RESOLUTIONS OF MODULES WITH INITIALLY LINEAR SYZYGIES

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Abstract. We introduce the class of modules with initially linear syzygies, a class including ideals with linear quotients, and study their minimal resolutions. Using a contracting homotopy for the resolutions, we see that the minimal resolution of a matroidal monomial ideal admits a DGA structure.

1. Introduction

Let $k$ be a field and $S = k[x_1, \ldots, x_n]$, the polynomial ring over $k$. In this paper we introduce the notion of modules with initially linear syzygies, and study the structure of their minimal resolutions. We show how to construct the minimal resolution of a module with initially linear syzygies using discrete Morse theory, and we then show that such modules are componentwise linear; finally, at the end of the paper we investigate multiplicative properties, and we show that the minimal free resolution of $S/I$ admits a differential graded algebra structure, where $I$ is either a stable monomial ideal or a squarefree matroidal ideal. The result for stable ideals is not new, it has been shown by Peeva [Pee96], but we obtain a simpler proof. The result for squarefree matroidal ideals is new, however. We finish the paper by calculating the product on a part of the minimal resolution of the ideal coming from the Fano matroid as an illustration of the results.

2. Modules with initially linear syzygies

Definition 1. A presentation of a finitely generated graded $S$-module $M$,

\[ 0 \rightarrow \text{Ker} \, \eta \rightarrow \bigoplus_i S \cdot g_i \xrightarrow{\eta} M \rightarrow 0, \]

is said to have initially linear syzygies with respect to a term order $\prec$ on the free module $\bigoplus_i S \cdot g_i$, if $\text{Ker} \, \eta \subseteq \bigoplus_i m \cdot g_i$, and if the initial module $\text{in}_\prec(\text{Ker} \, \eta)$ is generated by terms of the form $x_j g_i$.

We will say that $M$ has initially linear syzygies if it has such a presentation for some choice of generating set $\{g_1, \ldots, g_n\}$ and term order $\prec$.

Ideals with initially linear syzygies generalise ideals with linear quotients; let us recall that an ideal $I$ has linear quotients if there are elements $f_1, f_2, \ldots, f_n$ such that $I = (f_1, \ldots, f_n)$, and for each $1 < i \leq n$, the colon ideal $(f_1, \ldots, f_{i-1}) : f_i$ is generated by linear forms. It is not hard to see that a homogeneous ideal has linear quotients if and only if it has initially linear syzygies with respect to a position over term order, which is a term order on a free module $\bigoplus_i S \cdot g_i$ such that $g_i < g_j$ implies...
that $x^\alpha g_i \prec x^\beta g_j$ for all $\alpha$ and $\beta$. Ideals with linear quotients in turn generalise shellable monomial ideals, a concept introduced by Batzies and Welker [BW02]. A monomial ideal $I$ with minimal monomial generators $m_1, \ldots, m_t$ is shellable if there is a total order “$\sqsubseteq$” on $m_1, \ldots, m_t$ such that for $m_j, m_i$ with $m_j \sqsubseteq m_i$ there is an $m_k$ such that $m_k \sqsubseteq m_i$ and $x_g(m_k, m_i) m_i = \text{lcm}(m_k, m_i)$ divides $\text{lcm}(m_j, m_i)$ for some index $g(m_k, m_i)$. Thus a shellable monomial ideal has linear quotients and then also initially linear syzygies.

The families of monomial ideals we are particularly interested in are the stable ideals, and the squarefree matroidal ideals, which are ideals where the supports of the minimal generators form the bases of a matroid.

Since we will extensively use algebraic Morse theory, we will below briefly review our terminology. For details on algebraic Morse theory, see [JW09], [Jon03], [Koz05] and [Skö06]. By a based complex of $R$-modules we mean a chain complex $K_\bullet$ of $R$-modules together with a direct sum decomposition $K_n = \bigoplus_{\alpha \in I_n} K_\alpha$ where $\{I_n\}$ is a family of mutually disjoint index sets. For $f : \bigoplus_n K_n \rightarrow \bigoplus_n K_n$ a graded map, we write $f_{\alpha, \beta}$ for the component of $f$ going from $K_\alpha$ to $K_\beta$, and given a based complex $K_\bullet$, we construct a digraph $\Gamma_{K_\bullet}$ with vertex set $V = \bigcup_n I_n$ and with a directed edge $\alpha \rightarrow \beta$ whenever the component $d_{\beta, \alpha}$ is non-zero.

A partial matching on a digraph $D = (V, E)$ is a subset $A$ of the edges $E$ such that no vertex is incident to more than one edge in $A$. In this situation we define the new digraph $D^A = (V, E^A)$ to be the digraph obtained from $D$ by reversing the direction of each arrow in $A$. Given the matching $A$, we define the sets $A^+, A^-$ and $A^0$ by letting $A^+$ be the set of vertices that are targets of a reversed arrow from $A$; $A^-$ be the set of vertices that are sources of a reversed arrow from $A$; and $A^0$ to be the vertices that are not incident to an arrow from $A$. We call a partial matching $A$ supported on the digraph $\Gamma_{K_\bullet}$ a Morse matching if, for each edge $\alpha \rightarrow \beta$ in $A$, the corresponding component $d_{\beta, \alpha}$ is an isomorphism, and furthermore there is a well founded partial order $\prec$ on each $I_n$ such that $\gamma \prec \alpha$ whenever there is a path $\alpha^{(n)} \rightarrow \beta \rightarrow \gamma^{(n)}$ in $\Gamma_{K_\bullet}^A$.

3. The minimal resolution

In this section we start by observing that a finitely generated $S$-module $M$ has a free resolution given by a two-sided Koszul complex $G_\bullet$. The modules in this resolution are not even finitely generated, so it is far from being minimal. In this resolution we see that we can find a matching that gives us a projection that allows us to find the minimal resolution $F_\bullet$ of $M$ as a direct summand of $G_\bullet$, in the case when $M$ has initially linear syzygies. We then give a description of the differential in terms of reductions following Jüllenhack and Welker [JW09], and then show that in some cases the differential is of Eliahou–Kervaire type. We round off by showing that modules with initially linear syzygies are componentwise linear.

Let $M$ be a finitely generated $S$-module, let $V$ be the $k$-vector space with basis $e_1, \ldots, e_n$ and let $F_\bullet$ be the chain complex with modules $F_p = S \otimes \text{Alt}^p V \otimes M$. For an element $e_{i_1} \wedge \cdots \wedge e_{i_d}$ of $\text{Alt}^d V$ with $I = \{i_1, \ldots, i_d\} \subseteq [n]$ and $i_1 < \cdots < i_d$, we will write $e_I$, and we will also write $e_{1m}$ for the element $1 \otimes e_I \otimes m$; as $m$ ranges over a $k$-basis of $M$, these elements obviously form an $S$-basis for $F_n$. The differential $d_n : S \otimes \text{Alt}^n V \otimes M \rightarrow S \otimes \text{Alt}^{n-1} V \otimes M$ is defined on the basis elements by

$$d(e_{1m}) = \sum_{i \in I} e(i; I)(x_i e_{I\setminus i} m - e_{I\setminus i} x_i m),$$
where the sign $\varepsilon(i; I)$ is defined by
$$\varepsilon(i; I) = (-1)^{|\{j \in I, j < i\}|}.$$

**Lemma 1.** The complex $F_\bullet$ is a free resolution of $M$.

**Proof.** It is obvious that $H_0(F_\bullet) \simeq M$, so we just have to prove that $H_i(F_\bullet) = 0$ for $i \geq 1$.

For a $k$-vector space basis $B$ of $M$ we consider $F_\bullet$ as a based complex of $k$-vector spaces via the natural decomposition
$$S \otimes \text{Alt}^k V \otimes M \simeq \bigoplus_{\alpha \in \mathbb{N}^n, I \subseteq [n], m \in B} k \cdot x^\alpha e_I m.$$

For each $i$ we now define the following subset of the vertices in the digraph $\Gamma_{F_\bullet}$:
$$V_i = \{x^\alpha e_I m \mid \text{deg } x^\alpha + |I| = i\}.$$

Now, construct a partial matching $E_i$ on the subgraph $\Gamma_{F_\bullet}|_{V_i}$ consisting of the edges
$$E_i = \{x^\alpha e_I m \to x^\alpha x_i e_{I \setminus i} m \mid i = \min(\text{supp } \alpha \cup I)\}.$$

It should be clear that if $\alpha \to \beta \in E_i$ for $\alpha, \beta \in V_i$ then all $\gamma \in V_i$, $\gamma \neq \beta$ with an edge $\alpha \to \gamma$ are unmatched, so $E_i$ is a Morse matching on $\Gamma_{F_\bullet}|_{V_i}$, and since for all edges $\alpha \to \beta$ with $\alpha \in V_i$ and $\beta \in V_j$ we have $j \leq i$, we get from [Skö06, Lemma 7] that $\bigcup_i E_i$ is a Morse matching on $\Gamma_{F_\bullet}$. The claim of the lemma now follows from [Skö06, Theorem 1] since there are no $E$-critical vertices in degree 1 and higher. \qed

Now, we will construct a matching on $G_\bullet$, viewed as a based complex of $S$-modules, that will give us a splitting homotopy $\varphi$ of $G_\bullet$. Using the homotopy $\varphi$ we can then describe the minimal resolution of $M$. Given a presentation (1), we define for a basis element $g_i$ of the free module $\bigoplus_j S \cdot g_j$, its critical and non-critical indices by
$$\text{crit}(g_i) = \{j \mid x_j g_i \in \text{in}_{<}(\text{Ker } \eta)\}, \quad \text{ncrit}(g_i) = [n] \setminus \text{crit}(g_i).$$

Suppose $M$ has initially linear syzygies with a presentation as in (1), we then have a $k$-basis for $M$, (abusing notation by writing $x^\alpha g_i$ for $\eta(x^\alpha g_i)$),
$$\{x^\alpha g_i \mid \text{supp } \alpha \subseteq \text{ncrit } g_i\}.$$

We consider the resolution $F_\bullet$ as a based complex of $S$-modules via the decomposition
$$F_n \simeq \bigoplus_{I, \alpha, i, |I| = n, 	ext{supp } \alpha \subseteq \text{ncrit } g_i} S \cdot e_I x^\alpha g_i.$$

For each term $m$ in $\bigoplus_i S \cdot g_i$, we define a subset of the vertices of $\Gamma_F$
$$V_m = \{S \cdot e_I x^\alpha g_j \mid x_I x^\alpha g_j = m\},$$

and for each such $m$, we will now define a partial matching $E_m$ on the digraph $\Gamma_{F_\bullet}|_{V_m}$ by
$$E_m = \{S \cdot e_I x^\alpha g_j \to S \cdot e_{I \setminus i} x_i x^\alpha g_j \mid i = \max(|I \cup \text{supp } \alpha \cap \text{ncrit } g_j|)\}.$$

**Lemma 2.** The set $E = \bigcup_m E_m$ is a Morse matching on the digraph $\Gamma_{F_\bullet}$. The set of unmatched vertices consists of all $S \cdot e_I g_j$ where $I \subseteq \text{crit } g_j$. 

Proof. It is clear that $E_m$ is a partial matching on $\Gamma_{F_\bullet}|_{V_m}$ and along the same lines as in the proof of Lemma 1, if $\alpha \to \beta \in E_m$ for $\alpha, \beta \in V_m$ then all $\gamma \in V_m$ such that $\gamma \neq \beta$ and with an edge $\alpha \to \gamma$ are unmatched. Thus, there is no oriented cycle in the finite digraph $\Gamma_{F_\bullet}|_{V_m}$, and by [Skö06, Lemma 1], $E_m$ is a Morse matching on $\Gamma_{F_\bullet}|_{V_m}$. Whenever $\alpha \to \beta \in E_m$ with $\alpha \in V_m^1$ and $\beta \in V_m^2$ we have $m_2 \preceq m_1$, so by [Skö06, Lemma 7], $\bigcup_m E_m$ is a Morse matching on the full graph $\Gamma_{F_\bullet}$. □

With the above result, we can now define an $S$-linear splitting homotopy $\varphi$ on the resolution $F_\bullet$ that allows us to construct a smaller resolution $G_\bullet$. We will first give an explicit recursive definition of $\varphi$. In the definition we use the following notation due to Knuth: When $P$ is some proposition, then $[P] = 1$ if $P$ is true, and $[P] = 0$ if $P$ is false. Let us start by defining the two helper functions $\iota$ and $\varphi_0$ by

$$
\iota(\alpha) = \max(\text{supp } \alpha)
$$

$$
\varphi_0(e_{I}x^\alpha g_j) = [\iota(\alpha) > \max(\bigcup I \cap \text{ncrit } g_i)] \varepsilon(\iota(\alpha); I \cup \iota(\alpha)) \cdot \varepsilon I \cup \iota(\alpha) \cdot e_{I}x^\alpha g_j
$$

then

$$
\varphi(e_{I}x^\alpha g_j) = \varphi_0(e_{I}x^\alpha g_j) - \varphi(d\varphi_0(e_{I}x^\alpha g_j) - e_{I}x^\alpha g_j).
$$

Let $\pi$ be the projection

$$
\pi : \bigoplus_{I, \alpha, j} S \cdot e_I x^\alpha g_j \to \bigoplus_{I, j, \alpha} S \cdot e_I g_j,
$$

Now we can define a complex $G_\bullet$ by letting

$$
G_n = \bigoplus_{I \subseteq \text{crit}(g_j)} S \cdot e_I g_j
$$

and defining the differential by $dG = \pi(dF - dF \varphi dF)$; we can then formulate the following result that generalises work of Batzies and Welker [BW02] and Herzog and Takayama [HT02].

**Theorem 1.** Let $M$ be a module with initially linear syzygies, then $G_\bullet$ is the minimal free resolution of $M$.

*Proof.* Follows from applying [Skö06, Theorem 2] to the matching $E$ and the resolution $F_\bullet$. □

From Theorem 1, we can now immediately deduce the following two corollaries giving the projective dimension and Castelnuovo–Mumford regularity of a module with initially linear syzygies.

**Corollary 1.** If $M$ is minimally generated by $m_1, \ldots, m_r$, and $M$ has initially linear syzygies then $p \dim M = \max_j |\text{crit } e_j|$

**Corollary 2.** If $M$ is minimally generated by $m_1, \ldots, m_r$, and $M$ has initially linear syzygies then $\text{reg } M = \max_{i,j}(\deg m_i - \deg m_j)$

By specialising the last corollary we get:

**Corollary 3.** A graded module generated by homogeneous elements of the same degree with initially linear syzygies has a linear resolution.
Another, non-recursive, way of describing the differential in $F_*$ is in terms of reductions, as used by Jöllenbeck and Welker [JW09] in their description of the differential in the Anick resolution. In the Morse digraph $\Gamma_{\text{crit}}^E$, we define an elementary reduction path to be a zig-zag path of length 2

$$\alpha_0 \to \beta \to \alpha_1,$$

where $\alpha_0, \alpha_1$ are in degree $i$, and $\beta$ is in degree $i-1$ if $\alpha_0$ is in $E^0 \cup E^+$, and in degree $i+1$ if $\alpha \in E^-$. To such a path we assign the corresponding elementary reduction which is the map

$$\rho_{\alpha_1, \alpha_0} = \begin{cases} d^{-1}_{\beta, \alpha_1} \circ d_{\beta, \alpha_0}, & \text{if } \alpha_0 \in E^0 \cup E^+, \\ d_{\alpha_1, \beta} \circ d_{\alpha_0, \beta}, & \text{if } \alpha_0 \in E^-. \end{cases}$$

The matching condition implies that there is at most one elementary reduction path from $\alpha_0$ to $\alpha_1$. In general, we let a reduction path be a composition of zero or more elementary reduction paths

$$p = \alpha_0 \to \beta_1 \to \alpha_1 \to \cdots \to \beta_n \to \alpha_n, \quad n \geq 0,$$

and the corresponding reduction to be the composition

$$\rho_p = \rho_{\alpha_n, \alpha_{n-1}} \circ \rho_{\alpha_{n-1}, \alpha_{n-2}} \circ \cdots \circ \rho_{\alpha_1, \alpha_0}.$$

We denote the set of all reduction paths from $\alpha$ to $\beta$ by $[\alpha \rightsquigarrow \beta]$. From [Skö06, Lemma 5] we can now conclude that for a basis element $e_{I \forall g_\lambda}$, we can write

$$d_\lambda(e_{I \forall g_\lambda}) = \sum_{e_{I \forall g_\lambda}} \sum_{p \in [\epsilon \forall x^\beta \forall g_\lambda \mapsto e_{K \forall g_\lambda}]} \rho_p d_{e_{I \forall x^\alpha \forall g_\lambda} 
\to e_{I \forall x^\beta \forall g_\lambda}}(e_{I \forall g_\lambda}).$$

In our case we can see that we can divide the elementary reduction paths originating in a vertex in $E^-$ into two types. We say that an elementary reduction path is of type 1 when it is of the form

$$e_{I \forall x^\alpha} g_k \to e_{I \forall x^\beta} g_k,$$

where $j$ is the maximal element in $(\text{supp} \alpha \cup I) \cap \text{ncrit} \ g_k$. The corresponding reduction map is

$$\rho(e_{I \forall x^\alpha} g_k) = \varepsilon(j; I \cup j) \varepsilon(i; I \cup j) x^\alpha i e_{(I \cup j) \cup j} x^\beta i g_k.$$

An elementary reduction path is of type 2 if it is of the form

$$e_{I \forall x^\alpha} g_k \to e_{I \forall x^\beta} g_k \to e_{(I \cup j) \cup j} x^\beta i g_k,$$

where $x^\beta i g_k$ appears with nonzero coefficient $\lambda_{i,j,k, \alpha, \beta, l}$ in $\text{nf}((x_i x^\alpha/x_j) g_k)$, and $j$ is the maximal element in $(\text{supp} \alpha \cup I) \cap \text{ncrit} \ g_k$. The corresponding reduction map is

$$\rho(e_{I \forall x^\alpha} g_k) = -\varepsilon(j; I \cup j) \varepsilon(i; I \cup j) \lambda_{i,j,k, \alpha, \beta, l} e_{(I \cup j) \cup j} x^\beta i g_k.$$

We will now see that in well-behaved cases, there is a more explicit description of the differential.
Definition 2. If $M$ has a presentation with initial linear syzygies such that for every generator $g$ we have that

$$\text{crit}(\text{nf}(x_i g)) \subseteq \text{crit}(g),$$

we say that $M$ is crit-monotone. (We interpret $\text{crit}(\sum_{j \in J} p_j e_j)$ as $\bigcup_{j \in J} \text{crit}(e_j)$ if there are no redundant terms in the sum.)

It is easy to see that stable monomial ideals are crit-monotone, and in [HT02], Herzog and Takayama prove that matroidal ideals are crit-monotone, and they show that the differential in the minimal resolution of crit-monotone monomial ideals is of Eliahou–Kervaire type. Below, we will generalise their result to general crit-monotone modules with initially linear syzygies.

For a basis element $e_I g$ with $I = \{i_1, \ldots, i_n\}$ we define maps $d^L_j, d^R_j$, the left and right components of the differential, for $1 \leq j \leq n$ by

$$d^L_j(e_I g) = x_{i_j} e_{I \setminus i_j} g$$

and

$$d^R_j(e_I g) = \sum_k [I \setminus i_j \subseteq \text{crit}(g_k)] p_k e_{I \setminus i_j} g_k,$$

where $\text{nf}(x_i g) = \sum_k p_k g_k$ for $p_k \in S$ and $g_k$ a basis element, and we again make use of the Knuth notation.

Theorem 2. Let $M$ be a crit-monotone module with initially linear syzygies, then the differential $d$ in the minimal resolution $G_\bullet$ is given in degree $n$ by

$$d = \sum_{i=1}^n (-1)^{i-1} (d^L_i - d^R_i)$$

Proof. Consider an elementary reduction path, $e_I x^\alpha g_i \rightarrow e_J x^\beta g_j$, it is then easy to see that

$$|J \cap \text{crit } g_j| \leq |I \cap \text{crit } g_i|.$$ 

This observation together with the fact that we are only interested in reduction paths ending in vertices $e_J g$ with $J \subseteq \text{crit } g$, gives that the only non-trivial reduction paths appearing in the sum (2) are concatenations of elementary reduction paths of type 1 that are of the form

$$e_I x^\alpha g_k \rightarrow e_{I \cup J} \frac{x^\alpha}{x_j} g_k \rightarrow e_I x^\alpha g_k,$$

which have coefficient $x_j$ in the corresponding reduction. The reduction paths of length 0 that appear in the sum (2) thus contribute $\sum_{i=1}^n (-1)^{i-1} d^L_i$ to the differential, and the concatenations of reduction paths mentioned above contribute $-\sum_{i=1}^n (-1)^{i-1} d^R_i$, which sums to the desired formula. \hfill $\square$

Herzog and Hibi have introduced the concept of componentwise linear ideals [HH99]. For a graded module $M$ we let $M_{(d)}$ be the module generated by all homogeneous elements of degree $d$ in $M$; using this notation we say that $M$ is componentwise linear if the modules $M_{(d)}$ have linear resolutions for all $d$.

Theorem 3. If the finitely generated graded module $M$ has initially linear syzygies, then $M_{(d)}$ has initially linear syzygies for all $d$. 
Proof. By the hypothesis, there are homogeneous elements $m_1, m_2, \ldots, m_g$, that generate $M$, and a presentation

$$0 \to \text{Ker } \eta \to \bigoplus_{i=1}^g S \cdot e_i \xrightarrow{\eta} M \to 0$$

with $\eta(e_i) = m_i$, together with a term order ‘$<$’ on $F = \bigoplus_{i=1}^g S \cdot e_i$ such that Ker $\eta$ has a Gröbner basis $G$ consisting of initially linear terms.

We will now construct an explicit Gröbner basis $G_d$ for the syzygies of $M(d_g)$, and we will start by constructing a presentation of $M(d_g)$. This module is minimally generated by the images of the elements $x^\alpha e_i$ of degree $d$ in $M$ that are irreducible with respect to $G$, and has a $k$-basis given by all the irreducible elements $x^\alpha e_i$ where deg $e_i \leq d$. Thus, we consider the free module with basis elements $t^\alpha e_i$ where deg $t^\alpha + \deg e_i = d$ and $x^\alpha e_i$ irreducible with respect to $G$, and define the map $\eta_d$ by $\eta_d(t^\alpha e_i) = x^\alpha \eta(e_i) = x^\alpha m_i$. We now have the presentation

$$0 \to \text{Ker } \eta_d \to \bigoplus S \cdot t^\alpha e_i \xrightarrow{\eta_d} M(d_g) \to 0,$$

where the direct sum ranges over all $(\alpha, i)$ such that $x^\alpha e_i$ is irreducible with respect to $G$ and has degree $d$.

Now, we define the term order $\prec_d$ on the free module $F_d = \bigoplus S \cdot t^\alpha e_i$ by letting $x^\alpha \cdot t^\beta e_i \preceq_d x^\gamma \cdot t^\delta e_j$ if

$$x^{\alpha + \beta} e_i \prec x^{\gamma + \delta} e_j, \text{ or}$$

$$x^{\alpha + \beta} e_i = x^{\gamma + \delta} e_j, \text{ and } t^\beta \preceq_{\text{revlex}} t^\delta.$$

Let $G_d$ be a set consisting of two types of elements. First, for every element $x_a \cdot e_i - \sum_j g_j \cdot e_j \in G$, and every monomial $x^\alpha$ of degree $d - \deg e_i$ such that $\text{supp } \alpha \subseteq \text{ncrit } e_i$ we consider $x_a x^\alpha e_i - \sum_j \sum_k h_{j,k} \cdot e_k$ such that $\text{nf} (x^\alpha g_j e_j) = \sum_k h_{j,k} \cdot e_k = \sum_{j,k,l} c_{j,k,l} x^{\beta_{j,k,l}} e_k$ and we let $x_a \cdot t^\alpha e_i - \sum_j \sum_k c_{j,k,l} x^{\beta_{j,k,l}} t^{\delta_{j,k,l}} e_k \in G_d$, where $x^{\beta_{j,k,l}} = x^{\beta_{j,k,l}'} x^{\beta_{j,k,l}''} e_k$ in such a way that deg $x^{\beta_{j,k,l}} e_j = d$. Second, for every $e_i$ and every $\alpha$ such that $\text{supp } \alpha \subseteq \text{ncrit } e_i$ and deg $x^\alpha = d - \deg e_i - 1$, we have the elements $x_a \cdot t_b x^\alpha e_i - x_b \cdot t_a x^\beta e_i$ for each $a, b \in \text{ncrit}(e_i)$.

The claim is now that $G_d$ is a Gröbner basis for $\text{Ker } \eta_d$. To start with, it is clear that $G_d$ lies in the kernel of $\eta_d$, so it is sufficient to prove that for all $i \geq d$, there is a bijection between the terms of degree $i$ in $F_d$ which are irreducible with respect to $G_d$, and the terms of degree $i$ in $F$ which are of the form $x^\alpha e_j$ where deg $e_j \leq d$ and are irreducible with respect to $G$. The irreducible terms in $F_d$ of degree $i$ are all $x^\alpha t^\beta e_j$ such that $\text{supp } (\alpha + \beta) \subseteq \text{ncrit } e_j$ and $\max \text{supp } \alpha \leq \min \text{supp } \beta$, and from this we can conclude that the map $x^\alpha t^\beta e_j \mapsto x^{\alpha + \beta} e_i$ is a bijection. □

An immediate consequence of the preceding theorem is the following corollary that generalises [SV08], where it is shown that a homogeneous ideal with linear quotients is componentwise linear.

Corollary 4. A graded module with initially linear syzygies is componentwise linear.

Proof. Follows directly from Theorem 3 and Corollary 3. □
4. A Contracting Homotopy

We will now consider the minimal resolution $G_\bullet$ of a quadratic monomial ideal $M$ with initially linear syzygies as a based complex of $k$-modules by the natural decomposition

$$G_n = \bigoplus_{\alpha, I, j} k \cdot x^\alpha e_I g_j.$$ 

From a Morse matching on the digraph $\Gamma_{G_\bullet}$ we will construct a contracting homotopy $c$ on $G_\bullet$, that is a $k$-linear map satisfying

$$dc + cd = 1 - \eta.$$ 

The contracting homotopy will be used in the next section to show the existence of a DGA structure on $G_\bullet$ in the case when $M$ is a stable monomial ideal or a squarefree matroidal monomial ideal. The matching is constructed as follows. For $\beta \in \mathbb{N}^n$, let $V_{\beta, j}$ be the subset of $V$ consisting of all $x^\alpha e_I g_j$ such that $\deg_{g_j} x^\alpha e_I = \beta$. Exactly as in the proof of Lemma 1, we construct a partial matching $B_{\beta, j}$ on $\Gamma_{G_\bullet}|_{V_{\beta, j}}$

$$B_{\beta, j} = \{ x^\alpha e_I g_j \rightarrow x^\alpha x_i e_I \setminus i g_j \mid i = \min(\supp \alpha \cup I) \cap \text{crit } g_j \}.$$ 

Lemma 3. The set $B = \bigcup_{\beta, j} B_{\beta, j}$ is a Morse matching on $\Gamma_{G_\bullet}$.

Proof. The same argument as for Lemma 1 shows that each $B_{\beta, j}$ is a Morse matching on $\Gamma_{G_\bullet}|_{V_{\beta, j}}$, and we note that whenever there is an edge from a vertex in $V_{\beta, j}$ to a vertex in a different set $V_{\gamma, k}$ we have $x^\gamma e_k \prec x^{\beta} e_j$. \qed

Let

$$\tilde{i}(\alpha, j) = \min(\supp \alpha \cap \text{crit}(g_j)),$$

$$c_0(x^\alpha e_I g_j) = [\tilde{i}(\alpha, j) < \min I] \sum_{x^\alpha e_I g_j \rightarrow x^{\alpha} x_i e_I \setminus i g_j} c_0(x^\alpha e_I g_j).$$

We can now define the map $c$ by

$$c(x^\alpha e_I g_j) = c_0(x^\alpha e_I g_j) - c_0(x^\alpha e_I g_j) - x^\alpha e_I g_j.$$ 

A consequence of the above lemma is:

Corollary 5. The $k$-linear map $c$ is a contracting homotopy on $G_\bullet$ such that $\text{Im}(c)$ is spanned by the elements

$$\{ x^\alpha e_I g_j \mid \min((\supp x^\alpha \cup I) \cap \text{crit } g_j) \in I \}.$$ 

Proof. This follows from [Sk06, Lemma 6]. \qed

For monomial ideals with crit-monotone presentations, we can say a bit more about the contracting homotopy, by again using reductions for our description. We will define the set of $c$-critical indices of a basis element $e_I g_j$ by

$$c\text{-crit}(e_I g_j) = \{ i \mid i \in \text{crit } g_j, i < \min I \}.$$ 

We have the following formula for the homotopy acting on a basis element:

$$c(x^\alpha e_I g_j) = \sum_{x^\delta e_J g_l \in B^{-}} \sum_{p \in \{ x^\alpha e_I g_j \rightarrow x^\delta e_J g_l \}} c_0\rho_p(x^\alpha e_I g_j).$$
The composition with $c_0$ means that only elementary reductions paths $\alpha_0 \rightarrow \beta_1 \rightarrow \alpha_1$ where $\alpha_1$ is in $B^-$ will contribute to the result. These reduction paths are of the form
\[ x^\alpha e_1 g_\beta \rightarrow \frac{x^\alpha}{x_i} e_{I \cup \{i\}} g_\beta \rightarrow \frac{x^\alpha}{x_i} x^\delta e_1 g_\gamma, \]
where $i = \min(\text{c-crit}(e_1 g_\beta) \cap \text{supp} x^\alpha)$, $x^\delta g_\gamma = \text{nf}(x_i g_\beta)$, and $I \subseteq \text{crit}(g_\gamma)$. We can also note that for each $k$-basis element $x^\alpha e_1 g_\beta$, there is at most one elementary reduction path emanating from it. This means that the terms that occur in $c(x^\alpha e_1 g_\beta)$ are all of the form $\frac{m}{n} e_{\alpha,i} g_\gamma$ where $m$ divides $x^\alpha$.

We can now define a $k$-linear function $\rho$ by setting
\[ \rho(x^\alpha e_1 g_\beta) = \begin{cases} \frac{x^\delta}{x_i} x^\beta e_1 g_\gamma, & \text{if } J \neq \emptyset \text{ with } i = \min J, \\ 0, & \text{if } J = \emptyset. \end{cases} \]
where $J = \text{supp} x^\alpha \cap \text{c-crit}(e_1 g_\beta)$ and $\text{nf}(x_i g_\beta) = x^\delta g_\gamma$.

From these observations we can now deduce the following lemma.

**Lemma 4.** Let $M$ be a crit-monotone monomial ideal with initially linear syzygies. The contracting homotopy $c$ is given by
\[ c(x^\alpha e_1 g_\beta) = \sum_j c_0 \rho^j (x^\alpha e_1 g_\beta), \]
where $\rho$ is defined as above.

5. **DGA structures on resolutions**

In this section we will construct a differential graded algebra structure on the minimal resolution of $S/I$ where $I$ is either a stable monomial ideal or a square-free matroidal ideal. We can thereby extend, with a simpler proof, the result of Peeva [Pee96] showing the existence of a DGA structure on the minimal resolution of $S/I$ where $I$ is a stable ideal.

Let $\tilde{G}_\bullet$ be the resolution of $S/I$ obtained by splicing the resolution $G_\bullet \rightarrow I$ with $0 \rightarrow I \rightarrow S \rightarrow S/I \rightarrow 0$. Thus, we have
\[ \tilde{G}_\bullet : 0 \rightarrow G_n \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow S \rightarrow 0, \]
and we can extend the contracting homotopy $c$ defined on $G_\bullet$ to $\tilde{G}_\bullet$ by setting $c(x^\alpha) = x^{\alpha-\beta} g_\beta$, where $x^\beta$ is the smallest monomial generator of $I$ with respect to $\prec$ that divides $x^\alpha$. It is easy to see that with this definition, $c^2(x^\alpha) = 0$, and therefore $c^2 = 0$.

We will now define a map $\mu : \tilde{G}_\bullet \otimes \tilde{G}_\bullet \rightarrow \tilde{G}_\bullet$ so $\mu$ is the multiplication in a DGA structure on $\tilde{G}_\bullet$. The technique we will use to establish this result rests upon the following Lemma, which is a special case of [ML63, Theorem IX.6.2].

**Lemma 5.** Suppose that $X_\bullet$ and $Y_\bullet$ are complexes of $S$-modules, where $X_n = S \otimes_k V_n$ and $Y_n = S \otimes_k W_n$ for $k$-spaces $V_n$ and $W_n$, $n \geq 0$. Furthermore, suppose that $Y_\bullet$ is acyclic, with a contracting homotopy $c$ satisfying $c^2 = 0$. Then, every $S$-linear map $\varphi_0 : X_0 \rightarrow Y_0$ has a unique lifting to a chain map $\varphi : X_\bullet \rightarrow Y_\bullet$ satisfying $\varphi(V_n) \subseteq \text{Im}(c)$. This map is defined inductively by
\[ \varphi_{n+1}(\bar{x}) = c \varphi_n d(\bar{x}), \quad \bar{x} \in V_{n+1}. \]
We call elements of \( V_n \subseteq S \otimes_k V_n \) reduced; the reduced elements of \( \tilde{G}_* \) are thus the \( S \)-basis elements of \( \tilde{G}_* \).

Thus, we define our map \( \mu \) on the reduced elements \( \bar{x} \) and \( \bar{y} \) of degrees \( m \) and \( n \) respectively by:

\[
\mu(\bar{x} \otimes \bar{y}) = c\mu(d(\bar{x}) \otimes \bar{y}) = c\mu(d(\bar{x}) \otimes \bar{y}) + (-1)^m c\mu(\bar{x} \otimes d(\bar{y})).
\]

Now, consider the composition

\[
\tilde{G}_* \xrightarrow{\sim} S \otimes S \tilde{G}_* \xrightarrow{\iota \otimes 1} \tilde{G}_* \otimes S \tilde{G}_* \xrightarrow{\mu} \tilde{G}_*.
\]

which is the identity in degree 0, and since \( \mu(1 \otimes \bar{x}) \in \text{Im}(c) \); by Lemma 5, this is then the identity in all degrees, so \( 1 \in \tilde{G}_0 \) is a multiplicative identity element.

Furthermore, letting \( \tau \) be the twist morphism, \( \tau(x \otimes y) = (-1)^{mn} y \otimes x \) where \( x \) and \( y \) are homogeneous of degrees \( m \) and \( n \) respectively, we have that \( \mu \) and \( \mu \circ \tau \) both are chain maps \( \tilde{G}_* \otimes \tilde{G}_* \rightarrow \tilde{G}_* \) that in degree 0 are given by \( \mu(1 \otimes 1) = \mu \circ \tau(1 \otimes 1) \). Since for reduced elements \( \bar{x} \) and \( \bar{y} \) we have that \( \mu \circ \tau(\bar{x} \otimes \bar{y}) \in \text{Im}(c) \), Lemma 5 gives that \( \mu = \mu \circ \tau \), so \( \mu \) is graded commutative. Thus, to show that \( \mu \) gives a DGA structure to \( \tilde{G}_* \), it remains to show that \( \mu \) is associative, that is, that \( \mu(1 \otimes \mu) = \mu(\mu \otimes 1) \).

Recall that for a basis element \( e_I g_j \) we have \( c\text{-crit}(e_I g_j) = \{ i \mid i \in \text{crit} g_j, i < \min I \} \), and we now extend this to the whole of \( \tilde{G}_* \) by letting

\[
c\text{-crit}(\sum_i p_i \cdot e_i g_j) = \bigcup_i c\text{-crit}(e_i g_j),
\]

where we have no redundant terms in the sum. We are now in a position to formulate and prove the lemma that we will use to show associativity. We will in the following write \( x \star y \) for \( \mu(x \otimes y) \).

**Lemma 6.** If, for all basis elements \( e_I g_i, e_J g_j \), we have

\[
c\text{-crit}(e_I g_i \star e_J g_j) \subseteq c\text{-crit}(e_I g_i) \cap c\text{-crit}(e_J g_j),
\]

then \( \star \) is associative.

**Proof.** Since \( 1 \star x = x \) for all \( x \in \tilde{G}_* \), we only have to show that

\[
e_I g_i \star (e_J g_j \star e_K g_k) = (e_I g_i \star e_J g_j) \star e_K g_k
\]

for all basis elements \( e_I g_i, e_J g_j \) and \( e_K g_k \), and this now follows if we can show that

\[
e_I g_i \star (e_J g_j \star e_K g_k) \in \text{Im} c,
\]

holds for all \( e_I g_i, e_J g_j \) and \( e_K g_k \), since then by the graded commutativity of \( \star \) we would also get that

\[
(e_I g_i \star e_J g_j) \star e_K g_k \in \text{Im} c
\]

and by Lemma 5 we can conclude that they are equal. Now, suppose that \( x^\alpha e_L g_l \) occurs as a term in \( e_I g_i \star e_K g_k \), then, by the condition of the lemma, no variable occurring in \( x^\alpha \) will be c-critical in \( e_I g_i \star e_L g_l \), and thus (6) follows, and the proof is complete.

We will now in a series of lemmata show that the conditions of Lemma 6 are satisfied for the minimal resolution of stable and squarefree matroidal ideals. We start with the case where one of the basis elements have minimal degree.
Lemma 7. Let $g_a$ and $e_1 g_β$ be basis elements in the minimal resolution of a stable ideal. We then have
\[ c\text{-crit}(c(x^α e_1 g_β)) \subseteq c\text{-crit}(g_a). \]

Proof. By Lemma 4, we have that
\[ c(x^α e_1 g_β) = \sum_j c_0(x^{α_j} e_1 g_β_j) \]
where $α_j$ and $β_j$ satisfy $x^{α_j} = x^α v_j/u_j$ and $v_j g_β_j = nf(u_j g_β)$ for some monomial $u_j$ of degree $j$ dividing $x^α$. Now, if $c_0(x^{α_j} e_1 g_β_j) \neq 0$, then by the crit-monotonicity of the stable ideals, $c_0(x^{α_{0}} e_1 g_β_j) = x^{α_{0}}/x_k e_{1∪k} g_β_j$, for a $k ∈ supp(x^α/u_j)$, and thus
\[ c\text{-crit}(x^{α_{0}}/x_k e_{1∪k} g_β_j) = [1, k - 1] \subseteq \text{crit}(g_a). \]

Before showing the corresponding result for squarefree matroidal ideals, we have to say something about the term order we use.

Lemma 8. Let $M$ be a squarefree matroidal ideal in $S$, then $M$ is shellable with respect to a lexicographic ordering.

Proof. Essentially the same as the proof given by Herzog and Takayama [HT02, Lemma 1.3] for the revlex order.

From this we can conclude that the order given by $m g_a < n g_β$ whenever $x^α <_\text{lex} x^β$ gives us initially linear syzygies.

Lemma 9. Let $g_a$ and $e_1 g_β$ be basis elements in the minimal resolution of a squarefree matroidal ideal. We then have
\[ c\text{-crit}(c(x^α e_1 g_β)) \subseteq c\text{-crit}(g_a). \]

Proof. Again, by Lemma 4, we have that
\[ c(x^α e_1 g_β) = \sum_i c_0(x^{α_i} e_1 g_β_i) \]
where $α_j$ and $β_j$ satisfy $x^{α_j} = x^α v_j/u_j$ and $v_j g_β_j = nf(u_j g_β)$ for some monomial $u_j$ of degree $j$ dividing $x^α$. Now assume that $j ∈ \text{crit}(g_β_j)$ for some $j$ such that $j$ is less than all elements in $\text{crit}(g_β_j) \cap \text{supp} x^{α_{0\ldots k}}$. From the observation that $x^{α_i} x^{β_i} = x^α x^β$ for all $i$ we can conclude that
\[ \text{nf}(x_j x^{α_k} g_β_j) = \text{nf}(x_j x^ α g_β). \]

We can reduce $x_j x^{α_k} g_β_j$ to $x_l x^{α_k} g_β_l'$ for some $l > j$, and there cannot be any later reduction of the form $x_m g_σ → x_j g_σ'$ in a chain of reductions starting in $x_l x^{α_k} g_β_l'$, since that would imply that $m < j$, and that $x_m ∈ \text{crit}(g_β_j)$, and then we would have $m ∈ \text{supp} x^{α_k}$ which contradicts the choice of $j$. Thus we have for $x^α g_σ = \text{nf}(x_j x^ β g_a)$ that $j ∈ \text{supp} τ$, and thus, by the crit-monotonicity, $j ∈ \text{crit}(g_a)$. □

Lemma 10. Let $g_a$ and $e_1 g_β$ be basis elements in the minimal resolution of a stable ideal or a squarefree matroidal ideal. We then have
\[ c\text{-crit}(c(x^α e_1 g_β)) \subseteq c\text{-crit}(e_1 g_β). \]

Proof. By Lemma 4, we can see that the elements that appear in $c(x^α e_1 g_β)$ are of the form $x^β e_{α∪1} g_β$, and by the crit-monotonicity we know that $\text{crit}(g_β) \subseteq \text{crit}(g_β)$, hence the statement follows. □
We now turn to the case where the first basis elements in the product has non-minimal degree.

**Lemma 11.** Let $e_I g_\alpha$ and $e_J g_\beta$ be two basis elements in $\tilde{G}$, with $I \neq \emptyset$. If the inclusion
\[
\text{c-crit}(e_K g_\gamma \ast e_L g_\delta) \subseteq \text{c-crit}(e_K g_\gamma) \cap \text{c-crit}(e_L g_\delta),
\]
holds for all pairs of basis elements $e_K g_\gamma$ and $e_L g_\delta$ where $|K| + |L| < |I| + |J|$, then
\[
\text{c-crit}(c(d(e_I g_\alpha) \ast e_J g_\beta)) \subseteq \text{c-crit}(e_I g_\alpha) \cap \text{c-crit}(e_J g_\beta).
\]

**Proof.** By the crit-monotonicity, the differential $d$ can be written as
\[
d = \sum_j (-1)^{j-1} d^L_j - \sum_j (-1)^{j-1} d^R_j,
\]
so we consider the terms $c(d^L_i(e_I g_\alpha) \ast e_J g_\beta)$ and $c(d^R_i(e_I g_\alpha) \ast e_J g_\beta)$ separately.

If $i = 1$, we have
\[
c(d^L_1(e_I g_\alpha) \ast e_J g_\beta) = c(x_i e_{I \setminus I} g_\alpha \ast e_J g_\beta), \quad i = \min I.
\]
Suppose that $m e_K g_\gamma$ is a term in the product $e_I \setminus I g_\alpha \ast e_J g_\beta$; by assumption, we then have that $c-crit(e_K g_\gamma) \subseteq c-crit(e_I g_\alpha)$ and that $\text{supp} I \cap c-crit(e_K g_\gamma) = \emptyset$. If $c(x_i m e_K g_\gamma)$ is nonzero, then, by Lemma 4 we get that
\[
c(x_i m e_K g_\gamma) = m e_{I \setminus I} g_\gamma
\]
and thus,
\[
c-crit(c(x_i m e_K g_\gamma)) = c-crit(e_{I \setminus I} g_\gamma)
\]
\[
= c-crit(e_K g_\gamma) \cap |1, i - 1|
\]
\[
\subseteq c-crit(e_{I \setminus I} g_\alpha) \cap c-crit(e_J g_\beta) \cap |1, i - 1|
\]
\[
= c-crit(e_I g_\alpha) \cap c-crit(e_J g_\beta).
\]
If $i > 1$, then we have
\[
c(d^L_i(e_I g_\alpha) \ast e_J g_\beta) = c(x_i e_{I \setminus I} g_\alpha \ast e_J g_\beta), \quad i > \min I.
\]
If $m e_K g_\gamma$ occurs in the product $e_I \setminus I g_\alpha \ast e_J g_\beta$, then, since $i \notin c-crit(e_{I \setminus I} g_\alpha)$, the hypothesis of the lemma gives us that $i \notin c-crit(e_K g_\gamma)$, and therefore $c(x_i m e_K g_\gamma) = 0$.

Now, we look at $d^R_i$; for all $i$ we have
\[
c(d^R_i(e_I g_\alpha) \ast e_J g_\beta) = c(n e_{I \setminus I} g_\gamma \ast e_J g_\beta)
\]
for some monomial $n$ with $\text{supp} n \cap c-crit(e_{I \setminus I} g_\gamma) = \emptyset$. Let $m e_K g_\delta$ be a term in the product $e_I \setminus I g_\alpha \ast e_J g_\beta$, by assumption $c-crit(m e_K g_\delta) \subseteq c-crit(e_I \setminus I g_\gamma)$ and since $\text{supp} n \cap c-crit(e_K g_\gamma) = \emptyset$ and $\text{supp} n \cap c-crit(e_K g_\gamma) = \emptyset$, we can conclude that $c(m n e_K g_\gamma) = 0$. \hfill $\square$

**Theorem 4.** The minimal resolution of $M$ where $M$ is a squarefree matroidal ideal or a stable ideal has a DGA structure.

**Proof.** By Lemma 6 it suffices to show that
\[
c-crit(e_I g_\alpha \ast e_J g_\beta) \subseteq c-crit(e_I g_\alpha) \cap c-crit(e_J g_\beta)
\]
holds for all basis elements $e_I g_\alpha$ and $e_J g_\beta$, and by the definition of the product, we thus need to verify the relations

\begin{align}
(8) \quad c\text{-crit}(c(d(e_I g_\alpha) \ast e_J g_\beta)) &\subseteq c\text{-crit}(e_I g_\alpha) \cap c\text{-crit}(e_J g_\beta), \\
(9) \quad c\text{-crit}(c(d(e_J g_\beta) \ast e_I g_\alpha)) &\subseteq c\text{-crit}(e_J g_\beta) \cap c\text{-crit}(e_I g_\alpha).
\end{align}

We proceed by induction on $|I| + |J|$. If $|I| = |J| = 0$, (8) and (9) hold by Lemmas 7, 9 and 10. Now for the case $|I| + |J| > 0$, the inclusion (8) holds if $|I| = 0$ by Lemmas 7, 9 and 10, and by induction and Lemma 11 if $|I| > 0$; and similarly for the inclusion (9). $\square$

We will finish the paper by looking at the multiplicative structure of the minimal resolution of a small squarefree matroidal ideal.

**Example 1.** The Fano matroid is the matroid on the ground set $\{1, 2, \ldots, 7\}$, where every 3-element set is a basis, except for the following sets: $\{1, 2, 3\}, \{1, 4, 7\}, \{1, 5, 6\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 5\}, \{3, 6, 7\}$. The Fano matroid is often visualised using the following diagram

where a curve is drawn through every 3-element circuit. Let $I$ be the ideal in $S = k[x_1, \ldots, x_7]$ generated by the monomials corresponding to the bases in the Fano matroid. This ideal then has $\binom{7}{3} - 7 = 28$ generators, so for space reasons we will not describe the full multiplication table on the minimal resolution of $S/I$. Instead we will look at the resolution of $S'/J$, where the ideal $J$ is generated by the monomials in $I$ whose support is contained in $\{1, 2, 3\}$ and $S'$ is the polynomial ring $k[x_1, \ldots, x_4]$. This is then going to be a subalgebra of the minimal resolution of $S/I$. Thus, $J = (x_1 x_2 x_4, x_1 x_3 x_4, x_2 x_3 x_4)$, and we have the following basis elements in the resolution:

| Degree | Basis elements |
|--------|----------------|
| 0      | 1              |
| 1      | $g_{124}, g_{134}, g_{234}$ |
| 2      | $e_2 g_{134}, e_1 g_{234}$ |

Most of the products are either zero for degree reasons, or trivial due to multiplication by the identity element, so we are left with the following products to calculate: $g_{124} \ast g_{134}, g_{124} \ast g_{234}$ and $g_{134} \ast g_{234}$:

\begin{align*}
g_{124} \ast g_{134} &= c(x_1 x_2 x_4 g_{134}) - c(x_1 x_3 x_4 g_{124}) \\
&= x_1 x_4 e_2 g_{134} - 0 \\
&= x_1 x_4 e_2 g_{134}.
\end{align*}
\[ g_{124} \ast g_{234} = c(x_1 x_2 x_4 g_{234}) - c(x_2 x_3 x_4 g_{124}) = x_2 x_4 e_1 g_{234} - 0 = x_2 x_4 e_1 g_{234}, \]
\[ g_{134} \ast g_{234} = c(x_1 x_3 x_4 g_{234}) - c(x_2 x_3 x_4 g_{134}) = x_3 x_4 e_1 g_{234} - x_3 x_4 e_2 g_{134}. \]

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