ON CERTAIN ISOMORPHISMS BETWEEN ABSOLUTE GALOIS GROUPS

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1. Introduction

Let $k$ be an algebraically closed field of characteristic zero, $L$ its finitely generated extension of transcendence degree $\geq 2$, and $L'$ another finitely generated extension of $k$. It is a result of Bogomolov [B2] that any isomorphism between $\text{Gal}(\overline{L}/L)$ and $\text{Gal}(\overline{L}'/L')$ is induced by an isomorphism of fields $\overline{L} \rightarrow \overline{L}'$ identifying $L$ with $L'$.

If the transcendence degree of $L$ over $k$ is one, the group $\text{Gal}(\overline{L}/L)$ is free, and therefore, its structure tells nothing about the field $L$.

Let $F$ be an algebraically closed extension of $k$ of transcendence degree one, and $G = G_{F/k}$ be the group of automorphisms over $k$ of the field $F$. Let the set of subgroups $U_{L} := \text{Aut}(F/L)$ for all subfields $L$ finitely generated over $k$ be the basis of neighborhoods of the unity in $G$.

Let $\lambda$ be a continuous automorphism of $G$. The purpose of this note is to show that if $\lambda$ induces an isomorphism $\text{Gal}(F/L) \xrightarrow{\sim} \text{Gal}(F/L')$ then the fields $L$ and $L'$ are isomorphic (see Theorem 4.2 below for a more precise statement).

1.1. Notations. For a field $F_1$ and its subfield $F_2$ we denote by $G_{F_1/F_2}$ the group of automorphisms of the field $F_1$ over $F_2$. Throughout the note $k$ is an algebraically closed field of characteristic zero, $F$ its algebraically closed extension of transcendence degree $1 \leq n < \infty$ and $G = G_{F/k}$. If $K$ is a subfield of $F$ then $\overline{K}$ denotes its algebraic closure in $F$.

For a topological group $H$ we denote by $H^{o}$ its subgroup generated by the compact subgroups, and by $H^{ab}$ the quotient of $H$ by the closure of its commutant.

For a smooth projective curve $C$ over a field, $\text{Pic}^{\geq m}(C)$ (resp., $\text{Pic}^{> m}(C)$) is the submonoid in $\text{Pic}(C)$ of sheaves of degree $\geq m$ (resp., $> m$).

2. A Galois-type correspondence

We consider a topology on $G$ with the basis of neighborhood of an automorphism $\sigma : F \xrightarrow{\sim} F$ over $k$ given by the cosets of the form $\sigma U_{L}$ for all subfields $L$ of $F$ finitely generated over $k$, where $U_{L} = \text{Aut}(F/L)$. This topology was introduced in [ΠI].

One checks that the topology is Hausdorff, locally compact, and totally decomposable.

Proposition 2.1 (ΠI, Lemma 1, Section 3). The map

$\{\text{subfields in } F \text{ over } k\} \rightarrow \{\text{closed subgroups in } G\}$ given by $K \mapsto \text{Aut}(F/K)$

is injective and induces one-to-one correspondences

- $\{\text{subfields } K \text{ of } F \text{ with } k \subseteq K \text{ and } F = \overline{K}\} \leftrightarrow \{\text{compact subgroups of } G\}$;
- $\left\{\text{subfields } K \text{ of } F \text{ finitely generated over } k \text{ with } F = \overline{K}\right\} \leftrightarrow \{\text{compact open subgroups of } G\}$.

The inverse correspondences are given by $G \supseteq H \mapsto F^{H} \subseteq F$. □

Denote by $G^{o}$ the subgroup of $G$ generated by the compact subgroups. Obviously, $G^{o}$ is an open normal subgroup in $G$. 1
3. Decomposition subgroups in abelian quotients

Let $n = 1$. We are going to show that for any continuous automorphism $\lambda$ of $G$ and any $L$ of finite type over $k$ one has $\lambda(U_L) = U_{L'}$ for some $L'$ isomorphic to $L$.

To do that we first need to construct decomposition subgroups in the abelian quotients $U_L^{ab}$. For a smooth projective model $C$ of $L$ over $k$ set $\Phi_L = \operatorname{Hom}(\operatorname{Div}^0(C), \hat{\mathbb{Z}}(1))$. By Kummer theory, $U_L^{ab} = \operatorname{Hom}(L^\times, \hat{\mathbb{Z}}(1))$, so, as $\operatorname{Pic}^0(C)$ is a divisible group, the short exact sequence $1 \rightarrow L^\times/k^\times \rightarrow \operatorname{Div}^0(C) \rightarrow \operatorname{Pic}^0(C) \rightarrow 0$ induces an embedding $\Phi_L \hookrightarrow U_L^{ab}$. One identifies $\Phi_L$ with the $\hat{\mathbb{Z}}$-module of the $\hat{\mathbb{Z}}(1)$-valued functions on $C(k)$ modulo the constant ones.

The next step is to get a description of $\Phi_L$ in terms of the Galois groups. Clearly, $U_L^{ab} = \Phi_{k(x)}$.

Lemma 3.1.  

- If $U$ is an open compact subgroup in $G$ then $N_G(U) = N_{G^0}(U)$.
- If, moreover, $N_G(U)/U$ has no abelian subgroups of finite index then $U = U_{k(x)}$ for some $x \in F - k$.
- For any $x \in L - k$ the transfer $U_{k(x)}^{ab} \rightarrow U_L^{ab}$ factors through $\Phi_L$.
- The span of images of the transfers $U_{k(x)}^{ab} \rightarrow U_L^{ab}$ for all $x \in L - k$ is dense in $\Phi_L$.

Proof.

- By Proposition 2.1 $U = U_L$ for a field $L$ finitely generated over $k$. Then the group $N_G(U_L)/U_L$ coincides with the group of automorphisms of the field $L$ over $k$. As the automorphism groups of projective curves of genus $> 1$ are finite, if $L$ is isomorphic to the function field of such a curve, then the normalizer of $U$ in $G$ is compact. As the automorphism groups of elliptic curves are generated by elements of order $\leq 4$ and contain abelian subgroups of index $\leq 6$, if $L$ is isomorphic to the function field of such a curve, then the normalizer of $U$ in $G$ is generated by its compact subgroups. This implies that if $N_G(U)/U$ has no abelian subgroups of finite index then $L$ should be the function field of a rational curve. As the automorphism group of the rational curve is generated by involutions, the normalizer of $U$ in $G$ is generated by its compact subgroups.

- The transfer is induced by the norm homomorphism $L^\times/k^\times \xrightarrow{\text{Nm}_{L/k(x)}} k(x)^\times/k^\times$, which is the restriction of the push-forward map $\operatorname{Div}^0(C) \xrightarrow{x} \operatorname{Div}^0(\mathbb{P}^1)$.

Since $k(x)^\times/k^\times = \operatorname{Div}^0(\mathbb{P}^1)$, the transfer factors through $\Phi_L$.

- Each point $p$ of a smooth projective model $C$ of $L$ over $k$ is a difference of very ample effective divisors on $C$. These divisors themselves are zero-divisors of some rational functions, i.e., there are surjective morphisms $x, y : C \rightarrow \mathbb{P}^1$ and a point $0 \in \mathbb{P}^1$ such that $x^{-1}(0) - y^{-1}(0) = p$. Then $\delta_p = x^*\delta_0 - y^*\delta_0 : C(k) \rightarrow \hat{\mathbb{Z}}(1)$ is a $\delta$-function of the point $p$ of $C$. As the span of $\delta$-functions is dense in the group $\Phi_L$, we are done. $\square$

For a point of $C(k)$ its decomposition subgroup in $\Phi_L \subset U_L$ consists of all functions supported on it. In the case $L = k(x)$ the decomposition subgroups in $U_{k(x)}^{ab}$ are parametrized by the set (which is isomorphic to $\mathbb{P}^1(k)$) of parabolic subgroups $P$ in $N_GU_{k(x)}/U_{k(x)}$. The subgroup $D_P$ consists of elements in $U_{k(x)}^{ab}$ fixed under the adjoint action of $P$. Clearly, $D_P \cong \hat{\mathbb{Z}}(1)$.

Each inclusion of subgroups $U_L \subset U_{k(x)}$ induces a homomorphism $U_L^{ab} \rightarrow U_{k(x)}^{ab}$. Consider the evident homomorphism $U_L^{ab} \xrightarrow{\varphi_L} \prod_{x \in L - k} U_{k(x)}^{ab}$. For any non-zero element $h$ of the group $U_L^{ab}$, considered as a homomorphism from the group $L^\times$, there is an element $x \in L^\times$ with $h(x) \neq 0$, so the image of $h$ in $U_{k(x)}^{ab}$ is non-zero, and thus, $\varphi_L$ is injective.
To construct decomposition subgroups for an arbitrary \( L \), consider such a subgroup \( D \cong \hat{\mathbb{Z}} \) in the target of \( \varphi_L \) that its projection to each of \( U_{k(x)}^{ab} \) is of finite index in some decomposition subgroup. Then our nearest goal is to show that the set of decomposition subgroups in \( U_{k(x)}^{ab} \) coincides with the set of maximal subgroups among \( \Phi_L \cap \varphi_L^{-1}(D) \).

**Lemma 3.2** ( = Lemma 5.2 of [B1] = Lemma 3.4’ of [B2]). Suppose that \( f \) is such a function on a projective space \( \mathbb{P} \) over an infinite field that the restriction of \( f \) to each projective line in \( \mathbb{P} \) is constant on the complement to a point on it.

Then \( f \) is a flag function, i.e., there is a filtration \( P_0 \subset P_1 \subset P_2 \subset \ldots \) of \( \mathbb{P} \) by projective subspaces such that \( f \) is constant on \( P_0 \) and on all strata \( P_{j+1} - P_j \).

**Lemma 3.3.** For any smooth projective curve \( C \) there is a constant \( N = N(C) \) such that for any \( \mathcal{L}, \mathcal{L}' \in \text{Pic}^{\geq N}(C) \) the natural map \( \Gamma(C, \mathcal{L}) \otimes \Gamma(C, \mathcal{L}') \rightarrow \Gamma(C, \mathcal{L} \otimes \mathcal{L}') \) is surjective.

**Proof.** Fix an invertible sheaf \( \mathcal{L}_0 \) on \( C \) of degree 1. By Serre vanishing theorem, there is such an integer \( N' \) that the sheaf \( (\mathcal{L}_0 \boxtimes \mathcal{L}_0)^{\otimes N'}(-\Delta) \) on \( C \times C \) is generated by its global sections, and therefore, for any \( \mathcal{L}, \mathcal{L}' \in \text{Pic}^{\geq N'}(C) \) the sheaf \( (\mathcal{L} \boxtimes \mathcal{L}')(-\Delta) \) is ample. Let \( N = N' + 2g \). Then by Kodaira vanishing theorem, for any \( \mathcal{L}, \mathcal{L}' \in \text{Pic}^{\geq N}(C) \) the short exact sequence

\[
0 \rightarrow (\mathcal{L} \boxtimes \mathcal{L}')(-\Delta) \rightarrow \mathcal{L} \boxtimes \mathcal{L}' \rightarrow \mathcal{L} \otimes \mathcal{L}' \rightarrow 0
\]

of sheaves on \( C \times C \) induces a surjection \( \Gamma(C, \mathcal{L}) \otimes \Gamma(C, \mathcal{L}') \rightarrow \Gamma(C, \mathcal{L} \otimes \mathcal{L}') \). \( \square \)

**Lemma 3.4.** If \( \varphi_L^{-1}(D) \) is in \( \Phi_L \) then it is a subgroup in a decomposition subgroup in \( U_{k(x)}^{ab} \).

**Proof.** Let \( f \in \varphi_L^{-1}(D) \cap \Phi_L \), i.e., \( f : C(k) \rightarrow \hat{\mathbb{Z}}(1) \) for a smooth projective model \( C \) of \( L \) over \( k \), and for any very ample invertible sheaf \( \mathcal{L} \) on \( C \) restrictions of the induced function \( f : |\mathcal{L}| \rightarrow \hat{\mathbb{Z}}(1) \) to projective lines in \( |\mathcal{L}| \) are “\( \delta \)-functions” on them. Then, by Lemma 3.2, \( f \) is a flag function. Therefore, the function \( \hat{f} : |\mathcal{L}|^\vee \rightarrow \hat{\mathbb{Z}}(1) \) given by \( H \mapsto f(\text{general point of } H) \) is a “\( \delta \)-function”.

Let \( g \) be the genus of \( C \). Consider the composition \( \hat{f}_\mathcal{L} : C(k) \rightarrow |\mathcal{L}|^\vee \rightarrow \hat{\mathbb{Z}}(1) \). It takes \( x \) to

\[
f(x) + f(\text{general point of } |\mathcal{L}|(x)).
\]

Since it is a “\( \delta \)-function”, and all the hyperplanes \( x + |\mathcal{L}(x)| \) in \( \mathcal{L} \) are pairwise distinct, there are such functions \( b_0 : \text{Pic}^{\geq 2g}(C) \rightarrow \hat{\mathbb{Z}}(1) \) and \( a : \text{Pic}^{\geq 2g}(C) \rightarrow C(k) \) that

\[
f(x) + f(\text{general point of } |\mathcal{L}|(x)) = b_0(\mathcal{L})\delta_{x,a(\mathcal{L})} + b_1(\mathcal{L}),
\]

where \( b_1 : \text{Pic}^{2g}(C) \rightarrow \hat{\mathbb{Z}}(1) \) is the function \( \mathcal{L} \mapsto f(\text{general point of } |\mathcal{L}|) \). Then

\[
f(x) = b_0(\mathcal{L})\delta_{x,a(\mathcal{L})} + b_1(\mathcal{L}) - b_1(\mathcal{L}(x)).
\]

By Lemma 3.3, for any \( \mathcal{L}, \mathcal{L}' \in \text{Pic}^{\geq N}(C) \) the image of the map \( |\mathcal{L}| \times |\mathcal{L}'| \rightarrow |\mathcal{L} \otimes \mathcal{L}'| \) of summation of divisors is not contained in any hyperplane in \( |\mathcal{L} \otimes \mathcal{L}'| \). Then a sum of a general divisor in \( |\mathcal{L}| \) and a general divisor in \( |\mathcal{L}'| \) is a general divisor in the linear system \( |\mathcal{L} \otimes \mathcal{L}'| \), so one has

\[
b_1(\mathcal{L} \otimes \mathcal{L}') = b_1(\mathcal{L}) + b_1(\mathcal{L}'),
\]

and therefore, for any sheaf \( \mathcal{L}_0 \) of degree zero one has

\[
b_1(\mathcal{L}') + b_1(\mathcal{L}_0 \otimes \mathcal{L}) = b_1(\mathcal{L}) + b_1(\mathcal{L}_0 \otimes \mathcal{L}'),
\]

so \( b_2(\mathcal{L}_0) := b_1(\mathcal{L}_0 \otimes \mathcal{L}) - b_1(\mathcal{L}) : \text{Pic}^0(C) \rightarrow \hat{\mathbb{Z}}(1) \) does not depend on \( \mathcal{L} \). It is easy to see that \( b_2 \) is a homomorphism, which therefore should be zero, since \( \text{Pic}^0(C) \) is a divisible group. From this we conclude that \( b_1(\mathcal{L}) = b_1(\deg \mathcal{L}) \), and finally, \( f(x) = b_0(\mathcal{L})\delta_{x,a(\mathcal{L})} + b_2(\mathcal{L}) \) is a \( \delta \)-function on \( C(k) \), i.e., corresponds to a point of \( C \), or to a decomposition subgroup in \( U_{k(x)}^{ab} \). \( \square \)
4. Automorphisms of subgroups between $G^0$ and $G$

**Lemma 4.1.** 1. Suppose that for a subgroup $H$ in $G$ containing $G^0$ (the restriction to $G^0$ of) a homomorphism $\lambda : H \rightarrow G$ induces the identity map of the set $\mathfrak{F}$ of compact open subgroups in $G$. Then $\lambda = \text{id}$.

2. The centralizer of $G^0$ in $G_{F/\mathbb{Q}}$ is trivial.

**Proof.** For any $\sigma \in H$ and any open compact subgroup $U$ one has $\sigma U \sigma^{-1} = \lambda(\sigma U \sigma^{-1}) = \lambda(\sigma) \lambda(U) \lambda(\sigma)^{-1} = \lambda(\sigma) U \lambda(\sigma)^{-1}$, so $\sigma^{-1} \lambda(\sigma) = 1$.

For a variety $X$ of dimension $n$ over $k$ without birational automorphisms and any $x \in F - k$ there is a subfield $L_x \subset F$ containing $x$ isomorphic to the function field of $X$. Then the normalizer of $U_{L_x}$ coincides with $U_{L_x}$, and the intersection of all $U_{L_x}$ is $\{1\}$, so $\sigma^{-1} \lambda(\sigma) = 1$. On the other hand, if $\tau \in G_{F/\mathbb{Q}}$ normalizes $U_{k(x, P(x)^{1/2})}$ for all polynomials $P$ over $k$, then $\tau \in G_{F/k}$, and therefore, $\tau = 1$. $\square$

Let $\mathfrak{F}$ be the set of compact open subgroups in $G^0$, and let $\mathbb{Q}(\chi)$ be the quotient of the free abelian group generated by $\mathfrak{F}$ by the relations $[U] = [U : U'] \cdot [U']$ for all $U' \subset U$. As the intersection of a pair of a compact open subgroups in $G$ is a subgroup of finite index in both of them, $\mathbb{Q}(\chi)$ is a one-dimensional $\mathbb{Q}$-vector space. The group $G$ acts on it by the conjugations. Let $\chi$ be the character of this representation of $G$.

One can get an explicit formula for $\chi$ as follows. Fix a subfield $L$ of $F$ finitely generated and of transcendence degree $n$ over $k$. Then for any $\sigma \in G$ one has $[U_L] = [L \sigma(L) : L] \cdot [U_{L \sigma(L)}]$ and $[U_{\sigma(L)}] = [L \sigma(L) : \sigma(L)] \cdot [U_{L \sigma(L)}]$, and therefore, $\chi(\sigma) = \frac{[L \sigma(L) : L]}{[L \sigma(L) : \sigma(L)]}$. This implies that $\chi : G \rightarrow \mathbb{Q}_+^\times$ is surjective, and its restriction to $G^0$ is trivial.

**Theorem 4.2.** Let $n = 1$, $H$ be a subgroup in $G$ containing $G^0$, and $N_{G_{F/\mathbb{Q}}}(H)$ its normalizer in $G_{F/\mathbb{Q}}$. Then $N_{G_{F/\mathbb{Q}}}(H) \subseteq N_{G_{F/\mathbb{Q}}}(G) = \{\text{automorphisms of } F \text{ preserving } k\}$, and the adjoint action of $N_{G_{F/\mathbb{Q}}}(H)$ on $H$ induces an isomorphism from $N_{G_{F/\mathbb{Q}}}(H)$ to the group of continuous open automorphisms of $H$. If $H \supseteq \ker \chi$ then $N_{G_{F/\mathbb{Q}}}(H) = N_{G_{F/\mathbb{Q}}}(G^0)$.

**Proof.** For each $U \in \mathfrak{F}$ let $\text{Div}_U^+$ be the free abelian semi-group, whose generators are decomposition subgroups in $U_{\text{ab}}$, and for each integer $d \geq 2$ let

$$\mathfrak{G}_U^{(d)} = \{U_L \supset U \mid [U_L : U] = d, L \cong k(t)\} \subset \mathfrak{F}.$$  

For a smooth projective model $C$ of $F^U$ the set $\mathfrak{G}_U^{(d)}$ is in bijection with the disjoint union of Zariski-open subsets in Grassmannians

$$\prod_{\mathcal{L} \in \text{Pic}^d(C)} \left( \text{Gr}(1, |\mathcal{L}|) - \bigcup_{x \in C(k)} \text{Gr}(1, x + |\mathcal{L}(x)|) \right).$$

One can define:

- an “invertible sheaf of degree $d$ without base points” $\mathcal{L}$, as a subset of $\mathfrak{G}_U^{(d)} \subset \mathfrak{F}$ consisting of elements equivalent under the relation generated by $U_1 \sim_U U_2$ if there are decomposition subgroups $D_a \subset U_1^\text{ab}$ and $D_b \subset U_2^\text{ab}$ such that their preimages in $U^\text{ab}$ contain the same collections of decomposition subgroups with the same indices of their images in $D_a$ and $D_b$;
- the “linear system” $|\mathcal{L}|$, as the set of maximal collections of elements of $\mathcal{L}$ “intersecting at a single point”, i.e., the subset of the free abelian semi-group $\text{Div}_U^+$;
- a “line presented in $\mathcal{L}$” in $|\mathcal{L}|$, as an element of $\mathcal{L} \subset \mathfrak{G}_U^{(d)}$, considered as a subset in $|\mathcal{L}|$;
- an arbitrary “line” in $|\mathcal{L}|$, as a subset in $|\mathcal{L}|$ of type $D + l$, where $D \in \text{Div}_U^+$ and $l$ is a line presented in the sheaf $\mathcal{L}(-D)$ without base points;
• an “$s$-subspace” in $|\mathcal{L}|$, as the union of all lines passing through a given point in $|\mathcal{L}|$ and intersecting a given “($s-1$)-subspace” in $|\mathcal{L}|$.

Now we remark that for any sufficiently big $d$ and any sheaf $\mathcal{L} \subset \mathfrak{Gr}^{(d)}_U$ the set $C_U$ of decomposition subgroups in $U^d$ can be canonically identified with the subset of $|\mathcal{L}|'$ consisting of those hyperplanes in $|\mathcal{L}|$ that each line on each of them is “absent in $\mathcal{L}$”. As $|\mathcal{L}|'$ has a canonical structure of a projective space (but not of a projective space over $k$), this gives us a canonical structure of a scheme on $C_U$. Let $\kappa_U$ be the function field of $C_U$.

Clearly, $\lambda(G^o) = G^o$ and the restriction of $\lambda$ to $G^o$ induces a bijection $\mathfrak{Gr}^{(d)}_U \sim \mathfrak{Gr}^{(d)}_{\lambda(U)}$ for each $d \geq 2$, and for any sheaf $\mathcal{L} \subset \mathfrak{Gr}^{(d)}_U$ it induces a map $|\mathcal{L}| \rightarrow |\lambda(\mathcal{L})|$ which transforms subspaces into subspaces of the same dimension, i.e., a collineation. As $\lambda$ induces a collineation $|\mathcal{L}|' \sim |\lambda(\mathcal{L})|'$, the fundamental theorem of projective geometry (see, e.g., [A]) implies that such $\lambda$ induces an isomorphism $C_U \sim C_{\lambda(U)}$ of schemes over $\mathbb{Q}$. This isomorphism does not depend on $d$ and $\mathcal{L}$, since it determines the collineations $|\mathcal{L}'| \sim |\lambda(\mathcal{L}')|$ for all $\mathcal{L}' \subset \mathfrak{Gr}^{(d)}$. Denote by $\sigma_U$ the induced isomorphism $\kappa_{\lambda(U)} \sim \kappa_U$.

For each subgroup $U'$ of finite index in $U$ the natural map $C_{U'} \rightarrow C_U$ is a morphism of schemes, and in particular, $\kappa_U$ is naturally embedded into $\kappa_{U'}$. The group $G^o$ acts on the field $\lim_{U \rightarrow \mathcal{L}} \kappa_{\lambda(U)} \sim F$ commuting with the $G^o$-action. Since the diagram

$$
\begin{align*}
C_{U'} & \rightarrow C_{\lambda(U')} \\
\downarrow & \quad \downarrow \\
C_U & \rightarrow C_{\lambda(U)}
\end{align*}
$$

commutes, the restriction of $\sigma_{U'}$ to $\kappa_U$ coincides with $\sigma_U$, and finally, we get an automorphism $\sigma$ of $F$ induced by $\lambda$. As $k$ is the only maximal algebraically closed subfield in its arbitrary finitely generated extension, $\sigma$ induces an automorphism of $k$, and therefore, normalizes $G^o$.

Then the restriction to $G^o$ of $\text{ad}(\sigma) \circ \lambda$ acts trivially on all of $\mathfrak{Gr}^{(d)}$. As any open compact subgroup is an intersection of elements of $\mathfrak{Gr}^{(d)}_U$, for $d$ big enough and $U'$ small enough, $\text{ad}(\sigma) \circ \lambda$ acts on $\mathfrak{S}$ also trivially. By Lemma 4.3, this implies that $\lambda = \text{ad}(\sigma^{-1})$. □

**Remark.** If $k$ is countable then the inverse of any continuous automorphism as in the statement of Theorem 1.2 is automatically continuous :

**Lemma 4.3.** If $k$ is countable, and $U \xrightarrow{\lambda} U'$ is a continuous surjective homomorphism of open subgroups in $G_{F/k}$ and $G_{F'/k'}$ then the image in $U'$ of an open subset in $U$ is open.

**Proof.** Let $U_L \subset U$ be an open compact subgroup. Then $U/U_L$ is a countable set surjecting onto the set $U'/\lambda(U_L)$. By Proposition 2.1, for the subfield $L' = F^{\lambda(U_L)}$ one has $L' = F$. If $\lambda(U_L)$ is not open then $L'$ is not finitely generated over $k'$, and therefore, $U'/\lambda(U_L)$ is not countable. □

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