Norm derivatives and geometry of bilinear operators

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Abstract
We study the norm derivatives in the context of Birkhoff–James orthogonality in real Banach spaces. As an application of this, we obtain a complete characterization of the left-symmetric points and the right-symmetric points in a real Banach space in terms of the norm derivatives. We obtain a complete characterization of strong Birkhoff–James orthogonality in $\ell^1_1$ and $\ell^\infty$ spaces. We also obtain a complete characterization of the orthogonality relation defined by the norm derivatives in terms of some newly introduced variation of Birkhoff–James orthogonality. We further study Birkhoff–James orthogonality, approximate Birkhoff–James orthogonality, smoothness and norm attainment of bounded bilinear operators between Banach spaces.

Keywords Birkhoff–James orthogonality · Norm derivatives · Bilinear operators · Smoothness · Norm attainment set

Mathematics Subject Classification 46B20 · 46G25

1 Introduction

The purpose of the present article is to explore the connection between the norm derivatives and Birkhoff–James orthogonality in real Banach spaces. We also study smoothness and norm attainment of bounded bilinear operators between Banach spaces. Let us first establish the relevant notations and the terminologies to be used in the present article.

Throughout the text, we use the symbols $X$, $Y$, $Z$ to denote real normed linear spaces. Let $B_X = \{ x \in X : \| x \| \leq 1 \}$ and let $S_X = \{ x \in X : \| x \| = 1 \}$ be the
unit ball and the unit sphere of $X$, respectively. Let $X^*$ denote the dual space of $X$. Given $x \in S_X$, $f \in S_{X^*}$, is said to be a supporting functional at $x$ if $f(x) = ||x||$. Let $J(x) = \{f \in S_{X^*} : f(x) = ||x||\}$ denote the collection of all supporting functionals at $x$. Note that for each $x \in S_X$, the Hahn–Banach theorem ensures the existence of at least one supporting functional at $x$.

If $x, y \in X$, then we say that $x$ is Birkhoff–James orthogonal to $y$, written as $x \perp_B y$, if $||x + \lambda y|| \geq ||x||$ for all $\lambda \in \mathbb{R}$. We refer the readers to the pioneering articles [3, 8] for more information in this regard. In [8, Theorem 2.1], James proved that $x \perp_B y$ if and only if there exists $f \in J(x)$ such that $f(y) = 0$. Given $0 \neq x, y \in X$, $x$ is said to be strongly orthogonal to $y$ in the sense of Birkhoff–James [15], written as $x \perp_{SB} y$, if $||x + \lambda y|| > ||x||$ for all $\lambda \neq 0$. For $x, y \in X$ and $\epsilon \in [0, 1)$, $x$ is said to be approximate $\epsilon$-orthogonal to $y$ [4], written as $x \perp_{\epsilon} y$, if $||x + \lambda y||^2 \geq ||x||^2 - 2\epsilon ||x|| ||\lambda y||$ for all $\lambda \in \mathbb{R}$. Observe that Birkhoff–James orthogonality is homogeneous.

Sain [13] characterized Birkhoff–James orthogonality of linear operators on finite-dimensional Banach spaces by introducing the notions of the positive part of $x$, denoted by $x^+$, and the negative part of $x$, denoted by $x^-$, for an element $x \in X$. For any element $y \in X$, we say that $y \in x^+$ ($y \in x^-$) if $||x + \lambda y|| \geq ||x||$ for all $\lambda \geq 0$ ($\lambda \leq 0$). It is easy to see that $x^\perp = \{y \in X : x \perp_B y\} = x^+ \cap x^-$. In general, Birkhoff–James orthogonality relation between two elements need not be symmetric. In other words, for any two elements $x, y \in X$, $x \perp_B y$ does not necessarily imply $y \perp_B x$. An element $x \in X$ is said to be left symmetric (right symmetric) [13] if $x \perp_B y$ ($y \perp_B x$) implies $y \perp_B x$ ($x \perp_B y$). James [9] proved that if dim $X \geq 3$ and Birkhoff–James orthogonality is symmetric then the norm is induced by an inner product.

An element $x \in S_X$ is said to be smooth point if $J(x) = \{f\}$ for some $f \in S_{X^*}$. A Banach space $X$ is said to be smooth if every $x \in S_X$ is a smooth point. The characterization of smooth points obtained by James [8] has been used in our study, which states that if $0 \neq x \in X$ is a smooth point in $X$ if and only if for any $y, z \in X$, $x \perp_B y$ and $x \perp_B z$ implies that $x \perp_B (y + z)$. It is well known that smoothness of $x \in S_X$ is equivalent to the Gâteaux differentiability of norm at $x$, i.e., $x \in S_X$ is a smooth point in $X$ if and only if $\rho(x, y) = \lim_{\lambda \to 0^+} \frac{||x + \lambda y|| - ||x||}{\lambda}$ exists for all $y \in X$. We recall the following standard notions of left-hand and right-hand Gâteaux derivative of norm at $x \in X$ in direction of $y \in X$:

$$\rho_+(x, y) = \lim_{\lambda \to 0^+} \frac{||x + \lambda y|| - ||x||}{\lambda}, \quad \rho_-(x, y) = \lim_{\lambda \to 0^-} \frac{||x + \lambda y|| - ||x||}{\lambda}.$$  

We will use the following properties of norm derivatives in this note (see [2, 11] for details):

(i) For all $x, y \in X$ and all $\alpha \in \mathbb{R}$,

$$\rho_\pm(\alpha x, y) = \rho_\pm(x, \alpha y) = \begin{cases} \alpha \rho_\pm(x, y) & \text{if } \alpha \geq 0 \\ \alpha \rho_\pm(x, y) & \text{if } \alpha < 0. \end{cases}$$

(ii) $\rho_+(x, y) = \rho_-(x, y)$ for all $y \in X$ if and only if $0 \neq x$ is a smooth point in $X$.

(iii) $\rho_\pm$ are continuous with respect to the second variable.
We note that there are several notions of orthogonality in a normed linear space which are equivalent only if the norm is induced by an inner product [1]. Indeed, the above-mentioned concepts of $\rho_{\pm}$ yields the following notions of orthogonality in normed linear spaces [5]. Given $x, y \in X$,

\[
x \perp_{\rho_{+}} y \iff \rho_{+}(x, y) = 0;
\]

\[
x \perp_{\rho_{-}} y \iff \rho_{-}(x, y) = 0;
\]

\[
x \perp_{\rho} y \iff \rho_{+}(x, y) + \rho_{-}(x, y) = 0.
\]

As mentioned in [5], the relations $\perp_{\rho_{+}}, \perp_{\rho_{-}}$ and $\perp_{\rho}$ are equivalent in an inner product space but are generally incomparable in a normed space which is not smooth. Nevertheless, we illustrate that it is possible to establish a connection between $\perp_{\rho_{+}}, \perp_{\rho_{-}}$ and Birkhoff–James orthogonality. It is worth mentioning that in some cases the function $\rho$ may be more convenient than the functions $\rho_{+}, \rho_{-}$. Indeed, the function $\rho$ played a significant role in the paper [22], where the mapping $\rho$ was helpful to solve an open problem posed in 2010 in the book [2]. To serve our purpose, we introduce the following variation of Birkhoff–James orthogonality, denoted by $\perp_{B^*}$. If $x, y \in S_X$ then we write $x \perp_{B^*} y$ if $x \perp_{B} y$ and $x \perp_{B} u$ for any $u$ of the form $u = ty + (1-t)x$, where $t \in (0, 1)$. For more details related to $\rho_{\pm}$ orthogonality and its variants see [12, 21, 23] and references cited therein.

If $X$ and $Y$ are normed linear (Banach/reflexive Banach) spaces then it is easy to see that $X \times Y$ is a normed linear (Banach/reflexive Banach) space equipped with the norm $\|(x, y)\|_\infty = \max\{\|x\|, \|y\|\}$ for all $(x, y) \in X \times Y$. Throughout this article we will consider $X \times Y$ with the aforesaid norm. Let $T : X \times Y \to Z$ be a bilinear operator, i.e., $T$ is linear in each argument. The norm of $T$ is defined as $\|T\| = \sup \{\|T(x, y)\| : (x, y) \in S_X \times S_Y\}$. Let $M_T$ denote the norm attainment set of $T$, i.e., $M_T = \{(x, y) \in S_X \times S_Y : \|T(x, y)\| = \|T\|\}$. $(x_n, y_n) \subset S_X \times S_Y$ is said to be a norming sequence for $T$ if $\|T(x_n, y_n)\| \to \|T\|$. We say that $T$ is bounded if $\|T\| < \infty$ and $T$ is compact if for all bounded sequences $\{(x_n, y_n)\} \subset X \times Y$, the sequence $\{T(x_n, y_n)\}$ has a convergent subsequence. Let $B(X \times Y, Z)$ $(K(X \times Y, Z))$ denote the space of all bounded (compact) bilinear operators from $X \times Y$ to $Z$.

The concept semi-inner product (s.i.p.) in normed linear spaces [10] plays an important role in our discussion of the geometry of bilinear operators. Giles [7] proved that for every normed linear space $X$, there exists a s.i.p. on $X$ which is compatible with the norm, i.e., $\|x, x\| = |x|^2$ for all $x \in X$. Also, for a normed linear space $X$, s.i.p. on $X$ which is compatible with the norm is unique if and only if $X$ is smooth. For $X = \ell^p_n, 1 < p < \infty$, the following is the unique s.i.p. on $X$ compatible with its norm:

\[
[x, y] = \frac{1}{\|y\|^{p-2}} \sum_{i=1}^{n} x_i y_i |y_i|^{p-2},
\]

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \ell^p_n \setminus \{0\}$.

Whenever we work with a s.i.p. in a normed linear space, we assume that the s.i.p. is compatible with the norm. We will use the following result related to s.i.p. and Birkhoff–James orthogonality in normed linear spaces.
Theorem 1.1 [7] Let $X$ be a normed linear space and let $x, y \in X$. If $x \perp_B y$ then there exists a s.i.p. $\langle \ , \ \rangle$ on $X$ such that $\langle y, x \rangle = 0$.

After this introductory part, this article is demarcated into two sections. The first section deals with the norm derivatives and its connections with geometry of the space and Birkhoff–James orthogonality. In the next section, we explore Birkhoff–James orthogonality, approximate Birkhoff–James orthogonality, smoothness and the norm attainment of bounded bilinear operators between Banach spaces. These results extend some of the recent observations regarding the analogous properties of bounded bilinear operators between Banach spaces [14].

2 Norm derivatives

We start with the following complete characterizations of the positive part and the negative part of an element in a normed linear space, in terms of the norm derivatives $\rho_\pm$ and s.i.p. on the given normed linear space.

Theorem 2.1 Let $X$ be a normed linear space and let $x, y \in X$. Then the following are equivalent:

(a) $\rho_+(x, y) \geq 0$.
(b) $y \in x^+$.
(c) There exists a s.i.p. $\langle \ , \ \rangle$ on $X$ such that $\langle y, x \rangle \geq 0$.

Proof We first prove the equivalence of (a) and (b). To prove that (a) implies (b) we consider the following two cases.

Case I: Let $\rho_+(x, y) > 0$. Then there exists sufficiently small $\lambda_0 > 0$ such that $\|x + \lambda y\| - \|x\| > 0$ for all $\lambda \in (0, \lambda_0]$. Now, by the convexity of the norm it follows that $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \geq 0$ and hence $u \in x^+$.

Case II: Let $\rho_+(x, y) = 0$. Then it follows from [6, Theorem 3.2] that $x \perp_B y$ and hence $y \in x^+$.

Now, we show that (b) implies (a). Observe that (b) implies, $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \geq 0$. Thus $\rho_+(x, y) = \lim_{\lambda \to 0^+} \frac{\|x + \lambda y\| - \|x\|}{\lambda} \geq 0$ and (a) follows.

Equivalence of (b) and (c) follows from [17, Theorem 2.4].

The following result follows from similar arguments as used in the above theorem.

Theorem 2.2 Let $X$ be a normed linear space and let $x, y \in X$. Then the following are equivalent:
(a) $\rho_-(x, y) \leq 0$.
(b) $y \in x^-$.
(c) There exists a s.i.p. [ , ] on $X$ such that $[y, x] \leq 0$.

In [20], Sain et al. obtained a complete characterization of the left-symmetric points and the right-symmetric points in a normed linear space in terms of the positive part and the negative part of the elements of the space. As a consequence of Theorems 2.1 and 2.2 and results obtained in [20], we obtain the following complete characterizations of the left-symmetric points and the right-symmetric points of a normed linear space in terms of the norm derivatives $\rho_\pm$.

**Corollary 2.1** Let $X$ be a normed linear space and let $x \in X$. Then the following are equivalent:

(a) $x$ is a left-symmetric point in $X$.
(b) Given $y \in X$ with $\rho_+(x, y) \geq 0$ implies that $\rho_+(y, x) \geq 0$.
(c) Given $y \in X$ with $\rho_-(x, y) \leq 0$ implies that $\rho_-(y, x) \leq 0$.

**Proof** Let (a) holds true and let $y \in X$ be such that $\rho_+(x, y) \geq 0$. Then it follows from Theorem 2.1 that $y \in x^+$. Now, it follows from [20, Theorem 2.1] that $x \in y^+$. Again, using Theorem 2.1 it follows that $\rho_+(y, x) \geq 0$ and thus (b) follows.

Let (b) holds true and let $y \in X$ with $\rho_-(x, y) \leq 0$. Then using the properties of the norm derivatives we get $\rho_+(x, -y) \geq 0$. Now, (b) implies $\rho_+(y, x) \geq 0$. Again, using the properties of the norm derivatives we get $\rho_-(y, x) \leq 0$. Thus (c) follows.

Let (c) holds true. To prove that (a) holds true [20, Theorem 2.1] implies, it is sufficient to show that given $y \in X$ such that $y \in x^-$ implies $x \in y^-$. Let $y \in X$ be such that $y \in x^-$, then it follows from Theorem 2.2 that $\rho_-(x, y) \leq 0$. Now, (c) implies that $\rho_-(y, x) \leq 0$ and again using Theorem 2.2 it follows that $x \in y^-$. Thus (a) follows from [20, Theorem 2.1].

Using similar arguments as in the above corollary and [20, Theorem 2.2] the following result follows.

**Corollary 2.2** Let $X$ be a normed linear space let $x \in X$. Then the following are equivalent:

(a) $x$ is a right-symmetric point in $X$.
(b) Given $y \in X$ with $\rho_+(y, x) \geq 0$ implies that $\rho_+(x, y) \geq 0$.
(c) Given $y \in X$ with $\rho_-(y, x) \leq 0$ implies that $\rho_-(x, y) \leq 0$.

We would like to give an explicit description of the sign of the norm derivatives in $\ell^p_\infty$ spaces, $1 \leq p \leq \infty$. Note that when $1 < p < \infty$, the corresponding space
is smooth and, therefore, the desired description follows using the unique s.i.p. expression in the said space.

**Theorem 2.3** Let \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \ell_p^n \setminus \{0\}, 1 < p < \infty \). Then

(a) \( \rho_+(x, y) \geq 0 \) if and only if \( \sum_{i=1}^{n} y_i x_i |x_i|^{p-2} \geq 0 \),

(b) \( \rho_-(x, y) \leq 0 \) if and only if \( \sum_{i=1}^{n} y_i x_i |x_i|^{p-2} \leq 0 \).

**Proof** If \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \ell_p^n \setminus \{0\}, 1 < p < \infty \), then the unique s.i.p. on \( \ell_p^n, 1 < p < \infty \), is given by

\[
[y, x] = \frac{1}{\|x\|^{p-2}} \sum_{i=1}^{n} y_i x_i |x_i|^{p-2}.
\]

Now, the proof of (a), (b) follows from Theorems 2.1 and 2.2, respectively. \(\square\)

When \( p = 1, \infty \), the corresponding \( \ell_p^n \) space is not smooth and there exist infinitely many s.i.p. on the space. Our next result addresses this issue by means of the direct computation. If \( t \in \mathbb{R} \) then \( \text{sgn} \ t \) denotes the sign function, i.e., \( \text{sgn} \ t = \frac{t}{|t|} \) for \( t \neq 0 \) and \( \text{sgn} \ 0 = 0 \). We first write a simple observation: if \( x, y \in \mathbb{R} \), where \( x \neq 0 \), then there exists sufficiently small \( \lambda_0 > 0 \) such that \( |x + \lambda y| = |x| + \lambda (\text{sgn} \ x)y \) for all \( \lambda \in [-\lambda_0, \lambda_0] \).

**Theorem 2.4** Let \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \ell_\infty^n \). Then

(a) \( \rho_+(x, y) \geq 0 \) if and only if there exists \( 1 \leq i_0 \leq n \) such that \( \|x\| = |x_{i_0}| \) and \( (\text{sgn} \ x_{i_0})y_{i_0} \geq 0 \).

(b) \( \rho_-(x, y) \leq 0 \) if and only if there exists \( 1 \leq i_0 \leq n \) such that \( \|x\| = |x_{i_0}| \) and \( (\text{sgn} \ x_{i_0})y_{i_0} \leq 0 \).

**Proof** (a) If \( x = 0 \) then the result follows trivially. Now, we assume that \( x \neq 0 \). We first prove the sufficient part. It follows from Theorem 2.1 that to prove the sufficient part it is enough to show that \( y \in x^+ \). Let \( 1 \leq i_0 \leq n \) be such that \( \|x\| = |x_{i_0}| \), \( (\text{sgn} \ x_{i_0})y_{i_0} \geq 0 \). Then for any \( \lambda \geq 0 \), we have,
\[ \|x + \lambda y\| = \max_{1 \leq i \leq n} |x_i + \lambda y_i| \]
\[ \geq |x_{i_0} + \lambda y_{i_0}| \]
\[ = |x_{i_0}| + \lambda (\text{sgn } x_{i_0})y_{i_0} | \]
\[ \geq |x_{i_0}| + \lambda (\text{sgn } x_{i_0})y_{i_0} \]
\[ = \|x\|. \]

Thus for all \( \lambda \geq 0, \|x + \lambda y\| \geq \|x\| \) and hence the sufficient part follows.

We now prove the necessary part. Let \( \rho_+ (x, y) \geq 0 \), then it follows from Theorem 2.1 that \( y \in x^+ \). Let \( \{j_1, \ldots, j_m\} \subseteq \{1, \ldots, n\}, m \leq n \), be the maximal subset such that \( \|x\| = |x_{i_k}| \) for all \( 1 \leq k \leq m \). Suppose on the contrary that \( (\text{sgn } x_{i_k})y_{j_k} < 0 \) for all \( 1 \leq k \leq m \). We can choose sufficiently small \( \lambda_1 > 0 \) and \( 1 \leq l \leq m \) such that for all \( \lambda \in (0, \lambda_1], \) we have,
\[ \|x + \lambda y\| = \max_{1 \leq i \leq n} |x_i + \lambda y_i| = |x_{j_l}| + \lambda (\text{sgn } x_{j_l})y_{j_l} < |x_{j_l}| = \|x\|. \]

This leads to a contradiction and thus the necessary part follows.

Proof of (b) follows from similar arguments as above. \( \square \)

**Theorem 2.5** Let \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \ell^n_1 \). Then \( \rho_+ (x, y) \geq 0 \) if and only either of the following hold true:

(a) \( \{ i : 1 \leq i \leq n, x_i \neq 0 \text{ and } y_i \neq 0 \} = \emptyset. \)

(b) \( \{ i : 1 \leq i \leq n, x_i \neq 0 \text{ and } y_i \neq 0 \} \neq \emptyset \text{ and } \sum_{i=1}^{n} (\text{sgn } x_i)y_i + \sum_{i=1}^{n} |y_i| \geq 0. \)

**Proof** For the necessary part assume that \( \rho_+ (x, y) \geq 0. \) Now, from Theorem 2.1, it follows that \( y \in x^+ \). If \( x, y \) are such that \( \{ i : 1 \leq i \leq n, x_i \neq 0 \text{ and } y_i \neq 0 \} = \emptyset \) then (a) follows. Now, let \( \{ i : 1 \leq i \leq n, x_i \neq 0 \text{ and } y_i \neq 0 \} \neq \emptyset. \) Suppose on the contrary that \( \sum_{i=1}^{n} (\text{sgn } x_i)y_i + \sum_{i=1}^{n} |y_i| < 0. \)

We can find sufficiently small \( \lambda_1 > 0 \) such that for all \( \lambda \in (0, \lambda_1], \) we have,
This contradicts that \( y \in x^+ \). Thus (b) follows.

Now we prove the sufficient part. Let (a) holds true. Then it follows easily that \( x \perp_B y \), i.e., \( \|x + \lambda y\| \geq \|y\| \) for all \( \lambda \in \mathbb{R} \). Now, it follows from the definition of \( \rho_+(x, y) \) that \( \rho_+(x, y) \geq 0 \). We now assume that (b) holds true. Observe that from Theorem 2.1, it is sufficient to show that \( y \in x^+ \). If \( \lambda \geq 0 \) then

\[
\|x + \lambda y\| = \sum_{i=1}^{n} |x_i + \lambda y_i| = \sum_{i=1}^{n} |x_i + \lambda y_i| + \lambda \sum_{i=1}^{n} |y_i|
\]

\[
\geq \sum_{i=1}^{n} (|x_i| + \lambda |y_i|) + \lambda \sum_{i=1}^{n} |y_i| = \|x\| + \lambda \left( \sum_{i=1}^{n} (|x_i| + \lambda |y_i|) \right)
\]

\[
\geq \|x\| + \lambda \left( \sum_{i=1}^{n} |y_i| \right) = \|x\| + \lambda \left( \sum_{i=1}^{n} |y_i| \right)
\]

Thus \( y \in x^+ \) and \( \rho_+(x, y) \geq 0 \). \( \square \)

Similar arguments yields the following analogous characterization of \( \rho_-(x, y) \leq 0 \) for \( x, y \in \ell_1^n \).

\( \Box \)
Theorem 2.6 Let \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \ell_1^n \). Then \( \rho_-(x, y) \leq 0 \) if and only either of the following hold true:

(a) \( \{ i : 1 \leq i \leq n, \ x_i \neq 0 \text{ and } y_i \neq 0 \} = \emptyset \).

(b) \( \{ i : 1 \leq i \leq n, \ x_i \neq 0 \text{ and } y_i \neq 0 \} \neq \emptyset \) and \( \sum_{i=1}^{n} (\text{sgn } x_i)y_i + \sum_{i=1}^{n} |y_i| \leq 0 \).

Sain et al. [15] observed that a normed linear space \( X \) is strictly convex if and only if \( x, y \in S_X \) and \( x \perp_B y \) implies that \( x \perp_{SB} y \). Thus for \( \ell_p^n \) spaces, where \( 1 < p < \infty \), Birkhoff–James orthogonality and strong Birkhoff–James orthogonality coincide. For \( \ell_1^n \) or \( \ell_\infty^n \), characterization of strong Birkhoff–James orthogonality is not known. Our next result provides a complete characterization of strong Birkhoff–James orthogonality, involving the norm derivatives \( \rho_{\pm} \) in \( \ell_1^n \) and \( \ell_\infty^n \) spaces.

Theorem 2.7 Let \( X = \ell_1^n \) or \( \ell_\infty^n \), and let \( x, y \in S_X \). Then \( x \perp_{SB} y \) if and only if the following two conditions hold true:

(a) \( \rho_+(x, y) > 0 \),

(b) \( \rho_-(x, y) < 0 \).

Proof Let us first prove the necessary part of the theorem. Let \( x \perp_{SB} y \). Then \( y \in x^+ \) and \( y \in x^- \). Now, it follows from Theorems 2.1 and 2.2 that \( \rho_+(x, y) \geq 0 \), \( \rho_-(x, y) \leq 0 \). Suppose on the contrary that \( \rho_+(x, y) = 0 \).

If \( X = \ell_1^n \) then we can find a sufficiently small \( \lambda_1 > 0 \) such that for all \( \lambda \in (0, \lambda_1] \), we have,

\[
\|x + \lambda y\| = \sum_{i=1}^{n} |x_i + \lambda y_i| + \lambda \sum_{i=1}^{n} |y_i| \\
= \sum_{i=1}^{n} |x_i| + \lambda (\text{sgn } x_i)y_i + \lambda \sum_{i=1}^{n} |y_i| \\
= \sum_{i=1}^{n} |x_i| + \lambda (\text{sgn } x_i)y_i + \lambda \sum_{i=1}^{n} |y_i| \\
= \|x\| + \lambda \left( \sum_{i=1}^{n} (\text{sgn } x_i)y_i + \sum_{i=1}^{n} |y_i| \right).
\]
If $X = \ell_1^\infty$ then we can find a sufficiently small $\lambda_2 > 0$, $1 \leq i_0 \leq n$, such that $\|x\| = |x_{i_0}|$ and for all $\lambda \in (0, \lambda_1]$, we have,

$$\|x + \lambda y\| = \max_{1 \leq i \leq n} |x_i + \lambda y_i|$$

$$= \max_{1 \leq i \leq n} |x_i| + \lambda |\text{sgn} x_i y_i|$$

$$= |x_{i_0}| + \lambda |\text{sgn} x_{i_0} y_{i_0}|$$

$$= \|x\| + \lambda |\text{sgn} x_{i_0} y_{i_0}|.$$  

Thus in both cases we can find a sufficiently small $\alpha > 0$ such that for all $\lambda \in (0, \alpha]$, we have,

$$\|x + \lambda y\| = \|x\| + \lambda a,$$

where

$$a = \left\{ \begin{array}{ll}
\sum_{i=1}^n (\text{sgn} x_i) y_i + \sum_{i=1}^n |y_i| & \text{if } x \in \ell_1^n \\
|y_i| & \text{if } x_i \neq 0 \\
\text{sgn} x_{i_0} y_{i_0} & \text{for some } 1 \leq i_0 \leq n \text{ if } x \in \ell_1^\infty.
\end{array} \right.$$  

Now, the assumption $\rho_+(x, y) = 0$ implies that $a = 0$ and thus $\|x + \lambda y\| - \|x\| = 0$ for all $\lambda \in (0, \alpha]$. This contradicts the assumption that $x \perp y$. Thus $\rho_+(x, y) > 0$. Using similar arguments we can show that $\rho_-(x, y) < 0$.

Let us now prove the sufficient part of the theorem. If $\rho_+(x, y) > 0$ then we can find a sufficiently small $\lambda_3 > 0$ such that for all $\lambda \in (0, \lambda_3]$, we have,

$$\|x + \lambda y\| - \|x\| > 0.$$  

Now, by the convexity of the norm we obtain that

$$\|x + \lambda y\| > \|x\|$$

for all $\lambda > 0$. Similar arguments as above show that if $\rho_-(x, y) < 0$ then we can find a sufficiently small $\lambda_4 < 0$ such that for all $\lambda \in [-\lambda_4, 0)$, we have,

$$\|x + \lambda y\| - \|x\| > 0.$$  

Now, again by the convexity of the norm we obtain that

$$\|x + \lambda y\| > \|x\|$$

for all $\lambda < 0$. Thus $\|x + \lambda y\| > \|x\|$ for all $\lambda \neq 0$. This proves the sufficient part of the theorem. \qed

As a corollary to the above result, we obtain the following complete characterization of strong Birkhoff orthogonality in $X \oplus_1 Y$, where $X = \ell_1^n$ or $\ell_1^\infty$, $Y = \ell_1^m$
or $\ell^m_\infty$ and $X \oplus_1 Y = \{(x, y) : x \in X, y \in Y\}$ is a Banach space with the norm $\|(x, y)\| = \|x\| + \|y\|$. 

**Corollary 2.3** Let $X = \ell^n_1$ or $\ell^n_\infty$, and let $Y = \ell^m_1$ or $\ell^m_\infty$. Let $(x, y), (u, v) \in X \oplus_1 Y$. Then $(x, y) \perp_{SB} (u, v)$ if and only if the following two conditions hold true:

(a) $\rho_+(x, u) + \rho_+(y, v) > 0$,
(b) $\rho_-(x, u) + \rho_-(y, v) < 0$.

We would like to conclude this section by obtaining complete characterizations of $\rho_+$ orthogonality and $\rho_-$ orthogonality in normed linear space. First we recall the following result from [11], presenting a relationship between functionals supporting $B_X$ at $x$ and $\rho_\pm(x, \cdot)$.

**Lemma 2.1** [11, Lemma 5.4.16] Let $X$ be a normed linear space and let $x \in S_X$. Then $f \in S_X$ is a supporting functional at $x$ if and only if for any $y \in X$,

$$\rho_-(x, y) \leq f(y) \leq \rho_+(x, y).$$

For our next result characterizing $\rho_+$ and $\rho_-$ orthogonality, we recall the definition of a normal cone in normed linear spaces. A subset $K$ of $X$ is said to be a normal cone if

1. $K + K \subset K$,
2. $\alpha K \subset K$ for all $\alpha \geq 0$ and
3. $K \cap (-K) = \{0\}$.

**Theorem 2.8** Let $X$ be a Banach space and let $x, y \in S_X$. Then the following results hold true:

(a) $\rho_+(x, y) = 0$ if and only if $(x) \perp_B y$.
(b) $\rho_-(x, y) = 0$ if and only if $x \perp_B y$.

**Proof** Let us first prove the necessary part of (a). Since $\rho_+(x, y) = 0$, by [6, Theorem 3.2], it follows that $x \perp_B y$. If $x$ is a smooth point then $x^\perp \cap \text{span}\{x, y\} = \{\alpha y : \alpha \in \mathbb{R}\}$. This situation is illustrated in Fig. 1 below. In this case, $(-x) \perp_B y$ is trivially true. Now, assume that $x$ is a non-smooth point. If $H = x^\perp \cap \text{span}\{x, y\}$ then by taking $\epsilon = 0$ in [18, Theorem 2.1], it follows that $H = K \cup (-K)$, where $K$ is a normal cone in $\text{span}\{x, y\}$. Let $K$ be generated by $v_1$ and $v_2$, i.e., $K = \{\alpha v_1 + \beta v_2 : \alpha \geq 0, \beta \geq 0\}$. This situation is illustrated in Fig. 2 below. We now prove that if $y \in K$ then $y = v_2$ and similar arguments will show that if $y \in -K$ then $y = -v_2$. Then $A = \{z \in S_X \cap K : z = -v_2, y \neq -v_2\}$. Let $y \in K$ and $y \neq v_2$. Then $A = \{z \in S_X \cap K : z = -v_2, y \neq -v_2\} \neq \emptyset$. Let $z \in A$. Then there exists
$f \in J(x)$ such that $f(z) = 0$. Clearly, the choice of $z$ implies that $f(y) > 0$. Thus, using Lemma 2.1, we arrive at a contradiction. Hence if $y \in K$ then $y = v_2$ and this proves that $(-x) \perp_B y$.

Let us now prove the sufficient part of (a). Let $(-x) \perp_B y$. Observe that by continuity of $\rho_+(x,.)$ in second variable, it follows that $\rho_+(x,z) = 0$ for some $z \in \text{span}\{x,y\}$. Now, by [6, Theorem 3.2], it follows that $z \in x^\perp \cap \text{span}\{x,y\}$ and by the homogeneity of Birkhoff–James orthogonality we may and do assume without loss of generality that $\|z\| = 1$. If $x$ is a smooth point then it follows trivially that $\{z \in x^\perp \cap \text{span}\{x,y\} : \|z\| = 1\} = \{\pm y\}$. Thus $\rho_+(x,y) = 0$ in this case. Now, we assume that $x$ is a non-smooth point. It follows from the homogeneity of Birkhoff–James orthogonality that $x \perp_B y$. If $H = x^\perp \cap \text{span}\{x,y\}$ then by taking $\epsilon = 0$ in [18, Theorem 2.1], it follows that $H = K \cup (-K)$, where $K$ is a normal cone in $\text{span}\{x,y\}$. Let $K$ be generated by $v_1$ and $v_2$, i.e., $K = \{\alpha v_1 + \beta v_2 : \alpha \geq 0, \beta \geq 0\}$. This situation is illustrated in Fig. 2. Clearly, $y = v_2$ if $y \in K$ and $y = -v_1$ if $y \notin K$. We now claim that $z = y$. Let $z \in K$ and suppose on the contrary that $z \neq y$, i.e., $z \neq v_2$. Then $z = \frac{\alpha v_1 + (1-\alpha)v_2}{\|v_2 + (1-\alpha)v_1\|}$, $t \in [0, 1)$. If $f \in J(x)$ is such that $f(v_2) = 0$ then clearly $f(z) > 0$. Now, using Lemma 2.1, we arrive at a contradiction. Hence $z = y$ and $\rho_+(x,y) = 0$.

Similarly, we can prove that $z = y$ when $z \in -K$.

Proof of (b) follows from similar arguments as above.

\[\square\]
3 Bounded bilinear operators

Throughout this section, we will consider normed linear spaces with dimension strictly greater than one. Our first result shows that if $\mathcal{T} \in \mathcal{K}(X \times Y, Z)$ is weak–weak continuous then the norm attainment set of $\mathcal{T}$ is non-empty, where $X$, $Y$ are reflexive Banach spaces and $Z$ is a normed linear space.

**Lemma 3.1** Let $X$, $Y$ be reflexive Banach spaces and let $Z$ be a normed linear space. If $\mathcal{T} \in \mathcal{K}(X \times Y, Z)$ is weak–weak continuous then $\mathcal{T}$ attains its norm.

**Proof** Without loss of generality we can assume that $\mathcal{T} \neq 0$. Let $\{(x_n, y_n)\} \subseteq S_X \times S_Y$ be a norming sequence for $\mathcal{T}$, i.e., $\|\mathcal{T}(x_n, y_n)\| \to \|\mathcal{T}\|$. Now, reflexivity of $X$ and $Y$ implies that $X \times Y$ is also reflexive. Thus there exists a weakly convergent subsequence of $\{(x_n, y_n)\}$, which we again denote by $\{(x_n, y_n)\}$. Let $(x_0, y_0) \in B_{X \times Y}$ be the weak limit of $\{(x_n, y_n)\}$. By our assumption, $\mathcal{T}$ is weak–weak continuous, hence $\{\mathcal{T}(x_n, y_n)\}$ weakly converges to $\mathcal{T}(x_0, y_0)$.

Now, compactness of $\mathcal{T}$ implies that there exists a convergent subsequence of $\{\mathcal{T}(x_n, y_n)\}$. Without loss of generality we again denote this convergent subsequence by $\{\mathcal{T}(x_n, y_n)\}$. Let $z \in Z$ such that $\mathcal{T}(x_n, y_n) \to z$.

Since $\mathcal{T}(x_0, y_0)$ is weak limit of $\{\mathcal{T}(x_n, y_n)\}$, we get $z = \mathcal{T}(x_0, y_0)$ and $\|z\| = \|\mathcal{T}(x_0, y_0)\|$. $(x_0, y_0) \in B_{X \times Y}$ implies that $\|x_0\| \leq 1$ and $\|y_0\| \leq 1$. We now claim that $\|x_0\| = \|y_0\| = 1$. If possible assume that $\|x_0\| < 1$. Since $\mathcal{T}$ is not the zero operator, it is easy to observe that $\|x_0\| > 0$ and $\|y_0\| > 0$. Now, $(\frac{x_0}{\|x_0\|}, \frac{y_0}{\|y_0\|}) \in S_X \times S_Y$ and it follows from the bilinearity of $\mathcal{T}$ that $\|\mathcal{T}(\frac{x_0}{\|x_0\|}, \frac{y_0}{\|y_0\|})\| = \frac{1}{\|x_0\|\|y_0\|}\|\mathcal{T}(x_0, y_0)\| > \|\mathcal{T}\|$, contradicting the definition of $\|\mathcal{T}\|$. Thus $\|x_0\| = 1$ and similar arguments will show that $\|y_0\| = 1$. Hence $(x_0, y_0) \in M_\mathcal{T}$ and this completes the proof of the lemma. \hfill $\square$

If $X$, $Y$, $Z$ are finite-dimensional Banach spaces then for any operator $\mathcal{T} \in \mathcal{B}(X \times Y, Z)$, it follows trivially that $M_\mathcal{T} \neq \emptyset$. For finite-dimensional Banach spaces $X$, $Y$, $Z$ and operators $\mathcal{T}, \mathcal{A} \in \mathcal{B}(X \times Y, Z)$, a complete characterization of $\mathcal{T} \perp_B \mathcal{A}$, involving the elements of $M_\mathcal{T}$, was obtained in [14, Theorem 2.4]. This characterization is an extension of the complete characterization of Birkhoff–James orthogonality of compact linear operators obtained in [17, Theorem 2.1] to the bilinear setting, in the finite-dimensional case. If $X$ is a reflexive Banach space and $Y$ is a normed linear space then for any compact linear operator $T$ from $X$ to $Y$, the norm attainment set of $T$ is non-empty. [17, Theorem 2.1] provides a complete characterization of $T \perp_B \mathcal{A}$ involving norm attaining elements of $T$, where $T, \mathcal{A}$ are compact linear operators from a reflexive Banach space $X$ to a normed linear space $Y$.

**Lemma 3.1** implies that if $\mathcal{T} \in \mathcal{K}(X \times Y, Z)$ is weak–weak continuous then $M_\mathcal{T} \neq \emptyset$, where $X$, $Y$ are reflexive Banach spaces and $Z$ is a normed linear space. Using this fact, our next result provides a complete characterization of $\mathcal{T} \perp_B \mathcal{A}$, where $X$, $Y$ are reflexive Banach spaces, $Z$ is a normed linear space and
$T, A \in \mathcal{K}(X \times Y, Z)$ are weak–weak continuous. We note that the following result extends [14, Theorem 2.4] and [17, Theorem 2.1].

**Theorem 3.1** Let $X, Y$ be reflexive Banach spaces and let $Z$ be a normed linear space. Let $T \in \mathcal{K}(X \times Y, Z)$ be weak–weak continuous. Then for any weak–weak continuous bilinear operator $A \in \mathcal{K}(X \times Y, Z)$, $T \perp_B A$ if and only if there exists $(x, y), (u, v) \in M_T$ such that $A(x, y) \in (T(x, y))^+$ and $A(u, v) \in (T(u, v))^-$.

**Proof** For the sufficient part, observe that if $A(x, y) \in (T(x, y))^+$ for some $(x, y) \in M_T$ then $\|T + \lambda A\| \geq \|T(x, y) + \lambda A(x, y)\| \geq \|T(x, y)\| = \|T\|$ for all $\lambda \geq 0$. Similarly if $A(u, v) \in (T(u, v))^-$ for some $(u, v) \in M_T$ then $\|T + \lambda A\| \geq \|T\|$ for all $\lambda \leq 0$.

For the necessary part, observe that for each $n \in \mathbb{N}$, $T + \frac{1}{n} A \in \mathcal{K}(X \times Y, Z)$ and $T + \frac{1}{n} A$ is weak–weak continuous. Thus from Lemma 3.1, it follows that $(x_n, y_n) \in M_{T + \frac{1}{n} A}$ for some $(x_n, y_n) \in S_X \times S_Y$. Since $X \times Y$ is a reflexive Banach space, $B_{X \times Y}$ is weakly compact. Without loss of generality we assume that $(x_n, y_n)$ converges weakly to $(x, y)$. Using arguments, as given in the proof of Lemma 3.1, we can show that $(x, y) \in M_T$. Also, by using the arguments in the proof of [17, Theorem 2.1], it follows that $A(x, y) \in (T(x, y))^+$. Similarly, considering the compact bilinear and weak–weak continuous operators $T - \frac{1}{n} A, n \in \mathbb{N}$, it follows that there exists some $(u, v) \in M_T$ and $A(u, v) \in (T(u, v))^-$.

If in addition to the assumptions of the above theorem, we assume that $M_T = \{(\pm x_0, \pm y_0)\}$ for some $(x_0, y_0) \in S_X \times S_Y$ then we obtain the following immediate corollary.

**Corollary 3.1** Let $X, Y$ be reflexive Banach spaces and let $Z$ be a normed linear space. Suppose $T, A \in \mathcal{K}(X \times Y, Z)$ are such that $T, A$ are weak–weak continuous and $M_T = \{((\pm x_0, \pm y_0))$ for some $(x_0, y_0) \in S_X \times S_Y$. Then $T \perp_B A$ if and only if $T(x_0, y_0) \perp_B A(x_0, y_0)$.

A complete characterization of Birkhoff–James orthogonality of bounded linear operators between normed linear spaces (without assuming any restriction on the norm attainment set of the operator) was obtained in [17, Theorem 2.4]. The following result shows that a similar result is also true for bounded bilinear operators. We would like to mention that in the bilinear case, the proof follows from similar arguments as given in [17, Theorem 2.4].

**Theorem 3.2** Let $X, Y, Z$ be normed linear spaces and let $T \in \mathcal{B}(X \times Y, Z)$. Then for any $A \in \mathcal{B}(X \times Y, Z)$, $T \perp_B A$ if and only if either of the conditions in (a) or in (b) hold true:

(a) There exists a sequence $\{(x_n, y_n)\}$ in $S_X \times S_Y$ such that $\|T(x_n, y_n)\| \longrightarrow \|T\|$ and $\|A(x_n, y_n)\| \longrightarrow 0$, as $n \longrightarrow \infty$. 

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(b) There exist two sequences \( \{(x_n, y_n)\}, \{(u_n, v_n)\} \) in \( S_X \times S_Y \) and two sequences of positive real numbers \( \{\epsilon_n\}, \{\delta_n\} \) such that

(i) \( \epsilon_n \to 0, \delta_n \to 0 \) as \( n \to \infty \).

(ii) \( \|T(x_n, y_n)\| \to \|T\| \) and \( \|T(u_n, v_n)\| \to \|T\| \), as \( n \to \infty \).

(iii) \( A(x_n, y_n) \in (T(x_n, y_n))^{\epsilon_n} \) and \( A(u_n, v_n) \in (T(u_n, v_n))^{-\delta_n} \) for all \( n \in \mathbb{N} \).

In [19, Theorem 3.3], Sain et al. observed that any non-zero compact linear operator from a reflexive Banach space \( X \) to a normed linear space \( Y \) is smooth if and only if \( M_T = \{\pm x_0\} \) for some \( x_0 \in S_X \) and \( T_x_0 \) is a smooth point in \( Y \). In [14, Theorem 2.5], Sain extended this result to smooth bounded bilinear operator from \( X \times Y \) to \( Z \), where \( X, Y, Z \) are finite-dimensional reflexive Banach spaces. Our next aim is to continue the study of smooth compact bilinear operators from \( X \times Y \) to \( Z \) with non-empty norm attainment set, where \( X, Y \) are infinite-dimensional reflexive Banach spaces and \( Z \) is any normed linear space. Observe that for any weak–weak continuous bilinear operator \( T \in \mathcal{K}(X \times Y, Z) \), it follows from Lemma 3.1 that \( M_T \neq \emptyset \), where \( X, Y \) are reflexive Banach spaces. For this reason we will restrict ourselves to weak–weak continuous compact bilinear operators. First we prove that if the norm attainment set for a smooth bilinear operator \( T \in \mathcal{B}(X \times Y, Z) \) is non-empty then \( M_T \) can have only four points.

**Theorem 3.3** Let \( X, Y, Z \) be normed linear spaces. If \( T \in \mathcal{B}(X \times Y, Z) \) is a smooth point and \( M_T \neq \emptyset \) then there exists \((x_0, y_0) \in S_X \times S_Y \) such that \( M_T = \{(\pm x_0, \pm y_0)\} \).

**Proof** Suppose on the contrary that \( M_T \neq \{(\pm x_0, \pm y_0)\} \) for any \((x_0, y_0) \in S_X \times S_Y \). Since \( M_T \neq \emptyset \), this implies that there exist \((x_1, y_1), (x_2, y_2) \in M_T \) such that either \( x_1 \neq \pm x_2 \) or \( y_1 \neq \pm y_2 \). Clearly, it is not possible that \( x_1, x_2 \) are linearly dependent in \( X \) and \( y_1, y_2 \) are linearly dependent in \( Y \). We now consider the following remaining cases.

**Case I:** We assume that \( x_1, x_2 \) are linearly independent in \( X \). In this case, we first assume that \( y_1, y_2 \) are linearly independent in \( Y \). There exist scalars \( a_0, b_0 \) such that \( x_1 \perp_B (a_0 x_1 + x_2) \) and \( y_1 \perp_B (b_0 y_1 + y_2) \). Let \( x'_1 = a_0 x_1 + x_2 \) and \( y'_1 = b_0 y_1 + y_2 \). Let \( H_1, G_1 \) be subspaces of \( 1 \)-codimension in \( X, Y \), respectively, such that \( x'_1 \in H_1 \), \( x_1 \perp_B H_1 \) and \( y'_1 \in G_1 \), \( y_1 \perp_B G_1 \). Suppose \( H_2, G_2 \) are subspaces of \( 1 \)-codimension in \( H_1, G_1 \), respectively, such that \( x'_2 \perp_B H_2 \) and \( y'_2 \perp_B G_2 \). Now, every element \( x \in X \), and \( y \in Y \) can be written uniquely as \( x = a_1 x_1 + \beta_1 x'_1 + h, \ y = a_2 y_1 + \beta_2 y'_1 + g \) for some scalars \( a_1, a_2, \beta_1, \beta_2, h \in H_2, \ g \in G_2 \). If \((x, y) \in S_X \times S_Y \), then \((x, y) = (a_1 x_1 + \beta_1 x'_1 + h, a_2 y_1 + \beta_2 y'_1 + g) \). Also, \( \|a_1 x_1 + \beta_1 x'_1 + h\| = \|a_1 x_1 + \beta_1 x'_1 + h\| = 1 \) and \( 1 = \|a_1 x_1 + \beta_1 x'_1 + h\| \geq \|\beta_1 x'_1 + h\| - \|a_1 x_1\| \geq |\beta_1| \|x'_1\| \). It follows from the linear independence of \( x_1 \) and \( x_2 \) that \( \|x'_1\| > 0 \). Thus \( |\alpha_1| = \|a_1 x_1\| \leq 1 \) and \( |\beta_1| \leq \frac{1}{\|x'_1\|}. \) Using similar arguments, we can show that \( |\alpha_2| \leq 1 \) and \( |\beta_2| \leq \frac{2}{\|y'_1\|}. \) Let \( \{h_\alpha : \alpha \in \Lambda_1\}, \{g_\beta : \beta \in \Lambda_2\} \) be a Hamel basis for \( H_2 \) and \( G_2 \), respectively. Then \( \{x_1, x'_2, h_\alpha : \alpha \in \Lambda_1\}, \{y_1, y'_2, g_\beta : \beta \in \Lambda_2\} \) form a Hamel basis for \( X \) and \( Y \), respectively. We now define a bilinear operator \( A \) from \( X \times Y \) to \( Z \) by:
\[ \mathcal{A}(x_1, y_1) = \mathcal{T}(x_1, y_1), \mathcal{A}(x_1, y_2') = b_0 \mathcal{T}(x_1, y_1), \mathcal{A}(x_1, g_\beta) = 0, \]
\[ \mathcal{A}(x_2', y_1) = a_0 \mathcal{T}(x_1, y_1), \mathcal{A}(x_2', y_2') = a_0 b_0 \mathcal{T}(x_1, y_1), \mathcal{A}(x_2', g_\beta) = 0, \]
\[ \mathcal{A}(h_a, y_1) = 0, \mathcal{A}(h_a, y_2') = 0, \mathcal{A}(h_a, g_\beta) = 0 \]

for all \( \alpha \in \Lambda_1 \) and \( \beta \in \Lambda_2 \).

We claim that \( \mathcal{A} \in \mathcal{B}(X \times Y, Z) \). Observe that if \((x, y) = (a_1 x_1 + \beta_1 x_2' + h, \alpha_2 y_1 + \beta_2 y_2' + g) \in S_X \times S_Y \) then
\[ \mathcal{A}(x, y) = a_1 \alpha_2 \mathcal{T}(x_1, y_1) + \alpha_1 \beta_2 b_0 \mathcal{T}(x_1, y_1) + \beta_1 \alpha_2 a_0 \mathcal{T}(x_1, y_1) \]
\[ + \beta_1 \beta_2 a_0 b_0 \mathcal{T}(x_1, y_1) \]
and thus \( \mathcal{A} \in \mathcal{B}(X \times Y, Z) \). Also, it follows from the definition of \( \mathcal{A} \) that \( \mathcal{A}(x_2, y_2) = 0 \).

We now define \( \mathcal{B} \in \mathcal{B}(X \times Y, Z) \) by \( \mathcal{B} = \mathcal{T} \setminus \mathcal{A} \). Observe that \( \mathcal{B}(x_1, y_1) = 0 \). Thus
\[ \mathcal{T}(x_2, y_2) \perp_B \mathcal{A}(x_2, y_2), \mathcal{T}(x_1, y_1) \perp_B \mathcal{B}(x_1, y_1) \]
and hence \( \mathcal{T} \perp_B \mathcal{A} \) and \( \mathcal{T} \perp_B \mathcal{B} \). Now, by the smoothness of \( \mathcal{T} \) we have \( \mathcal{T} \perp_B (\mathcal{A} + \mathcal{B}) \). This leads to a contradiction as \( \mathcal{A} + \mathcal{B} = \mathcal{T} \). Thus in this case we get
\( M_T = \{ (\pm x_0, \pm y_0) \} \) for some \((\pm x_0, \pm y_0) \in S_X \times S_Y \).

Now, we assume that \( y_1, y_2 \) are linearly dependent in \( Y \). Then either \( y_1 = y_2 \) or \( y_1 = -y_2 \). Suppose \( G_1 \) be a subspace of 1-codimension in \( Y \) such \( y_1 \perp_B G_1 \). If \( \{g_\beta : \beta \in \Lambda_2\} \) is a Hamel basis for \( G_1 \) then \( \{y_1, g_\beta : \beta \in \Lambda_2\} \) forms a Hamel basis for \( Y \). Using the same notations as in the previous case, let \( \{x_1, x_2', h_\alpha : \alpha \in \Lambda_1\} \) be a Hamel basis for \( X \). We now define a bilinear operator \( \mathcal{A} \) from \( X \times Y \) to \( Z \) by:
\[ \mathcal{A}(x_1, y_1) = \mathcal{T}(x_1, y_1), \mathcal{A}(x_1, g_\beta) = 0, \mathcal{A}(x_2', y_1) = a_0 \mathcal{T}(x_1, y_1), \]
\[ \mathcal{A}(x_2', g_\beta) = 0, \mathcal{A}(h_\alpha, y_1) = 0, \mathcal{A}(h_\alpha, g_\beta) = 0 \]
for all \( \alpha \in \Lambda_1 \) and \( \beta \in \Lambda_2 \). Using the arguments similar to the previous case we can show that \( \mathcal{A} \in \mathcal{B}(X \times Y, Z) \). We now define \( \mathcal{B} \in \mathcal{B}(X \times Y, Z) \) by \( \mathcal{B} = \mathcal{T} \setminus \mathcal{A} \). Observe that \( \mathcal{A}(x_2, y_2) = 0 \) and \( \mathcal{B}(x_1, y_1) = 0 \). Thus in this case also we obtain \( \mathcal{T} \perp_B (\mathcal{A} + \mathcal{B}) \). This leads to a contradiction as \( \mathcal{T} = \mathcal{A} + \mathcal{B} \). Thus \( M_T = \{ (\pm x_0, \pm y_0) \} \) for some \((\pm x_0, \pm y_0) \in S_X \times S_Y \).

Case II: We are only left with the case that \( x_1, x_2 \) are linearly dependent in \( X \) and \( y_1, y_2 \) are linearly independent in \( Y \). In this case also, using the arguments as in case I, it follows that \( M_T = \{ (\pm x_0, \pm y_0) \} \) for some \((\pm x_0, \pm y_0) \in S_X \times S_Y \).

This completes the proof of the theorem. \( \square \)

If in addition to the conditions of Lemma 3.1, we assume \( \mathcal{T} \) to be smooth then we immediately obtain the following result on the norm attainment set of \( \mathcal{T} \).

**Corollary 3.2** Let \( X, Y \) be reflexive Banach spaces and let \( Z \) be a normed linear space. Suppose \( \mathcal{T} \in \mathcal{K}(X \times Y, Z) \) is a smooth and weak–weak continuous bilinear operator. Then \( M_T(x_0, y_0) = \{ (\pm x_0, \pm y_0) \} \) for some \((x_0, y_0) \in S_X \times S_Y \).
We now prove the analogous result of [19, Theorem 3.3] and [14, Theorem 2.5] for weak–weak continuous compact bilinear operators from \(X \times Y\) to \(Z\), where \(X, Y\) are reflexive Banach spaces and \(Z\) is a normed linear space.

**Theorem 3.4** Let \(X, Y\) be reflexive Banach spaces and let \(Z\) be a normed linear space. Let \(T \in \mathcal{K}(X \times Y, Z)\) be weak–weak continuous. Then \(T\) is a smooth point in \(\mathcal{B}(X \times Y, Z)\) if and only if there exists some \((x_0, y_0) \in S_X \times S_Y\) such that \(M_T = \{(\pm x_0, \pm y_0)\}\) and \(T(x_0, y_0)\) is a smooth point in \(Z\).

**Proof** Let us first prove the sufficient part of the theorem. Let \(A_1, A_2 \in \mathcal{B}(X \times Y, Z)\) be such that \(T \perp_B A_1\) and \(T \perp_B A_2\). It follows from Corollary 3.1 that 

\[
T(x_0, y_0) \perp_B A_1(x_0, y_0) \quad \text{and} \quad T(x_0, y_0) \perp_B A_2(x_0, y_0).
\]

Now, from the smoothness of \(T(x_0, y_0)\), we obtain that 

\[
T(x_0, y_0) \perp_B (A_1(x_0, y_0) + A_2(x_0, y_0)).
\]

Since \((x_0, y_0) \in M_T\), this implies that \(T \perp_B (A_1 + A_2)\).

Let us now prove the necessary part of the theorem. It follows from Lemma 3.1 and Theorem 3.3 that \(M_T = \{(\pm x_0, \pm y_0)\}\) for some \((x_0, y_0) \in S_X \times S_Y\). Suppose on the contrary that \(T(x_0, y_0)\) is not a smooth point in \(Z\). Then there exist \(z_1, z_2 \in Z\) such that 

\[
T(x_0, y_0) \perp_B z_1 \quad \text{and} \quad T(x_0, y_0) \perp_B z_2.
\]

Suppose \(H_1, G_1\) are subspaces of 1-codimension in \(X, Y\), respectively such that \(x_0 \perp_B H_1\) and \(y_0 \perp_B G_1\). Let \(\{h_\alpha : \alpha \in \Lambda_1\}, \{g_\beta : \beta \in \Lambda_2\}\) be a Hamel basis for \(H_1\) and \(G_1\), respectively. Then \(\{x_0, h_\alpha : \alpha \in \Lambda_1\}, \{y_0, g_\beta : \beta \in \Lambda_2\}\) form a Hamel basis for \(X\) and \(Y\), respectively. Now, we define \(\mathcal{A}, B \in \mathcal{K}(X \times Y, Z)\) by

\[
\mathcal{A}(x_0, y_0) = z_1, \quad \mathcal{A}(x_0, g_\beta) = 0, \quad \mathcal{A}(h_\alpha, y_0) = 0, \quad \mathcal{A}(h_\alpha, g_\beta) = 0,
\]

\[
B(x_0, y_0) = z_2, \quad B(x_0, g_\beta) = 0, \quad B(h_\alpha, y_0) = 0, \quad B(h_\alpha, g_\beta) = 0
\]

for all \(\alpha \in \Lambda_1, \beta \in \Lambda_2\). From the definition of \(\mathcal{A}\) and \(B\), it follows that \(T \perp_B \mathcal{A}\) and \(T \perp_B B\). Now, from the smoothness of \(T\) we get that \(T \perp_B (\mathcal{A} + B)\). Observe that \(\mathcal{A}\) and \(B\) are also weak–weak continuous. Thus it follows from Theorem 3.1 that 

\[
T(x_0, y_0) \perp_B (\mathcal{A}(x_0, y_0) + B(x_0, y_0)).
\]

This contradicts that \(T(x_0, y_0) \perp_B (z_1 + z_2)\). Hence \(T(x_0, y_0)\) is a smooth point in \(Z\). \(\square\)

A complete characterization of smoothness of bounded linear operators between normed linear spaces (without assuming any restriction on the norm attainment set of the operator) in terms of operator Birkhoff–James orthogonality and s.i.p. was obtained in [19, Theorem 2.1]. We observe that the same characterization extends to the setting of bounded bilinear operators. For the bilinear case, the proof follows from similar arguments, as given in [19, Theorem 2.1]. Therefore, we omit the proof of the following result.

**Theorem 3.5** Let \(X, Y, Z\) be normed linear spaces and let \(T \in \mathcal{B}(X \times Y, Z)\) be non-zero. Then the following conditions are equivalent:

\(\square\)
Theorem 3.6 Let $X$, $Y$ be reflexive Banach spaces and let $Z$ be a Banach space. Let $T \in K(X \times Y, Z)$ be weak–weak continuous. Then for any weak–weak continuous bilinear operator $A \in K(X \times Y, Z)$, $T \perp_B^e A$ if and only if there exist $(x_1, y_1), (x_2, y_2) \in M_T$ such that $\|T(x_1, y_1) + \lambda A(x_1, y_1)\|^2 \geq \|T\|^2 - 2\varepsilon \lambda \|T\|\|A\|$ for all $\lambda \geq 0$ and $\|T(x_2, y_2) + \lambda A(x_2, y_2)\|^2 \geq \|T\|^2 - 2\varepsilon \lambda \|T\|\|A\|$ for all $\lambda \leq 0$.

Theorem 3.7 Let $X$, $Y$, $Z$ be normed linear spaces and let $T \in B(X \times Y, Z)$. Then for any $A \in B(X \times Y, Z)$, $T \perp_B^e A$ if and only if either of the conditions in (a) or in (b) hold true:

(a) There exists a sequence $\{(x_n, y_n)\}$ in $S_X \times S_Y$ such that $\|T(x_n, y_n)\| \rightarrow \|T\|$ and $\|A(x_n, y_n)\| \rightarrow \varepsilon \|A\|$ as $n \rightarrow \infty$.

(b) There exists two sequences $\{(x_n, y_n)\}, \{(u_n, v_n)\}$ in $S_X \times S_Y$ and two sequences of positive real numbers $\{\varepsilon_n\}, \{\delta_n\}$ such that

(i) $\varepsilon_n \rightarrow 0, \delta_n \rightarrow 0, \|T(x_n, y_n)\| \rightarrow \|T\|$ and $\|T(u_n, v_n)\| \rightarrow \|T\|$ as $n \rightarrow \infty$.

\[
\|T(x_n, y_n) + \lambda A(x_n, y_n)\|^2 \geq (1 - \varepsilon_n^2)\|T(x_n, y_n)\|^2
- 2\varepsilon \sqrt{1 - \varepsilon_n^2}\|T(x_n, y_n)\|\|\lambda A\|
\]
for all $\lambda \geq 0$.

(ii) $\|T(u_n, v_n) + \lambda A(u_n, v_n)\|^2 \geq (1 - \delta_n^2)\|T(u_n, v_n)\|^2
- 2\varepsilon \sqrt{1 - \delta_n^2}\|T(u_n, v_n)\|\|\lambda A\|
\]
for all $\lambda \leq 0$.

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