ON A QUESTION OF ERDŐS AND ULAM

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Abstract. Ulam asked in 1945 if there is an everywhere dense rational set, i.e. a point set in the plane with all its pairwise distances rational. Erdős conjectured that if a set $S$ has a dense rational subset, then $S$ should be very special. The only known types of examples of sets with dense (or even just infinite) rational subsets are lines and circles. In this paper we prove Erdős’ conjecture for algebraic curves, by showing that no irreducible algebraic curve other than a line or a circle contains an infinite rational set.

1. Introduction

We define a rational set to be a set $S \subset \mathbb{R}^2$ such that the distance between any two elements is a rational number. We are interested in the existence of infinite rational distance sets on algebraic curves.

On any line, one can easily find an infinite rational set that is in fact dense. It is an easy exercise to find an everywhere dense rational subset of the unit circle. On the other hand, it is not known if there is a rational set with 8 points in general position, i.e. no 3 on a line, no 4 on a circle. In 1945, Anning and Erdős [1] proved that any infinite integral set, i.e. where all distances are integers, must be contained in a line. Problems related to rational and integral sets became one of Erdős’ favorite subjects in combinatorial geometry [8] [9] [10] [11] [12] [13].

In 1945, when Ulam heard Erdős’ simple proof [6] of his theorem with Anning, he said that he believed there is no everywhere dense rational set in the plane, see Problem III.5. in [2] and also [7]. Erdős conjectured that an infinite rational set must be very restricted, but that it was probably a very deep problem [7][8]. Not much progress has been made on Ulam’s question. There were attempts to find rational sets on parabolas [4] [5], and there were some results on integral sets, in particular bounds were found on the diameter of integral sets [16] [22]. Very recently Kreisel and Kurz [19] found an integral set with 7 points in general position.

In this paper, we prove that lines and circles are the only irreducible algebraic curves that contain infinite rational sets. Our main tool is Faltings’ Theorem [14]. We will also show that if a rational set $S$ has infinitely many points on a line or on a circle, then all but 4 resp. 3 points of $S$ are on the line or on the circle. This answers questions of Guy, Problem D20 in [15], and Pach, Section 5.11 in [3].

2. Main result

Our main result is the following.

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Theorem 2.1. Every rational set of the plane has only finitely many points in common with an algebraic curve defined over \( \mathbb{R} \), unless the curve has a component which is a line or a circle.

The two special cases, line and circle, are treated in more detail in the next theorem.

Theorem 2.2. If a rational set \( S \) has infinitely many points on a line or on a circle, then all but 4 resp. 3 points of \( S \) are on the line or on the circle.

Note that there are infinite rational sets with all but 4 points on a line, and there are infinite rational sets with all but 3 points on a circle. The circle case follows from the line case by applying an inversion with rational radius and center one of the 4 points not on the line. A construction of Huff [17][20] gives an infinite rational set with all but 4 points on a line.

We can formulate our Theorem 2.1 in a different way by using the term curve-general position: we say that a point set \( S \) of \( \mathbb{R}^2 \) is in curve-general position if no algebraic curve of degree \( d \) contains more than \( d(d+3)/2 \) points of \( S \). Note that \( d(d+3)/2 \) is the number of points in general position that determine a unique curve of degree \( d \).

Corollary 2.3. If \( S \) is an infinite rational set in general position, then there is an infinite \( S' \subset S \) such that \( S' \) is in curve-general position.

Proof: Let \( S_5 \) consist of any five points in \( S \), and let \( T_5 \) be the set of finitely many points on the unique conic through those 5 points. Continue recursively; at step \( n \), add a point from \( S \setminus T_{n-1} \) to get \( S_n \). For each \( d \) such that \( d(d+3)/2 \leq n \), let \( T_n \) be the union of \( T_{n-1} \) and the set of points of \( S \) that are on a curve of degree \( d \) through any \( d(d+3)/2 \) points in \( S_n \). Since each \( T_n \) is finite, we can always add another point. Then the infinite union of the sets \( S_n \) is an infinite subset of \( S \) with the required property.

3. Proof of Theorem 2.1

3.1. General Approach. We will use the following theorem of Faltings [14].

Theorem (Faltings). A curve of genus \( \geq 2 \), defined over a number field, contains only finitely many rational points.

In this paper by curve (defined over a field \( K \subset \mathbb{R} \)) we usually mean the zero set in \( \mathbb{R}^2 \) of a polynomial in two variables with coefficients from \( K \). But when we consider the genus of a curve, we are actually talking about the projective variety defined by the polynomial. For definitions, see [21].

First suppose we have an infinite rational set \( S \) contained in a curve \( C \) of genus \( \geq 2 \), defined over \( \mathbb{R} \). We can move two points in \( S \) to \((0,0)\) and \((0,1)\), so that by Lemma 3.4 below the elements of \( S \) are of the form \((r_1,r_2\sqrt{k})\). Then by the remark after Lemma 3.4, the curve is defined over \( \mathbb{Q}(\sqrt{k}) \). By Faltings’ theorem, \( S \) must be finite.

Below we will show that if we have an infinite rational set \( S \) on a curve \( C_1 \) of genus 0 or 1, then all but finitely many of the points in \( S \) will in fact give points on a curve \( C_2 \) in \( \mathbb{R}^3 \) of genus \( \geq 2 \). More precisely, assuming \((0,0)\) and \((0,1)\) are in \( S \), a point \((r_1,r_2\sqrt{k})\) will give a point \((r_1,r_2\sqrt{K},r_3)\) on a curve \( C_2 \), with all the \( r_i \) rational. Again we conclude by Faltings’ theorem that the original set \( S \) could not have been infinite.
3.2. Two lemmata. Rationality of distances in $\mathbb{R}^2$ is clearly preserved by translations, rotations and uniform scaling, $(x, y) \mapsto (\lambda x, \lambda y)$ with $\lambda \in \mathbb{Q}$. More surprisingly, rational sets are preserved under certain central inversions. This will be an important tool in our proof below.

**Lemma 3.3.** If we apply inversion to a rational set $S$, with center a point $x \in S$ and rational radius, then the image of $S \setminus \{x\}$ is a rational set.

**Proof:** We may assume the center to be the origin and the radius to be 1. The properties of inversion are most easily seen in complex notation, where the map is $z \mapsto \frac{1}{z}$. Suppose we have two points $z_1, z_2$ with rational distances $|z_1|, |z_2|$ from the origin, and with $|z_2 - z_1|$ rational. Then

$$\frac{1}{|z_1|} - \frac{1}{|z_2|} = \frac{|z_2 - z_1|}{|z_1 z_2|} = \frac{|z_2 - z_1|}{|z_1||z_2|}$$

is also rational.

A priori, points in a rational set could take any form. However, after moving two of the points to two fixed rational points by translating, rotating, and scaling, the points are in fact almost rational points. The following simple lemma is well-known among researchers working with integer sets. As far as we know, it was proved first by Kemnitz [18].

**Lemma 3.4.** For any rational set $S$ there is a square-free integer $k$ such that if a similarity transformation $T$ transforms two points of $S$ into $(0, 0)$ and $(1, 0)$ then any point in $T(S)$ is of the form

$$(r_1, r_2 \sqrt{k}), \ r_1, r_2 \in \mathbb{Q}.$$  

Note that this implies that any curve of degree $d$ containing at least $d(d + 3)/2$ points from $T(S)$ is defined over $\mathbb{Q}(\sqrt{k})$.

3.5. Curves of genus 1. Let $C_1 : f(x, y) = 0$ be an irreducible algebraic curve of genus $g_1 = 1$ and degree $d \geq 3$. Suppose that there is an infinite set $S$ on $C_1$ with pairwise rational distances. Assume that the points $O = (0, 0)$ and $(1, 0)$ are on $C_1$ and in $S$, and that $O$ is not a singularity of $C_1$. Below we will be allowed to make any other assumptions on $C_1$ that we can achieve by translating, rotating or scaling it, as long as we also satisfy the assumptions above. In particular, we can use any of the infinitely many rotations about the origin that put another point of $S$ on the $x$-axis.

We wish to show that the intersection curve $C_2$ of the surfaces

$$X : f(x, y) = 0,$$

$$Y : x^2 + y^2 = z^2,$$

has genus $g_2 \geq 2$.

Consider $C_1$ as a curve in the $z = 0$ plane, and define the map $\pi : C_2 \to C_1$ by $(x, y, z) \mapsto (x, y)$, i.e. the restriction to $C_2$ of the vertical projection from the cone $Y$ to the $z = 0$ plane. The preimage of a point $(x, y)$ consists of the two points $(x, y, \pm \sqrt{x^2 + y^2})$, except when $x^2 + y^2 = 0$, which in $C^2$ happens on the two lines $x + iy = 0$ and $x - iy = 0$. Then we can determine (or at least bound from below) the genus of $C_2$ using the Riemann-Hurwitz formula [21] applied to $\pi$,

$$2g_2 - 2 \geq \deg \pi \cdot (2g_1 - 2) + \sum_{P \in C_2} (e_P - 1).$$
This is usually stated with equality for smooth curves, but we are allowing $C_1$ and $C_2$ to have singularities. To justify our use of it, observe that the map $\pi$ corresponds to a map $\tilde{\pi} : \tilde{C}_1 \to \tilde{C}_2$ between the normalizations of the curves, for which Riemann-Hurwitz holds. The normalizations have the same genera as the original curves, and $\tilde{\pi}$ has the same degree. Furthermore a ramification point of $\pi$ away from any singularities gives a ramification point of $\tilde{\pi}$. It is enough for our purposes to have this inequality, but there could be more ramification points for $\tilde{\pi}$, above where the singularities used to be.

Applying this formula with $g_1 = 1$, $d = 2$, we have

$$g_2 \geq 1 + \frac{1}{2} \sum_{P \in C_2} (e_P - 1),$$

so to get $g_2 \geq 2$, we only need to show that $\pi$ has some ramification point.

The potential ramification points are above the intersection points of $C_1$ with the lines $x \pm iy = 0$, of which there are $2d$ by Bézout’s theorem, counting with multiplicities. Such an intersection point $P$ can only fail to have a ramification point above it if the curve has a singularity at $P$, or if the curve is tangent to the line there. We will show that there are only finitely many lines through the origin on which one of those two things happens. Then certainly one of the infinitely many rotations of $C_1$ that we allowed above will give an intersection point of $C_1$ with $x \pm iy = 0$ that has a ramification point above it.

The intersection of a line $y = ax$ with $f(x, y) = 0$ is given by $p_a(x) = f(x, ax) = 0$, and if the point of intersection is a singularity or a point of tangency, then $p_a(x)$ has a multiple root. We can detect such multiple roots by taking the discriminant of $p_a(x)$, which will be a polynomial in $a$ that vanishes if and only if $p_a(x)$ has a multiple root. Hence for all but finitely many values of $a$ the line $y = ax$ has $d$ simple intersection points with $f(x, y) = 0$. So indeed there is an allowed rotation after which $\pi$ is certain to have a ramification point.

3.6. **Curves of genus 0, $d \geq 4$.** Let $C_1 : f(x, y) = 0$ be an irreducible algebraic curve of genus $g_1 = 0$, and again assume that it passes through the origin, but does not have a singularity there. Then Riemann-Hurwitz with the same map $\pi$ as above gives

$$g_2 \geq -1 + \frac{1}{2} \sum_{P \in C_2} (e_P - 1),$$

so to get $g_2 \geq 2$ we need to show that there are at least 5 ramification points. As above, we can ensure that the lines $x \pm iy$ each have $d$ simple points of intersection. Discounting the intersection point of the two lines, this gives $2d - 2$ ramification points. Hence if the degree of $f$ is $d \geq 4$ we are done.

3.7. **Curves of genus 0, $d = 2, 3$.** Let $d = 3$ and assume $f(x, y) = 0$ is not a line or a circle. Consider applying inversion with the origin as center to the curve. This is a birational transformation, so does not change the genus. Therefore, when inversion increases the degree of $f$ to above 4, we are done.

Algebraically, inversion in the circle around the origin with radius 1 is given by

$$(x, y) \mapsto \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$
and since this map is its own inverse, the curve \( f(x, y) = 0 \) is sent to the curve
\[
C_3 : (x^2 + y^2)^k \cdot f \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right) = 0,
\]
where \( k \leq d \) is the lowest integer that makes this a polynomial. This curve is irreducible if and only if the original curve is irreducible. Since \( f \) does not have a singularity at the origin, it has a linear term \( ax + by \) with \( a, b \) not both zero. After inversion this gives a highest degree term
\[
(ax + by)(x^2 + y^2)^{k-1}.
\]
In our situation, \( d = 3 \), so if \( k = 3 \), the curve \( C_3 \) has degree \( 2k - 1 = 5 \), and we are done.

The only other possibility is that \( k = 2 \), which happens if \( x^2 + y^2 \) divides the leading terms of \( f \). We will treat these cases in a completely different way.

If \( d = 2 \), then applying inversion will give a curve of degree 3, unless its leading terms are \( x^2 + y^2 \), which exactly means that it is a circle! So we treat this case by reducing it to the \( d = 3 \) case.

Since \( f \) has degree 3 and genus 0, it must have a singularity. The singularity need not be in our rational set, but it is always a rational point, so we can move it to the origin, while maintaining the almost-rational form of the points in our rational set. Then \( f \) must have the form
\[
(ax + by)(x^2 + y^2) + cx^2 + dy^2 + exy.
\]
Note that this is exactly what we get if we apply inversion to a quadratic that is not a circle and goes through the origin.

In fact, we can ensure that \((1,0)\) is on the curve again, so that \( a + c = 0 \). Then if we divide by \( c, f \) is of the form
\[
-x + by)(x^2 + y^2) + x^2 + dy^2 + exy.
\]
We can parametrize this curve using lines \( x = ty \), giving the parametrization
\[
y(t) = \frac{t^2 + et + d}{(t - b)(t^2 + 1)} =: \frac{p(t)}{q(t)}, \quad x(t) = t \cdot y(t).
\]
If we let \( t_j \) be a value of \( t \) that gives one of the points from our rational distance set, it follows that for infinitely many \( t \),
\[
(y(t) - y(t_j))^2 + (x(t) - y(t_j))^2 = \left( \frac{p(t)}{q(t)} - \frac{p(t_j)}{q(t_j)} \right)^2 + \left( t \cdot \frac{p(t)}{q(t)} - t_j \cdot \frac{p(t_j)}{q(t_j)} \right)^2
\]
is a square. Then we can multiply by \( q(t)^2 q(t_j)^2 \) to get infinitely many squares of the form
\[
(p(t)q(t_j) - p(t_j)q(t))^2 + (tp(t)q(t_j) - t_jp(t)q(t))^2.
\]
This polynomial has degree 6 in \( t \). It has a factor \((t - t_j)^2\), and a factor \( t^2 + 1 \), since taking \( t = \pm i \) gives (using \( q(\pm i) = 0 \))
\[
(p(\pm i)q(t_j))^2 + (\pm i \cdot p(\pm i)q(t_j))^2 = 0.
\]
Factoring these out, we get a quadratic polynomial \( Q_j(t) \) in \( t \). Its leading coefficient is
\[
(t_j^2 + 1)((b^2 + d^2)^2) t_j^2 + 2(b^2 e + db - d^2 b)t_j + b^2 e^2 + b^2 d^2 + d^2 + 2ebd),
\]
and its constant term is
\[
(t_j^2 + 1)((1 + (e + b^2)) t_j^2 + 2(bd - b + de)t_j + d^2 + b^2).
\]
These polynomials in \( t_j \) are not identically zero (if \( b \) and \( d \) were both 0, then \( f \) would be reducible), hence we can pick \( t_j \) so that they are not zero. Then in turn \( Q_j(t) \) is a proper quadratic polynomial, and since it is essentially a distance function in the real plane, it cannot have real roots, so it has two distinct imaginary roots.

Therefore our infinite rational set gives infinitely many solutions to equations

\[
z_j^2 = (t^2 + 1) \cdot Q_j(t).
\]

Multiplying three of these together, and dividing out \((t^2 + 1)^2\), we get infinitely many solutions to

\[
z^2 = (t^2 + 1)Q_1(t)Q_2(t)Q_3(t).
\]

If there are no multiple roots on the right, then this is a hyperelliptic curve of degree 8, so it has genus 3, hence cannot have infinitely many solutions, a contradiction.

The one thing we need to check is that we can choose the \( t_j \) so that the \( Q_j \) do not have roots in common. We need some notation: write

\[
Q_j(t) = c_2(t_j)t^2 + c_1(t_j)t + c_0(t_j),
\]

where

\[
c_2(t_j) = (1 + (e + b)^2)t_j^2 + 2(bd + de - b)t_j + d^2 + b^2
\]

\[
c_1(t_j) = 2(bd + de - b)t_j^2 + 2(b^2 + d^2 - bed - bd - be - d)t_j + 2(bd + b^2e - bd^2)
\]

\[
c_0(t_j) = (d^2 + b^2)t_j^2 + 2(b^2e + db - d^2b)t_j + b^2e^2 + b^2d^2 + d^2 + 2ebd.
\]

Suppose that for infinitely many \( t_j \) the polynomial \( Q_j(t) \) has the same roots \( x_1 \) and \( x_2 \). Then for each of those \( t_j \) we have

\[
c_1(t_j) = -(x_1 + x_2) \cdot c_2(t_j), \quad c_0(t_j) = x_1 \cdot x_2 \cdot c_2(t_j).
\]

If we look at the coefficients of the \( t_j \) terms in these equations, we see that

\[
-x_1 - x_2 = \frac{2(b^2 + d^2 - bed - bd - be - d)}{2(bd + de - b)} = -b - \frac{be + d - d^2}{bd + de - b}.
\]

\[
x_1 \cdot x_2 = \frac{2(b^2e + db - d^2b)}{2(bd + de - b)} = b \cdot \frac{be + d - d^2}{bd + de - b}.
\]

Here we can read off that the roots are \( x_1 = b \) and \( x_2 = \frac{be + d - d^2}{bd + de - b} \), which is a contradiction, since the roots had to be imaginary.

**4. Proof of Theorem 2.2**

We will prove that if a rational set has infinitely many points on a line, then it can have at most 4 points off the line. The corresponding statement for 3 points off a circle then follows by applying an inversion. More precisely, suppose we have a rational set \( S \) with infinitely many points on a circle \( C \) and at least 4 points off that circle. Assume that the origin is one of the points in \( S \cap C \), and apply inversion with the origin as center, and with some rational radius. That turns \( C \) into a line \( L \), and we get a rational set with infinitely many points on \( L \), and 4 other points. Moreover, the new origin can be added to \( S \), so that we get 5 points off the line, contradicting what we will prove below. To see that the new origin has rational distance to all points in \( S \), observe that in complex notation the distances \( |z| \) to the old origin were rational for all \( z \in S \), and that the distances to the new origin are \( 1/|z| \).
To prove the statement for a line, our main tool will again be Faltings’ theorem, but now applied to a hyperelliptic curve
\[ y^2 = \prod_{i=1}^{6} (x - \alpha_i), \]
which has genus 2 if and only if the \( \alpha_i \) are distinct.

Suppose we have a rational set \( S \) with infinitely many points on a line, say the \( x \)-axis, and 5 or more points off that line. Then we can assume that 3 of those points are above the \( x \)-axis, and that one of them is at \((0,1)\). Let the other two points be at \((a_1,b_1)\) and \((a_2,b_2)\). Note that we are taking 3 points on one side of the line, because we want to avoid having one point a reflection of another. If we had, say, \((a_1,b_1) = (0,-1)\), the argument below would break down.

Take a point \((x,0)\) of \( S \) on the \( x \)-axis, with \( x \neq 0, a_1, a_2 \). Then we have that
\[ x^2 + 1, \ (x - a_1)^2 + b_1^2, \ \text{and} \ (x - a_2)^2 + b_2^2 \]
are rational squares, so that we get a rational point \((x,y)\) on the curve
\[ y^2 = (x^2 + 1)((x - a_1)^2 + b_1^2)((x - a_2)^2 + b_2^2). \]
To show that this is indeed a curve of genus 2, we need to show that the right hand side does not have multiple roots. Since \( b_1, b_2 \neq 0 \), the 3 factors do not have real roots, hence each has two distinct imaginary roots. Since each has the same coefficient of the \( x^2 \) term, two of the quadratic polynomials could only have common roots if they were identical. But the coefficients of the \( x \) term are respectively 0, \(-2a_1\) and \(-2a_2\), so only the last two factors could be equal. But if \( a_1 = a_2 \), then \( b_1 \neq \pm b_2 \), hence the constant terms would be \( a_1^2 + b_1^2 \neq a_2^2 + b_2^2 \). We conclude that the 3 factors do not have common roots.

Therefore the curve has genus 2, and cannot contain infinitely many rational points, contradicting the fact that \( S \) has infinitely many points on the line.

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References

[1] N.H. Anning and P. Erdős, Integral distances, Bull. Amer. Math. Soc., (1945) 51, 598–600.
[2] S.M. Ulam, A Collection of Mathematical Problems, Interscience Tracts in Pure and Applied Mathematics, Number 8. Interscience Publishers (1960) XIII, 150 p.
[3] P. Brass, W. Moser, and J. Pach, Research Problems in Discrete Geometry. Springer, Berlin. 1st ed. (2005) XII, 499 p.
[4] G. Campbell, Points on \( y = x^2 \) at rational distance, Math. Comp., (2004) 73, 2093–2108.
[5] A. Choudhry, Points at Rational Distances on a Parabola, Rocky Mountain J. Math. Volume 36, Number 2 (2006), 413–424.
[6] P. Erdős, Integral distances., Bull. Amer. Math. Soc., (1945) 51, 996.
[7] P. Erdős, Ulam, the Man and the Mathematician J. Graph Theory 9 (1985) no. 4, 445–449. Also appears in Creation Math. 19 (1986), 13–16.
[8] P. Erdős, Some combinatorial and metric problems in geometry, Colloquium Mathematica Societatis János Bolyai 48 . Intuitive Geometry, Siódok, (1985) 167–177.
[9] P. Erdős, On Some Problems of Elementary and Combinatorial Geometry, Annali di Mathematica pura ed applicata, (IV), Vol. CHI, (1975), 99–108.
[10] P. Erdős, *Verchu niakoy geometritchesky zadatchy* (On some geometric problems, in Bulgarian), Fiz.-Mat. Spis. Bulg. Akad. Nauk. 5(38) (1962), 205–212.

[11] P. Erdős, *Combinatorial problems in geometry*, Math. Chronicle 12 (1983), 35–54.

[12] P. Erdős, *Néhány elemi geometriai problémáról* (On some problems in elementary geometry, in Hungarian), Köz. Mat. Lapok 61 (1980), 49–54.

[13] P. Erdős and G.B. Purdy, *Extremal problems in combinatorial geometry* in: Handbook of Combinatorics, (R.L. Graham, M. Grötschel, and L. Lovász eds.) Elsevier Science, (1995) 809–875.

[14] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern* (Finiteness theorems for abelian varieties over number fields), Invent. Math. 73 (3) (1983), 349–366.

[15] R. Guy, *Unsolved Problems in Number Theory* Problem Books in Mathematics Subseries: Unsolved Problems in Intuitive Mathematics, Vol.1, Springer, 3rd ed., (2004) XVIII, 438 p.

[16] H. Harborth, A. Kemnitz, and M. Möller. *An upper bound for the minimum diameter of integral point sets*, Discrete & Comput. Geom., (1993) 9(4):427–432.

[17] G.B. Huff, *Diophantine problems in geometry and elliptic ternary forms*, Duke Math. J. vol. 15 (1948) 443–453.

[18] A. Kemnitz. *Punktmengen mit ganzzahligen Abständen*, Habilitationsschrift, TU Braunschweig, 1988.

[19] T. Kreisel and S. Kurz, *There are integral heptagons, no three points on a line, no four on a circle*, Discrete & Computational Geometry, Online first: DOI 10.1007/s00454-007-9038-6

[20] W. D. Peeples Jr. *Elliptic curves and rational distance sets*, Proc. Am. Math. Soc., (1954) 5: 29–33.

[21] J. Silverman *The Arithmetic of Elliptic Curves*, Springer-Verlag, 1986

[22] J. Solymosi. *Note on integral distances*, Discrete & Comput. Geom., (2003) 30(2) 337–342.

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