}]}{\delta \tilde{Z}_{ij}} \right|_{Z=Z_0, \tilde{Z}=\tilde{Z}_0} = 0,
\]
Anticipating that spatial fluctuations of the fields $Z$ and $\tilde{Z}$ will be energetically penalised, we make an ansatz where the saddle point fields $Z_0$ and $\tilde{Z}_0$ connected to the $A$ and the $B$ sublattice, respectively, are spatially constant. Furthermore, the fact that the action is diagonal in the composite index $a = (m, \sigma)$, permits one to seek for solutions of the form $(Z_0)_{aa'}, (\tilde{Z}_0)_{aa'} \propto \delta_{aa'}$.

Substituting this Ansatz into the functional integral and varying the action we obtain the equations

$$\tilde{Z}_0 = Z_0 = -\frac{x}{2} \hat{z} \pm x \left(-1 + \frac{x^2}{4} \hat{z}^2\right)^{1/2},$$

where $x \equiv (2d - 1)^{-1/2}$. Close to the middle of the band, $z_m \ll 1$, the solutions of (21) compatible with the analyticity requirements imposed by the logarithms in the action are given by

$$\tilde{Z}_0 = Z_0 \simeq ixs.$$

As in the construction of the field theory of the weakly disordered sublattice model [32], the diagonal matrices (22) are not the only solutions of the saddle point equations. The isotropy of the $\hat{z} = 0$ action in the space of $a$-indices rather implies that the set of solutions is highly degenerate. In order to identify the structure of the entire saddle point manifold we proceed in a standard manner and investigate how a general chiral symmetry transformation (12) affects the diagonal saddle point solutions. From the structure of the action (14) it follows that under (12)

$$Z_0 = ixs^{-1/2}s^{-1/2} \rightarrow ixs^{-1/2}\bar{T}Ts^{-1/2},$$

$$\tilde{Z}_0 = ixs^{-1/2}s^{-1/2} \rightarrow ixs^{-1/2}\bar{V}Vs^{-1/2} \equiv ixs^{1/2}(\bar{T})^{-1}s^{1/2}. \quad (23)$$

Since the action of the theory is invariant under global chiral symmetry transformations, all pairs $(Z_0, \tilde{Z}_0)$ of the structure (23) are solutions of the saddle point equations, too. Notice that in (23) the $\text{Gl}(n|n)$-matrices $T$ and $\bar{T}$ appear in a product combination. Taking into account that $\text{GL}(n|n) \cdot \text{GL}(n|n) = \text{GL}(n|n)$, we conclude that the full extent of the saddle point manifold is given by $\text{GL}(n|n)$.\footnote{This is an example of what in QCD is known as the phenomenon of ‘chiral symmetry breaking’: Neglecting quark masses, the microscopic Yang-Mills action (corresponding to our $\psi$-action in the small $z$-limit), is invariant under $G \times G$ where $G$ is a symmetry group whose detailed structure depends on the flavour content of the theory etc. (In our formalism, its role is played by $\text{GL}(n|n)$.) Now, in the effective low energy QCD Lagrangian, the microscopic chiral symmetry is spontaneously broken down to $G \times G \to G$, very much like the symmetry group of our low energy $Z$-functional is $\text{GL}(n|n)$ rather than $\text{GL}(n|n) \times \text{GL}(n|n)$.}
Eq. (23) defines the maximum manifold of solutions of the saddle point equations. In order to complete the determination of the global structure of the theory, we next need to explore how the manifold (23) intersects with the domain of the field integration (as specified by (15), (16) and (17)). To this end, we represent a general pair of matrices $Z$ and $\tilde{Z}$ according to

$$(Z, \tilde{Z}) \equiv ix(PT, T^{-1}P),$$

where $P, T \in \text{GL}(n|n)$ and we have omitted the site index $(ij)$ for notational transparency.

In order to give the reparameterisation (24) some meaning, we a) have to be specific about the internal structure of the factor matrices $T$ and $P$, and b) tell how the integration measure $D(Z, \tilde{Z})$ is affected by the transformation. As for a), the restriction (16) leads to the relation

$$P_{\text{FF}} \in \text{GL}(n)/\text{U}(n), \quad T_{\text{FF}} \in \text{U}(n).$$

Similarly, the decompositions (17) and (18) imply that

$$P_{\text{BB}} \in \text{U}(n), \quad T_{\text{BB}} \in \text{GL}(n)/\text{U}(n).$$

The off-diagonal blocks $P_{\text{BF}}, P_{\text{FB}}, T_{\text{BF}}$ and $T_{\text{FB}}$ are independent Grassmann variables. Finally, counting the number of independent parameters, we see that the $(P,T)$ manifold covers the entire $(Z, \tilde{Z})$ domain.

As for the question of the measure, first note that Eqs. (23) and (24) imply that the saddle point configurations of the theory are specified by $P = \mathbb{1}$. Consequently, excitations in $P$ are massive, and can be treated in a quadratic approximation. Due to the fact that close to $P = \mathbb{1}$ the integration measure is approximately flat, we are spared addressing point b) explicitly. More precisely, the measure of the $P$-integration is flat Euclidean, whilst the $T$-integration is over the invariant measures on the manifolds (25) and (26), respectively.

Eqs. (24), (25) and (26) contain the answer to the question of the general structure of the target manifold of the field theory raised above: The maximal manifold of solutions of the saddle point equation is given by $\text{GL}(n|n)$. In the BB-sector, the intersection of this manifold with the actual domain of the field integration is given by $\text{GL}(n)/\text{U}(n) \subset \text{GL}(n)$, and in the FF-sector, by $\text{U}(n) \subset \text{GL}(n)$. The Grassmann sectors are not restricted. We note that these global structures have been worked out earlier [40], albeit in a more general manner based on Cartan’s complete classification of symmetric spaces. Apart from the fact that it is not obvious how that more abstract line of reasoning can operationally be adapted to the RF-model, the present treatment has the advantage of encompassing the massive sector of the theory (the $P$’s). As we will see later on, the RF-problem differs from standard (non-chiral) field theories of disordered systems, in that the massive $P$’s have a significant impact on the final structure of the functional integral.

Finally, we note that the original boson-replica [32] and fermion-replica [41] based approaches chiral disordered model systems led to field theories over $\text{GL}(Rn)/\text{U}(Rn)$ and $\text{U}(Rn)$, respectively. Here $R \to 0$ denotes the number of replicas. This field manifold can be interpreted as the replica analogue of the BB-sector of our supersymmetric construction.

We next employ the saddle point decomposition (24) and (26) to reduce the exact functional (21) to a simpler effective functional describing the long range behaviour of the model.
Technically, the procedure will be based on a continuum approximation in combination with a gradient expansion around the spatially constant saddle point manifold. It is important to realize that this evaluation scheme is, in fact, problematic: As compared to analogous treatments of weakly disordered models, the gradient expansion is not stabilised by the presence of the small parameter ‘disorder strength/Fermi energy’. Apart from the external parameters \( z_m \), all energy scales are rather of the same order: \( \mathcal{O}(1) \). With the non-Abelian versions of the RF-model, \( N > 1 \), the situation is less problematic, the reason being that the mean field analysis is stabilised by the parameter \( 1/N \). However, for the \( N = 1 \) random magnetic flux model, the expansion around the mean field configurations is not controlled by a small parameter. In this situation all we can do is provide some independent physical evidence for the belief that the field theory constructed below remains valid all the way down to \( N = 1 \).

Some arguments to that effect will be given towards the end of the paper.

Keeping these words of caution in mind, we now turn to the discussion of the continuum expansion of the theory. On general grounds, it may be anticipated that after integration over massive modes, the effective action takes the form

\[
S[P,T] \xrightarrow{\text{(\cdots)}} S_{\text{eff}}[T] = S_z[T] + S_{\text{fl}}[T] + S_m[T],
\]

(27)

where \( S_z[T] \) is the contribution due to finite energies \( \hat{z} \), \( S_{\text{fl}}[T] \) the action associated with fluctuations of the Goldstone modes (the \( T \)'s), and \( S_m[T] \) is a residual action induced by the interaction between massive and Goldstone modes, respectively. We next assume that the field configurations relevant to the long range physics are smooth (differentiable). The different contributions to the action can then be computed by a gradient expansion followed by a continuum limit. Albeit conceptually straightforward, the explicit formulation of this program happens to be somewhat cumbersome. For this reason, the derivation of the different contributions to the action has been deferred to Appendix \( C \). As a result we obtain

\[
S_{\text{eff}}[T] = -\frac{1}{2} \int \left( \frac{a^{2-d}}{b} \text{str}(\partial T^{-1} \partial T) + \frac{2N(2d-1)^{1/2}a^{-d}}{4d} \text{str}(\hat{z}(T + T^{-1})) + \right.
\]

\[
+ \frac{a^{2-d}}{c} \text{str}(T^{-1} \partial T) \text{str}(T^{-1} \partial T) \Big) + S_b[T],
\]

(28)

where \( a \) represents the lattice spacing, \( b^{-1} = N/4d \), \( c^{-1} = C/8d \), and \( C \) denotes a numerical constant defined in Eq. (C16). Further, the discrete lattice summations \( \sum_i \) have been converted to integrals over smooth field configurations \( T \). Eq. (28) represents the main result of this paper. \( S_b[T] \) represents a boundary action whose structure depends on the geometry of the lattice. For a \( d \)-dimensional hypercubic lattice with \( N_i \) sites in direction-i,

\[\text{We remark that this extrapolation has the same status as the applicability of the Pruisken action to the Integer Quantum Hall transition. Strictly, the field theory can be formally justified only in the limit of high Landau level index, or large conductance. Yet, studies [37] of the phenomenological Chalker-Coddington model, corresponding to a single Landau band, indicate that the same action controls the fixed point of the Hall transition when the conductance is } O(e^2/h).\]
Notice that the boundary action depends on the ‘parity’ of the site-numbers, i.e. on whether or not the number of sites in the different extensions of the lattice is even or odd. Both the implications of this fact and the topological content of the boundary action will be discussed below.

V. ANALYSIS OF THE FIELD THEORY

What kind of information can be extracted from the effective action (28)? A comparatively complete answer to that question may be given in the ‘zero dimensional’ case, i.e. the case of systems of linear extension $L$ in a regime where all energy parameters are such that $z_m \ll a^2/L^2$. (Recall that all energies are measured in units of the modulus of the hopping matrix elements.) The denotation ‘zero dimensional’ is motivated by the fact that, as in the case of conventional field theories of disordered systems, the boundary conditions to be imposed on $\partial \mathcal{T}$ imply that the kinetic energy associated with spatial fluctuations is quantised: For constant field configurations it vanishes, while the first non-constant configuration has a fluctuation energy of $O(NA^2/L^2)$, the ‘Thouless energy’ of the system\[.\] (Notice that in $d = 2$, the Thouless energy is small of $O(\Delta)$, where $\Delta$ is the mean level spacing. Thus, unlike in weakly disordered 2d systems, there is no extended region separating the onset of the zero mode regime $z \sim E_c \equiv a^2/L^2$ from the fundamentally quantum regime $z \sim \Delta$.) For energies $z_m \ll a^2/L^2$ spatially fluctuating configurations are effectively frozen out and the action simplifies to

$$S_0[T] = -\frac{i}{2} \rho_0 \text{str } (\hat{T} + \hat{T}^{-1}) + S_0[T],$$

(30)

where $T \in \text{GL}(n|n)$ is a single matrix (the zero mode), and

$$\rho_0 = \frac{N(2d - 1)^{1/2}L^2}{2\pi da^2}.$$  

(31)

To illustrate how the zero mode action can be applied to the analysis of physical observables, we consider the DoS (7) in the zero dimensional limit. Applying the general recipe for the computation of Green functions outlined above to the source field $\hat{J} = \sigma_3^{bf}$, the DoS is represented by

$$\rho(\epsilon) = \frac{\rho_0}{4} \text{Re } \int dT e^{-S_0[T]} \text{str } ((T + T^{-1})\sigma_3^{bf}).$$

(32)

\footnote{That $a^2/L^2$ plays the role of the Thouless energy $E_c \sim v_F l/L^2 = v_F^2 \tau/L^2$ ($v_F$: Fermi velocity, $l$: elastic mean free path, $\tau$: elastic scattering time) of weakly disordered systems can be seen from the following rough estimate: Remembering that all energies are measured in units of the hopping matrix element, $v_F \sim a$ in the middle of the band. Further, for strong disorder, $\tau \sim 1$ is of the order of the band width. This gives $E_c \sim a^2/L^2$.}
The integral depends sensitively on the presence of the boundary term $S_b[T]$, which in turn depends on the parity of the lattice. If the number of sites is odd (i.e., the number of lattice sites along each and every direction is odd), the boundary term takes the form

$$S_b[T] = \text{str} \ln T,$$

otherwise the term vanishes. In both cases the explicit computation of the zero-mode integral (33) for the particular case of $n = 1$ is detailed in Appendix D. Explicit calculation shows that, for a lattice with an even number of sites, the DoS takes the universal form

$$\rho(\epsilon) = \frac{\pi^2 \epsilon \rho_0^2}{2} \left( J_0^2(\pi \epsilon \rho_0) + J_1^2(\pi \epsilon \rho_0) \right),$$

(34)

while if the number of sites is odd,

$$\rho(\epsilon) = \frac{\pi^2 \epsilon \rho_0^2}{2} \left( J_N^2(\pi \epsilon \rho_0) - J_{N-1}(\pi \epsilon \rho_0)J_{N+1}(\pi \epsilon \rho_0) \right),$$

(35)

where $J_N$ denotes the Bessel function of order $N$.

In fact, both of these results have been anticipated in the literature. Firstly, Eqs. (34) and (35) have been derived earlier within the framework of a supersymmetric non-linear $\sigma$-model for the chiral random matrix ensemble ChGUE [15,42], and within pure random matrix theory [9–12]. Indeed, both expressions have been discussed in their connection with chiral symmetry breaking of the QCD Dirac operator [9–11,42]. Moreover, in the present context, the validity of Eq. (34) has been investigated numerically [43].

In fact, the coincidence between the low energy RF-field theory and the non-linear $\sigma$-model for chiral random matrices is not restricted to the mean DoS but rather extends to arbitrary $n$-point correlation functions. Equivalently, it can be said that in the low energy regime the RF-model is described by chiral random matrix theory, a fact that could have been anticipated from symmetry arguments alone.

Eq. (34) also contains the answer to the question of what happens to the DoS upon leaving the domain of perturbation theory (c.f. Eq. (4) and below): Ultimately (i.e. on a scale comparable with the mean level spacing) it drops down to zero. Notice, however, that the zero dimensional analysis cannot exclude the existence of divergent behaviour prevailing down to energy scales $\sim E_c$ (see below).

At the same time, Eq. (34) suggests that on energy scales $\epsilon > \rho_0^{-1}$, the DoS approaches the constant value $\rho_0$. This prediction must be taken with some care: In $d = 2$ it is meaningless because the domain of applicability of the zero mode approximation is set by $\rho_0^{-1}$ itself and the present analysis has nothing to say about energies larger than that. For $d > 3$, $\rho_0$ should have the significance of a ‘mean’ DoS of the random flux model. Although, indeed, Eq. (31) agrees parametrically with approximate expressions for the band center DoS of large dimensional hypercubic lattices [44], it is not clear whether $\rho_0$ has any true physical significance for the physics of the RF models. The reason is that (31) represents a bare parameter which can be, in principle, subject to substantial renormalization. As will be discussed below, the extent to which the parameter changes under the RG flow is, at present, not clear.

Leaving the zero-mode regime and returning to the more complicated, truly extended case, we first note that the field theory described by (28) is not entirely original but rather has a number of closely related precursors:
Gade’s analysis of the weakly disordered sublattice model led to a boson replica version of the present model, i.e. a theory over fields $T \in \text{GL}(nR)/\text{U}(nR)$, where $R \to 0$ is the number of replicas. The action for these fields coincided with (28), save for the important difference that, due to the weakness of the disorder, the coupling constant $b^{-1}$ was parametrically larger than one.

In high energy physics the action (28) has been suggested on phenomenological and symmetry grounds as relevant for the effective description of partially quenched QCD [45]. In that context, the base manifold is 4 + 1-dimensional whilst the fields $T \in \text{U}(n_f + 1|1)$, where $n_f$ is the number of quark flavours.

A similar field theory (albeit without a $(\text{tr}(T^{-1}\partial T))^2$ operator) was derived in connection with a model of Dirac fermions subject to a random non-Abelian vector potential [46]. That the random Dirac model is closely related to the RF-problem considered here follows from the fact that, when put on a lattice, Dirac fermions are equivalent to free fermions subject to an average $\pi$-flux per lattice cell. In other words, the Dirac model corresponds to a weak random flux model centered around an average $\pi$-flux background.

As for the field theoretical description, apart from being defined on a different field manifold $(\text{SU}^c(N)/\text{SU}(N))$, where $\text{SU}^c(N)$ is the complex extension of $\text{SU}(N)$, the model differs from (28) in that it contains a Wess-Zumino-Novikov-Witten (WZNW) operator (however, see Ref. [17]). While the symmetries of the fields $\in \text{GL}(n|n)$ would, in principle, admit a supersymmetric version of a WZWN operator, we believe that the presence of a WZNW action is induced by the Dirac nature of the clean spectrum implying that there is no room for such a contribution in our model. (C.f. the absence of a Pruisken term in the action of a weak disordered electron gas subject to a random, zero average magnetic field.) To check that conjecture explicitly, we have computed the action of an SU(2) instanton type configuration defined in the FF-sector of the theory. No imaginary contribution to the action was obtained, a signature for the absence of a WZWN contribution. For the application of a similar recipe to determine the coupling constant of topological contributions to the action, see Ref. [37].

Finally, a fermion-replica analogue of the action (28) was obtained in connection with a lattice model of a disordered $d$-wave superconductor [41,48,49]. The connection to the RF-model follows from the fact that the clean limit of the $d$-wave model is governed by the presence of Dirac type excitation spectra and can therefore be effectively described in terms of a model qualitatively analogous to that of $\pi$-flux lattice fermions.

Various conclusions regarding the physical behaviour of the RF-model, most notably about its localization behaviour, can be directly inferred from Ref. [32]. There it was shown that the conductance of the weakly disordered $2d$ model at the band center (which is essentially determined by the coupling constant $b^{-1}$) did not change under one-loop perturbative

---

6Notice that for $T \in \text{SU}^c(N)/\text{SU}(N)$, $\det(T) = 1 \Rightarrow \text{tr}(T^{-1}\partial T) = 0$ which explains the absence of the $(\text{tr}(T^{-1}\partial T))^2$ operator in this field theory.
renormalization. This observation suggests that a non-localised state might exist in the middle of the band. Since the stability of the perturbative RG merely relies on the smallness of the parameters $b, b^2/c \ll 1$, its results can be straightforwardly carried over to the $N \gg 1$ non-Abelian RF model: The one-loop renormalization indicates that for $N \gg 1$ the strongly disordered RF model exhibits metallic behaviour at the band center.

It is natural to ask as to what extent this prediction survives the limit $N \to 1$. Actually, one may adopt an even more hesitant attitude and ask whether the basic structure of the action (28) is stable as $N \to 1$ (c.f. comments made in Appendix C in connection with the gradient expansion). We believe that the answer to the second question is positive whilst the answer to the first one remains, to some extent, a matter of faith.

Addressing the second issue first, we note that, by a standard argument of operator relevancy, the infrared continuum action should be dominated by operators with the least number of derivatives. Of these, there are only two invariant combinations that are compatible with the basic symmetries of the model, namely the two operators appearing in (28). (E.g. the presence of an operator like $\sim \text{str}(\partial_i T^{-1} \partial_j T), i \neq j$ is ruled out by virtue of reflection symmetry, etc.). Thus, the basic structure of the action is dictated by symmetry alone. As for the coupling constants multiplying the operators of the action, the absence of dimensionful parameters in the RF problem fixes their value up to numerical factors about which we need not be concerned.

Now, turning to the more serious question of the validity the RG results in the Abelian case, from the results presented here, there appears to be only one useful test of consistency: The existence of a metallic state for $N = 1$ with strong disorder would at least be compatible with the behaviour in the limiting cases: $N = 1$ and weak disorder, and $N \gg 1$ with arbitrary disorder. Moreover, this conclusion does also seem to be consistent with the most recent numerical investigations [31]. More concrete analytical evidence in favour of band center delocalisation does not seem available.

Besides the conductance, the DoS of the extended model also behaves in an unusual manner: The analysis of Ref. [32] predicts divergent behaviour upon approaching the band center, where the detailed functional form of the divergence depends on the dimensionality of the system. In particular, in two dimensions the RG analysis predicts

$$\rho(\epsilon) \sim \frac{e^{-\kappa \sqrt{-\ln(|\epsilon|)}}}{|\epsilon|}.$$  

The scale below which the $|\epsilon|^{-1}$ divergence begins to dominate over the exponential factor is determined by the constant $\kappa$. Unfortunately the dependence of that constant on the relevant parameters of the model can not be reliably extracted from the perturbative RG analysis, i.e. the only statement that can safely be made is that a divergence forms somewhere upon approaching the band center. Ultimately, the divergent behaviour is cut-off by the onset of the non-perturbative ergodic regime discussed above. A quantitative understanding of both the formation of the divergence and its truncation in the close vicinity of the band center would necessitate a combination of an RG analysis with a non-perturbative treatment of the zero mode contribution, a program that is well beyond the scope of the present paper.

A third piece of information that can be extracted from Ref. [32] is how finite energy arguments $z_m \equiv \epsilon_m \pm i0$ affect the behaviour of the model: In the spirit of the RG, finite
energies have the significance of relevant perturbations. Thus, the band center represents a critical manifold whilst, for finite $z_m$, the model flows into a different universality class. More precisely, the band center field theory — a model of universality class AIII in the classification of Ref. [40] or, more commonly, a model of ChGUE-symmetry — develops into an ordinary GUE type model. This is accompanied by a reduction of the symmetry manifold from $\text{Gl}(n|n)$ to the standard GUE manifold $\text{GL}(n|n)/\text{Gl}(n_+|n_+) \times \text{GL}(n_-|n_-)$, where $n_+ + n_- = n$ and $n_{\pm}$ is the number of energy arguments with positive/negative imaginary part. (Note that this conclusion is in accord with the analysis in Ref. [4] where a continuum random field model far away from the band center was shown to be described by a supersymmetric unitary model.)

In that context, one may ask how the degrees of freedom of the model of reduced, unitary symmetry, commonly denoted by $Q$ [39], emerge from the fields $T$ of the present model upon crossing over to the unitary regime. In fact the full RF model could have been formulated in terms of $Q$-matrices (albeit $Q$’s of a symmetry different from the standard unitary symmetry) from the outset: I.e. by defining $Q = \tilde{T}\Lambda\tilde{T}^{-1}$, where $\Lambda = \sigma_3 \otimes s$, $\tilde{T} = \exp(iW\sigma_2)$ and the $\sigma_i$ operate in sublattice space. The matrices $\tilde{T}$ are then related to our $T$’s by the identity $T = \exp(iW)$. The price to pay for this more common $Q$-formulation of the theory would be a certain amount of redundancy: A description in terms of structures $Q \sim \tilde{T}\Lambda\tilde{T}^{-1}$ is the natural one for problems with coset space symmetry (e.g. the unitary coset space $\text{GL}(n|n)/\text{Gl}(n_+|n_+) \times \text{GL}(n_-|n_-)$). However, the present problem has group type symmetry ($\text{Gl}(n|n)$) implying that a modelling directly in terms of group degrees of freedom is more natural.

VI. SUMMARY AND DISCUSSION

Summarising, we have mapped both the Abelian ($N = 1$) and non-Abelian ($N > 1$) random flux model onto an effective continuum field theory. The model differs in only two respects from the field theory obtained earlier [22] for band center lattice fermions subject to weak off-diagonal disorder: It is a) based on the formalism of supersymmetry (as opposed to boson replicas), and b) governed by a different coupling constant $b^{-1} \sim N \times O(1)$ (as opposed to the large value obtained in the weakly disordered case).

The supersymmetric formulation of the theory allows one to rigorously establish contact with the formalism of chiral random matrix theory [9–14], and to demonstrate that the ergodic sector of the model is described by a random matrix ensemble of appropriate symmetry. Apart from this point, however, the fact that the present model is formulated in terms of supersymmetry, whilst the earlier analysis [22] utilised a replica formulation is arguably incidental. What is, perhaps, more relevant is that both a weakly and a maximally disordered model with sublattice symmetry are described in terms of the same effective field theory. The only difference lies in the value of the coupling constant $b^{-1}$. That this constant takes a large/small value in the case of weak/strong disorder follows readily from its significance as the bare conductance of the model. Given that the two limiting cases (very weak/very strong disorder) lead to the same effective action [22], it is tempting to conjecture that the present field theory universally describes the long range behaviour of disordered sublattice models, where the amount of disordering is reflected in the value of
the coupling constant $b^{-1}$.

Unfortunately, the small value of the coupling constant $b^{-1}$ in the strongly disordered case prevents us from directly transferring the results of the perturbative RG analysis of Ref. [32] to the RF case. E.g. we cannot exclude the existence of some fixed point separating the delocalised regime for weak disorder from a potentially localised regime of the fully random flux model. Thus, the present analysis is not able to reliably answer the long-standing question of the behaviour of the strongly disordered Abelian RF-model in the band center. What can be said with some confidence is that the non-Abelian versions of the RF-model do exhibit long range correlations ($\leftrightarrow$ delocalisation), this being a consequence of the fact that $b^{-1} \propto N$ implying that the RG is stable. (A more detailed account of the application of the present formalism to the physics of systems describable in terms of non-Abelian RF-models (or relatives thereof) will be presented elsewhere.)

Finally, we mention that the present analysis of the RF-model can be generalized so as to encompass other lattice problems with unitarily implemented disorder: Firstly, models away from the ‘strong coupling limit’ (models with a finite plaquette action like in (4)) can be dealt with at the expense of introducing extra ‘ghost components’ in the $\psi$-field [50]. Secondly, models where stochasticity is implemented in terms of subgroups of the U($N$), e.g. as in the QCD-Hamiltonian (SU($N$) $\subset$ U($N$)) or the random Ising Hamiltonian ($Z_2 \subset$ U(1)), can be analysed in terms of suitably adapted versions of the colour-flavour transformation [51]. Some of these generalisations are currently under investigation.

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APPENDIX A: CHOICE OF THE BOSONIC INTEGRATION CONTOUR

In the original formulation [3] of the colour flavour transformation, the integration over the bosonic block of the fields $Z$, 
\[
\int_{\mathcal{M}_B} dZ(\ldots)
\]
extended over a manifold $\mathcal{M}_B$ defined by 
\[
\mathcal{M}_B = \{(Z, \tilde{Z}) | \tilde{Z} = Z^\dagger, |\text{EV}(Z\tilde{Z})| \leq 1\}.
\] (A1)
Here $|\text{EV}(Z\tilde{Z})| \leq 1$ is a shorthand notation for the ‘modulus of all eigenvalues of the matrix $Z\tilde{Z}$ smaller than one’. (As we will be concerned solely with boson variables throughout, the subscript ‘BB’ in $Z$ has been omitted.)

The Hermitian connection (A1) between $Z$ and $\tilde{Z}$ ultimately finds its origin in the interpretation of these fields as parameter sets of bosonic spin coherent states. (A comprehensive discussion of the role of Hermiticity in connection with boson spin coherent states can be found in Perelomov’s book [52].) Yet it is not enforced by criteria of mathematical consistency. In order to encompass the low-lying modes of the RF-problem it is in fact necessary to abandon this connection and to consider pairs $(Z, \tilde{Z} \equiv Z^\dagger \neq Z^\dagger)$, where the matrix $Z^\dagger$ has been defined in Eq. (A1). More precisely, we need to analytically continue to a different
integration manifold $\mathcal{N}_B$ on which the connection between $Z$ and $\tilde{Z} = Z^\dagger$ is still one-to-one but different from Hermitian adjungation.

Apart from the Hermitian connection between $Z$ and $\tilde{Z}$, Eq. (14) contains a second condition, viz. $|\text{EV}(Z\tilde{Z})| \leq 1$. As opposed to Hermiticity, this criterion is indeed needed for mathematical reasons. The most straightforward way to see this is to observe that the BB-block of the logarithmic term appearing in the colour-flavour transformed action (14) is given by $\text{tr} \ln (1 - Z\tilde{Z})$. The condition $|\text{EV}(Z\tilde{Z})| \leq 1$ ensures that the cut of the logarithm is not crossed upon integration over $Z$. In fact, the condition can be relaxed to

$$\text{EV}(Z\tilde{Z}) \notin ]1, \infty].$$

To understand the connection between the integrations over $\mathcal{M}_B$ and $\mathcal{N}_B$, respectively, we reconsider the decomposition

$$\text{GL}(n) \ni Z = WV, \quad W \equiv \exp(H) \in \text{GL}(n)/U(n), \quad V = \exp(iH') \in U(n),$$

where $H$ and $H'$ are $n$-dimensional Hermitian generators. The BB-restricted integration over a function $f(Z, \tilde{Z})$ can then be written as

$$\int_{\mathcal{M}_B} f(Z, \tilde{Z}) = \int_{\text{GL}(n)/U(n)} \int_{\text{EV}(W) \leq 1} dW \int_{U(n)} dV f(WV, V^{-1}W),$$

where we have used the fact that $\tilde{Z} = Z^\dagger = (WV)^\dagger = V^{-1}W$, and that the eigenvalues of $Z\tilde{Z} = W^2$ depend only on the Hermitian factor matrix $W$. The symbols $f$ and $dV$ denote an integration over the invariant measures on the coset space $\text{GL}(n)/U(n)$ and the group $U(n)$, respectively. The basic idea of the analytic continuation is now to ‘exchange’ the role of the compact $U(n)$ and the non-compact $\text{GL}(n)/U(n)$ submanifolds, respectively. To this end, notice that both $W$ and $V$ can be unitarily diagonalised,

$$W = \exp(H) = U \exp(D)U^\dagger \sim dW = dU dD J(D),$$

$$V = \exp(iH') = U' \exp(iD')U'^\dagger \sim dV = dU' dD' J'(D').$$

Here $U, U' \in U(n)$ are the diagonalising matrices, $D = \text{diag}(D_1, \ldots, D_n)$ and $D' = \text{diag}(D'_1, \ldots, D'_n)$ are the diagonal matrices of real eigenvalues $D_i$ and $D'_i$. Finally, $J(D)$ and $J'(D')$ are Jacobians resulting from the change of integration variables and the condition EV($W$) $\leq 1$ translates to $D_i \leq 0$.

We next ‘Wick rotate’ the integration over the eigenvalues $D_i$ from the real to the imaginary axis. Since $\text{tr} \ln (1 - Z\tilde{Z}) = \sum_i \ln(1 - e^{2D_i})$, the $D_i$’s can safely be sent from the negative real to the imaginary axis without crossing singularities. (After the rotation the eigenvalues of $Z\tilde{Z}$ are given by $e^{2D_i}$. I.e. their modulus does not exceed unity.) As a result of this manipulation, the integration over $\text{GL}(n)/U(n)$ becomes one over $U(n)$ and vice versa. More precisely,

$$\int_{\mathcal{M}_B} f(Z, \tilde{Z}) \equiv \int_{\text{GL}(n)/U(n)} \int_{\text{EV}(W) \leq 1} dW \int_{U(n)} dV f(WV, V^{-1}W) \overset{\text{Wick}}{\rightarrow} \int_{U(n)} dW \int_{\text{GL}(n)/U(n)} dV f(WV, V^{-1}W) \equiv \int_{\mathcal{N}_B} f(Z, \tilde{Z}).$$

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The last line defines the new integration manifold $\mathcal{N}_B$. Notice that, on the new manifold, the connection between $Z$ and $\tilde{Z}$ is no longer provided by Hermitian adjunction. Rather, we have

$$Z = \exp(iH) \exp(H') \quad \tilde{Z} = \exp(-H') \exp(iH) \equiv Z^\dagger.$$ 

**APPENDIX B: ANALYTIC CONTINUATION OF THE ENERGY ARGUMENTS**

This Appendix is devoted to a discussion of the convergence problems that arise after the colour-flavour transformation in connection with the Gaussian integration over the bosonic field variables $\psi_B$. Since we will be exclusively concerned with the bosonic sector of the theory, all subscripts ‘B’ will be omitted throughout.

That convergence problems occur follows simply from the fact that the integration over $Z$ extends over field configurations with finite imaginary eigenvalues. Thus, it looks like the Gaussian integration over $\exp\left(\sum_i \psi_i \bar{s}_{1/2}\right)$ does not converge for all values of $Z_{ij}$ implying that the orders of integration over $\psi$ and $Z$ cannot be exchanged.

Below we are going to show that this difficulty can be surmounted by giving the energy arguments $z_m$ a finite imaginary part. More precisely, it is necessary to analytically continue from $z_m = \epsilon_m \pm i\delta$ to $z_m = \epsilon_m \pm 2i\delta$. Since no singularities ($z_m \in$ real axis) are crossed, the continuation manipulation is permitted and, after it has been redone on the level of the subsequent $Z$ integration, does not affect the value of the functional integral.

To understand that a shift into the deep complex region $|z_m| > 2d$ solves the convergence difficulties, we consider bilinear forms like $\bar{\phi}Z\phi$ where $Z$ is meant to be the BB-block of any matrix $Z_{ij}$, $\bar{\phi} \equiv \psi^{\dagger}_{1/2}$ and $\phi \equiv s^{1/2}\psi_i$. To formulate the following estimates, it is useful to represent scalar products like the one above in terms of a bra-ket notation, e.g. $\bar{\phi}Z\phi \equiv \langle \bar{\phi}|Z|\phi \rangle$. Now, for any normalized vector $|n\rangle$, $\langle n|ZZ^\dagger|n\rangle < 1$, a result that straightforwardly follows from the fact that the matrix $ZZ^\dagger$ is unitarily diagonalisable and has (see below Eq. (17)) eigenvalues smaller than one. Using this estimate and setting $|\phi\rangle \equiv \langle \phi|\phi\rangle^{1/2}|n\rangle$, we next obtain

$$|\psi^{\dagger}_{1/2}Zs^{1/2}\psi_i| = |\langle \phi|Z|\phi \rangle|^2 = 2,$$

$$= \langle \bar{n}|Z|n\rangle \langle n|Z^\dagger|\bar{n}\rangle \langle \bar{\phi}|\bar{\phi}\rangle \langle \phi|\phi \rangle \leq$$

$$\leq \sum_m \langle \bar{n}|Z|m\rangle \langle m|Z^\dagger|\bar{n}\rangle \langle \bar{\phi}|\bar{\phi}\rangle \langle \phi|\phi \rangle =$$

$$= \langle \bar{n}|ZZ^\dagger|\bar{n}\rangle \langle \bar{\phi}|\bar{\phi}\rangle \langle \phi|\phi \rangle \leq \langle \bar{\phi}|\bar{\phi}\rangle \langle \phi|\phi \rangle = (\psi^{\dagger}_{1/2}\psi_i)^2,$$

where $\{|m\}\}$ is meant to be an extension of $|n\rangle$ to an orthonormalised basis. This result implies that,

$$|\text{Im} \psi^{\dagger}_{1/2}Zs^{1/2}\psi_i| = \psi^{\dagger}_{1/2}\psi_i$$
and further
\[ |\text{Im} \sum_{j \in N_i} \psi_i^\dagger s^{1/2} Z_{ij} s^{1/2} \psi_i| \leq \sum_{j \in N_i} |\text{Im} \psi_i^\dagger s^{1/2} Z_{ij} s^{1/2} \psi_i| \leq 2d \psi_i^\dagger \psi_i. \]

On the other hand, after the analytic continuation of the energy arguments,
\[ |\text{Im} \psi_i^\dagger s^{1/2} \hat{z} s^{1/2} \psi_i| \geq 2d \psi_i^\dagger \psi_i. \]

Consequently, the sum of the energy and the \( Z \)-contribution to the bosonic bilinear form (B1) has a positive imaginary part, implying that the \( \psi \)-integration converges.

**APPENDIX C: GRADIENT EXPANSION**

This Appendix is devoted to the explicit derivation of the three contributions to the action (28). To simplify the notation we introduce the convention that all indices \( i, i', \ldots \in A \), whilst \( j, j', \ldots \in B \).

**1. Fluctuation action, \( S_\text{fl} \):**

Setting \( \hat{z} = 0 \), ignoring massive modes, \( P = 1 \), and substituting the parameterisation (24) into (20) we obtain
\[
S_\text{fl}[T] = N \sum_i \text{str ln} \left( \sum_{j \in N_i} T_{ij} \right) + N \sum_j \text{str ln} \left( \sum_{i \in N_j} T_{ij}^{-1} \right). \tag{C1}
\]

We next assume the lattice to be embedded in a continuum, and that the low energy field configurations \( \{ T_{ij} \} \) can be extended to smooth functions \( T \) with continuous support. Under these conditions, it makes sense to Taylor expand
\[
T_{ij} = T_i + a \frac{1}{2} \partial_{i \rightarrow j} T_i + \frac{a^2}{8} \partial_{i \rightarrow j}^2 T_i + \ldots \tag{C2}
\]

where \( a \) is the lattice spacing, \( \partial_{i \rightarrow j} \) the directional derivative along the link \( i \rightarrow j \), and \( T_i \) the function \( T \) evaluated at the coordinate of the lattice point \( i \). Substituting this expression into (C1) and taking into account the fact that \( \sum_{j \in N_i} \partial_{i \rightarrow j} T_i = 0 \), we obtain
\[
S_\text{fl}[T] = N \sum_i \text{str ln} \left( 2dT_i \right) + N \sum_j \text{str ln} \left( 2dT_j^{-1} \right) +
+ \frac{Na^2}{16d} \left( \sum_i \sum_{j \in N_i} T_{ij}^{-1} \partial_{i \rightarrow j}^2 T_i + \sum_j \sum_{i \in N_j} T_j \partial_{j \rightarrow i}^2 T_j^{-1} \right) + \mathcal{O}(\partial^4). \tag{C3}
\]

Up to second order in the number of derivatives (the accuracy of the present expansion scheme), the first two terms cancel each other:
\[ \sum_i \text{str} \ln (2dT_i) + \sum_j \text{str} \ln (2dT_j^{-1}) = \sum_i \text{str} \ln (T_i) - \sum_j \text{str} \ln (T_j) = \]
\[ = \sum_i \left( \text{str} \ln (T_i) - \frac{1}{2d} \sum_{j \in N_i} \text{str} \ln (T_j) \right) \cong \]
\[ \cong \sum_i \left( \text{str} \ln (T_i) - \frac{1}{2d} \sum_{j \in N_i} \text{str} \ln (T_i + a\partial_{i\to j}T_i + \frac{a^2}{2} \partial^2_{i\to j}T_i) \right) \cong \]
\[ \cong -\frac{1}{2d} \sum_{i,j \in N_i} \text{str}(-T_i^{-1}(\partial_{i\to j}T_i)T_i^{-1}(\partial_{i\to j}T_i) + T_i^{-1}\partial^2_{i\to j}T_i) = \]
\[ = -p\frac{a^2}{4d} \sum_{i,j \in N_i} \text{str}(\partial_{i\to j}T_i^{-1}\partial_{i\to j}T_i + T_i^{-1}\partial^2_{i\to j}T_i) = 0, \quad (C4) \]

where the last equality is based on an ‘integration by parts’. Using the fact that \(\sum_{j \in N_i} \partial^2_{i\to j} = 2\partial^2\) is twice the Laplace operator, and that in the continuum limit \(\sum_{i \in A} \rightarrow \frac{1}{2\alpha} \int\) the remaining two terms can be brought to the form

\[ S_{\text{fl}}[T] = \frac{Na^2}{8d} \left( \sum_i \text{str}(T_i^{-1}\partial^2T_i) + \sum_j \text{str}(T_j\partial^2T_j^{-1}) \right) \]
\[ \cong \frac{Na^2-d}{16d} \int \text{str} \left( T^{-1}\partial^2T + T\partial^2T^{-1} \right) = -\frac{Na^2-d}{8d} \int \text{str}(\partial T^{-1}\partial T). \quad (C5) \]

### 2. Frequency action, \(S_z\):

The frequency action is straightforwardly derived by expanding the action (20) to linear order in \(\hat{z}\). Switching directly to a continuum notation, we obtain

\[ S_z[T] = S[T] - S[T]_{\hat{z}=0} \equiv \frac{O(\hat{z})}{\hat{z}=0} \rightarrow -i \frac{N(2d-1)^{1/2}}{4da^d} \int \text{str} \left( \hat{z}(T + T^{-1}) \right). \quad (C6) \]

### 3. Integration over massive modes and the action, \(S_m\):

In order to derive the contribution \(S_m\) we proceed in a standard manner and decompose the full action according to

\[ S[P,T] \rightarrow S[X,T] = S[T] + S[X] + S_{\text{int}}[T,X], \quad (C7) \]

where we have introduced generators of the massive modes by \(P_{ij} = \exp(X_{ij})\). In (C7), \(S[T]\) is the pure Goldstone action with which we have already dealt, \(S[X]\) is purely massive, and \(S_{\text{int}}[X,T]\) is the contribution that couples massive and massless modes. As for \(S[X]\), the fact that the mass of the \(X\)’s is of \(O(N)\) (see below) justifies an expansion to quadratic order in \(X\). With \(S_{\text{int}}[X,T]\) the situation is more complicated in that we will have to
expect contributions $O(X^n \partial T, X^n(\partial T)^2)$ where $n$ may be arbitrarily large. In order to justify a truncation of this series at quadratic order, $n = 2$, we need to employ $N \gg 1$ as an expansion parameter. At first sight it seems that for $N = O(1)$, we are facing a serious problem; Cutting the series is not a controlled approximation, whilst attempting to explicitly compute contributions with $n \gg 1$ is clearly futile. However, as discussed in the main body of the text, the situation is not that bad. The structure of the final result for $S_m$ will be fixed by the symmetries of the action. The worst that may happen for $N = O(1)$ is a change in the coupling constants (from the numerical value derived below to a renormalised one).

Up to an accuracy of $O(X^2)$, the action $S_m$ is given by

$$S_m[T] = \left\langle S_{\text{int}}^{(2)}[X, T] - \frac{1}{2}(S_{\text{int}}^{(1)}[X, T])^2 \right\rangle,$$

where $S_{\text{int}}^{(n)}[X, T]$ represents the contribution to $S_{\text{int}}[X, T]$ that is of $n$th order in $X$ and

$$\langle \ldots \rangle \equiv \int D X e^{-S[X] \langle \ldots \rangle},$$

is the functional average over massive modes. The starting point of our concrete evaluation of these expressions is the $\hat{z} = 0$ action

$$S[X, T] = -N \sum_{(i,j)} \text{str} \ln (1 + x^2 e^{2X_{ij}}) +$$

$$+ N \sum_i \text{str} \ln \left( \sum_{j \in N_i} e^{X_{ij} T_{ij}} \right) + N \sum_j \text{str} \ln \left( \sum_{i \in N_j} T_{ij}^{-1} e^{X_{ij}} \right).$$

Expanding to second order in $X_{ij}$, we obtain

$$S_m[X] = -\frac{N(d - 1)}{2d^2} \sum_{(i,j)} \text{str}(X_{ij}^2) -$$

$$- \frac{N}{8d^2} \sum_{i,j,j' \in N_i} \text{str}(X_{ij} X_{ij'}) - \frac{N}{8d^2} \sum_{j,i,i' \in N_j} \text{str}(X_{ij} X_{i'j}),$$

$$S_m^{(1)}[X, T] = \frac{N}{2d} \sum_{(i,j)} \text{str}(A_{ij} X_{ij}) + \frac{N}{2d} \sum_{(i,j)} \text{str}(X_{ij} B_{ij}),$$

$$S_m^{(2)}[X, T] = -\frac{N}{4d} \sum_i \text{str} \left( \frac{1}{d} \sum_{j,j' \in N_i} X_{ij} A_{ij} X_{ij'} + \frac{1}{2d} \sum_{j,j' \in N_i} X_{ij} A_{ij} X_{ij'} - \sum_{j \in N_i} A_{ij} X_{ij}^2 \right) -$$

$$- \frac{N}{4d} \sum_j \text{str} \left( \frac{1}{d} \sum_{i,i' \in N_j} X_{ij} B_{ij} X_{i'j} + \frac{1}{2d} \sum_{i,i' \in N_j} B_{ij} X_{ij} B_{i'j} X_{i'j} - \sum_{i \in N_j} X_{ij}^2 B_{ij} \right).$$

That there are no terms of $O(X^n, T)$ follows from the fact that the $\hat{z} = 0$ action depends only on bilinears $Z \bar{Z}$.
Here we have introduced the quantities
\[ \begin{align*}
A_{ij} & \equiv T_i^{-1}T_{ij} - 1 \equiv \frac{a}{2} T_i^{-1} \partial_{i\to j} T_i, \\
B_{ij} & \equiv T_{ij}^{-1}T_j - 1 \equiv \frac{a}{2} T_j^{-1} \partial_{i\to j} T_j.
\end{align*} \tag{C12} \]

To prepare the perturbative integration we introduce the notation
\[ S_m[X] \equiv N \sum_{(i,j)} \str \left( X_{ij} K^{-1}_{ij,i'j'} X_{i'j'} \right), \tag{C13} \]
where the kernel \( K^{-1} \) is defined through (C10). The result of the integration over \( X \) will be essentially determined by the properties of the inverse \( K \). For our purposes, rather than computing this quantity explicitly, it suffices to realize its two main features: (a) \( K \) is short-ranged (decays exponentially on a scale set by the level spacing). This follows from the fact that \( K^{-1} \) is massive and ‘tridiagonal’ on the lattice of links: More specifically, \( K^{-1} \sim \Delta + m^2 \), where \( \Delta \) is the lattice Laplace operator and \( m^2 \) a mass of \( \mathcal{O}(1) \). (b) \( K \) is symmetric.

Functional expectation values of the contributions \( S^{(n)}_{\text{int}}[X,T] \) can now be computed by means of the contraction rules
\[ \begin{align*}
\langle \str (AX_{ij}) \str (A'X_{i'j'}) \rangle_X & = \frac{1}{N} K_{ij,i'j'} \str (AA'), \\
\langle \str (AX_{ij}A'X_{i'j'}) \rangle_X & = \frac{1}{N} K_{ij,i'j'} \str (A) \str (A').
\end{align*} \tag{C14} \]

Instead of working out of all possible contractions of the terms appearing in (C11) explicitly, it is more efficient to “pre-exclude” certain contributions by virtue of their infrared-irrelevancy or other reasons. Specifically, the contraction of \( (S^{(1)}_m[X,T])^2 \) gives rise to contributions of the type \( \sim \str(T_i^{-1} \partial_{i\to j} T_i T_j^{-1} \partial_{i'\to j'} T_{i'}). \) Since the only gradient operator compatible with the fundamental symmetries of the model is given by \( \sim \str(\partial T \partial T^{-1}) \), contributions of this type can lead to no more than a renormalization of the coupling constant in \( S_n[T] \). Moreover, the structure of the kernel \( K_{ij,i'j'} \) implies that the coupling constant does not change significantly, so that the contribution above can be ignored.

As for the contraction of \( S^{(2)}_m[X,T] \) the 1st, 3rd, 4th and 6th contribution to that part of the action vanish by supersymmetry. The 2nd and 5th term give
\[ S_m[T] = \langle S^{(2)}_{\text{int}}[X,T] \rangle = -\frac{a^2}{32d^2} \left( \sum_{i,j'j'' \in N_i} \str(T_i^{-1} \partial_{i\to j'} T_i) \str(T_i^{-1} \partial_{i\to j''} T_i) K_{ij,ij'} + \right. \\
\left. + \sum_{j,ii' \in N_i} \str(T_j^{-1} \partial_{i\to j} T_j) \str(T_j^{-1} \partial_{i'\to j} T_j) K_{ij,ij'} \right) \simeq \\
\simeq -\frac{a^2}{16d^2} \sum_{i,jj' \in N_i} \str(T_i^{-1} \partial_{i\to j} T_i) \str(T_i^{-1} \partial_{i\to j'} T_i) K_{ij,ij'}. \]

It is a straightforward matter to check that, by the symmetry of the kernel \( K \), only the contributions with \( j' = j \) and \( j' = oj \) survive, where \( oj \) is the site opposite (with respect to \( i \) to \( j \). Focusing on the two non-vanishing contributions, we obtain
\[ S_m[T] \simeq -\frac{a^2 C}{16d^2} \sum_{(i,j)} \text{str} \left( T_i^{-1} \partial_{i \rightarrow j} T_i \right) \text{str} \left( T_i^{-1} \partial_{i \rightarrow j} T_i \right) \simeq -\frac{C}{16d^2 a^{d-2}} \int \text{str} \left( T^{-1} \partial T \right) \text{str} \left( T^{-1} \partial T \right), \quad (C15) \]

where the constant
\[ C = K_{ij,ij} - K_{ij,i} o_{j} \quad (C16) \]
is independent of \( i \) by translational symmetry.

4. Boundary Action, \( S_b \)

Thus far our derivation of the action has implicitly assumed that the system we are dealing with is infinitely extended. (E.g. in deriving (C5) we have repeatedly integrated by parts with no account for boundary terms.) For finite size systems, however, the action does include boundary terms. Even for large systems, these terms must not be neglected the reason being that some of them are topological and hence may affect the phenomenology of the system on arbitrary length scales.

To derive the boundary action, we assume that our (hypercubic) system has \( N_i, i = x, y, \ldots \) lattice sites in \( x, y, \ldots \) direction (c.f. fig. 3.) As before, the lattice spacing is set by \( a \). As usual with field theories of disordered systems [39] the constraint that no current flow through the interfaces translates to the condition \( \partial_{\perp T} \big|_{\text{boundary}} = 0 \), where \( \partial_{\perp} \) is the derivative normal to the boundary. Under these conditions, the only source for potential boundary contributions are the first two terms in Eq. (C3)\(^8\) As shown in Eq. (C4), these two terms do not contribute to the bulk action. In order to explore their role at the boundaries, we imagine that the sites of the lattice are paired to doublets as indicated in fig. 3.

Considering the case of a 1d-chain first we thus obtain,
\[ S_b^{(d=1)}[T] = N \sum_i \text{str} \ln \left( 2dT_i \right) + N \sum_j \text{str} \ln \left( 2dT_j^{-1} \right) = \]
\[ = N \sum_{i=1,3,\ldots,R_1} \left( \text{str} \ln \left( T_i \right) - \text{str} \ln \left( T_{i+1}^{-1} \right) \right) + N \frac{1 - \left( - \right)^{N_1}}{2} \text{str} \ln \left( T_{N_1} \right) \simeq \]
\[ \simeq -N \frac{1}{2} \int_0^L dx \partial_x \text{str} \ln \left( T(x) \right) + N \frac{1 - \left( - \right)^{N_1}}{2} \text{str} \ln \left( T((N_1 - 1)a) \right) = \]
\[ = \frac{N}{2} \left( \text{str} \ln \left( T(0) \right) - \left( - \right)^{N_1} \text{str} \ln \left( T(L) \right) \right), \quad (C17) \]

\(^8\)For reasons of operator relevancy a boundary operator must not contain more than one derivative. Due to current conservation, normal derivatives are not permitted. It is a straightforward matter to check that under these conditions the first two contributions to (C3) are the only ones that qualify as potential sources of boundary terms.
where $R_1 = N_1 - 1$ for $N_1$ even and $R_1 = N_1 - 2$ for $N_1$ odd. That scheme can straightforwardly be generalized to higher dimensions. E.g. in $d = 2$, we obtain

$$S^{(d=2)}_b[T] = \frac{N}{4} \left( (-)^{N_1 + N_2} \text{str ln} (T(L_1, L_2)) - (-)^{N_1} \text{str ln} (T(L_1, 0)) - (-)^{N_2} \text{str ln} (T(0, L_2)) + \text{str ln} (T(0, 0)) \right).$$

The obvious generalisation to arbitrary dimensions is given by

$$S_b[T] = \frac{N}{2d} \sum_{s_i=0,1} (-)^{\sum_{i=1}^d (N_i+1)s_i} \text{str ln} (T(s_1 L_1, \ldots, s_d L_d)).$$

(C18)

Summing the contributions (C5), (C6), (C15) and (C18) we obtain the full continuum action Eq. (28).

**APPENDIX D: THE ZERO MODE INTEGRAL**

The computation of the integral (32) can be most readily performed by employing the ‘polar’ parameterisation of the supermatrices,

$$T = k a k^{-1}, \quad k = \exp \left( \nu \eta \right), \quad a = \exp \left( x i y \right),$$

where $\eta$ and $\mu$ are Grassmann variables and $x, y \in \mathbb{R}$. The invariant measure associated with (D1) is given by

$$\int dT f(T) \to \int d\eta d\mu \int_0^\infty dx \int_0^{2\pi} dy \frac{1}{\sinh^2 \left( \frac{x-i y}{2} \right)} f(\eta, \mu, x, y).$$

(D2)

With this definition, the DoS takes the form

$$\rho(\epsilon) = \rho_0 \left[ 1 + \frac{1}{8} \text{Re} \int_0^\infty dx \int_0^{2\pi} dy f(x, y) \frac{\cosh x - \cos y}{\sinh^2 \left( (x - iy)/2 \right)} e^{is_+(\cosh x - \cos y)} \right]$$

where $s_+ = \pi \rho_0 \epsilon_+$, and

$$f(x, y) = \begin{cases} e^{-x + iy} & \text{N\_sites odd} \\ 1 & \text{N\_sites even} \end{cases}$$

For a lattice with an even number of sites, one can show

$$\partial_s \rho = \rho_0 \frac{1}{2} \text{Re}(i) \int_0^\infty dx \int_0^{2\pi} dy (\cosh x \cos y - 1) e^{is_+(\cosh x - \cos y)}$$

Then, making use of the identities,

$$J_n(s) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(s \sin y - ny)} = \frac{2}{\pi} \int_0^\infty dx \cosh(nx) \sin(s \cosh x - \pi n/2),$$

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we find $\partial_s \rho = \rho_0 \pi^2 (J_0^2(s) - J_1^2(s))/2$. Finally, making use of the identity

$$\partial_s \left[ s \left( J_{n+1}^2(s) + J_n^2(s) \right) \right] = (2n + 1) \left( J_n^2(s) - J_{n+1}^2(s) \right),$$

one obtains the standard result from ChRMT shown in Eq. (34).

For the lattice with an odd number of sites, we have the identity

$$\frac{1}{2} \text{Re} \left( \frac{e^{-x+iy}}{\sinh^2((x-iy)/2)} + \frac{e^{x+iy}}{\sinh^2((x+iy)/2)} \right) = 2 \left[ 1 + \frac{\cosh x \cos y - 1}{(\cosh x - \cos y)^2} \right]$$

As a result, the expression for the DoS separates into two contributions $\rho(\epsilon) = \rho_{\text{even}}(\epsilon) + \delta\rho(\epsilon)$. The first reproduces the expression for the DoS with an even number of lattice sites (34) while the second generates the correction,

$$\delta\rho(\epsilon) = \frac{\rho_0}{2} \text{Re} \int_0^\infty \int_0^{2\pi} dx \int_0^{2\pi} dy (\cosh x - \cos y)e^{is_{x+iy}(\cosh x - \cos y)}$$

Making use of the identities above we find $\delta\rho(s) = -\pi^2 J_0(s) J_1(s)$ which, taken together gives the expression for the DoS shown in Eq. (35) for the case $N = 1$. For arbitrary $N$, a calculation analogous to the one above obtains the more general formula (33).
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FIGURES

FIG. 1. The two-dimensional random flux model.

FIG. 2. The generalized random flux model.

FIG. 3. Dashed: generic path contributing to (9). Solid: Self retracing path which survives phase averaging.
FIG. 4. Visualisation of the different coupling of the $U$'s and the $Z$'s, respectively, to the field variables $\psi$ (represented by the solid dots).

FIG. 5. One- and two-dimensional finite size lattice with $N_1=$ odd sites in 1-direction and $N_2=$ even sites in 2-direction (2d case). The dashed ovals indicate the pairing of lattice sites employed in the computation of the boundary action.