Representation of a one class function of two variables by bicontinued fractions

Nehai funksiya \( u(z, w) = f(z)h(w) \) визначена на компакті \( K = Z \times W \subset C^2 \). Досліджується задача зображення функцій такого класу добутком ланцюгових дробів по кожній змінній, які називають біланцюговим дробом. В роботі доведено нові властивості обернених похідних Тіле та коефіцієнтів розкладення складеної функції однієї змінної в ланцюговий дріб Тіле і правильний ланцюговий С–дріб. Зокрема доведено, що якщо \( D(z; g) \) розкладення функції ланцюговий дріб Тіле, то розкладення функції \( \hat{g}(z) = Cg(z), C = const \), має вигляд \( D(z; Cg) = CD(z; g) \). Крім того показано, що обернені похідні Тіле та біланцюгові дроби визначаються за формулами

\[
(2k) (g(Cz)) = (2k) g(v) \bigg| v = Cz, \quad (2k-1) (g(Cz)) = C^{-1} (2k-1) g(v) \bigg| v = Cz, \quad k \in \mathbb{N}.
\]

Отримані властивості обернених похідних та ланцюгових дробів разом із відомими їх властивостями дозволяють обґрунтувати можливість зображення функцій двох змінних розглянутиого класу біланцюговими дробами Тіле та біланцюговими С–дробами, встановити області збіжності та рівномірної збіжності отриманих зображень.

В якості ілюстрації розглянуто зображення біланцюговими дробами двох функцій \( u_1(z, w) = (\delta + \beta z)^{\alpha} \ln(\varepsilon + \gamma w), \) \( \delta, \beta, \gamma, \varepsilon \in \mathbb{C}, \gamma \neq 0 \) та \( u_2(z, w) = e^{\alpha z} \ln(\beta + \gamma w), \) \( \delta, \beta, \gamma, \varepsilon \in \mathbb{C}, \gamma \neq 0 \). Доведено, що отримані зображення біланцюговими дробами функцій \( u_1 \) збігаються до функції в області \( G_1 = \{ z \in \mathbb{C} : |arg(\beta + \gamma w) - arg(\beta + \gamma z^*)| < \pi \} \times \mathbb{C}\{-\beta/\gamma\} \) і на довільному компакті \( K \subset G_1 \) біланцюгові дроби збігаються рівномірно. Аналогічно, в області \( G_2 = C \times \{ w : w \in \mathbb{C}\{-\beta/\gamma\}, |arg(\beta + \gamma w) - arg(\beta + \gamma w^*)| < \pi \} \) побудовані біланцюгові дроби збігаються до функції \( u_2 \). На довільному компакті \( K \subset G_2 \) біланцюгові дроби будуть збігатися рівномірно. Запропонована в даний роботі методика природно поширюється на випадок функцій трьох та більше змінних, які належать до аналогічних класів.

Ключові слова: ланцюгові дроби, біланцюгові дроби, функції двох комплексних змінних, зображення функцій

Функція \( u(z, w) = f(z)h(w) \) определена на компакте \( K \subset C^2 \). Исследуется задача представления функций такого класса произведением двух непрерывных дробей, которое именуется бинепрерьной дробью. Доказаны некоторые свойства обратных производных Тиле, непрерывных дробей Тиле и правильных С–дробей. Показана возможность представления функций такого класса бинепрерьными дробами. Рассмотрены примеры, указанны области сходимости и равномерной сходимости полученных бинепрерьных дробей к функции.

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Let function \( u(z,w) = f(z)h(w) \) be defined on the compact set \( K \subset \mathbb{C}^2 \). We study the problem of representation of functions of this class by the product of two continued fractions, which is called a bicontinued fraction. Some properties of Thiele reciprocal derivatives, Thiele continued fractions and regular \( C \)-fractions are proved. The possibility of representation of functions of this class by bicontinued fractions is shown. Examples are considered, domains of convergence and uniform convergence of obtained bicontinued fractions to the function are indicated.

Key words: continued fractions, bicontinued fractions, functions of two complex variables, representation of functions

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1. Introduction

The functions of one real or complex variable can be approximate polynomials [1, 2, 3], splines [4], rational functions or Padé approximants [5, 6, 7]. Continued fractions are used to approximate functions too [8, 9, 10]. Methods of approximation functions of one variable are generalized to functions of many variables [14, 15], functionals and operators [16, 17].

The Thiele formula is an analogue of a Taylor formula in the theory of continued fractions. The Thiele formula has advantages over other methods of expansion of a function in continued fractions since coefficients of expansion are determined by reciprocal derivatives of a function [11, 12, 13].

Bicontinued fraction is a new concept that is introduced in this article. Let \( u \) be a function of two variables of the form \( u(z,w) = f(z)h(w) \). The problem of a function approximation by bicontinued fractions investigated in the paper. The new properties of Thiele continued fractions and Thiele reciprocal derivatives are proved. These properties are used when representing functions by bicontinued fractions.

2. Approximation of functions by continued fractions

Note that in addition to the set of natural numbers \( \mathbb{N} \), we also use the following sets of integer numbers \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( \mathbb{N}_2 = \mathbb{N} \setminus \{1\} \).

The necessary concepts of the theory of continued fractions [9] will be given below.

**Definition 1.** A continued fraction is an ordered pair \( \langle \langle \{a_n\}, \{b_n\}, \{f_n\} \rangle \rangle \), where \( \{a_n : a_n \in \mathbb{C}, a_n \neq 0, n \in \mathbb{N}\} \), \( \{b_n : b_n \in \mathbb{C}, n \in \mathbb{N}_0\} \) are sequences of complex numbers, and \( \{f_n : f_n \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, n \in \mathbb{N}_0\} \) is a sequence in the extended complex plane defined as follows: \( f_n = S_n(0), n \in \mathbb{N}_0, S_0(w) = s_0(w), S_n(w) = S_{n-1}(s_n(w)), n \in \mathbb{N}, s_0(w) = b_0 + w, s_n(w) = a_n/(b_n + w), n \in \mathbb{N} \).
The infinite continued fraction of form
\[ b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_i}{b_i + \cdots} + \cdots}} \]
we shall denote by one of the symbols \( b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_i}{b_i + \cdots} + \cdots}} \), or \( b_0 + K_{i=1}^{\infty} (a_i/b_i) \), or \( b_0 + K(a_i/b_i) \). Similarly, the finite continued fraction, \( n \)th approximant of infinite continued fraction
\[ b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_n}{b_n + \cdots} + \cdots}} \]
may be denoted by \( b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_n}{b_n + \cdots} + \cdots}} \), or \( b_0 + K_{i=1}^{n} (a_i/b_i) \).

**Definition 2.** Continued fractions \( b_0 + K(a_i/b_i) \) and \( d_0 + K(c_i/d_i) \) with \( n \)th approximants \( f_n = b_0 + K_{i=1}^{n} (a_i/b_i) \) and \( f_n^* = d_0 + K_{i=1}^{n} (c_i/d_i) \), respectively, are said to be equivalence if \( f_n = f_n^*, n \in \mathbb{N}_0 \). It is written as follows \( b_0 + K(a_i/b_i) \equiv d_0 + K(c_i/d_i) \).

**Theorem 1.** Continued fractions \( b_0 + K(a_i/b_i) \) and \( d_0 + K(c_i/d_i) \) are equivalent iff there exists a sequence of non–zero constants \( \{ r_i : r_0 = 1, r_i \neq 0, i \in \mathbb{N}_0 \} \) such that
\[ d_0 = b_0, \ c_i = r_{i-1} r_i a_i, \ d_i = r_i b_i, \ i \in \mathbb{N} \quad (2.1) \]

Let the function of complex variable \( g \) is defined on the compact set \( Z \subset \mathbb{C} \). Function \( g \) can be interpolated on set of nodes \( Z = \{ z_i : z_i \in Z, z_i \neq z_j, i, j = 0, n \} \) by a Thiele interpolation continued fraction (T–ICF) of the form
\[ D_n(z; g) = \frac{P_n(z; g)}{Q_n(z; g)} = b_0 + K \left( \frac{(z - z_{k-1})/b_k(z_0, z_1, \ldots, z_k; g)}{k=1} \right). \quad (2.2) \]

Canonical numerator \( P_n(z; g) \) and canonical denominator \( Q_n(z; g) \) of a T–ICF (2.2) are polynomials whose degrees satisfy inequalities \( \deg P_n(z; g) \leq \left[ \frac{n+1}{2} \right], \deg Q_n(z; g) \leq \left[ \frac{n}{2} \right] \).

By \( \rho_k[z_0, z_1, \ldots, z_k; g], k = 0, n \), denote the reciprocal difference of the \( k \)th order.
Similarly, we get that it follows:

We shall prove the relation (2.6) by induction. When

It is known [12, 19] that the reciprocal difference of the formulae

The reciprocal differences are calculated as follows

We first prove the auxiliary statement. The reciprocal differences of a function $z$ and function values in nodes. The reciprocal difference is the symmetric function of arguments $z_0, z_1, \ldots, z_k$.

**Theorem 2.** Let $D_n(z; g)$ be a T–ICF of a function $g$ and $C = \text{const.}$ Then

$$D_n(z; Cg) = C \cdot D_n(z; g).$$

**Proof.** We first prove the auxiliary statement. The reciprocal differences of a function $Cg$ satisfy the relation

$$
\rho_k[z_0, \ldots, z_k; Cg] = \begin{cases} 
C \rho_k[z_0, \ldots, z_k; g], & k = 2m, \\
C^{-1} \rho_k[z_0, \ldots, z_k; g], & k = 2m + 1.
\end{cases}
$$

We shall prove the relation (2.6) by induction. When $k = 0, 1$ then the relation holds. Let’s assume that formula (2.6) holds when $k = 2s − 1, 2s$. For $k = 2s + 1$ from (2.3) it follows

$$
\rho_{2s+1}[z_0, \ldots, z_{2s+1}; Cg] = \rho_{2s-1}[z_0, \ldots, z_{2s}; Cg] + \frac{z_{2s+1} - z_{2s}}{\rho_{2s}[z_0, \ldots, z_{2s-1}, z_{2s+1}; Cg] - \rho_{2s}[z_0, \ldots, z_{2s-1}, z_{2s}; Cg]} = \frac{1}{C} \rho_{2s+1}[z_0, \ldots, z_{2s+1}; g].
$$

Similarly, we get that $\rho_{2s+2}[z_0, \ldots, z_{2s+2}; Cg] = C \cdot \rho_{2s+2}[z_0, \ldots, z_{2s+2}; g]$. The formula (2.6) is proved.

It follows from (2.4) and (2.6) that the coefficients of a T–ICF (2.2) of a function $Cg$ satisfy the relation

$$
b_k(z_0, \ldots, z_k; Cg) = \begin{cases} 
C b_k(z_0, \ldots, z_k; g), & k = 2m, \\
C^{-1} b_k(z_0, \ldots, z_k; g), & k = 2m + 1.
\end{cases}
$$
Consider the a T–ICF of a function \( Cg \). We have
\[
D_n(z; Cg) = b_0(z_0; Cg) + \sum_{k=1}^{n} \frac{((z - z_{k-1})/b_n(z_0, z_1, \ldots, z_k; Cg))}{K_k(z_0, \ldots, z_k; Cg)}.
\]

We shall use the relation (2.7). Then we shall have that
\[
D_n(z; Cg) = Cb_0(z_0; g) + \sum_{k=1}^{n} \frac{((z - z_{k-1})/C^{(-1)^k}b_n(z_0, z_1, \ldots, z_k; g)))}{K_k(z_0, \ldots, z_k; Cg)}.
\]

If we perform equivalent transformations (2.1) when \( r_k = C^{(-1)^k+1}, k \in \mathbb{N} \) then we obtain (2.5).

Since the reciprocal difference of the kth order \( \rho_k[z_0, \ldots, z_k; g], k = 0, n, \) is a symmetric function of the arguments \( z_0, z_1, \ldots, z_k, \) then we can go to the limiting case.

**Definition 3.** If exist limit (finite number or infinity) of a reciprocal difference kth order \( \rho_k[z_0, \ldots, z_k; g] \) when the nodes \( z_0, \ldots, z_k \to z_s \), where \( z_s \in \mathbb{Z} \) then the limit value is called the Thiele reciprocal derivative of a kth order of a function \( g \) at the point \( z_s \) and is denoted \( \rho(z_s) \).

It follows from the definition that \( \rho(z_s) = \lim_{z_0, \ldots, z_k \to z_s} \rho_k[z_0, \ldots, z_k; g] \). The Thiele reciprocal derivatives are calculated by the recurrent formula [11]
\[
(1) g(z_s) = \frac{1}{g'(z_s)}, \quad (k) g(z_s) = \frac{k}{(k-1)g(z_s)} + (k-2)g(z_s), \quad k = 2, n. \tag{2.8}
\]

**Theorem 3 ([12]).** If the function \( g \) is analytic at \( z_s \in \mathbb{C} \) then Thiele reciprocal derivatives of a function \( g \) at \( z_s \in \mathbb{C} \) are defined as follows
\[
(2k) g(z_s) = \frac{H^{(2k)}_k(z_s)}{H^{(2)}_k(z_s)}, \quad (2k-1) g(z_s) = \frac{H^{(2k-1)}_k(z_s)}{H^{(2)}_k(z_s)}, \quad k \in \mathbb{N}, \tag{2.9}
\]
where Hankel determinants \( H^{(m)}_k(z_s) \) are equals
\[
H^{(m)}_0(z_s) = 1, \quad H^{(m)}_k(z_s) = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+k-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+k-1} & c_{m+k} & \cdots & c_{m+2k-2} \end{vmatrix} \neq 0, \quad c_s = \frac{g^{(s)}(z_s)}{s!}, \quad c_s = 0, \quad s < 0.
\]

**Theorem 4 ([13]).** (A) If determinants \( H^{(0)}_{k+1}(z_s), H^{(0)}_k(z_s), H^{(3)}_{k-1}(z_s), H^{(1)}_k(z_s) \), where \( k = 0, n, \) be non-zero then a function \( g \) at the point \( z_s \) has finite Thiele reciprocal derivatives up to \((2n)\)th order inclusive which may be calculated by formula (2.8) or by formula (2.9).
If \( H_n^{(0)}(z_s) \neq 0 \) and \( H_n^{(2)}(z_s) = 0 \) then \((2n)g(z_s) = 0\). If \( H_n^{(0)}(z_s) = 0 \) and \( H_n^{(2)}(z_s) \neq 0 \) then \((2n)g(z_s) = \infty\).

If \( H_n^{(1)}(z_s) \neq 0 \) and \( H_n^{(1)}(z_s) = 0 \) then \((2n)g(z_s) = 0\). If \( H_n^{(1)}(z_s) = 0 \) and \( H_n^{(1)}(z_s) \neq 0 \) then \((2n)g(z_s) = \infty\).

Let the function \( g \) have Thiele reciprocal derivatives at the point \( z_s \in Z \). Then we get the expansion of a function \( g \) into a Thiele continued fraction (T–CF) in some neighbourhood of the point \( z_s \), i.e.

\[
D(z; g) = b_0(z_s; g) + \sum_{k=1}^{\infty} \left( (z - z_s)/b_k(z_s; g) \right),
\]

where

\[
b_0(z_s; g) = g(z_s), \quad b_1(z_s; g) = (1)g(z_s), \quad b_k(z_s; g) = (k)g(z_s) - (k-2)g(z_s), \quad k \in \mathbb{N}_2.
\]

The Thiele reciprocal derivatives have properties that follows directly from properties of reciprocal differences. In particular

\[
(2n-1)(Cg(z_s)) = \frac{1}{C} \cdot (2n-1)(g(z_s)), \quad (2n)(Cg(z_s)) = C \cdot (2n)(g(z_s)), \quad C = \text{const}, \quad n \in \mathbb{N}.
\]

From Theorem 2, definition of a Thiele reciprocal derivatives, formulæ (2.11) and properties (2.12) it follows that

\[
D(z; Cg) = C D(z; g), \quad C = \text{const}.
\]

Theorem 5 ([13]). Let the function \( w = f(z) \) has a Thiele reciprocal derivative at the point \( z_0 \in G \) and let the function \( u = g(w) \) has a Thiele reciprocal derivative at the point \( w_0 \in E \), where \( w_0 = f(z_0) \), then the composed function \( F(z) = g(f(z)) \) has a Thiele reciprocal derivative at point \( z_0 \) and \((1)F(z_0) = (1)g(w_0) \cdot (1)f(z_0)\).

We shall prove the following property of Thiele reciprocal derivatives.

Theorem 6. Let the function \( g \) has Thiele reciprocal derivatives and \( C = \text{const} \) then

\[
(2k)(g(Cz)) = (2k)(g(v))\big|_{v=Cz}, \quad (2k)(g(Cz)) = C^{-1} \cdot (2k-1)(g(v))\big|_{v=Cz}, \quad k \in \mathbb{N}.
\]

Proof. From theorem 5 follows that \((1)(f(Cz)) = \frac{1}{C} \cdot (1)f(v)\big|_{v=Cz}\). It follows from here, from (2.8) and (2.12) that

\[
(2)(f(Cz)) = 2 \cdot (1)(f(Cz)) + f(Cz) = 2 \cdot (1)(\frac{1}{C})f(v) + f(v)\big|_{v=Cz} = (2)f(v)\big|_{v=Cz}.
\]

Suppose that the formulas (2.14) are true when \( k = \frac{1}{s} - 1 \). Then when \( k = s \) from (2.12) we have that

\[
(2s)(f(Cz)) = 2s \cdot (1)(f(Cz)) + (2s-2)(f(Cz)) = 2s \cdot (1)(\frac{1}{C})f(v) + (2s-2)f(v)\big|_{v=Cz} = (2s)f(v)\big|_{v=Cz}.
\]
The relation
\[
\left. (2s+1) (f(Cz)) \right|_{v=Cz} = (2s+1) \cdot \left. \left( (1)^{(1)} (f(v)) \right)^{(2s-1)} (f(v)) \right|_{v=Cz} = \frac{1}{C} \cdot \left. (2s+1) f(v) \right|_{v=Cz}.
\]
is similarly proved.

It follows from the Theorem 6 that
\[
b_{2n}(Cz^*; g) = b_{2n}(v; g) \big|_{v=Cz^*}, \quad b_{2n+1}(Cz^*; g) = \frac{1}{C} \cdot b_{2n+1}(v; g) \big|_{v=Cz^*}, \quad n \in \mathbb{N}_0. \tag{2.16}
\]
A T–CF (2.10) can be given by an equivalent continued fraction with partial denominators equal to one. We have
\[
F(z; g) = a_0(z^*; g) + \sum_{n=1}^{\infty} \frac{a_n(z^*; g)(z - z^*)}{b_n(z^*; g)}, \tag{2.17}
\]
The coefficients of a continued fraction (2.17) can be determined by the coefficients of a T-CF as follows
\[
a_0(z^*; g) = b_0(z^*; g), \quad a_1(z^*) = \frac{1}{b_1(z^*; g)}, \quad a_n(z^*; g) = \frac{1}{b_{n-1}(z^*; g)b_n(z^*; g)}, \quad n \in \mathbb{N}_2. \tag{2.18}
\]
Let the function \( g \) be expanded in a formal power series at the neighbourhood of a point \( z^* \). It is proved in [13] that (2.17) is equal to a regular continued C–fraction (C–CF). Since the a C–CF is corresponding to the formal power series, then a T–CF will also correspond to the formal power series. It’s easy to make sure that \( F(z; Cg) = CF(z; g), \quad C = \text{const.} \)

**Theorem 7** ([13]). Let the function \( g \) be expanded into a C–CF (2.17) in the some neighbourhood of a point \( z^* \in \mathbb{Z} \) and \( \lim_{n \to \infty} a_n(z^*; g) = a, \quad a \in \mathbb{C}, \quad a \neq 0 \).

(A) The continued fraction (2.17) converge to the function \( g \) which is meromorphic in the domain \( R_a = \{ z \in \mathbb{C} : |\arg(a(z - z^*) + 1/4)| < \pi \} \).

(B) The convergence will be uniform on an arbitrary compact set \( C \subset R_a \) which not contains poles of a function \( g \).

(C) The function \( g \) is holomorphic at the point \( z^* \).

**Theorem 8** ([13]). Let the function \( g \) be expanded into a C–CF (2.17) in the some neighbourhood of the point \( z^* \in \mathbb{Z} \), \( a_n(z^*; g) \neq 0 \) and \( \lim_{n \to \infty} a_n(z^*; g) = 0 \).

(A) The continued fraction (2.17) converge to the function \( g \).

(B) The convergence will be uniform on an arbitrary compact set \( Z \subset \mathbb{C} \) which not contains poles of a function \( g \).

(C) The function \( g \) is holomorphic at the point \( z^* \) and \( g(z^*) = a_0(z^*; g) \).
3. Representation of functions of two variables by bicontinued fractions

The results obtained in the previous section, allow us to specify a way of representation functions of two complex variables of the form

$$u(z, w) = f(z)h(w)$$

by the product of two continued fractions for each variable. We shall call this product a bicontinued fraction.

Let the function $u$ be defined on the compact set $K = \mathbb{Z} \times \mathbb{W} \subset \mathbb{C}^2$. Suppose that functions $f$ and $h$ are analytic on the compact sets $\mathbb{Z}$ and $\mathbb{W}$ respectively. Then, according to Theorem 3 and Theorem 4, each of these functions has Thiele reciprocal derivatives.

Let $\bar{w} \in \mathbb{W}$ be a fixed point and $H = h(\bar{w})$. The auxiliary function $F(z) = Hf(z)$ can be expanded into a T–CF about variable $z$ on the compact set $\mathbb{Z}$ into a T–CF about variable $z$

$$D(z; f) = H \left( b_0(z_\ast; f) + \sum_{n=1}^{\infty} \frac{((z - z_\ast)/b_n(z_\ast; f))}{b_n(z_\ast; f)} \right).$$

Similarly, let $\bar{z} \in \mathbb{Z}$ be a fixed point and $F = f(\bar{z})$, then the auxiliary function $\mathcal{H}(w) = Fh(w)$ can be expanded into a T–CF in the neighbourhood of the point $w_\ast \in \mathbb{W}$ about variable $w$.

$$D(z; h) = F \left( b_0(w_\ast; h) + \sum_{n=1}^{\infty} \frac{((w - w_\ast)/b_n(w_\ast; h))}{b_n(w_\ast; h)} \right).$$

Since the point $(\bar{z}, \bar{w})$ is an arbitrary point of the compact set $K$ then we get a representation of a function $u$ as the product of two T–CF in the neighbourhood of a point $(z_\ast, w_\ast) \in K$

$$B(z, w; fh) = \left( b_0(z_\ast; f) + \sum_{n=1}^{\infty} \frac{((z - z_\ast)/b_n(z_\ast; f))}{b_n(z_\ast; f)} \right) \times$$

$$\left( b_0(w_\ast; h) + \sum_{n=1}^{\infty} \frac{((w - w_\ast)/b_n(w_\ast; h))}{b_n(w_\ast; h)} \right).$$

(3.2)

We call this product a Thiele bicontinued fraction (T–BCF). Similarly, the function $u$ can be represented by a bicontinued C–fraction (C–BCF)

$$C(z, w; fh) = \left( a_0(z_\ast; f) + \sum_{n=1}^{\infty} \frac{a_n(z_\ast; f)(z - z_\ast)/1}{a_n(z_\ast; f)(z - z_\ast)/1} \right) \times$$

$$\left( a_0(w_\ast; h) + \sum_{n=1}^{\infty} \frac{a_n(w_\ast; h)(w - w_\ast)/1}{a_n(w_\ast; h)(w - w_\ast)/1} \right).$$

(3.3)

We shall consider examples of function representations by bicontinued fractions and we shall show the domains of convergence and domains of uniform convergence of such representations too.
Example 1. Consider the function \( u_1(z, w) = (\delta + \beta z)^{\alpha} \tan(\varepsilon + \gamma w) \), where \( \alpha \in \mathbb{C}\setminus\{Z\} \), \( \beta, \gamma, \delta, \varepsilon \in \mathbb{C}\setminus\{0\} \).

Theorem 9. (A) The function \( u_1(z, w) = (\delta + \beta z)^{\alpha} \tan(\varepsilon + \gamma w) \) can be represented:

1) by a T–BCF

\[
B(z, w; u_1) = \xi^\alpha \left( 1 + \frac{\alpha \beta(z - z_s)}{\xi} + \frac{(1 - \alpha)\beta(z - z_s)}{2} + \frac{(\alpha + 1)\beta(z - z_s)}{3\xi} + \cdots + \right.
\]
\[
+ \frac{(n - \alpha)\beta(z - z_s)}{2(n + 1)\xi} + \frac{(n + \alpha)\beta(z - z_s)}{1} + \frac{(\alpha + 1)^2\beta(z - z_s)}{1} + \cdots + \frac{\gamma\psi(w - w_s)}{\gamma(w - w_s)} + \frac{\gamma\psi(w - w_s)}{4n + 1} + \gamma\psi(w - w_s) + \frac{\gamma(w - w_s)}{(4n + 3)\psi} + \frac{\gamma(w - w_s)}{1} + \cdots \right).
\]

(3.4)

2) by a C–BCF

\[
C(z, w; u_1) = \xi^\alpha \left( 1 + \frac{\alpha \beta(z - z_s)}{\xi} + \frac{(1 - \alpha)\beta(z - z_s)}{2} + \frac{(\alpha + 1)\beta(z - z_s)}{3\xi} + \cdots + \right.
\]
\[
+ \frac{(n - \alpha)\beta(z - z_s)}{2(n + 1)\xi} + \frac{(n + \alpha)\beta(z - z_s)}{1} + \frac{(\alpha + 1)^2\beta(z - z_s)}{1} + \cdots + \frac{\gamma\psi(w - w_s)}{\gamma(w - w_s)} + \frac{\gamma\psi(w - w_s)}{4n + 1} + \gamma\psi(w - w_s) + \frac{\gamma(w - w_s)}{(4n + 3)\psi} + \frac{\gamma(w - w_s)}{1} + \cdots \right).
\]

(3.5)

in the some neighbourhood of a point \((z_s, w_s) \in \mathbb{C} \times G(\varepsilon, \gamma; w), \text{where } G(\varepsilon, \gamma; w) = \mathbb{C}\setminus\{\gamma w \neq \frac{k\pi}{2} - \varepsilon, k \in \mathbb{Z}\}\).

(B) The T–BCF (3.4) and the C–BCF (3.5) converge to a function \( u_1(z, w) \) on a domain \( R(\delta, \beta, z_s; z) \times G(\varepsilon, \gamma; w), \text{where } R(\delta, \beta, z_s; z) = \{ z \in \mathbb{C} : |\arg(\delta + \beta z) - \arg(\delta + \beta z_s)| < \pi \}\).

(C) Biconvergent fractions converge uniformly on a arbitrary compact set \( K \subset R(\delta, \beta, z_s; z) \times G(\varepsilon, \gamma; w) \).

Proof. It is proved in [13] that the coefficients of expansion function \((\delta + z)^{\alpha}\) into a T–CF in the neighbourhood of a point \( z_s \in \mathbb{C}\setminus\{-\delta\} \) are equals

\[
b_1(z_s; (\delta + z)^{\alpha}) = \frac{\delta + z_s}{\alpha(\delta + z_s)^{\alpha}}, \quad b_{2n}(z_s; (\delta + z)^{\alpha}) = \frac{2 \prod_{i=0}^{n-1}(\alpha + i)(\delta + z_s)^{\alpha}}{\prod_{i=1}^{n}(i - \alpha)}, \quad b_{2n+1}(z_s; (\delta + z)^{\alpha}) = \frac{(2n + 1) \prod_{i=1}^{n}(i - \alpha)}{\prod_{i=0}^{n}(\alpha + i)(\delta + z_s)^{\alpha - 1}}, \quad n \in \mathbb{N}.
\]

(3.6)
From (2.16) and (3.6) it follows that the coefficients of expansion of a function \((\delta + \beta z)^\alpha\) into a T–CF in the neighbourhood of a point \(z_* \in \mathbb{C}\setminus\{-\delta/\beta\}\) will be equals

\[
b_1(z_*; (\delta + \beta z)^\alpha) = \frac{1}{\alpha \beta (\delta + \beta z_*)^{\alpha - 1}}, \quad b_{2n}(z_*; (\delta + \beta z)^\alpha) = \frac{2 \prod_{i=0}^{n-1} (\alpha + i) (\delta + \beta z_*)^\alpha}{\prod_{i=1}^{n} (i - \alpha)},
\]

\[
b_{2n+1}(z_*; (\delta + \beta z)^\alpha) = \frac{(2n + 1) \prod_{i=0}^{n-1} (i - \alpha)}{\beta \prod_{i=0}^{n} (\alpha + i) (\delta + \beta z_*)^{\alpha - 1}}, \quad n \in \mathbb{N}.
\]

We make the notation \(\xi = \delta + \beta z_*\). Let the sequence \(\{r_i\}\) be defined as follows

\[
r_0 = 1, \quad r_1 = \frac{1 - \alpha}{\alpha \xi^\alpha}, \quad r_2 = \frac{\prod_{i=1}^{n-1} (i - \alpha)}{\xi^\alpha \prod_{i=0}^{n-1} (\alpha + i)}, \quad r_{2n+1} = \frac{\xi^\alpha \beta \prod_{i=0}^{n} (\alpha + i)}{\prod_{i=1}^{n} (i - \alpha)}.
\]

We shall be performed equivalent transforms (2.1) using the sequence \(\{r_i\}\). We shall obtain the expansion of a function \((\delta + \beta z)^\alpha\) in the neighbourhood of a point \(z_* \in \mathbb{C}\setminus\{-\delta/\beta\}\) into a T–CF

\[
D(z; (\delta + \beta z)^\alpha) = \xi^\alpha \left(1 + \frac{\alpha \beta (z - z_*)}{\xi} + \frac{(1 - \alpha) \beta (z - z_*)}{2} + \frac{(\alpha + 1) \beta (z - z_*)}{3 \xi} + \ldots \right) + \frac{(n - \alpha) \beta (z - z_*)}{2} + \frac{(n + \alpha) \beta (z - z_*)}{(2n + 1) \xi} + \ldots, \quad \xi = \delta + \beta z_*. \quad \text{(3.7)}
\]

We can rewrite a T–CF (3.7) in the form of a equivalent C–CF. From (2.18) we get

\[
F(z; (\delta + \beta z)^\alpha) = \xi^\alpha \left(1 + \frac{\alpha \beta (z - z_*)}{\xi} + \frac{(1 - \alpha) \beta (z - z_*)}{2 \xi} + \frac{(\alpha + 1) \beta (z - z_*)}{1 \xi} + \ldots \right) + \frac{(n - \alpha) \beta (z - z_*)}{1} + \frac{(n + \alpha) \beta (z - z_*)}{(4n + 2) \xi} + \ldots. \quad \text{(3.8)}
\]

Since

\[
\lim_{n \to \infty} \frac{(n - \alpha) \beta}{(4n - 2)(\delta + \beta z_*)} = \lim_{n \to \infty} \frac{(n + \alpha) \beta}{(4n + 2)(\delta + \beta z_*)} = \frac{\beta}{4(\delta + \beta z_*)}
\]

and

\[
\arg \left(\frac{(z - z_*) \beta}{4(\delta + \beta z_*)} + \frac{1}{4}\right) = \arg \left(\frac{\delta + \beta z}{4(\delta + \beta z_*)}\right) = \arg(\delta + \beta z) - \arg(\delta + \beta z_*)
\]

then by Theorem 7 we obtain that the continued fractions (3.7) and (3.8) will be converge to the function \((\delta + \beta z)^\alpha\) on the domain \(\mathbb{R}(\delta, \beta, z_*, z)\) and on an arbitrary compact set \(\mathbb{Z} \subset \mathbb{R}(\delta, \beta, z_*, z)\) the continued fractions will be converge uniformly.

It is known ([11, 13]) that the coefficients of the expansion of a function \(\tan w\) into a T–CF in the neighbourhood of a point \(w_* \in \mathbb{C}\setminus\{k\pi/2, k \in \mathbb{Z}\}\) are equals

\[
b_{4n+1}(w_*; \tan w) = \frac{4n + 1}{1 + \tan^2 w_*}, \quad b_{4n+2}(w_*; \tan w) = -\frac{1 + \tan^2 w_*}{\tan w_*}, \quad n \in \mathbb{N},
\]

\[
b_{4n+3}(w_*; \tan w) = \frac{(4n + 3) \tan^2 w_*}{1 + \tan^2 w_*}, \quad b_{4n+4}(w_*; \tan w) = \frac{1 + \tan^2 w_*}{\tan w_*}, \quad n \in \mathbb{N}_0.
\]
REPRESENTATION OF A ONE CLASS FUNCTIONS

It follows from (2.16) and (3.9) that coefficients of expansion of a function \(\tan(\varepsilon + \gamma w)\) into a T–CF in the neighbourhood of a point \(w_* \in W = \mathbb{C} \setminus \{(\frac{k\pi i}{2} - \varepsilon)/\gamma, k \in \mathbb{Z}\}\) will be equals

\[
\begin{align*}
b_{4n+1}(w_*; \tan(\varepsilon + \gamma w)) &= \frac{4n+1}{\gamma(1 + \psi^2)}, b_{4n+2}(w_*; \tan(\varepsilon + \gamma w)) = \frac{1 + \psi^2}{\psi}, \\
b_{4n+3}(w_*; \tan(\varepsilon + \gamma w)) &= \frac{(4n+3)\psi^2}{\gamma(1 + \psi^2)}, b_{4n+4}(w_*; \tan(\varepsilon + \gamma w)) = \frac{1 + \psi^2}{\psi},
\end{align*}
\]

were \(\psi = \tan(\varepsilon + \gamma w_*), n \in \mathbb{N}_0\). We define the elements of the sequence \(\{r_i\}\) as follows

\[
r_{4n+1} = \gamma(1 + \psi^2), r_{4n+2} = \frac{\psi}{1 + \psi^2}, r_{4n+3} = \frac{\gamma(1 + \psi^2)}{\psi}, r_{4n+4} = \frac{\psi}{1 + \psi^2}, n \in \mathbb{N}_0.
\]

After equivalent transforms (2.1) we obtain the expansion of a function \(\tan(\varepsilon + \gamma w)\) into a T–CF

\[
D(w; \tan(\varepsilon + \gamma w)) = \psi + \frac{\gamma(1 + \psi^2)(w - w_*)}{1} + \frac{\gamma\psi(w - w_*)}{\gamma(w - w_*)} + \frac{1}{3\psi} + \frac{1}{1} + \cdots + \frac{1}{4n+1} + \\
+ \frac{\gamma\psi(w - w_*)}{\gamma(w - w_*)} + \frac{\gamma(w - w_*)}{(4n+3)\psi} + \frac{1}{1} + \cdots (3.10)
\]

We find the coefficients of a C–CF using the formulas (2.18). We have expansion

\[
\tan(\varepsilon + \gamma w) = \psi + \frac{\gamma(1 + \psi^2)(w - w_*)}{1} + \frac{\gamma\psi(w - w_*)}{1} + \frac{1}{4n+1} + \\
+ \frac{\gamma\psi(w - w_*)}{4n+1} + \frac{\gamma\psi(w - w_*)}{1} + \frac{1}{(4n+3)\psi} + \frac{1}{1} + \cdots (3.11)
\]

Since

\[
\frac{\gamma\psi}{4n+1} \neq 0, \quad \frac{\gamma}{(4n+3)\psi} \neq 0, \quad \lim_{n \to \infty} \frac{\gamma\psi}{4n+1} = \lim_{n \to \infty} \frac{\gamma}{(4n+3)\psi} = 0, \quad w_* \in W,
\]

then from Theorem 8 follows that a T–CF (3.10) and a C–CF (3.11) converge to the function \(\tan(\varepsilon + \gamma w)\) in the domain \(G(\varepsilon, \gamma; w)\) and on an arbitrary compact set \(W \subset G(\varepsilon, \gamma; w)\) continued fractions converge uniformly.

The function \(u_1\) has the form (3.1), then this theorem follows from (3.2) and (3.3).

Remark 1. Let \(\alpha\) be integer number, i.e. \(\alpha \in \mathbb{Z}\), then a function \((\delta + z)^\alpha\) will be represented by a finite T–CF ([13]).
Example 2. Consider the function \( u_2(z, w) = e^{\alpha z} \ln(\beta + \gamma w) \), where \( \alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\} \).

**Theorem 10.** Let’s have a domain 
\[
L(\beta, \gamma, w_*; w) = \{w : w \in \mathbb{C} \setminus \{-\beta/\gamma\}, |\arg(\beta + \gamma w) - \arg(\beta + \gamma w_*)| < \pi\}.
\]

**(A)** A function \( u_2(z, w) = e^{\alpha z} \ln(\beta + \gamma w) \) can be represented:

1) by a \( T\)-BCF

\[
\begin{align*}
B(z, w; u_2) &= e^{\alpha z} \left(1 + \frac{\alpha(z - z_*)}{1} + \frac{\alpha(z - z_*)}{-2} + \frac{\alpha(z - z_*)}{-3} + \frac{\alpha(z - z_*)}{2} + \\
&+ \cdots + \frac{\alpha(z - z_*)}{(-1)^{n-1}(2n-1) + \frac{\alpha(z - z_*)}{(-1)^n2} + \cdots} \right) \left(\ln \tau + \frac{\gamma(w - w_*)}{\tau} + \frac{\gamma(w - w_*)}{2} + \frac{2\gamma(w - w_*)}{3\tau} + \frac{2\gamma(w - w_*)}{4\tau} + \cdots\right),
\end{align*}
\]

2) by a \( C\)-BCF

\[
\begin{align*}
C(z, w; u_2) &= e^{\alpha z} \left(1 + \frac{\alpha(z - z_*)}{1} - \frac{\alpha(z - z_*)}{2} + \frac{\alpha(z - z_*)}{1} + \frac{\alpha(z - z_*)}{1} + \\
&+ \frac{\alpha(z - z_*)}{10} + \cdots + \frac{\alpha(z - z_*)}{2(2n-1) + \frac{\alpha(z - z_*)}{2(2n+1)} + \cdots} \times \right) \left(\ln \tau + \frac{\gamma(w - w_*)}{1} + \frac{\gamma(w - w_*)}{2\tau(w - w_*)} + \frac{\gamma(w - w_*)}{3\tau^2(w - w_*)} + \cdots\right)
\end{align*}
\]

in the neighbourhood of a point \((z_*, w_*) \in \mathbb{C} \times L(\beta, \gamma, w_*; w)\).

**(B)** Bicontinued fractions (3.12) and (3.13) converge to the function \( u_2 \) in the domain \( \mathbb{C} \times L(\beta, \gamma, w_*; w) \).

**(C)** Bicontinued fractions (3.12) and (3.13) converge uniformly on the on an arbitrary compact set \( K \subset \mathbb{C} \times L(\beta, \gamma, w_*; w) \).

**Proof.** It is well-known ([18, 19]) that the coefficients of the expansion of a function \( e^z \) in the neighbourhood of a point \( z_* \in \mathbb{C} \) into a \( T\)-CF are equals

\[
b_{2n-1}(z_*; e^z) = (-1)^{n+1}(2n-1)e^{-z_*}, \quad b_{2n}(z_*; e^z) = (-1)^n2e^{z_*}, \quad n \in \mathbb{N}.
\]

It follows from (2.16) and (3.14) that the coefficients of the expansion of a function \( e^{\alpha z} \) into a \( T\)-CF in the neighbourhood of a point \( z_* \in \mathbb{C} \) will be equals

\[
b_{2n-1}(z_*; e^{\alpha z}) = \frac{(-1)^{n+1}}{\alpha}(2n-1)e^{-\alpha z_*}, \quad b_{2n}(z_*; e^{\alpha z}) = (-1)^n2e^{\alpha z_*}, \quad n \in \mathbb{N}.
\]
If the sequence \( \{ r_i \} \) is defined as follows \( r_{2n-1} = \alpha e^{az}, r_{2n} = e^{-az}; n \in \mathbb{N} \), then after equivalent transformations we obtain the expansion of a function \( e^{az} \) into a T–CF

\[
D(z; e^{az}) = e^{az} \left( 1 + \frac{\alpha(z - z_*)}{1} + \frac{\alpha(z - z_*)}{-2} + \frac{\alpha(z - z_*)}{-3} + \frac{\alpha(z - z_*)}{2} + \frac{\alpha(z - z_*)}{5} + \right.
\]
\[
+ \frac{\alpha(z - z_*)}{-2} + \frac{\alpha(z - z_*)}{-7} + \cdots + \frac{\alpha(z - z_*)}{-(2n-1)(2n-1)} + \frac{\alpha(z - z_*)}{-(1)^n 2} + \cdots. \tag{3.15}
\]

The expansion of a function into a C–CF will be written as follow

\[
F(z; e^{az}) = e^{az} \left( 1 + \frac{\alpha(z - z_*)}{1} + \frac{-\alpha(z - z_*)}{2(2n-1)} + \frac{\alpha(z - z_*)}{2(2n-1)} + \frac{\alpha(z - z_*)}{2(2n+1)} + \right.
\]
\[
+ \frac{\alpha(z - z_*)}{1} + \cdots + \frac{\alpha(z - z_*)}{2(2n+1)} + \frac{\alpha(z - z_*)}{1} + \cdots. \tag{3.16}
\]

Since

\[
a_{2n-1}(z_*; f) = \frac{-\alpha}{2(2n-1)} \neq 0, \quad a_{2n}(z_*; f) = \frac{\alpha}{2(2n+1)} \neq 0
\]

and in addition

\[
\lim_{n \to \infty} \frac{-\alpha}{2(2n-1)} = \lim_{n \to \infty} \frac{\alpha}{2(2n+1)} = 0,
\]

it follows from Theorem 8 that continued fractions (3.15) and (3.16) converge to the function \( e^{az} \) on complex plane \( \mathbb{C} \) and on an arbitrary compact set \( \mathbb{Z} \subset \mathbb{C} \) continued fractions converge uniformly.

In the monograph [13] was proved that the coefficients of the expansion of a function \( \ln(\beta + w) \) into a T–CF in the neighbourhood of a point \( w_* \in \mathbb{C} \setminus \{-\beta\} \) takes values

\[
b_{2n-1}(w_*; \ln(\beta + w)) = (2n-1)(\beta + w_*), \quad b_{2n}(w_*; \ln(\beta + w)) = \frac{2}{n}, \quad n \in \mathbb{N}. \tag{3.17}
\]

Similar to the previous case, it follows from (2.16) and (3.17) that the coefficients of the expansion of a function \( \ln(\beta + \gamma w) \) into a T–CF in the neighbourhood of a point \( w_* \in W = \mathbb{C} \setminus \{-\beta/\delta\} \) are defined by formulae

\[
b_{2n-1}(w_*; \ln(\beta + \gamma w)) = \frac{2n+1}{\gamma}(\beta + \gamma w_*), \quad b_{2n}(w_*; \ln(\beta + \gamma w)) = \frac{2}{n}, \quad n \in \mathbb{N}.
\]

Let the elements of the sequence \( \{ r_i \} \) be equals \( r_{2n-1} = \gamma, r_{2n} = n, n \in \mathbb{N} \). After the equivalent transforms (2.1) we get the extension of a function into a T–CF, i.e.

\[
D(w; \ln(\beta + \gamma w)) = \ln \tau + \frac{\gamma(w - w_*)}{\tau} + \frac{\gamma(w - w_*)}{2} + \frac{\gamma(w - w_*)}{3\tau} + \frac{2\gamma(w - w_*)}{2} + \frac{2\gamma(w - w_*)}{5\tau} + \cdots + \frac{n\gamma(w - w_*)}{2} + \frac{n\gamma(w - w_*)}{(2n+1)\tau} + \cdots, \quad \tau = \beta + \gamma w_. \tag{3.18}
\]
In accordance by (2.18) the extension of a function into a C–CF will be written as

\[ F(w; \ln(\beta + \gamma w)) = \ln \tau + \frac{\gamma}{2\tau} (w - w_*) + \frac{\gamma}{2\tau} (w - w_*) + \frac{2\gamma}{3\tau} (w - w_*) + \frac{\gamma}{2\tau} (w - w_*) + \frac{\gamma}{2\tau} (w - w_*) + \cdots \] (3.19)

Since

\[ \lim_{n \to \infty} \frac{n\gamma}{2(2n-1)(\beta + \gamma w_*)} = \lim_{n \to \infty} \frac{n\gamma}{2(2n+1)(\beta + \gamma w_*)} = \frac{\gamma}{4(\beta + \gamma w_*)} \]

and

\[ \arg \left( \frac{\gamma(w - w_*) + \frac{1}{4}}{4(\beta + \gamma w_*)} \right) = \arg (\beta + \gamma w) - \arg (\beta + \gamma w_*) , \]

then it follows from Theorem 7 that a T–CF (3.18) and a C–CF (3.19) converge to the function \( \ln(\beta + \gamma w) \) in the domain \( L(\beta, \gamma, w_*; w) \) and on an arbitrary compact set \( W \subset L(\beta, \gamma, w_*; w) \) continued fractions converge uniformly.

It follows from (3.2) and (3.3) that this theorem holds.

4. Conclusions

The new properties of Thiele reciprocal derivatives which proved in this work allow obtaining a representation of functions of two variables in the form of the product of two continued fractions. This technique can be naturally extended to the case of functions of three, four or more variables. Also, the properties proved in this work together with other properties of Thiele reciprocal derivatives allow us to find the extension of a function \( f(\alpha + \beta z) \) from the extension of a function \( f(z) \).

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