Scattering Amplitudes from Soft Theorems and Infrared Behavior

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We prove that soft theorems uniquely fix scattering amplitudes in a wide range of theories, including Yang-Mills, gravity, the non-linear sigma model, Dirac-Born-Infeld, dilatonic theories, extended theories like NLSM⊕φ3 or BI⊕YM, as well as some higher derivative corrections to these theories. We conjecture the same is true even when imposing more general soft behavior, simply by assuming the existence of soft operators, or by imposing gauge invariance/the Adler zero only up to a finite order in soft expansions. Besides reproducing known amplitudes, this analysis reveals a new higher order correction to the NLSM, and two interesting facts: the subleading theorem for the dilaton, and the subsubleading theorem for DBI follow automatically from the more leading theorems. These results provide motivation that asymptotic symmetries contain enough information to fully fix a holographic S-matrix.

MOTIVATION

Three related concepts are central to quantum gravity: holography, the S-matrix, and the black hole information paradox. The holographic principle states that a theory with gravity in the bulk may be described completely by a non-gravitational quantum field theory on the boundary. This motivates how Hawking entropy can be proportional to the area of a black hole instead of its volume.

The S-matrix on the other hand is the unique local gauge invariant observable of quantum gravity. The naive merger of quantum mechanics and general relativity leads to tensions with locality and unitarity, and the S-matrix should have a formulation which avoids these tensions. This is expected only because the S-matrix itself is a naturally holographic object in the first place: it describes the (in|out) matrix of states measured at asymptotic infinity.

Recently, the holographic nature of the S-matrix was made even more concrete by demonstrating the equivalence between asymptotic symmetries and soft theorems [1][4], opening new paths towards finding a holographic dual of flat space-time itself, some reviewed in [5]. Even more surprisingly, it was proposed that infrared considerations and asymptotic symmetries can have implications for the black hole information paradox [6][7]. This raises an apparently superficial question: how much information can soft particles actually carry?

The goal of this letter is to take inspiration from the above issues and ask a more well defined question, in the spirit of the S-matrix program:

How much of an amplitude can be fixed by soft theorems?

The naive answer is that the low energy (IR) behavior of amplitudes is completely disjoint from the high energy (UV) behavior, so soft particles can only carry some partial information, fixing only the IR part of an amplitude.

The following separation seems valid then:

\[ A = A_{\text{IR}} \text{(Soft theorem satisfying)} + A_{\text{UV}} \text{(Soft theorem avoiding)} \]  \hspace{1cm} (1)

But surprisingly, we find that the UV information is not inaccessible via soft theorems - it is simply hidden in several different soft limits. This implies that soft theorems are sufficient to fully fix scattering amplitudes! And in fact, even milder soft behavior can be used instead of the full soft theorems, and still we find the amplitudes are fixed. This enables us to discover scattering amplitudes starting purely from the assumption of soft operators, or what we will call “soft gauge invariance” or “soft Adler zero”.

REVIEW OF SOFT THEOREMS

Soft theorems describe a universal behavior of scattering amplitudes when the energies of one or more massless particles are taken to zero. This limit is taken by rescaling momenta with a soft parameter \( p^\mu \to zp^\mu \), and taking the \( z \to 0 \) limit. The soft theorems then imply a factorization of the form:

\[ A_n \to \left( z^\sigma S(0) + z^{\sigma+1} S(1) + \ldots \right) A_{n-1} \]  \hspace{1cm} (2)

where the \( S_i \) are called soft operators, and encode symmetries of the theory being considered.

Originally discovered for photons in [8] and extended to gravitons in [9], soft theorems have enjoyed a renewed interest, at least in part due to their uncovered equivalence to memory effects and asymptotic symmetries [10], and the discovery of a new soft theorem for gravitons [11]. These results subsequently lead to an investigation of soft theorems and asymptotic symmetries for many other theories [12][48]. Soft theorems were shown to follow from considerations of gauge invariance, locality, unitarity, [47][48], shift symmetries in the case of scalar theories [49][51], the CHY formalism [52][57], ambitwistors...
or transmutation operators [60]. Soft theorems for dilaton theories also hinted towards a hidden conformal symmetry in gravity [61].

Meanwhile, a different type of soft behavior, known as the Adler zero [62, 63] and generalizations thereof, was being exploited in the construction of various effective theories: the non-linear sigma model (NLSM), Dirac-Born-Infeld (DBI), the Galileon [64, 65] and special Galileon (SG), as well as vector Born-Infeld (BI) (see [62, 66] for overviews of these theories). This was done by constraining the theory space of possible effective theories [67, 70] by direct construction using newly enabled recursions [71, 72], and most recently through the “soft bootstrap” procedure [73]. This special behavior was also used to rule out possible counterterms in the bootstrap procedure [73]. This special behavior was also used to rule out possible counterterms in $\mathcal{N} = 8$ supergravity [74, 75], recently shown to be finite up to 5 loops in $D = 4$ [76].

Soft limits as formal Taylor series were also crucial to proving that various scattering amplitudes can be fixed by only three conditions: “weak” locality, gauge invariance/Adler zero, and minimal mass dimension [77, 78]. Similarly, in [72] it was conjectured that there exist unique objects satisfying locality and correct D-dimensional BCFW scaling [80, 82]. Remarkably, in all of these cases unitarity emerged as a direct consequence of other basic principles. The present article is a continuation of this previous work, this time investigating the constraints imposed by locality and various types of soft behavior.

\section*{FIXING AMPHITUDES WITH SOFT BEHAVIOR}

Our procedure works as follows. We start with a general local ansatz $B_n = B_n \delta(\sum_i p_i)$, which has some appropriate singularity structure, corresponding to propagators of tree diagrams. Then we take $m$ particles soft, obtaining a soft limit expansion

$$B_n \to z^\sigma B_0^n + z^{\sigma+1}B_1^n + z^{\sigma+2}B_2^n + \ldots .$$

Finally we demand that this matches the corresponding soft theorems (which exist up to a finite order $N$)

$$B_n \to (z^\sigma S_0 + z^{\sigma+1}S_1 + \ldots + z^N S_N)A_{n-m} + \ldots .$$

This leads to a system of equations which turns out to have a unique solution: $B_n = A_n$, the corresponding higher point amplitude.

But the constraints coming from IR behavior can be relaxed beyond what is dictated by soft theorems, and still we find that the UV part is fixed. We can consider what sort of general local functions $B_n$ and $B_{n-m}$ can satisfy a soft behavior of the type

$$B_n \to (z^\sigma S_0 + z^{\sigma+1}S_1 + \ldots + z^N S_N)B_{n-m} + \ldots .$$

This happens only if both $B_n$ and $B_{n-m}$ are scattering amplitudes! Since in this approach no previous knowledge of amplitudes is required, we can probe for new amplitudes, in particular higher order corrections, simply by assuming that such objects satisfy the same symmetries as the “base” theory.

Finally, we consider an even more general soft behavior, without any factorizing soft operators. In the soft expansion [3] we simply require $B^n_0$ up to $B^n_N$ to satisfy gauge invariance (for vector theories), or the Adler zero (for scalar theories) in the soft particles. Even this very weak condition is sufficient to fix $B_n = A_n$, but this time not including any higher order corrections. This is a direct improvement of the previous results in [77, 78], since now we require gauge invariance/Adler zero only up to a finite order.

In conclusion, we propose three new constraints pertaining purely to the IR behavior which, together with locality, completely fix various scattering amplitudes:

1. Soft theorems
2. Soft operators
3. Soft gauge invariance/soft Adler zero.

The above results seem to hold for all massless theories which satisfy soft theorems. This includes QED, Yang-Mills, gravity, NLSM, DBI, dilaton effective theory [41, 83–86], among others. Most such amplitudes are of course already known, but we do discover a novel amplitude: the two extra derivative correction to the NLSM. Higher corrections to the NLSM were computed in [87], but those start at four extra derivatives. We are also able to reproduce the mysterious extended theories found in [88], simply by imposing the corresponding soft operators on an appropriate ansatz.

It should be noted that it is not necessary to use all soft limits in order to fully fix an amplitude, some particles can be skipped. For example, we have explicitly checked up to $n = 5$ that imposing the soft theorems for only $(n-2)$ particles is sufficient to obtain the usual Yang-Mills amplitudes. This makes it clear that imposing soft theorems is not equivalent to a Feynman diagram construction. While this fact could be interesting to understand better, for simplicity we will always use the maximum number of relevant soft limits.

\section*{SOFT THEOREM AVOIDING TERMS AND ENHANCED SOFT LIMITS}

The argument for uniqueness from soft theorems is straightforward: assume there are two different objects, the amplitude $A_n$, and a general local function $B_n$, satisfying the same soft theorems, which exist up to some order $O(z^N)$. This implies the difference $B_n - A_n$ must behave as $O(z^{N+1})$ in all $n$ soft limits. Therefore, after
imposing the soft theorems on $B_n$, it must take a form

$$B_n \rightarrow [\text{piece satisfying soft theorems}] + [\mathcal{O}(z^{N+1}) \text{ terms}]$$

The question is then whether such $\mathcal{O}(z^{N+1})$ soft theorem avoiding terms can exist at the mass dimension being considered. If these terms do not exist, it follows that $B_n = A_n$, so there is a unique object satisfying the soft theorems.

Surprisingly, it turns out that for the most common amplitudes, and even their low lying derivative corrections, this is indeed the case. Their mass dimension is too low to allow any soft theorem avoiding terms, so the amplitude is fully captured by its low energy behavior.

Proving this is slightly more complicated than simple power counting, because momentum conservation can lead to non-trivial cancellations in soft limits. It is well known in fact that various scalar amplitudes do enjoy enhanced soft limit behavior, different from what naive power counting suggests. In particular four such solutions are known: NLSM, DBI, the Galileon vertex, and power counting suggests. In particular four such solutions, this is indeed the case. Their mass dimension is fully captured by its low energy behavior.

Therefore, depending on the theory, multiplicity, and mass dimension, imposing the soft theorems can lead to one of three outcomes

$$B_n = A_n,$$

$$B_n = \text{IR}[A_n] + f(e, 1/K) A^{\text{enh}},$$

$$B_n = \text{IR}[A_n] + [\text{trivial objects}],$$

where $\text{IR}[A_n]$ is the IR piece of $A_n$ that is under the control of soft theorems, and $f(e, 1/K)$ can be some function of polarization vectors and propagators, left-over after factoring out $A^{\text{enh}}$. By “trivial objects” in the third equation we mean independent terms which by simple power counting avoid all imposed soft theorems. Which $A^{\text{enh}}$ could appear depends on the theory being considered. For example, because of the ordered propagator structure, starting from a Yang-Mills ansatz we can only get $A^{\text{NLSM}}$ as the enhanced object.

Why imposing the soft operators also gives sensible answers is not so clear, as the above argument no longer works. However, we have checked in various cases that whenever uniqueness from soft theorems works, so does uniqueness from soft operators (sometimes up to possible non-physical solutions for the highest fixable corrections). Similarly, we have no proof for the soft gauge invariance/soft Adler zero claim.

In the next sections we discuss in more detail what these constraints imply for each theory.

**YANG-MILLS**

Yang-Mills amplitudes satisfy single soft theorems [12]

$$A_{n+1} \rightarrow \left( \frac{1}{z} S_0 + z^0 S_1 \right) A_n + \mathcal{O}(z),$$

with the soft operators given by

$$S_0 = \frac{\epsilon \cdot p_1}{q \cdot p_1} - \frac{\epsilon \cdot p_n}{q \cdot p_n}, \quad S_1 = e_{\mu} q_{\nu} \left( \frac{J_{1}^{\mu \nu}}{q \cdot p_1} - \frac{J_{n}^{\mu \nu}}{q \cdot p_n} \right),$$

and

$$J_{i}^{\mu \nu} = e^{\mu}_{i} \frac{\partial}{\partial e_{i}^{\nu}} + p^{\mu}_{i} \frac{\partial}{\partial p_{i}^{\nu}} - (\mu \leftrightarrow \nu).$$

The soft theorems hold even for the higher derivative corrections. These correspond to higher mass dimension operators in the general effective Lagrangian for Yang-Mills [89–91]

$$L = F^2 + a_0 F^3 + a_1 F_1^3 + a_2 F_2^3 + a_3 F_3^4 + a_4 F_4^4,$$

where the $F_i^4$ operators represent the four different possible contractions of field strengths.

Our goal is to obtain the various amplitudes by starting from a local general ansatz, $B_n(p^{n-2+\kappa})$, with cubic ordered propagators, and with $(n-2+\kappa)$ powers of momenta in the numerators. We introduce $\kappa$ to keep track of the extra number of derivatives in the operator considered: $\kappa = 0$ corresponds to the usual YM amplitude; $\kappa = 2$ corresponds to the amplitude with an $F^3$ operator insertion; for $\kappa = 4$ there are five different amplitudes, corresponding to an $(F^3)^2$ or an $F^3$-type operator insertion.

**Soft theorems**

The first fully interesting case to consider is at $n = 5$. Take a local function with $\kappa = 0$, $B_5(p^3)$, and impose

$$B_5(p^3) \rightarrow \frac{1}{z} S_0 + z^0 S_1 A_4,$$

repeating the process for every particle. We find a unique solution satisfying these constraints, namely $B_5 = A_5$, the five point Yang-Mills amplitude.

We can do the same starting with a higher mass dimension ansatz, $\kappa = 2$, using the known amplitude with an $F^3$ operator insertion as a “seed”. Like above, impose for each particle

$$B_5(p^5) \rightarrow \frac{1}{z} S_0 + z^0 S_1 A_5^{F^3},$$

and the solution is again unique: $B_5 = A_5^{F^3}$. 
We can go to an even higher mass dimension at \( \kappa = 4 \), again using the known lower point amplitudes:

\[
B_5(p^7) \rightarrow \frac{1}{\zeta} S_0 + z^0 S_1 (a_1 A_4^{(k=3)^2} + a_2 A_4^{F_3^4} + a_3 A_4^{F_3^4} + a_4 A_4^{F_3^4} + a_5 A_4^{F_3^4}).
\]

(16)

There are five solutions, as expected: \( B_5 = a_1 A_6^{(k=3)^2} + a_2 A_4^{F_3^4} + a_3 A_4^{F_3^4} + a_4 A_5^{F_3^4} + a_5 A_5^{F_3^4} \).

Increasing the mass dimension to \( \kappa = 6 \), we can finally encounter terms that trivially avoid the soft theorems, for example terms like:

\[
\frac{(e.e.e.e.p_5)(p_1.p_2.p_3.p_4)^2}{p_1.p_2.p_3.p_4}.
\]

(17)

In this case, the solution takes a less useful form:

\[
B_5(p^9) = \text{IR}[A_6^{(k=3)}] + \text{trivial } O(z) \text{ objects}.
\]

(18)

This behavior is true in general, except for two exceptions at low multiplicity. First, at \( n = 4 \) we can only obtain the \( \kappa = 0 \) case by imposing soft theorems. Higher values for \( \kappa \) allow terms like \( e_i.e_2.e_3.e_4.p_1.p_2 \), which scale as \( O(z) \) in all soft limits. The second exception is \( n = 6, \kappa = 4 \), as this is where the NLSM will appear. Imposing

\[
B_6(p^{10}) \rightarrow \frac{1}{\zeta} S_0 + z^0 S_1 A_6^{F_3^4},
\]

we find a solution:

\[
B_6(p^{10}) = \text{IR}[A_6^{(k=3)}] + f(e) A_6^{NLSM},
\]

(20)

where \( f(e) \) is a function only of the six polarization vectors. Note that locality dictates each diagram in our ansatz to have at least two two-particle poles, which is why a value of \( \kappa = 4 \) was needed to factor out a NLSM amplitude.

Except these two cases, we claim that soft theorems fully constrain amplitudes for \( \kappa = 0, 2, \) and \( 4 \). This pattern will be true for most of the other theories to be discussed: GR, NLSM, and DBI. For the dilaton we will actually be able to determine more and more corrections as we increase the multiplicity.

**Soft operators**

We can replace the known lower point amplitudes with general local functions, and the process still works. Remarkably, this ends up fixing the lower point functions as well!

Starting from general four and five point ansatze, impose:

\[
B_5(p^3) \rightarrow \frac{1}{\zeta} S_0 + z^0 S_1 B_4(2,3,4,5),
\]

(21)

e tc., for each particle. The solution is \( B_5 = A_5 \) as before, but we also recover \( B_4 = A_4 \).

This is still true for \( \kappa = 2 \), where we find \( B_5 = A_5^{F_3^4} \), and also \( B_4 = A_4^{F_3^4} \). However, with \( \kappa = 4 \), besides the usual amplitudes, we find an extra non-gauge invariant solution. It would be interesting to understand the origin of this object, and whether it persists at higher multiplicity.

As a result of these observations, we conjecture that soft operators fully fix the \( \kappa = 0, 2 \) amplitudes, and \( \kappa = 4 \) amplitudes up to possible extra non-physical solutions.

**Soft gauge invariance**

We can relax the IR behavior even further. Simply demanding that in the various soft expansions:

\[
B_5(p^3) \rightarrow \frac{1}{\zeta} B_5^{\kappa=1} z^0 B_5^{\kappa=1} \cdots,
\]

(22)

the functions \( B_5^{\kappa=1} \) and \( B_5^{\kappa=1} \) are gauge invariant in particle \( i \), we find a unique solution \( B_5 = A_5 \). We also conjecture that this remains true at higher multiplicity.

**GRAVITY**

Compared to Yang-Mills, gravity amplitudes satisfy one extra soft theorem [11]:

\[
A_{n+1} \rightarrow \left( \frac{1}{\zeta} S_0 + z^0 S_1 + z^1 S_2 \right) A_n + O(z),
\]

(23)

but also have a higher mass dimensions of \( p^{2n-4-\kappa} \), so that overall the IR behavior ends up being just as constraining as it was for Yang-Mills. Again we use \( \kappa \) to label different amplitudes, with \( \kappa = 0 \) the usual GR amplitudes. The whole discussion is very similar to that of Yang-Mills, the only difference is that soft theorem avoiding terms in this case must scale \( O(z^2) \), corresponding to either DBI amplitudes or Galileon vertices as possible exceptions to power counting.

This implies that for \( n = 5 \) and \( n > 6 \), soft theorems fully constrain the \( \kappa = 0, 2, \) and \( 4 \) amplitudes.

For \( n = 4 \) only the usual \( \kappa = 0 \) amplitude can be obtained via soft theorems, while at \( n = 6 \) for \( \kappa = 4 \) imposing the soft theorems determines the gravity amplitude up to a DBI amplitude:

\[
B_6 = \text{IR}[A_6^{(k=4)}] + f(e) A_6^{\text{DBI}}.
\]

(24)

Finally, we conjecture that the soft operator and soft gauge invariance constraints also work as they did in the Yang-Mills case.
NLSM

Similar things also work for the NLSM \[ \text{[62 68 92]} \]. Besides the single soft theorem known as the Adler zero, the NLSM also satisfies double soft theorems, which are the analogue of the single soft limits for Yang-Mills or gravity:

\[ A_n \rightarrow (z^0 S_0 + z S_1) A_{n-2}, \quad (25) \]

with explicit expressions for the soft factors given in \[ \text{[54]} \] for the adjacent limit, and \[ \text{[93]} \] for the non-adjacent limit, which will be crucial. In this case soft theorem avoiding terms must scale as \( O(z^2) \) in double soft limits.

The starting ansatz for the NLSM is also of the form \( B_n(p^{n-2+\kappa}) \), with numerators of mass dimension \( (n - 2 + \kappa) \), and poles corresponding to propagators of quartic diagrams. Taking into account the minimum two three-particle poles per diagram, simple mass dimension counting proves that for \( n \geq 8 \) soft theorems must fully constrain the \( \kappa = 0, 2 \), and 4 cases. It should be noted that \( \kappa = 4 \) only follows once we use the non-adjacent soft theorem. For \( n = 6 \) only the \( \kappa = 0, 2 \) cases can be obtained.

We can check that this reproduces the higher corrections to the NLSM computed in \[ \text{[87]} \], but those start out at \( \kappa = 4 \), so we have nothing check against for \( \kappa = 2 \). However, we can use the soft operator approach to find if any \( \kappa = 2 \) amplitudes exist.

Soft operators

As before, we conjecture that soft operators reproduce the known \( \kappa = 0 \) and \( \kappa = 4 \) solutions (for \( n > 6 \)). But more interestingly they can also produce the \( \kappa = 2 \) corrections that we were missing. By simply imposing

\[ B_6^{\kappa=2} \rightarrow (z^0 S_0 + z S_1) B_4^{\kappa=2}, \quad (26) \]

we obtain unique solutions, with the four point amplitude given by the cyclically invariant combination:

\[ A_4^{\kappa=2} = s_{12} s_{14}. \quad (27) \]

It is interesting to note that \( Z \)-theory produces higher order corrections all of which obey BCJ relations \[ \text{[87 94 97]} \], while the solution corresponding to \[ \text{[27]} \] does not.

Soft Adler zero

Transferring our insights from the previous examples, it turns out that we can also impose a “soft Adler zero” to obtain the NLSM amplitude. Taking a formal double limit in particles \( n \) and \( n - 1 \):

\[ B_n \rightarrow \frac{1}{z} B_{n-2}^{-1} + z^0 B_{n-2}^0 + z B_{n-2}^1, \quad (28) \]

we now impose that the three terms above have \( O(z) \) behavior when taking \( n \) and \( n - 1 \) soft. Repeating the procedure for the other particles, we obtain that \( B_n \) must be the NLSM amplitude.

Single soft theorems and extended NLSM

In \[ \text{[88]} \] it was discovered that various amplitudes contain hidden so called extended theories in their single soft limits. In the case of the NLSM, schematically this limit is:

\[ A_n = z \sum_i s_i n A_{n-1}^{\text{NLSM} \oplus \phi} (i) + O(z^2), \quad (29) \]

where \( A_{n}^{\text{NLSM} \oplus \phi} \) is an amplitude of pions interacting with a scalar bi-adjoint theory. The origin of this amplitude is easy to see in the CHY formulation, but otherwise remains mysterious - although progress has been made by finding its Feynman rules \[ \text{[50]} \].

Using our guiding principle of soft behavior we can derive even these amplitudes simply by assuming that each NLSM scalar obeys the NLSM soft theorems/operators.

(DIRAC-)BORN-INFELD

Scalar Dirac-Born-Infeld

DBI satisfies double soft theorems up to order \( O(z^3) \) \[ \text{[52]} \], so we are looking for the lowest mass dimension objects with \( O(z^4) \) in all double soft limits. This is again the Galileon vertex, which has a mass dimension \( (2n-2) \).

The DBI ansatz has a form \( B_n(p^{2n-4+\kappa}) \) with quartic propagators, so taking into account the minimum two three-particle poles for \( n \geq 8 \), this implies we can obtain the \( \kappa = 0, 2 \), and 4 cases from soft theorems before hitting Galileon vertices. We conjecture the same cases can be obtained from soft operators (with the usual caveats), and that the soft Adler zero implies the \( \kappa = 0 \) case.

Because DBI obeys soft theorems up to such a high order, an interesting stronger statement holds: \( \kappa = 0 \) DBI is completely fixed by just the leading and subleading theorems, implying that the subsleading theorem is in fact not independent. As will be discussed later, a similar fact is true for the dilaton theories.

Vector Born-Infeld

For BI amplitudes only the leading double soft theorem is known:

\[ A_n \rightarrow z F_{n}^{\mu \nu} F_{n-1}^{\rho \sigma} \sum_i \frac{p_i^\rho p_i^\sigma}{(p_n + p_{n-1}) \cdot p_i} A_{n-2}, \quad (30) \]
with $F^{\mu\nu}$ the field strength tensor. Unfortunately, simple counting shows that this is not sufficient to fix even the $\kappa = 0$ case. However, we can apply the extended single soft theorem of [88], schematically given by:

$$A_n \rightarrow z \sum_{i,j} S_{i,j} A^n_{\kappa-1,YM}(i,j) + \mathcal{O}(z^2) ,$$

(31)

and obtain

$$B_n = IR[A^\kappa_n] + f(e) A^DBI_n .$$

(32)

The full amplitude can be obtained by further imposing gauge invariance. Of course this procedure requires knowing the $A^n_{\kappa-1,YM}(i,j)$ amplitudes, which luckily can themselves be fixed by imposing the single soft theorems for the gluons and the double soft theorems for the BI photons.

**DILATON**

Due to its peculiar universality, the dilaton is also interesting to study from the perspective of its soft behavior. First, we should note there are two types of dilatons: a gravity dilaton [34–36, 38], and a dilaton associated to the breaking of conformal invariance [39, 85]. We will focus on the latter, with soft theorems: pure polynomials will reproduce explicit breaking of conformal invariance [39, 85].

We will consider the general solution

$$A_n \rightarrow (z^0 S_0 + z^1 S_1) A_{n-1} ,$$

(33)

explicitly given in [41][84].

Depending on how conformal invariance is broken, two classes of theories emerge, both of which we can recover. In the case of spontaneous breaking, the dilaton is a Goldstone boson in the spectrum of the theory, whereas in the case of explicit breaking it can be thought of as an external source. In this case dilaton amplitudes do not have poles associated to a propagating dilaton. Therefore we can control which case we consider by the form of our ansatz: pure polynomials will reproduce explicit breaking, whereas allowing poles reproduces the spontaneous case (such as the Coulomb branch of Yang-Mills or the $\mathcal{N} = (2, 0)$ theory [41]).

Using the previous reasoning, uniqueness from soft theorems (operators) follows if there are no terms with $\mathcal{O}(z^2)$ behavior in all single soft limits. For explicit breaking this corresponds to the Galileon vertex. For the spontaneous breaking, this is the DBI constraint again - only this time, it is well known that DBI shows up in dilaton amplitudes associated to spontaneous breaking. It is also not surprising that gravitons and the dilaton share DBI as the $A^{\text{enh}}$ object - the gravity dilaton itself satisfies in fact very similar soft theorems [35, 61].

### Explicit breaking

In this case we consider a polynomial ansatz $P_n(p^{4+\kappa})$. The mass dimension of the Galileon is $(2n - 2)$, so for $4 + \kappa < 2n - 2$ all amplitudes are fixed by soft theorems. A new feature here is that we can obtain higher and higher corrections as we increase the multiplicity. On the other hand, this also implies that the subleading theorem is not independent. The leading order theorem is sufficient as long as $4 + \kappa < n$, implying that for all of these cases scale invariance implies conformal invariance. Since we do not assume unitarity, this generalizes the result of [41], where this observation was first made.

The soft operator approach also reproduces the known results. For instance, at the lowest mass dimension $\kappa = 0$, by simply imposing:

$$P_n \rightarrow (z^0 S_0 + z^1 S_1) P_{n-1} ,$$

(34)

we find unique solutions of the form:

$$A_n^{\text{explicit}}(p^4) = \sum_{1 \leq k \leq n} s^2_{ij} .$$

(35)

Imposing [34] with $\kappa = 2$, for $n > 5$ we also find unique solutions. We have checked up to $n = 7$ that these reproduce the results in [85]. For the $B_6(p^6) \rightarrow B_4(p^4)$ case we obtain two solutions, with the known amplitude being a linear combination of the two. It is not clear if there is any physical reason for this.

For higher $\kappa$, we conjecture that soft operators work within the same bound as the soft theorems, that is as long as $4 + \kappa < 2n - 2$.

### Spontaneous breaking

For this case we also allow three-particle poles in an ansatz $B_n(p^{4+\kappa})$. For $\kappa = 0$ the solutions in the spontaneous case are identical to the explicit case. The first change is with $\kappa = 2$ at 6 points, when we reach a mass dimension high enough to form a soft theorem avoiding DBI amplitude. Imposing the soft theorems we obtain the general solution

$$B_6(p^6) = b_1 A_6^{\text{explicit}}(p^6) + b_2 A_6^{DBI} ,$$

(36)

as expected, since the DBI action appears manifestly in the dilaton effective action [84].

**GAUGE INVARIANCE = SOFT THEOREM = BCFW SCALING**

The main result of this letter has an interesting corollary - it implies that the conjectured unique BCFW-scaling object of [79] is in fact the scattering amplitude...
for Yang-Mills. Or reversely, that the scattering amplitude does indeed have the usual $O(z^{-1})$ BCFW behavior, and so the general dimension BCFW recursion is justified.

It was already shown in [19] that BCFW scaling implies the Weinberg soft factor, but the subleading piece does not follow directly. Instead, we simply invoke uniqueness: it can be checked that $S_1 A_{n-1}$ satisfies the usual BCFW scaling, so it must be the correct subleading piece. Therefore this BCFW object satisfies the usual leading and subleading soft theorems, and uniqueness from soft theorems implies that it must be the amplitude.

CONCLUSIONS

We have argued that the IR behavior of amplitudes contains surprisingly more information than expected. We prove that imposing soft theorems completely fixes a wide range of amplitudes, and conjecture that in some cases the same should hold when imposing the less constraining soft operators and soft gauge invariance/Adler zero. This promotes soft behavior of amplitudes to a guiding principle in its own right, able to discover interesting objects starting only from simple incarnations of various symmetries.

Recently, an infinite set of “partial” soft theorems were discovered for the photon, gluon, and graviton [98, 99]. It would be interesting to understand whether these can help to fully constrain even higher derivative corrections to the theories considered in this letter.

For both conceptual and practical purposes, it would be extremely useful to transform these uniqueness results into a general Inverse Soft Limit type construction [100–103], which would allow building amplitudes directly from soft factors. This in turn could make manifest yet a new facet of scattering amplitudes, that of asymptotic symmetries, perhaps leading to a purely holographic description of the S-matrix.

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