RANK 2 LOCAL SYSTEMS, BARSOTTI-TATE GROUPS, AND SHIMURA CURVES

RAJU KRISHNAMOORTHY

Abstract. We construct a descent-of-scalars criterion for \( K \)-linear abelian categories. Using advances in the Langlands correspondence due to Abe, we build a correspondence between certain rank 2 local systems and certain Barsotti-Tate groups on complete curves over a finite field. We conjecture that such Barsotti-Tate groups "come from" a family of fake elliptic curves. As an application of these ideas, we provide a criterion for being a Shimura curve over \( \mathbb{F}_q \). Along the way we formulate a conjecture on the field-of-coefficients of certain compatible systems.

Contents

1. Introduction 1
2. Conventions, Notation, and Terminology 4
3. Extension of Scalars and Galois descent for abelian categories 4
4. A 2-cocycle obstruction for descent 5
5. \( F \)-crystals 8
6. Coefficient objects, Compatible Systems and Companions 12
7. \( F \)-isocrystals and \( l \)-adic local systems on Curves 13
8. Kodaira-Spencer 15
9. Algebraization and Finite Monodromy 18
10. Application to Shimura Curves 19
References 21

1. Introduction

Let \( C/\mathbb{F}_q \) be a smooth affine curve with compactification \( \overline{C} \) and let \( \pi: E_C \to C \) be a non-isotrivial family of elliptic curves. Then \( R^1\pi_*\mathbb{Q}_l \) is a rank 2 \( l \)-adic local system on \( C \) with infinite monodromy around some \( \infty \in \overline{C}\setminus C \), cyclotomic determinant, and all Frobenius traces in \( \mathbb{Q} \). A startling consequence of Drinfeld’s first work on the Langlands correspondence is a converse:

**Theorem.** (Drinfeld) Let \( C/\mathbb{F}_q \) be a smooth affine curve with compactification \( \overline{C} \). Let \( \mathcal{L} \) be a rank 2 irreducible \( \overline{\mathbb{Q}}_l \)-local system on \( C \) such that

- \( \mathcal{L} \) has infinite monodromy around some \( \infty \in \overline{C}\setminus C \),
- \( \mathcal{L} \) has determinant \( \overline{\mathbb{Q}}_l(-1) \), and
- the field of Frobenius traces of \( \mathcal{L} \) is \( \mathbb{Q} \).

Then \( \mathcal{L} \) comes from a family of elliptic curves: there exists a map \( f \)

\[
\begin{array}{ccc}
C & \xrightarrow{f} & \mathcal{M}_{1,1} \\
\pi \downarrow & & \downarrow \\
\mathcal{E} & & \\
\end{array}
\]

such that \( \mathcal{L} \cong f^*(R^1\pi_*\mathbb{Q}_l) \). Here \( \mathcal{M}_{1,1} \) is the moduli of elliptic curves with universal elliptic curve \( \mathcal{E} \).

See [ST18, Proposition 19, Remark 20] for how to recover this result from Drinfeld’s work.
Definition 1.1. Let $D$ be an indefinite non-split quaternion algebra over $\mathbb{Q}$ of discriminant $d$ and let $\mathcal{O}_D$ be a fixed maximal order. Let $k$ be a field with $\text{char}(k) \nmid d$. A quaternionic abelian surface is a pair $(A, i)$ of an abelian surface $A/k$ together with an injective ring homomorphism $i : \mathcal{O}_D \to \text{End}_k(A)$.

$D$ has a canonical involution, which we denote $\iota$. Pick $t \in \mathcal{O}_D$ with $t^2 = -d$. Then there is another associated involution $* \text{ on } D$:

$$x^* := t^{-1}x^tt$$

There is a unique principal polarization $\lambda$ on $A$ such that the Rosati involution restricts to $*$ on $\mathcal{O}_D$. We refer to the triple $(A, \lambda, i)$ as a fake elliptic curve, suppressing the dependence on $t \in \mathcal{O}_D$.

Just as one can construct a modular curve parameterizing elliptic curves, there is a Shimura curve $X^D$ parameterizing fake elliptic curves with multiplication by $\mathcal{O}_D$. Over the complex numbers, these are compact hyperbolic curves. Explicitly, if one chooses an isomorphism $D \otimes \mathbb{R} \cong M_{2 \times 2}(\mathbb{R})$, consider the image of $\Gamma = \mathcal{O}_D^\times$ of elements of $\mathcal{O}_D^\times$ of norm 1 (for the standard norm on $\mathcal{O}_D$) inside of $SL(2, \mathbb{R})$. $\Gamma$ acts properly discontinuously and cocompactly on $\mathbb{H}$. The quotient $X^D = [\mathbb{H}/\Gamma]$ is the complex Shimura curve associated to $\mathcal{O}_D$. In fact, $X^D$ has a canonical integral model and may therefore be reduced modulo $p$ for almost all $p$ [Buz97]. In analogy of Theorem 1, we pose the following conjecture.

Conjecture 1.2. Let $C/\mathbb{F}_q$ be a smooth projective curve. Let $\mathcal{L}$ be a rank 2 irreducible $\overline{\mathbb{Q}}_l$-local system on $C$ such that

- $\mathcal{L}$ has infinite geometric monodromy,
- $\mathcal{L}$ has determinant $\overline{\mathbb{Q}}_l(-1)$, and
- the field of Frobenius traces of $\mathcal{L}$ is $\mathbb{Q}$.

Then, $\mathcal{L}$ comes from a family of fake elliptic curves: there exists an indefinite quaternion algebra $D/\mathbb{Q}$, a moduli space of fake elliptic curves $X^D$ with universal family $A \xrightarrow{\pi} X^D$, and a map $f$:

$$\begin{align*}
\xymatrix{ & A \\
C \ar[r]^f & X^D}
\end{align*}$$

such that $\mathcal{L}^{\otimes 2} \cong f^*(R^1\pi_*\overline{\mathbb{Q}}_l)$.

In this article we prove the following, which perhaps provides some evidence for 1.2.

Theorem. (Theorem 8.9) Let $C/\mathbb{F}_q$ be a smooth, geometrically connected, proper curve with $q$ a square. There is a natural bijection between the following two sets

- $\overline{\mathbb{Q}}_l$-local systems $\mathcal{L}$ on $C$ such that
  - $\mathcal{L}$ is irreducible of rank 2
  - $\mathcal{L}$ has trivial determinant
  - The Frobenius traces are in $\mathbb{Q}$
  - $\mathcal{L}$ has infinite image,
  - for $\mathcal{L}$ up to isomorphism

- $p$-divisible groups $\mathcal{G}$ on $C$ such that
  - $\mathcal{G}$ has height 2 and dimension 1
  - $\mathcal{G}$ is generically versally deformed
  - $D(\mathcal{G})$ has all Frobenius traces in $\mathbb{Q}$
  - $\mathcal{G}$ has ordinary and supersingular points,
  - for $\mathcal{G}$ up to isomorphism

such that if $\mathcal{L}$ corresponds to $\mathcal{G}$, then $\mathcal{L}(-1/2)$ is compatible with the $F$-isocrystal $D(\mathcal{G}) \otimes \mathbb{Q}$.

Theorem 8.9 arose from the following question: is there a “purely group-theoretic” characterization of proper Shimura curves over $\mathbb{F}_q$? As an application of our techniques, we have the following criterion, motivated largely by [Mar91, Moc96, Moc98, Xin13].

Theorem. (Theorem 10.11) Let $X \xleftarrow{f} Z \xrightarrow{g} X$ be an étale correspondence of smooth, geometrically connected, proper curves without a core over $\mathbb{F}_q$ with $q$ a square. Let $\mathcal{L}$ be a $\overline{\mathbb{Q}}_l$-local system on $X$ as in Theorem 8.9 with $f^*\mathcal{L} \cong g^*\mathcal{L}$ as local systems on $Z$. Suppose the $\mathcal{G} \to X$ constructed in Theorem 8.9 is everywhere versally deformed. Then $X$ is the reduction modulo $p$ of a Shimura curve.
Note that the criterion only involves varieties in characteristic $p$ and makes no mention of a “family of abelian varieties”. The lifting process involved in Theorem 10.11 is due to Xia [Xia13], and it gives one of the canonical lifts of Mochizuki [Moe96]. In [EMO01, Problem 10], Mochizuki asks when these canonical lifts are defined over $\mathbb{Q}$. The techniques of Section 10 give a criterion: see Corollary 10.8 and Remark 10.10.

In joint work-in-progress with Mao Sheng, we have a strategy to prove that in the context of Theorem 10.11, $\mathcal{L}$ indeed comes from a family of fake elliptic curves, verifying a very special case of Conjecture 1.2. More precisely, the method should imply that $X$ is the reduction modulo $p$ of a moduli space of fake elliptic curves and $\mathcal{L}$ is the induced universal system. See Remark 10.12 for a detailed outline of the strategy, which uses $p$-adic nonabelian Hodge theory.

Finally, as a basic observation from our analysis, we prove the following:

**Theorem.** (Theorem 9.5) Let $X$ be a smooth, geometrically connected curve over $\mathbb{F}_q$. Let $\mathcal{E} \in \mathbf{F-Isoc}^c(X)$ be an overconvergent $F$-isocrystal on $X$ with coefficients in $\mathbb{Q}_p$ that is rank 2, absolutely irreducible, and has trivial determinant. Suppose further that the field generated by Frobenius traces on $\mathcal{E}$ is $\mathbb{Q}$. Then $\mathcal{E}$ has finite monodromy.

We make some remarks about Theorem 8.9.

**Remark 1.3.** The condition that $q$ is a square is to ensure that the character $\overline{\chi}^{(1/2)}$ on $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q)$ has Frobenius acting as an integer. Therefore $\mathcal{L}^{(-1/2)}$ also has Frobenius traces in $\mathbb{Q}$ and determinant $\overline{\chi}^{(-1)}$, exactly as in Conjecture 1.2.

**Remark 1.4.** Drinfeld’s work on unramified $GL_2$ Langlands implies that if $\mathcal{L}$ is as in Conjecture 1.2, then there exists a family of surfaces over a Zariski open $U \subset C \times C$

$$g : S \to U$$

such that $(\mathcal{L} \boxtimes \mathcal{L}^*)|_U$ is a direct summand of $R^2g_*\overline{\mathcal{Q}}$. Our conjecture is that $\mathcal{L}$ may be realized inside a “motive of weight 1” over $C$. (See Question 9.1 for a related question.)

**Remark 1.5.** The principle input in Theorem 8.9 is Abe’s construction of $p$-adic companions on curves over a finite field. Given this, there are two new ingredients.

1. The $p$-adic companion $\mathcal{E} \in \mathbf{F-Isoc}^c(X)_{\overline{\mathbb{Q}}_p}$ of $\mathcal{L}$ may be descended to $\mathbb{Q}_p$; in other words, a certain Brauer obstruction “automatically” vanishes. This is contained in Lemma 7.2 and Proposition 7.4.

2. There is a canonical lattice inside of $\mathcal{E}$; this is characterized by the associated Barsotti-Tate group being generically versally deformed. This is contained in Lemma 8.8.

**Remark 1.6.** In particular, given $\mathcal{L}$ as in Theorem 8.9, there is a natural effective divisor (possibly empty) $R$ on $C$, supported at those $c \in |C|$ over which $\mathcal{G}$ is not versally deformed. If Conjecture 1.2 is true, $R$ is the ramification divisor of a generically separable map

$$C \to X^D$$

realizing $\mathcal{L}$ as coming from a family of fake elliptic curves. We do not know how to construct this effective divisor directly either from $\mathcal{L}$ or its $p$-adic companion $\mathcal{E}$ without constructing $\mathcal{G}$.

We now briefly discuss the sections. The goal of Sections 3-8 is to prove Theorem 8.9.

- Section 3 sketches extension-of-scalars and Galois descent for $K$-linear abelian categories.
- Section 4 constructs a Brauer-class obstruction for descending certain objects in an abelian $K$-linear category.
- Section 5 briefly reviews background on $F$-crystals and $F$-isocrystals. Many of the statements are surely well-known, but we couldn’t always find a reference.
- Section 6 is a brief discussion of coefficient objects and several results of Abe and Lafforgue.
- Section 7 contains the application of the descent criterion above in Proposition 7.4.
- Section 8 discusses the deformation theory of BT groups. We prove that there is a unique such $\mathcal{G}$ that is generically versally deformed, thereby proving Theorem 8.9.
BT groups are generally non-algebraic, but here we can show there are only finitely many that occur in Theorem 8.9 via the Langlands correspondence. In Section 9, we speculate on when height 2, dimension 1 BT groups \( G \) come from a family of abelian varieties.

Finally, in Section 10, we discuss applications to “characterizing” certain Shimura curves over \( \mathbb{F}_q \).

Acknowledgements. This work is based on Chapter 3 of my PhD thesis at Columbia University. I am very grateful to Johan de Jong, my former thesis advisor, for guiding this project and for countless inspiring discussions. Ching-Li Chai read my thesis very carefully and provided many valuable comments on the article. This work was partially funded by an NSF postdoctoral fellowship, Grant No. DMS-1605825.

2. Conventions, Notation, and Terminology

For convenience, we explicitly state conventions and notations. These are in full force unless otherwise stated.

1. A curve \( C/k \) is a separated geometrically integral scheme of dimension 1 over \( k \).
2. A variety \( X/k \) is a geometrically integral \( k \)-scheme of finite type.
3. A morphism of curves \( X \to Y \) over \( k \) is a morphism of \( k \)-schemes that is non-constant, finite, and generically separable.
4. \( p \) is a prime number and \( q = p^d \).
5. \( \mathbb{F} \) is a fixed algebraic closure of \( \mathbb{F}_p \).
6. If \( k \) is perfect, \( W(k) \) denotes the Witt vectors and \( \sigma \) the canonical lift of Frobenius. \( K(k) \cong W(k) \otimes \mathbb{Q} \).
7. All \( p \)-adic valuations are normalized such that \( v_p(p) = 1 \).
8. If \( C \) is a \( K \)-linear category and \( L/K \) is an algebraic extension, we denote by \( \mathcal{C}_L \) the base-changed category.
9. Let \( X/k \) be a smooth variety over a perfect field. Then the category \( \text{F-Isoc}^\dagger(X) \) is the \( \mathbb{Q}_p \)-linear Tannakian category of overconvergent \( F \)-isocrystals on \( X \).
10. If \( \mathcal{G} \to X \) is a \( p \)-divisible group, we denote by \( \mathbb{D}(\mathcal{G}) \) the contravariant Dieudonné crystal attached to \( \mathcal{G} \). We use the phrases \( p \)-divisible group and Barsotti-Tate group interchangeably.

3. Extension of Scalars and Galois descent for abelian categories

We discuss extension-of-scalars of \( K \)-linear categories; in the case of \( K \)-linear abelian categories, we state Galois descent. We also carefully discuss absolute irreducibility and semi-simplicity.

Definition 3.1. Let \( \mathcal{C} \) be a \( K \)-linear additive category, where \( K \) is a field. Let \( L/K \) be a finite field extension. We define the base-changed category \( \mathcal{C}_L \) as follows.

- Objects of \( \mathcal{C}_L \) are pairs \((M, f)\), where \( M \) is an object of \( \mathcal{C} \) and \( f : L \to \text{End}_K M \) is a homomorphism of \( K \)-algebras. We call such an \( f \) an \( L \)-structure on \( M \).
- Morphisms of \( \mathcal{C}_L \) are morphisms of \( \mathcal{C} \) that are compatible with the \( L \)-structure.

If \( \mathcal{C} \) is an abelian category, so is \( \mathcal{C}_L \) [Sos14]. Moreover, if \( \mathcal{C} \) is a \( K \)-linear Tannakian category (not necessarily assumed to be neutral), then \( \mathcal{C}_L \) is an \( L \)-linear Tannakian category [Del14, Théorème 5.4].

Remark 3.2. Deligne defines \( \mathcal{C}_L \) in a slightly different way [Del14, 5]. Given an object \( X \) of \( \mathcal{C} \) and a finite dimensional \( K \)-vector space \( V \), define \( V \otimes X \) to represent the functor \( Y \mapsto V \otimes_K \text{Hom}_K(Y, X) \). Given a finite extension \( L/K \), an object with \( L \)-structure is defined to be an object \( X \) of \( \mathcal{C} \) together with a morphism \( L \otimes X \to X \) in \( \mathcal{C} \) satisfying certain natural conditions. Then the base-changed category \( \mathcal{C}_L \) is defined to be the category of objects equipped with an \( L \)-structure and with morphisms respecting the \( L \)-structure.
Note that there is an induction functor: \( \text{Ind}_K^L : \mathcal{C} \rightarrow \mathcal{C}_L \). Let \( \{\alpha\} \) be a basis for \( L/K \).

\[
\text{Ind}_K^L M = \bigoplus_{\alpha} M, f
\]

where \( f \) is induced by the action of \( L \) on the \( \{\alpha\} \). In Deligne’s formulation, \( \text{Ind}_K^L M \) is just \( L \otimes \mathcal{M} \) and the \( L \)-structure is the multiplication map \( L \otimes_{K} L \otimes \mathcal{M} \rightarrow L \otimes \mathcal{M} \). The induction functor has a natural right adjoint: \( \text{Res}_K^L \) which simply forgets about the \( L \)-structure. If \( M \) is an object of \( \mathcal{C} \), we sometimes denote by \( M_L \) the object \( \text{Ind}_K^L M \) for shorthand. If \( C \) is a \( K \)-linear abelian category, induction and restriction are exact functors.

**Remark 3.3.** The induction functor allows us to describe more general extensions-of-scalars. Namely, let \( E \) be an *algebraic* field extension of \( K \). We define \( \mathcal{C}_E \) to be the \( 2 \)-colimit of the categories \( \mathcal{C}_L \) where \( L \) ranges over the subfields of \( E \) finite over \( K \). In particular, we can define a category \( \mathcal{C}_E \) and an induction functor

\[
\text{Ind}_K^E : \mathcal{C} \rightarrow \mathcal{C}_E
\]

**Definition 3.4.** Let \( \mathcal{C} \) be a \( K \)-linear abelian category. We say that an object \( M \) is *absolutely irreducible* if \( \text{Ind}_K^E(M) \) is an irreducible object of \( \mathcal{C}_E \).

If \( L/K \) is a Galois extension, note that we have a natural (strict) action of \( G := \text{Gal}(L/K) \) on the category \( \mathcal{C}_L \) by twisting the \( L \)-structure. That is, if \( g \in G \), \( \gamma(C, f) := (C, f \circ g^{-1}) \). The group \( G \) “does nothing” to maps: a map \( \phi : (C, f) \rightarrow (C', f') \) in \( \mathcal{C}_L \) is just a map \( \phi_K : C \rightarrow C' \) in \( \mathcal{C} \) that commutes with the \( L \)-actions, and \( g \in G \) acts by fixing the underlying \( \phi_K \) while twisting the underlying \( L \)-structures \( f, f' \): \( \gamma \phi : (C, f \circ g^{-1}) \rightarrow (C', f' \circ g^{-1}) \).

If \( \lambda \in L \) is considered as a scalar endomorphism of \( M \), then the endomorphism \( \gamma \lambda : \gamma M \rightarrow \gamma M \) is the scalar \( g(\lambda) \). If \( \mathcal{C} \) is a \( K \)-linear rigid abelian \( \otimes \mathcal{M} \) category, then this action is compatible with the inherited rigid abelian \( \otimes \mathcal{M} \) structure on \( \mathcal{C}_L \). Similarly, if \( F : \mathcal{C} \rightarrow \mathcal{D} \) is a \( K \)-linear functor between \( K \)-linear categories, we have a canonical extension functor \( F_L : \mathcal{C}_L \rightarrow \mathcal{D}_L \) and \( \gamma F_L(M) \cong F_L(\gamma M) \).

**Definition 3.5.** Suppose \( \mathcal{C} \) is a abelian \( K \)-linear category and \( L/K \) is a finite Galois extension with group \( G \). Let \( \mathcal{C}_L \) be the base-changed category. We define the *category of descent data*, \( (\mathcal{C}_L)^G \), as follows. The objects of \( (\mathcal{C}_L)^G \) are pairs \( (M, \{c_g\}) \) where \( M \) is an object of \( \mathcal{C}_L \) and the \( c_g : M \rightarrow \gamma M \) are a collection of isomorphisms for each \( g \in G \) that satisfies the cocycle condition. The morphisms of \( (\mathcal{C}_L)^G \) are maps \( (f_g : \gamma M \rightarrow \gamma N) \) that intertwine the \( c_g \).

**Lemma 3.6.** If \( \mathcal{C} \) is an abelian \( K \)-linear category and \( L/K \) is a finite Galois extension, then Galois descent holds. That is, \( \mathcal{C} \) is equivalent to the category \( (\mathcal{C}_L)^G \).

**Proof.** This is [Sos14, Lemma 2.7].

We say objects and morphisms in the essential image of \( \text{Ind}_K^L \) *descend.*

### 4. A 2-cocycle obstruction for descent

Suppose \( \mathcal{C} \) is an abelian \( K \)-linear category and \( L/K \) a finite Galois extension with group \( G \). Let \( \mathcal{C}_L \) be the base-changed category. Suppose an object \( M \in \text{Ob}(\mathcal{C}_L) \) is isomorphic to all of its twists by \( g \in G \) and that the natural map \( L \rightarrow \text{End}_{\mathcal{C}_L}M \) is an isomorphism. (This latter restriction will be relaxed later in the important Remark 4.7.) We will define a cohomology class \( \xi_M \in H^2(G, L^*) \) such that \( \xi_M = 0 \) if and only if \( M \) descends to \( K \). This construction is well-known in representation theory.

**Definition 4.1.** For each \( g \in G \) pick an isomorphism \( c_g : M \rightarrow \gamma M \). The function \( \xi_{M,c} : G \times G \rightarrow L^* \), depending on the choices \( \{c_g\} \), is defined as follows:

\[
\xi_{M,c}(g, h) = c_{gh}^{-1} \circ \gamma c_h \circ c_g \in \text{Aut}_{\mathcal{C}_L}(M) \cong L^*
\]

**Proposition 4.2.** The function \( \xi_{M,c} \) is a 2-cocycle.
Proof. We need to check
\[ g_1 \xi(g_2,g_3) \xi(g_1,g_2) = \xi(g_1g_2,g_3) \]
We may think of the right hand side as a scalar function \( M \to M \), which allows us to write it as
\[ \xi(g_1g_2,g_3) \xi(g_1,g_2) = c_{g_1g_2}^{-1} \circ \xi(g_2,g_3) \circ c_{g_1g_2} \circ c_{g_1}^{-1} \circ \xi(g_2) \circ c_{g_1}, \]
\[ = c_{g_1g_2}^{-1} \circ \xi(g_2,g_3) \circ c_{g_1} \]
\[ = c_{g_1g_2}^{-1} \circ \xi(g_2,g_3) \circ c_{g_1} \circ g_1 \]
\[ = c_{g_1}^{-1} \circ \xi(g_2,g_3) \circ c_{g_1} \]
\[ = \xi(g_1,g_2) \circ g_1 \]
\[ = \xi(g_1,g_2,g_3) \]

In the penultimate line, we may commute the \( c_{g_1} \) and \( g_1 \xi(g_2,g_3) \) because the latter is in \( L \). \( \square \)

Remark 4.3. If \( \xi_{M,c} = 1 \), then the collection \( \{ c_g \} \) form a descent datum for \( M \) which is effective by Galois descent for abelian categories, Lemma 3.6.

Proposition 4.4. Let \( C \) be an abelian \( K \)-linear category and \( L/K \) a finite Galois extension with group \( G \). Let \( M \in \text{Ob}(C_L) \) such that
\* \( M \cong g \cdot M \) for all \( g \in G \)
\* The natural map \( L \to \text{End}_L(M) \) is an isomorphism.

If \( \xi_{M,c} \) is a coboundary, then \( M \) is in the essential image of \( \text{Ind}_K^G \), i.e. \( M \) descends.

Proof. If \( \xi_{M,c} \) is a coboundary, there exists a function \( \alpha : G \to L^* \) such that
\[ \xi_{M,c}(g,h) = \frac{g \alpha(h) \alpha(g)}{\alpha(gh)} \]
Now, set \( c'_g = \frac{c_g}{\alpha(g)} : M \to \text{g} \cdot M \) and note that the \( c'_g \) are a descent datum for \( M \) because the associated \( \xi_{M,c'} = 1 \). \( \square \)

Proposition 4.5. Given \( M \in C_L \) as in Proposition 4.4 and two choices \( \{ c_g \} \) and \( \{ c'_g \} \) of isomorphisms, \( \xi_{M,c} \) and \( \xi_{M,c'} \) differ by a coboundary and thus give the same class in \( H^2(G,L^*) \). We may therefore unambiguously write \( \xi_M \) for the cohomology class associated to \( M \).

Proof. Note that \( (c'_g)^{-1} \circ c_g : M \to M \) is in \( L^* \). This ratio will be a function \( \alpha : G \to L^* \) exhibiting the ratio \( \frac{\xi_{M,c}}{\xi_{M,c'}} \) as a coboundary. \( \square \)

Corollary 4.6. Let \( C \) be an abelian \( K \)-linear category, let \( L/K \) be a finite Galois extension with group \( G \), and let \( C_L \) be the base-changed category. Let \( M \) be an object of \( C_L \) such that
\( \text{(1)} M \cong g \cdot M \) for all \( g \in G \) and
\( \text{(2)} \) the natural map \( L \to \text{End}_{C_L}(M) \) is an isomorphism.

Then the cocycle \( \xi_M \) (as in Definition 4.1) is 0 in \( H^2(K,C_m) \) if and only if \( M \) descends.

Proof. Combine Propositions 4.4 and 4.5. \( \square \)

Remark 4.7. We did not have to assume that \( L \to \text{End}_{C_L}(M) \) was an isomorphism for a cocycle to exist. A necessary assumption is that there exists a collection \( \{ c_g \} \) of isomorphisms such that \( \xi_{M,c}(g,h) \in L^* \) for all \( g, h \in G \). The key is that \( H^2 \) exists as long as the coefficients are abelian. Note, however, that in this level of generality there is no guarantee that cohomology class is unique: it depends very much on the choice of the isomorphisms \( \{ c_g \} \). Therefore, this technique will not be adequate to prove that objects do not descend; we can only prove than an object does descend by finding a collection \( \{ c_g \} \) whose associated \( \xi_c \) is a coboundary.
Remark 4.8. Note that if $\mathcal{C}_R$ is the category of real representations of a compact group and $\mathcal{C}_C$ is the complexification, namely the category of complex representations of a compact group, then this 2-cocycle has a more classical name: “Frobenius-Schur Indicator”. It tests whether an irreducible complex representation of a compact group with real character can be defined over $\mathbb{R}$. If not, the representation is called quaternionic.

Now, suppose $\mathcal{C}$ is a $K$-linear rigid abelian $\otimes$ category. If $M \in \text{Ob}(\mathcal{C}_L)$ such that $^gM \cong M$ for all $g \in G$ and $L \to \text{End}_{\mathcal{C}_L}(M)$ is an isomorphism, then the same is true for $M^\ast$. Moreover, choosing $\{c_g\}$ for $M$ gives the natural choice of $\{(c_g^{-1})\}$ for $M^\ast$ so $\xi_M^{-1} = \xi_{M^\ast}$.

In general, if $M$ and $N$ are as above with choices of isomorphisms $\{c_g : M \to ^gM\}$ and $\{d_g : N \to ^gN\}$ with associated cohomology classes $\xi_M$ and $\xi_N$ respectively, then we can cook up a cohomology class to possibly detect whether $M \otimes N$ descends, $\xi_M\xi_N$, using the isomorphism $c_g \otimes d_g : M \otimes N \to ^gM \otimes ^gN \cong ^g(M \otimes N)$. This is interesting because in general $M \otimes N$ might have endomorphism algebra larger than $L$; in particular, we weren’t guaranteed the existence of a cohomology class $\xi_{M\otimes N}$, as discussed in Remark 4.7.

Lemma 4.9. Let $\mathcal{C}$ be a $K$-linear rigid abelian $\otimes$ category, $L/K$ a finite Galois extension with group $G$, and $\mathcal{C}_L$ the base-changed category. Let $M \in \text{Ob}(\mathcal{C}_L)$ have endomorphism algebra $L$ and suppose $^gM \cong M$ for all $g \in G$. Then $\text{End}(M) \cong M \otimes M^\ast$ descends to $\mathcal{C}_K$.

Proof. As noted above, if $\xi$ is the cocycle associated to $M$, then $\xi^{-1}$ is the cocycle associated to $M^\ast$. Then $1 = \xi\xi^{-1}$ is a cocycle associated to $M \otimes M^\ast$, whence it descends. \hfill $\square$

Remark 4.10. A related classical fact: let $V$ be a finite dimensional complex representation $V$ of a compact group $G$. Then the representation $\text{End}(V) \cong V \otimes V^\ast$ is defined over $\mathbb{R}$.

We now give two criteria for descent. Though the second is strictly more general than the first, the hypotheses are more complicated and we found it helpful to separate the two.

Lemma 4.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a $K$-linear functor between abelian $K$-linear categories. Let $L/K$ be a finite Galois extension with group $G$ and let $F_L : \mathcal{C}_L \to \mathcal{D}_L$ be the base-changed functor. Let $M \in \text{Ob}(\mathcal{C}_L)$ be an object such that $^gM$ is isomorphic to $M$ for $g \in G$ and $\text{End}_{\mathcal{C}_L}(M) \cong L$. Suppose $\text{End}_{\mathcal{D}_L}(F_L(M)) \cong L$. Then $M$ descends to $\mathcal{C}$ if and only if $F_L(M)$ descends to $\mathcal{D}$.

Proof. The natural map $\text{Hom}_{\mathcal{C}_L}(M, ^gM) \to \text{Hom}_{\mathcal{D}_L}(F_L(M), ^gF_L(M))$ is an isomorphism. Hence choosing isomorphisms $\{d_g : F_L(M) \to ^gF_L(M) \cong F_L(^gM)\}$ is the same as choosing isomorphisms $\{c_g : M \to ^gM\}$. Therefore the cocycles $\xi_{F_L(M)}$ and $\xi_M$ are the same. \hfill $\square$

Lemma 4.12. Let $F : \mathcal{C} \to \mathcal{D}$ be a $K$-linear functor between $K$-linear abelian categories and let $L/K$ be a finite Galois extension with group $G$. Let $F_L : \mathcal{C}_L \to \mathcal{D}_L$ be the base-changed functor. Suppose $M \in \text{Ob}(\mathcal{C}_L)$ with $L \cong \text{End}_{\mathcal{C}_L}(M)$ and $^gM \cong M$ for all $g \in G$. Further suppose $F_L(M) \cong N_1 \oplus N_2$ satisfying the following two conditions:

- $L \cong \text{End}_{\mathcal{D}_L}(N_1)$
- there is no nonzero morphism $N_1 \to ^gN_2$ in $\mathcal{D}_L$ for any $g \in G$.

Then $M$ (and $N_2$) descend if and only if $N_1$ descends.

Proof. The composition

$$\text{Hom}_{\mathcal{C}_L}(M, ^gM) \to \text{Hom}_{\mathcal{D}_L}(F_L(M), ^gF_L(M)) \cong \text{Hom}_{\mathcal{D}_L}(N_1 \oplus N_2, ^gN_1 \oplus ^gN_2) \to \text{Hom}_{\mathcal{D}_L}(N_1, ^gN_1)$$

is a homomorphism of $L$-vector spaces. In fact, the map is nonzero because an isomorphism $c_g : M \to ^gM$ is sent to the isomorphism $F_L(c_g)$. By the second assumption, this projects to an isomorphism $N_1 \to ^gN_1$. By the first assumption on $N_1$, the total composition is an isomorphism. Therefore a collection $\{n_g : N_1 \to ^gN_1\}$ is canonically the same as a collection $\{m_g : M \to ^gM\}$ and thus we have $\xi_M = \xi_{N_1}$. \hfill $\square$
We now examine the relation between the rank of $M$ and the order of its induced Brauer class $[\xi_M] \in H^2(K, G_m)$ when $C$ is assumed to be a $K$-linear Tannakian category. Recall that Tannakian categories are not necessarily neutral, i.e., they do not always admit a fiber functor to $\text{Vect}_K$. For the remainder of this section, $K$ is supposed to be a field of characteristic 0. Recall that Tannakian categories have a natural notion of rank [De07]. If $P$ is an object of rank 1, there is a natural diagram

$$\text{End}(P) \cong P \otimes P^* \cong K$$

where the top arrow is evaluation (i.e. the trace) and the bottom arrow comes from the $K$-vector-space structure of $\text{End}(P)$. As $P \otimes P^*$ has rank 1, these two maps identify $\text{End}(P)$ isomorphically with $K$.

**Proposition 4.13.** Let $K$ be a field of characteristic 0 and let $C$ be a neutral $K$-linear Tannakian category. Let $L/K$ be a finite Galois extension and let $C_L$ be the base-changed category. Let $P \in \text{Ob}(C_L)$ be a rank-1 object. If $g_P \cong P$ for all $g \in G$ then $P$ descends.

**Proof.** Let $F : C \to \text{Vect}_K$ be a fiber functor and denote by $F_L$ the base-changed fiber functor. By definition, $P$ being rank 1 means that $F_L(P)$ is a rank 1 $L$-vector space, so $L \cong \text{End}(F_L(P)) \cong \text{End}(P)$. All vector spaces descend, so by Lemma 4.11, $P$ descends as well. □

**Lemma 4.14.** Let $K$ be a field of characteristic 0, let $C$ be a $K$-linear Tannakian category, and let $L/K$ be a finite Galois extension. Suppose $M \in \text{Ob}(C_L)$ has rank $r$, $gM \cong M$ for all $g \in G$, and $L \cong \text{End}_C M$. If $\bigwedge M$ descends (which is automatically satisfied if $C$ is neutral by Proposition 4.13), then $\xi_M$ is $r$-torsion in $H^2(G, L^*)$. In particular, there exists a degree $r$ extension of $K$ over which $M$ is defined.

**Proof.** Pick $\{c_g : M \to gM\}$ giving the cohomology class $\xi_M$. The isomorphisms

$$\{c_g^\otimes r : M^\otimes r \to (gM)^\otimes r \cong g(M^\otimes r)\}$$

preserve the space of anti-symmetric tensors and restrict to give isomorphisms

$$c_g^\otimes r : \bigwedge^r M \to g \bigwedge^r M$$

The cohomology class associated to $c_g^\otimes r$ is $\xi^r$, and this cohomology class is unique because $L \cong \text{End}(\bigwedge M)$ as $\bigwedge M$ is a rank 1 object. As we assumed $\bigwedge M$ descends, we deduce that $\xi^r = 0 \in H^2(G, L^*)$. □

## 5. $F$-crystals

In this section, we set out our conventions about $F$-crystals and $F$-isocrystals. We further recall several results which are surely well-known but which we could not find documented in the literature. We recommend that the reader skip this section and refer to it when necessary. For a meta-reference, see [Ked16].

Let $X/k$ be a smooth scheme of finite type over a perfect field $k$. Berthelot has defined the absolute crystalline site on $X$: for a “modern” reference, see [Sta18, TAG 0715]. (We implicitly take the crystalline site with respect to $W(k)$ without further comment; in other words, in the formulation of the Stacks Project, $S = \text{Spec}(W(k))$ with the canonical PD structure.) Let $\text{Crys}(X)$ be the category of crystals in finite coherent $\mathcal{O}_{X/W(k)}$-modules. By functoriality of the crystalline topos, the absolute Frobenius $\text{Frob}_X : X \to X$ gives a functor $\text{Frob}_X^* : \text{Crys}(X) \to \text{Crys}(X)$.

**Notation 5.1.** Let $X/k$ be a smooth scheme over a perfect field.

- An $F$-crystal in finite, locally free modules on $X$ is a pair $(M, F)$ where $M$ is a crystal in finite, locally free modules and $F : \text{Frob}_X^* M \to M$ is an isogeny. The $\mathbb{Z}_p$-linear category of $F$-crystals in finite, locally free modules is denoted as $F\text{-Crys}(X)$.
- A Dieudonné crystal on $X$ is a triple $(M, F, V)$ where $(M, F)$ is an $F$-crystal in finite, locally free modules and $V : M \to \text{Frob}_X^* M$ is an isogeny such that $F \circ V = p$ and $V \circ F = p$.
- The category $\textbf{F-Isoc}(X)$ denotes the $\mathbb{Q}_p$-linear Tannakian category of $F$-isocrystals. This is the same as the isogeny category of $F$-crystals in finite, coherent modules.
• The category $\mathbf{F-IsoC}(X)$ denotes the $\mathbb{Q}_p$-linear Tannakian category of overconvergent $F$-isocrystals.

We record the following elementary lemma, which provides an explicit description of $\mathbf{F-IsoC}(k)_L$ with $L/\mathbb{Q}_p$ an algebraic extension.

**Proposition 5.2.** Let $k$ be a perfect field and let $L/\mathbb{Q}_p$ be an algebraic extension. Let $K(k) := \text{Frac}(W(k))$ denote the Fraction field of ring of Witt vectors and let $\sigma$ denote the canonical lift of Frobenius to $K(k)$. Then the category $\mathbf{F-IsoC}(k)_L$ is equivalent to the category of pairs $(V, F)$ where $V$ is a finite free module over $K(k) \otimes_{\mathbb{Q}_p} L$ and $F : V \to V$ is a $\sigma \otimes 1$-linear bijective map. The rank of $(V, F)$ is the rank of $V$ as a free $K(k) \otimes L$-module.

**Remark 5.3.** Note that $K(k) \otimes L$ is not necessarily a field; rather, it is a direct product of fields and $\sigma \otimes Id$ permutes the factors. It is a field if and only if $L$ and $K(k)$ are linearly disjoint over $\mathbb{Q}_p$. This occurs, for instance, if $L$ is totally ramified over $\mathbb{Q}_p$, or if the maximal finite subfield of $k$ is $\mathbb{F}_p$.

**Proof.** First of all, we immediately reduce to the case when $L/\mathbb{Q}_p$ is a finite extension. By definition, an object $((V', F'), f)$ of $\mathbf{F-IsoC}(k)_L$ consists of $(V', F')$ an $F$-isocrystal on $k$ and a $\mathbb{Q}_p$-algebra homomorphism $f : L \to \text{End}_{\mathbf{F-IsoC}(k)}(V', F')$. Recall that $V'$ is a finite dimensional $K(k)$ vector space and $F'$ is a $\sigma$-linear bijective map. This gives $V'$ the structure of a finite module (not a priori free) over $K(k) \otimes_{\mathbb{Q}_p} L$. The bijection $F'$ commutes with the action of $L$, hence $F'$ is $\sigma \otimes 1$-linear. We need only prove that $V'$ is a free $K(k) \otimes_{\mathbb{Q}_p} L$-module.

Let $L^o$ be the maximal unramified subfield of $L$ and let $M$ be the maximal subfield of $L^o$ contained in $K(k)$. This notion is unambiguous: $M$ is the unramified extension of $\mathbb{Q}_p$ with residue field the intersection of the maximal finite subfield of $k$ and the residue field of the local field $L$. Note that $L$ and $K(k)$ are linearly disjoint over $M$, so $K(k) \otimes_M L$ is a field. Let $r$ be the degree of the extension $M/\mathbb{Q}_p$. Then $K(k) \otimes_{\mathbb{Q}_p} L$ is the direct product $\prod_{i=1}^r (K(k) \otimes_M L)$, and $\sigma \otimes 1$ permutes the factors transitively. Because of this direct product decomposition, $V'$ can be written as $\prod_{i=1}^r V'_i$ with each $V'_i$ a $K(k) \otimes_{\mathbb{Q}_p} L$-vector space. As $F'$ is $\sigma \otimes 1$ linear and bijective, $F'$ transitively permutes the factors $V'_i$ and hence the dimension of each $V'_i$ as a $(K(k) \otimes_M L)$ vector space is the same. This implies that $V'$ is a free $K(k) \otimes_{\mathbb{Q}_p} L$-module.

**Definition 5.4.** Fix once and for all a compatible family $(p^\frac{\tau}{r}) \in \mathbb{Q}_p$ of roots of $p$. Let $\overline{\mathbb{Q}}_p(-\frac{1}{r}, *p^\frac{\tau}{r})$ be the following rank 1 object in $\mathbf{F-IsoC}(\mathbb{F}_p)_{\overline{\mathbb{Q}}_p}$ (using the description furnished by Proposition 5.2)

$$\overline{\mathbb{Q}}_p < v >, *p^\frac{\tau}{r}$$

where $v$ is a basis vector. Abusing notation, given any perfect field $k$ of characteristic $p$, we similarly denote the pullback to $k$ by $\overline{\mathbb{Q}}_p(-\frac{1}{r}, *p^\frac{\tau}{r})$. In terms of Proposition 5.2, it is given by the rank 1 module $K(k) \otimes_{\mathbb{Q}_p} < v >$ with $F(v) = p^\tau v$ and extended $\sigma \otimes 1$-linearly.

**Corollary 5.5.** Let $M = (V, F)$ be a rank 1 object of $\mathbf{F-IsoC}(k)_L$ with unique slope $\lambda = \frac{\tau}{r}$. Then $r \mid \text{deg}[L : \mathbb{Q}_p]$.

**Proof.** Consider the object $M' = \text{Res}^L_{\mathbb{Q}_p} M$. This is an isoclinic $F$-isocrystal on $k$ of rank $\text{deg}[L : \mathbb{Q}_p]$ with slope $\lambda$. We may suppose $k$ is algebraically closed; then by the Dieudonné-Manin decomposition, $M'$ is isomorphic to the direct sum of several copies of the unique simple $F$-isocrystal $E^\lambda$ on $k$ with slope $\lambda$. The rank of $E^\lambda$ is $r$, so the rank of $M'$ is divisible by $r$.

We now specialize our discussion to $F$-isocrystals over finite fields. For the remainder of this section, $\mathbb{F}_q$ is a finite field of $q$ elements, $\mathbb{Z}_q := W(\mathbb{F}_q)$, and $\mathbb{Q}_q := \mathbb{Z}_q[1/p]$. As usual, $\sigma$ denotes a lift of the absolute Frobenius. When we use the phrase $p$-adic valuation, it is always normalized so that $v_p(p) = 1$.

**Proposition 5.6.** Let $(V, F)$ be an $F$-isocrystal on $\mathbb{F}_q$ with coefficients in $L$ a $p$-adic local field. Then $F^d$ is a $\mathbb{Q}_q \otimes L$-linear endomorphism of $V$. Let $P_F(t) \in \mathbb{Q}_q \otimes L[t]$ be given as

$$P_F(t) := \det(1 - F^d t)|_V$$

Then $P_F(t) \in L[t]$. 

Proof. Let \( \tilde{P} \) be the characteristic polynomial of \( F^d \). It is equivalent to prove \( \tilde{P} \in L[t] \). The ring automorphism \( \sigma \otimes 1 \) has order \( d \) so \( F^d \) is a linear endomorphism on the free \( \mathbb{Q}_q \otimes L \)-module \( V \). The polynomial \( \tilde{P} \) a priori has coefficients in \( \mathbb{Q}_q \otimes L \), so we must show that the coefficients of \( \tilde{P}(t) \) are invariant under \( \sigma \otimes 1 \). Recall that the coefficients of the characteristic polynomial of an operator \( A \) are, up to sign, the traces of the exterior powers of \( A \). As there is a notion of \( \mathcal{A} \) for \( F \)-isocrystals it is enough to show that trace\((F^d)\) is invariant under \( \sigma \otimes 1 \).

To do this, pick a \( \mathbb{Q}_q \otimes L \) basis \( \{v_i\} \) of \( V \). Let \( S := (s_{ij}) \) be the “matrix” of \( F \) in this basis, i.e. \( F(v_i) = \sum_j s_{ij}v_j \). Then an easy computation shows that the matrix of \( F^d \) in this basis is given by

\[
(\sigma^{d-1}1)(\sigma^{d-2}1)\cdots(\sigma1)(S)
\]

Then \( \sigma^{\otimes 1} \text{trace}(F^d) = \text{trace}(\sigma^{\otimes 1}(F^d)) = \text{trace}((S)(\sigma^{d-1}1)\cdots(\sigma1)(S)) = \text{trace}(F^d) \) because \( \text{trace}(AB) = \text{trace}(BA) \).

\[\square\]

Proposition 5.7. Let \((V, F)\) be an \( F \)-isocrystal on \( \mathbb{F}_q \) with coefficients in \( L \). Then \( F^d \) is a \( \mathbb{Q}_q \otimes L \)-linear endomorphism of \( V \) with characteristic polynomial \( P_{F^d}(t) \in L[t] \). The slopes of \((V, F)\) are \( \frac{1}{d} \) times the \( p \)-adic valuations of the roots of \( P_{F^d}(t) \).

Proof. This follows immediately from the fact that the diagram

\[
\bigoplus_{\lambda} \text{F-Isoc}(k)_{L}^{\lambda} \cong \text{F-Isoc}(k) \cong \bigoplus_{\lambda} \text{F-Isoc}(k)^{\lambda}_{L}
\]

respects the slope decomposition together with the remark after [Kat79, Lemma 1.3.4].

\[\square\]

We now classify rank 1 objects \((V, F) \in \text{Ob F-Isoc}(\mathbb{F}_q)_L\), again using the explicit description in Proposition 5.2. As discussed above, the eigenvalue of \( F^d \) is in \( L \). The slogan of Proposition 5.8 is that this eigenvalue determines \((V, F)\) up to isomorphism. This allows us to classify semi-simple \( F \)-isocrystals on \( \mathbb{F}_q \).

Let \( v \) be a free generator of \( V \) over \( \mathbb{Q}_q \otimes \mathbb{Q}_p \). Then \( F(v) = \lambda v \) with \( \lambda \in (\mathbb{Q}_q \otimes \mathbb{Q}_p)^* \) and \( F^d(v) = (Nm_{\mathbb{Q}_q \otimes \mathbb{Q}_p/L}(\lambda)v \) where the norm is taken with respect to the cyclic Galois morphism \( L \to \mathbb{Q}_q \otimes L \). Therefore, any \( \lambda \in L \) that is in the image of the norm map \( Nm : (\mathbb{Q}_q \otimes L)^* \to L^* \) can be realized as the eigenvalue of \( F^d \). To prove that \( F^d \) uniquely determines a rank 1 \( F \)-isocrystal, we need only prove that there is a unique rank 1 \( F \)-isocrystal with \( F^d \) the identity map.

Proposition 5.8. Let \((V, F)\) be a rank 1 \( F \)-isocrystal on \( \mathbb{F}_q \) with coefficients in \( L \). Suppose \( F^d \) is the identity map. Then \((V, F)\) is isomorphic to the trivial \( F \)-isocrystal, i.e. there is a basis vector \( v \in V \) such that \( F(v) = v \).

Proof. Suppose \( F(v) = \lambda v \) where \( \lambda \in (\mathbb{Q}_q \otimes L)^* \). Then \( F^d(v) = Nm_{\mathbb{Q}_q \otimes \mathbb{Q}_p/L}(\lambda)v \), where the Norm map is defined with respect to the Galois morphism of algebras \( L \to \mathbb{Q} \otimes L \). This Galois group is cyclic, generated by \( \sigma \otimes 1 \). By assumption, we have \( Nm_{\mathbb{Q}_q \otimes \mathbb{Q}_p/L}(\lambda) = 1 \). Now, \( F(\lambda v) = (\sigma^{\otimes 1} \lambda)v = \lambda v \) and we want to find an \( a \in (\mathbb{Q}_q \otimes L)^* \) such that \( F(av) = av \), i.e. \( \frac{1}{d} \cdot \frac{\lambda}{Nm_{\mathbb{Q}_q \otimes \mathbb{Q}_p/L}(\lambda)} = \lambda \) where \( \lambda \) has norm 1. This is guaranteed by Hilbert’s Theorem 90 for the cyclic morphism of algebras \( L \to \mathbb{Q} \otimes L \).

\[\square\]

Corollary 5.9. Let \( M = (V, F) \) be a rank-1 \( F \)-isocrystal on \( \mathbb{F}_q \) with coefficients in \( L \). If the eigenvalue of \( F^d \) lives in finite index subfield \( K \subset L \) and is a norm in \( K \) with regards to the algebra homomorphism \( K \to \mathbb{Q}_q \otimes \mathbb{Q}_p \), then \( M \) descends to \( K \).

Proof. Let the eigenvalue of \( F^d \) be \( \lambda \) and let \( a \in \mathbb{Q}_q \otimes K \) have norm \( \lambda \). Consider the rank 1 object \( E \) of \( \text{F-Isoc}(\mathbb{F}_q)_K \) given on a basis element \( e \) by \( F(e) = a^{-1}e \). Then \( E_L \otimes M \) has \( F^d \) being the identity map. Proposition 5.8 implies that \( E_L \otimes M \) is the trivial \( F \)-isocrystal, i.e. that \( M \cong (E_L)^* \otimes (E^*)_L \). Therefore \( M \) descends as desired.

\[\square\]

Example 5.10. Consider the object \( \mathbb{F}_p(-\frac{1}{2}) \) of \( \text{F-Isoc}(\mathbb{F}_p)_\mathbb{Q}_p \). The eigenvalue of \( F^2 \) is \( \rho \in \mathbb{Q}_p \), but \( \rho \) is not in the image of the norm map and hence the object does not descend to an object of \( \text{F-Isoc}(\mathbb{F}_p) \).

One can also see this by virtue of the fact that no non-integral fraction can occur as the slope of a rank 1 object of \( \text{F-Isoc}(k) \). Note that the object does in fact descend to \( \text{F-Isoc}(\mathbb{F}_p)_\mathbb{Q}_p(\sqrt{\rho}) \).
Remark 5.11. Consider $\mathbb{T}_p(-\frac{a}{2})$ as an object of $\mathbf{F-Isoc}(\mathbb{F}_p)_{\mathbb{T}_p}$ where $\frac{a}{2}$ is in lowest terms. It is isomorphic to its Galois twists by the group $\text{Gal}(\mathbb{T}_p/\mathbb{Q}_p)$ if and only if $b/d$ where $q = p^d$. Indeed, the isomorphism class of this $F$-isocrystal is determined by the eigenvalue of $F^d$, which is $p^{\frac{ad}{d}}$. This is in $\mathbb{Q}_p$ if and only if $b \mid d$.

Corollary 5.9 poses a natural question. Let $K$ be a $p$-adic local field and let $q = p^d$. Which elements of $K$ are in the image of the norm map?

$$\text{Norm}: (\mathbb{Q}_q \otimes K)^* \to K^*$$

If $K = \mathbb{Q}_p$, then $K^* \cong \mathbb{Z}_p^*$, and the image of the norm map is exactly $\mathbb{Z}_p^* \otimes \mathbb{Z}_p^*$. More generally, if $K/\mathbb{Q}_p$ is a totally ramified extension with uniformizer $\varpi$, then $K^* \cong \mathbb{Z}_p^* \otimes \mathcal{O}_K^*$. Let $q \otimes K$ is a field that is unramified over $K$, and the image of the norm map is exactly $\mathbb{Z}_p^* \otimes \mathcal{O}_K^*$. On the other extreme, if $K \cong \mathbb{Q}_q$, then the following proposition shows that the norm map is surjective.

**Proposition 5.12.** Let $L/K$ be a cyclic extension of fields, with $\text{Gal}(L/K)$ generated by $g$ and of order $n$. Consider the induced Galois morphism of algebras: $L \to L \otimes_K L$. The image of the norm map for this extension is surjective.

**Proof.** $L \otimes_K L \cong \prod_{x \in G} L$, where the $L$-algebra structure is given by the first (identity) factor. Then $g$ acts by cyclically shifting the factors and the norm of an element $(\ldots, l_x, \ldots)$ is just $\prod_{x \in G} l_x$. Therefore the norm map is surjective. $\square$

**Example 5.13.** We have the following strange consequence. Consider the object $\mathbb{T}_p(-\frac{1}{2})$ of $\mathbf{F-Isoc}(\mathbb{F}_p)_{\mathbb{T}_p}$.

It descends to $\mathbf{F-Isoc}(\mathbb{F}_p^2)_{\mathbb{T}_p}$. $F^2$ has unique eigenvalue $p$, and $p$ is a norm for the algebra homomorphism $\mathbb{Q}_p^2 \hookrightarrow \mathbb{Q}_p^2 \otimes \mathbb{Q}_p^2$ by Proposition 5.12, so we may apply Corollary 5.9.

In general the image of the norm map is rather complicated to classify. However, in our case we can say the following.

**Lemma 5.14.** Let $K$ be a $p$-adic local field and let $q = p^d$. Then $\mathcal{O}_K^*$ is in the image of the norm map $(\mathbb{Q}_q \otimes K)^* \to K^*$.

**Proof.** $\mathbb{Q}_q \otimes K \cong \prod K'$ where $K'$ is an unramified extension of $K$ and the norm of an element $\prod K'$ is the product of the individual norms of the components with respect to the unramified extension $K'/K$. On the other hand, the image of the norm map $K'^* \to K^*$ certainly contains $\mathcal{O}_K^*$ as $K'/K$ is unramified. $\square$

**Corollary 5.15.** Let $(V, F)$ be a rank-1 $F$-isocrystal on $\mathbb{F}_q$ with coefficients in $L$. If the eigenvalue of $F^q$ lives in finite index subfield $K \subset L$ and $(V, F)$ has slope 0, then $(V, F)$ descends to $K$.

**Proof.** Apply Corollary 5.9 and Lemma 5.14. $\square$

**Remark 5.16.** The slope 0 subcategory part of $\mathbf{F-Isoc}(k)$ is a neutral Tannakian category by [Cre87, Theorem 2.1]. Therefore one may alternatively use Proposition 4.13 to prove Corollary 5.15.

It is an easy exercise to check that the irreducible objects of $\mathbf{F-Isoc}(\mathbb{F}_q)_{\mathbb{T}_p}$ have rank 1. Let $L$ be a $p$-adic local field. As semi-simple and absolutely semi-simple objects of $\mathbf{F-Isoc}(\mathbb{F}_q)_L$ coincide by [Del14, Lemma 5.2], it follows from Proposition 5.8 that a semi-simple object of $\mathbf{F-Isoc}(\mathbb{F}_q)_L$ is determined up to isomorphism by $P_F(t)$.

**Proposition 5.17.** Let $L/K$ be a finite Galois extension with group $G$ with $K$ a $p$-adic local field. Let $(V, F)$ be an $F$-isocrystal on $\mathbb{F}_q$ with coefficients in $L$ and denote $(\sigma V, \sigma F) := g(V, F)$. Then

$$P_{\sigma F}(t) = g P_F(t)$$

**Proof.** First of all, $V$ is a finite free $\mathbb{Q}_q \otimes L$ module. Let $g \in G$ and consider the object $g(V, F)$. The underlying sets $V$ and $gV$ may be naturally identified, and if $v \in V$, we write $g v$ for the corresponding element of $gV$; here $l_g(v) = (g^{-1}(l)(v))$. Pick a free basis $\{v_i\}$ of $V$ and let the “matrix” of $F$ in this basis be $S$. Then the “matrix” of $\sigma F$ in the basis $\{g v_i\}$ is $g S$. Moreover, the actions of $G$ and $\sigma$ commute, so $(\sigma F)^d = g(F^d)$. Therefore, $P_{\sigma F}(t) = g P_F(t)$ as desired. $\square$
Corollary 5.18. Let \( L/L_0 \) be a Galois extension of \( p \)-adic local fields with group \( G \). Let \( E = (V, F) \) be a semi-simple object of \( \text{F-Isoc}(\mathbb{F}_q)_L \). Then \( P_F(t) \in L_0[t] \) implies that \( gE \cong E \) for all \( g \in G \).

Proof. Immediate from Proposition 5.17 and the fact that a semi-simple \( F \)-isocrystal is determined up to isomorphism by its characteristic polynomials.

6. Coefficient objects, Compatible Systems and Companions

For a more comprehensive introduction to this section, see [Ked16, Ked18].

Definition 6.1. [Ked18] Let \( X/\mathbb{F}_q \) be a smooth, geometrically connected variety. The category \( \text{Weil}(X) \) denotes the \( \mathbb{Q}_l \)-linear Tannakian category of lisse Weil sheaves on \( X \). A coefficient object is an object either of \( \text{Weil}(X)_{\bar{\mathbb{Q}}_l} \) or \( \text{F-Isoc}^{\dagger}(X)_{\bar{\mathbb{Q}}_p} \). We informally call the former the étale case and the latter the crystalline case. We say that a coefficient object has coefficients in \( K \) if may be descended to the appropriate category with coefficients in \( K \).

Given a coefficient object \( F \) and a closed point \( x \) of \( X \), the notation \( P_x(F, t) \) refers to the reverse characteristic polynomial of Frobenius of \( F \) at \( x \).

Definition 6.2. Let \( F \) be a coefficient object. We say \( F \) is algebraic if \( P_x(F, t) \in \mathbb{Q}[t] \) for all closed points \( x \in |X| \). Let \( E \) be a number field. We say \( F \) is \( E \)-algebraic if \( P_x(F, t) \in E[t] \) for all closed points \( x \in |X| \).

Let \( X \) be a normal geometrically connected variety over \( \mathbb{F}_q \). Let \( \mathcal{E} \) and \( \mathcal{E}' \) be semi-simple algebraic coefficient objects on \( X \) with coefficient fields \( K \) and \( K' \) respectively. Fix an isomorphism \( \iota: \overline{K} \rightarrow \overline{K}' \). We say \( \mathcal{E} \) and \( \mathcal{E}' \) are \( \iota \)-companions if \( P_x^{\epsilon}(\mathcal{E}, t) = P_x^{\epsilon}(\mathcal{E}', t) \) for all closed points \( x \) and \( X \).

Definition 6.3. Let \( X \) be a normal geometrically connected variety over \( \mathbb{F}_q \) and let \( E \) be a number field. Then an \( E \)-compatible system is a system of \( E_\lambda \)-coefficient objects \( (\mathcal{E}_\lambda)_{\lambda \mid p} \) over places \( \lambda \mid p \) of \( E \) such that for each closed point \( x \) of \( X \), we have

\[
P_x(\mathcal{E}_\lambda, t) \in E[t] \subset E_\lambda[t]
\]

and furthermore \( P_x(\mathcal{E}_\lambda, t) \) is independent of \( \lambda \). A complete \( E \)-compatible system \( (\mathcal{E}_\lambda) \) is an \( E \)-compatible system together with, for each \( \lambda \mid p \), an object

\[
\mathcal{E}_\lambda \in \text{F-Isoc}^{\dagger}(X)_{E_\lambda}
\]

such that for every place \( \lambda \) of \( E \) and for every closed point \( x \) of \( X \), \( P_x(\mathcal{E}_\lambda, t) \in E[t] \subset E_\lambda[t] \) is independent of \( \lambda \) and that \( (\mathcal{E}_\lambda) \) satisfies the following completeness condition:

**Condition.** Consider \( \mathcal{E}_\lambda \) as a \( \bar{\mathbb{Q}}_\lambda \)-coefficient object. Then for any \( \iota : \bar{\mathbb{Q}}_\lambda \rightarrow \bar{\mathbb{Q}}_{\lambda'} \), the \( \iota \)-companion to \( \mathcal{E}_\lambda \) is isomorphic to an element of \( (\mathcal{E}_\lambda) \).

Remark 6.4. The \( \iota \) in Definition 6.2 does not reflect the topology of \( \bar{\mathbb{Q}}_p \) or \( \bar{\mathbb{Q}}_l \); in particular, it need not be continuous. The completeness conditions require that “all possible companions exist”. Implicit in the definition is the result, due to Abe, Deligne, Esnault, Lafforgue, and Kedlaya that, for a given irreducible coefficient object \( \mathcal{E} \) and fixed place \( \lambda \), there are only finitely many \( \bar{\mathbb{Q}}_\lambda \)-companions to \( \mathcal{E} \).

The following fundamental result follows from the work on the Langlands correspondence due to Lafforgue and Abe [Laf02, Abe18a, Abe18b]

**Theorem 6.5.** (Abe, Lafforgue) Let \( C \) be a smooth curve over \( \mathbb{F}_q \).

1. Deligne’s companions conjecture [Del80, 1.2.10] is true for \( C \).
2. Let \( l \neq p \) be a prime. For every isomorphism \( \iota : \bar{\mathbb{Q}}_l \rightarrow \bar{\mathbb{Q}}_p \), there is a bijective correspondence

\[
\begin{align*}
\text{Local systems } \mathcal{L} \text{ on } C \text{ such that} & \quad \text{Overconvergent } F \text{-Isocrystals } \mathcal{E} \text{ on } C \text{ such that} \\
\bullet \mathcal{L} \text{ has coefficients in } \bar{\mathbb{Q}}_l & \quad \bullet \mathcal{E} \text{ has coefficients in } \bar{\mathbb{Q}}_p \\
\bullet \mathcal{L} \text{ is irreducible of rank } n & \quad \bullet \mathcal{E} \text{ is irreducible of rank } n \\
\bullet \mathcal{L} \text{ has finite determinant up to isomorphism} & \quad \bullet \mathcal{E} \text{ has finite determinant up to isomorphism}
\end{align*}
\]
depending on \(i\) such that \(\mathcal{L}\) and \(\mathcal{E}\) are \(i\)-compatible.

(3) Let \(\mathcal{E}\) be an irreducible coefficient object with finite order determinant. Then there exists a number field \(E\) such that \(\mathcal{E}\) is part of a complete \(E\)-compatible system.

### 7. F-isocrystals and \(l\)-adic local systems on Curves

Let \(C/F_q\) be a smooth connected curve and let \(\tau\) be a geometric point. Let \(\mathcal{L}\) be an irreducible \(l\)-adic local system on \(C\) with trivial determinant, which one may think of as a continuous representation \(\rho_1 : \pi_1(C, \tau) \to \text{SL}(n, \mathbb{Q}_l)).\) Theorem 6.5 tells us that we can find a number field \(E\) such that \(\rho_1\) fits into a complete \(E\)-compatible system.

**Example 7.1.** The number field \(E\) can in general be larger than the field extension of \(\mathbb{Q}\) generated by the coefficients of the characteristic polynomials of all Frobenius elements \(F_x\). For instance, let \(D\) be a non-split quaternion algebra over \(\mathbb{Q}\) that is split at \(\infty\) and let \(p\) be a prime where \(D\) splits. The Shimura curve \(X^D\) exists as a smooth complete stacky curve over \(\mathbb{F}_p\) and it admits a universal abelian surface \(f : \mathcal{A} \to X^D\) with multiplication by \(O_D\). It is an exercise to check that if \(l\) splits and only if \(R^1f_*\mathbb{Q}_l\) splits as \(\mathcal{L} \oplus \mathcal{L}\). In particular, even though all Frobenius traces are in \(\mathbb{Q}\), not all of the \(l\)-adic companions can be defined over \(\mathbb{Q}_l\). In other words, they do not form a \(\mathbb{Q}\)-compatible system.

Using our simple descent machinery, we construct criteria for the field-of-definition of a coefficient object to be as small as possible.

**Lemma 7.2.** Let \(L/\mathbb{Q}_p\) be a finite extension, \(C\) a smooth curve over \(\mathbb{F}_q\), and \(\mathcal{E}\) an absolutely irreducible overconvergent \(F\)-isocrystal on \(C\) of rank \(n\) with coefficients in \(L\). Suppose that there exists a closed point \(i : x \to C\) such that \(i^*\mathcal{E}\) has \(0\) as a slope with multiplicity \(1\). Suppose further that for all closed points \(x \in |C|, P_{\mathcal{E}}(x, t) \in L_0[t]\) for some \(p\)-adic subfield \(L_0 \subset L\). Then \(\mathcal{E}\) can be descended to an \(F\)-isocrystal with coefficients in \(L_0\).

**Proof.** By enlarging \(L\) we may suppose that \(L/L_0\) is Galois; let \(G = \text{Gal}(L/L_0)\). As \(\mathcal{E}\) is irreducible, for any \(g \in G, \mathcal{E} \cong g^*\mathcal{E}\) by Proposition 5.17 and [Abe18a, A.4.1]. Now let us use Lemma 4.12 for the restriction functor

\[i^* : \text{F-Isoc}^\dagger(C) \to \text{F-Isoc}(x)\]

Because \(0\) occurs as a slope with multiplicity \(1\) in \(i^*\mathcal{E}\) we can write \(i^*\mathcal{E} \cong N_1 \oplus N_2\) under the isoclinic decomposition. Here \(N_1\) has rank \(1\) and unique slope \(0\) while no slope of \(N_2\) is \(0\). There are no maps between \(N_1\) and any Galois twist of \(N_2\) because twisting an \(F\)-isocrystal does not change the slope. The endomorphism algebra of any rank-1 object in \(\text{F-Isoc}(\mathbb{F}_q)_L\) is \(L\). Now, \(\text{End}(\mathcal{E}) \cong L\) by Schur’s Lemma because \(\mathcal{E}\) is absolutely irreducible.

Finally, we must argue that \(N_1\) descends to \(L_0\). Let \(x = \text{Spec}(\mathbb{F}_p)\). The fact that the slope \(0\) occurs exactly once in \(i^*\mathcal{E}\) implies that the eigenvalue \(\alpha\) of \(F^d\) on \(N_1\) is an element of \(L_0\). Moreover, \(\alpha \in \mathcal{O}_{L_0}^\times\) because \(N_1\) is slope \(0\). By Corollary 5.15, \(N_1\) descends to \(L_0\). The hypotheses of Lemma 4.12 are all satisfied and we may conclude that \(\mathcal{E}\) (and \(N_2\)) descends to \(L_0\).

We remark that Lemma 7.2 is the \(p\)-adic analog of a proposition of Chin [Chi03, Proposition 7]. The following is a special case of [Kos17, Theorem 1.4]. For completeness, we give a short proof.

**Lemma 7.3.** Let \(C\) be a smooth curve over \(\mathbb{F}_q\), and \(\mathcal{E}\) an irreducible rank \(n\) overconvergent \(F\)-isocrystal on \(C\) with coefficients in \(\mathbb{Q}_p\) such that \(\mathcal{E}\) has trivial determinant. By Theorem 6.5 there is a number field \(E\) such that \(\mathcal{E}\) is part of a complete \(E\)-compatible system. In particular, for every \(\lambda\) there is a compatible overconvergent \(F\)-isocrystal \(\mathcal{E}_\lambda\) with coefficients in \(E_\lambda\).

Suppose that for all \(\lambda\) and for all closed points \(x \in |C|, i^*_{\mathcal{E}_\lambda}\) is an isoclinic \(F\)-isocrystal on \(x\). Then the representation has finite image: for instance, for every \(\lambda \not\mid p\), the associated \(\lambda\)-adic representation has finite image. Equivalently, the “motive” can be trivialized by a finite étale cover \(C' \to C\).

**Proof.** The eigenvalues of \(F_x\) are \(\lambda\)-adic units for all \(\lambda \not\mid p\) by [Ked18, Theorem 0.2.1]. On the other hand, for each \(\lambda\) and for every closed point \(x \in |C|, i^*_{\mathcal{E}_\lambda}\) being isoclinic and having trivial determinant implies the slopes of \(i^*_{\mathcal{E}_\lambda}\) are \(0\) and hence that the eigenvalues of \(F_x\) are \(\lambda\)-adic units. As eigenvalues
of $F_x$ are algebraic numbers, this implies that they are all roots of unity. Moreover, each of these roots of unity lives in a degree $n$ extension of $E$ and there are only finitely many roots of unity that live in such extensions: there are only finitely many roots of unity with fixed bounded degree over $\mathbb{Q}$. Therefore there are only finitely many eigenvalues of $F_x$ as $x$ ranges through the closed points of $C$.

Now, pick $\lambda \not| p$ and consider the associated representation $\rho_\lambda : \pi_1(C, \mathfrak{F}) \to SL(n, E_\lambda)$. By the above discussion, there exists some integer $k$ such that for every closed point $x \in |C|$, the generalized eigenvalues $\rho_\lambda(F_x)$ are $k^{th}$ roots of unity. But Frobenius elements are dense and $\rho_\lambda$ is a continuous homomorphism, so that the same is true for the entire image of $\rho_\lambda$. The image of $\rho_\lambda$ therefore only has finitely many traces.

Burnside proved that if $G \subset GL(n, \mathbb{C})$ has finitely many traces and the associated representation is irreducible, then $G$ is finite: see, for instance [Row08, 19.A.9]. Thus the entire image of $\rho_\lambda$ is finite. □

**Proposition 7.4.** Let $C$ be a smooth curve over $\mathbb{F}_q$ and let $\mathcal{L}$ be an absolutely irreducible rank 2 $l$-adic local system with trivial determinant, infinite image, and all Frobenius traces in a number field $E$. There exists a finite extension $F$ of $E$ such that $\mathcal{L}$ is a part of a complete $F$-compatible system. Not every crystalline companion $\delta_\lambda \in \text{F-Isoc}(C)_{E_\lambda}$ is everywhere isoclinic by Lemma 7.3. For each such not-everywhere-isoclinic $\delta_\lambda$ there exists

1. A positive rational number $\frac{r}{s}$
2. A finite field extension $\mathbb{F}_{q^t}/\mathbb{F}_q$ with $C'$ denoting the base change of $C$ to $\mathbb{F}_{q^t}$ ($q^t$ will be the least power of $q$ divisible by $p^r$)

such that the $F$-isocrystal $\mathcal{M} := \delta_\lambda \otimes \mathbb{Q}_p(-\frac{r}{s})$ on $C'$ descends to $E_\lambda$.

**Proof.** Theorem 6.5 imply that there is a finite extension $F/E$, without loss of generality Galois, such that $\mathcal{L}$ fits into a complete $F$-compatible system on $C$. As we assumed $\mathcal{L}$ had infinite image, Lemma 7.3 implies that there is a place $\lambda | p$ of $F$ together with an object $\delta_\lambda \in \text{F-Isoc}(C)_{E_\lambda}$ that is compatible with $\mathcal{L}$ and such that the general point is not isoclinic. We abuse notation and denote the restriction of $\lambda$ to $E$ by $\lambda$ again. The object $\delta_\lambda$ is isomorphic to its twists by $\text{Gal}(F_\lambda/E_\lambda)$ by [Abe18a, A.4.1] because $P_{\lambda x}((x, t) \in E[t]$ for all closed points $x \in X$. Pick a closed point $i : x \to C$ such that $i^*\delta_\lambda$ has slopes $(-\frac{r}{s}, \frac{s}{r})$. Let $q^t$ be the smallest power of $q$ that is divisible by $p^r$ and let $C'$ denote the base change of $C$ to $\mathbb{F}_{q^t}$.

Consider the twist $\mathcal{M} := \delta_\lambda \otimes \mathbb{Q}_p(-\frac{r}{s})$, thought of as an $F$-isocrystal on $C'$ with coefficients in $\mathbb{Q}_p$. We have enlarged $q$ to $q^t$ where $p^t | q^t$; Remark 5.11 says that $\mathbb{Q}_p(-\frac{r}{s})$, considered as object of $\text{F-Isoc}(\mathbb{F}_{q^t}/\mathbb{Q}_p)$, is isomorphic to all of its twists by $\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$. Therefore, $E_\mathcal{M}((x, t) \in E_\lambda[t]$ for all closed points $x$. It again follows from [Abe18a, A.4.1] that $\mathcal{M}$ is isomorphic to all of its Galois twists by $\text{Gal}(\mathbb{Q}_p/E_\lambda)$. At the point “$x$”, the slopes are now $(0, \frac{2r}{s})$. Apply Lemma 7.2 to descend $\mathcal{M}$ to the field of traces $E_\lambda$, as desired. □

**Remark 7.5.** The point of Proposition 7.4 is that the field of definition of $\mathcal{M}$ is $E_\lambda$, a completion of the field of traces. Proposition 7.4 has the following slogan: for every not-everywhere-isoclinic crystalline companion of $\mathcal{L}$, there exists a twist such that the Brauer obstruction vanishes. This is in contrast to the $l$-adic case, where there is no $à$ priori reason an $l$-adic Brauer obstruction should vanish.

We now specialize to the case where $\mathcal{L}$ is a rank 2 $l$-adic local system with trivial determinant, infinite image, and having all Frobenius traces in a number field $E$ where $p$ splits completely. Proposition 7.4 implies that, up to extension of the ground field $\mathbb{F}_q$, we can find an $F$-isocrystal $\delta$ with coefficients in $\mathbb{Q}_p$ that is compatible with $\mathcal{L}$ up to a twist and is not everywhere isoclinic. Moreover, by construction, there is a point $x$ such that the slopes of $\delta_x$ are $(0, \frac{2r}{s})$. On the one hand the slope of the determinant of $\delta_x$ is necessarily an integer because the coefficients of $\delta$ are $\mathbb{Q}_p$. Therefore $\frac{2r}{s}$ is a positive integer. On the other hand, [Laf11, Corollaire 2.2] implies that $\frac{2r}{s} \leq 1$, so $\frac{2r}{s} = \frac{1}{2}$ and $\det(\delta) \equiv \mathbb{Q}_p(-1)$. We record this analysis in the following important corollary.

**Corollary 7.6.** Let $C$ be a curve over $\mathbb{F}_q$ and let $\mathcal{L}$ be an absolutely irreducible rank 2 $l$-adic local system with trivial determinant, infinite image, and all Frobenius traces in $\mathbb{Q}$. Suppose $q$ is a square.
Then there exists a unique absolutely irreducible overconvergent $F$-isocrystal $\mathcal{E}$ with coefficients in $\mathbb{Q}_p$ that is compatible with $\mathcal{L} \otimes \mathbb{Q}_l(-\frac{1}{2})$. By construction, $\mathcal{E}$ is generically ordinary i.e. there exists a closed point $x \in |C|$ such that $\mathcal{E}_x$ has slopes $(0,1)$.

**Proof.** Only the uniqueness needs to be proved. As $q$ is a square, the character $\overline{\mathbb{Q}_l}(\frac{1}{2})$ in fact descends to a character $\mathbb{Q}_l(\frac{1}{2})$ (because $q$ is a quadratic residue mod $l$) and the coefficients of the characteristic polynomials of the Frobenius elements on $\mathcal{L} \otimes \mathbb{Q}_l(-\frac{1}{2})$ are all in $\mathbb{Q}$. By [Abel18a, A.4.1], any absolutely irreducible $F$-isocrystal on $C$ is uniquely determined by these characteristic polynomials as there is a unique embedding $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}_p}$. □

**Lemma 7.7.** Let $C$ be a complete curve over a perfect field $k$, and $\mathcal{E}$ an absolutely irreducible rank-2 $F$-isocrystal on $C$ with coefficients in $\mathbb{Q}_p$. Suppose $\mathcal{E}$ has determinant $\mathbb{Q}_p(-1)$. Then there is a closed point $x \in |C|$ such that $\mathcal{E}_x$ is isoclinic.

**Proof.** As we are assuming $\mathcal{E}$ is rank 2, has coefficients in $\mathbb{Q}_p$, and determinant $\mathbb{Q}_p(-1)$, if either of the slopes of $\mathcal{E}_x$ were non-integral, the slopes would have to be $(\frac{1}{2}, \frac{1}{2})$. If the slopes were integral, they are integers $(a,b)$ that sum to 1. On other hand, [Laf11, Corollaire 2.2] shows that in this case, the slopes must be $(0,1)$. Assume that $\mathcal{E}_x$ is never isoclinic. Then for every closed point $x$ of $|C|$, $\mathcal{E}_x$ has slopes $(0,1)$. The slope filtration [Kat79, 2.6.2] is therefore non-trivial, which contradicts the irreducibility of $\mathcal{E}$. □

Combining Lemma 7.7 with the $\mathbb{Q}_p$ companion $\mathcal{E}$ constructed in Corollary 7.6, we see that $\mathcal{E}$ is generically ordinary and has (finitely many) supersingular points. In particular, the slopes of $\mathcal{E}_x$ for $x \in |C|$ are either $(0,1)$ or $(\frac{1}{2}, \frac{1}{2})$.

**Corollary 7.8.** Let $C$ be a complete curve over $\mathbb{F}_q$ and let $\mathcal{L}$ be an absolutely irreducible rank 2 $l$-adic local system with trivial determinant, infinite image, and all Frobenius traces in $\mathbb{Q}$. There is a BT group $\mathcal{G}$ on $C$ with the following properties.

- $\mathcal{G}$ has height 2 and dimension 1.
- $\mathcal{G}$ has slopes $(0,1)$ and $(\frac{1}{2}, \frac{1}{2})$ (i.e., $\mathcal{G}$ is generically ordinary with supersingular points).
- The Dieudonné crystal $\mathbb{D}(\mathcal{G})$ is compatible with $\mathcal{L}(-\frac{1}{2})$.

$\mathcal{G}$ has the following weak uniqueness property: the $F$-isocrystal $\mathbb{D}(\mathcal{G})$ is unique.

**Proof.** First of all, we have constructed an absolutely irreducible $\mathcal{E} \in \mathbf{F-Isoc}(C)$ with determinant $\mathbb{Q}_p(-1)$ so that $\mathcal{E}$ is compatible with $\mathcal{L}(-1/2)$ and the slopes of $\mathcal{E}$ are in the interval $[0,1]$ in Corollary 7.6. It follows from [KP21, Lemma 5.8] that $\mathcal{E}$ underlies a (non-unique) Dieudonné crystal $\mathcal{M}$. (This argument is essentially due to Katz, see [Kat79, Theorem 2.6.1]. For another argument, due to Crew, see [Ked16, Remark 2.3].) Using [J95, Main Theorem], there exists a BT group $\mathcal{G}$ on $C$ so that $\mathbb{D}(\mathcal{G}) \cong \mathcal{M}$. Moreover, by Lemma 7.7, it follows that $\mathcal{E}$ and hence $\mathcal{G}$ has both ordinary and supersingular points.

The weak uniqueness comes from the following facts: an absolutely irreducible overconvergent $F$-isocrystal with trivial determinant is uniquely specified by Frobenius eigenvalues, $\mathbb{D}(\mathcal{G})$ is absolutely irreducible because it has both ordinary and supersingular points, and there is a unique embedding $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}_p}$. □

We emphasize that the BT group $\mathcal{G}$ constructed in Corollary 7.8 is not unique.

8. **Kodaira-Spencer**

This section discusses the deformation theory of Barsotti-Tate groups in order to refine Corollary 7.8. It will also be useful in the applications to Shimura curves. Throughout this section, we use the terms “Barsotti-Tate group”, “BT group”, and “$p$-divisible group” interchangeably. The main references are Illusie [II85], Xiang [Xia13], and de Jong [J95, Section 2.5].

Let $S$ be a smooth scheme over a perfect field $k$ of characteristic $p$. Given a BT group $\mathcal{G}$ over $S$ and a closed point $x \in |S|$, there is a map of formal schemes $u_x : S_x \to \text{Def}(\mathcal{G}_x)$ to the (equal characteristic) universal deformation space of $\mathcal{G}_x$. If $\mathcal{G}$ has dimension $d$ and codimension $c$, then the
dimension of the universal deformation space is $cd$ [Ill85, Corollary 4.8 (i)]. In particular, in the case $\mathcal{G}$ has height 2 and dimension 1, $\text{Def}(\mathcal{G})$ is a one-dimensional formal scheme, exactly as in the familiar elliptic modular case.

Let $\mathcal{G} \to S$ be a Barsotti-Tate group. Then $\mathbb{D}(\mathcal{G})$ is a Dieudonné crystal which we may evaluate on $S$ to obtain a vector bundle $\mathbb{D}(\mathcal{G})_S$ of rank $c + d = \text{ht}(\mathcal{G})$. Let be $\omega$ the Hodge bundle of $\mathcal{G}$ (equivalently, of $\mathcal{G}[p]$): if $e : S \to \mathcal{G}[p]$ is the identity section, then $\omega := e^*\Omega_{\mathcal{G}[p]/S}$. Let $\alpha$ be the dual of the Hodge bundle of the Serre dual $\mathcal{G}^t$. There is the Hodge filtration

$$0 \to \omega \to \mathbb{D}(\mathcal{G})_S \to \alpha \to 0$$

We remark that $\omega$ has rank $d$ and $\alpha$ has rank $c$. The Kodaira-Spencer map is as follows

$$KS : T_S \to \text{Hom}_{O_S}(\omega, \alpha) \cong \omega^* \otimes \alpha$$

Illusie [Ill85, A.2.3.6] proves that $KS_S$ is surjective if and only if $u_x$ is formally smooth, see also [dJ95, 2.5.5]. This motivates the following definition.

**Definition 8.1.** Let $S/k$ be a smooth scheme over a perfect field $k$ of characteristic $p$. Let $\mathcal{G} \to S$ be a Barsotti-Tate group and $x \in |X|$ a closed point. We say that $\mathcal{G}$ is **versally deformed at** $x$ if either of the following equivalent conditions hold

- The fiber at $x$ of $KS$, $KS_x : T_{S,x} \to (\omega^* \otimes \alpha)_x$, is surjective.
- The map $u_x : S^\wedge_{/x} \to \text{Def}(\mathcal{G})_x$ is formally smooth.

**Definition 8.2.** Let $S/k$ be a smooth scheme over a perfect field $k$ of characteristic $p$. Let $\mathcal{G} \to S$ be a Barsotti-Tate group. We say that $\mathcal{G}$ is **generically versally deformed** if, for every connected component $S_i$, there exists a closed point $x_i \in |S_i|$ such that $\mathcal{G}$ is versally deformed at $x_i$.

**Remark 8.3.** If $\mathcal{G} \to S$ is generically versally deformed, there exists a dense Zariski open $U \subset X$ such that $\mathcal{G}|_U \to U$ is **everywhere versally deformed**. This is because the condition “a map of vector bundles is surjective” is an open condition.

**Example 8.4.** Let us recall Igusa level structures, as in [Ulm90]. Let $Y(1) = M_{1,1}$. There is a universal elliptic curve $E \to Y(1)$. Let $\mathcal{G} = E[p^\infty]$ be the associated $p$-divisible group over $Y(1)$. Here, $\mathcal{G}$ is height 2, dimension 1, and everywhere versally deformed on $Y(1)$. Let $X$ be the cover of $Y(1)$ that trivializes the finite flat group scheme $\mathcal{G}[p][et]$ away from the supersingular locus of $Y(1)$. $X$ is branched exactly at the supersingular points. Pulling back $\mathcal{G}$ to $X$ yields a BT group that is generically versally deformed but not everywhere versally deformed on $X$.

**Example 8.5.** Let $S = \mathcal{A}_{2,1} \otimes \mathbb{F}_p$, the moduli of principally polarized abelian surfaces. Then the BT group of the universal abelian scheme, $\mathcal{G} \to S$, is nowhere versally deformed: the formal deformation space of a height 4 dimension 2 BT group is 4, whereas dim($S$) = 3. Serre-Tate theory relates this to the fact that most formal deformations of an abelian variety of dimension dim($A$) > 1 are not algebraizable.

**Lemma 8.6.** (Xia’s Frobenius Untwisting Lemma) Let $S$ be a smooth scheme over a perfect field $k$ of characteristic $p$. Let $\mathcal{G} \to S$ be a BT group of height 2 and dimension 1. Then $\mathcal{G} \to S$ has trivial KS map if and only if there exists a BT group $\mathcal{H}$ on $S$ such that $\mathcal{H}^{(p)} \cong \mathcal{G}$.

**Proof.** This is [Xia13, Theorem 6.1].

Xia’s Lemma 8.6 allows us to set up the following useful equivalence for characterizing when a height 2 dimension 1 BT group $\mathcal{G}$ is generically versally deformed.

**Lemma 8.7.** Let $C$ be a smooth curve over a perfect field $k$ of characteristic $p$. Let $\mathcal{G}$ be a height 2, dimension 1 BT group over $C$. Let $\eta$ be the generic point of $C$. Suppose $\mathcal{G}_\eta$ is ordinary. Then the following are equivalent

1. The KS map is 0.
2. There exists a BT group $\mathcal{H}$ on $C$ such that $\mathcal{H}^{(p)} \cong \mathcal{G}$. 
(3) There exists a finite flat subgroup scheme \( N \subset G \) over \( C \) such that \( N \) has order \( p \) and is generically étale.

(4) The connected-étale exact sequence \( G[p^\circ] \to G[p] \to G[p]^{\text{ét}} \) over the generic point splits.

**Proof.** The equivalence of (1) and (2) is Lemma 8.6. Now, let us assume (2). Then there is a Verschiebung map

\[
V_{\mathcal{H}} : \mathcal{G} \cong \mathcal{H}^{(p)} \to \mathcal{H}
\]

whose kernel is generically étale and has order \( p \) because we assumed \( \mathcal{G} \) (and hence \( \mathcal{H} \)) were generically ordinary. Therefore (3) is satisfied. Conversely, given (3), \( N \) is \( p \)-torsion [TO70]. Therefore we have a factorization:

\[
\begin{xy}
  0 \ar[r] & \mathcal{G} / N \ar[r] & \mathcal{G} \ar[r] & \mathcal{G} / N \ar[r] & 0
\end{xy}
\]

Set \( \mathcal{H} = \mathcal{G} / N \). As we assumed \( N \) was generically étale and \( \mathcal{G} \) was generically ordinary, the map \( \mathcal{G} \to \mathcal{G} / N \) may be identified with Verschiebung: \( \mathcal{H}^{(p)} \to \mathcal{H} \). In particular, \( \mathcal{H}^{(p)} \cong \mathcal{G} \) as desired.

Let us again assume (3). Then \( N_\eta \subset G_\eta[p] \) projects isomorphically onto \( G_\eta[p]^{\text{ét}} \). Therefore the connected-étale sequence over the generic point splits. To prove the converse, simply take the Zariski closure of the section to \( G[p] \to G[p]^{\text{ét}} \) inside of \( G[p] \) to get \( N \) (this will be a finite flat group scheme because \( C \) is a smooth curve.) \( \square \)

**Lemma 8.8.** Let \( C \) be a smooth curve over a perfect field \( k \) of characteristic \( p \). Suppose \( \mathcal{G} \) and \( \mathcal{G}' \) are height 2, dimension 1 BT groups over \( C \) that are generically versally deformed and generically ordinary. Suppose further that their Dieudonné isocrystals are isomorphic: \( D(\mathcal{G}) \otimes \mathbb{Q} \cong D(\mathcal{G}') \otimes \mathbb{Q} \). Then \( \mathcal{G} \) and \( \mathcal{G}' \) are isomorphic.

**Proof.** The isocrystals being isomorphic implies that there is an isogeny \( D(\mathcal{G}) \to D(\mathcal{G}') \). Then [dJ95, Main Theorem] implies that their is an associated isogeny \( \phi : \mathcal{G} \to \mathcal{G}' \). By “dividing by \( p \)”, we may ensure that \( \phi \) does not restrict to 0 on \( \mathcal{G}[p] \). Now suppose for contradiction that \( \phi \) is not an isomorphism, i.e. that it has a kernel. Then \( \phi|_{\mathcal{G}[p]} \) also has a nontrivial kernel.

We have the following diagram of connected-generically étale sequences.

\[
\begin{xy}
  0 \ar[r] & \mathcal{G}[p]^{\circ} \ar[r] & \mathcal{G}'[p]^{\circ} \ar[r] & 0
\end{xy}
\]

\[
\begin{xy}
  0 \ar[r] & \mathcal{G}[p] \ar[r] & \mathcal{G}'[p] \ar[r] & 0
\end{xy}
\]

\[
\begin{xy}
  0 \ar[r] & \mathcal{G}[p]^{\text{ét}} \ar[r] & \mathcal{G}'[p]^{\text{ét}} \ar[r] & 0
\end{xy}
\]

As we have assumed \( \mathcal{G} \) is generically versally deformed, the kernel of \( \phi|_{\mathcal{G}[p]} \) cannot be generically étale by (3) of Lemma 8.7. Thus the kernel must be the connected group scheme \( \mathcal{G}[p]^{\circ} \) because the order of \( \mathcal{G}[p] \) is \( p^2 \). We therefore get a nonzero map \( \mathcal{G}[p]^{\text{ét}} \to \mathcal{G}'[p] \). Now \( \mathcal{G}[p]^{\text{ét}} \) has order \( p \) and is generically étale by definition, so by (3) of Lemma 8.7, \( \mathcal{G}' \) is not generically versally deformed, which contradicts our hypothesis. \( \square \)
Theorem 8.9. Let \( C \) be a smooth, geometrically irreducible, complete curve over \( \mathbb{F}_q \). Suppose \( q \) is a square. There is a natural bijection between the following two sets.

\[
\begin{align*}
\text{\( \mathbb{Q}_l \)-local systems } \mathcal{L} \text{ on } C \text{ such that} & \quad \text{\( p \)-divisible groups } \mathcal{G} \text{ on } C \text{ such that} \\
\{ & \begin{array}{ll}
\bullet \text{ \( \mathcal{L} \) is irreducible of rank } 2 \\
\bullet \text{ \( \mathcal{L} \) has trivial determinant} \\
\bullet \text{ The Frobenius traces are in } \mathbb{Q} \\
\bullet \text{ \( \mathcal{L} \) has infinite image, up to isomorphism}
\end{array} \\
\} & \leftrightarrow \begin{array}{ll}
\{ & \begin{array}{ll}
\bullet \text{ \( \mathcal{G} \) has height } 2 \text{ and dimension } 1 \\
\bullet \text{ \( \mathcal{G} \) is generically versally deformed} \\
\bullet \text{ \( \mathcal{G} \) has all Frobenius traces in } \mathbb{Q} \\
\bullet \text{ \( \mathcal{G} \) has ordinary and supersingular points, up to isomorphism}
\end{array}
\end{array}
\end{align*}
\]

such that if \( \mathcal{L} \) corresponds to \( \mathcal{G} \), then \( \mathcal{L} \otimes \mathbb{Q}_l(-1/2) \) is compatible with the \( F \)-isocrystal \( \mathbb{D}(\mathcal{G}) \otimes \mathbb{Q} \).

Proof. Given such an \( \mathcal{L} \), we can make a BT group \( \mathcal{G} \) as in Corollary 7.8. Xia’s Lemma 8.6 ensures that we can modify \( \mathcal{G} \) to be generically versally deformed by Frobenius “untwisting”; this process terminates because there are both supersingular and ordinary points, so the map to the universal deformation space cannot be identically 0. This BT group is unique up to (non-unique) isomorphism by Lemma 8.8.

To construct the map in the opposition direction, just reverse the procedure. Given such a \( \mathcal{G} \), first form Dieudonné isocrystal \( \mathbb{D}(\mathcal{G}) \otimes \mathbb{Q} \). This is an absolutely irreducible \( F \)-isocrystal because there are both ordinary and supersingular points. Twisting by \( \mathbb{Q}_p(1/2) \) yields an irreducible object of \( F\text{-Isoc}^1(\mathbb{X}, \mathbb{D}(\mathcal{G}) \otimes \mathbb{Q}) \) that has trivial determinant. By Theorem 6.5, for any \( l \neq p \) there is a compatible \( \mathcal{L}_l \) that is absolutely irreducible and has all Frobenius traces in \( \mathbb{Q} \). This \( \mathcal{L}_l \) is unique: there is only one embedding of \( \mathbb{Q} \) in \( \mathbb{Q}_l \) and \( \mathcal{L}_l \) is uniquely determined by the Frobenius traces. Finally, \( \mathcal{L}_l \) has infinite image: as \( \mathcal{G} \) had both ordinary and supersingular points, \( \mathbb{D}(\mathcal{G}) \otimes \mathbb{Q} \) cannot be trivialized on a finite étale cover. \( \square \)

Remark 8.10. Theorem 8.9 has the following strange corollary. Let \( \mathcal{L} \) and \( C \) be as in the theorem. Then there is a natural effective divisor on \( C \) associated to \( \mathcal{L} \): the points where the associated \( \mathcal{G} \) is not versally deformed, together with their multiplicity. This divisor is trivial if and only if \( \mathcal{G} \to C \) is everywhere versally deformed. We wonder if this has an interpretation on the level of cuspidal automorphic representations.

9. Algebraization and Finite Monodromy

In Theorem 8.9, the finiteness of the number of such local systems (a theorem whose only known proof goes through the Langlands correspondence) implies the finiteness of such BT groups. In general, BT groups on varieties are far from being algebraic: for instance, over \( \mathbb{F}_p \) there are uncountably many BT groups of height 2 and dimension 1 as one can see from Dieudonné theory. However, here they are constructed rather indirectly from a motive via the Langlands correspondence. All examples of such rank 2 local systems that we can construct involve abelian schemes and we are very interested in the following question.

Question 9.1. Let \( X \) be a smooth projective variety over \( \mathbb{F}_q \) and let \( \mathcal{G} \to X \) be a height 2, dimension 1 \( p \)-divisible group with ordinary and supersingular points. Is there an embedding as follows, where \( A \to X \) is an abelian scheme?

\[
\begin{array}{ccc}
\mathcal{G} & \longrightarrow & A \\
\downarrow & & \downarrow \\
& X & 
\end{array}
\]

Remark 9.2. We explain the hypotheses Question 9.1. That \( X \) is complete ensures that the convergent \( F \)-isocrystal \( \mathbb{D}(\mathcal{G}) \otimes \mathbb{Q} \) is automatically overconvergent. The existence of both ordinary and supersingular points ensures that the Dieudonné isocrystal \( \mathbb{D}(\mathcal{G}) \otimes \mathbb{Q} \) is absolutely irreducible. That \( \mathbb{D}(\mathcal{G}) \otimes \mathbb{Q} \) is absolutely irreducible with cyclotomic determinant ensures that the Frobenius traces are algebraic numbers.
Remark 9.3. In light of [KP21], it follows that Question 9.1 reduces to the question for a sufficiently ample curve \( C \subset X \) together with the \( p \)-adic companions conjecture for \( \mathcal{D}(\mathcal{G}) \otimes \mathbb{Q} \). In particular, if the field generated by Frobenius traces of \( \mathcal{D}(\mathcal{G}) \) is isomorphic to \( \mathbb{Q} \), then Question 9.1 for \((X, \mathcal{G})\) reduces to the question for \((C, \mathcal{G})\), where \( C \subset X \) is a curve that is the smooth complete intersection of smooth ample divisors.

To motivate the next conjecture, recall that modular curves are not complete. On the other hand, the universal local systems on Shimura curves parameterizing fake elliptic curves cannot all be defined over \( \mathbb{Q}_l \), as we saw in Example 7.1. In other words, they do not form a \( \mathbb{Q} \)-compatible system.

**Conjecture 9.4.** Let \( X \) be a complete curve over \( \mathbb{F}_q \). Suppose \( \{ \mathcal{L}_i \}_{i \neq \mathbb{F}_p} \) is a \( \mathbb{Q} \)-compatible system of absolutely irreducible rank 2 local systems with trivial determinant. Then they have finite monodromy.

In particular, if Conjecture 1.2 is true for \((X, \mathcal{L})\), then Conjecture 9.4 is also true: if the monodromy were not finite, then as in Example 7.1 it would follow that there exists an \( l' \) so that the \( \mathbb{Q}_{l'} \)-companion to \( \mathcal{L}_i \) cannot be defined over \( \mathbb{Q}_l \), contradicting the assumption. Therefore Conjecture 9.4 may be used to falsify Conjecture 1.2.

Using our techniques we can prove the related Theorem 9.5: it is a straightforward application of [Laf11, Corollaire 2.2]. Note that, in the context of Conjecture 9.4, the hypothesis of Theorem 9.5 is stronger exactly one way: namely, one assumes that the \( p \)-adic companion of \( \{ \mathcal{L}_i \}_{i \neq \mathbb{F}_p} \) has coefficients in \( \mathbb{Q}_p \). Note also that Theorem 9.5 does not assume that \( X \) is complete.

**Theorem 9.5.** Let \( X \) be a curve over \( \mathbb{F}_q \). Let \( \mathcal{E} \in \mathcal{F}_{\text{Isoc}}(X) \) be an overconvergent \( F \)-isocrystal on \( X \) with coefficients in \( \mathbb{Q}_p \) that is rank 2, absolutely irreducible, and has finite determinant. Suppose further that the field of traces of \( \mathcal{E} \) is \( \mathbb{Q} \). Then \( \mathcal{E} \) has finite monodromy.

**Proof.** We claim that \( \mathcal{E} \) is isoclinic at every closed point \( x \in |X| \). Indeed, by [Laf11, Corollaire 2.2], the slopes of \( \mathcal{E}_x \) differ by at most 1, forbidding slopes of the form \((-a, a)\) for \( 0 \neq a \in \mathbb{Z} \). As the coefficients of \( \mathcal{E} \) are \( \mathbb{Q}_p \), any fractional slope must appear more than once. As there is a unique embedding \( \mathbb{Q} \to \mathbb{Q}_p \), this implies that any \( p \)-adic companion to \( \mathcal{E} \) is isomorphic to \( \mathcal{E} \) itself by [Abe18a, A.4.1]. Therefore, we may conclude by Lemma 7.3. \( \square \)

**Remark 9.6.** Note that Theorem 9.5 uses Lemma 7.3 which critically uses [Ked18, Theorem 0.2.1], i.e., a partial resolution to Deligne’s companions conjecture. In particular, we use that \( \mathcal{E} \) lives in a complete compatible system.

**Corollary 9.7.** Let \( X \) be a curve over \( \mathbb{F}_q \). Let \( \mathcal{E} \in \mathcal{F}_{\text{Isoc}}(X) \) be an overconvergent \( F \)-isocrystal on \( X \) (with coefficients in \( \mathbb{Q}_p \)) that is rank 2, absolutely irreducible, and has determinant \( \mathbb{Q}_p(i) \) for \( i \in 2\mathbb{Z} \). Suppose further that the field of traces of \( \mathcal{E} \) is \( \mathbb{Q} \). Then \( \mathcal{E} \) has finite monodromy.

**Proof.** As \( i \) is even, \( \mathcal{E}(\frac{1}{2}) \in \mathcal{F}_{\text{Isoc}}(X) \), i.e. \( \mathcal{E}(\frac{1}{2}) \) has coefficients in \( \mathbb{Q}_p \). Apply Theorem 9.5. \( \square \)

10. **Application to Shimura Curves**

In this section, we indicate a criterion for an étale correspondence of projective hyperbolic curves over \( \mathbb{F}_q \) to be the reduction modulo \( p \) of some Shimura curves over \( \mathbb{C} \). Our goal was to find a criterion that was as “group theoretic” as possible.

**Definition 10.1.** Let \( X \leftarrow Z \to Y \) be a correspondence of smooth curves over \( k \). We say it has no core if \( k(X) \cap k(Y) \) has transcendence degree 0 over \( k \).

For much more on the theory of correspondences without a core, see our article [Kri18]. In general, correspondences of curves do not have cores. However, in the case of étale correspondences over fields of characteristic 0, we have the following remarkable result of Margulis, see [Mar91, Theorem 27] and [Moe98, Proposition 2.4].

**Theorem 10.2.** (Margulis) If \( X \leftarrow Z \to Y \) is a finite étale correspondence of smooth hyperbolic curves without a core over a field \( k \) of characteristic 0, then \( X, Y, \) and \( Z \) are all Shimura (arithmetic) curves. In particular, all of the curves and maps can be defined over \( \overline{k} \).
Remark 10.3. Hecke correspondences of modular/Shimura curves furnish examples of étale correspondences without a core.

Our strategy will therefore be to find an additional structure on an étale correspondence without a core such that the whole picture canonically lifts from \( \mathbb{F}_q \) to characteristic 0 and apply Margulis’ theorem. To make this strategy work, we need two inputs.

Lemma 10.4. Let \( X \leftrightarrow Z \rightarrow Y \) be an étale correspondence of projective hyperbolic curves over \( \mathbb{F} \) without a core. If the correspondence lifts to \( W(\mathbb{F}) \), then the lifted correspondence is étale and has no core. In particular, \( X, Z, \) and \( Y \) are the reductions modulo \( p \) of Shimura curves.

Proof. That the lifted correspondence is étale follows from open-ness of the étale locus. It has no core.

Xia proved the following theorem [Xia13, Theorem 1.2], which may be thought of as “Serre-Tate canonical lift” in for hyperbolic curves.

Theorem 10.5. Let \( k \) be a perfect of characteristic \( p \) and let \( X \) be a smooth proper hyperbolic curve over \( k \). Let \( \mathcal{G} \rightarrow X \) be a height 2, dimension 1 BT group over \( X \). If \( \mathcal{G} \) is everywhere versally deformed on \( X \), then there is a unique curve \( \tilde{X} \) over \( W(k) \) which is a lift of \( X \) and admits a lift \( \tilde{\mathcal{G}} \) of \( \mathcal{G} \). Furthermore, the lift \( \tilde{\mathcal{G}} \) is unique.

Remark 10.6. Note that the hypothesis of Theorem 10.5 implies that \( \mathcal{G} \rightarrow X \) is generically ordinary. This is why we call it an analog to the Serre-Tate canonical lift.

Example 10.7. Let \( D \) be a non-split quaternion algebra over \( \mathbb{Q} \) that is split at \( \infty \) and let \( p \) be a finite prime where \( D \) splits. The Shimura curve \( X^D \) parametrizing fake elliptic curves for \( \mathcal{O}_D \) exists as a smooth complete curve (in the sense of stacks) over \( \mathbb{Z}[\frac{1}{2p}] \) and hence over \( \mathbb{F}_p \). Let \( \mathcal{G} \rightarrow X \) be a smooth proper hyperbolic curve over \( \mathbb{F}_p \). It admits a universal abelian surface \( f: A \rightarrow X^D \) with multiplication by \( \mathcal{O}_D \). As \( D \otimes \mathbb{Q}_p \cong M_{2 \times 2}(\mathbb{Q}_p) \), one can use Morita equivalence, i.e. apply the idempotent

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]

on the height 4 dimension 1 BT group \( A[p^\infty] \) to get a height 2 dimension 1 BT group \( \mathcal{G} \), and in fact \( A[p^\infty] \cong \mathcal{G} \oplus \mathcal{G} \). Here \( \mathcal{G} \) is everywhere versally deformed on \( X^D \). Moreover, \( \mathcal{G} \) has both supersingular and ordinary points. For similar discussion, see [GK05, Section 5.1].

On the other hand, as in Example 8.4, let \( X^D_{Ig} \) be the Igusa cover which trivializes \( \mathcal{G}[p]^{et} \) on the ordinary locus. This cover is branched over the supersingular locus of \( \mathcal{G} \). Pulling back \( \mathcal{G} \) to \( X^D_{Ig} \) yields a BT group which is generically versally deformed but not everywhere versally deformed.

Corollary 10.8. Let \( X \leftrightarrow Z \rightarrow X \) be an étale correspondence of projective hyperbolic curves without a core over a perfect field \( k \). Let \( \mathcal{G} \rightarrow X \) be a BT group of height 2 and dimension 1 that is everywhere versally deformed. Suppose further that \( f^*\mathcal{G} \cong g^*\mathcal{G} \). Then \( X \) and \( Z \) are the reduction modulo \( p \) of Shimura curves.

Proof. Any lift \( \tilde{X} \) of \( X \) naturally induces lifts \( \tilde{Z}_f \) and \( \tilde{Z}_g \) of \( Z \) because \( f \) and \( g \) are étale and the goal is to find a lift \( \tilde{X} \) such that \( \tilde{Z}_f \cong \tilde{Z}_g \). Note that \( f^*\mathcal{G} \) and \( g^*\mathcal{G} \) are everywhere versally deformed on \( Z \) because \( f \) and \( g \) are étale. By Theorem 10.5, the pairs \( (X, \mathcal{G}), (Z, f^*\mathcal{G}), \) and \( (Z, g^*\mathcal{G}) \) canonically lift to \( W(k) \). As \( f^*\mathcal{G} \cong g^*\mathcal{G} \), the lifts of \( (Z, f^*\mathcal{G}) \) and \( (Z, g^*\mathcal{G}) \) are isomorphic and we get an étale correspondence of curves over \( W(k) \):

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\tilde{f}} & \tilde{Z} \\
\downarrow \tilde{g} & & \downarrow \tilde{g} \\
\tilde{X} & \xrightarrow{f} & \tilde{X}
\end{array}
\]

Finally by Lemma 10.4, \( X \) and \( Z \) are the reductions modulo \( p \) of Shimura curves. \( \square \)
Example 10.9. Let $X = X^D$ be a Shimura curve parametrizing fake elliptic curves with multiplication by $O_D$, as in Definition 1.1, with $D$ having discriminant $d$. Let $N$ be prime to $d$ and let $Z = X^D_p(N)$, i.e., a moduli space of pairs $(A_1 \to A_2)$ of fake elliptic curves equipped with a “cyclic isogeny of fake-degree $N$” [BT13, Section 2.2]. There exist Hecke correspondences

$$\begin{array}{c}
\quad Z \\
\downarrow \pi_1 \\
X \\
\uparrow \pi_2 \\
\quad X
\end{array}$$

Moreover, as in Example 7.1, as long as $(p, dN) = 1$, this correspondence has good reduction at $p$ and the universal $p$-divisible group splits as $\mathcal{G} \oplus \mathcal{G}$ on $X$. Here $\mathcal{G} \to X$ is everywhere versally deformed and $\pi^*_1 \mathcal{G} \cong \pi^*_2 \mathcal{G}$. In particular, there are examples where the conditions of Corollary 10.8 are met.

Remark 10.10. Xia’s lifting theorem yields one of the canonical lifts of Mochizuki. In particular, as Shimura curves are defined over $\overline{\mathbb{Q}}$, Corollary 10.8 provides a condition under which the a canonical lift of $X$ is defined over $\overline{\mathbb{Q}}$. This yields a very partial response to [EM001, Problem 10].

Theorem 10.11. Let $X \to Z \to X$ be an étale correspondence of smooth, geometrically connected, complete curves without a core over $\overline{\mathbb{F}}_q$ with $q$ a square. Let $\mathcal{L}$ be a $\overline{\mathbb{Q}}_l$-local system on $X$ as in Theorem 8.9 such that $f^* \mathcal{L} \cong g^* \mathcal{L}$ as local systems on $Z$. Suppose the $\mathcal{G} \to X$ constructed via Theorem 8.9 is everywhere versally deformed. Then $X$ and $Z$ are the reductions modulo $p$ of Shimura curves.

Proof. The uniqueness statement in Theorem 8.9 immediately implies that $f^* \mathcal{G} \cong g^* \mathcal{G}$. Apply Corollary 10.8. \qed

Remark 10.12. Finally, we explain a strategy to show that in the context of Theorem 10.11, $X$ is the reduction modulo $p$ of a moduli space of fake elliptic curves and furthermore that $\mathcal{L}$ is the universal local system. In particular, this strategy will imply that under the hypotheses of Theorem 10.11, $\mathcal{L}$ comes from a family of fake elliptic curves. This is joint work-in-progress with M. Sheng.

The canonical lift constructed by Xia is uniformizing in the sense of Mochizuki [Moc96]. This notion has recently been reinterpreted in the context of $p$-adic nonabelian Hodge theory. In particular, in Theorem 10.5 consider the generic fiber $\tilde{X}_{K(k)}$. By virtue of the existence of the $p$-divisible group, we obtain a rank 2 crystalline representation $\pi_1(\tilde{X}_{K(k)}) \to \text{GL}_2(\mathbb{Z}_p)$ which is uniformizing in the sense that under the $p$-adic Simpson correspondence of [LSZ19], the associated Higgs bundle is uniformizing.

We briefly outline the strategy, which involves a general result about Shimura curves. Let $Y$ be an integral model of a Shimura curve over $W(k)$ associated to a totally real field $E$ and a quaternion algebra $D$. (The totally real field $E$ is isomorphic to $\mathbb{Q}$ if and only if $Y$ is a moduli space of fake elliptic curves.) We prove that there exists a positive integer $f$ such that the uniformizing Higgs bundle on $Y$ is $f$-periodic using Deligne’s theory of strange models. In particular, the uniformizing Higgs bundle corresponds to a crystalline representation $\pi_1(\tilde{Y}_{K(k)}) \to \text{GL}_2(\mathbb{Z}_p)$, or equivalently a Fontaine-Faltings module with endomorphism structure $(V, \nabla, \varphi, \Fil, i)$. By forgetting the filtration, one obtains an $F$ crystal in finite, locally free modules and multiplication by $\mathbb{Z}_p[i]$ on the special fiber $Y$. We must simply show that the field generated by the Frobenius traces of the associated object of $\mathbf{F-Isoc}(Y)_{Q_{\text{et}}}$ is isomorphic to $E \subset Q_{p^f}$, the reflex field of the Shimura curve. This should again follow from the explicit computation with strange models.

Given the above statement about the Frobenius traces, it will follow that in the context of Theorem 10.11, $(X, \mathcal{L})$ comes from a universal family of fake elliptic curves together with the associated local system. We emphasize that while this strategy will only work in the restrictive context of Theorem 10.11, it would provide some evidence for Conjecture 1.2.

References

[Abe18a] Tomoyuki Abe. Langlands correspondence for isocrystals and the existence of crystalline companions for curves. J. Amer. Math. Soc., 31(4):921–1057, 2018. doi:10.1090/jams/898.
[Abe18] Tomoyuki Abe. Langlands program for p-adic coefficients and the petits camarades conjecture. J. Reine Angew. Math., 734:59–69, 2018. doi:10.1515/crelle-2015-0045.

[BT13] Benjamin Bakker and Jacob Tsimerman. On the Frey-Mazur Conjecture over low genus curves. arXiv preprint arXiv:1309.6568, 2013.

[Buz97] Kevin Buzzard. Integral models of certain Shimura curves. Duke Math. J., 87(3):591–612, 1997. doi:10.1215/S0012-7094-97-08719-6.

[Chi03] Chieu-Wyhe Chin. Independence of l in Lafforgue’s theorem. Adv. Math., 180(1):64–86, 2003. doi:10.1016/S0001-8708(02)00082-8.

[Cre87] Richard Crew. F-isocrystals and p-adic representations. In Algebraic geometry, Boudoin, 1985 (Brunsweik, Mainz, 1985), volume 46 of Proc. Sympos. Pure Math., pages 111–138. Amer. Math. Soc., Providence, RI, 1987. doi:10.1090/pspum/046.2/927977.

[Del80] Pierre Deligne. La conjecture de Weil. II. Inst. Hautes Études Sci. Publ. Math., (52):137–252, 1980. URL: http://www.numdam.org/item?id=PMIHES_1980__52__137_0.

[Del07] Pierre Deligne. Catégories tannakiennes. In The Grothendieck Festschrift, pages 111–195. Springer, 2007.

[Del14] Pierre Deligne. Semi-simplicité de produits tensoriels en caractéristique p. Inventiones mathematicae, 197(3):587–611, 2014.

[JS95] A. J. de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. Inst. Hautes Études Sci. Publ. Math., (82):5–96 (1995), 1995. URL: http://www.numdam.org/item?id=PMIHES_1995__82__5_0.

[EMO01] S. J. Edixhoven, B. J. J. Moonen, and F. Oort. Open problems in algebraic geometry. Bull. Sci. Math., 125(1):1–22, 2001. doi:10.1016/S0007-4497(00)01075-7.

[FK05] Eyal Z Goren and Payman L Kassaei. The canonical subgroup: a “subgroup-free” approach. arXiv preprint math/0502401, 2005.

[Ill85] Luc Illusie. Déformations de groupes de Barsotti-Tate (d’après A. Grothendieck). Astérisque, (127):151–198, 1985. Seminar on arithmetic schemes: the Mordell conjecture (Paris, 1983/84).

[Kat79] Nicholas M. Katz. Slope filtration of F-crystals. In Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I, volume 63 of Astérisque, pages 113–163. Soc. Math. France, Paris, 1979. Available at http://web.math.princeton.edu/~nmk/old/f-crystals.pdf.

[Ked16] Kiran S Kedlaya. Notes on isocrystals. arXiv preprint arXiv:1606.01321, 2016.

[Ked18] Kiran S Kedlaya. Étale and crystalline companions I. arXiv preprint arXiv:1811.00204v1, 2018.

[Kos17] Teruhisa Koshikawa. Overconvergent unit-root F-isocrystals and isotriviality. Math. Res. Lett., 24(6):1707–1727, 2017. doi:10.4310/MRL.2017.v24.n6.a7.

[KP21] Raju Krishnamoorthy and Ambrus Pál. Rank 2 local systems and Abelian varieties. Sel. Math. New Ser., 27(4):1–40, 2021. doi:10.1007/s00029-021-00669-8.

[Kr18] Raju Krishnamoorthy. Correspondences without a core. Algebra Number Theory, 12(5):1173–1214, 2018. doi:10.2140/ant.2018.12.1173.

[Laf02] Laurent Lafforgue. Chtoucas de Drinfeld et correspondance de Langlands. Invent. Math., 147(1):1–241, 2002. doi:10.1007/s0022201001174.

[Laf11] Vincent Lafforgue. Estimées pour les valuation p-adiques des valeurs propres des opérateurs de Hecke. Bull. Soc. Math. France, 139(4):455–477, 2011. doi:10.24033/bsmf.2614.

[LSZ19] Guitang Lan, Mao Sheng, and Kang Zuo. Semistable Higgs bundles, periodic Higgs bundles and representations of algebraic fundamental groups. J. Eur. Math. Soc. (JEMS), 21(10):3035–3112, 2019. doi:10.4171/JEMS/897.

[Mar91] G. A. Margulis. Discrete subgroups of semisimple Lie groups. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), volume 18, 1991. Springer-Verlag, Berlin.

[Moc96] Shinichi Mochizuki. A theory of ordinary p-adic curves. Publ. Res. Inst. Math. Sci., 32(6):957–1152, 1996. doi:10.2977/prims/1195145686.

[Moc98] Shinichi Mochizuki. Correspondences on hyperbolic curves. J. of Pure Appl. Algebra, 131(3):227–244, 1998. doi:10.1016/S0022-4049(97)00078-9.

[Row08] Louis Halle Rowen. Graduate algebra: noncommutative view, volume 91 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008. doi:10.1090/gsm/091.

[Sos14] Pawel Sosna. Scalar extensions of triangulated categories. Appl. Categ. Structures, 22(1):211–227, 2014. doi:10.1007/s10485-012-9297-0.

[ST18] Andrew Snowden and Jacob Tsimerman. Constructing elliptic curves from galois representations. Compositio Mathematica, 154(10):2045–2054, 2018. doi:10.1112/S0010437X18007315.

[Sta18] The Stacks Project Authors. Stacks Project. https://stacks.math.columbia.edu, 2018.

[TO70] John Tate and Frans Oort. Group schemes of prime order. In Annales scientifiques de l’École Normale Supérieure, volume 3, pages 1–21, 1970.

[Ulm90] Douglas Ulmer. On universal elliptic curves over Igusa curves. Invent. Math., 99(2):377–391, 1990. doi:10.1007/BF01234424.

[Xia13] Jie Xia. On the deformation of a Barsotti-Tate group over a curve. arXiv preprint arXiv:1303.2954, 2013.
Gaußstraße 20, Wuppertal

krishnamoorthy@alum.mit.edu