Effects of periodic potentials on the critical velocity of superfluid Fermi gases in the BCS-BEC crossover

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We study the effects of an external periodic potential on the critical velocity of a superfluid Fermi gas in the crossover between the Bardeen-Cooper-Schrieffer (BCS) phase and Bose-Einstein condensation (BEC). We numerically solve the Bogoliubov-de Gennes equations to model a three-dimensional (3D) gas of ultracold atoms in the superfluid phase flowing through a 1D optical lattice. We find that when the recoil energy is comparable to the Fermi energy, the presence of the periodic potential reduces the effect of pair-breaking excitations. This behavior is a consequence of the peculiar band structure of the quasiparticle energy spectrum in the lattice. When the lattice width is much larger than the Fermi energy, the periodic potential makes pairs of atoms to be strongly bound even in the BCS regime and pair-breaking excitations are further suppressed. We have also found that when the recoil energy is comparable to or larger than the Fermi energy, the critical velocity due to long-wavelength phonon excitations shows a non-monotonic behavior along the BCS-BEC crossover.

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I. INTRODUCTION

Ultracold atom gases in optical lattices have been continuously attracting great interest for the last ten years. Recent developments in the field of ultracold atom gases provide a new research arena in the physics of quantum fluids: by using Feshbach resonances of ultracold Fermi atoms, one can study the crossover from the Bardeen-Cooper-Schrieffer (BCS) phase to a Bose-Einstein condensate (BEC) of molecules. In the current research frontier, superfluidity of ultracold Fermi gases in optical lattices is very intriguing problem, which has interesting connections with similar issues in solid state physics, nuclear physics, and astrophysics.

The critical velocity of superflow due to energetic instability is one of the most important properties of superfluids, which has been pioneered by Landau. If the velocity of superflow exceeds some critical value, the kinetic energy of the superfluid can be dissipated by creating excitations. In uniform superfluid Fermi gases in the BCS-BEC crossover, excitations which cause the energetic instability are of two types: fermionic pair-breaking excitations in the BCS regime and long-wavelength phonon excitations in the BEC regime. In the unitary regime both mechanisms are suppressed and the critical velocity shows a maximum value.

Recently, effects of periodic potentials on the critical velocity of Fermi superfluids has been studied experimentally. This experiment has stimulated theoretical investigations of this problem. Most of them has focused on the BCS regime in tight-binding approximation. The purpose of the present work is to obtain an understanding of the critical velocity from a unified point of view covering all regions along the BCS-BEC crossover and both the strong and weak lattice regime. To this purpose, we use the Bogoliubov-de Gennes (BdG) equations. This theory accounts for both types of excitations which are relevant in this problem. In our previous work, we already used it for a gas at unitarity; here we extend the calculations in order to explore the whole crossover region. As a main result, we find that, when the lattice height is comparable to or much larger than the Fermi energy, the periodic potential reduces the effect of pair-breaking excitations. This is due to the periodic structure of the quasiparticle energy spectrum in the Brillouin zone and the formation of the bound molecules induced by the lattice. Another main result is that when the recoil energy is comparable to or larger than the Fermi energy, the critical velocity due to long-wavelength phonon excitations shows a non-monotonic behavior along the BCS-BEC crossover. These effects are unique for Fermi superfluids in periodic potentials and do not exist in the case of single barrier potentials.

This paper is organized as follows. In Sec. II we explain the basic formalism employed in the present work. Then we show the results in Sec. III. Finally, summary and outlook are given in Sec. IV.

II. BASIC FORMALISM

We want to study the effect of the periodic potential on the Landau critical velocity of Fermi superfluids in the whole BCS-BEC crossover, in situations where the Fermi energy is larger or smaller than the lattice height. For
In this aim, we need to use a theoretical framework which can account for the formation of bound molecules induced by the periodic potential, which is important when the lattice height is larger than the Fermi energy [15, 20]; the same formalism must also account for pair-tunneling processes, which are important on the BEC side of the resonance [20, 21]. A suitable approach consists of the numerical solution of the Bogoliubov-de Gennes (BdG) equations:

\[
\begin{pmatrix}
H'(r) & \Delta(r) \\
\Delta^*(r) & -H'(r)
\end{pmatrix}
\begin{pmatrix}
u_i(r) \\
v_i^*(r)
\end{pmatrix} = \epsilon_i \begin{pmatrix}
u_i(r) \\
v_i^*(r)
\end{pmatrix},
\]

(1)

where \(u_i\) and \(v_i\) are quasiparticle amplitudes and \(\epsilon_i\) the corresponding eigen-energies. The single-particle hamiltonian is \(H'(r) = -\hbar^2 \nabla^2 /2m + V_{\text{ext}} - \mu\), where \(m\) is the atom mass and \(V_{\text{ext}}(r)\) the external potential. The order parameter (or gap parameter) \(\Delta(r)\) and the chemical potential \(\mu\), appearing in Eq. (1), are variational parameters determined from the gap equation,

\[
\Delta(r) = -g \sum_i u_i(r) v_i^*(r),
\]

(2)

together with the constraint

\[
n_0 = \frac{2}{V} \sum_i \int |v_i(r)|^2 \, dr,
\]

(3)

enforcing the conservation of the average density \(n_0\). Here \(g\) is the coupling constant for the contact interaction and \(V\) is the volume of the system. The BdG eigenfunctions obey the normalization condition \(\int d^3r [u_i^*(r)u_j(r) + v_i^*(r)v_j(r)] = \delta_{i,j}\). Finally, the energy density \(e\) can be calculated as

\[
e = \frac{1}{V} \int d^3r \sum_i [2(\mu - \epsilon_i)|v_i(r)|^2 + \Delta^*(r)u_i(r)v_i^*(r)].
\]

(4)

In the present study, we consider a three-dimensional superfluid Fermi gas, which is uniform in the \(x\) and \(y\) directions and subject to a one-dimensional optical lattice along \(z\):

\[
V_{\text{ext}}(z) = sE_R \sin^2 q_{B}z \equiv V_0 \sin^2 q_{B}z.
\]

(5)

Here \(V_0 \equiv sE_R\) is the lattice height, \(s\) is the laser intensity in dimensionless units, \(E_R = \hbar^2 q_{B}^2 /2m\) is the recoil energy, \(q_{B} = \pi /d\) is the Bragg wave vector, and \(d\) is the lattice constant. For practical reasons, throughout this paper, we set \(s = 1\) except for special cases, which we shall mention explicitly. The ratio between the Fermi energy and the lattice height is then varied by changing the average density of the gas.

In the presence of a supercurrent with wave vector \(Q = P /h\) moving in the direction of the periodic potential, one can write the gap parameter in the form

\[
\Delta(r) = e^{iQz} \tilde{\Delta}(z),
\]

(6)

where \(\tilde{\Delta}(z)\) is a complex function with period \(d\). Therefore, from the gap equation, we see that the eigenfunctions of Eq. (1) must have the Bloch form \(u_i(r) = \tilde{u}_i(z)e^{iQz}e^{ik_zr}\) and \(v_i(r) = \tilde{v}_i(z)e^{-iQz}e^{ik_zr}\). The wave vector \(k_z\) lies in the first Brillouin zone (i.e., \(-q_{B} \leq k_z \leq q_{B}\)) and \(\tilde{u}_i\) and \(\tilde{v}_i\) are periodic in \(z\) with period \(d\). We also define the quasi-momentum \(P_{\text{edge}}\) and quasi-wavenumber \(Q_{\text{edge}}\) at the edge of the Brillouin zone for \(P\) and \(Q\) as \(P_{\text{edge}} = hQ_{\text{edge}} \equiv hQ_{B}/2\). (Note that the edge of the first Brillouin zone for \(P\) is at \(P_{\text{edge}} = q_{B}/2\) while that for \(k_z\) is at \(q_{B}\).) This Bloch decomposition transforms Eq. (1) into the following BdG equations for \(\tilde{u}_i\) and \(\tilde{v}_i\):

\[
\begin{pmatrix}
\tilde{H}'(z) & \tilde{\Delta}(z) \\
\tilde{\Delta}^*(z) & -\tilde{H}'(z)
\end{pmatrix}
\begin{pmatrix}
\tilde{u}_i(z) \\
\tilde{v}_i(z)
\end{pmatrix} = \epsilon_i \begin{pmatrix}
\tilde{u}_i(z) \\
\tilde{v}_i(z)
\end{pmatrix},
\]

(7)

where

\[
\tilde{H}'(z) = \frac{\hbar^2}{2m} \left[ k_z^2 + k_y^2 + (-i\partial_z + Q + k_z^2) \right] + V_{\text{ext}}(z) - \mu.
\]

(8)

Here, the label \(i\) represents the wave vector \(k\) as well as the band index. In order to remove the ultraviolet divergences in the BdG equations with contact potentials, we use the regularization scheme proposed by Refs. [23, 24]. Since we need to calculate the second derivatives of the energy with respect to the density and the quasi-momentum, we use large values of the cutoff energy \(E_C\), especially in the BEC side, where the size of the pair is much smaller than the average inter-atomic distance (details are described in Sec. III).

As discussed in Refs. [1, 11], the energetic instability of superfluids of dilute Fermi gases can be caused by two processes [22]: the creation of long-wavelength superfluid phonon excitations or fermionic pair-breaking excitations. The critical velocity by the former process can be determined by the hydrodynamic analysis of the excitations [16, 28, 30]. Starting from the continuity equation and the Euler equation, and linearizing with respect to the perturbations of the density and the velocity fields, we obtain the dispersion relation of the long-wavelength phonon,

\[
\omega(q) = \frac{\partial^2 e}{\partial n_0 \partial P} q + \sqrt{\frac{\partial^2 e}{\partial n_0^2} \frac{\partial^2 e}{\partial P^2} |q|}.
\]

(9)

Here, \(\hbar\omega\) and \(q\) are the energy and the wavenumber of the excitations, \(n_0\) and \(P\) are the average density and the quasimomentum of the superfluids. The energetic instability occurs when \(\omega(q)\) becomes negative:

\[
\frac{\partial^2 e}{\partial n_0 \partial P} = \frac{\partial^2 e}{\partial n_0^2} \frac{\partial^2 e}{\partial P^2}.
\]

(10)

In practice, we calculate the energy density \(e(n_0, P)\) for a given \(n_0\) and \(P\) from Eq. (1) using the solution of the BdG equations [1]. Then the critical quasimomentum \(P_c\) at which the energetic instability occurs is determined by
and $k_F = (3\pi^2 n_0)^{1/3}$ are the Fermi energy and momentum, respectively, of a uniform noninteracting Fermi gas of density $n_0$. For each value of $E_F/E_R$, we solve the BdG equations for several values of the parameter $1/k_F a_s$ along the crossover from the BCS to the BEC side, namely $1/k_F a_s = -1, -0.5, 0, 0.5, 1$, and $1$, where $a_s$ is the $s$-wave scattering length of atoms.

In the $x$ and $y$ directions, we assume periodic boundary conditions with period $L = 12\pi k_F^{-1}$. We set the cutoff energy $E_C$ as follows: for $E_F/E_R = 2.5$, $E_C = 40E_F$ in the BCS side and $E_C = 100E_F$ at unitarity and in the BEC side; for $E_F/E_R = 1$, $E_C = 50E_F$ in the BCS side and $E_C = 100E_F$ at unitarity and in the BEC side; for $E_F/E_R = 0.1$, $E_C = 250E_F$ in the BCS side, $E_C = 350E_F$ at unitarity, and $E_C = 500E_F$ in the BEC side.

A. Density profiles and gap parameter

In Fig. 1 we show the density profile $n(z)$ at $P = 0$ along the BCS-BEC crossover for $E_F/E_R = 1$ and $s = 1$. Here, $n(z)$ is normalized by the average density $n_0$.

From Eq. (10) evaluated at the critical quasi-momentum determined by this condition, we obtain a critical velocity for the pair-breaking excitations. The actual critical velocity of the system is the lowest between the ones obtained from the above two conditions [32]. We finally note that the gas becomes unstable also when some excitation energy starts to have a non-zero imaginary part. This corresponds to a dynamical instability, which causes an exponential growth of the amplitude of the perturbation. To address the problem of dynamical instability, short-wavelength bosonic excitations should be also properly included. This is beyond the scopes of the present work, in which we instead focus on the energetic instability. Results of the critical velocity for dynamical instability due to long-wavelength excitations are given in Appendix A.

III. RESULTS

We study the three cases of $E_F/E_R = 2.5, 1$, and 0.1 with a fixed value of $s = 1$ except for a few cases which we shall mention explicitly. Here $E_F = \hbar^2 k_F^2/(2m)$

$$\frac{1}{k_F a_s} = -1, 0, 1$$

FIG. 1: (Color online) Density profile $n(z)$ at $P = 0$ in the BCS-BEC crossover for $E_F/E_R = 1$ and $s = 1$. Here, $n(z)$ is normalized by the average density $n_0$.

$$v_c = \frac{1}{n_0} \left( \frac{\partial c}{\partial P} \right)_{P_c}.$$  \hspace{1cm} (11)

On the other hand, the critical velocity due to the pair-breaking fermionic excitations can be determined by looking at the quasiparticle energy spectrum $\epsilon_i$. The energetic instability by the pair-breaking excitations occurs when some quasiparticle energy $\epsilon_i$ starts to be negative:

$$\epsilon_i \leq 0.$$  \hspace{1cm} (12)
fected by the presence of the lattice if $E_F/E_R \ll 1$ and $s \lesssim 1$, even though the lattice height $V_0$ is large compared to the Fermi energy $E_F$. On the other hand, the sufficient condition for pair-breaking excitations to be affected by the periodic potential due to the formation of bound molecules is $E_F/V_0 \ll 1$. This condition can be satisfied either by decreasing $E_F/E_R$ or by increasing $s$. When $E_F/E_R \simeq 1$, the critical velocity for pair-breaking excitations is also affected by a peculiar band structure of the quasi-particle spectrum.

Our results for the critical velocity $v_c$ are shown in Fig. 3 for $s = 1$. Let us first concentrate on the results at high density, $E_F/E_R = 2.5$, in panel (a). The open circles correspond to the critical velocity for long-wavelength phonons, which exhibits a non-monotonic behavior. In particular, in the BCS regime (negative $1/k_Fa_s$) this critical velocity is strongly reduced compared to the one in a uniform gas (see, e.g., Fig. 8 in Ref. [10]). This is a peculiar effect of the lattice. However, for the parameters of Fig. 3(a), the actual critical velocity in the BCS regime is still given by fermionic pair-breaking excitations (filled squares). The latter are almost unaffected by the lattice and therefore, the overall behavior of the critical velocity in the crossover is qualitatively similar to that of a uniform gas, already discussed in Ref. [10]: in the BCS regime, $v_c$ increases when approaching unitarity ($1/k_Fa_s = 0$), because the intra-pair attraction becomes stronger and thus the amplitude of the gap parameter increases; in the opposite BEC regime ($1/k_Fa_s > 0$), the critical velocity is given by long-wavelength phonons and an increase of the inter-pair repulsion leads to a larger sound speed and, again, a critical velocity $v_c$ increases towards unitarity. As a consequence, $v_c$ takes a maximum value at $1/k_Fa_s \simeq 0$.

When the recoil energy is comparable to the Fermi energy, the periodic potential causes qualitative changes in the results of the critical velocity. For $E_F/E_R = 1$ [Fig. 3(b)], we observe that, at $1/k_Fa_s = -0.5$, the critical velocity is given by long-wavelength phonon excitations rather than pair-breaking excitations even in the BCS regime. We understand this effect as mainly due to a peculiar band structure of the quasiparticle energy spectrum. In Fig. 3(c), we show the lowest band of the quasiparticle energy spectrum $\epsilon_1$ for the first radial branch with $k^2_{\perp} \equiv k^2_x + k^2_y = 0$ at $P = 0$ and $1/k_Fa = -0.5$. In general, the quasiparticle spectrum near the center of the Brillouin zone, at $|k_z| \simeq 0$, is only weakly affected by the periodicity of the system and hence the change of $\epsilon_1$ with increasing $P$ is close to that in the uniform system, given by the Doppler term $Phk_z/m$. On the other hand, close to the zone edge, at $k_z \simeq \pm q_0$, the change of $\epsilon_1$ with increasing $P$ is much smaller than $Phk_z/m$ because of the periodicity of the Brillouin zone ($\epsilon_1$ at $k_z = \pm q_0$ must be identical). In the case of $E_F/E_R = 1$, the minimum of $\epsilon_1$ is indeed located close to the edge of the Brillouin zone, unlike the other two cases of $E_F/E_R = 2.5$ and 0.1. Therefore, the reduction of the minimum value of $\epsilon_1$ with increasing $P$ is relatively small for $E_F/E_R = 1$ and this is why we find that $v_c$ is determined by phononic instead of fermionic excitations in this case.

For smaller density ($E_F/E_R = 0.1$), we observe a significant increase of $v_c$ in the whole crossover [Fig. 3(c)]. In our previous article [16], we already showed that, in a unitary Fermi superfluid with $E_F/E_R \ll 1$, the phononic critical velocity is almost unaffected by the lattice, if $s \lesssim 1$, and remains close to the speed of sound of a uniform gas with the same density [34]. This can be understood by recalling that phonons always have wavelength larger than the healing length of the superfluid, which is the order of $k_F^{-1}$ or greater. When $E_F/E_R \ll 1$,
FIG. 3: (Color online) Critical velocity $v_c$ of the energetic instability for $E_F/E_R = 2.5$ (a), 1 (b), and 0.1 (c) with $s = 1$ in the BCS-BEC crossover. Open circles and filled squares show the critical velocity due to long-wavelength phonons and fermionic pair-breaking excitations, respectively. The horizontal dotted line in panel (c) represents the value of the sound velocity $c_s^{(0)}$ of uniform system at unitarity, $c_s^{(0)}/v_F = (1 + \beta)^{1/2}/\sqrt{3} \approx 0.443$. The red solid lines and the black dashed lines are guides to the eye.

the healing length becomes much larger than the lattice spacing $d = \pi/q_B$, and this makes the phonons insensitive to the lattice itself. In the present work, we find that the same is true even away from unitarity, leading to a larger critical velocity in the whole crossover [empty circles in this figure].

FIG. 4: (Color online) Lowest band of the quasiparticle energy spectrum $\epsilon_i$ for $P = 0$ and $1/k_F a_s = -0.5$. Here, we show the first radial branch with $k_z^2 \equiv k_x^2 + k_y^2 = 0$. For $E_F/E_R = 1$, the minimum of $\epsilon_i$ is located close to the Brillouin zone edge $k_z = q_B$.

FIG. 5: (Color online) Critical current $j_c = n_0 v_c$ for the same cases in Fig. 3. Here we show the lowest value of $j_c$, given either by long-wavelength phonon excitations or by fermionic pair-breaking excitations. The curves connecting symbols are guides to the eye.
FIG. 6: (Color online) Lowest band of the quasiparticle energy spectrum $\varepsilon_i$, for large lattice height with $s = 5$ and $E_F/E_R = 0.1$ (i.e., $E_F/V_0 = 0.02$) in the BCS regime at $1/k_Fa_s = -1$. Here, we show the first radial branch with $k_z^2 \equiv k_0^2 + k_y^2 = 0$, which always gives the smallest values of $\varepsilon_i$ in this case. The inset shows the amplitude $|\Delta(z)|$ of the order parameter at $P = 0$. The horizontal dotted line shows the amplitude of the order parameter for the uniform system at the same values of $1/k_Fa_s = -1$.

The critical velocity due to pair-breaking excitations (filled squares) is also increased because a lattice strength $V_0$ much larger than the Fermi energy gives a stronger attraction between paired atoms.

In Fig. 5 we show the critical current $j_c = n_0v_c$ for the same cases of Fig. 4. Due to the low density at $E_F/E_R = 0.1$, the critical current is much smaller than the other cases even though $v_c$ in units of $v_B = q_B/m$ for $E_F/E_R = 0.1$ is comparable to that of $E_F/E_R = 1$.

C. Dependence on the lattice height

All results shown in Figs. 4, 5, have been obtained by fixing the lattice height $s = 1$ and vary the density in order to change the key parameter $E_F/E_R$. If we increase $s$ keeping the average density fixed, the superfluid flow is suppressed and $v_c$ is also reduced in general.

A systematic analysis as a function of $s$, which would be natural from the experimental viewpoint, is computationally very demanding and is beyond the scopes of this work. The choice of $s = 1$ is not accidental, however. It turns out, in fact, that around $s = 1$ the effects of the lattice on the critical velocity are the most pronounced as far as the interplay between pair-breaking and long-wavelength bosonic excitations is concerned. For lower values of $s$ these two types of excitations behave qualitatively the same as in a uniform superfluid, as function of $1/k_Fa_s$, being scarcely affected by the lattice, at least within the range of $E_F/E_R$ considered in this work. On the other hand, at larger $s$ the pair-breaking instability is quickly suppressed and long-wavelength excitations becomes dominant along the crossover.

The reason why a strong lattice prevents the pair-breaking processes can be understood by looking at Fig. 6 where we show the quasiparticle energy spectra at various values of $P$ in the case of $s = 5$, $E_F/E_R = 0.1$ (i.e., $E_F/V_0 = 0.02$), and $1/k_Fa_s = -1$. The spectrum for $P = 0$ shows a quadratic dependence of $k_z$ with a positive curvature around $k_z = 0$ and there are no minima at $k_z \neq 0$. Even though the figure represents a case in the deep BCS regime, the structure of $\varepsilon_i$ is consistent with the formation of bound pairs. In the inset of the same figure, we show the amplitude $|\Delta(z)|$ of the gap parameter at $P = 0$. First, we note that the minimum value of $|\Delta(z)|$ at $z/d = \pm 1$ is smaller than, but still comparable to the value of $|\Delta|$ in the uniform case, suggesting that the system is indeed in the superfluid phase. More importantly, one sees a large enhancement of $|\Delta(z)|$, near $z = 0$, compared to the uniform system, which shows the formation of bosonic bound molecules. A consequence of this lattice induced molecular formation is that the energetic instability due to pair-breaking excitations is suppressed and does not occur at any values of $P$.

For the same parameters of Fig. 4, the energetic instability due to long-wavelength phonons instead occurs at $P = 0.226$ and the corresponding critical velocity at $1/k_Fa_s = -1$ is $v_c = 0.0662v_F$. We also find $v_c = 0.0429v_F$ at unitarity and $v_c = 0.0302v_F$ at $1/k_Fa_s = 1$. This means that, in the whole crossover, the critical velocity at $s = 5$ is largely reduced compared to the red line in Fig. 4(c) for $s = 1$ and does not exhibit a maximum anymore. The behavior of $v_c$ as a function of $s$ at unitarity has been already discussed in Ref. 16.

IV. SUMMARY AND OUTLOOK

We have studied the effects of a periodic potential on the Landau critical velocity of a Fermi superfluid in the BCS-BEC crossover. We have considered a 3D superfluid Fermi gas flowing in a 1D periodic potential produced by an optical lattice. Using the Bogoliubov-de Gennes equations, we have obtained a unifying picture both for weak and strong lattices and in the whole BCS-BEC crossover. We have found that, when the recoil energy is comparable to the Fermi energy, energetic instability due to fermionic pair-breaking excitations can be less effective as a consequence of the periodic structure of the quasiparticle energy spectrum. When the lattice height is much larger
than the Fermi energy, pair-breaking excitations are prevented because the lattice potential gives a stronger attraction between paired atoms, eventually forming bound bosonic molecules. We have also found that, when the recoil energy is comparable to or larger than the Fermi energy, the critical velocity due to the long wavelength phonon excitations is drastically reduced by the lattice in the BCS regime leading to its non-monotonic behavior along the BCS-BEC crossover.

A further interesting issue regards the possible existence of roton-like minima in the bosonic dispersion curve. This excitations are obtained at low filling fractions and within a tight-binding attractive Hubbard model \[14, 15, 26, 27\]. The roton-like minima arise from strong charge-density-wave fluctuations. These fluctuations are expected to be less favored in our system, where the gas is uniform in the transverse directions. However, if such roton-like excitations exist also in the our case (3D gas in a 1D lattice), they would lower the critical velocity in the BCS regime and for strong lattices. To address this issue, one should use, for instance, a quasiparticle random phase approximation (QRPA) on top of the stationary solution of the Bogoliubov-de Gennes equations. This is an interesting challenge for future investigations.

Finally, we would like to discuss a similarity between the present system and nuclear “pasta” phases \[36–38\] in crusts of neutron stars. The pasta nuclei are those of exotic shapes such as rod-like and slab-like structures. In neutron star crusts, the pasta nuclei are immersed in background electrons and a gas of dripped neutrons, which is regarded to be in the superfluid phase. The setup considered in the present work resembles superfluid neutrons in the pasta phase with slab-like nuclei, which are in the normal phase and provide a 1D periodic potential for superfluid neutrons \[39\].

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Appendix A: Dynamical instability within hydrodynamic analysis

In this Appendix, we show the results of the critical velocity \(v_c\) for dynamical instability due to long-wavelength excitations. This instability occurs when a non-zero imaginary part appears in the excitation energy \(\bar{\varepsilon}\). The condition is

\[
\frac{\partial^2 \varepsilon}{\partial n^2} \frac{\partial^2 \varepsilon}{\partial P^2} < 0. \tag{A1}
\]

Even though the long-wavelength excitations in general do not give the lowest critical velocity of the dynamical instability, this condition becomes useful at \(V_0 \gg E_F\), when the \(P\)-dependence of \(\varepsilon\) is almost sinusoidal and the critical velocity depends rather weakly on the wavelength of the excitations. In this case, the above condition gives an onset of the dynamical instability at \(P = P_{\text{edge}}/2\), which coincides with the condition \(\varepsilon(P + hq) - \varepsilon(P) = \varepsilon(P) - \varepsilon(P - hq)\) for \(V_0 \gg \Delta\). This corresponds to the energy and momentum conservation for two particles decaying into two different Bloch states with \(\varepsilon(P \pm hq)\).

In Fig. 7, we show the critical velocity for the dynamical instability determined by the condition \(\text{A1}\). The most striking feature is the large values of \(v_c\) in the case of \(s = 1\) and \(E_F/E_R = 0.1\). As in the case of the energetic instability, this is due to the fact that phonons are insensitive to the lattice when \(s \lesssim 1\) and \(E_F/E_R \ll 1\), i.e., the healing length is much larger than the lattice spacing. Note that, even for the same value of \(E_F/E_R = 0.1\), the critical velocity can be rather small provided \(s \gg 1\). This tendency is confirmed by the results for \(s = 5\) and \(E_F/E_R = 0.1\) shown by crosses in Fig. 8.
FIG. 7: (Color online) Critical velocity \( v_c \) for dynamical instability due to long-wavelength excitations in the BCS-BEC crossover. The solid lines are guides to the eye. The inset shows the magnification of the region of \( v_c/v_F = 0 - 0.3 \).

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tary Fermi gas [41, 42, 43] (see also Refs. [40, 45, 46]).

Here, because $E_F/E_R \gg 1$, we can safely employ the hy-
drodynamic theory with the local density approximation
given in Ref. [16] to estimate the critical velocity. Using
Eq. (3) of Ref. [16] together with $\beta = -0.58$ [35] and the
above parameters, we estimate $v_c \sim 10^{-3} - 10^{-2}v_F$ with
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