A Remark on the Geometry of Elliptic Scrolls and Bielliptic Surfaces

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Dedicated to the memory of F. Serrano

0 Introduction

The union of two quintic elliptic scrolls in $\mathbb{P}^4$ intersecting transversally along an elliptic normal quintic curve is a singular surface $Z$ which behaves numerically like a bielliptic surface. In the appendix to the paper [ADHP] where the equations of this singular surface were computed, we proved that $Z$ defines a smooth point in the appropriate Hilbert scheme and that $Z$ cannot be smoothed in $\mathbb{P}^4$. Here we consider the analogous situation in higher dimensional projective spaces $\mathbb{P}^{n-1}$, where, to our surprise, the answer depends on the dimension $n-1$. If $n$ is odd the union of two scrolls cannot be smoothed, whereas it can be smoothed if $n$ is even. We construct an explicit smoothing.

1 Elliptic scrolls

To every elliptic normal curve $E \subset \mathbb{P}^{n-1}$ of degree $n$ and every point $P \in E$ one can associate a translation scroll $S = S(E, P)$ by defining $S$ as the union of all secants of $E$ joining the points $x$ and $x + P$, ($x \in E$). If $n \geq 5$ and $P$ is not a 2-torsion point, then $S$ is a singular surface of degree $2n$ with $E$ as its singular locus. If, however, $P$ is a non-zero 2-torsion point, then $S$ becomes a smooth scroll of degree $n$ (the secants spanned by $x, x + P$ and $x - P, x$ coincide). Varying the point $P$ does, of course, not define a flat degeneration: as a scheme the general translation scrolls degenerates to a multiplicity-2 scheme whose support is the smooth scroll of degree $n$ (cf. [HVdV]). In this paper we are interested in the degree $n$ scrolls defined by non-zero 2-torsion points. We will simply call these scrolls degree $n$ elliptic scrolls. (If $n = 5$ this is the unique irregular smooth scroll in $\mathbb{P}^4$.) Our first aim is to determine these scrolls as abstract surfaces. Here, as in the sequel, we shall notice a difference between the cases $n$ even and $n$ odd.

We shall first treat the $n$ odd case. For this purpose we fix an elliptic curve $F$. Recall that there is a unique $\mathbb{P}^1$– bundle over $F$ with invariant $e = -1$ (in the sense of [Ha, Chapter V]). It was already observed in [A, p. 451] that this is the symmetric product $S^2F$ where the $\mathbb{P}^1$–bundle structure is given by summation

\[
\pi : S^2F \to F
\]

\[
\{x, y\} \mapsto x + y.
\]
Fix an origin 0 of $F$, and let $p : F \times F \to S^2F$ be the natural projection. The curve

$$F_0 = p(F \times \{0\})$$

is a section of the $\mathbb{P}^1$–bundle $S^2F$. We choose the point $p(0,0)$ as its origin and, by abuse of notation, shall denote it again by 0. If $\{P_i; i = 1, 2, 3\}$ are the non-zero 2-torsion points of $F$, then the curves

$$\Delta_i = \{(x, x + P_i); x \in F\} (\cong F) \quad (i = 1, 2, 3)$$

are mapped 2:1 under $p$ to 2-sections $F_i \subset S^2F$. As abstract curves $F_i = F/\langle P_i \rangle$. We shall choose the point $0_i = p(0, P_i) = p(P_i, 0)$ as the origin of $F_i$. The group of 2-torsion points of $\Delta_i$ is mapped to two points $(0_i, Q_i)$. Note that $F_i/\langle Q_i \rangle \cong F$. Every fibre $f$ of $\pi$ intersects $F_i$ in two points which differ by $Q_i$. We shall denote the fibre of $\pi$ over $P \in F$ by $f_P$ and put $S = S^2F$. The following formulae follow immediately from the above description:

1. $\mathcal{O}_S(F_0)|_{F_0} = \mathcal{O}_{F_0}(0)$
2. $\mathcal{O}_S(F_i) = \mathcal{O}_S(2F_0 - f_{P_i})$
3. $\mathcal{O}_S(F_i)|_{F_i} = \mathcal{O}_{F_i}(0_i - Q_i)$
4. $K_S = \mathcal{O}(-2F_0 + f_0)$.

**Proposition 1.1** Assume $n \geq 5$ odd. The line bundle $\mathcal{O}_S(H) = \mathcal{O}_S(F_0 + (\frac{n-1}{2}) f_0)$ is very ample and embeds $S$ as smooth surface of degree $n$ in $\mathbb{P}^{n-1}$. This surface is the translation scroll of the elliptic normal curves $F_i, i = 1, 2, 3$ defined by the 2-torsion points $Q_i$. Conversely every translation scroll of an elliptic normal curve of degree $n$ by a 2-torsion point arises in this way.

**Proof.** Very ampleness of $\mathcal{O}_S(H)$ follows e.g. from [Ha, Exercise V.2.12]. A straightforward calculation using Riemann-Roch shows $h^0(\mathcal{O}_S(H)) = n$. Since $H.F_i = n$ and $h^1(\mathcal{O}_S(H - F_i)) = 0$ (the latter can be seen e.g. by Kodaira vanishing), the 2-sections $F_i$ are mapped to elliptic normal curves of degree $n$. By construction $S$ is then the translation scroll defined by the pair $(F_i, Q_i)$. Conversely given any pair $(F_i, Q_i)$ consisting of an elliptic curve and a 2-torsion point, then $F_i$ is a 2-section of $S^2F$ with $F = F_i/\langle Q_i \rangle$ such that the rulings of $S^2F$ cut $F_i$ in two points differing by $Q_i$. \qed

We now turn to the case $n$ even where we assume $n \geq 6$. Let $Y \subset \mathbb{P}^{n-1}$ be a degree $n$ scroll given by a pair $(E, Q_i)$ where $E$ is an elliptic normal curve of degree $n$ and $P_i$ a non-zero 2-torsion point of $E$. Then $Y$ can also be constructed as follows: Embed $E$ as a normal curve of degree $n + 1$ in $\mathbb{P}^n$ and let $X$ be the degree $(n + 1)$-scroll given by $E \subset \mathbb{P}^n$ and the point $Q_i$. Projection from say the origin $0 \in E$ then maps $X$ to $Y$. More precisely, the surface $X$ which is the blow-up of $X$ in 0 is mapped to $Y$. Under this projection map the fibre of $X$ over the origin 0 is contracted, whereas the map is bijective otherwise. In fact this is the geometric realization of an elementary transformation of $X$.

**Proposition 1.2** The degree $n$ scroll $Y$ is smooth. It is isomorphic to the $\mathbb{P}^1$–bundle $\mathbb{P}(\mathcal{E})$ over the elliptic curve $F = E/\langle Q_i \rangle$ where $\mathcal{E} = \mathcal{O}_F \oplus \mathcal{O}_F(0 - P_i)$ and $P_i$ is the image of the 2-torsion points $Q_j, j \neq i$ under the projection to $F$. 

The embedding is given by the complete linear system defined by the line bundle \( \mathcal{O}_Y(H) = \mathcal{O}_Y(F_0 + \frac{n}{2} f_0) \) where \( F_0 \) (by abuse of notation) is the image of the section \( F_0 \) of \( X \).

**Proof.** Recall the situation on \( X = S^2 F \) where \( E \) is the 2-section \( F_i \). There are two sections of \( X \) which intersect \( F_i \) (transversally) in the point \( p \) of \( P_i \). There are two sections of \( X \) which intersect \( F_i \) (transversally) in the point \( p \) of \( P_i \). These sections become disjoint sections of the \( \mathbb{P}^1 \)-bundle \( \mathbb{P}(E) \) which is defined by the elementary transformation with centre \( p(0, P_i) \). We shall denote these sections again by \( F_0 \) resp. \( F_{P_i} \). Since after the elementary transformation \( F^2_0 = F^2_{P_i} = 0 \) it follows that we can assume that \( E = \mathcal{O} \otimes \mathcal{M} \) where \( \mathcal{M} \) has degree 0. It follows from the elementary transformation that the normal bundle of \( F_0 \) in \( \mathbb{P}(E) \) is isomorphic to \( \mathcal{O}_F(0 - P_i) \). This shows the claim about \( E \). The above description of the elementary transformation immediately gives \( \mathcal{O}_{\mathbb{P}(E)}(H) = \mathcal{O}_{\mathbb{P}(E)}(F_0 + \frac{n}{2} f_0) \). Clearly \( H^2 = n \). Since \( h^0(\mathcal{O}_{\mathbb{P}(E)}(H)) = n \) and since \( H \) is very ample [Ha, Exercise V.2.12] the proposition follows. \( \square \)

**Remark** Using the adjunction formula we immediately obtain the following results:

\[
K_Y = \mathcal{O}_Y(-2F_0) \otimes \mathcal{O}_Y(f_0 - f_{P_i})
\]

The curve \( E \) is again a 2-section of \( Y \) with self-intersection number \( E^2 = 0 \). Since \( E \) and \( F_0 \) do not intersect we obtain

\[
\mathcal{O}_Y(E) = \mathcal{O}_Y(2F_0)
\]

and combining (5) and (6) gives

\[
\mathcal{O}_Y(E) = \mathcal{O}_Y(-K) \otimes \mathcal{O}_Y(f_0 - f_{P_i}).
\]

Note that an analogous formula holds for \( E = F_i \) on \( S = S^2 F \).

### 2 Rigidity for \( n \) odd

We fix an elliptic normal curve \( E \) in \( \mathbb{P}^{n-1} \) of odd degree \( n \) and two non-zero 2-torsion points \( P_i \neq P_j \) on \( E \). These define degree \( n \) elliptic scrolls \( X_i \) and \( X_j \). The union \( Z = X_i \cup X_j \) of these scrolls is a singular surface of degree \( 2n \) whose singular locus is the curve \( E \), which is a double curve of \( Z \). Numerically \( Z \) is a bielliptic surface, its dualizing sheaf is a line bundle \( \omega_Z \) with \( \omega_Z^2 = \mathcal{O}_Z \). In the case \( n = 5 \) those surfaces were considered in connection with abelian and bielliptic surfaces in [ADHP], where their equations were determined. It was also shown [ADHP, appendix] that they define smooth points in their Hilbert scheme and that they are rigid, in the sense that every small deformation of \( Z \) is again of the same type. In particular, these surfaces cannot be smoothed. In this section we shall see that this is the same for all odd degrees, whereas, surprisingly, the situation is very different for \( n \) even.

Our first aim is to study the normal bundle of the degree \( n \) scroll in \( \mathbb{P}^{n-1} \). Let \( X_i \) be one of these scrolls. Then we have the following exact sequence for the rulings \( f \) of this scroll:
Lemma 2.1

(i) \( h^0(N_{X_i/p^{n-1}}) = n^2 \)

(ii) \( h^j(N_{X_i/p^{n-1}}) = 0 \) for \( j \geq 1 \).

Proof. Since \( h^j(K_X^{-1}) = 0 \) for all \( j \), it follows from sequence (8) that the claim is equivalent to \( h^0(Q) = n^2 \) and \( h^j(Q) = 0 \) for \( j \geq 1 \). The defining sequences for \( Q \) and the normal bundle \( N_{X_i/p^{n-1}} \) together with Riemann-Roch give

\[
\chi(Q) = \frac{1}{2}(c_1^2(Q) - 2c_2(Q)) + \frac{1}{2}c_1(Q)(-K_X) + (n-4)\chi(O_X) = n^2
\]

Hence it is enough to prove that \( h^j(Q) = 0 \) for \( j \geq 1 \). By Serre duality \( h^2(Q) = h^0(Q^* \otimes K_X) = 0 \) since \( Q^* \otimes K_X|f = (n-4)O_f(-3) \). To prove vanishing of \( h^1(Q) \) we first remark that \( Q(-1) \) is trivial on the fibres \( f \) and hence \( Q(-1) = \pi^*F \) where \( F \) is a rank \( n-4 \) bundle on \( F \). Since \( T_{p^{n-1}}(-1) \) is generated by global sections the same is true for \( N_{X_i/p^{n-1}}(-1) \) and hence also for \( Q(-1) \) and \( F \). But now, using the classification of vector bundles on elliptic curves \( [4] \) it follows that \( h^1(F(D)) = 0 \) for every divisor \( D \) on \( F \) of positive degree.

Recall that \( H = F_0 + (\frac{n+1}{2})f_0 \). Let \( Q' = Q(-1) \otimes (\frac{n-1}{2})f_0 \). Then \( h^1(Q') = h^1(F(\frac{n-1}{2})) = 0 \). Finally \( h^1(Q) = 0 \) follows from the exact sequence

\[
0 \to Q' \to Q \to Q|_{F_0} \to 0
\]

since \( Q|_{F_0} = F(\frac{n+1}{2})f_0 \) and \( h^1(F(\frac{n+1}{2})f_0) = 0 \).

\[
\square
\]

Lemma 2.2 \( h^j(N_{X_i/p^{n-1}}(-F_i)) = 0 \) for all \( j \).

Proof. Twisting (8) with \( O_{X_i}(-F_i) \) we obtain the exact sequence

\[
0 \to K_X^{-1}(-F_i) \to N_{X_i/p^{n-1}}(-F_i) \to Q(-F_i) \to 0.
\]

The line bundle \( K_X^{-1}(-F_i) = O_{X_i}(f_{P_i} - f_0) \) has no cohomology. Since for the restriction to a ruling \( Q(-F_i)|_{f} = (n-4)O_f(-1) \) it follows that \( h^0(Q(-F_i)) = h^2(Q(-F_i)) = 0 \). Finally we obtain \( -h^1(Q(-F_i)) = \chi(Q(-F_i)) = 0 \).

\[
\square
\]

We now turn to the normal bundle \( N_{Z/p^{n-1}} \) of \( Z \).
Proposition 2.3  
(i) \( h^0(N_{Z/P^{n-1}}) = n^2 \)
(ii) \( h^j(N_{Z/P^{n-1}}) = 0 \) for \( j \geq 1 \).

Proof. The line bundle
\[
T = N_{E/X_i} \otimes N_{E/X_j} = \mathcal{O}_E(20 - Q_i - Q_j)
\]
is a non trivial 2-torsion bundle. As in [CLM] we have the following exact sequences

\[
\begin{align*}
(9) & \quad 0 \to N_{X_i/P^{n-1}} \to N_{Z/P^{n-1}}|_{X_i} \to T \to 0 \\
(10) & \quad 0 \to N_{X_i/P^{n-1}}(-E) \to N_{Z/P^{n-1}}|_{X_i} \otimes \mathcal{O}_{X_i}(-E) \to T \otimes \mathcal{O}_{X_i}(-E) \to 0 \\
(11) & \quad 0 \to N_{Z/P^{n-1}}|_{X_i} \otimes \mathcal{O}_{X_i}(-E) \to N_{Z/P^{n-1}} \to N_{Z/P^{n-1}}|_{X_i} \to 0.
\end{align*}
\]

By formula (3) \( T \otimes \mathcal{O}_{X_i}(-E) = \mathcal{O}_E(0 - Q_j) \). Together with Lemma 2.2 it follows from (10) that
\[ h^j(N_{Z/P^{n-1}}|_{X_i} \otimes \mathcal{O}_{X_i}(-E)) = 0 \text{ for all } j. \]

From sequence (9) and Lemma 2.1 we obtain
\[ h^0(N_{Z/P^{n-1}}|_{X_i}) = n^2, \quad h^j(N_{Z/P^{n-1}}|_{X_i}) = 0 \text{ for } j \geq 1. \]
The result now follows from sequence (11). \( \Box \)

Theorem 2.4 (Rigidity) The component of the Hilbert scheme of surfaces containing \( Z \) is smooth of dimension \( n^2 \) at \([Z]\). Every small deformation of \( Z \) is again a union of two degree \( n \) elliptic scrolls intersecting transversally along an elliptic normal curve.

Proof. The statement about smoothness and the dimension of the Hilbert scheme follows immediately from Proposition 2.3. Let \( X(2, n) \) be the modular curve parametrizing elliptic curves with a level \( n \) structure and a non-zero 2-torsion point. Every point of \( X^0(2, n) = X(2, n) \setminus \{ \text{cusps} \} \) gives rise to a union \( Z \) of two degree \( n \) elliptic scrolls. Indeed the elliptic curves with level \( n \) structure are in 1:1 correspondence with Heisenberg invariant elliptic normal curves in \( \mathbb{P}^{n-1} \). Given a non-zero 2 torsion point we have exactly two other such points. We can use these two points to construct \( Z \). Conversely every point \( Z \) arises in this way up to a change of coordinates. Let \( \mathcal{H} \) be the component of the Hilbert scheme containing a given surface \( Z \). Then we have a natural map
\[ \Phi : X^0(2, n) \times \text{PGL}(n, \mathbb{C}) \to \mathcal{H}. \]

Since every elliptic normal curve has a finite automorphism group this map is finite and hence surjective in a neighbourhood of \([Z]\). \( \Box \)

Remark  Of course there exist global deformations of \( Z \) which are not a union of two degree \( n \) elliptic scrolls. E.g. \( Z \) can degenerate into non-reduced union of \( n \)-planes. For a discussion of possible degenerations in the case of \( \mathbb{P}^4 \), i.e. \( n = 5 \) see [ADHPR, section 9].
3 Smoothing for $n$ even

The case $n$ even is subtly different from the case $n$ odd, as can already be seen in the computation of the cohomology of the normal bundle of the union of two scrolls. Again we fix an elliptic normal curve $E$ in $\mathbb{P}^{n-1}$ of degree $n$ and two non-zero 2-torsion points $P_i \neq P_j$ defining degree $n$ elliptic scrolls $X_i$ and $X_j$. Let $Z = X_i \cup X_j$. As before we find for the normal bundle of $X_i$ an exact sequence

$$0 \rightarrow K^{-1}_{X_i} \rightarrow N_{X_i/\mathbb{P}^{n-1}} \rightarrow Q \rightarrow 0$$

with $Q|_f = (n-4)\mathcal{O}_f(1)$. In this case, however, $h^0(K^{-1}_{X_i}) = h^1(K^{-1}_{X_i}) = 1$. Nevertheless the arguments of Lemma 2.2 still go through and give

$$(12) \quad h^j(N_{X_i/\mathbb{P}^{n-1}}(-E)) = 0 \text{ for all } j.$$ 

We also have an exact sequence

$$(13) \quad 0 \rightarrow N_{X_i/\mathbb{P}^{n-1}}(-E) \rightarrow N_{X_i/\mathbb{P}^{n-1}} \rightarrow N_{X_i/\mathbb{P}^{n-1}}|_E \rightarrow 0.$$ 

Since $N_{X_i/\mathbb{P}^{n-1}}(-1)$ is globally generated we can conclude as in the proof of Lemma 2.1 that $h^j(N_{X_i/\mathbb{P}^{n-1}}|_E) = 0$ for $j \geq 1$. But then sequence (13) together with Riemann-Roch (numerically the cases $n$ odd and $n$ even behave in exactly the same way) gives

$$(14) \quad h^0(N_{X_i/\mathbb{P}^{n-1}}) = n^2, \quad h^j(N_{X_i/\mathbb{P}^{n-1}}) = 0 \text{ for } j \geq 1$$

which is as in the degree $n$ odd case. The main difference between the two cases lies in the fact that $N_{E/X_i} = N_{E/X_j} = \mathcal{O}_E$ and hence

$$(15) \quad T = N_{E/X_i} \otimes N_{E/X_j} = \mathcal{O}_E.$$ 

It now follows from sequences (9) and (11) together with formula (14) that $h^1(N_{Z/\mathbb{P}^{n-1}}|_{X_i}) = 1$. Since it follows by sequence (10) and by (12) that $h^2(N_{Z/\mathbb{P}^{n-1}}|_{X_i} \otimes \mathcal{O}_{X_i}(-E)) = 0$ we find that $h^1(N_{Z/\mathbb{P}^{n-1}}) > 0$, contrary to the degree $n$ odd case. Moreover sequences (9)–(11) show that $h^0(N_{Z/\mathbb{P}^{n-1}} = n^2 + 1$ or $n^2 + 2$ and $h^1(N_{Z/\mathbb{P}^{n-1}}) = 1$ or $2$ respectively. In fact we shall see later (Corollary 3.3) that $h^0(N_{Z/\mathbb{P}^{n-1}}) = n^2 + 1$.

We now want to construct an explicit embedded smoothing of the singular surface $Z$ to a bielliptic surface. Since $\omega_Z^2 = \mathcal{O}_Z$ it is natural to look at bielliptic surfaces of type 1 or 2) in the Bagnera-de Franchis list [3, List VI.20]. It is easy to see by Reider’s method (cf. [5]) that bielliptic surfaces of type 1) cannot be embedded in $\mathbb{P}^{n-1}$ for $n \leq 8$. Hence we shall now turn our attention to bielliptic surfaces of type 2). Recall that these surfaces are of the form $S = E \times F/G$ where $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $E$ by translation with 2-torsion points and on $F$ by $x \mapsto -x, x \mapsto x + \epsilon, \epsilon$ a 2-torsion point of $F$. We shall first show that these surfaces can be embedded as surfaces of degree $2n$ in $\mathbb{P}^{n-1}$. This will then give us the right idea for the construction of the degenerations.

**Proposition 3.1** Every bielliptic surface $S$ of type 2) can be embedded as a linearly normal surface of degree $2n$ in $\mathbb{P}^{n-1}(n \geq 6)$. 

**Proof.** Let $\pi : E \times F \to S$ be the projection map and set $A = \pi(E), B = \pi(F)$. Then $A.B = 4$. By [Se, Proposition 1.7] the element $B/2$ is in $\text{NS}(S)$ and we can consider the divisor

$$H = A + \frac{n}{4}B.$$ 

(Since $n$ is even this is indeed a divisor on $S$). Then $H^2 = 2n, H.A = n$ and $H.B = 4$. It is easy to check that $H$ is ample and Riemann-Roch together with Kodaira vanishing gives $h^0(O_S(H)) = n$. It is a straightforward application of Reider’s theorem to prove that $H$ is very ample. Hence the complete linear system defined by $H$ embeds $S$ as a linearly normal surface of degree $2n$ in $\mathbb{P}^{n-1}$. □

**Remark** The line bundle $\pi^*O_S(H)$ has degree $n$ on $E$ and degree 4 on $F$.

**Theorem 3.2** *(Smoothing)* Let $Z = X_i \cup X_j$ be a union of two degree $n$ elliptic scrolls in $\mathbb{P}^{n-1}(n \geq 6, \text{ even})$. Then there exists a flat family of surfaces $(Z_t)_{t \in \mathcal{T}}$ in $\mathbb{P}^{n-1}$ such that $Z_0 = Z$ and $Z_t$ for $t \neq 0$ is a linearly normal smooth bielliptic surface of degree $2n$.

**Proof.** We fix the elliptic curve $E = X_i \cap X_j$ and the two non-zero 2-torsion points $P_i$ and $P_j$ which define $X_i$ and $X_j$. Let $F = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau)$ be another elliptic curve which we consider variable. Let $S(4) \to X(4)$ be the Shioda modular surface of level 4. We consider the family $F = (F_t)_{t \in \mathcal{T}}$ where $t$ varies in some neighbourhood of a cusp of $X(4)$, say $i\infty$, where $t = e^{\pi i \tau / 4}$. Then $F_0$ is a 4-gon of rational curves and the 2-torsion points $Q_0 = 0, Q_1 = 1/2, Q_2 = \tau/2$ and $Q_3 = (1 + \tau)/2$ of $F_t = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau)$ define 4 sections of $F$ which intersect the singular fibre $F_0$ as indicated below

![Figure 1: Singular fibre of $F$](image)

The action of the 2-torsion points on smooth fibres extends to an action on $F$. Similarly the involution $\iota : x \mapsto -x$ on smooth fibres extends to an involution on $F$. We denote the section of $F$ given by $Q_2$ by $\varepsilon$. Then $\varepsilon$ acts on $F_0$ by rotation with $180^\circ$, i.e. it identifies $F_0$ and $F_2$, resp. $F_1$ and $F_3$. The involution $\iota$ interchanges $F_1$ and $F_3$, resp. induces involutions on $F_0$ and $F_2$ with fixed points $Q_0, Q_1$ and $Q_2, Q_3$. 
We now consider the product $\mathcal{X} = E \times \mathcal{F}$ which is naturally fibred over $T$ with fibre $X_t = E \times F_i$. We define an action of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathcal{X}$ as follows: The element $g_1 = (1, 0)$ acts on $E$ by $x \mapsto x + P_1$ and on $F$ by $x \mapsto x + \varepsilon$, whereas $g_2 = (0, 1)$ acts on $E$ by $x \mapsto x + P_2$ and on $F$ by $x \mapsto -x$. Then $g_1 g_2 = (1, 1)$ acts on $E$ by $x \mapsto x + (P_1 + P_2)$ and on $F$ by $x \mapsto -x + \varepsilon$. The total space $\mathcal{F}$ and hence $\mathcal{X}$ is smooth and $G$ acts freely on $\mathcal{X}$. Then $Z = \mathcal{X}/G$ is smooth and $Z_t = X_t/G$ is a bielliptic surface of type 2) for $t \neq 0$. The singular surface $Z_0$ has the following properties:

- $Z_0$ consists of two components $Z_0^0$ and $Z_0^1$, namely the images of $E \times F^0$ (resp. $E \times F^2$) and $E \times F^1$ (resp. $E \times F^3$).
- The singular locus $E \times \text{Sing} F_0$ of $X_0$ is mapped to an irreducible curve isomorphic to $E$ (and again denoted by $E$). This curve $E = Z_0^0 \cap Z_0^1$ is the singular locus of $Z_0$.
- $Z_0^0$ and $Z_0^1$ are $\mathbb{P}^1$ – bundles over the elliptic curves $E/\langle P_i \rangle$, resp. $E/\langle P_3 \rangle$ and the singular curve $E$ is a bisection of both $\mathbb{P}^1$ – bundles.
- The curves $E \times \{Q_i\}, i = 0, \ldots, 3$ are mapped to two disjoint sections $C_0^0$ and $C_0^1$ of $Z_0^0$ with $(C_0^0)^2 = (C_0^1)^2 = 0$.
- Similarly we can consider the sections of $\mathcal{F}$ given by the points $\hat{Q}_i = Q_i + \pi/4$. These sections again intersect $F_0$ in 4 points, which this time lie on $F^1$ and $F^3$ (see again figure 1). These curves $E \times \{Q_i\}$ map to two sections $C_0^0$ and $C_1^1$ on $Z_0^0$ with $(C_0^0)^2 = (C_1^1)^2 = 0$.

The next step is to construct a suitable line bundle $\mathcal{L}$ on $\mathcal{X}$ which descends to $Z$. First consider the degree $n$ line bundle $\mathcal{L}_0 = \mathcal{O}_E(nF_0)$ on $E$. Then the group $G$, which operates on $E$ by translation with 2-torsion points leaves $\mathcal{L}_0$ fixed as a line bundle. However, if we want to lift the action of $G$ to the bundle $\mathcal{L}$ itself we might have to extend the group $G$ depending on the commutator of lifts $g_1^{C_0}$ and $g_2^{C_0}$ of $g_1$ and $g_2$ to $\mathcal{L}_0$. By general Heisenberg theory

$$\left[ g_1^{C_0} , g_2^{C_0} \right] = \left( e^{2\pi i/n} \right)^{n/4} = e^{2\pi i n/4}$$

which is either 1 or $-1$ depending on whether $n \equiv 0 \mod 4$ or not. Hence if $n \equiv 0 \mod 4$ then the action of $G$ on $E$ lifts to an action on $\mathcal{L}_0$, whereas if $n \equiv 2 \mod 4$ we have to extend $G$ to the level 2 Heisenberg group $H$ which is a central extension

$$1 \to \{ \pm 1 \} \to H \to G \to \{ 0 \}.$$ 

Next we consider the sections $D_i$, resp. $\hat{D}_i$ of $\mathcal{F}$ given by the points $Q_i$, resp. $\hat{Q}_i$. Let $\mathcal{L}_1 = \mathcal{O}_F(D_0 + D_2 + D_0 + D_2)$. We claim that the action of $G$ on $\mathcal{F}$ lifts to $\mathcal{L}_1$, i.e. that $\left[ g_1^{\mathcal{L}_1} , g_2^{\mathcal{L}_1} \right] = 1$. It is enough to check this on a general fibre $F_i$ of $\mathcal{F}$. For this let $s$ be a section of $\mathcal{L}_1$ vanishing on $D_0 + D_2 + D_0 + D_2$. Since this divisor is invariant under $G$ it follows that both $g_1^{\mathcal{L}_1}$ and $g_2^{\mathcal{L}_2}$ map $s_i = s_i|_{F_i}$ to a multiple of itself and hence commute. If $n \equiv 0 \mod 4$ we can set $\mathcal{L} = \mathcal{L}_0 \boxtimes \mathcal{L}_1$. Then $G$ acts on $\mathcal{L}$ and since $G$ acts freely on $\mathcal{X} = E \times \mathcal{F}$ we obtain a line bundle $\mathcal{L} = \mathcal{L}/G$ on $Z$. In the case $n \equiv 2 \mod 4$ we must replace $\mathcal{L}$ by some suitable other line bundle. Let $\mathcal{M}_1 = \mathcal{O}_F(D_0 - D_1)$. Then $\mathcal{M}_1$ is
invariant under $G$, but $G$ does not lift to an action on $\mathcal{M}_1$. In fact we claim that $\left[\gamma_1^{\mathcal{M}_1}, \gamma_2^{\mathcal{M}_2}\right] = -1$. For this consider the function

$$f(\tau, z) = \frac{\vartheta(\tau, z + \frac{1}{2}(\tau + 1))}{\vartheta(\tau, z + \frac{1}{2}z)}$$

where $\vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} e^{2\pi i (\frac{1}{2}n^2 \tau + nz)}$ is the standard theta function. This is a meromorphic section of $M$ at least for $t = e^{2\pi i/4} \neq 0$. The claim then follows from the identity

$$\frac{\vartheta(\tau, -z - \frac{1}{2}(\tau + 1) + \frac{1}{2}z)}{\vartheta(\tau, -z - \frac{1}{2}(\tau + 1) - \frac{1}{2}z)} = -\frac{\vartheta(\tau, -z - \frac{1}{2}(\tau + 1) - \frac{1}{2}z)}{\vartheta(\tau, -z - \frac{1}{2}(\tau + 1) + \frac{1}{2}z)}$$

which follows immediately from [1, pp. 49,50], formulae (Θ1)-(Θ3). Now consider $\mathcal{L} = \mathcal{L}_0 \boxtimes (\mathcal{L}_1 \otimes \mathcal{M}_1)$. This time the action of $G$ on $X$ lifts a priori to an action of $H$ on $\mathcal{L}$. But by construction the centre of $H$ acts by $(-1)^2 = 1$, i.e. trivially. Hence we obtain again an action of $G$ on $\mathcal{L}$ and we can take $\mathcal{L} = \mathcal{L}/G$.

It remains to verify that $\tilde{\mathcal{L}}$ has the desired properties. Let $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}|_{Z_0}$. For $t \neq 0$ by Proposition 3.3 $\tilde{\mathcal{L}}_t$ embeds $Z_t$ as a linearly normal bielliptic surface (which by construction is of type 2). We have to show that $\tilde{\mathcal{L}}_0$ embeds $Z_0$ as the union of the two scrolls $X_i$ and $X_j$. Let $\tilde{\mathcal{L}}_0 = \tilde{\mathcal{L}}_0|_{Z_0^i}$ for $i = 0, 1$. By construction $\tilde{\mathcal{L}}_0^i$ has degree $n/2$ on the sections $C_0^j$ and $C_1^j$, degree 1 on the rulings and degree $n$ on the bisection $E$. Hence $\tilde{\mathcal{L}}_0^i \equiv O_{Z_0^j}(C_0^j + \frac{1}{2}f)$. Thus $h^0(\tilde{\mathcal{L}}_0^i) = n$ and the restriction map $H^0(Z_0^i, \tilde{\mathcal{L}}_0^i) \rightarrow H^0(E, \tilde{\mathcal{L}}_0^i|_E)$ is an isomorphism. In particular we find that

$$h^0(Z_0, \tilde{\mathcal{L}}_0) = h^0(Z_0^0, \tilde{\mathcal{L}}_0^0) + h^0(Z_0^1, \tilde{\mathcal{L}}_0^1) - h^0(E, \tilde{\mathcal{L}}_0|_E) = n = h^0(Z_t, \tilde{\mathcal{L}}_t).$$

Moreover the restriction map $H^0(Z_0, \tilde{\mathcal{L}}_0) \rightarrow H^0(Z_0^i, \tilde{\mathcal{L}}_0^i)$ is an isomorphism and $\tilde{\mathcal{L}}_0$ embeds each of the component $Z_0^i$ as a degree $n$ elliptic scroll. By construction the image scrolls are the translation scrolls of the embedded elliptic normal curve $E$ defined by the 2-torsion points $P_i$ and $P_j$. This gives the claim. \(\square\)

**Remark** The difference between the cases $n \equiv 0 \mod 4$ and $n \equiv 0 \mod 2$ is easily understood in terms of the geometry of bielliptic surfaces. In the first case $\frac{n}{4}B$ is an integer multiple of $B$ and hence effective. In the second case $B/2$ is an element of the Neron-Severi group of $S$, but is not effective.

**Corollary 3.3** If $n$ is even, then

(i) $h^0(N_{Z/\mathbb{P}^{n-1}}) = n^2 + 1$ (and hence $h^1(N_{Z/\mathbb{P}^{n-1}}) = 1$),

(ii) the Hilbert scheme containing the surface $Z$ is smooth at $[Z]$ where it has dimension $n^2 + 1$.

**Proof.** The bielliptic surfaces of type 2) define a component of the Hilbert scheme containing $Z$ which is of dimension at least $n^2 + 1$. Hence, if we can
prove (i), then assertion (ii) is an immediate consequence. In view of our earlier computations it is, therefore, enough to exclude the case \( h^0(N_{Z/P^{n-1}}) = n^2 + 2 \).

Consider the diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
H^0(N_{X_i/P^{n-1}}) \\
\downarrow^\beta \\
0 \rightarrow H^0(N_{Z/P^{n-1}}|_{X_j} \otimes O_{X_j}(-E)) \rightarrow H^0(N_{Z/P^{n-1}}) \xrightarrow{\alpha} H^0(N_{Z/P^{n-1}}|_{X_i}) \\
\downarrow \\
H^0(T)
\end{array}
\]

It follows from (12) and sequence (10) that \( h^0(N_{Z/P^{n-1}}|_{X_j} \otimes O_{X_j}(-E)) = 1 \). Hence, if \( h^0(N_{Z/P^{n-1}}) = n^2 + 2 \) then, since \( h^0(N_{Z/P^{n-1}}|_{X_j}) = n^2 + 1 \) (by (14) and sequence (9)), the map \( \alpha \) must be surjective. In particular \( \text{im}(\alpha) \supset \text{im}(\beta) \).

On the other hand the sequence

\[
0 \rightarrow N_{E/X_i} \rightarrow N_{E/P^{n-1}} \rightarrow N_{X_i/P^{n-1}}|_{E} \rightarrow 0
\]

yields rise to a diagram

\[
\begin{array}{ccc}
H^0(N_{E/X_i}) & \rightarrow & H^0(N_{E/P^{n-1}}) \\
\| & & \| \\
C & \xrightarrow{\cong} & C \\
\uparrow & & \uparrow \\
H^0(N_{X_i/P^{n-1}}) & \rightarrow & H^1(N_{E/X_i}) \rightarrow 0.
\end{array}
\]

Here the vertical isomorphism is a consequence of (12). In particular we can find a section \( s \in H^0(N_{X_i/P^{n-1}}) \) which does not lift to \( H^0(N_{E/P^{n-1}}) \). We claim that such a section cannot be in the image of \( \alpha \). To see this, assume that there exists a section \( \tilde{s} \in H^0(N_{Z/P^{n-1}}) \) with \( \alpha(\tilde{s}) = \beta(s) \). Then \( s \) and \( \tilde{s} \) define first order deformations \( X_i \) and \( Z \) of \( X_i \) and \( Z \) over \( \text{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2) \) such that \( X_i \subset Z \). A straightforward local calculation then shows that \( Z = X_i \cup X_j \) where \( X_j \) is a first order deformation of \( X_j \). Moreover \( X_i \cap X_j = E \) is a first order deformation of \( E \). In particular \( X_i \) contains a first order deformation of \( E \) which contradicts our choice of the section \( s \). This proves the claim. \( \square \)

References

[A] M.F. Atiyah, *Vector bundles over an elliptic curve*, Proc. Lond. Math. Soc. 3 VII 27 (1957), 414–452.

[ADHPR] A. Aure, W. Decker, K. Hulek, S. Popescu and K. Ranestad, *Syzygies of abelian and bielliptic surfaces*, Preprint 1996.

[B] A. Beauville, *Complex algebraic surfaces*, LMS Lecture Notes Series 68, CUP 1983.
C. Ciliberto, A. Lopez and R. Miranda *Projective degenerations of K3 surfaces, Gaussian maps and Fano threefolds*, Invent. Math. 114 (1993), 641–667.

R. Hartshorne, *Algebraic Geometry*, Springer 1977.

K. Hulek and A. Van de Ven, *The Horrocks-Mumford bundle and the Ferrand construction*, manuscripta math. 50 (1985), 313–335.

F. Serrano, *Divisors on bielliptic surfaces and embeddings in $\mathbb{P}^4$*, Math. Z. 203 (1990), 527–533.

J. Igusa, *Theta Functions*, Springer 1972.

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