AN INTEGRATED FIRST-ORDER THEORY OF POINTS AND INTERVALS OVER LINEAR ORDERS (PART II)

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Abstract. There are two natural and well-studied approaches to temporal ontology and reasoning: point-based and interval-based. Usually, interval-based temporal reasoning deals with points as a particular case of duration-less intervals. A recent result by Balbiani, Goranko, and Sciavicco presented an explicit two-sorted point-interval temporal framework in which time instants (points) and time periods (intervals) are considered on a par, allowing the perspective to shift between these within the formal discourse. We consider here two-sorted first-order languages based on the same principle, and therefore including relations, as first studied by Reich, among others, between points, between intervals, and inter-sort. We give complete classifications of its sub-languages in terms of relative expressive power, thus determining how many, and which, are the intrinsically different extensions of two-sorted first-order logic with one or more such relations. This approach roots out the classical problem of whether or not points should be included in an interval-based semantics. In this Part II, we deal with the cases of all dense and the case of all unbounded linearly ordered sets.

1. Introduction

The relevance of temporal logics in many theoretical and applied areas of computer science and AI, such as theories of action and change, natural language analysis and processing, and constraint satisfaction problems, is widely recognized. While the predominant approach in the study of temporal reasoning and logics has been based on the assumption that time points (instants) are the primary temporal ontological entities, there has also been significant activity in the study of interval-based temporal reasoning and logics over the past two

1998 ACM Subject Classification: [Theory of computation]: Logic — Modal and temporal logics; [Computing methodologies]: Artificial intelligence — Knowledge representation and reasoning — Temporal reasoning.

Key words and phrases: Interval based temporal logics; expressivity; Allen’s relations.

We would like to thank Dr. Davide Bresolin, of the University of Padova, for his help. This paper is the second part of a 2-parts paper.
decades. The variety of binary relations between intervals in linear orders was first studied systematically by Allen [AH87, All83, AF94], who explored their use in systems for time management and planning. Allen’s work and much that follows from it is based on the assumption that time can be represented as a dense line, and that points are excluded from the semantics. At the modal level, Halpern and Shoham [HS91] introduced the multi-modal logic HS that comprises modal operators for all possible relations (known as Allen’s relations [All83]) between two intervals in a linear order, and it has been followed by a series of publications studying the expressiveness and decidability/undecidability and complexity of the fragments of HS, e.g., [BMM+14, BMG+14]. Many studies on interval logics have considered the so-called ‘non-strict’ interval semantics, allowing point-intervals (with coinciding endpoints) along with proper ones, and thus encompassing the instant-based approach, too; more recent ones, instead, started to treat pure intervals only. Yet, little has been done so far on the formal treatment of both temporal primitives, points and intervals, in a unified two-sorted framework. A detailed philosophical study of both approaches, point-based and interval-based, can be found in [vB91] (see also [CM00]). A similar mixed approach has been studied in [AH89]. [MH06] contains a study of the two sorts and the relations between them in dense linear orders. More recently, a modal logic that includes different operators for points and interval has been presented in [BGST1].

The present paper provides a systematic treatment of point and interval relations (including equality between points and between intervals treated on the same footing as the other relations) at the first-order level. Our work is motivated, among other observations, by the fact that natural languages incorporate both ontologies on a par, without assuming the primacy of one over the other, and have the capacity to shift the perspective smoothly from instants to intervals and vice versa within the same discourse, e.g.: when the alarm goes on, it stays on until the code is entered, which contains two instantaneous events and a non-instantaneous one. Moreover, there are various temporal scenarios which neither of the two ontologies alone can grasp properly since neither the treatment of intervals as the sets of their internal points, nor the treatment of points as ‘instantaneous’ intervals, is really adequate. The technical identification of intervals with sets of their internal points, or of points as instantaneous intervals leads also to conceptual problems like the confusion of events and fluents. Instantaneous events are represented by time intervals and should be distinguished from instantaneous holding of fluents, which are evaluated at time points: therefore, the point $a$ should be distinguished from the interval $[a, a]$, and the truths in these should not necessarily imply each other. Finally, we note that, while differences in expressiveness have been found between the strict and non-strict semantics for some interval logics (see [MGMS11], for example), so far, no distinction in the decidability of the satisfiability has been found. Therefore, we believe that an attempt to systemize the role of points, intervals, and their interaction, would make good sense not only from a purely ontological point of view, but also from algorithmic and computational perspectives.

**Previous Work and Motivations.** As presented in the early work of van Benthem [vB91] and Allen and Hayes [AH85], interval temporal reasoning can be formalized as an extension of first-order logic with equality with one or more relations, and the properties of the resulting language can be studied; obviously, the same applies when relations between points are considered too. In this paper we ask the question: interpreted over linear orders, how many and which expressively different languages can be obtained by enriching first-order logic with relations between intervals, between points, and between intervals and points?
Since, as we shall see, there are 26 different relations (including equality of both sorts) between points, intervals, and points and intervals, \(2^{26}\) is an upper bound on this number. (It is worth noticing that in [MH06] the authors distinguish 30 relations, instead of 26; this is due to the fact that the concepts of the point a starting the interval \([a,b]\) and meeting it are considered to be different.) However, since certain relations are definable in terms of other ones, the actual number is less and in fact, as we shall show, much less. The answer also depends on our choices of certain semantic parameters, specifically, the class of linear orders over which we construct our interval structures. In this paper, in Part I [CDS], we consider the classification problem relative to:

(i) the class of all linear orders;

(ii) the class of all weakly discrete linear orders (i.e., orders in which every point with a successor/predecessor has an immediate one).

In Part II of this paper we consider:

(iii) the class of all dense linear orders;

(iv) the class of all unbounded linear orders;

Apart from the intrinsic interest and naturalness of this classification problem, its outcome has some important repercussions, principally in the reduction of the number of cases that need to be considered in other problems relating to these languages. For example, it reduces the number of representation theorems that are needed: given the dual nature of time intervals (i.e., they can be abstract first-order individuals with specific characteristics, or they can be defined as ordered pairs over a linear order), one of the most important problems that arises is the existence or not of a representation theorem. Consider any class of linear orders: given a specific extension of first-order logic with a set of interval relations (such as, for example, meets and during), does there exist a set of axioms in this language which would constrain (abstract) models of this signature to be isomorphic to concrete ones? Various representation theorems exist in the literature for languages that include interval relations only: van Benthem [vB91], over rationals and with the interval relations during and before, Allen and Hayes [AH85], for the dense unbounded case without point intervals and for the relation meets, Ladkin [Lad78], for point-based structures with a quaternary relation that encodes meeting of two intervals, Venema [Ven91], for structures with the relations starts and finishes, Goranko, Montanari, and Sciavicco [GMS03], for linear structures with meets and met-by, Bochman [Boc90], for point-interval structures, and Coetzee [Coe09] for dense structure with overlaps and meets. Clearly, if two sets of relations give rise to expressively equivalent languages, two separate representations theorems for them are not needed. In which cases are representation theorems still outstanding? Preliminary works that provide similar classifications appeared in [CS11] for first-order languages with equality and only interval-interval relations, and in [CDS12] for points and intervals (with equality between intervals treated on a par with the other relations) but only over the class of all linear orders. Finally, a complete study of first-order interval temporal logics enables a deeper understanding of their modal counterparts based on their shared relational semantics.

Structure of the paper. This paper is structured as follows. This paper is structured as follows. Section 2 provides the necessary preliminaries, along with an overview of the general methodology used in this paper. Part I of this paper dealt with definability and undefinability in the classes Lin and Dis, from which we start in order to tackle, in Section 3, the study the expressive power of the language by analyzing the definability properties of each basic relation in the class Den, and in Section 4 the corresponding undefinability
Table 1: Interval-interval relations, a.k.a. Allen’s relations. The equality relation is not depicted.

results in this case. Then, in Section 5 and Section 6, respectively, we present the same analysis in the unbounded case, before concluding. It is worth reminding that most of the results presented here are a consequence of those presented in Part I, to which we shall refer whenever necessary.

2. Basics

2.1. Syntax and semantics. Given a linear order $\mathbb{D} = \langle D, < \rangle$, we call the elements of $D$ points (denoted by $a, b, \ldots$) and define an interval as an ordered pair $[a, b]$ of points in $D$, where $a < b$. Abstract intervals will be denoted by $I, J, \ldots$, and so on. Now, as we have mentioned above, there are 13 possible relations, including equality, between any two intervals. From now on, we call these interval-interval relations. Besides equality, there are 2 different relations that may hold between any two points (before and after), called hereafter point-point relations, and 5 different relations that may hold between a point and an interval and vice-versa: we call those interval-point and point-interval relations, respectively, and we use the term mixed relations to refer to them indistinctly. Interval-interval relations are exactly Allen’s relations [All83]; point-point relations are the classical relations on a linear order, and mixed relations will be explained below. Traditionally, interval relations are represented by the initial letter of the description of the relation, like $m$ for meets. However, when one considers more relations (like point-point and point-interval relations) this notation becomes confusing, and even more so in the presence of more relations, e.g. when one wants to consider interval relations over a partial order. We introduce the following notation to resolve this issue: an interval $[a, b]$ induces a partition of $\mathbb{D}$ into five regions (see [Lig91]): region 0 which contains all points less than $a$, region 1 which contains $a$ only, region 2 which contains all the points strictly between $a$ and $b$, region 3 which contains only $b$ and region 4 which contains the points greater than $b$. Likewise, a point $c$ induces a partition of $\mathbb{D}$ into 3 pieces: region 0 contains all the points less than

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1This paper is focused on linear orders only; nevertheless, it is our intention to complete this study to include the treatment of partial orders also, and, at this stage, we want to make sure that we will be able to keep the notation consistent.
Table 2: Interval-point relations.

| Relation | Description |
|----------|-------------|
| $[a, b] \triangledown_p c \iff b = c$ | $c \neq a$ and $c \neq b$ |
| $[a, b] \bullet_p c \iff b < c$ | $c < b$ |
| $[a, b] \leq_p c \iff a < c < b$ | $c \in (a, b)$ |
| $[a, b] \mid_p c \iff a = c$ | $c = a$ |
| $[a, b] \varnothing_p c \iff c < a$ | $c \notin (a, b)$ |

$c$, region 2 contains only $c$, and region 4 contains all the points greater than $c$. Interval-interval relations will be denoted by $I_k k'_{ii} J$ (where the subscript $ii$ refers to interval-interval relations), where $k, k' \in \{0, 1, 2, 3, 4\}$, and $k$ represent the region of the partition induced by $I$ in which the left endpoint of $J$ falls, while $k'$ is the region of the same partition in which the right endpoint of $J$ falls; for example, $I_{34_{ii}} J$ is exactly Allen’s relation meets. Similarly, interval-point relations will be denoted by $I_k k'_{ip} a$ (where the subscript $ip$ stands for interval-point relations), where $k$ represents the position of $a$ with respect to $I$; for example, $I_{4p} a$ is the relation before. Analogously, point-point relations will be denoted by the symbol $k_{pp}$, and point-interval relations by the symbol $k k'_{pi}$. For point-point relations it is more convenient to use $<$ instead of $4_{pp}$, and $>$ instead of $0_{pp}$. In Tab. 1 we show six of the interval-interval relations, along with its original nomenclature and symbology, and in Tab. 2 we show the interval-point relations. Finally, we consider a equality per sort, using $=_{ii}$ to denote $13_{ii}$ (equality between intervals), and $=_{pp}$ to denote $2_{pp}$ (the equality between points). Now, given any of the mentioned relations $r$, its inverse, generically denoted by $\bar{r}$, can be obtained by inverting the roles of the objects in the case of non-mixed relations; for example, the inverse of the relation $2_{2_{ii}}$ (Allen’s relation contains) is the relation $0_{4_{ii}}$ (Allen’s relation during). On the other hand, mixed relations present a different situation: the inverse of a point-interval relation is an interval-point relation; thus, for example, the inverse of $3_{ip}$ is $0_{2_{pi}}$. Finally, notice that some combinations are forbidden: for instance, the relation $2_{2_{pi}}$ makes no sense, as all intervals have a non-zero extension.

**Definition 1.** We shall denote by $R$ the set of all above described relations; $I \subset R$ the subset of all 13 interval-interval relations (Allen’s relations) including the relation $=_{ii}$; $M \subset R$ the subset of all mixed relations; $P \subset R$ the subset of all point-point relations including the relation $=_{pp}$. Clearly, $R = I \cup M \cup P$.

**Definition 2.** In the following, we denote by:

(i) $\mathcal{L}$ the class of all linear orders;

(ii) $\mathcal{D}$ the class of all dense linear orders, that is, the class of all linear orders where there exists a point in between any two distinct points;

(iii) $\mathcal{D}$ the class of all weakly discrete linear orders, that is, the class of all linear orders where each point, other than the least (resp., greatest) point, if there is one, has a direct predecessor (resp., successor) – by a direct predecessor of a we of course mean a point $b$ such that $b < a$ and for all points $c$, if $c < a$ then $c \leq b$, and the notion of a direct successor is defined dually;

(iv) $\mathcal{U}$ the class of all unbounded linear orders, that is, the class of all linear order such that for every point $a$ there exists a point $b > a$ and a point $c < a$. 
Definition 3. Given a linear order $\mathbb{D}$, and given the set $I(\mathbb{D}) = \{[a, b] \mid a, b \in \mathbb{D}, a < b\}$ of all intervals built on $\mathbb{D}$:

- a \textbf{concrete interval structure of signature $S$} is a relational structure $\mathcal{F} = \langle I(\mathbb{D}), r_1, r_2, \ldots, r_n \rangle$, where $S = \{r_1, \ldots, r_n\} \subseteq \mathcal{I}$, and

- a \textbf{concrete point-interval structure of signature $S$} is a two-sorted relational structure $\mathcal{F} = \langle \mathbb{D}, I(\mathbb{D}), r_1, r_2, \ldots, r_n \rangle$, where $S = \{r_1, \ldots, r_n\} \subseteq \mathcal{R}$.

Since all relations between intervals, points, and all mixed relations are already implicit in $S \subseteq \mathcal{FO}$, we shall repeatedly apply the following definition and (rather standard) procedure.

2.2. (Un)definability and Truth Preserving Relations. We describe here the most important tools that we use to classify the expressive power of our (sub-)languages.

Definition 4. Let $S \subseteq \mathcal{R}$, and $C$ a class of linear orders. We say that $FO + S$ \textit{defines} $r \in \mathcal{R}$ over $C$, denoted by $FO + S \rightarrow_C r$, if there exists $FO + S$-formula $\varphi(x, y)$ such that $\varphi(x, y) \leftrightarrow r(x, y)$ is valid on the class of concrete point-interval structures of signature $(S \cup \{r\})$ based on $C$.

By $FO + S \rightarrow r$ we denote the fact that $FO + S \rightarrow_{\text{Lin}} r$ (and hence $FO + S \rightarrow_C r$ for every $C \in \{\text{Lin, Den, Dis, Unb, Fin}\}$). Obviously, $FO + S \rightarrow r$ for all $r \in S$.

Definition 5. Let $S, S' \subseteq \mathcal{R}$ and $C$ a class of linear orders. We say that $S$ is:

- $S'$-\textit{complete over $C$} (resp., $S'$-\textit{incomplete over $C$}) if and only if $FO + S \rightarrow_C r$ for all $r \in S'$ (resp., $FO + S \nrightarrow_C r$ for some $r \in S'$), and

- \textit{minimally $S'$-complete over $C$} (resp., \textit{maximally $S'$-incomplete over $C$}) if and only if it is $S'$-complete (resp., $S'$-incomplete) over $C$, and every proper subset (resp., every proper superset) of $S$ is $S'$-incomplete (resp., $S'$-complete) over the same class.

The notion of (minimally) $r$-completeness and (maximally) $r$-incompleteness over $C$ is immediately deduced from the above one, by taking $S' = \{r\}$ and denoting the latter simply by $r$. Moreover, one can project the above definitions over some interesting subsets of $\mathcal{R}$, such as $\mathcal{I}, \mathcal{M}$ or $\mathcal{P}$, obtaining relative completeness and incompleteness.

Let $C' \subseteq C$ be two classes of linear orders. Notice that if $FO + S \rightarrow_C r$ then $FO + S \rightarrow_{C'} r$ and, contrapositively, that if $FO + S \nrightarrow_{C'} r$ then $FO + S \nrightarrow_C r$. So specifically, if $S$ is $S'$-complete over $C$, then it is also $S'$-complete over $C'$. Also, if $S$ is $S'$-incomplete over $C'$, then it is also $S'$-incomplete over $C$. Notice however, that minimality and maximality of complete and incomplete sets does not necessarily transfer between super and subclasses in a similar way. In what follows, in order to prove that $FO + S \nrightarrow_C r$ for some $r$ and some class $C$, we shall repeatedly apply the following definition and (rather standard) procedure.

Definition 6. Let $\mathcal{F} = \langle \mathbb{D}, I(\mathbb{D}), S \rangle$ and $\mathcal{F}' = \langle \mathbb{D}', I(\mathbb{D}'), S' \rangle$ be concrete structures where $S \subseteq \mathcal{R}$. A binary relation $\zeta \subseteq (\mathbb{D} \cup I(\mathbb{D})) \times (\mathbb{D}' \cup I(\mathbb{D}'))$ is called a \textit{surjective $S$-truth preserving relation} if and only if:

(i) $\zeta$ respects sorts, i.e., $\zeta = \zeta_p \cup \zeta_i$, where $\zeta_p \subseteq \mathbb{D} \times \mathbb{D}'$ and $\zeta_i \subseteq I(\mathbb{D}) \times I(\mathbb{D}')$;
(ii) \( \zeta \) respects the relations in \( S \), i.e., if \((a, a'), (b, b') \in \zeta_p \) and \((I, I'), (J, J') \in \zeta_i \), then:
(a) \( r(a, b) \) if and only if \( r(a', b') \) for every point-point relation \( r \in S \);
(b) \( r(I, a) \) if and only if \( r(I', a') \) for every interval-point relation \( r \in S \);
(c) \( r(I, J) \) if and only if \( r(I', J') \) for every interval-interval relation \( r \in S \);

(iii) \( \zeta \) is total and surjective, i.e.:
(a) for every \( a \in \mathbb{D} \) (resp., \( I \in \mathbb{I}(\mathbb{D}) \)), there exist \( a' \in \mathbb{D}' \) (resp., \( I' \in \mathbb{I}(\mathbb{D}') \)) such that \((a, a') \in \zeta_p \) (resp., \((I, I') \in \zeta_i \));
(b) for every \( a' \in \mathbb{D}' \) (resp., \( I' \in \mathbb{I}(\mathbb{D}') \)), there exist \( a \in \mathbb{D} \) (resp., \( I \in \mathbb{I}(\mathbb{D}) \)) such that \((a, a') \in \zeta_p \) (resp., \((I, I') \in \zeta_i \)).

If we add to Definition 6 the requirement that that \( \zeta \) should be functional, we obtain nothing but the definition of an isomorphism between two-sorted first-order structures or, equivalently, an isomorphism between single sorted first-order structures with predicates added for ‘point’ and ‘interval’ (see e.g. [Hod93]). As one would expect, surjective \( S \)-truth preserving relations preserve the truth of all first-order formulas in signature \( S \). This is stated in Theorem [8] below. The reason why we consider only interval-point relations instead of all mixed relations is that, as we shall explain, we can limit ourselves to work without inverse relations, and point-interval relations are the inverse of interval-point ones.

**Definition 7.** If \( \zeta \) is a surjective \( S \)-truth preserving relation, we say that \( \zeta \) breaks \( r \not\in S \) if and only if there are:

(i) \((a, a'), (b, b') \in \zeta_p \) such that \( r(a, b) \) but \( \neg r(a', b') \), if \( r \) is point-point, or
(ii) \((a, a') \in \zeta_p \) and \((I, I') \in \zeta_i \) such that \( r(I, a) \) but \( \neg r(I', a') \), if \( r \) is interval-point, or
(iii) \((I, I'), (J, J') \in \zeta_i \) such that \( r(I, J) \) but \( \neg r(I', J') \), if \( r \) is interval-interval.

The following result is, as already mentioned, a straightforward generalization of the classical result on the preservation of truth under isomorphism between first-order structures, and it is proved by an easy induction on formulas, using clause (ii) of Definition 6 to establish the base case for atomic formulas and clause (iii) for the inductive step for the quantifiers.

**Theorem 8.** If \( \zeta = \zeta_p \cup \zeta_i \) is a surjective \( S \)-truth preserving relation between \( \mathcal{F} = (\mathbb{D}, \mathbb{I}(\mathbb{D}), S) \) and \( \mathcal{F}' = (\mathbb{D}', \mathbb{I}(\mathbb{D}'), S) \), and \( a_1, \ldots, a_k \in \mathbb{D}, a'_1, \ldots, a'_k \in \mathbb{D}, I_1, \ldots, I_l \in \mathbb{I}(\mathbb{D}) \), and \( I'_1, \ldots, I'_l \in \mathbb{I}(\mathbb{D}') \) are such that \((a_j, a'_j) \in \zeta_p \) for \( 1 \leq j \leq k \), and \((I_j, I'_j) \in \zeta_i \) for \( 1 \leq j \leq l \), then for every \( FO + S \) formulas \( \varphi(x^1_p, \ldots, x^k_p, y^1_i, \ldots, y^l_i) \) with free variables \( x^1_p, \ldots, x^k_p, y^1_i, \ldots, y^l_i \), we have that

\[
\mathcal{F} \models \varphi(a_1, \ldots, a_k, I_1, \ldots, I_l) \quad \text{if and only if} \quad \mathcal{F}' \models \varphi(a'_1, \ldots, a'_k, I'_1, \ldots, I'_l).
\]

Thus, to show that \( FO + S \nvdash r \) for a given \( r \in \mathcal{R} \), it is sufficient to find two concrete point-interval structures \( \mathcal{F} \) and \( \mathcal{F}' \) and a surjective \( S \)-truth preserving relation \( \zeta \) between \( \mathcal{F} \) and \( \mathcal{F}' \) which breaks \( r \). For the readers’ convenience, let us refer to surjective \( S \)-truth preserving relations as simply \( S \)-relations.

Although there are other constructions that could be used to show that relations are not definable in \( FO + S \), e.g. elementary embeddings or Ehrenfeucht-Fraïssé games, we have found \( S \)-relations sufficient for our purposes in this paper.

2.3. **Strategy.** The main objective of this paper is to establish all expressively different subsets of \( \mathcal{R} \) (and, then, of \( J, \mathcal{W} \) or \( P \)) over the mentioned classes of linear orders. To this end, for each \( r \in \mathcal{R} \) we compute all expressively different minimally \( r \)-complete and all
maximally $r$-incomplete subsets of $\mathfrak{R}$, from which we can easily deduce all expressively different minimally $r$-complete and maximally $r$-incomplete subsets of $\mathfrak{J, M}$ and $\mathfrak{P}$; minimally $\mathfrak{R}$- (resp., $\mathfrak{J}$-, $\mathfrak{M}$-, $\mathfrak{P}$-) complete and maximally incomplete subsets are, then, deduced as a consequence of the above results. The set $\mathfrak{R}$ contains, as we have mentioned, 26 different relations. This means that there are $2^{26}$ potentially different extensions of first-order logic to be studied. Clearly, unless we design a precise strategy that allows us to reduce the number of results to be proved, the task becomes cumbersome.

As a first simplification principle observe that, since we are working within first-order logic, all inverses of relations are explicitly definable, and hence we only need to assume as primitive a set which contains all relation up to inverses, which implies that point-interval relations can be omitted if we consider all interval-point ones. Accordingly, let $\mathfrak{R}$ be the set of interval-interval relations given in Tab. 2, and let $\mathfrak{P}$ be the set of interval-point relations given in Tab. 2, and let $\mathfrak{P} = \{<, =\}$. Lastly let $\mathfrak{R} = \mathfrak{J} \cup \mathfrak{M} \cup \mathfrak{P}$.

In order to further reduce the number of results to be presented, consider what follows. The order dual of a structure $\mathcal{F} = (\mathcal{D}, \mathcal{I}(\mathcal{D}))$ is the structure $\mathcal{F}^\partial = (\mathcal{D}^\partial, \mathcal{I}(\mathcal{D}^\partial))$ based on the order dual $\mathcal{D}$ (obtained by reversing the order) of the underlying linear order $\mathcal{D}$. All classes considered in this paper are closed under taking order duals.

**Definition 9.** The **reversible relations** are exactly the members of the set $\{0_i, 1_i, 3_i, 4_i, 14_i, 03_i\}$. The relations belonging to the complement $\mathfrak{R}^+ \setminus \{0_i, 1_i, 3_i, 4_i, 14_i, 03_i\}$ are called **symmetric**; if, in addition, $r = 2p$ or $r = 04_i$, then $r$ is said self-symmetric. If $r = 0_i$ (resp., $r = 1_i, r = 14_i$), its reverse is $r = 4_i$ (resp., $r = 3_i, r = 03_i$), and the other way around. Finally, the **symmetric $S'$** of a subset $S \subseteq \mathfrak{R}^+$ is obtained by replacing every reversible relation in $S$ with its reverse. We shall use the notation $S \sim S'$ to indicate that sets $S$ and $S'$ are symmetric.

This definition is motivated by the following easily verifiable facts. Let $r \in \mathfrak{R}^+$, $\mathcal{F}$ be a structure, and $x$ and $y$ be elements of $\mathcal{F}$ of the appropriate sorts for $r$; then:

(i) if $r$ is a reversible relation, with reverse $r'$, then $\mathcal{F} \models r(x,y)$ if and only if $\mathcal{F}^\partial \models r'(x,y)$;

(ii) if $r$ is self-symmetric, then $\mathcal{F} \models r(x,y)$ if and only if $\mathcal{F}^\partial \models r(x,y)$;

(iii) if $r$ is a symmetric, but not self-symmetric, relation, then $\mathcal{F} \models r(x,y)$ if and only if $\mathcal{F}^\partial \models r(y,x)$.

The following crucial lemma capitalizes on these facts.

**Lemma 10.** Let $S, S' \subseteq \mathfrak{R}^+$ be such that $S \sim S'$. If $r$ is a symmetric relation, then $FO + S \rightarrow r$ if and only if $FO + S' \rightarrow r$. Moreover, if $r$ is a reversible relation with reverse $r'$, then $FO + S \rightarrow r$ if and only if $FO + S' \rightarrow r'$.
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\[ D, D', \ldots \] (generic) linearly ordered sets

\( x_p, y_p, \ldots \) first-order variables for points

\( x_i, y_i, \ldots \) first-order variables for intervals

\( x, y, \ldots \) first-order variables of any sort

before, \ldots relations in text are emphasized

\[ F, F', \ldots \] (generic) concrete (point-)interval structures

\( S, S', \ldots \) (generic) subsets of \( \mathbb{R} \)-relations

\( \zeta (\zeta_p, \zeta_i) \) surjective relation (for points, for intervals)

\( Id_p(Id_i) \) 'identity' relation over points (intervals)

\( C, C' \) (generic) class of linearly ordered sets

\( FO + S \rightarrow_C r \) \( S \) defines \( r \) (w.r.t. the class \( C \))

\( S \sim S' \) \( S \) and \( S' \) are symmetric

\( a \in D \) \( a \) is a point of \( D \), where \( D = (D, <) \)

\( S \) in the text, a new proof case is underlined

\( r \) generic relation

\( mcs (mcs(r)) \) minimally complete set (minimally \( r \)-complete set)

\( MIS (MIS(r)) \) maximally incomplete set (maximally \( r \)-incomplete set)

| Table 3: Notational conventions used in this paper. |
|-----------------------------------------------------|
| A notation not used in this paper. |

**Proof.** Let \( S, S' \subset \mathbb{R}^+ \) such that \( S \sim S' \). For any \( FO + S \) formula \( \varphi \) that defines a given relation (and, therefore, with exactly two free variables), let \( \varphi' \) denote the formula obtained from \( \varphi \) by replacing every occurrence of a reversible relation with its reverse, and by swapping the arguments of every symmetric, but not self-symmetric, relation (occurrences of every self-symmetric relation are left unchanged). Induction on formulas then shows that \( F \models \varphi(x, y) \) (after substituting \( x, y \) with elements of the appropriate sorts) if and only if \( F^\partial \models \varphi'(x, y) \), for any structure \( F \). The base case of the induction is taken care of by the three observations preceding this lemma. Now, suppose that a \( FO + S \) formula \( \varphi(x, y) \) defines a symmetric relation \( r \). We claim that \( \varphi' \) also defines \( r \). Let \( F \) be an arbitrary structure of signature \( S \cup \{ r \} \). Then \( F^\partial \models \varphi(x, y) \leftrightarrow r(x, y) \), and hence \( F \models \varphi'(x, y) \leftrightarrow r(y, x) \) if \( r \) is not self-symmetric, and \( F \models \varphi'(x, y) \leftrightarrow r(x, y) \) otherwise. Next, suppose that the \( FO + S \) formula \( \varphi(x, y) \) defines a reversible relation \( r' \). We claim that \( \varphi' \) defines its reverse \( r' \). Let \( F \) be an arbitrary structure of signature \( S \cup \{ r \} \). Then \( F^\partial \models \varphi(x, y) \leftrightarrow r(x, y) \), and, hence, \( F \models \varphi'(x, y) \leftrightarrow r'(x, y) \).

In conclusion, we can limit our attention to 14 out of 26 relations by disregarding the inverses of relations in \( \mathbb{R}^+ \), and we do not need to explicitly analyze complete and incomplete sets for \( 3_p, 4_p \), and \( 0 \bar{3}_i \) as those correspond exactly to symmetric of complete and incomplete sets for \( 0_p, 1_p \), and \( 14_i \), respectively. This means that only 11 relations are to be analyzed (which we can refer to as explicit relations).

Even under the mentioned simplifications, there is a huge number of results to be presented and displayed. Let \( r \) be anyone of the explicit relations. In order to correctly identifying all minimally \( r \)-complete sets (\( mcs(r) \)), we need to know all maximally \( r \)-incomplete sets (\( MIS(r) \)) over the same class, and the other way around. To this end, we proceed in the following way:
Once the above process is completed for every relation, we can then easily deduce all minimally $\mathcal{R}^+$-complete and all maximally $\mathcal{R}^+$-incomplete sets, to complete the picture. A similar procedure works for $\mathcal{I}^+$, $\mathcal{M}^+$, and $\mathcal{P}^+$.

(1) fixed a class of linearly ordered sets and an explicit relation $r$, we first guess the $r$-complete subsets of $\mathcal{R}^+$, obtaining a first approximation of the definability rules for $r$;

(2) then, we apply the algorithm in Fig. [1] which uses the set of $r$-complete subsets of $\mathcal{R}^+$ (the parameter def rules) to obtain a first approximation of the maximally $r$-incomplete sets;

(3) after that, we prove that every $R_1, R_2, \ldots, R_k$ listed as a maximally $r$-incomplete set is actually $r$-incomplete, and, if not, we repeat from step 1, using the acquired knowledge to update the set of $r$-complete subsets of $\mathcal{R}^+$;

(4) at this point, the sets $S_1, S_2, \ldots, S_{k'}$ listed at step 1 are, actually, all minimally $r$-complete: for each $i$, $S_i$ is $r$-complete by definition, and if there was a $r$-complete set $S \subset S_i$, then for some $R_j$ listed as maximally $r$-incomplete set we could not prove its $r$-incompleteness. Therefore, $S_1, S_2, \ldots, S_{k'}$ are all minimally $r$-complete, and, as a consequence, $R_1, R_2, \ldots, R_k$ are all maximally $r$-incomplete.

Once the above process is completed for every relation, we can then easily deduce all minimally $\mathcal{R}^+$-complete and all maximally $\mathcal{R}^+$-incomplete sets, to complete the picture. A similar procedure works for $\mathcal{I}^+$, $\mathcal{M}^+$, and $\mathcal{P}^+$.

| $=p$    | $=i$ | $<$   | $0_p$ | $0_{ip}$ | $1_p$ | $2_ip$ |
|---------|------|-------|-------|----------|-------|--------|
| $\{\emptyset\}$ | $\{\emptyset, 2_ip\}$ | $\{\emptyset, 1_p\}$ | $\{1_p, <\}$ | $\{0_p, <\}$ | $\{0_p, 3_p\}$ | $\{0_p, 4_p\}$ |
| $\{1_p\}$ | $\{0_p, 3_p\}$ | $\{0_p, 3_p\}$ | $\{1_p, 2_ip\}$ | $\{1_p, 3_p\}$ | $\{2_ip, 3_p\}$ | $\{1_p, 3_ip\}$ |
| $\{3_ip\}$ | $\{0_p, 4_ip\}$ | $\{0_p, 3_ip\}$ | $\{1_p, 4_ip\}$ | $\{1_p, 3_ip\}$ | $\{2_ip, 4_ip\}$ | $\{1_ip, 3_ip\}$ |
| $\{4_ip, 3_ip\}$ | $\{4_ip, 3_ip\}$ | $\{4_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 4_ip\}$ | $\{2_ip, 3_ip\}$ |
| $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ |
| $\{2_ip, 4_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ | $\{2_ip, 3_ip\}$ |
| $\{3_ip, 4_ip\}$ | $\{3_ip, 4_ip\}$ | $\{3_ip, 4_ip\}$ | $\{3_ip, 4_ip\}$ | $\{3_ip, 4_ip\}$ | $\{3_ip, 4_ip\}$ | $\{3_ip, 4_ip\}$ |
| $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ |
| $\{0_ip, 04_i\}$ | $\{0_ip, 04_i\}$ | $\{0_ip, 04_i\}$ | $\{0_ip, 04_i\}$ | $\{0_ip, 04_i\}$ | $\{0_ip, 04_i\}$ | $\{0_ip, 04_i\}$ |
| $\{0_ip, 03_i\}$ | $\{0_ip, 04_i\}$ | $\{0_ip, 04_i\}$ | $\{0_ip, 04_i\}$ | $\{0_ip, 04_i\}$ | $\{0_ip, 04_i\}$ | $\{0_ip, 04_i\}$ |
| $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ | $\{0_ip, 03_i\}$ |

Table 4: The spectrum of the $\text{mcs}(r)$, for each $r \in \mathcal{M}^+ \cup \{=p, =i, <\}$. - Class: Lin (review).
Proof. Tab. 6 is correct.

Definability for the Relations show in boldface those complete sets that are stricter than some complete set for a given

Suppose, first, that which case the right-hand side of the definition is vacuously true, or there are infinitely many

Table 5: The spectrum of the mcs(r), for each r ∈ ℤ⁺ \ {=₁}. - Class: Den (review).

The most common notational conventions used in the paper are listed in Tab. 3

3. Completeness Results in the Class Den

As it turns out, only a few of the results that appear in Tab. 3 and Tab. 5 are to be refined to obtain all minimally complete sets under the density assumption. In the following, we show in boldface those complete sets that are stricter than some complete set for a given relation, or completely new, and we prove them explicitly modulo symmetry.

3.1. Definability for the Relations =₁, =ₗ < in Den.

Lemma 11. Tab. 6 is correct.

Proof. Consider the following definition:

\[ x_p = p y_p \leftrightarrow \forall x_i(x_i:0_p x_p \leftrightarrow x_i0_p y_p) \{0_p\} \]

Suppose, first, that \( x_p = p y_p \); we have that, either there exists no interval in the model, in which case the right-hand side of the definition is vacuously true, or there are infinitely many
Table 6: The spectrum of the \textit{mcs}(r), for each $r \in \{=_{p},=_{i},<\}$. - Class: Den.

Intervals (as the underlying linear order is dense), and every interval $x_i$ clearly has both $x_p$ and $y_p$ in relation $0_p$ with it or none of them, again satisfying the right-hand side. If, on the other hand, $\varphi(x_p, y_p)$ is satisfied, then, if there is no interval, then $x_p$ and $y_p$ coincide, and if there are infinitely many intervals, the only way to guarantee that the intervals that see both $x_p$ and $y_p$ via $0_p$ are exactly the same is to assign the same point to $x_p$ and $y_p$, as we wanted to prove. Now, consider the relation $=_{i}$, and the following definitions:

\[
x_i =_{i} y_i \leftrightarrow \begin{cases} 
\forall x_p(x_i2ip;x_p \leftrightarrow y_i2ip) & \{2ip\} \\
\forall z_i(x_i24ii z_i \leftrightarrow y_i24ii z_i) \land \forall z_i(z_i24ii x_i \leftrightarrow z_i24ii y_i) & \{24ii\} \\
\forall z_i(x_i04ii z_i \leftrightarrow y_i04ii z_i) \land \forall z_i(z_i04ii x_i \leftrightarrow z_i04ii y_i) & \{04ii\} \\
\forall z_i(x_i44ii z_i \leftrightarrow y_i44ii z_i) \land \forall z_i(z_i44ii x_i \leftrightarrow z_i44ii y_i) & \{44ii\}
\end{cases}
\]

As for $\{2ip\}$, observe that two intervals over a dense linear order are equal if and only if they have the same internal points. Consider, now, the set $\{24ii\}$. It is clear that if $x_i =_{i} y_i$ then $\varphi(x_i, y_i)$ holds. For the converse, suppose that $[a, b] \neq [c, d]$. Then either $[a, b]$ has an internal point which is not an internal point of $[c, d]$ or $[c, d]$ has an internal point which is not an internal point of $[a, b]$. In both cases either the first or the second conjunct of $\varphi$ does not hold. The remaining cases are treated with similar arguments. Let us now focus on the relation $<$ and the corresponding definitions:

\[
x_p < y_p \leftrightarrow (\neg \exists x_i(x_i0ip y_p) \land \exists y_i(y_i0ip x_p)) \lor \exists x_i(x_i0ip x_p \land \neg (x_i0ip y_p)) \quad \{0_p\}
\]

Consider the set $\{0_p\}$. Suppose that $\mathcal{F} \models \varphi(x_p, y_p)$. If the first disjunct of $\varphi$ holds then $y_p$ is the greatest point of the model, while $x_p$ is not, which gives $x_p < y_p$. Assume the second disjunct of $\varphi$ holds. Then the interval $x_i = [a, b]$ is such that $x_p = c < a$, but $y_p \geq a$, that
defines =

\text{Den}-Relations in $$\mathfrak{m}^+$$-relations. We focus on mixed relations, as well.

\text{Den} - Relations in $$\mathfrak{m}^+$$-relations. We focus on mixed relations, as well.

\begin{table}
\begin{center}
\begin{tabular}{|l|l|l|}
\hline
$0_{ip}$ & $1_{ip}$ & $2_{ip}$ \\
\hline
$\{i_{ip}, <\}$ & $\{0_{ip}\}$ & $\{0_{ip}, 3_{ip}\}$ \\
$\{2_{ip}, <\}$ & $\{2_{ip}, 3_{ip}\}$ & $\{0_{ip}, 4_{ip}\}$ \\
$\{i_{ip}, 2_{ip}\}$ & $\{2_{ip}, 4_{ip}\}$ & $\{0_{ip}, 14_{ii}, 24_{ii}\}$ \\
$\{i_{ip}, 3_{ip}\}$ & $\{2_{ip}, <\}$ & $\{0_{ip}, 14_{ii}, 44_{ii}\}$ \\
$\{i_{ip}, 4_{ip}\}$ & $\{3_{ip}, 14_{ii}\}$ & $\{0_{ip}, 24_{ii}, 14_{ii}\}$ \\
$\{2_{ip}, 3_{ip}\}$ & $\{3_{ip}, 24_{ii}, 03_{ii}\}$ & $\{0_{ip}, 03_{ii}\}$ \\
$\{2_{ip}, 4_{ip}\}$ & $\{3_{ip}, 03_{ii}, 04_{ii}\}$ & $\{0_{ip}, 34_{ii}\}$ \\
$\{3_{ip}, 14_{ii}\}$ & $\{3_{ip}, 03_{ii}, 44_{ii}\}$ & $\{0_{ip}, 04_{ii}\}$ \\
$\{3_{ip}, 24_{ii}\}$ & $\{3_{ip}, 34_{ii}\}$ & $\{0_{ip}, 03_{ii}\}$ \\
$\{4_{ip}, 14_{ii}\}$ & $\{4_{ip}, 24_{ii}, 03_{ii}\}$ & $\{0_{ip}, 14_{ii}, 24_{ii}\}$ \\
$\{4_{ip}, 34_{ii}\}$ & $\{4_{ip}, 03_{ii}, 44_{ii}\}$ & $\{0_{ip}, 14_{ii}, 44_{ii}\}$ \\
$\{4_{ip}, 24_{ii}\}$ & $\{4_{ip}, 03_{ii}, 44_{ii}\}$ & $\{0_{ip}, 24_{ii}, 44_{ii}\}$ \\
$\{4_{ip}, 14_{ii}\}$ & $\{4_{ip}, 34_{ii}\}$ & $\{0_{ip}, 04_{ii}\}$ \\
$\{4_{ip}, 24_{ii}, 03_{ii}\}$ & $\{4_{ip}, 24_{ii}, 44_{ii}\}$ & $\{0_{ip}, 03_{ii}\}$ \\
$\{4_{ip}, 03_{ii}, 44_{ii}\}$ & $\{4_{ip}, 34_{ii}\}$ & $\{0_{ip}, 04_{ii}\}$ \\
\hline
\end{tabular}
\end{center}
\caption{The spectrum of the mcs($r$), for each $r \in \mathfrak{m}^+$. - Class: \text{Den}.}
\end{table}

is $x_p < y_p$. Conversely, suppose that $x_p = a, y_p = b$, and that $a < b$. As the structure is dense, if $b$ is the last point of the model, we have no interval starting at $b$ but infinitely many intervals $x_i$ such that $x_i < a$, and if $b$ is not the last point of the model, then every $x_i$ starting at $b$ satisfies the second part of the definition. All other new definitions follow directly from the results for the class Lin combined with the denseness hypothesis. For example, the set $\{I_{ip}, 0_{4_{ii}}\}$ defines = under the density hypothesis, and, then $\{I_{ip}, 0_{4_{ii}}, =_i\}$ defines < in the class of all linear orders, and therefore in Den as well.

3.2. \textbf{Definability for} $\mathfrak{m}^+$-\textbf{Relations in} Den. To continue studying how the density hypothesis influences the ability of the sub-languages to express our relations, we focus now on mixed relations.

\textbf{Lemma 12.} Tab. \ref{tab:mcs} is correct.

\textbf{Proof.} As for the $\{a_{ip}\}$-completeness, only one new definition is necessary:

$$x_i a_{ip} y_p \iff \exists k_p (\forall z_p (x_i 2_{ip} z_p \rightarrow k_p \wedge y_p < k_p) \wedge x_i < k_p) \{2_{ip}, <\}$$
Let us prove that the set \( \{ z_p, < \} \) is \( a_{ip} \)-complete. If \( \mathcal{F} \models \varphi(x_i, y_p) \), then there exists a point \( k_p \) smaller than every point contained in \( x_i \) (there are infinitely many such points because we are in a dense structure); and \( y_p \) is smaller than \( k_p \), so it must be before \( x_i \). If, on the other hand \( x_i = [a, b] \) and \( y_p = c < a \), we take \( k_p = a \) to satisfy all requirements. The remaining \( a_{ip} \)-complete sets in the upper part of the table can be proved correct as we did in the previous section, taking into account the fact the new \(<\)-complete sets that we gain under the density hypothesis. It is important to notice that all other definitions are now consequences of those already proved. As for proving that \( \{ a_{ip} \} \) is \( I_{ip} \)-complete, observe that by the previous lemma this set defines \(<\), and consider the following definition:

\[
\forall z_p(x_i a_{ip} z_p \rightarrow z_p < y_p) \land \exists t_p(y_p < t_p) \land \{ a_{ip} \} \\
\forall u_i(w_i a_{ip} y_p \rightarrow \forall u_p(w_i a_{ip} u_p \rightarrow x_i a_{ip} u_p)).
\]

The left to right direction of the above definition is immediate. Suppose, on the contrary, that it is not the case that \( x_i I_{ip} y_p \). Then, assume that \( x_i = [a, b] \); because of the first conjunct, it must be the case that \( a \leq y_p \), and therefore that \( a < y_p \). If \( y_p \) is instantiated to the final point of the model, then the second conjunct is falsified, and therefore there must be an interval \([c, d]\) such that \([c, d] a_{ip} y_p\) because the underlying linear order is dense. Again, because of the denseness hypothesis, there is a point \( e \) such that \( y_p < c < e \), and hence, \([c, d] a_{ip} e\), but, at the same time, \([a, b] a_{ip} e\) is false, contradicting the third conjunct. Now, the remaining definitions for the relation \( I_{ip} \) are straightforward consequences of the previous results. Finally, we prove that \( \{ I_{ip}, \varphi_{ii} \} \) is \( 2_{ip} \)-complete by means of the following definability equation:

\[
x_i 2_{ip} y_p \iff \exists z_i(z_i I_{ip} y_p \land z_i \varphi_{ii} x_i). \quad \{ I_{ip}, \varphi_{ii} \}
\]

The \( 2_{ip} \)-completeness of \( \{ I_{ip}, \varphi_{ii} \} \) is extremely simple. Assuming \( \mathcal{F} \models \varphi([a, b], c) \), there must be an interval \( z_i = [c, d] \) contained in \([a, b]\), which implies \( a < c < b \), as we wanted. Conversely, if \( a < c < b \), since the structure is dense, we can find \( d \) such that \( a < c < d < b \), and \( z_i = [c, d]\) is a witness for \( \varphi \). All other definitions are consequences of the latter and the previous results.

\[\square\]

3.3. **Definability for Relations in \( \mathfrak{T}^+ \setminus \{=\} \) in Den.** To conclude this analysis of dense linear orders, we prove that Tab. [S] is correct.

**Lemma 13.** Tab. [S] is correct.

**Proof.** Starting with \( \{ 14_{ii} \} \)-completeness, we only need to give the following equation:

\[
x_i 14_{ii} y_i \iff \forall z_i(z_i 44_{ii} x_i \leftrightarrow z_i 44_{ii} y_i) \land \exists z_i(y_i 44_{ii} z_i \land \neg(x_i 44_{ii} z_i)). \quad \{ 44_{ii} \}
\]

To prove that \( \{ 44_{ii} \} \) is \( 14_{ii} \)-complete, assume first \( \mathcal{F} \models \varphi([a, b], [c, d]) \). If \( a < c \), then we can find a point \( e \) with \( a < e < c \) and the first conjunct of \( \varphi \) fails witnessed by the interval \( z_i = [a, e] \). Similarly \( a > c \) also gives a contradiction, so we obtain \( a = c \). The second
Table 8: The spectrum of the mcs(r), for each $r \in \mathbb{I}^+ \setminus \{=\}$. - Class: \textit{Den}.

conjunct of $\varphi$ gives an interval $z_i = [c, f]$ such that $d < e$ and $e \leq b$, and so we have $d < b$.

If we assume that $a = c < b < d$, we can take $z_i = [d, e]$ where $b < e < d$ to witness the second conjunct of $\varphi$ and the first conjunct is satisfied trivially. As for $34_{ii}$, once again, one new definition is enough:

$$x_i34_{ii} \lor 44_{ii}y_i \leftrightarrow \neg(x_i24_{ii}y_i) \land \exists z_i(x_i24_{ii}z_i \land z_i24_{ii}y_i). \quad \{24_{ii}\}$$

We will prove that $\{24_{ii}\}$ is $34_{ii}$-complete when the structure is dense using the weaker relation $34_{ii} \lor 44_{ii}$ (recall from Part I, where we proved that on linear domains the relation $34_{ii} \lor 44_{ii}$ is $34_{ii}$-complete). If $\mathcal{F} \models \varphi([a, b], [c, d])$, there must be some interval $z_i$ \textit{overlapped} by $x_i$ and overlapping $y_i$, which implies $a < c$ and $b < d$ and since $x_i$ cannot overlap $y_i$, we obtain $b \leq c$ as required. Conversely, assume that $[a, b]34_{ii} \lor 44_{ii} [c, d]$. Then using the density assumption we can take $z_i = [e, f]$ where $a < e < b$ and $c < f < d$, to witness $\varphi$. No new definitions are needed for $24_{ii}$ and only one new definition is needed for $04_{ii}$:

$$x_i04_{ii}y_i \leftrightarrow \forall z_p(y_i2_{ii}z_p \rightarrow x_i2_{ii}z_p) \land \exists z_p(x_i2_{ii}z_p \land \neg (y_p2_{ii}z_p)) \land \{2_{ii}\}$$

$$\neg \exists z_i(\forall z_p((x_i2_{ii}z_p \land \neg(y_p2_{ii}z_p)) \rightarrow z_i2_{ii}z_p)).$$
Proof. Lemma 14. designed to work not only in \( \text{Lin} \) but also in \( \text{Dis} \); nevertheless, all proofs in Part I must be revisited here, as they were designed to work not only in \( \text{Lin} \) but also in \( \text{Dis} \), and therefore are not valid in \( \text{Den} \).

Table 9: MIS \((r)\), for each \( r \in \mathbb{N}^+ \); upper part: sets for which we give an explicit construction; lower part: symmetric ones. - Class: Den.

To prove that \( \{2_p\} \) is \( 04_{ii} \)-complete assuming density, suppose, first, that \( \mathcal{F} \models \varphi([a, b], [c, d]) \). Then \([c, d]\) must contain every point contained by \([a, b]\) and a point not contained in \([a, b]\). So we have \( c \leq a \leq b \leq d \) and \([a, b] \neq [c, d]\). To prove that \( a \neq c \) and \( d \neq b \), it is enough to observe that the third conjunct of \( \varphi \) states that there is no interval \( z_i \) which contains all points contained by \( x_i \) but not \( y_i \). If, on the other hand, \([a, b]04_{ii}[c, d]\), any point \( e \) such that \( a < e < c \) (which exists because the structure is dense) witnesses the \( z_p \) of the second conjunct of \( \varphi \), while the first and third conjuncts hold trivially. Finally, no new equations are required for the case \( 44_{ii} \).

4. Incompleteness Results In The Class Den

We can now turn our attention to the maximal incomplete sets for relations in \( \mathbb{N}^+ \). Notice that for some \( r \in \mathbb{N}^+ \) some \( r \)-incomplete set in the class \( \text{Lin} \) is also maximally \( r \)-incomplete in the class \( \text{Den} \); nevertheless, all proofs in Part I must be revisited here, as they were designed to work not only in \( \text{Lin} \) but also in \( \text{Dis} \), and therefore are not valid in \( \text{Den} \).

Lemma 14. Tab. [9] is correct.

Proof. Let \( S \) be \( \{2_p\} \cup 3^+ \): proving that it is \( =_p \)-incomplete is almost immediate. Indeed, it suffices to take \( D \) and \( D' \) both equal to the subset of \( Q \) of all points between 0 and 1, \( \zeta = (\zeta_p, \zeta_i) \), where \( \zeta_i = Id_i \) (the identical relation on intervals), \( \zeta_p = \{(0, 1')\} \) plus the identical relation on points to have a surjective truth-preserving relation that breaks \( =_p \).

Proving that \( \{=_p, <, a_p, l_p\} \) is \( =_i \)-incomplete is equally easy: it suffices to take \( D = D' = Q, \zeta_p = Id_p, \) and \( \zeta_i = Id_i \) plus \( \zeta_i([0, 2], [0', 1']) \). Assume, now, \( S \) to be \( \{=_p, =_i, l_p, 14_{ii}\} \); we need to prove its \(<, a_p \)-incompleteness in the dense case. Let \( D = D' = Q \), and define \( \zeta = \zeta_p \cup \zeta_i \) as follows: \((a, -a') \in \zeta_p \) for every \( a \in Q \), and \((a, b), [-a', -a' + |b' - a'|] \in \zeta_i \) for every \([a, b] \in \mathbb{I}(Q)\), so that the length of every interval is preserved while their
beginning point are reflected over 0; as this relation breaks both < and \(a_p\), the latter cannot be expressed in this language. Let now \(S\) be \(\{=_p, 2_p\} \cup \mathbb{J}^+\): we can prove that it is <, \(a_p\), \(l_p\), \(i_p\), \(4_p\), \(4_p\)-incomplete. To this end, it suffices to take, once again, \(D\) and \(D'\) both equal to the subset of \(Q\) of all points between 0 and 1, \(\zeta = (\zeta_p, \zeta_i)\), where \(\zeta_i = Id_i\) (the identical relation on intervals), \(\zeta_p = \{(0,1'), (1,0')\}\) plus the identical relation on every other point to have a surjective truth-preserving relation that breaks the relations under analysis. As for \(S=\{=, =_i, <, 3_p, 4_p, 04_i\}\), we can prove its \(a_p\), \(l_p\), \(2_p\), \(i_p\)-incompleteness, where \(i \in \mathbb{J}^+ \setminus \{=, l, 03_i\}\) by defining two \(Q\)-based structures and a relation between them defined as the identity between points and as \(\zeta_i([a, b]) = [a' - |b' - a'|, b']\), obtaining (as we did for the same set of relations on \(Lin\), using, in that case, a pseudo-discrete structure) a relation that maps every interval to the interval with the same ending point but twice the length. In this way, all relations in \(S\) are respected. The \(m\)-incompleteness of \(\{=, <\} \cup \mathbb{J}^+\) for each \(m \in \mathbb{M}^+\) can be proved by taking again \(D = D' = Q\), \(\zeta = (\zeta_p, \zeta_i)\), where \(\zeta_i = Id_i\) and \(\zeta_p(a) = a' + 1\) for every \(a \in Q\), which clearly respects all interval-interval relations, and both equality and relative ordering between points, but breaks every relation between points and intervals. When \(S\) is \(\{=, =_i, <, 04_i, 3_p\}\), we have to prove that it is \(14_i\)-incomplete. Consider two structures based on \(Q\), and let \(\zeta = (\zeta_p, \zeta_i)\) be defined as \(\zeta_p = Id_p\), and \(\zeta_i = Id_i\) except for the interval \([-1, 0]\), which is mapped to \([-1, 1]\). When \(S\) is \(\{=, =_i, 2_p, 04_i\}\), we have to prove that it is \(i\)-incomplete in the dense case, where \(i \in \mathbb{J}^+ \setminus \{04_i, =_i\}\). Consider two structures based on \(Q\), and let \(\zeta = (\zeta_p, \zeta_i)\) be defined as \(\zeta_p(a) = -a'\) for every point and \(\zeta_i([a, b]) = [-b', -a']\) for every interval. Clearly, containment is respected for both sorts; nevertheless, all other interval-interval relations are broken. Finally, when \(S\) is \(\{=, =_i, <, 04_i\}\), we have to prove that it is \(i\)-incomplete, where \(i \in \mathbb{J}^+ \setminus \{=, 04_i\}\). Consider two structures based on \(Q\), and let \(\zeta = (\zeta_p, \zeta_i)\) be defined as \(\zeta_p(a) = Id_p\), and \(\zeta_i([a, b]) = [-b, -a]\); again, we respect containment between intervals, and the relative ordering between points is respected as well (since points are not affected by the construction), and we break every other interval-interval relation.

\[\square\]

5. Completeness Results in the Class Unb

The ability of fragments of our language to define relations when the underlying linear order is unbounded (but not necessarily discrete or dense) differs from the dense/discrete cases only slightly. Following the same schema, we now focus on the definability part, again, pointing out the differences with the linear case.

5.1. Definability for the Relations \(=, =_i, <\) in Unb.

**Lemma 15.** Tab. 10 is correct.

**Proof.** Starting with \(=\), we have now that every mixed relation is \(=\)-complete, as follows:

\[x_p =_p y_p \iff \forall x_i(x_i, m \ x_p \leftrightarrow x_i \ m \ y_p), \ \{m\}, \ m \in \mathbb{M}^+\]

Observe that this definition is the same that we have used in the dense case for \(a_p\) and \(4_p\); the difference is that now, because of the unboundedness hypothesis, the argument also works for \(2_p\), as we prove now. Suppose, first, that \(x_p =_p y_p\); it is trivial to see that every
Table 10: The spectrum of the \textit{mcs}(r), for each $r \in \{=p, =i, <\}$. - Class: Unb.

| $=p$ | $=i$ | $<$ |
|------|------|------|
| $\{<\}$ | $\{0p, 3p\}$ | $\{0ip\}$ |
| $\{0ip\}$ | $\{0p, 4p\}$ | $\{1p, 2p\}$ |
| $\{2ip\}$ | $\{1p, 3p\}$ | $\{1p, 3p\}$ |
| $\{1ip\}$ | $\{1p, 4p\}$ | $\{1ip, 24ii\}$ |
| $\{3ip\}$ | $\{14ii\}$ | $\{1p, 34ii\}$ |
| $\{4ip\}$ | $\{03ii\}$ | $\{1ip, 44ii\}$ |
| $\{}$ | $\{34ii\}$ | $\{2p, 3p\}$ |
| $\{}$ | $\{04ii\}$ | $\{2ip, 14ii\}$ |
| $\{}$ | $\{44ii\}$ | $\{2ip, 24ii\}$ |
| $\{}$ | $\{2ip, 03ii\}$ | $\{2ip, 34ii\}$ |
| $\{}$ | $\{3p, 14ii\}$ | $\{3p, 4ii\}$ |
| $\{}$ | $\{3ip, 24ii\}$ | $\{3ip, 4ii\}$ |
| $\{}$ | $\{3ip, 04ii\}$ | $\{3ip, 44ii\}$ |
| $\{}$ | $\{3ip, 44ii\}$ | $\{3ip, 44ii\}$ |
| $\{}$ | $\{4p\}$ | $\{}$ |

interval \(x_i\) clearly has both \(x_p\) and \(y_p\) in relation \(_{2p}\) with it or none of them, satisfying the definition. Suppose, on the contrary that \(x_p = a\) and \(y_p = b\) are not equal, and, without loss of generality we can assume that \(x_p < y_p\). Since the underlying domain is unbounded, there must be a point \(c < a\), and therefore, the interval \([c, b]\) is such that \([c, b]_{2pa}\) but it is not the case that \([c, b]_{2pb}\), falsifying the right-hand side. Notice that this argument does not work on dense structures that are left/right bounded, such as \([0, 1] \subset \mathbb{Q}\). Now, consider the relation \(=i\), and the following definitions:

\[
x_i =_i y_i \leftrightarrow \begin{cases} 
\forall z_i(x_i, 04ii z_i \leftrightarrow y_i, 04ii z_i) \land \forall z_i(z_i, 04ii x_i \leftrightarrow z_i, 04ii y_i) & \{04ii\} \\
\forall z_i(x_i, 44ii z_i \leftrightarrow y_i, 44ii z_i) \land \forall z_i(z_i, 44ii x_i \leftrightarrow z_i, 44ii y_i) & \{44ii\}
\end{cases}
\]

Consider the set \(\{04ii\}\). If \(x_i =_i y_i\) we immediately have \(\varphi(x_i, y_i)\). Conversely, suppose that \([a, b] \neq i [c, d]\), so either \(a \neq c\) or \(b \neq d\). Consider, w.l.g., the case \(a \neq c\), specifically \(a < c\). We then choose a point \(e > \max\{b, d\}\), which does exist because the underlying domain is unbounded. Now, we have that the interval \([c, d]_{04ii[a, e]}\), but it is not the case that \([a, b]_{04ii[a, e]}\), falsifying the right-hand side. The other case is treated with a similar argument. Let us now focus on \(<\): six new definitions are needed:
Consider, first the set \( \{0_p\} \). Suppose that \( \mathcal{F} \models \varphi(a,b) \), and, by contradiction, that \( a \geq b \). If \( a = b \), then every interval \([c,d]\) such that \([c,d]0_p a\) must be such that \([c,d]0_p b\) as well, contradicting the second conjunct, and, if \( b < a \), then there exists an interval \([a,c]\) such that \([a,c]0_p b\) but it is not the case that \([a,c]0_p a\), contradicting the first conjunct. On the other hand, suppose that \( a < b \), and suppose that \([c,d]0_p b\), i.e., \( b < c \), and, hence, \( a < c \), therefore \([c,d]a_0 b\), satisfying the first conjunct; moreover, any interval \([b,c]\): clearly \([b,c]0_p a\) but it is not the case that \([b,c]0_p b\), satisfying the first conjunct. Consider, now, the set \( \{1_p, 4_p\} \).

Suppose that \( \mathcal{F} \models \varphi(a,b) \), and, by contradiction, that \( a \geq b \). If \( a = b \), then every interval starting at \( a \) must also start at \( b \), falsifying the first conjunct; if, on the other hand, \( b < a \), then for every interval of the type \([a,c]\), there exists an interval \([c,d]\) such that \([b,a]1_p b\), \([b,a]44_i[c,d]\), but it is not the case that \([a,c]44_i[c,d]\), contradicting the second conjunct. For the contrary, assume that \( a < b \). Then \([a,b]1_p a\) but it is not the case that \([a,b]1_p b\), satisfying the first conjunct; also, the interval \([a,b]\) is such that for any interval \([b,c]\) and any interval \([d,e]\) with \( d > c \) it is the case \([a,b]44_i[d,e]\), satisfying the second conjunct. As for \( \{1_p, 4_p\} \), suppose that \( \mathcal{F} \models \varphi(a,b) \), and, by contradiction, that \( a \geq b \). If \( a = b \), then for any interval containing \( a \) also contains \( b \), and vice versa, making the first three conjuncts not simultaneously satisfiable. If, on the other hand, \( b < a \), any interval containing \( b \) must start before \( a \), and therefore no interval containing \( a \) can overlap an interval containing \( b \), making \( \varphi(a,b) \) false. Conversely, assume that \( a < b \). We can take two interval \([c,b]\) such that \( c < a \) and \([a,d]\) such that \( b < d \) to witness \( \varphi(a,b) \), as we wanted. As for \( \{2_p, 4_p\} \), suppose that \( \mathcal{F} \models \varphi(a,b) \), and, by contradiction, that \( a \geq b \). If \( a = b \), then \( a \) and \( b \) are contained by the same intervals, making the first three conjuncts not simultaneously satisfiable. If, on the other hand, \( b < a \), then any interval containing \( a \) but not \( b \) must start at \( b \) or after, and therefore it cannot start an interval containing \( b \), falsifying \( \varphi(a,b) \). Conversely, suppose that \( a < b \); then, the intervals \([a,b]\) and \([a,c]\) for some \( c > b \) witness \( \varphi(a,b) \), as we wanted. Every other definition is now straightforward.

\[ \Box \]

5.2. Definability for \( M^+\)-Relations in \( Unb \).

**Lemma 16.** Tab. [11]

**Proof.** To justify all new complete sets for \( a_p \), only one new explicit definition is actually needed:
| $0_p$       | $1_p$       | $2_p$       |
|-------------|-------------|-------------|
| $\{1_p, 2_p\}$ | $\{0_p\}$   | $\{0_p, 3_p\}$ |
| $\{1_p, 3_p\}$ | $\{2_p\}$   | $\{0_p, 4_p\}$ |
| $\{1_p, 4_p\}$ | $\{2_p, 4_p\}$ | $\{0_p, 14, 44\}$ |
| $\{1_p, 24\}$ | $\{2_p, 14\}$ | $\{0_p, 24\}$ |
| $\{1_p, 34\}$ | $\{2_p, 24, 04\}$ | $\{0_p, 03\}$ |
| $\{1_p, 04\}$ | $\{2_p, 04\}$ | $\{0_p, 04\}$ |
| $\{1_p, 44\}$ | $\{2_p, 34\}$ | $\{1_p, 3_p\}$ |
| $\{1_p, \leq\}$ | $\{2_p, 44\}$ | $\{1_p, 14, 44\}$ |
| $\{2_p, 3_p\}$ | $\{3_p\}$ | $\{1_p, 4_p\}$ |
| $\{2_p, 4_p\}$ | $\{3_p, 14\}$ | $\{1_p, 24\}$ |
| $\{2_p, 14\}$ | $\{3_p, 24, 03\}$ | $\{1_p, 03\}$ |
| $\{2_p, 24, 04\}$ | $\{3_p, 03\}$ | $\{1_p, 34\}$ |
| $\{2_p, 24, 44\}$ | $\{3_p, 04\}$ | $\{1_p, 04\}$ |
| $\{2_p, 03\}$ | $\{3_p, 34\}$ | $\{1_p, 14\}$ |
| $\{2_p, 34\}$ | $\{4_p, 14\}$ | $\{1_p, 24\}$ |
| $\{2_p, 04, \leq\}$ | $\{4_p, 24, 03\}$ | $\{3_p, 04\}$ |
| $\{2_p, 44, \leq\}$ | $\{4_p, 03\}$ | $\{3_p, 03, 44\}$ |
| $\{3_p, 14\}$ | $\{4_p, 03, 44\}$ | $\{3_p, 34\}$ |
| $\{3_p, 24, 03\}$ | $\{4_p, 14\}$ | $\{4_p, 24\}$ |
| $\{3_p, 03, 04\}$ | $\{4_p, 04\}$ | $\{4_p, 04\}$ |
| $\{3_p, 03, 44\}$ | $\{4_p, 34\}$ | $\{4_p, 34\}$ |
| $\{3_p, 34\}$ | $\{4_p, 14\}$ | $\{4_p, 24\}$ |
| $\{4_p, 14\}$ | $\{4_p, 24\}$ | $\{4_p, 04\}$ |
| $\{4_p, 44\}$ | $\{4_p, 03, 44\}$ | $\{4_p, 34\}$ |

Table 11: The spectrum of the $\text{mcs}(r)$, for each $r \in \mathfrak{M}^+$. - Class: Unb.

$$x_i0_p y_p \iff \exists z_i(x_i0_(4_i)z_i \land \forall k_p(z_i2_p k_p \rightarrow \neg(k_p < y_p))). \quad \{2_p, 04, \leq\}$$

Proceeding as always, suppose that $\mathcal{F} \models \varphi([a, b], c)$. If $a \leq c$, since $a$ is contained in $z_i = [d, e]$, where $d < a$ and $b < e$, which exists because the underlying structure is unbounded, we have a contradiction; so $c < a$. If, on the other hand, $a < c$, then we take $z_i = [c, d]$, where $d > b$ exists by hypothesis, to satisfy the definition. As for $1_p$, we show that $\{0_p\}$ is $1_p$-complete by noticing that $\{0_p\}$ is $\leq, =_p$-complete and by using the following, straightforward definition:

$$x_i1_p y_p \iff \forall z_p(x_i0_p z_p \rightarrow z_p < y_p) \land \forall z_p(\neg(x_i0_p z_p) \rightarrow (y_p < z_p \lor y_p =_p z_p)). \quad \{1_p\}$$

Finally, we have two new definitions for $2_p$:

$$x_i2_p y_p \iff \forall z_p(x_i0_p z_p \rightarrow z_p < y_p) \lor \forall z_p(\neg(x_i0_p z_p) \rightarrow (y_p < z_p \lor y_p =_p z_p)). \quad \{2_p\}$$
Definability for Relations in $I_4$.

Let $z = a < c < b$. If, on the other hand, $a < c < b$, then we have the interval $[c, a]$ that contradicts the definition; $c$ cannot be $a$, and if $c \geq b$, then every interval $[c, d]$, where $d > c \geq b$ exists by hypothesis, again, leads to a contradiction. Thus, $a < c < b$. If, on the other hand, $a < c < b$ then $z_i = [c, b]$ witnesses the existential quantifier in the definition.

5.3. Definability for Relations in $I_3 \setminus \{=\}$ in Unb.

**Lemma 17.** Tab. 12 is correct.

**Proof.** The entire classification for interval-interval relations on an unbounded linear order, as depicted in Tab. 12, can be justified by just two new definitions, namely for one new $I_4$-complete plus one new $I_3$-complete set. Moreover, for the former we have that $\{44_i\}$ becomes $I_4$-complete, but both its definition and the relative argument can be borrowed.
that this set is able to express the weaker relation
Lemma 18.

Table 13: \( \text{MIS}(r) \), for each \( r \in \mathcal{R}^+ \); upper part: sets for which we give an explicit construction; lower part: symmetric ones. - Class: \( \text{Unb} \).

unchanged from the dense case. Thus, we are left with proving one new \( 34_{ii} \)-completeness, which is \( \{2, 0_{ii}, 04_{ii}\} \). We do so by proving, as we did in Lemma 3.1 of Part I of this paper, that this set is able to express the weaker relation \( 34_{ii} \cup 44_{ii} \):

\[
x_{i}34_{ii} \cup 44_{ii} y_{i} \leftrightarrow \exists z_{i}, k_{i}(x_{i} 04_{ii} z_{i} \land y_{i} 04_{ii} k_{i} \land z_{i} 24_{ii} k_{i}) \land \exists z_{i}(x_{i} 04_{ii} z_{i} \land \neg (y_{i} 04_{ii} z_{i})) \land \exists k_{i}(y_{i} 04_{ii} k_{i} \land \neg (x_{i} 04_{ii} k_{i})) \land \\
\neg (x_{i} 24_{ii} y_{i}) \land \neg (y_{i} 24_{ii} x_{i}) \land \neg (x_{i} 04_{ii} y_{i}) \land \neg (y_{i} 04_{ii} x_{i}).
\]

If \( \mathcal{F} \models \varphi[a, b], [c, d] \) over an unbounded structure, we can eliminate all wrong assignments for \( x_{i} \) and \( y_{i} \). First, observe that \( x_{i} \) and \( y_{i} \) cannot overlap nor contain each other. Next, if \( y_{i} \) ends before \( x_{i} \) or at its beginning point, it would be impossible to place \( z_{i} \) and \( k_{i} \). Finally, if \( x_{i} \) starts or finishes \( y_{i} \), or the other way around, we have a contradiction with the second or the third requirement of \( \varphi \). This implies that \( x_{i} \text{ meets or is before } y_{i} \). Conversely, if \( [a, b]34_{ii} \lor 44_{ii}[c, d] \), then we can take \( z_{i} = [e, d] \), where \( e < a \), and \( k_{i} = [a, f] \), where \( d < f \), and the existence of \( e \) and \( f \) is guaranteed by the assumption of unboundness.

6. Incompleteness Results in The Class \( \text{Unb} \)

We can now turn our attention to the maximal incomplete sets for relations in \( \mathcal{R}^+ \). Notice that for some \( r \in \mathcal{R}^+ \), some \( r \)-incomplete set in the class \( \text{Den} \) is also maximally \( r \)-incomplete in the class \( \text{Unb} \), and it has been proven so by means of a dense unbounded counterexample; in these cases, we can borrow the same argument unchanged.

Lemma 18. Tab. 13 is correct.
Proof. Let \( S \) be \( \mathcal{J}^+ \); proving that it is \( =_p \)-incomplete is very easy. Indeed, it suffices to take \( \mathbb{D} = \mathbb{D}' = \mathbb{Q} \), \( \zeta = (\zeta_p, \zeta_i) \), where \( \zeta_i = Id_i \) (the identical relation on intervals), \( \zeta_p = \{(0, 1')\} \) plus the identical relation on points to have a surjective truth-preserving relation that breaks \( =_p \). The \( =_i \)-incompleteness of \( \{=_p, <, 0_p, I_p\} \) is justified with the same argument used in the dense case (which was based on \( \mathbb{Q} \)). We can then prove that also \( \{=_p, <, 2_i, 24_{ii}\} \) is \( =_i \)-incomplete, by taking \( \mathbb{D} = \mathbb{D}' = \mathbb{Z} \), \( \zeta_p = Id_p \), and \( \zeta_i = Id_i \) plus \( \zeta_i([1, 2], [0', 1']) \).

For the \( <, 0_p \)-incompleteness of \( \{=_i, =_p, I_p, 14_{ii}\} \) we can recycle the argument used for the dense case (again, based on \( \mathbb{Q} \)). The \( < \)-incompleteness of \( \{=_p\} \cup \mathcal{J}^+ \) can be proved by taking \( \mathbb{D} = \mathbb{D}' = \mathbb{Q} \), \( \zeta = (\zeta_p, \zeta_i) \), where \( \zeta_i = Id_i \) and \( \zeta_p(a) = -a' \) for every \( a \in \mathbb{Q} \), which clearly respects all interval-interval relations and the equality between points. As for proving that \( \{=_p, <\} \cup \mathcal{J}^+ \) is \( m \)-incomplete for each \( m \in \mathfrak{M}^+ \) we can recycle the same argument as in the dense case, as it was based on the set \( \mathbb{Q} \). When \( S \) is \( \{=_p, =_i, 2_p, 4_{ii}\} \), we have to prove that it is \( <, 0_p, I_p, i \)-incomplete, where \( i \in \mathcal{J}^+ \setminus \{\mathfrak{a}_{4_{ii}}, =_i\} \). Consider two structures based on \( \mathbb{Q} \), and let \( \zeta = (\zeta_p, \zeta_i) \) be defined as \( \zeta_p(a) = -a' \) for every point and \( \zeta_i([a, b]) = [-b', -a'] \) for every interval. Clearly, containment is respected for both sorts; nevertheless, \( <, 0_p \) and \( I_p \) and all interval-interval relations, except \( \mathfrak{a}_{4_{ii}} \), are broken.

Once again, we have already proved that \( \{=_i, =_p, <, 3_p, 4_{ii}, 0_{3_{ii}}\} \) is \( 0_p, I_p, 2_p, i \)-incomplete, where \( i \in \mathcal{J}^+ \setminus \{\mathfrak{a}_{3_{ii}}, =_i\} \) when we were treating the dense case, and the same holds for the \( m \)-incompleteness of \( \{=_p, <\} \cup \mathcal{J}^+ \), where \( i \in \mathcal{J}^+ \) and \( m \in \mathfrak{M}^+ \). Consider now the \( 0_p, I_p, i \)-incompleteness of \( \{=_i, =_p, <, 2_p, 24_{ii}\} \), where \( i \in \mathcal{J}_p \cup \{24_{ii}, =_i\} \). Take \( \mathbb{D} = \mathbb{D}' = \mathbb{Z} \), \( \zeta_p = Id_p \) and \( \zeta_i([a, a+1]) = [a'-1, a'+2] \) plus the identical relation over every other interval; since the only intervals affected by \( \zeta \) are unitary, the relation \( 24_{ii} \) cannot be broken, and since such interval do not have internal points, the relation \( 2_i \) cannot be broken either.

Once more, the \( 4_{ii} \)-incompleteness of \( \{=_i, =_p, <, 0_p, I_p\} \) comes directly from the dense case, and the same holds for the \( i \)-incompleteness of \( \{=_i, =_p, <, 0_{4_{ii}}\} \), where \( i \in \mathcal{J}^+ \setminus \{\mathfrak{a}_{4_{ii}}\} \), which concludes the proof.

\[ \square \]

7. Harvest: The Complete Picture for Den and Unb

We are now capable to identify all expressively different subsets of \( \mathfrak{R}^+ \) under the hypotheses of linearity+denseness and linearity+unboundedness. Unlike Part I, we limit ourselves to list the maximally incomplete sets and the minimally complete sets for each of the two cases in the full language only.

**Theorem 19.** If a set of relations is listed:

- as \( \text{mcs}(\mathfrak{R}^+) \) in Tab. 14 left column (resp., right column), then it is minimally \( \mathfrak{R}^+ \)-complete (resp., maximally \( \mathfrak{R}^+ \)-incomplete) in the class of all dense linearly ordered sets.

- as \( \text{mcs}(\mathfrak{R}^+) \) in Tab. 15 left column (resp., right column), then it is minimally \( \mathfrak{R}^+ \)-complete (resp., maximally \( \mathfrak{R}^+ \)-incomplete) in the class of all unbounded linearly ordered sets.
Table 14: Minimally $\mathcal{R}^+$-complete and maximally $\mathcal{R}^+$-incomplete sets. - Class: Den.

8. Conclusions

We considered here the two-sorted first-order temporal language that includes relations between intervals, points, and inter-sort, and we treated equality between points and between intervals as any other relation, with no special role. Under four different assumptions on the underlying structure, namely, linearity only, linearity+discreteness, linearity+density, and linearity+unboundedness, we asked the question: which relation can be first-order defined by which subset of all relations? As a result, we identified all possible inter-definability between relations, all minimally complete, and all maximally incomplete subsets of relations. These inter-definability results allow one to effectively compute all expressively different subsets of relations, and, with minimal effort, also all expressively different subsets of relations for the interesting sub-languages of interval relations only or mixed relations only. Two out of four interesting classes of linearly ordered sets are treated in Part I of this paper, while the remaining two are dealt with in the present one (Part II). There are several aspects of temporal reasoning in computer science to which this extensive study can be related:
- first-order logic over linear orders extended with temporal relations between points, intervals and mixed, are the very foundation of modal logics for temporal reasoning, and it is necessary to have a complete understanding of the former in order to deal with the latter;
- automated reasoning techniques for interval-based modal logics are at their first stages; an uncommon, but promising approach is to treat them as pure modal logics over particular Kripke-frames, whose first-order properties are, in fact, representation theorems such as those (indirectly) treated in this paper. As a future work, we also plan to systematically study the area of representation theorems;

| mcs                  | MIS                        |
|----------------------|----------------------------|
| \{0_{ip}, 2_{ip}\}   | \{0_{ip}, 1_{ip}, 14_{ii}, =_{i}, =_{p}, <\} |
| \{0_{ip}, 3_{ip}\}   | \{2_{ip}, 24_{ii}, =_{i}, =_{p}, <\} |
| \{0_{ip}, 4_{ip}\}   | \{2_{ip}, 04_{ii}, =_{i}, =_{p}\} |
| \{0_{ip}, 24_{ii}\}  | \{4_{ip}, 03_{ii}, =_{i}, =_{p}, <\} |
| \{0_{ip}, 03_{ii}\}  | \{14_{ii}, 24_{ii}, 03_{ii}, 34_{ii}, 04_{ii}, 44_{ii}, =_{i}, =_{p}, <\} |
| \{0_{ip}, 34_{ii}\}  | \{2_{ip}, 04_{ii}, <\} |
| \{0_{ip}, 04_{ii}\}  | \{2_{ip}, 44_{ii}\} |
| \{0_{ip}, 44_{ii}\}  | \{3_{ip}, 14_{ii}\} |
| \{1_{ip}, 2_{ip}\}   | \{3_{ip}, 24_{ii}\} |
| \{1_{ip}, 3_{ip}\}   | \{3_{ip}, 34_{ii}\} |
| \{1_{ip}, 4_{ip}\}   | \{3_{ip}, 04_{ii}\} |
| \{1_{ip}, 24_{ii}\}  | \{3_{ip}, 44_{ii}\} |
| \{1_{ip}, 03_{ii}\}  | \{4_{ip}, 14_{ii}\} |
| \{1_{ip}, 34_{ii}\}  | \{4_{ip}, 24_{ii}\} |
| \{1_{ip}, 44_{ii}\}  | \{4_{ip}, 34_{ii}\} |
| \{1_{ip}, 04_{ii}\}  | \{4_{ip}, 04_{ii}\} |
| \{1_{ip}, 44_{ii}\}  | \{4_{ip}, 44_{ii}\} |

Table 15: Minimally $\mathcal{R}^+$-complete and maximally $\mathcal{R}^+$-incomplete sets. - Class: Unb.
• the decidability of pure first-order theories extended with interval relations is well-known [Lad]; nevertheless, these results hinge on the decidability of MFO[$\leq$], while we believe that they could be refined both algorithmically and computationally;
• the study of other related languages, important in artificial intelligence, can benefit from our results, such as first-order and modal logics for spatial reasoning where basic objects are, for example, rectangles.

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