LOCAL WELL-POSEDNESS AND FINITE TIME BLOWUP FOR FOURTH-ORDER SCHröDINGER EQUATION WITH COMPLEX COEFFICIENT

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Abstract. We consider the fourth-order Schrödinger equation
\[ i\partial_t u + \Delta^2 u + \mu \Delta u + \lambda |u|^\alpha u = 0, \]
where \( \alpha > 0, \mu = \pm 1 \) or 0 and \( \lambda \in \mathbb{C} \). Firstly, we prove local well-posedness in \( H^4(\mathbb{R}^N) \) in both \( H^4 \) subcritical and critical case: \( \alpha > 0, (N - 8)\alpha \leq 8 \). Then, for any given compact set \( K \subset \mathbb{R}^N \), we construct \( H^4(\mathbb{R}^N) \) solutions that are defined on \( (-T, 0) \) for some \( T > 0 \), and blow up exactly on \( K \) at \( t = 0 \).

1. Introduction. In this paper, we study the following fourth-order nonlinear Schrödinger equation of the power type nonlinearity
\[
\begin{align*}
  i\partial_t u + \Delta^2 u + \mu \Delta u + \lambda |u|^\alpha u &= 0, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^N, \\
  u|_{t=0} &= \phi,
\end{align*}
\]
where \( \alpha > 0, \lambda \in \mathbb{C}, \mu \in \mathbb{R} \) is essentially given by \( \mu = \pm 1 \) or \( \mu = 0 \), and \( u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C} \) is a complex-valued function. Cauchy problem (1.1)-(1.2) may be considered as a generation of the classical Schrödinger equation
\[ i\partial_t u + \Delta u + \lambda |u|^\alpha u = 0 \]
with the complex coefficient \( \lambda \in \mathbb{C} \), which in turn is a particular case of the complex Ginzburg-Landau equation on \( \mathbb{R}^N \)
\[ \partial_t u = e^{i\theta} \Delta u + \zeta |u|^\alpha u \]
where \( |\theta| \leq \frac{\pi}{2} \) and \( \zeta \in \mathbb{C} \). Moreover the equation (1.4) in fact is a generic modulation equation that describes the nonlinear evolution of patterns at near-critical conditions. See for instance [17, 34, 40].

The fourth-order Schrödinger equation was introduced by Karpman [21], and Karpman-Shagalov [22] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. The study of nonlinear fourth-order Schrödinger equation with the power type nonlinearity as in (1.1) has been attracted a lot of interest in the past decade, see e.g. [6, 14, 19, 20, 32, 33, 37] and references therein.

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One main purpose of this article is to establish the local well-posedness for the Cauchy problem (1.1)-(1.2) in $H^4(\mathbb{R}^N)$. In [37], Pausader established the local well-posedness of (1.1) in both $H^2(\mathbb{R}^N)$ subcritical and critical cases, namely, $f(u) = \lambda |u|^\alpha u$ with $\lambda \in \mathbb{R}$ and $0 < \alpha, (N-4)\alpha \leq 8$. Moreover, he established the global well-posedness and the scattering for the radial datum in the defocusing energy-critical case. In this paper, we show that if the initial datum $\phi \in H^4$, then this regularity can be propagated. These results are very similar to the classical nonlinear Schrödinger case, see e.g. [9, 23].

**Theorem 1.1** (Subcritical case). Assume $\phi \in H^4(\mathbb{R}^N)$, $N \geq 1$, $\lambda \in \mathbb{C}, \mu = \pm 1$ or $0, \alpha > 0$ and $(N - 4)\alpha < 8$. There exist $T_{\text{max}}, T_{\text{min}} \in (0, \infty]$ and a unique maximum solution $u \in C((-T_{\text{min}}, T_{\text{max}}), H^4(\mathbb{R}^N))$ to the Cauchy problem (1.1)-(1.2). Moreover, the following properties hold,

1. If $T_{\text{max}} < \infty$, then
   \[
   \lim_{t \uparrow T_{\text{max}}} \|u(t)\|_{H^4} = \infty.  \tag{1.5}
   \]
   A corresponding conclusion is reached if $T_{\text{min}} < \infty$.
2. The solution $u \in H^1_{\text{loc}}((-T_{\text{min}}, T_{\text{max}}), L^r(\mathbb{R}^N))$ for any biharmonic admissible pair $(q, r) \in \Lambda_b$. (See Section 2 for the definition of $\Lambda_b$.)
3. The solution $u$ depends continuously on $\phi$ in the following sense: if $\phi_n \to \phi$ in $H^4(\mathbb{R}^N)$ and $u_n$ denotes the solution of (1.1) with the initial value $\phi_n$, then $u_n \to u$ in $C([-B, A], H^4(\mathbb{R}^N))$ and in $H^1((-B, A), L^r(\mathbb{R}^N))$ for any biharmonic admissible pair $(q, r) \in \Lambda_b$ and every $-T_{\text{min}} < -B < 0 < A < T_{\text{max}}$.

**Theorem 1.2** (Critical case). Assume $\phi \in H^4(\mathbb{R}^N)$, $N \geq 9$, $\lambda \in \mathbb{C}, \mu = \pm 1$ or $0$ and $\alpha = \frac{8}{N-8}$. There exist $T_{\text{max}}, T_{\text{min}} \in (0, \infty]$ and a unique maximum solution $u \in C((-T_{\text{min}}, T_{\text{max}}), H^4(\mathbb{R}^N))$ to the Cauchy problem (1.1)-(1.2). Moreover, the following properties hold,

1. If $T_{\text{max}} < \infty$, then
   \[
   \left\| u(t) \right\|_{L^{2N-8}(\mathbb{R}^N; \mathbb{R}^{2N(N-4)/8})} = \infty.  \tag{1.6}
   \]
   A corresponding conclusion is reached if $T_{\text{min}} < \infty$.
2. The solution $u \in H^1_{\text{loc}}((-T_{\text{min}}, T_{\text{max}}), L^r(\mathbb{R}^N))$ for any biharmonic admissible pair $(q, r) \in \Lambda_b$.
3. The solution $u$ depends continuously on $\phi$ in the following sense: if $\phi_n \to \phi$ in $H^4(\mathbb{R}^N)$ and $u_n$ denotes the solution of (1.1) with the initial value $\phi_n$, then $u_n \to u$ in $C([-B, A], H^4(\mathbb{R}^N))$ and in $W^{1,q}((-B, A), L^r(\mathbb{R}^N))$ for any biharmonic admissible pair $(q, r) \in \Lambda_b$ and every $-T_{\text{min}} < -B < 0 < A < T_{\text{max}}$.

We prove Theorems 1.1 and 1.2 in the spirit of [23, 24]. See also Cazenave, Fang and Han [9]. The fundamental ingredient is the choice of the metric space where the Banach fixed-point argument can be applied. We note that obtaining $H^4$ estimates by differentiating the equation four times in space would require that the nonlinearity is sufficiently smooth. Instead, we differentiate the equation once in time and then deduce $H^4$ estimates by using the equation and the estimates of $\partial_t u$. See Sections 4 and 5 for more details.

Another main result of this paper concerns the finite time blowup of the fourth-order Schrödinger equation (1.1). The existence of blowup solutions for the equation (1.1) has been strongly supported by a series of numerical studies done by Fibich, Ilan and Papanicolaou [18] for mass-critical and mass-supercritical powers $\alpha \geq \frac{8}{N}$. Recently, it has been proved rigorous in Boulenger [6], under the natural criteria
from the well-known blowup results for the classical nonlinear Schrödinger equation, that the blowup solutions exist for the radial data in $H^2(\mathbb{R}^N)$ with the negative initial energy. For more blowup results for the fourth-order Schrödinger equation, we refer to [2, 3, 4, 14, 20] and references therein.

In this paper, we establish the following finite time blowup result for the fourth-order Schrödinger equation (1.1).

**Theorem 1.3.** Under the conditions

$$\alpha > 0, \: (N - 8)\alpha \leq 8, \: \text{Im} \lambda < 0,$$

for any nonempty compact subset $K \subset \mathbb{R}^N$, there exist $S \in (-1, 0)$ and a solution $u \in C((S, 0), H^4(\mathbb{R}^N)) \cap C^1((S, 0), L^2(\mathbb{R}^N))$ to the equation (1.1) which blows up at time $0$ exactly on $K$ in the following sense.

1. If $x_0 \in K$ then for any $r > 0$,

$$\lim_{t \uparrow 0} \|u(t)\|_{L^2(|x - x_0| < r)} = \infty.$$  \hspace{1cm} (1.8)

2. If $U$ is an open subset of $\mathbb{R}^N$ such that $K \subset U$, then

$$\lim_{t \uparrow 0} \|\Delta^2 u(t)\|_{L^2(U)} = \infty, \: \lim_{t \uparrow 0} \|\partial_t u(t)\|_{L^2(U)} = \infty.$$  \hspace{1cm} (1.9)

3. If $\Omega$ is an open subset of $\mathbb{R}^N$ such that $\Omega \cap K = \emptyset$, then

$$\sup_{t \in (S, 0)} \|u(t)\|_{H^1(\Omega)} < \infty, \: \sup_{t \in (S, 0)} \|\partial_t u(t)\|_{L^2(\Omega)} < \infty.$$  \hspace{1cm} (1.10)

**Remark 1.1.** It follows from (1.8) and (1.9) that both $\|u(t)\|_2$, $\|\Delta^2 u(t)\|_2$ and $\|\partial_t u(t)\|_2$ blow up when $t \uparrow 0$. Moreover, the estimate (1.8) can be refined. More precisely, it follows from (8.7), (8.4) and (8.6) that

$$(1 - t)^{-\frac{N}{2} + \frac{\lambda}{2}} \lesssim \|u(t)\|_{L^2(|x - x_0| < r)} \lesssim (1 - t)^{-\frac{N}{2}},$$

where $k > N\alpha$ is given by (6.2).

Similar blowup results has been established for the classical Schrödinger equation. With the restriction $\alpha \geq 2$, it is proved in Cazenave, Martel and Zhao [12] that under the assumption $(N - 2)\alpha \leq 4$ and $\text{Im} \lambda = -1$, the finite time blowup occurs. More precisely, they introduce the ansatz $U_0$ satisfying

$$i\partial_t U_0 + \lambda |U_0|^\alpha U_0 = 0,$$

then using energy estimates and compactness arguments to complete the proof. After that, by refining the initial ansatz $U_0$ inductively, the blow-up result is extended to the whole range of $H^1$ subcritical powers and arbitrary $\text{Im} \lambda < 0$ in Cazenave, Han and Martel [10]. This result is then extended into $H^2$-subcritical case: $\alpha > 0$ and $(N - 4)\alpha < 4$ in [27] with the additional technical assumption $-\text{Im} \lambda > \frac{\alpha}{2} |\text{Re} \lambda|$.

We prove Theorem 1.3 by applying the strategy of [12]. More precisely, we consider the sequence $\{u_n\}_{n \geq 1}$ of solutions of (1.1) with the initial datum $u_n(\frac{n}{N}) = U_j(-\frac{n}{N})$, where $U_j$ is a refined blowup profile defined in Lemma 6.2. Since $U_j(\frac{n}{N}) \in H^4(\mathbb{R}^N)$ by (6.2) and (6.18), it follows from Theorems 1.1 and 1.2 that $u_n$ is defined on $(s_n, -\frac{1}{n})$ for some $s_n < -\frac{1}{n}$. We then define $\varepsilon_n(t) = u_n(t) - U_j(t)$. Next, by the energy arguments, we show that $\{\varepsilon_n\}_{n \geq 1}$ is uniformly bounded in $L^\infty((S, \tau), H^4)$ ($S$ is given by Proposition 7.1) for any $\tau \in (S, 0)$. In Section 8, we find $\varepsilon \in L^\infty((S, 0), H^2) \cap W^{1, \infty}((S, 0), L^2)$ and a subsequence of $\{\varepsilon_n\}_{n \geq 1}$ that converges weakly to $\varepsilon$ by the compactness argument. Next, we define $u(t) = U_j(t) + \varepsilon(t)$,
which turns out to be a $H^4$ solution to the equation (1.1). Finally, note that $\varepsilon$ is bounded in $L^\infty([S,0), H^4(\mathbb{R}^N)) \cap C^1([S,0), L^2(\mathbb{R}^N))$ and $U_J$ blows up at time 0 exactly on $K$, we deduce that $u(t)$ also blows up at time 0 exactly on $K$.

The solution $u$ given by Theorem 1.3 blows up at $t = 0$ like the function $U_J$ defined in Lemma 6.2. Since the function $U_0$ defined by (6.6) satisfying $i\partial_t U_0 + \lambda |U_0|^a U_0 = 0$, and $U_J$ is a refinement of $U_0$, we see that the solution $u$ displays an ODE-type blowup. We recall that there are many ODE-type blowup results for several other nonlinear equations, refer to [15, 11, 31, 36] for results in the parabolic context, refer to [1, 30, 39] for the nonlinear wave equations.

The rest of the paper is organized as follows. In Section 2, we fix notations and recall preliminary results. In Section 3, we establish the nonlinear estimate. In Sections 4 and 5, we prove Theorem 1.1 in the case $N \geq 9$ and Theorem 1.2 respectively. In Section 6, we introduce the blow-up ansatz and the corresponding estimates. Section 7 is devoted to the construction of a sequence of solutions of (1.1) close to the blow-up ansatz and some a priori estimates of the approximate solutions. In Section 8, we complete the proof of Theorem 1.3 by passing to the limit in the approximate solutions. Finally, an appendix is devoted to the proof of unconditional uniqueness theorem and Theorem 1.1 in the case $1 \leq N \leq 8$.

2. Preliminary. If $X,Y$ are nonnegative quantities, we sometimes use $X \lesssim Y$ to denote the estimate $X \leq CY$ for some positive constant $C$. Pairs of conjugate indices are written as $p$ and $p'$, $1 \leq p \leq \infty$, $1/p + 1/p' = 1$. We use $L^p(\mathbb{R}^N)$ to denote the Lebesgue space of functions $f : \mathbb{R}^N \rightarrow \mathbb{C}$ whose norm

$$\|f\|_{L^p} := \left( \int_{\mathbb{R}^N} |f(x)|^p dx \right)^{\frac{1}{p}} \tag{2.1}$$

where $1 \leq p \leq \infty$ is finite, with the usual modifications when $p = \infty$. Given $k \in \mathbb{N}, 1 \leq p \leq \infty$, we use $H^{k,p}(\mathbb{R}^N)$ or $H^{k,p}$ to denote the usual Sobolev space. We also use the space-time Lebesgue spaces $L^\gamma(I; L^p(\mathbb{R}^N))$ which are equipped with the norm

$$\|f\|_{L^\gamma(I; L^p(\mathbb{R}^N))} := \left( \int_I \|f\|_{L^p(\mathbb{R}^N)}^\gamma dt \right)^{\frac{1}{\gamma}}$$

for any space-time slab $I \times \mathbb{R}^N$, with the usual modification when either $\gamma$ or $p$ is infinity. For $1 \leq q < \infty$, Banach space $X$, we also introduce the vector-valued Lebesgue spaces $L^q(I; X)$ which are equipped with the norm

$$\|u\|_{L^q(I; X)} := \left( \int_I \|u\|_X^q dt \right)^{\frac{1}{q}}$$

for any time slab $I \subset \mathbb{R}$, with the usual modification when $q$ is infinity. Furthermore, the Banach space of functions $u \in H^{1,p}(I; X)$ satisfies $u, \partial_t u \in L^p(I; X)$. For $\alpha > 0$, we denote

$$1_{\alpha > 1} = \begin{cases} 0, & \text{if } 0 < \alpha \leq 1, \\ 1, & \text{if } \alpha > 1, \end{cases} \quad \text{and} \quad 1_{0 < \alpha < 1} = \begin{cases} 1, & \text{if } 0 < \alpha < 1, \\ 0, & \text{if } \alpha \geq 1. \end{cases}$$

Following standard notations, we introduce Schrödinger admissible pairs as well as the corresponding Strichartz’s estimate for the fourth-order Schrödinger equation.
Definition 2.1. A pair of Lebesgue space exponents \((\gamma, \rho)\) is called biharmonic Schrödinger admissible for the equation (1.1) if \((\gamma, \rho) \in \Lambda_b\) where

\[
\Lambda_b = \{ (\gamma, \rho) : 2 \leq \gamma, \rho \leq \infty, \frac{4}{\gamma} + \frac{N}{\rho} = \frac{N}{2}, (\gamma, \rho, N) \neq (2, \infty, 4) \}.
\]

Lemma 2.2 (Strichartz estimates for fourth-order NLS, [37]). Suppose that \((\gamma, \rho), (a, b) \in \Lambda_b\) are any two biharmonic admissible pairs, then for any function \(u \in L^2(\mathbb{R}^4)\) and \(h \in L^\alpha(I, L^\nu(\mathbb{R}^4))\) if \(\mu > 0\), suppose also \(|I| \leq 1\), we have

\[
\|e^{i(t\Delta + \mu \Delta)}u\|_{L^\gamma(I, L^\rho)} \leq C\|u\|_{L^2},
\]

\[
\left\| \int_0^t e^{i(t-s)(\Delta^2 + \mu \Delta)}h \, ds \right\|_{L^\gamma(I, L^\rho)} \leq C\|h\|_{L^\alpha(I, L^\nu)}
\]

where the integrates with respect to time are all over compact interval \(I \subset \mathbb{R}\).

In the rest of this paper, we fix \(\lambda \in \mathbb{C}, \mu = \pm 1\) or 0. We define

\[
\gamma = \frac{8(\alpha + 2)}{(N-8)\alpha}, \quad \rho = \frac{N(\alpha + 2)}{N + 4\alpha},
\]

when \(N \geq 9\) and \(0 < \alpha \leq \frac{8}{N-8}\). Then it is easy to check that \((\gamma, \rho) \in \Lambda_b\) is a biharmonic admissible pair, which will be frequently used along the paper. Moreover we define the map

\[
(Su)(t) = e^{i(t\Delta^2 + \mu \Delta)}\phi + i\lambda \int_0^t e^{i(t-s)(\Delta^2 + \mu \Delta)}[|u|^\alpha u](s) \, ds.
\]

Note that \(Su\) satisfies

\[
\begin{cases}
  i\partial_t (Su) + \Delta^2 (Su) + \mu \Delta (Su) + \lambda |u|^\alpha u = 0, \\
  (Su)(0) = \phi
\end{cases}
\]

and that

\[
\partial_t (Su) = i e^{i(t\Delta^2 + \mu \Delta)}[(\Delta^2 + \mu \Delta) \phi + \lambda |\phi|^\alpha \phi] + i\lambda \int_0^t e^{i(t-s)(\Delta^2 + \mu \Delta)}[|u|^\alpha u](s) \, ds.
\]

Given any \(\phi \in H^4\) and \(0 < T < \infty\), we define

\[
F(\phi, T) = \|e^{i(t\Delta^2 + \mu \Delta)(\Delta^2 + \mu \Delta)}\phi\|_{L^\gamma([0, T], L^\rho)} + \|e^{i(t\Delta^2 + \mu \Delta)}|\phi|^\alpha \phi\|_{L^\gamma([0, T], L^\nu)} + \|e^{i(t\Delta^2 + \mu \Delta)}\phi\|_{L^\gamma([0, T], L^\nu)}.
\]

It follows from Strichartz’s estimate (2.2) and Sobolev’s embedding \(H^4 \hookrightarrow L^{2(\alpha + 1)}\) that

\[
F(\phi, T) \lesssim \|\Delta^2 + \mu \Delta\|_{L^2} + \||\phi|^\alpha \phi\|_{L^2} + \|\phi\|_{H^4}
\]

\[
\lesssim \|\phi\|_{H^4} + \|\phi\|_{H^4}^{\alpha + 1}.
\]

So that we have

\[
\lim_{T \to 0} F(\phi, T) = 0
\]

by the dominate convergence theorem. Moreover, it follows from Strichartz’s estimate (2.2) and Sobolev’s embedding \(H^4 \hookrightarrow L^{2(\alpha + 1)}\) again that the map \((\phi, T) \mapsto \)}
where

\[ F(\phi, T) \text{ is continuous } H^4 \times (0, \infty) \rightarrow (0, \infty). \] Therefore, if \( \phi_n \rightarrow \phi \) in \( H^4 \) as \( n \rightarrow \infty \), then

\[ \sup_{n \geq 1} F(\phi_n, T) \rightarrow 0. \] (2.11)

3. **Nonlinear estimates.** The goal of this section is to establish the nonlinear estimates that we will need to prove Theorems 1.1 and 1.2. Throughout this section, we fix \( \lambda \in \mathbb{C}, \mu = \pm 1 \) or \( 0, N \geq 9, 0 < \alpha \leq \frac{8}{N} \) and \( I = [0, T] \) with \( T > 0 \).

Before starting the Lemmas, it is useful to introduce following numbers. Let \( q_0, p_0 \) be given by the equations

\[ \frac{\alpha + 2}{2(\alpha + 1)} = \frac{\alpha + 1}{q_0} + \frac{1}{\rho}, \] (3.1)

\[ \frac{1}{\rho} = \frac{\alpha}{2(\alpha + 1)} + \frac{1}{p_0}. \] (3.2)

Since \( \rho = \frac{N(\alpha + 2)}{N+4\alpha} \) by (2.4), it is easily seen that

\[ \frac{\alpha + 2}{2(\alpha + 1)} - \frac{4}{N} - \frac{1}{\rho} = \frac{8 - (N - 8)\alpha}{2(\alpha + 1)} \geq 0, \] (3.3)

\[ \frac{\alpha + 2}{2(\alpha + 1)} - \frac{1}{\rho} - \frac{1}{\rho} = \frac{4\alpha}{N} < 0, \] (3.4)

and

\[ \frac{1}{p_0} - \frac{1}{\rho} + \frac{4}{N} = \frac{8 - (N - 8)\alpha}{2N(\alpha + 1)} \geq 0. \] (3.5)

It follows from (3.1)-(3.5) that \( \frac{1}{\rho} \geq \frac{1}{q_0} \geq \frac{1}{\rho} - \frac{4}{N} \), and \( \frac{1}{\rho} > \frac{1}{p_0} \geq \frac{1}{\rho} - \frac{4}{N} \). Therefore, we deduce the embedding

\[ \|u\|_{L^{q_0}(I, L^p)} \lesssim \|u\|_{H^{1, \gamma}(I,L^p)}. \] (3.6)

**Lemma 3.1.** Assume that \( u^1, u^2 \in L^q(I, H^{4, \alpha}) \cap H^{1, \gamma}(I, L^\rho) \) are two solutions to the equation (1.1) with initial data \( \phi^1, \phi^2 \) respectively, there exist \( \rho < \tilde{\rho}_1, \tilde{\rho}_2 < \infty \) such that for any \( R > 0 \) we have

\[
\sup_{(q, r) \in \Lambda_b} \|Su^1 - Su^2\|_{H^{1, \gamma}(I, L^r)} \lesssim \left( \|\phi^1\|_{H^{q_1}}^\alpha + \|\phi^2\|_{H^{q_1}}^\alpha + 1 \right) \|\phi^1 - \phi^2\|_{H^4} + \left( T^{1 - \frac{n+2}{q_2}} + \frac{2 \alpha \alpha}{\gamma \gamma_1} \right) \|\partial_t u^1\|_{L^{(k-1)(q_1, \gamma)}(I, L^p)} \|u^1 - u^2\|_{H^{1, \gamma}(I, L^p)} + \left( T^{1 - \frac{n+2}{q_2}} + \frac{2 \alpha \alpha}{\gamma \gamma_1} \right) \|\partial_t u^2\|_{L^{(k-1)(q_1, \gamma)}(I, L^p)} \|u^1 - u^2\|_{H^{1, \gamma}(I, L^p)}
\]

\[
+ \left( T^{1 - \frac{n+2}{q_2}} + \frac{2 \alpha \alpha}{\gamma \gamma_1} \right) \|\partial_t u^2\|_{L^{(k-1)(q_1, \gamma)}(I, L^p)} \|u^1 - u^2\|_{H^{1, \gamma}(I, L^p)}
\]

\[
+ \left( T^{1 - \frac{n+2}{q_2}} + \frac{2 \alpha \alpha}{\gamma \gamma_1} \right) \|\partial_t u^2\|_{L^{(k-1)(q_1, \gamma)}(I, L^p)} \|u^1 - u^2\|_{H^{1, \gamma}(I, L^p)}
\]

\[
\times \|u^1 - u^2\|_{L^{(k-1)(q_1, \gamma)}(I, L^p)}, \] (3.7)

where

\[
(\partial_t u^2) R = \begin{cases} \|\partial_t u^2\|, & \text{if } |\partial_t u^2| \geq R, \\ 0 & \text{if } |\partial_t u^2| < R. \end{cases}
\]
Moreover, we have
\[
\|Su^1\|_{H^{1,\gamma}(I,L^q)} \lesssim F(\phi^1, T) + T^{1-\frac{\alpha+2}{\omega}} \|u^1\|_{L^\infty(I,H^{\frac{\alpha}{\omega+2}})} \|u^1\|_{H^{1,\gamma}(I,L^p)}, \tag{3.8}
\]

Proof. Firstly, we prove (3.8). It follows from equations (2.5), (2.7), Strichartz estimate (2.3), Hölder’s inequality and Sobolev’s embedding $H^{4+\rho} \to L^{\frac{\omega+2}{\alpha}}$ that
\[
\|Su^1\|_{H^{1,\gamma}(I,L^q)} \lesssim F(\phi^1, T) + \|u^1\|_{H^{1,\gamma}(I,L^p)},
\]

which gives (3.8).

We now prove (3.7). Using the same method as that used to derive (3.9), we obtain
\[
\sup_{(q,r)\in\Lambda_k} \|Su^1 - Su^2\|_{L^q(I,L^r)} \lesssim \|\phi^1 - \phi^2\|_{L^q} + \|u^1\|_{H^{1,\gamma}(I,L^p)} \lesssim \|\phi^1\|_{L^\infty(I,H^{\frac{\alpha}{\omega+2}})} \|u^1\|_{H^{1,\gamma}(I,L^p)}, \tag{3.10}
\]
and
\[
\sup_{(q,r)\in\Lambda_k} \|\partial_t(Su^1) - \partial_t(Su^2)\|_{L^q(I,L^r)} \lesssim \|u^1\|_{H^{1,\gamma}(I,L^p)} \lesssim \|u^1\|_{H^{1,\gamma}(I,L^p)}, \tag{3.11}
\]
where we used the Strichartz’s estimate (2.2). Applying Hölder’s inequality and Sobolev’s embedding $H^4 \to L^{2(\alpha+1)}$, we conclude that
\[
\|\phi^1\|_{L^2} \lesssim \|\phi^2\|_{L^2} \lesssim \|u^1\|_{H^{1,\gamma}(I,L^{\frac{\omega+2}{\alpha}})} \|u^1\|_{H^{1,\gamma}(I,L^{\frac{\omega+2}{\alpha}})}, \tag{3.12}
\]
Our next step is to control the last term in (3.11). Note that $\partial_t (|u|^\alpha) = \frac{\alpha+2}{\omega+2}|u|^\alpha \partial_t u + \frac{\alpha}{\omega+2} |u|^\alpha u^2 \partial_t \bar{u}$, we have by applying the inequality (6.21) below
\[
\|\partial_t \left( |u|^\alpha u - |u^2|^\alpha \partial_t u \right) \|_{L^q} \lesssim \|u^1\|_{H^{1,\gamma}(I,L^p)} \|u^1\|_{H^{1,\gamma}(I,L^p)}, \tag{3.13}
\]
Using Hölder’s inequality and the embedding $H^{4+\rho} \to L^{\frac{\omega+2}{\alpha}}$, we deduce that
\[
\|\|u^1\|_{H^{1,\gamma}(I,L^p)} \|u^1\|_{H^{1,\gamma}(I,L^p)}, \tag{3.14}
\]
By (3.10)–(3.14), the proof of (3.7) reduces to control $\|\partial_t u^2 F(u^1, u^2)\|_{L^{1,\gamma}(I,L^p)}$. We decompose $\partial_t u^2 = (\partial_t u^2)^R + (\partial_t u^2)^L$, where $(\partial_t u^2)^R = |\partial_t u^2| - (\partial_t u^2)^R$. Since $F(u^1, u^2) \lesssim \|u^1\|_{\alpha} + \|u^2\|_{\alpha}$ by (3.13), it follows from Hölder’s inequality and
Assume that the following two Lemmas.

This inequality together with (3.13) and Hölder’s inequality yields

Our final step is to estimate $\| (\partial_t u^2)^R F (u^1, u^2) \|_{L^{\gamma}(I, L^{\rho})}$. We consider two cases.

Suppose $0 < \alpha \leq 1$. Let $\tilde{\rho}_1 = \frac{\alpha}{\rho - (\alpha + 1)}$. Then it is easy to check that $\frac{1}{\tilde{\rho}_1} = \frac{\alpha}{\rho} + \frac{1}{\rho}$ and $\rho < \tilde{\rho}_1 < \infty$. This together with $| (\partial_t u^2)^R | \leq R$ yields

It now follows from (3.15) and Hölder’s inequality in time that

This inequality together with (3.13) and Hölder’s inequality yields

We turn now to the case $\alpha > 1$. Let $\tilde{\rho}_2$ be given by the equation $\frac{1}{\tilde{\rho}_2} = (\alpha - 1) \left( \frac{1}{\rho} - \frac{4}{N} \right) + \frac{1}{\rho} + \frac{1}{\tilde{\rho}_2}$. Then it is easy to check that $\rho < \tilde{\rho}_2 < \infty$. Using the same method as that used to derive (3.16), we obtain

It now follows from (3.13), (3.17), Hölder’s inequality and Sobolev’s embedding $H^{4, \rho} \hookrightarrow L^{\frac{2N}{N-2}}$ that

This completes the proof of Lemma 3.1.

In the process of the fixed-point argument, one first obtains estimates of $\partial_t u$ by time differentiation. Next, one obtains estimates of $\Delta^2 u$ by using the equation and estimates of $\partial_t u$. Thus, one needs to estimate $|u|^{\alpha} u$. To this end, we establish the following two Lemmas.

**Lemma 3.2.** Assume that $u \in L^\gamma(I, L^{q_0}) \cap H^{1, \gamma}(I, L^\rho)$ with $u(0) = \phi \in H^4$, where $q_0$ is defined in (3.1). Then we have $|u|^{\alpha} u \in C(I, L^2)$ and

$$
\| |u|^{\alpha} u \|_{L^\infty(I, L^2)} \lesssim \| \phi \|_{H^4} + T^{(\alpha+1)} (\frac{1}{\rho-1} - \frac{1}{\rho}) \| u \|_{L^\gamma(I, L^{q_0})} \| \partial_t u \|_{L^{\gamma}(I, L^{\rho})}. \tag{3.19}
$$
Proof. We first show that $|u|^α u ∈ C(I,L^2)$. Since

$$||u|^α u - |v|^α v| ≲ ||u|^α+1 u - |v|^α+1 v||_∞^{α+1}$$

(see Lemma 2.3 in [9]), it suffices to prove that $|u|^α+1 u ∈ C(I,L^{2(α+1)/α+2})$. Since $\frac{α+2}{2(α+1)} = \frac{α+1}{α+2} + \frac{1}{α}$ by (3.1), it follows from Hölder’s inequality that

$$\left|\partial_t \left( |u|^α u \right) \right|_{L^1(I,L^{2(α+1)/α+2})} \lesssim T^{1 - \frac{α+2}{2(α+1)}} ||u||^α_{L^2(I,L^{2(α+1)/α+2})} ||\partial_t u||_{L^γ(I,L^p)}, \hspace{1cm} (3.20)$$

which implies that $\partial_t \left( |u|^α+1 u \right) ∈ L^1(I,L^{2(α+1)/α+2})$. On the other hand, since $|φ|^α+1 φ ∈ L^{2(α+1)/α+2}$ by Sobolev’s embedding $H^4 \hookrightarrow L^{2(α+1)}$, we can apply (3.20) and the Fundamental Theorem of Calculus

$$|u|^α+1 u = |φ|^α+1 φ + \int_0^t \partial_s \left( |u|^α+1 u(s) \right) ds \hspace{1cm} (3.21)$$

to obtain $|u|^α+1 u ∈ C\left(I,L^{2(α+1)/α+2}\right)$, which in turn implies that $|u|^α u ∈ C(I,L^2)$.

Estimate (3.19) is now a consequence of (3.20), (3.21) and the inequality

$$|||u|^α u||_{L^2(I,L^2)} \lesssim |||u|^α+1 u||_{L^2(I,L^{2(α+1)/α+2})},$$

\[\blacksquare\]

Lemma 3.3. Assume $u ∈ L^γ(I,L^{p_0}) ∩ L^γ(I,L^{p_0}) ∩ H^1γ(I,L^p)$ with $u(0) = φ ∈ H^4$, where $q_0,p_0$ are defined in (3.1) and (3.2). Then we have

$$||u|^α u||_{L^γ(I,L^p)} \lesssim T^{α(\frac{1}{p_0} - \frac{1}{α})} \left( ||u||^α_{L^γ(I,L^{p_0}) ∩ L^γ(I,L^p)} + ||\partial_t u||^α_{L^γ(I,L^p)} \right)$$

$$+ ||φ||^α_{H^4} F(φ,T).$$

Proof. We assume first that $φ = 0$. Since $\frac{1}{p} = \frac{α}{2(α+1)} + \frac{1}{p_0}$ by (3.2), it follows from Hölder’s inequality that

$$||u|^α u||_{L^γ(I,L^p)} \lesssim ||||u||^α u||_{L^γ(I,L^{p_0})} \lesssim ||u||^α_{L^γ(I,L^{p_0})} \cdot ||u||_{L^γ(I,L^p)} \cdot \hspace{1cm} (3.22)$$

Our next step is to estimate $||u||^α_{L^γ(I,L^{2(α+1)})}$. Since $u(0) = φ = 0$, we see that

$$||u||^α_{L^γ(I,L^{2(α+1)})} = \left|\left|\int_0^t \partial_s \left( |u|^α u(s) \right) ds \right|_{L^γ(I,L^{2(α+1)}/α+2)} \right.$$}

$$\lesssim \left|\left|\partial_t \left( |u|^α u \right) \right|_{L^γ(I,L^{2(α+1)/α+2})} \right.$$}

This inequality together with (3.20) yields

$$||u||^α_{L^γ(I,L^{2(α+1)})} ≲ T^{α(\frac{1}{p_0} - \frac{1}{α})} \left( ||u||^α_{L^γ(I,L^{p_0}) ∩ L^γ(I,L^p)} + ||\partial_t u||^α_{L^γ(I,L^p)} \right).$$

(3.23)
It now follows from (3.22), (3.23) and Young’s inequality \( |x|^{\alpha + 1} + |y|^{\alpha + 1} \) that Lemma 3.3 holds in the case \( \phi = 0 \).

We now consider the case \( \phi \neq 0 \). Let \( v(t) = u(t) - e^{it(\Delta^2 + \mu \Delta)} \phi \). It follows from the elementary inequality \( |x|^{\alpha + 1} \lesssim |x|^{\alpha + 1} + |y|^{\alpha + 1} \) that

\[
\| |u|^{\alpha} u \|_{L^\gamma(I, L^p)} = \| u \|_{L^{\alpha+1}(I, L^{\alpha+1})}^{\alpha+1} 
\lesssim \| v \|_{L^{\alpha+1}(I, L^{\alpha+1})}^{\alpha+1} + \| e^{it(\Delta^2 + \mu \Delta)} \phi \|_{L^{\alpha+1}(I, L^{\alpha+1})}^{\alpha+1} 
\lesssim \| v \|_{L^\gamma(I, L^p)}^{\alpha} \| e^{it(\Delta^2 + \mu \Delta)} \phi \|_{L^\gamma(I, L^p)}.
\] (3.24)

Since \( v(0) = 0 \), we can apply the results established in the previous case to obtain

\[
\| |v|^{\alpha} v \|_{L^\gamma(I, L^p)} \lesssim T^{\alpha(\frac{\alpha}{\alpha+2} - \frac{\alpha}{p})} \left( \| v \|_{L^{\alpha+1}(I, L^{\alpha+1})}^{\alpha+1} \| \partial_t v \|_{L^\gamma(I, L^p)}^{\alpha+1} \right).
\] (3.25)

Moreover, since \( v(t) = u(t) - e^{it(\Delta^2 + \mu \Delta)} \phi \), we deduce from (3.25) and the elementary inequality \( |x|^{\alpha + 1} \lesssim |x|^{\alpha + 1} + |y|^{\alpha + 1} \) again that

\[
\| |v|^{\alpha} v \|_{L^\gamma(I, L^p)} \lesssim T^{\alpha(\frac{\alpha}{\alpha+2} - \frac{\alpha}{p})} \left( \| u \|_{L^{\alpha+1}(I, L^{\alpha+1})}^{\alpha+1} + \| \partial_t u \|_{L^\gamma(I, L^p)}^{\alpha+1} + F(\phi, T) \right).
\] (3.26)

where we used the embedding (3.6) to control \( L^\gamma(I, L^{\alpha_0}) \cap L^\gamma(I, L^{\rho_0}) \) norm of \( e^{it(\Delta^2 + \mu \Delta)} \phi \). On the other hand, applying the same method as that used to derive (3.22), we obtain

\[
\| e^{it(\Delta^2 + \mu \Delta)} \phi \|_{L^\gamma(I, L^p)} \lesssim \phi \|_{H^4} F(\phi, T),
\] (3.27)

where we used the embedding (3.6) and Sobolev’s embedding \( H^4 \hookrightarrow L^{2(\alpha+1)} \) in the last inequality.

Combining (3.24), (3.26) and (3.27), we obtain Lemma 3.3 in the case \( \phi \neq 0 \).}

The next Lemma is the key ingredient in our proof of the local existence and continuous dependence.

**Lemma 3.4.** Assume that \( u^1, u^2 \in L^\infty(I, H^4) \cap L^\gamma(I, H^{4, \rho}) \cap H^{1, \gamma}(I, L^\rho) \), then we have

\[
\| u^1 \|_{L^\infty(I, L^2)} \lesssim T^{\frac{\alpha}{\alpha+2} - \frac{\alpha}{p}} \left( \| u^1 \|_{L^{\alpha+1}(I, H^4)} + \| u^2 \|_{L^{\alpha+1}(I, H^4)} \right)
\times \| u^1 - u^2 \|_{L^\gamma(I, H^{4, \rho})} \| \partial_t u^1 - \partial_t u^2 \|_{L^\gamma(I, L^\rho)},
\] (3.28)
and
\[ \| u^1 \|^\alpha u^1 - | u^2 |^\alpha u^2 \|_{L^2(I,L^{2N/4})} \]
\[ \lesssim T^{1/2-\frac{\alpha}{2}} \left( \| u^1 \|_{L^\infty(I,H^4)} \| L^\gamma(I,B^4_{p,2}) \| + \| u^2 \|_{L^\infty(I,H^4)} \| L^\gamma(I,B^4_{p,2}) \| \right) \times \| u^1 - u^2 \|_{L^\gamma(I,H^{4,p})}. \] (3.29)

Proof. We first prove (3.28). It follows from H"older's inequality and Sobolev's embedding $H^4 \hookrightarrow L^{2(\alpha+1)}$ that
\[ \left( \| u^1 \|_{L^\infty(I,H^4)} + \| u^2 \|_{L^\infty(I,H^4)} \right) \| u^1 - u^2 \|_{L^\infty(I,L^{2(\alpha+1)})}. \] (3.30)
This inequality together with (3.23) and the embedding (3.6) yields (3.28).

We now prove (3.29). Since $\rho = \frac{N(\alpha+2)}{N+4\alpha}$ and $\alpha \leq \frac{8}{N-4}$, we have by direct calculation
\[ \frac{N-4}{2N} - \frac{\alpha}{2} \left( \frac{1}{2} - \frac{4}{N} \right) - \left( \frac{\alpha}{2} + 1 \right) \left( \frac{1}{\rho} - \frac{4}{N} \right) = \frac{8-(N-8)\alpha}{4N} \geq 0, \]
and
\[ \frac{N-4}{2N} - \frac{\alpha}{2} \cdot \frac{1}{2} - \left( \frac{\alpha}{2} + 1 \right) \frac{1}{\rho} = -\frac{\alpha}{4} - \frac{2(\alpha+1)}{N} < 0. \]
So that the straight line of points $(x,y)$ satisfying $y = -\frac{\alpha}{\alpha+2} x + \frac{N-4}{N(\alpha+2)}$ passes through the square (including edges) with vertices $(\frac{1}{2} - \frac{4}{N}, \frac{1}{\rho} - \frac{4}{N})$, $(\frac{1}{2}, \frac{1}{\rho} - \frac{4}{N})$ and $(\frac{1}{2}, \frac{1}{p_2} - \frac{4}{N})$. This implies that there exist $p_1, p_2$ satisfying $\frac{1}{2} \geq \frac{1}{p_1} \geq \frac{1}{2} - \frac{4}{N}, \frac{1}{p_2} \geq \frac{1}{p_2} > \frac{1}{\rho} - \frac{4}{N}$ and $\frac{N+4}{2N} = \frac{N}{2} \frac{1}{p_1} + (\frac{1}{2} + 1) \frac{1}{p_2}$. Hence we can deduce from H"older’s inequality and Sobolev’s embedding $H^4 \hookrightarrow L^{p_1}, H^4 \rho \hookrightarrow L^{p_2}$ that
\[ \left( \| u^1 \|^\alpha u^1 - u^2 \|_{L^{2N/4}} \right. \]
\[ \lesssim \| u^1 \|_{L^\infty(I,H^4)} \| u^1 \|_{L^\gamma(I,B^4_{p,2})} \| u^1 - u^2 \|_{L^\gamma(I,B^4_{p,2})}, \]
This inequality together with H"older’s inequality in time gives
\[ \left( \| u^1 \|^\alpha (u^1 - u^2) \right. \]
\[ \lesssim T^{1/2-\frac{\alpha}{4}} \| u^1 \|_{L^\infty(I,H^4)} \| u^1 \|_{L^\gamma(I,B^4_{p,2})} \| u^1 - u^2 \|_{L^\gamma(I,B^4_{p,2})}. \] (3.31)
Similarly,
\[ \left( \| u^2 \|^\alpha (u^1 - u^2) \right. \]
\[ \lesssim T^{1/2-\frac{\alpha}{4}} \| u^2 \|_{L^\infty(I,H^4)} \| u^2 \|_{L^\gamma(I,B^4_{p,2})} \| u^1 - u^2 \|_{L^\gamma(I,B^4_{p,2})}. \] (3.32)
Combining (3.31) and (3.32), we obtain (3.29). \(\square\)
4. **Proof of Theorem 1.1 in the case $N \geq 9$.** In this section, we use a contraction mapping argument based on the nonlinear estimates established in the previous Section to prove Theorem 1.1. The case $1 \leq N \leq 8$ is more simpler to deal with, since Sobolev’s embedding theorem $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ holds for any $2 \leq p < \infty$. We append its proof in the last section. In this section, we consider only the case $N \geq 9$.

We start with the following proposition.

**Proposition 4.1.** Assume $\lambda \in \mathbb{C}, \mu = \pm 1$ or $0, N \geq 9, 0 < \alpha < \frac{8}{N-5}, \phi \in H^4$ and $M \geq 2C_1 \left( \|\phi\|_{H^4}^2 + 1 \right) \|\phi\|_{H^4}^\alpha + 1$, where $C_1$ is the constant in (4.9). Then for any $0 < T \leq c(M)$, where $c(M)$ is the constant in (4.1), the Cauchy problem (1.1)-(1.2) admits a unique solution $u \in C \left( [0, T], H^4 \right) \cap L^\gamma \left( [0, T], H^4, \rho \right)$ with $\|u\|_{H^4 \cap [0, T], L^\gamma} \leq M$. Moreover, we have the further regularity, $u, u_t, \Delta^2 u \in C \left( [0, T], L^2 \left( \mathbb{R}^N \right) \right) \cap L^9 \left( (0, T), L^r \left( \mathbb{R}^N \right) \right)$ for every biharmonic admissible pair $(q, r) \in \Lambda_b$.

**Proof.** To begin with, we fix $c(M) > 0$ such that

$$
(C_1 + C_2) c(M)^\alpha \left( \frac{N-5}{2} \right) M^\alpha = \frac{1}{2},
$$

where $C_2$ are the constant (4.10).

Set $I = [0, T]$ and consider the metric space

$$
X_{T,M} = \left\{ u \in H^1(I, L^p) \cap L^\gamma(I, H^4) : \|u\|_{H_1(I, L^p)} \cap L^\gamma(I, H^4) \leq M \right\}
$$

(4.2)

It follows that $X_{T,M}$ is a complete metric space when equipped with the distance

$$
d(u, v) = \|u - v\|_{L^\gamma(I, L^p)}.
$$

(4.3)

To obtain a solution $u \in X_{T,M}$ to the equation (1.1), it suffices to show that the map $S$, defined in (2.5), is a contraction on the space $X_{T,M}$.

We first show that $S$ maps $X_{T,M}$ into itself. By (2.9) and (3.8), we have for any $u \in X_{T,M}$

$$
\|Su\|_{H^1(I, L^p)} \leq \|\phi\|_{H^4} + \phi\|_{H^4}^{\alpha + 1} + T^{1 - \frac{\alpha}{N-5}} \|u\|_{L^\gamma(I, H^4, \rho)}^{\alpha} ||u||_{H^1(I, L^p)}.
$$

(4.4)

Our next step is to estimate $\|Su\|_{L^\gamma(I, H^4, \rho)}$. It follows from the equation (2.6) that, for every $u \in X_{T,M}$,

$$
\|\Delta^2(Su)\|_{L^\gamma(I, L^p)} \leq \|\Delta(Su)\|_{L^\gamma(I, L^p)} + \|\Delta^{2}(Su)\|_{L^\gamma(I, L^p)} + \|\Delta^2 u\|_{L^\gamma(I, L^p)}.
$$

(4.5)

Moreover, it follows from Hölder’s inequality and Gagliardo-Nirenberg’s inequality that

$$
\|\Delta(Su)\|_{L^\gamma(I, L^p)} \leq C \left\|\Delta^2(Su)\right\|_{L^\gamma(I, L^p)}^{\frac{1}{2}} \left\|\Delta^2(Su)\right\|_{L^\gamma(I, L^p)}^{\frac{1}{2}} \leq C \left\|\Delta^2(Su)\right\|_{L^\gamma(I, L^p)}^{\frac{1}{2}} \left\|\Delta^2(Su)\right\|_{L^\gamma(I, L^p)}^{\frac{1}{2}} \leq \frac{1}{2} \|\Delta^2(Su)\|_{L^\gamma(I, L^p)} + \frac{1}{2} C^2 \|Su\|_{L^\gamma(I, L^p)},
$$

(4.6)

where we used Cauchy-Schwartz’s inequality in the last step. Estimates (4.5) and (4.6) imply that

$$
\|\Delta^2(Su)\|_{L^\gamma(I, L^p)} \leq \|\Delta(Su)\|_{L^\gamma(I, L^p)} + \|Su\|_{L^\gamma(I, L^p)} + \|Su\|_{L^\gamma(I, L^p)} + \|\Delta^2 u\|_{L^\gamma(I, L^p)}.
$$

(4.7)
On the other hand, applying Lemma 3.3, (2.9) and (3.6), we conclude that
\[
\|u\| \leq T^\alpha \left( \|u\|_{L^\beta(t,L^\nu)}^{\alpha+1} + \|\partial_t u\|_{L^\beta(t,L^\nu)}^{\alpha+1} \right) \|\phi\|_{H^4}^{\alpha+1} + \|\phi\|_{H^4}^{2(\alpha+1)}. \tag{4.8}
\]
It now follows from (4.4), (4.7) and (4.8) that, for \( u \in X_{T,M} \)
\[
\|Su\|_{H^{1,\gamma}(0,T) \cap L^{4,\rho}(0,T)} \leq C_1 \left( \|\phi\|_{H^4}^{2(\alpha+1)} + 1 \right) \|\phi\|_{H^4} + C_1 T^{\alpha} \left( \frac{\alpha}{2} \right) M^{\alpha+1}. \tag{4.9}
\]
Since \( 0 < T \leq c(M) \), we can apply (4.1) and (4.9) to obtain
\[
\|Su\|_{H^{1,\gamma}(0,T) \cap L^{4,\rho}(0,T)} \leq M.
\]
Our next aim is the desired Lipschitz property of \( S \) with respect to the metric \( d \) defined in (4.3). Given \( u, v \in X_{T,M} \), we can apply the same method as that used to derive (3.10) to obtain
\[
d(Su, Sv) \leq T^{1-\frac{\alpha+2}{\alpha}} \left( \|u\|_{L^\beta(t,L^\nu)} + \|v\|_{L^\beta(t,L^\nu)} \right) \|u-v\|_{L^\beta(t,L^\nu)} \leq \frac{1}{2} d(u,v). \tag{4.10}
\]
So we prove that \( S \) is a contraction on the space \( X_{T,M} \) for \( 0 < T \leq c(M) \). Using Banach’s fixed-point argument, we deduce that there exists a unique solution \( u \in L^\infty([0,T],H^4) \cap L^\gamma([0,T],H^{4,\rho}) \) to the Cauchy problem (1.1)-(1.2).

Now we prove some further regularity properties.

Firstly, we prove that \( u \in H^{1,\gamma}([0,T),L^r(\mathbb{R}^N)) \) for every biharmonic admissible pair \( (q,r) \in A_\delta \) and \( u, \partial_t u \in C([0,T],L^2(\mathbb{R}^N)) \). Note that we have (see (2.5) and (2.7))
\[
u(t) = e^{it(\Delta^2+\mu\Delta)}\phi + i\lambda \int_0^t e^{i(t-s)(\Delta^2+\mu\Delta)} \|u\|^{\alpha} u(s) \, ds, \tag{4.11}
\]
and
\[
\partial_t u = i\lambda e^{it(\Delta^2+\mu\Delta)} \left[ (\Delta^2 + \mu\Delta) \phi - \lambda |\phi|^{\alpha} \phi \right] - i\lambda \int_0^t e^{i(t-s)(\Delta^2+\mu\Delta)} \partial_s \|u\|^{\alpha} u(s) \, ds.
\]
Since \( \phi, (\Delta^2 + \mu\Delta) \phi - \lambda |\phi|^{\alpha} \phi \in L^2(\mathbb{R}^N) \) and \( \|u\|^{\alpha} u \in H^{1,\gamma'}([0,T),L^r(\mathbb{R}^N)) \) (see the proof of (3.9)), we deduce from Strichartz’s estimates (2.3) that \( u \in H^{1,\gamma}([0,T),L^r(\mathbb{R}^N)) \) for every biharmonic admissible pair \( (q,r) \in A_\delta \) and \( u, \partial_t u \in C([0,T],L^2(\mathbb{R}^N)) \).

Next we prove that \( \Delta^2 u \in C([0,T],L^2(\mathbb{R}^N)) \). For any \( t_1, t_2 \in [0,T] \), we have by using equation (1.1)
\[
\|\Delta^2 u(t_1) - \Delta^2 u(t_2)\| \lesssim \|\partial_t u(t_1) - \partial_t u(t_2)\| + \|u(t_1) - u(t_2)\|.
\tag{4.12}
\]
Using the same method as that used to derive (4.7), we deduce that
\[
\|\Delta^2 u(t_1) - \Delta^2 u(t_2)\| \lesssim \|\partial_t u(t_1) - \partial_t u(t_2)\| + \|u(t_1) - u(t_2)\|.
\tag{4.13}
\]
Since \( u, \partial_t u \) and \( \|u\|^{\alpha} u \in C([0,T],L^2(\mathbb{R}^N)) \) (see Lemma 3.2 for the continuity of \( \|u\|^{\alpha} u \)), we deduce from (4.13) that \( \Delta^2 u \in C([0,T],L^2(\mathbb{R}^N)) \). In particular, we have \( u \in L^\infty([0,T],H^4(\mathbb{R}^N)) \).
Our final step is to show that $\Delta^2 u \in L^q \left((0,T), L^r \left(\mathbb{R}^N\right) \right)$ for every admissible pair $(q,r) \in \Lambda_b$. Similarly to (4.7), we have
\[
\|\Delta^2 u\|_{L^q((0,T),L^r)} \lesssim \|u\|_{H^{1,\alpha}((0,T),L^r)} + \|u^\alpha\|_{L^q((0,T),L^r)}
\] (4.14)

On the other hand, since $u \in L^q \left((0,T), H^{4,\rho} \left(\mathbb{R}^N\right) \right) \cap L^\infty \left((0,T), H^4 \left(\mathbb{R}^N\right) \right) \cap H^{1,\gamma} \left((0,T), L^\rho \left(\mathbb{R}^N\right) \right)$, it follows from Lemma 3.4 that $|u|^\alpha u \in L^2 \left((0,T), L^\frac{2N}{\alpha+2} \left(\mathbb{R}^N\right) \right) \cap L^\infty \left((0,T), L^2 \left(\mathbb{R}^N\right) \right)$. This together with Hölder’s inequality implies that $|u|^\alpha u \in L^q \left((0,T), L^r \left(\mathbb{R}^N\right) \right)$ for every admissible pair $(q,r) \in \Lambda_b$. Therefore, from (4.14), we see that $\Delta^2 u \in L^q \left((0,T), L^r \left(\mathbb{R}^N\right) \right)$ for every admissible pair $(q,r) \in \Lambda_b$.

The proof of Proposition 4.1 is now completed. \qed

**Proof of Theorem 1.1 in the case $N \geq 9$.** We now resume the proof of Theorem 1.1. We consider only the positive time direction. A corresponding conclusion for reverse direction follows similarly. Firstly, we deduce from Proposition 4.1 that there exist $T > 0$ and a unique solution $u \in C \left([0,T], H^4 \right)$ to the Cauchy problem (1.1)-(1.2) with $u, u_t, \Delta^2 u \in C \left([0,T], L^2 \left(\mathbb{R}^N\right) \right) \cap L^q \left([0,T], L^r \left(\mathbb{R}^N\right) \right)$ for every biharmonic admissible pair $(q,r) \in \Lambda_b$. Using the uniqueness in Appendix and the standard procedure as in Chapter 4 of [7], we can extend $u$ to a maximal solution $u \in C \left([0,T_{\text{max}}], H^4 \right)$ to the Cauchy problem (1.1)-(1.2) with $u, u_t, \Delta^2 u \in C \left([0,T_{\text{max}}], L^2 \left(\mathbb{R}^N\right) \right) \cap L^q \left([0,T_{\text{max}}], L^r \left(\mathbb{R}^N\right) \right)$ for every biharmonic admissible pair $(q,r) \in \Lambda_b$. Moreover, since the solution $u$ of Proposition 4.1 is constructed on an interval depending on $\|\phi\|_{H^4}$, we deduce the blowup alternative (1.5).

In the rest of this section, we prove the continuous dependence of the solution map.

For any $0 < A < T_{\text{max}}(\phi)$, we set
\[
M = 4C_1 \left(\|u\|_{L^\infty(\left[0,A\right],H^4)}^2 + 1\right) \|u\|^\alpha_{L^\infty(\left[0,A\right],H^4)} + \|u\|_{L^\infty(\left[0,A\right],H^{4,\rho}) \cap H^{1,\gamma}(\left[0,A\right],L^\rho)} + 2
\] (4.15)
where $C_1$ is the constant in (4.9). Moreover, we fix $T > 0$ sufficiently small such that
\[
T \leq c(M), \left(C_3 + C_4\right) T^{1-\frac{\alpha+2}{\beta+1}} + C_5 T^{\frac{\alpha}{\beta+1}} \left(M^{\alpha + \frac{\alpha(2\alpha+1)}{\beta+1}} \right) \leq \frac{1}{2}
\] (4.16)
where $c(M)$ is the constant in Proposition 4.1 and $C_3, C_4, C_5$ are the constants in (4.20), (4.21) and (4.28), respectively.

Since $\phi_n \to \phi$ in $H^4$, we know that there exists a positive $n_1$ such that $M \geq 2C_1 \left(\|\phi_n\|_{H^4}^2 + 1\right) \|\phi_n\|_{H^4}^\alpha + 1$ for every $n \geq n_1$. Then we can deduce from Proposition 4.1 that for every $n \geq n_1$, the following equation
\[
u_n = e^{it(\Delta^2 + \mu \Delta)}\phi_n + i\lambda \int_0^t e^{i(t-s)(\Delta^2 + \mu \Delta)} [\|u_n\|^{\alpha} u_n] \, ds
\] (4.17)
admits a unique solution $u_n \in C(I,H^4) \cap L^\gamma(I,B^4_{p,2})$ with $I = [0,T]$ and $u_n$ is uniformly bounded,
\[
\|u_n\|_{L^\infty(I,H^{4,\rho}) \cap H^{1,\gamma}(I,L^\rho)} \leq M, \text{ for } \forall n \geq n_1.
\] (4.18)

The proof of the continuous dependence will proceed by a series of Claims.

**Claim 4.1.** Given any $(q,r) \in \Lambda_b$, we have $\|u_n - u\|_{H^{1,\gamma}(I,L^r)} \to 0$ as $n \to \infty$. 
Proof. We first show that
\[
\lim_{n \to \infty} \| u_n - u \|_{L^q(I, L^r) \cap L^\gamma(I, L^p)} = 0.
\]
(4.19)
Applying the same method as that used to derive (3.10), we conclude that
\[
\| u_n - u \|_{L^q(I, L^r) \cap L^\gamma(I, L^p)} \\
\quad \leq C_3 \| \phi_n - \phi \|_{L^2} + C_3 T^{1 - \frac{4 + \alpha}{4 + \alpha} - \frac{3}{2 \gamma}} M^\alpha \| u_n - u \|_{L^q(I, L^r)}.
\]
(4.20)
where we used the boundedness of \( u_n, u \) in (4.15) and (4.18). This inequality together with the smallness of \( T \) in (4.16) yields (4.19).

Next, it follows from Lemma 3.1, (4.19) and the boundedness of \( u_n, u \) again, we have, for any \( R > 0 \)
\[
\limsup_{n \to \infty} \| u_n - u \|_{H^{1, \gamma}(I, L^r) \cap H^{1, \gamma}(I, L^p)} \\
\quad \leq T^{1 - \frac{4 + \alpha}{4 + \alpha} - \frac{3}{2 \gamma}} M^\alpha \limsup_{n \to \infty} \| \partial_t u_n - \partial_t u \|_{L^q(I, L^r)}.
\]
(4.21)
Since \( C_4 T^{1 - \frac{4 + \alpha}{4 + \alpha} - \frac{3}{2 \gamma}} M^\alpha \leq \frac{1}{2} \) by (4.16), it follows from (4.21) that \( \limsup_{n \to \infty} \| u_n - u \|_{H^{1, \gamma}(I, L^r)} = 0 \).

This finishes the proof of Claim 4.1.

\[\square\]

Claim 4.2. \( \| u_n - u \|_{L^\gamma(I, H^{1, \gamma})} \to 0 \) as \( n \to \infty \).

Proof. Using the same method as that used to derive (4.7), we obtain
\[
\| u_n - u \|_{L^\gamma(I, H^{1, \gamma})} \quad \leq \quad \| u_n - u \|_{H^{1, \gamma}(I, L^r)} + \| u_n \|_{H^{1, \gamma}(I, L^r)}.
\]
(4.22)
Next, we estimate \( \| u_n \|_{H^{1, \gamma}(I, L^r)} \). It follows from Hölder’s inequality that
\[
\| u_n \|_{H^{1, \gamma}(I, L^r)} \quad \leq \quad \left( \| u_n \|_{L^{(\alpha+1)}(I, L^{(\alpha+1)}(I, L^r))} + \| u_n \|_{L^{(\alpha+1)}(I, L^{(\alpha+1)}(I, L^r))} \right) \| u_n - u \|_{L^{(\alpha+1)}(I, L^r)}.
\]
(4.23)
Moreover, applying Lemma 3.3 and the embedding (3.6), we deduce that
\[
\| u_n - u \|_{L^{(\alpha+1)}(I, L^r)} \quad \leq \quad T^{(\alpha - \frac{4}{2})} \left( \| u_n - u \|_{L^{(\alpha+1)}(I, H^{1, \gamma})} + \| \partial_t u_n - \partial_t u \|_{L^{(\alpha+1)}(I, H^{1, \gamma})} \right)
\]
(4.24)
\[
+ \| \phi_n - \phi \|_{H^{1, \gamma}} F(\phi_n - \phi, T).
\]
Similarly, we have
\[
\| u_n \|_{L^\gamma(I, L^r)} \quad \leq \quad T^{(\alpha - \frac{4}{2})} \left( \| u_n \|_{L^{(\alpha+1)}(I, H^{1, \gamma})} + \| \partial_t u_n \|_{L^{(\alpha+1)}(I, H^{1, \gamma})} \right)
\]
(4.25)
and
\[ |||u^n|||_{L^\gamma(I,L^\rho)} \lesssim T^{\alpha \left( \frac{4}{1+\alpha} - \frac{1}{2} \right)} \left( ||u||_{L^\gamma(I,H^{4,\rho})}^{\alpha + 1} + ||\partial_t u||_{L^\gamma(I,H^{4,\rho})}^{\alpha + 1} \right) + \phi|||_{H^4} F(\phi,T). \] (4.26)

Estimates (4.23), (4.24), (4.25) and (4.26) imply
\[ |||u^n|||_{L^\gamma(I,L^\rho)} \lesssim T^{\alpha \left( \frac{4}{1+\alpha} - \frac{1}{2} \right)} \left( M^\alpha + M^{\frac{\alpha(2\alpha+1)}{\alpha+1}} \right) ||u_n - u||_{L^\gamma(I,H^{4,\rho})} + C(M,T) \left( ||\partial_t u_n - \partial_t u||_{L^\gamma(I,L^\rho)} + ||\phi_n - \phi|||_{H^4} \right), \] (4.27)
where we also used (2.9) and the boundedness of \( u_n \) in (4.15), (4.18). Applying (4.22) and (4.27), we obtain
\[ ||u_n - u||_{L^\gamma(I,H^{4,\rho})} \leq C_5 T^{\frac{\alpha}{\alpha+1}} \left( \frac{4}{1+\alpha} - \frac{1}{2} \right) \left( M^\alpha + M^{\frac{\alpha(2\alpha+1)}{\alpha+1}} \right) ||u_n - u||_{L^\gamma(I,H^{4,\rho})} + C(M,T) \left( ||\partial_t u_n - \partial_t u||_{L^\gamma(I,L^\rho)} + ||\phi_n - \phi|||_{H^4} \right). \] (4.28)

Note that \( C_5 T^{\frac{\alpha}{\alpha+1}} \left( \frac{4}{1+\alpha} - \frac{1}{2} \right) \left( M^\alpha + M^{\frac{\alpha(2\alpha+1)}{\alpha+1}} \right) \leq \frac{1}{2} \) by (4.16), it follows from Claim 4.1 and (4.28) that Claim 4.2 holds. \( \square \)

Claim 4.3. Given any \((q,r) \in \Delta_0\), we have \( ||u_n - u||_{L^\gamma(I,H^{4,\rho})} \to 0 \) as \( n \to \infty \).

Proof. Firstly, we prove that \( u_n \) is uniformly bounded in \( L^\infty(I,H^4) \),
\[ \sup_{n \geq 1} ||u_n||_{L^\infty(I,H^4)} < \infty. \] (4.29)

Analogously to (4.4) and (4.7), we have
\[ ||u_n||_{H^{1,\infty}(I,L^2)} \lesssim ||\phi_n||_{H^4}^\alpha + ||\phi_n||_{H^4}^{\alpha + 1} + T^{1 - \frac{\alpha+2}{\alpha+1}} \left( ||u||_{L^\infty(I,H^{4,\rho})} \right) ||u_n||_{H^{1,\gamma}(I,L^\rho)}, \] (4.30)
and
\[ ||\Delta^2 u_n||_{L^\infty(I,L^2)} \lesssim ||\partial_t u_n||_{L^\infty(I,L^2)} + ||u_n||_{L^\infty(I,L^2)} + ||u_n||_{L^\infty(I,L^2)} \] (4.31)

Since \( u_n \) is uniformly bounded in \( L^\gamma(I,H^{4,\rho}) \cap H^{1,\gamma}(I,L^\rho) \) by (4.18), it follows from (4.30), (4.31) and Lemma 3.2 that (4.29) holds.

We now resume the proof of Claim 4.3. Analogously to (4.7), we have
\[ ||u_n - u||_{L^\gamma(I,H^{4,\rho})} \lesssim ||u_n - u||_{H^{1,\gamma}(I,L^\rho)} + \left( ||u_n||_{L^\infty(I,L^\rho)} \right) ||u_n||_{L^\gamma(I,L^\rho)}. \] (4.32)

Next, we prove that
\[ \lim_{n \to \infty} ||u_n||_{L^\infty(I,L^\rho)} = 0. \] (4.33)
Since \( ||u_n - u||_{L^\gamma(I,H^{4,\rho})} \to 0 \) as \( n \to \infty \) by Claim 4.2, we can apply Lemma 3.4 to deduce that
\[ \lim_{n \to \infty} \left( ||u_n||_{L^\infty(I,L^2)} \right) = 0, \] (4.34)
where we also used (4.15), (4.18) and (4.29). Applying (4.34) and Hölder’s inequality, we obtain (4.33).

Claim 4.3 is now an immediate consequence of (4.32), (4.33) and Claim 4.1. \( \square \)
We now resume the proof of the continuous dependence. It follows from Claims 4.1 and 4.3 that \( u_n \to u \) in \( L^q \left( [0, T], H^{1, q} \right) \cap H^{1, q} \left( [0, T], L^r \right) \) for any biharmonic admissible pair \((q, r) \in \Lambda_0\). In particular, we have \( \| u_n (T) - u(T) \|_{H^4} \underset{n \to \infty}{\to} 0 \). Arguing as previously, we deduce that the solution \( u_n \) exists on the interval \([T, 2T]\) for \( n \geq n_2 \) and that \( u_n \to u \) in \( L^q \left( [T, 2T], H^{4, q} \right) \cap H^{1, q} \left( [T, 2T], L^r \right) \) for any biharmonic admissible pair \((q, r) \in \Lambda_0\). Iterating finitely many times like this, we get the continuous dependence on the interval \([0, A]\). \( \square \)

5. Proof of Theorem 1.2. In this section, we prove Theorem 1.2. For the convenience of the reader, we briefly sketch the proof. Indeed, readers seeking a fuller treatment of certain details can consult Section 4. Throughout this section, we denote \( I = [0, T] \) with \( T > 0 \).

We first give an analogue of Proposition 4.1.

**Proposition 5.1.** Assume that \( \lambda \in \mathbb{C}, \mu = \pm 1 \) or \( 0, N \geq 0, \alpha = \frac{8}{N-8}, \phi \in H^4 \) and \( M \) be sufficiently small such that

\[
(C_6 + C_7) M^\alpha \leq \frac{1}{2}, \tag{5.1}
\]

where \( C_6, C_7 \) are the constants in (5.6) and (5.7), respectively. Let \( T > 0 \) and suppose further that

\[
C_6 \left( 1 + \| \phi \|_{H^4}^8 \right) F (\phi, T) \leq \frac{M}{2}. \tag{5.2}
\]

It follows that there exists a unique solution \( u \in C \left( [0, T], H^4 \right) \cap L^\gamma \left( [0, T], H^{4, \rho} \right) \) to the Cauchy problem (1.1)-(1.2) with \( \| u \|_{H^{1, \gamma} \left( I, L^p \right) \cap L^\gamma \left( I, H^{4, \rho} \right)} \leq M \). Moreover, we have \( u, u_t, \Delta^2 u \in C \left( [0, T], L^2 (\mathbb{R}^N) \right) \cap L^q \left( [0, T], L^r (\mathbb{R}^N) \right) \) for every biharmonic admissible pair \((q, r) \in \Lambda_0\).

**Proof.** We look for a fixed point of the map \( S \) on the space \( X_{T, M} \), where \( S, X_{T, M} \) were defined in (2.5) and (4.2) respectively.

We first show that \( S \) maps \( X_{T, M} \) into itself. Note that \( \gamma = \alpha + 2 \) in the critical case \( \alpha = \frac{8}{N-8} \), so that by Lemma 3.1 we have

\[
\| S u \|_{H^{1, \gamma} \left( I, L^p \right)} \lesssim F (\phi, T) + \| u \|_{L^\gamma \left( I, H^{4, \rho} \right)} \| u \|_{H^{1, \gamma} \left( I, L^p \right)} . \tag{5.3}
\]

Our next step is to estimate \( \| S u \|_{L^\gamma \left( I, H^{4, \rho} \right)} \). Similarly to (4.7), we have

\[
\| \Delta^2 (S u) \|_{L^\gamma \left( I, L^p \right)} \lesssim \| S u \|_{H^{1, \gamma} \left( I, L^p \right)} + \| u \|_{L^\gamma \left( I, L^p \right)} . \tag{5.4}
\]

Moreover, it follows from Lemma 3.3 and the embedding (3.6) that

\[
\| u \|_{L^\gamma \left( I, L^p \right)} \lesssim \left( \| u \|_{L^{\gamma + 1} \left( I, H^{4, \rho} \right)} + \| \partial_t u \|_{L^{\gamma + 1} \left( I, L^p \right)} \right) + \| \phi \|_{H^4}^8 F (\phi, T) . \tag{5.5}
\]

It now follows from (5.3), (5.4) and (5.5) that, for any \( u \in X_{T, M} \),

\[
\| S u \|_{H^{1, \gamma} \left( I, L^p \right) \cap L^\gamma \left( I, H^{4, \rho} \right)} \leq C_6 F (\phi, T) + C_6 M^{\alpha + 1} + C_6 \| \phi \|_{H^4}^8 F (\phi, T) . \tag{5.6}
\]

Applying (5.1), (5.2) and (5.6), we conclude that

\[
\| S u \|_{H^{1, \gamma} \left( I, L^p \right) \cap L^\gamma \left( I, H^{4, \rho} \right)} \leq M .
\]
On the other hand, given \( u, v \in X_{T,M} \), we can apply the same method as that used to derive (4.10) to obtain
\[
d(Su, Sv) \lesssim \left( \|u\|_{L^\infty(I, H^4, \rho)}^\alpha + \|v\|_{L^\infty(I, H^4, \rho)}^\alpha \right) \|u - v\|_{L^\alpha(I, L^\rho)} \leq C \alpha M^\alpha d(u, v). 
\] (5.7)

So we prove that \( S \) is a contraction on the space \( X_{T,M} \). Using Banach’s fixed-point argument, we obtain a unique solution \( u \in C ([0, T], H^4) \cap L^\gamma ([0, T], H^4, \rho) \) to the Cauchy problem (1.1)-(1.2). Moreover, arguing as in the last part of Proposition 4.1, we obtain the further regularity properties. This finishes the proof of Proposition 5.1.

**Proof of Theorem 1.2.** We now resume the proof of Theorem 1.2. We consider only the positive time direction. A corresponding conclusion for reverse direction follows similarly. We proceed as in the proof of Theorem 1.1: using Proposition 5.1 and the uniqueness in Appendix, we obtain a unique maximal solution \( u \in C ([0, T_{\max}), H^4) \) to the Cauchy problem (1.1)-(1.2) with \( u, u_t, \Delta^2 u \in C ((0, T_{\max}), L^q ((\mathbb{R}^N)) \cap L^q_{\text{loc}} ((0, T_{\max}), L^r (\mathbb{R}^N)) \) for every biharmonic admissible pair \((q, r) \in A_b\).

It remains to prove the blowup alternative (1.6). Suppose by contradiction that \( T_{\max} < \infty \)
\[
\|u\|_{L^\gamma ((0, T_{\max}), L^{\frac{\rho \alpha}{\rho - 2}})} < \infty. 
\] (5.8)
In fact, in the \( H^4 \) critical case \( \alpha = \frac{8}{N-8} \), we have \( \gamma = \frac{2N-8}{N-8} \) and \( \frac{\rho \alpha}{\rho - 2} = \frac{2N(N-4)}{(N-8)^2} \).

We reach a contradiction by showing that the solution \( u \) can be extended beyond the maximal interval \((0, T_{\max})\). Firstly, we claim that
\[
\|u\|_{H^{1,\gamma}((0, T_{\max}), L^\rho) \cap H^{1,\infty}((0, T_{\max}), L^2)} < \infty, 
\] (5.9)
and
\[
\|\Delta^2 u\|_{L^\gamma((0, T_{\max}), L^\rho)} < \infty. 
\] (5.10)
Indeed, it follows from monotone convergence theorem that there exists a \( T_0 \in [0, T_{\max}) \) such that
\[
C_b \|u\|^\alpha_{L^\gamma ([0, T_0], L^{\frac{\rho \alpha}{\rho - 2}})} \leq \frac{1}{2}, 
\]
where \( C_b \) is the constant in (5.12). Changing \( u (\cdot) \) to \( u (T_0 + \cdot) \) and \( \phi \) to \( u (T_0) \), we can assume that \( T_0 = 0 \), so that
\[
C_b \|u\|^\alpha_{L^\gamma ([0, T_{\max}), L^{\frac{\rho \alpha}{\rho - 2}})} \leq \frac{1}{2}. 
\] (5.11)
On the other hand, it follows from (2.5), (2.7), (2.9) and Strichartz’ estimate that
\[
\|u\|_{H^{1,\gamma}((0, T), L^\rho) \cap H^{1,\infty}((0, T), L^2)} 
\leq C_b (\|\phi\|_{H^4} + \|\phi\|_{H^4}^{\alpha+1} + C_b \|u\|_{L^\gamma ([0, T], L^{\frac{\rho \alpha}{\rho - 2}})}^\alpha) \|\phi\|_{H^{1,\gamma}((0, T), L^\rho)} 
\] (5.12)
for all \( 0 < T < T_{\max} \). This inequality together with (5.11) yields (5.9).

Our next goal is to prove (5.10). Applying the same method as that used to derive (4.7), we obtain
\[
\|\Delta^2 u\|_{L^\gamma ([0, T_{\max}), L^\rho)} \lesssim \|u\|_{H^{1,\gamma}((0, T_{\max}), L^\rho)} + \|u\|^\alpha_{L^\gamma ([0, T_{\max}), L^\rho)}. 
\] (5.13)
Note that \( q_0 = p_0 = \frac{\alpha}{\rho - 2} \) in the critical case \( \alpha = \frac{8}{N-2} \) by (3.1), (3.2), (3.3), (3.5) and the definition of \( \rho \) in (2.4). we can apply (5.13), Lemma 3.3, (5.8) and (5.9) to obtain (5.10).

We now apply (5.8) and (5.9) to derive a contradiction. Using the same method as that used to derive (4.29), we deduce that \( u \in L^\infty([0, T_{\text{max}}], H^4) \). Moreover, applying Strichartz’ estimate (2.2) and Sobolev’s embedding \( H^4 \hookrightarrow L^{2(\alpha+1)} \), we conclude that

\[
\| e^{i(t-T_\varepsilon)\Delta} u(T_\varepsilon) \|_{L^2} + \| e^{i(t-T_\varepsilon)\Delta} u(T_\varepsilon) \|_{L^4} < \infty,
\]

for any \( 0 < T_\varepsilon < T_{\text{max}} \). It now follows from the monotone convergence theorem that we can choose \( T_\varepsilon \) approaches to \( T_{\text{max}} \) such that

\[
C_6 (1 + \| u(T_\varepsilon) \|_{H^4}^\alpha) \left( \| e^{i(t-T_\varepsilon)\Delta} u(T_\varepsilon) \|_{L^2} + \| e^{i(t-T_\varepsilon)\Delta} u(T_\varepsilon) \|_{L^4} + \| e^{i(t-T_\varepsilon)\Delta} u(T_\varepsilon) \|_{L^\infty} \right)
\]

where the constants \( M, C_6 \) are the constants in (5.1) and (5.6) respectively. Finally, applying the monotone convergence theorem again, we deduce that there exists \( \delta_0 > 0 \) such that

\[
C_6 (1 + \| u(T_\varepsilon) \|_{H^4}^\alpha) \left( \| e^{i(t-T_\varepsilon)\Delta} u(T_\varepsilon) \|_{L^2} + \| e^{i(t-T_\varepsilon)\Delta} u(T_\varepsilon) \|_{L^4} + \| e^{i(t-T_\varepsilon)\Delta} u(T_\varepsilon) \|_{L^\infty} \right)
\]

where \( M, C_6 \) are the constants in (5.1) and (5.6) respectively. Finally, applying the monotone convergence theorem again, we deduce that there exists \( \delta_0 > 0 \) such that

\[
C_6 (1 + \| u(T_\varepsilon) \|_{H^4}^\alpha) \left( \| e^{i(t-T_\varepsilon)\Delta} u(T_\varepsilon) \|_{L^2} + \| e^{i(t-T_\varepsilon)\Delta} u(T_\varepsilon) \|_{L^4} + \| e^{i(t-T_\varepsilon)\Delta} u(T_\varepsilon) \|_{L^\infty} \right)
\]

It now follows from Proposition 5.1 and (5.16) that there exists a unique solution \( v \in C([T_\varepsilon, T_{\text{max}} + \delta_0], H^4) \) to the equation (1.1) with initial datum \( u(T_\varepsilon) \) at time \( t = T_\varepsilon \). Moreover, by the uniqueness established in Appendix, we see that \( u = v \) on \([T_\varepsilon, T_{\text{max}}] \), which implies that the solution can be extended beyond the maximal interval \((0, T_{\text{max}}] \). This is a contradiction, thereby completing the proof of the blowup alternative (1.6).

In the rest of this section, we prove the continuous dependence of the solution map.

Let \( M > 0 \) satisfy

\[
(C_6 + C_7 + C_9 + C_{10}) M^\alpha + C_{11} \left( M^\alpha + M^{\frac{\alpha(2\alpha+1)}{\alpha+1}} \right) \leq \frac{1}{2} \quad (5.17)
\]

where \( C_6, C_7, C_9, C_{10}, C_{11} \) are the constants in (5.6), (5.7), (5.19), (5.20) and (5.21) respectively. Moreover, we fix \( T > 0 \) sufficiently small such that

\[
C_6 (1 + \| \phi \|_{H^4}^\alpha) F(\phi, T) \leq M \quad (5.18)
\]
Proof. Using the same method as that used to derive (4.28), we obtain $u \to 0$. This finishes the proof of Claim 5.1.

Since $\phi_n \to \phi$ in $H^4$, we can find a positive $n_1$ such that $C_6 (1 + \|\phi_n\|_{H^4}) F(\phi_n, T) \leq \frac{M}{R}$ for every $n \geq n_1$. Then it follows from Proposition 5.1 that for every $n \geq n_1$, the equation (4.17) admits a unique solution $u_n \in C(I, H^4) \cap L^{2}(I, B^{3,2}_{2})$ and $u_n$ is uniformly bounded,

$$\|u_n\|_{L^{2}(I, H^{4,\rho}) \cap H^{1,\gamma}(I, L^p)} \leq M, \quad \forall \ n \geq n_1,$$

where $M$ is the constant in (5.17). Furthermore, we also have the boundedness for $u$,

$$\|u\|_{L^{2}(I, H^{4,\rho}) \cap H^{1,\gamma}(I, L^p)} \leq M.$$

The proof of the continuous dependence will proceed by the following claims.

**Claim 5.1.** Given any $(q, r) \in \Lambda_b$, we have $\|u_n - u\|_{H^{1,q}(I, L^r)} \to 0$ as $n \to \infty$.

*Proof.* We first show that $\|u_n - u\|_{L^q(I, L^r)} \to 0$ as $n \to \infty$. Analogously to (4.20), we obtain

$$\|u_n - u\|_{L^q(I, L^r)} \lesssim \|\phi_n - \phi\|_{L^2} + (\|u_n\|_{H^{1,q}(I, H^4)} + \|u\|_{H^{1,q}(I, H^4)}) \|u_n - u\|_{L^{2}(I, L^p)}.$$

$$\leq C_9 \|\phi_n - \phi\|_{L^2} + C_9 M^\alpha \|u_n - u\|_{L^{2}(I, L^p)}.$$

(5.19)

Since $C_9 M^\alpha \leq \frac{1}{2} C_7$ by (5.17), it follows from (5.19) that $\|u_n - u\|_{L^q(I, L^r)} \lesssim \|\phi_n - \phi\|_{L^2} \to 0$ as $n \to \infty$.

Next, it follows from Lemma 3.1 and the boundedness of $u_n, u$, we have, for any $R > 0$

$$\limsup_{n \to \infty} \|u_n - u\|_{H^{1,q}(I, L^r) \cap H^{1,\gamma}(I, L^p)} \lesssim \left( M^\alpha \limsup_{n \to \infty} \|u_n - u\|_{H^{1,\gamma}(I, L^p)} + M^\alpha \|\phi_n - \phi\|_{L^2} \right) R^{\|\partial u\|_{L^q(I, L^p)}}.$$

Since $R > 0$ is arbitrary, we can let $R \to \infty$ to obtain

$$\limsup_{n \to \infty} \|u_n - u\|_{H^{1,q}(I, L^r) \cap H^{1,\gamma}(I, L^p)} \leq C_{10} M^\alpha \limsup_{n \to \infty} \|u_n - u\|_{H^{1,\gamma}(I, L^p)}$$

(5.20)

Since $C_{10} M^\alpha \leq \frac{1}{2} C_7$ by (5.17), it follows from (5.20) that $\limsup_{n \to \infty} \|u_n - u\|_{H^{1,q}(I, L^r)} = 0$. This finishes the proof of Claim 5.1.

**Claim 5.2.** $\|u_n - u\|_{L^r(I, H^{4,\rho})} \to 0$ as $n \to \infty$.

*Proof.* Using the same method as that used to derive (4.28), we obtain

$$\|u_n - u\|_{L^r(I, H^{4,\rho})} \leq C_{11} \left( M^\alpha + M^{\frac{\alpha}{2\alpha + 1}} \right) \|u_n - u\|_{L^{2}(I, H^{4,\rho})} + C(M, T) \left( \|\partial_t u_n - \partial_t u\|_{L^{2}(I, L^p)} + \|\phi_n - \phi\|_{H^{4,\rho}} \right).$$

(5.21)

Since $C_{11} \left( M^\alpha + M^{\frac{\alpha}{2\alpha + 1}} \right) \leq \frac{1}{2}$ by (5.18), it follows from Claim 5.1 and (5.21) that Claim 5.2 holds.

**Claim 5.3.** Given any $(q, r) \in \Lambda_b$, we have $\|u_n - u\|_{L^q(I, H^{4,\rho})} \to 0$ as $n \to \infty$.

*Proof.* Using the same method as that used to derive Claim 4.3, we obtain Claim 5.3.
It follows from Claims 5.1 and 5.3 that the continuous dependence of the solution map holds on the interval \([0, T]\). Finally, by a standard iteration argument, we obtain the continuous dependence on \([0, A]\).

6. The blow-up ansatz. The rest of this paper is devoted to the proof of Theorem 1.3. In this section, we construct inductively an appropriate blow-up ansatz. The first candidate \(U_0\) is defined by (6.6) below. \(U_0\) is a natural candidate, since it is an explicit blowing-up solution of the ODE \(i\partial_t U_0 + \lambda|U_0|^\alpha U_0 = 0\). Moreover, the error term \(\Delta^2 U_0 + \mu \Delta U_0\) is lower order than both \(i\partial_t U_0\) and \(\lambda|U_0|^\alpha U_0\). (See Lemma 6.1 below.) Since \(\Delta^2 U_0 + \mu \Delta U_0\) is of order \((-t)^{-\frac{\delta}{2}}|U_0| \lesssim (-t)^{-\frac{\delta}{2} - \frac{\beta}{2}}\), the error term is not integrable in time near the singularity when \(\alpha\) is small. To treat any subcritical or critical \(\alpha\), we refine the approximate solution in the spirit of Cazenave-Han-Martel [10] to reduce the singularity of the error term at any order of \((-t)\). See (6.12)-(6.13) for more details. Throughout this section, we assume

\[
J = \left[ \frac{2}{\alpha} + 4\sigma \right] + 1 \tag{6.1}
\]

and

\[
k = \max\{4J + 6, [N\alpha] + 1\} \tag{6.2}
\]

with

\[
\sigma = \max\{\frac{4}{\delta}, \frac{4}{\alpha(1-\delta)}, \frac{2^{\alpha+2}|\lambda|M(-\alpha\Im \lambda)^{-1}}{\min\{\alpha, 1\}}(1-\delta)^{-1}, g(\alpha)\}, \tag{6.3}
\]

where

\[
\delta = \min\{\frac{1}{10}, \frac{\alpha}{\alpha + 4}, f(\alpha)\}, \tag{6.4}
\]

\(M\) is the constant in Lemma 6.3, \(f(\alpha)\) and \(g(\alpha)\) are defined by

\[
f(\alpha) = \begin{cases} 
1, & \text{if } 0 < \alpha \leq 1, \\
\frac{\alpha-1}{2\alpha+2}, & \text{if } \alpha > 1,
\end{cases}
\]

\[
g(\alpha) = \begin{cases} 
0, & \text{if } 0 < \alpha \leq 1, \\
\frac{2}{(\alpha-1)-2\alpha}, & \text{if } \alpha > 1.
\end{cases}
\]

Let \(K\) be any nonempty compact set of \(\mathbb{R}^N\) included in the ball of center \(0\) and radius \(R > 0\). It is well-known that there exists a smooth function \(Z : \mathbb{R}^N \to [0, \infty)\) which vanishes exactly on \(K\) (see Lemma 1.4 in [35]). Define the function \(A : \mathbb{R}^N \to [0, \infty)\) by

\[
A(x) = (Z(x)\chi(|x|) + (1 - \chi(|x|))|x|)^k
\]

where

\[
\chi \in C^\infty(\mathbb{R}, \mathbb{R}), \quad \chi(s) = \begin{cases} 
1, & 0 \leq s \leq R, \\
0, & s \geq 2R,
\end{cases}
\]

\(\chi'(s) \leq 0 \leq \chi(s) \leq 1, \quad s \geq 0.
\]

It follows that the function \(A \in C^{k-1}(\mathbb{R}^N, \mathbb{R})\), vanishes exactly on \(K\), satisfies

\[
\begin{cases} 
A \geq 0 \text{ and } |\partial^\beta A| \leq A^{1 - \frac{|eta|}{k}}, & \text{on } \mathbb{R}^N \text{ for } |\beta| \leq k - 1, \\
A(x) = |x|^k, & \text{for } x \in \mathbb{R}^N, |x| \geq 2R.
\end{cases} \tag{6.5}
\]

Set

\[
U_0(t, x) = (-\Im \lambda)^{-\frac{\delta}{2}}(-\alpha t + A(x))^{-\frac{\delta}{2} + i\frac{\lambda(x)}{\alpha A(x)}}, \quad t < 0, x \in \mathbb{R}^N. \tag{6.6}
\]

From (1.7), (6.2), (6.5) and (6.6), we have

\[
U_0 \text{ is } C^\infty \text{ in } t < 0 \text{ and } C^{k-1} \text{ in } x \in \mathbb{R}^N,
\]

\[
i\partial_t U_0 + \lambda|U_0|^\alpha U_0 = 0, \quad t < 0, x \in \mathbb{R}^N, \tag{6.7}
\]
\[|U_0| = (-\text{Im}\lambda)^{-\frac{3}{2}} (-\alpha t + A(x))^{-\frac{1}{2}} \leq (-\alpha \text{Im}\lambda)^{-\frac{3}{2}} (-t)^{-\frac{1}{2}}, \quad (6.8)\]

and
\[\partial_t |U_0| = -\text{Im}\lambda |U_0|^{\alpha + 1} \geq 0.\]

Next, we collect the estimates on \(U_0\) which are from [10, 27].

**Lemma 6.1.** Assume (1.7), (6.2), (6.5), and let \(U_0\) be given by (6.6). If \(p \geq 1\), then
\[\|U_0(t)\|_{L^p} \lesssim (t)^{-\frac{1}{2}}\quad (6.9)\]
for \(-1 \leq t < 0\). In addition, for every \(\rho \in \mathbb{R}, \ell \in \mathbb{N}\) and \(|\beta| \leq k - 1\),
\[|\partial_x^\beta (|U_0|^\rho)| \lesssim |U_0|^{\rho + \frac{1}{2} |\beta|} \lesssim (t)^{-\frac{1}{2}\rho} |U_0|^\rho,\]
\[|\partial_x^\beta (|U_0|^{\rho - 1} U_0)| \lesssim |U_0|^{\rho + \frac{1}{2} |\beta|} \lesssim (t)^{-\frac{1}{2}\rho} |U_0|^\rho,\]
\[|\partial_t \partial_x^\beta (|U_0|^{\rho - 1} U_0)| \lesssim |U_0|^{-1 + \rho + \frac{1}{2} |\beta|} \lesssim (t)^{-1 - \frac{1}{2}\rho} |U_0|^\rho,\]
for all \(x \in \mathbb{R}^N, t < 0\), and
\[U_0 \in C^\infty ((-\infty, 0), H^{k-1}(\mathbb{R}^N)).\]

Furthermore, for any \(x_0 \in \mathbb{R}^N\) such that \(A(x_0) = 0\), for any \(r > 0\), \(-1 \leq t < 0\) and \(1 \leq p \leq \infty\),
\[(-t)^{-\frac{1}{2} + \frac{\rho}{2\ell}} \lesssim \|U_0(t)\|_{L^p([x-x_0] \leq r)}. \quad (6.11)\]

Next, we refine the approximate solution in the spirit of Cazenave-Han-Martel [10]. More precisely, we consider the linearization of the equation (6.7),
\[i\partial_tw + \lambda \left( \frac{\alpha + 2}{2} |U_0|^\alpha w + \frac{\alpha}{2} |U_0|^{\alpha - 2} U_0^2 \overline{w} \right) = 0. \quad (6.12)\]
Equation (6.12) has two particular solutions \(w = iU_0\) and \(w = \partial_t U_0 = i\lambda |U_0|^\alpha U_0\). By means of variation of parameters, it is not hard to see that the corresponding nonhomogeneous equation
\[i\partial_tw + \lambda \left( \frac{\alpha + 2}{2} |U_0|^\alpha w + \frac{\alpha}{2} |U_0|^{\alpha - 2} U_0^2 \overline{w} \right) + G = 0\]
has the solution \(w = \mathcal{P}(G)\), where
\[\mathcal{P}(G) = \int_0^t \left[ |U_0|^{-\alpha - 2} \text{Im}(\overline{U_0}G) \right] (s)ds - \frac{i}{\text{Im}\lambda} \int_0^t |U_0|^{-2} \text{Re}(\overline{U_0}G) (s)ds.\]

We define \(U_j, w_j, \mathcal{E}_j\) by
\[w_0 = iU_0, \quad \mathcal{E}_0 = i\partial_t U_0 + \Delta^2 U_0 + \mu \Delta U_0 + \lambda |U_0|^\alpha U_0 = \Delta^2 U_0 + \mu \Delta U_0\]
and then recursively
\[w_j = \mathcal{P}(\mathcal{E}_{j-1}), \quad U_j = U_{j-1} + w_j, \quad \mathcal{E}_j = i\partial_t U_j + \Delta^2 U_j + \mu \Delta U_j + \lambda |U_j|^\alpha U_j \quad (6.13)\]
for \(j \geq 1\), as long as they make sense. We then have the following estimates.

**Lemma 6.2.** Assume (1.7), (6.1), (6.2), (6.5), and let \(U_0, U_j, w_j, \mathcal{E}_j\) be given by (6.6) and (6.13). There exists \(-1 < T < 0\) such that the following estimates hold
\[|\partial_x^\beta (U_j - U_0)| \lesssim (t)^{1 - \frac{1}{2} |\beta| + 4} |U_0|, \quad 0 \leq |\beta| \leq k - 1 - 4J, \quad (6.14)\]
\[|\partial_x^2 \mathcal{E}_j| \lesssim (t)^{1 - \frac{1}{2} + \frac{1}{4} - \frac{|\beta| + 4}{k}} |U_0|, \quad 0 \leq |\beta| \leq k - 5 - 4J, \quad (6.15)\]
for all complex numbers $u, v$ and with $\alpha > 0$.

Moreover, we can decompose $E_J = i\partial_t U_J + \Delta^2 U_J + \mu \Delta U_J + \lambda |U_J|^\alpha U_J$.

Proof. We claim that for all $0 \leq |\beta| \leq k - 1 - 4j$, then

1. If $0 \leq |\beta| \leq k - 1 - 4j$, then
   
   $$|\partial_t^j \partial_x^\beta w_j| \lesssim (-t)^j (1 - t^\alpha) \frac{|\beta|}{|\beta|} |U_0|, \quad t < 0, x \in \mathbb{R}^N,$$
   $$|\partial_t^j \partial_x^\beta w_j| \lesssim (-t)^{j+1} (1 - t^\alpha) \frac{|\beta|}{|\beta|} |U_0|, \quad t < 0, x \in \mathbb{R}^N,$$
   $$|\partial_t^j (U_j - U_0)| \lesssim (-t)^j (1 - t^\alpha - \frac{|\beta|}{|\beta|}) |U_0|, \quad t < 0, x \in \mathbb{R}^N,$$

2. If $0 \leq |\beta| \leq k - 5 - 4j$, then
   
   $$|\partial_t^j \partial_x^\beta E_j| \lesssim (-t)^j (1 - t^\alpha - \frac{|\beta|}{|\beta|}) |U_0|, \quad t < 0, x \in \mathbb{R}^N,$$
   $$|\partial_t^j E_j| \lesssim (-t)^{j+1} (1 - t^\alpha - \frac{|\beta|}{|\beta|}) |U_0|,$$
   $$\frac{1}{2} |U_0| \leq |U_j| \leq 2 |U_0|, \quad t < 0, x \in \mathbb{R}^N,$$
   $$U_j \in C^1 \left( (T, 0), H^{k-1-4j} \left( \mathbb{R}^N \right) \right),$$
   $$|\partial_t U_j| \lesssim (-t)^{-1} |U_0|.$$

In fact, the proof is an obvious adaptation of [10, 27]. More precisely, we just need to replace all $\frac{2}{j}$ with $\frac{4}{j}$ throughout the proof of Lemma 3.2 in [10] and Lemma 2.2 in [27] by considering the presence of $\Delta^2 U_j$. Finally, note that $0 \leq J \leq \frac{k - 6}{4}$ by (6.2), we complete the proof of Lemma 6.2 by setting $j = J$. \hfill $\square$

Lemma 6.3. Assume $\alpha > 0, p \in \mathbb{R}, l \in \mathbb{R}$ and $p + l > 0$. There exists a constant $M \geq 1$ such that for all complex numbers $u, v \in \mathbb{C}$,

$$|u + v|^p (u + v)^l - |u|^p u^l \leq M \left( |v|^{p+l} + 1_{p+l>1} |u|^{p+l-1} |v| \right)$$

and

$$|u + v|^p (u + v)^l - |u|^p u^l \leq M \left( |u|^{p+l-1} |v| + 1_{p+l>1} |v|^{p+l} \right).$$

Moreover, we can decompose

$$\partial_t ( |u + v|^\alpha (u + v) - |u|^\alpha u) = I_1 (u, v) + I_2 (u, v),$$

with

$$|I_1 (u, v)| \leq M |u|^\alpha |\partial_t v|$$

and

$$|I_2 (u, v)| \leq M \left( |u|^{\alpha-1} |v| + 1_{\alpha>1} |v|^\alpha \right) |\partial_t u|$$

$$+ \left( |v|^\alpha + 1_{\alpha>1} |u|^{\alpha-1} |v| \right) |\partial_v v|$$

for all complex numbers $u, v \in \mathbb{C}$. 

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Proof. The proof of the estimate (6.21) is standard, we omit its proof by simplicity. We now prove (6.22). It suffices to show that for any \( z \in \mathbb{C} \), we have
\[
\left| (1 + z)^p (1 + z)^l - 1 \right| \leq M \left( |z| + 1_{p+l>1} |z|^{p+l} \right). \tag{6.26}
\]
Let \( z \in \mathbb{C} \), \( |z| \geq \frac{1}{2} \). Note that \( |z|^{p+l} \lesssim |z| \) if \( p + l \leq 1 \), so that
\[
\left| (1 + z)^p (1 + z)^l - 1 \right| \lesssim |z|^{p+l} + 1 \lesssim |z| + 1_{p+l>1} |z|^{p+l}.
\]
For \( |z| \leq \frac{1}{2} \), we write
\[
|1 + z|^p (1 + z)^l - 1 = \int_0^1 \frac{d}{d\theta} \left[ |1 + \theta z|^p (1 + \theta z)^l \right] d\theta. \tag{6.27}
\]
Note that
\[
\frac{d}{d\theta} \left[ |1 + \theta z|^p (1 + \theta z)^l \right] = \left( \frac{p}{2} + l \right) |1 + \theta z|^p (1 + \theta z)^{l-1} z + \frac{p}{2} |1 + \theta z|^{p-2} (1 + \theta z)^{l+1} z,
\]
and \( \frac{1}{2} \leq |1 + \theta z| \leq \frac{3}{2} \) for any \( \theta \in [0, 1] \), \( |z| \leq \frac{1}{2} \), we have
\[
\left| \frac{d}{d\theta} \left[ |1 + \theta z|^p (1 + \theta z)^l \right] \right| \leq (p + l) |1 + \theta z|^{p+l-1} |z| \lesssim |z|.
\]
This inequality together with (6.27) gives (6.22). We turn now to the proof of (6.23)-(6.25). Since
\[
\partial_t (|u|^n u) = \frac{\alpha + 2}{2} |u|^n \partial_t u + \frac{\alpha}{2} |u|^{n-2} u^2 \partial_i \nabla,
\]
we can decompose
\[
\partial_t (|u + v|^\alpha (u + v) - |u|^n u) = I_1 + I_2,
\]
where
\[
I_1 (u, v) = \frac{\alpha + 2}{2} |u|^{\alpha} \partial_t v + \frac{\alpha}{2} |u|^{\alpha-2} u^2 \partial_t v
\]
and
\[
I_2 (u, v) = \frac{\alpha + 2}{2} (|u + v|^\alpha - |u|^\alpha) (\partial_t u + \partial_t v)
\]
\[
+ \frac{\alpha}{2} (|u + v|^{\alpha-2} (u + v)^2 - |u|^{\alpha-2} u^2) (\partial_t u + \partial_t v). \tag{6.28}
\]
Clearly, \( |I_1| \leq (\alpha + 1) |u|^{\alpha} |\partial_t v| \). Moreover, applying (6.28), (6.21) and (6.22), we obtain the estimate of \( I_2 \) in (6.25).
Choosing \( M \) larger enough, we complete the proof of Lemma 6.3 \( \square \)

7. Construction and estimates of approximate solutions. In this section, we construct a sequence of solutions \( u_n \) of (1.1), close to the ansatz \( U_J \) in Lemma 6.2, which will eventually converge to the blowing-up solution of Theorem 1.3. We will estimate \( \varepsilon_n = u_n - U_J \) by the energy method and Strichartz estimate. More precisely, we estimate
\[
(-t)^{-\sigma} \| \varepsilon_n \|_2 + (-t)^{-(1-\delta)\sigma} \| \Delta^2 \varepsilon_n \|_2 + (-t)^{-(1-\frac{d}{2})\sigma} \| \partial_t \varepsilon_n \|_2
\]
for some appropriate parameters \( \sigma, \delta \) defined in (6.3) and (6.4).
Let the ansatz \( U_J \) and \( T < 0 \) be given by Lemma 6.2. Since \( 4J \leq k - 6 \) by (6.2), we see that \( U_J \left( -\frac{1}{n} \right) \in H^1 (\mathbb{R}^N) \) by (6.18). Therefore, we can deduce from
Theorems 1.1 and 1.2 that there exist \( s_n < -\frac{1}{n} \) and a unique maximal solution \( u_n \in C \left( \left( s_n, -\frac{1}{n} \right], H^4(\mathbb{R}^N) \right) \) to the following nonlinear fourth-order Schrödinger equation
\[
\begin{aligned}
\begin{cases}
   i\partial_t u_n + \Delta^2 u_n + \mu \Delta u_n + \lambda |u_n|^\sigma u_n = 0, \\
   u_n (-\frac{1}{n}) = U_J \left( -\frac{1}{n} \right),
\end{cases}
\end{aligned}
\]  
(7.1)
with the blowup alternative that if \( s_n > -\infty \), then
\[
\|u_n(t)\|_{H^4} \xrightarrow{t \downarrow s_n} \infty,
\]  
(7.2)
in the subcritical case \( \alpha > 0 \), \( (N-8)\alpha < 8 \), and
\[
\|u_n(t)\|_{L^{\frac{2N-4}{N-\sigma}}((s_n, -\frac{1}{n}]; L^{\frac{2N(N-4)}{(N-\sigma)^2}})} = \infty
\]  
(7.3)
in the critical case \( \alpha = \frac{8}{N-3}, N \geq 9 \). Let \( \varepsilon_n \in C \left( \left( \max \{ s_n, T \}, -\frac{1}{n} \right], H^4(\mathbb{R}^N) \right) \) be defined by
\[
u_n = U_J + \varepsilon_n.
\]  
(7.4)
We then have the following estimate.

**Proposition 7.1.** Assume \( T \leq T_0 < 0 \) satisfies
\[
C_{11} (-T_0)^{-1+\min\{\alpha,1\}(1-\delta)\sigma} + C_{12} (-T_0)^{\alpha(1-\delta)\sigma} \leq \frac{1}{2},
\]
where \( C_{11}, C_{12} \) are the constants in (7.19) and (7.25) respectively. There exist \( T_0 \leq S < 0 \) and \( n_0 > -\frac{1}{N} \) such that \( s_n < S \), for all \( n \geq n_0 \). Moreover,
\[
\|\varepsilon_n(t)\|_{L^2} \leq (-t)^\sigma, \|\Delta^2 \varepsilon_n(t)\|_{L^2} \leq (-t)^{(1-\delta)\sigma}, \|\partial t \varepsilon_n(t)\|_2 \leq (-t)^{(1-\frac{1}{2})\sigma}
\]  
(7.5)
for all \( n \geq n_0 \) and \( t \in [S, -\frac{1}{n}] \).

**Proof.** Throughout the proof, we write \( \varepsilon \) instead of \( \varepsilon_n \) to simplify the notation. It follows from (6.13) and (7.4) that
\[
\begin{cases}
   i\partial_t \varepsilon + \Delta^2 \varepsilon + \mu \Delta \varepsilon + \lambda \left( |U_J + \varepsilon|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J \right) + \mathcal{E}_J = 0, \\
   \varepsilon(-\frac{1}{n}) = 0.
\end{cases}
\]  
(7.6)
Let
\[
\tau_n = \inf \{ t \in [\max\{T_0, s_n\}, -\frac{1}{n}]; \|\varepsilon(s)\|_{L^2} \leq (-s)^\sigma, \|\Delta^2 \varepsilon(s)\|_2 \leq (-s)^{(1-\delta)\sigma}, \|\partial t \varepsilon(s)\|_2 \leq (-s)^{(1-\frac{1}{2})\sigma} \text{ for all } t < s \leq -\frac{1}{n} \}.
\]  
(7.7)
Since \( \varepsilon(-\frac{1}{n}) = 0 \), we see that \( s_n \leq \tau_n < -\frac{1}{n} \). In what follows, we define \( I = (t, -\frac{1}{n}) \) with \( \tau_n < t < -\frac{1}{n} \).

Firstly, we claim that
\[
\|\partial_t \varepsilon\|_{L^q(I; L^r)} + \|\partial_t \varepsilon\|_{L^q(I; L^r)} \lesssim (-t)^{(1-2\delta)\sigma},
\]  
(7.8)
where \( (\gamma, \rho) \) is the admissible pair defined in (2.4) and
\[
(q, r) = \left( \frac{16(\alpha + 1)}{\alpha}, \frac{4(\alpha + 1)}{\alpha + 2} \right) \in \Lambda_b
\]  
(7.9)
is another given biharmonic admissible pair. In fact, note that
\[
\varepsilon(t) = i \int_t^{-\frac{1}{n}} e^{i(t-s)\left( \Delta^2 + \mu \Delta \right)} \left[ \lambda \left( |U_J + \varepsilon|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J \right) + \mathcal{E}_J \right](s) \, ds
\]  
(7.10)
by the equation (7.6) and that
$$
\partial_t \varepsilon(t) = -i e^{i(t + \frac{1}{n}) (\Delta^2 + \mu \Delta)} E_J \left( -\frac{1}{n} \right) + i \int_t^\infty e^{i(t - s)} [\partial_s (|U_J + \varepsilon|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J) + \partial_s E_J] (s) \, ds.
$$

Using Strichartz’s estimate (2.3) and Lemma 6.3, we deduce that
$$
\| \partial_t \varepsilon \|_{L^s(\mathbb{R}^n \cap L^p(\mathbb{R}^n))} \lesssim \| E_J \left( -\frac{1}{n} \right) \|_2 + \| I_1 (U_J, \varepsilon) \|_{L^4(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)} + \| I_2 (U_J, \varepsilon) \|_{L^6(\mathbb{R}^n) \cap L^6(\mathbb{R}^n)} + \| \partial_t E_J \|_{L^2(\mathbb{R}^n)}. \quad (7.11)
$$

Next, it follows from (6.8), (6.9), (6.15)-(6.17), (7.7) and Lemma 6.3 that
$$
\| \partial_t \varepsilon \|_{L^s(\mathbb{R}^n \cap L^p(\mathbb{R}^n))} \lesssim (\frac{1}{n}) \| E_J \|_2 \lesssim (1 - \frac{1}{n} + J(1 - \frac{1}{n} - \frac{4}{3} \lesssim (-t)^{-\frac{1}{2} + J(1 - \frac{1}{n} - \frac{4}{3} \lesssim (-t)^{(1 - 2\delta)\sigma}, \quad (7.12)
$$

$$
\| I_1 (U_J, \varepsilon) \|_{L^4(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)} \lesssim \| U_J \|_{L^4(\mathbb{R}^n)} \| \partial_t \varepsilon \|_{L^2(\mathbb{R}^n)} \lesssim (\frac{1}{n}) \| E_J \|_2 \lesssim (-t)^{(1 - 2\delta)\sigma}, \quad (7.13)
$$

$$
\| \partial_t E_J \|_{L^2(\mathbb{R}^n)} \lesssim (-t)^{-\frac{1}{2} + J(1 - \frac{1}{n} - \frac{4}{3} \lesssim (-t)^{(1 - 2\delta)\sigma}, \quad (7.14)
$$

where we also used (6.1)-(6.3) to ensure $-1 - \frac{1}{n} + J(1 - \frac{1}{n} - \frac{4}{3} \geq (1 - 2\delta) \sigma$. It remains to estimate $\| I_2 (U_J, \varepsilon) \|_{L^s(\mathbb{R}^n \cap L^p(\mathbb{R}^n))}$. Note that
$$
| I_2 (U_J, \varepsilon) | \lesssim \left( |U_J|^{\alpha - 1} |\varepsilon| + 1_{\alpha > 1} |\varepsilon|^\alpha \right) \| \partial_t U_J \| + \left( |\varepsilon|^\alpha + 1_{\alpha > 1} |U_J|^{\alpha - 1} |\varepsilon| \right) \| \partial_t \varepsilon \| \quad (7.15)
$$

by (6.25) and that
$$
|U_J|^{\alpha - 1} \| \partial_t U_J \| \lesssim (-t)^{-1} |U_0|\alpha \quad (7.16)
$$

by (6.17) and (6.19). It follows from (7.15), (7.16), (7.17) and H"{o}lder’s inequality that
$$
\| I_2 (U_J, \varepsilon) \|_{L^s(\mathbb{R}^n \cap L^p(\mathbb{R}^n))} \lesssim \left( \frac{1}{n} \right) \| U_0 \|_{L^2(\mathbb{R}^n \cap \mathbb{R}^n)} + 1_{\alpha > 1} \left( (-t)^{-1} \| \varepsilon \|_{L^2(\mathbb{R}^n \cap \mathbb{R}^n)} \right)^2 + \| \varepsilon \|_{L^2(\mathbb{R}^n \cap \mathbb{R}^n)} \| \partial_t \varepsilon \|_{L^2(\mathbb{R}^n \cap \mathbb{R}^n)} \quad (7.17)
$$

Applying (7.17), (6.9), (7.7) and Sobolev’s embedding $H^4 \hookrightarrow L^{2\alpha + 2} \cap L^r$, we conclude that
$$
\| I_2 (U_J, \varepsilon) \|_{L^s(\mathbb{R}^n \cap L^p(\mathbb{R}^n))} \lesssim \left( (-t)^{-2 + (1 - \delta)\sigma} + 1_{\alpha > 1} \left((-t)^{-1} \| \varepsilon \|_{L^2(\mathbb{R}^n \cap \mathbb{R}^n)} \right)^2 + \left((-t)^{\alpha(1 - \delta)\sigma} \| \partial_t \varepsilon \|_{L^2(\mathbb{R}^n \cap \mathbb{R}^n)} + 1_{\alpha > 1} \left((-t)^{-1} \| \varepsilon \|_{L^2(\mathbb{R}^n \cap \mathbb{R}^n)} \right)^2 + \left((-t)^{\alpha(1 - \delta)\sigma} \| \partial_t \varepsilon \|_{L^2(\mathbb{R}^n \cap \mathbb{R}^n)} \right) \right) \| \partial_t \varepsilon \|_{L^2(\mathbb{R}^n \cap \mathbb{R}^n)} \quad (7.18)
$$

Next, taking $L^s(\mathbb{R}^n(\mathbb{R}^n \cap L^p(\mathbb{R}^n)) \cap \mathbb{R}^n \cap \mathbb{R}^n$ norm in the above inequality, and then applying H"{o}lder’s inequality, we obtain
$$
\| I_2 (U_J, \varepsilon) \|_{L^s(\mathbb{R}^n \cap L^p(\mathbb{R}^n))} \lesssim (-t)^{-1} \frac{1}{n} + (1 - \delta)\sigma \| \partial_t \varepsilon \|_{L^s(\mathbb{R}^n \cap L^p(\mathbb{R}^n))} \quad (7.19)
$$

$$
\lesssim (-t)^{(1 - 2\delta)\sigma} + (-t)^{-1} \| \partial_t \varepsilon \|_{L^s(\mathbb{R}^n \cap L^p(\mathbb{R}^n))} \quad (7.18)
$$
where we used (6.3) to ensure \(-1 - \frac{1}{q} + (1 - \delta) \sigma \geq (1 - 2\delta) \sigma\). Putting (7.11)-(7.18) together, we deduce that
\[
\|\partial_t \varepsilon\|_{L^q(I,L^r) \cap L^{\gamma}(I,L^\rho)} \\
\leq C_{11} (t)^{-1+\frac{2\delta}{\sigma}} + C_{11} (t)^{-\frac{1+\min(\alpha,1)(1-\delta)}{\sigma}} \|\partial_t \varepsilon\|_{L^q(I,L^r)}.
\] (7.19)

Since \(C_{11} (t)^{-1+\min(\alpha,1)(1-\delta)} \leq \frac{1}{2}\), we obtain (7.8).

Next, we claim that \(s_n < \tau_n\). In fact, the subcritical case \(\alpha > 0\), \((N-8)\alpha < 8\) follows immediately from the blowup alternative (7.2) and the definition of \(\tau_n\) in (7.7). For the critical case \(\alpha = \frac{N}{N-8}, N \geq 9\), it suffices to prove that
\[
\|\varepsilon\|_{L^{2N-8}(I, L^{2N/(N-8)}(\mathbb{R}^N))} \leq C < \infty
\] (7.20)

by considering the blowup alternative (7.3).

We now prove (7.20). Applying the equation (7.6), Lemma 6.3 and the same argument used to prove (4.14), we conclude that
\[
\left\|\Delta^2 \varepsilon\right\|_{L^q(I,L^r)} \\
\lesssim \|\varepsilon\|_{H^{1,\gamma}(I,L^r)} + \|\|U_j\|^{\alpha} + \|\varepsilon\|^\alpha\|\varepsilon\|_{L^\gamma(I,L^r)} + \|E_j\|_{L^\gamma(I,L^r)}.
\] (7.21)

Moreover, using the same method as that used to derive (3.22) we obtain
\[
\left\|\|\varepsilon\|^\alpha\varepsilon\right\|_{L^{\gamma}(I,L^r)} \lesssim \|\varepsilon\|_{L^{\gamma}(I,L^r)}^{\alpha} \left(\|\varepsilon\|_{L^q(I,L^r)} + \|\Delta^2 \varepsilon\|_{L^q(I,L^r)}\right),
\] (7.22)

where we used the embedding (3.6). Applying (7.22), Sobolev's embedding \(H^1 \hookrightarrow L^\rho \cap L^{2(\alpha+1)}\) and (7.7), we deduce that
\[
\left\|\|\varepsilon\|^\alpha\varepsilon\right\|_{L^{\gamma}(I,L^r)} \lesssim (t)^{-\alpha(1-\delta)\sigma} \left(\|\varepsilon\|^\alpha\varepsilon\right) + \left\|\Delta^2 \varepsilon\right\|_{L^q(I,L^r)}
\lesssim (t)^{-\frac{1}{\alpha} + (1-\delta)\sigma} + (t)^{-\alpha(1-\delta)\sigma} \left\|\Delta^2 \varepsilon\right\|_{L^q(I,L^r)},
\] (7.23)

On the other hand, since \(|U_j|^{\alpha} \lesssim (t)^{-1}\) by (6.8) and (6.17), we can apply (6.9), (6.15), Sobolev's embedding \(H^1 \hookrightarrow L^p\) and (7.7) to obtain
\[
\|\|U_j\|^{\alpha}\varepsilon\|_{L^{\gamma}(I,L^r)} + \|E_j\|_{L^{\gamma}(I,L^r)} \\
\lesssim \|(-s)^{-1+\frac{1}{\alpha}}\|_{L^\gamma(I)} + \|\|E_j\|^\gamma\|_{L^\gamma(I)} \\
\lesssim (t)^{-\frac{1}{\alpha} + (1-\delta)\sigma}.
\] (7.24)

where we used (6.1)-(6.3) to ensure \(-\frac{1}{\alpha} + J \left(1 - \frac{1}{\alpha}\right) - 4 \frac{1}{\rho} \geq -1 + (1 - \delta) \sigma\) in the last inequality. It follows from (7.8), (7.21), (7.23) and (7.24) that
\[
\|\Delta^2 \varepsilon\|_{L^\gamma(I,L^r)} \leq C_{12} (t)^{-1+\frac{2\delta}{\sigma}} + C_{12} (t)^{\alpha(1-\delta)\sigma} \|\Delta^2 \varepsilon\|_{L^\gamma(I,L^r)}.
\] (7.25)

Since \(C_{12} (t)^{\alpha(1-\delta)\sigma} \leq \frac{1}{2}\), we deduce that
\[
\|\Delta^2 \varepsilon\|_{L^\gamma(I,L^r)} \lesssim (t)^{-1+\frac{2\delta}{\sigma}}.
\]

This inequality together with Sobolev's embedding \(\dot{H}^{\frac{4}{\rho}}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N(1-\delta)}{N-8}}(\mathbb{R}^N)\) yields (7.20).

We now resume the proof of Proposition 7.1. We first estimate \(\|\varepsilon(t)\|_{L^2}\). Multiplying (7.6) by \(\Im\) and then taking the imaginary part, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\varepsilon(t)\|_{L^2}^2 = -\Im \left( \lambda \int \|U_j + \varepsilon\|^\alpha(U_j + \varepsilon) - \Im \varepsilon \right) - \int \varepsilon \bar{E}_j \varepsilon.
\]
Using Lemma 6.3, we deduce that
\[
\frac{1}{2} \frac{d}{dt} \|\varepsilon(t)\|_{L^2}^2 \geq -|\lambda|M \int (|U_J|^\alpha + |\varepsilon|^\alpha)|\varepsilon|^2 - \|E_J\|_{L^2}^2 \|\varepsilon\|_{L^2}.
\] (7.26)

By (6.8) and (6.17)
\[
\int |U_J|^\alpha |\varepsilon|^2 \leq 2^\alpha (-\alpha \text{Im} \lambda)^{-1} (-t)^{-1} \|\varepsilon\|_{L^2}^2. 
\] (7.27)

Moreover, it follows from Sobolev’s embedding \(H^1(\mathbb{R}^N) \hookrightarrow L^{\alpha+2}(\mathbb{R}^N)\), (7.7) and (6.15) that
\[
\int |\varepsilon|^{\alpha+2} \lesssim \|\varepsilon\|_{H^1}^{\alpha+2} \lesssim (-t)^{(\alpha+2)(1-\delta)} \lesssim (-t)^{2\sigma}, \quad (7.28)
\]
and
\[
\|E_J\|_{L^2} \|\varepsilon\|_{L^2} \lesssim (-t)^{J(1-\frac{4}{\alpha})-\frac{4}{\alpha}+\sigma} \lesssim (-t)^{2\sigma}, \quad (7.29)
\]
where we used (6.1)–(6.3) to ensure \(\min\{\alpha+2, 1-\delta\} \geq 2\sigma\).

Applying (7.26), (7.27), (7.28) and (7.29), we conclude that
\[
\frac{d}{dt} \|\varepsilon(t)\|_{L^2}^2 \geq -2^{\alpha+1} (-\alpha \text{Im} \lambda)^{-1} |\lambda|M (-t)^{-1} \|\varepsilon\|_{L^2}^2 - C(-t)^{2\sigma},
\]
and so
\[
\frac{d}{dt} \left( (-t)^{-\sigma} \|\varepsilon(t)\|_{L^2}^2 \right) = \sigma (-t)^{-\sigma-1} \|\varepsilon(t)\|_{L^2}^2 + (-t)^{-\sigma} \frac{d}{dt} \|\varepsilon(t)\|_{L^2}^2 \\
\geq \left[ \sigma - 2^{\alpha+1} (-\alpha \text{Im} \lambda)^{-1} |\lambda| M \right] (-t)^{-\sigma-1} \|\varepsilon(t)\|_{L^2}^2 - C(-t)^{\sigma}.
\]

Using \(\varepsilon(-\frac{1}{n}) = 0\) and \(\sigma \geq 2^{\alpha+1} (-\alpha \text{Im} \lambda)^{-1} |\lambda| M\) by (6.3), we can integrate the above inequality on the interval \((t, -\frac{1}{n})\) to obtain
\[
\|\varepsilon(t)\|_{L^2} \leq C_{13} (-t)^{\frac{1}{2} + \sigma} \quad (7.30)
\]
for all \(t \in (-\frac{1}{n}, -\frac{1}{n})\).

Our next step is to estimate \(\|\Delta^2 \varepsilon\|_2\). Analogously to (7.21), we can use the equation (7.6) and Lemma 6.3 to obtain
\[
\|\Delta^2 \varepsilon\|_2 \lesssim \|\partial_t \varepsilon\|_2 + \|\varepsilon\|_2 + \|(|U_J|^\alpha + |\varepsilon|^\alpha)|\varepsilon|_2 + \|E_J\|_2.
\]
It follows from (6.8), (6.9), (6.15), (6.17), (7.7) and the embedding \(H^1 \hookrightarrow L^{2(\alpha+1)}\) that
\[
\|\Delta^2 \varepsilon\|_2 \lesssim (-t)^{\left(1-\frac{4}{\alpha}\right)+\sigma} + (-t)^{-1+\sigma} + (-t)^{(\alpha+1)(1-\delta)+\sigma} + (-t)^{-\frac{4}{\alpha}+J(1-\frac{4}{\alpha})-\frac{4}{\alpha}}
\]
\[
\leq C_{14} (-t)^{(1-\delta)\sigma}, \quad (7.31)
\]
where we used (6.1)–(6.3) to ensure \(\min\{-1+\sigma, -\frac{4}{\alpha}+J(1-\frac{4}{\alpha}-\frac{4}{\alpha})\} \geq (1-\delta)\sigma\).

We now estimate \(\|\partial_t \varepsilon\|_{L^2}\). Applying \(\partial_t\) to the equation (7.6), and then multiplying it by \(\overline{\varepsilon}\), integrating by parts, we obtain by applying Lemma 6.3
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t \varepsilon\|_2^2 = -\text{Im} \lambda \int \partial_t (|U_J + \varepsilon|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J) \overline{\partial_t \varepsilon} - \text{Im} \int \partial_t E_J \overline{\partial_t \varepsilon}
\]
\[
= -\text{Im} \lambda \int I_1(U_J, \varepsilon) \overline{\partial_t \varepsilon} - \text{Im} \lambda \int I_2(U_J, \varepsilon) \overline{\partial_t \varepsilon} - \text{Im} \int \partial_t E_J \overline{\partial_t \varepsilon}. \quad (7.32)
\]
These formal calculations can be justified by standard approximation arguments, see e.g. Proposition 3.1 in [27]. Applying Cauchy-Schwartz inequality, (6.9), (6.16) and (7.7), we obtain

$$\left| \int \partial_t \mathcal{E}_J \overline{\partial_t \varepsilon} \right| \lesssim (-t)^{-1+J(1-\frac{2}{r})-\frac{1}{2}+\frac{1}{2})^\sigma \lesssim (-t)^{2\sigma}, \quad (7.33)$$

where we used (6.1) to ensure \(-1 + J(1 - \frac{4}{r}) - \frac{4}{r} + (1 - \frac{4}{r})\sigma \geq 2\sigma\). On the other hand, it follows from (6.8), (6.17) and (6.24) that

$$|\text{Im} \lambda \int I_1(U_J, \varepsilon) \overline{\partial_t \varepsilon} | \leq |\lambda| M \int |U_J|^\alpha |\partial_t \varepsilon|^2 \leq 2^\alpha |\lambda| M(-\alpha \text{Im} \lambda)^{-1} \|\partial_t \varepsilon\|^2_2. \quad (7.34)$$

It remains to estimate \(-\text{Im} \lambda \int I_2(U_J, \varepsilon) \overline{\partial_t \varepsilon} \). Using (7.15), (7.16), (6.17), (6.19) and Hölder’s inequality, we conclude that

$$\left| -\text{Im} \lambda \int I_2(U_J, \varepsilon) \overline{\partial_t \varepsilon} \right| \lesssim (-t)^{-1} \|U_0\|_{\infty} \|\varepsilon\|_2 \|\partial_t \varepsilon\|_2 + 1_{\alpha > 1} (-t)^{-1} \|\varepsilon\|_{2\alpha+2} \|U_0\|_r \|\partial_t \varepsilon\|_r$$

$$+ \|\varepsilon\|_{2\alpha+2} \|\partial_t \varepsilon\|^2_2 + 1_{\alpha > 1} \|U_0\|_{2\alpha+2} \|\varepsilon\|_{2\alpha+2} \|\partial_t \varepsilon\|^2_2 \quad (7.35)$$

Moreover, it follows from (7.35), (6.9), (7.7) and Sobolev’s embedding \(H^1 \hookrightarrow L^{2\alpha+2} \cap L^r\) that

$$\left| -\text{Im} \lambda \int I_2(U_J, \varepsilon) \overline{\partial_t \varepsilon} \right| \lesssim (-t)^{-2+2(2-\frac{4}{r})} + 1_{\alpha > 1} (-t)^{-1-\frac{2}{r}+\alpha(1-\delta)\sigma} \|\partial_t \varepsilon\|_r$$

$$+ (-t)^{\alpha(1-\delta)\sigma} \|\partial_t \varepsilon\|^2_2 + 1_{\alpha > 1} (-t)^{-1+\frac{1}{2}+(1-\delta)\sigma} \|\partial_t \varepsilon\|^2_2$$

$$\lesssim (-t)^{-2+2(2-\frac{4}{r})} + 1_{\alpha > 1} (-t)^{-2+\alpha(1-\delta)\sigma} \|\partial_t \varepsilon\|_r$$

$$+ (-t)^{-1+\min(\alpha,1)(1-\delta)\sigma} \|\partial_t \varepsilon\|^2_2. \quad (7.36)$$

It now follows from (7.32), (7.33), (7.34) and (7.36) that

$$\frac{d}{dt} \|\partial_t \varepsilon\|^2_2 \geq -2^{\alpha+1} |\lambda| M(-\alpha \text{Im} \lambda)^{-1} (-t)^{-1} \|\partial_t \varepsilon\|^2_2 - C(-t)^{-2+2(2-\frac{4}{r})\sigma}$$

$$- C_1 1_{\alpha > 1} (-t)^{-2+\alpha(1-\delta)\sigma} \|\partial_t \varepsilon\|_r - C(-t)^{-1+\min(\alpha,1)(1-\delta)\sigma} \|\partial_t \varepsilon\|^2_2,$$

so that

$$\frac{d}{dt} \left[ (-t)^{-\frac{1}{2} \min(\alpha,1)(1-\delta)\sigma} \|\partial_t \varepsilon\|^2_2 \right]$$

$$\geq \left( \frac{1}{2} \min(\alpha,1)(1-\delta)\sigma - 2^{\alpha+1} |\lambda| M(-\alpha \text{Im} \lambda)^{-1} \right)$$

$$\times (-t)^{-1-\frac{2}{r} \min(\alpha,1)(1-\delta)\sigma} \|\partial_t \varepsilon\|^2_2 - C_1 1_{\alpha > 1} (-t)^{-2+\alpha(1-\delta)\sigma} \|\partial_t \varepsilon\|_r$$

$$- C(-t)^{-2+2(2-\frac{4}{r})\sigma} - \frac{1}{2} \min(\alpha,1)(1-\delta)\sigma \|\partial_t \varepsilon\|^2_2. \quad (7.37)$$

Using \(\varepsilon (\frac{1}{4}) = 0\) and \(\frac{1}{2} \min(\alpha,1)(1-\delta)\sigma \geq 2^{\alpha+1} |\lambda| M(-\alpha \text{Im} \lambda)^{-1}\) by (6.3), we can integrate the above inequality on \((t, -\frac{4}{r})\), and then apply Hölder’s inequality.
to obtain
\[\|\partial_t \varepsilon\|_2^2 \lesssim (-t)^{-1+\frac{2}{3}+\frac{\sigma}{4}+\min\{\alpha, 1\}(1-\delta)}\|\partial_t \varepsilon\|_{L_t^\infty L^r}^2\]
\[\lesssim (-t)^{-1+\frac{2}{3}+\min\{\alpha, 1\}(1-\delta)}\|\partial_t \varepsilon\|_2^2\]
where in the last inequality we used (7.8).

Note that
\[\min\left\{1 + \left(\frac{2 - \delta}{2}\right)\sigma, 1 + \min\{\alpha, 1\}(1-\delta)\sigma + 2(1-2\delta)\sigma\right\}\geq 2\left(1 - \frac{\delta}{2}\right)\sigma + \frac{\delta\sigma}{4},\quad \alpha > 1\]
by (6.4) and (6.3), so that
\[\|\partial_t \varepsilon\|_2^2 \leq C_{15} (-t)^{2\left(1-\frac{\delta}{2}\right)\sigma + \frac{\sigma}{4}}.\]  

(7.37)

Set \(S \in [T_0, 0]\) satisfying
\[C_{13}(-S)^{\frac{\sigma}{4}} \leq \frac{1}{2},\quad C_{14}(-S)^{\frac{\sigma}{4}} \leq \frac{1}{2},\quad C_{15}(-S)^{\frac{\sigma}{4}} \leq \frac{1}{2}.\]  

(7.38)

It follows from (7.30), (7.31), (7.37) and (7.38) that for \(n\) sufficiently large, we have \(S < -\frac{1}{n}\) and
\[\|\varepsilon\|_{L^2} \leq \frac{1}{2}(-t)^{\sigma}, \quad \|\Delta^2 \varepsilon\|_{L^2} \leq \frac{1}{2}(-t)^{(1-\delta)\sigma}, \quad \|\partial_t \varepsilon\|_2 \leq \frac{1}{2}(-t)^{(1-\frac{\delta}{2})\sigma},\]  

(7.39)

for all \(\max\{\tau_n, S\} < t < -\frac{1}{n}\). Suppose \(S \leq \tau_n\), then (7.39) holds for all \(\tau_n < t < -\frac{1}{n}\). This contradicts the definition (7.7) of \(\tau_n\) since \(s_n < \tau_n\). Therefore, we deduce that \(\tau_n < S\) and (7.39) holds for all \(S < t < -\frac{1}{n}\), which completes the proof of Proposition 7.1. \(\square\)

8. Proof of Theorem 1.3. Firstly, we show that there exists a solution \(u \in C((S, 0), H^4(\mathbb{R}^N) \cap C^1([S, 0), L^2(\mathbb{R}^N)))\) to the equation (1.1). Given \(\tau \in (S, 0)\), it follows from (7.5) that \(\{\varepsilon_n\}_{n \geq \frac{1}{m}}\) is bounded in \(L^\infty([S, \tau], H^4(\mathbb{R}^N)) \cap W^{1,\infty}([S, \tau], L^2(\mathbb{R}^N))\).

Therefore, after possibly extracting a subsequence, there exists \(\varepsilon \in L^\infty([S, \tau], H^4(\mathbb{R}^N)) \cap W^{1,\infty}([S, \tau], L^2(\mathbb{R}^N))\) such that
\[\varepsilon_n \rightharpoonup \varepsilon, \quad \text{weak}^* \quad \text{in} \quad L^\infty([S, \tau], H^4(\mathbb{R}^N)),\]  

(8.1)

\[\partial_t \varepsilon_n \rightharpoonup \partial_t \varepsilon, \quad \text{weak}^* \quad \text{in} \quad L^\infty([S, \tau], L^2(\mathbb{R}^N)).\]  

(8.2)

On the other hand, note that \(H^4(\Omega) \hookrightarrow L^{\alpha+2}(\Omega) \hookrightarrow L^2(\Omega)\) and \(\{\varepsilon_n\}_{n \geq \frac{1}{m}}\) is uniformly bounded in \(L^\infty([S, \tau], H^4(\mathbb{R}^N)) \cap W^{1,\infty}([S, \tau], L^2(\mathbb{R}^N))\), we have (after extracting a subsequence)
\[\varepsilon_n \rightharpoonup \varepsilon \quad \text{in} \quad L^\infty([S, \tau], L^{\alpha+2}(\Omega)),\]  

(8.3)

by Aubin-Lions Theorem, see Simon [38].

Since \(\tau \in (S, 0)\) is arbitrary, a standard argument of diagonal extraction shows that there exists \(\varepsilon \in L^\infty_{loc}([S, 0), H^4(\mathbb{R}^N)) \cap W^{1,\infty}_{loc}([S, 0), L^2(\mathbb{R}^N))\), such that (after
extracting a subsequence) (8.1), (8.2) and (8.3) hold for all \( S < \tau < 0 \). Moreover, (7.5), (8.1), (8.2) and (8.3) imply that
\[
\|\varepsilon(t)\|_{L^2} \leq (-t)^\sigma, \quad \|\Delta^2 \varepsilon(t)\|_{L^2} \leq (-t)^{(1-\delta)\sigma}, \quad \|\partial_t \varepsilon(t)\|_{L^2} \leq (-t)^{(1-\frac{2}{N})\sigma},
\]
for \( S \leq t < 0 \). In addition it follows easily from (7.6) and the convergence properties (8.1), (8.2) and (8.3) that
\[
i \partial_t \varepsilon + \Delta^2 \varepsilon + \mu \Delta \varepsilon + \lambda ([U_J + \varepsilon]^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J) + \mathcal{E}_J = 0
\]
in \( L^\infty_{loc}([S,0), L^2(\mathbb{R}^N)) \). Therefore, setting
\[
u(t) = U_J(t) + \varepsilon(t), \quad S \leq t < 0,
\]
we see that \( u \in L^\infty_{loc}([S,0), H^4(\mathbb{R}^N)) \cap H^1_{loc}([S,0), L^2(\mathbb{R}^N)) \) and that
\[
i \partial_t u + \Delta^2 u + \mu \Delta u + \lambda |u|^\alpha u = 0, \quad \text{in} \quad L^\infty_{loc}([S,0), L^2(\mathbb{R}^N))
\]
by (6.20), (8.5) and (8.6). By the local existence in \( H^4(\mathbb{R}^N) \) and the uniqueness in \( L^\infty_{loc}H^4_2 \), we conclude that \( u \in C([S,0), H^4(\mathbb{R}^N)) \cap C^1([S,0), L^2(\mathbb{R}^N)) \).

Next, we prove properties (1.8)-(1.10) in Theorem 1.3. We first prove (1.10). Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) such that \( \Omega \cap K = \emptyset \). It follows from (6.5) that \( A > 0 \) on \( \Omega \) and \( A(x) = |x|^k \) when \( |x| > 2R \); and so there exists a constant \( c > 0 \), such that \( A(x) \geq c(1 + |x|)^k \) on \( \Omega \). Moreover using (6.6) and (6.10), we deduce that
\[
|U_0| \lesssim A(x)^{-\frac{k}{N}} \lesssim (1 + |x|)^{-\frac{k}{N}}, \quad \|U_0|^\alpha |U_0| \lesssim (1 + |x|)^{-k(1+\frac{1}{N})}, \quad \text{on} \quad \Omega,
\]
and
\[
|\Delta U_0| \lesssim (1 + |x|)^{-\frac{k}{N} - 2}, \quad |\Delta^2 U_0| \lesssim (1 + |x|)^{-\frac{k}{N} - 4}, \quad \text{on} \quad \Omega.
\]
Since \( (1 + |x|)^{-\frac{k}{N}} \in L^2(\mathbb{R}^N) \) by (6.2), we deduce from (6.14), (6.15), (6.17) and the equation (6.20) that
\[
\limsup_{t \to 0} \|U_J\|_{H^4(\Omega)} < \infty, \quad \limsup_{t \to 0} \|\partial_t U_J\|_{L^2(\Omega)} < \infty.
\]
It now follows from (8.6) and the \( L^\infty([S,0), H^4(\mathbb{R}^N)) \cap C^1([S,0), L^2(\mathbb{R}^N)) \) boundedness of \( \varepsilon \) in (8.4) that (1.10) holds.

Our next aim is to prove (1.8). Let now \( x_0 \in K \) and \( r > 0 \), it follows from (6.9), (6.11) and (6.17) that
\[
(-t)^{-\frac{k}{N} + \frac{2\sigma}{N}} \lesssim \|U_J(t)\|_{L^2(|x-x_0| < r)} \lesssim (-t)^{-\frac{k}{N}}.
\]
Using (8.6) and the \( L^2 \) boundedness of \( \varepsilon \) from (8.4), we deduce that
\[
\|u(t)\|_{L^2(|x-x_0| < r)} \gtrsim \|U_J(t)\|_{L^2(|x-x_0| < r)} - \|\varepsilon(t)\|_{L^2(|x-x_0| < r)} \gtrsim (-t)^{-\frac{k}{N} + \frac{2\sigma}{N}} - (-t)^{\sigma},
\]
which proves (1.8) as \( -\frac{k}{N} + \frac{N}{2\sigma} < 0 \) by (6.3).

Finally, we come to the proof of (1.9). Since \( k \) satisfies \( (2 + \frac{8\alpha}{N})\alpha < 2N \) by (6.2), we can choose a constant \( p \) satisfying
\[
p > 2 + \frac{8\alpha}{k} \quad \text{and} \quad p(\alpha - N) < 2N.
\]
Hence, we can apply (6.9), (6.11), (6.17) and Gagliardo-Nirenberg’s inequality to obtain
\[
(-t)^{-\frac{k}{N} + \frac{2\sigma}{N}} \lesssim \|U_J\|_{L^p(U)} \lesssim \|\Delta^2 U_J\|_{L^2(U)}^{\frac{N}{2}(\frac{N}{2} - \frac{k}{N})} \|U_J\|_{L^2(U)}^{1 - \frac{N}{2}(\frac{N}{2} - \frac{k}{N})} \lesssim \|\Delta^2 U_J\|_{L^2(U)}^{\frac{N}{2}(\frac{N}{2} - \frac{k}{N})} (-t)^{-\frac{k}{N}(1 - \frac{N}{2}(\frac{N}{2} - \frac{k}{N})).
\]
so that
\[ (-t)^{\frac{2}{(p-2)} - \frac{1}{2}} \lesssim \| \Delta^2 U_J \|_{L^2(U)}. \]
This inequality together with (8.8) implies that
\[ \lim_{t \to 0} \| \Delta^2 U_J \|_{L^2(U)} = \infty. \quad (8.9) \]
On the other hand, note that \( U_j = U_{j-1} + w_j \) by (6.13), so that \( \partial_t U_J = \partial_t w_0 + \cdots + \partial_t w_J \). Then it follows from Lemma 6.2 that
\[
\| \partial_t U_J \|_{L^2(U)} \gtrsim (-t)^{-1} |U_0| - (-t)^{-1+(1-\frac{1}{2})}|U_0| - \cdots - (-t)^{-1+J(1-\frac{1}{2})}|U_0| 
\gtrsim (-t)^{-1}|U_0|
\]
provided \( t < 0 \) is sufficiently small. This inequality together with (6.11) gives
\[ \| \partial_t U_J \|_{L^2(U)} \gtrsim (-t)^{-1-\frac{1}{2} + \frac{N}{2}}, \]
which implies that
\[ \lim_{t \to 0} \| \partial_t U_J \|_{L^2(U)} = \infty. \quad (8.10) \]
Combining (8.9), (8.10) and the boundedness of \( \varepsilon(t) \) in (8.4), we obtain (1.9), thereby completing the proof of Theorem 1.3.

9. Appendix.

9.1. Proof of Theorem 1.1 in the case \( 1 \leq N \leq 8 \). We fix \( \varepsilon > 0 \) sufficiently small such that
\[ (2 - \alpha) \varepsilon \leq 2\alpha, \quad \varepsilon < 2\alpha. \]
Then we define
\[ r_\varepsilon = 2 + \varepsilon, \quad q_\varepsilon = \frac{8(2 + \varepsilon)}{N\varepsilon}. \quad (9.1) \]
It is straightforward to verify that \((q_\varepsilon, r_\varepsilon) \in \Lambda_0\) is a biharmonic admissible pair and \( 2 \leq \frac{q_\varepsilon}{r_\varepsilon - 2} \alpha < \infty \), which implies that \( H^4 \hookrightarrow L^{r_\varepsilon - 2} \alpha \).

Choosing \( M = 2C_{10}(\| \phi \|_{H^4} + \| \phi \|_{H^4}^{\alpha+1}) \), and \( T > 0 \) sufficiently small such that
\[ (C_{16} + C_{17}) T^{1-\frac{2}{p}} M^\alpha \leq \frac{1}{2} \quad (9.2) \]
where the constants \( C_{16}, C_{17} \) are defined in (9.9) and (9.10), respectively. Set \( J = [-T, T] \), and consider the metric space
\[ Z_{T,M} = \{ u \in L^\infty(J, H^4) \cap H^{1,q_\varepsilon}(J, L^{r_\varepsilon}) \cap H^{1,\infty}(J, L^2) : \| u \|_{L^\infty(J, H^4) \cap L^{r_\varepsilon}(J, L^{r_\varepsilon})} + \| \partial_t u \|_{L^\infty(J, L^2) \cap L^{q_\varepsilon}(J, L^{q_\varepsilon})} \leq M \} \quad (9.3) \]
It follows that \( Z_{T,M} \) is a complete metric space when equipped with the distance
\[ d(u, v) = \| u - v \|_{L^\infty(J, L^2)} + \| u - v \|_{L^{q_\varepsilon}(J, L^{q_\varepsilon})}. \quad (9.4) \]
In what follows, we show that the mapping \( S \), defined in (2.5), is a strict contraction mapping on the space \( Z_{T,M} \).
We first show that $S$ maps $Z_{T,M}$ into itself. Applying (2.5), (2.7), (2.9), Strichartz’s estimate (2.3), Hölder’s inequality and Sobolev’s embedding theorem $H^4(\mathbb{R}^N) \hookrightarrow L^{\frac{4N}{N-4}}(\mathbb{R}^N)$, we conclude that

$$
\|Su\|_{L^{\infty}(J,L^2) \cap L^\infty(J,L^{r_*})} + \|\partial_t(Su)\|_{L^{\infty}(J,L^2) \cap L^\infty(J,L^{r_*})}
\lesssim F(\phi, T) + \|u\|^{\alpha}(\|u\| + \|\partial_t u\|)\|_{L^{r}(J,L^{r_*)}}
\lesssim \|\phi\|_{H^4} + \|u\|^{\alpha+1} + T^{1-\frac{4}{4r_\alpha}} \|u\|^{\alpha}_{L^{\infty}(J,H^4)} \|\partial_t u\|_{L^{\infty}(J,L^{r_*})}.
$$

(9.5)

Our next step is to estimate $\|Su\|_{L^{\infty}(J,H^s)}$. Using the same method as that used to derive (4.13), we obtain

$$
\|\Delta^2(Su)\|_{L^{\infty}(J,L^2)} \lesssim \|\partial_t(Su)\|_{L^{\infty}(J,L^2)} + \|Su\|_{L^{\infty}(J,L^2)} + \|u\|^{\alpha}u\|_{L^{\infty}(J,L^2)}.
$$

(9.6)

To estimate $\|Su\|_{L^{\infty}(J,L^2)}$, we apply the Fundamental Theorem of Calculus

$$
|u|^{\alpha}u = \phi^{\alpha}u + \int_0^t \partial_s|u|^{\alpha}u(s)ds,
$$

(9.7)

Hölder’s inequality and Sobolev’s embedding $H^4 \hookrightarrow L^{2(\alpha+1)} \cap L^{\frac{2\alpha}{\alpha-2}}$ to obtain

$$
\|u\|^{\alpha}u\|_{L^{\infty}(J,L^2)} \lesssim \|\phi\|^{\alpha}u\|_{L^2} + \|\partial_t[u]|^{\alpha}u\|_{L^1(J,L^2)}
\lesssim \|\phi\|^{\alpha+1}_{L^{2}(\alpha+1)} + \|u\|^{\alpha}_{L^{\infty}(J,L^{\frac{2\alpha}{\alpha-2}})} \|\partial_t u\|_{L^{\infty}(J,L^{r_*})}
\lesssim \|\phi\|^{\alpha+1}_{H^4} + T^{1-\frac{4}{4r_\alpha}} \|u\|^{\alpha}_{L^{\infty}(J,H^4)} \|\partial_t u\|_{L^{\infty}(J,L^{r_*})}.
$$

(9.8)

Applying (9.5), (9.6) and (9.8), we see that if $u \in X_{T,M}$, then

$$
\|Su\|_{L^{\infty}(J,H^s) \cap L^\infty(J,L^{r_*})} + \|\partial_t (Su)\|_{L^{\infty}(J,L^2) \cap L^\infty(J,L^{r_*})}
\leq C_{16}(\|\phi\|_{H^4} + \|\phi\|^{\alpha+1}_{H^4}) + C_{16}T^{1-\frac{4}{4r_\alpha}} M^{\alpha+1} \leq M.
$$

(9.9)

Our next aim is the desired Lipschitz property of $S$ with respect to the metric $d$ defined in (9.4). It follows from Strichartz’s estimate (2.3), Hölder’s inequality and Sobolev’s embedding $H^4(\mathbb{R}^N) \hookrightarrow L^{\frac{4N}{N-4}}(\mathbb{R}^N)$ that, given $u, v \in Z_{T,M}$,

$$
d(Su, Sv) \lesssim \|u|^{\alpha}u - |v|^{\alpha}v\|_{L^{\infty}(J,L^{r_*})}
\lesssim T^{1-\frac{4}{4r_\alpha}} (\|u\|^{\alpha}_{L^{\infty}(J,H^4)} + \|v\|^{\alpha}_{L^{\infty}(J,H^4)}) \|u - v\|_{L^{\infty}(J,L^{r_*})}
\leq C_{17} T^{1-\frac{4}{4r_\alpha}} M^{\alpha} d(u, v) \leq \frac{1}{2} d(u, v).
$$

(9.10)

Therefore, we obtain a unique solution $u$ in $Z_{T,M}$ by Banach’s fixed point theorem. Moreover, using the same method used in Section 4 and the uniqueness in Appendix, we can extend $u$ to a maximal solution $u \in C([0,T_{\text{max}}), H^4)$ to the equation (1.1) with

$$
u, u_t, \Delta^2 u \in C([0,T_{\text{max}}), L^2(\mathbb{R}^N)) \cap L^q_{\text{loc}}((0,T_{\text{max}}), L^r(\mathbb{R}^N))
$$

for every biharmonic admissible pair $(q, r) \in \Lambda_b$. Finally, the blowup alternative (1.5) and the continuous dependence are proved as in Section 4. This completes the proof of Theorem 1.1 in the case $1 \leq N \leq 8$. □
9.2. Unconditional uniqueness. In this subsection, we prove the unconditional uniqueness in $H^4(\mathbb{R}^N)$ for the Cauchy problem (1.1)-(1.2). The proof is an obvious adaptation of Propositions 4.2.3 and 4.2.5 in [7].

**Theorem 9.1** (Critical case). Suppose $\lambda \in \mathbb{C}$, $\mu = \pm 1$ or $0$, $N \geq 9$, $\alpha = \frac{8}{N-4}$, $T > 0$, and $u^1, u^2 \in C([0,T], H^4(\mathbb{R}^N))$ are two solutions of (1.1) with the same initial data $\phi \in H^4$. It follows that $u^1 = u^2$.

**Proof.** Set

$$S = \sup \{ \tau \in [0,T]; u^1(t) = u^2(t) \text{ for } 0 \leq t \leq \tau \},$$

so that $0 \leq S \leq T$. Uniqueness follows if we show that $S = T$. Assume by contradiction that $S < T$. Changing $u^1(\cdot), u^2(\cdot)$ to $u^1(S+\cdot), u^2(S+\cdot)$, we are reduced to the case $S = 0$, so that

$$\inf_{(q,r) \in \Lambda_h} \| u^1 - u^2 \|_{L^q((0,\tau), L^r)} > 0 \text{ for all } 0 < \tau \leq T. \tag{9.11}$$

On the other hand, it follows from the equation (4.11) (for both $u^1$ and $u^2$) that

$$u^1(t) - u^2(t) = i\lambda \int_0^t e^{i(t-s)(\Delta^2 + \mu \Delta)} \left[ |u^1(s)|^\alpha u^1(s) - |u^2(s)|^\alpha u^2(s) \right] ds.$$

Applying Strichartz’s estimate (2.3), we deduce that

$$\sup_{(q,r) \in \Lambda_h} \| u^1 - u^2 \|_{L^q((0,\tau), L^r)} \lesssim \| g \|_{L^2((0,\tau), L^{\frac{2N}{N+4}})} \tag{9.12}$$

for every $0 < \tau \leq T$, where $g = |u^1|^\alpha + |u^2|^\alpha$. Since $u^1, u^2 \in C([0,T], H^4)$, we deduce from Sobolev’s embedding $H^4 \hookrightarrow L^{\frac{N}{N+4}}$ that $u^1, u^2 \in C([0,T], L^{\frac{N}{N+4}})$, which implies

$$g \in C([0,T], L^{\frac{N}{N+4}}). \tag{9.13}$$

Given any $R > 0$, we set

$$g^R = \begin{cases} |g|, & \text{if } |g| \geq R \\ 0, & \text{if } |g| < R \end{cases}, \quad g_R = |g| - g^R.$$

It is not difficult to show (by dominated convergence theorem, using (9.13)) that

$$\| g^R \|_{L^\infty((0,T), L^{\frac{N}{N+4}})} := \varepsilon_R \xrightarrow{R \to \infty} 0.$$

Moreover, since $|g_R| \leq R$,

$$\| g_R \|_{L^\infty((0,T), L^{\frac{N}{N+4}})} \leq R^{\frac{1}{2}} \| g \|_{L^\infty((0,T), L^{\frac{N}{N+4}})^{\frac{1}{2}}} =: C(R) < \infty,$$

for all $R > 0$. Therefore, given any $0 < \tau \leq T$, we deduce from Hölder’s inequality that

$$\| g^R (u^1 - u^2) \|_{L^2((0,\tau), L^{\frac{2N}{N+4}})} \leq \varepsilon_R \| u^1 - u^2 \|_{L^2((0,\tau), L^{\frac{2N}{N+4}})} \tag{9.14}$$

and

$$\| g_R (u^1 - u^2) \|_{L^2((0,\tau), L^{\frac{2N}{N+4}})} \lesssim C(R) \tau^{\frac{1}{2}} \| u^1 - u^2 \|_{L^\infty((0,\tau), L^2)}. \tag{9.15}$$
It follows from (9.12), (9.14) and (9.15) that
\[
\sup_{(q,r)\in \Lambda_b} \left\| u^1 - u^2 \right\|_{L^q((0,\tau),L^r)} \\
\leq C_{18} \left[ \varepsilon_R + C(R) \tau^{1+\frac{4}{N}} \right] \sup_{(q,r)\in \Lambda_b} \left\| u^1 - u^2 \right\|_{L^q((0,\tau),L^r)}. \tag{9.16}
\]

We first fix \( R \) sufficiently large so that \( C_{18}\varepsilon_R \leq \frac{1}{4} \). Then we choose \( 0 < \tau_0 \leq T \) sufficiently small so that \( C_{18}C(R)\tau_0^{1+\frac{4}{N}} \leq \frac{1}{4} \), and we deduce from (9.16) that
\[
\sup_{(q,r)\in \Lambda_b} \left\| u^1 - u^2 \right\|_{L^q((0,\tau),L^r)} = 0.
\]
This contradicts (9.11) and proves the uniqueness. \( \square \)

**Theorem 9.2** (Subcritical case). Suppose \( \lambda \in \mathbb{C}, N \geq 1, \mu = \pm 1 \) or \( 0 < \alpha, (N - 8)\alpha < 8, T > 0 \), and \( u^1, u^2 \in L^\infty([0,T],H^4(\mathbb{R}^N)) \) are two solutions of (1.1) with the same initial data \( \phi \in H^4 \). It follows that \( u^1 = u^2 \).

**Proof.** Similarly to the critical case, it suffices to prove that there exists \( 0 < \tau \leq T \) such that \( u^1(t) = u^2(t) \) for any \( 0 \leq t \leq \tau \). Let
\[
q = \frac{8(\alpha + 2)}{N\alpha}, \quad r = \alpha + 2.
\]
Then it is easy to check that \( q > 2 \) and \( (q,r) \in \Lambda_b \). Moreover, since \( u^1, u^2 \in L^\infty([0,T],H^4) \), we deduce from the embedding \( H^4 \hookrightarrow L^r \) that \( u^1, u^2 \in L^q([0,T],L^r) \). Next, applying Strichartz’s estimate and Hölder’s inequality, we conclude that
\[
\left\| u^1 - u^2 \right\|_{L^q((0,\tau),L^r)} \\
\leq \left\| |u^1|^\alpha u^1 - |u^2|^\alpha u^2 \right\|_{L^q((0,\tau),L^r)} \\
\leq \tau^{1-\frac{4}{N}} \left( \left\| u^1 \right\|_{L^\infty((0,\tau),L^r)}^{\alpha} + \left\| u^2 \right\|_{L^\infty((0,\tau),L^r)}^{\alpha} \right) \left\| u^1 - u^2 \right\|_{L^q((0,\tau),L^r)}
\]
for any \( 0 < \tau \leq T \). Moreover, it follows from Sobolev’s embedding \( H^4 \hookrightarrow L^r \) that
\[
\left\| u^1 - u^2 \right\|_{L^q((0,\tau),L^r)} \leq C_{19}\tau^{1-\frac{4}{N}} \left( \left\| u^1 \right\|_{L^\infty((0,\tau),H^4)}^{\alpha} + \left\| u^2 \right\|_{L^\infty((0,\tau),H^4)}^{\alpha} \right) \times \left\| u^1 - u^2 \right\|_{L^q((0,\tau),L^r)}. \tag{9.17}
\]
Then, we choose \( 0 < \tau_0 \leq T \) sufficiently small so that
\[
C_{19}\tau_0^{1-\frac{4}{N}} \left( \left\| u^1 \right\|_{L^\infty((0,T),H^4)}^{\alpha} + \left\| u^2 \right\|_{L^\infty((0,T),H^4)}^{\alpha} \right) \leq \frac{1}{2},
\]
and we deduce from (9.17) that \( u^1(t) = u^2(t) \) for any \( 0 \leq t \leq \tau_0 \). This finishes the proof of Theorem 9.2. \( \square \)

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