A Macdonald refined topological vertex

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Abstract

We consider the refined topological vertex of Iqbal et al (2009 J. High Energy Phys. JHEP10(2009)069), as a function of two parameters \(x, y\), and deform it by introducing the Macdonald parameters \(q, t\), as in the work of Vuletić on plane partitions (Vuletić M 2009 Trans. Am. Math. Soc. 361 2789–804), to obtain ‘a Macdonald refined topological vertex’. In the limit \(q \to t\), we recover the refined topological vertex of Iqbal et al and in the limit \(x \to y\), we obtain a \(qt\)-deformation of the original topological vertex of Aganagic et al (2005 Commun. Math. Phys. 25 425–78). Copies of the vertex can be glued to obtain \(qt\)-deformed 5D instanton partition functions that have well-defined 4D limits and, for generic values of \(q, t\), contain infinite-towers of poles for every pole present in the limit \(q \to t\).

Keywords: topological vertex, instanton partition functions, conformal blocks, conformal field theory

(Some figures may appear in colour only in the online journal)

1. Introduction

We start with a motivation for the study of topological vertices, then collect basic definitions from combinatorics, and basic facts related to the known topological vertices, their connection to partition functions of weighted plane partitions, and outline the purpose of the present work.
1. Motivation

The topological vertex is a combinatorial object that has come to play an increasingly important and surprisingly ubiquitous role in mathematical physics since its introduction, almost 25 years ago\(^4\). In theoretical high energy physics, it is the building block of topological string partition functions on local toric Calabi–Yau threefolds. In enumerative combinatorics, it is the generating function of topological invariants. In 2D quantum integrable models, it is the most elementary component in the structure of the correlation functions. To motivate the present work, we briefly explain the latter point. The most important problem in 2D critical phenomena is the computation of the correlation functions. This problem was essentially solved in a series of breakthroughs.

1.1. From 2D correlation functions to conformal blocks. In 1984, Belavin, Polyakov and Zamolodchikov [9] showed that correlation functions in conformally-invariant 2D critical phenomena split into holomorphic and anti-holomorphic conformal blocks, which are fully-determined by the constraints imposed by the infinite-dimensional 2D conformal symmetry. However, aside from simple cases such as 4-point blocks that can be computed in terms of hypergeometric functions, computing conformal blocks is far from an easy problem, if one is interested in explicit expressions.

1.1.2. From conformal blocks to matrix elements. In 2009, Alday, Gaiotto and Tachikawa [10] conjectured that 2D conformal blocks in conformal field theories with Virasoro symmetry are equal to 4D instanton partition functions\(^5\), which have been computed in power-series form using localisation [12]\(^6\). This conjecture has since been proven [14], and extended to conformal field theories with higher-rank infinite-dimensional algebras [15, 16]. From the point of view of computing conformal blocks, the simplification that Alday et al bring is that, by introducing an auxiliary scalar free field (whose contributions factorise at the end of a calculation) such that the conformal algebra is slightly extended, any conformal block splits into a product of matrix elements of a primary fields between arbitrary states. These matrix elements are basic Nekrasov partition functions that are completely known in convergent power series form\(^7\). But, since the matrix elements of Alday et al are instanton partition functions, they split even further, which takes us to topological strings.

1.1.3. From matrix elements to topological vertices. In 2003, Aganagic et al [3] showed that instanton partition functions split into topological vertices\(^8\). In 2005, Awata and Kanno [20, 21] introduced a refined version of the topological vertex, and in 2007, Iqbal et al [1] introduced another, equivalent version\(^9\). As we discuss in more detail below, the original topological vertex of Aganagic et al can be used to construct conformal blocks in Gaussian conformal

\(^4\)For reviews of topological strings that include an introduction to the topological vertex, see \([4–7]\), and for a thorough discussion of the refined topological vertex, see \([8]\).

\(^5\)For reviews of the Alday–Gaiotto–Tachikawa conjecture and developments based on it, see \([11]\).

\(^6\)For reviews of localisation methods, see \([13]\).

\(^7\)These power series may not converge as fast as, let’s say, Zamolodchikov’s recursion relations for the 4-point conformal blocks [17, 18], but they are known as simple power sums labeled by Young diagrams. Taking products of these matrix elements leads to an explicit power series expression for any conformal block, while Zamolodchikov’s recursion relations are (currently) known only for the 4-point blocks.

\(^8\)Soon thereafter, Okounkov, Reshetikhin and Vafa [19] showed that a topological vertex is the generating function of plane partitions that satisfy specific boundary conditions, as will be discussed in the sequel. In other words, a topological vertex splits into weighted 3D cells, and the difficult problem of computing 2D correlation functions is reduced to the combinatorial problem of counting 3D cells with specific boundary conditions!

\(^9\)This is the version that we refer to as ‘the refined topological vertex’ in this work.
field theory, while the refined topological vertex can be used to construct conformal blocks in more general conformal field theories.

1.1.4. A further-refined topological vertex. In the present work, we derive a topological vertex with further refinement. The geometric meaning of these refinements (in terms of topological strings on some Calabi–Yau threefold) remains to be understood, and their physical applications (what is computed when one glues many copies of these vertices) remains to be studied. Since the new refined topological vertex reduces to the known refined vertex in a suitable limit, it is clear that it leads to deformations of the results obtained using the refined vertex. However, the precise physical meaning of these deformations is outside the scope of this work.

1.2. Notation

1.2.1. Boldface variables. \( \mathbf{u} \) and also \( \mathbf{y} \) is the set of positive, non-zero integers \( \{1, 2, \cdots \} \). \( \mathbf{x} = (x_1, x_2, \cdots) \) and \( \mathbf{y} = (y_1, y_2, \cdots) \) are sets of possibly infinitely-many variables, \( \mathbf{a}_- = (a_{-1}, a_{-2}, \cdots) \) and \( \mathbf{a}_+ = (a_1, a_2, \cdots) \) are the free-boson creation and annihilation modes, \( \mathbf{Y} = (Y_1, Y_2) \), \( \mathbf{V} = (V_1, V_2) \) and \( \mathbf{W} = (W_1, W_2) \) are pairs of Young diagrams, and \( \emptyset = (\emptyset, \emptyset) \) is a set of two empty Young diagrams.

1.2.2. Primed variables. Given the variables \( (x, y, q, t, \cdots) \), we use \( (x', y', q', t', \cdots) \) for the inverse variables, \( x' = x^{-1}, \cdots \). Given a set of variables \( \mathbf{x} = (x_1, x_2, \cdots) \), we use \( \mathbf{x}' = (x_1', x_2', \cdots) \) for the set of inverse variables. The Young diagram \( \mathbf{Y}' \) is the transpose of \( \mathbf{Y} \).

1.2.3. Parameters. Our parameters \( (x, y) \) are the parameters \( (q, t) \) in [1], and our \( (q, t) \) are the same Macdonald parameters \( (q, t) \) in [2].

1.3. Combinatorics

1.3.1. Young diagrams. A Young diagram \( \mathbf{Y} = (y_1, \cdots) \), \( y_i \geq y_{i+1} \geq 0 \), is a 2D graphical representation of a partition of an integer \( |\mathbf{Y}| = \sum_{i=1}^{\infty} y_i \), see figure 1. It consists of rows \( (y_1, y_2, \cdots) \), the i-row consists of \( y_i \) cells with coordinates \( (i, j) \) that is \( (\square_{i,1}, \square_{i,2}, \cdots, \square_{i,y_i}) \). A generic cells \( \square_{i,j} \in \mathbf{Y} \) has coordinates \( 1 \leq i \leq y_1, 1 \leq j \leq N \), where \( N \) is the number of non-zero rows in \( \mathbf{Y} \). The Young diagram \( \mathbf{Y}' = (y_1', y_2', \cdots, y_{y_1}') \) is the transpose of \( \mathbf{Y} \), where \( y_1 \) is the length of the top row in \( \mathbf{Y} \).

1.3.2. The arms and legs of a cell. Consider a cell \( \square_{i,j} \) with coordinates \( (i, j) \) in \( \mathbb{R}^2 \), not necessarily inside a Young diagram \( \mathbf{Y} \). We define the lengths of the arm \( A_{\square} \) and the leg \( L_{\square} \) of \( \square_{i,j} \) with respect to the Young diagram \( \mathbf{Y} \), to be,

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10 More precisely, the conventions that we use to construct a topological vertex are such that our \( x \) is \( t \), and our \( y \) is \( q \) in [1].
where $y_j'$ is the length of the $j$-row in $Y'$, which is the $j$-row in $Y$. We also use the notation,

\[
A^+ = A + 1, \quad L^+ = L + 1
\] (1.2)

1.3.3. Remark. $A_{\square,Y}$ and $L_{\square,Y}$ are both negative for $\square \notin Y$.

1.3.4. Sequences. Given a Young diagram $Y$ that consists of an infinite sequence of rows $Y = \left\{ y_1, y_2, \cdots \right\}$, such that only finitely-many row-lengths are non-zero, together with an infinite sequence of integers $\iota = \left\{ \iota, \iota + 1, \cdots \right\}$, and two variables $u$ and $v$, we define the exponentiated sequences $u^\iota$ and $v^\pm Y$, and the product sequences $u^\iota v^\pm Y$ and $u^{\iota - 1} v^\pm Y$, etc.

\[
\begin{align*}
&u^\iota = \left\{ u, u^2, \cdots \right\}, \quad u^{\iota - 1} = \left\{ 1, u, \cdots \right\}, \\
&v^\pm Y = \left\{ v^{\pm y_1}, v^{\pm y_2}, \cdots \right\}, \\
&u^\iota v^\pm Y = \left\{ u v^{\pm y_1}, u^2 v^{\pm y_2}, \cdots \right\}, \\
&u^{\iota - 1} v^\pm Y = \left\{ v^{\pm y_1}, uv^{\pm y_2}, \cdots \right\},
\end{align*}
\] (1.3)

1.3.5. Plane partitions. Consider a Young diagram $Y$ and assign each cell $\square_{i,j}$ a non-negative integer $h_{i,j}$, such that,

\[
h_{i,j} \geq h_{i+1,j}, \quad h_{i,j} \geq h_{i,j+1}
\] (1.4)

If we consider $h_{i,j}$ as a height variable that counts the number of boxes placed on top of the cell $\square_{i,j}$, we obtain a plane partition $\pi$, see figure 1(a).

1.3.6. Plane partitions, Young diagrams and vertex operators. A plane partition $\pi$ can be sliced diagonally into a set of interlacing Young diagrams [22], see figure 2(b) In [22], Okounkov and Reshetikhin introduced a method to construct plane partitions via the action of vertex operators\(^{11}\), $\Gamma_+ \left\{ x_i^j \right\}, i \in \left\{ \cdots, -2, -1 \right\}$, and $\Gamma_- \left\{ x_i \right\}, i \in \left\{ 0, 1, 2, \cdots \right\}$ [22]. These vertex operators act on a Young diagram to generate a sum over all possible interlacing Young diagrams. Setting the variables $x_i$ to powers of a single parameter $x$, the weight of a box in a

\(^{11}\) Expressions for (more general versions of) these operators are given in section 2.
plane partition, one generates plane partitions, where each partition is weighted by the number of its boxes\textsuperscript{12}.

1.4. Topological vertices

1.4.1. The original topological vertex. In [3], Aganagic, Klemm, Marino and Vafa introduced the topological vertex as a building block of topological string partition functions\textsuperscript{13}, which are equal to instanton partition functions in 5D $\mathcal{N} = 2$ supersymmetric Yang–Mills theories with Nekrasov parameters $\epsilon_1$ and $\epsilon_2$, such that $\epsilon_1 + \epsilon_2 = 0$ [12, 23–26].

1.4.2. The topological vertex and plane partitions. In [19], Okounkov, Reshetikhin and Vafa showed that the topological vertex can be constructed in terms of the vertex operators, and that the topological vertex is a partition function that counts the number of plane partitions bounded by three Young diagrams $Y_1$, $Y_2$ and $Y_3$. It depends on the parameter $x$, the weight of a box in a plane partition.

1.4.3. Perpendicular versus diagonal boundaries. We consider plane partitions that live in the positive octant of $\mathbb{R}^3$ with coordinates $x, y, z \geq 0$. These plane partitions are bounded by three Young diagrams, $Y_1$, $Y_2$ and $Y_3$ that intersects the $x$-, $y$- and $z$-axis respectively. There are different possible choices for the exact ways that these Young diagrams intersect the respective axes. For example, the Young diagram $Y_1$ on the right boundary can be either perpendicular to the $x$-axis, or parallel to the main diagonal in the $xy$-plane. In the present work, we choose $Y_1$ and $Y_2$ to intersect the $x$-axis and the $y$-axis respectively, while parallel to the main diagonal in the $xy$-plane, and choose $Y_3$ to intersect the $z$-axis while parallel to the $xy$-plane. Any other choice, particularly the choice in which $Y_1$, $Y_2$ and $Y_3$ intersect the $x$-, $y$- and $z$-axis respectively, diagonally, leads expressions for the topological vertices with extra ‘framing’ factors that cancel out in the final result when computing instanton partition functions, conformal blocks, etc.

\textsuperscript{12} In [22], the box-weight parameter is $q$. In the present work, we use $x$, and reserve $q$ for one of the two Macdonald deformation parameters.

\textsuperscript{13} For a pedagogical introduction to topological strings and the topological vertex, see [4–8].
1.4.4. The refined topological vertex of Awata and Kanno. In [20, 21], Awata and Kanno introduced a refined topological vertex, that depends on two parameters, as a building block of refined topological string partition functions. These partition functions are equal to instanton partition functions in 5D \( \mathcal{N} = 2 \) supersymmetric Yang–Mills theories with Nekrasov parameters such that \( \epsilon_1 + \epsilon_2 \neq 0 \) [12, 27, 28]. This construction is not based on vertex operators and will not be used in this work.\(^{14}\)

1.4.5. The refined topological vertex of Iqbal, Kozcaz and Vafa. In [1], Iqbal, Kozcaz and Vafa constructed a refined topological vertex using the same vertex operators, but in the final expressions, the parameters \( x_i \) of \( \Gamma_− \left( x_i \right) \) are set to powers of a parameter \( x \), while those of \( \Gamma_+ \left( x'_i \right) \) are set to powers of a different parameter \( y \neq x \).\(^{15}\) In the limit \( y \to x \), the refined topological vertex reduces to the original topological vertex of [3].

The refined topological vertex is also a partition function that counts the number of plane partitions bounded by three Young diagrams \( Y_1, Y_2 \) and \( Y_3 \). It depends on two parameters \( x \) and \( y \) since the weight of a box in a plane partition is \( x \) or \( y \) depending on its position in the plane partition.

1.4.6. Remark. In [29], Awata, Feigin and Shiraishi showed that the refined topological vertex of Awata and Kanno [20, 21], and that of Iqbal et al [1] are the same object, expressed in two different symmetric function bases. In the sequel, we use 'the refined topological vertex' to refer to that of Iqbal et al.

1.5. MacMahon’s partition function and its refinements

1.5.1. MacMahon’s partition function. Choosing the Young diagrams that label the boundaries of the topological vertex to be empty, \( Y_1 = Y_2 = Y_3 = \emptyset \), where \( \emptyset \) is the trivial Young diagram with no cells, the topological vertex reduces to MacMahon’s partition function of the set of random plane partitions \( \pi \in \mathcal{P} \),

\[
\sum_{\pi \in \mathcal{P}} x^{\mid \pi \mid} = \prod_{i=1}^{\infty} \left( \frac{1}{1 - x^i} \right)^i, \tag{1.5}
\]

1.5.2. The \( y \)-refined MacMahon partition function. Setting the Young diagrams that label the boundaries of the refined topological vertex to be empty, \( Y_1 = Y_2 = Y_3 = \emptyset \), the refined topological vertex reduces to a one-parameter \( y \)-refinement of MacMahon’s partition function,

\[
\sum_{\pi \in \mathcal{P}} x^{\sum_{i=1}^{\infty} |Y_i|} y^{\sum_{j=1}^{\infty} |Y_j|} = \prod_{i,j=1}^{\infty} \left( \frac{1}{1 - x^i y^{j-1}} \right), \tag{1.6}
\]

where \( |Y_i| \) is the number of cells in the Young diagram \( Y_i \) at the \( i \)-diagonal slice of the plane partition \( \pi \) in the set of random plane partitions \( \mathcal{P} \). In the limit \( y \to x \), the right hand side of equation (1.6) reduces to that of equation (1.5).

\(^{14}\) We further comment on the refined topological vertex of Awata and Kanno in (6.4).

\(^{15}\) In [1], the box-weight parameters are \( \{q, t\} \). In the present work, we use \( \{x, y\} \) and, following [2], we reserve \( \{q, t\} \) for the Macdonald deformation parameters.
1.5.3. The qt-MacMahon partition functions. In [2], Vuletić introduced a Macdonald \( (q,t) \)-refinement of MacMahon’s plane-partition partition function,

\[
\sum_{\pi \in \mathcal{P}} F_{\pi}^{qt} x^{\|\pi\|} = \prod_{i,\{n+1\} = 1}^{\infty} \left( \frac{1 - x^i q^n t}{1 - x^i q^n} \right)^i,
\]

where \( F_{\pi}^{qt} \) is a function of \( q \) and \( t \) that specifies the dependence of a plane partition \( \pi \) on the parameters \( \{q,t\} \). The precise form of \( F_{\pi}^{qt} \) is not needed here, and can be found in [2]. In the limit \( q \to t \), the right hand side of equation (1.7) reduces to that of equation (1.5).

1.5.4. Remarks on the literature. In [30, 31], the qt-vertex operators of Shiraishi et al [32] were used to derive the right hand side of equation (1.7). A proof of the left hand side of equation (1.7) requires identities that involve Macdonald functions. A proof of these identities was not attempted in [30], but relevant identities were derived in [31]. A version of these operators was discussed in section 6.2 of [8], but was not used to build a topological vertex of the type discussed in the present work. In the limit \( t \to -1 \), equation (1.7) reduces to the partition function of strict plane partitions of [33, 34], a vertex-operator derivation of which was obtained in [30]. In this special case, the vertex operators are based on neutral free-fermions. A topological vertex that reduces to this \( t = -1 \)-weighted partition function was derived in [35].

1.6. The present work

1.6.1. Combining the \( y \)-refinement and the qt-refinement. It is natural to expect that one can combine the \( y \)-refinement of Iqbal et al and the qt-refinement of Vuletić. In the present work, we show that this is straightforward to do, and that the answer is,

\[
\sum_{\pi \in \mathcal{P}} F_{\pi}^{qt} x^{\sum_{i=1}^{\infty} |Y_i|} y^{\sum_{j=1}^{\infty} |Y_{-j}|} = \prod_{i,j,\{n+1\} = 1}^{\infty} \left( \frac{1 - x^i y^{j-1} q^n t}{1 - x^i y^{j-1} q^n} \right),
\]

where the left hand side is written in a way that makes it clear that the \( y \)-deformation of Iqbal et al and the qt-deformation of Vuletić are independent\(^{16}\).

1.6.2. The Macdonald kernel function. The product expression on the right hand side of equation (1.8) is a specialization of the kernel function of the Macdonald operator \([36–38]\).

1.6.3. Ding–Iohara–Miki algebra. The free-field realization of the Macdonald operator naturally gives rise to Ding–Iohara–Miki algebra \([39–41]\). This points to a connection between the qt-deformation of the refined topological vertex and Ding–Iohara–Miki algebra.

1.6.4. A Macdonald refined topological vertex. More generally, it is natural to expect that the Macdonald refined MacMahon function, equation (1.8) is the \( Y_1 = Y_2 = Y_3 = \emptyset \) limit of a Macdonald refined topological vertex\(^{17}\). In the present work is the derivation of this object.

\(^{16}\) The observation that the \( y \)-refinement of Iqbal et al \([1]\), and the qt-deformation of Vuletić are ‘orthogonal’, in the sense that they change the weights of the plane partitions in independent ways, is the starting point of this work.

\(^{17}\) One can also think of this object as a doubly-refined topological vertex. However, as we will see, the Macdonald parameters \( \{q,t\} \) appear in a distinct way from the \( y \)-refinement parameter of Iqbal et al.
16.5. Notation. We use \( \mathcal{O}_{Y,Y,Y} \) for the original topological vertex of Aganagic et al., \( \mathcal{R}_{Y,Y,Y} \) for the refined topological vertex of Iqbal et al., \( \mathcal{R}_{Y,Y,Y}^{AK} \) for the refined topological vertex of Awata and Kanno, and \( \mathcal{M}^{qt}_{Y,Y,Y} \) for the Macdonald refined topological vertex derived in the present work, which we refer to as ‘the Macdonald vertex’.

16.6. Limits of the Macdonald vertex. \( \mathcal{M}^{qt}_{Y,Y,Y} \) depends on \( \{x,y,q,t\} \). In the limit \( q \to t \), we recover \( \mathcal{R}_{Y,Y,Y} \). In the limit \( x \to y \), we obtain a \( qt \)-version of \( \mathcal{O}_{Y,Y,Y} \).

16.7. 4D limits of the Macdonald vertex. All of the above topological vertices can be glued to form instanton partition functions of 5D supersymmetric Yang–Mills theories. It is possible to take a 4D limit of each of these topological vertices. We consider the 4D of \( \mathcal{M}^{qt}_{Y,Y,Y} \) in section 7.

1.7. On the choice of parameters

In [19], the parameter that counts the number of boxes in a plane partition is \( q \). In [1], the parameter that refines this counting is \( t \). In [2], the parameter that counts the number of boxes is \( s \), there is no refinement in the sense of [1], but there are Macdonald-type deformation parameters \( \{q,t\} \). In the present work, the parameter that counts the number of boxes is \( x \), its refinement in the sense of Iqbal et al [1] is \( y \). We find this choice of variables natural in the sense that our \( \{x,y\} \) are related to the arguments of the vertex operators, which are position variables. Our deformation parameters \( \{q,t\} \) are those of [2]. They are Macdonald-type parameters in the sense that our \( qt \)-bosons are in bijection with the power-sum symmetric functions that the Macdonald functions are expanded in, our \( qt \)-Heisenberg commutation relations are in agreement with the power-sum inner products, and in the limit \( q \to t \), both parameters drop out of all expressions.

1.8. Outline of contents

In section 2, we recall basic facts related to Macdonald functions, in section 3, we do the same for free-bosons and vertex operators, and in section 4, we recall the isomorphism of power-sum symmetric functions and the generators of the Heisenberg algebra which are free-boson mode operators. In section 5, we construct the Macdonald refined topological vertex, and in section 6, we make remarks on its structure. In section 7, we glue four Macdonald refined topological vertices to construct the Macdonald refined \( U \left( \begin{array}{c} 2 \end{array} \right) \) ‘strip’ partition function, that is the building block of the Macdonald refined topological string partition functions, then in section 8, we compute the 4D limits of the objects computed in section 6. In section 9, we glue two strips to obtain the \( {\cal N} = 2 \) \( U \left( \begin{array}{c} 2 \end{array} \right) \) instanton partition function that is equal to a 4-point \( qt \)-conformal block on a sphere, and take its 4D limit, then in section 10, we compute the
\[ N = 2^* U \left( \begin{array}{c} 2 \\ \end{array} \right) \] instanton partition function that is equal to a 1-point \( q^\ell \)-conformal block on a torus, and take its 4D limit. In section 10, we conclude with a number of remarks.

2. Macdonald symmetric functions

Our starting point is Macdonald function theory. We list basic definitions related to Macdonald symmetric functions that are used in subsequent sections. We give detailed references to [42], and refer to it for a complete presentation. The aim of this section is to outline the derivation of the Cauchy identities in (2.25)–(2.27).

2.1. Notation

Let \( Y \) be a Young diagram that consists of \( m \) non-zero parts, \( Y = \left( \begin{array}{c} y_1, \cdots, y_m \end{array} \right) \), and \( \mathbf{x} \) a set of \( n \) variables, \( \mathbf{x} = \left( \begin{array}{c} x_1, \cdots, x_n \end{array} \right) \), such that \( m \leq n \). We use the notation,

\[
\mathbf{x}^Y_\mathcal{I} = \left( \begin{array}{c} x_{i_1}^{y_1}, \cdots, x_{i_m}^{y_m} \end{array} \right),
\]

where \( \mathcal{I} = \left( \begin{array}{c} i_1, \cdots, i_m \end{array} \right) \) is defined as follows. We start from the set of \( n \) integers, \( \mathbf{n} = \left( \begin{array}{c} 1, \cdots, n \end{array} \right) \), for example, \( \mathbf{n} = \left( \begin{array}{c} 1, 2, 3, 4 \end{array} \right) \), choose a subset of \( m \) integers \( \mathbf{m} \), such that \( \mathbf{m} \subseteq \mathbf{n} \), for example, \( \mathbf{m} = \left( \begin{array}{c} 1, 2, 4 \end{array} \right) \), and consider a permutation \( \mathcal{I} \) of \( \mathbf{m} \), for example, \( \mathcal{I} = \left( \begin{array}{c} 2, 4, 1 \end{array} \right) \). The sum on the right hand side of equation (2.1) is over all distinct permutations \( \mathcal{I} \), of all distinct subsets \( \mathbf{m} \subseteq \mathbf{n} \).

2.2. The monomial symmetric functions

The monomial symmetric function \( m_Y (\mathbf{x}) \) is,

\[
m_Y (\mathbf{x}) = \sum_{\mathcal{I}} \mathbf{x}^Y_\mathcal{I}, \tag{2.2}
\]

where the sum runs over all distinct permutations of the set \( \mathcal{I} \). For example,

\[
m_0 (\mathbf{x}) = 1, \quad m_1 (\mathbf{x}) = \sum_i x_i, \quad m_2 (\mathbf{x}) = \sum_i x_i^2, \quad m_{443} \left( x_1, \cdots, x_n \right) = \sum_{\mathcal{I}} x_{i_1}^4 x_{i_2}^4 x_{i_3}^3, \tag{2.3}
\]

where the sum in the last example is over all distinct permutations \( \mathcal{I} \), of all distinct subsets \( \mathbf{m} \subseteq \mathbf{n} = \left\{ 1, 2, \cdots, n \right\} \) of cardinality \( |\mathbf{m}| \geq 3 \), and \( i \neq j \neq k \in \mathbf{m} \).

2.3. The power-sum symmetric functions

Given a possibly-infinite set of variables \( \mathbf{x} = \left( \begin{array}{c} x_1, x_2, \cdots \end{array} \right) \), the power-sum function \( p_n (\mathbf{x}) \), \( n \in \left\{ 0, 1, \cdots \right\} \), is,

\[
18 \text{ In the special case of } \mathbf{m} = \mathbf{n}, \mathcal{I} \text{ is a permutation of } \mathbf{m}, \text{ and the sum over } \mathcal{I} \text{ is a sum over all distinct permutations of } \mathbf{m}.
19 \text{ Chapter I, p 18, equation (2.1), in [42].}
20 \text{ Chapter I, p 23, in [42].}
\[ p_0(x) = 1, \quad p_n(x) = \sum_i x_i^n = m_n(x), \quad n \in \{1, 2, \cdots\}, \quad (2.4) \]

and the power-sum function
\[ p_Y(x) \] indexed by \( Y = \{y_1, y_2, \cdots\} \) is, \( (2.5) \)

### 2.4. The qt-inner product of the power-sum functions

Consider the ring of symmetric functions in a set of variables \( x = \{x_1, x_2, \cdots\} \) with coefficients in the field of rational functions in two variables \( \{q, t\} \). In this case, the power-sum functions are orthogonal in the sense of the inner product, \( (2.6) \)

\[ (p_{Y_1}(x) \mid p_{Y_2}(x))_{qt} = z_{qt}^{Y_1} \delta_{Y_1 Y_2}, \quad z_{qt}^{Y_1} = \left( \frac{1}{1-q^{y_1}} \right) \prod_{i=1}^{y'_1} \left( \frac{1 - q^{y_i}}{1 - t^{y_i}} \right) \]

#### 2.4.1. Remark

The inner product in equation \( (2.6) \) can be understood as follows. For every power-sum function \( p_Y(x) \), there is a dual differential operator \( D_Y(x) \) in \( x = \{x_1, x_2, \cdots\} \), such that acting with \( D_Y(x) \) on \( p_Y(x) \), then setting \( x_1 = x_2 = \cdots = 0 \), one obtains the right hand side of equation \( (2.6) \).

### 2.5. An identity

The power-sum functions \( p_n(x) \) satisfy the identity, \( (2.7) \)

\[ \prod_{n=1}^{\infty} \exp \left( \frac{1}{n} \left( \frac{1-t^n}{1-q^n} \right) p_n(x) p_n(y) \right) = \prod_{i,j} \sum_{n=1}^{\infty} \left( \frac{1-x_i y_j q^{n t}}{1-x_i y_j q^n} \right). \]

#### 2.5.1. Remark

The right hand side of \( (2.7) \) is the Macdonald kernel. It specializes to the right hand side of equation \( (1.8) \).

### 2.6. The Macdonald function

\[ P_Y^{qt}(x) \] \( y = \{y_1, y_2, \cdots\} \) is the unique symmetric function in \( x = \{x_1, x_2, \cdots\} \), where the cardinality \(|x| \geq y_1\), that can be written as, \( (2.8) \)

\[ \sum_{x \in \{x_1, x_2, \cdots\}} \left( \prod_{i=1}^{\infty} \exp \left( \frac{1}{n} \left( \frac{1-t^n}{1-q^n} \right) p_n(x) p_n(y) \right) \right). \]

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21 Chapter I, p 24, in [42].
22 Chapter VI, p 225, equation (4.11) in [42].
23 Chapter I, p 75–76, in [42].
24 Chapter VI, p 310, in [42].
25 Chapter VI, p 322, in [42].
\[ P_{Y_1}^{qt}(x) = \sum_{Y_1 \succeq Y_2} u_{Y_1, Y_2}^{qt} m_{Y_1}(x), \quad u_{Y_1, Y_2}^{qt} = 1. \] (2.8)

where \( m_Y(x) \) is the monomial symmetric function labelled by \( Y \), and satisfies the orthogonality relation,

\[
\langle P_{Y_1}^{qt}(x) | P_{Y_2}^{qt}(x) \rangle = 0, \quad \text{for} \quad Y_1 \neq Y_2
\] (2.9)

2.6.1 Remarks. The inner product in equation (2.9) can be understood as follows. Expanding \( P_{Y_1}^{qt}(x) \) in terms of power-sum functions \( p_n, n \in \{0, 1, \ldots\} \), equation (2.9) follows from equation (2.6).

2.7. The dual Macdonald function

\[ Q_{Y_1}^{qt}(x) \] is defined in terms of \( P_{Y_1}^{qt}(x) \) as\(^{26,27}\),

\[
Q_{Y_1}^{qt}(x) = b_Y^{qt} P_{Y_1}^{qt}(x), \quad b_Y^{qt} = \prod_{\square \in Y} \left( \frac{1 - q^{A_{\square}^+} r^{L_{\square}^+}}{1 - q^{A_{\square}^+} r^{L_{\square}^+}} \right)
\] (2.10)

2.8. The \( qt \)-involution

There is a \( qt \)-operator \( \omega_{qt} \) that acts on power-sum functions, \( p_n(x) \) as\(^{28}\),

\[
\omega_{qt} \left( p_n(x) \right) = \left(-\right)^{n-1} \left( \frac{1 - q^n}{1 - r^n} \right) p_n(x).
\] (2.11)

which is a \( qt \)-involution in the sense that the action of \( \omega_{qt} \) is followed by the action of \( \omega_{tq} \), and \textit{vice versa}, so that,

\[
\omega_{tq} \left( \omega_{qt} \left( p_n(x) \right) \right) = p_n(x),
\] (2.12)

and consequently, \( \omega_{qt} \) acts on the Macdonald functions as\(^{29}\),

\[
\omega_{qt} \left( P_{Y_1}^{qt}(x) \right) = Q_{Y_1'}^{qt}(x), \quad \omega_{qt} \left( Q_{Y_1}^{qt}(x) \right) = P_{Y_1'}^{qt}(x)
\] (2.13)

2.8.1 Remark. \( \omega_{qt} \) exchanges \( P_{Y_1}^{qt}(x) = Q_{Y_1'}^{qt}(x) \), which includes \( Y = Y' \), and \( q = t \).

2.9. The \( qt \)-inner product of the Macdonald functions

The Macdonald functions satisfy the \( qt \)-inner product\(^{30}\),

\[
\langle P_{Y_1}^{qt}(x) | Q_{Y_2}^{qt}(x) \rangle = \delta_{Y_1, Y_2}.
\] (2.14)

\(^{26}\) Chapter VI, p 323, equation (4.12), in [42].
\(^{27}\) Chapter VI, p 339, equation (6.19), in [42].
\(^{28}\) Chapter VI, p 312, equation (2.14) in [42].
\(^{29}\) Chapter VI, p 327, in [42].
\(^{30}\) Chapter VI, p 324, in [42].
that is, \( P_{\gamma}^{q,t} \{ x \} \) and \( Q_{\gamma}^{q,t} \{ x \} \) span adjoint bases with respect to the \( qt \)-inner product, equation (2.14). Following [42], \( P_{\gamma}^{q,t} \{ x \} \) form a complete orthogonal basis, and \( Q_{\gamma}^{q,t} \{ x \} \) form an adjoint complete orthogonal basis, in the ring of symmetric functions in \( x \), with coefficients in the field of rational functions in \( \{ q, t \} \).

2.10. Cauchy identities for the Macdonald and dual Macdonald functions

\( P_{\gamma}^{q,t} \{ x \} \) and \( Q_{\gamma}^{q,t} \{ x \} \) satisfy the Cauchy identity\(^{31}\),

\[
\sum_{Y} P_{\gamma}^{q,t} \{ x \} Q_{\gamma}^{q,t} \{ y \} = \prod_{i,j=1}^{\infty} \frac{1 - x_{i} y_{j} q^{n+1}}{1 - x_{i} y_{j} q^{n}} \quad (2.15)
\]

Using the involution \( \omega_{qt} \), equation (2.13), on the Macdonald functions in the \( x \)-variables in equation (2.15), then in the \( y \)-variables\(^{32}\),

\[
\sum_{Y} Q_{\gamma}^{q,t} \{ x \} Q_{\gamma}^{q,t} \{ y \} = \sum_{Y} P_{\gamma}^{q,t} \{ x \} P_{\gamma}^{q,t} \{ y \} = \prod_{i,j=1}^{\infty} \left( 1 + x_{i} y_{j} \right) \quad (2.16)
\]

2.10.1. Remark. From equations (2.15) and (2.16)\(^{33}\),

\[
\omega_{qt} \left( \prod_{i,j=1}^{\infty} \frac{1 - x_{i} y_{j} q^{n+1}}{1 - x_{i} y_{j} q^{n}} \right) = \prod_{i,j=1}^{\infty} \left( 1 + x_{i} y_{j} \right) \quad (2.17)
\]

2.11. Notation

In the sequel, we use\(^{34,35}\),

\[
\Pi_{q,t} \{ x, y \} = \prod_{i,j=1}^{\infty} \frac{1 - x_{i} y_{j} q^{n+1}}{1 - x_{i} y_{j} q^{n}} , \quad \Pi_{0} \{ x, y \} = \prod_{i,j=1}^{\infty} \left( 1 + x_{i} y_{j} \right) \quad (2.18)
\]

2.12. Structure constants

The product of two Macdonald functions can be expanded in the form\(^{36}\),

\[
P_{\gamma_{1}}^{q,t} \{ x \} P_{\gamma_{2}}^{q,t} \{ x \} = \sum_{\gamma} \xi_{\gamma_{1}}^{q,t} \gamma_{1} P_{\gamma_{2}}^{q,t} \{ x \} , \quad (2.19)
\]

\(^{31}\) Chapter VI, p 324, equation (4.13), in [42].
\(^{32}\) Chapter VI, p 329, equation (5.4), in [42].
\(^{33}\) Chapter VI, p 313, equation (2.18), in [42].
\(^{34}\) Chapter VI, p 309, equation (2.5), in [42].
\(^{35}\) Chapter VI, p 352, in [42].
\(^{36}\) Chapter VI, p 343, equation (7.1)', in [42].
which can be used as a definition of the structure constants \( f^{qt}_{Y_1} \). Using the involution \( \omega_{qt} \), the product of two dual Macdonald functions, \( Q_{Y_1} \) and \( Q_{Y_2} \), can be expanded\(^{37}\),

\[
Q^{qt}_{Y_1}(x) \cdot Q^{qt}_{Y_2}(x) = \sum_{Y_3} f^{qt}_{Y_1 Y_3} Q^{qt}_{Y_3}(x) \quad (2.20)
\]

From equations (2.19) and (2.20)\(^{38}\),

\[
f^{qt}_{Y_1 Y_2} = \left( \frac{b^{qt}_{Y_1}}{b^{qt}_{Y_2}} \right) f^{qt}_{Y_1 Y_2} \quad (2.21)
\]

where \( b^{qt}_{Y} \) is defined in equation (2.10). The structure constant \( f^{qt}_{Y_1 Y_2} \) can be written as an inner product\(^{39}\),

\[
f^{qt}_{Y_1 Y_2} = \langle Q^{qt}_{Y_1}(x) | P^{qt}_{Y_2}(x) P^{qt}_{Y_2}(x) \rangle \quad (2.22)
\]

2.13. The skew Macdonald and skew dual Macdonald functions

\( P^{qt}_{Y_1/Y_2}(x) \) is defined as\(^{40}\),

\[
P^{qt}_{Y_1/Y_2}(x) = \left( \frac{b^{qt}_{Y_2}}{b^{qt}_{Y_1}} \right) Q^{qt}_{Y_1/Y_2}(x) \quad (2.23)
\]

and \( Q^{qt}_{Y_1/Y_2}(x) \) is defined as\(^{41}\),

\[
Q^{qt}_{Y_1/Y_2}(x) = \sum_{Y_3} f^{qt}_{Y_1 Y_3} Q^{qt}_{Y_3/Y_2}(x) \quad (2.24)
\]

2.14. Cauchy identities for skew Macdonald and dual skew Macdonald functions

The skew Macdonald and dual skew Macdonald functions satisfy the Cauchy identity\(^{42}\),

\[
\prod_{\lambda, \mu} \left( \frac{1 - x_\lambda y_\mu q^{n+1}}{1 - x_\lambda y_\mu q^n} \right) \sum_{Y} P^{qt}_{Y_1/Y}(x) Q^{qt}_{Y_2/Y}(y) = \sum_{Y} P^{qt}_{Y_1/Y_2}(x) Q^{qt}_{Y_2/Y_1}(y) \quad (2.25)
\]

Applying the involution, equation (2.13), to the \( x \)-symmetric functions in equation (2.25)\(^{43}\),

\[
\prod_{\lambda, \mu} \left( 1 + x_\lambda y_\mu \right) \sum_{Y} Q^{qt}_{Y_1/Y}(x) Q^{qt}_{Y_2/Y}(y) = \sum_{Y} Q^{qt}_{Y_1/Y_2}(x) Q^{qt}_{Y_2/Y_1}(y) \quad (2.26)
\]

\(^{37}\) Chapter VI, p 344, in [42].

\(^{38}\) Chapter VI, p 344, equation (7.3), in [42].

\(^{39}\) Chapter VI, p 343, equation (7.1), in [42].

\(^{40}\) Chapter VI, p 344, equation (7.6)', in [42].

\(^{41}\) Chapter VI, p 344, equation (7.5), in [42].

\(^{42}\) Chapter VI, p 352, in [42].

\(^{43}\) Chapter VI, p 352, in [42].
Repeating the exercise on the $y$-symmetric functions,
\[
\prod_{i,j=1}^{\infty} \left( 1 + x_i y_j \right) \sum_{Y} \mathcal{P}_{Y/\gamma_1}^{q/t} \left( x \right) \mathcal{P}_{Y/\gamma_2}^{t/q} \left( y \right) = \sum_{Y} \mathcal{P}_{Y/\gamma_1}^{q/t} \left( x \right) \mathcal{P}_{Y/\gamma_2}^{t/q} \left( y \right)
\]  

(2.27)

2.14.1. Remark. In each of equations (2.26) and (2.27), a Macdonald or dual Macdonald function that depends on $\{q,t\}$, is multiplied by a similar function that depends on $\{t,q\}$. These identities are important in the sequel. In fact, one can, in a sense, trace all constructions in this work to the Cauchy identity in equation (2.25).

3. qt-free bosons and qt-vertex operators

Our next point is the theory of qt-Heisenberg algebra and qt-vertex operators which are simple deformations of free bosons and vertex operators. We recall basic definitions related to these objects, which are necessary to obtain the correct Macdonald kernel function, a specialization of which is the Macdonald refined Macmahon partition function on the right hand side of equation (3.7).

3.1. Remarks on earlier works

All calculations in [1, 19, 22] use vertex operators based on 2D charged free-fermions, without deformation parameters. In [30], vertex operators based on $t$-free-fermions were used to generate $t$-plane partitions related to Hall–Littlewood symmetric functions\(^4\). These $t$-plane partitions are the $q \to 0$ limit of the $qt$-plane partitions introduced in [2], the same way that Hall–Littlewood functions are $q \to 0$ limits of Macdonald functions. The computations were tedious, and an extension using vertex operators based on $qt$-free-fermions to generate the $qt$-plane partitions related to Macdonald polynomials, of the type introduced in [2], was not attempted. In the present work, we choose to work exclusively in terms of 2D free-bosons.

3.2. qt-free bosons

Following Shiraishi, Kubo, Awata and Odake [32], given two variables $q$ and $t$, $|q| < 1$ and $|t| < 1$, we introduce the qt-free boson operators that satisfy the qt-Heisenberg algebra,
\[
[a_m^{qt}, a_n^{qt}] = n \left( \frac{1 - q^{2|n|}}{1 - t^{2|m|}} \right) \delta_{m+n,0}
\]  

(3.1)

Working in terms of the qt-free boson operators, left-state $\langle a_Y |$, labelled by a Young diagram $Y = (y_1, y_2, \cdots)$, is generated from the left vacuum state,
\[
\langle a_Y | = \langle 0 | a_{y_1}^{q,t} a_{y_2}^{q,t} \cdots,
\]

(3.2)

while the right-state $| a_Y \rangle$ labelled by the same Young diagram $Y$ is generated from the right vacuum state,
\[
| a_Y \rangle = a_{-y_1}^{q,t} a_{-y_2}^{q,t} \cdots | 0 \rangle
\]

(3.3)

\(^4\)To simplify the presentation, we say $t$-free fermion, $t$-plane partitions, etc for $t$-deformed free fermions, $t$-weighted plane partitions, etc
Using the $q_t$-Heisenberg commutation relations, the inner product of $|a_{Y_1}^{qt}\rangle$ and $|a_{Y_2}^{qt}\rangle$ is,
\[
\langle a_{Y_1}^{qt}|a_{Y_2}^{qt}\rangle = z_{Y_1}^{Y_2}\delta_{Y_1Y_2}.
\] (3.4)

3.3. $q_t$-vertex operators

Following Shiraishi et al [32], we define the $q_t$-vertex operators,
\[
\Gamma_{q_t}^+\left(x\right) = \exp\left\{ -\sum_{n=1}^{\infty} \frac{x^{-n}}{n} \left( 1 - t^n \right) a_{x^{-n}}^{q_t} \right\}, \quad \Gamma_{q_t}^-\left(x\right) = \exp\left\{ -\sum_{n=1}^{\infty} \frac{x^n}{n} \left( 1 - t^n \right) a_{x^n}^{q_t} \right\}
\] (3.5)

3.3.1. Remark. $\Gamma_{q_t}^+\left(x\right)$ depends on the inverse variables $x', \left(x^2\right)', \cdots$, while $\Gamma_{q_t}^+\left(x\right)$ depends on $x, x^2, \cdots$. We use this convention to produce known formulas related to symmetric functions and to plane partitions without modification.

3.4. $q_t$-vertex operator commutation relations

From the $q_t$-Heisenberg commutation relations, equation (3.1), and the dependence of $\Gamma_{q_t}^\pm$ on their variables, equation (3.5), we obtain the commutation relations,
\[
\Gamma_{q_t}^+\left(x'\right) \Gamma_{q_t}^-\left(y\right) = \prod_{n=0}^{\infty} \left( 1 - \frac{x'yq^n}{1 - x'yq^n} \right) \Gamma_{q_t}^-\left(y\right) \Gamma_{q_t}^+\left(x'\right)
\] (3.6)

3.5. On the relation to Ding–Iohara–Miki algebra

The $q_t$-operators and Heisenberg algebra in equation (3.1), and the $q_t$-vertex operators in equation (3.6), and related operators, are identical to those that appear in free-field realisation of the Macdonald operator [32, 41, 43–45], and of Ding–Iohara–Miki algebra [36–38]. The vertex operators $\Gamma_{q_t}^+\left(x\right)$ and $\Gamma_{q_t}^-\left(x\right)$ are the vertex operators $\phi^+\left(z\right)$ and $\phi^-\left(z\right)$ in equation (2.12) in [36], respectively.

3.6. From $q_t$-vertex operators to the $q_t$-MacMahon function

From the $q_t$-vertex operator commutation relations, equation (3.6),
\[
\langle 0 | \prod_{i=1}^{\infty} \Gamma_{q_t}^+\left(x^{-i}\right) \prod_{j=1}^{\infty} \Gamma_{q_t}^-\left(y^{j-1}\right) | 0 \rangle = \prod_{i,j=1}^{\infty} \frac{1 - x'Iy^{-i}q^{n+1}}{1 - x'Iy^{-i}q^{n+1}},
\] (3.7)

which is the result in equation (1.8). The right hand side of equation (3.7) is a specialization of that in remark 2.7 in [36] on the Macdonald kernel function.
4. The power-sum/Heisenberg correspondence

From identities that involve Macdonald functions, we obtain identities that involve operator-valued Macdonald functions that act on states labelled by Macdonald functions. The aim of this section is to derive equation (4.19), which are used in section 5 to derive the Macdonald vertex.

4.1. An isomorphism

Comparing 1. the inner product of power-sum functions in the Macdonald basis, equation (2.6), and 2. the inner product of the right and left-states, equation (3.4), we deduce that the symmetric power-sum function basis is isomorphic to the Fock space spanned by the left-states \( \langle aY | \), as well as that spanned by the right-states \( | aY \rangle \), where \( Y \) is an arbitrary partition. More precisely, we have the correspondence,

\[
p_n \left( x \right) \mapsto -a_n, \quad n \geq 1,
\]

(4.1)

and also the correspondence,

\[
p_n \left( x \right) \mapsto -a_{-n}, \quad n \geq 1,
\]

(4.2)

Since the power-sum functions form a complete basis, we can think of the Macdonald functions as functions of the power-sum functions, then formally replace the latter with Heisenberg generators to obtain operator-valued Macdonald functions that act on left and right vacuum states to produce left and right Macdonald states. We use this formal substitution to start from known Macdonald symmetric function identities and obtain identities that involve operator-valued Macdonald functions acting on left and right-states labelled by Macdonald functions.

4.2. The action of operator-valued Macdonald functions on states

We start from the Cauchy identity for skew Macdonald and skew dual Macdonald functions, equation (2.25), which we recall here,

\[
\prod_{i,j=1}^{\infty} \frac{1-x_i y_j q^{n+1}}{1-x_i y_j q^n} \sum_Y P_{Y_i/Y_{1}}^{q_{t}} \left( x \right) Q_{Y_{2}/Y_{1}}^{q_{t}} \left( y \right) = \sum_Y P_{Y_{1}/Y_{2}}^{q_{t}} \left( x \right) Q_{Y_{1}/Y_{2}}^{q_{t}} \left( y \right)
\]

(4.3)

Using equations (2.18) and (2.7),

\[
\exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - x^n \right) p_n \left( x \right) p_n \left( y \right) \right) \sum_Y P_{Y_i/Y_{1}}^{q_{t}} \left( x \right) Q_{Y_{1}/Y_{2}}^{q_{t}} \left( y \right) = \sum_Y P_{Y_{1}/Y_{2}}^{q_{t}} \left( x \right) Q_{Y_{1}/Y_{2}}^{q_{t}} \left( y \right)
\]

(4.4)

4.2.1 The action of \( \Gamma^{q_{t}}_{+} \) on a left-state. Using the power-sum/Heisenberg correspondence, equation (4.2), on \( p_n \left( x \right) \), on the right hand side of equation (4.4), we introduce free-boson mode operators that act as creation operators on a left-state, to obtain the operator-valued Macdonald Cauchy identity,
\[ \exp \left( \sum_{n=1}^{\infty} -\frac{1}{n} \left( \frac{1 - t^n}{1 - q^n} \right) a_n p_n \left( y \right) \right) \sum_{y} P_{Y/Y}^{qY/Y} \left( a_+ \right) Q_{Y/Y}^{qY/Y} \left( y \right) \]
\[ = \sum_{y} P_{Y/Y}^{qY/Y} \left( a_+ \right) Q_{Y/Y}^{qY/Y} \left( y \right) . \]  
\[ \text{(4.5)} \]

where \( P_{Y/Y}^{qY/Y} \left( a_+ \right) \) and \( P_{Y/Y}^{qY/Y} \left( a_+ \right) \) are obtained by expanding \( P_{Y/Y}^{qY/Y} \left( x \right) \) and \( P_{Y/Y}^{qY/Y} \left( x \right) \) in the power-sum basis, then using the power-sum/Heisenberg correspondence, equation (4.1). From the definition of the \( \Gamma_{+}^{QT} \) vertex operators, equation (3.5),
\[ \prod_{i=1}^{\infty} \Gamma_{+}^{QT} \left( y_i \right) \sum_{y} P_{Y/Y}^{qY/Y} \left( a_+ \right) Q_{Y/Y}^{qY/Y} \left( y \right) = \sum_{y} P_{Y/Y}^{qY/Y} \left( a_+ \right) Q_{Y/Y}^{qY/Y} \left( y \right) \]
\[ \text{(4.6)} \]

Using the \( qY \)-Heisenberg commutation relation, equation (3.1), and acting with each side of equation (4.11) on a left vacuum state,
\[ \sum_{y} \langle P_{Y/Y}^{qY/Y} \left| Q_{Y/Y}^{qY/Y} \right( y \right) \prod_{i=1}^{\infty} \Gamma_{+}^{QT} \left( y_i \right) = \sum_{y} \langle P_{Y/Y}^{qY/Y} \left| Q_{Y/Y}^{qY/Y} \right( y \right) \]
\[ \text{(4.7)} \]

where \( \langle P_{Y/Y}^{qY/Y} \rangle \) is a state in the free-boson Fock space obtained by the action of the operator-valued Macdonald function labelled by the skew Young diagram \( Y_i/Y_2 \).
\[ \langle \emptyset | P_{Y/Y}^{qY/Y} \left( a_+ \right) \rangle = \langle P_{Y/Y}^{qY/Y} \left| \right\rangle \]  
\[ \text{(4.8)} \]

Setting \( Y_2 = \emptyset \) in equation (4.7), we force \( Y = \emptyset \) in the sum on the left hand side,
\[ \langle P_{Y/Y}^{qY/Y} \mid \prod_{i=1}^{\infty} \Gamma_{+}^{QT} \left( y_i \right) = \sum_{y} \langle P_{Y/Y}^{qY/Y} \mid Q_{Y/Y}^{qY/Y} \left( y \right) \]
\[ \text{(4.9)} \]

4.2.2. The action of \( \Gamma_{-}^{QT} \) on a right-state. Using the power-sum/Heisenberg correspondence, equation (4.2), on \( p_n \left( y \right) \), to introduce free-boson mode operators that act as creation operators on a right-state, to obtain the operator-valued Macdonald Cauchy identity,
\[ \exp \left( \sum_{n=1}^{\infty} -\frac{1}{n} \left( \frac{1 - t^n}{1 - q^n} \right) p_n \left( x \right) a_{-n} \right) \sum_{y} P_{Y/Y}^{qY/Y} \left( x \right) Q_{Y/Y}^{qY/Y} \left( a_- \right) \]
\[ = \sum_{y} P_{Y/Y}^{qY/Y} \left( x \right) Q_{Y/Y}^{qY/Y} \left( a_- \right) \]
\[ \text{(4.10)} \]

From the definition of the \( \Gamma_{-}^{QT} \) vertex operators, equation (3.5),
\[ \prod_{i=1}^{\infty} \Gamma_{-}^{QT} \left( x_i \right) \sum_{y} P_{Y/Y}^{qY/Y} \left( x \right) Q_{Y/Y}^{qY/Y} \left( a_- \right) = \sum_{y} P_{Y/Y}^{qY/Y} \left( x \right) Q_{Y/Y}^{qY/Y} \left( a_- \right) \]
\[ \text{(4.11)} \]

Acting with each side of equation (4.11) on a right vacuum state,
\[ \prod_{i=1}^{\infty} \Gamma_{-}^{QT} \left( x_i \right) \sum_{y} P_{Y/Y}^{qY/Y} \left( x \right) |Q_{Y/Y}^{qY/Y} \right) = \sum_{y} P_{Y/Y}^{qY/Y} \left( x \right) |Q_{Y/Y}^{qY/Y} \right), \]
\[ \text{(4.12)} \]
where $|Q^q_{Y_1/Y_2}\rangle$ is a state in the free boson Fock space obtained by the action of the operator-valued Macdonald function labelled by the skew Young diagram $Y_1/Y_2$,

$$Q^q_{Y_1/Y_2}\left(a_-\right)|\emptyset\rangle = |Q^q_{Y_1/Y_2}\rangle$$

Setting $Y_1 = \emptyset$ in equation (4.12), we force $Y = \emptyset$ in the sum on the left hand side,

$$\prod_{i=1}^{\infty} \Gamma^q_{Y/Y_i} \left(x_i\right) |Q^q_{Y}\rangle = \sum_{Y} P^q_{Y/Y_i} \left(x\right) |Q^q_{Y}\rangle$$

(4.14)

4.2.3. The action of $\Gamma^q_+$ on a left-state, and $\Gamma^q_-$ on a right-state

Using equations (4.9) and (4.14), then equation (2.15),

$$\langle P^q_{Y_1} | \prod_{i=1}^{\infty} \Gamma^q_{Y} \left(x_i\right) \prod_{j=1}^{\infty} \Gamma^q_{Y} \left(y_j\right) |Q^q_{Y_2}\rangle$$

$$= \sum_{Y} Q^q_{Y_1/Y_2} \left(x\right) P^q_{Y_1/Y_2} \left(y\right) = \prod_{i,j, \{n+1\}} \left(1 - x_i y_j q^n\right) \sum_{Y} Q^q_{Y_1/Y_2} \left(x\right) P^q_{Y_1/Y_2} \left(y\right)$$

(4.15)

On the other hand, using the $qt$-vertex operator commutation relation, equation (3.6), then inserting a complete set of orthonormal states, the left hand side of equation (4.15) can be re-written as,

$$\langle P^q_{Y_1} | \prod_{i=1}^{\infty} \Gamma^q_{Y} \left(x_i\right) \prod_{j=1}^{\infty} \Gamma^q_{Y} \left(y_j\right) |Q^q_{Y_2}\rangle$$

$$= \prod_{i,j, \{n+1\}} \left(1 - x_i y_j q^n\right) \langle P^q_{Y_1} | \prod_{j=1}^{\infty} \Gamma^q_{Y} \left(y_j\right) \prod_{i=1}^{\infty} \Gamma^q_+ \left(x_i\right) |Q^q_{Y_2}\rangle$$

$$= \prod_{i,j, \{n+1\}} \left(1 - x_i y_j q^n\right) \sum_{Y} \langle P^q_{Y_1} | \prod_{j=1}^{\infty} \Gamma^q_{Y} \left(y_j\right) |Q^q_{Y}\rangle \langle P^q_{Y} \prod_{i=1}^{\infty} \Gamma^q_+ \left(x_i\right) |Q^q_{Y_2}\rangle$$

(4.16)

Comparing equations (4.15) and (4.16),

$$\left\langle P^q_{Y_1} | \prod_{i=1}^{\infty} \Gamma^q_+ \left(x_i\right) \right| Q^q_{Y_2}\rangle = P^q_{Y_1/Y_2} \left(x\right),$$

$$\langle P^q_{Y_1} | \prod_{i=1}^{\infty} \Gamma^q_+ \left(x_i\right) \left| Q^q_{Y_2}\rangle\right. = Q^q_{Y_1/Y_2} \left(x\right) \left| Q^q_{Y_2}\rangle\right.$$  

(4.17)

Since the states $\langle P^q_{Y_1} |$ form a basis of left-states, and $|Q^q_{Y_2}\rangle$ form a basis of right-states, and given the $qt$-inner product equation (2.14),

$$\left\langle P^q_{Y_1} | \prod_{i=1}^{\infty} \Gamma^q_+ \left(x_i\right) \right| = \sum_{Y} \langle P^q_{Y_1} \left| a_y \right. \rangle \left(\prod_{i=1}^{\infty} \Gamma^q_+ \left(x_i\right) \right) = \sum_{Y} \langle P^q_{Y_1} \left| a_y \right. \rangle \left(\prod_{i=1}^{\infty} \Gamma^q_+ \left(x_i\right) \right)$$

(4.18)
where \( \alpha_Y \left( x \right) \) and \( \beta_Y \left( x \right) \) are expansion coefficients that carry the dependence on the variables \( \left( x \right) \) and \( \left( y \right) \), while the expansion is in the set of Young diagrams \( Y \). Using equation (4.17), we determine \( \alpha_Y \left( y \right) \) and \( \beta_Y \left( x \right) \),

\[
\langle P_{qt}^{Y_1} | Y \rangle \prod_{i=1}^{\infty} \Gamma_{qt}^{\pm} \left( x_i \right) | Q_{qt}^{Y_2} \rangle = \sum_{Y} \langle P_{qt}^{Y_1} | P_{qt}^{Y_1/y} \left( y \right) \rangle \prod_{i=1}^{\infty} \Gamma_{qt}^{\pm} \left( x_i \right) | Q_{qt}^{Y_2} \rangle,
\]

which we need in the derivation of the Macdonald topological vertex in section 5.

5. The Macdonald vertex

Following [1, 19], we use the operator-valued Macdonald function identities derived in section 4 to construct the Macdonald vertex that represents the matrix element of a state labelled by a Young diagram \( Y_3 \), between a left-state labelled by a Macdonald function \( P_{qt}^{Y_1} \left( x \right) \), a right-state labelled by a dual Macdonald function \( Q_{qt}^{Y_2} \left( y \right) \).

5.1. Four steps

We construct the Macdonald vertex \( M_{qt}^{Y_1 Y_3 Y_2} \left( x,y \right) \), that is labelled by three finite Young diagrams \( Y_1, Y_2, Y_3 \), and depends on two parameters \( \left( q,t \right) \), in four steps. 1 We consider the infinitely-extended profile of \( Y_3 \), and construct an infinite sequence of vertex operators, \( \left( \prod_{\text{Maya}} \Gamma_{qt}^{+} \right) \) that encodes this profile, 2 We commute the \( qt \)-vertex operators in \( \left( \prod_{\text{Maya}} \Gamma_{qt}^{+} \right) \) so that all \( \Gamma_{qt}^{-} \) vertex operators are on the left, and all \( \Gamma_{qt}^{+} \) vertex operators are on the right, and in the process, we pick up a product of \( \left( x,y,q,t \right) \)-dependent factors, 3 we compute the expectation value of the resulting configuration of vertex operators between a Macdonald state labelled by \( Y_1 \) on the left, and another labelled by \( Y_2 \) on the right, using the operator-valued Macdonald function identities, and finally, 4 we normalize the result so that \( M_{qt}^{\emptyset \emptyset \emptyset} \left( x,y \right) = 1 \). But before we do that, we need to recall simple correspondences between Young diagrams, Maya diagrams, and sequences of vertex operators, and make a number of remarks on the choice of arguments in sequences of vertex operators.

5.2. A Young diagram/Maya diagram/vertex operator correspondence

Given a Young diagram \( Y \), we position it as in figure 3, consider its infinite profile, map the segments of the profile to black and white stones as follows,

\[
/ = | , \quad \backslash = \bullet ,
\]

(5.1)
to obtain a Maya diagram [46]. From the Maya diagram, obtain an infinite sequence of \( qt \)-vertex operators, by mapping the black and white stones to vertex operators, so that we end up with the correspondences,

\[
/ = \bigcirc = \Gamma_{qt}^{+} , \quad \backslash = \bullet = \Gamma_{qt}^{-} ,
\]

(5.2)
5.3. The choice of arguments in sequences of vertex operators

To reproduce the refined topological vertex of Iqbal et al from the Macdonald (refined) topological vertex in the limit $q \to t$, we choose the arguments in the $q t$-vertex operators as in [1],

\[
\begin{align*}
\Gamma_{qt}^+ & \left(x^{-i}y^3, i \in \left\{ \cdots, -2, -1 \right\} \right), \\
\Gamma_{qt}^- & \left(y^{j-1}x^{-y_3}, j \in \left\{ 0, 1, \cdots \right\} \right),
\end{align*}
\]  

where $y_3, i$ is the length of the $i$-row of the Young diagram $Y_3$ that labels the upper boundary of the vertex, and $y_3', j$ is the length of the $j$-row of the transpose Young diagram $Y'_3$.

5.3.1. Remarks on the arguments in the vertex operators with reference to figure 3. The arguments in the infinite sequences of vertex operators, that we need to construct, can be explained, with reference to figure 3 as follows. 1 For $Y_3 = \emptyset$, the assignment of arguments simplify,

\[
\begin{align*}
\Gamma_{qt}^+ & \left(x^i \right), i \in \left\{ \cdots, -2, -1 \right\}, \\
\Gamma_{qt}^- & \left(y^j \right), j \in \left\{ 0, 1, \cdots \right\}
\end{align*}
\]  

2 Consider the infinitely-extended profile of $Y_3 = \emptyset$. One can think of the left-half of this profile as the negative $x$-axis in a Cartesian coordinate system in $\mathbb{R}^2$, and of the right-half as the positive $y$-axis. 3 Split the negative $x$-axis into $x$-segments of unit length each, and label them with $i \in \left\{ \cdots, -2, -1 \right\}$, and split the positive $y$-axis into $y$-segments of unit length each, and label them with $j \in \left\{ 0, 1, \cdots \right\}$. 4 Consider the infinite extended profile of $Y_3 \neq \emptyset$, as in figure 3. Finitely-many $x$-segments and the $y$-segments are shifted away from their original positions, but they remain parallel to the latter. 5 The $x$-segments correspond to white stones, which in turn correspond to $\Gamma_{qt}^+$ vertex operators. The $y$-segments correspond to black stones, which in turn correspond to $\Gamma_{qt}^-$ vertex operators. 6 The argument of $\Gamma_{qt}^+$ that corresponds to an $x$-segment whose $Y_3 = \emptyset$-position is $i \in \left\{ \cdots, -2, -1 \right\}$ on the negative $x$-axis, and was shifted by $y_3$ away from that position is $x^{-i}y^3$. 7 The argument of $\Gamma_{qt}^-$ that corresponds

![Figure 3. The Young diagram/Maya diagram correspondence for $Y = \left\{ 4, 3, 3, 2 \right\}$. The infinite profile of the Young diagram is indicated with a heavy line. The integer below a stone is its position in the Maya diagram. The apex of the inverted Young diagram is located between positions $-1$ and $0.$](image-url)
to a y-segment whose \( y_3 = \emptyset \) position is \( \{ j - 1 \}, j \in \{ 1, 2, \ldots \} \), on the positive y-axis, and was shifted by \( y_3^t \) away from that position is \( \{ y_j^{t-1} \ \Gamma y_j^t \} \). We label the positions of the y-segments with \( \{ j - 1 \}, j \in \{ 1, 2, \ldots \} \), to conform with the conventions common in the literature.

### 5.3.2. Example

The Young diagram/Maya diagram correspondence in figure 3 leads to the vertex-operator sequence,

\[
\prod_{\text{Maya}(y)} \Gamma^{q_f \pm} = \cdots \Gamma^{q_f +} (x^{-5}) \Gamma^{q_f +} (x^{-4}) \Gamma^{q_f +} (y x^{-4}) \Gamma^{q_f +} (x^{-4} y^2) \Gamma^{q_f +} (y^2 x^{-3}) \\
\Gamma^{q_f +} (x^{-3} y^3) \Gamma^{q_f +} (x^{-2} y^3) \Gamma^{q_f +} (y^3 x^{-1}) \Gamma^{q_f +} (x^{-1} y^4) \Gamma^{q_f +} (y^4) \cdots
\]

(5.5)

To obtain the ‘unnormalized’ Macdonald vertex, \( M_{Y_1, Y_2, Y_3}^{qt, \text{unnorm}} \), we need to evaluate the sequence \( \prod_{\text{Maya}(y)} \Gamma^{q_f \pm} \) between a left-state labelled by \( Y_1 \) and a right-state labelled by \( Y_2 \).

\[
M_{Y_1, Y_2, Y_3}^{qt, \text{unnorm}} (x, y) = \langle P_{Y_2}^{q_f} | \prod_{\text{Maya}(y)} \Gamma^{q_f \pm} | Q_{Y_2}^{q_f} \rangle
\]

(5.6)

The final, normalised Macdonald vertex, \( M_{Y_1, Y_2, Y_3}^{qt} \), is,

\[
M_{Y_1, Y_2, Y_3}^{qt} (x, y) = \frac{M_{Y_1, Y_2, Y_3}^{\text{unnorm}} (x, y)}{M_{\emptyset, \emptyset, \emptyset}^{\text{unnorm}} (x, y)}
\]

(5.7)

where \( M_{\emptyset, \emptyset, \emptyset}^{\text{unnorm}} (x, y) \) is easily identified with the \( (x, y, q, r) \)-MacMahon partition function on the right hand side of equation (1.8). We are now in a position to construct the Macdonald vertex following the four steps outlined at the start of this section.

### 5.4. Step 1

We prepare an infinite sequence of \( q_t \)-vertex operators \( \prod_{\text{Maya}(y)} \Gamma^{q_f \pm} \) that encodes the infinite profile of the finite Young diagram \( Y_3 \).

### 5.5. Step 2

Given \( \prod_{\text{Maya}(y)} \Gamma^{q_f \pm} \), there are two ways to ‘straighten’ the sequence in preparation for evaluating its expectation value between states, we can either 1 perform a finite number of commutations to put all \( \Gamma^{q_f +} \) vertex operators on the right, and all \( \Gamma^{q_f -} \) vertex operators on the left, or 2 perform an infinite number of commutations to put all \( \Gamma^{q_f +} \) vertex operators on
the right, and all $\Gamma^{q/i}_+$ vertex operators on the left. In the present work, we choose the latter in order to obtain expressions that reduce to those in the literature in appropriate limits. From equation (3.6),
$$\Gamma^{q/i}_+\left(x^{-i}y^{\gamma_0}\right)\Gamma^{q/i}_-\left(y^{j-1}x^{\gamma_j'}\right) = \prod_{n=0}^{\infty}\left(1 - x^{-i}\gamma_{n}y^{j+i}\gamma_{n}q^n t\right)\Gamma^{q/i}_+\left(y^{j-1}x^{-i}\gamma_{j}\right)\Gamma^{q/i}_-\left(x^{-i}y^{\gamma_j}\right),$$
(5.8)
where $\gamma_{3,i} = \gamma_{3,i} + 1$, and $\gamma_{3,i}$ is the length of the $i$-row in $Y_j$. In the limit $q \to t$, the factor on the right hand side of equation (5.8) reduces to that in the corresponding commutation relation that appears in the derivation of the refined topological vertex in [1]. If we think of $\Gamma^{q/i}_+$ as attached to a segment $\big/$ in the extended profile of $Y_j$, and $\Gamma^{q/i}_-$ as attached to an adjacent segment $\big\backslash$, to the right of the former, then the commutation relation, equation (5.8) describes replacing the adjacent pair $\big\backslash\big/$, with the pair $\big\backslash\big/\big\backslash$, thereby adding a cell to $Y_j$, to generate a larger Young diagram. The exponents that appear in the factor on the right hand side of equation (5.8) have simple interpretations,
$$\gamma_{3,i} - j = A_{\square}, \quad \gamma_{3,i} - i = L_{\square},$$
(5.9)
where $A_{\square}$ and $L_{\square}$ are the arm-length and the leg-length of the cell $\square$ that is added to $Y_j$ via the commutation in equation (5.8), to generate a larger Young diagram, that is $\square \notin Y_j$. Inserting the sequence $\left(\prod_{\Delta\gamma Y_j}\gamma_{3,i}\right)$ between a left-state $\langle P_{Y_j}^{q/i} \big|$, and a right-state $\big| Q_{Y_j}^{q/i} \rangle$, then commuting the (infinitely-many) $\Gamma^{q/i}_+$ vertex operators to the right of the $\Gamma^{q/i}_-$ vertex operators,
$$\langle P_{Y_j}^{q/i} \big| \prod_{i=1}^{\infty}\Gamma^{q/i}_+\left(x^{-i}y^{\gamma_{i}}\right)\prod_{j=1}^{\infty}\Gamma^{q/i}_-\left(y^{j-1}x^{\gamma_{j}}\right) \big| Q_{Y_j}^{q/i} \rangle = \prod_{n=0}^{\infty}\left(1 - x^{-i}\gamma_{n}y^{j+i}\gamma_{n}q^n t\right)\langle P_{Y_j}^{q/i} \big| \prod_{i=1}^{\infty}\Gamma^{q/i}_+\left(y^{j-1}x^{-i}\gamma_{j}\right)\prod_{j=1}^{\infty}\Gamma^{q/i}_-\left(x^{-i}y^{\gamma_j}\right) \big| Q_{Y_j}^{q/i} \rangle \quad (5.10)$$
5.6. Step 3
Using equation (4.19),
$$\langle P_{Y_j}^{q/i} \big| \prod_{i=1}^{\infty}\Gamma^{q/i}_+\left(x^{-i}y^{\gamma_{i}}\right)\prod_{j=1}^{\infty}\Gamma^{q/i}_-\left(y^{j-1}x^{\gamma_{j}}\right) \big| Q_{Y_j}^{q/i} \rangle = \prod_{n=0}^{\infty}\left(1 - x^{-i}\gamma_{n}y^{j+i}\gamma_{n}q^n t\right)\sum_{y} P_{Y_j/y}^{q/i} \left(\tau^{j-1}x^{-\gamma_{j}}\right) Q_{Y_j/y}^{q/i} \left(x^{\gamma_{j}}y^{-\gamma_{j}}\right) \quad (5.11)$$
where $\tau = \{1, 2, \cdots\}$, $j = \{1, 2, \cdots\}$, and the arguments in $P_{Y_j/y}^{q/i} \left(\gamma^{j-1}x^{-\gamma_{j}}\right)$ and $Q_{Y_j/y}^{q/i} \left(x^{\gamma_{j}}y^{-\gamma_{j}}\right)$ should be understood in the sense of section 1.3.4
5.7. Step 4

To normalize the expression in equation (5.11) such that \( \mathcal{M}_{q,t}^{0,0} = 1 \), we divide it by the \( qt \)-Macmahon partition function, the right hand side of equation (1.8). Using the identity,

\[
\left( \prod_{i\in Y_1} \frac{1 - x^{-L_{i\cap}} y^{-A_{i\cap}} q^n t}{1 - x^{-L_{i\cap}} y^{-A_{i\cap}} q^n} \right) \left( \prod_{i,j=1}^{\infty} \frac{1 - x^{i} y^{-j} q^n t}{1 - x^{i} y^{-j} q^n} \right)^{-1} = \left( \prod_{i\in Y_3} \frac{1 - x^{L_{i\cap}} y^{-A_{i\cap}} q^n t}{1 - x^{L_{i\cap}} y^{-A_{i\cap}} q^n} \right),
\]

which follows from equations (2.8) and (2.11) in [21]. The result of the above steps is that the Macdonald vertex is,

\[
\mathcal{M}_{Y_1,Y_3}^{q,t} (x,y) = \prod_{n=0}^{\infty} \left( \prod_{i\in Y_1} \frac{1 - x^{-L_{i\cap}} y^{-A_{i\cap}} q^n t}{1 - x^{-L_{i\cap}} y^{-A_{i\cap}} q^n} \right) \sum_{y} P_{Y_1/Y}^{q,t} \left( y^{-1} x^{-Y_1} \right) Q_{Y_3/Y}^{q,t} \left( x^{t} y^{-Y_3} \right),
\]

(5.13)

where \( t = \{1,2,\cdots\} \), and the arguments in \( P_{Y_1/Y}^{q,t} \left( y^{-1} x^{-Y_1} \right) \) and \( Q_{Y_3/Y}^{q,t} \left( x^{t} y^{-Y_3} \right) \) are in the sense of section 1.3.4. In the limit \( q \rightarrow t \), all dependence on \( (q,t) \) drops out and we recover,

\[
\mathcal{R}_{Y_1,Y_3} (x,y) = \prod_{i\in Y_1} \frac{1}{1 - x^{-L_{i\cap}} y^{-A_{i\cap}}} \sum_{y} s_{Y_1/Y} \left( y^{-1} x^{-Y_1} \right) s_{Y_3/Y} \left( x^{t} y^{-Y_3} \right),
\]

(5.14)

which, in our choice of variables, is the refined topological vertex of Iqbal et al [1], in the ‘diagonal slicing’ in the sense of section 1.4.3. We do not consider the ‘perpendicular slicing’ in the sense of section 1.4.3, since the additional factors involved cancel out when we glue topological vertices to form instanton partition functions. In the limit \( y \rightarrow x \), we recover the corresponding expression for the original vertex of [3].

6. Remarks on the structure of the Macdonald vertex

We collect a number of remarks on the structure of the Macdonald vertex.

6.1. Not all legs are on equal footing

A topological vertex has three external legs. The \( x \)-leg, labelled by a Young diagram \( Y_1 \), and associated with the negative \( x \)-coordinates in the plane partition representation, and the \( y \)-leg, labelled by a Young diagram \( Y_2 \), and associated with the positive \( y \)-coordinates in the plane partition representation, are ‘non-preferred legs’. The remaining leg, labelled by the Young diagram \( Y_3 \), which is encoded in the sequence of vertex operators used to construct the vertex. Given the asymmetric way that the vertex is constructed, not all legs are on equal footing, and the third leg labelled by \( Y_3 \) is usually called ‘the preferred leg’.

\textsuperscript{45} Our \( (x,y) \) are \( (q,t) \) in [1], and our convention for \( Y_3 \) is the transpose of the corresponding Young diagram in [1].
6.2. The planar representation of the vertex

There are various, equivalent ways to represent the Macdonald vertex, \( M_{Y_1 Y_2 Y_3}^{q \ell} \{ x, y \} \), or any topological vertex for that matter, as a trivalent vertex in a diagram. For the purposes of section 6, it is convenient to use the conventions in figure 4. The \(-x\)-leg, associated with the negative \( x\)-coordinates in the plane partition representation of the Macdonald vertex, comes vertically from top of the diagram, labelled by \( Y_1 \). Turning clockwise around the vertex, one encounters the \(+y\)-leg, associated with the positive \( y\)-coordinates in the plane partition representation, going to the bottom, labelled by a Young diagram \( Y_2 \). The preferred leg is horizontal, labelled by \( Y_3 \).

6.3. The arguments and the diagrams

The \(-x\)-leg is associated with the Macdonald function \( P_{Y_1}^{q \ell} \), and when \( Y_3 = \emptyset \), the argument of \( P_{Y_1}^{q \ell} \) is \( \{ x \} \). The \(+y\)-leg is associated with the dual Macdonald function \( Q_{Y_2}^{q \ell} \), and when \( Y_3 = \emptyset \), the argument of \( Q_{Y_2}^{q \ell} \) is \( \{ y \} \). In this sense, \( Y_3 \) is a ‘mixing Young diagram’ that mixes the arguments of the Macdonald functions. When \( Y_3 \neq \emptyset \), the argument of \( P_{Y_1}^{q \ell} \) is no longer purely \( x \), and the argument of \( Q_{Y_2}^{q \ell} \) is no longer purely \( y \). In the planar representation of \( M_{Y_1 Y_2 Y_3}^{q \ell} \{ x, y \} \), in figure 4, \( Y_1 \) labels the vertical leg, \( Y_2 \) the diagonal leg, and \( Y_3 \) the horizontal (preferred) leg. These remarks be useful to check consistency when we glue Macdonald vertices.

6.4. The Macdonald vertex versus the refined topological vertex of Awata and Kanno [20, 21]

Using the notation,

\[
|Y| = \left( y_1 + y_2 + \cdots \right), \quad |Y_1/Y_2| = |Y_1| - |Y_2|, \quad \|Y\| = \left( y_1^2 + y_2^2 + \cdots \right),
\]

\[
\rho = \left( -\frac{1}{2}, -\frac{3}{2}, \cdots \right),
\]

the refined topological vertex of Awata and Kanno [20, 21] is,
\[ R_{\mathcal{Y}_1 \mathcal{Y}_2}^{\lambda_1 \lambda_2} (x, y) = \left( -1 \right)^{\frac{|\lambda_1|}{2} + |\lambda_2| + \frac{1}{2} |\mathcal{Y}_1| + \frac{1}{2} |\mathcal{Y}_2|} p^{\lambda_1} (y) \sum_{\mathcal{Y}} \left( \frac{1}{\mathcal{Y}} \right)^{\frac{1}{2} |\mathcal{Y}|} \left( \mathcal{Y} \right)^{\frac{1}{2} |\mathcal{Y}|} \mathcal{M}_{\mathcal{Y}_1 \mathcal{Y}_2}^{q,t} (x, y) \mathcal{M}_{\mathcal{Y}_1 \mathcal{Y}_2}^{q,t} (y, x) \mathcal{M}_{\mathcal{Y}_1 \mathcal{Y}_2}^{q,t} (x, y), \]  

where \( \iota_{AK} \) is an involution that acts on the power-sum functions as,

\[ \iota_{AK} (p_n) = -p_n. \]  

Since the arguments in the Macdonald symmetric functions in \( \mathcal{M}_{\mathcal{Y}_1 \mathcal{Y}_2}^{q,t} (x, y) \) depend on the variables \( (x, y) \), while Macdonald parameters \( (q, t) \) are additional, independent parameters, \( \mathcal{M}_{\mathcal{Y}_1 \mathcal{Y}_2}^{q,t} (x, y) \) is not the same as the refined topological vertex of Awata and Kanno.

7. The Macdonald \( U \left( \frac{1}{2} \right) \) strip partition function

We glue four Macdonald vertices to obtain the Macdonald-analogue of the normalized contribution of the bifundamental hypermultiplet to the 5D \( U \left( \frac{1}{2} \right) \) instanton partition function. The
rules that we use are the same as those used in [1] plus an additional rule specific to Macdonald vertices that requires that we glue a vertex that depends on \(q, t\) to a vertex that depends on \(t, q\), and vice versa.

7.1. From topological vertices to partition functions

Topological vertices are simple combinatorial objects that can be glued to form arbitrarily complicated 5D instanton partition functions in 5D supersymmetric gauge theories [47, 48]. These in turn are equal to off-critical deformations of conformal blocks in 2D conformal field theories [12, 27, 28]. In the 4D-limit discussed in section 8, one recovers 4D instanton partition functions that are equal to critical conformal blocks [10]. Let us focus on instanton partition functions in \(U_1 \times \cdots \times U_{N+1}\) gauge theories, \(N \in \{0, 1, \ldots\}\) in these cases, the 5D instanton partition functions are equal to off-critical conformal blocks in diagonal Virasoro \(\times U(1)\)-conformal field theories. These instanton partition functions can be built in two steps.

7.1.1. Gluing vertices to form a strip. One starts by gluing four topological vertices to form a partition function that, following Iqbal and Kashani-Poor [23, 24], we call the \(U(2)\) ‘strip’ partition function, see figure 5. This gluing is non-trivial in the sense that it requires the use of non-trivial symmetric function identities. At the level of 2D conformal field theory, the 5D strip is equal to the matrix element of a primary field between two arbitrary states in an off-critical deformation of the conformal field theory. In the 4D-limit discussed in section 8, we recover the corresponding critical expressions.

7.1.2. Gluing strips to form an instanton partition function. One glues \((N + 1)\) copies of the strip partition function to obtain the instanton partition function of a \(U_1 \times \cdots \times U_{N+1}\) 5D supersymmetric gauge theory. This gluing is essentially trivial in the sense that one proceeds as follows. 1 Assign Kähler parameters to the would-be internal legs, 2 Take the product of the strip partition functions, with the same partitions on the external legs that we wish to glue into internal legs, and 3 Sum over all partitions that label the new internal legs, weighted with the assigned Kähler parameters, without further evaluation. In this section, we focus on the gluing of four Macdonald vertices to obtain the Macdonald analogue of the \(U(2)\) strip partition function, which is the non-trivial step in the computation of the full instanton partition functions and conformal blocks.

72. \(\mathcal{M}_{Y_1, Y_2, Y_3}^{q_1, q_2, q_3} (x, y)\) and \(\mathcal{M}_{Y_1, Y_2, Y_3}^{q_1, q_2, q_3} (y, x)\)

In gluing Macdonald vertices, we need two versions of Macdonald vertices that differ in the way that we assign the variables. The first, \(\mathcal{M}_{Y_1, Y_2, Y_3}^{q_1, q_2, q_3} (x, y)\), is that defined in equation (5.13). The second, \(\mathcal{M}_{Y_1, Y_2, Y_3}^{q_1, q_2, q_3} (y, x)\), is obtained from the first by swapping the variable assignments \(x \leftrightarrow y\), and \(q \leftrightarrow t\).
73. Four vertices make a $U(2)$ strip

The $U(2)$ strip diagram is formed by gluing four topological vertices along their non-preferred $x$- and $y$-legs, as in figure 5. Each of these vertices is of type $\mathcal{M}_{Y_i, Y_j}^{q_t} \left( x, y \right)$ or $\mathcal{M}_{Y_i, Y_j}^{t_q} \left( y, x \right)$ as follows. As in figure 5, there are two external legs to the left, assigned partitions $\left\{ V_1, V_2 \right\}$, two external legs to the right, assigned partitions $\left\{ W'_1, W'_2 \right\}$. The internal lines are assigned exponentiated Kähler parameters $Q_i$, $i = 1, 2, 3$, and partitions $Y_1$, $Y_2$, and $Y_3$, from top to bottom. The vertical external legs are assigned empty partitions, and no Kähler parameters.

74. The rules of gluing

We consider two versions of the Macdonald vertex, $\mathcal{M}_{Y_i, Y_j}^{q_t} \left( x, y \right)$ and $\mathcal{M}_{Y_i, Y_j}^{t_q} \left( y, x \right)$, and glue them as follows. 1 In gluing vertices along their non-preferred legs, that is the $x$-legs and $y$-legs, we can only glue a $\mathcal{M}_{Y_i, Y_j}^{q_t} \left( x, y \right)$ vertex and $\mathcal{M}_{Y_i, Y_j}^{t_q} \left( y, x \right)$ vertex, 2 An $x$-leg can be only glued to an $x$-leg, and a $y$-leg can be only glued to a $y$-leg, 3 In gluing two legs, if one is assigned a partition $Y$, the other must be assigned the transpose partition $Y'$, and finally 4 The gluing is weighted by an exponentiated Kähler parameter $Q$, which contributes a factor $\left( -Q \right)^{\left| Y \right|}$ to the result.

75. The normalized $U(2)$ strip $S_{VW\Delta}^{\text{norm}}$

We compute the partition function that corresponds to the strip diagram, then normalize the result. In the context of the original, and the refined topological vertices, this calculation is standard, see for instance [8, 49]. In the present work, we follow the computation in [50]. We write the normalized $U(2)$ strip partition function as,

$$S_{VW\Delta}^{\text{norm}} = \frac{S_{VW\Delta}}{S_{\emptyset \emptyset \Delta}},$$

(7.1)

where $\emptyset = \left\{ \emptyset, \emptyset \right\}$, and $\emptyset$ is the empty Young diagram\footnote{We write $S_{VW\Delta}^{\text{norm}}$ rather than $S_{VW\Delta}^{\text{norm}} \left( x, y \right)$ and do the same for the factors of $S_{VW\Delta}^{\text{norm}} \left( x, y \right)$ discussed below, as that simplifies the notation without leading to ambiguities. On the other hand, we write $\mathcal{M}_{Y_i, Y_j}^{q_t} \left( x, y \right)$ to distinguish it from $\mathcal{M}_{Y_i, Y_j}^{t_q} \left( y, x \right)$ as in equation (7.2).}. The numerator on the right hand side of equation (7.1) is,

$$S_{VW\Delta} = \sum_{y_1, y_2, y_3} \left( -Q_1 \right)^{|y_1|} \left( -Q_2 \right)^{|y_2|} \left( -Q_3 \right)^{|y_3|} \mathcal{M}_{\emptyset, y_1}^{q_t} \left( x, y \right) \mathcal{M}_{y_1, y_2, y_3}^{t_q} \left( y, x \right) \mathcal{M}_{y_2, y_3, W'_1}^{q_t} \left( x, y \right) \mathcal{M}_{y_3, W'_1, W'_2}^{t_q} \left( y, x \right),$$

(7.2)

where $Q_i$, $i = 1, 2, 3$ are exponentiated Kähler parameters. We glue $x$-legs to $x$-legs and $y$-legs to $y$-legs, so the dependence of the vertices on $\left( x, y \right)$, and consequently the dependence on $\left( q, t \right)$ as well, must alternate along the strip. The denominator on the right hand side of equation (7.1) is the partition function of the same strip diagram, but with all external partition pairs empty.
7.6. Evaluating the numerator $S_{VW\Delta}$

Using the notation,

$$Z^{qt}_y(x, y) = \prod_{n=0}^{\infty} \left( \prod_{\square \in Y} \frac{1 - x^{L_{\square, x}} q^n}{1 - x^{L_{\square, x}} y A_{\square, x} q^n} \right),$$  \hspace{1cm} (7.3)

together with the fact that,

$$P^{qt}_{0/y} (x) = P^{qt}_{0/q} (x) = 1, \quad Q^{qt}_{0/y} (x) = Q^{qt}_{0/q} (x) = 1,$$  \hspace{1cm} (7.4)

and the definition of the Macdonald vertex in equations (5.13) and (7.2) becomes,

$$S_{VW\Delta} = Z^{qt}_{V_1} (x, y) Z^{qt}_{W_1} (y, x) Z^{qt}_{V_2} (x, y) Z^{qt}_{W_2} (y, x)$$

$$\sum_{Y_1, Y_2, Y_3} (-Q_1)^{|Y_1|} (-Q_2)^{|Y_2|} (-Q_3)^{|Y_3|}$$

$$Q^{qt}_{Y_1/Y_2} (x^t, y^{-V_1}) \left( \sum_{X_4} Q^{qt}_{Y_4/Y_3} (y^j x^{-W_1^2}) P^{qt}_{E_4/Y_4} (x^{t-1} y^{-W_1}) \right)$$

$$\left( \sum_{X_5} P^{qt}_{Y_5/Y_4} (y^j x^{-W_2}) Q^{qt}_{Y_1/Y_5} (x^t y^{-V_2}) \right) Q^{qt}_{Y_1/Y_3} (y^j x^{-W_3}),$$  \hspace{1cm} (7.5)

where the arguments in the Macdonal and dual Macdonald functions should be understood in the sense of section 1.3.4.

7.6.1. Remark. The first Macdonald vertex from the top contributes $Q^{qt}_{Y_1} (x^t y^{-V_1})$ to the sum on the right hand side of equation (7.5), since the Macdonald function associated with the vertical $x$-leg of this vertex, which is external, is trivial, and $P^{qt}_{0} = 1$. Similarly, the fourth Macdonald vertex from the top contributes $Q^{qt}_{Y_1} (y^j x^{-W_3})$, since the Macdonald function associated with the vertical $x$-leg of this vertex, which is also external, is also trivial $P^{qt}_{0} = 1$.

Due to alternating the $\{x, y\}$ and $\{q, t\}$ dependence of adjacent Macdonal vertices, the external legs of both vertices are $x$-legs.

7.6.2. The $Y_1$-sum. From the definition of the skew Macdonal function,

$$Q^{[Y_1/Y_2]} (x) = P^{qt}_{Y_1/Y_2} (Q x). \quad Q^{[Y_1/Y_2]} (x) = P^{qt}_{Y_1/Y_2} (Q x).$$  \hspace{1cm} (7.6)

and using the Cauchy identity, equation (2.26),

$$\sum_{Y_1} (-Q_1)^{|Y_1|} Q^{qt}_{Y_1} (x^t y^{-V_1}) Q^{qt}_{Y_1/Y_4} (y^j x^{-W_1^2})$$

$$= \prod_{i,j=1}^{\infty} \left( 1 - Q_1 x^{-w_{i,j} y^{-v_{i,j} + j}} \right) Q^{qt}_{Y_1} \left(-Q_1 x^t y^{-V_1} \right)$$  \hspace{1cm} (7.7)
76.3. The $Y_3$-sum. The $Y_3$-sum is evaluated similarly to the $Y_1$-sum,

$$
\sum_{Y_3} \left(-Q_3\right)^{|Y_3|} Q_{2}^\prime Q_{3} Q_{r}^\prime \left( y^{j} x^{-w'} \right) Q_{3} Q_{r}^\prime \left( x^{j} y^{-v} \right) = \prod_{i,j=1}^{\infty} \left( 1 - Q_3 x^{-w_{i,j}\tau + i} y^{-v_{i,j}\tau + j} \right) Q_{3}^Q Q_{r}^\prime \left( -Q_3 x_{3}^{j} x_{-v} \right) \tag{7.8}
$$

76.4. The $Y_2$-sum. Using the Cauchy identity, equation (2.27), we re-write this in terms of a sum over a new set of partition $Y_6$,

$$
\sum_{Y_2} \left(-Q_2\right)^{|Y_2|} \sum_{Y_6} P_{Y_2}^{\varphi} \left( y^{j} x^{-v} \right) P_{Y_6}^{\varphi} \left( x^{j} y^{-v} \right) = \left(-Q_2\right)^{|Y_2|} \sum_{i,j=1}^{\infty} \left( 1 - Q_2 x^{-v_{i,j}^{+\tau} + i} y^{-w_{i,j}^{+\tau} + j} \right) \sum_{Y_6} P_{Y_6}^{\varphi} \left( y^{j} x^{-v} \right) P_{Y_6}^{\varphi} \left( -Q_2 x^{j} y^{-v} \right), \tag{7.9}
$$

where $v_{i,j}^{+\tau} = v_{i,j} + 1$, and $w_{i,j}^{+\tau} = w_{i,j} + 1$. We use this notation below.

76.5. The $Y_\ell$-sum. Using the Cauchy identity, equation (2.25),

$$
\sum_{Y_\ell} \left(-Q_\ell\right)^{|Y_\ell|} Q_{12}^Q Q_{r}^\prime \left( y^{j} x^{-v} \right) P_{Y_\ell}^{\varphi} \left( x^{j} y^{-v} \right) = \prod_{i} Q_{12} \left( Q_{12}^{x} y^{-v}, y^{j} x^{-v} \right) Q_{r}^\prime \left( Q_{12}^{x} y^{-v} \right), \tag{7.10}
$$

where $Q_{12} = Q_1 Q_2$, and the arguments in $\Pi^{\varphi} \left( Q_{12}^{x} y^{-v}, y^{j} x^{-v} \right)$ are meant in the following sense.

76.6. Constructing $\Pi^{\varphi} \left( Q_{12}^{x} y^{-v}, y^{j} x^{-v} \right)$. Consider a scalar $Q_{12}$, two sequences $\ell = \{1, 2, \cdots\}$ and $\ell' = \{1, 2, \cdots\}$, and two Young diagrams $V_1 = \{v_{1,1}, V_{1,2}, \cdots\}$ and $V_2 = \{v_{2,1}, v_{2,2}, \cdots\}$. Consider the transpose $V_2' = \{v_{2,1}', v_{2,2}', \cdots\}$, and form the product sequences,

$$
Q_{12}^{x} y^{-v} = \left( Q_{12} x y^{-v_1}, Q_{12} x^2 y^{-v_2}, \cdots \right), \quad y^{j} x^{-v} = \left( x^{-v_1}, x y^{-v_2}, \cdots \right) \tag{7.11}
$$

$\Pi^{\varphi} \left( Q_{12}^{x} y^{-v}, y^{j} x^{-v} \right)$ is understood in the sense of equation (2.18), but with the elements of $x = \{x_1, x_2, \cdots\}$ in equation (2.18) replaced by the elements of $Q_{12}^{x} y^{-v}$ in equation (7.11), and the elements of $y = \{y_1, y_2, \cdots\}$ in equation (2.18) replaced by the elements of $y^{j} x^{-v}$ in equation (7.11).
76.7. The $Y_6$-sum. Similarly,
\[
\sum_{Y_6} Q_{Y_6}^{tq} \left( -Q_3 y^t x^{-w_i^t} \right) P_{Y_6}^{tq} \left( -Q_2 x^{t-1} y^{-w_i} \right) = \Pi^{tq} \left( Q_{23} x^{t-1} y^{-w_i}, y^t x^{-w_i^t} \right) Q_{Y_6}^{tq} \left( -Q_3 y^t x^{-w_i^t} \right),
\]
where $Q_{23} = Q_2 Q_3$.

76.8. The $Y_6$-sum. Finally, using the Cauchy identity, equation (2.16), the $Y_6$-sum introduced in equation (7.9) is evaluated,
\[
\sum_{Y_6} Q_{Y_6}^{tq} \left( Q_{12} x^t y^{-v_i} \right) Q_{Y_6}^{tq} \left( -Q_3 y^t x^{-w_i^t} \right) = \prod_{i,j=1}^{\infty} \left( 1 - Q_{123} x^{-w_i}, y^{v_i}, y^{-v_i}, y^{v_i}, y^{-v_i}, y^{w_i} \right),
\]
where $Q_{123} = Q_1 Q_2 Q_3$.

76.9. Remark. The factors collected on the right sides of equations (7.7)–(7.9) and (7.13) are independent of $(q,t)$.

77. The evaluated numerator $S_{VW\Delta}$

From equations (7.5) and (7.7)–(7.13), the evaluated numerator of the $U\left\{2\right\}$ strip partition function is,
\[
S_{VW\Delta} = A_{VW\Delta} B_{VW\Delta} C_{VW\Delta},
\]
where,
\[
A_{VW\Delta} = Z_{V_1}^{q_t} \left( x, y \right) Z_{V_2}^{q_t} \left( x, y \right) Z_{W_1}^{q_t} \left( x, y \right) Z_{W_2}^{q_t} \left( x, y \right),
\]
\[
B_{VW\Delta} = \prod_{i,j=1}^{\infty} \left( 1 - Q_{1} x^{-v_i} y^{v_i}, y^{-v_i}, y^{v_i}, y^{-v_i}, y^{v_i}, y^{-v_i}, y^{w_i} \right) \left( 1 - Q_{2} x^{-v_i} y^{v_i}, y^{-v_i}, y^{v_i}, y^{-v_i}, y^{w_i} \right) \left( 1 - Q_{3} x^{-v_i} y^{v_i}, y^{-v_i}, y^{v_i}, y^{-v_i}, y^{w_i} \right),
\]
\[
C_{VW\Delta} = \prod_{i,j=1}^{q_t} \left( Q_{12} x^t y^{-v_i} y^{-v_i}, y^t x^{-v_i}, y^{-v_i}, y^{v_i}, y^{-v_i}, y^{v_i}, y^{-v_i}, y^{w_i} \right) \Pi^{tq} \left( Q_{23} x^{t-1} y^{-w_i}, y^t x^{-w_i^t} \right).
\]

78. The evaluated denominator $S_{\emptyset\emptyset\Delta}$

From equation (7.1), the evaluated denominator of the $U\left\{2\right\}$ strip partition function is,
\[
S_{\emptyset\emptyset\Delta} = A_{\emptyset\emptyset\Delta} B_{\emptyset\emptyset\Delta} C_{\emptyset\emptyset\Delta},
\]
where $A_{\emptyset\emptyset\Delta}$, $B_{\emptyset\emptyset\Delta}$ and $C_{\emptyset\emptyset\Delta}$ are easily read off equations (7.15) and (7.17) by setting $V = \emptyset$ and $W = \emptyset$. In particular, $A_{\emptyset\emptyset\Delta} = 1$.
7.9. A normalized infinite product

Given two two Young diagrams, \( V = \{ v_1, v_2, \cdots \} \), \( W = \{ w_1, w_2, \cdots \} \), we define the normalized infinite product\(^{47}\),

\[
\prod_{i,j=1}^{\infty} \left( 1 - \frac{Q x^{i-v_j} y^{j-1-w_i}}{1 - Q x^{i-j} y^{j-1}} \right) = \prod_{i,j=1}^{\infty} \left( 1 - \frac{Q x^{i-v_j} y^{j-1-w_i}}{1 - Q x^{i-j} y^{j-1}} \right),
\]

which is normalized in the sense of the normalized strip partition function, equation (7.1).

7.10. From a normalized infinite product to a finite product

Following \([21, 51]\)^{48}, we re-write the normalized infinite product in equation (7.19) as,

\[
\prod_{i,j=1}^{\infty} \left( 1 - \frac{Q x^{i-v_j} y^{j-1-w_i}}{1 - Q x^{i-j} y^{j-1}} \right) = \prod_{\Box \in V} \left( 1 - \frac{Q x^{A_{\Box W} x^{L_{\Box W}}} y^{L_{\Box W}}} {1 - Q x^{A_{\Box W} x^{L_{\Box W}}} y^{L_{\Box W}}} \right) \prod_{\Box \in W} \left( 1 - \frac{Q x^{A_{\Box W} x^{L_{\Box W}}} y^{L_{\Box W}}} {1 - Q x^{A_{\Box W} x^{L_{\Box W}}} y^{L_{\Box W}}} \right),
\]

where, given a cell \( \Box \in V \), \( A_{\Box W} \) is the arm-length, and \( L_{\Box W} \) is the leg-length of \( \Box \), measured with respect to the profile of a Young diagram \( W \), that may be identical to, or different from \( V \). In particular, for \( \Box \notin W \), \( A_{\Box W} \) and \( L_{\Box W} \) are negative.

7.10.1. Remark.
The product on the left hand side of equation (7.20) is normalized in the sense of equation (7.19), but the products on the right hand side are not. Equation (7.20) expresses the normalized infinite products on the left hand side to the non-normalized finite products on the right hand side.

7.11. The evaluated, normalized \( S_{VWW}^{\text{norm}} \)

From equations (7.15)–(7.17), the normalised \( U \) \( (2) \) strip partition function is,

\[
S_{VWW}^{\text{norm}} = \left( A_{VWW}^{\text{norm}} \right) \left( B_{VWW}^{\text{norm}} \right) \left( C_{VWW}^{\text{norm}} \right) = A_{VWW}^{\text{norm}} B_{VWW}^{\text{norm}} C_{VWW}^{\text{norm}},
\]

where the factors are as follows.

7.11.1. \( A_{VWW}^{\text{norm}} \). This is the product of four factors that depend on \( q,t \), generated by normal ordering the \( qt \)-vertex operators in the Macdonald vertex,

\[
A_{VWW}^{\text{norm}} = Z_{V_i}^{q,t} \left( x, y \right) Z_{V_j}^{q,t} \left( x, y \right) Z_{W_i}^{q,t} \left( y, x \right) Z_{W_j}^{q,t} \left( y, x \right),
\]

where \( Z_{V_i}^{q,t} \left( x, y \right) \) is defined in equation (7.3)

\(^{47}\) The fact that this product is normalized is indicated by \( \checkmark \) at the top right corner of the factor. We use this notation in the sequel.

\(^{48}\) Equations (2.8)–(2.11) in [21].
714. **$B_{VW\Delta}^{\text{norm}}$**. This is the product of four factors that do not depend on $(q,t)$, generated in performing sums over states labelled by partitions,

$$
B_{VW\Delta}^{\text{norm}} = \prod_{\ell=1}^{\infty} \left( 1 - Q_{1} x^{-w_{1}^\ell + t_{1}, y^{-v_{1}^\ell + t_{1}}} \right) \left( 1 - Q_{2} x^{-w_{2}^\ell + t_{2}, y^{-v_{2}^\ell + t_{2}}} \right) \left( 1 - Q_{3} x^{-w_{3}^\ell + t_{3}, y^{-v_{3}^\ell + t_{3}}} \right) \left( 1 - Q_{123} x^{-w_{123}^\ell + t_{123}, y^{-v_{123}^\ell + t_{123}}} \right)
$$

Using equation (7.20),

$$
B_{VW}^{\text{norm}} = \prod_{\ell \in \mathcal{V}_{1}} \left( 1 - Q_{1} x^{-L_{1} \ell, y^{-A_{1} \ell}} \right) \prod_{\ell \in \mathcal{W}_{1}} \left( 1 - Q_{1} x^{-\ell \mathcal{W}_{1}, y^{A_{1} \ell}} \right) \prod_{\ell \in \mathcal{V}_{2}} \left( 1 - Q_{2} x^{-L_{2} \ell, y^{-A_{2} \ell}} \right) \prod_{\ell \in \mathcal{W}_{2}} \left( 1 - Q_{2} x^{-\ell \mathcal{W}_{2}, y^{A_{2} \ell}} \right) \prod_{\ell \in \mathcal{V}_{3}} \left( 1 - Q_{3} x^{-L_{3} \ell, y^{-A_{3} \ell}} \right) \prod_{\ell \in \mathcal{V}_{3}} \left( 1 - Q_{3} x^{-\ell \mathcal{W}_{3}, y^{A_{3} \ell}} \right) \prod_{\ell \in \mathcal{W}_{3}} \left( 1 - Q_{123} x^{-L_{123} \ell, y^{-A_{123} \ell}} \right) \prod_{\ell \in \mathcal{W}_{3}} \left( 1 - Q_{123} x^{-\ell \mathcal{W}_{3}, y^{A_{123} \ell}} \right),
$$

(7.24)

715. **$C_{VW\Delta}^{\text{norm}}$**. This is the product of two factors that depend on $(q,t)$, generated in performing sums over states labelled by partitions,

$$
C_{VW\Delta}^{\text{norm}} = \frac{\prod_{i,f,n=1}^{q} \left( Q_{12} x^{t_{1}, y^{1-1} x_{1}^{1}}, y_{1}^{1-1} y_{1}^{1-1} y_{1}^{1-1} \right) \Pi_{q}^{t} \left( Q_{23} x^{t_{1}, y^{1-1} y_{1}^{1-1} y_{1}^{1-1} \ell} \right)}{\Pi_{q}^{t} \left( Q_{23} x^{t_{1}, y^{1-1} \ell} \right)},
$$

(7.25)

where the arguments of $\Pi_{q}^{t}$ in equation (7.25) are explained in section 7.6.6. Using equation (7.19),

$$
\frac{\prod_{i,f,n=1}^{q} \left( Q_{x^{1}, y^{1-1} x_{1}^{1}}, y_{1}^{1-1} \right)}{\prod_{i,f,n=1}^{q} \left( Q_{x^{1}, y^{1-1} \ell} \right)} = \prod_{i,f,n=1}^{q} \left( 1 - Q x^{i-1} y^{j-1-w_{1} \ell, q^{n-1} \ell} \right) \left( 1 - Q x^{i-1} y^{j-1-w_{1} \ell, q^{n-1} \ell} \right)
$$

(7.26)

Equation (7.25) becomes,

$$
C_{VW\Delta}^{\text{norm}} = \prod_{i,f} \left( \frac{1 - Q_{12} x^{i-1} y^{j-1-w_{1} \ell, q^{n-1} \ell} \Pi_{q}^{t} \left( 1 - Q_{23} x^{i-1} y^{j-1-w_{1} \ell, q^{n-1} \ell} \right)}{\prod_{i,f,n=1}^{q} \left( Q_{23} x^{t_{1}, y^{1-1} \ell} \right)} \prod_{i,f,n=1}^{q} \left( 1 - Q_{12} x^{i-1} y^{j-1-w_{1} \ell, q^{n-1} \ell} \right) \prod_{i,f,n=1}^{q} \left( Q_{23} x^{t_{1}, y^{1-1} \ell} \right)
$$

(7.27)
7.12. Factorization of the qt-strip into qt-independent and qt-dependent factors

While \( \mathcal{M}_{Y_1 Y_2}^{qt} (x, y) \), equation (5.13), does not factorize into \( \mathcal{R}_{Y_1 Y_2} (x, y) \), equation (5.14), times a qt-dependent factor, the strip partition function \( \mathcal{S}_{VW}^{norm} \), equation (7.21), does.

7.12.1. Factorization of \( \mathcal{A}_{VW}^{norm} \) into qt-independent and qt-dependent factors. From equation (7.22), \( \mathcal{A}_{VW}^{norm} \) is a product of four copies of \( Z_Y^{qt} (x, y) \), defined in equation (7.3). Each of these factorizes,

\[
Z_Y^{qt} (x, y) = \left( \prod_{\square \in V_l} \frac{1}{1 - x^{L_{\square, r} r y^{A_{\square, r}}} y^{A_{\square, r}}} \right) \left( \prod_{n=1}^{\infty} \prod_{\square \in F} \frac{1 - x^{L_{\square, r} r y^{A_{\square, r}}} q^{n-1} t}{1 - x^{L_{\square, r} r y^{A_{\square, r}}} q^n} \right),
\]

where the first factor, on the right hand side of equation (7.28), appears in equation (5.14), and second factor \( \rightarrow 1 \) in the limit \( q \rightarrow t \). Thus,

\[
\mathcal{A}_{VW}^{norm} = \left( \mathcal{A}_{VW}^{norm} \right) (0, 0) \left( \mathcal{A}_{VW}^{norm} \right) (q, t),
\]

\[
\left( \mathcal{A}_{VW}^{norm} \right) (0, 0) = \prod_{n=1}^{2} \prod_{\square = \square_l} \frac{1}{1 - x^{L_{\square, r} r y^{A_{\square, r}}} y^{A_{\square, r}}} \left( \prod_{\square \in W_l} \frac{1}{1 - x^{L_{\square, r} r y^{A_{\square, r}}} q^n} \right),
\]

\[
\left( \mathcal{A}_{VW}^{norm} \right) (q, t) = \prod_{n=1}^{2} \prod_{\square = \square_l} \frac{1}{1 - x^{L_{\square, r} r y^{A_{\square, r}}} y^{A_{\square, r}}} q^{n-1} t \left( \prod_{\square = \square_l} \frac{1}{1 - x^{L_{\square, r} r y^{A_{\square, r}}} q^n} \right).
\]

7.12.2. \( B_{VW}^{norm} \) has no qt-dependent factors. From equation (7.24), \( B_{VW}^{norm} \) has no qt-dependence,

\[
B_{VW}^{norm} = \left( B_{VW}^{norm} \right) (0, 0)
\]

7.12.3. The factorization of \( C_{VW}^{norm} \) into qt-independent and qt-dependent factors. From equation (7.27),

\[
C_{VW}^{norm} = \left( C_{VW}^{norm} \right) (0, 0) \left( C_{VW}^{norm} \right) (q, t),
\]

\[
\left( C_{VW}^{norm} \right) (0, 0) = \prod_{\square \in V_l} \frac{1}{1 - Q_{12} x^{L_{\square, r} r y^{A_{\square, r}}}} \prod_{\square \in V_2} \frac{1}{1 - Q_{12} x^{L_{\square, r} r y^{A_{\square, r}}}} \prod_{\square \in W_l} \frac{1}{1 - Q_{23} x^{L_{\square, r} r y^{A_{\square, r}}}} \prod_{\square \in W_2} \frac{1}{1 - Q_{23} x^{L_{\square, r} r y^{A_{\square, r}}}}.
\]

(7.34)
which is the factor that appears when using $R_{Y_1,Y_2}$ in $[50]^{49}$,

$$
\left( c_{\text{norm}}^{\text{VW}} \right)^{(q,t)} = \prod_{n=1}^{\infty} \prod_{\Box \in V_1} \left( 1 - Q_{12} x^{t \cdot \Box_{12}} y^{-A_{\Box_{12}}} q^n t q^n \right) \prod_{\Box \in V_2} \left( 1 - Q_{12} x^{t \cdot \Box_{12}} y^{-A_{\Box_{12}}} q^n t q^n \right)
$$

\[ (7.35) \]

which $\to 1$, in the limit $q \to t$.

**713. Comments on the structure of $S^{\text{norm}}_{\text{VW}}$**

**713.1. The $q$-t-independent terms.**

$$
\left( c_{\text{norm}}^{\text{VW}} \right)^{(0,0)} = \left( A_{\text{norm}}^{\text{VW}} \right)^{(0,0)} \left( B_{\text{norm}}^{\text{VW}} \right)^{(0,0)} \left( c_{\text{norm}}^{\text{VW}} \right)^{(0,0)} .
$$

\[ (7.36) \]

is the $q$-t-independent factor that we obtain if we use $R_{Y_1,Y_2}$ to compute $S^{\text{norm}}_{\text{VW}}$. Since we choose each of the parameters $\{x,y,q,t\}$ to be $< 1$, all exponents on the right hand sides of equations (7.30) and (7.31) are non-negative, and at least one exponent in each factor is non-zero. $A_{\text{norm}}^{\text{VW}}$ has no poles. $B_{\text{norm}}^{\text{VW}}$ in equation (7.32), has no poles. Each pair of exponents in the same factor on the right hand sides of equation (7.34) is such that, one exponent may (or may not) be positive while the other is negative. This is because in each factor, $\Box$ is in one diagram $Y_1$, so it has positive arm-length and leg-length with respect to the profile of $Y_1$, but may (or may not) be outside the other diagram $Y_2$. When $\Box \not\in Y_2$, it has a negative arm-length and leg-length with respect to the profile of $Y_2$. These cases contribute to the right hand side of equation (7.34), and $\left( c_{\text{norm}}^{\text{VW}} \right)^{(0,0)}$ has poles$^{51}$. These are the poles obtained using $R_{Y_1,Y_2}$ when copies of $\left( c_{\text{norm}}^{\text{VW}} \right)^{(0,0)}$ are glued to form 5D instanton partition functions, the poles in $\left( c_{\text{norm}}^{\text{VW}} \right)^{(0,0)}$ correspond to BPS states in a 5D $\mathcal{N} = 2 U (2)$ gauge theory. In the 2D interpretation of the instanton partition functions, they correspond to states that flow in the internal channels of off-critical deformations of conformal blocks $[27, 28]$.

**713.2. The $q$-t-dependent terms.**

$$
\left( c_{\text{norm}}^{\text{VW}} \right)^{(q,t)} = \left( A_{\text{norm}}^{\text{VW}} \right)^{(q,t)} \left( B_{\text{norm}}^{\text{VW}} \right)^{(q,t)} \left( c_{\text{norm}}^{\text{VW}} \right)^{(q,t)} .
$$

\[ (7.37) \]

$^{49}$ Up to a change in notation that transposes Young diagrams, exchanging arm-lengths and leg-lengths.

$^{50}$ See equations (3.17)-(3.19) in [50], but note the differences in notation. In particular, $Q_{i}, i \in \{ 1, M, 2 \}$, $A_{\Box_{12}, Y}$ and $L_{\Box_{12}, Y}^+$ in [50], become $Q_{i}, i \in \{ 1, 2, 3 \}$, $A_{\Box_{12}, Y}$ and $L_{\Box_{12}, Y}^+$ in the present work. Also $A_{\Box_{12}, Y}$ and $L_{\Box_{12}, Y}^+$ in [50] are expanded as $A_{\Box_{12}, Y}^+ L_{\Box_{12}, Y}^+$ in the present work, cancelling factors of $x^t$ and $y^t$ whose analogues appear in [50].

$^{51}$ See detailed discussion in [52].
is the additional $q_t$-dependent factor in $S^\text{norm}_{\mathbf{W}_\Delta}$ that we obtain when we use $\mathcal{M}^\text{qt}_{Y_1,Y_2}\{x,y\}$, $\left(A^\text{norm}_{\mathbf{W}_\Delta}\right)^{q,t}$ and $\left(C^\text{norm}_{\mathbf{W}_\Delta}\right)^{q,t}$ have no poles for the same reasons that $\left(A^\text{norm}_{\mathbf{W}_\Delta}\right)^{0,0}$ and $\left(C^\text{norm}_{\mathbf{W}_\Delta}\right)^{0,0}$ do, while $\left(C^\text{norm}_{\mathbf{W}_\Delta}\right)^{q,t}$ has poles for the same reasons that $\left(C^\text{norm}_{\mathbf{W}_\Delta}\right)^{0,0}$ does. From equation (7.35), one can see that, for generic positive values of $\left(q,t\right)$, $\left(C^\text{norm}_{\mathbf{W}_\Delta}\right)^{q,t}$ contributes an infinite tower of poles for each pole in $\left(C^\text{norm}_{\mathbf{W}_\Delta}\right)^{0,0}$. For non-generic positive choices of $\left(q,t\right)$, one can cancel some of these new poles, but infinite towers remain. When copies of $\left(C^\text{norm}_{\mathbf{W}_\Delta}\right)^{q,t}$ with $q_t$-dependent factors are glued to form 5D $q_t$-deformed instanton partition functions, the $\left(C^\text{norm}_{\mathbf{W}_\Delta}\right)^{q,t}$ will contribute. To have a better look at these contributions, we take the 4D limit of $S^\text{norm}_{\mathbf{W}_\Delta}$.

8. The 4D limit of the $U(2)$ strip partition function

We take the 4D limit of the $U(2)$ strip partition function $\mathcal{M}^\text{qt}_{Y_1,Y_2}\{x,y\}$ is a $q_t$-deformation of $\mathcal{R}_{Y_1,Y_2}\{x,y\}$ and as such, it is a building block of $q_t$-deformed 5D instanton partition functions. In the absence of the $\{q,t\}$-parameters, one takes the 4D limit by writing the parameters $\{x,y\}$ and the exponentiated Kähler parameters $Q_i, i = 1, 2, 3$ as,

$$x = e^{R \epsilon_1}, y = e^{-R \epsilon_2}, Q_i = e^{R \Delta_i}, i \in \{1, 2, 3\},$$

where $R$ is the circumference of the $M$-theory circle, $\{\epsilon_1, \epsilon_2\}$ are Nekarsov’s equivariant deformation parameters,

$$\epsilon_1 < 0 < \epsilon_2,$$

and $\Delta_i, i \in \{1, 2, 3\}$ are Kähler parameters related to mass and Coulomb parameters in a 4D supersymmetric gauge theory, that can be related via the AGT correspondence with conformal dimensions of primary fields in a 2D conformal field theory [10]. In the presence of the $\{q,t\}$-parameters, we choose to expand these parameters without loss of generality in terms of $R$ as,

$$q = e^{-R \epsilon_3}, t = e^{-R \epsilon_4},$$

where $\{\epsilon_3, \epsilon_4\}$ are real, non-negative parameters of the same dimensions as $\{\epsilon_1, \epsilon_2\}$

8.1. Remark

If there is another parameter, of the same dimension as $R$, to expand $\{q,t\}$ in, then the difference of the two expansions can be absorbed in a redefinition of $\{\epsilon_3, \epsilon_4\}$.
8.2. Using the factorization to obtain a well-defined 4D limit

To obtain a well-defined 4D limit that reduces to the known 4D limit for $q = t$, we use the factorization into $q^t$-independent factors, and $q^t$-dependent factors that $\rightarrow 1$, in the limit $\epsilon_3 \rightarrow \epsilon_4$, as in section 7.12.

8.3. $A^{\text{norm, 4D}}_{V,W,\Delta}$

The 4D limit of $Z^q_{\mathbf{f}}(x,y)$ is,

$$Z^q_{\mathbf{f}}(x,y) \big|_{R\to 0} = R^{-|F|} \left( \prod_{\square \in F} -\epsilon_1 L^+_{\square,y} + \epsilon_2 A^\square_{\mathbf{f},y} \right) \left( \prod_{n=0}^{\infty} \frac{n \epsilon_3 + \epsilon_4 - \epsilon_1 L^+_{\square,y} + \epsilon_2 A^\square_{\mathbf{f},y}}{(n+1)} \right),$$

(8.4)

where all factors in the double-product on the right hand side $\rightarrow 1$, in the limit $\epsilon_3 \rightarrow \epsilon_4$.

Similarly, the 4D limit of $Z^q_{\mathbf{f}}(y,x)$ is,

$$Z^q_{\mathbf{f}}(y,x) \big|_{R\to 0} = R^{-|F|} \left( \prod_{\square \in Y} -\epsilon_1 L^+_{\square,y} + \epsilon_2 A^\square_{\mathbf{f},y} \right) \left( \prod_{n=0}^{\infty} \frac{n \epsilon_4 + \epsilon_3 - \epsilon_1 L^+_{\square,y} + \epsilon_2 A^\square_{\mathbf{f},y}}{(n+1)} \right),$$

(8.5)

From equations (8.4) and (8.5),

$$A^{\text{norm, 4D}}_{V,W,\Delta} = \left( A^{\text{norm, 4D}}_{V,W,\Delta} \right)_{(0,0)} \left( A^{\text{norm, 4D}}_{V,W,\Delta} \right)_{(q,t)},$$

(8.6)

$$\left( A^{\text{norm, 4D}}_{V,W,\Delta} \right)_{(0,0)} = R^{-|V|+|Y|+|W|} \left( \prod_{\square \in V} \frac{1}{-\epsilon_1 L^+_{\square,y} + \epsilon_2 A^\square_{\mathbf{f},y}} \prod_{\square \in Y} \frac{1}{-\epsilon_1 L^+_{\square,y} + \epsilon_2 A^\square_{\mathbf{f},y}} \right).$$

(8.7)

$$\left( A^{\text{norm, 4D}}_{V,W,\Delta} \right)_{(q,t)} = \left( \prod_{n=0}^{\infty} \prod_{\square \in V} \frac{n \epsilon_3 + \epsilon_4 - \epsilon_1 L^+_{\square,y} + \epsilon_2 A^\square_{\mathbf{f},y}}{(n+1)} \prod_{\square \in Y} \frac{n \epsilon_4 + \epsilon_3 - \epsilon_1 L^+_{\square,y} + \epsilon_2 A^\square_{\mathbf{f},y}}{(n+1)} \right).$$

(8.8)

8.4. $B^{\text{norm, 4D}}_{V,W,\Delta}$

In the limit $R \to 0$,

$$\prod_{i,j=1}^{\infty} \left( 1 - Q_{x^{i-1} y^{j-1}} \right) \bigg|_{R\to 0} = R^{|V|+|Y|} \left( \prod_{\square \in V} \frac{-\Delta + \epsilon_1 L^+_{\square,w} - \epsilon_2 A^\square_{\mathbf{f},y}}{-\Delta - \epsilon_1 L^+_{\square,w} + \epsilon_2 A^\square_{\mathbf{f},y}} \right) \prod_{\square \in Y} \left( \frac{-\Delta + \epsilon_1 L^+_{\square,w} - \epsilon_2 A^\square_{\mathbf{f},y}}{-\Delta - \epsilon_1 L^+_{\square,w} + \epsilon_2 A^\square_{\mathbf{f},y}} \right).$$

(8.9)
Consequently,

\[
\mathcal{S}^\text{norm, 4D}_\mathbf{V} \Delta = \mathcal{S}^\text{norm, 4D}_\mathbf{V} \Delta \left(0, 0\right) = R^2 \left(\left|V_1\right| + \left|V_2\right| + \left|W_1\right| + \left|W_2\right|\right) \\
\prod_{\Delta \in V_1} \left(\Delta_1 - \epsilon_1 L_{\mathbf{V} \Delta W_1} + \epsilon_2 A_{\mathbf{V} \Delta V_1}\right) \prod_{\epsilon_1 \mathbf{W}_1} \left(\Delta_1 + \epsilon_1 L_{\mathbf{V} \Delta V_1} + \epsilon_2 A_{\mathbf{V} \Delta W_1}\right) \\
\prod_{\Delta \in V_2} \left(\Delta_2 - \epsilon_1 L_{\mathbf{V} \Delta W_1} - \epsilon_2 A_{\mathbf{V} \Delta V_2}\right) \prod_{\epsilon_1 \mathbf{W}_1} \left(\Delta_2 + \epsilon_1 L_{\mathbf{V} \Delta V_2} + \epsilon_2 A_{\mathbf{V} \Delta W_1}\right) \\
\prod_{\Delta \in W_2} \left(\Delta_3 - \epsilon_1 L_{\mathbf{W} \Delta W_2} + \epsilon_2 A_{\mathbf{W} \Delta V_2}\right) \prod_{\epsilon_1 \mathbf{W}_1} \left(\Delta_3 + \epsilon_1 L_{\mathbf{W} \Delta V_2} - \epsilon_2 A_{\mathbf{W} \Delta W_1}\right) \\
\prod_{\Delta \in W_1} \left(\Delta_{123} - \epsilon_1 L_{\mathbf{W} \Delta W_1} + \epsilon_2 A_{\mathbf{W} \Delta V_1}\right) \prod_{\epsilon_1 \mathbf{W}_1} \left(\Delta_{123} + \epsilon_1 L_{\mathbf{W} \Delta V_1} - \epsilon_2 A_{\mathbf{W} \Delta W_1}\right),
\]

where \(\Delta_{123} = \Delta_1 + \Delta_2 + \Delta_3\).

\[8.10\]

**8.4.1 \(\mathcal{C}^\text{norm, 4D}_\mathbf{V} \Delta\)**. Finally, the 4D limit of \(\mathcal{C}^\text{norm, 4D}_\mathbf{V} \Delta\) is,

\[
\mathcal{C}^\text{norm, 4D}_\mathbf{V} \Delta = \mathcal{C}^\text{norm, 4D}_\mathbf{V} \Delta \left(0, 0\right) = \mathcal{C}^\text{norm, 4D}_\mathbf{V} \Delta \left(0, r\right)
\]

\[
\left(\mathcal{C}^\text{norm, 4D}_\mathbf{V}
\right) \left(0, 0\right) = R^2 \left(\left|V_1\right| + \left|V_2\right| + \left|W_1\right| + \left|W_2\right|\right) \\
\prod_{\Delta \in V_1} \left(\Delta_{12} - \epsilon_1 L_{\mathbf{V} \Delta V_1} + \epsilon_2 A_{\mathbf{V} \Delta V_1}\right) \prod_{\epsilon_1 \mathbf{W}_1} \left(\Delta_{12} + \epsilon_1 L_{\mathbf{V} \Delta V_1} + \epsilon_2 A_{\mathbf{V} \Delta W_1}\right) \\
\prod_{\Delta \in V_2} \left(\Delta_{23} - \epsilon_1 L_{\mathbf{V} \Delta W_2} + \epsilon_2 A_{\mathbf{V} \Delta V_2}\right) \prod_{\epsilon_1 \mathbf{W}_1} \left(\Delta_{23} + \epsilon_1 L_{\mathbf{V} \Delta V_2} + \epsilon_2 A_{\mathbf{V} \Delta W_1}\right) \\
\prod_{\Delta \in W_1} \left(\Delta_{123} - \epsilon_1 L_{\mathbf{W} \Delta W_1} + \epsilon_2 A_{\mathbf{W} \Delta V_1}\right) \prod_{\epsilon_1 \mathbf{W}_1} \left(\Delta_{123} + \epsilon_1 L_{\mathbf{W} \Delta V_1} - \epsilon_2 A_{\mathbf{W} \Delta W_1}\right),
\]

where \(\Delta_{12} = \Delta_1 + \Delta_2\), and \(\Delta_{23} = \Delta_2 + \Delta_3\).

\[8.12\]

\[8.13\]

**8.4.2. Comparing notation.** The expressions in equations (8.7), (8.10) and (8.12) in the present work, were computed in equations (4.3) and (4.4) in [50], using \(R_{V_1 V_2 W_1}\), \(x, y\), and matched in the 4D limit with the Nekrasov partition functions, as in [52]. The notation used in the present work translates to that used in [50], denoted with ‘old’, as follows,

\[
\epsilon_1 = \epsilon^{\text{old}}_2, \quad \epsilon_2 = \epsilon^{\text{old}}_1, \quad V_i = V_i^{\text{old}}, \quad W_i = W_i^{\text{old}}, \quad i \in \{1, 2\},
\]

\[
\Delta_1 = -\Delta^{\text{old}} - \frac{1}{2} \epsilon^{\text{old}}_2 + \frac{1}{2} \epsilon^{\text{old}}_1, \quad \Delta_2 = -\Delta^{\text{old}} + \frac{1}{2} \epsilon^{\text{old}}_2 - \frac{1}{2} \epsilon^{\text{old}}_1, \quad \Delta_3 = -\Delta^{\text{old}} - \frac{1}{2} \epsilon^{\text{old}}_2 + \frac{1}{2} \epsilon^{\text{old}}_1
\]

\[8.14\]
8.4.3. The 4D limit is well-defined. Putting equations (8.6)–(8.11) together, all factors of $R$ cancel out and we obtain a well-defined 4D limit,

$$S_{VW}^{\text{norm, 4D}}(x,y) = A_{VW}^{\text{norm, 4D}} B_{VW}^{\text{norm, 4D}} C_{VW}^{\text{norm, 4D}}$$

(8.15)

8.5. Comments on the structure of $S_{VW}^{\text{norm, 4D}}$ in the 4D limit

8.5.1. The $q_t$-independent terms. The $q_t$-independent product,

$$\left( A_{VW}^{\text{norm, 4D}} \right)_{(0,0)} = \left( A_{VW}^{\text{norm, 4D}} \right)_{(0,0)} \left( B_{VW}^{\text{norm, 4D}} \right)_{(0,0)} \left( C_{VW}^{\text{norm, 4D}} \right)_{(0,0)},$$

(8.16)

is the result obtained using $\mathcal{R}_{Y_i Y_j}$ $(x,y)$, as in equations (4.2)–(4.4) in [50]. The first thing to note is that the overall factors of $R$ in equations (8.7), (8.10) and (8.12), cancel when we compute $S_{VW}^{\text{norm, 4D}}$ as in equation (8.16) Aside from that, $A_{VW}^{\text{norm, 4D}}$ equation (8.7), has no poles since $\epsilon_i < 0 < \epsilon_i, i \in \{2, 3, 4\}$, $A_{\square, Y} \geq 0$ and $L_{\square, Y} \geq 0$, for $\square \in Y$. $B_{VW}^{\text{norm, 4D}}$ equation (8.12), has no poles, and $\left( C_{VW}^{\text{norm, 4D}} \right)_{(0,0)}$, equation (8.12), has the same poles obtained using $\mathcal{R}_{Y_i Y_j}$ $(x,y)$.\(^{53}\)

8.5.2. The $q_t$-dependent terms.

$$\left( A_{VW}^{\text{norm, 4D}} \right)_{(q,t)} = \left( A_{VW}^{\text{norm, 4D}} \right)_{(0,0)} \left( B_{VW}^{\text{norm, 4D}} \right)_{(q,t)} \left( C_{VW}^{\text{norm, 4D}} \right)_{(q,t)},$$

(8.17)

is the additional $q_t$-dependent factor in $S_{VW}^{\text{norm, 4D}}$ that we obtain when we use $\mathcal{M}_{Y_i Y_j}$ $(x,y)$, then take the 4D limit. $A_{VW}^{\text{norm, 4D}}$ equation (8.7), has no poles for the same reasons as in 5D. $B_{VW}^{\text{norm, 4D}}$ equation (8.10), has no poles, and $\left( C_{VW}^{\text{norm, 4D}} \right)_{(q,t)}$, equation (8.13), has an infinite tower of poles for every pole in $\left( C_{VW}^{\text{norm, 4D}} \right)_{(0,0)}$ for generic positive values of $\epsilon_3$ and $\epsilon_4$. A fraction of these infinite towers cancel for non-generic values of $\epsilon_3$ and $\epsilon_4$, but not all.

8.5.3. Remark. At the level of the strip, the additional $q_t$-dependent terms, containing infinitely-many new poles, appear in overall, multiplicative factors. As we show, in section 9, when we glue strips to form instanton partition functions, these terms are no longer overall multiplicative, but give different weights to the different terms summed over to form the instanton partition functions.

9. The $U(2)$ instanton partition function and its 4D limit

We glue two strips along non-preferred legs to obtain a 5D $q_t$-instanton partition function, then take its 4D limit.

\(^{52}\) As in the 5D case, in equations (4.2)–(4.4) in [50], slightly different notation was used. In particular, $\Delta_i, i \in \{1, M, 2\}$ in [50], become $\delta_i, i \in \{1, 2, 3\}$ in the present work. Other changes are discussed in footnote 50.

\(^{53}\) The pole structure of rational expressions as in equation (8.12) were discussed in detail in [52].
9.1. Two strips

Following the conventions used figure 6, the left-strip has a partition pair \( (\emptyset, \emptyset) \) on the left, \( (\bar{Y}_U, \bar{Y}_D) \) on the right, and contributes,

\[
\mathcal{S}^{\text{norm}}_{\emptyset \emptyset \Delta_1} = A^{\text{norm}}_{\emptyset \emptyset \Delta_1} \cdot B^{\text{norm}}_{\emptyset \emptyset \Delta_1} \cdot C^{\text{norm}}_{\emptyset \emptyset \Delta_1} \tag{9.1}
\]

where \( \Delta_1 = (\Delta_1, \bar{\Delta}_1, \Delta_1) \). The right-strip has a partition pair \( (Y_U, Y_D) \) on the left, \( (\emptyset, \emptyset) \) on the right, and contributes,

\[
\mathcal{S}^{\text{norm}}_{\bar{Y} \bar{Y} \Delta_2} = A^{\text{norm}}_{\bar{Y} \bar{Y} \Delta_2} \cdot B^{\text{norm}}_{\bar{Y} \bar{Y} \Delta_2} \cdot C^{\text{norm}}_{\bar{Y} \bar{Y} \Delta_2} \tag{9.2}
\]

where \( \Delta_2 = (\Delta_1, \bar{\Delta}_1, \Delta_1) \).

9.2. The \( U(2) \) \( qt \)-instanton partition function

Gluing the left-strip and the right-strip in figure 6 along non-preferred legs that share a common partition pair \( (\bar{Y}_U, Y_D) \), weighted by \( (-Q_C)^{|\bar{Y}_C|+|Y_D|} \), where \( Q_C \) is an exponentiated Kähler parameter, we obtain the \( U(2) \) \( qt \)-instanton partition function,

\[
\mathcal{W}_{\Delta_1 \Delta_2} = \sum_{Y_U, Y_D} (-Q_C)^{|Y_U|+|Y_D|} \mathcal{S}^{\text{norm}}_{\bar{Y} \bar{Y} \Delta_1} \mathcal{S}^{\text{norm}}_{\bar{Y} \bar{Y} \Delta_2} \tag{9.3}
\]

Figure 6. The web diagram of the 5D \( U(2) \) instanton partition function, and the 2D \( qt \)-conformal block.
It is useful to split the summand $S_{\text{norm}}^{\text{Y} \Delta L} S_{\text{norm}}^{\text{Y} \Delta R}$ into a factor \( S_{\text{norm}}^{\text{Y} \Delta L} \{q, t\} \) that does not depend on \( q, t \), and a factor \( S_{\text{norm}}^{\text{Y} \Delta R} \{q, t\} \) that depends on \( q, t \), and \( \rightarrow 1 \) in the limit \( q \rightarrow t \),

\[
\left( S_{\text{norm}}^{\text{Y} \Delta L} S_{\text{norm}}^{\text{Y} \Delta R} \right) \{0, 0\} = \prod_{\square \in Y_U} \left( 1 - x \chi_{L \in \rho} y^{A_{\Delta L}} \chi_{L \in \rho} y^{A_{\Delta L}} \right) \prod_{\bullet \in Y_D} \left( 1 - x \chi_{L \in \rho} y^{A_{\Delta L}} \chi_{L \in \rho} y^{A_{\Delta L}} \right)
\]

(9.4)
9.3. The 4D limit of the $\mathbf{U}(2)$ instanton partition function

To take the 4D limit of the instanton partition function, we expand the variables,

$$x = e^{R_t}, \quad y = e^{-R_t}, \quad q = e^{R_t}, \quad t = e^{R_t}, \quad Q_i = e^{R_{t_i}}, \quad i = \left\{1, 2, 3, 4, L, R, U, D\right\}. \quad (9.6)$$

and readily find,

$$\left\{\begin{array}{l}
\Delta_{1+} L_{t,0} - e_2 A_{t,0}^+ Y_D \\
\Delta_{1-} e_1 L_{t,0} + e_2 A_{t,0}^+ Y_D \\
\Delta_3 - e_1 L_{t,0} + e_2 A_{t,0} Y_D \\
\Delta_3 + e_1 L_{t,0} + e_2 A_{t,0} Y_D \\
\Delta_{12} + e_1 L_{t,0} - e_2 A_{t,0} Y_D \\
\Delta_{12} - e_1 L_{t,0} - e_2 A_{t,0} Y_D \\
\Delta_{12} + e_1 L_{t,0} + e_2 A_{t,0} Y_D \\
\Delta_{12} - e_1 L_{t,0} - e_2 A_{t,0} Y_D \\
\end{array}\right. \quad (9.7)$$

$$\left\{\begin{array}{l}
m_1 + e_1 - e_1 L_{t,0} + e_2 A_{t,0} Y_D \\
m_1 + e_1 - e_1 L_{t,0} + e_2 A_{t,0} Y_D \\
\Delta_3 - e_1 L_{t,0} + e_2 A_{t,0} Y_D \\
\Delta_3 + e_1 L_{t,0} + e_2 A_{t,0} Y_D \\
\Delta_{12} - e_1 L_{t,0} - e_2 A_{t,0} Y_D \\
\Delta_{12} + e_1 L_{t,0} + e_2 A_{t,0} Y_D \\
\Delta_{12} - e_1 L_{t,0} + e_2 A_{t,0} Y_D \\
\Delta_{12} + e_1 L_{t,0} - e_2 A_{t,0} Y_D \\
\end{array}\right. \quad (9.8)$$
9.4. Comments on the structure of $\mathcal{W}_{\Delta_1, \Delta_3}$

Equation (9.7) is identical to that obtained using the 4D Nekrasov instanton partition functions, such as in [52]. Equations (9.5) and (9.8) show that, for generic values of $\left(q, t\right)$, the $qt$-dependent contributions to instanton partition function bring in new poles, that are also present in the 4D limit. In section 10, we consider these poles in the context of the $\mathcal{N} = 2^*$ instanton partition function, or equivalently the 1-point function on a torus, in a 2D Virasoro conformal field theory.

10. The $\mathcal{N} = 2^*$ instanton partition function and its 4D limit

We compute the $\mathcal{N} = 2^*$ instanton partition function, or equivalently the 1-point $qt$-conformal block of a Virasoro conformal field theory.

To compute the 1-point conformal block on the torus, we start from the strip partition function $s_{\mathcal{W}_{\Delta_1, \Delta_3}}$, set $V = \mathcal{W}$, introduce a gluing exponentiated Kähler parameter $Q$ that contributes a factor $\left(-\mathcal{O}_V\right)^{\left|V_1\right|+\left|V_2\right|}$, and sum over $V = \left\{V_1, V_2\right\}$.

$$
W_{\Delta_1}^{\mathcal{N} = 2^*} = \sum_{V} \left(-\mathcal{O}_V\right)^{\left|V_1\right|+\left|V_2\right|} s_{\mathcal{W}_{\Delta_1, \Delta_3}} = \sum_{V} \left(-\mathcal{O}_V\right)^{\left|V_1\right|+\left|V_2\right|} \left(s_{\mathcal{W}_{\Delta_1, \Delta_3}}\right)^{\left(0,0\right)} \left(s_{\mathcal{W}_{\Delta_1, \Delta_3}}\right)^{\left(q,t\right)},
$$

(10.1)

$$
\left(s_{\mathcal{W}_{\Delta_1, \Delta_3}}\right)^{\left(0,0\right)} = \prod_{\square \in V_1} \left\{1 - Q_1 x^{-L_{\square}^1 v_1} y^{A\square_1 v_1} q^{-n_1} t\right\} \left\{1 - Q_2 x^{L_{\square}^2 v_1} y^{A\square_1 v_1} q^{n_1} t\right\} \left\{1 - Q_3 x^{-L_{\square}^3 v_1} y^{A\square_1 v_1} q^{n_1} t\right\},
$$

(10.2)

$$
\prod_{\square \in V_1} \left\{1 - Q_1 x^{-L_{\square}^1 v_1} y^{A\square_1 v_1} q^{-n_1} t\right\} \left\{1 - Q_2 x^{L_{\square}^2 v_1} y^{A\square_1 v_1} q^{n_1} t\right\} \left\{1 - Q_3 x^{-L_{\square}^3 v_1} y^{A\square_1 v_1} q^{n_1} t\right\},
$$

(10.3)

10.1. The 4D limit of $W_{\Delta_1}^{\mathcal{N} = 2^*}$

Setting,

$$
x = e^{R\epsilon_1}, \ y = e^{-R\epsilon_2}, \ q = e^{-R\epsilon_3}, \ t = e^{-R\epsilon_4}, \ Q_i = e^{R\Delta_i}, \ i = \left(1, 2, 3\right),
$$

(10.4)

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where $R$ is the circumference of the $M$-theory circle, the 4D limit of equation (10.1) is,

$$S_{VV\Delta}^{4D} = \left( S_{VV\Delta}^{4D} \right)_{(0,0)} \left( S_{VV\Delta}^{4D} \right)_{(q,t)},$$

(10.5)

$$\left( S_{VV\Delta}^{4D} \right)_{(0,0)} = \prod_{\square \in V_1} \prod_{\blacksquare \in V_2} \begin{cases} \Delta_1 - \epsilon_1 L_{\square, Y_1} + \epsilon_2 A_{\square, Y_1} & \Delta_1 + \epsilon_1 L_{\square, Y_1} - \epsilon_2 A_{\square, Y_1} \\ \Delta_2 + \epsilon_1 L_{\blacksquare, Y_1} + \epsilon_2 A_{\blacksquare, Y_1} & \Delta_2 - \epsilon_1 L_{\blacksquare, Y_1} + \epsilon_2 A_{\blacksquare, Y_1} \\ \Delta_3 - \epsilon_1 L_{\square, Y_1} - \epsilon_2 A_{\square, Y_1} & \Delta_3 + \epsilon_1 L_{\square, Y_1} + \epsilon_2 A_{\square, Y_1} \\ \Delta_{123} - \epsilon_1 L_{\square, Y_1} + \epsilon_2 A_{\square, Y_1} & \Delta_{123} + \epsilon_1 L_{\square, Y_1} - \epsilon_2 A_{\square, Y_1} \end{cases},$$

(10.6)

$$\left( S_{VV\Delta}^{4D} \right)_{(q,t)} = \prod_{n=1}^{\infty} \prod_{\square \in V_1} \prod_{\blacksquare \in V_2} \begin{cases} n\epsilon_3 + \epsilon_4 - \epsilon_1 L_{\square, Y_1} + \epsilon_2 A_{\square, Y_1} & n\epsilon_4 + \epsilon_3 - \epsilon_1 L_{\square, Y_1} + \epsilon_2 A_{\square, Y_1} \\ n\epsilon_3 + \epsilon_4 - \epsilon_1 L_{\blacksquare, Y_1} + \epsilon_2 A_{\blacksquare, Y_1} & n\epsilon_4 + \epsilon_3 - \epsilon_1 L_{\blacksquare, Y_1} + \epsilon_2 A_{\blacksquare, Y_1} \\ \Delta_{12} - n\epsilon_3 - \epsilon_4 - \epsilon_1 L_{\square, Y_1} + \epsilon_2 A_{\square, Y_1} & \Delta_{12} - n\epsilon_4 - \epsilon_3 - \epsilon_1 L_{\blacksquare, Y_1} + \epsilon_2 A_{\blacksquare, Y_1} \\ \Delta_{12} - n\epsilon_3 - \epsilon_4 + \epsilon_1 L_{\square, Y_1} - \epsilon_2 A_{\square, Y_1} & \Delta_{12} - n\epsilon_4 - \epsilon_3 + \epsilon_1 L_{\blacksquare, Y_1} - \epsilon_2 A_{\blacksquare, Y_1} \end{cases},$$

(10.7)

10.2. Comments on the structure of $W_{\Delta}^{\mathcal{N}=2^*}$

As in the case of the 5D and 4D $\mathcal{N} = 2$ $q t$-instanton partition functions that correspond to a 2D 4-point conformal blocks on the sphere, equations (10.2)–(10.7) show that the 5D and 4D $\mathcal{N} = 2^*$ $q t$-instanton partition functions that correspond to a 2D 1-point conformal block on the torus include $q t$-dependent terms that bring in new poles. In the following, we take the limit of $W_{\Delta}^{\mathcal{N}=2^*}$, which leads to an $\mathcal{N} = 4$ instanton partition function.

10.3. The $\mathcal{N} = 4$ limit of the $\mathcal{N} = 2^*$ instanton partition function

In the conventions used in the present work, this limit is obtained by setting,

$$\Delta_1 = \epsilon_2, \quad \Delta_2 = 2a - \epsilon_2, \quad \Delta_3 = \epsilon_2,$$

(10.8)

where $a$ is the Coulomb parameter of the $U(2)$ gauge group, or equivalently the charge of the Virasoro primary field that flows in the internal channel of the 1-point conformal block on
the torus. In 2D conformal field theory terms, this limit corresponds to choosing the vertex operator insertion in the 1-point conformal block on the torus to be the identity operator. In the absence of Macdonald parameters, the result is the character of the Virasoro highest weight representation that flows in the internal channel. We would like to check that, starting from $M_{Y_{1}Y_{2}Y_{3}}^{q,t}(x,y)$, we reproduce this result in the limit $q \to t$, and check the $q,t$-corrections. Using the parameters of equations (10.8) in (10.6),

$$\left(-1\right)^{|V_{1}|+|V_{2}|} \left(\mathcal{S}_{VVV}^{4D_{\Delta_{0}}}\right)_{(0,0)} = 1$$

(10.9)

If there were no $q,t$-dependent terms, equations (10.1) and (10.9) would combine to give,

$$W_{\Delta^{N=2^{*}}_{q,t}} = \sum_{V} \left(Q\right)^{|V_{1}|+|V_{2}|},$$

(10.10)

which is the correct $R_{Y_{1}Y_{2}Y_{3}}(x,y)$ result. However, using $M_{Y_{1}Y_{2}Y_{3}}^{q,t}(x,y)$, there is a $q,t$-dependent term,

$$\left(\mathcal{S}_{VVV}^{4D_{\Delta_{0}}}\right)_{(q,t)}$$

(10.11)

10.4. Comments on the structure of $W_{\Delta^{N=2^{*}}}^{q,t}$

In the $q \to t$ limit, the right hand side of equation (10.10) is an unevaluated version of the character of the Virasoro highest weight representation that flows in the internal channel of the 1-point function of the identity vertex operator on the torus, and $Q$ plays the role of the indeterminate $q$ that appears $q$-series expressions of Virasoro characters. In the presence of $q,t$-terms, this is modified to,

$$W_{\Delta^{N=2^{*}}}^{q,t} = \sum_{V} \left(Q\right)^{|V_{1}|+|V_{2}|} \left(\mathcal{S}_{VVV}^{4D_{\Delta_{0}}}\right)_{(q,t)},$$

(10.12)

where $\left(\mathcal{S}_{VVV}^{4D_{\Delta_{0}}}\right)_{(q,t)}$ is defined in equation (10.11), which has the interpretation of 'an extended Virasoro character', with infinitely-many additional states weighted by the factors on the right hand side of equation (10.11)

$^{54}$The derivation of the parameters in equation (10.8) is based on equation (6.2) in [50].
11. Remarks

We conclude with a number of remarks on relevant literature, and open questions.

11.1. Macdonald processes

The present work starts from the observation that there are two ways to refine MacMahon’s partition function of plane partition, one due to Iqbal et al [1], and one due to Vuletić [2], and these refinements are orthogonal in the sense that one can impose one or the other or both at the same time. The refinement of Iqbal et al is natural in the sense that it is the right modification that leads to instanton partition functions that are in turn related to conformal blocks in 2D conformal field theories with $c \neq 1$. The refinement of Vuletić is natural in the sense that it is the right modification that leads to Macdonald processes, a very rich topic in stochastic processes studied by A Borodin and collaborators in [53–56], and references therein. We anticipate that the $qt$-vertex operators used in the present work, in addition to the $y$-refinement of Iqbal et al, can be useful in the study of Macdonald processes.

11.2. The physical meaning of the $qt$-deformation is an open problem

The Macdonald vertex $M_{Y_1Y_2Y_3}^{\theta^I}(x, y)$ depends explicitly on four parameters $(x, y, q, t)$, and implicitly on $R$, the radius of the M-theory circle. The original topological vertex $O_{Y_1Y_2Y_3}(x, y)$ is obtained in the limit $x \to y$, and $q \to t$. In the limit $R \to \infty$, gluing copies of $O_{Y_1Y_2Y_3}(x, y)$, one obtains 4D instanton partition functions, with Nekrasov parameters $\epsilon_1 + \epsilon_2 = 0$, which are equal to Virasoro conformal blocks with central charge $c = 1$.

The refined topological vertex $R_{Y_1Y_2}(x, y)$ is obtained by keeping $x \neq y$, and taking the limit $q \to t$. In the limit $R \to \infty$, gluing copies of $R_{Y_1Y_2}(x, y)$, one obtains 4D instanton partition functions, with Nekrasov parameters $\epsilon_1 + \epsilon_2 \neq 0$, which are equal to Virasoro conformal blocks with central charge $c \neq 1$. The point we wish to make is that, in 2D terms, in the limit $R \to 0$, the $y$-deformation of Iqbal et al takes one from a 2D conformal field theory to another 2D conformal field theory, without affecting criticality. At finite $R$, the above remarks remain the same, but the instanton partition functions are 5D, and the corresponding 2D objects are off-critical deformations of 2D conformal blocks. With reference [43] and related works, $qt$-deformation definitely affects criticality.

To discuss the physical meaning of the $qt$-deformations, it is simpler to work in the $R \to 0$ limit. In each of equations (8.6)–(8.11), there is a leading factor that remains intact as $\epsilon_3 \to \epsilon_4$, and infinitely-many factors, each of which $\to 1$, in the limit $\epsilon_3 \to \epsilon_4$. We need to understand the $qt$-dependent terms.

11.2.1. The 2D interpretation. The $qt$-dependent factors, with their additional poles, are reminiscent of factors that appear in computations of $qt$-deformed expectation values in works such as [43]. In the latter works, the $qt$-deformed expectation values are typically given in terms of contour integrals, and it is tempting to expect that the $qt$-conformal blocks in the present work are evaluations of the contour integrals, possibly in the context of statistical mechanical models based on Virasoro algebra in the presence of an additional $U(1)$ algebra, which typically greatly simplifies evaluations. More precise statements on this issue are beyond the scope of this work.
11.2.2. The instanton partition function interpretation. The infinitely-many additional poles are reminiscent of Kaluza–Klein particles that decouple as $q \to t$. If this is the case, then given that $\mathcal{R}_{Y_1Y_2} \begin{pmatrix} x, y \end{pmatrix}$ leads to 5D instanton partition functions, $\mathcal{M}_{Y_1Y_2} \begin{pmatrix} x, y \end{pmatrix}$ leads to 6D instanton partition functions on Calabi–Yau threefolds given in terms of planar web diagrams.

The problem with this 6D interpretation is that, in the $\mathcal{R}_{Y_1Y_2} \begin{pmatrix} x, y \end{pmatrix}$ literature [26], and in more recent works [57–63], 6D instanton partition functions are obtained by gluing copies of $\mathcal{R}_{Y_1Y_2} \begin{pmatrix} x, y \end{pmatrix}$ to form planar web diagrams, then gluing external legs to make closed loops and summing over all states that can propagate along these loops. This leads to expressions that are identified with 6D instanton partition functions. If the instanton partition functions obtained using $\mathcal{M}_{Y_1Y_2} \begin{pmatrix} x, y \end{pmatrix}$ are 6D objects, then one needs to understand how they relate to the 6D expressions of [57–63]. This is another open problem that is beyond the scope of this work.

11.3. Towards an elliptic extension of the Macdonald vertex

In section 1.6.2, we noted that the Macdonald vertex is related to Ding–Iohara–Miki algebra via the fact that the former is built from the same $q$-vertex operators as the latter. In [36–38], Saito presented an elliptic extension of the Ding–Iohara–Miki algebra, based on $p\, q$-vertex operators that depend on an additional ‘nome’ parameter $p$.

It is straightforward to extend the construction of the Macdonald vertex to an elliptic topological vertex that depends on $p$, such that we recover the Macdonald vertex in the limit $p \to 0$, apart from the fact that it is not clear what the elliptic analogues of the Macdonald symmetric functions are.

It is straightforward to propose an ad hoc solution to this problem by conjecturing that a class of symmetric functions that play the role of eigenstates of Saito’s $p\, q$-vertex operators exist, the same way that the Macdonald functions are eigenstates of the $q$-vertex operators. As Saito’s $p\, q$-vertex operators depend on two free boson fields, the required elliptic Macdonald functions would be labelled by two Young diagrams. This construction needs careful motivation that lies beyond the scope of this work [64].

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