Abstract

Deep models trained through maximum likelihood have achieved state-of-the-art results for survival analysis. Despite this training scheme, practitioners evaluate models under other criteria, such as binary classification losses at a chosen set of time horizons, e.g., Brier score (BS) and Bernoulli log likelihood (BLL). Models trained with maximum likelihood may have poor BS or BLL since maximum likelihood does not directly optimize these criteria. Directly optimizing criteria like BS requires inverse-weighting by the censoring distribution. However, estimating the censoring model under these metrics requires inverse-weighting by the failure distribution. The objective for each model requires the other, but neither are known. To resolve this dilemma, we introduce Inverse-Weighted Survival Games. In these games, objectives for each model are built from re-weighted estimates featuring the other model, where the latter is held fixed during training. When the loss is proper, we show that the games always have the true failure and censoring distributions as a stationary point. This means models in the game do not leave the correct distributions once reached. We construct one case where this stationary point is unique. We show that these games optimize BS on simulations and then apply these principles on real world cancer and critically-ill patient data.

1 Introduction

Survival analysis is the modeling of time-to-event distributions and is widely used in healthcare to predict time from diagnosis to death, risk of disease recurrence, and changes in level of care. In survival data, events, known as failures, are often right-censored, i.e., only a lower bound on the time is observed, for instance, when a patient leaves a study before failing. Under certain assumptions, maximum likelihood estimators are consistent for survival modeling [Kalbfleish and Prentice, 2002].

Recently, deep survival models have obtained state-of-the-art results [Ranganath et al., 2016, Alaa and van der Schaar, 2017, Katzman et al., 2018, Kvamme et al., 2019, Zhong and Tibshirani, 2019]. Common among these are discrete-time models [Yu et al., 2011, Lee et al., 2018, Fotso, 2018, Lee et al., 2019, Ren et al., 2019, Kvamme and Borgan, 2019b, Kamran and Wiens, 2021, Goldstein et al., 2020, Sloma et al., 2021] even when data are continuous because they can borrow classification architectures and flexibly approximate continuous densities [Miscouridou et al., 2018].

Though training is often based on maximum likelihood, criteria such as Brier score (BS) and Bernoulli log likelihood (BLL) have been used to evaluate survival models [Haider et al., 2020]. The BS and BLL are classification losses adapted for survival by treating the model as a binary classifier at various time horizons.
horizons (will the event occur before or after 5 years?) [Kvamme and Borgan, 2019b, Lee et al., 2019, Steingrimsson and Morrison, 2020]. BS can also be motivated by calibration (section 3) which is valuable because survival probabilities are used to communicate risk [Sullivan et al., 2004]. However BS and BLL are challenging to estimate because they require inverse probability of censoring-weighting (IPCW), which depends on the true censoring distribution [Van der Laan et al., 2003].

Though consistent, due to finite data, maximum likelihood may lead to models with poor IPCW estimates featuring the other model. Inspired by game theory [Neumann and Morgenstern, 2007, Letcher et al., 2019], we ask: should the censoring model’s re-weighting role in the failure objective be considered part of the censoring objective? We find the answer to be no. In each step of training, each model follows gradients of its loss with the other model held fixed to compute weights.

When the loss is proper (e.g. BS, BLL) [Gneiting and Raftery, 2007], we show that games have the true failure and censoring distributions as a stationary point. This means the models in the game do not leave the correct distributions once reached. We then describe one case where this stationary point is unique. Finally, we show that inverse-weighted game training achieves better BS and BLL than maximum likelihood methods on simulations and real world cancer and ill-patient data.²

2 Notation and background on IPCW

Notation. Let $T$ be a failure time with cdf $F(t) = P(T \leq t)$, density $f$, survival function $\overline{F} = 1 - F$, and model $F_{\theta_F}$. Let $C$ be a censoring time with cdf $G$, density $g$, $\overline{G} = 1 - G$, and model $G_{\theta_C}$. This means $G(t) = P(C > t)$. Let $\overline{G}(t^-)$ denote $P(C \geq t)$. We observe features $X$, time $U = \min(T, C)$ and $\Delta = 1 [T \leq C]$. For discrete models over $K$ times, let $\theta_{Ct} = P_0(T = t)$ and $\theta_{Ct} = P_0(C = t)$.

Models. We focus on deep discrete models like those studied in Lee et al. [2018], Kvamme and Borgan [2019b]. The model maps inputs $X$ to a categorical distribution over times. When the observations are continuous, a discretization scheme is necessary. Following Kvamme and Borgan [2019b], Goldstein et al. [2020], we set bins to correspond to quantiles of observed times. We represent all times by the lower boundary of their respective interval.

Assumptions. We assume i.i.d. data and random censoring: $T \independent C \mid X$ [Kalbfleisch and Prentice, 2002]. We also require the censoring positivity assumption [Gerds et al., 2013]. Let $f = dF$. Then:

$$\exists \epsilon \text{ s.t. } \forall x \forall t \in \{t \leq t_{\text{max}} \mid f(t|x) > 0\}, \quad \overline{G}(t^-|x) \geq \epsilon > 0,$$

i.e. it is possible that censoring events occur late-enough for us to observe failures up until a maximum time $t_{\text{max}}$. Truncating at a maximum time is necessary in practice for continuous distributions because datasets may have no samples in the tails, leading to practical positivity violations [Gerds et al., 2013]. This truncation happens implicitly for categorical models by choosing the last bin.

In this work, we model the censoring distribution. This task is dual to the original survival problem: the roles of censoring and failure times are reversed. Therefore, to observe censoring events properly, we also require a version of eq. (1) to hold with the roles of $F$ and $G$ reversed (appendix A).

IPCW estimators. Inverse probability of censor-weighting (IPCW) is a method for estimation under censoring [Van der Laan et al., 2003, Bang and Robins, 2005]. Consider the marginal mean $\mathbb{E}[T]$. IPCW reformulates such expectations in terms of observed data. Using IPCW, we can show that:

$$\mathbb{E}[T] = \mathbb{E}_{X} \mathbb{E}_{T \mid X} \mathbb{E}_{C \mid X} \left[ \mathbb{1}_{[T \leq C | X]} T \right] = \mathbb{E}_{X} \mathbb{E}_{T \mid X} \mathbb{E}_{C \mid X} \left[ \mathbb{1}_{[T \leq C | X]} \frac{T}{\overline{G}(T^- | X)} \right] = \mathbb{E}_{T \mid C \mid X} \left[ \frac{\Delta U}{\overline{G}(U^- | X)} \right]$$

²Code is available at https://github.com/rajesh-lab/Inverse-Weighted-Survival-Games
We derive this fully in appendix B. The second equality holds because $\Delta = 1 \implies U = T$ and means we can identify $E[T|X]$ provided that we know $G$ and that random censoring and positivity hold.

### 3 Time-dependent survival evaluations

Brier score (BS) [Brier and Allen, 1951] is proper for classification, meaning that it has a minimum at the true data distribution [Gneiting and Raftery, 2007]. The BS is often adapted for survival evaluations [Lee et al., 2019, Kvamme et al., 2019, Haider et al., 2020]. For time $t$, it computes differences between the CDF and true event status at $t$, turning survival analysis into a classification problem at a given time horizon:

$$BS(t; \theta) = E \left[ \left( F_{\theta_r}(t|X) - 1 [T \leq t] \right)^2 \right]$$

(2)

BS is often used as a proxy for marginal calibration error [Kumar et al., 2018, Lee et al., 2019], which measures differences between CDF levels $\alpha \in [0, 1]$ and observed proportions of datapoints with $F_{\theta}(T|X) < \alpha$ [Demler et al., 2015]. This usage of BS stems from its decomposition into calibration plus a refinement (discriminative) term [DeGroot and Fienberg, 1983].

Unfortunately one cannot compute BS unmodified since $1 [T \leq t]$ is unobserved for a point censored before $t$. IPCW BS [Graf et al., 1999, Gerds and Schumacher, 2006] estimates BS($t$) under censoring:

$$BS(t; \theta) = E \left[ \frac{\bar{F}_{\theta_r}(t|X)^2 1[U \leq t]}{G(U^-|X)} + \frac{F_{\theta_r}(t|X)^2 1[U > t]}{G(t|X)} \right]$$

(3)

eq (3) is equivalent to eq. (2) (appendix C). Negative Bernoulli log likelihood (BLL) is similar, but with log loss (appendix D). BS and BLL are proper for classification at each time $t$, so their sum or integral over $t$ is still proper (appendix H).

**Proper objectives differ.** Though negative log likelihood (NLL), BS and BLL all have the same true distribution at optimum with infinite data, they may yield significantly different solutions in practice. For example, NLL-trained models may not achieve good BS [Kvamme and Borgan, 2019a]. In fig. 1, we show test set NLL and BS for 5 models trained with NLL at different learning rates on Gamma-simulated data (described in section 5.1). NLL does not align with BS: models that have low NLL may not have low BS. Model 4 has the lowest NLL but not the lowest BS. When a practitioner requires good performance under BS or BLL, they should optimize directly for those metrics.

**Re-weighting dilemma.** Censoring introduces challenges because we must use IPCW to estimate BS and BLL. Crucially, the $G$ in eq. (3) is the true censoring distribution rather than a model, but during training, we only have access to models. This poses a dilemma: can the models be used in re-weighting estimates during training to successfully optimize these criteria under censoring?
A reasonable attempt to solve the dilemma is to jointly optimize the sum of $F_\theta$ and $G_\theta$’s losses where each model re-weights the other’s loss. The expectation is that both models will improve over training and yield reliable IPCW estimates for each other. Concretely, consider this for eq. (3) plus the same objective with the roles of $F_\theta$ and $G_\theta$ reversed. Unfortunately, there exist solutions to this optimization problem with smaller loss than for the distributions from which the data was generated, making this summed objective improper for the pair of distributions. In fig. 2, we plot this for IPCW BS($t = 1$) for models over two timesteps as a function of each model’s single parameter.

To address this phenomenon, we introduce Inverse-Weighted Survival Games. In these games, a failure player and censor player simultaneously minimize their own loss function. The failure and censoring model are featured in both loss functions and playing the game results in a trained failure and censoring model. We show in experiments that these games produce models with good BS, BLL, and concordance relative to those trained with maximum likelihood.

For simplicity, we present the games for marginal categorical models. The analysis can be extended to conditional parameterizations with the usual caveats shared by maximum likelihood. Our experiments explore the conditional setting.

4.1 Basic definition of game

We follow the setup in Letcher et al. [2019]. A differentiable $n$-player game consists of $n$ players each with loss $\ell_i$ and parameter (or state) $\theta_i$. Player $i$ controls only $\theta_i$ and aims to minimize $\ell_i$. However, each $\ell_i$ is a function of the whole state $\theta = (\theta_1, \theta_{-i})^T$. The simultaneous gradient is the gradient of the losses w.r.t. each players’ respective parameters:

$$\xi(\theta) = [\nabla_{\theta_1} \ell_1, \ldots, \nabla_{\theta_n} \ell_n]$$

The dynamics of the game refers to following $-\xi$. The solution concepts in games are equilibria (the game analog of optima) and stationary points. One necessary condition for equilibria is finding a stationary point $\theta^*$ such that $\xi(\theta^*) = 0$. The simplest algorithm follows the dynamics to find stationary points and is called simultaneous gradient descent. With learning rate $\eta$,

$$\theta \leftarrow \theta - \eta \xi(\theta).$$

This can be interpreted as each player taking their best possible move at each instant.
4.2 Constructing survival games

We specify an Inverse-Weighted Survival Game as follows. First, choose a loss $L$ used to construct the losses for each player. Next, derive the IPCW form $L_I$ that can be used to compute $L$ under censoring: for true failure and censoring distributions $F^*$ and $G^*$, the losses $L$ and $L_I$ are related through $L_I(F_{\theta_T}; G^*) = L(F_{\theta_T})$ and $L_I(G_{\theta_C}; F^*) = L(G_{\theta_C})$. The loss functions for the two players are defined as:

$$\ell_F(\theta) \triangleq L_I(F_{\theta_T}; G_{\theta_C}), \quad \ell_G(\theta) \triangleq L_I(G_{\theta_C}; F_{\theta_T})$$

(4)

Compared to eq. (3), we have replaced the true re-weighting distributions with models. Finally, the failure player and censor player minimize their respective loss functions w.r.t. only their own parameters:

failure player: $\min_{\theta_T} \ell_F$, \hspace{1cm} censor player: $\min_{\theta_C} \ell_G$

One example of these games is the IPCW BS$(t)$ game, derived in appendix C. With $\Delta = 1 - \Delta$,

$$\ell_F(\theta) = \mathbb{E} \left[ \frac{F_{\theta_T}(t)^2 \Delta 1[U \leq t]}{G_{\theta_C}(U)} + \frac{F_{\theta_T}(t)^2 1[U > t]}{G_{\theta_C}(t)} \right]$$

$$\ell_G(\theta) = \mathbb{E} \left[ \frac{G_{\theta_C}(t)^2 \Delta 1[U \leq t]}{F_{\theta_T}(U)} + \frac{G_{\theta_C}(t)^2 1[U > t]}{F_{\theta_T}(t)} \right].$$

(5)

In section 4.3, we show that this formulation (fig. 2(b)) has formal advantages over the optimization in fig. 2(a) for particular choices of $L$.

**Multiple Timesteps.** The example is specified for a given $t$, but the games can be designed for multiple timesteps. We use BS for a $K$ timestep model to demonstrate. BS$(K)$ is 0 for any model: the left terms contain $F_{\theta_T}$ and $G_{\theta_C}$, which are both 0 when evaluated at $K$; in the right terms, $1[U > K]$ is always 0. One option is to define summed games with:

$$\ell_F = \sum_{t=1}^{K-1} \ell_F^t, \quad \ell_G = \sum_{t=1}^{K-1} \ell_G^t$$

The summed game is shown in algorithm 1. Alternatively, instead of the sum, it is possible to find solutions for all timesteps with respect to one pair of models $(F_{\theta_T}, G_{\theta_C})$. For $K$-1 timesteps this can be formalized as a 2$(K-1)$-player game: there is a failure player and censor player for the loss at each $t$:

$t^{th}$-failure player: $\min_{\theta_{T_t}} \ell_{F_t}^t$, \hspace{1cm} $t^{th}$-censor player: $\min_{\theta_{C_t}} \ell_{G_t}^t$

We study theory that applies to both approaches in section 4.3, namely that the true failure and censoring distribution are stationary points in both types of games. We prove additional properties about uniqueness of the stationary point for a special case of the multi-player game in section 4.4. Summed games are more stable to optimize in practice because they optimize objectives at all time steps w.r.t all parameters, while multiplayer games can only improve each time step’s loss w.r.t. one parameter. We study the summed games empirically in section 5.

**Algorithm 1** Following Gradients in Summed Games

| Input: | Choice of losses $\ell_F, \ell_G$, learning rate $\gamma$ |
|--------|----------------------------------------------------------|
| Initialize: | $T$ model parameters $\theta_T$ and $C$ model parameters $\theta_C$ randomly |
| repeat | $g_T = 0$ and $g_C = 0$ |
| for $t = 1$ to $K - 1$ do | $g_T = g_T + d\ell_F^t/d\theta_T$ |
| | $g_C = g_C + d\ell_G^t/d\theta_C$ |
| end for | $\theta_T \leftarrow \theta_T - \gamma g_T$ and $\theta_C \leftarrow \theta_C - \gamma g_C$ |
| until convergence | |
| Output: | $\theta_T, \theta_C$ |
4.3 IPCW games have a stationary point at data distributions

Among a game’s stationary points should be the true failure and censoring distributions.

**Proposition 1.** Assume \( \exists \theta^*_T \in \Theta_T, \exists \theta^*_C \in \Theta_C \) such that \( F^* = F_{\theta^*_T} \) and \( G^* = G_{\theta^*_C} \). Assume the game losses \( \ell_F, \ell_C \) are based on proper losses \( L \) and that the games are only computed at times for which positivity holds. Then \((\theta^*_T, \theta^*_C)\) is a stationary point of the game eq. (4).

The proof is in appendix I. The result holds for summed and multi-player games using BS, BLL, or other proper scoring rules such as AUC. When games are built from such objectives, the set of solutions includes \((\theta^*_T, \theta^*_C)\) and models do not leave this correct solution when reached. Under the stated assumptions, this result holds for discrete and continuous distributions. However, as mentioned in section 2, in practice a truncation time must be picked to ensure the assumptions are met for continuous distributions.

4.4 Uniqueness for Discrete Brier Games

We provide a stronger result for the BS game in eq. (5) when solving all timesteps with multi-player games: its only stationary point is located at the true failure and censoring distributions.

**Proposition 2.** For discrete models over \( K \) timesteps, assuming that \( \theta^*_T, \theta^*_C > 0 \), the solution \((\theta^*_T, \theta^*_C)\) is the only stationary point for the multi-player BS game shown in algorithm 2 for times \( t \in \{1, \ldots, K - 1\} \).

The proof is in appendix J. To illustrate this, fig. 2(b) shows that, unlike the minimization in fig. 2(a), the IPCW BS game moves to the correct solution at its unique stationary point.

5 Experiments

We run experiments on a simulation with conditionally Gamma times, a semi-simulated survival dataset based on MNIST, several sources of cancer data, and data on critically-ill hospital patients.

**Losses.** We build categorical models in 3 ways: the standard NLL method (eq. (8)), the IPCW BS game and the negative IPCW BLL game.

**Metrics.** For these models we report BS (uncensored for simulations and Kaplan-Meier (KM)-weighted for real data), BLL (also uncensored or weighted), concordance which measures the proportion of pairs whose predicted risks are ordered correctly [Harrell Jr et al., 1996], and NLL. We report mean and standard deviation of the metrics over 5 different seeds. In all plots, the middle solid line represents the mean and the shaded band represents the standard deviation.

**Model Description.** In all experiments except for MNIST, we use a 3-hidden-layer ReLU network that outputs 20 categorical bins (more bin choices in appendix G.1). For MNIST we first use a small convolutional network and follow with the same fully-connected network.

Model and training details including learning rate and model selection can be found in appendix F.

5.1 Simulation Studies

**Data.** We draw \( X \in \mathbb{R}^{32} \sim \mathcal{N}(0, 10I) \) and \( T \sim \text{Gamma(\text{mean} = \mu_t)} \) where \( \mu_t \) a log-linear function of \( X \). The censoring times are also gamma with mean \( 0.9 * \mu_t \). Both distributions have constant variance \( 0.05 \). It holds that \( T \perp \perp C \mid X \). Each random seed draws a new dataset.

**Results.** fig. 3 demonstrates that the games optimize the true uncensored BS, and, though more slowly with respect to training size, log-likelihood does too. The games have better test-set performance on all metrics for small training size. All methods converge on similar performance when there is enough data (though enough is highly-dependent on dimensionality and model class).

---

4 Though often reported, the time-dependent concordance\((t)\) is not proper [Blanche et al., 2019].
Figure 3: Test set evaluation metrics (y-axis) on the Gamma simulation versus number of training points (x-axis) for three methods. Each point in the plot represents the evaluation metric value of a fully trained model with that number of training points. Lower is better for all the metrics except for concordance.

Figure 4: Calibration curves [Avati et al., 2019] comparing game-training and NLL-training.

Calibration. We include a qualitative investigation of model calibration on the gamma simulation trained with 2000 datapoints. fig. 4 shows that the BS game achieves near-perfect calibration while the two likelihood-based methods suffer some error. This is expected since likelihood-based methods do not directly optimize calibration while BS does (section 3).

5.2 Semi-simulated studies

Data. Survival-MNIST [Gensheimer, 2019, Pölsterl, 2019] draws times conditionally on MNIST label $Y$. This means digits define risk groups and $T \perp X \mid Y$. Times within a digit are i.i.d. The model only sees the image pixels $X$ as covariates so it must learn to classify digits (risk groups) to model times. We follow Goldstein et al. [2020] and use Gamma times, $T \sim \text{Gamma}(\text{mean} = 10 \ast (Y + 1))$. We set the variance constant to 0.05. Lower digit labels $Y$ yield earlier event times. $C$ is drawn similarly but with mean $9.9 \ast (Y + 1)$. Each random seed draws a new dataset.

Results. This experiment demonstrates that better NLL does not correspond to better performance on BS, BLL, and concordance. Similarly to the previous experiment, fig. 5 shows that game methods attain better uncensored test-set BS and BLL on survival-MNIST than likelihood-based training does. The games likewise attain higher concordance. NLL training performs better at the metric it directly optimizes. This experiment also establishes that it is possible to optimize through deep convolutional models with batch norm and pooling using the game training methods.
Figure 5: Test set evaluation metrics (y-axis) on survival-MNIST versus number of training points (x-axis) for three methods. Each point in the plot represents the evaluation metric value of a fully trained model with that number of training points. Lower is better for all the metrics except for concordance.

5.3 Real Datasets

Data. We use several datasets used in recent papers [Chen, 2020, Kvamme et al., 2019] and available in the python packages DeepSurv [Katzman et al., 2018] and PyCox [Kvamme et al., 2019], and the R Survival [Therneau, 2021]. The datasets are:

- Molecular Taxonomy of Breast Cancer International Consortium (METABRIC) [Curtis et al., 2012]
- Rotterdam Tumor Bank (ROTT) [Foekens et al., 2000] and German Breast Cancer Study Group (GBSG) [Schumacher et al., 1994] combined into one dataset (ROTT & GBSG)
- Study to Understand Prognoses Preferences Outcomes and Risks of Treatment (SUPPORT) [Knaus et al., 1995] which includes severely ill hospital patients

For more description see appendix F. For real data, there is no known ground truth for the censoring distribution, which means evaluation requires assumptions. Following the experiments in Kvamme et al. [2019], we assume that censoring is marginal estimate with KM to evaluate models.6

Results. On METABRIC, games attain lower (better) KM-weighted BS and BLL than NLL-training when the number of datapoints is small, and have better concordance and NLL though they do not directly optimize them. As data size increases, all methods converge to similar performance. On ROTT & GBSG, the trend is similar: games optimize the BS and BLL more rapidly as a function of training set size than NLL-training does. Again, all methods converge to similar performance in all metrics when the number of datapoints is large enough. All methods perform similarly on SUPPORT.

Caveats. First, though popular, these survival datasets are low-dimensional, so any of the objectives can perform well on the metrics with just several hundred points. We see that this is distinct from MNIST, where thousands of points were required to improve performance. Second, though possible, it may not be true that censoring is marginal on these datasets, which would mean that the BS and BLL results only have their interpretation conditional on a particular set of covariates. Our method is also correct when the censoring is marginal though. Lastly, no method is stable for all metrics, for all training sizes, on all seeds for all datasets.

6 This is also the route taken in the R packages Survival [Therneau, 2021], PEC [Mogensen et al., 2012], and riskRegression [Gerds et al., 2020].
estimating the target (failure model objective or failure model itself) can benefit from a coupled estimation procedure where the nuisance parameter (censoring model) is also trained simultaneously. The failure model needs the censoring distribution to compute BS but censoring estimation needs the failure model, and despite this circular dependence, we characterize a case where the game training leads to the true data generating distributions.

Double Robust Censoring Unbiased Transformations. For functions \( h \), Rubin and van der Laan [2007] estimate conditional mean \( \mathbb{E}[h(T, X) | X] \) under censoring using a double-robust estimator: given estimates of the conditional failure and censoring cumulative distribution functions (CDFs) \( \hat{F}(t | X) \) and \( \hat{G}(c | X) \), the estimator of \( \mathbb{E}[h(T, X) | X] \) is unbiased when either nuisance CDF is correct. However, here we are concerned with estimating a quantity to be used as a loss for learning \( \hat{F} \). We therefore presumably do not already have an estimate of \( \hat{F} \) to be used in a doubly-robust estimator.

Censoring Unbiased Losses for Deep Learning. Steingrimsson and Morrison [2020] build failure model loss functions based on the estimators from Rubin and van der Laan [2007]. Their BS loss extends IPCW BS estimation to the doubly-robust case and to our knowledge is the first instance of
ICPCW-based estimation procedures being used in a general purpose way to define loss functions for deep survival analysis.

However, their censoring distribution is estimated once before training and held fixed rather than incorporated into a joint training procedure as in the games in this work. The fixed censoring estimate is implemented by KM, which assumes a marginal censoring distribution. Making the marginal assumption when censoring is truly conditional should not yield a performant model under the BS criteria since the training objective does not directly estimate or optimize the true BS that would be measured under no censoring. When marginal censoring does hold, KM estimation, which is non-parametric, may be a simpler and stable choice versus the game, depending on sample size, data variance, and conditional parameterization assumptions. But since it is in general unknown if censoring is marginal, we use conditional models which are also correct under marginal censoring.

7 Discussion

In this work, we propose a new training method for survival models under censored data. We argue that on finite data, it is important to close the gap between training methodology and the desired evaluation criteria. We showed in the experiments that better NLL does not correspond to better performance on BS, BLL, and concordance, all evaluations of interest in survival analysis.

The main trend in our experimental results was that data size matters: smaller meant the game methods performed better than NLL and enough data meant that they perform similarly, which is expected since all objectives are proper. However enough data is hard to define: it depends on dimensionality and on the data generating distribution and model class. It is a great direction to build more precise understanding on how objectives behave differently even when they have the same optimum on infinite data: though likelihood is known to be asymptotically efficient for survival analysis, more analysis is necessary for comparing likelihood and Brier score’s trade-offs on small sample sizes.

In the experiments, we focus on categorical models. On the other hand, proposition 1 applies to continuous distributions as well, provided that positivity can be satisfied. However, this is rare in practice because most survival data has a final follow-up time, and even before this time there may be very few samples with late times [Gerds et al., 2013]. For this reason, working with continuous distributions requires picking a truncation time and playing games only up to that time.

Evaluation on real data under censoring requires assumptions. It is important to further consider how to better assess test-set performance on metrics such as BS, BLL, and concordance. Because concordance is not proper [Blanche et al., 2019], we do not build objectives from it here, but it too is not invariant to censoring. Regarding games, we showed properties about stationary points. More analysis is necessary to describe important convergence properties of optimizing these games.

Social Impact. Survival models are deployed in hospital settings and have high impact on public health. In this work, we saw benefits of a new training approach for these models, but no training method is a panacea. Practitioners of survival analysis must take great care to consider various training and validation approaches, as well as consider possible test distribution shifts, prior to deployment.

Acknowledgments and Disclosure of Funding

This work was supported by:

- NIH/NHLBI Award R01HL148248
- NSF Award 1922658 NRT-HDR: FUTURE Foundations, Translation, and Responsibility for Data Science.
- NSF Award 1514422 TWC: Medium: Scaling proof-based verifiable computation
- NSF Award 1815633 SHF
References

A. M. Alaa and M. van der Schaar. Deep multi-task gaussian processes for survival analysis with competing risks. In Proceedings of the 31st International Conference on Neural Information Processing Systems, pages 2326–2334. Curran Associates Inc., 2017.

P. K. Andersen, O. Borgan, R. D. Gill, and N. Keiding. Statistical models based on counting processes. Springer Science & Business Media, 2012.

A. Avati, T. Duan, S. Zhou, K. Jung, N. H. Shah, and A. Y. Ng. Countdown regression: Sharp and calibrated survival predictions. In A. Globerson and R. Silva, editors, Proceedings of the Thirty-Fifth Conference on Uncertainty in Artificial Intelligence, UAI 2019, Tel Aviv, Israel, July 22-25, 2019, page 28. AUAI Press, 2019. URL http://auai.org/uai2019/proceedings/papers/28.pdf.

H. Bang and J. M. Robins. Doubly robust estimation in missing data and causal inference models. Biometrics, 61(4):962–973, 2005.

P. Blanche, J.-F. Dartigues, and H. Jacqmin-Gadda. Review and comparison of roc curve estimators for a time-dependent outcome with marker-dependent censoring. Biometrical Journal, 55(5):687–704, 2013.

P. Blanche, M. W. Kattan, and T. A. Gerds. The c-index is not proper for the evaluation of-year predicted risks. Biostatistics, 20(2):347–357, 2019.

G. W. Brier and R. A. Allen. Verification of weather forecasts. In Compendium of meteorology, pages 841–848. Springer, 1951.

G. H. Chen. Deep kernel survival analysis and subject-specific survival time prediction intervals. In Machine Learning for Healthcare Conference, pages 537–565. PMLR, 2020.

V. Chernozhukov, D. Chetverikov, M. Demirer, E. Duflo, C. Hansen, W. Newey, and J. Robins. Double/debiased machine learning for treatment and structural parameters, 2018.

C. Curtis, S. P. Shah, S.-F. Chin, G. Turashvili, O. M. Rueda, M. J. Dunning, D. Speed, A. G. Lynch, S. Samarajiwa, Y. Yuan, et al. The genomic and transcriptomic architecture of 2,000 breast tumours reveals novel subgroups. Nature, 486(7403):346–352, 2012.

M. H. DeGroot and S. E. Fienberg. The comparison and evaluation of forecasters. Journal of the Royal Statistical Society: Series D (The Statistician), 32(1-2):12–22, 1983.

O. V. Demler, N. P. Paynter, and N. R. Cook. Tests of calibration and goodness-of-fit in the survival setting. Statistics in medicine, 34(10):1659–1680, 2015.

J. A. Foekens, H. A. Peters, M. P. Look, H. Portengen, M. Schmitt, M. D. Kramer, N. Brûnner, F. Jänicke, M. E. Meijer-van Gelder, S. C. Henzen-Logmans, et al. The urokinase system of plasminogen activation and prognosis in 2780 breast cancer patients. Cancer research, 60(3):636–643, 2000.

D. J. Foster and V. Syrgkanis. Orthogonal statistical learning. arXiv preprint arXiv:1901.09036, 2019.

S. Fotso. Deep neural networks for survival analysis based on a multi-task framework. arXiv preprint arXiv:1801.05512, 2018.

B. Gensheimer, Michael F.and Narasimhan. A scalable discrete-time survival model for neural networks. PeerJ 7:e6257, 2019.

T. A. Gerds and M. Schumacher. Consistent estimation of the expected brier score in general survival models with right-censored event times. Biometrical Journal, 48(6):1029–1040, 2006.

T. A. Gerds, M. W. Kattan, M. Schumacher, and C. Yu. Estimating a time-dependent concordance index for survival prediction models with covariate dependent censoring. Statistics in Medicine, 32 (13):2173–2184, 2013.
T. A. Gerds, P. Blanche, T. H. Scheike, R. Mortensen, M. Wright, N. Tollenaar, J. Muschelli, U. B. Mogensen, and B. Ozenne. Package ‘riskregression’, 2020.

T. Gneiting and A. E. Raftery. Strictly proper scoring rules, prediction, and estimation. *Journal of the American statistical Association*, 102(477):359–378, 2007.

M. Goldstein, X. Han, A. M. Puli, A. Perotte, and R. Ranganath. X-cal: Explicit calibration for survival analysis. *Advances in Neural Information Processing Systems*, 33, 2020.

I. J. Goodfellow, J. Pouget-Abadie, M. Mirza, B. Xu, D. Warde-Farley, S. Ozair, A. Courville, and Y. Bengio. Generative adversarial networks. *arXiv preprint arXiv:1406.2661*, 2014.

E. Graf, C. Schmoor, W. Sauerbrei, and M. Schumacher. Assessment and comparison of prognostic classification schemes for survival data. *Statistics in medicine*, 18(17-18):2529–2545, 1999.

H. Haider, B. Hoehn, S. Davis, and R. Greiner. Effective ways to build and evaluate individual survival distributions. *Journal of Machine Learning Research*, 21(85):1–63, 2020.

F. E. Harrell Jr, K. L. Lee, and D. B. Mark. Multivariable prognostic models: issues in developing models, evaluating assumptions and adequacy, and measuring and reducing errors. *Statistics in medicine*, 15(4):361–387, 1996.

H. Hung and C.-T. Chiang. Estimation methods for time-dependent auc models with survival data. *Canadian Journal of Statistics*, 38(1):8–26, 2010a.

H. Hung and C.-t. Chiang. Optimal composite markers for time-dependent receiver operating characteristic curves with censored survival data. *Scandinavian journal of statistics*, 37(4):664–679, 2010b.

J. D. Kalbfleisch and R. L. Prentice. *The Statistical Analysis of Failure Time Data*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., 2nd edition, 2002.

F. Kamran and J. Wiens. Estimating calibrated individualized survival curves with deep learning. 2021.

J. L. Katzman, U. Shaham, A. Cloninger, J. Bates, T. Jiang, and Y. Kluger. Deepsurv: personalized treatment recommender system using a cox proportional hazards deep neural network. *BMC medical research methodology*, 18(1):24, 2018.

W. A. Knaus, F. E. Harrell, J. Lynn, L. Goldman, R. S. Phillips, A. F. Connors, N. V. Dawson, W. J. Fulkerson, R. M. Califf, N. Desbiens, et al. The support prognostic model: Objective estimates of survival for seriously ill hospitalized adults. *Annals of internal medicine*, 122(3):191–203, 1995.

A. Kumar, S. Sarawagi, and U. Jain. Trainable calibration measures for neural networks from kernel mean embeddings. In *International Conference on Machine Learning*, pages 2805–2814, 2018.

H. Kvamme and Ø. Borgan. The brier score under administrative censoring: Problems and solutions. *arXiv preprint arXiv:1912.08581*, 2019a.

H. Kvamme and Ø. Borgan. Continuous and discrete-time survival prediction with neural networks. *arXiv preprint arXiv:1910.06724*, 2019b.

H. Kvamme, Ørnulf Borgan, and I. Scheel. Time-to-event prediction with neural networks and cox regression. *Journal of Machine Learning Research*, 20(129):1–30, 2019. URL http://jmlr.org/papers/v20/18-424.html.

C. Lee, W. R. Zame, J. Yoon, and M. van der Schaar. Deephit: A deep learning approach to survival analysis with competing risks. In *Thirty-Second AAAI Conference on Artificial Intelligence*, 2018.

C. Lee, W. Zame, A. Alaa, and M. Schaar. Temporal quilting for survival analysis. In *The 22nd international conference on artificial intelligence and statistics*, pages 596–605. PMLR, 2019.

A. Letcher, D. Balduzzi, S. Racaniere, J. Martens, J. Foerster, K. Tuyls, and T. Graepel. Differentiable game mechanics. *The Journal of Machine Learning Research*, 20(1):3032–3071, 2019.
X. Miscouridou, A. Perotte, N. Elhadad, and R. Ranganath. Deep survival analysis: Nonparametrics and missingness. In Machine Learning for Healthcare Conference, pages 244–256, 2018.

U. B. Mogensen, H. Ishwaran, and T. A. Gerds. Evaluating random forests for survival analysis using prediction error curves. Journal of Statistical Software, 50(11):1–23, 2012. URL https://www.jstatsoft.org/v50/i11.

J. v. Neumann and O. Morgenstern. Theory of Games and Economic Behavior. Princeton University Press, 2007.

S. Pölsterl. Sebastian pölsterl, Jul 2019. URL https://k-d-w.org/blog/2019/07/survival-analysis-for-deep-learning/.

R. Ranganath, A. Perotte, N. Elhadad, and D. Blei. Deep survival analysis. arXiv preprint arXiv:1608.02158, 2016.

K. Ren, J. Qin, L. Zheng, Z. Yang, W. Zhang, L. Qiu, and Y. Yu. Deep recurrent survival analysis. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 33, pages 4798–4805, 2019.

D. Rubin and M. J. van der Laan. A doubly robust censoring unbiased transformation. The international journal of biostatistics, 3(1), 2007.

M. Schumacher, G. Bastert, H. Bojar, K. Hübner, M. Olschewski, W. Sauerbrei, C. Schmoor, C. Beyerle, R. Neumann, and H. Rauschecker. Randomized 2 x 2 trial evaluating hormonal treatment and the duration of chemotherapy in node-positive breast cancer patients. German breast cancer study group. Journal of Clinical Oncology, 12(10):2086–2093, 1994.

M. Sloma, F. Syed, M. Nemati, and K. S. Xu. Empirical comparison of continuous and discrete-time representations for survival prediction. In Survival Prediction-Algorithm, Challenges and Applications, pages 118–131. PMLR, 2021.

J. A. Steingrimsson and S. Morrison. Deep learning for survival outcomes. Statistics in medicine, 39(17):2339–2349, 2020.

L. M. Sullivan, J. M. Massaro, and R. B. D’Agostino Sr. Presentation of multivariate data for clinical use: The Framingham study risk score functions. Statistics in medicine, 23(10):1631–1660, 2004.

T. M. Therneau. A Package for Survival Analysis in R, 2021. URL https://CRAN.R-project.org/package=survival. R package version 3.2-11.

H. Uno, T. Cai, L. Tian, and L.-J. Wei. Evaluating prediction rules for t-year survivors with censored regression models. Journal of the American Statistical Association, 102(478):527–537, 2007.

M. J. Van der Laan and S. Rose. Targeted learning: causal inference for observational and experimental data. Springer Science & Business Media, 2011.

M. J. Van Der Laan and D. Rubin. Targeted maximum likelihood learning. The international journal of biostatistics, 2(1), 2006.

M. J. Van der Laan, M. Laan, and J. M. Robins. Unified methods for censored longitudinal data and causality. Springer Science & Business Media, 2003.

M. Wolbers, P. Blanche, M. T. Koller, J. C. Witteman, and T. A. Gerds. Concordance for prognostic models with competing risks. Biostatistics, 15(3):526–539, 2014.

S. Yadowsky, S. Basu, and L. Tian. A calibration metric for risk scores with survival data. In Machine Learning for Healthcare Conference, pages 424–450, 2019.

C.-N. Yu, R. Greiner, H.-C. Lin, and V. Baracos. Learning patient-specific cancer survival distributions as a sequence of dependent regressors. In Advances in Neural Information Processing Systems, pages 1845–1853, 2011.

C. Zhong and R. Tibshirani. Survival analysis as a classification problem. arXiv preprint arXiv:1909.11171, 2019.
A Notation, Assumptions, and Likelihoods in More Detail

A.1 Notation

Let $T$ be a failure time with CDF $F$. $T$’s survival function is defined by $\overline{F} = 1 - F$. We denote failure models by $F_{\theta_T}$. Let $C$ be a censoring time with CDF $G$, survival function $\overline{G}$, and model $G_{\theta_C}$. Under right-censoring, define $U = \min(T, C)$, $\Delta = 1[T \leq C]$ and we observe $(X_i, U_i, \Delta_i)$. We use $\overline{G}(t^-)$ to denote $P(C \geq t)$.

A.2 Assumptions

We assume i.i.d. data and random censoring: $T \perp \!\!\!\!\perp C \mid X$ [Kalbfleisch and Prentice, 2002]. Derivations in this work also require the censoring positivity assumption [Gerds et al., 2013]. Let $f = dF$ (a failure density) and $g = dG$ (a censoring density). Then we assume

$$\exists \epsilon \text{ s.t. } \forall x \forall t \in \{t \leq t_{\text{max}} \mid f(t|x) > 0\}, \quad \overline{G}(t^-|x) \geq \epsilon > 0,$$

for some truncation time $t_{\text{max}}$. Truncating at a maximum time is necessary in practice for continuous distributions because datasets may have no samples in the tails, leading to practical positivity violations [Gerds et al., 2013]. This truncation happens implicitly for categorical models by choosing the bins.

To observe censoring events properly, we also require a version of eq. (1) to hold with the roles of $F$ and $G$ reversed:

$$\exists \epsilon \text{ s.t. } \forall x \forall t \in \{t \leq t_{\text{max}} \mid g(t|x) > 0\}, \quad \overline{F}(t|x) \geq \epsilon > 0.$$

$t_{\text{max}}$ should be chosen so that these two conditions hold.

A.3 Likelihoods

As mentioned, we assume data are i.i.d. and censoring is random $T \perp \!\!\!\!\perp C \mid X$. Under these assumptions, the likelihood, by definition [Andersen et al., 2012], is:

$$L(\theta_T, \theta_C) = \prod_i \left[ f_{\theta_T}(U_i) \overline{G}_{\theta_C}(U_i^-) \right]^{\Delta_i} \left[ g_{\theta_C}(U_i) F_{\theta_T}(U_i) \right]^{1-\Delta_i}, \quad (8)$$

When a failure is observed, $\Delta_i = 1[T_i \leq C_i] = 1$ so we compute the failure density or mass $f$ at the observed time $U_i = T_i$. In this case, the only thing we know about the censoring time is $C_i \geq T_i = U_i$. We therefore compute $P(C_i \geq T_i) = P(C_i ≥ U_i) = 1 - G_{\theta_C}(U_i^-) = \overline{G}_{\theta_C}(U_i^-)$. Likewise, when a censoring time is observed, $\Delta_i = 0$ so we compute the censoring density or mass $g$ at the observed censoring time $U_i = C_i$. In this case, the only thing we know about the failure time is that $T_i > C_i$. We therefore compute $P(T_i > C_i) = P(T_i > U_i) = 1 - F(U_i) = \overline{F}(U_i)$.

Under the additional assumption of non-informativeness -that $F$ and $G$ don’t share parameters and therefore $\theta_T, \theta_C$ are distinct- the $g/G$ terms are constant wrt $\theta_T$ and the $f/F$ terms are constant wrt $\theta_C$. In this case, when one is modeling failures, they can use the partial failure likelihood:

$$L(\theta_T)_{\text{partial}} = \prod_i \left[ f_{\theta_T}(U_i) \right]^{\Delta_i} \left[ \overline{F}_{\theta_T}(U_i) \right]^{1-\Delta_i}$$

And when one is modeling censoring they can use the partial censoring likelihood:

$$L(\theta_C)_{\text{partial}} = \prod_i \left[ G_{\theta_C}(U_i^-) \right]^{\Delta_i} \left[ g_{\theta_C}(U_i) \right]^{1-\Delta_i}.$$

A.4 Failure partial likelihood depends on true censoring distribution

We now show that the failure partial likelihood’s scale depends on the true sampling distribution of censoring times, even if the censoring model has dropped as a constant in the objective. The expected likelihood is:

$$\mathbb{E}_{T \sim F_{\theta_T}^*, C \sim G_{\theta_C}^*, U = \min(T, C) : \Delta = 1[T \leq C]} \left[ f_{\theta_T}(U)^{1[\Delta = 1]} \overline{F}_{\theta_T}(U)^{1[\Delta = 0]} \right]$$
The reason is that $\Delta$ and $U$ depend on $T$ and $C$ (therefore on $F_{\theta_1^T}$ and $G_{\theta_1^C}$). We now constructively show that the failure model’s NLL can vary with the true censoring distribution. Let us consider a marginal survival analysis problem (no features) and random censoring. The log NLL is:

$$
\mathbb{E}_{F_{\theta_1^T},G_{\theta_1^C}} [\Delta \log f_{\theta_1^T}(U)] + \mathbb{E}_{F_{\theta_1^T},G_{\theta_1^C}} [(1 - \Delta) \log \overline{F}_{\theta_1^T}(U)]
$$

Now consider an $F_{\theta_1^T}$ whose support starts at time 1 (e.g. uniform over 1,2,3) and $G_{\theta_1^C}$ such that there is probability $\rho$ that $C = 0$ and probability $1 - \rho$ that $C$ take a value above the support of $T$ (e.g. >3). Points are therefore only censored at time 0 or uncensored.

$$
\mathbb{E}_{F_{\theta_1^T},G_{\theta_1^C}} [\Delta \log f_{\theta_1^T}(U)] + \mathbb{E}_{F_{\theta_1^T},G_{\theta_1^C}} [(1 - \Delta) \log \overline{F}_{\theta_1^T}(U)]
= (1 - \rho) \mathbb{E}_{F_{\theta_1^T}} [\log f_{\theta_1^T}(T)] + \rho \mathbb{E}_{G_{\theta_1^C}} [\log \overline{F}_{\theta_1^T}(0)]
= (1 - \rho) \mathbb{E}_{F_{\theta_1^T}} [\log f_{\theta_1^T}(T)] + \rho \mathbb{E}_{G_{\theta_1^C}} [\log 1]
= (1 - \rho) \mathbb{E}_{F_{\theta_1^T}} [\log f_{\theta_1^T}(T)] + \rho \mathbb{E}_{G_{\theta_1^C}} [0]
= (1 - \rho) \mathbb{E}_{F_{\theta_1^T}} [\log f_{\theta_1^T}(T)]
$$

This quantity depends on $\rho$. This shows that the failure model’s NLL depends on the true sampling distribution of censoring times.

**B IPCW Primer**

IPCW is a technique for estimation under censoring \cite{Gerds2006}. Consider estimating the marginal mean of $T : \mathbb{E}[T] = \mathbb{E}_X \mathbb{E}_T|X[T]$. $T$ is not observed for all datapoints. Instead, we observe $U = \min(T, C)$ and $\Delta = \mathbb{I}[T \leq C]$. IPCW reformulates such expectations in terms of observed data. Using this method, we can show that:

$$
\mathbb{E}_X \mathbb{E}_T[X|T] = \mathbb{E}_X \mathbb{E}_T[X|T] \left[ \frac{\mathbb{E}_C[X|T] \mathbb{I}[T \leq C]}{\mathbb{E}_C[X|T] \mathbb{I}[T \leq C]} T \right]
= \mathbb{E}_X \mathbb{E}_T[X|C] \left[ \frac{\mathbb{I}[T \leq C]}{\mathbb{E}_C[X|T] \mathbb{I}[T \leq C]} T \right]
= \mathbb{E}_T \mathbb{E}_X \mathbb{E}_C \left[ \frac{\mathbb{I}[T \leq C]}{\mathbb{E}_C[X|T] \mathbb{I}[T \leq C]} T \right]
= \mathbb{E}_T \mathbb{E}_X \mathbb{E}_C \left[ \frac{\mathbb{I}[T \leq C]}{G(T|X)} T \right]
= \mathbb{E}_U \mathbb{E}_X \left[ \frac{\Delta U}{G(U|X)} T \right]
$$

We have used $C'$ in the denominator to emphasize that it is not a function of $C$ in the integral over the numerator indicator once that expectation is moved out. We have used random censoring to go from $\mathbb{E}_T[X|E_C]$ to the joint $\mathbb{E}_T,C[X]$. The last equality changes from the complete data distribution to the observed distribution and holds because $\Delta = 1 \implies U = T$. This means we can estimate the
expectation, provided that we know $G$ and that random censoring and positivity (eq. (1)) hold. In practice, we must learn the censoring distribution, a challenging task as it is also censored.

Graf et al. [1999] develop the IPCW BS. Gerds and Schumacher [2006] extend it to conditional censoring and Kvamme and Borgan [2019a] specialize to administrative censoring. Gerds et al. [2013], Wolbers et al. [2014] develop the IPCW concordance. IPCW estimators for several forms of area under curve (AUC) have been studied in Hung and Chiang [2010a,b], Blanche et al. [2013, 2019], Uno et al. [2007]. Yadlowsky et al. [2019] derive an IPCW estimator for binary survival calibration.

C Deriving IPCW Brier Scores

We derive the IPCW BS introduced by Graf et al. [1999], Gerds and Schumacher [2006]. In the below let F-BS be the F model’s BS and let F-BS-CW be its censor-weighted version. The censor-weighted failure BS:

$$
F-BS-CW(t) = \mathbb{E}_{T,C} \left[ \frac{(1 - F_\theta(t))^2 \mathbb{1}[T \leq C] \mathbb{1}[U \leq t]}{P_\theta(C' \geq U)} + \frac{F_\theta(t)^2 \mathbb{1}[U > t]}{P_\theta(C' > t)} \right]
$$

where $U = \min(T, C)$ and $F_\theta = P_\theta(T \leq \cdot)$. Its relationship to the regular BS is:

$$
F-BS(t) = \mathbb{E}_T \left[ \left( F_\theta(t) - 1 \mathbb{1}[T \leq t] \right)^2 \right]
$$

$$
= \mathbb{E}_T \left[ (1 - F_\theta(t))^2 \mathbb{1}[T \leq t] + F_\theta(t)^2 \mathbb{1}[T > t] \right]
$$

$$
= \mathbb{E}_T \left[ \frac{E_C \mathbb{1}[T \leq C]}{E_C' \mathbb{1}[T \leq C']} (1 - F_\theta(t))^2 \mathbb{1}[T \leq t] + \frac{E_C \mathbb{1}[C > t]}{E_C' \mathbb{1}[C' > t]} F_\theta(t)^2 \mathbb{1}[T > t] \right]
$$

$$
= \mathbb{E}_{T,C} \left[ \frac{(1 - F_\theta(t))^2 \mathbb{1}[T \leq C] \mathbb{1}[T \leq t]}{P_\theta(C' \geq T)} + \frac{F_\theta(t)^2 \mathbb{1}[T > t] \mathbb{1}[C > t]}{P_\theta(C' > t)} \right]
$$

$$
= \mathbb{E}_{T,C} \left[ \frac{(1 - F_\theta(t))^2 \mathbb{1}[T \leq C] \mathbb{1}[U \leq t]}{P_\theta(C' \geq U)} + \frac{F_\theta(t)^2 \mathbb{1}[U > t]}{P_\theta(C' > t)} \right]
$$

$$
= F-BS-CW(t)
$$

The expectation comes out due to $T \perp C$. The last line follows from $T \leq C \implies U = T$ (in the left term) and $1[T > t] \mathbb{1}[C > t] = 1[U > t]$ (in the right term). Define likewise the failure-weighted censor BS

$$
G-BS-CW(t) = \mathbb{E}_{T,C} \left[ \frac{(1 - G_\theta(t))^2 \mathbb{1}[C < T] \mathbb{1}[U \leq t]}{P_\theta(T' > U)} + \frac{G_\theta(t)^2 \mathbb{1}[U > t]}{P_\theta(T' > t)} \right]
$$

where $G_\theta = P_\theta(C \leq \cdot)$. The relationship to the censoring distribution’s BS is:

$$
G-BS(t) = \mathbb{E}_C \left[ (G_\theta(t) - 1 \mathbb{1}[C \leq t])^2 \right]
$$

$$
= \mathbb{E}_C \left[ (1 - G_\theta(t))^2 \mathbb{1}[C \leq t] + G_\theta(t)^2 \mathbb{1}[C > t] \right]
$$

$$
= \mathbb{E}_C \left[ \frac{E_T \mathbb{1}[C < T]}{E_T' \mathbb{1}[C < T']} (1 - G_\theta(t))^2 \mathbb{1}[C \leq t] + \frac{E_T \mathbb{1}[T > t]}{E_T' \mathbb{1}[T' > t]} G_\theta(t)^2 \mathbb{1}[C > t] \right]
$$

$$
= \mathbb{E}_{T,C} \left[ \frac{(1 - G_\theta(t))^2 \mathbb{1}[C < T] \mathbb{1}[C \leq t]}{P_\theta(T' > C)} + \frac{G_\theta(t)^2 \mathbb{1}[T > t] \mathbb{1}[C > t]}{P_\theta(T' > t)} \right]
$$

$$
= \mathbb{E}_{T,C} \left[ \frac{(1 - G_\theta(t))^2 \mathbb{1}[C < T] \mathbb{1}[U \leq t]}{P_\theta(T' > U)} + \frac{G_\theta(t)^2 \mathbb{1}[U > t]}{P_\theta(T' > t)} \right]
$$

$$
= G-BS-CW(t)
$$
The expectation comes out due to $T \perp C$. The last line follows from $C < T \implies U = C$ (in the left term) and $1[T > t]1[C > t] = 1[U > t]$ (in the right term).

D Negative Bernoulli Log Likelihood

Negative BLL is similar to BS, but replaces the squared error with negated log loss:

$$\text{NBLL}(t; \theta) = \mathbb{E}_{T, C, X} \left[ -\log(F_{\theta_T}(t | X))1[T \leq t] - \log(F_{\theta_C}(t | X))1[T > t] \right]$$

IPCW BLL can likewise be written as [Kvamme et al., 2019]:

$$\text{F-NBLL-CW}(t; \theta) = \mathbb{E}_{T, C, X} \left[ -\log(F_{\theta_T}(t | X))\Delta 1[U \leq t] + \frac{-\log(F_{\theta_C}(t | X))1[U > t]}{G(t | X)} \right]$$

E Game Algorithm

Algorithm 2 Following Gradients in Multi-Player Games

Input: Choice of losses $\ell_F, \ell_G$, learning rate $\gamma$

Initialize $\theta_{T t}$ and $\theta_{C t}$ randomly for $t = 1, \ldots, K - 1$

repeat
  // for each parameter of each player
  for $t = 1$ to $K - 1$ do
    $g_{T t} \leftarrow \frac{d\ell_F}{d\theta_{T t}}$
    $g_{C t} \leftarrow \frac{d\ell_G}{d\theta_{C t}}$
  end for

  // for each parameter of each player
  for $t = 1$ to $K - 1$ do
    $\theta_{T t} \leftarrow \theta_{T t} - \gamma g_{T t}$
    $\theta_{C t} \leftarrow \theta_{C t} - \gamma g_{C t}$
  end for

until convergence

F Experiments

F.1 Data

Gamma Simulation We draw $x$ from a 32 dimensional multivariate normal $\mathcal{N}(0, 10I)$. We simulate conditionally gamma failure times with mean $\mu_t$ a log-linear function of $x$ with coefficients for each feature drawn Unif$(0, 0.1)$. The censoring times are also conditionally gamma with mean $0.9 * \mu_t$. Both distributions have constant variance $0.05$. $\alpha, \beta$ parameterization of the gamma is recovered from mean, variance by $\alpha = \mu^2/\sigma^2$ and $\beta = \mu/\sigma^2$. $T$ and $C$ are conditionally independent given $X$. Each random seed draws a new dataset.

We report metrics as a function of training size. We use training sizes $[200, 400, 600, 800, 1000]$. We use validation size 1024 and testing size 2048.

Survival MNIST Survival-MNIST [Gensheimer, 2019, Pölsterl, 2019] draws times conditionally on MNIST label $Y$. This means digits define risk groups and $T \perp X | Y$. Times within a digit are i.i.d. The model only sees the image pixels $X$ as covariates so it must learn to classify digits (risk groups) to model times. The PyCox package [Kvamme et al., 2019] uses Exponential times. We follow Goldstein et al. [2020] and use Gamma times. $T$’s mean is $10 * (Y + 1)$ so that lower labels $Y$ mean sooner event times. We set the variance constant to $0.05$. $C$ is drawn similarly but with $9.9 * (Y + 1)$. Each random seed draws a new dataset.

We report metrics as a function of training size. We use training sizes $[512, 1024, 2048, 4096, 8192, 10240]$. We use validation size 1024 and testing size 2048.
Real Data  We report results on

- **SUPPORT** [Knaus et al., 1995] which includes severely ill hospital patients. There are 14 features. We split into 5,323 for training, 1,774 for validation, and 1,776 for testing.
- **METABRIC** [Curtis et al., 2012]. There are 9 features. We split into 1,142 for training, 380 for validation, and 382 for testing.
- **ROTT** [Foekens et al., 2000] and GBSG [Schumacher et al., 1994] combined into one dataset (ROTT. & GBSG). There are 7 features. We split into 1,339 for training, 446 for validation, and 447 for testing.

For more description see Therneau [2021], Katzman et al. [2018], Chen [2020].

In the main text, we report results on a subset of these datasets with metrics as a function of training size. We use training sizes [10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 175, 200]. We use validation size 300 and always use the entire testing set. We standardize all real data with the training set mean and standard deviation.

F.2 Models

In all experiments except for MNIST, we use a 3-hidden-layer ReLU network. The hidden sizes are [128, 64, 64] for the Gamma simulation and [128, 256, 64] for the real data. We output 20 categorical bins. See appendix G.1 for different choices of number of bins, which did not show any significant differences in results. For MNIST we first use a small convolutional network and follow with the same fully-connected network, but using hidden sizes [512, 256, 64].

F.3 Training

We use learning rate 0.001 in all experiments for all losses using the Adam optimizer. We train for 300 epochs for the simulated data and 200 for the real data. For all data and all losses, this was enough to overfit on the training data. We use no weight decay or dropout.

F.4 Model Selection

We select the best model on the validation set using the following approach:

1. Save the $F$ and $G$ models from all the epochs in $F$-set and $G$-set.
2. Randomly choose a model $\tilde{F}$ in the $F$-set.
3. Use $\tilde{F}$ as the weight for $\ell_G$. Find the model $\tilde{G}$ from $G$-set to minimize $\ell_G$ weighted by $\tilde{F}$.
4. Use $\tilde{G}$ as the weight for $\ell_F$. Find the model $\tilde{F}$ from $F$-set to minimize $\ell_F$ weighted by $\tilde{G}$.
5. Repeat steps 3 and 4 until convergence.

Once converged, we use $\tilde{F}$ and $\tilde{G}$ as our best model to evaluate at the test set. The above approach plays as similar role as the game. Instead of gradient descent, this time we select a model from a set to play the game. We first fix $F$ to find the best $G$ based on $\ell_G$ and then fix $G$ to find the best $F$ based on $\ell_F$. 
G  Ablations

G.1  Changing number of bins on MNIST

Changing number of categorical bins (K) in $[10, 20, 30, 40, 50]$. Cannot directly compare between two choices of K due to changing meaning of likelihood/BS/Concordance but can compare NLL and BS-Game at each K. Trends similar across all choices of K.

(a) Uncensored BS  (b) Uncensored Neg BLL  (c) Concordance  (d) Categorical NLL

Figure 9: 10 bins. NLL (Blue). BS-Game (Orange).

Figure 10: 20 bins. NLL (Blue). BS-Game (Orange).

Figure 11: 30 bins. NLL (Blue). BS-Game (Orange).

Figure 12: 40 bins. NLL (Blue). BS-Game (Orange).

H  Proof of Summed or Integrated Brier Score to be proper

Proposition 3. Assume we have a list of time $t_1, \ldots, t_K$. Assume the true distribution for $T$ is $F^* = F_{\theta^*}$ in eq. (2). We have:

- The summed BS $\sum_{i=1}^{K} BS(t_i; \theta)$ is proper, i.e., it has one minimizer at the true parameters $\theta_{\min}$. 


• The integrated BS $\int_{t_1}^{t_K} BS(t; \theta) dt$ is proper, i.e., it has one minimizer at the true parameters $\theta^*_T$.

Proof. Since BS$(t)$ is proper, it has one minimizer at $\theta^*_T$, i.e., for $\theta_T \neq \theta^*_T$, BS$(t; \theta_T^*) \leq$ BS$(t; \theta_T)$ for all $t$. Since this holds for all $t$, we then have:

$$\sum_{i=1}^{K} BS(t_i; \theta_T^*) \leq \sum_{i=1}^{K} BS(t_i; \theta_T).$$

This means that the summed Brier Score at $\theta_T^*$ is smaller than at any other $\theta_T$. The summed BS has one minimizer at the true parameters $\theta_T^*$, i.e., it is proper. Since the BS inequality holds for all $t$, we also have

$$\int_{t_1}^{t_K} BS(t; \theta_T^*) dt \leq \int_{t_1}^{t_K} BS(t; \theta_T) dt$$

This means that the integrated Brier Score at $\theta_T^*$ is smaller than at any other $\theta_T$. The integrated BS has one minimizer at the true parameters $\theta_T^*$, i.e., it is proper. \hfill \Box

I Proof of proposition 1

Here we prove that the true solution is a stationary point of the game. We restate the proposition here.

Proposition. Assume $\exists \theta_T^* \in \Theta_T, \exists \theta_C^* \in \Theta_C$ such that $F^* = F_{\theta_T^*}$ and $G^* = G_{\theta_C^*}$. Assume the game losses $\ell_F, \ell_G$ are based on proper losses $L$ and that the games are only computed at times for which positivity holds. Then $(\theta_T^*, \theta_C^*)$ is a stationary point of the game eq. (4).

$$\ell_F(\theta) = L_I(F_{\theta_T}; G_{\theta_C}), \quad \ell_G(\theta) = L_I(G_{\theta_C}; F_{\theta_T})$$

(4)

Proof. In $\ell_F(\theta)$, by the definition of the IPCW estimator, when $\theta_C = \theta_C^*$, $L_I(F_{\theta_T}; G_{\theta_C}) = L(F_{\theta_T})$. Due to the fact that $L$ is proper, $\theta_T^*$ is a minimizer for $L(F_{\theta_T})$. Then at $(\theta_T, \theta_C) = (\theta_T^*, \theta_C^*)$, we have

$$\frac{d\ell_F(\theta)}{d\theta_T} \bigg|_{\theta_T = \theta_T^*, \theta_C = \theta_C^*} = \frac{dL_I(F_{\theta_T}; G_{\theta_C})}{d\theta_T} \bigg|_{\theta_T = \theta_T^*, \theta_C = \theta_C^*} = \frac{dL(F_{\theta_T})}{d\theta_T} \bigg|_{\theta_T = \theta_T^*} = 0$$

Similarly for $\ell_G(\theta)$, we have

$$\frac{d\ell_G(\theta)}{d\theta_C} \bigg|_{\theta_C = \theta_C^*, \theta_T = \theta_T^*} = \frac{dL_I(G_{\theta_C}; F_{\theta_T})}{d\theta_C} \bigg|_{\theta_C = \theta_C^*, \theta_T = \theta_T^*} = \frac{dL(G_{\theta_C})}{d\theta_C} \bigg|_{\theta_C = \theta_C^*} = 0$$

Since the two gradients are zero, the game will stay at the true parameters. Therefore, $(\theta_T^*, \theta_C^*)$ is a stationary point of the game eq. (4). \hfill \Box

J Proof of proposition 2

Here we prove that under one construction of the game in algorithm 2, the true solution is the unique stationary point of the game. We restate the proposition here.
**Proposition.** Consider discrete distributions over \( K \) times. Let \( \theta_T = \{\theta_{T1}, \ldots, \theta_{T(K-1)}\} \), \( \theta_{Tt} = F_0(T = t), F_0(t) = \sum_{k=1}^t \theta_{Tk} \), and likewise for \( C, \theta_C \). Assuming that \( \theta_{Tt} > 0 \) and \( \theta_{Ct} > 0 \), the solution \((\theta_{Tt}, \theta_{Ct})\) is the only stationary point for the multi-player BS game shown in algorithm 2 for times \( t \in \{1, \ldots, K - 1\} \).

**Proof.** We show by induction on the time \( t \) of the IPCW BS game that the simultaneous gradient equations are only satisfied at \( \hat{\theta}_t = \theta_{Tt}^* \) and \( \hat{\theta}_C = \theta_{Ct}^* \). There is a lot of arithmetic but eventually it comes down to \( t \) substitution of one variable for another \((t)\) assuming all previous timestep parameters are correct \((\text{induction})\) \((3)\) finding the zeros of a quadratic \((4)\) showing that one of the two solutions is the correct parameter and the other is invalid.

**Note:** this proof uses the notation that \( \hat{\theta} \) is a model parameter and \( \theta^* \) is the correct one.

**J.1 BS(1) (base case)**

We can compute the expectations defining F-BS-CW(1) and G-BS-CW(1) in closed form. That gives us:

\[
\begin{align*}
\text{F-BS-CW}(1) &= \theta_{T1}^*(1 - \hat{\theta}_{T1})^2 + (1 - \theta_{T1}^*)(1 - \theta_{C1}^*) \frac{\hat{\theta}_{T1}^2}{1 - \theta_{C1}} \\
\text{G-BS-CW}(1) &= \theta_{C1}^*(1 - \theta_{T1}^*)(1 - \theta_{C1}^*)^2 + (1 - \theta_{T1}^*)(1 - \theta_{C1}^*) \frac{\hat{\theta}_{C1}^2}{1 - \theta_{T1}^*}
\end{align*}
\]

The derivatives are

\[
\begin{align*}
\frac{d\text{F-BS-CW}(1)}{d\theta_{T1}} &= 2 \frac{(1 - \theta_{T1}^*)(1 - \theta_{C1}^*)}{1 - \theta_{C1}} \hat{\theta}_{T1} - 2(1 - \theta_{T1}^*)\theta_{T1}^* = 0 \\
\frac{d\text{G-BS-CW}(1)}{d\theta_{C1}} &= 2 \frac{(1 - \theta_{T1}^*)(1 - \theta_{C1}^*)}{1 - \theta_{T1}^*} \hat{\theta}_{C1} - 2 \frac{(1 - \theta_{T1}^*)(1 - \hat{\theta}_{C1}^*)\theta_{C1}^*}{1 - \theta_{T1}^*} = 0
\end{align*}
\]

We can take each derivative equation and write one variable in terms of the other. First, taking \( \frac{d\text{F-BS-CW}}{d\theta_{T1}} \) and writing \( \hat{\theta}_{T1} \) in terms of \( \theta_{C1}^* \):

\[
\frac{d\text{F-BS-CW}(1)}{d\theta_{T1}} = 2 \frac{(1 - \theta_{T1}^*)(1 - \theta_{C1}^*)}{1 - \theta_{C1}} \hat{\theta}_{T1} - 2(1 - \theta_{T1}^*)\theta_{T1}^* = 0
\]

then implies

\[
\begin{align*}
\frac{(1 - \theta_{T1}^*)(1 - \theta_{C1}^*)}{1 - \theta_{C1}} \hat{\theta}_{T1} &= (1 - \theta_{T1}^*)\theta_{T1}^* \\
\frac{(1 - \theta_{T1}^*)(1 - \theta_{C1}^*)}{1 - \theta_{C1}} \hat{\theta}_{T1} + \theta_{T1}^* \hat{\theta}_{T1} &= \theta_{T1}^* \\
\left(\frac{(1 - \theta_{T1}^*)(1 - \theta_{C1}^*)}{1 - \theta_{C1}} + \theta_{T1}^*\right) \hat{\theta}_{T1} &= \theta_{T1}^* \\
\hat{\theta}_{T1} &= \frac{\theta_{T1}^*}{\left(\frac{(1 - \theta_{T1}^*)(1 - \theta_{C1}^*)}{1 - \theta_{C1}} + \theta_{T1}^*\right)}
\end{align*}
\]

Now solving for \( \hat{\theta}_{C1} \) in the G-BS-CS derivative:

\[
\frac{d\text{G-BS-CW}(1)}{d\theta_{C1}} = 2 \frac{(1 - \theta_{T1}^*)(1 - \theta_{C1}^*)}{1 - \theta_{T1}} \hat{\theta}_{C1} - 2 \frac{(1 - \theta_{T1}^*)(1 - \hat{\theta}_{C1}^*)\theta_{C1}^*}{1 - \theta_{T1}^*} = 0
\]

which implies

\[
\begin{align*}
\frac{(1 - \theta_{T1}^*)(1 - \theta_{C1}^*)}{1 - \theta_{T1}} \hat{\theta}_{C1} &= (1 - \theta_{T1}^*)(1 - \hat{\theta}_{C1}^*)\theta_{C1}^* \\
\frac{(1 - \theta_{T1}^*)(1 - \theta_{C1}^*)}{1 - \theta_{T1}} \hat{\theta}_{C1} = \frac{(1 - \theta_{T1}^*)(1 - \hat{\theta}_{C1}^*)\theta_{C1}^*}{1 - \theta_{T1}} \theta_{C1}^*
\end{align*}
\]
Given \(1 - \theta^*_T \neq 0\) and \(1 - \hat{\theta}_T \neq 0\), we have
\[
(1 - \theta^*_C)(\hat{\theta}_C) = (1 - \hat{\theta}_C)\theta^*_C
\]
which gives us \(\hat{\theta}_C = \theta^*_C\). Given \(1 - \theta^*_T \neq 0\) and \(1 - \hat{\theta}_T \neq 0\), the above derivative equations jointly imply
\[
\hat{\theta}_T = \left(\theta^*_T\right)\left(\frac{(1 - \theta^*_T)(1 - \theta^*_C)}{1 - \hat{\theta}_C} + \theta^*_T\right)^{-1}, \quad \hat{\theta}_C = \theta^*_C
\]
Substituting \(\hat{\theta}_C = \theta^*_C\) in the formula for \(\hat{\theta}_T\) in terms of \(\hat{\theta}_C\), we have
\[
\hat{\theta}_T = \left(\theta^*_T\right)\left(\frac{(1 - \theta^*_T)(1 - \theta^*_C)}{1 - \theta^*_C} + \theta^*_T\right)^{-1} = \frac{\theta^*_T}{(1 - \theta^*_T) + \theta^*_T} = \theta^*_T
\]
Therefore, under the assumptions, for the BS(1) case, we have the only stationary point at the two true 1st-timestep parameters: \(\hat{\theta}_T = \theta^*_T\) and \(\hat{\theta}_C = \theta^*_C\).

J.2 Induction step

We can proceed by induction over timesteps. Claim: given \(P_\theta(T \leq a) = P^*(T \leq a)\) and \(P_\theta(C \leq a) = P^*(C \leq a), \quad a = 1, \ldots, k\), the stationary point of the game BS(k+1) has to satisfy \(P_\theta(T = k + 1) = P^*(T = k + 1)\) and \(P_\theta(C = k + 1) = P^*(C = k + 1)\) i.e. \(\hat{\theta}_{T,k+1} = \theta^*_{T,k+1}\) and \(\hat{\theta}_{C,k+1} = \theta^*_{C,k+1}\). We first simplify F-BS-CW.

\[
\text{F-BS-CW}(k + 1) = \mathbb{E}_{T,C} \left[ \frac{(1 - F_\theta(k + 1))^2 \mathbb{I}[T \leq C] \mathbb{I}[U \leq k + 1]}{P_\theta(C' \geq U)} + \frac{F_\theta(k + 1)^2 \mathbb{I}[U > k + 1]}{P_\theta(C' > k + 1)} \right]
\]
We simplify each term of F-BS-CW separately. The left term of F-BS-CW is

\[
\frac{\mathbb{E}_{T,C} (1 - F_\theta(k + 1))^2 \mathbb{1}[T \leq C] \mathbb{1}[U \leq k + 1]}{P_\theta(C' \geq U)} = P_\theta(T > k + 1)^2 \frac{\mathbb{1}[T \leq C] \mathbb{1}[U \leq k + 1]}{P_\theta(C' \geq U)}
\]

\[
= P_\theta(T > k + 1)^2 \sum_{a=1}^{K} \sum_{b=a}^{K} P^*(T = a) P^*(C = b) \frac{1[a \leq b] \mathbb{1}[\min(a, b) \leq k + 1]}{P_\theta(C' \geq \min(a, b))}
\]

(condition $1[a \leq b]$ moves from indicator to sum limits and $\min(a, b) = a$)

\[
= P_\theta(T > k + 1)^2 \sum_{a=1}^{K} \sum_{b=a}^{K} P^*(T = a) P^*(C = b) \frac{1[a \leq k + 1]}{P_\theta(C' \geq a)}
\]

(condition $1[a \leq k + 1]$ moves from indicator to sum limit)

\[
= P_\theta(T > k + 1)^2 \sum_{a=1}^{k+1} \sum_{b=a}^{K} P^*(T = a) P^*(C = b) \frac{1[a \leq k + 1]}{P_\theta(C' \geq a)}
\]

[induction hypothesis: $P_\theta(C \leq a) = P^*(C \leq a), \ a = 1, \ldots, k \implies P_\theta(C > a) = P^*(C > a), \ a = 1, \ldots, k$]

\[
= P_\theta(T > k + 1)^2 \sum_{a=1}^{k+1} P^*(T = a) \cdot 1
\]

\[
= P_\theta(T > k + 1)^2 P^*(T \leq k + 1)
\]

\[
= (1 - \sum_{i=1}^{k} \hat{\theta}_{T,i} - \hat{\theta}_{T,(k+1)})^2 \sum_{i=1}^{k+1} \theta^*_{T,i}
\]

[induction hypothesis: $P_\theta(T \leq a) = P^*(T \leq a), \ a = 1, \ldots, k$]

\[
= (1 - \sum_{i=1}^{k} \hat{\theta}^*_{T,i} - \hat{\theta}_{T,(k+1)})^2 \sum_{i=1}^{k+1} \theta^*_{T,i}
\]

\[
= (1 - p - x)^2 (p + t)
\]

\[
\Delta \mathcal{A}, \text{ where } p = \sum_{i=1}^{k} \hat{\theta}^*_{C,i}, \ g = \sum_{i=1}^{k} \theta^*_{C,i}, \ x = \hat{\theta}_{T,(k+1)}, \ y = \hat{\theta}_{C,(k+1)}, \ t = \theta^*_{T,(k+1)} C = \theta^*_{C,(k+1)}.
\]
The right term of F-BS-CW is

$$\mathbb{E}_{T,C} \frac{F_\theta(k+1)^2 \mathbbm{1}[U > k + 1]}{P_\theta(C' > k + 1)}$$

$$= \frac{F_\theta(k+1)^2}{P_\theta(C' > k + 1)} \mathbb{E}_{T,C} \mathbbm{1}[U > k + 1]$$

$$= \frac{F_\theta(k+1)^2}{P_\theta(C' > k + 1)} P^*(T > k + 1)P^*(C > k + 1)$$

$$= \frac{(\sum_{i=1}^{k+1} \hat{\theta}_{T,i})^2}{1 - \sum_{i=1}^{k+1} \hat{\theta}_{C,i}} (1 - \sum_{i=1}^{k+1} \theta_{T,i}) (1 - \sum_{i=1}^{k+1} \theta_{C,i})$$

$$\text{induction hypothesis: } P_\theta(T \leq a) = P^*(T \leq a) \text{ and } P_\theta(C \leq a) = P^*(C \leq a), \quad a = 1, \ldots, k$$

$$= \frac{(\sum_{i=1}^{k} \theta_{T,i} + \hat{\theta}_{T(k+1)})^2}{1 - \sum_{i=1}^{k} \theta_{C,i} - \hat{\theta}_{C(k+1)}} (1 - \sum_{i=1}^{k} \theta_{T,i}) (1 - \sum_{i=1}^{k} \theta_{C,i})$$

$$= \frac{(p + x)^2}{1 - q - y} (1 - p - t)(1 - q - c) \equiv B$$

where again \( p = \sum_{i=1}^{k} \theta_{T,i}, q = \sum_{i=1}^{k} \theta_{C,i}, x = \hat{\theta}_{T(k+1)}, y = \hat{\theta}_{C(k+1)}, t = \theta_{T(k+1)}, c = \theta_{C(k+1)} \).

To summarize, F-BS-CW\((k + 1) = A + B:\)

$$F-BS-CW(k + 1) = (1 - p - x)^2(p + t) + \frac{(p + x)^2}{1 - q - y} (1 - p - t)(1 - q - c)$$

Then we simplify G-BS-CW.

$$G-BS-CW(k + 1) = \mathbb{E}_{T,C} \left[ \frac{(1 - G_\theta(k+1)^2 \mathbbm{1}[C < T] \mathbbm{1}[U \leq k + 1]}{P_\theta(T' > U)} + \frac{G_\theta(k+1)^2 \mathbbm{1}[U > k + 1]}{P_\theta(T' > k + 1)} \right]$$
The left term of G-BS-CW

\[
\mathbb{E}_{T,C} \frac{(1 - G_\delta(k + 1))^2 \mathbbm{1} [C < T] \mathbbm{1} [U \leq k + 1]}{P_\delta(T' > U)} = (1 - G_\delta(k + 1))^2 \mathbb{E}_{T,C} \frac{\mathbbm{1} [C < T] \mathbbm{1} [U \leq k + 1]}{P_\delta(T' > U)}
\]

\[
= (1 - G_\delta(k + 1))^2 \sum_{a = 1}^K \sum_{b = a + 1}^K P^*(C = a) P^*(T = b) \mathbbm{1} [a < b] \mathbbm{1} [\min(a, b) \leq k + 1] \frac{P_\delta(T' > \min(a, b))}{P_\delta(T' > a)}
\]

condition \( \mathbbm{1} [a < b] \) moves from indicator to sum limits and \( \min(a, b) = a \)

\[
= (1 - G_\delta(k + 1))^2 \sum_{a = 1}^K \sum_{b = a + 1}^K P^*(C = a) P^*(T = b) \mathbbm{1} [a \leq k + 1] \frac{P_\delta(T' > a)}{P_\delta(T' > a)}
\]

condition \( \mathbbm{1} [a \leq k + 1] \) moves from indicator to sum limits

\[
= (1 - G_\delta(k + 1))^2 \sum_{a = 1}^k \sum_{b = a + 1}^k P^*(C = a) P^*(T = b) \quad \text{split sum over } a \text{ into two terms: 1 through } k, \text{ and } k+1, \text{ recall } b \text{ starts at } a+1
\]

\[
= (1 - G_\delta(k + 1))^2 \left( \sum_{a = 1}^k \sum_{b = a + 1}^k P^*(C = a) P^*(T = b) \frac{P_\delta(T' > a)}{P_\delta(T' > a)} + \sum_{b = k + 2}^K P^*(C = k + 1) P^*(T = b) \frac{P_\delta(T' > k + 1)}{P_\delta(T' > k + 1)} \right)
\]

\[
= (1 - G_\delta(k + 1))^2 \left( \sum_{a = 1}^k P^*(C = a) P^*(T = a + 1) \frac{P_\delta(T' > a)}{P_\delta(T' > a)} + \sum_{b = k + 2}^K P^*(C = k + 1) P^*(T = k + 1) \frac{P_\delta(T' > k + 1)}{P_\delta(T' > k + 1)} \right)
\]

induction hypothesis: \( P_\delta(T \leq a) = P^*(T \leq a), \ a = 1, \ldots, k \iff P_\delta(T > a) = P^*(T > a), \ a = 1, \ldots, k \)

\[
= (1 - G_\delta(k + 1))^2 \left( \sum_{a = 1}^k P^*(C = a) + \frac{P^*(C = k + 1) P^*(T > k + 1)}{P_\delta(T' > k + 1)} \right)
\]

\[
= (1 - G_\delta(k + 1))^2 \left( \sum_{i = 1}^k \hat{\theta}_{C,i} - \hat{\theta}_{C(k+1)} \right)^2 \left( \sum_{i = 1}^k \hat{\theta}_{C,i} + \frac{\hat{\theta}_{C(k+1)}(1 - \hat{\theta}_{T(k+1)} - \sum_{i = 1}^k \hat{\theta}_{T,i})}{1 - \sum_{i = 1}^k \hat{\theta}_{T,i} - \hat{\theta}_{T(k+1)}} \right)
\]

induction hypothesis: \( P_\delta(T \leq a) = P^*(T \leq a) \) and \( P_\delta(C \leq a) = P^*(C \leq a), \ a = 1, \ldots, k \)

\[
= (1 - G_\delta(k + 1))^2 \left( \sum_{i = 1}^k \hat{\theta}_{C,i} - \hat{\theta}_{C(k+1)} \right)^2 \left( \sum_{i = 1}^k \hat{\theta}_{C,i} + \frac{\hat{\theta}_{C(k+1)}(1 - \hat{\theta}_{T(k+1)} - \sum_{i = 1}^k \hat{\theta}_{T,i})}{1 - \sum_{i = 1}^k \hat{\theta}_{T,i} - \hat{\theta}_{T(k+1)}} \right)
\]

By symmetry with \( B \), the right term is

\[
\mathbb{E}_{T,C} \frac{G_\delta(k + 1) \mathbbm{1} [U > k + 1]}{P_\delta(T' > k + 1)} = \frac{(q + y)^2}{1 - p - x} (1 - q - c)(1 - p - t) \triangleq D
\]
Again using \( p = \sum_{i=1}^{K} \theta_{Ti}, q = \sum_{i=1}^{K} \theta_{Ci}, x = \hat{\theta}_{T(k+1)}, y = \hat{\theta}_{C(k+1)}, t = \theta_{T(k+1)}, \), we have
\[
G-BS-CW(k + 1) = C + D = (1 - q - y)^2(q + \frac{c(1 - t - p)}{1 - p - x}) + \frac{(q + y)^2}{1 - p - x}(1 - q - c)(1 - p - t)
\]
The numerator and the denominator are both negative. If \( q + y \) that satisfies the above solution, then
\[
\frac{\partial G-\text{wt-BS}(k+1)}{\partial x} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial x} = -2(1 - p - x)(p + t) + 2\frac{(p + x)}{1 - q - y}(1 - p - t)(1 - q - c)
\]
\[
\frac{\partial F-\text{wt-GBS}(k+1)}{\partial y} = \frac{\partial C}{\partial y} + \frac{\partial D}{\partial y} = -2(1 - q - y)(q + \frac{c(1 - t - p)}{1 - p - x}) + 2\frac{(q + y)}{1 - p - x}(1 - q - c)(1 - p - t) = 0
\]
It’s a system of quadratic equations with two unknowns. The system has analytical solutions. Solving the above equations for \( x, y \) by Mathematica (it is quite a long derivation manually), the solutions are
\[
x = t, y = c
\]
or
\[
x = \left(1/(1 - q + q^2 + q)\right)(c - qcp - qt + q^2t + ct)
- \left(p(-1 + q + c) - q^2p - cp + qt - q^2t - ct\right)/(1 - q)(p + t)
+ \left(q(\frac{c(1 - t - p)}{1 - p - x}) - q^2p - cp + qt - q^2t - ct\right)/(1 - q)(p + t)
- \left(t(-1 + q + c + q^2p - cp + qt - q^2t - ct\right)/(1 - q)(p + t)
+ \left(q(-1 + q + c + ap - q^2p - cp + qt - q^2t - ct\right)/(1 - q)(p + t)
\]
\[
y = \left(1 + q + c + gp - q^2p - cp + qt - q^2t - ct\right)/(1 - q)(p + t)
\]
To check if this second solution is valid, it would need to be the case that \( q + y < 1 \) because we only consider \( k + 1 \) \( < \) \( K \). If we ask mathematica to simplify \( q+y \) that satisfies the above solution, then this holds:
\[
q + y = \frac{-1 + q - c(-1 + p + t)}{(1 - q)(p + t)}
\]
The numerator and the denominator are both negative. If \( k + 1 \) \( < \) \( K \) (we know BS at \( K \) is 0 and also we only have \( K \)-1 parameters), the numerator minus denominator
\[
-1 + q - c(-1 + p + t) - (-1 + q)(p + t) = (-1 + q)(1 - p - t) - c(-1 + p + t)
= (1 - q + c)(1 - p - t) < 0
\]
Therefore,
\[
\sum_{i=1}^{K} \theta_{Ci}^* + \hat{\theta}_{C(k+1)} = q + y > 1
\]
This is invalid. So
\[
x = t, y = c
\]
is the only solution, i.e., \( \hat{\theta}_{T(k+1)} = \theta_{T(k+1)}^*; \hat{\theta}_{C(k+1)} = \theta_{C(k+1)}^* \). By induction, we conclude that
\[
\hat{\theta}_{Ti} = \theta_{Ti}^*, \hat{\theta}_{Ci} = \theta_{Ci}^*, i = 1, \ldots, K - 1
\]
By \( \hat{\theta}_{TK} = 1 - \sum_{i=1}^{K-1} \hat{\theta}_{Ti} \) and \( \hat{\theta}_{CK} = 1 - \sum_{i=1}^{K-1} \hat{\theta}_{Ci} \), we have
\[
\hat{\theta}_{TK} = \theta_{TK}^*; \hat{\theta}_{CK} = \theta_{CK}^*
\]
Therefore,
\[
\hat{\theta}_{Ti} = \theta_{Ti}^*, \hat{\theta}_{Ci} = \theta_{Ci}^*, i = 1, \ldots, K
\]
is the only stationary point for the game.