Exact Penalties for Decomposable Optimization Problems

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Abstract: We consider a general decomposable convex optimization problem. By using right-hand side allocation technique, it can be transformed into a collection of small dimensional optimization problems. The master problem is a convex non-smooth optimization problem. We propose to apply the exact non-smooth penalty method, which gives a solution of the initial problem under some fixed penalty parameter and provides the consistency of lower level problems. The master problem is suggested to be solved by a two-speed subgradient projection method, which enhances the step-size selection. Preliminary results of computational experiments confirm its efficiency.

Key words: Convex optimization, right-hand side decomposition, exact non-smooth penalty method, subgradient projection method, two-speed step-size choice.

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1 Introduction

The general optimization problem consists in finding the minimal value of some goal function $f : \mathbb{R}^n \to \mathbb{R}$ on a feasible set $D \subseteq \mathbb{R}^n$. For brevity, we write this problem as

$$\min_{x \in D} f(x),$$

(1)

its solution set is denoted by $D^*$ and the optimal value of the function by $f^*$, i.e.

$$f^* = \inf_{x \in D} f(x).$$

It is well known that solution methods that take into account peculiarities of particular problems show better computational results than general purpose oriented ones. In

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particular, most large dimensional optimization problems have a special structure which admits decomposition. There exist various decomposition approaches allowing one to replace the initial large scale problem with a sequence of relatively small and simple ones. Besides, they justify certain decentralized control principles for large organization and industrial systems; see e.g. [1, 2, 3, 4] and the references therein. Together with the known price (Dantzig-Wolfe) and variable (Benders) decomposition the so-called right-hand side (Kornai-Liptak) decomposition approach was proved to be rather efficient in Optimization; see [5, 1, 2]. Its main idea is that one can attain the global optimal value in a complex system by some upper level allocation procedure of joint resource shares, whereas subsystems (elements) are free in choosing their best actions within these shares. In particular, this method enables us to reduce a large scale smooth convex optimization problem to a non-smooth optimization one, whose cost function is convex but not strictly or strongly convex. In addition, the upper level optimization problem may appear indefinite for some rather natural resource share allocations, which makes its solution quite complicated.

In this paper, we propose to apply the exact non-smooth penalty method together with the right-hand side decomposition and obtain a two-level optimization problem, which yields a solution of the initial problem under some fixed penalty parameter, unlike the smooth penalty method, and provides the consistency of constraints of lower level problems for all the share allocations. Nevertheless, for a successful implementation of this decomposition method we have to solve the upper level non-smooth convex optimization problem with a suitable simple method since computation of any subgradient of the cost function requires a solution of independent partial optimization problems. We suggest a modification of the simplest subgradient projection method, which consists in utilization of a two-speed step-size procedure. It does not require any a priori information and provides the same minimal memory and computational expenses per iteration, but enables us to generate step-sizes adaptively and reduce the number of improper steps. We carried out series of computational experiments that confirmed efficiency of the proposed methods.

2 Decomposition via Shares Allocation

We will utilize the following partition of the basic $n$-dimensional Euclidean space

$$\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_l},$$

so that $n = \sum_{i \in I} n_i$ with $I = \{1, \ldots, l\}$. Similarly, for each $x \in \mathbb{R}^n$ we determine $x = (x_i)_{i \in I}$ where $x_i = (x_{i1}, \ldots, x_{in_i})^T \in \mathbb{R}^{n_i}$ for $i \in I$. Next, for each $i \in I$ let $X_i$ be a set in $\mathbb{R}^{n_i}$, $h_i : X_i \rightarrow \mathbb{R}^m$ a mapping with components $h_{ij} : X_i \rightarrow \mathbb{R}$ for $j = 1, \ldots, m,$
and \( f_i : X_i \to \mathbb{R}, i \in I \) be some functions. We define the feasible set as follows:

\[
D = \left\{ x \in X \left| \sum_{i=1}^{l} h_i(x_i) \leq b \right. \right\},
\]

where \( X = X_1 \times \ldots \times X_l \), \( b \) is a fixed vector in \( \mathbb{R}^m \), besides, let

\[
f(x) = \sum_{i=1}^{l} f_i(x_i).
\]

We will take the following set of basic assumptions.

(A1) Each set \( X_i \) is convex and closed and each function \( f_i : X_i \to \mathbb{R} \) is convex and continuous for \( i \in I \). Also, \( h_{ij} : X_i \to \mathbb{R}, j = 1, \ldots, m, i \in I \) are convex continuous functions.

Then (1), (2), and (3) give a general decomposable convex optimization problem.

Let us now introduce the set of partitions of the right-hand side constraint vector \( b \):

\[
U = \left\{ u \in \mathbb{R}^{ml} \left| \sum_{i=1}^{l} u_i = b \right. \right\},
\]

where \( u = (u_1, \ldots, u_l)^\top, u_i \in \mathbb{R}^m, i = 1, \ldots, l \). Here \( u_i \) determines the \( i \)-th share of \( b \). The right-hand side decomposition method is based on inserting an additional upper control level for finding the optimal shares whereas the lower level optimization problem is decomposed into \( l \) independent partial problems in \( \mathbb{R}^{n_i} \) for \( i \in I \); see [5 1]. More precisely, the approach consists first in the simple equivalent transformation of the feasible set by inserting auxiliary variables:

\[
D = \left\{ x \in X \left| \exists u \in U, h_i(x_i) \leq u_i, \ i \in I \right. \right\},
\]

Then problem (1), (2), and (3) is replaced by the upper level optimization problem

\[
\min_{u \in U} \to \tilde{\mu}(u) = \sum_{i=1}^{l} \tilde{\mu}_i(u_i),
\]

where

\[
\tilde{\mu}_i(u_i) = \inf \{ f_i(x_i) \mid x_i \in X_i, h_i(x_i) \leq u_i \}
\]

is the marginal value function for the \( i \)-th partial problem, \( i \in I \).

Observe that the initial optimization problem has \( n \) variables and \( m \) complex constraints, whereas problem (1) has \( ml \) variables and \( m \) simple constraints, besides, computation of the value of each function \( \tilde{\mu}_i(u_i) \) requires a solution of the optimization problem having \( n_i \) variables and \( m \) simplified constraints. The preferences of this substitution are obvious if \( n \) is too large. The main drawback of this approach is that the
partial optimization problems in (4) may have no solutions for some feasible partitions, e.g. due to inconsistent constraints. Therefore, this approach needs certain modifications. For instance, it can be improved by applying a suitable penalty method. Let $P_i(w_i) = P_i(x_i, u_i)$ be a general penalty function for the set

$$W_i = \{ w_i = (x_i, u_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^m \mid h_i(x_i) \leq u_i \},$$

i.e.

$$P_i(x_i, u_i) \begin{cases} = 0, & \text{if } (x_i, u_i) \in W_i, \\ > 0, & \text{if } (x_i, u_i) \notin W_i; \end{cases}$$

for $i \in I$. Then problem (1)–(5) can be approximated by a sequence of penalized problems of the form

$$\min_{u \in U} \tilde{\eta}(u, \tau) = \sum_{i=1}^I \tilde{\eta}_i(u_i, \tau),$$

(6)

where

$$\tilde{\eta}_i(u_i, \tau) = \inf\{ f_i(x_i) + \tau P_i(x_i, u_i) \mid x_i \in X_i \}, \quad i \in I,$$

(7)

$\tau > 0$ is a penalty parameter. Unlike (5), the partial optimization problems in (7) always have solutions under general assumptions. The above approach for separable convex optimization problems was suggested in [6, 7, 8] where the known smooth penalty functions were taken. Then the functions $\tilde{\eta}_i$ appear differentiable, however, finding a solution of problem (1), (2), and (3) requires tending the parameter $\tau$ to infinity, but large values of $\tau$ give rather difficult auxiliary problem (6)–(7). For this reason, we intend to apply the exact (non-smooth) penalty functions $P_i(x_i, u_i)$ since penalized problem (6)–(7) for some fixed $\tau$ large enough then becomes equivalent to the initial problem. This property of non-smooth penalties is well known for general optimization problems; see [9].

3 Saddle Point Problems

The suggested approach will be based on relationships with saddle point problems. For this reason, we establish some properties of these problems associated with the optimization problem (1), (2), and (3). We define first its Lagrange function

$$M(x, \lambda) = \sum_{i=1}^I [f_i(x_i) + \langle \lambda, h_i(x_i) \rangle] - \langle \lambda, b \rangle,$$

and the saddle point problem: Find a pair $(x^*, \lambda^*) \in X \times \mathbb{R}_+^m$ such that

$$\forall \lambda \in \mathbb{R}_+^m, \quad M(x^*, \lambda) \leq M(x^*, \lambda^*) \leq M(x, \lambda^*) \quad \forall x \in X,$$

(8)

where

$$\mathbb{R}_+^m = \{ v \in \mathbb{R}^m \mid v_i \geq 0 \quad i = 1, \ldots, m \}.$$
Problem (8) is equivalent to the system:

\[
\sum_{i=1}^{l} [f_i(x_i) - f_i(x_i^*) + \langle \lambda^*, h_i(x_i) - h_i(x_i^*) \rangle] \geq 0 \quad \forall x \in X, \tag{9}
\]

\[
\lambda^* \geq 0, \quad b - \sum_{i=1}^{l} h_i(x_i^*) \geq 0, \quad \langle \lambda^*, b - \sum_{i=1}^{l} h_i(x_i^*) \rangle = 0. \tag{10}
\]

By using the suitable Karush-Kuhn-Tucker saddle point theorem for problem (1), (2), and (3) (see e.g. [10, Section 28]), we obtain that it is equivalent to (8) under certain regularity type conditions. So, together with (A1) we will take the following basic assumption.

(A2) There exists a saddle point \((x^*, \lambda^*) \in X \times \mathbb{R}_m^+\) in (8).

Under these assumptions we have the minimax equality

\[
f^* = \inf_{x \in X} \sup_{\lambda \in \mathbb{R}_m^+} M(x, \lambda) = \sup_{\lambda \in \mathbb{R}_m^+} \inf_{x \in X} M(x, \lambda); \tag{11}
\]

see e.g. [3, Ch. 1, Corollary 4.1].

Similarly, we can define the Lagrange function for the same optimization problem with the auxiliary share variables

\[
L(x, u, y) = \sum_{i=1}^{l} [f_i(x_i) + \langle y_i, h_i(x_i) - u_i \rangle],
\]

and the corresponding saddle point problem: Find a triple \((x^*, u^*, y^*) \in X \times U \times \mathbb{R}_m^l\) such that

\[
\forall y \in \mathbb{R}_m^l, \quad L(x^*, u^*, y) \leq L(x^*, u^*, y^*) \leq L(x, u, y^*) \quad \forall x \in X, \forall u \in U. \tag{12}
\]

Problem (12) is equivalent to the system:

\[
\sum_{i=1}^{l} [f_i(x_i) - f_i(x_i^*) + \langle y_i^*, h_i(x_i) - h_i(x_i^*) + u_i^* - u_i \rangle] \geq 0 \quad \forall x \in X, \tag{13}
\]

\[
y_i^* = (1/l) \sum_{s=1}^{l} y_s^*, \quad i \in I, \tag{14}
\]

\[
y_i^* \geq 0, \quad u_i^* - h_i(x_i^*) \geq 0, \quad \langle y_i^*, u_i^* - h_i(x_i^*) \rangle = 0, \quad i \in I. \tag{15}
\]

Observe that (14) implies

\[
y_i^* = y_j^* \quad \forall i \neq j, \tag{16}
\]

i.e. all the dual variables in (12) coincide.
**Proposition 1** Suppose (A1) and (A2) are fulfilled.

(i) If a pair \((x^*, \lambda^*) \in X \times \mathbb{R}^m_+\) is a saddle point in (8), then there exists a point \(u^* \in U\) such that the triple \((x^*, u^*, y^*)\) where \(y^*_i = \lambda^*, \ i \in I\), is a saddle point in (12).

(ii) If a triple \((x^*, u^*, y^*) \in X \times U \times \mathbb{R}^m_+\) is a saddle point in (12), then a pair \((x^*, \lambda^*)\) is a saddle point in (8) where \(\lambda^* = y^*_i\) for any \(i \in I\).

**Proof.** Let \((x^*, \lambda^*) \in X \times \mathbb{R}^m_+\) be a saddle point in (8). Set \(\tilde{u}_i = h_i(x^*_i), \ i \in I\). Then (10) gives \(\sum_{i=1}^I \tilde{u}_i \leq b\). Now define

\[
u^*_i = (1/l) \left[ b - \sum_{s=1}^I \tilde{u}_s \right] + \tilde{u}_i \text{ and } y^*_i = \lambda^*, \ i \in I,
\]

then (14) holds and (10) gives (15). Also, (9) gives (13). Hence, \((x^*, u^*, y^*)\) is a saddle point in (12).

Conversely, let \((x^*, u^*, y^*)\) be a saddle point in (12). Due to (16), we can set \(\lambda^* = y^*_i\) for any \(i \in I\). Also, (13) gives (9), whereas (15) gives (10). Hence, \((x^*, \lambda^*)\) is a saddle point in (8).

Combining the above assertions and (11) we also obtain the minimax equality

\[
f^* = \inf_{x \in X, u \in U} \sup_{y \in \mathbb{R}^m_+} L(x, u, y) = \sup_{y \in \mathbb{R}^m_+} \inf_{x \in X, u \in U} L(x, u, y).
\]

We can take a parametric vector \(t \in \mathbb{R}^m_+\) and define the reduced set of dual variables:

\[
Y_t = \{ y \in \mathbb{R}^m_+ \mid y_i \leq t, \ i \in I \}.
\]

Then, by analogy with (12), we can define the modified saddle point problem: Find a triple \((x^*, u^*, y^*) \in X \times U \times Y_t\) such that

\[
\forall y \in Y_t, \quad L(x^*, u^*, y) \leq L(x^*, u^*, y^*) \leq L(x, u, y^*) \quad \forall x \in X, \forall u \in U.
\]

If \((x^*, \lambda^*) \in X \times \mathbb{R}^m_+\) is a saddle point in (8) and \(\lambda^* \leq t\), then it follows from Proposition 1 that the corresponding triple \((x^*, u^*, y^*)\), which is a saddle point in (12), where \(y^*_i = \lambda^*, \ i \in I\), will be a saddle point in (18). Moreover, (17) now implies the minimax equality

\[
f^* = \inf_{x \in X, u \in U} \sup_{y \in Y_t} L(x, u, y) = \sup_{y \in Y_t} \inf_{x \in X, u \in U} L(x, u, y).
\]

Therefore, saddle points in (18) are also related to solutions of the optimization problem (1), (2), and (3).

**Proposition 2** Let (A1) and (A2) be fulfilled and let \((x^*, \lambda^*) \in X \times \mathbb{R}^m_+\) be a saddle point in (8). If \(\lambda^* \leq t\), then there exists a point \(u^* \in U\) such that the triple \((x^*, u^*, y^*)\) where \(y^*_i = \lambda^*, \ i \in I\), is a saddle point in (12) and (18) and the minimax equality (19) holds true.
4 Exact Decomposable Penalty Method

Let us select the primal optimization problem in (19):

$$\min_{x \in X, u \in U} \rightarrow \sup_{y \in Y} L(x, u, y).$$

Since

$$\sup_{y \in Y} L(x, u, y) = \sup_{y \in Y} \sum_{i=1}^{l} \left[ f_i(x_i) + \langle y_i, h_i(x_i) - u_i \rangle \right]$$

$$= \sum_{i=1}^{l} \max_{0 \leq y_i \leq t} \left[ f_i(x_i) + \langle y_i, h_i(x_i) - u_i \rangle \right]$$

$$= \sum_{i=1}^{l} \left[ f_i(x_i) + P_i(x_i, u_i, t) \right],$$

where $P_i(x_i, u_i, t) = \langle t, [h_i(x_i) - u_i]_+ \rangle$ for $i \in I$, $[v]_+$ denotes the projection of $v \in \mathbb{R}^m$ onto the non-negative orthant $\mathbb{R}^m_+$, we can rewrite (20) as follows:

$$\min_{x \in X, u \in U} \rightarrow \sum_{i=1}^{l} \left[ f_i(x_i) + P_i(x_i, u_i, t) \right].$$

This is nothing but the non-smooth penalty method problem for the initial optimization problem (1), (2), and (3) with the auxiliary share variables. We now give the basic equivalence properties.

**Theorem 1** Let (A1) and (A2) be fulfilled and let $(x^*, \lambda^*) \in X \times \mathbb{R}^m_+$ be a saddle point in (8).

(i) If $\lambda^* \leq t$, then $x^*$ is a solution of problem (21), which has the optimal value $f^*$.

(ii) If $\lambda^* < t$, then any solution $(\bar{x}, \bar{u})$ of problem (21) solves also problem (1), (2), and (3).

**Proof.** Part (i) follows directly from Proposition 2. Next, set

$$W = \{(x, u) \in X \times U \mid h_i(x_i) \leq u_i, i \in I\}$$

and take any $t \in \mathbb{R}^m_+$ so that problem (21) has a solution pair $(x(t), u(t))$. Then

$$f(x(t)) \leq \sum_{i=1}^{l} \left[ f_i(x_i(t)) + P_i(x_i(t), u_i(t), t) \right]$$

$$\leq \inf_{(x,u) \in W} \sum_{i=1}^{l} \left[ f_i(x_i) + P_i(x_i, u_i, t) \right] = \inf_{x \in D} f(x) = f^*. \quad (22)$$
Take now any \( t' > \lambda^* \), then there exists \( t'' \) such that \( t' > t'' > \lambda^* \). For brevity, set 
\[ x' = x(t'), \quad u' = u(t') \quad \text{and} \quad x'' = x(t''), \quad u'' = u(t'') \], these elements exist due to (i). From Proposition 2 it follows that there exists a point \( u^* \in U \) such that the triple \((x^*, u^*, y^*)\) where \( y^*_i = \lambda^* \), \( i \in I \), is a saddle point in (12) and (18) both for \( t = t' \) and \( t = t'' \), moreover, the minimax equalities (17) and (19) both for \( t = t' \) and \( t = t'' \) hold true. Combining these relations together with (22) we obtain

\[
\sum_{i=1}^l \left[ f_i(x'_i) + P_i(x'_i, u'_i, t') \right] \leq f^* = L(x^*, u^*, y^*) \\
\leq L(x', u', y^*) \leq \sup_{y \in Y_{t''}} L(x', u', y) = \sum_{i=1}^l \left[ f_i(x'_i) + P_i(x'_i, u'_i, t'') \right] .
\]

It follows that

\[
\sum_{i=1}^l \langle t' - t'', h_i(x'_i) - u'_i \rangle \leq 0,
\]

hence, \((x', u') \in W \) and \( x' \in D \). Due to (22), this gives \( x' \in D^* \).

These results can be viewed as a specialization the known properties of non-smooth penalty functions in general optimization problems (see [9]) to decomposable optimization problems.

Since the variables \( x \) and \( u \) in (21) are not contained in joint constraints, we can apply the sequential minimization and take the equivalent optimization problem

\[
\min_{u \in U} \rightarrow \mu(u, t) = \sum_{i \in I} \mu_i(u_i, t),
\] (23)

where

\[
\mu_i(u_i, t) = \inf_{x_i \in X_i} \{ f_i(x_i) + P_i(x_i, u_i, t) \}, \quad i \in I,
\] (24)

which gives precisely the exact (non-smooth) decomposable penalty function method; cf. (4)–(5) and (6)–(7). Now Theorem 1 guarantees that a solution of (23)–(24) yields a solution of (4)–(5) if \( \lambda^* < t \), but now the \( i \)-th partial problem (24) does not contain inconsistent constraints for any partition of the right-hand side vector \( b \). As in (6)–(7), calculation of the marginal value function for each partial problem can be made independently and separately of the others. Hence, we have derived another two-level decomposition method that has certain preferences over the other right-hand side decomposition methods.

Under the assumptions in (A1) the functions \( \mu_i, \ i \in I \), are convex in \( u_i \) (see e.g. [3] Ch. 1, Theorem 5.11), hence so is \( \mu \) in \( u \). At the same time, \( \mu \) need not be differentiable in \( u \) in general. Its subdifferential \( \partial_u \mu(u, t) \) in \( u \) can be found by the proper formula for the composite convex functions; see [11] Ch. 3, Theorem 2.6] and [3] Ch. 1, Theorem 5.11]. However, we now give a specialization of this formula for
calculations of a subgradient for each marginal value function in (24), which is more suitable for utilization in iterative solution methods for problem (23)–(24). In fact, due to the minimax equality we have

\[ \mu_i(u_i, t) = \inf_{x_i \in X_i} \{ f_i(x_i) + P_i(x_i, u_i, t) \} = \inf_{x_i \in X_i} \sup_{0 \leq y_i \leq t} \{ f_i(x_i) + \langle y_i, h_i(x_i) - u_i \rangle \} \]

\[ = \sup_{0 \leq y_i \leq t} \inf_{x_i \in X_i} \{ f_i(x_i) + \langle y_i, h_i(x_i) - u_i \rangle \} = \sup_{0 \leq y_i \leq t} \sigma_i(y_i, u_i). \]

Take arbitrary points \( u'_i, u''_i \in \mathbb{R}^m \). Let

\[ y'_i = \arg \max_{0 \leq y_i \leq t} \sigma_i(y_i, u'_i), \quad y''_i = \arg \max_{0 \leq y_i \leq t} \sigma_i(y_i, u''_i). \]

Then

\[ \mu_i(u''_i, t) - \mu_i(u'_i, t) = \sigma_i(y''_i, u''_i) - \sigma_i(y'_i, u'_i) \geq \sigma_i(y'_i, u''_i) - \sigma_i(y'_i, u'_i) \]

\[ = \langle -y'_i, u''_i - u'_i \rangle. \]

This means that \(-y'_i \in \partial_{u_i} \mu_i(u'_i, t)\). Therefore, the calculation of some subgradient of \( \mu_i(u'_i, t) \) in \( u_i \) reduces to finding a solution of the problem

\[ \max_{0 \leq y_i \leq t} \to \inf_{x_i \in X_i} \{ f_i(x_i) + \langle y_i, h_i(x_i) \rangle - \langle y_i, u'_i \rangle \}, \quad (25) \]

which is precisely the modified dual optimization problem to (5) with the additional upper bound \( y_i \leq t \).

### 5 Step-size Strategies for Subgradient Projection Methods

Together with the above example there exist a great number of some other significant applications of convex minimization problems containing just non-differentiable functions; see [1, 12, 13, 14, 15, 16] and the references therein. For this reason, their theory and methods were developed rather well. We recall that most applications admit calculation of only one arbitrary taken element from the subdifferential of a non-differentiable function at any point. During a rather long time, most efforts were concentrated on developing more powerful and rapidly convergent methods such as space dilation and bundle type ones. However, significant areas of applications related to decision making in industrial, transportation, information and communication systems, having large dimensionality and inexact data together with scattered necessary
information force one to avoid complex transformations and even line-search procedures, which are involved in all the mentioned methods. Let us take the problem of minimizing a convex, but not necessarily differentiable function \( \varphi : E \to \mathbb{R} \) on a convex set \( V \subseteq E \) in a finite-dimensional Euclidean space \( E \), or briefly,

\[
\min_{v \in V} \to \varphi(v). \tag{26}
\]

Its solution set is denoted by \( V^* \) and the optimal value of the function by \( \varphi^* \). Next, the constraint set is supposed to be rather simple in the sense that the projection of a point \( x \) onto \( V \), which is denoted by \( \pi_V(x) \), is not very expensive.

Then one can apply the simplest and most popular subgradient projection method, which provides convergence to a solution under the so-called divergent series step-size rule. Its iteration computation expenses and accuracy requirements are rather low, but its convergence may be rather slow; see e.g. \([13\ Ch. 2, §§1–2] \) and \([14\ Ch. 5, §3] \). There are several ways to speed up convergence of the subgradient methods via utilization of a priori information such as the optimal value or some condition numbers. However, it is usually difficult to calculate these values exactly, whereas taking inexact estimates may again lead to slow convergence. For this reason, we will try to improve convergence of the subgradient projection method within the divergent series step-size rule.

In this section, we will take the following assumptions.

\textbf{(B1)} The set \( V \subseteq E \) is convex and closed and the function \( \varphi : V \to \mathbb{R} \) is convex and continuous. Also, the set \( V^* \) is nonempty.

\textbf{(B2)} There exist a number \( C < +\infty \) such that

\[
\|g\| \leq C, \quad \forall g \in \partial \varphi(v), \quad \forall v \in V.
\]

Now we write the simplest subgradient projection method for problem (26):

\[
v^{k+1} = \pi_V[v^k - \theta_k g^k], \quad g^k \in \partial \varphi(v^k), \tag{27}
\]

where

\[
\theta_k > 0, \quad \sum_{k=0}^{\infty} \theta_k = \infty, \quad \sum_{k=0}^{\infty} \theta_k^2 < \infty. \tag{28}
\]

The method stops if \( g^k = 0 \) or \( v^{k+1} = v^k \), then \( v^k \in V^* \), but we suppose this situation does not occur. Its convergence properties can be formulated as follows (see \([12\ Ch. 1, §3, p.47] \), \([17\ Ch. 3, Theorem 4.5] \), and \([3\ Ch. 5, §2] \)).

\textbf{Proposition 3} Let \( \textbf{(B1)} \) and \( \textbf{(B2)} \) be fulfilled and let the sequence \( \{v^k\} \) be generated in conformity with (27)–(28). Then

\[
\lim_{k \to \infty} v^k = v^* \in V^*.
\]
Besides, if we replace the subgradient \( g^k \) in (27) with the normed subgradient \( q^k = (1/\|g^k\|)g^k \), the assertion of Proposition 3 remains true without (B2).

The slow convergence of the sequences generated by method (27)–(28) is mainly due to the divergent series condition in (28) that is non-adaptive to iterates since nonlinear functions may behave in a different manner on different parts of their domains. Usually, the sequence of step-sizes \( \{\theta^k\} \) has a unique rate of decrease, e.g.

\[
\theta^k = \theta/(k + 1)^\gamma, \quad \theta > 0,
\]

where \( \tau \in (0.5, 1] \). Then the sequence \( \{\theta^k\} \) may contain many rather large improperly chosen step-sizes that do not in fact decrease the distance to a solution and cause the slow convergence. In order to reduce the number of these improper step-sizes we propose to apply a two-speed step-size procedure within (28).

Namely, choose an index sequence \( \{i_s\} \) such that

\[
i_0 = 0, \quad 0 < i_{s+1} - i_s \leq d < \infty, \quad s = 0, 1, \ldots,
\]

and a sequence \( \{\beta_s\} \) such that

\[
\beta_s > 0, \quad \sum_{s=0}^{\infty} \beta_s = \infty, \quad \sum_{s=0}^{\infty} \beta_s^2 < \infty.
\]

Then apply the subgradient projection method (27) where

\[
\theta^k = \begin{cases} 
\beta_s, & \text{if } k = i_s, \\
\nu \theta^k_{k-1}, & \text{if } i_s < k < i_{s+1},
\end{cases} \quad \forall s = 0, 1, \ldots, \quad \nu \in (0, 1). \tag{31}
\]

It follows that the more rapid step-size decrease at iterates between \( i_s \) and \( i_{s+1} \) allows us to find more suitable steps, but the divergent series condition in (30) prevents from the very small steps. It should be noted that the proposed two-speed step-size procedure differs from the relaxation type versions of the subgradient projection method (see [17, Ch. 3, §4]) since our version does not provide any monotone decrease of the goal function at each iteration and does not involve any line-search.

At the same time, it is not difficult to show that (29)–(31) imply (28). In fact,

\[
\sum_{k=0}^{\infty} \theta^k \geq \sum_{s=0}^{\infty} \beta_s = \infty
\]

and

\[
\sum_{k=0}^{\infty} \theta^2_k \leq d \sum_{s=0}^{\infty} \beta_s^2 < \infty.
\]

Therefore, the assertion of Proposition 3 holds true for the subgradient projection method (27), (29)–(31).

Next, if we write problem (23)–(24) in the format (26), then (A1)–(A2) imply (B1)–(B2). Since the partial subgradients are calculated as negative solutions of problems (25), we can take \( C = \sqrt{7}\|t\| \).
6 Computational Experiments

In order to check the performance of the proposed methods we carried out preliminary series of computational experiments. We chose two classes of test problems and implemented all the methods mentioned below in Delphi with double precision arithmetic.

6.1 Non-smooth test problem

Our first goal was to compare method (27), (29)–(31) with other simple subgradient methods. We took for comparison the well-known non-smooth test problem of form (26) with \( V = E \) from [18], where

\[
\varphi(v) = \max_{i=1,\ldots,m} \eta_i(v), \quad \eta_i(v) = b_i \sum_{j=1}^{n} (v_j - a_{ij})^2, \quad i = 1, \ldots, m,
\]

with \( n = 5 \) and \( m = 10 \). The coefficients of the quadratic functions are given by the vector \( b = (1, 5, 10, 2, 4, 3, 1.7, 2.5, 6, 3.5)\top \) and by the transposed matrix

\[
A\top = \begin{pmatrix}
0 & 2 & 1 & 1 & 3 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 2 & 4 & 2 & 2 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 0 \\
0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 0 \\
0 & 3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0
\end{pmatrix};
\]

see also [13, 14]. The problem has the optimal value \( \varphi^* = 22.60016 \). We chose the starting point \( v^0 = (0, 0, 0, 0, 1)\top \).

We chose the simplest subgradient method (27) where

\[
\theta_k = \theta/(k + 1), \quad \theta > 0.
\]

It was abbreviated as (SGM). For method (27), (29)–(31), which was abbreviated as (SGMTS), we chose the rules

\[
\beta_s = \theta/(s + 1), \quad \theta > 0, \quad i_{s+1} - i_s = d.
\]

We also took the same method (27) where

\[
\theta_k = \theta/\sqrt{k + 1}, \quad \theta > 0. \tag{32}
\]

It was abbreviated as (SGMSQ). In addition, the so-called subgradient method of simple double averaging from [19] was used. It can be written as follows:

\[
u^{k+1} = \mu_k v^k + (1 - \mu_k)y^k, \quad \mu_k = (k + 1)/(k + 2),
\]

\[
y^k = v^0 - \theta_k p^k, \quad p^k = \sum_{i=0}^{k} g^i, \quad g^i \in \partial\varphi(v^i),
\]

We chose the simplest subgradient method (27) where

\[
\theta_k = \theta/(k + 1), \quad \theta > 0.
\]

It was abbreviated as (SGM). For method (27), (29)–(31), which was abbreviated as (SGMTS), we chose the rules

\[
\beta_s = \theta/(s + 1), \quad \theta > 0, \quad i_{s+1} - i_s = d.
\]

We also took the same method (27) where

\[
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\[
u^{k+1} = \mu_k v^k + (1 - \mu_k)y^k, \quad \mu_k = (k + 1)/(k + 2),
\]

\[
y^k = v^0 - \theta_k p^k, \quad p^k = \sum_{i=0}^{k} g^i, \quad g^i \in \partial\varphi(v^i),
\]

We chose the simplest subgradient method (27) where

\[
\theta_k = \theta/(k + 1), \quad \theta > 0.
\]

It was abbreviated as (SGM). For method (27), (29)–(31), which was abbreviated as (SGMTS), we chose the rules

\[
\beta_s = \theta/(s + 1), \quad \theta > 0, \quad i_{s+1} - i_s = d.
\]

We also took the same method (27) where

\[
\theta_k = \theta/\sqrt{k + 1}, \quad \theta > 0. \tag{32}
\]

It was abbreviated as (SGMSQ). In addition, the so-called subgradient method of simple double averaging from [19] was used. It can be written as follows:

\[
u^{k+1} = \mu_k v^k + (1 - \mu_k)y^k, \quad \mu_k = (k + 1)/(k + 2),
\]

\[
y^k = v^0 - \theta_k p^k, \quad p^k = \sum_{i=0}^{k} g^i, \quad g^i \in \partial\varphi(v^i),
\]
where $\theta_k$ was chosen as in (32). We abbreviate this method as (DASG).

We compared all the methods for different accuracy $\varepsilon$ with respect to the goal function deviation

$$\Delta(v) = \varphi(v) - \varphi^*.$$ 

We took the same starting step $\theta = 0.1$ for all the methods. For (SGMTS), we took the ratio $\nu = 0.7$ and set $d = 25$. The results are given in Table 1 where we indicate the total number of iterations (it) (or the total number of subgradient calculations) for attaining the desired accuracy $\varepsilon$.

The implementation of (SGMTS) showed rather rapid convergence in comparison with the other methods. Convergence of (SGM) appeared rather slow, but stable. At the same time, (SGM) showed better convergence properties than the subgradient method utilizing rule (32), even after the averaging procedure.

### 6.2 Decomposable linear programming test problems

Afterwards, we took decomposable linear programming test examples of form (1), (2), and (3). The main goal was to investigate convergence of the subgradient projection methods to a solution of the non-smooth master problem (23)-(24). More precisely, the initial problem was the following:

$$\max \rightarrow \left\{ \sum_{i=1}^{l} \langle c_i, x_i \rangle \left| \sum_{i=1}^{l} A_i x_i \leq b, x_i \geq 0, i = 1, \ldots, l \right. \right\}.$$ 

It can be treated as a model describing the economic system which deals in $n$ commodities and $m$ pure factors of production. These common factors are utilized by $l$ producers, so that the $i$-th producer chooses an output vector $x_i \in \mathbb{R}^{n_i}$, his/her consumption rates are described by an $m \times n_i$ matrix $A_i$, whereas the vector $c_i \in \mathbb{R}^{n_i}$ denotes prices of his/her outputs, the vector $b \in \mathbb{R}^m$ denotes inventories of common factors. Therefore, the system should choose the outputs for maximizing the income value.
This problem is rewritten equivalently as follows:

\[
\min \rightarrow \left\{ \sum_{i=1}^{l} \langle -c_i, x_i \rangle \mid A_i x_i \leq u_i, \ x_i \geq 0, \ i = 1, \ldots, l, \ u \in U \right\},
\]

which corresponds to the format (11)–(15), where

\[ f_i(x_i) = \langle -c_i, x_i \rangle, \ h_i(x_i) = A_i x_i, \ X_i = \mathbb{R}^{n_i}_+, \ i = 1, \ldots, l. \]

Hence, we can apply the exact non-smooth decomposable penalty method to this problem and solve its basic problem (23)–(24) with a suitable subgradient projection method. In this case, the calculation of some subgradient of the function \( \mu_i(u'_i, t) \) in \( u_i \) reduces to finding a solution of the partial linear programming problem

\[
\min \rightarrow \left\{ \langle u_i, y_i \rangle \mid A_i^T y_i \geq c_i, \ 0 \leq y_i \leq t \right\}.
\]

Next, iterate (27) with \( V = U \) is rewritten equivalently as follows:

\[
v^{k+1} = v^k - \theta_k \bar{g}^k, \quad \bar{g}^k_i = g^k_i - (1/l) \sum_{i=1}^{l} g^k_s, \ i = 1, \ldots, l, \quad g^k \in \partial \varphi(v^k).
\]

We chose \( m = 2 \) and \( n_i = 2 \) for each \( i = 1, \ldots, l \) and took different values of \( l \). Then the master problem (6) has \( 2l \) variables. The elements of the above matrices and vectors were generated to be positive with the help of trigonometric functions. The upper bound \( t \) was somewhat different for varying elements. In all the cases, we took the same starting point \( u^0 \) with \( u^0_i = (1/l)b \) for \( i \in I \).

Since methods (SGMSQ) and (DASG) appeared too slow, we chose the subgradient projection method (33) with the rule

\[
\theta_k = \theta/(k + 2), \quad \theta > 0,
\]

which was also abbreviated as (SGM), and method (33), (29)–(31), which was abbreviated as (SGMTS), where we chose the rules

\[
\beta_s = \theta/(s + 2), \quad \theta > 0, \quad i_{s+1} - i_s = d.
\]

We fixed the value \( \theta = 5 \) for both the methods. Since the optimal values of test problems were unknown we show the number of iterations (it) and the best attained value of the goal function (f). The results are given in Tables 2 and 3.

The implementation of (SGMTS) for different dimensionalities showed also more rapid convergence in comparison with (SGM).

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Table 2: Computations by (SGM)

|      | $l = 2$ | $l = 10$ | $l = 20$ | $l = 50$ |
|------|--------|---------|---------|---------|
|      | it     | $f$     | it      | $f$     | it     | $f$     | it     | $f$     |
| 0    | -6.3536 0 | -6.5946 | 0 | -7.8872 | 0 | -7.9186 |
| 50   | -7.3354 50 | -14.0226 | 50 | -12.8349 | 50 | -11.928 |
| 100  | -14.6558 100 | -14.911 | 100 | -14.6894 |
| 100  | 400 | -15.1555 | 350 | -16.2088 | 1950 | -16.7793 |
| $t_1$ | 2.62 | 3.92 | 2.62 | 6.01 | 2.66 | 6.37 | 2.67 | 6.41 |
| $t_2$ | 2.62 | 3.92 | 2.62 | 6.01 | 2.66 | 6.37 | 2.67 | 6.41 |

Table 3: Computations by (SGMTS)

|      | $l = 2$ | $l = 10$ | $l = 20$ | $l = 50$ |
|------|--------|---------|---------|---------|
|      | $\nu = 0.2$ | $d = 10$ | $\nu = 0.8$ | $d = 25$ | $\nu = 0.9$ | $d = 40$ | $\nu = 0.9$ | $d = 100$ |
|      | it     | $f$     | it      | $f$     | it     | $f$     | it     | $f$     |
| 0    | -6.3536 0 | -6.5946 | 0 | -7.8872 | 0 | -7.9186 |
| 50   | -7.0786 50 | -14.8631 | 50 | -14.7241 | 50 | -13.058 |
| 100  | -7.2158 100 | -15.952 | 100 | -15.2835 | 100 | -14.7465 |
| 200  | -7.3739 - | -150 | -18.519 | 250 | -19.5964 |
| $t_1$ | 2.62 | 3.92 | 2.62 | 6.01 | 2.66 | 6.37 | 2.67 | 6.41 |
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