The Termination of Algorithms for Computing Gröbner Bases

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Abstract

The F5 algorithm Faugère (2002) is generally believed as one of the fastest algorithms for computing Gröbner bases. However, its termination problem is still unclear. Recently, an algorithm GVW Gao et al. (2010) and its variant GVWHS Volny IV (2011) have been proposed, and their efficiency are comparable to the F5 algorithm. In the paper, we clarify the concept of an admissible module order. For the first time, the connection between the reducible and rewritable check is discussed here. We show that the top-reduced S-Gröbner basis must be finite if the admissible monomial order and the admissible module order are compatible. Compared with Volny IV (2011), this paper presents a complete proof of the termination and correctness of the GVWHS algorithm. What is more, it can be seen that the GVWHS is in fact an F5-like algorithm. Different from the GVWHS algorithm, the F5B algorithm may generate redundant sig-polynomials. Taking into account this situation, we prove the termination and correctness of the F5B algorithm. And we notice that the original F5 algorithm in Faugère (2002) slightly differs from the F5B algorithm in the insertion strategy on which the F5-rewritten criterion is based. Exploring the potential ordering of sig-polynomials computed by the original F5 algorithm, we propose an F5GEN algorithm with a generalized insertion strategy, and prove the termination and correctness of it. Therefore, we have a positive answer to the long standing problem of proving the termination of the original F5 algorithm.

Key words: Termination, Gröbner basis, GVWHS, F5

1. Introduction

In cryptography, the cipher of a cryptosystem sometimes can be transformed into a system of equations. Solving a set of multivariate polynomial equations (nonlinear and
randomly chosen) over a finite field is an NP-hard problem Garey and Johnson (1979). Based on which, Albrecht et al. Albrecht et al. (2011) constructed a Polly-Cracker-style cryptosystem. However, in much more cases a designer has to embed some kind of trapdoor function to enable efficient decryption and signing. Although the structure of the cipher is hidden, the equations are so special that one can exploit them via Gröbner basis based techniques to attack the cryptosystem.

In 1965 Buchberger’s Buchberger (1965) thesis he described the appropriate framework for the study of polynomial ideals, with the introduction of Gröbner bases. Since then, Gröbner basis has become a fundamental tool of computational algebra and it has found countless applications in coding theory, cryptography and even directions of Physics, Biology and other sciences.

Although Buchberger presented several improvements to his algorithm for computing Gröbner bases in Buchberger (1979), the efficiency is not so good. Recent years have seen a surge in the number of algorithms in computer algebra research, but efficient ones are few. Faugère Faugère (2002) proposed the idea of signatures and utilized two powerful criteria to avoid useless computation in the F5 algorithm. Faugère and Joux broke the first Hidden Field Equation (HFE) Cryptosystem Challenge (80 bits) by using the F5 algorithm in Faugère and Joux (2003). The proof of the termination in Faugère (2002) was labeled as a conjecture in Stegers (2005). However, Gash Gash (2009) pointed out that there exists an error in the proof of the termination of the F5 algorithm, and he proposed another conjecture for it. It will be shown in this paper that the conjecture is still wrong. In Arri and Perry (2011), a simpler algorithm was constructed to prove the termination, but the proof unfortunately has flaws due to the abuse of the monomial order and the module order mentioned in this paper. Though the F5 algorithm seems to terminate for any polynomial ideals, the proof of it has been admitted as an open problem in Sun and Wang (2011b), Eder and Perry (2011), Eder et al. (2011). Recently, signature-based algorithms like the GVW algorithm and its variant the GVWHS algorithm are proposed in Gao et al. (2010), Volny IV (2011). The algorithms are claimed to terminate if the monomial order and the module order are “compatible”, but readers can hardly find a direct proof. The relation between the reducible and rewritable check, which was not considered before, is studied in the paper, and the finiteness of the top-reduced S-Gröbner basis for a polynomial ideal is proved if the “compatible” property is satisfied. Then we give a complete proof of the termination of the GVWHS algorithm. Besides, through reformulation, the GVWHS algorithm can be seen as an F5-like algorithm (with a different insertion strategy). Though the F5B algorithm (F5 algorithm in Buchberger’s style) may generate redundant sig-polynomials, by analyzing the similarity with the GVWHS algorithm, we prove the termination of the F5B algorithm. Moreover, the termination of the F5GEN algorithm (F5 algorithm with a generalized insertion strategy) is also proved later on. Moreover, by employing an appropriate insertion strategy for the F5GEN algorithm, the proof of the correctness and termination of the original F5 algorithm is self-evident.

The paper is organized as follows. We start by settling basic notations in Section 2. In Section 3, we present a new definition of the admissible module order. Then two admissible orders and their connection are described in Section 4 and the top-reduced S-Gröbner basis for a polynomial ideal is proved to be finite. Based on this finiteness, we propose a new proof of the termination of the GVWHS algorithm in Section 5 and point out that the GVWHS algorithm is a variant of F5 algorithm by introducing the intermediate F5G
algorithm (F5 algorithm in GVWHS’s style). In Section 6, a simpler version of the F5B algorithm in Sun and Wang (2011a) is presented and proved. Considering the different insertion strategy between the F5 algorithm and the F5B algorithm in this paper, we prove the correctness and termination of the F5GEN algorithm in Section 7.

2. Preliminaries

Let \( r \) be a (binary) relation on a set \( M \), one may associate the **strict part** \( r_s = r \setminus r^{-1} \), and let \( N \subseteq M \). Then an element \( a \) of \( N \) is called \( r \)-minimal in \( N \) if there is no \( b \in N \) with \( b r a \). A **strictly descending \( r \)-chain** in \( M \) is an infinite sequence \( \{a_n\}_{n \in \mathbb{N}} \) of elements of \( M \) such that \( a_{n+1} \geq r a_n \) for all \( n \in \mathbb{N} \). If there is another relation \( t \) satisfying \( r \subseteq t \), then \( t \) is called an **extension** of \( r \). The relation \( r \) is called **well-founded** if every non-empty subset \( N \) of \( M \) has an \( r \)-minimal element, \( r \) is a **well-order** on \( M \) if \( r \) is a well-founded linear order on \( M \). For more concepts not presented here, refer to Becker et al. (1993).

Let \( R = k[x_1, \ldots, x_n] \) be the polynomial ring over the field \( k \) with \( n \) variables. We define an admissible order on the monoid \( M = \{ \prod_{i=1}^{n} x_i^{a_i} \mid a_i \in \mathbb{N} \} \).

**Definition 1.** An **admissible monomial order** \( \leq_m \) is a linear order on \( M \) that satisfies the following conditions.

1. \( 1 \leq_m m \) for all \( m \in M \).
2. \( m_1 \leq_m m_2 \) implies \( m_1 \cdot s \leq_m m_2 \cdot s \) for all \( s, m_1, m_2 \in M \).

It can be seen that the admissible order \( \leq_m \) is a well-order on \( M \). Sometimes we write \( =_m \) for \( = \). For any \( p \in R \), without confusion, we denote the leading monomial of \( p \) by \( lm(p) \), the leading coefficient by \( lc(p) \), and the leading term by \( lt(p) = lc(p)lm(p) \) with respect to the order \( \leq_m \).

Let \( \mathcal{I} \) be the ideal generated by the set \( \mathcal{F} = \{ f_1, \ldots, f_d \} \subseteq R \), that is,

\[
\mathcal{I} = \langle f_1, \ldots, f_d \rangle = \{ p_1 f_1 + \ldots + p_d f_d \mid p_1, \ldots, p_d \in R \}.
\]

Consider the following \( R \)-submodule of \( R^d \times R \):

\[
SP = \{ (u, p) \in R^d \times R \mid u \cdot f = p \},
\]

where \( f = (f_1, \ldots, f_d) \in R^d \), and \( e_i \) is \( i \)-th unit vector of \( R^d \) such that the free \( R \)-module \( R^d \) is generated by the set \( \Sigma = \{ e_1, \ldots, e_d \} \). The element \( sp \) in \( SP \) we call a sig-polynomial. A subset \( \text{Syz} = \{ (u, 0) \in SP \} \) is defined the **syzygy submodule** for \( SP \), and \( \text{NSP} = SP \setminus \text{Syz} \) is called the set of **non-syzygy sig-polynomials**. Let \( (u_1, p_1) \) and \( (u_2, p_2) \) be two non-syzygy sig-polynomials in \( SP \). A syzygy \( (p_2 u_1 - p_1 u_2, 0) \) is called a **principal syzygy**.

3. The admissible module order

Below is a fundamental tool for a clearer understanding of termination of algorithms for computing Gröbner bases.

**Definition 2.** Let \( \preceq \) be a quasi-order on \( M \) and let \( N \subseteq M \). Then a subset \( B \) of \( N \) is called a **Dickson basis**, or simply basis of \( N \) w.r.t. \( \preceq \) if for every \( a \in N \) there exists some \( b \in B \) with \( b \preceq a \). We say that \( \preceq \) has the **Dickson property**, or is a **Dickson quasi-order**, if every subset \( N \) of \( M \) has a finite basis w.r.t. \( \preceq \).
If \( \preceq \) is a (Dickson) quasi-order on \( M \), then we call \( (M, \preceq) \) a (Dickson) quasi-ordered set. Let now \( (M, \preceq) \) and \( (N, \preceq) \) be quasi-ordered sets, then a quasi-order \( \preceq' \) on Cartesian product \( M \times N \) is defined as follows:

\[
(a, b) \preceq' (c, d) \Leftrightarrow a \preceq b \text{ and } c \preceq d,
\]

for all \((a, b), (c, d) \in M \times N\). The direct product of the quasi-order sets \((M, \preceq)\) and \((N, \preceq)\) is denoted by \((M \times N, \preceq')\). The Dickson property can be derived as follows.

**Lemma 3.** [Becker et al. (1993)] Let \((M, \preceq)\) and \((N, \preceq)\) be Dickson quasi-ordered sets, and let \((M \times N, \preceq')\) be their direct product. Then \((M \times N, \preceq')\) is a Dickson quasi-ordered set.

The immediate corollary is that \((\mathbb{N}^n, \preceq')\), the direct product of \(n\) copies of the natural numbers \((\mathbb{N}, \preceq)\) with their natural ordering is a Dickson partially ordered set. This is Dickson's lemma, and another version of which is given below by an isomorphism.

**Lemma 4** (Dickson’s lemma). [Becker et al. (1993)] The divisibility relation \( | \) on \( M \) is a Dickson partial order on \( M \). More explicitly, every non-empty subset \( S \) of \( M \) has a finite subset \( B \) such that for all \( s \in S \), there exists \( t \in B \) with \( t | s \).

Let \( M_d = \{m_1 e_i \mid m \in M, i \in \{1, \ldots, d\} \} \) be the \( M \)-monomodule of \( R^d \). The definition of the divisibility relation \( | \) on \( M_d \) is

\[
m_1 e_i | m_2 e_j \iff m_1 | m_2 \text{ and } i = j \in \{1, \ldots, d\}.
\]

By an abuse of notation, we still denote \( | \) instead of \( |' \). Since \((M, |)\) is a Dickson partial ordered set, by decomposing \( M_d \) into \( \bigcup M e_i \), \((M_d, |)\) is also a Dickson partial ordered set. On \( M_d \), we will define the admissible order similarly.

**Definition 5.** An admissible module order \( \leq_s \) is a linear order on \( M_d \) that satisfies the following conditions.

1. \( e_i \leq_s m e_i \) for all \( m e_i \in M_d \).
2. \( m_1 e_i \leq_s m_2 e_i \) implies \( m_1 \cdot s e_i \leq_s m_2 \cdot s e_i \) for all \( s \in M, m_1 e_i, m_2 e_i \in M_d \).

For convenience, \( =_s \) is replaced by \( = \). In fact, the admissible order \( \leq_s \) implies the following properties.

**Proposition 6.** The admissible module order \( \leq_s \) is a well-order on \( M_d \), and it extends the order \( | \) on \( M_d \), i.e., \( m_1 e_i | m_2 e_i \) implies \( m_1 e_i \leq_s m_2 e_i \), for all \( m_1 e_i, m_2 e_i \in M_d, i \in \{1, \ldots, d\} \).

**Proof.** If \( m_1 e_i | m_2 e_i \) in \( M_d \), then there exists \( t \in M \) with \( t \cdot m_1 e_i =_s m_2 e_i \). Since \( e_i \leq_s m_2 e_i \), this implies

\[
m_1 e_i = 1 \cdot m_1 e_i \leq_s t \cdot m_1 e_i = m_2 e_i.
\]

This shows that \( \leq_s \) extends \( | \) on \( M_d \). By Dickson’s lemma, \( \leq_s \) is a Dickson partial order on \( M_d \). And \( \leq_s \) is a well-order on \( M \) as it is a linear order. \( \square \)

It should be noticed that \( \leq_s \) may or may not be related to \( \leq_m \). The compatible property [Kreuzer (2000)] between \( \leq_m \) and \( \leq_s \) is used for the proof of termination for
the GVWHS algorithm in [Volny IV (2011)]: \( \sigma_e \leq \tau e_j \) if and only if \( \sigma \leq \tau \). And in [Arri and Perry (2011)], this property is implicitly used in the proof of termination. The following section will show that this relation is indispensable for the proof of finiteness.

For any \( \text{sp} = (u, p) \in \mathcal{SP} \), let \( \text{lm} \leq (u) = \mu e_k \) be the signature of \((u, p)\) and \( \text{lm} \leq (p) \) the leading monomial of \((u, p)\). By an abuse of notation, we write \( \text{lm} \) for \( \text{lm} \leq (u) \) and \( \text{lm} \leq (p) \) if no misunderstanding occurs. We call \( k = \text{idx}(u) = \text{idx}(\text{sp}) \) the index and call \( \mu \) the monomial of the signature. The set of the signatures of elements in \( \mathcal{SP}^* = \mathcal{SP} \setminus \{(0, 0)\} \) is denoted by \( \text{sig}(\mathcal{SP}^*) \).

4. Properties of sig-polynomials

**Definition 7.** Define a map

\[
LM : \text{NSP} \rightarrow \mathcal{M}_d \times \mathcal{M}
\]

\[
(u, p) \rightarrow (s, m) = (\text{lm}(u), \text{lm}(p)),
\]

and three orders \( \prec_{m,s} \), \( \prec_{s,m} \) and \( \mid \text{super} \) on the image \( LM(\text{NSP}) \) in the following way:

\[
(s', m') \prec_{m,s} (s, m) \Leftrightarrow \lambda \cdot m' = m \text{ and } \lambda \cdot s' <_s s,
\]

\[
(s'', m'') \prec_{s,m} (s, m) \Leftrightarrow \lambda \cdot s'' = s \text{ and } \lambda \cdot m'' <_m m,
\]

\[
(s^*, m^*) \mid \text{super} (s, m) \Leftrightarrow \lambda \cdot m^* = m \text{ and } \lambda \cdot s^* = s,
\]

where \( sp, sp', sp'' \in \text{NSP}, \lambda \in \mathcal{M} \), and by the relation \( <_s \) (\( <_m \)) is meant the strict part of the associated admissible module order.

Under the map \( LM \), the image of a sig-polynomial is called a **leading pair**. We can generalize two orders \( \prec_{s,m} \) and \( \mid \text{super} \) on \( LM(\mathcal{SP}^*) \) by adding the following definitions.

\[
(s', 0) \prec_{s,m} (s, m) \Leftrightarrow s' \mid s,
\]

\[
(s^*, 0) \mid \text{super} (s'', 0) \Leftrightarrow s^* \mid s'\text{,}
\]

where the sig-polynomials above are all in \( \mathcal{SP}^* \) and \( m \neq 0 \).

Without confusion, denote \( \mid \) on \( LM(\mathcal{SP}^*) \) instead of \( \mid \text{super} \) too. Now, a special kind of reduction is introduced as follows.

**Definition 8 (Top-Reduction).** Let \((u, p) \in \mathcal{SP}^* \) be a sig-polynomial and \( B \subseteq \mathcal{SP}^* \) a set of sig-polynomials. \( sp \) is called to be **top-reducible** by \( B \), if there exists a sig-polynomial \((u', p') \in B \) satisfying one of the three conditions below,

1. \( LM(u', p') \prec_{m,s} LM(u, p) \), for \( \text{lm}(p) \neq 0 \),
2. \( LM(u', p') \prec_{s,m} LM(u, p) \), for \( \text{lm}(p) \neq 0 \),
3. \( LM(u', p') \mid LM(u, p) \).

Otherwise, \((u, p) \) is **top-irreducible** by \( B \). Such a top-reduction is called **regular**, if item 1 or 2 is satisfied, and **super** otherwise.
Proof. (1) The order \( \prec_{m,s} (\prec_{s,m}) \) is strictly well-founded partial-order on \( \text{LM}(\text{NSP}) \) \( (\text{LM}(\text{SP}^*)) \).

(2) Let \( S_p \) be the set of \( \prec_{m,s} \)-minimal elements in \( \text{LM}(\text{NSP}) \) and \( S_q \) the set of \( \prec_{s,m} \)-minimal elements in \( \text{LM}(\text{SP}^*) \), then \( S_q = S_p \oplus S_{syz} \), where \( S_{syz} = \{(s, m) \in S_q \mid m = 0\} \).

Below follows a natural corollary.

Corollary 10. Let \( sp \) be a non-syzygy sig-polynomial in \( \text{SP}^* \). \( sp \) is ts-rewritable by \( \text{SP}^* \) if and only if it is tm-reducible by \( \text{SP}^* \).

It can be seen that \( \text{ISP} = \{sp \in \text{SP}^* \mid \text{LM}(sp) \in S_q\} \) is the set of all sig-polynomials which are not ts-rewritable by \( \text{SP}^* \). Super top-reducing elements further in \( \text{ISP} \) results the subset of all top-irreducible sig-polynomials called the **top-reduced S-Gröbner**

\(^1\) The term “ts-rewriting” has the similar meaning as the “M-pair” in Volny IV (2011).
basis $TSG$ for $SP$. The signature of a top-irreducible sig-polynomial is defined by the **top-irreducible signature** for $SP$. Besides, by two equivalent sig-polynomials $sp$ and $sp'$ we mean $sp' \neq sp$ such that $LM(sp') = LM(sp)$. If we store only one for equivalent sig-polynomials in $TSG$, for fixed orders $\leq_m$ and $\leq_s$, the top-reduced S-Gröbner basis $TSG$ is uniquely determined by the module $SP$ up to equivalence. Those top-reducible sig-polynomials in $SP \setminus TSG$ are also called redundant sig-polynomials.

Since $(M, \|)$ and $(M_d, \|)$ are Dickson partial ordered sets, by Lemma 3, we have $(M_d \times M, \|)$ is also a Dickson partial ordered set of which the order $\|^{*}$ is defined as follows:

$$(s_1, m_1) \ |^{*} (s_2, m_2) \iff s_1 | s_2 \text{ and } m_1 | m_2,$$

where $(s_1, m_1), (s_2, m_2)$ are in $M_d \times M$.

**Lemma 11.** Let $(s_1, m_1)$ and $(s_2, m_2)$ be two arbitrary leading pairs in $LM(SP^{*})$ such that $(s_1, m_1) \ |^{*} (s_2, m_2)$. If the admissible monomial order $\leq_m$ and the admissible module order $\leq_s$ are compatible, then $(s_1, m_1)$ and $(s_2, m_2)$ are comparable with respect to one of the three orders $\leq_{m,s}, \leq_{s,m}$ and $\|$.

**Proof.** Let $s$ and $m$ be two monomials in $M$ such that $s = s_2 / s_1$ and $m = m_2 / m_1$. There are three cases as follows.

1. If $m = s$, then $(s_1, m_1) \ | (s_2, m_2)$.
2. If $s \leq_m m$, then $s m_1 < m_2 m_2$, and $(s_1, m_1) \prec_{s,m} (s_2, m_2)$.
3. If $s < m$, as $\leq_m$ and $\leq_s$ are compatible, $m s_1 \leq_s s s_1 = s_2$, and $(s_1, m_1) \prec_{m,s} (s_2, m_2)$.

Therefore, $(s_1, m_1)$ and $(s_2, m_2)$ are comparable with respect to one of the three orders $\leq_{m,s}, \leq_{s,m}$ and $\|$. $\square$

The finiteness of the top-reduced S-Gröbner basis is due to the following fact.

**Theorem 12.** The divisibility relation $|$ is a Dickson partial order on $LM(SP^{*})$. Moreover, the top-reduced S-Gröbner basis for $SP$ is finite.

**Proof.** It is straightforward to verify that $|$ is reflexive, transitive and antisymmetric. Since $|$ is a Dickson partial order on $LM(SP^{*})$, the $|$-minimal elements in $LM(SP^{*})$ are finite. Because the leading pair of a top-irreducible sig-polynomial is $|$-minimal in $LM(SP^{*})$ by Lemma 11. So there are a finite number of top-irreducible sig-polynomials in $TSG$ up to equivalence. $\square$

It can be seen that the “compatible” property is indispensable for the finiteness of the top-reduced S-Gröbner basis $TSG$. Hence in the remaining sections of this paper, we will assume that the admissible monomial order $\leq_m$ and the admissible module order $\leq_s$ are compatible. Suppose two sig-polynomials $sp_1 = (u_1, p_1), sp_2 = (u_2, p_2) \in NSP$. Let

$$m = lcm(lm(p_1), lm(p_2)), \quad m_1 = \frac{m}{lm(p_1)}, \quad m_2 = \frac{m}{lm(p_2)}.$$

If $m_1 lm(u_1) > s m_2 lm(u_2)$, then

- $cp = m_1 (u_1, p_1) = (m_1 u_1, m_1 p_1)$ is called a J-pair of $sp_1$ and $sp_2$;
- $sp_1$ (sp$_2$) is called the first (second) component of cp;
- $m_1$ and $m_2$ are called the multipliers of $sp_1$ and $sp_2$. 

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5. The GVWHS Algorithm

As in [Volny IV (2011)], it can be deduced that if $e_1, \ldots, e_d$ are top-irreducible signatures. Let $sp = (e_i, g)$ be a sig-polynomial in $\text{NSP}$, where $1 \leq i \leq d$. If $LM(sp)$ is not $\prec_{s,m}$-minimal in $LM(SP^*)$, there must exist a sig-polynomial $sp' = (e_i', g') \in SP^*$ whose leading pair is $\prec_{s,m}$-minimal. As $sp'$ cannot be super-top-reduced by $SP^*$, $sp'$ is top-irreducible sig-polynomial, and it is top-irreducible signature.

For a signature $s$, we denote by $SP_{\leq_s(s)}$ the subset of sig-polynomials in $SP$ of which the signatures are smaller than or equal to $s$ with respect to the order $\leq_s$, and denote by $SG_{\leq_s(s)}$ the S-Grobner basis for $SP_{\leq_s(s)}$. We have the following theorem which is similar but stronger than [Volny IV (2011) Th. 4.11].

**Theorem 13.** Let $s$ be a signature in $\text{sig}(SP^*)$ such that $s \neq e_i$ for any $1 \leq i \leq d$. $s$ is top-irreducible if and only if $s$ is the signature of a J-pair $cp$ of two non-syzygy top-irreducible sig-polynomials with smaller signatures and $cp$ is not ts-rewritable by $SG_{\prec_s(s)}$.

**Proof.** As $\prec_s$ is well-founded, and $e_1, \ldots, e_d$ are top-irreducible, there exists a top-irreducible sig-polynomial $(u_k, g_k)$ such that $m_k(u_k, g_k)$ has signature $s$ and $m_k(u_k, g_k)$ is not ts-rewritable by $SG_{\prec_s(s)}$, where $m_k \geq m$. Because $s$ is a top-irreducible signature, $LM(m_ku_k, m_kg_k)$ is not $\prec_{s,m}$-minimal in $LM(SP^*)$. Hence there is a non-syzygy top-irreducible sig-polynomial $(u_i, g_i) \in SG_{\prec_s(s)}$ tm-reducing $m_k(u_k, g_k)$. Denote by $m'_k(u_k, g_k)$ the J-pair of $(u_k, g_k)$ and $(u_j, g_j)$, where $m'_k | m_k$.

Assume for a contradiction that $m'_k$ properly divides $m_k$. Because $m'_k(u_k, g_k)$ is not ts-rewritable by $SG_{\prec_s(s)}$, after a sequence of tm-reduction on $m'_k(u_k, g_k)$, we get a tm-irreducible sig-polynomial $(u_i, g_i)$ and $g_i \neq 0$. $(u_i, g_i)$ is equivalent to $m_l(u_i, g_i)$, a monomial multiple of some top-irreducible sig-polynomial $(u_i, g_i) \in SG_{\prec_s(s)}$, where $m_l \geq m$. Thus, $(u_i, g_i)$ ts-rewrites $m'_k(u_k, g_k)$ and hence $m_k(u_k, g_k)$, a contradiction. Therefore, $m'_k = m_k$, that is, $cp = (m_ku_k, m_kg_k)$ is the J-pair of two non-syzygy top-irreducible sig-polynomials with smaller signatures such that $s = lm(m_ku_k) = lm(u')$ and $cp$ is not ts-rewritable by $SG_{\prec_s(s)}$.

For the forward direction, assume for a contradiction that $s$ is not top-irreducible. Then $LM(u, g)$ is $\prec_{s,m}$-minimal in $LM(SP^*)$. By Proposition 9, $LM(u, g)$ is also $\prec_{s,m}$-minimal in $LM(SP^*)$ as $g \neq 0$, a contradiction.

First, we present the GVWHS algorithm, which is modified slightly from the algorithm mentioned in [Volny IV (2011)]. The subset of non-syzygy sig-polynomials in $SG$ is denoted by $G_1$ and $\text{sig}(G_1)$ is the set of signatures of sig-polynomials in $G_1$. Let $S$ be a set of polynomials (sig-polynomials), $\text{sort}(S, \leq_m (\leq_s))$ means that we arrange $S$ by ascending leading monomials (signatures) of polynomials (sig-polynomials) with respect to the order $\leq_m (\leq_s)$.

The only difference compared with the basic algorithm in [Volny IV (2011)] is that we discard the Grobner basis for the syzygy module when the algorithm terminates. Note that the proof for correctness in [Volny IV (2011)] is not complete. Suppose $s$ is a top-irreducible signature, it must be proved, as in Theorem 13, that there exists a J-pair of $cp$ such that $cp$ is an M-pair and $s_{cp} = s$.

**Theorem 14.** For any finite subset $F$ of polynomials in $R$, the GVWHS algorithm terminates after finitely many steps and it creates a Grobner basis for the ideal $I = \langle F \rangle$. 
Algorithm 1 The GVWHS algorithm

1: inputs:
    F = \{f_1, \ldots, f_d\} \in R, a list of polynomials
    \leq_m an admissible monomial order on M
    \leq_s an admissible module order on M_d which is compatible with \leq_m

2: outputs:
    \mathcal{G}_1, a Gröbner basis for I = < f_1, \ldots, f_d >
    interreduce F and F := sort(\{f_1, \ldots, f_d\}, \leq_m)

3: init1:
    CPs := \{(e_1, f_1), \ldots, (e_d, f_d)\} and SG = \{(f_i e_j - f_j e_i, 0) | 1 \leq i < j \leq d\}

4: while CPs \neq \emptyset do
    5: cp := min\{\{cp \in CPs\} \leq_s\} and CPs := CPs \setminus \{cp\}
    6: if cp is not ts-rewritable by \mathcal{S}G then
    7: \quad \mathcal{S}G \leftarrow \mathcal{S}G \cup \{cp\}
    8: \quad \mathcal{S}G := \mathcal{S}G \cup \{sp = (u, g) \mid g \neq 0\}
    9: \quad \text{J-pair for each distinct signature of minimal leading monomial}
    10: \quad SG := SG \cup \{sp = (u, g) \mid (u, g) \in G_1\}
    11: \quad SG := SG \cup \{sp = (u, g) \mid (u, g) \in \text{Syz}\}

Proof.} We proceed by induction on the top-irreducible signature s. Because \leq_s is an admissible module order on M_d, the smallest signature of sig-polynomials in SP^* must be one of the top-irreducible signatures e_1, \ldots, e_d, denoted by e_i. The case s = e_i is trivial. As CPs is initialized with \{(e_1, f_1), \ldots, (e_d, f_d)\}, during the first while-loop, (e_i, f_i) is added into \mathcal{S}G, which is the S-Gröbner basis \mathcal{S}G_{\leq_s(e_i)}.

Let s > e_i, and suppose that \mathcal{S}G created by the GVWHS algorithm is \mathcal{S}G_{\leq_s(e_i)} after finitely many while-loops. If s = e_j, where 1 \leq j \leq d, j \neq i, there exists a cp = (e_j, f_j) at line 6 and cp is not ts-rewritable by \mathcal{S}G_{\leq_s(e_j)}. Tm-reducing cp repeatedly by \mathcal{S}G_{\leq_s(e_j)} at line 8 results a top-irreducible sig-polynomial sp with signature e_j because e_j is top-irreducible. Thus, \mathcal{S}G_{\leq_s(e_j)} can be obtained. If s \neq e_j, we can also obtain a J-pair cp' with signature s at line 6 and cp' is not ts-rewritable by \mathcal{S}G_{\leq_s(s)} by Theorem 13. After that, a top-irreducible sig-polynomial sp with signature s is created and \mathcal{S}G = \mathcal{S}G_{\leq_s(s)}. Because top-irreducible signatures are finite in SP, after finitely many steps, \mathcal{S}G = \mathcal{S}G_{\leq_s(s_{\text{max}})} is the S-Gröbner basis.

By Theorem 13, the remaining J-pairs in CPs, if any, are all sig-polynomials with top-irreducible signatures and they will be ts-rewritten by \mathcal{S}G. Therefore, the algorithm terminates and generates \mathcal{S}G, an S-Gröbner basis for SP, and the output is a Gröbner basis for the ideal I = < F >.

In the remaining part of this section, we aim to reformulate the GVWHS algorithm into an F5G algorithm (F5-like algorithm in GVWHS's style) and find out the connection between the GVWHS algorithm and the F5 algorithm. It is, we shall see, an F5-like algorithm with a different insertion strategy. Before proceeding to prove the termination of the F5G algorithm, we introduce another order as follows.
Define an order \( \preceq_l \) on \( LM(\mathcal{SP}^*) \) in the following way:

\[
(\mu e_i, m) \preceq_l (\mu' e_i, m') \iff \mu' \leq_m \mu
\]

Note that the order \( \preceq_l \) is not defined when two elements in \( LM(\mathcal{SP}^*) \) are with different signatures, so \( \preceq_l \) is a well-founded quasi-order on \( LM(\mathcal{SP}^*) \). Particularly, if we restrict \( \preceq_l \) on the subset \( \{ (\mu e_i, m) \in LM(\mathcal{SP}^*) \mid i = i_0, 1 \leq i_0 \leq d \} \), then \( \preceq_l \) is a well-order on it. Moreover, if \( (\mu e_i, m) \prec_s m (\mu' e_i, m') \) or \( (\mu' e_i, m') \prec_m, s (\mu e_i, m) \), we have \( (\mu e_i, m) \prec_l (\mu' e_i, m') \).

Below is the pseudo code of the F5G algorithm in Buchberger’s style which is similar to the algorithm in Sun and Wang [2011a]. We detach the set \( \mathcal{PSyz} \) of principal syzygies from \( \mathcal{SG} \), and the remainder is denoted by \( \mathcal{SG}' \). That is to say, \( \mathcal{SG} = \mathcal{PSyz} \cup \mathcal{SG}' \). As is known that there may exist syzygies in \( \mathcal{SG}' \), so by \( \mathcal{G} \) is meant the set of non-syzygy sig-polynomials in \( \mathcal{SG}' \). The notations are similar with those in the GVWHS algorithm.

### Algorithm 2 The F5G Algorithm (F5-like algorithm in GVWHS’s style)

1. **Inputs:**
   - \( \mathcal{F} = \{ f_1, \ldots, f_d \} \in \mathcal{R} \), a list of polynomials
   - \( \leq_m \), an admissible monomial order on \( \mathcal{M} \)
   - \( \leq_s \), an admissible module order on \( \mathcal{M}_d \) which is compatible with \( \leq_m \)
   - \( \preceq_l \), an order on \( LM(\mathcal{SP}^*) \)

2. **Outputs:**
   - \( \mathcal{G}_1 \), a Gröbner basis for \( \mathcal{I} = \langle f_1, \ldots, f_d \rangle \)
3. **Interreduce \( \mathcal{F} \) and \( \mathcal{F} := \text{sort}(\{ f_1, \ldots, f_d \}, \leq_m) \), \( F_i = (e_i, f_i) \) for \( i = 1, \ldots, d \)
4. **Init2:**
   - \( \text{CPs} := \text{sort}(\{ J - \text{pair}[F_i, F_j] \mid 1 \leq i < j \leq d \}, \leq_s) \), \( \mathcal{SG}' = \{ F_i \mid i = 1, \ldots, d \} \) and \( \mathcal{PSyz} = \{ (f_i e_j - f_j e_i, 0) \mid 1 \leq i < j \leq d \} \)
5. **While \( \text{CPs} \neq \emptyset \) do:
   - \( \text{cp} := \text{min}(\{ \text{cp} \in \text{CPs}, \leq_s \}) \) and \( \text{CPs} := \text{CPs} \setminus \{ \text{cp} \} \)
   - if \( \text{cp} \) is neither ts-rewritable by \( \mathcal{PSyz} \) nor F5-rewritable by \( \mathcal{SG}' \) then
     - \( \text{cp} \rightarrow \mathcal{SG}' \rightarrow \text{cp} = (u, g) \)
   - if \( g \neq 0 \) then
     - \( \text{CPs} := \text{sort}(\text{CPs} \cup \{ J - \text{pair}(sp, sp') \mid \forall sp' \in \mathcal{G}_1, sp' \neq sp \}, \leq_s) = \{ m\mathcal{SG}'(k) \} \) and store only one J-pair for each distinct signature of which the first component has maximum index \( k \) in \( \mathcal{SG}' \)
   - \( \mathcal{PSyz} := \mathcal{PSyz} \cup \{ (gu - g'u, 0) \mid (u, g) \in \mathcal{G}_1 \} \) and discard those super top-reducible in \( \mathcal{PSyz} \)
6. **Return:**
   - \( \{ g \mid (u, g) \in \mathcal{SG}' \setminus \mathcal{Syz} \} \)

It is important to note that the index \( k \) mentioned at line 11, different from the index of a sig-polynomial, points to the sig-polynomial of the \( k \)-th position in \( \mathcal{SG}' \).

Let \( sp_j, sp_i \) be two sig-polynomials in \( \mathcal{SG}' \) and let \( cp = tsp_i, cp' = tsp_j \) be two J-pairs with the same signature. From the insert_by_decreasing_l function, we know that \( sp_j \) appears later in \( \mathcal{SG}' \) than \( sp_i \) if \( LM(sp_j) \prec_l LM(sp_i) \). In this case, \( cp' \) is discarded as its first component \( sp_j \) is ahead of the first component \( sp_i \) of \( cp \). Hence line 11 of the F5G algorithm is equivalent to storing only one J-pair for each distinct signature of
Algorithm 3 F5-rewritable
1: inputs:
   \( cp = m(u_r, g_k) \in \mathcal{SP} \)
   \( SG' := SG'(i) = \{(u_1, g_1), \ldots, (u_r, g_r)\} \)
2: outputs:
   true if \( m u_k \) is F5-rewritable by another sig-polynomial in \( SG' \)
3: find the first index \( j_b \) and the last index \( j_e \) in \( SG' \) such that \( idx(sp) = idx(SG'(j_b)) = idx(SG'(j_e)) \)
4: for \( i = j_e \) to \( j_b \) do
5:   if \( lm(u_i) | lm(mu_k) \) then
6:     return \( i \not= k \)
7: return false

Algorithm 4 insert_by_decreasing_j
1: inputs:
   \( sp \), a sig-polynomial
   \( SG' := SG'(i) = \{(u_1, g_1), \ldots, (u_r, g_r)\} \)
   \( \preceq \), an order on \( LM(\mathcal{SP}) \)
2: find the first index \( j_b \) and the last index \( j_e \) in \( SG' \) such that \( idx(sp) = idx(SG'(j_b)) = idx(SG'(j_e)) \)
3: for \( i = j_e \) to \( j_b \) do
4:   if \( LM(SG'(i)) \succeq \_ LM(sp) \) then
5:     insert \( sp \) into \( SG' \) after \( SG'(i) \)
6:     return
7: insert \( sp \) into \( SG' \) before \( SG'(j_b) \)
8: return

minimal leading monomial at line 10 of the GVWHS algorithm. Even more, the F5G algorithm adopts the same criterion as the GVWHS algorithm for finding redundant sig-polynomials.

Lemma 16. During an execution of the while-loop, let \( cp_0 \) be the J-pair chosen at line 6 in the F5G algorithm, and let \( CPs_0 \) be the value of \( CPs \), \( PSyz_0 \) the value of \( PSyz \), and \( SG'_0 \) the value of \( SG' \) at line 6. The criteria of line 7 in the F5G algorithm are equivalent to the statement of judging whether \( cp_0 \) is not ts-rewritable by \( PSyz_0 \cup SG'_0 \).

Proof. Assume that the J-pair \( cp_0 = m(u_k, g_k) = msp_k \) is ts-rewritable by \( SG'_0 \) in the F5G algorithm. We may find \( sp_j = (u_i, g_i) \in SG'_0 \) ts-rewrite \( cp_0 \) and \( lm(m_i u_i) = lm(m_k u_k) \), where \( m_i > m_k \). Since \( LM(sp_j) \not\prec_i LM(sp_k) \) means \( i > k \), \( cp_0 \) is F5-rewritable by \( sp_i \) in the F5G algorithm. That is to say, \( cp_0 \) can not pass the criteria of line 7 in the F5G algorithm.

If \( cp_0 = msp_k \in CPs_0 \) is not ts-rewritable by \( SG'_0 \). Assume for a contradiction that \( cp_0 \) is F5-rewritable by \( sp_j \in SG'_0 \), \( j > k \). We know \( LM(sp_j) \not\prec_i LM(sp_k) \), or else the J-pair \( cp_0 \) had been discarded by line11 of the F5G algorithm. So \( LM(sp_j) \not\prec_i LM(sp_k) \), which means \( lm(m_j u_j) = lm(m_k u_k) \) and \( lm(m_j g_j) < lm(m_j g_k) \), where \( m_j > m_k \). Hence \( cp_0 \) is ts-rewritable by \( sp_j \), a contradiction. □
Note that Lemma 16 does not apply to the algorithms we will discuss later since the insertion strategy of the F5G is used for the proof. In Theorem 13, two components of a J-pair have to be top-irreducible. As a matter of fact, a generalized lemma follows.

**Lemma 17.** If \( s \) is the signature of a J-pair \( cp = msp_k = m(u_k, g_k) \) of two non-syzygy sig-polynomials \( sp_k \) and \( sp_j \) (with smaller signatures) and \( cp \) is not ts-rewritable by \( SG_{<, (s)} \), then \( s \) is a top-irreducible signature of \( SP \).

**Proof.** Assume for a contradiction that \( lm(mu_k) \) is not a top-irreducible signature. Then \( LM(cp) = s_m \)-minimal in \( LM(SP^*) \). But there exists \( m'sp_j = m(u_j, g_j) \) such that \( lm(m'u_j) < s_lm(mu_k) \) and \( lm(m'g_j) = lm(mg_k) \), that is, \( cp \) is tm-reducible by \( SG_{<, (s)} \), a contradiction. \( \Box \)

**Theorem 18.** For any finite subset \( F \) of polynomials in \( R \), the F5G algorithm terminates after finitely many steps and it creates a Gröbner basis for the ideal \( I = < F > \).

**Proof.** Due to Lemma 16, we will use the criterion of judging whether \( cp \) is not ts-rewritable by \( PSyz \cup SG' \) instead. Similar to the corresponding proof of the GVWHS algorithm, we proceed by induction on the top-irreducible signature \( s \). Because \( s_\prec \) is an admissible module order on \( M_{dj} \), the smallest signature of sig-polynomials in \( SP^* \) must be one of the top-irreducible signatures \( e_1, \ldots, e_d \), denoted by \( e_i \). The case \( s = e_i \) is trivial. As \( SG' \) is initialized with \( \{ (e_1, f_1), \ldots, (e_d, f_d) \} \), \( SG' \) is the S-Gröbner basis for \( SP_{\leq, (e_i)} \).

Let \( s > e_i \), and suppose that \( PSyz \cup SG' \) created by the F5G algorithm is \( PSyz_{\leq, (e_i)} \cup SG'_{<, (e_j)} = SG_{<, (s)} \) after finitely many while-loops. If \( s = e_j \), where \( 1 \leq j \leq d, j \neq i \), there is only one sig-polynomial \( (e_j, f_j) \) in \( SG'_{<, (e_j)} \) with top-irreducible signature \( e_j \). And if \( (e_j, f_j) \) is tm-irreducible by \( SG_{<, (e_j)} \), \( PSyz_{<, (e_j)} \cup SG'_{<, (e_j)} \) is \( SG_{<, (e_j)} \). If \( (e_j, f_j) \) is tm-reducible by \( SG_{<, (e_j)} \), during an execution of the while-loop, line 6 will create a J-pair \( cp = (e_j, f_j) \) and \( cp \) is not ts-rewritable by \( SG_{<, (e_j)} \). Tm-reducing \( cp \) repeatedly by \( SG_{<, (e_j)} \) at line 8 results in a top-irreducible sig-polynomial \( sp \) with signature \( e_j \) because \( e_j \) is top-irreducible. Thus, \( SG_{\leq, (e_j)} \) can be obtained. If \( s \neq e_j \), we can also obtain a J-pair \( cp' \) with signature \( s \) at line 6 and \( cp' \) is not ts-rewritable by \( SG'_{<, (s)} \) by Theorem 13. After that, a top-irreducible sig-polynomial \( sp \) with signature \( s \) will be created. Because top-irreducible signatures are finite in \( SP \), after finitely many steps, \( PSyz \cup SG' = SG_{\leq, (s_{max})} \) is the S-Gröbner basis.

By Lemma 17, the remaining J-pairs in \( CPs \), if any, are all ts-rewritable by \( SG \). Therefore, the algorithm terminates and generates an S-Gröbner basis \( PSyz \cup SG' \) for \( SP \), and the output is a Gröbner basis for the ideal \( I = < F > \). \( \Box \)

6. **The termination and correctness of the F5B Algorithm**

We present two variants of the F5 algorithm here and in the next section, both of which share the same F5-rewritten criterion with that in the F5G algorithm. So we do not write the F5-rewritable function in detail again.

For two non-syzygy components \( sp_1 \) and \( sp_2 \) of a J-pair, let \( m_1 \) and \( m_2 \), respectively, be their multipliers. A much simpler version than the F5B algorithm (F5 algorithm
The F5B algorithm here does not apply F5-rewritable check for \( m_2 m_2 \) nor in the tm-reduction of the J-pair. Omitting these influences neither the termination nor the correctness of the F5B algorithm in \cite{Sun2011}. For details, one can refer to \cite{Sun2011b} and \cite{Eder2011}.

**Algorithm 5** The F5B Algorithm (F5 algorithm in Buchberger’s style)

1. **inputs:**
   - \( F = \{ f_1, \ldots, f_d \} \in R \), a list of polynomials
   - \( \preceq_m \), an admissible monomial order on \( M \)
   - \( \preceq_s \), an admissible module order on \( M_d \) which is compatible with \( \preceq_m \)
   - \( \preceq_L \), an order on \( LM(SP) \)
2. **outputs:**
   - \( G_1 \), a Gröbner basis for \( I = \langle f_1, \ldots, f_d \rangle \)
3. interreduce \( F \) and \( F := \text{sort}(\{ f_1, \ldots, f_d \}, \preceq_m) \), \( F_i = (e_i, f_i) \) for \( i = 1, \ldots, d \)
4. **init2:**
   - \( \text{CPs} := \text{sort}(\{ J\text{-pair}[F_i, F_j] \mid 1 \leq i < j \leq d \}, \preceq_s) \), \( SG' = \{ F_i \mid i = 1, \ldots, d \} \) and \( PSyz = \{ (f_i e_j - f_j e_i, 0) \mid 1 \leq i < j \leq d \} \)
   - while \( \text{CPs} \neq \emptyset \) do
     5. \( cp := \min(\{ cp \in \text{CPs} \}, \preceq_s) \) and \( \text{CPs} := \text{CPs}\backslash\{cp\} \)
     6. if \( cp \) is neither ts-rewritable by \( PSyz \) nor F5-rewritable by \( SG' \) then
       8. \( cp \mapsto sp = (u, g) \)
     9. \( SG' := \text{insert_by_index}(sp, SG') \)
     10. if \( g \neq 0 \) then
         11. \( \text{CPs} := \text{sort}(\text{CPs} \cup \{ J\text{-pair}(sp, sp') \mid \forall sp' \in G_1, sp' \neq sp \}, \preceq_s) = \{ mSG'(k) \} \) and store only one J-pair for each distinct signature of which the first component has maximum index \( k \) in \( SG' \)
   12. \( PSyz := PSyz \cup \{ (gu - g, 0) \mid (u, g) \in G_1 \} \) and discard those super top-reducible in \( PSyz \)
13. **return** \( \{ g \mid (u, g) \in SG' \backslash Syz \} \)

**Algorithm 6** insert_by_index

1. **inputs:**
   - \( sp \), a sig-polynomial
   - \( SG' := SG'(i) = \{ (u_1, g_1), \ldots, (u_r, g_r) \} \)
2. find the last index \( j_e \) in \( SG' \) such that \( idx(sp) = idx(SG'(j_e)) \)
3. insert \( sp \) into \( SG' \) after \( SG'(j_e) \)
4. **return**

Instead of using an auxiliary number for each sig-polynomial in \cite{Sun2011}, the F5B algorithm here realizes the same rewritable check by adjusting the order of sig-polynomials in \( SG' \). One can find that the real difference between the F5B and F5G algorithms is the insertion of elements in \( SG' \). The reason why line 11 does not affect the correctness of the algorithm lies in the fact that the first component of the discarded J-pair appears earlier in \( SG' \) than that of the stored J-pair.
Definition 20. Let \((sp_i, sp_j)\) and \((sp_k, sp_p)\) be two misplaced pairs. And define \((sp_i, sp_j) \prec_{pm} (sp_k, sp_p)\), if one of the following cases is satisfied.

1. \(LM(sp_i) \prec_i LM(sp_k)\)
2. \(LM(sp_i) \prec_i LM(sp_k)\) and \(LM(sp_j) \prec_i LM(sp_i)\)

If each J-pair \(msp_j\) is either ts-rewritable by \(PSyz\) or F5-rewritable by \(SG'\), we call the misplaced pair \((sp_i, sp_j)\) correct.
Theorem 21. For any finite subset $\mathbf{F}$ of polynomials in $R$, the F5B algorithm terminates after finitely many steps and it creates a Gröbner basis for the ideal $I = < \mathbf{F} >$.

Proof. We still proceed by induction on the top-irreducible signature $s$. If $s = e_i$ is the smallest top-irreducible signature, the initialized $\text{PSyz} \cup S\mathcal{G}$ is the S-Gröbner basis $S\mathcal{G}_{\leq, (e_i)}$.

Let $s > e_i$, and suppose that $\text{PSyz} \cup S\mathcal{G}'$ created by the F5B algorithm is $\text{PSyz}_{\leq, (s)} \cup S\mathcal{G}_{\leq, (s)}' = S\mathcal{G}_{\leq, (s)}$ after finitely many while-loops. If $s = e_j$, where $1 \leq j \leq d$, $j \neq i$, there is only one sig-polynomial $(e_j, f_j)$ in $S\mathcal{G}_{\leq, (e_i)}$ with top-irreducible signature $e_j$. And if $(e_j, f_j)$ is tm-irreducible by $S\mathcal{G}_{\leq, (e_i)}$, $\text{PSyz}_{\leq, (e_i)} \cup S\mathcal{G}_{\leq, (e_j)}' = S\mathcal{G}_{\leq, (e_j)}$. If $(e_j, f_j)$ is tm-irreducible by $S\mathcal{G}_{\leq, (e_i)}$, during an execution of the while-loop, line 6 will create a J-pair $cp = (e_j, f_j)$ and $cp$ is neither ts-rewritable by $\text{PSyz}_{\leq, (e_j)}$ nor F5-rewritable by $S\mathcal{G}_{\leq, (e_j)}$. Tm-reducing $cp$ repeatedly by $S\mathcal{G}_{\leq, (e_i)}$ at line 8 results a top-irreducible sig-polynomial $sp$ with signature $e_j$ because $e_j$ is top-irreducible. Thus, $S\mathcal{G}_{\leq, (e_j)}$ can be obtained. If $s \neq e_j$, we can also obtain a J-pair $cp'$ with signature $s$ at line 6 and $cp'$ is neither ts-rewritable by $\text{PSyz}_{\leq, (s)}$ nor F5-rewritable by $S\mathcal{G}_{\leq, (s)}$ by Lemma 19. After that, a top-irreducible sig-polynomial $sp$ with signature $s$ will be created. Because top-irreducible signatures are finite in $SP$, after finitely many steps, the algorithm generates an S-Gröbner basis $\text{PSyz} \cup S\mathcal{G}' = S\mathcal{G}$ for $SP$.

If there are J-pairs in CPs at this time, a new $cp'' = m(u_k, g_k) = msp_k$ may pass the criteria and thus generating a new tm-irreducible sig-polynomial $sp_n$ in $S\mathcal{G}'$. There must exist a top-irreducible $sp_k$ in $S\mathcal{G}'$ such that $sp_k$ can super top-reduce $sp_n$ and $(sp_n, sp_k)$ is a misplaced pair. That is, $LM(sp_n) \triangleq LM(sp_n) \prec_1 LM(sp_k)$ and $h < k < n$. On the one hand, the J-pairs of $sp_n$ and other possible sig-polynomials, be of the form $m'sp_n$ or not, will generate tm-irreducible sig-polynomials, say, $sp_p$ with $\leq$-smaller leading pairs if it passes the criteria of the F5B algorithm. Since the leading pair of $sp_p$ is equal to that of a top-irreducible sig-polynomial and the top-irreducible sig-polynomials in $SP$ are finite, this process of creating a J-pair and generating a sig-polynomial always terminates. On the other hand, after finite steps, the misplaced pair $(sp_n, sp_k)$ will be corrected. Though an insertion of a new tm-irreducible sig-polynomial may produce other misplaced pairs, the $\prec_{pm}$-maximum misplaced pair of $S\mathcal{G}'$ without being corrected is gradually decreasing with respect to the order $\prec_{pm}$. As there are finite pairs not $\prec_{pm}$-equal, the algorithm will terminate finally and output a Gröbner basis for $I = < \mathbf{F} >$. $\square$

7. Proof of the termination of the F5 algorithm

In the original F5 algorithm in [Faugère (2002)], the input polynomials in $F = \{f_1, \ldots, f_d\}$ are homogeneous, and after initialization, sig-polynomials are $(e_1, f_1), \ldots, (e_d, f_d)$. A property follows: If $sp = (u, g) \in \text{NSP}$ and $idx(u) = i$, $1 \leq i \leq d$, then $\deg(lm(u)) + \deg(lm(f_i)) = \deg(g)$. We define the **g-weighted degree** the same with that in [Gao et al. 2011]: The g-weighted degree $gw = \deg$ of a sig-polynomial $sp = (u, g)$ is equal to $\deg(lm(u)) + \deg(lm(f_{idx(u)}))$. Therefore, selecting critical pairs of the minimal degree in the original F5 algorithm equals selecting J-pairs of the minimal g-weighted degree. For an admissible monomial order $\leq_m$, we define the admissible module order $\leq_{x_0}$ as follows.

We say that $x^\alpha e_i <_{x_0} x^\beta e_j$ if
(1) $i < j$, 
(2) $i = j$ and $gw - deg(x^0 e_i) < gw - deg(x^0 e_j)$, 
(3) $i = j$, $gw - deg(x^0 e_i) = gw - deg(x^0 e_j)$ and $x^0 < x^0$. 

Particularly, we have $x^0 e_i = s_0 x^0 e_j$ if $i = j$ and $x^0 = x^0$. 

Sure enough, the order $\leq s_0$ is an admissible module order. By using this order $\leq s_0$, we can understand the reformulation of the original F5 algorithm easier. In [Faugère (2002)], Faugère builded up an array Rule to store the ordering of sig-polynomials on which the F5-rewritten criterion is based. As presented in the following pseudo code, we will just discard the Rule and store the ordering directly in $\mathcal{SG}'$. 

Though the F5B and original F5 algorithms share the same F5-rewritten criterion, the ordering in $\mathcal{SG}'$ of the F5B algorithm slightly differs from that in Rule of the F5 algorithm. In the F5B algorithm, let $sp_1$ and $sp_2$ be two sig-polynomials of the same index in $\mathcal{SG}'$. If $s_{sp_1} < s_{sp_2}$, $sp_1$ must appear earlier in $\mathcal{SG}'$ than $sp_2$. This is also interpreted as an isRewritten criterion in [Hashemi and M.-Alizadeh (2011)]. However, in the original F5 algorithm, the claim is not true for sig-polynomials. Since the Rule is updated not only in the Spol function of [Faugère (2002)] but also in the TopReduction function, at the end of each run though the while-loop, the newly added sig-polynomials in Rule have the same index. Moreover, the g-weighted degrees of them are equal as the input polynomials of the original F5 algorithm are homogeneous. Then there is no guarantee that the sig-polynomials are arranged in $\leq s_0$-descending order (note that the original F5 algorithm insert new sig-polynomials at the beginning of Rule). By running several examples, this non-monotony in Rule is verified. 

Nevertheless, a weaker relation exists between sig-polynomials in Rule. During an execution of the while-loop in the original F5 algorithm, let $d$ be the minimal degree of critical pairs. The sig-polynomials added in Rule are all of g-weighted degree $d$ in the Spol and TopReduction functions. Hence if two sig-polynomials $sp_1$ and $sp_2$ in Rule are of the same index satisfying $gw - deg(sp_1) < gw - deg(sp_2)$, then $sp_1$ appears earlier in Rule than $sp_2$. Besides, if a J-pair $cp$ of two non-syzygy sig-polynomials $sp_3$ and $sp_4$ passes criteria of the original F5 algorithm and it is F5-reduced to $sp_5$, then $sp_5$ appears later than $sp_3$ and $sp_4$. Here the latter property plays an important part in the proof below. 

The following is the F5GEN algorithm (F5 algorithm with a generalized insertion strategy) derived from the original one in [Faugère (2002)]. It use the same F5-rewritten criterion as the previous ones. Here we still omit F5-rewritable check when tm-reducing J-pairs as in [Faugère (2002)]. 

In the insert F5GEN function of the F5GEN algorithm, we can restrict an appropriate strategy of insertion such that the ordering in $\mathcal{SG}'$ is the same as that in Rule of the original F5 algorithm. The idea for constructing signature-based algorithms also for non-homogeneous polynomial ideals has been mentioned in Eder and Perry’s earlier papers. We shall see that this F5GEN algorithm here is true for any polynomial ideals both homogeneous and non-homogeneous, admissible module orders other than $\leq s_0$ and the weak condition of ordering in $\mathcal{SG}'$ mentioned in the above pseudo code. But once the input polynomials are homogeneous and the admissible module order $\leq s_0$ is chose, the F5GEN algorithm with an appropriate strategy of insertion will simulate the original F5 algorithm accurately. Together with the analysis of equivalence between the original F5 algorithm and the F5B algorithm in [Sun and Wang (2011a)], the proof of termination

\footnote{Here, F5-reducing means using F5-rewritable check and tm-reducing.}
Algorithm 7 The F5GEN Algorithm (F5 algorithm with a generalized insertion strategy)

1: inputs:
   \( F = \{f_1, \ldots, f_d\} \in R \), a list of polynomials
   \( \leq_m \) an admissible monomial order on \( M \)
   \( \leq_s \), an admissible module order on \( M_d \) which is compatible with \( \leq_m \)
   \( \leq_l \), an order on \( LM(SP^*) \)

2: outputs:
   \( G_1 \), a Gröbner basis for \( I = \langle f_1, \ldots, f_d \rangle \)

3: interreduce \( F \) and \( F := \text{sort}(\{f_1, \ldots, f_d\}, \leq_m), F_i = (e_i, f_i) \) for \( i = 1, \ldots, d \)

4: init2:
   \( CPs := \text{sort}(\{J - \text{pair}[F_i, F_j] \mid 1 \leq i < j \leq d\}, \leq_s), SG' = \{F_i \mid i = 1, \ldots, d\} \) and
   \( PSyz = \{(f_i e_j - f_j e_i), 0 \mid 1 \leq i < j \leq d\} \)

5: while \( CPs \neq \emptyset \) do

6: \( cp := \min(\{cp \in CPs\}, \leq_s) \) and \( CPs := CPs \setminus \{cp\} \)

7: if \( cp \) is neither ts-rewritable by \( PSyz \) nor F5-rewritable by \( SG' \) then

8: \( sp := \text{assoc}(CPs \cup \{J - \text{pair}(sp, sp') \mid \forall sp' \in G_1, sp' \neq sp\}, \leq_s) = \{mSG'(k)\} \) and store only one J-pair for each distinct signature of which the first component has maximum index \( k \) in \( SG' \)

9: \( \text{PSyz} := \text{PSyz} \cup \{(g u_i - g_i u, 0) \mid (u_i, g_i) \in G_1\} \) and discard those super top-reducible in \( PSyz \)

10: return \( \{g \mid (u, g) \in SG' \setminus \text{Syz}\} \)

Algorithm 8 insert_F5GEN

1: inputs:
   \( sp \), a sig-polynomial
   \( SG' := SG'(i) = \{(u_1, g_1), \ldots, (u_r, g_r)\} \)
   \( cp = m(u_k, g_k) \), the J-pair which is tm-reduced to \( sp \)

2: find the first index \( j_b \) and the last index \( j_e \) in \( SG' \) such that \( idx(sp) = idx(SG'(j_b)) = idx(SG'(j_e)) \)

3: insert \( sp \) into \( SG' \) after \( SG'(i) \), where \( j_b - 1 \leq i \leq j_e \), such that \( sp \) appears later in \( SG' \) than \( sp_k = (u_k, g_k) \)

4: return

and correctness for this F5GEN algorithm can be used to prove the termination and correctness of the original algorithm in Faugère (2002).

Lemma 22. Let \( s \) be a signature in \( \text{sig}(SP^*) \) such that \( s \neq e_i \), for any \( 1 \leq i \leq d \). During an execution of the while-loop in the F5GEN algorithm, let \( PSyz_{<, (s)} \) and \( SG'_{<, (s)} \) be the values of \( PSyz \) and \( SG' \). If \( s \) is top-irreducible, then \( s \) is the signature of a J-pair \( cp \) of two non-syzygy sig-polynomials in \( SG'_{<, (s)} \) with smaller signatures and \( cp \) is neither ts-rewritable by \( PSyz_{<, (s)} \) nor F5-rewritable by \( SG'_{<, (s)} \).
Proof. By Theorem 13, there exists a J-pair \(cp' = m'(u_k, g_k)\) of two non-syzygy top-irreducible sig-polynomials with smaller signatures such that \(s = lm(m'u_k)\) and \(cp'\) is not ts-rewritable by \(SG_{<s}(s)\). If \(cp'\) is F5-rewritable by \(SG'_{<s}(s)\), let \((u_j, g_j)\) be the last non-syzygy sig-polynomial in \(SG'_{<s}(s)\) with the signature dividing \(s\) according to the structure of the F5GEN algorithm. As \(s\) is top-irreducible signature, \(m(u_j, g_j)\) can be tm-reduced by some non-syzygy top-irreducible sig-polynomial \((u_l, g_l)\) in \(SG'_{<s}(s)\). Denote by \(m^*(u_j, g_j)\) the J-pair of \((u_j, g_j)\) and \((u_l, g_l)\), where \(m^* | m\).

Assume for a contradiction that \(m^*\) properly divides \(m\). It can be deduced that \(m^*(u_j, g_j)\) is neither ts-rewritable by \(PSyz_{<s}(s)\) nor F5-rewritable by \(SG'_{<s}(s)\). After a sequence of tm-reduction on \(m^*(u_j, g_j)\), we get a tm-irreducible sig-polynomial \((u_v, g_v)\).

Because of the insertion strategy of the F5GEN algorithm, \((u_v, g_v)\) must appear later in \(SG'_{<s}(s)\) than \((u_j, g_j)\), which contradicts the fact that \((u_j, g_j)\) F5-rewrites \(cp'\). Therefore, \(m^* = m\), that is, \(cp = m(u_j, m(g_j))\) is the J-pair of two non-syzygy sig-polynomials with smaller signatures such that \(s = lm(mu_j) = lm(u')\) and \(cp\) is neither ts-rewritable by \(PSyz_{<s}(s)\) nor F5-rewritable by \(SG'_{<s}(s)\). □

For the original F5 algorithm, Gash [Gash (2009)] made a conjecture that there is not a sig-polynomial in \(SG'\) super top-reducible by another one. But this can not be satisfied sometimes. Here we can not guarantee the reverse direction of Lemma 22 is satisfied too. It is highly possible that there exist a misplaced pair \((sp_i, sp_j)\) in \(SG'\) as the insertion strategy of the F5 algorithm (it can be seen as an implementation of the F5GEN algorithm) is different from the F5G. From the proof of Theorem 21, we know that \(LM(sp) \prec_l LM(sp') \prec_l LM(sp'')\) if the sig-polynomial \(sp\) is the result tm-reduced from a J-pair of \(sp'\) and \(sp''\). A J-pair of the form \(msp_j\) may pass the criteria and be reduced to \(m' sp_i\) since \(LM(sp_i) \prec_l LM(sp_j)\). The sig-polynomial \(sp_i\) is added earlier in \(SG'\) than \(sp_j\) and \(sp_i\) can not be selected in the F5-rewritable function. So both \(sp_i\) and \(m'sp_i\) are kept in \(SG'\), a contradiction. One can verify this situation by running several examples.

Theorem 23. For any finite subset \(F\) of polynomials in \(R\), the F5GEN algorithm terminates after finitely many steps and it creates a Gröbner basis for the ideal \(I = \langle F \rangle\).

Proof. Again, we proceed by induction on the top-irreducible signature \(s\) and let \(e_i\) be the smallest top-irreducible signature. The case \(s = e_i\) is trivial.

Let \(s > e_i\), and suppose that \(PSyz \cup SG'\) created by the F5GEN algorithm is \(SG'_{<s}(s)\) after finitely many while-loops. If \(s = e_j\), \(SG'_{<s}(e_j)\) can be obtained in similar fashion with the proof of the F5B algorithm. If \(s \neq e_j\), we can also obtain a J-pair \(cp'\) with signature \(s\) at line 6 and \(cp'\) is neither ts-rewritable by \(PSyz_{<s}(s)\) nor F5-rewritable by \(SG'_{<s}(s)\) by Lemma 22. After that, a top-irreducible sig-polynomial \(sp\) with signature \(s\) will be created. Because top-irreducible signatures are finite in \(SP\), after finitely many steps, the algorithm generates an S-Gröbner basis \(PSyz \cup SG' = SG\) for \(SP\).

If there are J-pairs in \(CPs\) at this time, the leading pair of a newly generated sig-polynomial which is tm-reduced from the J-pair \(cp\), is \(\leq_l\)-smaller than two components of \(cp\). On the one hand, by the insertion strategy of the F5GEN algorithm and leading pair of generated sig-polynomials are \(\leq_l\)-equal to that of top-irreducible sig-polynomials, a branch of creating a J-pair and generating a sig-polynomial will end finitely. On the other hand, after finite steps, a misplaced pair will be corrected. Though an insertion of a new
A signature-based algorithm may produce other misplaced pairs, the \( \prec_{pm} \)-maximum misplaced pair of \( SG' \) without being corrected is gradually decreasing with respect to the order \( \prec_{pm} \). As there are finite pairs not \( \prec_{pm} \)-equal, the algorithm will terminate finally and output a Gröbner basis for \( I = \langle F \rangle \). Therefore, for any finite set of homogeneous polynomials, the original F5 algorithm in [Faugère, 2002] terminates finitely and it creates a Gröbner basis for the polynomial ideal.

8. Conclusion

This paper presents a clear proof of the termination of the GVWHS, F5B and F5 algorithms under the condition that the admissible monomial order and the admissible module order are compatible. Of course, there exist some optimizations for improving the efficiency, like recording \((\text{lm}(u), g)\) for each \((u, g)\) in the implementation. These optimizations do not affect the correctness and termination. One may find out that the F5G, F5B and original F5 algorithms are implementations of the F5GEN algorithm with different insertion strategy. That means, the GVWHS algorithm is just an F5-like algorithm. Moreover, with this proved F5GEN algorithm, researchers can shift their focus on the different variants of the F5GEN algorithm and find out the fastest one.

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