Remarks on a fractional diffusion transport equation with applications to the dissipative quasi-geostrophic equation

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April 12, 2011

Abstract

In this article we study Hölder regularity for solutions of a transport equation based in the dissipative quasi-geostrophic equation. Following an idea of A. Kiselev and F. Nazarov presented in [10], we will use the molecular characterization of local Hardy spaces $h^r$ in order to obtain information on Hölder regularity of such solutions. This will be done by following the evolution of molecules in a backward equation. We will also study global existence, Besov regularity for weak solutions and a maximum principle (or positivity principle).

Keywords: quasi-geostrophic equation, Hölder regularity, Hardy spaces.

1 Introduction

The dissipative quasi-geostrophic equation has been studied by many authors, not only because of its own mathematical importance, but also as a 2D model in geophysical fluid dynamics and because of its close relationship with other equations arising in fluid dynamics. See [5], [14] and the references given there for more details. This equation has the following form:

\[
\tag{QG}_\alpha
\begin{array}{l}
\partial_t \theta(x, t) = \nabla \cdot (u \theta)(x, t) - \Lambda^{2\alpha} \theta(x, t) \\
\theta(x, 0) = \theta_0(x)
\end{array}
\]

for $0 < \alpha < 1$ and $t \in [0, T]$. Here $\theta$ is a real-valued function and the velocity $u$ is determined by means of Riesz transforms in the following way:

\[u = (-R_2 \theta, R_1 \theta).\]

Recall that Riesz Transforms $R_j$ are given by $\widehat{R_j \theta}(\xi) = \frac{i \xi_j}{|\xi|^2} \hat{\theta}(\xi)$ for $j = 1, 2$ and that $\Lambda^{2\alpha} = (-\Delta)^{\alpha}$ is the Laplacian's fractional power defined by formula

\[\Lambda^{2\alpha} \hat{\theta}(\xi) = |\xi|^{2\alpha} \hat{\theta}(\xi)\]

where $\hat{\theta}$ denotes the Fourier transform of $\theta$.

It is classical to consider three cases in the analysis of dissipative quasi-geostrophic equation following the values of diffusion parameter $\alpha$. Case $1/2 < \alpha$ is called sub-critical since diffusion factor is stronger than nonlinearity. In this case, weak solutions were constructed by S. Resnick in [16] and P. Constantin & J. Wu showed in [4] that smooth initial data gives a smooth global solution.

The critical case is given when $\alpha = 1/2$. Here, P. Constantin, D. Córdoba & J. Wu studied in [6] global existence in Sobolev spaces, while global well-posedness in Besov spaces has been treated by H. Abidi & T. Hmidi in [1]. Also in this case and more recently, A. Kiselev, F. Nazarov & A. Volberg showed in [11] that any regular periodic data generates a unique $C^\infty$ solution.

Finally, the case when $0 < \alpha < 1/2$ is called super-critical, partially because it is harder to work with than the two other cases, but mostly because the diffusion term is weaker than the nonlinear term. Heuristically, it is natural to expect less regularity results than in the other two cases, however we will show here how to study regularity in the super-critical case. In this last case, weak solutions for initial data in $L^p$ or in $H^{-1/2}$ were studied by F. Marchand in [15].

In this article, and following L. Caffarelli & A. Vasseur in [2], we will study a special version of the dissipative quasi-geostrophic equation $\{(QG)\}_\alpha$. The idea is to replace the Riesz Transform-based velocity $u$ by a new velocity $v$ to
obtain the following $n$-dimensional fractional diffusion transport equation for $0 < \alpha \leq 1/2$:

$$
(T)_\alpha \left\{ \begin{array}{l}
\partial_t \theta(x, t) = \nabla \cdot (v \theta)(x, t) - \Lambda^{2\alpha} \theta(x, t) \\
\theta(x, 0) = \theta_0(x) \\
\text{div}(v) = 0.
\end{array} \right.
$$

(1)

In (1) functions $\theta$ and $v$ are such that $\theta : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ and $v : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$. Remark that the velocity $v$ is now a given data for the problem and we will always assume that $v$ is divergence free and belongs to $L^\infty((0, T]; \text{bmo} (\mathbb{R}^n))$.

We fix once and for all the parameter $\alpha \in [0, 1/2]$ in order to study the critical and the super-critical case.

The main theorem presented in this article studies the regularity of the solutions of the fractional diffusion transport equation $(T)_\alpha$ with $\alpha \in [0, 1/2]$:

**Theorem 1 (Hölder regularity)** Let $\theta_0$ be a function such that $\theta_0 \in L^\infty(\mathbb{R}^n)$. If $\theta(x, t)$ is a solution for the equation $(T)_\alpha$ with $\alpha \in [0, 1/2]$, then for all time $0 < t < T$, we have that $\theta(x, t)$ belongs to the Hölder space $C^\gamma(\mathbb{R}^n)$ with $0 < \gamma < 2\alpha$.

The conclusion of this theorem is a surprising fact: it asserts that, even when the diffusion term is weaker than the transport term in equation (1), we still obtain a smoothing effect.

Let us say a few words about the proof of this theorem. A classical result of harmonic analysis states that Hölder spaces $C^\gamma(\mathbb{R}^n)$ can be paired with local Hardy spaces $h^\sigma(\mathbb{R}^n)$. Therefore, if we prove that the duality bracket

$$
(\theta(\cdot, t), \psi_0) = \int_{\mathbb{R}^n} \theta(x, t)\psi_0(x)dx
$$

(2)

is bounded for every $\psi_0 \in h^\sigma(\mathbb{R}^n)$, with $t \in [0, T]$, we obtain that $\theta(\cdot, t) \in C^\gamma(\mathbb{R}^n)$. One of the main features of Hardy spaces is that they admit a characterization by molecules (see definition 1.1 below), which are rather simple functions, and this allows us to study the quantity (2) only for such molecules.

This dual approach was originally given in the torus $\mathbb{T}^n$ by A. Kiselev & F. Nazarov in [10], but only on the critical case $\alpha = 1/2$: it is proved in this article that it is possible to obtain a small regularity gain by studying the evolution of a certain class of well suited functions.

The main novelty of this paper, besides the generalization to $\mathbb{R}^n$ and the use of Hardy spaces, is the treatment of the super-critical case $0 < \alpha < 1/2$ for the equation $(T)_\alpha$ with initial data in $L^\infty(\mathbb{R}^n)$.

Let us stress that the study of super-critical case is made possible by carefully choosing the molecules used in the decomposition of Hardy spaces. Broadly speaking and following [17] p. 130, a molecule is a function $\psi$ so that

(i) $\int_{\mathbb{R}^n} \psi(x)dx = 0$

(ii) $|\psi(x)| \leq r^{-n/\sigma} \min\{1, r^{\beta_n}/|x - x_0|^{\beta_n}\}$

with $\beta > 1$ and $x_0 \in \mathbb{R}^n$, where the parameter $r \in [0, +\infty]$ stands for the size of the molecule $\psi$.

Since we are going to work with local Hardy spaces, we will introduce a size threshold in order to distinguish small molecules from big ones in the following way:

**Definition 1.1 (r-molecules)** Let $\alpha \in [0, 1/2]$, set $\frac{n}{\sigma + 2\alpha} < \sigma < 1$, define $\gamma = n(\frac{\sigma}{2} - 1)$ and fix a real number $\omega$ such that $\gamma < \omega < 2\alpha$. An integrable function $\psi$ is an $r$-molecule if we have

- **Small molecules** $(0 < r < 1/2)$:

  $$
  \int_{\mathbb{R}^n} |\psi(x)||x - x_0|^\omega dx \leq r^{\omega - \gamma}, \text{ for } x_0 \in \mathbb{R}^n
  $$

  (concentration condition)

  $$
  ||\psi||_{L^\infty} \leq \frac{1}{r^{\gamma + \gamma}}
  $$

  (height condition)

  $$
  \int_{\mathbb{R}^n} \psi(x)dx = 0
  $$

  (moment condition)
• Big molecules \((1/2 \leq r < +\infty)\):

In this case we only require conditions (3) and (4) for the \(r\)-molecule \(\psi\) while the moment condition (5) is dropped.

It is interesting to compare this definition of molecules to the one used in [10]. In our molecules the parameter \(\gamma\) reflects explicitly the relationship between Hardy and Hölder spaces (see theorem 3 below). However, the most important fact relies in the parameter \(\omega\) which give us the additional flexibility that will be crucial in the sequel.

**Remark 1.1** Note that the point \(x_0 \in \mathbb{R}^n\) can be considered as the “center” of the molecule.

**Remark 1.2** Conditions (3) and (4) are an easy consequence of condition (ii) and they both imply the estimate

\[ ||\psi||_{L^1} \leq C r^{-\gamma} \]

thus every \(r\)-molecule belongs to \(L^p(\mathbb{R}^n)\) for \(1 < p < +\infty\).

The main interest for using molecules relies in the possibility of transferring the regularity problem to the evolution of such molecules:

**Proposition 1.1 (Transfer property)** Let \(\alpha \in [0, 1/2]\). Let \(\psi(x, s)\) be a solution of the backward problem

\[
\begin{aligned}
\partial_s \psi(x, s) &= -\nabla \cdot [v(x, t - s)\psi(x, s)] - \Lambda^{2\alpha} \psi(x, s) \\
\psi(x, 0) &= \psi_0(x) \in L^1 \cap L^\infty(\mathbb{R}^n) \\
div(v) &= 0 \quad \text{and} \quad v \in L^\infty([0, T]; bmo(\mathbb{R}^n))
\end{aligned}
\]  

(6)

If \(\theta(x, t)\) is a solution of (1) with \(\theta_0 \in L^\infty(\mathbb{R}^n)\) then we have the identity

\[
\int_{\mathbb{R}^n} \theta(x, t)\psi(x, 0)dx = \int_{\mathbb{R}^n} \theta(x, 0)\psi(x, t)dx.
\]

(7)

**Proof.** We first consider the expression

\[
\partial_s \int_{\mathbb{R}^n} \theta(x, t - s)\psi(x, s)dx = \int_{\mathbb{R}^n} -\partial_s \theta(x, t - s)\psi(x, s) + \partial_s \psi(x, s)\theta(x, t - s)dx.
\]

Using equations (1) and (6) we obtain

\[
\begin{align*}
\partial_s \int_{\mathbb{R}^n} \theta(x, t - s)\psi(x, s)dx &= \int_{\mathbb{R}^n} -\nabla \cdot [(v(x, t - s)\theta(x, t - s)) \psi(x, s)] + \Lambda^{2\alpha} \theta(x, t - s)\psi(x, s) \\
&\quad - \nabla \cdot [(v(x, t - s)\psi(x, s))] \theta(x, t - s) - \Lambda^{2\alpha} \psi(x, s)\theta(x, t - s)dx.
\end{align*}
\]

Now, using the fact that \(v\) is divergence free, we have that expression above is equal to zero, so the quantity

\[
\int_{\mathbb{R}^n} \theta(x, t - s)\psi(x, s)dx
\]

remains constant in time. We only have to set \(s = 0\) and \(s = t\) to conclude.

\[
\square
\]

This proposition says that, in order to control \(\langle \theta(\cdot, t), \psi_0 \rangle\), it is enough (and much simpler) to study the bracket \(\langle \theta_0, \psi(\cdot, t) \rangle\). Let us explain in which sense this transfer property is useful: in the bracket \(\langle \theta_0, \psi(\cdot, t) \rangle\) we have much more informations than in the bracket (2).

**Proof of the theorem 1.** Once we have the transfert property, the proof of theorem 1 is quite simple and it reduces to a \(L^1\) estimate for molecules. Indeed, assume that, for all molecular initial data \(\psi_0\), we have a \(L^1\) control for \(\psi(\cdot, t)\) a solution of (6), then theorem 1 follows easily: applying proposition 1.1 and the fact that \(\theta_0 \in L^\infty(\mathbb{R}^n)\) we have

\[
||\langle \theta(\cdot, t), \psi_0 \rangle|| = \left| \int_{\mathbb{R}^n} \theta(x, t)\psi_0(x)dx \right| = \left| \int_{\mathbb{R}^n} \theta(x, 0)\psi(x, t)dx \right| \leq ||\theta_0||_{L^\infty} ||\psi(\cdot, t)||_{L^1} < +\infty.
\]

(8)

From this, we obtain that \(\theta(\cdot, t)\) belongs to the Hölder space \(C^\gamma(\mathbb{R}^n)\).
We need now to study the control of the $L^1$ norm of $\psi(\cdot, t)$ and we divide our proof in two steps following the molecule’s size. For big initial molecules, i.e. if $r \geq 1/2$, the control needed is very simple: apply the maximum principle (10) below and the remark 1.2 to obtain

$$\|\theta_0\|_{L^\infty} \|\psi(\cdot, t)\|_{L^1} \leq \|\theta_0\|_{L^\infty} \|\psi_0\|_{L^1} \leq C \frac{1}{r} \|\theta_0\|_{L^\infty},$$

but, since $r \geq 1/2$, we have that $|\langle \theta(\cdot, t), \psi_0 \rangle| < +\infty$ for all big molecules.

In order to finish the proof of the theorem, it only remains to treat the $L^1$ control for small molecules and this will be done in the section 3 where we prove that for all $0 < r < 1/2$ we have a good control over the quantity $\|\psi(\cdot, t)\|_{L^1}$.

Thus, finally, getting back to (8) we obtain that $|\langle \theta(\cdot, t), \psi_0 \rangle|$ is always bounded for $0 < t < T$ and for any molecule $\psi_0$: we have proved by a duality argument the theorem 1.

Let us recall now that for (smooth) solutions of equations $(QG)_\alpha$ and $(T)_\alpha$ above we have a remarkable property of maximum principle. This was proved in [7] and in [15] and it gives us the next inequalities.

$$\|\theta(\cdot, t)\|_{L^p} + p \int_0^t \int_{\mathbb{R}^n} |\theta(x, s)|^{p-2} \theta(x, s) \lambda^{2\alpha} \theta(x, s) dx ds \leq \|\theta_0\|_{L^p} \quad (2 \leq p < +\infty) \tag{9}$$

or more generally

$$\|\theta(\cdot, t)\|_{L^p} \leq \|\theta_0\|_{L^p} \quad (1 \leq p \leq +\infty) \tag{10}$$

These estimates are extremely useful and they are the starting point of several works. Indeed, the study of inequality (9) helps us incidentally to solve a question pointed out by F. Marchand in [15] concerning weak solution’s global regularity:

**Theorem 2 (Weak solution’s regularity)** Let $2 \leq p < +\infty$ and $\alpha \in [0, 1/2)$. If $\theta_0 \in L^p(\mathbb{R}^n)$ is an initial data for $(QG)_\alpha$ or $(T)_\alpha$ equations, then the associated weak solution $\theta(\cdot, t)$ belongs to $L^{\infty}([0, T]; L^p(\mathbb{R}^n)) \cap L^p([0, T]; \dot{B}^{2\alpha/p}_p(\mathbb{R}^n))$.

These three theorems are the core of the paper, however, for the sake of completeness, we will prove some other interesting results concerning the equation (1).

The plan of the article is the following: in the section 2 we recall some facts concerning the molecular characterization of local Hardy spaces and some other facts about Hölder and bmo spaces. In section 3 we study the $L^1$-norm control for molecules and in section 4 we study existence and unicity of solutions with initial data in $L^p$ with $2 \leq p < +\infty$ and we prove theorem 2. Section 5 is devoted to a positivity principle that will be useful in our proofs and section 6 studies existence of solution with $\theta_0 \in L^\infty$. Finally, section 7 applies these results to the 2D-quasi-geostrophic equation $(QG)_\alpha$.

## 2 Molecular Hardy spaces, Hölder spaces and bmo

Hardy spaces have several equivalent characterizations (see [3], [8] and [17] for a detailed treatment). In this paper we are mainly interested in the molecular approach to define local Hardy spaces $h^\sigma$ with $0 < \sigma < 1$:

**Definition 2.1 (Local Hardy spaces $h^\sigma$)** Let $0 < \sigma < 1$. The local Hardy space $h^\sigma(\mathbb{R}^n)$ is the set of distributions $f$ that admits the following molecular decomposition:

$$f = \sum_{j \in \mathbb{N}} \lambda_j \psi_j \tag{11}$$

where $(\lambda_j)_{j \in \mathbb{N}}$ is a sequence of complex numbers such that $\sum_{j \in \mathbb{N}} |\lambda_j|^\sigma < +\infty$ and $(\psi_j)_{j \in \mathbb{N}}$ is a family of r-molecules in the sense of definition 1.1 above. The $h^\sigma$-norm\footnote{It is not actually a norm since $0 < \sigma < 1$. Local Hardy spaces are, however, complete metric spaces for the distance $d(f, g) = \|f - g\|_{h^\sigma}$. More details can be found in [8] and [17].} is then fixed by the formula

$$\|f\|_{h^\sigma} = \inf \left\{ \left( \sum_{j \in \mathbb{N}} |\lambda_j|^\sigma \right)^{1/\sigma} : f = \sum_{j \in \mathbb{N}} \lambda_j \psi_j \right\}$$

where the infimum runs over all possible decompositions (11).


Local Hardy spaces have many remarkable properties and we will only stress here, before passing to duality results concerning $h^\sigma$ spaces, the fact that Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is dense in $h^\sigma(\mathbb{R}^n)$. For further details see [9], [17], [8] and [5].

Now, let us take a closer look of the dual space of local Hardy spaces. In [8] D. Goldberg proved the next important theorem:

**Theorem 3 (Hardy-Hölder duality)** Let $\frac{n+2\alpha}{n+2} < \sigma < 1$ and fix $\gamma = n\left(\frac{1}{\sigma} - 1\right)$. Then the dual of local Hardy space $h^\sigma(\mathbb{R}^n)$ is the Hölder space $\mathcal{C}^\gamma(\mathbb{R}^n)$ fixed by the norm

$$
\|f\|_{\mathcal{C}^\gamma} = \|f\|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}.
$$

This result allows us to study in terms of Hardy spaces the Hölder regularity of functions and it would be applied to solutions of the $n$-dimensional fractional diffusion transport equation (1).

**Remark 2.1** Since $0 < \sigma < 1$, we have $\sum_{j \in \mathbb{N}} |\lambda_j| \leq \left(\sum_{j \in \mathbb{N}} |\lambda_j|^\sigma\right)^{1/\sigma}$ thus for testing Hölder continuity of a function $f$ it is enough to study the quantities $||f, \psi_j||$ where $\psi_j$ is an $r$-molecule.

We finish this section by recalling some useful facts about $bmo$ space used to characterize velocity $v$. This space is defined as locally integrable functions $f$ such that

$$
\sup_{|B| \leq 1} \frac{1}{|B|} \int_B |f(x) - f_B| dx < M \quad \text{and} \quad \sup_{|B| > 1} \frac{1}{|B|} \int_B |f(x)| dx < M
$$

for a constant $M$;

where we noted $B(R)$ a ball of radius $R > 0$ and $f_B = \frac{1}{|B(R)|} \int_{B(R)} f(x) dx$. The norm $\|\cdot\|_{bmo}$ is then fixed as the smallest constant $M$ satisfying these two conditions.

We will use the next properties for a function belonging to $bmo$:

**Proposition 2.1** Let $f \in bmo$, then

1) for all $1 < p < +\infty$, $f$ is locally in $L^p$ and \( \frac{1}{|B|} \int_B |f(x) - f_B|^p dx \leq C \|f\|_{bmo}^p \)

2) for all $k \in \mathbb{N}$, we have $|f_{2^k B} - f_B| \leq C k \|f\|_{bmo}$ where $2^k B(R) = B(2^k R)$.

**Proposition 2.2** Let $f$ be a function in $bmo(\mathbb{R}^n)$. For $k \in \mathbb{N}$, define $f_k$ by

$$
f_k(x) = \begin{cases} 
-k & \text{if } f(x) \leq -k \\
f(x) & \text{if } -k \leq f(x) \leq k \\
k & \text{if } k \leq f(x).
\end{cases}
$$

(12)

Then $(f_k)_{k \in \mathbb{N}}$ converge weakly to $f$ in $bmo(\mathbb{R}^n)$.

For a proof of these results and more details concerning Hardy, Hölder and $bmo$ spaces see [3], [8], [12], [9] and [17].

## 3 $L^1$ control for small molecules

As said in the introduction, we need to control in time the $L^1$-norm of the solutions $\psi(\cdot, t)$ of the backward problem (6) for all initial data $\psi_0$ where $\psi_0$ is a small $r$-molecule. This will be achieved by iteration in several steps. The first step explains the molecules’ deformation after a very small time $s_0 > 0$. The second step takes as starting point the results of the first step and gives us the deformation for another small time $s_1$: this allows us to obtain a good $L^1$ control for time $s_0 + s_1$. Once this is achieved, it is enough to iterate the second step and in this way we obtain the wished result for all time $0 < t < T$.

### 3.1 Small time molecule’s evolution: First iteration

The following theorem shows how the molecular properties are deformed with the evolution for a small time $s_0$.
Theorem 4 Let $\alpha \in [0, 1/2]$. Set $\sigma, \gamma$ and $\omega$ three real numbers such that $\frac{n}{n+2\alpha} < \sigma < 1$, $\gamma = n(\frac{1}{\sigma} - 1)$ and $0 < \gamma < \omega < 2\alpha$. Let $\psi(x, s_0)$ be a solution of the problem

\[
\begin{aligned}
\partial_{s_0}\psi(x, s_0) &= -\nabla \cdot (v \psi)(x, s_0) - \Lambda^{2\alpha}\psi(x, s_0) \\
\psi(x, 0) &= \psi_0(x) \\
die(v) &= 0 \text{ and } v \in L^{\infty}([0, T]; \mathbb{R}^n) \text{ with } \sup_{s_0 \in [0, T]} \|v(\cdot, s_0)\|_{bmo} \leq \mu
\end{aligned}
\] (13)

If $\psi_0$ is a small $r$-molecule in the sense of definition 1.1 for the local Hardy space $h^{\sigma}(\mathbb{R}^n)$, then, there exists two positive constants $\delta, K$ such that

\[
0 < \frac{\delta}{K} < \omega - \gamma \quad (14)
\]

and a positive constant $\epsilon$ such that, for all $0 < s_0 \leq \epsilon r$ small, we have the following estimates

\[
\int_{\mathbb{R}^n} |\psi(x, s_0)||x - x(s_0)|^\omega \, dx \leq \left( \frac{r}{r + Ks_0} \right)^{\delta/K} (r + Ks_0)^{\omega - \gamma} \quad (15)
\]

\[
\|\psi(\cdot, s_0)\|_{L^\infty} \leq \frac{1}{\left( r^{2\alpha + \frac{\omega}{\omega + \gamma}} + c_0s_0 \right)^{\frac{1}{\omega + \gamma}}} \quad (16)
\]

\[
\|\psi(\cdot, s_0)\|_{L^1} \leq \frac{v_n}{\left( r^{2\alpha + \frac{\omega}{\omega + \gamma}} + c_0s_0 \right)^{\frac{1}{\omega + \gamma}}} \quad (17)
\]

where $v_n$ denotes the $n$-dimensional unit ball volume.

The new molecule’s center $x(s_0)$ used in formula (15) is fixed by

\[
\begin{aligned}
x'(s_0) &= v_{B_r} = \frac{1}{|B_r|} \int_{B_r} v(y, s_0) \, dy \\
x(0) &= x_0
\end{aligned}
\] (18)

where $B_r = B(x(s_0), r)$.

This remains true as long as $0 < r + Ks_0 < 1$.

Remark 3.1 This definition for the point $x(s_0)$ reflects the molecule’s center transport using velocity $v$.

The proof of this theorem follows the next scheme: the small concentration condition (15), which is be proved in the proposition 3.1, implies the height condition (16) (proved in proposition 3.2). Once we have these two conditions, the $L^1$ estimate (17) will follow easily and it is proved in proposition 3.3.

Proposition 3.1 (Small time Concentration condition) Under the hypothesis of theorem 4, if $\psi_0$ is a small $r$-molecule, then the solution $\psi(x, s)$ of (13) satisfies

\[
\int_{\mathbb{R}^n} |\psi(x, s_0)||x - x(s_0)|^\omega \, dx \leq \left( \frac{r}{r + Ks_0} \right)^{\delta/K} (r + Ks_0)^{\omega - \gamma} \quad (19)
\]

for $x(s_0) \in \mathbb{R}^n$ fixed by formula (18), with $0 \leq s_0 \leq \epsilon r$.

Proof. Let us write $\Omega(x - x(s_0)) = |x - x(s_0)|^\omega$ and $\psi(x) = \psi_+(x) - \psi_-(x)$ where the functions $\psi_{\pm}(x) \geq 0$ have disjoint support. We will note $\psi_{\pm}(x, s_0)$ solutions of (13) with $\psi_{\pm}(x, 0) = \psi_{\pm}(x)$.

At this point, we assume the following positivity principle which is proved in section 5:

Theorem 5 Let $1 \leq p \leq +\infty$, if initial data $\psi_0 \in L^p(\mathbb{R}^n)$ is such that $0 \leq \psi_0(x) \leq M$, then the associated solution $\psi(x, s_0)$ of (13) satisfies $0 \leq \psi(x, s_0) \leq M$ for all $s_0 \in [0, T]$.

Thus, by linearity and using the above theorem we have that

\[
|\psi(x, s_0)| = |\psi_+(x, s_0) - \psi_-(x, s_0)| \leq \psi_+(x, s_0) + \psi_-(x, s_0)
\]

and we can write

\[
\int_{\mathbb{R}^n} |\psi(x, s_0)|\Omega(x - x(s_0)) \, dx \leq \int_{\mathbb{R}^n} \psi_+(x, s_0)\Omega(x - x(s_0)) \, dx + \int_{\mathbb{R}^n} \psi_-(x, s_0)\Omega(x - x(s_0)) \, dx
\]
so we only have to treat one of the integrals on the right side above. We have:

\[
I = \left| \partial_s \int_{\mathbb{R}^n} \Omega(x-x(s_0))\psi_+(x,s_0)dx \right|
\]

\[
= \left| \int_{\mathbb{R}^n} \partial_s \Omega(x-x(s_0))\psi_+(x,s_0) + \Omega(x-x(s_0)) \left[ -\nabla \cdot (v\psi_+(x,s_0)) - \Lambda^{2\alpha} \psi_+(x,s_0) \right] dx \right|
\]

\[
= \left| \int_{\mathbb{R}^n} -\nabla \Omega(x-x(s_0)) \cdot x'(s_0)\psi_+(x,s_0) + \Omega(x-x(s_0)) \left[ -\nabla \cdot (v\psi_+(x,s_0)) - \Lambda^{2\alpha} \psi_+(x,s_0) \right] dx \right|
\]

Using the fact that \( v \) is divergence free, we obtain

\[
I = \left| \int_{\mathbb{R}^n} \nabla \Omega(x-x(s_0)) \cdot (v-x'(s_0))\psi_+(x,s_0) - \Omega(x-x(s_0))\Lambda^{2\alpha} \psi_+(x,s_0) dx \right|.
\]

Finally, using the definition of \( x'(s_0) \) given in (18) and replacing \( \Omega(x-x(s_0)) \) by \(|x-x(s_0)|^\omega \) we obtain

\[
I \leq c \int_{I_1} |x-x(s_0)|^{\omega-1}|v-\overline{v}_{B_r}|\psi_+(x,s_0)|dx + c \int_{I_2} |x-x(s_0)|^{\omega-2\alpha}|\psi_+(x,s_0)|dx.
\] (20)

**Remark 3.2** Note that when \( 0 < \alpha < 1/2 \) the power of the factor \(|x-x(s_0)|\) is different in \( I_1 \) and \( I_2 \), so computations are a little bit easier when \( \alpha = 1/2 \). However, the other case is not complicated to deal with.

We will study separately each one of the integrals \( I_1 \) and \( I_2 \) by the next lemmas:

**Lemma 3.1** For integral \( I_1 \) we have the estimate

\[
I_1 \leq C r^{\omega-1-\gamma} \mu.
\]

**Proof.** We begin by considering the space \( \mathbb{R}^n \) as the union of a ball with dyadic coronas centered on \( x(s_0) \), more precisely we set \( \mathbb{R}^n = \bigcup_{k \in \mathbb{N}} E_k \) where

\[
E_0 = \{ x \in \mathbb{R}^n : |x-x(s_0)| \leq r \} = B_r,
\]

\[
E_k = \{ x \in \mathbb{R}^n : r2^{k-1} \leq |x-x(s_0)| \leq r2^k \} \text{ for } k \geq 1,
\]

(i) Estimations over the ball \( B_r \).

Applying Hölder inequality on integral \( I_1 \) we obtain

\[
\int_{B_r} |x-x(s_0)|^{\omega-1}|v-\overline{v}_{B_r}|\psi_+(x,s_0)|dx \leq \left( \frac{1}{p} + \frac{1}{q} + \frac{1}{z} = 1 \text{ and } p, z, q > 1 \right) \leq \left( \frac{1}{p} + \frac{1}{q} + \frac{1}{z} = 1 \text{ and } p, z, q > 1 \right)
\]

\[
\times \left| |x-x(s_0)|^{\omega-1}|\psi_+(x,s_0)| \right|_{L^p(B_r)} \times \left| v-\overline{v}_{B_r} \right|_{L^q(B_r)} \left| \psi_+(x,s_0) \right|_{L^z(B_r)}
\]

where \( \frac{1}{p} + \frac{1}{q} + \frac{1}{z} = 1 \) and \( p, z, q > 1 \). We treat each one of the previous terms separately:

- First observe that for \( 1 < p < n/(1-\omega) \) we have

\[
\| |x-x(s_0)|^{\omega-1} \|_{L^p(B_r)} \leq C r^{n/p+\omega-1}.
\]

- By hypothesis we have \( v(\cdot, s_0) \in \text{bmo} \), thus

\[
\| v-\overline{v}_{B_r} \|_{L^q(B_r)} \leq C |B_r|^{1/z} \| v(\cdot, s_0) \|_{\text{bmo}}.
\]

since \( \sup_{s_0 \in [0,T]} \| v(\cdot, s_0) \|_{\text{bmo}} \leq \mu \) we write

\[
\| v-\overline{v}_{B_r} \|_{L^q(B_r)} \leq C r^{n/z} \mu.
\]

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Finally, by the maximum principle (10) for \( L^q \) norms we have \( \| \psi^+ (\cdot, s_0) \|_{L^q (B_r)} \leq \| \psi^+ (\cdot, 0) \|_{L^q} \); hence using the fact that \( \psi_0 \) is an \( r \)-molecule and remark 1.2 we obtain
\[
\| \psi^+ (\cdot, s_0) \|_{L^q (B_r)} \leq C \left( \frac{r^{-\gamma}}{r^{\alpha + \gamma}} \right)^{1/q}.
\]

We gather all this inequalities in order to obtain the following estimation for (22):
\[
\int_{B_r} \frac{|x - x(s_0)|^{\omega - 1}}{v - Spr} \| \psi^+ (x, s_0) \| dx \leq C r^{\omega - 1 - \gamma} \mu.
\]

(ii) Estimations for the dyadic corona \( E_k \).

Let us note \( I_k \) the integral
\[
I_k = \int_{E_k} \frac{|x - x(s_0)|^{\omega - 1}}{v - Spr} \| \psi^+ (x, s_0) \| dx.
\]
Since over \( E_k \) we have\(^2\) \( |x - x(s_0)|^{\omega - 1} \leq C 2^{k(\omega - 1)} r^{\omega - 1} \) we write
\[
I_k \leq C 2^{k(\omega - 1)} r^{\omega - 1} \left( \int_{B_{2^k}} \frac{|v - Spr} {\| \psi^+ (x, s_0) \|} \| \psi^+ (x, s_0) \| dx + \int_{E_k} \frac{\| \psi^+ (x, s_0) \|} {\| \psi^+ (x, s_0) \|} \| \psi^+ (x, s_0) \| dx \right)
\]
where we noted \( B_{2^k} = B(x(s_0), 2^k r) \), then
\[
I_k \leq C 2^{k(\omega - 1)} r^{\omega - 1} \left( \int_{B_{2^k}} \frac{|v - Spr} {\| \psi^+ (x, s_0) \|} \| \psi^+ (x, s_0) \| dx + \right.
\]
\[
\left. + \int_{B_{2^k}} \frac{\| \psi^+ (x, s_0) \|} {\| \psi^+ (x, s_0) \|} \| \psi^+ (x, s_0) \| dx \right)
\]
Now, since \( v(\cdot, s_0) \in bmo \), using proposition 2.1 we have
\[
\| \frac{\psi^+ (x, s_0)} {\| \psi^+ (x, s_0) \|} \|_{bmo} \leq C k \mu
\]
and we can write
\[
I_k \leq C 2^{k(\omega - 1)} r^{\omega - 1} \left( \int_{B_{2^k}} \frac{|v - Spr} {\| \psi^+ (x, s_0) \|} \| \psi^+ (x, s_0) \| dx + C k \mu \| \psi^+ (\cdot, s_0) \|_{L^\infty} \right)
\]
\[
\leq C 2^{k(\omega - 1)} r^{\omega - 1} \left( \| \psi^+ (\cdot, s_0) \|_{L^\infty} \| v - Spr \|_{L^{\frac{\omega - 1}{a_0 + n\omega - 1}}} + C k \mu r^{-\gamma} \right)
\]
where we used Hölder inequality with \( 1 < a_0 < \frac{n}{\omega + (\omega - 1)} \) and maximum principle for the last term above. Using again the properties of \( bmo \) spaces we have
\[
I_k \leq C 2^{k(\omega - 1)} r^{\omega - 1} \left( \| \psi^+ (\cdot, 0) \|_{L^\infty} \| v - Spr \|_{L^{\frac{\omega - 1}{a_0 + n\omega - 1}}} + C k \mu r^{-\gamma} \right).
\]
Let us now apply estimates given by hypothesis over \( \| \psi^+ (\cdot, 0) \|_{L^1} \), \( \| \psi^+ (\cdot, 0) \|_{L^\infty} \) and \( \| v(\cdot, s_0) \|_{bmo} \) to obtain
\[
I_k \leq C 2^{k(n - n/a_0 + n\omega - 1) r^{\omega - 1 - \gamma} \mu} + C 2^{k(\omega - 1) k \mu r^{\omega - 1 - \gamma}}.
\]
Since \( 1 < a_0 < \frac{n}{\omega + (\omega - 1)} \), we have \( n - n/a_0 + (\omega - 1) < 0 \), so that, summing over each dyadic corona \( E_k \), we obtain
\[
\sum_{k \geq 1} I_k \leq C r^{\omega - 1 - \gamma} \mu.
\]
Finally, gathering together estimations (23) and (24) we obtain the desired conclusion.

\( \blacksquare \)

**Lemma 3.2** For integral \( I_2 \) in inequality (20) we have the following estimate
\[
I_2 \leq C r^{\omega - 1 - \gamma}.
\]
\(^2\)recall that \( 0 < \gamma < \omega < 2a \leq 1 \).
Proof. As for lemma 3.1, we consider $\mathbb{R}^n$ as the union of a ball with dyadic coronas centered on $x(s_0)$ (cf. (21)).

(i) Estimations over the ball $B_r$.

We apply now the Hölder inequality in the integral above with $1 < a_1 < n/(2\alpha - \omega)$ and $\frac{1}{a_1} + \frac{1}{\omega} = 1$ in order to obtain

$$
\int_{B_r} |x - x(s_0)|^{\omega - 2\alpha} \psi_+(x, s_0) dx \leq \|x - x(s_0)|^{\omega - 2\alpha} \|_{L^{a_1}(B_r)} \|\psi_+(\cdot, s_0)\|_{L^{1/a_1}(B_r)}
$$

Using hypothesis over $\|\psi_+(\cdot, 0)\|_{L^1}$ and $\|\psi_+(\cdot, 0)\|_{L^{\infty}}$ we obtain

$$
\int_{B_r} |x - x(s_0)|^{\omega - 2\alpha} \psi_+(x, s_0) dx \leq C r^{\omega - 2\alpha - \gamma}.
$$

At this point we distinguish two cases.

- If $\alpha = 1/2$, i.e. for the critical case, there is nothing to do.

- For $0 < \alpha < 1/2$, recall that we are working with small molecules so we have $0 < r < 1/2$ and $r^{\omega - 2\alpha - \gamma} \leq r^{\omega - 1 - \gamma}$.

Thus, in any case we can write:

$$
\int_{B_r} |x - x(s_0)|^{\omega - 2\alpha} \psi_+(x, s_0) dx \leq C r^{\omega - 1 - \gamma}. \tag{25}
$$

(ii) Estimations for the dyadic corona $E_k$.

Here we have

$$
\int_{E_k} |x - x(s_0)|^{\omega - 2\alpha} \psi_+(x, s_0) dx \leq C 2^{k(\omega - 2\alpha)} r^{\omega - 2\alpha} \int_{E_k} |\psi_+(x, s_0)| dx \leq C 2^{k(\omega - 2\alpha)} r^{\omega - 2\alpha} \|\psi_+(\cdot, s_0)\|_{L^1}
$$

It is a this step that the flexibility of molecules is essential. Indeed, in the definition 1.1 we have fixed $0 < \gamma < \omega < 2\alpha$ so we have $\omega - 2\alpha < 0$ and thus, summing over $k \geq 1$, we obtain

$$
\sum_{k \geq 1} \int_{E_k} |x - x(s_0)|^{\omega - 2\alpha} \psi_+(x, s_0) dx \leq C r^{\omega - 2\alpha - \gamma}.
$$

Repeating the same argument used before (i.e. the fact that $0 < r < 1/2$), we finally get

$$
\sum_{k \geq 1} \int_{E_k} |x - x(s_0)|^{\omega - 2\alpha} \psi_+(x, s_0) dx \leq C r^{\omega - 1 - \gamma}. \tag{26}
$$

To finish the proof of lemma 3.2 we glue together estimates (25) and (26).

\[\square\]

Now we continue the proof of proposition 3.1. Using lemmas 3.1 and 3.2 and getting back to estimate (20) we have

$$
\left| \partial_{s_0} \int_{\mathbb{R}^n} \Omega(x - x(s_0)) \psi_+(x, s_0) dx \right| \leq C r^{\omega - 1 - \gamma} (\mu + 1)
$$

This last estimation is compatible with the estimate (19) for $0 \leq s_0 \leq \epsilon r$ small enough: just fix $K$ and $\delta$ such that

$$
C (\mu + 1) \leq K (\omega - \gamma) - \delta. \tag{27}
$$

Indeed, since the time $s_0$ is very small, we can linearize the right hand side of (19) in order to obtain

$$
\phi = \left( \frac{r}{r + K s_0} \right)^{\gamma/K} (r + K s_0)^{\omega - \gamma} \approx r^{\omega - \gamma} \left( 1 + [K (\omega - \gamma) - \delta] \frac{s_0}{r} \right).
$$

Now, taking the derivative with respect to $s_0$ in the above expression we have

$$
\phi' \approx r^{\omega - 1 - \gamma} (K (\omega - \gamma) - \delta)
$$

and condition (27) follows.
Now we will give a slightly different proof for the maximum principle of A. Córdoba & D. Córdoba. Indeed, the following proof only relies on the concentration condition proved in the lines above.

**Proposition 3.2 (Small time Height condition)** Under the hypothesis of theorem 4, if \( \psi(x, s_0) \) satisfies concentration condition (19), then we have the next height condition

\[
\| \psi(\cdot, s_0) \|_{L^\infty} \leq \frac{1}{\left( \frac{r^{2\alpha}}{1 + C_0} \right)^{\frac{\alpha}{2\alpha}}}.
\]  

**(Proof.** To begin with, we observe that this inequality is trivially verified if

\[
\| \psi(\cdot, 0) \|_{L^\infty} \leq \frac{1}{2^{\rho n + \gamma}}.
\]

Indeed, since time \( s_0 \) is small we can obtain (28) from this estimate. So, without loss of generality, we can assume that

\[
\frac{1}{2^{\rho n + \gamma}} \leq \| \psi(\cdot, 0) \|_{L^\infty} \leq \frac{1}{r^{n + \gamma}}.
\]

Assume now that molecules we are working with are smooth enough. Following an idea of [7] (section 4 p.522-523), we will note \( \mathcal{R} \) the point of \( \mathbb{R}^n \) such that \( \psi(\mathcal{R}, s_0) = \| \psi(\cdot, s_0) \|_{L^\infty} \). Thus we can write

\[
\frac{d}{ds_0} \| \psi(\cdot, s) \|_{L^\infty} \leq - \int_{\mathbb{R}^n} \frac{|\psi(\mathcal{R}, s) - \psi(y, s)|}{|\mathcal{R} - y|^{n + 2\alpha}} dy \leq 0.
\]

For simplicity, we will assume that \( \psi(\mathcal{R}, s_0) \) is positive.

Let us consider the corona centered in \( \mathcal{R} \) defined by

\[
\mathcal{C}(R_1, R_2) = \{ y \in \mathbb{R}^n : R_1 \leq |\mathcal{R} - y| \leq R_2 \}
\]

where \( R_2 = \rho R_1 \) with \( \rho > 2 \) and where \( R_1 \) will be fixed later. Then:

\[
\int_{\mathbb{R}^n} \frac{\psi(\mathcal{R}, s_0) - \psi(y_0, s_0)}{|\mathcal{R} - y|^{n + 2\alpha}} dy \geq \int_{\mathcal{C}(R_1, R_2)} \frac{\psi(\mathcal{R}, s_0) - \psi(y_0, s_0)}{|\mathcal{R} - y|^{n + 2\alpha}} dy.
\]

Define the sets \( B_1 \) and \( B_2 \) by \( B_1 = \{ y \in \mathcal{C}(R_1, R_2) : \psi(\mathcal{R}, s_0) - \psi(y_0, s_0) \geq \frac{1}{2} \psi(\mathcal{R}, s_0) \} \) and \( B_2 = \{ y \in \mathcal{C}(R_1, R_2) : \psi(\mathcal{R}, s_0) - \psi(y_0, s_0) < \frac{1}{2} \psi(\mathcal{R}, s_0) \} \) such that \( \mathcal{C}(R_1, R_2) = B_1 \cup B_2. \)

We obtain the estimations

\[
\int_{\mathcal{C}(R_1, R_2)} \frac{\psi(\mathcal{R}, s_0) - \psi(y_0, s_0)}{|\mathcal{R} - y|^{n + 2\alpha}} dy \geq \int_{B_1} \frac{\psi(\mathcal{R}, s_0) - \psi(y_0, s_0)}{|\mathcal{R} - y|^{n + 2\alpha}} dy \geq \frac{\psi(\mathcal{R}, s_0)}{2 R_1^{n + 2\alpha}} |B_1| = \frac{\psi(\mathcal{R}, s_0)}{2 R_1^{n + 2\alpha}} (|\mathcal{C}(R_1, R_2)| - |B_2|).
\]

Since \( R_2 = \rho R_1 \) one has

\[
\int_{\mathcal{C}(R_1, R_2)} \frac{\psi(\mathcal{R}, s_0) - \psi(y_0, s_0)}{|\mathcal{R} - y|^{n + 2\alpha}} dy \geq \frac{\psi(\mathcal{R}, s_0)}{2 \rho^{n + 2\alpha} R_1^{n + 2\alpha}} \left( v_n (\rho^n - 1) R_1^n - |B_2| \right)
\]

where \( v_n \) denotes the \( n \)-dimensional unit ball volume.

Now, we will estimate with the next lemma quantity \( |B_2| \) in terms of \( \psi(\mathcal{R}, s_0) \) and \( R_1. \)

**Lemma 3.3** For the set \( B_2 \) we have the following estimations

1) if \( |\mathcal{R} - x(s_0)| > 2 R_2 \) then \( C_1 \psi(\mathcal{R}, s_0)^{-1} R_1^{-1} \geq |B_2|. \)

2) if \( |\mathcal{R} - x(s_0)| < R_1 / 2 \) then \( C_1 \psi(\mathcal{R}, s_0)^{-1} R_1^{-1} \geq |B_2|. \)

3) if \( R_1 / 2 \leq |\mathcal{R} - x(s_0)| \leq 2 R_2 \) then \( (C_2 R_1^{n - \omega} \psi(\mathcal{R}, s_0)^{-1})^{1/2} \geq |B_2|. \)
Recall that for the molecule’s center \( x_0 \in \mathbb{R}^n \) we noted it’s transport by \( x(s_0) \) which is defined by formula (18).

**Proof.** For all these estimates, our starting point is the concentration condition (19). Thus, since \( 0 < r < 1/2 \) and since \( 0 < r + Ks_0 < 1 \) we have

\[
C_0 \geq \int_{\mathbb{R}^n} |\psi(y, s_0)||y - x(s_0)|^\omega \, dy \geq \int_{B_2} |\psi(y, s_0)||y - x(s_0)|^\omega \, dy \geq \frac{\psi(x, s_0))}{2} \int_{B_2} |y - x(s_0)|^\omega \, dy.
\]  

(32)

We just need to estimate the last integral following the cases given by the lemma. The first two cases are very similar. Indeed, if \( |\mathbb{R} - x(s_0)| > 2R_2 \) then we have

\[
\min_{y \in B_2 \subset C(R_1, R_2)} |y - x(s_0)|^\omega \geq R_2^\omega = \rho^\omega R_1^\omega
\]

while for the second case, if \( |\mathbb{R} - x(s_0)| < R_1/2 \), one has

\[
\min_{y \in B_2 \subset C(R_1, R_2)} |y - x(s_0)|^\omega \geq \frac{R_1^\omega}{2}.\]

Applying these results to (32) we obtain \( C_0 \geq \psi(x, s_0))\rho^\omega R_1^\omega |B_2| \) and \( C_0 \geq \psi(x, s_0))\frac{R_1^\omega}{2}|B_2| \), and since \( \rho > 2 \) we have the desired estimate

\[
\frac{C_1}{\psi(x, s_0))R_1^\omega} \geq \frac{2C_0}{\rho^\omega \psi(x, s_0))R_1^\omega} \geq |B_2|
\]

with \( C_1 = 2^{\omega+1}C_0 \).

For the last case, since \( R_1/2 \leq |\mathbb{R} - x(s_0)| \leq 2R_2 \) we can write using the Cauchy-Schwarz inequality

\[
\int_{B_2} |y - x(s_0)|^\omega \, dy \geq |B_2|^2 \left( \int_{B_2} |y - x(s_0)|^{-\omega} \, dy \right)^{-1}
\]

(34)

Now, observe that in this case we have \( B_2 \subset B(x(s_0), 5R_2) \) and then

\[
\int_{B_2} |y - x(s_0)|^{-\omega} \, dy \leq \int_{B(x(s_0), 5R_2)} |y - x(s_0)|^{-\omega} \, dy \leq v_n(5\rho R_1)^{n-\omega}.
\]

Getting back to (34) we obtain

\[
\int_{B_2} |y - x(s_0)|^\omega \, dy \geq |B_2|^2 v_n^{-1}(5\rho R_1)^{-n+\omega}
\]

We use this estimate in (32) to obtain

\[
\frac{C_2R_1^{n/2-\omega/2}}{\psi(x, s_0)^{1/2}} \geq |B_2|,
\]

(35)

where \( C_2 = (2C_05^{n-\omega}v_n\rho^{n-\omega})^{1/2} \). The lemma is proved.

With estimates (33) and (35) at our disposal we can write

(i) if \( |\mathbb{R} - x(s_0)| > 2R_2 \) or \( |\mathbb{R} - x(s_0)| < R_1/2 \) then

\[
\int_{C(R_1, R_2)} \frac{\psi(x, s_0) - \psi(y, s_0)}{|\mathbb{R} - y|^{n+2\alpha}} \, dy \geq \frac{\psi(x, s_0)}{2\rho^{n+2\alpha} R_1^{n+2\alpha}} \left( v_n(\rho^n - 1) R_1^n - \frac{C_1}{\psi(x, s_0))R_1^\omega} \right)
\]

(ii) if \( R_1/2 \leq |\mathbb{R} - x(s_0)| \leq 2R_2 \)

\[
\int_{C(R_1, R_2)} \frac{\psi(x, s_0) - \psi(y, s_0)}{|\mathbb{R} - y|^{n+2\alpha}} \, dy \geq \frac{\psi(x, s_0)}{2\rho^{n+2\alpha} R_1^{n+2\alpha}} \left( v_n(\rho^n - 1) R_1^n - \frac{C_2R_1^{n/2-\omega/2}}{\psi(x, s_0)^{1/2}} \right)
\]

Now, if we set \( R_1 = \psi(x, s_0)^{1/\omega} \) and if \( \rho \) is big enough, we obtain for cases (i) and (ii) the following estimate for (31):

\[
\int_{C(R_1, R_2)} \frac{\psi(x, s_0) - \psi(y, s_0)}{|\mathbb{R} - y|^{n+2\alpha}} \, dy \geq C \psi(x, s_0)^{1+\frac{2\alpha}{\omega}}
\]

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where \( C = C(n) \) is a constant. Hence, and for all possible cases considered before, we have the next estimate for (30):

\[
\frac{d}{ds_0} \| \psi(\cdot, s_0) \|_{L^\infty} \leq -C \psi(\cdot, s_0)^{1 + \frac{2\alpha}{n+2}} = -C \| \psi(\cdot, s_0) \|_{L^\infty}^{1 + \frac{2\alpha}{n+2}}.
\]

Solving this inequality, one has

\[
\| \psi(\cdot, s_0) \|_{L^\infty} \leq \frac{\| \psi(\cdot, 0) \|_{L^\infty}}{(1 + Cs_0 \| \psi(\cdot, 0) \|_{L^\infty}^{\frac{2\alpha}{n+2}})^{\frac{n+2}{n+2\alpha}},}
\]

so, using the assumption (29) we obtain:

\[
\| \psi(\cdot, s_0) \|_{L^\infty} \leq \frac{1}{r^{n+\gamma} (1 + c_0 s_0 \left( \frac{1}{r^{n+\gamma}} \right)^{\frac{2\alpha}{n+2\alpha}})^{\frac{n+2\alpha}{n+2}}} = \frac{1}{\left( \frac{2\alpha}{n+2\alpha} + c_0 s_0 \right)^{\frac{n+2\alpha}{n+2}}}
\]

and therefore, the proof of proposition 3.2 is finished for regular molecules. In order to obtain the global result, remark that, for viscosity solutions (52), we have that \( \Delta \theta(\tau, s_0) \leq 0 \) at the points \( \tau \) where \( \theta(\cdot, s_0) \) reaches its maximum value.

\[\blacksquare\]

**Remark 3.3** By this procedure we obtain a \( L^\infty \) control as long as the concentration condition (19) is bounded by some constant: this will allow us to let evolve the system for larger times.

We treat now the last part of theorem 4:

**Proposition 3.3 (First \( L^1 \) estimate)** If \( \psi(x, s_0) \) is a solution of the problem (13), then we have the following \( L^1 \)-norm estimate:

\[
\| \psi(\cdot, s_0) \|_{L^1} \leq \frac{v_n}{(r^{2\alpha \frac{n+2\alpha}{n+2\gamma} + c_0 s_0)^{\frac{n+2\alpha}{n+2}}}.
\]

**Proof.** We write

\[
\int_{\mathbb{R}^n} |\psi(x, s_0)| dx = \int_{\{|x-x(s_0)|<D\}} |\psi(x, s_0)| dx + \int_{\{|x-x(s_0)|\geq D\}} |\psi(x, s_0)| dx
\]

\[
\leq v_n D^n \| \psi(\cdot, s_0) \|_{L^\infty} + D^{-\omega} \int_{\mathbb{R}} |\psi(x, s_0)||x-x(s_0)|^\omega dx
\]

Now using (28) and (19) one has:

\[
\int_{\mathbb{R}^n} |\psi(x, s_0)| dx \leq v_n \frac{D^n}{(r^{2\alpha \frac{n+2\alpha}{n+2\gamma} + c_0 s_0)^{\frac{n+2\alpha}{n+2}}} + D^{-\omega} \left( \frac{r}{r + K s_0} \right)^{\delta/K} (r + K s_0)^{\omega-\gamma}
\]

where \( v_n \) denotes the volume of the unit ball. To continue, it is enough to choose correctly the real parameter \( D \) to obtain

\[
\int_{\mathbb{R}^n} |\psi(x, s_0)| dx \leq v_n \left( \frac{r}{r + K s_0} \right)^{\delta/K} \left( \frac{r + K s_0}{r^{2\alpha \frac{n+2\alpha}{n+2\gamma} + c_0 s_0} \right)^{\frac{n+2\alpha}{n+2}}
\]

Now, since \( 0 < (r + K s_0) < 1 \) we finally have

\[
\| \psi(\cdot, s_0) \|_{L^1} \leq \frac{v_n}{(r^{2\alpha \frac{n+2\alpha}{n+2\gamma} + c_0 s_0)^{\frac{n+2\alpha}{n+2}}}
\]

\[\blacksquare\]

### 3.2 Molecule’s evolution: Second iteration

In the previous section we have obtained a good \( L^1 \)-norm control for time \( 0 \to s_0 \) with \( s_0 \) very small. In order to have larger time behaviour, we need to iterate this work, we will obtain an estimate from \( s_0 \to s_1 \) with \( s_1 \) a new time small enough.

It is important to explain why we need two iterations. The first one gives us the molecule deformation in small time from initial molecular data, the second one is an attempt to recover the molecular properties from this initial deformation in order to obtain the wished \( L^1 \) control for larger times.
Theorem 6 Let $\alpha \in [0, 1/2]$. Set $\gamma$ and $\omega$ two real numbers such that $0 < \gamma < \omega < 2\alpha$. Let $0 < s_1 \leq T$ and let $\psi(x, s_1)$ be a solution of the problem

\[
\begin{align*}
\partial_s \psi(x, s_1) &= -\nabla \cdot (v \psi)(x, s_1) - \Lambda^{2\alpha} \psi(x, s_1) \\
\psi(x, 0) &= \psi(x, s_0) \quad \text{with } s_0 > 0 \\
div(v) &= 0 \quad \text{and } v \in L^\infty([0, T]; bmo(\mathbb{R}^n)) \quad \text{with } \sup_{s_1 \in [s_0, T]} \|v(\cdot, s_1)\|_{bmo} \leq \mu
\end{align*}
\]

If $\psi(x, s_0)$ satisfies the three following conditions

\[
\int_{\mathbb{R}^n} |\psi(x, s_0)||x - x(s_0)|^\omega dx \leq \left( \frac{r}{r + Ks_0} \right)^{\delta/K} (r + Ks_0)^{\omega - \gamma}
\]

\[
\|\psi(\cdot, s_0)\|_{L^\infty} \leq \frac{1}{r^{2\alpha + \omega + \gamma} + c_0 s_0}^{\frac{n+\omega}{\omega}}
\]

\[
\|\psi(\cdot, s_0)\|_{L^1} \leq \frac{v_n}{r^{2\alpha + \omega + \gamma} + c_0 s_0}^{\frac{n+\omega}{\omega}}
\]

with $s_0$ such that $0 < r + Ks_0 < 1$. Then, there exists positive constants $\delta, K, \epsilon$, with $0 < \delta/K < \omega - \gamma$ such that, for all $0 < s_1 \leq \epsilon r^4$ small, we have the following estimates

\[
\int_{\mathbb{R}^n} |\psi(x, s_1)||x - x(s_1)|^\omega dx \leq \left( \frac{r^\epsilon (r + Ks_0)\eta}{r^\epsilon (r + Ks_0)^\eta + Ks_1} \right)^{\delta/K} (r^\epsilon (r + Ks_0)^\eta + Ks_1)^{\omega - \gamma}
\]

\[
\|\psi(\cdot, s_1)\|_{L^\infty} \leq \left( \frac{r^\epsilon (r + Ks_0)^\eta + Ks_1}{r^{2\alpha + \omega + \gamma} + c_0 s_0 + s_1} \right)^{\frac{n+\omega}{\omega}}
\]

\[
\|\psi(\cdot, s_1)\|_{L^1} \leq \frac{v_n}{r^{2\alpha + \omega + \gamma} + c_0 s_0 + s_1}^{\frac{n+\omega}{\omega}}
\]

with

\[
\frac{(n+\gamma)}{(1+\gamma - \omega)(n+\omega)} < \epsilon < 2 \quad \text{and} \quad 0 < \eta \ll 1.
\]

This remains true as long as $0 < r^\epsilon (r + Ks_0)^\eta + Ks_1 < 1$.

Remark 3.4

- Note that the initial data $\psi(x, s_0)$ of problem (37) is given by the molecular deformation of theorem 4.
- The new molecule’s center $x(s_1)$ used in formula (38) is fixed by

\[
\begin{align*}
x'(s_1) &= \mathbb{B}_f(x(s_1), f) = \mathbb{B}_f = \frac{1}{|B_f|} \int_{B_f} v(y, s_1) dy \\
x(0) &= x(s_0)
\end{align*}
\]

And here we noted $B_f = B(x(s_1), f)$ with $f = f(r, s_0)$ a real valued function given by

\[
f(r, s_0) = \frac{r^\epsilon (r + Ks_0)^\eta + Ks_1}{r^{2\alpha + \omega + \gamma} + c_0 s_0 + s_1}.
\]

Remark that with this definition of $f$, with the value of $\epsilon$ given in (41) and with the fact that $(r + Ks_0) < 1$ we have for all $r$ the bounds $0 < f(r, s_0) < 1$. This fact is important for the sequel.

As we are going to see, theorem 6 will follow from a perturbation of these molecular conditions explained in propositions 3.1 and 3.2: even if $\psi(\cdot, s_1)$ is not exactly a molecule, we are able to prove some similar estimates, close enough to the concentration condition and to the height condition, in order to obtain the wished $L^1$ control.

Proposition 3.4 (Concentration condition)

Under the hypothesis of theorem 6, if $\psi(\cdot, s_0)$ is an initial data then the solution $\psi(x, s_1)$ of (37) satisfies

\[
\int_{\mathbb{R}^n} |\psi(x, s_1)||x - x(s_1)|^\omega dx \leq \left( \frac{r^\epsilon (r + Ks_0)^\eta}{r^\epsilon (r + Ks_0)^\eta + Ks_1} \right)^{\delta/K} (r^\epsilon (r + Ks_0)^\eta + Ks_1)^{\omega - \gamma}
\]

for $x(s_1) \in \mathbb{R}^n$ fixed by formula (42), with $0 \leq s_1 \leq \epsilon r^4$. 

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Proof. Let us write $\Omega(x - x(s_1)) = |x - x(s_1)|^{\omega}$ and $\psi(x) = \psi_+(x) - \psi_-(x)$ where the functions $\psi_\pm(x) \geq 0$ have disjoint support. Thus, by linearity and using the positivity theorem we have

$$|\psi(x, s_1)| = |\psi_+(x, s_1) - \psi_-(x, s_1)| \leq \psi_+(x, s_1) + \psi_-(x, s_1)$$

and we can write

$$\int_{\mathbb{R}^n} |\psi(x, s_1)| \Omega(x - x(s_1))dx \leq \int_{\mathbb{R}^n} \psi_+(x, s_1) \Omega(x - x(s_1))dx + \int_{\mathbb{R}^n} \psi_-(x, s_1) \Omega(x - x(s_1))dx$$

so we only have to treat one of the integrals on the right side above. We have:

$$I = \int_{\mathbb{R}^n} \partial_{s_1} \Omega(x - x(s_1)) \psi_+(x, s_1)dx$$

$$= \int_{\mathbb{R}^n} \partial_{s_1} \Omega(x - x(s_1)) \psi_+(x, s_1) + \Omega(x - x(s_1)) [-\nabla \cdot (v \psi_+(x, s_1)) - \Lambda^{2\alpha} \psi_+(x, s_1)] dx$$

$$= \int_{\mathbb{R}^n} \nabla \Omega(x - x(s_1)) \cdot x'(s_1) \psi_+(x, s_1) + \Omega(x - x(s_1)) [-\nabla \cdot (v \psi_+(x, s_1)) - \Lambda^{2\alpha} \psi_+(x, s_1)] dx$$

Using the fact that $v$ is divergence free, we obtain

$$I = \int_{\mathbb{R}^n} \nabla \Omega(x - x(s_1)) \cdot (v - x'(s_1)) \psi_+(x, s_1) - \Omega(x - x(s_1)) \Lambda^{2\alpha} \psi_+(x, s_1) dx.$$

Finally, using the definition of $x'(s_1)$ given in (42) and replacing $\Omega(x - x(s_1))$ by $|x - x(s_1)|^{\omega}$ we obtain

$$I \leq c \int_{I_1} |x - x(s_1)|^{\omega - 1} |v - \nabla B_f| \psi_+(x, s_1) dx + c \int_{I_2} |x - x(s_1)|^{\omega - 2\alpha} |\psi_+(x, s_1)| dx. \quad (45)$$

We will study separately each one of the integrals $I_1$ and $I_2$ by the next lemmas:

**Lemma 3.4** For integral $I_1$ we have the estimate

$$I_1 \leq C r^{(\omega - 1 - \gamma)(r + K s_0)^\eta(\omega - 1 - \gamma)} \mu,$$

where $1 < a_0 < \frac{n}{n + (\omega - 1)}$ and $\frac{n}{n + \omega - 1} < q < +\infty$.

Proof. We begin by considering the space $\mathbb{R}^n$ as the union of a ball with dyadic coronas centered on $x(s)$, more precisely we set $\mathbb{R}^n = \bigcup_{k \in \mathbb{N}} E_k$ where

$$E_0 = \{x \in \mathbb{R}^n : |x - x(s_1)| \leq f = B_f, \quad (46)$$

$$E_k = \{x \in \mathbb{R}^n : f2^{k-1} < |x - x(s_1)| \leq f2^k \} \quad \text{for } k \geq 1.$$

(i) Estimations over the ball $B_f$.

Applying Hölder inequality on integral $I_1$ we obtain

$$I_{1,E_0} = \int_{B_f} |x - x(s_1)|^{\omega - 1} |v - \nabla B_f| \psi_+(x, s_1) dx \leq \|x - x(s_1)|^{\omega - 1}\|_{L^p(B_f)}$$

$$\times \|v - \nabla B_f\|_{L^q(B_f)} \|\psi_+(\cdot, s_1)\|_{L^r(B_f)}$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and $p, q, r > 1$. We treat each one of the previous terms separately:

- Observe that for $1 < p < n/(1 - \omega)$ we have

  $$|x - x(s_1)|^{\omega - 1}\|_{L^p(B_f)} \leq C f^{n/p + \omega - 1}.$$

- By hypothesis we have $v(\cdot, s_1) \in bmo$, thus

  $$\|v - \nabla B_f\|_{L^q(B_f)} \leq C |B_f|^{1/2} \|v(\cdot, s_1)\|_{bmo}.$$

since $\sup_{s_t \in [s_0, T]} \|v(\cdot, s_1)\|_{bmo} \leq \mu$ we write

  $$\|v - \nabla B_f\|_{L^q(B_f)} \leq C f^{n/2} \mu.$$
• Finally, by the maximum principle (10) for \( L^q \) norms we have \( \| \psi_+(\cdot, s_1) \|_{L^q(B_f)} \leq \| \psi(\cdot, s_0) \|_{L^q}^{1/q} \| \psi(\cdot, s_0) \|_{L^\infty}^{1-1/q} \).

We gather all this inequalities in order to obtain the following estimation for \( I_{1, E_0} \):

\[
I_{1, E_0} = \int_{B_f} |x - x(s_1)|^{\omega-1} |v - \nabla_{B_f} | \psi_+(x, s_1) | dx \leq C\mu f^{n(1-1/q) + \omega-1} \| \psi(\cdot, s_0) \|_{L^1}^{1/q} \| \psi(\cdot, s_0) \|_{L^\infty}^{1-1/q}.
\]

(ii) Estimations for the dyadic corona \( E_k \).

Let us note \( I_{1, E_k} \) the integral

\[
I_{1, E_k} = \int_{E_k} |x - x(s_1)|^{\omega-1} |v - \nabla_{B_f} \psi_+(x, s_1) | dx.
\]

Since over \( E_k \) we have\(^3\) \( |x - x(s_1)|^{\omega-1} \leq C2^{k(\omega-1)} f^{\omega-1} \) we write

\[
I_{1, E_k} \leq C2^{k(\omega-1)} f^{\omega-1} \int_{E_k} |v - \nabla_{B(f^{2k})} | \psi_+(x, s_1) | dx + \int_{E_k} \| \nabla_{B_f} - \nabla_{B(f^{2k})} \| \psi_+(x, s_1) | dx
\]

\[
\leq C2^{k(\omega-1)} f^{\omega-1} \left[ \int_{B(f^{2k})} \| \psi_+(\cdot, s_1) \|_{L^\infty} |v - \nabla_{B(f^{2k})} \|_{L^\omega} |dx + C k \mu \| \psi_+(\cdot, s_0) \|_{L^1} \right]
\]

Now, since \( \psi(\cdot, s_1) \in \text{bmo} \), using proposition 2.1 we have

\[
\| \nabla_{B_f} - \nabla_{B(f^{2k})} \| \leq C k \| \psi(\cdot, s_1) \|_{\text{bmo}} \leq C k \mu
\]

and we can write

\[
I_{1, E_k} \leq C2^{k(\omega-1)} f^{\omega-1} \left[ \| \psi_+(\cdot, s_1) \|_{L^\infty} |v - \nabla_{B(f^{2k})} \|_{L^\omega} + C k \mu \| \psi_+(\cdot, s_0) \|_{L^1} \right]
\]

where we used Hölder inequality with \( a_0 > 1 \) and maximum principle for the last term above. Using again the properties of \( \text{bmo} \) spaces we have

\[
I_{1, E_k} \leq C2^{k(\omega-1)} f^{\omega-1} \left[ \| \psi_+(\cdot, s_0) \|_{L^1}^{1/a_0} \| \psi_+(\cdot, s_0) \|_{L^\infty}^{1-1/a_0} |B(f^{2k})|^{1-1/a_0} |v(\cdot, s_1) |_{\text{bmo}} + C k \mu \| \psi_+(\cdot, s_0) \|_{L^1} \right].
\]

Since \( 1 < a_0 < \frac{n}{n+(\omega-1)} \), we have \( n(1-1/a_0) + (\omega - 1) < 0 \), so that, summing over each dyadic corona \( E_k \), we obtain

\[
\sum_{k \geq 1} I_{1, E_k} \leq C \mu \left( f^{n(1-1/a_0) + \omega-1} \| \psi(\cdot, s_0) \|_{L^1}^{1/a_0} \| \psi(\cdot, s_0) \|_{L^\infty}^{1-1/a_0} + f^{\omega-1} \| \psi(\cdot, s_0) \|_{L^1} \right).
\]

We finally obtain the following inequalities:

\[
I_1 = I_{1, E_0} + \sum_{k \geq 1} I_{1, E_k} \leq C \mu f^{n(1-1/q) + \omega-1} \| \psi(\cdot, s_0) \|_{L^1}^{1/q} \| \psi(\cdot, s_0) \|_{L^\infty}^{1-1/q} \tag{a}
\]

\[
+ C \mu \left( f^{n(1-1/a_0) + \omega-1} \| \psi(\cdot, s_0) \|_{L^1}^{1/a_0} \| \psi(\cdot, s_0) \|_{L^\infty}^{1-1/a_0} + f^{\omega-1} \| \psi(\cdot, s_0) \|_{L^1} \right) \tag{b}
\]

Now, using the value of \( \varepsilon \) explicited in (41), the hypothesis on the initial data \( \psi(\cdot, s_0) \) and the definition of \( f \) given in (43), we will prove that each one of the terms (a), (b) and (c) above is bounded by the quantity

\[
r^{\omega-1}\gamma (r + K \varepsilon)^{\eta(\omega-1)}.
\]

Indeed:

\(^3\)recall that \( 0 < \gamma < \omega < 2 \alpha \leq 1 \).
• for the first term (a), we have:
\[ f^{n(1-1/q) + \omega-1} \|\psi(\cdot, s_0)\|_{L^q}^{1/q} \|\psi(\cdot, s_0)\|_{L^\infty}^{1-1/q} \leq \left[ \left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{1/q} \right]^{1/q} \left[ \left( r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0 \right) - \frac{\lambda_0}{\omega} \right]^{1-1/q} \]
and we prove that the expression of right-hand side above is bounded by \( r^{\varepsilon(\omega-1-\gamma)} (r + Ks_0)^{\eta(\omega-1-\gamma)} \).

\[ f^{n(1-1/q) + \omega-1} \left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{1/q} \left( r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0 \right) \leq r^{\varepsilon(\omega-1-\gamma)} (r + Ks_0)^{\eta(\omega-1-\gamma)} \]

Now, we use the definition of the function \( f \):
\[
\left[ \frac{r^{\varepsilon(1+\gamma-\omega)} (r + Ks_0)^{\frac{r(1+\gamma-\omega)}{\omega} - 1}}{\left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{1-1/q}} \right] f^{n(1-1/q) + \omega-1} \leq r^{\varepsilon(\omega-1-\gamma)} (r + Ks_0)^{\eta(\omega-1-\gamma)}
\]

Since \( n(1-1/q) + \omega - 1 > 0 \) we can write
\[
1 \leq r^{\varepsilon(1+\gamma-\omega)} \left( r + Ks_0 \right)^{-\eta(1+\gamma-\omega)} \left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{(1-1/q) \frac{\lambda_0}{\omega}}
\]
\[
\iff 1 \leq r^{\varepsilon(1+\gamma-\omega)} (r + Ks_0)^{-\eta(1+\gamma-\omega)} \left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{\frac{\lambda_0}{\omega}}
\]
Recall now that \( 0 < (r + Ks_0) < 1 \) and note that \( \eta(1+\gamma-\omega) > 0 \), then we have \( (r + Ks_0)^{-\eta(1+\gamma-\omega)} > 1 \) in order to prove the previous estimate it is enough to study the following inequality
\[
1 \leq r^{\varepsilon(1+\gamma-\omega)} \left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{\frac{\lambda_0}{\omega}}
\]
Since \( \varepsilon(1+\gamma-\omega) > \frac{\lambda_0}{\omega} \) and since \( 0 < r < 1/2 \), it is clear that the previous estimate is true for all \( r > 0 \). Hence, we obtain that
\[
(a) \leq r^{\varepsilon(\omega-1-\gamma)} (r + Ks_0)^{\eta(\omega-1-\gamma)}.
\]

• For the second term (b) we have
\[
 f^{n(1-1/a_0) + \omega-1} \|\psi(\cdot, s_0)\|_{L^1}^{1/a_0} \|\psi(\cdot, s_0)\|_{L^\infty}^{1-1/a_0} \leq f^{n(1-1/a_0) + \omega-1} \times \left[ \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right]^{1/a_0} \left[ \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right]^{1-1/a_0}
\]
Recalling that \( n(1-1/a_0) + \omega - 1 < 0 \), we need to prove the following estimate
\[
 f^{n(1-1/a_0) + \omega-1} \left[ \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right]^{1/a_0} \left[ \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right]^{1-1/a_0} \leq r^{\varepsilon(\omega-1-\gamma)} (r + Ks_0)^{\eta(\omega-1-\gamma)}
\]
\[
\iff r^{\varepsilon(1+\gamma-\omega)} (r + Ks_0)^{\eta(1+\gamma-\omega)} \left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{\frac{\lambda_0}{\omega}} (1-1/a_0) \leq r^{\varepsilon(1+\gamma-\omega)} (r + Ks_0)^{\eta(1+\gamma-\omega)} \left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{\frac{\lambda_0}{\omega}} (1-1/a_0)
\]
\[
\iff r^{\varepsilon(1+\gamma-\omega)} (r + Ks_0)^{\eta(1+\gamma-\omega)} \left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{\frac{\lambda_0}{\omega}} (1-1/a_0) \leq r^{\varepsilon(1+\gamma-\omega)} (r + Ks_0)^{\eta(1+\gamma-\omega)} \left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{\frac{\lambda_0}{\omega}} (1-1/a_0)
\]
We need to prove
\[
1 \leq r^{\varepsilon(1+\gamma-\omega)} (r + Ks_0)^{-\eta(1+\gamma-\omega)} \left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{\frac{\lambda_0}{\omega}} (1-1/a_0)
\]
\[
\iff 1 \leq r^{\varepsilon(1+\gamma-\omega)} (r + Ks_0)^{-\eta(1+\gamma-\omega)} \left( \frac{r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0}{\frac{\lambda_0}{\omega}} \right)^{\frac{\lambda_0}{\omega}} (1-1/a_0)
\]
Again, since \( 0 < (r + Ks_0) < 1 \) and since \( \eta(1+\gamma-\omega) > 0 \), it is enough to study the following inequality
\[
1 \leq r^{\varepsilon(1+\gamma-\omega)} \left( r^{2\alpha} + \frac{\lambda_0}{\omega} + c_0 s_0 \right)^{\frac{\lambda_0}{\omega}}
\]
but we know already that this estimate is always true (for \( r \) small), so we obtain that
\[
(b) \leq r^{\varepsilon(\omega-1-\gamma)} (r + Ks_0)^{\eta(\omega-1-\gamma)}.
\]
Finally, for the last term (c) we write
$$f^{\omega-1}\|\psi(\cdot, s_0)\|_{L^1} \leq f^{\omega-1}(r^{2\alpha_n+\gamma} + c_0 s_0)^{-\omega}$$
and we prove that the expression of right-hand side above is bounded by $r^{(1+\gamma-\omega)}(r + K s_0)^{\eta(\omega-1-\gamma)}$:
$$f^{\omega-1}(r^{2\alpha_n+\gamma} + c_0 s_0)^{-\omega} \leq r^{(1+\gamma-\omega)}(r + K s_0)^{\eta(1+\gamma-\omega)}$$
which is nothing but the definition of the function $f$ given in (43).

Gathering these estimate on (a), (b) and (c), and getting back to (47) we finally obtain
$$I_1 \leq C f^{\omega-1}(r + K s_0)^{\eta(\omega-1-\gamma)} \mu.$$The lemma 3.4 is proved.

Lemma 3.5 For integral $I_2$ in inequality (45) we have the following estimate
$$I_2 \leq C f^{\omega-1}(r + K s_0)^{\eta(\omega-1-\gamma)}.$$where $1 < a_1 < n/(2\alpha - \omega)$ and $\frac{1}{a_1} + \frac{1}{b_1} = 1$.

Proof. As for lemma 3.1, we consider $\mathbb{R}^n$ as the union of a ball with dyadic coronas centered on $x(t)$ (cf. (46)).

(i) Estimations over the ball $B_f$.
Since $1 < a_1 < n/(2\alpha - \omega)$, applying Hölder inequality and maximum principle we have
$$I_{2,E_0} = \int_{B_f} |x - x(s_1)|^{\omega-2\alpha} |\psi_+(x, s_1)| dx \leq \|x - x(s_1)|^{\omega-2\alpha} L^1(B_f) \|\psi_+(\cdot, s_1)\|_{L^1(B_f)} \leq C f_n^{a_1+\omega-2\alpha} \|\psi_+(\cdot, s_0)\|_L^{1/a_1} \|\psi_+(\cdot, s_0)\|_{L^\infty}^{1/a_1}.$$

At this point we distinguish two cases.
• If $\alpha = 1/2$, i.e. for the critical case, there is nothing to do.

• For $0 < \alpha < 1/2$, recall that we have, by the definition of the function $f$ the estimate $0 < f < 1$ and $f^{\omega-2\alpha-\gamma} \leq f^{\omega-1-\gamma}$. Thus, we can write:
$$I_{2,E_0} \leq C f_n^{a_1+\omega-1} \|\psi_+(\cdot, s_0)\|_L^{1-1/a_1} \|\psi_+(\cdot, s_0)\|_{L^\infty}.$$

(ii) Estimations for the dyadic corona $E_k$.
Here we have
$$I_{2,E_k} = \int_{E_k} |x - x(s_1)|^{\omega-2\alpha} |\psi_+(x, s_1)| dx \leq C f_n^{2(k-\omega-2\alpha)} \int_{E_k} |\psi_+(x, s_1)| dx \leq C f_n^{2(k-\omega-2\alpha)} \|\psi_+(\cdot, s_1)\|_{L^1} \leq C f_n^{2(k-\omega-2\alpha)} \|\psi_+(\cdot, s_0)\|_{L^1}.$$

Since $0 < \gamma < \omega < 2\alpha$ we have $\omega - 2\alpha < 0$ and thus, summing over $k \geq 1$, we obtain
$$\sum_{k \geq 1} I_{2,E_k} = \sum_{k \geq 1} \int_{E_k} |x - x(s_1)|^{\omega-2\alpha} |\psi_+(x, s_1)| dx \leq C f^{\omega-2\alpha} \|\psi_+(\cdot, s_0)\|_{L^1}.$$Repeating the same argument used before (i.e. the fact that $0 < f < 1$), we finally get
$$\sum_{k \geq 1} I_{2,E_k} \leq C f^{\omega-1} \|\psi_+(\cdot, s_0)\|_{L^1}. \quad (49)$$
To finish the proof of lemma 3.2 we glue together (48) and (49) and we obtain

\[ I_2 = I_{2,E_0} + \sum_{k \geq 1} I_{2,E_k} \leq C \left( \frac{f^{n/a_1+\omega-1} \|\psi(\cdot, s_0)\|^1_{L^{1/a_1}} \|\psi(\cdot, s_0)\|^1_{L^\infty}}{(d)} + f^{\omega-1} \|\psi(\cdot, s_0)\|_{L^1} \right) \]

Now, we prove that the next condition holds:

\[ \|\psi(\cdot, s_1)\|_{L^\infty} \leq 1 \]

\[ (r^{2\alpha n+\gamma n+\omega} + c_0 s_0 - \varepsilon)^{\frac{\gamma}{2\alpha}} \]

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For the term (d) we write

\[ f^{n/a_1 + \omega - 1} \|\psi(\cdot, s_0)\|^1_{L^{1/a_1}} \|\psi(\cdot, s_0)\|^1_{L^\infty} \leq f^{n/a_1 + \omega - 1} \times \left[ (r^{2\alpha n+\gamma n+\omega} + c_0 s_0 - \varepsilon)^{\frac{\gamma}{2\alpha}} \right]^{1/a_1} \]

and thus, we need to prove

\[ f^{n/a_1 + \omega - 1} (r^{2\alpha n+\gamma n+\omega} + c_0 s_0) - \varepsilon (1 - 1/a_1) - \varepsilon^{\frac{\gamma}{2\alpha}} 1/a_1 \leq r^{c(\omega-1)\gamma} (r + K s_0)^{\eta(\omega-1)\gamma} \]

Now, we use the definition of the function f:

\[ \left[ \frac{1 + \varepsilon \gamma}{1 + \varepsilon \gamma} (r + K s_0)^{\omega(1+\gamma - \varepsilon)} (r^{2\alpha n+\gamma n+\omega} + c_0 s_0) - \varepsilon^{\frac{\gamma}{2\alpha}} \right]^{1/a_1} \]

\[ \Longleftrightarrow 1 \leq r^{-c(1+\gamma - \varepsilon)} \left( r + K s_0 \right)^{-\eta(1+\gamma - \varepsilon) \frac{\gamma}{2\alpha}} (r^{2\alpha n+\gamma n+\omega} + c_0 s_0) \frac{1}{\varepsilon} \]

\[ \Longleftrightarrow 1 \leq r^{-c(1+\gamma - \varepsilon)} (r + K s_0)^{-\eta(1+\gamma - \varepsilon) \frac{\gamma}{2\alpha}} \]

We use now the fact that 0 < (r + K s_0) < 1 and \( \eta(1 + \gamma - \omega) > 0 \) to reduce our study to the known estimate

\[ 1 \leq r^{-c(1+\gamma - \varepsilon)} \left( r^{2\alpha n+\gamma n+\omega} + c_0 s_0 \right)^{\frac{1}{\varepsilon}} \]

which lead us to the inequality

\[ (d) \leq r^{c(\omega-1)\gamma} (r + K s_0)^{\eta(\omega-1)\gamma} \]

To finish we remark that (c) is bounded by \( r^{c(\omega-1)\gamma} (r + K s_0)^{\eta(\omega-1)\gamma} \): just apply the same arguments used to prove the part (c) above.

Finally, we obtain

\[ I_2 = I_{2,E_0} + \sum_{k \geq 1} I_{2,E_k} \leq C r^{c(\omega-1)\gamma} (r + K s_0)^{\eta(\omega-1)\gamma} \]

The lemma 3.5 is proved.

Now we continue the proof of proposition 3.4. Using lemmas 3.4 and 3.5 and getting back to estimate (45) we have

\[ \left| \partial_{s_1} \int_{\mathbb{R}^n} \Omega(x - x(s_1)) \psi_+(x, s_1) dx \right| \leq C r^{c(\omega-1)\gamma} (r + K s_0)^{\eta(\omega-1)\gamma} (\mu + 1) \]

This last estimation is compatible with the estimate (44) for 0 ≤ s_1 ≤ \varepsilon r^4 small enough: just fix K and \( \delta \) such that

\[ C (\mu + 1) \leq K (\omega - \gamma) - \delta. \]

Remark that this is exactly the condition (27).

Now we write down the maximum principle for a small time s_1 but with an initial condition \( \psi(\cdot, s_1) \), with \( s_0 > 0 \).

**Proposition 3.5 (Height condition)** Under the hypothesis of theorem 6, if \( \psi(x, s_1) \) satisfies concentration condition (44), then we have the next height condition

\[ \|\psi(\cdot, s_1)\|_{L^\infty} \leq \frac{1}{\left( r^{2\alpha n+\gamma n+\omega} + c_0 (s_0 + s_1) \right)^{\frac{\gamma}{2\alpha}}}. \]

(50)

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Proof. The proof follows essentially the same lines of the proposition 3.2. Indeed, since we have the concentration condition (44), we obtain
\[
\frac{d}{ds_1} \|\psi(\cdot, s_1)\|_{L^\infty} \leq -C\psi(\cdot, s_1)^{1+\frac{2\gamma}{2}} = -C\|\psi(\cdot, s_1)\|_{L^\infty}^{1+\frac{2\gamma}{2}}.
\]
Solving this inequality one has
\[
\|\psi(\cdot, s_1)\|_{L^\infty} \leq \frac{\|\psi(\cdot, s_0)\|_{L^\infty}}{(1 + Cs_1 \|\psi(\cdot, s_0)\|_{L^\infty}^{\frac{2\gamma}{2}})}^{\frac{2}{2+\gamma}}.
\]
Now, using the estimate (36) we have
\[
\|\psi(\cdot, s_1)\|_{L^\infty} \leq \frac{1}{(r^{2\alpha} + c_0(s_0 + s_1))^{\frac{2+\gamma}{2\alpha}}}.
\]

The crucial part of the proof of theorem 6 is given by the next proposition which give us a control on the \(L^1\)-norm for a time \(s_0 + s_1\).

**Proposition 3.6 (Second \(L^1\)-norm estimate)** Under the hypothesis of theorem (6) we have
\[
\|\psi(\cdot, s_1)\|_{L^1} \leq \frac{v_n}{(r^{2\alpha} + c_0(s_0 + s_1))^{\frac{2+\gamma}{2\alpha}}}
\]

Proof. We write:
\[
\|\psi(\cdot, s_1)\|_{L^1} = \int_{\{x \in \mathbb{R}^n : |x - x(s_1)| < B\}} |\psi(x, s_1)|dx + \int_{\{x \in \mathbb{R}^n : |x - x(s_1)| \geq B\}} |\psi(x, s_1)|dx
\]
\[
\leq v_n B^n \|\psi(\cdot, s_1)\|_{L^\infty} + B^{-\omega} \int_{\mathbb{R}^n} |\psi(x, s_1)||x - x(s_1)|^\omega dx.
\]
Now, we use the estimates (44) and (50) to obtain:
\[
\|\psi(\cdot, s_1)\|_{L^1} \leq v_n B^n \frac{1}{(r^{2\alpha} + c_0(s_0 + s_1))^{\frac{2+\gamma}{2\alpha}}} + B^{-\omega} \left(\frac{r^\gamma (r + Ks_0)^{\gamma}}{r^\gamma (r + Ks_0)^{\gamma} + Ks_1}\right)^{\delta/K}\left(\frac{r^\gamma (r + Ks_0)^{\gamma} + Ks_1}{r^\gamma (r + Ks_0)^{\gamma} + Ks_1}\right)^{\omega-\gamma}
\]
Choosing carefully \(B\) we have
\[
\|\psi(\cdot, s_1)\|_{L^1} \leq v_n \left(\frac{r^\gamma (r + Ks_0)^{\gamma}}{r^\gamma (r + Ks_0)^{\gamma} + Ks_1}\right)^{\delta/K} \frac{r^\gamma (r + Ks_0)^{\gamma} + Ks_1}{(r^{2\alpha} + c_0(s_0 + s_1))^{\frac{2+\gamma}{2\alpha}}} \leq \frac{v_n}{(r^{2\alpha} + c_0(s_0 + s_1))^{\frac{2+\gamma}{2\alpha}}}.
\]

3.3 The last iteration
In the sections 3.1 and 3.2 we studied respectively the evolution of small molecules from time 0 to a small time \(s_0\) and from this time \(s_0\) to a larger time \(s_0 + s_1\) and we obtained a good \(L^1\) control for such molecules.

It is now possible to reapply theorem 6 in order to obtain a larger time \(L^1\) control: just use as initial data the conditions (38)-(40). The calculus of the second iteration will be exactly the same (it will be however necessary to adapt at each step the function \(f\) given in (43)), and we will pass from time \(s_0 + s_1\) to a bigger time \(s_0 + s_1 + s_2\) and the \(L^1\) estimate will be of the following form:
\[
\|\psi(\cdot, s_2)\|_{L^1} \leq \frac{v_n}{(r^{2\alpha} + c_0(s_0 + s_1 + s_2))^{\frac{2+\gamma}{2\alpha}}}
\]

note in particular that this estimate is bounded for all \(r > 0\).

We obtain a \(L^1\) control and we have finished the proof of theorem 1.
4 Existence and uniqueness for $L^p$ initial data

In this section we will study existence and uniqueness for weak solution of equation (1) for $0 < \alpha < 1/2$ with initial data $\theta_0 \in L^p(\mathbb{R}^n)$ where $p \geq 2$. Remark that equation (1) differs mainly from the backward equation (6) by the sign of the velocity. Since the velocity $v$ is a data for the problem, it is equivalent to consider $-v$ instead of $v$, thus, for simplicity, we fix velocity's sign in the next way:

$$\begin{align*}
\partial_t \theta(x,t) + \nabla \cdot (v \theta)(x,t) + \Lambda^{2\alpha} \theta(x,t) &= 0 \\
\theta(x,0) &= \theta_0(x) \in L^p(\mathbb{R}^n) \\
div(v) &= 0 \quad \text{and} \quad v \in L^\infty([0,T];bmo(\mathbb{R}^n)).
\end{align*}$$

(51)

4.1 Viscosity Solutions

The term *Viscosity Solutions* is taken from [7] and it refers to weak solutions of (51) which are the weak limit, as $\varepsilon \rightarrow 0$, of a sequence of solutions of problems

$$\begin{align*}
\partial_t \theta(x,t) + \nabla \cdot (v_\varepsilon \theta)(x,t) + \Lambda^{2\alpha} \theta(x,t) &= \varepsilon \Delta \theta(x,t) \\
\theta(x,0) &= \theta_0(x) \\
div(v) &= 0 \quad \text{and} \quad v \in L^\infty([0,T];L^\infty(\mathbb{R}^n)).
\end{align*}$$

(52)

where $v_\varepsilon$ is defined by $v_\varepsilon = v \ast \omega_\varepsilon$ with $\omega_\varepsilon(x) = \varepsilon^{-n}\omega(x/\varepsilon)$ and $\omega \in C_0^\infty(\mathbb{R}^n)$ is a function such that $\int_{\mathbb{R}^n} \omega(x)dx = 1$.

**Remark 4.1** Observe that we fixed here the velocity $v$ such that $v \in L^\infty([0,T];L^\infty(\mathbb{R}^n))$. This is not very restrictive because by proposition 2.2 we can construct a sequence $v_\varepsilon \in L^\infty$ that converge weakly to $v$ in $bmo$.

Problem (52) admits the following equivalent integral representation:

$$\theta(x,t) = e^{\varepsilon t \Delta} \theta_0(x) - \int_0^t e^{\varepsilon (t-s) \Delta} \nabla \cdot (v_\varepsilon \theta)(x,s)ds - \int_0^t e^{\varepsilon (t-s) \Delta} \Lambda^{2\alpha} \theta(x,s)ds$$

(53)

For a proof of this fact see [15] or [12]. We will use then the Picard contraction scheme and for this we will consider the space $L^\infty([0,T];L^p(\mathbb{R}^n))$ with the following norm

$$\|f\|_{L^\infty L^p} = \sup_{t \in [0,T]} \|f(\cdot,t)\|_{L^p}$$

**Theorem 7 (Local existence)** Let $0 < \alpha < 1/2$, $2 \leq p < +\infty$ and let $\theta_0$ and $v$ be two functions such that $\theta_0 \in L^p(\mathbb{R}^n)$, div($v$) = 0 and $v \in L^\infty([0,T');L^\infty(\mathbb{R}^n))$.

If initial data satisfies $\|\theta_0\|_{L^p} \leq K$ and if $T'$ is a time small enough such that

$$C \left( \frac{T'^{1/2}}{\varepsilon^{1/2}} \|v\|_{L^\infty L^\infty} + \frac{T'^{\alpha}}{\varepsilon^{\alpha}} \right) \leq 1/2,$$

then (53) has a unique solution $\theta \in L^\infty([0,T'];L^p(\mathbb{R}^n))$ on the closed ball $\overline{B}(0,2K) \subset L^\infty([0,T');L^p(\mathbb{R}^n))$.

**Proof.** We note $L_\varepsilon(\theta)$ and $N_\varepsilon^v(\theta)$ the quantities

$$L_\varepsilon(\theta)(x,t) = \int_0^t e^{\varepsilon (t-s) \Delta} \Lambda^{2\alpha} \theta(x,s)ds \quad \text{and} \quad N_\varepsilon^v(\theta)(x,t) = \int_0^t e^{\varepsilon (t-s) \Delta} \nabla \cdot (v_\varepsilon \theta)(x,s)ds.$$

For the first expression we have:

**Lemma 4.1** If $f \in L^\infty([0,T'];L^p(\mathbb{R}^n))$, then

$$\|L_\varepsilon(f)\|_{L^\infty L^p} \leq C \sqrt{\frac{T'}{\varepsilon}} \|f\|_{L^\infty L^p}$$

(54)
To apply the Picard contraction scheme, let us now construct a sequence of functions in the following way

Using estimates (54), (55) and (56) we have

where we noted \( h_t \) the heat kernel\(^4\) on \( \mathbb{R}^n \). Then we have the estimates

For the term \( N^\varepsilon_n \) we have:

**Lemma 4.2** If \( f \in L^\infty([0, T]; L^p(\mathbb{R}^n)) \) and if \( v \in L^\infty([0, T]; L^\infty(\mathbb{R}^n)) \), then

**Proof.** We write:

Finally, since \( e^{t \Delta} \) is a contraction operator, estimate \( \| e^{t \Delta} f \|_{L^p} \leq \| f \|_{L^p} \) is valid for all function \( f \in L^p(\mathbb{R}^n) \) with \( 1 \leq p \leq +\infty \), for all \( t > 0 \) and all \( \varepsilon > 0 \). Thus, we have

To apply the Picard contraction scheme, let us now construct a sequence of functions in the following way

and we take the \( L^\infty L^p \)-norm of this expression to obtain

Using estimates (54), (55) and (56) we have

Thus, if \( \| \theta_0 \|_{L^p} \leq K \) and with the definition of \( T' \), we have by iteration that \( \| \theta_{n+1} \|_{L^\infty L^p} \leq 2K \): the sequence \( (\theta_n)_{n \in \mathbb{N}} \) constructed from initial data \( \theta_0 \) belongs to the closed ball \( \overline{B}(0, 2K) \).

In order to finish this proof, let us show that \( \theta_n \rightarrow \theta \) in \( L^\infty([0, T']; L^p(\mathbb{R}^n)) \). For this we write

\(^4\)which is nothing but a gaussian.
and using previous lemmas we have
\[ \|\theta_{n+1} - \theta_n\|_{L^\infty L^p} \leq C \left( \frac{T^{1/2}}{\varepsilon^{1/2}} \|v\|_{L^\infty L^\infty} + \frac{T^{1-\alpha}}{\varepsilon^\alpha} \right) \|\theta_n - \theta_{n-1}\|_{L^\infty L^p} \]
so, by iteration we obtain
\[ \|\theta_{n+1} - \theta_n\|_{L^\infty L^p} \leq \left[ C \left( \frac{T^{1/2}}{\varepsilon^{1/2}} \|v\|_{L^\infty L^\infty} + \frac{T^{1-\alpha}}{\varepsilon^\alpha} \right) \right]^n \|\theta_1 - \theta_0\|_{L^\infty L^p} \]
hence, with the definition of \( T' \) it comes \( \|\theta_{n+1} - \theta_n\|_{L^\infty L^p} \leq \left( \frac{1}{2} \right)^n \|\theta_1 - \theta_0\|_{L^\infty L^p} \). Finally, if \( n \to +\infty \), the sequence \( (\theta_n)_{n \in \mathbb{N}} \) convergences towards \( \theta \) in \( L^\infty([0, T^*]; L^p(\mathbb{R}^n)) \). Since it is a Banach space we deduce unicity for the solution \( \theta \) of problem (53)

\[ \text{Corollary 4.1} \quad \text{The solution constructed above depends continuously on initial data } \theta_0. \]

\[ \text{Proof.} \quad \text{Let } \phi_0 \in L^p(\mathbb{R}^n) \text{ be an initial data and let } \phi \text{ be the associated solution. We write} \]
\[ \theta(x, t) - \phi(x, t) = e^{t\Delta}(\theta_0(x) - \phi_0(x)) - L_\varepsilon(\theta(x, t) - \phi(x, t)) - N_\varepsilon^v(\theta(x, t) - \phi(x, t)) \]
Taking \( L^\infty L^p \)-norm in formula above and applying the same previous calculations one obtains
\[ \|\theta - \phi\|_{L^\infty L^p} \leq \|\theta_0 - \phi_0\|_{L^p} + C_0 \|\theta - \phi\|_{L^\infty L^p} \] (57)
This shows continuous dependence of the solution since \( C_0 = \left( C \left( \frac{T^{1/2}}{\varepsilon^{1/2}} \|v\|_{L^\infty L^\infty} + \frac{T^{1-\alpha}}{\varepsilon^\alpha} \right) \right) \leq 1/2 \).

\[ \text{Remark 4.2} \quad \text{If } \theta \text{ and } \phi \text{ are two different solutions of problem (52) with } \theta_0 \in L^p(\mathbb{R}^n), \text{ then formula (57) shows us that they actually coincide on } [0, T_1] \text{ with } T_1 \text{ a real number such that} \]
\[ C \left( \frac{T^{1/2}}{\varepsilon^{1/2}} \|v\|_{L^\infty L^\infty} + \frac{T^{1-\alpha}}{\varepsilon^\alpha} \right) < 1 \]
Once we obtain a local result, global existence easily follows by a simple iteration since problems studied here (equations (1), (51) or (52)) are linear as velocity \( v \) does not depend on \( \theta \).

\[ \text{Remark 4.3} \quad \text{Solutions } \theta(\cdot, \cdot) \text{ constructed above depends on } \varepsilon \text{ and it will be more convenient to note them as } \theta(\cdot, \cdot). \]
For the time being, we will just note them \( \theta(\cdot, \cdot) \).

We study now the regularity of solutions constructed by this method.

\[ \text{Theorem 8} \quad \text{Solutions of approximated problem (52) are smooth.} \]

\[ \text{Proof.} \quad \text{By iteration we will prove that} \]
\[ \theta \in \bigcap_{0 < T_0 < T_1 < t < T_2 < T^*} L^\infty([0, t]; W^{k+1, p}(\mathbb{R}^n)) \quad \text{for all } k \geq 0. \]
Remark that this is true for \( k = 0 \). So let us assume that it is also true for \( k > 0 \) and we will show that it is still true for \( k + 1 \).

Set \( t \) such that \( 0 < T_0 < T_1 < t < T_2 < T^* \) and let us consider the next problem
\[ \theta(x, t) = e^{t(T_0 - T_0)}(\theta(x, T_0) - \theta(x, T_0)) - \int_{T_0}^t e^{(t-s)}(v_x \theta)(x, s)ds - \int_{T_0}^t e^{(t-s)}(\Delta^2 \theta(x, s)ds \]
We have then the following estimate
\[ \|\theta\|_{L^\infty W^{k+1, p}} \leq \|e^{t(T_0 - T_0)}(\cdot, T_0)\|_{L^\infty W^{k+1, p}} \]
\[ + \left\| \int_{T_0}^t e^{(t-s)}(v_x \theta)(\cdot, s)ds \right\|_{L^\infty W^{k+1, p}} + \left\| \int_{T_0}^t e^{(t-s)}(\Delta^2 \theta)(\cdot, s)ds \right\|_{L^\infty W^{k+1, p}} \]
Now, we will treat separately each one of the previous terms.
(i) For the first one we have
\[ \| e^{(t-T_0)} \Delta \theta (\cdot, T_0) \|_{W^{k+1,p}} = \| \theta (\cdot, T_0) * \Lambda_{k+1} h_{\varepsilon(t-T_0)} \|_{L^p} \leq \| \theta (\cdot, T_0) \|_{L^p} \| \Lambda_{k+1} h_{\varepsilon(t-T_0)} \|_{L^1} \]
where \( h_t \) is the heat kernel, so we can write
\[ \| e^{(t-T_0)} \Delta \theta (\cdot, T_0) \|_{L^\infty W^{k+1,p}} \leq C \| \theta (\cdot, T_0) \|_{L^p} \sup \left\{ \| \varepsilon(t-T_0) \|^{-\frac{k+1}{p}} ; 1 \right\} \]

(ii) For the second term, one has
\[ I = \left\| \int_{T_0}^t e^{(t-s)} \nabla \cdot (v_\varepsilon \theta) (\cdot, s) ds \right\|_{W^{k+1,p}} \leq \int_{T_0}^t \| \nabla \cdot (v_\varepsilon \theta) * h_{\varepsilon(t-s)} \|_{W^{k+1,p}} ds \]
\[ \leq C \int_{T_0}^t \| v_\varepsilon \theta (s, s) \|_{W^{k+1,p}} \sup \left\{ \| \varepsilon(t-s) \|^{-\frac{k+1}{p}} ; 1 \right\} ds \]
Remark that we have here the estimations below for \( N \geq k/2 \)
\[ \| v_\varepsilon \theta (\cdot, s) \|_{W^{k,p}} \leq \| v(\cdot, s) \|_{C^N} \| \theta (\cdot, s) \|_{W^{k,p}} \leq C \varepsilon^{-N} \| v(\cdot, s) \|_{L^\infty} \| \theta (\cdot, s) \|_{W^{k,p}} \]
hence, we can write
\[ I \leq C \| v \|_{L^\infty} \| \theta \|_{L^\infty W^{k,p}} \int_{T_0}^t \varepsilon^{-N} \sup \left\{ \| \varepsilon(t-s) \|^{-\frac{k+1}{p}} ; 1 \right\} ds \]

(iii) Finally, for the last term we have
\[ \left\| \int_{T_0}^t e^{(t-s)} \Delta^{2\alpha} \theta (\cdot, s) ds \right\|_{W^{k+1,p}} \leq C \int_{T_0}^t \| \theta (\cdot, s) \|_{W^{k+2\alpha,p}} \| \varepsilon(t-s) \|^{-\frac{k+1}{2\alpha}} ds \]
\[ \left\| \int_{T_0}^t e^{(t-s)} \Delta^{2\alpha} \theta (\cdot, s) ds \right\|_{L^\infty W^{k+1,p}} \leq C \| \theta \|_{L^\infty W^{k+2\alpha,p}} \int_{T_0}^t \sup \left\{ \| \varepsilon(t-s) \|^{-\frac{k+1}{2\alpha}} ; 1 \right\} ds. \]
Now, with formulas (i)-(iii) at our disposal, we have that the norm \( \| \theta \|_{L^\infty W^{k+1,p}} \) is controlled for all \( \varepsilon > 0 \): we have proved spatial regularity.

Time regularity follows since we have
\[ \frac{\partial^k \theta (x, t)}{\partial t^k} + \nabla \cdot \left( \frac{\partial^k \theta (x, t)}{\partial t^k} \right) (x, t) + \Lambda^{2\alpha} \left( \frac{\partial^k \theta (x, t)}{\partial t^k} \right) (x, t) = \varepsilon \Delta \left( \frac{\partial^k \theta (x, t)}{\partial t^k} \right) (x, t). \]

\[ \square \]

### 4.2 Maximum principle and Besov regularity

As a motivation for theorem 10 below, we rewrite in the following lines the proof of the maximum principle.

**Theorem 9 (Maximum Principle)** Let \( \alpha \in [0, 1/2], \) \( 2 \leq p < +\infty \) and let \( \theta \) be a smooth solution of equation (52). Then we have the following estimation
\[ \| \theta (\cdot, t) \|_{L^p} \leq \| \theta_0 \|_{L^p}. \]  

**Proof.** We write
\[ \frac{d}{dt} \| \theta (\cdot, t) \|_{L^p} = p \int_{\mathbb{R}^n} |\theta|^{p-2} \theta \left( \varepsilon \Delta \theta - \nabla \cdot (v_\varepsilon \theta) - \Lambda^{2\alpha} \theta \right) dx = p \varepsilon \int_{\mathbb{R}^n} |\theta|^{p-2} \theta \Delta \theta dx + p \int_{\mathbb{R}^n} |\theta|^{p-2} \theta \Lambda^{2\alpha} \theta dx \]
where we used the fact that \( \text{div}(v) = 0 \). Thus, we have
\[ \frac{d}{dt} \| \theta (\cdot, t) \|_{L^p} = p \varepsilon \int_{\mathbb{R}^n} |\theta|^{p-2} \theta \Delta \theta dx + p \int_{\mathbb{R}^n} |\theta|^{p-2} \theta \Lambda^{2\alpha} \theta dx = 0, \]
and integrating in time we obtain
\[ \| \theta (\cdot, t) \|_{L^p} = p \varepsilon \int_0^t \int_{\mathbb{R}^n} |\theta|^{p-2} \theta \Delta \theta dx ds + p \int_0^t \int_{\mathbb{R}^n} |\theta|^{p-2} \theta \Lambda^{2\alpha} \theta dx ds = \| \theta_0 \|_{L^p}. \]  
To finish, we have the next lemma
Lemma 4.3 The quantities
\[-p\epsilon \int_{\mathbb{R}^n} |\theta|^{p-2}\theta \Delta \theta dx \quad \text{and} \quad p \int_0^t \int_{\mathbb{R}^n} |\theta|^{p-2}\theta \Lambda^{2\alpha} \theta dx ds\]
are both positive.

Proof. For the first expression, since \(e^{s\Delta}\) is a contraction semi-group we have \(\|e^{s\Delta} f\|_{L^p} \leq \|f\|_{L^p}\) for all \(s > 0\) and all \(f \in L^p(\mathbb{R}^n)\). Thus \(F(s) = \|e^{s\Delta} f\|_{L^p}\) is decreasing in \(s\); taking the derivative in \(s\) and evaluating in \(s = 0\) we obtain the desired result. For the second expression a proof can be found in [7] (the positivity lemma p.516). However, we will give another proof of this fact with theorem 10 below.

Getting back to (59), we have that all these quantities are bounded and positive, so theorem 9 follows easily.

Remark 4.4 This maximum principle (58) is still valid for \(1 \leq p \leq +\infty\). See [15] for a proof.

As said in the introduction, the study of expression (59) above lead us to a result concerning weak solution’s regularity which is announced in theorem 2. More precisely we have

Theorem 10 (Besov Regularity) Let \(2 \leq p < +\infty\) and let \(f : \mathbb{R}^n \to \mathbb{R}\) be a function such that
\[\int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) \Lambda^{2\alpha} f(x) dx < +\infty \quad \text{then} \quad f \in \dot{B}^{2\alpha/p}_{p,p}(\mathbb{R}^n).\]

Proof. To begin with, assume that \(f\) is a positive function and let us show that we have the following estimates:
\[\|f\|^p_{\dot{B}^{2\alpha/p}_{p,p}} \leq C \|f^{p/2}\|^2_{\dot{B}^{2\alpha}_{2,2}} \leq C' \int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) \Lambda^{2\alpha} f(x) dx \quad (60)\]

In this case, we use the

Lemma 4.4 For \(0 < \epsilon \leq 1\) and for all \(a, b > 0\) we have
\[|a^\epsilon - b^\epsilon| \leq |a - b|^\epsilon.\]

Proof. The proof of this lemma is straightforward, since \(d_\epsilon(x, y) = |x - y|^\epsilon\) is a distance over \(\mathbb{R}\) and we have
\[d_\epsilon(a, 0) - d_\epsilon(b, 0) \leq d_\epsilon(a, b).\]

Hence, applying this lemma with \(\epsilon = 2/p\), \(a = f(x)^{p/2}\) and \(b = f(y)^{p/2}\), one has
\[|f(x) - f(y)| \leq |f(x)|^{p/2} - f(y)|^{p/2}|^{2/p} \]
which implies
\[\|f\|^p_{\dot{B}^{2\alpha/p}_{p,p}} \simeq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+2\alpha}} dxdy \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)|^{p/2} - f(y)|^{p/2}|^2}{|x - y|^{n+2\alpha}} dxdy \simeq \|f^{p/2}\|^2_{\dot{B}^{2\alpha}_{2,2}}\]
and this give us the first part of (60). For the second part we have
\[\|f^{p/2}\|^2_{\dot{B}^{2\alpha}_{2,2}} = \|f^{p/2}\|^2_{\dot{B}^{2\alpha}_{2,2}} = \int_{\mathbb{R}^n} |\Lambda^{\alpha} f^{p/2}(x)|^2 dx = \int_{\mathbb{R}^n} f^{p/2}(x) \Lambda^{2\alpha} f^{p/2}(x) dx, \quad (61)\]
by the self-adjointness of operator \(\Lambda^{\alpha}\).

We consider now the semi-group \((e^{-\tau \Lambda^{2\alpha}})_{\tau \geq 0}\). Since \(p \geq 2\), using Jensen inequality\(^5\) we obtain the estimate
\[\|e^{-\tau \Lambda^{2\alpha}} f\|_{L^p} \leq \left(e^{-\tau \Lambda^{2\alpha}} f^{p/2}\right)^{2/p}.\]
Thus, integrating this inequality we obtain \(\|e^{-\tau \Lambda^{2\alpha}} f\|^p_{L^p} \leq \|e^{-\tau \Lambda^{2\alpha}} f^{p/2}\|^2_{\dot{B}^{2\alpha}_{2,2}}\). Finally, taking the derivative with respect to \(\tau\) and evaluating in \(\tau = 0\) one obtains
\[\int_{\mathbb{R}^n} f^{p/2}(x) \Lambda^{2\alpha} f^{p/2}(x) dx \leq -\int_{\mathbb{R}^n} f^{p/2}(x) \Lambda^{2\alpha} f^{p/2}(x) dx \]
\(^5\)see [13] for the details concerning the semi-group \((e^{-\tau \Lambda^{2\alpha}})_{\tau \geq 0}\).
Hence, getting back to (61) it comes \( \|f^{p/2}\|_{B^0_{p,2}}^2 \leq \int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) \Lambda^{2\alpha} f(x) dx \) and estimates (60) are proved.

Let us now prove the general case. For this, we write \( f(x) = f_+(x) - f_-(x) \) where \( f_\pm(x) \) are positives functions with disjoint support. We have:

\[
\int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) \Lambda^{2\alpha} f(x) dx = \int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \Lambda^{2\alpha} f_+(x) dx + \int_{\mathbb{R}^n} f_-(x)^{p-2} f_-(x) \Lambda^{2\alpha} f_-(x) dx \]

\[ - \int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \Lambda^{2\alpha} f_-(x) dx - \int_{\mathbb{R}^n} f_-(x)^{p-2} f_-(x) \Lambda^{2\alpha} f_+(x) dx < +\infty \]

We only need to treat the two last integrals, and in fact we just need to study one of them since the other can be treated in a similar way. So, for the third integral we have

\[
\int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \Lambda^{2\alpha} f_-(x) dx = \int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \int_{\mathbb{R}^n} \frac{f_-(x) - f_-(y)}{|x - y|^{n+2\alpha}} dy dx \]

\[ = \int_{\mathbb{R}^n} f_+(x)^{p-2} \int_{\mathbb{R}^n} \frac{f_+(x) - f_-(y)}{|x - y|^{n+2\alpha}} dy dx \]

However, since \( f_+ \) and \( f_- \) have disjoint supports we obtain the following estimate:

\[
\int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \Lambda^{2\alpha} f_-(x) dx = - \int_{\mathbb{R}^n} f_+(x)^{p-2} \int_{\mathbb{R}^n} \frac{f_+(x) - f_-(y)}{|x - y|^{n+2\alpha}} dy dx \leq 0
\]

This quantity is negative as all the terms inside the integral are positive. With this observation we see that the last terms of (62) are positive and we have

\[
\int_{\mathbb{R}^n} f_+(x)^{p-2} f_+(x) \Lambda^{2\alpha} f_+(x) dx + \int_{\mathbb{R}^n} f_-(x)^{p-2} f_-(x) \Lambda^{2\alpha} f_-(x) dx \leq \int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) \Lambda^{2\alpha} f(x) dx < +\infty
\]

Then, using the first part of the proof we have \( f_\pm \in B^{\alpha/p,p}_{\infty} \) and since \( f = f_+ - f_- \) we conclude that \( f \) belongs to the Besov space \( B^{\alpha/p,p}_\infty \).

We have proved the following general estimate

\[
\|f\|_{B^{\alpha/p,p}_\infty}^p \leq C \int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) \Lambda^{2\alpha} f(x) dx
\]

\[ \Box \]

**Remark 4.5** From this inequality one easily deduces positivity of this last integral. This constitutes another proof for the positivity lemma of [7] for \( 2 \leq p < +\infty \). Another proof, far more general of theorem 10 is given in [13].

To obtain weak solutions of (51) with initial data in \( L^p \), \( p \geq 2 \), we will now pass to the limit by taking \( \varepsilon \rightarrow 0 \).

We have obtained a family of regular functions \( (\theta^{(\varepsilon)})_{\varepsilon > 0} \in L^\infty([0,T]; L^p(\mathbb{R}^n)) \) which are solutions of (52) and satisfy the uniform bound

\[
\|\theta^{(\varepsilon)}(\cdot, t)\|_{L^p} \leq \|\theta_0\|_{L^p}
\]

Since \( L^\infty([0,T]; L^p(\mathbb{R}^n)) = (L^1([0,T]; L^q(\mathbb{R}^n)))' \), with \( \frac{1}{p} + \frac{1}{q} = 1 \), we can extract from those solutions \( \theta^{(\varepsilon)} \) a subsequence \( (\theta_k)_{k \in \mathbb{N}} \) which is \( \ast \)-weakly convergent to some function \( \theta \) in the space \( L^\infty([0,T]; L^p(\mathbb{R}^n)) \), which implies convergence in \( D'(\mathbb{R}^+ \times \mathbb{R}^n) \). However, this weak convergence is not sufficient to assure the convergence of \( (v_\varepsilon \theta_k) \) to \( v \theta \). For this we use the remarks that follows.

First, using remark 4.1 we can consider a sequence \( (v_k)_{k \in \mathbb{N}} \) with \( v_k \) as in formula (12) such that \( v_k \rightharpoonup v \) weakly in \( bmo \). Secondly, combining (58) and Theorem 10 we obtain that solutions \( \theta_k \) belongs to the space \( L^\infty([0,T]; L^p(\mathbb{R}^n)) \cap L^1([0,T]; B^{\alpha/p,p}_\infty(\mathbb{R}^n)) \) for all \( k \in \mathbb{N} \).

To finish, fix a function \( \varphi \in C_0^\infty([0,T] \times \mathbb{R}^n) \). Then we have the fact that \( \varphi \theta_k \in L^1([0,T]; B^{\alpha/p,p}_\infty(\mathbb{R}^n)) \) and \( \partial_t \varphi \theta_k \in L^1([0,T]; B^{\alpha/p,p}_\infty(\mathbb{R}^n)) \). This implies the local inclusion, in space as well as in time, \( \varphi \theta_k \in \bar{W}^{\alpha/p,p}_{t,x} \subset \bar{W}^{\alpha/p,2}_{t,x} \) so we can apply classical results such as the Rellich's theorem to obtain convergence of \( v_k \theta_k \) to \( v \theta \).

Thus, we obtain existence and unicity of weak solutions for the problem (51) with an initial data in \( \theta_0 \in L^p(\mathbb{R}^n) \), \( 2 \leq p < +\infty \). Moreover, since such solutions satisfy inequality (58) we have that these solutions \( \theta(x,t) \) belongs to the space \( L^\infty([0,T]; L^p(\mathbb{R}^n)) \cap L^p([0,T]; B^{\alpha/p,p}_\infty(\mathbb{R}^n)) \).
5 Positivity principle

We prove in this section theorem 5. Recall that by hypothesis we have $0 \leq \psi_0 \leq M$ and $\psi_0 \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq +\infty$. To begin with, we fix two constants, $\rho, R$ such that $R > 2\rho > 0$. Then we set $\alpha_{0,R}(x)$ a function equals to $M/2$ over $|x| \leq 2R$ and equals to $\psi_0(x)$ over $|x| > 2R$ and we write $\beta_{0,R}(x) = \psi_0(x) - \alpha_{0,R}(x)$, so by construction we have

$$
\psi_0(x) = \alpha_{0,R}(x) + \beta_{0,R}(x)
$$

with $\|\alpha_{0,R}\|_{L^\infty} \leq M$ and $\|\beta_{0,R}\|_{L^\infty} \leq M/2$. Remark that $\alpha_{0,R}, \beta_{0,R} \in L^p(\mathbb{R}^n)$.

Now fix $v \in L^\infty([0,T];bmo(\mathbb{R}^n))$ such that $\text{div}(v) = 0$ and consider the equations

$$
\begin{align*}
\partial_t \alpha_R(x,t) + \nabla \cdot (v \alpha_R)(x,t) + \Lambda^{2\alpha} \alpha_R(x,t) &= 0 \\
\partial_t \beta_R(x,t) + \nabla \cdot (v \beta_R)(x,t) + \Lambda^{2\alpha} \beta_R(x,t) &= 0
\end{align*}
$$

(63)

with $\alpha \in ]0,1/2].$

Using the maximum principle and by construction we have the following estimates for $t \in [0,T]$:

$$
\begin{align*}
\|\alpha_R(\cdot,t)\|_{L^p} &\leq \|\alpha_{0,R}\|_{L^p} + CM R^{n/p} (1 < p < +\infty) \\
\|\alpha_R(\cdot,t)\|_{L^\infty} &\leq \|\alpha_{0,R}\|_{L^\infty} \leq M. \\
\|\beta_R(\cdot,t)\|_{L^\infty} &\leq \|\beta_{0,R}\|_{L^\infty} \leq M/2.
\end{align*}
$$

(64)

Lemma 5.1 The function $\psi(x,t) = \alpha_R(x,t) + \beta_R(x,t)$, where $\alpha_R(x,t)$ and $\beta_R(x,t)$ are solutions of the systems (63), is the unique solution for the problem

$$
\begin{align*}
\partial_t \psi(x,t) + \nabla \cdot (v \psi)(x,t) + \Lambda^{2\alpha} \psi(x,t) &= 0 \\
\psi(x,0) = \alpha_{0,R}(x) + \beta_{0,R}(x).
\end{align*}
$$

(65)

Proof. Using hypothesis over $\alpha_R(x,t)$ and $\beta_R(x,t)$ and the linearity of equation (65) we have that the function $\psi_R(x,t) = \alpha_R(x,t) + \beta_R(x,t)$ is a solution for this equation. Unicity is assured by the maximum principle and by the continuous dependence from initial data given corollary 4.1, thus we can write $\psi(x,t)$.

To continue, we will need an auxiliary function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x) = 0$ for $|x| \geq 1$ and $\phi(x) = 1$ if $|x| \leq 1/2$ and we set $\varphi(x) = \phi(x/R)$.

Now, we will estimate the $L^p$-norm of $\varphi(x)(\alpha_R(x,t) - M/2)$ with $p > n/2\alpha$.

Remark 5.1 Although some of the following calculations are valid for $1 \leq p \leq +\infty$, we will need at the end the fact that $p > n/2\alpha$.

We write:

$$
\begin{align*}
\partial_t \|\varphi(\cdot)(&\alpha_R(\cdot,t) - M/2)\|_{L^p}^p = p \int_{\mathbb{R}^n} |\varphi(\cdot)(\alpha_R(\cdot,t) - M/2)|^{p-2} (\varphi(\cdot)(\alpha_R(\cdot,t) - M/2)) \\
&\times \partial_t (\varphi(\cdot)(\alpha_R(\cdot,t) - M/2)) dx
\end{align*}
$$

(66)

We observe that we have the next identity for the last term above

$$
\begin{align*}
\partial_t (\varphi(\cdot))(\alpha_R(\cdot,t) - M/2) &= -\nabla \cdot (\varphi(\cdot) \psi(\cdot) - M/2) - \Lambda^{2\alpha} (\varphi(\cdot) \alpha_R(\cdot,t) - M/2) \\
&+ (\alpha_R(\cdot,t) - M/2) v \cdot \nabla \varphi(x) \\
&+ [\Lambda^{2\alpha}, \varphi] \alpha_R(\cdot,t) - M/2 \Lambda^{2\alpha} \varphi(x)
\end{align*}
$$

(67)

We observe that we have the next identity for the last term above

$$
\begin{align*}
\partial_t (\varphi(\cdot))(\alpha_R(\cdot,t) - M/2) &= -\nabla \cdot (\varphi(\cdot) \psi(\cdot) - M/2) - \Lambda^{2\alpha} (\varphi(\cdot) \alpha_R(\cdot,t) - M/2) \\
&+ (\alpha_R(\cdot,t) - M/2) v \cdot \nabla \varphi(x) \\
&+ [\Lambda^{2\alpha}, \varphi] \alpha_R(\cdot,t) - M/2 \Lambda^{2\alpha} \varphi(x)
\end{align*}
$$

(67)
where we noted \([\Lambda^{2\alpha}, \varphi]\) the commutator between \(\Lambda^{2\alpha}\) and \(\varphi\). Thus, using this identity in (66) and the fact that \(\text{div}(v) = 0\) we have
\[
\partial_t \|\varphi(\cdot, t) - M/2\|_{L^p}^p = -p \int_{\mathbb{R}^n} |\varphi(x)(\alpha_R(x, t) - M/2)|^{p-2} (\varphi(x)(\alpha_R(x, t) - M/2)) \times \Lambda^{2\alpha}(\varphi(x)(\alpha_R(x, t) - M/2)) dx
\]
\[
+ p \int_{\mathbb{R}^n} |\varphi(x)(\alpha_R(x, t) - M/2)|^{p-2} (\varphi(x)(\alpha_R(x, t) - M/2)) \times ((\Lambda^{2\alpha}, \varphi)\alpha_R(x, t) - M/2\Lambda^{2\alpha}\varphi(x)) dx.
\]
Remark that integral (67) is positive so one has
\[
\partial_t \|\varphi(\cdot, t) - M/2\|_{L^p} \leq p \int_{\mathbb{R}^n} |\varphi(x)(\alpha_R(x, t) - M/2)|^{p-2} (\varphi(x)(\alpha_R(x, t) - M/2)) \times ((\Lambda^{2\alpha}, \varphi)\alpha_R(x, t) - M/2\Lambda^{2\alpha}\varphi(x)) dx.
\]
Using Hölder inequality and integrating the previous expression we have
\[
\|\varphi(\cdot)(\alpha_R(\cdot, t) - M/2)\|_{L^p} \leq \|\varphi(\cdot)(\alpha_R(\cdot, 0) - M/2)\|_{L^p}
\]
\[
+ \int_0^t \|([\Lambda^{2\alpha}, \varphi]\alpha_R(\cdot, s))\|_{L^p} + \|M/2\Lambda^{2\alpha}\varphi\|_{L^p} ds.
\]
The first term of the right side is null since over the support of \(\varphi\) we have identity \(\alpha_R(x, 0) = M/2\). For the second term \([\|\Lambda^{2\alpha}, \varphi]\alpha_R(\cdot, s)\|_{L^p}\) we have the two following cases:

- If \(\alpha = 1/2\), we have the estimate below given by Calderón’s commutator (see [9]) and by the maximum principle

\[
\|\Lambda^{2\alpha}, \varphi\|_{\alpha_R(\cdot, s)} \leq CR^{-1}\|\alpha_R(\cdot, s)\|_{L^p} \leq CR^{-1}\|\alpha_0, R\|_{L^p}.
\]

- For \(0 < \alpha < 1/2\) we will need the next lemma

**Lemma 5.2** For \(1 \leq p \leq +\infty\) we have the estimation

\[
\|\Lambda^{2\alpha}, \varphi\|_{\alpha_R(\cdot, s)} \leq CR^{-2\alpha}\|\alpha_0, R\|_{L^p} \quad (0 < \alpha < 1/2).
\]

**Proof.** We begin with the case \(p = +\infty\) and we write
\[
[\Lambda^{2\alpha}, \varphi]\alpha_R(y, s) = \int_{\{|x-y| > R\}} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+2\alpha}} \alpha_R(y, s) dy + \int_{\{|x-y| \leq R\}} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+2\alpha}} \alpha_R(y, s) dy.
\]

Thus, using the maximum principle, we can write
\[
\|([\Lambda^{2\alpha}, \varphi]\alpha_R(\cdot, s))\|_{L^\infty} \leq CR^{-2\alpha}\|\alpha_R(\cdot, s)\|_{L^\infty} \leq CR^{-2\alpha}\|\alpha_0, R\|_{L^\infty}.
\]

For the case \(p = 1\) we have:
\[
\int_{\mathbb{R}^n} |([\Lambda^{2\alpha}, \varphi]\alpha_R(y, s))| dy \leq C\|\varphi\|_{L^\infty} \int_{\{|x-y| > R\}} \frac{|\alpha_R(y, s)|}{|x-y|^{n+2\alpha}} dy + R^{-1} \int_{\mathbb{R}^n} \int_{\{|x-y| \leq R\}} \frac{|\alpha_R(y, s)|}{|x-y|^{n+2\alpha-1}} dy dx.
\]
Using again the maximum principle we obtain
\[ \|\Lambda^{2\alpha} \varphi \alpha R(\cdot, s)\|_{L^1} \leq CR^{-2\alpha} \|\alpha R(\cdot, s)\|_{L^1} \leq CR^{-2\alpha} \|\alpha_0, R\|_{L^1}. \]

Finally, the case \( 1 < p < +\infty \) is given by interpolation.

Now, getting back to the last term of (68) we have by definition of \( \varphi \) the estimate \( \|M/2\Lambda^{2\alpha} \varphi\|_{L^p} \leq CMR^{n/p - 2\alpha} \). We thus have
\[ \|\varphi(\cdot)(\alpha R(\cdot, t) - M/2)\|_{L^p} \leq CR^{-2\alpha} \int_0^t \left( \|\alpha_0, R\|_{L^p} + MR^{n/p} \right) ds. \]

Observe that we have at our disposal estimate (64), so we can write
\[ \|\varphi(\cdot)(\alpha R(\cdot, t) - M/2)\|_{L^p} \leq CR^{-2\alpha} \left( \|\psi_0\|_{L^p} + MR^{n/p} \right) \]
Using again the definition of \( \varphi \) one has
\[ \left( \int_{B(0, \rho)} |\alpha R(\cdot, t) - M/2|^p dx \right)^{1/p} \leq CR^{-2\alpha} \left( \|\psi_0\|_{L^p} + MR^{n/p} \right). \]
Thus, if \( R \to +\infty \) and since \( p > n/2\alpha \), we have \( \alpha(x, t) = M/2 \) over \( B(0, \rho) \).

Hence, by construction we have \( \psi(x, t) = \alpha_R(x, t) + \beta_R(x, t) \) where \( \psi \) is a solution of \( (T)_\alpha \) with initial data \( \psi_0 = \alpha_0, R + \beta_0, R \), but, since over \( B(0, \rho) \) we have \( \alpha(x, t) = M/2 \) and \( \|\beta(\cdot, t)\|_{L^\infty} \leq M/2 \), one finally has the desired estimate \( 0 \leq \psi(x, t) \leq M \).

6 Existence of solutions with a \( L^\infty \) initial data

The proof given before for the positivity principle allows us to obtain the existence of solutions for fractional diffusion transport equation (1) with \( \alpha \in [0, 1/2] \) when the initial data \( \theta_0 \) belongs to the space \( L^\infty(\mathbb{R}^n) \). Indeed, let us fix \( \theta_0^R = \theta_0 1_{B(0, R)} \) with \( R > 0 \) so we have \( \theta_0^R \in L^p(\mathbb{R}^n) \) for all \( 1 \leq p \leq +\infty \). Following section 4, there is a unique solution \( \theta^R \) for the problem
\[
\begin{align*}
\partial_t \theta^R + \nabla \cdot (\nu \theta^R) + \Lambda^{2\alpha} \theta^R &= 0 \\
\theta^R(x, 0) &= \theta_0^R(x) \\
div(v) &= 0 \quad \text{and} \quad v \in L^\infty([0, T]; \text{bmo}(\mathbb{R}^n)).
\end{align*}
\]

such that \( \theta^R \in L^\infty([0, T]; L^p(\mathbb{R}^n)) \) with \( p > n/2\alpha \) and by the maximum principle we have:
\[ \|\theta^R(\cdot, t)\|_{L^\infty} \leq \|\theta_0^R\|_{L^\infty} = \|\theta_0\|_{L^\infty}. \]

Now define \( \varphi(x) = \phi(x/2R) \) with \( \phi \in C_0^\infty(\mathbb{R}^n) \) such that \( \phi(x) = 0 \) for \( |x| \geq 1 \) and \( \phi(x) = 1 \) for \( |x| \leq 1/2 \). Then, following the same ideas used to obtain formula (66) we have
\[ \|\varphi \theta^R(\cdot, t)\|_{L^p} \leq \|\varphi \theta_0^R\|_{L^p} + C \int_0^t R^{-2\alpha} \|\theta^R(\cdot, s)\|_{L^p} \|\varphi \theta^R(\cdot, s)\|_{L^p} ds. \]

Now, using the definition of \( \varphi \) and its support properties one has
\[ \|\theta^R(\cdot, t)\|_{L^p} \leq R^{n/p} \|\theta_0^R\|_{L^p} + CR^{n/p - 2\alpha} \|\theta_0\|_{L^p}^2. \]

Taking the limit \( p \to +\infty \) and making \( R \to +\infty \) we finally have
\[ \|\theta(\cdot, t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}. \]

This shows that for an initial data \( \theta_0 \in L^\infty(\mathbb{R}^n) \) there exists an associated solution \( \theta \in L^\infty([0, T]; L^\infty(\mathbb{R}^n)) \).
7 Application to the 2D-quasi-geostrophic equation

We have worked so far with a velocity given by a general function $v \in L^\infty([0,T];bmo(\mathbb{R}^n))$, let us now treat super-critical case of the 2D-quasi-geostrophic equation with $u = (-R_2 \theta, R_1 \theta)$; where $R_j$ are the Riesz transforms.

Fix $\theta_0$ an initial data belonging to $L^p \cap L^\infty(\mathbb{R}^2)$, with $p \geq 2$. Following [15] we have the existence of a solution $\theta(\cdot, t)$ for the equation $(QG)_\alpha (\alpha \in [0, 1/2])$ with $\theta(\cdot, t) \in L^p \cap L^\infty(\mathbb{R}^2)$ for $t \in [0, T]$.

Since Riesz transforms are bounded in $L^p$ and since they are bounded from $L^\infty$ into $BMO$, we have a uniform bound of the velocity $u$ in terms of the $bmo$ norm: we can apply theorem 1 to obtain H"older regularity for the solution of 2D-quasi-geostrophic equation.

Acknowledgments. I would like to thank Pierre Gilles Lemarié-Rieusset for scientific advice and motivating work environment.

References

[1] H. Abidi & T. Hmidi. On the global well-posedness of the critical quasi-geostrophic equation. arXiv. http://arxiv.org/pdf/0702215 (2007).

[2] L. Caffarelli & A. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. arXiv. http://arxiv.org/pdf/math/0608447v1 (2006).

[3] R. Coifmann & G. Weiss. Extensions of Hardy spaces and their use in analysis, Bull Amer. Math. Soc., Vol 83, N° 4, (1977).

[4] P. Constantin & J. Wu. Behavior of solutions of 2D quasi-geostrophic equations. SIAM J. Math. Anal. 30, 937-948 (1999).

[5] P. Constantin & J. Wu. Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation. Annales de l’Institut Henri Poincaré. Analyse non linéaire. Vol 25, N°6, 1103-1110 (2008).

[6] P. Constantin, D. Cordoba & J. Wu. On the critical dissipative quasi-geostrophic equation. arXiv. http://arxiv.org/pdf/math.AP/0103040 (2001).

[7] A. Cordoba & D. Cordoba. A maximum principle applied to quasi-geostrophic equations, Commun. Math. Phys. 249, 511-528 (2004).

[8] D. Goldberg. A local version of real Hardy spaces. Duke Mathematical Journal, Vol 46, N°1, (1979).

[9] L. Grafakos. Classical and Modern Fourier Analysis. Prentice Hall (2004).

[10] A. Kiselev & F. Nazarov. A variation on a theme of Caffarelli and Vasseur. arXiv. http://arxiv.org/pdf/0908.0923 (2009).

[11] A. Kiselev, F. Nazarov & Volberg. Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. Invent. math. 167, 445–453 (2007).

[12] P. G. Lemarié-Rieusset. Recent developments in the Navier-Stokes problem. Chapman & Hall/CRC (2002).

[13] P. G. Lemarié-Rieusset & D. Chamorro. Quasi-geostrophic equation, nonlinear Bernstein inequalities and α-stable processes. Preprint Université d’Evry (2010).

[14] F. Marchand. Propagation of Sobolev regularity for the critical dissipative quasi-geostrophic equation. Asymptotic Analysis, Vol. 49, N°3-4, 275-293, (2006).

[15] F. Marchand. Existence and regularity of weak solutions to the quasi-geostrophic equations in the spaces $L^p$ or $H^{-1/2}$, Commun. Math. Phys. 277, 45-67 (2008).

[16] S. Resnick. Dynamical problems in nonlinear advective partial differential equations. Ph.D. Thesis, University of Chicago (1995).

[17] E. M. Stein. Harmonic Analysis. Princeton University Press (1993).

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