OPTIMAL ASSET PORTFOLIO WITH STOCHASTIC VOLATILITY UNDER THE MEAN-VARIANCE UTILITY WITH STATE-DEPENDENT RISK AVERSION

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Abstract. This paper studies the portfolio optimization of mean-variance utility with state-dependent risk aversion, where the stock asset is driven by a stochastic process. The sub-game perfect Nash equilibrium strategies and the extended Hamilton-Jacobi-Bellman equations have been used to derive the system of non-linear partial differential equations. From the economic point of view, we demonstrate the numerical evaluation of the suggested solution for a special case where the risk aversion rate is proportional to the wealth value. Our results show that the asset driven by the stochastic volatility process is more general and reasonable than the process with a constant volatility.

1. Introduction. Portfolio optimization is a challenging problem in investment and risk management. The objective is to optimize the proportions of the assets to be held in a portfolio according to certain criterion on the expected rate of investment return and/or the level of financial risk. By quantifying the criterion by certain utility function, the problem can be formulated as maximizing the expected value of the utility function at certain pre-determined time in future such as at the end of certain investment period. Over the last couple of decades, various types of utility functions have been used in financial modeling and portfolio optimization. In the work of [20], in developing an equilibrium asset and option pricing model, it is assumed that the the representative investor seeks to maximize the expected value of the utility function over a period of time and a constant relative risk aversion (CRRA) utility function is used; in [8], an instantaneous utility function with habit forming performance is proposed in formulating the preference of the representative investor; while in [2], a mean-variance utility function is used for a dynamic asset allocation problem. In this paper, we focus on the mean-variance (MV) utility.

Since Markowitz [15] first introduces the asset optimization analysis for mean-variance, a large volume of literature has been published concerning this topic, and the one period case is often taken into consideration. However, in practice, this is not reasonable because the optimal portfolio optimization problems are multi-period

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optimizing problems, and should be modeled within the multi-period framework. On the other hand, the mean-variance based portfolio optimization in the multi-period framework is time inconsistent as the Bellman Optimization principle does not hold in this case [4].

Commonly, two approaches are used to deal with this kind of time-inconsistent optimal control problems. One of the approaches is called pre-committed, namely the initial point \((0, x_0)\) is fixed and the control law \(u^*\) is chosen to maximize \(Q(0, x_0, u)\). Here we simply regard that the optimal law we choose at initial time will be the best choice in the later time point \((s, X_s)\) for the functional \(Q(s, X_0, u)\).

Kydland and Prescott[11] discuss the economical meaning of this pre-committed strategy. While Richardson[18], followed by Bajeux-Besnainou[1], may be the first one to explore the portfolio optimization under mean-variance in continuous time even though he just sets one single stock with a constant risk-free rate. By Li and Ng[12], the original MV problem can be transformed into a stochastic linear-quadratic control problem. Further research mainly focuses on model extensions and improvements[13]-[19], such as the inclusion of cost transaction by Dai, Xu, Zhou[7].

An alternative approach is to formulate the time inconsistent problem in the framework of game theory. Game theory approaches have a long history since Markowitz [15] first introduces the mean-variance portfolio optimization analysis. Along this road, Goldman [9], Krusell and Smith [10], Peleg and Yaari[17] and Pol lak [16] have also formulated their problems within the game theory framework. Particularly, in the MV analysis, Basak and Chabakauri [2] firstly studied the game theoretic approach towards portfolio optimization in continuous time. Later, Bjork and Murgoci [4] extend Basak’s model by relaxing the risk aversion rate in mean-variance utility from a constant to a state dependent one. This is a very provoking development both from modeling and economic significance. More and more researchers are attracted to game theoretic approach because it is more rigorous.

For this paper, we take the Game theory approach to solve the time inconsistent problem. In brief, our control problem in multiple periods is regarded as a game among multiple players which represent our present favor and the incarnations of our further tastes. Our objective is to find the sub-game perfect Nash equilibrium point for this game. The main contributions of our paper are as follows. First, we relax the constant volatility in Bjork and Murgoci [4] into a stochastic one which is more economically relevant. Secondly, we derive the general extended Hamilton-Jacobi-Bellman equation by taking stochastic volatility into account. Thirdly, we establish an Euler discretization scheme for numerical solution of the underlying equations, and then numerically demonstrate the effect of stochastic volatility on asset allocation under a special case of state dependent risk aversion.

2. The portfolio optimization model and HJB equations. We consider two assets in our model, including the stock and the risk-free bond. Let \(B_t\) and \(r\) be the investment and the investment return rate of the risk-free bond respectively, then

\[ dB_t = rB_t dt, \quad B_0 = 1. \] (1)

Assume that the stock is governed by the following stochastic process:

\[ dS_t = \alpha S_t dt + \sigma(V_t)S_t dW^S_t. \] (2)
Similar to the square root process introduced by Cox Ingersoll and Ross\[6\], we assume the dynamic stochastic volatility to \( \sigma(V_t) = \sqrt{V_t} \) and the \( V_t \) is governed by:

\[
dV_t = \kappa(\theta - V_t)dt + \varepsilon\sqrt{V_t}dW_t^V, \tag{3}
\]

where \( \kappa, \theta, \varepsilon \) are constants. For \( \kappa, \theta > 0 \), (3) is a continuous time first-order autoregressive process where the randomly moving volatility is elastically pulled toward a central location or long-term value, \( \theta \). The parameter \( \kappa \) indicates the speed of adjustment, \( W_t^S \) and \( W_t^V \) are dependent Wiener Processes with the coefficient \( \rho \).

Denote the total wealth as \( X_t \) with the initial wealth \( X_0 = 0 \), and define \( u_t \) as the amount of wealth invested into the selected stock \( S_t \). Since \( S_t \) is a stochastic process, the total wealth process \( X_t \) is also stochastic and governed by

\[
dX_t = \left[ rX_t + (\alpha - r)u_t \right]dt + u_t\sqrt{V_t}dW_t^S. \tag{4}
\]

We choose the mean-variance utility with state-dependent risk aversion \( \gamma(x) \) as the objective function, namely

\[
J(t, x, v, u) = E[X_T] - \frac{\gamma(x)}{2} Var[X_T], \tag{5}
\]

Our purpose is to find the best control law \( \hat{u} \) to maximize the mean-variance utility at \( t = T \), that is to maximize the expected return with a penalty term for the risk, namely

\[
Q(t, X_t, V_t) = \max_{u \in U} \{ E_{t,x,v}[X_T^U] - \frac{\gamma(x)}{2} Var_{t,x,v}[X_T^U] \}, \tag{6}
\]

where \( U \in [0, X_t] \) is the admissible control space.

**Remark 1.** Our model generalizes the work of Bjork\[4\] by allowing the volatility to be stochastic rather than constant, making the model more realistic.

Following the work of Bjork, Murgoci and Zhou\[4\] we use the game theoretic formulation to solve the above defined optimization problem. First, we provide the definition of the so-called optimal equilibrium control. Let

\[
u_{\Delta t}(s, \cdot) = \begin{cases} u, & \text{for } t \leq s < t + \Delta t, \\ \hat{u}(s, \cdot), & \text{for } t + \Delta t \leq s \leq T, \end{cases}
\]

where \( u \in \mathbb{R}^k \), \( \Delta t > 0 \), and \( t \in [0, T] \) is arbitrarily chosen. If

\[
\lim_{\Delta t \to 0} \frac{J(t, x, v, \hat{u}) - J(t, x, v, u_{\Delta t})}{\Delta t} \geq 0,
\]

for all \( u \in \mathbb{R}^k \), and \( (t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \), we say that \( \hat{u} \) is the equilibrium control and we have the equilibrium value function as follows

\[
Q(t, x, v) = J(t, x, v, \hat{u}).
\]

**Remark 2.** In the standard time-consistent setting, the equilibrium control is the same as the optimal control.

In this work, we extend the work of Bjork and Murgoci\[4\] to the case with stochastic volatility. Let \( A \) be the infinitesimal generator. For any fixed \( u \in U \), we introduce the corresponding controlled infinitesimal generator \( A^u \) defined by

\[
A_u^u = \frac{\partial}{\partial t} + [rx + (\alpha - r)u] \frac{\partial}{\partial x} + [\kappa(\theta - v)] \frac{\partial}{\partial v} + \frac{1}{2} vu^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2} v^2 \varepsilon^2 \frac{\partial^2}{\partial v^2} + \varepsilon uv \rho \frac{\partial^2}{\partial x \partial v}. \tag{7}
\]

Based on the work of Bjork and Murgoci\[5\], we derive the following result:
Theorem 2.1. The extended HJB equations for the Nash equilibrium problem by taking into account stochastic volatility are as follows

\[ Q_t + \sup_{u} \{ rx + (\alpha - r)uQ_x + [\kappa(\theta - v)]Q_v + \frac{1}{2}u^2vQ_{xx} + \frac{1}{2}v^2Q_{vv} \]
\[ + \varepsilon uvQ_{xv} - f_t - [rx + (\alpha - r)u][f_x(t, x, v, y) + f_y(t, x, v, y)] \]
\[ - [\kappa(\theta - v)]f_v - \frac{1}{2}u^2v(f_{xx} + 2f_{xv} + f_{yy}) - \frac{1}{2}\varepsilon^2vQ_{xv} - \varepsilon uv\rho(f_{xv} + f_{vy}) \]
\[ + f^u_t + [rx + (\alpha - r)u]f^u_g(t, x, v) + [\kappa(\theta - v)]f^u_v + \frac{1}{2}u^2v^2f^u_{xx} + \frac{1}{2}\varepsilon^2v^2f_{vv} \]
\[ + \varepsilon uv\rho f^u_{xv} - [rx + (\alpha - r)u]G_x - [\kappa(\theta - v)]G_v \]
\[ - \frac{1}{2}u^2v(G_{xx} + 2G_{xg}g_x + G_{gg}(g_x)^2) - \frac{1}{2}\varepsilon^2v(G_{vv} + 2G_{vg}g_v + G_{gg}(g_v)^2) \]
\[ - \varepsilon uv\rho(G_{xv} + G_{xg}g_v + G_{gg}g_x + G_{gg}(g_x)(g_v) = 0, \]  

\[ A^u f^u(t, x, v) = 0, 0 \leq t \leq T; \]  

\[ A^u g(t, x, v) = 0, 0 \leq t \leq T; \]  

\[ Q(T, x, v) = F(x, x, v) + G(x, x, v), \]  

\[ f^u(T, x, v) = F(y, x, v), \]  

\[ g(T, x, v) = x. \]  

In the above theorem, \( f(\cdot) \) is different from \( f^u(\cdot) \). The former is the function of variables \( t, x, v, y \), i.e. \( f(t, x, v, y) \), while the later is a function of \( t, x, v \), i.e. \( f^u(t, x, v) = f(t, x, v, \hat{y}) \) where \( \hat{y} \) is fixed. When the parameter \( y \) is fixed, \( f(t, x, v, y) \) is equal to \( f^u(t, x, v) \). And in our specific problem, the functions \( F, G, f, g \) are as follows.

\[ f^u(t, x, v, y) = E_t, x, v[F(y, X_T^u)], \]  

\[ g^u(t, x, v) = E_t, x, v[X_T^u], \]  

\[ F(y, X_T) = X_T - \frac{\gamma(y)}{2}(X_T)^2, \]  

\[ G(y, g) = \frac{\gamma(y)}{2}g^2. \]

The details of the derivation of Theorem 2.1 are given in Appendix A.

3. The optimal strategy under state-dependent risk aversion and stochastic volatility. For state-dependent risk aversion \( \gamma(x) \), from (17), we have

\[ G_x = \frac{\gamma'(x)}{2}g^2, \]  

\[ G_{xx} = \frac{\gamma''(x)}{2}g^2, \]  

\[ G_{xg} = \gamma'(x)g, \]  

\[ G_{gg} = \gamma(x). \]
By substituting the above equations into the HJB system in Theorem 2.1, we obtain the following extended HJB system:

\[
Q_t + \sup_u \left\{ \left[ r x + (\alpha - r) u \right] [Q_x - f_y - \frac{\gamma'(x)}{2} g^2] + \left[ \kappa(\theta - v) \right] Q_v + \frac{1}{2} u^2 v [Q_{xx} - 2 f_{xy} - f_{yy} - \frac{\gamma''(x)}{2} g^2 - 2 \gamma'(x) g g_x - \gamma(x) g_x^2] \right\} + \frac{1}{2} \varepsilon^2 v Q_{vv} + \varepsilon u \nu [Q_{xv} - f_{vy} - \gamma'(x) g g_v - \gamma(x) g_x g_v] \right\} = 0, \tag{22}
\]

\[
f_t + \left[ r x + (\alpha - r) \tilde{u} \right] f_x + \left[ \kappa(\theta - v) \right] f_v + \frac{1}{2} v \tilde{u}^2 f_{xx} + \frac{1}{2} v \varepsilon^2 f_{vv} + \varepsilon u \nu f_{xv} = 0, \tag{23}
\]

\[
g_t + \left[ r x + (\alpha - r) \tilde{u} \right] g_x + \left[ \kappa(\theta - v) \right] g_v + \frac{1}{2} v \tilde{u}^2 g_{xx} + \frac{1}{2} v \varepsilon^2 g_{vv} + \varepsilon u \nu g_{xv} = 0, \tag{24}
\]

\[
f(T, x, v, x) = x - \frac{\gamma(x)}{2} x^2, \tag{25}
\]

\[
g(T, x, v) = x, \tag{26}
\]

As \( \tilde{u} \) is the optimal equilibrium control law, we have

\[
Q(t, x, v) = E_{t, x, v}[X_T^0] - \frac{\gamma(x)}{2} \text{Var}_{t, x, v}[X_T^0], \tag{27}
\]

\[
f(t, x, v, x) = E_{t, x, v}[X_T^0] - \frac{\gamma(x)}{2} E_{t, x, v}[X_T^0]^2, \tag{28}
\]

\[
g(t, x, v) = E_{t, x, v}[X_T^0], \tag{29}
\]

\[
Q(t, x, v) = f(t, x, v, x) + \frac{\gamma(x)}{2} g^2(t, x, v). \tag{30}
\]

From the accordance of equation (30), we obtained the relationships between \( Q, f \) and \( g \):

\[
Q_t = f_t + \gamma g g_t, \tag{31}
\]

\[
Q_x = f_x + f_y + \frac{\gamma'}{2} g^2 + \gamma g g_x, \tag{32}
\]

\[
Q_v = f_v + \gamma g g_v, \tag{33}
\]

\[
Q_{xx} = f_{xx} + f_{yy} + 2 f_{xy} + \frac{\gamma''}{2} g^2 + 2 \gamma' g g_x + \gamma g_x^2 + \gamma g g_{xx}, \tag{34}
\]

\[
Q_{xv} = f_{xv} + f_{vy} + \gamma' g g_v + \gamma g_x g_v + \gamma g g_{xv}, \tag{35}
\]

\[
Q_{vv} = f_{vv} + g g_v^2 + \gamma g g_{vv}. \tag{36}
\]

By taking the above equations back into (22), we have:

\[
f_t + \gamma(x) g g_t + \sup_u \left\{ \left[ r x + (\alpha - r) u \right] [f_x + \gamma(x) g g_x] + \left[ \kappa(\theta - v) \right] [f_v + \gamma(x) g g_v] \right\} + \frac{1}{2} u^2 v [f_{xx} + \gamma(x) g g_{xx}] + \frac{1}{2} \varepsilon^2 v [f_{vv} + \gamma(x) g g_{vv}] + \varepsilon u \nu [f_{xv} + \gamma(x) g g_{xv}] \right\} = 0, \tag{37}
\]

Let

\[
Z = \left[ r x + (\alpha - r) u \right] [f_x + \gamma(x) g g_x] + \left[ \kappa(\theta - v) \right] [f_v + \gamma(x) g g_v] + \frac{1}{2} u^2 v [f_{xx} + \gamma(x) g g_{xx}] + \frac{1}{2} \varepsilon^2 v [f_{vv} + \gamma(x) g g_{vv}] + \varepsilon u \nu [f_{xv} + \gamma(x) g g_{xv}], \tag{38}
\]
Then according to the envelope theorem, we have

\[
\frac{\partial Z}{\partial u} \bigg|_{\hat{u}} = 0,
\]

(39)

Hence, by solving (38-39), we obtain the expression of the equilibrium control law \( \hat{u} \), as given in the following theorem.

**Theorem 3.1.** Under state-dependent risk aversion and stochastic volatility, the optimal control strategy is

\[
\hat{u} = -\frac{1}{v} \frac{(\alpha - r)(f_x + \gamma(x)gg_x) + \varepsilon v p(f_{xx} + \gamma(x)gg_{xx})}{f_{xx} + \gamma(x)gg_{xx}},
\]

(40)

where \( f \) and \( g \) are given by the partial equation (23-26).

**Remark 3.** Theorem 3.1 is also applicable to the case of a constant risk aversion and stochastic volatility. In this special case, the \( \gamma(x) \) in (40) is replaced by the constant risk aversion \( \gamma \).

**Remark 4.** Compared with the result obtained under constant volatility in [4], we have an term \( \varepsilon v p(f_{xx} + \gamma(x)gg_{xx}) \) in (40) because of the stochastic volatility. Obviously, our results include the work of [4] under constant volatility as a special case.

4. **Optimal control under ‘natural’ risk aversion.** Based on the dimensional analysis of the two terms in the mean-variance utility as in [4], we assume the risk aversion to take the following form:

\[
\gamma(x) = \frac{\gamma}{x},
\]

(41)

Thus, we have:

\[
\gamma'(x) = -\frac{\gamma}{x^2},
\]

(42)

\[
\gamma''(x) = \frac{2\gamma}{x^3}.
\]

(43)

By substituting the above into (37), the extended HJB system for this case becomes

\[
f_t + \frac{\gamma}{x} gg_t + \sup_u \{[r + (\alpha - r)u][f_x + \frac{\gamma}{x} gg_x] + [\kappa(\theta - v)][f_v + \frac{\gamma}{x} gg_v] \\
+ \frac{1}{2} u^2 v[f_{xx} + \frac{\gamma}{x} gg_{xx}] + \frac{1}{2} \varepsilon^2 v[f_{vv} + \frac{\gamma}{x} (g_v)^2 + \frac{\gamma}{x} gg_{vv}] \\
+ \varepsilon u p[f_{xv} + \frac{\gamma}{x} gg_{xv}] \} = 0,
\]

(44)

Using similar steps as (38-39), we obtain

\[
\hat{u} = -\frac{1}{v} \frac{(\alpha - r)(f_x + \frac{\gamma}{x} gg_x) + \varepsilon v p(f_{xx} + \frac{\gamma}{x} gg_{xx})}{f_{xx} + \frac{\gamma}{x} gg_{xx}},
\]

(45)

Since \( X_t \) is the Markov process, it conforms to the Geometric Brownian Motion. From the martingale theorem, we conjecture that

\[
E_{t,x,v}(X_{T}^{\hat{u}}) = p(t)x,
\]

(46)

\[
E_{t,x,v}[(X_{T}^{\hat{u}})^2] = q(t)x^2,
\]

(47)
where $x$ is a function of $t$ and $v$. Thus,

$$f(t, x, v, y) = E_{t, x, v}[X_T^U] - \frac{\gamma}{2y} E_{t, x, v}[X_T^U]^2,$$

$$g(t, x, v) = E_{t, x, v}[X_T^U].$$

All the above settings lead to the Ansatz,

$$f(t, x, v, y) = p(t) x - \frac{\gamma}{2y} q(t) x^2,$$

$$g(t, x, v) = p(t) x.$$

Thus,

$$f_t(t, x, v, y) = p'(t) x - \frac{\gamma}{2y} q'(t) x^2,$$

$$f_x(t, x, v, y) = p(t) - \frac{\gamma}{y} q(t),$$

$$f_{xx}(t, x, v, x) = -\frac{\gamma}{x} q(t),$$

$$f_v = f_{xv} = f_{vv} = 0,$$

$$g_t(t, x, v) = p'(t) x,$$

$$g_x(t, x, v) = p(t),$$

$$g_v = g_{xx} = g_{xv} = g_{vv} = 0.$$

Substituting all the above equations back into (45), we obtain

$$\hat{u} = \frac{1}{v} \frac{(\alpha - r)[p + \gamma(p^2 - q)] x}{\gamma q},$$

where $p$, $q$ can be obtained by substituting the above equations back into (23) and (24)

$$p' x - \frac{\gamma}{2y} q' x^2 + (r x + \frac{(\alpha - r)^2[p + \gamma(p^2 - q)] x}{\gamma q v})(p - \frac{\gamma}{y} q x)$$

$$+ \frac{1}{2} v \frac{(\alpha - r)^2}{\nu^2 q^2} [p + \gamma(p^2 - q)]^2 x^2 (-\frac{\gamma}{x} q) = 0,$$

$$p' x + (r x + \frac{(\alpha - r)^2[p + \gamma(p^2 - q)] x}{\gamma q v}) p = 0.$$

By splitting the above two equations, we have the following ODE for the determination of $p$ and $q$

$$p' + (r + \frac{(\alpha - r)^2[p + \gamma(p^2 - q)]}{\gamma q v}) p = 0,$$

$$q' + \{2(r + \frac{(\alpha - r)^2[p + \gamma(p^2 - q)]}{\gamma q v}) + \frac{(\alpha - r)^2}{\nu^2 q^2} [p + \gamma(p^2 - q)]^2\} q = 0,$$

$$p(T) = 1,$$

$$q(T) = 1.$$

Alternatively, we can write $\hat{u}$ by the expression

$$\hat{u} = \lambda(t, v) x,$$
For the equation (63-66), we focus directly on the function \( \lambda(t, v) \) using integral equations:

\[
\lambda(t, v) = \frac{(\alpha - r)}{\gamma q v} \left\{ e^\int_t^T [r + \beta \lambda(s, v) + \nu \lambda^2(s, v)] ds + \gamma e^\int_t^T \nu \lambda^2(s, v) ds - \gamma \right\},
\]

where \( p = e^{\int_t^T [r + (\alpha - r) \lambda(s, v)] ds} \) and \( q = e^{\int_t^T (r + (\alpha - r) \lambda(s, v) + \frac{1}{2} \nu \lambda^2(s, v)) ds} \).

**Remark 5.** For the equation (63-66), we focus directly on the function \( \lambda(t, v) \) in the Ansatz \( \hat{u} = \lambda(t, v) x \) to derive a single integral equation for \( \lambda \). Particularly, by writing the wealth process \( X \) as:

\[
dX^u_t = [r + (\alpha - r) \lambda(t, v)] X^u_t dt + \lambda(t, v) X^u_t \sqrt{V^u_t} dW^S_t,
\]

and using the results from (46), (47) and (68), we can solve for the expression of \( p \) and \( q \) in (69).

5. **Numerical examples.** In this section, we present the numerical results for the suggested model by discretizing the wealth process using the Euler discretization scheme:

\[
\begin{align*}
X_{t+dt} &= X_t + [rX_t + (\alpha - r)u_t] dt + u_t \sqrt{v_t^l} dt Z_X, \\
v_{t+dt} &= v_t^+ + \kappa (\theta - v_t^+) dt + \epsilon \sqrt{v_t^l} dt Z_v, \\
v_t^+ &= \max(0, v_t), \\
Z_s = \rho Z_1 + \sqrt{1 - \rho^2} Z_2, \\
Z_1, Z_2 &\sim N(0, 1).
\end{align*}
\]

In addition, we implement the special case of risk aversion \( \gamma(x) = \frac{2}{x} \) with some choices of \( \gamma \) where \( \lambda(t, v) \) is calculated from (69). Our focus is on the effect of the stochastic volatility on the portfolio mean-variance optimization problem. Figure 1-3 plots the value of the portfolio, the amount of money invested in stocks, risk aversion parameters and the dynamic of the volatility versus time.

In Figures 1 and 2, we compare the investing strategies for the slow mean reverting volatility and the fast mean reverting volatility with \( \kappa = 0.01 \) and \( \kappa = 20 \) against constant volatility respectively. The large \( \kappa \) drives the volatility to be mean reverting around its long-run volatility more often. For the slower mean reversion rate case, the difference in the amount of money invested into stocks is more visible than the case of constant volatility. Next, we demonstrate the effects of varying the correlation between wealth and volatility. As can be seen in Figure 3, increasing the magnitude of \( \rho \) causes a strong correlation (in this case, negative) between the amount invested into stock as well as the value of wealth and volatility movement. Particularly, for \( \rho = -0.9 \), the amount invested into stocks decreases significantly during the high volatility time and increases significantly during the low volatility period comparing with \( \rho = -0.5 \) and \( \rho = -0.1 \). We also present 100,000 Monte Carlo simulations to simulate the wealth movement to 1-year period at Figure 4. We see that the terminal values of the wealth are empirically normal with \( \mu = 1.128 \) and \( \sigma = 0.009 \).
Figure 1. Wealth, money, risk aversion and fast stochastic volatility versus time for $T = 1, \alpha = 0.15, r = 0.05, \gamma(x) = \frac{4}{x}, \kappa = 0.01, \theta = 0.04, \epsilon = 0.1$ and $\rho = -0.9$.

Figure 2. Wealth, money, risk aversion and fast stochastic volatility versus time for $T = 1, \alpha = 0.15, r = 0.05, \gamma(x) = \frac{4}{x}, \kappa = 20, \theta = 0.04, \epsilon = 0.1$ and $\rho = -0.9$.

Figure 3. Wealth, money, risk aversion and fast stochastic volatility versus time for $T = 1, \alpha = 0.15, r = 0.05, \gamma(x) = \frac{4}{x}, \kappa = 20, \theta = 0.04, \epsilon = 0.1$ with different $\rho$. 
From the iterated conditioning, we obtain

$$E_{t,X_t,V_t}[F(y,X_t^U)] + G(y,E_{t,X_t,V_t}[X_t^U]) = 0$$

where $F$ and $G$ have the expression in (16) and (17). For $s > t$,

$$J(s, X_s, V_s, U) = E_{s,X_s,V_s}[F(Y_s,X_s^U)] + G(Y_s,E_{s,X_s,V_s}[X_s^U])$$

From the Markovian structure and the definitions of (16) and (17),

$$E_{s,X_s,V_s}[F(Y_s,X_s^U)] = f^U(s, X_s, V_s, Y_s)$$

$$E_{s,X_s,V_s}[X_s^U] = g^U(s, X_s, V_s)$$

Then, (73) can be written as follows,

$$J(s, X_s, V_s, U) = f^U(s, X_s, V_s, Y_s) + G^U(Y_s, g^U(s, X_s, V_s)).$$

Taking expectations on both sides gives us,

$$E_{t,X_t,V_t}[J(s, X_s, V_s, U)] = E_{t,X_t,V_t}[f^U(s, X_s, V_s, Y_s)]$$

$$+ E_{t,X_t,V_t}[G(Y_s, g^U(s, X_s, V_s))],$$

and going back to the definition of (72), we have

$$E_{t,X_t,V_t}[J(s, X_s, V_s, U)] = J(t, X_t, V_t, U) + E_{t,X_t,V_t}[f^U(s, X_s, V_s, Y_s)]$$

$$- E_{t,X_t,V_t}[F(y,X_t^U)] + E_{t,X_t,V_t}[G(Y_s, g^U(s, X_s, V_s))] - G(y,E_{t,X_t,V_t}[X_t^U]),$$

From the iterated conditioning, we obtain

$$E_{t,X_t,V_t}[F(y,X_t^U)] = E_{t,X_t,V_t}[E_{s,X_s,V_s}[F(y,X_s^U)]]$$

$$= E_{t,X_t,V_t}[f^U(s, X_s, V_s, y_s)],$$

and that

$$E_{t,X_t,V_t}[X_t^U] = E_{t,X_t,V_t}[E_{s,X_s,V_s}[X_s^U]] = E_{t,X_t,V_t}[g^U(s, X_s, V_s)].$$

**Appendix A. Derivation of the extended HJB equations.** As for the HJB equations in [4], by (5), we have

$$Q(t,X_t,V_t) = \sup_{u \in U} J(t,X_t,V_t,U),$$

where

$$J(t,X_t,V_t,U) = E_{t,X_t,V_t}[F(y,X_t^U)] + G(y,E_{t,X_t,V_t}[X_t^U]),$$
Substituting (79) and (80) back into (78), we obtain
\[
E_{t,X,V}[J(s,X_s,V_s,U)] - J(t,X_t,V_t) - E_{t,X,V}[f^U(s,X_s,V_s)] + E_{t,X,V}[f^U(s,X_s,V_s,y)] - E_{t,X,V}[G(Y_s,g^U(s,X_s,V_s))] + G(y,E_{t,X,V}[g^U(s,X_s,V_s)])
\]
(81)
Then
\[
\sup_{u \in U} \{E_{t,X,V}[J(s,X_s,V_s,U)] - J(t,X_t,V_t) - E_{t,X,V}[f^U(s,X_s,V_s)] + E_{t,X,V}[f^U(s,X_s,V_s,y)] - E_{t,X,V}[G(Y_s,g^U(s,X_s,V_s))] + G(y,E_{t,X,V}[g^U(s,X_s,V_s)])\} = 0.
\]
(82)
From (71) and the definition of the control law, we know \(U\) coincides with the equilibrium law \(\bar{u}\) in \([s,T]\), then we have the following formula,
\[
J(s,X_s,V_s,\bar{u}) = Q(s,X_s,V_s),
\]
(83)
\[
f^U(s,X_s,V_s,y) = f(s,X_s,V_s,y),
\]
(84)
\[
g^U(s,X_s,V_s) = g(s,X_s,V_s).
\]
(85)
Thus, (82) can be written as
\[
\sup_{u \in U} \{E_{t,X,V}[Q(s,X_s,V_s)] - Q(t,X_t,V_t) - E_{t,X,V}[f(s,X_s,V_s,y)] - E_{t,X,V}[G(Y_s,g(s,X_s,V_s))] + G(y,E_{t,X,V}[g^U(s,X_s,V_s)])\} = 0.
\]
(86)
By denoting
\[
E_{t,X,V}[Q(s,X_s,V_s)] - Q(t,X_t,V_t) = A^uQ,
\]
(87)
\[
E_{t,X,V}[f(s,X_s,V_s,y)] = A^uf,
\]
(88)
\[
E_{t,X,V}[f(s,X_s,V_s,y)] = A^uf^v,
\]
(89)
\[
E_{t,X,V}[G(Y_s,g(s,X_s,V_s))] = A^uG,
\]
(90)
\[
G(y,E_{t,X,V}[g^U(s,X_s,V_s)]) = H^ug.
\]
(91)
Then from the extended HJB equation in [5], we developed to the one with stochastic volatility as follows,
\[
Q_t + \sup_u \{[r + \alpha - \rho)v]Q_x + [\kappa(\theta - v)]Q_v + \frac{1}{2}u^2vQ_xx + \frac{1}{2}\varepsilon^2vQ_vv + \varepsilon uv\rho Q_xv - A^uf + A^uf^v - A^uG(x,v,g) + H^ug\} = 0,
\]
(92)
we have
\[
-A^uf + A^uf^v = -f_t - [r + \alpha - \rho)v](f_x + f_y) - [\kappa(\theta - v)]f_v
- \frac{1}{2}u^2v(f_{xx} + 2f_{xy} + f_{yy}) - \frac{1}{2}\varepsilon^2vfvv - \varepsilon uv\rho(f_{xv} + f_{yv}) + f^v_t
+ [r + \alpha - \rho)v]f^v_x + [\kappa(\theta - v)]f^v_y + \frac{1}{2}u^2v^2f^v_{xx} + \frac{1}{2}\varepsilon^2v^2f^v_{vv} + \varepsilon uv\rho f^v_{xv},
\]
(93)
\[- A^u G + H^u g = -G_g g_t - [r x + (\alpha - r)u](G_x + G_g g_x) \]
\[- [\kappa(\theta - v)](G_v + G_g g_v) - \frac{1}{2} u^2 v(G_{xx} + 2G_{xg} g_x + G_g g_{xx} + G_g g_g (g_x)^2) \]
\[- \frac{1}{2} \varepsilon^2 v[G_{vv} + 2G_{vg} g_v + G_g g_{vv} + G_g g_v g_v] + G_g g_t \]
\[- [r x + (\alpha - r)u]G_g g_x + [\kappa(\theta - v)]G_g g_v + \frac{1}{2} u^2 vG_g g_{xx} + \frac{1}{2} \varepsilon^2 vG_g g_v v \]
\[- \varepsilon u v p G_g g_x v. \]

Substituting (93) and (94) back into (92), we obtain Theorem 2.1.

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