A (classical) Representation for a Spin-S Entity as a compound system in $\mathbb{IR}^3$ consisting of $2S$ Individual Spin-1/2 Entities.

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Abstract

We generalize the results of [7] for the coherent states of a spin-1 entity to spin-S entities with $S > 1$ and to non-coherent spin states: through the introduction of 'hidden correlations' (see [8]) we introduce a representation for a spin-S entity as a compound system consisting of $2S$ 'individual' spin-1/2 entities, each of them represented by a 'proper state', and such that we are able to consider a measurement on the spin-S entity as a measurement on each of the individual spin-1/2 entities. If the spin-S entity is in a maximal spin state, the $2S$ individual spin-1/2 entities behave as a collection of indistinguishable but separated entities. If not so, we have to introduce the same kind of hidden correlations as required for a hidden correlation representation of a compound quantum systems described by a symmetrical superposition. Moreover, by applying the Majorana representation of [11] and Aerts' representation for a spin-1/2 entity of [3], this hidden correlation representation yields a classical mechanistic representation of a spin-S entity in $\mathbb{IR}^3$.

Key words: spin-S, compound systems, hidden correlations, Majorana-representation.

1 Introduction.

In this paper, we represent a spin-S entity as a compound system consisting of $2S$ individual spin-1/2 entities between which there exist the kind of correlations that we have introduced in [8], called hidden correlations. Such a hidden correlation representation implies that a measurement on the spin-S entity can be represented as $2S$ measurements on the individual spin-1/2 entities, in the sense that a measurement on one of the individual spin-1/2 entities induces a change of the proper state of the remaining not yet measured individual spin-1/2 entities. A more general conceptual framework in which we can consider these hidden correlation representations has already been presented in [8]. However, without having to repeat much of this framework we have made this paper formally self-consistent. The concepts of individual entities and proper states introduced in [8] will be used in this paper (without any further explanation) in order to distinguish between the proper states of the individual entities in the compound system, i.e., the individual spin-1/2 entities, and the state of the entity that corresponds with the compound system itself, i.e., the spin-S entity.

As mentioned in the abstract, we generalize the results of [7] for the coherent states of a spin-1 entity to spin-S entities with $S > 1$ and to non-coherent spin states. In fact, because of its simple geometry, the spin-1 representation in [7] illustrates in a very simple way the true nature of the representation introduced

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1Since in this paper the focus is on purely formal results of which some do also stand without a preferred approach to quantum measurements, we will avoid as much as possible to refer to any interpretative (and thus speculative) framework, in contrary to [8]. However, it is indeed possible to embed the results of this paper within the conceptual framework of [8] in a straightforward way. Necessary remarks on this embedment will be included in this paper as footnotes.

2However, there are more profound reasons besides clarity to distinguish between states and proper states, and entities and individual entities. For a detailed discussion on this matter we refer to [8]. If we consider the results that we will obtain in this paper relative to the more general framework introduced in [8], it follows that from a purely formal point of view the introduction of the notion of proper state is not really required since for the representation introduced in this paper it is possible to represent the proper states of the individual spin-1/2 entities as the states of a spin-1/2 entity. In the language of [8] this means that the measurements on spin-S entities do not include a hard act of creation.
in this paper. Although the representation is also valid for non-coherent spin states, we first deduce all results for the specific case of coherent spin states. There are two main reasons for this. At first, it makes all proofs and derivations extremely transparent and natural, with the state space as the primal mathematical object. Secondly, there are serious reasons to think that the only real physical spin states are the coherent ones, which makes this partial representation for the coherent spin states the essential one (see for example [1]).

The existence of the representation introduced in this paper can be brought back to the existence of the following three sub-representations:

1. It is possible to represent the states of the spin-S entity as 2S spin-1/2 states in a one to one way by applying Majorana’s representation of [11].
2. It is possible to represent the transition probabilities of spin-S entities in \((\otimes \mathbb{C}^2)^{2S} \cong \mathbb{C}^{4S}\), i.e., in the tensor product in which compound systems of 2S individual spin-1/2 entities are described. This is proved in section 2.2 of this paper.
3. Every measurement on a compound system described in the tensor product of finite number of Hilbert spaces can be represented as a collection of measurements on the individual entities in this compound system, each of them represented by a proper state, and on which we introduce hidden correlations (this has been proved in [8], where we have made an explicit construction of a hidden correlation representation for compound quantum systems described in a tensor product of Hilbert spaces). We perform this last step of the representation in section 2.3.

These three sub-representations together yield the representation of a spin-S entity as 2S individual spin-1/2 entities. Moreover, we can go one step further. If we combine our representation with Aerts’ classical mechanistic model system of [3] for a spin-1/2 entity in the three dimensional space of reals we obtain a classical mechanistic model system in the three dimensional space of reals for all spin-S entities (this is discussed in section 4 of this paper).

As already mentioned above, one of the main ingredients in this paper is the Majorana representation. In his famous paper of 1932, Majorana showed that the study of the angular momentum \(J\) of a quantum entity in a varying magnetic field can be reduced to the study of 2\(J\) angular momenta with value 1/2, i.e., we only have to consider 2\(J\) representative points on a sphere. Although this particular representation proved its advantage in experimental applications, to our knowledge, it has never been used in issues that deal with the fundamental interpretation of the mathematical structures encountered in the quantum framework, and in particular in quantum measurement theory. We also remark that all the to the present author’s knowledge known proofs of the Majorana theorem deduce the existence of a representation as spin-1/2 momentum operators from the algebraic properties of the momentum operators, and thus, one never uses a representation of the spin-S states as spin-1/2 states through an explicit canonical embedding. A reason for this is probably that most authors even seem to refuse to attach a definite interpretative meaning to the representation (see for example 6, 7th line of the third paragraph). Therefore we think that the specific proof of the Majorana theorem in this paper contributes to the understanding and transparency of the Majorana theorem itself.

2 A representation for coherent spin states.

As we mentioned in the introduction, we proceed in three steps: in the first subsection we present the Majorana representation; in the next one we construct an explicit procedure to relate the transition probabilities of the spin-S states to the proper states of 2S individual spin-1/2 entities in a compound system; finally, due to the specific nature of this procedure, we are able to introduce deterministic correlations between the individual spin-1/2 entities in order to consider a measurement on a spin-S entity as 2S measurements on the individual spin-1/2 entities in the compound system, where the initial proper states of the individual entities is given by the Majorana representation.

2.1 Decomposition of the spin-S state space into spin-1/2 states.

First we study the transition probabilities of measurements on spin-1/2 entities, and then we do the same for spin-S entities. This will enable us to introduce the Majorana representation in an explicit way for...
coherent spin-S states. In fact, we consider a transition probability as it is defined in \[2\] or \[13\], i.e., if \(\Sigma\) is the state space of a physical entity, then the transition probability is a map \(P: \Sigma \times \Sigma \to [0,1]\) which is such that for a fixed state \(p\), \(P(p,q)\) is the probability to obtain as the outcome of a measurement the one that corresponds with the state \(q\). In the case of quantum entities the transition probability is for every measurement given by the square of the Hilbert space in-product of the unit vectors which represent the states. The states which represent the possible outcomes of a measurement will be called outcome states.

### 2.1.1 The transition probability for spin-1/2 states.

In this section, we represent the states of a spin-1/2 entity on the Poincaré sphere \(S\). First we need to calculate the transition probability for spin-1/2 entity. The transition probability depends only on the relative position of the Stern-Gerlach apparatus in which we prepare the entity and the second Stern-Gerlach with which we measure the spin. We represent such a measurement by the Euler angles \(\alpha, \beta, \gamma\), and denote it as \(e_{\alpha,\beta,\gamma}\). If the initial state corresponds with a spin quantum number \(m = \frac{1}{2}\) we denote it as \(p_+^0\), and if it corresponds with a spin quantum number \(m = -\frac{1}{2}\) we denote it as \(p_-^0\). We represent \(p_+^0\) by the vector \(\psi_+^0 = (1,0) \in \mathbb{C}^2\) and \(p_-^0\) by \(\psi_-^0 = (0,1) \in \mathbb{C}^2\). The eigenvectors corresponding to a measurement \(e_{\alpha,\beta,\gamma}\) are the same as the ones we obtain when we rotate the initial states by an active rotation characterized by the Euler angles \(\alpha, \beta, \gamma\). This active rotation is represented by the unitary operator acting on \(\mathbb{C}^2\) that corresponds with the following matrix (the derivation of this matrix can be found in \[14\]):

\[
M_{\alpha,\beta,\gamma} = \begin{pmatrix}
\cos\frac{\beta}{2} e^{i\frac{\alpha}{2}} & \sin\frac{\beta}{2} e^{-i\frac{\gamma}{2}} \\
\sin\frac{\beta}{2} e^{i\frac{\gamma}{2}} & \cos\frac{\beta}{2} e^{-i\frac{\alpha}{2}}
\end{pmatrix}
\]

Thus, for the measurement \(e_{\alpha,\beta,\gamma}\) we have a set of outcome states represented by the following eigenvectors:

\[
\psi_{+}^{\alpha,\beta,\gamma} = M_{\alpha,\beta,\gamma}\psi_{+}^0 = (e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} e^{i\frac{\gamma}{2}} \sin\frac{\beta}{2}) e^{i\frac{\alpha}{2}}
\]

\[
\psi_{-}^{\alpha,\beta,\gamma} = M_{\alpha,\beta,\gamma}\psi_{-}^0 = (e^{-i\frac{\alpha}{2}} \sin\frac{\beta}{2} e^{i\frac{\gamma}{2}} \cos\frac{\beta}{2}) e^{i\frac{\alpha}{2}}
\]

The vectors in eq.\[2\] and eq.\[3\] that correspond to different values of \(\gamma\) (for fixed \(\alpha\) and \(\beta\)) represent the same states. As a consequence, we omit the superscript \(\gamma\) in the notations for the vectors and the measurements. We represent the states corresponding to the vectors in eq.\[2\] and eq.\[3\] respectively by \(p_{+}^{\alpha,\beta}\) and \(p_{-}^{\alpha,\beta}\). Thus we have \(p_{+}^{0,\alpha,\beta} = p_{+}^{0}\) and \(p_{-}^{0,\alpha,\beta} = p_{-}^{0}\). We also have:

\[
p_{+}^{\alpha,\beta} = p_{+}^{\alpha,\pi,\pi,\beta}
\]

We denote the transition probability to obtain a state \(p_{+}^{\alpha,\beta}\) in a measurement \(e_{\alpha,\beta}\) on an entity in a state \(p_{+}^0\) as \(P_{+,+}^{\alpha,\beta}\), and the probability to obtain \(p_{-}^{\alpha,\beta}\) in a measurement \(e_{\alpha,\beta}\) on an entity in a state \(p_{-}^0\) as \(P_{+,+}^{\alpha,\beta}\). Analogously we define \(P_{-,+}^{\alpha,\beta}\) and \(P_{-,+}^{\alpha,\beta}\). We have:

\[
P_{+,+}^{\alpha,\beta} = |\langle\psi_{+}^{0}|\psi_{+}^{\alpha,\beta}\rangle|^2 = \cos^2\frac{\beta}{2} = \frac{1 + \cos\beta}{2}
\]

\[
P_{-,+}^{\alpha,\beta} = \frac{1 + \cos\beta}{2}
\]

\[
P_{+,+}^{\alpha,\beta} = \sin^2\frac{\beta}{2} = \frac{1 - \cos\beta}{2}
\]

Following eq.\[4\] we find that the set of states of a spin-1/2 entity is given by:

\[
\Sigma_2 = \{p_{+}^{\alpha,\beta}|\alpha \in [0, 2\pi], \beta \in [0, \pi]\}
\]
2.1.2 Representation of the spin-1/2 states on $S$.

Let $S$ be a unit sphere in $\mathbb{R}^3$ with its center in the origin. We represent every state $p_{\alpha,\beta}^S \in \Sigma_{\frac{1}{2}}$ by the point in $S$ with coordinates $(\cos \beta, \sin \beta, \cos \beta)$. It is clear (as a consequence of the definition of the Euler angles), that the representation of $\Sigma_{\frac{1}{2}}$ in $S$ is one to one and onto. The state $p_{\alpha,\beta}^+\alpha,\beta,\gamma$ corresponds with the point on the sphere in the direction corresponding with the Euler angles $\alpha, \beta, \gamma$. This active rotation is represented by a unitary operator acting on $\mathbb{C}^{2S+1}$. We proceed along the same lines as in the previous section. We denote a measurement characterized by the Euler angles $\alpha, \beta, \gamma$ as $\psi_{\alpha,\beta,\gamma}$. If the initial state of the entity corresponds with a magnetic spin quantum number $M$, we represent $\psi_{\alpha,\beta,\gamma}^0 = (0, M, i, j)$, where $i$ and $j$ take values in $\{-S, -S + 1, \ldots, S - 1, S\}$ (this indexation simplifies the expression and will be used throughout the paper). The eigenvectors corresponding to a measurement $\psi_{\alpha,\beta,\gamma}$ are again the same as the ones we obtain when we rotate the initial states by an active rotation characterized by the Euler angles $\alpha, \beta, \gamma$. This active rotation is represented by a unitary operator acting on $\mathbb{C}^{2S+1}$ that corresponds with the following matrix (the explicit expression is again derived in [13]):

$$M_{\alpha,\beta,\gamma}^S = e^{-i\alpha}e^{-i\beta}C_{i,j}^S \sum_k (-1)^k (\cos \frac{\theta}{2})^{2S+1-j-2k} (\sin \frac{\theta}{2})^{j-i+2k} (S - j - k)! (S + i - k)!(k + j - i)! k!$$

(again $i$ and $j$ take values in $\{-S, -S + 1, \ldots, S - 1, S\}$), where:

$$C_{i,j}^S = \sqrt{(S + i)!(S - i)!(S + j)! (S - j)!}$$

and where the summation goes over all $k$ such that all exponents are non-negative. This implies:

$$\max\{0, M - M'\} \leq k \leq \min\{S - M', S + M\}$$

Thus, for the measurement $\psi_{\alpha,\beta,\gamma}^0$ we have the following collection of eigenvectors that correspond with the possible outcome states $\{p_M^\alpha\beta\}$, labeled by the different possible values that can be taken by $M$ (again there is no dependence of the states on $\gamma$):

$$\forall M: \psi_{\alpha,\beta,\gamma}^M = M_{\alpha,\beta,\gamma}^S \psi_{\alpha,\beta,\gamma}^0$$

and explicitly we have:

$$\psi_{\alpha,\beta,\gamma}^M = e^{-i\alpha}C_{i,j}^S \sum_k (-1)^k (\cos \frac{\theta}{2})^{2S+1-j-2k} (\sin \frac{\theta}{2})^{j-i+2k} (S - j - k)! (S + i - k)!(k + j - i)! k! \psi_{\alpha,\beta,\gamma}^0$$

As a consequence, one easily verifies that we have the following relations between these states:

$$p_{\alpha,\beta}^M = p_{\alpha,\beta}^{M+1}$$

and in particular:

$$p_0^{\alpha,\beta} = p_0^{\alpha,\beta+1}$$

Moreover, the above written equations are the only ones that relate identical states for different values of $M$ and different directions characterized by the Euler angles. We have the following transition probability for a change of state of $p_M^0$ into $p_M'^\alpha\beta$:

$$P_{M,M'}^{\alpha,\beta} = |\langle \psi_{M'}^0 | \psi_{M}^{\alpha,\beta} \rangle|^2 = \left| C_{M,M'}^{\alpha,\beta} \sum_k (-1)^k (\cos \frac{\theta}{2})^{2S+1-j-2k} (\sin \frac{\theta}{2})^{j-i+2k} (S - j - k)! (S + i - k)!(k + j - i)! k! \right|^2$$

So again the transition probabilities depend only on the angle $\beta$. 


2.1.4 Representation of the spin-S states on $S$.

As a first step in considering the spin-$S$ entity as $2S$ individual spin-1/2 entities in a compound system we introduce a representation for its states as proper states of the individual entities in the compound system:

We will represent the state $p^\alpha_\beta_M$ of a spin-$S$ entity by the following proper states of the $2S$ individual spin-1/2 entities in the compound system:

\[
\begin{align*}
S + M & \text{ proper states } p^\alpha_\beta_M \\
S - M & \text{ proper states } p^\alpha_\beta_{-M}
\end{align*}
\] (17)

One easily verifies that, due to eq.(14) and eq.(15), this representation is one to one. Since every spin-1/2 state corresponds with a point on a sphere, we have represented the state $p^\alpha_\beta_M$ of a spin-$S$ entity by $2S$ points on sphere: $S + M$ points on the sphere in the direction corresponding with the Euler angles $\alpha, \beta$ and $S - M$ points in the opposite direction. This is the Majorana representation for the specific case of coherent spin-$S$ states. It is onto on the possible arrangements of $2S$ points on a sphere which are such that all points are in one of the two opposite locations on the sphere. As a consequence, the representation is completely determined by the angles $\alpha, \beta$ and the value of $M$.

2.2 Relating the spin-S transition probabilities to the spin-1/2 transition probabilities.

As already announced, in this section we show in which way the transition probability of spin-$S$ states can be related to the spin-1/2 transition probabilities. Let us consider the Majorana representation of a spin-$S$ state $p^0_M$. The quantum description of a compound system consisting of $2S$ individual spin-1/2 entities, $S + M$ in a proper state $\psi^0_0$ and $S - M$ in a proper state $\psi^0_0$, can be written as the following product:

\[
\psi^0_0 \otimes \ldots \otimes \psi^0_0 \otimes \ldots \otimes \psi^0_0
\]

For this expression we introduce the following reduced notation:

\[
(\otimes \psi^0_0)^{S+M} (\otimes \psi^0_0)^{S-M}
\] (18)

A way to abstract from the ordering of the proper states of the individual spin-1/2 entities in the product is symmetrization (for the moment, we don’t attach any physical significance to this symmetrization). To do this, we have to sum over all possible permutations of the proper states in the product. If we denote all distinguishable permutations\(^7\) of $2S$ elements by $\Pi$, we can write this as:

\[
\sum_{\pi \in \Pi} \pi (\otimes \psi^0_0)^{S+M} (\otimes \psi^0_0)^{S-M}
\] (19)

Due to this symmetrization, we loose the normalization:

\[
| \sum_{\pi \in \Pi} \pi (\otimes \psi^0_0)^{S+M} (\otimes \psi^0_0)^{S-M} |^2 = \sum_{\pi, \pi' \in \Pi} \pi (\otimes \psi^0_0)^{S+M} (\otimes \psi^0_0)^{S-M} \pi' (\otimes \psi^0_0)^{S+M} (\otimes \psi^0_0)^{S-M}
\]

\[
= \sum_{\pi \in \Pi} 1 = \frac{(2S)!}{(S+M)!(S-M)!}
\]

since there are $(2S)!$ permutations of $2S$ elements of which $(S+M)!(S-M)!$ are indistinguishable due to permutation of identical spin-1/2 states. We introduce the following abbreviation:

\[
N = \sqrt{\frac{(2S)!}{(S+M)!(S-M)!}}
\] (20)

\(^7\)Permutations can be indistinguishable if we permute identical elements.
Let us calculate the transition probability of the above constructed (normalized) symmetrical superposition to a product state that corresponds with $S + M'$ individual spin-1/2 entities in a proper state $p^{\alpha,\beta}_+ \text{ and } S - M'$ in a proper state $p^{\alpha,\beta}_-$. 

\[
\left|\left(\frac{1}{N} \sum_{\pi \in \Pi} \pi \left(\otimes \psi^0_+^{S+M} \otimes \psi^{-\beta}_-^{S-M} \right) \left| \otimes \psi^\alpha_+^{S+M'} \otimes \psi^\beta_-^{S-M'} \right)\right|\right|^2
\]

\[
= \left|\frac{1}{N} \sum_{\pi \in \Pi} \pi \left(\left(\otimes \psi^0_+^{S+M} \otimes \psi^{-\beta}_-^{S-M} \right) \left| \otimes \psi^\alpha_+^{S+M'} \otimes \psi^\beta_-^{S-M'} \right)\right)\right|^2
\]

\[
= \left|\frac{1}{N} \sum_{k} a_k (\psi^0_+ | \psi^\alpha_+^{S+M'} b_+ + \psi^0_+ | \psi^\beta_-^{S-M'} b_- + (\psi^0_+ | \psi^\alpha_+^{S+M'} b_+ - (\psi^0_+ | \psi^\beta_-^{S-M'} b_- \right|^2
\]

where the exponents have to fulfill the following equations:

\[
b_{++} + b_{+-} = S + M
\]

\[
b_{+-} + b_{-+} = S - M
\]

\[
b_{++} + b_{--} = S + M'
\]

\[
b_{+-} + b_{-+} = S - M'
\]

of which only three are independent (which justifies the summation over $k$). We can parameterize the solutions in the following way:

\[
b_{++} = S + M - k
\]

\[
b_{+-} = S + M' - b_{++} = M' - M + k
\]

\[
b_{+-} = S + M - b_{++} = k
\]

\[
b_{--} = S - M' - b_{-+} = S - M' - k
\]

Since all exponents should be non-negative, $k$ only takes integer values that fulfill:

\[
\max \{0, M - M'\} \leq k \leq \min \{S - M', S + M\}
\]

The constants $a_k$ are equal to the number of permutations that lead to exponents that correspond to this value of $k$, and thus they are the product of:

1. the number of possibilities to select $b_{++}$ times the factor $\otimes \psi^\alpha_+^{S+M'}$ out of the $S + M'$ ones
2. the number of possibilities to select $b_{+-}$ times the factor $\otimes \psi^\beta_-^{S-M'}$ out of the $S - M'$ ones

Thus we have:

\[
a_k = \frac{(S + M)!}{(S + M - k)! (M' - M + k)! k! (S - M' - k)!}
\]

After substituting all this we obtain:

\[
\left|\left(\frac{1}{N} \sum_{\pi \in \Pi} \pi \left(\otimes \psi^0_+^{S+M} \otimes \psi^{-\beta}_-^{S-M} \right) \left| \otimes \psi^\alpha_+^{S+M'} \otimes \psi^\beta_-^{S-M'} \right)\right|\right|^2
\]

\[
= \sum_k \sqrt{(S + M)!} \sqrt{(S - M)!} \sqrt{(S + M)'!(S - M)'!} \left(-1\right)^k \left(\cos \beta\right)^{2S+M-M'-2k} \left(-\sin \beta\right)^{2k} \frac{(2S)!}{(S + M'!)(S - M')!} \frac{(2S)!}{(S + M)! (S - M')!}
\]

This equation is equal to eq. (16) up to a constant:

\[
\left|\sqrt{(S + M)!} \sqrt{(S - M)!} \right|^2 = \frac{(2S)!}{(S + M)! (S - M')!}
\]

This are exactly the number of distinguishable orderings of the vectors in the product state:

\[
(\otimes \psi^\alpha_+^{S+M'} \otimes \psi^\beta_-^{S-M'})
i.e., all possible representations of the Majorana representation for the outcome state $p_{M_0}^{0,\beta}$ as a product state in $(\otimes \mathbb{C}^2)^{2S}$. Thus we are able to recover the spin-S transition probabilities in the tensor product space of the $2S$ Hilbert spaces related to the individual spin-1/2 entities which the compound system described in $(\otimes \mathbb{C}^2)^{2S}$ consists of:

$$
|\langle \psi_M^{0}\mid \psi_M^{\alpha,\beta}\rangle|^2 = \frac{1}{N} \sum_{\pi \in \Pi} \pi \left(\langle \otimes \psi_+^{0}\rangle^{S+M} \langle \otimes \psi_-^{0}\rangle^{S-M} \mid H_{M'}^{\alpha,\beta}\right) \left(\langle \otimes \psi_+^{0}\rangle^{S+M'} \langle \otimes \psi_-^{0,\beta}\rangle^{S-M'}\right) \right|^2
$$

or, when we denote by $H_{M'}^{\alpha,\beta}$ the subspace of $(\otimes \mathbb{C}^2)^{2S}$ spanned by the different vectors:

$$
\pi' \left(\langle \otimes \psi_+^{\alpha,\beta}\rangle^{S+M'} \langle \otimes \psi_-^{\alpha,\beta}\rangle^{S-M'}\right)
$$

that correspond with the different permutations $\pi'$:

$$
|\langle \psi_M^{0}\mid \psi_M^{\alpha,\beta}\rangle|^2 = \frac{1}{N} \sum_{\pi \in \Pi} \pi \left(\langle \otimes \psi_+^{0}\rangle^{S+M} \langle \otimes \psi_-^{0}\rangle^{S-M} \mid H_{M'}^{\alpha,\beta}\right) \left(\langle \otimes \psi_+^{0}\rangle^{S+M'} \langle \otimes \psi_-^{0,\beta}\rangle^{S-M'}\right) \right|^2
$$

(32)

where $\langle \psi\mid H_{M'}^{\alpha,\beta}\rangle$ stands for the projection of the vector $\psi$ on the subspace $H_{M'}^{\alpha,\beta}$. As a consequence we also have:

$$
\sum_M \left(\frac{1}{N} \sum_{\pi \in \Pi} \pi \left(\langle \otimes \psi_+^{0}\rangle^{S+M} \langle \otimes \psi_-^{0}\rangle^{S-M} \mid H_{M'}^{\alpha,\beta}\right) \left(\langle \otimes \psi_+^{0}\rangle^{S+M'} \langle \otimes \psi_-^{0,\beta}\rangle^{S-M'}\right)\right)^2 = 1
$$

(33)

We can summarize this as follows:

Let:

$$
(\otimes \psi_+^{0})^{S+M} (\otimes \psi_-^{0})^{S-M} (\otimes \psi_+^{\alpha,\beta})^{S+M'} (\otimes \psi_-^{\alpha,\beta})^{S-M'} \in (\otimes \mathbb{C}^2)^{2S}
$$

(34)

respectively correspond with the Majorana representation of $\psi_M^{0} \in \mathbb{C}^{2S+1}$ and of $\psi_M^{\alpha,\beta} \in \mathbb{C}^{2S+1}$. We realize the spin-S transition probabilities in $(\otimes \mathbb{C}^2)^{2S}$ if we represent $\psi_M^{0}$ by:

$$
\frac{1}{N} \sum_{\pi \in \Pi} \pi \left(\langle \otimes \psi_+^{0}\rangle^{S+M} \langle \otimes \psi_-^{0}\rangle^{S-M}\right)
$$

(35)

and $\psi_M^{\alpha,\beta}$ by the subspace spanned by:

$$
\left\{ \pi \left(\langle \otimes \psi_+^{\alpha,\beta}\rangle^{S+M'} \langle \otimes \psi_-^{\alpha,\beta}\rangle^{S-M'}\right) \mid \pi \in \Pi \right\}
$$

(36)

We also make the following important observation:

Due to the absence of an ordering for the proper states of the individual spin-1/2 entities that appear in the Majorana representation of a spin-S state, there is a one to one onto correspondence between:

1. the spin-S states $\psi_M^{0} \in \mathbb{C}^{2S+1}$ themselves
2. the symmetrized superpositions of $(\otimes \mathbb{C}^2)^{2S}$ given by eq.3
3. the subspaces $H_M^{0}$ of $(\otimes \mathbb{C}^2)^{2S}$ obtained through permutation of the order in the products $(\otimes \psi_+^{0})^{S+M} (\otimes \psi_-^{0})^{S-M}$

A straightforward consequence of this is that the direct sum $\bigoplus_M H_M^{\alpha,\beta}$ is exactly $(\otimes \mathbb{C}^2)^{2S}$ i.e., these collections of subspaces correspond in a one to one way with the orthonormal bases of $\mathbb{C}^{2S+1}$ that correspond with coherent spin-S states. This particular correspondence between the transition probability of a spin-S entity and the transition probabilities of the $2S$ individual spin-1/2 entities in a compound quantum system will enable us to represent measurements on a spin-S entity as a measurement on each of the $2S$ individual entities in a compound system by applying the construction of 3.
2.3 Decomposition of a measurement on a spin-S entity in a measurement on each of the 2$S$ individual spin-1/2 entities through the introduction of hidden correlations.

Also here we proceed in two steps, first we introduce hidden correlations on the spin-1/2 entities, and then we try to get rid of the density matrices representing the proper states which do not correspond with spin-1/2 states.

2.3.1 Introduction of hidden correlations on the spin-1/2 entities.

In section 2.2 we have shown that we are able to obtain the transition probabilities of a spin-S entity in the tensor product space $(\otimes C^2)^{2S}$, which is the quantum mechanical representation space for the description of compound systems consisting of 2$S$ individual spin-1/2 entities: we identify the initial state of a measurement on the spin-S entity with a symmetrical superposition of the states of the spin-1/2 entities in its Majorana representation and the possible outcome states with the union of all possible product states in $(\otimes C^2)^{2S}$ that correspond with different distinguishable orderings of the spin-1/2 states in their respective Majorana representation. This clearly allows us to apply the representation for tensor product states by means of hidden correlations introduced in [8]:

>We say that there exist 'hidden correlations' between individual entities in a collection $\{S_\nu\}_\nu$ if: 1. a measurement on one individual entity $S_\alpha$ induces a change of the proper state of the other individual entities 2. This change of proper state depends deterministically on the proper state transition of $S_\alpha$ 3. After this measurement, $S_\alpha$ cannot be influenced anymore by measurements on other entities.

For compound quantum systems represented by a tensor product we obtained the following results:

Every compound quantum system $S$ consisting of a finite collection of individual entities $\{S_\nu\}_\nu$, and described by $\Psi_S \in \otimes_\nu \mathcal{H}_\nu$ has a hidden correlation representation: 1. Initially, every $S_\alpha$ is in a proper state $\omega_\alpha$. 2. If $S_\lambda$ takes the state $\phi_\lambda$ due to a measurement on it after we have already performed measurements on $S_\alpha, \ldots, S_\nu$, then the proper state of every not yet measured individual entity $S_\mu$ changes to $\omega_{\mu \lambda \alpha \ldots \nu} = \phi_\lambda$. Both $\omega_\alpha$ and $\omega_{\mu \lambda \alpha \ldots \nu}$, which are explicitly defined in [8], are represented by a density matrix which in general does not correspond with a spin-1/2 state.

We have a compound system consisting of 2$S$ individual spin-1/2 entities described by eq.(35) and a measurement on this compound system of which an outcome is represented by:

$$\bigcup_{M'} \{ \pi \left( (\otimes \psi_+^{0,\beta})^{S+M'} (\otimes \psi_-^{0,\beta})^{S-M'} \right) \mid \pi \in \Pi \}$$

(37)

where we identify all eigenvectors with the same value of $M'$ as the same outcome (since they correspond with the same Majorana representation of a spin-S state). Thus, we are able to represent every measurement on the spin-S entity as a series of 2$S$ consecutive measurements on the individual entities such that every measurement on one of these entities induces a transition of the proper state of the not yet measured entities. There is one important feature of the representation introduced in [8] which, at first sight poses some problems for the application within the context of this paper. Namely, the state transitions induced by hidden correlation might change the initial proper states which correspond in a one to one way with spin-1/2 states into proper states represented by a density matrix which doesn’t. Nonetheless, as we show in the following section, this situation can always be avoided. In the case of maximal spin states, i.e., $|M| = S$, all proper states in the Majorana representation are the same and thus, the representative symmetrical superposition is a product, which is its own biorthogonal decomposition:

$$(\otimes \psi_+^0)^{2S} = \psi_+^0 \otimes (\otimes \psi_+^0)^{2S-1}$$

(38)

According to the procedure outlined in [8], we have to construct a map:

$$T : \mathbb{C}^2 \to (\otimes C^2)^{2S-1}$$

(39)

to obtain the transitions of the proper states of the individual entities when a measurement on one of them gives an outcome state $p_+^{0,\beta}$ or $p_-^{0,\beta}$:

$$T(\psi_+^{0,\beta}) = T(\psi_-^{0,\beta}) = (\otimes \psi_+^0)^{2S-1}$$

(40)

8By these spin-1/2 states we mean the so called 'pure states'.
The proper state transition of one of the not yet measured individual entities can be found through a map $R$ that relates these vectors $T(\psi^{S,\beta}_+)$ and $T(\psi^{\alpha,\beta}_+)$ with a density matrix, by considering the coefficients in their biorthogonal decomposition (see [8]). We only find density matrices corresponding with $p^\alpha_\omega$. Since these were also the initial proper states of these individual entities, they are not influenced by the measurement. Thus, the $2S$ individual entities behave as a collection of indistinguishable but separated spin-1/2 entities.

### 2.3.2 Correspondence of the density matrices representing the proper states with spin-1/2 states.

As mentioned above, due to the results of [9] one might think that one is forced to consider density matrices for the proper states of the individual entities. This is not true since we only demand a global probabilistic correspondence with subsets of possible outcomes (namely the outcome states corresponding with the vectors that span $\mathcal{H}_M^{\alpha,\beta}$ and not for every individual entity in the compound system). In other words: we are not able to distinguish the different outcome states of the individual entities in the compound system, we only can obtain some knowledge on how many are in a proper state represented by $\psi^{\alpha,\beta}_M$ and how many are in a proper state represented by $\psi^{\alpha,\beta}_S$. As a consequence we have a weaker constraint, which will allow us to decompose the density matrices into spin-1/2 states. Due to the results of [9] and [10] we know that every density matrix can easily be decomposed in $2S$ 'pure' states ($S \geq 1$). We illustrate this for the specific case of the initial proper states. If we have an initial state represented by eq.(35) we find the proper states of the individual entities through a biorthogonal decomposition (see [9]):

$$a_+ \psi^0_+ \otimes \frac{1}{N^+} \sum_{\pi \in \Pi} \pi ((\otimes \psi^0_+)^{S+M-1}(\otimes \psi^0_-)^{S-M}) + a_- \psi^0_- \otimes \frac{1}{N^-} \sum_{\pi \in \Pi} \pi ((\otimes \psi^0_+)^{S+M}(\otimes \psi^0_-)^{S-M-1})$$

where:

$$a_+ = \frac{N^+}{N} = \sqrt{(S+M)!(S-M)!}/(2S)! \quad a_- = \frac{N^-}{N} = \sqrt{(S+M)!(S-M)!}/(2S)!$$

The corresponding density matrix $\omega_M$ is:

$$\begin{pmatrix}
\frac{S+M}{2S} & 0 \\
0 & \frac{S-M}{2S}
\end{pmatrix}$$

in the base $\{\psi^0_+, \psi^0_-, \psi^1_-, \psi^2_-, \ldots, \psi^{-S}_-, \psi^{S-1}_-, \psi^{S}_-\}$. Due to the symmetry of eq.(35), we obtain the same density matrix for every individual entity. Clearly, all these density matrices can be decomposed in $2S$ spin-1/2 states, namely $S+M$ times $\psi^0_+$ and $S-M$ times $\psi^0_-$. Thus, as could be expected, the $2S$ identical density matrices $\omega_M$ are probabilistically equivalent with the spin-1/2 states in the Majorana representation from which we started, since we cannot distinguish them. In a similar way, the other density matrices that appear in the procedure can be equivalently replaced by spin-1/2 states. An explicit realization of this decomposition of density matrices for the specific case of coherent spin-1 states can be found in [9].

### 3 Generalization to non-coherent states.

It is possible to extend the procedure of the previous sections to non-coherent spin states. Unfortunately in doing so, we lose the transparency of the procedure and also the explicit reference to the spatial structure. We proceed along the following steps:

1. **For every spin-$S$ state there exists a unique representation as $2S$ spin-1/2 states. An explicit procedure to do this is presented in the appendix at the end of this chapter. Also the outcome states of a measurement**
can be represented by their Majorana representation (as is done in section 2.1). 2. We symmetrize the product state that corresponds to this 2S spin-1/2 states (that are representative for the initial state) in the sense of section 2.2. We obtain a representation of the initial state in $(\otimes \mathbb{C}^2)^{2S}$. 3. We apply the construction of [3] in order to introduce hidden correlations on the 2S individual entities (in the sense of section 2.3). In doing so, we obtain the exact probabilities of a spin-S system when represented as a compound system consisting of 2S individual spin-1/2 entities.

Of course, this rather straightforward induction of the procedure presented in the previous section is not yet an explicit proof. However, the explicit proof doesn’t contribute in any way to any deeper insights and is therefore omitted.

4 A classical mechanistic representation in $\mathbb{R}^3$.

In section 3 of [3] we have explained a procedure that enables the construction of classical mechanistic representations for hidden correlation representations of general physical entities. For the spin-S entities introduced in this paper, we can realize such a classical mechanistic representations easily: in [3], Aerts has been able to construct a very simple classical mechanistic model system (i.e., a model system in which appear only Kolmogorovian probability measures) in the three dimensional space of reals for a measurement on a spin-1/2 entity with the states also represented on the Poincaré sphere; thus, if we represent the measurements on the individual spin-1/2 entities that appear in our representation by Aerts' model system, our representation gives rise to a classical mechanistic model system in the three dimensional space of reals for all spin-S entities.

5 Conclusion.

We are able to represent a measurement on a spin-S entity as a measurement on each of 2S individual spin-1/2 entities in a compound system if we introduce hidden correlations, i.e., a measurement on one of the individual entities induces a transition of the proper state of the other ones. If the spin-S entity is in a maximal spin state ($|M| = S$), the 2S individual entities behave as a collection of indistinguishable but separated quantum entities. If not so, the kind of correlations that we have to introduce are the same ones as for compound quantum systems described in the tensor product by a symmetrical superposition.

6 Appendix: a Bacry-like procedure for the Majorana representation.

Remarkably, although by leading authors in the field of angular momentum techniques it is claimed that Bacry’s proof is equivalent with Majorana’s (see [1]), it seems that this is not completely true\textsuperscript{10}. Nonetheless, as we will show in this appendix, Bacry’s approach reveals a new way to ‘generate’ the Majorana representation when it is adapted in a proper way to coincide with Majorana’s representation.

Bacry’s procedure for representing spin-S states as 2S spin-1/2 states is the following. Let $\psi \in \mathbb{C}^{2S+1}$ be a vector representative for the spin-S state $p$, and let $\{\psi_{-S}, \ldots, \psi_S\}$ be the coordinates of $\psi$ in a given basis (we repeat that the indices take values in $\{-S, -S+1, \ldots, S-1, S\}$). We can consider the following polynomial in $x$:

$$K^B_{\psi}[x] = \sum_{i} \psi_i x^{S-i}$$

(43)

For every such polynomial with complex coefficients, there exists a factorization in factors of first order in $x$, and these 2S factors are unique up to a multiplicative constant, i.e., they uniquely determine a ray in $\mathbb{C}^2$. Thus, every factor can be identified with one unique spin-1/2 state. Since also the polynomial $K^B_{\psi}[x]$ is determined up to a multiplicative constant (because $\psi$ is) we can relate to the spin-S state $p$ 2S spin-1/2 states in a unique way. Unfortunately, these spin-1/2 states are not the ones of the Majorana representation which we discussed in section 2.1.4. Consider the spin-1/2 states in the majorana

\textsuperscript{10}We remark that this difference has lead the present author to the incorrect remark that his representation of states of a spin-1 entity as two points on a sphere is differed from the Majorana representation (see [4] footnote 2).
representation of a spin-S state \( p_{M}^{\alpha,\beta} \). If we proceed along the lines of the above mentioned procedure we find the following polynomial representative for \( p_{+}^{\alpha,\beta} \):

\[
K_{+}^{\alpha,\beta}[x] = e^{i\frac{\alpha-I\gamma}{2}}\cos\frac{\beta}{2}x + e^{i\frac{\alpha-I\gamma}{2}}\sin\frac{\beta}{2}
\]

(44)

and for \( p_{-}^{\alpha,\beta} \):

\[
K_{-}^{\alpha,\beta}[x] = -e^{i\frac{\alpha+I\gamma}{2}}\sin\frac{\beta}{2}x + e^{i\frac{\alpha+I\gamma}{2}}\cos\frac{\beta}{2}
\]

(45)

Thus we find:

\[
K_{p_{M}}^{Bacry}[x] = (K_{+}^{\alpha,\beta}[x])^{M+S}(K_{-}^{\alpha,\beta}[x])^{M-S}
\]

\[
= \sum_{i} \psi_{i}(p_{M}^{\alpha,\beta})x^{2S+1-i}
\]

where we have (the computation proceeds along the same lines as in section 2.2):

\[
\psi_{i}(p_{M}^{\alpha,\beta}) = \sum_{k} (-1)^{k}(S+M')!(S-M')!(\cos\frac{\beta}{2})^{2S+M-i-2k}(-\sin\frac{\beta}{2})^{i-M+2k}(S+i-k)!(S-i-k)!k!e^{-i(I\alpha-M\gamma)}
\]

(46)

and this leads to a vector \( (\psi_{i}(p_{M}^{\alpha,\beta})) \), which defers definitely from \( \psi_{i}^{\alpha,\beta} \) given by eq.(9). Nonetheless, if in stead of considering the polynomial \( K_{p_{M}}^{Bacry}[x] \) we consider a modified polynomial \( K_{p_{M}}^{Majorana}[x] \), we do find the decomposition that corresponds with the Majorana representation. By comparing eq.(44) and eq.(46) one easily sees that:

\[
\psi_{i}^{\alpha,\beta} = \sqrt{(S+M)!(S-M)!} \psi_{i}(p_{M}^{\alpha,\beta})
\]

(47)

Thus, for \( \psi \in \mathbb{C}^{2S+1} \), representative for the spin-S state \( p \), we have to consider the following polynomial in \( x \):

\[
K_{p}^{Majorana}[x] = \sum_{i} \sqrt{(S+i)!(S-i)!} \psi_{i}x^{S-i}
\]

(48)

in stead of \( K_{p_{M}}^{Bacry}[x] \) in order to obtain the Majorana representation. Since this procedure for finding a Majorana representation for spin-S states goes for any spin-S state (and not only for coherent ones), we can use it to generalize our model introduced in section 2.3 for coherent spin-S states to non-coherent spin-S states.

References

[1] T. Aaberge, Helv. Phys. Acta 67, 127 (1994).
[2] L. Accardi, Nuovo Cimento 34, 161 (1982).
[3] D. Aerts, J. Math. Phys. 27, 202 (1986).
[4] H. Bacry, J. Math. Phys. 15, 1686 (1974).
[5] L.C. Biedenharn and J.D. Louck, Angular Momentum in Quantum Physics, Addison-Wesley, Reading (1981).
[6] F. Bloch and I.I. Rabi, Rev. Mod. Phys. 17, 237 (1945).
[7] B. Coecke, Helv. Phys. Acta 68, 396 (1995).
[8] B. Coecke, Found. Phys. 28, 1109 (1998).
[9] L.P. Hughston, R. Jozsa and W.K. Wooters, Phys. Lett. 183A, 14 (1993).
[10] E.T. Jaynes, Phys. Rev. 108, 171 (1957).
[11] E. Majorana, Nuovo Cimento 9, 43 (1932).
[12] B. Mielnik, Comm. Math. Phys. 9, 55 (1968).

[13] W. Pauli, Die Allgemeinen Prinzipien der Wellenmechanic, Handbuch der Physik Vol. V, Part I, Springer-Verlag, Berlin (1958).

[14] C. Piron, Foundations of Quantum Physics, W.A. Benjamin, London (1976).

[15] J. Schwinger, Trans. N.Y. Acad. Sci. [II] 38, 170 (1977).

[16] E.P. Wigner, Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra, Academic press, London (1959).