Design and Performance Analysis of a New Class of Rate Compatible Serial Concatenated Convolutional Codes

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Abstract

In this paper, we provide a performance analysis of a new class of serial concatenated convolutional codes (SCCC) where the inner encoder can be punctured beyond the unitary rate. The puncturing of the inner encoder is not limited to inner coded bits, but extended to systematic bits. Moreover, it is split into two different puncturings, in correspondence with inner code systematic bits and parity bits. We derive the analytical upper bounds to the error probability of this particular code structure and address suitable design guidelines for the inner code puncturing patterns. We show that the percentile of systematic and parity bits to be deleted strongly depends on the SNR region of interest. In particular, to lower the error floor it is advantageous to put more puncturing on inner systematic bits. Furthermore, we show that puncturing of inner systematic bits should be interleaver dependent. Based on these considerations, we derive design guidelines to obtain well-performing rate-compatible SCCC families. Throughout the paper, the performance of the proposed codes are compared with analytical bounds, and with the performance of PCCC and SCCC proposed in the literature.

Index Terms

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I. INTRODUCTION

Rate-compatible codes were introduced for the first time in [1], where the concept of punctured codes was extended to the generation of a family of rate-compatible punctured convolutional (RCPC) codes. The rate-compatibility restriction requires that the rates are organized in a hierarchy, where all code bits of a high rate punctured code are used by all the lower rate codes. Based on RCPC codes, Hagenauer proposed an ARQ strategy which provides a flexible way to accommodate code rate to the error protection requirements, or varying channel conditions. Furthermore, rate-compatible codes can be used to provide unequal error protection (UEP). The concept of rate-compatible codes has then been extended to parallel and serial concatenated convolutional codes [2–4].

Recently, a new class of hybrid serial concatenated codes was proposed in [5] with bit error performance between that of PCCC and SCCC. A similar concept has been presented in [6] to obtain well performing rate-compatible SCCC families. To obtain rate-compatible SCCCs, the puncturing is limited to inner coded bits. However, in contrast to standard SCCC, codes in [6] are obtained puncturing both inner parity bits and systematic bits, thereby obtaining rates beyond the outer code rate. With this assumption, puncturing is split into two puncturing patterns, for both systematic and parity bits. This particular code structure offers very good performance over a range of rates, including very high ones, and performs better than standard SCCC.

The optimization problem of this particular code structure consists in optimizing these two puncturing patterns and finding the optimal proportion of inner code systematic and parity bits to be punctured to obtain a given rate. Some design criteria to obtain good rate-compatible SCCC families are discussed in [6]. However, the considerations in [6] are limited to heuristic design guidelines, with no theoretical analysis support. Thus, a deeper and more formal insight on the performance of this new class of SCCCs is required, in order to provide suitable design guidelines aimed at the code optimization.

In this paper, we provide a performance analysis of this new class of concatenated codes. By properly redrawing the SCCC as a parallel concatenation of two codes, we derive the analytical upper bounds to the error probability using the concept of uniform interleaver. We then propose
suitable design criteria for the inner code puncturing patterns, and to optimize the proportion of inner systematic and parity bits to be deleted. We show that the optimal percentage of bits to be punctured depends on the SNR region of interest. In particular, it is shown that to improve the performance in the error floor region, it is advantageous to increase the proportion of surviving inner code parity bits, as far as a sufficient number systematic bits is kept. Moreover, the optimal puncturing of the inner code systematic bits depends on the outer encoder and, thus, it must be interleaver dependent. Finally, based on these considerations, we address design guidelines to obtain well-performing SCCC families.

The paper is organized as follows. In the next section, we describe the new class of concatenated codes addressed in the paper. In Section III, the upper bounds to the residual bit error probability and frame error probability of this new class of codes are derived and design criteria are outlined. Design guidelines to obtain well-performing SCCC families are discussed in Section IV. In Section V, simulation results are compared with the analytical upper bounds. Finally, in Section VI we draw some conclusions.

II. A NEW CLASS OF SERIAL CONCATENATED CONVOLUTIONAL CODES

Throughout the paper we shall refer to the encoder scheme shown in Fig. 1.

We consider the serial concatenation of two systematic recursive convolutional encoders. To obtain high rates both encoders are punctured. However, in contrast to standard SCCC where high rates are obtained by concatenating an extensively punctured outer encoder with an inner encoder of rate \( R_{i_c} \leq 1 \) such that the rate of the SCCC, \( R_{SCCC} \), is at most equal to the rate of the outer encoder (\( R_{SCCC} \leq R_{o_c} \)), the inner encoder in Fig. 1 can be punctured beyond the unitary rate, i.e., the overall code rate \( R_{SCCC} \) can be greater than the outer code rate \( R_{o_c} \). Moreover, as made evident in the figure, puncturing is not directly applied to the inner code sequence but split into two different puncturings, in correspondence to inner code systematic bits and inner code parity bits (\( P_{s_i} \) and \( P_{p_i} \), respectively). Assuming an inner mother code of rate \( 1/n \), the rate of the resulting SCCC is given by

\[
R_{SCCC} = R_{o_c}' R_{i_c} = R_{i_c} \frac{1}{\rho_s + (n-1)\rho_p}
\]

where \( R_{o_c}' \) is the outer code rate after applying the fixed puncturing pattern \( P_o \), and \( \rho_s \) (\( \rho_p \)) is the systematic permeability (parity permeability) rate, defined as the proportion of inner code
systematic bits (parity bits) which are not punctured. Given a certain desired $R_{SCCC}$, $\rho_s$ and $\rho_p$ are related by

$$\rho_s = \frac{R_o^{d}}{R_{SCCC}} - (n - 1)\rho_p. \quad (2)$$

This particular code structure offers superior performance to that of standard SCCC, especially for high-rates. Notice that for high rates, the exhaustive puncturing of the outer code leads to a poor code in terms of free distance, thus leading to a higher error floor. On the contrary, the code structure discussed here, keeps the interleaver gain for low rates also in the case of very high rates, since the heavy puncturing is moved to the inner encoder. Moreover it is well suited for rate-compatible schemes.

It is clear that the performance of the overall SCCC code depends on puncturing patterns $P_o$, $P_s^i$ and $P_p^i$, and, subsequently, on the permeability rates $\rho_s$ and $\rho_p$, which should be properly optimized. In [6], some heuristic design guidelines were given to select $\rho_s$ and $\rho_p$, leading to well-performing families of rate-compatible SCCCs. However, the work in [6] lacks in providing formal analysis to clarify the behavior of this code structure and to provide a unique framework to properly select $\rho_s$ and $\rho_p$. The aim of this paper is to address design guidelines to clarify some relevant aspects of this new code structure, and to provide the clues for the code optimization.

The design of concatenated codes with interleavers involves the choice of the interleaver and the constituent encoders. The joint optimization, however, seems to lead to prohibitive complexity problems. In [7] Benedetto and Montorsi proposed a method to evaluate the error probability of parallel concatenated convolutional codes (PCCC) independently from the interleaver used. The method consists in a decoupled design, in which one first designs the constituent encoders, and then tailors the interleaver on their characteristics. To achieve this goal, the notion of uniform interleaver was introduced in [7]; the actual interleaver is replaced with the average interleaver\(^1\). The use of the uniform interleaver drastically simplifies the performance evaluation of Turbo Codes. Following this approach, the best constituent encoders for serial code construction are found in [8], where the analysis in [7] was extended to SCCCs, giving design criteria for constituent encoders.

In the next section, we gain some analytical insight into the code structure of Fig. 1 to address design guidelines to properly select $\rho_s$, $P_s^i$ and $\rho_p$, $P_p^i$. To this purpose, we derive the analytical

\(^1\)This average interleaver is actually the weighted set of all interleavers.
upper bounds to the bit and frame error probability, following the concept of uniform interleaver used in [7] and [8] for PCCC and SCCC. However, we do not treat the code structure of Fig. 1 as a standard SCCC, so we cannot directly apply the considerations in [8]. Indeed, the treatment in [8] would consider the inner encoder (with its puncturing) as a unique entity, therefore diluting the contribution of the inner code systematic bits and parity bits to the bound. Instead, our idea is to decouple the contribution of the inner systematic bits and inner parity bits to the error probability bound to better identify how to choose $\rho_s$, $P^s$ and $\rho_p$, $P^p$. In fact, we shall show that to obtain good SCCC codes in the form of Fig. 1 the selection of the inner code puncturing directly depends on the outer code, which has a crucial effect on performance. This dependence cannot be taken into account by the upper bounds derived in [8] for SCCC.

III. Analytical Upper Bounds to the Error Probability

Following the derivations in [7] and [8] for PCCC and SCCC, in this section we derive the union bound of the bit error probability for the code construction of Fig. 1.

Recalling [8], the bit error probability of a SCCC can be upper bounded through

$$P_b(e) \leq \sum_{w=w_{m}}^{N R^C_o} \frac{w}{N R^C_o} A_{w, H}^{C_s}(w, H) \left|_{H=e^{-\frac{R_{SCCC} E_b}{N_0}}} \right.$$  

$$= \sum_{l=h_m}^{N R^C_o} \sum_{w=w_{m}}^{N R^C_o} \frac{w}{N R^C_o} A_{w, H}^{C_s} e^{-\frac{h R_{SCCC} E_b}{N_0}}$$  

where $w_{m}$ is the minimum weight of an input sequence generating an error event of the outer code, $N$ is the interleaver length, and $h_m$ is the minimum weight of the codewords of the SCCC, $C_s$, of rate $R_{SCCC}$. $A_{w, H}^{C_s}(w, H)$ is the Conditional Weight Enumerating Function (CWEF) of the overall SCCC code. For a generic serially concatenated code, consisting of the serial concatenation of an outer code $C_o$ with an inner code $C_i$ through an interleaver, the CWEF of the overall SCCC code $A_{w, H}^{C_s}$ can be calculated replacing the actual interleaver with the uniform interleaver and exploiting its properties. The uniform interleaver transforms a codeword of weight $l$ at the output of the outer encoder into all distinct $\left(\frac{N}{l}\right)$ permutations. As a consequence, each codeword of the outer code $C_o$ of weight $l$, through the action of the uniform interleaver, enters the inner encoder generating $\left(\frac{N}{l}\right)$ codewords of the inner code $C_i$. The CWEF of the overall SCCC code can then be evaluated from the knowledge of the CWEFs of the outer and inner
codes; the coefficients $A_{w,h}^{C_s}$ are given by

$$A_{w,h}^{C_s} = \sum_{l=0}^{N} \binom{N}{l} \frac{A_{o,l}^{C_o} \times A_{i,l}^{C_i}}{A_{w,l,j}^{C''_o} \times A_{l,m}^{C'_i}}$$

(4)

where $A_{w,l}^{C_o}$ and $A_{l,h}^{C_i}$ are the coefficients of the CWFEs of the outer and inner codes, respectively.

This is basically the same result obtained in [8]. However, and this is the key novelty of our analysis, to evaluate the performance of the code structure of Fig. 1 instead of proceeding as in [8] using (4), it is more suitable to refer to Fig. 2, which properly redraws the encoder of Fig. 1 for the derivation of the upper bound. Fig. 2 allows us to decouple the contributions of the inner code puncturings $P_s^i$ and $P_p^i$ to the error probability bound. Call $C''_o$ the code obtained from the puncturing of the outer code $C_o$ through $P_o$ and $P'$, with $P' = \Pi^{-1}[P_s^o]$, i.e., the de-interleaved version of $P_s^o$, $C'_o$ the code obtained from the puncturing of the outer code $C_o$ through $P_o$, and $C'_i$ the inner encoder $C_i$ generating only parity bits punctured through $P_p^i$, which is fed with an interleaved version of codewords generated by $C''_o$. Now, the serial concatenated code structure under consideration can be interpreted as the parallel concatenation of the code $C''_o$ and $C'_i$. Therefore, the SCCC codeword weight $h$ can be split into two contributions $j$ and $m$, corresponding to the output weights of the codewords generated by encoder $C''_o$ and by encoder $C'_i$, respectively, such that $h = j + m$. With reference to Fig. 2 equation (4) can then be rewritten as

$$A_{w,h}^{C_s} = A_{w,j+m}^{C_s} = \sum_{l=d^{i'}_l}^{N} \sum_{j=d^{i''}_j}^{N/R_o''} \frac{A_{w,l,j}^{C''_o} \times A_{l,m}^{C'_i}}{\binom{N}{l}}$$

(5)

where $d^{i'}_l$ is the free distance of the code $C'_o$ and $d^{i''}_j$ is the free distance of the code $C''_o$. In (5), $R_o''$ is the rate of the code $C''_o$, $A_{w,l,j}^{C''_o}$ indicates the number of codewords of $C''_o$ of weight $j$ associated with a codeword of $C'_o$ of weight $l$ generated from an information word of weight $w$.

3Notice that, in abuse of notation, we have maintained the terminology outer encoder and inner encoder in Fig. 2 though they do not strictly act as outer and inner encoders. However, we believe that this notation reflects better the correspondence with Fig. 1.
and $A^c_{w, l, j}$ indicates the number of codewords of $C'_i$ of weight $m$ associated with a codeword of $C'_o$ of weight $l$.

$A^c_{w, l, j}$ and $A^c_{l, m}$ can be expressed as

$$
A^c_{w, l, j} = \sum_{n_{o''} = 1}^{n_{M}} \left( \frac{N/p}{n_{o''}} \right) A^o_{w, l, j, n_{o''}}
$$

$$
A^c_{l, m} = \sum_{n_{o'} = 1}^{n_{M}} \left( \frac{N/p}{n_{o'}} \right) A^{o'}_{l, m, n_{o'}}
$$

(6)

where the coefficient $A^o_{w, l, j, n_{o''}}$ represents the number of code $C'_o$ sequences of weight $j$, associated with a codeword of $C'_o$ of weight $l$ generated from an information word of weight $w$, and number of concatenated error events $n_{o''}$. In (6), $n_{M}$ is the largest number of error events concatenated in a codeword of the code $C'_o$ of output weight $j$ associated with a codeword of $C'_o$ of weight $l$ and an information word of weight $w$: $n_{M}$ is a function of $w$, $l$ and $j$ that depends on the encoder. Also in (6), the coefficient $A^{o'}_{l, m, n_{o'}}$ represents the number of code $C'_i$ sequences of weight $m$, input weight $l$, and number of concatenated error events $n_{o'}$. As for $n_{M}$, $n_{M}$ is the largest number of error events concatenated in a codeword of the code $C'_i$ of output weight $m$ generated from an information word of weight $l$.

Substituting (6) in (5), the value of the coefficients $A^c_{w, j + m}$ is upper bounded as

$$
A^c_{w, j + m} \leq \sum_{l = d_j}^{N/R_c^o} \sum_{j = d_i}^{N/R_c^o} \sum_{n_{o''} = 1}^{n_{M}} \sum_{n_{o'} = 1}^{n_{M}} \left( \frac{N/p}{n_{o''}} \right) \left( \frac{N/p}{n_{o'}} \right) \left( \frac{N}{l} \right) \cdot A^o_{w, l, j, n_{o''}} A^{o'}_{l, m, n_{o'}}
$$

$$
\leq \sum_{l = d_j}^{N/R_c^o} \sum_{j = d_i}^{N/R_c^o} \sum_{n_{o''} = 1}^{n_{M}} \sum_{n_{o'} = 1}^{n_{M}} \left( \frac{N^{n_{o''} + n_{o'} - 1} l^1}{l!} \right) \cdot A^o_{w, l, j, n_{o''}} A^{o'}_{l, m, n_{o'}}
$$

(7)

Finally, substituting (7) into (3), we obtain the upper bound for the bit error probability,

$$
P_b(e) \leq \sum_{j = m = h_m}^{N/R_c^o} e^{-\frac{(j + m) R_{SCCC} E_k}{N_0}} \cdot \sum_{w = w_0}^{N/R_c^o} \sum_{l = d_i}^{N/R_c^o} \sum_{j = d_i}^{N/R_c^o} \sum_{n_{o''} = 1}^{n_{M}} \sum_{n_{o'} = 1}^{n_{M}} \left( \frac{N^{n_{o''} + n_{o'} - 1} l^1}{l!} \right) \frac{w}{p^{n_{o''} + n_{o'}! n_{o'}! n_{o''}!}} \frac{w}{R_c} A^o_{w, l, j, n_{o''}} A^{o'}_{l, m, n_{o'}}
$$

(8)
Equivalently, the upper bound for the frame error probability is given by
\[
P_f(e) \leq \sum_{j+m = h_m}^{N/R_i} e^{-\frac{(j+m)R_{SCCC} E_b}{N_0}}
\]
\[
\cdot \sum_{w=w_{w_i}}^{N/R_i'} N \sum_{l=d_l}^{N/R_i''} \sum_{j=d_j}^{n''} \sum_{n'=1}^{n'} \sum_{l'=1}^{l'} N^{n'' + n' - l} \frac{l!}{w_{w_i} l_{l_i} n_{n_i} A_{w,l,j,n} A_{l,m,n}}
\]
(9)

For large \(N\) and for a given \(h = j + m\), the dominant coefficient of the exponentials in (8) and (9) is the one for which the exponent of \(N\) is maximum [8]. This maximum exponent is defined as
\[
\alpha(h = j + m) \triangleq \max_{w,i,l} \{n'' + n' - l - 1\}
\]
(10)

For large \(E_b/N_0\), the dominating term is \(\alpha(h_m)\), corresponding to the minimum value \(h = h_m\),
\[
\alpha(h_m) \leq 1 - d_f
\]
(11)
and the asymptotic bit error rate performance is given by
\[
\lim_{E_b/N_0 \to \infty} P_b(e) \leq B N^{1-d_f} \text{erfc} \left( \sqrt{\frac{h_m R_{SCCC} E_b}{N_0}} \right)
\]
(12)
where \(B\) is a constant that depends on the weight properties of the encoders, and \(N\) is the interleaver length.

On the other hand, the dominant contribution to the bit and frame error probability for \(N \to \infty\) is the largest exponent of \(N\), defined as
\[
\alpha_M \triangleq \max_h \alpha(h = j + m) = \max_{w,i,l,h} \{n'' + n' - l - 1\}
\]
(13)

We consider only the case of recursive convolutional inner encoders. In this case, \(\alpha_M\) is given by
\[
\alpha_M = - \left\lfloor \frac{d_f}{2} + 1 \right\rfloor
\]
(14)
and
\[
\lim_{N \to \infty} P_b(e) \leq K N^{\alpha_M} \text{erfc} \left( \sqrt{\frac{h(\alpha_M) R_{SCCC} E_b}{N_0}} \right)
\]
(15)
where again \(K\) is a constant that depends on the weight properties of the encoders and \(h(\alpha_M)\) is the weight associated to the highest exponent of \(N\).
Now, denoting by $d_{\text{f},\text{eff}}$ the minimum weight of inner code $C_i'$ sequences generated by input sequences of weight 2, we obtain the following results for the weight $h(\alpha_M)$ associated to the highest exponent of $N$: \[
 h(\alpha_M) = \frac{d_{\text{f}}' d_{\text{f},\text{eff}}'}{2} + d^{\text{fe}}(d_{\text{f}}') \quad \text{if} \ d_{\text{f}}' \ \text{even} \]
\[
 h(\alpha_M) = \frac{(d_{\text{f}}' - 3)d_{\text{f},\text{eff}}'}{2} + h^{(3)}_{m} + d^{\text{fe}}(d_{\text{f}}') \quad \text{if} \ d_{\text{f}}' \ \text{odd} \]
where $d^{\text{fe}}(d_{\text{f}}')$ is the minimum weight of $C_o''$ code sequences corresponding to a $C_o'$ code sequence of weight $d_{\text{f}}'$ and $h^{(3)}_{m}$ is the minimum weight of sequences of the inner code $C_i'$ generated by a weight-3 input sequence.

Finally, since $d^{\text{fe}}(d_{\text{f}}') \geq d^{\text{fe}}$ (asymptotic with respect to $N$) we can also write
\[
 h(\alpha_M) \geq \frac{d_{\text{f}}' d_{\text{f},\text{eff}}'}{2} + d^{\text{fe}} \quad \text{if} \ d_{\text{f}}' \ \text{even} \]
\[
 h(\alpha_M) \geq \frac{(d_{\text{f}}' - 3)d_{\text{f},\text{eff}}'}{2} + h^{(3)}_{m} + d^{\text{fe}} \quad \text{if} \ d_{\text{f}}' \ \text{odd} \]

From (15) and (16) we obtain the following result for the error probability:
\[
P_b(e) \leq C_{\text{even}} N^{-d_{\text{f}}'/2} \text{erfc} \left( \sqrt{\left( \frac{d_{\text{f}}' d_{\text{f},\text{eff}}'}{2} + d^{\text{fe}}(d_{\text{f}}') \right) \frac{R_{\text{SCCC}} E_b}{N_0}} \right) \tag{18}
\]
if $d_{\text{f}}'$ is even, and
\[
P_b(e) \leq C_{\text{odd}} N^{-\frac{d_{\text{f}}'+1}{2}} \text{erfc} \left( \sqrt{\left( \frac{(d_{\text{f}}' - 3)d_{\text{f},\text{eff}}'}{2} + h^{(3)}_{m} + d^{\text{fe}}(d_{\text{f}}') \right) \frac{R_{\text{SCCC}} E_b}{N_0}} \right) \tag{19}
\]
if $d_{\text{f}}'$ is odd. Constants $C_{\text{even}}$ and $C_{\text{odd}}$ can be derived as in [8] for SCCC.

We observe that the coefficient $h(\alpha_M)$ increases with $d_{\text{f},\text{eff}}'$, $d^{\text{fe}}(d_{\text{f}}')$ and also with $h^{(3)}_{m}$ in the case of odd $d_{\text{f}}'$. This suggests that, to improve the performance, one should choose a suitable combination of $C_o''$ and $C_i'$ such that $h(\alpha_M)$ is maximized, and the puncturing patterns $P_o$, $P'$ and $P_i^p$ (and subsequently permeabilities $\rho_o$ and $\rho_p$) should be selected accordingly. Moreover, such a combination depends on the value of $d_{\text{f}}'$. For instance, if $d_{\text{f}}' = 4$ the term $d_{\text{f},\text{eff}}'$ appears to be dominant with respect to $d^{\text{fe}}(d_{\text{f}}')$, since it is multiplied by a factor two ($d_{\text{f},\text{eff}}'/2$), whereas for $d_{\text{f}}' = 2$ both contributions are equally weighted.

Notice also that the contribution of the code $C_o''$ to $h(\alpha_M)$, given by $d^{\text{fe}}(d_{\text{f}}')$, corresponds to the contribution of the inner code systematic part in Fig. [1]. Therefore, since $d^{\text{fe}}(d_{\text{f}}')$ depends on
the outer code, to optimize the puncturing pattern $P^s_i (P^s_i = \Pi[P'])$ of the inner code systematic bits, one must take into account this dependence.

We can draw from (18) and (19) some important design considerations:

- As for traditional SCCC, $P_o$ should be chosen to optimize the outer code distance spectrum.
- The coefficient that multiplies the signal to noise ratio $E_b/N_0$ increases with $d_{i, \text{eff}}^{d_{i}}$ and $d_{o}^{d_{i}}(d_{i}^{d_{i}})$. Thus, we deduce that $P'$ and $P^p_i$ should be chosen so that $h(\alpha_M)$ is maximized. This implies to select a suitable combination of permeabilities $\rho_s$ and $\rho_p$. For a fixed pair $\rho_s$ and $\rho_p$, $P^p_i$ must be optimized to yield the best encoder $C'_i$ IOWEF. Furthermore, $P'$ (i.e. $P^s_i$) must be selected to optimize $d_{o}^{d_{i}}(d_{i}^{d_{i}})$. If we consider (16) instead of (17), the criterion is equivalent to optimize the distance spectrum of $C''_o$. Notice that this is equivalent to optimize the outer code $C_o$ punctured through $P_o$ and $P'$ with permeability $\rho_s$. Then, $P^s_i$ must be set to the interleaved version of $P'$, i.e., $P^s_i = \Pi[P']$. Therefore, $P^s_i$ turns out to depend on the outer code, and thus, it is also interleaver dependent. We stress the need to optimize $P^s_i$ according to this dependence.

A complementary analysis tool for the design of concatenated schemes would be to consider the EXIT charts or equivalent plots [15, 16]. These analysis techniques explain very well the behavior of iterative decoding schemes in the low SNR region (convergence region) and often lead to design rules that are in contrast with those outlined in this section, which are more suited for the analysis in the error floor region. Unfortunately, EXIT chart analysis is mainly based on Monte Carlo simulations and does not allows to extract useful code design parameters. For this reason we have not included this technique in the paper. The reader however should be warned that for the careful design of concatenated schemes both aspects must be considered and this implies that comparison of the designed schemes through simulation cannot be avoided. This fact also allow to justify some differences in the simulation results which are not evident from the uniform interleaver analysis. A convergence analysis of this class of SCCC will be discussed in a forthcoming paper.

IV. RATE-COMPATIBLE SERIAL CONCATENATED CONVOLUTIONAL CODES

Rate-compatible serial concatenated convolutional codes are obtained by puncturing inner code bits with the constraint that all the code bits of a high rate code must be kept in all lower rate codes. Depending on the puncturing pattern, the resulting code may be systematic (none
of the systematic bits are punctured), partially systematic (a fraction of the systematic bits are punctured) or non-systematic (all systematic bits are punctured). In [9] it was argued that a systematic inner code performs better than a partially systematic code. This result was assumed in [4] and [10] to build rate-compatible SCCCs limiting puncturing to inner parity bits. This assumption, however, is not valid for all SNRs. Indeed, keeping some systematic bits may be beneficial for speed up iterative decoding convergence. Since puncturing is limited to inner parity bits, the rate of the SCCC satisfies the constraint \( R_{SCCC} \leq R'_o \). As already stated, in contrast to [4] and [10] we do not restrict puncturing to parity bits, but extend it also to systematic bits, thus allowing \( R_{SCCC} \) beyond the outer code rate \( R'_o \), which provides a higher flexibility.

Assuming an outer encoder puncturing pattern fixed (\( P_o \) in Fig. 1), the design of well-performing rate-compatible SCCCs in the form of Fig. 1 limits to optimize the inner code puncturing patterns for systematic and parity bits according to the design criteria outlined in the previous section, with the constraint of rate-compatibility. Applying these design rules, optimal SCCC families can be found considering inner systematic and inner parity bits separately:

- To find the optimum puncturing pattern for inner code parity bits, start puncturing the inner mother code parity bits one bit at a time, fulfilling the rate-compatibility restriction. Define as \( d_w \) the minimum weight of inner codewords generated by input words with weight \( w \), and by \( N_w \) the number of nearest neighbors (multiplicities) with weight \( d_w \). Select at each step the candidate puncturing pattern \( P_{p} \) for the inner code parity bits as the one optimizing its IOWEF, i.e., yielding the optimum values for \((d_w, N_w)\) for \( w = 2, \ldots, w_{\text{max}} \) (first \( d_w \) is maximized and then \( N_w \) is minimized).

- Select the candidate puncturing pattern \( P' \) as the one yielding the best outer code (punctured through \( P_o \) and \( P' \) ) output weight enumerating function (OWEF). Namely, to find the optimum puncturing pattern for inner code systematic bits, start puncturing the outer mother code output bits one bit at a time, fulfilling the rate-compatibility restriction.

Define as \( A_d \) the number of nearest neighbors (multiplicities) with output distance \( d \) of the outer code. Select at each step the candidate puncturing pattern \( P' \) as the one yielding the optimum values for \( A_d \), i.e., the one which sequentially optimize the values \( A_d \) for
Since also outer code information bits are punctured, the invertibility\(^3\) of the outer code at each step must be guaranteed. At the end, since the systematic bits at the input of the inner encoder are an interleaved version of the outer encoder output bits, take the best puncturing pattern \(P'\) and apply its interleaved version \(P_s = \Pi[P']\) to inner code systematic bits (see Figs. 1 and 2).

V. SIMULATION RESULTS AND COMPARISON WITH ANALYTICAL BOUNDS

The performance of rate-compatible SCCCs mainly depend on its overall rate \(R_{SCCC}\) and on the selected combination of \(\rho_s\) and \(\rho_p\). In this Section, based on the considerations drawn in Section III and IV, we discuss how to properly select \(\rho_s\) and \(\rho_p\). We compare through simulation several rate-compatible puncturing schemes, with different interleaver lengths, and compare the performance of the proposed codes with the upper bounds to the error probability.

We consider the serial concatenation of two rate-1/2, 4-states, systematic recursive encoders, with generator polynomials \((1, 5/7)\) in octal form. The outer encoder is punctured to rate 2/3 by applying a fixed puncturing pattern. In particular, two puncturing patterns \(P_o\) have been taken into account, namely \(P_{o,1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\) and \(P_{o,2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\). The overall code rate is, thus, \(R_{SCCC} = 1/3\). Higher rates are then obtained by puncturing the inner encoder through puncturing patterns \(P_i^s\) and \(P_i^p\) for systematic and parity bits, respectively, as previously discussed. The free distance of the outer encoder, \(d_{o,free}\), when puncturing pattern \(P_{o,1}\) is applied, is odd and equal to 3, whereas for \(P_{o,2}\), \(d_{i,o}^p\) is even and equal to 4. Some considerations must be done at this point:

1) If \(d_{i,o}^p = 3\), \(\alpha_M = -\left[\frac{d_{i,o}^p+1}{2}\right] = -2\). In this case, the minimum weight of inner code input sequences that yields \(\alpha_M = -2\) (since \(n^{o''} = n^{i''} = 1\)) is \(l_{\min} = 3\), and \(h(\alpha_M) = h_{i,o}^{(3)} + d^{o''}(d_{i,o}^p)\). However, this value of \(\alpha_M\) is achieved also by the inner input weights \(l = 4\) and \(l = 6\), leading to a slight modification of (16). In fact, \(l = 4\) yields \(\alpha_M = -2\) (since \(n^{o''} = 1\) and \(n^{i''} = 2\)), and \(h(\alpha_M) = 2d_{i,o,eff}^p + d^{o''}(d_{i,o}^p + 1)\). Also \(l = 6\) yields \(\alpha_M = -2\) (since \(n^{o''} = 2\) and \(n^{i''} = 3\)), and \(h(\alpha_M) = 3d_{i,o,eff}^p + 2d^{o''}(d_{i,o}^p)\). Notice that even when \(l > l_{\min}\) yields the maximum value of \(\alpha_M = -2\), the design rules stated in Section IV are still valid, leading in every case to the maximization of \(h(\alpha_M)\).

\(^{3}\) A code is said to be invertible if, knowing only the parity-check symbols of a code vector, the corresponding information symbols can be uniquely determined [11].
2) If $d_0 = 4$, $\alpha_M = \left\lfloor \frac{d_0 + 1}{2} \right\rfloor = -2$. In this case, only the minimum weight of the inner code input sequences $l_{\min} = 4$ yields $\alpha_M = -2$ (since $n^o = 1$ and $n^i = 2$), and $h(\alpha_M) = 2d_{i,eff}^o + d_{o}^o(d_0^o)$.

The algorithm to find the optimal (where optimal is intended to be according to the criterion addressed in Section IV) puncturing patterns $P_i^p$ and $P_i^s = \Pi[P']$ for inner code parity and systematic bits, respectively, works sequentially, by puncturing one bit at a time in the optimal position, subject to the constraint of rate compatibility. This sequential puncturing is performed starting from the lowest rate code (i.e., the baseline rate-1/3 code), and ending up at the highest possible rate. In Table II the puncturing pattern $P_i^p$ for inner code parity bits is shown. To find this pattern, a frame length $K = 200$ and an interleaver length $N = K/R_c = 300$ have been assumed. The puncturing pattern has been found by optimizing the inner code IOWEF, as explained in the previous section. This puncturing pattern yields the optimum values of $(d_w, N_w)$ for $w = 2, \ldots, w_{\max}$ and for each puncturing position. The puncturing positions of $P_i^p$ go from 1 to the interleaver length $N$. The evolution of the values $(d_w, N_w)$ with the number of punctured inner parity bits for $w = 2$ are reported in Fig. 3. Notice that $d_w, \forall w$ (not only for $w = 2$), is a non-increasing function of the number of punctured bits, and there are some $d_w = 0$ with a corresponding $N_w \neq 0$, which means that the corresponding code $C_i'$ is not invertible. Notice also that $N_2$, given a value of $d_2$, is an increasing function of the number of punctured bits.

In Table II the puncturing pattern $P'$, the interleaved version of which, $\Pi[P']$, is meant for inner code systematic bits, is shown, having applied the fixed puncturing pattern $P_o$ to the outer code. This puncturing pattern yields the best outer code (punctured through $P_o$ and $P'$) output weight enumerating function (OWEF) for each puncturing position. The puncturing positions go from 1 to $2K$, being $K$ the frame length. The number of punctured bits go from 0 to $K/2$, i.e., the rate of the outer code punctured through $P_o$ and $P'$ is assumed to go from 2/3 (no puncturing is applied to the systematic bits) to 1. The reason to limit the rate of $C_o''$ up to 1 is that further puncturing results in a significant performance degradation. The puncturing pattern $P'$ for inner code systematic bits having applied $P_o$ is shown in Table II.

We have also performed an optimization of the inner code systematic bits puncturing pattern $P_i^s = \Pi[P']$ restricting the puncturing to outer code parity bits only, thus yielding to an overall systematic SCCC. The puncturing pattern $P'$, having applied the fixed puncturing pattern $P_o$ to
systematic bits, is reported in Table IV. It is worth to point out that the performances obtained by restricting the puncturing to outer code systematic bits are very similar to those obtained without this restriction.

In Table V are listed the parameters $h_m^{(3)}$, $d_{off}'(d_{on})$, $h(\alpha M)$, $h_m$, and the multiplicity $N_{hm}$ of the codewords at distance $h_m$, for different values of the parity permeability $\rho_p$ for an SCCC of overall code rate $R_{SCCC} = 2/3$, being the outer encoder punctured through $P_{o,1}$, and the inner encoder punctured through $P_{i}'$, reported in Table I and $P_i^s = \Pi[P']$, where $P'$ is reported in Table II. Notice that being $R_{oc}' = 2/3$ in (2), to obtain a rate $R_{SCCC} = 2/3$ code $\rho_s$ and $\rho_p$ must be related by

$$\rho_s = 1 - \rho_p \quad (20)$$

For instance, the code with $\rho_p = 20/300$ has been obtained by applying the puncturing pattern of Table I to inner code parity bits, selecting the first $280 = N(1 - \rho_p)$ puncturing positions in Table I and applying the interleaved version of the puncturing pattern of Table II to inner code systematic bits, selecting the first $20 = N(1 - \rho_s)$ puncturing positions in Table II, so that $\rho_s + \rho_p = 1$ (see (20)).

The frame length selected for this example is $K = 200$. The corresponding interleaver length $N$ is given by $K/R_{oc}' = 300$. The different values of $\rho_p$ are listed as rational numbers with denominator $N$ (since the maximum number of inner parity bits which are not punctured is $N$). For all permeabilities $h_m^{(3)} = 0$, thus $h(\alpha M)$ is completely dominated by $d_{off}'(d_{on})$.

The union bound (9) on the residual Frame Error Rate (FER) of the codes listed in Table V is plotted in Fig. 4. The markers used in Fig. 4 correspond to those listed in Table V. It is shown that the error floor is lowered by increasing $\rho_p$, i.e., the proportion of surviving inner code parity bits. The higher error floor is obtained for $\rho_p = 20/300$, whereas increasing $\rho_p$ leads to better performance in the error floor region. Nevertheless, it should be stressed that a sufficient number of systematic bits should be preserved in order to ensure a good behavior for high $E_b/N_0$ values. This can be observed for the curve $\rho_p = 100/300$, which shows a worse slope. Indeed, for asymptotic values of $E_b/N_0$, the performance is dominated by $h_m$, the minimum weight of code sequences. Therefore, the best performance for very high signal-to-noise ratios $E_b/N_0$ is obtained for $\rho_p = 20/300$ (curve with ’□' in Fig. 4), since the corresponding code has $h_m = 3$, whereas the worst performance is obtained for $\rho_p = 100/300$ (curve with ’o' in Fig. 4), since
the corresponding code has $h_m = 1$.

In Fig. 5 we compare simulation results of the rate-2/3 SCCC of Table V with the analytical upper bounds for several values of $\rho_p$. The curves are obtained with a log-map SISO algorithm and 10 decoding iterations. These results are obtained considering a random interleaver of length $N = 3000$ and applying the puncturing patterns of Tables I and II periodically. The simulation results show a very good agreement with the analytical bounds and confirm that lower error floors can be obtained by increasing $\rho_p$. For example, the code $\rho_p = 8/30$ shows a gain of 1.4 dB at FER = $10^{-5}$ w.r.t. the code $\rho_p = 2/30$. However, this gain tends to vanish for very high $E_b/N_0$, where the term $h_m$ is predominant (note the of the two curves).

On the other hand, the performance in the waterfall region can be explained in part looking at the cumulative function $\sum d_{(A)}^{C_S}$ of the output distance spectrum of the serial concatenated codes. The codes for which the cumulative function of the average distance spectrum is minimum perform better at low SNRs, since, in this region, the higher distance error events have a nontrivial contribution to error performance. The cumulative functions of the codes listed in Table V are traced in Fig. 6. The worst performance for low signal-to-noise ratios $E_b/N_0$ is obtained for $\rho_p = 20/300$ (curve with '□' in Fig. 4), since the corresponding code has the maximum cumulative function of the average distance spectrum, whereas the best performance is obtained for $\rho_p = 100/300$ (curve with '◦' in Fig. 4), since the corresponding code has the minimum cumulative function of the average distance spectrum. This is in agreement with the simulation results of Fig. 5.

For comparison purposes, we also report in Fig. 5 the performance of the rate-2/3 PCCC proposed in [12] and the rate-2/3 SCCC proposed in [4]. The PCCC code in [12] is a code of similar complexity of the SCCC codes proposed here obtained by optimally puncturing the mother code specified in the wideband code-division multiple-access (WCDMA) and CDMA2000 standards, consisting of the parallel concatenation of two rate-1/2, 8-states, convolutional encoders. The SCCC code in [4] is the same as our baseline code (two rate-1/2, 4-states, systematic recursive encoders), but puncturing is limited to inner code parity bits. As it can be observed in Fig. 5, the proposed SCCC code shows a significant gain in the error floor region w.r.t. the code in [12]. On the other hand, the code in [4] performs much worse than our code, since all inner code systematic bits are maintained after puncturing.

In Table VI are listed the parameters $d_i^{\prime}, d_i^{\prime\prime}, h(\alpha_M), h_m$ and the multiplicity $N_{h_m}$.
of the codewords at distance $h_m$, for different values of $\rho_p$, being the outer encoder punctured through $P_p^i$, reported in Table I and $P_i^s = \Pi[P']$, where $P'$ is reported in Table II. The frame length selected for this example is always $K = 200$ ($N = 300$).

Fig. 7 gives the union bound (9) on the residual Frame Error Rate of the codes listed in Table VI. The markers used in Fig. 7 are listed in Table VI. Similar performance to the codes of Fig. 4 (obtained applying $P_o$, and the puncturing patterns of Tables I and II) are observed. The bounds are congruent with the parameters reported in Table VI. All the codes with $\rho_p > 20/300$ have $h(\alpha_M) = h_m = 2$. Then, the performance are dominated by the multiplicity of $N_{h_m}$ which diminishes as $\rho_p$ increases, i.e., the number of inner code parity bits which are not punctured is increased. Therefore, to enhance performance in the error floor region one should put more puncturing on inner code systematic bits. In fact, the hierarchy of the curves in Fig. 7 corresponds to the hierarchy of $N_{h_m}$ in Table VI. Finally, the curve corresponding to $\rho_p = 20/300$ shows the worst performance in the region of interest, where the multiplicity $N_{h_m}$ is the dominant term. However, for very high $E_b/N_0$, being the performance mainly dominated by $h_m$ (equal to three), the curve corresponding to $\rho_p = 20/300$ shows the best performance.

Fig. 8 shows the simulated performance of the SCCCs with rate $R_{SCCC} = 9/10$ in terms of residual FER vs. $R_{outer} = K \rho_s$, for different values of $E_b/N_0$. The curves show that the higher the SNR, and hence the lower the target FER, the heavier should be the puncturing on inner systematic bits, i.e., the lower should be $\rho_s$. On the contrary, for higher error probabilities it is advantageous to keep more systematic bits.

Finally, in Fig. 9 we compare the simulated performance of the SCCCs with rate $R_{SCCC} = 9/10$ with the analytical upper bounds for several values of $\rho_p$. The curves show that the higher the $E_b/N_0$, the heavier should be the puncturing on inner systematic bits, i.e., the higher should be $\rho_p$. Nevertheless, it should be stressed that some of the inner systematic bits must be maintained in order to allow convergence of the decoding process. For comparison purposes, we also report in the same figure the performance of the rate-9/10 PCCC proposed in [12]. A gain of 2 dB at FER $10^{-5}$ is obtained for the code $\rho_p = 160/2220$ w.r.t. the code in [12].

From the analytical upper bounds and these examples we may conclude that performance strongly depend on the puncturing patterns, and also on the spreading of the puncturing over the inner code systematic bits and parity bits. To lower the error floor, it is advantageous to put more puncturing on inner code systematic bits, resulting in a lower error floor and, in general,
in a faster convergence (see the curves marked with filled circles in Fig. 5).

VI. Conclusions

In this paper we have proposed a method for the design of rate-compatible serial concatenated convolutional codes (SCCC).

To obtain rate-compatible SCCC's, the puncturing has not been limited to inner parity bits only, but has also been extended to inner systematic bits, puncturing the inner encoder beyond the unitary rate. A formal analysis has been provided for this new class of SCCC by deriving the analytical upper bounds to the error probability. Based on these bounds, we have derived suitable design guidelines for this particular code structure to optimize the inner code puncturing patterns. In particular, it has been shown that the puncturing of the inner code systematic bits depends on the outer code and, therefore, it is also interleaver dependent. Moreover, the performance of a SCCC for a given rate can be enhanced in the error-floor region by increasing the proportion of surviving inner code parity bits, as far as a sufficient number of systematic bits is preserved.

The code analyzed in this paper, due to its simplicity and versatility, has been chosen for the implementation of a very high speed (1Gbps) Adaptive Coded Modulation modem for satellite application. The interested reader can find implementation details in [17].

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## Table I

Puncturing positions for inner code parity bits.

| Index | Puncturing position |
|-------|---------------------|
| 1 - 10 | 299 0 5 294 276 77 96 257 139 24 |
| 11 - 20 | 47 224 264 126 54 151 17 174 192 106 |
| 21 - 30 | 161 241 212 89 250 36 285 113 236 63 |
| 31 - 40 | 205 82 269 68 217 31 229 179 144 12 |
| 41 - 50 | 156 131 133 139 169 118 230 239 42 255 |
| 51 - 60 | 75 158 165 13 122 196 183 279 260 21 |
| 61 - 70 | 51 298 292 9 1 220 253 233 148 28 209 |
| 71 - 80 | 110 272 212 89 250 36 283 113 236 63 |
| 81 - 90 | 140 127 191 240 72 117 201 46 265 225 |
| 91 - 100 | 16 249 81 213 32 290 180 57 95 166 |
| 101 - 110 | 147 232 40 275 25 256 88 282 133 |
| 111 - 120 | 206 186 153 295 227 43 268 35 123 69 |
| 121 - 130 | 234 195 58 162 50 4 143 20 105 192 |
| 131 - 140 | 114 216 237 261 13 228 130 136 60 177 |
| 141 - 150 | 98 203 287 184 252 91 139 66 273 120 |
| 151 - 160 | 75 55 29 40 210 198 84 280 189 247 |
| 161 - 170 | 292 150 99 176 61 154 5 297 230 18 |
| 171 - 180 | 263 111 219 141 167 48 239 125 11 193 |
| 181 - 190 | 70 34 271 254 208 79 103 285 182 138 |
| 191 - 200 | 227 164 22 45 242 128 115 94 52 143 |
| 201 - 210 | 6 207 235 139 258 25 87 107 278 172 |
| 211 - 220 | 234 15 38 223 296 71 132 188 119 59 |
| 221 - 230 | 204 248 134 83 178 284 138 2 33 100 |
| 231 - 240 | 262 214 235 274 72 25 63 291 121 199 44 |
| 241 - 250 | 171 146 90 10 246 132 56 108 222 163 |
| 251 - 260 | 74 255 181 211 30 277 194 293 93 149 |
| 261 - 270 | 116 80 266 7 53 238 37 137 175 231 |
| 271 - 280 | 67 202 14 160 288 112 239 41 86 218 |
| 281 - 290 | 124 185 19 155 281 243 97 49 129 220 |
| 291 - 300 | 26 270 168 62 190 76 231 104 207 142 |
### TABLE II
Puncturing positions for inner code systematic bits and fix puncturing pattern $P_{0,1}$.

| Index  | Puncturing position |
|--------|---------------------|
| 1 - 10 | 101 193 285 341 49 145 241 369 313 |
| 11 - 20| 73 169 217 25 265 121 385 325 85 357 |
| 21 - 30| 297 181 229 37 133 253 13 61 157 205 |
| 31 - 40| 108 276 345 389 309 89 373 329 196 40 |
| 41 - 50| 148 244 8 64 124 220 172 292 260 360 |
| 51 - 60| 96 20 396 281 184 136 232 52 333 76 |
| 61 - 70| 160 208 112 305 257 377 349 33 317 80 |
| 71 - 80| 68 392 176 128 212 45 353 152 236 300 |
| 81 - 90| 105 16 201 68 365 272 140 5 321 225 |
| 91 - 100| 92 165 29 288 380 188 336 249 274 48 |
### TABLE III
Puncturing positions for inner code systematic bits and fix puncturing pattern $P_{i,2}$.

| Index | Puncturing position |
|-------|---------------------|
| 1 - 10 | 1 398 10 272 105 176 338 226 58 138 |
| 11 - 20 | 305 369 35 203 83 251 154 320 120 290 |
| 21 - 30 | 352 386 16 209 64 232 170 41 266 39 |
| 31 - 40 | 184 0 344 299 146 89 257 376 128 314 |
| 41 - 50 | 216 48 360 162 112 282 23 192 240 72 |
| 51 - 60 | 330 392 136 280 26 194 74 242 328 50 |
| 61 - 70 | 218 378 114 160 306 358 264 90 18 186 |
| 71 - 80 | 144 370 288 235 57 337 106 8 211 168 |
| 81 - 90 | 385 322 122 258 66 296 42 152 362 248 |
| 91 - 100 | 200 96 312 52 130 178 346 274 224 80 |
TABLE IV
PUNCTURING POSITIONS FOR INNER CODE SYSTEMATIC BITS CORRESPONDING TO OUTER CODE PARITY BITS AND FIX PUNCTURING PATTERN $P_{0,1}$.

| Index   | Puncturing position |
|---------|---------------------|
| 1 - 10  | 1 397 117 233 201 273 53 137 237 365 |
| 11 - 20 | 33 301 89 177 137 253 217 9 381 73 |
| 21 - 30 | 317 349 41 105 285 189 145 229 261 169 |
| 31 - 40 | 125 393 53 329 85 361 21 297 205 101 |
| 41 - 50 | 377 37 313 69 345 249 5 277 161 221 |
| 51 - 60 | 185 121 141 289 385 233 65 337 29 93 |
| 61 - 70 | 257 173 355 213 305 13 109 155 389 321 |
| 71 - 80 | 43 193 281 245 129 81 389 197 49 325 |
| 81 - 90 | 269 17 149 291 373 93 181 77 309 133 |
| 91 - 100| 225 33 341 357 209 64 293 113 265 165 |
### TABLE V

Parameters of the rate $R_{SCCC} = 2/3$ code with interleaver length $N$ and the first fix puncturing pattern $P_{o,1}$

| $\rho_P$ | $h_m^{(3)}$ | $d^{(\rho)}(d_i^2)$ | $h(\alpha_M)$ | $h_m$ | $N_{h_m}$ | Markers |
|----------|--------------|---------------------|----------------|-------|-----------|---------|
| 20/300   | 0            | 3                   | 3              | 3     | 3.60E-01  | □       |
| 40/300   | 0            | 2                   | 2              | 2     | 4.81E-03  | +       |
| 60/300   | 0            | 2                   | 2              | 2     | 7.12E-03  | ×       |
| 80/300   | 0            | 2                   | 2              | 2     | 5.28E-03  | △       |
| 100/300  | 0            | 1                   | 1              | 1     | 1.40E-04  | ○       |
### TABLE VI
PARAMETERS OF THE RATE $R_{SCCC} = 2/3$ CODE WITH INTERLEAVER LENGTH $N$ AND THE SECOND FIX PUNCTURING PATTERN $P_{\rho,2}$

| $\rho_p$  | $d_{t,\text{eff}}$ | $d_t' \gamma(d_t' \gamma)$ | $h(\alpha_M)$ | $h_m$ | $N_{h_m}$ | Markers |
|-----------|---------------------|-----------------------------|----------------|-------|-----------|---------|
| 20/300    | 0                   | 3                           | 3              | 3     | 3.32E-01  | □       |
| 40/300    | 0                   | 2                           | 2              | 2     | 5.24E-03  | +       |
| 60/300    | 0                   | 2                           | 2              | 2     | 4.12E-03  | ×       |
| 80/300    | 0                   | 2                           | 2              | 2     | 1.98E-03  | △       |
| 100/300   | 0                   | 2                           | 2              | 2     | 8.47E-04  | ○       |