On codes over the finite non chain ring $A = \mathbb{F}_4 + v\mathbb{F}_4$, $v^2 = v$ and its covering radius of codes with Bachoc weight

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Abstract. In this paper, some lower and upper bounds on the covering radius of codes over the finite non chain ring $A = \mathbb{F}_4 + v\mathbb{F}_4$, $v^2 = v$ with respect to Bachoc weight is given. Also, the covering radius of various Block Repetition Codes of same and different length over the finite non chain ring $A = \mathbb{F}_4 + v\mathbb{F}_4$, $v^2 = v$ is obtained.

1. Introduction

Codes over finite commutative rings have been studied for almost 50 years. The main motivation of studying codes over rings is that they can be associated with codes over finite fields through the Gray map. Recently, coding theory over finite commutative non-chain rings is a hot research topic. The coding theory over finite commutative non-chain rings is first introduced by Bestumiyia et al [2]. There has much interest in codes over finite rings in recent years, especially the rings $\mathbb{Z}_{2k}$ where $2k$ denotes the ring of integers modulo $2k$. In particular, codes over $\mathbb{Z}_4$ have been widely studied [1, 3–6]. Good binary linear and non-linear codes can be obtained from codes over $\mathbb{Z}_4$ via the gray map.

The covering radius is an important geometric parameter of codes. It not only indicates the maximum error correcting capability of codes, but also relates to some practical problems such as the data compression and transmission. Studying of the covering radius of codes has attracted many coding scientists for almost 30 years. The covering radius of linear codes over binary finite fields was studied in [11].

Recently, the covering radius of codes over finite rings has been studied. In 1999, Sole et al gave many upper and lower bounds on the covering radius of a code over $\mathbb{Z}_4$ with different distances. The covering radius of codes over $\mathbb{Z}_2 + u\mathbb{Z}_2$ with $u^2 = 0$ and $\mathbb{Z}_2 + v\mathbb{Z}_2$, $v^2 = v$ was studied in [7–9] and [13].

However, to the best of my knowledge, there are few studies on the covering radius of codes over finite non chain rings, especially over the non chain ring $A$.

In this paper, define the covering radius of codes with Bachoc distance over $A$. Then, study the covering radius of repetition codes over $A$. The rest of this paper is organized as follows.

In section 2, some preliminaries and notations on the non chain ring $A$ is given. In section 3, definition of the covering radius of codes with Bachoc distance over $A$ is given. In section 4, some lower and upper
Hamming and Bachoc weights among all non-zero codewords of the Gray map, is a quaternary code of length 2.

A linear Gray map and a generator matrix are defined as:

\[ A \oplus B = \{ a + vb | a, b \in \mathbb{F}_4 \} \]

Let \( A \) be the number of non-zero coordinates. Throughout this paper, \( A \) denotes the commutative ring \( \mathbb{F}_4 + v\mathbb{F}_4 = \{ a + vb|a, b \in \mathbb{F}_4 \} \) with \( v^2 = v \) and take \( \mathbb{F}_4 = \{ 0, 1, w, w + 1 \} \) where \( w^2 = w + 1 \). The ring \( A \) is a finite non-chain ring with 16 elements. Any element of \( A \) can be uniquely expressed as \( c = a + vb \), where \( a, b \in \mathbb{F}_4 \).

A linear code \( C \) of length \( n \) over \( A \) is an additive subgroup of \( A^n \). An element of \( C \) is called a codeword of \( C \) and a generator matrix of \( C \) is a matrix whose rows generate \( C \). The Hamming weight \( w_H(x) \) of a vector \( x \) in \( A^n \) is the number of non-zero coordinates.

Let \( A = \mathbb{F}_4 + v\mathbb{F}_4 \) be a ring. Define \( A^* = \{ x = a + vb|0 \neq a, b \in \mathbb{F}_4 \} \) where \( v^2 = v \). The weight \( w \) on \( A \) is defined by

\[ w_b(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in A^* \\ 2 & \text{if } x \in A \setminus (A^* \cup \{0\}) \end{cases} \]

in [10].

The Bachoc distance between the codewords \( x \) and \( y \) in \( A^n \) is defined as \( d_B(x, y) = w_b(x - y) \).

The Hamming and Bachoc distances \( d_H(x, y) \) and \( d_B(x, y) \) between two vectors \( x \) and \( y \) are \( w_H(x - y) \) and \( w_B(x - y) \) respectively. The minimum Hamming and Bachoc weights \( (d_H \) and \( d_B) \) of \( C \) are the smallest Hamming and Bachoc weights among all non-zero codewords of \( C \) respectively.

A linear Gray map \( \phi \) from \( A^n \rightarrow \mathbb{F}_4^n \times \mathbb{F}_4^n \) is the coordinates-wise extension of the function from \( A \) to \( \mathbb{F}_4 \times \mathbb{F}_4 \). The image \( \phi(c) \) of a linear code \( C \) over \( A \) of length \( n \) by the Gray map, is a quaternary code of length \( 2n \).

Let \( A \) and \( B \) be two linear codes. The operations \( \otimes \) and \( \oplus \) are defined as \( A \otimes B = \{(a,b)|a \in A, b \in B\} \) and \( A \oplus B = \{a + vb|a \in A, b \in B\} \). A generator matrix for a nonzero linear code \( C \) over \( A \) can be put into the following form:

\[
G = \begin{bmatrix}
I_{k_1} & (1+v)B_1 & vA_1 & (1+v)A_2 + vB_2 & (1+v)A_3 + vB_3 \\
0 & vI_{k_2} & 0 & vA_4 & 0 \\
0 & 0 & (1+v)I_{k_3} & 0 & (1+v)B_4
\end{bmatrix},
\]

where \( A_i \) and \( B_i \) are \( \mathbb{F}_4 \) matrix for all \( 1 \leq i, j \leq 4 \). Let \( C \) be a linear code over \( A \). Then, \( |C| = 16^{k_1}4^{k_2}4^{k_3} \).

3. Covering Radius of Codes

Let \( d \) be a Bachoc distance in \( A \). The covering radius of a code \( C \) over \( A \) with respect to a Bachoc weight \( d \) is given by

\[
r_d(C) = \max_{u \in A^*} \left\{ \min_{c \in C} \{d(u, c)\} \right\}.
\]

The following result of Mattson [11] is useful for computing covering radius of codes over rings generalized easily from codes over finite fields.
**Proposition 3.1.** If $C_0$ and $C_1$ are codes over $A$ generated by matrices $G_0$ and $G_1$ respectively and if $C$ is the code generated by

$$G = \begin{pmatrix} 0 & G_1 \\ G_0 & A \end{pmatrix},$$

then $r_d(C) \leq r_d(C_0) + r_d(C_1)$ and the covering radius of $D$ (concatenation of $C_0$ and $C_1$) satisfy the following

$$r_d(D) \geq r_d(C_0) + r_d(C_1),$$

for all distance $d$ over $A$.

4. Covering radius of repetition codes

A $q$-ary repetition code $C$ over a finite field $F_q = \{a_0 = 0, a_1 = 1, a_2, a_3, \ldots, a_{q-1}\}$ is an $[n, 1, n]$ code $C = \{a \in F_q\}$, where $a = a_1 a_2 \cdots a_n$. The covering radius of $C$ is $\lceil \frac{m(q-1)}{q} \rceil$ [12].

Using this, it can be seen easily that the covering radius of block of size $n$ repetition code $[n(q-1), 1, n(q-1)]$ generated by $G = [11 \cdots 1 a_2 a_2 \cdots a_{q-1} a_{q-1} \cdots a_{q-1} 1]$ is $\lceil \frac{n(q-1)^2}{q} \rceil$, since it will be equivalent to a repetition code of length $(q-1)n$.

Consider the repetition code over $A$. There are two types of repetition codes of length $n$ viz.

1. unit repetition code $C_1 : [n, 1, n, n]$ generated by $G_1 = [11 \cdots 1]$
2. zero repetition code

   - $C_{uv} : (n, 2, n, 2n)$ generated by $G_{uv} = [uv uv \cdots uv]$
   - $C_v : (n, 4, n, 2n)$ generated by $G_v = [v v \cdots v]$
   - $C_2 : (n, 8, n, n)$ generated by $G_2 = [2 2 \cdots 2]$

The following result determines the covering radius with respect to Bachoc distance.

**Theorem 4.1.** Let $C_i$ be a zero divisor repetition codes in $G_{1 \leq i \leq uv, v, 2}$. Prove that the following

1. $n \leq r(C_{uv}) \leq 2n$,
2. $r(C_v) = \frac{5n}{4}$ and
3. $n \leq r(C_2) \leq \frac{5n}{4}$.

**Proof.** Let $x = \underbrace{uv uv \cdots uv}_{2} \in A^n$. The code $C_i = \{c_k = k(G_i), k \in A^n\}$, where $i \in \{uv, v, 2\}$. In a generator matrix $G_{uv}$, $C_{uv} = \{c_k = k(G_{uv}), k \in A^n\}$, that is $C_{uv} = \{c_0 = 00 \cdots 0, c_1 = uv uv \cdots uv\}$, then

$$d(x, c_0) = wt(x - c_0) = \frac{n}{2} wt(uv) = n$$

and

$$d(x, c_1) = wt(x - c_1) = \frac{n}{2} wt(uv) = n.$$
Therefore, \(d(x, C_{\text{rep}}) = \min(n, n) = n\). By the definition of covering radius, thus \(r(C_{\text{rep}}) \geq n\).

Let \(x\) be any word in \(A^n\). Let us take \(x \) has \(\omega_i, \forall i \in \{0, 1, 15\}\) coordinates as \(j\)'s, \(j \in A\), and \(\sum_{i=0}^{15} \omega_i = n\). Then
\[
d(x, c_0) = n - \omega_0 + \omega_4 + \omega_8 + \omega_{12}
\]
and
\[
d(x, c_1) = n - \omega_4 + \omega_0 + \omega_4 + \omega_{12}.
\]
Thus \(d(x, C_{\text{rep}}) = \min\{d(x, c_0), d(x, c_1)\}\), which implies that \(d(x, C_{\text{rep}}) \leq 2n\) and \(r(C_{\text{rep}}) \leq 2n\). Hence, \(n \leq r(C_{\text{rep}}) \leq 2n\).

Similarly, \(r(C_{\text{rep}}) = \frac{3n}{2}\) and \(n \leq r(C_{\text{rep}}) \leq \frac{5n}{2}\).

**Theorem 4.2.** Let \(C_1\) be a unit repetition code in \(G_1\), then \(r(C_1) = \frac{9n}{8}\).

**Proof.** In Theorem 4.1, \(r(C_1) \leq \frac{9n}{8}\). In \(G_1\), \(c_k = k(G_1), k \in A\).

Let \(x = c_0 c_1 c_2 c_3 c_4 c_5 c_6 c_7 c_8 c_9 c_{10} c_{11} c_{12} c_{13} c_{14} c_{15}\) \(\in A^n\), where \(t = \lfloor \frac{n}{15} \rfloor\), then \(d(x, c_i) = n + 2t\), where \(i \in A\). Therefore \(r(C_1) \geq \min\{d(x, c_i), i \in A\} \geq \frac{n}{8}\) and hence \(r(C_1) = \frac{9n}{8}\).

**Example 4.3.** Consider the repetition codes of length \(n = 4\) over \(A = \mathbb{F}_4 + v\mathbb{F}_4\), where \(v^2 = v\). Then \(C_{\text{rep}} = \{k(vwvwvw) | k \in A\}\) and \(C_2 = \{k(2222) | k \in A\}\). Further, that \(r(C_1) = 8\) and \(r(C_2) = 5\). The covering radii of these repetition codes with respect to Bachoc distance have achieved the upper bound that appeared in Theorem 4.1.

**Example 4.4.** Consider the repetition codes of length \(n = 4\) over \(A = \mathbb{F}_4 + v\mathbb{F}_4\), where \(v^2 = v\). Then \(C_1 = \{k(0000) | k \in A\}\) and \(C_1 = \{k(1111) | k \in A\}\). Further, that \(r(C_1) = 6\) and \(r(C_1) = \frac{9}{2}\). The covering radii of these repetition codes with respect to Bachoc distance have achieved the bound that appeared in Theorem 4.1 and 4.2.

4.1. Covering radius of repetition code with same length

In order to determine the covering radius of \(A\) with two blocks repetition code each of size \(n\). \(B\text{Rep}^{2n}\) :

\[
[2n, 1, n, n, \ldots, ]\text{ generated to the matrix } G = \begin{bmatrix}
G_1 & G_{\text{rep}}
\end{bmatrix}.
\]

**Theorem 4.5.** Let \(C\) be a code over \(A\) generated by the matrix

\[
G = \begin{bmatrix}
G_1 & G_{\text{rep}}
\end{bmatrix},
\]

then \(\frac{17n}{8} \leq r(\text{BRep}^{2n}) \leq \frac{20n}{8}\).

**Proof.** By Theorem 4.1 and 4.2, the Proposition 3.1 and the given generator matrix \(G\), we get

\[
r(\text{BRep}^{2n}) \geq \frac{17n}{8}
\]

\(\text{(1)}\)

Let \(x = (g_1, g_2) \in A^{2n}\) and let us take in \(g_1\), \(i\) appears \(r_i\) times, \(i \in A\) and in \(g_2\), \(i\) appears \(s_i, i \in A\) with \(\sum_{i=0}^{15} r_i = n = \sum_{i=0}^{15} s_i\) and \(c_i = i(G_1G_{\text{rep}}), i \in A\). Then
\[
d(x, c_i) = 2n - r_i + r_{4+i} + r_{8+i} + r_{12+i} - s_{8+i} + s_{4+i} + s_{2+i} + s_{1+i} + s_{12+i} + (i-1)
\]
and hence
\[
d(x, B\text{Rep}^{2n}) = \min\{d(x, c_i), i \in A\},
\]
which implies that, \(d(x, B\text{Rep}^{2n}) \leq \frac{20n}{8}\) and thus
\[
r(\text{BRep}^{2n}) \leq \frac{20n}{8}
\]
\(\text{(2)}\)

By the Equations (1) and (2), so \(\frac{17n}{8} \leq r(\text{BRep}^{2n}) \leq \frac{20n}{8}\).

\(\Box\)
In three blocks repetition codes each of size $n$. $B\text{Rep}^{3n} : [3n, 1, n, 2n]$ generated by

$$G = [11 \cdots 1 \underbrace{vv \cdots vv}_{n} \cdots v].$$

The proof of the theorem 4.6 and 4.7 is similar to above theorem 4.5.

**Theorem 4.6.** Let $C$ be a code over $A$ generated by the matrix

$$G = [G_1 G_{vv} G_v]_n.$$

Then $\frac{29n}{8} \leq r(B\text{Rep}^{3n}) \leq 4n.$

In $A$, the four blocks each of size $n$ repetition code $B\text{Rep}^{4n} : [4n, 1, n, 2n]$ generated by

$$G = [G_1 G_{vv} G_v G_2].$$

The following result is obtain

**Theorem 4.7.** Let $C$ be a code over $A$ generated by the matrix

$$G = [G_1 G_{vv} G_v G_2].$$

Then, $\frac{27n}{8} \leq r(B\text{Rep}^{3n}) \leq \frac{42n}{8}.$

### 4.2. Covering radius of repetition code with different length

The two different length of Block repetition code of size $m$ and $n$ is $B\text{Rep}^{m+n} : [m + n, 1, m, \min\{m, m + n\}]$ generated to the matrix $G = [G_1 G_{vv}].$

**Theorem 4.8.** $\frac{9m + 8n}{8} \leq r(B\text{Rep}^{m+n}) \leq \frac{9m + 16n}{8}.$

**Proof.** By theorem 4.1 and 4.2 and by the above generator matrix

$$r(B\text{Rep}^{m+n}) \geq \frac{9m + 8n}{8}. \quad (3)$$

Let $z = (x|y) \in A^{m+n}$ where $x \in A^m$ and $y \in A^n$. Then,

$$d(z, B\text{Rep}^{m+n}) \leq \frac{9m + 16n}{8}.$$

Thus

$$r(B\text{Rep}^{m+n}) \leq \frac{9m + 16n}{8}. \quad (4)$$

From equation (3) and (4), $\frac{9m + 8n}{8} \leq r(B\text{Rep}^{m+n}) \leq \frac{9m + 16n}{8}.$ \qed
In a three different Block repetition code of length is \( m, n \) and \( o \) is \( BRep^{m+n+o} : [m+n+o, 1, m, \min(m, m+n+o)] \) generated by

\[
G = \begin{bmatrix}
m & n & o
\end{bmatrix}.
\]

Then, the following result is determined

**Theorem 4.9.** \( \frac{2m+3n+2o}{2} \leq r(BRep^{m+n+o}) \leq \frac{9m+16n+12o}{8} \).

Four different Block repetition code of length are \( m, n, o \) and \( p \):

\[
BRep^{m+n+o+p} : [m+n+o+p, 1, m, \min(2m, m+n+o+p)] \text{ generated by}
\]

\[
G = \begin{bmatrix}
m & n & o & p
\end{bmatrix}.
\]

Obtain the following theorem

**Theorem 4.10.** \( \frac{8m+12n+8o+9p}{8} \leq r(BRep^{m+n+o+p}) \leq \frac{9m+16n+12o+10p}{8} \).

**Covering radius for different order of the zero divisor repetition code**

**Same length**

- \( \frac{5n}{2} \leq r(C_{cd}) \leq \frac{7n}{2} \) in the generator matrix \( G_{cd} = \begin{bmatrix} n & n \\ n & n \end{bmatrix} : (2n, 4, n, 2n) \)

- \( 2n \leq r(C_{cd}) \leq \frac{13n}{4} \) in the generator matrix \( G_{cd} = \begin{bmatrix} n & n \\ n & n \end{bmatrix} : (2n, 8, n, 2n) \)

- \( \frac{5n}{2} \leq r(C_{cd}) \leq \frac{11n}{4} \) in the generator matrix \( G_{cd} = \begin{bmatrix} n & n \\ n & n \end{bmatrix} : (2n, 8, n, 2n) \)

- \( \frac{7n}{2} \leq r(C_{cd}) \leq \frac{13n}{4} \) in the generator matrix \( G_{cd} = \begin{bmatrix} n & n \\ n & n \end{bmatrix} : (2n, 8, n, 2n) \)

**Different length**

- \( \frac{2m+3n}{2} \leq r(C_{cd}) \leq \frac{4m+3n}{2} \) in the generator matrix \( G_{cd} = \begin{bmatrix} m & n \\ m & n \end{bmatrix} : (m+n, 4, m, \min(2m, 2m+n)) \)

- \( m+n \leq r(C_{cd}) \leq \frac{8m+5n}{4} \) in the generator matrix \( G_{cd} = \begin{bmatrix} m & n \\ m & n \end{bmatrix} : (m+n, 8, m, \min(2m, 2m+n)) \)

- \( \frac{4m+3n}{2} \leq r(C_{cd}) \leq \frac{6m+5n}{4} \) in the generator matrix \( G_{cd} = \begin{bmatrix} m & n \\ m & n \end{bmatrix} : (m+n, 8, m, \min(2m, 2m+n)) \)

- \( \frac{2m+3n+2o}{2} \leq r(C_{cd}) \leq \frac{8m+6n+5o}{4} \) in the generator matrix \( G_{cd} = \begin{bmatrix} m & n & o \\ m & n & o \end{bmatrix} : (m+n+o, 8, n, \min(2m, 2m+n)) \)

- \( \frac{2m+3n+2o}{2} \leq r(C_{cd}) \leq \frac{8m+6n+5o}{4} \) in the generator matrix \( G_{cd} = \begin{bmatrix} m & n & o \\ m & n & o \end{bmatrix} : (m+n+o, 8, n, \min(2m, 2m+n)) \)

- \( \frac{2m+3n+2o}{2} \leq r(C_{cd}) \leq \frac{8m+6n+5o}{4} \) in the generator matrix \( G_{cd} = \begin{bmatrix} m & n & o \\ m & n & o \end{bmatrix} : (m+n+o, 8, n, \min(2m, 2m+n)) \)
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