Enumerative properties of restricted words and compositions

Andrew MacFie

2019.Feb.04
Abstract

Words and integer compositions are fundamental combinatorial objects. In each case, the object is a finite sequence of terms over a particular set. Relevant properties, sometimes called “parameters”, are the length of the sequence and, for integer compositions, the sum of the sequence.

There has been interest within enumerative combinatorics in counting words and compositions, especially restricted variations where the objects satisfy extra conditions. “Local” restrictions are related to contiguous subsequences, for example Smirnov words where adjacent letters must be different. For integer compositions or words over an ordered alphabet, a “subword pattern avoidance” restriction requires all contiguous subsequences of a fixed length to not satisfy a certain relative ordering. For example, we may count compositions not containing a strictly increasing contiguous subsequence of length three. “Global” restrictions, on the other hand, are related to arbitrary subsequences. A “subsequence pattern avoidance” restriction requires all subsequences of a fixed length to not satisfy a certain relative ordering.

Beyond sequences we may also consider objects with different structures, and interpret local and global restrictions appropriately. We say “cyclically restricted” finite sequences are those where the last and first terms are considered adjacent for the purposes of the restriction, i.e. the restriction wraps around from the end to the start. “Circular” objects are the orbits of finite sequences under circular shifts, so all circular shifts of a finite sequence are considered the same object.

We can generalize integer compositions by replacing the semigroup of positive integers with a different additive semigroup, giving the broader concept of a “composition over a semigroup”, i.e. a finite sequence with a certain sum over the semigroup. Beyond the positive integers, we focus on semigroups which are finite groups – where such compositions are in fact also “words” in the group theory sense. Compositions over a finite group are relatively little-studied in combinatorics but turn out to be amenable to combinatorial analysis in analogy to both words and integer compositions.

In this document we achieve exact and asymptotic enumeration of words, compositions over a finite group, and/or integer compositions characterized by local restrictions and, separately, subsequence pattern avoidance. We also count cyclically restricted and circular objects. This either fills gaps in the current literature by e.g. considering particular new patterns, or involves general progress, notably with locally restricted compositions over a finite group. We associate these compositions to walks on a covering graph whose structure is exploited to simplify asymptotic expressions. Specifically, we show that under certain conditions the number of locally restricted compositions of a group element is asymptotically independent of the group element. For some problems our results extend to the case of a positive number of subword pattern occurrences (instead of zero for pattern avoidance) or convergence in distribution of the normalized number of occurrences. We typically apply the
more general propositions to concrete examples such as the familiar Carlitz compositions or simple subword patterns.
## Contents

1 Introduction

2 Locally restricted compositions  
  2.1 Compositions over a finite group  
  2.2 Note on minimization of transfer matrices  
  2.3 Note on weighted trees

3 Locally cyclically restricted compositions  
  3.1 Compositions over a finite group  
  3.2 Note on integer compositions

4 Locally restricted compositions with symmetry  
  4.1 Circular compositions  
  4.2 Note on counting palindromic compositions

5 Subsequence pattern avoidance  
  5.1 Words and integer compositions  
    5.1.1 Pairs of generalized patterns of length 3  
    5.1.2 Some partially ordered patterns with 2 letters  
    5.1.3 Note on counting with symmetries  
  5.2 Note on compositions over $\mathbb{Z}_k$

6 Conclusion

A Notation

References
1 Introduction

If $\Xi$ is a finite set (sometimes called an alphabet), a word $w$ over $\Xi$ is a sequence $w = (w(1), \ldots, w(m))$ where $w(i) \in \Xi$ for each $i$. In particular, if $|\Xi| = k$ we call $w$ a $k$-ary $m$-word. Without loss of generality, if $|\Xi| = k$ we assume $\Xi = [k] = \{1, \ldots, k\}$, which is an alphabet with a total order. The terms that make up a word are called letters. Of course the number of all $k$-ary $m$-words is $k^m$.

If $(S, +)$ is a semigroup, an $m$-composition of $s \in S$ over $S$ is a sequence $x = (x(1), \ldots, x(m))$ where $x(i) \in S$ for each $i$, and $\Sigma x = x(1) + \cdots + x(m) = s$. If $S$ is finite, a composition over $S$ and a word over $S$ mean the same thing; the difference is that we use the word composition in the context where we pay attention to the sum of the sequence. The prototypical compositions are integer compositions, where $(S, +) = (\mathbb{Z}_{\geq 0}, +)$. The terms that make up a composition are called parts. A simple exercise gives that the number of $m$-compositions of $n$ over the positive integers is $\binom{n-1}{m-1}$.

A subword of a finite sequence is a contiguous subsequence, so $(a, a, c, b)$ is a subword of $(a, a, a, a, c, b, b)$. For any kind of finite sequences we may sometimes use the shorthand word notation $(a, a, a, a, c, b, b) = a^4 cb^2 = aaaacbb$.

We say a directed graph (digraph) is a pair $(V, E)$, where $V$ is a finite set (the “vertices”), and $E \subseteq V \times V$ is a binary relation (the “directed edges” or “arcs”). If a digraph is specified only by its arcs, the vertices are taken to be all those which appear in an arc. A weighted digraph is a digraph $(V, E)$ together with a vertex weight function $W : V \to S$, where $S$ is a fixed semigroup. Words and compositions are both finite sequences over a set. Equivalently, we may regard them as directed paths (digraphs with an ordered set of vertices and arcs from predecessors to successors) where vertices take weights from the set (which is taken without loss of generality to be a semigroup). The benefit of this view comes when generalizing beyond directed paths to different types of weighted digraphs.

Our goal, ultimately, is to count weighted digraphs. And specifically, we are interested in counting how many of these objects satisfy a certain restriction. Below, we describe a general concept of pattern occurrence and avoidance in weighted digraphs which we can use to express restricted weighted digraph families. The familiar definitions of patterns in words and compositions (e.g. [37]) are available as special cases.

If $\Gamma$ is a digraph, we write $V(\Gamma)$ and $E(\Gamma)$ for the sets of vertices and arcs of $\Gamma$. Given digraphs $\Gamma_1, \Gamma_2$, a digraph homomorphism is a function $h : V(\Gamma_1) \to V(\Gamma_2)$ such that for any two vertices $u, v \in V(\Gamma_1)$, we have

$$(u, v) \in E(\Gamma_1) \implies (h(u), h(v)) \in E(\Gamma_2).$$

If $\Gamma_1, \Gamma_2$ have weight functions $W_1, W_2$, a weighted homomorphism is a homomorphism $h$ such that $W_1(v) = W_2(h(v))$ for all $v \in V(\Gamma_1)$. A (weighted) isomorphism is a bijective (weighted) homomorphism.
A one-vertex subdivision of a digraph $(V, E)$ is a new digraph $(V', E')$, where $V' = V \cup \{v\}$ and for some $(v_1, v_2) \in E$, we have
$$E' = \{(v_1, v), (v, v_2)\} \cup E \setminus \{(v_1, v_2)\}.$$ A weighted one-vertex subdivision is one where the weight function is not modified except for the new vertex $v$, which may take any weight. In general, a subdivision of a digraph is a digraph obtained by 0 or more one-vertex subdivisions. In the context of weighted graphs, subdivisions are always weighted.

**Example 1.1.** Figure 1 shows digraphs $\Gamma, \Gamma'$. The digraph $\Gamma'$ is a subdivision of $\Gamma$ obtained by adding the vertices $u_5, u_6$ which are shown in bold.

Given weighted digraphs $\Gamma, P$, a subdivision $P'$ of $P$, and a subgraph $\Gamma_1$ of $\Gamma$, if we have a weighted isomorphism $f : V(P') \rightarrow V(\Gamma_1)$, we say that $f_{|V(P)}$ is the match of $f$ with respect to $P$. A local occurrence of $P$ in $\Gamma$ is the match with respect to $P$ of some weighted $f$ from $P$ to a subgraph of $\Gamma$. A global occurrence of $P$ is the match with respect to $P$ of some weighted $f$ from any subdivision $P'$ of $P$ to a subgraph of $\Gamma$.

That is, global occurrences may map adjacent vertices in $P$ to non-adjacent vertices in $\Gamma$ while local occurrences cannot. The semigroup $S$ of weights is always the same for $\Gamma$ and $P$.

**Example 1.2.** Figure 2 shows weighted digraphs $\Gamma$ and $P$. There exist no local occurrences of $P$ in $\Gamma$ but many global occurrences. One global occurrence is given by matching the vertices
$$u_1 \mapsto v_1, u_2 \mapsto v_2, u_3 \mapsto v_3.$$ Another is given by
$$u_1 \mapsto v_1, u_2 \mapsto v_5, u_3 \mapsto v_6.$$
The size of a digraph is the number of arcs it contains. A **digraph pattern** is a set \( \mathcal{P} \) of weighted digraphs such that the sizes of digraphs in \( \mathcal{P} \) form a bounded set. The elements \( P \in \mathcal{P} \) may be called **pattern instances** and for our purposes, the instances of \( \mathcal{P} \) are always one or more different weightings of a single digraph. A digraph \( \Gamma \) has \( r \) occurrences of \( \mathcal{P} \) if the total, over all instances \( P \in \mathcal{P} \), of the number of occurrences of \( P \) in \( \Gamma \) is \( r \). Avoiding a digraph pattern means having 0 occurrences, and avoidance of a set of patterns means avoiding each of the patterns.

Given an arbitrary finite vertex set, say \( V = [n] \), an **unlabeled** weighted digraph \( \tilde{\Gamma} \) on \( V \) is an equivalence class of weighted digraphs on \( V \) where \( \Gamma_1 \) and \( \Gamma_2 \) are equivalent iff there is a weighted isomorphism from \( \Gamma_1 \) to \( \Gamma_2 \). Relabeling vertices has no effect on digraph pattern matching because it does not affect the structure of the digraph or its weights, so we may speak of the number of occurrences of a pattern in an unlabeled weighted digraph.

The sum, a.k.a. total, of a weighted digraph \( \Gamma \) is \( \sum_{v \in V(\Gamma)} W(v) \). This expression is always well defined for abelian semigroups \( S \). For non-abelian \( S \) we must have labeled digraphs, and the vertices must have a fixed total order which determines the order of summation.

**Example 1.3.** Define the set of directed paths

\[
\left\{ \{(j, j+1) : 1 \leq j < n \} : n \geq 1 \right\},
\]

and define the set of directed cycles

\[
\left\{ \{(j, j+1) : 1 \leq j < n \} \cup \{(n, 1)\} : n \geq 1 \right\}.
\]
With the terminology of [37] we may make the following identifications. Weighted paths are words or compositions, and weighted cycles are cyclic words or compositions (where the last term is considered to precede the first for pattern occurrence purposes). Unlabeled weighted cycles over the vertex sets \([n]\) correspond to circular words or compositions. △

In the remainder, “path” means directed path and “cycle” means directed cycle.

With the above concepts laid down we are able to describe a wide taxonomy of counting problems which all ask, how many weighted digraphs are there with \(r\) occurrences of a digraph pattern \(P\)? The primary dimensions of this taxonomy follow.

- **Class of digraphs.** There are many options for the kind of digraph that we are counting. Paths and cycles are the most basic, but others could be used: regular, planar, bipartite, et cetera. The elements of \(P\) are also digraphs and can come in any form. We largely focus on digraph patterns \(P\) that consist of various weightings of a path.

- **Labeled vs. unlabeled.** The digraphs we count may be either labeled or unlabeled. Usually counting the labeled case is a prerequisite for the unlabeled case.

- **Local vs. global occurrences.** If we ask for \(r\) occurrences, we are either talking about local or global occurrences.

- **Track size, total, or both.** When counting words, any algebraic structure of the alphabet is ignored, unlike with compositions. Similarly for digraphs, we may or may not keep track of the total.

- **Choice of semigroup.** Any semigroup could be used as long as it gives finite counts, e.g. the number of 6-compositions of 17 over \(\mathbb{Z}\) is infinite.

The remainder of this document contains solutions to a selection of problems from the taxonomy just described. We largely defer discussions of the relevant prior literature to the sections that follow due to their heterogeneity. However, we mention here the 2010 book [37] by Heubach and Mansour which is a useful reference for many of the topics of this document, especially for exact (as opposed to asymptotic) counting. The remaining sections of this document are organized as follows.

We begin in §2.1 with local occurrences in weighted paths, where the semigroup \(S\) is a finite group. A weighted path with no local occurrences of some pattern is known as a locally restricted composition, assuming we track the total. (In this and subsequent sections we generally use the familiar concepts, such as “compositions”, although we refer to weighted digraphs when useful.) We find, under conditions, that the number of locally restricted compositions of a group element is asymptotically independent of the group element. We reach the same conclusion for compositions containing \(r > 0\) local occurrences of a pattern. After verifying these conditions for a variety of examples, we show that under similar conditions the number of local occurrences in a random composition is
asymptotically normal. In §2.2 we make a note on when the matrices used in the
transfer matrix method can be reduced in size for computational and practical
benefits. This section and others with heading “Note on…” are extended
remarks which briefly introduce relevant lines of research. The problem of
counting directed rooted trees is noted in §2.3, also in the context of local
pattern occurrence.

Next, §3.1 is similar to §2.1 but counts digraphs which are cycles rather
than paths, which correspond to objects known as locally cyclically restricted
compositions. Again we find that under conditions the asymptotic number
of such compositions of a finite group element does not depend on the group
element, and we show asymptotic normality of the number of local pattern
occurrences. In §3.2 we note how to count locally cyclically restricted integer
compositions, i.e. cycles weighted by \( \mathbb{Z}_{>0} \), in the framework of locally restricted
integer compositions of \( [4] \).

The results of §4.1 together with Moebius inversion allow us to count circular
locally restricted compositions over a finite group which is done in §4.1. As in
Example 1.3, circular objects correspond to unlabeled weighted cycles. And
in §4.2 we note how to count “undirected” locally restricted compositions, i.e.
unlabeled weighted undirected paths.

“Subsequence patterns” and “generalized patterns” are types of patterns that
are used in the context of ordered semigroups \( S \). In the language of this
section they are digraph patterns made up of all paths that have a certain size
and a certain relative ordering among the vertex weights. “Partially ordered
patterns” can be used to represent a set of subsequence patterns. Subsequence
and partially ordered patterns are used in the context of global occurrences,
while generalized patterns actually specify which edges may be subdivided
and which may not. In §5.1.1 we count weighted paths, specifically words or
integer compositions, that avoid different pairs of generalized patterns. The
counting results in §5.1.2 are concerned with words or integer compositions
that avoid a family of partially ordered patterns (roughly, patterns where
the maximal weights must be at the beginning or end). In §5.1.3 we note
how to adapt results for subsequence pattern avoidance in words to circular
words (unlabeled weighted cycles) or “undirected” words (unlabeled weighted
undirected paths). We make a note on subsequence pattern avoidance in objects
other than words and integer compositions in §5.2, namely in compositions
over \( \mathbb{Z}_k \). Our technique involves using the multisection formula together with
results for integer compositions, and we apply it to an example partially ordered
pattern.

Finally, we list open problems in §6. A glossary of notation is found in Appendix
A.
2 Locally restricted compositions

A locally restricted composition is one that avoids a certain set of length-$l$ sequences as subwords. Over the integers, these objects have been studied successfully in a number of papers by Bender et al. [4, 5, 7]. In fact, those works include somewhat more general restrictions, where a subword may or may not be allowed based on the residue of its position in a composition, and special rules can apply to parts near the beginning or end. Under some conditions, asymptotics for the number of locally restricted integer compositions were given in [4]. That paper also established a normal limiting distribution for the number of occurrences of a subword in a uniform random integer composition. The later papers [5, 7] focused on the probability distributions of part sizes and other parameters.

Given two sequences $x, y$ of the same length over ordered sets, we say $x$ and $y$ are order isomorphic iff $x(i) < x(j) \iff y(i) < y(j)$ for all $i, j$. A subword pattern $\tau$ is a word over $[k]$. Assume the length of $\tau$ is $l$. An occurrence of $\tau$, as a subword pattern, in $x$ is a sequence of indices $i, \ldots, i + l - 1$ such that $(x(i), \ldots, x(i + l - 1))$ is order isomorphic to $\tau$.

Mansour and others in [31, 18, 37] count integer compositions by number of occurrences of specific subword patterns such as 123 and 112. These results are less general than those obtained by Bender and collaborators but give simpler expressions. The umbral technique in [72] is also used to explicitly count locally restricted objects.

**Remark 2.1.** In the language of §1, compositions are weighted directed paths where we keep track of the total weight. Weighted undirected paths may be counted in a similar manner. △

2.1 Compositions over a finite group

Locally restricted compositions over finite fields and even finite abelian groups were counted in [28] under some conditions, and in less generality in the preceding papers mentioned therein. Over $\mathbb{Z}_k$, the method used in that paper involves obtaining the relevant generating function $F(z)$ for integer compositions over $[k]$, and working with $\sum_{j \equiv s \mod k} [z^j] F(z)$. For other finite abelian groups, the method is extended to multivariate generating functions. Below we give an alternative counting method that expands the range of applicability beyond abelian groups, addressing a problem posed in [28]. We begin this section considering compositions over a finite semigroup $(S, +)$ and eventually specialize to finite groups.

**Definition 2.1.** Let $\Xi$ be a finite set, and let $n$ be a positive integer. The $n$-dimensional de Bruijn graph (actually a digraph) on $\Xi$ has vertex set $V = Seq_n(\Xi)$ and includes the arc from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$ iff 

$$(u_2, \ldots, u_n) = (v_1, \ldots, v_{n-1}).$$
Let $\sigma$ be a positive integer which we call the span, and let $D$ be a subgraph of the $\sigma$-dimensional de Bruijn graph on $S$. Then $D$ is called a de Bruijn subgraph. This digraph $D$ is associated with a set of locally restricted compositions as follows. A walk in a digraph is a sequence of vertices, not necessarily distinct, where for any subword $(u, v)$ there is an arc from $u$ to $v$. An $m$-composition over $S$ is legal according to $D$ iff it takes the form

$$w_1 \sim (w_2(\sigma), \ldots, w_{m-\sigma+1}(\sigma)) = (w_1(1), \ldots, w_1(\sigma), w_2(\sigma), \ldots, w_{m-\sigma+1}(\sigma)),$$

where $w_1, \ldots, w_{m-\sigma+1}$ is a walk in $D$. In other words, we build compositions from $D$ by starting at any vertex, and taking a walk in which we append the last element of each vertex we visit after the first. Additionally, we may designate sets of start and end vertices which are the allowed vertices for walks to start and end at.

We write the set of all $m$-compositions of $s$ that are legal according to $D$ with start set $\Psi$ and finish set $\Phi$ as $\mathcal{P}_s(m; D, \Psi, \Phi)$. We may also write this with $s, m, \Psi$, or $\Phi$ omitted to remove those conditions, e.g. $\mathcal{P}_s(D, \Psi, \Phi) = \cup_m \mathcal{P}_s(m; D, \Psi, \Phi)$. Also, $p_s(m; D, \Psi, \Phi) = |\mathcal{P}_s(m; D, \Psi, \Phi)|$, $P_s(z; D, \Psi, \Phi) = \sum_{m \geq 0} p_s(m; D, \Psi, \Phi) z^m$.

Define a new digraph $D_x$ with vertex set $V(D) \times S$ such that $((u, s), (v, t)) \in E(D_x)$ iff $(u, v) \in E(D)$ and $s + v(\sigma) = t$. We call $D_x$ the derived digraph (of $D$). We define the start set $\Psi_x \subseteq V(D_x)$ to contain all $(v, s)$ such that $\sum v = s$ and $v \in \Psi$. For each $s \in S$ the finish set $\Phi_s \subseteq V(D)$ for $s$ contains all vertices $(v, s)$ where $v \in \Phi$.

Fix an ordering on $V(D_x)$ so we can define an adjacency matrix $M_x$ of $D_x$. Let $\psi_x \in \mathbb{R}^{|V(D_x)|}$ be the indicator vector for $\Psi_x$, and let $\phi_s \in \mathbb{R}^{|V(D_x)|}$ be the indicator vector for $\Phi_s$.

**Proposition 2.1.** For $m \geq \sigma$, we have

$$p_s(m; D, \Psi, \Phi) = \psi_x \trans M_x^{m-\sigma} \phi_s.$$  

The generating function $P_s(z; D, \Psi, \Phi)$ is rational.

**Proof.** Let $W_s$ be a walk in $D_x$ starting in $\Psi_x$ and ending in $\Phi_s$ in the form

$$W_s = ((w_1, t), \ldots, (w_{m-\sigma+1}, s)),$$

so the $D$-vertices corresponding to $W_s$ are $w_1, \ldots, w_{m-\sigma+1}$. We say the $m$-composition of $s$ defined by $W_s$ is

$$w_1 \sim (w_2(\sigma), \ldots, w_{m-\sigma+1}(\sigma)).$$

By the definition of $D_x$, the $m$-compositions corresponding to a walk $W_s$ in $D_x$ are exactly those $m$-compositions allowed by $D$ with total $s$. That is, the compositions defined by $D_x$ and $D$ are the same, but $D_x$ also directly keeps track of the total.
Figure 4: A base digraph $D$ (left) and derived digraph $D \times$ (right) representing Carlitz compositions over $\mathbb{Z}_3$. Here all vertices of $D$ are allowed start and finish vertices. Vertices in $D \times$ that are allowed start vertices are shown with a double circle.

Counting walks in a digraph via the adjacency matrix is a well-known procedure. The result follows from the relation

$$[M^q]_{i,j} = \sum_{k=1}^{|V(D\times)|} [M]_{i,k}[M^{q-1}]_{k,j}$$

which means walks of length $q + 1$ from vertex $i$ to $j$ are walks of length 2 from $i$ to $k$, merged with a walk of length $q$ from $k$ to $j$. This is known as the transfer matrix method; background may be found in [68].

We have

$$P_s(z; D, \Psi, \Phi) = \sum_{m \geq \sigma} \psi^\top_M \sigma \phi z^m + P(z) = z^\sigma \psi^\top_M (I - zM) \phi + P(z),$$

where $P(z)$ is a polynomial which counts the appropriate $m$-compositions with $m < \sigma$. The entries of $(I - zM) \phi$ lie in the field of fractions of $\mathbb{Q}[z]$, i.e. the rational functions $\mathbb{Q}(z)$.

Example 2.1. Carlitz compositions are those where adjacent parts must be different. Figure 4 shows an example of $D \times$ for Carlitz compositions over $\mathbb{Z}_3$.

Let us order the vertices of $D \times$ as

$$(0), (0), ((1), 1), ((2), 0), ((0), 2), ((1), 0), ((2), 2), ((0), 1), ((1), 2), ((2), 1).$$
Figure 5: Uniform-randomly generated Carlitz 100-compositions of 0 (above) and 1 (below) over \( \mathbb{Z}_3 \). (The vertical axis represents the value of a part.)

Then we get

\[
M_x = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}, \psi_x = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \phi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

\[\psi_x^\top M_x^{3-1} \phi_0 = 6.\]

So the number of Carlitz 3-compositions of 0 in \( \mathbb{Z}_3 \) is 6. \( \triangle \)

**Remark 2.2.** The following procedure generates a walk in \( D_x \) of length \( m - \sigma + 1 \), where all such walks are equally probable:

1. Pick a start vertex \( v_1 \) weighted by the number of \((m - \sigma + 1)\)-walks from that vertex to a finish vertex.

2. Given the current vertex \( v_i \), select an out-neighbor where such neighbors are weighted by the number of \((m - \sigma + 1 - i)\)-walks from the neighbor vertex to a finish vertex.

Naturally, using the correspondence between walks and compositions, this gives a method of random generation for locally restricted compositions. Figure 5 shows an example with Carlitz compositions. Other examples of the method are found throughout this section. \( \triangle \)

If \( D_x \) is strongly connected and aperiodic, then we can obtain a highly-precise asymptotic expression for \( p_x(m; D, \Psi, \Phi) \), \( m \to \infty \), via Proposition 2.1 and the Perron-Frobenius theorem. (A digraph is aperiodic iff the set of all cycle lengths has no common divisor besides 1.) We now give some general facts about the strong connectedness of \( D_x \).

If \( D \) is not strongly connected then certainly \( D_x \) is not strongly connected either. However, if \( D \) decomposes into disconnected strong components, then naturally we are able to simply count with each component separately and add. In the following, we assume \( D \) is strongly connected (and nonempty).

Unfortunately, if \( D \) is strongly connected, \( D_x \) is not necessarily strongly connected. Say \( D \) is the digraph given in Figure 6 over \( \mathbb{Z}_4 \) with span \( \sigma = 2 \).
Figure 6: A digraph $D$ with vertices in $\mathbb{Z}_4^2$.

In $D_x$, there is a path from $((3,0), 3)$ to $((1,3), 3)$, but there is no path from $((1,3), 0)$ to $((1,3), 3)$.

If the entirety of $D_x$ is not strongly connected then we would hope it is simply a disjoint union of strong components. This is not true for general finite semigroups $S$. For example, if there is $s^* \in S$ satisfying

$$\forall s \in S : s^* + s = s + s^* = s^*,$$

then walks in the digraph $D_x$ will get “stuck” at $s^*$ and some connected vertices will not be strongly connected. We do obtain this desideratum, however, if $S$ is a group, as we show eventually below. In the following we assume that $S = G$ is a group.

**Definition 2.2.** Let $D_B$ be an arbitrary digraph, referred to as the base digraph. Let $G$ be a finite group, and let $\alpha : E(D_B) \to G$ map arcs of $D_B$ to group elements. Together, $D_B$ and $\alpha$ are known as a voltage graph. We define the derived digraph $D_\alpha$ such that $V(D_\alpha) = V(D_B) \times G$ and $((u,a),(v,b))$ is an arc iff $(u,v) \in E(D_B)$ and $a + \alpha(u,v) = b$.

The digraphs $D_x$ directly give derived digraphs in the sense of voltage graphs, specifically “right derived ordinary voltage graphs”, if we associate the group element $u(\sigma)$ to all incoming arcs to $u$ in the base digraph $D$.

**Remark 2.3.** Let $(V_1, E_1)$ and $(V_2, E_2)$ be graphs. Then $(V_2, E_2)$ is a covering graph of $(V_1, E_1)$ iff there is a surjection $f : V_2 \to V_1$ such that for each $v \in V_2$, the restriction $f_{|N[v]}$ is a bijection. In that case, $f$ is called a covering map. We note that derived graphs can be seen as a covering graphs of the base graph, but directed. The book [32] provides a basic introduction to covering graphs in Chapter 2. Covering graphs are more generally known as covering spaces in topology.

**Lemma 2.1.** The derived digraph $D_\times$ is a disjoint union of strong components.

**Proof.** Select a vertex $(u,a)$, and take another vertex $(v,b)$ such that there is a path $(u,a) \rightarrow (v,b)$ in $D_\times$. The long right arrow notation denotes the existence of a path from one vertex to another. Since $D$ is strongly connected, there is a path $(v,b) \rightarrow (u,c)$ in $D_\times$ for some $c \in G$. This implies that $(u,a) \rightarrow (u,c)$. We are done if we can show that $(u,c) \rightarrow (u,a)$. 


Since there is a path \((u, a) \rightarrow (u, c)\), we know that for any positive integer \(j\), there is a path \((u, a) \rightarrow (u, a + j(-a + c))\), which is found by repeating the path in \(D\). In a finite digraph we will eventually get \(g > j > 0\) with \(a + j(-a + c) = a + g(-a + c)\), thus \(j(-a + c) = g(-a + c)\) and \((g - j)(-a + c) = 0\). We conclude that

\[
(u, a) \rightarrow (u, a + (-a + c)) \\
\rightarrow \ldots \\
\rightarrow (u, a + (g - j)(-a + c)) = (u, a).
\]

**Lemma 2.2.** For each \(v \in V(D)\) and \(a, b \in G\), there is a digraph automorphism \(f\) on \(D\) with \(f(v, a) = (v, b)\). In particular, the strong components of \(D\) are isomorphic.

**Proof.** Let \(f : V(D) \times G \rightarrow V(D) \times G\) be defined \(f(v, c) = (v, b - a + c)\). We have \(f(v, a) = (v, b)\), and clearly \(f\) is a bijection. Take an arc from \((u, c)\) to \((w, d)\). Then \(c + w = d\), so \(b - a + c + w = b - a + d\), so there is also an arc from \(f(u, c)\) to \(f(w, d)\). This automorphism is mentioned in [32, §2.2.1].

The second claim follows since every strong component contains a vertex \((v, c)\) for some \(c \in G\), which follows from the strong connectedness of \(D\).

Aperiodicity of \(D\) does not follow from aperiodicity of \(D\), as shown in Example 2.2, so it must be verified separately.

**Example 2.2.** The condition of aperiodicity of \(D\) cannot be transferred from \(D\). For example, if \(a \in G\) has order at least 3 and if \(E(D) = \{(a, a)\}\) then \(D\) is periodic. Figure 7 shows a less trivial counterexample digraph \(D\).

The following basic result applies the Perron-Frobenius theorem to asymptotic counting.

**Proposition 2.2.** Let \(M\) be a nonzero \(n \times n\) adjacency matrix of a strongly connected and aperiodic digraph. Then if \(\alpha, \beta \in \mathbb{R}^n\), we have

\[
\alpha^\top M^m \beta = (\alpha \cdot v_\lambda)(u_\lambda \cdot \beta)\lambda^m(1 + O(\theta^m)), \quad m \to \infty,
\]

where \(\lambda \geq 1\) is the largest-magnitude eigenvalue of \(M\), \(v_\lambda\) is a positive \(\lambda\)-eigenvector of \(M\), \(u_\lambda\) is a positive \(\lambda\)-eigenvector of \(M^\top\) such that \(v_\lambda \cdot u_\lambda = 1\), and \(0 \leq \theta < 1\).

**Proof.** By [63, Proposition 2.4], any largest-magnitude eigenvalue \(\lambda\) of \(M\) satisfies \(|\lambda| \geq 1\). We use a few other facts from linear algebra covered in e.g. [71, 53]. By the Perron-Frobenius theorem, we conclude there is a unique largest-magnitude eigenvalue \(\lambda > 0\) and \(\lambda\) has multiplicity 1. Furthermore, \(M\) has Jordan decomposition

\[
M = \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}^{-1},
\]

where 

\[
\begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} u_\lambda & * \\ * & * \end{bmatrix}
\]

and 

\[
\begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} v_\lambda & * \\ * & v_\lambda \end{bmatrix}
\]

with 

\[
v_\lambda \cdot u_\lambda = 1.
\]
Figure 7: An aperiodic strongly connected digraph $D$ (above) with vertices in $\mathbb{Z}_4^2$ such that $D_x$ (one component shown below) has period 2. Examples with connected $D_x$ exist as well, such as the above $D$ over $\mathbb{Z}_8$ with 7 replacing 3 and 3 replacing 2.

where $v_\lambda$ is a positive $\lambda$-eigenvector of $M$, $u_\lambda$ is a positive $\lambda$-eigenvector of $M^T$, and $B$ is a block-diagonal matrix with spectral radius $0 \leq r < \lambda$. The fact that $v_\lambda \cdot u_\lambda = 1$ follows once we note that the first and last matrices in a Jordan decomposition are inverses. Taking powers, we have

$$M^m = \begin{bmatrix} v_\lambda \\ \ast \end{bmatrix} \begin{bmatrix} \lambda^m & 0 \\ 0 & B^m \end{bmatrix} \begin{bmatrix} u_\lambda \\ \ast \end{bmatrix},$$

where $B^m = O(r^m)$. The result is now immediate with $\theta = r/\lambda$.

The essential idea of Proposition 2.2 is quite classical, see e.g. [25, Corollary V.1].

It is sometimes useful to know more about the growth rate of the number of walks.

**Proposition 2.3.** Let $M$ be an $n \times n$ adjacency matrix of a strongly connected and aperiodic digraph. If $n \geq 2$ then $M^m = \Theta(B^m)$, where $B > 1$.

**Proof.** Let $v_1$ be a vertex in the digraph. Since the digraph is aperiodic, there are two distinct cycles $C_1$ and $C_2$ starting from $v_1$; let their lengths be $c_1, c_2$. Let $\ell = \text{lcm}(c_1, c_2)$. Construct a walk of length $\alpha \ell$ by choosing $\alpha$ segments of length $\ell$ where each segment is either $C_1$ repeated or $C_2$ repeated. Then
the number of walks of length \( \alpha \ell \) is at least \( (2^{1/\ell})^{\alpha \ell} \). By Proposition 2.2 the number of walks of length \( m \) is \( \Theta(B^m) \) so we must have \( B > 1 \).

**Definition 2.3.** A de Bruijn subgraph \( D \) is regular iff \( D \) is strongly connected, contains at least 2 vertices, and its derived digraph \( D_x \) is aperiodic.

**Proposition 2.4.** Suppose \( D \) is regular and \( p(m; D, \Psi, \Phi) \sim A \cdot B^m \). Then either \( p_s(m; D, \Psi, \Phi) = 0 \) or

\[
p_s(m; D, \Psi, \Phi) = C_s \cdot B^m(1 + O(\theta^m)), \quad m \to \infty
\]

where \( C_s > 0 \) can be computed from \( D_x \) and \( 0 \leq \theta < 1 \).

**Proof.** Since the strong components of \( D_x \) are isomorphic by Lemma 2.1 and Lemma 2.2, they each have the same adjacency matrix and the same eigenvalues. The only difference between a composition of \( s \) and an arbitrary composition is the allowed finish vertices. Thus by Proposition 2.1 and Proposition 2.2 we conclude \( p_s(m; D, \Psi, \Phi) = C_s \cdot B^m(1 + O(\theta^m)) \) where \( C_s = 0 \) only if \( p_s(m; D, \Psi, \Phi) = 0 \). The latter case occurs if there is no strong component of \( D_x \) containing vertices from both \( \Psi_x \) and \( \Phi_s \). \( \square \)

The asymptotics of \( p_s(m; D, \Psi, \Phi) \) are now established. However, in some cases we can usefully simplify the constants involved.

**Definition 2.4.** If \( A \) is an \( m \times n \) matrix and \( B \) is a \( p \times q \) matrix, then the Kronecker product \( A \otimes B \) is the \( mp \times nq \) matrix \( C \) such that \([C]_{p(r-1)+v,q(s-1)+w} = [A]_{v,w}[B]_{r,s} \). Visually,

\[
A \otimes B = \begin{bmatrix}
[A]_{1,1}B & \cdots & [A]_{1,n}B \\
\vdots & \ddots & \vdots \\
[A]_{m,1}B & \cdots & [A]_{m,n}B
\end{bmatrix}.
\]

Basic properties of the Kronecker product are discussed in [38, Chapter 4]. We quote a couple of relevant facts.

**Proposition 2.5** (Lemma 4.2.10 in [38]). Let \( F \) be a field. Let \( A \in F^{m \times n}, B \in F^{p \times q}, C \in F^{m \times k}, D \in F^{q \times r} \). Then \((A \otimes B)(C \otimes D) = AC \otimes BD\).

**Proposition 2.6** (Equation 4.2.8 in [38]). Let \( F \) be a field. We have

\[
A \otimes (B + C) = A \otimes B + A \otimes C
\]

for all \( A \in F^{m \times n} \) and \( B, C \in F^{p \times q} \).

We next establish the structure of derived digraphs \( D_x \) in terms of the Kronecker product.

**Lemma 2.3.** Say \( |V(D)| = \alpha \) and fix a vertex ordering \( v_1, \ldots, v_\alpha \). Let \( M \) be the adjacency matrix of \( D \) with respect to this ordering. Also fix an ordering on \( G = \{a_1, a_2, \ldots, a_\beta\} \) where \( a_1 = 0 \). Finally, define a vertex ordering on \( D_x \) as

\[
(v_1, a_1), \ldots, (v_\alpha, a_1), (v_1, a_2), \ldots, (v_\alpha, a_2), \ldots, (v_1, a_\beta), \ldots, (v_\alpha, a_\beta).
\]
Let $M_x$ be the adjacency matrix of $D_x$ with respect to this ordering. For each $a \in G$, define the $\alpha \times \alpha$ matrix $M_a$ and $\beta \times \beta$ matrix $P_a$ such that

$$[M_a]_{i,j} = [v_j(\sigma) = a, (v_i, v_j) \in E(D)],$$

$$[P_a]_{i,j} = [a_i + a = a_j].$$

Then $M_x = \sum_{a \in G} P_a \otimes M_a$ and $M = \sum_{a \in G} M_a$.

**Proof.** We have

$$[M_x]_{i+a(j-1),k+a(l-1)} = [(v_i, v_k) \in E(D), a_j + v_k(\sigma) = a_l]$$

and

$$\left[ \sum_{a \in G} P_a \otimes M_a \right]_{i+a(j-1),k+a(l-1)} = \sum_{a \in G} [P_a]_{j,l} [M_a]_{i,k}$$

$$= \sum_{a \in G} [a_j + a = a_l] [v_k(\sigma) = a, (v_i, v_k) \in E(D)]$$

$$= [(v_i, v_k) \in E(D), a_j + v_k(\sigma) = a_l].$$

Clearly

$$[M]_{i,j} = [(v_i, v_j) \in E(D)] = \sum_{a \in G} [v_j(\sigma) = a, (v_i, v_j) \in E(D)] = \sum_{a \in G} [M_a]_{i,j}. \quad \Box$$

**Theorem 2.1.** Assume

- for some $v \in V(D)$ we have that for all $a \in G$ there is a legal composition starting and ending with $v$ with total $a$, and
- for some $u \in V(D)$ the set

$$\{ m : \exists \text{ a walk } x = (u,v,\cdots,w,u) \text{ of length } m+1, \sum x = \sum u \},$$

has a GCD of $1$, where $\sum x$ is the total of the composition corresponding to $x$.

Assume $p(m; D, \Psi, \Phi) \sim A \cdot B^m$. Then

$$p_a(m; D, \Psi, \Phi) = \frac{A}{|G|} \cdot B^m(1 + O(\theta^m)), \quad m \to \infty, 0 \leq \theta < 1.$$ 

**Proof.** From the first condition we know there is a single strong component i.e. $D_x$ is strongly connected. The second condition ensures that this component is aperiodic. This allows us to conclude that the Perron-Frobenius theorem applies directly to $D_x$.

Let $M, M_x, M_a, P_a$ be as in Lemma 2.3

Let $\lambda > 0$ be the dominant eigenvalue of $M$, let $v_\lambda$ be an associated positive eigenvector, and let $u_\lambda$ be an associated positive left eigenvector (eigenvector of $M^T$). Let $\xi \in \mathbb{R}^\beta$ be the all-1 vector $[1 \ 1 \ \cdots \ 1]$. 

14
We claim that $\xi \otimes v_\lambda$ is an eigenvector of $M_\infty$ with eigenvalue $\lambda$. First, by Proposition 2.5 $(P_a \otimes M_a)(\xi \otimes v_\lambda) = P_a \xi \otimes M_a v_\lambda$. Since $P_a$ is a permutation matrix, we have $P_a \xi = \xi$. Thus

\[
M_\infty(\xi \otimes v_\lambda) = \left( \sum_{a \in G} P_a \otimes M_a \right) (\xi \otimes v_\lambda) \\
= \sum_{a \in G} (P_a \otimes M_a)(\xi \otimes v_\lambda) \\
= \sum_{a \in G} P_a \xi \otimes M_a v_\lambda \\
= \sum_{a \in G} \xi \otimes M_a v_\lambda.
\]

By Proposition 2.6 $\sum_{a \in G} \xi \otimes M_a v_\lambda = \xi \otimes \sum_{a \in G} M_a v_\lambda$. We conclude

\[
M_\infty(\xi \otimes v_\lambda) = \xi \otimes \sum_{a \in G} M_a v_\lambda \\
= \xi \otimes \left( \sum_{a \in G} M_a \right) v_\lambda \\
= \xi \otimes M v_\lambda \\
= \xi \otimes \lambda v_\lambda \\
= \lambda \xi \otimes v_\lambda.
\]

Similarly we have that $\xi \otimes u_\lambda$ is a left eigenvector for $M_\infty$ with eigenvalue $\lambda$.

By Proposition 2.2, we know that

\[
p_a(m; D, \Psi, \Phi) = \psi_\infty^\top M_\infty^{m-\sigma} \phi_a = C_a \lambda^m (1 + O(\theta^m)),
\]

where $C_a = c(\psi_\infty \cdot (\xi \otimes v_\lambda))((\xi \otimes u_\lambda) \cdot \phi_a)$ for some fixed scaling factor $c > 0$.

Suppose $a = a_i$; then $\phi_a = e_i \otimes \phi$. Thus

\[
(\xi \otimes u_\lambda) \cdot \phi_a = (\xi \otimes u_\lambda) \cdot (e_i \otimes \phi) = (\xi \otimes e_i) \otimes (u_\lambda \cdot \phi) = u_\lambda \cdot \phi.
\]

Since $u_\lambda \cdot \phi$ does not depend on $a$, the proof is now complete.

**Corollary 2.1.** Assume the conditions of Theorem 2.1. Construct a probability space from $P(m; D, \Psi, \Phi)$ and the uniform probability measure. Then for $a \in G$, let $\mathbb{P}_m(a)$ be the probability that an element drawn randomly from $P(m; D, \Psi, \Phi)$ has total $a$. We have

\[
\mathbb{P}_m(a) \to \frac{1}{|G|}, \quad m \to \infty,
\]

or in other words, $\mathbb{P}_m$ converges strongly to the uniform measure on $G$.

**Proof.** Direct from Theorem 2.1.

**Example 2.3.** We show a case where the strong connectedness condition in Theorem 2.1 is required. Let $D$ be the digraph given in Figure 8 where $G = \mathbb{Z}_2$. Assume $\Psi = \Phi = V(D)$. 

15
Figure 8: A base digraph $D$ (above) and derived digraph $D_x$ with 2 strong components (below). The vertices of $D$ are 4-tuples over $\mathbb{Z}_2$. 
Let \(M^{(1)}, M^{(2)}\) be adjacency matrices of the two strong components of \(D_x\), under a particular vertex ordering. We have

\[
M^{(1)} = M^{(2)} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

From a Jordan decomposition, we get a left eigenvalue

\[
\lambda = \frac{1}{3} \left( 0.368841, 0.286991, 0.223305, 0.173751, 0.135194, 0.105193 \right),
\]

and a right eigenvector

\[
v_\lambda = [1.2852, 0.366538, 0.471074, 0.605423, 0.77809, 1.0].
\]

Also

\[
\psi^{(1)}_x = [1, 1, 1, 1, 1, 0]^T, \quad \phi_0^{(1)} = [1, 0, 1, 1, 1, 1]^T,
\]

\[
\phi_1^{(1)} = [0, 1, 0, 0, 0, 0]^T,
\]

and

\[
\psi^{(2)}_x = [0, 0, 0, 0, 0, 1]^T, \quad \phi_0^{(2)} = [0, 1, 0, 0, 0, 0]^T,
\]

\[
\phi_1^{(2)} = [1, 0, 1, 1, 1, 1]^T.
\]

We compute

\[
p_0(m; D) = \psi^{(1)}_x^T (M^{(1)})^{m-4} \phi_0^{(1)} + \psi^{(2)}_x^T (M^{(2)})^{m-4} \phi_0^{(2)}
\]

\[
\sim \lambda^{m-4}(\psi^{(1)}_x \cdot v_\lambda)(u_\lambda \cdot \phi_0^{(1)}) + (\psi^{(2)}_x \cdot v_\lambda)(u_\lambda \cdot \phi_0^{(2)})
\]

\[
\doteq \lambda^{m-4}(3.50632 \cdot 1.00628 + 1.0 \cdot 0.286991)
\]

\[
= 3.81533 \cdot \lambda^{m-4}
\]

and

\[
p_1(m; D) = \psi^{(1)}_x^T (M^{(1)})^{m-4} \phi_1^{(1)} + \psi^{(2)}_x^T (M^{(2)})^{m-4} \phi_1^{(2)}
\]

\[
\sim \lambda^{m-4}(\psi^{(1)}_x \cdot v_\lambda)(u_\lambda \cdot \phi_1^{(1)}) + (\psi^{(2)}_x \cdot v_\lambda)(u_\lambda \cdot \phi_1^{(2)})
\]

\[
\doteq \lambda^{m-4}(3.50632 \cdot 0.286991 + 1.0 \cdot 1.00628)
\]

\[
= 2.01256 \cdot \lambda^{m-4}.
\]

So indeed Theorem 2.1 does not hold. \(\triangle\)

**Lemma 2.4.** Assume \(G\) is abelian. Define a \(D_x\)-automorphism \(f\) by \(f(v,b) = (v,a+b)\) for some \(a \in G\). If \(f\) maps any vertex to its own strong component, then \(f\) maps all vertices to their own strong component.
Proof. Let \((u, c), (v, b)\) be arbitrary vertices and say \((v, b) \rightarrow (v, a + b)\) in \(D \times\). We seek to show that \((u, c) \rightarrow (u, a + c)\).

There is some \((u, d)\) in the same strong component as \((v, b)\) and so \((u, d) \rightarrow (u, a + d)\).

The automorphism \(g(w, r) = (w, c - d + r)\) maps \((u, d)\) to \((u, c)\) and \((u, a + d)\) to \((u, c - d + a + d) = (u, a + c)\).

Thus \((u, d) \rightarrow (u, a + d)\) implies \(g(u, d) = (u, c) \rightarrow g(u, a + d) = (u, a + c)\).

This is illustrated in Figure 9.

The following is a useful characterization of strong connectedness of \(D \times\) for abelian \(G\).

**Lemma 2.5.** Assume \(G\) is abelian. Let \(A\) be a generating set for \(G\), i.e. \(\langle A \rangle = G\). If for all \(a_i \in A\) there is a vertex \((v, 0) \in V(D \times)\) such that \((v, 0) \rightarrow (v, a_i)\) in \(D \times\), then \(D \times\) is strongly connected.

**Proof.** We show that for any \(u \in V(D)\) and \(r \in G\), the vertices \((u, 0)\) and \((u, r)\) are in the same strong component of \(D \times\).

Say \(r = j(1)a_1 + \cdots + j(p)a_p\) for \(a_i \in A, j(i) \in \mathbb{Z}_{\geq 0}\). We know from Lemma 2.4 that the \(D \times\)-automorphism \(f_j(v, s) = (v, a_j + s)\) maps strong components to themselves. Thus the composition \(f_r = f^{(1)}_j \circ \cdots \circ f^{(p)}_j\) also maps strong components to themselves. We conclude that \((u, 0)\) and \(f_r(u, 0) = (u, j(1)a_1 + \cdots + j(p)a_p) = (u, r)\) belong to the same strong component.

We now consider some examples of \(D\).

We generalize Carlitz compositions as follows. A sequence \(x \in \text{Seq}_m(G)\) is a \(d\)-\textit{Carlitz composition} iff every subword \(x(i), \ldots, x(i + d)\) contains no repeated part. Thus Carlitz compositions are 1-Carlitz. We note that unlike for integer compositions, we generally allow the identity element as a part. We also note
that this definition is consistent with [25] but different from Definition 4.33 in [37] p. 115]. Words with no equal adjacent letters are also called Smirnov words as in [25] Example III.24.

**Lemma 2.6.** There is a de Bruijn subgraph $D$ with span $\sigma = d + 1$ representing $d$-Carlitz compositions such that $D_\times$ is strongly connected and aperiodic, provided $|G| \geq d + 2$.

**Proof.** Take as vertex set for $D$ all $(d + 1)$-tuples of distinct elements of $G$. The allowed start and finish vertices are all of $V(D)$. The strong connectedness of $D$ is established in (the proof of) [25] Corollary 2.

To show strong connectedness of $D_\times$, we fix a vertex $(a_1, \ldots, a_{d+1}) \in V(D)$ and for any $s \in G$ exhibit a walk from $((a_1, \ldots, a_{d+1}), 0)$ to $((a_1, \ldots, a_{d+1}), s)$. Let $n$ be the order of $\Sigma a = a_1 + \cdots + a_{d+1}$. We consider two cases.

Case 1: $s \notin \{a_1, \ldots, a_{d+1}\}$. The first step is to $(a_2, \ldots, a_{d+1}, s)$. Follow this by the $(d + 1)$-step path to $(a_1, \ldots, a_{d+1})$ Take the $(d + 1)$-step path back to $(a_1, \ldots, a_{d+1})$ exactly $n - 1$ times. The total of this walk is

$$s + \Sigma a + (n - 1)\Sigma a = s,$$

thus $((a_1, \ldots, a_{d+1}), 0) \rightarrow ((a_1, \ldots, a_{d+1}), s)$ in $D_\times$.

Case 2: $s = a_j$ for some $1 \leq j \leq d + 1$. Let $b$ represent some element of $G$ not in $\{a_1, \ldots, a_{d+1}\}$. Begin with the $(d + 1)$-step path to $(b, a_1, \ldots, a_{j-1}, a_{j+1}, a_{d+1})$. Let $n_1$ be the order of $b + a_1 + \cdots + a_{j-1} + a_{j+1} + a_{d+1}$. Follow the $(d + 1)$-step path back to $(b, a_1, \ldots, a_{j-1}, a_{j+1}, a_{d+1})$ exactly $n_1 - 1$ times. Traverse one arc to $(a_1, \ldots, a_{j-1}, a_{j+1}, a_{d+1}, s)$, then follow the $(d + 1)$-step path to $(a_1, \ldots, a_{j-1}, a_{j+1}, a_{d+1}, b)$. Let $n_2$ be the order of $a_1 + \cdots + a_{j-1} + a_{j+1} + a_{d+1} + b$. Take the $(d + 1)$-step path back to $(a_1, \ldots, a_{j-1}, a_{j+1}, a_{d+1}, b)$ exactly $n_2 - 1$ times. Finally take the $(d + 1)$-step path to $(a_1, \ldots, a_{d+1})$ and cycle $(a_1, \ldots, a_{d+1})$ the suitable number of times. The total of this walk is $0 + s + 0 + 0 = s$ so $((a_1, \ldots, a_{d+1}), 0) \rightarrow ((a_1, \ldots, a_{d+1}), s)$ in $D_\times$.

We now turn to aperiodicity. Take a $D$-vertex $u = (a_1, \ldots, a_{d+1})$ that does not contain the part $0$. We give two closed walks starting from $u$ with total $0$ and lengths that differ by $1$.

Let $n$ be the order of $a_1 + \cdots + a_{d+1}$. The first walk repeats the $(d + 1)$-step cycle back to $u$ exactly $n$ times. The second walk first takes a step to $(u_2, \ldots, u_{d+1}, 0)$ followed by the $(d + 1)$-step path to $u$. Then we cycle back to $u$ exactly $n - 1$ times.

**Proposition 2.7.** The number of $d$-Carlitz $m$-compositions of $s \in G$ over a finite group $G$ is

$$\frac{1}{|G|^d (|G| - d)^{m-d} (1 + O(\theta^m))}, \quad m \to \infty, 0 \leq \theta < 1,$$

provided $|G| \geq d + 2$.  

19
Figure 10: Uniform-randomly generated 2-Carlitz 100-compositions of 0 (above) and 1 (below) over $\mathbb{Z}_5$. (The vertical axis represents the value of a part.)

Table 1: Exact counts of Carlitz $m$-compositions of $a$ over $S_3$.

| $m$ | $a$ | id | (123) | (12) |
|-----|-----|-----|-------|------|
| 3   |     | 27  | 24    | 25   |
| 4   |     | 134 | 128   | 120  |
| 5   |     | 613 | 631   | 625  |
| 6   |     | 3096| 3102  | 3150 |
| 7   |     | 15667| 15604 | 15625|
| 8   |     | 78224| 78263 | 78000|
| 9   |     | 390513| 390681| 390625|
| 10  |     | 1952696| 1952402| 1953750|
| 11  |     | 9765817| 9765529| 9765625|
| 12  |     | 48830424| 48831663| 48825000|
| 13  |     | 244140763| 244140556| 244140625|
| 14  |     | 1220690096| 1220686202| 1220718750|
| 15  |     | 6103512717| 6103517079| 6103515625|
| 16  |     | 30517650374| 30517659188| 30517500000|

Proof. With Lemma 2.6 we conclude that the conditions of Theorem 2.1 are satisfied.

In $D$ each vertex has an out-degree of $|G| - d$. This allows us to count walks in $D$ directly. We have $V(D) = |G|^{d+1}$. Thus the number of $m$-compositions represented by $D$ is $|G|^{d+1}(|G| - d)^{m-d-1} = |G|^d(|G| - d)^{m-d}$. We conclude by applying Theorem 2.1.

Figure 10 shows randomly generated 2-Carlitz 100-compositions over $\mathbb{Z}_5$. Table 1 gives counts for Carlitz $m$-compositions of $a$ over $S_3$.

As in [28] we say an $m$-composition $x$ is locally $d$-Mullen iff no nonempty subword of $x$ of length at most $d$ has total 0.

Proposition 2.8. The number of locally $d$-Mullen $m$-compositions of $a \in G$ over a finite group $G$ is

$$
\frac{1}{|G|} (|G| - 1)^{d-1} (|G| - d)^{m-d+1} (1 + O(\theta^m)), \quad m \to \infty, 0 \leq \theta < 1,
$$

provided $|G| \geq d + 2$. 

20
Proof. Let $D^{(1)}$ be the digraph of span $d$ representing locally $(d + 1)$-Mullen compositions, and let $D^{(2)}$ be the derived digraph. Let $D^{(2)}$ be the digraph of span $d + 1$ representing $d$-Carlitz compositions. We note that $|V(D^{(1)})| = |V(D^{(2)})| = |G|^{d+1}$, recalling that the part 0 is never allowed in a locally Mullen composition. Define a function $f : V(D^{(1)}) \to V(D^{(2)})$ as follows:

\[ f((v_1, \ldots, v_d), a) = \left( a - \sum_{j=1}^{d} v_j, a - \sum_{j=2}^{d} v_j, \ldots, a - v_d, a \right). \]

A computation gives us that $f$ is a graph isomorphism from $D^{(1)}$ to $D^{(2)}$. Thus the strong connectedness and aperiodicity of $D^{(2)}$ established in Lemma 2.6 hold for $D^{(1)}$ as well and Theorem 2.1 applies.

A part $x(i)$ in a locally $d$-Mullen compositions must not take the value $0, -x(i - 1), -x(i - 1) - x(i - 2)$, etc. and these values are distinct since

\[ n > n', \sum_{j=1}^{n} (-x(i - j)) = \sum_{j=1}^{n'} (-x(i - j)) \implies \sum_{j=n'+1}^{n} (-x(i - j)) = 0. \]

The number of locally $d$-Mullen $m$-compositions with any total is then $(|G| - 1)^{d-1}(|G| - d)^{m-d+1}$ and Theorem 2.1 gives the result. \hfill \Box

Proposition 2.9. The number of $m$-compositions of $s \in G$ over a finite group $G$ such that the sum of any $d + 1$ consecutive parts is not 0 is

\[ |G|^{d-1}(|G| - 1)^{m-d}(1 + O(\theta^m)), \quad m \to \infty, 0 \leq \theta < 1, \]

provided $d \leq |G| - 2$.

Proof. Define the appropriate $D$ so that $V(D)$ contains all $(d + 1)$-tuples of vertices that do not sum to 0. The strong connectedness of $D$ is established in (the proof of) [28, Corollary 2].

Let $u = (a_1, \ldots, a_{d+1})$ be an arbitrary vertex in $V(D)$, and let $s$ be an element of $G$. We seek a path (or a walk) from $(u, 0)$ to $(u, s)$ in $D_x$.

Let $b \in G$ satisfy the system

\[ a_1 + \cdots + a_d + b \neq 0 \]
\[ a_2 + \cdots + a_d + b + s \neq 0. \]

This gives at least $|G| - 2$ possible values for $b$.

Let $b' \in G$ satisfy the system

\[ a_j + \cdots + a_d + b + s + b' + a_2 + \cdots + a_{j-2} \neq 0, \quad 3 \leq j \leq d + 1 \]
\[ s + b' + a_2 + \cdots + a_d \neq 0 \]
\[ b' + a_2 + \cdots + a_{d+1} \neq 0. \]

This gives at least $|G| - d - 1$ possible values for $b'$. \hfill \Box
Starting from \( u \), we take a \((d + 1)\)-step walk to \((a_1, \ldots, a_d, b)\). Let \( n_1 \) be the order of \( a_1 + \ldots + a_d + b \). We cycle back to \((a_1, \ldots, a_d, b)\) exactly \( n_1 - 1 \) times. Now we take one step by appending \( s \). Then we take a \((d + 1)\)-step walk to \( b', a_2, \ldots, a_{d+1} \) and cycle that vertex the appropriate number of times. Finally walk to and cycle \( a_1, \ldots, a_{d+1} \). The total of this walk is \( 0 + s + 0 + 0 = s \). We conclude that \( D_\times \) is strongly connected.

To establish aperiodicity, let \( u = (a_1, \ldots, a_{d+1}) \) be a \( D \)-vertex satisfying the following. Set \( a_d \) so that \( a_1 + \cdots + a_d \neq 0 \). Set \( a_{d+1} \) so that for \( i = 1, \ldots, d \) we have \( \Sigma a - a_i \neq 0 \). Thus for \( i = 1, \ldots, d + 1 \) we have \( \Sigma a - a_i \neq 0 \). There are at least \(|G| - d\) possible values for \( a_{d+1} \). Then we may take the same approach as in the proof of Proposition \([2.7]\) where we consider two cycles from \( u \), one with an extra 0 inserted.

We have \(|V(D)| = |G|^d(|G| - 1)\) and each vertex has out-degree \(|G| - 1\). Thus there are \(|G|^d(|G| - 1)(|G| - 1)^{m-d-1} = |G|^d(|G| - 1)^{m-d}\) walks in \( D \) defining an \( m \)-composition. Applying Theorem \([2.1]\) gives the result. \( \square \)

Figure 11 shows uniformly-randomly generated 100-compositions over \( \mathbb{Z}_5 \) such that no part may be followed by its (additive) inverse.

**Proposition 2.10.** Let \( p_a(m) \) be the number of \( m \)-compositions of \( a \in G \) such that the sum of any \( d + 1 \) consecutive parts is not 0. Then for \( a \neq 0, b \neq 0 \), we have \( p_a(m) = p_b(m) \). If \( m \) is not a multiple of \( d + 1 \), then \( p_0(m) = p_a(m) \).

**Proof.** Let \( x = (x(1), \ldots, x(m)) \) be an \( m \)-composition. Let \( y(i) = \sum_{n=1}^{i} x(n) \).

Clearly \( x \) uniquely determines \( y \) and vice versa. Also, \( x \) has total \( a \) iff \( y(m) = a \).

Let \( y^{(j)}(i) = y((i - 1)(d + 1) + j - 1) \) for \( j \in [d + 1] \). Then \( x \) satisfies the condition iff each \( y^{(j)} \) is Carlitz and \( y^{(d+1)}(1) \neq 0 \).

First assume \( m \) is not a multiple of \( d + 1 \), so \( y(m) \) is the last part of some \( y^{(j)}, j \neq d + 1 \). Let \( \pi : G \rightarrow G \) be defined \( \pi(b) = b \) for all \( b \not\in \{0, a\} \), and \( \pi(a) = 0, \pi(0) = a \). Then if we apply \( \pi \) to \( y^{(j)} \) within \( y \) and take differences, we get a new \( x' \) which satisfies the condition and has total 0. Thus \( p_0(m) = p_a(m) \).

Second, if \( m \) is a multiple of \( d + 1 \), the previous \( \pi \) does not work since it may change whether \( y^{(d+1)}(1) \neq 0 \). However, if we take some bijective \( \pi' : G \rightarrow G \) which fixes 0 and swaps two nonzero elements \( a \) and \( b \), and apply it to \( y^{(d+1)} \) in \( y \) we conclude \( p_a(m) = p_b(m) \). \( \square \)
Table 2: Exact counts of \( m \)-compositions of \( a \) with no part followed by its inverse, over \( Q_8 \) (written as a subgroup of \( S_8 \)).

Table 2 gives counts for \( m \)-compositions of \( a \) over the quaternion group \( Q_3 \) such that no part may be followed by its inverse.

**Example 2.4.** We examine restrictions where all parts are simply required to lie in a fixed set \( \Xi \). We assume without loss of generality that the subset \( \Xi \) generates \( G \). When working with permutation groups, the meaning of “composition” as in “integer composition” is actually the same as in “functional composition”.

If \( \Xi = G \) then the number of compositions of \( a \) is always \( |G|^{m-1} \) since the first \( m - 1 \) parts are arbitrary and the last part is uniquely determined. However if \( \Xi \subset G \) this is no longer the case.

The digraph \( D \) with vertex set \( \Xi \) is clearly strongly connected, and it is straightforward to see that \( D_x \) is strongly connected as well.

For any cycle with final edge labeled \( a \) in the Cayley graph constructed from \( \Xi \), there is a cycle of equal length at \(((a),0)\) in \( D_x \). This implies that \( D_x \) is aperiodic iff the Cayley graph is aperiodic.

One way to ensure an aperiodic Cayley graph is to include \( 0 \in \Xi \). In general Cayley graphs are not aperiodic e.g. only an even number of transposition permutations can equal the identity since the identity is an even permutation.

We turn to the problem of counting compositions with \( r > 0 \) occurrences of a pattern. In preparation, we quote the fundamental fact of rational generating function asymptotics which is applied a few times in the remainder.

**Theorem 2.2** (Theorem IV.9 in [25]). \textit{If} \( f(z) \) \textit{is a rational function that is}...
analytic at 0 and has poles $\alpha_1, \alpha_2, \ldots, \alpha_m$, then there exist $m$ polynomials $\Pi_j(x)$ such that for sufficiently large $n$ we have $[z^n]f(z) = \sum_{j=1}^{m} \Pi_j(n)\alpha_j^{-n}$ where the degree of $\Pi_j$ is the order of the pole of $f$ at $\alpha_j$, minus one.

**Theorem 2.3.** Let $\tilde{D}$ be a de Bruijn graph. Let $U \subset V(\tilde{D})$ and $\Psi, \Phi \subseteq V(\tilde{D})$ all be nonempty and suppose $D = \tilde{D} - U$ is regular with strongly connected derived digraph $D_x$.

Let $\mu$ be the minimum number of occurrences of $U$ (as subwords) in a composition in $\mathcal{P}(\tilde{D}, \Psi, \Phi)$ that has at least 1 occurrence of $V(D)$. Assume that for all sufficiently large values of $m$ there exist compositions in $\mathcal{P}(m; \tilde{D}, V(D), V(D))$ with exactly 1 occurrence of $U$, and that $p(m; D, V(D), V(D)) \sim A \cdot B^m$.

If $r \geq \max(\mu, 1), \mu \geq 0$ then the number of $m$-compositions of $a \in G$ starting in $\Psi$ and finishing in $\Phi$ with exactly $r$ occurrences of $U$ is

$$p_a(m, r; D, \Psi, \Phi) = m^{\delta(\mu, \Phi)} A_{r, \mu} \cdot B^m \left(1 + O(m^{-1})\right), A_{r, \mu} > 0, B > 1, \quad m \to \infty.$$  

**Proof.** Define an occurrence segment as a composition $w$ of length at least $\sigma$ where every part in $w$ is involved in an occurrence of $U$. A detour in $\tilde{D}$ is an occurrence segment $w = (w(1), \ldots, w(\ell))$ where there is an arc from $V(D)$ to $(w(1), \ldots, w(\sigma))$ and an arc from $(w(\ell - \sigma + 1), \ldots, w(\ell))$ to $V(D)$. The occurrence segment $w$ gives a left semi-detour if there is an arc from $V(D)$ to $(w(1), \ldots, w(\sigma))$ and $w$ gives a right semi-detour if there is an arc from $(w(\ell - \sigma + 1), \ldots, w(\ell))$ to $V(D)$.

Fix elements of $v_\Psi \in \Psi$ and $v_\Phi \in \Phi$ as start and finish segments. If $v_\Psi \notin U$, set $\tilde{v}_\Psi = v_\Psi$, and if $v_\Psi \in U$, set $\tilde{v}_\Psi$ to some right semi-detour with $v_\Psi$ at the beginning. If $v_\Psi \notin U$, set $\tilde{v}_\Phi = v_\Phi$ and if $v_\Psi \in U$, set $\tilde{v}_\Phi$ to a left semi-detour with $v_\Phi$ at the end. Fix a further sequence of detours $d_1, \ldots, d_n$ so that the total number of occurrences of $U$ in all (semi-)detours is $r$. For $m$ sufficiently large, an $m$-composition with $r$ occurrences of $U$ has the form

$$x = \tilde{v}_\Psi y_1 d_1 y_2 d_2 \cdots y_n d_n y_{n+1} \tilde{v}_\Phi,$$

where each $y_i$ is a non-empty composition such that no parts of $y_i$ are involved in an occurrence of $U$ in $x$. We further fix $a_1, \ldots, a_{n+1} \in G$ such that $\sum y_i = a_i$ implies $\sum x = a$. Let the total length of the $y_i$ be $m - \delta$.

Given all of the fixed objects, the $y_i$ are subject to start and finish constraints, totals, and a total length $m - \delta$. Let $c^{(\delta)}(z)$ be the generating function counting possible $y_i$ where $z$ marks length. Then

$$c^{(\delta)}(z) = z^\sigma \psi_i^\top \left( \sum_{j \geq 0} M^j \psi_j \right) \phi_i + P_i(z),$$

where $M^j$ is the adjacency matrix of $D_x$ and $\psi_i$ and $\phi_i$ are the appropriate start and finish vectors. The term $P_i(z)$ is a polynomial which counts the appropriate $m$-compositions with $m < \sigma$. The number of sequences $y_1, \ldots, y_{n+1}$ is then

$$[z^{m-\delta}] \prod_{i=1}^{n+1} c^{(\delta)}(z).$$
By Theorem 2.1 and Theorem 2.2,
\[ \prod_{i=1}^{n+1} c^{(i)}(z) = A' \left( \frac{1}{(1-Bz)^{n+1}} + O((1-Bz)^{-n}) \right), \quad z \to 1, \]
and
\[ [z^{m-\delta}] \prod_{i=1}^{n+1} c^{(i)}(z) = m^n A^n \cdot B^m (1 + O(m^{-1})), \quad m \to \infty. \]

There is a finite set of possible values for the objects we fixed and from the assumptions we know \( n \) attains the value \( r - \mu \), so we conclude the result. \( \square \)

**Proposition 2.11.** Let \( D \) be a strongly connected, aperiodic digraph with at least 2 vertices. Let \( A \subseteq V(D) \times V(D) \) be a nonempty set of allowed start-finish vertex pairs. Let \( \Xi \subset V(D) \) be a nonempty set of designated vertices such that \( D - \Xi \) is strongly connected. Let \( X_m \) be the number of vertices of \( \Xi \) in a uniform random walk of length \( m \) in \( D \) where the initial and final vertices are found as a pair in \( A \). Then \( E(X_m) \sim c_1m, \Var(X_m) \sim c_2m \) where \( c_1, c_2 > 0 \) do not depend on \( A \), and
\[ \frac{X_m - E(X_m)}{\sqrt{\Var(X_m)}} \Rightarrow N(0,1). \]

**Proof.** Say \( V(D) = n \) and fix an ordering \( v_1, \ldots, v_n \) on \( V(D) \). We use \( u \) as an indeterminate to mark occurrences of \( \Xi \). Let \( C \) be an \( n \times n \) matrix where
\[ [C]_{i,j} = [(v_i, v_j) \in A](u[v_i \in \Xi] + [v_i \notin \Xi]), \]
and let \( T \) be an \( n \times n \) matrix where
\[ [T]_{i,j} = [(v_i, v_j) \in E(D)](u[v_j \in \Xi] + [v_j \notin \Xi]). \]
The matrix \( T \) is known as the transfer matrix. Then \([u^r] \sum_{i,j} [C]_{i,j} [T^{m-1}]_{i,j} \) is the number of walks of length \( m \) in \( D \) with \( r \) occurrences of \( \Xi \) with start and finish vertices allowed by \( A \).

Theorem 1 in [10] establishes limiting distributions for secondary parameters in the context of the transfer matrix method. It can be applied to obtain the result if we verify that there is a vertex \( v \in V(D) \) and positive integer \( k \) such that there are walks from \( v \) to \( v \) of length \( s \) with differing numbers of terms in \( \Xi \). Suppose \( v \notin \Xi \). Let \( W \) be a sufficiently long walk from \( v \) to \( v \) in \( D - \Xi \). There are walks from \( v \) to \( v \) which visit \( \Xi \), and by aperiodicity of \( D \) such a walk exists of the exact length of \( W \), so we are finished. \( \square \)

Clearly if \( \Xi = \emptyset \) or \( \Xi = V(D) \) the number of occurrences of \( \Xi \) is trivial. We note that de Bruijn graphs are always aperiodic and strongly connected.

**Example 2.5.** We look at compositions over \( \mathbb{Z}_2 \) and keep track of occurrences of \( \Xi = \{(0, 0), (1, 1)\} \). The relevant de Bruijn graph \( D \) has span \( \sigma = 2 \). For the following we fix a particular ordering \( v_1 \) on the vertices of \( D_x \). We use the indeterminates \( u \) and \( z \) to mark length and total. We define start vector
\[ \psi = z^2[u \hspace{1cm} u \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} 1 \hspace{1cm} 1]^{\top}, \]
where $\psi(i)$ is 0 for non-start vertices, $z^\sigma u$ for start vertices corresponding to $\Xi$, and $z^\sigma$ otherwise. The finish vector for compositions of 0 is

$$\phi_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$  

The matrix $C$ from Proposition 2.11 is then $\psi \phi_0^T$. The transfer matrix is

$$T = \begin{bmatrix}
    u & 0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 & 0 & u & 0 \\
    0 & 0 & 0 & 1 & 0 & u & 0 \\
    u & 0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 1 & 0 & u & 0 & 0 \\
    0 & u & 0 & 0 & 0 & 0 & 1 \\
    0 & u & 0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 1 & 0 & u & 0 & 0 
\end{bmatrix},$$

where $T_{i,j} = 0$ if $(v_i, v_j) \notin E(D_x)$, $T_{i,j} = u$ if $(v_i, v_j) \in E(D_x)$ and $v_j$ corresponds to $\Xi$, and $T_{i,j} = 1$ otherwise. We define

$$P(z,u) = z^{\sigma - 1} \psi^T (I - zT)^{-1} \phi_0,$$

getting that $[z^m u^r] P(z,u)$ is the number of $m$-compositions of 0 over $\mathbb{Z}_2$ with $r$ occurrences of $\Xi$.

Let $X_m^{(0)}$ be the number of occurrences of $\Xi$ in a uniform-random $m$-composition of 0. We have

$$E(X_m^{(0)}) = \frac{[z^m]D_u P(z,u)|_{u=1}}{[z^m]P(z,1)} = \frac{1}{2} m + O(1),$$

and

$$\text{Var}(X_m^{(0)}) = \frac{[z^m]D_u^2 P(z,u)|_{u=1}}{[z^m]P(z,1)} + \frac{[z^m]D_u P(z,u)|_{u=1}}{[z^m]P(z,1)} - \left( \frac{[z^m]D_u P(z,u)|_{u=1}}{[z^m]P(z,1)} \right)^2 = \frac{1}{4} m + O(1).$$

So Proposition 2.11 (and an application of Slutsky’s theorem) entail

$$\frac{X_m^{(0)} - \frac{1}{2} m}{\frac{1}{2} \sqrt{m}} \Rightarrow N(0,1). \quad \triangle$$

Asymptotic joint distributions and local limit phenomena are derivable, under conditions, based on [10] and/or [11]. One can also analyze additional parameters (longest runs, etc.) in analogy to the existing local restriction theory. However in these matters as in Proposition 2.11 one expects to get results identical to those for words (disregarding the total) since arbitrary start and finish requirements do not affect asymptotic distributions.

**Lemma 2.7.** Let the greatest letter in a subword pattern $\tau$ be $j^*$. Assume $k \geq 2$ and $k \geq j^*$. If $\tau$ has length $p + 1 \geq 2$ and $\tau$ is not $1^p 2$ and its symmetries ($1^2 p, 2^p 1$, and $21^p$), there is a strongly connected de Bruijn subgraph $D$ with span $\sigma \geq p$ whose walks represent words over $[k]$ that avoid $\tau$.  

26
Proof. The patterns $1^p2$ do not satisfy this because $1^p$ and $2^p$ are both allowed but there is no allowed sequence of the form $1^p w 2^p$ where $w$ is some word.

Let $\tau = (\tau(1), \ldots, \tau(\sigma + 1))$. Let $D$ be the de Bruijn subgraph of span $\sigma$ representing $k$-ary words avoiding $\tau$. Let $x = (x(1), \ldots, x(\sigma))$ and $y = (y(1), \ldots, y(\sigma))$ be vertices of $D$. We proceed by cases, establishing either that $x \rightarrow y$ and $y \rightarrow x$ or $1^\sigma \rightarrow x$ and $x \rightarrow 1^\sigma$.

Case 1: $j^* = 1$. If $x(\sigma) \neq y(1)$, then the concatenation $xy$ is allowed. Otherwise, take $c \neq x(\sigma) = y(1)$ and then $xcy$ is allowed.

Case 2: $j^* \geq 3$. Assume WLOG $\tau(1) > 1$. Then $1^\sigma x$ is always allowed. If $\tau(\sigma) > 1$ then $x1^\sigma$ is allowed too. Otherwise $\tau(\sigma) = 1$ and $xk^\sigma 1^\sigma$ is allowed.

Case 3: $j^* = 2$. Assume WLOG $\tau(1) = 2$. Again $1^\sigma x$ is allowed. If $\tau(\sigma) = 2$ then $x1^\sigma$ is allowed too. If $\tau(\sigma) = 1$ and $\tau$ is not monotonic then $xk^\sigma 1^\sigma$ is allowed. Finally, if $\tau = 2^p1^q$ with $p, q > 1$ then $x(k1)^p 1^{p-1}$ is allowed.

This shows that a satisfactory digraph exists with span $\sigma$. It is now easily seen that a digraph with greater span would also be strongly connected. \qed

Lemma 2.8. Let $G$ be a totally ordered finite group and let $\tau$ be a subword pattern of length at least 2 other than $1^p 2$ and its symmetries ($12^p, 2^p1$, and $21^p$). If $j^*$ is the greatest letter in $\tau$, assume $|G| \geq \max(3, j^*)$. The de Bruijn subgraph $D$ with span $\sigma = |\tau|$ representing compositions over $G$ avoiding $\tau$ is such that $D_\times$ is strongly connected and aperiodic.

Proof. Let $a, b \in G$ be distinct and both nonzero.

We show strong connectedness of $D_\times$. Let $c = -a$. If $\tau = 1^\sigma$, then the composition $0^{\sigma-1}a0b0c0^{\sigma-1}a$ exhibits a path $(0^{\sigma-1}a, 0) \rightarrow (0^{\sigma-1}a, b)$. Otherwise, if $\tau \neq i^p j^q$ where $p, q \geq 1$ and $i \neq j$, then $0^\sigma a0^\sigma$ is allowed and therefore $(0^\sigma, 0) \rightarrow (0^\sigma, a)$ in $D_\times$. Finally, if $\tau = i^p j^q$, let $n$ be the order of $\sigma a$. Then $0^\sigma a^n a^0$ shows $(0^\sigma, 0) \rightarrow (0^\sigma, a)$.

We show aperiodicity of $D_\times$. The vertex $(0^\sigma, 0)$ exists in $D_\times$ and has a loop iff $\tau \neq 1^\sigma$. For the pattern $1^\sigma$ where $\sigma \geq 3$, the two sequences $b^\sigma-1b^\sigma-10$ and $b^\sigma-10b^\sigma-10$ are allowed and correspond to walks $(b^\sigma-10, 0, b^\sigma-10, (\sigma-1)b)$ with lengths differing by 1. Lastly, if $\tau = 11$, the compositions $abab$ and $ab0ab$ represent paths from $((a, b), 0)$ to $((a, b), a+b)$ with lengths differing by 1. \qed

Theorem 2.4. Let $G$ be a finite group with a total order and let $\tau$ be a subword pattern of length at least 2 not $1^p 2$ or its symmetries ($12^p, 2^p1$, and $21^p$). If $j^*$ is the greatest letter in $\tau$, assume $|G| \geq \max(3, j^*)$. The number of $m$-compositions of $a \in G$ containing $r$ occurrences of $\tau$ is

$$A_r m^r B^m (1 + O(m^{-1})), \quad A_r > 0, B > 1, m \rightarrow \infty.$$ 

If $X_m^{(a)}$ is the number of occurrences of $\tau$ in a uniform random $m$-composition of $a \in G$ then

$$\frac{X_m^{(a)} - E(X_m^{(a)})}{\sqrt{\text{Var}(X_m^{(a)})}} \Rightarrow N(0, 1).$$
Figure 12: Uniform-randomly generated 100-compositions of 0 (above) and 1 (below) over $\mathbb{Z}_5$ which avoid 132. (The vertical axis represents the value of a part.)

Proof. To satisfy the requirements of Theorem 2.3 we must show that there are arbitrarily long compositions with a single occurrence of $\tau$. Let $a$ be the minimal element of $G$ and suppose $b > a$. If $\tau = 1^p$, then $\cdots ababa^p baba \cdots$ is such a composition. If $\tau \neq q^p$, let $y$ be an occurrence of $\tau$ with minimal-valued parts; then $a \cdots aya \cdots a$ is such a composition. We can now apply Lemma 2.8, Theorem 2.3, Proposition 2.3, and Proposition 2.11 to conclude the result.

We are immediately able to modify results where they are available for words containing subwords patterns.

**Proposition 2.12.** Let $G$ be a totally ordered group with $|G| = k$, e.g. $\mathbb{Z}_k$ where $0 < 1 < \cdots < k - 1$. Define

$$C(y) = [y^r] \frac{1}{1 - y - \sum_{p=3}^d \sum_{j=0}^{p-3} \binom{k}{p+j} y^{p+j}(q-1)^{p-2} q^{p-1} y^{p-j}}$$

as in [37, p. 112]. Let $\rho > 0$ be the radius of convergence of $C(y)$, and let $A_r = \lim_{y \to \rho} ((1 - y/\rho)^{r+1} C(y))$.

The number of $m$-compositions of $a \in G$ containing $r$ occurrences of the subword pattern 123 is

$$\frac{1}{k} A_r m^r \left( \frac{1}{\rho} \right)^m (1 + O(m^{-1})), \quad m \to \infty.$$ 

Proof. Theorem 4.30 in [37] states that $[y^m] C(y)$ is the number of $m$-compositions with any total containing $r$ occurrences of 123. The result then follows from Theorem 2.4.

Table 3 shows counts of $m$-compositions of $a$ avoiding $\tau = 132$ over $\mathbb{Z}_5$. Figure 12 gives randomly selected compositions avoiding 132 over $\mathbb{Z}_5$. Figure 13 gives the same for compositions avoiding 121.

### 2.2 Note on minimization of transfer matrices

Given a transfer matrix (adjacency matrix) $T$, we may compute counting sequences by raising $T$ to a power, and if $T$ has a simple enough structure, we may even be able to extract a closed form expression for the counting
Table 3: Exact counts of $m$-compositions of $a$ avoiding 132 over $\mathbb{Z}_5$.

| $m$ | $a$ | 0    | 1    | 2    |
|-----|-----|------|------|------|
| 2   | 3   | 5    | 5    | 5    |
| 3   | 4   | 23   | 23   | 23   |
| 4   | 5   | 105  | 105  | 105  |
| 5   | 6   | 478  | 477  | 477  |
| 7   | 8   | 2171 | 2171 | 2171 |
| 9   | 10  | 9869 | 9868 | 9868 |
| 10  | 11  | 44861| 44861| 44861|
| 12  | 13  | 203930| 203930| 203930|
| 14  | 15  | 927032| 927032| 927033|
| 16  | 17  | 4214147| 4214147| 4214147|
| 18  | 19  | 19156861| 19156861| 19156865|
| 20  | 21  | 87084158| 87084158| 87084158|
| 22  | 23  | 395871195| 395871195| 395871198|
| 24  | 25  | 1799569607| 1799569609| 1799569610|
| 26  | 27  | 8180566793| 8180566793| 8180566793|

Figure 13: Uniform-randomly generated 100-compositions of 0 (above) and 1 (below) over $\mathbb{Z}_5$ which avoid 121. (The vertical axis represents the value of a part.)
sequence, or learn other information. A simple transfer matrix corresponds to a simple digraph $D$. Locally restricted compositions over a finite set constitute a regular language, so equivalently we may say we are interested in simple finite automata. In those terms, a question arises: Given a finite automaton $A$, when is it possible to find a smaller automaton $A'$ such that $A$ and $A'$ are equivalent for counting purposes?

**Definition 2.5.** As in [57], we say two deterministic finite automata (DFAs) $A, A'$ are weakly equivalent iff for each integer $m \geq 0$, the automata $A$ and $A'$ accept the same number of words of length $m$.

Our question is largely answered by an algorithm given in [57, § 4.2], which we refer to as the Ravikumar-Eisman algorithm. The algorithm is given a DFA $A$ and returns a weakly equivalent DFA $A'$ with the same number or fewer states. While the Ravikumar-Eisman algorithm is not guaranteed to find the smallest such $A'$, it is shown to be practically useful and no better technique is currently available. Roughly speaking, the Ravikumar-Eisman algorithm works by finding states which are equivalent in the weak sense (there are equal numbers of accepted words of length $m$ starting at each state for each $m$); these states are then merged.

**Example 2.6.** Figure 14 shows a naive automaton for Carlitz/Smirnov words on the alphabet $\{a, b, c\}$. In fact this automaton is minimal in the usual sense. However, there is a length-preserving bijection between 3-ary Carlitz words and the language accepted by the automaton in Figure 15, which is returned by the Ravikumar-Eisman algorithm.

For general $k$-ary Carlitz words, we still need only a 2-state DFA $A'$ rather than the naive $k + 1$ states. Suppose we number the start state of $A'$ as 1 and the other state as 2. Let $f_i(m), i = 1, 2$ be the number of $m$-words accepted by $A'$ if state $i$ were the start state. Either by converting to a regular grammar or using the transfer matrix method we get $f_2(m) = (k - 1)^m, f_1(m) = kf_2(m - 1)$ which allows us to conclude (the obvious) $f_1(m) = k(k - 1)^{m-1}$. △

![Figure 14: A DFA that accepts 3-ary Carlitz words over \{a, b, c\}.](image-url)
If we generalize $k$-ary Carlitz words to avoidance of the subword pattern $1^p$, a naive automaton $A$ with $k^p + O(1)$ states has weakly equivalent automaton $A'$ with $p + O(1)$ states including, for each $1 \leq i < p$, a state representing words ending with a run of length $i$. Similar phenomena are seen for other subword patterns, with the general theme that simpler patterns have simpler automata.

We can also consider the following refinement of weak equivalence for multivariate counting.

**Definition 2.6.** Take two DFAs $A$ and $A'$ that recognize a language over a $k$-ary alphabet. Then $A$ and $A'$ are completely weakly equivalent iff for all $j_1, \ldots, j_k$, the DFAs $A$ and $A'$ accept an equal number of words with $j_i$ letters $i$, for $1 \leq i \leq k$.

There is a brute-force algorithm for exact minimization of DFAs according to complete weak equivalence: Given a DFA $A$, enumerate all smaller DFAs $A'$ in ascending order by size. Extract the multivariate rational generating functions for $A$ and $A'$ where there is an indeterminate symbol marking each letter, and compare by subtracting and testing for 0.

A straightforward generalization of the faster Ravikumar-Eisman algorithm to the multivariate problem depends on generalizing Lemmas A1 and A2 in [69] from sequences of real numbers to sequences of real polynomials. We give this generalization.

**Lemma 2.9.** Let $\Xi$ be a finite nonempty index set. For all $\xi \in \Xi$ let $A_\xi : \mathbb{Z}_{\geq 0} \to \mathbb{R}[x_1, \ldots, x_k]$ satisfy

$$A_\xi(n + 1) = \sum_{t \in \Xi} c_{\xi,t} A_t(n), \quad n \geq 0$$

where $c_{\xi,t} \in \mathbb{R}[x_1, \ldots, x_k]$. Then each $A_\xi$ satisfies a linear difference (recurrence) equation of degree $|\Xi|$ or less with coefficients in $\mathbb{R}[x_1, \ldots, x_k]$.

**Proof.** The proof of Lemma A1 in [69] requires no modification to prove this result, except that linear algebra is done over $\mathbb{R}(x_1, \ldots, x_k)$ rather than $\mathbb{R}$. \hfill $\Box$

**Lemma 2.10.** Let $A, B : \mathbb{Z}_{\geq 0} \to \mathbb{R}[x_1, \ldots, x_k]$ be sequences satisfying linear difference equations of degrees $a$ and $b$ with coefficients in $\mathbb{R}[x_1, \ldots, x_k]$. If for $0 \leq n \leq a + b$ we have $A(n) = B(n)$ then the sequences $A(n)$ and $B(n)$ are identical for all $n$.

**Proof.** As above for [69, Lemma A2]. \hfill $\Box$

Figure 15: A DFA weakly equivalent to one that accepts 3-ary Carlitz words.
2.3 Note on weighted trees

The number of unweighted binary plane trees avoiding certain local structures is found in [60]. In that paper, §5 gives an algorithm to compute a system of algebraic equations specifying the relevant generating function. An extension to ternary and m-ary trees is in [26] §3. Enumeration of unweighted trees by number of local occurrences (not just avoidance) is done in [19]. The paper [21] considers global pattern avoidance, still in unweighted trees.

We say locally restricted trees weighted by a finite group are those that avoid subgraphs from a fixed set $\Xi$ of weighted trees, where the set $\Xi$ has a maximum size. We specifically consider rooted trees where there is a directed edge from parent to child. Plane trees correspond, for example, to the family of trees where the vertex set is $[m]$, parents are less than children, and all vertices at a given depth form a contiguous interval of integers. For non-abelian groups, there must be a stipulated ordering on tree vertices such as depth-first search, in order to define the total of a tree. We require that the order be recursive, in the sense that the total of the tree must be a sum made from the weight of the root and the sums of subtrees rooted at children of the root. Variations on trees with similar enumerative properties include functional graphs, directed acyclic graphs, and cactus graphs.

Example 2.7. Say we have a tree $T$ with weights from $\mathbb{Z}_2$ where the first 2 levels are of the form

$$
\begin{array}{c}
0 \\
/ \\
/ \\
0 & 0
\end{array}
$$

If $T$ avoids parent-child-grandchild paths with the same weight on all 3 vertices, sibling subtrees are independent but no level-3 vertex of $T$ may have the weight 0.

An alternative visual representation of weighted trees for groups with some total order uses a color behind vertices where darker means greater. This plotting technique works well for larger trees because of higher visibility, although it is less precise. An example is shown in Figure 16.

It is straightforward (albeit a little cumbersome) to show that the generating function $T_a(z)$, counting locally restricted trees with total $a \in G$, where $z$ marks number of vertices, is expressible in terms of a system of algebraic equations. This is, of course, common in tree enumeration as in examples in [60], [25] §1.5], [22], et cetera. The theory of coefficients of algebraic functions in, e.g. [25] §VII.6.1 may be applied, under conditions, to derive the usual $[z^n]T_a(z) \sim A_a m^{-1/2} B^n, A_a > 0, B > 1$. The works [19, 22] conduct analysis of the number of pattern occurrences in uniform random unweighted trees and show convergence in distribution to the standard normal after normalization; we expect that their method applies similarly in the present context.

Finally, we note that computing expansions of multivariate algebraic series
Newton iteration is a relatively efficient option. The package Genfunlib [10] for Mathematica implements Newton iteration as the command `CoefsByNewton`, but only for single equations, so there is a preliminary step of eliminating all but one component of the original system. An example expansion follows of the solution to $f(z, u) = u + z(f(z, u)^2 + f(z, u))$.

```
In[1]:= CoefsByNewton[
  f[z, u] == u + z (f[z, u]^2 + f[z, u]),
  f[z, u], {z, 0, 5}]
```

```
Out[1]= u + (u+u^2)z + (u+3 u^2+2 u^3)z^2
+ (u+6 u^2+10 u^3+5 u^4)z^3 + (u+10 u^2+30 u^3+35 u^4+14 u^5)z^4
+ (u+15 u^2+70 u^3+140 u^4+126 u^5+42 u^6)z^5 + O[z]^6
```
3 Locally cyclically restricted compositions

Remark 3.1. In the language of §1, cyclically restricted compositions are weighted directed cycles where we track the total weight. Weighted undirected cycles may be counted in a similar manner. Sets of cycles, i.e., 2-regular graphs, are also closely related. △

3.1 Compositions over a finite group

In §2.1 we represented compositions by walks on any de Bruijn subgraph \( D \) over \( \text{SEQ}_\sigma(G) \). Within the current section and §4.1 we slightly specialize the possibilities for \( D \) as follows. If \( \sigma \geq 1 \) is the span, let \( \bar{D} \) be the \( \sigma \)-dimensional de Bruijn graph on \( G \), let \( U \subset V(\bar{D}) \), and let \( D = \bar{D} - U \). Note that given such a digraph \( D \), the set \( U \) is uniquely determined.

An \( m \)-composition \( x \) is locally cyclically restricted according to \( D \) iff
\[
(x(1), \ldots, x(m), x(1), \ldots, x(\sigma - 1))
\]
avoids \( U \) as subwords. A number of observations about this definition should be made. First, \( m \)-compositions where \( m < \sigma \) do not correspond to walks over \( D \) but may or may not be cyclically restricted according to \( D \). Second, for \( m < \sigma \), we are technically departing from the isomorphic nature of pattern occurrences in the language of §1 and really this corresponds to homomorphic pattern occurrences. We do not remark on this point further.

Let \( \mathcal{C}_a(m; D) \) be the set of all \( m \)-compositions of \( a \) that are cyclically restricted according to \( D \), and define
\[
c_a(m; D) = |\mathcal{C}_a(m; D)|, \quad C_a(z; D) = \sum_{m \geq 0} c_a(m; D)z^m.
\]

Lemma 3.1. For \( v \in V(D) \), let \( \Sigma'v = v(1) + \cdots + v(\sigma - 1) \). For \( m \geq \sigma \) we have
\[
c_a(m; D) = \sum_{\substack{v \in V(D) \\text{for } u \in \mathcal{N}^{-}(v)}} p_{a+\Sigma'v}(m + \sigma - 1; D, \{v\}, \{u\}),
\]
for \( a \in G \).

Proof. If \( m \geq \sigma \), consider a walk \( w_1, \ldots, w_m \) in \( D_x \), where \( w_1 = (v, \Sigma v) \) and \((w_m, (v, a + \Sigma v)) \in E(D_x)\). Let \((x(1), \ldots, x(m + \sigma - 1))\) be the composition represented by the walk. Then \((x(1), \ldots, x(m))\) is precisely an \( m \)-composition of \( a \) which is cyclically restricted according to \( D \). □

Proposition 3.1. Fix an ordering on \( V(D_x) \) and let \( M_\times \) be the adjacency matrix of \( D_x \). For \( (v, a) \in V(D_x) \), let \( \xi_{v,a} \in \mathbb{R}^{[V(D_\times)]} \) be the indicator vector
for vertex \((v,a)\). Then for \(m \geq \sigma\),
\[
c_a(m; D) = \sum_{v \in V(D)} \sum_{u \in N^-(v)} (\xi_v, \Sigma_v)^\top M_\infty^{m-1} \xi_{u,a+\Sigma v}.
\]

**Proof.** This follows from Lemma 3.1 and Proposition 2.1. \(\square\)

**Proposition 3.2.** Assume \(D\) is regular. We have either 
\[
c_a(m; D) = 0
\]
for \(A, B > 0, 0 \leq \theta < 1\).

**Proof.** This follows from Lemma 3.1 and Proposition 2.4. \(\square\)

**Proposition 3.3.** Assume \(D_\infty\) is strongly connected and aperiodic. Then 
\[
c_a(m; D) = A \cdot B^m(1 + O(\theta^m))
\]
where \(A\) does not depend on \(a \in G\).

**Proof.** This follows from Lemma 3.1 and Theorem 2.1. \(\square\)

Let \(x = (x(1), \ldots, x(m))\) and \(y = (y(1), \ldots, y(\sigma))\) be compositions. A local cyclic occurrence of \(y\) in \(x\) is an occurrence of \(y\) as a subword in 
\((x(1), \ldots, x(m), x(1), \ldots, x(\sigma - 1))\).

**Theorem 3.1.** Assume \(U\) is nonempty and suppose \(D = \bar{D} - U\) is regular with strongly connected derived digraph \(D_\infty\).

For \(u \in V(\bar{D})\), let \(\mu(u)\) be the minimum number of occurrences of \(U\) in a composition in \(\mathcal{P}(\bar{D}, \{u\}, N^-(u))\) with at least 1 occurrence of \(V(D)\). Let \(\mu\) be the minimal such \(\mu(u)\). Assume for all sufficiently large values of \(m\) there exist compositions in \(\mathcal{P}(m; \bar{D}, V(D), V(D))\) with exactly 1 occurrence of \(U\), and that
\[
p(m; D, V(D), V(D)) \sim A \cdot B^m.
\]

If \(r \geq \max(\mu, 1), \mu \geq 0\) then the number of \(m\)-compositions of \(a \in G\) with exactly \(r\) cyclic occurrences of \(U\) is
\[
c_a(m, r; D) = m^{r-\mu} A_{r,\mu} \cdot B^m(1 + O(m^{-1}))\quad m \to \infty.
\]

**Proof.** The result follows from Lemma 3.1 and Theorem 2.3. \(\square\)

**Theorem 3.2.** Assume \(U\) is nonempty. The number of cyclic occurrences of \(U\) in a uniform random \(m\)-composition of \(a \in G\) is asymptotically normal with mean and variance asymptotic to those of the number of occurrences of \(U\) in a uniform random word over \(G\).

**Proof.** This follows directly from Proposition 2.11. \(\square\)

We consider some examples of cyclic restrictions.
Proposition 3.4. A composition \( x = (x(1), \ldots, x(m)) \) is a cyclic Carlitz composition iff \( (x(1), \ldots, x(m), x(1)) \) is a Carlitz composition. The number of cyclic Carlitz \( m \)-compositions of \( a \in G \) over a finite group \( G \) is
\[
\frac{(|G| - 1)^m}{|G|} (1 + O(\theta^m)), \quad m \to \infty, 0 \leq \theta < 1,
\]

provided \( |G| \geq 3 \).

Proof. First let us consider cyclic Carlitz \( m \)-words over \([k]\). Assume the first letter is \( k \). A cyclic Carlitz (or Smirnov) word is then a sequence of pairs of a single letter \( k \) followed by a non-empty Carlitz word on \([k-1]\). Let \( \bar{H}_k = kz/(1 - (k - 1)z) \) be the ordinary generating function for non-empty Carlitz words on \([k]\). Thus if \( F_k(z) \) is the ordinary generating function for cyclic Carlitz words on \([k]\), we have
\[
F_k(z) = k \frac{z \bar{H}_{k-1}(z)}{1 - z \bar{H}(z)} = k \frac{(k - 1)z^2}{(z + 1)(1 - (k - 1)z)}
\]
and
\[
[z^m]F_k(z) = (k - 1)^m + k(-1)^m + (-1)^{m+1}, \quad m > 1.
\]
The above derivation is a special case of Theorem 4 in [34]. It remains to recall from Proposition 2.7 that there is a digraph \( D \) representing Carlitz compositions such that \( D \times \) is aperiodic and strongly connected; Proposition 3.3 applies.

Let \( \Xi = \{\xi_1, \xi_2, \ldots\} \) be an ordered set. A sequence \( w = (w(1), \ldots, w(m)) \) over \( \Xi \) is \( p \)-smooth iff for all \( i = 1, \ldots, m - 1 \) if we have \( w(i) = \xi_j, w(i + 1) = \xi_k \) then \( |k - j| \leq p \). Additionally, \( w \) is \( p \)-smooth cyclic iff \( (w(1), \ldots, w(m), w(1)) \) is \( p \)-smooth. Research Direction 6.5 in [37, p. 239] asks for an explicit formula for the number of \( p \)-smooth cyclic \( k \)-ary words of length \( m \).

We apply Proposition 3.3 in the case \( p = 1 \).

Proposition 3.5. Let \( \mathbb{Z}_k \) have ordering \( 0, 1, \ldots, k-1 \). Let
\[
C(z) = 1 + \frac{kz(1 + 3z)}{(1 + z)(1 - 3z)} - \frac{2(k + 1)z}{(1 + z)(1 - 3z)} \frac{U_{k-1}(1/z)}{U_k(1/z)}
\]
be the ordinary generating function for \( k \)-ary 1-smooth cyclic words as in [37, Exercise 6.10] and [44, 43] where \( U_k \) is the \( k \)th Chebyshev polynomial of the second kind.

Let \( \rho > 0 \) be the radius of convergence of \( C(z) \), and let \( A = \lim_{z \to \rho} (1 - z/\rho)C(z) \). Then the number of 1-smooth cyclic \( m \)-compositions of \( i \in \mathbb{Z}_k \) is asymptotic to
\[
\frac{1}{k} A \cdot \left( \frac{1}{\rho} \right)^m (1 + O(\theta^m)), \quad m \to \infty, 0 \leq \theta < 1.
\]
Table 4: Exact counts of \( m \)-compositions of \( a \) cyclically avoiding 132 over \( \mathbb{Z}_5 \).

| \( m \) | \( a = 0 \) | \( a = 1 \) |
|-------|-----------|-----------|
| 3     | 19        | 19        |
| 4     | 85        | 85        |
| 5     | 390       | 385       |
| 6     | 1763      | 1763      |
| 7     | 8023      | 8016      |
| 8     | 36469     | 36469     |
| 9     | 165790    | 165790    |
| 10    | 753660    | 753660    |
| 11    | 3426039   | 3426039   |
| 12    | 15574231  | 15574231  |
| 13    | 70798118  | 70798118  |
| 14    | 321837325 | 321837325 |
| 15    | 1463023035| 1463023045|
| 16    | 6650677797| 6650677797|

Figure 17: Uniform-randomly generated 100-compositions of 0 (above) and 1 (below) over \( \mathbb{Z}_5 \) which cyclically avoid 132. (The vertical axis represents the value of a part.)

**Proof.** Let \( \tilde{D} \) be the de Bruijn graph on \( \mathbb{Z}_k^2 \), let \( U \subseteq \mathbb{Z}_k^2 \) be all \((a, b)\) which are not smooth, and let \( D = \tilde{D} - U \). In the derived graph \( D_\chi \), for any \( i \in \mathbb{Z} \), we exhibit a walk from \((0, 0), i\) to \((0, 0), 0\) by taking the following sequence of elements of \( \mathbb{Z}_k \). First, take 0, 1, 2, \ldots, \(-i - 1\), \(-i\), \(-i - 1\), \ldots, 2, 1, 0. Let \( c = 1 + 2 + \cdots + (-i) - 1 \). The sum of elements on this sequence is \( 2c - i \). Let \( n \) be the order of \( 2c \) in \( \mathbb{Z}_k \). Repeat the following \( n - 1 \) times: 0, 1, 2, \ldots, \(-i - 2\), \(-i - 1\), \(-i - 1\), \(-i - 2\), \ldots, 0, 0. The grand total of these sequences concatenated is \( 2c - i + (n - 1)c = -i \) and thus there is a walk in \( D_\chi \) starting at \((0, 0), i\) and ending at \((0, 0), 0\). The digraph \( D_\chi \) is clearly aperiodic since there is a loop at the vertex \((0, 0), 0\). Thus we may apply Proposition 3.3.

Remark 3.2. Wheel graphs are a variation on cycles with similar enumerative properties. A wheel graph consists of a cycle \( C \) with a vertex \( v \) added and (directed) edges from \( v \) to each vertex in \( C \). △
3.2 Note on integer compositions

Let \( x = (x(1), \ldots, x(m)) \) be an integer composition, i.e. \( x(i) \in \mathbb{Z}_{>0}, 1 \leq i \leq m \). To define locally restricted integer compositions with span \( \sigma \in \mathbb{Z}_{>0} \) we use a local restriction function \( R : \mathbb{Z}_{>0}^* \to \{0, 1\} \) which encodes the \( \sigma \)-tuples that are allowed as a subword inside an integer composition. If \( \text{SEQ}(\mathbb{Z}_{>0}) \) is the set of all integer compositions of any length, define \( \bar{R} : \text{SEQ}(\mathbb{Z}_{>0}) \to \{0, 1\} \) so that \( \bar{R}(x) = 1 \) iff \( R(x(i), x(i+1), \ldots, x(i+\sigma-1)) = 1 \) for all \( 1 \leq i \leq m - \sigma + 1 \), in which case \( x \) is allowed according to \( R \). As an expedient it is also helpful to define an infinite digraph \( \mathcal{D} \) with vertex set \( V(\mathcal{D}) = R^{-1}(1) \) and where \( (u, v) \in E(\mathcal{D}) \) iff \( \bar{R}(uv) = 1 \). Note that a walk in \( \mathcal{D} \) represents a composition obtained by concatenating the vertices; as such, walks do not represent all restricted compositions, only those whose length is a multiple of \( \sigma \). In this way the infinite digraph \( \mathcal{D} \) is interpreted differently from the de Bruijn graphs used in other sections. We assume there is some vertex ordering \( V(\mathcal{D}) = \{v_1, v_2, \ldots\} \).

We define the transfer operator \( T(z) \) formally as the infinite matrix where \( [T(z)]_{i,j} = ((v_i, v_j) \in V(\mathcal{D}))z^{\Sigma v_i + \Sigma v_j} \).

A research direction suggested in \([34, \S 4]\) is developing a framework for locally cyclically restricted integer compositions. The framework for locally restricted integer compositions in \([4]\) can be used with little modification. In this section we follow the definitions of \([4]\) with some simplifications.

We say that \( x \) is a composition which is cyclically restricted by \( R \) iff 
\[
\bar{R}(x(1), \ldots, x(m), x(1), \ldots, x(\sigma)) = 1.
\]

The endpoint operator \( E(z, y) \) is a formal infinite matrix defined by
\[
[E(z, y)]_{i,j} = \sum_{k \geq 1} y^{2\sigma} [T(z)]_{k,i} \sum_{x} z^{\Sigma v_j + 2\Sigma x + \Sigma v_k} y^{2\sigma + 2|x|},
\]
where the second sum ranges over compositions \( x \) with length in \( \{0, \ldots, \sigma - 1\} \) such that \( \bar{R}(v_jxv_k) = 1 \). The endpoint operator plays the role of the start and finish vectors of \([4]\).

In this section, \( \sum_{i,j \geq 1} f_{i,j}(z) = \lim_{N \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{N} f_{i,j}(z) \). The analogous definition is used for triple sums. (By \([39]\) Theorem 4.48 we have the basic fact that \( \sum_{i \geq 1} \sum_{j \geq 1} |s_{i,j}| = \sum_{i,j \geq 1} |s_{i,j}| \) for \( s_{i,j} \in \mathbb{C} \), provided at least one side converges.)

**Proposition 3.6.** Let \( S(z, y) = \sum_{j \geq 0} (y^{2\sigma})^j T(z)^j \). Define
\[
C(z^2, y^2) = \sum_{i,j \geq 1} [S(z, y)]_{i,j} [E(z, y)]_{i,j}.
\]

Then for \( m \geq 3\sigma \), the coefficient \([z^m y^m]\) \( C(z, y) \) is the number of integer \( m \)-compositions of \( n \) that are cyclically restricted according to \( R \).
Proof. We have
\[
\sum_{i,j \geq 1} [S(z, y)]_{i,j} [E(z, y)]_{i,j} = \sum_{k,i,j \geq 1} y^\sigma z^{\Sigma v_k} y^{2\sigma} [T(z)]_{k,i} [S(z, y)]_{i,j} y^\sigma z^{\Sigma v_j} \sum_x z^{2\Sigma x} y^{2|x|}.
\]
Consider a term
\[
y^\sigma z^{\Sigma v_k} y^{2\sigma} [T(z)]_{k,i} [S(z, y)]_{i,j} y^\sigma z^{\Sigma v_j} \sum_x z^{2\Sigma x} y^{2|x|}.
\]
This is the generating function for restricted compositions of the form
\[
v_k v_i w v_j x v_k,
\]
where \(w\) is a concatenation of vertices in \(V(D)\), with \(x^2\) marking total sum and \(y^2\) marking length, and such that the final \(v_k\) does not count. Summing over all \(i, j, k\) enumerates cyclically restricted \(m\)-compositions where \(m \geq 3\sigma\).

We now wish to extract asymptotic information from \(C(z, y)\). The technicalities that arise come almost entirely from the transfer operator rather than the endpoint operator, and this analysis is available in [4] which uses advanced tools from functional analysis that generalize finite dimensional matrix theory. Here we simply provide the required minor adaptations of [4]. The use of \(C(z^2, y^2)\) rather than \(C(z, y)\) is a technical requirement for some manipulations of the infinite matrices.

**Definition 3.1.** We say a local restriction function \(R\) is regular iff the following hold.

1. The infinite digraph \(D\) contains at least two vertices, is strongly connected, and is aperiodic.
2. There is \(p \in \mathbb{Z}_{>0}\) and (possibly equal) vertices \(u, v\) such that
   \[\gcd\{m - n : m, n \in \Xi\} = 1,\]
   where
   \[\Xi = \{\Sigma x_1 + \cdots + \Sigma x_{p-1} : u, x_1, \ldots, x_{p-1}, v \text{ is a walk in } D\}.\]

**Definition 3.2.** Let \(\Omega \subseteq \mathbb{C}\) be a domain. Then \(\mathcal{M}(\Omega)\) is the set of infinite matrices \(M(z)\) such that each entry \([M(z)]_{i,j}\) is holomorphic in \(\Omega\) and such that for every compact \(K \subseteq \Omega\) there exists \(c > 0\) with
\[
\sum_{i,j \geq 1} ||[M(z)]_{i,j}||^2 \leq c, \quad \forall z \in K.
\]

**Proposition 3.7.** Let \(R\) be a regular local restriction function, and let \(T(z)\) and \(E(z, y)\) be the associated transfer and endpoint operators. Let \(S(z, 1) = \sum_{j \geq 0} T(z)^j\). Then \(S(z, 1)\) has radius of convergence \(r\) satisfying \(0 < r < 1\). There exists a domain \(\Omega \supseteq \{z : |z| \leq r, z \neq \pm r\}\) such that \(S(z, 1), E(z, 1) \in \mathcal{M}(\Omega)\).
Theorem 14.6] implies that since \( \Omega \) can be taken to be a subset of \( \{ z : \text{spr}(T(z)) < 1 \} \), where \( \text{spr}(T(z)) \) is the spectral radius of \( T(z) \).

\[ \square \]

Proposition 3.8. Let \( \Omega \subseteq \mathbb{C} \) be a domain. If \( S(z), E(z) \in \mathcal{M}(\Omega) \), then \( \sum_{i,j \geq 1} |S(z)|_{i,j} |E(z)|_{i,j} \) is holomorphic on \( \Omega \).

Proof. This follows from [4, Proposition 4 (b)] if we suitably interpret \( S(z) \) and \( E(z) \) as vectors and \( \sum_{i,j \geq 1} |S(z)|_{i,j} |E(z)|_{i,j} \) as their dot product. The proof is omitted in [4] so we include a direct proof here for good measure.

We use \( c(z) \) to denote the sum \( \sum_{i,j \geq 1} |S(z)|_{i,j} |E(z)|_{i,j} \). Let \( K \subset \Omega \) be compact, and let \( c_1, c_2 > 0 \) satisfy

\[
\sum_{i,j \geq 1} |(S(z))_{i,j}|^2 \leq c_1, \quad \sum_{i,j \geq 1} |(E(z))_{i,j}|^2 \leq c_2
\]

for all \( z \in K \). Then by the Cauchy-Schwarz inequality,

\[
\left( \sum_{i,j \geq 1} |(S(z))_{i,j} |(E(z))_{i,j}| \right)^2 \leq c_1 c_2, \quad \forall z \in K.
\]

This implies that the partial sums \( \sum_{i=1}^{N} \sum_{j=1}^{N} |S(z)|_{i,j} |E(z)|_{i,j} \) of \( c(z) \) are uniformly bounded on compact \( K \) since

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} |(S(z))_{i,j} |(E(z))_{i,j}| \leq \sum_{i=1}^{N} \sum_{j=1}^{N} |(S(z))_{i,j} |(E(z))_{i,j}| \leq \sum_{i,j \geq 1} |(S(z))_{i,j} |(E(z))_{i,j}| \leq \sqrt{c_1 c_2}.
\]

In particular, we have pointwise absolute convergence of \( c(z) \) on \( \Omega \).

By [61, Theorem 10.28] it suffices to show that \( c(z) \) converges uniformly on compact subsets of \( \Omega \). If \( \mathcal{F} \) is the set of all partial sums of \( c(z) \), then [61, Theorem 14.6] implies that since \( \mathcal{F} \) is uniformly bounded on compact subsets of \( \Omega \), every sequence in \( \mathcal{F} \) has a subsequence that converges uniformly on compact subsets of \( \Omega \). Let \( s^{(n)} \) be a sequence in \( \mathcal{F} \) that converges pointwise to \( c(z) \).

For a given compact \( K \), every subsequence of \( s^{(n)}_{|K} \) that converges uniformly must converge to the same limit, namely \( c_K(z) \). Therefore \( s^{(n)}_{|K} \) itself converges uniformly.

\[ \square \]

Proposition 3.9. Assume \( R \) is a regular local restriction function with transfer operator \( T(z) \), endpoint operator \( E(z, y) \), and generating function \( C(z, y) \). Let \( S(z, 1) = \sum_{j} T(z)^j \) be an infinite matrix with radius of convergence \( 0 < r < 1 \).

If \( C(z^2, 1) = \sum_{i,j} [S(z, 1)]_{i,j} |E(z, 1)|_{i,j} \) is holomorphic for \( |z| \leq r \) with possible exceptions \( z = \pm r \), then \( C(z, 1) \) has radius of convergence \( \sqrt{r} \) and is analytic for \( |z| = \sqrt{r} \) except for a simple pole at \( z = \sqrt{r} \).
Proof. See Theorems 1–3 and their proofs in [4]. □

Proposition 3.10. If $R$ is a regular local restriction function with generating function $C(z,y)$ enumerating locally cyclically restricted integer compositions, then

$$[z^n]C(z,1) = A \cdot B^n(1 + O(\theta^n)), \quad n \to \infty, A > 0, B > 1, 0 \leq \theta < 1.$$  

Proof. This is direct from Propositions 3.7, 3.9 and an application of meromorphic generating function asymptotics. Since $C(z,1)$ has a single pole on $|z| = r$, there is $\epsilon > 0$ such that $C(z,1)$ is meromorphic on $|z| \leq r + \epsilon$ and we may employ [25, Theorem IV.10]. □

We consider some examples.

Lemma 3.2. There is a regular local restriction function associated with integer compositions avoiding a non-trivial (length $\geq 2$) subword pattern $\tau$, as long as $\tau$ is not $12^p - 1$ or its symmetries ($12^p$, $2p1$, and $21^p$).

Proof. We verify the conditions in Definition 3.1 for the corresponding local restriction function $R$ of span $\sigma = |\tau|$. We consider condition 1. Aperiodicity is particularly easy: unless $\tau = 1^p$, there is a vertex $1^p$ with a loop. If $\tau = 1^p$, the vertex $1^{|\sigma/2|}2^{|\sigma/2|}$ has a loop. Strong connectedness follows from Lemmas 2.7 and 2.8.

For condition 2, we look at the following cases. For $\tau = 1^p$, take $v_1 = v_2 = 1^{p-1}$. Two walks of equal length with sum differing by $1$ are given by $1^{p-1}2, 3^{p-1}4, 1^{p-1}2$ and $1^{p-1}2, 3^{p-1}2, 1^{p-1}2$. For $\tau = 1^p2^q$ with $p, q \geq 2$, we take the walks $1^p2^q$, $23 \cdots \sigma(\sigma + 1), 1^p$ and $1^p2^q$, $23 \cdots \sigma(\sigma + 2), 1^p$. For other $\tau$, two suitable walks are given by $1^p, 1^{p-1}2, 2^p, 1^p$ and $1^p, 1^{p-2}22, 2^p, 1^p$. □

Example 3.1. By Corollary 5 in [34], the generating function for cyclic Carlitz compositions with $z$ marking the total is

$$C(z) = \frac{\sum_{n=1}^{\infty} \frac{z^n}{(z^{n+1})^2}}{1 - \sum_{n=1}^{\infty} \frac{z^n}{z^{n+1}}} + \sum_{n=1}^{\infty} \frac{z^{2n}}{z^n + 1}.$$  

Proposition 3.10 implies the following. Let $\rho > 0$ be the radius of convergence of $C(z)$, and let $A = \lim_{z \to \rho}(1 - z/\rho)C(z)$. We have $[z^n]C(z) = A \cdot (1/\rho)^n (1 + O(\theta^n))$. △

Research Direction 4.4 in [37] begins as follows. “We say that a sequence (composition, word, partition) $s_1 \cdots s_m$ cyclically avoids a subword $\tau = \tau_1 \cdots \tau_k$ if $s_1 \cdots s_m s_1 \cdots s_{k-1}$ avoids $\tau$. For example, the composition 33412 avoids the subword 123, but does not cyclically avoid 123 (since 3341233 contains 123).” The problem is to find the generating function for the number of compositions of $n$ that cyclically avoid a subword pattern of length $k$. Lemma 3.2 implies that we usually get a regular local restriction function for subword pattern avoidance. We consider the patterns 122 and 321.
Example 3.2. Compositions cyclically avoiding 122 over \([k]\) take the following form. Either there is no part \(k\), the composition only contains \(k\), or there is at least one \(k\) and at least one other part. In this third case, the subwords between any parts \(k\) are nonempty 122-avoiding integer compositions over \([k - 1]\) and so is the composition obtained by concatenating the subword after the final \(k\) and the subword before the first \(k\).

Let \(C_k(z, u)\) be the generating function for nonempty cyclic 122-avoiding compositions where \(z\) marks total and \(u\) marks length, and let \(P_k(z, u)\) be the generating function for nonempty 122-avoiding compositions. The above reasoning yields

\[
C_k(z, u) = C_{k-1}(z, u) + \frac{uz^k}{1 - uz^k} + uz^k \frac{1}{1 - P_{k-1}(z, u)uz^k}(uD_u + 1)P_{k-1}(z, u).
\]

The generating function \(P_k(z, u)\) is given in \([37, \text{Theorem 4.35}]\) as

\[
P_k(z, u) = \left(1 - \sum_{j=1}^{k} z^j u \prod_{i=j+1}^{k} (1 - z^{2i} u^2)\right)^{-1} - 1.
\]

Let \(C(z) = \lim_{k \to \infty} C_k(z, 1)\). Since \(\tau = 122\) does not correspond to a strongly connected digraph \(D\), Proposition \(3.10\) does not apply to \(C(z)\). The coefficients \([z^n]C(z)\) for \(n = 1, \ldots, 10\) are 1, 2, 4, 8, 13, 28, 52, 101, 196, 383. \(\triangle\)

Example 3.3. For \(\tau = 321\), we consider two counting sequences. Let \(^{\sim}21\) be the pattern 21 except that it only counts if it appears at the beginning of a composition. We count compositions over \([k]\) that avoid both \(321\) and \(^{\sim}21\). Such a composition either has no parts \(k\) or has at least one \(k\). In the latter case, say the composition can be written \(\sigma_1 k\sigma'\), where \(\sigma_1\) is a composition on \([k - 1]\) and avoids \(\{321, ^{\sim}21\}\), and \(\sigma'\) is a composition on \([k]\) avoiding \(\{321, ^{\sim}21\}\). If the composition \(\sigma_1\) is empty then either \(\sigma'\) is empty or \(k\sigma' = kk\sigma''\) where \(\sigma''\) is a composition on \([k]\) avoiding \(\{321, ^{\sim}21\}\). This method proceeds similarly to the proof of Lemma 4.29 in \([37]\). Let \(\bar{P}_k(z, u)\) be the generating function for compositions avoiding \(\{321, ^{\sim}21\}\) where \(z\) marks total and \(u\) marks length. This gives

\[
\bar{P}_k(z, u) = \bar{P}_{k-1}(z, u) + (\bar{P}_{k-1}(z, u) - 1)uz^k \bar{P}_k(z, u) + uz^k + u^2 z^{2k} \bar{P}_k(z, u).
\]

Now we go back to compositions cyclically avoiding just 321. Case 1: The composition has no part \(k\). Case 2: There are at least 2 parts \(k\). Such a composition can be written \(\sigma_1 k\sigma' k\sigma_2\), where \(\sigma'\) avoids \(\{321, ^{\sim}21\}\) and \(\sigma_2\sigma_1\) is a composition over \([k - 1]\) avoiding \(\{321, ^{\sim}21\}\). Case 3: There is 1 part \(k\). Then the composition is \(\sigma_1 k\sigma_2\) where \(\sigma_2\sigma_1\) is a composition over \([k - 1]\) avoiding \(\{321, ^{\sim}21\}\). If \(C_k(z, u)\) is the generating function for compositions cyclically avoiding 321, we have

\[
C_k(z, u) = C_{k-1}(z, u) + uz^k \bar{P}_k(z, u)uz^k(uD_u + 1) \bar{P}_{k-1}(z, u)
+ uz^k(uD_u + 1) \bar{P}_{k-1}(z, u).
\]

If we let \(C(z) = \lim_{k \to \infty} C_k(z, 1)\) then again, Proposition \(3.10\) and Lemma \(3.2\) imply \([z^n]C(z) \sim A \cdot B^n\) with \(A, B > 0\) determined by \(C(z)\). \(\triangle\)
The method of random generation given in Remark 2.2 achieves an exact uniform distribution but for compositions over an infinite set such as $\mathbb{Z}_{>0}$ its performance becomes poor. Instead we employ a Markov chain Monte Carlo (MCMC) method inspired by the article [48] which concerns pattern-avoiding permutations.

The method is as follows. Let $\tau$ be a permutation pattern, i.e. where no letters are repeated, and assume the length of $\tau$ is at least 3. Let $n, m > 0$ be fixed, where $n$ represents a total and $m$ represents a length. (The length $m$ can itself be randomly chosen first using exact counting.) Assume $X_0$ is an $m$-composition of $n$ with at most 2 distinct part sizes. Given $X_h$, $h \geq 0$, we generate $X_{h+1}$ as follows. Let $j, k$ be independently selected uniformly at random from $[m]$. If $X_h(j) = 1$ or $j = k$, then $X_{h+1} = X_h$. Otherwise, let $Y$ be the following composition. We have $Y(j) = X_h(j) - 1$, $Y(k) = X_h(k) + 1$, and $Y(i) = X_h(i)$ for $i \neq j, k$. If $Y$ avoids $\tau$, then $X_{h+1} = Y$, otherwise $X_{h+1} = X_h$. A composition avoiding 123 generated by this procedure is shown in Figure 18.

**Proposition 3.11.** The limiting distribution of the Markov chain $X_h$ is uniform over $m$-compositions of $n$ that avoid $\tau$.

**Proof.** By the theory of Markov chains [12, Ex. 8.20], it suffices to show that $X_h$ is aperiodic, irreducible, and has symmetric transition probabilities. Let $p(x, y)$ be the transition probability from a composition $x$ to $y$. Aperiodicity is clear since $p(x, x) > 0$. For symmetry, if $x \neq y$ are compositions with $p(x, y) > 0$, then $p(x, y) = p(y, x) = 1/m^2$. With symmetry established, irreducibility requires that for any $x$ there is a sequence of transitions with nonzero probability that lead from $x$ to, say, $X_0$. We construct such a sequence. Repeat the following until there are at most 2 distinct part sizes, at which point reaching $X(0)$ is clearly possible. Let $y$ be the current composition and let $K = |\tau|$. Let $j$ be the index of the maximum part in $y$; if this is not unique, take the least such index if $(K, K - 1)$ is a subsequence of $\tau$ and take the greatest such index if $(K - 1, K)$ is a subsequence of $\tau$. Let $k$ be the index of the minimum part in $y$; if this is not unique, take the least such index if $(1, 2)$ is a subsequence of $\tau$, and take the greatest such index if $(2, 1)$ is a subsequence of $\tau$. Decrement $y(j)$ and increment $y(k)$. \(\square\)
4 Locally restricted compositions with symmetry

Here we consider locally restricted compositions with symmetry, which corresponds to local patterns in unlabeled weighted digraphs, in the language of §1. In this section, groups $G$ are assumed to be abelian, since the order of the parts in a composition is no longer well defined.

Although we do not directly invoke it here, general counting with symmetry typically involves Burnside’s lemma.

**Lemma 4.1 (Burnside).** The number of orbits of a permutation group $S$ on a set $X$ is

$$|X/S| = \frac{1}{|S|} \sum_{s \in S} \text{fix}(s),$$

where $\text{fix}(s)$ is the number of fixed points of $s$.

Further background may be found in [47, §6].

4.1 Circular compositions

As in §3.1 here $G$ is a finite group and $\bar{D}$ is a $\sigma$-dimensional de Bruijn graph over $G$. We speak of digraphs $D = \bar{D} - U$ for some $U \subset V(\bar{D})$.

**Lemma 4.2.** Assume $x$ is a composition and $x = u \cdots u = u^d$. If $u$ is a subword containing $r$ cyclic occurrences of $U$, then $x$ contains $dr$ cyclic occurrences of $U$.

**Proof.** A cyclic occurrence of $U$ in a composition is fully determined by the starting index. All cyclic occurrences of $U$ in $x$ must correspond to an occurrence in some $u$, and vice versa. \qed

The circular shift of the finite sequence $(x(1), \ldots, x(m))$ is

$$(x(j), x(j + 1), \ldots, x(m), x(1), x(2), \ldots, x(j)),$$

for some $1 \leq j \leq m$. A circular composition is an equivalence class of cyclically restricted compositions where the equivalence is under circular shift. For example, there are two possible circular Carlitz 3-compositions over $\mathbb{Z}_3$, each with the same total:

$$\{012, 201, 120\}, \{021, 210, 102\}.$$

Let $\hat{\mathcal{C}}_a(m; D)$ be the set of all circular $m$-compositions of $a$ that are cyclically restricted according to $D$, and define

$$\hat{c}_a(m; D) = |\hat{\mathcal{C}}_a(m; D)|, \quad \hat{C}_a(z; D) = \sum_{m \geq 0} \hat{c}_a(m; D)z^m.$$
Let $P = \mathbb{Z}_{>0} \times G$ be the poset where $(j, a) \preceq (k, b)$ iff $j|k$ and $(k/j)a = b$. The Moebius function $\mu_P$ of $P$ is defined recursively by $\mu_P(s, s) = 1$ for $s \in P$ and $\mu_P(s, u) = -\sum_{s \preceq t < u} \mu_P(s, t)$ for $s < u$ in $P$. A finite sequence is aperiodic iff it is not equal to any of its circular shifts.

**Proposition 4.1.** We have

$$\tilde{c}_a(m; D) = \sum_{(d, b) \leq (m, a)} \frac{1}{d} \sum_{(d', b') \leq (d, b)} c_{b'}(d', D) \mu_P((d', b'), (d, b)).$$

**Proof.** Let $\text{acyc}(m, a)$ be the number of aperiodic cyclically restricted $m$-compositions of $a$. For any $m$-composition $x$ of $a \in G$, we have $x = u \cdots u = u^{m/d}$ for some aperiodic $u$ and some $d$ which divides $m$, by [66, Theorem 2.3.4]. Thus by Lemma 4.2,

$$c_a(m; D) = \sum_{(d, b) \geq (m, a)} \text{acyc}(d, b).$$

By the Moebius inversion formula [68, Proposition 3.7.1],

$$\text{acyc}(m, a) = \sum_{(d, b) \leq (m, a)} c_b(d; D) \mu_P((d, b), (m, a)).$$

Now, a circular composition consists of all possible shifts of some composition $x = u^{m/d}$ where $u$ is aperiodic, by [66, Theorem 2.4.2], so

$$\tilde{c}_a(m; D) = \sum_{(d, b) \leq (m, a)} \frac{1}{d} \text{acyc}(d, b),$$

which gives the result. □

**Theorem 4.1.** Assume $D$ is regular and $c_a(m; D) \sim A_a \cdot B^m$ for $a \in G$. We have

$$\tilde{c}_a(m; D) = \frac{1}{m} A_a \cdot B^m (1 + O(\omega^m)), \quad m \to \infty, 0 \leq \omega < 1.$$ 

All but an exponentially small proportion of $\tilde{C}_a(m; D)$ and $C_a(m; D)$ are aperiodic. If $D$ satisfies the assumptions of Theorem 2.1, then $A_a$ does not depend on $a$.

**Proof.** From above we know

$$\tilde{c}_a(m; D) = \sum_{(d, b) \leq (m, a)} \frac{1}{d} \text{acyc}(d, b),$$

where $\text{acyc}(m, a)$ is the number of aperiodic cyclically restricted $m$-compositions of $a$. We claim that $\tilde{c}_a(m; D) \sim \frac{1}{m} \text{acyc}(m, a) \sim \frac{1}{m} c_a(m; D)$.

From Proposition 3.2 we have $c_a(m; D) = A_a \cdot B^m (1 + O(\theta^m))$, where $B > 1$. Now

$$\sum_{(d, b) \neq (m, a)} \frac{1}{d} \text{acyc}(d, b) \leq \sum_{(d, b) \neq (m, a)} \frac{1}{d} c_b(d; D)$$

$$\leq |G| \frac{m}{2} A_a \cdot B^{m/2} \left(1 + O(\max(B^{-m/2}, \theta^{m/2}))\right).$$
On the other hand, we have
\[ c_a(m; D) \geq \text{acyc}(m, a) = c_a(m; D) - \sum_{(d, b) \prec (m, a)} \text{acyc}(d, b) \geq c_a(m; D) - \sum_{(d, b) \prec (m, a)} c_b(d; D), \]
and so
\[ - \sum_{(d, b) \prec (m, a)} c_b(d; D) \leq \text{acyc}(m, a) - c_a(m; D) \leq 0. \]

Thus \(\text{acyc}(m, a) = A_a \cdot B^m(1 + O(\omega^m))\) where \(\omega = \max(\theta, B^{-1/2})\).

**Theorem 4.2.** Assume \(U\) is nonempty and suppose \(D = \bar{D} - U\) is regular with strongly connected derived digraph \(D_x\).

For \(u \in V(\bar{D})\), let \(\mu(u)\) be the minimum number of occurrences of \(U\) in a composition in \(P(\bar{D}, \{u\}, N^{-}(u))\) with at least 1 occurrence of \(V(D)\). Let \(\mu\) be the minimal such \(\mu(u)\). Assume for all sufficiently large values of \(m\) there exist compositions in \(P(m; \bar{D}, V(D), V(D))\) with exactly 1 occurrence of \(U\), and that \(p(m; D, V(D), V(D)) \sim A \cdot B^m\).

If \(r \geq \max(\mu, 1), \mu \geq 0\) then the number of circular \(m\)-compositions of \(a \in G\) with exactly \(r\) cyclic occurrences of \(U\) is
\[ \tilde{c}_a(m, r; D) = m^{r-\mu-1}A_{r, \mu} \cdot B^m(1 + O(m^{-1})) \quad m \to \infty. \]

**Proof.** Let \(Q = \mathbb{Z}_{\geq 0} \times G \times \mathbb{Z}_{\geq 0}\) be a poset where \((j_1, a, j_2) \preceq (k_1, b, k_2)\) iff \(j_1 \mid k_1, (k_1/j_1)a = b, \) and \((k_1/j_1)j_2 = k_2\). The Moebius function \(\mu_Q\) of \(Q\) is defined recursively by \(\mu_Q(s, s) = 1\) for \(s \in Q\) and \(\mu_Q(s, u) = -\sum_{s \preceq t < u} \mu_Q(s, t)\) for \(s < u\) in \(Q\). By analogy to Proposition 4.1 we have
\[ \tilde{c}_a(m, r) = \sum_{(d_1, b, d_2) \preceq (m, a, r)} \frac{1}{d} \sum_{(d'_1, b', d'_2) \preceq (d_1, b, d_2)} c_{b'}(d'_1, d'_2; D) \mu_Q((d'_1, b', d'_2), (d_1, b, d_2)). \]

Following the proof of Theorem 4.1, the dominant term is \(m^{-1}c_a(m, r; D)\), so we conclude with reference to Theorem 3.1.

**Definition 4.1.** A mixture of two random variables \(X, Y\) with weights \(0 \leq p, 1 - p \leq 1\) is a random variable \(Z\) such that the distribution functions satisfy 
\[ F_Z(x) = pF_X(x) + (1 - p)F_Y(x), x \in \mathbb{R}. \]

The following lemma is an expedient used to show when normalized convergence in distribution holds up to low-probability events.

**Lemma 4.3.** Let \(X_n, Y_n \geq 0\) be \(L^2\) random variables for \(n \in \mathbb{Z}_{\geq 0}\). Let \(Z_n\) be a mixture of \(X_n\) and \(Y_n\) with weights \(p_n\) and \(1 - p_n\), where \(p_n \to 1\). Assume that \(E(X_n)\) or \(E(Z_n)\) are bounded away from 0, and that \(\text{Var}(Z_n)\) or \(\text{Var}(X_n)\) are bounded away from 0, and that
\[ (1 - p_n) \left( E(Y_n^2) + E(Y_n)E(X_n) + E(X_n^2) \right) = o(1). \]
Then we have $(X_n - E(X_n))/\sqrt{\text{Var}(X_n)} \Rightarrow F$ iff $(Z_n - E(Z_n))/\sqrt{\text{Var}(Z_n)} \Rightarrow F$, and $E(X_n) \sim E(Z_n)$, $\text{Var}(X_n) \sim \text{Var}(Z_n)$.

Proof. We have $E(Z_n) = p_n E(X_n) + (1 - p_n) E(Y_n)$, and in general

$$E(Z_n^2) = 2 \int_0^\infty xP(Z_n > x)dx$$
$$= 2 \int_0^\infty xp_n P(X_n > x) + x(1 - p_n) P(Y_n > x)dx$$
$$= p_n E(X_n^2) + (1 - p_n) E(Y_n^2)$$

by [59] Ex. 22b. Now

$$\text{Var}(Z_n) = E(Z_n^2) - E(Z_n)^2$$
$$= p_n E(X_n^2) + (1 - p_n) E(Y_n^2) - (p_n E(X_n) + (1 - p_n) E(Y_n))^2$$
$$= p_n \text{Var}(X_n) + (1 - p_n) \text{Var}(Y_n) + p_n(1 - p_n)(E(X_n) - E(Y_n))^2.$$ 

From the assumptions, we know $p_n \text{Var}(X_n) \sim \text{Var}(X_n)$ since $\text{Var}(X_n)$ is bounded away from 0. Also, $(1 - p_n) \text{Var}(Y_n) \leq (1 - p_n) E(Y_n^2) = o(1)$. And

$$p_n(1 - p_n)(E(X_n) - E(Y_n))^2 \leq (1 - p_n) 2(E(X_n)^2 + E(Y_n)E(X_n) + E(Y_n)^2)$$
$$\leq (1 - p_n) 2(E(X_n^2) + E(Y_n)E(X_n) + E(Y_n^2))$$
$$= o(1).$$

Thus

$$E(Z_n) \sim E(X_n) \text{ and } \text{Var}(Z_n) \sim \text{Var}(X_n).$$

Let $C(F) \subseteq \mathbb{R}$ be the points where $F$ is continuous. By Slutsky’s theorem we have

$$\frac{X_n - E(X_n)}{\sqrt{\text{Var}(X_n)}} \Rightarrow F \text{ iff } \frac{X_n - E(Z_n)}{\sqrt{\text{Var}(Z_n)}} \Rightarrow F$$

and

$$\frac{Z_n - E(Z_n)}{\sqrt{\text{Var}(Z_n)}} \Rightarrow F \text{ iff } \frac{Z_n - E(X_n)}{\sqrt{\text{Var}(X_n)}} \Rightarrow F.$$

And thus for $x \in C(F),$

$$\lim_{n \to \infty} F_{Z_n}\left(\sqrt{\text{Var}(Z_n)}x + E(Z_n)\right) = \lim_{n \to \infty} p_n F_{X_n}\left(\sqrt{\text{Var}(Z_n)}x + E(Z_n)\right)$$
$$+ (1 - p_n) F_{Y_n}\left(\sqrt{\text{Var}(Z_n)}x + E(Z_n)\right)$$
$$= \lim_{n \to \infty} p_n F_{X_n}\left(\sqrt{\text{Var}(Z_n)}x + E(Z_n)\right)$$
$$= \lim_{n \to \infty} F_{X_n}\left(\sqrt{\text{Var}(Z_n)}x + E(Z_n)\right)$$
$$= \lim_{n \to \infty} F_{X_n}\left(\sqrt{\text{Var}(X_n)}x + E(X_n)\right). \quad \square$$
Then the number of cyclic occurrences of asymptotic to those of the number of occurrences of word over where we apply the asymptotics of the Stirling subset numbers \([54]\). The first which is the number of parts missing between the minimum and maximum positions and aperiodic circular compositions have the same distribution of number of occurrences of \(G\) in a uniform random \(m\)-composition, whether circular or not, is aperiodic. Clearly aperiodic compositions and aperiodic circular compositions have the same distribution of number of occurrences of \(U\).

**Lemma 4.3** can then be applied to conclude \((X_n^2 - E(X_n^2))/\text{Var}(X_n^2) \Rightarrow N(0,1)\) and from there \((X_n^3 - E(X_n^3))/\text{Var}(X_n^3) \Rightarrow N(0,1)\). \(\square\)

Some examples of circular objects follow.

**Example 4.1.** For a composition \(x = (x(1), \ldots, x(m))\), we define

\[
gap(x) = \max_i x(i) - \min_i x(i) + 1 - |\{x(i) : i = 1, 2, \ldots, m\}|,
\]

which is the number of parts missing between the minimum and maximum parts of \(x\). If \(\gap(x) = 0\) we say \(x\) is gap-free. Research Direction 3.1 parts (3) and (4) in [37, p. 86] ask for an explicit generating function for the number of circular compositions/words \(x\) such that \(\gap(x) = \ell\).

Let \(c(m)\) be the number of gap-free \(k\)-ary words and let \(\tilde{c}(m)\) be the number of circular gap-free words. The number of gap-free \(k\)-ary words with \(j\) distinct letters is \((k - j + 1)j!\left(\text{\binom{m}{j}}\right)\). Thus

\[
c(m) = \sum_{j=1}^{k} (k - j + 1)j! \left(\text{\binom{m}{j}}\right) \sim \sum_{j=1}^{k} (k - j + 1)j^m \sim k^m,
\]

where we apply the asymptotics of the Stirling subset numbers \([54]\). The first letter in a gap-free \(m\)-word is arbitrary if the remaining \((m - 1)\)-word has \(k\) distinct letters. The number of such words is \(k! \left(\text{\binom{m-1}{k}}\right) \sim k^{m-1}\), so the first letter is arbitrary in almost all gap-free words. Thus for an abelian finite group \(G\), the number of gap-free \(m\)-compositions of \(a\) is \(c_a(m) \sim k^{m-1}\).

Using the familiar Moebius function \(\mu\), as in [9], we have

\[
\tilde{c}(m) = \sum_{d|m} \frac{1}{d} \sum_{d'|d} \mu(d/d') c(d') \sim \frac{1}{m} k^m.
\]
And the number of circular gap-free $m$-compositions of $a \in G$ is $\tilde{c}_a(m) \sim \frac{1}{m}k^{m-1}$.

Example 4.2. Considering avoidance of the subword pattern 132, for any total $a$ there will be 1 composition with 1 part, namely $(a)$. For $m \geq 2$, some compositions are grouped into non-trivial equivalence classes. For $m = 1, \ldots, 5$, the numbers of 132-avoiding circular $m$-compositions of 0 over $\mathbb{Z}_5$ are 1, 3, 7, 23, 82, and the counts for $m$-compositions of 1 are 1, 3, 7, 23, 77.

4.2 Note on counting palindromic compositions

An unlabeled undirected weighted path of length $m$ restricted according to $D$ is equivalent to an unordered pair $\{x, \bar{x}\}$ where $x, \bar{x} \in \mathcal{P}(m; D)$, or the singleton $\{x\}$ if $x \in \mathcal{P}(m; D)$ and $x = \bar{x}$. These may also be called undirected words. For simplicity we assume all vertices of $D$ are allowed as start and finish vertices.

Proposition 4.2. Assume $D$ is such that $x \in \mathcal{P}(D) \implies \bar{x} \in \mathcal{P}(D)$. Let $\Xi = \{\xi : \xi\bar{\xi} \in \mathcal{P}(D), \xi \in \mathcal{V}(D)\}$. If $m \geq 2\sigma$ is even, the number of $G$-weighted undirected paths of length $m$ with total $a$ restricted by $D$ is

$$p_a(m; D) = \frac{1}{2}p_a(m; D) + \sum_{b : 2b = a} \frac{1}{2}p_b(m/2; D, V(D), \Xi).$$

For $c \in G$, let $\Xi_c = \{\xi : \xi c\bar{\xi} \in \mathcal{P}(D), \xi \in \mathcal{V}(D)\}$. If $m \geq 2\sigma$ is odd, the number of $G$-weighted undirected paths of length $m$ with total $a$ restricted by $D$ is

$$p_a(m; D) = \frac{1}{2}p_a(m; D) + \frac{1}{2} \sum_{c \in G} \sum_{b : 2b + c = a} p_b((m - 1)/2; D, V(D), \Xi_c).$$

Proof. The number of undirected paths is determined by dividing by 2, with an adjustment for palindromic compositions: those $x$ such that $x = \bar{x}$. If $m$ is even, the set of palindromic $m$-compositions is in correspondence with $\mathcal{P}(m/2; D, V(D), \Xi)$ and

$$p_a(m; D) = \frac{1}{2} \left( p_a(m; D) - \sum_{b : 2b = a} p_b(m/2; D, V(D), \Xi) \right) + \sum_{b : 2b = a} p_b(m/2; D, V(D), \Xi).$$

The case of even $m$ is similar.

The analogous result for integer compositions is found in [4, §11].
5 Subsequence pattern avoidance

Given a word $w$ over $[k]$, the reduction of $w$, written $\text{red}(w)$, is obtained by replacing the $j$th smallest letters of $w$ with $j$'s, for all $j$. For example, $\text{red}(46632) = 34421$. A subsequence pattern, sometimes called a classical pattern, is a word over some $[k]$ written with hyphens between letters: $1-1-1-3-2-2 \in [3]^7 = \text{SEQ}_7([3])$. Given words $w$ of length $m$ and $\tau$ of length $l$, an occurrence of $\tau$, as a subsequence pattern, in $w$ is a sequence of indices $1 \leq i_1 < \cdots < i_l \leq m$ such that $\text{red}(w(i_1), \ldots, w(i_l)) = \tau$.

A partially ordered pattern is similar to a subsequence pattern except that not all letters are comparable. The letters in a partially ordered pattern are from a partially ordered alphabet; letters shown with the same number of primes are comparable to each other (e.g. $1''$ and $2''$), while letters shown without primes are comparable to all letters of the alphabet. An occurrence of a partially ordered pattern in a word $w$ is a distinguished subsequence of terms of $w$ such that the relative order of two entries in the subsequence need be the same as that of the corresponding letters in the pattern only if the corresponding letters in the pattern are comparable; e.g. the partially ordered pattern $1'-1''-2$ is found in the word $42213$ three times as $42213$, $42213$ and $42213$ (the subsequences of length three in which the third letter is larger than the first two).

A generalized pattern is again similar to a subsequence pattern except there may or may not be a hyphen between adjacent letters. If there is no hyphen, those two letters can only match with adjacent letters in a word. For example, if $\tau = 11-2$, then $424135$ has no occurrences of $\tau$ but $244135$ has the occurrence $244135$.

Subsequence patterns were first studied in the context of permutations [67] but are now adapted to different objects. The number of $k$-ary words of length $m$ avoiding a given subsequence or generalized pattern has been studied for a number of different patterns [14, 15, 23, 37, 49, 56, 58, 50, 40]. Specifically, exact results for the avoidance of various subsequence patterns with at most 2 distinct letters were found in [17]. For partially ordered pattern-based enumeration for words and other objects, see [16, 35, 41, 42]. The article [27] counts words with $r \geq 0$ occurrences of a some simple subsequence patterns.

Occurrences of subsequence, partially ordered, and generalized patterns are defined for compositions as they are for words. The counting question simply changes to, how many compositions with length $m$ and total $n$ avoid the pattern?

A generating function counting integer compositions avoiding some 3-letter patterns is given in the note [62], which is a simplification of earlier work in [3]. A recurrence relation is also given in [1]. Compositions avoiding the remaining 3-letter patterns, and pairs of 3-letter patterns are counted in [36]. That paper also looks at the subsequence pattern $1^p-2-1^q$. Partially ordered patterns in compositions are considered in [35]. Compositions avoiding a generalized pattern of length 3 are counted using generating functions in [37 § 5.3].
Remark 5.1. Let \( p_k(m, r) \) be the number of \( k \)-ary \( m \)-words with \( r \) occurrences of the pattern \( 1 \cdots 1 = 1^r \). A simple argument shows

\[
\sum_{m,r \geq 0} p_k(m, r) \frac{z^m}{m!} u^r = \left( \sum_{i \geq 0} u^{(i)} \frac{z^i}{i!} \right)^k.
\]

The recent paper [64] sheds light on expressions of this form, by establishing integral representations such as

\[
\sum_{n \geq 0} g_n q^n z^n = \frac{1}{\sqrt{2\pi}} \int_0^\infty \left( \sum_{b=\pm 1} G \left( e^{bt \sqrt{2\log(q)} z} \right) \right) e^{-t^2/2} dt,
\]

where \( G(z) = \sum_{n \geq 0} g_n z^n \). Enumerative applications of these representations have yet to be explored. \( \triangle \)

The literature on subsequence pattern avoidance in words and compositions notably lacks results which are general in nature, giving information about all patterns. One might ask, for example, whether the generating function counting words avoiding a pattern is always algebraic, holonomic, et cetera.

Remark 5.2. In the language of §1, we deal with paths avoiding global occurrences of digraph patterns. Undirected paths and directed and undirected cycles are approached in a similar manner.

For a weighted digraph \( \Gamma \), let \( s(\Gamma) \) be the symmetric closure of \( \Gamma \), i.e. the underlying undirected graph. Given a weighted path \( \Gamma_p \), and digraph pattern instance \( P \) which is also a weighted path, an occurrence of \( P \) in \( s(\Gamma_p) \) is either an occurrence of \( P \) in \( \Gamma_p \) or an occurrence of \( P^{-1} \) in \( \Gamma_p \), where \( P^{-1} \) is \( P \) with arcs reversed. Let \( c(\Gamma_p) \) be the directed cycle formed by adding an arc to \( \Gamma_p \). Then an occurrence of \( P \) in \( c(\Gamma_p) \) is the occurrence of some circular shift of \( P \) in \( \Gamma_p \). Occurrences of \( P \) in \( s(c(\Gamma_p)) \) are occurrences of circular shifts and/or reversals of \( P \) in \( \Gamma_p \). \( \triangle \)

5.1 Words and integer compositions

This section fills some gaps in the literature on words and integer compositions that avoid a pattern. Our main tools are recurrence relations and generating functions, and we use various standard counting techniques.

Remark 5.3. The random sampling in this section is performed by exploiting the structure of recurrence relations. The method achieves exact uniform sampling and makes use of two rules, one for addition and one for multiplication. Assume there are three classes of objects, \( A, B, C \) and the number of objects in each are \( a, b, c \). We have the relation \( a = b + c \) if \( A = B \cup C \). Then to draw an object uniformly randomly from \( A \), we may draw an object from \( B \) with probability \( b/(b+c) \) or an object from \( C \) with probability \( c/(b+c) \). Now if \( A = B \times C \), we have \( a = bc \). Here we may draw uniformly at random from \( A \) by independently drawing from both \( B \) and \( C \). This simple method is often applicable where we have a recurrence relation, in which case we recurse until reaching a base case. \( \triangle \)
5.1.1 Pairs of generalized patterns of length 3

While we do not consider every possible pair of generalized patterns of length 3 in this section, we give a number of representative examples. We expect similar techniques apply to most of the remaining such pattern pairs.

The pair \{11-2, 12-3\}

We use the generating function \(P_k(w|z, u)\) to enumerate integer compositions over \([k]\) starting with the subword \(w\) and avoiding \{11-2, 12-3\}, where \(z\) marks the total and \(u\) marks the length. We write \(P_k(z, u)\) for \(P_k(e|z, u)\) where \(e\) is the sequence of length 0, and we write \(P(z, u)\) to refer to \(\lim_{k \to \infty} P_k(z, u)\).

Proposition 5.1. We have

\[
P(z, u) = \frac{1}{1 - uz} \prod_{i \geq 2} \left(1 - uz^i \prod_{j=1}^{i-1} (1 + uz^j) \right)^{-1}.
\]

Proof. We follow the proof of Theorem 5.21 [37] at least in spirit.

Take a composition \(x\) over \([k]\) that avoids \{11-2, 12-3\}. Assume \(x\) begins with the part \(i\) (where \(1 \leq i \leq k\)). Then \(x\) is either \(i\) by itself or begins \((i, j, \ldots)\) for some part \(j\). Now if \(j < i\), then the first part of \(x\) cannot be involved in an occurrence of the pattern set, so the composition \((j, x(3), \ldots, x(m))\) is arbitrary as long as it avoids the pattern set. On the other hand, if \(j \geq i\), no later parts may be greater than \(j\) so the composition \((x(3), \ldots, x(m))\) is an arbitrary composition over \([j]\) avoiding the pattern set. This gives, for \(k \geq 1, 1 \leq i \leq k,

\[
P_k(i|z, u) = z^i u + \sum_{j=1}^{i-1} P_k(ij|z, u) + \sum_{j=i}^{k} P_k(ij|z, u)
\]

\[
= z^i u + z^i u \left( \sum_{j=1}^{i-1} P_k(j|z, u) + \sum_{j=i}^{k} z^j u P_j(z, u) \right).
\]

Define \(G_k(i) = P_k(i|z, u) - P_{k-1}(i|z, u)\) for \(k \geq 2\) and \(1 \leq i < k\). By Equation (5.1) we have \(G_k(i) = z^i u \left( \sum_{j=1}^{i-1} G_k(j) + z^k u P_k(z, u) \right)\) for \(1 \leq i < k\). It can then be seen by induction that \(G_k(i) = u^2 z^{i+k} P_k(z, u) \prod_{j=1}^{i-1} (1 + uz^j), 1 \leq i < k\). We naturally define \(G_k(k) = uz^k P_k(z, u)\). Induction or a combinatorial argument...
Table 5: Counts of the \( m \)-compositions of \( n \) avoiding \{12-2, 12-3\}.

| \( n \) | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-------|----|----|----|----|----|----|----|----|----|----|----|
| 0     | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1     | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 2     | 0  | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3     | 0  | 1  | 2  | 1  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4     | 0  | 1  | 3  | 2  | 1  | 0  | 0  | 0  | 0  | 0  | 0  |
| 5     | 0  | 1  | 4  | 5  | 2  | 1  | 0  | 0  | 0  | 0  | 0  |
| 6     | 0  | 1  | 5  | 8  | 6  | 2  | 1  | 0  | 0  | 0  | 0  |
| 7     | 0  | 1  | 6  | 12 | 12 | 6  | 2  | 1  | 0  | 0  | 0  |
| 8     | 0  | 1  | 7  | 17 | 20 | 15 | 6  | 2  | 1  | 0  | 0  |
| 9     | 0  | 1  | 8  | 23 | 33 | 28 | 16 | 6  | 2  | 1  | 0  |
| 10    | 0  | 1  | 9  | 29 | 50 | 50 | 35 | 16 | 6  | 2  | 1  |

also show that for \( k \geq 2 \) we have

\[
P_k(z, u) - P_{k-1}(z, u) = \sum_{i=1}^{k-1} G_k(i) + G_k(k) \\
= u^k P_k(z, u) \sum_{i=1}^{k-1} u^i z^i \prod_{j=1}^{i-1} (1 + uz^j) + u^k P_k(z, u) \\
= u^k P_k(z, u) \left( -1 + \prod_{j=1}^{k-1} (1 + uz^j) \right) + u^k P_k(z, u) \\
= u^k P_k(z, u) \prod_{j=1}^{k-1} (1 + uz^j),
\]

so

\[
P_k(z, u) = \left( 1 - u^k \prod_{j=1}^{k-1} (1 + uz^j) \right)^{-1} P_{k-1}(z, u).
\]

With the initial condition \( P_1(z, u) = 1/(1 - uz) \) we have

\[
P_k(z, u) = \frac{1}{1 - uz} \prod_{i=2}^{k} \left( 1 - u^i \prod_{j=1}^{i-1} (1 + uz^j) \right)^{-1}.
\]

We conclude the result by letting \( k \to \infty \). \( \square \)

Figure [19] shows randomly generated compositions avoiding \{12-2, 12-3\}. Table 5 show initial counts.

The pair \( \{21-2, 2-12\} \)

We count \( k \)-ary words avoiding the set of generalized patterns \( \{21-2, 2-12\} \). Note that it is not true that letters 1 must be found only in contiguous blocks...
Figure 19: Uniform-randomly generated compositions of 150 avoiding \{12-2, 12-3\}.

at the very beginning and/or very end of a word. For example, 312 avoids the patterns and has the least letter in the middle.

Let \( p_k(m) \) be the number of \( k \)-ary \( m \)-words that avoid \{21-2, 2-12\}.

**Proposition 5.2.** For \( k \geq 1 \) we have

\[
p_k(m) = \frac{\prod_{i=1}^{k-1}(2i - 1)}{(2k - 2)!} m^{2k-2} + O(m^{2k-3}), \quad m \to \infty.
\]

**Proof.** Take a \( k \)-ary word \( w \) avoiding \{21-2, 2-12\}. There are \( p_{k-1}(m) \) such words with no letters \( k \). We assume the greatest letter present in \( w \) is \( k \). All copies of \( k \) must be contiguous in order to avoid the patterns. If we delete these copies of \( k \) from \( w \), the remaining word has the same structure but is a word over \([k - 1]\). If there are \( b \) letters \( k \), there are \( m - b + 1 \) possible positions of the contiguous run of these letters. Thus

\[
p_k(m) = p_{k-1}(m) + \sum_{b=1}^{m} (m - b + 1) p_{k-1}(m - b), \quad k \geq 1, m \geq 0,
\]

and \( p_0(m) = [m = 0] \). Passing to the generating function \( P_k(z) = \sum_{m \geq 0} p_k(m) z^m \) gives

\[
P_k(z) = P_{k-1}(z) + \frac{z}{1 - z} D_z(z P_{k-1}(z)), \quad P_0(z) = 1.
\]

By induction for \( k \geq 1 \), \( P_k(z) \) has a unique singularity at 1 where it has a pole of order \( 2k - 1 \) and

\[
P_k(z) = \frac{\prod_{i=1}^{k-1}(2i - 1)}{(1 - z)^{2k-1}} + O((1 - z)^{-(2k-2)}), \quad z \to 1.
\]

Using Theorem 2.2 we extract asymptotics for the coefficients of \( P_k(z) \) and conclude the result. \( \Box \)

Table 6 gives initial counts of words avoiding this pattern set \{21-2, 2-12\}, and

Table 20 has randomly generated examples.
| $k$ | $m$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|-----|----|----|----|----|----|----|----|----|----|----|----|
| 2   |     | 1  | 2  | 4  | 7  | 11 | 16 | 22 | 29 | 37 | 46 | 56 |
| 3   |     | 1  | 3  | 9  | 24 | 56 | 116| 218| 379| 619| 961| 1431|
| 4   |     | 1  | 4  | 16 | 58 | 186| 526| 1324|3011|6283|12196|22276|

Table 6: Counts of the $k$-ary $m$-words avoiding \{21-2, 2-12\}.

Figure 20: Uniform-randomly generated 10-ary (above) and 40-ary (below) 100-words avoiding \{21-2, 2-12\}.
The pair \{11-2, 12-1\}

Let $P_A(w|z, u)$ be the generating function for integer compositions over the finite set $A \subset \mathbb{Z}_{>0}$, starting with the subword $w$, that avoid the pattern set \{11-2, 12-1\}, where $z$ marks total and $u$ marks length. The generating function $P_A(z, u)$ refers to $P_A(e|z, u)$ where $e$ is the empty word. We use the notation $M(A, i) = \{j : j \in A, j \leq i\}$.

**Proposition 5.3.** We have

$$P_A(z, u) = 1 + \sum_{i \in A} P_A(i|z, u),$$

and

$$P_A(i|z, u) = z^i u + \sum_{j \in A, j < i} z^i u P_A(j|z, u) + z^{2i} u^2 P_{M(A, i)}(z, u) + \sum_{j \in A, j > i} z^i u P_{A\{i\}}(j|z, u).$$

**Proof.** Let $x$ be a composition over $A$ avoiding \{11-2, 12-1\}. If $x$ begins with the part $i \in A$, either $x = (i)$ or $x = (i, j, \ldots)$. In the latter case we may have $j < i, j = i$, or $j > i$, so we get

$$P_A(i|z, u) = z^i u + \sum_{j \in A, j < i} P_A(ij|z, u) + P_A(ii|z, u) + \sum_{j \in A, j > i} P_A(ij|z, u).$$

If $j < i$ then $i$ is part of an occurrence only if $j$ is. So if we delete $i$ the remaining composition is arbitrary. If $j = i$ then $x = (i, i, \ldots)$. In order to avoid 11-2 the composition remaining after deleting $ii$ is arbitrary as long as no parts are above $i$. And finally if $j > i$ we may delete $i$ and have an arbitrary composition starting with $j$ as long as the part $i$ does not appear. Thus we have

$$z^i u + \sum_{j \in A, j < i} P_A(ij|z, u) + P_A(ii|z, u) + \sum_{j \in A, j > i} P_A(ij|z, u) = z^i u + \sum_{j \in A, j < i} z^i u P_A(j|z, u) + z^{2i} u^2 P_{M(A, i)}(z, u) + \sum_{j \in A, j > i} z^i u P_{A\{i\}}(j|z, u).$$

Table 7 shows initial counts of compositions avoiding \{11-2, 12-1\}, and Figure 21 shows randomly-generated objects.

The pair \{12-3, 3-21\}

Let $p_k(m)$ be the number of $k$-ary $m$-words that avoid the pattern set \{12-3, 3-21\}.

**Proposition 5.4.** For $k \geq 2$ we have $p_k(m) \sim A_k \cdot (\sqrt{k-1} + 1)^m$, $m \to \infty$. 

56
Table 7: Counts of the $m$-compositions of $n$ avoiding \{11-2, 12-1\}.

\[
\begin{array}{ccccccccccc}
\hline
n \backslash m & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 1 & 4 & 4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 1 & 5 & 8 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
7 & 0 & 1 & 6 & 11 & 7 & 2 & 1 & 1 & 0 & 0 & 0 \\
8 & 0 & 1 & 7 & 17 & 11 & 4 & 2 & 1 & 1 & 0 & 0 \\
9 & 0 & 1 & 8 & 24 & 24 & 10 & 4 & 2 & 1 & 1 & 0 \\
10 & 0 & 1 & 9 & 30 & 42 & 16 & 6 & 4 & 2 & 1 & 1 \\
\hline
\end{array}
\]

Proof. Let $w$ be a $k$-ary word avoiding \{12-3, 3-21\}. Then either $w$ contains no letters $k$, or $w$ can be written as the concatenation

\[w' k^{i_1} w_1 \cdots k^{i_r} w_r k^{s} w'',\]

where $r \geq 0$, $w'$ is a word over $[k-1]$ that avoids \{12, 3-21\}, $w''$ is a word on $[k-1]$ that avoids \{21, 12-3\}, and the $w_i$ are words on $[k-1]$ that avoid \{12, 21\}.

Nonempty words avoiding \{12, 21\} clearly have one distinct letter repeated some number of times. Words avoiding \{21, 12-3\} are either empty, have one distinct letter, or have exactly one increase and no decreases.

This translates to

\[P_k(z) = P_{k-1}(z) + G_{k-1}(z) \frac{1}{1 - (z/(1-z))} H_{k-1}(z) \frac{z}{1-z} G_{k-1}(z),\]

where $G_k(z)$ counts words avoiding \{21, 12-3\} (or \{12, 3-21\}), so

\[G_k(z) = 1 + \sum_{i=1}^{k} \frac{z}{1-z} + \sum_{i=1}^{k-1} \frac{z}{1-z} \sum_{j=i+1}^{k} \frac{z}{1-z} = 1 + k \frac{z}{1-z} + \frac{k^2 - k}{2} \frac{z^2}{(1-z)^2},\]

and $H_k(z) = kz/(1-z)$ counts nonempty words avoiding \{12, 21\}. 

\[57\]
Table 8: Counts of $k$-ary $m$-words avoiding \{12-3, 3-21\}.

Iterating the recurrence relation, we have

$$P_k(z) = \frac{1}{1 - z} + \sum_{j=2}^{k} G_{j-1}(z) \frac{z}{1 - z} \frac{1}{1 - H_{j-1}(z) z/(1 - z)}.$$  

We examine the factor

$$\frac{1}{1 - H_{j-1}z/(1 - z)} = \frac{-z^2 + 2z - 1}{jz^2 - 2z^2 + 2z - 1}$$

The root of $jz^2 - 2z^2 + 2z - 1$ with smallest absolute value is $z = \frac{1}{\sqrt{j-1}+1}$. For $j \geq 2$, this value is a simple pole less than 1 and decreasing (toward 0). By Theorem 2.2 we conclude the statement.

Table 8 shows initial coefficients of $P_k(z)$.

5.1.2 Some partially ordered patterns with 2 letters

Here we consider the family of partially ordered patterns of the form $2^p.1′\cdots.1^{(q)}.2^r = 2\cdots 2.1′.1′\cdots 1^{(q)}.2\cdots 2$. We break into cases depending on the values of $p, q, r$.

Case $p, q, r \geq 1$

Let $h_k(n, m)$ be the number of integer $m$-compositions of $n$ over $[k]$ that avoid $2^p.1′\cdots 1^{(q)}.2^r$ where $p, q, r \geq 1$.

**Proposition 5.5.** We have the recurrence relation

$$h_k(n, m) = \sum_{b=0}^{m} [0 \leq b < p + r \text{ or } b > m - q] \binom{m}{b} h_{k-1}(n - bk, m - b)$$

$$+ \sum_{b=p+r}^{m-q} \sum_{t=M}^{M+q-1} \binom{w-2}{M-2} \binom{m-t+1}{(p-1) + (r-1) + 1} h_{k-1}(n - bk, m - b),$$

for $m, n, \geq 0$ and $k \geq 2$. For $k = 1$, we have $h_1(n, m) = \lfloor n = m \rfloor$. 

58
Proof. Assume \( k \geq 2 \) and let \( b \) be the number of letters \( k \) in a word \( w \). If \( b \leq p + r - 1 \) or \( b \geq m - q + 1 \), these letters cannot be part of an occurrence, so their positions do not matter, thus there are \( \binom{n}{b}h_{k-1}(n - bk, m - b) \) such words \( w \). If \( b \geq p + r \), then between the \( p \)th \( k \) from the left and the \( r \)th \( k \) from the right there must be at most \( q - 1 \) letters that are not \( k \). Let \( t \) be the number of all letters between the \( p \)th \( k \) from the left and the \( r \)th \( k \) from the right, and let \( M = b - (p - 1) - (r - 1) \) be the number of letters \( k \) among those letters. Then there are 
\[
\sum_{b=p+r}^{m-q-1} \sum_{t=M}^{t+2} \binom{t-2}{m-t+1} \binom{m-t+1}{(p-1)+(r-1)+1} h_{k-1}(n - bk, m - b),
\]
possible ways of placing the letters \( k \) in \( w \). the first binomial coefficient chooses the letters \( k \) between the \( p \)th from left and \( r \)th from right, and the second chooses the position of the remaining letters \( k \) as well as the position of the \( p \)th from the left.

So for \( m \geq p + r \), this gives 
\[
h_k(n, m) = \sum_{b=0}^{m} [0 \leq b < p + r \text{ or } b > m - q] \binom{m}{b} h_{k-1}(n - bk, m - b)
+ \sum_{b=p+r}^{m-q} \sum_{t=M}^{t+2} \binom{t-2}{m-t+1} \binom{m-t+1}{(p-1)+(r-1)+1} h_{k-1}(n - bk, m - b),
\]
as desired. It can be verified that the recurrence is valid as well for the values \( 0 \leq m < p + r \).

We note that \( h_k(n, m) \) is a function of \( p + r \) rather than \( p \) and \( r \) independently.

Case \( p = 1, q = 2, r = 1 \)

For the special case \( \tau = 2-1'-1''-2 \) we illustrate an asymptotic analysis. We further simplify by ignoring totals and counting words. Let \( H_k(z) = \sum_{m \geq 0} h_k(m)z^m \) where \( h_k(m) \) is the number of \( k \)-ary \( m \)-words that avoid \( 2-1'-1''-2 \).

Proposition 5.6. If \( k \geq 2 \) we have \( h_k(m) = \frac{A_k}{(3(k-1))!}m^{3(k-1)}(1+O(m^{-1})) \), \( m \to \infty \), where \( A_k = \prod_{j=1}^{k-1}(1 + 3(j-1)) \).
Table 9: Counts of the $k$-ary words of length $m$ avoiding $2\cdot 1'-1''\cdot 2$.

| $k$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|-----|----|----|----|----|----|----|----|----|----|----|
| 1   | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 2   | 1  | 2  | 4  | 8  | 15 | 26 | 42 | 64 | 93 | 130|
| 3   | 1  | 3  | 9  | 27 | 76 | 196| 462| 844| 2692|40163|
| 4   | 1  | 4  | 16 | 64 | 242| 844| 2692|7852| 21043|52184|
| 5   | 1  | 5  | 25 | 125| 595 |2635|10743|40163|137738|434798|

Figure 22: Uniform-randomly generated $k$-ary 100-words where $k = 3, 4, 5, 20$ (top to bottom) avoiding $2\cdot 1'-1''\cdot 2$.

**Proof.** By Proposition 5.5 we know

$$h_k(m) = h_{k-1}(m) + mh_{k-1}(m-1)$$

$$+ \sum_{b=2}^{m-2} ((m - b + 1) + (m - b)(b - 1)) h_{k-1}(m - b)$$

$$+ mh_{k-1}(1) + h_{k-1}(0)$$

$$= \sum_{b=0}^{m} ((m - b + 1) + (m - b)(b - 1)) h_{k-1}(m - b)$$

Passing to generating functions, we have

$$H_k(z) = \frac{1}{1 - z} H_{k-1}(z) + \frac{z^2}{(1 - z)^2} H'_{k-1}(z)$$

for $k \geq 2$, and $H_1(z) = 1/(1 - z)$.

By induction $H_k(z)$ is rational with unique singularity at $z = 1$ and $H_k(z) \sim A_k \frac{1}{(1-z)^{3k-1}}, z \to 1$ so by Theorem 2.2 we conclude the statement.

Table 9 shows initial coefficients of $H_k(z)$. Figure 22 has uniform-randomly generated words avoiding $2\cdot 1'-1''\cdot 2$. 

60
Case \( p = 0 \)

If we have \( q, r \geq 1 \) but we allow \( p = 0 \), we have the pattern \( \tau = 1^r 1^{q-2} \). Words avoiding \( \tau \) were counted in [27, §2]; integer compositions were left as an open problem. Let \( h_k(n, m) \) be the number of integer \( m \)-compositions of \( n \) over \([k]\) that avoid \( \tau \).

**Proposition 5.7.** For \( k \geq 2 \) and \( m \geq q + r \) we have

\[
h_k(n, m) = \sum_{j=1}^{q} \binom{q}{j} h_k(n - jk, m - j)(-1)^{j+1}
+ \sum_{b=0}^{r-1} \binom{m-q}{b} h_{k-1}(n - bk, m - b),
\]

and \( h_1(n, m) = [n = m] \).

**Proof.** For the range \( m \geq q + r \), we recursively count \( m \)-compositions \( x \) avoiding \( \tau \) by first counting \( x \) such that at least one of the first \( q \) letters is \( k \). By the principle of inclusion-exclusion, the number of such \( x \) is

\[
\sum_{j=1}^{q} N_j (-1)^{j+1},
\]
where \( N_j \) is the sum, over all \( j \)-subsets of the first \( q \) positions, of the number of compositions \( x \) with \( k \)'s in the positions given by the subset. The quantity \( N_j \) is given by

\[
N_j = \binom{q}{j} h_k(n - jk, m - j),
\]

since inserting \( j \) copies of \( k \) into any of the first \( q \) positions of an \((m-j)\)-composition is reversible and does not affect the number of occurrences of \( \tau \).

Now we count the compositions \( x \) that have no letters \( k \) in their first \( q \) positions. Let \( b \) be the number of letters \( k \) in \( x \). If \( b \leq r - 1 \), then there are not enough letters \( k \) to be part of a pattern, so there are

\[
\sum_{b=0}^{r-1} \binom{m-q}{b} h_{k-1}(n - bk, m - b),
\]
compositions of this kind.

If \( b \geq r \) then there is at least one occurrence of \( \tau \). Thus we have, for \( m \geq q + r, k \geq 2 \),

\[
h_k(n, m) = \sum_{j=1}^{q} \binom{q}{j} h_k(n - jk, m - j)(-1)^{j+1}
+ \sum_{b=0}^{r-1} \binom{m-q}{b} h_{k-1}(n - bk, m - b).
\]

For \( m < q + r \), we have \( h_k(n, m) = [z^n u^m] \left( \sum_{j=1}^{k} z^j u \right)^m \). \( \square \)
Figure 23: Uniform-randomly generated compositions of 150 avoiding 1'-1''-2.

We expect similar techniques to those used for $p, q, r \geq 1$ and $p = 0$ apply to count avoidance of $\tau$ where $\tau$ involves the letter 2 and mutually incomparable symbols $1^{(j)}$.

Figure 23 shows randomly generated compositions avoiding 1'-1''-2.

5.1.3 Note on counting with symmetries

Reversal (unlabeled undirected paths)

As in §4.2 we say that an undirected word is an unordered pair $\{w, \bar{w}\}$ where $w \neq \bar{w}$ or simply $\{w\}$ if $w = \bar{w}$. The pair $\{w, \bar{w}\}$ avoids a pattern $\tau$ iff both $w$ and $\bar{w}$ avoid $\tau$. For a subsequence pattern $\tau$, we define the set $folds(\tau)$ to be all possible words $\tau'$ obtained by the following procedure. Split $\tau$ into two subwords $\tau = \tau_1\tau_2$. Take words $\tau'$ such that $\tau_1$ and $\bar{\tau}_2$ are subsequences of $\tau'$.

**Proposition 5.8.** Let $\tilde{p}_k(m; T)$ be the number of undirected $k$-ary $m$-words that avoid all subsequence patterns in the set $T$. Then for even $m$ we have

$$
\tilde{p}_k(m; \{\tau\}) = \frac{1}{2} p_k(m; \{\tau, \bar{\tau}\}) + \frac{1}{2} p_k(m/2; folds(\tau))
$$

**Proof.** An undirected word is either palindromic or not. If not, it corresponds to a pair of 2 distinct directed words. If it is palindromic, it takes the form $u\bar{u}$ where $u$ avoids $folds(\tau)$. Also,

$$
\tilde{p}_k(m; \{\tau\}) = \frac{1}{2} \left( p_k(m; \{\tau, \bar{\tau}\}) - p_k(m/2; folds(\tau)) \right) + p_k(m/2; folds(\tau)). \quad \Box
$$

The case of odd $m$ is less straightforward. Burstein [13] counts a number of examples of words avoiding the set $\{\tau, \bar{\tau}\}$ for where $\tau$ is a short subsequence pattern with no repeated letters. Avoiding $folds(\tau)$ becomes quite restrictive but is not necessarily impossible. Any non-decreasing word avoids $folds(2\text{-}1\text{-}3)$. 

62
Circular shift (unlabeled cycles)

A circular word is an equivalence class of words where two words are equivalent if one is a circular shift of the other.

As in Remark 5.2, a word $w$ cyclically avoids a subsequence pattern $\tau$ if all circular shifts of $w$ avoid the pattern. Alternatively, $w$ cyclically avoids $\tau$ if $w$ avoids all circular shifts of $\tau$. We observe that we do not have property that if $u$ cyclically avoids a pattern so does $uu$. For example, to cyclically avoid $1-2-3$, we avoid the set $T = \{1-2-3, 3-1-2, 2-3-1\}$, and if $u = 321$, then $u$ avoids the pattern but $uu = 321321$ contains $2-3-1$.

We define the set $\operatorname{merges}_i(\tau)$ to contain all $\tau'$ produced by the following procedure. If $i$ is an integer satisfying $1 \leq i < |\tau|$, we consider any circular shift $\tau^*$ of $\tau$ expressed as a concatenation of subwords $\tau^* = \tau_1 \cdots \tau_i$, some of which may be empty. For each $\tau^*$, include any word $\tau'$ such that each $\tau_j$ is a subsequence of $\tau'$.

Proposition 5.9. Let $p_k(m; T)$ be the number of $k$-ary $m$-words avoiding subsequence patterns in $T$, and let $\tilde{c}_k(m; T)$ be the same for circular $m$-words. Let $t$ be the number of distinct letters in the subsequence pattern $\tau$. We have

$$\tilde{c}_k(m; \{\tau\}) = \sum_{j|m} \frac{1}{j} \sum_{d|j} \mu(j/d) \left( [m/d < |\tau|] p_k(d; \operatorname{merges}_{m/d}(\tau)) + [m/d \geq |\tau|] \alpha(d) \right),$$

where $\alpha(d) = \sum_{j=1}^{t-1} \binom{k}{j} \binom{d}{j}$ is the number of all $k$-ary $d$-words with fewer than $t$ distinct letters, and $\mu$ is the Moebius function.

Proof. The period of an $m$-word $w$ is the least integer $n$ such that $w = u^{m/n}$ for some word $u$. Fix $m$, and for $j \leq m$, define $f_j$ to be the number of $k$-ary $m$-words cyclically avoiding $\tau$ with period dividing $j$. Then by Moebius inversion the number of words with period exactly $j$ is $\sum_{d|j} \mu(d) f_{j/d}$, and so the number of all circular words is

$$\sum_{j|m} \frac{1}{j} \sum_{d|j} \mu(d) f_{j/d} = \sum_{j|m} \frac{1}{j} \sum_{d|j} \mu(j/d) f_d.$$

If an $m$-word $w$ has period dividing $d$, then it has the form $w = u^{m/d}$ for a subword $u$ of length $d$. If $m/d \geq |\tau|$ then $w$ cyclically contains $\tau$ iff $u$ contains at least as many distinct letters as there are in $\tau$. And if $m/d < |\tau|$, then $w$ cyclically avoids $\tau$ iff $u$ avoids $\operatorname{merges}_{m/d}(\tau)$. \qed

5.2 Note on compositions over $\mathbb{Z}_k$

The problem of counting compositions over a group that avoid a subsequence pattern has not been addressed in prior literature, but was suggested in [28]. Here we present a general technique illustrated for the pattern $1'-2-1''$. 

63
Proposition 5.10. Let $P_k^{(a)}(y)$ be the generating function for compositions of $a$ over $\mathbb{Z}_k$ avoiding the pattern set \{1-3-2, 2-3-1, 1-2-1\} (alternatively the partially ordered pattern 1'-2-1''), where $y$ marks length. Then

$$[y^m]P_k^{(a)}(y) = \frac{1}{k}m^{2k-2} + O(m^{2k-3}), \quad m \to \infty.$$ 

Proof. Let $P_k(x,y)$ be the generating function for integer compositions over the part set $[k]$ avoiding $1'-2-1''$, where $x$ marks total and $y$ marks length. Example 5.62 in [37] provides the expression

$$P_k^{(a)}(x,y) = \frac{1}{k^d} \sum_{d=1}^{k^d} \prod_{b=d}^{k^b} (1 - x^d y)^2.$$ 

The multisection formula [30, Ex. 1.1.9] for power series $F(z) = \sum f_n z^n$ is

$$\frac{1}{k} \sum_{j=0}^{k-1} e^{-2\pi i j a/k} F(e^{2\pi i j/k} z) = \sum_{n \equiv a \pmod{k}} f_n z^n.$$ 

Using the multisection formula we have

$$P_k^{(a)}(y) = \frac{1}{k} \sum_{c=1}^{k} e^{-2\pi i c a/k} P_k(e^{2\pi i c/k}, y)$$

$$= \frac{1}{k} \sum_{c=1}^{k} e^{-2\pi i c a/k} \left( \frac{1}{\prod_{d=1}^{k^d} (1 - e^{2\pi i c d/k} y)^2} - \sum_{d=1}^{k} e^{2\pi i d c/k} y \prod_{b=d}^{k^b} (1 - e^{2\pi i b c/k} y)^2 \right).$$

Thus $P_k^{(a)}(y)$ is rational. We claim that any pole of $P_k^{(a)}(y)$ other than $y = 1$ has order at most $2(k-1)$. All poles other than $y = 1$ would come from terms where $c \neq k$. If $c \neq k$, then for $d = 1, \ldots, k$, the factor $e^{2\pi i c d/k}$ takes on at least 2 different values e.g. at $d = 1$ and $d = k$, so we conclude the claim. We claim the pole at $y = 1$ has order $2k - 1$. Such poles can only come from the term $c = k$. This term is

$$\frac{1}{\prod_{d=1}^{k} (1 - y)^2} - \sum_{d=1}^{k} \frac{y}{\prod_{b=d}^{k} (1 - y)^2} = \frac{1}{\prod_{d=1}^{k} (1 - y)^2} - \frac{y}{\prod_{d=1}^{k} (1 - y)^2}$$

$$- \sum_{d=2}^{k} \frac{y}{\prod_{b=d}^{k} (1 - y)^2} = \frac{1}{(1 - y)^{2k-1}} + O((1 - y)^{-2(k-1)}).$$

We conclude the second claim and the proposition follows by applying Theorem 2.2. \qed

Table 10 shows initial coefficients of $P_k^{(a)}(y)$.

Similar analysis can potentially be performed e.g. for compositions avoiding a length-3 permutation pattern as enumerated in [37, Theorem 5.7] and those avoiding the pattern 1-1-2 as enumerated in Theorem 5.13 in [37, p. 139].
| $a$ | $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|-----|---|---|---|---|---|---|---|---|---|---|----|
| 0   |     | 1 | 1 | 4 | 12| 32| 71| 150| 287| 517| 877| 1436|
| 1   |     | 0 | 1 | 4 | 13| 34| 76| 154| 294| 526| 893| 1450|
| 2   |     | 0 | 1 | 4 | 12| 32| 74| 152| 288| 518| 883| 1440|
| 3   |     | 0 | 1 | 4 | 13| 32| 75| 154| 294| 522| 891| 1450|

Table 10: Exact counts of $m$-compositions of $a$ over $\mathbb{Z}_4$ avoiding 1‘-2-1”.

**Remark 5.4.** Note that if we wanted to count compositions mod $k$ using a recurrence relation that recurses on $k$, we have the following problem. While we can create a composition over $[k]$ by creating one over $[k - 1]$ and inserting some copies of $k$, we must know the total of the composition over $[k - 1]$ as a value mod $k$, not mod $k - 1$. So the alphabet and modulus have to be tracked separately. \(\triangle\)
6 Conclusion

As a conclusion, this section mentions some relevant problems which are as yet unsolved. The book [37] contains a variety of proposed research problems many of which are also unsolved.

Consider locally restricted compositions where the parts come from a finite generating set of an infinite group. In the framework of §2.1, the base graph $D$ is finite but the derived graph $D_\times$ is infinite. Take for example the infinite group $\mathbb{Z}$ with generating set $\{-1, 0, 1\}$. Unrestricted compositions over $\{-1, 0, 1\}$ with total, say, 0 no longer form a regular language but do form a context-free language, recognizable by a pushdown automaton. There may be difficulty from finitely generated non-abelian groups, however, due to the fact that the word problem is undecidable and therefore not even context-free. For finitely-generated abelian groups, it is possible that the number of locally restricted compositions of $a$ is asymptotically independent of $a$ but it is no longer possible for each total $a$ to be asymptotically equally likely since the group is infinite. The recent paper [20] explores other problems involving finitely-generated groups and enumeration.

Suppose we have a group $G$ which is infinite but we also have a weight function $W : G \rightarrow \mathbb{Z}_{>0}$. As long as each preimage $W^{-1}(n)$ is finite, we can define the number of locally restricted compositions of $a$ over $G$ with a given total weight $n$. It is plausible that approaches in the above sections and [4] can be applied to this counting problem.

Circular integer compositions (unlabeled cycles weighted by positive integers) appeared in the early paper [65]. More recently, the enumerative study of locally restricted circular integer compositions has progressed in [33, 34] which study Carlitz compositions and restrictions on the set of parts. The conclusion section of [34] suggests expanding to general local restrictions as in [4]. Other recent work [29] has looked at part sizes in circular integer compositions.

We can also consider a set of colored parts $(j, c)$, where $j \in \mathbb{Z}_{>0}$ and $c$ comes from a set of colors (with no particular algebraic structure). A colored integer composition of $n$ is a sequence $((j_1, c_1), \ldots, (j_m, c_m))$ where $\sum_i j_i = n$. Questions about restricted integer compositions can be asked again for colored integer compositions and for colored versions of compositions over a finite group. Some results and open problems mentioned in [6] are relevant.

In the sections above on local restrictions, we focus on “implicit” results that cover a wide range of particular restrictions. What this does not provide is a “simple” formula for exact counts, or explicit constants within asymptotic expressions. So there is the possibility of finding (more) explicit, but less general, expressions to complement our implicit ones. For a deep discussion of the meaning of explicitness in enumeration, see [70, 55].

In §5 we count $k$-ary words of length $m$ avoiding a subsequence pattern set $T$. A further parameter can be tracked, namely the length $p$ of the longest contiguous run of a single letter. This problem for subword pattern avoidance
is addressed in [8] but there is no previous work for subsequence patterns. It is quickly deduced that this is roughly equivalent to counting words that avoid $T$ and also avoid the subword pattern $1^p$. Combining subsequence and subword pattern avoidance presents a challenge.

Finite mappings from a set to itself correspond to functional digraphs, which have a well-known structure [24]. Research such as [2] has enumerated functional digraphs with a kind of local restriction: the indegree of each vertex (a.k.a. the number of its preimages) must lie within a fixed set $\Xi$. More recent papers such as [52] consider the distribution of the least common multiple $T$ and product $B$ of the cycle lengths in restricted functional digraphs, for particular $\Xi$. These values $T, B$ are related to the sequence of iterations of the mapping. Problems remain such as expanding to more general $\Xi$. 
A Notation

Stirling subset numbers (Stirling numbers of the second kind) [31] p. 258: \( \binom{m}{k} \)

Finite cyclic groups: \( \mathbb{Z}_k = \{0, \ldots, k - 1\} \)

Positive integers: \( \mathbb{Z}_{>0} = \{1, 2, 3, \ldots\} \)

Non-negative integers: \( \mathbb{Z}_{\geq0} = \{0, 1, 2, \ldots\} \)

Transitive closure of edge relation: \( \rightarrow \) (long arrow)

Closed neighborhood: \( N[v] \), open neighborhood (excludes \( v \) unless there is a loop): \( N(v) \)

In-neighborhoods [13]: \( N^{-}(v) \), \( N^{-}[v] \), out-neighborhoods: \( N^{+}(v) \), \( N^{+}[v] \)

Matrix/vector literal: \( \begin{bmatrix} 1 & 2 \end{bmatrix} = [1 \quad 2]^\top \)

Matrix entry: \( [M]_{i,j} \)

Disjoint union: \( \cup \)

Concatenation of finite sequences: \( (a, b) \cdots (c, d) = (a, b, c, d) \)

Finite sequence short form: \( 1^423^2 = (1, 1, 1, 1, 2, 3, 3) \)

Sum of finite sequence \( a \): \( \Sigma a \) (capital sigma)

Reversal of finite sequence: if \( a = (a(1), \ldots, a(m)) \), then \( \bar{a} = (a(m), \ldots, a(1)) \)

All \( n \)-tuples over set \( \Xi \) [23]: \( \text{SEQ}_n(\Xi) \)

All finite sequences over set \( \Xi \): \( \text{SEQ}(\Xi) = \cup_n \text{SEQ}_n(\Xi) \)

Subset: \( \subseteq \), strict subset: \( \subset \)

Normal distribution function with mean \( \mu \), variance \( \sigma^2 \): \( N(\mu, \sigma^2) \)

Iverson bracket, 1 if the statement \( \phi \) is true and 0 otherwise [45]: \( [\phi] \)

Cardinality: \( |\Xi| \)

First \( k \) positive integers: \( \{k\} = \{n : 1 \leq n \leq k, n \in \mathbb{Z}\} \)

Convergence in distribution, weak convergence: \( \Rightarrow \)

There is \( c > 0 \) such that for all sufficiently large \( n \), \( |f(n)| \leq cg(n) \): \( f(n) = O(g(n)) \)

Asymptotic equivalence, \( f(n) \) asymptotic to \( g(n) \): \( \lim_{n \to \infty} f(n)/g(n) = 1 \iff f(n) \sim g(n) \)

Partial derivative of power series with respect to indeterminate \( u \): \( D_u f \)

Falling factorial: \( n^\underline{k} = n(n - 1) \cdots (n - k + 1) \)
References

[1] Michael H Albert, Robert EL Aldred, Mike D Atkinson, C Handley, and D Holton. Permutations of a multiset avoiding permutations of length 3. *European Journal of Combinatorics*, 22(8):1021–1031, 2001.

[2] James Arney and Edward Bender. Random mappings with constraints on coalescence and number of origins. *Pacific Journal of Mathematics*, 103(2):269–294, 1982.

[3] MD Atkinson, SA Linton, and LA Walker. Priority queues and multisets. *the Electronic Journal of Combinatorics*, 2(1):R24, 1995.

[4] Edward A Bender and E Rodney Canfield. Locally restricted compositions ii. General restrictions and infinite matrices. *the Electronic Journal of Combinatorics*, 16(1):R108, 2009.

[5] Edward A Bender, Rodney Canfield, and Zhicheng Gao. Locally restricted compositions iv. nearly free large parts and gap-freeness. *Discrete Mathematics & Theoretical Computer Science*, 2012.

[6] Edward A Bender and Zhicheng Gao. Locally restricted sequential structures and runs of a given subcomposition in random locally restricted integer compositions. In preparation.

[7] Edward A Bender and Zhicheng Gao. Part sizes of smooth supercritical compositional structures. *Combinatorics, Probability and Computing*, 23(5):686–716, 2014.

[8] Edward A Bender and Zhicheng Gao. Locally restricted sequential structures and runs of a subcomposition in integer compositions. *arXiv preprint arXiv:1605.04353*, 2016.

[9] Edward A Bender and Jay R Goldman. On the applications of mobius inversion in combinatorial analysis. *American Mathematical Monthly*, pages 789–803, 1975.

[10] Edward A Bender, L Bruce Richmond, and SG Williamson. Central and local limit theorems applied to asymptotic enumeration. iii. matrix recursions. *Journal of Combinatorial Theory, Series A*, 35(3):263–278, 1983.

[11] Alberto Bertoni, Christian Choffrut, Massimiliano Goldwurm, and Violetta Lonati. On the number of occurrences of a symbol in words of regular languages. *Theoretical Computer Science*, 302(1-3):431–456, 2003.

[12] Patrick Billingsley. *Probability and Measure*. John Wiley & Sons, 2008.

[13] John Adrian Bondy and Uppaluri Siva Ramachandra Murty. *Graph Theory with Applications*, volume 290. North-Holland, 1976.

[14] P. Brändén and T. Mansour. Finite automata and pattern avoidance in words. *J. Comb. Theory Ser. A*, 110(1):127–145, 2005.
[15] Alexander Burstein. *Enumeration of words with forbidden patterns*. PhD thesis, University of Pennsylvania, 1998.

[16] Alexander Burstein and Sergey Kitaev. Partially ordered patterns and their combinatorial interpretations. *Pure Math. and Appl. (PU.M.A.)*, 19(2-3):27–38, 2008.

[17] Alexander Burstein and Toufik Mansour. Words restricted by patterns with at most 2 distinct letters. *The Electronic Journal of Combinatorics*, 9(2), 2002.

[18] Alexander Burstein and Toufik Mansour. Counting occurrences of some subword patterns. *Discrete Math. Theor. Comput. Sci.*, 6(1):1–11, 2003.

[19] Frédéric Chyzak, Michael Drmota, Thomas Klausner, and Gerard Kok. The distribution of patterns in random trees. *Combinatorics, Probability and Computing*, 17(1):21–59, 2008.

[20] François Dahmani, David Futer, and Daniel T Wise. Growth of quasiconvex subgroups. In *Mathematical Proceedings of the Cambridge Philosophical Society*, pages 1–26. Cambridge University Press, 2018.

[21] Michael Dairyko, Lara Pudwell, Samantha Tyner, and Casey Wynn. Non-contiguous pattern avoidance in binary trees. *The Electronic Journal of Combinatorics*, 19(3):P22, 2012.

[22] Michael Drmota. *Random Trees: An Interplay Between Combinatorics and Probability*. Springer Science & Business Media, 2009.

[23] G. Firro and T. Mansour. Restricted k-ary words and functional equations. *Discrete Appl. Math.*, 157(4):602–616, 2009.

[24] Philippe Flajolet and Andrew M Odlyzko. Random mapping statistics. In *Workshop on the Theory and Application of of Cryptographic Techniques*, pages 329–354. Springer, 1989.

[25] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, Cambridge, 2009.

[26] Nathan Gabriel, Katherine Peske, Lara Pudwell, and Samuel Tay. Pattern avoidance in ternary trees. *Journal of Integer Sequences*, 15(2):3, 2012.

[27] Zhicheng Gao, Andrew MacFie, and Daniel Panario. Counting words by number of occurrences of some patterns. *The Electronic Journal of Combinatorics*, 18(1):P143, 2011.

[28] Zhicheng Gao, Andrew MacFie, and Qiang Wang. Counting compositions over finite abelian groups. *The Electronic Journal of Combinatorics*, 25(2):P2.19, 2018.

[29] Meghann Moriah Gibson, Matthew Just, and Hua Wang. Note on restricted parts in cyclic compositions. *INTEGERS*, 18:2, 2018.

[30] Ian P Goulden and David M Jackson. *Combinatorial Enumeration*. Courier Corporation, 2004.
Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics*. Addison-Wesley Publishing Company, Reading, MA, second edition edition, 1994. A foundation for computer science.

Jonathan L. Gross and Thomas W. Tucker. *Topological Graph Theory*. Dover, 1987.

Petros Hadjicostas. Cyclic compositions of a positive integer with parts avoiding an arithmetic sequence. *Journal of Integer Sequences*, 19(2):3, 2016.

Petros Hadjicostas. Cyclic, dihedral and symmetrical carlitz compositions of a positive integer. *Journal of Integer Sequences*, 20(2):3, 2017.

Silvia Heubach, Sergey Kitaev, and Toufik Mansour. Partially ordered patterns and compositions. *Pure Math. and Appl. (PU.M.A.)*, 17(1-2):1–12, 2007.

Silvia Heubach and Toufik Mansour. Avoiding patterns of length three in compositions and multiset permutations. *Advances in Applied Mathematics*, 36(2):156–174, 2006.

Silvia Heubach and Toufik Mansour. *Combinatorics of Compositions and Words*. Chapman & Hall/CRC, 2010.

Roger A. Horn and Charles R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, 1994. Corrected reprint of the 1991 original.

John Hunter. An introduction to real analysis. Department of Mathematics, University of California at Davis, 2014.

Vít Jelínek and Toufik Mansour. Wilf-equivalence on k-ary words, compositions, and parking functions. *The Electronic Journal of Combinatorics*, 16(R58):1, 2009.

Sergey Kitaev. Segmented partially ordered generalized patterns. *Theor. Comp. Sci.*, 349(3):420–428, 2005.

Sergey Kitaev and Toufik Mansour. Partially ordered generalized patterns and k-ary words. *Ann. Combin.*, 7(2), 2003.

Arnold Knopfmacher, Toufik Mansour, and Augustine Munagi. Smooth compositions and smooth words. *Pure Math. and Appl. (PU.M.A.)*, 22(2):209–226, 2011.

Arnold Knopfmacher, Toufik Mansour, Augustine Munagi, and Helmut Prodinger. Staircase words and chebyshev polynomials. *Applicable Analysis and Discrete Mathematics*, pages 81–95, 2010.

Donald E Knuth. Two notes on notation. *The American Mathematical Monthly*, 99(5):403–422, 1992.

Andrew MacFie. Genfunlib, April 2014.
[47] Andrew MacFie. Software for enumerative and analytic combinatorics. abs/1601.02683, 2016.

[48] Neal Madras and Hailong Liu. Random pattern-avoiding permutations. Algorithmic Probability and Combinatorics, AMS, Providence, RI, pages 173–194, 2010.

[49] Toufik Mansour. Restricted 132-avoiding $k$-ary words, Chebyshev polynomials, and continued fractions. Adv. in Appl. Math., 36(2):175–193, 2006.

[50] Toufik Mansour and Mark Shattuck. On avoidance of patterns of the form $\sigma\tau$ by words over a finite alphabet. Discrete Mathematics & Theoretical Computer Science, Vol. 17 no.2, Sep 2015.

[51] Toufik Mansour and Basel Sirhan. Counting $l$-letter subwords in compositions. Discrete Mathematics and Theoretical Computer Science, 8:285–297, 2006.

[52] Rodrigo SV Martins, Daniel Panario, Claudio Qureshi, and Eric Schmutz. Periods of iterations of mappings over finite fields with restricted preimage sizes. arXiv preprint arXiv:1701.09148, 2017.

[53] Carl D Meyer. Matrix Analysis and Applied Linear Algebra, volume 71. Siam, 2000.

[54] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders. NIST Digital Library of Mathematical Functions. Release 1.0.15, 2017.

[55] Igor Pak. Complexity problems in enumerative combinatorics. abs/1803.06636, 2018.

[56] Lara Pudwell. Enumeration schemes for pattern-avoiding words and permutations. PhD thesis, Rutgers, The State University of New Jersey, 2008.

[57] Bala Ravikumar and Gerry Eisman. Weak minimization of dfa–an algorithm and applications. Theoretical computer science, 328(1-2):113–133, 2004.

[58] A. Regev. Asymptotics of the number of $k$-words with an $l$-descent. The Electronic Journal of Combinatorics, 5:R15, 1998.

[59] Sidney I Resnick. A Probability Path. Springer Science & Business Media, 2013.

[60] Eric S Rowland. Pattern avoidance in binary trees. Journal of Combinatorial Theory, Series A, 117(6):741–758, 2010.

[61] Walter Rudin. Real and Complex Analysis. Tata McGraw-Hill Education, 1987.
[62] Carla D Savage and Herbert S Wilf. Pattern avoidance in compositions and
multiset permutations. *Advances in Applied Mathematics*, 36(2):194–201,
2006.

[63] Helmut H. Schaefer. *Banach lattices and positive operators*. Springer-
Verlag, New York-Heidelberg, 1974. Die Grundlehren der mathematischen
Wissenschaften, Band 215.

[64] Maxie D. Schmidt. Square series generating function transformations.
*Journal of Inequalities and Special Functions*, 8(2):125–156, 2017.

[65] Rocco Servedio and Yeong-Nan Yeh. A bijective proof on circular com-
positions. *Bulletin of the Institute of Mathematics Academia Sinica*,
23:283–294, 1995.

[66] Jeffrey Shallit. *A Second Course in Formal Languages and Automata
Theory*. Cambridge University Press, 2008.

[67] Richard P Stanley. Increasing and decreasing subsequences and their
variants. In *International Congress of Mathematicians*, volume 1, pages
545–579, 2007.

[68] Richard P. Stanley. *Enumerative Combinatorics*, volume 1 of *Cambridge
Studies in Advanced Mathematics*. Cambridge University Press, Cambridge,
second edition edition, 2012.

[69] Richard Edwin Stearns and Harry B. Hunt III. On the equivalence
and containment problems for unambiguous regular expressions, regular
grammars and finite automata. *SIAM Journal on Computing*, 14(3):598–
611, 1985.

[70] H. Wilf. What is an answer? *J. Amer. Math. Monthly*, 89(5):289–292,
1982.

[71] Roberto Zanasi. System and control theory, 2017. University of Modena
and Reggio Emilia.

[72] Doron Zeilberger. The umbral transfer-matrix method. i. foundations.
*Journal of Combinatorial Theory, Series A*, 91(1-2):451–463, 2000.