Using Smith Normal Forms and $\mu$-Bases to Compute All the Singularities of Rational Planar Curves

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Abstract

We prove the conjecture of Chen, Wang and Liu in [8] concerning how to calculate the parameter values corresponding to all the singularities, including the infinitely near singularities, of rational planar curves from the Smith normal forms of certain Bezout resultant matrices derived from $\mu$-bases.

Keywords: rational planar curve, singularities, infinitely near points, blow up, intersection multiplicity, $\mu$-basis.

1 Introduction

The nature and number of the singularities of planar algebraic curves contain a great deal of information about the geometry and topology of these curves. Therefore much has been written about how to compute these singularities [1], [2], [3], [4], [5], [6]. Recently, Chen, Wang, and Liu presented a conjecture concerning how to use the Smith normal forms of certain Bezout resultant matrices derived from $\mu$-bases to calculate the parameter values corresponding to all the singularities, including the infinitely near singularities, of rational planar curves [8]. The goal of this paper is to prove their conjecture.

This paper is a sequel to our paper on $\mu$-Bases and Singularities of Rational Planar Curves [11], where we show how to compute the parameters
corresponding to each singularity, including the infinitely near singularities, of rational planar curves using certain Bezout resultant matrices derived from \( \mu \)-bases. Here we shall show how to use the approach established in that paper to prove the conjecture of Chen, Wang, and Liu.

We proceed in the following fashion. In Section 2, we review the notions of \( \mu \)-bases for rational planar curves and Smith normal forms of polynomial matrices. In Section 3 we summarize the main results in [11] on the computation and analysis of the singularities, together with their infinitely near singularities, of rational planar curves. In Section 4 we state the conjecture of Chen, Wang and Liu. Section 5 is devoted to a proof of our main result. Here we focus on a single singularity. First we compute the Smith normal forms of two Hybrid Bezout matrices, one of which provides all the parameters of the infinitely near singularities while the other provides all the parameters of the original singularity. We then combine these two Smith normal forms together by invoking companion matrices to factor the \( k \)-th determinant factors of the Bezout matrix that appears in our main theorem and thereby complete our proof. We close in Section 6 with a more detailed discussion of the relationship between our main theorem and the conjecture of Chen, Wang and Liu.

2 Preliminaries: \( \mu \)-bases and Smith normal forms

We begin by reviewing some preliminary concepts which we shall need in the statement and proof of our main result.

2.1 \( \mu \)-bases

Let \( \mathbb{R}[s, u] \) be the set of homogeneous polynomials in the homogeneous parameter \( s : u \) with real coefficients. A parametrization for a degree \( n \) rational planar curve is usually written in homogeneous form as

\[
P(s, u) = (a(s, u), b(s, u), c(s, u)),
\]

where \( a(s, u), b(s, u), c(s, u) \) are degree \( n \) homogeneous polynomials in \( \mathbb{R}[s, u] \). To avoid the degenerate case where \( P(s, u) \) parameterizes a line, we shall assume that the three homogeneous polynomials \( a(s, u), b(s, u), c(s, u) \) are relatively prime and linearly independent. Moreover, throughout this paper we will assume that the parametrization \( P(s, u) \) is generically one-to-one.

A polynomial vector \( L(s, u) = (A(s, u), B(s, u), C(s, u)) \) is a syzygy of the parametrization \( P(s, u) \) if

\[
L(s, u) \cdot P(s, u) = A(s, u)a(s, u) + B(s, u)b(s, u) + C(s, u)c(s, u) \equiv 0.
\]
The set $M_p$ of all syzygies of a rational planar curve $P(s,u)$ is a module over the ring $\mathbb{R}[s,u]$, called the syzygy module. The syzygy module $M_p$ is known to be a free module with two generators [7].

**Definition 2.1** Two syzygies $p(s,u)$ and $q(s,u)$ are called a $\mu$-basis for the rational planar curve $P(s,u)$ if $p$ and $q$ form a basis for $M_p$, i.e., every syzygy $L(s,u) \in M_p$ can be written as

$$L(s,u) = \alpha(s,u)p(s,u) + \beta(s,u)q(s,u), \quad (3)$$

where $\alpha(s,u), \beta(s,u) \in \mathbb{R}[s,u]$.

Note that since we are using homogeneous polynomials, Definition [2.1] implicitly implies the following degree constraint of the elements of a $\mu$-basis [15]:

$$\text{deg}(p) + \text{deg}(q) = \text{deg}(P).$$

Every rational planar curve has a $\mu$-basis. Moreover, there is a fast algorithm for computing $\mu$-bases based on Gaussian elimination [7].

$\mu$-bases have many advantageous properties. For example, we can recover the parametrization of the rational planar curve $P(s,u)$ from the outer product of a $\mu$-basis:

$$p(s,u) \times q(s,u) = kP(s,u), \quad (4)$$

where $k$ is a nonzero constant. We can also retrieve the implicit equation $f(x,y,w) = 0$ of the rational planar curve $P(s,u)$ by taking the resultant of a $\mu$-basis:

$$f(x,y,w) = \text{Res}_{s,u}(p(s,u) \cdot X, q(s,u) \cdot X), \quad (5)$$

where $X = (x,y,w)$ [7].

### 2.2 Smith normal forms

The statement and proof of our main results concern matrices whose entries are polynomials. To study these matrices, we are going to employ Smith normal forms. The definition and main properties of Smith normal forms are reviewed below and summarized in Definitions 2.2–2.4 and Propositions 2.1–2.4; for further details and proofs, see [14].

**Definition 2.2** A polynomial matrix $P \in M_{m \times m}(\mathbb{R}[t])$ is said to be invertible if $\det(P) = c \in \mathbb{R}$ and $c \neq 0$. 

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Proposition 2.1 The following elementary row matrices in $M_{m \times m}(\mathbb{R}[t])$ are invertible:

1. $E_{ij}$: interchange rows $i$ and $j$ of the identity matrix $I_m$;
2. $E_i(\lambda)$: multiply row $i$ of $I_m$ by $\lambda \in \mathbb{R}, \lambda \neq 0$;
3. $E_{ij}(f)$: add $f$ times row $j$ of $I_m$ to row $i$, $f \in \mathbb{R}[t]$.

Similarly the elementary column matrices $F_{ij}$, $F_i(\lambda)$, $F_{ij}(f)$ in $M_{m \times m}(\mathbb{R}[t])$ are invertible.

Proposition 2.2 Each invertible polynomial matrix $P \in M_{m \times m}(\mathbb{R}[t])$ is a product of elementary matrices.

Proposition 2.3 For every nonzero polynomial matrix $A \in M_{m \times m}(\mathbb{R}[t])$ with $r = \text{rank}(A)$, there exist invertible polynomial matrices $P, Q \in M_{m \times m}(\mathbb{R}[t])$ such that

\[
PAQ = \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_r \\
0 & \alpha_2 & \cdots & \alpha_{r-1} \\
& & \ddots & 0 \\
& & & \ddots \\
& & & & \alpha_r
\end{pmatrix}, \tag{6}
\]

where $\alpha_1, \ldots, \alpha_r \in \mathbb{R}[t]$ are polynomials with $\alpha_k | \alpha_{k+1}$ for $1 \leq k < r$.

Definition 2.3 The matrix in (6) is called the Smith normal form of the polynomial matrix $A$. We shall denote the Smith normal form of $A$ by $S(A)$. Note that Smith normal forms of polynomial matrices are unique up to constant multiples of the entries.

Definition 2.4 Let $A, B \in M_{m \times m}(\mathbb{R}[t])$. Then $A$ is said to be equivalent to $B$ over $\mathbb{R}[t]$ if and only if there are invertible matrices $P, Q \in M_{m \times m}(\mathbb{R}[t])$ such that $PAQ = B$.

Proposition 2.4 Equivalent matrices $A, B \in M_{m \times m}(\mathbb{R}[t])$ have the same Smith normal forms.
Proposition 2.5 \[17\] Let $A, B \in M_{m\times m}(\mathbb{R}[t])$ be nonsingular matrices, and denote by $\alpha_k, \beta_k, \gamma_k$ the $k$-th invariant factor of $A$, $B$, and $AB$, respectively. Then

$$\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k} \beta_{j_1} \beta_{j_2} \cdots \beta_{j_k} \gamma_{i_1+j_1-1} \gamma_{i_2+j_2-2} \cdots \gamma_{i_k+j_k-k},$$

where the integer subscripts satisfy

$$1 \leq i_1 < i_2 < \cdots < i_k, \quad 1 \leq j_1 < j_2 < \cdots < j_k, \quad i_k + j_k \leq k + m.$$ 

In the proof of our main result in Section 5, we shall need the following lemma.

Lemma 2.1 Let $F(s,t), G(s,t)$ be two bivariate polynomials of the same degree $m$ in $t$. Then the Smith normal form of the Bezout matrix $B_t( (s-t)F, (s-t)G)$ is

$$\text{diag}(\alpha_1(s), \cdots, \alpha_{m-1}(s), 0)$$

if and only if the Smith normal form of the Bezout matrix $B_t(F,G)$ is

$$\text{diag}(\alpha_1(s), \cdots, \alpha_{m-1}(s)).$$

Proof. Since the two polynomials $(s-t)F(s,t), (s-t)G(s,t)$ are of the same degree in $t$, we can just consider the symmetric Bezout matrix, which is the coefficient matrix of a Bezoutian. Thus

$$\begin{vmatrix} (s-t)F(s,t) & (s-t)G(s,t) \\ (s-\alpha)F(s,\alpha) & (s-\alpha)G(s,\alpha) \end{vmatrix} = (s-t)(s-\alpha) \begin{vmatrix} F(s,t) & G(s,t) \\ F(s,\alpha) & G(s,\alpha) \end{vmatrix}$$

$$= (s-t)(s-\alpha)(1, t, \cdots, t^{m-1})B_t(F,G)(1, \alpha, \cdots, \alpha^{m-1})^T.$$ 

Let $q_{ij}$ be the entries in the matrix $B_t((s-t)F, (s-t)G), i, j = 1, \cdots, m+1$, and let $p_{ij}$ be the entries in the matrix $B_t(F,G), i, j = 1, \cdots, m$. Also
set \( p_{i,m+1} = p_{m+1,j} = 0 \) for any \( i, j = 1, \cdots, m+1 \). Then from (7), for 
\( i, j = 2, \cdots, m+1 \)

\[
q_{ij} = s^2 p_{ij} - s(p_{i,j-1} + p_{i-1,j}) + p_{i-1,j-1}. \tag{8}
\]

The right-hand side of equation (8) represents row and column operations on the matrix

\[
\begin{pmatrix}
0 & 0 \\
0 & B_t(F,G)
\end{pmatrix}
\]

But by Proposition 2.4, row and column operations do not alter the Smith normal form because by Proposition 2.1, elementary matrices are invertible. Therefore, the Smith normal form of the Bezout matrix \( B_t((s-t)F, (s-t)G) \) is

\[
\begin{pmatrix}
S(B_t(F,G)) & 0 \\
0 & 0
\end{pmatrix},
\]

where \( S(B_t(F,G)) \) is the Smith normal form of the Bezout matrix \( B_t(F,G) \).

Definition 2.3 deals with Smith normal forms of univariate polynomial matrices. But in our work on rational curves, we deal mainly with homogeneous polynomials. Therefore next we provide a definition for Smith normal forms of matrices whose entries are homogeneous polynomials.

**Definition 2.5** Let \( A(t,v) \in M_{m \times m}(\mathbb{R}[t,v]) \). Suppose that the Smith normal form of the matrix \( A(t,1) \) is

\[
\text{diag}(d_m(t), \tilde{d}_m(t)\tilde{d}_{m-1}(t), \cdots, \tilde{d}_m(t)\cdots\tilde{d}_1(t)),
\]

and the Smith normal form of the matrix \( A(1,v) \) is

\[
\text{diag}(\tilde{d}_m(v), \tilde{d}_m(v)\tilde{d}_{m-1}(v), \cdots, \tilde{d}_m(v)\cdots\tilde{d}_1(v)).
\]

Define

\[
d_i(t,v) \triangleq \text{LCM}(\tilde{d}_i(t,v), \hat{d}_i(t,v)), \tag{9}
\]

where \( \tilde{d}_i(t,v) \) and \( \hat{d}_i(t,v) \) are the homogenizations of the polynomials \( \tilde{d}_i(t) \) and \( \hat{d}_i(t) \). Then the Smith normal form of the matrix \( A(t,v) \) is given by

\[
\text{diag}(d_m(t,v), d_m(t,v)d_{m-1}(t,v), \cdots, d_m(t,v)\cdots d_1(t,v)).
\]
The $k$-th determinant factor of a polynomial matrix $A$ is the GCD of the $k \times k$ minors of the matrix $A$. The following result is proved in \[8\].

**Proposition 2.6** \[8\] Suppose that $A(t,v) \in M_{m \times m}(\mathbb{R}[t,v])$. Let $D_k(t,v)$ be the determinant factors of $A(t,v)$, and let $d_k(t,v)$ be defined as in Definition 2.5. Then

\[
D_k(t,v) = d_m(t,v)^k d_{m-1}(t,v)^{k-1} \cdots d_{m-k+1}(t,v)^2 d_{m-k+1}(t,v).
\] (10)

Proposition 2.6 implies that the following definition is equivalent to Definition 2.5.

**Definition 2.6** Suppose that $A(t,v) \in M_{m \times m}(\mathbb{R}[t,v])$. Let $D_k(t,v)$ denote the determinant factors of $A(t,v)$, and let $\alpha_i = \frac{D_i}{D_{i-1}}$, $i = 2, \cdots, m$ and $\alpha_1 = D_1$. Then the the Smith normal form of $A(t,v)$ is given by

\[
\text{diag}(\alpha_1, \cdots, \alpha_m).
\]

Later we shall use the following property of Smith normal forms.

**Corollary 2.1** Let $F(s,u; t,v), G(s,u; t,v)$ be two bihomogeneous polynomials of the same degree $m$ in $t,v$. Then the Smith normal form of the Bezout matrix $B_{t,v}((sv-tu)F, (sv-tu)G)$ is

\[
\text{diag}(\alpha_1(s,u), \cdots, \alpha_{m-1}(s,u), 0)
\]

if and only if the Smith normal form of the Bezout matrix $B_{t,v}(F, G)$ is

\[
\text{diag}(\alpha_1(s,u), \cdots, \alpha_{m-1}(s,u)).
\]

Proof. Let

\[
B_1(s,u) \triangleq B_{t,v}((sv-tu)F, (sv-tu)G), \quad B_2(s,u) \triangleq B_{t,v}(F, G).
\]

By Lemma 2.1,

\[
S(B_1(s,1)) = \text{diag}(S(B_2(s,1)), 0), \quad S(B_1(1,u)) = \text{diag}(S(B_2(1,u)), 0).
\]

Therefore by Definition 2.5

\[
S(B_1(s,u)) = \text{diag}(S(B_2(s,u)), 0).
\]

\[\blacksquare\]
3 Previous results on singularities of rational planar curves

In this section we review some results on the singularities of rational planar curves derived in [11].

Let \( P(t,v) \) be a rational planar curve with a \( \mu \)-basis \( p(t,v), q(t,v) \), and define
\[
F(s,u; t,v) \triangleq \frac{p(s,u) \cdot P(t,v)}{sv - tu}, \\
G(s,u; t,v) \triangleq \frac{q(s,u) \cdot P(t,v)}{sv - tu}.
\]

Notice that \( F(s,u; t,v) \) and \( G(s,u; t,v) \) are polynomials, since \( p(t,v) \cdot P(t,v) = q(t,v) \cdot P(t,v) \equiv 0 \).

Proposition 3.1 [11] A parameter pair \((s^*, u^*; t^*, v^*)\) is a common root of \( F(s,u; t,v) \) and \( G(s,u; t,v) \) if and only if the two parameters \((s^*, u^*)\) and \((t^*, v^*)\) correspond to the same singularity on the curve \( P(t,v) \).

Notation 1 Let \( Q \) be a singular point on the rational planar curve \( P(s,u) \), and let \( (s_i, u_i), i = 1, \ldots, k \) be all the distinct parameters corresponding to \( Q \). We denote the intersection multiplicity of \( F(s,u; t,v) = 0 \) and \( G(s,u; t,v) = 0 \) at the singularity \( Q \) by
\[
I_Q(F,G) \triangleq \sum_{ij} I_{S_{ij}}(F,G),
\]
where \( S_{ij} = (s_i, u_i; s_j, u_j) \), and \( I_{S_{ij}}(F,G) \) is the intersection multiplicity of the two curves \( F(s,u; t,v) = 0 \) and \( G(s,u; t,v) = 0 \) at the parameter pair \((s_i, u_i; s_j, u_j)\).

Proposition 3.2 [11] Let \( \nu_Q^* \) denote the multiplicity of an infinitely near point \( Q^* \) of a singularity \( Q \) on the curve \( P(t,v) \). Then
\[
I_Q(F,G) = \nu_Q^* - 1,
\]
where the sum is taken over all the infinitely near points \( Q^* \) of the point \( Q \) including \( Q \) itself.
Let \( \tilde{p}(s,u) \) and \( \tilde{q}(s,u) \) be any pair of syzygies of the rational curve \( P(s,u) \) that are linearly independent for any parameters corresponding to the point \( Q \), and define

\[
\tilde{F}(s,u; t,v) \triangleq \frac{\tilde{p}(s,u) \cdot P(t,v)}{sv - tu}, \\
\tilde{G}(s,u; t,v) \triangleq \frac{\tilde{q}(s,u) \cdot P(t,v)}{sv - tu}.
\]  

Then the intersection multiplicity at \( Q \) of \( \tilde{F} \) and \( \tilde{G} \) is the same as the intersection multiplicity at \( Q \) of \( F \) and \( G \).

**Proposition 3.3** \cite{11}

\[
I_Q(\tilde{F}, \tilde{G}) = I_Q(F, G).
\]

In \cite{11}, in order to study the intersections of the two algebraic curves \( F(s,u; t,v) = 0 \) and \( G(s,u; t,v) = 0 \), we focus on one singularity \( Q \) of the rational planar curve \( P(t,v) \). We then move the point \( Q \) to the origin \((0,0,1)\) so that the parametrization of the curve \( P(t,v) \) has the form

\[
P(t,v) = (a(t,v)h(t,v), b(t,v)h(t,v), c(t,v)),
\]  

where \( \gcd(a,b) = \gcd(h,c) = 1 \) and \( h(t,v) \) is the inversion formula for the singular point \( Q \). That is, the roots of \( h(t,v) \) provide all the parameter values with proper multiplicity corresponding to the singularity \( Q \). From a pair of obvious syzygies

\[
M(s,u) \triangleq (-b, a, 0), \quad L(s,u) = (c, 0, -ah)
\]  

we construct two additional polynomials

\[
M(s,u; t,v) \triangleq \frac{M(s,u) \cdot P(t,v)}{sv - tu} = \frac{a(s,u)b(t,v) - b(s,u)a(t,v)}{sv - tu}h(t,v), \\
L(s,u; t,v) \triangleq \frac{L(s,u) \cdot P(t,v)}{sv - tu} = \frac{c(s,u)a(t,v)h(t,v) - c(t,v)a(s,u)h(s,u)}{sv - tu}.
\]
By Proposition 3.3 in order to examine $I_Q(F,G)$, we can turn to $I_Q(M,L)$. But $I_Q(M,L)$ breaks into two parts: $I_Q(M,L)$ and $I_Q(h,L)$, where $I_Q(h,L)$ gives all the parameters for the original singular point $Q$ while $I_Q(M,L)$ gives all the parameters corresponding to the infinitely near singularities of the point $Q$. Indeed we have the following results.

**Proposition 3.4** [11] Let $r$ be the order of the singularity $Q$. Then
\[ I_Q(h,L) = r(r-1). \]

**Proposition 3.5** [11] Let $\nu^Q_*$ denote the order of the infinitely near singularity $Q^*$ of $Q$. Then
\[ I_Q(M,L) = \sum_{Q^*} \nu^Q_* (\nu^Q_* - 1), \]
where the sum is taken over all the infinitely near singularities $Q^*$ of $Q$ not including $Q$ itself.

## 4 The conjecture of Chen, Wang and Liu

In order to state the conjecture of Chen, Wang and Liu, we first need to introduce the notion of an *inversion formula*.

**Definition 4.1** Let $(s_i, u_i), i = 1, \ldots, r$ be all the parameters corresponding to the point $Q$ on the curve $P(s,u)$, i.e., $P(s_i, u_i) = Q, i = 1, \ldots, r$. Then a polynomial $h(s,u)$ whose roots are $(s_i, u_i), i = 1, \ldots, r$ is an inversion formula for the point $Q$. Similarly, for an infinitely near singularity $Q^*$ on the $k$-th blow-up curve $P^k(s,u)$, an inversion formula for the point $Q^*$ is a polynomial $h(s,u)$ whose roots are all the parameters on the parametrization $P^k(s,u)$ corresponding to the point $Q^*$, i.e., $P^k(s_i, u_i) = Q^*, i = 1, \ldots, r$. Generally, the inversion formula for $Q^*$ must be a factor of the inversion formula for $Q$.

**Remark 4.1** [4] Let $Q$ be a singularity on a rational planar curve $P(s,u)$ with a $\mu$-basis $p(s,u), q(s,u)$. An inversion formula for $Q$ is given by
\[ h(s,u) = \gcd(p(s,u) \cdot Q, q(s,u) \cdot Q). \]
We are now ready to state the conjecture of Chen, Wang and Liu. Let $P(t,v)$ be a rational planar curve of degree $n$ with a $\mu$-basis $p(s,u)$, $q(s,u)$, and let $B(t,v)$ be the Hybrid Bezout resultant matrix [13] of the two polynomials $p(s,u) \cdot P(t,v)$ and $q(s,u) \cdot P(t,v)$ with respect to $(s,u)$. Suppose that the Smith normal form of the matrix $B(t,v)$ is
\[
\text{diag}(d_{n-\mu}(t,v), d_{n-\mu}(t,v)d_{n-\mu-1}(t,v), \ldots, d_{n-\mu}(t,v) \cdots d_2(t,v), 0).
\]
Then Chen, Wang and Liu state the following conjecture [8].

Conjecture
\[
d_r(t,v) = h_r(t,v) \prod_{i \geq r} \psi_i^r(t,v),
\]
where $h_r(t,v)$ is the inversion formula of all the order $r$ singularities on the curve $P(t,v)$, and $\psi_i^r(t,v)$ is the inversion formula for all the order $r$ infinitely near singularities in the neighborhood of order $i \geq r$ singular points on $P(t,v)$.

By [7] we can turn to prove the following result, which is equivalent to the conjecture of Chen, Wang and Liu.

**Theorem 4.2** Let $B(s,u)$ be the Bezout resultant matrix of the two polynomials $p(s,u) \cdot P(t,v)$ and $q(s,u) \cdot P(t,v)$ with respect to $(t,v)$. Then the Smith normal form of the matrix $B(s,u)$ is
\[
\text{diag}(1, \ldots, 1, d_{n-\mu}(s,u), d_{n-\mu}(s,u)d_{n-\mu-1}(s,u), \ldots, d_{n-\mu}(s,u) \cdots d_2(s,u), 0),
\]
where $d_r(s,u), r = 2, \ldots, n - \mu$ are defined in Equation (17).

**Theorem 4.3** is equivalent to the following result.

**Theorem 4.3** Let $B_{t,v}(F,G)$ be the Bezout resultant matrix of the two polynomials
\[
F(s,u; t,v) = \frac{p(s,u) \cdot P(t,v)}{sv - tu}, \quad G(s,u; t,v) = \frac{q(s,u) \cdot P(t,v)}{sv - tu}
\]
with respect to $(t,v)$. Then the Smith normal form of the matrix $B_{t,v}(F,G)$ is
\[
\text{diag}(1, \ldots, 1, d_{n-\mu}(s,u), d_{n-\mu}(s,u)d_{n-\mu-1}(s,u), \ldots, d_{n-\mu}(s,u) \cdots d_2(s,u)),
\]
where $d_r(s,u), r = 2, \ldots, n - \mu$ are defined in Equation (17).

The equivalence of Theorem 4.2 and Theorem 4.3 follows from Corollary 2.1 Therefore, instead of proving Theorem 4.2 we shall prove Theorem 4.3.
5 The proof of Theorem 4.3

We are going to prove Theorem 4.3 by applying an approach similar to the analysis in [11]. We begin in subsection 5.1 by reducing Theorem 4.3 to the computation of the Smith normal form of the Bezout resultant matrix of two polynomials constructed from a pair of syzygies of the curve. In subsection 5.2, we decompose this Smith normal form into two Smith normal forms, one of which provides all the parameters of the infinitely near singularities while the other provides all the parameters of the original singularity. Then in subsection 5.3, we introduce companion matrices to factor the Bezout resultant matrices and finally we use this factorization to combine the two Smith normal forms together in subsection 5.4 to complete the proof.

5.1 Reducing to the Smith normal form of $B_{t,v}(M,L)$

We are going to show that all the information in the Smith normal form of the Bezout resultant matrix $B_{t,v}(F,G)$ is contained in the Smith normal form of the Bezout resultant matrix $B_{t,v}(M,L)$, where $M(s,u;t,v)$ and $L(s,u;t,v)$ are the two polynomials defined in Equation (15). Before we continue, a word about our notation.

Remark 5.1 For a polynomial $d(s,u)$, we use $d^Q(s,u)$ to denote all the factors of $d(s,u)$ whose roots are parameters corresponding to the point $Q$. For example, if the inversion formula for the point $Q$ is $s^2(s+u)$, and $d(s,u) = s(s+u)^2(s-u)$, then $d^Q(s,u) = s(s+u)^2$. Consequently, if $A$ is a polynomial matrix and

$$S(A) = \text{diag}(f_1, \cdots, f_n),$$

then we shall write

$$S^Q(A) \triangleq \text{diag}(f_1^Q, \cdots, f_n^Q)$$

to denote the Smith normal form of $A$ restricted to $Q$.

Let $\bar{F}, \bar{G}$ be the two polynomials in Equation (12) generated from a pair of syzygies of the curve $P(t,v)$ that are always independent for any parameter $(s,u)$ corresponding to the point $Q$. Then we have the following matrix version for Proposition 3.3

Theorem 5.2

$$S^Q(B_{t,v}(F,G)) = S^Q(B_{t,v}(\bar{F}, \bar{G})).$$
Proof. Let the Smith normal form of the Bezout resultant matrix $B_{t,v} (\tilde{F}, G)$ be
\[
\text{diag}(f_1(s,u), \cdots, f_{n-1}(s,u)),
\]
and let the Smith normal form of the Bezout resultant matrix $B_{t,v} (F,G)$ be
\[
\text{diag}(f_1(s,u), \cdots, f_{n-1}(s,u)).
\]
It suffices to prove that
\[
f_i^Q = \tilde{f}_i^Q.
\]
Since $\tilde{p}(s,u), \tilde{q}(s,u)$ are a pair of syzygies and $p(s,u), q(s,u)$ are a $\mu$-basis, there are polynomials $\alpha(s,u), \beta(s,u), \gamma(s,u), \delta(s,u)$ such that
\[
\tilde{p}(s,u) = \alpha(s,u)p(s,u) + \beta(s,u)q(s,u)
\]
\[
\tilde{q}(s,u) = \gamma(s,u)p(s,u) + \delta(s,u)q(s,u).
\]
Therefore
\[
\tilde{F}(s,u;t,v) = \alpha(s,u)F(s,u;t,v) + \beta(s,u)G(s,u;t,v)
\]
\[
\tilde{G}(s,u;t,v) = \gamma(s,u)F(s,u;t,v) + \delta(s,u)G(s,u;t,v).
\]
Also note that $\deg_{t,v}(F) = \deg_{t,v}(G) = n - 1$, and $\deg_{t,v}(\tilde{F}) = \deg_{t,v}(\tilde{G}) = n - 1$, where $n = \deg(P)$. So the Bezout resultant matrix $B_{t,v}(\tilde{F}, G)$ is the coefficient matrix of the following Bezoutian:
\[
\begin{vmatrix}
\alpha(s,u)F(s,u;t,v) + \beta(s,u)G(s,u;t,v) & \gamma(s,u)F(s,u;t,v) + \delta(s,u)G(s,u;t,v) \\
\alpha(s,u)F(s,u;\bar{t},\bar{v}) + \beta(s,u)G(s,u;\bar{t},\bar{v}) & \gamma(s,u)F(s,u;\bar{t},\bar{v}) + \delta(s,u)G(s,u;\bar{t},\bar{v})
\end{vmatrix}_{t\bar{v} - tv}
\]
\[
= \begin{vmatrix}
F(s,u;t,v) & G(s,u;t,v) \\
F(s,u;\bar{t},\bar{v}) & G(s,u;\bar{t},\bar{v})
\end{vmatrix}_{t\bar{v} - tv}
\]
\[
\begin{vmatrix}
\alpha(s,u) & \gamma(s,u) \\
\beta(s,u) & \delta(s,u)
\end{vmatrix}_{t\bar{v} - tv}.
\]
Hence
\[
B_{t,v}(\tilde{F}, G) = (\alpha(s,u)\delta(s,u) - \beta(s,u)\gamma(s,u))B_{t,v}(F,G),
\]
so
\[
f_i(s,u) = (\alpha(s,u)\delta(s,u) - \beta(s,u)\gamma(s,u))\tilde{f}_i(s,u).
\]
Now \((\alpha(s,u)\delta(s,u) - \beta(s,u)\gamma(s,u)) = 0\) if and only if the two syzygies \(\bar{p}, \bar{q}\) are linear dependent. But by assumption
\[
\gcd(\alpha(s,u)\delta(s,u) - \beta(s,u)\gamma(s,u), h(s,u)) = 1,
\]
where \(h(s,u)\) is the inversion formula for the singularity \(Q\). Therefore
\[
f^Q_1 = \tilde{f}^Q_1.
\]

Hence to prove Theorem 4.3, we need to focus only on the Smith normal form of the Bezout matrix \(B_{t,v}(\bar{F}, \bar{G})\) constructed from another pair of syzygies of the curve. Now suppose \(Q = (0,0,1)\) is an order \(r\) singularity on the curve \(P(s,u)\). Then the degree \(n\) curve \(P(s,u)\) has a parametrization:
\[
P(s,u) = (a(s,u)h(s,u), b(s,u)h(s,u), c(s,u)),
\]
where \(\gcd(a,b) = \gcd(h,c) = 1\) and the roots of \(h(s,u)\) are all the parameters corresponding to the singularity \(Q\). Moreover, we can perform a coordinate transformation so that \(\gcd(a,h) = 1\). As before, we first transfer all the information for the singularity \(Q\) from \(B_{t,v}(F,G)\) to \(B_{t,v}(M,L)\), where \(M(s,u;t,v)\) and \(L(s,u;t,v)\) are defined in Equation (15). Indeed by Theorem 5.2 we get the following result.

**Corollary 5.1**
\[
S^Q(B_{t,v}(F,G)) = S^Q(B_{t,v}(M,L)).
\]

### 5.2 The Smith normal forms of \(B_{t,v}(M, L)\) and \(B_{t,v}(h, L)\)
Since \(M(s,u;t,v) = \overline{M}(s,u;t,v)h(t,v)\), to study the Bezout matrix \(B_{t,v}(M, L)\), we shall next turn to the Smith normal forms of the matrices \(B_{t,v}(M, L)\) and \(B_{t,v}(h, L)\). Note that here we are switching from a Bezout matrix to two Hybrid Bezout matrices because \(\deg_{t,v}(M) = \deg_{t,v}(L) = \deg_{t,v}(\overline{M}) + \deg_{t,v}(h)\).

For brevity we shall assume that the singularity \(Q\) has ordinary infinitely near singularities only in its first neighborhood. The more general cases can be treated similarly (see Remark 5.8 below).
When we blow up the original curve $P(s, u)$ in (19) (see [11] for details), we get the curve

$P^1(s, u) = (a^2h, bc, ca)$.

Let $F^1(s, u; t, v)$ and $G^1(s, u; t, v)$ be the two algebraic curves constructed from a $\mu$-basis for the new curve $P^1(s, u)$ in the same way as we define $F(s, u; t, v)$ and $G(s, u; t, v)$ from a $\mu$-basis for $P(s, u)$. Note that $\deg_{t,v}(F^1) = \deg_{t,v}(G^1) = \deg(P^1) - 1 = 2n - r - 1$, where $r$ is the order of the singularity $Q$.

**Theorem 5.3**

$S^Q(B_{t,v}(F^1, G^1))$

$$\begin{pmatrix}
1 & \cdots & 1 \\
\vdots & & \vdots \\
\psi_r(s, u) & \psi_r(s, u) & \psi_r(s, u) \\
\psi_r(s, u) & \psi_r(s, u) & \psi_r(s, u) \\
\vdots & \vdots & \vdots \\
\prod_{i=2}^r \psi_i(s, u) & \prod_{i=2}^r \psi_i(s, u) & \prod_{i=2}^r \psi_i(s, u) \\
\end{pmatrix},$$

(20)

where $\psi_i(s, u)$ are the inversion formulas for all the order $i$ infinitely near singularities of $Q$.

Proof. Let the Smith normal form of $B_{t,v}(F^1, G^1)$ be

$$\text{diag}(f_{2n-r}, f_{2n-r}f_{2n-r-1}, \cdots, f_{2n-r} \cdots f_2).$$

Then by a result similar to Corollary 4 in [8] (See Theorem A.1 in the Appendix) and Corollary 2.1,

$$\psi_i(s, u) = f_i(s, u),$$

(21)

so

$$\psi_i(s, u) = f_i(s, u).$$

(22)

Suppose that there are $m_i$ infinitely near singularities of order $i$ related to the point $Q$ on the new curve $P^1(s, u)$. Then by Proposition 3.2 and Equation 22,

$$\sum_{Q^*} I_{Q^*}(F^1, G^1) = \sum_{i=1}^r m_i \times i \times (i - 1) = \sum_{i=1}^r \deg(\psi_i) \times (i - 1) \leq \sum_{i=1}^r \deg(f_i^Q) \times (i - 1),$$

(23)
where the sum is taken over all the order $i$ infinitely near singularities of $Q$ not including $Q$ itself. On the other hand, since all the factors $f_i^Q$ contribute to the intersection number $\sum_{Q^*} I_{Q^*}(F^1, G^1)$, and for each $i$, the factor $f_i^Q$ appears in the last $i - 1$ positions of the Smith normal form of the matrix $B_{t,v}(F^1, G^1)$,

$$\sum_{Q^*} I_{Q^*}(F^1, G^1) \geq \sum_{i=1}^r \deg(f_i^Q) \times (i - 1), \quad (24)$$

Hence Equation (23) and Equation (24) yield

$$\deg(f_i^Q) = \deg(\psi_i), \quad i = 1, \ldots, r.$$ 

Therefore by Equation (22)

$$f_i^Q(s, u) = \psi_i(s, u), \quad i = 1, \ldots, r.$$ 

Also since the orders of the infinitely near singularities of $Q$ are less than or equal to the order of $Q$,

$$f_i^Q(s, u) = 1 \quad \text{for} \quad i > r.$$ 

We shall next transfer the information on the singularity $Q$ from the Smith normal form of $B_{t,v}(F^1, G^1)$ to the Smith normal form of $B_{t,v}(M, L)$. 

**Theorem 5.4**

$$S^Q(B_{t,v}(M, L)) = S^Q(B_{t,v}(F^1, G^1))_{(n-1)\times(n-1)},$$

where the subscript $(n-1)\times(n-1)$ means the $(n-1)\times(n-1)$ submatrix in the lower right corner.

Proof. For the blow up curve

$$P^1(s, u) = (a^2h, bc, ca),$$

we have a pair of syzygies

$$S^1(s, u) \triangleq (0, a, -b), \quad T^1(s, u) \triangleq (c, 0, -ah). \quad (25)$$
Construct two polynomials from $S_1(s,u)$ and $T_1(s,u)$:

$$S_1(s,u; t,v) \triangleq \frac{S_1(s,u) \cdot P_1(t,v)}{sv - tu} = \frac{a(s,u)b(t,v) - b(s,u)a(t,v)}{sv - tu} c(t,v),$$

$$T_1(s,u; t,v) \triangleq \frac{T_1(s,u) \cdot P_1(t,v)}{sv - tu} = \frac{c(s,u)a(t,v)h(t,v) - c(t,v)a(s,u)h(s,u)}{sv - tu} a(t,v).$$

Since $S_1, T_1$ are a pair of syzygies for the curve $P_1(s,u)$, by Theorem 5.2

$$S^Q(B_{t,v}(S_1, T_1)) = S^Q(B_{t,v}(F_1, G_1)).$$

Comparing the expressions for $M, L$ in Equation (15) with the expressions for $S_1, T_1$ in Equation (26), and recalling that $\gcd(h,c) = \gcd(a,h) = 1$, we conclude that

$$S^Q(B_{t,v}(M, L)) = S^Q(B_{t,v}(F_1, G_1)).$$

From the previous two theorems we know that the Smith normal form of the Bezout matrix $B_{t,v}(M, L)$ provides the parameters for all the infinitely near singularities of the singular point $Q$. Next we shall show that the Smith normal form of $B_{t,v}(h(t,v), L(s,u; t,v))$ provides all the parameters for the singularity $Q$ itself.

**Theorem 5.5**

$$S^Q(B_{t,v}(h(t), L(s,u; t,v))) = \text{diag}(1, \cdots, 1, h(s,u), \cdots, h(s,u)).$$

Proof. Denote by

$$B \triangleq B_{t,v}((sv - tu)h(t,v), (sv - tu)L).$$

By Lemma 2.11 we only need to prove that

$$S^Q(B) = \text{diag}(1, \cdots, 1, h(s,u), \cdots, h(s,u), 0).$$
Since
\[(sv - tu)L(s, u; t, v) = c(s, u)a(t, v)h(t, v) - c(t, v)a(s, u)h(s, u),\]
and the polynomials \(c(s, u)a(t, v)h(t, v)\) and \(c(t, v)a(s, u)h(s, u)\) are both of degree \(n\) in \((t, v)\), we have
\[B \approx B_1 + B_2,\]
where
\[B_1 = c(s, u)B_{t,v}(h(t, v)(sv - tu), a(t, v)h(t, v)),\]
\[B_2 = a(s, u)h(s, u)B_{t,v}(h(t, v)(sv - tu), c(t, v)),\]
and the notation \(\approx\) means that the first \(r + 1\) rows of the matrices \(B\) and \(B_1 + B_2\) are the same, while the entries in the last \(n - r - 1\) rows of the matrix \(B_1 + B_2\) are equal to twice the corresponding entries in the matrix \(B\). We can examine the Smith normal form of the matrix \(B_1 + B_2\) instead of the Smith normal form of the matrix \(B\) because we are interested only in the polynomials in the Smith normal form, so the multiplication by two does not matter.

Let \(H_k(s, u)\) be an order \(k\) submatrix of the matrix \(B_1 + B_2\). Then \(H_k(s, u) = H_{k,1}(s, u) + H_{k,2}(s, u)\), where \(H_{k,1}(s, u)\) and \(H_{k,2}(s, u)\) are order \(k\) submatrices of \(B_1\) and \(B_2\), so by (16)

\[
\det(H_k(s, u)) = \det(H_{k,1}(s, u)) + \sum_{j=1}^{k-1} \Gamma_n^j \det(H_{1}/H_{2}^j) + \det(H_{k,2}(s, u)), \quad (28)
\]
where \(\Gamma_n^i \det(H_{1}/H_{2}^j)\) is the sum of the combination of determinants in which \(i\) rows of \(H_{1,k}\) are replaced by the corresponding rows of the matrix \(H_{k,2}\).

Note that
\[\text{rank}(B_1) = n - r\]
because \(\gcd(h(t, v)(sv - tu), a(t, v)h(t, v)) = h(t, v)\). Hence
\[
\det(H_{n-r+1,1}(s, u)) \equiv 0.
\]

Also note that every element in \(B_2\) is a multiple of \(a(s, u)h(s, u)\). Therefore from Equation (28) we know that
\[a(s, u)h(s, u)| \det(H_{n-r+1}(s, u)).\]

Since the \(k\)-th determinant factor \(D_k(s, u)\) of the matrix \(B_1 + B_2\) is the GCD of the \(k \times k\) minors of \(B_1 + B_2\),
\[a(s, u)h(s, u)|D_{n-r+1}(s, u). \quad (29)\]
Suppose that the Smith normal form of the matrix $B_1 + B_2$ is

$$S(B_1 + B_2) = \text{diag}(f_n, f_{n-1}, \ldots, f_2, f_1),$$

By Theorem A.1 in the Appendix, for any root $(s, u)$ of the polynomial $h$, $f_i(s, u) \neq 0$ for $i > r$. Hence since $f_{i+1}|f_i$, it follows by Equation (29) that

$$h(s, u)|f_i, \quad i = 1, \ldots, r.$$  

Note that $\det(B) \equiv 0$ because $\gcd((sv-tu)h(t, v), (sv-tu)L) \neq 1$. Moreover by Proposition 3.4 the intersection multiplicity

$$I_Q(h(t, v), L(s, u; t, v)) = r(r-1) = \deg(h) \times (r-1),$$

so

$$S^Q(B) = \text{diag}(1, \ldots, 1, h(s, u), \ldots, h(s, u), 0).$$

Now we have computed the Smith normal forms of the Bezout matrices $B_{t,v}(\overline{M}, L)$ and $B_{t,v}(h, L)$. Since $\det(B_{t,v}(M, L)) = \text{Res}_{t,v}(M, L)$,

$$\det(B_{t,v}(M, L)) = \det(B_{t,v}(\overline{M}, L)) \det(B_{t,v}(h, L)).$$

Unfortunately

$$B_{t,v}(M, L) \neq B_{t,v}(\overline{M}, L)B_{t,v}(h, L),$$

so

$$S(B_{t,v}(M, L)) \neq S(B_{t,v}(\overline{M}, L))S(B_{t,v}(h, L)),$$

Therefore we need some additional preparation to combine $S(B_{t,v}(\overline{M}, L))$ and $S(B_{t,v}(h, L))$ to get $S(B_{t,v}(M, L))$.

### 5.3 Companion matrices and factorization of Hybrid Bezout matrices

In this subsection we shall introduce companion matrices to factor Hybrid Bezout matrices and prepare for the later recombination of $S(B_{t,v}(\overline{M}, L))$ and $S(B_{t,v}(h, L))$.

In this subsection $\mathbb{D}$ is an integral domain of characteristic zero.
Definition 5.1 Let $P(t)$ be a degree $n$ polynomial in $\mathbb{D}[t]$:

$$P(t) = p_0 t^n + p_1 t^{n-1} + \cdots + p_n, \quad p_0 \neq 0.$$ 

The companion matrix of $P(t)$ is defined by:

$$\Delta_P = \begin{pmatrix}
0 & 0 & \cdots & 0 & -p_n \\
p_0 & 0 & \cdots & 0 & -p_{n-1} \\
0 & p_0 & \cdots & 0 & -p_{n-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & p_0 & -p_1
\end{pmatrix}.$$

The following proposition states the well known relationship between companion matrices and the resultant of two univariate polynomials.

Proposition 5.1 \cite{12, 13} Let $P, Q$ be two polynomials in $\mathbb{D}[t]$ with $m = \deg(Q) \leq \deg(P) = n$. Then

$$\text{Res}(Q, P) = p_0^m \det(Q(\Delta_{P/p_0})),$$

where $Q(\Delta_{P/p_0})$ refers to the evaluation of the polynomial $Q$ at the matrix $\Delta_{P/p_0}$.

By Proposition 5.1, $\text{Res}(QR, P) = \text{Res}(Q, P) \text{Res}(R, P)$. Generally, however, $B(QR, P) \neq B(Q, P)B(R, P)$, but the following proposition provides a resultant matrix which can be factored in this way.

Proposition 5.2 \cite{12} Let $P, Q, R$ be polynomials in $\mathbb{D}[t]$ satisfying $\deg(Q) + \deg(R) \leq \deg(P)$. Let

$$H(Q, P) \triangleq J_n \cdot Q(\Delta_P^{t/p_0}) \cdot J_n,$$

where

$$J_n = \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{pmatrix}_{n \times n}.$$

Then

$$H(QR, P) = H(Q, P) \cdot H(R, P).$$
Proof.

\[ H(Q, P) \cdot H(R, P) = J_n \cdot Q(\Delta^t_{P/p_0}) \cdot J_n \cdot J_n \cdot R(\Delta^t_{P/p_0}) \cdot J_n \]
\[ = J_n \cdot Q(\Delta^t_{P/p_0}) \cdot R(\Delta^t_{P/p_0}) \cdot J_n \]
\[ = J_n \cdot QR(\Delta^t_{P/p_0}) \cdot J_n \]
\[ = H(QR, P). \]

The following factorization shows the relationship between Hybrid Bezout resultant matrices and the companion resultant matrices defined in Proposition 5.2.

Proposition 5.3 \[12\] Let \( P, Q \) be two polynomials in \( \mathbb{D}[t] \) with \( m = \deg(Q) \leq \deg(P) = n \). Let \( B(Q, P) \) be the Hybrid Bezout resultant matrix of \( P \) and \( Q \) with respect to \( t \), and let \( H(Q, P) \) be the matrix defined in Equation (30). Then

\[ B(Q, P) = T_m \cdot H(Q, P), \]

where

\[
T_m = \begin{pmatrix}
p_0 & \cdots & p_{m-1} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
p_0 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & \cdots & \cdots 
\end{pmatrix}_{n \times n},
\]

and \( Q(\Delta^t_{P/p_0}) \) refers to the evaluation of the polynomial \( Q \) at the transpose of the matrix \( \Delta_{P/p_0} \).

Theorem 5.6 Let \( f, g, h \) be polynomials in \( \mathbb{D}[t] \) with \( \deg(f) = m, \deg(g) = n \) and \( \deg(h) \geq m + n \). Denote by \( \alpha_k, \beta_k, \gamma_k \) the \( k \)-th invariant factors of the Hybrid Bezout matrices \( B(f, h), B(g, h) \) and \( B(fg, h) \). Then

\[
\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k} \beta_{j_1} \beta_{j_2} \cdots \beta_{j_k} |h_0|^{i_1 + j_1 - 1} \gamma_{i_2 + j_2 - 2} \cdots \gamma_{i_k + j_k - k - l},
\]

where \( h_0 \) is the leading coefficient of the polynomial \( h(t) \), and \( l \) is some non-negative integer.

Proof. By Proposition 5.3

\[ B(f, h) = T_m \cdot H(f, h), \ B(g, h) = T_n \cdot H(g, h), \]

\[ B(fg, h) = T_{m+n} \cdot H(fg, h). \]
and
\[ B(fg, h) = T_{m+n} \cdot H(fg, h). \]

Since by Proposition 5.2,
\[ H(fg, h) = H(f, h) \cdot H(g, h), \]

we have
\[ T_{m+n} \cdot T_n^{-1} \cdot B(f, h) \cdot T_n^{-1} \cdot B(g, h) = B(fg, h). \quad (31) \]

Note that now our equality holds over the \( \mathbb{D}[t, h_0^{-1}] \). The entries in the matrices \( T_n^{-1} \) and \( T_n^{-1} \) have denominators \( h_0^n \) and \( h_0^n \), so multiplying both sides of Equation (31) by \( h_0^m + h_0^n \) to clear these denominators yields
\[ T_{m+n} \cdot (h_0^m T_n^{-1}) \cdot B(f, h) \cdot (h_0^n T_n^{-1}) \cdot B(g, h) = h_0^{m+n} B(fg, h). \quad (32) \]

Now the left hand side of Equation (32) is a product of polynomials. From Proposition 5.5 we get directly that
\[ \alpha_1 \alpha_2 \cdots \alpha_k \beta_j \gamma_j | h_0^l \gamma_{i_1+j_1-1} \gamma_{i_2+j_2-2} \cdots \gamma_{i_k+j_k-k} \]
for some \( l \).

5.4 Joining \( S(B_{t,v}(M, L)) \) and \( S(B_{t,v}(h, L)) \)

Now we are ready to join \( S(B_{t,v}(M(s, u; t, v), L(s, u; t, v))) \) and \( S(B_{t,v}(h(t, v), L(s, u; t, v))) \) together to compute \( S(B_{t,v}(M, L)) \).

**Theorem 5.7**
\[
S^Q(B_{t,v}(M, L)) = \text{diag}(1, \cdots, 1, h(s, u) \psi_r(s, u), h(s, u) \psi_r(s, u) \psi_{r-1}(s, u), \cdots, \\
\quad h(s, u) \prod_{i=2}^{r} \psi_i(s, u)),
\]

where \( \psi_i(s, u) \) is the inversion formula for all the order \( i \) infinitely near singularities of \( Q \).

**Proof.** Denote the Smith normal form of \( B_{t,v}(M, L) \) by
\[ \text{diag}(\bar{g}_1, \cdots, \bar{g}_{n-1}), \]
the Smith normal form of \( B_{t,v}(h, L) \) by
\[ \text{diag}(\bar{g}_1, \cdots, \bar{g}_{n-1}), \]

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and the Smith form of $B_{t,v}(M, L)$ by
\[
\text{diag}(g_1, \cdots, g_{n-1}).
\]

By Theorem 5.4 we know that
\[
\overline{g}_Q^i = \begin{cases} 
1, & \text{for } 1 \leq i < n - r + 1 \\
\prod_{k=2}^{n-i+1} \psi_k(s, u), & \text{for } n - r + 1 \leq i \leq n - 1.
\end{cases}
\] (33)

Also by Theorem 5.5 we know that
\[
\tilde{g}_Q^i = \begin{cases} 
1, & \text{for } 1 \leq i < n - r + 1 \\
h(s, u), & \text{for } n - r + 1 \leq i \leq n - 1.
\end{cases}
\] (34)

By Proposition 2.5, for $i = 1, \cdots, r - 1$ we have
\[
\overline{g}_1 \overline{g}_2 \cdots \overline{g}_{n-r} \overline{g}_{n-r+1} \overline{g}_2 \cdots \overline{g}_{n-r} \overline{g}_{n-r+i} \prod_{k=2}^{\psi_{r-i+1}} h(s, u).
\] (35)

for some non-negative integer $l$, where $l_0(s, u)$ is the leading coefficient of the polynomial $L(s, u; t, v)$ in $(t, v)$. But
\[
l_0(s, u) = lc(ah)c(s, u) - lc(c)h(s, u),
\]
where $lc$ means the leading coefficient of the polynomial. Thus
\[
\gcd(l_0(s, u), h(s, u)) = 1.
\]

Therefore, when restricted to the point $Q$, we have
\[
\overline{g}_1 \overline{g}_2 \cdots \overline{g}_{n-r} \overline{g}_{n-r+1} \overline{g}_2 \cdots \overline{g}_{n-r} \overline{g}_{n-r+i} \prod_{k=2}^{\psi_{r-i+1}} h(s, u).
\] (36)

which by (33) and (34) is equivalent to
\[
\psi_2 \cdots \psi_{r-i+1} h|g_{n-r+i}.
\] (37)

Since $\det(B_{t,v}(M, L)) \det(B_{t,v}(h, L)) = B_{t,v}(M, L)$, we immediately get
\[
\psi_2 \cdots \psi_{r-i+1} h = g_{n-r+i}, \ i = 1, \cdots, r - 1.
\] (38)

up to a constant multiple. The proof is then complete.

By Theorem 5.7 and Corollary 5.1 for all the singularities on the curve $P(s, u)$, we finally have
\[
d_k(s, u) = \prod_Q d_k^Q = h_k(s, u) \prod_{i \geq k} \psi_i^j(s, u).
\]

Theorem 4.3 is now proved.
Remark 5.8  At the beginning of Section 5.2, we assume that all the singularities on the original curve \( P(s, u) \) can be totally resolved after one blow-up of the curve. Actually our proof works in general where an arbitrary number of \( k \) blow-ups are needed to totally resolve all the singularities on the curve \( P(s, u) \). The proof is by induction. Suppose that a singularity \( Q \) on the curve \( P(s, u) \) has infinitely near singularities in the \( k \)-th neighborhood. We start from the \( k \)-th blow-up curve, whose singularities have no infinitely near singularities. Then by our proof, the Smith normal form of the Bezout matrix of the two polynomials \( F^{k-1} \) and \( G^{k-1} \) constructed from a \( \mu \)-basis for the \( k-1 \)-st blow-up curve contains all the singularities on the \( k \)-th blow-up curve (given by \( S(B(M^{k-1}, L^{k-1})) \)) together with all the basic singularities on the \( k-1 \)-st blow-up curve itself (given by \( S(B(h^{k-1}, L^{k-1})) \)). We can continue with this method proceeding by induction until we reach the top of the singularity tree. (See Figure 1).

\[
\begin{align*}
S(B(F,G)) &= S(B(M,L)) \\
S(B(F^1,G^1)) &= S(B(M^1, L)) \\
S(B(F^2,G^2)) &= S(B(M^1, L^1)) \\
&\vdots \\
S(B(F^k,G^k)) &= S(B(M^k, L^k)) \\
S(B(h,L)) &= S(B(h^k,L^k))
\end{align*}
\]

Figure 1: Proof by induction on the height of the singularity tree.
6 A Discussion of the Conjecture

Let \( P(t, v) = (a(t, v), b(t, v), c(t, v)) \) be a rational planar curve with a \( \mu \)-basis \( p(s, u), q(s, u) \), and let \( L_1(s, u) = (c(s, u), 0, -a(s, u)), L_2(s, u) = (0, c(s, u), -b(s, u)) \) be a pair of obvious syzygies of the curve \( P(t, v) \). Then to compute all the singularities on the curve \( P(t, v) \), we can compute the Smith normal form for any one of the following four Bezout resultant matrices.

1. \( B_{s,u}(p(s, u) \cdot P(t, v), q(s, u) \cdot P(t, v)) \)
2. \( B_{t,v}(p(s, u) \cdot P(t, v), q(s, u) \cdot P(t, v)) \)
3. \( B_{s,u}(L_1(s, u) \cdot P(t, v), L_2(s, u) \cdot P(t, v)) \)
4. \( B_{t,v}(L_1(s, u) \cdot P(t, v), L_2(s, u) \cdot P(t, v)) \)

Note that only matrix 1 which is the focus of the conjecture of Chen et al. is a Hybrid Bezout matrix, while the other three matrices are Bezout matrices. Theoretically, it is easier to study Bezout matrices than Hybrid Bezout matrices — in fact, one of the main obstructions to proving the conjecture of Chen, Wang and Liu is to prove an analogue of Theorem 5.2 for Hybrid Bezout matrices; which is why we study the Bezout matrix in 2 rather than the Hybrid Bezout matrix in the conjecture.

The Smith normal forms of these four Bezout matrices are:

1. \( \text{diag}(d_{n-\mu}(t, v), d_{n-\mu}(t, v)d_{n-\mu-1}(t, v), \ldots, d_{n-\mu}(t, v) \cdots d_2(t, v), 0) \)
2. \( \text{diag}(1, \ldots, 1, d_{n-\mu}(s, u), d_{n-\mu}(s, u)d_{n-\mu-1}(s, u), \ldots, d_{n-\mu}(s, u) \cdots d_2(s, u), 0) \)
3. \( c(s, u)\text{diag}(1, \ldots, 1, d_{n-\mu}(s, u), d_{n-\mu}(s, u)d_{n-\mu-1}(s, u), \ldots, d_{n-\mu}(s, u) \cdots d_2(s, u), 0) \)
4. \( c(s, u)\text{diag}(1, \ldots, 1, d_{n-\mu}(s, u), d_{n-\mu}(s, u)d_{n-\mu-1}(s, u), \ldots, d_{n-\mu}(s, u) \cdots d_2(s, u), 0) \).

The conjecture of Chen, Wang and Liu deals with the Hybrid Bezout matrix of \( F(s, u; t, v) \) and \( G(s, u; t, v) \) with respect to the parameter \( (s, u) \), while what we actually proved is a result closely related this conjecture dealing with the larger Bezout matrix of \( F(s, u; t, v) \) and \( G(s, u; t, v) \) with respect to the parameter \( (t, v) \). Geometrically, since by Proposition 3.1 the parameter pair \( (s, u; t, v) \) is an intersection point of the two algebraic
curves $F(s,u,t,v) = 0$ and $G(s,u,t,v) = 0$ if and only if the parameter pair $(t,v,s,u)$ is also an intersection point of the two algebraic curve $F(s,u,t,v) = 0$ and $G(s,u,t,v) = 0$, taking the Hybrid Bezout matrix of $F(s,u,t,v)$ and $G(s,u,t,v)$ with respect to parameter $(s,u)$ or taking the Bezout matrix of $F(s,u,t,v)$ and $G(s,u,t,v)$ with respect to $(t,v)$ should give the same Smith normal form except that the latter matrix has larger size. However, currently we lack a rigorous algebraic proof for the equivalence of these two Smith normal forms. Also note that although the Hybrid Bezout matrix in the conjecture of Chen et al. is smaller than the Bezout matrix in our main theorem — $(n - \mu) \times (n - \mu)$ vs. $n \times n$ — the entries in our Bezout matrix with respect to $(t,v)$ are lower degree polynomials than the polynomial entries in the Hybrid Bezout matrix with respect to $(s,u)$ of Chen et al. — degree $n - \mu$ vs. degree $n$.

We can also compute all the singularities of the curve $P(t,v)$ from the Smith normal forms of the Bezout matrices of $L_1(s,u) \cdot P(t,v)$ and $L_2(s,u) \cdot P(t,v)$ either with respect to the parameter $(s,u)$ or with respect to the parameter $(t,v)$. Note that both $L_1(s,u) \cdot P(t,v) = c(s,u)a(t,v) - a(s,u)c(t,v)$ and $L_2(s,u) \cdot P(t,v) = c(s,u)b(t,v) - b(s,u)c(t,v)$ are antisymmetric with respect to the parameters $(s,u)$ and $(t,v)$. Therefore the Smith normal forms 3 and 4 are the same up to a sign. Here, however, we need to remove the extra factor $c(s,u)$ or $c(t,v)$ from the Smith normal forms 3 or 4 to get the true singularities of the curve $P(t,v)$ because for any root $(s^*, u^*)$ of the polynomial $c(s,u)$, $\gcd(L_1(s^*, u^*) \cdot P(t,v), L_2(s^*, u^*) \cdot P(t,v)) = c(t,v)$.

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Appendix

Let $P(t,v)$ be a rational planar curve with a $\mu$-basis $p(s,u), q(s,u)$. Suppose that all the singularities on the rational planar curve $P(t,v)$ have no infinitely near singularities. Then we have the following result closely related to Corollary 4 in [Chen, Wang and Liu] which can be derived from a very similar approach.

**Theorem A.1** The Smith normal form of the Bezout matrix $B_{t,v}(p(s,u) \cdot P(t,v), q(s,u) \cdot P(t,v))$ is

$$\text{diag}(1, \cdots, 1, h_{n-\mu}(s,u), h_{n-\mu}h_{n-\mu-1}, \cdots, h_{n-\mu} \cdots h_2(s,u), 0),$$

where $h_i(s,u)$ are the products of the inversion formulas of all the order $i$ singularities on the curve $P(t,v)$.

To prove Theorem A.1 we shall prepare with the following theorems. One can compare the outline of our proof with the proofs from Lemma 2 to Theorem 5 in [Chen, Wang and Liu].

**Theorem A.2** Let $Q = P(s_0, u_0)$ be an order $r$ singularity on the curve $P(s,u)$. Then

$$h(s,u) = \gcd(p(s_0, u_0) \cdot P(s,u), q(s_0, u_0) \cdot P(s,u))$$

is an inversion formula for the point $Q$.

Proof. Without loss of generality, we can assume that $Q = (0,0,1)$. Let

$$p(s,u) = (p_1(s,u), p_2(s,u), p_3(s,u)), \quad q(s,u) = (q_1(s,u), q_2(s,u), q_3(s,u)).$$

Then

$$p_3(s_0, u_0) = p(s_0, u_0) \cdot Q = p(s_0, u_0) \cdot P(s_0, u_0) = 0,$$

$$q_3(s_0, u_0) = q(s_0, u_0) \cdot Q = q(s_0, u_0) \cdot P(s_0, u_0) = 0. \quad (E.1)$$

Since $Q = (0,0,1)$, the curve $P(s,u)$ has the parametrization

$$P(s,u) = (a(s,u)h(s,u), b(s,u)h(s,u), c(s,u)),$$

where $\gcd(a,b) = \gcd(h,c) = 1$. Hence by Equation (E.1)

$$\gcd(p(s_0, u_0) \cdot P(s,u), q(s_0, u_0) \cdot P(s,u))$$

$$= \gcd(p_1(s_0, u_0)ah + p_2(s_0, u_0)bh + p_3(s_0, u_0)c,$$

$$q_1(s_0, u_0)ah + q_2(s_0, u_0)bh + q_3(s_0, u_0)c)$$

$$= \gcd(p_1(s_0, u_0)ah + p_2(s_0, u_0)bh, q_1(s_0, u_0)ah + q_2(s_0, u_0)bh)$$

$$= k\lambda \quad \text{for some polynomial } k.$$
We claim that \( k \) is a constant. Otherwise suppose that \((s^*, u^*)\) is a root of \( k \). Then
\[
\begin{align*}
p_1(s_0, u_0)a(s^*, u^*) + p_2(s_0, u_0)b(s^*, u^*) &= 0 \\
q_1(s_0, u_0)a(s^*, u^*) + q_2(s_0, u_0)b(s^*, u^*) &= 0.
\end{align*}
\]
This means that the two vectors \( p(s_0, u_0) \) and \( q(s_0, u_0) \) are linearly dependent, which is impossible. Hence \( k \) is a constant. Therefore, up to a constant multiple
\[
\begin{align*}
h(s, u) &= \gcd(p(s_0, u_0) \cdot P(s, u), q(s_0, u_0) \cdot P(s, u)).
\end{align*}
\]

In the following theorems we denote the Bezout matrix \( B_{t,v}(p(s, u) \cdot P(t, v), q(s, u) \cdot P(t, v)) \) by \( B(s, u) \).

**Theorem A.3** The point \( P(s_0, u_0) \) is an order \( r \) singular point if and only if \( \text{rank}(B(s_0, u_0)) = n - r \), where \( r = \deg(P) \).

Proof. By Theorem A.2 the point \( P(s_0, u_0) \) is an order \( r \) singular point if and only if \( \deg(\gcd(p(s_0, u_0) \cdot P(s, u), q(s_0, u_0) \cdot P(s, u))) = r \). By the standard properties of Bezout resultant matrices, the degree of this gcd is \( r \) if and only if \( \text{rank}(B(s_0, u_0)) = n - r \).

Recall that the order \( k \) determinant factor of a matrix is the GCD of all the order \( k \) minors of the matrix. Let \( D_k \) be the determinant factors of the matrix \( B(s, u) \), \( k = 1, \ldots, n \).

**Theorem A.4** Let \( h(s, u) \) be an inversion formula of an order \( r \) singular point \( Q \) on the curve \( P(t, v) \). Then \( h(s, u)|D_{n-r+1} \).

Proof. Without loss of generality, we can assume that \( Q = (0, 0, 1) \). Let
\[
\begin{align*}
p(s, u) &= (p_1(s, u), p_2(s, u), p_3(s, u)), & q(s, u) &= (q_1(s, u), q_2(s, u), q_3(s, u)).
\end{align*}
\]
Then by Remark 4.1
\[
\gcd(p_3, q_3) = \gcd(p(s, u) \cdot Q, q(s, u) \cdot Q) = h(s, u). \quad (E.3)
\]
Now let
\[
P(t, v) \triangleq \sum_{i=0}^{n} (\lambda_{1i}, \lambda_{2i}, \lambda_{3i}) t^i u^{n-1},
\]
where \((\lambda_1, \lambda_2, \lambda_3)\) are constant vectors. Then

\[
p(s, u) \cdot P(t, v) = \sum_{i=0}^{n}(\lambda_1p_1 + \lambda_2p_2 + \lambda_3p_3)t^iv^{n-i} = \sum_{i=0}^{n}a_i(s, u)t^iv^{n-i}
\]

\[
q(s, u) \cdot P(t, v) = \sum_{i=0}^{n}(\lambda_1q_1 + \lambda_2q_2 + \lambda_3q_3)t^iv^{n-i} = \sum_{i=0}^{n}b_i(s, u)t^iv^{n-i}.
\]

By the construction of the Bezout matrices, the elements \(b_{ij}\) in the Bezout matrix \(B_{t,v}(p(s, u) \cdot P(t, v), q(s, u) \cdot P(t, v))\) are [13]:

\[
b_{ij} = \sum_{k=1}^{m_{ij}} \alpha_{j+k-1}\beta_{i-k} - \alpha_{i-k}\beta_{j+k-1}, \quad (E.4)
\]

where \(m_{ij} = \min\{i, n+1-j\}\). Therefore by Equations \((E.3)\) and \((E.4)\), the Bezout matrix \(B(s, u)\) can be written as

\[
B(s, u) = h(s, u)G(s, u) + H(s, u), \quad (E.5)
\]

where \(G(s, u)\) and \(H(s, u)\) are matrices of size \(n \times n\), and \(H(s, u)\) has the form

\[
H(s, u) = (q_2(s, u)p_1(s, u) - p_2(s, u)q_1(s, u))H_0, \quad (E.6)
\]

where \(H_0\) is a constant matrix. Next we shall examine the rank of \(H(s, u)\). To do this we need to examine the rank of the constant matrix \(H_0\).

Let \((s_0, u_0)\) be a root of \(h(s, u)\). Then by Theorem A.3 and Equation \((E.5)\),

\[
\text{rank}(H(s_0, u_0)) = \text{rank}(B(s_0, u_0)) = n - r. \quad (E.7)
\]

Since the \(\mu\)-basis elements \(p(s_0, u_0) = (p_1(s_0, u_0), p_2(s_0, u_0), 0)\) and \(q(s_0, u_0) = (q_1(s_0, u_0), q_2(s_0, u_0), 0)\) are linearly independent,

\[
q_2(s_0, u_0)p_1(s_0, u_0) - p_2(s_0, u_0)q_1(s_0, u_0) \neq 0.
\]

Hence Equation \((E.7)\) and Equation \((E.6)\) yield \(\text{rank}(H_0) = n - r\). Therefore, the polynomial matrix \(H(s, u)\) has rank \(n - r\).

Let \(B_{n-r+1}(s, u)\) be a size \(n - r + 1\) submatrix of \(B(s, u)\). Then

\[
B_{n-r+1}(s, u) = h(s, u)G_{n-r+1}(s, u) + H_{n-r+1}(s, u).
\]

Therefore,

\[
\det(B_{n-r+1}) = h^{n-r+1}\det(G_{n-r+1}) + \cdots + \det(H_{n-r+1}).
\]
Since \( \text{rank}(H(s,u)) \equiv n - r \), \( \det(H_{n-r+1}) \equiv 0 \). Therefore, \( h|\det(B_{n-r+1}) \).

Hence \( h|D_{n-r+1} \).

Once we have Theorem A.4, we can apply the same approach as in the rest of proofs of [Chen, Wang and Liu] to derive Theorem A.1.

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