A sequential optimality condition for Mathematical Programs with Cardinality Constraints

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Abstract

In this paper we propose an Approximate Weak stationarity (AW-stationarity) concept designed to deal with Mathematical Programs with Cardinality Constraints (MPCaC), and we proved that it is a legitimate optimality condition independently of any constraint qualification. Such a sequential optimality condition improves weaker stationarity conditions, presented in a previous work. Many research on sequential optimality conditions has been addressed for nonlinear constrained optimization in the last few years, some works in the context of MPCC and, as far as we know, no sequential optimality condition has been proposed for MPCaC problems. We also establish some relationships between our AW-stationarity and other usual sequential optimality conditions, such as AKKT, CAKKT and PAKKT. We point out that, despite the computational appeal of the sequential optimality conditions, in this work we are not concerned with algorithmic consequences. Our aim is purely to discuss theoretical aspects of such conditions for MPCaC problems.

Keywords. Mathematical programs with cardinality constraints, Sequential optimality conditions, Weak stationarity, Constraint qualification, Nonlinear programming.

Subclass. 90C30, 90C33, 90C46

1 Introduction

In this paper we propose a sequential optimality condition, associated to the weak stationarity condition presented in our previous work [12], designed to deal with Mathematical Programs with Cardinality Constraints (MPCaC) given by

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in X, \\
& \quad \|x\|_0 \leq \alpha,
\end{align*}
\]

where \(f : \mathbb{R}^n \to \mathbb{R}\) is a continuously differentiable function, \(X \subset \mathbb{R}^n\) is a set given by equality and/or inequality constraints, \(\alpha > 0\) is a given natural number and \(\|x\|_0\) denotes the cardinality of the vector \(x \in \mathbb{R}^n\), that is, the number of nonzero components of \(x\). We assume that \(\alpha < n\), since

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otherwise the cardinality constraint would be innocuous. Note, however, that if \( \alpha \) is too small, the cardinality constraint may be too restrictive leading to an empty feasible set. Furthermore, the main difference between the problem (11) and a standard nonlinear programming problem is that the cardinality constraint, despite of the notation, is not a norm, nor continuous neither convex.

One reformulation to deal with this difficult cardinality constraint consists of addressing its continuous counterpart [8]

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in X,
\quad e^T y \geq n - \alpha, \\
\quad x_i y_i = 0, & \quad i = 1, \ldots, n, \\
\quad 0 \leq y_i \leq 1, & \quad i = 1, \ldots, n,
\end{align*}
\]

which will be referred to as relaxed problem and, with some abuse of terminology, will be indicated as MPCaC as well. It can be seen that these problems are equivalent in the sense that global solutions of (11) correspond, in a natural way, to global solutions of (2) and, if \( x^* \in \mathbb{R}^n \) is a local minimizer of (11), then every feasible pair \((x^*, y^*)\) is a local minimizer of (2).

In [12] we proposed new and weaker stationarity conditions for this class of problems, by means of a unified approach that goes from the weakest to the strongest stationarity. Indeed we cannot assert about KKT points for MPCaC problems, since some standard constraint qualifications are violated. This occurs in view of the complementarity constraints \( x_i y_i = 0, \quad i = 1, \ldots, n \). However, the weaker condition proposed in [12], called \( W_I \)-stationarity, even being weaker than KKT, is not a necessary optimality condition. Therefore, we propose in this work an Approximate Weak stationarity (\( AW \)-stationarity) concept, which will be proved to be a legitimate optimality condition, independently of any constraint qualification.

In the last few years, special attention has been paid to the so-called sequential optimality conditions for nonlinear constrained optimization [1, 2, 4, 5, 6, 13, 16]. Sequential optimality conditions are intrinsically related to the stopping criteria of numerical algorithms, and their study aims at unifying the theoretical convergence analysis associated with the corresponding algorithm. Within this context, for instance, the augmented Lagrangian method (see [7] and references therein) has been extensively analyzed, being shown to satisfy weak sequential conditions, thus giving rise to strong convergence results.

Sequential optimality conditions are necessary for optimality, i.e., a local minimizer of the problem under consideration verifies such a condition, independently of the fulfillment of any constraint qualification (CQ). The approximate Karush-Kuhn-Tucker (AKKT) is one of the most popular of these conditions, and it was defined in [2] and [13]. Another two sequential optimality conditions for standard nonlinear programming, both stronger than AKKT, are positive approximate KKT (PAKKT) [1] and complementary approximate KKT (CAKKT) [6]. Whenever it is proved that an AKKT (or CAKKT or PAKKT) point is indeed a Karush-Kuhn-Tucker (KKT) point under a certain CQ, any algorithm that reaches AKKT (or CAKKT or PAKKT) points (e.g. augmented Lagrangian-type methods) automatically has the theoretical convergence established assuming the same CQ. This paves the grounds for the aforementioned unification.

Sequential optimality conditions have also been proposed for nonstandard optimization [3, 10, 11, 15]. In the context of Mathematical Programs with Equilibrium Constraints (MPECs) and motivated by AKKT, it was introduced in [15] the MPEC-AKKT condition with a geometric appeal and in [3], new conditions were established for Mathematical Problems with Complementarity
Constraints (MPCCs), namely AW-, AC- and AM-stationarity. The latter one was compared with the sequential condition present in [15].

Even though there is a considerable literature devoted to sequential conditions for standard nonlinear optimization and even for specific problems (MPCC and MPEC), to the best of our knowledge, no sequential optimality condition has been proposed for MPCaC problems. Such problems are very degenerate because of the problematic complementarity constraints $x_i y_i = 0$ and therefore the known sequential optimality conditions may not be suitable to deal with them. Thereby, we propose a sequential optimality condition, namely AW-stationarity, associated to $W_I$-stationarity and designed to deal with MPCaC problems. This condition is based on the one proposed in [3] for MPCC problems. The main contribution of this paper is that AW-stationarity is indeed a necessary optimality condition, without any constraint qualification assumption. We also establish some relationships between our AW-stationarity and other well known sequential optimality conditions. In particular, and surprisingly, we prove that AKKT fails to detect good candidates for optimality for every MPCaC problem.

We stress that, despite the algorithmic appeal of the sequential optimality conditions, in this work we are neither concerned with applications nor with computational aspects or algorithmic consequences. Our aim is to discuss theoretical aspects of such conditions for MPCaC problems.

The paper is organized as follows: in Section 2 we establish the notation, some definitions and results concerning standard nonlinear programming and recall the weak stationarity concept proposed in our previous work [12]. Section 3 presents the main results of this paper, concerning sequential optimality conditions for MPCaC. In Section 4 we provide some relationships between approximate stationarity for standard nonlinear optimization and AW-stationarity. Concluding remarks are presented in Section 5.

**Notation.** Throughout this paper, for vectors $x, y \in \mathbb{R}^n$, $x * y$ denotes the Hadamard product between $x$ and $y$, that is, the vector obtained by the componentwise product of $x$ and $y$. In the same way, the “min” in the vector $\min\{x, y\}$ is taken componentwise. We also use the following sets of indices: $I_{00}(x, y) = \{i \mid x_i = 0, y_i = 0\}$, $I_{\pm 0}(x, y) = \{i \mid x_i \neq 0, y_i = 0\}$, $I_{0+}(x, y) = \{i \mid x_i = 0, y_i \in (0, 1)\}$, $I_{01}(x, y) = \{i \mid x_i = 0, y_i = 1\}$, $I_{0>}(x, y) = \{i \mid x_i = 0, y_i > 0\}$ and $I_0(x) = \{i \mid x_i = 0\}$. For a vector-valued function $\xi : \mathbb{R}^n \to \mathbb{R}^s$, denote $I_\xi(x) = \{i \mid \xi_i(x) = 0\}$, the set of active indices, and $\nabla \xi = (\nabla \xi_1 \ldots \nabla \xi_s)$, the transpose of the Jacobian of $\xi$.

## 2 Preliminaries

In this section we recall some basic definitions and results related to standard nonlinear programming (NLP), as well as the weak stationarity concept proposed in our previous work [12].

Consider first the problem

$$
\begin{align*}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & g(x) \leq 0, \\
& h(x) = 0,
\end{align*}
$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable functions. The feasible set of the problem (3) is denoted by

$$
\Omega = \{x \in \mathbb{R}^n \mid g(x) \leq 0, h(x) = 0\}.
$$
Definition 2.1 We say that \( x^* \in \Omega \) is a global solution of the problem (3), that is, a global minimizer of \( f \) in \( \Omega \), when \( f(x^*) \leq f(x) \) for all \( x \in \Omega \). If \( f(x^*) \leq f(x) \) for all \( x \in \Omega \) such that \( \|x - x^*\| \leq \delta \), for some constant \( \delta > 0 \), \( x^* \) is said to be a local solution of the problem.

A feasible point \( x^* \in \Omega \) is said to be stationary for the problem (3) if there exists a vector \( \lambda = (\lambda^g, \lambda^h) \in \mathbb{R}_+^m \times \mathbb{R}^p \) (Lagrange multipliers) such that

\[
\nabla f(x^*) + \sum_{i=1}^m \lambda_i^g \nabla g_i(x^*) + \sum_{i=1}^p \lambda_i^h \nabla h_i(x^*) = 0,
\]

\[
(\lambda^g)^T g(x^*) = 0.
\]

The function \( L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R} \) given by

\[
L(x, \lambda^g, \lambda^h) = f(x) + (\lambda^g)^T g(x) + (\lambda^h)^T h(x)
\]

is the Lagrangian function associated with the problem (3).

The conditions (5a)–(5b) are known as Karush-Kuhn-Tucker (KKT) conditions and, under certain qualification assumptions, are satisfied at a minimizer.

2.1 Constraint qualifications

There are a lot of constraint qualifications, that is, conditions under which every minimizer satisfies KKT. In order to discuss some of them, let us recall the definition of cone, which plays an important role in this context.

We say that a nonempty set \( C \subset \mathbb{R}^n \) is a cone if \( td \in C \) for all \( t \geq 0 \) and \( d \in C \). Given a set \( S \subset \mathbb{R}^n \), its polar is the cone

\[
S^\circ = \{ p \in \mathbb{R}^n \mid p^T x \leq 0, \forall x \in S \}.
\]

Associated with the feasible set of the problem (3), we have the tangent cone

\[
T_{\Omega}(\bar{x}) = \left\{ d \in \mathbb{R}^n \mid \exists (x^k) \subset \Omega, (t_k) \subset \mathbb{R}_+ : t_k \to 0 \text{ and } \frac{x^k - \bar{x}}{t_k} \to d \right\}
\]

and the linearized cone

\[
D_{\Omega}(\bar{x}) = \left\{ d \in \mathbb{R}^n \mid \nabla g_i(\bar{x})^T d \leq 0, i \in I_g(\bar{x}) \text{ and } \nabla h(\bar{x})^T d = 0 \right\}.
\]

The following basic result says that we may ignore inactive constraints when dealing with the tangent and linearized cones.

Lemma 2.2 Consider a feasible point \( \bar{x} \in \Omega \), an index set \( J \supset I_g(\bar{x}) \) and

\[
\Omega' = \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0, i \in J, h(x) = 0 \}.
\]

Then, \( T_{\Omega}(\bar{x}) = T_{\Omega'}(\bar{x}) \) and \( D_{\Omega}(\bar{x}) = D_{\Omega'}(\bar{x}) \).

Proof. Note first that \( \bar{x} \in \Omega' \) since \( \Omega \subset \Omega' \). Moreover, since \( g_i(\bar{x}) < 0 \) for \( i \notin J \), there exists \( \delta > 0 \) such that \( B(\bar{x}, \delta) \cap \Omega' = B(\bar{x}, \delta) \cap \Omega \). Thus, \( T_{\Omega'}(\bar{x}) = T_{\Omega}(\bar{x}) \) because the conditions \( t_k \to 0 \) and \( (x^k - \bar{x})/t_k \to d \) imply that \( x^k \to \bar{x} \). The equality between the linearized cones is straightforward, as the active indices corresponding to \( \Omega \) and \( \Omega' \) coincide.

Now we relate the cones of feasible sets when some variables do not appear in the constraints.
Lemma 2.3 Consider the general feasible set $\Omega$, defined in (4), and the set

$$\Omega' = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid g(x) \leq 0, h(x) = 0\}.$$ 

Given a feasible point $(\bar{x}, \bar{y}) \in \Omega'$, we have

$$T_{\Omega'}(\bar{x}, \bar{y}) = T_{\Omega}(\bar{x}) \times \mathbb{R}^m \quad \text{and} \quad D_{\Omega'}(\bar{x}, \bar{y}) = D_{\Omega}(\bar{x}) \times \mathbb{R}^m.$$ 

As a consequence,

$$T_{\Omega'}^o(\bar{x}, \bar{y}) = T_{\Omega}^o(\bar{x}) \times \{0\} \quad \text{and} \quad D_{\Omega'}^o(\bar{x}, \bar{y}) = D_{\Omega}^o(\bar{x}) \times \{0\}.$$ 

Proof. The relation between the tangent cones follows directly from the definition. Moreover, if $\zeta(x, y) = g(x)$ and $\xi(x, y) = h(x)$ represent the constraints that define $\Omega'$ and $d = (\alpha, \beta)$, then

$$\nabla\zeta_i(x, y)^T d = \nabla g_i(x)^T \alpha \quad \text{and} \quad \nabla\xi_j(x, y)^T d = \nabla h_j(x)^T \alpha,$$

which gives the second claim. Finally, the last statement of the lemma follows from the fact that $(S \times \mathbb{R}^m)^o = S^o \times \{0\}$ for any set $S \subset \mathbb{R}^n$.

The two weakest constraint qualifications are defined below.

Definition 2.4 We say that Abadie constraint qualification (ACQ) holds at $\bar{x} \in \Omega$ if $T_{\Omega}(\bar{x}) = D_{\Omega}(\bar{x})$. If $T_{\Omega}^o(\bar{x}) = D_{\Omega}^o(\bar{x})$, we say that Guignard constraint qualification (GCQ) holds at $\bar{x}$.

In the following lemma we analyze GCQ for simple complementarity constraints.

Lemma 2.5 Consider the set

$$\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y \geq 0, x \ast y = 0\}.$$ 

Given $(\bar{x}, \bar{y}) \in \Omega$, there holds $T_{\Omega}^o(\bar{x}, \bar{y}) = D_{\Omega}^o(\bar{x}, \bar{y})$.

Proof. Denote the constraints that define $\Omega$ by $\zeta(x, y) = -y$ and $\xi(x, y) = x \ast y$. Given $d = (u, v) \in D_{\Omega}(\bar{x}, \bar{y})$, we claim that the vectors $d_1 = (u, 0)$ and $d_2 = (0, v)$ belong to $T_{\Omega}(\bar{x}, \bar{y})$. Indeed, since

$$\bar{y} i u + \bar{x} i v = \nabla\zeta_i(\bar{x}, \bar{y})^T d = 0$$

for all $i = 1, \ldots, n$, we have $u_{I^o_>} > 0$ and $v_{I^o_+} = 0$, where we used the simplified notation $I^o_0 = I^o_>(\bar{x}, \bar{y})$ and $I^o_{\pm} = I^o_{\pm}(\bar{x}, \bar{y})$. Thus, the sequences $t_k = 1/k$ and $(x^k, y^k) = (\bar{x} + t_k u, \bar{y})$ satisfy $y^k \geq 0, x^k_{I^o_>} = 0, y^k_{I^o_{\pm} \cup I^o_0} = 0$, which means that $(x^k, y^k) \subset \Omega$, and

$$\left(\frac{t_k}{t_k}ight) \rightarrow d_1,$$

proving that $d_1 \in T_{\Omega}(\bar{x}, \bar{y})$. Now, defining $(z^k, w^k) = (\bar{x} + \bar{y} + t_k v)$, we have

$$\left(\frac{z^k - \bar{x}, w^k - \bar{y}}{t_k}ight) \rightarrow d_2 \quad \text{and} \quad z^k \ast w^k = 0.$$ 

Furthermore, for $i \in I^o_{\pm}$, we have $w^k_i > 0$ for all sufficiently large $k$. On the other hand, if $i \in I^o_0$, then the constraint $\zeta_i$ is active and hence, $-v_i = \nabla\zeta_i(\bar{x}, \bar{y})^T d \leq 0$, giving $w^k_i = \bar{y}_i + t_k v_i \geq 0$. Thus, $(z^k, w^k) \subset \Omega$, which yields $d_2 \in T_{\Omega}(\bar{x}, \bar{y})$, proving the claim. Finally, given $p \in T_{\Omega}^o(\bar{x}, \bar{y})$ we conclude that $p^T d = p^T d_1 + p^T d_2 \leq 0$, proving that $p \in D_{\Omega}^o(\bar{x}, \bar{y})$. 

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2.2 Sequential optimality conditions for standard NLP

The goal of this section is to present some well known approximate optimality conditions for nonlinear constrained optimization [2, 4, 5, 6, 13].

Definition 2.6 Let \( \bar{x} \in \mathbb{R}^n \) be a feasible point for the problem (3). We say that \( \bar{x} \) is an Approximate KKT (AKKT) point if there exist sequences \((x^k) \subset \mathbb{R}^n\) and \((\lambda^k) = (\lambda^{g,k}, \lambda^{h,k}) \subset \mathbb{R}_+^m \times \mathbb{R}^p\) such that \(x^k \to \bar{x}\),

\[
\begin{align*}
\nabla_x L(x^k, \lambda^{g,k}, \lambda^{h,k}) &\to 0, \\
\min \{-g(x^k), \lambda^{g,k}\} &\to 0.
\end{align*}
\]

We have below two stronger conditions than AKKT.

Definition 2.7 Let \( \bar{x} \in \mathbb{R}^n \) be a feasible point for the problem (3). We say that \( \bar{x} \) is a Complementary Approximate KKT (CAKKT) point if there exist sequences \((x^k) \subset \mathbb{R}^n\) and \((\lambda^k) = (\lambda^{g,k}, \lambda^{h,k}) \subset \mathbb{R}_+^m \times \mathbb{R}^p\) such that \(x^k \to \bar{x}\),

\[
\begin{align*}
\nabla_x L(x^k, \lambda^{g,k}, \lambda^{h,k}) &\to 0, \\
\lambda^{g,k} g(x^k) &\to 0 \quad \text{and} \quad \lambda^{h,k} h(x^k) &\to 0.
\end{align*}
\]

Remark 2.8 Note that if \((\alpha^k) \subset \mathbb{R}_+^n\) and \((\beta^k) \subset \mathbb{R}\) are sequences satisfying \(\alpha^k \beta^k \to 0\) and \(\beta^k \to \beta \leq 0\), then \(\min\{-\beta^k, \alpha^k\} \to 0\). Indeed, if \(\beta < 0\), we have \(\alpha^k \to 0\) and hence \(\alpha^k < -\beta^k\) for all \(k\) sufficiently large, giving \(\min\{-\beta^k, \alpha^k\} = \alpha^k \to 0\). On the other side, if \(\beta = 0\), we also conclude that \(\min\{-\beta^k, \alpha^k\} \to 0\), since \(\alpha^k \geq 0\). This means that condition (8a) implies (7b), and thus CAKKT implies AKKT.

Another known sequential optimality condition relates to the sign of the multipliers.

Definition 2.9 Let \( \bar{x} \in \mathbb{R}^n \) be a feasible point for the problem (3). We say that \( \bar{x} \) is a Positive Approximate KKT (PAKKT) point if there exist sequences \((x^k) \subset \mathbb{R}^n\) and \((\lambda^k) = (\lambda^{g,k}, \lambda^{h,k}) \subset \mathbb{R}_+^m \times \mathbb{R}^p\) such that \(x^k \to \bar{x}\),

\[
\begin{align*}
\nabla_x L(x^k, \lambda^{g,k}, \lambda^{h,k}) &\to 0, \\
\min \{-g(x^k), \lambda^{g,k}\} &\to 0 \\
\lambda^{g,k} g_i(x^k) &> 0 \quad \text{if} \quad \limsup_{k \to \infty} \frac{\lambda^{g,k}}{\delta_k} > 0, \\
\lambda^{h,k} h_j(x^k) &> 0 \quad \text{if} \quad \limsup_{k \to \infty} \frac{|\lambda^{h,k}|}{\delta_k} > 0,
\end{align*}
\]

where \(\delta_k = \|(1, \lambda^k)\|_{\infty}\).

As well known in the literature, all the three sequential conditions above are necessary optimality conditions without any constraint qualification.
2.3 Weak stationarity for MPCaC

In this section we recall the weaker stationarity concept and some related results, established in [12], for MPCaC. As we have seen in that work, except in special cases, e.g., when \( X \) is given by linear constraints, we do not have the fulfillment of constraint qualifications for the relaxed problem (2). So, the standard KKT conditions are not necessary optimality conditions, fact that in turn justifies the study of weaker conditions.

For ease of presentation consider the functions \( \theta : \mathbb{R}^n \rightarrow \mathbb{R}, G, H, \tilde{H} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) given by

\[
\theta(y) = n - \alpha - e^T y, \quad G(x) = x, \quad H(y) = -y \quad \text{and} \quad \tilde{H}(y) = y - e.
\]

Then we can rewrite the relaxed problem (2) as

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) \leq 0, h(x) = 0, \\
& \quad \theta(y) \leq 0, \\
& \quad H(y) \leq 0, \tilde{H}(y) \leq 0, \\
& \quad G(x)^* H(y) = 0.
\end{align*}
\]

(10)

Given a feasible point \((\bar{x}, \bar{y})\) for the problem (10) and a set of indices \( I \) such that

\[
I_{0+}(\bar{x}, \bar{y}) \cup I_{01}(\bar{x}, \bar{y}) \subset I \subset I_0(\bar{x}),
\]

(11)

we have that \( i \in I \) or \( i \in I_{00}(\bar{x}, \bar{y}) \cup I_{\pm 0}(\bar{x}, \bar{y}) \) for all \( i \in \{1, \ldots, n\} \). Thus, \( G_i(\bar{x}) = 0 \) or \( H_i(\bar{y}) = 0 \). This suggests to consider an auxiliary problem by removing the problematic constraint \( G(x)^* H(y) = 0 \) and including other ones that ensure the null product. We then define the \( I \)-Tightened Nonlinear Problem at \((\bar{x}, \bar{y})\) by

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) \leq 0, h(x) = 0, \\
& \quad \theta(y) \leq 0, \\
& \quad G_i(x) = 0, \quad i \in I, \\
& \quad H_i(y) \leq 0, \quad i \in I_{0+}(\bar{x}, \bar{y}) \cup I_{01}(\bar{x}, \bar{y}), \\
& \quad H_i(y) = 0, \quad i \in I_{00}(\bar{x}, \bar{y}) \cup I_{\pm 0}(\bar{x}, \bar{y}), \\
& \quad \tilde{H}(y) \leq 0.
\end{align*}
\]

(12)

This problem will be also indicated by \( \text{TNLP}_I(\bar{x}, \bar{y}) \) and, when there is no chance for ambiguity, it will be referred simply to as \( \text{tightened problem} \). Note that we tighten only those constraints that are involved in the complementarity constraint \( G(x)^* H(y) = 0 \), by incorporating the equality constraints \( G_i \)’s and converting the active inequalities \( H_i \)’s into equalities.

The following lemma is a straightforward consequence of the definition of the tightened problem \( \text{TNLP}_I(\bar{x}, \bar{y}) \).

**Lemma 2.10** Consider the tightened problem (12). Then,

1. the inequalities defined by \( \tilde{H}_i, \ i \in I_{0+}(\bar{x}, \bar{y}) \cup I_{01}(\bar{x}, \bar{y}), \) are inactive at \((\bar{x}, \bar{y})\);

2. \((\bar{x}, \bar{y})\) is feasible for \( \text{TLNP}_I(\bar{x}, \bar{y}) \);
3. every feasible point of (12) is feasible for (10);
4. if \((\bar{x}, \bar{y})\) is a global (local) minimizer of (10), then it is also a global (local) minimizer of \(\text{TNLP}_I(\bar{x}, \bar{y})\).

The Lagrangian function associated with \(\text{TNLP}_I(\bar{x}, \bar{y})\) is the function

\[
\mathcal{L}_I : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^{\lvert I \rvert} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}
\]

given by

\[
\mathcal{L}_I(x, y, \lambda^g, \lambda^h, \lambda^G, \lambda^H) = L(x, \lambda^g, \lambda^h) + \lambda^\theta \theta(y) + (\lambda^G)^T G_I(x) + (\lambda^H)^T H(y) + (\lambda^\tilde{H})^T \tilde{H}(y),
\]

where \(L\) is the Lagrangian defined in (6).

Note that the tightened problem, and hence its Lagrangian, depends on the index set \(I\), which in turn depends on the point \((\bar{x}, \bar{y})\). It should be also noted that

\[
\nabla_{x,y} \mathcal{L}_I(x, y, \lambda) = \begin{pmatrix}
\nabla_x L(x, \lambda^g, \lambda^h) + \sum_{i \in I} \lambda^G_i e_i \\
-\lambda^\theta e - \lambda^H + \lambda^\tilde{H}
\end{pmatrix}.
\]

(13)

The weaker stationarity concept proposed in [12] is presented below.

**Definition 2.11** Consider a feasible point \((\bar{x}, \bar{y})\) of the relaxed problem (10) and a set of indices \(I\) satisfying (11). We say that \((\bar{x}, \bar{y})\) is \(I\)-weakly stationary (\(W_I\)-stationary) for this problem if there exists a vector

\[
\lambda = (\lambda^g, \lambda^h, \lambda^\theta, \lambda^G, \lambda^H, \lambda^\tilde{H}) \in \mathbb{R}^m_+ \times \mathbb{R}^p_+ \times \mathbb{R}^{\lvert I \rvert} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+
\]

such that

1. \(\nabla_{x,y} \mathcal{L}_I(\bar{x}, \bar{y}, \lambda) = 0\);
2. \((\lambda^g)^T g(\bar{x}) = 0\);
3. \(\lambda^\theta \theta(\bar{y}) = 0\);
4. \(\lambda^H_i = 0\) for all \(i \in I_{0+}(\bar{x}, \bar{y}) \cup I_{01}(\bar{x}, \bar{y})\);
5. \((\lambda^H)^T \tilde{H}(\bar{y}) = 0\).

**Remark 2.12** The first item of Definition 2.11 is nothing else than the gradient KKT condition for the tightened problem (12). Items (2), (3) and (5) represent the standard KKT complementarity conditions for the inequality constraints \(g(x) \leq 0\), \(\theta(y) \leq 0\) and \(\tilde{H}(y) \leq 0\), respectively, of the tightened problem. Item (4) also represents KKT complementarity conditions for the constraints \(H_i(y) \leq 0\), \(i \in I_{0+}(\bar{x}, \bar{y}) \cup I_{01}(\bar{x}, \bar{y})\), in view of Lemma 2.10(1).

As an immediate consequence of Remark 2.12 we have the following characterization of \(W_I\)-stationarity for the relaxed problem in terms of stationarity for the tightened problem.
Proposition 2.13 Let \((\bar{x}, \bar{y})\) be a feasible point of the relaxed problem \((11)\). Then, \((\bar{x}, \bar{y})\) is \(W_I\)-stationary if and only if it is a KKT point for the tightened problem \((12)\).

Note that in view of Proposition 2.13 we could have defined \(W_I\)-stationarity simply as KKT for the tightened problem \((12)\). Nevertheless, we prefer as in Definition 2.11 in order to have its last condition (4) explicitly, instead of hidden in the complementarity condition. This way of stating weak stationarity is also similar to that used in the MPCC setting, see [3, 9].

In the next result we justify why Definition 2.11 is considered a weaker stationarity concept for the relaxed problem.

Theorem 2.14 Suppose that \((\bar{x}, \bar{y})\) is a KKT point for the relaxed problem \((10)\). Then \((\bar{x}, \bar{y})\) is \(W_I\)-stationary.

At this point we could ask if \(W_I\)-stationarity, being weaker than KKT, is a necessary optimality condition. That is, can we ensure that a minimizer of the relaxed problem is \(W_I\)-stationary for some index set \(I\) satisfying \((11)\)? The answer is no, as illustrated in the following example.

Example 2.15 Consider the MPCaC and the associated relaxed problem given below.

\[
\begin{align*}
\text{minimize} & \quad x_1 \\
\text{subject to} & \quad (1 - x_1)^3 + x_2^3 \leq 0, \\
& \quad \|x\|_0 \leq 2,
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad x_1 \\
\text{subject to} & \quad (1 - x_1)^3 + x_2^3 \leq 0, \\
& \quad y_1 + y_2 + y_3 \geq 1, \\
& \quad x_i y_i = 0, i = 1, 2, 3, \\
& \quad 0 \leq y_i \leq 1, i = 1, 2, 3.
\end{align*}
\]

The point \(x^* = (1, 0, 0)\) is a global solution of MPCaC and \((x^*, y^*)\), with \(y^* = (0, 1, 0)\), is a global solution of the relaxed problem. For the points \(x^*\) and \((x^*, y^*)\) we have

\[I_0 = \{2, 3\}, \ I_{01} = \{2\}, \ I_{\pm 0} = \{1\}, \ I_{00} = \{3\} \quad \text{and} \quad I_{0+} = \emptyset.\]

So, there are two choices for \(I\) that satisfy \((11)\): \(I' = \{2\}\) or \(I'' = \{2, 3\}\). Let us analyze each one of them.

For \(I = I'\) we have

\[
\nabla_x L(x^*, \lambda^0) + \sum_{i \in I} \lambda_i^0 e_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_2^0 \\ \lambda_3^0 \end{pmatrix}.
\]

Since this expression does not vanish, taking into account \((13)\), we see that the pair \((x^*, y^*)\) is not \(W_I\)-stationary.

Now, for \(I = I''\) we have

\[
\nabla_x L(x^*, \lambda^0) + \sum_{i \in I} \lambda_i^0 e_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_2^0 \\ \lambda_3^0 \end{pmatrix}.
\]

Again, the above expression does not vanish and then \((x^*, y^*)\) is not \(W_I\)-stationary.

In view of Example 2.15 and motivated to find a necessary optimality condition for MPCaC problems, we propose in the next section the concept of approximate weak stationarity, which will be satisfied at every minimizer, independently of any constraint qualification.
3 Sequential optimality conditions for MPICaC

In order to define our sequential optimality condition, consider the function
\[ L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \]
defined by
\[
\mathcal{L}(x, y, \lambda^g, \lambda^h, \lambda^\theta, \lambda^G, \lambda^H, \lambda^{\tilde{H}}) = L(x, \lambda^g, \lambda^h) + \lambda^\theta \theta(y) + (\lambda^G)^T G(x) + (\lambda^H)^T H(y),
\]
where \( L \) is the Lagrangian defined in [6].

Note that \( \mathcal{L} \) resembles the Lagrangian \( L_I \), associated with TNLP\(_I(x, y)\). The only difference is that the term \((\lambda^G)^T G_I(x)\) was replaced by \((\lambda^G)^T G(x)\). Here it will be convenient to see that
\[
\nabla_{x,y,L}(x, y, \lambda) = \begin{pmatrix}
\nabla_x L(x, \lambda^g, \lambda^h) + \sum_{i=1}^n \lambda_i^G \nabla G_i(x) \\
\lambda^\theta \theta(y) + \sum_{i=1}^n \lambda_i^H \nabla H_i(y) + \sum_{i=1}^n \lambda_i^{\tilde{H}} \nabla \tilde{H}_i(y)
\end{pmatrix}.
\]

**Definition 3.1** Let \((\bar{x}, \bar{y})\) be a feasible point of the relaxed problem [17]. We say that \((\bar{x}, \bar{y})\) is Approximately Weakly stationary (AW-stationary) for this problem if there exist sequences \((x^k, y^k) \subset \mathbb{R}^n \times \mathbb{R}^n\) and
\[
(\lambda^k) = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{\theta,k}, \lambda^{G,k}, \lambda^{H,k}, \lambda^{\tilde{H},k}) \subset \mathbb{R}^m_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+
\]
such that
1. \((x^k, y^k) \rightarrow (\bar{x}, \bar{y})\);
2. \(\nabla_{x,y,L}(x^k, y^k, \lambda^k) \rightarrow 0\);
3. \(\min\{-g(x^k), \lambda^{g,k}\} \rightarrow 0\);
4. \(\min\{-\theta(y^k), \lambda^{\theta,k}\} \rightarrow 0\);
5. \(\min\{|G_i(x^k)|, |\lambda_i^{G,k}|\} \rightarrow 0\) for all \(i = 1, \ldots, n\);
6. \(\min\{-H_i(y^k), |\lambda_i^{H,k}|\} \rightarrow 0\) for all \(i = 1, \ldots, n\);
7. \(\min\{-\tilde{H}(y^k), \lambda^{\tilde{H},k}\} \rightarrow 0\).

**Remark 3.2** Definition 3.1 resembles AKKT condition, where [3], [4] and [7] represent the approximate complementarity conditions for the inequality constraints \(g(x) \leq 0\), \(\theta(y) \leq 0\) and \(\tilde{H}(y) \leq 0\), respectively and [6] is related to the last complementarity condition in \(W_I\)-stationarity. As a matter of fact, AW-stationarity is equivalent to AKKT for TNLP\(_I\), as we shall see ahead in Theorem 4.5.
Let us review Example 2.15 in light of the above definition. We have seen that the minimizer is not $W_I$-stationary, but now we can see that it is $AW$-stationary.

**Example 3.3** Consider the problem given in Example 2.15. We claim that the global solution of the relaxed problem, $(x^*, y^*)$, is $AW$-stationary. Indeed, consider the sequences $(x^k, y^k) \subset \mathbb{R}^3 \times \mathbb{R}^3$ and

$$\lambda^k = (\lambda^g, k, \lambda^G, k, \lambda^H, k, \lambda^H, k) \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$$

defined by $x^k = (1 + 1/k, 0, 0)$, $y^k = (0, 1, 0)$, $\lambda^g, k = k^2/3$, $\lambda^G, k = 0$ and $\lambda^H, k = \lambda^H, k = 0$. Then, we have $(x^k, y^k) \to (x^*, y^*)$ and

$$\nabla_x L(x^k, \lambda^{g, k}) + \sum_{i=1}^n \lambda^{G, k} \nabla G_i(x^k) = \begin{pmatrix} 1 - 3\lambda^{g, k}(1 - x_1^k)^2 \\ 0 \\ 2\lambda^{g, k} x_3^k \end{pmatrix} = 0.$$ 

So, in view of (14), we obtain the first two items of Definition 3.1. Now, note that $g(x^k) \to g(x^*) = 0$ and $\theta(y^k) \to \theta(y^*) = 0$, which in turn imply that

$$\min \{-g(x^k), \lambda^{g, k}\} \to 0 \quad \text{and} \quad \min \{-\theta(y^k), \lambda^{g, k}\} \to 0,$$

giving items (3) and (4). The relation $\min \{G_i(x^k), |\lambda^{G, k}_i|\} \to 0$ is immediate. Besides, since $H(y^k) \to H(y^*) \leq 0$, $\lambda^{H, k} = 0$, $H(y^k) \to H(y^*) \leq 0$ and $\lambda^{H, k} = 0$, we have $\min \{-H_i(y^k), |\lambda^{H, k}_i|\} \to 0$, obtaining items (5), (6) and (7).

Now we shall prove that the above example reflects a general result, that is, every minimizer of an MPCA problem is $AW$-stationary. We start the theoretical analysis with two simple facts. The first one says that the expression $\sum_{i=1}^n \lambda^{G, k}_i \nabla G_i(x^k)$ could be replaced by $\sum_{i \in I_0} \lambda^{G, k}_i \nabla G_i(x^k)$. The second fact states that $AW$-stationarity is weaker than $W_I$-stationarity, and consequently weaker than KKT, in view of Theorem 2.14.

**Lemma 3.4** Let $(\bar{x}, \bar{y})$ be an $AW$-stationary point for the relaxed problem (10), with corresponding sequences $(x^k, y^k)$ and $(\lambda^k)$. Then,

$$\nabla_x L(x^k, \lambda^{g, k}, \lambda^{h, k}) + \sum_{i \in I_0} \lambda^{G, k}_i \nabla G_i(x^k) \to 0.$$ 

**Proof.** In view of (14), we have, in particular,

$$\nabla_x L(x^k, \lambda^{g, k}, \lambda^{h, k}) + \sum_{i=1}^n \lambda^{G, k}_i \nabla G_i(x^k) \to 0.$$  \hspace{1cm} (15)

For $i \notin I_0$, we have $\lim_{k \to \infty} G_i(x^k) = G_i(\bar{x}) = \bar{x}_i \neq 0$. Therefore, we can assume without loss of generality that there exists $\epsilon > 0$ such that $|G_i(x^k)| \geq \epsilon$ for all $k$. Since $\min \{|G_i(x^k)|, |\lambda^{G, k}_i|\} \to 0$, we obtain $|\lambda^{G, k}_i| \to 0$ and hence,

$$\sum_{i \notin I_0} \lambda^{G, k}_i \nabla G_i(x^k) \to 0.$$ 

By subtracting this from (15), we conclude the proof. \qed

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Lemma 3.5  Let \((\bar{x}, \bar{y})\) be a \(W_1\)-stationary point for the relaxed problem (10), in the sense of Definition 2.11. Then \((\bar{x}, \bar{y})\) is AW-stationary for this problem.

Proof. Consider a vector

\[ \lambda = (\lambda^g, \lambda^h, \lambda^\theta, \lambda^G, \lambda^H, \lambda^\tilde{H}) \in \mathbb{R}_+^m \times \mathbb{R}^p \times \mathbb{R}_+ \times \mathbb{R}^{|I|} \times \mathbb{R}^n \times \mathbb{R}_+^n \]

satisfying Definition 2.11. Then, the (constant) sequences \((x^k, y^k) \subseteq \mathbb{R}^n \times \mathbb{R}^n\) and

\[(\lambda^k) = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{\theta,k}, \lambda^{G,k}, \lambda^{H,k}, \lambda^{\tilde{H},k}) \subseteq \mathbb{R}_+^m \times \mathbb{R}^p \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+^n,
\]

defined by

\[(x^k, y^k) = (\bar{x}, \bar{y}), \quad (\lambda^{g,k}, \lambda^{h,k}, \lambda^{\theta,k}, \lambda^{G,k}, \lambda^{H,k}, \lambda^{\tilde{H},k}) = (\lambda^g, \lambda^h, \lambda^\theta, \lambda^G, \lambda^H, \lambda^\tilde{H}) \]

and \(\lambda^{G,k}_i = 0\) for \(i \notin I\) and \(k \in \mathbb{N}\), satisfy Definition 3.1. \(\Box\)

Remark 3.6  We point out here that, in contrast to \(W_1\)-stationarity, which conveniently depends on the set \(I\), our sequential optimality condition is independent of any set \(I\). This is a desirable feature since AW-stationarity has a certain amount of algorithmic appeal. In practice, one is able to use such conditions as a stopping criterion for an algorithm designed to solve the MPCaC problem.

Before proving our main sequential optimality results, let us see some preliminary lemmas. To this end, consider the augmented problem

\[
\begin{align*}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & g(x) \leq 0, h(x) = 0, \\
& \theta(y) \leq 0, \\
& w^G - G(x) = 0, w^H + H(y) = 0, \\
& \tilde{H}(y) \leq 0, \\
& w \in W, \\
\end{align*}
\]

(16)

where \(W = \{w = (w^G, w^H) \in \mathbb{R}^n \times \mathbb{R}_+^n \mid w^G \cdot w^H = 0\}\).

This problem will be crucial in the analysis. In the next two lemmas we establish the equivalence between the relaxed problem (10) and this augmented problem. Moreover, there is a suitable reason to write the constraints \(H(y) \leq 0\) and \(G(x) \cdot H(y) = 0\) of (10) in the format \(w \in W\). Such a strategy will enable us to apply Lemma 2.5 to obtain a suitable reason qualification for an auxiliary problem ahead.

Lemma 3.7  Let \((x^*, y^*)\) be a local (global) minimizer of the relaxed problem (10). Given \(w^* \in \mathbb{R}^n \times \mathbb{R}_+^n\), if the point \((x^*, y^*, w^*)\) is feasible for the augmented problem (16), then it is a local (global) minimizer of this problem. In particular, this holds for \(w^* = (G(x^*), -H(y^*))\).

Proof. First, the relation between local minimizers. In view of the equivalence of norms, we consider \(\| \cdot \|_\infty\), for convenience. By hypothesis, there exists \(\delta > 0\) such that if \((x, y)\) is feasible for (10) and \(\|(x, y) - (x^*, y^*)\|_\infty \leq \delta\), then \(f(x) \leq f(x^*)\). Suppose that \((x^*, y^*, w^*)\) is feasible for the problem (16) and consider an arbitrary feasible point \((x, y, w)\) for this problem such that \(\|(x, y, w) -
\((x^*, y^*, w^*)\|_\infty \leq \delta\). Then, the pair \((x, y)\) is feasible for \((11)\) and \(\|(x, y) - (x^*, y^*)\|_\infty \leq \delta\). Hence, 
\(f(x^*) \leq f(x)\) and, therefore, \((x^*, y^*, w^*)\) is a local minimizer of \((10)\). Note that \((x^*, y^*, w^*)\), with \(w^* = (G(x^*), -H(y^*))\), is trivially feasible. Finally, if we ignore the neighborhoods in the argument above, we obtain the relation between global minimizers.

For the sake of completeness we prove below the converse of Lemma 3.7.

**Lemma 3.8** Let \((x^*, y^*, w^*)\) be a local (global) minimizer of \((10)\). Then \((x^*, y^*)\) is a local (global) minimizer of \((10)\).

**Proof.** By the feasibility of \((x^*, y^*, w^*)\) we have that \((x^*, y^*)\) is feasible for \((11)\),

\[(w^*)^G = G(x^*) \quad \text{and} \quad (w^*)^H = -H(y^*).\]  
(17)

Consider \(\delta_1 > 0\) such that \(f(x^*) \leq f(x)\) for all feasible point \((x, y, w)\) of \((16)\), satisfying \(\|(x, y, w) - (x^*, y^*, w^*)\|_\infty \leq \delta_1\). Let \(\delta_2 > 0\) be such that

\[\|G(x) - G(x^*)\|_\infty \leq \delta_1 \quad \text{and} \quad \|H(y) - H(y^*)\|_\infty \leq \delta_1\]  
(18)

for all \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n\) with \(\|(x, y) - (x^*, y^*)\|_\infty \leq \delta_2\). Define \(\delta = \min\{\delta_1, \delta_2\}\) and take \((x, y)\), feasible for \((10)\), such that \(\|(x, y) - (x^*, y^*)\|_\infty \leq \delta\). Thus we have \((18)\), which in view of \((17)\) can be rewritten as \(\|w - w^*\|_\infty \leq \delta_1\), with \(w = (G(x), -H(y))\). Therefore, \((x, y, w)\) is feasible for \((16)\) and

\[\|(x, y, w) - (x^*, y^*, w^*)\|_\infty \leq \delta_1,\]

implying that \(f(x^*) \leq f(x)\).

Now, let us see the global optimality. So, assume that \((x^*, y^*, w^*)\) is a global minimizer of \((10)\). Then \((x^*, y^*)\) is feasible for \((10)\). Furthermore, given an arbitrary feasible point \((x, y)\), we have that \((x, y, w)\), with \(w = (G(x), -H(y))\), is feasible for \((16)\). Therefore, \(f(x^*) \leq f(x)\).

**Lemma 3.9** Suppose that \((x^*, y^*)\) is a local minimizer of the relaxed problem \((10)\). Then, given an arbitrary norm \(\|\cdot\|\), there exists \(\delta > 0\) such that \((x^*, y^*, w^*)\), with \(w^* = (G(x^*), -H(y^*))\), is the unique global minimizer of the problem

\[
\begin{align*}
\text{minimize} & \quad f(x) + \frac{1}{2}\|(x, y) - (x^*, y^*)\|^2_2 \\
\text{subject to} & \quad g(x) \leq 0, h(x) = 0, \\
& \quad \theta(y) \leq 0, \\
& \quad w^G - G(x) = 0, w^H + H(y) = 0, \\
& \quad H(y) \leq 0, \\
& \quad w \in W, \\
& \quad \|(x, y, w) - (x^*, y^*, w^*)\| \leq \delta.
\end{align*}
\]  
(19)

**Proof.** By Lemma 3.7 we have that \((x^*, y^*, w^*)\) is a local minimizer of \((10)\). Consider \(\delta > 0\) such that if \((x, y, w)\) is feasible for \((16)\) and

\[\|(x, y, w) - (x^*, y^*, w^*)\| \leq \delta,\]  
(20)

then \(f(x^*) \leq f(x)\). Note that \((x^*, y^*, w^*)\) is feasible for \((19)\). Moreover, given any feasible point \((x, y, w)\) of \((19)\), we have that it is also feasible for \((16)\) and satisfies \((20)\). Hence,

\[f(x^*) + \frac{1}{2}\|(x^*, y^*) - (x^*, y^*)\|^2_2 = f(x^*) \leq f(x) \leq f(x) + \frac{1}{2}\|(x, y) - (x^*, y^*)\|^2_2,\]
proving that \((x^*, y^*, w^*)\) is a global minimizer of (19).

Now, suppose that \((\bar{x}, \bar{y}, \bar{w})\) is also a global minimizer of (19). Then,

\[
f(\bar{x}) + \frac{1}{2}\| (\bar{x}, \bar{y}) - (x^*, y^*) \|_2^2 \leq f(x^*) + \frac{1}{2}\| (x^*, y^*) - (x^*, y^*) \|_2^2 = f(x^*) \leq f(\bar{x}),
\]

where the last inequality follows from the fact that \((\bar{x}, \bar{y}, \bar{w})\) is feasible for (16) and satisfies (20). Therefore, \((\bar{x}, \bar{y}) = (x^*, y^*)\), and hence

\[
\bar{w} = (G(\bar{x}), -H(\bar{y})) = (G(x^*), -H(y^*)) = w^*;
\]

proving the uniqueness.

The next result shows that our stationarity concept, given in Definition 3.1, is a legitimate optimality condition, independently of any constraint qualification. This is a requirement for them to be useful in the analysis of algorithms.

**Theorem 3.10** If \((x^*, y^*)\) is a local minimizer of the relaxed problem (11), then it is an AW-stationary point, in the sense of Definition 3.1.

**Proof.** Defining \(w^* = (G(x^*), -H(y^*))\), we conclude from Lemma 3.9 that there exists \(\delta > 0\) such that the point \((x^*, y^*, w^*)\) is the unique global minimizer of the problem (19), with \(\| \cdot \|_2\) in the last constraint. Define the (partial) infeasibility measure associated with this problem as

\[
\phi(x, y, w) = \frac{1}{2} \left( \| g^+(x) \|_2^2 + \| h(x) \|_2^2 + \| \theta^+(y) \|_2^2 + \| w^G - G(x) \|_2^2 + \| w^H + H(y) \|_2^2 + \| \tilde{H}^+(y) \|_2^2 \right),
\]

consider a sequence \(\rho_k \to \infty\) and let \((x^k, y^k, w^k)\) be a global minimizer of the penalized problem

\[
\begin{align*}
\text{minimize} & \quad f(x) + \frac{1}{2}\| (x, y) - (x^*, y^*) \|_2^2 + \rho_k \phi(x, y, w) \\
\text{subject to} & \quad w \in W, \\
& \quad \| (x, y, w) - (x^*, y^*, w^*) \|_2^2 \leq \delta^2,
\end{align*}
\]

(21)

which is well defined because the objective function is continuous and the feasible set is compact. Since \(\| (x^k, y^k, w^k) - (x^*, y^*, w^*) \|_2 \leq \delta\), we can assume without loss of generality that the sequence \((x^k, y^k, w^k)\) converges to some point \((\bar{x}, \bar{y}, \bar{w})\). We claim that \((\bar{x}, \bar{y}, \bar{w}) = (x^*, y^*, w^*)\). Note first that \((x^*, y^*, w^*)\) is feasible for (21) and \(\phi(x^*, y^*, w^*) = 0\). So, by the optimality of \((x^k, y^k, w^k)\) we have

\[
f(x^k) + \frac{1}{2}\| (x^k, y^k) - (x^*, y^*) \|_2^2 + \rho_k \phi(x^k, y^k, w^k) \leq f(x^*),
\]

(22)

implying that \(\phi(x^k, y^k, w^k) \to 0\), because \(\rho_k \to \infty\). This in turn implies that \(\phi(\bar{x}, \bar{y}, \bar{w}) = 0\), giving \(g^+(\bar{x}) = 0\), \(h(\bar{x}) = 0\), \(\theta^+(\bar{y}) = 0\), \(\bar{w}^G = G(\bar{x})\), \(\bar{w}^H = -H(\bar{y})\) and \(\tilde{H}^+(\bar{y}) = 0\). Moreover, as the sequence \((x^k, y^k, w^k)\) is feasible for (21), its limit point \((\bar{x}, \bar{y}, \bar{w})\) satisfies \(\bar{w} \in W\), because \(W\) is a closed set, and \(\| (\bar{x}, \bar{y}, \bar{w}) - (x^*, y^*, w^*) \| \leq \delta\). Therefore, \((\bar{x}, \bar{y}, \bar{w})\) is feasible for (19). Furthermore, from (22) we obtain

\[
f(x^k) + \frac{1}{2}\| (x^k, y^k) - (x^*, y^*) \|_2^2 \leq f(x^*).
\]
Furthermore, from (24b) and (25b), we have proving item (2).

Noting that the partial gradients of \( \psi \) are given by

\[
\nabla \psi(x^k, y^k, w^k) = \begin{cases} 
\nabla f(x^k) + (x^k - x^*) + \rho_k \nabla_x \psi(x^k, y^k, w^k) = 0 & (24a) \\
(y^k - y^*) + \rho_k \nabla_y \psi(x^k, y^k, w^k) = 0 & (24b) \\
\rho_k \nabla_w \psi(x^k, y^k, w^k) = 0 & (24c) \\
\rho_k \nabla_{w^*} \psi(x^k, y^k, w^k) - \mu_{H,k} + \mu_{G,k} = 0 & (24d) \\
\mu_{H,k} + \mu_{G,k} = 0 & (24e)
\end{cases}
\]

Noting that the partial gradients of \( \varphi \) are given by

\[
\nabla \varphi(x, y, w) = \begin{cases} 
\nabla f(x) + \nabla g(x) + \nabla h(x) + \nabla G(x) = 0 & (25a) \\
\nabla g(x) + \nabla h(x) + \nabla G(x) = 0 & (25b) \\
\nabla g(x) + \nabla h(x) + \nabla G(x) = 0 & (25c)
\end{cases}
\]

and defining \( \lambda^k \) as

\[
\lambda^g = \rho_k \nabla g^*(x^k), \quad \lambda^h = \rho_k \nabla h(x^k), \quad \lambda^\theta = \rho_k \nabla \theta^*(y^k), \\
\lambda^G = \rho_k (G(x) - w^G), \quad \lambda^H = \rho_k (w^H + H(y)), \quad \lambda^\tilde{H} = \rho_k \tilde{H}^+(y),
\]

we see immediately that \( \lambda^g \geq 0, \lambda^\theta \geq 0 \) and \( \lambda^\tilde{H} \geq 0 \). Moreover, using (24a) and (25a), we obtain

\[
\nabla x \mathcal{L}(x^k, y^k, \lambda^k) = \nabla f(x^k) + \rho_k \nabla_x \psi(x^k, y^k, w^k) = x^* - x^k \rightarrow 0.
\]

Furthermore, from (24b) and (25b), we have

\[
\nabla_y \mathcal{L}(x^k, y^k, \lambda^k) = \rho_k \nabla_y \psi(x^k, y^k, w^k) = y^* - y^k \rightarrow 0,
\]

proving item (2).
Let us prove item (3). By the feasibility of \((x^*, y^*)\) we have \(g_i(x^*) \leq 0\) for all \(i = 1, \ldots, m\). If \(g_i(x^*) = 0\), then \(\min \{-g_i(x^k), \lambda^g_i k\} \rightarrow 0\) since \(g_i(x^k) \rightarrow 0\) and \(\lambda^g_i k \geq 0\). On the other hand, if \(g_i(x^*) < 0\), we may assume that \(g_i(x^k) < 0\) for all \(k\). Thus, \(g^+_i(x^k) = 0\), yielding \(\lambda^g_i k = \rho_k g^+_i(x^k) = 0\). Therefore, \(\min \{-g_i(x^k), \lambda^g_i k\} = 0\). Items (4) and (7) can be proved by the same reasoning.

Now, note that by (24c), (24d) and (25c) we have
\[
\lambda^H_i k = \mu^H_i k - \mu^0_i k * w^G_i k = 0
\]
Therefore, using the fact that \(w^k i \in W\), we obtain
\[
\lambda^G_i k \cdot w^G_i k \quad \text{and} \quad \lambda^H_i k = \mu^H_i k - \mu^0_i k * w^G_i k = 0
\]
for all \(i = 1, \ldots, m\). Furthermore, given \(i \notin I_0(x^*)\), we have
\[
w^G_i k \rightarrow (w^*_i)^G = G_i(x^*) = x^*_i \neq 0,
\]
implying that \(\lambda^G_i k = 0\) for all \(k\) large enough. So, \(\min \{|G_i(x^k)|, |\lambda^G_i k|\} \rightarrow 0\). On the other hand, if \(i \in I_0(x^*)\), we have \(G_i(x^k) \rightarrow G_i(x^*) = x^*_i = 0\), and hence, \(\min \{|G_i(x^k)|, |\lambda^G_i k|\} \rightarrow 0\), proving item (5).

To prove the next item, note that using (24d), (24e) and the fact that \(w^k i \in W\),
\[
\lambda^H_i k \cdot w^H_i k = \mu^H_i k \cdot w^H_i k - \mu^0_i k \cdot w^G_i k \cdot w^H_i k = 0
\]
By the feasibility of \((x^*, y^*)\), we have \(H(y^*) \leq 0\). In the case \(H_i(y^*) < 0\), there holds
\[
w^H_i k \rightarrow (w^*_i)^H = -H_i(y^*) > 0,
\]
giving \(\lambda^H_i k = 0\) for all \(k\) large enough. Thus, \(\min \{-H_i(y^k), |\lambda^H_i k|\} = 0\). On the other hand, if \(H_i(y^*) = 0\), we have \(H_i(y^k) \rightarrow H_i(y^*) = 0\), and consequently, \(\min \{-H_i(y^k), |\lambda^H_i k|\} \rightarrow 0\), proving item (6) and completing the proof.

\section{Relations to other sequential optimality conditions}

In this section we discuss the relationships between approximate stationarity for standard nonlinear optimization and \(AW\)-stationarity.

As well known, every minimizer of an optimization problem is AKKT (see Definition 2.6). However, and surprisingly, we start by proving that the AKKT condition fails to detect good candidates for optimality for every MPCaC problem.

\textbf{Theorem 4.1} Every feasible point \((\bar{x}, \bar{y})\) for the relaxed problem (17) is AKKT.

\textbf{Proof}. We need to prove that there exist sequences \((x^k, y^k) \subset \mathbb{R}^n \times \mathbb{R}^n\) and
\[
(\mu^g_i k, \mu^h_i k, \mu^0_i k, \mu^H_i k, \mu^0_i k, \mu^H_i k, \mu^0_i k, \mu^H_i k) \subset \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^+ \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n
\]
such that \((x^k, y^k) \to (\bar{x}, \bar{y})\) and
\[
\left( \nabla_x L(x^k, \mu^g, \mu^h, \mu^{\theta, k}) \right) + \left( \begin{array}{c} 0 \\ \mu^\theta, k \nabla \theta(y^k) \end{array} \right) + \sum_{i=1}^{n} \left( \begin{array}{c} \mu^H, k \nabla H_i(y^k) \end{array} \right) + \sum_{i=1}^{n} \left( \begin{array}{c} \mu^\xi, k \nabla H_i(y^k) \end{array} \right) \to 0, \tag{28a}
\]
\[
\min \{-g(x^k), \mu^g\} \to 0, \quad \min \{-\theta(y^k), \mu^\theta, k\} \to 0, \tag{28b}
\]
\[
\min \{-H(y^k), \mu^H, k\} \to 0, \quad \min \{-\tilde{H}(y^k), \mu^\tilde{H}, k\} \to 0. \tag{28c}
\]

Let \(b = \nabla f(\bar{x})\) and define \(x^k = \bar{x}, \mu^g, k = 0, \mu^h, k = 0, \mu^\theta, k = 0, \mu^H, k = 0\) and
\[
y^k_i = \bar{y}_i; \quad \mu^H, k = 0, \quad \mu^\xi, k = \frac{b_i}{y^k_i} \quad \text{for } i \in I_{0+}(\bar{x}, \bar{y}) \cup I_{1+}(\bar{x}, \bar{y}),
\]
\[
y^k_i = \frac{b_i}{k}, \quad \mu^H, k = 0, \quad \mu^\xi, k = k \quad \text{for } i \in I_{00}(\bar{x}, \bar{y}),
\]
\[
y^k_i = -\text{sign}(\bar{x}_i)b_i \quad \mu^\xi, k = -\text{sign}(\bar{x})_i k \quad \mu^H, k = -\mu^\xi, k x^k_i \quad \text{for } i \in I_{\pm 0}(\bar{x}, \bar{y}).
\]
Thus we have \(\mu^H, k \geq 0, (x^k, y^k) \to (\bar{x}, \bar{y}),\)
\[
\nabla x_i L(x^k, \mu^g, k, \mu^h, k) - \mu^\xi, k y^k_i = b_i - \mu^\xi, k y^k_i \to 0,
\]
and
\[
-\mu^\theta, k - \mu^H, k + \mu^H, k - \mu^\xi, k x^k_i \to 0
\]
for all \(i = 1, \ldots, n\), giving (28a). Moreover, it is easy to see that (28b) and (28c) also hold.

Another sequential optimality condition for standard NLP is PAKKT (Definition 2.9). It is stronger than AKKT, but not stronger than AW-stationarity. The next example shows that PAKKT for the relaxed problem does not imply AW-stationarity, even under strict complementarity.

**Example 4.2** Consider the MPCaC and the corresponding relaxed problem given below.

\[
\begin{align*}
\text{minimize} & \quad x_2 \\
\text{subject to} & \quad x_1^2 \leq 0, \\
& \quad \|x\|_0 \leq 1,
\end{align*}
\]
\[
\begin{align*}
\text{minimize} & \quad x_2 \\
\text{subject to} & \quad x_1^2 \leq 0, \\
& \quad y_1 + y_2 \geq 1, \\
& \quad x_i y_i = 0, \quad i = 1, 2, \\
& \quad 0 \leq y_i \leq 1, \quad i = 1, 2.
\end{align*}
\]

Given \(a > 0\), we claim that the point \((\bar{x}, \bar{y})\), with \(\bar{x} = (0, a)\) and \(\bar{y} = (1, 0)\), is PAKKT but not AW-stationary. Indeed, for the first statement, consider the sequences \((x^k, y^k) \subset \mathbb{R}^2 \times \mathbb{R}^2\) and
\[
(x^k) = (\lambda^g, k, \lambda^\theta, k, \mu^H, k, \lambda^\xi, k) \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathbb{R}^2
\]
given by \(x^k = (1/k^3, a), \quad y^k = (1, -1/k), \quad \lambda^g, k = k^2, \quad \lambda^\theta, k = 0, \quad \mu^k = (0, ak), \quad \lambda^H, k = (0, 0) \quad \text{and} \quad \lambda^\xi, k = (0, k)\). Then we have \((x^k, y^k) \to (\bar{x}, \bar{y})\) and, denoting \(\xi(x, y) = x \ast y\), the gradient of the
Lagrangian of the relaxed problem reduces to
\[
\begin{pmatrix}
\nabla f(x^k) \\
0
\end{pmatrix}
+ \lambda^g, k
\begin{pmatrix}
\nabla g(x^k) \\
0
\end{pmatrix}
+ \mu^k_2 \nabla H_2(y^k) + \lambda^{\xi, k}_2 \nabla \xi_2(x^k, y^k)
\]
\[
= \begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}
+ \begin{pmatrix}
2\lambda^{g, k}x_1^k \\
0 \\
0 \\
0
\end{pmatrix}
+ \begin{pmatrix}
0 \\
0 \\
\lambda^{\xi, k}_2 y_1^k \\
-\mu^k_2 \\
\lambda^{\xi, k}_2 x_2^k
\end{pmatrix}
= \begin{pmatrix}
2/k \\
0 \\
0 \\
0
\end{pmatrix}
\to 0,
\]
proving (9a). Now, note that \( g(x^k) \to g(x) = 0 \) and \( \theta(y^k) \to \theta(y) = 0 \), which in turn imply that
\[
\min \{-g(x^k), \lambda^{g, k}\} \to 0 \quad \text{and} \quad \min \{-\theta(y^k), \lambda^{\theta, k}\} \to 0. \tag{30}
\]
Moreover, we have \(-\tilde{H}(y^k) \to -\tilde{H}(y) \geq 0 \) and \( \lambda^{H, k} = (0, 0) \), giving
\[
\min \{-\tilde{H}(y^k), \lambda^{H, k}\} \to 0. \tag{31}
\]
Furthermore, since \(-H_1(y^k) \to -H_1(y) \geq 0 \) and \( \mu^{k} = 0 \) and \(-H_2(y^k) = y_2^k \to 0 \), we have
\[
\min \{-H(y^k), \mu^{k}\} \to 0. \tag{32}
\]
Conditions (30), (31) and (32) prove the approximate complementarity (9b). Moreover, we have \( \delta_k = \|(1, \gamma^k)\|_{\infty} = k^2 \) for all \( k \) large enough,
\[
\limsup_{k \to \infty} \frac{\lambda^{g, k}}{\delta_k} > 0 \quad \text{and} \quad \lambda^{g, k} g(x^k) > 0.
\]
For the remaining multipliers the lim sup is zero and so we conclude that (9c) and (9d) hold, proving that Definition 2.9 is satisfied, that is, \((\bar{x}, \bar{y})\) is PAKKT.

Now, let us see that \((\bar{x}, \bar{y})\) is not AW-stationary. For this purpose, assume that the sequences \((x^k, y^k) \subset \mathbb{R}^2 \times \mathbb{R}^2\) and
\[
(\lambda^k) = (\lambda^{g, k}, \lambda^{\theta, k}, \lambda^{G, k}, \lambda^{H, k}, \lambda^{H, k}) \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2
\]
are such that \((x^k, y^k) \to (\bar{x}, \bar{y})\) and \( \min \{|G_2(x^k)|, |\lambda^{G, k}_2|\} \to 0 \). Then, since
\[
|G_2(x^k)| = |x_2^k| \to a > 0,
\]
we obtain \( \lambda^{G, k}_2 \to 0 \). Therefore, the expression
\[
\nabla_x L(x^k, \lambda^{g, k}) + \sum_{i=1}^2 \lambda^{G, k}_i \nabla G_i(x^k) = \begin{pmatrix}
2\lambda^{g, k}x_1^k + \lambda^{G, k}_1 \\
1 + \lambda^{G, k}_2
\end{pmatrix}
\]
cannot converge to zero. Thus, taking into account (14), item (3) of Definition 3.1 does not hold and hence \((\bar{x}, \bar{y})\) is not AW-stationary.

In contrast to AKKT and PAKKT, the other classical sequential optimality condition, CAKKT (Definition 2.7), does imply AW-stationarity, as we can see in the next result.
**Theorem 4.3** If $(\bar{x}, \bar{y})$ is a CAKKT point for the relaxed problem (10), then it is AW-stationary.

**Proof.** In view of Definition 2.7 there exist sequences $(x^k, y^k) \subset \mathbb{R}^n \times \mathbb{R}^n$ and 

\[
(\lambda^g,k, \lambda^h,k, \lambda^\theta,k, \mu^k, \lambda^H,k, \lambda^\xi,k) \subset \mathbb{R}^m_+ \times \mathbb{R}^p \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+
\]

such that 

\[
\begin{align*}
\nabla_x L(x^k, \lambda^g,k, \lambda^h,k, 0) + \sum_{i=1}^n \left( \lambda^\theta,k \nabla \theta(y^k) + \sum_{i=1}^n \left( \mu^k_i \nabla H_i(y^k) \right) 
\left. \right|_{\lambda^H,k \cdot x^k \to 0, \lambda^h,k \cdot h(x^k) \to 0, \lambda^\theta,k \cdot \theta(y^k) \to 0, \mu^k \cdot H(y^k) \to 0, \lambda^H,k \cdot H(y^k) \to 0, \lambda^\xi,k \cdot G(x^k) \cdot H(y^k) \to 0. \right) \right.
\end{align*}
\]

So, we may define

\[
\lambda^H,k = \mu^k + \lambda^\xi,k \cdot G(x^k) \quad \text{and} \quad \lambda^G,k = \lambda^\xi,k \cdot H(y^k)
\]

to obtain item (2) of Definition 3.1 from (33a). Items (3), (4) and (7) follow from (33b), (33c) and Remark 2.8. Let us prove item (5). For $i \in I_0$, there holds 

\[
G_i(x^k) \to G_i(\bar{x}) = \bar{x}_i = 0.
\]

Thus, \(\min\{|G_i(x^k)|, |\lambda_i^G,k|\} \to 0\). If $i \notin I_0$, we have $G_i(x^k) \to \bar{x}_i \neq 0$, which in view of (33d) yields 

\[
\lambda_i^G,k = \lambda_i^\xi,k H_i(y^k) \to 0.
\]

Therefore, \(\min\{|G_i(x^k)|, |\lambda_i^G,k|\} \to 0\) for all $i = 1, \ldots, n$. Finally, in order to prove item (6), note that (33c) and (33d) give 

\[
\lambda_i^H,k H_i(y^k) = \mu_i^k H_i(y^k) + \lambda_i^\xi,k G_i(x^k) H_i(y^k) \to 0.
\]

So, applying the argument of Remark 2.8 with $\alpha^k = |\lambda_i^H,k|$ and $\beta^k = H_i(y^k)$, we obtain 

\[
\min\{-H_i(y^k), |\lambda_i^H,k|\} \to 0
\]

for all $i = 1, \ldots, n$. Therefore, $(\bar{x}, \bar{y})$ is AW-stationary for the problem (10). \(\square\)

**Remark 4.4** Despite being stronger, we emphasize that the sequential optimality condition CAKKT is not so suitable to deal with MPCaC problems as AW-stationarity. The goal of considering CAKKT is to obtain, under certain constraint qualifications, KKT points for standard nonlinear programming problems. However, as we have been discussed, MPCaC are very degenerate problems because of the problematic complementarity constraint $G(x) \cdot H(y) = 0$. This means that we cannot expect to find strong stationary points for this class of problems and thereby making AW-stationarity a good tool for dealing with them.
To finish this section, we relate our sequential optimality condition to the tightened problem. The following result is a sequential version of Proposition 2.13.

**Theorem 4.5** Let \((\bar{x}, \bar{y})\) be a feasible point of the relaxed problem \([10]\). Then \((\bar{x}, \bar{y})\) is AW-stationary if and only if it is an AKKT point for the tightened problem \(TNLP_{I_0}(\bar{x}, \bar{y})\) defined in \([12]\).

**Proof.** Suppose first that \((\bar{x}, \bar{y})\) is AW-stationary. Then, in view of Lemma 3.3, we conclude that there exist sequences \((x^k, y^k) \subset \mathbb{R}^n \times \mathbb{R}^n\) and

\[
(\lambda^k) = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{\theta,k}, \lambda^{G,k}, \lambda^{H,k}, \lambda^{\tilde{H},k}) \subset \mathbb{R}^m_{+} \times \mathbb{R}^p_{+} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n
\]

such that \((x^k, y^k) \to (\bar{x}, \bar{y})\),

\[
\nabla_x L(x^k, \lambda^{g,k}, \lambda^{h,k}) + \sum_{i \in I_0} \lambda^{G,k}_i \nabla G_i(x^k) \to 0, \quad (34a)
\]

\[
\lambda^{\theta,k} \nabla \theta(y^k) + \sum_{i=1}^n \lambda^{H,k}_i \nabla H_i(y^k) + \sum_{i=1}^n \lambda^{\tilde{H},k}_i \nabla \tilde{H}_i(y^k) \to 0, \quad (34b)
\]

\[
\min\{-g(x^k), \lambda^{g,k}\} \to 0, \quad \min\{-\theta(y^k), \lambda^{\theta,k}\} \to 0, \quad (34c)
\]

\[
\min\{-H_i(y^k), |\lambda^{H,k}_i|\} \to 0, \quad i = 1, \ldots, n, \quad \min\{-\tilde{H}(y^k), \lambda^{\tilde{H},k}\} \to 0. \quad (34d)
\]

For \(i \in I_{0+} \cup I_{01}\) we have \(H_i(y^k) \to H_i(\bar{y}) = -\bar{y}_i < 0\). Therefore, we can assume without loss of generality that there exists \(\epsilon > 0\) such that \(-H_i(y^k) \geq \epsilon\) for all \(k\). So, using (34d), we obtain \(|\lambda^{H,k}_i| \to 0\), which in turn implies that

\[
\sum_{i \in I_{0+} \cup I_{01}} \lambda^{H,k}_i \nabla H_i(y^k) \to 0.
\]

By subtracting this from (34b), we obtain

\[
\lambda^{\theta,k} \nabla \theta(y^k) + \sum_{i \in I_{00} \cup I_{\pm 0}} \lambda^{H,k}_i \nabla H_i(y^k) + \sum_{i=1}^n \lambda^{\tilde{H},k}_i \nabla \tilde{H}_i(y^k) \to 0.
\]

So, we can redefine \(\lambda^{H,k}_i, i \in I_{0+} \cup I_{01}\), to be zero, without affecting (34b). Therefore, taking into account (34a), (34c), the second part of (34d) and the fact that \(\min\{-H_i(y^k), \lambda^{H,k}_i\} = 0\) for \(i \in I_{0+} \cup I_{01}\), we conclude that \((\bar{x}, \bar{y})\) is AKKT for \(TNLP_{I_0}(\bar{x}, \bar{y})\), which we recall here for convenience,

\[
\text{minimize } f(x)
\]

subject to

\[
g(x) \leq 0, h(x) = 0, \quad \theta(y) \leq 0, \quad G_i(x) = 0, i \in I_0, \quad H_i(y) \leq 0, i \in I_{0+} \cup I_{01}, \quad H_i(y) = 0, i \in I_{00} \cup I_{\pm 0}, \quad \tilde{H}(y) \leq 0.
\]
To prove the converse, suppose that \((\bar{x}, \bar{y})\) is AKKT for TNLP\(_I(x, y)\). Then there exist sequences \((x^k, y^k) \subset \mathbb{R}^n \times \mathbb{R}^n\) and

\[
(\lambda^k) = (\lambda^{g,k}, \lambda^{h,k}, \lambda^{G,k}_I, \lambda^{H,k}, \lambda^{H,k}) \subset \mathbb{R}^m_+ \times \mathbb{R}^p_+ \times \mathbb{R}^{|I_0|} \times \mathbb{R}^n \times \mathbb{R}^n_+,
\]

with \(\lambda^{H,k}_i \geq 0\) for \(i \in I_{0+} \cup I_{01}\), such that \((x^k, y^k) \to (\bar{x}, \bar{y})\),

\[
\nabla_x L(x^k, \lambda^{g,k}, \lambda^{h,k}) + \sum_{i \in I_0} \lambda^{G,k}_i \nabla G_i(x^k) \to 0, 
\]

\[
\lambda^{g,k} \nabla \theta(y^k) + \sum_{i=1}^n \lambda^{H,k}_i \nabla H_i(y^k) + \sum_{i=1}^n \lambda^{H,k}_i \nabla H_i(y^k) \to 0, 
\]

\[
\min\{-g(x^k), \lambda^{g,k}\} \to 0, \quad \min\{-\theta(y^k), \lambda^{g,k}\} \to 0, 
\]

\[
\min\{-H_i(y^k), \lambda^{H,k}_i\} \to 0, \quad i \in I_{0+} \cup I_{01}, \quad \min\{-\tilde{H}(y^k), \lambda^{H,k}\} \to 0.
\]

Extending the sequence \((\lambda^{G,k}_I)\) from \(\mathbb{R}^{|I_0|}\) to \(\mathbb{R}^n\) by letting \(\lambda^{G,k}_i = 0\) for \(i \notin I_0\), we can rewrite \((35a)\) as

\[
\nabla_x L(x^k, \lambda^{g,k}, \lambda^{h,k}) + \sum_{i=1}^n \lambda^{G,k}_i \nabla G_i(x^k) \to 0.
\]

Moreover, for \(i \in I_0\), there holds \(G_i(x^k) \to G_i(\bar{x}) = \bar{x}_i = 0\). Thus,

\[
\min\{|G_i(x^k)|, |\lambda^{G,k}_i|\} \to 0
\]

for all \(i = 1, \ldots, n\). Besides, for \(i \in I_{00} \cup I_{\pm 0}\), we have \(H_i(y^k) \to H_i(\bar{y}) = -\bar{y}_i = 0\), which implies \(\min\{-H_i(y^k), |\lambda^{H,k}_i|\} \to 0\). Therefore, in view of \((35d)\) and the fact that \(\lambda^{H,k}_i \geq 0\) for \(i \in I_{0+} \cup I_{01}\), we have

\[
\min\{-H_i(y^k), |\lambda^{H,k}_i|\} \to 0
\]

for all \(i = 1, \ldots, n\). Thus, from \((35b), (35c), (35d), (36), (37)\) and \((38)\), we conclude that \((\bar{x}, \bar{y})\) satisfies the conditions of Definition 3.1 that is, \((\bar{x}, \bar{y})\) is an AW-stationary point for the problem \((10)\). \(\square\)

5 Conclusion

In this paper we have presented a sequential optimality condition, namely Approximate Weak stationarity (AW-stationarity), for Mathematical Programs with Cardinality Constraints (MPCaC). This condition improves \(W_I\)-stationarity, which was established in our previous work [12].

Several theoretical results were presented, such as: AW-stationarity is a legitimate optimality condition independently of any constraint qualification; every feasible point of MPCaC is AKKT; the equivalence between the AW-stationarity and AKKT for the tightened problem TNLP\(_I(x, y)\). In addition, we have established some relationships between our AW-stationarity and other usual sequential optimality conditions, such as AKKT, CAKKT and PAKKT, by means of properties, examples and counterexamples.

It should be mentioned that, even though the computational appeal of the sequential optimality conditions, in this work we were not concerned with algorithmic consequences, which is subject of ongoing research.
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