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A regularity criterion for the 3D MHD equations in terms of the gradient of the pressure in the multiplier spaces

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Abstract In this paper, we consider the regularity criterion for the 3D MHD equations and prove that if the gradient of the pressure belongs to $L^{2r}(0, T; \dot{X}_r(\mathbb{R}^3))$ with $0 \leq r \leq 1$, then the solution is smooth. Notice that we extend the result given by Gala (Appl Anal 92:96–103, 2013).

1 Introduction
In this paper, we consider the following three-dimensional (3D) magnetohydrodynamic (MHD) equations:

$$\begin{align*}
    u_t + (u \cdot \nabla)u - (b \cdot \nabla)b - \Delta u + \nabla \pi &= 0, \\
    b_t + (u \cdot \nabla)b - (b \cdot \nabla)u - \Delta b &= 0, \\
    \nabla \cdot u &= 0, \\
    \nabla \cdot b &= 0, \\
    u(0) = u_0, \quad b(0) = b_0,
\end{align*}$$

where $u = (u_1, u_2, u_3)$ is the fluid velocity field, $b = (b_1, b_2, b_3)$ is the magnetic field, $\pi$ is a scalar pressure, and $(u_0, b_0)$ are the prescribed initial data satisfying $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in the distributional sense. Physically, (1) governs the dynamics of the velocity and magnetic fields in electrically conducting fluids, such as plasmas,
liquid metals, and salt water. Moreover, (1)\(_1\) reflects the conservation of momentum, (1)\(_2\) is the induction equation, and (1)\(_3\) specifies the conservation of mass.

Besides its physical applications, the MHD system (1) is also mathematically significant. Duvaut and Lions [3] constructed a global weak solution to (1) for initial data with finite energy. However, the issue of regularity and uniqueness of such a given weak solution remains a challenging open problem. Many sufficient conditions (see e.g., [1,2,4–7,9,11–16,18,20–28] and the references therein) were derived to guarantee the regularity of the weak solution.

We are interested in the regularity condition in terms of the pressure, the pressure gradient or its partial components. Let us now list some finest results up to date.

- In [6], the author improved [25], and established the fundamental Serrin-type regularity criterion in terms of the pressure,
  \[
  \pi \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \leq \infty;
  \]
  or the pressure gradient,
  \[
  \nabla \pi \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 3, \quad 1 < q \leq \infty;
  \]
  that is, if one of the above two conditions holds on (0, T) with 0 < T < \infty, then the solution is smooth on (0, T).

- Jia and Zhou [13] used intricate decomposition technique and delicate inequalities to obtain the following regularity criterion:
  \[
  \partial_3 \pi \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad 3 \leq q < \infty. \tag{2}
  \]

Applying a more subtle decomposition technique (see [23, Remark 3]), Zhang et al. [23] consider the range \(3/2 \leq q \leq 3\).

In this paper, we would like to make a further contribution in this direction. We shall extend the smoothness condition

\[
\nabla \pi \in L^{2r}(0, T; \dot{X}_r), \quad 0 \leq r \leq 1
\]

for the Navier–Stokes equations (see [8]) to the MHD equations (1), where \(\dot{X}_r\) is the multiplier spaces (see Sect. 2 below). However, due to strong coupling of the velocity field with the magnetic field, we could only be able to prove the following regularity condition for (1),

\[
\nabla \pi \in L^{2r}(0, T; \dot{X}_r), \quad 0 \leq r \leq 1.
\]

Before stating the main result, let us recall the weak formulation of the MHD equations (1).

**Definition 1.1** Let \((u_0, b_0) \in L^2(\mathbb{R}^3)\) with \(\nabla \cdot u_0 = \nabla \cdot b_0 = 0\), and \(T > 0\). A measurable \(\mathbb{R}^3\)-valued pair \((u, b)\) is called a weak solution to (1) with initial data \((u_0, b_0)\), provided that the following three conditions hold:

1. \(u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)), \ b \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))\);
2. (1)\(_{1,2,3,4}\) are satisfied in the distributional sense;
3. the energy inequality
   \[
   \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla (u, b)\|_{L^2}^2 \, ds \leq \|(u_0, b_0)\|_{L^2}^2,
   \]
   holds for almost every \(t \geq 0\).

Now, we are ready to announce the main result of the paper.
Let \((u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)\) with \(\nabla \cdot u_0 = \nabla \cdot b_0 = 0\), and \(T > 0\). Assume that \((u, b)\) is a given weak solution pair of the MHD system (1) with initial data \((u_0, b_0)\) on \((0, T)\). If

\[
\nabla \pi \in L^{\frac{2}{r}}(0, T; \dot{X}_r), \quad 0 \leq r \leq 1,
\]

then, the solution pair \((u, b)\) is smooth on \((0, T)\).

Noticing that \(\dot{X}_0 \cong BMO\) (see Sect. 2 for the definition, and [10,19] for the equivalence relation), we have the following corollary.

**Corollary 1.3** Let \((u_0, b_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)\) with \(\nabla \cdot u_0 = \nabla \cdot b_0 = 0\), and \(T > 0\). Assume that \((u, b)\) is a given weak solution pair of the MHD system (1) with initial data \((u_0, b_0)\) on \((0, T)\). If

\[
\nabla \pi \in L^1(0, T; BMO),
\]

then the solution pair \((u, b)\) is smooth on \((0, T)\).

In the rest of the paper, we make some preliminaries in Sect. 2 and prove Theorem 1.2 in Sect. 3.

### 2 Preliminaries

In this section, we recall the definition and fine properties of the multiplier spaces \(\dot{X}_r\) (see e.g., [10,19]).

**Definition 2.1** For \(0 \leq r < 3/2\), the homogeneous space \(\dot{X}_r\) is defined as the space of \(f \in L^2_{loc}(\mathbb{R}^3)\) such that

\[
\|f\|_{\dot{X}_r} \equiv \sup_{\|g\|_{\dot{H}^{r}} \leq 1} \|fg\|_{L^2} < \infty,
\]

where \(\dot{H}^{r}(\mathbb{R}^3)\) is the space of distributions \(u\) such that

\[
\|u\|_{\dot{H}^{r}} \equiv \|(-\Delta)^{\frac{r}{2}} u\|_{L^2} < \infty.
\]

We have the following scaling properties:

\[
\|f(\cdot + x_0)\|_{\dot{X}_r} = \|f\|_{\dot{X}_r}, \quad \forall \; x_0 \in \mathbb{R}^3,
\]

\[
\|f(\lambda \cdot)\|_{\dot{X}_r} = \frac{1}{\lambda^r} \|f\|_{\dot{X}_r}, \quad \forall \; \lambda > 0.
\]

When \(r = 0\), we have

\[
\dot{X}_0 \cong BMO,
\]

where \(BMO\) is the homogeneous space of bounded mean oscillations associated with semi-norm (see [17])

\[
\|f\|_{BMO} = \sup_{\lambda > 0} \frac{1}{\lambda^3} \int \int \left| f(x) - \frac{1}{B_r(y)} \int_{B_r(y)} f(z) \, dz \right| \, dy.
\]

Furthermore, for \(0 < r < \frac{3}{2}\), we have the following strict imbeddings:

\[
L^2(\mathbb{R}^3) \subset \dot{X}_r(\mathbb{R}^3),
\]

which could be justified simply as

\[
\|fg\|_{L^2} \leq \|f\|_{L^2} \|g\|_{L^{\frac{6}{6-r}}} \quad \text{(Hölder inequality)}
\]

\[
\leq C \|f\|_{L^2} \|g\|_{\dot{H}^{r}} \quad \text{(Sobolev imbeddings)}
\]

\[
\leq C \|f\|_{L^2} \quad \forall \; g \in \dot{H}^{r}(\mathbb{R}^3) \text{ with } \|g\|_{\dot{H}^{r}} \leq 1.
\]
3 Proof of Theorem 1.2

In this section, we are ready to prove Theorem 1.2.

First, let us convert (1) into a symmetric form. Writing

\[ \omega^\pm = u \pm b, \]

we find by adding and subtracting (1)_1 with (1)_2,

\[
\begin{align*}
\omega_+^2 + (\omega_+ \cdot \nabla) \omega_+^2 - \Delta \omega_+^2 + \nabla \pi &= 0, \\
\omega_-^2 + (\omega_- \cdot \nabla) \omega_-^2 - \Delta \omega_-^2 + \nabla \pi &= 0, \\
\nabla \cdot \omega_+^2 &= \nabla \cdot \omega_-^2 = 0, \\
\omega_+^0(0) &= \omega_0^+ \equiv u_0 + b_0, \\
\omega_-^0(0) &= \omega_0^- \equiv u_0 - b_0.
\end{align*}
\]

Multiplying (5)_1 with \(|\omega_+|^2 \omega_+^2\) and (5)_2 with \(|\omega_-|^2 \omega_-\), and integrating over \(\mathbb{R}^3\), we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \omega_+^2 \|_{L^2}^2 + \frac{1}{2} \| \nabla |\omega_+|^2 \|_{L^2}^2 + \| |\omega_+| \cdot |\nabla \omega_+| \|_{L^2}^2 &= \int_{\mathbb{R}^3} \nabla \pi \cdot |\omega_+|^2 \omega_+^2 \, dx \\
&= I.
\end{align*}
\]

We may now estimate \(I\), applying Hölder inequality,

\[
I \leq \| \nabla \pi \cdot |\omega_+^2| \|_{L^2} \| \omega_+^2 \|_{L^2} \\
\leq \| \nabla \pi \|_{L^r} \| \omega_+^2 \|_{L^s} \| \omega_+^2 \|_{L^2} \\
\leq \| \nabla \pi \|_{L^r} \| \omega_+^2 \|_{L^s} \| \nabla |\omega_+|^2 \|_{L^2} \| \omega_+^2 \|_{L^2} \\
\leq C \| \nabla \pi \|_{L^r} \| \omega_+^2 \|_{L^s} \| \nabla |\omega_+|^2 \|_{L^2}^2 + \frac{1}{4} \| \nabla |\omega_+|^2 \|_{L^2}^2.
\]

Notice that the weak solution \((u, b) \in L^\infty(0, T; L^2(\mathbb{R}^3))\), then we have \(\omega_+ \in L^\infty(0, T; L^2(\mathbb{R}^3))\). Also, since \(\frac{2(1 - r)}{2 - r} \leq 2\), we may invoke Young inequality to dominate \(I\) further as

\[
I \leq C \| \nabla \pi \|_{L^r} \left[ \| \omega_+^2 \|_{L^2}^2 + 1 \right] + \frac{1}{4} \| \nabla |\omega_+|^2 \|_{L^2}^2.
\]

Substituting (7) into (6), and applying Gronwall inequality, we deduce

\[
\| \omega_+^2(t) \|_{L^4}^4 \leq \| \omega_0^+ \|_{L^4}^4 \exp \left[ C \int_0^t \| \nabla \pi(s) \|_\infty \, ds \right] < \infty.
\]

Thus,

\[
\omega_+^2 \in L^\infty(0, T; L^4(\mathbb{R}^3)) \Rightarrow u = \frac{1}{2} (\omega_+ + \omega_-) \in L^\infty(0, T; L^4(\mathbb{R}^3)).
\]

The classical Serrin-type regularity criterion, as in [11], then concludes the Proof of Theorem 1.2.

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