SIMPLE TRANSITIVE 2-REPRESENTATIONS FOR TWO NONFIAT 2-CATEGORIES OF PROJECTIVE FUNCTORS

V. Mazorchuk$^1$ and X. Zhang$^2$ UDC 512.5

It is shown that any simple transitive 2-representation of the 2-category of projective endofunctors for the quiver algebra of $k(\bullet \to \bullet)$ and for the quiver algebra of $k(\bullet \to \bullet \to \bullet)$ is equivalent to a cell 2-representation.

1. Introduction and Description of the Results

Classification problems are interesting and important problems in the classical representation theory. Thus, the classifications of various classes of simple or indecomposable modules over different classes of algebras play a significant role in the development and applications of the contemporary representation theory.

Higher representation theory is a recent direction of representation theory originating from the papers [2, 3, 18, 19]. It is of especial interest in the higher representation theory to study the so-called finitary 2-categories because they are natural 2-analogs of finite-dimensional algebras. The initial abstract study of finitary 2-categories and the corresponding 2-representation theory was carried out in [12–17, 20].

As an outcome of this investigation, one interesting and important class of 2-representations called simple transitive 2-representations was defined in [16]. These 2-representations are natural 2-analogs of ordinary simple modules over algebras. Therefore, the problem of classification of simple transitive 2-representations is natural and interesting. In several cases, it turns out that simple transitive 2-representations can be classified; see, e.g., various results in [5, 16, 17, 22, 23]. We also refer the reader to [6, 7, 11, 21] for the related questions and applications. In particular, in [7], the classification of simple transitive 2-representations for the 2-category of Soergel bimodules over the coinvariant algebra of symmetric group was essentially used for the classification of projective functors in the parabolic category $\mathcal{O}$ for $\mathfrak{sl}_n$.

The most basic example of 2-category is the 2-category $\mathcal{C}_A$ of projective functors for a finite-dimensional algebra $A$ over an algebraically closed field $k$ defined in [12] (Subsection 7.3). In [14, 17], it was shown that categories of the form $\mathcal{C}_A$ essentially exhaust a natural class of “simple” finitary 2-categories with weak involutions. For these 2-categories, it was shown [16, 17] that simple transitive 2-representations are exactly the cell 2-representations defined in [12]. The existence of a weak involution on a 2-category restricts the classification result to the case where $A$ is a self-injective algebra.

The aim of the present paper is to classify simple transitive 2-representations of $\mathcal{C}_A$ for the smallest possible nonself-injective algebra, namely, for the path algebra $A$ of the quiver $1 \to 2$, over an algebraically closed field $k$. It turns out that our approach can be also extended, with a much higher amount of technical work, to the quiver algebra of $k(\bullet \to \bullet \to \bullet)$, where, as usual, the dotted arrow depicts the corresponding zero relation.

Our main result is the following theorem (we refer the reader to Section 2 for the detailed description of all definitions):

$^1$ Uppsala University, Uppsala, Sweden.
$^2$ East China Normal University, Shanghai, China.

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Theorem 1. For $A = \mathbb{k}(\bullet \rightarrow \bullet)$ or $A = \mathbb{k}(\bullet \rightarrow \bullet)$, any simple transitive 2-representation of the 2-category $\mathcal{C}_A$ is equivalent to a cell 2-representation.

Despite the fact that the formulation of Theorem 1 is quite similar to the corresponding statement for the case where $A$ is self-injective considered in [16, 17], our approach to the proof is noticeably different because the general approach outlined in [16, 17] does not work. However, our approach has numerous similarities with the approach proposed in [23] and is, for the most part, based on the careful analysis of all possible cases.

In Section 2, we collect all necessary preliminaries for the 2-representation theory of the 2-category $\mathcal{C}_A$.

In Sect. 3, we prove some general results about 2-representations of $\mathcal{C}_A$ under the additional assumption that the algebra $A$ possesses a nonzero projective injective module.

In Sections 4, 5, and 6, we present the proof of Theorem 1 for the case where $A = \mathbb{k}(\bullet \rightarrow \bullet)$. In more detail, the preliminary information about $\mathcal{C}_A$ for $A = \mathbb{k}(\bullet \rightarrow \bullet)$ can be found in Section 4. Section 5 contains combinatorial results for some integer matrices allowing one to formulate three essentially different cases investigated in what follows. In Section 6, we prove Theorem 1 for $A = \mathbb{k}(\bullet \rightarrow \bullet)$.

In Sections 7, 8, 9, and 10, we present the proof of Theorem 1 in the second case for the algebra $A = \mathbb{k}(\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet)$. In more detail, in Section 7 we present the preliminaries for $\mathcal{C}_A$. Sections 8 and 9 are devoted to the determination of an integer matrix that captures the combinatorics of a faithful simple transitive 2-representation of $\mathcal{C}_A$. Finally, in Section 10, we complete the proof of Theorem 1 for the algebra $\mathbb{k}(\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet)$.

2. 2-Category $\mathcal{C}_A$ and its 2-Representations

2.1. Notation and Conventions. Throughout the paper, we work over an algebraically closed field $\mathbb{k}$ and abbreviate $\mathcal{C}_k$ by $\mathcal{C}$. Unless otherwise explicitly stated, a module is understood as a left module. We compose maps from the right to the left. For a 1-morphism $F$, by $\text{id}_F$ we denote the identity 2-morphism for $F$.

2.2. 2-Category $\mathcal{C}_A$. We refer the reader to [9–11] for the general information about 2-categories. A 2-category is a category enriched over the monoidal category $\text{Cat}$ of small categories.

Let $A$ be a connected basic finite-dimensional $\mathbb{k}$-algebra and let $C$ be a small category equivalent to $A\text{-mod}$. Consider a 2-category $\mathcal{C}_A$ (that depends on $C$) defined as follows:

$\mathcal{C}_A$ has a single object $\mathbb{i}$ identified with $C$;

the 1-morphisms in $\mathcal{C}_A$ are endofunctors of $C$ isomorphic to functors given by tensoring with $A\text{-}A$-bimodules from the additive closure of both $A A$ and $A A \otimes A A$;

2-morphisms in $\mathcal{C}_A$ are natural transformations of functors.

The 2-category $\mathcal{C}_A$ is finitary in the sense of [12] (Subsection 2.2).

2.3. 2-Representations $\mathcal{C}_A$. We now consider the 2-category $\mathcal{C}_A$-afmod of all finitary 2-representations of $\mathcal{C}_A$. In this 2-category,

an object is a strict additive functorial action of $\mathcal{C}_A$ denoted by $M$ or similarly, on an additive, idempotent split, Krull–Schmidt, $\mathbb{k}$-linear category $M(\mathbb{i})$ with finitely many isomorphism classes of indecomposable objects and finite-dimensional morphism spaces;

1-morphisms are 2-natural transformations;

2-morphisms are modifications.
We refer the reader to [14] for details. Two 2-representations are called equivalent provided that there is a 2-natural transformation between them whose restriction to each object is an equivalence of categories.

We also consider a 2-category \( C_A - \text{mod} \) defined similarly by using functorial action on categories equivalent to the module categories of finite-dimensional \( k \)-algebras. There is the diagrammatically defined Abelianization 2-functor

\[
\overline{\cdot} : \ C_A - \text{afmod} \rightarrow \ C_A - \text{mod}.
\]

Given a functorial action of \( C_A \) on some \( M(\bar{1}) \), as above, the 2-functor \( \overline{\cdot} \) defines (componentwise) a functorial action of \( C_A \) upon the Abelian category \( \overline{M(\bar{1})} \) whose objects are diagrams of the form \( X \rightarrow Y \) over \( M(\bar{1}) \) and morphisms are given by the obvious commutative squares in which one mods out the projective homotopy relations. We refer the reader to [13] (Subsection 4.2) for details.

A finitary 2-representation of \( C_A \) is called transitive provided that, for any indecomposable objects \( X \) and \( Y \) in \( M(\bar{1}) \), there is a 1-morphism \( F \) in \( C_A \) such that \( Y \) is isomorphic to a direct summand of \( M(F) \) \( X \).

A transitive 2-representation \( M \) is called simple provided that \( M(\bar{1}) \) does not have nonzero proper \( C_A \)-invariant ideals.

For simplicity, we often use the module notation \( F \) \( X \) instead of the representation notation \( M(F) \) \( X \).

Note that every strict monoidal category can be regarded as a 2-category with one object. With this identification, the proposed notion of 2-representation corresponds to the notion of strict monoidal functor. It is possible to treat 2-representations (with fixed target) as homomorphism 2-categories in an appropriate version of the 3-category \( 2\text{-Cat} \) (here, our choice of the level of strictness for transformations corresponds to strong transformations in the language of [9] (see [14], Subsection 2.3, for details)).

### 2.4. Cells in \( C_A \)

Let \( 1 = e_1 + e_2 + \ldots + e_n \) be a primitive decomposition of \( 1 \in A \). Up to isomorphism, the indecomposable 1-morphisms in \( C_A \) are given by tensoring with \( A A_A \) or with \( A A e_i \otimes e_j A_A \), where \( i, j = 1, 2, \ldots, n \). We fix a representative \( F_0 \) in the isomorphism class of 1-morphisms that corresponds to tensoring with \( A A_A \). For \( i, j = 1, 2, \ldots, n \), we fix a representative \( F_{ij} \) in the isomorphism class of 1-morphisms that corresponds to tensoring with \( A A e_i \otimes e_j A_A \). The set of isomorphism classes of indecomposable 1-morphisms in \( C_A \) has the natural structure of multisemigroup (see [13], Section 3) and [8]. The combinatorics of this structure is encoded into the so-called left, right, and two-sided cells (see [13], Section 3). Two 1-morphisms \( F \) and \( G \) belong to the same left cell provided that there exist 1-morphisms \( K_1 \) and \( K_2 \) such that \( F \) is isomorphic to a direct summand of \( K_1 \circ G \) and \( G \) is isomorphic to a direct summand of \( K_2 \circ F \). The right and two-sided cells are defined similarly by using the compositions on the right or on both sides, respectively.

For \( C_A \), the two-sided cells are

\[
\mathcal{J}_0 := \{F_0\} \quad \text{and} \quad \mathcal{J} := \{F_{ij} : i, j = 1, 2, \ldots, n\}.
\]

The two-sided cell \( \{F_0\} \) is both a left cell and a right cell. The other left cells are

\[
\{F_{ij} : i = 1, 2, \ldots, n\}, \quad j = 1, 2, \ldots, n.
\]

The other right cells are

\[
\{F_{ij} : j = 1, 2, \ldots, n\}, \quad i = 1, 2, \ldots, n.
\]

As usual, we have

\[
F_{ij} \circ F_{st} = F_{it} \oplus \dim(e_j A e_i).
\]
Further, we set
\[ F := \bigoplus_{i,j=1}^n F_{ij} \]
and note that
\[ F \circ F \cong F^\oplus \dim(A). \quad (2) \]

All 1-morphisms in the additive closure of F are called projective endofunctors of \( \mathcal{C} \) and, similarly, for \( A\text{-mod} \).

As usual, we say that a pair \((F_{ij}, F_{st})\) of 1-morphisms is a pair of adjoint 1-morphisms provided that there exist 2-morphisms
\[ \alpha : F_{ij} \circ F_{st} \to F_0 \quad \text{and} \quad \beta : F_0 \to F_{st} \circ F_{ij} \]
such that
\[ (\alpha \circ_0 \text{id}_{F_{ij}}) \circ_1 (\text{id}_{F_{ij}} \circ_0 \beta) = \text{id}_{F_{ij}} \quad \text{and} \quad (\text{id}_{F_{st}} \circ_0 \alpha) \circ_1 (\beta \circ_0 \text{id}_{F_{st}}) = \text{id}_{F_{st}}. \]

The 2-category \( \mathcal{C}_A \) is \( J \)-simple in a sense that any nonzero two-sided 2-ideal of \( \mathcal{C}_A \) contains the identity 2-morphisms for all 1-morphisms given by projective endofunctors (see [1, 13]).

2.5. **Cell 2-Representations.** The first example of finitary 2-representation of \( \mathcal{C}_A \) is the principal 2-representation
\[ \mathbf{P} := \mathcal{C}_A(i, -). \]
This representation has a unique maximal \( \mathcal{C}_A \)-invariant ideal and the corresponding quotient is the cell 2-representation \( C_L \), where \( L = \{ F_0 \} \).

For any other left cell \( L \), the additive closure of elements in \( L \) gives a 2-subrepresentation of \( \mathbf{P} \). This 2-subrepresentation also has a unique maximal \( \mathcal{C}_A \)-invariant ideal and the corresponding quotient is the cell 2-representation \( C_L \). The indicated cell 2-representation is equivalent to the defining action of \( \mathcal{C}_A \) on the category \( A\text{-proj} \) of projective objects in \( A\text{-mod} \) (see [12] for details).

2.6. **Matrices in the Grothendieck Group.** Let \( M \) be a finitary 2-representation of \( \mathcal{C}_A \) and let \( X_1, X_2, \ldots, X_k \) be a fixed complete and irredundant list of representatives of the isomorphism classes of indecomposable objects in \( M(i) \). For a 1-morphism \( G \) in \( \mathcal{C}_A \), by \([G]\) we denote a \( k \times k \) matrix with nonnegative integer coefficients in which, for \( i, j = 1, 2, \ldots, k \), the coefficient corresponding to the intersection of the \( i \)th row with the \( j \)th column gives the number of indecomposable direct summands of \( M(G) X_j \), which are isomorphic to \( X_i \). Note that
\[ [G \oplus H] = [G] + [H] \quad \text{and} \quad [G \circ H] = [G][H]. \]

2.7. **Action on Simple Transitive 2-Representations.** The following statement was proved in [16] (Lemma 12):

**Lemma 1.** Let \( M \) be a simple transitive 2-representation of \( \mathcal{C}_A \). Then, for any nonzero object \( X \in M(i) \), the object \( FX \) is projective in \( M(i) \).

The following statement was proved in [16] (Lemma 13):

**Lemma 2.** Let \( B \) be a finite-dimensional \( k \)-algebra and let \( G \) be an exact endofunctor of \( B\text{-mod} \). Assume that \( G \) sends each simple object of \( B\text{-mod} \) into a projective object. Then \( G \) is a projective functor.
3. Existence of a Projective-Injective Module Guarantees the Exactness of the Action

3.1. Exactness of the Action of Some Projective Functors. Let $\mathcal{M}$ be a simple transitive 2-representation of $\mathcal{C}_A$. Consider its Abelianization $\overline{\mathcal{M}}$. For $\overline{\mathcal{M}}(\mathbf{1})$, let $L_1, L_2, \ldots, L_k$ be a complete and irredundant list of representatives of the isomorphism classes of simple objects. For $i \in \{1, 2, \ldots, k\}$, by $P_i$ we denote the indecomposable projective cover of $L_i$. At the same time, by $I_i$ we denote the indecomposable injective envelope of $L_i$.

**Lemma 3.** Let $Q$ be a finite-dimensional $\mathbb{k}$-algebra and let $K$ be a right exact endofunctor of $Q$-mod. Then the following conditions are equivalent:

(a) The functor $K$ sends projective objects into projective objects.

(b) The right adjoint $K'$ of $K$ is exact.

**Proof.** By adjunction, for a projective generator $P \in Q$-mod, we have a natural isomorphism

$$\text{Hom}_Q(KP, -) \cong \text{Hom}_Q(P, K')_-. \quad (3)$$

If $KP$ is projective, then the left-hand side of (3) is exact. Hence, the right-hand side is also exact. As $P$ is a projective generator, the functor $\text{Hom}_Q(P, -)$ detects any nonzero homology. This means that $K'$ is exact. Therefore, (a) implies (b).

Conversely, assume that $K'$ is exact. Then the right-hand side of (3) is exact. Hence, the left-hand side is exact. This means that $KP$ is projective. Therefore, (b) implies (a). The claim follows.

**Lemma 4.** Assume that there exist $s, t \in \{1, 2, \ldots, n\}$ such that the left $A$-modules $Ae_s$ and $\text{Hom}_A(e_t A, \mathbb{k})$ are isomorphic. Then, for any $i \in \{1, 2, \ldots, n\}$, the pair $(F_{it}, F_{si})$ is a pair of adjoint 1-morphisms.

**Proof.** The functor $F_{it}$ is given by tensoring with the $A$-$A$-bimodule $Ae_i \otimes e_t A$. The right adjoint of this functor is thus the functor $\text{Hom}_A(Ae_i \otimes e_t A, -)$. By the computation performed in [12] (Subsection 7.3), the exact functor $\text{Hom}_A(Ae_i \otimes e_t A, -)$ is isomorphic to the functor of tensoring with the $A$-$A$-bimodule

$$\text{Hom}_A(e_t A, \mathbb{k}) \otimes e_i A.$$

By assumption, the injective $A$-module

$$I_t \cong \text{Hom}_A(e_t A, \mathbb{k})$$

is isomorphic to the projective $A$-module $Ae_s$. Therefore, $\text{Hom}_A(e_t A, \mathbb{k}) \otimes e_i A$ is isomorphic to $Ae_s \otimes e_i A$. This means that $F_{si}$ is isomorphic to the right adjoint of $F_{it}$. The claim follows.

**Corollary 1.** Assume that there exist $s, t \in \{1, 2, \ldots, n\}$ such that the left $A$-modules $Ae_s$ and $\text{Hom}_A(e_t A, \mathbb{k})$ are isomorphic. Then, for any $i \in \{1, 2, \ldots, n\}$ and any 2-representation $\mathcal{N}$ of $\mathcal{C}_A$, the pair $(\mathcal{N}(F_{it}), \mathcal{N}(F_{si}))$ is a pair of adjoint functors.

**Proof.** This fact directly follows from Lemma 4 and the definitions.

**Corollary 2.** Assume that there exist $s, t \in \{1, 2, \ldots, n\}$ such that the left $A$-modules $Ae_s$ and $\text{Hom}_A(e_t A, \mathbb{k})$ are isomorphic. Then, for any $i \in \{1, 2, \ldots, n\}$ and any finitary 2-representation $\mathcal{N}$ of $\mathcal{C}_A$, the functor $\mathcal{N}(F_{si})$ is exact.

**Proof.** This follows from the definitions by combining Lemma 3 with Corollary 1.
3.2. Auxiliary Lemma.

Lemma 5. Let $Q$ be a finite dimensional $k$-algebra and let $K$, $H$, and $G$ be three endofunctors of $Q$-mod. Assume that:

(a) $H$ is a projective functor;
(b) $K$ is right exact;
(c) $K$ sends projective objects to projective objects;
(d) $K \circ H \hookrightarrow G$.

Then $G$ is a projective functor.

Proof. By assumption (a), the functor $H$ is given by tensoring with the $Q$-$Q$-bimodule $X \otimes Y$ for some projective left $Q$-module $X$ and some projective right $Q$-module $Y$. By assumption (b), $K$ is given by tensoring with some $Q$-$Q$-bimodule $V$. By using assumption (d), the $Q$-$Q$-bimodule specifying the functor $G$ can be represented as follows:

$$V \otimes_Q (X \otimes Y) \cong (V \otimes_Q X) \otimes Y.$$  \hspace{1cm} (4)

By assumption (c), $V \otimes_Q X$ is a projective left $Q$-module. This implies that (4) is a projective $Q$-$Q$-bimodule and, hence, $G$ is a projective functor.

3.3. Exactness of the Action.

Proposition 1. Assume that there exist $s, t \in \{1, 2, \ldots, n\}$ such that the left $A$-modules $Ae_s$ and $\text{Hom}_k(e_tA, k)$ are isomorphic. Let $M$ be a simple transitive 2-representation of $\mathcal{C}_A$. Then the functor $M(F)$ is exact.

Proof. Let $B$ be a finite dimensional algebra such that $M(\mathcal{i})$ is equivalent to $B$-mod.

For $i \in \{1, 2, \ldots, n\}$, we consider a 1-morphism $F_{si}$. By Corollary 2, the functor $M(F_{si})$ is exact. By Lemma 1, $M(F_{si})$ sends any object in $M(\mathcal{i})$ to a projective object. Therefore, by Lemma 2, $M(F_{si})$ is a projective endofunctor of $B$-mod.

Thus, for any $j \in \{1, 2, \ldots, n\}$, we get

$$F_{js} \circ F_{si} \cong F_{ji}^{\otimes k},$$

where $k = \dim(e_s Ae_s) > 0$. Therefore, $M(F_{ji}^{\otimes k})$ is a projective functor for $B$-mod by Lemma 5. By additivity, $M(F_{ji})$ is a projective functor also for $B$-mod. In particular, $M(F_{ji})$ is exact. The claim follows.

4. The algebra $k(\bullet \rightarrow \bullet)$

Let $k$ be an algebraically closed field. Denote by $A$ the path algebra, over $k$, of the quiver $1 \overset{\alpha}{\rightarrow} 2$. The algebra $A$ has a basis $e_1$, $e_2$, and $\alpha$ and the multiplication table $(x, y) \mapsto x \cdot y$ is given by

| $x \setminus y$ | $e_1$ | $e_2$ | $\alpha$ |
|----------------|-------|-------|----------|
| $e_1$         | $e_1$ | 0     | 0        |
| $e_2$         | 0     | $e_2$ | $\alpha$|
| $\alpha$      | $\alpha$ | 0     | 0        |
Note that $e_1Ae_2 = 0$, as $A$, contains no paths from 2 to 1. We also note that the left $A$-modules $Ae_1$ and $\text{Hom}_k(e_2A, k)$ are isomorphic.

Let $C$ be a small category equivalent to $A$-$\text{mod}$. Consider the corresponding finitary 2-category $\mathcal{C}_A$. Up to isomorphism, the indecomposable 1-morphisms in $\mathcal{C}_A$ are $F_0$ and $F_{ij}$, where $i, j = 1, 2$. Note that relation (2) for $A$ reads as $F \circ F = F^{\oplus 3}$. In view of (1), the table of compositions for the functors $F_{ij}$ (up to isomorphisms) is as follows:

| $\circ$ | $F_{11}$ | $F_{12}$ | $F_{21}$ | $F_{22}$ |
|---------|----------|----------|----------|----------|
| $F_{11}$ | $F_{11}$ | $F_{12}$ | 0        | 0        |
| $F_{12}$ | $F_{11}$ | $F_{12}$ | $F_{11}$ | $F_{12}$ |
| $F_{21}$ | $F_{21}$ | $F_{22}$ | 0        | 0        |
| $F_{22}$ | $F_{21}$ | $F_{22}$ | $F_{21}$ | $F_{22}$ |

(5)

We set $\mathcal{J}_0 := \{F_0\}$ and $\mathcal{J} := \{F_{ij} : i, j = 1, 2\}$. Note that the 2-category $\mathcal{C}_A$ is not weakly fiat in a sense of [13, 17] because the algebra $A$ is not self-injective.

As $\mathcal{C}_A$ is $\mathcal{J}$-simple and $A$ has the trivial center, the only proper nonzero quotient of $\mathcal{C}_A$ contains just the identity 1-morphism (up to isomorphism) and its scalar endomorphisms (cf. [14]). Therefore, this quotient is fiat with strongly regular $\mathcal{J}$-classes and, hence, it has a unique, up to equivalence, simple transitive 2-representation, namely, $C_{\mathcal{L}_0}$, where $\mathcal{L}_0 = \mathcal{J}_0$ (see [16], Theorem 18). This means that, in order to prove Theorem 1 for $A$, it is sufficient to consider the faithful 2-representations of $\mathcal{C}_A$.

From the formula

$$\text{Hom}_{A\times A}(Ae_i \otimes e_j A, Ae_s \otimes e_t A) \cong e_i Ae_s \otimes e_t Ae_j,$$

for all $i, j, s, t \in \{1, 2\}$, we get the following table of $\text{Hom}_{\mathcal{C}_A}(X, Y)$ (up to isomorphism), where $X$ and $Y$ are indecomposable 1-morphisms:

| $X \setminus Y$ | $F_{11}$ | $F_{12}$ | $F_{21}$ | $F_{22}$ |
|-----------------|----------|----------|----------|----------|
| $F_{11}$        | $k$      | $k$      | 0        | 0        |
| $F_{12}$        | 0        | $k$      | 0        | 0        |
| $F_{21}$        | $k$      | $k$      | $k$      | $k$      |
| $F_{22}$        | 0        | $k$      | 0        | $k$      |

(7)

5. Integer Matrices for $k(\bullet \to \bullet)$

5.1. Integer Matrices Satisfying $M^2 = 3M$. In this section, we classify all square matrices $M$ with positive integer coefficients satisfying $M^2 = 3M$.

**Proposition 2.** Let $M$ be a $k \times k$ matrix, for some $k$, with positive integer coefficients, satisfying $M^2 = 3M$. Then $M$ is one of the following matrices:

$$M_1 := (3), \quad M_2 := \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \quad M_3 := \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \quad M_4 := \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$
\[ M_5 := \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad M_6 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]

**Proof.** Clearly, we have \( M_i^2 = 3M_i \), for each \( i = 1, 2, 3, 4, 5, 6 \). Hence, it is necessary to show that no other square matrix with positive integer coefficients satisfies \( M^2 = 3M \).

Let \( M \) be a \( k \times k \) matrix, for some \( k \), with positive integer coefficients satisfying \( M^2 = 3M \). Then \( M \) is diagonalizable (as \( x^2 - 3x \) has no multiple roots) and the only possible eigenvalues for \( M \) are 0 and 3. From the Perron–Frobenius theorem, it follows that the Perron–Frobenius eigenvalue 3 must have multiplicity one. Therefore, \( M \) has rank one and trace three. Since all entries in \( M \) are positive integers, we get \( k \leq 3 \).

If \( k = 1 \), then, clearly, \( M = M_1 \).

If \( k = 3 \), then all diagonal entries in \( M \) are 1. Since all \( 2 \times 2 \) minors in \( M \) should have determinant zero and positive integer entries, all entries in \( M \) are equal to 1 and, thus, \( M = M_6 \).

If \( k = 2 \), then the two diagonal entries in \( M \) are 1. Since the determinant of \( M \) is equal to zero, the remaining two entries are also 1 and 2. Therefore, \( M = M_i \) for some \( i \in \{2, 3, 4, 5\} \).

Proposition 2 is proved.

### 5.2. The Matrix \([F]\) for a Faithful Simple Transitive 2-Representation.

Let \( M \) be a finitary, simple, transitive, and faithful 2-representation of \( \mathcal{C}_A \). Let \( M := [F] \) be the matrix of \( M(F) \) and, for \( i, j = 1, 2 \), let \( M_{ij} := [F_{ij}] \) be the matrix of \( M(F_{ij}) \). Note that

\[ M = M_{11} + M_{12} + M_{21} + M_{22}. \]

The symmetric group \( S_k \) acts upon \( \text{Mat}_{k \times k}(\mathbb{Z}) \) by conjugation with permutation of matrices. This action corresponds to the permutation of basis elements, whenever the matrix upon which we act represents an endomorphism of some free \( \mathbb{Z} \)-module. We call this action the **permutation action**.

**Proposition 3.** In order to respect the multiplication rule (5), up to the permutation action, there exist the following three possibilities:

(a) \( M = M_2 \) and

\[
M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix};
\]

(b) \( M = M_3 \) and

\[
M_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};
\]

(c) \( M = M_6 \) and

\[
M_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]
\[ M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \]

**Proof.** As \( M \) is simple, transitive, and faithful, we conclude that \( M \) has positive integer entries. As \( F \circ F = F^{\oplus 3} \), we have \( M = M_i \) for some \( i \in \{1, 2, 3, 4, 5, 6\} \) by Proposition 2. Since \( M \) is the sum of four nonzero matrices (corresponding to all \( F_{ij} \)) each of which has nonnegative integer entries, we get \( M \neq M_1 \). The case \( M = M_4 \) is reduced to the case \( M = M_3 \) by swapping the basis elements. The case \( M = M_5 \) is reduced to the case \( M = M_2 \) by swapping the basis elements. It is easy to check that the cases (a), (b), and (c) listed in the formulation of the proposition satisfy (5).

Assume that \( M = M_2 \). Note, in view of (5), that \( F_{11}, F_{12} \) and \( F_{22} \) are idempotent, while \( F_{21} \) is nilpotent. Therefore, \( M_{11}, M_{12}, M_{22} \) must have nonzero diagonals, while the diagonal for \( M_{21} \) should be equal to zero. It follows from the equality \( M_{11}M_{22} = 0 \) that \( M_{11} \) and \( M_{22} \) cannot have common diagonal entries. In any case, this means that \( M_{12} \) has the nonzero diagonal entry in the left upper corner. We first assume the following:

\[
M_{11} = \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}.
\]

From \( M_{11}M_{21} = 0 \), we get

\[
M_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix}.
\]

However, this contradicts the equality \( M_{11}M_{12} = M_{12} \). We now assume that

\[
M_{11} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}.
\]

From \( M_{11}M_{21} = M_{11}M_{22} = 0 \), we find

\[
M_{11} = \begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & 1 \\ * & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & 0 \\ * & 1 \end{pmatrix}.
\]

Finally, from \( M_{12}M_{22} = M_{12} \), we obtain

\[
M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.
\]

Assume that \( M = M_3 \). In view of (5), we note that \( F_{11}, F_{12}, \) and \( F_{22} \) are idempotent, while \( F_{21} \) is nilpotent. Therefore, \( M_{11}, M_{12}, \) and \( M_{22} \) must have nonzero diagonals, while the diagonal for \( M_{21} \) should be equal to zero. It follows from \( M_{11}M_{22} = 0 \) that \( M_{11} \) and \( M_{22} \) cannot have common diagonal entries. In any case, this means
that $M_{12}$ has the nonzero diagonal entry in the left upper corner. We first assume the following:

$$M_{11} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}.$$  

From $M_{11}M_{21} = 0$, we get

$$M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & * \\ 0 & 1 \end{pmatrix}.$$  

However, this contradicts the relation $M_{21}M_{12} = M_{22}$. Thus, we assume the following:

$$M_{11} = \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}.$$  

Finally, from $M_{11}M_{21} = M_{11}M_{22} = 0$, we conclude that

$$M_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$  

From $M_{11}M_{12} = M_{12}$, we obtain

$$M_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$  

**Assume that** $M = M_6$. In view of (5), we note that $F_{11}$, $F_{12}$, and $F_{22}$ are idempotent, while $F_{21}$ is nilpotent. Therefore $M_{11}$, $M_{12}$, and $M_{22}$ must have nonzero diagonals, while the diagonal of $M_{21}$ must be equal to zero. Therefore, up to permutations of basis vectors, we can assume that

$$M_{11} = \begin{pmatrix} 1 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 0 & * & * \\ * & 1 & * \\ * & * & 0 \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 1 \end{pmatrix}.$$  

Thus, in view of the fact that $M_{11}M_{21} = M_{11}M_{22} = 0$ we conclude that the last column of $M_{11}$ must be zero and the first rows of both $M_{21}$ and $M_{22}$ must also be equal to zero. Since the $M_{ij}$'s add up to $M$, the rightmost
element in the first row of $M_{12}$ must be equal to 1:

\[
M_{11} = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 0 & * & 1 \\ * & 1 & * \\ * & * & 0 \end{pmatrix},
\]

\[
M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & * \\ * & * & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ * & * & 1 \end{pmatrix}.
\]

It follows from the relation $M_{11}M_{12} = M_{12}$ that the second row of $M_{11}$ cannot be zero. This yields

\[
M_{11} = \begin{pmatrix} 1 & * & 0 \\ 1 & 0 & 0 \\ * & * & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 0 & * & 1 \\ 0 & 1 & * \\ * & * & 0 \end{pmatrix},
\]

\[
M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ * & * & 1 \end{pmatrix}.
\]

Further, $M_{11}M_{12} = M_{12}$ implies that the first and the second rows of $M_{12}$ must coincide. Moreover, the first element in the third row of $M_{12}$ must be zero and, in addition, the third row in $M_{11}$ and, thus, also in $M_{12}$ must be equal to zero:

\[
M_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 1 \end{pmatrix}.
\]

Hence, from $M_{21}M_{11} = M_{21}$, we get

\[
M_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Finally, from \( M_{12} M_{21} = M_{11} \), we obtain

\[
M_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
M_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.
\]

Proposition 3 is proved.

6. Proof of Theorem 1 for \( \mathbb{k} \bullet \rightarrow \bullet \)

Let \( M \) be a simple transitive 2-representation of \( \mathcal{C}_A \). Let \( B \) be a basic finite dimensional algebra such that \( M(i) \) is equivalent to \( B \)-proj.

As the left \( A \)-modules \( Ae_1 \) and \( \text{Hom}_k(e_2 A, \mathbb{k}) \) are isomorphic, it follows from Proposition 1 that the functor \( \mathbf{M}(F) \) is exact. Thus, Lemmas 1 and 2 imply that \( \mathbf{M}(F) \) is a projective endofunctor of \( B \text{-mod} \).

Case 1. Assume that \( M = M_3 \) and, hence, the \( M_{ij} \)’s are given by Proposition 3(b). Let \( \mathbf{M} \) be the Abelianization of \( M \). As usual, we write \( P_1 \) and \( P_2 \) for the indecomposable projectives in \( \mathbf{M}(\mathbf{i}) \) and \( L_1 \) and \( L_2 \) for their respective simple tops. Let \( e_1 \) and \( e_2 \) be the corresponding primitive idempotents in \( B \). For \( i, j = 1, 2 \), by \( G_{ij} \) we denote an endofunctor of \( \mathbf{M}(\mathbf{i}) \) that corresponds to tensoring with \( B e_i \otimes e_j B \).

From the form of \( M_{21} \), we see that \( F_{21} \) acts via \( G_{12} \). Similarly, \( F_{22} \) acts via \( G_{11} \). From the matrices \( M_{21} \) and \( M_{22} \), it follows that

\[
[P_1 : L_1] = 1, \quad [P_1 : L_2] = 0, \quad [P_2 : L_1] = 0, \quad [P_2 : L_2] = 1.
\]

This means that \( B \cong \mathbb{k} \oplus \mathbb{k} \). Therefore, all \( G_{ij} \) are isomorphisms between the corresponding \( \mathbb{k} \text{-mod} \) components. Thus, it directly follows from the matrices \( M_{12} \) and \( M_{21} \) that there are no nonzero homomorphisms from \( F_{21} \) to \( F_{12} \). This contradicts (7) and, therefore, Case 1 cannot occur.

Case 2. Assume that \( M = M_2 \) and, hence, the \( M_{ij} \)’s are given by Proposition 3(a). Let \( \mathbf{M} \) be the Abelianization of \( M \). As usual, we write \( P_1 \) and \( P_2 \) for the indecomposable projectives in \( \mathbf{M}(\mathbf{i}) \) and \( L_1 \) and \( L_2 \) for their respective simple tops. Let \( e_1 \) and \( e_2 \) be the corresponding primitive idempotents in \( B \). For \( i, j = 1, 2 \), by \( G_{ij} \) we denote an endofunctor of \( \mathbf{M}(\mathbf{i}) \) that corresponds to tensoring with \( B e_i \otimes e_j B \).

From the form of \( M_{11} \), we see that \( F_{11} \) acts via \( G_{11} \). Similarly, \( F_{21} \) acts via \( G_{21} \).
From the form of $M_{12}$, we see that $F_{12}$ acts either via $G_{12}$, or via $G_{11}$, or via $G_{12} \oplus G_{11}$. However, we already know that the matrix of $G_{11}$ is $M_{11}$. This gives us the following possibilities: $G_{12}$ or $G_{12} \oplus G_{11}$ for $F_{12}$.

Assume that $F_{12}$ acts via $G_{12} \oplus G_{11}$. We already know the matrix of $G_{11}$ and, hence, the matrix of $G_{12}$ is
\[
\begin{pmatrix}
0 & 1 \\
0 & 0 \\
\end{pmatrix}.
\]

This fact and the matrix $M_{11}$ imply that
\[
G_{11} P_1 \cong P_1, \quad G_{11} P_2 = 0, \quad G_{12} P_1 = 0, \quad G_{12} P_2 \cong P_2.
\]

Therefore,
\[
[P_1 : L_1] = 1, \quad [P_1 : L_2] = 0, \quad [P_2 : L_1] = 0, \quad [P_2 : L_2] = 1
\]
and we have
\[
B \cong k \oplus k.
\]

This leads to the same contradiction as in Case 1 above. Therefore, $F_{12}$ acts via $G_{12}$. Similarly, it is possible to show that $F_{22}$ acts via $G_{22}$.

From the matrices obtained for all $G_{ij}$’s, it follows that the Cartan matrices of $A$ and $B$ coincide, which means that $A$ and $B$ are isomorphic (this is a special feature for our case but the algebra $A$ is very small and, therefore, this claim is clear). Furthermore, all $F_{ij}$’s act via the corresponding $G_{ij}$. Thus, it follows from the ordinary arguments (see [16], Proposition 9) that $\mathcal{M}$ is equivalent to a cell 2-representation of $\mathcal{C}_A$.

**Case 3.** Assume that $M = M_6$ and, thus, the $M_{ij}$’s are given by Proposition 3(c). Let $\overline{M}$ be the Abelianization of $M$. As usual, we write $P_1$, $P_2$, and $P_3$ for the indecomposable projectives in $M(\mathfrak{a})$ and $L_1$, $L_2$, and $L_3$ for their respective simple tops. Let $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$ be the corresponding primitive idempotents in $B$. For $i, j = 1, 2, 3$, by $G_{ij}$ we denote an endofunctor of $\overline{M}(\mathfrak{a})$ that corresponds to tensoring with $B\epsilon_i \otimes \epsilon_j B$.

From the form of $M_{21}$, we see that $F_{21}$ acts via $G_{31}$. From the form of $M_{11}$, we see that $F_{11}$ acts via $G_{11} \oplus G_{21}$. This yields
\[
[P_1 : L_1] = 1, \quad [P_2 : L_1] = [P_3 : L_1] = 0.
\]

From the form of $M_{22}$, we see that $F_{22}$ acts either via $G_{32}$, or via $G_{33}$, or via $G_{32} \oplus G_{33}$. In the last case, we conclude that the matrices of $G_{32}$ and $G_{33}$ are, respectively,
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
\]

Thus, we get
\[
[P_2 : L_2] = 1, \quad [P_1 : L_2] = [P_3 : L_2] = 0
\]
and
\[
[P_3 : L_3] = 1, \quad [P_1 : L_3] = [P_2 : L_3] = 0.
\]
This implies that
\[ B \cong k \oplus k \oplus k \]
and leads to a contradiction similar to the contradiction in Case 1.

**Subcase 3.1.** Assume that \( F_{22} \) acts via \( G_{32} \). This yields
\[ [P_2 : L_2] = [P_3 : L_2] = 1, \quad [P_1 : L_2] = 0. \]
In view of the equality \([P_1 : L_2] = 0\), we obtain \( \epsilon_2 \epsilon_1 = 0 \). This means that
\[ \text{Hom}_{B-B}(B \epsilon_3 \otimes \epsilon_1 B, B \epsilon_3 \otimes \epsilon_2 B) = 0, \]
i.e., \( \text{Hom}(G_{31}, G_{32}) = 0 \) but this contradicts the inequality \( \text{Hom}_F(F_{21}, F_{22}) \neq 0 \); see (7).

**Subcase 3.2.** Assume that \( F_{22} \) acts via \( G_{33} \). This implies that
\[ [P_3 : L_3] = [P_2 : L_3] = 1, \quad [P_1 : L_3] = 0. \]
From \([P_1 : L_3] = 0\), we obtain \( \epsilon_3 \epsilon_1 = 0 \). This means that
\[ \text{Hom}_{B-B}(B \epsilon_3 \otimes \epsilon_1 B, B \epsilon_3 \otimes \epsilon_3 B) = 0, \]
i.e., \( \text{Hom}(G_{31}, G_{33}) = 0 \) but this contradicts \( \text{Hom}_F(F_{21}, F_{22}) \neq 0 \); see (7). The proof is now completed.

7. **The Algebra** \( k(\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet)/(\beta \alpha) \)

Let \( k \) be an algebraically closed field. By \( A \) we denote the path algebra, over \( k \), for the quiver
\[ k \left( \begin{array}{ccc} 1 & \alpha & 2 \\ \alpha & 3 \end{array} \right) \]
modulo the relations \( \beta \alpha = 0 \).

The algebra \( A \) has the basis \( e_1, e_2, e_3, \alpha, \) and \( \beta \) and the multiplication table \((x, y) \mapsto x \cdot y\) of the following form:

| \(x\setminus y\) | \(e_1\) | \(e_2\) | \(e_3\) | \(\alpha\) | \(\beta\) |
|-----------------|------|------|------|------|------|
| \(e_1\)        | \(e_1\) | 0    | 0    | 0    | 0    |
| \(e_2\)        | 0    | \(e_2\) | 0    | \(\alpha\) | 0    |
| \(e_3\)        | 0    | 0    | \(e_3\) | 0    | \(\beta\) |
| \(\alpha\)     | \(\alpha\) | 0    | 0    | 0    | 0    |
| \(\beta\)      | 0    | \(\beta\) | 0    | 0    | 0    |

Note that \( e_1 A e_2 = 0, \ e_1 A e_3 = 0, \ e_2 A e_3 = 0, \) and \( e_3 A e_1 = 0 \). We also note that the left \( A \)-modules \( A e_1 \) and \( \text{Hom}_k(e_2 A, k) \) are isomorphic and the left \( A \)-modules \( A e_2 \) and \( \text{Hom}_k(e_3 A, k) \) are isomorphic, as well.
Let \( C \) be a small category equivalent to \( A\text{-mod} \). Consider the corresponding finitary 2-category \( \mathcal{C}_A \). Up to isomorphism, the indecomposable 1-morphisms in \( \mathcal{C}_A \) are \( F_0 \) and \( F_{ij} \), where \( i, j = 1, 2, 3 \). Note that relation (2) for \( A \) has the form

\[
F \circ F \cong F^0 F^5.
\]

Using (1), we get the following table of compositions for the functors \( F_{ij} \) (up to isomorphisms):

| \( \circ \) | \( F_{11} \) | \( F_{12} \) | \( F_{13} \) | \( F_{21} \) | \( F_{22} \) | \( F_{23} \) | \( F_{31} \) | \( F_{32} \) | \( F_{33} \) |
|---|---|---|---|---|---|---|---|---|---|
| \( F_{11} \) | \( F_{11} \) | \( F_{12} \) | \( F_{13} \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( F_{12} \) | \( F_{11} \) | \( F_{12} \) | \( F_{13} \) | \( F_{11} \) | \( F_{12} \) | \( F_{13} \) | 0 | 0 | 0 |
| \( F_{13} \) | 0 | 0 | 0 | \( F_{11} \) | \( F_{12} \) | \( F_{13} \) | \( F_{11} \) | \( F_{12} \) | \( F_{13} \) |
| \( F_{21} \) | \( F_{21} \) | \( F_{22} \) | \( F_{23} \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( F_{22} \) | \( F_{21} \) | \( F_{22} \) | \( F_{23} \) | \( F_{21} \) | \( F_{22} \) | \( F_{23} \) | 0 | 0 | 0 |
| \( F_{23} \) | 0 | 0 | 0 | \( F_{21} \) | \( F_{22} \) | \( F_{23} \) | \( F_{21} \) | \( F_{22} \) | \( F_{23} \) |
| \( F_{31} \) | \( F_{31} \) | \( F_{32} \) | \( F_{33} \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( F_{32} \) | \( F_{31} \) | \( F_{32} \) | \( F_{33} \) | \( F_{31} \) | \( F_{32} \) | \( F_{33} \) | 0 | 0 | 0 |
| \( F_{33} \) | 0 | 0 | 0 | \( F_{31} \) | \( F_{32} \) | \( F_{33} \) | \( F_{31} \) | \( F_{32} \) | \( F_{33} \) |

(8)

We set \( \mathcal{J}_0 := \{ F_0 \} \) and \( \mathcal{J} := \{ F_{ij} : i, j = 1, 2, 3 \} \) and note that the 2-category \( \mathcal{C}_A \) is not weakly fiat in the sense of [13, 17] because the algebra \( A \) is not self-injective.

Since \( \mathcal{C}_A \) is \( \mathcal{J} \)-simple and \( A \) has the trivial center, the only proper nonzero quotient of \( \mathcal{C}_A \) contains just the identity 1-morphism (up to isomorphisms) and its scalar endomorphisms (cf. [14]). Therefore, this quotient is fiat with strongly regular \( \mathcal{J} \)-classes and, hence, it has a unique, up to equivalence, simple transitive 2-representation, namely \( C_{\mathcal{L}_0} \), where \( \mathcal{L}_0 = \mathcal{J}_0 \) (see [16], Theorem 18). This means that, in order to prove Theorem 1 for \( A \), it is sufficient to consider the \textit{faithful} 2-representations of \( \mathcal{C}_A \).

From (6), we get the following table of \( \text{Hom}_{\mathcal{C}_A(1)}(X, Y) \) (up to isomorphisms), where \( X \) and \( Y \) are indecomposable 1-morphisms:

| \( X \setminus Y \) | \( F_{11} \) | \( F_{12} \) | \( F_{13} \) | \( F_{21} \) | \( F_{22} \) | \( F_{23} \) | \( F_{31} \) | \( F_{32} \) | \( F_{33} \) |
|---|---|---|---|---|---|---|---|---|---|
| \( F_{11} \) | \( k \) | \( k \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( F_{12} \) | 0 | \( k \) | \( k \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( F_{13} \) | 0 | 0 | \( k \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( F_{21} \) | \( k \) | \( k \) | 0 | \( k \) | \( k \) | 0 | 0 | 0 | 0 |
| \( F_{22} \) | 0 | \( k \) | \( k \) | 0 | \( k \) | \( k \) | 0 | 0 | 0 |
| \( F_{23} \) | 0 | 0 | \( k \) | 0 | \( k \) | 0 | 0 | 0 | 0 |
| \( F_{31} \) | 0 | 0 | 0 | \( k \) | \( k \) | 0 | \( k \) | \( k \) | 0 |
| \( F_{32} \) | 0 | 0 | 0 | 0 | \( k \) | \( k \) | 0 | \( k \) | \( k \) |
| \( F_{33} \) | 0 | 0 | 0 | 0 | 0 | \( k \) | 0 | 0 | \( k \) |
8. Integer Matrices for $k(\begin{array}{c} \alpha \\ \beta \end{array} \rightarrow \begin{array}{c} \bullet \\ \bullet \end{array})/(\beta \alpha)$

8.1. Integer Matrices Satisfying $M^2 = 5M$. In this section, we classify all square matrices $M$ with positive integer coefficients satisfying the relation $M^2 = 5M$.

Proposition 4. Let $M$ be, for some $k$, a $k \times k$ matrix with positive integer coefficients satisfying $M^2 = 5M$. Then, up to the permutation action, $M$ is one of the following matrices:

\[
N_1 := (5), \quad N_2 := \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}, \quad N_3 := \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}, \quad N_4 := \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \\
N_5 := \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}, \quad N_6 := \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}, \quad N_7 := \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}, \quad N_8 := \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}, \\
N_9 := \begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}, \quad N_{10} := \begin{pmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \quad N_{11} := \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \quad N_{12} := \begin{pmatrix} 2 & 4 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \\
N_{13} := \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \quad N_{14} := \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}, \quad N_{15} := \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad N_{16} := \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad N_{17} := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \\
N_{18} := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad N_{19} := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
\]

We now mention an important distinction from Proposition 2: in order to make our list shorter, Proposition 4 gives classification only up to the permutation action.

Proof. Clearly, we have $N_i^2 = 5N_i$, for each $i = 1, 2, \ldots, 16$. Hence, it is necessary to show that any other square matrix with positive integer coefficients satisfying $M^2 = 5M$ can be reduced to one of the matrices presented above by the permutation action.

Let $M$ be a $k \times k$ matrix, for some $k$, with positive integer coefficients satisfying $M^2 = 5M$. Then $M$ is diagonalizable (as $x^2 - 5x$ has no multiple roots) and the only possible eigenvalues for $M$ are 0 and 5. It follows from the Perron–Frobenius theorem that the Perron–Frobenius eigenvalue 5 must have multiplicity one. Therefore, $M$ has rank one and trace five. Since all entries in $M$ are positive integers, we get $k \leq 5$. By using the permutation action, we can assume that the entries on the main diagonal of $M$ weakly decrease from the top left corner to the bottom right corner.
If $k = 1$, then, clearly, $M = N_1$.

If $k = 2$, then the diagonal of $M$ is either $(4, 1)$ or $(3, 2)$. In the first case, as the determinant of $M$ is zero, the two remaining entries are either 2 and 2 or 4 and 1. This gives $M = N_2$, $M = N_3$, or $M = N_4$. In the second case, as the determinant of $M$ is zero, the two remaining entries are either 2 and 3 or 1 and 6. This gives $M = N_5$, $M = N_6$, $M = N_7$, or $M = N_8$.

If $k = 3$, then the diagonal of $M$ is either $(3, 1, 1)$ or $(2, 2, 1)$. In the first case, $M$ has rank one and, hence, the determinant of any $2 \times 2$ minor in $M$ is equal to zero. This means that all entries that are neither in the first row, nor in the first column are equal to 1. If the first row contains more than one entry different from 1, then all entries in this row are equal to 3, and we get $M = N_{10}$. If the first column contains more than one entry different from 1, then all entries in this column are equal to 3, and we get $M = N_9$.

In the second case, we can write

$$M = \begin{pmatrix} 2 & m_{12} & m_{13} \\ m_{21} & 2 & m_{23} \\ m_{31} & m_{32} & 1 \end{pmatrix}.$$ 

Then

$$m_{32}m_{23} = 2, \quad m_{31}m_{13} = 2, \quad \text{and} \quad m_{21}m_{12} = 4.$$ 

Hence, both $(m_{32}, m_{23})$ and $(m_{31}, m_{13})$ are in $\{ (1, 2), (2, 1) \}$. We can choose them independently and, therefore, the fact that $M$ has rank one uniquely determines the pair $(m_{21}, m_{12})$. This gives us

$$M = N_{11}, \quad M = N_{12}, \quad \text{and} \quad M = N_{13}$$

and also the possibility

$$M = N_{12}' := \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix},$$

which reduces to $M = N_{12}$ by the permutation action.

If $k = 4$, then the diagonal of $M$ is $(2, 1, 1, 1)$. As $M$ has rank one, the determinant of any $2 \times 2$ minor in $M$ is equal to zero. This means that all entries that are neither in the first row, nor in the first column are equal to 1. If the first row contains more than one entry different from 1, then all entries in this row are equal to 2, and we get $M = N_{16}$. If the first column contains more than one entry different from 1, then all entries in this column are equal to 2, and we obtain $M = N_{15}$.

If $k = 5$, then all diagonal entries in $M$ are equal to 1. Since all $2 \times 2$ minors in $M$ must have determinants equal to zero and positive integer entries, this means that all entries in $M$ are equal to 1 and, thus, $M = N_{14}$.

Proposition 4 is proved.

### 8.2. Filtering “Easy Cases” Out.

Let $M$ be a finitary, simple, transitive, and faithful 2-representation of $\mathcal{C}_A$. Let $M := [F]$ be the matrix of $M(F)$ and, for $i, j = 1, 2, 3$, let $M_{ij} := [F_{ij}]$ be the matrix of $M(F_{ij})$. We have $M_{ij} \neq 0$, for all $i, j = 1, 2, 3$. By Proposition 4, up to the permutation action, we obtain $M = N_i$ for some $i \in \{ 1, 2, \ldots, 16 \}$ as in Proposition 4. Note that the trace of $M$ is five.

As usual, the “position $(i, j)$” is called the intersection of the $i$th row and the $j$th column of the matrix.
In what follows, we assume that \( M(i) \) is equivalent to \( B \text{-mod} \) for some basic algebra \( B \). Let \( L_1, L_2, \ldots, L_k \) be a complete and irredundant list of representatives of the isomorphism classes of simple objects in \( \overline{M}(i) \). For \( i \in \{1, 2, \ldots, k\} \), by \( P_i \) we denote the indecomposable projective cover of \( L_i \) and by \( I_i \) we denote the indecomposable injective envelope of \( L_i \). The matrices \( M_{ij} \) are given with respect to the indicated fixed ordering of isomorphism classes of simple objects.

**Lemma 6.**

(i) All diagonal elements in \( M_{13}, M_{21}, M_{31}, \text{ and } M_{32} \) are zero.

(ii) Each of the matrices \( M_{11}, M_{12}, M_{22}, M_{23}, \text{ and } M_{33} \) has exactly one entry equal to 1 on the diagonal and all other diagonal entries are equal to zero.

**Proof.** It follows from (8) that \( F_{ij} \) is idempotent if and only if

\[
(i, j) \in \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}
\]

and \( F_{ij}^2 = 0 \), otherwise. As the trace of a nonzero idempotent with nonnegative coefficients is nonzero, each idempotent \( M_{ij} \) has the trace not smaller than one. Since the trace of \( M \) is equal to five, we conclude that the trace of all idempotents \( M_{ij} \) is equal to 1. This proves claim (ii). Claim (i) follows from claim (ii) because \( M \) is the sum of the \( M_{ij} \)'s.

**Corollary 3.** The matrix \( M \) cannot be equal to \( N_i \), where \( i \in \{1, 2, \ldots, 10\} \).

**Proof.** Each matrix \( N_i \), where \( i \in \{1, 2, \ldots, 10\} \), contains a diagonal element larger than or equal to 3. If \( M = N_i \) is possible, then this diagonal element is nonzero for at least three idempotents \( M_{ij} \). However, in this case, any product of any two matrices of this kind must be nonzero. At the same time, it follows from (8) that, for any three different idempotents \( F_{ij} \), one of the products of two of these elements is equal to zero. The obtained contradiction completes the proof.

**8.3. Auxiliary Adjunction.** In what follows, we need the following simple observations:

**Lemma 7.** Let \( D \) be a finite-dimensional algebra and let \( (G, H) \) be an adjoint pair of right exact endofunc- tors of \( D \text{-mod} \). Also let \( L \) and \( L' \) be simple \( D \)-modules and let \( P \) and \( P' \) be their corresponding indecomposable projective covers. Assume that \( L' \) appears in the top of \( GP \). Then \( HP' \neq 0 \).

**Proof.** By adjunction, we have

\[
0 \neq \text{Hom}_B(GP, L') \cong \text{Hom}_B(P, HL'),
\]

which implies that \( HL' \neq 0 \). As \( H \) is right exact, this yields \( HP' \neq 0 \).

**Lemma 8.** We have the following pairs of adjoint 1-morphisms in \( \mathcal{C}_A \):

\[
(F_{33}, F_{23}), \quad (F_{23}, F_{22}), \quad (F_{22}, F_{12}), \quad (F_{12}, F_{11}).
\]

**Proof.** Since both the left \( A \)-modules \( Ae_1 \) and \( \text{Hom}_k(e_2 A, k) \) are isomorphic and the left \( A \)-modules \( Ae_2 \) and \( \text{Hom}_k(e_3 A, k) \) are also isomorphic, the claim follows from Lemma 4.
8.4. Idempotent Integral Matrices of Rank One. We now recall \{see [4] (Theorem 2)\} that, up to the permutation action, the idempotent matrices of rank one with nonnegative integer entries have the form

\[
\begin{pmatrix}
0 & v & uw^t \\
0 & 1 & w^t \\
0 & 0 & 0
\end{pmatrix},
\]

where 0 denotes the null matrix (of the appropriate size) and \(v\) and \(w\) are arbitrary vectors with nonnegative integer entries. In particular, if the diagonal entry 1 is in the \(i\)th row, then the entire matrix can be written as the product of its \(i\)th column by its \(i\)th row.

8.5. Filtering Matrices \(N_{14}, N_{15}\) and \(N_{16}\) Out.

Proposition 5. The matrix \(M\) cannot be equal to \(N_{14}, N_{15}\), or \(N_{16}\).

Proof. Assume that the diagonals of the matrices \(M_{11}\) and \(M_{12}\) are different. This means that \(M_{11}\) has 1 in the \(i\)th row, that \(M_{12}\) has 1 in the \(j\)th row, and that \(i \neq j\).

Then \(F_{12} P_{j}^i\) has \(P_{j}\) as a direct summand. Therefore, combining Lemmas 7 and 8, we conclude that \(M_{11}\) must have a nonzero entry in the position \((i, j)\).

As \(M_{11}\) has a nonzero entry in the position \((i, j)\) and the matrix \(M_{12}\) has 1 in the position \((j, j)\), we conclude that \(M_{11} M_{12}\) has a nonzero entry in the position \((i, j)\). It follows from (8) that

\[M_{11} M_{12} = M_{12},\]

which means that \(M_{12}\) has a nonzero entry in the position \((i, j)\). This, in fact, means that the case \(M = N_{14}\) is impossible.

Assume that \(M = N_{15}\). Then, necessarily, \(j = 1\). By applying exactly the same argument as above to \(M_{22}\) and \(M_{23}\), we conclude that \(M_{23}\) has 1 in the position \((1, 1)\). As follows from (8), this contradicts \(M_{23} M_{12} = 0\). Therefore, \(M = N_{15}\) is not possible.

Assume that \(M = N_{16}\). Then necessarily \(i = 1\). By applying exactly the same argument as above to \(M_{22}\) and \(M_{23}\), we show that \(M_{22}\) has 1 in the position \((1, 1)\). As follows from (8), this contradicts \(M_{11} M_{22} = 0\). Therefore, \(M = N_{16}\) is impossible. This completes the proof.

The remaining cases for \(M\) are studied on the case-by-case basis.

9. Filtering Matrices \(N_{11}\) and \(N_{12}\) Out

9.1. Statement. The main aim of this section is to prove the following proposition:

Proposition 6. The matrix \(M\) can be equal neither to \(N_{11}\), nor to \(N_{12}\).

We start with the following observation:

Lemma 9. The only unordered pairs of idempotent 1-morphisms of the form \(F_{ij}\) such that the product of any two elements in the pair is nonzero are

\[
\{F_{11}, F_{12}\}, \quad \{F_{12}, F_{22}\}, \quad \{F_{22}, F_{23}\}, \quad \{F_{23}, F_{33}\}.
\]

Proof. This assertion directly follows from (8).
9.2. Proof for $M = N_{11}$. We arrange matrices $M_{ij}$, where $i, j = 1, 2, 3$, as follows:

\[
\begin{array}{ccc}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}.
\end{array}
\]  

(10)

Assume that $M = N_{11}$. The diagonal elements in $N_{11}$ are $(2, 2, 1)$. Therefore, two pairs of idempotent matrices of the form $M_{ij}$ must have common diagonal elements. Any product of matrices in any pair of this kind must be nonzero. Therefore, in view of Lemma 9, it is necessary to consider three cases:

**Case 1.** First, we suppose that the pairs of idempotent matrices sharing diagonal elements are $\{F_{11}, F_{12}\}$ and $\{F_{22}, F_{23}\}$. Up to the permutation action, it is possible to assume that

\[
\begin{pmatrix}
1 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
1 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 1 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 1
\end{pmatrix}.
\]

By using all possible zero products that appear in (8), we conclude that the $M_{ij}$’s look as follows:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & * & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
* & 0 & 0 \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
* & 1 & 0 \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
* & * & 0 \\
0 & * & 1
\end{pmatrix}.
\]
Since all matrices must be nonzero and add up to $M$, we obtain

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & * & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & * & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
* & 1 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

This contradicts $M_{13}M_{31} = M_{11}$, which is a consequence of (8). Therefore, this case is not possible.

**Case 2.** Suppose now that the pairs of idempotent matrices sharing diagonal elements are $\{F_{12}, F_{22}\}$ and $\{F_{23}, F_{33}\}$. Up to the permutation action, we can assume that the $M_{ij}$'s have the following form:

\[
\begin{bmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 1
\end{bmatrix},
\begin{bmatrix}
1 & * & * \\
* & 0 & * \\
* & * & 0
\end{bmatrix},
\begin{bmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{bmatrix},
\begin{bmatrix}
* & 1 & * \\
* & 0 & * \\
* & * & 0
\end{bmatrix}.
\]

By using all possible zero products appearing in (8), we can show that the $M_{ij}$'s have the following form:

\[
\begin{bmatrix}
0 & 0 & * \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & * \\
0 & 0 & 0 \\
* & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & * & 0 \\
0 & 0 & 0 \\
0 & * & 0
\end{bmatrix}.
\]
In this case, $P_2$ is a direct summand of $F_{23}P_2$. It follows from Lemmas 7 and 8 that

$$F_{22}P_2 \neq 0,$$

which is a contradiction. Therefore, this case is also impossible.

**Case 3.** First, we suppose that the pairs of idempotent matrices sharing diagonal elements are \{F_{11}, F_{12}\} and \{F_{23}, F_{33}\}. Up to the permutation action, we can assume that the $M_{ij}$’s have the following form:

\[
\begin{pmatrix}
1 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
1 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 1
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix}
\]

By using all possible zero products that appear in (8), for the $M_{ij}$’s, we obtain

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
* & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & * \\
0 & 0 & 0 \\
* & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & 0 \\
0 & 1 & *
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & 0 \\
0 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & 0 \\
0 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & 0 \\
0 & 1 & *
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & 0 \\
0 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & 0 \\
0 & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
* & 0 & 0 \\
0 & * & 0
\end{pmatrix}
\]
In this case, $P_2$ is a direct summand of $F_{23} P_2$. It follows from Lemmas 7 and 8 that $F_{22} P_2 \neq 0$, which is a contradiction. Therefore, this case cannot occur either.

This completes the proof of Lemma 9 for $M = N_{11}$.

9.3. Proof for $M = N_{12}$. Let

$$N'_{12} := \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$ 

The matrix $N'_{12}$ can be reduced to $M = N_{12}$ by the permutation action. However, it is convenient to use the freedom of permutation action in a different way (see in what follows). In view of Lemma 9, we have three cases to consider.

Case 1. Suppose first that the pairs of idempotent matrices sharing diagonal elements are $\{F_{11}, F_{12}\}$ and $\{F_{22}, F_{23}\}$. Thus, by using the permutation action and all possible zero products that appear in (8), we conclude that the $M_{ij}$’s look as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ * & 1 & 0 \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & * \\ 0 & * & 0 \\ 0 & * & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $M = N_{12}$, then the $(1, 3)$-entry of $M_{13}$ is equal to 2, while the $(3, 1)$-entry of $M_{31}$ is equal to 1. As the $(1, 1)$-entry of $M_{11}$ is 1, we arrive at a contradiction with $M_{13} M_{31} = M_{11}$, which follows from (8). This implies that $M = N'_{12}$ and, in view of the fact that $M_{12} M_{21} = M_{11}$, we find

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
* & 0 & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
* & * & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
* & * & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 1 & * \\
0 & * & 0 \\
* & * & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 1 & 2 \\
0 & * & 0 \\
0 & 0 & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 1
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
.
\]

Since the \( M_{ij} \)’s must add up to \( M \), we get

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 0 \\
3 & 1 & 0 \\
* & * & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
1 & 0 & 0 \\
* & * & 0 \\
0 & 1 & 1
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
.
\]

This contradicts \( M_{23}M_{31} = M_{21} \), which follows from (8). Therefore, this case cannot occur.

**Case 2.** In this case, we suppose that the pairs of idempotent matrices sharing diagonal elements are \( \{F_{12}, F_{22}\} \) and \( \{F_{23}, F_{33}\} \). This gives the same contradiction as in Case 2 from Subsection 9.2. Therefore, this case is also impossible.

**Case 3.** Suppose first that the pairs of idempotent matrices sharing diagonal elements are \( \{F_{11}, F_{12}\} \) and \( \{F_{23}, F_{33}\} \). This gives the same contradiction as in Case 3 from Subsection 9.2. Therefore, this case also cannot occur. This completes the proof of Lemma 9.

10. **Proof of Theorem 1 for \( \mathbb{k}(\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet)/(\beta \alpha) \)**

10.1. **Finding the Matrices.** Combining Proposition 4 with Corollary 3 and Propositions 5 and 6, we get \( M = N_{13} \). We now arrange our matrices as in (10).

We need the following simple and general observation:

**Lemma 10.** Let \( M \) be any of the \( N_m \)’s and let \( i, j \in \{1, 2, 3\} \). If, for some \( s \), the column \( s \) in the matrix \( M_{ij} \) is nonzero, then the column \( s \) is nonzero in \( M_{ij} \) for any \( t \in \{1, 2, 3\} \).
**Proposition 7.** The only possibility for the $M_{ij}$’s is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

**Proof.** In view of Lemma 9, it is necessary to consider three cases.

**Case 1.** Suppose that the pairs of idempotent matrices sharing diagonal elements are \{F_{12}, F_{22}\} and \{F_{23}, F_{33}\}. This gives the same contradiction as in Case 2 from Subsection 9.2. Therefore, this case cannot occur.

**Case 2.** Suppose that the pairs of idempotent matrices sharing diagonal elements are \{F_{11}, F_{12}\} and \{F_{23}, F_{33}\}. This gives the same contradiction as in Case 3 from Subsection 9.2. Therefore, this case cannot occur either.

**Case 3.** Suppose that the pairs of idempotent matrices sharing diagonal elements are \{F_{11}, F_{12}\} and \{F_{22}, F_{23}\}. Then, by using the permutation action and all possible zero products that appear in (8), we conclude that the $M_{ij}$’s have the following form:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & * & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & * & * \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
* & 0 & 0 \\
* & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
* & * & 0 \\
* & * & 0
\end{pmatrix}.
\]
Since all matrices must be nonzero and add up to $M$ and, in addition,

$$M_{13}M_{31} = M_{11}, \quad M_{21}M_{12} = M_{22}, \quad \text{and} \quad M_{21}M_{13} = M_{23}$$

[given by (8)], we conclude that

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
* & * & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
* & * & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
* & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Comparing the first and the second columns in $M_{32}$ with the same columns in $M_{22}$ and also the third column in $M_{33}$ with the third column in $M_{23}$ and using Lemma 10, we get exactly the arrangement presented in the formulation of proposition.

Proposition 7 is proved.

10.2. Connecting to the Cell 2-Representation. We now know that the $M_{ij}$’s have the form specified in Proposition 7. For $i, j = 1, 2, 3$, we denote the corresponding indecomposable projective endofunctor of $\mathbf{M}(1)$ by $G_{ij}$.

From the form of $M_{i1}$, where $i = 1, 2, 3$, we conclude that $F_{i1}$ acts via $G_{i1}$ (up to isomorphisms). Moreover, we also have $[P_i : L_1] = \delta_{i,1}$.

From the form of $M_{12}$, we see that $F_{12}$ acts either via $G_{12}$, or via $G_{11}$, or via $G_{12} \oplus G_{11}$. However, we already know that $G_{11}$ has the matrix $M_{11}$. This leaves us only the following possibilities:

$$G_{12} \quad \text{or} \quad G_{12} \oplus G_{11}.$$ 

Assume that $F_{12}$ acts via $G_{12} \oplus G_{11}$. Then the matrix of $G_{12}$ is

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

This implies that

$$[P_i : L_2] = \delta_{i,2}, \quad \text{for} \quad i = 1, 2, 3.$$
According to (9), there is a nonzero 2-morphism \( \alpha : F_{21} \to F_{12} \). Since \( M \) is faithful, \( \mathbf{M}(\alpha) \) is nonzero. We now evaluate the latter:

- \( P_3 \) is zero as \( P_3 \) is annihilated by both \( F_{21} \) and \( F_{12} \);
- \( P_2 \) is zero as \( P_2 \) is annihilated by \( F_{21} \);
- \( P_1 \) is zero as \( F_{12} P_1 \cong P_1 \), \( F_{21} P_1 \cong P_2 \), and we also have

\[
\text{Hom}_{\mathbf{M}(\alpha)}(P_2, P_1) = [P_1 : L_2] = 0,
\]

by the previous paragraph.

Therefore, \( \mathbf{M}(\alpha) \) must be equal to zero. This is a contradiction. Consequently, \( F_{12} \) acts via \( G_{12} \), which also implies that

\[
[P_1 : L_2] = 1.
\]

This means that \( F_{i2} \) acts via \( G_{i2} \) for \( i = 1, 2, 3 \).

A similar argument shows that \( F_{i3} \) acts via \( G_{i3} \) for \( i = 1, 2, 3 \) and that

\[
[P_2 : L_3] = [P_3 : L_3] = 1, \quad [P_3 : L_1] = [P_3 : L_2] = 0.
\]

This means that \( B \cong A \) and that each \( F_{ij} \) acts via the corresponding \( G_{ij} \). Thus, by using standard arguments (see [16], Proposition 9), we conclude that \( \mathbf{M} \) is equivalent to a cell 2-representation of \( \mathcal{E}_A \). The claim of Theorem 1 for the algebra \( k(\bullet \xrightarrow{1} \bullet \xrightarrow{1} \bullet) \) follows.

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