1. Introduction

Semiinfinite cohomology of Lie algebras appeared in mathematics more than 10 years ago (see [F]), and yet it belongs to the area of homological algebra existing partly in the form of folklore. A remarkable breakthrough was achieved by A. Voronov (see [V]) who managed to define the semiinfinite cohomology in the derived category setting. Yet the general definition of semiinfinite cohomology of associative algebras has been unknown.

1.1. One of the aims of this paper is to give a rigorous construction of the functor of semiinfinite cohomology of arbitrary (graded) associative algebras defined in the corresponding derived categories of graded modules. The basic setup here includes a graded associative algebra $A$ with two graded subalgebras $B$ and $N$ such that $A = B \otimes N$ as a graded vector space. These conditions are satisfied in particular in the case of the universal enveloping algebra of a graded Lie algebra, but the general case is much wider. We will show that the semiinfinite cohomology of the universal enveloping algebra coincides with the corresponding Lie algebra semiinfinite cohomology (see [V], [F]).

Our construction of the standard complex for the calculation of the semiinfinite cohomology suggests that semiinfinite cohomology could be realized by some combination of the standard Tor and Ext functors. Our considerations do not give such a realization, yet it exists in suitably chosen triangulated categories. This will be explained elsewhere.

1.2. Let us describe the structure of the paper. In the second section we recall several basic facts about quantum groups at roots of unity. In the third section we give the definition of the semiinfinite cohomology of associative algebras and prove some basic results about them. In the fourth and fifth sections we consider an example of the algebra $A$ equal to the finite dimensional Hopf algebra $u$ (finite quantum group) introduced by G.Lusztig in [L1]. In this case the semiinfinite homology $\text{Tor}_C^2(k, Y)$ appeared in [FiS]. They calculated the cohomology of the space of configurations of points on the projective line $\mathbb{P}^1$ with coefficients in the sheaf $\mathcal{Y}$ equal to the localization of the module $Y$ to the point $0 \in \mathbb{P}^1$.

It is known that the semisimple Lie algebra $\mathfrak{g}$ acts on the cohomology of the trivial module over the finite quantum group with the same Cartan matrix (see [GK]). In the fourth section we show that this fact holds for semiinfinite cohomology of the trivial module and calculate the character of this $\mathfrak{g}$-module. Unfortunately even in the simplest case the representation itself remains unknown. B.Feigin has proposed a conjecture describing the $\mathfrak{g}$-module of semiinfinite cohomology of the trivial module over the finite quantum group in terms of distributions on the nilpotent cone of $\mathfrak{g}$. In Appendix A we prove some facts confirming the conjecture on the level of characters.

The main result of the fifth section is Theorem 5.3, stating that conformal blocks are naturally embedded into the semiinfinite Tor spaces (see the exact statement in section 5). This Theorem along with the results of the papers [Fi], [FiS] sheds light on the conjecture of Feigin, Schechtman and Varchenko about
the integral representation of conformal blocks (see [FSV]). Namely, the above results imply the following statement: the local system of conformal blocks on the space of configurations of points on $\mathbb{P}^1$ is a direct summand in the direct image of some perverse sheaf on a larger configuration space. The perverse sheaf itself is the Goresky-MacPherson extension of a one-dimensional local system. The example 5.3.2 shows that the local system of conformal blocks is in general a \textit{proper} direct summand in the direct image of the above Goresky-MacPherson sheaf.

In Appendix B we present several Voronov’s results on semiinfinite homological algebra (see [V]) and make an attempt to realize semiinfinite cohomology of associative algebras as a two-sided derived functor (in spirit of [V], 3.9).

This paper grew out of attempts to understand the natural general setting for semiinfinite cohomology. I was introduced to the subject by B. Feigin back in 1993. He also formulated the conjectural answer for the semiinfinite cohomology of finite quantum groups. Thus the present paper owes its very existence to B. Feigin. I am also greatly indebted to M. Finkelberg and L. Positselsky for many helpful discussions. I would like to thank D. Timashev for bringing the paper [H] to my attention.

\section*{Notation}

Throughout the paper we use the following notation.

\begin{itemize}
  \item \((a)_{i,j=1}^r\) is a Cartan matrix of the finite type
  \item \(d_1, \ldots, d_r \in \{1, 2, 3\}\) such that \((d_ia_{ij})\) is symmetric
  \item \(R\) is the root system corresponding to \((a_{ij})\)
  \item \(R^\pm\) is the set of positive (resp. negative) roots
  \item \(\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha\) is the sum of positive roots
  \item \(\Sigma = \{\alpha_1, \ldots, \alpha_r\}\) is the set of simple roots
  \item \(\text{ht} \beta = \sum_{i=1}^r b_i\), where \(\beta = \sum_{i=1}^r b_i \alpha_i \in R^+\) is the height function on the set of the positive roots
  \item \(X\) is the weight lattice of \(R\)
  \item \(Y\) is the root lattice of \(R\), i.e. \(Y = \mathbb{Z} \alpha_1 \oplus \ldots \oplus \mathbb{Z} \alpha_r \rightarrow X\)
  \item \(Y^\pm\) is the subsemigroup in \(Y\) generated by the set \(\Sigma\) (resp. by \(-\Sigma\))
  \item \((\cdot | \cdot)\) is a scalar product defined on \(\Sigma \subset X\) by the formula \((\alpha_i | \alpha_j) = d_ia_{ij}\)
  \item \(\alpha^\vee_1, \ldots, \alpha^\vee_r\) are the simple coroots
  \item \(W\) is the Weyl group corresponding to \(R\)
  \item \(g\) is the semisimple Lie algebra with the Cartan matrix \((a_{ij})\)
  \item \(G\) is the simply connected Lie group with Lie algebra \(g\)
  \item \(Q(v)\) is the field of rational functions in the indeterminate \(v\)
  \item \([m]_d^l = \prod_{j=1}^m \frac{a_{i,j}}{a_{i,j} - v^{d - j}} \in Q(v)\) where \(m, d \in \mathbb{N}\)
  \item \([m]_d^t = \prod_{j=1}^t \frac{v^{d(m-j+1)} - v^{d-m+1}}{v^{d-m+1} - v^{-d}} \in Q(v)\) where \(m \in \mathbb{Z}, t, d \in \mathbb{N}\)
\end{itemize}

\section{Quantum groups at roots of unity}

In this section we collect several well-known facts about finite quantum groups that we will need later.

\subsection{Quantum groups at roots of unity}

For every symmetrizable Cartan matrix \((a)_{i,j=1}^r\) Drinfeld and Jimbo constructed a Hopf algebra \(U_1\) over the field \(Q(v)\) of rational functions with the generators \(E_i, F_i, K_i, K_i^{-1}, i = 1, \ldots, r\), and the
following relations:

\[ K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1 \]
\[ K_i E_j = v^{d_i a_i j} E_j K_i, \quad K_i F_j = v^{-d_i a_i j} F_j K_i \]
\[ E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{v^{di} - v^{-di}} \]
\[ \sum_{r+s=1-a_{ij}} (-1)^s \left[ \frac{1 - a_{ij}}{s} \right] E_i^r E_j^s = 0 \text{ if } i \neq j \]

(see also [L1]). This algebra is called the quantum universal enveloping algebra of the corresponding Kac-Moody Lie algebra or the quantum group.

2.2. Given an integer \( p > 1 \) prime to the nonzero elements of the fixed Cartan matrix, we choose a primitive \( p \)-th root of unity \( \zeta \) and set \( k = \mathbb{Q}(\zeta) \). De Concini and Kac introduced the \( k \)-algebra \( U_2 \) with generators \( E_i, F_i, K_i, K_i^{-1} \) and relations similar to the ones for \( U_1 \) but with \( v \) replaced by \( \zeta \) (see [DCK], 1.5).

Denote the subalgebra in \( U_2 \) generated by all \( E_i \) (resp. by all \( F_i \), resp. by all \( K_i \) and \( K_i^{-1} \)) by \( U_2^+ \) (resp. by \( U_2^- \), resp. by \( U_2^0 \)). Then \( U_2^0 \) is the algebra of Laurent polynomials in the commuting variables \( K_i \). We define the following \( X \)-grading on \( U_2 \): \( \deg E_i = \alpha_i, \deg F_i = -\alpha_i, \deg K_i = 0 \).

Like in classical universal enveloping algebras, for every \( \beta \in R^+ \) one can define the root elements \( E_\beta \in (U_2)_\beta \) and \( F_\beta \in (U_2)^{\beta} \), in particular \( E_{\alpha_i} = E_i \), \( F_{-\alpha_i} = F_i \). (see [L1]).

2.2.1. Lemma: (see [DCK], 1.7) The elements

\[ \prod_{\beta \in R^+} E_{\beta}^n(\beta) \prod_{\beta \in R^+} K_{\beta}^s(i) \prod_{\beta \in R^+} F_{\beta}^{-m(\beta)} \]

are linearly independent and generate \( U_2 \) as a vector space. Here \( n(\beta) \) and \( m(\beta) \) run through nonnegative integers and \( s(i) \) run through arbitrary integers. \( \square \)

2.2.2. Corollary: The multiplication in \( U_2 \) defines an isomorphism of the vector spaces \( U_2^+ \otimes U_2^0 \otimes U_2^- \longrightarrow U_2. \) \( \square \)

All \( E_{\beta}^p \) and \( F_{\beta}^{-p}, \beta \in R^+, \) and \( K_{\beta}^p, \quad i = 1, \ldots, r, \) belong to the center of the algebra \( U_2 \) (see [DCK], 3.1). Denote the ideals generated by the elements \( \{ E_{\beta}^p, F_{-\beta}^{-p} | \beta \in R^+ \} \) in the algebras \( U^+, U^-, \) and \( U \) respectively by \( I^+, I^- \) and \( I \). Obviously \( I^\pm = I \cap U_{2^\pm} \).

Set \( U = U_2/I. \) Let \( U^\pm \) and \( U^0 \) be the images of the corresponding subalgebras under the projection. Note that \( U^+ \) (resp. \( U^- \)) is \( Y^+ \)-graded (resp. \( Y^- \)-graded). We obtain the base in \( U \) that consists of monomials similar to those of the previous Lemma but with \( 0 \leq m(\beta), n(\beta) \leq p \) for every \( \beta \in R^+ \) and arbitrary \( s(i) \in \mathbb{Z}. \)

2.3. There exists an alternative version of the quantum group defined by Lusztig (see [L1], 8.1). Namely Lusztig considers the \( \mathbb{Q}[v, v^{-1}] \)-subalgebra \( U_{\mathbb{Z}} \subset U_1 \) generated by the elements

\[ E_i^{(n)} := E_i^n/|n|!_{d_i}, \quad F_i^{(n)} := F_i^n/|n|!_{d_i}, \quad K_i^{\pm 1}, \quad i = 1, \ldots, r, \quad n \geq 0. \]

It is analogous to the classical integral form of the universal enveloping algebras due to Kostant.

By definition set \( U_3 = k \otimes \mathbb{Q}[v, v^{-1}] U_{\mathbb{Z}}. \) Then \( U_3 \) contains elements \( E_i, F_i, K_i \) and \( K_i^{-1}, \quad i = 1, \ldots, r, \) that satisfy the basic relations for the generators of \( U_2. \) Thus there exists a homomorphism \( f : U_2 \longrightarrow U_3 \) mapping the generators of \( U_2 \) to the corresponding elements of \( U_3. \)
its image is a finite dimensional subalgebra \( u \subset U_3 \) (see [L1], 8.2). The kernel of \( f \) contains \( I \), and we obtain the surjective map: \( U \twoheadrightarrow u \). Both algebras are \( X \)-graded, and the map preserves the grading.

2.3.1. **Lemma:** (see [AJS], 1.3) The following statements hold for \( U \) and \( u \):

(i) \( U = U^+ \otimes U^0 \otimes U^- \) as a vector space;
(ii) \( u = u^+ \otimes u^0 \otimes u^- \) as a vector space;
(iii) both maps \( U^\pm \twoheadrightarrow u^\pm \) are isomorphisms of algebras;
(iv) \( u^0 \) is equal to the quotient algebra of \( U^0 \) by the ideal generated by all \( K_i^{2p} - 1 \). \( \square \)

Thus we obtain PBW-type bases in \( u \). We call both \( U \) and \( u \) the finite quantum groups. The algebra \( u^- \otimes u^0 \) (resp. \( u^0 \otimes u^+ \)) is called the negative (resp. the positive) Borel subalgebra in \( u \) and is denoted by \( b^- \) (resp. by \( b^+ \)).

2.4. Using PBW-type bases, Kac, De Concini and Procesi defined some remarkable filtrations on \( U_2, U \) and \( u \) that generalize the usual PBW-filtrations on universal enveloping algebras (see [DCKP]).

Consider the lexicographically ordered set \( S = \mathbb{Z}_+^{2N+1} \), where \( N \) is the number of the positive roots. A filtration of a vector space by subspaces numbered by the set \( S \) is called \( S \)-filtration.

2.4.1. We fix a convex order on the set of the positive roots \( R^+ \). Roughly speaking the convex property means that the \( q \)-commutator of two root vectors \( E_\alpha \) and \( E_\beta \) in \( U_1 \) consists of monomials formed only from root vectors between \( \alpha \) and \( \beta \) in the order (see e.g. [DCKP] for the exact definition). We denote the monomial

\[
E_{s_1}^{s_{N_1}} \cdots E_{s_{N}}^{s_{N}} K_1^{b_1} \cdots K_r^{b_r} F_{i_1}^{t_1} \cdots F_{i_N}^{t_N}
\]

by \( M_{(s,t,\alpha)} \), where \( \alpha = \sum_{j=1}^r b_j \alpha_j \). We define the total height

\[
d_0(M_{(s,t,\alpha)}) = \sum_{i=1}^N (s_i + t_i) \text{ht} \beta_i,
\]

where \( \text{ht} \beta \) is the height of the root \( \beta \), and the total degree

\[
d(M_{(s,t,\alpha)}) = (s_N, \ldots, s_1, t_1, \ldots, t_N, d_0(M_{(s,t,\alpha)})) \in \mathbb{Z}_+^{2N+1}.
\]

We introduce the \( S \)-filtration \( F \) on \( U_2 \) by the total degree.

2.4.2. **Lemma:** (see e.g. [DCKP], 4.2) The associated polygraded algebra \( gr U_2 \) for the \( S \)-filtration on \( U_2 \) is generated by the elements \( E_\alpha, F_{-\alpha} \) (\( \alpha \in R^+ \)) and \( K_i^{\pm 1} \) (\( i = 1, \ldots, r \)) satisfying the following relations:

\[
K_i E_\beta = \zeta^{(\alpha|\beta)} E_\beta K_i, \quad K_i F_{-\beta} = \zeta^{-(\alpha|\beta)} F_{-\beta} K_i, \quad \beta \in R^+;
\]

\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i;
\]

\[
E_\alpha F_{-\beta} = F_{-\beta} E_\alpha, \quad \alpha, \beta \in R^+;
\]

\[
E_\alpha E_\beta = \zeta^{-(\alpha|\beta)} E_\beta E_\alpha, \quad F_{-\alpha} F_{-\beta} = \zeta^{(\alpha|\beta)} F_{-\beta} F_{-\alpha}
\]

if \( \alpha, \beta \in R^+ \) and \( \alpha > \beta \) in the convex order on \( R^+ \). \( \square \)

The filtration \( F \) defines \( \mathbb{Z}_+^{2N+1} \)-filtrations on \( U \) and \( u \) and \( \mathbb{Z}_+^{N+1} \)-filtrations on \( u^\pm, u^+ \otimes u^0, u^0 \otimes u^- \) etc.

We denote them by \( F \) as well.
2.4.3. **Corollary:** The graded algebra $\text{gr}^FU$ is generated by the elements $E_{\alpha}$, $F_{-\alpha}$, $\alpha \in R^+$, and $K_i^\pm$, $i = 1, \ldots, r$, subject to the relations from Lemma 2.4.2 and the following relation: $E_{\alpha}^p = F_{-\alpha}^p = 0$ for every $\alpha \in R^+$. \(\square\)

Recall that a finite dimensional associative algebra $A$ is called Frobenius if the dual module to the right regular module over $A$ is isomorphic to the left regular $A$-module: $A_L \cong \text{Hom}_k(A_R, k)$. The nondegenerate bilinear pairing $A \times A \rightarrow k$ induced by this identification is called the trace on $A$.

2.4.4. **Lemma:** Let $A$ be a finite dimensional filtered algebra such that the top component of the corresponding graded algebra is one dimensional, and $grA$ is Frobenius with the trace defined as follows:

$$(x, y) \mapsto (x \cdot y)_{\text{top}} \in (grA)_{\text{top}} \cong k.$$  

Here $\cdot$ denotes the multiplication in $grA$. Then $A$ is also Frobenius. The trace on $A$ is defined as follows:

$$(x, y) \mapsto xy \mapsto (xy)_{\text{top}} \in (grA)_{\text{top}} \cong k.$$  

The product here is the product in $A$. \(\square\)

2.4.5. **Lemma:** The algebras $u$, $u^+$ and $u^-$ are Frobenius.

**Proof.** By Lemma 2.4.3 the algebras $gru^\pm$ satisfy the conditions of the Lemma 2.4.4. Thus $u^\pm$ are Frobenius. The trace on $u$ is defined in [Xi], 2.9. \(\square\)

2.4.6. We will need a filtration on $u$ that differs a little from $F$. We set

$$d_1(M_{(s,t,\alpha)}) = \sum_{j=1}^N s_j \text{ht} \beta_j$$

and introduce the partial degree

$$\tilde{d}(M_{(s,t,\alpha)}) = (s_N, \ldots, s_1, d_1(M_{(s,t,\alpha)})).$$

Let $F'$ be the $\mathbb{Z}^{N+1}$-filtration of $u$ by the partial degree. Note that $F'$ coincides with $F$ on $u^+$ and it coincides on $u^-$ with the natural $\mathbb{Z}$-grading obtained from the $Y^-$-grading. Thus $grF'(u^-) = u^-$, $grF'(u^+) = grF(u^+)$. Recall that an augmented subalgebra $B \subset A$ is called normal if the left and the right ideals in $A$ generated by the augmentation ideal $\overline{B}$ in $B$ coincide. Then $A//B$ denotes the quotient algebra of $A$ by the two sided ideal. The algebras $u$, $u^\pm$, $grF(u)$ and $grF(u^\pm)$ are naturally augmented: the augmentation is provided by the map

$$E_{\alpha} \mapsto 0, \quad F_{-\alpha} \mapsto 0, \quad K_i \mapsto 1, \quad \alpha \in R^+, \quad i = 1, \ldots, r.$$  

2.4.7. **Lemma:** $F'$ defines a filtration on $u$ compatible with the multiplication in $u$.

$$grF'(u) = u^- \otimes u^0 \otimes grF(u^+)$$

as a vector space, $u^-$, $u^0$ and $grF(u^+)$ are subalgebras in $grF'(u)$. Finally, $grF(u^+)$ is normal in $grF'(u)$.

**Proof.** The first statement is an easy consequence of Lemma 2.4.2 since if two monomials $M_{(s,t,\alpha)}$ and $M_{(u,v,\beta)}$ are of equal total degrees then their partial degrees also coincide: $\tilde{d}(M_{(s,t,\alpha)}) = \tilde{d}(M_{(u,v,\beta)})$.

As $F$ and $F'$ coincide on $u^+$, $grF(u^+)$ is a subalgebra in $grF'(u)$.

The decomposition of $grF'(u)$ into tensor product follows from the similar statement for $u$ (see Lemma 2.3.3).
To prove that elements of $u^-$ commute with elements of $gr^F(u^+)$ in $gr^{F'}(u)$ note that for each $i = 1, \ldots, r$
\[ [F_i, M_{(s, 0, 0)}] \in u^0 \otimes u^+ \text{ and } d_1([F_i, M_{(s, 0, 0)}]) < d_1(M_{(s, 0, 0)}) \]. □

2.4.8. **Lemma:** The algebra $gr^{F'}(u)$ is Frobenius.

**Proof.** As a vector space $gr^{F'}(u) = u^- \otimes u^0 \otimes gr^F(u^+)$. Set the linear form, inducing the trace on $gr^{F'}(u)$, equal to the tensor product of the linear forms inducing the traces on the components. □

2.5. Now we are going to describe the categories of $u$-modules we will work with. $C$ is the category of $X$-graded left $u$-modules $M = \bigoplus_{\lambda \in X} \lambda M$, dim $\lambda M < \infty$, such that $K_i$ acts on $\lambda M$ by multiplication by $\zeta^{(\alpha_\gamma')_\lambda}$, where $\alpha_\gamma'$ are the simple coroots,
\[ E_i : \lambda M \rightarrow \lambda + \alpha_i M, \quad F_i : \lambda M \rightarrow \lambda - \alpha_i M. \]

Morphisms in $C$ are the morphisms of $X$-graded $u$-modules that preserve gradings. $C^{(\gamma)}$ is the category of right $X$-graded $u$-modules $N = \bigoplus_{\lambda \in X} \lambda N$, dim $\lambda N < \infty$, such that $K_i$ acts on $\lambda N$ by multiplication by $\zeta^{-(\alpha_\gamma')_\lambda}$,
\[ E_i : \lambda N \rightarrow \lambda + \alpha_i N, \quad F_i : \lambda N \rightarrow \lambda - \alpha_i N, \]
with morphisms that preserve $X$-gradings. One can define the categories $\mathcal{C}(b)$, $\mathcal{C}(gr^{F'}(u))$ and $\mathcal{C}^{(\gamma)}(gr^{F'}(u))$ in a similar way.

We define the twisting functors by elements of the weight lattice $\lambda \in pX$ on the category $C$:
\[ M \mapsto M(\lambda), \quad \text{where } \mu M(\lambda) := \lambda + \mu M, \]
with the same action of $u$. We denote the space $\bigoplus_{\beta \in X} \text{Hom}_C(L, M(p\beta))$ by $\text{Hom}_u(L, M)$.

We will also need the categories of finite dimensional left $u^\pm$-modules denoted by $u^\pm - \text{Mod}$.

We call the $u$-module $M_\lambda^+ := u \otimes_{b^-} k_{\lambda}$ (resp. the $u$-module $M_\lambda^- := u \otimes_{b^+} k_{\lambda}$) the positive (resp. negative) left Verma module with the highest (resp. lowest) weight $\lambda$. Here $k_{\lambda}$ is one-dimensional $u^0$-module placed in the $X$-grading $\lambda$, the trivial action of $u^\pm$ equips it with the structure of a $b^\pm$-module.

Contragradient Verma modules $M_\lambda^{-*}$ are defined in the following way: $M_\lambda^{-*} = \text{Hom}_{b^-}(u, k_{\lambda})$ with the natural left action of $u$. We will need the following statement.

2.5.1. **Lemma:** (see [AJS], 4.10, 4.12)

(i) $M_\lambda^+ = M_{\lambda + (p-1)2\rho}^{-*}$;

(ii) $\text{Hom}_C(M_\lambda^-, M_\mu^+) = 0$ if $\lambda \neq \mu + (p-1)2\rho$;

(iii) $\text{Hom}_C(M_\lambda^-, M_{\lambda + (p-1)2\rho}^+) = k$. □

2.5.2. Every Verma module $M_\lambda^-$ has a unique simple quotient module $L_\lambda$ with the highest weight $\lambda \in X$, and this way one obtains the full list of simple modules in the category $C$ (see [AJS], 4.1).
3. **Semiinfinite cohomology of modules over associative algebras**

Consider a free abelian group $X$ of the finite rank $r$ and its subgroup $Y$ generated by a set $\Sigma \subset X$ consisting of $r$ elements, such that the elements of $\Sigma$ form a base of the vector space $X \otimes \mathbb{Q}$. Denote the subsemigroup in $Y$ generated by the set $\Sigma$ (resp. by the set $-\Sigma$) by $Y^+$ (resp. by $Y^-$). In this section we do not suppose that $X$ and $Y$ are the weight and the root lattices corresponding to some root system, but our notation is adopted to that case.

Let $v : X \rightarrow \mathbb{Q}$ be a linear function defined as follows: for every $\alpha \in \Sigma$ $v(\alpha) = 1$, $v$ is extended on $X$ by linearity.

### 3.1. Suppose we have an $Y$-graded associative algebra $A$ with the $Y$-graded subalgebras $B$ and $N$ satisfying the following conditions:

- $(i) \ N$ is graded by $Y^+$;
- $(ii) \ N_0 = k$;
- $(iii) \ \dim N_\beta < \infty$ for any $\beta \in Y^+$;
- $(iv) \ B$ is graded by $Y^-$;
- $(v) \ \text{the multiplication in } A \ \text{defines the isomorphisms of } Y\text{-graded vector spaces } B \otimes N \rightarrow A \text{ and } N \otimes B \rightarrow A.$

In particular $N$ is naturally augmented.

Note that our conditions are satisfied both for finite quantum groups and for universal enveloping algebras of graded Lie algebras. In the latter case the decomposition into tensor product of negative and positive parts is given by the natural decomposition of the Lie algebra into the direct sum of its negatively and positively graded parts.

### 3.2. Consider the category $A\text{-mod}$ of $X$-graded left $A$-modules with morphisms that preserve $X$-gradings. The corresponding category of right $A$-modules is denoted by $-\text{mod}-A$. We will need the following subcategories in the category of complexes $\text{Kom}(A\text{-mod})$.

For an $X$-graded module $M$ denote by $\text{supp} \ M$ the set $\{ \alpha \in X \ | \ \alpha \ M \neq 0 \}$. Denote by $X^+Q(\beta)$ (resp. $X^-Q(\beta)$) the convex cone in $X \otimes \mathbb{Q}$, generated by $\Sigma$ (resp. $-\Sigma$) with the vertex in $\beta \in X \otimes k$.

For $s_1, s_2, t_1, t_2 \in \mathbb{Z}$, $s_1, s_2 > 0$, the set

\[
\{(p, q) \in \mathbb{Q}^{\otimes 2} | s_1 p + q \geq t_1, s_2 p - q \geq t_2\}
\]

(resp. the set

\[
\{(p, q) \in \mathbb{Q}^{\otimes 2} | s_1 p + q \leq t_1, s_2 p - q \leq t_2\}
\]

is denoted by $Q^+(s_1, s_2, t_1, t_2)$ (resp. by $Q^-(s_1, s_2, t_1, t_2)$).

Consider the category $C^+(A)$ (resp. $C^-(A)$) in $\text{Kom}(A\text{-mod})$, consisting of complexes of $X$-graded $A$-modules $M = \bigoplus_{q \in \mathbb{Z}, \lambda \in X} \lambda M^q$ satisfying the following conditions:

- $(U)$ there exist $s_1, s_2, t_1, t_2 \in \mathbb{Z}$, $s_1, s_2 > 0$, such that $v(\text{supp} \ M^*) \subset Q^+(s_1, s_2, t_1, t_2)$ and for any $(p, q) \in v(\text{supp} \ M^*)$ the set $v^{-1}(p, q)$ is finite; resp.
- $(D)$ there exist $s_1, s_2, t_1, t_2 \in \mathbb{Z}$, $s_1, s_2 > 0$, such that $v(\text{supp} \ M^*) \subset Q^-(s_1, s_2, t_1, t_2)$ and for any $(p, q) \in v(\text{supp} \ M^*)$ the set $v^{-1}(p, q)$ is finite.

Here the set $v(\text{supp} \ M^*)$ is considered as a subset of a $(p, q)$-plane: $v(\lambda, q) := (v(\lambda), q)$.
3.3. Now we are going to define the standard complex for the computation of semiinfinite cohomology of $A$-modules. Consider the right $N$ module

$$N^* = \operatorname{Hom}_k(N, k) = \bigoplus_{\beta \in X^+} \operatorname{Hom}_k(\beta N, k).$$

The right action of $N$ is defined as follows:

$$n \cdot f(m) = f(nm), \quad n, m \in N, \quad f \in N^*.$$

Voronov calls the right $A$-module $S_A = N^* \otimes_A \operatorname{End}_A(S_A)$ the right semiregular representation (see [V]). Obviously $S_A$ is isomorphic to $N^* \otimes_B \operatorname{End}_B(S_A)$ as a right $B$-module.

For two $X$-graded $B$-modules $L$ and $M$ denote the space $\bigoplus_{\lambda \in X} \operatorname{Hom}_B(L, M_{\langle \lambda \rangle})$, where $\langle \lambda \rangle$ is the grading shift, by $\operatorname{Hom}_B(L, M)$.

3.3.1. \textbf{Lemma}: $S_A = \operatorname{Hom}_B(A, B)$ as a right $A$-module.

\textbf{Proof}. Define the pairing $\phi : S_A \times A \longrightarrow B$ as follows:

$$\phi(f \otimes a_1, a_2) = f_1(a_1 a_2),$$

where $f_1$ denotes $f$ used by the first argument in $A \otimes B$. The required isomorphism is provided by $\phi$. \qed

Set $A^2 = \operatorname{End}_A(S_A)$. The functors of induction and coinduction provide the natural inclusions of algebras $N \hookrightarrow A^2$ and $B \hookrightarrow A^2$. Clearly up to a certain completion $A^2 = B \otimes N$ as a vector space.

3.3.2. \textbf{Lemma}: The subspace $A^2 = B \otimes N \subset A^2$ is a $Y$-graded subalgebra. \qed

Thus $S_A$ becomes an $A^2 - A$ bimodule.

3.3.3. Recall the bar construction for the algebra $A$ with respect to the subalgebra $B \subset A$. The standard bar resolution $\bar{\operatorname{Bar}}^-(A, B, M) \in \operatorname{Kom}(A\text{-mod})$ of an $A$-module $M$ is defined as follows:

$$\bar{\operatorname{Bar}}^{-n}(A, B, M) = A \otimes_B \ldots \otimes_B A \otimes_B M \text{ (n + 1 times),}$$

$$d(a_0 \otimes \ldots \otimes a_n \otimes v) = \sum_{s=0}^{n-1} (-1)^s a_0 \otimes \ldots \otimes a_s a_{s+1} \otimes \ldots \otimes v + (-1)^n a_0 \otimes \ldots \otimes a_{n-1} \otimes a_n v.$$

Here $a_0, \ldots, a_n \in A$, $v \in M$.

3.3.4. \textbf{Lemma}: The subspace $\bar{\operatorname{Bar}}^-(A, B, M)$:

$$\bar{\operatorname{Bar}}^{-n}(A, B, M) = \{ a_0 \otimes \ldots \otimes a_n \otimes v \in \bar{\operatorname{Bar}}^{-n}(A, B, M) \mid \exists s \in \{1, \ldots, n\} : a_s \in B \}$$

is a subcomplex in $\bar{\operatorname{Bar}}^-(A, B, M)$. \qed

The quotient $\bar{\operatorname{Bar}}^+(A, B, M) = \bar{\operatorname{Bar}}^-(A, B, M) / \bar{\operatorname{Bar}}^-(A, B, M)$ is called the restricted bar resolution of the $A$-module $M$ with respect to the subalgebra $B$. The ideal of augmentation in $N$ is denoted by $\overline{N}$. 

3.3.5. Lemma:

(i) \( \text{Bar}^{-n}(A, B, M) = N \otimes N^{\otimes n} \otimes M \) as a left \( N \)-module;

(ii) \( \text{Bar}^*(A, B, M) = \text{Bar}^*(N, k, M) \) as a complex of \( N \)-modules. In particular \( \text{Bar}^*(A, B, M) \) is a \( N \)-free resolution of the \( A \)-module \( M \). \( \square \)

For any \( M^* \in C^*(A) \) consider the total complex of its bar resolution \( \text{Bar}^*(A, B, M^*) \) and the complex of \( A^2 \)-modules \( S_A \otimes_A \text{Bar}^*(A, B, M^*) \)

**Remark:** Since \( S_A \) is \( B \)-free and \( \text{Bar}^*(A, B, M^*) \) is \( N \)-free

\[ H^*(S_A \otimes_A \text{Bar}^*(A, B, M^*)) = \text{Tor}^*(S_A, M^*). \]

For a complex of left \( A^2 \)-modules \( L^* \in C^*(A^2) \) we denote the total complex of its restricted bar resolution with respect to the subalgebra \( N \in A^2 \) by \( \text{Bar}^*(A^2, N, L^*) \).

3.3.6. Definition: Let \( L^* \in C^*(A^2), M^* \in C^*(A) \). The standard complex for the computation of semiinfinite Ext functor \( C^{\infty+*}(L^*, M^*) \) is defined as follows:

\[ C^{\infty+*}(L^*, M^*) = \text{Hom}_A^*(\text{Bar}^*(A^2, N, L^*), S_A \otimes_A \text{Bar}^*(A, B, M^*)). \]

By definition set \( \text{Ext}^{\infty+*}_A(L^*, M^*) = H^*(C^{\infty+*}(L^*, M^*)). \)

Note that unlike usual Ext and Tor functors semiinfinite cohomology exists both in negative and positive homological degrees even for \( L \) and \( M \) being complexes-objects (see [GeM]).

As a vector space \( C^{\infty+*}(L^*, M^*) = \bigoplus_{n,m} \text{Hom}_k(\overline{B}^{\otimes n} \otimes L^*, N^{\otimes m} \otimes M^*). \) Here \( \overline{B} \) denotes the space \( B/k \). Since both arguments of the semiinfinite Ext functor are \( X \)-graded, both the standard complex \( C^{\infty+*}(L^*, M^*) \) and its cohomology are also \( X \)-graded:

\[ \gamma C^{\infty+*}(L^*, M^*) = \bigoplus_{\alpha - \beta = \gamma; n, m} \text{Hom}_k(\beta(\overline{B}^{\otimes n} \otimes L^*), \alpha(N^{\otimes m} \otimes M^*)), \]

\[ \gamma \text{Ext}^{\infty+*}_A(L^*, M^*) = H^*(\gamma C^{\infty+*}(L^*, M^*)). \]

From now on the zeroth \( X \)-grading component of \( \text{Ext}^{\infty+*}_A(L^*, M^*) \) is denoted by \( \text{Ext}^{\infty+*}_A(L^*, M^*). \)

3.4. Consider the filtrations \( (1)F \) (resp. \( (11)F \)) on \( C^{\infty+*}(L, M) \), \( L \in A^2 \)-mod, \( M \in A \)-mod, by the number \( n \) (resp. \( m \)). The \( E_0 \)-terms of the corresponding spectral sequences are as follows:

\[ (1)E_0^{pq} = \text{Hom}_A((A^2 \otimes A^2/N)(A^2/N) \otimes_N \ldots \otimes_N (A^2/N) \otimes_N L, S_A \otimes_A \text{Bar}^q(A, B, M)) = \text{Hom}_N((A^2/N) \otimes_N \ldots \otimes_N (A^2/N) \otimes_N L, S_A \otimes_B (A/B) \otimes_B \ldots \otimes_B (A/B) \otimes_B M) \]

\[ = \text{Hom}_k(\overline{B}^{\otimes p} \otimes L, (A/B) \otimes_B \ldots \otimes_B (A/B) \otimes_B M) \]

\[ = \text{Hom}_B(\text{Bar}^{-p}(B, k, L), (A/B) \otimes_B \ldots \otimes_B (A/B) \otimes_B M) \]

with the differential being that in \( \text{Bar}^*(B, k, L) \), and

\[ (11)E_0^{pq} = \text{Hom}_k((A^2/N) \otimes_N \ldots \otimes_N (A^2/N) \otimes_N L, k) \otimes_N \text{Bar}^{-q}(N, k, M) \]
with the differential being that in Bar\textsuperscript{*}(N, k, M). Thus the $E_1$ terms are as follows:

\[(I) \ E_1^{p,q} = \text{Ext}^p_B(L, \bigotimes_B (A/B) \otimes_B \cdots \otimes_B (A/B) \otimes_B M),\]

\[(II) \ E_1^{p,q} = \text{Tor}^N_q(\text{Hom}_k((A^q/N) \otimes_N \cdots \otimes_N (A^q/N) \otimes_N L, k), M).\]

Both spectral sequences are $X$-graded.

3.4.1. **Lemma:** Let $L \in A^\dagger - \text{mod}$, supp $L \in X^-_Q(\lambda)$, $M \in A - \text{mod}$, supp $M \in X^+_Q(\mu)$. Then for a fixed $\beta \in X$ both $\beta^{((I) \ E^{p,q})}$ and $\beta^{((II) \ E^{p,q})}$ converge.

**Proof.** Since $B$ is $Y^{-}$-graded, there exists $\beta_0 \in X$ such that for every $p \geq 0$ supp $\overline{B}^p \otimes L$ belongs to $X^-_Q(\beta_0)$. The $X$-graded space $\bigoplus_{q \geq 0} N^\otimes q \otimes M$ satisfies the condition (U). Thus in a fixed $X$-grading component $\beta$ both spectral sequences of the complex $\beta C \mathfrak{F}^+\texttt{*}(L, M)$ are situated in the part of the $(p,q)$-plane which is bounded in $p$ from the left and in $q$ both from the left and from the right. $\square$

Recall that an object $M \in A$-mod is injective (resp. projective) relative to the subalgebra $N$ if for every complex of $A$-modules $C\texttt{*}$ such that $C\texttt{*}$ is homotopic to zero as a complex of $N$-modules $H\texttt{*}(\text{Hom}_A(C\texttt{*}, M)) = 0$. (resp. $H\texttt{*}(\text{Hom}_A(M, C\texttt{*})) = 0$).

3.4.2. **Lemma:** The following facts hold for $L\texttt{*} \in C^\dagger(A^\dagger)$, $M\texttt{*} \in C^\dagger(A)$:

(i) if $M\texttt{*}$ is $N$-projective, then

\[\text{Ext}^+_A(L\texttt{*}, M\texttt{*}) = H\texttt{*}(\text{Hom}_A(\text{Bar}^\texttt{*}(A^\dagger, N, L\texttt{*}), S_A \otimes_A M\texttt{*}));\]

(ii) if $L\texttt{*}$ is $B$-projective, then

\[\text{Ext}^+_A(L\texttt{*}, M\texttt{*}) = H\texttt{*}(\text{Hom}_A(L\texttt{*}, S_A \otimes_A \text{Bar}^\texttt{*}(A, B, M\texttt{*})));\]

(iii) if $M\texttt{*}$ is both $N$-projective and $A$-injective relative to $N$, then

\[\text{Ext}^+_A(L\texttt{*}, M\texttt{*}) = H\texttt{*}(\text{Hom}_A(L\texttt{*}, S_A \otimes_A M\texttt{*})).\]

**Proof.** (i) Consider the canonical mapping $\varphi : \text{Bar}^\texttt{*}(A, B, M\texttt{*}) \rightarrow M\texttt{*}$. Then Cone\textsuperscript{*} $\varphi$ is an exact complex of $N$-projective $A$-modules satisfying (U). In particular it is homotopic to zero as a complex of $N$-modules. As a complex of $N$-modules

\[S_A \otimes_A \text{Cone}^\texttt{*} \varphi = N^\texttt{*} \otimes_N \text{Cone}^\texttt{*} \varphi,\]

where $N^\texttt{*}$ is considered as a $N$-bimodule. Thus $S_A \otimes_A \text{Cone}^\texttt{*} \varphi$ is also homotopic to zero over $N$. Now by Shapiro lemma $A^\dagger$-modules induced from $N$-modules are relatively projective, so Bar\textsuperscript{*}(A^\dagger, N, L\texttt{*}) consists of relatively projective modules. Let

\[\tilde{\varphi} : C \mathfrak{F}^+\texttt{*}(L\texttt{*}, M\texttt{*}) \rightarrow \text{Hom}_A^\texttt{*}(\text{Bar}^\texttt{*}(A^\dagger, N, L\texttt{*}), S_A \otimes_A M\texttt{*})\]

be the morphism complexes corresponding to $\varphi$. Then

\[\text{Cone}^\texttt{*} \tilde{\varphi} = \text{Hom}_A^\texttt{*}(\text{Bar}^\texttt{*}(A^\dagger, N, L\texttt{*}), S_A \otimes_A \text{Cone}^\texttt{*} \varphi).\]

We prove that the latter complex is exact. Consider the bigrading on Cone\textsuperscript{*} $\tilde{\varphi}$:

\[\text{Cone}^{m,n} \tilde{\varphi} = \bigoplus_{p+q=m} \text{Hom}_A^\texttt{*}(\text{Bar}^p(A^\dagger, N, L\texttt{*}), \text{Cone}^n \varphi).\]
Our grading conditions provide that the spectral sequence of the bigraded complex converges. On the other hand
\[ E_1^{m, n} = \bigoplus_{p + s = m} H^s(\text{Hom}_{A^t}(\text{Bar}^p(A^t, N, L^s), \text{Cone}^s \varphi)) = 0. \]

(ii) The proof is similar to the previous one.

(iii) It is sufficient to prove that the mapping
\[ \text{Hom}_{A^t}(L^*, S_A \otimes_A M^*) \longrightarrow \text{Hom}_{A^t}(\text{Bar}^*(A^t, N, L^*), S_A \otimes_A M^*) \]
is a quasiisomorphism. We are going to show that if \( M \) is both \( N \)-projective and injective relatively to \( N \) then \( S_A \otimes_A N \) is injective over \( A^t \).

First note that the functor \( S_A \otimes_A * \) takes \( N \)-free modules to \( N \)-cofree ones, thus it takes \( N \)-projectives to \( N \)-injectives.

The functor \( \text{Hom}_{A^t}(S_A, *) \) is the right conjugate functor for \( S_A \otimes_A * \). It is left exact and is well defined on \( N \)-modules since it can be written as follows: \( M \mapsto \text{Hom}_N(N^*, M) \). Thus \( S_A \otimes_A * \) preserves relative injectiveness.

Finally note that a \( A^t \) module that is both \( N \)-injective and relatively injective is also \( A^t \)-injective. Thus for a finite exact complex \( P^* \) the complex \( \text{Hom}_{A^t}(P^*, S_A \otimes_A M^*) \) is exact. It remains to check the convergence of a spectral sequence similar to the one from the first statement. \( \square \)

3.5. Theorem: Semiinfinite Ext functor is well defined on the corresponding derived categories:
\[ \text{Ext}_{A^t}^\pm(L^*, M^*) : D^\pm(A^t) \times D^\pm(A) \longrightarrow D(\text{Vect}). \]
Here \( D^\pm(A^t) \) (resp. \( D^\pm(A) \)) denotes the localization of the category \( C^\pm(A^t) \) (resp. \( C^\pm(A) \)) by the class of quasiisomorphisms.

Proof. We are to prove that \( \text{Ext}_{A^t}^\pm(L^*, M^*) = 0 \) for \( L^* \in C^\pm(A^t), \ M^* \in C^\pm(A) \) if either of the arguments is exact. Suppose \( M^* \) is exact, the proof in the other case is quite similar.

We fix \( \lambda \in X \). Consider the following bigrading on \( C^\pm(L^*, M^*) \):
\[ C^\pm_{p, q}(L^*, M^*) = \bigoplus_{m-n+s=p} \text{Hom}_k((\overline{B})^\otimes m \otimes L^{-s}), (N)^\otimes n \otimes M^q). \]
The spectral sequence of the bicomplex \( \lambda C^\pm_{p, q}(L^*, M^*) \) converges since it is situated in the part of the \((p, q)\)-plane bounded in \( p \) from the left and in \( q \) both from the left and from the right. Thus the total complex \( C^\pm_{p, q}(L^*, M^*) \) is exact. \( \square \)

3.5.1. Remark: The last two statements show that one can use an arbitrary resolution of \( M^* \) that is both \( N \)-projective and injective relatively to \( N \) for the computation of \( \text{Ext}_{A^t}^\pm(L^*, M^*) \).

4. Calculation of the character of \( \text{Ext}_{C}^\pm(k, k) \).

In this section we are going to define a \( g \)-module structure on the semiinfinite cohomology of the trivial \( u \)-module. We will also calculate the character of this \( g \)-module.

From now on \( X \) and \( Y \) denote the weight and the root lattice respectively, the linear function \( v \) coincides with \( \text{ht} \).

We will need the subcategories in the category of complexes \( \text{Kom}(C) \), satisfying the condition (U) (resp. (D)) from the previous section (the category \( C \) is defined in 2.4). These categories are denoted by \( C^\uparrow \) (resp. by \( C^\downarrow \)).
4.1. Fix the triangular decomposition of the finite quantum group: $u = b^- \otimes u^+$, i.e. $A = u$, $B = b^-$, $N = u^+$ in the notations of [3.3].

4.1.1. Lemma: $u^2 = u^+$.

Proof. The algebra $u^+$ is Frobenius (see 2.4.4), thus the right $u$-module $S_u$ is isomorphic to the right regular $u$-module. But for any finite dimensional algebra the algebra of endomorphisms of the right regular module is isomorphic to the algebra itself. □

Note that the category $C$ differs from the category $u$-mod thus to define semiinfinite cohomology of $u$-modules one has to introduce a $X$-graded algebra $A$ such that $C \cong A$-mod (the algebra $A$ is constructed in particular in [AJS], Remark 1.4) and to consider $\text{Ext}_{A}\text{-mod}(\cdot, \cdot)$ or to define semiinfinite cohomology in the category $C$ explicitly.

Note that for $M \in C$ the complexes $\text{Bar}^\ast(u, b^+, M)$ and $\text{Bar}^\ast(u, b^-, M)$ belong to $\mathcal{K}/\mathcal{F}(C)$.

4.1.2. Definition: For $L^\ast \in C^\downarrow$, $M^\ast \in C^\uparrow$ the semiinfinite Ext functor is defined as follows:

$$\text{Ext}^{\mathbb{Z}^+\ast}(L^\ast, M^\ast) := H^\ast(\text{Hom}_{u}^\ast(\text{Bar}^\ast(u, b^+, L^\ast), \text{Bar}^\ast(u, b^-, M^\ast))).$$

Note that by Lemma 4.1.1 the module $S_u$ is isomorphic to the right regular $u$-module, so the definition is a direct analogue of 3.3.4. In particular the statements of Lemma 3.4.2 remain true.

4.1.3. The subalgebra $u^+ \subset b^+$ (resp. $u^- \subset b^-$) is normal. Clearly $b^+/u^+ = u^0$ is semisimple being the group algebra of the group $\mathbb{Z}/2p\mathbb{Z}$. In particular a $u$-module $L$ is $b^+$-projective (resp. $b^-$-projective) if and only if it is $u^+$-projective (resp. $u^-$-projective).

4.2. Consider the following $u^+$-free $u^-$-cofree resolution of a $U_3$-module $M^\ast \in C^\uparrow(U_3)$:

$$R^\ast(M^\ast) := \text{Bar}^\ast(U_3, B_3^+, k)^\ast \otimes \text{Bar}^\ast(U_3, B_3^-, M^\ast).$$

Here $B_3^\pm \subset U_3$ denotes the positive and the negative Borel subalgebras in the "big" quantum group $U_3$. The definition of the tensor product over the base field uses the standard Hopf algebra structure on $U_3$.

The left $U_3$-module structure on

$$\text{Bar}^\ast(U_3, B_3^+, k)^\ast := \bigoplus_{\lambda \in X} \text{Hom}_k(\lambda \text{ Bar}^\ast(U_3, B_3^+, k), k)$$

is defined using the antipode in $U_3$.

Evidently $R^\ast(M^\ast)$ satisfies the condition of Lemma 3.4.2(iii), in particular it satisfies the condition (U). Thus it can be used for the computation of semiinfinite cohomology of $u$-modules.

4.2.1. Lemma: Let $M^\ast \in C^\uparrow(U_3)$. Then $\bigoplus_{\lambda \in X} \text{Ext}^{\mathbb{Z}^+\ast}(k, M^\ast\langle p\lambda \rangle)$ admits a structure of $g$-module.

Proof. In [L1], Theorem 8.10, it is proved that the algebra $u$ is normal in $U_3$, and the quotient algebra $U_3/\mathfrak{u}$ is isomorphic to the universal enveloping algebra of the semisimple Lie algebra $\mathfrak{g}$. Thus by Shapiro lemma

$$\bigoplus_{\lambda \in X} \text{Ext}^{\mathbb{Z}^+\ast}(k, M^\ast\langle p\lambda \rangle) = H^\ast(\text{Hom}^\ast_{\mathfrak{u}}(k, R^\ast(M^\ast))) = H^\ast(\text{Hom}^\ast_{U_3}(U_3/\mathfrak{u}, R^\ast(M^\ast))).$$

The left $U_\mathfrak{g}$-module $U_3/\mathfrak{u}$ is naturally equipped with a right action of the quotient algebra $U_3/\mathfrak{u} = U(\mathfrak{g})$ commuting with the left action of $U_3$. Thus the semiinfinite Ext spaces carry the natural structure of $g$-modules. □
4.3. The rest of this section is devoted to the computation of the character of \( \text{Ext}^\infty_{\mathcal{F}}(k, k<p\lambda>). \) The problem is that we do not know “the minimal” \( u^-\)-free resolution of the trivial \( u^-\)-module \( k.\)

4.3.1. **Conjecture**: There exists a resolution \( R_{\min}^+(k) \) of the trivial \( u^-\)-module \( k, \) filtered by Verma modules \( M_\lambda^- \), such that the character of the space spanned by the highest weight vectors \( v_\lambda \in M_\lambda^- \) is given by the formal series

\[
ch(t) := \sum_{\mu \in W} e^{w(\mu) - p\lambda t^2} \prod_{\alpha \in R^+} (1 - e^{-p\alpha t^2}).
\]

Here \( \{e^\alpha\} \) is the standard notation for the \( R^-\)-grading, and \( t \) is the variable denoting the homological degree. \( \square \)

The word “minimal” is explained by the following result of Ginzburg and Kumar.

4.3.2. **Lemma**: (see [GK], Theorem 2.5.)

\[
ch(\text{Ext}^+_u(k, k), t) = \frac{\sum_{\mu \in W} e^{-w(\mu) + p\lambda t^2}}{\prod_{\alpha \in R^+} (1 - e^{p\alpha t^2})} \quad \square
\]

One can construct \( R_{\min}^+(k) \) for \( u(sl_2) \) explicitly, and one easily obtains the following statement.

4.3.3. **Lemma**:

\[
ch(\bigoplus_{\lambda \in \mathbb{Z}} \text{Ext}^+_{\mathcal{C}(u(sl_2))}(k, k(p\lambda)), t) = \frac{e^{p\alpha}(t + t^{-1})}{(1 - e^{p\alpha t^2})(1 - e^{p\alpha t^{-2}})}
\]

Here \( \alpha \) is the only positive root of \( sl_2. \) \( \square \)

4.4. To obtain a character formula for semiinfinite cohomology over other finite quantum groups we are going to use the filtrations \( F^\nu \) on quantum groups defined in 2.4.6. We begin with the calculation of \( \bigoplus_{\lambda \in X} \text{Ext}^+_{\mathcal{C}(\mathfrak{gr}^F(u))}(k, k(p\lambda)), \) using the following statement.

4.4.1. **Lemma**: There exists a \( \text{gr}(u^+)\)-free resolution of the trivial \( \mathfrak{gr}^F(u)\)-module \( k \) with a space of \( \mathfrak{gr}(u^+)\)-generators \( \bigotimes_{\alpha \in R^+} A(\xi^\alpha) \otimes \bigotimes_{\alpha \in R^+} S(\eta^\alpha). \) Here \( \{\xi^\alpha\} \) denotes the exterior algebra generators of the homological degree \(-1\) and the \( X\)-grading \( \alpha, \) \( \{\eta^\alpha\} \) denotes the symmetric algebra generators of the homological degree \(-2\) and the \( X\)-grading \( p\alpha. \)

**Proof.** Consider the subalgebras \( k(x_\alpha) \), \( \alpha \in R^+ \), in \( \mathfrak{gr}(u^+) \) generated by the images of the root elements from \( u^+. \) Each algebra \( k(x_\alpha) \) is isomorphic to \( k[x_\alpha]/(x_\alpha^2) \), and \( \mathfrak{gr}^F(u^+) \) is the twisted tensor product of the algebras \( k(x_\alpha). \) That is, \( \mathfrak{gr}^F(u^+) \cong \bigotimes_{\alpha \in R^+} k(x_\alpha) \) as a vector space, and the following relations are satisfied:

\[
(\ast) \quad x_\alpha x_\beta = \xi^{(\alpha, \beta)} x_\beta x_\alpha
\]

when \( \alpha < \beta \) in the chosen convex ordering on the set of positive roots.

Moreover as proved by Lemma 2.4.7 elements from \( u^- \) commute with elements of \( \mathfrak{gr}^F(u^+) \) in \( \mathfrak{gr}^F(u). \) Thus it is enough to construct \( R_{\min}^+(k) \) for every algebra \( k(x_\alpha) \) and to take tensor product of these resolutions over the set of positive roots.
For the latter algebra the required resolution looks as follows:

\[ 0 \rightarrow k(x_{\alpha})x_{\alpha}^p \rightarrow k(x_{\alpha})x_{\alpha} \rightarrow k(x_{\alpha}) \xrightarrow{c} k \rightarrow 0. \]

The algebra \( gr^F(u^+) \) acts on the tensor product of such complexes by the set of positive roots by the commutation rule (*), the action of \( u^t \) on it comes from the \( X \)-grading and \( u^t \) acts on the complex by zero. □

We denote this resolution by \( R_{\text{min,gr}^F(u)}^{-•}(k) \).

### 4.4.2. Proposition:

\[
ch \left( \bigoplus_{\lambda \in X} \Ext^•_{C(gr^F(u))}(k, k(p\lambda)), t \right) = t^{-\dim p} \prod_{\alpha \in \mathbb{R}^+} (1 - e^{p\alpha t^2})(1 - e^{p\alpha t^{-2}}).
\]

Here as before \( t \) denotes the homological grading, \( \{e^\alpha\} \) is the standard notation for the \( X \)-grading. The equality is understood as an equality of power series in variables numbered by the set of the generators of \( X \) with coefficients in \([k[t, t^{-1}]]\).

**Proof.** \( gr^F(u^+) \) is a Frobenius algebra, hence \( gr^F(u)^{\sharp} = gr^F(u) \), and

\[
\bigoplus_{\lambda \in X} \Ext^•_{C(gr^F(u))}(k, k(p\lambda)) = H^•(\text{Hom}^•_{gr^F(u)}(P^{-•}(k), R_{\text{min,gr}^F(u)}^{-•}(k))).
\]

Here \( P^{-•}(k) \) is an arbitrary \( u^- \)-free resolution of \( k \) belonging to \( C^• \). The fact that \( gr^F(u^+) \) is Frobenius also implies that \( R_{\text{min,gr}^F(u)}^{-•}(k) \) consists of \( gr^F(u^-) \)-cogenerators, and the space of cogenerators

\[
\bigotimes_{\alpha \in \mathbb{R}^+} \Lambda(\xi^\alpha) \otimes \bigotimes_{\alpha \in \mathbb{R}^+} S(\eta^\alpha) \otimes k_{(p-1)2\rho}.
\]

Consider the spectral sequence of the bicomplex

\[
\text{Hom}^•_{gr^F(u)}(P^{-•}(k), R_{\text{min,gr}^F(u)}^{-•}(k)).
\]

The term \( E_1 \) looks as follows:

\[
\bigoplus_{\lambda \in X} \Ext^•_{C(b^-)}(k, \bigotimes_{\alpha \in \mathbb{R}^+} \Lambda(\xi^\alpha) \otimes k_{(p-1)2\rho}(p\lambda)) \otimes \bigotimes_{\alpha \in \mathbb{R}^+} S(\eta^\alpha),
\]

where the space \( \bigotimes_{\alpha \in \mathbb{R}^+} \Lambda(\xi^\alpha) \otimes k_{(p-1)2\rho} \) is the direct sum of one dimensional \( b^- \)-modules.

### 4.4.3. Lemma: (see [GK], Theorem 2.5.)

(i) \( \bigoplus_{\lambda \in X} \Ext^•_{C(b^-)}(k, k_{\mu}(p\lambda)) = 0 \) when \( \mu \neq w(\rho) - \rho \);

(ii) for \( \mu = w(\rho) - \rho \)

\[
ch \left( \bigoplus_{\lambda \in X} \Ext^•_{C(b^-)}(k, k_{\mu}(p\lambda)), t \right) = \frac{t^l(w)}{\prod_{\alpha \in \mathbb{R}^+} (1 - e^{p\alpha t^{-2}})}.
\]

□
4.4.4. Lemma: (see [J], part II, Lemma 12.10) Let $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ be the two sets of pairwise distinct positive roots such that

$$\alpha_1 + \ldots + \alpha_k \equiv \beta_1 + \ldots + \beta_m \mod (pX).$$

Then for $p > 2(h - 1)$

$$\alpha_1 + \ldots + \alpha_k = \beta_1 + \ldots + \beta_m.$$

Here $h$ denotes the Coxeter number of the root system (see [J], p.262).

For any element of the Weyl group $w \in W$ there exists a unique element $w' \in W$, such that $w(\rho) + w'(\rho) = 0$. In [GK], 2.5 it is proved that for every $w' \in W$

$$\dim_{\rho - w'(\rho)} \bigotimes_{\alpha \in R^+} \Lambda(\xi^{\alpha}) = 1.$$

Thus for every $w \in W$

$$\dim_{w(\rho) - \rho} \bigotimes_{\alpha \in R^+} \Lambda(\xi^{\alpha}) \otimes k_{-2\rho} = 1.$$

Noting that $l(w') = \dim n^- - l(w)$ if $w(\rho) + w'(\rho) = 0$ we see that all the nonzero entries of the term $E_1$ of the spectral sequence are of the same parity whence it degenerates, and the proposition follows from Lemmas 4.4.3 and 4.4.4.

4.5. Theorem:

$$ch\left( \bigoplus_{\lambda \in X} \text{Ext}^{\infty +}_{\mathcal{C}^+}(k, k_{\langle p\lambda \rangle}), t \right) = t^{l - \dim n^-} e^{2p\rho} \sum_{w \in W} t^{2(w)} \prod_{\alpha \in R^+} \left( 1 - e^{p\alpha t} \right)^2 \left( 1 - e^{p\alpha t^{-2}} \right).$$

Proof. The spectral sequence arising from the filtration of the complex $C^{\infty +}(k, k)$, which is induced by the filtration $\mathcal{P}'$ on $u$, degenerates as all the nonzero semiinfinite cohomology of the trivial module over $\text{gr}^F(u)$ are of the same parity.

5. Tor$_{\mathcal{P}^+\cdot}(L, M)$ AND CONFORMAL BLOCKS

In this section we are going to compare our semiinfinite cohomology of quantum groups with the functor defined by Finkelberg and Schechtman.

5.1. First we define the semiinfinite homology of quantum groups.

5.1.1. Definition: For $M \in \mathcal{C}$, $L \in \mathcal{C}^{(r)}$ we set

$$\text{Tor}_{\mathcal{P}^+\cdot}(L, M) = 0(H^\bullet(P_{u^-}(L) \otimes u P_{u^+}^\bullet(M))).$$

Here $P_{u^-}^\bullet(L) \in \mathcal{C}^{(r)\downarrow}$ is a $u^-$-free resolution of $L$, $P_{u^+}^\bullet(M) \in \mathcal{C}^\uparrow$ is a $u^+$-free resolution of $X$; and $0(H^\bullet(\ldots))$ denotes the zeroth $X$-grading component.

Like in the third section, one can easily check that the definition doesn’t depend on the choice of resolutions. Alternatively, this is a corollary of the following comparison Lemma:
5.1.2. **Lemma:** \( \text{Ext}^r_{\mathcal{C}}(L, M) = \text{Tor}^r_{\mathcal{C}}(M^*, L)^* \).

**Proof.** The statement follows immediately from the definition of semi-infinite homology and the standard relation between \( \text{Hom}_u \) and \( \otimes_u \). \( \square \)

In particular, semi-infinite Tor is well defined as a functor on the corresponding derived categories of \( u \)-modules (see 3.5). We will need the following reformulation of the statement of Lemma 3.4.2(iii) on the language of semi-infinite Tor functor.

5.1.3. **Lemma:** For \( L^* \in \mathcal{C}^{(r)}_1 \), \( M^* \in \mathcal{C}^1 \) if \( M^* \) is both \( u^+ \)-projective and \( u \)-injective relative to \( u^- \) then \( \text{Tor}^r_{\mathcal{C}}(L^*, M^*) = (0)(H^*(L^* \otimes_u M^*)) \). \( \square \)

Note that since both algebras \( u \) and \( u^+ \) are Frobenius (see 2.4.4), the condition of the previous lemma means simply that \( M^* \in \mathcal{C}^1 \) consists of \( u \)-projective modules.

5.1.4. **Remark:** M. Finkelberg and V. Schechtman gave a geometric definition of semi-infinite Tor functor in the category \( \mathcal{C} \) (see [FiS], part IV). In [Ar] and [FiS], part IV, it is proved that the geometric definition coincides with the one presented here.

5.2. Semi-infinite homology plays an important part in the calculation of conformal blocks. Recall the definition of conformal blocks (see e.g. [A]).

Suppose for simplicity that our Cartan matrix is symmetric. Let \( \gamma \in R^\vee \) be the highest root, and denote by \( \Delta \in \mathcal{X} \) the first alcove,

\[
\Delta = \{ \lambda \in X | \langle \gamma, \gamma + \rho \rangle < p, \langle \alpha_i^\vee, \lambda + \rho \rangle > 0, i = 1, \ldots, r \}.
\]

For \( \lambda_1, \ldots, \lambda_n \in \Delta \) the conformal blocks \( (L_{\lambda_1}, \ldots, L_{\lambda_n}) \) are defined as the maximal trivial direct summand in \( L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_n} \).

5.3. **Theorem:** There exists a natural inclusion

\[
\varphi : (L_{\lambda_1}, \ldots, L_{\lambda_n}) \hookrightarrow \text{Tor}^r_{\mathcal{C}}(k, L_{\lambda_1} \otimes \ldots \otimes L_{\lambda_n} \otimes L_{(p-1)2\rho}).
\]

The proof follows immediately from the definition of the conformal blocks and the following statement.

5.3.1. **Proposition:** \( \text{Tor}^r_{\mathcal{C}}(k, L_{(p-1)2\rho}) = k \).

**Proof.** Using 5.1.2 we are going to prove the corresponding statement for semi-infinite cohomology.

Choose left resolutions of \( k \) and \( L_{(p-1)2\rho} \) beginning with \( M_0^+ \) and \( M_{(p-1)2\rho}^- \) respectively and satisfying the conditions (U) and (D) respectively. Such resolutions exist, for example one can take the standard resolution \( \text{Bar}^*(u, b^-, \overline{M}_0^-) \) of the kernel of the canonical projection \( M_0^+ \rightarrow k \) (resp. the standard resolution \( \text{Bar}^*(u, b^+, \overline{M}_{(p-1)2\rho}^-) \) of the kernel of the canonical projection \( M_{(p-1)2\rho}^- \rightarrow L_{(p-1)2\rho} \)). Denote these resolutions by \( P(-)^*(L_{(p-1)2\rho}) \) and \( P(\rightarrow)^*(k) \) respectively.

Using (2.5.1(iii)) we see that \( \text{Hom}_C(P(-)^*(L_{(p-1)2\rho}), P(\rightarrow)^*(k)) \) is nonzero only when \( n = m = 0 \), and

\[
\text{Hom}_C(P(-)^0(L_{(p-1)2\rho}), P(\rightarrow)^0(k)) = k.
\]

5.3.2. The inclusion \( \varphi \) in general is not bijective. Consider the following example: \( \mathfrak{g} = \mathfrak{sl}_2, p = 5, n = 4, \lambda_1 = \lambda_2 = 2, \lambda_3 = \lambda_4 = 3 \) (we have identified \( X \) with \( \mathbb{Z} \), so \( \rho = 1 \), and \( (p-1)2\rho = 8 \)).
5.3.3. Lemma: \( \text{Tor}^{\mathcal{C}}_{-1}(k, P_0) = k \), where \( P_0 \) is the projective covering of the trivial module.

Proof. The statement follows easily from 5.1.3. \( \square \)

As the maximal trivial direct summand of \( P_0 \) is zero, it is enough to find this module among the direct summands of \( L_2 \otimes L_2 \otimes L_3 \otimes L_3 \otimes L_3 \). If we find a projective direct summand \( P \) in \( V = L_2 \otimes L_2 \otimes L_3 \otimes L_3 \) with the highest weight 0, then \( V \otimes L_8 \) will contain a projective direct summand with the highest weight 8, i.e. \( P_0 \).

It is well known (see [L2], Proposition 7.1), that all the modules \( L_\lambda, \lambda \in \Delta \), lift to the simple \( U_3 \)-modules. It follows from the results of \([A]\) that for \( g = \mathfrak{sl}_2 \), \( \lambda_1, \ldots, \lambda_n \in \Delta \), the module \( L_{\lambda_1} \otimes \cdots \otimes L_{\lambda_n} \) is a direct sum of projective \( U_3 \)-modules and simple \( U_3 \)-modules with highest weights in \( \Delta \).

Thus the \( U_3 \)-module \( L_2 \otimes L_2 \otimes L_3 \otimes L_3 \) contains the idencomposable projective \( U_3 \)-module \( \tilde{P}_8 \) (which is the projective covering of the simple \( U_3 \)-module \( \tilde{L}_8 \)) with the highest weight 10 as an \( U_3 \)-direct summand. One can check easily that when restricted to \( u \) the module \( \tilde{P}_8 \) contains as a direct summand \( P_{-2} \) — the projective covering of the \( u \)-module \( L_{-2} \). But the highest weight of \( P_{-2} \) is 0.

We conclude that \( L_2 \otimes L_2 \otimes L_3 \otimes L_3 \otimes L_8 \) contains a direct summand \( P_0 \). Hence the semiinfinite Tor is strictly bigger than conformal blocks.

5.4. Now we construct a certain duality on semiinfinite Tor spaces which corresponds to Poincaré duality in Finkelberg-Schechtman interpretation.

5.4.1. Denote by \( \tilde{u} \) the finite quantum group defined in the same way as \( u \), but with \( \zeta \) replaced by \( \zeta^{-1} \) in the defining relations.

5.4.2. Lemma: The map \( \phi : E_i \mapsto F_i, F_i \mapsto E_i, K_i \mapsto K_i^{-1} \) defines an antiisomorphism of algebras \( \phi : u \rightarrow \tilde{u} \). \( \square \)

Denote by \( \tilde{\mathcal{C}} \) the category of left \( \tilde{u} \)-modules satisfying the conditions of the type 3.2. Define the functor

\[
D : \mathcal{C} \longrightarrow \tilde{\mathcal{C}}^{\text{opp}}, \quad D(M) = \text{Hom}_k(M, k)
\]

with the natural left action of \( \tilde{u} \) constructed as follows:

\[
u \cdot f(m) := f(\phi^{-1}(\nu)m), \quad \text{where} \ u \in \tilde{u}, f \in \text{Hom}_k(M, k).
\]

One can check directly that the supports of the modules \( M \in \mathcal{C} \) and \( D(M) \in \tilde{\mathcal{C}} \) coincide.

5.4.3. Proposition: There exists a nondegenerate pairing

\[
\langle \ , \ \rangle : \text{Tor}^{\mathcal{C}}_{-1}(k, M) \times \text{Tor}^{\mathcal{C}}_{-1}(k, D(M)) \longrightarrow k.
\]

Proof. By 5.1.2

\[
\text{Tor}^{\mathcal{C}}_{-1}(k, M) = (\text{Ext}^{\mathcal{C}}_{\mathcal{C}}(M, k))^*, \quad \text{Tor}^{\mathcal{C}}_{-1}(k, D(M)) = (\text{Ext}^{\mathcal{C}}_{\mathcal{C}}(D(M), k))^*.
\]

thus it is enough to construct a nondegenerate pairing

\[
\text{Ext}^{\mathcal{C}}_{\mathcal{C}}(M, k) \times \text{Ext}^{\mathcal{C}}_{\mathcal{C}}(D(M), k) \longrightarrow k.
\]

Choose a resolution \( R^\bullet(k) \in \mathcal{C}^k \) consisting of projective \( u \)-modules. Then by 5.1.3

\[
\text{Ext}^{\mathcal{C}}_{\mathcal{C}}(M, k) = 0(H^\bullet(\text{Hom}^\bullet_u(M, R^\bullet(k)))), \quad \text{and by definition of } D
\]

\[
\text{Ext}^{\mathcal{C}}_{\mathcal{C}}(D(M), k) = 0\left(H^\bullet\left(\text{Hom}^\bullet_u(D(M), D(R^\bullet(k)))\right)\right) = 0(\text{Hom}^\bullet_u(R^\bullet(k), M)).
\]
We may assume that $R^\bullet(k)$ consists of modules of the form $L = \text{Coind}_u(V)$. For such a module consider the canonical isomorphism

$$μ : 0(\text{Hom}_u(M, L)) \xrightarrow{\sim} 0(\text{Hom}_u(V, M)) \xrightarrow{\sim} 0(\text{Hom}_u(\text{Ind}_u(V), M))^* \xrightarrow{\sim} 0(\text{Hom}_u(L, M))^*$$

The last equality uses the fact that $\text{Ind}_u(V) = \text{Coind}_u(V)$ since all the algebras $u^+, u^-$ and $u$ are Frobenius.

We are to check that $μ$ commutes with morphisms of $u$-modules $L_1 \to L_2$. But the general statement follows easily from the statement for $L_1$ and $L_2$ being $u$-free. So we are to check that $μ : \text{Hom}_u(M, L)^* \xrightarrow{\sim} \text{Hom}_u(M, R^\bullet(k))$ is an isomorphism of right $u$-modules. This can be verified directly. So $μ$ induces a nondegenerate pairing of the complexes:

$$\langle , \rangle : \text{Hom}_u(R^\bullet(k), M) \times \text{Hom}_u(M, R^\bullet(k)) \to k.$$

that becomes the required pairing on the semiinfinite Tor spaces. □

**Appendix A. Distributions on the nilpotent cone**

In this section we are going to give a geometric interpretation of the character of the semiinfinite cohomology of the trivial module over the finite quantum group.

A.1. Consider the nilpotent cone $N \subset \mathfrak{g}$. $N$ is a singular affine algebraic variety containing the positive nilpotent subalgebra $n \subset N \subset \mathfrak{g}$. Denote by $\mathcal{F}(N)$ the space of complex algebraic functions on $N$. The adjoint action of $\mathfrak{g}$ preserves $N$ and induces a representation of $\mathfrak{g}$ in $\mathcal{F}(N)$. Our considerations are parallel to the following result due to Ginzburg and Kumar [GK].

A.1.1. Proposition: (see [GK], Theorem 5) (i) $\bigoplus_{\lambda \in X} \text{Ext}_{\mathfrak{g}}^{2n+1}(k, k(p\lambda)) = 0$ for any $n \geq 0$;

(ii) $\bigoplus_{\lambda \in X} \text{Ext}_{\mathfrak{g}}^{2n}(k, k(p\lambda)) = \mathcal{F}^n(N)$ as $\mathfrak{g}$-modules. Here the grading on the right hand side is the grading by homogeneous degree of functions on $N$. □

B. Feigin has proposed the following conjecture.

A.1.2. Conjecture: The $\mathfrak{g}$-module $\bigoplus_{\lambda \in X} \text{Ext}_{\mathfrak{g}}^{2n+1}(k, k(p\lambda))$ is isomorphic to the $\mathfrak{g}$-module of distributions on $N$ with support in $n \subset N$. □

To formulate the exact statement we will need several well known facts about the geometry of the nilpotent cone.

We will need the Grothendieck-Springer resolution of $N$. Choose a maximal torus $T \subset G$ and a Borel subgroup $T \subset B \subset G$. Consider the flag variety $B = G/B$. Let $\mathfrak{b} \subset \mathfrak{g}$ be the Lie algebra of $B$, and let $n$ be its nilpotent radical.
A.1.3. **Lemma:** (see [CG], 3.1.36) (i) The natural map
\[ \sigma : T^*(B) \rightarrow N, \]
where \( T^*(B) \) is the cotangent bundle to \( B \), is a resolution of singularities of \( N \).
(ii) \[ \sigma^{-1}(n) = \bigcup_{w \in W} T_{C_w}^* B \subset T^*(B), \]
where \( T_{C_w}^* B \) denotes the conormal bundle to a \( B \)-orbit \( C_w \). \[ \square \]

Denote the union of the conormal bundles to the \( B \)-orbits \( C_w \) by \( S \).

A.2. We use some results and methods due to Kempf [K]. The notion of cohomology with support in a locally closed subvariety was investigated in that paper. Kempf also introduced the action of a Lie algebra on local cohomology in the case when the corresponding Lie group acts on the ambient space.

We denote the local cohomology of \( X \) with support in \( Y \subset X \) and with coefficients in a sheaf \( F \) by \( H^\bullet_{\sigma}(X,F) \).

A.2.1. **Lemma:**
(i) \( \mathfrak{g} \) acts naturally on \( H^\bullet_{\sigma}(N,\mathcal{O}_N) \). Here \( \mathcal{O}_N \) denotes the structure sheaf of \( N \).
(ii) There is a natural isomorphism of \( \mathfrak{g} \)-modules \( H^\bullet_{\sigma}(N,\mathcal{O}_N) \cong H^\bullet_{\sigma}(T^*(B),\mathcal{O}_{T^*(B)}) \).

**Proof.** (i) Follows from [K], Lemma 11.1.
(ii) The existence of the isomorphism of vector spaces in question follows from the fact that \( R^0\sigma_*\mathcal{O}_{T^*(B)} = \mathcal{O}_N \), and \( R^i\sigma_*\mathcal{O}_{T^*(B)} = 0 \). The isomorphism commutes with the \( \mathfrak{g} \)-action since \( \sigma \) is \( G \)-equivariant. \[ \square \]

We will calculate the character of the \( \mathfrak{g} \)-module \( H^\bullet_{\sigma}(T^*(B),\mathcal{O}_{T^*(B)}) \). Denote by \( s \) the homological grading. The grading by the homogeneous degree is denoted by \( t \).

A.2.2. **Theorem:**
\[ ch \left( H^\bullet_{\sigma}(T^*(B),\mathcal{O}_{T^*(B)}), t \right) = e^{2p} \sum_{w \in W} \prod_{\alpha \in R^+} \frac{t^{\ell(w)}}{(1 - e^{\alpha t})(1 - e^{\alpha t^{-1}})^{s^{\dim n^-}}}, \]
the equality here is the equality of power series in the variables numbered by the generators of \( X \) with coefficients in \( k[t,t^{-1}] \).

**Proof.** The statement follows immediately from the Lemmas A.2.4, A.2.5 and A.2.6. \[ \square \]

Comparing the answer with Theorem A.3.1 we obtain the following fact.

A.2.3. **Corollary:** Up to a shift of grading in \( t \) the character of the semiinfinite cohomology of the trivial module over the finite quantum group coincides with the character of \( H^\bullet_{\sigma}(N,\mathcal{O}_N) \). \[ \square \]
A.2.4. **Lemma:** \( \text{ch} \left( H^*_T \left( T^*(B), O_{T^*(B)} \right), t \right) = \sum_{w \in W} \text{ch} \left( H^*_T \left( T^*(B), O_{T^*(B)} \right), t \right). \)

**Proof.** Fix a total linear order on the set \( \{ C_w \mid w \in W \} \) compatible with the natural partial order by inclusion.

We introduce a the filtration on \( S \) by subspaces \( S_w := \bigcup_{l(w') < w} T^*_w B. \)

We prove by induction that

\[
\text{ch} \left( H^*_S \left( T^*(B), O_{T^*(B)} \right), t \right) = \sum_{w' \leq w} \text{ch} \left( H^*_T \left( C_{w'}, (B), O_{T^*(B)} \right), t \right).
\]

For \( w = 1, C_w = pt \), the statement is evident.

If \( w \) follows \( w' \) directly in the total linear order then \( S_{w'} \) is closed in \( S_w \) and \( S_w \setminus S_{w'} = T^*(C_w) \). Then by definition of local cohomology there exists a long exact sequence

\[
\ldots \longrightarrow H^i_{S_{w'}} \left( T^*(B), O_{T^*(B)} \right) \longrightarrow H^i_{S_w} \left( T^*(B), O_{T^*(B)} \right) \longrightarrow H^i_{C_w} \left( T^*(B), O_{T^*(B)} \right) \longrightarrow H^{i+1}_{S_w} \left( T^*(B), O_{T^*(B)} \right) \longrightarrow \ldots
\]

The cohomology of a nonsingular variety with support in a nonsingular (locally closed) subvariety is nonzero only in one degree equal to the codimension of the subvariety, thus the statement is proved. \( \square \)

A.2.5. **Lemma:** (see \([K], 6.4\)) There exists a \( T \)-equivariant neighbourhood of a \( B \)-orbit \( P(C_w) \) such that

(i) \( P(C_w) \cong \prod_{\alpha \in R^+, w(\alpha) \in R^-} V(\rho - w(\rho)) \times \prod_{\alpha \in R^-, w(\alpha) \in R^-} V(\rho - w(\rho)) \)

as topological spaces with the action of \( T \). Here \( V(\lambda) \) denotes the one dimensional representation of \( T \) of weight \( \lambda \);

(ii) under this identification \( C_w \) corresponds to the first factor. \( \square \)

A.2.6. **Lemma:**

\[
\sum_{w \in W} l^{(w)} \left( \prod_{\alpha \in R^+, w(\alpha) \in R^+} (1 - e^\alpha t) \prod_{\alpha \in R^+, w(\alpha) \in R^-} (1 - e^\alpha t^{-1}) \right) = \left( \prod_{\alpha \in R^+} (1 - e^\alpha t) \right) \left( \sum_{w \in W} l^{(w)} \right). \square
\]

**APPENDIX B. SEMIINFINITE COHOMOLOGY AS A DERIVED FUNCTOR**

Here we discuss the notion of a \( K \)-semijective complex introduced by Voronov (see \([V]\)) and relations between our approach to semiinfinite cohomology of associative algebras and Voronov’s investigation of Lie algebras’ semiinfinite cohomology.

It turns out that Voronov’s results remain true not only in the case of graded Lie algebras but also in a more general setting of an associative algebra \( A \) equipped with a triangular decomposition \( A = B \otimes N \). So we remain in the situation of \([V]\).

We begin with recalling several basic results of semiinfinite homological algebra.
B.1. Let $\mathcal{O}(A)$ be the category of $X$-graded $A$-modules $M = \bigoplus_{\lambda \in X} \chi M$ satisfying the following condition: there exist $\beta_1, \ldots, \beta_s \in X$ such that $\text{supp} M \subset \bigcup_{i=1}^s X^+ Q(\beta_i)$, with morphisms being morphisms of $A$-modules that preserve $X$-gradings. The corresponding category of $X$-graded $N$-modules is denoted by $\mathcal{O}(N)$.

We denote the homotopy category of unbonded complexes over $\mathcal{O}(A)$ (resp., over $\mathcal{O}(N)$) with morphisms being morphisms of complexes modulo nillhomotopies, by $K(A)$ (resp., by $K(N)$).

B.1.1. **Definition:** (see [V], 3.3) An object $S^\bullet \in K(A)$ is called $K$-semijective if

(i) it is $K$-projective over $N$, i.e. $\text{Hom}_{K(N)}(S^\bullet, V^\bullet) = 0$ for any (possibly unbounded) complex $V^\bullet \in K(N)$;

(ii) it is $K$-injective over $A$ relative to $N$, i.e. $\text{Hom}_{K(A)}(V^\bullet, S^\bullet) = 0$ for any complex $V^\bullet \in K(A)$ such that $\text{Hom}_{K(N)}(V^\bullet, V^\bullet) = 0$ (in particular $V^\bullet$ is acyclic).

Note that our definition is “turned upside down” with respect to the Voronov’s one.

Note also that $A$-modules that are both $N$-projective and $A$-injective relative to $N$ (see 2.4.2) evidently are $K$-semijective. We call such $A$-modules semijective (without $K$).

B.1.2. **Lemma:** (i) Any bounded from above complex of $N$-projective $A$-modules is $K$-projective over $N$;

(ii) any bounded from below complex of $A$-injective relative to $N$ modules is $K$-injective relative to $N$;

(iii) in particular any bounded complex of semijective modules is $K$-semijective.

The following statement is the main achievement of semiinfinite homological algebra.

B.2. **Theorem:** (see [V], Theorem 3.3) Let $K(SJ(A))$ be the homotopical category of $K$-semijective complexes over $\mathcal{O}(A)$. $\mathcal{D}(A)$ denotes the unbounded derived category of complexes over $\mathcal{O}(A)$. Then the functor of localization by the class of quasiisomorphisms provides a natural equivalence of triangulated categories

$$K(SJ(A)) \xrightarrow{\sim} \mathcal{D}(A).$$

B.2.1. Consider the following resolution $R^\bullet(M)$ of an $A$-module $M \in \mathcal{O}(A)$:

$$R^\bullet(M) := \text{Hom}_A^\bullet(\text{Bar}^\bullet(A, N, A), \text{Bar}^\bullet(A, B, M)),$$

where $A$ in $\text{Bar}^\bullet(A, N, A)$ is considered as a $A-N$ bimodule. Evidently $R^\bullet(M) \in \mathcal{C}^\uparrow(A)$.

B.2.2. **Lemma:** $R^\bullet(M) \in \mathcal{C}^\uparrow(A)$ is $K$-semijective.

**Proof.** First note that $\text{Bar}^\bullet(A, N, A)$ is homotopically equivalent to $A$ as a $A-N$ bimodule — the homotopy is provided by the map

$$a_0 \otimes \ldots \otimes a_n \otimes a \rightarrow a_0 \otimes \ldots \otimes a_n \otimes a \otimes 1.$$

Thus $R^\bullet(M)$ is homotopically equivalent to

$$\text{Hom}_A^\bullet(A, \text{Bar}^\bullet(A, B, M)) = \text{Bar}^\bullet(A, B, M)$$

as a $N$-module. In particular for any exact complex $V^\bullet \in \mathcal{O}(A)$

$$\text{Hom}_{K(N)}(R^\bullet(M), V^\bullet) = \text{Hom}_{K(N)}(\text{Bar}^\bullet(A, B, M), V^\bullet) = 0.$$
Thus $R^\bullet(M)$ is $K$-projective over $N$. To prove that $R^\bullet(M)$ is $A$-injective relative to $N$ note that for a complex of $A$-modules $V^\bullet$ homotopically equivalent to zero over $N$

\[ \text{Hom}_A(V^\bullet, \text{Hom}_A(\text{Bar}^\bullet(A, N, A), \text{Bar}^\bullet(A, B, M))) = \\]
\[ \text{Hom}_A(\text{Bar}^\bullet(A, N, A) \otimes A V^\bullet, \text{Bar}^\bullet(A, B, M)) = \] 
\[ \text{Hom}_A(\text{Bar}^\bullet(A, N, V^\bullet), \text{Bar}^\bullet(A, B, M)). \]

Since $V^\bullet \cong 0$ in $K(N)$, each line $\text{Bar}^p(A, N, V^\bullet)$ of the bicomplex $\text{Bar}^\bullet(A, N, V^\bullet)$ is homotopically equivalent to zero over $A$. As $\text{Bar}^p(A, N, V^\bullet) \neq 0$ only for $p \leq 0$, the total complex of $\text{Bar}^\bullet(A, N, V^\bullet)$ is also homotopically equivalent to zero over $A$.

Thus $\text{Hom}_{K(A)}(V^\bullet, \text{Hom}_A(\text{Bar}^\bullet(A, N, A), \text{Bar}^\bullet(A, B, M))) = 0$, and we are done. \qed

Thus by Theorem 3.4.2(iii) one can treat $\text{Ext}^\bullet_{\text{Bar}}(L, M)$ as an exotic derived functor of the functor $M \mapsto \text{Hom}_{\text{Bar}}(L, S_A \otimes A M)$ (cf. [V], 3.9).

In particular in [V], 3.2.1 it is proved that in the case of $A = U(a)$ for some graded Lie algebra $a$ the algebra $A^d$ differs from $A$ by a 2-cocycle of the Lie algebra $a$. One can check directly that the functor $\text{Hom}_{\text{Bar}}(k, S_A \otimes A *)$ coincides with the functor of semiinvariants defined in [V], 3.6.

B.2.3. Recall the construction of the standard complex for the computation of Lie algebra semiinfinite cohomology.

For a graded module $M$ over a graded Lie algebra $a = \bigoplus_{n \in \mathbb{Z}} a_n$ the standard resolution with respect to the graded Lie subalgebra $b \subset a$ looks as follows:

\[ \text{St}^\bullet(a, b, M) := (U(a) \otimes U(b) \Lambda^\bullet(a/b)) \otimes M. \]

Here the $b$-module $\Lambda(a/b)$ is just the direct sum of the exterior powers of the $b$-representation in $a/b$, tensor product of $a$-modules over the base field is defined using the Hopf algebra structure on $U(a)$, the differential is written as follows:

\[ d((u \otimes \pi_1 \wedge \ldots \wedge \pi_n) \otimes m) = \sum_{i=1}^n (-1)^i (ua_i \otimes \pi_1 \wedge \ldots \wedge \pi_{i-1} \wedge \pi_{i+1} \wedge \ldots \wedge \pi_n) \otimes m + \sum_{i<j} (-1)^{i+j} (u \otimes [a_i, a_j] \wedge \pi_1 \wedge \ldots \wedge \pi_i \wedge \pi_{i+1} \wedge \ldots \wedge \pi_{j-1} \wedge \pi_{j+1} \wedge \ldots \wedge \pi_n) \otimes m. \]

Here $\pi_i \in a/b$. One can check that the differential is correctly defined.

Consider the triangular decomposition of the Lie algebra $a$:

\[ a^{\leq 0} := \bigoplus_{n \leq 0} a_n, \quad a^{> 0} := \bigoplus_{n > 0} a_n, \quad a = a^{\leq 0} \oplus a^{> 0} \]

as a vector space.

Clearly $\text{St}^\bullet(a, a^{\leq 0}, M)$ belongs to $C^+(U(a))$.

Let $a^d$ be the central extension of $a$ such that $U(a^d) = U(a^d)$. Then, as before,

\[ K^\bullet_{U(a)}(k, M) = \text{Hom}_{U(a^d)}(\text{St}^\bullet(a^d, a^{d>0}, k), S_{U(a)} \otimes U(a) \text{St}^\bullet(a, a^{\leq 0}, M)), \]
\[ \text{Ext}^\bullet_{U(a)}(k, M) = H^\bullet(K^\bullet_{U(a)}(k, M)). \]

The complex $K^\bullet_{U(a)}(k, M)$ is exactly the standard complex for the computation of Lie algebra semiinfinite cohomology consisting of semiinfinite exterior powers (see e.g. [V], 2.5). That gives another proof of coincidence of semiinfinite Ext functor and Lie algebra semiinfinite cohomology.
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