INVARANTS OF KNOTTED SURFACES FROM LINK HOMOLOGY AND BRIDGE TRISECTIONS

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Abstract. Meier and Zupan showed that every surface in the four-sphere admits a bridge trisection and can therefore be represented by three simple tangles. This raises the possibility of applying methods from link homology to knotted surfaces. We use link homology to construct an invariant of knotted surfaces (up to isotopy) which distinguishes the unknotted sphere from certain knotted spheres. We also construct an invariant of a bridge trisected surface which takes the form of an $A_{\infty}$-algebra. Both invariants are defined by a novel connection between $A_{\infty}$-algebras and Manolescu and Ozsváth’s hyperboxes of chain complexes.

Introduction

The last two decades has seen an explosion of link homology theories. These gadgets assign chain complexes to link diagrams so that the homology group of the complex is a link invariant. The ur-theory is Khovanov homology, whose Euler characteristic is the famous Jones polynomial. Khovanov’s construction spawned many variants and generalizations. All of these theories are expected to have a certain functoriality property: cobordisms of links (in $S^3 \times I$) should induce maps of homology groups which depend only on the isotopy class of the cobordism.

A closed surface $W \subset S^4$ can be viewed as a cobordism between two unknotts $U$ by puncturing $W$ twice. So for any functorial link homology theory $H$ one can assign to $W$ the map $F_W: H(U) \to H(U)$. The homology of the unknot is usually quite simple and $F_W$ can be encoded by a scalar. Sadly, these numbers typically know no more than the genus of $W$. So this recipe, a standard one in the study of topological field theories, does not bear fruit in link homology.

In this paper, we apply link homology to Meier and Zupan’s bridge trisections [16], a new perspective on knotted surfaces. Just as every link in $S^3$ can be split into two trivial tangles, so every surface-link in $S^4$ can be split into three trivial systems of disks. These systems meet along three tangles. The upshot is that every isotopy class
of surface in \(S^4\) can be represented by a trio of trivial tangles which pair together into unlinks. A diagram of three such tangles is called a triplane diagram.

Meier and Zupan construct a full set of Reidemeister-type moves between triplane diagrams. Two of the moves also preserve the isotopy type of the surface and its trisection. The other, stabilization, pushes the surface through the splitting surface. One subtlety of the theory is that “destabilization is necessary”: it is not the case that any two bridge trisections of a surface are stably equivalent. (This is in contrast to bridge splittings of a link, Heegaard splittings of a three-manifold, and trisections of a four-manifold.) Therefore it is important to develop tools which recognize when a bridge trisection may be stabilized.

Bridge trisections offer a new opportunity to apply knot-theoretic tools to surface-links. What can link homology say about knotted surfaces through bridge trisections? More specifically,

- Can link-homological techniques produce invariants of (isotopy classes of) knotted surfaces via bridge trisections?
- Can link-homological techniques produce obstructions to destabilization of triplane diagrams?

There is some tension between these questions: the first asks for tools which are invariant under stabilization while the second asks for tools which detect stabilization.

The answer to the first question is “yes.”

**Theorem 1.** Let \(\mathcal{K}\) be a smooth, oriented, knotted surface in \(S^4\). There is a \(\mathbb{Z}/2\mathbb{Z}\)-valued invariant of oriented knotted surfaces \(q(\mathcal{K})\) defined via link homology which can be computed from a triplane diagram for \(\mathcal{K}\).

Write \(S^2\) for the unknotted sphere and \(\mathcal{K}_p\) for the spun \((2,p)\)-torus link. \(q(S^2) = 0\) and \(q(\mathcal{K}_p) = 1\) for all \(p \in \mathbb{Z}\).

If \(p\) is odd then \(\mathcal{K}_p\) is a knotted sphere. So \(q\) can distinguish isotopy classes of knotted spheres. (See Theorems 6.13 and 7.3 for a precise statement.)

Let \(t = (t_1, t_2, t_3)\) be a triplane diagram. To define \(q\), we study the group

\[
A(t) = \bigoplus_{i,j=1}^3 \text{CKh}(\tilde{t}_i t_j)
\]

where \(\tilde{t}_j\) denotes the mirror of \(t_j\). Each link \(t_i \tilde{t}_j\) is an unlink, so the homology of \(A(t)\) is not interesting. This group can be given the structure of an associative algebra following Khovanov in [10]. If instead of Khovanov homology we use Szabó homology, we obtain an \(A_\infty\)-algebra. Szabó homology is a combinatorial link homology theory which interpolates between the world of link homology and Floer homology [24]. (We discuss this connection below.) For a link diagram \(\mathcal{D}\) write \(\text{CSz}(\mathcal{D})\) for the Szabó chain group of \(\mathcal{D}\).
**Theorem 2.** Let $t$ be a triplane diagram for $K$. Then

$$\mathcal{A}(t) = \bigoplus_{i,j=1}^{3} \text{CSz}(t_i \bar{t}_j)$$

has the structure of an $A_\infty$-algebra over the ring $\mathbb{F}[U]$. Suppose that $t'$ is a triplane diagram for $K$ which presents the same bridge trisection as $t$. Then $\mathcal{A}(t)$ and $\mathcal{A}(t')$ are $A_\infty$-chain homotopic.

(See Definition 3.4 and Theorem 4.1 for precise statements.) $\mathcal{A}(t)$ is not an invariant of $K$ up to isotopy – the chain homotopy type of $\mathcal{A}(t)$ changes under stabilization. Therefore it may be useful in obstructing destabilization.

Szabó homology and Bar-Natan homology were combined in [21] to produce a link homology theory over $\mathbb{F}[U,W]$, where $U$ and $W$ are formal variables. The combined theory, which we call $\text{CS}(D)$, is not as well-behaved as Szabó homology. If one tries to repeat the proof of the theorem above, one obtains some sort of “perturbed” $A_\infty$-algebra. This structure is just well-behaved enough to admit a sane triple multiplication

$$\mu_3': \text{CS}(t_i \bar{t}_{i+1}) \otimes \text{CS}(t_{i+1} \bar{t}_{i+2}) \otimes \text{CS}(t_{i+2} \bar{t}_i) \to \text{CS}(t_i \bar{t}_i)$$

for $i \in \{1, 2, 3\}$. The homology of $\text{CS}(t_i \bar{t}_{i+1})$ has a unique generator with greatest grading. Writing $\theta_{i(i+1)}$ for a representative of this homology class, we study the coefficient of

$$\mu_3'(\theta_{i(i+1)} \otimes \theta_{(i+1)(i+2)} \otimes \theta_{(i+2)i})$$

at $\theta_{ii}$ for each $i$. They can be compared by considering the canonical cobordism $t_i \bar{t}_i \to U^{2b}$, the $2b$-component unlink, shown in Figure 15. $q(t)$ is defined as the sum of the images of the three triple products under these cobordism maps. Invariance of $q$ follows from the same techniques as $\mathcal{A}(t)$ and careful study of stabilization.

We have many questions about $q$. It is additive under split union, but it’s behavior under connected sum is a mystery. Many open questions about knotted surfaces involve connected sum, so understanding this behavior is very important. We would also like to find non-trivial knots on which $q(K) = 0$ (if there are any!).

In general, a resolution of a triplane diagram is not a triplane diagram. However, for a triplane diagram presented as the half-plat closures of three braids, the “braidlike” resolution is a triplane diagram. The computation of $q(K_p)$ suggests that this resolution may determine $q(t)$. It would be interesting to understand the topological meaning of this resolution.

**Algebraic techniques.** To prove the second theorem, we use the language of hyperboxes developed by Manolescu and Ozsváth in [14]. Hyperboxes were developed in the context of Heegaard Floer homology, but they are a purely algebraic concept.
To motivate the construction, let’s first see how to build an associative algebra from a triplane diagram $t$. Suppose that each tangle in $t$ is the half-plat closure of a braid.

There is a multiplication map

$$\text{CKh}(t_{i\bar{j}}) \otimes \text{CKh}(t_{j\bar{k}}) \rightarrow \text{CKh}(t_{i\bar{j}j\bar{k}}) \cong \text{CKh}(t_{i\bar{k}})$$

defined using the cobordism shown in Figure 1 (This sort of cobordism was studied in [10].) The first step consists of $b$ one-handle attachments and the second consists of some Reidemeister 2 moves. Extend this to a map

$$\mu_2: A(t) \otimes A(t) \rightarrow A(t)$$

by linearity and the rule that $\mu_2$ is zero on summands like

$$\text{CKh}(t_{i\bar{j}}) \otimes \text{CKh}(t_{k\bar{l}}), \quad j \neq k.$$ 

$\mu_2$ is a chain map $A(t)$ is a differential graded algebra. Now suppose one wants to define a map

$$\mu_3: A(t)^{\otimes 3} \rightarrow A(t)$$

which is a homotopy between $\mu_2 \circ (\text{Id} \otimes \mu_2)$ and $\mu_2 \circ (\mu_2 \otimes \text{Id})$. (In fact these two maps are equal – forget that for a moment.) $\mu_3$ should represent an isotopy between the two cobordisms in Figure 2: it shuffles $b$ one-handle attachments past $b$ other one-handle attachments. So it is somehow built from $b^2$ smaller homotopies.

Hyperboxes are a nice way to organize these homotopies. To see this in the abstract, consider the diagram

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3.$$
This “factored mapping cone” contains at least as much information as the mapping cone of \( f_2 \circ f_1 \circ f_0 \). This a one-dimensional hyperbox of chain complexes.

Now suppose that \( f_i \) is chain homotopic to some map \( f'_i \) for each \( i \). An attentive algebraic topology student can write down a chain homotopy between \( f_2 \circ f_1 \circ f_0 \) and \( f'_2 \circ f'_1 \circ f'_0 \). What if these homotopies come with their own factorizations? (This is the case when we move a one-handle past \( b \) other one-handles.) The resulting structure is a two-dimensional hyperbox of chain complexes.

One can turn a one-dimensional hyperbox into a mapping cone by simply composing the maps. Similarly, one can turn a two-dimensional hyperbox into a two-dimensional cubical complex which represents a chain homotopy. In general, this process is called compression. The map \( \mu_k: \mathcal{A}^{\otimes k}(t) \to \mathcal{A}(t) \) is defined by setting up a \( k \)-dimensional hyperbox using \( (k-1) \) families of one-handle attachments. The diagonal maps are defined using Szabó homology. The compression of this hyperbox is a \( k \)-dimensional cubical complex. The \( k \)-dimensional diagonal map is \( \mu_k \).

Let \( x \in \mathcal{A}(t)^{\otimes k} \) be a simple tensor. To show that the \( \mu \) maps satisfy the \( A_\infty \)-relations we must show that all of these hyperboxes are suitably related. We call such an assignment \( x \mapsto H_x \) a system of hyperboxes. From a system of hyperboxes one can build an \( A_\infty \)-algebra. To prove Theorem 2, one must define maps of \( A_\infty \)-algebras which are constructed from systems of hyperboxes. These maps can be defined via maps of systems of hyperboxes.

**Theorem 3.** This construction amounts to a functor from the homotopy category of systems of hyperboxes to the homotopy category of \( A_\infty \)-algebras.
The precise statement is Theorem 8.19. These constructions are heavy in vocabulary and notation which is not used elsewhere in the paper. Therefore the exact statement and proof have been banished to the final, self-contained section of the paper. Along the way we prove some other new results which may be interesting to the hyperbox *cognoscenti*. John Baldwin, Cotton Seed and I use them in [2].

**Connections to Floer homology and trisections of four-manifolds.** Bridge trisections are the knot-theoretic analogues of Gay and Kirby’s *trisections of four-manifolds* [7]. Every smooth, compact four-manifold may be split into three boundary sums of $S^1 \times B^3$. These pieces meet along connected sums of $S^1 \times S^2$. The triple intersection of these three-manifolds is a surface. This same surface is a Heegaard splitting for each $\#^k(S^1 \times S^2)$. So the $\sharp^k S^1 \times B^3$ are analogous to the disk systems and the $\#^k S^1 \times S^2$ are analogous to the tangles. (In fact, bridge trisections are most naturally defined in terms of the unique genus 0 trisection of $S^4$.) The analogue of a triplane diagram is a *trisection diagram*. Let $\Sigma$ be a compact surface of genus $g$. Let $\alpha$, $\beta$, and $\gamma$ be families of $g$ curves so that each triple $(\Sigma, \alpha, \beta)$, $(\Sigma, \beta, \gamma)$, $(\Sigma, \gamma, \alpha)$ is a Heegaard diagram for $\# S^1 \times S^2$. From this data one can build a trisected four-manifold, and every trisection can be represented in this way.

Such diagrams appeared in the Ozsváth and Szabó’s *Heegaard Floer homology* as **Heegaard triples** [17]. Ozsváth and Szabó use these diagrams to define an invariant $\Phi(X)$ of closed four-manifolds in [18]. Therefore it is reasonable to ask if $\Phi(X)$ can be computed from a trisection diagram for $X$. But $\Phi(X)$ is defined in terms of a special sequence of Heegaard triples, and the relationship between these diagrams and trisection diagrams is not clear at the time of this writing. This circle of ideas has produced enthusiasm as well as skepticism among the Heegaard Floer faithful.

We take Theorems 1 and 2 as proof of concept for the usefulness of Heegaard Floer homology in application to trisections. Conjecturally, $\text{Sz}(L) \cong \widehat{\text{HF}}(\Sigma(-L)) \otimes \widehat{\text{HF}}(S^1 \times S^2)$, where $\Sigma(-L)$ stands for the double cover of $S^3$ branched along the mirror of $L$. This was first conjectured by Szabó, and Seed provided substantial numerical and structural evidence [22]. Meier and Zupan show that a bridge trisection of $K$ induces a trisection on the double cover of $S^4$ branched along $K$. Modulo Seed and Szabó’s conjecture, this gives a four-dimensional interpretation of $A(t)$. Whether or not the conjecture is true, the techniques of this paper apply directly to the Heegaard Floer homology of branched double covers via our work in [20]. (See also the discussion following Definition 3.4.2)

To interpret $q(t)$ in this language, one must first interpret the Bar-Natan perturbation of Khovanov and Szabó homology. One such interpretation is given in the introduction of [21]. Another is that the formal variable $U$ in the Bar-Natan perturbation is analogous to the formal variable $U$ in Heegaard Floer homology which counts intersections with a basepoint. (For example, both theories are in
some sense trivial if one formally inverts $U$. In light of this and Manolescu and Ozsváth’s computation of $\Phi(X)$ via hyperboxes, we conjecture that $q(K)$ is related to the Ozsváth-Szabó four-manifold invariant of the double cover of $S^4$ branched along $K$. If this is true, one should be able to reinterpret $q(t)$ in terms of a “plus” and “minus” theory. Alternatively, one should be able to define an invariant from trisection diagrams “handle-by-handle” via hyperboxes.

**Signs.** We work over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, see Remark 2.2 for more on signs.

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1. **Topological background**

1.1. **Braids and plat closures.** Let $\beta$ be a $2b$-strand braid. In this paper braid typically extend from left to right. For a tangle $t$ write $\overline{t}$ for the mirror image of $t$ so that $\overline{\beta} = \beta^{-1}$. Let $p_b$ be the crossingless tangle shown in Figure 3.

**Definition.** The link $p_b \beta p_b$ is called the *plat closure of $\beta$*. The $(0, 2b)$-tangle $p_n \beta$, also denoted $\overline{\beta}$, is called the *half-plat closure of $\beta$.*

![Figure 3. The tangle $p_n$.](image)

The braid group $B_{2b}$ acts (on the right) on the set of tangles with $2n$ ordered basepoints: $t \cdot \beta = t\beta$.

**Definition.** The *Hilden subgroup* $N_{2b} \subset B_{2b}$ is the stabilizer of $p_b$. 
So if \( h, h' \in N_{2b} \) and \( \beta \in B_{2b} \) then the plat closures of \( \beta \) and \( h\beta h' \) are isotopic. Birman answered gave necessary and sufficient conditions for \( \beta \) and \( \beta' \) to have isotopic plat closures [4]. The situation for half-plat closures is much simpler.

**Lemma 1.1.** Let \( \beta, \beta' \in B_{2b} \). \( \beta \) and \( \beta' \) are isotopic if and only if \( \beta'\beta^{-1} \in N_{2b} \).

**Proof.** \( p_b\beta' = p_b\beta \) if and only if \( p_b\beta'\beta^{-1} = p_b \). \( \square \)

Hilden [8] found a generating set for \( N_{2b} \). See also [20] and [5].

1.2. **Knotted surfaces, bridge trisections, and triplane diagrams.** In the three-dimensional world, an \( n \)-bridge sphere for a knot \( K \) is a sphere \( S \) which cuts \( K \) into two trivial tangles and so that \( S \cap K \) is a collection of \( n \) points. “Trivial” means that all the arcs can be simultaneously isotoped to lie on \( S \). The four-dimensional equivalent of a trivial tangle is a trivial disk system.

**Definition.** A trivial \( c \)-disk system is a pair \( (X, C) \) where \( X \) is a four-ball and \( C \subset X \) is a collection of \( c \) properly embedded disks which can be simultaneously isotoped to lie on \( \partial X \).

A fundamental property of trivial disk systems is that \( (X, C) \) is determined up to isotopy rel boundary by the unlink \( \partial X \cap C \), see [9]. So bisections of surfaces are not very interesting: the disk systems on each side must be identical. Gay and Kirby’s trisections of four-manifolds [7] motivates the following definition of Meier and Zupan in [16].

**Definition.** A \((b; c)\)-bridge trisection of a knotted surface \( K \subset S^4 \) is a collection of three \( c \)-disk systems \( (X_1, C_1), (X_2, C_2), \) and \( (X_3, C_3) \), so that

- \( (X_1, X_2, X_3) \) is the standard genus 0 trisection of \( S^4 \). (See [7].)
- \( C_1 \cup C_2 \cup C_3 = K \).
- The tangle \( T_{ij} = C_i \cap C_j \) is a trivial \( b \)-tangle in the three-ball \( B_{ij} = X_i \cap X_j \) for all distinct \( i \) and \( j \).

A \((b; c_1, c_2, c_3)\)-bridge trisection is defined similarly, with \( (X_i, C_i) \) a trivial \( c_i \)-disk system.

**Theorem ([16]).** Every knotted surface in \( S^4 \) admits a bridge trisection.

The sphere at the core of the genus 0 trisection of \( S^4 \) will be called the bridge sphere for \( K \). The tangles \( T_{ij} \) intersect the bridge sphere in \( 2b \) points. The link \( T_{ij} T_{jk} \) is an unlink for any \( i, j, k \). Any (cyclically ordered) triple of tangle diagrams \( (t_1, t_2, t_3) \) satisfying these two conditions is called a triplane diagram. By definition, every bridge trisection can be represented by a triplane diagram. From this perspective, it sometimes makes more sense to write \( c_{ij} \) for the number of components in \( t_i \) rather than \( c_i \). Meyer and Zupan show that every triplane diagram is the triplane diagram...
of some bridge trisection and determine a complete set of Reidemeister-type moves for triplane diagrams.

**Theorem ([16]).** Two triplane diagrams represent isotopic surfaces if and only if they are related by a sequence of the following triplane moves.

**Interior Reidemeister move:** a Reidemeister move on any of the three tangles performed in the complement of a neighborhood of the bridge sphere.

**Braid transposition:** the addition of an Artin generator of the braid group $B_{2b}$ or its inverse to the ends of all three tangles.

**Stabilization and destabilization:** Let $i, j, \text{ and } k$ be distinct. Suppose that $t_i\overline{t}_j$ has a crossingless component $C$. Let $\gamma$ be an arc so that $\partial\gamma$ lies on $C$, the interior of $\gamma$ does not intersect $t_i\overline{t}_j$, and $\gamma$ meets the bridge sphere in a single point called $p$. The stabilization of $t$ along $\gamma$ is the result of surgering along $\gamma$ to obtain two new tangles, $t'_i$ and $t'_j$, then adding a small bit to $t_k$ at $p$ to obtain $t'_k$. Destabilization is the reverse process.

![Figure 4. Stabilization along the arc $\gamma$. On top left, the crossingless component $C$ of $t_i\overline{t}_j$. On the bottom left, the bottoms of the strands of $t_k$ and the point $p$. On the right, the results of stabilization.](image)

The first two moves correspond to isotopies of the trisection, i.e. isotopies of the surface through trisections which do not pass through the bridge sphere. Stabilization corresponds to pushing part of the surface through the spine and thus it changes the isotopy type of the trisection. In contrast to some other Reidemeister-type theorems, destabilization really is necessary: there are triplane diagrams for the same isotopy class of surface which are not isotopic after any number of stabilizations.

Every trivial tangle may be written as the half-plat closure of a braid. Later we will restrict ourselves to studying such diagrams. In this case “interior Reidemeister
moves” may be replaced by braid isotopies and multiplication by elements of $N_{2b}$. The unknotted sphere in $S^4$ is the unique surface with a bridge number one triplane diagram. For another example, see Figure 23.

Some basic features of a knotted surface can be easily read off from a triplane diagram. For example, if $t$ is a $(b; c_1, c_2, c_3)$- triplane diagram, then the Euler characteristic of the surface it represents is given by $c_1 + c_2 + c_3 - b$.

**Definition.** An orientation of the triplane diagram $(t_1, t_2, t_3)$ is a choice of orientation on each tangle so that $t_1\overline{t}_2$, $t_2\overline{t}_3$, and $t_3\overline{t}_1$ are oriented as links.

The following proposition is well-known to experts.

**Proposition.** Let $t$ be a triplane diagram for $K$. The set of orientations on $K$ is in bijection with the set of orientations of $t$.

## 2. Szabó’s link homology theory

This section introduces Szabó homology following the presentation in [21]. (The theory first appeared in [24].) Write $V$ for the algebra $\mathbb{F}[X]/(X^2)$. It is standard to write $v_+$ for 1 and $v_-$ for $X$. For a crossingless link diagram $D$ with $k$ components, define

$$\text{CKh}(D) = V^\otimes k.$$ 

Concretely, $\text{CKh}(D)$ is the vector space with a basis given by the labelings of the components of $D$ by the symbols + and −. These labelings are the canonical generators of $\text{CKh}(D)$. Now suppose that $D$ has $c$ crossings. Figure 5 shows the two ways to resolved a crossing. The set of resolutions of $D$ is thus indexed by $\{0, 1\}^c$ (after ordering the crossings). For $I \in \{0, 1\}^c$ write $D(I)$ for the resolution of $D$ according to $I$. The collection of these diagrams is the cube of resolutions. The Khovanov chain group of $D$ is defined as

$$\text{CKh}(D) = \bigoplus_{I \in \{0, 1\}^c} \text{CKh}(D(I)).$$

![Figure 5. A crossing, its 0-resolution, and its 1-resolution.](image)
One can define Khovanov homology over $\mathbb{F}[W]$ or $\mathbb{F}[U,W]$ where $U$ and $W$ are formal variables. The definition is exactly the same except that $\text{CKh}(\mathcal{D}(I))$ is generated by $\mathbb{F}[U,W]$-linear combinations of the canonical generators.

There is a partial order on the cube of resolutions induced by the order on $\{0, 1\}$. Write $\|I - J\|$ for the $\ell^\infty$ distance between $I$ and $J$. If $I < J$, then $\mathcal{D}(J)$ may be obtained from $\mathcal{D}(I)$ by $\|I - J\|$ diagrammatic one-handle attachments. These one-handle attachments can be described by surgery arcs in $\mathcal{D}(I)$. An oriented surgery arc is called a decoration. A planar diagram with $k$ decorations is called a $k$-dimensional configuration. Orienting the surgery arcs in the all-zeroes resolution $I_0$ of $\mathcal{D}$ orients them in every other resolution. We will always assume that orientations of decorations on other resolutions are induced in this way.

So for $I < J$ and $\|I - J\| = k$ there is a $k$-dimensional configuration $\mathcal{C}(I, J)$ which describes how to obtain $\mathcal{D}(J)$ from $\mathcal{D}(I)$. The collection of circles in $\mathcal{C}(I, J)$ which intersect decorations is the active part of $\mathcal{C}(I, J)$. The other circles form the passive part. We will often conflate these circles and their labels, e.g. in the next paragraph.

To define a link homology theory, one cooks up a map $F_{\mathcal{C}(I, J)} : \text{CKh}(I) \to \text{CKh}(J)$ for each $I$ and $J$. Each of these maps acts by the identity on the passive part of $\text{CKh}(I)$. In other words, writing $\text{CKh}(I) = \text{CKh}_{\text{active}}(I) \oplus \text{CKh}_{\text{passive}}(I)$, the restriction of $F_{\mathcal{C}(I, J)}$ to $\text{CKh}_{\text{passive}}(I)$ is the identity map. This property is called the extension rule.

**Definition 2.1.** Let $\mathcal{C}(I, J)$ be a $k$-dimensional configuration from $\mathcal{D}(I)$ to $\mathcal{D}(J)$.

- The Khovanov configuration map $\mathcal{K}_C$ is defined via the Frobenius algebra structure on $\mathbb{F}[X]/(X^2)$. If $k > 1$ then $\mathcal{K}_C = 0$. If $k = 1$, then acts by multiplication or co-multiplication depending on the number of active circles.
- The Szabó configuration map $\mathcal{S}_C$ is defined in [21]. If $\mathcal{C}$ is one-dimensional then $\mathcal{S}_C = 0$. It is important that $\mathcal{S}$ satisfies the disconnected rule: if the union of the active part of $\mathcal{C}$ and the decorations has more than one connected component, then $\mathcal{S}_C = 0$. If $\mathcal{C}$ has a degree one circle which is $v_+$-labeled in $x$, then $\mathcal{S}_C(x) = 0$.

For a link diagram $\mathcal{D}$ define

$$
d_{\text{Kh}}, d_{\text{Sz}} : \text{CKh}(\mathcal{D}) \to \text{CKh}(\mathcal{D})$$

$$d_{\text{Kh}} = \sum_{I < J} \mathcal{K}_{\mathcal{C}(I, J)}$$

$$d_{\text{Sz}} = \sum_{I < J} W^{\|I - J\| - 1} \mathcal{S}_{\mathcal{C}(I, J)}$$
CKh(\mathcal{D}) has two gradings. Let \( x \in \text{CKh}(\mathcal{D}(I)) \) be a canonical generator. The homological and quantum gradings of \( x \) are
\[
\begin{align*}
h(x) &= \|I\| - n_- \\
q(x) &= \tilde{q}(x) + \|I\| + n_+ - 2n_-
\end{align*}
\]
These can be combined into the \( \Delta \)-grading \( h - q/2 \). (This grading is typically called \( \delta \).) Give \( \mathcal{W} \) the \((h, q)\)-grading \((-1, -2)\).

**Theorem (24).** \( d_{\text{Kh}} + d_{\text{Sz}} \) is a differential of degree \((1, 0)\) on \( \text{CKh}(\mathcal{D}) \). The graded chain homotopy type of \((\text{CKh}(\mathcal{D}), d_{\text{Kh}} + d_{\text{Sz}})\) is a link invariant.

Note that the \( q \)-grading in Section 3 is slightly different.

**2.1. Cobordisms.** It will be important to understand how \( \text{CSz} \) interacts with cobordisms of links. Let \( \Sigma \subset \mathbb{R}^3 \times I \) be a properly embedded link cobordism from \( L_0 \) to \( L_1 \). Let \( \mathcal{D}_0 \) and \( \mathcal{D}_1 \) be diagrams for \( L_0 \) and \( L_1 \). Write \( \Sigma \) as a composition of elementary cobordisms: handle attachments, planar isotopies, and Reidemeister moves. To each of these cobordisms we assign a map. The map
\[
F_\Sigma : \text{CSz}(\mathcal{D}_0) \to \text{CSz}(\mathcal{D}_1)
\]
ought to be the composition of these elementary maps. (Of course it is not clear \textit{a priori} that \( F_\Sigma \) is independent of the decomposition of \( \Sigma \) into elementary pieces.)

A 1-handle attachment can be specified diagrammatically by a planar arc \( \gamma \) with its endpoints on \( \mathcal{D}_0 \). Orient this arc. Put a crossing in \( \mathcal{D}_0 \) along \( \gamma \) as in Figure 6 and call the resulting diagram \( \mathcal{D} \). The 0-resolution of the new crossing yields \( \mathcal{D}_0 \) and the 1-resolution yields \( \mathcal{D}_1 \). The complex \( \text{CS}(\mathcal{D}) \) is the mapping cone of a map \( h_\gamma \) from \( \text{CS}(\mathcal{D}_0) \) to \( \text{CS}(\mathcal{D}_1) \). This is the map assigned to the 1-handle attachment along \( \gamma \). Loosely, to compute \( h_\gamma(x) \) for a canonical generator \( x \), compute \((d_{\text{Kh}} + d_{\text{Sz}})\) but insert \( \gamma \) as a decoration into each configuration.

**Figure 6.** Adding a crossing along an arc.

0- and 2-handle attachments are much simpler. A 0-handle attachment adds a crossingless, closed component to a diagram. It is easy to show that
\[
\text{CSz}(\mathcal{D} \cup \circ) \cong \text{CSz}(\mathcal{D}) \otimes \text{CSz}(\circ).
\]

\footnote{There is some subtlety here – see Section 2.3 of [1] – but it is not relevant to this paper.}
The 0-handle attachment map is the map $\text{CSz}(\mathcal{D}) \to \text{CSz}(\mathcal{D} \cup \circ)$ induced by
\[ x \mapsto x \otimes v_+ \]
on simple tensors. The 2-handle attachment map is the dual map induced by
\[ x \otimes v_- \mapsto x. \]

In [20] we showed that Szabó homology fits into the cobordism-theoretic framework developed by Bar-Natan to study Khovanov homology [3]. Bar-Natan assigns to each link diagram a complex in a certain cobordism category. From this complex one can recover the Khovanov chain group by applying a certain functor $F_{\text{Kh}}$. Bar-Natan’s Reidemeister invariance “maps” are sums of certain cobordisms in $\mathbb{R}^2 \times I$. The functor $F_{\text{Kh}}$ carries these cobordisms to Khovanov’s Reidemeister invariance maps. One upshot of [20] is that Bar-Natan’s maps, interpreted as in the previous paragraphs, induce Reidemeister invariance maps on Szabó homology.

For example, Bar-Natan’s Reidemeister 2 map is shown in Figure 6 of [3]. Take the Szabó chain group of each diagram to obtain $\text{CSz}(\mathcal{D})$ (on top) and $\text{CSz}(\mathcal{D}')$ (on the bottom). The sum of the cobordisms maps induces a map of Szabó complexes.

In [20] we showed that $\text{CSz}$ is functorial: diagrammatic descriptions of isotopic cobordisms induce chain homotopic maps\(^2\). This implies that one-handle attachment maps whose arcs are disjoint must commute up to homotopy. Let $\gamma$ and $\gamma'$ be disjoint arcs of attachment. Write $h_{\gamma \cup \gamma'}$ for the map which counts configurations with both $\gamma$ and $\gamma'$. Then
\[ h_{\gamma} \circ h_{\gamma'} + h_{\gamma'} \circ h_{\gamma} = h_{\gamma \cup \gamma'} \circ \partial + \partial \circ h_{\gamma \cup \gamma'}. \]

This follows from studying the complex assigned to the diagram given by replacing both $\gamma$ and $\gamma'$ by crossings. The maps $h_{\gamma}$ is defined via a mapping cone. We can view $h_{\gamma'}$ as a “map of mapping cones,” one of whose components is $h_{\gamma \cup \gamma'}$. The entire complex is an iterated mapping cone. With $n$ disjoint arcs of attachment one obtains an $n$-dimensional iterated mapping cone. This same argument with an analogous diagram can be used to show that Reidemeister 2 maps with disjoint support commute up to homotopy

Remark 2.2. Khovanov homology is only functorial in characteristic two. To extend the techniques of this and the next section to characteristic zero, one must first extend those of [20].

3. An invariant of bridge trisections

In this section we show how to construct an $A_\infty$-algebra $\mathcal{A}(t)$ from a triplane diagram $t$. The main algebraic step is to construct a system of hyperboxes from $t$.\(^2\)

\(^2\)We actually proved this about $\text{CSz}$ with $W = 1$, but the proof extends to the polynomial version without trouble.
We define systems of hyperboxes and show how they are related to $A_\infty$-algebras (and remind the reader of the basics of $A_\infty$-algebras) in Section 8.

3.1. The system of hyperboxes.

**Definition 3.1.** Let $t$ be a triplane diagram. We say that $t$ is in *plat form* if all three tangles are planar isotopic to half-plat closures of braids.

Every $(n,0)$-tangle is isotopic (but not necessarily planar isotopic) to a half-plat closure. Therefore every trisection diagram may be put into plat form.

**Definition 3.2.** Suppose that $t$ is a triplane diagram in plat form. The *canonical surgery arcs* on $t$ are the arcs connecting the plats in $t_j$ and $t_j$, oriented towards $t_j$. Number these arcs with 1 to $n$ from top to bottom. For a diagram of the form

$$t_{i_1}t_{i_2} \coprod t_{i_3}t_{i_4} \coprod \cdots \coprod t_{i_k}t_{i_{k+1}}$$

There are $(k-1)$ families of canonical surgery arcs, defined similarly. They are the red, dotted arcs in Figure 7.

![Figure 7](image)

Let $s = (s_1, \ldots, s_{k+1})$ be a sequence of length $k+1 \geq 2$ in $\{1, 2, 3\}$. Define

$$\mathcal{D}_s = (t_{s_1}t_{s_2}) \coprod (t_{s_2}t_{s_3}) \coprod \cdots \coprod (t_{s_k}t_{s_{k+1}}).$$

For simplicity, let us first consider a triplane diagram in plat form with no crossings. (This implies that each tangle is the plat closure of the identity braid, but we won’t use that fact.) If $|s| = 2$, then set $H_s = \text{CSz}(\mathcal{D}_s)$. If $|s| > 2$, then there are $k-2$ families of canonical surgery arcs on $\mathcal{D}_s$, each with $b$ arcs. For each coordinate

$$\delta = (d_1, \ldots, d_{k-2}) \in [0, b]^{k-1}$$

there is a diagram $\mathcal{D}_{s,\delta}$ given by performing surgery along the first $d_i$ arcs in the $i$-th family. For example, $\mathcal{D}_{s,(0,\ldots,0)} = \mathcal{D}_s$ and $\mathcal{D}_{s,(1,\ldots,1)} = t_{s_1}t_{s_k}$. Let

$$C_s = \bigoplus_{\delta \in [0,n]^{k-2}} \text{CSz}(\mathcal{D}_{s,\delta})\{-kb + \|\delta\|\}$$
where \{-\} denotes a shift in the \(q\)-grading. Fix a coordinate \(\delta\). A direction vector \(\epsilon\) from \(\delta\) picks out \(|\epsilon|\) canonical arcs: for each \(i\) so that \(\epsilon_i = 1\), take the \((d_i + 1)\)-st arc from the \(i\)-th collection. In other words, use the “next” arcs in each axis with a 1.

Let \(C_{s,\delta,\epsilon}\) be the configuration whose underlying diagram is \(D_{s,\delta}\) and whose decorations are the arcs picked out by \(\epsilon\). Set
\[
D^\epsilon_{\delta} : \text{CSz}(D_{s,\delta})\{ -kb + |\delta| \} \to \text{CSz}(D_{s,\delta+\epsilon})\{ -kb + |\delta| + |\epsilon| \}
\]
\[
D^\epsilon_{\delta} = \mathcal{S}_{C_{s,\delta,\epsilon}} D_{s,\delta} = \sum_{\delta,\epsilon} D^\epsilon_{\delta}.
\]

A coordinate vector \(\delta\) and direction \(\epsilon\) define a cube which we call the \((\delta,\epsilon)\)-cube of \(C_s\). If we had shifted the \(h\)-grading by \(|\delta|\), then the cube would be the Szabó complex of the link given by replacing each decoration from \(\epsilon\) with a positive crossing. So instead of \(D^\epsilon_{\delta}\) having \(\Delta\)-degree 1 like the Szabó differential it has degree is \(1 - |\epsilon|\).

Therefore \(H_s = (C_s, D_s)\) is a hyperbox of chain complexes with grading given by \(\Delta\).

When \(t\) has crossings there are a few additional complications. First, the diagram \(D_{s,\delta,\epsilon}^1\) is isotopic to \(t_{1t_n}\) but not equal to it. One such isotopy is given by a sequence of Reidemeister 2 moves which cancel inverse Artin generators. For a fixed braid word \(\beta_i\) order these from the inside out: if \(\beta = \sigma_{i_1} \cdots \sigma_{i_j}\), then
\[
\beta^{-1} \beta = \sigma_{i_j}^{-1} \cdots \sigma_{i_1}^{-1} \sigma_{i_1} \cdots \sigma_{i_j}.
\]

and the first cancellation is between \(\sigma_{i_j}^{-1}\) and \(\sigma_{i_1}\). Suppose that the braid word underlying \(t_i\) has length \(\ell_i\). Form a \((k-1)\)-dimensional hyperbox of link diagrams of size
\[
(b + \ell_{i_1}, \ldots, b + \ell_{i_{k-1}}).
\]

For each \(\delta = (d_1, \ldots, d_{k-1})\) there is a diagram \(D_{s,\delta}\) given as follows: if \(d_i \leq n\), then perform surgery along the first \(d_i\) canonical arcs between \(t_{i+1}\) and \(t_{i+2}\), just as in the crossingless case. If \(d_i = b + m_i\) with \(m_i > 0\) then perform all \(n\) surgeries, then perform the first \(m_i\) Reidemeister 2 moves. For such a \(\delta\) define
\[
|\delta| = \|(\min\{\epsilon_1, d_1\}, \ldots, \min\{\epsilon_n, d_n\})\|.
\]

Set
\[
C_s = \bigoplus_{\epsilon \in (d_1+\epsilon_1, \ldots, d_n+\epsilon_n)^{k-2}} \text{CSz}(D_{s,\epsilon})\{ -kb + |\delta| \}.
\]

Suppose that all of the coordinates of \(\delta\) are less than \(n\). The \((\delta,\epsilon)\)-cube defines a diagram \(D_{\delta,\epsilon}\) by replacing the canonical surgery arcs with with positive crossings as in the crossingless case. Note that these edge maps count configurations using the canonical surgery decoration along with any number of “internal” decorations...
from the crossings of $t_{si} \overline{t}_{si+1}$. Nevertheless the argument above shows that they have $\Delta$-degree $\|\epsilon\| - 1$.

If $\delta$ has some coordinates greater than or equal to $n$, then $\epsilon$ may pick out Reidemeister 2 moves in addition to surgery arcs. The cube is an iterated mapping cone as described in Section 2.1. The map $D_\delta$ is a component of the iterated mapping cone of these moves. The Reidemeister 2 moves have $\Delta$-degree zero, and the grading behavior follows.

**Proposition 3.3.** Let $t$ be a triplane diagram in plat form. The recipe above defines a system of hyperboxes of chain complexes $\mathcal{H}(t)$ over

$$C = \bigoplus_{i,j=1}^{3} \text{CSz}(t_i \overline{t}_j) \{ -b \}.$$ 

graded by $\Delta = q - 2h$.

**Proof.** We have shown that each $H_s$ is a hyperbox. We must check that the assignment $s \mapsto H_s$ defines a system of hyperboxes. The only thing to check for the first condition is the grading. Suppose that $C_s$ has dimension $(k - 1)$. Then

$$C_{s,\delta} = \left( \bigotimes \text{CSz}(t_{si} \overline{t}_{si+1}) \right) \{ -kb + \lfloor \delta \rfloor \}.$$ 

for some subsequence $s_{ij}$ of $s$. There are $k - \lfloor \delta \rfloor / b$ factors in this decomposition. Therefore

$$C_{s,\delta} = \bigotimes \text{CSz}(t_{si} \overline{t}_{si+1}) \{ -b \}$$

as required.

The face condition follows from the extension rule and the disconnected rule. Let $F$ be the face of $H_s$ between the $\epsilon$ and $\epsilon'$ corners. The extension rule implies that the map assigned to a handle attachment along a canonical surgery arc acts as the identity on the fixed sequence of $s$. Let $c$ and $c'$ be distinct subsequences in $c(\epsilon, \epsilon')$. Let $\gamma$ and $\gamma'$ be canonical surgery arcs which are attached as part of $c$ and $c'$, respectively. A configuration in $F$ involving both $\gamma$ and $\gamma'$ must be disconnected. In other words, there is a hyperbox $F'$ so that

$$F \cong H_c \otimes F'.$$

Therefore $H_s$ satisfies the face condition, cf. Lemma 8.10 part (2). $\square$

**Definition 3.4.** Let $\mathcal{A}(t)$ be the $A_\infty$-algebra over $\mathbb{F}[W]$ constructed from Propositions 3.3 and 8.13. The underlying group is

$$\bigoplus_{i,j=1}^{3} \text{CSz}(t_i \overline{t}_j) \{ -n \}$$
For \( y \in C_s \), \( \mu_k(y) \) is the image of \( y \) under the longest diagonal map of \( \hat{H}_s \) applied to \( y \).

The only part of argument which uses the language of Szabó homology is the extension rule, which holds up to homotopy for any strong Khovanov-Floer theory \([20][1]\). If the theory is conic with a vanishing differential, then the rule holds on the nose. So Proposition \( 3.3 \) holds word for word after replacing \( CSz \) by any conic, strong Khovanov-Floer theory whose internal differential vanishes. The upshot is that we can construct an \( A_\infty \)-algebra with underlying space

\[
\bigoplus_{i,j=1}^{3} \text{CF}(t_i\bar{t}_j)
\]

where \( \text{CF}(t_i\bar{t}_j) \) is a Heegaard Floer complex for the branched double cover of \( t_i\bar{t}_j \) (with the right choice of Heegaard diagrams). In this case the maps \( \mu_i \) count holomorphic polygons in Heegaard multi-diagrams rather than configurations.

3.2. An example: the unknot. Let us compute \( A(t) \) in the case that \( t \) is the crossingless, bridge number 1 triplane diagram for the unknotted sphere in \( S^4 \). Each summand of \( A(t) \) has rank two and \( \mu_1 = 0 \).

Let \( x \) be a simple tensor of length \( k > 2 \). There is a hyperbox \( H_x \) underlying \( \mu_k(x) \). The active part of any connected configuration which appears in \( H_x \) consists of \( k \) circles connected to each other in a line. The map assigned to such a configuration is zero. The map assigned to a disconnected configuration is zero. We conclude that there are no non-zero configurations of dimension greater than one, and therefore all diagonal \( \mathcal{S} \) maps in \( H_x \) are zero. It follows that \( \mu_k = 0 \) for \( k > 2 \).

Let \( t' \) be a stabilization of \( t \). One can show that \( \mu_3 \) does not vanish on \( A(t') \). So stabilization can dramatically change the character of \( A \). (In any case the rank of \( A(t') \) and its homology must be greater than that of \( A(t) \) and its homology, respectively.)

4. Invariance of the algebra

**Theorem 4.1.** Let \( t \) be a triplane diagram in plat form. The \( A_\infty \)-chain homotopy type of \( A(t) \) is an invariant of the trisection presented by \( t \).

**Proof.** Let \( t' \) be a triplane diagram in plat form for the same trisection. The remainder of this section is dedicated to showing that \( A(t) \simeq A(t') \) if \( t \) and \( t' \) differ by a

- braid isotopy (Proposition 4.3)
- Hilden moves (Proposition 4.4)
- bridge sphere transposition (Proposition 4.5)

In light of Meier and Zupan’s invariance statement (page 9), this suffices to prove the theorem. \( \square \)
To prove the theorem we will construct a map of systems $\rho$ for each move. We will prove that if $\rho$ and $\rho'$ are constructed from a move and its inverse, then $\rho' \circ \rho \simeq \text{Id}$ as a map of systems. By Theorem 8.19, these two maps induces maps of $A_\infty$-algebras $f$ and $f'$ so that $f' \circ f \simeq \text{Id}$. The following lemma will also be useful, especially in concert with Proposition 8.23.

**Proposition 4.2.** Let $D$ be a link diagram. Suppose that $D'$ is a subset of $D$ which is a braid. Suppose further that $D'$ is isotopic, as a braid, to the identity braid. Write $D''$ for the link diagram which results from replacing $D'$ with the identity braid.

Let $R$ and $R'$ be two sequences of Reidemeister moves supported in $D'$ which transform $D'$ into the identity braid. $R$ and $R'$ induce maps

$$F_R: \text{CSz}(D) \to \text{CSz}(D'')$$
$$F_{R'}: \text{CSz}(D) \to \text{CSz}(D'')$$

These maps are chain homotopic. The homotopy can be described by a cobordism whose support lies in $D'$.

**Proof.** This statement holds if one replaces $\text{CSz}(D)$ by $[D]$, Bar-Natan’s cobordism-theoretic link homology theory, see Lemmas 8.6 and 8.9 of [3]. Proposition 5.5 of [20] states that each conic, strong Khovanov-Floer theory “factors through” this $[\cdot]$; there is a functor $F_{CSz}$ so that $F_{CSz}([D]) = \text{CSz}(D)$. This functor carries handle attachments to handle attachment maps and Reidemeister maps to Reidemeister maps. (It was essentially described in Section 2.1.) So any relation that holds between Reidemeister maps on $[D]$ holds for $\text{CSz}(D)$. □

**Proposition 4.3.** Let $\beta$ and $\beta'$ be braid words which represent equal elements of $B_{2b}$. Let $t = (\bar{\beta}, \bar{\beta}_2, \bar{\beta}_3)$ and $t' = (\bar{\beta}', \bar{\beta}_2, \bar{\beta}_3)$. Then $A(t) \simeq A(t')$.

**Proof.** It suffices to prove the theorem in the case that $\beta$ and $\beta'$ differ by a single relation in $B_{2b}$: commuting two Artin generators, a braided Reidemeister 2 move, or triple-point move.

We begin with commuting two generators. Let $s$ be a sequence in $\{1, 2, 3\}$. Write $H_s$ and $H'_s$ for the hyperboxes assigned to $s$ by $\mathcal{H}(t)$ and $\mathcal{H}(t')$. If $s$ does not include 1, then $H_s = H'_s$. If $s$ does, then the maps on corresponding edges are identical except that the order of the two Reidemeister moves has been swapped. Proposition 4.2 implies that $\mathcal{H}(t)$ and $\mathcal{H}(t')$ are internally homotopic. Proposition 8.23 implies that $A(t) \simeq A(t)$.

Suppose that $\beta$ and $\beta'$ differ by a single braided Reidemeister 2 move. Without loss of generality suppose that $\beta'$ has two more crossings than $\beta$, and call the added crossings new. Let $s$ be a sequence which contains 1. Then $H'_s$ is larger than $H_s$: any axis which corresponds to 1 is two edges longer than the same axis in $H_s$ because it has to undo the new crossings. The other segments of $H'_s$ all correspond to segments
of $H_s$ in an obvious way. Elementarily enlarge $H_s$ so that it has the same shape as $H'_s$ and so that corresponding pieces line up. (In other words, the new cancellations in $H'_s$ correspond to elementary enlargements in $H_s$.) Call this new box $H_s$.

Define a map $\rho_s : H_s \to H'_s$ cube-by-cube as follows. Let $C$ and $C'$ be corresponding cubes of $H_s$ and $H'_s$. $C$ belongs to some face $F$ of $C_s$, and likewise for $C'$. Suppose for the moment that the contraction sequence of $F$ has a single element and $C$ has no fixed sequence. There are three possibilities.

- Neither $C$ nor $C'$ involves a canceling any crossings. In other words, $C = \text{CSz}(D)$ and $C' = \text{CSz}(D')$ where $D'$ may differ from $D$ by some Reidemeister 2 moves. Define $\rho_s|_C$ to be the composition of these Reidemeister 2 maps.

- $C$ and $C'$ do involve undoing some crossings, but not the new ones. $C$ and $C'$ are (iterated) mapping cones of Reidemeister 2 maps on $\text{CSz}(D)$ and $\text{CSz}(D')$, where $D$ and $D'$ are link diagrams which differ by Reidemeister 2 moves. Define $\rho_s|_C$ to be the cone of the new Reidemeister 2 map on the mapping cone. (In other words, iterate the mapping cone again).

- $C'$ undoes some of the new crossings. Define $\rho_s|_C$ by the schematic in Figure 8. Each vertex represents an $(n-1)$-dimensional cube, the Szabó complex of a link. A portion of that link is shown in the vertex. The vertical arrows are components of $\rho_s|_C$. The horizontal axis is an axis which corresponds to a 1 in $s$, so each horizontal arrow represents a map in $C_s$ or $C'_s$. The Reidemeister moves on the solid arrows define maps of complexes. It follows from functoriality of Szabó homology that the solid arrows commute up to homotopy. The homotopies are also given by sums of handle attachment maps; they are indicated on the schematic by the dotted arrows. If $C'$ undoes multiple pairs of new crossings, then define $\rho_s|_C$ by iterating Figure 8 like an iterated mapping cone.

![Figure 8](image-url)
In general, $C$ lives in a face whose contraction sequence $c$ has multiple elements. Write $f$ for the fixed sequence. By definition,
\[
C = \left( \bigotimes_{c' \in c(c,c')} C_{c'} \right) \otimes \left( \bigotimes_{f' \in f} C_{f'} \right).
\]
where $C_{c'}$ are cubes of hyperboxes which live in faces whose contraction sequences have a single element. Define $\rho_s|_C$ by
\[
\rho_s = \left( \bigotimes_{c' \in c} \rho_{c'} \right) \otimes \left( \bigotimes_{f' \in f} \rho_{f'} \right)
\]
where $\rho_f$ is the mapping cone of the usual Reidemeister map on Szabó homology.

Define $\rho'_s$ by reversing all the Reidemeister 2 moves. We aim to show that $\rho_s \circ \rho'_s \simeq \text{Id}$ by a chain homotopy $J_s$. If $D_s$ is unchanged by the Reidemeister 2 moves, then $\rho_s$ and $\rho'_s$ are identity maps and $J_s = 0$. If $s$ has a one-element contraction sequence and no fixed sequence then $\rho' \circ \rho$ is, on each cube $C$, the composition of a Reidemeister map and its inverse. Define $J_s$ on $C$ as the identity plus the homotopy $j$ between $\rho_s|_C \circ \rho'_s|_C$ and $\text{Id}_C$, see Figure 9. (This is the closest thing to the “mapping cone” of a chain homotopy.)

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (C) at (0,0) {$C$};
  \node (C') at (1,0) {$C$};
  \node (Id) at (0,-2) {$\text{Id}$};
  \node (Id') at (1,-2) {$\text{Id}$};
  \node (C') at (1,-4) {$C$};

  \draw[->] (C) to node[above] {$\rho' \circ \rho$} (C');
  \draw[dashed] (C) to node[right] {$j$} (Id);
  \draw[->] (Id') to node[above] {$\text{Id}$} (C');
  \draw[->] (C) to node[below] {$\text{Id}$} (C');
\end{tikzpicture}
\caption{Figure 9}
\end{figure}

If $F$ has a larger contraction sequence, then define $J_s|_F$ by equation 20 with $F'_s = (\rho' \circ \rho)_s'$ and $G'_s = \text{Id}_s'$. In other words,
\[
J_s|_F = \left( \bigotimes_{s' \in c} \bigoplus_{s''_1,\ldots,s''_r} (\rho' \circ \rho)_{s''_1} \otimes \cdots \otimes (\rho' \circ \rho)_{s''_r} \otimes J_{s''_1} \otimes \text{Id}_{s''_1} \otimes \cdots \otimes \text{Id}_{s''_r} \right) \otimes \text{Id}.
\]
Here $J_{s''_1}$ is the map for the singleton contraction sequence $(s''_1)$. Following Definition 8.17, $J_s$ constitutes a chain homotopy between $\text{Id}$ and $\rho' \circ \rho$.

The triple point move has an additional wrinkle. Define $\rho_s$ cube-by-cube as above: if $s$ does not contain 1 then $\rho_s = \text{Id}$. For other $s$, $H_s$ and $H'_s$ have the same size. Call
the crossings affected by the triple point move altered. It is straightforward to define maps between cubes which do not cancel altered crossings.

The Reidemeister 3 move shuffles the order of the altered crossings. To correct for this, enlarge \( H_s \) on each axis which involves an altered crossing immediately before that crossing is to be canceled. Consider the hyperbox \( H''_s \) in which the elementary extension is changed to a pair of Reidemeister 3 moves on the affected crossings and the maps after the extension are changed to agree with those of \( H'_s \). By Proposition 4.2, \( H''_s \simeq H_s \) and therefore the systems \( \mathcal{H} \) and \( \mathcal{H}' \) are internally homotopic.

Elementarily enlarge \( H'_s \) in the same position (but do not replace any of the maps afterwards). Construct a map \( \rho: H'_s \to H''_s \) using the Reidemeister 2 recipe: if corresponding cubes \( C \) and \( C' \) do not involve undoing altered crossings, then define \( \rho_s|_C \) using the Reidemeister 3 maps. For cubes which undo altered crossings, the diagrams agree (near those crossings) and so the map can be defined using the same recipe. In the extended region, the map is defined by the two-dimensional schematic in Figure 10.

\[ \begin{array}{ccc}
R3 & \xrightarrow{\text{Id}} & R3 \\
\downarrow & & \downarrow \\
\text{Figure 10. The schematic for the map } \rho \text{ in the extension region. The top is } H''_s \text{ and the bottom is } H'_s. \text{ In this case the square is commutative on the nose so no homotopy is necessary.} & & \\
\end{array} \]

**Proposition 4.4.** Suppose that \( \overline{\beta} \) and \( \overline{\beta}' \) differ by a Hilden move. Let \( t = (\overline{\beta}, \overline{\beta}_2, \overline{\beta}_3) \) and \( t' = (\overline{\beta}', \overline{\beta}_2, \overline{\beta}_3) \). \( A(t) \) is chain homotopic to \( A(t') \).

**Proof.** There is some \( h \) in the Hilden subgroup \( N_{2n} \subset B_{2n} \) (Section 1.1) so that \( \beta h \) is isotopic to \( \beta' \). There is a diagrammatic cobordism

\[ \overline{\beta} \rightarrow \overline{\beta} h \]

whose support is disjoint from \( \beta \); this follows from the motion group interpretation of \( N_h \). (See, for example, [2].) This cobordism is the composition of a sequence of Reidemeister moves, so it induces a chain homotopy equivalence on all the relevant Szabó chain groups.

It suffices to consider the case in which \( h \) is one of Hilden’s generators for \( N_h \). The support of each generator is a small neighborhood of the plats, so the crossings added
by the generator will always be canceled first. These canceling Reidemeister 2 moves have support disjoint from all the canonical surgery arcs (except for the ones which are pushed around by the Hilden move). Let $\mathcal{H}''$ be the system which is identical to $\mathcal{H}'$ except that the new cancellations happen immediately after the handle attachments between the affected plats. So the new order is: attach plats above and including the ones affected by the Hilden moves, cancel the crossings introduced by Hilden moves, attach the rest of the plats, cancel the rest of the crossings. $\mathcal{H}''$ is internally chain homotopic to $\mathcal{H}'$.

Now enlarge $H_s$ so that it has the same shape as $H''_s$ and so that the enlargements sit at the same positions as the cancellations of the new handles. We still call the result as $H_s$. We must cook up a map for each of Hilden’s generators. The recipe is basically the same as for the Reidemeister maps: the map will always be the identity in the region “after” the new crossings have been canceled. The map will be a composition of Reidemeister moves which realize the Hilden move in the region “before” the relevant plats are connected. The interesting part of the map is the region in which the relevant plats are connected and the new crossings are canceled.

$t_i$ is the easiest. The upwards maps are the obvious Reidemeister maps. Consider the cobordism $h \circ r$, using the notation from Figure 11. This cobordism is isotopic to one which begins with a Reidemeister 2 move on (say) the left tangle, then connects the two plats. (This uses movie move 7 and 13 in [2].) The support of the Reidemeister 2 move is disjoint from the canonical surgery arc, so the maps assigned to those two cobordisms commute up to homotopy. This shows that $h \circ r \simeq r' \circ h$ in Bar-Natan’s cobordism category. Therefore the homotopy can be chosen to be a map defined by handle attachments. This is the dotted arc on the left of Figure 12.

Figure 12 shows the argument for $s_i$. (We have changed the order of handle attachment, but of course the two systems are internally homotopic.) The first square commutes (up to homotopy) because the bottom plats can be passed under the upper plats by isotopies which are disjoint from $\gamma$. For the second square, repeatedly use movie move 15 to show that $h' \circ r$ is equivalent to the cobordism in Figure 13. This cobordism is a composition of Reidemeister 2 moves with a disjoint canonical handle
attachment. Swap the order of these two maps to obtain \( r' \circ h' \). The next square and all the ones after it commute up to homotopy by Proposition 4.2.

The arguments for the generators \( r_1 \) and \( r_2 \) are basically identical to \( s_i \): use movie move 15 to write a cobordism as a sequence of Reidemeister 2 moves and disjoint canonical handle attachments.

Just as in the previous proof, these definitions amount to a map of hyperboxes \( H_s \rightarrow H'_s \) for any sequence \( s \). They constitute maps of systems of hyperboxes \( \rho \) because they are defined by cobordisms. They are invertible, up to homotopy, because all of their components are Reidemeister moves: one can reverse the maps and run the same argument. Therefore \( \rho' \circ \rho \simeq \text{Id} \). These homotopies are also given by cobordisms, so together they form a homotopy between maps of systems. □

**Proposition 4.5.** Suppose that \( t \) and \( t' \) are triplane diagrams in plat form which differ by a braid transposition. Then \( A(t) \simeq A(t') \).

The proof is essentially the same as Propositions 4.3 and 4.4.

5. **Hybrid form and the homology algebra**

5.1. **Hybrid form.** Say that a triplane diagram is in hybrid form if each tangle is a disjoint union of the plat closure of a \((2b - 2k)\) braid and \(k\) crossingless arcs. \(k\) need
not be the same for each tangle. (In fact $k$ is not even unique.) Order the set of plats and arcs using the same rule as plat form: $\gamma > \gamma'$ if the highest point of $\gamma$ is higher than that of $\gamma'$. Note that any crossingless diagram is in hybrid form.

Now define $A(t)$ as above except that some crossings may need to be canceled before some canonical one-handles are attached.

**Proposition 5.1.** The construction of $A(t)$ extends to diagrams in hybrid form. Suppose that $t$ and $t'$ are diagrams in either form which present isotopic trisections. Then $A(t) \simeq A(t')$.

**Proof.** The argument that $A(t)$ is an $A_\infty$-algebra is the same as before.

Suppose that $t_1$ can be written as the plat closure of the braid $\beta_1 \in B_{2b-2k}$ and $k$ crossingless arcs. We show that it is isotopic to the union of the plat closure of a braid $\beta'_1$ and $k - 1$ crossingless arcs, which is constructed as follow. Let $a$ be the maximum crossingless arc. If $a$ is not nested inside $\beta_1$ or vice-versa, then one can just take $\beta'_1$ to include the two strands of $a$. So assume that either $a$ is nested in $\beta$ or vice-versa. There is an isotopy of $t'_1$ which moves $a$ to an arc $a'$ and does not move any other strands so that $\beta_1 \cup a' = \beta'_1$. Slide the maximum of the arc to sit above the other plats (if $\beta_1$ is nested inside $a$) or below the other plats (if $a$ is nested inside $\beta_1$), see Figure 14. This can be arranged so that all the new crossings occur behind the old crossings. Write $t'$ for this new triplane diagram.

An argument like the proof of Proposition 4.4 shows that $A(t) \simeq A(t')$. \hfill \Box

![Figure 14](image-url) Transforming hybrid form into plat form.
5.2. The homology algebra. In this section we show that, for connected surfaces, the associative algebra $H(\mathcal{A}(t))$ is essentially determined by the genus of $K$.

**Theorem 5.2.** Let $t$ and $t'$ be $b; c_{12}, c_{23}, c_{31}$ triplane diagrams in plat or hybrid form. Then $H(\mathcal{A}(t)) \cong H(\mathcal{A}(t'))$.

**Proof.** Write $A$ for $H(\mathcal{A}(t))$ and write $m$ for multiplication on $A$. The group $\text{Kh}(t_{i\bar{i}}t_{j\bar{j}})$ has a unique generator of highest quantum grading called $\Theta_{ij}$. Choose a basepoint on each component of $t_{i\bar{i}}t_{j\bar{j}}$ for each $i$ and $j$. There is a basepoint action on $\text{CKh}(t_{i\bar{i}}t_{j\bar{j}})$, see [11], which satisfies

$$m((X_p)_*\Theta_{ij} \otimes \Theta_{jk}) = (X_p)_*m(\Theta_{ij} \otimes \Theta_{jk}).$$

In general the interaction between basepoint maps and Reidemeister maps can be complicated, but $t_{i\bar{i}}t_{j\bar{j}}$ is always an unlink so

$$(\rho \circ X_p)_* = (X_{p'} \circ \rho)_*$$

where $\rho$ is a Reidemeister map, $p$ is a basepoint, and $p'$ is a basepoint on the same component following the Reidemeister map. (This follows because $X_p \simeq X_{p'}$ when $p$ and $p'$ lie on the same component of a link). So $\text{Kh}(t_{i\bar{i}}t_{j\bar{j}})$ is a cyclic module over the ring

$$\Lambda_{ij} = \mathbb{F}[X_1, \ldots, X_{c_{ij}}]/(X_1^2, \ldots, X_{c_{ij}}^2)$$

with generator $\Theta_{ij}$. If $S$ is connected, then the cobordisms underlying $m$ are connected for $i, j, k$ distinct. It follows that

(1) $$m((X_p)_*\Theta_{ij} \otimes \Theta_{jk}) = (X_p)_*m(\Theta_{ij} \otimes \Theta_{jk}).$$

It follows that $m((X_p)_*\Theta_{ij} \otimes \Theta_{jk}) = X_1 \cdots X_b$ if it is non-zero. From this and the degree of $m$ it follows that

(2) $$m(\Theta_{ij} \otimes \Theta_{jk}) = \sum_{j=1}^{c_{ik}} \left( \prod_{i=1}^{c_{jk}} X_i \cdots \hat{X}_j \cdots X_{c_{ik}} \right)$$

if it is non-zero (where the hat denotes omission).

Assume for a moment that $t$ is in plat form. Consider the modules $\text{CKh}(\beta_i\beta_j)$ as defined by Khovanov in [10]. The map $m$ is (a restriction of) the isomorphism

$$\text{Kh}(\beta_i\beta_j) \otimes_{H^{2n}} \text{Kh}(\beta_j\beta_k) \cong \text{Kh}(\beta_i\beta_k).$$

so $m(\Theta_{ij} \otimes \Theta_{jk}) \neq 0$.

Now suppose that $t$ and $t'$ are both $(b; c_{12}, c_{23}, c_{31})$-trisections. Choose a bijection between the components of $t_{i\bar{i}}t_{j\bar{j}}$ and $t'_{i\bar{i}}t'_{j\bar{j}}$ for all $i$ and $j$ so that the bijection between $t_{i\bar{i}}t_i$ and $t'_{j\bar{j}}t'_j$ is given by mirroring. This induces a graded isomorphism $\text{Kh}(t_{i\bar{i}}t_{j\bar{j}}) \cong \text{Kh}(t'_{i\bar{i}}t'_{j\bar{j}})$. Equations (1) and (2) show that this isomorphism extends to a map of algebras. \qed
Remark 5.3. The isomorphism above ignores some extra structure on $H(A(t))$. Instead of choosing a basepoint on each component, one could set the basepoints to be the bottoms of the tangles in $t$, i.e. $S \cap K$. The ring

$$\Lambda = \mathbb{F}[X_1, \ldots, X_{2n}] / (X_1^2, \ldots, X_{2n}^2)$$

acts on the modules $\text{Kh}(t_i \tilde{t}_j)$. This action “knows” $n$ and $c_{ij}$, $c_{jk}$, and $c_{ki}$ and therefore it knows the genus of $S$. It would be nice to prove Theorem 5.2 by exhibiting an isomorphism of $\Lambda$-modules which extends to an isomorphism of algebras.

This theorem has precedents in independent work of Jacob Rasmussen [19] and Kokoro Tanaka [25]. There is a natural way to obtain an invariant of closed surfaces in $S^4$ from Khovanov homology: puncture the surface at its top and bottom, find a movie presentation $\Sigma$ of the punctured surface, and compute the map

$$F_{\Sigma} : \text{Kh}(U) \rightarrow \text{Kh}(U)$$

where $U$ is the unknot. The Khovanov homology of $U$ is generated by $\Theta$ and $\Theta^{-}$. The invariant of $\Sigma$ is the $\Theta^{-}$-coordinate of $F_{\Sigma}(\Theta)$. Functoriality implies that this map does not depend on the particular movie presentation. For grading reasons this map must vanish if $K$ is not a torus, and Rasmussen and Tanaka showed that the map takes the same value on any torus. Tanaka proved a similar statement for Bar-Natan’s deformation.

Remark 5.4. Instead of studying $H(A(t))$, one could form a differential graded algebra $A(t)$ using the same construction as $A(t)$ but with only the Khovanov differential. Equivalently, consider

$$A(t) \otimes_{\mathbb{F}[W]} \mathbb{F}[W] / (W).$$

Theorem 5.2 does not imply that $A(t)$ is determined by combinatorial data. It might be interesting to study Massey products on this algebra.

Theorem 5.2 and the following theorem of Kadeishvili allow us to make $A(t)$ more concrete as an invariant.

**Theorem** (Kadeishvili). Let $A$ be an $A_{\infty}$-algebra. There is an $A_{\infty}$-structure on $H(A)$ so that $\mu_1 = 0$, $\mu_2 = m_2^*$, and $A$ is $A_{\infty}$-quasi-isomorphic to $A$.

An $A_{\infty}$-algebra with $\mu_1 = 0$ is called minimal. It follows that there is a minimal model for $A(t)$ with rank $2^{3c} + 2^{3n}$. From two triplane diagrams with the same combinatorial data, we obtain two minimal $A_{\infty}$-algebras which are isomorphic as associative algebras: the invariant is an $A_{\infty}$-structure on this algebra.
5.3. Units. A unit in an $A_\infty$-algebra is an element $\iota$ so that
\begin{align*}
\mu_1(\iota) &= 0 \\
\mu_2(\iota \otimes x) &= \mu_2(x \otimes \iota) = x \\
\mu_n(x_1 \otimes \cdots \otimes \iota \otimes \cdots \otimes x_n) &= 0
\end{align*}
for all $x$ in the algebra and $n > 2$. An $A_\infty$-algebra which has a unit is called unital.

**Proposition 5.5.** If $t$ is crossingless then $\mathcal{A}(t)$ is unital.

**Proof.** There is a unique element with highest grading in $\text{CSz}(t_i t_i)$ for each $i \in \{1, 2, 3\}$. Call this element $\iota_i$. Define $\iota = \iota_1 + \iota_2 + \iota_3$.

The operation $\mu_2$ is associative because $\mu_1 = 0$. So $\mathcal{A}(t)$ is closely related to Khovanov’s arc algebra $H^{2b}$ [10]. Each $\text{CSz}(t_i t_j)$ with $i \neq j$ is a submodule of the module assigned to $t_i t_j$, and $\text{CSz}(t_i t_i)$ is a subring of $H^{2b}$. $\iota_i$ is clearly the unit in the subring. It follows that $\mu_2(x \otimes \iota) = \mu_2(\iota \otimes x) = x$ for all $x \in \mathcal{A}(t)$.

Let $y = x_1 \otimes \cdots \otimes x_n$ be a simple tensor so that $x_j = \iota$ for some $j$. We aim to show that $\mu_n(y) = 0$. Let $H_y$ be the hyperbox used to compute $\mu_n(y)$. Consider a configuration in which one of the circles of $x_j$ is active. That circle has either degree one or in-degree and out-degree one. Every Szabó and Bar-Natan configuration map vanishes if that circle is labeled $v_+$. It follows from the definition of compression that $\mu_n$ vanishes on $y$. \hfill \Box

For diagrams with crossings, $\mathcal{A}(t)$ is not necessarily unital. $t_i t_i$ has an obvious isotopy to a crossingless bridge splitting $t_i' t_i'$. It seems reasonable that one could exchange $\text{CSz}(t_i t_i)$ for $\text{CSz}(t_i' t_i')$ in $\mathcal{A}(t)$ to obtain a unital $A_\infty$-algebra which is $A_\infty$-chain homotopy equivalent to $\mathcal{A}(t)$.

6. The Bar-Natan perturbation and the isotopy class invariant

In Section 3 we applied Szabó homology to a system of hyperboxes to obtain an $A_\infty$-algebra. In this section we include Bar-Natan’s perturbation to obtain an isotopy class invariant. The summary is this: there is a link invariant $\text{CS}(D)$, defined in [21] and recalled in Section 6.1, which combines Bar-Natan’s perturbation of Khovanov homology to Szabó homology. The cobordism maps in this theory are highly non-local. The recipes of Section 3 and 4 with CS standing in for CSz produce “perturbed $A_\infty$-algebras.” If $D$ is a diagram of an oriented unlink then $\text{HS}(D)$, the homology of $\text{CS}(D)$, has a unique class in maximal grading. Write $\theta_{ij} \in \text{CS}(t_i t_j)$ for a representative of this homology class. In Section 6.4 we study these classes and their behavior under the perturbed multiplication $\mu_2'$ and triple multiplication $\mu_3'$. We are particularly interested in
\begin{equation*}
\mu_3'((\theta_{(i+1)(i+2)} \otimes \theta_{(i+1)(i+2)} \otimes \theta_{(i+2)i}) \in \text{CS}(t_i t_i)
\end{equation*}
for $i = 1, 2, 3$. (Throughout we take indices mod three.) There is a cobordism

$$t_i \bar{t}_i \to U^{2b}$$

shown in Figure 15. Write $c_i$ for the map induced by this closure cobordism. This map is studied in Section 6.3. (We will usually drop the $i$). Write $\Theta$ for the homology class in $\text{HS}(U^{2b})$ with greatest grading. In Section 6.5 we show that the coefficient of

$$\sum_{i=1}^{3} cl_i(\mu_i(\theta_{i(i+1)} \otimes \theta_{i(i+1)(i+2)} \otimes \theta_{i(i+2)}))$$

at $\Theta$ is a well-defined invariant of the trisection presented by $t$. In principle this invariant takes values in $F[U,W]$ but for grading reasons it really lives in $F$. This invariant may change under stabilization, but a related sum called $q(t)$ does not. In the Section 7 we will show that $q(t)$ is non-trivial. We will also show that the computation of $q(t)$ is somewhat less laborious than this exposition might suggest.

The entire endeavor rests on the gradings, and therefore we only work with oriented surfaces. As all the link diagrams we work with our unlinks, it does not seem so far-fetched that the techniques might work for non-orientable surfaces. Note that Lemma 6.8 and the discussion suggests that $q(K)$ should be zero for any surface with odd Euler characteristic.

6.1. The Bar-Natan perturbation. Let $U$ be a formal variable. The Bar-Natan perturbation of Khovanov homology comes from replacing the Frobenius algebra $F[X]/(X^2)$ by $F[X,U]/(X^2 - U)$. By setting $U = 0$ one recovers the Khovanov chain group. The resulting link homology theory has many interesting applications. (Over $\mathbb{Q}$, it is essentially equivalent to the famous Lee perturbation.)

Bar-Natan’s differential does not commute with the Szabó differential, so one cannot simply add them together to get a combined link homology theory. Sarkar, Seed, and Szabó make these theories compatible in [21]. Recall that Szabó homology is defined via a function $S$ which assigns linear maps to configurations. The Bar-Natan configuration map, $\mathcal{B}$, is defined as follows. Let $C$ be a configuration. Construct a graph whose vertices are in bijection with the active circles of $C$. Put an edge between two vertices for each decoration connecting the underlying circles. Call $C$ a tree if its graph is a union of trees. Call $C$ a dual tree if the configuration dual to $C$ is a tree. (The dual of a configuration is given by first performing surgery along all the decorations, then rotating all the arcs by ninety degrees counterclockwise.)

Let $x \in \text{CKh}(\mathcal{D}(I))$ be a canonical generator. $\mathcal{B}_C(x) = 0$ unless $C$ is a disjoint union of $v_-$-labeled trees and $v_+$-labeled dual trees. On a $v_-$-labeled tree $\mathcal{B}_C$ is defined by

$$\mathcal{B}_C(v_- \otimes \cdots \otimes v_-) = v_-$$
and $B_C(x) = 0$ for any other labeling. On a $v_+$-labeled dual tree, define

$$B_C(1) = 1 \otimes \cdots \otimes 1$$

and $B_C(x) = 0$ for any other labeling. Then

$$d_{BN} = \sum_{I<J} U_W^{|I-J|-1} B_{C(I,J)}.$$

Give $U$ the $(h,q)$-grading $(0, -2)$. If $C(I, J)$ is one-dimensional then $B_{C(I,J)}$ agrees with the usual Bar-Natan perturbation.

**Theorem** ([21]). $d_{BN}$ is a differential which commutes with $d_{Kh}$ and $d_{Sz}$. The chain homotopy type of $\text{CKh}(D) \otimes \mathbb{F}[U, W]$ equipped with $d_{Kh} + d_{BN}$ or $d_{Kh} + d_{BN} + d_{Sz}$ is a graded link invariant.

Write $\partial' = d_{Kh} + d_{Sz} + d_{BN}$ for the total differential and $\partial'' = d_{Kh} + d_{BN}$. Write $\text{CS}(D)$ for the theory using $\partial'$ and $\text{CBN}(D)$ for the theory using $\partial''$.

$\text{CS}$ and $\text{CBN}$ are not strong Khovanov-Floer theories ([20]) because they do not satisfy the K"unneth formula:

$$\text{CS}(D \coprod D') \not\cong \text{CS}(D) \otimes \text{CS}(D')$$

(and likewise for $\text{CBN}$) even though the two sides are isomorphic as $\mathbb{F}[U, W]$-modules. Also, if $\Sigma$ and $\Sigma'$ are cobordisms from $D_0$ to $D_1$ and $D'_0$ to $D'_1$, respectively, then

$$\text{CS}(W \otimes W') \not\cong \text{CS}(W) \otimes \text{CS}(W').$$

(and likewise for $\text{CBN}$). Nevertheless, the argument for functoriality of Szabó homology applies to Bar-Natan homology, *mutatis mutandis*. The theories satisfy every other condition to be conic, strong Khovanov-Floer theories. If $D'$ is crossingless – i.e. if $\text{CS}(D')$ has a vanishing differential – then

$$\text{CS}(D \coprod D') \cong \text{CS}(D) \otimes \text{CS}(D')$$

$$\text{CBN}(D \coprod D') \cong \text{CBN}(D) \otimes \text{CBN}(D')$$

The K"unneth formula for diagrams is only ever used in [20] in this situation. The K"unneth formula for cobordisms is only used to prove the $S$, $T$, and $4Tu$ relations of [3]. One only needs to prove these relations for cobordisms of the form $\text{Id} \otimes W'$ where $\text{Id}$ is the product cobordism $D \times I$ and $W'$ is a cobordism of crossingless diagrams. In lieu of a K"unneth formula, the cobordism maps satisfy the following:

$$F_{\text{Id} \coprod W'} = G_{W'} \otimes F_{W'},$$

where

$$G_{W'} : \text{CS}(D) \to \text{CS}(D)$$
depends in some way on the topology of $W'$. But if $F_{W'} = 0$, then $F_{\text{Id}W'} = G_{W'} \otimes 0 = 0$. So the $S$, $T$, and $4Tu$ relations hold for CS. Therefore the proof survives even without the K"unneth formula.

**Theorem 6.1.** CS and CBN are functorial link invariants. The Reidemeister invariance maps and homotopies are described by Bar-Natan's cobordism maps.

6.2. **The perturbed $A_\infty$-algebra.** Let’s try to apply the arguments of Section 3 to CS. There is a map

$$\mu'_2: \text{CS}(t_1\bar{t}_j) \otimes \text{CS}(t_j\bar{t}_k) \to \text{CS}(t_i\bar{t}_k)$$

given by a sequence of one-handle attachment maps. Now let $s = (s_1, s_2, s_3, s_4)$. Construct a hyperbox of chain complex $H'_s$ exactly as in Section 3: each cube is the CS complex of a link diagram. But this hyperbox cannot be part of a system of hyperboxes. For consider the $(1,0)$-face of $H'_s$. Label the canonical surgery arcs in the first column $\gamma_1, \ldots, \gamma_n$. In Szabó homology, the composition of maps along that face is

$$(h_{\gamma_n} \circ \cdots \circ h_{\gamma_1}) = (h_{\gamma_n} \circ \cdots \circ h_{\gamma_1})|_{\text{CS}_z(D(s_1, s_2, s_3))} \otimes \text{Id}$$

because, for example,

$$h_{\gamma_1} = h_{\gamma_1}|_{\text{CS}_z(D(s_1, s_2, s_3))} \otimes \text{Id}.$$

But in CS,

$$h_{\gamma_1} = h_{\gamma_1}|_{\text{CS}(D(s_1, s_2, s_3))} \otimes \text{Id} + h_{\gamma_1, BN}|_{\text{CS}(D(s_1, s_2, s_3))} \otimes U^{-1}Wd_{BN}$$

where $h_{\gamma, BN}$ denotes the component of $h_\gamma$ counts Bar-Natan configurations. We use $U^{-1}$ to reduce the $U$-exponent of a term with positive $U$-exponent. The key point is that $\mathcal{B}$ assigns non-zero maps to disjoint unions of Bar-Natan configurations. So attaching a one-handle to a diagram “activates” the Bar-Natan differential in other parts. To see that the exponent is correct, recall that a $k$-dimensional Bar-Natan configuration is counted with coefficient $UW^{k-1}$. Observe that

$$UW^{k+\ell-1} = U^{-1}W(UW^{k-1})(UW^{\ell-1}).$$

Counting $(k + \ell)$-dimensional configurations by counting $k$- and $\ell$-dimensional configurations separately undercounts by a factor of $W$ and overcounts by a factor of $U$. We will use this idea frequently and refer to it as the *gluing principle*. Therefore, in
(h_{\gamma_n} \circ \cdots \circ h_{\gamma_1}) = (h_{\gamma_n} |_{CS(\mathcal{D}(s_1, s_2, s_3))} \otimes \text{Id} + h_{\gamma_1, \text{BN}} |_{CS(\mathcal{D}(s_1, s_2, s_3))} \otimes U^{-1} W d_{\text{BN}}) \circ \cdots \circ (h_{\gamma_n} |_{CS(\mathcal{D}(s_1, s_2, s_3))} \otimes \text{Id} + h_{\gamma_n, \text{BN}} |_{CS(\mathcal{D}(s_1, s_2, s_3))} \otimes U^{-1} W d_{\text{BN}})

(3) = (h_{\gamma_n} \circ \cdots \circ h_{\gamma_1}) \otimes \text{Id} + \sum_{i=1}^{n} (h_{\gamma_n} \circ \cdots \circ h_{\gamma_i, \text{BN}} \circ \cdots \circ h_{\gamma_1}) \otimes U^{-1} W d_{\text{BN}}

(4)

Write the second term as

$$U^{-1} W \overline{\mu}_2 \otimes d_{\text{BN}}$$

It follows that the map induced by the (1, 0)-face is

$$\mu'_2 \otimes \text{Id} + \otimes \overline{\mu}_2 \otimes U^{-1} W d_{\text{BN}}.$$ 

To be concrete, $\overline{\mu}_2$ is a multiple of the component of $\mu'_2$ which counts, at some point, a Bar-Natan configuration. The same argument applies to CBN – call the map $\overline{\mu}'_2$. 

All in all, one can still define $\mu'_{3}$ or $\mu''_{3}$ by compressing the hyperbox assigned to $s$.

**Definition 6.2.** Let $t$ be a triplane diagram in plat or hybrid form. Let $\mathcal{A}_S(t)$ be the vector space

$$\mathcal{A}_S = \bigoplus_{i,j=1}^{3} \text{CS}(t_i \overline{t}_j)$$

equipped with the operations $\partial$, $\mu'_2$, and $\mu'_{3}$. Let $\mathcal{A}_{\text{BN}}(t)$ be the vector space

$$\mathcal{A}_{\text{BN}} = \bigoplus_{i,j=1}^{3} \text{CBN}(t_i \overline{t}_j)$$

equipped with the operations $\partial'$, $\mu'_2$, and $\mu''_{3}$.

**Proposition 6.3.** $\mathcal{A}_S(t)$ satisfies the following equations:

$$\partial'^2 = 0$$

$$\partial' \circ \mu'_2 + \mu'_2 (\partial' \otimes \text{Id}) + \mu'_2 (\text{Id} \otimes \partial') = U^{-1} W \mu'_2 (d_{\text{BN}} \otimes d_{\text{BN}})$$

$$\mu'_{3} \circ (\text{Id} \otimes \text{Id} \otimes \partial' + \text{Id} \otimes \partial' \otimes \text{Id} + \partial' \otimes \text{Id} \otimes \text{Id}) + \delta' \circ \mu'_{3} + \mu'_2 (\mu'_2 \otimes \text{Id} + \text{Id} \otimes \mu'_2)$$

$$= U^{-1} W \mu'_2 (\mu'_2 \otimes d_{\text{BN}} + d_{\text{BN}} \otimes \mu'_2)$$

$$+ U^{-1} W (\mu'_{3} (d_{\text{BN}} \otimes d_{\text{BN}} \otimes \text{Id}) + \mu'_{3} (d_{\text{BN}} \otimes \text{Id} \otimes d_{\text{BN}})$$

$$+ \mu'_{3} (\text{Id} \otimes d_{\text{BN}} \otimes d_{\text{BN}})) + U^{-2} W^2 \mu'_{3} (d_{\text{BN}} \otimes d_{\text{BN}} \otimes d_{\text{BN}})$$

$\mathcal{A}_{\text{BN}}(t)$ satisfies the same equations with $\partial''$, $\mu''_{3}$ and $\overline{\mu}'_2$ replacing $\partial'$, $\mu'_{3}$, and $\overline{\mu}_2$. 

We call these equations the *perturbed $A_n$-equations* for $n = 1, 2, 3$; compare with equation (12).

**Proof.** The first equation follows from the definition of CS. The second equation follows from the fact that $\mu'_2$ is a chain map. The term on the right comes from the part of $\partial$ (or $\partial'$) which does not respect the Künneth formula.

The third equation comes from the fact that $\mathcal{H}_s$ is a chain complex for length four $s$. Each term represents part of the square of the differential from the initial vertex of $\mathcal{H}_s$ to its terminal vertex. The terms on the right all come from the failure of the Künneth formula for CS. In the final term, Bar-Natan configurations are split into three pieces, so one must apply the gluing principle twice. □

**Proposition 6.4.** Let $t$ and $t'$ be triplane diagrams in hybrid or plat form which differ by a single bridge sphere transposition or interior Reidemeister move. Then there is a collection of maps

$$\rho'_i : \mathcal{A}_S(t)^{\otimes i} \to \mathcal{A}_S(t')$$

so that $\rho'_1$ is the direct sum of the Reidemeister maps on CS. These maps satisfy the following relations. First,

$$\rho'_1 \circ \partial + \partial \circ \rho'_1 = 0.$$ 

Also,

$$\rho'_2(\partial \otimes \text{Id}) + \rho'_2(\text{Id} \otimes \partial) + \partial \circ \rho'_2 + \mu'_2(\rho'_1 \otimes \rho'_1) + \rho'_1 \mu'_2 = U^{-1}W \rho'_2(d_{BN} \otimes d_{BN}) + U^{-1}W \mu'_2 R_{1,1}$$

and

$$\rho'_3(\partial \otimes \text{Id} \otimes \text{Id}) + \rho'_3(\text{Id} \otimes \partial \otimes \text{Id}) + \rho'_3(\text{Id} \otimes \text{Id} \otimes \partial) + \partial \rho'_3 + \rho'_2(\mu'_2 \otimes \text{Id}) + \rho'_2(\text{Id} \otimes \mu'_2) + \rho'_1 \mu'_3 + \mu'_2(\rho'_1 \otimes \rho'_2) + \mu'_2(\rho'_2 \otimes \rho'_1) + \mu'_3(\rho_1 \otimes \rho_1 \otimes \rho_1)$$

$$= U^{-1}W (\rho'_2(d_{BN} \otimes d_{BN} \otimes \text{Id}) + \rho'_3(d_{BN} \otimes \text{Id} \otimes d_{BN}) + \rho'_3(\text{Id} \otimes d_{BN} \otimes d_{BN}))$$

$$+ U^{-2}W^2 \rho'_3(d_{BN} \otimes d_{BN} \otimes d_{BN})$$

$$+ U^{-1}W (\rho'_2(\tilde{\mu}_2 \otimes d_{BN}) + \rho'_2(d_{BN} \otimes \tilde{\mu}_2))$$

$$+ U^{-1}W (\mu'_2(R_{2,1} + R_{1,2})).$$

Here

$$R_{1,1} : \mathcal{A}_S(t)^{\otimes 2} \to \mathcal{A}_S(t)^{\otimes 2}$$

$$R_{2,1}, R_{1,2} : \mathcal{A}_S(t)^{\otimes 3} \to \mathcal{A}_S(t')^{\otimes 2}$$

are linear maps derived from $\rho'_2 \otimes \rho'_1$ and $\rho'_1 \otimes \rho'_2$. The same holds for $\mathcal{A}_{BN}$.

The point is that the maps $R_{i,j}$ describe parts of the Reidemeister maps which do not respect the Künneth principle. Their exact form is not important.
Proof. To prove Theorem 4.1 we showed constructed maps of systems of hyperboxes $H(t) \to H(t')$. For each sequence $s$ we constructed a map of hyperboxes $\rho_s$. The $A_\infty$-relations are derived from the fact that $\hat{\rho}_s$ is a chain complex.

Write $H_s$ and $H'_s$ for the hyperboxes assigned to $s$ by $A_S(t)$ and $A_S(t')$. Define $\rho'_s : H_s \to H'_s$ using the same handle attachment maps as $\rho_s$. If $s$ has length $(i + 2)$, then define $\rho'_1$ to be the long diagonal in $\hat{\rho}_s$. (We only study $\rho'_1$, $\rho'_2$, and $\rho'_3$.) The relations satisfied by these maps stem from the fact that $\hat{\rho}_s$ is a chain complex.

If $s$ has length two then $\rho_s$ is defined as the Reidemeister map $CS(t_1 \bar{t}_1) \to CS(t'_1 \bar{t}'_1)$. This proves equation (6). For equation (7), study the two-dimensional cubical complex $\hat{\rho}_s$. The terms on the right, except the last line, come from the Bar-Natan differential’s disrespect for the Künneth principle. Define $R_{1,1}$ to be the part of $\rho_1 \otimes \rho_1$ which does not respect the Künneth principle.

Equation 7 follows from the same sort of reasoning. \hfill $\square$

6.3. The closure cobordisms. Consider the cobordism from $t_i \bar{t}_i$ to $U^{2b}$ shown in Figure 15: it begins with canonical handle attachments and finishes with cancellations. Choose the cancellations so that $t_i \cap \bar{t}_i$ is in the complement of all these moves. This cobordism induces a map

$$cl_i : HS(t_i \bar{t}_i) \to HS(U^{2b}).$$

Our goal in this section is to understand the relationship between $cl_i$ and the multiplication operations. Figure 16 shows how to put all of these maps on the same footing. Define the double closure map

$$cl_{ik} : CS(t_i \bar{t}_k) \otimes CS(t_k \bar{t}_i) \to U^n$$

as follows. Form a hyperbox using the two families of canonical surgery arcs in Figure 16. Include also the appropriate Reidemeister moves between cancellable crossings as in Section 3. (One has to choose which set of arcs comes first.) $cl_{ik}$ is
defined exactly as any of the \( \mu_3 \) maps: compress this hyperbox and look at the long diagonal map. Define the triple cobordism map \( \text{cl}_{ijk} \) by using all three families.

**Figure 16**

**Proposition 6.5.** Let \( x_1 \in \text{CS}(t_1\overline{t}_2) \), \( x_2 \in \text{CS}(t_j\overline{t}_k) \), and \( x_3 \in \text{CS}(t_3\overline{t}_1) \). Then

\[
(8) \quad \text{cl}_i(\mu'_2(x \otimes y)) + \text{cl}_j(\mu'_2(y \otimes x)) = \text{cl}_{ij}( (\partial \otimes \text{Id} + \text{Id} \otimes \partial + U^{-1}Wd_{BN} \otimes d_{BN})(x \otimes y)).
\]

and

\[
\sum_{i=1}^{3} \text{cl}_i(\mu'_2(x_i \otimes x_{i+1} \otimes x_{i+2})) + \sum_{i=1}^{3} \text{cl}_{i(i+2)}(x_i \otimes \mu'_2(x_{i+1} \otimes x_{i+2}) + \mu'_2(x_i \otimes x_{i+1}) \otimes x_{i+2})
\]

\[
= \text{cl}_{ijk}( (\partial \otimes \text{Id} \otimes \text{Id} + \text{Id} \otimes \partial \otimes \text{Id} + \text{Id} \otimes \text{Id} \otimes \partial)(x_1 \otimes x_2 \otimes x_3))
\]

\[
+ U^{-1}W \text{cl}_{ijk}( (d_{BN} \otimes d_{BN} \otimes \text{Id} + \text{Id} \otimes d_{BN} \otimes d_{BN} + d_{BN} \otimes \text{Id} \otimes d_{BN})(x_1 \otimes x_2 \otimes x_3))
\]

\[
+ \text{terms divisible by } W^2 + \text{perturbation terms}
\]

**Proof.** Same as every proof in Section 6.2. \( \square \)

6.4. **Top generators.** Let \( t \) be an \((b; c_{12}, c_{23}, c_{31})\) triplane diagram for \( K \). \( \text{HS}(t_i\overline{t}_j)\{−b\} \) and \( \text{H}(\text{CBN}(t_i\overline{t}_j)\{−b\}) \) have a unique elements \( \Theta_{ij} \) with quantum grading 0. Write \( \theta_{ij} \) for a homogeneous cycle representative of \( \Theta_{ij} \). Each chain complex has an inner product induced by the basis of canonical generators. Write \( \langle −, − \rangle \) for this inner product. The closure cobordisms are natural with respect to top generators in the
sense that $\langle \text{cl}(\theta), \Theta \rangle = U^b$ and if $\langle \text{cl}(x), \Theta \rangle$ has non-trivial coefficient on $U^b$ then $\langle x, \theta \rangle = 1$.

Lemma 6.6. Let $x \in \text{CS}(t_i \bar{t}_j)$. Let $d$ be any combination of $d_{\text{Kh}}$, $d_{\text{BN}}$, and $d_{\text{Sz}}$. Then

$$\langle \mu'_2(x \otimes d\theta_{jk}), \theta_{ik} \rangle = \langle \mu''_2(x \otimes d\theta_{jk}), \theta_{ik} \rangle = 0.$$

Proof. Suppose $t_i \bar{t}_j$ has $\ell$ crossings. The homological grading is really a projection of the $\mathbb{Z}^\ell$-grading which assigns to a canonical generator the resolution of its underlying diagram. The proof that the differential and handle attachment maps are homologically graded relies on the fact that these maps are increasing in this “resolution grading.” The resolution grading of $\mu_2(x \otimes d\theta_{jk})$ cannot be the same as $\theta_{ik}$ because $d\theta_{jk}$ is has positive homological grading.

Lemma 6.7. Let $x \in \text{CKh}(t_i \bar{t}_j)$ be a canonical generator. Then

$$\langle \mu'_2(x \otimes \theta_{jk}), \theta_{ik} \rangle$$

and

$$\langle \mu''_2(x \otimes \theta_{jk}), \theta_{ik} \rangle$$

have no terms divisible by $W$ but not $W^2$.

Proof. We prove the formula for $\mu'_2$ and come back to $\mu''_2$ at the end. $\mu'_2(x \otimes \theta_{jk})$ can only have such a term if $x$ has homological degree $(-1)$ – otherwise the power of $W$ would be larger. Because $\mu'_2$ is a map of fixed degree, the $W$-degree one part of

(9) $$\langle \mu'_2(x \otimes \theta_{jk}), \theta_{ik} \rangle$$

must be a monomial of the form $aWU^b$ for some $a \in \mathbb{F}$ and $b \in \mathbb{Z}$. Now

$$\partial \mu'_2(x \otimes \theta) = \mu'_2(\partial x \otimes \theta) + \mu'_2(x \otimes \partial \theta) + W \mu'_2(d_{\text{BN}} x \otimes d_{\text{BN}} \theta).$$

The third and fourth terms vanish at $\theta_{ij}$ by Lemma 6.6. In fact $\langle \partial \mu'_2(x \otimes \theta), \theta_{ik} \rangle$ vanishes as well. For suppose that $\partial \mu'_2(x \otimes \theta) = a\theta_{ik} + y \neq 0$ with $a \in \mathbb{F}[U, W]$. It follows that $y$ is also a cycle. If $y$ is a boundary then $[\theta_{jk}] = 0$. If $y$ is not a boundary, then it represents a class with lower grading. But then $a[\theta_{jk}] = [y]$, which is impossible. It follows that

$$\langle \mu'_2(\partial x \otimes \theta_{jk}), \theta_{ik} \rangle = 0$$

Now suppose that $x$ is a canonical generator in homological degree $-1$. The terms of $\mu'_2(x \otimes \theta_{jk})$ which are divisible by $W$ but not $W^2$ come from two-dimensional configurations in which one decoration comes from a crossing in $t_i \bar{t}_j$ and the other comes from a surgery arc. Let $c$ be a crossing which supports such a configuration. Write $\partial_c$ for the component of $\partial$ which involves $c$. Then

$$\langle \mu'_2(\partial_c x \otimes \theta), \theta_{ik} \rangle$$
counts two-dimensional configurations which have been divided into two one-dimensional configurations. (Compare to “broken polygons” in, for example, Heegaard Floer homology.) Let $C$ be a two-dimensional configuration which uses $c$. Write $C = c_\cdot$ and $C = c'$ for the one-dimensional configurations which make up $C$. We claim that the $WU_b$ coefficient of
\[ \langle \mu'_2, C(x \otimes \theta_{ik}) \rangle \]
is the same as the $U^{b+1}$-coefficient of
\[ \langle \mu'_2, C'(d_c(x) \otimes \theta_{ik}) \rangle \]
where $\mu'_2, C$ stands for the part of $\mu'_2$ which uses the exact configuration $C$. If $C$ is a Bar-Natan configuration then this follows from the gluing principle. If $C$ is a Szabó configuration then one must check all the non-trivial, two-dimensional configurations.

Summing over all the crossings, the $WU_b$-coefficient of
\[ \langle \mu'_2, C'(x \otimes \theta_{ik}) \rangle \]
is the same as the $U^{b+1}$-coefficient of
\[ \mu'_2(\partial x \otimes \theta_{ik}) \]
which must be zero. For $A_{BN}(t)$ the argument is the same except that one can ignore the Szabó configurations.

\[ \Box \]

**Lemma 6.8.** Fix representatives $\theta_{ij}$, $\theta_{jk}$, and $\theta_{ki}$. Then
\[ \langle \mu'_2(\theta_{ij} \otimes \theta_{jk}), \theta_{ik} \rangle = U^{b+\frac{c_{ik} - c_{ij} - c_{jk}}{2}}. \]

**Proof.** Consider the cobordism $W$ underlying $\text{cl}_i(\mu'_2(\mu'_2(\theta_{ij} \otimes \theta_{jk}) \otimes \theta_{ki}))$. It has $c_{ij} + c_{jk} + c_{ki}$ zero-handles, $3b$ one-handles, and $2b$ two-handles, so $\chi(W) = \chi(t)$. (In the language of Section 11.2 of [3], we should also think of $W$ as carrying $2b$ dots.)

Tanaka [25] showed that the "Khovanov-Jacobsson" number of an orientable surface is essentially determined by the genus of that surface following Carter, Saito, and Satoh [6]. It is straightforward to show that, if every zero-handle is decorated with a dot, then this number is (over any base field), where $g$ is the genus of $W$, by directly computing for a nicely embedded surface in $\mathbb{R}^3$. So $\langle \text{cl}_i(\mu'_2(\mu'_2(\theta_{ij} \otimes \theta_{jk}) \otimes \theta_{ki})), \Theta \rangle \neq 0$. It follows that $\langle \mu'_2(\mu'_2(\theta_{ij} \otimes \theta_{jk}) \otimes \theta_{ki}), \theta_{ij} \rangle \neq 0$.

The power of $U$ is determined by the degree of $\mu'_2$.

---

3Here is a sketch of a more conceptual argument. The component of the Bar-Natan perturbation in $W$-degree zero appears in Szabó’s theory as the chain homotopy $h$ which shows that the choice of decorations does not matter (up to homotopy) in defining $\text{CSz}(\mathcal{D})$. It is straightforward to check that, if $\mathcal{S}_C(x) \neq 0$, then $\mathcal{S}_C'(x) = 0$ where $C'$ is any configuration given by reversing a decoration in $C$. The component of $\mu'_2, C'(d_c(x) \otimes \theta_{ik})$ in the relevant degree is precisely $\partial h + h \partial$. 
For crossingless links, the formula above can be worked out by counting the number of merges and splits in $\Xi$. It’s a fun exercise to check that the formulas coincide.

The formula implies that $\mu_2(\theta_{ij} \otimes \theta_{jk})$ has coefficient 0 at $\theta_{ik}$ if $b + c_{ij} - c_{jk} + c_{ki}$ is odd. One can check this directly for the standard diagrams of projective planes. If $i$, $j$, and $k$ are distinct then

$$a_{ijk} = -\frac{\chi(K)}{2} + c_{ik}.$$  

So the coefficient vanishes if $\chi(K)$ is odd. Note also that

$$a_{ii} = \frac{b + b - 2c_{ij}}{2} = b - c_{ij}$$

and $a_{iii} = 0$. Therefore $\langle \mu_2(\theta_{ij} \otimes \mu_2(\theta_{jk} \otimes \theta_{ki})), \theta_{ii} \rangle$ is a multiple of $U^{b-\chi(K)/2}$. The triple product $\langle \mu_3^\prime(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}), \theta_{ii} \rangle$ is a multiple of $U^{b-\chi(K)/2-1}W$.

**Lemma 6.9.** The coefficient of $\text{cl}_i \mu_3^\prime(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki})$ at $\Theta$ does not depend on the choice of homogeneous representatives for $\theta_{jk}$ or $\theta_{ii}$, and similarly for CBN.

**Proof.** Let $\partial y$ be a boundary in $\text{CS}(t_{i,j,k})$ with homological degree 0. Then

$$\mu_3^\prime(\theta_{ij} \otimes \partial y \otimes \theta_{ki}) = \mu_2^\prime(\mu_2(\theta_{ij} \otimes y) \otimes \theta_{ki}) + \mu_2^\prime(\theta_{ij} \otimes \mu_2(\theta_{ij} \otimes y) \otimes \theta_{jk})$$

$$+ \partial \mu_3(\theta_{ij} \otimes y \otimes \theta_{ki})$$

$$+ U^{-1}W(\mu_3^\prime(d_{BN}\theta_{ij} \otimes d_{BN}y \otimes \theta_{ki}) + \mu_2^\prime(d_{BN}\theta_{ij} \otimes y \otimes d_{BN}\theta_{ki})$$

$$+ \mu_2^\prime(d_{BN}y \otimes d_{BN}\theta_{ki}) + \mu_2^\prime(\bar{\mu}_2(y \otimes \theta_{ki})) + U^{-2}W^2\mu_3^\prime(d_{BN}\theta_{ij} \otimes d_{BN}y \otimes d_{BN}\theta_{ki}).$$

The first line on the right side vanishes at $\theta_{ii}$ by Lemma 6.7. The second line is in the kernel of $u_i$. The remaining lines vanish at $\theta_{ii}$ by Lemma 6.6. The same argument applies to CBN. \qed

### 6.5. Triple product invariants.

Consider the product

$$\mu_3^\prime(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}).$$

It is computed via a certain hyperbox $H^\prime_{(i,j,k,i)}$. Write $\bar{H}$ for the same hyperbox but with all the higher Bar-Natan differentials removed. It is not clear that this is a hyperbox if $t$ has crossings: it is just a diagram of chain complexes and maps. Write $\bar{\mu}_3$ for the operation induced by this family of hyperboxes. Note

$$\mu_3^\prime = \mu_3^\prime + \bar{\mu}_3$$

because $\mu_3^\prime$ is a sum over the diagonals in $H^\prime_s$. The maps assigned to one-dimensional edges are the same in all three theories. A diagonal contributes to $\mu_3^\prime$ if it is either a Bar-Natan configuration or a Szabó configuration. (If it is both, then it counts twice.)
The Bar-Natan diagonals are counted by \( \mu''_3 \) and the Szabó diagonals are counted by \( \bar{\mu}_3 \).

**Definition 6.10.** Let \( t \) be a triplane diagram in plat or hybrid form. Choose representatives \( \theta_{ij} \in CS(t_i t_j) \). Define \( q(t) \in \mathbb{F} \) to be the \( U^{2b-x(K)/2-1}W \)-coefficient of

\[
\sum_{i=1}^{3} \text{cl}_i(\bar{\mu}_3(\theta_{i(i+1)} \otimes \theta_{(i+1)(i+2)} \otimes \theta_{(i+2)i})), \Theta).
\]

Define \( q'(t) \) by replacing \( \bar{\mu}_3 \) with \( \mu'_3 \).

**Proposition 6.11.** \( q'(t) \) and \( q(t) \) do not depend on the choice of \( \theta \)'s.

**Proof.** (of Proposition 6.11) Define \( q'' \) identically to \( q \) using \( \mu''_3 \) for \( \mu'_3 \). We leave it to the reader to show that \( q'' \) and \( \mu''_3 \) satisfy Proposition 6.5. It follows from the discussion above that

\[
q(t) = q'(t) + q''(t).
\]

So it suffices to show that both \( q'(t) \) and \( q''(t) \) are well-defined. We will prove the Proposition first for \( \mu'_3 \). Let \( \partial x \) be a boundary in homological degree zero of \( CS(t_1 t_2) \). Then

\[
\mu'_3(\theta_{31} \otimes \partial x \otimes \theta_{23}) = 0
\]

by Lemma 6.7. We claim that the \( W \)-degree one part of

\[
\langle \text{cl}(\mu'_3(\partial x \otimes \theta_{23} \otimes \theta_{31}) + \mu'_3(\theta_{23} \otimes \theta_{31} \otimes \partial x)), \Theta \rangle
\]

is zero. Observe that

\[
\mu'_3(\partial x \otimes \theta_{ij} \otimes \theta_{jk}) = \mu'_2(x \otimes \mu'_2(\theta_{jk} \otimes \theta_{ki})) + \mu'_2(\mu'_2(x \otimes \theta_{jk} \otimes \theta_{ki}))
\]

\[
+ U^{-1}W(\mu'_2((d_{BN} \otimes d_{BN} \otimes \text{Id} + d_{BN} \otimes \text{Id} + d_{BN} + \text{Id} \otimes d_{BN} + d_{BN})(x \otimes \theta_{jk} \otimes \theta_{ki})))
\]

\[
+ U^{-1}W(\mu'_2(d_{BN}x \otimes \bar{\mu}_2(\theta_{jk} \otimes \theta_{ki})) + \mu'_2(\bar{\mu}_2(x \otimes \theta_{jk} \otimes d_{BN}\theta_{ki})))
\]

+ terms divisible by \( W^2 \).

The underlined term is the only one on the right which could contribute to \( q' \): for the first line, by Lemma 6.7, and for the second and third, by Lemma 6.6. It suffices to show that

\[
\langle \text{cl}_i(\mu'_2(d_{BN}x \otimes \bar{\mu}_2(\theta_{jk} \otimes \theta_{ki}))), \Theta \rangle = \langle \text{cl}_j(\mu'_2(\bar{\mu}_2(\theta_{jk} \otimes \theta_{ki}) \otimes d_{BN}x)), \Theta \rangle.
\]

Apply equation (8) to the underlined term, writing \( y = \bar{\mu}_2(\theta_{jk} \otimes \theta_{ki}) \)

\[
\text{cl}_i(\mu'_2(d_{BN}x \otimes y)) + \text{cl}_j(\mu'_2(y \otimes d_{BN}x)) = \text{cl}_{ij}(\partial d_{BN}x \otimes y + d_{BN}x \otimes \partial y + U^{-1}W d_{BN}^2 x \otimes d_{BN}y)).
\]

The right side clearly cannot contribute to \( q' \), and this completes the proof for \( \mu' \). The main difficulty in adapting the proof to \( \mu''_3 \) is that the \( \theta \)'s are \( \partial \)-cycles, not...
∂'-cycles. Nevertheless, Lemma 6.7 holds. Equation (11) will have a few more terms like \( \mu'_3(x \otimes \partial' \theta_{jk} \otimes \theta_{ki}) \) but it is straightforward to show that these are irrelevant. The remainder of the proof applies directly.

**Proposition 6.12.** Suppose that \( t \) and \( t' \) present isotopic trisections. Then \( q(t) = q(t') \) and similarly for \( q' \).

**Proof.** As in the last proof, it suffices to show that \( q' \) and \( q'' \) are invariant. We begin with \( \mu'_3 \) with the intention of applying Proposition 6.4. Observe that \( \rho'_2(\theta \otimes \theta) \) cannot contribute to \( q \) by essentially the same argument as Lemma 6.6. This argument applies to \( \rho'_2(x \otimes y) \) where \( x \) and \( y \) are any two cycles in homological degree zero. In light of Lemma 6.6, equation (7) reduces to

\[
\mu'_3(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}) + \rho'_1 \mu'_3(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}) = U^{-1} W(\mu'_2(R_{2,1} + R_{1,2}))
\]

The right side is divisible by \( W^2 \) because the maps \( R_{2,1} \) and \( R_{1,2} \) count at least one two-dimensional configuration. The cobordism underlying \( cl_i \) is the composition of some Reidemeister 2 moves and some one-handle attachments. Therefore it is natural with respect to top generators. It follows that

\[
\langle cl_i(\mu'_3(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki})), \Theta \rangle = \langle cl_i(\mu'_3(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki})), \Theta \rangle
\]

and therefore

\[
\langle cl_i(\mu'_3(\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki})), \Theta \rangle = \langle cl_i(\mu'_3(\theta'_{ij} \otimes \theta'_{jk} \otimes \theta'_{ki})), \Theta \rangle.
\]

As in the previous proof, the only caution in adapting the proof to \( \mu''_3 \) is that the \( \theta \)s are not \( \partial' \)-cycles.

### 6.6. Stabilization

The rest of this section is devoted to proving the following theorem.

**Theorem 6.13.** Suppose that \( t \) presents the surface \( K \). Let \( t' \) be a stabilization of \( t \). Then \( q(t') = q(t) \). Therefore \( q(t) \) is an invariant of the surface \( K \) up to isotopy.

We do not expect that \( q' \) is invariant under stabilization. Stabilization is not symmetric in the tangles of \( t \). To keep things concrete, we focus initially on

\[
\mu'_3(\theta_{12} \otimes \theta_{23} \otimes \theta_{31})
\]

and distinguish between central stabilization – stabilization which splits a component of \( t_2 \overline{t}_3 \) – and edge stabilization – stabilization which splits a component of \( t_1 \overline{t}_2 \) or \( t_3 \overline{t}_1 \). Of course these descriptors apply just as well with the indices rotated.

The following lemma is helpful in constraining the possibilities for configurations.

**Lemma 6.14.** Let \( x \) be a simple tensor which has nontrivial coefficient in \( \theta_{12} \otimes \theta_{23} \otimes \theta_{31} \). Suppose that

\[
\langle cl_1 \mu'_3(x), \Theta \rangle \neq 0.
\]
Then all the circles which meet the bridge sphere in the resolution underlying $x$ are $v_+$-labeled. The same applies with the indices cyclically rotated.

Proof. The key observation is that the points where $t_1$ meets the bridge sphere are in the complement of all the Reidemeister 2 moves. Therefore they are only subject to one-handle attachments. Their labels are constricted by the filtration rule: if they start $v_-$, they will always be $v_-$. If they are, then they cannot contribute to $\Theta$. $\square$

6.6.1. Central stabilization. Let $t'$ be the triplane diagram given by splitting a crossingless component in $t_2\bar{t}_3$. Write

$$\theta = \theta_{12} \otimes \theta_{23} \otimes \theta_{31}$$

$$\theta' = \theta'_{12} \otimes \theta'_{23} \otimes \theta'_{31}.$$

Let $H$ be the hyperbox underlying the computation of $\mu'_3(\theta)$. It is two-dimensional with shape $(b,b)$. As in Section 8.2.1, that $\mu'_3(\theta)$ is the sum of $b^2$ maps applied to $\theta$, one for each diagonal of $H$. Similarly, the hyperbox $H'$ underlying $\mu'_3(\theta')$ has shape $(b+1,b+1)$, and $\mu'_3(\theta')$ is the sum of $(b+1)^2$ maps applied to $\theta'$.

The arc of stabilization can be specified by the $\Xi$ arcs which lie directly above it. Suppose that these are the $n_1$-st (on the left) and $n_2$-st (on right) arcs. Call the $(n_1+1)$-st and the $(n_2+1)$-st $\Xi$ handles in $t'$ the new surgery arcs. See Figure 17.

Now we can divide the squares of $H\theta'$ into five sectors. Identify each square with the coordinates of its bottom-left vertex and consider the $(j_1,j_2)$ square.

- If $j_1 \leq n_1$ and $j_2 \leq n_2$, then the square is pre-stabilization.
- If $j_1 > n_1 + 1$ and $j_2 > n_2 + 1$, then the square is post-stabilization.
- If $j_1 > n_1 + 1$ or $j_2 > n_2 + 1$ but not both, then the square is a side square.
- The unique square with $j_1 = n_1 + 1$ and $j_2 = n_2 + 1$ is called the square of stabilization.
- The other squares with either $j_1 = n_1 + 1$ or $j_2 = n_2 + 1$, but not both, are called new squares.

The squares could be put into two categories, old and new, as shown in Figure 18. We will compare $\mu'_3(\theta')$ to $\mu'_3(\theta)$ by showing that new squares and the square of stabilization contribute nothing to $\mu'_3(\theta')$, while the other squares make the same contribution times $H$. Let $p'_{j_1,j_2}$ be the path which uses the diagonal in the square with lower left corner at $(j_1,j_2)$. Conflate this path with the map along this path. Write $p_{j_1,j_2}$ (resp. $p'_{j_1,j_2}$) for the composition of all the maps in $p_{j_1,j_2}$ (resp. $p'_{j_1,j_2}$) up to and including the diagonal.

Lemma 6.15. In this context,

$$\sum_{(j_1,j_2) \text{ an old square}} p'_{j_1,j_2}(\theta') = \mu'_3(\theta').$$
If \((j_1, j_2)\) is the square of stabilization then \(p_{j_1,j_2}(\theta') = 0\).

**Proof.** Suppose that \(p'_{j_1,j_2}\) is pre-stabilization. Observe that the active part of the configuration at \(H'_{(j_1,j_2)}\) is identical to that of \(H_{(j_1,j_2)}\). It follows that

\[
p'_{j_1,j_2}(\theta') = p_{j_1,j_2}(\theta) \otimes v_+
\]

where the extra \(v_+\) factor is identified with the new component of \(t'_2\bar{t}'_3\). (This component is always passive in every configuration in \(p'\).) Figure 19 shows that the only difference between \(p_{j_1,j_2}(\theta)\) and \(p'_{j_1,j_2}(\theta')\) after the diagonal is that \(p'_{j_1,j_2}\) has an extra merge and then split between a \(v_+\)-labeled, crossingless component and some strand. By Lemma 6.14 we may assume that this strand is also \(v_+\)-labeled. Therefore

\[
\langle p'_{j_1,j_2}(\theta'), \Theta_{ii}' \rangle = \langle p_{j_1,j_2}(\theta), \theta_{ii} \rangle.
\]

If \(p'_{j_1,j_2}\) is post-stabilization then the same analysis applies. The only difference is that the extra merge and split occur before the diagonal.

**Figure 17**
Figure 18. The division of $H'$ into old and new squares after a $(6,3)$-stabilization (of a crossingless diagram). The blue squares are old, the red squares are new, and the green square is the square of stabilization.

Suppose that $p'_{j_1,j_2}$ is a side square. The analysis is more or less the same. There is an extra merge before the diagonal, and an extra split after. The key observation is that the configuration underlying the diagonal map does not involve one of the new handles and therefore its active part is identical to that of either $p_{j_1,j_2-1}$ or $p_{j_1-1,j_2}$. Therefore

$$\langle p'_{j_1,j_2}(\Theta'), \Theta'_{ii} \rangle = U \langle p_{j_1-1,j_2}(\theta), \Theta_{ii} \rangle$$

or

$$\langle p'_{j_1,j_2}(\Theta'), \Theta'_{ii} \rangle = U \langle p_{j_1,j_2-1}(\theta), \Theta_{ii} \rangle.$$  

If $p'_{j_1,j_2}$ corresponds to the square of stabilization, then the active part of this configuration contains a $v_+$-labeled circle with in- and out-degree 1. It follows that the diagonal map is zero. □

Lemma 6.16. $\langle \sum_{(j_1,j_2) \text{ a new square}} p'_{j_1,j_2}(\theta'), \Theta' \rangle = 0$.

Proof. Any side square to the south or west of the square of stabilization carries a configuration with a degree one, $v_+$-labeled component. Therefore it cannot contribute. The situation for north and east squares is represented in Figure 20. The red decoration is always “dual degree one” - it’s head and tail lie on the same component, and it can be isotoped to lie in a small neighborhood of a point of that component. It follows any two-dimensional configuration the red decoration must be of type E. The map assigned to such a configuration will put a $v_-$ on a component.
(a) Before the new handle attachments in a central stabilization.

(b) Immediately after the new handle attachments in a central stabilization.

(c) The corresponding configuration in the non-stabilized triplane diagram. The active part of this configuration is identical to the Figure (b).

\textbf{Figure 19}

which meets the bridge sphere on the outside. By Lemma 6.14 this configuration cannot contribute to $\Theta'$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure20}
\caption{Two new configurations in a central stabilization.}
\end{figure}

6.6.2. \textit{Edge stabilization.}

\textbf{Lemma 6.17.} \textit{Using the same terminology as above,}

$$\sum_{(j_1, j_2) \text{ on old square}} p'_{j_1,j_2}(\theta') = \mu'_3(\theta').$$

\textit{If} $(j_1, j_2)$ \textit{is the square of stabilization, then} $p'_{j_1,j_2}(\theta') = 0$. 
Proof. The proof for old squares is the same as that of Lemma 6.15 with Figure 21 in place of Figure 19. The new decorations add a merge and split, this time between different components. The merge involves a $v_+-$labeled, crossingless component. The split is on a circle which meets the bridge sphere on the outside, so it the circle must be $v_+-$labeled. The configuration underlying the square of stabilization has a degree one, $v_+-$labeled circle and therefore the corresponding map vanishes. □

\[ \sum_{(j_1,j_2)}^p j_1, j_2 (\theta'_1, \theta'_2), \Theta' \]

Figure 21. Corresponding post-stabilization configurations for a side stabilization.

Lemma 6.18. \[ \sum_{(j_1,j_2)}^p j_1, j_2 (\theta'_1, \theta'_2), \Theta' \] = 0.

Proof. For one side stabilization, the configuration underlying any square to the north or south of the square of stabilization involves a degree one, $v_+-$labeled circle. For the other side, the same holds to the east and west.

The situation for the other axis is represented in Figure 22. This representation is valid if one insists that the new, red decoration comes last in the ordering of handle attachments. Any two-dimensional configuration which contributes to $\Theta'$ must be connected, so it must involve one of the black arrows. But such an arrow will always lie either entirely above the red arrow or entirely below it. Therefore the configuration is of type E. The corresponding map will place $v_-$labels which are unacceptable in light of Lemma 6.14. □

\[ \sum_{i=1}^{3} cl_i (\bar{m}u_3 (\theta'_{i+1} \otimes \theta'_{i+1} \otimes \theta'_{i+2})), \Theta' \] = 0

Proof of Theorem 6.13. We have shown that

The bridge number of $t'$ is one greater than that of $t$. It follows that $q(t) = q(t')$. □
7. Examples and non-triviality of \( q \)

7.1. **Unknotted spheres.** Write \( S^2 \) for the unknotted sphere in \( S^4 \). Let \( t \) be the unique crossingless, bridge number one diagram from \( S^2 \). Every higher-dimensional configuration in computing \( q(t) \) has a degree one, \( v_+ \)-labeled circle. Therefore \( q(t) = 0 \) and \( q(S^2) = 0 \).

7.2. **Split union and unlinked spheres.**

**Proposition 7.1.** Let \( t = t_0 \coprod t_1 \) be a split tri-plane diagram. Then
\[
q(t) = q(t_0) + q(t_1).
\]

**Proof.** Write \( t = (t_1,t_2,t_3) \) and \( t_i = (t_{1,i},t_{2,i},t_{3,i}) \) for \( i = 0 \) or 1. Suppose that \( t_i \) has bridge number \( b_i \). Let \( x \in \text{CS}(t_1t_2) \otimes \text{CS}(t_2t_3) \otimes \text{CS}(t_3t_1) \). Then \( x = x_1 \otimes x_2 \otimes x_3 \) with \( x_j \in \text{CS}(t_jt_{j+1}) \) and \( x_j = x_{j,0} \otimes x_{j,1} \) where \( x_{j,i} \in \text{CS}(t_{j,i}t_{j+1,i}) \). (The CS chain group satisfies the Künneth formula even though the CS complex does not.)

Let \( H_x \) be the hyperbox underlying the computation of \( \mu'_3(x) \). The \((j_1,j_2)\)-square of \( H_x \) has a disconnected configuration if either \( j_1 \geq 1 \) and \( j_2 < b_2 \) or if \( j_1 < b_1 \) and \( j_2 \geq b_2 \); these are the configurations which involve a decoration from both sides of the split union. It follows that \( \mu'_3(x) \) is a sum over the configurations which contributed to \( q(t_0) \) and \( q(t_1) \). It follows that
\[
\text{cl}_1(\mu'_3(x)) = \text{cl}_1(\mu'_3(x_0) \otimes (\mu'_2 \otimes (\mu'_2 \otimes \text{Id}))(x_1)) + \text{cl}_1((\mu'_2 \otimes (\mu'_2 \otimes \text{Id}))(x_0) \otimes \mu'_3(x_1))
= U^{2b_t - \chi(t_0)/2} \text{cl}_1(\mu'_3(x_1)) + U^{2b_t - \chi(t_1)/2} \text{cl}_1(\mu'_3(x_0)).
\]

Now suppose that \( \langle \text{cl}_1(\mu'_3(x_0)), \Theta \rangle = aU^{2b_t - \chi(t_0)/2 - 1}W \) with \( a \in \mathbb{F} \). Then
\[
\langle \text{cl}_1(\mu'_3(x_0) \otimes (\mu'_2 \otimes (\mu'_2 \otimes \text{Id}))(x_1))), \Theta \rangle = aU^{2b_t + \chi(t_0)/2 + 2b_t + \chi(t_1) - 1}W = aU^{2b_t + \chi(t)/2 - 1}.
\]

The claimed formula follows. \( \square \)

**Corollary 7.2.** Let \( \mathcal{K} = \mathcal{K}_0 \coprod \mathcal{K}_1 \) be a split union of knotted surfaces. Then \( q(\mathcal{K}) = q(\mathcal{K}_0) + q(\mathcal{K}_1) \).
Let $\mathcal{K}$ be a a collection of unlinked, unknotted spheres. Then $q(\mathcal{K}) = 0$.

It is interesting to compare this corollary with Proposition 4.1 of [16]. There it is shown that if $c_{ij} = b$ for some $i \neq j$ then $t$ presents an unlink of $b$ unknotted spheres. In this case $t$ is equivalent to a triplane diagram $t'$ in which $t_i\bar{t}_j$ is the plat closure of the $2b$-strand identity braid. The argument that $q(S^2) = 0$ shows that $q(t') = 0$.

7.3. Spins. In [16], Meier and Zupan systematically produce bridge trisection diagrams for spins of knots. Figure 23 shows a tri-plane diagram $t_p$ for the spun $(2,p)$-torus link. We call this surface-link $\mathcal{K}_p$. If $p$ is odd then $t_p$ presents a knotted sphere. If $p$ is even and non-zero then $t_p$ presents a non-trivial link of an unknotted sphere and torus. These diagrams have $b = 4$ and $\chi = 2$.

\[ \text{Figure 23. The tri-plane diagram } t_p \text{ for the spun } (2,p)\text{-torus link.} \]

**Theorem 7.3.** $q(t_p) = 1$ and therefore $q(\mathcal{K}_p) = 1$ for all $p \in \mathbb{Z}$.

Therefore $q$ can distinguish between some knotted and unknotted spheres. First we show that $\text{poly}(t_p) = q(t_0)$ for all $p$. Then we compute $q(t_0)$ directly. It is interesting to think about the topological meaning of the first step.

Let’s begin with a few observations about computing $q(t)$ in general. A resolution of $t_i\bar{t}_j$ may be written $I\bar{J}$ where $I$ is a resolution of $t_i$ and $J$ is a resolution of $t_j$. Consider the cancellation

$\beta_i\bar{\beta}_j\beta_j\bar{\beta}_k \rightarrow \beta_i\bar{\beta}_k$.

Let $x \in \text{CS}(t_i\bar{\beta}_j\beta_j\bar{\beta}_k)$ be a simple tensor in resolution $I\bar{J}\bar{J}'K$. Studying the Reidemeister 2 maps (Figure 6 of [3]), we see that the corresponding map

$\text{CS}(t_i\bar{\beta}_j \coprod \beta_j\bar{\beta}_k) \rightarrow \text{CS}(t_i\bar{t}_k)$

vanishes unless $J = J'$. Therefore we only need to consider canonical summands of $\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}$ in which $\theta_{ij}$ and $\theta_{jk}$ have mirrored underlying resolutions on $t_j$. The same applies to $\theta_{jk}$ and $\theta_{ki}$ with respect to $t_k$ and to $\theta_{ij}$ and $\theta_{ki}$ with respect to $t_i$. So we may ignore any summand of $\theta_{ij} \otimes \theta_{jk} \otimes \theta_{ki}$ whose resolution is not of the form $I\bar{J}\bar{J}'K\bar{I}$. As $\bar{\mu}_3$ is non-decreasing in the underlying resolution of simple tensors, we may ignore the decorations from the crossings of the tangles.
Proof of Theorem 7.3. We show that \( q(K_0) = 1 \) by direct computation in the next lemma.

Observe that each \( t_i \xi_{i+1} \) can be unwound to a crossingless diagram using only Reidemeister 2 moves. This means that a resolution \( I \) of \( t_1 \) determines that of \( t_2 \) and therefore of \( t_3 \), i.e.

\[
\mu_3(\theta_{12} \otimes \theta_{23} \otimes \theta_{31}) = \sum I \mu_3(\theta_{12}; I, \theta_{23}; I, \theta_{31}; I).
\]

In some sense \( t_0 \) is a “resolution” of \( t_p \) for any \( p \). Write \( I_0 \) for the corresponding resolution of \( t_1 \). The key claim is that the top generator in this resolution is the only one which \( q(t) \).

![Figure 24](image.png)

**Figure 24.** \( \theta_{12} \otimes \theta_{23} \otimes \theta_{31} \) for \( t_0 \). Every component is \( v_+ \)-labeled.

Figure 25 shows a resolution which differs from \( I_0 \) at only two crossings. The circles which do not meet the bridge sphere must be \( v_+ \)-labeled in any summand of \( \theta \). Lemma 6.14 shwos that the other circles must also be \( v_+ \)-labeled. Call this generator \( \theta \).

![Figure 25](image.png)

\( \mu_3'(\theta) \) is computed via a hyperbox which involves handle attachments and the components of two Reidemeister 2 maps which each cap off a circle. The map is non-zero only if these circles are \( v_- \)-labeled. The first Reidemeister 2 move will cap off a circle (which may be \( v_- \)-labeled) and attach a handle between the two circles which
do not meet the bridge sphere in Figure 25. (Any two-dimensional configuration using this decoration will have a degree one, $v_+$-labeled circle.) It follows that the second Reidemeister 2 map will cap off a $v_+$-labeled circle. We conclude that $\mu_3'(\theta) = 0$. This argument applies to any resolution which differs from $I_0$ at only one crossing in $t_2$ or $t_3$.

The other possibility is that one of $t_2$ or $t_3$ differs from $I_0$ at multiple crossings. One half of such a generator is shown in Figure 26. The dotted circles are $v_-$-labeled while the rest are $v_+$-labeled. The structure of the Reidemeister 2 maps guarantees that if one circles is $v_-$-labeled, then the opposite circle must be $v_+$-labeled. There is another summand in the top generator in which these labels are swapped, and the contributions from these two cancel. On the other hand if all the circles are $v_+$-labeled, then the argument from the previous paragraph applies.

In summary,

$$\langle \mu_3'(\theta_{12}, \theta_{23}, \theta_{31}), U^2 W \theta_{11} \rangle = \langle \mu_3'(\theta_{12;I_0}, \theta_{23;I_0}, \theta_{31;I_0}), U^2 W \theta_{11} \rangle.$$  

The same argument applies to $\mu_3'(\theta_{23} \otimes \theta_{31} \otimes \theta_{12})$ and $\mu_3'(\theta_{31} \otimes \theta_{12} \otimes \theta_{23})$. It follows that $q(t_p) = q(t_0)$.

**Lemma 7.4.** $q(t_0) = 1$.

**Proof.** $t_0$ is the split union of an unknotted sphere and a diagram $t'$ of a torus. Proposition 7.1 implies that we may ignore the sphere. The hyperboxes for $q(t')$ are shown in Figures 27–29. (Actually the hyperboxes compute $\mu_3'$, but the action of $cl_i$ is straightforward.) The little diagrams in the squares are the active parts of the configuration underlying that diagonal. Any square with a degree one circle contributes nothing: that circle must be $v_+$-labeled, so the corresponding Szabó map vanishes. This means we may ignore the leftmost column and bottom row of each figure.

Consider the remaining four configurations in Figure 27. Starting from the top right and moving clockwise, they are of type 1, 6, 12, and 14, referring to the list of two-dimensional configurations in [24]. Only the first contributes to $q$. Do the same
for Figure 28, the configurations are of type 1, 14, 8, and 7. Both the first and third contribute to $q$. Finally, for Figure 29, the configurations are of type 13, 9, 16, and 9 again. So none of them contribute. It follows that $q(t_0) = 1$. 

\[\square\]
Figure 27. The hyperbox to compute $\mu_3'(\theta_{12} \otimes \theta_{23} \otimes \theta_{31})$ in Lemma 7.4.
Figure 28. The hyperbox to compute $\mu_3'(\theta_{23} \otimes \theta_{31} \otimes \theta_{12})$ in Lemma 7.4
Figure 29. The hyperbox to compute $\mu_3'(\theta_{31} \otimes \theta_{12} \otimes \theta_{23})$ in Lemma 7.4.
8. Hyperboxes of chain complexes and $A_\infty$ algebras

This section is dedicated to establishing the algebraic framework for our $A_\infty$ algebras. This framework extends the constructions of Manolescu and Ozsváth in their mammoth paper on Heegaard Floer homology [15]. We show that from a collection of hyperboxes satisfying some coherence conditions one can build an $A_\infty$-algebra. Theorem 8.19 states this construction is functorial up to homotopy. Corollary 8.8 and Sections 8.2.2 and 8.3 are new results which we hope will be useful in future work. The construction is extended to include $A_\infty$-bimodules in [2].

8.1. $A_\infty$-algebras. See for an introduction to $A_\infty$-algebras, [12] for an exhaustive resource, and [23] for something in between. We avoid tricky sign conventions by working over a ring of characteristic two $\mathbb{R}$.

Definition. An $A_\infty$-algebra over $\mathbb{R}$ is a $\mathbb{Z}$-graded $\mathbb{R}$-module $A$ and a collection of maps

$$\mu_k: A^\otimes k \rightarrow A, \quad k \geq 1$$

of degree $2 - k$ which satisfy, for each $n \geq 1$, the $A_n$-relation:

(12) $$\sum_{i+j+k=n} \mu_{i+1+k}(\text{Id}^\otimes i \otimes \mu_j \otimes \text{Id}^\otimes k) = 0.$$  

Equation (12) states that $\mu_1$ is a differential on $A$, that $\mu_2$ is a chain map, and that $\mu_3$ is a chain homotopy between $\mu_2 \circ (\mu_2 \otimes \text{Id})$ and $\mu_2 \circ (\text{Id} \otimes \mu_2)$. So $A$ is a “dga up to homotopy.”

Definition. Let $A$ and $B$ be $A_\infty$-algebras over $\mathbb{R}$. A map of $A_\infty$-algebras is a collection of maps

$$f_k: A^\otimes k \rightarrow B$$

of degree $1 - k$ which satisfy, for each $n \geq 1$,

(13) $$\sum_{i+j+k=n} f_{i+1+j}(\text{Id}^\otimes i \otimes \mu_j \otimes \text{Id}^\otimes k) = \sum_{i_1 + \cdots + i_r = n} \mu_r(f_{i_1} \otimes \cdots \otimes f_{i_r}).$$

The identity map is the map with $f_1 = \text{Id}$ and $f_i = 0$ for $i > 1$.

If $f: A \rightarrow B$ and $g: B \rightarrow C$ are maps of $A_\infty$-algebras, their composition $(g \circ f)$ is defined by

$$(g \circ f)_n = \sum_{i_1 + \cdots + i_r = n} f_r(g_{i_1} \otimes \cdots \otimes g_{i_r}).$$

Definition. Let $f, g: A \rightarrow B$ be maps of $A_\infty$-algebras. A $A_\infty$-chain homotopy between $f$ and $g$ is a collection of linear maps

$$h_k: A^\otimes k \rightarrow B$$
so that

\begin{equation}
    f_n - g_n = \sum_{i_1 + \cdots + i_r + j_1 + \cdots + j_s = n} \mu_{r+1+s}(f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_s})
\end{equation}

\begin{equation}
    + \sum_{a+b+c=n} h_{a+1+c}(\text{Id}^\otimes a \otimes \mu_b \otimes \text{Id}^\otimes c)
\end{equation}

A map which is \(A_{\infty}\)-homotopic to the identity map is called a \textit{chain homotopy equivalence}.

8.2. Hyperboxes of chain complexes. Here we recall the construction of Manolescu and Ozsváth [15]. Let \(d = (d_1, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n\). Write \(E(d)\) for the box with dimensions \(d\) and its initial corner at the origin. Write \(E(n)\) for the \(n\)-dimensional hypercube: the \(n\)-dimensional box with dimensions \((1, \ldots, 1)\).

\textbf{Definition.} An \(n\)-dimensional hyperbox of chain complexes of shape \(d\) is a collection of graded chain complexes

\[\bigoplus_{\delta \in E(d)} C_\delta\]

and a collection of linear maps

\[D^\delta_{\epsilon}: C^\delta \to C^{\delta+\epsilon}\]

with \(\delta \in E(d)\) and \(\epsilon \in E_n\) so that:

- If \(\delta + \epsilon \notin E(d)\) then \(D^\delta_{\epsilon} = 0\).
- The map \(D^0_{\delta}: C^\delta \to C^\delta\) is the differential on \(C^\delta\).
- The map \(D^\delta_{\epsilon}\) has degree \(1 - \|\epsilon\|\).
- Each hypercube of any dimension is a chain complex. In other words,

\begin{equation}
    \sum_{\epsilon + \epsilon' \leq (1, \ldots, 1)} D^\delta_{\epsilon' + \epsilon} \circ D^\delta_{\epsilon} = 0
\end{equation}

for all \(\delta \in E(d)\).

Informally, a hyperbox is a collection of cubical chain complexes which have been stacked. Note that the last condition does not imply that a hyperbox of chain complexes is itself a chain complex! We will write \(H = (C, D)\) \(C = \bigoplus C_\delta\) and \(D = \bigoplus D^\delta_{\epsilon}\) for a generic hyperbox of chain complexes. Here are some examples.

A \textbf{0-dimensional hyperbox of chain complexes} is a chain complex.

\footnote{In [15], \(D^\delta_{\epsilon}\) is required to have degree \(\|\epsilon\| - 1\). One could alternatively refer to our definition as a hyperbox of \textit{co}chain complexes, but this isn’t a difference worth emphasizing.}
A 1-dimensional hyperbox of chain complexes of size \((d)\) is a collection of chain complexes \(C_i\) for \(i \in \{0, \ldots, d\}\) and chain maps \(f_i: C_i \to C_{i+1}\). In the above notation, \(C = \bigoplus_{i=0}^{d} C_i\) and \(D\) is the direct sum of all the differentials on the \(C_i\) and the maps \(f_i\). (Equation \[16\] implies that the \(f_i\) must be chain maps.) So a one-dimensional hyperbox can be thought of as the mapping cone of a factored chain map.

A 2-dimensional hyperbox of chain complexes of size \((d_1, d_2)\) is a collection of chain complexes \(\{C^{i,j}\}\) for \(0 \leq i \leq d_1\) and \(0 \leq j \leq d_2\) along with maps

\[
\begin{align*}
&f_{i,j}^{(1,0)}: C_{i,j} \to C_{i+1,j} \\
&f_{i,j}^{(0,1)}: C_{i,j} \to C_{i,j+1} \\
&f_{i,j}^{(1,1)}: C_{i,j} \to C_{i+1,j+1}
\end{align*}
\]

(horizontal maps)

(vertical maps)

(diagonal maps)

Equation \[16\] implies that the horizontal and vertical maps are chain maps. It also implies that \(f_{i,j}^{(1,1)}\) is a homotopy between \(f_{i+1,j}^{(0,1)} \circ f_{i,j}^{(1,0)}\) and \(f_{i,j+1}^{(1,0)} \circ f_{i,j}^{(0,1)}\).

A hypercube of complexes of dimension \(n\) is a cubical chain complex with diagonal maps. Such complexes underlie many spectral sequences in low-dimensional topology.

Let \(H = (C, D)\) be a hyperbox of chain complexes of shape \(d = (d_1, \ldots, d_n)\). Our standing notation will be that \(\delta \in E(d)\) (a “coordinate vector”) and \(\epsilon \in E_n\) (a “direction vector”). \(\delta\) is called a corner if every coordinate in \(\delta\) is either maximal or zero. In other words, there is some \(\epsilon \in E_n\) so that

\[
\delta = (d_1 \epsilon_1, \ldots, d_n \epsilon_n).
\]

We call \(\delta\) (or the chain complex at \(\delta\)) the \(\epsilon\)-corner. We will sloppily append coordinates to a vector by writing e.g. \((\delta, 1)\) for \((\delta_1, \ldots, \delta_n, 1)\).

**Definition 8.1.** Let \(0^1 H\) and \(1^1 H\) be hyperboxes of chain complexes. A map of hyperboxes \(F: 0^1 H \to 1^1 H\) is a hyperbox of size \((d, 1)\) so that the \(d_{n+1} = 0\) face of \(F\) is \(0^1 H\) and the \(d_{n+1} = 1\) face of \(F\) is \(1^1 H\) with grading shifted up by 1.

A map of hyperboxes is determined by the edges with positive \((n+1)\)-st coordinate, i.e. the maps

\[
F_\delta^\epsilon: 0^1 C_\delta \to 1^1 C_{\epsilon+\delta}
\]

of degree \(\|\epsilon\|\) satisfying

\[
(17) \quad \sum(D_{\epsilon+\epsilon'}^\delta \circ F_{\delta}^\epsilon + F_{\epsilon+\delta}^\epsilon' \circ D_{\delta}^\epsilon) = 0
\]

for all \(\epsilon \in E(d, 0)\), and all \(\epsilon'\) with \((n+1)\)-coordinate 1 so that \(\epsilon + \epsilon' \leq \epsilon_1\). Conversely, a collection of maps from \(C\) to \(C'\) satisfying these relations defines a map of hyperboxes.
Let $F: 0H \rightarrow 1H$ and $G: 1H \rightarrow 2H$ be maps of hyperboxes. Their composition $G \circ F: 0H \rightarrow 2H$ is defined by

\[(G \circ F)^{\epsilon'}: 0C^{\epsilon} \rightarrow 2C^{\epsilon + \epsilon'} \]

(18)

\[(G \circ F)^{\epsilon'} = \sum_{\epsilon'' \leq \epsilon'} G^{\epsilon'' - \epsilon'} \circ F^{\epsilon''} \]

In terms of boxes: glue $F$ and $G$ together along their common face to obtain a hyperbox of shape $(d, 2)$. To obtain a hyperbox of shape $(d, 1)$, compose all possible combinations of maps in the $(n + 1)$-st direction. One can give similar definitions for homotopies, homotopy equivalences, and quasi-isomorphisms in the category of hyperboxes of chain complexes. In fact, if one thinks of two maps $F$ and $G$ as hyperboxes, then a chain homotopy can be thought of as a map of these hyperboxes:

**Definition 8.2.** Let $F, G: 0H \rightarrow 1H$. A *chain homotopy* from $F$ to $G$ is a hyperbox $J$ of size $(d, 1, 1)$ so that

- the $\delta_n^{+2} = 0$ face of $J$ is $F$.
- the $\delta_n^{+2} = 1$ face of $J$ is $G$.
- if $\epsilon = (0, \ldots, 0, 1)$ then $J_\delta^{0} = \text{Id}$.
- if $\epsilon = (\epsilon', 0, 1)$ and $\epsilon' \neq 0$ then $J_0^{\epsilon'} = 0$.

The last two points can be summarized as “the restriction of $J_\delta$ to a face with $\delta_n^{+1} = 0$ is the identity map.”

**A map of zero-dimensional hyperboxes** is the mapping cone of a chain map.

**A map of $n$-dimensional hypercubes** is an $(n + 1)$-dimensional hypercube, i.e. the mapping cone of a map of cubical complexes. A chain homotopy of maps of hypercubes is equivalent to a chain homotopy of chain maps of cubical complexes.

8.2.1. *Compression.* There is a recursive recipe called *compression* for building a cubical chain complex from a hyperbox of chain complexes. Let $H = (C, D)$ be a hyperbox of chain complexes of shape $d = (d_1, \ldots, d_n)$. Let $\hat{C}$ be the cubical complex whose underlying space is the sum of the corners of $H$. One can construct a differential $\hat{D}$ on $\hat{C}$ from $H$. The hypercube $\hat{H} = (\hat{C}, \hat{D})$ is the *compression* of $H$. This recipe was first described by Manolescu and Ozsváth. We take an alternative view due to Liu [13].

Let $H$ be a one-dimensional hyperbox of shape $(d)$. Define

\[
\begin{align*}
\hat{C} &= C_0 \oplus C_d \\
\hat{D}_0^1 &= f_{d-1} \circ \cdots \circ f_0 \\
\hat{D}_0^0 &= D_0^0, \quad \hat{D}_1^0 = D_0^0
\end{align*}
\]
and $\hat{H} = (\hat{C}, \hat{D})$.

Let $H$ be an $n$-dimensional hyperbox with shape $(d_1, \ldots, d_n)$ and $d_n > 1$. We can think of $H$ as $d_n$ hyperboxes of shape $(d_1, \ldots, d_{n-1}, 1)$ attached along faces. Label these hyperboxes as $H_1^n, H_2^n$, and so on. Each of these boxes is a map of hyperboxes of dimension $n - 1$.

**Definition 8.3.** Define $H^n$ to be the map

$$H^n = H^n_{d_n} \circ \cdots \circ H^n_1.$$  

$H^n$ is the **partial compression of $H$ along the $n$-th axis**, or just the **$n$-th partial compression**. It has shape $(d_1, \ldots, d_{n-1}, 1)$. If $d_n = 1$, then $H^n = H$.

**Definition 8.4.** Let $H$ be an $n$-dimensional hyperbox. Define

$$\hat{H} = H^{n,n-1,\ldots,1}.$$  

In other words, $\hat{H}$ is the result of $n$ partial compressions starting with the $n$-th and ending with first. $\hat{H}$ is a hypercube of dimension $n$.

It’s a helpful exercise to describe $\hat{H}$ more explicitly for a two-dimensional hyperbox $H$. (And these compressions play a starring role in Section 6.) First suppose that $H$ of shape $(d, 1)$. Then $H^2 = H$. Think of $H$ as $d$ maps of one-dimensional hyperboxes: to compute $\hat{H} = H^{2,1}$, compose those maps. So

$$\hat{D}^{(1,0)}_{(0,0)} = D^{(1,0)}_{(0,0)}.$$  

Next,

$$\hat{D}^{(1,0)}_{(0,0)} = D^{(1,0)}_{(d-1,0)} \circ \cdots \circ D^{(1,0)}_{(1,0)} \circ D^{(1,0)}_{(0,0)}$$

$$\hat{D}^{(1,0)}_{(0,0)} = D^{(1,0)}_{(d-1,1)} \circ \cdots \circ D^{(1,0)}_{(1,1)} \circ D^{(1,0)}_{(0,1)}.$$  

To understand $\hat{D}^{(1,1)}_{(0,0)}$ we study equation (18). The map is a sum of maps, one for each path in $H$ from the initial vertex to the terminal vertex. Such a path can only include one diagonal edge – in fact, it’s totally determined by that edge. Therefore we can describe $\hat{D}^{(1,1)}_{(0,0)}$ by the schematic in Figure 30. The thick blue diagonal represents a kind of step which can only appear once.

**Figure 30**

Now suppose instead that $H$ has shape $(d_1, d_2)$. Then $H^2$ is a hyperbox of shape $(d_1, 1)$. Each square in $H^2$ is the compression of a hyperbox of size $(1, d_2)$. Therefore
its diagonal maps is given by Figure 31, the mirror of Figure 30. Now apply the procedure above remembering that the blue diagonal stands for this shape.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure31.png}
\caption{Figure 31}
\end{figure}

The result is that $\hat{H}$ has underlying space

$$C_{(0,0)} \oplus C_{(1,0)} \oplus C_{(0,1)} \oplus C_{(1,1)}.$$ 

The vertical (resp. horizontal) maps are compositions of vertical (resp. horizontal) maps in $H$. The diagonal map is a sum of maps, one for each diagonal edge in $H$. This diagonal completely describes a path from $(0,0)$ to $(d_1,d_2)$ of the shape in Figure 32. Readers familiar with hyperboxes will recognize that this description of $\hat{H}$ agrees with Manolescu and Ozsváth’s.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure32.png}
\caption{Figure 32}
\end{figure}

**Proposition 8.5** (13). Liu’s definition of compression in 13 agrees with Manolescu and Ozsváth’s in 13.

If $G: H \rightarrow H'$ is a map of hyperboxes, then $\hat{G}: \hat{H} \rightarrow \hat{H}'$. Suppose that $F: H' \rightarrow H''$ is another map of hyperboxes. It would be nice if

$$\hat{F} \circ \hat{G} = \hat{F} \circ \hat{G}.$$ 

But this formula is false. Write $FG$ for the hyperbox given by gluing together $F$ and $G$ along the appropriate faces so that $(FG)^{n+1} = F \circ G$ and therefore $\hat{FG} = \hat{F} \circ \hat{G}$. 

58 ADAM SALTZ
On the other hand, $\hat{F} \circ \hat{G}$ is computed by first fully compressing both $F$ and $G$. In summary,

$$\hat{F} \circ \hat{G} = (FG)^{n,\ldots,1,n+1}$$

$$\hat{F} \circ \hat{G} = (FG)^{n+1,n,\ldots,1}.$$ 

Nevertheless,

**Lemma 8.6.** $\hat{F} \circ \hat{G} \simeq \hat{F} \circ \hat{G}$.

**Proof.** We work by induction. Suppose that $H$, $H'$, and $H''$ are $n$-dimensional with $n > 1$ and assume that the theorem holds for $(n-1)$-dimensional hyperboxes. We can think of $H$, $H'$, and $H''$ as one-dimensional hyperboxes in the category of hyperboxes -- a hyperhyperbox. At each vertex is an $(n-1)$-dimensional hyperbox of chain complexes. In this language $FG$ as a two-dimensional hyperhyperbox. The two ways to compress this hyperhyperbox yield $FG^{n+1,n}$ and $FG^{n,n+1}$. By hypothesis, these two are chain homotopy equivalent.

Consider $FG$ as a two-dimensional hyperhyperbox of size $(d_{n-1}, 2)$. The vertices are compressed along the $n$-th axis and the maps are adjusted accordingly. Repeating the argument above, $FG^{n,n+1,n-1}$ is chain homotopy equivalent to $FG^{n,n-1,n+1}$. Continue $n - 2$ more times to prove the theorem.

Now we prove the base case: if $FG$ is a hyperbox of size $(d, 2)$ then $FG^{2,1} \simeq FG^{1,2}$. If $d = 1$ there is nothing to show, so assume $d > 1$. The vertical and horizontal maps agree, so we only need to study the two diagonal maps. Call them $p$ and $q$ where $p$ follows the scheme in Figure 32. Both $p$ and $q$ are sums of maps along certain paths in the cube, one for each diagonal. Write $p_{i,j}$ and $q_{i,j}$ for the terms in $p_{i,j}$ and $q_{i,j}$ which use the diagonal from the vertex $(i, j)$. There is a unique, increasing path from $(0, 0)$ to $(d, 2)$ which uses the diagonals at both $(i, 0)$ and $(j, 1)$. Call the linear map obtained by composing maps along this path $h_{i,j}$. Define

$$h = \sum_{i<j} h_{i,j}$$

so that

$$h_{i,j} \circ D^{(0,0)}(0,0) + D^{(0,0)}(d,2) \circ h_{i,j} = f^{(1,0)}_{(d-1,2)} \circ \cdots \circ f^{(1,0)}_{(i+1,2)} \circ \left( f^{(1,0)}_{(j,2)} \circ f^{(0,1)}_{(j,1)} + f^{(0,1)}_{(j+1,1)} \circ f^{(1,0)}_{(j,1)} \right) \circ f^{(1,0)}_{(j-1,1)} \circ \cdots \circ f^{(1,0)}_{(i+1,1)} \circ f^{(1,1)}_{(i,0)} \circ f^{(1,0)}_{(i-1,0)} \circ \cdots \circ f^{(1,0)}_{(1,0)} \circ f^{(1,0)}_{(0,0)}$$
This equation is easiest to understand in the visual calculus of Figure 33. The thickened, blue edges represent directions which can only be used once. The dot represents the internal differential. (In the algebra of songs this is the result of playing \{\}.) From there one can check that

\[
\sum_{i<j} h_{i,j} \circ D_{(0,0)} + D_{(d,2)} \circ h_{i,j} = p + q
\]

as in the proof of Stokes’ theorem from multivariable calculus.

\[\hat{F} \simeq \hat{G}.\]
Proof. Let $J : F \to G$ be the homotopy. Then $\hat{J}$ is the mapping cone of a chain map $\text{Id} + j : \hat{F} \to \hat{G}$. Meanwhile $\hat{F}$ and $\hat{G}$ are mapping cones of chain maps $f$ and $g$. The square of the differential on $\hat{J}$ is

$$\text{Id} \circ f + g \circ \text{Id} + j \circ D + D \circ j.$$ 

□

**Corollary 8.8.** Compression of hyperboxes is a functor from the homotopy category of hyperboxes to the homotopy category of chain complexes.

8.2.2. Two tensor products. For plain old chain maps $f$ and $g$,

$$\text{cone}(f \otimes g) \neq \text{cone}(f) \otimes \text{cone}(g).$$

Suppose that $\otimes$ is some kind of tensor product operation on hyperboxes. If $F : H_0 \to H_0'$ and $G : H_1 \to H_1'$ are maps of hyperboxes, then $F \otimes G$ is analogous to $\text{cone}(f) \otimes \text{cone}(G)$ rather than $\text{cone}(f \otimes g)$. Therefore there are two different tensor products: $\otimes$ for hyperboxes in general and a separate operation, $\bar{\otimes}$, for maps.

Let $H$ be a hyperbox of chain complexes of dimension $n$ and shape $d$. Let $H'$ be a hyperbox of chain complexes of dimension $n'$ and shape $d'$. Define $H \bar{\otimes} H'$ to be the hyperbox of dimension $n + n'$ and shape $(d, d')$ whose underlying space is

$$(C \otimes C')(\delta, \delta') = C_\delta \otimes C_{\delta'}.$$ 

and whose maps $D^\otimes$ are defined as follows:

$$D^\otimes_{(\delta, \delta')} = \begin{cases} D_\delta \otimes \text{Id}_{H'_\delta'}, & \epsilon' = (0, \ldots, 0) \\ \text{Id}_{H_\delta} \otimes D_{\delta'}' & \epsilon = (0, \ldots, 0) \\ 0 & \text{otherwise} \end{cases}$$

**Lemma 8.9.** $H \otimes H'$ is a hyperbox.

Proof. Each cube of $H \otimes H'$ is the ordinary tensor product of cubical complexes. □

**Lemma 8.10.**

1. $H \otimes (H' \otimes H'') = (H \otimes H') \otimes H''$.
2. $\hat{H} \otimes \hat{H}' = \hat{H} \otimes \hat{H}'$. Equivalently, the differential on $\hat{H} \otimes \hat{H}'$ has form $\hat{D} \otimes \text{Id} + \text{Id} \otimes \hat{D}'$.

Proof. The first assertion holds because it holds for ordinary cubical chain complexes.

The underlying groups of $\hat{H} \otimes \hat{H}'$ and $\hat{H} \otimes \hat{H}'$ are identical. The differential on $H \otimes H'$ can be written

$$D^\otimes = \text{Id} \otimes D' + D \otimes \text{Id}$$

where $D$ is the differential on $C$ and $D'$ is the differential on $C'$. We claim this holds for $(H \otimes H')^{m+n-\ldots-k}$ with $0 \leq k \leq m + n$. Suppose it holds for $k = i$. To construct the next partial compression, one thinks of the boxes in the $(i-1)$-st direction as
maps of hyperboxes and composes them. This direction belongs to either \( H \) or \( H' \). By hypothesis, all of the maps have the form \( \text{Id}_H \otimes g \) or \( f \otimes \text{Id}_{H'} \). It follows that the differential on the fully compressed hyperbox has form \( \text{Id} \otimes D' + D \otimes \text{Id} \).

Any chain complex \( (C'', d) \) can be thought of as a zero-dimensional hyperbox. Therefore \( H \otimes C'' \) is a hyperbox with the same shape as \( H \) and \( (C \otimes C')_{\delta} = C_{\delta} \otimes C' \)

\[
D_{\delta, \epsilon}^\otimes = \begin{cases} 
D_{\delta}^0 \otimes d & \epsilon = \epsilon_0 \\
D_{\delta}^1 \otimes \text{Id} & \text{otherwise}
\end{cases}
\]

Now we define the tensor product of maps of hyperboxes. Let \( F: H_0 \to H_0' \) and \( G: H_1 \to H_1' \) be maps of hyperboxes with shapes \( (d, 1) \) and \( (d', 1) \), respectively. Define \( F \otimes G \) to be the map \( H_0 \otimes H_1 \to H_0' \otimes H_1' \) defined by

\[
(F \otimes G)^{\epsilon \epsilon'}_{\epsilon_2 \epsilon_3} = F^{\epsilon \epsilon}_\epsilon \otimes G^{\epsilon' \epsilon'}_{\epsilon_2 \epsilon_3}.
\]

**Lemma 8.11.**

1. If \( F \) and \( G \) are maps of cubical complexes, then \( F \otimes G \) is the usual tensor product of chain maps.
2. \( F \otimes G \) really is a map of hyperboxes \( \widehat{H}_0 \otimes \widehat{H}_1 \) to \( \widehat{H}_0' \otimes \widehat{H}_1' \).
3. \( (F \otimes G) \circ (F' \otimes G') = (F \circ F') \otimes (G \circ G') \).
4. \( \text{Id} \otimes \text{Id} = \text{Id} \).
5. As a map of chain complexes, \( \widehat{F} \otimes \widehat{G} = \widehat{F \otimes G} \).

**Proof.** The first and second statements are straightforward. The third and fourth follow from thinking cube-by-cube. The fifth follows from the second point of Lemma 8.10. The sixth is essentially a consequence of the fact that

\[
(f \otimes f') \circ (g \otimes g') = (f \circ g) \otimes (f' \circ g')
\]

for linear maps.

---

8.3. **From hyperboxes to \( A_\infty \)-algebras.** First, some notation for sequences and subsequences. Let \( s = (s_0, \ldots, s_{k+1}) \) be a sequence of natural numbers. Write \( |s| \) for the length of \( s \). Let \( \{C_{ij} : i, j \in \mathbb{N}\} \) be a collection of chain complexes. Set

\[
C_s = C_{s_0 s_1} \otimes C_{s_1 s_2} \otimes \cdots \otimes C_{s_k s_{k+1}}.
\]

For a sequence \( s = (s_0) \) of length one, define \( C_s = C_{s_0 s_0} \).

Write \( s' \subset s \) if \( s' \) is a sequence, \( s_0, s_{k+1} \in s' \), and there is an order-preserving injection \( s' \hookrightarrow s \). For a 0-1 sequence \( \epsilon \), write \( s(\epsilon) \) for the subsequence of \( s \) which
contains $s_i$ precisely if $\epsilon_i = 0$. For example,

\[
s((0, \ldots, 0)) = s \\
s((1, \ldots, 1)) = (s_0, s_k + 1).
\]

Let $\epsilon$ and $\epsilon'$ be two 0-1 sequences so that $\epsilon < \epsilon'$. Consider each maximal contiguous subsequence of 0s in $\epsilon$ which do not appear in $\epsilon'$. For example, in

\[
\epsilon = (0, 1, 0, 0, 1, 0, 1, 0, 0)
\]

\[
\epsilon' = (1, 1, 1, 0, 1, 0, 1, 1, 1)
\]

the maximal subsequences are underlined. Let $c(\epsilon, \epsilon')$ be the set which contains, for each underlined sequence, all the corresponding elements of $s$ and the surrounding ones. For example,

\[
s = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)
\]

\[
s(\epsilon) = (1, 2, 4, 5, 7, 9, 10, 11, 12)
\]

\[
s(\epsilon') = (1, 5, 7, 9, 12)
\]

\[
c(\epsilon, \epsilon') = ((1, 2, 4, 5), (9, 10, 11, 12))
\]

We call $c(\epsilon, \epsilon')$ the contraction sequence of $\epsilon$ and $\epsilon'$. The fixed sequence $f(\epsilon, \epsilon')$ of $\epsilon$ and $\epsilon'$ is a sequence of pairs of elements from $s(\epsilon)$. Its elements are contiguous pairs of elements of $s(\epsilon)$ which do not appear in $c(\epsilon, \epsilon')$. So in the running example,

\[
f(\epsilon, \epsilon') = ((5, 7), (7, 9)).
\]

The key point is that the contraction and fixed sequence describe a decomposition of $C_s$:

\[
C_s = \left( \bigotimes_{\epsilon' \in c(\epsilon, \epsilon')} C_{\epsilon'} \right) \otimes \left( \bigotimes_{f' \in f(\epsilon, \epsilon')} C_{f'} \right).
\]

8.3.1. Systems of hyperboxes.

**Definition 8.12.** Let $C = \{C_{i,j}\}$ be a collection of chain complexes indexed by $\mathbb{N} \times \mathbb{N}$. A system of hyperboxes over $C$ is an assignment of a $(k - 1)$-dimensional hyperbox $H_s$ to each sequence $s$ with $|s| = k$ satisfying the following properties:

- The $\epsilon$-corner of $H_s$ is $C_{s(\epsilon)}$.
- Let $F$ be the face of $H_s$ between the $\epsilon$- and $\epsilon'$-corners. Then

\[
F \cong \left( \bigotimes_{\epsilon' \in c(\epsilon, \epsilon')} H_{\epsilon'} \right) \otimes \left( \bigotimes_{f' \in f(\epsilon, \epsilon')} C_{f'} \right)
\]

where the isomorphism only permutes tensor factors.
Observe that the initial corner of \( H_s \) is \( C_s \) and the terminal corner is \( C_{s_0 s_k} \). The second condition is called the face condition. It is a locality condition for the component maps – maps which combine one part of a tensor product should not affect the other parts (cf. the Künneth formula in many link homology theories).

Let \( x \in C^\otimes^{k+1} \) be a simple tensor. We say that \( x \) is admissible if \( x \in C_s \) for some \( s \). We say that \( s \) is the underlying sequence of \( x \) and write \( H_s \) for \( H_s \). Let \( \mu_k(x) \in C_{s_0 s_k+1} \) be the image of \( x \) under the longest diagonal map on \( \hat{H}_x \). If \( x \) is not admissible, then set \( \mu_k(x) = 0 \). Extend by linearity to obtain a map

\[
\mu_k: C^\otimes k \to C
\]

for \( k \geq 1 \).

**Proposition 8.13.** \((C, \{\mu_i\})\) is an \( A_\infty \)-algebra.

For a system of hyperboxes \( \mathcal{H} \) we call this \( A_\infty \)-algebra \( \mathcal{A}(\mathcal{H}) \).

**Proof.** Suppose that \( x \) is admissible of length \( k \) with corresponding sequence \( s \). Each diagonal from the origin of \( \hat{H}_x \) corresponds to a 0-1 sequence \( \epsilon \) and therefore to a contraction sequence \( c(\epsilon, \epsilon') \). If \( c(\epsilon, \epsilon') \) has more than one element, then the corresponding diagonal map vanishes because of the face condition and Lemma 8.10.

Therefore the differential on \( \hat{H}_x \), applied to \( C_s \), is equal to

\[
\sum_{i+j+\ell = k} \text{Id}^\otimes i \otimes \mu_j \otimes \text{Id}^\otimes \ell.
\]

Therefore the component of \( \hat{D}^2 \) which extends from the \((0, \ldots, 0)\)-corner to the \((1, \ldots, 1)\)-corner of \( \hat{H}_x \) is

\[
\sum_{i+j+\ell = 1} \mu_{i+\ell+1} \circ (\text{Id}^\otimes i \otimes \mu_j \otimes \text{Id}^\otimes \ell) = 0.
\]

This is precisely Equation 12. If \( x \) is not admissible, then

\[
(\text{Id}^\otimes i \otimes \mu_j \otimes \text{Id}^\otimes) (x)
\]

is admissible only if \( \mu_j \) is applied to a non-admissible simple tensor. So

\[
\mu_{i+1+k-j} \left( \left( \text{Id}^\otimes i \otimes \mu_j \otimes \text{Id}^\otimes(k-i-j) \right) (x) \right) = 0
\]

one way or another.

\( \mu_i \) is computed from the compression of an \((i-1)\)-dimensional hyperbox. The compression of a hyperbox is a hyperbox, so it’s diagonal has degree \( 1 - i \). This is precisely the degree required in the definition of an \( A_\infty \)-algebra. \( \square \)
This construction constitutes a functor between the homotopy category of systems of hyperboxes of chain complexes and the homotopy category of $A_{\infty}$-algebras. Before proving that we should construct the former category.

**Definition 8.14.** Let $\mathcal{H}$ and $\mathcal{H}'$ be systems of hyperboxes over $C$ and $C'$ so that $H_s$ and $H'_s$ have the same shape for each $s$. A map of systems of hyperboxes $g$ is a collection of maps of hyperboxes

$$G_s: H_s \to H'_s$$

which satisfies the following face condition. Let $F$ be the $(\epsilon, \epsilon')$-face of $H_s$. Then $G_s$ must satisfy

$$G_s|_F = \left( \bigotimes_{s' \in f(\epsilon, \epsilon')} G'_{s'} \right) \otimes \left( \bigotimes_{f' \in f(\epsilon, \epsilon')} G_{f'} \right)$$

Note that for $f' \in f(\epsilon, \epsilon')$, $G_{f'}: C_{f'} \to C'_{f'}$ is an ordinary chain map (or more precisely, its mapping cone). Loosely, $G_s$ acts on the fixed part of $H_s$ by chain maps.

For any $\mathcal{H}$ there is an identity map $Id: \mathcal{H} \to \mathcal{H}$ where $G_s$ is the identity map for all $s$.

**Definition 8.15.** Let $G: \mathcal{H} \to \mathcal{H}'$ and $G': \mathcal{H}' \to \mathcal{H}''$ be maps of systems of hyperboxes. Define

$$(G' \circ G)_s = G'_s \circ G_s.$$

**Lemma 8.16.** Definition 8.15 actually defines a map of systems

$$(G' \circ G): \mathcal{H} \to \mathcal{H}''$$

and $(Id \circ G) = G$ and $(G \circ Id) = G$.

**Proof.** Follows directly from Lemma 8.11. \qed

**Definition 8.17.** A chain homotopy between maps of systems $g, g': \mathcal{H} \to \mathcal{H}'$ is a collection of hyperboxes

$$J_s: G_s \to G'_s$$

whose length one maps are identity maps and which also satisfies the following face condition. Let $F$ be a face of $J_s$ in which the last component changes. Suppose that the $d_{|s|+1} = d_{|s|+2} = 0$ face of $F$ is the $(\epsilon, \epsilon')$ face of $H_s$. Then

$$J_s|_F = \left( \bigotimes_{s' \in f(\epsilon, \epsilon')} \bigoplus_{i=1}^r \left( F_{s''_1} \otimes \cdots \otimes F_{s''_i} \otimes J_{s''_i} \otimes G_{s''_i+1} \otimes \cdots \otimes G_{s''_r} \right) \right) \otimes \left( \bigotimes_{f' \in f(\epsilon, \epsilon')} Id_{f'} \right).$$

$A_{\infty}$-algebras and their maps do not form a category: composition is only associative up to homotopy. So it is not possible to upgrade this statement by removing the mentions of “homotopy.”
A map of systems $\mathcal{H} \to \mathcal{H}'$ induces a map of $A_\infty$-algebras $g: \mathcal{A}(\mathcal{H}) \to \mathcal{A}(\mathcal{H}')$ in the following way. Let $x \in \mathcal{A}(\mathcal{H})$ be a simple tensor of length $n$. There is a corresponding map of hyperboxes $G_x: H_x \to H_x'$. Define $g_n(x)$ to be the image of $x$ under the longest diagonal map on $\hat{G}_x$. Extend $g_n$ to all of $C^\otimes n$ by linearity.

Lemma 8.18. $g = \{g_n\}_{n=1}^\infty$ is a map of $A_\infty$-algebras.

Proof. We aim to show that $\{g_n\}_{n=1}^\infty$ satisfies Equation 14. Consider the differential on $\hat{G}_x$ where $x$ is a simple tensor of length $n$. Its restriction to the initial vertex of $\hat{G}_x$ can be written as

$$\sum_{i+j+\ell=n} \text{Id}^\otimes i \otimes \mu_j \otimes \text{Id}^\otimes \ell + \sum_{i_1 + \cdots + i_r = n} g_{i_1} \otimes \cdots \otimes g_{i_r}.$$ 

The first term is the part of $\hat{G}_x$ corresponding to diagonals which do not change the $(n+1)$-st coordinate. The second term corresponds to diagonals which do change that coordinate; it has that form by Lemma 8.11, part (5). Therefore the component of the square of the differential on $\hat{G}_x$ from the initial vertex to the terminal vertex is

$$\sum_{i+j+\ell=n} g_{j+1} \circ (\text{Id}^\otimes i \otimes \mu_j \otimes \text{Id}^\otimes \ell) + \mu_r \left( \sum_{i_1 + \cdots + i_r = n} g_{i_1} \otimes \cdots \otimes g_{i_r} \right) = 0.$$ 

□

Theorem 8.19. Proposition 8.13 and the construction above define a functor from the homotopy category of systems of hyperboxes to the homotopy category of $A_\infty$-algebras.

Proof. Proposition 8.13 and the discussion above the theorem define the maps on objects and morphisms.

Suppose that $\mathfrak{f}$ and $\mathfrak{g}$ are chain homotopic maps of systems of hyperboxes. Let $J_s$ be the homotopy between $F_s$ and $G_s$. Let $x \in C_s$ be an admissible simple tensor. Define $j_n(x)$ to be the longest diagonal map on $\hat{J}_x$ applied to $x$. These maps satisfy equation (14) by the same arguments as above: consider the differential from the initial vertex of $\hat{J}_x$. It has three types of terms: the identity, application of $f_n$, the differential on $C_s$, and diagonals which change both the $(|s|+1)$-st and $(|s|+2)$-nd coordinates. (Any other diagonal map is zero because $J_s$ is a homotopy.) Now consider the component of the square of the differential from the initial to the final vertex of $\hat{J}_x$. The terms are of the form $\text{Id} \circ f_n$, $g_n \circ \text{Id}^\otimes |s|$, $j_n(\text{Id}^\otimes a \otimes \mu_i \otimes \text{Id}^\otimes b)$ and $\mu_i(f_i \otimes \cdots \otimes f_i \otimes j_k \otimes g_{i_1} \otimes \cdots \otimes g_{i_r})$. Therefore these maps satisfy equation (14). (If $x$ is inadmissible then equation (14) holds by the same argument as Proposition 8.13.)

Let $g$ and $g'$ be maps of systems so that $g' \circ g$ is defined. The map of $A_\infty$-algebras induced by $G' \circ G$ is $A_\infty$-chain homotopic to $g' \circ g$ by Corollary 8.8, Definition 8.14 and Lemma 8.11, part 5.
It is straightforward to check that the identity map of systems induces the identity map of $A_\infty$-algebras. \hfill \Box

**Remark 8.20.** For the sake of concreteness, this section only discussed systems of hyperboxes in which summands are indexed by pairs of natural numbers, but of course one could repeat the construction over any set. For example, in Section 3 we work with the set \{1, 2, 3\}.

**Remark 8.21.** For inadmissible $x$ we defined $\mu_k(x) = 0$ for lack of better options. It might seem more natural to say that multiplication is not even defined on $x$. This amounts to constructing an $A_\infty$-category rather than $A_\infty$-algebra. One ought to be able to adapt all the definitions of this section to build an $A_\infty$-category from a system of hyperboxes in which the objects are natural numbers, $\text{Hom}(i, j) = C_{i, j}$, and composition is given by the multiplication maps. Maps of systems induces $A_\infty$-functors, and so on.

### 8.4. Internal homotopy and enlargements

There is another way in which two systems of hyperboxes could yield homotopy equivalent $A_\infty$-algebras. Here is what we have in mind: suppose that $f \simeq g$. It may hold that $f = f_1 \circ f_0$ and $g = g_1 \circ g_0$ that $f_0$ and $g_0$ are not chain homotopic. (For example, they may have different codomains.) So the compressions of the hyperboxes $C_0 \to C_1 \to C_2$ and $C_0 \to C'_1 \to C_2$ are chain homotopic even if the hyperboxes themselves are not.

**Definition 8.22.** Let $\mathcal{H}$ and $\mathcal{H}'$ be systems of hyperboxes over $C$. Suppose that these is an integer $\ell$ so that

- $C_s \cong C'_s$ for all $s$.
- If $s$ does not contain $\ell$, then $H_s \cong H'_s$ by a map of hyperboxes which extends the isomorphisms in the previous bullet point.
- If $s$ contains $\ell$, then think of $H_s$ and $H'_s$ as a sequence of maps glued together along the dimension containing $\ell$. The composition of these maps is chain homotopic by a homotopy so that the homotopy vanishes on any tensor product whose underlying sequence does not contain $\ell$.

We say that $\mathcal{H}$ and $\mathcal{H}'$ are **internally chain homotopic**.

**Proposition 8.23.** If $\mathcal{H}$ and $\mathcal{H}'$ are internally homotopic and all the chain complexes in both systems are finite-dimensional over $R$, then $\mathcal{A}(\mathcal{H}) \simeq \mathcal{A}(\mathcal{H}')$.

**Proof.** We will define a map $f$ of algebras. Let $x$ be a simple tensor of length $k$ whose sequence $s$ does not contain $\ell$. Define $f_1(x)$ to be the isomorphism in the first bullet point and $f_k(x) = 0$ if $k > 1$.

Now suppose that the last occurrence of $\ell$ in $s$ is at the $i$-th entry. The key observation is that $H^{k, \ldots, i+1}_s$ and $H'^{k, \ldots, i+1}_s$ are homotopic as maps of hyperboxes. This
follows from the functoriality of compression. So there is a map

$$F_s: H^{k,\ldots,i+1}_s \to H'^{k,\ldots,i+1}_s$$

whose length one edges are isomorphims. With the assumption that all the chain complexes are finite-dimensional, a standard argument shows that $F_s$ has an inverse $G_s$ up to homotopy. It follows that

$$H^{k,\ldots,i+1}_s \simeq H'^{k,\ldots,i+1}_s$$
as hyperboxes. Let $f_s$ be the longest diagonal map in $\hat{F}_s$. By our previous arguments, the sum of all the $f_s$ defines a map $f$ of $A_\infty$-algebras. Let $g$ be the reverse map defined by hyperboxes $G_s$. We have $G_s \circ F_s \simeq \text{Id}$. It follows that $g \circ f \simeq \text{Id}$. □

Here is a definition from [15] which inspired the proposition above.

**Definition 8.24.** Let $H = (C, D)$ be a hyperbox of chain complexes of shape $d$ and dimension $n$. Fix $k \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, d_k\}$. Set $\tau_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is in the $kT$ position. Define $d^+$ by

$$d^+_i = \begin{cases} 
    d_i & i \neq k \\
    d_i + 1 & i = k.
\end{cases}$$

Define $H^+ = (C^+, D^+)$ to be the hyperbox of shape $d + \tau_k$ with

$$C^{+,\tau} = \begin{cases} 
    C^\delta & \delta_k \leq j \\
    C^{\delta - \tau_k} & \delta_k \geq j + 1
\end{cases}$$

and

$$D^{+,\tau}_\delta = \begin{cases} 
    D^\delta & \delta_k + \epsilon_k \leq j \\
    \text{Id} & \delta_k = j, \epsilon = \tau_k \\
    0 & \delta_k = j, \epsilon_k = 1, ||\epsilon|| > 1 \\
    D^{\delta - \tau_k} & \delta > 0.
\end{cases}$$

We say that $H^+$ is the *elementary enlargement* of $H$ at $(j, k)$.

It is straightforward to show that $\hat{H}^+ \simeq \hat{H}$. It also follows from Proposition 8.23.

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