Dynamic Resource Allocation in the Cloud with Near-Optimal Efficiency

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Abstract

Cloud computing has motivated renewed interest in resource allocation problems with new consumption models. A common goal is to share a resource, such as CPU or I/O bandwidth, among distinct users with different demand patterns as well as different quality of service requirements. To ensure these service requirements, cloud offerings often come with a service level agreement (SLA) between the provider and the users. An SLA specifies the amount of a resource a user is entitled to utilize. In many cloud settings, providers would like to operate resources at high utilization while simultaneously respecting individual SLAs. There is typically a tradeoff between these two objectives: for example, utilization can be increased by shifting away resources from idle users to “scavenger” workload, but with the risk of the former then becoming active again.

We study this fundamental tradeoff by formulating a resource allocation model that captures basic properties of cloud computing systems, including SLAs, highly limited feedback about the state of the system, and variable and unpredictable input sequences. Our main result is a simple and practical algorithm that achieves near-optimal performance on the above two objectives. First, we guarantee nearly optimal utilization of the resource even if compared to the omniscient offline dynamic optimum. Second, we simultaneously satisfy all individual SLAs up to a small error. The main algorithmic tool is a multiplicative weight update algorithm, and a duality argument to obtain its guarantees.

1 Introduction

Cloud computing has motivated renewed interest in resource allocation, manifested in new consumption models (e.g., AWS spot pricing), as well as the design of resource-sharing platforms [1, 2]. These platforms need to support a heterogenous set of users, also called tenants, that share the same physical computing resource, e.g., CPU, memory, I/O bandwidth. Providers such as Amazon, Microsoft and Google offer cloud services with the goal of benefiting from economies of scale. However, the inefficient use of resources – over-provisioning on the one hand or congestion on the other – could result in a low return on investment or in loss of customer goodwill, respectively. Hence, resource allocation algorithms are key for efficiently utilizing cloud resources.

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To ensure quality of service, cloud offerings often come with a service level agreement (SLA) between the provider and the users. An SLA specifies the amount of resource the user is entitled to consume. Perhaps the most common example is renting a virtual machine (VM) that guarantees an explicit amount of CPU, memory, etc. Naturally, VMs that guarantee more resources are priced higher. In this context, a simple allocation policy is to assign each user the resources specified by the SLAs. However, such an allocation can be wasteful, as users may not need the resource at all times. In principle, a dynamic allocation of resources can increase the total efficiency of the system. However, allocating resources dynamically without carefully accounting for SLAs can lead to user dissatisfaction.

Recent scheduling proposals address these challenges through work-maximizing yet fair schedulers [3, 4]. However, such schedulers do not have explicit SLA guarantees. On the other hand, other works focus on enforcing SLAs [5, 6, 7], but do not explicitly optimize the use of extra resources.

Our goal in this work is to design algorithms that guarantee both high utilization and the satisfaction of individual SLAs. To that end, we formulate a basic model for online dynamic resource allocation. We focus on a single divisible resource, such as CPU or I/O bandwidth, that has to be shared among many users. Each user also has an SLA that specifies the fraction of the resource that it expects to obtain. The actual demand of the user is in general time-varying, and may exceed the fraction specified in the SLA. As in many real systems, the demand is not known in advance, but rather arrives in an online manner. Arriving demand is either processed or queued up, depending on the resource availability. In many real-world scenarios, it is difficult to measure the actual demand size (see, e.g., [8]). Accordingly, we assume that the system (and underlying algorithm) receives only a simple binary feedback per user at any given time: whether the user queue is empty (the user’s work arriving so far has been completed), or not. This is a plausible assumption in many systems, because one can observe workload activity, yet anticipating how much of the resource a job will require is more difficult.

While online dynamic resource allocation problems have been studied in different contexts and communities (see Section 1.3 for an overview), our work aims to address the novel aspects of this problem arising in the cloud computing paradigm, particularly the presence of SLAs, the highly limited feedback about the state of the system, and a desired robustness over arbitrary input sequences. For the algorithm design itself, we pay close attention to practicality: Our algorithm involves fairly simple computations that can be implemented with minimal overhead of space or time. This algorithm achieves nearly optimal utilization of the resource, as well as approximately satisfying the SLA of each individual user. We see two main use-cases for these algorithms:

- In enterprise settings (“private cloud”), different applications or organizations share the same infrastructure. These often have SLAs, but providers would still like to maximize the ROI by maximizing utilization [9].
- In public clouds, users buy VMs, which are offered at different “sizes” (which is practically the SLA). Here, high utilization is also important. Consequently, cloud providers offer “best-effort” alternatives, such as Azure Batch (MS) or Spot instances (AWS). Our work can be viewed as a principled way to accommodate such services, and even give VMs better service than expected, an important consideration as public cloud products gradually become commoditized.
1.1 The Model

We consider the problem of having multiple tenants or users sharing a single resource, such as CPU, I/O or networking bandwidth. For simplicity, we assume that the total resource capacity is normalized to 1. We have $N$ users sharing the resource, a finite, but unknown, discrete time horizon indexed $t = 1, \ldots, T$, and an underlying queuing system. For each user $i$, we are also given an expected share of resource $\beta(i) \geq 0$ such that $\sum_{i=1}^{N} \beta(i) \leq 1$. The input is an online sequence of workloads $L_1, \ldots, L_T \in \mathbb{R}_{+}^N$, where $L_t(i) \geq 0$ corresponds to $i$’s workload at time $t$. The system maintains a queue $Q_t(i)$, denoting $i$’s remaining work at time $t$. In our model, the decision maker does not have any access to the values of the queues or the workloads. This allows us to consider settings where the job sizes are not known in advance and minimal information is available about the underlying system, a regular occurrence in many cloud applications. At time $t$, the following happens:

1. **Feedback:** The decision maker observes which queues are empty (the set of users $i$ with $Q_t(i) = 0$), and which are not ($Q_t(i) > 0$).
2. **Decision:** The decision maker updates user resource allocations $h_t(i)$, satisfying $\sum_i h_t(i) \leq 1$.
3. **Update:** The load $L_t(i)$ for each $i$ arrives and each user processes as much of the work from the queue plus the arriving workload as possible. The work completed by user $i$ in step $t$ is
   \[ w_t(i) := \min\{h_t(i), L_t(i) + Q_t(i)\}. \]
   The queues at the end of the time step are updated accordingly:
   \[ Q_{t+1}(i) = \max\{0, L_t(i) + Q_t(i) - h_t(i)\}. \]

We assess the performance of any algorithm based on two measures.

1. **Work Maximization.** The algorithm should maximize the total work completed over all users, and thus utilize the resource as much as possible.
2. **SLA Satisfaction.** The algorithm should (approximately) satisfy the SLAs in the following manner. The work completed by user $i$ up to any time $1 \leq t \leq T$ should be no less than the work completed for this user up to $t$ if it were given a constant fraction $\beta(i)$ of the resource over the whole horizon.

Achieving each of the criteria on their own is straightforward. A greedy strategy that takes away resources from an idle user and gives them to any user whose queue is non-empty is approximately work-maximizing; see appendix C. To satisfy the SLAs, we give each user a static assignment of $h_t(i) := \beta(i)$ for all $t$. Naturally, the two criteria compete with each other; the following examples illustrate why these simple algorithms do not satisfy both.

**Example 1.1 (Maximizing utilization may imply an SLA violation)** Consider two users with equal SLAs, $\beta(1) = \beta(2) = 0.5$. Let $L_t(1) = 1$ for all $t$ and $L_1(2) = 0$ and $L_t(2) = 1$ for all $t \geq 2$. A greedy strategy could set the static allocation $h_t(1) = 1$ and $h_t(2) = 0$ for all $t$. This maximizes the completed work and fully utilizes the resource, but user 2’s SLA is violated; in fact, this user’s work is not processed at all.

3
Example 1.2 (Focusing solely on the SLA may imply poor utilization) Consider again two users with the same SLAs, \( \beta(1) = \beta(2) = 0.5 \). Let \( L_t(1) = 0.2 \) and \( L_t(2) = 0.8 \) for all \( t \leq T \). The static allocation \( h_t(1) = h_t(2) = 0.5 \) for all \( t \) satisfies the SLAs but is inefficient. At any time \( t \), the total work completed under this allocation is \( 0.7t \); the alternative static allocation \( h'_t(1) = 0.2 \), \( h'_t(2) = 0.8 \) maximizes utilization, completing \( t \) units of work at time \( t \), and satisfies user 1’s SLA.

1.2 Our Results and Contributions

Our main contribution is to give a simple and efficient online algorithm that achieves approximate work maximization as well as approximate SLA satisfaction even in the limited feedback model that we consider. For work maximization, we analyze the performance by comparing it to the optimal offline dynamic allocation that knows all the data up front. In contrast, our online algorithm receives limited feedback even in an online setting. Thus, our aim is to minimize the quantity

\[
\text{work}_{h^*_1, \ldots, h^*_T} - \text{work}_\text{alg} = \sum_{t=1}^{T} \sum_{i} w^*_t(i) - \sum_{t=1}^{T} \sum_{i} w_t(i),
\]

where \( w^*_t = (w^*_t(1), \ldots, w^*_t(N)) \) is the work performed by the optimal allocations \( h^*_1, \ldots, h^*_T \) at time \( t \). The objective of the decision maker is to minimize this quantity by constructing a sequence of good allocations that approach the best allocations in hindsight. Note that our benchmark is dynamic, rather than the more common static offline optimum usually considered in regret minimization.

Similarly, for SLA satisfaction, our benchmark is the total work done for a user if it were given \( \beta(i) \) resources for each time \( 1 \leq t \leq T \). We give bi-criteria online algorithms that achieve nearly the same performance as the benchmarks if the resources for the latter are slightly more constrained than that of the algorithm.

**Theorem 1** For any \( 0 < \varepsilon < \frac{1}{T} \), SLAs \( \beta = (\beta(1), \ldots, \beta(N)) \) satisfying \( \beta(i) \geq 2\varepsilon N \), and online loads \( L_1, \ldots, L_T \in \mathbb{R}^N_{\geq 0} \). Algorithm \( \square \) achieves the following guarantees.

1. **Approximate Work Maximization.** Let \( h^*_1, \ldots, h^*_T \in [0, 1]^N \) be the optimal offline sequence of allocations such that \( \sum_i h^*_t(i) = 1 \) for all \( 1 \leq t \leq T \). Then

\[
\text{work}_\text{alg} \geq (1 - \varepsilon) \text{work}_{h^*_1, \ldots, h^*_T} - O\left(N\varepsilon^{-2}\log(N/\varepsilon)\right).
\]

2. **Approximate SLA Satisfaction.** There exists \( \tilde{s} = \tilde{s}(N, \varepsilon) = O\left(N^2\varepsilon^{-3}\log(N/\varepsilon)\right) \) such that for any user \( i \) and time \( t \), if we take \( h'_1, \ldots, h'_T \in [0, 1]^N \) to be any sequence of allocations with \( h'_t(i) \leq \beta(i) \), then

\[
\sum_{t=1}^{T} w_t(i) \geq \left(1 - \frac{\varepsilon^2}{2N}\right) \sum_{t=1}^{T} w'_t(i) - \beta_i \tilde{s},
\]

where \( w'_t \) is the work performed by the allocations \( h'_1, \ldots, h'_T \).

Our algorithm follows two different multiplicative weight strategies depending on the limited feedback received. If the users with empty queues are assigned more than a fraction \( \varepsilon \) of the resource,
the decision maker updates the allocations greedily by penalizing these users. On the other hand, if \( 1 - \varepsilon \) or more of the resource is assigned to users with non-zero queues, and thus most of the resource is being utilized, the decision maker redistributes the resource by penalizing those users assigned more than their corresponding SLAs. The penalizing rates are distinct in the two cases and are chosen appropriately. We show that this efficient heuristic strategy is enough to achieve approximate work maximization and SLA satisfaction.

The analysis of the algorithm relies on a dual fitting approach. For work maximization, we can write the offline dynamic optimal allocation as a solution to a linear program and then construct feasible dual solutions whose objective is close to the algorithm’s resource utilization. A crucial ingredient of the multiplicative weight algorithm is the use of entropic projection on the \textit{truncated} simplex, which ensures every user gets at least a fraction \( \varepsilon/N \) of the resource at all times. Intuitively, this means any user with a non-empty queue will be assigned his SLA requirement in a few steps.

The rest of the work is organized as follows. In Section 2, we present the preliminaries and the multiplicative weight algorithm. Section 3 contains the proof of Theorem 1 in the bi-criteria form. We split the proof into two parts: work maximization in Section 3.2 and SLA satisfaction in Section 3.3. In Appendix D, we show a general lower bound for all online deterministic algorithms that allocate a divisible resource with the same feedback as our model. The lower bound shows that the guarantee for work maximization in Theorem 1 is nearly tight for certain ranges of parameters.

1.3 Related Work

There has been significant work in resource allocation problems arising from cloud computing applications, both from a practical as well as a theoretical standpoint \cite{7, 2, 9, 6, 5, 8, 1, 10, 11}. A focus of many of these works has been to understand the tradeoffs between efficiency and ensuring guarantees to individual users. A closely related topic is fairness in resource allocation \cite{3, 4}.

More broadly, online resource allocation is a classical problem and has been studied for networking in adversarial as well as stochastic settings; we refer the readers to the books \cite{12, 13, 14} on the topic.

Multiplicative weight update algorithms have been widely studied in optimization \cite{15, 16}, online convex optimization \cite{17}, online competitive analysis \cite{18} and learning theory \cite{19, 20}. Our results bear some resemblance to regret analysis \cite{17, 21}, where the benchmark is the optimal offline static policy.

2 Algorithm

2.1 Preliminaries

For \( N \geq 1 \), we identify the set of users \( \{1, \ldots, N\} \). For \( 0 < \varepsilon < 1 \), we call an allocation \( \mathbf{h} = (h(1), \ldots, h(N)) \in [0, 1]^N \) a \((1 - \varepsilon)\)-allocation if \( \sum_i h(i) \leq 1 - \varepsilon \).

For any \( t \), we define the set of active users at that time as the set of users with non-empty queue, and denote this set by \( A_t \). Observe that \( h_t(i) = w_t(i) \) for all active users. Let \( B_t = [N] \setminus A_t \) be the sets of users with empty queues at time \( t \); we call these users inactive. \( A_t \) and \( B_t \) correspond to the feedback given to the decision maker.
Given an allocation $h_t$, we define $D_t = \{ i : h_t(i) \leq \beta(i) \}$ to be the set of users below their SLAs and $U_t = [N] \setminus D_t$ to be the set of users above their SLAs.

We assume without loss of generality that the allocations set by the decision maker always add up to 1. We propose an algorithm that uses a multiplicative weight strategy to penalize a subset of users by multiplying their allocation by factor less than one. Because the allocations do not sum to 1, we in fact project onto the truncated simplex,

$$\Delta_\varepsilon = \{ x = (x(1), \ldots, x(N)) : \|x\|_1 = 1, x(i) \geq \varepsilon/N, \forall i \}.$$  

To fix notation, let $\pi_{\Delta_\varepsilon}(\cdot)$ be the projection function onto $\Delta_\varepsilon$ using KL-divergence, i.e., $\pi_{\Delta_\varepsilon}(y) := \text{argmin}_{x \in \Delta_\varepsilon} \sum_i x(i) \log(x(i)/y(i))$, where $y = (y(1), \ldots, y(N)) \in \mathbb{R}_+^N$. In Appendix [A] we show how to efficiently compute this projection. The following proposition states some basic facts that are useful in our analysis. The proof appears in Appendix [B].

**Proposition 1** Let $y \in \mathbb{R}_+^N$, $x = \pi_{\Delta_\varepsilon}(y)$, and $S = \{ i : x(i) = \frac{\varepsilon}{N} \}$. Then:

(a) If $y(1) \leq y(2) \leq \cdots \leq y(N)$, then $S = \{1, \ldots, k\}$ for some $k \geq 0$.

(b) $x(i) = y(i)e^{\mu_i}C$, where $C = \left(\frac{1}{\sum_{j \in S} y(j)}\right)$, $\mu_i \geq 0$ for all $i$ and $\mu_i = 0$ for $i \notin S$.

(c) $x$ can be computed in $O(N \log N)$ time.

### 2.2 The Multiplicative Weight Algorithm

We propose an algorithm that follows a multiplicative weight strategy, described in Algorithm [1]

**Algorithm 1: Multiplicative Weight Update Algorithm**

**Input:** Parameters $0 < \varepsilon \leq \frac{1}{10}$, $\eta > 0$.

1. **Initialization:** $h_1$ any distribution over $\Delta_\varepsilon$, $\eta' = \ln \left(1 + \frac{\varepsilon^2}{N}(1 - e^{-\eta})\right)$.

2. for $t = 1, \ldots, T$ do

   3. Read active and inactive users $A_t$ and $B_t$.

   4. Set $D_t = \{ i : h_t(i) \leq \beta(i) \}$ and $U_t = [N] \setminus D_t$.

   5. Update allocation:

      - If $\sum_{i \in A_t} h_t(i) < 1 - \varepsilon$

        $$\widehat{h}_{t+1}(i) = \begin{cases} 
        h_t(i) & \text{if } i \in A_t \\
        h_t(i)e^{-\eta} & \text{if } i \in B_t 
        \end{cases}.$$  \hspace{1cm} \text{(rule 1)}

      - If $\sum_{i \in A_t} h_t(i) \geq 1 - \varepsilon$, then

        $$\widehat{h}_{t+1}(i) = \begin{cases} 
        h_t(i) & \text{if } i \in D_t \\
        h_t(i)e^{-\eta'} & \text{if } i \in U_t 
        \end{cases}.$$  \hspace{1cm} \text{(rule 2)}

6. $h_{t+1} \leftarrow \pi_{\Delta_\varepsilon}(\widehat{h}_{t+1})$. 


Intuitively, if the decision maker has enough resources to distribute \( \sum_{i \in A_t} h_t(i) < 1 - \varepsilon \), then she follows a greedy strategy, penalizing users with empty queues. Otherwise, if almost all the resources are being used \( \sum_{i \in A_t} h_t(i) \geq 1 - \varepsilon \), then she must redistribute the resources in order to satisfy the SLAs, by penalizing the users whose allocation exceeds their SLA. The penalty parameter \( \eta' \) used in the second case is chosen so that these penalties are comparatively less severe than in the first case.

3 Analysis

To give the analysis of the algorithm and prove Theorem 1, we prove the following stronger guarantees about Algorithm 1. We compare its performance to the optimal offline dynamic strategy that uses at most \( 1 - \frac{6}{10} \varepsilon \) fraction of the resources at each time step.

**Theorem 2** Given loads \( L_1, \ldots, L_T \), for any \( \varepsilon > 0 \) and \( \eta > 0 \) such that \( \varepsilon < \frac{1}{10} \), Algorithm 1 guarantees

\[
\text{work}_{h_1^*, \ldots, h_T^*} - \text{work}_{\text{alg}} \leq 2N \frac{\varepsilon^2 \ln(N/\varepsilon)}{1 - e^{-\eta}},
\]

where \( h_1^*, \ldots, h_T^* \) is the optimal offline sequence of \( (1 - \frac{6}{10} \varepsilon) \)-allocations.

The first guarantee of Theorem 1 regarding work conservation now follows simply from Theorem 2. Given any offline dynamic policy \( h_1, \ldots, h_T \) such that \( \sum_t h_t(i) = 1 \), we define \( h_t := (1 - \varepsilon) h_t \) which satisfies the assumption of Theorem 2. Now we have

\[
\text{work}_{\text{alg}} \geq \text{work}_{h_1, \ldots, h_T} - 72N \frac{\varepsilon^2 \ln(6N/\varepsilon)}{1 - e^{-\eta}} \geq (1 - \varepsilon) \cdot \text{work}_{h_1, \ldots, h_T} - 144N \frac{\varepsilon^2 \ln(N/\varepsilon)}{1 - e^{-\eta}}
\]

where the first inequality follows from Theorem 2 and the second inequality follows since the work done by the dynamic policy \( h_1, \ldots, h_T \), say \( w_1, \ldots, w_T \), induces a feasible work \( \overline{w}_1, \ldots, \overline{w}_T \) under the dynamic policy \( \overline{h}_1, \ldots, \overline{h}_T \). However, the real work done by policy \( \overline{h}_1, \ldots, \overline{h}_1 \) is at least the work done by \( \overline{w}_1, \ldots, \overline{w}_T \) which is \( (1 - \varepsilon) \) work \( h_1, \ldots, h_T \).

Similarly, for SLA satisfaction, we prove a similar stronger bi-criteria result which implies the SLA guarantee in Theorem 1.

**Theorem 3** Fix \( \eta \geq 1 \). Take any SLAs \( \beta(1), \ldots, \beta(N) \) such that \( \beta(i) \geq 2 \frac{\varepsilon^2}{27N} \). Then, there exists \( \bar{s} = \bar{s}(\varepsilon, N) \) such that for any user \( i \) and time \( t \), if we take \( h'_1, \ldots, h'_T \) to be any allocations such that \( h'_t(i) = \left( 1 - \frac{\varepsilon^2}{27N} \right) \beta(i) \), the work done by Algorithm 1 for user \( i \) satisfies

\[
\sum_{\tau=1}^{t+\bar{s}} w_{\tau}(i) \geq \sum_{\tau=1}^{t} w'_{\tau}(i),
\]

where \( w'_t \) is the work done by the allocations \( h'_1, \ldots, h'_T \).
3.1 The Offline Formulation

Before presenting the main results, we state the offline LP formulation of the maximum work problem for \((1 - \varepsilon)\)-allocations. We denote by \(w_t = (w_t(1), \ldots, w_t(N))\) the work done for each user at time \(t\). Given loads \(L_1, \ldots, L_T\), the offline formulation and its dual LP are given in Figure 1. As written, the dual LP includes a change of variable; see Appendix B for details. Constraints (1) state that the work done by the allocation cannot be greater than the loads. Constraints (2) limit the work performed at time \(t\) to \(1 - \varepsilon\) or less. The LP \((D_\varepsilon)\) will be of special importance in the analysis. Using our algorithm, we will construct a dual feasible solution.

Before proceeding, we present a characterization of the solution of \((P_\varepsilon)\). Observe that \((P_\varepsilon)\) is feasible and bounded since the feasible region is a non-empty polytope. Let \(v_{P_\varepsilon}\) be the optimal value of \((P_\varepsilon)\). The following proposition gives a simple characterization of \(v_{P_\varepsilon}\); the proof appears in Appendix B.

**Proposition 2** \(v_{P_\varepsilon} = \min_{0 \leq t \leq T} \left( \sum_{s=1}^{t} \sum_{i} L_s(i) + (1 - \varepsilon)(T - t) \right)\).

3.2 Work Maximization

In this section we prove Theorem 2. Our first Lemma characterizes the implications of the update rules for users in both cases. The proof follows from a careful analysis of the dynamics using the KL-divergence and appears in Appendix B.

The first result of the lemma says that when rule 1 is applied, all active users, i.e., users in \(A_t\), receive a multiplicative boost in their allocations. On the other hand, the second result of the lemma states that when rule 2 is applied, all users with allocation over their SLAs, i.e., users in \(U_t\), mildly decrease their allocations.

**Lemma 1** Let \(c = \varepsilon N (1 - e^{-\eta})\). Then Algorithm 1 satisfies the following guarantee:

1. Suppose \(\sum_{i \in A_t} h_t(i) < 1 - \varepsilon\). If \(i \in A_t\), then \(h_{t+1}(i) \geq h_t(i)(1 + c)\).
2. Suppose \(\sum_{i \in A_t} h_t(i) \geq 1 - \varepsilon\). If \(i \in U_t\), then \(h_{t+1}(i) \geq h_t(i)(1 - 2\varepsilon c)\).

**Proof of Theorem 2** Given loads \(L_1, \ldots, L_T \in \mathbb{R}_+^N\), consider the following \(\{0,1\}\)-matrix \(M\) of dimension \(N \times T\) that encodes the information about the status of queues obtained while running...
Algorithm \( M_{i,t} \) for horizon \( T \):
\[
M_{i,t} = \begin{cases} 
0 & \text{if queue is empty at } t, Q_t(i) = 0, \\
1 & \text{if queue is not empty at } t, Q_t(i) > 0.
\end{cases}
\]

Let \( \tilde{s} = \frac{\ln(N/s)}{\varepsilon c} \), where \( c \) is defined in Lemma \( \ref{lemma:tilde_s} \). Now, pick \( s^* \) to be the maximum non-negative integer \( s \) (could be 0) such that
\[
\sum_{t=1}^{s} \sum_{i} L_t(i) \leq \sum_{t=1}^{s+\tilde{s}} \sum_{i} w_t(i) \quad \tag{7}
\]

**Claim 3.1** Consider any block of time \([r, r + \tilde{s}]\) where \( r > s^* \), then there exists a user \( i \) such that \( M_{i,r'} = 1 \) for all \( r' \in [r, r + \tilde{s}] \).

**Proof** Suppose not. Then we claim that \( r \) satisfies condition (7). Consider any user \( i \) and let \( r' \in [r, r + \tilde{s}] \) be such that \( M_{i,r'} = 0 \). Then work done by the user up to time \([r + \tilde{s}]\) is at least
\[
\sum_{t=1}^{r+\tilde{s}} w_t(i) \geq \sum_{t=1}^{r'} w_t(i) = \sum_{t=1}^{r'} L_t(i) \geq \sum_{t=1}^{r} L_t(i).
\]
Now summing over all \( i \), we get the desired contradiction. \( \square \)

We now prove the following claim that shows that the algorithm ensures that, on average, the total resource utilization after \( s^* \) is close to \( 1 - 6\varepsilon \). The proof of the Claim relies on Lemma \( \ref{lemma:tilde_s} \) and appears in Appendix \( \ref{appendix:proof_of_main_result} \).

**Claim 3.2** Let \( L = [r, r + \tilde{s}] \) where \( r > s^* \) be a consecutive block of \( \tilde{s} \) timesteps and let \( L' = \{ t \in L : \sum_{j \in A_t} h_t(j) < 1 - \varepsilon \} \) be the time steps in \( L \) with low utilization. Then \( |L'| \leq 5\varepsilon \tilde{s} \) and therefore,
\[
\sum_{t=s^*+1}^{T} \sum_{i} w_t(i) \geq (1 - 6\varepsilon)(T - s^*) - \tilde{s}.
\]
Now, consider the following feasible dual solution of \((D_{6\varepsilon})\): \( \gamma_t(i) = 1, \beta_t = 0 \) for all users \( i \) and \( t = 1, \ldots, s^* \), and \( \gamma_t(i) = 0, \beta_t = 1 \) for all users \( i \) and \( t = s^* + 1, \ldots, T \). Observe that \( \sum_{t=1}^{T} \beta_t = T - s^* \). For \( h^*_1, \ldots, h^*_T \) optimal \((1 - 6\varepsilon)\)-allocations we obtain
\[
\text{work}_{h^*_1, \ldots, h^*_T} \leq v_{\text{dual}}(\gamma_1, \ldots, \gamma_T, \beta_1, \ldots, \beta_T) \quad \text{(weak duality)}
\]
\[
= \sum_{t=1}^{s^*} \sum_{i} L_t(i) + (1 - 6\varepsilon)(T - s^*)
\leq \sum_{t=1}^{s^*+\tilde{s}} \sum_{i} w_t(i) + (1 - 6\varepsilon)(T - s^*) \quad \text{(s* choice)}
\leq \sum_{t=1}^{s^*} \sum_{i} w_t(i) + \tilde{s} + \sum_{t=s^*+1}^{T} \sum_{i} w_t(i) + \tilde{s} \quad \text{(Claim 3.2)}
\leq \text{work}_{\text{alg}} + 2N\varepsilon^{-2} \frac{\ln(N/\varepsilon)}{1 - e^{-\eta}}.
\]
where we have used \( \sum_{t=s^*+1}^{T} w_t(i) \leq \tilde{s} \) and the definition of \( \tilde{s} \). \( \square \)

By optimizing for \( \epsilon \) when the horizon \( T \) is known, we obtain the following regret-style guarantee.
Corollary 1 For $\eta \geq 1$ and $\varepsilon = \frac{N^{1/3}}{T^{1/3}}$, Algorithm 1 guarantees

\[ \text{work}_{h_1, \ldots, h_T} - \text{work}_{\text{alg}} \leq O(N^{1/3}T^{2/3} \log(NT)), \]

where $h_1^*, \ldots, h_T^*$ are the optimal offline dynamic 1-allocations.

Corollary 1 follows directly from Theorem 1 and states that our upper bound is tight up to a logarithmic factor by the right choice of $\varepsilon$. Indeed, Algorithm 1 has a canonical hard input: overload one user. If at each time, we load 1 to user 1 and 0 to the rest of users, we will have a loss of $\varepsilon (1 - \frac{1}{N}) T \geq \frac{2}{T}$ compared against the best offline dynamic 1-allocations. By choosing $\varepsilon = \frac{N^{1/3}}{T^{1/3}}$, we obtain the lower bound.

### 3.3 SLA Satisfaction

In this section, we prove Theorem 3. Recall that $e^{3\eta} = 1 + \frac{e^2}{N}(1 - e^{-\eta})$, $D_t = \{i : h_t(i) \leq \beta(i)\}$ and $U_t = [N] \setminus D_t$. Observe that for $\sum_{i \in A_t} h_t(i) \geq 1 - \varepsilon$, the set $U_t$ corresponds to the set of penalized users and the set $D_t$ corresponds to the set of non-penalized users. Analogous to Lemma 1, we have the following Lemma whose proof appears in Appendix 1.

**Lemma 2** Suppose $\varepsilon \leq \frac{1}{10}$, $\eta \geq 1$ and $\sum_{i \in A_t} h_t(i) \geq 1 - \varepsilon$. If $U_t \cap \overline{S}_{t+1} \neq \emptyset$, then, for any $i \in D_t$ we have

\[ h_{t+1}(i) \geq h_t(i)(1 + c') \]

with $c' = \frac{e^3}{4N^2}$.

**Proposition 3** Suppose that $\sum_{i \in A_t} h_t(i) \geq 1 - \varepsilon$ and for all users $i$ we have $\beta(i) \geq 2\frac{c'}{N}$. Then, $U_t \cap S_{t+1} = \emptyset$.

This last result implies that if the algorithm is using almost all the resources but some user is penalized, then we will always have $U_t \cap \overline{S}_{t+1} \neq \emptyset$. Under this hypothesis, Lemma 2 will be true.

**Proof of Theorem 3** Let $\tilde{s} = \left\lceil \frac{\log(N/\varepsilon)}{\log(1+c')} \right\rceil$, where $c'$ is defined in Lemma 2. Now, we proceed by induction on $t$ to prove that $\sum_{\tau = 1}^{t+\tilde{s}} w_r(i) \geq \sum_{\tau = 1}^{t} w_r'(i)$, where $w_r'$ is the work done by the allocations $h_1', \ldots, h_T'$.

Clearly, the case $t = 0$ is direct.

Take $t \geq 1$ and suppose the result true for $t - 1$. If there exists $r \in [t, t+\tilde{s}]$ such that user $i$’s queue is empty, then

\[ \sum_{\tau = 1}^{t+\tilde{s}} w_r(i) \geq \sum_{\tau = 1}^{r} w_r(i) = \sum_{\tau = 1}^{t} L_r(i) \geq \sum_{\tau = 1}^{t} w_r'(i). \]

Therefore, assume that for all $\tau \in [t, t+\tilde{s}]$ we have that user $i$’s queue is non-empty. By induction hypothesis

\[ \sum_{\tau = 1}^{t-1+\tilde{s}} w_r(i) \geq \sum_{\tau = 1}^{t-1} w_r'(i). \]

In order to complete the proof, we need to prove that $w_{t+\tilde{s}}(i) \geq w_r'(i)$. We proceed as follows. Suppose that for all $\tau \geq t$ we have $w_r(i) < \beta(i)$. Since $h_{t}(i) = w_r(i)$ (queue is non-empty) then, there is a user $j_r \neq i$ with $h_r(j_r) > \beta(j_r)$. Therefore, $h_r(i)$ always increases multiplicative by a
rate of $(1+c)$ if $\sum_{k \in A_t} h_{\tau}(k) < 1 - \varepsilon$ by using Lemma 1 or at a rate $(1+c')$ if $\sum_{k \in A_t} h_{\tau}(k) \geq 1 - \varepsilon$ by Lemma 2. In any case, the rate is at least $(1+c')$. Therefore,

\[ w_{t+\tilde{s}}(i) \geq \frac{\varepsilon}{N} (1+c') \tilde{s} \geq 1 \geq \beta(i). \]

A contradiction. From the previous analysis we obtain the existence of $\tau^* \in [t, t + \tilde{s}]$ such that $w_{\tau^*}(i) \geq \beta(i)$. By Lemma 2 and using the previous arguments, we can show that the allocation $h_{\tau^*}(i)$ will never go below $(1-\varepsilon c)\beta(i)$ for all $\tau \geq \tau^*$. In particular $w_{t+\tilde{s}}(i) \geq (1-\varepsilon c)\beta(i) \geq w_t'(i)$. □

The following corollary states that the queues of each user will not be much larger as if they would have received almost their corresponding SLAs.

**Corollary 2** For any user $i$ and any dynamic allocation such that $h_t'(i) \leq (1-\varepsilon c)\beta(i)$ we have $Q_{t}'(i) + \beta(i) \cdot \tilde{s} \geq Q_t(i)$, where $Q$ is the underlying queue used by algorithm 1 and $Q'$ is the queue used by the allocations $H_t'$.

**Proof** By Theorem 3 we obtain $\sum_{\tau=1}^{t} w_{\tau}(i) \geq \sum_{\tau=1}^{t-\tilde{s}} w_{\tau}'(i) \geq \sum_{\tau=1}^{t} w_{\tau}'(i) - \beta(i) \tilde{s}$, where we have used that $w_{\tau}(i) < \beta(i)$. Since $\sum_{\tau=1}^{t} w_{\tau}(i) = \sum_{\tau=1}^{t} L_{\tau}(i) - Q_{\tau}(i)$ we obtain the result. □

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A Projecting on $\Delta_\varepsilon$

**Proposition 4** Let $y \in \mathbb{R}^N_+$ and let $x = \pi_{\Delta_\varepsilon}(y)$. If $S = \{i : x(i) = \frac{\varepsilon}{N}\}$, then:
(a) If $y(1) \leq y(2) \leq \cdots \leq y(N)$, then $S = \{1, \ldots, k\}$ for some $k \geq 0$.

(b) $x(i) = y(i)e^{\mu_i}C$, where $C = \left(\frac{1-\frac{\varepsilon}{N}|S|}{\sum_{j \in y(S)} 1}\right)$, $\mu_i \geq 0$ for all $i$ and $\mu_i = 0$ for $i \notin S$.

(c) $x$ can be computed in $O(N \log N)$ time.

**Proof** Let $y \in \mathbb{R}_+^N$. The projection of $y$ on $\Delta_e$ corresponds to the solution of the non-linear problem

$$\text{(Q)} \quad \min \sum_i x(i) \ln \left(\frac{x(i)}{y(i)}\right) \quad \begin{array}{l} \sum_i x(i) = 1 \\ x(i) \geq \varepsilon/N \end{array}$$

Its lagrangian [22] is

$$\mathcal{L}(x, \lambda, \mu) = \sum_i x(i) \ln \left(\frac{x(i)}{y(i)}\right) - \lambda \left(\sum_i x(i) - 1\right) - \sum_i \mu_i \left(x(i) - \frac{\varepsilon}{N}\right).$$

Using the FO conditions:

$$\forall i : x(i) = y(i)e^{\mu_i + \lambda - 1} = y(i)e^{\mu_i}C.$$  

and the SO conditions:

$$\mu_i \geq 0, \forall i, \quad \text{and} \quad x(i) > \frac{\varepsilon}{N} \implies \mu_i = 0.$$  

Let $S = \{i : x(i) = \varepsilon/N\}$ and $T = [N] \setminus S$. Then, using $\sum_i x(i) = 1$ we obtain

$$e^{\lambda - 1} = \frac{1 - \frac{\varepsilon}{N}|S|}{\sum_{i \in T} y(i)}.$$  

This proves part (b). Now, suppose we have $y(1) \leq \cdots \leq y(N)$. If $i, j \in T$, then $x(i) = y(i)e^{\lambda - 1}$ and $x(j) = y(j)e^{\lambda - 1}$ and then

$$x(i) \leq x(j) \iff y(i) \leq y(j).$$  

That is, in $T$ the variables preserve their ordering.

If $i \in S$ and $j \in T$, then $y(i)e^{\lambda - 1 + \mu_i} = x(i) = \frac{\varepsilon}{N} < x(j) = y(j)e^{\lambda - 1}$, which implies $y(i) < y(j)$ using that $\mu_i \geq 0$. Now, let $k = \min\{i \in T\}$ which is a well-defined number using constraint $\sum_{i=1}^N x(i) = 1$. We claim that for any $j \geq k$, $j \in T$, that is, $T$ corresponds to the interval $[k, N]$. By contradiction, suppose that $j > k$ does not belong to $T$, then $y(j) < y(k)$ by previous calculus. However $y(j) \geq y(k)$ by the ordering of $y$. A contradiction. With this, the algorithm to project is clear, we sort $y$ and then we test increasingly the possible set $S = \{1, \ldots, k - 1\}$ for $k = 1, \ldots, N$ and select the best candidate. This proves (a). In the following paragraphs we prove that the first feasible solution found in this process is the right one.

Observe that once $S = \{1, \ldots, k\}$ is feasible, then $S' = \{1, \ldots, j\}$ remains feasible for all $j \geq k$. Indeed, if $S = \{1, \ldots, k\}$ is feasible, then

$$1 = \frac{\varepsilon}{N}k + \sum_{i \in T} x(i).$$  

13
Now, increasing $S$ to $S' = \{1, \ldots, k+1\}$ means that we pick $x(k+1) > \frac{\varepsilon}{N}$ and we decrease it to $\frac{\varepsilon}{N}$. Therefore, $x(k+2), \ldots, x(N)$ must increase. Therefore, $S'$ remains feasible. The proof for general case $j \geq k$ follows by induction.

Now, we claim that if $S = \{1, \ldots, k\}$ is feasible, then $S' = \{1, \ldots, k+1\}$ cannot have better optimal value. Indeed,

$$
\sum_{i=1}^{k+1} \frac{\varepsilon}{N} \ln \frac{\varepsilon}{Ny(i)} + \left(1 - \frac{\varepsilon}{N}(k+1)\right) \ln \frac{1 - \frac{x(k+1)}{N}}{\sum_{i \geq k+2} y(i)} - \sum_{i=1}^{k} \frac{\varepsilon}{N} \ln \frac{\varepsilon}{Ny(i)} - \left(1 - \frac{\varepsilon}{N}k\right) \ln \frac{1 - \frac{x(k)}{N}}{\sum_{i \geq k+1} y(i)}
$$

$$
= \frac{\varepsilon}{N} \ln \frac{\varepsilon}{Ny(k+1)} + \left(1 - \frac{\varepsilon}{N}(k+1)\right) \ln \frac{1 - \frac{x(k+1)}{N}}{\sum_{i \geq k+2} y(i)} - \left(1 - \frac{\varepsilon}{N}k\right) \ln \frac{1 - \frac{x(k)}{N}}{\sum_{i \geq k+1} y(i)}
$$

The function $f(x) = x \ln x$ is convex. Now, pick $x = \frac{\varepsilon}{Ny(k+1)}$, $y = \frac{1 - \frac{x(k+1)}{N}}{\sum_{i \geq k+2} y(i)}$ and $\lambda = \frac{y(k+1)}{\sum_{i \geq k+1} y(i)}$. Then

$$
\lambda x + (1 - \lambda)y = \frac{y(k+1)}{\sum_{i \geq k+1} y(i)} \left(\frac{\varepsilon}{N} \ln \frac{\varepsilon}{Ny(k+1)}\right) + \sum_{i \geq k+2} y(i) \left(1 - \frac{\varepsilon}{N}(k+1)\right) \frac{1 - \frac{x(k+1)}{N}}{\sum_{i \geq k+2} y(i)}
$$

$$
= \frac{1 - \frac{x(k+1)}{N}}{\sum_{i \geq k+1} y(i)}.
$$

Then, using the convexity of $f$ we obtain the result. This implies that the first feasible prefix $S$ that we find is the optimal one. Therefore, by ordering $y$ in $O(N \log N)$ time and then running binary search we can find $S$ in $O(N \log N)$ time. This finished the proof of (c). \qed

## B Omitted Proofs

### B.1 Proofs of Section 3.1

Here we present dual stated in the offline formulation of the maximum work problem. We have the LP

$$
\max \sum_{i=1}^{N} \sum_{t=1}^{T} w_t(i)
\begin{align*}
(P_\varepsilon) \quad \sum_{t=1}^{T} w_s(i) & \leq \sum_{s=1}^{t} L_s(i) \quad \forall t, i \ (1) \\
\sum_{i=1}^{N} w_t(i) & \leq 1 - \varepsilon \quad \forall t \ (2) \\
w_t & \geq 0 \quad \forall t
\end{align*}
$$

Using the variables $\alpha_t(i)$ for constraint (1) and $\beta_t$ for constraint (2) we obtain the dual

$$
\min \sum_{i=1}^{N} \sum_{t=1}^{T} \alpha_t(i) \sum_{s=1}^{t} L_s(i) + (1 - \varepsilon) \sum_{t=1}^{T} \beta_t
\begin{align*}
(D_\varepsilon) \quad \sum_{i=1}^{N} \alpha_s(i) + \beta_t & \geq 1 \quad \forall t, i \ (1') \\
\alpha, \beta & \geq 0
\end{align*}
$$

Using the change of variable $\gamma_t(i) = \sum_{s=t}^{N} \alpha_s(i)$ we obtain the stated dual

$$
\min \sum_{i=1}^{N} \sum_{t=1}^{T} L_t(i) \gamma_t(i) + (1 - \varepsilon) \sum_{t=1}^{T} \beta_t
\begin{align*}
(D_\varepsilon) \quad \gamma_t & \geq \gamma_{t+1} \quad \forall t \\
\beta, \gamma & \geq 0
\end{align*}
$$

Proof of Proposition 2
We prove each inequality separately. Let \(0 \leq t^* \leq T\) be such that \(\sum_{s=1}^{t^*} \sum_i L_s(i) + (1-\varepsilon)(T-t^*) = \min_{0 \leq t \leq T} \sum_{s=1}^{t} \sum_i L_s(i) + (1-\varepsilon)(T-t)\). Consider the dual solution \((\beta, \gamma)\) such that \(\gamma_t = 1, \beta_t = 0\) for \(t = 1, \ldots, t^*\) and \(\gamma_t = 0, \beta_t = 1\) for \(t = t^* + 1, \ldots, T\). Then, by weak duality,

\[
v_{P_\varepsilon} \leq v_{\text{dual}}(\beta, \gamma) = \sum_{s=1}^{t^*} \sum_i L_s(i) + (1-\varepsilon)(T-t^*) = \min_{0 \leq t \leq T} \sum_{s=1}^{t} \sum_i L_s(i) + (1-\varepsilon)(T-t).
\]

Now, consider the greedy algorithm that, in each iteration, gives enough allocation to the users in order to complete their work starting with user 1, then user 2, and so on. We restrict the algorithms' allocations to \((1-\varepsilon)\)-allocations. We denote by \(w_t\) the vector of work done at time \(t\). Let \(t^*\) be the maximum non-negative \(t\) such that \(\sum_i w_t(i) < 1 - \varepsilon\). Observe that \(\sum_{s=1}^{t^*} \sum_i w_s(i) = \sum_{s=1}^{t^*} \sum_i L_s(i)\). Then

\[
\min_{0 \leq t \leq T} \sum_{s=1}^{t} \sum_i L_s(i) + (1-\varepsilon)(T-t) \leq \sum_{s=1}^{t^*} \sum_i L_s(i) + (1-\varepsilon)(T-t^*) = \text{work}_{\text{greedy}} \leq v_{P_\varepsilon},
\]

since \(v_{P_\varepsilon}\) is the optimal solution.

\[\square\]

**Remark 1** This max-min result shows that the greedy algorithm is optimal for solving \((P_\varepsilon)\) and also shows how to compute the dual variables. Finally, solving \((P_\varepsilon)\) can be done efficiently in \(O(NT)\) by running the greedy algorithm.

### B.2 Proofs of Section 3.2

In what follows, we denote by \(S_t\) the users with allocation \(\frac{x}{N}\) at time \(t\).

**Proof of Lemma 1**

1. Suppose \(B_t \cap \overline{S}_{t+1} \neq \emptyset\), then,

\[
\sum_{i \in S_{t+1}} \hat{h}_{t+1}(i) = \sum_{i \in A_t \cap S_{t+1}} h_t(i) + e^{-\eta} \sum_{i \in B_t \cap \overline{S}_{t+1}} h_t(i)
= \sum_{i \in S_{t+1}} h_t(i) + (1-e^{-\eta}) \sum_{i \in B_t \cap \overline{S}_{t+1}} h_t(i)
= 1 - \sum_{i \in S_{t+1}} h_t(i) + (1-e^{-\eta}) \sum_{i \in B_t \cap \overline{S}_{t+1}} h_t(i)
\leq 1 - \frac{\varepsilon}{N} |S_{t+1}| + (1-e^{-\eta}) \frac{\varepsilon}{N}
\]

and therefore,

\[
\frac{1 - \frac{\varepsilon}{N} |S_{t+1}|}{\sum_{i \in S_{t+1}} \hat{h}_{t+1}(i)} \geq \frac{1 - \frac{\varepsilon}{N} |S_{t+1}|}{1 - \frac{\varepsilon}{N} |S_{t+1}| - \frac{\varepsilon}{N}(1-e^{-\eta})}
\geq \frac{1}{1 - \frac{\varepsilon}{N}(1-e^{-\eta})}
\geq 1 + \frac{\varepsilon}{N}(1-e^{-\eta}) \quad (\frac{1}{1-x} \geq 1 + x \text{ for } x \in (0, 1))
\]
Now, assume $B_t \subseteq S_{t+1}$. Then, $A_t \supseteq \overline{S}_{t+1}$. With this, we obtain
\[ \sum_{i \in \overline{S}_{t+1}} \hat{h}_{t+1}(i) \leq \sum_{i \in A_t} h_t(i) < 1 - \varepsilon. \]

Therefore,
\begin{align*}
\frac{1 - \frac{\varepsilon}{N}|S_{t+1}|}{\sum_{i \in \overline{S}_{t+1}} \hat{h}_{t+1}(i)} & \geq \frac{1 - \frac{\varepsilon}{N}|S_{t+1}|}{1 - \varepsilon} \\
& = 1 + \frac{\varepsilon(1 - \frac{1}{N}|S_{t+1}|)}{1 - \varepsilon} \\
& \geq 1 + \frac{\varepsilon}{(1 - \varepsilon)N} \quad (|S_{t+1}| \leq N - 1)
\end{align*}

Since \( \frac{\varepsilon}{(1 - \varepsilon)N} \geq \frac{\varepsilon}{N} (1 - e^{-\eta}) \), by proposition we conclude that
\[ h_{t+1}(i) = h_t(i)e^{\mu_i} \frac{1 - \frac{\varepsilon}{N}|S_{t+1}|}{\sum_{k \in \overline{S}_{t+1}} \hat{h}_{t+1}(k)} \geq h_t(i) \left( 1 + \frac{\varepsilon}{N} (1 - e^{-\eta}) \right), \]

and the allocation increases multiplicatively.

2. Since \( \sum_{i \in A_t} h_t(i) \geq 1 - \varepsilon \), \( h_t(i) > \beta(i) \) imply \( \hat{h}_{t+1}(i) = h_t(i)e^{-\eta'} \). Therefore, by proposition the projection holds
\[ h_{t+1}(i) = \hat{h}_{t+1}(i)e^{\mu_i}C \geq h_t(i)e^{-\eta'} C \]

where we have used \( \mu_i \geq 0 \). Recall that \( C = \frac{1 - \frac{\varepsilon}{N}|S_{t+1}|}{\sum_{k \in \overline{S}_{t+1}} \hat{h}_{t+1}(k)} \), so it is enough to lower bound the expression \( e^{-\eta'} C \). Applying rule 2 we have
\begin{align*}
\sum_{k \in \overline{S}_{t+1}} \hat{h}_{t+1}(k) &= e^{\eta'} \sum_{k \in D_t \cap S_{t+1}} h_t(k) + \sum_{k \in U_t \cap S_{t+1}} h_t(k) \\
&= (e^{\eta'} - 1) \sum_{k \in D_t \cap \overline{S}_{t+1}} h_t(k) + \sum_{k \in U_t \cap \overline{S}_{t+1}} h_t(k) \\
&= (e^{\eta'} - 1) \sum_{k \in D_t \cap \overline{S}_{t+1}} h_t(k) + 1 - \sum_{k \in \overline{S}_{t+1}} h_t(k) \\
&\leq 1 - \frac{\varepsilon}{N}|S_{t+1}| + (e^{\eta'} - 1).
\end{align*}

Hence
\begin{align*}
e^{-\eta'} C &= \frac{1 - \frac{\varepsilon}{N}|S_{t+1}|}{e^{\eta'} \sum_{k \in \overline{S}_{t+1}} \hat{h}_{t+1}(k)} \\
&\geq \frac{1 - \frac{\varepsilon}{N}|S_{t+1}|}{1 - \varepsilon + (e^{\eta'} - 1)} \\
&\geq \frac{1 - \varepsilon}{e^{\eta'} - \varepsilon} \\
&\geq 1 - 2\varepsilon c. \quad \text{(by the choice of } \eta')
\end{align*}
Proof of Claim 3.2 Now, let \([s^* + 1, \ldots, T]\) and let us divide this interval into blocks of length \(\tilde{s}\) with a possible last piece of length of length at most \(\tilde{s}\). Let \(L\) be one of these blocks and let \(i\) be the user given by claim 3.1, that is, \(M_{i,r} = 1\) for all \(r \in L\). Consider

\[ L' = \{ t \in L : \sum_{j \in A_t} h_t(j) < 1 - \varepsilon \}. \]

Observe that \(L'\) are the times in the block \(L\) where rule 1 applies. Therefore, using Lemma 1, user \(i\) increases her allocation multiplicatively in \(L'\) by a factor of \((1 + c)\). Observe that for \(t \notin L'\), user’s allocation can increase or decrease depending on \(h_t(i)\). However, by Lemma 2 we know that \(h_t(i)\) will not decrease by a huge amount. Let \(k' = |L'|\), then \(i\) increases her allocation for \(k'\) times and decreases it for at most \(\tilde{s} - k'\) times. Therefore, \(k'\) maximum value is such that

\[ \frac{\varepsilon}{N}(1 + c)^{k'}(1 - 2\varepsilon)^{\tilde{s} - k'} = 1 \]

and therefore,

\[
 k' \leq \frac{\ln(N/\varepsilon) + \tilde{s} \ln(1 - 2\varepsilon)^{-1}}{\ln((1 + c)/(1 - 2\varepsilon))} \\
 \leq \frac{1 + c}{c(1 + 2\varepsilon)} \ln(N/\varepsilon) + \frac{2\varepsilon(1 + c)}{(1 + 2\varepsilon)(1 - 2\varepsilon) \tilde{s}} \\
 \leq \frac{\varepsilon(1 + c)}{(1 + 2\varepsilon) \tilde{s}} + \frac{2\varepsilon(1 + c)}{(1 + 2\varepsilon)(1 - 2\varepsilon) \tilde{s}} \\
 = \varepsilon \left( 1 + \frac{2}{1 - 2\varepsilon} \right) \frac{1}{2} \tilde{s} \\
 \leq 5\varepsilon \tilde{s}. 
\]

Hence, \(L'\) is at most a fraction of \(\tilde{s}\) and with this

\[
 \sum_{t \in L} \sum_{i} w_t(i) \geq (1 - \varepsilon)(\tilde{s} - k') \geq (1 - \varepsilon)(1 - 5\varepsilon) \tilde{s} \geq (1 - 6\varepsilon)|L|.
\]

Summing over all blocks we conclude the desired result. □

B.3 Proof of Section 3.3

Proof of Lemma 2 Following the proof of the first part in Lemma 1 we obtain

\[ h_{t+1}(i) \geq h_t(i) \left( 1 + \frac{\varepsilon}{N}(1 - e^{-\eta'}) \right). \]
Therefore,
\[
\frac{\varepsilon}{N}(1 - e^{-\eta'}) = \frac{\varepsilon}{N} \left(1 - \frac{1}{1 + \frac{\varepsilon^2}{N}(1 - e^{-\eta})}\right) \\
= \frac{\varepsilon}{N} \frac{\varepsilon^2(1 - e^{-\eta})}{1 + \frac{\varepsilon^2}{N}(1 - e^{-\eta})} \\
\geq \frac{\varepsilon^3}{4N^2}.
\]

\[\square\]

**Proof of Proposition**

First, we prove that if \(i \in S_{t+1}\), then \(h_t(i) \leq \frac{\varepsilon}{N}e^{\eta'}\). Indeed, by proposition we have
\[
\frac{\varepsilon}{N} = h_{t+1}(i) = h_t(i)e^{\mu_tC} \geq h_t(i)e^{-\eta'C}.
\]

Observe that
\[
\sum_{j \in S_{t+1}} \hat{h}_{t+1}(j) \leq \sum_{j \in S_{t+1}} h_t(j) = 1 - \sum_{j \in S_{t+1}} h_t(j) \leq 1 - \frac{\varepsilon}{N}|S_{t+1}|,
\]
and from here we deduce \(C \geq 1\). Therefore, \(h_t(i) \leq \frac{\varepsilon}{N}e^{\eta'}\). Now, \(i \in B_t\) implies \(h_t(i) > \beta(i)\) and since
\[
\frac{\varepsilon}{N}e^{\eta'} \leq \frac{\varepsilon}{N}\left(1 + \frac{\varepsilon^2}{N}\right) < 2\frac{\varepsilon}{N} \leq \beta(i)
\]
we obtain that \(i \notin S_{t+1}\).

\[\square\]

**C Greedy Online Algorithm**

In this section, we prove that the following greedy allocation strategy is almost optimal in work maximization. The algorithm divides the users into 3 categories: \(A\) non-empty queue users with non-zero allocation, \(B\) non-empty queue users with zero allocation and \(I\) empty queue users with zero allocation. At time \(t\), a user \(i \in A\) is left in \(A\) if she still has non-empty queue, otherwise we will move her to \(I\); a user \(i \in I\) will be moved to \(B\) if her queue is becomes non-empty, otherwise she will remain in \(I\); finally, if all users from \(A\) are moved to \(I\), then we will move all \(B\) to \(A\), otherwise we will leave \(B\) untouched. In any case, we will distribute uniformly among the users that remain in \(A\).

Users move from \(A\) to \(I\), \(I\) to \(B\) and \(B\) to \(A\). Let \(w_t\) be the work done by the algorithm and let \(w_t'\) be the optimal offline work.

**Theorem 4** For any loads \(L_1, \ldots, L_T \in \mathbb{R}_{\geq 0}^{N}\), and any \(\varepsilon > 0\), this greedy Algorithm guarantees
\[
\sum_{t=1}^{T} \sum_{i} w_t(i) + \frac{2N^2}{\varepsilon} \geq \sum_{t=1}^{T} \sum_{i} w_t'(i)
\]
where \(w_t'\) is the work done by the optimal offline sequence of \((1 - 2\varepsilon/N)\)-allocations.

18
Proof Let $t^*$ be the maximum $t \geq 0$ such that $\sum_{s=1}^{t+\lfloor N^2 \epsilon \rfloor} \sum_i w_s(i) \geq \sum_{s=1}^t \sum_i L_s(i)$. By claim 3.1, we know that each interval $[r, r + N^2/\epsilon]$, with $r > t^*$, has a user with no-empty queue. As in claim 3.2 we divide the interval $[t^* + 1, T]$ into blocks of length $N^2/\epsilon$ with a last block of length at most $N^2/\epsilon$. Pick any of these blocks, say $L$, and let $L' = \{ t \in L : \sum_i w_t(i) < 1 \}$. It is easy to see that $|L'| \leq 2N$ and therefore, summing over all block, we have $\sum_{i \geq t^* + 1} \sum_i w_t(i) + N^2/\epsilon \geq (T - t^*)(1 - 2\epsilon/N)$. The conclusion follows applying weak duality in $(P_{2\epsilon/N})$. □

Remark 2 Against the best 1-allocations we can optimize $\epsilon$ and obtain $\epsilon = \sqrt{NT}$. This greedy strategy will be $O(\sqrt{NT})$ far from the optimal dynamic work. Observe that this matches the lower bound in Theorem 5.

D Lower Bound

Theorem 5 For any online deterministic algorithm $A$ setting at each time 1-allocations, with an underlying queuing system, and with the same limited feedback as Algorithm 1 there exists a sequence of online loads $L_1, \ldots, L_T$ such that

$$\text{work}_{h^*_1, \ldots, h^*_T} - \text{work}_A = \Omega \left( \sqrt{T} \right),$$

where $h^*_1, \ldots, h^*_T$ are the optimal offline dynamic 1-allocations.

Proof We consider the case with $N = 2$ users, the general case reduces to $N = 2$ by loading jobs only in two users. Let $A$ be an online algorithm for allocating a divisible resource for 2 users and with underlying queuing system and limited feedback. Without loss of generality, we can assume that the allocations sets by $A$ always add up to 1 at every time step.

We will construct a sequence of loads $L_t = (L_t(1), L_t(2))$ that at every time will add up to 1. This will ensure that the overall work done by the optimal offline dynamic policy will be $T$. On the other hand, we will show that this sequence of loads will lead to large queue length for at least one of the users. The main ingredient is to use the fact the algorithm receives limited feedback about the state of the system, i.e., which users have an empty queue. In particular, this implies that if there are two distinct set of load vectors $L_t$ and $L'_t$ for some interval $t \in [r, s]$ such that the queues remain non-empty on both these sequences, then the resource allocation to the users in the two load sequences must be identical.

We will divide the time window $[1, T]$ into phases. Each phase will begin with a configuration of queues, say $Q = (Q(1), Q(2))$, where one of the queues is empty and the other one nonempty. We set $q = Q(1) + Q(2)$ and we denote by $q_i$ the $q$ at phase $i$. We define $q_0 = 0$. We will prove that at the end of each phase $i \geq 1, q_{i+1} \geq q_i + \frac{1}{T}$ with all $q_{i+1}$ cumulated in one queue and the other queue empty.

Initially, the algorithm has a fixed deterministic allocation $h_1 = (h_1(1), h_1(2))$. If $h_1(1) \leq h_1(2)$, then we load $L_1 = (0, 1)$. Otherwise, we load $L_1 = (1, 0)$. In any case, we have $q_1 \geq \frac{1}{2}$ and all $q_i$ in one queue.

Now, we will describe how the general phases work. For the sake of simplicity, we will describe the phase starting at time $t = 1$. We have queue configuration $Q = (Q(1), Q(2))$ with $q > 0$. By the initial phase, we can assume $q \geq 1/2$. Moreover, we can assume that only one of the queues is
nonempty, this point will be clear after we describe how the phase works and it is clearly true for phase 1. Phase \( i \) with \( q = q_i \) will last at most \( 2q_i + 2 \) time steps.

Suppose that \( Q(1) = 0 \) and \( Q(2) > 0 \). If \( h_1(1) = 1 \), then we load \( L_1 = (0, 1) \) and the phase ends with \( q \) increased by 1 and user 1’s queue empty. Therefore, we can assume \( h_1(1) < 1 \). Our first load will be \( L_1 = (h_1(1) + \varepsilon, h_2(2) - \varepsilon) \) with \( 0 < \varepsilon < \frac{1}{4} \) small enough that exists since \( h_1(1) < 1 \). The following loads will be \( L_t = h_t \), the allocation of \( \mathcal{A} \) at time \( t \). Observe that the first load will ensure that both users see nonempty queues until the end of the phase. Moreover, user 1 always have exactly \( \varepsilon \) remaining in her queue.

- If there is a time \( \tau^* \in [1, 2q + 1] \) such that \( h_{\tau^*}(1) \geq 1/2 \), then we change the load at time \( \tau^* \) for \( L'_{\tau^*} = (0, 1) \). This will increment \( q \) by at least \( 1/2 - \varepsilon \geq 1/4 \) and the phase ends. Observe that \( Q_{\tau^*+1}(1) = 0 \).

- We can assume now that for the loads \( L_t = h_t \) we always have \( h_t(1) < 1/2 \) for all \( t \in [2, 2q+1] \). We change the loads to \( L'_t = (1, 0) \) until time \( \tau^* \) in which user 2 empties her queue. Recall that the feedback of the algorithm is only the set of empty queues at every time step. Thus the behavior of \( \mathcal{A} \) under \( L_t \) and \( L'_t \) will be the same until time \( \tau^* \). Now, we change load \( L_{\tau^*} \) by \( L'_{\tau^*} = (1 - h_{\tau^*}(2) + Q_{\tau^*}(2) - \varepsilon', h_{\tau^*}(2) - Q_{\tau^*}(2) + \varepsilon') \) with \( 0 < \varepsilon' < 1/4 \) small enough. This will ensure that queue 2 will be exactly \( \varepsilon' \). Now, in an extra step, we load \( L'_{\tau^*+1} = (1, 0) \). Again, we have \( q \) increased by \( 1/4 \) and this ends the phase. Observe that \( Q_{\tau^*+2}(2) = 0 \).

The analysis is similar for \( Q(1) > 0 \) and \( Q(2) = 0 \). Observe that at the end of each phase, only one queue is nonempty and the other one is empty. In any case, we have the desired increment. With this, we can set the following recurrence,

\[
q_0 = 0, \quad q_{i-1} + \frac{1}{4} \leq q_i \quad \forall i \geq 1.
\]

We deduce that \( q_i \geq i/4 \). Now, let \( m \) be the number of phases. By construction, each phase last at most \( 2q_i + 2 \). Then

\[
T \leq \sum_{i=1}^{m} (2q_i + 2) \leq 40q_m^2,
\]

where we have used \( 4q_m \geq 4q_i \geq i \geq 1 \). From here we deduce that \( q_m \geq \sqrt{T/40} \).

Now, the work done by the algorithm and the unfulfilled work in the queues must add up the overall load. Then

\[
q_m + \text{work}_A = T = \text{work}_{n_1^* \ldots n_T^*},
\]

from which we obtain the result. \( \square \)