Dehn surgery, homology and hyperbolic volume

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If a closed, orientable hyperbolic 3–manifold $M$ has volume at most 1.22 then $H_1(M; \mathbb{Z}_p)$ has dimension at most 2 for every prime $p \neq 2, 7$, and $H_1(M; \mathbb{Z}_2)$ and $H_1(M; \mathbb{Z}_7)$ have dimension at most 3. The proof combines several deep results about hyperbolic 3–manifolds. The strategy is to compare the volume of a tube about a shortest closed geodesic $C \subset M$ with the volumes of tubes about short closed geodesics in a sequence of hyperbolic manifolds obtained from $M$ by Dehn surgeries on $C$.

1 Introduction

We shall prove:

**Theorem 1.1** Suppose that $M$ is a closed, orientable hyperbolic 3–manifold with volume at most 1.22. Then $H_1(M; \mathbb{Z}_p)$ has dimension at most 2 for every prime $p \neq 2, 7$, and $H_1(M; \mathbb{Z}_2)$ and $H_1(M; \mathbb{Z}_7)$ have dimension at most 3. Furthermore, if $M$ has volume at most 1.182, then $H_1(M; \mathbb{Z}_7)$ has dimension at most 2.

The bound of 2 for the dimension of $H_1(M; \mathbb{Z}_p)$ is sharp when $p$ is 3 or 5. Indeed, the manifolds $m003(-3,1)$, and $m007(3,1)$ from the list given in [10] have respective volumes 0.94\ldots and 1.01\ldots, and their integer homology groups are respectively isomorphic to $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_6$.

Apart from these two examples, the only example known to us of a closed, orientable hyperbolic 3–manifold with volume at most 1.22 is the manifold $m003(-2,3)$ from the list given in [10]. These three examples suggest that the bounds for the dimension of $H_1(M; \mathbb{Z}_p)$ given by Theorem 1.1 may not be sharp for $p \neq 3, 5$.

The proof of Theorem 1.1 depends on several deep results, including a strong form of the “log 3 Theorem” of Anderson, Canary, Culler and Shalen [4, 8]; the Embedded Tube Theorem of Gabai, Meyerhoff and N Thurston [9]; the Marden Tameness Conjecture,
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recently proved by Agol [1] and by Calegari and Gabai [7]; and an even more recent result due to Agol, Dunfield, Storm and W Thurston [3]. The strategy of our proof is to compare the volume of a tube about a shortest closed geodesic $C \subset M$ with the volumes of tubes about short closed geodesics in a sequence of hyperbolic manifolds obtained from $M$ by Dehn surgeries on $C$.

After establishing some basic conventions in Section 2, we carry out the strategy described above in Sections 3–6, for the case of manifolds which are “non-exceptional” in the sense that they contain shortest geodesics with tube radius greater than $(\log 3)/2$.

In Section 5, for the case of non-exceptional manifolds with volume at most $1.22$, we establish a bound of 3 for the dimension of $H_1(M; \mathbb{Z}_p)$ for any prime $p$. In Section 6, again for the case of non-exceptional manifolds with volume at most $1.22$, we establish a bound of 2 for the dimension of $H_1(M; \mathbb{Z}_p)$ for any odd prime $p$. In Section 7 we use results from [9] to handle the case of exceptional manifolds, and complete the proof of Theorem 1.1.

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2 Definitions and conventions

2.1 If $g$ is a loxodromic isometry of hyperbolic 3–space $\mathbb{H}^3$ we shall let $A_g$ denote the hyperbolic geodesic which is the axis of $g$. The cylinder about $A_g$ of radius $r$ is the open set $Z_r(g) = \{ x \in \mathbb{H}^3 \mid \text{dist}(x, A_g) < r \}$.

2.2 Suppose that $M$ is a complete, orientable hyperbolic 3–manifold. Let us identify $M$ with $\mathbb{H}^3/\Gamma$, where $\Gamma \cong \pi_1(M)$ is a discrete, torsion-free subgroup of $\text{Isom}_+ \mathbb{H}^3$. If $C$ is a simple closed geodesic in $M$ then there is a loxodromic isometry $g \in \Gamma$ with $A_g/\langle g \rangle = C$. For any $r > 0$ the image $Z_r(g)/\langle g \rangle$ of $Z_r(g)$ under the covering projection is a neighborhood of $C$ in $M$. For sufficiently small $r > 0$ we have

$$\{ h \in \Gamma \mid h(Z_r(g)) \cap Z_r(g) \neq \emptyset \} = \langle g \rangle.$$

Let $R$ denote the supremum of the set of $r$ for which this condition holds. We define tube$(C) = Z_R(g)/\langle g \rangle$ to be the maximal tube about $C$. We shall refer to $R$ as the tube radius of $C$, and denote it by tuberad$(C)$.

2.3 If $C$ is a simple closed geodesic in a closed hyperbolic 3–manifold $M$, it follows from [13], [2] that $M - C$ is homeomorphic to a hyperbolic manifold $N$ of finite volume having one cusp. The manifold $N$, which by Mostow rigidity is unique up to isometry, will be denoted drill$_C(M)$.
2.4 If \( C \) is a shortest closed geodesic in a closed hyperbolic 3–manifold \( M \), ie, one such that \( \text{length}(C) \leq \text{length}(C') \) for every other closed geodesic \( C' \), then in particular \( C \) is simple, and the notions of 2.2 and 2.3 apply to \( C \).

2.5 Suppose that \( N = \mathbb{H}^3/\Gamma \) is a non-compact orientable complete hyperbolic manifold of finite volume. Let \( \Pi \cong \mathbb{Z} \times \mathbb{Z} \) be a maximal parabolic subgroup of \( \Gamma \) (so that \( \Pi \) corresponds to a peripheral subgroup under the isomorphism of \( \Gamma \) with \( \pi_1(N) \)). Let \( \xi \) denote the fixed point of \( \Pi \) on the sphere at infinity and let \( B \) be an open horoball centered at \( \xi \) such that \( \{ g \in \Gamma \mid gB \cap B \neq \emptyset \} = \Pi \). Then \( \mathcal{H} = B/\Pi \), which we identify with the image of \( B \) in \( N \), is called a cusp neighborhood in \( N \).

If \( \mathcal{H} \) is a cusp neighborhood in \( N = \mathbb{H}^3/\Gamma \) then the inverse image of \( \mathcal{H} \) under the covering projection \( \mathbb{H}^3 \to N \) is a union of disjoint open horoballs. The cusp neighborhood \( \mathcal{H} \) is maximal if and only there exist two of these disjoint horoballs whose closures have non-empty intersection.

2.6 If \( N \) is a complete, orientable hyperbolic manifold of finite volume, \( \hat{N} \) will denote a compact core of \( N \). Thus \( \hat{N} \) is a compact 3–manifold whose boundary components are all tori, and the number of these tori is equal to the number of cusps of \( N \).

3 Drilling and packing

Lemma 3.1 Suppose that \( M \) is a closed, orientable hyperbolic 3–manifold, and that \( C \) is a shortest geodesic in \( M \). Set \( N = \text{drill}_C(M) \). If \( \text{tuberad}(C) \geq (\log 3)/2 \) then \( \text{vol} N < 3.0177 \text{ vol} M \).

Proof The proof is based on a result due to Agol, Dunfield, Storm and W Thurston [3]. We let \( L \) denote the length of the geodesic \( C \) in the closed hyperbolic 3–manifold \( M \), and we set \( R = \text{tuberad}(C) \) and \( T = \text{tube}(C) \). Proposition 10.1 of [3] states that

\[
\text{vol} N \leq (\coth^3 2R)(\text{vol} M + \frac{\pi}{2}L \tanh R \tanh 2R).
\]

Note that

\[
\text{vol} T = \pi L \sinh^2 R = \left( \frac{\pi}{2}L \tanh R \right) (2 \sinh R \cosh R) = \left( \frac{\pi}{2}L \tanh R \right) (\sinh 2R).
\]

Thus

\[
\text{vol} N \leq (\coth^3 2R) \left( \text{vol} M + \frac{	ext{vol} T \tanh 2R}{\sinh 2R} \right)
= (\coth^3 2R) \left( \text{vol} M + \frac{\text{vol} T}{\cosh 2R} \right).
\]
In the language of [16], the quantity \((\text{vol} T) / (\text{vol} M)\) is the density of a tube packing in \(\mathbb{H}^3\). According to [16, Corollary 4.4], we have \((\text{vol} T) / (\text{vol} M) < 0.91\). Hence \(\text{vol} N < f(x) \cdot \text{vol}(M)\), where \(f(x)\) is defined for \(x \geq 0\) by

\[
f(x) = (\coth^3 2x) \left( 1 + \frac{0.91}{\cosh 2x} \right).
\]

Since \(f(x)\) is decreasing for \(x \geq 0\), and since a direct computation shows that \(f(0.5495) = 3.01762\ldots\), we have \(\text{vol} N < 3.0177 \cdot \text{vol} M\) whenever \(R \geq 0.5495\).

It remains to consider the case in which \(0.5495 > R \geq \frac{\log 3}{2} = 0.5493\ldots\). In this case we use [16, Theorem 4.3], which asserts that the tube-packing density \((\text{vol} T) / (\text{vol} M)\) is bounded above by \((\sinh R)g(R)\), where \(g(x)\) is defined for \(x > 0\) by

\[
g(x) = \frac{\arcsin \frac{1}{2 \cosh r}}{\arcsinh \frac{\tanh r}{\sqrt{3}}}.
\]

Since \(g(x)\) is clearly a decreasing function for \(x > 0\), and since \(\sinh R\) is increasing for \(x > 0\), we have

\[
(\text{vol} T) / (\text{vol} M) < (\sinh 0.5495)g((\log 3)/2) = 0.90817\ldots
\]

Hence \(\text{vol} N < f_1(x) \cdot \text{vol}(M)\), where \(f_1(x)\) is defined for \(x \geq 0\) by

\[
f_1(x) = (\coth^3 2x) \left( 1 + \frac{0.90817}{\cosh 2x} \right).
\]

Again, \(f_1(x)\) is decreasing for \(x \geq 0\), and we see by direct computation that \(f_1((\log 3)/2) = 3.017392\ldots\). Hence we have \(\text{vol} N < 3.0174 \cdot \text{vol} M\) in this case.

**Lemma 3.2** Suppose that \(M\) is a closed, orientable hyperbolic 3–manifold such that \(\text{vol} M \leq 1.22\), and that \(C\) is a shortest geodesic in \(M\). Set \(N = \text{drill}_C(M)\). If \(\text{tuberad}(C) > (\log 3)/2\) then the maximal cusp neighborhood in \(N\) has volume less than \(\pi\).

**Proof** We let \(d(\infty) = .853276\ldots\) denote Böröczky’s lower bound [6] for the density of a horoball packing in hyperbolic space. It follows from the definition of the density of a horoball packing that the volume of a maximal cusp neighborhood in \(N\) is at most \(d(\infty) \cdot \text{vol} N\). **Lemma 3.1** gives \(\text{vol} N < 3.0177 \cdot 1.22 < \pi/d(\infty)\), and the conclusion follows. \(\square\)
4 Filling

As in [4], we shall say that a group is *semifree* if it is a free product of free abelian groups; and we shall say that a group \( \Gamma \) is *\( k \)--semifree* if every subgroup of \( \Gamma \) whose rank is at most \( k \) is semifree. Note that \( \Gamma \) is 2--semifree if and only if every rank-2 subgroup of \( \Gamma \) is either free or free abelian.

The following improved version of [4, Theorem 6.1] is made possible by more recent developments.

**Theorem 4.1** Let \( k \geq 2 \) be an integer and let \( \Phi \) be a Kleinian group which is freely generated by elements \( \xi_1, \ldots, \xi_k \). Let \( z \) be any point of \( \mathbb{H}^3 \) and set \( d_i = \text{dist}(z, \xi_i \cdot z) \) for \( i = 1, \ldots, k \). Then we have

\[
\sum_{i=1}^{k} \frac{1}{1 + e^{d_i}} \leq \frac{1}{2}.
\]

In particular there is some \( i \in \{1, \ldots, k\} \) such that \( d_i \geq \log(2k-1) \).

**Proof** If \( \Gamma \) is geometrically finite this is included in [4, Theorem 6.1]. In the general case, \( \Gamma \) is topologically tame according to [1] and [7], and it then follows from [15, Theorem 1.1], or from the corresponding result for the free case in [14], that \( \Gamma \) is an algebraic limit of geometrically finite groups; more precisely, there is a sequence of geometrically finite Kleinian groups \( (\Gamma_j)_{j \geq 1} \) such that each \( \Gamma_j \) is freely generated by elements \( \xi_{1j}, \ldots, \xi_{kj} \), and \( \lim_{j \to \infty} \xi_{ij} = \xi_i \) for \( i = 1, \ldots, k \). Given any \( z \in \mathbb{H}^3 \), we set \( d_{ij} = \text{dist}(z, \xi_{ij} \cdot z) \) for each \( j \geq 1 \) and for \( i = 1, \ldots, k \). According to [4, Theorem 6.1], we have

\[
\sum_{i=1}^{k} \frac{1}{1 + e^{d_{ij}}} \leq \frac{1}{2}
\]

for each \( j \geq 1 \). Taking limits as \( j \to \infty \) we conclude that

\[
\sum_{i=1}^{k} \frac{1}{1 + e^{d_i}} \leq \frac{1}{2}.
\]

\( \square \)

Let us also recall the following definition from [4, Section 8]. Let \( \Gamma \) be a discrete torsion-free subgroup of \( \text{Isom}_+ (\mathbb{H}^3) \). A positive number \( \lambda \) is termed a *strong Margulis number* for \( \Gamma \), or for the orientable hyperbolic 3--manifold \( N = \mathbb{H}^3 / \Gamma \), if whenever \( \xi \) and \( \eta \) are non-commuting elements of \( \Gamma \), we have

\[
\frac{1}{1 + e^{\text{dist}(\xi \cdot z, z)}} + \frac{1}{1 + e^{\text{dist}(\eta \cdot z, z)}} \leq \frac{2}{1 + e^{\lambda}}.
\]

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The following improved version of [4, Proposition 8.4] is an immediate consequence of Theorem 4.1.

**Corollary 4.2**  Let \( \Gamma \) be a discrete subgroup of \( \text{Isom}_+(\mathbb{H}^3) \). Suppose that \( \Gamma \) is 2–semifree. Then \( \log 3 \) is a strong Margulis number for \( \Gamma \).

**Lemma 4.3**  Let \( N \) be a non-compact finite-volume hyperbolic 3–manifold. Suppose that \( S \) is a boundary component of the compact core \( \hat{N} \), and \( \mathcal{H} \) is the maximal cusp neighborhood in \( N \) corresponding to \( S \). If infinitely many of the manifolds obtained by Dehn filling \( \hat{N} \) along \( S \) have 2–semifree fundamental group then \( \mathcal{H} \) has volume at least \( \pi \).

**Proof**  Suppose that \( (N_i) \) is an infinite sequence of distinct hyperbolic manifolds obtained by Dehn filling \( \hat{N} \) along \( S \), and that \( \pi_1(N_i) \) is 2–semifree for each \( i \).

Thurston’s Dehn filling theorem [5, Appendix B], implies that for each sufficiently large \( i \), the manifold \( N_i \) admits a hyperbolic metric; that the core curve of the Dehn filling \( N_i \) of \( \hat{N} \) is isotopic to a geodesic \( C_i \) in \( N_i \); that the length \( L_i \) of \( C_i \) tends to 0 as \( i \to \infty \); and that the sequence of maximal tubes \( (\text{tube}(C_i))_{i \geq 1} \) converges geometrically to \( \mathcal{H} \). In particular

\[
\lim_{i \to \infty} \text{vol}(\text{tube}(C_i)) = \text{vol } \mathcal{H}.
\]

According to Corollary 4.2, \( \log 3 \) is a strong Margulis number for each of the hyperbolic manifolds \( N_i \). It therefore follows from [4, Corollary 10.5] that \( \text{vol } \text{tube}(C_i) > V(L_i), \)

where \( V \) is an explicitly defined function such that \( \lim_{x \to 0} V(x) = \pi \). In particular, this shows that

\[
\text{vol } \mathcal{H} \geq \lim_{i \to \infty} V(L_i) \geq \pi. \quad \square
\]

5  **Non-exceptional manifolds, arbitrary primes**

5.1  A closed hyperbolic 3–manifold \( M \) will be termed \textit{exceptional} if every shortest geodesic in \( M \) has tube radius at most \( (\log 3)/2 \).

In this section we shall prove a result, Proposition 5.3, which gives a bound of 3 for the dimension of \( H_1(M; \mathbb{Z}_p) \) for any prime \( p \) when \( M \) is a non-exceptional manifold with volume at most 1.22.
Lemma 5.2  Suppose that $M$ is a compact, irreducible, orientable 3–manifold, such that every non-cyclic abelian subgroup of $\pi_1(M)$ is carried by a torus component of $\partial M$. Suppose that either

(i) $\dim H_1(M; \mathbb{Q}) \geq 3$, or
(ii) $M$ is closed and $\dim H_1(M; \mathbb{Z}_p) \geq 4$ for some prime $p$.

Then $\pi_1(M)$ is 2–semifree.

Proof  Let $X$ be any subgroup of $\pi_1(M)$ having rank at most 2. According to [11, Theorem VI.4.1], $X$ is free, or free abelian, or of finite index in $\pi_1(M)$. If $\dim H_1(M; \mathbb{Q}) \geq 3$, it is clear that $X$ has infinite index in $\pi_1(M)$. If $M$ is closed and $H_1(M; \mathbb{Z}_p) \geq 4$ for some prime $p$, then Proposition 1.1 of [17] implies that every 2–generator subgroup of $\pi_1(M)$ has infinite index. Thus in either case $X$ is either free or free abelian. This shows that $\pi_1(M)$ is 2–semifree.

Proposition 5.3  Suppose that $M$ is a closed, orientable, non-exceptional hyperbolic 3–manifold such that $\text{vol } M \leq 1.22$. Then $H_1(M; \mathbb{Z}_p)$ has dimension at most 3 for every prime $p$.

Proof  Since $M$ is non-exceptional, there is a shortest geodesic $C$ in $M$ with $R = \text{tubercad}(C) > (\log 3)/2$. We set $N = \text{drill}_C(M)$. Let $\mathcal{H}$ denote the maximal cusp neighborhood in $N$. Since $R > (\log 3)/2$, Lemma 3.2 implies that $\text{vol } \mathcal{H} < \pi$.

Now assume that $\dim H_1(M; \mathbb{Z}_p) \geq 4$ for some prime $p$. There is an infinite sequence $(M_i)$ of manifolds obtained by distinct Dehn fillings of $\hat{N}$ such that $H_1(M_i; \mathbb{Z}_p)$ has dimension at least 4 for each $i$. (For example, if $(\lambda, \mu)$ is a basis for $H_1(\partial \hat{N}, \mathbb{Z}_p)$ such that $\lambda$ belongs to the kernel of the inclusion homomorphism $H_1(\partial \hat{N}, \mathbb{Z}_p) \to H_1(\hat{N}, \mathbb{Z}_p)$, we may take $M_i$ to be obtained by the Dehn surgery corresponding to a simple closed curve in $\partial \hat{N}$ representing the homology class $\lambda + ip\mu$.) It follows from Thurston’s Dehn filling theorem [5, Appendix B] that for sufficiently large $i$ the manifold $M_i$ is hyperbolic. Hence by case (ii) of Lemma 5.2, the fundamental group of $M_i$ is 2–semifree for sufficiently large $i$. Thus Lemma 4.3 implies that $\text{vol } \mathcal{H} \geq \pi$, a contradiction.

6 Non-exceptional manifolds, odd primes

Proposition 6.3, which is proved in this section, gives a bound of 2 for the dimension of $H_1(M; \mathbb{Z}_p)$ for any odd prime $p$ when $M$ is a non-exceptional manifold with volume at most 1.22.
Definition 6.1 Let $N$ be a connected manifold, $\star \in N$ a base point, and $Q$ a subgroup of $\pi_1(N, \star)$. We shall say that a connected based covering space $r : (N', \star') \to (N, \star)$ carries the subgroup $Q$ if $Q \leq r_2(\pi_1(N', \star')) \leq \pi_1(N, \star)$.

Lemma 6.2 Suppose that $\mathcal{H}$ is a maximal cusp neighborhood in a finite-volume hyperbolic 3–manifold $N$. Let $\star$ be a base point in $\mathcal{H}$, and let $P \leq \pi_1(N, \star)$ denote the image of $\pi_1(\mathcal{H}, \star)$ under inclusion. Then there is an element $\beta$ of $\pi_1(N, \star)$ with the following property:

(†) For every based covering space $r : (N', \star') \to (N, \star)$ which carries the subgroup $\langle P, \beta \rangle$ of $\pi_1(N, \star)$, there is a maximal cusp neighborhood $\mathcal{H}'$ in $N'$ which is isometric to $\mathcal{H}$.

Proof. We write $N = \mathbb{H}^3/\Gamma$, where $\Gamma$ is a discrete, torsion-free subgroup of $\text{Isom}(\mathbb{H}^3)$. Let $q : \mathbb{H}^3 \to N$ denote the quotient map and fix a base point $\star'$ which is mapped to $\star$ by $q$. The components of $q^{-1}(\mathcal{H})$ are horoballs. Let $B_0$ denote the component of $q^{-1}(\mathcal{H})$ containing $\star'$. The stabilizer $\Gamma_0$ of $B_0$ is mapped onto the subgroup $P$ of $\pi_1(N, \star)$ by the natural isomorphism $\iota : \Gamma \to \pi_1(N, \star)$.

Since $\mathcal{H}$ is a maximal cusp, there is a component $B_1 \neq B_0$ of $q^{-1}(\mathcal{H})$ such that $\overline{B_1} \cap \overline{B_0} \neq \emptyset$. We fix an element $g$ of $\Gamma$ such that $g(B_0) = B_1$, and we set $\beta = \iota(g) \in \pi_1(N, \star)$.

To show that $\beta$ has property (†), we consider an arbitrary based covering space $r : (N', \star') \to (N, \star)$ which carries the subgroup $\langle P, \beta \rangle$ of $\pi_1(N, \star)$. We may identify $N'$ with $\mathbb{H}^3/\Gamma'$, where $\Gamma'$ is some subgroup of $\Gamma$ containing $\langle \Gamma_0, g \rangle$.

Since $\Gamma_0 \subset \Gamma'$, the cusp neighborhood $\mathcal{H}$ lifts to a cusp neighborhood $\mathcal{H}'$ in $N'$. In particular $\mathcal{H}'$ is isometric to $\mathcal{H}$. The horoballs $B_0$ and $B_1 = g(B_0)$ are distinct components of $(q')^{-1}(\mathcal{H}')$, where $q' : \mathbb{H}^3 \to N'$ denotes the quotient map. Since $g \in \Gamma'$ and $\overline{B_1} \cap \overline{B_0} \neq \emptyset$, the cusp neighborhood $\mathcal{H}'$ is maximal.

Proposition 6.3 Suppose that $M$ is a closed, orientable, non-exceptional hyperbolic 3–manifold such that $\text{vol } M \leq 1.22$. Then $H_1(M; \mathbb{Z}_p)$ has dimension at most 2 for every odd prime $p$.

Proof. Since $M$ is non-exceptional, we may fix a shortest geodesic $C$ in $M$ with $R = \text{tuberculad}(C) > (\log 3)/2$. We set $N = \text{drill}_C(M)$. Let $\mathcal{H}$ denote the maximal cusp neighborhood in $N$. Since $R > (\log 3)/2$, Lemma 3.2 implies that $\text{vol } \mathcal{H} < \pi$.

As in the statement of Lemma 6.2, we fix a base point $\star \in \mathcal{H}$, and we denote by $P \leq \pi_1(N, \star)$ the image of $\pi_1(\mathcal{H}, \star)$ under inclusion. We fix an element $\beta$ of $\pi_1(N, \star)$ having property (†) of Lemma 6.2. We set $Q = \langle P, \beta \rangle \leq \pi_1(N, \star)$.
Suppose that $\dim H_1(M;\mathbb{Z}_p) \geq 3$ for some prime $p$. We shall prove the proposition by showing that this assumption leads to a contradiction if $p$ is odd.

It follows from Poincaré duality that the image of the inclusion homomorphism $\alpha : H_1(\partial \hat{N};\mathbb{Z}_p) \to H_1(\hat{N};\mathbb{Z}_p)$ has rank 1. Hence the image of $P$ under the natural homomorphism $\pi_1(N,\star) \to H_1(N;\mathbb{Z}_p)$ has dimension 1. It follows that the image $\hat{Q}$ of $Q$ under this homomorphism has dimension either 1 or 2. In the case $\dim \hat{Q} = 1$ we shall obtain a contradiction for any prime $p$. In the case $\dim \hat{Q} = 2$ we shall obtain a contradiction for any odd prime $p$.

First consider the case $\dim \hat{Q} = 1$. We have assumed $\dim H_1(M;\mathbb{Z}_p) \geq 3$. Thus there is a $\mathbb{Z}_p \times \mathbb{Z}_p$–regular based covering space $(N',\star')$ of $(N,\star)$ which carries $Q$. By property ($\dagger$), there is a maximal cusp neighborhood $\mathcal{H}'$ in $N'$ which is isometric to $\mathcal{H}$. In particular $\mathrm{vol} \mathcal{H}' < \pi$.

Since in particular $(N',\star')$ carries $P$, the boundary of the compact core $\hat{N}$ lifts to $\hat{N}'$. As $N'$ is a $p^2$–fold regular covering, it follows that $\hat{N}'$ has $p^2 \geq 4$ boundary components.

It follows from Thurston’s Dehn filling theorem [5, Appendix B] that there are infinitely many hyperbolic manifolds obtained by Dehn filling one boundary component of $\hat{N}'$. If $Z$ is any hyperbolic manifold obtained by such a filling, then $Z$ has at least three boundary components, and it follows from case (i) of Lemma 5.2 that $\pi_1(Z)$ is $2$–semifree. It therefore follows from Lemma 4.3 that each maximal cusp neighborhood in $N'$ has volume at least $\pi$. Since we have seen that $\mathrm{vol} \mathcal{H}' < \pi$, this gives the desired contradiction in the case $\dim \hat{Q} = 1$.

It remains to consider the case in which $\dim \hat{Q} = 2$ and the prime $p$ is odd. Since we have assumed that $\dim H_1(M;\mathbb{Z}_p) \geq 3$, there is a $p$–fold cyclic based covering space $(N',\star')$ of $(N,\star)$ which carries $Q$. Since $N'$ carries $P$, the boundary of the compact core $\hat{N}$ lifts to $\hat{N}'$, and as $N'$ is a $p$–fold regular covering, it follows that $\hat{N}'$ has $p$ boundary components.

We claim that the inclusion homomorphism $\alpha' : H_1(\partial \hat{N}',\mathbb{Z}_p) \to H_1(\hat{N}',\mathbb{Z}_p)$ is not surjective. To establish this, we consider the commutative diagram

\[
\begin{array}{ccc}
H_1(\partial \hat{N}';\mathbb{Z}_p) & \xrightarrow{\alpha'} & H_1(N';\mathbb{Z}_p) \\
\downarrow & & \downarrow r_* \\
H_1(\partial \hat{N};\mathbb{Z}_p) & \xrightarrow{\alpha} & H_1(N;\mathbb{Z}_p)
\end{array}
\]

where $r : N' \to N$ is the covering projection. Since $(N',\star')$ carries $Q$ we have $\hat{Q} \subset \mathrm{Im} r_*$. Hence surjectivity of $\alpha'$ would imply $\hat{Q} \subset \mathrm{Im} \alpha$. This is impossible: we
observed above that $\text{Im} \alpha$ has rank 1, and we are in the case $\dim \hat{Q} = 2$. Thus $\alpha'$ cannot be surjective.

Since $\hat{N}'$ has $p$ boundary components, it follows from Poincaré duality that $\dim \text{Im} \alpha' = p \geq 3$. Since $\alpha'$ is not surjective and $p$ is an odd prime, it follows that $\dim H_1(N'; \mathbb{Z}_p) \geq p + 1 \geq 4$.

Since $(N', \star')$ carries $Q$, some subgroup $Q'$ of $\pi_1(N', \star')$ is mapped isomorphically to $Q$ by $r_\sharp$. In particular $Q'$ has rank at most 3. Since $\dim H_1(N'; \mathbb{Z}_p) \geq 4$, there is a $p^2$–fold cyclic based covering space $(N'', \star'')$ of $(N', \star')$ which carries $Q'$. Hence $(N'', \star'')$ is a $p^2$–fold (possibly irregular) based covering space of $(N, \star)$ which carries $Q$. By property (†), there is a maximal cusp neighborhood $\mathcal{H}''$ in $N''$ which is isometric to $\mathcal{H}$. In particular $\text{vol} \mathcal{H}'' < \pi$.

Since $P \leq Q$, there is a component $T$ of $\partial \hat{N''}$ such that $Q'$ contains a conjugate of the image of $\pi_1(T)$ under the inclusion homomorphism $\pi_1(T) \to \pi_1(N')$. Hence $T$ lifts to the $p$–fold cyclic covering space $N''$ of $N'$. It follows that the covering projection $r' : N'' \to N'$ maps $p \geq 3$ components of $(r')^{-1}(\partial \hat{N''})$ to $T$. As $\hat{N''}$ has at least three boundary components, $\hat{N''}$ must have at least five boundary components.

Hence if $Z$ is any hyperbolic manifold obtained by Dehn filling one boundary component of $\hat{N''}$, we have $\dim H_1(Z; \mathbb{Q}) \geq 4 > 3$, and it follows from case (i) of Lemma 5.2 that $\pi_1(Z)$ is 2–semifree. It therefore follows from Lemma 4.3 and Thurston’s Dehn filling theorem that each maximal cusp neighborhood in $N''$ has volume at least $\pi$. Since we have seen that $\text{vol} \mathcal{H}'' < \pi$, we have the desired contradiction in this case as well. □

7 Exceptional manifolds

Our treatment of exceptional manifolds begins with Proposition 7.1 below, the proof of which will largely consist of citing material from [9]. In order to state it we must first introduce some notation.

For $k = 0, \ldots, 6$ we define constants $\tau_k$ as follows:

\[
\begin{align*}
\tau_0 &= 0.4779 \\
\tau_1 &= 1.0756 \\
\tau_2 &= 1.0527 \\
\tau_3 &= 1.2599 \\
\tau_4 &= 1.2521 \\
\tau_5 &= 1.0239 \\
\tau_6 &= 1.0239 
\end{align*}
\]
For $k = 0, \ldots, 6$ let $E_k$ be the 2–generator group with presentation

$$E_k = \langle x, y : r_{1,k}, r_{2,k} \rangle,$$

where the relators $r_{1,k} = r_{1,k}(x, y)$ and $r_{2,k} = r_{2,k}(x, y)$ are the words listed below (in which we have set $X = x^{-1}$ and $Y = y^{-1}$):

- $r_{1,0} = xyXyyXxxyxyy$,
- $r_{2,0} = XyxyXyxxyxyy$,
- $r_{1,1} = XXyXYXyXyyXyy$, $r_{2,1} = XXyyXyxxyxyxxyy$,
- $r_{1,2} = XxyXyxXYxxyXyy$, $r_{2,2} = XXyXXyXyxyxyy$,
- $r_{1,3} = XXyxyXxyXyXxyXyyXyy$, $r_{2,3} = XXyxyXyxxyxyXYxxyXyyXyy$,
- $r_{1,4} = XXyxyXXyXxyXyyXYxXyyXYxXyy$, $r_{2,4} = XXyxyXXxyXyxXyyXYxXyyXYxXyy$,
- $r_{1,5} = XyXXyXYxxyXYxxy$, $r_{2,5} = XyxyXYyXXyXyxyxy$,
- $r_{1,6} = YYXYyXXyXYxXyy$, $r_{2,6} = YYXyXXyXyxyXyy$.

The group $E_0$ is the fundamental group of an arithmetic hyperbolic 3–manifold which is known as Vol3. This manifold, which was studied in [12], is described as m007(3,1) in the list given in [10], and can also be described as the manifold obtained by a $(-1,2)$ Dehn filling of the once-punctured torus bundle with monodromy $-R^2L$.

**Proposition 7.1** Suppose that $M$ is an exceptional closed, orientable hyperbolic 3–manifold which is not isometric to Vol3. Then there exists an integer $k$ with $1 \leq k \leq 6$ such that the following conditions hold:

1. $M$ has a finite-sheeted cover $\tilde{M}$ such that $\pi_1(\tilde{M})$ is isomorphic to a quotient of $E_k$; and
2. there is a shortest closed geodesic $C$ in $M$ such that $\text{vol}(\text{tube}(C)) \geq \tau_k$.

**Proof** This is in large part an application of results from [9], and we begin by reviewing some material from that paper.

We begin by considering an arbitrary simple closed geodesic $C$ in a closed, orientable hyperbolic 3–manifold $M = \mathbb{H}^3/\Gamma$. As we pointed out in 2.2, there is a loxodromic
isometry \( f \in \Gamma \) with \( A_f / \langle f \rangle = C \). If we set \( R = \text{tuberad}(C) \) and \( Z = \ZR(f) \), it follows from the definitions that \( \text{tube}(C) = Z / \langle f \rangle \), that \( h(Z) \cap Z = \emptyset \) for every \( h \in \Gamma - \langle f \rangle \), and that there is an element \( w \in \Gamma - \langle f \rangle \) such that \( w(Z) \cap Z \neq \emptyset \).

Let us define an ordered pair \((f, w)\) of elements of \( \Gamma \) to be a GMT pair for the simple geodesic \( C \) if we have (i) \( A_f / \langle f \rangle = C \), (ii) \( w \notin \langle f \rangle \), and (iii) \( w(Z) \cap Z \neq \emptyset \). Note that since \( \langle f \rangle \) must be a maximal cyclic subgroup of \( \Gamma \), condition (ii) implies that the group \( \langle f, w \rangle \) is non-elementary.

Set \( Q = \{(L, D, R) \in C^3 : \text{Re} L, \text{Re} D > 0\} \). For any point \( P = (L, D, R) \in Q \) we will denote by \((f_P, w_P)\) the pair \((f, w) \in \text{Isom}_+^{\mathbb{H}^3} \times \text{Isom}_+^{\mathbb{H}^3}, \) where \( f, w \in PGL_2(\mathbb{C}) = \text{Isom}_+^{\mathbb{H}^3} \) are defined by

\[
f = \begin{bmatrix} e^{L/2} & 0 \\ 0 & e^{-L/2} \end{bmatrix}
\]

and

\[
w = \begin{bmatrix} e^{R/2} & 0 \\ 0 & e^{-R/2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{D/2} & 0 \\ 0 & e^{-D/2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.
\]

With this definition, \( f_P \) has (real) translation length \( \text{Re} L \), and the (minimum) distance between \( A_f \) and \( w(A_f) \) is \( (\text{Re} D)/2 \).

In [9, Section 1], it is shown that if \((f, w)\) is a GMT pair for a shortest geodesic \( C \) in a closed, orientable hyperbolic 3–manifold and \( \text{tuberad}(C) \leq (\log 3)/2 \), then \((f, w)\) is conjugate by some element of \( \text{Isom}^{\mathbb{H}^3} \) to a pair of the form \((f_P, w_P)\) where \( P \in Q \) is a point such that \( \exp(P) \doteq (e^L, e^D, e^R) \) lies in the union \( X_0 \cup \cdots \cup X_6 \) of seven disjoint open subsets of \( C^3 \) that are explicitly defined in [9, Proposition 1.28].

For every \( k \) with \( 0 \leq k \leq 6 \) and every point \( P = (L, D, R) \) such that \( \exp(P) \in X_k \), it follows from [9, Definition 1.27 and Proposition 1.28] that

(I) the isometries \( r_{1,k}(f_P, w_P) \) and \( r_{2,k}(f_P, w_P) \) have translation length less than \( \text{Re} L \);

and it follows from [9, Table 1.1] that

(II) \( \pi \text{Re}(L) \sinh^2(\text{Re}(D)/2) > \tau_k \).

According to [9, Proposition 3.1], if \( C \) is a shortest geodesic in a closed, orientable hyperbolic 3–manifold, and if some GMT pair for \( C \) has the form \((f_P, w_P)\) for some \( P \) with \( \exp(P) \in X_0 \), then \( M \) is isometric to \( \text{Vol3} \).

Now suppose that \( M \) is an exceptional closed, orientable hyperbolic 3–manifold. Let us choose a shortest closed geodesic \( C \) in \( M \). By the definition of an exceptional manifold, \( C \) has tube radius \( \leq (\log 3)/2 \). Hence the facts recalled above imply that \( C \) has a GMT pair of the form \((f_P, w_P)\) for some \( P \) such that \( \exp(P) \in X_k \) for some \( k \) with \( 0 \leq k \leq 6 \);
and furthermore, that if \( M \) is not isometric to \( \text{Vol}3 \), then \( 1 \leq k \leq 6 \). We shall show that conclusions (1) and (2) hold with this choice of \( k \).

For \( i = 1, 2 \) it follows from property (I) above that the element \( r_{i,k}(f, \omega) \) has real translation length less than the real translation length \( \text{Re} L \) of \( f \). Since \( C \) is a shortest geodesic in \( M \), it follows that the conjugacy class of \( r_{i,k}(f, \omega) \) is not represented by a closed geodesic in \( M \). As \( M \) is closed it follows that \( r_{i,k}(f, \omega) \) is the identity for \( i = 1, 2 \). Hence the subgroup of \( \Gamma \) generated by \( f \) and \( \omega \) is isomorphic to a quotient of \( \mathcal{E}_k \). Since we observed above that \( \langle f, \omega \rangle \) is non-elementary, there is a non-abelian subgroup \( Y \) of \( \pi_1(M) \) which is isomorphic to a quotient of \( \mathcal{E}_k \). In particular \( Y \) has rank 2, and it cannot be a free group of rank 2 since the relators \( r_{1,k} \) and \( r_{2,k} \) are non-trivial. Hence by [11, Theorem VI.4.1] we must have \( |\pi_1(M) : Y| < \infty \). This proves (1).

Finally, we recall that

\[ \text{vol tube}(C) = \pi(\text{length}(C)) \sinh^2(\text{tuberad}(C)) = \pi(\text{Re} L) \sinh^2((\text{Re} D)/2). \]

Hence (2) follows from (II).

We shall also need the following slight refinement of [17, Proposition 1.1].

**Proposition 7.2** Let \( p \) be a prime and let \( M \) be a closed 3–manifold. If \( p \) is odd assume that \( M \) is orientable. Let \( X \) be a finitely generated subgroup of \( \pi_1(M) \), and set \( n = \dim H_1(X; \mathbb{Z}_p) \). If \( \dim H_1(M; \mathbb{Z}_p) \geq \max(3, n + 2) \), then \( X \) has infinite index in \( \pi_1(M) \). In fact, \( X \) is contained in infinitely many distinct finite-index subgroups of \( \pi_1(M) \).

**Proof** In this proof, as in [17, Section 1], for any group \( G \) we shall denote by \( G_1 \) the subgroup of \( G \) generated by all commutators and \( p \)-th powers, where \( p \) is the prime given in the hypothesis. Since \( \dim H_1(X; \mathbb{Z}_p) = n \) we may write \( X = E X_1 \) for some rank-\( n \) subgroup \( E \) of \( X \).

We first assume that \( n \geq 1 \). Set \( \Gamma = \pi_1(M) \). Let \( \mathcal{S} \) denote the set of all finite-index subgroups \( \Delta \) of \( \Gamma \) such that \( \Delta \supseteq X \) and \( \dim H_1(\Delta; \mathbb{Z}_p) \geq n + 2 \). The hypothesis gives \( \Gamma \in \mathcal{S} \), so that \( \mathcal{S} \neq \emptyset \). Hence it suffices to show that every subgroup \( \Delta \in \mathcal{S} \) has a proper subgroup \( D \) such that \( D \in \mathcal{S} \).

Any group \( \Delta \in \mathcal{S} \) may be identified with \( \pi_1(\tilde{M}) \) for some finite-sheeted covering space \( \tilde{M} \) of \( M \). In particular, \( \tilde{M} \) is a closed 3–manifold, and is orientable if \( p \) is odd. Since \( \Delta \in \mathcal{S} \) we have \( X \leq \Delta = \pi_1(\tilde{M}) \) and \( \dim H_1(\tilde{M}; \mathbb{Z}_p) = \dim H_1(\Delta; \mathbb{Z}_p) \geq n + 2 \). Now set \( D = E \Delta_1 \leq \Delta \). Applying [17, Lemma 1.5], with \( \tilde{M} \) in place of \( M \), we deduce that \( D \) is a proper, finite-index subgroup of \( \Delta \), and that \( \dim H_1(D; \mathbb{Z}_p) \geq 2n + 1 \geq n + 2 \).
On the other hand, since $\Delta \in S$, we have $X \leq \Delta$, and hence $X = E X_1 \leq E \Delta_1 = D$. It now follows that $D \in S$, and the proof is complete in the case $n \geq 1$.

If $n = 0$ then, since $\dim H_1(M; \mathbb{Z}_p) \geq 3$, there exists a finitely generated subgroup $X' \geq X$ such that $H_1(X'; \mathbb{Z}_p)$ has dimension 1. The case of the Lemma which we have already proved shows that $X'$ has infinite index. Thus $X$ has infinite index as well. \qed

**Corollary 7.3** Let $p$ be a prime and let $M$ be a closed, orientable 3–manifold. Let $X$ be a finite-index subgroup of $\pi_1(M)$, and set $n = \dim H_1(X; \mathbb{Z}_p)$. Then $\dim H_1(M; \mathbb{Z}_p) \leq \max(2, n + 1)$.

**Lemma 7.4** Suppose that $M$ is an exceptional hyperbolic 3–manifold with volume at most 1.22. Then $H_1(M; \mathbb{Z}_p)$ has dimension at most 2 for every prime $p \neq 2, 7$, and $H_1(M; \mathbb{Z}_2)$ and $H_1(M; \mathbb{Z}_7)$ have dimension at most 3. Furthermore, if $M$ has volume at most 1.182, then $H_1(M; \mathbb{Z}_7)$ has dimension at most 2.

**Proof** If $M$ is isometric to Vol3 then $\pi_1(M)$ is generated by two elements, and the conclusions follow. For the rest of the proof we assume that $M$ is not isometric to Vol3, and we fix an integer $k$ with $1 \leq k \leq 6$ such that conditions (1) and (2) of Proposition 7.1 hold.

By condition (2) of Proposition 7.1, we may fix a shortest closed geodesic $C$ in $M$ such that vol($T$) $\geq \tau_k$, where $T =$ tube($C$). It follows from a result of Przeworski’s [16, Corollary 4.4] on the density of cylinder packings that vol($T$) $< 0.91$ vol $M$, and so vol $M$ $> \tau_k / 0.91$. If $k = 3$ we have $\tau_k / 0.91 > 1.22$, and we get a contradiction to the hypothesis. Hence $k \in \{1, 2, 4, 5, 6\}$.

Furthermore, we have $\tau_1 / 0.91 > 1.182$. Hence if vol $M$ $\leq 1.182$ then $k \in \{2, 4, 5, 6\}$.

By condition (1) of Proposition 7.1, $\pi_1(M)$ has a finite-index subgroup $X$ which is isomorphic to a quotient of $\mathcal{E}_k$. From the defining presentations of the groups $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_4$, $\mathcal{E}_5$, and $\mathcal{E}_6$, we find that $H_1(\mathcal{E}_1; \mathbb{Z})$ is isomorphic to $\mathbb{Z}_7 \oplus \mathbb{Z}_7$, that $H_1(\mathcal{E}_2; \mathbb{Z})$ and $H_1(\mathcal{E}_4; \mathbb{Z})$ are isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$, while $H_1(\mathcal{E}_5; \mathbb{Z})$ and $H_1(\mathcal{E}_6; \mathbb{Z})$ are isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. (One can check that the two groups $\mathcal{E}_5$ and $\mathcal{E}_6$ are isomorphic to each other.) In particular, since $k \in \{1, 2, 4, 5, 6\}$ we have $\dim H_1(\mathcal{E}_k; \mathbb{Z}_p) \leq 1$ for any prime $p \neq 2, 7$, and $\dim H_1(\mathcal{E}_k; \mathbb{Z}_p) \leq 2$ for $p = 2$ or 7. As $X$ is isomorphic to a quotient of $\mathcal{E}_k$, it follows that $\dim H_1(X; \mathbb{Z}_p) \leq 1$ for any prime $p \neq 2, 7$, and $\dim H_1(X; \mathbb{Z}_p) \leq 2$ for $p = 2$ or 7. Hence by Corollary 7.3, we have $\dim H_1(M; \mathbb{Z}_p) \leq 2$ for $p \neq 2, 7$, and $\dim H_1(M; \mathbb{Z}_p) \leq 3$ for $p = 2, 7$.

It remains to prove that if vol $M$ $\leq 1.182$ then $\dim H_1(M; \mathbb{Z}_7) \leq 2$. We have observed that in this case $k \in \{2, 4, 5, 6\}$. By the list of isomorphism types of the
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$H_1(E_k;\mathbb{Z})$ given above, it follows that $\dim H_1(E_k;\mathbb{Z}_7) = 0 < 1$. Hence in this case the argument given above for $p \neq 2, 7$ goes through in exactly the same way to show that $\dim H_1(M;\mathbb{Z}_7) \leq 2$. □

Proof of Theorem 1.1 For the case in which $M$ is non-exceptional, the theorem is an immediate consequence of Propositions 5.3 and 6.3. For the case in which $M$ is exceptional, the assertions of the theorem are equivalent to those of Lemma 7.4. □

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