An integrable 3D lattice model with positive Boltzmann weights

Vladimir V Mangazeev\textsuperscript{1,2}, Vladimir V Bazhanov\textsuperscript{1,2} and Sergey M Sergeev\textsuperscript{1,3}

\textsuperscript{1} Department of Theoretical Physics, Research School of Physics and Engineering, Australian National University, Canberra, ACT 0200, Australia
\textsuperscript{2} Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200, Australia
\textsuperscript{3} Faculty of Education, Science, Technology and Mathematics, University of Canberra, Bruce, ACT 2601, Australia

E-mail: Vladimir.Mangazeev@anu.edu.au

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Abstract
In this paper we construct a three-dimensional (3D) solvable lattice model with non-negative Boltzmann weights. The spin variables in the model are assigned to edges of the 3D cubic lattice and run over an infinite number of discrete states. The Boltzmann weights satisfy the tetrahedron equation, which is a 3D generalization of the Yang–Baxter equation. The weights depend on a free parameter $0 < q < 1$ and three continuous field variables. The layer-to-layer transfer matrices of the model form a two-parameter commutative family. This is the first example of a non-trivial solvable 3D lattice model with non-negative Boltzmann weights.

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1. Introduction

The tetrahedron equation \cite{1, 2} is a three-dimensional (3D) analogue of the Yang–Baxter equation. It implies the commutativity of layer-to-layer transfer matrices \cite{3} for 3D lattice models of statistical mechanics and field theory and, thus, generalizes the most fundamental integrability structure of exactly solvable models in two dimensions \cite{4}.

Historically, the first solution of the tetrahedron equation was proposed by Zamolodchikov \cite{1, 2}. It was subsequently proven by Baxter \cite{5} and further studied in \cite{6–13}. A generalization of this solution to any number of discrete spin states, $N \geq 2$, was found in \cite{14, 15}. Next, a different solution to the tetrahedron equations, with spins having infinitely many discrete states, was originally constructed in \cite{16} and then further generalized in \cite{17} for the case of continuous spin variables. Subsequently, all the above solutions were again rederived from a common point of view based on rather remarkable geometric considerations \cite{18}. It is worth mentioning also that known 3D integrable models helped to reveal some hidden structures of quantum groups, in particular, the ‘rank-size’ duality \cite{14, 16}.
Nevertheless, despite all these fascinating mathematical connections the topic of 3D integrability has never really attracted any notable attention in statistical mechanics, since all solutions of the tetrahedron equations, hitherto obtained, always had negative (and even complex) entries and therefore could not be directly interpreted as Boltzmann weights of physical model of statistical mechanics.

In this paper we break this unremarkable tradition and obtain the first solution of the tetrahedron equation, which has only real non-negative weights. The spin variables in the model are assigned to the edges of a 3D cubic lattice and have an infinite number of discrete states, labelled by non-negative integers. Therefore, every vertex of the lattice can occur in an infinite number of configurations, determined by spin arrangements on the six edges attached to the vertex. Not all these arrangements are allowed, as there are two constraints on the values of the edge spins at the vertex (similar to the arrow conservation law in the two dimensional (2D) ice model [19]). Forbidden arrangements are assigned with vanishing weights, however for all allowed ones the weights are real and positive. The idea of the very existence of such solution was previously pronounced by one of us in [20] on the basis of analytical properties of the Lagrangian function of associated classical integrable discrete systems.

In section 2 we present the new solution of the tetrahedron relation and prove its positivity. Various properties of this solution are discussed in section 3. The vertex weights depend on a single parameter $0 < q < 1$ and three ‘field variables’, similar to those of the 2D six-vertex model. The partition function for periodic boundary condition is defined in section 4. The commuting layer-to-layer transfer matrices are constructed in section 5. The ‘rank-size’ duality is considered in section 5.4.

2. A positive solution to the tetrahedron equation

2.1. Operator maps of the $q$-oscillator algebras and the functional tetrahedron equation

Remarkably, the derivation of the new solution of the tetrahedron equation, which we present here, only requires rather minor modifications to already existing results [16]. Consider the $q$-oscillator algebra

\[ \text{Osc}_q: \quad ka^\pm = q^\pm a^\pm k, \quad qa^+ a^- - q^{-1}a^- a^+ = q - q^{-1}, \quad (1) \]

generated by the three elements $k$, $a^+$ and $a^-$ and impose an additional relation

\[ k^2 = q(1 - a^+ a^-) \equiv q^{-1}(1 - a^- a^+), \quad (2) \]

which is consistent with (1). We will always assume that $0 < q < 1$ and that the element $k$ is invertible\(^4\). Below we will need to use several matrices acting in a tensor product of two 2D vector spaces $\mathbb{C}^2 \otimes \mathbb{C}^2$. Any such matrix can be conveniently represented as a two by two block matrix with 2D blocks where the matrix indices related to the second vector space numerate the blocks while the indices of the first space numerate matrix elements inside the blocks. With these conventions define an operator-valued matrix, acting in $\mathbb{C}^2 \otimes \mathbb{C}^2$,

\[ L(k, a^\pm) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & k & a^+ & 0 \\
0 & 0 & a^- & -k \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (3) \]

whose elements belong to the algebra (1).

\(^4\) The sign of $k$ is fixed by the representations (16), (17) below.
In [16] the problem of solving the tetrahedron equation was reduced to finding matrix representations of a certain operator map of the tensor cube of the algebra (1) to itself,

\[ \mathcal{R}_{123} : \text{Osc}_q \otimes \text{Osc}_q \otimes \text{Osc}_q \rightarrow \text{Osc}_q \otimes \text{Osc}_q \otimes \text{Osc}_q. \]  

Let \( k_i, a_i^\pm, i = 1, 2, 3 \), denote the generators in the first, second and third factors this product, respectively, and

\[ k_i' = \mathcal{R}_{123}(k_i), \quad a_i^{\pm} = \mathcal{R}_{123}(a_i^{\pm}), \quad i = 1, 2, 3, \]  

denote their images under the map (4).

To construct this map introduce three operator-valued matrices \( L_{\alpha, \beta}(k_1, a_1^\pm), L_{\alpha, \gamma}(k_2, a_2^\pm) \) and \( L_{\beta, \gamma}(k_3, a_3^\pm) \) acting in a tensor product of three 2D vector spaces \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \), labelled by \( \alpha, \beta \) and \( \gamma \), respectively. The matrix \( L_{\alpha, \beta}(k_1, a_1^\pm) \) acts non-trivially only in the first two spaces \( \alpha \) and \( \beta \), where it is defined by (3) and coincides with the identity operator in the third vector space \( \gamma \). Its matrix elements belong to the first \( q \)-oscillator algebra in the direct product (4). The matrices \( L_{\alpha, \gamma}(k_2, a_2^\pm) \) and \( L_{\beta, \gamma}(k_3, a_3^\pm) \) are defined in a similar way.

The map (4),(5) is uniquely defined (up to a sign of \( k_2' \)) by the following matrix equation,

\[ L_{\alpha, \beta}(k_1, a_1^\pm) L_{\alpha, \gamma}(k_2, a_2^\pm) L_{\beta, \gamma}(k_3, a_3^\pm) = L_{\beta, \gamma}(k_3', a_3'^\pm) L_{\alpha, \gamma}(k_2', a_2'^\pm) L_{\alpha, \beta}(k_1', a_1'^\pm). \]  

Explicitly, the map \( \mathcal{R}_{123} \) reads

\[ k_i' a_i^{\pm} = k_i a_i^{\pm} + k_i a_i^\pm a_i^\mp, \]

\[ a_i^{\pm} = a_i^\pm a_i^\mp - k_i k_j a_i^\pm, \]

\[ k_i' a_i^\pm = k_i a_i^\pm + k_i a_i^\pm a_i^\mp, \]  

where

\[(k_2')^2 = k_1^2 k_2^2 + k_1 k_3 (q^{-1} a_1^\pm a_2^\mp + qa_1^\pm a_2^\mp) + k_2^2 + k_3^2 - (q^{-1} + q)k_1^2 k_3^2 \]  

while \( k_1' \) and \( k_2' \) are given by the relations

\[ k_1' k_2' = k_1 k_2, \quad k_2' k_3' = k_2 k_3. \]

Consider now the direct product of six \( q \)-oscillator algebras,

\[ \mathcal{A} = \text{Osc}_q \otimes \text{Osc}_q \otimes \cdots \otimes \text{Osc}_q, \]  

labelled consequently by \( i = 1, 2, \ldots, 6 \) and introduce an abbreviated notation

\[ L_{\alpha, \beta}^{(i)} = L_{\alpha, \beta}(k_i, a_i^\pm) \quad i = 1, 2, \ldots, 6. \]

Then, using (5) one can rewrite (6) in the form

\[ L_{\alpha, \beta}^{(1)} L_{\alpha, \gamma}^{(2)} L_{\beta, \gamma}^{(3)} = \mathcal{R}_{123}(L_{\alpha, \beta}^{(4)} L_{\alpha, \gamma}^{(5)} L_{\beta, \gamma}^{(6)}), \]  

which shows that the application of the map \( \mathcal{R}_{123} \) is equivalent to reversing the order of the three \( L \)-operators. It is not difficult to see, that using (10) four times, one can reverse the order of the following six-fold product

\[ L_{\alpha, \beta}^{(1)} L_{\alpha, \gamma}^{(2)} L_{\beta, \gamma}^{(3)} L_{\alpha, \delta}^{(4)} L_{\beta, \delta}^{(5)} L_{\alpha, \gamma}^{(6)} = T(L_{\gamma, \alpha}^{(6)} L_{\beta, \alpha}^{(5)} L_{\gamma, \beta}^{(4)} L_{\alpha, \gamma}^{(3)} L_{\alpha, \gamma}^{(2)} L_{\alpha, \beta}^{(1)}), \]  

where the matrices \( L_{\alpha, \beta}^{(1)}, L_{\alpha, \gamma}^{(2)}, \) etc, act in the tensor product of four vector spaces \( \mathbb{C}^2 \), labelled \( \alpha, \beta, \gamma \) and \( \delta \), while their matrix elements belong to the \( q \)-oscillator algebras (8). Remarkably, the required map \( T \) can be decomposed into elementary moves (10) in two different ways,

\[ T = \mathcal{R}_{123} \circ \mathcal{R}_{145} \circ \mathcal{R}_{246} \circ \mathcal{R}_{356} \]  

5 This equation is sometimes called the local Yang–Baxter equation [21]. Note, that it is not equivalent to the Yang–Baxter equation. Even though it has the same matrix structure, the \( L \)-matrices in the LHS and RHS of (6) are different. Moreover, they have operator-valued (rather than number-valued) matrix elements.
and

$$T = R_{356} \circ R_{246} \circ R_{145} \circ R_{123}. \quad (13)$$

Taking into account that the matrix elements of the product in the LHS of (11) span the full basis in (8) one obtains the functional tetrahedron equation

$$R_{123} \circ R_{145} \circ R_{246} \circ R_{356} = R_{356} \circ R_{246} \circ R_{145} \circ R_{123}. \quad (14)$$

Note, that this equation can be verified by direct calculations of compositions of the maps in both sides, using the explicit expressions (7). For further details of the derivation we refer the reader to the original publication [16].

2.2. Recurrence relations and positivity

Consider an infinite-dimensional oscillator Fock space, spanned by the set of vectors $|n\rangle$, $n = 0, 1, 2, \ldots, \infty$, with the natural scalar product

$$\langle m|n \rangle = \delta_{m,n}, \quad N|n \rangle = n|n \rangle, \quad \langle n|N = \langle n|n, \quad (15)$$

where we have introduced the ‘occupation number’ operator $N$. The algebra (1) has two irreducible highest weight representations acting the this space, which we denote $\mathcal{F}_q^\pm$. The representation $\mathcal{F}_q^+$ is defined as

$$k = q^{N+1/2}, \quad (16a)$$

and

$$a^-|0 \rangle = 0, \quad a^+|n \rangle = (1 - q^{2+2n})|n + 1 \rangle, \quad a^-|n \rangle = |n - 1 \rangle, \quad \langle 0|a^+ = 0, \quad \langle n|a^+ = (n - 1)(1 - q^{2n}), \quad \langle n|a^- = \langle n + 1|, \quad (16b)$$

with $n = 0, 1, 2, \ldots$. Similarly, the representation $\mathcal{F}_q^-$ is defined as

$$k = q^{-N-1/2}, \quad (17a)$$

and

$$a^+|0 \rangle = 0, \quad a^-|n \rangle = |n + 1 \rangle, \quad a^+|n \rangle = (1 - q^{-2n})|n - 1 \rangle, \quad \langle 0|a^- = 0, \quad \langle n|a^- = (n - 1)(1 - q^{-2n}), \quad \langle n|a^+ = \langle n + 1|, \quad (17b)$$

with $n = 0, 1, 2, \ldots$. Following [16] we realise the map (5) as an internal automorphism

$$R_{123}(x) = R_{123} \circ R_{123}^{-1}, \quad R_{123, x} \in \text{Osc}_q \otimes \text{Osc}_q \otimes \text{Osc}_q, \quad (18)$$

of the direct product of the three oscillator algebras. Obviously, there are eight possible ways, $\mathcal{F}_q^\sigma_1 \otimes \mathcal{F}_q^\sigma_2 \otimes \mathcal{F}_q^\sigma_3$, with $\sigma_1, \sigma_2, \sigma_3 = \pm$, to choose a Fock representation in this product. Once the representation is chosen the matrix elements of the operator $R$ can be calculated using the explicit form of the map (7). Note, that this procedure uniquely defines the matrix elements of $R$ (to within an overall normalization), since the representations $\mathcal{F}_q^\pm$ are irreducible.

The problem of finding $R$ for the case when all three representations coincide with $\mathcal{F}_q^\pm$ has already been solved in [16, 17]. In this paper we consider another symmetric case, when all three representations coincide with $\mathcal{F}_q^+$ and demonstrate some rather remarkable positivity properties of the resulting operator $R$.

First, using (7), let us derive recurrence relations for the matrix elements

$$R_{n_i, n_i'}|n_1, n_2, n_3 \rangle = \langle n_1, n_2, n_3|R|n_1', n_2', n_3' \rangle, \quad n_i, n_i' = 0, 1, 2, \ldots, \infty, \quad i = 1, 2, 3. \quad (19)$$

of the operator $R$, where $|n_1, n_2, n_3 \rangle = |n_1 \rangle \otimes |n_2 \rangle \otimes |n_3 \rangle$ denotes states in $\mathcal{F}_q^- \otimes \mathcal{F}_q^- \otimes \mathcal{F}_q^-$. Equation (7c) imply that the indices $n_i$ and $n_i'$ obey two constraints

$$n_1 + n_2 = n_1' + n_2', \quad n_2 + n_3 = n_2' + n_3'. \quad (20)$$
for all non-zero matrix elements in (19). Therefore all these elements only depend on four
independent discrete variables, for which we choose $n_2, q^{-2n_1'}, q^{-2n_2'}$ and $q^{-2n_3'}$. It follows then,
that the matrix (19) can be represented in the form
\[
R_{n_1', n_2', n_3'}^{n_1, n_2, n_3} = \delta_{n_1+n_2, n_1'+n_2'} \delta_{n_2+n_3, n_2'+n_3'} q^{(n_2+1)-(n_2'-1)(n_2-n_2') \over (q^2; q^2)_n} Q_{n_2'}(q^{-2n_1'}, q^{-2n_2'}, q^{-2n_3'}),
\]
(21)
where $n_i, n_i' = 0, 1, 2, 3, \ldots$ and we have introduced a set of (yet unknown) functions $Q_{n_2'}(x, y, z)$ depending on the three variables $x = q^{-2n_1'}, y = q^{-2n_2'}$ and $z = q^{-2n_3'}$. The specific $q$-dependent factor in (21), involving the Pochhammer symbol
\[
(x; q)_n = \prod_{k=0}^{n-1}(1 - xq^k),
\]
has been chosen to ensure that the functions $Q_{n_2'}(x, y, z)$ are polynomials in $x, y, z$ with coefficients which are themselves polynomials in the variable $q$ (this immediately follows from (26), see below). Next, substituting the formula (21) into (18), (5) and (7), specialized for the representation (17), one can derive a set of recurrence relation for $Q_{n_2'}(x, y, z)$. First, consider the simplest case $n_2 = 0$. Taking the ‘+’ signs (lower signs) in (7a) and calculating matrix elements of both sides of these equations sandwiched between the states $\langle n_1, 0, 0 \rangle$ and $\langle n_1', 0, 0 \rangle$, one obtains a set of three simple relations
\[
Q_0(xq^{-2}, y, z) = Q_0(x, yq^{-2}, z) = Q_0(x, y, zq^{-2}) = Q_0(x, y, z).
\]
(23)
Thus, for the normalization
\[
R_{0,0,0}^{0,0,0} = 1,
\]
(24)
one can set
\[
Q_0(x, y, z) \equiv 1, \quad \forall x, y, z = 1, q^{-2}, q^{-4}, q^{-6} \ldots .
\]
(25)
More generally, equation (5) implies the following recurrence relation,
\[
Q_{n+1}(x, y, z) = (x - 1)(y - 1)Q_n(x, y, z) + xz(1 - 1)q^{2n}Q_n(x, y, z).
\]
(26)
In particular, the next two polynomials read
\[
Q_1(x, y, z) = 1 - (x + z) + xyz,
\]
\[
Q_2(x, y, z) = (1 - x)(1 - yq^2)(1 - zq^2) - x^2q^4(1 - y^2) - zq^2(1 + q^2)(1 - y)(1 - z).
\]
(27)
Actually, it is not too difficult to solve (26) with the initial condition (25) and derive an explicit formula valid for all values of $n$,
\[
Q_n(x, y, z) = (x; q^2)_n 2\phi_1 \left( \begin{array}{c} q^{-2n} \\ q^{-2n} \\ q^{-2n} \end{array} \right)_n \frac{q^{2-2n} x^2}{xy}, \frac{q^{2-2n} x}{x}, q^{2}, q^{2}, q^{2n}.
\]
(28)
where $2\phi_1$ is the truncated generalized hypergeometric function, defined as
\[
2\phi_1(p^{-n}, b; c; p, z) \overset{\text{def}}{=} \sum_{k=0}^{n} \frac{(p^{-n}; p)_k b(c; p)_k}{(p; p)_k} z^k, \quad n \geq 0.
\]
(29)
Let us now formulate our main statement.

**Theorem.** For any non-negative integers $n, n_1', n_2', n_3' \geq 0$, and any real $q$ in the interval $0 < q < 1$, the special values of the polynomials $Q_n$
\[
Q_n(q^{-2n_1'}, q^{-2n_2'}, q^{-2n_3'}) \geq 0, \quad \forall n, n_1', n_2', n_3' \in \mathbb{Z}_{\geq 0}
\]
are always non-negative.
Proof. First, notice that for \( x, y, z \in \{1, q^{-2}, q^{-4}, q^{-6}, \ldots \} \) the coefficients in front of \( Q_4(xq^2, yq^2, zq^2) \) and \( Q_6(x, yq^2, zq^2) \) in (26) are non-negative. Then, a proof by induction simply follows from (26) and the initial condition (25).

Taking this result into account, one immediately concludes that all matrix elements of the \( R \)-matrix given (21) and (28) are non-negative provided \( 0 < q < 1 \).

3. Properties of the \( R \)-matrix

3.1. Matrix elements

Some care should be taken when calculating the \( R \)-matrix, defined by (21) and (28), for \( n_2 > n'_2 \). In this case the third argument of the hypergeometric function is equal to a non-positive power of \( q \), where the function \( \phi_1 \) will have a pole. However, this pole is exactly cancelled by a zero coming from the pre-factor in the RHS of (28) and the result will always be finite (see, e.g., the first two polynomials (27). In fact, it is easy to rewrite the formula (21) in a form which does not have any poles

\[
R'_{n_1, n_2, n_3} = \delta_{n_1+n_2, n'_1} \delta_{n_2+n_3, n'_2} q^{6(n_2+n_3)} \delta_{n'_3, n_1-n_2-n_3} \sum_{r=0}^{n_2} \frac{(q^{-2n'1}; q^2)_r (q^{-2n'2}; q^2)_r}{(q^2; q^2)_r^2} q^{-2r(n_3+n'_3+1)}.
\]

A few first matrix elements read,

\[
R'_{000} = 1, \quad R'_{010} = q^{-1}, \quad R'_{100} = q^{-2}, \quad R'_{110} = q^{-2} - 1, \quad R'_{121} = q^{-7} + q^{-5} - q^{-1},
\]

\[
R'_{021} = (q^{-6} - 1)(q^{-6} - 1), \quad R'_{231} = q^{-14} + (q^{-6} + q^{-4})(q^{-6} - 1).
\]

3.2. Symmetry properties

Introduce the following constant matrices acting in the direct product of the three Fock spaces,

\[
P_{13}|n_1, n_2, n_3) = |n_3, n_2, n_1), \quad S_3|n_1, n_2, n_3) = q^{-n'_3} (q^2; q^2)_n)|n_1, n_2, n_3).
\]

The 12-element symmetry group of the \( R \)-matrix (31) is generated by two transformations

\[
P_{13} R_{123} P_{13} = R_{123}
\]

and

\[
P_{12} (R_{123})^T P_{12} = q^{N_2-N_3} S_3 R_{123} S_3^{-1},
\]

where the superscript \( f_3 \) denotes the matrix transposition in the third space.

3.3. Tetrahedron equation

The \( R \)-matrix (31) satisfies the tetrahedron equation

\[
R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123},
\]

which is corollary of (14) and (18). It involves operators acting in six Fock spaces, where \( R_{ijk} \) acts non-trivially in the \( i \)-th, \( j \)-th and \( k \)-th spaces, but acts as the identity in the other three spaces. In matrix form equation (36) reads

\[
\sum_{j, k} R'_{i, j, k} R''_{j, k, l} R'''_{k, l, m} R''''_{l, m, n} = \sum_{j, k} R''''_{l, m, n} R'''_{k, l, m} R''_{j, k, l} R'_{i, j, k},
\]

(37)
This equation contains summations over the six ‘internal’ indices $n_i$, running over all non-negative integer values. However, due to the presence of two $\delta$-functions in (31) there are only four independent summations in both sides of (37). Moreover, for any fixed values of the ‘external’ indices $n_i$ and $n'_i$ all the summation variables in (37) are bounded from above and below. So, there are no convergence problems in (37), since all sums there are finite.

As the reader might have noticed, that equation (31) defines a constant solution of the tetrahedron equation (36), as all four $R$-matrices therein are exactly the same. Below we will introduce additional continuous parameters into (31), which will play the role of the spectral parameters similar to those in 2D solvable models.

Let $\lambda_i, \mu_i, i = 1, 2, \ldots, 6$, be positive real numbers. Using the conservation laws (20), it is easy to check that if $R_{ijk}$ satisfies (36), then so do the ‘dressed’ $R$-matrices

$$R'_{ijk} = \left( \frac{\mu_i}{\mu_j} \right)^{N_i} R_{ijk} \left( \frac{\lambda_i}{\lambda_j} \right)^{N_j} \left( \frac{\mu_i}{\mu_j} \right)^{N_k},$$

where the indices $(i, j, k)$ take four sets of values appearing in (36). Note that the twelve parameters $\lambda_i, \mu_i$ enter the four equations (38) only via eight independent ratios, so these equations define a solution of (36) containing eight continuous parameters. Even though that at the first sight these new degrees of freedom appear to be trivial, they allow to define a very non-trivial family of commuting layer-to-layer transfer matrices (see section 5).

In addition to (38) the tetrahedron equation is, obviously, invariant under diagonal similarity transformations

$$R'_{ijk} = \left( c_i \right)^{N_i} R_{ijk} \left( c_j \right)^{N_j} \left( c_k \right)^{N_k},$$

where $c_1, c_2, \ldots, c_6$ are arbitrary positive constants. However, these transformations will not play any role in the following, since they are nonessential for periodic boundary conditions.

3.4. Asymptotic behaviour

In preparation for considerations of the layer-to-layer transfer matrices with periodic boundary conditions we need to study an asymptotic behaviour of the matrix elements of the $R$-matrix (31) for large values of the indices $n_i, n'_i$. Indeed these indices run over an infinite number of non-negative integers values, so the convergence of sums involving these matrix elements will need to be investigated.

Consider the recurrence relation (26) in the limit of large positive values of $n$ and $x, y, z$. Simple estimates shows that the second term in the RHS (26) will be dominant in this limit. Then in the leading order one gets

$$Q_{n+1}(x, y, z) \simeq x y z q^{2n} Q_n(x, y q^2, z).$$

Solving this equation, estimating corrections from the sub-leading terms in (26) and substituting the result into (21), one obtains

$$R_{n_1, n_2, n_3, n_4}^{n'_1, n'_2, n'_3} = \delta_{n_1+n_2+n_3+n_4} \delta_{n_2+n_3+n_4} \delta_{n_1+n_2+n_3+n_4} q^{-n_1 n_2 - n_1 n_3 - n_2 n_3 + O(n)},$$

where $n_1, n_2, n_3, n'_1, n'_2, n'_3 \to \infty$ and we have assumed that these variables are of the same order of magnitude. Note, that the leading term in the asymptotics (41) exactly coincides with the most singular term of the sum (31) in the limit $q \to 0$ (it comes from $r = n_2$ term of the sum).

4. Partition function

In this section we define a solvable 3D model of statistical mechanics with non-negative Boltzmann weights. Consider the cubic lattice of size $L \times M \times N$, with height $L$, width $M$
Figure 1. An arrangement of the edge-spin states around the vertex with the coordinates $(l, m, n)$. The corresponding Boltzmann weight is given by an element of the $R$-matrix.

In this paper we consider the case of the periodic boundary conditions in all three directions $i_1, m, n = i_L, m, n$, $jl, m, n = jl, M+1, n$, $kl, m, 1 = kl, m, N+1$, (42)

First, note that the $\delta$-functions in (31) lead to the following local conservation laws for each vertex

\[ i_{l,m,n} + j_{l,m,n} = i_{l+1,m,n} + j_{l+1,m,n}, \quad jl, m,n + k_{l,m,n} = jl, m+1,n + k_{l,m,n+1}, \quad \forall l, m, n \] (43)

in any allowed spin arrangement on the whole lattice. As an immediate consequence there will be many ‘global’ conservation laws for various sums of spins on 1D chains and 2D layers of edges of the same type. For example, the set of all horizontal coordinate planes divides the whole lattice into $L$ layers. Each of these layers will contain $M \times N$ vertical edges with the spins $\{i_{l,m,n}\}, m = 1, \ldots, M, n = 1, \ldots, N$. Then for any allowed spin arrangement on the whole lattice the sum of these $i$-type spins

\[ \mathcal{I} = |\{i\}| = \sum_{n=1}^{N} \sum_{m=1}^{M} i_{l,m,n}, \] (44)

and depth $N$. Here we assume that the lattice axes are oriented along the vertical and two horizontal directions, ‘left to right’ and ‘front to back’. The vertices of the lattice are labelled by the coordinates $(l, m, n)$, where $l = 1, \ldots, L, m = 1, \ldots, M, n = 1, \ldots, N$. The edges of the lattice carry fluctuating spin variables taking arbitrary non-negative integer values. In the previous sections these spin variables (oscillator occupation numbers in the Fock spaces) were denoted $n_1, n_2, n_3, \ldots$. Here it will be more convenient to use the symbols $i, j, k$, indexed by the coordinates of the adjacent vertex, as shown in figure 1. The spins $i_{l,m,n}$ are associated with the vertical edges ($i$-type spins), the spins $j_{l,m,n}$ with the horizontal left-to-right edges ($j$-type spins) and the spins $k_{l,m,n}$ with the horizontal front-to-back edges ($k$-type spins). Each vertex configuration is assigned with a Boltzmann weight given by an element of the $R$-matrix (31), also shown in figure 1. Actually, we will use the ‘dressed’ solution (38) of the tetrahedron equation, which includes additional edge weights. However, due to multiple conservation laws for the spin variables (see below) these edge weights can be equivalently redistributed among different edges of the lattice and it is not necessary to have them for all the edges. For our purposes it will be convenient to keep the edge weights only on the boundary edges$^6$.

$^6$ The situation is similar to the 2D six-vertex model, where one can move vertical and horizontal fields to a single column and a single row of the lattice.

$^8$
will be the same for all horizontal layers, i.e., it will not depend on the vertical coordinate \( l \). Similarly, define another two sums

\[
J = \sum_{n=1}^{N} \sum_{l=1}^{L} j_{l,m,n}, \quad K = \sum_{m=1}^{M} \sum_{l=1}^{L} k_{l,m,n}
\]  

(45)

for the \( j \)-type and \( k \)-type spins. In addition, we will also use the following 1D sums of spins

\[
T^{(M)}_n = \sum_{m=1}^{M} i_{l,m,n}, \quad J^{(L)}_n = \sum_{l=1}^{L} j_{l,m,n}, \quad K^{(L)}_m = \sum_{l=1}^{L} k_{l,m,n}.
\]

(46)

Note that due (43) the above sums depend only on one coordinate, instead of two. For instance, the first of these sum \( T^{(M)}_n \) does not depend on \( l \). This means that the sum of spins on a row of vertical edges, obtained from each other by translations in the front-to-back direction, does not depend on the height of this row in the lattice. Equipped with these definitions, introduce a function of spins

\[
U = \sum_{l=1}^{L} J^{(N)}_l K^{(M)}_l + \sum_{m=1}^{M} T^{(N)}_m K^{(L)}_m + \sum_{n=1}^{N} T^{(M)}_n J^{(L)}_n,
\]

(47)

which is expressed only in terms of the above spin sums.

Recall that the spins in the model run over an infinite number of values (all non-negative integers), therefore, there could be potential convergence problems for the partition function with the periodic boundary conditions (42). To better understand the situation let us estimate the leading asymptotics of the product of the vertex weights over all lattice vertices

\[
\mathcal{P} = \prod_{l,m,n} P_{i_{l,m,n}}{, \ j_{l,m,n}}{, \ k_{l,m,n}},
\]

(48)

for a generic spin configuration, when all the spins are large

\[
i_{l,m,n} \sim j_{l,m,n} \sim k_{l,m,n} \sim O(\Lambda), \quad \Lambda \to \infty,
\]

(49)

but kept of the same order of magnitude, so that their ratios remain finite. Using the asymptotics (41), the periodic boundary condition (42) and the local conservation laws (43), one can show that

\[
\log \mathcal{P} / \log q \sim -U + S(|d|) + O(\Lambda),
\]

(50)

where \( U \) is defined in (47) and the second term

\[
S(|d|) = \sum_{l,m,n} \left\{ \sum_{s=1}^{m-1} \left( \sum_{r=1}^{l-1} d_{r,m,n} + \sum_{t=1}^{n-1} d_{l,m,t} \right) - \sum_{r=1}^{l} d_{r,m,n} + \sum_{t=1}^{n} d_{l,m,t} \right\},
\]

(51)

depends only on a set of differences of the spins

\[
d_{l,m,n} = i_{l+1,m,n} - i_{l,m,n} = j_{l,m,n} - j_{l,m+1,n} = k_{l,m,n+1} - k_{l,m,n}.
\]

(52)

Remarkably, thanks to (43), there are three alternative expressions for the above differences, so that they can be solely associated with either \( i \)-type, \( j \)-type or \( k \)-type spins. Also, it is worth noting that the quantity (51) can be written in the form

\[
S(|d|) = 2 \sum_{C} d_{l_{1,m_1,n_1}} d_{l_{2,m_2,n_2}},
\]

(53)
where sum is taken over a set coordinates satisfying the conditions
\[ \mathcal{C} : 1 \leq l_1 \leq l_2 \leq (L - 1), \quad 1 \leq m_2 \leq m_1 \leq (M - 1), \quad 1 \leq n_1 \leq n_2 \leq (N - 1). \]  

Now, we are ready to define the partition function of the model. First, define a restricted partition function,
\[ Z_I = \sum_{|I| = I} \sum_{(j,k)} q^{ij} u^j w^k \prod_{l,m,n} (q^{b_{l,m,n} k^{l,m,n}}, j_{l,m,n}, k_{l,m,n}), \quad I = 0, 1, 2, \ldots \]  

Note, that the quantity \( U \) is positive, it is quadratic in spins and diverge like \( U \sim O(\Lambda^2) \). The second term \( S(|d|) \), is also quadratic in spins, but for a fixed value of \( I \) it remains finite, when \( \Lambda \rightarrow \infty \). Indeed, according to (51) and (52) this term can be expressed only in terms of differences of the \( i \)-type spin. However, since the total sum of these spins is fixed to \( I \), one concludes that \( S(|d|) \sim O(I^2) \), independent of \( \Lambda \). Thus, if the parameter \( \mu > 1 \) the summand in the formula (55) vanishes exponentially for large spins (remind, that \( q < 1 \)) and the sum over \( j \)- and \( k \)-type spins therein will converge. Next, for \( \mu = 1 \) there might be growing terms in the exponent of (56), which are linear in spin. However, such terms are not dangerous, since they can be damped by choosing sufficiently small parameters \( v \) and \( w \) in (55). More detailed estimates suggest that (55) converges for
\[ \mu = 1, \quad v < 1, \quad w < 1. \]  

In the next sections we will show that the partition function (55) corresponds to an integrable 3D model, in the sense that the corresponding layer-to-layer transfer matrices form a two-parameter commutative family. The full partition function is defined
\[ Z = \sum_{I=0}^{\infty} u^I Z_I, \quad u < 1 \]  

where the parameter \( u \) is related to the vertical edge weights. The convergence of the sum (58) requires an additional study (it could require an additional \( I \)-dependent dumping factor).

5. Commuting family of layer-to-layer transfer matrices

5.1. Definition of the transfer matrix

The purpose of this section is to define a commuting family of layer-to-layer transfer matrices, associated with the partition function (55). Consider a particular horizontal layer of the lattice shown in figure 2, corresponding to some fixed value of the height \( l \) and assume periodic boundary conditions in both horizontal directions.

Redenote the spins, associated with this layer, by dropping the coordinate \( l \) from the indices
\[ i_{l,m,n} \rightarrow i_{m,n}, \quad i_{l+1,m,n} \rightarrow i_{m,n}, \quad j_{l,m,n} \rightarrow j_{m,n}, \quad k_{l,m,n} \rightarrow k_{m,n}. \]
The layer-to-layer transfer matrix is defined as
\[ T_{ij}^{\mu}(v,w) = \sum_{jk} q_{\mu}^{JK} v^J w^K \left( \prod_m q_{\mu I}^{jm} j_{m1} \right) \left( \prod_n q_{\mu I}^{jn} j_{n1} \right) \left( \prod_{mn} q_{\mu I}^{jm} j_{m1} \right), \]
where \( I^{(N)} \) and \( I^{(M)} \) are defined in (46) and
\[ J = \sum_{n=1}^N j_n, \quad K = \sum_{m=1}^M k_m. \]

Figure 2. The layer-to-layer transfer matrix.

Note that for the periodic boundary conditions in horizontal directions \( J \) is independent of \( m \), \( K \) is independent of \( n \), while \( I^{(N)} \) and \( I^{(M)} \) are the same for all horizontal layers. Let \( \langle \Psi_I | \) and \( | \Psi_I \rangle \) be vectors, describing the superposition (with coefficient one) of all \( i \)-type spin states, obeying the total sum constraint \( |I| = I \). Then the partition function (55) can be written as
\[ Z_I = \langle \Psi_I | T(v,w) | \Psi_I \rangle. \]

Below we will show that the transfer matrices (60) form a two-parameter commutative family,
\[ [T(v,w), T(v',w')] = 0, \quad \forall v, w, v', w'. \]

5.2. Composite Yang–Baxter equation

It is well known that any edge-spin model on the cubic lattice can be viewed as a 2D model on the square lattice with an enlarged space of states for the edge spins (see [16] for additional explanations). Consider a line of vertices in the front-to-back direction and let
\[ i = \{i_1, i_2, \ldots, i_N\}, \quad i' = \{i'_1, i'_2, \ldots, i'_N\}, \quad \text{etc} \]
denote multi-spin variables, describing the states of external edges of similar types, as shown in figure 3. Also, let \( k_1, k_2, \ldots, k_N \) denote states of the internal edges along the line in the front-to-back direction, where we assume the periodic boundary conditions \( k_{N+1} = k_1 \). Also, it is useful to introduce the variables
\[ I = \sum_{n=1}^N i_n, \quad I' = \sum_{n=1}^N i'_n, \quad J = \sum_{n=1}^N j_n, \quad J' = \sum_{n=1}^N j'_n. \]
Figure 3. A front-to-back line of the cubic lattice.

Define a composite weight

$$ S_{i,j}^{k,l}(w) = \sum_{\{k\}} w^{k_1} \prod_{n=1}^{N} R_{i_n,j_n,k_{n+1}}^{i_n,j_n,k_n} \quad (66) $$

where \( w \) is an arbitrary (positive) parameter. The presence of the delta functions in (31) leads to two 'global' conservation laws for the multi-spin variables

$$ I = I', \quad J = J', \quad (67) $$

and also determines a local structure of non-zero matrix elements of \( S \),

$$ S_{i,j}^{k,l}(w) \sim \text{const} \prod_{n=1}^{N} \delta_{i_n+j_n, k_n+l_n}. \quad (68) $$

Standard arguments [3] relating the tetrahedron and Yang–Baxter equations allows one to conclude that the composite \( R \)-matrix (66) satisfies the Yang–Baxter equation

$$ \sum_{\{i,j,k\}} S_{i,j}^{k,l}(w) S_{i,k}^{j,l}(w') S_{j,l}^{i,k}(w) = \sum_{\{i,j,k\}} S_{j,k}^{i,l}(w') S_{k,l}^{j,i}(w) S_{l,i}^{k,j}(w). \quad (69) $$

This equation states an equality of two linear operators, acting in a direct product of three identical infinite-dimensional vector spaces \( \mathcal{F}_q^{-N} \otimes \mathcal{F}_q^{-N} \otimes \mathcal{F}_q^{-N} \), spanned on the vectors \( |i_1, \ldots, i_N\rangle \otimes |j_1, \ldots, j_N\rangle \otimes |\bar{i}_1, \ldots, \bar{j}_N\rangle \), \( i_n, j_n, \bar{j}_n = 0, 1, 2, \ldots, \infty \), \( n = 1, 2, \ldots, N \). \quad (70)

The relations (67) imply a conservation of sums of spins in each of the three spaces

$$ I = I' = I'', \quad J = J' = J'', \quad \bar{J} = \bar{J}' = \bar{J}''. \quad (71) $$

Therefore, equation (69) reduces to a direct sum of an infinite number of Yang–Baxter equations, corresponding to particular values of \( I, J, \bar{J} = 0, 1, 2, \ldots, \infty \). Further, with (68) it is easy to see that equation (69) is not affected by the replacement

$$ S_{i,j}^{k,l}(w) \rightarrow (q^{\mu} v/v_0)^{j_n,i_n} S_{i,j}^{k,l}(w), \quad S_{i,k}^{j,l}(w') \rightarrow (q^{\mu} v/v_0)^{j_n,i_n} S_{i,k}^{j,l}(w'). \quad (72) $$

usually referred to as an introduction of 'horizontal fields'.

From the results of [16] it is clear that the composite \( R \)-matrix (66) should be closely related to the \( R \)-matrices associated with the affine quantum algebra \( U_q(\hat{sl}(N)) \). These models
were discovered in the early 1980s [22–25] and have since found numerous applications in integrable systems. They are related to an anisotropic deformation of the \( \mathfrak{sl}(n) \)-invariant Heisenberg magnets [26–29]. In the simplest \( N = 2 \) case, these models include the most general six-vertex model [30] and all its higher-spin descendants. Indeed, following the arguments of [16] one can identify the subspace

\[
\pi_I = \left\{ i_1, i_2, \ldots, i_N : \sum_{n=1}^{N} i_n = I \right\}
\]  

(73)

with the rank \( I \) symmetric tensor representation of \( U_q(\widehat{\mathfrak{sl}}(N)) \). More detailed analysis shows that the composite \( R \)-matrix (66) can be viewed as an infinite direct sum:

\[
\mathcal{S}(w) = \bigoplus_{I;J=0}^{\infty} \mathbf{R}_{I;J}^{(I;J)}(w)
\]  

(74)

of the \( U_q(\widehat{\mathfrak{sl}}(N)) \) \( R \)-matrices, \( \mathbf{R}_{I;J}^{(I;J)} \), intertwining the symmetric tensor representations \( \pi_I \) and \( \pi_J \). It is worth noting that in this setting these matrices have some specific normalization uniquely determined by the definition (66) and the solution of the tetrahedron equation (31).

As an illustration consider the case \( N = 2 \). Then using (31), (66) and (74) one obtains7 [31]

\[
\left[ \mathbf{R}_{I,J}^{(I;J)}(\lambda) \right]_{i_1,j_1}^{i_2,j_2} = \delta_{i_1+j_2,i_1'+j_1'} q^{2(I-i_1)(J-j_1) - I-2J} q^{2(i_1'-j_1') - 2J} \frac{(q^2;q^2)_{I-i_1} \lambda^{I-i_1}}{(q^2;q^2)_{I-i_1}}
\]

\[
\times \left( \lambda^2 q^{J-J'; I-q} \sum_{m,l=0}^{I-I} \frac{(-1)^k q^{2k(i_1'-j_1') - 2J(K-J-J')}}{(q^2;q^2)_{I-i_1}} \right)
\]

\[
\times \left( q^{2J-J'; q} \sum_{m,l=0}^{J-J'} \frac{(-1)^k q^{2k(i_1'-j_1') - 2J(K-J-J')}}{(q^2;q^2)_{I-i_1}} \right)
\]

(75)

where \( m(i, j) = \min(i, j) \) and

\[
w = \lambda^2, \quad 0 \leq i_1, j_1, 0 \leq i_1, j_1 \leq J.
\]  

(76)

This is a general expression for the \('higher-spin'\) \( R \)-matrix of the six-vertex model, with \((I+1)\)- and \((J+1)\)-state spins on the vertical and horizontal edges, respectively. Note, in particular, that for \( I = J = 1 \) the formula (75) reduces to the \( R \)-matrix of the six-vertex model [32]

\[
\left[ \mathbf{R}_{1,1}^{(I;J)}(\lambda) \right]_{10}^{01} = \mathbf{R}_{1,1}^{(1;1)}(\lambda) = q\lambda - (q\lambda)^{-1},
\]

\[
\left[ \mathbf{R}_{1,1}^{(I;J)}(\lambda) \right]_{10}^{10} = \mathbf{R}_{1,1}^{(1;1)}(\lambda) = \lambda^{-1},
\]

\[
\left[ \mathbf{R}_{1,1}^{(I;J)}(\lambda) \right]_{10}^{00} = \mathbf{R}_{1,1}^{(1;1)}(\lambda) = q - q^{-1}.
\]  

(77)

The derivation of (75) and its connections to other solutions of the Yang–Baxter equation, related with the six-vertex model are given in [31].

### 5.3. Inhomogeneous case and commutativity

Let us again refer to figure 2 and introduce multi-spin variables

\[
i_m = \{i_{m,1}, i_{m,2}, \ldots, i_{m,N} \}, \quad j_m = \{j_{m,1}, j_{m,2}, \ldots, j_{m,N} \}, \quad \text{etc}
\]  

(78)

7 In writing (75) we have changed the overall normalization factor.
describing states of spins on lines of similar edges, extended in the front-to-back direction. Introduce also two set of positive real numbers, \( \{v\} = \{v_1, v_2, \ldots, v_n\} \) and \( \{w\} = \{w_1, w_2, \ldots, w_m\} \), such that

\[
v_1 v_2 \cdots v_n = 1, \quad w_1 w_2 \cdots w_m = 1.
\]

These numbers will parameterize inhomogeneities of the model. Consider the transfer matrix

\[
T^{(v)}_{[w]} (v, w|\{v\}, \{w\}) = \sum_{(j)} q^M \left( \prod_n q^{I_{(j)}(v/v_n)^{I_{(j)}}} \right) \prod_{m=1}^M q^{S_{m,0} J_{m+1} (q^{\mu_{(m)} v_{m+1}^{\mu_{(m)}}}) w/w_m}
\]

where \( I^{(N)} \) and \( J^{(M)} \) are defined in (46), \( J \) is defined in (61) and the constants \( \mu_1, \mu_2, \mu_3 > 1 \) are real. It is not difficult to see that (80) reduces to (60) if

\[
v_1 = v_2 = \ldots = v_n = 1, \quad w_1 = w_2 = \ldots = w_m = 1, \quad \mu_1 = \mu_2 = \mu_3 = \mu.
\]

The parameters in (80) have the following interpretation from the point of view of a 2D lattice model with the composite weights (66). The parameter \( w \) is the spectral parameter, associated with the horizontal direction. The constants \( w_m \) provide a set of spectral parameters, associated with the vertical direction (usually called inhomogeneities of the spectral parameter). According to the decomposition (74) the transfer matrix is infinite sum of various ‘fusion’ transfer matrices with different representations in the auxiliary space. The parameter \( v \) can be viewed as a ‘fugacity’ weighing different symmetric tensor representations. Finally, from the 2D point of view the constants \( v_n \) manifest themselves as ‘horizontal fields’.

The transfer matrices (80) commute, provided they have the same values of \( \mu_1, \mu_2, \mu_3, \{v_1, \ldots, v_n\} \) and \( \{w_1, \ldots, w_n\} \). The proof, of course, follows from (69), but requires some additional explanations. There are two places where (80) differs from the standard expression of the transfer matrix. First, the spectral parameter \( w \) is multiplied by different powers of the variable \( q \), depending on the certain sums of \( i \)- and \( j \)-type spins. However, as explained before, these sums are conserved quantities. Therefore, this modification merely affects the relative value of the spectral parameter in different diagonal blocks of the transfer matrix and, obviously, cannot affect the commutativity. Further, the factor in the parenthesis in (80), involving the product over \( n \), essentially reduces to the transformation (72), since \( T^{(M)} \) is a conserved quantity for the periodic boundary condition in the left-to-right direction.

5.4. Rank-size duality

As mentioned above the composite weight (74) decomposes into an infinite direct sum of the \( R \)-matrices, corresponding to symmetric tensor representations of the quantized affine algebra \( U_q (\hat{sl}_N) \). Therefore the transfer matrix (80) is also an infinite sum of transfer matrices, related to \( U_q (\hat{sl}_N) \). First, there is a sum over all symmetric tensor representations \( \pi_J, J = 0, 1, 2, \ldots, \infty \), corresponding to the auxiliary (horizontal) space and then a direct sum over all possible representations

\[
\mathcal{H} = \pi^{(N)}(1) \otimes \pi^{(N)}(2) \otimes \cdots \otimes \pi^{(N)}(M)
\]

in the quantum (vertical) space of the chain of the length \( M \). These representations will be labelled by the sequence of integers

\[
I^{(N)} = \{I^{(N)}_1, I^{(N)}_2, \ldots, I^{(N)}_M\}, \quad I^{(N)}_m = 0, 1, 2, \ldots
\]
where $T^{(N)}_m$ is defined in (46). Then the transfer matrix (80) can be written as

$$T(v, w | \{v\}, \{w\}) = \bigoplus_{p^N(N)} \sum_{J=0}^{\infty} v^J T^{(N)}_{p^N, J}(w | \{v\}, \{w\}),$$

(84)

where $w$ is the spectral parameter, $\{w\}$ defines its inhomogeneities, $v$ stands for the ‘horizontal fugacity’ and $\{v\}$ defines the horizontal fields. However, using the symmetry relations (34) and (35) one can swap the left-to-right and front-to-back directions (and also reverse the vertical direction) and then rewrite the transfer matrix (84) in the form

$$T(v, w | \{v\}, \{w\}) = \bigoplus_{p^M(N)} \sum_{K=0}^{\infty} w^K \left[T^{(M)}_{p^M, K}(v | \{w\}, \{v\})\right]^T$$

(85)

where the rank of the algebra $N$ is exchanged with length of the chain (and vice versa), the spectral parameter is exchanged with the horizontal fugacity and the set of the spectral parameter inhomogeneities is exchanged with the set of horizontal field. The superscript $T$ denotes the transposition in the quantum space and

$$I^{(M)} = \{I^{(M)}_1, I^{(M)}_2, \ldots, I^{(M)}_N\}, \quad I^{(M)}_n = 0, 1, 2, \ldots$$

(86)

where $I^{(M)}_n$ is defined in (46). This remarkable relation is called the rank-size duality. Other instances of this duality were previously found in [14, 16]. Somewhat similar phenomena arise in quantum spin tubes and spin ladders [33]. It would be extremely interesting to understand this duality further, in particular, to study its implications to the algebraic and analytic structure of the Bethe ansatz.

### 6. Conclusion

In this paper we constructed a solution of the tetrahedron equation which only contains non-negative matrix elements. It is given explicitly by (31) and (38). Various properties of this solution, including symmetry relations, are discussed in section 3. Further, in section 4 we have defined a solvable model of statistical mechanics on a regular cubic lattice with periodic boundary conditions. Its partition function is given by (55). The layer-to-layer transfer matrices of the model form a two-parameter commutative family and possess a remarkable rank-size duality equations (84), (85), previously discovered in [14, 16].

Further properties of the proposed model will be considered elsewhere. It appears that the constructions of this work give new insights into the algebraic structure of the 2D integrable models associated with quantum affine algebra $U_q(\hat{sl}(N))$. In particular, even in the simplest case of $N = 2$, related to the six-vertex model, one can obtain many new and rather explicit expressions for associated solutions of the Yang–Baxter equation. These questions are considered in our forthcoming paper [31].

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