Memory effects in quantum dynamics modelled by quantum renewal processes

Nina Megier, Manuel Ponzi, Andrea Smirne and Bassano Vacchini

1 Dipartimento di Fisica “Aldo Pontremoli”, Università degli Studi di Milano, via Celoria 16, 20133 Milan, Italy
2 Istituto Nazionale di Fisica Nucleare, Sezione di Milano, via Celoria 16, 20133 Milan, Italy

Simple, controllable models play an important role to learn how to manipulate and control quantum resources. We focus here on quantum non-Markovianity and model the evolution of open quantum systems by quantum renewal processes. This class of quantum dynamics provides us with a phenomenological approach to characterise dynamics with a variety of non-Markovian behaviours, here described in terms of the trace distance between two reduced states. By adopting a trajectory picture for the open quantum system evolution, we analyse how non-Markovianity is influenced by the constituents defining the quantum renewal process, namely the time-continuous part of the dynamics, the type of jumps and the waiting time distributions. We focus not only on the mere value of the non-Markovianity measure, but also on how different features of the trace distance evolution are altered, including times and number of revivals.

I. INTRODUCTION

Quantum phenomena are deemed to be the main ingredients of the next technological breakthroughs. Quantum correlations, quantum coherences and quantum non-Markovianity are the key resources to outperform classical protocols in many tasks, within the contexts of, e.g., communication [1–2], teleportation [3], cryptography [4], metrology [5] and thermodynamics [6], thus providing the pillars for future progresses in quantum technology [7]. Even though the developments of quantum theory already started at the beginning of the last century, a deep and thorough understanding of the above-mentioned features in view of their operational exploitation is still being developed [8–10]. This is why simple, controllable models play an important role to learn how to manipulate and control the quantum resources.

In this article, we will focus on the analysis of a Markov property in the quantum setting and on the description of a class of open quantum system dynamics featuring memory effects and allowing for a phenomenological treatment. The Markov property is a concept from the theory of classical stochastic processes, where a clear definition of Markov process can be introduced in terms of conditional probability distributions. This notion is connected with the memorylessness of the process, i.e. the fact that the future of the process is independent of its history. As stochastic processes are used to model reality in many different fields of research, as finance, biology, chemistry and social science, this is a highly relevant and often recurring concept [11]. Stochastic processes naturally appear in the description of (open) classical systems where, at least in principle, the stochasticity can be always traced back to the lack of knowledge on the underlying total Hamiltonian and the initial conditions [12–14]. The extension of the classical formalism to the theory of open quantum systems is not straightforward, due to the invasive nature of the quantum measurements. As a consequence, many different, nonequivalent definitions of quantum Markov process were introduced, all of them aimed to reveal the occurrence of memory effects in quantum evolutions. In this respect, the notion of memory in the quantum realm still calls for a full physical interpretation. Some hints in this direction come from the framework of quantum thermodynamics, where, e.g., the connection between non-Markovianity and irreversible entropy production has been explored [13–14].

We will point out how the class of quantum renewal processes can be used as a phenomenological tool to describe dynamics with different non-Markovian behaviours. Our study complements other approaches, whose starting point is rather a microscopic description specifying a reference total Hamiltonian. In particular, strategies aimed at controlling the non-Markovianity of the dynamics have explored the manipulation of the system-environmental coupling [15–17], or the modification of the reduced system itself [18–19]. The possibility of delaying the occurrence of non-Markovianity [20], and enhancing it by means of feedback control [21] has been also investigated.

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The existence of an underlying microscopical description of the evolution ensures that the reduced dynamics is indeed physical, i.e. the corresponding dynamical map \( \Lambda_t \) which maps the initial reduced density operator \( \rho(0) \) to a density operator at later time \( t: \rho(t) = \Lambda_t[\rho(0)] \), is completely positive and trace preserving (CPTP) \([1](\)\). The density operator yields the probability distributions in quantum physics, so that trace preservation of the dynamics keeps the

\[\rho_{\text{tot}}(0) = \rho(0) \otimes \rho_E(0)\]

\[\text{nina.megier@mi.infn.it}

\[\text{This follows from the assumption that initially the reduced system and its environment are not correlated, i.e. the initial total state is factorised: }\]

\[\rho_{\text{tot}}(0) = \rho(0) \otimes \rho_E(0).\]
Where and only if the trace distance is not a monotonous function of time, i.e., there exist a couple of initial states \( \rho \), \( \sigma \) such that

\[
D(\rho, \sigma) = \frac{1}{2} \text{Tr} |\rho - \sigma| = \frac{1}{2} \sum_i |v_i|,
\]

where the times are ordered: \( t_{n+1} \geq t_n \geq \ldots \geq t_1 \geq 0 \), i.e., once we know the value \( x_n \) of the stochastic process at time \( t_n \), the past history prior to \( t_n \) does not affect the predictions about the value of the process at any future time \( t_{n+1} \). Due to the invasive nature of quantum measurements, the extension of this definition to the quantum regime is not straightforward and many different, non-equivalent definitions of quantum Markovianity have been introduced. In most of them Markovianity is a property of the dynamical map \( \Lambda_t \) itself, such as \( \text{(C)P-} \)
divisibility, the change of the volume of accessible states and monotonicity of the trace distance as a quantifier of state distinguishability; the latter is the one we will adopt here. On the other hand, other approaches, such as the process matrix formalism, ground the notion of quantum Markovianity on conditional probabilities associated with sequences of measurements, going beyond the single-time description of the open system dynamics and calling for multi-time correlations.

The definition of non-Markovianity we use here is based on the change of distinguishability between system states, quantified in the original paper by means of the trace distance between two reduced states in the course of the evolution. The trace distance between two quantum states \( \rho, \sigma \) is defined as

\[
D(\rho, \sigma) = \frac{1}{2} \text{Tr} |\rho - \sigma| = \frac{1}{2} \sum_i |v_i|,
\]

where \( v_i \) are the eigenvalues of the operator \( \rho - \sigma \). The quantum dynamics fixed by the map \( \Lambda_t \) is non-Markovian if and only if the trace distance is not a monotonous function of time, i.e., there exist a couple of initial states \( \rho(0) \) and \( \sigma(0) \) and a time \( t > 0 \) for which

\[
\frac{d}{dt} D(\rho(t), \sigma(t)) > 0,
\]

where \( \rho(t) = \Lambda_t \rho(0) \) and \( \sigma(t) = \Lambda_t \sigma(0) \).

The rest of the article is organized as follows. In Sect. II we introduce the concept of non-Markovianity for stochastic processes. After this, we describe a possible definition of quantum non-Markovianity based on the monotonicity of the trace distance between two reduced states, which we adopt in the whole article. In Sect. III we continue with the presentation of the renewal processes in the classical and the quantum domain, while Sect. IV is devoted to the phenomenological description of reduced dynamics. In addition, despite their simplicity, quantum renewal processes can show a wide range of non-Markovian behaviours, which we will analyse in details in the following.

II. MEMORY EFFECTS IN QUANTUM DYNAMICS

We say that a stochastic process \( X(t), t \geq 0 \), taking values in a discrete set \( \{x_i\}_{i \in \mathbb{N}} \) is Markov if the corresponding conditional probability distributions satisfy for any finite \( n \) the following inequalities

\[
p_{i|n}(x_{n+1}, t_{n+1}| x_n, t_n; \ldots; x_1, t_1) = p_{i|1}(x_{n+1}, t_{n+1}| x_n, t_n),
\]

The quantum dynamics fixed by the map \( \Lambda_t \) is non-Markovian if and only if the trace distance is not a monotonous function of time, i.e., there exist a couple of initial states \( \rho(0) \) and \( \sigma(0) \) and a time \( t > 0 \) for which

\[
\frac{d}{dt} D(\rho(t), \sigma(t)) > 0,
\]

where \( \rho(t) = \Lambda_t \rho(0) \) and \( \sigma(t) = \Lambda_t \sigma(0) \).

2 The dynamical map \( \Lambda_t \) is CP-divisible if the map \( \Lambda_{t,s} \), defined as \( \Lambda_t = \Lambda_{t,s} \Lambda_s \) is CPTP for all \( 0 \leq s \leq t \).
FIG. 1: Sketch of the information backflow in an open quantum system dynamics, which is at the basis of the notion of quantum non-Markovianity used in this paper: initially the reduced states $\rho$, $\sigma$ approach each other since the information is flowing out of the reduced system, to the environment or to the correlations between the system and the environment (left); on the other hand, an information backflow makes the two states diverge from each other at a later time (right), as can be witnessed via proper state distinguishability quantifiers.

Importantly, since the trace distance is contractive under the action of any (C)PTP map $\phi$:

$$D(\phi(\rho), \phi(\sigma)) < D(\rho, \sigma),$$

(C)P-divisibility [25, 45, 46, 53] implies monotonicity of the trace distance and thus Markovianity according to the definition above, while the converse does not hold [49]. The trace-distance based definition of non-Markovianity provides a clear-cut interpretation in terms of the information flow between the open quantum system and the environment as the key element possibly leading to the occurrence of memory effects in the dynamics. In addition, this picture allows us to trace back the exchange of information between the open system and the environment to the correlations established by their mutual interaction [54-59], see Fig. 1. Initially the whole information is contained in the reduced system, however, due to the system-environment interaction, some information gets transferred to external degrees of freedom in the course of the evolution. Such information can be stored both in the environment and in the system-environmental correlations. In Markovian dynamics the information flow is unidirectional, i.e. the information is always flowing from the open system to the outside world and any couple of reduced states get closer and closer with the passing of time. On the other hand, for non-Markovian evolutions some information backflow occurs, which is witnessed by an increase of the distance between pairs of reduced states on certain intervals of time. Let us stress that this viewpoint was recently strengthened, as it was shown that also different distinguishability measures between two quantum states, including entropic quantities, can be used to quantify the information flow; it appears in particular that the quantum Jensen-Shannon divergence is a natural entropic quantifier of information backflow [60]. Additionally, a connection between monotonic contractivity of a generalisation of the trace distance and P-divisibility exists [53, 61], providing a common background to these approaches to non-Markovianity, which however goes beyond the scope of this work.

Relying on the trace distance, it is then possible to define a measure of the degree of non-Markovianity of a quantum dynamics. The idea is to integrate over all the revivals of the trace distance over the duration of the dynamics, i.e., to quantify the overall amount of information flown back to the reduced system. In addition, since we want the non-Markovianity measure to be a property of the dynamical map, while the change of the trace distance, Eq. (3), will generally depend on the chosen initial states $\rho(0)$ and $\sigma(0)$, the non-Markovianity measure involves an optimisation over all the possible couples of initial states [48]:

$$N = \max_{\rho(0), \sigma(0)} \int_{dD(\rho(s), \sigma(s))/ds > 0} \frac{d}{ds} D(\rho(s), \sigma(s)) ds.$$  

It was shown in [62] that the optimal pair of states, i.e. the one achieving the maximum in the non-Markovianity measure, lies on the boundary of the states space and is made of orthogonal states. In particular, for qubit states this means that the optimal pair consists of pure states that can be represented as a pair of antipodal points on the Bloch sphere.

III. RENEWAL PROCESSES: CLASSICAL AND QUANTUM

Here we investigate a class of open quantum system dynamics, quantum renewal processes, which are a generalisation of a classical concept. Firstly, we briefly review semi-Markov processes, of which renewal processes are a subset, and then provide a formulation of the relevant notions in the quantum realm [63, 64].
As discussed in Sect. [4], the characterization of a Markovian time evolution is essentially fixed by the GKSL theorem, determining the expression of the generator of the dynamics. An equivalent result for an arbitrary dynamics featuring non-Markovian effects is not known, and only very specific results have been obtained. The main difficulty lies in providing evolution equations whose solutions are indeed CPTP maps. These so-called master equations can be recast in two forms, either time local, i.e. with the functional expression [22]

$$\frac{d}{dt} \rho(t) = \mathcal{L}(t)[\rho(t)],$$  \hspace{2cm} (6)

or time non-local, that is in the form [65] [66]

$$\frac{d}{dt} \rho(t) = \int_0^t \mathcal{K}(t-s)[\rho(s)].$$  \hspace{2cm} (7)

The superoperators $\mathcal{L}(t)$ and $\mathcal{K}(t)$ are generally related [67] [20], though in a highly non-trivial way. Moreover general conditions on their expression warranting CPTP are not known, except for special cases. In this contribution we make reference to a large class of well-defined evolutions obtained building on an analogy with classical non-Markovian stochastic processes.

A semi-Markov process is a continuous time random jump process, for which the jump probabilities are possibly site dependent but independent from each other. The probability distribution of the time between the jumps is called waiting time distribution (WTD) and provides a probability density over the positive real line

$$f(s) \geq 0, \quad \int_0^\infty ds f(s) = 1. \hspace{2cm} (8)$$

If the WTD is exponentially distributed, then the semi-Markov process reduces to a continuous time Markov chain. Otherwise, for general distributions, the memory about the time already spent in the state affects the subsequent statistics of the process, which is then non-Markovian. The transition probabilities $T_{mn}(t)$ from the state $m$ to the state $n$ obey the equation [71]

$$\frac{d}{dt} T_{mn}(t) = \int_0^t d\tau \sum_k [W_{mk}(\tau)T_{kn}(t-\tau) - W_{km}(\tau)T_{mn}(t-\tau)], \hspace{2cm} (9)$$

where the matrices $W_{mk}(t)$ are fixed by the (possibly state-dependent) WTDs $f_i(t)$, along with the corresponding survival probabilities $g_i(t)$ defined by

$$g_i(t) = 1 - \int_0^t ds f_i(s), \hspace{2cm} (10)$$

and the semi-Markov matrix $\Pi$ whose entries are the jump probabilities between sites; in particular, denoting as $\hat{x}(u)$ the Laplace transform of $x(t)$, one has $\hat{W}_{mk}(u) = \Pi_{mk}\hat{f}_k(u)\hat{g}_k^{-1}(u)$. Moreover, note that a semi-Markov process can also be seen as the merging of a renewal process and a Markovian jump process. In a renewal process, the events, here the transitions among states, occur randomly in time and the time intervals between successive events are independent. Accordingly, the evolution depends only on the current site and the time elapsed since arriving at it. In the case of the standard renewal process all waiting times are identical, while for a so-called modified process the first $k$ waiting time distributions are different.

The notion of trajectory is one of the basic concepts in the description of classical stochastic processes. Indeed, in abstract terms a stochastic process can always be characterized by a suitable measure over a sample space of trajectories. Recovering a notion of trajectory is less straightforward in the quantum case, where the object of interest is the reduced density matrix $\rho(t)$, but this can actually be done in the context of open quantum systems [23] [72]. More specifically, the dynamics we are considering allow for an interpretation in terms of an average over trajectories in the space of operators. All the trajectories start in the same initial state $\rho(0)$, and then in each trajectory the times at which the system state undergoes discontinuous changes, the so-called jumps, are random variables. Accordingly, the reduced density operator of the open quantum system can be obtained by a weighted sum of all possible trajectories, corresponding to fixed jump times. We will see that this point of view also helps us understand the dependence of the non-Markovianity on the specific parameters.
Quantum renewal processes are a subclass of quantum semi-Markov processes \cite{31, 11, 69, 73, 74}, for which the time evolution reads

\[ \rho(t) = p_0(t)\mathcal{F}_0(t)\rho(0) + \sum_{n=0}^{\infty} \int_0^t dt_2 \cdots \int_0^{t_2} dt_1 p_n(t; t_n, \ldots, t_1)\mathcal{F}_n(t-t_n)\mathcal{E}_n \cdots \mathcal{F}_2(t_2-t_1)\mathcal{E}_1(t_1)\rho(0), \]  

(11)

where the CPTP maps \( \mathcal{E}_n \) describe the jumps, while the CPTP maps \( \mathcal{F}_n(t) \) give the time-continuous evolutions between the jumps, and \( p_n(t; t_n, \ldots, t_1)dt_n \cdots dt_1 \) is the probability that the jumps occur (solely) around fixed times \( t_1, \ldots, t_n \). Note the close analogy to the classical description recalled above. In the case of the standard process the jump times are independent and identically distributed, i.e. each waiting time has the same probability distribution and they are all mutually independent. In a modified process, instead, the probability distributions for the first jumps can differ from each other and the following ones.

To obtain the quantum renewal processes from the general quantum semi-Markov processes one fixes the time evolution between the jumps to be of GKS-L form \cite{75}. What is more, one also introduces only two kinds of jumps: anterior \( \mathcal{J} \) and subsequent \( \mathcal{E} \) with respect to the time continuous evolution. Consequently, in quantum renewal processes one focuses on the stochastic distribution of the jumps, as in the case of classical renewal processes. Accordingly, we obtain the following form of the open quantum system density operator at time \( t \):

\[ \rho(t) = p_0(t)e^{\mathcal{E}t}\rho(0) + \sum_{n=0}^{\infty} \int_0^t dt_2 \cdots \int_0^{t_2} dt_1 p_n(t; t_n, \ldots, t_1)e^{\mathcal{M}(t-t_n)}\mathcal{E}e^{\mathcal{E}(t_n-t_{n-1})\mathcal{J}} \cdots \mathcal{E}e^{\mathcal{E}t_1}\mathcal{J}\rho(0). \]  

(12)

Here, we use a “left-ordering”, as explained in \cite{31}, since a particular ordering of the operators has to be chosen in order to construct the quantum evolution from the classical counterpart. We also set in the following \( \mathcal{M} = \mathcal{L} \) (the time continuous evolution is always the same) and \( \mathcal{J} = \mathcal{I} \). With this, the above mentioned trajectories correspond to the dynamical maps \( e^{\mathcal{E}(t-t_n)}\mathcal{E}e^{\mathcal{E}(t_n-t_{n-1})} \cdots \mathcal{E}e^{\mathcal{E}t_1} \), which contribute to the overall evolution with weights \( p_n(t; t_n, \ldots, t_1)dt_n \cdots dt_1 \).

For the standard quantum renewal process the same WTD \( f(t) \) governs the whole stochasticity of the jumps’ times,

\[ p_n(t; t_n, \ldots, t_1) = g(t-t_n) \cdots f(t_2-t_1)f(t_1), \]

where \( g(t) \) is the corresponding survival probability. When the renewal process is modified the first \( k \) WTDs can be different,

\[ p_n(t; t_n, \ldots, t_1) = \begin{cases} g_{n+1}(t-t_n)f_n(t_n-t_{n-1}) \cdots f_1(t_1), & n \leq k \\ g(t-t_n) \cdots f(t_{k+1}-t_k)f(t_k-t_{k-1}) \cdots f_1(t_1), & n > k. \end{cases} \]

(13)

(14)

Here we will investigate how the non-Markovianity of the dynamics, in terms of the monotonicity of the trace distance, depends on the choice of the involved operators, describing intermediate continuous evolutions and jumps, as well as the chosen probability distribution for the jumps. We will observe a rich variety of possible behaviours and analyse the influence of particular parameters to control the strength, time of occurrence and precise manifestation of quantum non-Markovianity.

### IV. TRAJECTORY PICTURE

In general there exist infinitely many different decompositions of a reduced dynamics, i.e. different ways to write the reduced density operator in the form

\[ \rho(t) = \sum_{i \in I} p_i(t)\rho_i(t), \]

(15)

where \( I \) can be countable or uncountable set. In this representation the prefactors \( p_i(t) \) can be interpreted as probabilities or probabilities densities, i.e. they are positive and normalized, and the operators \( \rho_i(t) \) are themselves proper density operators, i.e. trace one and positive semi-definite. If the operators can be obtained by the action of CPTP maps \( \Delta_i \) applied on the very same initial state \( \rho(0) \), each \( \rho_i(t) \) can be associated to a different trajectory, whose occurrence probability is indeed given by the corresponding \( p_i(t) \). There exist two main types of decompositions directly linked to a trajectory picture of the dynamics: time-continuous, as exemplary quantum state diffusion \cite{76, 73}, and so called jump unravelings \cite{79, 80}. As recalled above, also quantum renewal processes have a direct decomposition in terms of trajectories, which are defined at the level of the density operators, see in particular Eq. (12). Finally,
note that an important question connected with the trajectory description of the reduced dynamics is the existence of a continuous measurement interpretation associated with it [22, 81, 85].

The construction of a particular trajectory can take place in two different ways. In the first method one firstly fixes the time interval \([0, T]\) of interest and then draws the jumps’ times according to the WTDs. After each drawing if the sum of waiting times exceeds \(T\) one terminates the process. Then the generation of the trajectory is obtained by inserting the jumps at the given times. In the second method the generation of the trajectory and drawing of the jump times take place in parallel. The time interval \([0, T]\) is divided into small intervals of length \(\Delta t\), and at each intermediate midpoint one determines randomly if the jump takes place or not, with the probability fixed by the corresponding waiting time distribution. In this second approach, fixing the time interval \([0, T]\) in advance is in principle not necessary as one can decide along the trajectory when to stop the evolution. Note, that for a modified renewal process only the first method is applicable for the case in which the last \(k\) waiting time distributions are different, a situation which was introduced in [33] under the name of inverse time operator ordering. The same is true when the last time-continuous evolution is different from the preceding ones, \(\mathcal{M} \neq \mathcal{L}\) in Eq. (12), or in processes starting with a jump rather than with a time continuous evolution, \(J \neq 1\) in Eq. (12). In all these situations one has to know beforehand, i.e. before one starts to generate the trajectory, how many jumps occur in the investigated time interval \([0, T]\), to know which waiting time distribution or which time evolution has to be used to generate the trajectory at a particular point of time. In this paper, for simplicity, we restrict ourselves to cases where both methods to generate the trajectory can be implemented. We will see that the trajectory point of view in describing the evolution let us better understand the influence of the particular parameters on the non-Markovianity of the corresponding dynamical map.

The quantum renewal processes, due to the non-trivial interplay between the operatorial and stochastic contributions, can manifest a wide range of non-Markovian behaviours. However, if one assumes that all WTDs coincide, i.e. the quantum renewal process is unmodified, and are given by an exponential distribution

\[
f(t) = \mu e^{-\mu t},
\]

where \(\mu\) is the corresponding rate, the issue simplifies significantly. In this case, a simple connection between the WTD \(f(t)\) and the associated survival probability \(g(t)\) exists: \(f(t) = \mu g(t)\). As shown in [75], the corresponding memory kernel in the Laplace picture reads

\[
\tilde{K}(u) = \mathcal{L} + (\mathcal{E} - 1)f(u - \mathcal{L})\tilde{g}^{-1}(u - \mathcal{L}).
\]

Accordingly, in time domain we obtain for this case

\[
K(t) = \delta(t)[\mathcal{L} + \mu(\mathcal{E} - 1)],
\]

no matter what the generator \(\mathcal{L}\) and the jump operator \(\mathcal{E}\) are. This memory kernel corresponds to a quantum dynamical semigroup, and, accordingly, the underlying evolution is Markovian.

To go beyond this case, we analyse how the time continuous dynamics, type of jumps and waiting time distributions influence qualitatively and quantitatively the non-Markovianity of the corresponding process. We focus on qubit evolutions, so that the trace distance between two quantum states equals the half of the Euclidian distance of these states when depicted on the Bloch ball. Recall that any qubit state can be written as

\[
\rho = \frac{1}{2}(1 + \tilde{r} \cdot \tilde{\sigma}),
\]

with the vector \(\tilde{\sigma}\) consisting of the Pauli matrices, \(\tilde{\sigma}^T = (\sigma_1, \sigma_2, \sigma_3)\), and \(\tilde{r}^T = (x, y, z)\) defining the Bloch vector representation of the state \(\rho\). Accordingly, the trace distance between two qubit states evolving via a quantum renewal process can be calculated as

\[
\mathcal{D}(\rho(t), \rho^2(t)) = \frac{1}{2} \lim_{N \to \infty} \sqrt{\left(\frac{1}{N} \sum_{n=1}^{N} \Delta_\xi(t)\right)^2 + \left(\frac{1}{N} \sum_{n=1}^{N} \Delta_\eta(t)\right)^2 + \left(\frac{1}{N} \sum_{n=1}^{N} \Delta_\zeta(t)\right)^2},
\]

where the sums are running over realisations of the stochastic process governed by the associated WTDs, and \(\Delta_i(t)\) corresponds to difference of the \(i\)-coordinates in \(n\)-th realisation, e.g.

\[
\Delta_\xi(t) = x^n_\xi(t) - x^n_\xi(t),
\]

which we call an \(x\)-trajectory. Consequently, the trace distance between two states is not an average trace distance between the corresponding random trajectories and calculating the trace distance has to occur after generating the
whole set of trajectories. Note that to have non-monotonicity in the trace distance, a non-monotonicity of the absolute value of at least one of the coordinates is necessary. This is the case not only when one of the coordinates is non-monotonous, but also when it changes its sign. This can only happen when some realisations of the trajectories include a sign change. This is however not a sufficient condition, as we will see in the following.

We now set the different elements of the quantum renewal processes fixing the resulting trajectories and average dynamics.

A. Intermediate evolutions

We choose the time continuous evolution to be unital

\[ \mathcal{L}[\rho] = \sum_{k=1}^{3} \frac{1}{2} \gamma_k (\sigma_k \rho \sigma_k - \rho), \]  

(22)

with \( \gamma_j \geq 0 \) and

\[ e^{\mathcal{L}t}[\sigma_i] = e^{-t\lambda_i} \sigma_i, \quad \lambda_i = \gamma_j + \gamma_k, \quad \text{for } i \neq j \neq k. \]  

(23)

Choosing a unital dynamical map does not affect the trace distance measure of non-Markovianity, which is anyhow insensitive to translations of the Bloch sphere \([42, 60, 86]\). As the time-continuous evolution introduced above describes a monotonic contraction of the Bloch sphere, we do not expect that it introduces any memory effects. Indeed, the dynamical map \( e^{\mathcal{L}t} \) is not only Markovian according to the distinguishability criterion introduced in \([52]\), but it is a CP-divisible semigroup. We will see that a greater "strength" of this dephasing evolution - corresponding to larger values of the \( \lambda \)'s introduced in Eq. (23) - will result in smaller non-Markovianity of the associated quantum renewal process.

B. Jumps

As said above, the quantum non-Markovianity will not occur if for all realisations of the stochastic process the coordinates, Eq. (21), are monotonic and do not change sign. Accordingly, a jump channel which only consists of a contraction (and possibly translation, which, however, cannot be detected by the trace distance condition - see comment above) will necessarily lead to a Markovian dynamics. An example of such a channel is the amplitude damping (AD) channel \( \mathcal{E}_{AD} \), with Kraus operators

\[ K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \]  

(24)

which shrinks the Bloch ball and translates it along the \( z \)-axes by factors determined by the decay rate \( \gamma \). Consequently, no non-Markovianity is detected, no matter what probability distribution drives the stochasticity of the jumps' times. In particular, also for a choice of classically non-Markovian waiting time distributions, as Erlang distributions introduced later, one still obtains Markovian evolution, according to the trace distance criterion.

Consequently, the next step is to choose a jump channel that results in changing the sign of the trajectories. We have chosen the \( x \)-Pauli channel composed with AD:

\[ \mathcal{E}_{x-AD} = \mathcal{E}_x \circ \mathcal{E}_{AD}, \]

with

\[ \mathcal{E}_x[\rho] = \sigma_x \rho \sigma_x. \]

The Pauli channels describe a \( \pi \) rotation about the corresponding axis and in particular we focus here on the composition of the AD channel with the \( x \)-PC. For this jump channel we will, indeed, manage to detect non-Markovianity, depending on the choice of parameters determining the dynamics.

Note, that, as the superoperators \( \mathcal{E}_x \) and \( \mathcal{E}_{AD} \) do not commute, the jump channels \( \mathcal{E}_{x-AD} \) and \( \mathcal{E}_{AD-x} = \mathcal{E}_{AD} \circ \mathcal{E}_x \) are different. Generally speaking, the latter possibility leads to a slightly greater non-Markovianity measure, as the jumps occur before the disruptive AD channel. Nonetheless, the qualitative behaviour for both of the choices is similar, and for simplicity we restrict here to \( \mathcal{E}_{x-AD} \).
C. Waiting time distributions

As noticed earlier, when the underlying WTDs are exponentials and the process is unmodified, i.e. all WTDs are the same, the evolution is Markovian, independently of the choice of the jump channel. This is the case even if the trajectories are non-monotonic and the sign changes take place, so, in particular, for the channel $\mathcal{E}_{x-AD}$ investigated by us. However, the situation drastically changes if we allow for a modified quantum renewal process. Even if all the WTDs are exponentials, but the first $k$-th of them have different rates, we can observe a high variety of different behaviours. In particular, the number of revivals strongly depends on the choice of the rates.

There is, however, no need to restrict our choice of WTDs to exponentials. To go beyond this case, we also analyse the quantum renewal process dynamics where the stochasticity of the jumps is governed by the Erlang WTD, which reads in the Laplace domain \[ f_r(u) = \left( \frac{\mu}{\mu + u} \right)^r, \] \(^{(25)}\)

from which one can see that it is the convolution of $r$ exponential distributions with the same rate parameter $\mu$. The ratio $r/\mu$ fixes the mean waiting time while the variance reads $r/\mu^2$. Accordingly, for the Erlang WTDs the mean value and the variance can be independently varied, as contrasted with the exponential WTD, where the mean waiting time $1/\mu$ fixes the variance.

We will see that in the case of Erlang WTDs even the unmodified process can lead to non-Markovianity.

\[ y(t) \]

\[ \Delta_\phi(t) \]

FIG. 2: Left: Value of the non-Markovianity measure of a quantum renewal process as given in Eq. (5) in the dependence on the choice of initial orthogonal pure states, identified by the extremes of a diameter in the Bloch sphere; it clearly appears that optimal pairs lie on a vertical equator. Right: $y$-component for the trajectory in a particular realization of the process corresponding to an initial optimal pair $|\phi_{1/2}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$. We are here considering a $\mathcal{E}_{x-AD}$ jump channel and parameters $\gamma = 0.3$, $\mu = 1$, $\mu_1 = 10$, $\lambda_1 = \lambda_2 = \lambda_3 = 0.9$. Here and in the following we work in arbitrary units.

V. NON-MARKOVIANITY OF QUANTUM RENEWAL PROCESSES

As mentioned in Sect. II, occurrence and strength of memory effects depend on the chosen pair of initial states. This is clarified in Fig. 2, left, where the value of the non-Markovianity measure for the case of the jump operator

\[^{3}\] In book [87] the Erlang distribution is called the special Erlangian distribution.
$E_{x-\text{AD}}$ is plotted as a function of the direction identifying a pair of pure orthogonal states, corresponding to points on the Bloch sphere. It clearly appears that the maximum is attained for states $|\phi_{1/2}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |i\rangle)$. We will therefore in the following consider always this pair of initial states lying on the $y$-axes. Note that for this choice $\Delta_x(t) = \Delta_z(t) = 0$, corresponding to the fact that the $x$ and $z$ components of the Bloch vector of the two evolving states remain equal, so that the only relevant parameter in the continuous time evolution is the rate $\lambda_2$. This behaviour is due to our particular choice of the jump channel, leading to a rotation about the $x$-axis. A typical trajectory of the $y$-component of the Bloch vector is depicted in Fig. 2, right, characterized by sign changes which determine possible non monotonicity of the trace distance obtained as in Eq. (20). In our analysis we will not only investigate the mere change of the non-Markovianity measure, but also the way the trace distance evolution is altered: number of revivals, times of revivals and other qualitative features.

### A. Exponential WTD - general results

Here, we focus on the behaviour of the trace distance in the case of exponential WTDs. Accordingly, beside the dependence on the dephasing rate of the continuous time evolution $\lambda_2$ and the decay rate $\gamma$ corresponding to the strength of AD jumps, the non-Markovianity is also influenced by the rates $\mu_i$ fixing the exponential WTDs.

![FIG. 3: The number of revivals of the trace distance for a quantum renewal process with $E_x$ jump channel in its dependence on the value of the rates fixing the WTDs. The maximal number of revivals for the modified quantum renewal process with $k$ exponential WTDs equals $k - 1$ (here we take $k = 3$ and $k = 4$ from left to right; $\lambda_2 = 0.9$, $\mu = 1$ and $\mu_3 = 3$ (right panel) in arbitrary units). The white lines mark the boundaries between parameter regions corresponding to processes whose trajectories exhibit different number of jumps.](image)

![FIG. 4: The trace distance, testifying non-Markovianity when showing a non-monotonic behavior, for a quantum renewal process with exponential WTDs. In the left panel jumps are realized by means of a $E_x-\text{AD}$ jump channel, and one can appreciate the reduction of the revivals for growing damping. In the middle panel jumps are given by $E_x-\text{AD}$ and stronger dephasing in the intermediate time evolution again suppresses non-Markovianity. The right panel, with jump operator $E_x$, shows how a larger number of revivals does not necessarily lead to a higher non-Markovianity measure. Across the panels $\lambda_2 = 0.9$, $\mu = 3$ and $\mu_1 = 13$, apart from the last panel with $\mu_1 = 30$.](image)
The number of revivals, i.e., time intervals where the trace distance grows, strongly depends on the number of different WTDs and on the corresponding rates. It can be observed that for a process with $k$-WTDs the maximal number of revivals is $k - 1$ and can only be reached if the following relation between the rates is satisfied:

$$\mu_1 > \mu_2 > \ldots > \mu_{k-1} > \mu.$$  \hspace{1cm} (26)

This fact is investigated in Fig. 3, where we report the number of revivals for a modified process with $E_x$ jump channel and with 3 WTDs (left) or 4 WTDs (right) in dependence on the rate values. Note that throughout the manuscript we work in arbitrary units. The different coloured areas correspond to different numbers of revivals, clearly growing with the number of WTDs and depending on the corresponding rates. The presence of amplitude damping in the jump decreases the parameter range corresponding to higher number of revivals. At the same time the AD reduces the value of the non-Markovianity measure. This is put into evidence in Fig. 4, left, where the behaviour of the trace distance is plotted together with the estimate for the associated non-Markovianity measure, corresponding to the sum of the revival heights. A similar effect is obtained by increasing the strength of the dephasing rate $\lambda_2$ describing the time continuous dynamics, as shown in Fig. 4, middle, where only $E_x$ determines the jumps.

We further stress that a higher number of revivals does not necessarily lead to a higher non-Markovianity measure, see Fig. 4, right. Non-Markovianity is enhanced when the rate of the first WTD is much larger than the rate of the following one, $\mu_1 \gg \mu_2$, allowing for a larger revival. Subsequent rates play a less relevant role, since, on average, the dephasing has become more effective by the time the corresponding jump occurs.

The different role of $\gamma$ and $\lambda_2$ is visible by comparing Fig. 4 left and Fig. 4 middle, noticing that only $\gamma$ affects the value of the (first) revival time. Their different influence at the level of the trajectories is visualised in Fig. 4. As one can observe, an increase of the decay rate implies that the height of the jumps decreases, while it does not affect the previous time continuous dynamics. This is different in the case of varying $\lambda_2$, where both the extension and the starting point of the jumps is changed and the influence on the revival time after averaging over all trajectories is wiped out.

One can also understand the necessity of the hierarchy given in (26), to have the maximal number of revivals, as well as their maximum number $k - 1$. When the condition is satisfied, then (approximately) the first, second, , , $k - 1$ jumps do not influence each other. With this we mean that the $n$-th jump occurs when $n - 1$ jumps have already taken place in most of the trajectories. Accordingly, the $k - 1$ first jumps are connected with the revival of the trace distance, while the following jumps do not result in the revivals. The reason is that for the exponential WTDs the mean value and the variance cannot be modified independently and are such that for an unmodified process the trace distance is monotonically decreasing, as was shown in Sect. IV. This will be different in the case of Erlang WTD, which we discuss in Sects. VC and WD. When the condition (26) is not satisfied, the number of the revivals for a modified process with $k$ different WTDs will be smaller than $k - 1$.

All revivals depicted till now started when the trace distance assumed value zero, i.e., when at the associated time the evolved states are the same. This can be seen as a special realization of non-Markovian behaviour, since in this case the dynamical map is neither invertible nor divisible. This is, however, not always the case. We observe that a revival occurs for larger values of the trace distance when the condition $\mu_2 \gg \mu_1 > \mu$ (3-WTDs process) is satisfied, see Fig. 4, left, where the $\mu_2$ is varied, and right, where $\mu$ is altered. The mean waiting time of the second jump is small enough with respect to the first jump to prevent the trace distance to reach zero, and the following third jumps occur too late to change this tendency. Note, that in this case the maximal number of revivals, $k - 1$, cannot be reached.
As already elaborated, in the considered case the time continuous dynamics between the jumps does not strongly affect the qualitative picture of non-Markovianity. It is therefore of interest to consider the effect of jumps and modified waiting time distributions alone, setting $E(t) = 0$, see Eq. (12). In this case the density operator follows the evolution:

$$\rho(t) = \sum_{n=0}^{\infty} p_n(t) E^n \rho(0),$$  \hspace{1cm} (27)

where $p_n(t)$ is the probability of having exactly $n$ jumps till time $t$, i.e. no statements about the times of the particular jumps are made as contrasted with $p_n(t; t_n, \ldots, t_1)$ in Eq. (13). As the influence of AD jump was also mainly in decreasing the non-Markovianity measure, with the same argument we take $E \rightarrow E_\epsilon$ so that we have idempotency of the jump transformation $E^2 = 1$. Accordingly, the sum in Eq. (27) can be split in two terms, one with even $n$ and one with odd $n$, see [41] for an analogous discussion with the $\sigma_z$-Pauli channel:

$$\rho(t) = (p_{even}(t) + p_{odd}(t) E) \rho(0).$$  \hspace{1cm} (28)

The difference between the matrices $\rho_1(t)$ and $\rho_2(t)$ then simply becomes

$$\rho_1(t) - \rho_2(t) = \left( p_{even}(t) \Delta_{11} - p_{odd}(t) \Delta_{10} \right),$$  \hspace{1cm} (29)

where $\Delta_{ij}$ gives the difference of the associated components of the operators $\rho_1(t)$ and $\rho_2(t)$ in the $\sigma_z$ basis. With the choice of the optimal states, $|\phi_{1/2}\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$, we obtain for the trace distance

$$D(\rho_1(t), \rho_2(t)) = |p_{even}(t) - p_{odd}(t)| = |\eta(t)|,$$  \hspace{1cm} (30)

which is the absolute value of the difference between the probability of the even number of jumps and odd number of jumps. As distinct from investigations in [41], here we take into account also case of modified processes, where first $k$ WTDs are different from the following one. The quantities $p_{even}(t)$ and $p_{odd}(t)$ take then in Laplace picture the form

$$\hat{p}_{even}(u) = \hat{g}_1(u) + \hat{f}_1(u) \hat{f}_2(u) \hat{g}_3(u) + \ldots + \hat{f}_1(u) \ldots \hat{f}_k(u) \left( \frac{1 + \hat{f}(u)}{2} - (-)^k \frac{1 - \hat{f}(u)}{2} \right) \frac{1}{1 - \hat{f}^2(u)} \hat{g}(u),$$  \hspace{1cm} (31)

$$\hat{p}_{odd}(u) = \hat{f}_1(u) \hat{g}_2(u) + \hat{f}_1(u) \hat{f}_2(u) \hat{f}_3(u) \hat{g}_4(u) + \ldots + \hat{f}_1(u) \ldots \hat{f}_k(u) \left( \frac{1 + \hat{f}(u)}{2} + (-)^k \frac{1 - \hat{f}(u)}{2} \right) \frac{1}{1 - \hat{f}^2(u)} \hat{g}(u).$$  \hspace{1cm} (32)
In the case of the exponential WTDs we can accordingly go beyond the Markovian case of an exponential distribution corresponding to \( q(t) = e^{-\mu t} \). For the simplest case of 2 WTDs one obtains
\[
q(t) = \frac{2(\mu - \mu_1)e^{-\mu_1 t} + \mu_1 e^{-2\mu t}}{2\mu - \mu_1}.
\]  
(33)

The expression of \( q(t) \) for a larger number of WTDs retains the same form, i.e. a weighted sum of \( k \) exponentials \( e^{-2\mu_1}, e^{-2\mu_2}, \ldots, e^{-2\mu_k} \). Non-monotonicity of the absolute value of the function \( q(t) \) can arise in two ways: non-monotonicity of \( q(t) \) itself or its sign change. Note that these are not independent, as \( q(t) \) convergences to zero for \( t \to \infty \). Accordingly, with every sign change at least one local maximum or minimum has to follow. On the other hand, a local maximum (minimum) can occur without sign change, but then need to be followed by a minimum (maximum).

For the case of two waiting time distributions one can analytically verify that the maximal number of revivals is one, and that revivals take place at
\[
t = -\frac{1}{2\mu - \mu_1} \ln \frac{2(\mu_1 - \mu)}{\mu_1},
\]  
(34)

where the condition \( \mu_1 > \mu \) has to be satisfied. This corresponds to the requirement obtained for the dynamics considered in Sect. [V.A], Eq. (26), which, however, could feature an intermediate time continuous evolution and a jump transformation containing AD. Note that the time \( t \) is smaller than the mean jump time of the first jump \( 1/\mu_1 \) for \( \mu_1 > 2\mu \), otherwise it is larger. For larger \( k \) in general no closed-form formula for the number or the times of revivals can be given, as the exponential function is transcendental. Nonetheless, thanks to the Descartes’ rule of signs the maximal number of revivals \( (q(t) = 0) \) equals the number of the sign changes of the prefactors of the exponential functions, where the rates are put in ascending (or descending) order [88]. The sign change can happen maximally \( k - 1 \) times for \( k \)-terms, which explains the observation we have made earlier in Sect. [V.A]. Note, that the same argument could be used for the derivative of \( q(t) \), connected with the occurrence of local maxima/minima. However, the maximal number of revivals \( k - 1 \) can only happen when all of the revivals are at zero distance, as the non-monotonicity of \( q(t) \) without sign change involves one minimum and one maximum per revival. Note that consequently for a process with 2 WTDs the revival can only occur because of the sign change of \( q(t) \), i.e. at zero trace distance.

C. Erlang WTD - general results

Considering WTDs that can lead to non-Markovianity for unmodified processes, the maximal possible number of revivals can get larger. This can be observed by taking into account an Erlang distribution, whose WTD is given by Eq. (25), governing the randomness of the jump times. For Erlang distributions with fixed mean value, the higher the shape parameter \( r \) or the larger the rate \( \mu \), the narrower the distribution. Accordingly, with growing \( r \) or \( \mu \) the revivals of the trace distance can be seen more and more like independent phenomena. In this case the jumps do not ”destructively interfere” with each other and the time intervals of the jumps are almost disjoint. This explains the increase of the non-Markovianity measure with higher shape parameter \( r \) or larger rate \( \mu \), as one can see in the simulations in Fig. 7 left. This slightly influences also the time of the revivals, and the higher the shape parameter, the closer this time is to the mean value of the first WTD.

![FIG. 7: Behavior of the trace distance for the case of a quantum renewal process with a \( \mathcal{E}_{x-AD} \) jump channel and WTDs given by Erlang distributions. In the left and right panel we see that revivals increase with the shape parameter \( r_1 \) (\( \mu = 4, r = 2, r_1/\mu_1 = 1/2 \) left and \( \mu = 12, r = 6, r_1/\mu_1 = 2/3 \) right). In the middle panel we see dependence on the parameter \( r \) with fixed \( \mu_1 = 3, r_1 = 2 \) and \( r/\mu = 1/2 \).](image)
Also in the case of the Erlang WTDs the trace distance revivals do not necessarily occur when the trace distance takes the value zero. This behaviour was observed for modified renewal processes. The small $r$ of the first WTD and the large $r$ of the subsequent WTD boost the phenomenon, see Fig. 7, middle and right. Note, that contrary to the case of the exponential WTDs, here the revival can occur at non-zero trace distance also for the simplest modified process, i.e. with 2 distinct WTDs.

D. Erlang WTD - purely jump dynamics

For the limiting case of no time continuous evolution in between the jumps $E_x$, relying on Eq. (30) for the trace distance between the optimal pair of states, one can analytically show that an infinite number of revivals is possible. The difference of the probability of the even and odd number of jumps for an unmodified process is given in Laplace domain by

$$\tilde{q}(u) = \frac{(\mu + u)^r - \mu^r}{u(\mu^r + (\mu + u)^r)}.$$ (35)

In particular, for $r = 2$, so for WTD given by a convolution of two exponential functions with the same rate, we obtain

$$q(t) = e^{-\mu t}(\sin(\mu t) + \cos(\mu t)),$$ (36)

which obviously leads to an infinite number of revivals, always occurring at the zero trace distance. For the modified process, with two different WTD and when both shape parameters equal two, $r = r_1 = 2$, one obtains

$$q(t) = \frac{1}{(2\mu_1^2 - 2\mu_1 \mu + \mu_1^2)^2} \left(2(\mu_1 - \mu) e^{-\mu_1 t} \left(\mu_1^3 - 3\mu_1^2 \mu + 2\mu_1 \mu^2 - 2\mu^3 + 3t\mu_1 + 4\mu_1 \mu^2 - 2\mu^3\right)\right)$$

$$- \mu_1^2 e^{-\mu t} \left((2\mu - \mu_1)^2 - 2\mu^2\right) \cos(\mu t) - (2\mu_1^2 - 2\mu_1) \sin(\mu t)) \right).$$ (37)

Accordingly, we have a term characterised by an oscillation, which is damped with a damping rate $\mu$, and a polynomial of the first order in $t$, damped with a damping rate $\mu_1$. From Fig. 8 we see that for $\mu = 1$ and $r = r_1 = 2$, if the rate of the first waiting time distribution $\mu_1$ is between zero and a value close to one, no revivals take place. This can be understood from Eq. (37), since if the rate $\mu$ is larger than the rate of the first WTD, the oscillatory part is strongly suppressed. However, for this regime the polynomial part stays always positive, and no revivals occur. Otherwise, we have an infinite number of revivals.

FIG. 8: Plot of the function $q(t)$ as in Eq. (37). The non-monotonicity of this function determines non-Markovianity in the model, see Eq. (30). The function $q(t)$ corresponds to the difference between the probability of having an even or an odd number of jumps as a function of time and WTDs’ rates. We consider two Erlang WTDs with shape parameters $r = r_1 = 2$ and $\mu = 1$, so that the vertical black line corresponds an unmodified renewal process. Note the periodic change of values along the vertical axis determining an infinite number of revivals.
VI. CONCLUSIONS AND OUTLOOKS

In this work, we have analysed a simple and versatile class of quantum dynamics, the quantum renewal processes, focusing on the different kinds of non-Markovian behavior that can be obtained by controlling their defining properties. Quantum renewal processes naturally allow for a representation of the dynamics in terms of an average over stochastic trajectories and we have here investigated the influence that the time-continuous part of the trajectories, the type of the jumps and the waiting time distributions have on the quantitative and qualitative features of the trace distance evolution. In particular, we focused not only on the measure of non-Markovianity, but also on relevant modifications of the trace distance evolution, as the number, times of occurrence and extension of the revivals. Among others, the revivals of the trace distance can be significantly altered or even enhanced when dealing with modified renewal processes, where there is a difference between a certain number of initial waiting time distributions and the subsequent ones, or if one considers Erlang waiting time distributions, which are classically non-Markovian and can lead to higher number of revivals than the exponential ones.

Our analysis shows that the trajectory picture of quantum renewal processes yields a deeper insight into how to manipulate the trace distance evolution, for a varied class of dynamics built on the analogy with classical stochastic processes. Indeed, it will be of interest to explore to which extent the trajectory viewpoint can be a convenient starting point to engineer non-Markovianity also in more complex and general quantum dynamics, pointing to different features of the evolution that can be addressed to enhance or suppress the presence of memory effects.

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