Dynamics of An Underdamped Josephson Junction Ladder

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Abstract

We show analytically that the dynamical equations for an underdamped ladder of coupled small Josephson junctions can be approximately reduced to the discrete sine-Gordon equation. As numerical confirmation, we solve the coupled Josephson equations for such a ladder in a magnetic field. We obtain discrete-sine-Gordon-like IV characteristics, including a flux flow and a “whirling” regime at low and high currents, and voltage steps which represent a lock-in between the vortex motion and linear “phasons”, and which are quantitatively predicted by a simple formula. At sufficiently high anisotropy, the fluxons on the steps propagate ballistically.

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The discrete sine-Gordon equation has been used by several groups to model so-called hybrid Josephson ladder arrays [1-3]. Such an array consists of a ladder of parallel Josephson junctions which are inductively coupled together, e.g. by superconducting wires [4]. The sine-Gordon equation then describes the phase differences across the junctions. In an applied magnetic field, this equation predicts remarkably complex behavior, including flux flow resistance below a certain critical current, and a field-independent resistance above that current arising from so-called “whirling” modes [2]. In the flux flow regime, the fluxons in this ladder propagate as localized solitons, and the IV characteristics exhibit voltage plateaus arising from the locking of solitons to linear “spin wave” modes. At sufficiently large values of the anisotropy parameter \( \eta_J \) defined later, the solitons may propagate “ballistically” on the plateaus, i.e. may travel a considerable distance even after the driving current is turned off.

In this Letter, we show that this behavior is all found in a model in which the ladder is treated as a network of coupled small junctions arranged along both the edges and the rungs of the ladder. This model is often used to treat two-dimensional Josephson networks [5], and includes no inductive coupling between junctions, other than that produced by the other junctions. To confirm our numerical results, we derive a discrete sine-Gordon equation from our coupled-network model. Thus, these seemingly quite different models produce nearly identical behavior for ladders. By extension, they suggest that some properties of two-dimensional arrays might conceivably be treated by a similar simplification. In simulations [6-9], underdamped arrays of this type show some similarities to ladder arrays, exhibiting the analogs of both voltage steps and whirling modes.

Our ladder consists of coupled superconducting grains, the \( i^{th} \) of which has order parameter \( \Phi_i = \Phi_0 e^{i\theta_i} \). Grains \( i \) and \( j \) are coupled by resistively-shunted Josephson junctions (RSJ’s) with current \( I_{ij} \), shunt resistance \( R_{ij} \) and shunt capacitance \( C_{ij} \), with periodic boundary conditions (see Fig. 1).

The phases \( \theta_i \) evolve according to the coupled RSJ equations \( \hbar \dot{\theta}_i / (2e) = V_i, \sum_j M_{ij} \dot{V}_j = \)
\[
I_i^{\text{ext}}/I_c - \sum_j (R/R_{ij})(V_i - V_j) - \sum_j (I_{ij}/I_c) \sin(\theta_{ij} - A_{ij}).
\]
Here the time unit is \( t_0 = \hbar/(2eRI_c) \), where \( R \) and \( I_c \) are the shunt resistance and critical current across a junction in the x-direction (see Fig. 1); \( I_i^{\text{ext}} \) is the external current fed into the \( i^{\text{th}} \) node; the spatial distances are given in units of the lattice spacing \( a \), and the voltage \( V_i \) in units of \( I_cR \).

Finally, \( A_{ij} = (2\pi/\Phi_0) \int_{i}^{j} \mathbf{A} \cdot \mathbf{dl} \), where \( \mathbf{A} \) is the vector potential. We assume \( N \) plaquettes in the array, and postulate a current \( I \) uniformly injected into each node on the outer edge and extracted from each node on the inner edge of the ring. We also assume a uniform transverse magnetic field \( B \equiv f\phi_0/a^2 \), and use the Landau gauge \( \mathbf{A} = -Bx \hat{y} \). We solve these equations numerically using a fourth-order Runge-Kutta algorithm.

We now show that this model reduces approximately to a discrete sine-Gordon equation for the phase differences. Label each grain by \((x, y)\) where \( x/a = 0, \ldots, N-1 \) and \( y/a = 0, 1 \). Subtracting the equation of motion for \( \theta(x, 1) \) from that for \( \theta(x, 2) \), and defining \( \Psi(x) = \frac{1}{2}[\theta(x, 1) + \theta(x, 2)] \), \( \chi(x) = [\theta(x, 2) - \theta(x, 1)] \), we obtain a differential equation for \( \chi(x) \) which is second-order in time. This equation may be further simplified using the facts that \( A_{x,y,x+1,y} = 0 \) in the Landau gauge, and that \( A_{x,1;x,2} = -A_{x,2;x,1} \), and by defining the discrete Laplacian \( \chi(x + 1) - 2\chi(x) + \chi(x - 1) = \nabla^2 \chi(x) \). Finally, using the boundary conditions, \( I^{\text{ext}}(x, 2) = -I^{\text{ext}}(x, 1) \equiv I \), and introducing \( \varphi(x) = \chi(x) - A_{x,2;x,1} \), we obtain

\[
(1 - \eta^2 \nabla^2)\beta \ddot{\varphi} = i - (1 - \eta^2 \nabla^2)\dot{\varphi} - \sin(\varphi) + 2\eta^2 \times \sum_{i=\pm 1} \cos(\Psi(x) - \Psi(x + i)) \sin(\varphi(x) - \varphi(x + i)/2),
\]

where we have defined a dimensionless current \( i = I/I_{cy} \), and anisotropy factors \( 2\eta^2_r = R_y/R_x \), \( 2\eta^2_c = C_x/C_y \), and \( 2\eta^2_j = I_{cx}/I_{cy} \).

We now neglect all combined space and time derivatives of order three or higher. Similarly, we set the cosine factor equal to unity (this is also checked numerically to be valid a posteriori) and linearize the sine factor in the last term, so that the final summation can be expressed simply as \( \nabla^2 \varphi \). With these approximations, eq. (1) reduces to discrete driven sine-Gordon equation with dissipation:
\[ \beta \ddot{\varphi} + \dot{\varphi} + \sin(\varphi) - \eta_j^2 \nabla^2 \varphi = i, \]  

(2)

where \( \beta = 4\pi e I_{cy} R_y^2 C_y / h \).

To confirm the accuracy of this reduction, we have solved the coupled Josephson equations on an \( 8 \times 1 \) (N=8) ring. Fig. 1 shows the resulting IV characteristics with \( \beta = 33, \eta_J = 0.71 \), and several values of \( f \). \( \langle V \rangle \) denotes the space and time-averaged voltage differences across the \( y \)-junctions. There are two regimes. The first is the “flux flow regime” where \( \langle V \rangle(I, f) \) is roughly proportional to \( f \) (up to about \( f = 3/8 \)) and to \( I \). In this regime, for each \( f \), \( \langle V \rangle(I, f) \) also exhibits a series of voltage steps (see below). The second regime corresponds to “resistance steps” in which \( \langle V \rangle = RI \) independent of \( f \). This regime is dominated by the whirling modes, and is also discussed further below.

Fig. 2 shows an expanded low-current regime for \( f = 1/8, \beta = 61 \), and several \( \eta_J \)'s. The voltage steps are very prominent. The IV characteristics are hysteretic on each of the steps, as shown with broken lines for \( \eta_J^2 = 1.25 \). Similar hysteretic steps are well known in numerical studies of the discrete sine-Gordon equation [1,2]. The critical current for the onset of voltage varies from about \( 0.2 I_c \) at \( \eta_J = 0.71 \) to \( \approx 0 \) for \( \eta_J > 1 \), in the less discrete regime, similar to results obtained in [3].

**Soliton behavior.** In the absence of damping and driving, the continuum version of eq. (2) has, among other solutions, the sine-Gordon soliton [10], given by

\[ \varphi_s(x, t) \sim 4 \tan^{-1} \left[ \exp \left\{ \frac{(x - v_v t)}{\sqrt{\eta_J^2 - \beta v_v^2}} \right\} \right] \]  

(3)

where \( v_v \) is the velocity. The phase in this soliton rises from \( \sim 0 \) to \( \sim 2\pi \) in a width \( d_k \sim \sqrt{\eta_J^2 - \beta v_v^2} \).

Fig. 3 shows the local phase difference \( \varphi(x, t) \) for the Josephson ladder at two currents in the flux-flow regime \((I/I_c = 0.22 \) and 0.41\)), as well as one in the “whirling” regime, \( I/I_c = 0.8(f = 1/8, \beta = 61, \eta_J^2 = 1.25) \). The first two show clear soliton-like behavior, namely, an increase of \( \varphi(x, t) \) in steps of \( \approx 2\pi \) over a time interval \( d_k/v_v \), as predicted by the sine-Gordon equation. The passage of the kink is accompanied by ripples arising from phason excitations. Typically, an integer number of ripple periods is found between successive kink
passages. The local voltage \( \dot{\varphi}(0, t) \) (shown as broken lines) in the flux-flow regime is due to such sine-Gordon solitons, modified by coupling to phasons. The sharp peaks in the local voltage correspond to the vortex passage (we have confirmed this by snapshots of the vortex motion in our simulations), while the exponentially decaying smaller peaks correspond to the phasons which couple to the vortex. In the whirling regime at \( I/I_c = 0.8 \), \( \langle V \rangle \propto I \), and the local voltage oscillates sinusoidally in time. Each oscillation period again corresponds to the passage of a vortex through a given junction, as can be confirmed by the fact that the \( v_v \) thus found is consistent with the independently computed \( \langle V \rangle \), via the Josephson relation.

The step positions in Fig. 2 are determined by a locking of the vortex motion to the phasons. The phason dispersion relation is determined by linearizing the left-hand side of eq. (2) with \( i = 0 \). The result is \( \varphi_m(x, t) \propto \exp(-(t/2\beta)e^{i(k_m x - \omega_m t)}) \), where \( \omega_m = \pm \sqrt{1 + 4\eta_J^2 \sin^2(k_m/2)}/\sqrt{\beta} \), as obtained previously by several groups [1,2]. The allowed wave vectors \( k_m \) are determined by periodic boundary conditions: \( k_m = 2\pi m/N, m = 0, 1, 2, 3, 4, \ldots \). To obtain the locking condition, note that the vortex circulates the ladder with frequency \( \omega_v = 2\pi v_v/(Na) \). A resonance will occur if there are an integer number of phason cycles per vortex cycle. This condition gives \( \omega_m = n\omega_v \) with \( n = 1, 2, 3, \ldots \), or

\[
\frac{1}{\sqrt{\beta} \langle V \rangle} = \frac{n}{\sqrt{1 + 4\eta_J^2 \sin^2(\pi m/N)}}.
\]  

(4)

All the voltage steps we have found in Fig. 2 satisfy this condition. At \( \eta_J = 0.71 \), for example, we identify resonances in the range \( 3 < n < 15, m = 1 \), from a high-resolution \( IV \) characteristic and its derivatives. Presumably, the resonances corresponding to higher \( n \) are weaker because the phasons are damped, relaxing over a time \( 2\beta \). At larger \( \eta_J \), we can identify only a few values of \( n \). The values of \( n \) for each step can also be found by enumerating the number of phason wavelengths between successive vortex passages, as in Fig. 3. In the inset of Fig. 2 we compare the positions of the steps thus located to the predictions of eq. (4). In all cases, only the \( m = 1 \) mode is necessary to account for the resonances.

The transition to the resistive state (“Region II”) occurs at \( n_{\text{min}} = 4, 2, 2, 1 \) for \( \eta_J^2 = \)
This can also be understood from the kink-phason resonance picture. To a phason mode, the passage of a kink of width $d_k$ will appear like the switching on of a step-like driving current over a time of order $d_k/v_v$. The kink will couple to the phasons only if $d_k/v_v \geq \pi/\omega_1$, the half-period of the phason, or equivalently

$$\frac{1}{\sqrt{\beta v_v}} \geq \sqrt{1 + \pi^2} = \frac{3.3}{\eta_J}.$$  \hspace{1cm} (5)

This condition agrees very well with our numerical observations, even though it was obtained by considering soliton solutions from the continuum sine-Gordon equation.

The fact that the voltage in regime I is approximately linear in $f$ can be qualitatively understood from the following argument. Suppose that $\varphi$ for $Nf$ fluxons can be approximated as a sum of well-separated solitons, each moving with the same velocity and described by $\varphi(x,t) = \sum_{j=1}^{Nf} \varphi_j$, where $\varphi_j = \varphi_s(x - x_j, t)$. Since the solitons are well separated, we can use following properties: $\sin[\sum_j \varphi_j] = \sum_j \sin \varphi_j$ and $\int \dot{\varphi}_j \dot{\varphi}_i dx \propto \delta_{ij}$. By demanding that the energy dissipated by the damping of the moving soliton be balanced by that the driving current provides ($\propto \int dx \dot{\varphi}(x)$), one can show that the $Nf$ fluxons should move with the same velocity $v$ as that for a single fluxon driven by the same current. In the “whirling” regime, the $f$-independence of the voltage can be understood from a somewhat different argument. Here, we assume a periodic solution of the form $\varphi = \sum_{j=1}^{Nf} \varphi_w(x - \tilde{v}t - j/f)$ moving with an unknown velocity $\tilde{v}$ where $\varphi_w(\xi)$ describes a whirling solution containing one fluxon. Then using the property $\varphi(x + m/f) = \varphi(x) + 2\pi m$, one can show after some algebra that $\sin[\sum_j^{Nf} \varphi_w(x - \tilde{v}t - j/f)] = \sin[Nf \varphi_w(x - \tilde{v}t)]$. This means that $Nf \varphi_w(x - \tilde{v}t)$ is a solution to eq. (2) as is $\varphi_w(x - vt)$. Finally, using the approximate property $\varphi_w(\xi) \sim \xi$ of the whirling state, one finds $\tilde{v} = v/(Nf)$, leading to an $f$-independent voltage.

Ballistic soliton motion and soliton mass. A common feature of massive particles is their “ballistic motion,” defined as inertial propagation after the driving force has been turned off. Such propagation has been reported experimentally [11] but as yet has not been observed numerically in either square or triangular lattices [4,5]. In the “flux-flow” regime at $\eta_J = 0.71$, we also find no ballistic propagation, presumably because of the large pinning energies.
produced by the periodic lattice. (The critical current for soliton depinning at \( \eta_J = 0.71 \) is about \( 0.2I_c \), about twice that calculated for a square lattice [12].) However, at \( \eta_J > 1 \), we do observe ballistic motion in the flux flow region I. As an example, Fig. 4 shows \( V(t) \) and \( dV/dt \) at junction 0, for \( \eta_J^2 = 5 \), \( I/I_c = 0.41 \) (on the \( n = 1 \) voltage step), after the driving current is switched off at \( t = 0 \). The washboard-like ridges of \( V \) on a decreasing background, and perhaps more clearly, the spikes in \( dV/dt \), correspond to a vortex passing through this junction. The increasing distance between peaks indicates that the vortex is slowing down. The vortex circulates at least five times around the ring before stopping - a fact which is also verified by direct observation of the vortex motion in real time. Qualitatively similar behavior was also observed for slower vortices on the lower current steps so long as \( \eta_J > 1 \).

This behavior can be understood by noting that increasing \( \eta_J \) increases the width of the kink \( d_k \), and thereby effectively makes the ladder seem less discrete. The fluxon at large \( \eta_J \) therefore has a much lower depinning current and can propagate ballistically. This suggests the interesting experimental possibility that one can tune the vortex pinning current and effective mass by manipulation of the anisotropy in the Josephson coupling strength.

We can define the fluxon mass in our ladder by equating the charging energy \( E_c = C/2 \sum_{ij} V_{ij}^2 \) to the kinetic energy of a soliton of mass \( M_v \): \( E_{kin} = \frac{1}{2} M_v v_v^2 \) [7]. Since \( E_c \) can be directly calculated in our simulation, while \( v_v \) can be calculated from \( \langle V \rangle \), this gives an unambiguous way to determine \( M_v \). For \( \eta_J^2 = 0.5 \), we find \( E_c/C \sim 110(\langle V \rangle /I_c R)^2 \), in the flux-flow regime (region I of Fig. 1). This gives \( M_v^I \sim 3.4C \phi_0^2/a^2 \), more than six times the usual estimate for the vortex mass in a 2D square lattice [13]. Similarly, the vortex friction coefficient \( \gamma \) can be estimated by equating the rate of energy dissipation, \( E_{dis} = 1/2 \sum_{ij} V_{ij}^2/R_{ij} \), to \( \frac{1}{2} \gamma v_v^2 \). This estimate yields \( \gamma^I \sim 3.4 \phi_0^2/(Ra^2) \), once again more than six times the value predicted for 2D arrays [7]. This large dissipation explains the absence of ballistic motion for this anisotropy [14,15]. At larger values \( \eta_J^2 = 5 \) and 2.5, a similar calculation gives \( M_v^I \sim 0.28 \) and \( 0.34 \phi_0^2/(Ra^2) \), \( \gamma^I \sim 0.28 \) and \( 0.34 \phi_0^2/(Ra^2) \). These lower values of \( \gamma^I \), but especially the low pinning energies, may explain why ballistic motion is possible at these values of \( \eta_J \).
To conclude, we have shown analytically that the dynamics of an underdamped Josephson ladder can be approximately reduced to those of a discrete sine-Gordon equation. As confirmation, we showed numerically that the ladder exhibits many phenomena previously reported in a discrete sine-Gordon system. These include separate flux-flow and whirling regimes in the IV characteristics, voltage steps in the flux flow regime which arise from locking of vortices to phason excitations on the ladder, and ballistic vortex propagation on the steps at sufficiently high anisotropy. It would be of interest to construct anisotropic ladders in order to verify some of these predictions, and to attempt to extend these arguments to 2D arrays.

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FIGURES

FIG. 1. Calculated IV curves for various values of $f$ in an $8 \times 1$ ladder ring. $f \equiv B a^2/\phi_0 = 1/8$ is the number of flux quanta per plaquette. We use $\beta = 33$ and $\eta J = 0.71$. Plotted voltages are divided by $f$. Inset: schematic of ring topology used in simulation. A uniform current is injected into the inner grains and drawn out from the outer grains.

FIG. 2. Calculated IV curves for $f = 1/8$, $\beta = 61$, and several values of the anisotropy parameter $\eta J$. The inset shows the voltages of steps corresponding to locking number $n$ determined from our numerical result compared to those calculated from eq. 4 (solid lines).

FIG. 3. Local phase difference $\varphi_i(t)$ (solid lines) and voltage $V_i(t)$ (Broken lines) for $i = 0$, $f = 1/8$, $\eta J^2 = 1.25$, $\beta = 61$, and three values of the applied current: $I/I_c = 0.22$ and $0.41$, corresponding to currents on voltage plateaus with $n = 3, 2$; and $I/I_c = 0.8$, corresponding to a current in the whirling regime. Note the different scales in the axes for the whirling regime.

FIG. 4. Plot of $V_0(t)$ and $dV_0(t)/dt$ at $x = 0$ after driving current is turned off at time $t = 0$. We use $\eta J^2 = 5$, and $I/I_c = 0.41$, corresponding to an $n = 1$ voltage plateau.