A new 3-component Degasperis-Procesi hierarchy

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Abstract. We study the bi-Hamiltonian structures for the hierarchy of a 3-component generalization of the Degasperis-Procesi (3-DP) equation. We show that all Hamiltonian functionals in the hierarchy are homogenous, and Hamiltonian functionals of the hierarchy in the negative direction are local. We construct two different Liouville transformations by construct a reciprocal transformation. The associated system for the first one is a reduction of a negative flow in a modified Yajima-Oikawa hierarchy. The associated system for the second one passes the Painlevé test, besides the Hamiltonian structures of the 3-DP equation under this Liouville transformation are considered. In addition, we consider a limit for the 3-DP equation.

1. Introduction

The Camassa-Holm (CH) equation

\[ m_t + um_x + 2u_xm = 0, \quad m = u - u_{xx}, \quad (1) \]

has attract much attention in recent years \cite{1}, because it is derived from dispersive shallow-water motion and the remarkable discovery of peakon solutions. The peakon solutions are interesting in general analysis of PDEs and beautiful in itself \cite{2}. The CH equation is integrable from the point of view of admitting a Lax pair and possessing a bi-Hamiltonian structure \cite{11, 12}. Besides it is linked to the first negative flow in the KdV hierarchy by a reciprocal transformation \cite{4, 5, 6}. Many other algebraic and geometric properties of the CH equation are studied (see e.g. \cite{7}).

By applying the method of asymptotic integrability to a family of third order dispersive PDE, the DP equation

\[ m_t + um_x + 3u_xm = 0, \quad m = u - u_{xx}, \quad (2) \]

admits peakons is proposed by Degasperis and Procesi \cite{8}. The DP equation admits a Lax pair and a bi-Hamiltonian structure. An infinite sequence of conservation laws for the equation are also obtained. A reciprocal transformation is constructed to connect it with a negative flow in the Kaup-Kupershmidt hierarchy \cite{9}. Subsequently, many other equations of CH type are proposed and studied such as the Novikov equation, a 2-component CH equation and the Geng-Xue equation (see e.g. \cite{10, 11, 12, 13, 14, 15}).
Recently, a $3 \times 3$ matrix spectral problem
\[
\varphi_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 + \lambda u & 0 & v \\ \lambda w & 0 & 0 \end{pmatrix} \varphi,
\] (3)
for a 3-component CH type hierarchy is proposed by Geng and Xue [16], which may be reduced to those of the CH equation, the DP equation, the Novikov equation and the Geng-Xue equation. They also derive a hierarchy corresponding to the spectral problem (3) with the trivial flow is choose as $(u,v,w)^T = (0,v,-w)^T$, and the first negative flow in which reads as
\[
\begin{align*}
    u_t &= -vp_x + u_x q + \frac{3}{2} u q_x - \frac{3}{2} u(p_x r_x - pr), \\
    v_t &= 2vq_x + v_x q, \\
    w_t &= vr_x + w_x q + \frac{3}{2} w q_x + \frac{3}{2} w(p_x r_x - pr), \\
    u &= p - p_{xx}, \\
    w &= r_{xx} - r, \\
    v &= \frac{1}{2}(q_{xx} - 4q + p_{xx} r_x - r_{xx} p_x + 3p_x r - 3p r_x).
\end{align*}
\] (4)
This system can be reduced to the CH equation as $p = r = 0$. It admits a bi-Hamiltonian structure and an infinite sequence of conserved quantities [16, 17]. Subsequently, by considering reductions of a 4-component CH type system, we discuss another 3-component CH type system admitting the spectral problem [18]
\[
\phi_x = \begin{pmatrix} 0 & 0 & 1 \\ \lambda m_1 & 0 & \lambda m_3 \\ 1 & \lambda m_2 & 0 \end{pmatrix} \phi.
\] (5)
This 3-component CH type system also admits a bi-Hamiltonian structure and infinitely many conserved quantities. A reciprocal transformation is applied to connect it with the first negative flow in a generalized MKdV hierarchy (a modified Yajima-Oikawa hierarchy [19, 20]), and the associated system is shown to pass the standard Painlevé test of WTC [21]. Notice that the spectral problem (5) is gauge linked to the spectral problem (3) [18], in the following we will show the corresponding flows are connected directly.

In this paper, we will consider a new 3-DP hierarchy associated with the following spectral problem
\[
\varphi_x = U\varphi, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ 1 + \lambda u^2 & 0 & v \\ \lambda w & 0 & 0 \end{pmatrix},
\] (6)
which is obtained by replacing $u$ in (3) with $u^2$ for convenience. This hierarchy is different from the hierarchy selected by Geng and Xue because we choose the trivial flow as $(u,v,w)^T = (0,v,-w)^T$ and the Hamiltonian functionals are obtained using the spectral problem (6). The first positive flow in this hierarchy is reads as
\[u_t + (u p r)_x = 0,\]
The 3-DP equation (7) is reduced to the DP equation, the Novikov equation and the Geng-Xue equation as $u = 0, r = 1$, as $u = 0, p = r$ and as $u = 0$ respectively. We construct infinitely many conserved quantities and study the bi-Hamiltonian structures for the 3-DP hierarchy. We construct two Liouville transformations for the 3-DP equation. The first one can be used to connect the 3-DP equation with a negative flow in a modified Yajiama-Oikawa hierarchy. The second one is to connected the 3-DP equation with a negative generalized mKdV equation passing the Painlevé test.

The outline of this paper is as follows. In Section 2, we construct infinitely many conserved quantities for the 3-DP equation using the spectral problem (6). We analyze the homogeneous and local properties of the Hamiltonian functionals in the 3-DP hierarchy. In Section 3, we construct a Liouville transformation to connect the 3-DP equation with the first negative flow in a modified Yajiama-Oikawa hierarchy. In Section 4, we construct another Liouville transformation for the 3-DP equation (7) to test its Painlevé property, and the Hamiltonian structures for the 3-DP equation under the reciprocal transformation are also considered. Besides the two 3-component CH type system possessing the spectral problem (5) and (6) are obtained. In Section 5, a limit of the 3-DP equation are studied.

2. Conserved quantities and bi-Hamiltonian structure of the 3-DP hierarchy

2.1. Conserved quantities

The 3-DP equation (7) arises as the compatibility condition for the linear system

$$
\varphi_x = U \varphi, \quad \varphi_t = V \varphi,
$$

where

$$
V = \begin{pmatrix}
\frac{1}{3\lambda} + pr_x & -pr_x & \frac{p}{\lambda} \\
p_x r_x - \lambda u^2 pr & \frac{1}{3\lambda} - p_x r & \frac{p_x}{\lambda} - vpr \\
-\lambda wpr - r_x & r & p_x r - pr_x - \frac{2}{3\lambda}
\end{pmatrix}.
$$

Based on the Lax pair (8), infinitely many conserved densities or conservation laws for the 3-DP equation can be constructed. For example, setting $\rho = (\ln \varphi_3)_x$ and expanding it in powers of $\lambda$, as pointed out in [16], one may able to obtain an infinite sequence of conserved densities for (7) from coefficients of $\rho$ by solving

$$
(\partial + \rho)[(\frac{\rho}{w})_x + \frac{\rho^2}{w}] - (1 + \lambda u^2) \frac{\rho}{w} - \lambda v = 0.
$$

However, it is not easy to solve (9) and the expansion of $\rho$ in [16] can be generalized. In what follows, we will consider a better formulation for computations and get exacts
Furthermore, we can also consider the expansions of \( a, b \) which lead to an infinite sequence of conserved quantities and the first three read as

\[
\begin{align*}
\rho &= \lambda wa, \\
\rho &= \lambda w^2 a.
\end{align*}
\]

One can easily show that \( a \) and \( b \) satisfy the following two equations

\[
\begin{align*}
a_x &= b - \lambda wa^2, \quad (10) \\
b_x &= (1 + \lambda u^2)a + v - \lambda wab. \quad (11)
\end{align*}
\]

Solving the above equations by expanding \( a, b \) as

\[
\begin{align*}
a &= \sum_{j \geq 0} a_j \lambda^j, \\
b &= \sum_{j \geq 0} b_j \lambda^j,
\end{align*}
\]

we get

\[
\begin{align*}
a_0 &= b_0, \\
a_1 &= b_1 - wa^2, \\
a_i &= b_i - w \sum_{k=0}^{i-1} a_k a_{i-k-1}, \\
a_{lox} &= b_{lox} = a_0 + v, \\
b_{lox} &= b_0 = a_0 + v, \\
b_1 &= a_1 + u^2 a_0 - wa_0 b_0, \\
b_i &= a_i + u^2 a_{i-1} - w \sum_{k=0}^{i-1} a_k b_{i-k-1}, (i \geq 2).
\end{align*}
\]

A direct computation shows that

\[
\begin{align*}
a_0 &= -p, \\
a_1 &= (1 - \partial^2)^{-1}(u^2 p + 3 wpp_x + w^2 p^2), \\
a_i &= (1 - \partial^2)^{-1}[w \sum_{k=0}^{i-1} a_k b_{i-k-1} + (w \sum_{k=0}^{i-1} a_k a_{i-k-1})x - u^2 a_{i-1}], \\
b_i &= a_{ix} + w \sum_{k=0}^{i-1} a_k a_{i-k-1}, (i \geq 2),
\end{align*}
\]

which lead to an infinite sequence of conserved quantities and the first three read as

\[
\begin{align*}
\Gamma_1 &= -\int pwdx, \\
\Gamma_2 &= \int [u^2 pr + wpp_x r - wp^2 r_x]dx, \\
\Gamma_3 &= \int [a_1(3wpr_x + wrp - u^2 r) - w^2 p^3 r]dx.
\end{align*}
\]

Furthermore, we can also consider the expansions of \( a, b \) in the form

\[
a = \sum_{j \geq 1} a_j \lambda^{-\frac{j}{2}}, \quad b = \lambda^{\frac{1}{2}} \sum_{j \geq 1} b_j \lambda^{-\frac{j}{2}},
\]

which are different from the expansions in \[16\]. Similarly, we have

\[
\begin{align*}
a_1 &= uw^{-1}, \\
a_2 &= \frac{1}{2} u^{-2} v - \frac{3}{2} (uw)^{-1} u_x + w^{-2} w_x, \\
b_{i+1} &= -u^{-1}(b_{ix} - a_{i+1} + w \sum_{k=2}^{i} a_k b_{i+2-k}), \\
a_{i+1} &= \frac{1}{2} u^{-1}(b_{i+1} - a_{ix} - w \sum_{k=2}^{i} a_k a_{i+2-k}).
\end{align*}
\]

Then the first four conserved quantities may be obtained, which are

\[
\gamma_1 = \int udx,
\]
\[ \Upsilon_2 = \frac{1}{2} \int u^{-2} v w dx, \]
\[ \Upsilon_3 = \frac{1}{4} \int u^{-5} (\frac{1}{2} u^2 u_x^2 + u^2 w v_x + 2u^4 - \frac{3}{2} v^2 w^2 - u^2 v w_x)dx, \]
\[ \Upsilon_4 = \frac{1}{2} \int [-u^{-4} (v w + v_x w_x) + 2u^{-6} (u u_x (vw)_x - w^2 v w_x - 2vw u^2) + u^{-8} (3w^2 v^2 w_x + w^3 v^3)]dx. \]

2.2. Hamiltonian structure

In this subsection, we will study the 3-DP hierarchy in view of bi-Hamiltonian structure. To begin with, the 3-DP equation (7) is generated by the two conserved quantities \( \Gamma_1, \Gamma_2 \), and we have the following result.

**Theorem 1** The 3-DP equation (7) is a bi-Hamiltonian system, namely, it may be written as

\[
\begin{pmatrix}
  u \\
  v \\
  w 
\end{pmatrix}_t = J \begin{pmatrix}
  \frac{\delta H_1}{\delta u} \\
  \frac{\delta H_1}{\delta v} \\
  \frac{\delta H_1}{\delta w}
\end{pmatrix} = K \begin{pmatrix}
  \frac{\delta H_2}{\delta u} \\
  \frac{\delta H_2}{\delta v} \\
  \frac{\delta H_2}{\delta w}
\end{pmatrix},
\]

where

\[
J = \begin{pmatrix}
  \frac{1}{2} \partial & 0 & 0 \\
  0 & 0 & 1 - \partial^2 \\
  0 & \partial^2 - 1 & 0
\end{pmatrix},
\]
\[
K = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & \frac{3}{2} v \partial^{-1} v & -u^2 - \frac{3}{2} v \partial^{-1} w \\
  0 & u^2 - \frac{3}{2} w \partial^{-1} v & \frac{3}{2} w \partial^{-1} w
\end{pmatrix} - 2\Omega (\partial^3 - 4\partial)^{-1} \Omega^*,
\]

herein

\[
\Omega = (\partial u, \frac{1}{2} v \partial + \partial v, \frac{1}{2} w \partial + \partial w)^T,
\]

\[H_1 = -\Gamma_1,\]
\[H_2 = -\Gamma_2.\]

Since the Hamiltonian operator has been verified, one can prove the theorem easily. Then a recursion operator for 3-DP hierarchy is gotten as \( R = KJ^{-1} \). In fact the first positive flow in the hierarchy is just \( (u, v, w)_t = R(0, v, -w)^T \), and the other positive flows in the hierarchy may be obtained as

\[
\begin{pmatrix}
  u \\
  v \\
  w 
\end{pmatrix}_{t_n} = J \begin{pmatrix}
  \frac{\delta H_{n+1}}{\delta u} \\
  \frac{\delta H_{n+1}}{\delta v} \\
  \frac{\delta H_{n+1}}{\delta w}
\end{pmatrix} = K \begin{pmatrix}
  \frac{\delta H_n}{\delta u} \\
  \frac{\delta H_n}{\delta v} \\
  \frac{\delta H_n}{\delta w}
\end{pmatrix}, \quad n = 1, 2, \ldots
\]

Similarly, we can also construct infinitely many negative flows

\[
\begin{pmatrix}
  u \\
  v \\
  w 
\end{pmatrix}_{t_{-n}} = K \begin{pmatrix}
  \frac{\delta H_{-(n+1)}}{\delta u} \\
  \frac{\delta H_{-(n+1)}}{\delta v} \\
  \frac{\delta H_{-(n+1)}}{\delta w}
\end{pmatrix} = J \begin{pmatrix}
  \frac{\delta H_{-n}}{\delta u} \\
  \frac{\delta H_{-n}}{\delta v} \\
  \frac{\delta H_{-n}}{\delta w}
\end{pmatrix}, \quad n = 1, 2, \ldots
\]
with the first two Hamiltonian functionals given by $H_{-1} = -2\Upsilon_2$, $H_{-2} = -2\Upsilon_4$. Then the first negative flow in the hierarchy is obtained by using the Hamiltonian functionals $H_{-1}, H_{-2}$, that is

$$
\begin{align*}
    u_t - \left(\frac{vw}{u^2}\right)_x &= 0, \\
v_t - \left(\frac{v}{u^2}\right)_{xx} + \frac{v}{u^2} &= 0, \\
w_t - \frac{w}{u^2} + \left(\frac{w}{u^2}\right)_{xx} &= 0.
\end{align*}
$$

(18)

It is worth to note that $\Upsilon_1$ and $\Upsilon_3$ are the Casimir functionals of the Hamiltonian operators $\mathcal{J}$ and $\mathcal{K}$ respectively.

Since the structure of Hamiltonian functionals $H_n$s in the 3-DP hierarchy is largely unknown, like the cases in \cite{22, 23}, we will consider the homogeneous and local property of them. Introducing $\theta = (u, v, w)^T$ and $X_n[\theta] = \frac{\delta H_n}{\delta \theta}$, then recursive relation in the positive direction

$$
\mathcal{J} \frac{\delta H_{n+1}}{\delta \theta} = \mathcal{K} \frac{\delta H_n}{\delta \theta}, \quad n = 1, 2, ..., 
$$

yields an infinite sequence of variational derivatives for the Hamiltonian functionals $H_n$s

$$
X_{n+1}[\theta] = \mathcal{J}^{-1} \mathcal{K} X_n[\theta], \quad n = 1, 2, ....
$$

(19)

Similarly, the variational derivatives for the Hamiltonian functionals $H_{-n}$s in the negative direction are given by

$$
X_{-(n+1)}[\theta] = \mathcal{K}^{-1} \mathcal{J} X_{-n}[\theta], \quad n = 1, 2, ....
$$

Proposition 1 The variational derivatives $X_n[\theta]$ are homogeneous in the sense that

$$
X_n[\epsilon \theta] = \epsilon^{2n-1} X_n[\theta], \quad n \geq 1,
$$

(20)

and

$$
H_n[\epsilon \theta] = \frac{1}{2n} \int X_n[\theta] \cdot \theta dx, \quad n \geq 1.
$$

(21)

Proof: When $n = 1$, the formulate (20) holds clearly. Now suppose (20) also holds for $n = k$, that is

$$
X_k[\epsilon \theta] = \epsilon^{2k-1} X_k[\theta].
$$

Then for $n = k + 1$, we have

$$
X_{k+1}[\epsilon \theta] = \mathcal{J}^{-1}[\epsilon \theta] \mathcal{K}[\epsilon \theta] X_k[\epsilon \theta] = \epsilon^2 \mathcal{J}^{-1}[\theta] \mathcal{K}[\theta] X_k[\theta],
$$

which implies that

$$
X_{k+1}[\epsilon \theta] = \epsilon^{2k+1} X_{k+1}[\theta].
$$

In addition, for any $n \geq 1$, we have

$$
H_n[\theta] = \int_0^1 \int X_n[\epsilon \theta] \cdot \theta dxd\epsilon = \frac{1}{2n} \int X_n[\theta] \cdot \theta dx,
$$

then the Hamiltonian functionals $H_n$s are also homogeneous with

$$
H_n[\epsilon \theta] = \epsilon^{2n} H_n[\theta], \quad n = 1, 2, ....
$$
The recursive formula for $H_n$ yields infinitely many Hamiltonian functionals in the positive direction, and $H_1$ and $H_2$ are local. However, $H_n, n \geq 3$ becomes nonlocal. For example $H_3 = -\Gamma_3$, which is shown to be nonlocal.

**Proposition 2** The variational derivatives $X_{-n}[\theta]$s satisfy
\[ X_{-n}[\epsilon\theta] = \epsilon^{1-2n}X_{-n}[\theta], \quad n = 1, 2, ... \] (22)
while
\[ H_{-n}[\theta] = \frac{1}{2 - 2n} \int X_{-n}[\theta] \cdot \theta dx, \] (23)
and $H_{-n}$s are all local.

The formulae (22) and (23) may be proven by taking the process before, and we will prove the local property of $H_{-n}$ below.

**Lemma 1** ([22, 23, 24]) If a differential function $M[\theta]$ satisfies
\[ \int M[\theta] dx = 0 \]
for all $\theta$, then there exists a unique differential function $N[\theta]$ up to addition of a constant such that $M[\theta]$ is the total x-derivative $M[\theta] = (N[\theta])_x$. Define
\[ X_{-k}[\theta] = (A_k, B_k, C_k)^T \]
\[ E_k = (\partial^3 - 4\partial)^{-1}(u\partial, \frac{3}{2}v\partial + \frac{1}{2}v_x, \frac{3}{2}w\partial + \frac{1}{2}w_x)X_{-k}[\theta], \quad k \geq 1, \]
When $n = 1$, $X_{-1}[\theta]$ is local since
\[ X_{-1}[\theta] = (2\frac{vu}{u^3}, -\frac{w}{u^2}, -\frac{v}{u^2})^T. \]
Suppose when $n = k$, $X_{-k}[\theta]$ is local, that is $A_k, B_k, C_k$ are all local. Then for $n = k + 1$, we have
\[ X_{-(k+1)}[\theta] = \mathcal{K}^{-1}\mathcal{J}X_{-k}[\theta] = (\mathcal{K}^{-1}\mathcal{J})^kX_{-1}[\theta], \]
which is equal to
\[ \mathcal{K}X_{-(k+1)}[\theta] = \mathcal{J}X_{-k}[\theta]. \] (24)
This shows that
\[ E_{k+1} = \frac{1}{4u}A_k, \] (25)
\[ \frac{3}{2}v\partial^{-1}(vB_{k+1} - wC_{k+1}) - u^2C_{k+1} + (3v\partial + 2v_x)E_{k+1} = (1 - \partial^2)C_k, \] (26)
\[ u^2B_{k+1} - \frac{3}{2}w\partial^{-1}(vB_{k+1} - wC_{k+1}) + (3w\partial + 2w_x)E_{k+1} = (\partial^2 - 1)B_k. \] (27)

Then we will prove the local property of $X_{-(k+1)}$ in two steps. The first step is to prove that $B_{k+1}$ and $C_{k+1}$ are local. Since $A_k, B_k, C_k$ are all local, we can obtain...
immediately from (26) and (27) that \(B_{k+1}\) and \(C_{k+1}\) are local, if there exist a differential function \(M_k\) such that
\[
vB_{k+1} - wC_{k+1} = \frac{w}{u^2}(1 - \partial^2)C_k + \frac{v}{u^2}(\partial^2 - 1)B_k - \frac{3vw}{2u^2} \partial \frac{A_k}{u} - \frac{(vw)_x}{2u^3} A_k
\]
\[
= M_kx.
\]
Then according to the Lemma 1, we only need to prove
\[
Y_1 = \int \left[ \frac{w}{u^2}(1 - \partial^2)C_k + \frac{v}{u^2}(\partial^2 - 1)B_k - \frac{3vw}{2u^2} \partial \frac{A_k}{u} - \frac{(vw)_x}{2u^3} A_k \right] dx = 0.
\]
In fact
\[
Y_1 = \int \left[ \frac{w}{u^2}(1 - \partial^2)C_k + \frac{v}{u^2}(\partial^2 - 1)B_k - \frac{3vw}{2u^2} \partial \frac{A_k}{u} - \frac{(vw)_x}{2u^3} A_k \right] dx
\]
\[
= \int \left[ C_k(1 - \partial^2) + B_k(\partial^2 - 1) + \frac{v}{u^2} + A_k(\frac{vw}{u^2})_x \right] dx
\]
\[
= \int \left( A_k \right) \cdot J \left( \frac{2vw}{u^2} - \frac{v}{u} \right) dx
\]
\[
= \int X_{-k}[\theta] \cdot JX_{-1}[\theta] dx,
\]
and
\[
\int X_{-k}[\theta] \cdot JX_{-1}[\theta] dx = \int (\mathcal{K}^{-1}J)^{k-1}X_{-1}[\theta] \cdot JX_{-1}[\theta] dx
\]
\[
= - \int X_{-1}[\theta] \cdot J\mathcal{K}^{-1}(\mathcal{J}^{k-1})X_{-1}[\theta] dx
\]
\[
= - \int (\mathcal{K}^{-1}J)^{k-1}X_{-1}[\theta] \cdot JX_{-1}[\theta] dx
\]
\[
= - \int X_{-k}[\theta] \cdot JX_{-1}[\theta] dx.
\]
Therefore \(Y_1 = 0\), and hence \(B_{k+1}\) and \(C_{k+1}\) are local.

The next step is to prove that \(A_{k+1}\) is local. From (25), we obtain
\[
A_{k+1x} = \frac{1}{u} \left[ (\partial^3 - 4\partial) \frac{A_k}{4u} - (3v\partial + 2v_x)B_{k+1} - (3w\partial + 2w_x)C_{k+1} \right].
\]
Since \(B_{k+1}\) and \(C_{k+1}\) are all local, so \(A_{k+1}\) is local if the right part of the above equality is a total \(x\)-derivative \(N_{kx}\) for a differential function \(N_k\). Therefore \(A_{k+1}\) is local if
\[
Y_2 = \int \left( \frac{1}{u} \left[ (\partial^3 - 4\partial) \frac{A_k}{4u} - (3v\partial + 2v_x)B_{k+1} - (3w\partial + 2w_x)C_{k+1} \right] \right) dx = 0.
\]

**Lemma 2** Define
\[
\mathcal{D} = \begin{pmatrix}
\frac{3}{2}v\partial^{-1}v & -u^2 - \frac{3}{2}v\partial^{-1}w \\
\frac{3}{2}w\partial^{-1}v & 2u - \frac{3}{2}w\partial^{-1}w
\end{pmatrix},
\]
we have
\[
\mathcal{D}^{-1} = \frac{1}{u^2} \begin{pmatrix}
\frac{3}{2}w\partial^{-1}w & u^2 + \frac{3}{2}w\partial^{-1}v \\
\frac{3}{2}w\partial^{-1}w & u^2 + \frac{3}{2}w\partial^{-1}v
\end{pmatrix} \frac{1}{u^2}.
\]
To make the expressions compact, we introduce some new notations as:

\[
Z_1 = (1 - \partial^2)C_k - (3v\partial + 2v_x)\frac{A_k}{4u}, \quad Z_2 = (\partial^2 - 1)B_k - (3vw\partial + 2w_x)\frac{A_k}{4u},
\]

\[
Z_3 = \frac{v_x}{u^3} - \frac{3u_xv}{2u^4} - \frac{3vw^2}{2u^4}, \quad Z_4 = -\frac{w_x}{u^3} + \frac{3u_xw}{2u^4} - \frac{3vw^2}{2u^5}.
\]

Using the Lemma 2 to solve \(B_{k+1}\) and \(C_{k+1}\) from (26) and (27), we arrive at

\[
Y_2 = \int \left( \frac{1}{u} \left[ (\partial^3 - 4\partial) \frac{A_k}{4u} - (3v\partial + 2v_x)B_{k+1} - (3w\partial + 2w_x)C_{k+1} \right] \right) dx
\]

\[
= \int \left[ \frac{A_k}{4u} (\partial^3 - 4\partial) \frac{1}{u} + B_{k+1} \left( \frac{v_x}{u} - \frac{3u_xv}{u^2} \right) + C_{k+1} \left( \frac{w_x}{u} - \frac{3u_xw}{u^2} \right) \right] dx
\]

\[
= \int \left[ \frac{A_k}{4u} (\partial^3 - 4\partial) \frac{1}{u} + \left( \frac{v_x}{u^3} - \frac{3u_xw}{2u^4} \right) \right] \frac{3}{2} v^{-1} \left( \frac{wZ_1 + vZ_2}{u^2} \right) + Z_2 \right] \right] dx
\]

\[
= \int \left[ \frac{A_k}{4u} (\partial^3 - 4\partial) \frac{1}{u} + Z_2 Z_3 + Z_1 Z_4 \right] dx
\]

\[
= \int \left( \frac{A_k}{4u} \right) \frac{1}{u} + (3w\partial + 2w_x)Z_3 + (3v\partial + 2v_x)Z_4)
\]

\[
= \int \left( \frac{A_k}{4u} \right) \frac{1}{u} + B_k (\partial^2 - 1) Z_1 + C_k (1 - \partial^2) Z_2 \right) dx
\]

\[
= \int \left( \frac{A_k}{4u} \right) \frac{1}{u} + \frac{3u_xw}{2u^4} + \frac{w}{u^3} \left[ \frac{3u_xw}{2u^4} - \frac{3u_xw}{u^4} + \frac{1}{u^2} \right] \right) dx
\]

\[
= \int \left[ -2X_{-k} \theta \right] \frac{1}{u} \frac{\partial Y_3}{\partial \theta} \right) dx
\]

\[
= \int 2(\kappa^{-1} \mathcal{I})^{k-1} X_{-1} \theta \frac{\partial Y_3}{\partial \theta} \right) dx
\]

\[
= \int -2(\kappa^{-1} \mathcal{I})^k X_{-1} \theta \cdot K \frac{\partial Y_3}{\partial \theta} \right) dx
\]

\[
= \int -2(\kappa^{-1} \mathcal{I})^k X_{-1} \theta \cdot (0, 0, 0)^T d\tau \right) dx
\]

\[
= 0.
\]

Therefore \(A_{k+1}\) is local. Consequently, we prove \(X_{-n}\)s are all local, then using the Lemma 4.4 in [23] (see also [22, 24]), we show \(H_{-n}\)s are local.

3. First Liouville transformation for the 3-DP equation

Note that the 3-DP equation (17) has a conserved density \(u\), then the corresponding conservation law may be obtained as

\[
u_t = (\text{upr}),
\]

which defines a reciprocal transformation

\[
dy = udx - upr dt, \quad d\tau = dt.
\]

(28)
In the following, we will connect the three-component CH type system (7) with the negative flow in a modified Yajima-Oikawa hierarchy. Writing the matrix spectral problem (6) in scalar form and under a change of variables, we arrive at

$$\varphi_{1yy}u^2 + uu_y\varphi_{1y} = (1 + \lambda u^2)\varphi_1 + u\varphi_3,$$

$$\varphi_{3y} = \lambda uu^{-1}\varphi_1.$$  \hspace{1cm} (29)

After a gauge transformation $$\varphi_1 = u^{-\frac{1}{2}}\phi_1$$, we get

$$\phi_{1yy} + \left(\frac{1}{4}u_y^2 - \frac{u_y}{2u} - \frac{1}{u^2}\right)\phi_1 = \lambda \phi_1 + v\varphi^{-\frac{3}{2}}_3,$$

$$\varphi_{3y} = \lambda uu^{-\frac{3}{2}}\phi_1.$$  \hspace{1cm} (30)

Similarly, the auxiliary problem in (8) is changed to

$$\phi_{1r} = \left(\frac{1}{3\lambda} + \frac{1}{2}up yr - \frac{1}{2}up yr\right)\phi_1 + \frac{1}{\lambda}pu^\frac{1}{2}\varphi_3,$$

$$\varphi_{3r} = ru^\frac{1}{2}\phi_1 - (ru^\frac{1}{2})\phi_1 + (up yr - up yr - \frac{2}{3\lambda})\varphi_3.$$  \hspace{1cm} (31)

To obtain the associated Lax pair for (8), we rewrite (31-32) and (33-34) in matrix form by setting

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_{1y} \\ \varphi_3 \end{pmatrix},$$

$$\varphi_{x} = M\varphi, \hspace{0.5cm} \varphi_{r} = N\varphi,$$  \hspace{1cm} (35)

where

$$M = \begin{pmatrix} 0 & 1 & 0 \\ \lambda + Q_3 & 0 & Q_1 \\ \lambda Q_2 & 0 & 0 \end{pmatrix},$$

$$N = \begin{pmatrix} \frac{1}{3\lambda} + \frac{1}{2}(q_1 q_2y - q_2 q_1y) & 0 & \frac{q_1}{\lambda} \\ \frac{1}{2}(Q_2 q_1 + Q_1 q_2) & \frac{1}{3\lambda} + \frac{1}{2}(q_1 q_2y - q_2 q_1y) & q_2 \\ -q_2y & \frac{q_1}{\lambda} & q_2 q_1y - q_1 q_2y - \frac{2}{3\lambda} \end{pmatrix},$$

herein

$$Q_1 = vu^{-\frac{3}{2}}, \hspace{0.5cm} Q_2 = vu^{-\frac{3}{2}}, \hspace{0.5cm} Q_3 = \frac{u_{yy}}{2u} - \frac{1}{4}u_y^2 + \frac{1}{u^2},$$

$$q_1 = pu^\frac{1}{2}, \hspace{0.5cm} q_2 = ru^\frac{1}{2}.$$  \hspace{1cm} (36)

The compatibility of the Lax pair (35) yields nothing but just the transformed system for (7), that is

$$Q_{1r} = \frac{3}{2}Q_1(q_1 q_2y - q_2 q_1y) - q_1, \hspace{0.5cm} S_1 = 0,$$

$$Q_{2r} = \frac{3}{2}Q_2(q_2 q_1y - q_1 q_2y) + q_2, \hspace{0.5cm} S_2 = 0,$$

$$Q_{3r} = \frac{1}{2}(Q_2 q_1 + Q_1 q_2)y + Q_1 q_2y + Q_2 q_1y,$$  \hspace{1cm} (36)

where

$$S_1 = q_{1yy} - Q_3 q_1 + Q_1, \hspace{0.5cm} S_2 = q_{2yy} - Q_3 q_2 + Q_2.$$
Consequently, under the Liouville transformation

\[
y = \int_{-\infty}^{x} u(\xi) d\xi,
\]

\[
\begin{pmatrix}
Q_1(y) \\
Q_2(y) \\
Q_3(y)
\end{pmatrix} =
\begin{pmatrix}
u u^{-3/2} \\
w u^{-3/2} \\
\frac{u}{2\alpha} - \frac{3w^2}{4\alpha} + \frac{1}{w^2}
\end{pmatrix},
\]  
(37)

the three-component CH type system (7) and its the Lax pair (8) is transformed to the associated system (36) and its Lax pair (35) respectively. Moreover the spectral problem in (35) is nothing but that of a modified Yajima-Oikawa hierarchy which has been studied in [25].

Next, we will find the relation between the associated 3-DP equation (36) and a negative flow in a modified Yajima-Oikawa hierarchy associated with the spectral problem in the Lax pair (35). As pointed out in [25], the modified Yajima-Oikawa hierarchy has a generalized bi-Hamiltonian structure with the Hamiltonian pair given by

\[
\mathcal{J}_1 =
\begin{pmatrix}
-\frac{3}{2} Q_1 \partial_y^{-1} Q_1 & \frac{3}{2} Q_2 \partial_y^{-1} Q_2 + 1 & 0 \\
\frac{3}{2} Q_2 \partial_y^{-1} Q_1 - 1 & -\frac{3}{2} Q_2 \partial_y^{-1} Q_2 & 0 \\
0 & 0 & -\frac{1}{2} \partial_y^3 + \partial_y Q_3 + Q_3 \partial_y
\end{pmatrix}
\]

\[
\mathcal{K}_1 =
\begin{pmatrix}
0 & \partial^2 - Q_3 & 0 \\
Q_3 - \partial^2 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{1}{2} \Theta \partial^{-1} \Theta^*,
\]

where \(\Theta = \left(\frac{1}{2} Q_1 \partial_y + \partial_y Q_1, \frac{1}{2} Q_2 \partial_y + \partial_y Q_2, -\frac{1}{2} \partial_y^3 + Q_3 \partial_y + \partial_y Q_3\right)^T\). Therefore the recursion operator of the modified Yajima-Oikawa hierarchy is obtained as \(\mathcal{R} = \mathcal{K}_1 \mathcal{J}_1^{-1}\). A negative flow is obtained as

\[
\begin{pmatrix}
Q_1 \\
Q_2 \\
Q_3
\end{pmatrix}_\tau = \mathcal{R}^{-1} \begin{pmatrix}
Q_1 \\
-Q_2 \\
0
\end{pmatrix},
\]  
(38)

it may be reduced to

\[
\begin{pmatrix}
Q_1 \\
Q_2 \\
Q_3
\end{pmatrix}_\tau = \mathcal{J}_1 \begin{pmatrix}
-q_2 \\
-q_1 \\
q_1 q_2
\end{pmatrix},
\mathcal{K}_1 \begin{pmatrix}
-q_2 \\
-q_1 \\
q_1 q_2
\end{pmatrix} = \begin{pmatrix}
Q_1 \\
-Q_2 \\
0
\end{pmatrix},
\]

which is equivalent to

\[
Q_{1\tau} = \frac{3}{2} Q_1 \partial_y^{-1} (Q_1 q_2 - Q_2 q_1) - q_1, \quad Z_1 = 0,
\]

\[
Q_{2\tau} = \frac{3}{2} Q_2 \partial_y^{-1} (Q_2 q_1 - Q_1 q_2) + q_2, \quad Z_2 = 0,
\]

\[
Q_{3\tau} = 2Q_3 (q_1 q_2)_y + Q_3 q_1 q_2 - \frac{1}{2} (q_1 q_2)_{yyy}, \quad Z_3 = 0,
\]  
(39)

where

\[
Z_1 = \frac{1}{4} (\frac{3}{2} Q_1 \partial_y + Q_1 y)[S_1 q_2 + S_2 q_1 + 2 \partial_y^{-1} (S_2 q_1 y + S_1 q_2)_y] - S_1,
\]
where

\[ Z_2 = \frac{1}{4} \left( \frac{3}{2} Q_2 \partial_y + Q_2 y \right) \left[ S_1 q_2 + S_2 q_1 + 2 \partial_y^{-1} \left( S_2 q_{1y} + S_1 q_{2y} \right) \right] + S_2, \]

\[ Z_3 = \frac{1}{4} \left( -\frac{1}{2} \partial_y^3 + Q_2 \partial_y + \partial_y Q_3 \right) \left[ S_1 q_2 + S_2 q_1 + 2 \partial_y^{-1} \left( S_2 q_{1y} + S_1 q_{2y} \right) \right]. \]

Notice that \( S_1 = S_2 = 0 \) implies \( Z_1 = Z_2 = Z_3 = 0 \), therefore we infer that the associated CH type system \((36)\) is just a reduction of the first negative flow in a modified Yajima-Oikawa hierarchy \((39)\).

### 4. Second Liouville for the 3-DP equation

#### 4.1. A Liouville transformation

As pointed out in \([21]\), the 3-CH type system \((40)\) is Liouville linked to the first negative flow in a generalized MKdV hierarchy, and the spectral problem \((3)\) is gauge linked to the spectral problem \((6)\). It would seem to be a reasonable guess that the two system are equal. In fact the 3-component CH type system in \([18]\) should be corrected as

\[
\begin{align*}
    m_{1t} + u_2 g m_{1x} - m_3 (u_2 f - u_2 g) - m_1 (3u_2 f - m_3 u_2) &= 0, \\
    m_{2t} + u_2 g m_{2x} + m_3 (3u_2 f + m_3 u_2) &= 0, \\
    m_{3t} + u_2 g m_{3x} - m_3 (2u_2 f + u_2 g - m_3 u_2) &= 0, \\
    m_2 &= u_2 - u_2 x, \quad m_1 - m_3 x = g - g_2, \quad f = m_3 - g_2,
\end{align*}
\]

then a directly calculation shows that \((40)\) is connected to \((7)\) via

\[ u = (m_2 m_3)^{\frac{1}{3}}, \quad v = m_2, \quad w = m_1 - m_3 x, \quad p = u_2, \quad r = g. \]

To verify the Painlevé property of the 3-DP equation \((7)\), following the steps in \([21]\), we reconsider spectral problem \((29,30)\) as

\[
\begin{align*}
    \varphi_{1yy} + \frac{u_y}{u} \varphi_{1y} - \frac{1}{u^2} \varphi_1 - \mu \frac{v}{u^2} \psi_2 &= 0, \\
    \psi_{2y} - \mu \frac{u^2}{v} \varphi_{1y} - \mu \left[ \frac{u^2}{v} \varphi_{1} + \frac{u}{w} \right] \varphi_1 &= 0
\end{align*}
\]

by defining \( \mu = \lambda^2 \) and \( \psi_2 = \mu \frac{u^2}{v} \varphi_1 + \frac{1}{\mu} \varphi_3 \). Setting \( a = \frac{u}{w} e^{-\partial_y^{-1}(vw_{x-3})} \) and making a gauge transformation \( \varphi_1 = a \phi_1, \psi_2 = \frac{u}{w} a \phi_2 \), the spectral problem \((42,43)\) is transformed to

\[
\begin{align*}
    \phi_{1yy} - Q_2 \phi_{1y} - Q_1 \phi_1 &= \mu \phi_2, \\
    \phi_{2y} - Q_3 \phi_2 &= \mu \phi_{1y},
\end{align*}
\]

where

\[
\begin{align*}
    Q_1 &= (3 \frac{v_y}{v} u + \frac{u_y}{u} \frac{w}{w}) u^3 + 3 \frac{u_y v_y}{w^2} u^3 - \frac{v^2 w^2}{w^3} u^3 + \frac{1 + 2 u u_{yy}}{u^3} - \frac{4 u_y^2}{u^3} - \frac{v_{yy}}{v}, \\
    Q_2 &= 2 \frac{v w}{u^3} - 2 \frac{v_y}{v} + 3 \frac{u_y}{u}, \\
    Q_3 &= \frac{v w}{u^3}.
\end{align*}
\]
Analogously, the auxiliary problem in (8), under the transformation (28), may be changed to

\[ \phi_1 \tau = \frac{1}{\mu q_1 \phi_2 + \frac{1}{3\mu^2} \phi_1}, \quad (46) \]

\[ \phi_2 \tau = \frac{1}{\mu} (q_2 \phi_{1y} + [1 - q_2 y + (Q_3 - Q_2)q_2] \phi_1) + (q_1 - \frac{2}{3\mu^2}) \phi_2, \quad (47) \]

where

\[ q_1 = \frac{u^2}{v}, \quad q_2 = \frac{rv}{u}. \]

Consequently, the Lax pair (8) for the 3-DP equation (7) is changed to

\[ \Phi_x = \begin{pmatrix} 0 & 1 & 0 \\ Q_1 & Q_2 & \mu \\ 0 & \mu & Q_3 \end{pmatrix} \Phi, \quad (48) \]

\[ \Phi_r = \begin{pmatrix} \frac{1}{3\mu^2} & 0 & \frac{u}{\mu} \\ 0 & \frac{1}{3\mu^2} + q_1 & \frac{Q_3 q_1 + q_2}{\mu} \\ \frac{1 - q_2 y + (Q_3 - Q_2)q_2}{\mu} & q_2 & q_1 - \frac{2}{3\mu^2} \end{pmatrix} \Phi. \quad (49) \]

Then the compatibility condition of the associated Lax pair (48-49) yields the associated 3-DP equation

\[ Q_{1 \tau} = q_2 y + (Q_2 - Q_3)q_2 + Q_1 q_1 - 1, \]

\[ Q_{2 \tau} = Q_3 q_1 - q_2 + 2 q_{1y}, \quad S_1 = 0, \]

\[ Q_{3 \tau} = q_2 - Q_3 q_1, \quad S_2 = 0, \quad (50) \]

where

\[ S_1 = [(\partial_y + Q_2 - Q_3)(\partial_y + Q_3) - Q_1]q_1 + 1, \]

\[ S_2 = [(\partial_y - Q_3)(\partial_y + Q_2 - Q_3) - Q_1]q_2 + Q_3. \]

In fact the associated 3-DP equation (50) may be also obtained by applying the reciprocal transformation (28) to the 3-DP equation (7). More precisely, the 3-DP equation is transformed to the associated 3-DP equation through the Liouville transformation

\[ \left\{ \begin{array}{l}
    y = I(x, \theta^{(n)}) = \int_{-\infty}^{x} u(\xi)d\xi, \\
    Q_1(y) = P_1(x, \theta^{(n)}), \\
    Q_2(y) = P_2(x, \theta^{(n)}), \\
    Q_3(y) = P_3(x, \theta^{(n)})
  \end{array} \right., \quad (51) \]

where

\[ P_1 = u^{-2}(1 + \left[ \frac{v w}{u^2} + 2 \frac{u_x}{u} - \frac{v_x}{v} \right] x - \left[ \frac{v w}{u^2} + 2 \frac{u_x}{u} - \frac{v_x}{v} \right]^2), \]

\[ P_2 = u^{-3} (2v w - 2u^2 v_x v^{-1} + 3 u u_x), \]

\[ P_3 = v w u^{-3}, \]

with \( v = p - p_{xx}, w = r - r_{xx} \).
In words, the 3-DP equation (1) and its Lax pair (3) are Liouville linked to the associated 3-DP equation (50) and its Lax pair (48-49) respectively. Furthermore, the CH type systems are not pass the Painlevé test [9, 27], however, it is not hard to verify that the associated 3-DP equation (50) passes the Painlevé test. Powers of the leader terms for \( q_1, q_2, Q_1, Q_2, Q_3 \) is \(-1, -2, -2, -1, -1\) respectively, and the resonances are \( j = -2, -1, 1, 2, 3, 4, 5 \).

The spectral problem [48] can be considered as a spectral problem for a generalized MKdV hierarchy, because it may be reduced to that of the MKdV hierarchy as \( Q_1 = Q_2 = 0 \). Further, a scalar spectral problem for (48) can be gotten by eliminating \( \phi_2 \) from the spectral problem (44-45), that is

\[
L \phi_1 = \lambda \phi_1, \quad L = \partial_y^{-1}(\partial_y - Q_3)(\partial_y^2 - Q_2 \partial_y - Q_1),
\]

which is connected to a modified Yajima-Oikawa hierarchy admitting the spectral problem [26]

\[(\partial_y^2 + u_1 \partial_y + v_1 + \partial_y^{-1} w_1)\phi_1 = \lambda \phi_1 \tag{52}\]

with the fields related by a Miura transformation

\[
\begin{pmatrix}
  u_1 \\
  v_1 \\
  w_1
\end{pmatrix} = F(Q_1, Q_2, Q_3) = \begin{pmatrix}
  -Q_2 - Q_3 \\
  Q_2 Q_3 - Q_1 + Q_{3y} \\
  Q_1 Q_3 - (Q_2 Q_3)_y - Q_{3yy}
\end{pmatrix}. \tag{53}
\]

Since the Hamiltonian pair for the above modified Yajima-Oikawa hierarchy is known, a Hamiltonian pair for the generalized MKdV hierarchy is gotten as

\[
\mathcal{J}_2 = F'^{-1} \mathcal{J}_1 (F'^{-1})^\dagger, \quad \mathcal{K}_2 = F'^{-1} \mathcal{K}_1 (F'^{-1})^\dagger, \tag{54}
\]

where

\[
\mathcal{J}_1 = \begin{pmatrix}
  0 & 0 & 2 \partial_y \\
  0 & 2 \partial_y & \partial_y^2 + u_1 \partial_y \\
  2 \partial_y & -\partial_y^2 + \partial_y u_1 & 0
\end{pmatrix},
\]

\[
\mathcal{K}_1 = \begin{pmatrix}
  6 \partial_y & \ast & \ast \\
  4 u_1 \partial_y & 2 \partial_y^3 + 2 u_1 \partial_y u_1 + \partial_y v_1 + v_1 \partial_y & \chi_1 \\
  2 \partial_y^3 - 2 \partial_y u_1 \partial_y + 2 v_1 \partial_y & \chi_2 & \ast
\end{pmatrix},
\]

herein

\[
\chi_1 = 2 w_1 \partial_y + \partial_y w_1 - (\partial_y^3 - \partial_y u_1 \partial_y + v_1 \partial_y)(\partial_y - u_1),
\]

\[
\chi_2 = \partial_y u_1 w_1 + u_1 w_1 \partial_y + w_1 \partial_y^2 - \partial_y^2 w_1
\]

and the omitted terms are determined by skew-symmetry. Therefore a recursion operator for the generalized MKdV hierarchy is obtained as \( \mathcal{R} = F'^{-1} \mathcal{K}_1 \mathcal{J}_1^{-1} F' \), and the first negative flow in the generalized MKdV hierarchy may be written as

\[
\begin{pmatrix}
  Q_1 \\
  Q_2 \\
  Q_3
\end{pmatrix}_\tau = F'^{-1} \mathcal{J}_1 \begin{pmatrix}
  A \\
  B \\
  C
\end{pmatrix}, \quad F'^{-1} \mathcal{K}_1 \begin{pmatrix}
  A \\
  B \\
  C
\end{pmatrix} = 0 \tag{55}
\]
for functions $A(y, \tau), B(y, \tau), C(y, \tau)$. To find the relation between the associated 3-DP equation (50) and the negative flow (55) in the generalized MKdV hierarchy, we can take

$$
\begin{aligned}
A &= \frac{1}{4}(\partial_y - u_1)\partial^{-1}_y S_1 - \frac{1}{2}\partial_y^{-1}S_2 - Q_3q_{1y} - Q_3^2q_1, \\
B &= \frac{1}{2}\partial_y^{-1}S_1 - Q_3q_1, \\
C &= -q_1,
\end{aligned}
$$

(56)

here and in the sequel the integration constants are assume to zero. Then the negative flow (55) in the generalized MKdV hierarchy is reduced to

$$
\begin{pmatrix}
Q_{1r} \\
Q_{2r} \\
Q_{3r}
\end{pmatrix} =
\begin{pmatrix}
q_{2y} + (Q_2 - Q_3)q_2 + Q_1q_1 - 1 \\
Q_3q_1 - q_2 + 2q_{1y} \\
q_2 - Q_3^2q_1
\end{pmatrix},
\begin{pmatrix}
G_1 \\
G_2 \\
G_3
\end{pmatrix} = 0,
$$

(57)

where

$$
\begin{aligned}
G_1 &= (Q_1 + \frac{1}{2}Q_{1y}\partial^{-1}_y)S_1 + S_2y - Q_2S_2, \\
G_2 &= \frac{1}{2}(\partial_y + \partial_y Q_2\partial^{-1}_y)S_1 + 2S_2, \\
G_3 &= \frac{1}{2}\partial_y Q_3\partial^{-1}_y S_1 + S_2.
\end{aligned}
$$

Consequently, the associated 3-DP equation (50) is a reduction of the first negative flow (55) in the generalized MKdV hierarchy, since $S_1 = 0, S_2 = 0$ yields $G_1 = G_2 = G_3 = 0$.

### 4.2. Hamiltonian structure behavior under the Liouville transformation

According to [28], if two soliton equations are linked by a Liouville transformation, Hamiltonian structures and conserved quantities of them can be related. In this part we will consider the Hamiltonian structures of the 3-DP equation (7) under the Liouville transformation (51). To this end, let $\vartheta = (Q_1, Q_2, Q_3)^T$, then from the point of view of Hamiltonian structures, we have

$$
\begin{aligned}
\theta_t &= \mathcal{B}(\vartheta)\frac{\delta H}{\delta \vartheta}\mathcal{B}(\vartheta)E_{\vartheta}h, \\
\vartheta_t &= \check{\mathcal{B}}(\vartheta)\frac{\delta \tilde{H}}{\delta \vartheta}\check{\mathcal{B}}(\vartheta)E_{\vartheta}\check{h},
\end{aligned}
$$

(58)

(59)

where

$$
H = \int h(x, \vartheta^{(n)})dx, \quad \tilde{H} = \int \check{h}(y, \vartheta^{(n)})dy,
$$

$E_{\vartheta}, E_{\vartheta}$ are the corresponding Euler operators, and $H[\vartheta^{(n)}] = \check{H}[\vartheta^{(n)}]$. Defining $\Lambda(\vartheta, \theta) = \vartheta - (P_1, P_2, P_3)^T$, hence it is easy to see that

$$
\vartheta_t = -T_1\theta_t, \quad T_1 = \Lambda_{\vartheta},
$$

(60)

where $\Lambda_{\theta}$ is Fréchet derivative for the vector variable. Then a direct computation shows that

$$
T_1 =
\begin{pmatrix}
Q_{1y}I'[u] - P_1'[u] & Q_{1y}I'[v] - P_1'[v] & Q_{1y}I'[w] - P_1'[w] \\
Q_{2y}I'[u] - P_2'[u] & Q_{2y}I'[v] - P_2'[v] & Q_{2y}I'[w] - P_2'[w] \\
Q_{3y}I'[u] - P_3'[u] & Q_{3y}I'[v] - P_3'[v] & Q_{3y}I'[w] - P_3'[w]
\end{pmatrix}.
$$
Furthermore, the action of Euler operator under a change of variables is given by

\[ E_0 h = T_2 E_0 \tilde{h}, \quad (61) \]

where

\[
T_2 = \begin{pmatrix}
    P^+_{1,v}(I_x) - I^+_v(P_{1x}) & P^+_{2,v}(I_x) - I^+_v(P_{2x}) & P^+_{3,v}(I_x) - I^+_v(P_{3x}) \\
    P^+_{1,w}(I_x) - I^+_w(P_{1x}) & P^+_{2,w}(I_x) - I^+_w(P_{2x}) & P^+_{3,w}(I_x) - I^+_w(P_{3x}) \\
    P^+_{2,w}(I_x) - I^+_w(P_{2x}) & P^+_{3,w}(I_x) - I^+_w(P_{3x}) & \end{pmatrix}.
\]

**Lemma 3** Under the transformation \((51)\), we have the following formulae:

\[ T_1 = O \text{diag}(u^{-1}, v^{-1}, vu^{-3}), \quad T_2 = -\text{diag}(1, uv^{-1}, vu^{-2})O^\dagger, \]

where

\[ O = \begin{pmatrix}
    Q_1y \partial_y^{-1} + 2Q_1 + 2(Q_2 - \partial_y)(\partial_y - Q_3) & (\partial_y - Q_2)(\partial_y - Q_3) & Q_2 - \partial_y \\
    \partial_y Q_2 \partial_y^{-1} + 4Q_3 - 3\partial_y & 2(\partial_y - Q_3) & -2 \\
    Q_3y \partial_y^{-1} + 3Q_3 & -Q_3 & -1 \end{pmatrix}. \]

**Lemma 4** Let \( \vartheta \) and \( \theta \) are related by the transformations \((51)\), then the following identities hold:

\[ \frac{1}{v}(1 - \partial_x^2)\frac{v}{u^2} = \Theta_1 \equiv Q_1 - (\partial_y - Q_2)(\partial_y + Q_3), \quad (62) \]

and

\[ \frac{1}{u^2}(\partial_x^2 - 4\partial_x)\frac{1}{u} = \Theta_2 \equiv (\partial_y - Q_2)\partial_y(\partial_y + Q_2) - 2Q_1\partial_y - 2\partial_yQ_1. \quad (63) \]

The two Lemmas above can be proved through a straightforward computation. Hence the main results can be summarized as:

**Theorem 2** The associated 3-DP equation is a bi-Hamiltonian system, namely, it can be written as

\[
\left( \begin{array}{c}
Q_1 \\
Q_2 \\
Q_3
\end{array} \right)_t = \tilde{J} \left( \begin{array}{c}
\frac{\delta \tilde{H}_1}{\delta Q_1} \\
\frac{\delta \tilde{H}_1}{\delta Q_2} \\
\frac{\delta \tilde{H}_1}{\delta Q_3}
\end{array} \right) = \tilde{K} \left( \begin{array}{c}
\frac{\delta \tilde{H}_2}{\delta Q_1} \\
\frac{\delta \tilde{H}_2}{\delta Q_2} \\
\frac{\delta \tilde{H}_2}{\delta Q_3}
\end{array} \right),
\]

where

\[
\tilde{J} = O \left( \begin{array}{ccc}
\frac{1}{2}\partial_y & 0 & 0 \\
0 & 0 & \Theta_1 \\
0 & -\Theta_1^\dagger & 0
\end{array} \right) O^\dagger, \quad \quad (64)
\]

\[
\tilde{K} = \left( \begin{array}{ccc}
-Q_1\partial_y - \partial_yQ_1 & (\partial_y - Q_2)\partial_y & (\partial_y - Q_2)\partial_y \\
-\partial_y(\partial_y + Q_2) & 2\partial_y & 2\partial_y \\
-\partial_y(\partial_y + Q_2) & 2\partial_y & 0
\end{array} \right), \quad \quad (65)
\]

and

\[
\tilde{H}_1 = \int Q_3q_1 dy, \quad \quad \tilde{H}_2 = \int [Q_3q_1(q_1q_2 - q_2q_1) + q_1q_2(Q_2 - 2Q_3)] - q_1q_2 dy.
\]
**Proof:** Substituting (60,61) into (58-59), a Hamiltonian pair for the associated 3-DP equation is obtained as
\[
\tilde{J} = -T_1 JT_2, \quad \tilde{K} = -T_1 KT_2, \quad (66)
\]
with Hamiltonian functionals of the 3-DP equation and the associated 3-DP equation connected by the formula (61).

To obtain bi-Hamiltonian structure of the associated 3-DP equation, we should calculate \(\tilde{J}\) and \(\tilde{K}\) in the new variable \(y\). Using conjugation of operator to the identity (62), we can easily check that
\[
\frac{v}{w^3} (\partial_x^2 - 1) \frac{u}{v} = (\partial_y - Q_3) (\partial_y + Q_2 - Q_3) - Q_1. \quad (67)
\]
Hence (64) is gotten immediately by substituting the equalities (62) and (67) into (66).

Let us introduce
\[
\mathcal{P} = \begin{pmatrix}
\frac{3}{2} \partial_y - \frac{1}{2} Q_2 + Q_3 \\
\frac{3}{2} Q_3 \partial_y - \frac{1}{2} Q_2 Q_3 + Q_3 y - \frac{Q_3^2}{3}
\end{pmatrix},
\]
then we have
\[
\tilde{K} = -T_1 KT_2 \\
= \mathcal{O} \text{diag}(u^{-1}, v^{-1}, vu^{-3}) \mathcal{K} \text{diag}(1, uv^{-1}, vu^{-2}) \mathcal{O}^\dagger \\
= \mathcal{O} \begin{pmatrix}
0 & 0 & 0 \\
0 & -\frac{3}{2} \partial_y - 1 - \frac{3}{2} \partial_y^{-1} Q_3 \\
0 & 1 - \frac{3}{2} Q_3 \partial_y^{-1} & \frac{3}{2} Q_3 \partial_y^{-1} Q_3
\end{pmatrix} \mathcal{O}^\dagger - 2 \mathcal{O} \mathcal{P} u (\partial^3 - 4 \partial)^{-1} u^2 \mathcal{P}^\dagger \mathcal{O}^\dagger \\
= \mathcal{O} \begin{pmatrix}
0 & 0 & 0 \\
0 & -\frac{3}{2} \partial_y - 1 - \frac{3}{2} \partial_y^{-1} Q_3 \\
0 & 1 - \frac{3}{2} Q_3 \partial_y^{-1} & \frac{3}{2} Q_3 \partial_y^{-1} Q_3
\end{pmatrix} \mathcal{O}^\dagger - \frac{1}{2} \Theta_2 (\partial_y - Q_2) \partial_y + \frac{1}{2} \Theta_2^{-1} (\partial_y - Q_2) \partial_y + (\partial_y - Q_2) \partial_y \\
= \begin{pmatrix}
-\frac{1}{2} (\partial_y - Q_2) \partial_y + (\partial_y + Q_2) \\
-\partial_y (\partial_y + Q_2) \\
-\partial_y (\partial_y + Q_2)
\end{pmatrix} + \frac{1}{2} \Theta_2 (\partial_y - Q_2) \partial_y + (\partial_y - Q_2) \partial_y \\
= \begin{pmatrix}
-Q_1 \partial_y - \partial_y Q_1 + (\partial_y - Q_2) \partial_y + (\partial_y - Q_2) \partial_y \\
-\partial_y (\partial_y + Q_2) + 2 \partial_y \\
-\partial_y (\partial_y + Q_2) + 2 \partial_y
\end{pmatrix} + \frac{1}{2} \Theta_2 (\partial_y - Q_2) \partial_y + (\partial_y - Q_2) \partial_y \\
by \text{using the identity } (64) \text{ and } \mathcal{O} \mathcal{P} = -\left(\frac{1}{2} \Theta_2, 0, 0\right)^T. \text{ So } (65) \text{ is obtained.}
\]

5. A limit system

The limits of the CH type equations might also contain some important models. For example, the Hunter-Saxton equation, which can describe wave motion in a nematic liquid crystal [29], may be consider as a limit of the CH equation [30]. The Ostrovsky equation, which appears as the description of high-frequency waves in a relaxing medium [31], can be obtained as a short wave limit of the DP equation [11]. In this section, we will consider a limit of the associated 3-DP equation (17).

Under the transformation
\[
x \to \varepsilon x, \quad t \to \varepsilon t, \quad u \to \varepsilon^3 u, \quad (68)
\]
a limit for the 3-DP equation may be obtained in the limit \( \epsilon \to 0 \) as:

\[
\begin{align*}
  u_t + (u p_r)_x &= 0, \\
  v_t + 3v p_x r + v_r p_x r + u^2 p &= 0, \\
  w_t + 3w p r_x + w_r p_r - u^2 r &= 0, \\
  v &= -p_{xx}, \quad w = -r_{xx}.
\end{align*}
\] (69)

The limit system (69) is also integrable in the sense of admitting bi-Hamiltonian structure and a Lax pair. The bi-Hamiltonian structure can be obtained by applying the transformation (68) to that of the 3-DP equation, that is

\[
\begin{pmatrix}
  u \\
  v \\
  w
\end{pmatrix}_t = \mathcal{J}_1 \begin{pmatrix}
  \delta \bar{H}_2 \\
  \delta \bar{H}_1 \\
  \delta \bar{H}_2
\end{pmatrix} = \mathcal{K}_1 \begin{pmatrix}
  \delta \bar{H}_1 \\
  \delta \bar{H}_1 \\
  \delta \bar{H}_1
\end{pmatrix},
\] (70)

where

\[
\mathcal{J}_1 = \begin{pmatrix}
  \frac{1}{2} \partial & 0 & 0 \\
  0 & 0 & -\partial^2 \\
  0 & \partial^2 & 0
\end{pmatrix},
\]

\[
\mathcal{K}_1 = \begin{pmatrix}
  0 & 0 & \frac{3}{2} v \partial^{-1} v \\
  0 & -u^2 - \frac{3}{2} v \partial^{-1} w & \frac{3}{2} w \partial^{-1} w \\
  0 & w^2 - \frac{3}{2} w \partial^{-1} v & \frac{3}{2} w \partial^{-1} w
\end{pmatrix} - 2 \Omega \partial^{-3} \Omega^*,
\]

with the functionals given by

\[
\bar{H}_1 = \int p_x r_x dx,
\]

\[
\bar{H}_2 = \int (p p_x r_{xx} + p p_x r_x^2 - u^2 p r) dx.
\]

Taking the same limit to the Lax pair (8) with \( \lambda \to \epsilon \lambda \), a Lax pair for the limit system (69) is obtained as:

\[
\varphi_x = \begin{pmatrix}
  0 & 1 & 0 \\
  \lambda u^2 & 0 & v \\
  \lambda w & 0 & 0
\end{pmatrix} \varphi,
\]

\[
\varphi_t = \begin{pmatrix}
  \frac{1}{3 \lambda} + p r_x & -pr & \frac{p_r}{\lambda} - v r \\
  p x r_x - \lambda u^2 p r & \frac{1}{3 \lambda} - p x r & \frac{p_r}{\lambda} - v r \\
  -\lambda w p r - r x & p x r - pr_x - \frac{2}{3 \lambda} & \lambda
\end{pmatrix} \varphi.
\]

Moreover, the limit of the 3-DP equation is also reciprocal connected to the first negative flow in the generalized MKdV hierarchy by taking the similar process before.

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References

[1] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661-1664.
[2] A. N. W. Hone, H. Lundmark and J. Szmigielski, Explicit multipeakon solutions of Novikov’s cubically nonlinear integrable Camassa-Holm type equation, Dyn. PDE 6 (2009) 253-289.

[3] R. Camassa, D. D. Holm and J. M. Hyman, A new integrable shallow water equation, Adv. Appl. Mech. 31 (1994) 1-33.

[4] B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation, Phys. D 95, (1996) 229-243.

[5] J. Lenells, The correspondence between KdV and Camassa-Holm, IMRN 71 (2004) 3797-3811.

[6] H. P. McKean, The Liouville correspondence between the Korteweg-de Vries and the Camassa-Holm hierarchies, Commun. Pure Appl. Math. 56 (2003) 998-1015.

[7] J. Song, C. Qu and Z. Qiao, A new integrable two-component system with cubic nonlinearity, J. Math. Phys. 52 (2011) 013503.

[8] A. Degasperis and M. Procesi, Asymptotic integrability, in Symmetry and Perturbation Theory edited by Degasperis A and Gaeta G World Scientific (1999) pp23-37.

[9] A. Degasperis, D. D. Holm and A. N. W. Hone, A new integrable equation with peakon solutions, Theor. Math. Phys. 133 (2002) 1463-1474.

[10] V. S. Novikov, Generalisations of the Camassa-Holm equation, J. Phys. A 42 (2009) 342002.

[11] A. N. W. Hone and J. P. Wang, Prolongation algebras and Hamiltonian operators for peakon equations, Inverse Problem 19 (2003) 129-145.

[12] X. G. Geng and B. Xue, An extension of integrable peakon equations with cubic nonlinearity, Nonlinearity 22 (2009) 1847-1856.

[13] A. N. W. Hone and J. P. Wang, Integrable peakon equations with cubic nonlinearity, J. Phys. A 41 (2008) 372002.

[14] P. J. Olver and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-waves solutions having compact support, Phys. Rev. E 53 (1996) 1900-1906.

[15] M. Chen, S. Q. Liu and Y. J. Zhang, A two-component generalization of the Camassa-Holm equation and its solutions, Lett. Math. Phys. 75 (2006) 1-15.

[16] X. G. Geng and B. Xue, A three-component generalization of Camassa-Holm equation with N-peakon solutions, Adv. Math. 226 (2011) 827-839.

[17] N. H. Li and Q. P. Liu, Bi-Hamiltonian structure of a three-component Camassa-Holm type equation, J. Nonlinear Math. Phys. 20 (2013) 126-134.

[18] N. H. Li, Q. P. Liu and Z. Popowicz, A four-component Camassa-Holm type hierarchy, J. Geom. Phys. 85 (2014) 29-39.

[19] N. Yajima and M. Oikawa, Formation and interaction of sonic-Langmuir solitons, Prog. Theor. Phys. 56 (1976) 1719-1739.

[20] Q. P. Liu, Modifications of k-constrained KP hierarchy, Phys. Lett. A 187 (1994) 373-381.

[21] N. H. Li, Reciprocal link for a three-component Camassa-Holm type equation, J. Nonlinear Math. Phys. 23 (2016) 150-156.

[22] J. Lenells, Conservation laws of the Camassa-Holm equation, J. Phys. A 38 (2005) 869-880.

[23] J. Kang, X. Liu, P. J. Olver and C. Qu, Liouville correspondence between the modified KdV hierarchy and its dual integrable hierarchy, J Nonlinear Sci. 26 (2016) 141-170.

[24] P. J. Olver, Applications of Lie Groups to Differential Equations, 2nd ed., Springer, Berlin, 1993.

[25] X. G. Geng and F. Li, A hierarchy of new nonlinear evolution equations associated with a 3 × 3 matrix spectral problem, Chin. Phys. Lett. 26 (2009) 050201.

[26] W. Oevel and W. Strampp, Constrained KP hierarchy and bi-Hamiltonian structures, Commun. Math. Phys. 157 (1993) 51-81.

[27] P. G. Estévez and J. Prada, Hodograph transformations for a Camassa-Holm hierarchy in 2+1 dimensions, J. Phys. A 38 (2005) 1287-1297.

[28] J. C. Brunelli and S. Sakovich, Hamiltonian structures for the Ostrovsky-Vakhnenko equation, Commun. Nonlin. Sci. Numer. Simul. 18 (2013) 56-62.

[29] J. Hunter and R. Saxton, Dynamics of director fields, SIAM J. Appl. Math. 51 (1991) 1498-1521.

[30] H. H. Dai and M. Pavlov, Transformations for the Camassa-Holm equation, its high-frequency
limit and the Sinh-Gordon equation, J. Phys. Soc. Japan \textbf{67} (1998) 3655-3657.
[31] V. A. Vakhnenko, Solitons in a nonlinear model medium, J. Phys. A \textbf{25} (1992) 4181-4187.