LAZER-MCKENNA CONJECTURE FOR HIGHER ORDER ELLIPTIC PROBLEM WITH CRITICAL GROWTH

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Abstract. This paper is concerned with the following problem involving critical Sobolev exponent and polyharmonic operator:

\[
\begin{aligned}
&(-\Delta)^m u = u^{m^* - 1} + \lambda u - s_1 \varphi_1, \quad \text{in } B_1, \\
u \in \mathcal{D}^{m,2}_0(B_1),
\end{aligned}
\]

where \(B_1\) is the unit ball in \(\mathbb{R}^N\), \(s_1\) and \(\lambda > 0\) are two positive parameters, \(\varphi_1 > 0\) is the eigenfunction of \((-\Delta)^m, \mathcal{D}^{m,2}_0(B_1)\) corresponding to the first eigenvalue \(\lambda_1\) with \(\max_{y \in B_1} \varphi_1(y) = 1\), \(u_+ = \max(u, 0)\) and \(m^* = \frac{2N}{N - 2m}\). By using the Lyapunov-Schmidt reduction method, we prove that the number of solutions for \((P)\) is unbounded as the parameter \(s_1\) tends to infinity, therefore proving the Lazer-McKenna conjecture for the higher order operator equation with critical growth.

1. Introduction. We consider the following polyharmonic elliptic problem with critical Sobolev exponent:

\[
\begin{aligned}
&(-\Delta)^m u = u^{m^* - 1} + \lambda u - s_1 \varphi_1, \quad \text{in } B_1, \\
u \in \mathcal{D}^{m,2}_0(B_1),
\end{aligned}
\]

where \(B_1\) is the unit ball in \(\mathbb{R}^N\), \(s_1\) and \(\lambda > 0\) are parameters, \(\varphi_1 > 0\) is the eigenfunction of \((-\Delta)^m, \mathcal{D}^{m,2}_0(B_1)\) corresponding to the first eigenvalue \(\lambda_1\) with \(\max_{y \in B_1} \varphi_1(y) = 1\), \(u_+ = \max(u, 0)\) and \(m^* = \frac{2N}{N - 2m}\), \(N \geq 2m + 1\), \(m\) is positive integer and \(\mathcal{D}^{m,2}_0(B_1)\) is the completion of \(C_0^\infty(B_1)\) with respect to the norm induced by the scalar product:

\[
(u, v) = \begin{cases}
\int_{B_1} \Delta^{\frac{m}{2}} u \cdot \Delta^{\frac{m}{2}} v, & \text{if } m \text{ is even}, \\
\int_{B_1} \nabla \Delta^{\frac{m-1}{2}} u \cdot \nabla \Delta^{\frac{m-1}{2}} v, & \text{if } m \text{ is odd}.
\end{cases}
\]
Problem (1) is the higher order case of the Ambrosetti-Prodi type problem with critical exponent:
\[
\begin{cases}
-\Delta u = g(u) - \bar{s}\varphi_1(x) & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega,
\end{cases}
\] (3)
where $\Omega$ is a bounded domain in $\mathbb{R}^N$, and $g(t)$ satisfies $\lim_{t \to -\infty} \frac{g(t)}{t} = \nu < \lambda_1$, $\lim_{t \to +\infty} \frac{g(t)}{t} = \mu > \lambda_1$. Here $\mu = +\infty$ and $\nu = -\infty$ are allowed. It is known that the location of $\nu, \mu$ with respect to the eigenvalues of $-\Delta$ with Dirichlet boundary condition in a bounded domain $\Omega$ plays an important role in the study of the existence and multiplicity of solutions for the problem (3). It has been a problem of considerable interest since the early 1970s, many interesting results have been achieved, see for example [1, 4, 5, 8, 9, 10, 11, 14, 15, 16, 20, 21, 22, 23] and references therein. In particular, in the view of the results in [13, 23], Lazer and McKenna [14] conjectured that if $\mu = +\infty$ and $g(t)$ does not grow too fast at infinity, then (3) has an unbounded number of solutions as $\bar{s} \to +\infty$.

In case of $m = 1$, by using a partially numerical method, Breuer, McKenna and Plum [3] proved that if $g(t) = t^2$ and $\Omega \subset R^2$ is the unit square, then (3) has at least four solutions; Dancer and Yan [6] proved that if $g(t) = |t|^p$ with $p \in (1, +\infty)$ for $N = 2$ and $p \in (1, \frac{N+2}{N-2})$ for $N \geq 3$, then the Lazer-McKenna conjecture is true. While, in [7], they proved that the Lazer-McKenna conjecture is also true if $g(t) = t^p + \lambda t$, $\lambda \in (-\infty, \lambda_1)$, $N \geq 3$ and $p \in (1, \frac{N+2}{N-2})$. In the critical case, that is $p = \frac{N+2}{N-2}$, Li, Yan and Yang [17] proved that if $N \geq 7$ and $\lambda \in (0, \lambda_1)$, $s_1 > 0$, the number of the solutions is unbounded as $s_1 \to +\infty$ (see also [18, 24] for $N \geq 6$). Note that the solutions constructed in [17, 18] are different from those constructed in [24], where the previous one blows up near the maximum points of the function $\varphi_1(y)$ in $\Omega$, while the solutions obtained in [24] have several peaks clustering near a boundary point which is a maximum of the function $-\frac{\partial \varphi_1(x)}{2m}$.

Motivated by the above work, the aim of the present paper is going to discuss the Lazer-McKenna conjecture for the case of any $m > 1$ with critical nonlinearities. That is the following problem:
\[
\begin{cases}
(-\Delta)^m u = u^{m^*_1-1} + \lambda u - s_1 \varphi_1, & \text{in } B_1, \\
u \in D^{m, 2}_0(B_1),
\end{cases}
\] (4)
As far as we know, no such type of results have been obtained for polyharmonic operator problems (4). However, we do not know whether the similar result is true for the general domain $\Omega$ due to the lack of positivity of the first eigenfunction. As a consequence, we prove the Lazer-McKenna conjecture for polyharmonic equation (4) in $B_1$. Even this, due to the lack of of the comparison principle for polyharmonic operator, some new technique idea are needed. For example, instead of maximum principle, we turn to use Green’s representation to obtain the essential estimate in the energy expansion.

Let $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots$ be the eigenvalue of the operator $(-\Delta)^m$ with Dirichlet boundary condition in $B_1$ the unit ball in $\mathbb{R}^N$ (see [12]). We assume that $\lambda$ and $s_1$ satisfy one of the following conditions:
(\Lambda_1) $\lambda \in (0, \lambda_1)$ and $s_1 > 0$;
(\Lambda_2) $\lambda \in (\lambda_i, \lambda_{i+1})$ for some $i \geq 1$ and $s_1 < 0$.

Our main result is the following:
Theorem 1.1. Assume \( N \geq 6m + 1, \lambda \) and \( s_1 \) satisfy \((\Lambda_1)\) or \((\Lambda_2)\). Then the number of the solutions for \((4)\) is unbounded as \(|s_1| \to +\infty\).

Noting that if \((\Lambda_1)\) or \((\Lambda_2)\) holds. Then \((4)\) has a non-negative solution \(u_{s_1} = \frac{s_1}{N-1} \varphi_1\). Moreover, if \(u_{s_1} + u\) is a solution of \((4)\), then \(u\) satisfies:

\[
\begin{cases}
(\Delta)^m u = (u - s_1 \varphi_1)^{m-1} + \lambda u, \text{ in } B_1,
\end{cases}
\]

where \(s = \frac{s_1}{N-1} > 0\).

Therefore in order to prove Theorem 1.1, it is sufficient to prove the number of solutions of \((5)\) is unbounded as \(s \to +\infty\).

Let

\[U_{\bar{x},\bar{\mu}}(y) = \frac{c_0 \bar{\mu}^{N-2m}}{(1 + \bar{\mu}^2 |y - \bar{x}|^2)^{N-2m}}, \text{ for } \bar{x} \in \mathbb{R}^N, \bar{\mu} > 0,
\]

be the unique positive solution (up to translation and scaling) of the following equation

\[(-\Delta)^m U = U^{m-1} \text{ in } \mathbb{R}^N,
\]

where \(c_0 > 0\) is a constant depending on \(m\) and \(N\).

Let \(PU_{\bar{x},\bar{\mu}}\) be the projection of \(U_{\bar{x},\bar{\mu}}\) on \(B_1\), i.e.

\[
\begin{cases}
(\Delta)^m PU_{\bar{x},\bar{\mu}} = U_{\bar{x},\bar{\mu}}^{m-1}, \text{ in } B_1,
\end{cases}
\]

\[PU_{\bar{x},\bar{\mu}} \in D_0^{m,2}(B_1).
\]

In the following of the paper, we will use the notations:

\[U = U_{0,1}, \quad \frac{\partial U}{\partial \bar{\mu}} = \frac{\partial U_{0,\bar{\mu}}}{\partial \bar{\mu}} \bigg|_{\bar{\mu} = 1} \quad \text{and} \quad \frac{\partial U}{\partial \bar{x}_i} = \frac{\partial U_{\bar{x},\bar{\mu}}}{\partial \bar{x}_i} \bigg|_{\bar{x} = 0}, \quad i = 1, \cdots, N.
\]

We have the following existence result for problem \((5)\).

Theorem 1.2. Assume that \(N \geq 6m + 1, \lambda, s_1\) satisfy either \((\Lambda_1)\) or \((\Lambda_2)\). Then for any integer \(k \geq 1\), there exists a \(s_k > 0\), depending on \(k\), such that for any \(s \geq s_k\), \((5)\) has a solution of the form \(u_s = \sum_{j=1}^{k} PU_{x_{s,j},\mu_{s,j}} + \omega_{s,k}\), satisfying as \(s \to +\infty\),

1. \(x_{s,j} \to x_j\) with \(\varphi_1(x_j) = \max_{y \in \Omega} \varphi_1(y), j = 1, \cdots, k;\)
2. \(s^{2/(N-6m)}|x_{s,i} - x_{s,j}| \to +\infty, i \neq j;\)
3. \(\mu_{s,j}s^{-2/(N-6m)} \to \bar{c} > 0, j = 1, \cdots, k;\)
4. \(\omega_{s,k} \in D_0^{m,2}(B_1), \|\omega_{s,k}\| \to 0.\)

It is easy to see that Theorem 1.1 is a direct consequence of Theorem 1.2.

Before the end of this introduction, let us outline the proof of Theorem 1.2 more precisely.

For any integer \(k > 0, x_i \in \Omega, i = 1, \cdots, k, \mu_i \in \mathbb{R}_+^1, i = 1, \cdots, k\), we define

\[E_{x_i,\mu_i,k} = \left\{ \omega : \omega \in D_0^{m,2}(B_1), \left\langle \omega, \frac{\partial (PU_{x_{j},\mu_{j}})}{\partial \mu_{j}} \right\rangle = \left\langle \omega, \frac{\partial (PU_{x_{j},\mu_{j}})}{\partial x_{j,h}} \right\rangle = 0 \right\} \]

\[h = 1, \cdots, N, j = 1, \cdots, k\].
Denote $T_0 = \left(\frac{(N-2m)B_2}{4m\lambda B_0}\right)^{2/(N-6m)}$, where $B_0 = \frac{1}{2} \int_{\mathbb{R}^N} U^2 \, dy$ and $B_2 = \int_{\mathbb{R}^N} U^{m^* - 1} \, dy$.

Set

$$M_s = \{ (x, \mu) : \mu_j \in \left[ (T_0 - Ls^{-\theta})s^{\frac{2}{N-8m}}, (T_0 + Ls^{-\theta})s^{\frac{2}{N-8m}} \right], \varphi_1(x_i) \geq 1 - s^{-\theta},
|x_i - x_j| \geq \left( \frac{1}{s^{(N-8m)/(N-6m)-\theta}} \right)^{1/(N-2m)}, i, j = 1, \ldots, k, i \neq j \},$$

where $x = (x_1, \ldots, x_k), \mu = (\mu_1, \ldots, \mu_k), L > 0$ is a fixed large constant and $\theta > 0$ is a fixed small constant.

Recall that the functional corresponding to (5) is defined as

$$I_s(u) = \begin{cases} 
\frac{1}{2} \int_{B_1} (|\Delta \frac{m}{2} u|^2 - \lambda u^2) - \frac{1}{m} \int_{B_1} (u - s\varphi_1)^{m^*}, & \text{if } m \text{ is even}, \\
\frac{1}{2} \int_{B_1} (|\nabla \frac{m}{2} u|^2 - \lambda u^2) - \frac{1}{m} \int_{B_1} (u - s\varphi_1)^{m^*}, & \text{if } m \text{ is odd}.
\end{cases}$$

To prove Theorem 1.2, follow the idea of [17], we first reduce the problem to a finite-dimensional problem. Then we prove that there exists a $C^1$ map $\omega_{s,x,\mu}$ from $M_s$ to $D_0^{m,s}(B_1)$, such that $\omega_{s,x,\mu} \in E_{s,x,k}$, and

$$\frac{\partial J_s(x, \mu, \omega_{s,x,\mu})}{\partial \omega} = \sum_{j=1}^k A_j \frac{\partial (PU_{x_j,\mu_j})}{\partial \mu_j} + \sum_{j=1}^k \sum_{h=1}^N B_{jh} \frac{\partial (PU_{x_j,\mu_j})}{\partial x_{j,h}},$$

for some constants $A_j$ and $B_{jh}$, where

$$J_s(x, \mu, \omega) = I_s \left( \sum_{j=1}^k PU_{x_j,\mu_j} + \omega \right).$$

Note that in order to show that $\sum_{j=1}^k PU_{x_j,\mu_j} + \omega$ is a solution of (5), it is sufficient to find a suitable $(x_s, \mu_s) \in M_s$ such that $(x_s, \mu_s)$ is a critical point of the function $J_s(x, \mu, \omega(s, x, \mu))$, or the corresponding constants $A_j$ and $B_{jh}$ are all equal to zero. For this purpose, we turn to consider the function $K(x, \mu) = J_s(x, \mu, \omega_{s,x,\mu}), \ (x, \mu) \in M_s$. And reduce the problem of finding the critical point of $J_s$ to the problem of finding the critical point of $K$. However, we may allow (see Section 3) that $K(x, \mu)$ has a saddle point in $M_s$. Hence the classical maximization procedure can not applied directly. To overcome this difficulty, follow the idea of [18], we first prove that there is a $C^1$ map $\mu(x)$ such that for each fixed $x$, $\frac{\partial K(x, \mu(x))}{\partial \mu_j} = 0, \ j = 1, \ldots, k$. Then we prove that $K(x, \mu)$ has a critical point $(x, \mu(x))$ by using a maximization procedure for the function $K(x, \mu(x))$.

Our paper is organized as follows. In Section 2, we reduce the problem of finding solutions for (5) to a finite-dimensional problem. The proof of the main existence result will be given in Section 3. We put some essential estimates and the energy expansions in the Appendices A and B, respectively.
2. Finite-dimensional reduction. In this section, we will reduce the problem of finding a $k$--peak solution for (5) to a finite-dimension problem. We define

$$
\varepsilon_{ij} = \frac{1}{\mu_i^{(N-2m)/2} \mu_j^{(N-2m)/2} |x_i - x_j|^{N-2m}}, \text{ for any } i \neq j.
$$

It is easy to see that for $(x, \mu) \in M_s$, $|x_i - x_j| \geq \left(\frac{1}{\mu_i^{(N-2m)/2} \mu_j^{(N-2m)/2} x_i - x_j |^{N-2m}}\right)^{1/(N-2m)}$, $i \neq j$, implies

$$
\varepsilon_{ij} \leq s^{-\theta - 4m/(N-6m)} \text{ for } i \neq j.
$$

For each $(x, \mu) \in M_s$, we expand $J_s(x, \mu, \omega)$ at $\omega = 0$ as follows:

$$
J_s(x, \mu, \omega) = \langle l_s, \omega \rangle + \frac{1}{2} \langle Q_s \omega, \omega \rangle + R_s(\omega),
$$

where $l_s \in E_{x, \mu, k}$ satisfying

$$
\langle l_s, \omega \rangle = \sum_{j=1}^k \mu_j^{1/2} (1 + \sigma) \left( \frac{1}{\mu_j^{(N-2m)/4}} \right) + \varepsilon_{ij} \left( \frac{1}{\mu_j^{(N-2m)/4}} \right), \forall \omega \in E_{x, \mu, k},
$$

and $Q_s$ is a bounded linear map from $E_{x, \mu, k}$ to $E_{x, \mu, k}$ satisfying

$$
\langle Q_s \omega, \eta \rangle = \langle \omega, \eta \rangle - \lambda \int_{B_1} \mu_j^{1/2} (1 + \sigma) \left( \frac{1}{\mu_j^{(N-2m)/4}} \right) + \varepsilon_{ij} \left( \frac{1}{\mu_j^{(N-2m)/4}} \right) \omega dy, \forall \omega, \eta \in E_{x, \mu, k},
$$

and

$$
R_s(x, \mu)(\omega) = -\frac{1}{m^*} \int_{B_1} \left( \sum_{j=1}^k \mu_j^{1/2} (1 + \sigma) \left( \frac{1}{\mu_j^{(N-2m)/4}} \right) + \varepsilon_{ij} \left( \frac{1}{\mu_j^{(N-2m)/4}} \right) \right)^{m^* - 2} \omega dy, \forall \omega, \eta \in E_{x, \mu, k},
$$

and

$$
R_s(x, \mu)(\omega) = -\frac{1}{m^*} \int_{B_1} \left( \sum_{j=1}^k \mu_j^{1/2} (1 + \sigma) \left( \frac{1}{\mu_j^{(N-2m)/4}} \right) + \varepsilon_{ij} \left( \frac{1}{\mu_j^{(N-2m)/4}} \right) \right)^{m^* - 2} \omega dy,
$$

Lemma 2.1. We have

$$
\langle l_s, \omega \rangle = O \left( \sum_{j=1}^k \frac{1}{\mu_j^{2m}} + \sum_{j=1}^k \frac{s^{1/(1+\sigma)/2} \mu_j^{(1+\sigma)/(N-2m)/4} + \sum_{i \neq j} \varepsilon_{ij}^{1/(1+\sigma)/2} \mu_j^{(1+\sigma)/(N-2m)/4}} \right) ||\omega||,
$$

where $\sigma > 0$ is a constant.
Proof. For any $\omega \in D_{0}^{m,2}(B_{1})$, we have

$$\langle l_{x}, \omega \rangle$$

$$= \int_{B_{1}} \left( \sum_{j=1}^{k} U_{x_{j}, \mu_{j}}^{m-1} - \left( \sum_{j=1}^{k} P U_{x_{j}, \mu_{j}} - s \varphi_{1} \right)^{m-1}_{+} \right) \omega dy - \lambda \int_{B_{1}} \sum_{j=1}^{k} P U_{x_{j}, \mu_{j}} \omega dy$$

$$= \int_{B_{1}} \left( \sum_{j=1}^{k} \left( P U_{x_{j}, \mu_{j}} - s \varphi_{1} \right)^{m-1}_{+} - \left( \sum_{j=1}^{k} P U_{x_{j}, \mu_{j}} - s \varphi_{1} \right)^{m-1}_{+} \right) \omega dy$$

$$= \sum_{j=1}^{k} \int_{B_{1}} \left( U_{x_{j}, \mu_{j}}^{m-1} - \left( P U_{x_{j}, \mu_{j}} - s \varphi_{1} \right)^{m-1}_{+} \right) \omega dy - \lambda \int_{B_{1}} \sum_{j=1}^{k} P U_{x_{j}, \mu_{j}} \omega dy.$$  \hfill (10)

Firstly, for any $j = 1, \cdots, k$,

$$\left| \int_{B_{1}} P U_{x_{j}, \mu_{j}} \omega dy \right| \leq \int_{B_{1}} U_{x_{j}, \mu_{j}} |\omega| dy$$

$$\leq \left( \int_{B_{1}} U^{2N/(N+2m)}_{x_{j}, \mu_{j}} dy \right)^{(N+2m)/2N} ||\omega||$$ \hfill (11)

Then, by Lemma A.1, we have

$$\left| \int_{B_{1}} \left( U_{x_{j}, \mu_{j}}^{m-1} - \left( P U_{x_{j}, \mu_{j}} - s \varphi_{1} \right)^{m-1}_{+} \right) \omega dy \right|$$

$$\leq \left| \int_{B_{1}} \left( U_{x_{j}, \mu_{j}}^{m-1} - \left( U_{x_{j}, \mu_{j}} - s \varphi_{1} \right)^{m-1}_{+} \right) \omega dy \right| + O \left( \int_{B_{1}} U^{m-2}_{x_{j}, \mu_{j}} \psi_{x_{j}, \mu_{j}} |\omega| dy \right)$$

$$\leq \left( \int_{B_{1}} \left| U_{x_{j}, \mu_{j}}^{m-1} - \left( U_{x_{j}, \mu_{j}} - s \varphi_{1} \right)^{m-1}_{+} \right| dy \right)^{2N/(N+2m)} ||\omega||^{(N+2m)/2N}$$

$$+ O \left( \frac{1}{\mu_{j}^{(N+2m)/2}} \right) ||\omega||.$$ \hfill (12)

$$\leq C \left( \int_{B_{1}} \left( U_{x_{j}, \mu_{j}}^{m-1_{-\sigma}} \right)^{2N/(N+2m)} dy \right)^{(N+2m)/2N} s^{\frac{1}{2}+\sigma} ||\omega||$$

$$+ O \left( \frac{1}{\mu_{j}^{(N+2m)/2}} \right) ||\omega||$$

$$= O \left( \frac{1}{\mu_{j}^{(N+2m)/2}} + \left( s \mu_{j}^{-(N-2m)/2} \right)^{\frac{1}{2}+\sigma} \right) ||\omega||.$$
Lemma 2.2. There exists a constant $\rho > 0$ independent of $s$ and $(x, \mu) \in M_s$, such that

$$||Q_s \omega|| \geq \rho ||\omega||, \quad \forall \omega \in E_{x, \mu, k}.$$  

Proof. The proof is similar to the proof of Lemma 2.3 in [17], we just sketch it.

We argue by contradiction. Suppose that there are $s_n \to +\infty$, $(x_n, \mu_n) \in M_{s_n}$ and $\omega_n \in E_{x_n, \mu_n, k}$ such that

$$||Q_{s_n} \omega_n|| = o(1) ||\omega_n||,$$

where $o(1) \to 0$ as $n \to +\infty$. Without loss of generality, we may assume $||\omega_n|| = 1$.

By the standard blow-up argument, we can prove that for any $R > 0$, 

$$\int_{B_{\mu_n^{-1} R(x_n, j)}} \omega_n^2 = o(1), \quad j = 1, \cdots, k,$$

which implies that

$$\int_{B_1} \left( \sum_{j=1}^{k} P U_{x_n, j, \mu_n, j} - s \varphi_1 \right)^{m^*-2} \omega_n^2 = o(1).$$

Combining (14) and (15), we obtain

$$\langle \omega_n, \eta \rangle - \lambda \int_{B_1} \omega_n \eta = o(1) ||\eta||, \quad \forall \eta \in E_{x_n, \mu_n, k}.$$  

Since $\omega_n$ is bounded in $D^{m,2}_0(B_1)$, we can assume that there exists an $\omega^* \in D^{m,2}_0(B_1)$ such that 

$$\omega_n \rightharpoonup \omega^* \text{ weakly in } D^{m,2}_0(B_1).$$

It follows from (16) that

$$\langle \omega^*, \eta \rangle - \lambda \int_{B_1} \omega^* \eta = 0, \quad \forall \eta \in D^{m,2}_0(B_1).$$
From (A1) or (A2), $\lambda$ is not an eigenvalue. Thus $\omega^* = 0$ from (17).

Thus $\int_{B_1} \omega_n^2 = o(1)$, which together with (16) gives $||\omega_n|| = o(1)$. This is a contradiction. \hfill $\Box$

**Proposition 1.** There exists a $s_k > 0$, such that for each $s \geq s_k$, there exists a $C^1$-map $\omega_{s,x,\mu} : M_s \to D_{0}^{m,2}(B_1)$, such that $\omega_{s,x,\mu} \in E_{x,\mu,k}$ and

$$\frac{\partial J_s(x, \mu, \omega_{s,x,\mu})}{\partial \omega} = \sum_{i=1}^{k} A_i \frac{\partial (PU_{s_i,\mu})}{\partial \mu_i} + \sum_{i=1}^{k} \sum_{h=1}^{N} B_{ih} \frac{\partial (PU_{s_i,\mu})}{\partial x_{i,h}},$$

for some constants $A_i$ and $B_{ih}$. Moreover, we have

$$||\omega_{s,x,\mu}|| = O \left( \sum_{i=1}^{k} \frac{1}{\mu_{i}^{2m}} + \sum_{i=1}^{k} \frac{s^{(1+\sigma)} / (1+\sigma)(N-2m)/4}{\mu_{i}^{1+\sigma}} + \sum_{i \neq j} \varepsilon_{ij}^{1+\sigma} \right),$$

where $\sigma > 0$ is some constant.

**Proof.** We have

$$J_s(x, \mu, \omega) = J_s(x, \mu, 0) + \langle l_s, \omega \rangle + \frac{1}{2} \left( Q_s \omega, \omega \right) + R_s(\omega),$$

where $l_s, Q_s, R_s$ are defined respectively in (7), (8), (9). A direct calculation shows that $R_s(\omega)$ satisfies $R_s^{(j)}(\omega) = O \left( ||\omega||^{m-j} \right), \quad j = 0, 1, 2$. Thus to find a critical point for $J_s(x, \mu, \omega)$ in $E_{x,\mu,k}$ is equivalent to solving

$$l_s + Q_s \omega + R_s(\omega) = 0. \quad (18)$$

By Lemma 2.2, we obtain that $Q_s$ is invertible in $E_{x,\mu,k}$ and there is a constant $C > 0$ such that $||Q_s^{-1}|| \leq C$. By implicit function theorem there is a $\omega_{s,x,\mu} \in E_{x,\mu,k}$ such that (18) holds. Moreover, $||\omega_{s,x,\mu}|| \leq C||l_s||$. On the other hand, by Lemma 2.1, we have

$$||l_s|| \leq C \left( \sum_{i=1}^{k} \frac{1}{\mu_{i}^{2m}} + \sum_{i=1}^{k} \frac{s^{(1+\sigma)} / (1+\sigma)(N-2m)/4}{\mu_{i}^{1+\sigma}} + \sum_{i \neq j} \varepsilon_{ij}^{1+\sigma} \right).$$

Combining the above argument, we finish the proof. \hfill $\Box$

3. **Proof of the main result.** Let $K(x, \mu) = J_s(x, \mu, \omega_{s,x,\mu}), \quad (x, \mu) \in M_s,

$$V_k = \{ x : x = (x_1, \cdots, x_k), x_i \in \Omega, |x_i - x_j| \geq \frac{1}{s(2N-8m)/(N-6m)-\sigma}, \forall i \neq j, \phi_{1}(x_i) \geq 1 - s^{-\theta}, i = 1, \cdots, k \},$$

where $\omega_{s,x,\mu}$ is the map obtained in Proposition 1.

From Proposition 1 and 4, we have

$$K(x, \mu) = J_s(x, \mu, 0) + O(||\omega_{s,x,\mu}||^2)$$

$$= kA - \sum_{j=1}^{k} \left( \frac{B_0 \lambda}{\mu_{j}} - \frac{B_2 \phi_1(x_j) s}{\mu_{j}^{(N-2m)/2}} \right) - \sum_{i < j} (B_i + \tilde{B}(x_i, x_j)) \varepsilon_{ij}$$

$$+ O \left( \sum_{j=1}^{k} \left( \frac{1}{\mu_{j}^{2m+\sigma}} + \frac{s}{\mu_{j}^{(N-2m)/2+1}} + \left( \frac{s}{\mu_{j}^{(N-2m)/2}} \right)^{1+\sigma} \right) + \sum_{i < j} \varepsilon_{ij}^{1+\sigma} \right). \quad (19)$$
Proposition 2. There is a $C^1$ map $\mu(x) = (\mu_1(x), \cdots, \mu_k(x))$ from $V_k$ to $\mathbb{R}^k$ such that
\[
\frac{\partial K(x, \mu(x))}{\partial \mu_i} = 0, \quad i = 1, \cdots, k.
\] (20)
Moreover, $\mu_i(x) = (T_0 + O(s^{-\theta}))s^{2/(N-6m)}$.

Proof. Let
\[
\mu_i = T_i s^{2/(N-6m)}, \quad T_i \in [T_0 - Ls^{-\theta}, T_0 + Ls^{-\theta}].
\]
For any $x \in V_k$, noting that $\varphi_1(x_i) = 1 + O(s^{-\theta})$. It follows from (47) that $\frac{\partial K(x, \mu(x))}{\partial \mu_i} = 0$ is equivalent to
\[
\frac{2m\lambda B_0}{T_i^{2m+1}} - \frac{N - 2m}{2} \frac{B_2}{T_i^{(N-2m)/2+1}} = O(s^{-\theta}), \quad i = 1, \cdots, k.
\] (21)

It is easy to see that (21) has a solution $\tilde{T} = (\tilde{T}_{s,1}, \cdots, \tilde{T}_{s,k})$ in
\[ W_k = \{ \tilde{T} = (\tilde{T}_1, \cdots, \tilde{T}_k) : \tilde{T}_i \in [T_0 - Ls^{-\theta}, T_0 + Ls^{-\theta}] \} \]

By Lemma 9, we see that the matrix $\left( \frac{\partial^2 K(x, \mu)}{\partial \mu_i \partial \mu_j} \right)$ is positive for any $\tilde{T}$ in $W_k$.

Thus (20) has a unique solution in $W_k$ and by the uniqueness of the solution of (20) in $W_k$, we deduce that $\mu(x) = C^1$. \qed

Proof of Theorem 1.2. Let $\tilde{K}(x) = K(x, \mu(x))$, $x \in V_k$, where $\mu(x)$ is the map obtained in Proposition 2. We consider the following problem:
\[
\max_{x \in V_k} \tilde{K}(x).
\]

Let $x_s \in V_k$ be a maximum point of $\tilde{K}(x)$ in $V_k$. We will prove that $x_s$ is an interior point of $\tilde{K}(x)$, thus $x_s$ is a critical point of $\tilde{K}(x)$.

Let $x_0$ be a maximum point of $\varphi_1$. Choose $\tilde{x}_{s,j} \in B_{s^{-\theta}}(x_0)$, $j = 1, \cdots, k$, satisfying $|\tilde{x}_{s,i} - \tilde{x}_{s,j}| \geq c's^{-\theta}$, $\forall i \neq j$, where $c' > 0$ is a small constant. Then
\[
\varphi_1(x_{s,j}) = 1 + O(|\tilde{x}_{s,j} - x_0|^2) = 1 + O(s^{-2\theta}).
\]

For this $\tilde{x}_s = (\tilde{x}_{s,1}, \cdots, \tilde{x}_{s,k}) \in V_k$, from (19), we have
\[
\tilde{K}(\tilde{x}_s) = kA - k \left( \frac{\lambda B_0}{T_0^{2m}} - \frac{B_2}{T_0^{(N-2m)/2}} \right) s^{-4m/(N-6m)} + O(s^{-4m/(N-6m) - 2\theta}). \] (22)

In the following, we proceed a contradiction argument.

Suppose that there exists $i \neq j$ such that $|x_{s,i} - x_{s,j}| = \left( \frac{1}{(N-6m)/(N-6m) - \frac{1}{2m}} \right) \frac{1}{s^{2m}}$.

Denote
\[
\mu_i(x_s) = \tilde{T}_{s,i} s^{2/(N-6m)}, \quad \tilde{T}_{s,i} \in [T_0 - Ls^{-\theta}, T_0 + Ls^{-\theta}].
\]

We have
\[
- \frac{\lambda B_0}{T_{s,i}^{2m}} + \frac{B_2}{T_{s,i}^{(N-2m)/2}} = - \frac{\lambda B_0}{T_0^{2m}} + \frac{B_2}{T_0^{(N-2m)/2}} + O(s^{-2\theta}),
\] (23)
since $T_0$ is the unique minimum point of $- \frac{\lambda B_0}{T^{2m}} + \frac{B_2}{T^{(N-2m)/2}}$. 


By (19), we have
\[\hat{K}(x_s) \leq kA - \sum_{i=1}^{k} \left( \frac{\lambda B_0}{\mu_i^{1/m}(x_s)} - \frac{B_2}{\mu_i^{(N-2m)/2}(x_s)} \right) s^{-4m/(N-6m)} \]
\[- B_3 \varepsilon_{ij} + O(s^{-4(1+\sigma)/(N-6m)}) \]
\[= kA - k \left( \frac{\lambda B_0}{T_0^{2m}} - \frac{B_2}{T_0^{(N-2m)/2}} \right) s^{-4m/(N-6m)} \]
\[- B_3 \varepsilon_{ij} + O(s^{-4m/(N-6m)-\theta}) \]
\[= kA - k \left( \frac{\lambda B_0}{T_0^{2m}} - \frac{B_2}{T_0^{(N-2m)/2}} \right) s^{-4m/(N-6m)} \]
\[- B_3 s^{-4m/(N-6m)-\theta} + O(s^{-4m/(N-6m)-\theta}) \]
\[< \hat{K}(\tilde{x}_s).\]

This is a contradiction since \(x_s\) is a maximum point of \(\hat{K}\) in \(V_k\).

Suppose that \(\varphi_1(x_{s,i}) = 1 - s^{-\theta}\) for some \(i = 1, \cdots, k\).

From (19) and (23), we obtain
\[\hat{K}(x_s) \leq kA + \sum_{i=1}^{k} \left( - \frac{\lambda B_0}{T_0^{2m}} + \frac{B_2}{T_0^{(N-2m)/2}} \right) s^{-4m/(N-6m)} + O(s^{-4m/(N-6m)-\theta}) \]
\[- B_2 \varepsilon_{ij} + O(s^{-4m/(N-6m)-\theta}) \]
\[= kA + \sum_{i=1}^{k} \left( - \frac{\lambda B_0}{T_0^{2m}} + \frac{B_2}{T_0^{(N-2m)/2}} \right) s^{-4m/(N-6m)} \]
\[- B_2 s^{-4m/(N-6m)-\theta} + O(s^{-4m/(N-6m)-\theta}) \]
\[= kA + k \left( - \frac{\lambda B_0}{T_0^{2m}} + \frac{B_2}{T_0^{(N-2m)/2}} \right) s^{-4m/(N-6m)} \]
\[- B_2 s^{-4m/(N-6m)-\theta} + O(s^{-4m/(N-6m)-\theta}) \]
\[< \hat{K}(\tilde{x}_s).\]

This is a contradiction.

Thus we have proved that \(x_s\) is an interior point of \(V_k\). As a result, \(x_s\) is a critical point of \(\hat{K}(x)\).

\[\square\]

Appendix A. Some basic estimates. Let \(x_j \in \Omega\) and \(\mu_j > 0, j = 1, \cdots, k\).

In this section, we assume that \(d(x_j, \partial \Omega) \geq \delta > 0\), where \(\delta\) is a small constant, \(\mu_i \mu_j |x_i - x_j|^2\) is large for \(i \neq j\), \(s \mu_j^{-4(N-2m)/2}\) is small, and \(0 < c_1 \leq \frac{\mu_i}{\mu_j} \leq c_2 < +\infty\).

Denote \(\psi_{x_j,\mu_j} = U_{x_j,\mu_j} - PU_{x_j,\mu_j}\). We have the following estimates.

**Lemma A.1.** Assume that \(d(x_j, \partial \Omega) \geq \delta > 0\). We have the following estimates for \(\psi_{x_j,\mu_j}\):
\[\psi_{x_j,\mu_j}(y) = O \left( \frac{1}{\mu_j^{(N-2m)/2}} \right).\]
\[
\frac{\partial \psi_{x_j,\mu_j}(y)}{\partial \mu_j} = O \left( \frac{1}{\mu_j^{(N-2m)/2+1}} \right), \quad \frac{\partial^2 \psi_{x_j,\mu_j}(y)}{\partial \mu_j^2} = O \left( \frac{1}{\mu_j^{(N-2m)/2+2}} \right),
\]

and
\[
\frac{\partial \psi_{x_j,\mu_j}(y)}{\partial x_j, h} = O \left( \frac{1}{\mu_j^{(N-2m)/2}} \right), \quad h = 1, \ldots, N.
\]

**Proof.** By Green’s representation, we have
\[
\psi_{x_j,\mu_j}(y) = U_{x_j,\mu_j}(y) - PU_{x_j,\mu_j}(y)
\]
\[
= \int_{\mathbb{R}^N} F(y, x)U_{x_j,\mu_j}^{m^*}(x)dx - \int_{B_1} G(y, x)U_{x_j,\mu_j}^{m^*}(x)dx
\]
\[
= \int_{B_1} H(y, x)U_{x_j,\mu_j}^{m^*}(x)dx + \int_{B_1^c} F(y, x)U_{x_j,\mu_j}^{m^*}(x)dx
\]
\[
: = \psi_1 + \psi_2,
\]
where \( F(y, x) = \frac{\delta_0}{|y-x|^{N-2m}} \) is the fundamental solution of \((-\Delta)^m\) in \(\mathbb{R}^N\), \(G(y, x)\) is the Green function of \((-\Delta)^m\) in \(B_1\), \(H(y, x)\) is the regular part of \(G(y, x)\).

By Taylor’s expansion, we have
\[
\psi_1 = \int_{B_1} H(y, x_j)U_{x_j,\mu_j}^{m^*}(x)dx + \int_{B_1} \langle \nabla H(y, x_j), x - x_j \rangle U_{x_j,\mu_j}^{m^*}(x)dx
\]
\[
+ O \left( \int_{B_1} |x - x_j|^2 U_{x_j,\mu_j}^{m^*}(x)dx \right)
\]
\[
= H(y, x_j) \int_{B_1} U_{x_j,\mu_j}^{m^*}(x)dx + \int_{B_1 \setminus B_{d_j}(x_j)} \langle \nabla H(y, x_j), x - x_j \rangle U_{x_j,\mu_j}^{m^*}(x)dx
\]
\[
+ O \left( \int_{B_1} |x - x_j|^2 U_{x_j,\mu_j}^{m^*}(x)dx \right),
\]
where \(d_j = \text{dist}(x_j, \partial B_1)\).

On the one hand, we have
\[
\int_{B_1} U_{x_j,\mu_j}^{m^*}(x)dx = \int_{\mathbb{R}^N} U_{x_j,\mu_j}^{m^*}(x)dx - \int_{B_1^c} U_{x_j,\mu_j}^{m^*}(x)dx
\]
\[
= \frac{1}{\mu_j^{(N-2m)/2}} \int_{\mathbb{R}^N} U_{x_j,\mu_j}^{m^*}dx + O \left( \frac{1}{\mu_j^{(N+2m)/2} \partial_{x_j}^2} \right).
\]

By change of variables, we have
\[
\int_{B_1 \setminus B_{d_j}(x_j)} \langle \nabla H(y, x_j), x - x_j \rangle U_{x_j,\mu_j}^{m^*}(x)dx = O \left( \frac{1}{\mu_j^{(N+2m)/2} \partial_{x_j}^2} \right),
\]
\[
\int_{B_1} |x - x_j|^2 U_{x_j,\mu_j}^{m^*}(x)dx = O \left( \frac{1}{\mu_j^{(N+2m)/2} \partial_{x_j}^2} \right),
\]
Combining (26), (27) and (28), we obtain that
\[
\psi_1 = \frac{H(y, x_j)}{\mu_j^{(N-2m)/2}} \int_{\mathbb{R}^N} U_{x_j,\mu_j}^{m^*}dx + H(y, x_j)O \left( \frac{1}{\mu_j^{(N+2m)/2} \partial_{x_j}^2} \right) + O \left( \frac{1}{\mu_j^{(N+2m)/2} \partial_{x_j}^2} \right).
\]
For $\psi_2$, we have

$$\psi_2 = O \left( \frac{1}{\mu_j^{(N+2m)/2} d_j^m} \right).$$

Thus, we obtain that

$$\psi_{x_j, \mu_j}(y) = \frac{H(y, x_j)}{\mu_j^{(N-2m)/2}} \int_{\mathbb{R}^N} U_j^{m-1} dx + H(y, x_j) O \left( \frac{1}{\mu_j^{(N+2m)/2} d_j^m} \right) + O \left( \frac{1}{\mu_j^{(N+2m)/2} d_j^m} \right).$$

Similarly, we have

$$\frac{\partial \psi_{x_j, \mu_j}}{\partial \mu_j}(y) = (m^* - 1) \left[ \int_{B_1} H(y, x) U_j^{m^*-2}(x) \frac{\partial U_{x_j, \mu_j}}{\partial \mu_j}(x) dx 
+ \int_{B_1^c} F(y, x) U_j^{m^*-2}(x) \frac{\partial U_{x_j, \mu_j}}{\partial \mu_j}(x) dx \right]$$

$$= (m^* - 1) \left[ \int_{B_1} H(y, x_j) U_j^{m^*-2}(x) \frac{\partial U_{x_j, \mu_j}}{\partial \mu_j}(x) dx 
+ \int_{B_1 \setminus B_{\delta_j}(x_j)} |\nabla H(y, x), x - x_j| U_j^{m^*-2}(x) \frac{\partial U_{x_j, \mu_j}}{\partial \mu_j}(x) dx 
+ O \left( \int_{B_1} |x - x_j| 2 U_j^{m^*-2}(x) \frac{\partial U_{x_j, \mu_j}}{\partial \mu_j}(x) dx \right) 
+ O \left( \frac{1}{\mu_j^{(N+2m)/2+1} d_j^m} \right) \right].$$

$$= (m^* - 1) \frac{H(y, x_j)}{\mu_j^{(N-2m)/2+1}} \int_{\mathbb{R}^N} U_j^{m^*-2} \frac{\partial U_{0, \mu}}{\partial \mu} \bigg|_{\mu = 1}$$

$$+ H(y, x_j) O \left( \frac{1}{\mu_j^{(N+2m)/2+1} d_j^m} \right) + O \left( \frac{1}{\mu_j^{(N+2m)/2+1} d_j^m} \right)$$

$$= O \left( \frac{1}{\mu_j^{(N-2m)/2+1}} \right).$$

$$\frac{\partial^2 \psi_{x_j, \mu_j}}{\partial \mu_j^2}(y) = (m^* - 1)(m^* - 2) \left[ \int_{B_1} H(y, x) U_j^{m^*-3}(x) \left( \frac{\partial U_{x_j, \mu_j}}{\partial \mu_j} \right)^2(x) dx 
+ \int_{B_1^c} F(y, x) U_j^{m^*-3}(x) \left( \frac{\partial U_{x_j, \mu_j}}{\partial \mu_j} \right)^2(x) dx \right]$$

$$+ (m^* - 1) \left[ \int_{B_1} H(y, x) U_j^{m^*-2}(x) \frac{\partial^2 U_{x_j, \mu_j}}{\partial \mu_j^2}(x) dx \right].$$
\[ + \int_{B_1} F(y, x) U_{x_j, \mu_j}^{m^*-2} \frac{\partial^2 U_{x_j, \mu_j}}{\partial \mu_j^2}(x) dx \]

\[ = (m^*-1)(m^*-2) \int_{B_1} H(y, x) U_{x_j, \mu_j}^{m^*-3}(x) \left( \frac{\partial U_{x_j, \mu_j}}{\partial \mu_j} \right)^2(x) dx \]

\[ + (m^*-1) \int_{B_1} H(y, x) U_{x_j, \mu_j}^{m^*-2} \frac{\partial^2 U_{x_j, \mu_j}}{\partial \mu_j^2}(x) dx + O \left( \frac{1}{\mu_j^{(N+2m)/2+2} d_j^m} \right) \]

\[ = (m^*-1)(m^*-2) \int_{B_1} \frac{H(y, x)}{\mu_j^{(N-2m)/2+2}} \int_{R^N} U_{x_j, \mu_j}^{m^*-3} \left( \frac{\partial U_{0, \mu}}{\partial \mu} \bigg|_{\mu=1} \right)^2 dx \]

\[ + (m^*-1) \int_{B_1} \frac{H(y, x)}{\mu_j^{(N-2m)/2+2}} \int_{R^N} U_{x_j, \mu_j}^{m^*-2} \frac{\partial^2 U_{0, \mu}}{\partial \mu^2} \bigg|_{\mu=1} dx \]

\[ + H(y, x_j) O \left( \frac{1}{\mu_j^{(N+2m)/2+2} d_j^m} \right) + O \left( \frac{1}{\mu_j^{(N+2m)/2+2} d_j^m} \right) \]

\[ = O \left( \frac{1}{\mu_j^{(N-2m)/2+2}} \right), \]

and

\[ \frac{\partial \psi_{x_j, \mu_j}}{\partial x_j, h}(y) \]

\[ = \frac{\partial H(y, x_j)}{\partial x_j, h} \frac{1}{\mu_j^{(N-2m)/2}} \int_{R^N} U_{x_j, \mu_j}^{m^*-1}(x) dx \]

\[ + \frac{\partial H(y, x_j)}{\partial x_j, h} O \left( \frac{1}{\mu_j^{(N+2m)/2} d_j^m} \right) + O \left( \frac{1}{\mu_j^{(N+2m)/2} d_j^m} \right) \]

\[ = O \left( \frac{1}{\mu_j^{(N-2m)/2}} \right). \]

\[ \square \]

**Appendix B. Energy expansions.**

**Proposition 3.** Assume \( N \geq 6m + 1 \). Then

\[ I(PU_{x_j, \mu_j}) = A - \frac{B_0 \lambda}{\mu_j^{2m}} + \frac{B_2 \phi_1(x_j) s}{\mu_j^{(N-2m)/2}} \]

\[ + O \left( \frac{1}{\mu_j^{N-2m}} + \left( \frac{s}{\mu_j^{(N-2m)/2}} \right)^{1+\sigma} + \frac{s}{\mu_j^{(N-2m)/2+1}} \right), \]

where

\[ B_0 = \frac{1}{2} \int_{R^N} U^2 dy, \quad B_2 = \int_{R^N} U^{m^*-1} dy, \]

\( \sigma \) is some positive constant and
Proof. Without loss of generality, we assume $m$ is even. 

\[
A = \begin{cases} 
\frac{1}{2} \int_{\mathbb{R}^N} |\Delta \frac{\partial}{\partial x} U|^2 dy - \frac{1}{m^*} \int_{\mathbb{R}^N} U^{m^*} dy, & \text{if } m \text{ is even}, \\
\frac{1}{2} \int_{\mathbb{R}^N} |\Delta \frac{m-1}{m} U|^2 dy - \frac{1}{m^*} \int_{\mathbb{R}^N} U^{m^*} dy, & \text{if } m \text{ is odd}. 
\end{cases}
\] (32)

By Lemma A.1, we have

\[
I(\text{PU}_{x_j,\mu_j}) = \frac{1}{2} \int_{B_1} |\Delta \frac{\partial}{\partial x} \text{PU}_{x_j,\mu_j}|^2 - \lambda (\text{PU}_{x_j,\mu_j})^2 dy \\
- \frac{1}{m^*} \int_{B_1} (\text{PU}_{x_j,\mu_j} - s\varphi_1)^{m^*} dy \\
= \frac{1}{2} \int_{B_1} |\Delta \frac{\partial}{\partial x} \text{PU}_{x_j,\mu_j}|^2 dy - \lambda \int_{B_1} (\text{PU}_{x_j,\mu_j})^2 dy \\
- \frac{1}{m^*} \int_{B_1} (\text{PU}_{x_j,\mu_j} - s\varphi_1)^{m^*} dy \\
:= I_1 - I_2 - I_3. 
\] (33)

By Lemma A.1, we have

\[
I_1 = \frac{1}{2} \int_{B_1} \text{PU}_{x_j,\mu_j}^{m^*-1} \text{PU}_{x_j,\mu_j} dy \\
= \frac{1}{2} \int_{B_1} \text{PU}_{x_j,\mu_j}^{m^*} dy - \frac{1}{2} \int_{B_1} \text{PU}_{x_j,\mu_j}^{m^*-1} \psi_{x_j,\mu_j} dy \\
= \frac{1}{2} \int_{\mathbb{R}^N} \text{PU}_{x_j,\mu_j}^{m^*} dy + O\left(\frac{1}{\mu_j^{N-2m}}\right) \\
= \frac{1}{2} \int_{\mathbb{R}^N} |\Delta \frac{\partial}{\partial x} U|^2 dy + O\left(\frac{1}{\mu_j^{N-2m}}\right), 
\] (34)

\[
I_2 = \frac{\lambda}{2} \int_{B_1} U^2_{x_j,\mu_j} + O\left(\lambda \int_{B_1} \text{PU}_{x_j,\mu_j} \psi_{x_j,\mu_j} + \lambda \int_{B_1} \psi_{x_j,\mu_j}^2 \right) \\
= \frac{\lambda}{2} \int_{\mathbb{R}^N} U^2_{x_j,\mu_j} + O\left(\lambda \int_{B_1} U^2_{x_j,\mu_j} + \lambda \int_{B_1} U \psi_{x_j,\mu_j} + \lambda \int_{B_1} \psi_{x_j,\mu_j}^2 \right) \\
= \frac{\lambda}{2} \int_{\mathbb{R}^N} U^2_{x_j,\mu_j} + O\left(\frac{\lambda}{\mu_j^{N-2m}}\right) \\
= \frac{\lambda}{2\mu_j^{2m}} \int_{\mathbb{R}^N} U^2 + O\left(\frac{\lambda}{\mu_j^{N-2m}}\right). 
\] (35)

For $I_3$, we have

\[
\int_{B_1} (\text{PU}_{x_j,\mu_j} - s\varphi_1)^{m^*} dy \\
= \int_{B_1} (U_{x_j,\mu_j} - s\varphi_1)^{m^*} dy + O\left(\int_{B_1} (U_{x_j,\mu_j} - s\varphi_1)^{m^*-1} \psi_{x_j,\mu_j} dy \right) \\
= \int_{B_1} (U_{x_j,\mu_j} - s\varphi_1)^{m^*} dy + O\left(\frac{1}{\mu_j^{N-2m}}\right).
\]
where $B$ is a bounded positive function.

**Proposition 4.** Assume $\sigma > 1$. The results follows from (34), (35) and (36).

where we used for any positive numbers $a$ and $b$, 

$$(a - b)^+ = a^m - m^* a^{m^* - 1} b + O(a^{m^* - 1} b^1 + \sigma),$$

where $\sigma > 0$ is any fixed constant in $(0, 1)$. Thus,

$$I_3 = \frac{1}{m^*} \int_{\mathbb{R}^N} U^{m^*} dy - B_2 s^3 \sum_{j} \frac{\sigma}{\mu^*_j} \varphi_1(x_j) y_j + O \left( \frac{1}{\mu^*_j} \frac{s}{\mu^*_j} \frac{1}{\mu^*_j} \frac{1}{\mu^*_j} \frac{1}{\mu^*_j} \right).$$

The results follows from (34), (35) and (36).

**Proposition 4.** Assume $N \geq 6m + 1$. We have

$$I \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} \right) = kA - \sum_{j=1}^{k} \left( \frac{B_0 \lambda}{\mu^*_j} - \frac{B_2 \varphi_1(x_j)}{\mu^*_j} \right) - \sum_{i \neq j} \frac{B_3 + \tilde{B}(x_i, x_j)}{\mu^*_j} \varepsilon_{ij} + O \left( \sum_{i < j} \frac{1}{\mu^*_j} \frac{s}{\mu^*_j} \frac{1}{\mu^*_j} \frac{1}{\mu^*_j} \right),$$

where $B_3 > 0$ and $\sigma > 0$ are some constants and $\tilde{B}(x_i, x_j) = \varepsilon_{ij}^{-1} \int_{B_1} U_{x_i, \mu_i} U_{x_j, \mu_j} dy$ is a bounded positive function.

**Proof.** Assume $m$ is even.

$$I \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} \right) = \sum_{j=1}^{k} I(PU_{x_j, \mu_j}) + \frac{1}{2} \sum_{i \neq j} \int_{B_1} \left( \Delta^{m^*}_2 PU_{x_j, \mu_j} \Delta^{m^*}_2 PU_{x_j, \mu_j} - \lambda PU_{x_i, \mu_i} PU_{x_j, \mu_j} \right) dy$$

$$- \frac{1}{m^*} \int_{B_1} \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} - s \varphi_1 \right)^{m^*} - \sum_{j=1}^{k} \left( PU_{x_j, \mu_j} - s \varphi_1 \right)^{m^*} dy.$$
For $i \neq j$, similar to the estimate in [2], we have
\[
\int_{B_1} \Delta \delta \, PU_{x_i, \mu_i} \Delta \delta \, PU_{x_j, \mu_j} \, dy
= \int_{B_1} U_{x_i, \mu_i}^{m^*} \, PU_{x_j, \mu_j} \, dy
= \int_{B_1} U_{x_i, \mu_i}^{m^*} \, PU_{x_j, \mu_j} \, dy - \int_{B_1} U_{x_i, \mu_i}^{m^*} \, \phi_{x_j, \mu_j} \, dy
= \int_{B_1} U_{x_i, \mu_i}^{m^*} \, PU_{x_j, \mu_j} \, dy + O \left( \frac{1}{\mu_i^{N-2m}} + \frac{1}{\mu_j^{N-2m}} \right)
= c_0^m B_2 \varepsilon_{ij} + O \left( \varepsilon_{ij}^{1+\sigma} + \frac{1}{\mu_i^{N-2m}} + \frac{1}{\mu_j^{N-2m}} \right).
\]
\[
\int_{B_1} PU_{x_i, \mu_i} \, PU_{x_j, \mu_j} \, dy
= \int_{B_1} U_{x_i, \mu_i} \, PU_{x_j, \mu_j} \, dy + O \left( \frac{1}{\mu_i^{N-2m}} + \frac{1}{\mu_j^{N-2m}} \right)
= B(x_i, x_j) \varepsilon_{ij} + O \left( \frac{1}{\mu_i^{N-2m}} + \frac{1}{\mu_j^{N-2m}} \right).
\]
By Lemma A.1, we have
\[
\int_{B_1} \left( \sum_{j=1}^k PU_{x_j, \mu_j} - s \phi_1 \right)^{m^*} \, dy
= \int_{B_1} \left( \sum_{j=1}^k U_{x_j, \mu_j} - s \phi_1 \right)^{m^*} \, dy + O \left( \sum_{j=1}^k \frac{1}{\mu_j^{N-2m}} \right)
= \int_{B_1} \left( \sum_{j=1}^k U_{x_j, \mu_j} \right)^{m^*} \, dy - \sum_{j=1}^k \left( U_{x_j, \mu_j} - s \phi_1 \right)^{m^*} \, dy
+ O \left( \sum_{j=1}^k \frac{1}{\mu_j^{N-2m}} \right).
\]
\[
= m^* c_0^m B_2 \sum_{i \neq j} \varepsilon_{ij} + O \left( \sum_{i \neq j} \varepsilon_{ij}^{1+\sigma} + \sum_{j=1}^k \frac{1}{\mu_j^{(1+\sigma)(N-2m)/2}} + \sum_{j=1}^k \frac{1}{\mu_j^{N-2m}} \right).
\]
Combining Proposition 3, we finish the proof. \qed
Proposition 5. For any \( i = 1, \cdots, k \), we have

\[
\frac{\partial J_s(x, \mu, 0)}{\partial \mu_i} = \frac{2mB_0\lambda}{\mu_i^{2m+1}} - \frac{N - 2m}{2} \frac{B_2\varphi_1(x_1)}{\mu_i^{(N-2m)/2+1}}
\]

\[
+ \frac{1}{\mu_i} O \left( \frac{1}{\mu_i^{N-2m}} + \left( \frac{s}{\mu_i^{(N-2m)/2}} \right)^{1+\sigma} + \frac{s}{\mu_i^{(N-2m)/2+1}} + \sum_{i \neq j} \varepsilon_{ij} \right).
\]

Proof. We have

\[
\frac{\partial J_s(x, \mu, 0)}{\partial \mu_i} = \left( \int_{B_1} \Delta \varphi \sum_{j=1}^k PU_{x_1, \mu_j} \frac{\partial (PU_{x, \mu_i})}{\partial \mu_i} \right)
\]

\[
= \int_{B_1} \Delta \varphi \sum_{j=1}^k PU_{x_1, \mu_j} \Delta \varphi \frac{\partial (PU_{x, \mu_i})}{\partial \mu_i} dy
\]

\[
- \lambda \int_{B_1} \left( \sum_{j=1}^k PU_{x_1, \mu_j} \right) \frac{\partial (PU_{x, \mu_i})}{\partial \mu_i} dy
\]

\[
- \int_{B_1} \left( \sum_{j=1}^k PU_{x_1, \mu_j} - s\varphi_1 \right) m_*^{-1} \frac{\partial (PU_{x, \mu_i})}{\partial \mu_i} dy
\]

\[
= \int_{B_1} \Delta \varphi PU_{x_1, \mu_i} \Delta \varphi \frac{\partial (PU_{x, \mu_i})}{\partial \mu_i} dy - \lambda \int_{B_1} PU_{x_1, \mu_i} \frac{\partial (PU_{x, \mu_i})}{\partial \mu_i} dy
\]

\[
- \int_{B_1} \left( PU_{x_1, \mu_i} - s\varphi_1 \right) m_*^{-1} \frac{\partial (PU_{x, \mu_i})}{\partial \mu_i} dy + O \left( \sum_{j \neq i} \varepsilon_{ij} \right),
\]

(38)

and

\[
\int_{B_1} \Delta \varphi PU_{x_1, \mu_i} \Delta \varphi \frac{\partial (PU_{x, \mu_i})}{\partial \mu_i} dy - \lambda \int_{B_1} PU_{x_1, \mu_i} \frac{\partial (PU_{x, \mu_i})}{\partial \mu_i} dy
\]

\[
= \frac{2mB_0\lambda}{\mu_i^{2m+1}} + \frac{1}{\mu_i} O \left( \frac{1}{\mu_i^{N-2m}} \right).
\]

(39)

On the other hand, from Lemma A.1, we have

\[
\int_{B_1} \left( PU_{x_1, \mu_i} - s\varphi_1 \right) m_*^{-1} \frac{\partial (PU_{x, \mu_i})}{\partial \mu_i} dy
\]

\[
= \int_{B_1} \left( U_{x_1, \mu_i} - s\varphi_1 \right) m_*^{-1} \frac{\partial U_{x, \mu_i}}{\partial \mu_i} dy + O \left( \frac{1}{\mu_i^{N-2m+1}} \right)
\]

\[
= \int_{B_1} U_{x_1, \mu_i} m_*^{-1} \frac{\partial U_{x, \mu_i}}{\partial \mu_i} dy - (m_* - 1) \int_{B_1} U_{x_1, \mu_i} m_*^{-2} s\varphi_1 \frac{\partial U_{x, \mu_i}}{\partial \mu_i} dy
\]

\[
+ \frac{1}{\mu_i} O \left( \frac{s^{1+\sigma}}{\mu_i^{(1+\sigma)(N-2m)/2}} + \frac{1}{\mu_i^{N-2m}} \right)
\]

\[
= -(m_* - 1) \int_{B_1} U_{x_1, \mu_i} m_*^{-2} s\varphi_1 \frac{\partial U_{x, \mu_i}}{\partial \mu_i} dy + \frac{1}{\mu_i} O \left( \frac{s^{1+\sigma}}{\mu_i^{(1+\sigma)(N-2m)/2}} + \frac{1}{\mu_i^{N-2m}} \right)
\]
The result follows from (38), (39) and (40).

For any Proposition 7.

Similarly, we can prove the following:

**Proposition 6.** For any $i = 1, \cdots, k$, $h = 1, \cdots, N$, we have

$$\frac{\partial J_s(x, \mu, 0)}{\partial x_i, h} = \mu_i O \left( \frac{1}{\mu_i^{1+\sigma}} \left( \frac{1}{\mu_i^{N-2m}/2} + \frac{1}{\mu_i^{N-2m}/2+1} + \sum_{i \neq j} \varepsilon_{ij} \right) \right),$$

where $x_i = (x_{i,1}, \cdots, x_{i,N})$.

**Proposition 7.** For any $i = 1, \cdots, k$, we have

$$\frac{\partial^2 J_s(x, \mu, 0)}{\partial \mu_i^2} = \mu_i O \left( \frac{1}{\mu_i^{1+\sigma}} \left( \frac{1}{\mu_i^{N-2m}/2} + \frac{1}{\mu_i^{N-2m}/2+1} + \sum_{i \neq j} \varepsilon_{ij} \right) \right) \sum_{j=1}^{k} \frac{1}{\mu_j^2}.$$

If $i \neq j$, then

$$\frac{\partial^2 J_s(x, \mu, 0)}{\partial \mu_i \partial \mu_j} = O \left( \sum_{h \neq i} \varepsilon_{hi} \right) \sum_{l=1}^{k} \frac{1}{\mu_l^2}.$$

**Proof.** Assume $m$ is even.

$$\frac{\partial^2 J_s(x, \mu, 0)}{\partial \mu_i^2} = I'_s \left( \sum_{l=1}^{k} \frac{\partial (PU_{x_i, \mu_l})}{\partial \mu_i}, \frac{\partial (PU_{x_i, \mu_l})}{\partial \mu_i} \right)$$

$$+ \left( I'_s \left( \sum_{l=1}^{k} PU_{x_i, \mu_l}, \frac{\partial^2 (PU_{x_i, \mu_l})}{\partial \mu_i^2} \right) \right)$$

$$= \int_{B_1} \left| \Delta \frac{\partial (PU_{x_i, \mu_l})}{\partial \mu_i} \right|^2 dy - \lambda \int_{B_1} \left( \frac{\partial (PU_{x_i, \mu_l})}{\partial \mu_i} \right)^2 dy$$

$$- (m^* - 1) \int_{B_1} \left( \sum_{l=1}^{k} PU_{x_i, \mu_l} - s\varphi_1 \right)^{m^*-2} \left( \frac{\partial (PU_{x_i, \mu_l})}{\partial \mu_i} \right)^2 dy \quad (42)$$

$$+ \int_{B_1} \Delta \frac{\partial (PU_{x_i, \mu_l})}{\partial \mu_i} \Delta \frac{\partial^2 (PU_{x_i, \mu_l})}{\partial \mu_i^2} dy$$

$$- \lambda \int_{B_1} \left( \sum_{l=1}^{k} PU_{x_i, \mu_l} \right) \frac{\partial^2 (PU_{x_i, \mu_l})}{\partial \mu_i^2} dy$$

$$- \int_{B_1} \left( \sum_{l=1}^{k} PU_{x_i, \mu_l} - s\varphi_1 \right)^{m^*-1} \frac{\partial^2 (PU_{x_i, \mu_l})}{\partial \mu_i^2} dy.$$


Firstly, we have

\[
\int_{B_1} \left| \frac{\Delta \bar{\varphi}}{\partial \mu_i} \right|^2 dy + \int_{B_1} \frac{\Delta \bar{\varphi}}{\partial \mu_i} \left( \sum_{l=1}^{k} \frac{\partial (PU_{x_1, \mu_l})}{\partial \mu_i} \right) dy
\]

\[
= (m^* - 1) \int_{B_1} U_{x_1, \mu_l}^{m^*-2} \frac{\partial U_{x_1, \mu_l}}{\partial \mu_i} dy + \int_{B_1} \bar{U}_{x_1, \mu_l}^{m^*-1} \frac{\partial^2 (PU_{x_1, \mu_l})}{\partial \mu_i^2} dy + O \left( \sum_{l \neq i} \frac{\varepsilon_{li}}{\mu_i^2} \right)
\]

\[
= (m^* - 1) \int_{\mathbb{R}^N} U_{x_1, \mu_i}^{m^*-2} \left( \frac{\partial U_{x_1, \mu_i}}{\partial \mu_i} \right)^2 dy + \int_{\mathbb{R}^N} \bar{U}_{x_1, \mu_i}^{m^*-1} \frac{\partial^2 U_{x_1, \mu_i}}{\partial \mu_i^2} dy + O \left( \sum_{l \neq i} \frac{\varepsilon_{li}}{\mu_i^2} + \frac{1}{\mu_i^{N-2m+2}} \right)
\]

\[
= O \left( \sum_{l \neq i} \frac{\varepsilon_{li}}{\mu_i^2} + \frac{1}{\mu_i^{N-2m+2}} \right).
\]

Secondly,

\[
(m^* - 1) \int_{B_1} \left( \sum_{l=1}^{k} PU_{x_1, \mu_l} - s \varphi_1 \right)^{m^*-2} \left( \frac{\partial (PU_{x_1, \mu_l})}{\partial \mu_i} \right)^2 dy
\]

\[
+ \int_{B_1} \left( \sum_{l=1}^{k} PU_{x_1, \mu_l} - s \varphi_1 \right)^{m^*-1} \frac{\partial^2 (PU_{x_1, \mu_l})}{\partial \mu_i^2} dy
\]

\[
= (m^* - 1) \int_{\mathbb{R}^N} U_{x_1, \mu_i}^{m^*-2} \left( \frac{\partial U_{x_1, \mu_i}}{\partial \mu_i} \right)^2 dy + \int_{\mathbb{R}^N} \bar{U}_{x_1, \mu_i}^{m^*-1} \frac{\partial^2 U_{x_1, \mu_i}}{\partial \mu_i^2} dy
\]

\[
- s \varphi_1 (x_i) (m^* - 1)(m^* - 2) \int_{\mathbb{R}^N} U_{x_1, \mu_i}^{m^*-3} \left( \frac{\partial U_{x_1, \mu_i}}{\partial \mu_i} \right)^2 dy
\]

\[
+ (m^* - 1) \int_{\mathbb{R}^N} U_{x_1, \mu_i}^{m^*-2} \frac{\partial^2 U_{x_1, \mu_i}}{\partial \mu_i^2} dy + O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \frac{k}{\mu_j} \frac{s}{(N-2m)^{2+\sigma}} + \sum_{j} \left( s \mu_j^{-(N-2m)/2+\sigma} + \sum_{l \neq j} \varepsilon_{ij} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^{2}} \right).
\]

that is

\[
(m^* - 1) \int_{B_1} \left( \sum_{l=1}^{k} PU_{x_1, \mu_l} - s \varphi_1 \right)^{m^*-2} \left( \frac{\partial (PU_{x_1, \mu_l})}{\partial \mu_i} \right)^2 dy
\]

\[
+ \int_{B_1} \left( \sum_{l=1}^{k} PU_{x_1, \mu_l} - s \varphi_1 \right)^{m^*-1} \frac{\partial^2 (PU_{x_1, \mu_l})}{\partial \mu_i^2} dy
\]

\[
= \frac{1}{m^*} \frac{\partial^2}{\partial \mu_i^2} \int_{\mathbb{R}^N} U_{x_1, \mu_i}^{m^*-1} dy - s \varphi_1 (x_i) \frac{\partial^2}{\partial \mu_i^2} \int_{\mathbb{R}^N} U_{x_1, \mu_i}^{m^*-2} dy
\]
Finally,
\[
\begin{align*}
\lambda \int_{B_1} \left( \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i} \right)^2 dy + \lambda \int_{B_1} \left( \sum_{l=1}^k PU_{x_l,\mu_l} \right) \frac{\partial^2 (PU_{x_i,\mu_i})}{\partial \mu_i^2} dy \\
= \lambda \int_{B_1} \left( \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i} \right)^2 dy + \lambda \int_{B_1} PU_{x_i,\mu_i} \frac{\partial^2 (PU_{x_i,\mu_i})}{\partial \mu_i^2} dy \\
+ O \left( \sum_{j=1}^k \frac{1}{\mu_j^{2m+1}} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^k \frac{1}{\mu_j} \\
= \frac{1}{2} \lambda \frac{\partial^2}{\partial \mu_i^2} \int_{\mathbb{R}^N} U_{x_i,\mu_i}^2 dy + O \left( \sum_{j=1}^k \frac{1}{\mu_j^{N-2m+2}} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^k \frac{1}{\mu_j^2} \\
= \frac{2m(2m+1)B_0 \lambda}{\mu_i^{2m+2}} + O \left( \sum_{j=1}^k \frac{1}{\mu_j^{N-2m+2}} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^k \frac{1}{\mu_j^2} \\
\end{align*}
\]

The result follows from (42) \sim (46).

**Proposition 8.** Assume \( N \geq 6m + 1 \). Then
\[
\frac{\partial K(x, \mu)}{\partial \mu_i} = 2m \lambda B_0 \mu_i^{2m+1} + \frac{N - 2m}{2} \frac{B_2 \varphi_1(x_i)s}{\mu_i^{(N-2m)/2+1}} \\
+ O \left( \sum_{j=1}^k \frac{1}{\mu_j^{2m+1}} + \sum_{j=1}^k \frac{s}{\mu_j^{(N-2m)/2+1}} + \sum_{j=1}^k \frac{(s \mu_j^{-(N-2m)/2})^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij}}{\mu_j} \right) \sum_{j=1}^k \frac{1}{\mu_j}. 
\]

**Proposition 9.** Assume \( N \geq 6m + 1 \). Then
\[
\frac{\partial^2 K(x, \mu)}{\partial \mu_i^2} = -\frac{2m(2m+1)\lambda B_0}{\mu_i^{2m+2}} + \frac{(N - 2m)(N - 2m + 2)}{4} \frac{B_2 \varphi_1(x_i)s}{\mu_i^{(N-2m)/2+2}} \\
+ O \left( \sum_{j=1}^k \frac{1}{\mu_j^{2m+2}} + \sum_{j=1}^k \frac{s}{\mu_j^{(N-2m)/2+2}} + \sum_{j=1}^k \frac{(s \mu_j^{-(N-2m)/2})^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij}}{\mu_j} \right) \sum_{j=1}^k \frac{1}{\mu_j^2},
\]
and if \( i \neq j \),

\[
\frac{\partial^2 K(x, \mu)}{\partial \mu_i \partial \mu_j} = O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \frac{s_j}{\mu_j^{(N-2m)/2+1}} + \sum_{j=1}^{k} \left( s_j \mu_j^{-(N-2m)/2} \right)^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^{2}}.
\]

To prove Proposition 8 and Proposition 9, we need the following lemmas.

**Lemma B.1.** Let \( \omega_{s,x,\mu} \) be the function obtained in Proposition 1. Then for any fixed \( i = 1, \cdots, k \) and \( h = 1, \cdots, N \),

\[
\frac{\partial^j J_s(x, \mu, \omega_{s,x,\mu})}{\partial \mu_i} = 2mB_0 \lambda \mu_i^{2m+1} \frac{N - 2m}{2} \frac{B_2 \varphi_1(x_i)s}{\mu_i^{(N-2m)/2+1}} + \frac{1}{\mu_i} O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \frac{s_j}{\mu_j^{(N-2m)/2+1}} + \sum_{j=1}^{k} \left( s_j \mu_j^{-(N-2m)/2} \right)^{1+\sigma} + \sum_{j \neq l} \varepsilon_{jl} \right).
\]

\[
\frac{\partial^j J_s(x, \mu, \omega_{s,x,\mu})}{\partial x_i,h} = \mu_i O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \frac{s_j}{\mu_j^{(N-2m)/2+1}} + \sum_{j=1}^{k} \left( s_j \mu_j^{-(N-2m)/2} \right)^{1+\sigma} + \sum_{j \neq l} \varepsilon_{jl} \right).
\]

**Proof.** We just prove (48), the proof of (49) is similar. From Proposition 5, we have

\[
\frac{\partial^j J_s(x, \mu, \omega_{s,x,\mu})}{\partial \mu_i} = \left( I_s' \left( \sum_{j=1}^{k} PU_{x_j,\mu_j} + \omega_{s,x,\mu} \right), \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i} \right)
\]

\[
= \left( I_s' \left( \sum_{j=1}^{k} PU_{x_j,\mu_j} \right), \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i} \right) - \lambda \int_{B_1} \omega_{s,x,\mu} \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i} dy
\]

\[
- \int_{B_1} \left[ \left( \sum_{j=1}^{k} PU_{x_j,\mu_j} + \omega_{s,x,\mu} - s \varphi_1 \right)^{m_{s-1}} - \left( \sum_{j=1}^{k} PU_{x_j,\mu_j} - s \varphi_1 \right)^{m_{s-1}} \right] \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i} dy
\]

\[
= 2mB_0 \lambda \frac{N - 2m}{2} \frac{B_2 \varphi_1(x_i)s}{\mu_i^{(N-2m)/2+1}} - \lambda \int_{B_1} \omega_{s,x,\mu} \frac{\partial (PU_{x_i,\mu_i})}{\partial \mu_i} dy.
\]
\[- \int_{B_1} \left[ \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} + \omega_{s,x,\mu} - s\varphi_1 \right)^{m^*-1} + \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} - s\varphi_1 \right)^{m^*-1} \right] \partial(PU_{x_i, \mu_i}) \, dy \]

\[+ \frac{1}{\mu_i} O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \frac{s}{\mu_j^{(N-2m)/2+1}} + \sum_{j=1}^{k} \left( s\mu_j^{-(N-2m)/2} \right)^{1+\sigma} + \sum_{j \neq l} \varepsilon_{jl} \right) \]

On the one hand,

\[\left| \int_{B_1} \omega_{s,x,\mu} \frac{\partial(PU_{x_i, \mu_i})}{\partial \mu_i} \, dy \right| \]

\[\leq \frac{C}{\mu_i} \int_{B_1} |\omega_{s,x,\mu}| |U_{x_i, \mu}| \, dy \]

\[\leq \frac{C}{2m+1} ||\omega_{s,x,\mu}|| \]

\[= \frac{1}{\mu_i} O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \frac{s}{\mu_j^{(N-2m)/2+1}} + \sum_{j=1}^{k} \left( s\mu_j^{-(N-2m)/2} \right)^{1+\sigma} + \sum_{j \neq l} \varepsilon_{jl} \right), \]

and

\[- \int_{B_1} \left[ \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} + \omega_{s,x,\mu} - s\varphi_1 \right)^{m^*-1} - \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} - s\varphi_1 \right)^{m^*-1} \right] \partial(PU_{x_i, \mu_i}) \, dy \]

\[= (m^* - 1) \int_{B_1} \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} - s\varphi_1 \right)^{m^*-2} \omega_{s,x,\mu} \partial(PU_{x_i, \mu_i}) \, dy \]

\[+ O \left( \frac{1}{\mu_i} \int_{B_1} \sum_{j=1}^{k} U_{x_j, \mu_j}^{m^*-2} |\omega_{s,x,\mu}|^2 \, dy \right) \]

\[= (m^* - 1) \int_{B_1} \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} - s\varphi_1 \right)^{m^*-2} \omega_{s,x,\mu} \partial(PU_{x_i, \mu_i}) \, dy \]

\[+ \frac{1}{\mu_i} O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \frac{s}{\mu_j^{(N-2m)/2+1}} + \sum_{j=1}^{k} \left( s\mu_j^{-(N-2m)/2} \right)^{1+\sigma} + \sum_{j \neq l} \varepsilon_{jl} \right). \]
Similar to the estimates before, we have

\[
\int_{B_i} \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} - s \varphi_1 \right)^{m-2} \omega_{s, x, \mu} \frac{\partial (PU_{x_i, \mu_i})}{\partial \mu_i} dy
\]

\[
\int_{B_i} \left( \sum_{j=1}^{k} U_{x_j, \mu_j} - s \varphi_1 \right)^{m-2} \omega_{s, x, \mu} \frac{\partial (U_{x_i, \mu_i})}{\partial \mu_i} dy + O \left( \frac{1}{\mu_i^{4m+1}} \| \omega_{s, x, \mu} \| \right)
\]

\[
\int_{B_i} \left[ \left( \sum_{j=1}^{k} U_{x_j, \mu_j} - s \varphi_1 \right)^{m-2} - U_{x_i, \mu_i}^{m-2} \right] \omega_{s, x, \mu} \frac{\partial U_{x_i, \mu_i}}{\partial \mu_i} dy
\]

\[
+ \frac{1}{\mu_i} O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \frac{s}{\mu_j^{(N-2m)/2+1}} + \sum_{j=1}^{k} \left( s \mu_j^{-(N-2m)/2} \right)^{1+\sigma} + \sum_{j \neq l} \varepsilon_{jl} \right)
\]

\[
= \frac{1}{\mu_i} O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \frac{s}{\mu_j^{(N-2m)/2+1}} + \sum_{j=1}^{k} \left( s \mu_j^{-(N-2m)/2} \right)^{1+\sigma} + \sum_{j \neq l} \varepsilon_{jl} \right).
\]

(53)

Combining (50) ~ (53), we obtain (48).

\[\square\]

**Lemma B.2.** Let \( A_i \) and \( B_{ih} \) be the constants obtained in Proposition 1. Then we have

\[ A_i = O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} s \mu_j^{-(N-2m)/2} \right) \sum_{j=1}^{k} \mu_j, \]

and

\[ B_{ih} = O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} s \mu_j^{-(N-2m)/2} \right) \sum_{j=1}^{k} \frac{1}{\mu_j}. \]

**Proof.** From Proposition 1 and Lemma B.1, we know that \( A_i \) and \( B_{ih} \) satisfy

\[
\sum_{j=1}^{k} \left\langle \frac{\partial PU_{x_j, \mu_j}}{\partial \mu_j}, \frac{\partial PU_{x_i, \mu_i}}{\partial \mu_i} \right\rangle A_j + \sum_{j=1}^{k} \sum_{h=1}^{N} \left\langle \frac{\partial PU_{x_j, \mu_j}}{\partial x_{jh}}, \frac{\partial PU_{x_i, \mu_i}}{\partial \mu_i} \right\rangle B_{jh}
\]

\[= \left\langle \frac{\partial J_s}{\partial \omega}, \frac{\partial PU_{x_i, \mu_i}}{\partial \mu_i} \right\rangle = \frac{1}{\mu_i} O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} s \mu_j^{-(N-2m)/2} + \sum_{j \neq l} \varepsilon_{jl} \right); \]

\[
\sum_{j=1}^{k} \left\langle \frac{\partial PU_{x_j, \mu_j}}{\partial \mu_j}, \frac{\partial PU_{x_i, \mu_i}}{\partial x_{im}} \right\rangle A_j + \sum_{j=1}^{k} \sum_{h=1}^{N} \left\langle \frac{\partial PU_{x_j, \mu_j}}{\partial x_{jh}}, \frac{\partial PU_{x_i, \mu_i}}{\partial x_{im}} \right\rangle B_{jh}
\]

\[= \left\langle \frac{\partial J_s}{\partial \omega}, \frac{\partial PU_{x_i, \mu_i}}{\partial x_{im}} \right\rangle = \mu_i O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} s \mu_j^{-(N-2m)/2} + \sum_{j \neq l} \varepsilon_{jl} \right). \]

A direct calculation shows that

\[
\left\langle \frac{\partial PU_{x_i, \mu_i}}{\partial \mu_j}, \frac{\partial PU_{x_i, \mu_i}}{\partial \mu_i} \right\rangle = \begin{cases} 
(c_0 + o(1)) \frac{1}{\mu_i^{4m}} \sim s^{-4/(N-6m)}, & i = j, \\
o \left( \frac{\varepsilon_{ij}}{\mu_i \mu_j} \right) \sim o(s^{-4/(N-6m)}), & i \neq j.
\end{cases}
\]

(56)
The proof of Proposition 8. From Proposition 1, Lemma B.1 and Lemma B.2, we have

\[
\frac{\partial K(x, \mu)}{\partial \mu_i} = \frac{\partial J_s(x, \mu, \omega_s, x, \mu)}{\partial \mu_i} - \sum_{h=1}^{N} B_{ih} \left( \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \frac{s}{\mu_j^{(N-2m)/2+1}} + \sum_{j=1}^{k} (s^{N/2} - (N-2m)^2)^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij} \right) \right) \frac{1}{\mu_j}.
\]

To obtain the expansion of the second derivatives of \( K(x, \mu) \), from (59), we find that

\[
\frac{\partial^2 K(x, \mu)}{\partial \mu_i \partial \mu_j} = \frac{\partial}{\partial \mu_j} \left( \sum_{l=1}^{k} PU_{x_l, \mu_l} + \omega_{s, x, \mu} \right) \frac{\partial (PU_{x_l, \mu_l})}{\partial \mu_i} - A_i \frac{\partial}{\partial \mu_j} \left( \frac{\partial^2 PU_{x_l, \mu_l}}{\partial \mu^2_i} + \omega_{s, x, \mu} \right) - \sum_{h=1}^{N} B_{ih} \left( \frac{\partial^2 PU_{x_l, \mu_l}}{\partial \mu_i \partial x_{ih}} + \omega_{s, x, \mu} \right) - \sum_{h=1}^{N} \frac{\partial B_{ih}}{\partial \mu_j} \left( \frac{\partial^2 PU_{x_l, \mu_l}}{\partial \mu_i \partial x_{ih}} + \omega_{s, x, \mu} \right).
\]

In the following, we estimate each term of (60).

**Lemma B.3.** We have

\[
\left\| \frac{\partial \omega_{s, x, \mu}}{\partial \mu_i} \right\| = O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} \left( s^{N/2} - (N-2m)^2 \right)^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij} \right) \frac{1}{\mu_j}.
\]

**Proof.** The proof is similar to [17], we just sketch it. Choose \( a_{jh} \) and \( b_j \) such that

\[
\omega := \frac{\partial \omega_{s, x, \mu}}{\partial \mu_i} - \sum_{j=1}^{k} \sum_{h=1}^{N} a_{jh} \frac{\partial PU_{x_j, \mu_j}}{\partial \mu_j} - ds \sum_{j=1}^{k} b_j \frac{\partial PU_{x_j, \mu_j}}{\partial \mu_j} \in E_{x, \mu, k}.
\]

Noting that \( \omega_{s, x, \mu} \in E_{x, \mu, k} \),

\[
\left\langle \frac{\partial \omega_{s, x, \mu}}{\partial \mu_i}, \frac{\partial PU_{x_j, \mu_j}}{\partial \mu_j} \right\rangle = \left\{ \begin{array}{ll} 0, & i \neq j, \\ \frac{1}{\mu_i^2} O(||\omega_{s, x, \mu}||), & i = j, \end{array} \right.
\]

The results follow by solving (54) and (55).
and

\[
\left\langle \frac{\partial \omega_{s,x,\mu}}{\partial \mu_i}, \frac{\partial PU_{x_j,\mu_j}}{\partial x_{jh}} \right\rangle = \begin{cases} 0, & \text{if } i \neq j \\ O(|\omega_{s,x,\mu}|^2), & \text{if } i = j, \end{cases}
\] (63)

we obtain that

\[
a_{jh} = \sum_{j=1}^{k} \frac{1}{\mu_j} O(|\omega_{s,x,\mu}|^2), \quad b_j = O(|\omega_{s,x,\mu}|^2).
\] (64)

We know that \( \omega_{s,x,\mu} \) satisfies

\[
\left\langle \sum_{j=1}^{k} PU_{x_j,\mu_j} + \omega_{s,x,\mu}, \eta \right\rangle
\]
\[
= \sum_{j=1}^{k} \sum_{h=1}^{N} B_{jh} \left\langle \frac{\partial PU_{x_j,\mu_j}}{\partial x_{jh}}, \eta \right\rangle + \sum_{j=1}^{k} A_j \left\langle \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}, \eta \right\rangle, \quad \forall \eta \in D_0^{m,2}(B_1).
\]

Differentiating with respect to \( \mu_i \), we obtain that

\[
I''_s \left( \sum_{j=1}^{k} PU_{x_j,\mu_j} + \omega_{s,x,\mu}, \frac{\partial \omega_{s,x,\mu}}{\partial \mu_i}, \eta \right)\]
\[
= \sum_{j=1}^{k} \sum_{h=1}^{N} B_{jh} \left( \frac{\partial^2 PU_{x_j,\mu_j}}{\partial \mu_i \partial x_{jh}}, \eta \right) + A_i \left( \frac{\partial^2 PU_{x_i,\mu_i}}{\partial \mu_i^2}, \eta \right) + \sum_{j=1}^{k} \sum_{h=1}^{N} \frac{\partial B_{jh}}{\partial \mu_i} \left( \frac{\partial PU_{x_j,\mu_j}}{\partial x_{jh}}, \eta \right) + \sum_{j=1}^{k} \frac{\partial A_j}{\partial \mu_i} \left( \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}, \eta \right), \quad \forall \eta \in D_0^{m,2}(B_1).
\] (65)

A direct computation leads to for \( \forall \eta \in D_0^{m,2}(B_1) \),

\[
I''_s \left( \sum_{j=1}^{k} PU_{x_j,\mu_j} + \omega_{s,x,\mu}, \frac{\partial PU_{x_i,\mu_i}}{\partial \mu_i}, \eta \right)
\]
\[
= O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} \frac{(N-2m+2)^{3/2}}{(1+\sigma)^{3/2}} \sum_{i \neq j} \varepsilon_{ij}^{(1+\sigma)/2} \sum_{j=1}^{k} \frac{1}{\mu_j^2} |\eta|^2 \right)
\] (66)

and

\[
I''_s \left( \sum_{j=1}^{k} PU_{x_j,\mu_j} + \omega_{s,x,\mu}, \frac{\partial \omega_{s,x,\mu}}{\partial x_{ih}}, \eta \right)
\]
\[
= O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} \frac{(N-2m+2)^{3/2}}{(1+\sigma)^{3/2}} \sum_{i \neq j} \varepsilon_{ij}^{(1+\sigma)/2} \sum_{j=1}^{k} \frac{1}{\mu_j^2} |\eta|^2 \right)
\] (67)

Combining (65) and (66), from Lemma B.2, we obtain that for \( \forall \eta \in E_{x,\mu,k} \),

\[
I''_s \left( \sum_{j=1}^{k} PU_{x_j,\mu_j} + \omega_{s,x,\mu}, \frac{\partial \omega_{s,x,\mu}}{\partial x}, \eta \right)
\]
Lemma B.4. We have

\[ \text{Combining (61) \sim (68), we obtain that for } \forall \eta \in E_{x,\mu,k}, \]

\[
I''_s \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} + \omega_{s,x,\mu} \right) (\tilde{\omega}, \eta)
\]

\[
= O \left( \sum_{j=1}^{k} \frac{1}{\mu_{2m}^j} + \sum_{j=1}^{k} (s \mu_j^{-(N-2m)/2})^{(1+\sigma)/2} + \sum_{i \neq j} \varepsilon_{ij}^{(1+\sigma)/2} \right) \sum_{j=1}^{k} \frac{1}{\mu_j} ||\eta||. \tag{69}
\]

Using Lemma 2.2, we have

\[
\rho||\tilde{\omega}|| \leq \sup_{\eta \neq 0, \eta \in E_{x,\mu,k}} \frac{|I''_s \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} + \omega_{s,x,\mu} \right) (\tilde{\omega}, \eta)|}{||\eta||}
\]

\[
= O \left( \sum_{j=1}^{k} \frac{1}{\mu_{2m}^j} + \sum_{j=1}^{k} (s \mu_j^{-(N-2m)/2})^{(1+\sigma)/2} + \sum_{i \neq j} \varepsilon_{ij}^{(1+\sigma)/2} \right) \sum_{j=1}^{k} \frac{1}{\mu_j}. \tag{70}
\]

The result follows from (61), (64) and (70) \qed

From Lemma B.3 and (66), we have the following lemma.

Lemma B.4. We have

\[
I''_s \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} + \omega_{s,x,\mu} \right) \left( \frac{\partial \omega_{s\text{-}x,\mu}}{\partial \mu_i}, \frac{\partial (PU_{x_j, \mu_j})}{\partial \mu_j} \right)
\]

\[
= O \left( \sum_{j=1}^{k} \frac{1}{\mu_{2m+\sigma}^j} + \sum_{j=1}^{k} (s \mu_j^{-(N-2m)/2})^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^2}. \tag{71}
\]

By Lemma B.2 and B.3, (60) leads to

\[
\frac{\partial^2 K(x, \mu)}{\partial \mu_i \partial \mu_j}
\]

\[
= \frac{\partial}{\partial \mu_j} \left( I'_s \left( \sum_{i=1}^{k} PU_{x_i, \mu_i} + \omega_{s,x,\mu} \right), \frac{\partial (PU_{x_i, \mu_i})}{\partial \mu_i} \right)
\]

\[
- \frac{\partial A_i}{\partial \mu_j} \left( \frac{\partial^2 PU_{x_i, \mu_i}}{\partial^2 \mu_i}, \omega_{s,x,\mu} \right) - \sum_{h=1}^{N} \frac{\partial B_{ih}}{\partial \mu_j} \left( \frac{\partial^2 PU_{x_i, \mu_i}}{\partial \mu_i \partial x_{ih}}, \omega_{s,x,\mu} \right)
\]

\[
+ O \left( \sum_{j=1}^{k} \frac{1}{\mu_{2m+\sigma}^j} + \sum_{j=1}^{k} (s \mu_j^{-(N-2m)/2})^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^2}. \tag{72}
\]

Let \( r_s \) be the constant such that \( U_{0,\mu_i}(r_s) = 2s \). Then

\[
r_s \sim \frac{1}{(s \mu_i^{(N-2m)/2})^{1/(N-2m)}}.
\]

Next we have
Lemma B.5. If $i \neq j$, then

$$I''_s \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} + \omega_{s,x,\mu} \right) \left( \frac{\partial PU_{x_i, \mu_i}}{\partial \mu_i}, \frac{\partial PU_{x_j, \mu_j}}{\partial \mu_j} \right) = O(\varepsilon_{ij}) \sum_{l=1}^{k} \frac{1}{\mu_l^2}.$$ 

Moreover,

$$I''_s \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} + \omega_{s,x,\mu} \right) \left( \frac{\partial PU_{x_i, \mu_i}}{\partial \mu_i}, \frac{\partial PU_{x_j, \mu_j}}{\partial \mu_j} \right)$$

$$= I''_s \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} \right) \left( \frac{\partial PU_{x_i, \mu_i}}{\partial \mu_i}, \frac{\partial PU_{x_j, \mu_j}}{\partial \mu_j} \right)$$

$$- (m^* - 1)(m^* - 2) \int_{B_{r_s}(x_i)} U_{x_i, \mu_i}^{m^* - 3} \omega_{s,x,\mu} \left( \frac{\partial U_{x_i, \mu_i}}{\partial \mu_i} \right)^2 \, dy$$

$$+ O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} \sum_{j=1}^{\sigma} (\mu_j^{N-2m})^{1/2+1} + \sum_{j=1}^{k} (s_{ij} - (N-2m)/2)^{1+\sigma} \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^2}.$$ 

Proof. The proof is similar to [17], we sketch it. We only prove the case $i = j$, the other case can be proved similarly. We have

$$I''_s \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} \right) \left( \frac{\partial PU_{x_i, \mu_i}}{\partial \mu_i}, \frac{\partial PU_{x_j, \mu_j}}{\partial \mu_j} \right)$$

$$= I''_s \left( \sum_{j=1}^{k} PU_{x_i, \mu_j} \right) \left( \frac{\partial PU_{x_i, \mu_i}}{\partial \mu_i}, \frac{\partial PU_{x_i, \mu_j}}{\partial \mu_j} \right) - (m^* - 1) \int_{B_i} g_s(y) \, dy$$

(73)

where

$$g_s(y)$$

$$= \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} - s \varphi_1 + \omega_{s,x,\mu} \right)^{m^* - 2}$$

$$- \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} - s \varphi_1 \right)^{m^* - 2} \left( \frac{\partial PU_{x_i, \mu_i}}{\partial \mu_i} \right)^2.$$ 

By Hölder inequality, we have

$$\int_{B_i \setminus B_{r_s}(x_i)} g_s(y) \, dy$$

$$\leq \frac{C}{\mu_i^2} \int_{B_i \setminus B_{r_s}(x_i)} \left| \omega_{s,x,\mu} \right|^{m^* - 2} U_{x_i, \mu_i}^2$$

$$\leq \frac{C}{\mu_i^2} \left( \int_{B_i \setminus B_{r_s}(x_i)} U_{x_i, \mu_i}^{m^*} \right)^{2/m^*} \left| \omega_{s,x,\mu} \right|^{m^* - 2}$$

$$\leq \frac{C}{\mu_i^2} \frac{1}{\mu_i^{N-2m}} \frac{N-2m}{r_s^{N-2m}} \left| \omega_{s,x,\mu} \right|^{m^* - 2}.$$
Proof. We have

\[
\leq C \frac{s}{\mu^2 \mu_1^2} \left\| \omega_{s,x,\mu} \right\| \left( N - 2m \right)^{2} \left( N - 2m \right)^{2} - 2
\]

\[
= O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \left( s \mu_j^2 \left( N - 2m \right)^{2} \right)^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}}, \quad (74)
\]

and

\[
\int_{B_{r_s}(x_i)} g_s(y) dy
\]

\[
= (m^* - 2) \int_{B_{r_s}(x_i)} \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \left( s \mu_j^2 \left( N - 2m \right)^{2} \right)^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}}, \quad (75)
\]

The result follows from (73), (74) and (75).

Lemma B.6. We have

\[
\left\langle I_s \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} + \omega_{s,x,\mu} \right), \frac{\partial^2 (PU_{x_i, \mu_i})}{\partial \mu_i^2} \right\rangle
\]

\[
= \left\langle I_s \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} \right), \frac{\partial^2 (PU_{x_i, \mu_i})}{\partial \mu_i^2} \right\rangle + (m^* - 1)(m^* - 2) \int_{B_{r_s}(x_i)} U_{x_i, \mu_i}^{m^* - 3} \omega_{s,x,\mu} \left( \frac{\partial U_{x_i, \mu_i}}{\partial \mu_i} \right)^2 dy
\]

\[
+ O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \left( s \mu_j^2 \left( N - 2m \right)^{2} \right)^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}}, \quad (76)
\]

Proof. We have

\[
\left\langle I_s \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} + \omega_{s,x,\mu} \right), \frac{\partial^2 (PU_{x_i, \mu_i})}{\partial \mu_i^2} \right\rangle
\]

\[
= \left\langle I_s \left( \sum_{j=1}^{k} PU_{x_j, \mu_j} \right), \frac{\partial^2 (PU_{x_i, \mu_i})}{\partial \mu_i^2} \right\rangle + \left( \omega_{s,x,\mu}, \frac{\partial^2 (PU_{x_i, \mu_i})}{\partial \mu_i^2} \right) - \int_{B_1} g_{s,1}(y) dy
\]
From Lemma B.4, B.5 and B.6, we have
\[ g_{s,1}(y) = \left( \sum_{j=1}^{k} PU_{x,\mu_j} - s \varphi_1 + \omega_{s,\mu} \right)_{+}^{m^* - 1} - \left( \sum_{j=1}^{k} PU_{x,\mu_j} - s \varphi_1 \right)_{+}^{m^* - 1} \frac{\partial^2 (PU_{x,\mu_j})}{\partial \mu_i^2}. \]

Similar to Lemma B.5, we have
\[
\left\langle \omega_{s,\mu}, \frac{\partial^2 (PU_{x,\mu_i})}{\partial \mu_i^2} \right\rangle - \int_{B_1} g_{s,1}(y) dy \\
= (m^* - 1)(m^* - 2) \int_{B_1} U_{x,\mu_i}^{m^* - 3} \omega_{s,\mu} \left( \frac{\partial U_{x,\mu_i}}{\partial \mu_i} \right)^2 \\
+ O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \left( \frac{s}{(N-2m)/2 + 1} + \sum_{j=1}^{k} (s\mu_j^{-2m})^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^2} \right). 
\]
and
\[
\int_{B_1} \lambda \omega_{s,\mu} \frac{\partial^2 (PU_{x,\mu_i})}{\partial \mu_i^2} \\
= O \left( \frac{||\omega_{s,\mu}||}{\mu_i^{2m+2}} \right) \\
= O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \left( \frac{s}{(N-2m)/2 + 1} + \sum_{j=1}^{k} (s\mu_j^{-2m})^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^2} \right). 
\]
Noting that
\[
\int_{B_1 \backslash C_{x}} U_{x,\mu_i}^{m^* - 3} \omega_{s,\mu} \left( \frac{\partial U_{x,\mu_i}}{\partial \mu_i} \right)^2 \\
= O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} (s\mu_j^{-2m})^{1+\sigma} + \sum_{i \neq j} \varepsilon_{ij} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^2}, 
\]
the result follows from (77), (78) and (79).

**Lemma B.7.** Let \( A_i \) and \( B_{ih} \) be the constants obtained in Proposition 1. Then we have
\[
\frac{\partial A_i}{\partial \mu_j} = O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} s\mu_j^{-2m} + \sum_{j \neq l} \varepsilon_{jl} \right), 
\]
and
\[
\frac{\partial B_{ih}}{\partial \mu_j} = O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} s\mu_j^{-2m} + \sum_{j \neq l} \varepsilon_{jl} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^2}. 
\]

**Proof.** From Lemma B.4, B.5 and B.6, we have
\[
\frac{\partial}{\partial \omega} \left( \frac{\partial J_{x}}{\partial \omega}, \frac{\partial PU_{x,\mu_i}}{\partial \mu_i} \right) = \sum_{i=1}^{k} \frac{1}{\mu_i^2} O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} s\mu_j^{-2m} + \sum_{j \neq l} \varepsilon_{jl} \right). 
\]
Similarly,
\[
\frac{\partial}{\partial \mu_j} \left( \frac{\partial J_S}{\partial \omega} \frac{\partial P U_{x_i, \mu_i}}{\partial x_i} \right) = O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} s\mu_j^{-(N-2m)/2} + \sum_{j \neq l} \varepsilon_{jl} \right). \tag{81}
\]
Differentiating (54) and (55) with respect to \( \mu_j \), we have
\[
\sum_{i=1}^{k} \left( \frac{\partial P U_{x_i, \mu_i}}{\partial \mu_j} \right) \frac{\partial A_i}{\partial \mu_j} + \sum_{i=1}^{k} \sum_{h=1}^{N} \left( \frac{\partial P U_{x_i, \mu_i}}{\partial x_i} \right) \left( \frac{\partial P U_{x_i, \mu_i}}{\partial \mu_i} \right) \frac{\partial B_{ih}}{\partial \mu_j} = \frac{1}{\mu_j^2} O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} s\mu_j^{-(N-2m)/2} + \sum_{j \neq l} \varepsilon_{jl} \right), \tag{82}
\]
and
\[
\sum_{i=1}^{k} \left( \frac{\partial P U_{x_i, \mu_i}}{\partial \mu_j} \right) \frac{\partial A_i}{\partial \mu_j} + \sum_{i=1}^{k} \sum_{h=1}^{N} \left( \frac{\partial P U_{x_i, \mu_i}}{\partial x_i} \right) \left( \frac{\partial P U_{x_i, \mu_i}}{\partial \mu_i} \right) \frac{\partial B_{ih}}{\partial \mu_j} = O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m}} + \sum_{j=1}^{k} s\mu_j^{-(N-2m)/2} + \sum_{j \neq l} \varepsilon_{jl} \right). \tag{83}
\]
The result follows by solving (82) and (83).

The proof of Proposition 9. From (72), Lemma B.5 \sim B.7, we have
\[
\frac{\partial^2 K(x, \mu)}{\partial \mu_x \partial \mu_j} = I_x'' \left( \sum_{l=1}^{k} P U_{x_l, \mu_l} + \omega_{x, x, \mu} \right) \left( \frac{\partial (P U_{x_l, \mu_l})}{\partial \mu_i} \frac{\partial (P U_{x_l, \mu_l})}{\partial \mu_j} \right) + I_x' \left( \sum_{l=1}^{k} P U_{x_l, \mu_l} + \omega_{x, x, \mu} \right) \frac{\partial^2 (P U_{x_l, \mu_l})}{\partial \mu_j \partial \mu_i} + O \left( \sum_{j=1}^{k} \frac{1}{\mu_j^{2m+\sigma}} + \sum_{j=1}^{k} \frac{s}{\mu_j^{(N-2m)/2+1}} + \sum_{j=1}^{k} (s\mu_j^{-(N-2m)/2})^{1+\sigma} + \sum_{j \neq l} \varepsilon_{jl} \right) \sum_{j=1}^{k} \frac{1}{\mu_j^2} \tag{84}
\]

The result follows from Proposition 7.

\[\square\]

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