DIOPHANTINE APPROXIMATION IN PRESCRIBED DEGREE

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ABSTRACT. We investigate approximation to a given real number by algebraic numbers and algebraic integers of prescribed degree. We deal with both best and uniform approximation, and highlight the similarities and differences compared with the intensely studied problem of approximation by algebraic numbers (and integers) of bounded degree. We establish the answer to a question of Bugeaud concerning approximation to transcendental real numbers by quadratic irrational numbers, and thereby we refine a result of Davenport and Schmidt from 1967. We also obtain several new characterizations of Liouville numbers, and certain new insights on inhomogeneous Diophantine approximation. As an auxiliary side result, we provide an upper bound for the number of certain linear combinations of two given relatively prime integer polynomials with a linear factor. We conclude with several open problems.

Keywords: exponents of Diophantine approximation, Wirsing’s problem, geometry of numbers, continued fractions
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1. INTRODUCTION

1.1. Outline and notation. The famous Dirichlet Theorem asserts that for any real number $\zeta$ and any parameter $X > 1$, the estimate

$$|\zeta - \frac{p}{q}| \leq X^{-1} q^{-1}$$

has a rational solution $p/q$ with $1 \leq q \leq X$. A well-known immediate implication is the fact that for any $\zeta \in \mathbb{R} \setminus \mathbb{Q}$ there exist infinitely many rational numbers $p/q$ that satisfy

$$|\zeta - \frac{p}{q}| \leq q^{-2}.$$  

We will also refer to a situation as in (1) as uniform approximation, whereas (2) is a result on best approximation. There has been much research on generalizations of (1) and (2) concerning approximation to a real number by algebraic numbers degree at most $n$, for some given positive integer $n$. Define the height $H(P)$ of a polynomial $P$ as the maximum absolute value among its coefficients. For an algebraic real number $\alpha$, define its height by $H(\alpha) = H(P)$ with $P$ the minimal polynomial of $\alpha$ with coprime integral coefficients. Wirsing [46] proposed the following generalization of (2). For any real number $\zeta$ not algebraic of degree at most $n$, and any $\epsilon > 0$, does the inequality

$$|\zeta - \alpha| \leq H(\alpha)^{-n-1+\epsilon}$$

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1.
have infinitely many solutions in algebraic real numbers \( \alpha \) of degree at most \( n \)? Roughly speaking, is any number not algebraic of degree at most \( n \) approximable to degree \( n + 1 \) by algebraic numbers of degree at most \( n \)? Wirsing could only prove the claim for the exponents \(-n(n + 3)/2 + \epsilon\), and despite some effort there has been little improvement since. The current best exponents due to Tishchenko [45] is still of the form \(-n/2 - C_n\) for constants \( C_n < 4 \). It is well-known that the corresponding uniform claim, as in [11], is in general false when \( n \geq 2 \), see for example [14, Theorem 2.4]. In case of \( \zeta \) an algebraic number of degree at least \( n + 1 \), the claim \( (3) \) is true as a consequence of Schmidt’s Subspace Theorem. Schmidt [42] raised the question whether \( (3) \) can be sharpened by replacing the \( \epsilon \) in the exponent by some multiplicative constant. Observe that this refined version is no longer guaranteed even for algebraic numbers. However, Davenport and Schmidt [19] already in 1967 verified the stronger version in the special case \( n = 2 \). Indeed, they showed that for any \( \zeta \) not algebraic of degree at most two and some \( c = c(\zeta) > 0 \), the estimate

\[
(4) \quad |\zeta - \alpha| \leq cH(\alpha)^{-3}
\]

has infinitely many rational or quadratic irrational real solutions \( \alpha \). Any constant \( c > \frac{160}{9} \max\{1, \zeta^2\} \) can be chosen. In this paper we investigate approximation to a real number by algebraic numbers of prescribed degree \( n \geq 2 \). This topic, in contrast to bounded degree, has been rather poorly investigated. Some results on this topic are due to Bugeaud and Teulie [8, 15, 44]. A variant of Wirsing’s problem, or Schmidt’s version of it, studied by Bugeaud [8, Problem 23, Section 10.2] is to restrict the degree of \( \alpha \) equal to \( n \). Bugeaud himself had recently expressed doubts [10, Problem 2.9.2] on a positive answer, even for \( n = 2 \), rooting in the fact that the claim is in general false for \( \alpha \) cubic algebraic integers instead of quadratic numbers, as shown by Roy [30, 31]. However, our first new result shows that for \( n = 2 \) the problem does indeed have an affirmative answer, and thereby we refine the result of Davenport and Schmidt.

**Theorem 1.1.** Let \( \zeta \) be a real number not rational or quadratic irrational. Then, for some effectively computable constant \( c = c(\zeta) \), there exist infinitely many quadratic irrational real numbers \( \alpha \) for which the inequality \( (4) \) is satisfied.

It is worth noticing that the constant \( c \) we can provide for Theorem 1.1 will probably be larger than \( \frac{160}{9} \max\{1, \zeta^2\} \) in [44], however the explicit computation is cumbersome and we do not attempt to carry it out. We continue to discuss Theorem 1.1 in Section 2 below and there we also provide more new results concerning approximation by quadratic irrationals. In Section 3 we study approximation to real numbers by algebraic numbers of exact degree \( n \), for arbitrary \( n \geq 2 \). We propose a problem related to the Wirsing problem and solve it for \( n = 3 \). Moreover, we illustrate the difference between approximation in bounded versus exact degree. Indeed, many classical results turn out to be false when the degree is fixed. Section 4 is devoted to approximation to real numbers by algebraic integers. Davenport and Schmidt [20] wrote a pioneering paper on this topic in 1969, and more recent results can be found in [8, 15, 44] again. We will discuss these contributions in Section 4. It is tempting to believe that approximation by algebraic integers of degree \( n + 1 \) is closely related to approximation by algebraic numbers of degree \( n \). However, Roy [31, 30] constructed counterexamples to an intuitive conjecture for cubic
integers. Related results can also be found in the book of Cassels \[16\], and the more recent paper \[12\] by Bugeaud and Laurent, which investigates inhomogeneous approximation in a wide generality. Our new contribution to this topic is related to \[12\] and yields a new characterization of Liouville numbers, that are numbers for which we may choose an arbitrarily large negative exponent in the right hand side of \(1\). Proofs, unless very short, are carried out in Section \[5\]. For some proofs it is convenient to use the concept of the parametric geometry of numbers introduced by Schmidt and Summerer \[43\]. Finally we will gather several open problems in Section \[6\].

We enclose some notation which will simplify the formulation of our results. For a ring \(R\), we will denote \(R_{\leq n}[T]\) the set of polynomials of degree at most \(n\) with coefficients in \(R\), and similarly define \(R_{=n}[T]\) and \(R_{\geq n}[T]\). The most important instances will be \(\mathbb{Z}_{\leq n}[T]\) and \(\mathbb{Z}_{=n}[T]\). We denote by \(\mathbb{A}_{\leq n}\) and \(\mathbb{A}_n\) the set of non-zero real algebraic numbers of degree at most \(n\) and equal to \(n\), respectively. Similarly, \(\mathbb{A}^\text{mt}_{\leq n}\) and \(\mathbb{A}^\text{mt}_n\) denote the sets of non-zero real algebraic integers of degree at most or exactly \(n\), respectively. We will write \(A \ll B\) when \(A \leq c(.)B\) for a constant \(c\) that may depend on the subscript arguments, and \(A \ll B\) when the constant \(c\) is absolute. Moreover \(A \asymp B\) and \(A \asymp B\) will be short notation for \(A \ll B\ll A\) and \(A \ll B \ll A\), respectively.

### 1.2. Classical and new exponents

We will formulate most of our results in terms of classical exponents of Diophantine approximation and certain variations. We now define all these exponents and discuss their basic properties.

Let \(w^*_n(\zeta)\) and \(\hat{w}^*_n(\zeta)\) respectively denote the supremum of real numbers \(w^*\) such that the system

\[
H(\alpha) \leq X, \quad 0 < |\zeta - \alpha| \leq H(\alpha)^{-1}X^{-w^*}, \tag{5}
\]

has a solution \(\alpha \in \mathbb{A}_{\leq n}\) for arbitrarily large \(X\), and all large \(X\), respectively. Let \(w^*_n(\zeta)\) and \(\hat{w}^*_n(\zeta)\) be defined similarly with \(\alpha \in \mathbb{A}_{=n}\) instead of \(\alpha \in \mathbb{A}_{\leq n}\). We call the exponents without ”hat” best approximation constants and the ones with ”hat” uniform exponents. Indeed, for \(n = 1\) this concept relates to the largest possible exponents in \(2\) and \(1\) for given \(\zeta\), respectively. In this notation, Wirsing’s problem asks whether \(w^*_n(\zeta) \geq n\) holds for any transcendental real number, and Theorem \[1.1\] implies \(w^*_n(\zeta) \geq n\). We obviously have the relations

\[
0 \leq w^*_n(\zeta) \leq w^*_n(\zeta), \quad 0 \leq \hat{w}^*_n(\zeta) \leq \hat{w}^*_n(\zeta). \tag{6}
\]

The classical exponents moreover satisfy

\[
w^*_1(\zeta) \leq w^*_2(\zeta) \leq \cdots, \quad 1 = \hat{w}^*_1(\zeta) \leq \hat{w}^*_2(\zeta) \leq \cdots, \tag{7}
\]

where the only non-obvious identity \(1 = \hat{w}^*_1(\zeta)\) due to Khintchine \[24\] shows that the exponent of \(X\) in \(1\) cannot be improved for any irrational real \(\zeta\). In contrast, the analogous claims to \(1\) for exact degree are in general false, as we will show in Theorem \[3.8\] below.

Define \(w_n(\zeta)\) and \(\hat{w}_n(\zeta)\) respectively as the supremum of real numbers \(w\) for which the system

\[
H(P) \leq X, \quad 0 < |P(\zeta)| \leq X^{-w} \tag{8}
\]
has a solution \( P \in \mathbb{Z}_{\leq n}[T] \), for arbitrarily large \( X \) and all large \( X \), respectively. Similarly, let \( w_n(\zeta) \) and \( \hat{w}_n(\zeta) \) be the supremum of \( w \in \mathbb{R} \) such that \([8]\) has an irreducible solution \( P \in \mathbb{Z}_{\leq n}[T] \) for arbitrarily large values of \( X \), and all large \( X \), respectively. The requirement for the polynomials in the definitions of \( w_n(\zeta) \) and \( \hat{w}_n(\zeta) \) to be irreducible is natural, otherwise we would have trivial equality with the corresponding classical exponents. Indeed, for any \( Q(T) \in \mathbb{Z}_{=d}[T] \) with \( d < n \), the polynomial \( \tilde{Q}(T) = T^{n-d}Q(T) \) has degree precisely \( n \), the same height \( H(\tilde{Q}) = H(Q) \) and satisfies \( \tilde{Q}(\zeta) \asymp_{n,\zeta} Q(\zeta) \).

We again have the obvious relations

\[ 0 \leq w_n(\zeta) \leq w_n(\zeta), \quad 0 \leq \hat{w}_n(\zeta) \leq \hat{w}_n(\zeta), \]

and the classical exponents are non-decreasing

\[ w_1(\zeta) \leq w_2(\zeta) \leq \cdots, \quad 1 = \hat{w}_1(\zeta) \leq \hat{w}_2(\zeta) \leq \cdots. \]

The identity \( \hat{w}_1(\zeta) = 1 \) is due to Khintchine [24] again. On the other hand, the estimates in \([10]\) are again in general false for the corresponding exponents of exact degree. Furthermore any Sturmian continued fraction defined as in \([11]\) also provides a counterexample for certain indices. Indeed, for these numbers we have both \( w_2(\zeta) > w_3(\zeta) \) and \( \hat{w}_2(\zeta) > \hat{w}_3(\zeta) \), as follows from the results in [39]. See also [38, Theorem 2.1 and Theorem 2.2]. In contrast to the open Wirsing problem, for every transcendental real number \( \zeta \), a multi-dimensional variant of Dirichlet’s Theorem states

\[ w_n(\zeta) \geq \hat{w}_n(\zeta) \geq n. \]

In fact, for any \( X \geq 1 \) we find \( P \in \mathbb{Z}_{\leq n}[T] \) of height at most \( X \) such that \( |P(\zeta)| \leq cX^{-n} \), for an explicit constant \( c = c(\zeta) \). The essential problem in the Wirsing conjecture is that a lower bound for \( w_n^*(\zeta) \) also requires that the derivative \( P'(\zeta) \) of the polynomials \( P \) inducing \([11]\) at \( \zeta \) are not too small in absolute value. This is no longer clear when \( n \geq 2 \). Again \([11]\) turns out to be false for the corresponding exponents of exact degree, even \( \hat{w}_n(\zeta) = 0 \) does occur for certain \( \zeta \) when \( n \geq 2 \). We will deal with the problem if \( w_n(\zeta) \geq n \) holds for any \( \zeta \). Finally we point out that \([8, \text{Lemma A.8}]\) links the exponents in the form

\[ w_n^*(\zeta) \leq w_n(\zeta) \leq w_n^*(\zeta) + n - 1, \quad \hat{w}_n^*(\zeta) \leq \hat{w}_n(\zeta) \leq \hat{w}_n^*(\zeta) + n - 1, \]

\[ w_n^*(\zeta) \leq w_n(\zeta) \leq w_n^*(\zeta) + n - 1, \quad \hat{w}_n^*(\zeta) \leq \hat{w}_n(\zeta) \leq \hat{w}_n^*(\zeta) + n - 1. \]

The left inequalities are easy to infer. They use the elementary fact that if \( P \) is the minimal polynomial of \( \alpha \), then \( |P(\zeta)| = |P(\alpha) - P(\zeta)| \leq |P'(\zeta)| \cdot |\alpha - \zeta| \) for some \( z \) between \( \alpha \) and \( \zeta \), but on the other hand \( |P'(\zeta)| \ll_{n,\zeta} H(P) \) when \( z \) is close to \( \zeta \). Hence \( |P(\zeta)| \ll_{n,\zeta} H(P)|\alpha - \zeta| \) and the claims follow. The right inequalities are more difficult to show.

2. Approximation by quadratic irrational numbers

We recall that a transcendental real number is called \( U \)-number in Mahler’s classification of real numbers when \( w_n(\zeta) = \infty \) for some \( n \geq 1 \). More precisely, \( \zeta \) is called \( U_m \)-number when \( m \) is the smallest index for which \( w_m(\zeta) = w_m(\zeta) = \infty \). The \( U_1 \)-numbers are also called Liouville numbers. The set of \( U \)-numbers is the disjoint union of the sets of \( U_m \)-numbers over \( m \geq 1 \).
**Theorem 2.1.** Let $\zeta$ be a real number which satisfies $\hat{w}_2(\zeta) > 2$. Then we have
\[
w_2(\zeta) = w_2(\zeta), \quad \hat{w}_2(\zeta) = \hat{w}_2(\zeta),
\]
and
\[
w^*_2(\zeta) = w^*_2(\zeta).
\]
If additionally $\hat{w}^*_2(\zeta) \geq 2$ holds, we have $\hat{w}^*_2(\zeta) = \hat{w}^*_2(\zeta)$ as well. Moreover, $\zeta$ is not a $U$-number.

We will see in Theorem 3.8 below that all identities in Theorem 2.1 are in general false when we drop the assumption $\hat{w}_2(\zeta) > 2$. On the other hand, as indicated in Section 1.2, there are plenty of numbers, including Sturmian continued fractions, that satisfy the hypothesis $\hat{w}_2(\zeta) > 2$. The last claim of Theorem 2.1 can be regarded as an extension of [1, Théorème 5.3] in the special case $n = 2$, where the possibility that $\zeta$ is a $U_m$-number for $m \leq n = 2$ was not ruled out. In fact the semi-effective exponential upper bounds for the growth of the sequence $(w_n(\zeta))_{n \geq 1}$ from [1, Théorème 4.2] apply to any $\zeta$ which satisfies $\hat{w}_2(\zeta) > 2$. See also [10, Corollary 4.6] for a stronger upper bound for the exponent $w_3(\zeta)$ of the form $w_3(\zeta) < 15/(\hat{w}_2(\zeta) - 2)^2$ as soon as $\hat{w}_2(\zeta) > 2$. We will discuss generalizations of these phenomena in Problem 3 in Section 6 below.

With Theorem 2.1 we can simultaneously refine $w^*_2(\zeta) \geq 2$ from Theorem 1.1 and
\[
w^*_2(\zeta) \geq \hat{w}_2(\zeta)(\hat{w}_2(\zeta) - 1)
\]
recently discovered by Moshchevitin [28, Theorem 2].

**Theorem 2.2.** For any real number $\zeta$ not rational or quadratic irrational, we have
\[
w^*_2(\zeta) \geq \hat{w}_2(\zeta)(\hat{w}_2(\zeta) - 1) \geq 2.
\]

**Proof.** When $\hat{w}_2(\zeta) > 2$, we deduce (17) from (15) and (16). On the other hand, in case of $\hat{w}_2(\zeta) = 2$, the claim becomes $w^*_2(\zeta) \geq 2$ and is implied by Theorem 1.1.

We conclude this section with remarks on Theorem 1.1. As indicated in Section 1.1, a variation of Theorem 1.1 concerning approximation by cubic algebraic integers turns out to be false. Indeed, a non-empty subclass of Roy’s extremal numbers introduced in [31] are not approximable by cubic algebraic integers to the expected order three. More precisely, it was shown in [30] that for some extremal numbers $\zeta$ and any $\alpha \in \mathbb{A}^{int}_{\leq 3}$ we have
\[
|\zeta - \alpha| \gg_{\zeta} H(\alpha)^{-\theta}, \quad \theta = \frac{\sqrt{5} + 3}{2} = 2.6180\ldots < 3.
\]

We also refer to Moshchevitin [29] and Roy [34] for a negative answer to a somehow related two-dimensional problem introduced by W.M. Schmidt [41] concerning small evaluations of linear forms with sign restrictions.

### 3. Approximation in higher prescribed degree

**3.1. The problem** $w_{=n}(\zeta) \geq n$. In the case of general $n \geq 2$, we first want to state a basic observation relating classical and new best approximation exponents.
Lemma 3.1. Let $n \geq 1$ be an integer and $\zeta$ be any real number. For the best approximation exponents we have the identities

\begin{equation}
(19) \quad w_n(\zeta) = \max\{w_{=1}(\zeta), \ldots, w_{=n}(\zeta)\}, \quad w^*_n(\zeta) = \max\{w^*_1(\zeta), \ldots, w^*_n(\zeta)\}.
\end{equation}

Proof. The estimates

\begin{equation}
w_n(\zeta) \geq \max\{w_{=1}(\zeta), \ldots, w_{=n}(\zeta)\}, \quad w^*_n(\zeta) \geq \max\{w^*_1(\zeta), \ldots, w^*_n(\zeta)\}
\end{equation}

are an obvious consequence of (9), (10) and (6), (7), respectively. The reverse right inequality follows from pigeon hole principle, since clearly infinitely many $\alpha$ are an obvious consequence of (9), (10) and (6), (7), respectively. The reverse left estimate follows similarly, when we also take into account that the polynomials in the definition of $w^*_n(\zeta)$ must have the same degree. The reverse left estimate follows as noticed by Wirsing [46, Hilfssatz 4].

Note that the argument of Lemma 3.1 no longer applies to the uniform exponents $\widehat{w}_n(\zeta), \widehat{w}^*_n(\zeta)$. Indeed, the uniform identities analogous to (19) turn out to be false in general when $n \geq 2$, as we will see in Theorem 3.8 below.

We discuss the problem, motivated by the Dirichlet Theorem (11), if we can fix the degree of infinitely many involved polynomials to be exactly $n$ and still obtain the lower bound $n$. In other words we want to know if $w_{=n}(\zeta) \geq n$ holds for any integer $n \geq 1$ and any transcendental real $\zeta$. From Lemma 3.1 and (11) we only know that $w_{=k}(\zeta) \geq n$ for some $1 \leq k \leq n$. The problem seems to be reasonably easier than asking for $w^*_{=n}(\zeta) \geq n$ related to the Wirsing problem as no information on derivatives of the polynomials is required. For $n = 2$, Theorem 1.1 and (13) settles the claim. For $n = 3$, we can still show the answer is affirmative. For $n \geq 4$, we can no longer provide a definite answer. However, we establish a sufficient criterion concerning the irreducibility of certain integer polynomials.

Theorem 3.2. Let $\zeta$ be a transcendental real number. We have

\begin{equation}
w_{=3}(\zeta) \geq \widehat{w}_3(\zeta) \geq 3.
\end{equation}

For $n \geq 4$ an integer, suppose the following claim holds. For any $\epsilon > 0$ and a constant $c = c(n, \epsilon)$, and any pair of coprime polynomials $P, Q \in \mathbb{Z}_{\leq n}$ there exist non-zero integers $a, b$ with $\max\{|a|, |b|\} \leq c \cdot \max\{H(P), H(Q)\}^\epsilon$, such that $aP + bQ$ is irreducible. Then we have

\begin{equation}
w_{=n}(\zeta) \geq \widehat{w}_n(\zeta) \geq n.
\end{equation}

The proof of (20) relies on a proof of the involved irreducibility criterion in the particular case $n = 3$, which is of some interest on its own. For the sequel we write

\begin{equation}
R_p(T) = Q(T) + pP(T), \quad S_p(T) = P(T) + pQ(T),
\end{equation}

for given integer polynomials $P, Q$ and an integer $p$.

Theorem 3.3. Let $n \geq 2$ be an integer. Let $P \in \mathbb{Z}_{\leq n-1}[T]$ non-zero and $Q \in \mathbb{Z}_{=n}[T]$ such that

- $P, Q$ have no common linear factor over $\mathbb{Z}[T]$,
- $Q(0) = 0$, and
- $\frac{Q(T)}{TP(T)}$ is non-constant.
Let \( X = \max\{H(P), H(Q)\} \). Then for any \( \delta > 0 \), there exists a constant \( c = c(n, \delta) > 0 \) not depending on \( P, Q \) such that the number of prime numbers \( p \) for which either of the polynomials \( R_p \) or \( S_p \) has a linear factor, is less than \( cX^\delta \). Thus, for any \( \varepsilon > 0 \) there exists \( d = d(n, \varepsilon) > 0 \) and a prime number \( p \leq dX^\varepsilon \) for which both \( R_p \) and \( S_p \) have no linear factor. Moreover, for any prime \( p > nX^{n+1} \) the polynomials \( R_p \) and \( S_p \) both have no linear factor.

**Remark 1.** The theorem is formally true for \( n = 1 \) as well but the assumptions on \( P, Q \) cannot be satisfied (which must be the case as the claim is obviously false). Notice also that when \( Q(0) = 0 \), the other two conditions are clearly necessary for the conclusion. We propose the natural generalization to drop the condition \( Q(0) = 0 \) upon replacing the third condition on \( P, Q \) by the assumption that the rational function \( Q(T)/P(T) \) is not a linear polynomial. Our proof of Theorem 3.3 cannot be modified in a straightforward way to settle this conjecture. The situation also becomes more difficult if we allow \( P \in \mathbb{Z}[n][T] \).

The bound for the number of \( R_p \) and \( S_p \) with linear factor can be significantly reduced if we assume that the constant coefficient \( b_0 \) of \( P \) and the leading coefficient \( a_n \) of \( Q \) both do not have an untypically large number of divisors \( \tau(b_0), \tau(a_n) \). More precisely, the proof shows an upper bound of the form \( \ll n^\delta \tau(b_0)\tau(a_n) \log X \) for the number of \( R_p \) or \( S_p \) with linear factor. If we assume \( \tau(N) \ll \log N \) which is true on average (see [3, Theorem 3.3]) for both \( N = a_n \) and \( N = b_0 \), the bound becomes \( \ll n^\delta (\log X)^3 \). We emphasize that the first and third condition on \( P, Q \) in Theorem 3.3 are in particular satisfied whenever \( P \) and \( Q \) have no common factor.

**Corollary 3.4.** The conclusion of Theorem 3.3 holds if we assume \( P \in \mathbb{Z}_{\leq n-1}[T] \) and \( Q \in \mathbb{Z}_{=n}[T] \) have no common factor and \( Q(0) = 0 \), instead of the three conditions on \( P, Q \) there.

For \( n \geq 4 \), the imposed condition in Theorem 3.2 seems to be rather weak as well, however we do not have a rigorous proof. Our proof of Theorem 3.3 already requires some results from analytic number theory. We quote some related irreducibility results. It was shown by Cavachi [17] that for given coprime \( Q \in \mathbb{Z}_{=n}[T] \), \( P \in \mathbb{Z}_{\leq n-1}[T] \), among the polynomials \( S_p \) for \( p \) a prime number, only finitely many are reducible. Subsequent papers even provided effective lower bounds for \( p \), in dependence of the degrees and heights of \( P \) and \( Q \), such that for any prime \( p \) exceeding this bound the irreducibility is settled [18, 6]. See also [1] for a recent generalization to prime powers. However, the bounds on \( p \) are comparable in size with our bound \( nX^{n+1} \), too weak for the purpose to prove (21) (in fact it is not hard to see that \( R_p \) indeed can be irreducible for some \( p \leq X^{1-\varepsilon} \) with \( \varepsilon > 0 \), start with a reducible polynomial \( U \in \mathbb{Z}_{=n}[T] \) and \( P \in \mathbb{Z}_{\leq n-1}[T] \) coprime to it and let \( Q = U - pP \) for large \( p \)). Concerning \( R_p \), no irreducibility results of this kind seem available. We finally remark that if \( w_n(\zeta) > w_{n-1}(\zeta) \) we obviously have \( w_n(\zeta) = w_n(\zeta) \geq n \) in view of (11). In particular, for fixed \( \zeta \) the claim (21) certainly holds for infinitely many indices \( n \), unless \( \zeta \) is a \( U_m \)-number for \( m \geq 2 \) (the case \( m = 1 \) will be covered by Corollary 3.11 below).

3.2. **Particular classes of numbers.** Our upcoming results will frequently use special classes of real numbers with continued fraction expansions of a special form. We refer to [8]...
Section 1.2] for an introduction to continued fractions. Let \( w \geq 1 \) be a real parameter. We define \( B_w \) as the class of numbers whose convergents \( p_i/q_i \) in the continued fraction expansion satisfy \( p_1 = 1, q_1 = 2, \) and the recurrence relation

\[
p_{i+2} = M \lfloor p_i^{w-1} \rfloor + p_i, \quad q_{i+2} = M \lfloor q_i^{w-1} \rfloor + q_i,
\]

where \( M \geq 1 \) is any positive integer parameter. This is, apart from some restriction on \( M \), the same class of numbers considered by Bugeaud [9]. He used the equivalent recursive definition

\[
ζ = [0; 2, M \lfloor q_1^{w-1} \rfloor, M \lfloor q_2^{w-1} \rfloor, \ldots],
\]

where \( q_1 = 2 \) and \( q_j \) are defined as the denominator of the \( j \)-th convergent to the real number \( ζ \). We naturally extend this concept to \( w = ∞ \) by defining \( B_∞ \) the set of numbers with

\[
\lim_{j→∞} \frac{\log a_{j+1}}{\log a_j} = ∞,
\]

for \( (a_j)_{j≥1} \) the sequence partial quotients associated to \( ζ \). Indeed, similar to (22), this assumption implies that every convergent is a very good approximation to \( ζ \). The set \( B_∞ \) was called strong Liouville numbers by LeVeque [26]. In fact, all our results below concerning \( B_∞ \) remain valid for the wider class of semi-strong Liouville numbers introduced by Alniaçik [2]. Bugeaud [9, Corollary 1] showed that when \( w ≥ 2n−1, \) and \( M \) is sufficiently large in terms of \( n \), then any number as in (22) satisfies

\[
w_1(ζ) = w_∗_1(ζ) = w_2(ζ) = \ldots = w_n(ζ) = w_∗_n(ζ) = w.
\]

For \( n ≥ 1 \) and \( w ∈ [n, ∞] \), we denote by \( D_{n,w} \) the class of real numbers that satisfy (24). Any set \( D_{n,∞} \) coincides with the set of Liouville numbers. Our proof of Theorem 3.7 below will show that actually one may take any integer \( M \geq 1 \) for the conclusion (24) when \( w ≥ 2n−1, \) in other words every \( ζ ∈ B_w \) satisfies (24) for \( w ≥ 2n−1. \) Moreover the claim obviously remains true when \( w = ∞. \) Thus we have

\[
B_w ⊆ D_{n,w}, \quad w ∈ [2n−1, ∞].
\]

For smaller parameters \( w ∈ [n, 2n−1) \), it has not been yet shown that \( D_{n,w} ≠ ∅, \) so our results below on sets \( D_{n,w} \) are in fact kind of conditional when \( w < 2n−1. \) However, we strongly believe that \( D_{n,w} ≠ ∅ \) always holds, maybe (25) even extends to \( w ∈ [n, ∞]. \) In this context we refer to the Main Problem formulated in [8, Section 3.4, page 61], which if true would directly imply this hypothesis. The numbers in \( D_{n,w} \) and \( B_w \) have biased approximation properties. Their special structure permits to determine most exponents of approximation, and it will turn out that they behave differently with respect to approximation by numbers of bounded degree than by numbers of prescribed degree.

3.3. Properties of \( B_w \) and \( D_{n,w}. \) It has been settled in [36, Theorem 5.1] and within the proof of [10, Theorem 5.6] respectively that for \( ζ ∈ D_{n,w} \) with \( w ∈ [n, ∞] \), the classical uniform exponents can be determined as

\[
\hat{w}_n(ζ) = n, \quad \hat{w}_n^*(ζ) = \frac{w}{w−n+1}.
\]
As a first new contribution we determine the uniform exponents of prescribed degree \( n \) for numbers in \( D_{n,w} \). For the sequel we agree on \( 1/\infty = 0 \) and \( 1/0 = +\infty \).

**Theorem 3.5.** Let \( n \geq 2 \) be an integer, \( w \in [n, \infty] \) and \( \zeta \in D_{n,w} \). Then

\[
\hat{w}_n(\zeta) = \tilde{w}_n^*(\zeta) = \frac{n}{w - n + 1}.
\]

The left identity contrasts \( \hat{w}_n(\zeta) > \tilde{w}_n^*(\zeta) \) for \( \zeta \in D_{n,w} \) with \( w > n \) from (26). For our next corollary, we define, as for example in [9], the spectrum of an exponent of approximation as the set of real values taken by it as the argument \( \zeta \) runs through the set of transcendental real numbers. We recall some facts. Metric results by Baker and Schmidt [1] and Bernik [5] imply that the spectra of \( w_n(\zeta) \) and \( w_n(\zeta) \) equal \( [n, \infty] \), and the spectra of \( w_n^*(\zeta) \) and \( w_n^*(\zeta) \) contain \([n, \infty]\). (In fact the inclusion of \( \{\infty\} \) requires also the existence of \( U_{n,\mu} \)-numbers, see [26].) Hence Wirsing’s problem is equivalent to asking whether the spectrum of \( w_n^*(\zeta) \) is identical to \([n, \infty]\). Concerning classical uniform exponents, it is known that the spectrum of \( \hat{w}_n \) is contained in \([n, \mu(n)] \) with

\[
\mu(2) = \frac{3 + \sqrt{5}}{2}, \quad \mu(3) = 3 + \sqrt{2}, \quad \mu(n) = n - \frac{1}{2} + \sqrt{n^2 - 2n + \frac{5}{4}}, \quad n \geq 4.
\]

The lower bounds arise from (11), the upper bounds are currently best known [14, Theorem 2.1]. For previously known results see Davenport and Schmidt [20]. The bound \( \mu(2) \) is optimal, Roy [31] proved equality \( \hat{w}_2(\zeta) = \mu(2) \) for certain \( \zeta \) he called extremal numbers. We refer to [10] for further references on the spectrum of \( \hat{w}_2 \). For the exponent \( \tilde{w}_n^* \), it follows from (7) and (13) that its spectrum is contained in \([1, \mu(n)] \), and furthermore we know [10] that it contains \([1, 2 - 1/n] \). Similarly, the spectra of the exponents \( \hat{w}_n(\zeta) \) and \( \tilde{w}_n^*(\zeta) \) are contained in \([0, \mu(n)] \).

If we let \( w \) in (22) vary in \([2n - 1, \infty]\), from Theorem 3.5 and (25) we derive new information on the spectra of the exponents \( \hat{w}_n(\zeta) \) and \( \tilde{w}_n(\zeta) \) and certain differences.

**Corollary 3.6.** Let \( n \geq 2 \) be an integer. The spectra of \( \hat{w}_n(\zeta) \) and \( \tilde{w}_n(\zeta) \) both contain the interval \([0, 1]\). The spectrum of \( \hat{w}_n - \tilde{w}_n \) contains \([n - 1, n] \) and the spectrum of \( \tilde{w}_n^* - \hat{w}_n^* \) contains \([1 - 1/n, 1]\).

**Proof.** For \( w \in [2n - 1, \infty] \) consider any \( \zeta \in D_{n,w} \), which is non-empty by (25). Combine the identities (26) and (27).

From the above we expect that the spectra of \( \hat{w}_n(\zeta) \) and \( \tilde{w}_n^*(\zeta) \) actually contain \([0, n] \), similar as for \( \hat{w}_n(\zeta) \) and \( \tilde{w}_n^*(\zeta) \) where we expect the interval \([1, n]\) to be included. Our next result establishes an estimation of \( w_n \) for the class of numbers as above.

**Theorem 3.7.** Let \( n \geq 2, w \in [2n - 1, \infty] \) and \( \zeta \in D_w \). Then

\[
w_n(\zeta) \leq \frac{nw}{w - n + 1}.
\]

**Remark 2.** Theorem leads to a new proof of [9, Corollary 1], that is (24) for \( D_w \) when \( w \geq 2n - 1 \), and allows for choosing arbitrary \( M \) in (22) (which we required for (25)). We further note that we derive explicit constructions of \( \zeta \) with prescribed exponent \( w_n(\zeta) \in [2n - 1, \infty] \). Indeed, if we put \( \xi = \sqrt[n]{\zeta} \) for \( \zeta \in D_w \) with \( w \in (2n - 1, \infty] \) the
Moreover, for \( w \in [2n - 1, \infty) \) we have
\[
\hat{w}_n(\zeta) \geq \hat{w}_n^*(\zeta) > 1 = \hat{w}_1(\zeta) = \max\{\hat{w}_1(\zeta), \ldots, \hat{w}_n(\zeta)\},
\]
such that the uniform inequalities analogous to (19) are both false in general.

If \( w > 2n - 1 \), we have simultaneously the strict inequalities
\[
w_n(\zeta) < w_n(\zeta), \quad w_n^*(\zeta) < w_n^*(\zeta), \quad \hat{w}_n(\zeta) < \hat{w}_n(\zeta), \quad \hat{w}_n^*(\zeta) < \hat{w}_n^*(\zeta).
\]
In particular, when \( w = \infty \) we have (32) simultaneously for all \( n \geq 2 \).

### 3.4 Previous results and consequences.
Davenport and Schmidt [20] established a link between approximation to a real number \( \zeta \) by algebraic numbers/Integers of bounded degree and simultaneous approximation to successive powers of \( \zeta \). For a convenient formulation of (variants of) their results we introduce the exponents of simultaneous approximation \( \lambda_n(\zeta), \hat{\lambda}_n(\zeta) \) defined by Bugeaud and Laurent [11]. They are given as the supremum of real \( \lambda \) such that the system
\[
1 \leq x \leq X, \quad \max_{1 \leq j \leq n} |\zeta^j x - y_j| \leq X^{-\lambda}
\]
has a solution \( (x, y_1, \ldots, y_n) \in \mathbb{Z}^{n+1} \) for arbitrarily large \( X \), and all large \( X \), respectively. Dirichlet’s Theorem implies \( \lambda_n(\zeta) \geq \hat{\lambda}_n(\zeta) \geq 1/n \). Khintchine’s transference principle [23] links the exponents \( w_n \) and \( \lambda_n \) in the form
\[
\frac{w_n(\zeta)}{(n - 1)w_n(\zeta) + n} \leq \lambda_n(\zeta) \leq \frac{w_n(\zeta) - n + 1}{n}.
\]
See German [21] for inequalities linking the uniform exponents. Upper bounds for \( \hat{\lambda}_n(\zeta) \) and \( \lambda_n(\zeta) \), respectively, translate into lower bounds for \( w_n^*(\zeta) \) and \( \hat{w}_n^*(\zeta) \), respectively.

**Theorem 3.9** (Davenport, Schmidt, Bugeaud, Teulie). Let \( n \geq 1 \) be an integer and \( \zeta \) be a real number not algebraic of degree at most \( n/2 \). Assume that there exist constants \( \lambda > 0 \) and \( c > 0 \), such that for certain arbitrarily large \( X \), the estimate
\[
1 \leq x \leq X, \quad \max_{1 \leq j \leq n} |x \zeta^j - y_j| \leq cX^{-\lambda}
\]
has no solution in an integer vector \( (x, y_1, \ldots, y_n) \). Then the inequality
\[
|\zeta - \alpha| \ll_{n, \zeta} H(\alpha)^{-1/\lambda - 1}
\]
has infinitely many solutions \( \alpha \in \mathbb{A}_{\zeta,n} \). Similarly, if (34) has no integral solution for all large \( X \), then
\[
H(\alpha) \leq X, \quad |\zeta - \alpha| \ll_{n, \zeta} X^{-1/\lambda - 1}
\]
has a solution $\alpha \in \mathbb{A}_{=n}$ for all large $X$. In particular, we have

\begin{equation}
(37) \quad w^*_n(\hat{\zeta}) \geq \frac{1}{\lambda_n(\hat{\zeta})}, \quad \hat{w}^*_n(\zeta) \geq \frac{1}{\lambda_n(\zeta)}.
\end{equation}

We omit the proof as the results are essentially known and consequence of the proofs of Davenport and Schmidt \cite{DavenportSchmidt} Lemma 1] and a slight variant of it by Bugeaud \cite{Bugeaud} Theorem 2.11]. See also the comment subsequent to the proof of \cite{8, Theorem 2.11}, and Bugeaud and Teulie \cite{15}, \cite{44}.

Theorem 2.11]. See also the comment subsequent to the proof of \cite{8, Theorem 2.11}, and Bugeaud and Schmidt \cite{20, Lemma 1} and a slight variant of it by Bugeaud \cite{8, resulting numerical bounds when combined with Theorem 3.9 become

\[ w^*_n(\zeta) \geq 2.3557 \ldots, \quad w^*_4(\zeta) \geq \frac{4}{\sqrt{73} - 7} = 2.5906 \ldots, \quad w^*_5(\zeta) \geq 3. \]

Theorem 3.9 will in fact be a crucial ingredient for the proofs of many of our new results. Below we present some of its immediate consequences when combined with some recent results from \cite{36} and our new results from Section 3.3. First we derive a new characterization of Liouville numbers.

**Corollary 3.10.** A real number $\zeta$ is a Liouville number if and only if

\begin{equation}
(38) \quad \hat{w}^*_n(\zeta) = \hat{w}^*_n(\zeta) = 0, \quad \text{for any } n \geq 2.
\end{equation}

In fact, if $\zeta$ is not a Liouville number, then $\hat{w}^*_n(\zeta) \geq \hat{w}^*_n(\zeta) > 0$ for any $n \geq 2$.

**Proof.** If $\zeta$ is a Liouville number then (38) follows from Theorem 3.9 with $w = \infty$. If otherwise $w_1(\zeta) = \lambda_1(\zeta) < \infty$, then (37) yields $\hat{w}^*_n(\zeta) \geq \lambda_n(\zeta)^{-1} \geq \lambda_1(\zeta)^{-1} > 0$. \hfill \qedsymbol

The implication (38) for Liouville numbers might appear strong at first view, but is somehow suggestive given the results on inhomogeneous approximation by Bugeaud and Laurent \cite{12}, see Section 4 below. Problem 7 in Section 6 below asks for a similar characterization involving the classical exponents $\lambda^*_n(\zeta)$. Our second corollary to Theorem 3.9 proves a strengthened version of Wirsing’s conjecture for numbers with large irrationality exponent.

**Corollary 3.11.** Let $\zeta$ be a real number and $n \geq 1$ an integer. Assume $w_1(\zeta) \geq n$ holds. Then we have $w^*_n(\zeta) \geq n$.

**Proof.** It was shown in \cite{36, Theorem 1.12} that $w_1(\zeta) \geq n$ implies $\hat{\lambda}_n(\zeta) = 1/n$. Hence the assertion is derived from Theorem 3.9. \hfill \qedsymbol

Observe Corollary 3.11 applies in particular to all numbers in any class $\mathcal{P}_{n,w}$ for $w \geq n$. Our last corollary establishes some more exponents for strong Liouville numbers.

**Corollary 3.12.** Let $\zeta \in \mathcal{B}_{\infty}$. Then $w_{=n}(\zeta) = w^*_n(\zeta) = n$ holds for all $n \geq 2$.

**Proof.** From Corollary 3.11 and (12) we know that $w_{=n}(\zeta) \geq w^*_n(\zeta) \geq n$. On the other hand, (29) with $w = \infty$ implies $w_{=n}(\zeta) \leq n$ for $n \geq 2$. \hfill \qedsymbol
We remark that we cannot expect the result to extend for arbitrary Liouville numbers. For the formulation of our final result in this section we need to define successive minima for \(1 \leq j \leq n + 1\), define \(w_{n,j}(\zeta)\) and \(\hat{w}_{n,j}(\zeta)\) respectively as the supremum of \(w\) such that (8) has \(j\) linearly independent solutions for arbitrarily large \(X\) and all large \(X\), respectively. We see that \(w_{n,1}(\zeta) = w_n(\zeta)\) and \(\hat{w}_{n,1}(\zeta) = \hat{w}_n(\zeta)\). Mahler showed that the identities

\[
\lambda_n(\zeta)^{-1} = \hat{w}_{n,n+1}(\zeta), \quad \hat{\lambda}_n(\zeta)^{-1} = w_{n,n+1}(\zeta)
\]

are valid for any transcendental real \(\zeta\). These are special cases of Mahler’s duality, see Schmidt and Summerer [43] and also [35, (1.24)] for more general versions. We show that in general we cannot replace the right hand sides \(1/\hat{\lambda}_n(\zeta) = w_{n,n+1}(\zeta)\) and \(1/\lambda_n(\zeta) = \hat{w}_{n,n+1}(\zeta)\) of (37), respectively, by the next larger successive minimum value \(w_{n,n}(\zeta)\) and \(\hat{w}_{n,n}(\zeta)\), respectively.

**Theorem 3.13.** Let \(n \geq 1\) be an integer and \(w > 2n - 1\). For \(\zeta \in \mathcal{D}_{n,w}\) we have

\[
\hat{w}^*_n(\zeta) = \hat{w}_n(\zeta) < \hat{w}_{n,n}(\zeta).
\]

For \(\zeta \in \mathcal{B}_w\) moreover

\[
w^*_n(\zeta) = w_n(\zeta) < w_{n,n}(\zeta).
\]

It is not clear whether the analogous inequalities for the classic exponents can be satisfied.

4. APPROXIMATION BY ALGEBRAIC INTEGERS

We define several new variants of the classical exponents, related to the approximation to a real number by algebraic integers.

**Definition 1.** Let \(\zeta\) be a real number and \(n \geq 1\) an integer. Let \(w^{\text{int}}_n(\zeta)\) (and \(w^{\text{int}}_n(\zeta)\) resp.) be the supremum of \(w\) such that (8) has a monic polynomial solution \(P \in \mathbb{Z}_{\leq n}\) (and an irreducible monic solution \(P \in \mathbb{Z}_{= n}\) resp.) for arbitrarily large \(X\). Similarly, define \(\hat{w}^{\text{int}}_n(\zeta)\) (and \(\hat{w}^{\text{int}}_n(\zeta)\) resp.) as above, with the respective properties satisfied for all large \(X\). Denote by \(w^{\text{int}}_n(\zeta)\) (and \(w^{\text{int}}_n(\zeta)\) resp.) the supremum of \(w^*\) such that (8) has a solution \(\alpha \in \mathbb{A}_{\leq n}\) (and \(\alpha \in \mathbb{A}^\text{int}_{= n}\) resp.) for arbitrarily large \(X\). Similarly, define \(\hat{w}^{\text{int}}_n(\zeta)\) (and \(\hat{w}^{\text{int}}_n(\zeta)\) resp.) as above, with the respective properties satisfied for all large \(X\).

By a similar argument as in [46, Hilfssatz 4] we may consider only irreducible polynomials within the definition of \(w^{\text{int}}_n(\zeta)\). On the other hand, we do not expect this to be true for the uniform exponents \(\hat{w}^{\text{int}}_n(\zeta)\), although we do not address the topic of counterexamples here. The irreducibility assumption on the polynomials with respect to the exponents of prescribed degree again avoids trivial identities, as in Section 1.2. The corresponding versions of the obvious relations (9), (7), (12), (10), (11) and (13) hold again, apart from \(w^{\text{int}}_1(\zeta) = \hat{w}^{\text{int}}_1(\zeta) = w^{\text{int}}_1(\zeta) = \hat{w}^{\text{int}}_1(\zeta) = 0\) unless \(\zeta \in \mathbb{Z}\). The monotonicity conditions will most likely again require bounded degree, however again we do not address counterexamples in exact degree. We should also notice the obvious facts

\[
w_n(\zeta) \geq w^{\text{int}}_n(\zeta), \quad \hat{w}_n(\zeta) \geq \hat{w}^{\text{int}}_n(\zeta), \quad w^*_n(\zeta) \geq w^{\text{int}}_n(\zeta), \quad \hat{w}^*_n(\zeta) \geq \hat{w}^{\text{int}}_n(\zeta).
\]
However, approximation by elements in \( \mathbb{A}_{\leq n}^{\text{int}} \) should rather be compared to approximation by elements in \( \mathbb{A}_{\leq n-1} \), as there is the same degree of freedom in the choice of coefficients for the corresponding minimal polynomials.

We quote a variant of Theorem 3.9 again essentially due to Davenport and Schmidt.

**Theorem 4.1** (Davenport, Schmidt). Let \( m, n \) be positive integers with \( m \geq n + 1 \), and \( \zeta \) be a real number not algebraic of degree at most \( n/2 \). Assume that there exist constants \( \lambda > 0 \) and \( c > 0 \) such that for arbitrarily large values of \( X \), the estimate (34) has no solution in an integer vector \((x, y, \ldots, y_n)\). Then, the inequality

\[
|\zeta - \alpha| \ll_m \zeta H(\alpha)^{-1/\lambda - 1}
\]

has infinitely many solutions \( \alpha \in \mathbb{A}_{\leq m}^{\text{int}} \). In particular, we have

\[
w_{m+1}^m(\zeta) \geq \frac{1}{\lambda_n(\zeta)}, \quad w_{n+1}^m(\zeta) \geq \frac{1}{\lambda_n(\zeta)}.
\]

Similarly, if (34) has no solutions for all large \( X \), then

\[
H(\alpha) \leq X, \quad |\zeta - \alpha| \ll_m \zeta X^{-1/\lambda - 1}
\]

has a solution \( \alpha \in \mathbb{A}_{\leq m}^{\text{int}} \) for all large \( X \). In particular we have

\[
\hat{w}^m_{m+1}(\zeta) \geq \frac{1}{\lambda_n(\zeta)}, \quad \hat{w}^m_{n+1}(\zeta) \geq \frac{1}{\lambda_n(\zeta)}.
\]

The claims (40) and (41) reproduce [20, Lemma 1], see also [3] Theorem 2.11 and [14]. The dual claims are obtained similarly; we will omit the proof. The claim is closely related to the very general main result in [12] by Bugeaud and Laurent on inhomogeneous approximation, of which we will discuss a special case below. Similarly to Theorem 3.9 known estimates for \( \lambda_n \) lead to lower bounds roughly of size \( n/2 \) for \( w^m_{m+1} \).

Recall (13) holds for special numbers. On the other hand, it is unknown and was posed as a problem in [3] and recently rephrased in [14], whether \( w^m_{m+1}(\zeta) \geq n \) holds for any transcendental real \( \zeta \) when \( n \geq 3 \). The analogue problem for \( w^m_{m+1}(\zeta) \) is open as well.

Again both answers are positive for a pair \( n, \zeta \) with the property \( \lambda_n(\zeta) = 1/n \), in view of Theorem 4.1. We notice the answer is also positive when \( \zeta \) allows sufficiently good rational approximations, analogously to Corollary 3.11.

**Corollary 4.2.** Let \( \zeta \) be a real number and \( n \geq 1 \) an integer. Assume \( w_1(\zeta) \geq n \) holds. Then we have \( w^m_{m+1}(\zeta) \geq n \) for any \( m \geq n + 1 \). In particular \( w^m_{m+1}(\zeta) \geq n \).

As Corollary 3.11, the claim follows directly from [36, Theorem 1.12] and Theorem 4.1. The main contribution of this paper concerning approximation by algebraic integers are bounds for the uniform constants \( \hat{w}^m_{m+1}(\zeta) \) and \( \hat{w}^m_{n+1}(\zeta) \), for special numbers \( \zeta \). Another characterization of Liouville numbers is obtained as a special case.

**Theorem 4.3.** Let \( n \geq 2 \) be an integer, \( w \in [n, \infty] \) and \( \zeta \in \mathcal{P}_{n,w} \). Then

\[
\frac{n - 1}{w - n + 2} \leq \hat{w}^m_{m+1}(\zeta) \leq \hat{w}^m_{n+1}(\zeta) \leq \frac{n}{w - n + 1}.
\]

In particular, a transcendental real number \( \zeta \) is a Liouville number if and only if

\[
\hat{w}^m_{m+1}(\zeta) = \hat{w}^m_{n+1}(\zeta) = 0, \quad \text{for any } n \geq 1.
\]
We point out that Theorem 1.3 can be interpreted in terms of inhomogeneous approximation, complementing [12]. Indeed, (15) yields that for \( n \geq 1, \varepsilon > 0 \), \( \zeta \) a Liouville number and \( \alpha \in \{ \zeta^{n+1}, \zeta^{n+2}, \ldots \} \) (and more generally any \( \alpha = Q(\zeta) \) for \( Q \in \mathbb{Q}_{\geq n+1}[T] \)), the system

\[
\max_{0 \leq j \leq n} |x_j| \leq X, \quad |\alpha + x_0 + \zeta x_1 + \cdots + \zeta^n x_n| \leq X^{-\varepsilon}
\]

has no solution for certain arbitrarily large values of \( X \). The main result in [12] shows the same for Lebesgue almost all \( \alpha \). The latter provided a major improvement on Cassels [16, Theorem 3 of Chapter III]. Thus our contribution in (15) can be interpreted as to provide explicit examples of \( \alpha \) for which the metric claim is satisfied. The metric result in [12] and the proof of (44) below furthermore suggest equality in the two left inequalities in (14) for any real \( \zeta \). Note that if otherwise \( \alpha = Q(\zeta) \) for \( Q \in \mathbb{Q}_{\leq n}[T] \), then for all large \( X \) and certain \( x_i \) the right expression in (46) is \( \ll Q X^{-n} \), as it is roughly speaking just a shift of the homogeneous problem.

5. PROOFS

The proof of Theorem 1.1 is based on the result of Davenport and Schmidt (4). To rule out that all good approximations to \( \zeta \) are rational we use the method from [36, Theorem 1.12], which we explicitly carry out again for the reason to be self-contained and the convenience of the reader. Recall \( \|\alpha\| \) denotes the distance of \( \alpha \in \mathbb{R} \) to the nearest integer.

**Proof of Theorem 1.1.** Assume for given \( \zeta \) the claim would be false. Then, by the result of Davenport and Schmidt, for some constant \( c = c(\zeta) \) there exist infinitely many rational numbers \( \alpha = \frac{y_0}{x_0} \) for which (4) holds (in particular \( \lambda_1(\zeta) \geq 2 \)). Now we essentially follow the proof of [36, Theorem 1.12] for \( n = 2 \). For a fraction \( \frac{y_0}{x_0} \) as above, we clearly may assume \( |\zeta - \frac{y_0}{x_0}| \leq 1 \), and thus the formula \( |\zeta^2 - \frac{y_0^2}{x_0^2}| = |\zeta - \frac{y_0}{x_0}| \cdot |\zeta + \frac{y_0}{x_0}| \leq (2|\zeta| + 1) \cdot |\zeta - \frac{y_0}{x_0}| \) implies

\[
(47) \quad \left| \zeta^j - \frac{y_0^j}{x_0^j} \right| \leq c_1 x_0^{-3}, \quad j \in \{1, 2\},
\]

for \( c_1 = \max\{(2|\zeta| + 1)c, 1\} \). Define \( X = x_0^2/(2c_1) \) and let \( 1 \leq x \leq X \) be an arbitrary integer. Since \( x \leq x_0^2/(2c_1) \leq x_0^2/2 < x_0^2 \), the integer \( x \) has a representation in base \( x_0 \) as

\[
x = b_0 + b_1 x_0, \quad b_i \in \{0, 1, 2, \ldots, x_0 - 1\}.
\]

Denote by \( i \in \{0, 1\} \) the smallest index with \( b_i \neq 0 \), and further let \( u = i + 1 \in \{1, 2\} \). Since \( x_0, y_0 \) are coprime and \( b_i \neq 0 \), we have

\[
(48) \quad \left\| \frac{y_0^u}{x_0^u} \right\| = \left\| b_i x_0^{u-1} \frac{y_0^u}{x_0^u} \right\| = \left\| \frac{b_i y_0^u}{x_0} \right\| \geq x_0^{-1}.
\]

On the other hand (47) yields

\[
(49) \quad \left| x \left( \zeta^u - \frac{y_0^u}{x_0^u} \right) \right| \leq X \left| \zeta^u - \frac{y_0^u}{x_0^u} \right| \leq \frac{x_0^2}{2c_1} \cdot c_1 x_0^{-3} = \frac{1}{2} x_0^{-1}.
\]
Combination of (48) and (49) and the triangular inequality give
\[ \max\{\|\xi x\|, \|\xi^2 x\|\} \geq \|\xi^\epsilon x\| \geq \frac{1}{2} x_0^{-1} = c' X^{-1/2}, \]
for the constant $c' = 1/\sqrt{8c_1}$ that again depends on $\xi$ only. Thus, since $x \leq X$ was arbitrary, the assumption (31) of Theorem 3.9 is satisfied for $\lambda = 1/2$ and the constant $c'$. Hence (35) applies, which yields precisely the claim. Since $c$ in (41) and thus $c_1$ is effective and the implied constant in (35) can be made effective as well, so is our constant. \hfill \Box

For the proof of Theorem 2.1 we recall the notion of best approximation polynomials of a given degree $n$ associated to a real number $\xi$. It can be defined as the sequence of integer polynomials $(P_i)_{i \geq 1}$ with the properties $1 \leq H(P_1) \leq H(P_2) \leq \cdots$ and $|P_i(\xi)|$ minimizes the value $|P(\xi)|$ among $P \in \mathbb{Z}_{\leq n}[T]$ of height $0 < H(P) \leq H(P_i)$. The polynomials involved in the definition of $w_n$ can obviously be chosen as best approximation polynomials. Furthermore every best approximation polynomial satisfies $|P_i(\xi)| \ll_{n,\xi} H(P_i)^{-n}$ by Dirichlet’s Theorem, see also the proof of [8, Lemma 8.1]. Moreover $|P_i(\xi)| \leq H(P_i)^{-w_n(\xi) + \epsilon}$ for any $\epsilon > 0$ and sufficiently large $i \geq i_0(\epsilon)$. We will utilize the estimates
\[ (50) \quad H(P_1 P_2) \asymp_n H(P_1) H(P_2) \]
for any polynomials $P_1, P_2 \in \mathbb{Z}_{\leq n}[T]$, sometimes referred to as Gelfand’s Lemma. See also [10] or [8, Lemma A.3]. We will apply [36, Theorem 5.1] for $n = 2$ several times, which asserts that $w_1(\xi) \geq n$ implies $\hat{w}_n(\xi) = n$, or equivalently $\hat{w}_n(\xi) > n$ implies $w_1(\xi) < n$.

Proof of Theorem 2.1. First we show (14). In view of the obvious inequalities (9), it suffices to show $w_{\leq 2}(\xi) \geq w_2(\xi)$ and $\hat{w}_{\leq 2}(\xi) \geq \hat{w}_2(\xi)$. Note that from our assumption $\hat{w}_2(\xi) > 2$ and [36, Theorem 5.1] we infer $w_1(\xi) < 2$. Hence, since any quadratic best approximation polynomial satisfies $|P(\xi)| \ll_{n,\xi} H(P)^{-2}$, no linear polynomial of large height can induce a quadratic best approximation polynomial. Moreover, essentially by (50), also no product $P = P_1 P_2$ of linear polynomials $P_i$ of large enough height $H(P)$ can be a best approximation. Indeed, if $\epsilon > 0$ and we write $H(P_1) H(P_2) =: H$, we have $H(P) \gg H$ by (50) but also
\[ |P(\xi)| = |P_1(\xi)| \cdot |P_2(\xi)| \geq H(P_1)^{-w_1(\xi) - \epsilon} H(P_2)^{-w_1(\xi) - \epsilon} \gg H^{-w_1(\xi) - \epsilon}. \]
If we choose $\epsilon = (2 - w_1(\xi))/2 > 0$ we again obtain a contradiction to $P$ being a best approximation polynomial. Thus any quadratic best approximation polynomial of sufficiently large height is irreducible of degree two. The deduction of (14) is now obvious. Next we show (15). Let $(\alpha_i)_{i \geq 1}$ be a sequence of rational or quadratic irrational numbers as in the definition of $w^*_i(\xi)$, with minimal polynomials $P_i$ respectively. By Theorem 1.1 we can assume $|\xi - \alpha_i| \ll H(\alpha_i)^{-3}$. With the standard estimate $|P_i(\xi)| \ll_{n,\xi} H(P_i) |\xi - \alpha_i|$ mentioned already in Section 1.2 we infer $|P_i(\xi)| \ll_{\xi} H(P_i)^{-2}$. If infinitely many among the polynomials $P_i$ were linear, we would have $w_1(\xi) \geq 2$ and hence again by [36, Theorem 5.1] we infer $\hat{w}_2(\xi) = 2$, contradicting the assumption. Hence all but finitely many $\alpha_i$ are quadratic irrational and (15) follows. The claim on $\hat{w}^*_2$ follows similarly.

For the last claim, observe that it was shown in [11, Théorème 5.3] that when $\hat{w}_n(\xi) > n$ and $\xi$ is a $U$-number, then it must be a $U_m$-number for $m \leq n$. Applied for $n = 2$, we have to exclude that $\xi$ is a $U_1$-number or a $U_2$-number. Now [36, Theorem 5.1] implies directly
that \( \zeta \) cannot be a \( U_1 \)-number. Similarly, for \( \zeta \) any \( U_2 \)-number, we obtain \( \hat{\omega}_2(\zeta) = 2 \) from [14 Corollary 2.5], contradicting our hypothesis.

We turn to the proofs of Section 3. We start with the proof of the polynomial criterion. We recall some notation and classical facts from analytic number theory. Let \( N \geq 1 \) be an integer. The Prime Number Theorem implies that there are \( \pi(N) \gg N/\log N \) primes up to \( N \), and the number of divisors \( \tau(N) \) of \( N \) is bounded by \( \tau(N) \ll N^{\epsilon} \) for arbitrarily small \( \epsilon > 0 \), see the book of Apostol [23 page 296]. Finally, the number of prime divisors \( \omega(N) \) of \( N \) is bounded by \( \omega(N) \ll \log N \) (more precisely \( \omega(N) \ll \log N/\log \log N \) with asymptotic equality when \( N \) is primorial, see Hardy and Wright [22]).

Proof of Theorem 3.3. Let \( P, Q \) as in the theorem with \( X = \max\{H(P), H(Q)\} \), and \( \delta \in (0, 1) \) be given. We may write

\[
(51) \quad Q(T) = a_1T + a_2T^2 + \cdots + a_nT^n, \quad P(T) = b_0 + b_1T + \cdots + b_{n-1}T^{n-1},
\]

with \( b_0, a_n \) non-zero. Indeed, \( b_0 \neq 0 \) since \( P \) has no common linear factor with \( Q \) and \( T|Q(T) \), and \( a_n \neq 0 \) since \( Q \) has exact degree \( n \) by assumption.

We first show the bounds for \( R_h \) with \( h \) a prime. Assume that \( h \) is prime and the polynomial \( R_h(T) = Q(T) + hP(T) \) has a linear factor \( q_hT - p_h \). Equivalently each \( R_h \) has a rational root \( p_h/q_h \), written in lowest terms. Inserting \( R_h(p_h/q_h) = 0 \) in (51) and multiplication with \( q_h^\delta \neq 0 \) yields

\[
(52) \quad a_1p_hq_h^{-1} + a_2p_h^2q_h^{-2} + \cdots + a_np_h^n + h(b_0q_h^n + b_1p_hq_h^{n-1} + \cdots + b_{n-1}p_h^{n-1}q_h) = 0.
\]

Since any expression apart from \( a_n|p_h^n \) contains the factor \( q_h \) and \( (p_h, q_h) = 1 \), we conclude \( q_h|a_n \). Similarly \( p_h|(h) \). Since \( a_n \neq 0 \), the quoted result from [3] above with \( \epsilon = \delta/3 \) yields that it has at most \( \tau(a_n) \ll \delta |a_n|^{\delta/3} \) divisors, so there appear at most \( \ll \delta |a_n|^{\delta/3} \leq H(Q)^{\delta/3} \ll X^{\delta/3} \) different denominators \( q_h \). For any such fixed \( q = q_h \), we estimate the number of primes \( h \) for which the polynomial \( R_h \) can have a root \( p_h/q_h \).

Recall \( p_h|(h) \), such that either \( p_h|b_0 \) or \( p_h = hs_h \) for some \( s_h|b_0 \). We treat both possible cases separately. Assume first \( p_h|b_0 \). Since \( b_0 \neq 0 \), there are at most \( \tau(b_0) \ll \delta |b_0|^{\delta/3} \leq H(Q)^{\delta/3} \ll X^{\delta/3} \) divisors of \( b_0 \), so there are at most \( \ll \delta X^{\delta/3} \) different \( p_h \). However, for given \( p_h, q = q_h \) the number \( h \) is uniquely determined by (52). Indeed, otherwise both expressions \( P(p_h/q_h) \) and \( Q(p_h/q_h) \) must vanish, contradicting the hypothesis that \( P, Q \) have no common linear factor. Hence also the number of \( h \) belonging to this class is \( \ll \delta X^{\delta/3} \).

Now assume \( p_h = hs_h \) with \( s_h|b_0 \). Since \( b_0 \neq 0 \) also \( s_h \neq 0 \). Then for our fixed \( q_h = q \), the relation (52) becomes after division by \( h \neq 0 \) and rearrangements

\[
(53) \quad h^n(a_ns^n + b_{n-1}s^{n-1}q + \cdots + h(a_2s^2q^{n-2} + sb_1q^{n-1}) + a_1sq^{n-1} + b_0q^n) = 0.
\]

Reducing modulo \( h \) we see that \( h|(q^{n-1}(s_a + b_0q)) \). Since \( h|p_h \) and \( (p_h, q) = 1 \) we cannot have \( h|q \), such that

\[
(54) \quad h|(s_ha_1 + b_0q).
\]

First assume \( s_ha_1 + b_0q \neq 0 \). Since \( q \) is fixed and \( s_h|b_0 \neq 0 \), there are at most \( \tau(b_0) \ll \delta |b_0|^{\delta/3} \ll X^{\delta/3} \) choices for \( s_h \), consequently there arise at most \( \ll \delta X^{\delta/3} \) different numbers \( N_s = s_ha_1 + b_0q \). Since \( q|a_n \) and \( s_h|b_0 \), all quantities \( s_h, a_1, b_0, q \) are bounded by \( X \) and thus \( |N_s| \leq 2X^2 \) for all \( N_s \neq 0 \). Hence each \( N_s \neq 0 \) has at most \( \omega(N_s) \ll \log |N_s| \leq
log(2X^2) \ll \log X \text{ prime divisors. So by (54) at most } \ll X^{6/3} \log X \text{ choices for } h \text{ arise in this way.}

Now assume \( s_ha_1 + b_0q = 0 \). Then from (53) after division by \( hs_h \neq 0 \) we obtain that \( h|(q^{n-2}(a_2s_h + b_1q)) \) and again since \( h \nmid q \) we conclude \( h|(a_2s_h + b_1q) \). In case of \( s_ha_2 + b_1q \neq 0 \), with the same argument as above we again obtain at most \( \ll X^{6/3} \log X \) choices for \( h \). If otherwise \( s_ha_2 + b_1q = 0 \), then we proceed as above to obtain \( h|(a_2s_h + b_2q) \).

Repeating this procedure up to \( h|(a_ns_h + b_{n-1}q) \) leads to in total at most \( \ll n,\delta X^{6/3} \log X \) choices of \( h \), unless we have \( s_ha_k + b_{k-1}q = 0 \) for all \( 1 \leq k \leq n \). Since \( a_ns_h \neq 0 \), the latter implies \( (a_1,a_2,\ldots,a_n) = \frac{2}{nX}(b_0,b_1,\ldots,b_{n-1}) \), in other words \( Q(T)/(TP(T)) = -q/s_h \) is constant, which we excluded by assumption.

Thus we indeed only have \( \ll n,\delta X^{6/3} \log X \) choices of primes \( h \) for which \( R_h \) has a root with given denominator \( q_h = q \). Since we have noticed that at most \( \ll X^{6/3} \) different \( q \) occur and \( \log X \ll X^{6/3} \), the total number of \( h \) for which \( R_h \) has a rational root is indeed \( \ll n,\delta X^6 \).

Similar arguments show that the number of primes \( l \) for which \( S_I = P + lQ \) has a linear factor are of order \( \ll n,\delta X^6 \) as well. We only sketch the proof. Inserting \( S_I(p_l/q_l) = 0 \) in (51) we obtain \( p_l|b_0 \) and hence only \( \ll n,\delta X^{6/3} \) many numerators \( p_l \) of roots \( p_l/q_l \) of \( S_I \) can appear. We again estimate the number of primes \( l \) with fixed \( p_l = p \). Since \( q_l|(lh\alpha_n) \), by a very similar recursive procedure as for \( R_h \), the number of such \( l \) is again of order \( \ll n,\delta X^{6/3} \log X \) unless \( a_n/b_{n-1} = a_{n-1}/b_{n-2} = \cdots = a_1/b_0 \), which is again excluded by assumption. The claim follows as above.

By Prime Number Theorem there are \( \gg Y/\log Y \gg Y^\delta \) primes up to \( Y \), if we choose \( Y = dX^\varepsilon \) for \( \varepsilon = 2\delta \) and suitable \( d = d(n,\delta) \) we avoid the primes in both sets of cardinality \( \ll n,\delta X^6 \) above.

Finally we show that \( R_h \) or \( S_I \) having a linear factor implies \( h \leq nX^{n+1} \) or \( l \leq nX^{n+1} \), respectively. We only show the claim for \( R_h \), the other case is very similar again. Recall that \( |q_h| \leq |b_0| \leq X \) since \( q_h|b_0 \). We showed that either \( p_h|b_0 \) or \( p_h = hs_h \) with \( s_h|b_0 \). In the first case \( |p_h| \leq X \), such that if we express \( h \) from (52) as a rational function then each expression in the numerator \( q_h^kQ(p_h/q_h) \) and denominator \( q_h^kP(p_h/q_h) \) is bounded in absolute value by \( X^{n+1} \). Hence the numerator has absolute value at most \( nX^{n+1} \), and the denominator at least 1 as we have already noticed it is non-zero. We conclude the bound \( h \leq nX^{n+1} \) in this case. In the latter case, from \( s_h|b_0 \neq 0 \) we infer \( |s_h| \leq X \), and from (54) we obtain the upper bound \( h \leq 2X^2 \) unless \( s_ha_1 + b_0q = 0 \). In this case the proof above showed \( h|(s_ha_2 + b_1q) \) as well such that again \( h \leq 2X^2 \) unless \( s_ha_2 + b_1q = 0 \). We iterate this argument, and by assumption there must exist \( k \leq n \) with \( s_ha_k + b_{k-1}q \neq 0 \). Hence we obtain the bound \( h \leq 2X^2 \leq nX^{n+1} \) in the second case anyway. \( \square \)

We recall that by Gelfond’s Lemma there exists a (small) constant \( K = K(n) > 0 \) such that for \( P,Q \in \mathbb{Z}_{\leq n} \) as soon as \( H(Q) < KH(P) \) we cannot have that \( P \) divides \( Q \). This argument was already used for the proof of [14, Theorem 2.3], which is indeed very similar to our proof below.

**Proof of Theorem 3.2.** Let \( \epsilon > 0 \). As already noticed in the proof of Lemma 3.1 there exist infinitely many irreducible polynomials \( P \in \mathbb{Z}_{\leq n}[T] \) with \( |P(\zeta)| \leq H(P)^{-m_n(T)+\epsilon} \). If the degree of \( P \) is \( n \) the claim follows trivially, so we may assume it is less. Consider \( P \)
fixed of large height and let \( X = H(P) \cdot K/2 \) with \( K = K(n) \) as above. Then
\[ H(P) \ll_n X, \quad |P(\zeta)| \ll_n X^{-\omega_n(\zeta)+\epsilon}. \]

Now by definition of \( \omega_n(\zeta) \) there exists \( R \in \mathbb{Z}_{\leq n}[T] \) such that
\[ H(R) \leq X, \quad |R(\zeta)| \leq X^{-\omega_n(\zeta)+\epsilon}. \]

Obviously \( R \neq P \) since \( H(P) > X > H(R) \). By construction in fact \( R \) cannot be a multiple of \( P \), and since \( P \) is irreducible, \( P, R \) have to be coprime. Again the claim of the theorem follows trivially from (56) if the degree of \( R \) is \( n \), so we may assume \( R \in \mathbb{Z}_{\leq n-1}[T] \). If \( d \leq n - 1 \) denotes the degree of \( R \), then let \( Q(T) = R(T)T^{n-d} \), which has degree \( n \). Since obviously \( P, Q \) are coprime as well and \( H(Q) = H(R) \leq X \), we can apply the hypothesis to find non-zero integers \( a, b \) of absolute value at most \( c(n, \epsilon)X^\epsilon \) such that \( S(T) = aQ(T) + bP(T) \) is irreducible. Since \( P \in \mathbb{Z}_{\leq n-1}[T] \), \( Q \in \mathbb{Z}_{=n}[T] \) and \( a \neq 0 \), we have \( S \in \mathbb{Z}_{\leq n}[T] \). Clearly \( |Q(\zeta)| \ll_{\zeta,n} |Q(\zeta)| \). Thus from (55) and (56) we infer
\[ H(S) \leq |a|H(Q) + |b|H(P) \leq \max\{|a|, |b|\} X \ll_n \epsilon X^{1+\epsilon} \]
and
\[ |S(\zeta)| \leq |a| \cdot |Q(\zeta)| + |b| \cdot |P(\zeta)| \leq \max\{|a|, |b|\} X^{-\omega_n(\zeta)+\epsilon} \ll_{n,\epsilon} X^{-\omega_n(\zeta)+2\epsilon}. \]

Hence (21) follows (conditionally) as \( \epsilon \) can be chosen arbitrarily small.

Finally, for (20) we readily check that \( P, Q \) satisfy the assumptions of Theorem 3.3. It yields \( a = 1 \), \( b = p \) with \( \max\{|a|, |b|\} = p \leq cX^\epsilon \) for which \( S = aQ + bP \) has no linear factor. Since \( S \) is cubic it must in fact be irreducible. \( \square \)

For the proof of Theorem 3.3 it is convenient to use the notion of parametric geometry of numbers introduced by Schmidt and Summerer [43]. We develop the theory only as far as needed for our concern and slightly modify their notation. We refer to [43] for more details. Keep \( \zeta \in \mathbb{R} \) and \( n \geq 1 \) an integer fixed. For a parameter \( Q > 1 \) and \( 1 \leq j \leq n+1 \), let \( \psi_{n,j}(Q) \) be the least value of \( \eta \) such that
\[ |x| \leq Q^{1+\eta}, \quad |\zeta^j x - y_j| \leq Q^{-\frac{1}{\eta}} \]
has \( j \) linearly independent solutions \((x, y_1, \ldots, y_n) \in \mathbb{Z}^{n+1}\). Then \( -1 \leq \psi_{n,j}(Q) \leq 1/n \) for any \( Q \) by Minkowski’s Theorem. Let
\[ \underline{\psi}_{n,j} = \liminf_{Q \to \infty} \psi_{n,j}(Q), \quad \overline{\psi}_{n,j} = \limsup_{Q \to \infty} \psi_{n,j}(Q). \]

Similarly denote by \( \psi^*_{n,j}(Q) \) the smallest number \( \eta \) such that
\[ H(P) \leq Q^{\frac{1}{\eta}}, \quad |P(\zeta)| \leq H(P)^{-1+\eta} \]
has \( j \) linearly independent solutions in \( P \in \mathbb{Z}[T] \) of degree at most \( n \). We have \( -1/n \leq \psi^*_{n,j}(Q) \leq 1 \) for every \( Q > 1 \). Then Mahler’s duality, whose special case (83) we mentioned, can be reformulated as \( |\psi_{n,j}(Q) + \psi^*_{n,n+2-j}(Q)| \ll 1/\log Q \) for \( 1 \leq j \leq n+1 \), and hence \( \overline{\psi}_{n,j} = -\psi^*_{n,n+2-j} \). It was shown in the remark on page 80 in [43] that the identity \( \psi^*_{n,1} = -n\psi^*_{n,n+1} \) is equivalent to equality in Khintchine’s inequality [33], that is
\[ \lambda_n(\zeta) = \frac{w_n(\zeta) - n + 1}{n}. \]
Recall also the notion of the successive minima exponents \( w_{n,j}, \hat{w}_{n,j} \) defined subsequent to Corollary 3.12.

**Proof of Theorem 3.5.** We will restrict to the case \( w < \infty \), the proof of the remaining case \( w = \infty \) works very similarly. By assumption \( \zeta \in D_{n,w} \), for any \( \epsilon > 0 \) the estimate
\[
|P(\zeta)| \leq H(P)^{-w+\epsilon}
\]
has a solution \( P(T) = aT + b \) with integers \( a, b \) of arbitrarily large height \( H(P) = \max\{|a|, |b|\} \). Then the polynomials \( P_0 = P, P_1 = TP, \ldots, P_{n-1} = T^{n-1}P \) have degree at most \( n \), satisfy \( H(P_i) = H(P) \) and
\[
|P_i(\zeta)| \ll_{n,\zeta} H(P_i)^{-w+\epsilon}, \quad 0 \leq i \leq n - 1.
\]
Moreover the \( P_i \) are obviously linearly independent. Thus \( \hat{w}_{n,n}(\zeta) \geq w \), and hence by assumption \( w_{n,1}(\zeta) = w_{n,2}(\zeta) = \cdots = w_{n,n}(\zeta) = w \). This fact can be translated in the language of the values \( \psi^*, \overline{\psi} \) defined above as \( -n\psi^*_n = \overline{\psi}_{n,n+1} \); see the remark in [43] and its proof quoted above. Mahler’s duality stated yields the equivalent claim \( \psi^*_{n,1} = -n\psi^*_{n,n+1} \). Hence there is equality in the right Khintchine inequality (33) as carried out above, that is
\[
\hat{w}^*_n(\zeta) = n \left( \frac{w(\zeta) - n + 1}{n} \right) = \frac{w - n + 1}{n}.
\]
Thus with Theorem 3.9 we have
\begin{equation}
\hat{w}^*_n(\zeta) \geq \frac{1}{\lambda_n(\zeta)} = \frac{n}{w - n + 1}.
\end{equation}
For the reverse inequality notice that on the other hand the span of \( \{P_0, \ldots, P_{n-1}\} \) contains only polynomial multiples of \( P_0 \) and thus no irreducible \( Q \in \mathbb{Z}_n[T] \) (even no irreducible polynomial of degree \( 2 \leq d \leq n \)). Thus if we consider parameters \( X \) of the form \( X = H(P_0) \) in (53), we conclude that
\begin{equation}
\hat{w}_n(\zeta) \leq \hat{w}_{n,n+1}(\zeta).
\end{equation}
Combination of the left estimate in the right inequality of (13), Mahler’s identity (39), (57) and (58) yields
\[
\frac{n}{w - n + 1} = \frac{1}{\lambda_n(\zeta)} \leq \hat{w}^*_n(\zeta) \leq \hat{w}_n(\zeta) \leq \hat{w}_{n,n+1}(\zeta) = \frac{1}{\lambda_n(\zeta)} = \frac{n}{w - n + 1}.
\]
Hence (27) follows. \( \square \)

**Remark 3.** The proof shows that any \( \zeta \in D_{n,w} \) provides equality in the right inequality of (37).

For the proof of Theorem 3.7 we recall [14, Lemma 3.1], where we drop the originally involved condition which is easily seen not to be required for the conclusion.

**Lemma 5.1.** Assume \( P \) and \( Q \) are coprime polynomials of degree \( m \) and \( n \) respectively, and \( \zeta \) is a real number. Then at least one of the inequalities
\begin{equation}
|P(\zeta)| \gg_{m,n,\zeta} H(P)^{-n+1}H(Q)^{-m}, \quad |Q(\zeta)| \gg_{m,n,\zeta} H(P)^{-n}H(Q)^{-m+1}
\end{equation}
holds.
Proof of Theorem 3.7. We will again only deal with the case \( w < \infty \), the case \( w = \infty \) can be treated very similarly using (23). So let \( w \in [2n - 1, \infty) \) and \( \zeta \in \mathcal{B}_w \). Let us assume \( \rho > 0 \) is fixed and \( Q \) is an irreducible polynomial of degree exactly \( n \) such that

\[
|Q(\zeta)| \leq H(Q)^{-t - \rho}, \quad t = \frac{n w}{w - n + 1}.
\]

For every convergent \( P_j/q_j \) to \( P_j(T) = q_jT - p_j \). Then, as pointed out in [1], we have \( |P_j(\zeta)| H(P_j)^{-w} \) and \( H(P_{j+1}) \approx H(P_j)^w \) for \( j \geq 1 \). Let \( \delta \) be the index for which \( H(P_{\delta}) = q_{\delta} \leq H(Q) < q_{\delta+1} = H(P_{\delta+1}) \), where we used \( \zeta \in (0, 1) \). Clearly \( P_j \) is coprime to \( Q \) for all \( j \geq 1 \), since \( P_j \) have degree one and \( Q \) is irreducible of degree \( n \geq 2 \). Thus we can apply Lemma 5.1 with \( m = 1, n \) and the pair of polynomials \( P_j, Q \). Let \( \delta > 0 \). In case of \( H(Q) \leq H(P_j)^{w - n + 1 - \delta} \), for \( j = \delta \) the left inequality of (59) is violated as it would lead to

\[
H(P_\delta)^{-w + \delta} = H(P_\delta)^{-n + 1} H(P_\delta)^{-(w - n + 1 - \delta)} 
\leq H(P_\delta)^{-n + 1} H(Q)^{-1} \approx n \zeta \quad \text{contradiction for large } \delta.
\]

Thus we must have \( |Q(\zeta)| \approx n \zeta \quad \text{for large } \delta \quad \text{contradicting the assumption (60)} \quad \text{for large } \delta \quad \text{since } t \geq n \). If otherwise \( H(Q) \geq H(P_\delta)^{w - n + 1 - \delta} \), then we apply Lemma 5.1 for the polynomials \( P_{\delta+1} \) and \( Q \). The left inequality in (59) leads to

\[
H(P_{\delta+1})^{-w} \approx n \zeta \quad |P_{\delta+1}(\zeta)| \approx n \zeta \quad H(P_{\delta+1})^{-n + 1} H(Q)^{-1} \geq H(P_{\delta+1})^{-n}
\]

contradiction to \( w > n \) for large \( \delta \). Similarly the right inequality in (59) leads to

\[
|Q(\zeta)| \approx n \zeta \quad H(P_{\delta+1})^{-n} \approx n \zeta \quad H(P_\delta)^{-nw} \approx n \zeta \quad H(Q)^{-nw/(w - n + 1 - \delta)}
\]

again a contradiction to (60) for large \( \delta \) if \( \delta \) was chosen small enough that we still have \( t + \rho > nw/(w - n + 1 - \delta) \). Hence there can only be finitely many solutions to (60) for any \( \rho > 0 \) and irreducible \( Q \in \mathbb{Z}[T] \). The claim (23) follows.

For the deduction of Theorem 3.8 we apply the identities (26) for \( \zeta \in \mathcal{B}_w \). In fact the lower bound

\[
\hat{w}_n^*(\zeta) = \frac{w_n(\zeta)}{w_n(\zeta) - n + 1} = \frac{w}{w - n + 1}
\]

established by Bugeaud and Laurent [11, Theorem 2.1], would suffice.

Proof of Theorem 3.8. Combination of (27) with (26) yields \( \hat{w}_n(\zeta) = n/(w - n + 1) < n = \hat{w}_n(\zeta) \), as soon as \( w > n \). Similarly, from (26) we infer \( \hat{w}_n^*(\zeta) = n/(w - n + 1) < w/(w - n + 1) = \hat{w}_n^*(\zeta) \). Thus we have shown (30). For (31), again (26) implies \( \hat{w}_n(\zeta) = w/(w - n + 1) > 1 \) strictly, as soon as \( w < \infty \). On the other hand, when \( w \geq 2n - 1 \), we readily check \( \hat{w}_n(\zeta) = n/(w - n + 1) \leq 1 \), and similarly \( \hat{w}_m(\zeta) \leq 1 \) for \( 1 \leq m \leq n \). The remaining estimates of (31) are obvious consequences of (7), (12) and (13) and the previous observation.

When \( w > 2n - 1 \), from (29) we infer \( w_n^*(\zeta) \leq w_n(\zeta) \leq nw/(w - n + 1) < 2n - 1 < w = w_n^*(\zeta) = w_n(\zeta) \), which shows the two most left inequalities of (32). The uniform inequalities in (32) were already established in (30) under the weaker condition \( w > n \).
Proof of Theorem 3.5. We have \( \hat{w}_{n,n}(\zeta) \geq 1 \) for any real \( \zeta \) not algebraic of degree at most \( n \). More generally the analogous exponent assigned to any \( \zeta \in \mathbb{R}^n \) that is \( \mathbb{Q} \)-linearly independent together with \( \{1\} \) is bounded below by 1. This follows from the results in [34]. On the other hand, \( \hat{w}_{n,n}(\zeta) = \hat{w}_n(\zeta) = n/(w-n+1) < 1 \) for \( \zeta \in \mathcal{D}_{n,w} \) when \( w > n \), by Theorem 3.5. Combination shows the first claim. For the second assertion, we first claim \( w_{n,n}(\zeta) = w_n(\zeta) = w \). For any convergent \( p/q \) to \( \zeta \in \mathcal{D}_{n,w} \) consider \( P(T) = qT - p \) and the derived linearly independent polynomials \( \{D_1, D_2, \ldots, D_n\} = \{P, TP, \ldots, T^{n-1}P\} \). We clearly have \( H(D_j) = H(P) \) and \( |D_j(\zeta)| \gg \frac{|P(\zeta)|}{n} \). This shows \( w_{n,n}(\zeta) \geq w_1(\zeta) = w \), the reverse inequality \( w = w_n(\zeta) \geq w_{n,n}(\zeta) \) is obvious. On the other hand, from Theorem 3.7 we obtain \( w_{n,n}(\zeta) = w_{n,n}(\zeta) = nw/(w-n+1) < w \) for \( \zeta \in \mathcal{D}_{n,w} \) with \( w > 2n-1 \). This concludes the proof of the second claim. \( \square \)

We prove Theorem 4.3 in a similar way as Theorem 3.5.

Proof of Theorem 4.3. Notice that the assumptions are precisely as in Theorem 3.5. For the left inequality, as in the proof of Theorem 3.5 we obtain \( \lambda_{n-1}(\zeta) = (w-(n-1)+1)/(n-1) \) for \( \zeta \in \mathcal{D}_{n,w} \) since \( w \geq n \geq n-1 \) (index shift \( n \) to \( n-1 \) compared to Theorem 3.5). Thus (34) indeed yields

\[
\hat{w}_{n,int}(\zeta) \geq \frac{1}{\lambda_{n-1}(\zeta)} = \frac{n-1}{w-n+2}.
\]

The most right inequality of (44) remains to be proved. For simplicity put \( v = n/(w-n+1) \). In the proof of Theorem 3.5 we noticed that \( \zeta \in \mathcal{D}_{n,w} \) satisfies \( \hat{w}_{n,n+1}(\zeta) = v \). More precisely, the proof showed that for any \( \epsilon > 0 \) there are arbitrarily large parameters \( X \) such that every solution \( P \in \mathbb{Z}_{\leq n}[T] \) of

\[
H(P) \leq X, \quad |P(\zeta)| \leq X^{-v-\epsilon}
\]

is a polynomial multiple of a linear polynomial \( Q(T) = aT + b \). Here \( -b/a \) is a very good rational approximation (in particular a convergent) to \( \zeta \). By elementary facts on continued fractions we clearly have \( a > 1 \) and \( (a,b) = 1 \). It follows from the Lemma of Gauss that every polynomial multiple \( U(T) = R(T)Q(T) \) of \( Q \) with arbitrary \( R \in \mathbb{Q}[T] \) which has integral coefficients \( U \in \mathbb{Z}[T] \), must actually arise from \( R \in \mathbb{Z}[T] \). Thus \( U(T) \) has leading coefficient divisible by \( a \) and hence is not monic. In other words, for parameters \( X \) as above every monic polynomial \( P \in \mathbb{Z}_{\leq n}[T] \) with \( H(P) \leq X \) must satisfy

\[
|P(\zeta)| \geq X^{-v-\epsilon}.
\]

The right inequality in (44) follows as we may let \( \epsilon \) tend to 0. The equivalence claim (45) for Liouville numbers follows immediately from the upper and lower bound in (44). \( \square \)

The proof more precisely shows the finiteness of solutions \( P \in \mathbb{Z}_{\leq n}[T] \) to (62) with bounded leading coefficient when \( \zeta \in \mathcal{D}_{n,w} \).

6. Some open problems

In this section we formulate selected open problems, mainly concerning our new exponents for approximation of exact degree. Some of them have already been addressed, explicitly or implicitly, in the course of the paper. First we discuss several variants of Wirsing’s problem which we introduced right at the beginning in Section 1.1.
Problem 1. Is it true that for any transcendental real \( \zeta \) and every \( n \geq 3 \) we have \( w^*_n(\zeta) \geq n \)? Does even the estimation
\begin{equation}
|\zeta - \alpha| \ll_{n, \zeta} H(\alpha)^{-n^{-1}}
\end{equation}
have infinitely many solutions \( \alpha \in A_n \)? Similarly, is it true that for every \( n \geq 3 \) we have \( w^*_{n+1}(\zeta) \geq n \), or more generally \( w^*_{m}(\zeta) \geq n \) for every \( m \geq n + 1 \)? What about refinements in the spirit of (63)?

Recall we have shown (63) for \( n = 2 \) in Theorem 1.1 whereas \( w^*_{3}(\zeta) < 2 \) for certain extremal numbers was pointed out in [18]. Next we discuss variants of the related natural question discussed in Section 3.1.

Problem 2. Do we have \( w_n(\zeta) \geq n \) for all \( n \geq 4 \) and any transcendental real number \( \zeta \)? Is it even true that the inequality
\begin{equation}
|P(\zeta)| \ll_{n, \zeta} H(P)^{-n}
\end{equation}
has infinitely many solutions \( P \in \mathbb{Z}[T] \)? What about \( w_{n+1}(\zeta) \geq n \) for \( n \geq 3 \)?

As pointed out we strongly believe the answer to be positive at least for \( w_n(\zeta) \). We cannot prove the stronger condition (64) even for \( n = 3 \), for if \( \tilde{w}_n(\zeta) = 3 \) the method of Theorem 3.2 would require a bound of order \( O(1) \) for the smallest suitable prime \( p \) in the auxiliary Theorem 3.3. On the other hand, observe that \( \tilde{w}_n(\zeta) < n \) holds for certain \( \zeta \), as follows from Theorem 3.5. In Theorem 3.8 we saw that the exponents of bounded degree can differ vastly from the exponents of exact degree. However, we may ask for a generalization of Theorem 2.1.

Problem 3. Assume \( n \geq 3 \) is an integer and \( \zeta \) is a transcendental real number with \( \tilde{w}_n(\zeta) > n \). Is it true that
\begin{align*}
w_n(\zeta) &= w_n(\zeta), \\
\tilde{w}_n(\zeta) &= \tilde{w}_n(\zeta), \\
w^*_n(\zeta) &= w^*_n(\zeta), \\
\tilde{w}^*_n(\zeta) &= \tilde{w}^*_n(\zeta)
\end{align*}
necessarily holds? Further, is it true that \( \zeta \) cannot be a \( U \)-number?

The claim could potentially be true in a trivial sense in case no number satisfies the condition. For \( n \geq 3 \) we cannot rule out that \( \zeta \) is a \( U_m \)-number of index \( 2 \leq m \leq n - 1 \), which is an empty range for \( n = 2 \) as in Theorem 2.1. The next question concerns the relation between approximation by algebraic numbers versus algebraic integers.

Problem 4. Let \( n \geq 1 \) be an integer. Does there exist transcendental real \( \zeta \) such that \( w_n(\zeta) < w^*_{n+1}(\zeta) \) or \( w^*_n(\zeta) < w^*_{n+1}(\zeta) \)? Similarly for \( \tilde{w}_n(\zeta) < \tilde{w}^*_{n+1}(\zeta) \) or \( \tilde{w}^*_n(\zeta) < \tilde{w}^*_{n+1}(\zeta) \). More general, determine the spectra of \( w_n(\zeta) - w^*_{n+1}(\zeta) \), \( w^*_n(\zeta) - w^*_{n+1}(\zeta) \), \( \tilde{w}_n(\zeta) - \tilde{w}^*_{n+1}(\zeta) \) and \( \tilde{w}^*_n(\zeta) - \tilde{w}^*_{n+1}(\zeta) \).

The estimate [18] for some extremal numbers showed that \( w^*_{n+1}(\zeta) < n \) is possible, at least for \( n = 2 \). It seems that conversely numbers which are very well approximable by algebraic integers have not been constructed yet for any degree.

Problem 5. For \( n \geq 1 \), construct real transcendental \( \zeta \) for which \( w^*_{n+1}(\zeta) > n \), or even \( w^*_{n+1}(\zeta) > n \).

The next problem is much more general and a complete answer seems out of reach.
**Problem 6.** Determine the spectra of the new exponents $w_n, w_n^*, \hat{w}_n, \ldots$

We have noticed that combination of Corollary 3.6 and Corollary 3.10 yields that a transcendental real number is a Liouville number if and only if $\hat{w}_n(\zeta) = 0$, or equivalently $\hat{w}_n^*(\zeta) = 0$, for all $n \geq 1$. Recall also the characterization (45) for Liouville numbers. A natural related question for the exponents $\hat{w}_n^*(\zeta)$ remains partly open.

**Problem 7.** Is a transcendental real number $\zeta$ a Liouville number if and only if

$$\hat{w}_n^*(\zeta) = 1, \quad n \geq 1,$$

holds?

As stated in Section 1.2, we have $\hat{w}_n^*(\zeta) \geq 1$ and any Liouville number has the property (65). On the other hand, the estimate (61) implies that $w_2(\zeta) = \infty$ is necessary for (65). We conclude that $\zeta$ must be either a Liouville number or a $U_2$-number. Our next problem is motivated by Theorem 4.3.

**Problem 8.** Determine $\hat{w}_n^{int}(\zeta)$ and $\hat{w}_n^{int}(\zeta)$ for $\zeta \in \mathcal{D}_{n,w}$, or at least for the special examples $\zeta \in \mathcal{B}_w$.

We have noticed below Theorem 4.3 that it is plausible to believe in equality with the left bound in (44). We conclude with another problem which stems from Theorem 4.3 concerning inhomogeneous approximation. We only state the case $n = 1$ explicitly.

**Problem 9.** Let $\zeta$ be a Liouville number. For $\alpha \in \mathbb{R}$ denote by $\hat{w}_1(\zeta, \alpha)$ the supremum of exponents $w$ for which

$$1 \leq x_1 \leq X, \quad |\alpha + x_0 + \zeta x_1| \leq X^{-w}$$

has a solution in integers $x_0, x_1$ for all large $X$. Does the spectrum of $\hat{w}_1(\zeta, \alpha)$ contain (or even equal) the interval $[0, 1]$?

As remarked above it follows from [12] that $\hat{w}_1(\zeta, \alpha) = 0$ for almost all $\alpha$, and by our results, including any $\alpha$ of the form $Q(\zeta)$ with $Q \in \mathbb{Q}_{\geq 2}[T]$. Moreover $\{1\}$ is contained in spectrum, and we may take $\alpha = Q(\zeta)$ with any $Q \in \mathbb{Q}_{\leq 1}[T]$. In contrast to $\hat{w}_1(\zeta) = \hat{w}_1(\zeta, 0) = 1$ for all $\zeta$, it seems that $\hat{w}_1(\zeta, \alpha) > 1$ cannot be excluded for arbitrary $\alpha$ with the current knowledge.

See also [8] Section 10.2 and [10] for several problems concerning the classic exponents $w_n, \hat{w}_n, w_n^*, \hat{w}_n^*, \lambda_n, \hat{\lambda}_n$ (some questions of the first reference have already been solved).

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