Tension mediated nonlinear coupling between orthogonal mechanical modes of nanowire resonators

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We study the nonlinear coupling between orthogonal flexural modes of doubly clamped InAs nanowire resonators. The two orthogonal modes are formed by the symmetry breaking and lifting of degeneracy of the fundamental mode. The presence of a Duffing nonlinearity emerges when a mode is driven to large amplitudes. In this regime the modes are coupled due to the tension induced from the large amplitude of oscillations and is reflected in the hysteretic response of the mode that is not strongly driven. We study the driven-driven response of the mechanical modes to elucidate the role of nonlinear mode coupling in such mechanical resonators. The dynamics of the coupled modes studied here could prove useful in technological applications such as nanowire based vectorial force sensing.

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I. INTRODUCTION

Understanding the coupling between different modes of a complex system is of great interest as it can be used to improve functionality such as enhancing the sensitivity of the system. Optomechanics, for example, has harnessed the coupling between a mechanical oscillator and an optical cavity to explore the limits of precise position measurements. Similarly, the coupling between two vibrational modes of nanomechanical systems has also been explored in recent times.

Coupling between mechanical modes has been studied in nanofabricated doubly clamped beams, nanofabricated cantilevers, carbon nanotubes, 2D materials, cantilevers of VLS (vapour liquid solid) grown nanowires and doubly clamped VLS nanowires. The origin of nonlinear intermodal coupling can be complex and system specific; efforts to understand the microscopic origin are ongoing across systems as tunability of this nonlinear coupling can provide additional functionality to devices.

VLS grown nanowires offer advantage of engineering the properties with unprecedented control. Recently, orthogonal vibrational modes of the cantilevers, singly clamped nanowires, made using VLS grown nanowires have been used to vectorially map the forces on a surface to provide maps of electric fields. The two orthogonal vibrational modes arise due to an unintentional breaking of symmetry of the degenerate fundamental mode. This pair of modes can provide complementary information about the nano electromechanical system (NEMS) and are, hence, of interest. We show that the Duffing nonlinearity of a mode that is driven to large amplitudes imprints itself as a hysteretic increase in frequency of the second mode. In this paper, we study in detail the inter-modal nonlinear coupling and show that it can be large and tunable as a function of the tension in the nanowires. Tunability due to tension is absent in singly clamped beams where nonlinear coupling has been studied recently. We compare the strength of coupling across systems and find that the coupling in our system is large and comparable to the coupling observed in carbon nanotube resonators. In addition, the key role of tension mediated intermodal coupling is demonstrated by the fact that the strength of the coupling in our doubly clamped nanowire devices is significantly larger than cantilever devices made using nanowires. The microscopic model we develop explains the experimental observations accurately.
II. EXPERIMENTAL METHODS

We start by briefly describing the nanofabrication of devices used in our experiments; details of fabrication are available in previous reports\textsuperscript{16,23}. InAs nanowires of \(\sim 10 \mu\text{m}\) length and 100 nm diameter were grown using MOCVD techniques. Intrinsic silicon wafers with 100 nm of thermally grown nitride were used as the substrate for device fabrication. A thin layer of electron beam (e-beam) resist was spun on the substrate. The thickness of this resist layer decides the height above the substrate by which the nanowire is suspended. The nanowires were removed from the growth substrate and then drop cast above the e-beam resist coated substrate and then covered by additional layers of e-beam resists. After patterning source, drain, and gate electrodes using standard e-beam lithography, the substrate was loaded in to a sputtering system with an \textit{in-situ} plasma etcher for conformal deposition of metal. Prior to metal deposition the sample was exposed to an argon plasma to remove the residual oxide on the nanowires. This allows the formation of Ohmic contact to the nanowire. Gold was sputtered on the substrate preceded by a thin layer of chromium for adhesion. The sputtered metal forms both the electrical contacts as well as mechanical anchors for suspending the device. Lift-off followed by drying the substrate completes the device fabrication process.

Figure 1 shows a false colored scanning electron microscope (SEM) image of a device along with the schematic of the circuit used in our experiments. The insulating nature of the intrinsic silicon substrates at low temperatures negates the effects of parasitic capacitances and allows us to carry out direct readout of the electrical signal at radio frequencies (\(rf\)). DC and \(rf\) signals were combined using a bias tee and applied to the gate electrode to actuate the mechanical oscillations in the resonator. The change in \(rf\) current through the device due to mechanical oscillations were then measured by a vector network analyzer (VNA). The signal from the VNA was used to drive and probe one mechanical mode in the linear regime, while another signal combined from a function generator (FG) was used to drive and pump the second mode to large amplitude oscillations. This allows us to investigate the driven response of one mechanical mode to large amplitude oscillations of the second mode. All experiments were performed with sample temperature at \(\sim 5\) K.

Figure 2(a) shows the response of the nanowire resonator measured using the network analyzer with the pump signals turned off for an applied DC gate voltage of -20 V. The lower frequency mode at \(\sim 59.5\) MHz is seen to have a large signal compared to
FIG. 1. Circuit schematic. SEM image of the device along with a schematic of the circuit used in our experiments is shown here. $V_{sd}$ is applied to the source electrode and the rf current through the nanowire is amplified and measured through the drain electrode. The gate electrode is positioned $\sim 500$ nm away from the nanowire. VNA: vector network analyzer, FG: function generator. The blue shaded area denotes the parts inside the cryostat. Scale bar corresponds to 2 $\mu$m.

The higher mode at $\sim 60.9$ MHz with quality factor of $\sim 2800$ for both modes. Here, the amplitude of mode 1 at resonance is estimated to be $\sim 0.8$ nm (see supplementary material) for a drive power of -30 dBm. For the remainder of the article we shall refer to the lower/higher frequency mode as mode 1/mode 2. The small frequency difference ($\sim 1.4$ MHz) between the two modes implies that they are not harmonics of the vibration, rather they correspond to the non-degenerate, orthogonal flexural modes of the resonator. These orthogonal modes are shown in the inset of Figure 2(a). The degeneracy of the orthogonal components of the fundamental mode can be broken by any asymmetry in the nanowire. Figure 2(b) shows the dispersion of the resonant modes with applied DC gate voltage. Mode 1 is seen to disperse negatively with increasing absolute value of the gate voltage. This is indicative of the capacitive softening\textsuperscript{16,25} effect of mode 1. Mode 2, however, is seen to have negligible dispersion with the gate voltage. The observed dispersion of the two modes to an applied DC gate voltage is indicative of their plane of oscillations. We deduce that mode 1 oscillates mainly in the plane of the substrate as the capacitive softening effect occurs in nanoelectromechanical systems (NEMS) when the oscillations are in the plane of the gate electrode. Mode 2, therefore, oscillates in a nearly vertical plane perpendicular...
FIG. 2. Electromechanical response. (a) The two modes of the resonator are separated by \( \sim 1.4 \) MHz for an applied DC gate voltage of -20 V. (b) Dispersion of the two modes with DC gate voltage. Color plot shows the calibrated response obtained from the VNA. Dashed lines are guides to the eye. Power used to drive the resonators is -30 dBm in both (a) and (b).

III. RESULTS AND DISCUSSION

Figure 3 shows the effect of a large driving force on mode 2 on the frequency response of mode 1. Here the linear response of mode 1 is probed with a weak signal from the VNA with -30 dBm power whereas the pump frequency, around mode 2, is a strong signal from the FG such that mode 2 is driven in the nonlinear Duffing regime. Both the frequencies are swept in the increasing direction. For lower pump powers (0 dBm, left panel in Figure 3) where the resonant amplitude of mode 2 is estimated to be 23 nm (see supplementary material for detailed calculations), mode 1 frequency is seen to increase slightly when the pump frequency is in the vicinity of mode 2. As the pump power is increased to 10 dBm (right panel in Figure 3) mode 1 frequency is seen to be affected over larger range of the pump frequencies. As the pump frequency is increased mode 1 frequency is seen to gradually increase and abruptly decrease to its original value beyond a certain pump
FIG. 3. **Nonlinear mode coupling.** (a)-(b) Experimentally obtained response of mode 1 as a function of pump signal frequency near mode 2. The pump signal power is 0 dBm in (a) and 10 dBm in (b) where the pump frequency is increased (forward sweep) in both cases. Frequency of mode 1 increases when the pump frequency is in the range of mode 2. For larger pump power, the window of coupled response is seen to be larger. The colour scale of high/low represents mode 1 being on/off resonance. Probe power is -30 dBm. (c)-(d) Simulation showing effect of magnitude of pump signal power on frequency shift of probe response for $v_{AC1} = 0.14$ V in (c) and $v_{AC2} = 0.30$ V in (d) frequency. The observed response indicates that these orthogonal modes are nonlinearly coupled. We now explore the form of this nonlinear coupling further.

The leaning of the resonant frequency and subsequent, abrupt jump in the response is characteristic of the Duffing nonlinearity in these devices. The Duffing nonlinearity is characterized by presence of terms of the form $\alpha u^3$ in the equation of motion that gives rise to bistability in the response\textsuperscript{17}. As the pump frequency matches mode 2, it drives mode 2 to large amplitudes until the point of bistability.

The response of mode 1, therefore, follows the Duffing response shape of mode 2. The frequency leaning of mode 1 arises from the deformation induced tension that accumulates on the nanowire as mode 2 oscillates with a large amplitude. The deformation induced tension is significant when the amplitude of oscillations becomes large. This tension can be written as

$$\frac{EA}{2L} \int_0^L \left( \frac{\partial \epsilon}{\partial z} \right)^2 dz, \tag{1}$$
FIG. 4. Hysteresis in mode coupling. (a)-(b) Experimentally obtained response of mode 1 as a function of the pump signal for (a) forward and (b) reverse sweep of the pump frequency around mode 2. Probe power is -20 dBm. (c)-(d) Simulation showing hysteresis behavior in coupling regime with \( \hat{v}_{AC2} = 0.26 \) V for forward sweep in (c) and reverse sweep in (d).

where \( E \) is the Young’s modulus, \( A \) and \( L \) are the cross sectional area and length of the beam respectively, and \( \epsilon \) is the mode shape as a function of the distance \( z \) along the length of the nanowire. The mode shape can be written in terms of the in-plane, \( U(z,t) \), and out-of-plane, \( V(z,t) \), components for mode 1 and mode 2 respectively. Then the governing partial differential equations of motion for each mode will include a term of the form\( ^6 \) (see supplementary material for full Euler-Bernoulli equations)

\[
\left[ \frac{EA}{2L} \int_0^L \left( \left( \frac{\partial U(z,t)}{\partial z} \right)^2 + \left( \frac{\partial V(z,t)}{\partial z} \right)^2 \right) dz \right] \frac{\partial^2 W}{\partial z^2}.
\]  

(2)

where \( W \) is either \( U(z,t) \) or \( V(z,t) \).

Along with the frequency leaning that accompanies the Duffing response, the bistable nature also gives rise to hysteresis in the response of a single mode with the direction of frequency sweeps. As the modes are coupled, we expect the hysteretic Duffing response of the pumped mode to reflect in the response of the probed mode as well. Figure 4\( ^4 \) shows the effect of pump sweep direction on the response of mode 1. The coupled response of mode 1 faithfully mimics the Duffing response of mode 2.

The equations of motion for our system can be written in a simplified form using the
modal coordinates, \( u \) and \( v \), as \[6,8\] (see supplementary material for details)

\[
\ddot{u} + cu + \omega_u^2 u + \alpha_u u^3 + \alpha_{uv} u^2 v = f_u(t) \tag{3}
\]

\[
\ddot{v} + cv + \omega_v^2 v + \alpha_v v^3 + \alpha_{uv} u^2 v = f_v(t). \tag{4}
\]

To arrive at equation [3] we have used a reduced order technique to solve the coupled partial differential equations (as described by Euler-Bernoulli theory) that govern the dynamics of our system. The factors \( \alpha_u, \alpha_v, \alpha_{uv} \propto \alpha \) where \( \alpha \) is the non-dimensional coefficient of nonlinear modal interaction governed by the geometry of the device (see supplementary material for details). The above equations of motion capture all the physics of our system. We solve these equations numerically to compare with the experimentally observed results. The bottom panels in Figures [3][4] show the simulated response of the nanowire resonator for corresponding experimental data in the top panels. The calculated response is seen to match well with the experimentally observed frequency shifts, bistability, and hysteresis.

In order to understand and compare our experiments with similar systems studied earlier we compare the key parameters in Table I. \( f_1 \) denotes the frequency of the lower of the two coupled modes, \( \Delta f \sim f_2 - f_1 \) is difference in the frequency of the modes and \( \frac{\Delta f_2}{\Delta f_1} \) is the rate of change of frequency of the un-pumped mode as a function of the pumped mode. Larger the nonlinear coupling larger is \( \Delta f_2/\Delta f_1 \) and to first order the slope of the nonlinear coupling induced response is linear; this is seen in Figure [3](b) and Figure [4](a). We find that the extent of nonlinear coupling is much larger in doubly clamped nanomechanical devices and this attests to the key role of tension in mediating the nonlinear coupling. Naturally, such a strong nonlinear coupling is not present in cantilevers made using nanowires\[12\]. We also find, from our calculations, that the coupling is the strongest between the two orthogonal modes of the fundamental (details provided in the supplementary material). It would be interesting to explore parametric degenerate and nondegenerate amplification using such coupled modes\[10\]. Especially since nanowire based vectorial AFM\[13,15\] has been demonstrated using parametric schemes for these coupled modes could help push sensitivity to new limits in vectorial scanning.

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### Table I. Nonlinear intermodal coupling across nanomechanical systems

| System used                                      | $f_1$   | $\Delta f \sim f_2 - f_1$ | $\frac{\Delta f}{\Delta f_1}$ |
|--------------------------------------------------|---------|-----------------------------|---------------------------------|
| Carbon nanotubes [1] (Castellanos-Gomez et al.)  | 180 MHz | $\sim$730 MHz               | $\sim$1                         |
| Nanofabricated doubly clamped beams [2] (Westra et al.) | 275 kHz | 200 kHz                     | $\sim$1                         |
| VLS grown nanowires cantilevers [12] (Cadeddu et al.) | $\sim$1 MHz | 6 kHz                      | $\sim$0.1                       |
| VLS grown doubly clamped nanowires (present work) | 50 MHz  | 1.6 MHz                     | $\sim$0.8                       |

### IV. CONCLUSIONS

In summary, we have studied the tension mediated nonlinear coupling between the orthogonal modes of an InAs nanowire resonator. Our experiments show that the Duffing nonlinearity of one mode affects the linear response of the second mode as observed from the abrupt and hysteretic frequency shifts. Our calculations deduce the role of large-oscillation induced tension on the observed dynamics of the system. Such coupled dynamics could be exploited to improve the performance and bandwidth of NEMS based sensors.

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Supplementary material: Tension mediated nonlinear coupling between orthogonal mechanical modes of nanowire resonators

S1. MATHEMATICAL MODELLING

We have performed numerical simulations to understand the mode coupling behaviour of InAs nanowire resonators. A nanowire can be modeled as cylindrical vibrating beam having same frequency in all direction of the planes of flexural vibration. But, in presence of small imperfection in cross-section, a single resonant frequency of the nanowire splits into two nearby frequencies in orthogonal principal planes\(^\text{S1}\). we have observed similar two nearby resonant frequencies in the experiments of the InAs nanowire. So, the InAs nanowire has been modelled as a wire of elliptical cross-section as shown in Figure\(^\text{S1}\). For electrostatic actuation, it is placed \(g_0\) distance apart from a gate electrode. The nanowire has mass density \(\rho\), length \(L\), semi-major axis \(r_u\), and semi-minor axis \(r_v\); further, the cross-section area is denoted as \(A\) and principal flexural stiffnesses as \(EI_u\) and \(EI_v\). The

![FIG. S1. A schematic diagram of electrostatically actuated InAs nanowire.](image)

flexural dynamics of the nanowire is described in terms of its displacements \(\hat{U}(\hat{z}, \hat{t})\) and \(\hat{V}(\hat{z}, \hat{t})\) along principal orthogonal directions (X-axis and Y-axis). Here, the variables \(\hat{z}\) and \(\hat{t}\) are distance coordinate along axis of the nanowire and time coordinate respectively. The purpose of using \((\cdot)\) on variables name is to distinguish their dimensional form from their non-dimensional form which are introduced in this section later. Using Euler-Bernoulli beam theory, the flexural dynamics of the electrostatically actuated InAs nanowire is
governed by following coupled partial differential equations:\(^{52,54}\)

\[
EI_u \frac{\partial^4 \hat{U}}{\partial z^4} + \rho A \frac{\partial^2 \hat{U}}{\partial t^2} + \hat{c} \frac{\partial \hat{U}}{\partial t} = \left[ \frac{EA}{2L} \int_0^L \left( \left( \frac{\partial \hat{U}}{\partial z} \right)^2 + \left( \frac{\partial \hat{V}}{\partial z} \right)^2 \right) dz + \hat{N} \right] \frac{\partial^2 \hat{U}}{\partial z^2} + \hat{F}^u_e
\]

\[
EI_v \frac{\partial^4 \hat{V}}{\partial z^4} + \rho A \frac{\partial^2 \hat{V}}{\partial t^2} + \hat{c} \frac{\partial \hat{V}}{\partial t} = \left[ \frac{EA}{2L} \int_0^L \left( \left( \frac{\partial \hat{U}}{\partial z} \right)^2 + \left( \frac{\partial \hat{V}}{\partial z} \right)^2 \right) dz + \hat{N} \right] \frac{\partial^2 \hat{V}}{\partial z^2} + \hat{F}^v_e
\]

(S1)

In Eq. (S1), the variable \( \hat{c} \) is introduced to account for viscous damping, whereas the purpose of the end force \( \hat{N} \) is to account for the presence of residual stress due to doubly-clamped or fixed boundaries. The nanowire is actuated by unit length forces \( \hat{F}^u_e \) and \( \hat{F}^v_e \) along \( x \)-direction and \( y \)-direction respectively. These forces build-up when we actuate the nanowire by providing DC and AC voltages between nanowire and the gate electrode. For simulations, we have assumed unit length force of an electrostatically actuated circular cross-section nanowire having radius \( R = \sqrt{R_u R_v} \) as actuation force of the InAs nanowire. In case of a cylindrical nanowire, which has circular cross-section, the electrostatic actuation force is developed in the direction of gate electrode:\(^{55}\)

\[
\hat{F}_e(\hat{r}_g, \hat{t}) = \frac{\pi \varepsilon_0 \hat{V}_g^2}{\sqrt{(g_0 + R - \hat{r}_g)^2 - R^2 \left[ \cosh^{-1} \left( \frac{g_0 + R - \hat{r}_g}{R} \right) \right]^2}}
\]

(S2)

here \( \hat{r}_g \) is the displacement of the nanowire in direction of the gate electrode. The variable \( \hat{V}_g \) is the applied voltage between the nanowire and gate electrode and \( \varepsilon_0 \) is vacuum permittivity. In view of experiments, we have two AC voltage excitations corresponding to probe frequency \( \hat{\omega}_{f1} \) and pump frequency \( \hat{\omega}_{f2} \) along with the DC voltage, \( \hat{V}_g = \left( \hat{V}_{g,DC} + \hat{V}_{g,AC_u} \cos(\hat{\omega}_{f1} \hat{t}) + \hat{V}_{g,AC_v} \cos(\hat{\omega}_{f2} \hat{t}) \right) \). During simulation, we have approximated \( \hat{F}_e \) to compute unit length forces \( \hat{F}^u_e \) along \( x \)-direction and \( \hat{F}^v_e \) along \( y \)-direction.

We have assumed the major and minor axes of the elliptical cross-section of nanowire are inclined with respect to direction of the gate electrode \( x_g \) (refer Fig. [S1]). The forces \( \hat{F}^u_e = \lambda_u \hat{F}_e \) and \( \hat{F}^v_e = \lambda_v \hat{F}_e \) are components of actuation force \( \hat{F}_e \) of a cylindrical nanowire, where \( \lambda_u \) and \( \lambda_v \) are constants.

For simplicity, we have transformed Eq. [S1] in non-dimensional form by introducing following non-dimensional variables defined as \( U = \hat{U}/g_0, \ V = \hat{V}/g_0, \ z = \hat{z}/L, \) and \( t = \hat{t}/T \) (\( T = \sqrt{\rho AL^4/EI_u} \) is a time constant). The non-dimensional form of Eq. [S1] is

\[
U''' + \hat{c} \hat{U}' + \hat{U} = \left[ \alpha \int_0^1 (U'^2 + V'^2) \ dz + \hat{N} \right] U'' + \hat{F}^u_e
\]

\[
\gamma V''' + \hat{c} \hat{V}' + \hat{V} = \left[ \alpha \int_0^1 (U'^2 + V'^2) \ dz + \hat{N} \right] V'' + \hat{F}^v_e
\]

(S3)
In Eq. (S3), the notations \((\cdot)’\) and \((\cdot)\) denote partial derivative with respect to \(z\) and \(t\) respectively. In Eq. (S3), \(N = \hat{N}L^2/EI_u\) is the non-dimensional form of the end force \(\hat{N}\) and the variable \(\gamma\) is the ratio of principal flexural stiffnesses. The non-dimensional coefficient \(\alpha = A_0^2/2I_u\) quantifies nonlinear modal interaction behaviour in the nanowire flexural dynamics. The parameter \(c\) is the non-dimensional damping coefficient. We have used the value of \(c\) corresponding to experimental quality factor in numerical calculations. The non-dimensional unit length forces are denoted by \(F_u^e\) and \(F_v^e\).

We have solved Eq. (S3) using Galerkin based reduced order model technique. The solution of Eq. (S3) has been assumed in form of

\[ U(z,t) = \phi(z)u(t) \quad \text{and} \quad V(z,t) = \psi(z)v(t). \quad \text{(S4)} \]

Here, \(\phi(z)\) and \(\psi(z)\) are mode shapes along principal directions corresponding to fundamental nearby natural frequencies, whereas \(u(t)\) and \(v(t)\) are modal coordinate displacements. The modeshape \(\phi(z)\) (along x-direction) has been calculated by solving characteristic equation \(\phi'''' = N\phi + \omega_u^2\phi\) and modeshape \(\psi(z)\) (along y-direction) has been calculated by solving \(\gamma\psi'''' = N\psi + \omega_v^2\psi\). Here, \(\omega_u\) and \(\omega_v\) are the two nearby fundamental natural frequencies. The assumed solution (S4) has been substituted in Eq. (S3), multiplied the equations with modeshapes, and integrated the equations from 0 to 1; we get

\[
\ddot{u} + c\dot{u} + \omega_u^2 u + \alpha_u u^3 + \alpha_{uv} uv^2 = f_u^e
\]
\[
\ddot{v} + c\dot{v} + \omega_v^2 v + \alpha_u u^2 v + \alpha_v v^3 = f_v^e
\]

(S5)

Various coefficients of the nonlinear terms of Eq. (S5) are

\[
\alpha_u = \alpha \left( \int_0^1 (\phi')^2 dz \right)^2, \quad \alpha_v = \alpha \left( \int_0^1 (\psi')^2 dz \right)^2, \quad \alpha_{uv} = \alpha \left( \int_0^1 (\phi')^2 dz \int_0^1 (\psi')^2 dz \right)
\]

In Eq. (S5), the values of electrostatic actuation forces are \(f_u^e = \int_0^1 F_u^e \phi dz\) and \(f_v^e = \int_0^1 F_v^e \psi dz\). By Taylor series expansion of the function (S2) of electrostatic forcing, one can deduce that electrostatic forcing functions \(f_u^e\) and \(f_v^e\) have first harmonic excitation terms, second harmonic excitation terms, and parametric excitation terms. It is because Eq. (S5) is a function of displacement coordinates and due to presence of square of voltage \(\hat{V}_g\), where \(\hat{V}_g\) has both DC and AC voltage components. But, when investigation is near fundamental natural frequency and the gap distance between the nanowire and gate electrode is relatively larger, first harmonic excitation term has dominant effect. So, we retain only first harmonic excitation term in electrostatic actuation force as

\[
f_u^e = f_u \cos(\omega_1 t) \quad \text{and} \quad f_v^e = f_v \cos(\omega_2 t)
\]
where,

\[ f_u = 2\lambda_u f_{\text{zero}}C_{\text{volt}} \hat{V}_{gDC} \hat{V}_{gAC1} \int_0^1 \phi dz, \quad \omega_{f1} = \hat{\omega}_{f1} T, \]

\[ f_v = 2\lambda_v f_{\text{zero}}C_{\text{volt}} \hat{V}_{gDC} \hat{V}_{gAC2} \int_0^1 \psi dz, \quad \omega_{f2} = \hat{\omega}_{f2} T, \]

\[ f_{\text{zero}} = \frac{\pi}{\sqrt{\left(1 + \frac{R}{g_0}\right)^2 - \left(\frac{R}{g_0}\right)^2 \left[\cosh^{-1} \left(1 + \frac{R}{g_0}\right)\right]^2}}, \quad C_{\text{volt}} = \frac{\varepsilon_0 L^4}{g_0^2 E I_u}. \]

As the driving force increases, the amplitude of the modes increase giving rise to mode coupling. The amplitude of motion of the modes can be estimated by using the expression for the electrostatic driving force given in Eq. 2. By approximating the displacement of the nanowire to be small compared to the gate electrode separation, the total amplitude of the driving force acting along the length of the nanowire can be written as

\[ F = \frac{2\pi \varepsilon_0 \hat{V}_{gDC} \hat{V}_{gAC} L}{\sqrt{(g_0 + R)^2 - R^2 \left[\cosh^{-1} \left(\frac{g_0 + R}{R}\right)\right]^2}}. \]

For a linear response, the amplitude of motion on resonance is then given by \( z = \frac{F Q}{m \omega_m^2} \), where \( m \) is the mass of the nanowire and \( \omega_m \) is the frequency of the mode. Using the dimensions of suspended length \( L = 3.3 \mu m \), semi-minor axis \( r_u = 57.7 \) nm, semi-major axis \( r_v = 59.3 \) nm, initial gap \( g_0 = 500 \) nm, and quality factor \( Q = 2800 \), the amplitude can be estimated to be \( \sim 0.8 \) nm for mode 1 for a drive power of -30 dBm, and \( \sim 23 \) nm for mode 2 for a drive power of 0 dBm and DC gate voltage of 20 V.

S2. NUMERICAL SIMULATION

We have solved Eq. (S5) to understand mode coupling behaviour of the InAs nanowire resonator. The nanowire resonator was simulated using the dimensions given above and an end force \( \hat{N} \) as built-in tension of magnitude 1.47 times first Euler-buckling load. We have chosen the values of \( r_u \) and \( r_v \) near experimentally measured radius of the nanowire, along with \( \hat{N} \), such that the nanowire has nearly same experimentally measured split resonant frequencies. In simulations, we actuate the nanowire with DC voltage \( \hat{V}_{gDC} = 20 \) V and vary magnitude of AC voltages \( \lambda_u \hat{V}_{gAC1} = \hat{v}_{ac1} \) and \( \lambda_v \hat{V}_{gAC2} = \hat{v}_{ac2} \) to investigate resonance behaviour. Figures S2(a) and S2(b) show the resonance behaviour around split resonant frequencies; here the lower and higher frequency modes are referred as mode 1 and mode 2 respectively. To obtain resonance in mode 1 oscillation, as shown in Fig. S2(a), we provide small magnitude of first AC voltage \( \hat{v}_{ac1} = 0.04 \) V in absence of second
AC voltage $\hat{v}_{ac2} = 0$ V. Similarly, for resonance in mode 2 oscillation, as shown in Fig. S2(b), we provide second AC voltage $\hat{v}_{ac2} = 0.04$ V in absence of first AC voltage $\hat{v}_{ac1} = 0$ V. The line shape of these resonance curves demonstrate that they are linear in nature, and it is due to small magnitude of harmonic excitation. However, the nanowire resonator is inherently nonlinear in nature due to doubly-clamped boundary conditions, and nonlinearity is accounted in the equation of motion (S5) with Duffing nonlinearity terms. The nonlinearity effects become dominant in resonance behaviour at higher amplitude of harmonic AC voltage excitation. Figures S3(a) and S3(b) depict nonlinear resonance curves of mode 2 oscillation of the nanowire. Here, we actuate the nanowire with higher magnitude of second AC voltage $\hat{v}_{ac2} = 0.26$ V in absence of first AC voltage $\hat{v}_{ac1} = 0$ V.

The amplitude of AC voltage excitation is same for both Figs. S3(a) and S3(b), but direction of frequency sweep of AC voltage $\hat{v}_{ac2} \cos(\hat{\omega}_{f2} t)$ is different. In Fig. S3(a), the frequency sweep is in forward direction, whereas it is in reverse direction for Fig. S3(b). The resonance curves show hysteresis behaviour with change in direction of frequency sweep and it reflects the presence of nonlinearity.

Till now, we have discussed resonance in mode 1 and mode 2 oscillations separately by providing single input AC voltage corresponding to either mode 1 frequency or mode 2 frequency in absence of other. Now, we discuss the mode coupling behaviour in the InAs nanowire by simultaneously actuating both mode 1 and mode 2 by providing both first
FIG. S3. Nonlinear resonance curve for higher magnitude of AC voltage (a) forward sweep (b) reverse sweep and second AC voltages. Figures S4(a) and S4(b) show mode 1 resonance behaviour of the nanowire when we actuate both first and second AC voltages. Here mode 1 is probed with a small magnitude of first AC voltage \( \hat{v}_{ac1} = 0.04 \text{ V} \), and second AC voltage actuation is a pump signal. Figure S4(a) shows mode 1 resonance for small magnitude of pump signal \( \hat{v}_{ac2} = 0.04 \text{ V} \). We can observe that there is shift in mode 1 resonant frequency with increase in magnitude of pump signal to \( \hat{v}_{ac2} = 0.26 \text{ V} \), as shown in Fig. S4(b). In both figures, excitation frequency of pump signal is 61 MHz. We explain the frequency shift in a simplified manner in the following way. When a nanowire is placed in an undeflected position, the axial force present in the nanowire is only due to the residual stress. During oscillation, additional axial force is developed in the nanowire due to deflection under restrained boundaries, and it is measured by following nonlinear coupling factor in Eq. S3:

\[
\alpha \int_0^1 \left( U'^2 + V'^2 \right) dz. \tag{S6}
\]

The motion of the nanowire along one mode influences the motion along other mode due to this nonlinear coupling factor. In Eq. S3, first equation mainly governs mode 1 oscillation, whereas second equation governs mode 2 oscillation. When we pump mode 2 with high magnitude of AC voltage (refer second equation of Eq. S3), oscillatory motion in mode 2 vibration \( V(z,t) = \psi(z)v(t) \) is set-up. The modal displacement of
mode 2, \(v(t)\), has harmonic response at the pump frequency \(\omega_{f2}\) as \(v(t) = A_v \cos(\omega_{f2} t)\), where \(A_v\) is amplitude of oscillation. Due to mode 2 oscillation, the additional axial force developed in the nanowire is proportional to square of \(v(t)\) or \((A_v \cos(\omega_{f2} t))^2\) or \((A_v^2/2)(1 + \cos(2\omega_{f2} t))\). The additional axial force affects the mode 1 resonance due to presence of the nonlinear coupling factor \((S6)\). Hence, the constant component of additional axial force, proportional to \(A_v^2/2\), adds up with the residual axial load and eventually shifts the resonant frequency of mode 1 vibration (refer first equation of Eq. \((S5)\)).

**S3. ANALYTICAL SOLUTION**

We provide analytical expressions for computing the resonant frequency shift in probe mode oscillation. The magnitude of this resonant frequency shift can be computed by expression

\[
\Delta_{probe} = \left(\sqrt{\omega_u^2 + \alpha_{uv} A_u^2 / 2}\right) - \omega_u. \tag{S7}
\]

Further, the amplitude of mode 2 oscillation \(A_v\) can be obtained analytically by solving equation of motion using perturbation method under following approximations. In the present investigation, mode 1 is excited with a low magnitude AC voltage in comparison to mode 2 excitation. So, when we solve Eq. \((S5)\), the effect of mode 1 vibration is negligible.
in computation of amplitude of mode 2, $A_v$. We have solved the second equation of Eq. (S5), after ignoring coupling terms $\alpha_{uv}^2 v^2$, using the perturbation technique, method of multiple scales. The perturbation solution can be presented as a frequency response equation to compute the amplitude $A_v$ as a function of frequency difference in pump signal $\Delta_{\text{pump}} = \omega_{f2} - \omega_v$, and it is given by

$$
\left(\frac{c}{2}\right)^2 + \left(\Delta_{\text{pump}} - \frac{3\alpha_v A_v^2}{8\omega_v}\right)^2 = \frac{f_v^2}{4\omega_v^2 A_v^2}.
$$

(S8)

Hence, by solving Eqs. (S7) and (S8), we can obtain probe signal frequency shift $\Delta_{\text{probe}}$ as a function of pump signal frequency difference $\Delta_{\text{pump}}$.

We present the effectiveness of Eqs. (S7) and (S8) in computation of frequency shift $\Delta_{\text{probe}}$ using Figs. S5(a) and (b). Here, Fig. S5(a) shows nonlinear resonance behaviour of mode 2 oscillation when magnitude of pump signal or second AC voltage $\hat{v}_{ac2} = 0.26$ V; it is obtained by solving Eq. (S8). By comparing Fig. S5(a) with Figs. S3(a) and (b), we can say that perturbation solution (S8) is in good agreement with numerical solution of Eq. (S5). We have further computed frequency shift $\Delta_{\text{probe}}$ using solution of Eq. (S8) and Eq. (S7), and the solution is presented here in Fig. S5(b) by showing relationship between $\Delta_{\text{probe}}$ and $\Delta_{\text{pump}}$. In this figure, we have also compared the analytical solution of Eq. (S7) with the numerical solution of Eq. (S5). There is good agreement between both solutions, and it demonstrates that the analytical expressions can serve as useful tool for computing frequency shift $\Delta_{\text{probe}}$.

In this work, modal interaction between first planar and first nonplanar modes of vibration has been investigated. However, researchers have earlier investigated similar modal interaction between first planar and higher planar modes. The derived expressions Eqs. (S7) and (S8) are also useful for obtaining the relationship between probe mode frequency shift $\Delta_{\text{probe}}$ (first planar mode) and pump mode frequency difference $\Delta_{\text{pump}}$ (higher planar mode). The motion of such a system is only governed by the first equation of Eq. (S3) because modal interaction between only planar modes are under consideration. From Eq. (S3), we get a system of two second order ordinary differential equations like Eq. (S5) by following Galerkin procedure as applied for derivation of Eq. (S5). Further, Eqs. (S7) and (S8) are applicable to obtain relationship between $\Delta_{\text{probe}}$ and $\Delta_{\text{pump}}$.

To compare the modal interaction between different modes using Eqs. (S7) and (S8) we define a tuning constant $K_{\text{tuning}} = \alpha_{uv} f_v^2 / c^2 \omega_v^2$ which approximately quantifies relative coupling strength in modal interaction. The motivation in defining this constant is as
follow. We can expand expression Eq. (S7) using Taylor series expansion and retain only linear term, and get that $\Delta_{\text{probe}}$ is directly proportional to $\alpha_{uv}$ and $A_{v}^2$. In linear resonance case, the amplitude at resonance $A_v$ is directly proportional to $f_v$ and inversely proportional to $c$ and $\omega_v$. Next, we have computed tuning constant $K_{\text{tuning}}$ for different cases of modal interaction in Table I. Here, we consider three cases of modal interaction for first planar mode with first nonplanar mode (Type 1), first planar mode with second planar mode (Type 2), and first planar mode with third planar mode (Type 3). It is interesting that tuning constant is highest for the case of Type 1 modal interaction. The tuning constant for Type 2 modal interaction is zero due to antisymmetric nature of second mode which makes $f_v = 0$. We can also deduce from Table I that tuning constant $K_{\text{tuning}}$ for Type 1 modal interaction is about 2.65 times greater than Type 3 modal interaction.

Relationship between $\Delta_{\text{probe}}$ and $\Delta_{\text{pump}}$ for case of Type 1 modal interaction is already shown in Fig. S5(b). We have further solved Eqs. (S7) and (S8) for investigating interaction of Type 3 modal interaction. We present the solution here in Figs. S6(a) and (b). Here, Fig. S6(a) shows nonlinear resonance behaviour of third planar mode oscillation when magnitude of pump signal or second AC voltage $\hat{v}_{ac2} = 0.26$ V as in case for Fig. S5(a). Further, the relationship between $\Delta_{\text{probe}}$ and $\Delta_{\text{pump}}$ for Type 3 modal interaction is
| Modal Interaction Type | Tuning Constant $(K_{tuning})$ |
|------------------------|-------------------------------|
| First planar and first nonplanar (Type 1) | 22.13 |
| First planar and second planar (Type 2) | 0 |
| First planar and third planar (Type 3) | 8.33 |

TABLE I. Magnitude of tuning constant $K_{tuning}$ for different types of modal interaction

FIG. S6. (a) Nonlinear resonance curve of planar third mode vibration when magnitude of second AC voltage $\hat{v}_{ac2} = 0.26$ V (b) Relationship between frequency tuning in probe signal $\Delta_{probe}$ (planar first mode) with frequency difference of pump signal $\Delta_{pump}$ (planar third mode) due to nonlinear coupling.

shown in Fig. S6(b). Under similar loading condition, we can compare coupling strength of Type 1 and Type 3 modal interaction by observing Figs. S5(b) and S6(b). From these figures, we can deduce that tuning capacity (maximum value of $\Delta_{probe}$) in Type 1 modal interaction is about 2.65 times than Type 3 modal interaction. It is consistent with comparison of tuning constant $K_{tuning}$ of Type 1 and Type 3 modal interactions.

We can ascertain more information on Type 1 modal interaction by analyzing the expression for the tuning constant, $K_{tuning} = \alpha_u f_v^2 / c^2 \omega_v^2$. It provides insight on the frequency shift behaviour of the probe mode $\Delta_{probe}$ for varying magnitude of AC voltage of the pump mode $\hat{v}_{ac2}$. The maximum value of frequency shift is approximately proportional to the square of the magnitude of $\hat{v}_{ac2}$. This is because $K_{tuning}$ is proportional to the
square of $f_v$, and $f_v$ is proportional to $\hat{v}_{ac}$. Figure S7 shows the calculated dependence of maximum frequency shift of probe mode on the magnitude of AC voltage of the pump mode.

S4. GATING RESPONSE

The semiconducting nanowires used in this work are $n$-type in nature. As the conductance of the device depends on the carrier density in the nanowire the gate voltage can deplete or saturate the conductance (see Figure S8(a)). Therefore, as the nanowire undergoes mechanical oscillations the conductance of the device itself is modified due to the modulated gate capacitance. The conductance modulation due to oscillations is large when the transconductance of the device is large. Therefore, the strength of measured resonance signal depends on the transconductance.

Figure S8 shows the resonant response of mode 2 of the nanowire resonator at $\pm 20$ V gate voltage showing the effect of a non-zero transconductance. As the magnitude of the gate voltage is same, the force acting on the resonator in both cases are equal. However, in regions (blue circle) where the transconductance of the device is high our measurement scheme is more effective in detecting the mechanical motion. Typical resonant response measured in experiments is seen in Figure S8(b) as a dip.
FIG. S8.  (a) The n-type field effect transistor response of the semiconducting nanowire is seen from the variation of the DC current through the nanowire as a function of the DC gate voltage. The slope of the curve is directly proportional to the transconductance of the device. The blue circle marks a region of high transconductance whereas the red circle marks a region of low transconductance. (b) The response of mode 2 measured at the gate voltages marked by the circles in (a) shows the effect of transconductance on the measured rf signal.
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