Non-stationary rotating black holes: Entropy and Hawking’s radiation

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Abstract

In this paper we derive a class of non-stationary rotating solutions including Vaidya-Bonnor-de Sitter, Vaidya-Bonnor-monopole and Vaidya-Bonnor-Kerr. The rotating Vaidya-Bonnor-de Sitter solution describes an embedded black hole that the rotating Vaidya-Bonnor black hole is embedded into the rotating de Sitter cosmological universe. In the case of the Vaidya-Bonnor-Kerr, the rotating Vaidya-Bonnor solution is embedded into the vacuum Kerr solution, and similarly, Vaidya-Bonnor-monopole. By considering the charge to be function of $u$ and $r$, we discuss the Hawking’s evaporation of the masses of variable-charged non-embedded, non-rotating and rotating Vaidya-Bonnor, and embedded rotating, Vaidya-Bonnor-de Sitter, Vaidya-Bonnor monopole and Vaidya-Bonnor-Kerr, black holes. It is found that every electrical radiation of variable-charged black holes will produce a change in the mass of the body without affecting the Maxwell scalar in non-embedded cases; whereas in embedded cases the Maxwell scalar, the cosmological constant, monopole charge and the Kerr mass are not affected by the radiation process. It is also found that during the Hawking’s radiation process, after the complete evaporation of masses of these variable-charged black holes, the electrical radiation will continue creating (i) negative mass naked singularities in non-embedded ones, and (ii) embedded negative mass naked singularities in embedded black holes. The surface gravity, entropy and angular velocity of the horizon are presented for each of these non-stationary black holes.

Keywords: Hawking’s radiation, Vaidya-Bonnor, Vaidya-Bonnor-de Sitter, Vaidya-Bonnor-monopole and Vaidya-Bonnor-Kerr black holes.

1. Introduction

The Hawking’s radiation [1] suggests that black holes which are formed by collapse, are not completely black, but emit radiation with a thermal spectrum. It means that, as the radiation carries away energy, the black holes must presumably lose mass and eventually disappear [2]. In an introductory survey Hawking and Israel [3] have discussed the black hole radiation in three possible ways with creative remarks –‘So far there is no good theoretical frame work with which to treat the final stages of a black hole but there seem to be three possibilities: (i) The black hole might disappear completely, leaving just
the thermal radiation that it emitted during its evaporation. (ii) It might leave behind a non-radiating black hole of about the Planck mass. (iii) The emission of energy might continue indefinitely creating a negative mass naked singularity'. In [4] these three possibilities of black hole radiation could be expressed in classical spacetime metrics, by considering the charge \( e \) of the electromagnetic field to be function of the radial coordinate \( r \) of non-stationary Reissner-Nordstrom as well as Kerr-Newman black holes. The variable-charge \( e(r) \) with respect to the coordinate \( r \) is followed from Boulware’s suggestion [5] that the stress-energy tensor may be used to calculate the change in the mass due to the radiation. According to Boulware’s suggestion, the energy momentum tensor of a particular black hole can be used to calculate the change in the mass in order to incorporate the Hawking’s radiation effects in classical spacetime metrics. This idea suggests to consider the stress-energy tensor of electromagnetic field of different forms or functions from those of stationary Reissner-Nordstrom, as well as Kerr-Newman, black holes as these two black holes do not seem to have any direct Hawking’s radiation effects. Thus, a variable charge in the field equations will have the different function of the energy momentum tensor of the charged black hole. Such a variable charge \( e \) with respect to the coordinate \( r \) in Einstein’s equations is referred to as an electrical radiation (or Hawking’s electrical radiation) of the black hole. Every electrical radiation \( e(r) \) of the non-rotating as well as rotating black holes leads to a reduction in its mass by some quantity. If one considers such electrical radiation taking place continuously for a long time, then a continuous reduction of the mass will take place in the black hole body whether rotating or non-rotating, and the original mass of the black hole will evaporate completely. At that stage the complete evaporation will lead the gravity of the object depending only on the electromagnetic field, and not on the mass. We refer to such an object with zero mass as an ‘instantaneous’ naked singularity - a naked singularity that exists for an instant and then continues its electrical radiation to create negative mass. So this naked singularity is different from the one mentioned in Steinmular et al. [6], Tipler et al. [7] in the sense that an ‘instantaneous’ naked singularity, discussed in [6, 7] exists only for an instant and then disappears.

It is emphasized that the time taken between two consecutive radiation is supposed to be so short that one may not physically realize how quickly radiation take place. Thus, it seems natural to expect the existence of an ‘instantaneous’ naked singularity with zero mass only for an instant before continuing its next radiation to create a negative mass naked singularity. This suggests that it may also be possible in the common theory of black holes that, as a black hole is invisible in nature, one may not know whether, in the universe, a particular black hole has mass or not, but electrical radiation may be detected on the black hole surface. Immediately after the complete evaporation of the mass, if one continues to radiate the remaining remnant, there will be a formation of a new mass. If one repeats the electrical radiation further, the new mass will increase gradually and then the spacetime geometry will represent the ‘negative mass naked singularity’. In order to study Hawking’s radiation in classical spacetime metrics, the Boulware’s suggestion leads us to consider the stress-energy tensors of electromagnetic field of different forms.
or functions from those of Reissner-Nordstrom, as well as Kerr-Newman, black holes as these two black holes do not seem to have any direct Hawking’s radiation effects. Thus, (i) the changes in the mass of black holes, (ii) the formation of ‘instantaneous’ naked singularities with zero mass and (iii) the creation of ‘negative mass naked singularities’ in stationary Reissner-Nordstrom as well as Kerr-Newman black holes [4] may presumably be the correct formulations in classical spacetime metrics of the three possibilities of black hole evaporation suggested by Hawking and Israel [3].

The aim of this paper is to study the relativistic aspect of Hawking’s radiation in non-stationary, embedded and non-embedded black holes by considering the charge \( e(u) \) to be variable with respect to the coordinate \( r \), i.e. \( e(u, r) \). This consideration of the variable charge \( e(u, r) \) will lead to a different energy momentum tensor than the original ones of both the non-stationary, embedded and non-embedded, black holes. Here, according to Cai, et. al. [8], the embedded black hole means that the rotating Vaidya-Bonnor black hole is embedded into the rotating de Sitter cosmological universe to produce the rotating embedded Vaidya-Bonnor-de Sitter black hole and so on. It is also noted that all the black holes which are extended from the Vaidya-Bonnor solutions, are all non-stationary black holes. For examples, the rotating Vaidya-Bonnor, rotating Vaidya-Bonnor-de Sitter, rotating Vaidya-Bonnor-monopole and rotating Vaidya-Bonnor-Kerr black holes, which are to be discussed later in this paper, are all non-stationary black holes. Here, using Wang-Wu functions, it is shown the derivation of embedded rotating black holes, Vaidya-Bonnor-de Sitter, Vaidya-Bonnor-monopole and Vaidya-Bonnor-Kerr in a simple analytic method. Then the relativistic aspect of Hawking radiation effects has been studied in these non-stationary black holes mentioned above. The results are summarized in the form of theorems as follows:

**Theorem 1** Every electrical radiation of non-stationary embedded and non-embedded black holes will produce a change in the mass of the bodies without affecting the Maxwell scalar.

**Theorem 2** The non-rotating and rotating Vaidya-Bonnor black holes will lead to the formation of instantaneous naked singularities with zero mass during the Hawking’s evaporation process of electrical radiation.

**Theorem 3** The non-stationary rotating embedded black holes will lead to the formation of instantaneous charged black holes, with accord to the nature of the background spaces, during the Hawking’s radiation process.

**Theorem 4** During the radiation process, after the complete evaporation of masses of both variable-charged non-rotating and rotating Vaidya-Bonnor black holes, the electrical radiation will continue indefinitely creating negative mass naked singularities.

**Theorem 5** During the radiation process, after the complete evaporation of masses of variable-charged non-stationary embedded black holes, the electrical radiation will continue indefinitely creating embedded negative mass naked singularities.
Theorem 6 If an electrically radiating non-stationary black hole is embedded into a space, it will continue to embed into the same space forever.

Theorem 7 Every embedded black hole, stationary or non-stationary, is expressible in Kerr-Schild ansatz.

It is found that the theorems 1, 2 and 4 are in favour of the first, second and third possibilities of the suggestions made by Hawking and Israel [3]. But theorem 4 provides a violation of Penrose’s cosmic-censorship hypothesis that ‘no naked singularity can ever be created’ [9]. Theorems 3, 5 and 6 show the various stages of the life of embedded non-stationary radiating black holes. Theorem 7 shows that every embedded black hole is a solution of Einstein’s field equations.

Here, it is more appropriate to use the phrase ‘change in the mass’ rather then ‘loss of mass’ as there is a possibility of creating new mass after the exhaustion of the original mass, if one repeats the same process of electrical radiation. This can be seen latter in this paper. Hawking’s radiation is being incorporated, in the classical general relativity describing the change in mass appearing in the classical space-time metrics. In section 2 we derive a class of rotating non-stationary solutions, including Vaidya-Bonnor-de Sitter, Vaidya-Bonnor-monopole and Vaidya-Bonnor-Kerr describing the embedded black holes. The surface gravity, entropy and angular velocity of the horizons are presented as these are the three important properties of black holes. In section 3 we study the relativistic aspect of Hawking’s radiation in non-stationary, embedded and non-embedded black holes by considering the charge $e(u)$ to be variable with respect to the coordinate $r$, i.e. $e(u,r)$. We conclude with the remarks and suggestions of our results presented in this paper in section 4. In an appendix we cite the NP quantities for a rotating metric with functions of two variables. We also present the general formula for the surface gravity and entropy on a horizon of a black hole. We use the differential form structure in Newman and Penrose (NP) formalism [10] developed by McIntosh and Hickman [11] in $(–2)$ signature.

2. Rotating non-stationary solutions

In this section we shall derive a class of non-stationary rotating solutions describing embedded black holes in general relativity, by using the Wang-Wu functions $q_n$ [12] given in (A11) for a spherically symmetric metric with functions of two variables (A1) [13].

(i) Rotating Vaidya-Bonnor-de Sitter solution

We shall combine the rotating Vaidya-Bonnor solution with the rotating de Sitter solution [13], if the Wang-Wu functions $q_n(u)$ are chosen as

$$q_n(u) = \begin{cases} M(u), & \text{when } n = 0 \\ -e^2(u)/2, & \text{when } n = -1 \\ \Lambda^*/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 0, -1, 3 \end{cases}$$

(2.1)
where $M(u)$ and $e(u)$ are related with the mass and the charge of rotating Vaidya-Bonnor solution. Thus, using this $q_n(u)$ in (A11) we obtain the mass function as

$$M(u, r) = M(u) - \frac{e^2(u)}{2r} + \frac{\Lambda^* r^3}{6}. \quad (2.2)$$

The line element describing a rotating Vaidya-Bonnor-de Sitter solution takes the form

$$ds^2 = \left[1 - R^{-2}\left\{2rM(u) + \frac{\Lambda^* r^4}{3} - e^2(u)\right\}\right] du^2 + 2du dr + 2aR^{-2}\left\{2rM(u) + \frac{\Lambda^* r^4}{3} - e^2(u)\right\} \sin^2 \theta \, du \, d\phi - 2a \sin^2 \theta \, dr \, d\phi - R^2 d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta \right\} R^{-2} \sin^2 \theta \, d\phi^2, \quad (2.3)$$

where $\Delta^* = r^2 - 2rM(u) - \Lambda^* r^4/3 + a^2 + e^2(u)$. Here, $a$ is the non-zero rotational parameter per unit mass. Then the other quantities for the metric are

$$\rho^* = \frac{1}{KR^2 R^2} \left\{ e^2(u) + \Lambda^* r^4 \right\},$$
$$p = \frac{1}{KR^2 R^2} \left\{ e^2(u) - \Lambda^* r^2 (r^2 + 2a^2 \cos \theta) \right\},$$
$$\mu^* = \frac{1}{KR^2 R^2} \left\{ 2r^2 \left\{ M(u),u - \frac{1}{r} e(u) \, e(u),u \right\} + a^2 \sin^2 \theta \left\{ M(u),u - \frac{1}{r} e(u) \, e(u),u \right\},_{,u} \right\},$$
$$\omega = \frac{-i a \sin \theta}{\sqrt{2} KR^2 R^2} \left\{ R M(u),u - 2 e(u) \, e(u),u \right\}, \quad (2.4)$$
$$\Lambda = \frac{\Lambda^* r^2}{6 R^2}, \quad (2.5)$$
$$\psi_2 = \frac{1}{KR^2 R^2} \left[ e^2(u) - R M(u) + \frac{\Lambda^* r^2}{3} a^2 \cos^2 \theta \right],$$
$$\psi_3 = \frac{-i a \sin \theta}{2\sqrt{2} R^2 R^2} \left[ (4r + R) M(u),u \right] - 4 e(u) \, e(u),u \right\],$$
$$\psi_4 = \frac{a^2 \sin^2 \theta}{2KR^2 R^2 R^2} \left[ R^2 \left\{ r M(u),u - e(u) e(u),u \right\},_u - 2r \left\{ r M(u),u - e(u) e(u),u \right\},_u \right\]. \quad (2.6)$$

The metric (2.3) will describe a cosmological black holes with the horizons at the values of $r$ for which $\Delta^* = 0$ having four roots $r_{++}, r_{+-}, r_{-+}$ and $r_{--}$ given in appendix (A15) and (A16). The first three values will describe respectively the event horizon, the Cauchy horizon and the cosmological horizon. The surface gravity of the horizon at $r = r_{++}$ is

$$K = -\left[ \frac{1}{r R^2} \left\{ r \left( r - M(u) - \frac{\Lambda^* r^3}{6} \right) + \frac{e^2(u)}{2} \right\} \right]_{r=r_{++}}. \quad (2.7)$$

Then the entropy and angular velocity of the horizon are respectively found as,

$$S = \pi (r^2 + a^2)|_{r=r_{++}}, \quad \text{and} \quad \Omega_{H} = \frac{a \left\{ 2 r M(u) + (\Lambda^* r^4 / 3) - e^2(u) \right\}}{\left\{ r^2 + a^2 \right\}^2}|_{r=r_{++}}. \quad (2.8)$$
In this rotating solution (2.3), the Vaidya-Bonnor null fluid is interacting with the non-null electromagnetic field on the de Sitter cosmological space. Thus, the total energy momentum tensor (EMT) for the rotating solution (2.3) takes the following form:

$$T_{ab} = T_{ab}^{(n)} + T_{ab}^{(E)} + T_{ab}^{(C)}, \quad (2.9)$$

where the EMTs for the rotating null fluid, the electromagnetic field and cosmological matter field are given respectively

$$T_{ab}^{(n)} = \mu^* \ell_a \ell_b + 2\omega \ell_{(a} m_{b)} + 2\varpi \ell_{(a m_b)}, \quad (2.10)$$

$$T_{ab}^{(E)} = 2\rho^{(E)} \{\ell_{(a n_b)} + m_{(a m_b)}\}, \quad (2.11)$$

$$T_{ab}^{(C)} = 2\{\rho^{(C)} \ell_{(a n_b)} + p^{(C)} m_{(a m_b)}\}. \quad (2.12)$$

where, $\mu^*$ and $\omega$ are given in (2.4) and $\rho^{(E)} = p^{(E)} = \frac{e^2(u)}{KR^2 R^2}$, $p^{(C)} = \frac{\Lambda^* r^2}{KR^2 R^2} - 2a^2 \cos^2 \theta$.

Now, for future use we may, without loss of generality, have a decomposition of the Ricci scalar $\Lambda$, given in (2.5), as

$$\Lambda = \Lambda^{(E)} + \Lambda^{(C)}, \quad (2.13)$$

where $\Lambda^{(E)}$ is the zero Ricci scalar for the electromagnetic field and $\Lambda^{(C)}$ is the non-zero cosmological Ricci scalar with $\Lambda^{(C)} = (\Lambda^* r^2/6 R^2)$. The appearance of $\omega$ shows that the Vaidya-Bonnor null fluid is rotating as the expression of $\omega$ in (2.4) involves the rotating parameter $a$ coupling with $M(u)$, both are non-zero quantities for a rotating Vaidya-Bonnor null radiating universe. If we set $a = 0$, we recover the non-rotating Vaidya-Bonnor-de Sitter solution and then the energy-momentum tensor (2.9) can be written in the form of Guth’s modification of $T_{ab}$ [14] as

$$T_{ab} = T_{ab}^{(n)} + T_{ab}^{(E)} + \Lambda^* g_{ab} \quad (2.14)$$

where $T_{ab}^{(E)}$ is the energy-momentum tensor for non-null electromagnetic field and $g_{ab}$ is the non-rotating Vaidya-Bonnor-de Sitter metric tensor. From this, without loss of generality, the EMT (2.9) can be regarded as the extension of Guth’s modification of energy-momentum tensor in rotating spaces. The rotating Vaidya-Bonnor-de Sitter metric can be written in Kerr-Schild form:

$$g_{ab}^{VBdS} = g_{ab}^{dS} + 2Q(u, r, \theta) \ell_a \ell_b \quad (2.15)$$

where $Q(u, r, \theta) = -\{r M(u) - e^2(u)/2\} R^{-2}$. Here, $g_{ab}^{dS}$ is the rotating de Sitter metric and $\ell_a$ is geodesic, shear free, expanding and non-zero twist null vector for both $g_{ab}^{dS}$ as well as $g_{ab}^{VBdS}$ given in (A2) with (A7). The above Kerr-Schild form can also be written on the rotating Vaidya-Bonnor background:

$$g_{ab}^{VBdS} = g_{ab}^{VB} + 2Q(r, \theta) \ell_a \ell_b \quad (2.16)$$
where $Q(r, \theta) = -(\Lambda^* r^4/6) R^{-2}$. These two Kerr-Schild forms (2.15) and (2.16) prove the non-stationary version of theorem 7 in the case of rotating Vaidya-Bonnor-de Sitter solution. If we set $M(u)$ and $e(u)$ are both constant, this Kerr-Schild form (2.16) will be that of Kerr-Newman-de Sitter black hole. The rotating Vaidya-Bonnor-de Sitter metric will describe a non-stationary spherically symmetric solution whose Weyl curvature tensor is algebraically special in Petrov classification possessing a geodesic, shear free, expanding and non-zero twist null vector $\ell_a$ given in (A2) below. One can easily recover a rotating Vaidya-de Sitter metric from this Vaidya-Bonnor-de Sitter solution by setting the charge $e(u) = 0$. If one sets $a = 0$, $e(u) = 0$ in (2.3), one can also obtain the standard non-rotating Vaidya-de Sitter solution [15]. Ghosh and Dadhich [16] have studied the gravitational collapse problem in non-rotating Vaidya-de Sitter space by identifying the de Sitter cosmological constant $\Lambda^*$ with the bag constant of the null strange quark fluid. Also if one sets $a = 0$ in (2.3) one can recover the non-rotating Vaidya-Bonnor-de Sitter black hole [17]. It certainly indicates that the embedded solution (2.3) can be derived by using Wang-Wu functions (2.1) in the rotating metric (A1). It is emphasized that the Vaidya-Bonnor-de Sitter solution, when $M(u)$ and $e(u)$ are set constants, will recover the Kerr-Newman-de Sitter black hole. We find that the Kerr-Newman-de Sitter black hole, obtained with the constant parameters $M$ and $e$, is different from the one derived by Carter [18], Mallet [19] and Xu [20] in terms of involving $\Lambda^*$.

(ii) Non-stationary rotating Vaidya-Bonnor-monopole solution

Here we shall study the rotating non-stationary Vaidya-Bonnor monopole solution by choosing the Wang-Wu functions $q_n(u)$ as

$$q_n(u) = \begin{cases} 
M(u), & \text{when } n = 0 \\
b/2, & \text{when } n = 1 \\
-e^2(u)/2, & \text{when } n = -1 \\
0, & \text{when } n \neq 0, 1, -1 
\end{cases} \quad (2.17)$$

where $M(u)$ and $e(u)$ are related with the mass and the charge of rotating Vaidya-Bonnor solution. Thus, by using this $q_n(u)$, we obtain the mass function as,

$$M(u, r) = M(u) + \frac{rb}{2} - \frac{e^2(u)}{2r}. \quad (2.18)$$

The line element describing a rotating Vaidya-Bonnor-monopole solution take the form

$$ds^2 = \left[1 - R^{-2}\{2rM(u) - e^2(u) + br^2\}\right] du^2 + 2 du dr + 2aR^{-2}\{2M(u) - e^2(u) + br^2\} \sin^2 \theta du d\phi - 2a \sin^2 \theta dr d\phi - R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta\} R^{-2} \sin^2 \theta d\phi^2, \quad (2.19)$$

where $\Delta^* = r^2(1 - b) - 2rM(u) + a^2 + e^2(u)$. Here, $a$ is the rotational parameter per unit mass, $b$ is the monopole constant, $e(u)$ represents the charge of Vaidya-Bonnor solution and $M(u)$ represents the mass function of rotating Vaidya-Bonnor null radiating fluid. Then the other quantities for the metric are

$$\rho^* = \frac{1}{K R^2 R^2 \{e^2(u) + br^2\}}.$$
The metric (2.19) will describe a black hole with the horizons at the values of \( r \) for which \( \Delta^* = 0 \) having two roots \( r_+ \) and \( r_- \):

\[
r_{\pm} = \frac{1}{1 - b} \left[ \frac{M(u)}{2} \pm \sqrt{\frac{M^2(u)}{4} - (1 - b) \{a^2 + e^2(u)\}} \right]
\]  

(2.23)

From this we observe that the value of \( b \) must lie in \( 0 < b < 1 \), with the horizon at \( r = r_+ \). Then the surface gravity of the horizon at \( r = r_+ \) is

\[
\mathcal{K} = \frac{-1}{r_+ R^2} \left[ r_+ \left( r - \frac{b}{2} \right) - M(u) \right] + \frac{e^2(u)}{2}.
\]  

(2.24)

The entropy and angular velocity of the horizon are respectively obtained as follows:

\[
S = \pi (r_+^2 + a^2), \quad \text{and} \quad \Omega_H = \frac{a \{2r M(u) - e^2(u) + b r^2\}}{\{r^2 + a^2\}^2} \bigg|_{r=r_+}.
\]  

(2.25)

In this rotating solution (2.19), the matter field describing monopole particles is interacting with the non-null electromagnetic field. Thus, the total energy momentum tensor (EMT) for the rotating solution (2.19) takes the following form:

\[
T_{ab} = T^{(n)}_{ab} + T^{(E)}_{ab} + T^{(m)}_{ab},
\]  

(2.26)

where the EMTs for the monopole matter field and electromagnetic field are given respectively

\[
\begin{align*}
T^{(n)}_{ab} &= \mu^* \ell_a \ell_b + 2 \omega \ell_a \overline{m b} + 2 \overline{\omega} \ell_a \overline{m b}, \\
T^{(E)}_{ab} &= 2 \rho^{(E)}(E) \ell_{(a} n_{b)} + m_{(a} \overline{m b)}, \\
T^{(C)}_{ab} &= 2 \rho^{(C)} \ell_{(a} n_{b)} + p^{(C)} m_{(a} \overline{m b)}.
\end{align*}
\]  

(2.27) - (2.29)

where \( \mu^* \) and \( \omega \) are unchanged given in (2.20) and other quantities are

\[
\rho^{(E)} = p^{(E)} = \frac{e^2(u)}{K R^2 R^2}, \quad \rho^{(m)} = \frac{b r^2}{K R^2 R^2}.
\]  

(2.27) - (2.29)
\[ p^{(m)} = \frac{1}{K R^2 R^2} \left( e^2(u) - ba^2 \cos^2 \theta \right). \tag{2.30} \]

Now, for feature use we may, without loss of generality, have a decomposition of the Ricci scalar \( \Lambda \), given in (2.21), as

\[ \Lambda = \Lambda^{(E)} + \Lambda^{(m)}, \tag{2.31} \]

where \( \Lambda^{(E)} \) is the zero Ricci scalar for the electromagnetic field and \( \Lambda^{(m)} \) is the non-zero Ricci scalar for monopole field with \( \Lambda^{(m)} = (b/12 R^2) \). One has also seen the interaction of the rotating parameter \( a \) with the monopole constant \( b \) in the expression of \( p^{(m)} \), which makes difference between the rotating as well as non-rotating monopole solutions. We also find that the solution (2.19) with \( M(u) = e(u) = 0 \) represents a rotating monopole solution, which is Petrov type D with the Weyl scalar \( \psi_2 \) given in (2.22) whose repeated principal null vector \( \ell_a \) is shear free, rotating and non-zero twist. The Vaidya-Bonnor-monopole metric can be written in Kerr-Schild form

\[ g_{ab}^{VBm} = g_{ab}^{m} + 2Q(u, r, \theta) \ell_a \ell_b \tag{2.32} \]

where \( Q(u, r, \theta) = -(rM(u) - e^2(u)/2) R^{-2} \). Here, \( g_{ab}^{m} \) is the rotating monopole solution and \( \ell_a \) is geodesic, shear free, expanding and non-zero twist null vector for both \( g_{ab}^{m} \) as well as \( g_{ab}^{VBm} \). The above Kerr-Schild form can also be written on the rotating Vaidya-Bonnor background as

\[ g_{ab}^{VBm} = g_{ab}^{VB} + 2Q(r, \theta) \ell_a \ell_b \tag{2.33} \]

where \( Q(r, \theta) = -(br^2/6) R^{-2} \). These two Kerr-Schild forms (2.32) and (2.33) show the proof of the non-stationary version of theorem 7 in the case of rotating Vaidya-Bonnor-monopole solution. If we set \( M(u) \) and \( e(u) \) are both constant, this Kerr-Schild form (2.32) will be that of Kerr-Newman-monopole black hole. The rotating Vaidya-Bonnor-monopole metric will describe a non-stationary spherically symmetric solution whose Weyl curvature tensor is algebraically special in Petrov classification possessing a geodesic, shear free, expanding and non-zero twist null vector \( \ell_a \). One can easily recover a rotating Vaidya-monopole metric from this Vaidya-Bonnor-monopole solution by setting the charge \( e(u) = 0 \). If one sets \( a = 0, \ e(u) = 0 \) in (2.19), one can also obtain the standard non-rotating Vaidya-monopole solution [12].

(iii) Non-stationary rotating Vaidya-Bonnor-Kerr solution

We shall combine the rotating Vaidya-Bonnor solution with the rotating Ker solution, if the Wang-Wu functions \( q_n(u) \) are chosen such that

\[ q_n(u) = \begin{cases} \tilde{m} + M(u), & \text{when } n = 0 \\ -e^2(u)/2, & \text{when } n = -1 \\ 0, & \text{when } n \neq 0, -1 \end{cases} \tag{2.34} \]

where \( \tilde{m} \) is the mass of Kerr black hole, and \( M(u) \) and \( e(u) \) are related with the mass and the charge of rotating Vaidya-Bonnor solution. Thus, using this \( q_n(u) \) in (A11) we
obtain the mass function
\[ M(u, r) = \tilde{m} + M(u) - \frac{e^2(u)}{2}. \tag{2.35} \]

The line element representing a rotating Vaidya-Bonnor-Kerr solution takes the form
\[ ds^2 = \left[ 1 - R^{-2}(2r(\tilde{m} + M(u)) - e^2(u)) \right] du^2 + 2du dr + 2aR^{-2}(2r(\tilde{m} + M(u)) - e^2(u)) \sin^2 \theta du d\phi - 2a \sin^2 \theta dr d\phi - R^2 d\theta^2 - \{r^2 + a^2\} - \Delta^* a^2 \sin^2 \theta \right] R^{-2} \sin^2 \theta d\phi^2, \tag{2.36} \]

where \( \Delta^* = r^2 - 2r\tilde{m} + M(u) + a^2 + e^2(u) \). Here, \( a \) is the non-zero rotational parameter per unit mass. Then the other quantities for the metric are
\[
\begin{align*}
\rho^* &= \frac{e^2(u)}{KR^2 R^2}, \\
p &= \frac{-e^2(u)}{KR^2 R^2}, \\
\psi_2 &= \frac{\tilde{m} + M(u)}{KR^2 R^2}, \\
\psi_3 &= \frac{-i \sin \theta}{2\sqrt{2R^2 R^2}} \left[ \left( 4r + R^2 \right) M(u, u) - 4e(u) e(u, u) \right], \\
\psi_4 &= \frac{-2a^2 \sin^2 \theta}{2R^2 R^2 R^2} \left[ \left( R^2 \right) r M(u, u) - e(u) e(u, u) \right] \left. \right|_{u = \psi_1}, \\
\end{align*}
\]
and \( \mu^* \) and \( \omega \) are remained the same, given in case 2(ii). The solution (2.36) will describe a black hole if \( \tilde{m} + M(u) > a^2 + e^2(u) \) with external horizon at \( r_+ = \tilde{m} + M(u) + \sqrt{[\tilde{m} + M(u)]^2 - \{a^2 + e^2(u)\}} \), internal Cauchy horizon at \( r_- = \tilde{m} + M(u) - \sqrt{[\tilde{m} + M(u)]^2 - \{a^2 + e^2(u)\}} \) and non stationary limit surface \( r \equiv e(u, \theta) \) \( = \tilde{m} + M(u) + \sqrt{[\tilde{m} + M(u)]^2 - a^2 \cos^2 \theta - e^2(u)} \). The surface gravity of the event horizon at \( r = r_+ \) is
\[ \kappa = -\left. \frac{1}{r R^2} \left[ r \sqrt{\tilde{m} + M(u)}^2 - a^2 - e^2(u) + \frac{e^2(u)}{2} \right] \right|_{r=r_+}. \tag{2.38} \]

The entropy and angular velocity of the black hole at the horizon are respectively found as
\[
\begin{align*}
S &= 2\pi \left\{ \tilde{m} + M(u) \right\} \left\{ \tilde{m} + M(u) + \sqrt{[\tilde{m} + M(u)]^2 - a^2 - e^2(u)} - e^2(u), \right. \\
\Omega_H &= \frac{a \left\{ 2r(\tilde{m} + M(u)) - e^2(u) \right\}}{(r^2 + a^2)^2} \bigg|_{r=r_+}. \tag{2.39} \\
\end{align*}
\]

In this rotating solution (2.36), the Vaidya null is interacting with the non-null electromagnetic field. Thus, the total energy momentum tensor (EMT) for the rotating solution (2.36) takes the following form:
\[ T_{ab} = T_{ab}^{(n)} + T_{ab}^{(E)} \tag{2.40} \]
where the EMTs for the null radiating fluid field and electromagnetic field are given respectively

\[ T^{(n)}_{ab} = \mu^* \ell_a \ell_b + 2 \omega \ell_a \overline{m_b} + 2 \overline{\omega} \ell_a m_b, \quad (2.41) \]

\[ T^{(E)}_{ab} = 2 \rho^*(E) \{ \ell_a m_b + m_a \overline{m_b} \}. \quad (2.42) \]

The appearance of non-vanishing \( \omega \) shows the null fluid is rotating as the expression of omega involves the rotating parameter \( a \) coupling with \( M, u \), both non-zero quantities for a rotating Vaidya-Bonnor null radiating universe. This rotating Vaidya-Bonnor-Kerr metric can be written in Kerr-Schild form as

\[ g_{ab}^{\text{VBK}} = g_{ab}^K + 2Q(u,r,\theta) \ell_a \ell_b \quad (2.43) \]

where \( Q(u,r,\theta) = -\{ rM - e^2(u)/2 \} R^{-2} \). Here, \( g_{ab}^K \) is the rotating Kerr metric and \( \ell_a \) is geodesic, shear free, expanding and non-zero twist null vector for both \( g_{ab}^K \) as well as \( g_{ab}^{\text{VBK}} \). The above Kerr-Schild form can also be written on the rotating Vaidya-Bonnor background

\[ g_{ab}^{\text{VB}} = g_{ab}^{\text{VB}} + 2Q(r,\theta) \ell_a \ell_b \quad (2.44) \]

where \( Q(r,\theta) = -r\tilde{m} R^{-2} \). These two Kerr-Schild forms (2.43) and (2.44) prove the non-stationary version of theorem 7 in the case of rotating Vaidya-Bonnor-Kerr solution. The rotating Vaidya-Bonnor-Kerr metric will describe a non-stationary spherically symmetric solution whose Weyl curvature tensor is algebraically special in Petrov classification possessing a geodesic, shear free, expanding and non-zero twist null vector \( \ell_a \). One can easily recover a rotating Vaidya-Kerr metric from this Vaidya-Bonnor-Kerr solution by setting the charge \( e(u) = 0 \).

3. Changing masses of non-stationary variable-charged black holes

In this section, by solving Einstein-Maxwell field equations with the variable-charge \( e(u,r) \), we develop the relativistic aspect of Hawking radiation in non-stationary classical space-time metrics. The calculation of Newman-Penrose (NP) spin coefficients is being carried out through the technique developed by McIntosh and Hickman [11] in (+,−,−,−) signature. In the formulation of this relativistic aspect of Hawking radiation, we do not impose any condition in the field equations except considering the charge \( e \) to be a function of the coordinates \( u \) and \( r \).

(i) Variable-charged non-rotating Vaidya-Bonnor solution

We consider the non-rotating variable-charged Vaidya-Bonnor solution with the assumption that the charge \( e \) of the body is a function of coordinate \( u \) and \( r \):

\[ ds^2 = \left(1 - \frac{2M(u)}{r} + \frac{e^2(u,r)}{r^2} \right) du^2 + 2du \, dr - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2). \quad (3.1) \]
Initially when $e(u, r) = e(u)$, this metric provides the non-rotating charged Vaidya-Bonnor solution [21]. Using the null tetrad vectors one can calculate the spin coefficients, Ricci scalars and Weyl scalars as follows:

$$
\kappa = \sigma = \nu = \lambda = \pi = \tau = \epsilon = 0,
\rho = \frac{1}{r}, \ \beta = -\alpha = \frac{1}{2\sqrt{2}r} \cot\theta,
\mu = -\frac{1}{2r} \left\{ 1 - \frac{2M(u)}{r} + \frac{e^2(u, r)}{r^2} \right\},
\gamma = \frac{1}{2r^2} \left\{ M(u) + e(u, r)e(u, r),_r - \frac{e^2(u, r)}{r^2} \right\}, \tag{3.2}
$$

$$
\phi_{00} = \phi_{01} = \phi_{10} = \phi_{20} = \phi_{02} = 0,
\phi_{11} = \frac{1}{4r^2} \left\{ e(u, r)e(u, r),_r - \frac{e(u, r)e(u, r),_r}{r^3} + \frac{e^2(u, r)}{2r^4} \right\},
\phi_{22} = -\frac{1}{12r^2} \left\{ e^2(u, r),_r + e(u, r),_r e(u, r),_r r \right\}, \tag{3.3}
$$

When the energy momentum tensor is of electromagnetic fields, then the Ricci tensor $R_{ab}$ is proportional to the Maxwell stress tensor [10] that is

$$
\phi_{AB} = k \phi_A \phi_B, \quad k = 8\pi G/c^2 \tag{3.5}
$$

with $A, B = 0, 1, 2$ and the NP Ricci scalar

$$
\Lambda \equiv \frac{1}{24} R_{ab} g^{ab} = 0. \tag{3.6}
$$

Hence, vanishing $\Lambda$ in (3.6) with (3.3) leads

$$
e^2(u, r) = 2r m_1(u) + C(u) \tag{3.7}
$$

where $m_1(u)$ and $C(u)$ are real functions of $u$. Then the Ricci scalar becomes

$$
\phi_{11} = \frac{C(u)}{2r^4}. \tag{3.8}
$$

Thus, the Maxwell scalar $\phi_1 = \frac{1}{2} F_{ab}(\ell^a n^b + m^a m^b)$ takes the form, by identifying the real function $C(u) = e^2(u)$,

$$
\phi_1 = \frac{1}{\sqrt{2}} e(u) r^{-2}, \tag{3.9}
$$

showing that the Maxwell scalar $\phi_1$ does not change its form by considering the charge $e$ to be a function of $u$ and $r$ in Einstein-Maxwell field equations. Here, by using equation (3.7) in (3.2) and (3.4), we have the resulting NP quantities

$$
\mu = -\frac{1}{2r} \left[ 1 - \frac{2}{r} \left\{ M(u) - m_1(u) \right\} + \frac{e^2(u)}{r^2} \right], \tag{3.10}
$$

$$
\gamma = \frac{1}{2r^2} \left\{ M(u) - m_1(u) \right\} - \frac{e^2(u)}{r}. \tag{3.10}
$$
\[ \psi_2 = -\frac{1}{r^3} \left[ \{M(u) - m_1(u)\} - \frac{e^2(u)}{r} \right]. \]  

(3.11)

and the metric (3.1) takes the form

\[ ds^2 = \left[ 1 - \frac{2}{r} \{M(u) - m_1(u)\} + \frac{e^2(u)}{r^2} \right] du^2 + 2du dr - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(3.12)

This means that the mass \( M(u) \) of non-rotating Vaidya-Bonnor black hole (3.1) is lost a quantity \( m_1(u) \) at the end of the first electrical radiation. This loss of mass is agreeing with Hawking’s discovery that the radiating objects must lose its mass [2]. On the lose of mass, Wald [22] has pointed that a black hole will lose its mass at the rate as the energy is radiated. If one considers the same process for second time taking \( e \) in (3.12) to be function of \( u \) and \( r \) with the mass \( M(u) - m_1(u) \) in Einstein-Maxwell field equations, then the mass may again be decreased by another real function \( m_2(u) \) (say); that is, after the second time radiation the total mass might become \( M(u) - \{m_1(u) + m_2(u)\} \). This is due to the fact, that the Maxwell scalar \( \phi_1 \) with condition (3.9) does not change its form after considering the charge \( e(u) \) to be function of \( u \) and \( r \) for the second time as \( \Lambda \) calculated from the Einstein-Maxwell field equations has to vanish for electromagnetic fields with \( e(u, r) \). Hence, if one repeats the same process for \( n \)-times considering every time the charge \( e(u) \) to be function of \( u \) and \( r \), then one would expect the solution to change gradually and the total mass becomes \( M(u) - \{m_1(u) + m_2(u) + m_3(u) + ... + m_n(u)\} \)

and therefore the metric (3.12) will take the form:

\[ ds^2 = \left[ 1 - \frac{2}{r} \mathcal{M}(u) + \frac{e^2(u)}{r^2} \right] du^2 + 2du dr - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  

(3.13)

where the mass of the black hole after the radiation of \( n \)-times would be

\[ \mathcal{M}(u) = M(u) - \{m_1(u) + m_2(u) + m_3(u) + ... + m_n(u)\}. \]  

(3.14)

This suggests that for every electrical radiation, the original mass \( M(u) \) of the non-rotating black hole may lose some quantity. Thus, it seems reasonable to expect that, taking Hawking’s radiation of black holes into account, such continuously lose of mass may lead to evaporate the original mass \( M(u) \). In case the black hole has evaporated down to the Planck mass, the mass \( M(u) \) may not exactly equal to the continuously lost quantities \( m_1(u) + m_2(u) + m_3(u) + ... + m_n(u) \). That is, according to the second possibility of Hawking and Israel [3] quoted above, there may left a small quantity of mass, say, Planck mass of about \( 10^{-5}g \) with continuous electrical radiation. Otherwise, when \( M(u) = m_1(u) + m_2(u) + m_3(u) + ... + m_n(u) \) for a complete evaporation of the mass, \( \mathcal{M}(u) \) would be zero, rather than, leaving behind a Planck-size mass black hole remnant. At this stage the non-rotating black hole geometry would have the electric charge \( e \) only, but no mass, that the line element would be of the form

\[ ds^2 = \left( 1 + \frac{e^2(u)}{r^2} \right) du^2 + 2du dr - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(3.15)

Here, the metric describes the non-rotating charged solution with imaginary roots \( r_{\pm} = \pm \sqrt{-\{e^2(u)\}} \), indicating naked singularity with zero mass. As the electrical radiation has to continue, this remnant will remain only for an instant.
Here, again its electrical radiation will be continue to create negative mass. [Steinmular, King and LoSota [6] refer a spherically symmetric star, which radiates away all its mass as ‘instantaneous’ naked singularity that exists only for an instant and then disappears. This ‘instantaneous’ naked singularity is also mentioned in [6]. The time taken between two consecutive radiations is supposed to be very short that one may not physically realize how quickly radiations take place. Thus it seems natural to expect the existence of ‘instantaneous’ naked singularity with zero mass only for an instant before continuing its next radiation to create negative mass naked singularity. It may also be possible in the reasonable theory of black holes that, as a black hole is invisible in nature, one may not know that in the universe, a particular black hole has mass or not, but electrical radiation may be detected on the black hole surface. So, there may be some radiating black holes without masses in the universe, where the gravity may depend only on the electric charge, i.e., $\psi_2 = e^2(u)/r^4$, not on the mass of the black holes. Just after the exhaustion of the mass, if one continues the remaining non-stationary solution (3.15) to radiate, there may be a formation of new mass $m^*_1(u)$ (say). If one repeats the electrical radiation further, the new mass might increase gradually and then, the metric (3.15) with the new mass would become

$$ds^2 = \left(1 + \frac{2}{r} M^*(u) + \frac{e^2(u)}{r^2}\right) du^2 + 2du dr - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

(3.16)

where the new mass is given by

$$M^*(u) = m^*_1(u) + m^*_2(u) + m^*_3(u) + m^*_4(u) + ...$$

(3.17)

Comparing the metrics (3.13) and (3.16) one could observe that the classical space-time (3.16) will describe a non-rotating spherical symmetric star with a negative mass $M^*(u)$. Such objects with negative masses are referred as naked singularities [1, 6, 7]. The metric (3.16) may be regarded to describe the incorporation of the third possibility of Hawking and Israel [3] in the case of non-rotating singularity. Here it is noted that the creation of negative mass naked singularity is mainly based on the continuous electrical radiation of the variable charge $e(u, r)$ in the energy momentum tensor of Einstein-Maxwell equations. This also indicates the incorporation of Boulware’s suggestion [5] that ‘the stress-energy tensor may be used to calculate the change in the metric due to the radiation’. This new mass $M^*(u)$ would never decrease, rather might increase gradually as the radiation continues forever. and the unaffected Maxwell scalar $\phi_1$ is given in (3.9). The metric (3.16) admits the energy momentum tensor with the mass $M^*$ showing effect of radiation. Thus, one has seen the changes in the mass of the non-rotating charged black hole after every radiation. Hence, it follows the theorem 1 cited above in the case of non-rotating variable-charged black hole.

(ii) Variable-charged rotating Vaidya-Bonnor Black hole

Here, we shall incorporate the Hawking radiation, how the variable-charged rotating black hole affect in the classical space-time metric when the electric charge $e$ is taken
as a function of \( u \) and \( r \) in the Einstein-Maxwell field equations. The line element with \( e(u, r) \) is

\[
\begin{align*}
    ds^2 &= [1 - R^{-2(2rM(u) - e^2(u, r))}] du^2 + 2du dr \\
    &+ 2aR^{-2(2rM(u) - e^2(u, r))} \sin^2 \theta du d\phi - 2a \sin^2 \theta dr d\phi \\
    &- R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta\} R^{-2} \sin^2 \theta d\phi^2, \\
\end{align*}
\]

(3.18)

where \( \Delta^* = r^2 - 2rM(u) + a^2 + e^2(u, r) \). This metric will also reduce to rotating Vaidya-Bonnor solution when \( e(u, r) = e(u) \) initially, then the Einstein-Maxwell field equations for the metric (3.18) can be solved to obtain the following change NP quantities

\[
\begin{align*}
    \gamma &= \frac{1}{2R R^2} \left[ \{r - M(u) + e(u, r)e(u, r),r\} R - \Delta^* \right], \\
    \nu &= \frac{1}{\sqrt{2R R^2}} \left[ i a \sin \theta \left\{ r M(u),u - e(u, r)e(u, r),u \right\} \right].
\end{align*}
\]

The Weyl scalars are

\[
\begin{align*}
    \psi_0 &= \psi_1 = 0, \\
    \psi_2 &= \frac{1}{2R R^2} \left[ \{ - R M(u) + e^2(u, r) \} - R e(u, r)e(u, r),r \\
    &+ \frac{1}{6} R R \left\{ e^2(u, r),r + e(u, r)e(u, r),r,r \right\} \right], \\
    \psi_3 &= -\frac{1}{2\sqrt{2R R^2}} \left[ 4 i a \sin \theta \left\{ r M(u),u - e(u, r)e(u, r),u \right\} \\
    &+ i a R \sin \theta \left\{ M(u) - e(u, r)e(u, r),r \right\},u \right\}], \\
    \psi_4 &= \frac{a^2 \sin^2 \theta}{2R R^2 R^2} \left\{ r M(u),u - e(u, r)e(u, r),u \right\},u \\
    &- \frac{a^2 r \sin^2 \theta}{R R R^2 R^2} \left\{ r M(u),u - e(u, r)e(u, r),u \right\} \\
    &- \frac{1}{R R R^2 R^2} \left\{ r M(u),u - e(u, r)e(u, r),u \right\},u \right\],
\end{align*}
\]

(3.19)

and the Ricci scalars:

\[
\begin{align*}
    \phi_{00} &= \phi_{01} = \phi_{10} = \phi_{20} = \phi_{02} = 0, \\
    \phi_{11} &= \frac{1}{4R R^2} \left[ 2 e^2(u, r) - 4r e(u, r)e(u, r),r + R^2 \left\{ e(u, r)e(u, r),r \right\},r \right], \\
    \phi_{12} &= \frac{i a \sin \theta}{2\sqrt{2R R^2}} \left[ R M(u),u - 2e(u, r)e(u, r),u + R \left\{ e(u, r)e(u, r),u \right\},r \right], \\
    \phi_{22} &= -\frac{1}{R^2 R^2} \left[ \left\{ r M(u),u - e(u, r)e(u, r),u \right\} \right], \\
    \phi_{20} &= -\frac{1}{2R^2 R^2} \left[ e^2(u, r),r + e(u, r)e(u, r),r,r \right].
\end{align*}
\]

(3.21)

(3.22)

For electromagnetic field the Ricci scalar \( \Lambda \) must vanish. Thus, the vanishing \( \Lambda \) of (3.22) implies that

\[
e^2(u, r) = 2r m_1(u) + C(u)
\]

(3.23)

where \( m_1(u) \) and \( C(u) \) are real functions of \( u \). Then, using this result in equation (3.21) we obtain the Ricci scalar

\[
\phi_{11} = \frac{C(u)}{2R R^2 R^2}.
\]

(3.24)
Accordingly, the Maxwell scalar will become, after identifying the function \( C(u) = e^2(u) \),

\[
\phi_1 = \frac{e(u)}{\sqrt{2 R R}}. \tag{3.25}
\]

This shows that the Maxwell scalar \( \phi_1 \) is unaffected by considering the charge \( e(u) \) to be a function \( u, r \) in the field equations. Hence, from the Einstein-Maxwell equations we obtain the changed NP quantities

\[
\mu = -\frac{1}{2 R R} \left[ r^2 - 2r \{ M(u) - m_1(u) \} + a^2 + e^2(u) \right], \tag{3.26}
\]

\[
\gamma = \frac{1}{2 R R} \left[ \left\{ r - (M(u) - m_1(u)) \right\} \sqrt{R} \right.
- \left\{ r^2 - 2r \left( M(u) - m_1(u) \right) + a^2 + e^2(u) \right\}], \tag{3.27}
\]

\[
\psi_2 = \frac{1}{R R R^2} \left[ - \{ M(u) - m_1(u) \} R + e^2(u) \right],
\]

\[
\psi_3 = -\frac{1}{2 \sqrt{R R R^2}} \left[ i a \sin \theta \left\{ (4r + \sqrt{R}) \left( M(u) - m_1(u) \right)_u \right\}
- 4 i a \sin \theta \left\{ e(u) e(u)_u \right\} \right],
\]

\[
\psi_4 = \frac{a^2 \sin^2 \theta}{2 R R R^2} \left[ r \left\{ M(u) - m_1(u) \right\}_{,u} - e(u) e(u)_u \right],
\]

\[
\phi_{11} = \frac{e^2(u)}{2 R^2 R^2}. \tag{3.28}
\]

Thus, the solution (3.18) with a new function \( m_1(u) \) takes the following form

\[
\begin{align*}
\frac{ds^2}{R^2} &= \left[ 1 - R^{-2} \left\{ 2r \left( M(u) - m_1(u) \right) - e^2(u) \right\} \right] du^2 + 2du dr \\
&+ 2aR^{-2} \left\{ 2r \left( M(u) - m_1(u) \right) - e^2(u) \right\} \sin^2 \theta du d\phi - 2a \sin^2 \theta dr d\phi \\
&- R^2 d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta \right\} R^{-2} \sin^2 \theta d\phi^2, \tag{3.29}
\end{align*}
\]

where \( \Delta^* = r^2 - 2r \{ M(u) - m_1(u) \} + a^2 + e^2(u) \). This introduction of mass \( m_1(u) \) in the metric (3.29) suggests that the first electrical radiation of rotating black hole may reduce the original gravitational mass \( M(u) \) by a quantity \( m_1(u) \). If one considers another radiation by taking \( e(u) \) to be a function of \( u, r \) with the mass \( M(u) - m_1(u) \), then the Einstein-Maxwell field equations yield to reduce this mass by another real function \( m_2(u) \); i.e., after the second radiation, the mass may become \( M(u) - \{ m_1(u) + m_2(u) \} \). Here again, the Maxwell scalar \( \phi_1 \) remains the same form after the second radiation also. Thus, if one considers the \( n \)-time radiations taking every time the charge \( e(u) \) to be function of \( u, r \), the Maxwell scalar \( \phi_1 \) will remain unaffected, but the metrics will be of the following form:

\[
\begin{align*}
\frac{ds^2}{R^2} &= \left[ 1 - R^{-2} \left\{ 2r \left( M(u) - e^2(u) \right) \right\} \right] du^2 + 2du dr \\
&+ 2aR^{-2} \left\{ 2r \left( M(u) - e^2(u) \right) \right\} \sin^2 \theta du d\phi - 2a \sin^2 \theta dr d\phi \\
&- R^2 d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta \right\} R^{-2} \sin^2 \theta d\phi^2, \tag{3.30}
\end{align*}
\]
where the total mass of the black hole, after the \( n \)-time radiations will take the form

\[
\mathcal{M}(u) = M(u) - \{m_1(u) + m_2(u) + m_3(u) + m_4(u) + \ldots + m_n(u)\}. \quad (3.31)
\]

Taking Hawking’s radiation of black holes, one might expect that the total mass of black hole may be radiated away just leaving \( \mathcal{M}(u) \) equivalent to Planck mass of about \( 10^{-5} \), that is, \( \mathcal{M}(u) \) may not be exactly equal to \( m_1(u) + m_2(u) + m_3(u) + m_4(u) + \ldots + m_n(u) \), but has a difference of about Planck-size mass, as in the case of non-rotating black hole given in (i). Otherwise, the total mass of black hole may be evaporated completely after continuous radiation when \( \mathcal{M}(u) = 0 \), that is, \( M(u) = m_1(u) + m_2(u) + m_3(u) + m_4(u) + \ldots + m_n(u) \). Here one may regard that the rotating variable-charged black hole might be radiated away completely all its mass just leaving the electrical charge \( e(u) \) only. One could observe this situation in the form of classical space-time metric as

\[
ds^2 = (1 + e^2(u) R^{-2}) \, du^2 + 2 du \, dr - 2 a e^2 R^{-2} \sin^2 \theta \, du \, d\phi - 2 a \sin^2 \theta \, dr \, d\phi - R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta \} R^{-2} \sin^2 \theta \, d\phi^2,
\]

with the charge \( e(u) \), but no mass, where \( \Delta^* = r^2 + a^2 + e^2(u) \). The metric (3.32) will describe a rotating ‘instantaneous’ naked singularity with zero mass. At this stage, the Weyl scalar \( \psi_2, \psi_3 \) and \( \psi_4 \) takes the form

\[
\psi_2 = \frac{e^2(u)}{R \, R^2} , \quad (3.33)
\]

\[
\psi_3 = \frac{\sqrt{2} a i \sin \theta}{R \, R^2} \{e(u)e(u),u\} , \quad (3.34)
\]

\[
\psi_4 = \frac{a^2 \sin^2 \theta}{2 R \, R^2 \, R^2} [2 r e(u)e(u),u - R^2 \{e(u)e(u),u\},u] . \quad (3.35)
\]

showing the gravity on the surface of the remaining solution depending only on the electric charge \( e(u) \); however, the Maxwell scalar \( \phi_1 \) remains the same as in (3.25). For future use, we mention the changed NP spin coefficients

\[
\mu = - \frac{1}{2 R^2} \{r^2 + a^2 + e^2(u)\} , \quad (3.36)
\]

\[
\gamma = - \frac{1}{2 R^2} \{rR - \{r^2 + a^2 + e^2(u)\}\} , \quad (3.37)
\]

\[
\phi_{11} = \frac{e^2(u)}{2 R^2 \, R^2} , \quad (3.37)
\]

\[
\phi_{12} = - \frac{i a \sin \theta}{\sqrt{2} R^2 \, R^2} \{e(u)e(u),u\} ,
\]

\[
\phi_{22} = \frac{1}{R^2 \, R^2} \{r e(u)e(u),u\}, + \frac{a^2 \sin^2 \theta}{R^2 \, R^2} \{e(u)e(u),u\},u . \quad (3.38)
\]

As the electrical radiation has to continue, the remaining remnant will remain only for an instant. Hence we refer to the solution (3.32) as an ‘instantaneous’ naked singularity with zero mass. It suggests that there may be rotating black holes in the universe whose masses are completely radiated; their gravity depend only on the electric charge of the
body and their metrics look like the one given in the equation (3.32). It appears that the idea of this evaporation of masses of radiating black holes may be agreed with that of Hawking’s evaporation of black holes. Unruh [23] has examined various aspects of black hole evaporation based on Schwarzschild metric. It is worthwhile to study the nature of such black holes (3.32). This might give a different nature, which one has not yet come across so far in the reasonable theory of black holes. Here, immediately after the exhaustion of the Vaidya-Bonnor mass, one may consider again the charge $e(u)$ to be function of $u$ and $r$ for next radiation in (3.32), so that one must get from the Einstein’s field equations the scalar $\Lambda$ as given in equation (3.22). Then the vanishing of this $\Lambda$ for electromagnetic field, there will be creation of a new mass (say $m^*_1(u)$) in the remaining space-time geometry. If this radiation process continues forever, the new mass will increase gradually as

$$\mathcal{M}^*(u) = m^*_1(u) + m^*_2(u) + m^*_3(u) + m^*_4(u) + \ldots$$

(3.39)

However, it appears that this new mass would never decrease. Then the space-time geometry takes the form

$$ds^2 = \left[1 + \frac{\Delta^*}{R^2}\right] du^2 + 2 du dr - 2 a R^{-2} \left[2 r \mathcal{M}^*(u) + e^2(u)\right] \sin^2 \theta du d\phi - 2 a \sin^2 \theta dr d\phi - R^2 d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta \right\} R^{-2} \sin^2 \theta d\phi^2,$$

(3.40)

where $\Delta^* = r^2 + 2r \mathcal{M}^*(u) + a^2 + e^2(u)$. The affected Weyl scalars and other NP coefficients are calculated from the Einstein-Maxwell field equations as

$$\psi_2 = \frac{1}{2 R R^2} \left\{ \mathcal{M}^*(u) R + e^2(u) \right\},$$

(3.41)

$$\psi_3 = \frac{a_i \sin \theta}{2 \sqrt{2 R R^2}} \left\{ (4 r + R) \mathcal{M}^*(u),_u + 4 e(u) e(u),_u \right\},$$

(3.42)

$$\psi_4 = - \frac{a^2 \sin^2 \theta}{2 R R^2 R^2} \left\{ r \mathcal{M}^*(u) + e(u) e(u),_u \right\},$$

(3.43)

$$\mu = - \frac{1}{2 R R^2} \left\{ r^2 + 2r \mathcal{M}^*(u) + a^2 + e^2(u) \right\},$$

(3.44)

$$\gamma = \frac{1}{2 R R^2} \left[ (r + \mathcal{M}^*(u)) R - \left\{ r^2 + 2r \mathcal{M}^*(u) + a^2 + e^2(u) \right\} \right],$$

with $\phi_1$ remained the same as in (3.25). The metric (3.40) may be regarded to describe a rotating negative mass naked singularity. We have presented the possible changes in the mass of the rotating charged black hole without affecting the Maxwell scalar $\phi_1$ and accordingly, metrics are cited for future use. Thus, this completes the proof of the theorem 4 for the rotating part of charged black hole.

(iii) Variable-charged rotating Vaidya-Bonnor-de Sitter Black hole
In this section we shall consider the variable-charged black hole embedded into de Sitter space. By solving Einstein-Maxwell field equations with the variable-charge $e(u, r)$ of rotating Vaidya-Bonnor black holes embedded in de Sitter space, we develop the relativistic aspect of Hawking radiation in classical spacetime metrics. We consider the charge $e$ to be function of coordinates $u$ and $r$ and the decomposition of the Ricci scalar $\Lambda \equiv (1/24) R_{ab} g^{ab}$ into two parts, without loss of generality, as follows

$$\Lambda = \Lambda^{(C)} + \Lambda^{(E)}, \quad (3.45)$$

where $\Lambda^{(C)}$ is the non-zero cosmological Ricci scalar and $\Lambda^{(E)}$ is the zero Ricci scalar of electromagnetic field for the rotating as well as non-rotating black holes. This decomposition of Ricci scalar $\Lambda$ is possible because the cosmological object and the electromagnetic field are two different matter fields of different physical nature, though they are supposed to exist on the same spacetime coordinates here. For our purpose of the paper, this type of decomposition of Ricci scalars $\Lambda$ will serve well in the study of Hawking’s radiation of black holes embedded into the de Sitter cosmological space. The line element of rotating Vaidya-Bonnor-de Sitter black hole with variable charge $e(u, r)$ is

$$ds^2 = \left[1 - R^{-2} \left\{2rM(u) - e^2(u, r) + \frac{\Lambda^* r^4}{3}\right\}\right] du^2 + 2 du dr$$

$$+ 2a R^{-2} \left\{2rM(u) - e^2(u, r) + \frac{\Lambda^* r^4}{3}\right\} \sin^2 \theta \, du \, d\phi - 2a \sin^2 \theta \, dr \, d\phi$$

$$- R^2 d\theta^2 - \left\{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta\right\} R^{-2} \sin^2 \theta \, d\phi^2, \quad (3.46)$$

where $\Delta^* = r^2 - 2rM(u) + a^2 + e^2(u, r) - \Lambda^* r^4/3$. This metric will recover the rotating Vaidya-Bonnor-de Sitter solution given in (2.3) when $e(u, r) = e(u)$ initially. Then the Einstein-Maxwell field equations for the metric (3.46) can be solved. So we obtain the changed NP quantities

$$\mu = -\frac{1}{2 \overline{R} R^2} \left\{r^2 - 2rM(u) + a^2 + e^2(u, r) - \frac{\Lambda^* r^4}{3}\right\}, \quad (3.47)$$

$$\gamma = \frac{1}{2 \overline{R} R^2} \left[\left\{r - M(u) + e(u, r) e(u, r), - \frac{\Lambda^* r^4}{3}\right\} \overline{R} \right. \right.$$

$$\left. - \left\{r^2 - 2rM(u) + a^2 + e^2(u, r) - \frac{2\Lambda^* r^4}{3}\right\}\right], \quad (3.48)$$

$$\psi_2 = \frac{1}{\overline{R} R^2} \left\{- RM(u) + e^2(u, r) - e(u, r) e(u, r), \overline{R} + \frac{\Lambda^* r^2}{3} a^2 \cos^2 \theta\right\}$$

$$+ \frac{1}{6 R^2} \left\{e(u, r) e(u, r),\right\}, \quad (3.49)$$

$$\phi_{11} = \frac{1}{2 R^2 R^2} \left\{e^2(u, r) - 2r e(u, r) e(u, r), - \Lambda^* r^2 a^2 \cos^2 \theta\right\}$$

$$+ \frac{1}{4 R^2} \left\{e^2(u, r), + e(u, r) e(u, r),\right\}, \quad (3.50)$$

$$\Lambda = \frac{\Lambda^* r^2}{6 R^2} - \frac{1}{12 R^2} \left\{e^2(u, r), + e(u, r) e(u, r),\right\}, \quad (3.51)$$

where $\psi_3$ and $\psi_4$ are same as (3.20) and $\phi_{12}, \phi_{22}$ in (3.21) of case 3(ii). Thus, we have seen that in each expression of $\phi_{11}$ and $\psi_2$ there is the cosmological $\Lambda^*$ term coupling.
with the rotation parameter \( a \). Hence, without loss of generality, it will be convenient here to have a decomposition of \( \phi_{11} \) into two parts - one for the cosmological Ricci scalar \( \phi_{11}^{(C)} \) and the other for the electromagnetic field \( \phi_{11}^{(E)} \) as in the case of \( \Lambda \) in (3.45), such that

\[
\phi_{11}^{(C)} = -\frac{1}{2 R^2 R^2} \Lambda^* r^2 a^2 \cos^2 \theta, \tag{3.52}
\]

\[
\phi_{11}^{(E)} = \frac{1}{2 R^2 R^2} \left\{ e^2(u, r) - 2r e(u, r)e(u, r)_r \right\},
+ \frac{1}{4R^2} \left\{ e(u, r) e(u, r)_r \right\}_r. \tag{3.53}
\]

Similarly, we also have the decomposition of \( \Lambda \) as

\[
\Lambda^{(C)} = \frac{\Lambda^* r^2}{6 R^2}, \tag{3.54}
\]

\[
\Lambda^{(E)} = -\frac{1}{12 R^2} \left\{ e^2(u, r)_r + e(u, r) e(u, r)_r \right\}. \tag{3.55}
\]

Now, the scalar \( \Lambda^{(E)} \) for electromagnetic field must vanish for this rotating metric. Thus, the vanishing \( \Lambda^{(E)} \) of the equation (3.55) yields that

\[
e^2(u, r) = 2 r m_1(u) + C(u) \tag{3.56}
\]

where \( m_1(u) \) and \( C(u) \) are real functions. Then, substituting this result in equation (3.53) we obtain the Ricci scalar for electromagnetic field

\[
\phi_{11}^{(E)} = \frac{C(u)}{2 R^2 R^2}. \tag{3.57}
\]

However, the cosmological Ricci scalar \( \phi_{11}^{(C)} \) remains the same form as in (3.52). Accordingly, the Maxwell scalar takes, after identifying the function \( C(u) \equiv e^2(u) \),

\[
\phi_1 = \frac{e(u)}{\sqrt{2 R^2}}. \tag{3.58}
\]

Hence, the affected NP quantities after substitution of \( e^2(u, r) \) (3.56) are

\[
\mu = -\frac{1}{2 R R^2} \left\{ r^2 - 2r \left( M(u) - m_1(u) \right) + a^2 + e^2(u) - \frac{\Lambda^* r^4}{3} \right\}, \tag{3.59}
\]

\[
\gamma = \frac{1}{2 R R^2} \left\{ \left( r - \left( M(u) - m_1(u) \right) - \frac{\Lambda^* r^4}{3} \right) R 
- \left\{ r^2 - 2r \left( M(u) - m_1(u) \right) + a^2 + e^2(u) - \frac{2\Lambda^* r^4}{3} \right\} \right\}, \tag{3.60}
\]

\[
\psi_2 = \frac{1}{R R^2 R^2} \left\{ - \left( M(u) - m_1(u) \right) R + e^2(u) + \frac{\Lambda^* r^2}{3} a^2 \cos^2 \theta \right\}, \tag{3.61}
\]

\[
\phi_{11} = \frac{1}{2 R^2 R^2} \left\{ e^2(u) - \Lambda^* r^2 a^2 \cos^2 \theta \right\}, \tag{3.61}
\]

\[
\Lambda = \Lambda^{(C)} = \frac{\Lambda^* r^2}{6 R^2}. \tag{3.62}
\]
We have seen the changes in $\mu$, $\gamma$, $\psi_2$, $\psi_3$ and $\psi_4$ but changes in $\psi_3$ and $\psi_4$ are same as (3.28) of case 3(ii) and there is no change in $\phi_1^{(C)}$ and $\Lambda^{(C)}$. Thus, the rotating solution (3.46) with a new real function $m_1(u)$ after the first radiation becomes

\[
\begin{align*}
   ds^2 &= \left[1 - R^{-2}\left\{2r\left(M(u) - m_1(u)\right) - e^2(u) + \frac{\Lambda^* r^4}{3}\right\}\right] du^2 + 2du dr \\
   &\quad + 2aR^{-2}\left\{2r\left(M(u) - m_1(u)\right) - e^2(u) + \frac{\Lambda^* r^4}{3}\right\} \sin^2\theta du d\phi - 2a \sin^2\theta dr d\phi \\
   &\quad - R^2d\theta^2 - \left\{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta\right\} R^{-2}\sin^2\theta d\phi^2,
\end{align*}
\]

where $\Delta^* = r^2 - 2r\left\{M(u) - m_1(u)\right\} + a^2 + e^2(u) - \Lambda^* r^4/3$. This suggests that the first electrical radiation of rotating black hole leads to a reduction of the gravitational mass $M(u)$ by a quantity $m_1(u)$ with the unaffected Maxwell scalar $\phi_1$ and the constant $\Lambda^{(C)}$. If we consider another radiation by taking $e(u)$ in (3.63) to be a function of $u$ and $r$ with the mass $M(u) - m_1(u)$ and the decomposition (3.45), then the Einstein-Maxwell field equations yield to reduce this mass by another quantity $m_2(u)$ (say); i.e., after the second radiation, the mass will become $M(u) - \{m_1(u) + m_2(u)\}$. Here again, the Maxwell scalar $\phi_1$ and the constant $\Lambda^{(C)}$ remain unaffected after the second radiation also. Thus, if we consider $n$ radiations every time taking the charge $e$ to be function of $u$ and $r$ with the decomposition of $\Lambda$, the Maxwell scalar $\phi_1$ will be the same, but the metric will take the form:

\[
\begin{align*}
   ds^2 &= \left[1 - R^{-2}\left\{2rM(u) - e^2(u) + \frac{\Lambda^* r^4}{3}\right\}\right] du^2 + 2du dr \\
   &\quad + 2aR^{-2}\left\{2rM(u) - e^2(u) + \frac{\Lambda^* r^4}{3}\right\} \sin^2\theta du d\phi - 2a \sin^2\theta dr d\phi \\
   &\quad - R^2d\theta^2 - \left\{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta\right\} R^{-2}\sin^2\theta d\phi^2,
\end{align*}
\]

where the total mass of the black hole, after the $n$ radiations will be of the form

\[
M(u) = M(u) - \{m_1(u) + m_2(u) + m_3(u) + m_4(u) + \ldots + m_n(u)\}.
\]

Taking Hawking’s radiation of black holes, we can expect that the total mass of black hole will be radiated away just leaving $\mathcal{M}(u)$ equivalent to Planck mass of about $10^{-5}$ g, that is, $M(u)$ may not be exactly equal to $m_1(u) + m_2(u) + m_3(u) + m_4(u) + \ldots + m_n(u)$, but has a difference of about Planck-size mass, as in the case of non-rotating black hole. Otherwise, the total mass of black hole will be evaporated completely after continuous radiation when $\mathcal{M}(u) = 0$, that is, $M(u) = m_1(u) + m_2(u) + m_3(u) + m_4(u) + \ldots + m_n(u)$. Here the rotating variable-charged black hole might completely radiate away its mass just leaving the electrical charge $e(u)$ and the cosmological constant $\Lambda^*$. We find this situation in the form of classical space-time metric as

\[
\begin{align*}
   ds^2 &= \left\{1 + \left(e^2(u) - \frac{\Lambda^* r^4}{3}\right) R^{-2}\right\} du^2 + 2du dr \\
   &\quad - 2a R^{-2}\left(e^2(u) - \frac{\Lambda^* r^4}{3}\right) \sin^2\theta du d\phi - 2a \sin^2\theta dr d\phi \\
   &\quad - R^2d\theta^2 - \left\{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta\right\} R^{-2}\sin^2\theta d\phi^2,
\end{align*}
\]
with the charge \( e(u) \) and the cosmological constant \( \Lambda^* \), but no mass, where \( \Delta^* = r^2 + a^2 + e^2(u) - \Lambda^* r^4/3 \). The metric (3.66) will describe a rotating ‘instantaneous’ charged cosmological black holes. At this stage, the Weyl scalar \( \psi_2 \) takes the form

\[
\psi_2 = \frac{1}{RR R^2} \left\{ e^2(u) + \frac{\Lambda^* r^2}{3} a^2 \cos^2 \theta \right\}, \tag{3.67}
\]

where as changed in \( \psi_3 \) and \( \psi_4 \) are remain same as in (3.34) and (3.35). At this stage we shall mention some important parameters of the charged cosmological black hole. That is, the surface gravity of the horizon at \( r = r_{++} \) is

\[
K = -\left[ \frac{1}{r R^2} \left\{ r \left( r - \frac{\Lambda^* r^3}{6} \right) + \frac{e^2(u)}{2} \right\} \right]_{r=r_{++}}, \tag{3.68}
\]

with the Hawking’s temperature \( T_H = K/2\pi \). The entropy and angular velocity of the black hole at the horizon are found as follows

\[
S = \pi (r^2 + a^2)_{r=r_{++}}, \quad \text{and} \quad \Omega_H = \frac{-a \left\{ e^2(u) - \Lambda^* r^4/3 \right\}}{(r^2 + a^2)^2} \bigg|_{r=r_{++}}. \tag{3.69}
\]

Here, the value of \( r_{\pm} \) may be obtained from appendix (A15). At this instant, the gravity on the surface of the remaining solution depending on the electric charge \( e(u) \) and the cosmological constant \( \Lambda^* \) coupling with the rotational parameter \( a \) can be seen in (3.67); however, the Maxwell scalar \( \phi_1 \) and the Ricci scalar \( \Lambda^{(C)} \) remain the same as in (3.58) and (3.54) respectively. The formation of charged cosmological black hole (3.66) leads the proof of one part of the theorem 3 cited above for the case embedded into de Sitter space. Here the idea of this complete evaporation of mass of radiating black hole embedded into the de Sitter space is in agreement with that of Hawking’s evaporation of black holes. It is worth studying the nature of such rotating black holes (3.66). This might give a different physical nature, which one has not been seen in common theory of black holes embedded into the de Sitter space. Here, we again consider the charge \( e(u) \) to be function of \( u \) and \( r \) for the next radiation in (3.66), so that we get, from the Einstein’s field equations, the scalar \( \Lambda^{(E)} \) as given in equation (3.55) and the same scalar \( \Lambda^{(C)} \) as in (3.54). Then the vanishing of this \( \Lambda^{(E)} \) for electromagnetic field will lead to create a new mass (say \( m_1^*(u) \)) in the remaining space-time geometry (3.66). For the second radiation, we again consider the charge \( e(u) \) to be function of \( u \) and \( r \) in the field equations with the mass \( m_1^*(u) \). Then the vanishing of \( \Lambda^{(E)} \) will lead to increase the new mass by another quantity \( m_2^*(u) \) (say) i.e., after the second radiation of (3.66) the new mass will be \( m_1^*(u) + m_2^*(u) \). If this radiation process continues further for a long time, the new mass will increase gradually as

\[
\mathcal{M}^*(u) = m_1^*(u) + m_2^*(u) + m_3^*(u) + m_4^*(u) + .... \tag{3.70}
\]

Then, the spacetime metric will take the form

\[
ds^2 = \left[ 1 + R^{-2} \left\{ 2r \mathcal{M}^*(u) + e^2(u) - \frac{\Lambda^* r^4}{3} \right\} \right] du^2 + 2du dr - 2a R^{-2} \left\{ 2r \mathcal{M}^*(u) + e^2(u) - \frac{\Lambda^* r^4}{3} \right\} \sin^2 \theta \, du \, d\phi - 2a \sin^2 \theta \, dr \, d\phi
\]
\[-R^2 d\theta^2 - \left\{ \left( r^2 + a^2 \right)^2 - \Delta^* a^2 \sin^2 \theta \right\} R^{-2} \sin^2 \theta \, d\phi^2, \tag{3.71} \]

where \( \Delta^* = r^2 + 2r M^* (u) + a^2 + e^2 (u) - \Lambda^* r^4 / 3 \). The changed NP quantities \( \psi_2, \psi_3, \psi_4 \) and \( \mu \) are as follows

\[
\psi_2 = \frac{1}{RRR^2} \{ M^* (u) R + e^2 (u) + \frac{\Lambda^* r^2}{3} a^2 \cos^2 \theta \}, \tag{3.72} \]

\[
\mu = - \frac{1}{2RR^2} \left\{ r^2 + 2r M^* (u) + a^2 + e^2 (u) - \frac{\Lambda^* r^4}{3} \right\}, \tag{3.73} \]

with \( \psi_3 \) and \( \psi_4 \), given as in (3.43), and \( \phi_1 \) and \( \Lambda^{(C)} \) remain same as in (3.58) and (3.54). Comparing the metrics (3.64) and (3.71), we find that the classical spacetime (3.71) describes a rotating negative mass naked singularity embedded into the de Sitter cosmological space. Thus, from the above it follows the proof of theorem 5 in the case of de Sitter space. We have also shown the possible changes in the mass of the rotating charged Vaidya-Bonnor-de Sitter black hole without affecting the Maxwell scalar \( \phi_1 \) as well as the cosmological constant \( \Lambda^* \) and accordingly, metrics are cited for future use. Thus, this completes the proof of the theorem 1 based on the rotating charged de Sitter black hole. Also since there is no effect on the cosmological constant \( \Lambda^* \) during Hawking’s evaporation process, it will always remain unaffected. That is, unless some external forces apply to remove the cosmological constant \( \Lambda^* \) from the spacetime geometries, it will continue to exist along with the electrically radiating objects, rotating or non-rotating forever. This leads to the proof of the theorem 6 cited above for the rotating black holes. The metric (3.71) can be written in Kerr-Schild form:

\[
g^{\text{NMDS}}_{ab} = g^{\text{ds}}_{ab} + 2Q(u, r, \theta) \ell_a \ell_b \]

where \( Q(u, r, \theta) = \{ r M(u) + e^2 (u)/2 \} R^{-2} \), where \( g^{\text{ds}}_{ab} \) is the rotating de Sitter cosmological metric.

(iv) Variable-charged rotating Vaidya-Bonnor-monopole Black hole

Here, we shall incorporate the Hawking radiation, how the rotating variable-charged black hole is affected in the classical space-time metric when the electric charge \( e \) is taken as a function of \( u \) and \( r \) in the Einstein-Maxwell field equations. The line element for the rotating Vaidya-Bonnor black hole with \( e(u, r) \) is

\[
ds^2 = [1 - R^{-2} \{ 2r M(u) - e^2 (u, r) + br \} ] du^2 + 2du \, dr
+ 2a R^{-2} \{ 2r M(u) - e^2 (u, r) + br \} \sin^2 \theta \, du \, d\phi - 2a \sin^2 \theta \, dr \, d\phi
- R^2 d\theta^2 - \left\{ r^2 + a^2 \right\} R^{-2} \sin^2 \theta \, d\phi^2, \tag{3.74} \]

where \( \Delta^* = r^2 (1 - b) - 2r M(u) + a^2 + e^2 (u, r) \). This metric will also reduce to rotating Vaidya-Bonnor-monopole solution when \( e(u, r) = e(u) \), initially. Then the Einstein-Maxwell field equations for the above metric with \( e(u, r) \) can be solved to obtain the affected quantities with the monopole constant \( b \):

\[
\psi_2 = \frac{1}{RRR^2} \left\{ - R M(u) + e^2 (u, r) \right\} - Re(u, r) e(u, r), \]

\[
\psi_3 = \frac{1}{RRR^2} \left\{ - R M(u) + e^2 (u, r) \right\} - Re(u, r) e(u, r),
\]

\[
\psi_4 = \frac{1}{RRR^2} \left\{ - R M(u) + e^2 (u, r) \right\} - Re(u, r) e(u, r),
\]

\[
\mu = \frac{1}{2RR^2} \left\{ r^2 + 2r M(u) + a^2 + e^2 (u) - \frac{\Lambda^* r^4}{3} \right\},
\]

\[
Q(u, r, \theta) = \{ r M(u) + e^2 (u)/2 \} R^{-2},
\]

\[
g^{\text{ds}}_{ab} = \{ r M(u) + e^2 (u)/2 \} R^{-2}.
\]
Accordingly, the Maxwell scalar may become, after identifying the function \( \psi_3 \) and \( \psi_4 \) are same as in (3.20) of case 3(ii). According to the total energy momentum tensor, we shall, without loss of generality, have the following decompositions for Vaidya-Bonnor-monopole:

\[
\begin{align*}
\phi_{11}^{(E)} &= \frac{1}{4 R^2 R^2} \left[ 2 e^2(u,r) - 4re(u,r) e(u,r)_r \right. \\
&\left. + R^2 \{e(u,r) e(u,r)_r \}, \right] \\
\phi_{11}^{(m)} &= \frac{1}{4 R^2 R^2} b (r^2 - a^2 \cos^2 \theta), \\
\Lambda^{(E)} &= - \frac{1}{12 R^2} \{e^2(u,r)_r + e(u,r) e(u,r)_r \}, \\
\Lambda^{(m)} &= \frac{1}{12 R^2}.
\end{align*}
\]  

(3.78)

For electromagnetic field, the Ricci scalar \( \Lambda^{(E)} \) given above must vanish. This yields

\[
e^2(u,r) = 2 rm_1(u) + C(u)
\]

(3.79)

where \( m_1(u) \) and \( C(u) \) are real functions of \( u \). Then, using this result in equation (3.77) we obtain the Ricci scalar

\[
\phi_{11}^{(E)} = \frac{C(u)}{2 R^2 R^2}.
\]

(3.80)

Accordingly, the Maxwell scalar may become, after identifying the function \( C(u) = e^2(u) \),

\[
\phi_1 = \frac{e(u)}{\sqrt{2 RR}}.
\]

(3.81)

Then, using the relation (3.79) in (3.75), we find the changed Weyl scalar \( \psi_2 \)

\[
\psi_2 = \frac{1}{RR R^2 R^2} \left[ - \{M(u) - m_1(u)\} R + e^2(u) - \frac{b}{6} \left( RR + 2 r a \cos \theta \right) \right],
\]

(3.82)

(3.83)

but changes in \( \psi_3 \) and \( \psi_4 \) are same as (3.28) of case 3(ii). There is no change in monopole constant \( b \). Thus, we have the line element with the change of mass as

\[
ds^2 = \left[ 1 - R^{-2} \left\{ 2r \left( M(u) - m_1(u) \right) - e^2(u) + 2r^2 \right\} \right] du^2 + 2du \, dr \\
+ 2a R^{-2} \{2r \left( M(u) - m_1(u) \right) - e^2(u) + 2r^2 \} \sin^2 \theta \, du \, d\phi - 2a \sin^2 \theta \, dr \, d\phi \\
- R^2 d\theta^2 - \{r^2 + a^2 - \Delta^* a^2 \sin^2 \theta \} R^{-2} \sin^2 \theta \, d\phi^2,
\]

(3.84)

where \( \Delta^* = r^2(1 - b) - 2r \{M(u) - m_1(u)\} + a^2 + e^2(u) \). This introduction of real function \( m_1(u) \) in the metric (3.84) suggests that the first electrical radiation of rotating
black hole may reduce the original gravitational mass $M(u)$ by a quantity $m_1(u)$. If one considers another radiation by taking $e(u)$ in (3.84) to be a function of $u$ and $r$ with the mass $M(u) - m_1(u)$, then the Einstein-Maxwell field equations yield to reduce this mass by another real function $m_2(u)$; i.e., after the second radiation, the mass may become $M(u) - m_1(u) - m_2(u)$. Here again, the Maxwell scalar $\phi_1$ remains the same form after the second radiation also. Thus, if one considers the $n$-time radiations taking every time the charge $e(u)$ to be function of $u$, the Maxwell scalar $\phi_1$ will remains the same. Taking Hawking’s radiation of black holes, one might expect that the total mass of black hole may be radiated away just leaving $M(u)$ equivalent to Planck mass of about $10^{-5} g$, that is, $M(u)$ may not be exactly equal to $m_1(u) + m_2(u) + m_3(u) + m_4(u) + \ldots + m_n(u)$, but has a difference of about Planck-size mass, as in the case of non-rotating black hole. Otherwise, the total mass of black hole may be evaporated completely after continuous radiation when $M(u) = 0$, that is, $M(u) = m_1(u) + m_2(u) + m_3(u) + m_4(u) + \ldots + m_n(u)$. Here one may regard that the rotating variable-charged black hole might be radiated completely away all its mass just leaving the electrical charge $e(u)$ and monopole constant $b$ only.

One could observe this situation in the form of classical space-time metric as

$$
ds^2 = \left[1 + R^{-2}\{e^2(u) - br^2\}\right] du^2 + 2du dr - 2ar^{-2}\{e^2(u) - br^2\} \sin^2 \theta du d\phi - 2a \sin^2 \theta dr d\phi - R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta\} R^{-2} \sin^2 \theta d\phi^2,
$$

(3.85)

with the charge $e(u)$ and the monopole constant $b$, but no mass, where $\Delta^* = r^2 (1 - b) + a^2 + e^2(u)$. The metric (3.85) will describe a rotating ‘instantaneous’ charged monopole black hole as the remnant of Vaidya-Bonnor-monopole space, otherwise the metric describes the rotating charged monopole black hole with the horizons at $r_\pm = \pm \frac{1}{\sqrt{b - 1}} \{a^2 + e^2(u)\}$. Hence one may refer to the solution (3.85) as an instantaneous’ charged black hole with the surface gravity,

$$\mathcal{K} = -\frac{1}{r_+ R^2} \left[r_+ \left\{r_+ \left(1 - \frac{b}{2}\right) + \frac{e^2(u)}{2}\right\}\right].
$$

(3.86)

The entropy and angular velocity of the horizon are respectively obtained by

$$\mathcal{S} = \pi (r_+^2 + a^2), \quad \text{and} \quad \Omega_H = \frac{-a \{e^2(u) - br^2\}}{(r^2 + a^2)^2} \bigg|_{r = r_+}.
$$

(3.87)

At this stage, the Weyl scalar $\psi_2$ takes the form

$$\psi_2 = \frac{1}{RR R^2} \left\{e^2(u) - \frac{b}{6} \left(R R + 2 r a i \cos \theta\right)\right\},
$$

(3.88)

and $\psi_3, \psi_4$ are remained same as in (3.34) and (3.35) showing the gravity on the surface of the remaining solution depending on the electric charge $e(u)$ and the monopole charge $b$ coupling with the rotational parameter $a$; however, the Maxwell scalar $\phi_1$ and the Ricci scalar $\Lambda^{(m)}$ remain the same as in (3.81) and (3.78) respectively. This completes the other part of the theorem 3 cited above for the case embedded into the monopole universe. It means that there may be rotating black holes in the universe whose masses
are completely radiated; their gravity depends on the electric charge of the body and the monopole charge $b$, and their metrics appear similar to that in (3.85). Here the idea of this complete evaporation of masses of radiating black holes embedded in the monopole space is in agreement with that of Hawking’s evaporation of black holes. Here, we again consider the charge $e(u)$ to be function of $u$ and $r$ for next radiation in (3.85), so that from the Einstein’s field equations, we get the scalar $\Lambda^{(E)}$ as given above and the same scalar $\Lambda^{(m)}$ as in (3.78). Then the vanishing of this $\Lambda^{(E)}$ for electromagnetic field will lead to create a new mass (say $m_1^*(u)$) in the remaining space-time geometry (3.85). For the second radiation, we again consider the charge $e(u)$ to be function of $u$ and $r$ in the field equations with the mass $m_1^*(u)$. Then the vanishing of $\Lambda^{(E)}$ will lead to increase the new mass by another quantity $m_2^*(u)$ (say) i.e., after the second radiation of (3.85) the new mass will be $m_1^*(u) + m_2^*(u)$. If this radiation process continues further for a long time, the new mass will increase gradually as

$$M^*(u) = m_1^*(u) + m_2^*(u) + m_3^*(u) + m_4^*(u) + ...$$  (3.89)

Then, the spacetime metric will take the form

$$ds^2 = \left[ 1 + R^{-2} \left\{ 2rM^*(u) + e^2(u) - b r^2 \right\} du^2 + 2du dr 
-2aR^{-2} \left\{ 2rM^*(u) + e^2(u) - b r^2 \right\} \sin^2 \theta du d\phi - 2a \sin^2 \theta dr d\phi 
-R^2 d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* \right\} R^{-2} \sin^2 \theta d\phi^2. \right]$$  \hspace{1cm} (3.90)

where $\Delta^* = r^2(1 - b) + 2rM^*(u) + a^2 + e^2(u)$. The changed NP quantities $\psi_2$, $\psi_3$ and $\psi_4$ are as follows

$$\psi_2 = \frac{1}{RRR^2} \left\{ M^*(u)R + e^2(u) - \frac{b}{6} \left( RR + 2ra \cos \theta \right) \right\}, \hspace{1cm} (3.91)$$

and $\psi_3$, $\psi_4$ are given in (3.43) and $\phi_1$ and $\Lambda^{(m)}$ are remained unchanged. Here we find that the classical spacetime (3.90) describes a rotating negative mass naked singularity embedded into the monopole space. Thus, from the above it follows the proof of the rotating part of theorem 5 that **during the radiation process, after the complete evaporation of masses of variable-charged non-stationary embedded black holes, the electrical radiation will continue indefinitely creating embedded negative mass naked singularities.**

We have also shown the possible changes in the mass of the rotating variable charged Vaidya-Bonnor-monopole black hole without affecting the Maxwell scalar $\phi_1$ as well as the monopole constant $b$, and accordingly, metrics are cited for future use. Thus, this completes the proof of other part of the theorem 1 for the embedded rotating Vaidya-Bonnor-monopole black hole. Also since there is no effect on the monopole constant $b$ during Hawking’s evaporation process, it will always remain unaffected. That is, unless some external forces apply to remove the monopole constant $b$ from the spacetime geometries, it will continue to exist along with the electrically radiating rotating objects forever. This leads to the proof of the theorem 6 for rotating embedded black holes that **if an electrically radiating non-stationary black hole is embedded into a space, it will...**

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continue to embed into the same space forever. The metric (3.90) can be expressed in Kerr-Schild form on the monopole background

\[ g^{\text{NMm}}_{ab} = g^m_{ab} + 2Q(u, r, \theta)\ell_a \ell_b \]

where \( Q(u, r, \theta) = \{r\mathcal{M}(u) + e^2(u)/2\}R^{-2} \). Here, \( g^m_{ab} \) is the rotating monopole metric and \( \ell_a \) is geodesic, shear free, expanding and non-zero twist null vector for both \( g^m_{ab} \) as well as \( g^{\text{NMm}}_{ab} \).

(v) Variable charged rotating Vaidya-Bonnor-Kerr black hole

Here, one may incorporate the Hawking radiation, how the rotating variable-charged black hole affect in the classical space-time metric when the electric charge \( e \) is taken as a function of \( u \) and \( r \) in the Einstein-Maxwell field equations. The line element rotating Vaidya-Bonnor-Kerr black hole with \( e(u, r) \) is

\[
\begin{align*}
\text{ds}^2 &= \left[ 1 - R^{-2}\left\{2r\left(\bar{m} + M(u)\right) - e^2(u, r)\right\}\right] du^2 + 2du dr \\
&\quad + 2aR^{-2}\left\{2r\left(\bar{m} + M(u)\right) - e^2(u, r)\right\}\sin^2\theta du d\phi \\
&\quad - 2\sin^2\theta dr d\phi - R^2d\theta^2 - \left\{(r^2 + a^2) - \Delta^* \sin^2\theta\right\}R^{-2}\sin^2\theta d\phi^2,
\end{align*}
\]

(3.92)

where \( \Delta^* = r^2 - 2r\bar{m} + M(u) + a^2 + e^2(u, r) \). This metric will also reduce to rotating Vaidya-Bonnor-Kerr solution when \( e(u, r) = e(u) \) initially, the Einstein-Maxwell field equations for the above metric with \( e(u, r) \) can be solved to obtain the following quantities:

\[
\begin{align*}
\Lambda &= -\frac{1}{12}R^2\left\{e^2(u, r), r + e(u, r) e(u, r), rr\right\}, \\
\psi_2 &= \frac{1}{RRR^2} \left[ - R\left\{\bar{m} + M(u)\right\} + e^2(u, r) + \frac{R R}{6} \left\{e(u, r) e(u, r), r\right\}\right],
\end{align*}
\]

(3.93)

(3.94)

where \( \psi_3 \) and \( \psi_4 \) are same as (3.20) and \( \phi_{11}, \phi_{12} \) and \( \phi_{22} \) are affected as in (3.21). The scalar \( \Lambda = \Lambda^E \) must vanish for this rotating metric. Thus, vanishing \( \Lambda^E \) of the equation (3.93) implies that

\[ e^2(u, r) = 2rm_1(u) + C(u) \]

(3.95)

where \( m_1(u) \) and \( C(u) \) are real functions of \( u \). Then, using this result, we obtain the Ricci scalar

\[
\phi_{11} = \frac{C(u)}{2R^2 R^2}.
\]

(3.96)

Accordingly, the Maxwell scalar may become, after identifying the function \( C(u) = e^2(u) \),

\[
\phi_1 = \frac{e(u)}{\sqrt{2RR}}.
\]

(3.97)

Then using the relation (3.95) in (3.94), we find the changed Weyl scalar

\[
\psi_2 = \frac{1}{RRR^2} \left[ - R\left\{\bar{m} + M(u) - m_1(u)\right\} + e^2(u)\right],
\]

(3.98)
and changes in Weyl scalars $\psi_3$ and $\psi_4$ are same as (3.28) of case 3(ii). Thus, we have the line element with the change of mass as

$$
\begin{align*}
  ds^2 &= \left[ 1 - R^{-2}\left\{ 2r \left( \bar{m} + M(u) - m_1(u) \right) - e^2(u) \right\} \right] du^2 + 2du\,dr \\
  &\quad + 2a R^{-2} \left\{ 2r \left( \bar{m} + M(u) - m_1(u) \right) - e^2(u) \right\} \sin^2\theta \, du \, d\phi \\
  &\quad - 2a \sin^2\theta \, dr \, d\phi - R^2 \, d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta \right\} R^{-2} \sin^2\theta \, d\phi^2
\end{align*}
$$

(3.99)

where $\Delta^* = r^2 - 2r \bar{m} + m_1(u) + a^2 + e^2(u)$. This introduction of real function $m_1(u)$ in the metric (3.99) suggests that the first electrical radiation of rotating black hole may reduce the original gravitational mass $M(u)$ by a quantity $m_1(u)$. If one considers another radiation by taking $e(u)$ in (3.99) to be a function of $u$ and $r$ with the mass $M(u) - m_1(u) + \bar{m}$, then the Einstein-Maxwell field equations yield to reduce this mass by another real function $m_2(u)$; i.e., after the second radiation, the mass may become $M(u) - \{ m_1(u) + m_2(u) \} + \bar{m}$. Here again, the Maxwell scalar $\phi_1$ remains the same form after the second radiation also. Thus, if one considers the $n$-time radiations taking every time the charge $e(u)$ to be function of $u$, the Maxwell scalar $\phi_1$ will be the same. Taking Hawking’s radiation of black holes, one might expect that the total mass of black hole may be radiated away just leaving $M(u)$ equivalent to Planck mass of about $10^{-5} g$ and $m$ remain same. that is, $M(u)$ may not be exactly equal to $m_1(u) + m_2(u) + m_3(u) + m_4(u) + ... + m_n(u)$, but has a difference of about Planck-size mass, as in the case of non-rotating black hole. Otherwise, the total mass of black hole may be evaporated completely after continuous radiation when $M(u) = m_1(u) + m_2(u) + m_3(u) + m_4(u) + ... + m_n(u)$, just leaving the mass $m$ and electrical charge $e(u)$ only. Thus, one could observe this situation in the form of classical space-time metric as

$$
\begin{align*}
  ds^2 &= \left[ 1 - R^{-2}\left\{ 2r \bar{m} - e^2(u) \right\} \right] du^2 + 2du\,dr \\
  &\quad + 2a R^{-2} \left\{ 2r \bar{m} - e^2(u) \right\} \sin^2\theta \, du \, d\phi \\
  &\quad - 2a \sin^2\theta \, dr \, d\phi - R^2 \, d\theta^2 \\
  &\quad - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta \right\} R^{-2} \sin^2\theta \, d\phi^2,
\end{align*}
$$

(3.100)

where $\Delta^* = r^2 - 2r \bar{m} + a^2 + e^2(u)$. The metric (3.100) will describe a rotating ‘instantaneous’ charged black hole, i.e. a rotating charged Kerr black hole with $\bar{m} > a^2 + e^2(u)$. At this stage, the Weyl scalar $\psi_2$ takes the form

$$
\psi_2 = \frac{1}{R \, R^2} \left\{ - R \, \bar{m} + e^2(u) \right\},
$$

(3.101)

and $\psi_3$, $\psi_4$ are remain same as (3.34) and (3.35). The surface gravity of the horizon at $r = r_+ = \bar{m} + \sqrt{\bar{m}^2 - \{a^2 + e^2(u)\}}$, is

$$
K = - \frac{1}{r_+ \, R^2} \left\{ r_+ \sqrt{\bar{m}^2 - a^2 - e^2(u)} + \frac{e^2(u)}{2} \right\}.
$$

(3.102)

Then we find the entropy and angular velocity of the horizon respectively as follows:

$$
S = 2\pi \bar{m} \left\{ \bar{m} + \sqrt{\bar{m}^2 - a^2 - e^2(u)} \right\} - e^2(u) \quad \text{and} \quad \Omega_H = \frac{a \, \left\{ 2r \, \bar{m} - e^2(u) \right\}}{(r^2 + a^2)^2} \bigg|_{r = r_+}
$$

(3.103)
showing the gravity on the surface of the remaining solution depending on the electric charge \( e(u) \) and \( \tilde{m} \); however, the Maxwell scalar \( \phi_1 \) remains the same as in (3.97). Here, one may consider again the charge \( e(u) \) to be function of \( u \) and \( r \) for next radiation in (3.100), so that one must get from the Einstein’s field equations the scalar \( \Lambda \) as given in equation (3.93). Then the vanishing of this \( \Lambda \) for electromagnetic field, there may be creation of a new mass (say \( m^*_1(u) \)) in the remaining space-time geometry. If this radiation process continues forever, the new mass may increase gradually as

\[
\mathcal{M}^*(u) = m^*_1(u) + m^*_2(u) + m^*_3(u) + m^*_4(u) + \ldots.
\]  

(3.104)

However, it appears that this new mass would never decrease. Then the space-time geometry may take the form

\[
\begin{align*}
 ds^2 & = \left[ 1 + R^{-2} \left\{ 2r \left( \mathcal{M}^*(u) - \tilde{m} \right) + e^2(u) \right\} \right] du^2 + 2du \, dr \\
 & - 2a R^{-2} \left\{ 2r \left( \mathcal{M}^*(u) - \tilde{m} \right) + e^2(u) \right\} \sin^2 \theta \, du \, d\phi - 2a \sin^2 \theta \, dr \, d\phi \\
 & - R^2 \sin^2 \theta \, d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta \right\} R^{-2} \sin^2 \theta \, d\phi^2,
\end{align*}
\]  

(3.105)

where, \( \Delta^* = r^2 + 2r \left( \mathcal{M}^*(u) - \tilde{m} \right) + a^2 + e^2(u) \). The Weyl scalar \( \psi_2 \) and other NP coefficients are calculated from the Einstein-Maxwell field equations as

\[
\psi_2 = \frac{1}{R R R R^2} \left[ R \left\{ \mathcal{M}^*(u) - \tilde{m} \right\} + e^2(u) \right],
\]  

(3.106)

with \( \psi_3, \psi_4 \), given in (3.43) and \( \phi_1 \) and \( \Lambda^{(E)} \) are remain unchanged. The metric (3.105) may be regarded to describe a rotating negative mass naked singularity embedded into Kerr black hole. We have presented the possible changes in the mass of the rotating charged black hole without affecting the Maxwell scalar \( \phi_1 \) and Kerr mass accordingly, metrics are cited for future use. Thus, this completes the proofs of other parts of the theorem 5 that the electrical radiation will continue indefinitely creating embedded negative mass naked singularities. The metric (3.105) can be written in Kerr-Schild form on the Kerr background as

\[
g^{\text{N MK}}_{ab} = g^K_{ab} + 2Q(u, r, \theta) \ell_a \ell_b
\]  

(3.107)

where \( Q(u, r, \theta) = \{ r \mathcal{M}(u) + e^2(u)/2 \} R^{-2} \). Here, \( g^K_{ab} \) is the rotating Kerr metric and \( \ell_a \) is geodesic, shear free, expanding and non-zero twist null vector for both \( g^K_{ab} \) as well as \( g^{\text{N MK}}_{ab} \).

3. Conclusion

In this paper, we have presented NP quantities for a rotating spherically symmetric metric with two variables \( u, r \) in the appendix (A1) below. With the help of these NP quantities, we have first derived a class of non-stationary rotating solutions including Vaidya-Bonnor-de Sitter, Vaidya-Bonnor-monopole, and Vaidya-Bonnor-Kerr. Then we studied the gravitational structure of the solutions by observing the nature of the energy
momentum tensors of respective spacetime metrics. These solutions describe embedded rotating black holes. For example, Vaidya-Bonnor black hole is embedded into the rotating de Sitter cosmological space to produce the Vaidya-Bonnor-de Sitter cosmological black hole and similarly, Vaidya-Bonnor-monopole and Vaidya-Bonnor-Kerr black holes. The embedded rotating solutions have also been expressed in terms of Kerr-Schild ansätze in order to indicate them as solutions of Einstein’s field equations. These ansätze show the extensions of those of Glass and Krisch [24] and Xanthopoulos [25]. This completes the proof of the theorem 7 that every embedded black hole, stationary or non-stationary, is expressible in Kerr-Schild ansatz. The theorem 7 is valid for both the stationary and non-stationary embedded black holes here.

The remarkable feature of the analysis of rotating solutions in this paper is that all the rotating solutions are non-stationary algebraically special of the Petrov classification of spacetime metric, possess the same null vector $\ell_a$, given in (A2), which is geodesic, shear free, expanding as well as non-zero twist (A7). From the study of the rotating solutions we find that some solutions after making rotation have disturbed their gravitational structures. For example, after making rotation, the Vaidya metric with $e(u) = \Lambda^* = 0$ in (2.19) becomes algebraically special in Petrov classification with the non-vanishing $\psi_2$, $\psi_3$ and $\psi_4$ (2.22), and a null vector $\ell_a$ which is geodesic, shear free, expanding and non-zero twist. Similarly, the rotating de Sitter solution (2.3) with $M(u) = e(u) = 0$ becomes Petrov type D spacetime metric, where the rotating parameter $a$ is coupled with the cosmological constant in the expressions of $\phi_{11} = -(1/2 R^2 R^2) \Lambda^* r^2 a^2 \cos^2 \theta$, and $\psi_2 = (1/3 R^2 R^2) \Lambda^* r^2 a^2 \cos^2 \theta$. The rotating monopole solution (2.18) with $M(u) = e(u) = 0$ possesses the energy momentum tensor with the monopole charge $b$ couples with the rotating parameter $a$ as in $p = (-1/2 R^2 R^2) b a^2 \cos^2 \theta$. The method adopted here with Wang-Wu functions might be another possible version for obtaining non-stationary rotating black hole solutions with visible energy momentum tensors describing the interaction of different matter fields with well-defined physical properties like Guth’s modification of $T_{ab}$ (2.14). It is believed that such interactions of different matter fields as in (2.9), (2.14) and (2.26) have not seen published before. We have also found the direct involvement of the rotation parameter $a$ in each expression of the surface gravity and the angular velocity, which shows the important of the study of non-stationary rotating embedded, black holes in order to understand the nature of different black holes located in the universe.

In section 3, we find that the changes in the masses of non-embedded as well as embedded black holes take place due to the vanishing of Ricci scalar of electromagnetic fields with the charge $e(u,r)$. It is also shown that the Hawking’s radiation can be expressed in classical spacetime metrics, by considering the charge $e(u)$ to be a function of $u$, and $r$ of Vaidya-Bonnor-de Sitter, Vaidya-Bonnor-monopole and Vaidy-Bonnor-Kerr black holes. That is, every electrical radiation produces a change in the mass of the non-stationary charged black holes. It may be concluded the proof of the theorem 1. These changes in the mass of black holes, non-embedded [non-rotating (3.1) and rotating (3.18)] and embedded into de Sitter, monopole and Kerr spaces, after every
electrical radiation, describe the relativistic aspect of Hawking’s evaporation of masses of black holes in the classical spacetime metrics. Thus, we find that the black hole evaporation process is due to the electrical radiation of the variable charge $e(u,r)$ in the energy momentum tensor describing the change in the mass in classical spacetime metrics which is in agreement with Boulware’s suggestion [7]. The Hawking’s evaporation of masses and the creation of non-embedded negative mass naked singularities (3.16), (3.40), and embedded ones (3.71), (3.90) and (3.105), are also due to the continuous electrical radiation. This suggests that, if one accepts the continuous electrical radiation to lead the complete evaporation of the original non-stationary mass of black holes, then the same radiation will also lead to the creation of new non-stationary mass to form negative mass naked singularities. This clearly indicates that an electrically radiating embedded black hole will not disappear completely, which is against the suggestion made in [2, 3, 5]. It is noted that we observe the different results from the study of embedded and non-embedded black holes. In the embedded cases here above, the presence of other quantities like the cosmological constant $\Lambda^*$ (3.66), the monopole constant $b$ (3.85) and the Kerr mass $\tilde{m}$ (3.100) presumably prevent the disappearance of embedded radiating black holes during the radiation process, and thereby, the formation of ‘instantaneous’ rotating charged black holes (3.66), (3.85) and (3.100). In non-embedded cases (3.15) and (3.32), the disappearance of a black hole during radiation process is unavoidable, however occurs for an instant with the formation of ‘instantaneous’ naked singularity with zero mass, before continuing its next radiation.

It appears that (i) the changes in the mass of black holes, (ii) the formation of ‘instantaneous’ naked singularities with zero mass and (iii) the creation of ‘negative mass naked singularities’ in non-embedded, non-rotating as well as rotating, Vaidya-Bonnor black holes, (3.16) and (3.40) are presumably the correct formulation in non-stationary classical spacetime metrics of the three possibilities of black hole evaporation suggested by Hawking and Israel [3]. However, the creation of ‘negative mass naked singularities’ may be a violation of Penrose’s cosmic censorship hypothesis [9]. It is found that (i) the changes in the masses of embedded black holes, (ii) the formation of ‘instantaneous’ rotating charged black holes (3.66), (3.85) and (3.100), and (iii) the creation of embedded ‘negative mass naked singularities’ in Vaidya-Bonnor-de Sitter, Vaidya-Bonnor-monopole and Vaidya-Bonnor-Kerr black holes might be the mathematical formulations in non-stationary classical spacetime metrics of the three possibilities of black hole evaporation [3]. All embedded black holes discussed here can be expressed in Kerr-Schild ansätze, accordingly their consequent negative mass naked singularities are also expressible in Kerr-Schild forms showing them as solutions of Einstein’s field equations. It is also observed that once a charged black hole is embedded into some spaces, it will continue to embed forever through out its Hawking evaporation process. For example, Vaidya-Bonnor black hole is embedded into the rotating de Sitter universe, it continues to embed as ‘instantaneous’ charged black hole in (3.66) and embedded negative mass naked singularity as in (3.71). There Hawking’s radiation does not affect the cosmological constant $\Lambda^*$ through out the evaporation process of Vaidya-Bonnor mass, and similarly,
in the cases of Vaidya-Bonnor-monopole as well as Vaidya-Bonnor-Kerr black holes we find that the monopole constant and Kerr mass remain unaffected. This means that the embedded negative mass naked singularities (3.71), (3.90) and (3.105) possess the total energy momentum tensors (2.9), (2.26) and (2.40) respectively with mass $M^*(u)$, as the change in the mass $M(u)$, due to continuous radiation, affects the energy momentum tensors. Thus, it may be concluded that once a black hole is embedded into some universe, it will continue to embed forever without disturbing the nature of matters of the back ground spaces. This completes the proof of theorem 6. If one accepts the Hawking continuous evaporation of charged black holes, the loss of mass and creation of new mass are the process of the continuous radiation. So, it may also be concluded that once electrical radiation starts, it will continue to radiate forever describing the various stages of the life of radiating black holes.

Also, we find from the above that the change in the mass of black holes, embedded or non-embedded, takes place due to the Maxwell scalar $\phi_1$, remaining unchanged in the field equations during continuous radiation. So, if the Maxwell scalar $\phi_1$ is absent from the space-time geometry, there will be no radiation, and consequently, there will be no change in the mass of the non-stationary black holes. Therefore, we cannot, theoretically, expect to observe such relativistic change in the mass of uncharged, non-rotating as well as rotating, Vaidya black holes. It is observed that the non-stationary classical space-time metrics discussed above would describe the possible life style of radiating embedded black holes at different stages during their continuous radiation. These embedded non-stationary classical spacetimes metrics describing the changing life style of black holes are different from the non-embedded non-stationary ones in various respects shown above. Here the study of these embedded solutions suggests the possibility that in an early universe there might be some non-stationary black holes, which might have embedded into some other spaces possessing different matter fields with well-defined physical properties.

Appendix

In this appendix we present a rotating metric with the function $M(u,r)$. The line element will be of the form [13]

$$ds^2 = \left\{1 - 2rM(u,r)R^{-2}\right\} du^2 + 2du dr + 4arM(u,r)R^{-2}\sin^2\theta du d\phi - 2a\sin^2\theta dr d\phi - R^2d\theta^2 - \left\{(r^2 + a^2)^2 - \Delta^*a^2\sin^2\theta\right\}R^{-2}\sin^2\theta d\phi^2,$$

(A1)

where $R = r + ia\cos\theta$ and $\Delta^* = r^2 - 2rM(u,r) + a^2$. Then the covariant complex null tetrad vectors for the metric can be chosen as follows

$$\ell_a = \delta^1_a - a\sin^2\theta\delta^4_a,$$

(A2)

$$n_a = \frac{\Delta^*}{2R^2}\delta^1_a + \delta^2_a - \frac{\Delta^*}{2R^2}a\sin^2\theta\delta^4_a,$$

$$m_a = -\frac{1}{\sqrt{2R}}\left\{-ia\sin\theta\delta^1_a + R^2\delta^3_a + i(r^2 + a^2)\sin\theta\delta^4_a\right\},$$

(A3)

$$\overline{m}_a = -\frac{1}{\sqrt{2R}}\left\{ia\sin\theta\delta^1_a + R^2\delta^3_a - i(r^2 + a^2)\sin\theta\delta^4_a\right\}.$$
Then the Newman-Penrose spin coefficients, the Ricci scalars and Weyl scalars for the metric (A1) are given below:

\[\kappa = \sigma = \lambda = \epsilon = 0,\]
\[\rho = \frac{1}{R}, \quad \mu = -\frac{\Delta^*}{2RR^2},\]
\[\alpha = \frac{(2ai - R\cos \theta)}{2\sqrt{2RR^2}}, \quad \beta = \frac{\cos \theta}{2\sqrt{2R}},\]
\[\pi = \frac{i\sin \theta}{\sqrt{2RR^2}}, \quad \tau = -\frac{i\sin \theta}{\sqrt{2R^2}},\]
\[\gamma = \frac{1}{\sqrt{2RR^2}} \left[(r - M - r M_r) R - \Delta^*\right],\]
\[\nu = \frac{1}{\sqrt{2RR^2}} i a r \sin \theta M_u,\]
\[\phi_{00} = \phi_{01} = \phi_{10} = \phi_{20} = \phi_{02} = 0,\]
\[\phi_{11} = \frac{1}{4RR^2} \left[4r^2M_r + R^2(-2M_r - r M_{rr})\right],\]
\[\phi_{12} = \frac{1}{2\sqrt{2RR^2}RR^2} \left[i a \sin \theta \{RM_u - r M_r R\}\right],\]
\[\phi_{21} = \frac{1}{2\sqrt{2RR^2}RR^2} \left[i a \sin \theta \{R M_u - r M_r R\}\right],\]
\[\phi_{22} = \frac{1}{2RR^2} \left[2r^2M_u + a^2 r M \sin^2 \theta\right],\]
\[\Lambda = \frac{1}{12RR^2} \left(2M_r + r M_{rr}\right),\]
\[\psi_0 = \psi_1 = 0,\]
\[\psi_2 = \frac{1}{RR^2} \left\{-R M + \frac{R}{6} M_r (4r + 2i a \cos \theta) - \frac{r}{6} R R M_{rr}\right\},\]
\[\psi_3 = -\frac{i a \sin \theta}{2\sqrt{2RR^2}RR^2} \left\{(4r + R) M_u + r R M_{uu}\right\},\]
\[\psi_4 = \frac{a^2 r \sin^2 \theta}{2RR^2 RR^2} \left\{R^2 M_{uu} - 2r M_u\right\}.\] (A4)

(A5)

From these NP spin coefficients we find that the rotating metric (A1) possesses, in general, a geodesic \((\kappa = \epsilon = 0)\), shear free \((\sigma = 0)\), expanding \((\dot{\theta} \neq 0)\) and non-zero twist \((\omega^2 \neq 0)\) null vector \(\ell_a\) [9] where

\[\dot{\theta} = -\frac{1}{2}(\rho + \overline{\rho}) = \frac{r}{R^2}, \quad \omega^* = \frac{1}{4}(\rho - \overline{\rho})^2 = -\frac{a^2 \cos^2 \theta}{R^2 R^2}.\] (A7)

The the energy momentum tensor will take the form

\[T_{ab} = \mu^* \ell_a \ell_b + 2 \rho^* \ell_{(a} m_{b)} + 2 \rho m_{(a} \overline{m}_{b)} + 2\omega \ell_{(a} \overline{m}_{b)} + 2 \overline{\omega} \ell_{(a} m_{b)},\] (A8)

with the following quantities

\[\mu^* = -\frac{1}{KR^2 RR^2} \left\{2r^2M_u + a^2 r \sin^2 \theta M_{uu}\right\},\]
\[\rho^* = \frac{1}{KR^2 RR^2} M_{rr},\]
\[p = -\frac{1}{R} \left\{2a^2 \cos^2 \theta\right\} M_r + \frac{r}{R^2} M_{rr}\].
\[ \omega = -\frac{ia \sin \theta}{\sqrt{2KR^2R^2}} \left( RM_{,u} - r \overline{R} M_{,ur} \right). \]  

(A9)

these quantities have the following relations with the Ricci scalars (A5)

\[ K \mu^* = 2\phi_{22}, \quad K \omega = -2\phi_{12}, \]

\[ K \rho^* = 2\phi_{11} + 6\Lambda, \quad K p = 2\phi_{11} - 6\Lambda. \]  

(A10)

Wang and Wu [12] have expanded the metric function \( M(u,r) \) for the non-rotating solution \((a = 0)\) in the power of \( r \)

\[ M(u,r) = \sum_{n=-\infty}^{+\infty} q_n(u) r^n, \]  

(A11)

where \( q_n(u) \) are arbitrary functions of \( u \). They consider the above sum as an integral when the ‘spectrum’ index \( n \) is continuous. Here using this expression in equations (A11) we can generate rotating metrics with \( a \neq 0 \) as follows:

\[ \mu^* = -\frac{r}{KR^2R^2} \sum_{n=-\infty}^{+\infty} \left\{ 2q_n(u,ru^n+1 + a^2\sin^2\theta q_n(u,ru^n) \right\}, \]

\[ \rho^* = \frac{2r^2}{KR^2R^2} \sum_{n=-\infty}^{+\infty} nq_n(u) r^{n-1}, \]  

(A12)

\[ p = -\frac{1}{KR^2} \sum_{n=-\infty}^{+\infty} nq_n(u) r^{n-1} \left\{ \frac{2a^2\cos^2\theta}{R^2} + (n-1) \right\}, \]

\[ \omega = -\frac{i a \sin \theta}{\sqrt{2KR^2R^2}} \sum_{n=-\infty}^{+\infty} (R-n\overline{R}) q_n(u,ru^n. \]

and other NP quantities can also be presented with this function \( q_n(u) \).

According to Carter [18] and York [27], we introduce a scalar \( K \) defined by the relation \( \nabla_b n^a = \mathcal{K} n^a \), where the null vector \( n^a \) in (2.5) above is parameterized by the coordinate \( u \), such that \( d/du = n^a \nabla_a \). Then this scalar can, in general, be expressed in terms of NP spin coefficient \( \gamma \) (A1) as follows:

\[ \mathcal{K} = n^b \nabla_b n^a \ell_a = -(\gamma + \overline{\gamma}). \]  

(A13)

On a horizon, the scalar \( \mathcal{K} \) is called the surface gravity of a black hole.

For future use we shall cite the four roots of the biquadratic equation

\[ \Delta^* \equiv r^2 - 2rM(u) - \Lambda^* r^4/3 + a^2 + e^2 = 0 \]  

(A14)

as follows for non-zero cosmological constant \( \Lambda^* \):

\[ [r^*_+] = +\frac{1}{2} \sqrt{\Gamma} \pm \frac{1}{2} \sqrt{\left\{ \frac{4}{\Lambda^*} - 32^{1/3} \chi - \frac{1}{32^{1/3} \Lambda^*} \right\} \left\{ P + \overline{P}^{2 - 4Q} \right\}^{1/3}} \]

\[ -\frac{1}{\Lambda^* \sqrt{\Gamma}} M(u) ; \]  

(A15)

\[ [r^*_+] = -\frac{1}{2} \sqrt{\Gamma} \pm \frac{1}{2} \sqrt{\left\{ \frac{4}{\Lambda^*} - 32^{1/3} \chi - \frac{1}{32^{1/3} \Lambda^*} \right\} \left\{ P + \overline{P}^{2 - 4Q} \right\}^{1/3}} \]

\[ -\frac{1}{\Lambda^* \sqrt{\Gamma}} M(u) ; \]
\[ + \frac{12}{\Lambda^* \sqrt{\Gamma}} M(u) \] (A16)

where

\[ P = 54 \left\{ 18 \Lambda^* M(u)^2 - 12 \Lambda^* (a^2 + e^2) - 1 \right\}, \quad Q = \left\{ 9 - 36 \Lambda^* (a^2 + e^2) \right\}^3, \]

\[ \chi = \frac{1 - 4 \Lambda^* (a^2 + e^2)}{\Lambda^* \left\{ P + \sqrt{P^2 - 4Q} \right\}^{1/3}}, \quad \Gamma = \frac{2}{\Lambda^*} + 32^{1/3} \chi + \frac{32^{1/3}}{\Lambda^*} \left\{ P + \sqrt{P^2 - 4Q} \right\}^{1/3}. \]

The calculation of these roots has been carried out by using ‘Mathematica’. The area of a horizon of black hole can be calculated as follows [9]:

\[ A = \int_0^\pi \int_0^{2\pi} \sqrt{\mathbf{g}_{\theta\theta} \mathbf{g}_{\phi\phi}} \, d\theta \, d\phi \bigg|_{\Delta^* = 0}, \] (A17)

depending on the values of the roots of \( \Delta^* = 0 \). Then the entropy on a horizon of a black hole may be obtained from the relation \( S = \frac{A}{4} \) [28].

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