Characteristic polynomials and zeta functions of equitably partitioned graphs

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Abstract Let $\pi = \{V_1, \ldots, V_r\}$ be an equitable partition of the vertex set of a directed graph (digraph) $X$. It is well known that the characteristic polynomial $\phi(X/\pi, x)$ of a quotient graph $X/\pi$ divides that of $X$, but the remainder part is not well investigated. In this paper, we define a deletion graph $X\setminus\pi$ over an equitable partition $\pi$, which is a signed directed graph defined for a fixed set of deleting vertices $\{\tau_i \in V_i, i = 1, \ldots, r\}$, and give a similarity transformation exchanging the adjacency matrix $A(X)$ which is compatible with the equitable partition for a block triangular matrix whose diagonal blocks are the adjacency matrix of the quotient graph and the deletion graph. In fact, we show the result for more general matrices including adjacency matrix of graphs, and as corollaries, we show the followings: (i) a decomposition formula of the reciprocal of the Ihara-Bartholdi zeta function over an equitably partitioned undirected graph into the quotient graph part and the deletion graph part, and (ii) Chen and Chen’s result ([CC17, Theorem 3.1]) on the Ihara-Bartholdi zeta functions on generalized join graphs, and (iii) Teranishi’s result [Ter03, Theorem 3.3].

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1. Introduction

A partition $\pi = \{V_1, \ldots, V_r\}$ of the vertex set $V(X)$ of a directed graph (digraph) $X$ (we allow multiple loops and edges) is called equitable if for each $i, j = 1, \ldots, r$, there is a integer $b_{ij}$ such that each vertex in the cell $V_i$ has exactly $b_{ij}$ (oriented) edges to vertices in the cell $V_j$. The adjacency matrix of the (weighted) quotient graph over $\pi$ is defined by the matrix $(b_{ij})$ and is denoted by $A(X/\pi)$. It is well known that the characteristic polynomial of the (weighted) quotient graph over $\pi$
(front divisor of $X$) divides the characteristic polynomial of the adjacency matrix $A(X)$ of the graph ([GR01, 9.3]), but the remainder part is not well investigated.

We define the deletion graph $X \setminus \pi$ over $\pi$ which is a signed directed graph defined for a fixed set of deleting vertices $\{v_i \in V, i = 1, \ldots, r\}$, and give a similarity transformation exchanging the adjacency matrix $A(X)$ which is compatible with the equitable partition for a block triangular matrix whose diagonal blocks are the adjacency matrices of the quotient graph and the deletion graph. Hence, we have that the remainder part is the characteristic polynomial of the adjacency matrix $A(X \setminus \pi)$ of the deletion graph. This answers the question posed by Deng and Wu [DW05, Section 5] of whether we can associate any geometrical meaning on the remainder part.

Moreover, we get the decomposition formula for the characteristic polynomials for the Laplacian matrix, and the reciprocal of the Ihara-Bartholdi zeta function. Since a generalized join (composition) (Schwenk [S74]) of regular graphs is a special case of an equitably partitioned graph, we get the decomposition formula of the reciprocal of the Ihara-Bartholdi zeta function of generalized joined graph by Chen and Chen [CC17] as a corollary of our Theorem.

Having an equitably partitioned graph is equivalent to having a covering projection ([DSW07, Lemma 3.1]), and when the graph is a covering of a voltage assignment, this is equivalent to having a free action (i.e. regular covering) ([GT77, Theorems 3,4], see also [DW05]). There is much in the literature on the decomposition of (a) the characteristic polynomial of the adjacency matrix of the graph and (b) the reciprocal of the Ihara-Bartholdi zeta functions. We list them. On the topic (a), there are results for graph covering with voltages in a finite group by Mizuno and Sato [MS95, Theorem 1], [MS97, Theorem 1] (see also [KL92], [Sat99 Theorem 24], [KL01], [FKL04]); for branched cover with branch index 1 by Deng and Wu [DW05, Theorem 4.2] assuming a semi-free action on digraph; for branched cover with branch index 1 by Deng, Sato and Wu [DSW07, Theorem 6.4]. On the topic (b), there are results for the reciprocal of the (weighted) Ihara-zeta (Bartholdi-zeta) function of a regular ($g$-cyclic $\Gamma$-, or irregular) cover by Mizuno and Sato [MS01, Theorem 5], [MS02, Theorem 7], [MS04, Theorem 4], and by Sato [Sat06, Theorem 3], [Sat07, Theorem 4].

When there is a symmetry (automorphism), so when the equitable partition is the orbit partition, [BFW16], [FSSW17],[FSW18] give a decomposition of any automorphism compatible matrix, which include the adjacency matrix, the Laplacian matrix, etc.

The remainder of the paper is organized as follows. In section 2, we give basic facts on equitably partitioned directed graphs. In section 3, we define the deletion graph. In section 4, we give our main theorem (Theorem 4.4) of a similarity transformation exchanging the adjacency matrix $A(X)$ for a block triangular matrix whose diagonal blocks are the adjacency matrix of the quotient graph and the deletion graph, giving the decomposition formula. In Section 5 and 6 we give applications of the decomposition formula to the reciprocal of the Ihara-Bartholdi zeta functions of equitably partitioned graphs, especially on generalized join graphs.
2. Equitable Partitions of Directed Graphs

For totally ordered sets \( U, V \) and a set \( W \), we denote by \( \text{Mat}(U \times V; W) \) the set of matrices indexed by \( U \times V \) whose components are in \( W \), that is, the set \( \{ M : U \times V \rightarrow W \} \) of mappings from \( U \times V \) to \( W \). For a square matrix \( M \), \( \phi(M, x) = \det(xI - M) \) is the characteristic polynomial of \( M \). Let \( X = (V(X), E(X)) \) be a finite directed (multi)graph with a set \( V(X) \) of vertices and a set \( E(X) \) of directed edges. If \( e \in E(X) \) implies \( \bar{e} \in E(X) \) (here \( \bar{e} \) is an inverse edge), then \( X \) can be considered as an undirected graph. We allow \( X \) to have multiple edges and multiple loops. For \( u, v \in V(X) \), we denote by \( u \rightarrow v \) if there is an edge that goes from \( u \) to \( v \), and \( u \sim v \) if \( u \rightarrow v \) and \( v \rightarrow u \). The adjacency matrix of \( X \) is denoted by \( A(X) \), that is, \( A(X)_{uv} \) is the number of edges from \( u \) to \( v \).

**Definition 2.1.** Let \( \pi = \{V_1, \ldots, V_r\} \) be a partition of the vertex set \( V(X) \) of a directed graph \( X \). For \( i, j = 1, \ldots, r \), for each vertex \( u \) in the cell \( V_i \), if the number \( b_{ij} \) of edges that goes from \( u \) to the vertices in \( V_j \) does not depend on the choice of \( u \), we say that \( \pi \) is an equitable partition. In this case, the multi-directed graph \( X/\pi \), called the (weighted) quotient (or front divisor) of \( X \) over \( \pi \), is such that the set of vertices is \( \pi \), and there are \( b_{ij} \) edges from \( V_i \) to \( V_j \) ([GR01, 9.3]). The adjacency matrix of \( X/\pi \) is given by \( A(X/\pi) = (b_{ij}) \in \text{Mat}(\pi \times \pi; \mathbb{Z}+) \).

**Example 2.2.**

(a) Let \( X \) be the following \( C_4 \): \( \pi = \{V_1, V_2\}, V_i = \{v_i^1, v_i^2\}, i = 1, 2, v_1^1 = \circ, v_2^2 = \blacksquare \).

(b) Next, consider the following directed graph. \( \pi = \{V_1, V_2\}, V_1 = \{v_1^1\} \, j = 1, 2, 3\}, V_2 = \{v_2^k\} \, k = 1, 2\}, v_1^1 = \circ, v_2^2 = \blacksquare \).
\( A(X) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \end{pmatrix} \), \( A(X/\pi) = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \),

\( \phi(A(X), x) = (x^2 - 4x + 1)x(x-1)(x-2), \phi(A(X/\pi), x) = (x^2 - 4x + 1) \).

(c) We recall examples from [GR01; Section 9.3]. Let \( X = J(5, 2, 0) \) be the Petersen graph. (i) Let \( \pi_1 = \{V_1, V_2\}, V_i = \{v_k^i | k = 1, \ldots, 5\}, i = 1, 2, v_1^1 = \Box, v_2^2 = \Box \).

\[ A(X) = \begin{pmatrix} A_1 & I_5 \\ I_5 & A_2 \end{pmatrix}, \]

\[ A_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad A(X/\pi_1) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \]

\( \phi(A(X), x) = (x-3)(x-1)^5(x+2)^4, \phi(A(X/\pi_1), x) = (x-3)(x-1) \).

(ii) Next consider the following distance partition \( \pi_2 = \{V_1, V_2, V_3\}, V_i = \{v_k^i\}, V_1 = \{v_1^1\}, V_2 = \{v_k^2 | k = 1, 2, 3\}, V_3 = \{v_l^3 | l = 1, \ldots, 6\}, v_1^1 = \Box, v_2^2 = \Box, v_3^3 = \Box \), \( k = 1, 2, 3, v_l^3 = \Box \), \( l = 1, \ldots, 6 \). \( V_i \) is the set of vertices such that their distances from \( v_1^1 \) are the same \( i-1 \).
Then \((M, \pi)\) is equitable means that \(\pi\) is an equitable partition, and 
\[
A(X)/\pi = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

For other examples of equitable partitions, see [GR01].

(d) Let \(X = H[X_1, \ldots, X_r]\) be the generalized join (composition) graph of \(X_1, \ldots, X_r\)
determined by \(H\) ([Sch74, 4.]), and assume \(X_i\) are \(k_i\)-regular ([Sch74, section 4]; [CC17]). That is, 
\(#V(H) = r, V(X) = \cup_{i=1}^{r} V(X_i)\) and for \(v^i_k \in V(X_i), v^i \in V(X_j),\)

\[
A(X)_{v^i_k v^j_l} = \begin{cases}
A(X)_{v^i_k v^i_l} & \text{if } i = j, \\
A(H)_{i j} & \text{if } i \neq j.
\end{cases}
\]

Then \(\pi = \{V(X_1), \ldots, V(X_r)\}\) is an equitable partition of \(X\) by letting

\[
A(X/\pi)_{i j} = \begin{cases}
k_i & \text{if } i = j, \\
A(H)_{i j} n_j & \text{if } i \neq j,
\end{cases}
\]

here \(n_j = \#V_j\).

**Definition 2.3.** Let \(\pi = \{V_1, \ldots, V_r\}\) be a partition of \(V(X)\) of a digraph \(X\) and let \(M \in \text{Mat}(V(X) \times V(X); \mathbb{C})\). We say that the pair \((M, \pi)\) is equitable if there exists \(B = (b_{i j}) \in \text{Mat}(\pi \times \pi; \mathbb{C})\) such that

\[
M|_{V_i \times V_j} \cdot 1 = b_{i j} \cdot 1, \ i, j = 1, \cdots, r,
\]

here \(1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{Z}^{\#V_j}\).

We denote by \(M/\pi := B\), and call it the quotient matrix of \(M\) over the partition \(\pi\). Note that \((A(X), \pi)\) is equitable means that \(\pi\) is an equitable partition, and 
\(A(X)/\pi = A(X/\pi)\).

**Remark 2.4.** Let \(\pi = \{V_1, \ldots, V_r\}\) be an equitable partition of \(V(X)\), and let \(M = \alpha A(X) + D\). Here, \(\alpha \in \mathbb{C}\) and \(D\) is a diagonal matrix such that

\[
D|_{V_i \times V_i} = d_i I_i, \quad I_i \in \text{Mat}(V_i \times V_i; \{0, 1\})\]

is the unit matrix.

Then \((M, \pi)\) is equitable with the quotient matrix

\[
M/\pi = \alpha A(X/\pi) + (\delta_{i j} d_i),
\]
here $\delta_{ij} = 1$ if $i = j$ and $= 0$ if $i \neq j$. Assume that $X$ is an undirected graph and let $D(X)$ be the degree matrix of $X$, that is, the diagonal matrix such that $D(X)_{uv}$ is the number of edges that go from $u$. Let $L(X) = -A(X) + D(X)$ (resp. $Q(X) = A(X) + D(X)$) be the Laplacian matrix (resp. the signless Laplacian matrix) of $X$, and let $Z_X(u, t)$ be the Ihara-Bartholdi zeta function (see section 5). Then by letting (a) $D = D(X)$, (i) $\alpha = -1$, (ii) $\alpha = 1$, (b) $D = D^2(X), \alpha = -t$, we have

\begin{align*}
(a) & \quad (i) \quad L(X)/\pi = L(X/\pi) = -A(X/\pi) + (\delta_{ij}d_i(X)), \\
& \quad (ii) \quad Q(X)/\pi = Q(X/\pi) = A(X/\pi) + (\delta_{ij}d_i(X)), \\
& \quad (b) \quad Z_{X/\pi}(u, t)^{-1} = s_1^{m-n} \det (-tA(X/\pi) + (\delta_{ij}d_i^Z(X))) ,
\end{align*}

here $d_i(X)$ is the degree of vertices in $V_i$, and $d_i^2(X) = s_1 + s_2d_i(X), s_1 = 1 - (1 - u)^2t^2, s_2 = (1 - u)t^2$.

3. The Deletion Graph over a Partition of a Graph

Definition 3.1. We define a signed directed graph $X = (V(X), SD(X))$ as follows: $SD(X)$ is a set of a signed directed edges with its adjacency matrix $A(X) \in \text{Mat}(V(X) \times V(X); \mathbb{Z})$ whose components are integers. That is, for $u, v \in V(X)$, if $A(X)_{uv} \geq 0$, there are $A(X)_{uv}$ positive edges from $u$ to $v$, if $A(X)_{uv} \leq 0$, there are $|A(X)_{uv}|$ negative edges from $u$ to $v$. $SD(X) = (D^+(X), D^-(X))$ is such that

\begin{align*}
D^+(X) &= \{(i, u, v) \mid A(X)_{uv} > 0, i = 1, \ldots, |A(X)_{uv}|\}, \\
D^-(X) &= \{(i, u, v) \mid A(X)_{uv} < 0, i = 1, \ldots, |A(X)_{uv}|\}.
\end{align*}

For signed directed graphs $X_i = (V(X_i), SD(X_i)), i = 1, 2$, we define the signed directed graphs $X_1 + X_2$ and $X_1 - X_2$ by the following:

\begin{align*}
V(X_1 + X_2) &= V(X_1 - X_2) = V(X_1) \cup V(X_2), \quad \text{and} \\
A(X_1 + X_2) &= \tilde{A}(X_1) + \tilde{A}(X_2) \quad \text{and} \quad A(X_1 - X_2) = \tilde{A}(X_1) - \tilde{A}(X_2),
\end{align*}

here $\tilde{A}(X_i)$ is the extension of $A(X_i)$ to $V(X_1) \cup V(X_2)$ by:

\[
\tilde{A}(X_i)_{uv} = \begin{cases}
A(X_i)_{uv} & \text{if } u, v \in V(X_i), \\
0 & \text{otherwise.}
\end{cases}
\]

Definition 3.2. Let $X = (V(X), E(X))$ be a directed graph, and $B, C \subset V(X), \tau \in V(X)$. We define the signed directed graph $B(\tau) \triangleright C$ by: $V(B(\tau) \triangleright C) = B \cup C$, and

\[
A(B(\tau) \triangleright C)_{uv} = \begin{cases}
A(X)_{uv} & \text{if } u \in B \text{ and } v \in C, \\
0 & \text{otherwise.}
\end{cases}
\]

Let $\pi = \{V_1, \ldots, V_r\}$ be a partition of vertices $V(X)$ of a directed graph $X$, and fix a set of vertices $\{\bar{V}_i \in V_i, i = 1, \ldots, r\}$. Let

\[
V' := V(X) \setminus \{\bar{V}_i\}_{i=1}^r, \quad V'_i := V_i \setminus \{\bar{V}_i\}.
\]
We define a signed directed graph \( X \setminus \pi \) and call it the deletion graph over the partition \( \pi \), as follows:

\[
X \setminus \pi := X|_{V'} - \sum_{i,j=1}^{r} (V'_i(\pi_i) \triangleright V'_j),
\]

here, \( X|_{V'} \) is the restriction of \( X \) to \( V' \).

Let \( P \in \text{Mat}(V(X) \times \pi; \{0,1\}) \) be the characteristic matrix, that is,

\[
P_{v_k^i} = \delta_{ij} \quad \text{for } v_k^i \in V_i.
\]

For \( M \in \text{Mat}(V(X) \times V(X); \mathbb{C}) \), define the deletion matrix over the partition \( \pi \), \( M \setminus \pi \in \text{Mat}(V' \times V'; \mathbb{C}) \) by the following:

\[
M \setminus \pi := M|_{V' \times V'} - P|_{V' \times \pi} \cdot M|_{\{\pi_i\}_{i=1}^{r} \times V'};
\]

here \( (P|_{V' \times \pi} \cdot M|_{\{\pi_i\}_{i=1}^{r} \times V'})_{v_k^i v_l^j} = \sum_{h=1}^{r} P_{v_k^h} M_{\pi_h v_l^j} = M_{\pi v_l^j} \).

The following holds.

**Proposition 3.3.**

\[
A(X) \setminus \pi = A(X \setminus \pi).
\]

(Proof) Let \( C'(X) = P|_{V' \times \pi} \cdot A(X)|_{\{\pi_i\}_{i=1}^{r} \times V'} \). Then for \( v_k^i \in V'_i := V_i \setminus \{\pi_i\} \) and \( v_l^j \in V'_j \), we have \( C'(X)_{v_k^i v_l^j} = A(X)_{\pi v_l^j} \); and

\[
A \left( \sum_{i,j=1}^{r} (V'_i(\pi_i) \triangleright V'_j) \right)_{v_k^i v_l^j} = A \left( (V'_i(\pi_i) \triangleright V'_j)_{v_k^i v_l^j} = A(X)_{\pi v_l^j}.
\]

Since \( A(X|_{V'}) = A(X)|_{V' \times V'} \), we have the assertion. \( \square \)

**Example 3.4.** Consider Example 2.2 (c) (i). Let \( \pi_1 = v_1^1 = \circ \), \( \pi_2 = v_5^2 = \Box \). Then

\[
P|_{V' \times \pi} \cdot A(X)|_{\{\pi_i\}_{i=1}^{r} \times V'} = \begin{pmatrix} 1_4 & 0 \\ 0 & 1_4 \end{pmatrix} \begin{pmatrix} 1_4 & 0 \\ 0 & 1_4 \end{pmatrix} = \begin{pmatrix} 0 & 1_4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1_4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} J_{23} & 0 \\ 0 & J_{14} \end{pmatrix}.
\]

(3.4)
here $J_{k_1 \ldots k_s}$ is the matrix such that $k_1, \ldots, k_s$-column vectors are $1$, and others are $0$. On the other hand,

$$A \left( \sum_{i,j=1}^{2} (V_i' \triangleright V_j) \right) = \begin{pmatrix} (V_1' \triangleright V_1) & (V_1' \triangleright V_2) \\ (V_1' \triangleright V_3) & (V_1' \triangleright V_6) \end{pmatrix} = \begin{pmatrix} J_{23} & 0 \\ 0 & J_{14} \end{pmatrix}.$$ 

Next, consider Example 2.2 (c) (ii). Let $\overline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\overline{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}$, $\overline{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 0 \end{pmatrix}$. Then

$$P_{|V' \times \pi} \cdot A(X)_{|\{\overline{v}_i\}_{i=1}^{r} \times V'} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 13 \\ 0 \\ 0 \\ 15 \end{pmatrix} = \begin{pmatrix} 0 \\ 15 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & J_5 \\ 0 & J_{24} \end{pmatrix}.$$ 

On the other hand,

$$A \left( \sum_{i,j=1}^{3} (V_i' \triangleright V_j) \right) = \begin{pmatrix} (V_1' \triangleright V_1) & (V_1' \triangleright V_2) & (V_1' \triangleright V_3) \\ (V_1' \triangleright V_4) & (V_1' \triangleright V_5) & (V_1' \triangleright V_6) \end{pmatrix} = \begin{pmatrix} 0 & J_5 \\ 0 & J_{24} \end{pmatrix}. $$

Hence, $A(X) \setminus \pi = A(X \setminus \pi)$. 

\[ \square \]

4. The Similarity Transformation

Lemma 4.1. Let $(M, \pi)$ be an equitable pair, here $M \in \text{Mat}(V(X) \times V(X); \mathbb{C})$ and $\pi = \{V_1, \ldots, V_r\}$ is a partition of vertex set $V(X)$ of a graph $X$. Then

$$MP = P \cdot M/\pi.$$ 

(Proof) $$(MP)_{v_i,j} = \sum_{v'_i \in V} M_{v_i,v'_i} P_{v'_i,j} = \sum_{i} M_{v_i,v'_i} = (M/\pi)_{ij},$$ and

$$(P \cdot M/\pi)_{v_i,j} = \sum_{h=1} P_{v_i,h} (M/\pi)_{h,j} = (M/\pi)_{ij}. \quad \square$$

Definition 4.2. Let $\pi = \{V_1, \ldots, V_r\}$ be a partition of vertices $V(X)$ of a graph $X$. Fix $\{\overline{v}_i \in V_i, i = 1, \ldots, r\}$. Define

$$Q = Q(\{\overline{v}_i\}_{i=1}^{r}) \in \text{Mat}(V(X) \times V'; \{0,1\}),$$
here \( V' = V(X) \backslash \{ \overline{v}_i \}_{i=1}^r \), by the following:

\[
Q_{v_i v_j} = \delta_{v_i v_j}.
\]

For instance, in Example 2.2 (a), letting \( \overline{v}_i = v^2_i, i = 1, 2 \),

\[
P = \begin{pmatrix}
1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix}, \quad \text{and } Q = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

Lemma 4.3. The column vectors of \((P, Q) \in \text{Mat}(V(X) \times (\pi \times V'); \{0, 1\})\) form a basis of \(C^{V(X)}\), that is, \((P, Q)\) is invertible.

(Proof) Assume

\[
\sum_{i=1}^r \lambda_i P_i + \sum_{v^j_i \in V_j} \mu_{v^j_i} Q_{v^j_i} = 0,
\]

where \(P_i\) (resp. \(Q_{v^j_i}\)) is the \(i\)-th (resp. \(v^j_i\)-th) column vector of \(P\) (resp. \(Q\)). For \(s \in \{1, \ldots, r\}, P_{\overline{v}_s} = \delta_{s},\) and since \(v^j_i \in V_j \subset V' = V(X) \backslash \{ \overline{v}_i \}_{i=1}^r\), we have \(v^j_i \neq \overline{v}_s\), implying \(Q_{v^j_i} = 0\). So, considering the entry corresponding to \(v^j_s\) we have \(\lambda_s = 0, s = 1, \ldots, r\). And we have \(\mu_{v^j_i} = 0, \forall v^j_i \in V', j = 1, \ldots, r\). □

The following is our main theorem.

Theorem 4.4. Let \((M, \pi)\) be an equitable pair, here \(M \in \text{Mat}(V(X) \times V(X); \mathbb{C})\) and \(\pi = \{V_1, \ldots, V_r\}\) is a partition of vertex set \(V(X)\) of a graph \(X\). Then letting \(\overline{P} = (P, Q)\) we have

\[
\overline{P}^{-1} M \overline{P} = \begin{pmatrix}
M/\pi & C \\
0 & M\backslash\pi
\end{pmatrix}, C = M_{\{\overline{v}_i\}_{i=1}^r \times V'},
\]

hence,

\[
\phi(M, x) = \phi(M/\pi, x) \cdot \phi(M\backslash\pi, x),
\]

here \(M\backslash\pi\) is defined for a fixed \(\{\overline{v}_i\}_{i=1}^r, \overline{v}_i \in V_i,\) and \(\phi(M\backslash\pi, x)\) does not depend on \(\{\overline{v}_i\}_{i=1}^r\). In particular, if \(\pi\) is equitable, then

\[
\phi (A(X), x) = \phi (A(X/\pi), x) \cdot \phi (A (X \backslash \pi), x).
\]

(Proof) We show that

\[
MQ = PC + Q \cdot M\backslash\pi.
\]

For \(v^j_k \in V, v^j_i \in V_{\overline{j}'} = V_{\overline{j}} \backslash \{ \overline{v}_j \},\)

(4.5) \[
(MQ)_{v^j_k v^j_i} = \sum_{w \in V} M_{v^j_k w} Q_{w v^j_i} = M_{v^j_k v^j_i}, \quad \text{and}
\]

\[
\phi (A(X), x) = \phi (A(X/\pi), x) \cdot \phi (A (X \backslash \pi), x).
\]
\[(PC + Q \cdot M\backslash \pi)_{v_k v_l} = \sum_{h=1}^{r} P_{v_k h} M_{\pi_h v_l} + \sum_{w \in V'} Q_{v_k w} (M\backslash \pi)_{w v_l}\]

\[
= \begin{cases} 
M_{v_k v_l} + (M\backslash \pi)_{v_k v_l} & \text{if } v_k \neq \pi_i \\
M_{\pi_i v_l} & \text{if } v_k = \pi_i.
\end{cases}
\]

If \(v_k \neq \pi_i\),
\[
(M\backslash \pi)_{v_k v_l} = (M|_{V' \times V'})_{v_k v_l} - \sum_{h=1}^{r} (P|_{V' \times \pi})_{v_k h} M_{\pi_h v_l}
\]

\[
= M_{v_k v_l} - M_{\pi_i v_l}.
\]

So that we have \((PC + Q \cdot M\backslash \pi)_{v_k v_l} = M_{v_k v_l} = (MQ)_{v_k v_l}\) (by (4.5)).

\[\square\]

**Example 4.5.** Let \(X, \pi\) be as in Example 2.2.

(a) Delete \(\{v_1, v_2\}\).

\[X \backslash \pi = X|_{\varnothing \varnothing} - \left(\left(\{\varnothing\} \bigcirc \varnothing\right) \triangleright \varnothing + \left(\{\varnothing\} \bigcirc \varnothing\right) \triangleright \varnothing\right)\]

\[+ \left(\{\varnothing\} \bigcirc \varnothing\right) \triangleright \varnothing + \left(\{\varnothing\} \bigcirc \varnothing\right) \triangleright \varnothing\)\]

\[= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \text{, here we denoted the negative edges by the red lines.}
\]

Hence, \(A(X \backslash \pi) = \begin{pmatrix}
-1 & 1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & 1
\end{pmatrix}\). By our Theorem, we have \(\phi(A(X), x)\)

\[= \begin{vmatrix}
x - 1 & -1 & -1 \\
-1 & x - 1 & -1 \\
-1 & -1 & x + 1
\end{vmatrix} = ((x - 1)^2 - 1) \cdot ((x + 1)^2 - 1) = x^2(x + 2)(x - 2).\]

(b) Delete \(\{\pi_1, \pi_2\}\).

\[X \backslash \pi = X|_{\varnothing \varnothing \varnothing} - \left(\left(\{\varnothing, \varnothing\} \bigcirc \varnothing\right) \triangleright \varnothing + \left(\{\varnothing, \varnothing\} \bigcirc \varnothing\right) \triangleright \varnothing\right)\]

\[+ \left(\{\varnothing\} \bigcirc \varnothing\right) \triangleright \varnothing + \left(\{\varnothing\} \bigcirc \varnothing\right) \triangleright \varnothing\)\]

\[= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \text{, hence, } A(X \backslash \pi) = \begin{pmatrix}
-1 & 1 & 1 \\
-1 & 1 & 0 \\
2 & -2 & -1
\end{pmatrix}\text{, implying}
\]
\[ \phi(A(X), x) = \begin{vmatrix} x-3 & -1 \\ -2 & x-1 \end{vmatrix} \cdot \begin{vmatrix} x+1 & -1 & -1 \\ 1 & x-1 & 0 \\ -2 & 2 & x+1 \end{vmatrix} = (x^2 - 4x + 1) \cdot x(x-1)(x+2). \]

(c) (i) By (3.4) we have
\[
A(X \backslash \pi_1) = A(X|V') - P|V' \times \pi_1 \cdot A(X)|_{\{\pi_i\}_{i=1}^2 \times V'}
\]
\[
= \begin{pmatrix} A'_1 & I_4 \\ I_4 & A'_2 \end{pmatrix} - \begin{pmatrix} J_{23} & 0 \\ 0 & J_{14} \end{pmatrix} = \begin{pmatrix} A_1 & I_4 \\ I_4 & A_2 \end{pmatrix},
\]
here \( A_1 = A'_1 - J_{23} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \) and \( J_{23} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \).

\[
A_2 = A'_2 - J_{14} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} - J_{14} = \begin{pmatrix} -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 \end{pmatrix}.
\]

So, \( \phi(A(X \backslash \pi_1), x) = \begin{vmatrix} xI_4 - A_1 & -I_4 \\ -I_4 & xI_4 - A_2 \end{vmatrix} = |(xI_4 - A_1)(xI_4 - A_2) - I_4| 
\]
\[
= |x^2I_4 - x(A_1 + A_2) + A_1A_2 - I_4|.
\]

Since \( A_1 + A_2 = A_1A_2 = -I_4 \), we have \( \phi(A(X \backslash \pi_1), x) = |(x^2 + x - 2)I_4| = (x^2 + x - 2)^4 \).

By \( \phi(A(X/\pi_1), x) = (x - 3)(x - 1)(x^2 + x - 2)^4 = (x - 1)^5(x + 2)^4(x - 3). \)

To get the characteristic polynomial of the adjacency matrix of the Petersen graph by the another method, see [GR01, section 9.1 and exercise 8.9].

5. Applications to the Ihara-Bartholdi zeta functions

In this section, we assume \( X = (V(X), E(X)) \) is an undirected graph, that is, \( E(X) \) is a set of symmetric directed edges. As corollaries of our Theorem, we have the decomposition of the characteristic polynomial of Laplacian matrix, and the reciprocal of the Ihara-Bartholdi zeta functions of equitably partitioned graphs.

Zeta functions of a graph are defined as follows. For an edge \( e \in E(X) \), we denote by \( o(e) \) (resp. \( t(e) \)) the origin (resp. terminus) of \( e \). A closed path in \( X \) is an sequence of edges \( C = (e_1, \ldots, e_k) \) with \( t(e_i) = o(e_{i+1}) \) for \( i \in \mathbb{Z}/k\mathbb{Z} \). We denote by \( k = |C|, \) the length of \( C \), and by \( cbc(C) = \#\{i \in \{1, \ldots, k\} \mid e_{i+1} = \overline{e_i}\} \), the cyclic bump count of \( C \). A cycle \([C]\) is the equivalence class of a closed path \( C \) under cyclic permutation of its edges (that is, \((e_1, \ldots, e_k) \sim (e_2, \ldots, e_k, e_1)\)). A cycle is prime if none of its representatives can be written as \( C^k \) for some \( k \geq 2 \). We denote by \( C \) the set of prime cycles. Bartholdi zeta function is defined by

\[
Z_X(u, t) = \prod_{[C] \in C} \frac{1}{1 - u^{cbc(C)\overline{t}[C]}},
\]
$Z_X(0, t) = Z_X(t)$ is the Ihara zeta function defined by Ihara [Iha66] in which he considered a zeta function of a regular graph and gave its reciprocal as a polynomial. It was generalized to general graphs by [Bas92] (see also [Ser80], [Has89], [Has90], [Sun96a], [Sun96b], [ST96], [FZ99], [KS00]). Bartholdi generalized Bass’s Theorem as the following.

**Theorem 5.1.** [Bar99] Let $X$ be a connected graph with $n$ vertices and $m$ (non-oriented) edges. Then the reciprocal of the Bartholdi-Ihara zeta function of $X$ is given by

$$Z_X(u, t)^{-1} = (1 - (1 - u)^2 t^2)^{m-n} \det \left( I_n - tA(X) + (1 - u)(D(X) - (1 - u)I_n)t^2 \right),$$

where $A(X)$ is the adjacency matrix of $X$, and $D(X) = (d_v)_{ij}$ is the degree matrix which is diagonal with $d_v = \deg v, V(X) = \{v_1, \ldots, v_n\}$.

For $u, t \in \mathbb{C}$, let

$$(5.6) \quad D^\sharp(X) = s_1I_n + s_2D(X) \in \text{Mat} \left( V(X) \times V(X); \mathbb{C} \right)$$

be the diagonal matrix, here $s_1 = 1 - (1 - u)^2 t^2$, $s_2 = (1 - u)t^2$. Then we have

$$Z_X(u, t)^{-1} = s_1^{m-n} \det(-tA(X) + D^\sharp(X)).$$

As a Corollary of Theorem 4.4, we have the following.

**Corollary 5.2.** Let $\pi = \{V_1, \ldots, V_r\}$ be an equitable partition of the vertex set $V(X)$ of a graph $X$ and fix a set of deleting vertices $\{\overline{v}_i \in V_i, i = 1, \ldots, r\}$. Let $D$ be a diagonal matrix such that $D|V_i \times V_i = d_iI_i$ (for instance, $D = O, D(X)$ or $D^\sharp(X)$).

Then we have

$$\phi(\alpha A(X) + D, x) = \phi(\alpha A(X/\pi) + (\delta_{ij} d_i)_i, x) \cdot \phi \left( \alpha A(X/\pi) + D|_{V' \times V'}, x \right).$$

In particular,

(a) $\phi(L(X), x) = \phi(-A(X/\pi) + D(X/\pi), x) \cdot \phi \left( -A(X/\pi) + D(X)|_{V' \times V'}, x \right)$,

(b) $Z_X(u, t)^{-1} = s_1^{m-n} \det(-tA(X/\pi) + D^\sharp(X/\pi)) \cdot \det \left( -tA(X/\pi) + D^\sharp(X)|_{V' \times V'} \right)$,

here $s_1 = 1 - (1 - u)^2 t^2$.

(Proof) Let $M = \alpha A(X) + D$. Then $M/\pi = \alpha A(X/\pi) + (\delta_{ij} d_i)$. For $v'_i \in V'_i := V_j \setminus \{\overline{v}_j\}$, $\left( D|_{\{v'_i\}_{i=1}^r \times V'} \right)_{v'_ie'_j} = D_{v'_ie'_j} = 0$. So, we have $M|_{\{v'_i\}_{i=1}^r \times V'} = \alpha A(X)|_{\{v'_i\}_{i=1}^r \times V'}$.

Hence,

$$M/\pi = \alpha A(X|_{V' \times V'}) + D|_{V' \times V'} - P|_{V' \times \pi} \cdot \alpha A(X)|_{\{v'_i\}_{i=1}^r \times V'} = \alpha(A(X/\pi) + D|_{V' \times V'}) = \alpha A(X/\pi) + D|_{V' \times V'} \quad \text{by Proposition 3.3}.$$


So, we have the first assertion. By letting (a) \( D = D(X) \), (i) \( \alpha = -1 \), (ii) \( \alpha = 1 \), (b) \( D = D^2(X) \), \( \alpha = -t \), we have the other assertions. \( \square \)

**Remark 5.3.** Note that \( D(X)|_{V \times V'} \neq D(X|_V) \) in general. Let \( X \) be a cycle \( C_4 \) in Example 2.2 (a). Then \( V' = \{\emptyset, [\text{II}]\} \), \( D(X) = 2I_4 \), and \( D(X)|_{V \times V'} = 2I_2 \), but \( D(X|_V) = D(C_2) = I_2 \).

### 6. Generalized Join Graphs

In this section, graphs considered are assumed to be simple. We consider the case when \( X \) is the generalized join (composition) of \( X_1, \ldots, X_r \) (see Example 2.2 (d)). The following Corollary of Theorem 4.4 includes results of \([\text{Sch74, Theorem 7}]\) and \([\text{CC17, Theorem 3.1}]\).

**Corollary 6.1.** Let \( X = H[X_1, \ldots, X_r] \) be a generalized join (composition) with each \( X_i \) being \( k_i \)-regular. Let \( D \in \text{Mat} (V(X) \times V(X); \mathbb{C}) \) be a diagonal matrix such that

\[
D|_{V(X_i) \times V(X_i)} = d_i I_i.
\]

Then, letting \( \pi = \{V(X_1), \ldots, V(X_r)\} \), for any \( \alpha \in \mathbb{C} \),

\[
(6.7) \quad \phi (\alpha A(X) + D, x) = \phi (\alpha A(X/\pi), x) \cdot \prod_{i=1}^{r} \frac{\phi (\alpha A(X_i) + d_i I_i, x)}{x - \alpha k_i - d_i}
\]

\[
(6.8) \quad \text{det} \left( -\alpha A(H) + \mathcal{D}(x) \right) \cdot \prod_{i=1}^{r} \frac{n_i \phi (\alpha A(X_i) + d_i I_i, x)}{x - \alpha k_i - d_i},
\]

here \( \mathcal{D}(x)_{ij} = \delta_{ij} \frac{1}{n_i} (x - \alpha k_i - d_i) \), \( n_i = \#V(X_i) \).

In particular,

(a) \( \phi (A(X), x) = \phi (A(X/\pi), x) \cdot \prod_{i=1}^{r} \frac{\phi (A(X_i), x)}{x - k_i} \)

\[
= \text{det} \left( -A(H) + \mathcal{D}(x) \right) \cdot \prod_{i=1}^{r} \frac{n_i \phi (A(X_i), x)}{x - k_i},
\]

here \( \mathcal{D}(x)_{ij} = \delta_{ij} \frac{1}{n_i} (x - k_i) \), and

(b) \( Z(u, t)^{-1} \)

\[
(6.9) \quad s_1^{m-n} \text{det} \left( -t A(X/\pi) + (\delta_{ij} d_i^2 (X)) \right) \cdot \prod_{i=1}^{r} \frac{\text{det}(-t A(X_i) + d_i^2 (X) I_i)}{-tk_i + d_i^2 (X)}
\]

\[
= s_1^{m-n} \text{det}(-t A(H) + \mathcal{D}) \prod_{i=1}^{r} n_i \frac{\text{det}(-t A(X_i) + d_i^2 (X) I_i)}{-tk_i + d_i^2 (X)},
\]

here \( \mathcal{D} \in \text{Mat} (r \times r; \mathbb{C}) \) is a diagonal matrix such that \( \mathcal{D}_{ii} = \frac{1}{n_i} (-tk_i + d_i^2 (X)) \in \mathbb{C}, d_i^2 (X) = s_1 + s_2 d_i (X) = s_1 + s_2 (k_i + \sum_{j \neq i} A(H)_{ij} n_j), s_1 = 1 - (1 - u)^2 t^2, s_2 = (1 - u)t^2, d_i (X) = \deg v, v \in V(X_i). \)
(Proof) Let $M = \alpha A(X) + D$.
(i) First we prove that

\begin{equation}
\phi(M \setminus \pi, x) = \prod_{i=1}^{r} \frac{\phi(\alpha A(X_i) + d_i I_i, x)}{x - \alpha k_i - d_i},
\end{equation}

which implies (6.7) by Theorem 4.4 and (2.3). By Example 2.2 (d) and Remark 2.4, $(M, \pi)$ is an equitable pair. Let $V'_i := V(X_i) \setminus \{\pi_i\}, i = i, \ldots, r$. As the notation in Theorem 4.4, for $i \neq j$, $v'_k \in V'_i, v'_l \in V'_j$, since \( P_{V'} \times \pi : A(X)_{\{\pi_i\}_{i=1}^{r} \times V'} \) we have $A(X \setminus \pi)_{v'_k v'_j} = A(X)_{v'_k v'_j} = A(H)_{ij} = 0$, which implies $(M \setminus \pi)_{v'_k v'_j} = \alpha A(X \setminus \pi)_{v'_k v'_j} + D_{v'_k v'_j} = 0$.

Hence, $M \setminus \pi = \begin{pmatrix} C_1 & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & C_r \end{pmatrix}$, here $C_i = (M \setminus \pi)_{V'_i \times V'_i}$. So,

\begin{equation}
\phi(M \setminus \pi, x) = \prod_{i=1}^{r} \phi(C_i, x).
\end{equation}

We prove that

\begin{equation}
\phi(C_i, x) = \phi(M_i, x) \quad \text{here } M_i = M_{V'_i \times V'_i} = \alpha A(X_i) + d_i I_i,
\end{equation}

which implies (6.10). For a subgraph $X_i$, let $\pi_i = \{V(X_i)\}$ be the trivial partition. Since $M_i 1 = (\alpha k_i + d_i) 1$, we have $M_i / \pi_i = \alpha k_i + d_i \in \mathbb{C}$. So, by Theorem 4.4 we have

\[ \phi(M_i, x) = (x - \alpha k_i - d_i) \phi(M_i \setminus \pi_i, x). \]

We show $M_i \setminus \pi_i = (M \setminus \pi)_{V'_i \times V'_i} = C_i$, which implies (6.12). Since

\[ \begin{pmatrix} (M \setminus \pi)_{V'_i \times V'_i} \end{pmatrix}_{v'_k v'_j} = (M \setminus \pi)_{v'_k v'_j} = (M_{V'_i \times V'_i} - P_{V'_i \times \pi} : M_i \{\pi_i\}_{i=1}^{r} \times V'_i)_{v'_k v'_j} = M_{v'_k v'_j} - M_{\pi_i v'_j}, \]

and

\[ (M_i \setminus \pi_i)_{v'_k v'_j} = (M_i |_{V'_i \times V'_i} - P |_{V'_i \times \pi_i} : M_i \{\pi_i\} \times V'_i)_{v'_k v'_j} = M_{v'_k v'_j} - M_{\pi_i v'_j}. \]

So, we have the assertion.

(ii) Next we prove that

\begin{equation}
\phi(M / \pi, x) = n_1 \cdots n_r \det(-\alpha A(H) + D(x)),
\end{equation}

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which implies (6.8). By (2.1) and (2.2),

\[(M/\pi)_{ij} = \alpha A(X/\pi)_{ij} + (D/\pi)_{ij}\]

(6.14)

\[
\begin{align*}
\text{if } i = j, & \quad \alpha k_i + d_i \\
\text{if } i \neq j, & \quad \alpha A(H)_{ij}n_j
\end{align*}
\]

Hence, letting

\[(6.15) \quad \rho_{ij} = \begin{cases} 
\frac{1}{n_i} (x - \alpha k_i - d_i) & \text{if } i = j, \\
-\alpha A(H)_{ij} & \text{if } i \neq j,
\end{cases} \]

we have \(xI - M/\pi = (\rho_{ij}n_j)\). Since

\[
(\rho_{ij}) = -\alpha A(H) + \nabla(x), \text{ here } \nabla(x)_{ij} = \delta_{ij} \frac{1}{n_i} (x - \alpha k_i - d_i),
\]

we have (6.13).

\[\square\]

**Remark 6.2.** We can generalize Corollary 6.1 to the following Teranishi’s result [Ter03, Theorem 3.3].

Let \(X = H[X_1, \ldots, X_r]\) be a generalized join (composition) and for each \(i = 1, \ldots, r\), let \(\pi_i = (V_{i}^{1}, \ldots, V_{i}^{n_i})\) be an equitable partition of \(V(X_i)\). Then letting \(\pi = (\pi_i)_{i=1}^r = (V_i^{\alpha_i})_{i=1}^r\), \(\pi\) is an equitable partition of \(V(X)\) and

\[
\phi(A(X), x) = \phi(A(X/\pi), x) \cdot \prod_{i=1}^{r} \phi(A(X_i/\pi_i), x)
\]

\[
= \phi(A(X/\pi), x) \cdot \prod_{i=1}^{r} \phi(A(X_i/\pi_i), x).
\]

When \(X_i\) is \(k_i\)-regular, letting \(\pi_i = \pi_i = \{V(X_i)\}\) the trivial partition, \(\pi_i\) is an equitable partition of \(V(X_i)\) with \(\phi(A(X_i/\pi_i), x) = x - k_i\), we get (6.7) for \(\alpha = 1, D = 0\), and similarly for all \(\alpha\) and \(D = (d_iI_i)\). The same proof also applies in the proof of Corollary 6.1, considering the partition \(\pi_i\) instead of the trivial partition \(\pi\).

**Remark 6.3.** As the notation in [CC17, Theorem 3.1],

\[
\gamma_i := 1 - tk_i + (1-u)(k_i + N_i - 1 + u)t^2
\]

\[
= -tk_i + 1 - (1 - u)^2t^2 + (1 - u)t^2(k_i + N_i) = -tk_i + s_1 + s_2d_i(X)
\]

(6.16)

\[
=-tk_i + d^2(X),
\]

here \(N_i = \sum_{j \neq i} A(H)_{ij}n_j\). Let \(M = -tA(X) + D^2(X)\). Letting \(\alpha = -t, D = D^2(X)\) in (6.14) we have \((M/\pi)_{ij} = \begin{cases} 
\gamma_i & \text{if } i = j, \\
-\rho_{ij}n_j & \text{if } i \neq j,
\end{cases} \)

here \(\rho_{ij} = A(H)_{ij}\). Let \(N_{ij} = \begin{cases} 
\gamma_i & \text{if } i = j, \\
-t\sqrt{n_i n_j} \rho_{ij} & \text{if } i \neq j.
\end{cases} \)
Then, by letting $\rho'_{ij} = \begin{cases} \frac{\gamma_i}{n_i} & \text{if } i = j, \\ -t\rho_{ij} & \text{if } i \neq j, \end{cases}$
we have $M/\pi = (\rho'_{ij} n_j)$ and $N = (\rho'_{ij} \sqrt{n_i n_j})$. Then
(6.17) $\det M/\pi = \det (-tA(X/\pi) + (\delta_{ij} d_i^Z(X))) = \det N$.

Let
$$M_{X_i}(F) = I_i - tA(X_i) + (1 - u)(F - (1 - u)I_i) t^2$$
$$= -tA(X_i) + s_1 I_i + s_2 F.$$  
Since $D(X_i) + N_i I_i = (k_i + N_i)I_i = d_i(X) I_i$, we have
(6.18) $M_{X_i}(D(X_i) + N_i I_i) = -tA(X_i) + (s_1 + s_2 d_i(X)) I_i = -tA(X_i) + d_i^Z(X) I_i$.
So, by (6.9),(6.16),(6.17),(6.18), we have
$$Z(u, t)^{-1} = (1 - (1 - u)^2 t^2)^{m-n} \det N \cdot \prod_{i=1}^r \frac{\det (M_{X_i}(D(X_i) + N_i I_i))}{\gamma_i},$$
which is [CC17, Theorem 3.1].

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