Asymptotics for multifactor Volterra type stochastic volatility models

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ABSTRACT

We study multidimensional stochastic volatility models in which the volatility process is a positive continuous function of a continuous multidimensional Volterra process that can be not self-similar. The main results obtained in this paper are a generalization of the results due, in the one-dimensional case, to Cellupica and Pacchiarotti (J. Theor. Probab. 34(2):682–727). We state some (pathwise and finite-dimensional) large deviation principles for the scaled log-price and as a consequence some (pathwise and finite-dimensional) short-time large deviation principles.

ARTICLE HISTORY

Received 10 June 2022
Accepted 27 August 2022

KEYWORDS

Large deviations; Volterra type Gaussian processes; multifactor stochastic volatility models

2010/2020

MATHEMATICS SUBJECT CLASSIFICATION

60F10; 60G15; 60G22

1. Introduction

The last few years have seen renewed interest in stochastic volatility models in which the volatility process is a positive continuous function $\sigma$ of a continuous stochastic process $\tilde{B}$, that we assume to be a Volterra type Gaussian process. The goal of this paper is to extend to the multidimensional case and in a more general asset a problem of large deviations for the log-price process of Volterra type stochastic volatility models, studied in the one-dimensional case in [1, 2] (homogeneous case) and in [3] (time-inhomogeneous case). Large deviations theory deals with the exponential decay of probabilities of “rare events,” i.e., events whose probability is very small. These probabilities are important in many fields of study, including statistics, finance, engineering, statistical physics, and chemistry, since they often give informations about the large fluctuations of a random system around its most probable trajectory.Jacquier et al. [4] prove a large deviations principle (LDP) for a scaled version of the log stock price process. In this same direction, Bayer et al. [5], Forde and Zhang [6], Horvath et al. [7] and most recently Friz et al. [8] (to name a few) prove large deviations principles for a wide range of one-dimensional stochastic volatility models. Some results concerning asymptotic for the log-price processes, in the multifactor setup can be found, for example, in [9] where a
sample path LDP for log-processes associated with general Volterra systems is studied and in [10] where a comprehensive sample path LDP for log-processes associated with multivariate time-inhomogeneous stochastic volatility models is stated. The main difference with the other models is that in this paper we can obtain short-time large deviations even if the Volterra process that appears in the variance is not self-similar (see Section 5). In the model considered in [1], the volatility is a positive continuous function of a continuous Volterra process. We generalize this model in a multifactor setup, in which the volatility is a matrix of continuous function of a multidimensional continuous Volterra process, and we find a pathwise LDP for the family of the multivariate log-price process. The generalization to the multidimensional case requires many technical details which as far as it seems to us cannot be cut out. So despite the many similarities with the previous paper, we are unable to write the proofs more concisely here. Some recent literature uses stochastic volatility models with jumps (see, e.g., [11–13] and references therein), but as far as we know, there are no large deviation results in these cases and this could be an interesting topic for a future work. In this paper, we consider a quite general model in which the dynamic of the multivariate log-price process is modeled by the following equations:

\[
\begin{align*}
\frac{dS_i(t)}{S_i(t)} &= \mu_i(\hat{B}(t)) \ dt + \sum_{\ell=1}^{p} \tilde{\sigma}_{i\ell}(\hat{B}(t)) \ dB_{\ell}(t) + \sum_{j=1}^{d} \sigma_{ij}(\hat{B}(t)) \ dW_j(t) \\
S_i(0) &= s_0^i
\end{align*}
\]

for every \(1 \leq i \leq d\), where \(s_0^i \in \mathbb{R}^d\) is the initial value, \(T > 0\) is the time horizon and the process \(\hat{B}\) is a non-degenerate continuous \(\mathbb{R}^p\)-valued multidimensional Volterra type process (see Definition 2.8). The process \(W\) is a \(d\)-dimensional standard Brownian motion independent from a \(p\)-dimensional standard Brownian motion \(B\) appearing in definition of the process \(\hat{B}\). This model has its own interest from a mathematical point of view due to its generality. Other large deviations results for multidimensional models can be found, for example, in [14] (for multidimensional diffusions) and [15] (for multidimensional stochastic Volterra equations). Furthermore, the processes \(S_n\), \(1 \leq i \leq d\), under suitable hypotheses on the coefficients can be interpreted as price processes of correlated risky assets. More precisely, the \(d\) components of the process \(S\) model the (dependent) prices of \(d\) assets on the market. In this case the stochastic differential equation should be written as

\[
\frac{dS_i(t)}{S_i(t)} = \mu_i(\hat{B}(t)) \ dt + \sum_{j=1}^{d} \Lambda_{ij}(\hat{B}(t)) \ d\hat{W}_j(t)
\]

with \(\Lambda_{ij}\) some suitable functions, for every \(1 \leq i, j \leq d\) and \(\hat{W}\) a \(d\)-dimensional Brownian motion. The matrix \(\Lambda = (\Lambda_{ij})_{i,j=1,...,d}\) is the volatility map and the matrix-valued process \((\Lambda(\hat{B}_t))_{t \in [0, T]}\) represents the joint volatility of the \(d\) assets. With a little abuse of language we call the process \(S\), defined in (1), the price process and \(Z\) defined as \(Z_i = \log S_i\) for \(1 \leq i \leq d\) the log-price associated. The initial condition for the log-process is denoted by \(x_0^i\). It is clear that \(x_0^i = \log s_0^i\).

It is assumed that \(\mu_i : \mathbb{R}^p \to \mathbb{R}\), \(\sigma_{ij} : \mathbb{R}^p \to \mathbb{R}\) and \(\tilde{\sigma}_{i\ell} : \mathbb{R}^p \to \mathbb{R}\) are continuous functions, for every \(1 \leq i, j \leq d\) and \(1 \leq \ell \leq p\).
The model is called uncorrelated when $\tilde{\sigma}_{i\ell} = 0$, for every $1 \leq i \leq d$ and $1 \leq \ell \leq p$, i.e., when the dynamic of $S$ is not driven by $B$, otherwise it is called correlated. It is easy to understand what we mean with “generalized model”; indeed, we find the one-dimensional model taking $d = p = 1$, $\tilde{\sigma}_{11} = \tilde{\rho} \sigma$ and $\sigma_{11} = \tilde{\rho} \sigma$. Notice that the one-dimensional case also generalize the model in [1].

We consider a suitable scaled version $(Z^n)_n$ (see Equation (17)) of the log-price process $Z$ and we obtain a sample path large deviation principle for the family of processes $((Z^n_t - x_0)_{t \in [0,T]} )_{n \in \mathbb{N}}$. A large deviation principle for $(Z^n_T - x_0)_{n \in \mathbb{N}}$ can be obtained with the same techniques, but it can be also obtained by contraction and this is the approach we follow here. We always suppose without loss of generality that $s^0_i = 1$ (and then $x^0_i = 0$), for every $1 \leq i \leq d$.

The paper is organized as follows. In Section 2 we recall some basic facts about large deviations for continuous Gaussian processes and we give the definition of multidimensional Volterra process. For some facts about large deviations for joint and marginal distributions we refer for simplicity to Section 3 in [1] that is a summary of the results we use and that are proved in [16]. In Sections 3 and 4 are contained the main results. More precisely in Section 3 we prove a large deviation principle for the log-price process in the uncorrelated model. In Section 4, we prove a large deviation principle for the log-price process in the correlated model (we will follow the same pattern as in [1]). In Section 5, following [17] we obtain a multidimensional short-time LDP.

2. LDP for multidimensional Volterra processes

We briefly recall some main facts on large deviation principles and Volterra processes we are going to use. For a detailed development of this very wide theory we can refer, for example, to the following classical references: the book of Varadhan [18], Chapitre II in Azencott [19], Section 3.4 in Deuschel and Strook [20], Chapter 4 (in particular Sections 4.1, 4.2 and 4.5) in Dembo and Zeitouni [21], for large deviation principles; [22, 23] for Volterra processes.

2.1. Large deviations

Definition 2.1. Let $E$ be a topological space, $\mathcal{B}(E)$ the Borel $\sigma$-algebra and $(\eta_n)_{n \in \mathbb{N}}$ a family of probability measures on $\mathcal{B}(E)$; let $\gamma : \mathbb{N} \to \mathbb{R}^+$ be a speed function, i.e., $\gamma_n \to +\infty$ as $n \to +\infty$. We say that the family of probability measures $(\eta_n)_{n \in \mathbb{N}}$ satisfies a LDP on $E$ with the rate function $I$ and the speed $\gamma_n$ if, for any open set $\Theta$,

$$- \inf_{x \in \Theta} I(x) \leq \liminf_{n \to +\infty} \frac{1}{\gamma_n} \log \eta_n(\Theta)$$

and for any closed set $\Gamma$

$$\limsup_{n \to +\infty} \frac{1}{\gamma_n} \log \eta_n(\Gamma) \leq - \inf_{x \in \Gamma} I(x).$$  

(2)

A rate function is a lower semicontinuous mapping $I : E \to [0, +\infty]$. A rate function $I$ is said good if $\{I \leq a\}$ is a compact set for every $a \geq 0$. 
Definition 2.2. Let $E$ be a topological space, $\mathcal{B}(E)$ the Borel $\sigma$-algebra and $(\eta_n)_{n \in \mathbb{N}}$ a family of probability measures on $\mathcal{B}(E)$; let $\gamma : \mathbb{N} \to \mathbb{R}^+$ be a speed function. We say that the family of probability measures $(\eta_n)_{n \in \mathbb{N}}$ satisfies a weak LDP (WLDP) on $E$ at the speed $\gamma_n$ if, for any $\delta > 0$,

$$\lim_{n \to +\infty} \operatorname{limsup}_{p=1} \frac{1}{\gamma_n} \log P(d_E(\tilde{Z}_n, Z_n) > \delta) = -\infty.$$ 

As far as the LDP is concerned exponentially equivalent measures are indistinguishable.

From now on, given $T > 0$, we will denote with $\mathcal{C}^p$ the set of $\mathbb{R}^p$-valued continuous functions on $[0, T]$, $\mathcal{C}$ if $p = 1$, endowed with the topology induced by the sup-norm $|| \cdot ||_{\infty}$, i.e., if $f = (f_1, \ldots, f_p)$ then

$$||f||_{\infty} = \sup_{0 \leq t \leq T} ||f(t)||,$$

where $|| \cdot ||$ is the euclidean norm in $\mathbb{R}^p$. We will denote with $\mathcal{C}^p_0$, $\mathcal{C}_0$ if $p = 1$, the subspace of $\mathbb{R}^p$-valued continuous functions on $[0, T]$ starting from zero. In what follows, we will always suppose our processes to be continuous.

Remark 2.3. We say that a family of continuous processes $((U^n(t))_{t \in [0, T]} )_{n \in \mathbb{N}}$, $U^n(0) = 0$ satisfies a LDP if the family of their laws satisfies a LDP on $\mathcal{C}^p$.

Let us conclude this section with some important definitions.

Definition 2.4. Let $(E, d_E)$ be a metric space (we consider on $E$ the Borel $\sigma$-algebra) and let $(Z^n)_{n \in \mathbb{N}}$ and $(\tilde{Z}^n)_{n \in \mathbb{N}}$ be two families of $E$-valued random variables. Then $(Z^n)_{n \in \mathbb{N}}$ and $(\tilde{Z}^n)_{n \in \mathbb{N}}$ are exponentially equivalent (at the speed $\gamma_n$) if for any $\delta > 0$,

$$\lim_{n \to +\infty} \operatorname{limsup}_{p=1} \frac{1}{\gamma_n} \log P(d_E(Z^n, \tilde{Z}^n) > \delta) = -\infty.$$ 

As far as the LDP is concerned exponentially equivalent measures are indistinguishable. See Theorem 4.2.13 in [21].

Let us give the definition of exponentially good approximation and some results about this topic. Here the main reference is [24].

Definition 2.5. Let $(E, d_E)$ be a metric space; consider the $E$-valued random variables $Z^n$ and $Z^{n,m}$. The families $(Z^{n,m})_{n \in \mathbb{N}}$ for $m \geq 1$ are called exponentially good approximations of $(Z^n)_{n \in \mathbb{N}}$ at the speed $\gamma_n$ if, for every $\delta > 0$,

$$\lim_{m \to +\infty} \lim_{n \to +\infty} \operatorname{limsup}_{p=1} \frac{1}{\gamma_n} \log P(d_E(Z^{m,m}, Z^n) > \delta) = -\infty.$$ 

Next theorem, Theorem 3.11 in [24], states that under a suitable condition if for each $m \geq 1$ the sequence $(Z^{n,m})_{n \in \mathbb{N}}$ satisfies a LDP with the rate function $I^m$, then also $(Z^n)_{n \in \mathbb{N}}$ satisfies a LDP with the rate function $I$, obtained in terms of the $I^m(\cdot)$'s.

Theorem 2.6. (Theorem 3.11 in [24]) Let $(Z^{n,m})_{n \in \mathbb{N}}$ for $m \geq 1$ be exponentially good approximations of $(Z^n)_{n \in \mathbb{N}}$ at the speed $\gamma_n$. Suppose that $(Z^{n,m})_{n \in \mathbb{N}}$ satisfies a LDP with the speed $\gamma_n$ and the good rate function $I^m$. Then $(Z^n)_{n \in \mathbb{N}}$ satisfies a LDP with the speed $\gamma_n$ and the good rate function $I$ given by

$$I(x) = \lim_{\delta \to 0} \lim_{m \to +\infty} \inf_{y \in B_\delta(x)} I^m(y) = \lim_{\delta \to 0} \lim_{m \to +\infty} \sup_{y \in B_\delta(x)} I^m(y),$$
with \( B_\delta(x) = \{ y \in E : d_E(x,y) < \delta \} \)

We also need the following proposition.

**Proposition 2.7.** (Proposition 3.16 in [24]) In the same hypotheses of Theorem 2.6, if

- \( I^m(x) \xrightarrow{m \to +\infty} J(x) \), for \( x \in E \);
- \( x_m \xrightarrow{m \to +\infty} x \) implies \( \liminf_{m \to +\infty} I^m(x_m) \geq J(x) \),

for some functional \( J(\cdot) \), then \( I(\cdot) = J(\cdot) \).

### 2.2. Multidimensional Volterra processes

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a probability space and \( B = (B(t))_{t \in [0,T]} \) a standard Brownian motion. Suppose \( \hat{B} = (\hat{B}(t))_{t \in [0,T]} \) is a centered (real) Gaussian process having the following Fredholm representation,

\[
\hat{B}(t) = \int_0^T K(t,s) \, dB(s), \quad 0 \leq t \leq T, \tag{3}
\]

where \( T > 0 \) and \( K \) is a measurable square integrable kernel on \([0,T]^2\) such that

\[
\sup_{t \in [0,T]} \int_0^T K(t,s)^2 \, ds < \infty.
\]

The modulus of continuity of the kernel \( K \) is defined as follows

\[
M(\delta) = \sup_{\{t_1, t_2 \in [0,T] : |t_1 - t_2| \leq \delta \}} \int_0^T |K(t_1,s) - K(t_2,s)|^2 \, ds, \quad 0 \leq \delta \leq T.
\]

The covariance function of the process \( \hat{B} \) is given by

\[
k(t,s) = \int_0^T K(t,u)K(s,u) \, du, \quad t, s \in [0,T].
\]

Let us define a Volterra process.

**Definition 2.8.** The process in (3) is called a Volterra type Gaussian process if the following conditions hold for the kernel \( K \):

(a) \( K(0,0) = 0 \) and \( K(t,s) = 0 \) for all \( 0 \leq t < s \leq T \);

(b) There exist constants \( c > 0 \) and \( \alpha > 0 \) such that \( M(\delta) \leq c \, \delta^\alpha \) for all \( \delta \in [0,T] \).

**Remark 2.9.** Condition (a) is a typical Volterra type condition for the kernel \( K \) and the integral representation in (3) becomes \( \hat{B}(t) = \int_0^t K(t,s) \, dB(s) \), for \( 0 \leq t \leq T \). So \( \hat{B} \) is adapted to the natural filtration generated by \( B \). Condition (b) guarantees the existence of a Hölder continuous modification of the process \( \hat{B} \).

In this section we introduce \( \mathbb{R}^p \)-valued Gaussian processes with independent components in which each component is a Volterra type Gaussian process.
Suppose that there exist kernels under suitable conditions on the covariance functions, a LDP for every $0 < i < p$, $K_i$ satisfies conditions (a), (b) in Definition 2.8.

Definition 2.10. The process $\hat{B} = (\hat{B}(t))_{t \in [0, T]}$ defined by:

$$\hat{B}(t) = (\hat{B}_0(t), ..., \hat{B}_p(t)) = \left( \int_0^T K_0(t, s) \, dB_0(s), ..., \int_0^T K_p(t, s) \, dB_p(s) \right)$$ (4)

for every $0 \leq t \leq T$, is a multidimensional Volterra type process if for every $1 \leq i \leq p$, $K_i$ satisfies conditions (a), (b) in Definition 2.8.

Remark 2.11. The process $\hat{B}$ defined in (4) is a centered Gaussian process with covariance function $k : [0, T] \times [0, T] \rightarrow \mathbb{R}^{p \times p}$ given by

$$k_{ij}(t, s) = \delta_{ij} \int_0^t K_i(t, u)K_j(s, u) \, du$$

for every $1 \leq i, j \leq p$ and $t, s \in [0, T]$. Moreover, $\hat{B}$ admits a continuous version with Hölder continuous sample paths of index $\gamma$ for every $\gamma < \alpha/2$.

Let $(\hat{B}_n)_{n \in \mathbb{N}}$ a family of processes such that, for every $n \in \mathbb{N}$, $(\hat{B}_n(t))_{t \in [0, T]}$ is a continuous Volterra type process of the form

$$\hat{B}_n(t) = (\hat{B}_0^n(t), ..., \hat{B}_p^n(t)) = \left( \int_0^T K_0^n(t, s) \, dB_0(s), ..., \int_0^T K_p^n(t, s) \, dB_p(s) \right)$$

with $K_i^n$ suitable kernels and covariance function

$$k_{ij}^n(t, s) = \delta_{ij} \int_0^t K_i^n(t, u)K_j^n(s, u) \, du \quad s, t \in [0, T], \quad i, j = 1, \ldots, p.$$ 

Suppose that there exist kernels $K_\ell$ such that, for every $\ell = 1, ..., p$,

$$\lim_{n \to +\infty} \frac{K_\ell^n(t, s)}{c_n} = K_\ell(t, s).$$ (5)

Under suitable conditions on the covariance functions, a LDP for $((c_n, B, \hat{B}_n))_{n \in \mathbb{N}}$ holds. More precisely, $((c_n, B, \hat{B}_n))_{n \in \mathbb{N}}$ satisfies a LDP on $(c_n^2)^2$ with the speed $c_n^2$ and the good rate function

$$I_{(B, \hat{B})}(f, g) = \begin{cases} \frac{1}{2} \int_0^T \|\hat{f}(s)\|^2 \, ds & (f, g) \in \mathcal{H}_{(B, \hat{B})}^p \\ +\infty & \text{otherwise} \end{cases}$$ (6)

where

$$\mathcal{H}_{(B, \hat{B})}^p = \{(f, g) \in (c_n^2)^2 : f \in H_{01}^{1, p}[0, T], g = \hat{f}\},$$ (7)

$$\hat{f} = (\hat{f}_1, ..., \hat{f}_p), \quad \hat{f}_\ell(t) = \int_0^t K_\ell(t, u)\hat{f}_\ell(u) \, du \quad t \in [0, T], \quad 1 \leq \ell \leq p,$$ (8)

$K_\ell$ is defined from Equation (5) and $H_{01}^{1, p}[0, T]$ is the Cameron-Martin space of the $p$-dimensional Brownian motion.

In this paper we are not interested to this problem and we assume that such LDP holds. For details about this LDP for Gaussian processes see [1, 25, 26].
**Assumption 2.12.** \( ((\varepsilon_n B^n)_{n \in \mathbb{N}}) \) satisfies a LDP on \( (\mathcal{C}_0^p)^2 \) with the speed \( \varepsilon_n^{-2} \) and the good rate function given by Equation (6).

**Example 2.13.** Let us consider two examples in which Assumption 2.12 is satisfied. We consider a multidimensional version of the models studied in [17].

Let us define the multidimensional log fractional Brownian motion as the process defined in (4) with

\[ K_n(t, s) = C(t - s)^{H - 1/2}(-\log (t - s))^{-a}, \quad \ell = 1, \ldots, p \]

for \( 0 \leq H \leq 1/2, \ a > 1 \) and \( C > 0 \) a positive constant.

For \( \eta_n \to 0 \) consider \( \hat{B}(t) = \hat{B}(\eta_n t) \). Then

\[ \hat{B}(\eta_n t) = \hat{B}(\eta_n t) \left( \int_0^t K^n(t, s) \ dB_1(s), \ldots, \int_0^t K^n(t, s) \ dB_p(s) \right), \]

where

\[ K^n(t, s) = \sqrt{\eta_n} K(\eta_n t, \eta_n s). \]

Then \( ((\varepsilon_n B^n)_{n \in \mathbb{N}}) \) satisfies a LDP on \( (\mathcal{C}_0^p)^2 \) with the speed \( \varepsilon_n^{-2} = \eta_n^{-2H}(-\log \eta_n)^{2p} \) and the good rate function defined in (5) with limit kernel that is the one of the Riemann-Liouville fractional Brownian motion.

In the same way let us define the multidimensional fractional Ornstein-Uhlenbeck process. Here,

\[ K_\ell(t, s) = K_{H\ell}(t, s) - a \int_s^t e^{-a(t-u)} K_{H\ell}(u, s) \ du, \quad \ell = 1, \ldots, p \]

for \( a > 0, \ H \in (0, 1) \) and \( K_{H\ell} \) the kernel of the fractional Brownian motion.

For \( \eta_n \to 0 \) consider \( \hat{B}(t) = \hat{B}(\eta_n t) \). Then \( ((\varepsilon_n B^n)_{n \in \mathbb{N}}) \) satisfies a LDP on \( (\mathcal{C}_0^p)^2 \) with the speed \( \varepsilon_n^{-2} = \eta_n^{-2H} \) and the good rate function defined in (5) with limit kernel that is the one of the fractional Brownian motion.

**Remark 2.14.** From Assumption 2.12 and the contraction principle, the family \( (\hat{B}^n)_{n \in \mathbb{N}} \) satisfies a LDP on \( \mathcal{C}_0^p \) with the speed \( \varepsilon_n^{-2} \) and the good rate function

\[ I_B(g) = \inf \left\{ \frac{1}{2} \int_0^T ||\hat{f}(s)||^2 \ ds : \hat{f} = g, \ f \in H_0^{1,p}[0, T] \right\}, \]

with the understanding \( I_B(g) = +\infty \) if the set is empty.

### 3. The uncorrelated model

The results of this section are an intermediate step in order to obtain the general case. Here it is assumed that the prices of the \( d \) assets are independent. The dynamic in the uncorrelated model of \( (S(t))_{t \in [0, T]} = (S_1(t), \ldots, S_d(t))_{t \in [0, T]} \) is the one in (1) with \( \sigma_{i\ell} = 0 \) for every \( 1 \leq i \leq d, \ 1 \leq \ell \leq p \), that is
for every $1 \leq i \leq d$, where, we recall, $s^0 = (s^0_1, ..., s^0_d) \in \mathbb{R}^d$ is the initial price, $T > 0$ is the time horizon and the process $\hat{B}$ is a non-degenerate continuous multidimensional Volterra process. The unique solution to the equation is the exponential

$$S_i(t) = s^0_i \exp \left\{ \int_0^t \left( \mu_i(\hat{B}(s)) - \frac{1}{2} \sum_{j=1}^d \sigma_{ij}(\hat{B}(s))^2 \right) ds + \int_0^t \sum_{j=1}^d \sigma_{ij}(\hat{B}(s)) dW_j(s) \right\}$$

for every $1 \leq i \leq d$ and $0 \leq t \leq T$ (for further details see Section IX-2 in [27]). Therefore, the log-price processes $X_i(t) = \log S_i(t)$, with $X_i(0) = x^0_i = \log s^0_i$ is

$$X_i(t) = x^0_i + \int_0^t \left( \mu_i(\hat{B}(s)) - \frac{1}{2} \sum_{j=1}^d \sigma_{ij}(\hat{B}(s))^2 \right) ds + \int_0^t \sum_{j=1}^d \sigma_{ij}(\hat{B}(s)) dW_j(s),$$

for every $1 \leq i \leq d$ and $0 \leq t \leq T$. For the sake of simplicity (it is not restrictive), from now on we assume that the initial conditions $s^0_i$ for the asset prices satisfy $s^0_i = 1$ and so $X^0_i = \log s^0_i = 0$ for every $1 \leq i \leq d$. Now, let $\varepsilon : \mathbb{N} \to \mathbb{R}_+$ be an infinitesimal function i.e., $\varepsilon_n \to 0$, as $n \to +\infty$. For every $n \in \mathbb{N}$, we will consider the following scaled version of the stochastic differential equations in (10)

$$\begin{cases}
    dS^n_i(t) = S^n_i(t)\mu_i(\hat{B}^n(t)) \, dt + \varepsilon_n S^n_i(t) \sum_{j=1}^d \sigma_{ij}(\hat{B}^n(t)) \, dW_j(t) & 0 \leq t \leq T \\
    S^n_i(0) = 1
\end{cases}$$

for every $1 \leq i \leq d$, where $(\hat{B}^n)_{n \in \mathbb{N}}$ is a family of multidimensional Volterra processes such that Assumption 2.12 is fulfilled. The log-price processes $X^n_i(t) = \log S^n_i(t)$ in the scaled model are given by

$$X^n_i(t) = \int_0^t \left( \mu_i(\hat{B}^n(s)) - \frac{1}{2} \varepsilon_n^2 \sum_{j=1}^d \sigma_{ij}(\hat{B}^n(s))^2 \right) ds + \varepsilon_n \int_0^t \sum_{j=1}^d \sigma_{ij}(\hat{B}^n(s)) dW_j(s)$$

for every $1 \leq i \leq d$ and $0 \leq t \leq T$.

We made the following assumptions on the coefficients.

**Assumption 3.1.** $\mu_i : \mathbb{R}^p \to \mathbb{R}$ and $\sigma_{ij} : \mathbb{R}^p \to \mathbb{R}$ are continuous functions, for every $1 \leq i, j \leq d$ and $\det(a(y)) \neq 0$ for every $y \in \mathbb{R}^p$ where $a = \sigma \sigma^T$.

Note that the hypotheses of continuity on the coefficients $\sigma_{ij}$ are quite mild. Similar hypotheses can be found in [10] and, for example, in the classical Bergomi model (see [28]), where the volatility is the exponential (i.e., a continuous function) of a Volterra process. The hypothesis on the determinant are the generalization to the multidimensional case of the request on the volatility map to be positive.

**Remark 3.2.** Under Assumption 3.1, there exists the inverse matrix $a^{-1}(y)$ for every $y \in \mathbb{R}^p$ and $a^{-1}_{ij}$ are continuous functions for every $i, j = 1, ..., d$. Furthermore the matrix $a$
is uniformly strictly positive definite on compact sets, i.e., for every unit vector \( x \in \mathbb{R}^d \) and \( K \subset \mathbb{R}^p \) compact there exists \( \alpha_K > 0 \) such that \( \inf_{y \in K} x^T a(y) x \geq \alpha_K > 0 \), where \( \alpha_K = \inf_{y \in K} \lambda_{\min}(y) \), being \( \lambda_{\min}(y) \) the minimum eigenvalue of \( a(y) \). Therefore the matrix \( a(y) - \alpha_K I \) is uniformly positive definite on \( K \). And in a similar way we have that the matrix \( \beta_K I - a(y) \), where \( \beta_K = \sup_{y \in K} \lambda_{\max}(y) \), being \( \lambda_{\max}(y) \) the maximum eigenvalue of \( a(y) \), is uniformly positive definite on \( K \). Note that under Assumption 3.1 we have the same properties for the matrix \( a^{-1} \).

The main result of this section is the following sample path LDP for the process \((X^n)_{n \in \mathbb{N}}\).

**Theorem 3.3.** Under Assumptions 3.1 and 2.12 a LDP with the speed \( \varepsilon_n^2 \) and the good rate function \( I_X(\cdot) \) defined in (15) holds for the family \((X^n)_{n \in \mathbb{N}}\) of processes defined in (11).

In order to prove this theorem we proceed as in [1]. Checking that hypotheses of Theorem 3.3 (Chaganty’s Theorem) in [1]) are fulfilled we obtain a WLDP for the family \((\hat{B}^n, X^n)_{n \in \mathbb{N}}\) with a rate function \( I(\cdot \ | \ \cdot) \); then we prove that \( I(\cdot \ | \ \cdot) \) is a good rate function and therefore (by Chaganty main result) the family \((X^n)_{n \in \mathbb{N}}\) satisfies a (full) LDP. Assumption 2.12 ensures that condition (i) in Theorem 3.3 is fulfilled more precisely, the family \((\hat{B}^n)_{n \in \mathbb{N}}\) satisfies a LDP on \( \mathcal{G}_0^p \) with the speed \( \varepsilon_n^2 \) and the good rate function \( I_{\hat{B}}(\cdot) \) given by (9). So we have to prove only condition (ii) in Theorem 3.3, that is the LDP continuity condition (see Definition 3.2 in [1]). For the multidimensional case we need some technical results on positive definite matrices which we postpone in the appendix.

We want to investigate the behavior of \( X^n \), when conditioned to the process \( \hat{B}^n \). More precisely, we want to establish a LDP for the family of the conditional processes \( X^n,\varphi = X^n | (\hat{B}^n(t) = \varphi(t) \quad 0 \leq t \leq T) \) as \( n \to +\infty \). For (almost) every \( \varphi \in \mathcal{G}_0^d \), we have (in law)

\[
X^n,\varphi(t) = \int_0^t \left( \mu_i(\varphi(s)) - \frac{1}{2} \varepsilon_n^2 \sum_{j=1}^d \sigma_{ij}(\varphi(s))^2 \right) ds + \varepsilon_n \int_0^t \sum_{j=1}^d \sigma_{ij}(\varphi(s)) \ dW_j(s) \tag{12}
\]

for every \( 1 \leq i \leq d \) and \( 0 \leq t \leq T \). It is enough to show that the conditions \( (a), (b) \) and \( (c) \) of LDP continuity condition are satisfied (condition \( (ii) \) of Chaganty’s Theorem). The proofs of the next results are quite similar to the proofs of Proposition 5.4, Proposition 5.6 and Lemma 5.7 in [1]. Anyway there are many technical issues related to the multidimensional case therefore we give here all the details. Let us define the following functional \( \Gamma \) that will be useful in the sequel.

For \( A \in \mathcal{G}^{d \times d} \) symmetric and positive definite (i.e., \( A(t) \) is a symmetric and positive definite matrix for every \( t \in [0, T] \)) and \( x \in \mathcal{G}^d \) define the functional

\[
\Gamma(x|A) = \begin{cases} 
\frac{1}{2} \int_0^T \dot{x}(t)^T A(t) \dot{x}(t) \ dt & x \in H_0^{1,d}[0,T] \\
+\infty & \text{otherwise}
\end{cases}
\]
Remark 3.4. For $x, y, z \in H_0^{1,d}[0, T]$ and $A, B \in \mathcal{G}^{d \times d}$ we have,

(i) $\Gamma(x|B) \geq \Gamma(x|A)$ if $B - A$ is positive definite;
(ii) $\Gamma(x + y|A) \leq 2\Gamma(x|A) + 2\Gamma(y|A)$;
(iii) $\Gamma(x + y + z|A) \leq 3\Gamma(x|A) + 3\Gamma(y|A) + 3\Gamma(z|A)$.

Proposition 3.5. Under Assumptions 3.1 the sequence of the conditional processes $(X^n_{\cdot, \varphi})_{n \in \mathbb{N}}$ satisfies the LDP continuity condition with the speed $\varepsilon_n^{-2}$ and the good rate function

$$J(x|\varphi) = \Gamma\left( x - \int_0^T \mu(\varphi(t)) \ dt | a^{-1}(\varphi) \right) = \begin{cases} \frac{1}{2} \int_0^T (\dot{x}(t) - \mu(\varphi(t)))^T a^{-1}(\varphi(t)) (\dot{x}(t) - \mu(\varphi(t))) \ dt & x \in H_0^{1,d}[0, T] \\ +\infty & \text{otherwise.} \end{cases}$$

(13)

Proof. (a) For every $\varphi \in \mathcal{G}_0^d$ we prove that $(X^n_{\cdot, \varphi})_{n \in \mathbb{N}}$ obeys a LDP by using the multidimensional version of Theorem 2.14 in [1] that still holds as a simple consequence of Theorem 3.1 in [14]. For every $n \in \mathbb{N}$ and $t \in [0, T]$, the coefficients are:

- $b_n(t) = \mu(\varphi(t)) - \frac{1}{2} \varepsilon_n^2 \text{Diag}(a(\varphi(t)))$ (with $\text{Diag}(C)$ we mean the diagonal of the matrix $C$) and so $b_n \to \mu(\varphi)$ in $\mathcal{G}^d$, as $n \to +\infty$.
- $\sigma_n(t) = \sigma(\varphi(t))$, for every $t \in [0, T]$, not depending on $n$.

Then the family $(X^n_{\cdot, \varphi})_{n \in \mathbb{N}}$ satisfies a LDP with the speed $\varepsilon_n^{-2}$ and the good rate function

$$J(x|\varphi) = \inf \left\{ \frac{1}{2} \int_0^T |\dot{y}(t)|^2 \ dt : \int_0^T \mu(\varphi(s)) \ ds + \int_0^T \sigma(\varphi(s)) \dot{y}(s) \ ds = x(t), \ y \in H_0^{1,d}[0, T] \right\}$$

with the usual understanding $J(x|\varphi) = +\infty$ if the set is empty. If $y \in H_0^{1,d}[0, T]$ then

$$\dot{x}(t) = \mu(\varphi(t)) + \sigma(\varphi(t)) \dot{y}(t) \quad \text{a.e., with} \ x(0) = 0.$$

Thanks to Remark A.1 (ii) and Assumption 3.1 we have that $\det(\sigma^{\varphi}) \neq 0$ and the rate function above simplifies to $J(\cdot|\varphi)$ defined in (13).

(b) Let $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{G}_0^d$ such that $\varphi_n \to \varphi$ in $\mathcal{G}_0^d$, as $n \to +\infty$. For every $n \in \mathbb{N}$ we consider the law of the $\mathbb{R}^d$-valued Gaussian diffusion process $(X^n_{t, \varphi_n(t)})_{t \in [0, T]}$ defined by (12), where the function $\varphi$ is replaced by $\varphi_n$. The coefficients now are:

- $b_n(t) = \mu(\varphi_n(t)) - \frac{1}{2} \varepsilon_n^2 \text{Diag}(a(\varphi_n(t)))$, for every $t \in [0, T]$;
- $\sigma_n(t) = \sigma(\varphi_n(t))$, for every $t \in [0, T]$.

From Remark A.2, $a_{ii}(\varphi_n) \to a_{ii}(\varphi)$ in $\mathcal{G}$, as $n \to +\infty$, for every $1 \leq i \leq d$. Therefore $b_n(t) \to \mu(\varphi(t))$ and $\sigma_n(t) \to \sigma(\varphi(t))$ so $b_n \to \mu(\varphi)$ in $\mathcal{G}^d$ and $\sigma_n \to \sigma(\varphi)$ in
$\mathcal{C}^{d \times d}$, as $n \to +\infty$. Thus $(X^n, \varphi_n)_{n \in \mathbb{N}}$ obeys a LDP with the speed $\varepsilon_n^{-2}$ and the good rate function $J(\cdot | \varphi)$ defined in (13).

(c) Now, we will check that $J(\cdot | \varphi)$ is lower semi-continuous as a function of $(\varphi, x) \in \mathcal{C}_0^P \times \mathcal{C}_0^d$. Let $(\varphi_n, x_n)_{n \in \mathbb{N}} \subset \mathcal{C}_0^P \times \mathcal{C}_0^d$ and $(\varphi, x) \in \mathcal{C}_0^P \times \mathcal{C}_0^d$ be functions such that $(\varphi_n, x_n) \to (\varphi, x)$ in $\mathcal{C}_0^P \times \mathcal{C}_0^d$, as $n \to +\infty$. If

$$\liminf_{n \to +\infty} J(x_n | \varphi_n) = \lim_{n \to +\infty} J(x_n | \varphi_n) = +\infty,$$

there is nothing to prove. Therefore, up to a subsequence, we can suppose that $(x_n)_{n \in \mathbb{N}} \subset H_0^{1,d}[0, T]$. Now, from Lemma A.4 and the definition of $\Gamma$, we have that for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \geq n_\varepsilon$

$$J(x_n | \varphi_n) = \Gamma\left(x_n - \int_0^1 \mu(\varphi_n(t)) \ dt \alpha^{-1}(\varphi_n)\right) \geq (1 - \varepsilon) \Gamma\left(x_n - \int_0^1 \mu(\varphi_n(t)) \ dt \alpha^{-1}(\varphi)\right)$$

$$= (1 - \varepsilon) J\left(x_n - \int_0^1 \mu(\varphi_n(t)) \ dt + \int_0^1 \mu(\varphi(t)) \ dt | \varphi\right).$$

Moreover thanks to Remark A.2, we have

$$x_n - \int_0^1 \mu(\varphi_n(t)) \ dt + \int_0^1 \mu(\varphi(t)) \ dt \to x$$

in $\mathcal{C}_0^d$, as $n \to +\infty$. The functional $J(\cdot | \varphi)$, being a rate function, is lower semi-continuous on $\mathcal{C}_0^d$, therefore

$$\liminf_{n \to +\infty} J(x_n | \varphi_n) \geq (1 - \varepsilon) \liminf_{n \to +\infty} J\left(x_n - \int_0^1 \mu(\varphi_n(t)) \ dt + \int_0^1 \mu(\varphi(t)) \ dt | \varphi\right)$$

$$\geq (1 - \varepsilon) J(x | \varphi)$$

for every $\varepsilon > 0$ and thus we can conclude. □

**Proposition 3.6.** Suppose that Assumption 2.12 holds and that $\sigma$ and $\mu$ satisfy Assumption 3.1. Then $(\tilde{B}^n, X^n)_{n \in \mathbb{N}}$ satisfies the WLDP with the rate function

$$I(\varphi, x) = I_B(\varphi) + J(x | \varphi)$$

for $(\varphi, x) \in \mathcal{C}_0^P \times \mathcal{C}_0^d$, where $I_B(\cdot)$ and $J(\cdot | \cdot)$ are defined in (9) and (13), respectively. Moreover, $(X^n)_{n \in \mathbb{N}}$ satisfies the LDP with the speed $\varepsilon_n^{-2}$ and the rate function

$$I_X(x) = \begin{cases} \inf_{f \in H_0^{1,p}[0, T]} \left\{ \frac{1}{2} ||f||_{H_0^{1,p}[0, T]}^2 + J(x | f) \right\} & x \in H_0^{1,d}[0, T] \\ +\infty & \text{otherwise} \end{cases}$$

where $\hat{f}$ is defined in (8).

**Proof.** Thanks to Remark 2.14, the family $(\tilde{B}^n)_n$ satisfies a LDP on $\mathcal{C}_0^P$ with the speed $\varepsilon_n^{-2}$ and the good rate function

$$I_B(g) = \inf\left\{ \frac{1}{2} \int_0^T ||\hat{f}(s)||^2 \ ds : \hat{f} = g, \ f \in H_0^{1,p}[0, T] \right\},$$
with the understanding \( I_B(g) = +\infty \) if the set is empty. Thanks to Proposition 3.5, \((X_n, \theta)\) where
\[
X_n, \theta = X^n(B^n(t) = \theta(t) \quad 0 \leq t \leq T)
\]
satisfies the LDP continuity condition. Therefore, we have that the family \((\hat{B}^n.X^n)_{n\in\mathbb{N}}\) satisfies the hypotheses of Chaganty’s Theorem and then enjoys the WLD with the rate function given by (14). Always by Chaganty’s Theorem the family of processes \((X^n)_{n\in\mathbb{N}}\) satisfies a LDP with the speed \(e_n^{-2}\) and the rate function
\[
IX(x) = \inf_{\phi \in \theta_0^d} I(\phi, x) = \inf_{\phi \in \theta_0^d} (I_B(\phi) + J(x|\phi)).
\]
The first term in the sum is infinite if does not exist \(f \in H_0^{1,p}[0, T]\) such that \(\phi = \hat{f}\). Therefore, we have that \(I_X(\cdot, \cdot)\) is given by Equation (15).

We proved that the family of log-price processes \((X^n)_{n\in\mathbb{N}}\) satisfies a LDP with the speed \(e_n^{-2}\) and the rate function \(I_X(\cdot, \cdot)\). The proof that \(I_X(\cdot, \cdot)\) is a good rate function follows from Lemma 2.6 in [16].

**Proposition 3.7.** The rate function \(I_X(\cdot, \cdot)\) is a good rate function.

**Proof.** It is enough to show (see also [1] for the details), that
\[
\bigcup_{\phi \in K_1} \{x \in C_0^d : J(x|\phi) \leq L\}
\]
is a compact subset of \(C_0^d\) for any \(L \geq 0\) and for any compact set \(K_1 \subset C_0^p\). Let \(K_1\) be a compact subset of \(C_0^p\); for \(\phi \in K_1\) define
\[
A_{\phi}^L = \{x \in C_0^d : J(x|\phi) \leq L\} = \{x \in H_0^{1,d}[0, T] : J(x|\phi) \leq L\}.
\]
\(A_{\phi}^L\) is a compact subset of \(C_0^d\). We want to show that every sequence in \(\bigcup_{\phi \in K_1} A_{\phi}^L\) has a convergent subsequence. Let \((x_n)_{n\in\mathbb{N}} \subset \bigcup_{\phi \in K_1} A_{\phi}^L\), then, for every \(n \in \mathbb{N}\), there exists \(\phi_n \in K_1\) such that \(x_n \in A_{\phi_n}^L\) (i.e., \(J(x_n|\phi_n) \leq L\)). Since \((\phi_n)_{n\in\mathbb{N}} \subset K_1\), up to a subsequence, we can suppose that \(\phi_n \to \phi\) in \(C_0^p\), as \(n \to +\infty\), with \(\phi \in K_1\). Now, we claim that there exists a constant \(N > 0\) such that, for every \(n \in \mathbb{N}\), \(J(x_n|\phi) \leq N\). Note that,
\[
J(x_n|\phi) = \Gamma \left( x_n - \int_0^T \mu(\phi(t)) \ dt | a^{-1}(\phi) \right)
\]
\[
= \begin{cases} 
\frac{1}{2} \int_0^T (\dot{x}_n(t) - \mu(\phi(t)))^T a^{-1}(\phi(t))(\dot{x}_n(t) - \mu(\phi(t))) \ dt & x \in H_0^{1,d}[0, T] \\
+\infty & \text{otherwise.} 
\end{cases}
\]
Therefore, adding and substracting the term \(\int_0^T \mu(\phi_n(t))dt\) and taking into account Remark 3.4(ii), for every \(n \in \mathbb{N}\), we have,
\[ J(x_n|\varphi) = \Gamma\left(x_n - \int_0 \mu(\varphi(t))dt|a^{-1}(\varphi)\right) = \Gamma\left(x_n - \int_0 \mu(\varphi_n(t))dt + \int_0 \mu(\varphi_n(t))dt|a^{-1}(\varphi)\right) \]

\[ - \int_0 \mu(\varphi(t))dt|a^{-1}(\varphi)) \]

\[ \leq 2\Gamma\left(x_n - \int_0 \mu(\varphi_n(t))dt|a^{-1}(\varphi)\right) + 2\Gamma\left(\int_0 (\mu(\varphi_n(t)) - \mu(\varphi(t)))dt|a^{-1}(\varphi)\right). \]

Thanks to Lemma A.5, there exists a constant \( M > 1 \) such that, for every \( n \in \mathbb{N} \), the matrix \( Ma^{-1}(\varphi_n) - a^{-1}(\varphi) \) is positive definite; thanks to Remark 3.2, there exists a constant \( \beta_\varphi > 0 \) such that \( \beta_\varphi I - a^{-1}(\varphi) \) is positive definite, therefore by using Remark 3.4(i) we have

\[ J(x_n|\varphi) \leq 2MI(x_n|\varphi_n) + \beta_\varphi \int_0^T (\mu(\varphi_n(t)) - \mu(\varphi(t)))^T (\mu(\varphi_n(t)) - \mu(\varphi(t)))dt \leq N, \]

since \( J(x_n|\varphi_n) \leq L \) and the integral is bounded. Then, since \( A^N_\varphi = \{x \in \mathcal{C}_0^d : J(x|\varphi) \leq N\} \) is a compact subset of \( \mathcal{C}_0^d \), up to a subsequence, we can suppose that \( x_n \to x \in A^N_\varphi \) in \( \mathcal{C}_0^d \), as \( n \to +\infty \). Moreover, from the semicontinuity of \( J(\cdot|\cdot) \), we have

\[ J(x|\varphi) \leq \liminf_{n \to +\infty} J(x_n|\varphi_n) \leq L, \]

that implies \( x \in A^L_\varphi \).

\[ \square \]

4. The correlated model

The dynamic of the asset price process \((S(t))_{t \in [0,T]} = (S_1(t), ..., S_d(t))_{t \in [0,T]}\) is modeled by (1).

Let \( Z_i(t) = \log S_i(t) \) be the \( i \)-th log-price process, then we have

\[ Z_i(t) = X_i(t) - \frac{1}{2} \int_0^t \sum_{\ell=1}^p \tilde{\sigma}_{i\ell}(\hat{B}(s))^2 ds + \int_0^t \sum_{\ell=1}^p \tilde{\sigma}_{i\ell}(\hat{B}(s)) dB_\ell(s) \quad (16) \]

for every \( 1 \leq i \leq d \) and \( 0 \leq t \leq T \), where, \((X(t))_{t \in [0,T]}\) is the log-price process defined in the previous section. Now, consider the following scaled version of the stochastic differential equation in (1)

\[
\begin{cases}
\frac{dS^n_i(t)}{S^n_i(t)} = \mu_i(\hat{B}^n(t)) \ dt + \tilde{e}_n \left( \sum_{\ell=1}^p \tilde{\sigma}_{i\ell}(\hat{B}^n(t)) \right) dB_\ell(t) + \sum_{j=1}^d \sigma_{ij}(\hat{B}^n(t)) \ dW_j(t) \\
S^n_i(0) = 1
\end{cases}
\]

for every \( 1 \leq i \leq d \) and \( 0 \leq t \leq T \), where \((\hat{B}^n_{i})_{n \in \mathbb{N}}\) is a family of multidimensional Volterra processes satisfying Assumption 2.12. Then, the \( i \)-th log-price process in the scaled model is

\[ Z^n_i(t) = X^n_i(t) - \frac{1}{2} \tilde{e}_n^2 \int_0^t \sum_{\ell=1}^p \tilde{\sigma}_{i\ell}(\hat{B}^n(s))^2 \ ds + \tilde{e}_n \int_0^t \sum_{\ell=1}^p \tilde{\sigma}_{i\ell}(\hat{B}^n(s)) dB_\ell(s) \quad (17) \]
where, for every $n \in \mathbb{N}$, $(X^n(t))_{t \in [0,T]}$ is the scaled log-price process defined in the previous section.

In what follows, we will want to prove a sample path LDP for the family of processes $(Z^n)_{n \in \mathbb{N}}$, defined in (17). The study of the correlated model is more complicated than the previous one. In fact, in this case, we should also study the behavior of the family of processes

$$
\left( \left( V^n_i(t) - \frac{1}{2} \varepsilon_n^2 \int_0^T \sum_{\ell=1}^p \tilde{\sigma}_{i\ell}(\tilde{B}^n(s))^2 \, ds \right)_{t \in [0,T]} \right)_{n \in \mathbb{N}},
$$

where

$$V^n_i(t) = \varepsilon_n \int_0^t \sum_{\ell=1}^p \tilde{\sigma}_{i\ell}(\tilde{B}^n(s)) \, dB_t(s)
$$

for every $1 \leq i \leq d$, $0 \leq t \leq T$. Notice that this process depends on the couple $(\varepsilon_n B, \tilde{B}^n)$, but we can’t directly apply Chaganty’s Theorem to the family

$$(\varepsilon_n B, \tilde{B}^n, Z^n)_{n \in \mathbb{N}}$$

since $V^n$ cannot be written as a continuous function of $(\varepsilon_n B, \tilde{B}^n)$ and so the LDP continuity condition is not fulfilled. To overcome this problem, as in [1], we introduce a new family of processes $(Z^{n,m})_{n \in \mathbb{N}}$, where for every $m \geq 1$, $V^n$ is replaced by a suitable continuous function of $(\varepsilon_n B, \tilde{B}^n)$. Thanks to the results obtained in the previous section, we prove that the hypotheses of Chaganty’s Theorem are fulfilled for the family $((\varepsilon_n B, \tilde{B}^n), Z^{n,m})_{n \in \mathbb{N}}$. Then, for every $m \geq 1$, $(Z^{n,m})_{n \in \mathbb{N}}$ satisfies a LDP with a certain good rate function $I^m$ (Section 4.1). Then, proving that the family $((Z^{n,m})_{n \in \mathbb{N}})_{m \geq 1}$ is an exponentially good approximation (see Definition 2.5) of $(Z^n)_{n \in \mathbb{N}}$, we obtain a LDP for the family $(Z^n)_{n \in \mathbb{N}}$ with the good rate function obtained in terms of the $I^m$’s (Section 4.2). Finally (in Section 4.3) we give an explicit expression for the rate function (not in terms of the $I^m$’s).

Suppose $\sigma \in \mathcal{C}^{d \times d}$ and $\mu \in \mathcal{C}^d$ satisfy Assumption 3.1. In this section, we need some more hypotheses on the coefficients.

**Assumption 4.1.** $\tilde{\sigma}_{i\ell} : \mathbb{R}^p \to \mathbb{R}$ are locally $\omega$-continuous functions (see Definition 6.1 in [1]) for $1 \leq i \leq d$ and $1 \leq \ell \leq p$.

**Assumption 4.2.** There exist constants $\alpha, M_1, M_2 > 0$, such that for every $i,j = 1, \ldots, d$, $1 \leq \ell \leq p$,

$$|\tilde{\sigma}_{i\ell}(x)| + |\sigma_{ij}(x)| + |\mu_i(x)| \leq M_1 + M_2 \|x\|^2, \quad x \in \mathbb{R}^p.$$

**Remark 4.3.** Under Assumption 4.2 if $\lambda_{\text{max}}(x)$ is the maximum eigenvalue of the matrix $a(x)$, then there exists a constant $M > 0$ such that $\lambda_{\text{max}}(x) \leq M \|x\|^{2\alpha}$ and therefore $\frac{1}{\lambda_{\text{max}}(x)} \geq \frac{1}{M \|x\|^{2\alpha}}$. Notice that $\frac{1}{\lambda_{\text{max}}(x)}$ is the minimum eigenvalue of the matrix $a^{-1}(x)$.

The main result of this section is the following theorem. Also in this section we proceed as in [1].
Theorem 4.4. Suppose that Assumptions 2.12, 3.1, 4.1 and 4.2 are fulfilled. Then, the family of processes \( (Z^n)_{n \in \mathbb{N}} \) satisfies a LDP with the speed \( e_n^{-2} \) and the good rate function
\[
\mathcal{I}_Z(x) = \begin{cases} 
\inf_{f \in H_0^{1,p}[0,T]} \mathcal{I}((f, \hat{f}), x) & x \in H_0^{1,d}[0,T] \\
+\infty & \text{otherwise}
\end{cases}
\]

where for every \( f \in H_0^{1,p}[0,T] \),
\[
\mathcal{I}((f, \hat{f}), x) = \frac{1}{2}||f||_{H_0^{1,p}[0,T]}^2 + J(x - \Phi(f, \hat{f}))
\]

where \( J( \cdot | \cdot ) \) is defined in (13) and \( \Phi \) is defined in (25).

4.1. LDP for the approximating families

In this section we suppose that Assumptions 3.1 and 4.1 are fulfilled. First we define the analogous of the function \( W_m \) in Equation (22) in [1].

For every \( m \geq 1 \), let us define the function \( U_m : \mathcal{C}_p \times \mathcal{C}_p \to \mathcal{C}_d \) as follows: for \( (f, g) \in \mathcal{C}_p \times \mathcal{C}_p \), \( t \in [0, T] \) and \( i = 1, \ldots, d \),
\[
\Phi^m_i(f, g)(t) = \sum_{\ell=1}^p \Psi^m_{i\ell}(f, g)(t),
\]
where
\[
\Psi^m_{i\ell}(f, g)(t) = \sum_{k=0}^{\left\lfloor \frac{m}{T} \right\rfloor - 1} \hat{\sigma}_{i\ell} \left( g \left( \frac{k + 1}{m} T \right) \right) \left[ f_{\ell} \left( \frac{k + 1}{m} T \right) - f_{\ell} \left( \frac{k}{m} T \right) \right] + \hat{\sigma}_{i\ell} \left( g \left( \left\lfloor \frac{mt}{T} \right\rfloor \frac{T}{m} \right) \right) \left[ f_{\ell}(t) - f_{\ell} \left( \left\lfloor \frac{mt}{T} \right\rfloor \frac{T}{m} \right) \right]
\]
and \( \lfloor \cdot \rfloor \) is the usual floor function. Let us collect some properties of the function \( \Phi^m(\cdot, \cdot) \) in the following remark.

Remark 4.5. It is clear that \( \Phi^m(\cdot, \cdot) \) is a continuous function on \( \mathcal{C}_p \times \mathcal{C}_p \) (where we are using the sup norm topology for both arguments).

Furthermore for every \( (f, g) \in H_0^{1,p}[0,T] \times \mathcal{C}_p \), the function \( \Phi^m(\cdot, \cdot) \) can be written in the following way:
\[
\Phi^m_i(f, g)(t) = \sum_{\ell=1}^p \int_0^T \hat{\sigma}_{i\ell} \left( g \left( \left\lfloor \frac{ms}{T} \right\rfloor \frac{T}{m} \right) \right) \dot{f}_{\ell}(s) \, ds,
\]
for every \( 1 \leq i \leq d \) and \( 0 \leq t \leq T \). Then thanks to the Cauchy-Schwarz inequality we have that for every \( f \in H_0^{1,p}[0,T] \), there exists \( M > 0 \), depending on \( f \), such that,
\[
\int_0^T ||\Phi^m(f, \hat{f})(t)||^2 dt \leq M ||f||_{H_0^{1,p}[0,T]}^2.
\]
For every \( m \geq 1 \), let us define the new family of processes \( \{(Z^{n,m}(t))_{t \in [0,T]}\}_{n \in \mathbb{N}} \), where

\[
Z^{n,m}(t) = \int_0^t \left( \mu_i(\hat{B}^n(s)) - \frac{1}{2} \sigma^2 \sum_{j=1}^d \sigma_{ij}(\hat{B}^n(s))^2 - \frac{1}{2} \epsilon_n^2 \sum_{t=1}^p \tilde{\sigma}_{it}(\hat{B}^n(s))^2 \right) ds \\
+ \Phi^m_i(\varepsilon_n B, \hat{B}^n)(t) + \varepsilon_n \int_0^t \sum_{j=1}^d \sigma_{ij}(\hat{B}^n(s)) dW_j(s),
\]

where \( \Phi^m_i(\cdot, \cdot) \) is defined in (20), for every \( 1 \leq i \leq d \) and \( 0 \leq t \leq T \). For every \( m \geq 1 \), we will prove a LDP for the family of processes \( \{(Z^{n,m})_{n \in \mathbb{N}}\} \). More precisely we prove the following theorem.

**Theorem 4.6.** Suppose that Assumptions 3.1 and 4.1 are fulfilled. For every \( m \geq 1 \), a LDP with the speed \( \varepsilon_n^{-2} \) and the good rate function \( I^m_\alpha(\cdot) \) given by (24) holds for the family \( \{Z^{n,m}\}_{n \in \mathbb{N}} \).

For this purpose we will check that hypotheses of Chaganty’s Theorem hold for the family of processes \( \{(\varepsilon_n B, \hat{B}^n), (Z^{n,m})\}_{n \in \mathbb{N}} \). By Assumption 2.12, we already know that \( \{(\varepsilon_n B, \hat{B}^n)\}_{n \in \mathbb{N}} \) satisfies a LDP with the speed \( \varepsilon_n^{-2} \) and the good rate function \( I_{B, \hat{B}}(\cdot, \cdot) \) given by (6). Fix \( m \geq 1 \) and \((f, g) \in \mathcal{C}_0^p \times \mathcal{C}_0^p \). Our next goal is to prove that the family of conditional processes

\[
Z^{n,m, (f,g)} = Z^{n,m}|(\varepsilon_n B(t) = f(t), \hat{B}^n(t) = g(t) \quad 0 \leq t \leq T)
\]
satisfies the LDP continuity condition (condition (ii) of Chaganty’s Theorem).

For every \((f, g) \in \mathcal{C}_0^p \times \mathcal{C}_0^p \) and \( t \in [0,T] \), in law, for every \( 1 \leq i \leq d \), we have

\[
Z^{n,m, (f,g)}(t) = \tilde{X}^{n,g}(t) + \Phi^m_i(f, g)(t)
\]
with \( \tilde{X}^{n,g} \) defined as

\[
\tilde{X}^{n,g}_i(t) = X^{n,g}_i(t) - \frac{1}{2} \epsilon_n^2 \int_0^t \sum_{t=1}^p \tilde{\sigma}_{it}(g(s))^2 ds
\]
and \( X^{n,g} \) is the conditional log price process in the uncorrelated model defined in (12).

The following proposition proves the LDP continuity condition for the family \( \{Z^{n,m, (f,g)}\}_{n \in \mathbb{N}} \).

**Proposition 4.7.** Fix \((f, g) \in \mathcal{C}_0^p \times \mathcal{C}_0^p \). Then, for every \( m \geq 1 \), the sequence of the conditional processes \( \{Z^{n,m, (f,g)}\}_{n \in \mathbb{N}} \) satisfies the LDP continuity condition with the speed \( \varepsilon_n^{-2} \) and the rate function

\[
\mathcal{J}^m(x|(f,g)) = J(x - \Phi^m(f,g)|g)
\]
where \( J(\cdot | g) \) is given by (13). Notice that \( \mathcal{J}^m(x|(f,g)) \) is finite if and only if \( x - \Phi^m(f,g) \in H^p_0, a[0,T] \).

**Proof.**

(a) Fix \((f, g) \in \mathcal{C}_0^p \times \mathcal{C}_0^p \). We have to prove that the sequence of the conditional processes \( \{Z^{n,m, (f,g)}\}_{n \in \mathbb{N}} \) satisfies the LDP continuity condition with the speed
and this completes the proof that, for every $g \in \mathcal{C}_0^p$ the family of processes $(X^{n,g})_{n \in \mathbb{N}}$ satisfies the same LDP as $(X^{n,g})_{n \in \mathbb{N}}$ then thanks to Equation (21) this is a simple applications of the contraction principle.

(b) Let $((f_n,g_n))_{n \in \mathbb{N}} \subset \mathcal{C}_0^p \times \mathcal{C}_0^p$ be such that $(f_n,g_n) \to (f,g)$ in $\mathcal{C}_0^p \times \mathcal{C}_0^p$, as $n \to +\infty$. We have to prove that the family of processes $(Z^{n,m}(f_n,g_n))_{n \in \mathbb{N}}$, for every $m \geq 1$, satisfies a LDP with the speed $\varepsilon_n^{-2}$ and the good rate function $\mathcal{J}^m(\cdot | (f,g))$ defined in (23).

If $g_n \to g$ in $\mathcal{C}_0^p$, as $n \to +\infty$, immediately follows that the family of processes $((X^{n,g_n})_{n \in \mathbb{N}}$ satisfies a LDP with the speed $\varepsilon_n^{-2}$ and the good rate function $J(\cdot | g)$. Combining this with contraction principle, we have that, for every $m \geq 1$, the family of processes

$$\big(\bar{X}^{n,g_n} + \Phi^m(f,g)\big)_{n \in \mathbb{N}}$$

satisfies a LDP with the speed $\varepsilon_n^{-2}$ and the good rate function $\mathcal{J}^m(\cdot | (f,g))$, defined in (23). Furthermore, for every $m \geq 1$, $\Phi^m(f_n,g_n) \to \Phi^m(f,g)$ in $(\mathcal{C}_0^p)^2$, as $n \to +\infty$, since $\Phi^m$ is a continuous function. Therefore, the families $(\bar{X}^{n,g_n} + \Phi^m(f_n,g_n))_{n \in \mathbb{N}}$ and $(\bar{X}^{n,g_n} + \Phi^m(f,g))_{n \in \mathbb{N}}$ are exponentially equivalent (see Definition 4.2.10 in [21]) and the statement is proved.

(c) We have to prove the lower semi-continuity of $\mathcal{J}^m(\cdot | (\cdot,\cdot))$ as a function of $(x, (f,g)) \in \mathcal{C}_0^d \times (\mathcal{C}_0^p \times \mathcal{C}_0^p)$. That is if the sequence $((f_n,g_n), x_n)_{n \in \mathbb{N}}$ is such that

$$((f_n,g_n), x_n) \to ((f,g), x)$$
in $\mathcal{C}_0^p \times \mathcal{C}_0^p \times \mathcal{C}_0^d$, as $n \to +\infty$, then, for every $m \geq 1$,

$$\liminf_{n \to +\infty} \mathcal{J}^m(x_n | (f_n,g_n)) \geq \mathcal{J}^m(x | (f,g)).$$

For every $m \geq 1$, $\Phi^m(\cdot,\cdot)$ is continuous on $\mathcal{C}_0^p \times \mathcal{C}_0^p$, therefore if $((f_n,g_n), x_n) \to ((f,g), x)$ in $(\mathcal{C}_0^p \times \mathcal{C}_0^p) \times \mathcal{C}_0^d$, as $n \to +\infty$, then

$$x_n - \Phi^m(f_n,g_n) \to x - \Phi^m(f,g)$$
in $\mathcal{C}_0^d$, as $n \to +\infty$. Then, by the lower semi-continuity of $J(\cdot | \cdot)$,

$$\liminf_{n \to +\infty} \mathcal{J}^m(x_n | (f_n,g_n)) = \liminf_{n \to +\infty} J(x_n - \Phi^m(f_n,g_n) | g_n) \geq J(x - \Phi^m(f,g) | g) = \mathcal{J}^m(x | (f,g))$$

and this completes the proof that, for every $(f,g) \in \mathcal{C}_0^p \times \mathcal{C}_0^p$ the family of processes $(Z^{n,m}(f,g))_{n \in \mathbb{N}}$ satisfies the LDP continuity condition for every $m \geq 1$.

\[\square\]

Proposition 4.8. Suppose $\sigma$, $\mu$ and $\tilde{\sigma}$ satisfy Assumptions 3.1 and 4.1. Then, for every $m \geq 1$, the family $((\varepsilon_nB,B^\mathbb{N},Z^{n,m}))_{n \in \mathbb{N}}$ satisfies a WLD with the speed $\varepsilon_n^{-2}$ and the rate function

$$I^m((f,g), x) = I_{B,B}(f,g) + J(x - \Phi^m(f,g) | g)$$

for $x \in \mathcal{C}_0^d$ and $(f,g) \in \mathcal{C}_0^p \times \mathcal{C}_0^p$, where $I_{B,B}(\cdot,\cdot)$ and $J(\cdot | \cdot)$ are defined in (6) and (13), respectively. Moreover, the family of processes $(Z^{n,m})_{n \in \mathbb{N}}$ satisfies a LDP with the
speed $\varepsilon_n^{-2}$ and the rate function

$$I^m_Z(x) = \begin{cases} \inf_{f \in H_0^1, \mu, \nu, \tau} \left\{ \frac{1}{2} ||f||^2_{H_0^1, \mu, \nu, \tau} + J(x - \Phi^m(f, \hat{f})) \right\} & \text{if } x \in H_0^1, \mu, \nu, \tau \\ +\infty, & \text{otherwise} \end{cases} \tag{24}$$

where $\hat{f}$ is defined in (8).

**Proof.** Thanks to Assumption 2.12 and Proposition 4.7, the family $((\varepsilon_nB, \hat{B}^n), (Z^{n,m})_{n \in \mathbb{N}})$ satisfies the hypothesis of Chaganty’s Theorem. Therefore $(Z^{n,m})_{n \in \mathbb{N}}$ satisfies a LDP with the speed $\varepsilon_n^{-2}$ and the rate function

$$I^m_Z(x) = \inf_{(f,g) \in \mathcal{G}_0} \left\{ I_{B,\hat{B}}(f,g) + \mathcal{J}^m(x | (f,g)) \right\}$$

for $x \in \mathcal{G}_0$. Combining the expressions of $I_{B,\hat{B}}(\cdot, \cdot)$ and $\mathcal{J}^m(\cdot | (\cdot, \cdot))$, the rate function $I^m_Z(\cdot)$ is given by (24). \qed

We have proved that, for every $m \geq 1$, the family of processes $(Z^{n,m})_{n \in \mathbb{N}}$ satisfies a LDP with the speed $\varepsilon_n^{-2}$ and the rate function $I^m_Z(\cdot)$. Now, we want to prove that, for every $m \geq 1$, the rate function $I^m_Z(\cdot)$ is actually a good rate function. The proof of the following proposition is the same as Proposition 6.11 in [1]. We give the details since some estimates are not immediate.

**Proposition 4.9.** For every $m \geq 1$, the rate function $I^m_Z(\cdot)$ is a good rate function.

**Proof.** As in Proposition 3.7 it is enough to show that the set

$$\bigcup_{(f,g) \in K_1} \{ x \in \mathcal{G}_0^d : \mathcal{J}^m(x | (f,g)) \leq L \} = \bigcup_{(f,g) \in K_1} A^L_{(f,g)},$$

is a compact subset of $\mathcal{G}_0^d$ for any $L \geq 0$ and for any level set $K_1$ of the rate function $I_{B,\hat{B}}(\cdot, \cdot)$. Let $K_1$ be a level set of $I_{B,\hat{B}}(\cdot, \cdot)$ and $L \geq 0$. If $(f,g) \in K_1$, then $g = \hat{f}$ and

$$A^L_{(f,g)} = A^L_{f, \hat{f}} = \{ x \in H_0^1, \mu, \nu, \tau : \mathcal{J}^m(x | (f, \hat{f})) \leq L \}.$$

For every $(f, \hat{f}) \in K_1$, $A^L_{f, \hat{f}}$ is a compact set of $\mathcal{G}_0^d$, since $\mathcal{J}^m(\cdot | (f, \hat{f}))$ is a good rate function. We want to show that every sequence in $\bigcup_{(f, \hat{f}) \in K_1} A^L_{f, \hat{f}}$ has a convergent subsequence. Let $(x_n)_{n \in \mathbb{N}} \subset \bigcup_{(f, \hat{f}) \in K_1} A^L_{f, \hat{f}}$, then, for every $n \in \mathbb{N}$, there exists $(f_n, \hat{f}_n) \in K_1$ such that $x_n \in A^L_{f_n, \hat{f}_n}$ (that is $\mathcal{J}^m(x_n | (f_n, \hat{f}_n)) \leq L$). Since $((f_n, \hat{f}_n))_{n \in \mathbb{N}} \subset K_1$, up to a subsequence, we can suppose $(f_n, \hat{f}_n) \to (f, \hat{f})$ in $\mathcal{G}_0^d \times \mathcal{G}_0^d$, as $n \to +\infty$, with $(f, \hat{f}) \in K_1$. Now, we claim that there exists a constant $N > 0$, such that, for every $n \in \mathbb{N}$,

$$\mathcal{J}^m(x_n | (f, \hat{f})) = \Gamma \left( x_n - \Phi^m(f, \hat{f}) - \int_0^1 \mu(\hat{f}(t))dt \right) \leq N.$$

From (iii) in Remark 3.4 and Lemma A.5, adding and subtracting $\Phi^m(f_n, \hat{f}_n) + \int_0^1 \mu(\hat{f}_n(t))dt$, we have that, for every $n \in \mathbb{N}$,
\[
J^m(x_n|f, \hat{f}) = \Gamma \left( x_n - \Phi^m(f, \hat{f}) - \int_0^\infty \mu(\hat{f}(t)) dt \right) \\
\leq 3M J^m(x_n|f_n, \hat{f}_n) + 3 \Gamma (\Phi^m(f_n, \hat{f}_n) - \Phi^m(f, \hat{f}) | a^{-1}(\hat{f})) \\
+ 3 \Gamma \left( \int_0^\infty (\mu(\hat{f}_n(t)) - \mu(\hat{f}(t))) dt \right) \\
\leq 3ML + 3 \Gamma (\Phi^m(f_n, \hat{f}_n) - \Phi^m(f, \hat{f}) | a^{-1}(\hat{f})) + 3 \Gamma \left( \int_0^\infty (\mu(\hat{f}_n(t)) - \mu(\hat{f}(t))) dt \right).
\]

The term \( \Gamma (\Phi^m(f_n, \hat{f}_n) - \Phi^m(f, \hat{f}) | a^{-1}(\hat{f})) \) is bounded by a constant independent of \( n \). Indeed, combining Remarks 3.2 and 4.5, there exists a constant \( M > 0 \) (it may also change from line to line) such that

\[
\Gamma (\Phi^m(f_n, \hat{f}_n) - \Phi^m(f, \hat{f}) | a^{-1}(\hat{f})) \leq M (||f_n||_{H^2_{0, \tau}}^2 + ||f||_{H^2_{0, \tau}}^2)
\]

and this quantity is bounded, since \((f, \hat{f}), (f_n, \hat{f}_n) \in K_1\), that is a level set of \( I_{B, B}(\cdot, \cdot) \).

Furthermore, reasoning as in Proposition 3.7, we have

\[
\Gamma \left( \int_0^\infty (\mu(\hat{f}_n(t)) - \mu(\hat{f}(t))) dt \right) \leq M.
\]

Therefore, there exists a constant \( N > 0 \) such that \( J^m(x_n|f, \hat{f}) \leq N \), that implies \( x_n \in A^N_{(f, \hat{f})} \), for every \( n \in \mathbb{N} \). Then, since \( A^N_{(f, \hat{f})} = \{ x \in C^d_0 : J^m(x|f, \hat{f}) \leq N \} \) is a compact subset of \( C^d_0 \), up to a subsequence, we can suppose that \( x_n \to x \in A^N_{(f, \hat{f})} \) in \( C^d_0 \), as \( n \to +\infty \). Moreover, from the lower semicontinuity of \( J^m(\cdot |f, \hat{f}) \),

\[
J^m(x|f, \hat{f}) \leq \liminf_{n \to +\infty} J^m(x_n|f_n, \hat{f}_n) \leq L,
\]

that implies \( x \in A^L_{(f, \hat{f})} \).

\[\Box\]

### 4.2. LDP for the log-price process

We have to Theorem 4.6 provides a LDP for the families \((Z_n^m)_{n \in \mathbb{N}}\) for every \( m \geq 1 \), but our goal is to get a LDP for the family \((Z_n)_{n \in \mathbb{N}}\).

prove that the sequence of processes \((\{Z_n^m\}_{m \geq 1})\) is an exponentially good approximation of \((Z_n)_{n \in \mathbb{N}}\).

Our next goal is to get a LDP for the family of log-prices \((Z_n^m)_{n \in \mathbb{N}}\), defined in (17). We will prove that, for every \( m \geq 1 \), the families of processes \((Z_n^m)_{n \in \mathbb{N}}\) are exponentially good approximations of \((Z_n)_{n \in \mathbb{N}}\).

**Proposition 4.10.** The families \((Z_n^m)_{n \in \mathbb{N}}\) are exponentially good approximations of \((Z_n)_{n \in \mathbb{N}}\) for every \( m \geq 1 \).

**Proof.** We need to check that, for every \( \delta > 0 \),

\[
\lim_{m \to +\infty} \limsup_{n \to +\infty} e^{2 \log P(||Z_n^m - Z_n^m||_\infty > \delta)} = -\infty.
\]
Now,
\[ ||Z_{n,m} - Z^n||_\infty = ||V^n - \Phi^m(\varepsilon_n B, \hat{B}^n)||_\infty = \sup_{t \in [0,T]} ||V^n(t) - \Phi^m(\varepsilon_n B, \hat{B}^n)(t)|| \]
with \( V^m \) and \( \Phi^m(\cdot, \cdot) \) defined in (18) and (20), respectively. Therefore
\[
\{ ||Z_{n,m} - Z^n||_\infty > \delta \} \subset \bigcup_{i=1}^{d} \left\{ \sup_{t \in [0,T]} |V^n_i(t) - \Phi^m_i(\varepsilon_n B, \hat{B}^n)(t)| > \delta / \sqrt{d} \right\}
\]
that implies
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} e_n^2 \log \mathbb{P}(||Z_{n,m} - Z^n||_\infty > \delta) \leq \lim_{m \to +\infty} \lim_{n \to +\infty} e_n^2 \sum_{i=1}^{d} \log \mathbb{P} \left( \sup_{t \in [0,T]} |V^n_i(t) - \Phi^m_i(\varepsilon_n B, \hat{B}^n)(t)| > \delta / \sqrt{d} \right) = -\infty
\]
where the last equality follows from Proposition 6.18 in [1].

We have proved the following theorem.

**Theorem 4.11.** Suppose that Assumptions 2.12, 3.1 and 4.1 are fulfilled. Then, the family of processes \( (Z^n)_n \in \mathbb{N} \), satisfies a LDP with the speed \( e_n^2 \) and the good rate function \( I_Z(\cdot, \cdot) \) given by
\[
I_Z(x) = \lim_{\delta \to 0} \liminf_{m \to +\infty} \inf_{y \in B_{0}(x)} I^m_Z(y) = \lim_{\delta \to 0} \limsup_{m \to +\infty} \inf_{y \in B_{\delta}(x)} I^m_Z(y),
\]
for \( x \in C_0^d \) with \( B_{\delta}(x) = \{ y \in C_0^d : ||x - y||_\infty < \delta \} \) and \( I^m_Z(\cdot, \cdot) \) defined in (24).

### 4.3. Identification of the rate function

In this section, we suppose that Assumptions 4.2, 4.1 and 3.1 are fulfilled.

Theorem 4.11 provides a LDP for the family of processes \( (Z^n)_n \in \mathbb{N} \) with a good rate function \( I_Z(\cdot, \cdot) \) obtained in terms of the \( I^m_Z \)'s. Our next goal is to give an explicit expression to the rate function \( I_Z(\cdot, \cdot) \). Let us define a function \( \Phi : C_0^p \times C_0^p \to C_0^d \) (the analogous of the function \( \Psi \) in Equation (35) in [1]) as
\[
\Phi^I(f, g)(t) = \begin{cases} 
\sum_{\ell=1}^{p} \Psi^\ell(f, g)(t) & (f, g) \in \mathcal{H}^p(B, \hat{B}) \\
0 & \text{otherwise}
\end{cases}
\]
(25)

where for every \( 1 \leq i \leq d, 1 \leq \ell \leq p \)
\[
\Psi^\ell(f, g)(t) = \begin{cases} 
\int_{0}^{t} \tilde{\sigma}_\ell(\hat{f}(s)) \hat{f}_\ell(s) \, ds & (f, g) \in \mathcal{H}^p(B, \hat{B}) \\
0 & \text{otherwise,}
\end{cases}
\]
and \( \mathcal{H}^p(B, \hat{B}) \) is defined in (7). Since for \( f_\ell \in H^1_0[0,T] \), we have that \( \hat{f}_\ell \in C_0 \), for every \( 1 \leq \ell \leq p \), the function \( \Phi(\cdot, \cdot) \) is finite on \( C_0^p \times C_0^p \) and \( \Phi(f, \hat{f}) \) is differentiable with a square integrable gradient, i.e., \( \Phi(f, \hat{f}) \in L^2_0[d[0,T]. \) Proceeding as in Remark 4.5, from
(25), we obtain that, for every \( f \in H^{1,p}_0[0, T] \),
\[
\int_0^T ||\Phi(f, \hat{f})(t)||^2 dt \leq M||f||^2_{H^{1,p}_0[0, T]}.
\]

Next lemma is the same as Lemma 2.13 in [2]. We give some details of the proof for the sake of completeness.

**Lemma 4.12.** For every \( L > 0 \), let \( D_L = \{ f \in H^{1,p}_0[0, T] : ||f||^2_{H^{1,p}_0[0, T]} \leq L \} \). Then one has,
\[
\lim_{m \to +\infty} \sup_{f \in D_L} ||\Phi(f, \hat{f}) - \Phi^m(f, \hat{f})||_\infty = 0.
\]

**Proof.** As in Lemma 22 in [29], that still holds if the arguments are vectorial functions, we have
\[
\lim_{m \to +\infty} \sup_{f \in D_L} \sup_{t \in [0, T]} \left| \tilde{\sigma}_{id}(\hat{f}(t)) - \tilde{\sigma}_{id}\left(\hat{f}\left(\frac{mt}{T} - \frac{i}{m}\right)\right) \right| = 0, \tag{26}
\]
for every \( 1 \leq i \leq d, \ 1 \leq \ell \leq p \). Furthermore, for \( 1 \leq i \leq d \),
\[
||\Phi_i(f, \hat{f}) - \Phi_i^m(f, \hat{f})||_\infty \leq \sum_{\ell=1}^p ||\Psi_{i\ell}(f, \hat{f}) - \Psi_{i\ell}^m(f, \hat{f})||_\infty.
\]

Therefore from Lemma 6.22 in [1] (the statement still holds when the argument are vectorial function), the lemma is proved. \( \square \)

We introduce the following functional
\[
\mathcal{I}_Z(x) = \begin{cases} 
\inf_{f \in H^{1,p}_0[0, T]} \mathcal{H}((f, \hat{f}), x) & x \in H^{1,d}_0[0, T] \\
+\infty & \text{otherwise}
\end{cases} \tag{27}
\]
where for every \( f \in H^{1,p}_0[0, T] \),
\[
\mathcal{H}((f, \hat{f}), x) = \frac{1}{2} ||f||^2_{H^{1,p}_0[0, T]} + J(x - \Phi(f, \hat{f})(\hat{f}))
\]
with \( J(\cdot | \cdot) \) and \( \Phi(\cdot, \cdot) \) defined in (13) and (25), respectively. Now, we will enunciate some remarks and lemmas in order to prove that \( \mathcal{I}_Z(\cdot) = I_Z(\cdot) \).

**Remark 4.13.** For \( x \in H^{1,d}_0[0, T] \) we have,
\[
\mathcal{I}_Z(x) = \inf_{f \in H^{1,p}_0[0, T]} \mathcal{H}((f, \hat{f}), x) \leq \mathcal{H}((0,0), x)
\]
\[
= \frac{1}{2} \int_0^T (\dot{x}(t) - \mu(0))^T a^{-1}(0)(\dot{x}(t) - \mu(0)) \ dt,
\]
therefore
\[
\mathcal{I}_Z(x) = \inf_{f \in D_{C_x}} \mathcal{H}((f, \hat{f}), x)
\]
where \( C_x = \int_0^T (\dot{x}(t) - \mu(0))^T a^{-1}(0)(\dot{x}(t) - \mu(0)) \ dt \) and \( D_{C_x} = \{ f \in H^{1,p}_0[0, T] : ||f||^2_{H^{1,p}_0[0, T]} \leq C_x \} \). Similarly, for \( x \in H^{1,d}_0[0, T] \), for every \( m \geq 1 \), we have
\[ I_2^m(x) = \inf_{f \in D_{C^s}} \mathcal{H}_m((f, \hat{f}), x) \]

where, we recall, \( I_2^m(\cdot) \) is the rate function defined in (24) and
\[
\mathcal{H}_m((f, \hat{f}), x) = \frac{1}{2} ||f||_{H_0^1, p}^2 + I(x - \Phi^m(f, \hat{f})\hat{f}).
\]

In order to prove that \( I_2(\cdot) = I_Z(\cdot) \), we have to verify that the hypotheses of Proposition 2.7 are fulfilled. We start by proving the convergence to \( I_Z(\cdot) \) of the rate functions \( I_2^m(\cdot) \)'s.

**Lemma 4.14.** For every \( x \in H_0^1, d[0, T] \), we have that
\[
\lim_{m \to +\infty} \sup_{t \in [0, T]} ||\Phi(f, \hat{f})(t) - \Phi^m(f, \hat{f})(t)||_\infty^2 = 0.
\]

**Proof.** For \( f \in D_{C^s} \), \( M \) a positive constant, we have
\[
|\Phi^m(f, \hat{f})(t) - \Phi(f, \hat{f})(t)|^2 
\leq \left( \sum_{t=1}^{p} |\hat{f}_t(t)| \sup_{t \in [0, T]} |\bar{\sigma}_d(\hat{f}(T/m)) - \bar{\sigma}_d(\hat{f}(t))| \right)^2
\leq M\sum_{t=1}^{p} |\hat{f}_t(t)| \sup_{t \in [0, T]} |\bar{\sigma}_d(\hat{f}(T/m)) - \bar{\sigma}_d(\hat{f}(t))|^2.
\]

for every \( 1 \leq i \leq d \). Therefore, the statements follows from Equation (26). \( \square \)

Now, we want to identify the rate function \( I_Z(\cdot) \) for the family of processes \( (Z^n(t))_{t \in [0, T]} \) with \( I_Z(\cdot) \). We start with the pointwise convergence (first point of the Proposition 2.7).

**Lemma 4.15.** For every \( x \in \mathcal{D}_0^d \),
\[
\lim_{m \to +\infty} I_2^m(x) = I_Z(x),
\]
where \( I_2^m(\cdot) \) and \( I_Z(\cdot) \) are defined in (24) and (27), respectively.

**Proof.** If \( x \not\in H_0^1, d[0, T] \), then, for every \( m \geq 1 \), \( I_2^m(x) = I_Z(x) = +\infty \). For \( x \in H_0^1, d[0, T] \), we have
\[
|I_2^m(x) - I_Z(x)| = \left| \inf_{f \in D_{C^s}} \mathcal{H}_m((f, \hat{f}), x) - \inf_{f \in D_{C^s}} \mathcal{H}((f, \hat{f}), x) \right| \leq \sup_{f \in D_{C^s}} |\mathcal{H}_m((f, \hat{f}), x) - \mathcal{H}((f, \hat{f}), x)|.
\]

Straightforward computations show that
\[
\mathcal{H}_m((f, \hat{f}), x) - \mathcal{H}((f, \hat{f}), x) = 
\Gamma(\Phi(f, \hat{f}) - \Phi^m(f, \hat{f})a^{-1}(\hat{f})) + \int_0^T (\Phi(f, \hat{f})(t) - \Phi^m(f, \hat{f})(t)a^{-1}(\hat{f}(t)))\dot{x}(t)
- \mu(\hat{f}(t)) - \Phi(f, \hat{f})(t) dt.
\]
Now, thanks to Remark 3.2, there exists a constant $M > 0$ such that

$$\sup_{f \in D_{c_s}} \Gamma(\Phi(f, \hat{f}) - \Phi^m(f, \hat{f})|a^{-1}(\hat{f})) \leq M \sup_{f \in D_{c_s}} \int_0^T ||\Phi(f, \hat{f})(t) - \Phi^m(f, \hat{f})(t)||^2 dt.$$

The last term goes to zero thanks to Lemma 4.14. Moreover, for every $x \in H^1_0[0, T]$, thanks to Remark A.6 there exists a constant $A_x > 0$ such that

$$\sup_{f \in D_{c_s}} ||a^{-1}(\hat{f})||_\infty \leq A_x, \quad \sup_{f \in D_{c_s}} ||\mu_i(\hat{f})||_\infty \leq A_x$$

for every $1 \leq i, j \leq d$. Therefore thanks to the Cauchy-Schwarz inequality, we have

$$\sup_{f \in D_{c_s}} \left| \int_0^T (\Phi(f, \hat{f})(t) - \Phi^m(f, \hat{f})(t)) t^{-1}(\hat{f}(t))(\hat{x}(t) - \mu(\hat{f}(t)) - \Phi(\hat{f}, \hat{f})(t)) dt \right| \leq$$

$$A_x \sup_{f \in D_{c_s}, i,j=1} \int_0^T |\Phi_i(f, \hat{f})(t) - \Phi^m_i(f, \hat{f})(t)||x_i(t) - \mu_i(\hat{f}(t)) - \Phi_j(f, \hat{f})(t)| dt \leq$$

$$A_x \sup_{f \in D_{c_s}} \left( \int_0^T ||\Phi(f, \hat{f})(t) - \Phi^m(f, \hat{f})(t)||^2 dt \right)^\frac{1}{2} \sup_{f \in D_{c_s}} \left( \int_0^T ||\hat{x}(t) - \mu(\hat{f}(t)) - \Phi(f, \hat{f})(t)||^2 dt \right)^\frac{1}{2}.$$

It is not hard to prove that there exists $R_x > 0$ such that,

$$\sup_{f \in D_{c_s}} \int_0^T ||\hat{x}(t) - \mu(\hat{f}(t)) - \Phi(f, \hat{f})(t)||^2 dt \leq R_x,$$

therefore, from Lemma 4.14, one has $\lim_{m \to +\infty} |I^n_Z(x) - I_Z(x)| = 0$. \qed

It remains to show that $x_m \to_{m \to +\infty} x$ implies $\lim \inf_{m \to +\infty} I^n_Z(x_m) \geq I_Z(x)$ (second point of the Proposition 2.7). For this purpose we need to prove that $I_Z(\cdot)$ is lower semicontinuous.

**Lemma 4.16.** The function $\Phi : C_0([0, T], \mathbb{R}^p) \times C_0([0, T], \mathbb{R}^p) \to \mathcal{C}_0^d$ is continuous on the set

$$B_L = \{(f, g) \in \mathcal{H}^p_{(B,B)} : ||f||^2_{H^p_0[0, T]} \leq L\}$$

for every $L > 0$.

**Proof.** Easily follows from Lemma 4.12 and the continuity of $\Phi_m$ (for every $m \geq 1$). \qed

In the next lemma we will prove that $\mathcal{J}(\cdot)(\cdot, \cdot)$ is lower semicontinuous as a function of $((f, g), x) \in B_L \times \mathcal{C}_0^d$.

**Lemma 17.** Let $((f_n, g_n), x_n) \in B_L \times \mathcal{C}_0^d$ be such that

$$(f_n, g_n), x_n \to (f, g), x$$

in $(C_0([0, T], \mathbb{R}^p) \times C_0([0, T], \mathbb{R}^p)) \times \mathcal{C}_0^d$, as $n \to +\infty$. Therefore,

$$\lim \inf_{n \to +\infty} \mathcal{J}(x_n|(f_n, g_n)) \geq \mathcal{J}(x|(f, g)),$$

i.e., the functional $\mathcal{J}(\cdot)(\cdot, \cdot)$ is lower semi-continuous as a function of $((f, g), x) \in B_L \times \mathcal{C}_0^d$. 
Proof. By hypothesis, \( ((f_n, g_n))_{n \in \mathbb{N}} \subset B_L \) is such that \( (f_n, g_n) \to (f, g) \) in the space \( C_0^d \times C_0^{d'} \), as \( n \to +\infty \). Therefore, since \( B_L \) is a compact set (it is a level set of the good rate function \( I_{(B, \tilde{B})} (\cdot, \cdot) \)), we have that \( (f, g) \in B_L \) and \( g = \hat{f} \). If \( \liminf_{n \to +\infty} \mathcal{J}(x_n|(f_n, g_n)) = \lim_{n \to +\infty} \mathcal{J}(x_n|(f_n, g_n)) = +\infty \) there is nothing to prove, hence we can suppose that \( (x_n)_{n \in \mathbb{N}} \subset H_0^{1, d}[0, T] \). Now, combining Lemma A.4 and Remark 3.4, we have that for every \( \varepsilon > 0 \) there exists \( n_\varepsilon \in \mathbb{N} \) such that for every \( n \geq n_\varepsilon \)

\[
\mathcal{J}(x_n|(f_n, \hat{f}_n)) = \Gamma \left( x_n - \Phi(f_n, \hat{f}_n) - \int_0^1 \mu(\hat{f}_n(t)) \ dt |a^{-1}(\hat{f}_n) \right) \\
> (1 - \varepsilon) \Gamma \left( x_n - \Phi(f_n, \hat{f}_n) - \int_0^1 \mu(\hat{f}_n(t)) \ dt |a^{-1}(\hat{f}_n) \right) \\
= (1 - \varepsilon) \mathcal{J} \left( x_n - \Phi(f_n, \hat{f}_n) - \int_0^1 \mu(\hat{f}_n(t)) \ dt + \Phi(f, \hat{f}) + \int_0^1 \mu(\hat{f}(t)) \ dt |f, \hat{f} \right)
\]

Moreover, from Remark A.2 and Lemma 4.16,

\[
x_n - \int_0^1 \mu(\hat{f}_n(t)) \ dt - \Phi(f_n, \hat{f}_n) + \int_0^1 \mu(\hat{f}(t)) \ dt + \Phi(f, \hat{f}) \to x
\]

in \( C_0^d \), as \( n \to +\infty \). For every \( f \in H_0^{1, d}[0, T] \), the functional

\[
\mathcal{J}(x|(f, \hat{f})) = \begin{cases} 
\Gamma(x - \Phi(f, \hat{f}) - \int_0^1 \mu(\hat{f}(t)) \ dt |a^{-1}(\hat{f}) & x \in H_0^{1, d}[0, T] \\
+\infty & \text{otherwise}
\end{cases}
\]

is lower semi-continuous, being the rate function of a LDP for the family \( (Z^n(f, \hat{f}))_{n \in \mathbb{N}} \), where \( Z^n(f, \hat{f})(t) = X_n(f, \hat{f})(t) + \Phi(f, \hat{f})(t) \) and \( X_n(f, \hat{f}) \) is defined in (22). Therefore we have that

\[
\liminf_{n \to +\infty} \mathcal{J}(x_n|(f_n, \hat{f}_n)) > (1 - \varepsilon) \mathcal{J}(x|(f, \hat{f}))
\]

for every \( \varepsilon > 0 \) and thus the thesis follows. \( \square \)

Lemma 4.18. The functional \( I_Z(\cdot, \cdot) \) defined in (27) is lower semi-continuous.

Proof. Let \( L > 0 \) be fixed. In order to show that \( I_Z(\cdot, \cdot) \) is lower semi-continuous, it is enough to prove that the level sets

\[
M_L = \{ x \in C_0^d : I_Z(x) \leq L \}
\]

\[
= \{ x \in H_0^{1, d}[0, T] : \inf_{f \in H_0^{1, d}[0, T]} \{ I_{B, \tilde{B}}(f, \hat{f}) + \mathcal{J}(x|(f, \hat{f})) \} \leq L \}
\]

are closed for every \( L > 0 \). The proof is the same as Lemma 6.29 in [1] (by using Lemma 4.17). \( \square \)

Next Lemma is the same as Lemma 6.30 in [1]. We give only some technical details of the proof since the multidimensional extension is not immediate.
Lemma 4.19. If \( x_m \to x \), as \( m \to +\infty \), in \( G_0^d \), then
\[
\liminf_{m \to +\infty} I_Z^m(x_m) \geq I_Z(x)
\]
where \( I_Z^m(\cdot) \) and \( I_Z(\cdot) \) are defined in (24) and (27), respectively.

Proof. If \( (x_m)_{m \geq m_0} \subset G_0^d \setminus H_0^1[0, T] \), for some \( m_0 > 0 \)
\[
\liminf_{m \to +\infty} I_Z^m(x_m) = \lim_{m \to +\infty} I_Z^m(x_m) = +\infty
\]
and there is nothing to prove, hence we can suppose that \( (x_m)_{m \in \mathbb{N}} \subset H_0^1[0, T] \). Now, there are two possibilities:

(i) \( \sup_{m \geq 1} \|x_m\|^2_{H_0^1[0, T]} < +\infty \);

(ii) \( \sup_{m \geq 1} \|x_m\|^2_{H_0^1[0, T]} = +\infty \).

The proof of the case (i), by using Lemma 4.18, is the same as in the one-dimensional case (see Lemma 6.30 in [1]). Now, let us consider the case (ii), hence, up to a subsequence, we can suppose that
\[
\lim_{m \to +\infty} \|x_m\|^2_{H_0^1[0, T]} = +\infty
\]
and we have to prove that \( \lim_{m \to +\infty} I_Z^m(x_m) = +\infty \). For every \( u > 0 \) we have,
\[
I_Z^m(x_m) = \min \left\{ \inf_{\|f\|_{H_0^1[0, T]}^2 \leq \|x_m\|^2_{H_0^1[0, T]}} \mathcal{H}_m((f, \hat{f}), x_m), \quad \inf_{\|f\|_{H_0^1[0, T]}^2 > \|x_m\|^2_{H_0^1[0, T]}} \mathcal{H}_m((f, \hat{f}), x_m) \right\}
\]
\[
\geq \min \left\{ \inf_{\|f\|_{H_0^1[0, T]}^2 \leq \|x_m\|^2_{H_0^1[0, T]}} \mathcal{J}_m(x_m, (f, \hat{f})), \quad \inf_{\|f\|_{H_0^1[0, T]}^2 > \|x_m\|^2_{H_0^1[0, T]}} I_{(B, B)}(f, \hat{f}) \right\}
\]
(29)
where \( I_{(B, B)}(\cdot, \cdot) \) and \( \mathcal{J}_m(\cdot, (f, \hat{f})) \) are defined, respectively, in (6) and (23). Now we consider the two infima in (29). For the second one we have,
\[
\inf_{\|f\|_{H_0^1[0, T]}^2 > \|x_m\|^2_{H_0^1[0, T]}} I_{(B, B)}(f, \hat{f}) = \inf_{\|f\|_{H_0^1[0, T]}^2 > \|x_m\|^2_{H_0^1[0, T]}} \frac{1}{2} \|f\|_{H_0^1[0, T]}^2 \geq \frac{1}{2} \|x_m\|^2_{H_0^1[0, T]}.
\]
(30)
Suppose now that \( \|f\|_{H_0^1[0, T]}^2 \leq \|x_m\|^2_{H_0^1[0, T]} \). From Remark 3.2, Remark 4.3 and the Cauchy-Schwarz inequality, there exists \( M > 0 \) such that
\[
\mathcal{J}_m(x_m, (f, \hat{f})) = \Gamma \left( x_m - \Phi^m(f, \hat{f}) - \int_0^T \mu(\hat{f}(t)) \, dt \right) \left( a^{-1}(\hat{f}) \right)
\]
\[
\geq \frac{1}{2M^2 \|f\|_{H_0^1[0, T]}^2} \int_0^T \left( \|\dot{x}_m(t) - \Phi^m(f, \hat{f})(t) - \mu(\hat{f}(t)) \|^2 \right) \, dt
\]
\[
\geq \frac{1}{2M^2 \|f\|_{H_0^1[0, T]}^2} \int_0^T \left( \|\dot{x}_m(t)\|^2 - 2\|\dot{x}_m(t)\|\|\Phi^m(f, \hat{f})(t) + \mu(\hat{f}(t))\| \right) \, dt.
\]
Furthermore
\[
\int_0^T \| \dot{x}_m(t) \| \| \dot{\Phi}^m(f, \hat{f})(t) + \mu(\hat{f}(t)) \| \, dt \leq \int_0^T \| \dot{x}_m(t) \| \| \Phi^m(f, \hat{f})(t) \| \, dt \\
+ \int_0^T \| \dot{x}_m(t) \| \| \mu(\hat{f})(t) \| \, dt
\]
and under Assumption 4.2, denoting with \( M \) a generic positive constant, we have that
\[
\int_0^T \| \Phi^m(f, \hat{f})(t) \|^2 \, dt \leq M \| \hat{f} \|_{\infty}^{2x} \| f \|^2_{L^p_{0, T}} \leq M \| f \|_{L^p_{0, T}}^{2x+2},
\]
and thanks to Remark A.6 and Assumption 4.2 we have
\[
\int_0^T \| \mu(\hat{f})(t) \|^2 \, dt \leq M \| \hat{f} \|_{\infty}^{2x} \leq M \| f \|_{L^p_{0, T}}^{2x}.
\]

Therefore, denoting with \( M_1 \) and \( M_2 \) two generic positive constants,
\[
\int_0^T \| \dot{x}_m(t) \| \| \Phi^m(f, \hat{f})(t) + \mu(\hat{f}(t)) \| \, dt \leq (M_1 \| f \|_{L^p_{0, T}}^{2x+1} + M_2 |f|_{L^p_{0, T}}^2) \| x_m \|_{L^p_{0, T}}
\]
Now we can choose \( u > 0 \) such that \((x+1)u < 1\), and since \|f\|_{L^p_{0, T}} \leq \|x_m\|_{L^p_{0, T}}\) for large \( m \) and a suitable \( c > 0 \), one has
\[
\mathcal{J}^m(x_m|(f, \hat{f})) \leq \frac{1}{2M^2 (1 - 3xu)} (\|x_m\|_{L^p_{0, T}}^{2x+1} - 2M_1 \|x_m\|_{L^p_{0, T}}^{1+3xu} - 2M_2 \|x_m\|_{L^p_{0, T}}^{(x+1)u+1})
\]
\[
\geq c \|x_m\|_{L^p_{0, T}}^{2(1 - 3xu)}.
\]
(31)

So (28) follows from (29), (30), (31) and then the proof is complete.

Let’s now check that a LDP holds for the family of log-price at final time \( T \), \((Z^n(T))_{n \in \mathbb{N}}\).

**Corollary 4.20.** The family of random variables \((Z^n(T))_{n \in \mathbb{N}}\) satisfies a LDP on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) with the speed \( \varepsilon_n^{-2} \) and the good rate function \((z \in \mathbb{R}^d)\),
\[
I_T(z) = \inf_{f \in H^{p}_{0, T}} \left\{ \frac{1}{2} \| f \|^2_{L^p_{0, T}} + \frac{1}{2} (z - \Phi(f, \hat{f})(T) - M_f)^T A_f^{-1} (z - \Phi(f, \hat{f})(T) - M_f) \right\},
\]
(32)

where

(i) the matrix \( A_f \in \mathbb{R}^{d \times d} \) is such that \((A_f)_{ij} = \int_0^T a_{ij}(\hat{f}(t)) \, dt, 1 \leq i, j \leq d\);
(ii) the vector \( M_f \in \mathbb{R}^d \) is such that \((M_f)_i = \int_0^T \mu_i(\hat{f}(t)) \, dt, 1 \leq i \leq d\).

**Proof.** We know, from Theorem 4.4, that a LDP on \( \mathcal{C}^0_{0, T} \) with the speed \( \varepsilon_n^{-2} \) and the good rate function \( I_Z(\cdot) \) holds for the family of processes \((Z^n)_{n \in \mathbb{N}}\), where \( I_Z \) is the functional defined in (27). Let \( H \) be the function defined by
Therefore, given $z$, it is not hard to prove that $H$ is a continuous function. Therefore, by the contraction principle,

$$I_T(z) = \inf_{x \in H_0^{i,d}[0,T]:x(T)=z} \inf_{f \in H_0^{i,p}[0,T]} \left\{ \frac{1}{2} ||f||_{L_0^{i,p}[0,T]}^2 + J(x - \Phi(f,\hat{f})(\hat{f})) \right\}$$

$$= \inf_{f \in H_0^{i,p}[0,T]} \inf_{x \in H_0^{i,d}[0,T]:x(T)=z} \left\{ \frac{1}{2} ||f||_{L_0^{i,p}[0,T]}^2 + J(x - \Phi(f,\hat{f})(\hat{f})) \right\}$$

Since the term $\frac{1}{2} ||f||_{L_0^{i,p}[0,T]}^2$ is not dependent on $x$, we only need to calculate

$$\inf_{x \in H_0^{i,d}[0,T]:x(T)=z} \frac{1}{2} \int_0^T (\dot{x}(t) - \mu(\hat{f}(t)))$$

$$- \Phi(f,\hat{f})(t)) a^{-1}(\hat{f}(t))(\dot{x}(t) - \mu(\hat{f}(t)) - \Phi(f,\hat{f})(t)) dt.$$ 

Therefore, given $z \in \mathbb{R}^d$, $f \in H_0^{i,p}[0,T]$ and $x \in H_0^{i,d}[0,T]$, set $u = \dot{x}$; we need to solve the variational calculus problem (see for example [30]) with functional

$$\mathcal{F}(u) = \int_0^T F(u) \ dt$$

$$= \frac{1}{2} \int_0^T (u(t) - \mu(\hat{f}(t)) - \Phi(f,\hat{f})(t)) a^{-1}(\hat{f}(t))(u(t) - \mu(\hat{f}(t)) - \Phi(f,\hat{f})(t)) dt$$

and integral constraint

$$\mathcal{G}(u) = \int_0^T G(u) \ dt = \int_0^T (u(t) - z) \ dt = 0$$

Observe that the constraint is such that

$$\int_0^T u(t) \ dt = \int_0^T \dot{x}(t) \ dt = x(T) = z.$$ 

The Euler-Lagrange equation associated to the problem is

$$\frac{\partial}{\partial u} (F + \lambda^T G) = a^{-1}(\hat{f}(t))(u(t) - \mu(\hat{f}(t)) - \Phi(f,\hat{f})(t)) + \lambda = 0,$$

and then we can conclude \hfill \square

5. Short-time large deviations

In this section we prove a multi-dimensional short-time LDP, when $\mu = 0$, following [17]. Let us denote $\hat{B}^n(t) = \hat{B}(\delta_n t)$ and suppose that the family $(\epsilon_n B, \hat{B}^n)$ satisfies a LDP with the speed $\epsilon_n^{-2}$ (see Example ...). Consider the family of processes $(Z(\epsilon_n t))_{t \in [0,1]}$, where $Z$ is the log-price process with $\mu = 0$. We have the following result.
Theorem 5.1. In the hypotheses of Theorem 4.4 the two families $((e_n\delta_n^{-1/2}Z(\delta_n t))_{t \in [0, T]}), n \in \mathbb{N}$ and $(Z^n)_{n \in \mathbb{N}}$ are exponentially equivalent (see Definition in DZ) and therefore satisfy the same LDP. In particular,

(i) the family $((e_n\delta_n^{-1/2}Z(\delta_n t))_{t \in [0, T]}), n \in \mathbb{N}$ satisfies a LDP with speed $\varepsilon_n^{-2}$ and good rate function given by (19) with $\mu = 0$;

(ii) the family of random variables $(e_n\delta_n^{-1/2}Z(\delta_n T))_{n \in \mathbb{N}}$ satisfies a LDP with speed $\varepsilon_n^{-2}$ and good rate function given by (32) with $\mu = 0$.

Proof. Consider now the process $(Z(\delta_n t))_{t \in [0, T]}$. Thanks to Theorem 4.4 we have a LDP for the family of processes $(Z^n)_{n \in \mathbb{N}}$. Consider now the process $(Z(\delta_n t))_{t \in [0, T]}$. In law we have,

$$Z_i(\delta_n t) = \int_0^{\delta_n t} \left( -\frac{1}{2} \sum_{j=1}^d \sigma_{ij}(\hat{B}(s))^2 - \frac{1}{2} \sum_{\ell=1}^p \tilde{\sigma}_{i\ell}(\hat{B}(s))^2 \right) ds + \int_0^{\delta_n t} \sum_{\ell=1}^p \tilde{\sigma}_{i\ell}(\hat{B}(s)) dB_{i}(s) + \int_0^{\delta_n t} \sum_{j=1}^d \sigma_{ij}(\hat{B}(s)) dW_j(s)$$

$$= \delta_n \int_0^t \left( -\frac{1}{2} \sum_{j=1}^d \sigma_{ij}^n(\hat{B}^n(s))^2 - \frac{1}{2} \sum_{\ell=1}^p \tilde{\sigma}_{i\ell}(\hat{B}(s))^2 \right) ds + \sqrt{\delta_n} \int_0^t \sum_{\ell=1}^p \tilde{\sigma}_{i\ell}(\hat{B}^n(s)) dB_{i}(s) + \sqrt{\delta_n} \int_0^t \sum_{j=1}^d \sigma_{ij}(\hat{B}^n(s)) dW_j(s)$$

for every $1 \leq i \leq d$ and $0 \leq t \leq T$. Now, if we repeat the proof of Theorem 2.6 in [17] we obtain that the family of processes $[((e_n\delta_n^{-1/2}Z(e_n t))_{t \in [0, T]}), n \in \mathbb{N}$ is exponentially equivalent (see Definition 2.4) to the family $(Z^n)_{n \in \mathbb{N}}$ and therefore satisfy a LDP with the speed $\varepsilon_n^{-2}$ and the rate function defined in (19) with $\mu = 0$.

Acknowledgements

The authors thank Paolo Pigato for some hints and comments about the financial aspect of the problem.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix A: Technical results

We collect here some (well known) facts on continuous functions and some technical results on positive definite matrices.

**Remark A.1.**

(i) Suppose \( f : \mathbb{R}^p \to \mathbb{R} \) is a continuous function and let \( \varphi_n, \varphi \in \mathcal{C}^p \) be functions such that \( \varphi_n \overset{\text{e}}{\to} \varphi \), as \( n \to +\infty \), then \( f^e \varphi_n \overset{\text{e}}{\to} f^e \varphi \), as \( n \to +\infty \).

(ii) Suppose \( f : \mathbb{R}^p \to \mathbb{R} \) is a continuous function and let \( \varphi_n \in C([0, T], \mathbb{R}^p) \) be a sequence of equi-bounded functions, i.e., there exist \( M > 0 \) such that for every \( n \in \mathbb{N} \), \( \| \varphi_n \|_\infty \leq M \), then there exist constants \( f_M, \bar{f}_M > 0 \) such that, for every \( n \in \mathbb{N} \) and for every \( t \in [0, T] \),

\[
0 < f_M \leq |f(\varphi_n(t))| \leq \bar{f}_M.
\]

The following properties immediately follow from **Remark A.1**, since a converging sequence of functions is an equi-bounded set in the space of continuous functions.

**Remark A.2.** Let \( (\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{C}^p \) and \( \varphi \in \mathcal{C}^p \) such that \( \varphi_n \overset{\text{e}}{\to} \varphi \) in \( \mathcal{C}^p \), as \( n \to +\infty \). Then,

(i) \( \mu(\varphi_n) \to \mu(\varphi) \) and therefore \( \int_0^T \mu(\varphi_n(s)) \, ds \to \int_0^T \mu(\varphi(s)) \, ds \) in \( \mathcal{C}^d \), as \( n \to +\infty \).

(ii) \( \sigma(\varphi_n) \to \sigma(\varphi) \) and \( a(\varphi_n) \to a(\varphi) \) in \( C([0, T], \mathbb{R}^{d \times d}) \), as \( n \to +\infty \), then for every \( 1 \leq i, j \leq d \), as \( n \to +\infty \), \( \int_0^T a_{ij}(\varphi_n(s)) \, ds \to \int_0^T a_{ij}(\varphi(s)) \, ds \) in \( C([0, T], \mathbb{R}) \).

(iii) Thanks to **Remarks 3.2** and A.1, \( a_{ij}^{-1}(\varphi_n) \to a_{ij}^{-1}(\varphi) \) in \( C([0, T], \mathbb{R}) \), as \( n \to +\infty \), for every \( 1 \leq i, j \leq d \).

Now, let us consider the following results.

**Lemma A.3.** Let \( (\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{C}^p \) and \( \varphi \in \mathcal{C}^p \) such that \( \varphi_n \overset{\text{e}}{\to} \varphi \) in \( \mathcal{C}^p \), as \( n \to +\infty \). Then, there exists a constant \( C_\varphi > 0 \) such that

\[
v^T a^{-1}(\varphi_n(t)) v \geq C_\varphi > 0 \quad \text{and} \quad v^T a^{-1}(\varphi(t)) v \geq C_\varphi > 0
\]

for every \( t \in [0, T] \), \( n \in \mathbb{N} \) and unit vector \( v \in \mathbb{R}^d \).

**Proof.** Since \( \varphi_n \overset{\text{e}}{\to} \varphi \) in \( \mathcal{C}^p \), then there exists \( N > 0 \) such that \( \| \varphi_n \|_\infty \leq N \), therefore the proof is a consequence of **Remarks 3.2** with \( C_\varphi = \inf_{y \in \mathbb{N}, N \varphi} \lambda_{\min}(y) > 0 \) and \( \lambda_{\min}(y) \) is the smallest eigenvalue of \( a^{-1}(y) \).

**Lemma A.4.** Let \( (\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{C}^p \) and \( \varphi \in \mathcal{C}^p \) such that \( \varphi_n \overset{\text{e}}{\to} \varphi \) in \( \mathcal{C}^p \), as \( n \to +\infty \). Then, for every \( \varepsilon > 0 \) there exists \( n_\varepsilon \in \mathbb{N} \) such that the matrix \( a^{-1}(\varphi_n(t)) - (1 - \varepsilon) a^{-1}(\varphi(t)) \) is (strictly) positive definite, for every \( n \geq n_\varepsilon \) and \( 0 \leq t \leq T \).
Therefore, for every $t \in [0, T]$ we have
\[ v^T(a^{-1}(\varphi_n(t)) - (1 - \varepsilon)a^{-1}(\varphi(t)))v = v^T(a^{-1}(\varphi_n(t)) - a^{-1}(\varphi(t)))v + \varepsilon v^Ta^{-1}(\varphi(t))v \]
\[ \geq v^T(a^{-1}(\varphi_n(t)) - a^{-1}(\varphi(t)))v + \varepsilon C_\varphi \]

Now, from (iii) in Remark A.2, there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \geq n_\varepsilon$ and $t \in [0, T]$ one has
\[ |v^T(a^{-1}(\varphi_n(t)) - a^{-1}(\varphi(t)))v| \leq \frac{\varepsilon C_\varphi}{2} \]

Therefore, for every $t \in [0, T]$ and $n \geq n_\varepsilon$
\[ v^T(a^{-1}(\varphi_n(t)) - (1 - \varepsilon)a^{-1}(\varphi(t)))v \geq \frac{\varepsilon C_\varphi}{2} > 0. \]

**Lemma A.5.** Let $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{C}^p$ and $\varphi \in \mathcal{C}^p$ such that $\varphi_n \to \varphi$ in $\mathcal{C}^p$, as $n \to +\infty$. Then, there exists $M > 1$ such that the matrix $M a^{-1}(\varphi_n(t)) - a^{-1}(\varphi(t))$ is strictly positive definite, for every $n \in \mathbb{N}$ and $0 \leq t \leq T$.

**Proof.** Let $v \in \mathbb{R}^d$ be a unit vector and $M > 1$. From Lemma A.3 for every $t \in [0, T]$,
\[ v^T(Ma^{-1}(\varphi_n(t)) - a^{-1}(\varphi(t)))v = (M - 1)v^T a^{-1}(\varphi_n(t))v + v^T(a^{-1}(\varphi_n(t)) - a^{-1}(\varphi(t)))v \]
\[ \geq (M - 1)C_\varphi + v^T(a^{-1}(\varphi_n(t)) - a^{-1}(\varphi(t)))v. \]

From (iii) in Remark A.2, there exists a constant $N > 0$ such that for every $n \in \mathbb{N}$
\[ |v^T(a^{-1}(\varphi_n(t)) - a^{-1}(\varphi(t)))v| \leq N \]

Therefore, for every $t \in [0, T]$ and $n \in \mathbb{N}$,
\[ v^T(Ma^{-1}(\varphi_n(t)) - a^{-1}(\varphi(t)))v \geq (M - 1)C_\varphi - N \]
thus, it is enough to choose $M > 1$ such that $(M - 1)C_\varphi - N > 0$. \hfill \Box

**Remark A.6.** For $L > 0$, denote by $D_L$ the closed ball in the Cameron Martin space, i.e.,
\[ D_L = \{ f \in H^L_{0, T} : \|f\|_{H^L_{0, T}}^2 \leq L \}. \]

Then for $f \in D_L$, from the Cauchy-Schwarz inequality,
\[ \|\hat{f}(t)\|^2 = \sum_{l=1}^{p} \hat{f}_l(t)^2 = \sum_{l=1}^{p} \left( \int_0^T K_l(t, s)\hat{f}_l(s) \, ds \right)^2 \leq \sum_{l=1}^{p} \int_0^T K_l(t, s)^2 \, ds \int_0^T \hat{f}_l(t)^2 \, ds. \]

Therefore (thanks to condition (b) in Definition 2.8) there exists a constant $M > 0$ such that
\[ \|\hat{f}(t)\|^2 \leq M\|f\|_{H^L_{0, T}}^2 \]
and therefore
\[ \sup_{f \in D_L} \sup_{t \in [0, T]} \|\hat{f}(t)\| \leq ML. \]