Extinction transition on diffusive substrate: a different universality class?

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Abstract

The extinction transition on a one dimensional heterogeneous substrate with diffusive correlations is studied. Diffusively correlated heterogeneity is shown to affect the location of the transition point, as the reactants adapt to the fluctuating environment. At the transition point the density decays like \( t^{-0.159} \), as predicted by the theory of directed percolation. However, the scaling function describing the behavior away from the transition shows significant deviations from the DP predictions; it is suggested, thus, that the off-transition behavior of the system is governed by local adaptation to favored regions.
I. INTRODUCTION

The extinction transition in the stochastic birth-death process is very important in many branches of science and serves also as a paradigmatic example of an out of equilibrium phase transition [1, 2, 3]. Given a homogeneous substrate and a single absorbing state, Grassberger [4] and Janssen [5] conjectured that the microscopic details of the stochastic process are irrelevant close to the extinction point and the transition is in the directed percolation (DP) universality class. The basic rationale behind this conjecture is that a spatially extended system splits, close to the transition, into active and inactive zones, where after each typical period of time there is certain probability for an active state to die, to survive, or to infect its inactive neighbors. If these regions are considered as lattice points on a $d$ dimensional array, the chance of an active site to survive or to infect its neighbors within a unit time is equivalent to the chance that a bond exists between lattice point at time $t$ and its neighbors in a subsequent replica of the system at $t+1$. Accordingly, the extinction transition happens when the bond density is exactly at threshold for an infinite cluster, and the transition belongs to the directed percolation universality class in $d+1$ dimensions.

The Grassberger-Janssen conjecture has proven to be extremely robust, and a large number of stochastic models that admit extinction transition were shown to belong to the DP equivalence class if the substrate is homogenous [1, 2]. It was further shown that spatio-temporal substrate noise (i.e., birth-death rates that fluctuate in space and time with only short range correlations) is an irrelevant perturbation close to the transition, so small noise is averaged out and leaves the DP transition unaffected [1]. Quenched (time independent) disorder, on the other hand, is a relevant perturbation [6] and seems to change the nature of the transition. In that case a Griffiths phase exists between the active and the inactive regions [7]. In the parameter region that corresponds to the Griffiths phase the survival of an active region depends on the local properties of the substrate, not on activation by neighboring regions. In particular, for each time scale the survival of active regions depends on the existence of spatial domains that admit high carrying capacity [7, 8]. Although stochastic fluctuations guarantee extinction for any finite sample, the time scale for that grows exponentially with the carrying capacity. This implies that exponentially rare spatial regions with high birth rate support the population for exponentially large times. An
optimization argument shows that in such a case the survival until \( t \) is dominated by rare spatial fluctuations of linear size \( L \sim \log(t) \); accordingly, the density falls algebraically with time. The Griffiths phase is located between an extinction region, where essentially no good islands exist, and the active phase, where good islands infect each other to yield a never dying process.

What happens, then, if the disorder is neither annealed nor quenched, but is diffusively correlated? Two contradictory arguments may be advanced. In terms of the renormalization group properties of Reggeon field theory, diffusive disorder is a relevant operator, so in principle one may expect that the transition will be in a different equivalence class. This was suggested, in fact, by Kree, Schaub and Schmittmann \([9]\), who then proceeded to develop a perturbative renormalization group based \( \epsilon \) expansion around four dimensions, a treatment that yield predictions of new (non-DP) critical exponents. One may doubt, however, the validity of the one-loop approximation used by these authors below \( 2d \), as the diffusion constant of the disorder flows to zero upon successive renormalization group decimation, thus the perturbative expansion admits only runaway solutions \([6]\).

On the other hand, it is clear that the basic intuitive justification for the Grassberger-Janssen conjecture is applicable as long as the favored regions are mobile \([10]\). Even if the process is confined to the sites where the diffusive catalysts exist, these catalysts move randomly in space, thus again any region may survive, become inactive or infect neighboring regions. One may suggest, accordingly, that even in the presence of diffusive catalysts the extinction transition will still be within the DP equivalence class.

Here we present numerical results that suggest a third answer: it seems that at the transition point the critical exponent that characterizes the extinction transition is indeed identical with the DP exponent. However, the subcritical and the supercritical behavior are not described by the directed percolation scaling function. The DP transition is characterized by a scaling function of the form

\[
\rho(t) \sim t^{-\delta} \Phi(\Delta t^{1/\nu}) 
\]

Where \( \Phi \) is a universal scaling function, \( \delta \) is the universal exponent (in 1d \( \delta = 0.159 \)) \( \Delta \) is the distance from the transition and \( \nu \) is the temporal correlation length. This implies that all the curves coincide when plotting \( \rho t^{\delta} \) vs. \( \Delta t^{1/\nu} \). In the presence of diffusive disorder, while the density decreases like \( t^{-\delta} \) where \( \delta \) takes the same numerical value predicted by the
FIG. 1: A sketch of the phase space in the $D - p_0$ plane (lower panel), and the actual results measured for the contact process with a diffusively correlated substrate. As $D$ approaches infinity the environmental stochasticity becomes uncorrelated in space and time and the system approaches the homogenous limit of the DP transition. The critical value of the birth rate, $p_c$, coincides with the value measures for a contact process on homogenous 1d substrate, as demonstrated in the upper panel. On the other hand as $D$ become smaller the effect of local adaptation is significant and the transition is shifted into the extinction region. As $D \to 0$ the transition point appears to coincide with the right edge of the Griffiths phase. In the upper panel the actual values of $P_c$, the critical values of $P_0$, are shown. For the same system parameters the quenched disorder transition (from the Griffiths phase to the active one) happens at $p_c = 0.746$, a value that corresponds to the $x$ axis here. Data were obtained from the MC simulation of the contact process with $L = 10^4$ and $d = 0.1$. 
DP theory, the scaling function does not exist, as will be shown below.

The model used here is a contact process (CP) on a fluctuating substrate. It take place on a 1d lattice with $L$ sites (with periodic boundary conditions), where any lattice site is either occupied by an agent or empty. In an elementary Monte-Carlo reaction step an agent is chosen at random and then attempts to multiply with probability $p$ or ”die” (be eliminated from the sample) with probability $1 - p$. If the agent does attempt to multiply, one of the two neighboring sites in chosen at random and becomes occupied if it is currently empty; if the site is already occupied, nothing happens.

To take into account the fluctuating environment, $p$ is taken to be a space-time dependent fluctuating quantity. At $t = 0$, sites are chosen with equal probability to be either ”good” ($p_i = p_0 + d$) or ”bad” ($p_i = p_0 - d$) where $i$ is a site index and $d$ is a constant. Subsequently, each elementary step is chosen to be either a reaction step, as described above, or a catalyst diffusion step. In a diffusion step, a randomly picked site switches its $p$ value with one of its nearest neighbors. The probability of a given step to be a reaction step is taken to be $r = \rho/(\rho + D)$, where $D$ is the catalyst diffusion rate, so that the probability to be a diffusion step is $1 - r$. After each elementary MC step the time counter has been advanced by $1/L(\rho + D)$.

The phase diagram is shown in Figure 1. For $D \to \infty$, the space-time disorder becomes uncorrelated and the system belongs to the DP universality class. At this parameter region the transition occurs at $p_c = 0.767$, the critical value for the contact process on a homogenous substrate [3, 11]. As $D$ becomes smaller, the catalysts spend more time in certain spatial regions. Agents in these regions produce more offspring in adjacent sites, and when the catalyst jumps to a neighboring site its probability to be occupied is larger than average. This implies that reactants actually ”adapt” to the instantaneous configuration of the catalysts, an effect that yields very strong proliferation in unbounded growth models [12]. Here the growth is bounded and the effect of adaptation is finite, still the transition point shifts leftward as exemplified in Fig. 1. The location of the extinction transition for the slowest diffusion we were able to measure is very close to the location of the transition from the Griffiths phase at $D = 0$; it seems plausible that the transition line converges to the right end of the Griffiths phase as $D$ approaches zero.

Figure 2 shows the approach to the transition from the active and from the inactive phase. For any crossing of the transition line checked, either by increasing the diffusion or
FIG. 2: Density vs. time (logarithmic scale) at the transition and in its vicinity. In the upper panel the dashed line (2) in figure 1 is followed and diffusion is increased with fixed \( p_0 = 0.767 \) that corresponds to the transition point of the pure contact process. The system is always in its active phase, as suggested by the adaptation argument. In the lower panel the separation line is crossed along arrow (1) of figure 1 \( (p_c = 0.749) \), and the system crosses from the active to the inactive phase. In both cases, and all other cases checked by us (values correspond to the data points in the upper panel of Fig. 1), the slope at the transition is 0.159, in agreement with the DP theory predictions. Datasets were obtained with lattice size \( L = 10^4 \) and \( d = 0.1 \) by changing \( p_0 \), the density at the transition decays like a power law with the DP critical exponent \( \delta = 0.159 \). However, the off-transition behavior does not obey the DP predictions. According to Eq. (1) a graph of \( \rho(t) t^\delta \) vs. \( \Delta t^{1/\eta} \) should be universal close to the transition, as discussed comprehensively by Lübeck [2]. Figure 3 shows that this, in fact, the case both
FIG. 3: The universal scaling function for uncorrelated spatio-temporal noise (inset) and for finite diffusion \((D = 0.225)\) of the catalysts. Different colors correspond to different distance \(\Delta\) from the transition, as explained in the legend.

on a homogenous substrate and for infinite diffusion (i.e., when the locations of the catalysts are uncorrelated in space and time). On the other hand for finite diffusion the long-time behavior fails to fit the universal curve; this implies that a one-parameter universal scaling function is not enough to describe the system.

Two types of noise appear in the problem at hand: the demographic stochasticity associated with the discretness of individual reactants and the environmental stochasticity associated with the diffusive wandering of the underlying catalysts. For an unbounded growth problem (when the carrying capacity of any spatial point is infinite) this system is equivalent to KPZ growth with correlated disorder, but the KPZ perturbative scheme fails for diffusively correlated disorder \([14]\) since local adaptation initiates localized colonies. Indeed, for the unbounded problem extinction happens almost surely (i.e., for any finite spatial domain reactants goes extinct at the long time limit), yet the average population grows faster than exponentially for \(d \leq 2\) \([13]\). This effect disappears in the case of limited growth, as the adapted colonies can no longer grow exponentially forever. Still, it seems that the favored regions dominate the system’s behavior above and below the transition. At the transition point survival is based on the ability of catalyst rich zones to ”infect” each other, and the microscopic details average out within the diverging correlation length. Once
\( \xi_\perp \) becomes finite different zones are effectively independent and the spatial heterogeneity dictates the local decay; in such a case local adaptation leads to longer survival times and the system behavior resembles the Griffiths phase, thus leading to deviations from the DP universal scaling function.

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