Generic behaviour of nonlinear sound waves near the surface of a star: smooth solutions

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(Dated: 18 January 2009, revised version 10 April 2009)

We are interested in the generic behaviour of nonlinear sound waves as they approach the surface of a star, here assumed to have the polytropic equation of state \( P = K \rho^n \). Restricting to spherical symmetry, and considering only the region near the surface, we generalise the methods of Carrier and Greenspan (1958) for the shallow water equations on a sloping beach to this problem. We give a semi-quantitative criterion for a shock to form near the surface during the evolution of generic initial data with support away from the surface. We show that in smooth solutions the velocity and the square of the sound speed remain regular functions of Eulerian radius at the surface.

I. INTRODUCTION

In numerical simulations of neutron stars in general relativity, the matter is often modelled as a perfect fluid. The simplest equation of state usually considered is the ideal gas equation of state

\[ P = (\Gamma - 1)e\rho, \]  

where \( P \) is the pressure, \( \rho \) the rest mass density and \( e \) the internal energy per rest mass. The polytropic index \( \Gamma \equiv 1 + 1/n > 1 \) is a constant.

If the entropy per rest mass is everywhere the same, the ideal gas equation of state reduces to the polytropic equation of state

\[ P = K \rho^n, \]  

where \( K \) is another constant depending on the entropy per rest mass. If the initial data are isentropic, the solution remains isentropic until a shock forms. For the polytropic equation of state with \( n > 0 \), spherically symmetric self-gravitating solutions with a regular centre (stars) have a surface, characterised by \( P = \rho = 0 \) at finite radius \( r = r_s \), where \( \rho \sim (r_s - r)^n \) near the surface.

Standard numerical methods for evolving stars fail at the surface because division by zero density occurs and the speed of sound goes to zero. For smooth solutions in spherical symmetry, this can be avoided by using Lagrangian coordinates, but in 3-dimensional (3D) simulations with high-resolution shock capturing (HRSC) methods, the standard practice is to match the star to a thin “atmosphere”, which is then artificially kept from accreting onto it. This method is likely to give qualitatively wrong results, as the wave structure of the Riemann problem that underlies HRSC methods is different if the right state is vacuum.

The failure of the numerical methods is related to the physical fact that the perfect fluid approximation must break down at the surface. This approximation includes the assumption that small fluid elements are in thermal equilibrium on dynamical timescales, but as the density goes to zero, the thermal timescale diverges while the fluid dynamical timescales are still determined by waves in the interior and remain finite. In reality, some kind of plasma physics approximation applies.

The premise of this paper is that a mathematically correct numerical implementation of the perfect fluid assumption is more correct than the use of an unphysical atmosphere, which at best introduces physically unmotivated approximations and at worst does not even have a continuum limit. In this paper we provide two mathematical results that should be useful in achieving this goal. We begin here with smooth solutions and leave shocks for later work.

Our preliminary question is whether smooth initial data representing an outgoing wave with compact support form a shock as the wave approaches the surface. That a shock forms is suggested by the fact that the sound speed goes to zero at the surface with \( c_s \sim \sqrt{r_s - r} \) (independently of the polytropic index \( n \)), so that any outgoing wave steepens. Sperhake [1] has investigated this numerically in general relativity in spherical symmetry and concludes that small amplitude waves do not shock but large amplitude waves do. In the Newtonian case in spherical symmetry this had already been proved by Pelinovsky and Petrukhin [3]. We improve on this result by deriving a semiquantitative criterion for a sound wave to remain regular as it approaches the surface.

Our main question is what kinematic boundary conditions can be used in a numerical simulation to represent the free boundary at the surface of the star. This has been addressed in general relativity by Sperhake [1] for nonlinear spherical perturbations, using Lagrangian coordinates, and by Passamonti [2] for linear non-spherical perturbations. Here we consider the nonlinear case in Eulerian coordinates.

To answer both questions we use the mathematical methods of a classic paper by Carrier and Greenspan [4] concerning the shallow water equations on a sloping beach. We begin by reviewing their results and extending them from the shallow water case \( n = 1 \) to the general polytropic case \( n > 0 \).
II. MATHEMATICAL SETUP

For simplicity we assume spherical symmetry. Near the surface of the star, gravity is typically weak. Furthermore, the formation of shocks does not require large fluid velocities. This suggests that Newtonian physics should be a good approximation for what we want to investigate. On a sufficiently small scale the spherical symmetry of the star reduces to planar symmetry, and the Newtonian gravitational acceleration \( g \) is dominated by the interior of the star, and can be approximated as constant in space and time. Finally, for smooth solutions, sufficiently close to the surface, the entropy gradient can be neglected compared to the density gradient in determining the pressure gradient. We can therefore approximate the ideal gas as isentropic, with equation of state (2). (This last approximation would not hold if a shock reached the surface.)

In the “radial” spatial coordinate \( x \) and time \( t \), with \( v \) the Eulerian fluid velocity in the \( x \) direction, the Euler and conservation equations are

\[
\begin{align*}
v_t + v v_x + \Gamma K \rho^{1/2} \rho_x &= -g, \\
\rho_t + v \rho_x + \rho v_x &= 0.
\end{align*}
\]

(3) (4)

Here \( \rho = 0 \) defines a free boundary \( x = x_s(t) \). Within the approximation of planar symmetry, \( x \) has an infinite range, with \( x < x_s(t) \) representing the interior of the star.

It is useful to replace the dependent variable \( \rho \) with the sound speed \( c \) given by \( c^2 = dP/d\rho = \Gamma K \rho^{1/2} \) to obtain

\[
\begin{align*}
v_t + v v_x + 2 n c c_x &= -g, \\
c_t + v c_x + \frac{1}{2n}c v_x &= 0.
\end{align*}
\]

(5) (6)

For \( n = 1 \), these equations are identical with the shallow water equations restricted to planar symmetry on a uniformly sloping beach, with \( x \) and \( v \) the horizontal position and velocity, \( \rho \sim c^2 \) the height of the water, \( g \) the effective horizontal gravitational acceleration, and \( x = x_s(t) \) the instantaneous shoreline [5].

The unique static solution of \((5,6)\) is

\[
v = 0, \quad c = \sqrt{-\frac{g \Delta}{n}},
\]

and hence \( \rho \sim (-x)^n \), where we have fixed a translation invariance by locating the surface at \( x = 0 \).

III. HOLOGRAPH TRANSFORM

The problem can be written as

\[
[\partial_t + (v \pm c) \partial_x] (v + gt \pm 2nc) = 0,
\]

(8)

and so admits the Riemann invariants \((v + gt) \pm 2nc\) with characteristic speeds \(v \pm c\). Carrier and Greenspan [4] (considering the shallow water case \( n = 1 \)) suggested a hodograph transform from independent variables \( t \) and \( x \) to independent variables \( \lambda \) and \( \sigma \) given by

\[
\begin{align*}
\lambda &= v + gt, \\
\sigma &= 2nc.
\end{align*}
\]

(9) (10)

(These definitions differ from [4] by a factor of 2.)

The resulting transformation of partial derivatives is

\[
\left( \frac{\partial_t}{\partial x} \right) = \Delta^{-1} \left( x_{\sigma} - x_{\lambda} \right) \left( \frac{\partial_{\lambda}}{\partial_{\sigma}} \right),
\]

(11)

where

\[
\Delta \equiv t_{\lambda} x_{\sigma} - x_{\lambda} t_{\sigma}.
\]

(12)

In particular, we have

\[
\begin{align*}
\left( \frac{\lambda t}{x_{\sigma}}, \frac{\sigma t}{x_{\lambda}} \right) &= \Delta^{-1} \left( x_{\sigma} - x_{\lambda} \right).
\end{align*}
\]

(13)

Clearly, the transformation is regular if and only if \( \Delta \neq 0, \pm \infty \).

Substituting (9-10) and (13) into (8), we obtain

\[
\begin{align*}
x_{\sigma} - (\lambda - gt) t_{\sigma} + \left( \frac{\sigma}{2n} \right) t_{\lambda} &= 0, \\
x_{\lambda} + \left( \frac{\sigma}{2n} \right) t_{\sigma} - (\lambda - gt) t_{\lambda} &= 0.
\end{align*}
\]

(14) (15)

This PDE system is not yet linear because of the appearance of \( gt \) in the coefficients of \( t_{\sigma} \) and \( t_{\lambda} \). However, from the two nonlinear first-order PDEs (14-15) one can derive a linear second-order PDE for \( t(\lambda, \sigma) \), namely

\[
t_{\lambda\lambda} = t_{\sigma\sigma} + \frac{2n + 1}{\sigma} t_{\sigma}.
\]

(16)

Trivially, \( \lambda \) taken as a function of \( \lambda \) and \( \sigma \) obeys the same PDE as \( t \), and by adding the two we obtain the autonomous linear wave equation

\[
v_{\lambda\lambda} = v_{\sigma\sigma} + \frac{2n + 1}{\sigma} v_{\sigma}
\]

(17)

for \( v(\lambda, \sigma) \). This is the key equation of this paper.

The problem has now been cast into linear form, and the free boundary \( x = x_s(t) \) has been mapped to the coordinate line \( \sigma = 0 \), with \( \sigma > 0 \) representing the interior of the star.

IV. A CRITERION FOR SHOCK FORMATION

From (13) with (9,10) we find

\[
\begin{align*}
v_x &= \frac{1}{g \Delta} v_\sigma, \\
c_x &= \frac{1}{2ng \Delta} (1 - v_\lambda),
\end{align*}
\]

(18) (19)
and so a shock forms from regular initial data as and only if \( \Delta \to 0 \). Using (9-10,14-15), the Jacobian \( \Delta \) defined by (12) can be expressed in terms of \( v \) alone as

\[
\Delta = -\frac{\sigma}{2ng^2} \left[ (1-v\lambda)^2 - v_x^2 \right].
\]

(20)

We see that the wave does not form a shock if the first derivatives of \( v \) in a solution of (17) remain sufficiently small, so that \( \Delta \) remains negative. Such solutions are easily obtained by rescaling the amplitude of any given solution.

We shall now consider small smooth initial data for (5,6) on the curve \( t = 0, x < 0 \). These correspond to Cauchy data for (17) on the curve given by \( \lambda = \lambda_0(\sigma) \), \( \sigma > 0 \). We require these data to obey

\[
(1-v\lambda)^2 - v_x^2 > 0 \tag{21}
\]

for all \( \rho > 0 \) on \( \lambda = \lambda_0(\sigma) \). This criterion is necessary for the existence of the equivalence between (5,6) and (17), and implies that there is no shock present in the initial data. We then formally evolve the data to \( \lambda > \lambda_0(\sigma) \) using (17). Setting aside the boundary at \( \sigma = 0 \), which we consider later, this solution exists because (17) is linear. However, if at any point in \( \lambda > \lambda_0(\sigma) \) the condition (21) is violated, the wave has developed a shock at some \( t > 0 \), and the solution of (17) does not have physical meaning for larger values of \( t \).

In order to translate initial data in coordinates \((x, t)\) to \((\sigma, \lambda)\), we consider smooth data with compact support away from the boundary and which are sufficiently weak (in the sense of close to the static star solution) that initially the solution can be approximated by a solution of the linearisation of (5,6) around the static star solution. We then evolve these data using (17), and so do not require them to remain small. We use (21) in this solution as the necessary and sufficient criterion for the absence of shocks.

Linearising (5,6) about the static solution (7), we obtain

\[
\delta v_{tt} = \left( -\frac{gx}{n} \right) \left( \delta v_{xx} + \frac{n+1}{x} \delta v_x \right).
\]

(22)

(We have written \( \delta v \) instead of \( v \) to stress that this is only an approximation valid for small \( v \).) The same equation can be obtained from (17) by the substitutions

\[
\lambda = gt, \quad \sigma = 2\sqrt{gmx}.
\]

(23)

This gives us a simple approximate relation between initial data for the linearisation of (5,6), and initial data for (17) (which is linear but contains the nonlinear dynamics).

A formal d’Alembert solution of (17) is

\[
v(\lambda, \sigma) = \sum_{k=0}^{\infty} \sum_{\pm} \sigma^{-n+\frac{\pm}{2}-k} f_{k+}^\pm(\lambda \pm \sigma)
\]

where \( f_0^\pm \) is free data and

\[
f_{k+}^\pm = \frac{(k+\frac{\pm}{2})^2 - n^2}{2(k+1)} \int f_k^\pm.
\]

(25)

A few remarks will put this result into context: This series converges at most in the sense of an asymptotic series as \( \sigma \to \infty \), and clearly diverges for sufficiently small \( \sigma \). Another formal d’Alembert solution exists which has ascending powers of \( \sigma \), but it does not interest us here. In the special case \( n = 1/2 \), either series reduces to the well-known d’Alembert solution of the spherical wave equation in 3 dimensions, while for \( n = -1/2 \) we obtain the d’Alembert solution of the 1-dimensional wave equation.

Consider now an isolated wave packet approaching the surface with initial position \( \sigma_0 \), width \( \sigma_1 \ll \sigma_0 \) and amplitude \( v_0 \), so that \( |v_0| \sim |v_\sigma| \sim v_0/\sigma_1 \) initially. In this regime, we can approximate

\[
v(\lambda, \sigma) \approx \sigma^{-n+\frac{\pm}{2}} f_0^\pm(\lambda + \sigma).
\]

(26)

The derivatives of \( v \) take their largest values when the wave packet turns around close to the surface. From the scaling properties of solutions of (17), this must happen at \( \sigma \sim \sigma_1 \), at which point its amplitude will be \( v_0(\sigma_1/\sigma_0)^{-n-1/2} \) in the approximation (26). Evaluating (21) at that point, we obtain a criterion for the wave never to form a shock, which is

\[
\frac{v_0}{\sigma_1} \lesssim \left( \frac{\sigma_1}{\sigma_0} \right)^{n+\frac{\pm}{2}}.
\]

(27)

Finally, expressing \( \sigma_0 \) and \( \sigma_1 \) in terms of the initial Eulerian position \( x_0 \) and length scale \( x_1 \) of the wave packet by using (23), we obtain the regularity criterion

\[
\frac{v_0}{\sqrt{g x_0}} \lesssim \left( \frac{x_1}{|x_0|} \right)^{n+(3/2)}
\]

(28)

In these estimates we neglect an unknown \( O(1) \) factor depending on the precise shape of the wave packet.

Although we have worked in the approximation of planar symmetry and constant \( g \), it is useful to express the parameter \( g \) in terms of \( v_* = \sqrt{2gr_*} \), which is the escape velocity at the surface of a spherical star, where \( r_* \) is its radius and \( g \) is the gravitational acceleration at its surface. We can then rewrite the estimate (28) as

\[
v_0 \lesssim \left( \frac{x_1}{|x_0|} \right)^{n+(3/2)} \left( \frac{|x_0|}{r_*} \right)^{\frac{3}{2}} v_*
\]

(29)

A numerical example will illustrate this: in a neutron star modelled as a polytrope with \( r_* \sim 10^4 m \), \( v_* \sim 10^8 m/s \) and \( n = 1 \), a sound wave of wavelength \( x_1 \sim 1 m \) deep in the interior \( (x_0 \sim -r_*) \) must have an amplitude of \( v_0 \lesssim 10^{-2} m/s \) to remain regular.
V. GENERIC BEHAVIOUR AT THE FREE BOUNDARY

The surface of the star is a free boundary characterised by the kinematic boundary conditions

\[ P(x_*(t), t) = 0, \]
\[ \frac{dx_*}{dt} = v(x_*(t), t). \] \hspace{1cm} (30)

These conditions are straightforward to implement in Lagrangian coordinates, but in 3D HRSC simulations we need their equivalent in Eulerian coordinates. For solutions which remain smooth, we obtain these by going through the hodograph transformation.

The general solution of Eq. (17) can be written as a linear superposition of solutions of the form

\[ v(\lambda, \sigma) = e^{i\omega\lambda} \sigma^{-n} J_{\pm n}(\omega\sigma). \] \hspace{1cm} (32)

As \( J_n(\sigma) \) is \( \sigma^n \) times a power series in positive even powers of \( \sigma \), the solution using \( J_n \) is an even regular function of \( \sigma \), while the solution using \( J_{-n} \) diverges as \( \sigma^{-2n} \) as \( \sigma \to 0 \). The regular solution can be selected by imposing the boundary condition

\[ v_\sigma = 0 \quad \text{at} \quad \sigma = 0, \] \hspace{1cm} (33)

which together with (17) makes a well-posed linear initial-boundary value problem. Clearly this condition is the required kinematic boundary condition for smooth solutions.

We now translate this back into the Eulerian variables \( c(x, t) \) and \( v(x, t) \). Assuming the wave does not form a shock, the square bracket in (20) is strictly positive, and so \( \Delta \sim \sigma \) at the boundary. Substituting \( \Delta \sim \sigma \) into (19) gives

\[ \sigma c_x \sim (1 - v_\lambda) \] \hspace{1cm} (34)

at the boundary. The right-hand side is even in \( \sigma \) because \( v \) is even in \( \sigma \) by the assumption of regularity. It follows, using (10), that \( (c^2)_x \) is a regular function of \( c^2 \), and hence \( c^2 \) is a regular function of \( x \).

Substituting \( \Delta \sim \sigma \) into (18) gives

\[ v_x \sim \sigma^{-1} v_\sigma \] \hspace{1cm} (35)

at the boundary. The right-hand side is again even in \( \sigma \). Hence \( v_x \) is an even function of \( c^2 \) and so, using our previous result, it is a regular function of \( x \). It follows that \( v \) is a regular function of \( x \).

It is clear that the \( \lambda \) or \( t \) dependence does not affect these results in the limit \( x \to x_*(t) \) or \( \sigma \to 0 \). We have therefore shown that as long as the solution remains regular, \( c^2 \) and \( v \) are regular functions of \( x \) and \( t \) at the surface. This is the desired kinematic free boundary condition. In particular, \( c^2 \sim x_*(t) - x \) at the moving surface of regular solutions, as in the static case.

Note that \( \rho \sim (c^2)^n \), so \( \rho \) is a regular function of \( x \) only if \( n \) is an integer. Note also that in general \( v \) and \( c^2 \) are neither even nor odd in \( x - x_* \).

VI. DISCUSSION

Building on the earlier work [3, 4], we have given various forms of an upper limit on the amplitude of nonlinear sound waves if they are to avoid forming a shock. This tells us in which physical regime a simple (non-shock capturing) numerical method will be valid because shocks do not occur. It may also be of direct astrophysical interest.

For solutions which remain regular as they are reflected at the free boundary, we have shown that the usual free boundary condition is equivalent to \( v \) and \( c^2 \) being regular functions of \( x \) and \( t \). This suggests an alternative numerical treatment of the stellar surface which does not require an unphysical atmosphere.

Our results were derived within the approximations of Newtonian physics, a constant gravitational field, a polytropic equation of state and planar symmetry (as the limit of spherical symmetry near the surface). As discussed in the introduction, these are all natural approximations to make, except for spherical symmetry. However, applying geometric optics to the linearised sound wave equation for the pressure perturbation \( \delta P \),

\[ \delta P_{tt} = \left( -\frac{g_x}{n} \right) \left( \delta P_{xx} + \frac{n + 1}{x} \delta P_x + \delta P_{yy} + \delta P_{zz} \right), \] \hspace{1cm} (36)

we find that its sound rays, without loss of generality restricted to the \( xy \) plane, are given by \( y(x) = a + \ln(1 + bx^2) \) for constants \( a \) and \( b \), and so in the geometric optics approximation sound waves approaching the surface \( x = 0 \) at any angle are refracted towards lower sound speed until they reach the surface at right angles. This provides some justification for the assumption that our results will also be qualitatively correct beyond the restriction to spherical (planar) symmetry.

Acknowledgments

We would like to thank Marvin Jones for discussions, and Michael Gabler for pointing out an error in the original version.

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