String Dualities from Matrix Theory: A Summary

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1. Introduction

In the last year a significant amount of evidence has accumulated in support of the conjecture of Banks, Fischler, Shenker and Susskind on the non-perturbative formulation of M-theory\textsuperscript{[1]}. The conjecture, however, needs to be extended when describing M-theory compactified down to seven or less non-compact spacetime dimensions. The details of the extension for M-theory on a 4-torus were discussed in \textsuperscript{[2,3]}, on a 5-torus in \textsuperscript{[3,4]}, on \(T^5/Z_2\) in \textsuperscript{[4]}, and a proposal for M-theory on K3 was put forward in \textsuperscript{[5]}. For a more systematic approach, see \textsuperscript{[6]}. In this lecture I will discuss some aspects of how the various perturbative and non-perturbative symmetries of string theory manifest themselves in Matrix theory when non-compact space-time is seven dimensiona\textsuperscript{2}.

The focus of these lectures will be the \(SL(5, Z)\) duality of M-theory on \(T^4\) and the duality between M theory compactified on K3 and the Heterotic string on \(T^3\). The results presented in this lecture were obtained in collaboration with M. Rozali and N. Seiberg. I will mainly follow the approach in \textsuperscript{[3,5]} and the reader is referred to additional details there. Some of the results in section 3 were derived also by \textsuperscript{[10,11]}. The constructions presented in this lecture are also similar to those in \textsuperscript{[12]}, but in the context of Matrix theory. Additional insight and details can be found in the lectures of N. Seiberg and E. Verlinde at this conference.

2. U-duality of M-theory on \(T^4\)

In dimension \(d < 4\), the Matrix description of M-theory on \(T^d\) is given by Super-Yang-Mills (SYM) on a “Dual” manifold (with 16 Susy)\textsuperscript{[7]}. This dual manifold is also a d-dimensional torus but the radii of the torus are different \textsuperscript{[1,7,8,12-17]}, and we will denote it by \(\tilde{T}^d\). In these dimensions this proposal does indeed constitute a complete proposal as these SYM are renormalizable and therefore exist without the need to add degrees of freedom. This conjecture, however, has to be extended when describing M-theory compactified\textsuperscript{[10]} on \(T^3\) (and other 4 or higher dimensional manifolds). The reason is that the “SYM on a dual torus” guarantees that the extreme IR of the base-space theory will be roughly correct, but does not provide any information on the UV of the theory. Indeed, 4+1 SYM is not renormalizable and one needs to specify its UV in order to make sense of such a proposal. The relevant extension was discussed in \textsuperscript{[10]} where it was suggested that M-theory on a 4-torus is given by the large N limit of the (2,0) supersymmetric field theory compactified on a 5-torus, which we will denote by \(T^5\).

The details of the extension are the following.\textsuperscript{3}
Let us begin with the 4+1 SYM picture. For simplicity, the torus will be taken to be rectangular. M-theory has 5 dimensionful parameters which are the eleven dimensional Planck scale $l_p$, the 4 periodicities of the torus $L_i$. The parameters of the gauge theory are the dual torus lengths $\tilde L_i$ and the gauge coupling $g^2$. They are given by

\[ g^2 = \frac{(2\pi)^3 l_i^3}{L_i \tilde{R}} \]

\[ \tilde L_i = \frac{(2\pi)^3 l_i^3}{L_i \tilde{R}} \]

where $R$ is the radius of the compactified light cone.

This gauge theory is not renormalizable and therefore not well defined. However, it can be used as a low energy effective theory which is valid at energies below $\frac{1}{g^2}$ (In this regime several tests of this theory as a formulation of M-theory were shown to be successful [18]). In order to define our 4+1 dimensional nonrenormalizable theory we need to give more information about its short distance degrees of freedom. In our case, we are guided by the $SL(5,\mathbb{Z})$ U-duality group to suggest that the desired definition of this theory is in terms of the (2,0) supersymmetric fixed point in six dimensions.

The (2,0) theory is an interacting quantum field theory at a fixed point of the renormalization group (for a review, see [19]). It was first discussed in the context of type IIB compactification on a singular K3 [20] and later in the context of $N$ nearby 5-branes in M-theory [21][22]. Here we use the same field theory, compactified on $T^5$, as a definition of M-theory [1].

We thus propose that M-theory on $T^4$ with radii $L_{1,2,3,4}$ is described by the large $N$ limit of the (2,0) theory compactified on a five torus $T^5$ (with an $SL(5,\mathbb{Z})$ invariant choice of spin structure [23]). Its five sizes are related to $L_{1,2,3,4}$ in the following way. In equation (1), the sizes of the four torus and the coupling of the 4+1 SYM were given. This SYM has an additional conserved $U(1)$ current given by $j = *(F \wedge F)$. This symmetry is to be identified with the Kaluza-Klein $U(1)$ symmetry of rotating around the small circle. This determines the circumference of the circle to be $\frac{2\pi}{g^2}$. In the 4+1 SYM an $n$ instantons configuration, which has energy $\frac{2\pi^2}{g^4}$, corresponds to a state with total momentum $n$ around the small circle and hence we identify its periodicity with $\frac{2\pi^2}{g^4}$. The 4+1 SYM description only identifies the charge of this $U(1)$ symmetry.

To summarize, we propose that the (2,0) theory is compactified on $T^5$ whose sizes are given by

\[ \tilde L_i = \frac{(2\pi)^3 l_i^3}{L_i \tilde{R}} \]

\[ \tilde L_5 = \frac{(2\pi)^3 l_i^3}{L_1 L_2 L_3 L_4 \tilde{R}}. \]

Our main focus is the manifestation of the U-duality group for this configuration, but let us briefly discuss some tests of this proposal. As a first test of this proposal we can restate what we had explained before. At energies much smaller than $\frac{1}{g^2}$ and for $\tilde L_5 \ll \tilde L_{1,2,3,4}$ our theory becomes the 4+1 dimensional SYM. This corresponds to the space-time 4-torus being much larger than the Planck-scale. The gap to the states that carry momentum around $\tilde L_5$ is proportional to the volume of the space-time torus. A process of scattering gravitons in a region smaller compared to the 4-torus of space-time will yield the correct results, up to terms that are suppressed by the volume of this 4-torus. In particular general relativity will be a correct local description. As another test one can describe the various (particle like) solitons in the theory as fluxes in the (2,0) theory.

The most striking feature of this proposal is that this definition of M-theory on $T^4$ makes the U-duality manifest. The U-duality group in this case is $SL(5,\mathbb{Z})$. It is simply the geometric duality group of $T^5$. This symmetry involves mixing the five radii $\tilde L$ in a way which is complicated as an action on the individual $L_i$. Since this U-duality group is manifest, so are its subgroups which appear in compactifications on lower dimensional tori.

It is interesting to discuss the interpretation of spacetime as we go from a given space-time 4-
torus and its image under an element of $SL(5, Z)$. A large space $T^4$ is given by a the $(2,0)$ theory on $T^4 \times S^1$ where the size of the $S^1$ is much smaller than any other size of the torus. We then obtain, after a Kaluza-Klein reduction, a weakly coupled 4+1 SYM and the weakly coupled Wilson lines are the positions of the 0-branes (and hence gravitons) on the physical space $T^4$ (i.e., space-time emerges as the moduli space of vacua of the quantum mechanical system).

Let us now take the 5-torus and follow a trajectory to an $SL(5, Z)$ dual point (which is not included in the $SL(4, Z)$ of the 4+1 SYM torus). For convenience, we can do this by enlarging the size of the 5-th circle and then shrinking one of the circles of the 4-torus. On the way we will pass a point in which all the 5 radii are of approximately equal size. At this point we are clearly not justified in performing a Kaluza-Klein reduction on any of the circles and we must analyze the theory in the full $(2,0)$ theory. We will argue that there is no reasonable space-time interpretation at this point.

It is easy to see that there is no valid semiclassical moduli space approximation for the $(2,0)$ theory. This can be seen is several complementary ways. Let us try and construct the light degrees of freedom on the moduli space by considering the constant modes of $B$ (the self dual 2-form in the $(2,0)$ theory). The self-duality condition forces these modes to be time independent and hence the theory has no light modes and hence no moduli space of vacua.

In fact, in quantum mechanics (unlike field theory with more than 2 space dimensions) there is never a moduli space of vacua. Only if the theory has a parameter, which can be interpreted as $\hbar$, can we expect an approximate notion of moduli space of vacua. Then, for $q = 0$ the classical theory can have many static solutions which we can identify as its moduli space of vacua, $\mathcal{M}$. When $\hbar$ is small we can study the full quantum theory by restricting the degrees of freedom to $\mathcal{M}$ and quantizing only them.

Returning to our $(2,0)$ theory on $\tilde{T}^5$, we realize that this theory does not have a free parameter like $\hbar$. The six dimensional theory, because of its self-duality, has fixed $\hbar = 1$. Hence, as we saw above, it cannot have a semiclassical limit with a moduli space of vacua. Instead, we can find a moduli space of vacua by creating an effective $\hbar$. One way to do that is to consider the limit of this theory with $\tilde{L}_5 \ll \tilde{L}_{1,2,3,4}$. Then, by going through the 4+1 dimensional SYM we find a quantum mechanical system whose moduli space is $T^4$ with sizes $L_{1,2,3,4} \sim \frac{1}{\tilde{L}_{1,2,3,4}}$. This interpretation of space-time is not natural, if we pursue it to the region where all the $\tilde{L}$’s are of the same order of magnitude. As any one of the $\tilde{L}$’s is much smaller than the others there is another natural interpretation of space time. This is the essence of U-duality in our construction. The natural interpretation of the theory in terms of space time changes as the five radii $\tilde{L}$ change.

3. M-theory on K3 and the Heterotic string on $T^3$

A similar problem occurs when trying to discuss the Matrix model description for M-theory on other 4-dimensional manifolds. As it is misleading to begin with the SYM prescription, our starting point will be the $(2,0)$ field theory. In this section we discuss compactifications of M-theory down to seven dimensions, that have 8 linearly realized supersymmetries (in the infinite momentum frame). These are M-theory on K3 and the Heterotic string on $T^3$. We obtain such a theory by compactifying the $(2,0)$ theory on a 5-dimensional base-space that breaks half the supersymmetry. The manifold that we use is the natural candidate $S^1 \times K3$ (the ‘‘ denotes that it is a different K3 than the space-time one). We will refer to this manifold as the base-space.

The Matrix model for M-theory on $T^4$ had a space-time interpretation only when the base-space $\tilde{T}^5$ degenerated in specific ways. We will see that the same is also true here but, unlike in the previous case, in different degenerations of the $S^1 \times K3$ base-space, we obtain different space-time interpretations. When the manifold degenerates to a 4 dimensional base-space in different ways we obtain M-theory on a large $K3$ and the Heterotic theory on a large $T^3$. As these configurations are smoothly connected, and we can follow the transition from one limit to another and the duality
between these theories is manifest (the technology involved is, in fact, similar to [30]).

### 3.1. M-Theory on K3

We begin by obtaining M-theory on K3 in Matrix theory [24]. Let us denote the size of the base-space $S^1$ by $\Sigma_1$ and the volume of the K3 by $V$. The limit of M-theory on a large K3 is obtained by

$$\Sigma_1, V \rightarrow 0 \quad \sum_i \Sigma_i \text{ fixed.} \quad (3)$$

The second requirement guarantees that the eleven dimensional Planck scale is fixed.

When $\Sigma_1$ is much smaller than any length scale of the $K3$, the IR of the base-space theory is approximated by a Kaluza-Klein reduction on the $S^1$, which is a weakly coupled SYM on $K3$ (the gauge coupling of the this SYM is $g^2 = \Sigma_1$). By a sort of T-duality, which we assume exist\(^5\). We then obtain a description of the moduli space of the theory in terms of 0-branes moving on a dual K3, which is the physical space-time $K3$.

In the case of orbifold limits of $K3$, the requirement that $S^1$ is smaller than any length scale of $K3$ cannot be satisfied, as there are 2-cycles of zero size. This results in the appearance of additional degrees of freedom in the effective 4+1 SYM. These are related to the 32 fermions that appear in [24-29] and will be discussed further in section 3.2.3. For now we restrict our attention to degrees of freedom that live in the bulk, far away from any orbifold points.

The remainder of this section will be devoted to a discussion of some supporting evidence for this proposal. Let us work in the limit that $K3$ is the orbifold $T^4/Z_2$, whose sizes are $\Sigma_2, \ldots, \Sigma_5$. The relation of the spacetime parameters to the SYM parameters are similar to those of M-theory on a 4-torus, and are given in [24,25]. These are

$$L_i^2 = \frac{2\pi R V}{\Sigma_1 \Sigma_i} \quad \frac{L_p^6}{(2\pi)^2} \Sigma_i \quad (4)$$

Where $L_i$ (i= 2,\ldots,5) are the spacetime lengths and $L_p$ is the eleven dimensional Planck length.

In the limit when $g^2 \rightarrow 0$ ($\hbar \rightarrow 0$) the theory becomes semi-classical and the Wilson lines define a moduli space. This moduli space is interpreted as the classical compactification manifold in spacetime. The weakly coupled 4+1 SYM description of this space is equivalent to the description of 0-branes moving on this manifold, as can be shown by an explicit T-duality. This description is valid when this spacetime manifold is much larger than $L_p$.

Another check that we have identified correctly the Matrix theory is to reproduce the moduli space of M theory on K3. As is often the case in the infinite momentum frame, modifications to the ground state of the theory are obtained by modifying the Hamiltonian. In our case we can modify the base space geometry. The different choices of base space geometry should give the spacetime moduli space.

We are therefore interested in 5 dimensional manifolds that break half the supersymmetry. There are several discrete choices that need to be made. Within one of these choices we are restricted to metrics on $S^1 \times K3$ with an $SU(2)$ holonomy. The only parameters of such a metric are the size of the $S^1$ and a choice of an Einstein (Ricci flat) metric on $K3$. The Moduli space is therefore locally $SO(3,19)/(SO(3) \times SO(19)) \times R^+$. This is the correct moduli space of M-theory on K3 (and of the Heterotic string on $T^3$) [31]. There are actually additional couplings but these are associated the compactness of the null directions in the DLCQ description of M-theory on K3, and should not be counted as parameters of M-theory on K3.

The model also has enhanced gauge symmetries at the correct points of moduli space. If the base-space has certain singularities, then the T-dual K3 has similar singularities. It was shown in [24,25] that the $N \rightarrow \infty$ Matrix theory then has the additional states that make up the additional gauge bosons.

Our main goal is to discuss the duality between M-theory on K3 and the Heterotic string on $T^4$ [11]. The construction on the Heterotic side
is more complicated and less complete as the
degeneration of the base-space is more complicated.
Nevertheless we can discuss some aspects of it,
and suggest how this duality comes about in Ma-
trix theory.

3.2. Heterotic Theory on $T^3$

3.2.1 Heterotic vacuum with $SU(2)^{16}$ Enhanced
gauge symmetry

The case which is easiest to analyze is when
$K3 = T^4/Z^2$ (which has 16 $A_1$ singularities).
This configuration is the one that is most closely
related to the configuration of $E_8$. Let us pick
one dimension of $T^4/Z_2$, say $\Sigma_5$, and take it to
zero, as well as the volume of the remaining space
$V = \Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4$. Again, we take $V$ and $\Sigma_5$ to zero
at a fixed ratio.

After the Kaluza-Klein reduction on $\Sigma_5$ we ob-
tain a SYM on $S^1 \times (T^3/Z_2)$ with a coupling
g$^2 = \Sigma_5$ and some boundary conditions on
the gauge fields (We will not discuss these boundary
conditions here but only state that for $N = 1$
which is the only case where we can check them
explicitly, they agree with $[33]$.). This model
therefore reproduces the Matrix description of the
Heterotic String on $T^3$ $[23,26]$. Note also that the
instanton number reverses its sign under the $Z_2$
action, which means that these boundary condi-
tions give the unique correct extension to the 5-
dimensional manifold $S^1 \times T^4/Z_2$.

We have obtained an Heterotic string theory
on $T^3$, and we can write its parameters in terms of
g$^2, \Sigma_1, \Sigma_3, \Sigma_4$. It is more instructive, however, to
write it in terms of the M-theory on $K3$ param-
eters. Doing so, one obtains

$$T_{\text{string}} = \frac{L_1 L_3 L_5}{(2\pi)^3 L_5^2}$$

$$\lambda_7 = \frac{(L_1 L_3 L_5)^3}{(2\pi)^2 L_5^2}$$

which are the seven dimensional Heterotic/M-
theory duality relations $[31]$. One can also repro-
duce more detailed formulas that relate the radii
of the $K3$ to those of the $T^3$ $[23]$.

6As was explained in $[23]$, the state with $N=1$
corresponds to a solitonic state in the M-theory. We are now in posi-
tion to see how the gauge boson changes from a pertur-
bitative state in the Heterotic string to a solitonic state in

3.2.2. A Conjecture Regarding the $E_8 \times E_8$ Case

We are interested in the Heterotic string on $T^3$
in its M-theory limit. i.e. when it is M-theory
on $S^1/Z_2 \times T^3$. We therefore expect to see a
well defined moduli space of this form only when
the space-time gauge symmetry is $E_8 \times E_8$, or a
subgroup of it.

The picture that we suggest is very similar to
that of Vafa and Morrison $[35]$. Let us check the
case in which the base-space $K3$ has two $E_8$ singu-
larities. The $K3$ can then be written as an elliptic
fibration over $P^1$. On the $P^1$ there are two singular
fibers which contain the $E_8$ singularities and four
additional singularities (where the fiber but
not the $K3$ degenerates). We are interested in the
limit in which a pair of the additional singularities
approach each $E_8$ loci. In that case the base be-
comes a thin long cylinder capped in the vicinity
of the $E_8$ singularity. Throughout the cylinder,
as long as we are away from the singularities, the
fiber has a constant complex structure parameter.

We are interested in reducing the (2,0) theory
on the small circle which is a part of the cylin-
drical base of the elliptic fibration. We take therefore
all the other dimensions of the base space to be
of the same order of magnitude, and larger than
this small circle. Note that the size of the fiber is
a parameter of the theory (unlike in F-theory). In
this configuration the (2,0) theory has a mass gap
and we can perform a Kaluza-Klein reduction on
the small circle. The base space of the resulting
4+1 SYM looks like $S^1 \times T^2 \times I$ where the first
$S^1$ is outside $K3$, the $T^2$ is the fiber and $I$ is an
interval, which is what is left from the cylinder
after the Kaluza-Klein reduction. On this space
we have a weakly coupled gauge theory with some
boundary conditions on the gauge fields and mat-
ter fields. We can now have four Wilson lines on
this space which give us the four compact space
coordinates. Unfortunately, we do not know how
to calculate the boundary conditions in this pic-
ture, so we can not verify that the moduli space of
Wilson lines is indeed $T^3 \times (S^1/Z_2)$.

One more comment is in order. When we de-
form away from the $E_8 \times E_8$ loci, the degenera-
tion of $K3$ is generically very complicated. There is no
guarantee that for such a degeneration there will
be any sensible description of the low energy in terms of any 4+1 SYM on a manifold. This is different from other approaches taken to the Heterotic Matrix theory in [14-19].

3.2.3 Additional Degrees of Freedom

Another important difference is the way that the fermions in the Heterotic Matrix model are treated. In our picture the fermions are to be understood as fermionization of the bosons $fB$ over shrunk cycles. As such they are localized at the singularities and are not allowed to move. Enhanced symmetry is obtained in a geometric way in which the mixing of the compact space parameters and the $E_8 \times E_8$ Wilson lines is apparent.

More important is the fact that we do not need to add these fermions by hand [27]. They are automatically provided by the $(2,0)$ definition of the theory. At no point of the discussion do we need to take a circle, in the Matrix description of M-theory on $T^4$, orbifold it and add 8-branes. Rather the 8-branes are generated in the effective space-time by the existence of additional degrees of freedom in a specific degeneration of the $(2,0)$ base-space. We know that there are 8-branes only through the existence of the 32 fermions. When we calculate any low energy scattering the fermions contribute to the scattering amplitude such that a low energy observer interprets the result as the existence of 8-branes.

3.3. Duality

To summarize both M on $K^3$ and the Heterotic string on $T^3$ are described by the same model. Hence, duality is manifest. In one limit of the geometry of the base-space we obtain the a description of the weakly coupled low-energy of the Heterotic string and in another limit that of M theory on $K^3$. The transition between these two limits, as expected, goes through a region in which the compact part of space-time is not well defined.

Acknowledgments

The results presented in this talk were obtained in collaboration with M. Rozali and N. Seiberg. Both myself and my collaborators would like to thank O. Aharony, P. Aspinwall, T. Banks, D. Berenstein, R. Corrado, J. Distler, M. Douglas, R. Entin, D. Finnel, W. Fischler, O. Ganor, P. Horava, S. Kachru, S. Sethi, S. Shenker and E. Witten for useful and illuminating discussions. This work was supported by NSF grant NSF PHY-9513835.

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