Performance Evaluation of a M/Geo[xy]/1 Queue with varying probabilities of success which Treats Two Like Jobs As a Single Entity

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Abstract. The developed model comprises of a single server capable of handling two different job types X and Y type job. Job Y takes more time for execution than job X. The objective is to construct a single server which would replace the standard M/M/2 queuing model. The method used to find the relative measures involves the cost equation. The properties of the service distribution are discussed in detail. The maximum likelihood estimates for the parameters are obtained. The results are analytically derived for the M/Geo[xy]/1 model. A comparison is done between the model proposed and the standard M/M/2 queue. From the numerical results, it is observed that the waiting time in queue increases as the number of cycles is increased but however it is more economical than the M/M/2 model with restriction on the number of time slices.

Key Words: Geometric distribution, moment generating function, moments, maximum likelihood estimates, waiting time in queue.

AMS Subject Classification: 68M20, 90B22, 60K25

1. Introduction

M/Geo/1 queue with Poisson nature of arrivals and service distribution G(x) named M/Geo[xy]/1 is studied in this paper. The server developed in the model is capable of handling two job types. Each job is executed for a fixed time slot/ slice. Let X and Y be the two different Poisson distributed job arrivals. A time sharing phenomenon with a common fixed time slot is adopted by the server. At the end of a time slot, X and Y complete service with probability p₁ and p₂ respectively.

2. Literature Review

In practice the Poisson arrivals and exponential service discipline queues do not suit to model the real time situations in many instances. Therefore, it becomes essential to extend the theory to general arrivals and general service disciplines. In [1], a multi-class multi-server queuing model is proposed. A conservation law for the waiting time distribution is established with numerical validation. Doo Ho Lee et al. study the probability generating function of the queue length and the sojourn time distribution of the
disaster arrivals. In [3], the asymptotic behavior of M/G/1 retrial queue with impatient customers is analyzed by confining the various parameters influencing it. The authors in [4] analyze a discrete-time multi-server queuing system using generating functions approach.

Herwig Bruneel and Tom Maertens in [5] model customers using geometric random variables in a general fashion. The generating function, the expected values of the system content and the waiting times have been analytically obtained. Polynomial approximation technique has been employed to simplify numerical validation. The excursions of the workload process of a G/G1/1 queues are studied in [6]. The stationary distribution of the process, its duration and associated properties are studied in [7]. In [8], a customer’s observation of waiting time delays in two different parallel queues and their strategies are analyzed. H. Li, T. Yang have employed matrix analytic method to study the system behavior in [9]. Decomposition of the discriminatory processor sharing models is given using the results got for egalitarian processor sharing queues in [10]. In [11], the authors develop M/G/1 queues with negative customers and disasters. The stationary queue length and busy period are dealt in depth. [12] gives, an in-depth study of the busy period and the number of customers served during the period for a GI/Geo/1 queue with multiple vacations.

3. Main Results
The probability mass function of the random variable denoting the number of cycles for completion of service is given by

\begin{align}
P_X(i) &= p_1 (1 - p_1)^{i-1} \quad 1 - p_1 = q_1 \text{ and } i = 1,2,\ldots \quad (1) \\
P_Y(j) &= p_2 (1 - p_2)^{j-1} \quad 1 - p_2 = q_2 \text{ and } j = 1,2,\ldots \quad (2)
\end{align}

Hence the service distribution is given by \( Z = X + Y \), where \( X \) and \( Y \) are independent of each other.

Figure 1. The M/Geo[xy]/1 queue with varying probabilities of success for the x and y type of jobs.
\[ f_{XY}(x, y) = P(X = x, Y = y) = P(X = x, Y > x) = p_1q_1^{x-1} \sum_{y=x+1}^{\infty} p_2q_2^{y-1} \quad x = 1,2,\ldots \]
\[ = p_1(q_1q_2)^{x-1} \quad x = 1,2,\ldots \]
\[ G(x) = \frac{p_1}{q_1}(q_1q_2)^x \quad x = 1,2,\ldots \]  
(3)

where \( p_1 + q_1 = 1 \) and \( p_2 + q_2 = 1 \).

### 3.1 Mean of the service distribution

The behavior of the discrete random variable is characterized by the distribution function or the probability mass function. In recent scenario a finite collection of values rather than the entire set of values are required for the study. Thus the measures of central tendency gain importance of which the mean or the first order moment is mainly used for such calculation purposes.

\[ E[G(x)] = \sum_{x=0}^{\infty} x \frac{p_1}{q_1}(q_1q_2)^x \]
\[ \text{Mean} = E[G(x)] = \frac{p_1q_2}{(1-q_1q_2)^2} \quad 0 < q_1, q_2 < 1. \]  
(4)

### 3.2 The second order moment \( E[G(x)^2] \) is given by

\[ E[G(x)^2] = \sum_{x=0}^{\infty} x^2 \frac{p_1}{q_1}(q_1q_2)^x \]
\[ = \frac{p_1}{q_1} \sum_{x=2}^{\infty} x(x-1)(q_1q_2)^x + \sum_{x=0}^{\infty} x \frac{p_1}{q_1}(q_1q_2)^x. \]
\[ E[G(x)^2] = \frac{2p_1}{q_1} \left( \frac{(q_1q_2)^2}{(1-q_1q_2)^3} \right) + \frac{p_1q_2}{(1-q_1q_2)^3} \quad 0 < q_1, q_2 < 1. \]

### 3.3 The variance of the distribution \( G(x) \)

Variance of \( G(x) = E[G(x)^2] - E[G(x)]^2 \)
\[ = \frac{2p_1}{q_1} \left( \frac{(q_1q_2)^2}{(1-q_1q_2)^3} \right) + \frac{p_1q_2}{(1-q_1q_2)^3} - \left( \frac{p_1q_2}{(1-q_1q_2)^2} \right)^2 \]
\[ = \left( \frac{p_1q_1q_2^2 + p_1q_2}{(1-q_1q_2)^3} \right) - \left( \frac{p_1q_2}{(1-q_1q_2)^2} \right)^2 \]
where \( 0 < q_1, q_2 < 1 \) and \( p_1 + q_1 = 1, p_2 + q_2 = 1. \)  
(5)
3.4 Coefficient of variation is given by

The formula for the square of the coefficient of variation is $c_v^2 = \frac{\text{var}(G(x))}{E[G(x)]^2}$

$$c_v^2 = \frac{\left(\frac{p_1q_1q_2^2 + p_1q_2}{(1 - q_1q_2)^3}\right) - \left(\frac{p_1q_2}{(1 - q_1q_2)^2}\right)^2}{\left(\frac{p_1q_2}{(1 - q_1q_2)^2}\right)^2}$$

$$c_v^2 = \frac{\left(\frac{p_1q_1q_2^2 + p_1q_2}{(1 - q_1q_2)^2}\right)}{\left(\frac{p_1q_2}{(1 - q_1q_2)^2}\right)^2} - 1 \quad (6)$$

3.5 Moment generating function is given by

$$M_{G(x)}(t) = E[e^{xt}] = \sum_{x=0}^{\infty} e^{xt} \frac{p_1}{q_1} (q_1q_2)^x$$

$$= \frac{p_1}{q_1\left(1 - q_1q_2e^t\right)} \quad (7)$$

4. Relative measures of the M/Geo[x]/1 model

Relative measures such as the average waiting time of a job, average number of jobs in queue/system, and mean queue length of the queue are analytically derived. The number of jobs waiting for service in the waiting line is equal to the work done by the system. For an arbitrary job in an M/G/1 system the number of jobs in the waiting line is equal to the work done by the system as seen by the job arrival. Therefore

$$E(\text{Number of jobs in the waiting line}) = E(\text{Work done by the system as seen by the job arrival})$$

As Poisson arrivals see time averages, we get $W_q = V$

where $V$ is the work in the system as seen by an arrival.

4.1 Mean waiting time of a job in queue before service - $W_q$

The average amount of time a job spends waiting in queue before service is given by

$$W_q = \frac{\lambda E[G(x)^2]}{2(1 - \lambda E[G(x)])}$$
\[ W_q = \frac{\lambda \left[ 2p_1q_1q_2^2 + p_1q_2\left(1 - q_1q_2\right) \right]}{2\left[1 - q_1q_2\right]\left(1 - q_1q_2\right)^2 - \lambda p_1p_2} \]  

(8)

4.2 Average length of the waiting line - \( L_q \)

The number of jobs waiting for service is given by

\[ L_q = \lambda W_q \]

\[ = \frac{\lambda^2 \left[ 2p_1q_1q_2^2 + p_1q_2\left(1 - q_1q_2\right) \right]}{2\left[1 - q_1q_2\right]\left(1 - q_1q_2\right)^2 - \lambda p_1p_2} \]  

(9)

4.3 Average amount of time spent by a job in the system - \( W \)

The total time of execution for a job arrival is the sum of the time spent in waiting line and the mean service rate is given by \( W = W_q + E[G(x)] \)

\[ W = \frac{\lambda E[G(x)]^2}{2\left[1 - \lambda E[G(x)]\right]} + E[G(x)] \]

\[ = \frac{\lambda^2 \left[ 2p_1q_1q_2^2 + p_1q_2\left(1 - q_1q_2\right) \right]}{2\left[1 - q_1q_2\right]\left(1 - q_1q_2\right)^2 - \lambda p_1p_2} + \frac{p_1p_2}{\left[1 - q_1q_2\right]^2} \]  

(10)

4.4 Average number of jobs in the system – \( L \)

The number of jobs found in the system at any time instant is given by

\[ L = \lambda W \]

\[ = \frac{\lambda^3 \left[ 2p_1q_1q_2^2 + p_1q_2\left(1 - q_1q_2\right) \right]}{2\left[1 - q_1q_2\right]\left(1 - q_1q_2\right)^2 - \lambda p_1p_2} + \frac{\lambda p_1p_2}{\left[1 - q_1q_2\right]^2} \]  

(11)

4.5 Maximum likelihood estimates of \( q_1 \) and \( q_2 \)

In the proposed model, the service distribution of the jobs is taken to be geometrically distributed with parameter \( q_1 \) for X job type and \( q_2 \) for Y job type. In this section the maximum likelihood estimates of these two parameters are derived. The function \( g'(x, p_1, q_1, p_2, q_2) = g(x, q_1, q_2) \). Let \( x'_1, x'_2, x'_3, x'_4, ..., x'_n \) be a random sample of size \( n \) from the given population.
Therefore the maximum likelihood value of \( x_n = 1 - 1/n_x \). An increase in the value of \( x \) causes \( q_1 \) to increase. Maximum value for \( q_1 \) is possible when there is an abundant number of X type jobs.

\[
\frac{\partial \log L}{\partial q_1} = 0 \implies q_1 = 1 - \frac{1}{n_x}.
\]

(12) Therefore maximizing this function \( L \) over both \( q_1 \) and \( q_2 \), it is observed that \( L \) increases as both \( q_1 \) and \( q_2 \) increase.

5. NUMERICAL VALIDATION

The M/Geo\([xy]/1\) model and the M/M/2 queue model are compared. The value of \( q_1 \) is taken as \( q_1 = 1 - \frac{1}{x} \) as the time slot is very small. From the Table 1, it is seen that the waiting time in queue increases as the number of cycles is increased but however it is more economical than the waiting time in queue for an M/M/2 model upto four time slices. It is also observed that as \( q_2 \) decreases the waiting time in the queue also decreases. The waiting time for M/M/2 queue is obtained taking the service rate \( \mu = 0.21 \) and using the formula

\[
W_q = \frac{\rho^2}{\mu(1-\rho^2)} \quad \text{here} \quad \rho = \frac{\lambda}{2\mu}.
\]

Table 1. Results for waiting time in queue (\( W_q \)) for M/Geo\([xy]/1\) queue upto five cycles and the standard M/M/2 queue.

| M/Geo\([xy]/1\) | \( x \ \lambda \) | 0.165   | 0.1809  | 0.1812  | 0.1923  | 0.1956  | 0.2054  |
|------------------|-----------------|--------|--------|--------|--------|--------|--------|
| \( q_2=0.9 \)    |                 |        |        |        |        |        |        |
| 1                | -0.0872         | -0.09724 | -0.09743 | -0.10465 | -0.10683 | -0.11339 |
| 2                | 0.103505        | 0.117151 | 0.117417 | 0.127492 | 0.130578 | 0.139999 |
| 3                | 0.447964        | 0.513289 | 0.514578 | 0.563874 | 0.579155 | 0.626347 |
| 4                | 0.854646        | 0.988636 | 0.991306 | 1.094167 | 1.126355 | 1.226687 |
| 5                | 1.263274        | 1.471483 | 1.475662 | 1.637519 | 1.688527 | 1.848625 |
| $q_2=0.8$ | 1 | -0.0872 | -0.09724 | -0.09743 | -0.10465 | -0.10683 | -0.11339 |
|---|---|---|---|---|---|---|
| | 2 | 0.037415 | 0.041927 | 0.042014 | 0.045288 | 0.046281 | 0.049285 |
| | 3 | 0.162769 | 0.182918 | 0.183307 | 0.197995 | 0.202463 | 0.216015 |
| | 4 | 0.259843 | 0.292198 | 0.292825 | 0.316439 | 0.323627 | 0.345442 |
| | 5 | 0.326873 | 0.36743 | 0.368215 | 0.397793 | 0.406793 | 0.434098 |
| $q_2=0.7$ | 1 | -0.0872 | -0.09724 | -0.09743 | -0.10465 | -0.10683 | -0.11339 |
| | 2 | 0.00609 | 0.00678 | 0.006793 | 0.007288 | 0.007437 | 0.007886 |
| | 3 | 0.058702 | 0.065345 | 0.065472 | 0.070234 | 0.07167 | 0.075988 |
| | 4 | 0.088828 | 0.098785 | 0.098976 | 0.106102 | 0.108248 | 0.114699 |
| | 5 | 0.104692 | 0.116299 | 0.116521 | 0.124812 | 0.127306 | 0.134797 |
| $q_2=0.6$ | 1 | -0.0872 | -0.09724 | -0.09743 | -0.10465 | -0.10683 | -0.11339 |
| | 2 | -0.00803 | -0.0089 | -0.00891 | -0.00953 | -0.00972 | -0.01027 |
| | 3 | 0.01682 | 0.018621 | 0.018656 | 0.019935 | 0.020319 | 0.021468 |
| | 4 | 0.028353 | 0.031354 | 0.031411 | 0.033538 | 0.034176 | 0.036082 |
| | 5 | 0.033427 | 0.03693 | 0.036996 | 0.039474 | 0.040216 | 0.042433 |
| M/M/2 $\mu=0.21$ | 0.869066 | 1.084616 | 1.08904 | 1.263025 | 1.318855 | 1.496904 |

The other relative measures can be obtained by using Little’s law.

6. SUMMARY

In this paper a two dimensional random variable is transformed into a one dimensional one by way of construction. No specific transformation technique is used. A queuing model $M/Geo[x/y]/1$ by name whose service discipline is mixture of two geometric distributions with parameters $q_1$ and $q_2$ is developed. The characteristics of the service discipline are studied in detail. The relative measures of the model are studied using the cost equation and is compared with the standard M/M/2 model. The model is more efficient than the standard M/M/2 model (with service rate 0.21) when the job processing requirement is less than or equal to four time slots. The main application of this model would be in designing time sharing automated systems in a network.

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