On the relation between metric and spin–2 formulations of linearized Einstein theory

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Abstract

A twenty–dimensional space of charged solutions of spin–2 equations is proposed. The relation with extended (via dilatation) Poincaré group is analyzed. Locally, each solution of the theory may be described in terms of a potential, which can be interpreted as a metric tensor satisfying linearized Einstein equations. Globally, the non–singular metric tensor exists if and only if 10 among the above 20 charges do vanish. The situation is analogous to that in classical electrodynamics, where vanishing of magnetic monopole implies the global existence of the electro–magnetic potentials. The notion of asymptotic conformal Yano–Killing tensor is defined and used as a basic concept to introduce an inertial frame in General Relativity via asymptotic conditions at spatial infinity. The introduced class of asymptotically flat solutions is free of supertranslation ambiguities.

1 Introduction

The linearized Einstein equations describing weak gravitational field can be formulated in terms of the metric tensor (see Section 2) or, in terms of the Weyl tensor, as a spin–2 field (see Section 3). Both formulations are globally equivalent if the topology of the spacetime is trivial. However, the linearized Einstein theory can only be applied in the asymptotically flat region, which has a nontrivial topology (a tube containing the strong field region has to be removed from Minkowski space). In this case both formulations are no longer globally equivalent. Similarly as in classical electrodynamics, where the vanishing of magnetic monopoles in topologically non–trivial regions implies the existence of magnetic vector–potential ([1], [2]), the necessary and sufficient condition for the equivalence of the two formulations of the linearized gravity is the vanishing of certain charges which we introduce in Section 4.

The new charges result in a natural way from a geometric formulation of the “Gauss law” for the gravitational charges, defined in terms of the Riemann tensor. We present
this formulation in Section 5. It leads to the notion of the \textit{conformal Yano–Killing tensor}. A conformal Yano–Killing (CYK) equation \((30)\) possesses twenty–dimensional space of solutions for flat Minkowski metric in four–dimensional spacetime \((n = 4)\). There is no obvious correspondence between ten–dimensional asymptotic Poincaré group and the twenty–dimensional space of CYK tensors. Only half of them (the four–momentum vector \(p_\mu\) and the angular momentum tensor \(j_{\mu\nu}\)) are Poincaré generators. This situation is analogous to that of electrodynamics, where, in topologically non–trivial regions, we have two charges (electric + magnetic) despite the fact that the gauge group is one–dimensional.

Let us notice, that for \(n = 2\) (\(n\) is a dimension of spacetime) the space of solutions of the equation \((30)\) is infinite and for \(n = 3\) the corresponding space is only four–dimensional. Possible dimensions we summarize in a table:

| Dimension of Spacetime | \(n = 2\) | \(n = 3\) | \(n = 4\) |
|------------------------|------------|------------|------------|
| Dimension of \((\text{pseudo})\text{euclidean group}\) | 3          | 6          | 10         |
| Dimension of Conformal Group | \(\infty\) | 10         | 15         |
| Dimension of Space of CYK Tensors | \(\infty\) | 4          | 20         |

The above table shows that there is no obvious relation between CYK tensors and the group.

On the other hand, in the case \(n = 4\), it is possible to connect CYK tensors with eleven–dimensional group of Poincaré transformations enlarged by dilatation (pseudo–similarity transformations). Eleven–dimensional algebra (space of Killing vectors) of this group allows us to construct (via the wedge product) all the CYK tensors in Minkowski spacetime.

A natural application of the above construction to the description of asymptotically flat spacetimes is proposed in sections 6 and 7. It allows us to define an asymptotic charge at spatial infinity without supertranslation ambiguities. The existence or nonexistence of the corresponding asymptotic CYK tensors can be chosen as a criterion for classification of asymptotically flat spacetimes. For example, the Taub–NUT spacetimes \([13]\) can be excluded assuming that the corresponding conserved quantity vanishes. Similarly, the Demiański solution \([14]\) corresponds to a non–vanishing charge \(d\) and can also be excluded this way (see Section 4).

We stress that the new 10 charges introduced in the present paper are different from 10 exact gravitationally–conserved quantities at null infinity, considered by E.T. Newman and R. Penrose \([8]\). The situation at null infinity (which we do not analyze in our paper) is much more delicate than the one at spatial infinity. To extend our definition to the null infinity, we will probably have to assume additional asymptotic conditions which guarantee the existence of the asymptotic CYK tensor.
2 Linearized gravity

Linearized Einstein theory (see e.g. [5] or [4]) can be formulated as follows. Einstein equation
\[ 2G_{\mu\nu}(g) = 16\pi T_{\mu\nu} \]
give, after linearization:
\[
h_{\mu\alpha}^{;\alpha} + h_{\nu\alpha}^{;\alpha} - h_{\mu\nu}^{;\alpha} - (\eta^{\alpha\beta} h_{\alpha\beta})_{;\mu\nu} - \eta_{\mu\nu} [h_{\beta\alpha}^{;\alpha\beta} - h_{\alpha}^{;\beta} - \eta^{\alpha\beta} h_{\alpha}^{;\beta}] = 16\pi T_{\mu\nu}
\]
(1)
where pseudoriemannian metric \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), \( \eta_{\mu\nu} \) is the flat Minkowski metric, \(";"\) denotes four–dimensional covariant derivative with respect to the metric \( \eta_{\mu\nu} \).

It is useful to define the following object:
\[
H_{\mu\alpha\beta} := h_{\alpha\beta}^{;\mu} + \eta_{\alpha\beta}^{;\mu} - h_{\alpha\beta}^{;\mu} - h_{\mu\alpha}^{;\beta} - \eta^{\alpha\beta} h_{\alpha\beta}^{;\mu} - \eta^{\alpha\beta} h_{\alpha\beta}^{;\mu} + \eta^{\gamma\gamma}(\eta_{\mu\nu}^{\alpha\beta} - \eta^{\alpha\beta} h_{\alpha\beta})
\]
which fulfills the following identities:
\[
H_{\mu\alpha\beta} = \eta_{\alpha\beta}^{;\mu} = H_{[\mu\alpha][\nu\beta]} \quad H^{\mu[\alpha\beta]} = 0
\]
The equation (1) may be rewritten as:
\[
H_{\mu\alpha\beta}^{;\alpha\beta} = 16\pi T_{\mu\nu}
\]
(2)
Let \( \Sigma = \{ x^0 = \text{const.} \} \) be a spacelike hyperplane. We can define the energy–momentum vector \( p_{\mu} \):
\[
16\pi p_{\mu} := 16\pi \int_{\Sigma} T_{\mu0} = \int_{\Sigma} H^{\mu0} = \int_{\Sigma} H^{\mu0} = \int_{\partial \Sigma} H^{\mu0} k_{\alpha} d^2 S_k
\]
(3)
and the angular momentum tensor \( j_{\mu\nu} \):
\[
16\pi j_{\mu\nu} := 16\pi \int_{\Sigma} J_{\mu\nu0} = 16\pi \int_{\Sigma} (x^{\mu} T_{\nu0} - x^{\nu} T_{\mu0}) = \int_{\partial \Sigma} (x^{\mu} H^{\nu0} - x^{\nu} H^{\mu0} - H^{\nu0} - H^{\mu0}) d^2 S_k
\]
(4)
where
\[
J^{\mu\nu\kappa} := x^{\mu} T^{\nu\kappa} - x^{\nu} T^{\mu\kappa}, \quad J^{\mu\nu\kappa,\kappa} = 0
\]
(5)
Here \( (x^{\mu}) \) is a global (pseudo-)cartesian coordinate system in the Minkowski space. Without coordinates, we may relate the above quantities to the Poincaré group generators:
\[
\mathcal{T}_{\mu} = \frac{\partial}{\partial x^{\mu}}, \quad \mathcal{L}_{\mu\nu} = x_{\mu} \frac{\partial}{\partial x^{\nu}} - x_{\nu} \frac{\partial}{\partial x^{\mu}} \quad (\text{where } x_{\mu} = \eta_{\mu\nu} x^{\nu})
\]
The index “\( \mu \)” in (3) refers to a translation Killing field \( \mathcal{T} \) and “\( \mu\nu \)” in (4) refers respectively to a generator \( \mathcal{L} \) of proper Lorentz transformations.
We have also conservation laws:
\[
\frac{d}{dt} \alpha + \int_{\partial \Sigma} T^{\alpha k} d^2S_k = 0
\]
\[
\frac{d}{dt} \mu^{\alpha} + \int_{\partial \Sigma} J^{\mu \alpha k} d^2S_k = 0
\]

Equation (1) is invariant with respect to the “gauge” transformation:
\[
h_{\mu \nu} \rightarrow h_{\mu \nu} + \xi_{\mu}^{\; \nu} + \xi_{\nu}^{\; \mu}
\]
where \(\xi_{\mu}\) is a covector field.

The 3+1 decomposition of (1) gives 6 dynamical equations for the space–space components \(h_{\mu \nu}\) of the metric (latin indices run from 1 to 3) and 4 equations which do not contain time derivatives of \(h_{\mu \nu}\). It is convenient to introduce the “ADM–momentum” \(P_{\mu \nu}\), by the following formula (for full nonlinear theory see [3]):

\[
2\Lambda^{-1} P_{\mu \nu} = \dot{h}_{\mu \nu} - (h_{0 k|\nu} + h_{0 |\mu| k}) + \eta_{\mu \nu}(2h_{0|k|m} - \dot{h})
\]

where \(h := \eta_{\mu \nu} h^{\mu \nu}\), \(\Lambda := (\det \eta_{\mu \nu})^{1/2}\), the symbol “|” denotes the three–dimensional covariant derivative and dot means, as usual, the time derivative. This way the 6 dynamical, second order equations can be written as the system of 12 first order (in time) equations:

\[
\dot{P}_{\mu \nu} = \Lambda \left( h_{\mu |k| m} + h_{|k| \nu} - h^{m}_{\nu k m} - h^{m}_{|m| k} + h^{0}_{0|k| \nu} - \eta_{\mu \nu} h^{0}_{0|m} \right) + 8\pi \Lambda T_{\nu \mu}
\]
\[
\dot{h}_{\mu \nu} = 2\Lambda^{-1} (P_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} P) + h_{0 |\nu| k} + h_{0 |k|\nu}
\]

where \(P = \eta^{\mu \nu} P_{\mu \nu}\).

The constraint equations can be written as follows:

\[
P^{\mu \nu}_{|t} = -8\pi \Lambda T^{0 \mu}
\]
\[
h^{\mu}_{|k|t} - h^{0}_{|k|} = 16\pi T^{0 \mu}
\]

Equations (10) and (11) are called the vector constraint and the scalar constraint respectively. As a consequence of (3) and (7) the gauge splits also into its time–like component \(\xi_{0}\) which acts on \(P_{\mu \nu}\):

\[
\Lambda^{-1} P_{\mu \nu} \rightarrow \Lambda^{-1} P_{\mu \nu} - \xi_{0|\mu} + \eta_{\mu \nu} \xi_{0|m}
\]

and a three–dimensional gauge \(\xi_{k}\) acting on the three–metric:

\[
\dot{h}_{\mu \nu} \rightarrow h_{\mu \nu} + \xi_{\mu|k} + \xi_{k|\mu}
\]

The Cauchy data \((g_{\mu \nu}, P^{\mu \nu})\) in \(\Sigma\) are equivalent to \((\overline{g}_{\mu \nu}, \overline{P}^{\mu \nu})\) if they can be related by the gauge transformation with \(\xi_{\mu}\) vanishing on \(\partial \Sigma\). The evolution of canonical variables \(P^{\mu \nu}\)
and $h_{kl}$ given by equations (8–9) is not unique unless the lapse function ($h^0_0$) and the shift vector ($h^0_k$) are specified.

To describe the dynamics we take as an example the volume $V \subset \Sigma$ contained between the two spheres of radius $r_0$ and $r_1$ respectively:

$$V = K(0, r_0, r_1)$$

We are interested in exterior vacuum solutions so we are assuming that $T_{\mu\nu} = 0$ in $V$. Limiting cases $r_0 \to 0$ and/or $r_1 \to \infty$ will also be considered. We use radial coordinates in $V$: $y^3 = r$, $y^1 = \theta$, $y^2 = \phi$ (spherical angles $(y^A) A = 1, 2$). Moreover, $y^0 = x^0$ denotes the time coordinate.

The entire gauge-independent information about the state of the gravitational field ($h_{kl}$, $P^{kl}$) is contained in the Riemann tensor $R_{\mu\nu\lambda\kappa}$ which, in linear approximation, can be expressed in the following way by $h_{\mu\nu}$

$$2R_{\mu\nu\lambda\kappa} = h_{\mu\kappa;\lambda\nu} + h_{\nu;\lambda\mu\kappa} - h_{\nu;\kappa\lambda\nu} - h_{\mu;\lambda\nu}$$

(12)

Let $a$ denotes the two-dimensional Laplace–Beltrami operator on a unit sphere $S(1)$. Moreover, $H := \eta^{AB}h_{AB}$, $\chi_{AB} := h_{AB} - \frac{1}{2}\eta_{AB}H$, $S := \eta^{AB}P_{AB}$, $S_{AB} := P_{AB} - \frac{1}{2}\eta_{AB}S$ according to the notation used in [6].

The following four objects built from the Riemann tensor can be expressed in terms of the Cauchy data:

$$\mathbf{x} = r^2\eta^{AC}\eta^{BD}R_{ABCD} = 2h^{33} + 2r\eta^{AB}\chi_{AB} + (rH)_3 - \frac{1}{2}aH$$

(13)

$$\mathbf{X} = 2\Lambda r^2\eta^{AC}\eta^{BD}R_{0DAB;C} = 2r^2S^{AB} + 2rP^{3A}_{|A} + aP^{33}$$

(14)

$$\mathbf{y} = r^2\varepsilon^{AC}R_{03AC} = 2\Lambda^{-1}r^2P^{3A}B\varepsilon_{AB}$$

(15)

$$\mathbf{Y} = 2\Lambda r^2\varepsilon^{AC}\eta^{BD}R_{3BCD;A} = \Lambda(a + 2)h^{3A}B\varepsilon_{AB} - r^2(\Lambda\chi^{CA}C_{AB}\varepsilon^{AB}),_3$$

(16)

where on each sphere $S(r)$ we denote by $\varepsilon^{AB}$ the Levi–Civita antisymmetric tensor such that $\Lambda^{12} = 1$ and by $"||"$ the two-dimensional covariant derivative related to the two-metric $\eta_{AB}$.

We are ready now to rewrite equations (3) and (4):

$$16\pi p^0 = \int_{\partial V} \Lambda(h^{3k}|_k - h^{3}) = \int_{\partial V} \frac{\Lambda}{r} \left[2h^{33} - (rH)_3\right] = \int_{\partial V} \frac{\Lambda}{r} \mathbf{x}$$

(17)

$$16\pi p^z = -2 \int_{\partial V} P^{3z} = -2 \int_{\partial V} \left[P^{33}\cos \theta + P^{3A}(r\cos \theta)_A\right] =$$

$$= 2 \int_{\partial V} \left(rP^{3A}_A - P^{33}\right) \cos \theta = \int_{\partial V} \mathbf{x} \cos \theta$$

(18)

$$16\pi s^z := 16\pi j^{xy} = -2 \int_{\partial V} P^3_\phi = -2 \int_{\partial V} P^3_A(r^2\varepsilon^{AB}\cos \theta)_B =$$
where \((x, y, z)\) are cartesian coordinates on \(\Sigma\) \((x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, z = r \cos \theta)\) and we show only typical components of four–vector \(p^\mu\) and tensor \(j^{\mu\nu}\).

We decompose the four objects \(x, y, X, Y\) into the monopole part, the dipole part and the radiation (wave) part. It can be easily checked that constraint equations (10) and (11) imply a specific radial dependence of the monopole and the dipole part:

\[
2\Lambda r^2 \eta^{AC} \eta^{BD} R_{ABCD} = \frac{4m}{r} + \frac{12k}{r^2} + x
\]

\[
2\Lambda r^2 \eta^{AC} \eta^{BD} R_{0DAB;C} = \Lambda \frac{12p}{r^2} + X
\]

\[
r^2 \varepsilon^{AC} R_{03AC} = \frac{12s}{r^2} + y
\]

\[
2\Lambda r^2 \varepsilon^{AC} \eta^{BD} R_{3BCD;A} = Y
\]

where \(x, X, y, Y\) are monopole– and dipole–free, \(m\) is a number (which we identify with the monopole function on \(S(1)\)) and \(k, p, s\) are three–dimensional vectors which we identify with the dipole functions on \(S(1)\). Let us notice that the monopole part of \(y\) and dipole part of \(Y\) vanish because of the last equalities in (13) and (16) but not from the definition in terms of the Riemann tensor. This observation will be analyzed in section 3.

Field equations given in Appendix imply the following time dependence of the charges:

\[
\dot{m} = \dot{p} = \dot{s} = 0
\]

\[
\dot{k} = p
\]

The dynamical part of the field is described by the four radiative degrees of freedom \(x, X, y, Y\). We proved in \([6]\) that Einstein equations are equivalent to the following dynamical equations:

\[
\dot{X} = \Lambda \Delta x
\]

\[
\dot{Y} = \Lambda \Delta y
\]

\[
\Lambda \dot{x} = X
\]

\[
\Lambda \dot{y} = Y
\]

(where by \(\Delta\) we denote the three–dimensional Laplacian) or – equivalently – to the pair of wave equations for the “true degrees of freedom” \(x\) and \(y\). We have shown in \([3]\) that the variables \(x, X, y, Y\) contain the entire gauge–free information about the unconstrained degrees of freedom.
3 Spin–2 field

We summarize the standard formulation of a spin–2 field $W_{\mu\nu\alpha\beta}$. This field can be interpreted as a Weyl tensor of linearized gravity (see [7] or [19]).

$$W_{\mu\nu\alpha\beta} = W_{\nu\beta\mu\alpha} = W_{[\mu\alpha][\nu\beta]}$$
$$W_{\mu[\alpha\nu\beta]} = 0$$
$$W_{\alpha\beta} = W^{\mu}_{\alpha\mu\beta} = 0$$

The above properties define spin–2 field.

The $\ast$–operation has the following properties:

$$(\ast W)_{\alpha\beta\gamma\delta} = \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} W^{\mu\nu}_{\gamma\delta}$$

$$(W^\ast)_{\alpha\beta\gamma\delta} = \frac{1}{2} W_{\alpha\beta}^{\mu\nu} \varepsilon_{\mu\nu\gamma\delta}$$

$$(\ast W^\ast)_{\alpha\beta\gamma\delta} = \frac{1}{4} \varepsilon_{\alpha\beta\mu\nu} W^{\mu\nu\rho\sigma} \varepsilon_{\rho\sigma\gamma\delta}$$

$$\ast W = W^\ast$$
$$\ast (\ast W) = \ast W^\ast = -W$$

and $\ast W$ is called dual spin–2 field.

Field equations:

$$\nabla_{\lambda} W_{\mu\nu\alpha\beta} = 0$$

are equivalent to

$$\nabla^\mu W_{\mu\nu\alpha\beta} = 0$$

or

$$\nabla_{\lambda} (\ast W_{\mu\nu\alpha\beta}) = 0$$

or

$$\nabla^\mu (\ast W_{\mu\nu\alpha\beta}) = 0$$

We have already assumed that $T^{\mu\nu} = 0$, so the Riemann tensor $R_{\mu\nu\alpha\beta}$ is equal to the Weyl tensor $W_{\mu\nu\alpha\beta}$. Let us introduce the following special solution of the above equations for which the radiative degrees of freedom $x, X, y, Y$ vanish:

$$\frac{4m}{r} + \frac{12k}{r^2} = x$$
$$\Lambda \frac{12p}{r^2} = \Xi$$
$$\frac{12s}{r^2} = y$$
$$0 = \Upsilon$$

$$W_{BC0A} = -\frac{3}{r^2} \varepsilon_{BC} \left( \frac{s_A}{r} - \varepsilon_A^{D} p_{D} \right)$$

$$W_{AB03} = \frac{6}{r^4} \varepsilon_{ABS} s$$
\[ W_{3A30} = \frac{3}{r^2} \left( \varepsilon_A^{D} s_{,D} + p_{,A} \right) \]

\[ W_{3AB0} = \frac{3}{r^4} \varepsilon_{AB} s \]

\[ W_{3030} = -\frac{2}{r^3} \left( m + \frac{3k}{r} \right) \]

\[ W_{0A03} = \frac{3}{r^3} k_{,A} \]

\[ W_{ABCD} = \frac{2}{r^3} \left( m + \frac{3k}{r} \right) (\eta_{AC} \eta_{BD} - \eta_{AD} \eta_{BC}) \]

\[ W_{3AB3} = -W_{0AB0} = \frac{\eta_{AB}}{r^3} \left( m + \frac{3k}{r} \right) \]

\[ W_{BC3A} = -\frac{3}{r^3} \varepsilon_{BC} \varepsilon_A^{D} k_{,D} \]

where \( W_{\mu\nu\lambda\kappa} \) is a spin–2 tensor in the flat Minkowski space.

It can be easily verified that the following metric tensor \( h_{\mu\nu} \) ("potential") gives the above spin–2 field (as a solution of equation (12))

\[ h_{00} = -\frac{2m}{r} + \frac{2k}{r^2} \]

\[ h_{0A} = -6p_{,A} - \frac{2}{r} \varepsilon_{A}^{B} s_{,B} \]

\[ h_{03} = \frac{6p}{r} \]

\[ h_{33} = \frac{2m}{r} + \frac{6k}{r^2} \] (25)

or in cartesian coordinates \((x^k)\):

\[ h_{00} = \frac{2m}{r} + \frac{2k_{,m} x^m}{r^2} \]

\[ h_{0k} = \frac{-6p_k}{r} - \frac{2}{r^3} \varepsilon_{klm} s^l x^m \]

\[ h^{kl} = \frac{x^k x^l}{r^2} \left( \frac{2m}{r} + \frac{6k_{,m} x^m}{r^3} \right) \] (26)
Applying linearized Einstein equation (1) to the metric tensor (26) we obtain the corresponding energy–momentum tensor as a distribution located in origin:

\[ T^{00} = m \delta - k^m \delta_m \]
\[ T^{0k} = p^k \delta + \frac{1}{2} \epsilon^{kml} s_l \delta_m \]
\[ T^{kl} = 0 \] (27)

where by \( \delta \) we have denoted the three–dimensional Dirac’s delta and \( \epsilon^{kml} \) is a three–dimensional antisymmetric tensor (\( \epsilon^{xyz} = 1 \)).

4 New charged solution

The theory presented in the previous Section has interesting charged solutions which do not admit any global metric tensor as a “potential”. This situation is very similar to the one in electrodynamics where the magnetic monopole solution does not admit any nonsingular, global vector potential.

Without assuming the existence of the “global potential” \( h_{\mu \nu} \) for the field \( W_{\mu \nu \alpha \beta} \), we can have a non–vanishing monopole charge \( b \) in \( \mathbf{Y} \) and a non–vanishing dipole charge \( d \) in \( \mathbf{Y} \). We have the following nonvanishing components of the Weyl tensor:

\[ \frac{4b}{r} = \mathbf{y} \quad \frac{12d}{r^2} = \mathbf{Y} \]
\[ W_{CD3A} = \frac{3}{r^2} \epsilon_{CD} d_A \]
\[ W_{0A30} = -\frac{3}{r^2} \epsilon_A d_C \]
\[ \dot{s} = d \]
\[ W_{B03A} = \frac{b}{r^3} \epsilon_{AB} \]
\[ W_{AB03} = \frac{2b}{r^3} \epsilon_{AB} \]
\[ \dot{b} = 0 \]

The above monopole charge \( b \) corresponds to the so called “NUT charge” (see [11]). It is also called the dual mass (see [12]). Both charges satisfy the Gauss law:

\[ \int_{S(r)} (r \mathbf{y})_{\beta} = 0 \]
as a consequence of
\[(r\vec{y}),_3 + r^3 \varepsilon^{CD} W_{CD0\|B} \eta^{AB} = 0\]
and
\[\int_{S(r)} (\vec{Y} \cos \theta),_3 = 0\]
from
\[(\vec{Y}),_3 + 2r^2 \Lambda \varepsilon^{BC} W^{3A}_{B3\|CA} = 0\]
The potentials can, however, be introduced locally. If we want to extend their definition for the entire spacetime, we end up necessarily with a “wire singularity”. The nonvanishing component of such a singular metric for the above monopole solution equals:

\[h_{\theta\phi} = 4b \cos \theta\]

If we choose the dipole \(d\) parallel to the \(z\)-axis i.e. \(d = d \cos \theta\) then the nonvanishing component of the corresponding singular metric for the dipole solution is the following:

\[h_{\theta\phi} = 2rd \sin \theta \cos \theta \quad \text{or} \quad h_{r\theta} = 2d(\sin^2 \theta \ln \tan \frac{\theta}{2} - \cos \theta)\]
The above form of the potential \(h_{\mu\nu}\) corresponds to the linearized part of the Demiański solution \([14]\) or rather to the “special Demiański solution” and \(d\) is the so called Demiański parameter \(c\) (see \([15]\) on pp. 172–173). We propose to call the dipole \(d\) the Demiański charge.

We can also introduce two more dipole charges \(q\) and \(w\) in \(\vec{y}\) and \(\vec{x}\) respectively:

\[6q = \vec{y} \quad 6w = \vec{x}\]

\[W_{BC0A} = \frac{3}{2r} \varepsilon_{BC} q_A\]

\[W_{AB03} = \frac{3}{r^2} \varepsilon_{AB} q\]

\[W_{3A03} = \frac{3}{2r} \varepsilon^D_A q_D\]

\[W_{3AB0} = \frac{3}{2r^2} \varepsilon_{AB} q\]

\[\dot{\vec{d}} = -\vec{q} = \dot{\vec{s}}\]

\[\dot{\vec{q}} = 0\]

\[W_{3030} = -\frac{3w}{r^2}\]
\[ W_{0A03} = -\frac{3}{2r} w, A \]

\[ W_{ABCD} = \frac{3w}{r^2} (\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}) \]

\[ W_{3AB3} = -W_{0AB0} = \frac{3w}{2r^2} \eta_{AB} \]

\[ W_{BC3A} = \frac{3}{2r} \varepsilon_{BC}\varepsilon_A D w, D \]

\[ \dot{p} = -w = \ddot{k} \]

\[ \ddot{w} = 0 \]

The charges \( q \) and \( w \) correspond to the metric tensors which do not vanish at spatial infinity \((h_{\mu\nu} = O(1))\).

The fully “charged” solution has the following form:

\[ 6w + \frac{4m}{r} + \frac{12k}{r^2} = \mathbf{x} \quad \Lambda \frac{12p}{r^2} = \mathbf{X} \]

\[ 6q + \frac{4b}{r} + \frac{12s}{r^2} = \mathbf{y} \quad \Lambda \frac{12d}{r^2} = \mathbf{Y} \]

\[ W_{BC0A} = \varepsilon_{BC} \left( \frac{3}{2r} q, A + \frac{3}{r^2} \varepsilon_A D p, D - \frac{3}{r^3} s, A \right) \]

\[ W_{AB03} = \varepsilon_{AB} \left( \frac{3q}{r^2} + \frac{2b}{r^3} + \frac{6s}{r^4} \right) \]

\[ W_{3A0} = -\frac{3}{2r} \varepsilon_A D q, D + \frac{3}{r^2} p, A + \frac{3}{r^3} \varepsilon_A D s, D \]

\[ W_{3AB0} = \varepsilon_{AB} \left( \frac{3q}{2r^2} + \frac{b}{r^3} + \frac{3s}{r^4} \right) \]

\[ W_{3003} = \frac{3w}{r^2} + \frac{2m}{r^3} + \frac{6k}{r^4} \]

\[ W_{A003} = \frac{3}{2r} w, A - \frac{3}{r^3} k, A - \frac{3}{r^2} \varepsilon_A D d, C \]

\[ W_{ABCD} = \left( \frac{3w}{r^2} + \frac{2m}{r^3} + \frac{6k}{r^4} \right) (\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}) = \left( \frac{3w}{r^2} + \frac{2m}{r^3} + \frac{6k}{r^4} \right) \varepsilon_{AB}\varepsilon_{CD} \]

\[ W_{3AB3} = -W_{0AB0} = \eta_{AB} \left( \frac{3w}{2r^2} + \frac{m}{r^3} + \frac{3k}{r^4} \right) \]
\[ W_{3ABC} = \varepsilon_{BC} \left( \frac{3}{2r} \varepsilon_A^D w_{iD} + \frac{3}{r^2} d_{iA} - \frac{3}{r^3} \varepsilon_A^D k_{iD} \right) \]

In terms of the so-called “electromagnetic” tensors (\(\mathcal{W}\)):

\[ E_{kl} = W_{0kl0} \quad H_{kl} = \frac{1}{2} W_{0kij} \varepsilon_{ij} \quad W_{klmn} = -\varepsilon_{kl} \varepsilon_{mn} E_{ij} \]

our solutions take the following form:

\[ E_{33} = W_{0303} = -\left( \frac{3w}{r^2} + \frac{2m}{r^3} + \frac{6k}{r^4} \right) \]

\[ E_{3A} = W_{0A03} = -\frac{3}{2r} w_{,A} + \frac{3}{r^2} \varepsilon_A^C d_{,C} + \frac{3}{r^3} k_{,A} \]

\[ E_{AB} = W_{0A0B} = \frac{1}{2} \left( \frac{3w}{r^2} + \frac{2m}{r^3} + \frac{6k}{r^4} \right) \eta_{AB} \]

\[ H_{33} = \frac{1}{2} W_{03AB} \varepsilon^{AB} = \frac{3q}{r^2} + \frac{2b}{r^3} + \frac{6s}{r^4} \]

\[ H_{3A} = W_{3B03} \varepsilon^B_A = \frac{3}{2r} q_{,A} + \frac{3}{r^2} \varepsilon_A^C p_{,C} - \frac{3}{r^3} s_{,A} \]

\[ H_{AB} = W_{0A3C} \varepsilon^C_B = -\frac{1}{2} \left( \frac{3q}{r^2} + \frac{2b}{r^3} + \frac{6s}{r^4} \right) \eta_{AB} \]

5 Conformal Yano–Killing tensors

Let \(Q_{\mu\nu}\) be an antisymmetric tensor field. Contracting the Weyl tensor \(W_{\mu\nu\kappa\lambda}\) with \(Q_{\mu\nu}\) we obtain a natural object which can be integrated over two–surfaces. The result does not depend on the choice of the surface if \(Q_{\mu\nu}\) fulfills the following condition introduced by Penrose (see \(\mathcal{W}\) and \(\mathcal{H}\)):

\[ Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \eta_{\sigma[\lambda Q_{\kappa]}^{\delta}\delta} = 0 \]  

(28)

We can rewrite equation (28) in a generalized form for \(n\)–dimensional spacetime with metric \(g_{\mu\nu}\):

\[ Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \frac{3}{n-1} g_{\sigma[\lambda Q_{\kappa]}^{\delta}\delta} = 0 \]  

(29)

It is easy to check that equation (29) is equivalent to the following one:

\[ Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1} \left( g_{\sigma\lambda} Q_{\nu\kappa;\nu} + g_{\kappa}(\lambda Q_{\sigma})^{\mu}_{\mu} \right) \]  

(30)
The tensor which fulfills the last equation will be called a \textit{conformal Yano Killing tensor} (or simply CYK). The CYK tensor is a natural “conformal” generalization of the Yano tensor (see \cite{16} and \cite{17}). More precisely, for any scalar function $f > 0$ and for a given metric $g_{\mu\nu}$ equation \eqref{30} for $Q_{\mu\nu}$ and $g_{\mu\nu}$ is equivalent to the same equation for $f^{3}Q_{\mu\nu}$ and the conformally related metric $f^{2}g_{\mu\nu}$. It is interesting to notice, that the “square” $A_{\mu\nu}$ of our tensor $Q_{\mu\nu}$:

$$A_{\mu\nu} := Q_{\mu}^{\lambda}Q_{\lambda\nu}$$

fulfills the following equation:

$$A_{(\mu\nu;\kappa)} = \frac{4}{n-1}g_{(\mu\nu\kappa)}Q_{\lambda\kappa}^{\delta\delta}$$  \hspace{1cm} (31)

which simply means that the symmetric tensor $A_{\mu\nu}$ is a conformal Killing tensor.

For our purposes we need to specify the formulae \eqref{29} and \eqref{30} to the special case of the flat four–dimensional Minkowski space ($g_{\mu\nu} = \eta_{\mu\nu}$, $n = 4$). In this simple situation the general solution of \eqref{30} or \eqref{28} assumes the following form in cartesian coordinates ($x^{\mu}$):

$$Q_{\mu\nu} = q_{\mu\nu} + 2u^{[\mu}x^{\nu]} - \varepsilon^{\mu\nu\kappa\lambda}v_{\kappa}x^{\lambda} - \frac{1}{2}k^{\nu\mu}x_{\lambda}x^{\lambda} + 2k^{[\lambda}x^{\nu]_{\mu}}x_{\lambda}$$  \hspace{1cm} (32)

where $q^{\mu\nu}$, $k^{\mu\nu}$ are constant antisymmetric tensors and $u^{\mu}$, $v^{\mu}$ are constant vectors.

It is easy to verify that the charge given by $Q_{\mu\nu}$ is well defined. Indeed, we have:

$$\int_{\partial V}W^{\mu\nu\lambda\kappa}Q_{\lambda\kappa}d\sigma_{\mu\nu} = \int_{V}(W^{\mu\nu\lambda\kappa}Q_{\lambda\kappa})_{,\nu}d\Sigma_{\mu} =$$

$$= \int_{V}(W^{\mu\nu\lambda\kappa}Q_{\lambda\kappa}) + W^{\mu\nu\lambda\kappa}Q_{\lambda\kappa,\nu})d\Sigma_{\mu} = 0$$

where the first term vanishes because of the field equations and the second term vanishes because of the symmetries of the Weyl tensor and because of equation \eqref{28}. The above equality implies that the flux of the quantity $(W^{\mu\nu\lambda\kappa}Q_{\lambda\kappa})$ through any two closed two–surfaces $S_{1}$ and $S_{2}$ is the same if there is a three–volume $V$ between them (i.e. if $\partial V = S_{1} - S_{2}$). We define the charge corresponding to the specific CYK tensor $Q$ as the value of this flux.

The above construction can be applied also to the dual $^{*}W$. It turns out that we do not get more charges from $^{*}W$ because the dual $Q^{*}$ has the same form \eqref{32} with the following interchange:

$$q \longleftrightarrow q^{*} \text{ } k \longleftrightarrow k^{*} \text{ } u \longleftrightarrow v$$

where $(Q^{*})_{\mu\nu} := \frac{1}{2}\varepsilon_{\mu\nu}^{\lambda\kappa}Q_{\lambda\kappa}$.

Let us observe that the solutions \eqref{32} form a twenty–dimensional vector space. This means that we have obtained 20 charges but only 14 remain when we pass to the limit at spatial infinity assuming that $W \sim \frac{1}{r}$ (the charge related to $q$ is identically zero). \textit{I don’t know any local argument (i.e. using only field equations) for the vanishing of this charge.}
Let $D$ be a generator of dilatations in Minkowski space. We have the following commutation relations between generators of pseudo–similarity group (Poincaré group extended by scaling transformation):

\[
\begin{align*}
T_\mu &= \frac{\partial}{\partial x^\mu}, \quad \mathcal{L}_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}, \quad D = x^\nu \frac{\partial}{\partial x^\nu}, \\
[T_\mu, T_\nu] &= 0, \\
[T_\mu, \mathcal{L}_{\alpha\beta}] &= \eta_{\mu\alpha} T_\beta - \eta_{\mu\beta} T_\alpha, \\
[T_\mu, D] &= T_\mu, \\
[D, \mathcal{L}_{\alpha\beta}] &= 0, \\
[\mathcal{L}_{\mu\nu}, \mathcal{L}_{\alpha\beta}] &= \eta_{\mu\alpha} \mathcal{L}_{\beta\nu} - \eta_{\mu\beta} \mathcal{L}_{\alpha\nu} + \eta_{\nu\alpha} \mathcal{L}_{\mu\beta} - \eta_{\nu\beta} \mathcal{L}_{\mu\alpha}.
\end{align*}
\]

As we already know the charges $k, s$ contain the information about $j^{\mu\nu}, m, p$ form a four–vector $p^\mu$ and similarly $b, d$ form a dual four–vector $b^\mu$. More precisely the following relations hold:

\[
16\pi w_{\mu\nu} := \int_{\partial\Sigma} W (T_\mu \wedge T_\nu)
\]

\[
16\pi w_{z0} = 2 \int_{S(r)} \Lambda W^{03}_{z0} = - \int_{S(r)} \frac{\cos \theta}{r} (\Lambda x),
\]

\[
16\pi w_{xy} = 2 \int_{S(r)} \Lambda W^{03}_{xy} = - \int_{S(r)} \frac{\cos \theta}{r} (\Lambda y),
\]

\[
16\pi^* w_{\mu\nu} := \int_{\partial\Sigma}^* W (T_\mu \wedge T_\nu) = \int_{\partial\Sigma} W^* (T_\mu \wedge T_\nu)
\]

\[
16\pi^* w_{z0} = 2 \int_{S(r)} \Lambda^* W^{03}_{z0} = \int_{S(r)} \Lambda \cos \theta \epsilon^{AB} W_{AB30} + \]

\[
+ \int_{S(r)} \Lambda r \cos \theta \epsilon^{AB} W_{AB0C} ||D \eta^{CD} = - \int_{S(r)} \frac{\cos \theta}{r} (\Lambda y),
\]

\[
16\pi p_\mu := \int_{\partial\Sigma} W (D \wedge T_\mu)
\]

\[
16\pi p_0 = 2 \int_{S(r)} \Lambda x^\mu W^{03}_{\mu0} = - \int_{S(r)} \frac{\Lambda x}{r}
\]

\[
16\pi p_z = 2 \int_{S(r)} \Lambda x^\mu W^{03}_{\mu z} = 2 \int_{S(r)} \Lambda t W^{03}_{0z} + \Lambda r W^{03}_{3z} = 16\pi t w_{0z} + \int_{S(r)} x \cos \theta
\]
\[ 16\pi b_\mu := \int_{\partial \Sigma} W (\mathcal{D} \wedge \mathcal{T}_\mu) \]

\[ 16\pi b_0 = 2 \int_{S(r)} \Lambda x^\mu W^0_{\mu 0} = - \int_{S(r)} \Lambda r \varepsilon^{AB} W_{AB 0} = \int_{S(r)} \frac{\Lambda}{r} \nabla \]

\[ 16\pi b_z = 2 \int_{S(r)} \Lambda x^\mu W^0_{\mu z} = 2 \int_{S(r)} \Lambda t^* W^0_{0z} + \Lambda r^* W^0_{0z} = 16\pi t^* w_{0z} + \int_{S(r)} \nabla \cos \theta \]

\[ 16\pi j_{\mu \nu} := \int_{\partial \Sigma} W (\mathcal{D} \wedge \mathcal{L}_{\mu \nu} - \frac{1}{2} \eta (\mathcal{D}, \mathcal{D}) \mathcal{T}_\mu \wedge \mathcal{T}_\nu) \]

\[ 16\pi j_{0z} = 2 \int_{S(r)} \Lambda x^\mu \left( r W^0_{0\mu} \cos \theta + t W^0_{0z} \right) - \int_{S(r)} \Lambda x^\mu x_\mu W^0_{0z} = \]

\[ = -8\pi (r^2 - t^2) w_{0z} - 16\pi t p_z + \int_{S(r)} \Lambda \mathbf{x} \cos \theta \]

\[ 16\pi j_{xy} = 2 \int_{S(r)} \Lambda x^\mu \left( W^0_{xy} - W^0_{y\mu} \right) - \int_{S(r)} \Lambda x^\mu x_\mu W^0_{0z} = \]

\[ = -8\pi (r^2 - t^2) w_{xy} - 16\pi t b_z + \int_{S(r)} \Lambda \mathbf{y} \cos \theta \]

\[ 16\pi j_{\mu \nu} := \int_{\partial \Sigma} *W (\mathcal{D} \wedge \mathcal{L}_{\mu \nu} - \frac{1}{2} \eta (\mathcal{D}, \mathcal{D}) \mathcal{T}_\mu \wedge \mathcal{T}_\nu) = \int_{\partial \Sigma} W^* (\mathcal{D} \wedge \mathcal{L}_{\mu \nu} - \frac{1}{2} \eta (\mathcal{D}, \mathcal{D}) \mathcal{T}_\mu \wedge \mathcal{T}_\nu) \]

The conservation law for the charge \( w_{\mu \nu} \) is a consequence of field equations:

\[ \int_{\partial \Sigma} W^\mu_{\lambda \kappa} \lambda d\sigma_{\mu \kappa} = \int_{\Sigma} (W^\mu_{\lambda \kappa} \lambda)_{\kappa \nu} d\Sigma_{\mu} = 0 \]

For \( p^\mu \) and \( b^\mu \) we obtain the conservation laws from the following observation:

\[ \int_{\partial \Sigma} x^\lambda W^\mu_{\lambda \kappa} \lambda d\sigma_{\mu \kappa} = \int_{\Sigma} (x^\lambda W^\mu_{\lambda \kappa} \lambda \nu + \delta^\lambda_\nu W^\mu_{\lambda \kappa}) d\Sigma_{\mu} = 0 \]

(the same holds for \( *W \)).

For \( j^{\mu \nu} \) the corresponding identities are as follows:

\[ \int_{\partial \Sigma} (x_\lambda W^{\mu \nu \lambda \kappa} \lambda \delta - x_\lambda W^{\mu \nu \lambda \delta} \lambda) \lambda d\sigma_{\mu \nu} = \]

\[ = \int_{\Sigma} (x_\lambda W^{\mu \nu \lambda \kappa} \lambda \delta - x_\lambda W^{\mu \nu \lambda \delta} \lambda) \lambda d\sigma_{\mu \nu} = \]

\[ = \int_{\Sigma} (x_\lambda W^{\mu \delta \lambda \kappa} \lambda \delta - x_\lambda W^{\mu \delta \lambda} \lambda \kappa) d\Sigma_{\mu} = \]

\[ = \int_{\Sigma} x_\lambda (W^{\mu \delta \lambda \kappa} \lambda + W^{\mu \delta \lambda} + W^{\mu \lambda \delta}) d\Sigma_{\mu} = 0 \]
6 Asymptotically flat spacetimes

Consider an asymptotically flat spacetime (at spatial infinity), fulfilling the (complete non-linear) Einstein equations. Suppose, moreover, that the energy–momentum tensor of the matter vanishes around spatial infinity ("sources of compact support"). This means that the Riemann tensor and the Weyl tensor do coincide outside of the world tube containing the matter. Let us analyze, for simplicity, this situation in terms of an asymptotically flat coordinate system (for nice geometric formulations of asymptotic flatness see e.g. [18] or [11]). We suppose that there exists an (asymptotically Minkowskian) coordinate system \((x^\mu)\):

\[
g_{\mu\nu} - \eta_{\mu\nu} \sim r^{-b} \quad g_{\mu\nu,\lambda} \sim r^{-b-1}
\]

where \(r := \sum_{k=1}^{3} (x^k)^2\) and typically \(b = 1\) (but \(1 \geq b > \frac{1}{2}\) is also possible – see [21]).

For a general asymptotically flat (AF) metric we cannot expect that the equations (28) and (30) admit any solution. Instead, we assume that the left–hand side of (28):

\[
Q_{\lambda\kappa\sigma} := \bar{Q}_{\lambda(\kappa;\sigma)} - \bar{Q}_{\kappa(\lambda;\sigma)} + g_{\sigma[\lambda} Q_{\kappa]}^{\delta} \delta
\]

has a certain asymptotic behaviour at spatial infinity

\[
Q_{\mu\nu\lambda} \sim r^{-c}
\]

On the other hand, suppose that \(Q_{\mu\nu}\) behaves asymptotically as follows:

\[
Q_{\mu\nu} \sim r^a \quad Q_{\lambda\kappa,\sigma} \sim r^{a-1}
\]

Moreover, suppose that the Riemann tensor \(R_{\mu\nu\kappa\lambda}\) behaves asymptotically as follows:

\[
R_{\mu\nu\kappa\lambda} \sim r^{-b-1-d}
\]

It can be easily checked (see e.g. [10]) that the vacuum Einstein equations imply the following equality:

\[
\nabla_\lambda \left( * R^{\alpha\beta\mu} Q_{\alpha\beta} \right) = -\frac{2}{3} R^{\mu\lambda\alpha\beta} Q_{\lambda\alpha\beta}
\]

(34)

The left–hand side of (34) defines an asymptotic charge provided that the right–hand side vanishes sufficiently fast at infinity. It is easy to check that, for this purpose, the exponents \(b, c, d\) have to fulfill the inequality

\[
b + c + d > 2
\]

(35)

In typical situation when \(b = d = 1\), the inequality (35) simply means that \(c > 0\). In this case a weaker condition is also possible (for example \(Q_{\mu\nu\lambda} \sim (\ln r)^{-1-\epsilon}\) with \(\epsilon > 0\)).

Let us define an asymptotic conformal Yano–Killing tensor (ACYK) as an antisymmetric tensor \(Q_{\mu\nu}\) such that \(Q_{\mu\nu,\lambda} \to 0\) at spatial infinity. For constructing the ACYK
tensor we can begin with the solutions (32) in flat Minkowski space. Asymptotic behaviour at infinity of these flat solutions explain why we expect for any ACYK tensor the following behaviour:

\[ Q_{\mu\nu} = (2)Q_{\mu\nu} + (1)Q_{\mu\nu} + (0)Q_{\mu\nu} \]

where \((2)Q_{\mu\nu} \sim r^2\), \((1)Q_{\mu\nu} \sim r\) and \((0)Q_{\mu\nu} \sim r^{1-c}\).

It is easy to verify that \(c \geq b + 1 - a\) and if \(b = 1\) than for \(a = 2\) we have \(c \geq 0\). This means that in a general situation there are no solutions of (33) with nontrivial \((2)Q_{\mu\nu}\) and \(c > 0\). This is the origin of the difficulties with the definition of the angular momentum. On the other hand it is easy to check that the energy–momentum four–vector and the dual one are well defined \((a = c = 1)\) and the condition \(b + d > 1\) can be easily fulfilled (typically \(b = d = 1\)).

7 Strong asymptotic flatness

Here, we propose a new, stronger definition of the asymptotic flatness. The definition is motivated by the above discussion.

Suppose that there exists a coordinate system \((x^\mu)\) such that:

\[
\begin{align*}
g_{\mu\nu} - \eta_{\mu\nu} &\sim r^{-1} \\
\Gamma^\kappa_{\mu\nu} &\sim r^{-2} \\
R_{\mu\nu\kappa\lambda} &\sim r^{-3}
\end{align*}
\]

In the space of ACYK tensors fulfilling the asymptotic condition

\[
Q_{\lambda(\kappa:\sigma)} - Q_{\kappa(\lambda:\sigma)} + g_{\sigma[\lambda} Q_{\kappa]}^\delta,\delta = Q_{\lambda\kappa\sigma} \sim r^{-1}
\]

we define the following equivalence relation:

\[
Q_{\mu\nu} \equiv Q'_{\mu\nu} \iff Q_{\mu\nu} - Q'_{\mu\nu} = O(1)
\]

for \(r \to \infty\). We assume that the space of equivalence classes defined by (36) and (37) has a finite dimension \(D\) as a vector space. The maximal dimension \(D = 14\) correspond to the situation where there are no supertranslation problems in the definition of an angular momentum. In the case of spacetimes for which \(D < 14\) the lack of certain ACYK tensor means that the corresponding charge is not well defined.
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Appendix — linear ADM equations

Linear Einstein equations are as follows:

\begin{align*}
2\Lambda^{-1}\dot{\rho}^{33} &= -h^0_rA - 2\rho^{-1}h^0_{03} + h^{33}r^A + 2\rho^{-2}h^{33} + 16\pi T^{33} + \\
&+ (H, 3 - 2h^{34}r|A - 2\rho^{-1}h^{33})_{33} + 2\rho^{-1}(H, 3 - 2h^{34}r|A - 2\rho^{-1}h^{33}) \\
&= 2\Lambda^{-1} \dot{\rho}^{33} = (h^0_{03} - r^{-1}h^0_0, 3 - h^0_r|m) + [\eta_{AB}(\frac{1}{2}H, 3 - h^{34}r|A - 2\rho^{-1}h^{33})]_{33} + \\
&+ 2\rho^{-1}\eta_{AB}(\frac{1}{2}H, 3 - h^{34}r|C - 2\rho^{-1}h^{33}) + (\frac{1}{2}h^C_rC + r^{-2}H)\eta_{AB} + \\
&-(h^{34}r|B + h^{34}r|A - \eta_{AB}h^{34}r|C)_{33} + h^{34}r|AB + r^{-2}\eta_{AB}h^{33} + \\
&+ (\chi^B_r, 3\eta_{CA}), 3 + \chi_{AB}r|C - \chi^C_A|BC - \chi^C_B|AC + 16\pi T_{AB} \\
&= 2\Lambda^{-1} (P^{33} - S) + 2h^{03, 3} \\
&\dot{h}_{33} = \Lambda^{-1} (P^{33} - S) + 2h^{03, 3} \\
&\dot{h}_{3A} = 2\Lambda^{-1} P^{3A} + h_{03}^{A} + h_0^{A} \\
&\dot{h}_{AB} = 2\Lambda^{-1} S_{AB} - \eta_{AB} \Lambda^{-1} P_{33} + h_{0A} |B + h_{0B} |A + 2\rho^{-1}\eta_{AB}h_{03} \\
&h^{kl} - h^{kl} |kl = (H, 3 - 2h^{34}r|A - 2\rho^{-1}h^{33})_{33} + 3\rho^{-1}(H, 3 - 2h^{34}r|A - 2\rho^{-1}h^{33}) \\
&+ h^{33}r^A + 2\rho^{-2}h^{33} + (\frac{1}{2}h^C_rC + r^{-2}H) - \chi^A_B |AB = -16\pi T^{00} \\
&P^{33} + P^{3A}r|A - r^{-1}S = 8\pi \Lambda T_{03} \\
&P_{3A}r|B = P_{3A} + S_A r|B + \frac{1}{2}S |A = 8\pi \Lambda T_{0A} \\
\end{align*}


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