Gauge invariance and non-constant gauge couplings

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Abstract

It is shown that space-time dependent gauge couplings do not completely break gauge invariance. We demonstrate this in various gauge theories.

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1 Introduction

Gauge invariance is certainly one of the most important guiding principles in modern particle physics. Among its key features is that it prevents the gauge fields from acquiring a direct mass term. However, this obstacle can be circumvented through the Higgs mechanism [1, 2, 3] where the mass of the gauge field is a space-time dependent quantity governed by a scalar field. When this scalar field falls into its ground state, the mass term becomes constant.

Another characteristic (or common belief) of gauge invariance is that it forces the gauge couplings to be constant. In this article, we would like to ask the question of whether gauge invariance is completely lost if the gauge couplings are space-time dependent quantities. The conclusion of this investigation is that some gauge invariance is still present.

The issue of non-constant gauge couplings arises, at least, in two contexts. The first is encountered in the procedure of renormalisation where at the quantum level the couplings are function of the energy scale (running couplings) or equivalently functions of distance. For example, in quantum electrodynamics the gauge coupling increases with energy (that is, with decreasing distance). Yet this quantum phenomenon is completely absent in the classical theory. It would, therefore, be desirable to see if this property (space-time dependence) of the gauge couplings could be implemented at the classical level.

The second domain where non-constant gauge couplings could be of relevance is in cosmology and astrophysics. Indeed, one might reasonably challenge the assumption that the electric charge (the gauge coupling of quantum electrodynamics), or other gauge couplings in non-Abelian gauge theories, should be constant at all times and in all regions of the Universe.

In this context, theories modelling the space-time dependence of some gauge couplings have already appeared in the literature [4, 5, 6, 7, 8]. A general review of the subject could also be found in [9]. They are based on the introduction of new scalar fields and their essence could be summarised by the Lagrangian [4]

\[ \mathcal{L} = -\frac{1}{4}e^{-2\varphi}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i\partial - eA - m]\psi + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \] (1.1)

Of course when \( \varphi = 0 \) (and assuming that \( V(0) = 0 \)), this Lagrangian reduces to the usual field theory of quantum electrodynamics.\(^1\) We have, for simplicity, omitted the gravity sector.

This theory is invariant under the gauge transformation

\[ \psi \rightarrow e^{-ie\alpha(x)}\psi \quad , \quad A_\mu \rightarrow A_\mu + \partial_\mu\alpha(x) \] (1.2)

and the equations of motion corresponding to the gauge field \( A_\mu \), the fermion field \( \psi \) and the scalar field \( \varphi \) are, respectively, given by

\[ \partial_\mu F^{\mu\nu} = e e^{2\varphi}\bar{\psi}\gamma^\nu\psi + 2\partial_\mu\varphi F^{\mu\nu} \]

\[ [i\partial - eA - m]\psi = 0 \]

\[ \partial^\mu\partial_\mu\varphi = -\frac{dV}{d\varphi} + \frac{1}{2}e^{-2\varphi}F_{\mu\nu}F^{\mu\nu} \quad . \] (1.3)

\(^1\)The fermionic contribution \( \bar{\psi}[i\partial - eA - m]\psi \) is not present in ref. [4].
We immediately notice that if one sets the fermion field \( \psi \) to zero then what remains of the first equation in (1.3) is

\[
\partial_\mu F^{\mu \nu} = 2 \partial_\mu \varphi F^{\mu \nu}.
\]  

(1.4)

Clearly, these are not Maxwell’s equations in the vacuum \((\partial_\mu F^{\mu \nu} = 0)\). Hence, the scalar field is present even if the electromagnetic interaction between the gauge field \( A_\mu \) and the charged fermion \( \psi \) is absent. Furthermore, there is an ‘asymmetry’ between the first two equations of (1.3). In the first equation, the coupling to the electromagnetic current \( \bar{\psi} \gamma^\nu \psi \) is \( e e^{2\varphi} \) while the electromagnetic coupling in the Dirac equation (the second equation) is \( e \).

In this paper we adopt the strategy of taking the existing gauge theories and simply replace the constant gauge couplings by non-constant ones and demand that gauge invariance holds. For instance, in the case of quantum electrodynamics, the electromagnetic interaction \( e \bar{\psi} A_\mu \psi \) (with the constant gauge coupling \( e \)) will be replaced by \( \tilde{e}(x) \bar{\psi} A_\mu \psi \). In this way, one has the same interaction terms (or vertices) between the gauge fields and the other fields as in the case of constant gauge couplings. We start by examining the case a relativistic charged particle interacting with an electromagnetic field. We then extend the analyses to quantum mechanics, quantum electrodynamics, non-Abelian gauge theories, the Abelian Higgs model and the electro-weak theory.

### 2 Coupling of a charged point particle to the electromagnetic field

The usual Lagrangian for a relativistic charged particle of mass \( m \) interacting with an electromagnetic field \((\vec{E}, \vec{B})\) is (see for instance [10])

\[
\mathcal{L} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} \vec{A}.\vec{v} - e\varphi.
\]  

(2.1)

The strength of this coupling is the constant \( e \) while \( c \) is the speed of light. The velocity vector \( \vec{v} = \frac{d\vec{r}}{dt} \) with \( \vec{r}(t) \) being the vector position of the particle.

The vector potential \( \vec{A} \) and the scalar potential \( \varphi \) are the quantities in terms of which the electric field \( \vec{E} \) and the magnetic field \( \vec{B} \) are defined. These are given by

\[
\begin{align*}
\vec{E} &= -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \varphi, \\
\vec{B} &= \vec{\nabla} \wedge \vec{A}.
\end{align*}
\]  

(2.2)

The electric and magnetic fields are invariant under

\[
\vec{A} \rightarrow \vec{A} - \vec{\nabla} \alpha , \quad \varphi \rightarrow \varphi + \frac{1}{c} \frac{\partial \alpha}{\partial t},
\]  

(2.3)

where \( \alpha = \alpha(\vec{r}, t) \) is an arbitrary function. Under this gauge transformation the Lagrangian transforms as

\[
\mathcal{L} \rightarrow \mathcal{L} - \frac{e}{c} \frac{d\alpha}{dt}.
\]  

(2.4)
The additional term $-\varepsilon c \frac{d\varepsilon}{dt}$ is a total derivative in time. Hence, the action $S = \int_{t_1}^{t_2} \mathcal{L} \, dt$ changes by a constant and the equations of motion are, as a consequence, invariant.

Suppose now that the strength of the coupling between the charged particle and the electromagnetic field is not constant. Namely, we consider the Lagrangian

$$\mathcal{L} = -m c^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{\varepsilon}{c} \vec{A} \cdot \vec{v} - \varepsilon \varphi,$$  \hspace{1cm} (2.5)

where the coupling $\varepsilon$ is a function of space and time. That is, $\varepsilon = \varepsilon(\vec{r}, t)$. The speed of light $c$ is assumed, throughout this paper, to be constant.

Since the definitions of the $\vec{E}$ and $\vec{B}$ have not changed, the gauge symmetry is still as in (2.3). Under the gauge transformation (2.3) the Lagrangian transforms again as

$$\mathcal{L} \rightarrow \mathcal{L} - \frac{\varepsilon}{c} \frac{d\alpha}{dt}.$$  \hspace{1cm} (2.6)

In order for gauge invariance to hold at the level of the action $S = \int_{t_1}^{t_2} \mathcal{L} \, dt$, we demand that this variation is a total differential in time. That is

$$\frac{\varepsilon}{c} \frac{d\alpha}{dt} = \frac{d\beta}{dt},$$  \hspace{1cm} (2.7)

for some function $\beta$. This requirement is fulfilled if the gauge parameter $\alpha$ is an arbitrary function of $\varepsilon(\vec{r}, t)$, namely $\alpha = \alpha(\varepsilon)$, and in this case we have

$$\beta(\varepsilon) = \int \frac{\varepsilon}{c} \frac{d\alpha}{d\varepsilon} \, d\varepsilon.$$  \hspace{1cm} (2.8)

We conclude that even if the coupling $\varepsilon$ is not constant, gauge invariance is not completely lost.

The Lagrangian (2.5) leads to the following equations of motion

$$\frac{d\vec{p}}{dt} = \frac{d}{dt} \left( \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \varepsilon \vec{E} + \frac{\varepsilon}{c} \vec{v} \wedge \vec{B} + \vec{F}_e,$$

$$\vec{F}_e = \frac{1}{c} \left( (\vec{A} \cdot \vec{v}) - c \varphi \right) \vec{v} - \frac{1}{c} \left( \vec{v} \cdot \vec{\nabla} \varepsilon \right) \frac{\partial \vec{A}}{\partial t}.$$  \hspace{1cm} (2.9)

The forces acting on the particle are the Lorentz force (but with a space-time dependent electric charge $\varepsilon$) plus another force $\vec{F}_e$ due to the fact that the gauge coupling $\varepsilon$ is not constant.

One might be tempted to simply redefine the vector potential $\vec{A}$ as $\vec{A} \rightarrow \frac{\varepsilon}{c} \vec{A}$ and the scalar potential $\varphi$ as $\varphi \rightarrow \frac{\varepsilon}{c} \varphi$, where $\varepsilon$ is a constant, in the Lagrangian (2.5) in order to absorb the space-time dependence of the coupling $\varepsilon$. This is indeed possible if the electromagnetic field is not dynamical. In the full theory, however, the gauge invariant Lagrangian for the electromagnetic field is as usual given by

$$\mathcal{L}_{\text{gauge}} = \frac{1}{8\pi} \int \left( E^2 - B^2 \right) \, dV,$$  \hspace{1cm} (2.10)
where \( dV = dx\,dy\,dz \) is the volume element. Hence a redefinition of the gauge fields would induce a change in the expressions of \( \vec{E} \) and \( \vec{B} \) which results in a non-standard kinetic term for the electromagnetic field.

Maxwell’s equation stem from the variation with respect to \( \vec{A} \) and \( \varphi \) of the full action

\[
S = \int \left[ -mc^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{\vec{E} \cdot \vec{v}}{c} - \vec{v} \varphi \right] dt + \frac{1}{8\pi} \int \left( E^2 - B^2 \right) dV dt .
\] (2.11)

In order to introduce the concept of the charge density, let us write

\[
\vec{e}(\vec{r}, t) = e\lambda(\vec{r}, t)
\] (2.12)

with \( e \) a constant (to be identified with the charge of the point particle). The charge density and the current density are then defined by

\[
e = \int \rho \, dV \quad \text{with} \quad \rho = e\delta(\vec{r} - \vec{r}_0) \quad \text{and} \quad \vec{j} = \rho\vec{v},
\] (2.13)

where \( \vec{r}_0 \) is the vector position of the charge \( e \) whose vector velocity is \( \vec{v} \).

In this way the action becomes

\[
S = \int \left( -mc^2\sqrt{1 - \frac{v^2}{c^2}} \right) dt + \int \left[ \frac{\lambda}{c} \rho \left( \vec{A} \cdot \vec{v} - c\varphi \right) + \frac{1}{8\pi} \int \left( E^2 - B^2 \right) \right] dV dt .
\] (2.14)

The field equations are found by demanding that \( \delta S = 0 \) under the variations \( \vec{A} \rightarrow \vec{A} + \delta\vec{A} \) and \( \varphi \rightarrow \varphi + \delta\varphi \) (assuming, of course, that the motion of the charge is known). This procedure yields

\[
\vec{\nabla} \cdot \vec{E} = 4\pi\lambda\rho
\]
\[
\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \lambda\vec{j} .
\] (2.15)

A continuity equation (conservation equation) is established by taking the divergence of the second equation ( \( \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0 \)). This is given by

\[
\frac{\partial}{\partial t} (\lambda\rho) + \vec{\nabla} \cdot (\lambda\vec{j}) = 0 .
\] (2.16)

Replacing \( \rho \) and \( \vec{j} \) by their expressions in (2.13) and \( \lambda \) by \( \vec{e}/e \) yields

\[
\frac{\partial \vec{e}}{\partial t} + (\vec{v} \cdot \vec{\nabla} \vec{e}) = 0 .
\] (2.17)

Hence the extra force \( \vec{F}_\vec{e} \) takes the form

\[
\vec{F}_\vec{e} = \frac{1}{c} \left( (\vec{A} \cdot \vec{v}) - c\varphi \right) \vec{\nabla} \vec{e} .
\] (2.18)
3 Quantum mechanics

The Schrödinger equation for a non-relativistic charged particle moving through an electromagnetic field is (see for instance [11])

\[ i\hbar \frac{\partial \psi}{\partial t} = \left[ \frac{1}{2m} \left( -i\hbar \vec{\nabla} - \frac{e}{c}\vec{A} \right)^2 + e\varphi \right] \psi \ . \tag{3.1} \]

This equation is invariant under the gauge transformation (2.3) if the wave function transforms as

\[ \psi \rightarrow e^{-i\alpha} \psi \ . \tag{3.2} \]

As a consequence, the probability density and the probability current are invariant.

Let us now assume that the strength of the gauge interaction, \( e \), is no longer a constant and consider, instead, the equation

\[ i\hbar \frac{\partial \psi}{\partial t} = \left[ \frac{1}{2m} \left( -i\hbar \vec{\nabla} - \tilde{e}\vec{A} \right)^2 + \tilde{e}\varphi \right] \psi \ , \tag{3.3} \]

where \( \tilde{e} = \tilde{e}(\vec{r}, t) \). We demand that this equation is still invariant under

\[ \vec{A} \rightarrow \vec{A} - \vec{\nabla} \alpha \ , \ \varphi \rightarrow \varphi + \frac{1}{c} \frac{\partial \alpha}{\partial t} \ , \ \psi \rightarrow e^{-i\beta} \psi \tag{3.4} \]

for some \( \alpha \) and \( \beta \) to be determined. In order for the above equation to remain invariant, one must have

\[ \vec{\nabla} \beta = \frac{\tilde{e}}{\hbar c} \vec{\nabla} \alpha \ , \ \frac{\partial \beta}{\partial t} = \frac{\tilde{e}}{\hbar c} \frac{\partial \alpha}{\partial t} \ . \tag{3.5} \]

These conditions have a solution if \( \alpha \) is an arbitrary function of \( \tilde{e} \), that is \( \alpha = \alpha(\tilde{e}) \), and \( \beta \) is given by

\[ \beta(\tilde{e}) = \int \frac{\tilde{e}}{\hbar c} \frac{d\alpha}{d\tilde{e}} \ d\tilde{e} \ . \tag{3.6} \]

We deduce here also that some gauge invariance is still present in the Schrödinger equation (3.3) with non-constant electromagnetic coupling.

4 Quantum electrodynamics

Quantum electrodynamics is a theory describing the interaction of charged fields with radiation. In the case of fermionic fields, the theory is given by the classical Lagrangian (see for instance [12, 13, 14])

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left[ i\partial_\mu - eA_\mu - m \right] \psi \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.1} \]

When the coupling \( e \) between the fermion field \( \psi \) and the radiation field \( A_\mu \) is constant, the theory is invariant under the gauge transformation

\[ \psi \rightarrow e^{-ie\alpha(x)} \psi \ , \ A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x) \ . \tag{4.2} \]
where $\alpha(x)$ is a completely arbitrary function of the space-time coordinates.

Let us now examine what becomes of this gauge invariance if the gauge coupling is a space-time dependent function. The Lagrangian is of the same form as before

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left[ i \partial - \tilde{e} \gamma^\mu A_\mu - m \right] \psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(4.3)

but now $\tilde{e} = \tilde{e}(x)$. In analogy with (4.2), we demand that this Lagrangian is invariant under

$$\psi \rightarrow e^{-i\beta(x)} \psi \quad , \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x)$$

(4.4)

for some functions $\alpha(x)$ and $\beta(x)$. It follows that the Lagrangian (4.3) remains invariant under the transformations (4.4) provided that $\alpha(x)$ and $\beta(x)$ satisfy

$$\partial_\mu \beta(x) = \tilde{e}(x) \partial_\mu \alpha(x) \quad \text{or} \quad d\beta = \tilde{e} d\alpha \quad .$$

(4.5)

This equation is consistent only if

$$\partial_\mu \tilde{e} \partial_\nu \alpha = \partial_\nu \tilde{e} \partial_\mu \alpha \quad \text{or} \quad d\tilde{e} \wedge d\alpha = 0 \quad .$$

(4.6)

This last condition has a solution if $\alpha(x) = \alpha(\tilde{e}(x))$. That is, $\alpha$ is an arbitrary function of the coupling $\tilde{e}(x)$. In this case $\beta$ is also a function of $\tilde{e}$ and is given by

$$\beta(\tilde{e}(x)) = \int \left( \frac{d\alpha}{d\tilde{e}} \right) \tilde{e} \quad .$$

(4.7)

This last equation can be written, after an integration by parts, as

$$\beta = \tilde{e} \alpha - \int \alpha d\tilde{e} \quad .$$

(4.8)

We can clearly see that if $\tilde{e}$ is independent of the space-time points (that is, $d\tilde{e} = 0$) then $\beta = \tilde{e} \alpha$ and the transformations (4.4) are the usual gauge transformations of ordinary quantum electrodynamics with a constant gauge coupling. To summarise, the Lagrangian (4.3), with $\tilde{e} = \tilde{e}(x)$, is invariant under the local transformations

$$\psi \rightarrow e^{-i \int \left( \frac{d\alpha}{d\tilde{e}} \right) \tilde{e}} \psi \quad , \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha \quad ,$$

(4.9)

where $\alpha(\tilde{e}(x))$ is arbitrary.

The equations of motion corresponding to the Lagrangian (4.3) are

$$\partial_\mu F^{\mu\nu} = \tilde{e} \overline{\psi} \gamma^\nu \psi$$

$$\left[ i \gamma^\mu \partial_\mu - \tilde{e} \gamma^\mu A_\mu - m \right] \psi = 0$$

(4.10)

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2 We could also have written $\beta = \int \tilde{e} d\alpha$. This means that $\tilde{e}$ is given in terms of $\alpha$. However, one is first given a Lagrangian (that is $\tilde{e}$) and then one looks for the symmetries of this Lagrangian (that is $\alpha$). This is why we prefer to say that the gauge function $\alpha$ is expressed in terms of the coupling $\tilde{e}$.
The first equation implies that the current
\[ J^\nu = \bar{\psi} \gamma^\nu \psi \]  
(4.11)
is conserved \((\partial_\nu J^\nu = 0)\). The corresponding conserved charge is
\[ Q = \int J^0 d^3x = \int \bar{\psi} \gamma^0 \psi d^3x = \int \bar{\psi} \gamma^0 \psi d^3x . \]  
(4.12)
As \(\bar{e}(x)\) cannot be taken out of the integral, the conserved quantity \(Q\) is the integral of the fermionic ‘probability density’ \(\bar{\psi} \gamma^0 \psi = (\sqrt{\bar{e}} \psi)^\dagger (\sqrt{\bar{e}} \psi)\). It is as if the ‘wave function’ is \(\sqrt{\bar{e}} \psi\) and not \(\psi\).

5 Non-Abelian gauge theories

Consider the pure Yang-Mills Lagrangian
\[ \mathcal{L} = -\frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) , \]
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu , A_\nu] . \]  
(5.1)
Here \(A_\mu = A_\mu^a T_a\) is a non-Abelian gauge field taking values, for example, in the Lie algebra \(SU(N)\) with commutation relations \([T_a , T_b] = if_{abc} T_c\) and \(\text{Tr}(T_a T_b) = \frac{1}{2} \delta_{ab}\). This theory, when the gauge coupling \(g\) is constant, is invariant under
\[ A_\mu \longrightarrow h A_\mu h^\dagger - \frac{i}{g} \partial_\mu hh^\dagger , \]  
(5.2)
where the group element \(h(x)\) is an arbitrary function in the Lie group corresponding to the Lie algebra \(SU(N)\).

Let us now assume that the strength of the coupling between the gauge fields is a space-time dependent quantity. The Lagrangian we consider is given by
\[ \mathcal{L} = -\frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) \]
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\bar{g} [A_\mu , A_\nu] , \]  
(5.3)
where \(\bar{g} = \bar{g}(x)\). We then demand that this Lagrangian is invariant under the gauge transformation
\[ A_\mu \longrightarrow U A_\mu U^\dagger - \frac{i}{\bar{g}} \partial_\mu UU^\dagger \]  
(5.4)
for some Lie group element \(U(x)\). Under this transformation, the field strength \(F_{\mu\nu}\) transforms as
\[ F_{\mu\nu} \longrightarrow UF_{\mu\nu} U^\dagger + \frac{i}{\bar{g}^2} \left( \partial_\mu \bar{g} \partial_\nu UU^\dagger - \partial_\nu \bar{g} \partial_\mu UU^\dagger \right) . \]  
(5.5)
If the field strength is to transforms as \(F_{\mu\nu} \longrightarrow UF_{\mu\nu} U^\dagger\) (in order for the gauge kinetic term \(-\frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu})\) to be invariant) then the condition
\[ \partial_\mu \bar{g} \partial_\nu UU^\dagger - \partial_\nu \bar{g} \partial_\mu UU^\dagger = 0 \quad \text{or} \quad d\bar{g} \wedge dUU^\dagger = 0 \]  
(5.6)
must hold. This relation is satisfied provided that the Lie group element $U$ is an arbitrary function of $\tilde{g}(x)$. That is, $U(x) = U(\tilde{g}(x))$. Therefore, the non-Abelian gauge symmetry is not totally lost if the gauge coupling $\tilde{g}$ is not constant.

The gauge coupling $\tilde{g}$ characterises the interaction of the non-Abelian gauge fields between themselves and at the same time it describes the strength of the interaction of these gauge fields with any other fields. Let, for instance, $\Psi$ and $\bar{\Psi}$ be a set of fermions, carrying an index of the Lie Algebra $SU(N)$, and transforming as

$$\Psi \rightarrow U\psi \quad , \quad \bar{\Psi} \rightarrow \bar{\Psi}U^\dagger \quad ,$$

where we have suppressed the Lie algebra indices. The gauge covariant derivative

$$\mathcal{D}_\mu \Psi = [\partial_\mu - i\tilde{g}(x)A_\mu] \Psi$$

transforms under (5.4) as

$$\mathcal{D}_\mu \Psi \rightarrow U (\mathcal{D}_\mu \Psi)$$

and the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\text{Tr} (F_{\mu\nu}F^{\mu\nu}) + \bar{\Psi}[i\gamma^\mu \mathcal{D}_\mu - m] \Psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\tilde{g}(x) [A_\mu , A_\nu] \quad ,$$

where the covariant derivative $\mathcal{D}_\mu$ is as defined in (5.8), is gauge invariant. This is the quantum chromodynamics Lagrangian with a space-time dependent gauge coupling $\tilde{g}$ (we have suppressed the Lie algebra indices in the second term).

### 6 The Abelian Higgs model

The Abelian Higgs model, with a space-time dependent gauge coupling, is described by the Lagrangian

$$\mathcal{L}_{U(1)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu \phi^* - i\tilde{e}A_\mu \phi^*) (\partial^\mu \phi + i\tilde{e}A^\mu \phi) - m^2\phi^* \phi - \lambda (\phi^* \phi)^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad .$$

The gauge coupling $\tilde{e}$ is a space-time dependent function\footnote{The parameters $m^2$ and $\lambda$ could also be space-time dependent quantities. This does not affect gauge symmetry.}. That is, $\tilde{e} = \tilde{e}(x)$. In this case, the Lagrangian $\mathcal{L}_{U(1)}$ is invariant under the gauge transformations

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha \quad , \quad \phi \rightarrow e^{-i\beta} \phi \quad ,$$

where $\alpha(x)$ is an arbitrary function of the gauge coupling $\tilde{e}$. That is $\alpha(x) = \alpha(\tilde{e}(x))$ and $\beta$ depends on $\alpha$ through

$$\beta(\tilde{e}(x)) = \int \left( \frac{d\alpha}{d\tilde{e}} \right) d\tilde{e} = \tilde{e} \alpha (\tilde{e}) - \int \alpha (\tilde{e}) d\tilde{e} \quad .$$

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Let us now investigate the physical content of the above theory. We start by parametrising the complex scalar field $\phi$ as

$$\phi = \rho e^{i\theta} .$$  

(6.4)

The gauge transformation $\phi \rightarrow e^{-i\beta} \phi$ is now given by

$$\theta \rightarrow \theta - \beta$$  

(6.5)

and $\rho(x)$ is unchanged as $\rho^2 = \phi^* \phi$ is gauge invariant. With this parametrisation, the Lagrangian becomes

$$L_{U(1)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e^2 \rho^2 \left( A_\mu + \frac{1}{e} \partial_\mu \theta + \frac{i}{e} \partial_\mu \ln \rho \right) \left( A^\mu + \frac{1}{e} \partial^\mu \theta - \frac{i}{e} \partial^\mu \ln \rho \right)$$

$$- m^2 \rho^2 - \lambda \rho^4 .$$  

(6.6)

In the Abelian Higgs model with constant gauge coupling, the physical content is determined by choosing the unitary gauge $\theta = 0$. However, in the case when $e$ is not constant, the gauge $\theta = 0$ cannot, in general, be reached. One attains the unitary gauge by choosing the arbitrary function $\beta(e(x))$ equal to $\theta(x)$ such that the transformed field $\theta(x) - \beta(e(x))$ vanishes. This is, in general, not possible as a function of four coordinates (that is $\beta(e(x))$ cannot be expressed in terms of a function of one variable only (that is $\beta(e(x))$).

If we insist on reproducing all the features of the Abelian Higgs model with constant gauge coupling, then we could demand that the field $\theta(x)$ is itself a function of $e(x)$. That is,

$$\theta(x) = \theta(e(x)) .$$  

(6.7)

After all, $\theta$ is not a physical field (see below). In this case, we could reach the unitary gauge $\theta(e(x)) = 0$. We will, however, proceed in a way which is equivalent to choosing the unitary gauge. It consists in working with gauge invariant variables. Let us define the gauge invariant vector field

$$V_\mu = A_\mu + \frac{1}{e} \partial_\mu \theta .$$  

(6.8)

Indeed, $V_\mu$ transforms as $V_\mu \rightarrow V_\mu + \partial_\mu \alpha - \frac{1}{e} \partial_\mu \beta$ and $\partial_\mu \alpha - \frac{1}{e} \partial_\mu \beta = 0$. Notice also that the term $\frac{1}{e} \partial_\mu \theta$, if $\theta(x) = \theta(e(x))$, can be written as $\partial_\mu \omega$, where $\omega(e(x)) = \int \left( \frac{1}{e} \frac{d\theta}{dx} \right) d\bar{e}$.

The Lagrangian of the Abelian Higgs model takes the form

$$L_{U(1)}^{\text{unitary}} = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + e^2 \rho^2 A_\mu A^\mu + \partial_\mu \rho \partial^\mu \rho - m^2 \rho^2 - \lambda \rho^4 .$$

(6.9)

The field $\theta(e(x))$ has disappeared and is, therefore, not a true degree of freedom. The mechanism of spontaneous symmetry breaking consists in expanding the scalar field $\rho$ as

$$\rho(x) = \rho_0 + \frac{\sigma(x)}{\sqrt{2}} .$$  

(6.10)

where the minimum of the potential $V(\rho) = m^2 \rho^2 + \lambda \rho^4$ is located at $\rho^2 = \rho_0^2 = -\frac{m^2}{2\lambda}$, with $m^2 < 0$ and $\lambda > 0$. The degrees of freedom are therefore a massive vector field $V_\mu$ with a
masse $M_V^2 = 2\rho_0^2 \tilde{c}^2$ and a massive scalar field $\sigma$ (the Higgs field) with mass $M_\sigma^2 = -m^2$. These are precisely the properties of the Abelian Higgs model with constant gauge coupling. However, the masse of the vector field $V_\mu$, when the gauge coupling is not constant, depends on space-time. We conclude that the mechanism of spontaneous symmetry breaking is, in this case, not sufficient to guarantee a constant mass for the vector field $V_\mu$.

7 The standard electro-weak theory

The standard electro-weak theory, with non-constant gauge couplings, is described by the Lagrangian (see for instance [12, 13, 14] for the case of constant gauge couplings)

$$\mathcal{L}_{SU(2)\times U(1)} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{leptons}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}.$$ (7.1)

The gauge part is

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2} \text{Tr} (W_\mu W^{\mu}) - \frac{1}{4} B_\mu B^{\mu}$$

$$W_\mu = \partial_\mu W - i\tilde{g}(x) [W_\mu, W_\nu]$$

$$B_\mu = \partial_\mu B - \partial_\nu B_\nu,$$ (7.2)

where the $SU(2)$ gauge field is $W_\mu = W^a T_a$ with the three matrices $T_a$ obeying the $SU(2)$ commutation relations $[T_a, T_b] = i\epsilon_{abc} T_c$ and $\text{Tr}(T_a T_b) = \frac{1}{2} \delta_{ab}$. In the $2 \times 2$ representation $T_a = \frac{1}{2} \sigma_a$, where $\sigma_a$ are the usual Pauli matrices. The $U(1)$ gauge potential is denoted $B_\mu$.

The $SU(2)$ gauge coupling $\tilde{g}$ is taken to be a space-time dependent function, namely $\tilde{g} = \tilde{g}(x)$. The $U(1)$ gauge coupling will be denoted $\tilde{g}' = \tilde{g}'(x)$ and is also a space-time dependent quantity.

The Lagrangian $\mathcal{L}_{\text{gauge}}$ is invariant under the gauge transformations

$$W_\mu \rightarrow UW_\mu U^\dagger - \frac{i}{\tilde{g}} (\partial_\mu U) U^\dagger$$

$$B_\mu \rightarrow B_\mu + \partial_\mu \alpha,$$ (7.3)

provided that the $SU(2)$ group element $U$ (with $UU^\dagger = 1$) is an arbitrary function of $\tilde{g}(x)$. That is, $U = U(\tilde{g}(x))$, as has been shown in section 5.

The leptons (for simplicity, we include only the electron and its neutrino) enter through

$$\mathcal{L}_{\text{leptons}} = i \bar{R} \gamma^\mu (\partial_\mu + i\tilde{g}' B_\mu) R + i L \gamma^\mu \left( \partial_\mu + \frac{i}{2} \tilde{g}' B_\mu - i\tilde{g} W_\mu \right) L$$ (7.4)

with

$$L \equiv \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}, \quad R \equiv e_R$$ (7.5)

and for a fermion $\psi$ we have $\psi_L = \frac{1}{2} (1 - \gamma_5) \psi$ and $\psi_R = \frac{1}{2} (1 + \gamma_5) \psi$. The left-handed neutrino is denoted $\nu_e$ while $e_L$ and $e_R$ refer, respectively, to the left and the right chiralities of the electron.

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When the $U(1)$ gauge coupling $\tilde{g}'$ is a space-time dependent function, the fermions transform under the $U(1)$ gauge symmetry as

$$L \rightarrow e^{-i\beta/2}L, \quad R \rightarrow e^{-i\beta}R,$$

where now the $U(1)$ gauge parameter $\alpha$ is an arbitrary function of the $U(1)$ gauge coupling $\tilde{g}'(x)$. That is, $\alpha = \alpha(\tilde{g}'(x))$ and $\beta$ is given by

$$\beta(\tilde{g}') = \int \left( \frac{\tilde{g}' d\alpha}{d\tilde{g}'} \right) d\tilde{g}'$$

as has been established in section 4.

On the other hand, under the $SU(2)$ gauge symmetry, the fermions transform as

$$L \rightarrow UL, \quad R \rightarrow R.$$  

Recall that $U$ is a function of $\tilde{g}(x)$.

The complex scalar field

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

enters the electro-weak theory through the Lagrangian

$$\mathcal{L}_{\text{Higgs}} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{m^2}{2} \Phi^\dagger \Phi - \frac{\lambda}{4} \left( \Phi^\dagger \Phi \right)^2$$

$$D_\mu \Phi = \left( \partial_\mu - \frac{i}{2} \tilde{g}' B_\mu - i \tilde{g} W_\mu \right) \Phi.$$  

(7.10)

The complex scalar field has the $U(1)$ gauge symmetry

$$\Phi \rightarrow e^{i\beta/2} \Phi$$

and the $SU(2)$ gauge transformation

$$\Phi \rightarrow U \Phi.$$  

(7.12)

Finally, the Yukawa part is given by

$$\mathcal{L}_{\text{Yukawa}} = -G_e \left( \bar{L} \Phi R + \bar{R} \Phi^\dagger L \right),$$

(7.13)

where $G_e$ is the electron Yukawa coupling constant.

So far, we have shown that it is possible to render the gauge couplings of the standard electro-weak theory space-time dependent while maintaining some gauge symmetry. However, we still have to examine the spectrum of this theory. Let us recall that in the case of constant gauge couplings, the simplest way to get the spectrum is to choose for the scalar field $\Phi$ the unitary gauge

$$\Phi = \begin{pmatrix} 0 \\ \eta + \sigma(x) \sqrt{2} \end{pmatrix}.$$  

(7.14)

\footnote{Gauge symmetry does not prevent the parameters $m^2$, $\lambda$ and $G_e$ to be space-time dependent variables.}
Here $\eta^2 = -m^2/\lambda$, with $m^2 < 0$, is the ground state energy for the scalar field $\Phi$ (the minimum of the potential).

On the other hand, for non-constant gauge couplings the above choice for the scalar field is, in general, not attainable. This is due to the fact that the $SU(2)$ gauge parameter is not an arbitrary function of space-time but an arbitrary function of the gauge coupling $\tilde{g}(x)$. In other words, starting from the gauge choice (7.14) one cannot reach all the scalar field configurations by means of a gauge transformation (it is, in general, not possible to adjust $\Phi(x)$ to a chosen gauge using a matrix $U(\tilde{g}(x))$ that depends on $x^\mu$ only through $\tilde{g}(x^\mu)$).

One way out of this is to assume that the the non-physical degrees of freedom contained in the scalar field $\Phi$ are a function of $\tilde{g}(x)$.

In order to see this, let us parametrise the scalar $\Phi$ as

$$
\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \rho \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix},
$$

(7.15)

where $\rho(x)$ is defined as

$$
\rho^2 = \Phi^\dagger \Phi = |\phi_1|^2 + |\phi_2|^2.
$$

(7.16)

The two complex fields $\chi_1$ and $\chi_2$ satisfy

$$
|\chi_1|^2 + |\chi_2|^2 = 1
$$

(7.17)

Since $\rho$ is a gauge invariant quantity, the $SU(2)$ gauge transformation acts only on the fields $\chi_1$ and $\chi_2$. It is these fields (that are not physical, as shown below) which we will assume to depend on the the $SU(2)$ gauge coupling $\tilde{g}(x)$. Namely,

$$
\chi_1(x) = \chi_1(\tilde{g}(x)) \quad , \quad \chi_2(x) = \chi_2(\tilde{g}(x))
$$

(7.18)

In this way a matrix $U(\tilde{g}(x))$ can be found to reach the unitary gauge (7.14). We will, however, choose to work with gauge invariant variables instead.

We start by noticing that given a vector $\Phi = \rho \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$, such that $|\chi_1|^2 + |\chi_2|^2 = 1$, one can always write it as

$$
\Phi = \rho X^\dagger \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$

(7.19)

where the matrix $X^\dagger$ belongs to the $SU(2)$ group ($XX^\dagger = 1$ and det$(X) = 1$) and is given by

$$
X^\dagger = \begin{pmatrix} \chi_2^* & \chi_1 \\ -\chi_1^* & \chi_2 \end{pmatrix}.
$$

(7.20)

The $SU(2)$ matrix $X^\dagger$ transforms as

$$
X^\dagger \rightarrow UX^\dagger
$$

(7.21)

in order for $\Phi$ to have the $SU(2)$ gauge transformation $\Phi \rightarrow U\Phi$. 

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Our next step is to introduce the two variables

\[ W^X_\mu = XW_\mu X^\dagger - \frac{i}{\tilde{g}} \partial_\mu XX^\dagger \]

\[ L^X = XL \]  

(7.22)

The U(1) gauge field \( B_\mu \) and fermionic singlet \( R \) remain unchanged, as they are not affected by the \( SU(2) \) gauge transformation. The \( SU(2) \) vector field \( W^X_\mu \) and the fermionic doublet \( L^X \) are gauge invariant under the \( SU(2) \) gauge symmetry, as can be verified by using the \( SU(2) \) gauge transformations of \( W_\mu \), \( L \) and \( X \).

In order to find the expression of the electro-weak Lagrangian in terms of the new variables (this procedure is totally equivalent to choosing the unitary gauge for which \( \Phi^X = \rho \begin{pmatrix} 0 \\ 1 \end{pmatrix} = X\Phi \)), we write the new gauge field \( W^X_\mu \) as

\[ W^X_\mu = \frac{1}{2} \begin{pmatrix} W^1_\mu \sigma_1 + W^2_\mu \sigma_2 + W^3_\mu \sigma_3 \\ W^1_\mu \sigma_1 - W^2_\mu \sigma_2 + W^3_\mu \sigma_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} W^3_\mu \\ W^+_\mu \\ W^-_\mu \end{pmatrix} \]  

(7.23)

where we have defined the two vector fields

\[ W^+_\mu = W^1_\mu + iW^2_\mu \quad , \quad W^-_\mu = W^1_\mu - iW^2_\mu \]  

(7.24)

The scalar sector in terms of the new variables is given by the Lagrangian

\[ \mathcal{L}_{\text{Higgs}} = \partial_\mu \rho \partial^\mu \rho + \frac{1}{4} \rho^2 \left[ (\tilde{g}^2 + \tilde{g}'^2) Z_\mu Z^\mu + \tilde{g}' W^+_\mu W^-_\mu \right] - \frac{m^2}{2} \rho^2 - \frac{\lambda}{4} \rho^4 \]  

(7.25)

where we have also introduced the two gauge variables

\[ A_\mu = \frac{\tilde{g}' W^3_\mu + \tilde{g} B_\mu}{(\tilde{g}^2 + \tilde{g}'^2)^{1/2}} \]

\[ Z_\mu = \frac{\tilde{g}' W^3_\mu - \tilde{g} B_\mu}{(\tilde{g}^2 + \tilde{g}'^2)^{1/2}} \]  

(7.26)

or, equivalently

\[ W^3_\mu = \frac{\tilde{g}' A_\mu + \tilde{g} Z_\mu}{(\tilde{g}^2 + \tilde{g}'^2)^{1/2}} \]

\[ B_\mu = \frac{\tilde{g}' A_\mu - \tilde{g} Z_\mu}{(\tilde{g}^2 + \tilde{g}'^2)^{1/2}} \]  

(7.27)

In order to find the expression of the fermionic Lagrangian in terms of the new variables, we introduce the notation

\[ L^X = \begin{pmatrix} N_e \\ \mathcal{E}_L \end{pmatrix} \quad , \quad R = \mathcal{E}_R \]  

(7.28)
Explicitly, we have

\[
L_{\text{leptons}} = i \bar{\mathcal{E}} \gamma^\mu \left( \partial_\mu + i \frac{\bar{g} g'}{(\bar{g}^2 + g'^2)^{1/2}} A_\mu \right) \mathcal{E} + i \bar{N}_e \gamma^\mu \partial_\mu N_e \\
+ \left[ \frac{1}{2} \left( \frac{g^2 + g'^2}{2} N_e \gamma^\mu N_e + \frac{\bar{g}^2 - g'^2}{2} \mathcal{E}_R \gamma^\mu \mathcal{E}_R - \frac{g^2 - g'^2}{2 (\bar{g}^2 + g'^2)^{1/2}} \mathcal{E}_L \gamma^\mu \mathcal{E}_L \right) \right] Z_\mu \\
+ \frac{\bar{g}}{2} \left[ \mathcal{E}_L \gamma^\mu \mathcal{W}_\mu^+ + \mathcal{N}_e \gamma^\mu \mathcal{W}_\mu^- \right],
\]

(7.29)

where we have defined \( \mathcal{E} = \mathcal{E}_L + \mathcal{E}_R \).

Similarly, the Yukawa part, in terms of the new variables, yields

\[
L_{\text{Yukawa}}^{U} = -G_\rho \left( \bar{\mathcal{E}}_L \mathcal{E}_R + \bar{\mathcal{E}}_R \mathcal{E}_L \right) = -G_\rho \bar{\mathcal{E}} \mathcal{E}.
\]

(7.30)

The gauge part of the electro-weak theory is given by

\[
L_{\text{gauge}} = -\frac{1}{2} \text{Tr} \left( W_{\mu\nu} W^{\mu\nu} \right) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} = -\frac{1}{2} \text{Tr} \left( W_{\mu\nu}^X W_{\mu\nu}^X \right) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}
\]

\[
W_{\mu\nu}^X = \partial_\mu W_{\nu}^+ - \partial_\nu W_{\mu}^- - i \bar{g}(x) \left[ W_{\mu}^X, W_{\nu}^X \right].
\]

(7.31)

The second equality holds because the matrix \( X \) is, by assumption, a function of \( \bar{g}(x) \). Using the expression of the matrix \( W_{\mu\nu}^X \) in (7.23), the field strength \( W_{\mu\nu}^X \) takes the form

\[
W_{\mu\nu}^X = \frac{1}{2} \left( \begin{array}{cc}
\mathcal{W}_{\mu\nu}^3 - \bar{g} H_{\mu\nu} & \nabla_\mu W_\nu^- - \nabla_\nu W_\mu^- \\
\nabla_\mu W_\nu^+ - \nabla_\nu W_\mu^+ & -\mathcal{W}_{\mu\nu}^3 + \bar{g} H_{\mu\nu}
\end{array} \right),
\]

(7.32)

where

\[
\mathcal{W}_{\mu\nu}^3 = \partial_\mu \mathcal{W}_{\nu}^3 - \partial_\nu \mathcal{W}_{\mu}^3
\]

\[
H_{\mu\nu} = -\frac{i}{2} \left( W_{\mu}^+ W_{\nu}^- - W_{\nu}^+ W_{\mu}^- \right)
\]

\[
\nabla_\mu W_\nu^+ = \left( \partial_\mu + i \bar{g} \mathcal{W}_{\mu}^3 \right) W_\nu^+
\]

\[
\nabla_\mu W_\nu^- = \left( \partial_\mu - i \bar{g} \mathcal{W}_{\mu}^3 \right) W_\nu^-.
\]

(7.33)

Notice that one has terms involving \( \partial_\mu W_\nu^3 \) and \( \partial_\mu B_\nu \) and upon replacing \( W_{\mu}^3 \) and \( B_{\mu} \) by their expressions in (7.27), one generates quantities involving the derivatives of \( \bar{g}(x) \) and \( \bar{g}'(x) \). Hence, if we want to have the same terms as in the case of the standard electro-weak theory with constant gauge couplings then we must demand that

\[
\frac{\bar{g}}{(\bar{g}^2 + g'^2)^{1/2}} = c, \quad \frac{\bar{g}'}{(\bar{g}^2 + g'^2)^{1/2}} = c',
\]

(7.34)

where \( c \) and \( c' \) are two constants. This means that the two couplings \( \bar{g} \) and \( \bar{g}' \) are related by \( \bar{g}/\bar{g}' = c/c' \), and one has only one space-time independent gauge coupling.
By an explicit calculation, and using the assumption (7.34), we find that the gauge part is given by

\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{4} \mathcal{F}_{\mu \nu} \mathcal{F}^{\mu \nu} - \frac{1}{4} \mathcal{Z}_{\mu \nu} \mathcal{Z}^{\mu \nu} - \frac{1}{4} \left( \nabla_\mu \mathcal{W}_\nu^+ - \nabla_\nu \mathcal{W}_\mu^+ \right) \left( \nabla_\mu \mathcal{W}_\nu^- - \nabla_\nu \mathcal{W}_\mu^- \right) + \frac{1}{2} \frac{\bar{g}}{(\bar{g}^2 + \bar{g}'^2)^{1/2}} \left( \bar{g} \mathcal{Z}_{\mu \nu} + \bar{g}' \mathcal{F}_{\mu \nu} \right) H^{\mu \nu} - \frac{1}{4} \bar{g}^2 H_{\mu \nu} H^{\mu \nu}, \]  

(7.35)

where

\[ \mathcal{F}_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]

\[ \mathcal{Z}_{\mu \nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu. \]  

(7.36)

are the field strengths corresponding to the gauge fields \( A_\mu \) and \( Z_\mu \).

To summarise, the full electro-weak Lagrangian \( \mathcal{L}_{SU(2) \times U(1)} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{leptons}} + \mathcal{L}_{\text{Yukawa}} \) with non-constant gauge couplings, subject to the assumption (7.34), contains the same terms as in the case of the electro-weak theory with constant gauge couplings. The fields \( \chi_1 \) and \( \chi_2 \) are unphysical as they have disappeared from the final theory (they have been absorbed by the non-Abelian gauge fields).

Furthermore, the Lagrangian \( \mathcal{L}_{SU(2) \times U(1)} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{leptons}} + \mathcal{L}_{\text{Yukawa}} \) is still invariant under the \( U(1) \) gauge symmetry

\[ A_\mu \rightarrow A_\mu + \partial_\mu \alpha, \]

\[ \mathcal{E} \rightarrow e^{-i\beta} \mathcal{E}, \]

\[ \mathcal{W}_\mu^+ \rightarrow e^{-i\beta} \mathcal{W}_\mu^+, \]

\[ \mathcal{W}_\mu^- \rightarrow e^{i\beta} \mathcal{W}_\mu^-, \]  

(7.37)

where \( \alpha = \alpha(\bar{e}) \) is an arbitrary function of \( \bar{e} = \frac{\bar{g}^2}{(\bar{g}^2 + \bar{g}'^2)^{1/2}} \) and \( \beta \) is given by

\[ \beta(\bar{e}) = \int \left( e \frac{d\alpha}{d\bar{e}} \right) d\bar{e}. \]  

(7.38)

The neutral fermion \( N_\nu \) and the neutral vector field \( Z_\mu \) are not affected by this gauge symmetry. We should also mention that, according to (7.27), \( \mathcal{W}_3^\pm \) transforms as \( \mathcal{W}_3^\pm \rightarrow \mathcal{W}_3^\pm + \frac{\bar{g}}{(\bar{g}^2 + \bar{g}'^2)^{1/2}} \partial_\mu \alpha \) leading to the transformations \( \nabla_\mu \mathcal{W}_\nu^+ \rightarrow e^{-i\beta} \nabla_\mu \mathcal{W}_\nu^+ \) and \( \nabla_\mu \mathcal{W}_\nu^- \rightarrow e^{i\beta} \nabla_\mu \mathcal{W}_\nu^- \). This shows that the Lagrangian \( \mathcal{L}_{\text{gauge}} \) in (7.35) is explicitly gauge invariant.

The spectrum of the theory described by the full Lagrangian \( \mathcal{L}_{SU(2) \times U(1)} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{leptons}} + \mathcal{L}_{\text{Yukawa}} \) is found by making the substitution \( \rho(x) = \eta + \frac{\sigma(x)}{\sqrt{2}} \). It contains: i) three massive vector fields \( (Z_\mu, \mathcal{W}_\mu^+, \mathcal{W}_\mu^-) \) and a massless gauge field \( A_\mu \). ii) A massive scalar field \( \sigma \) (the Higgs field). iii) A massive fermion \( \mathcal{E} \) (the electron) together with a massless one \( N_\nu \) (the neutrino). The latter does not couple to the massless vector field \( A_\mu \). The different masses are of course read from the quadratic parts of the Lagrangian. However, the masses of the vector fields \( (Z_\mu, \mathcal{W}_\mu^+, \mathcal{W}_\mu^-) \) are space-time dependent even after implementing the spontaneous symmetry breaking mechanism.
8 Conclusions

It is commonly stated that non-constant gauge couplings are incompatible with gauge invariance. We show in this paper that gauge invariance is not completely lost if the gauge couplings are not constant. This remark could be seen just as a mathematical curiosity in its own right but it might also have some physical consequences especially in cosmology.

It is certainly interesting to investigate the quantum properties of the various gauge field theories presented in this paper. The simplest of these theories is obviously the one described by the Lagrangian

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left[ i\partial - \bar{\epsilon}(x) A - m \right] \psi . \]  

The first question to be asked is how to deal with the space-time dependent gauge coupling \( \bar{\epsilon}(x) \)? We could regard \( \bar{\epsilon}(x) \) as a dynamical field having a Lagrangian of the form

\[ \mathcal{L}_{\bar{\epsilon}} = \frac{1}{2} \partial_\mu \bar{\epsilon} \partial^\mu \bar{\epsilon} - V(\bar{\epsilon}) , \]

where \( V(\bar{\epsilon}) \) is some potential energy. A dynamical field \( \bar{\epsilon} \) might be desirable from the point of view of cosmology and astrophysics (if one includes gravity). However, it is problematic at the level of quantum field theory. Indeed, the interaction term \( \bar{\epsilon} \bar{\psi} A \psi \) is, if \( \bar{\epsilon} \) possesses a kinetic term, a dimension five operator and leads to a non-renormalisable theory.

On the other hand, if the Lagrangian \( \mathcal{L}_{\bar{\epsilon}} \) is not included then one could view the non-constant gauge coupling \( \bar{\epsilon}(x) \) as a non-propagating background. The Feynman rules and the Feynman graphs are then exactly those of quantum electrodynamics with constant gauge coupling. In this case, we expect the theory to be renormalisable.

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