SPACE-TIME DISCRETIZATION FOR NONLINEAR PARABOLIC SYSTEMS WITH $p$-STRUCTURE

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Abstract. In this paper we consider nonlinear parabolic systems with elliptic part which can be also degenerate. We prove optimal error estimates for smooth enough solutions. The main novelty, with respect to previous results, is that we obtain the estimates directly without introducing intermediate semi-discrete problems. In addition, we prove the existence of solutions of the continuous problem with the requested regularity, if the data of the problem are smooth enough.

1. Introduction

In this paper we study the (full) space-time discretization of a parabolic problem with Dirichlet boundary conditions. Our method differs from most previous investigations in as much as we use no intermediate problems to prove an optimal error estimate. This result is achieved under certain natural regularity assumptions of the solution of the continuous problem. Moreover, we also prove this required regularity for the solution of the singular problem for large data, in the case of Dirichlet boundary conditions. We restrict ourselves to the three-dimensional setting, however, all results carry over to the general setting in $d$-dimensions.

More precisely, we consider for a sufficiently smooth bounded domain $\Omega \subset \mathbb{R}^3$ and a finite time interval $I := (0, T)$, for some given $T > 0$, the parabolic system

$$\frac{\partial u}{\partial t} - \text{div} S(Du) = f \quad \text{in } I \times \Omega,$$

$$u = 0 \quad \text{on } I \times \partial \Omega,$$

$$u(0) = u_0 \quad \text{in } \Omega,$$

where the elliptic operator $S$ has $(p, \delta)$-structure and depends only on the symmetric part of the gradient $Du$ of the vector-valued unknown $u : \Omega \to \mathbb{R}^3$. Of course, the whole theory also works with some simplifications if $S$ depends on the full gradient $\nabla u$ and in a $d$-dimensional setting with $d \geq 2$. The variational formulation of (parabolic$_p$) is (for smooth enough solutions) the following

$$\left( \frac{\partial u}{\partial t}(t), v \right) + \left( S(Du(t)), Dv \right) = \left( f(t), v \right) \quad \forall v \in V, \text{ a.e. } t \in I,$$

$$\left( u(0), v \right) = \left( u_0, v \right) \quad \forall v \in V,$$

where we will set, for reasons explained later, $V = (W_0^{1,p}(\Omega) \cap L^2(\Omega))^3$. We perform an error analysis for the fully implicit space-time discretization

$$(du_h^m, v_h) + \left( S(Du_h^m), v_h \right) = \left( f(t_m), v_h \right) \quad \forall v_h \in V_h, \quad m = 1, \ldots, M,$$

$$(u_0^h, v_h) = (u_0, v_h) \quad \forall v_h \in V_h,$$  

(1.2)

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where $d_t u^m := \kappa^{-1}(u^m - u^{m-1})$ is the backward difference quotient with $\kappa := \frac{T}{M}$, $M \in \mathbb{N}$ given, $t_m := m \kappa$, and where $V_h \subset V$ is an appropriate finite element space with mesh size $h > 0$. Precise definitions will be given below.

2. Notation and preliminaries

In this section we introduce the notation we will use. Moreover, we recall some technical results which will be needed in the proof of the main convergence result.

2.1. Function spaces. We use $c, C$ to denote generic constants, which may change from line to line, but are not depending on the crucial quantities. Moreover we write $f \sim g$ if and only if there exists constants $c, C > 0$ such that $c f \leq g \leq C f$.

We will use the customary Lebesgue spaces $(L^p(\Omega), \| \cdot \|_p)$ and Sobolev spaces $(W^{k,p}(\Omega), \| \cdot \|_{k,p})$, $k \in \mathbb{N}$. We do not distinguish between scalar, vector-valued or tensor-valued function spaces in the notation if there is no danger of confusion. However, we denote scalar functions by roman letters, vector-valued functions by small boldfaced letters and tensor-valued functions by capital boldfaced letters. If the norms are considered on a set $M$ different from $\Omega$, this is indicated in the respective norms as $\| \cdot \|_{p,M}$, $\| \cdot \|_{k,p,M}$. We equip $W_0^{1,p}(\Omega)$ (based on the Poincaré Lemma) with the gradient norm $\| \nabla \cdot \|_p$. We denote by $|M|$ the 3-dimensional Lebesgue measure of a measurable set $M$. The mean value of a locally integrable function $f$ over a measurable set $M \subset \Omega$ is denoted by $\langle f \rangle_M := \frac{1}{|M|} \int_M f \, dx$. Moreover, we use the notation $(f,g) := \int_M f g \, dx$, whenever the right-hand side is well defined.

2.2. Basic properties of the elliptic operator. For a tensor $P \in \mathbb{R}^{3 \times 3}$ we denote its symmetric part by $P_{\text{sym}} := \frac{1}{2}(P + P^\top) \in \mathbb{R}^{3 \times 3}_{\text{sym}} := \{ A \in \mathbb{R}^{3 \times 3} \mid P = P^\top \}$. The scalar product between two tensors $P, Q$ is denoted by $P \cdot Q$, and we use the notation $|P|^2 = P \cdot P$. We assume that the extra stress tensor $S$ has $(p, \delta)$-structure, which will be defined now. A detailed discussion and full proofs of the following assertions can be found in [14, 27].

Assumption 2.1. We assume that $S : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}$ belongs to $C^0(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}) \cap C^1(\mathbb{R}^{3 \times 3}\setminus\{0\}, \mathbb{R}^{3 \times 3}_{\text{sym}})$, satisfies $S(P) = S(P_{\text{sym}})$, and $S(0) = 0$. Moreover, we assume that $S$ has $(p, \delta)$-structure, i.e., there exist $p \in (1, \infty)$, $\delta \in [0, \infty)$, and constants $C_0, C_1 > 0$ such that

$$
\sum_{i,j,k,l=1}^3 \partial_{kl} S_{ij}(P)Q_{ij}Q_{kl} \geq C_0 (\delta + |P_{\text{sym}}|)^{p-2}|Q_{\text{sym}}|^2,
$$

(2.2a)

$$
|\partial_{kl} S_{ij}(P)| \leq C_1 (\delta + |P_{\text{sym}}|)^{p-2},
$$

(2.2b)

are satisfied for all $P, Q \in \mathbb{R}^{3 \times 3}$ with $A_{\text{sym}} \neq 0$ and all $i, j, k, l = 1, \ldots, 3$. The constants $C_0, C_1$, and $p$ are called the characteristics of $S$.

Remark 2.3. We would like to emphasize that, if not otherwise stated, the constants in the paper depend only on the characteristics of $S$ but are independent of $\delta \geq 0$.

Another important tool are shifted N-functions $\{ \varphi_a \}_{a \geq 0}$, cf. [14, 16, 27]. Defining for $t \geq 0$ a special N-function $\varphi$ by

$$
\varphi(t) := \int_0^t \varphi'(s) \, ds \quad \text{with} \quad \varphi'(t) := (\delta + t)^{p-2}t, \quad (2.4)
$$
we can replace $C_i(\delta + |P_{\text{sym}}|)^{p-2}$ in the right-hand side of (2.2) by $\tilde{C}_i \varphi''(|P_{\text{sym}}|)$, $i = 0, 1$. Next, the shifted functions are defined for $t \geq 0$ by

$$
\varphi_a(t) := \int_0^t \varphi'_a(s) \, ds \quad \text{with} \quad \varphi'_a(t) := \varphi'(a + t) \frac{t}{a + t}.
$$

(2.5)

Note that $\varphi_a(t) \sim (\delta + a + t)^{p-2}t^2$ and also $(\varphi_a)^*(t) \sim ((\delta + a)^{p-1} + t)^{p-2}t^2$, where the superscript denotes the complementary function. We will use also the Young inequality: for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$, such that for all $s, t, a \geq 0$ it holds

$$
t s \leq \varepsilon \, \varphi_a(t) + c_\varepsilon \, (\varphi_a)^*(s),
$$

(2.6)

Closely related to the extra stress tensor $S$ with $(p, \delta)$-structure is the function $F: \mathbb{R}^{3 \times 3} \to \mathbb{R}_{\text{sym}}^{3 \times 3}$ defined through

$$
F(P) := \delta + |P_{\text{sym}}|^2 |P_{\text{sym}}|.
$$

(2.7)

In the following lemma we recall several useful results, which will be frequently used in the paper. The proofs of these results and more details can be found in [14, 27, 16, 3].

**Proposition 2.8.** Let $S$ satisfy Assumption 2.1, let $\varphi$ be defined in (2.4), and let $F$ be defined in (2.7).

(i) For all $P, Q \in \mathbb{R}^{3 \times 3}$

$$
(S(P) - S(Q)) \cdot (P - Q) \sim |F(P) - F(Q)|^2,
$$

$$
\sim \varphi|P_{\text{sym}}|(|P_{\text{sym}} - Q_{\text{sym}}|),
$$

$$
\sim \varphi''(|P_{\text{sym}}| + |Q_{\text{sym}}|)|P_{\text{sym}} - Q_{\text{sym}}|^2,
$$

$$
S(Q) \cdot Q \sim |F(Q)|^2 \sim \varphi(|Q_{\text{sym}}|),
$$

$$
|S(P) - S(Q)| \sim \varphi'|P_{\text{sym}}|(|P_{\text{sym}} - Q_{\text{sym}}|).
$$

The constants depend only on the characteristics of $S$.

(ii) For all $\varepsilon > 0$, there exist a constant $c_\varepsilon > 0$ (depending only on $\varepsilon > 0$ and on the characteristics of $S$) such that for all $u, v, w \in W^{1, p}(\Omega)$

$$
(S(Du) - S(Dv), Dw - Dv) \leq \varepsilon \, ||F(Du) - F(Dv)||^2 + c_\varepsilon \, ||F(Du) - F(Dv)||^2,
$$

$$
(S(Du) - S(Dv), Dw - Dv) \leq \varepsilon \, ||F(Dw) - F(Dv)||^2 + c_\varepsilon \, ||F(Du) - F(Dv)||^2,
$$

and for all $P, Q \in \mathbb{R}^{3 \times 3}, t \geq 0$

$$
\varphi(P)(t) \leq c_\varepsilon \, \varphi|P|(t) + \varepsilon \, |F(Q) - F(P)|^2,
$$

$$
(\varphi(P))^*(t) \leq c_\varepsilon \, (\varphi|P|)^*(t) + \varepsilon \, |F(Q) - F(P)|^2.
$$

(iii) Let $\Omega$ be a bounded domain. Then, for all $H \in L^p(\Omega)$

$$
\int_\Omega |F(H) - \langle F(H) \rangle_\Omega|^2 \, dx \sim \int_\Omega |F(H) - F(\langle H \rangle_\Omega)|^2 \, dx,
$$

where the constants depend only on $p$.

There hold the following important equivalences, first proved in [29]. See also [8, Proposition 2.4].
Proposition 2.9. Assume that \( S \) has \((p, \delta)\)-structure. For \( i = 1, 2, 3 \) and for sufficiently smooth symmetric tensor fields \( Q \) we denote
\[
\mathbb{P}_i(Q) := \partial_i S(Q) \cdot \partial_i Q = \sum_{k,l,m,n=1}^3 \partial_{kl} S_{mn}(Q) \partial_{kl} Q_{mn} \cdot \partial_i Q_{mn}.
\]
Then we have for all smooth enough symmetric tensor fields \( Q \) and all \( i = 1, 2, 3 \)
\[
\mathbb{P}_i(Q) \sim \varphi''(|Q|)|\partial_i Q|^2 \sim |\partial_i F(Q)|^2, \tag{2.11}
\]
\[
\mathbb{P}_i(Q) \sim \frac{|\partial_i S(Q)|^2}{\varphi''(|Q|)}, \tag{2.12}
\]
where the constants only depend on the characteristics of \( S \).

2.3. Discretizations. For the time-discretization, given \( T > 0 \) and \( M \in \mathbb{N} \), we define the time step size as \( \kappa := T/M > 0 \), with the corresponding net \( I_d^M := \{t_m\}_{m=0}^M \), \( t_m := m \kappa \). We use the notation \( I_m := (t_{m-1}, t_m) \), with \( m = 1, \ldots, M \). For a given sequence \( \{v^m\}_{m=0}^M \) we define the backward differences quotient as
\[
d_t v^m := \frac{v^m - v^{m-1}}{\kappa}.
\]

For the spatial discretization we assume that \( \Omega \subset \mathbb{R}^3 \) is a polyhedral domain
with Lipschitz continuous boundary. Let \( T_h \) denote a family of shape-regular triangulations,
consisting of 3-dimensional simplices \( K \). We denote by \( h_K \) the diameter of \( K \)
and by \( \rho_K \) the supremum of the diameters of inscribed balls. We assume that \( T_h \)
is non-degenerate, i.e., \( \max_{K \in T_h} \frac{h_K}{\rho_K} \leq \gamma_0 \). The global mesh-size \( h \) is defined by
\[
h := \max_{K \in T_h} h_K. \quad \text{Let } S_K \text{ denote the neighborhood of } K, \text{ i.e., } S_K \text{ is the union of all simplices of } T_h \text{ touching } K. \quad \text{By the assumptions we obtain that } |S_K| \sim |K| \quad \text{and that the number of patches } S_K \text{ to which a simple belongs are both bounded uniformly in } h \text{ and } K.
\]

We denote by \( \mathcal{P}_h(T_h) \), with \( k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), the space of scalar or vector-valued functions, which are polynomials of degree at most \( k \) on each \( K \in T_h \). Given a triangulation \( T_h \) of \( \Omega \) with the above properties and given \( r_0 \leq r_1 \in \mathbb{N}_0 \) we denote by \( X_h^{r_1} \) the space
\[
X_h := \left\{ v \in (C(\overline{\Omega}))^3 \mid v \in \mathcal{P} \right\},
\]
with \( \mathcal{P}_r(T_h) \subset \mathcal{P} \subseteq \mathcal{P}_r(T_h) \). Note that there exists a constant \( c = c(r_1, \gamma_0) \) such that for all \( v_h \in X_h, K \in T_h, j \in \mathbb{N}_0 \), and all \( x \in K \) holds
\[
|\nabla^j v_h(x)| \leq c \int_K |\nabla^j v_h(y)| dy. \tag{2.13}
\]

For the weak formulation of the continuous and discrete problems we will use the following function spaces
\[
V := (W_0^{1,p}(\Omega) \cap L^2(\Omega))^3 \quad \text{and} \quad V_h := V \cap X_h.
\]

We also need some numerical interpolation operators. Rather than working with a specific interpolation operator we make the following assumptions:

Assumption 2.14. We assume that \( r_0 = 1 \) and that there exists a linear projection operator \( P_h : (W^{1,1}(\Omega))^3 \rightarrow X_h \) which

\[\text{Note that there is no summation convention over the repeated Latin lower-case index } i \text{ in } \partial_i S(Q) \cdot \partial_i Q.\]
(a) is locally $W^{1,1}$-stable in the sense that
\[
\int K |P_h w| \, dx \leq c \int |w| \, dx + c \int \frac{hK}{hK} |\nabla w| \, dx \quad \forall w \in (W^{1,1}(\Omega))^3, \forall K \in \mathcal{T}_h;
\]
(2.15)

(b) preserves zero boundary values, i.e., $P_h : (W^{1,1}_0(\Omega))^3 \to (W^{1,1}_0(\Omega))^3 \cap X_h$.

Note that, e.g., the Scott-Zhang operator (cf. [28]) satisfies this assumption. The properties of interpolation operators $P_h$ satisfying Assumption 2.14 are discussed in detail in [20, Sec. 4,5], [3, Sec. 3.2]. We collect the for us relevant properties in the next proposition.

**Proposition 2.16.** Let $P_h$ satisfy Assumption 2.14.

(i) Let $F(Dv) \in W^{1,2}(\Omega)$. Then there exists a constant $c = c(p, r_1, \gamma_0)$ such that
\[
\|F(Dv) - F(DP_h v)\|_2 \leq c h \|\nabla F(Dv)\|_2.
\]

(ii) Let $q \in [1,2)$ and $\ell = 1$ or $\ell = 2$ be such that $W^{\ell,q}(\Omega) \hookrightarrow \hookrightarrow L^2(\Omega)$. Then, there exists a constant $c = c(q, \ell, r_1, \gamma_0)$ such that for all $v \in W^{\ell,q}(\Omega)$ holds
\[
\|v - P_h v\|_2 \leq c h^{\ell+3(\frac{1}{2} - \frac{1}{q})} \|\nabla^\ell v\|_q.
\]

(iii) Let $F(Dv) \in W^{1,2}(\Omega)$ and $F(Dw) \in L^2(\Omega)$. Then, there exists a constant $c = c(p, r_1, \gamma_0)$ such that
\[
\int \varphi(Dv)(|DP_h v - DP_h w|) \, dx \leq c h^2 \|\nabla F(Dv)\|_2^2 + c \|F(Dv) - F(Dw)\|_2^2,
\]
where the constants depends only on $\gamma_0$ and $p$.

**Proof.** The first assertion is proved e.g. in [20, Cor. 5.8]. The second assertion is a generalization of the well known approximation property if on both sides there would be the same exponent $q$. Assertion (ii) will be proved in a more general context in the Appendix. Also assertion (iii), which is of more technical character, will be proved in the Appendix. \qed

**2.4. Main results.** Let us now formulate the main result, proving optimal convergence rates for the error between the solution $u$ of the continuous problem \textbf{(parabolic)} and the discrete solution $\{u_h\}_{m=0}^M$ of the space-time discretization \textbf{(1.2)}. Observe that the existence and uniqueness of the solution $\{u_h\}_{m=0}^M$ of the discrete problem \textbf{(1.2)} follows directly from the assumptions on the operator. Moreover, testing \textbf{(1.2)} with $\{u_h\}_{m=0}^M$ yields the energy estimate
\[
\max_{m=1,\ldots,M} \|u_h^m\|_2^2 + \kappa \sum_{m=1}^M \|F(Du_h^m)\|_2^2 \leq C,
\]
for some constant independent of $h, \kappa$.

**Theorem 2.17.** Let the tensor field $S$ in \textbf{(parabolic)} have $(p, \delta)$-structure for some $p \in [1,2]$, and $\delta \in [0,\infty)$ fixed but arbitrary, and let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain with Lipschitz continuous boundary. Assume that $f \in W^{1,2}(I; L^2(\Omega))$, $u_0 \in W^{1,2/p,2}(\Omega)$ and that the solution $u$ of \textbf{(parabolic)} satisfies \textbf{(1.1)} and
\[
F(Du) \in W^{1,2}(I \times \Omega).
\]
(2.18)
Let the space \( V_h \) be defined as above with \( r_0 = 1 \) and let \( \{ u_h^m \}_{m=0}^M \) be solutions of (1.2). Then there exists \( \kappa_0 \in (0, 1] \) such that for given \( h \in (0, 1) \), \( \kappa \in (0, \kappa_0) \), satisfying

\[ h^{4/p'} \leq \sigma_0 \kappa, \]

for some \( \sigma_0 > 0 \), we have the following error estimate

\[ \max_{m=1, \ldots, M} \| u_h^m - u(t_m) \|^2 + \kappa \sum_{m=1}^M \| F(Du_h^m) - F(Du(t_m)) \|^2 \leq c(h^2 + \kappa^2), \]

where the constant \( c \) depends only on the characteristics of \( S \), \( \| F(Du) \|_{W^{1,2}(I \times \Omega)} \), \( \| \partial_t f \|_{L^2(I;L^2(\Omega))} \), \( \| \partial_T f \|_{L^2(I;L^2(\Omega))} \), \( \| \partial_T u \|_{1,2, \gamma_0} \), \( r_1 \), \( \delta \), and \( \sigma_0 \).

**Remark 2.20.** An optimal error estimate for problem (parabolic\(_p\)) with a non-linearity depending on the full gradient \( \nabla u \) under slightly different assumptions has been proved in [13] for \( p > \frac{2d}{d+2} \). The case of evolutionary \( p \)-Navier-Stokes equations, where the nonlinearity depends on the symmetric gradient \( Du \), has been treated in [26, 17, 18, 6] in the case of space periodic boundary conditions. The evolutionary \( p \)-Stokes equations have been treated in [22] in the case of Dirichlet boundary conditions. All these results treat intermediate semi-discrete problems, for which a certain regularity has to be proved, to obtain the desired optimal convergence rates. This in fact limits the results in [26, 17, 18, 6] to the case of space periodic boundary conditions. Here we avoid such problems by proving the error estimate directly without using intermediate semi-discrete problems. The approach can be extended to the treatment of \( p \)-Navier-Stokes equations, which will be done in a forthcoming paper.

In [32, 30] the convergence of a fully implicite space-time discretization (without convergence rate but also with no assumptions of smoothness of the limiting problem) of the evolutionary \( p \)-Navier-Stokes equations in the case of Dirichlet boundary conditions is proved. The convergence of the same numerical scheme (1.2) towards a weak solution has been recently proved in [2] even for general evolution equations with pseudo-monotone operators.

We wish also to mention the recent results in [10] concerning the parabolic problem with a variable exponent.

The regularity assumed in (2.18) is natural in the sense that under certain circumstances the existence of such solutions can be proved.

**Theorem 2.21.** Let the tensor field \( S \) in (parabolic\(_p\)) have \((p, \delta, \gamma)\)-structure for some \( p \in (1, 2] \), and \( \delta \in [0, \infty) \) fixed but arbitrary, and let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with \( C^{2,1} \) boundary. Assume that

\[ f \in L^p(I;L^p(\Omega)) \cap W^{1,2}(I;L^2(\Omega)), \]

and

\[ u_0 \in W^{2,2}(\Omega) \cap W^{1,2}_{0}(\Omega), \text{ with } \text{div} S(Du_0) \in L^2(\Omega). \]

Then, the system (parabolic\(_p\)) has a unique regular solution, i.e., \( u \in L^p(I;W^{1,p}_{0}(\Omega)) \) fulfills

\[ \| u \|^2_{W^{1,\infty}(I;L^2(\Omega))} + \| F(Du) \|^2_{W^{1,2}(I \times \Omega)} \leq c_0, \]

where \( \alpha \) depends only on the characteristics of \( S \), \( \delta \), \( \gamma \), \( \Omega \), \( \| u_0 \|_{2,2} \), \( \| \text{div} S(Du_0) \|_{2,2} \), \( \| f \|^2_{L^p(I \times \Omega)} \), \( \| f \|^2_{L^2(I \times \Omega)} \), \( \| \partial_T f \|^2_{L^2(I \times \Omega)} \), and satisfies (1.1) with \( V = W^{1,p}_{0}(\Omega) \cap L^2(\Omega) \).

\(^2\)Note that the dependence of the constant \( c_0 \) on \( \delta \) is such that \( c_0(\delta) \leq c_0(\delta_0) \) for all \( \delta \leq \delta_0 \).
Remark 2.23. In the literature exist several regularity results which are related to Theorem 2.21. In most cases the regularity is studied for a scalar equation and/or in the steady case with a nonlinearity depending on the full gradient. The main difficulty of our problem is the regularity near the boundary in normal direction. Results in the interior are rather standard, since they can be considered as a special sub-case of the problem with space periodic boundary conditions (cf. [5] and references therein). Most results treating a nonlinearity depending on the symmetric gradient strongly rely on the non-degeneracy of the elliptic operator ($\delta > 0$) and more regular data, see e.g. [9]. In addition, some results concern the case $p > 2$ (cf. [25], [1]), while we are here considering the case $p \in (1, 2)$ which has some very special features already in the steady case.

The results proved here are not covered in the classical literature. A crucial fact is that our problem does not contain a divergence-free constraint. This allows us to prove optimal regularity results up to the boundary (cf. [8] for a treatment of the steady case). For recent results on a related parabolic system cf. [11], [12] and references therein.

The regularity $F(Du) \in W^{1,2}(I \times \Omega)$ can be formulated in terms of Bochner–Sobolev spaces. From [19, Thm. 33] and standard embedding results it follows

$$F(Du) \in L^{\infty}(I; L^3(\Omega)).$$

Since $|Du|^p/2 + \delta^2 \sim |F(Du)| + \delta^2$ we get, by Hölder’s inequality,

$$u \in L^{\infty}(I; W^{1,3p/2}(\Omega)).$$

In [5, Lemma 4.5] it is shown that

$$\frac{\|\nabla^2 u\|_{L^p}^2}{\delta} \leq c \|\nabla F(Du)\|_2^2 + \|\nabla u\|_{3p/2}^{2-p},$$

$$\frac{\|\partial_t \nabla u\|_{L^p}^2}{\delta} \leq c \|\partial_t F(Du)\|_2^2 + \|\nabla u\|_{3p/2}^{2-p}. $$

Thus, we also get

$$u \in L^2(I; W^2, \frac{\delta}{\delta t} (\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(I; W^1, \frac{\delta}{\delta t} (\Omega)), \quad \text{for} \quad p = 1, 2,$$

where the bounds depend only on $\|F(Du)\|_{W^{1,2}(I \times \Omega)}$ and $\delta_0$. This implies in particular that $u \in C(T; W^{1,3p/2}(\Omega))$.

3. On the numerical error

In this section we prove the error estimates from Theorem 2.17. To this end we need to derive the equation for the error and to use the discrete Gronwall lemma together with approximation properties coming from the fact that we have regular enough solutions, together with the assumption on the nonlinear operator S.

3.1. Approximation properties. Crucial properties to estimate the quasi-norm of the finite dimensional projections concern the time regularity of the continuous solution. In particular, the last term in the estimate from Proposition 2.16 (ii) for $v = v(t)$, $w = v(s)$ will give convergence rates with respect to time, under appropriate regularity assumptions on the partial derivative with respect to time. This is based on the following lemma which is in the same spirit as [4, Proposition 3.6] and which will be used several times in the sequel.
Lemma 3.1. Let be given \( f : I \to X \), where \( X \) is a Banach space and let \( f \) be strongly measurable. Let us assume that

\[
\frac{\partial f}{\partial t} \in L^2(I; X).
\]

Then, it holds

\[
\kappa \sum_{m=1}^{M} \int_{I_m} \int_{I_m} \| f(s) - f(t) \|_X^2 ds \, dt \leq \kappa^2 \left\| \frac{\partial f}{\partial t} \right\|_{L^2(I; X)}^2,
\]

\( (3.2) \)

Proof. We prove the first estimate from (3.2). We start, thanks to the Bochner theorem (see Yosida [33, Chap. V.5]), by estimating the difference as follows

\[
\| f(s) - f(t) \|_X = \left\| \int_{t}^{s} \frac{\partial f}{\partial \tau}(\rho) \, d\rho \right\|_X \leq \int_{t}^{s} \left\| \frac{\partial f}{\partial \tau}(\rho) \right\|_X \, d\rho \quad \forall s, t \in I.
\]

Hence, by Cauchy-Schwarz inequality, for all \( s, t \in I_m \subseteq I \), since \( |s - t| \leq \kappa \), it follows

\[
\| f(s) - f(t) \|_X \leq \left( \int_{t}^{s} \left\| \frac{\partial f}{\partial \tau}(\rho) \right\|_X \, d\rho \right)^2 \leq \int_{t}^{s} \left\| \frac{\partial f}{\partial \tau}(\rho) \right\|_X^2 \, d\rho \, |s - t|,
\]

\[
\leq \kappa \int_{t}^{s} \left\| \frac{\partial f}{\partial \tau}(\rho) \right\|_X^2 \, d\rho.
\]

From the latter, we get by a double integration

\[
\int_{I_m} \int_{I_m} \| f(s) - f(t) \|_X^2 \, ds \, dt \leq \kappa \int_{I_m} \int_{I_m} \left\| \frac{\partial f}{\partial \tau}(\rho) \right\|_X^2 \, d\rho \, ds \, dt = \kappa \int_{I_m} \left\| \frac{\partial f}{\partial \tau}(\rho) \right\|_X^2 \, d\rho,
\]

since the right-hand side does not depend on \( s \) and \( t \). Multiplying by \( \kappa \) and summing over \( m \) we get

\[
\kappa \sum_{m=1}^{M} \int_{I_m} \int_{I_m} \| f(s) - f(t) \|_X^2 \, ds \, dt \leq \kappa^2 \kappa \sum_{m=1}^{M} \left\| \frac{\partial f}{\partial \tau}(\rho) \right\|_X^2 \, d\rho = \kappa^2 \left\| \frac{\partial f}{\partial \tau} \right\|_{L^2(I; X)}^2.
\]

The second estimates follows in the same way observing that \( f, \frac{\partial f}{\partial \tau} \in L^2(I; X) \) implies \( f \in C(I; X) \), hence \( f(t_m) \) is well defined. Then, one can re-write the difference term as follows

\[
\| f(s) - f(t_m) \|_X = \left\| \int_{t_m}^{s} \frac{\partial f}{\partial \tau}(\rho) \, d\rho \right\|_X,
\]

and by using the same techniques as before one concludes the proof. \( \square \)

3.2. Error estimates. To prove the Theorem 2.17 we take the retarded averages of (1.1) over \( I_m, m = 1, \ldots, M \)

\[
(d_t \mathbf{u}(t_m), \mathbf{v}) + \int_{I_m} (S(D\mathbf{u}(s)), D\mathbf{v}) \, ds = \int_{I_m} (f(s), \mathbf{v}) \, ds,
\]

(3.3)
which is valid for all \( v \in V \). As usual, we subtract equation (3.3) from (1.2) to obtain the equation for the error

\[
(d_t(u_h^m - u(t_m)), v_h) + \int_{I_m} (S(Du_h^m) - S(Du(s)). Dv) \, ds = \int_{I_m} (f(t_m) - f(s), v) \, ds,
\]

valid for all \( m = 1, \ldots, M \) and for all \( v_h \in V_h \).

**Proposition 3.5.** Under the assumptions of Theorem 2.17 we have the following discrete inequality, valid for \( m = 1, \ldots, M \) and \( 0 < \kappa \leq 1 \)

\[
d_t \|u_h^m - u(t_m)\|^2 + c \|F(Du_h^m) - F(Du(t_m))\|^2 \\
\leq c h^2 \int_{I_m} \|\nabla^2 u(s)\|^2 \, ds + c \int_{I_m} \|F(Du(s)) - F(Du(t_m))\|^2 \, ds \\
+ c \frac{h^{2+4/p'}}{\kappa} \int_{I_m} \|\nabla u(t_m)\|^{2/p} \, ds \\
+ c \frac{h^{4/p'}}{\kappa} \int_{I_m} \|\nabla u(s)\|^{2/p} \, ds,
\]

(3.6)

To prove Proposition 3.5 we treat separately the terms resulting from using in (3.4) the legitimate test function

\[ v_h = u_h^m - P_h u(t_m). \]

We start with the term involving the discrete time-derivative.

**Lemma 3.7.** It holds that

\[
(d_t(u_h^m - u(t_m)), u_h^m - P_h u(t_m)) \\
\geq \frac{1}{2} d_t \|u_h^m - u(t_m)\|^2 + c \frac{h^{2+4/p'}}{\kappa} \int_{I_m} \|\nabla^2 u(s)\|^2 \, ds \\
- c \frac{h^{4/p'}}{\kappa} \|\nabla u(t_m)\|^{2/p} \\
- \frac{1}{2} d_t \|u_h^m - u(t_m)\|^2.
\]

**Proof.** We re-write in this case the test function as follows

\[ v_h = u_h^m - u(t_m) + u(t_m) - P_h u(t_m), \]

(3.8)

to obtain

\[
(d_t(u_h^m - u(t_m)), u_h^m - u(t_m)) = \frac{1}{2} d_t \|u_h^m - u(t_m)\|^2 + \frac{\kappa}{2} d_t \|u_h^m - u(t_m)\|^2.
\]

The remaining term is treated as follows

\[
(d_t(u_h^m - u(t_m)), u(t_m) - P_h u(t_m)) \leq \frac{\kappa}{4} \|d_t(u_h^m - u(t_m))\|^2 + \frac{1}{\kappa} \|u(t_m) - P_h u(t_m)\|^2.
\]

The first term is absorbed in the last term of the previous equality. Note that the second term can not be estimated as \( \|u(t_m) - P_h u(t_m)\|^2 \leq c h^{2+4/p'} \|\nabla^2 u(t_m)\|^{2/p} \) since the right-hand side might be infinite; in view of the regularity of \( u \) the norm \( \|\nabla^2 u(t)\|^{2/p} \) is only finite for almost everywhere \( t \in I \). Thus, we proceed differently and add and subtract (time) mean values. Using that \( P_h \int_{I_m} u(s) \, ds = \)
\[ \int_I P_h u(s) \, ds \], and Fubini’s theorem, we get
\[ \| u(t_m) - P_h u(t_m) \|_2 \]
\[ \leq \left\| u(t_m) - \int_I u(s) \, ds - P_h \left( u(t_m) - \int_I u(s) \, ds \right) \right\|_2 + \left\| \int_I u(s) - P_h u(s) \, ds \right\|_2. \]

Both terms are estimated using Proposition 2.16 (ii) and the regularities (2.24) to obtain
\[ \left\| u(t_m) - \int_I u(s) \, ds - P_h \left( u(t_m) - \int_I u(s) \, ds \right) \right\|_2 \leq c h^{2/\nu'} \| \nabla u(t_m) - \int_I \nabla u(s) \, ds \|_{\frac{4p}{p+4}}, \]
where we used that \( W^{1, \frac{4p}{p+4}}(\Omega) \hookrightarrow L^2(\Omega) \), valid for all \( p > 1 \), and
\[ \left\| \int_I u(s) - P_h u(s) \, ds \right\|_2 \leq c h^{2/\nu'} \int_I \| \nabla^2 u(s) \|_{\frac{4p}{p+4}} \, ds. \]

Putting the estimates together we obtain the assertion. \( \square \)

Next, we estimate the term with the \((p, \delta)\)-structure and obtain the following inequality:

**Lemma 3.9.** It holds that
\[
\int_I \left( S(Du_h^m) - S(Du(t_m)) , Du_h^m - DP_h u(t_m) \right) \, ds
\geq \| F(Du_h^m) - F(Du(t_m)) \|_2^2 - c h^2 \int_I \| \nabla F(Du(s)) \|_2^2 \, ds
\]
\[ - c \int_I \| F(Du(s)) - F(Du(t_m)) \|_2^2 \, ds. \]

**Proof.** We re-write the term with the \( p \)-structure as follows
\[
\int_I \left( S(Du_h^m) - S(Du(t_m)) + S(Du(t_m)) - S(Du(s)) , Du_h^m - DP_h u(t_m) \right) \, ds
\]
\[ = \int_I \left( S(Du_h^m) - S(Du(t_m)) , Du_h^m - DP_h u(t_m) \right) \, ds
\]
\[ + \int_I \left( S(Du(t_m)) - S(Du(s)) , Du_h^m - DP_h u(t_m) \right) \, ds =: A_1 + A_2, \]
and estimate the two terms separately.

**Estimate of \( A_1 \):** We have
\[
A_1 = \left( S(Du_h^m) - S(Du(t_m)) , Du_h^m - Du(t_m) \right) + \left( S(Du_h^m) - S(Du(t_m)) , Du(t_m) - DP_h u(t_m) \right) =: A_{1,1} + A_{1,2}. \]

The first term is giving the information
\[
A_{1,1} = \left( S(Du_h^m) - S(Du(t_m)) , Du_h^m - Du(t_m) \right) \geq c \| F(Du_h^m) - F(Du(t_m)) \|_2^2, \]
while the second can be estimated as follows, by adding and subtracting the average \( \int_{I_m} u(s) \, ds \) in the second entry. In fact, we have

\[
A_{1,2} = \int_{I_m} \left( S(Du_h^n) - S(Du(t_m)), Du(s) - DP_h u(t_m) \right) \, ds \\
+ \int_{I_m} \left( S(Du_h^n) - S(Du(t_m)), Du(t_m) - Du(s) \right) \, ds =: B_1 + B_2.
\]

By Proposition 2.8 it follows

\[
|B_2| \leq \varepsilon \| F(Du(t_m)) - F(Du_h^n) \|_2^2 + c \varepsilon \int_{I_m} \| F(Du(s)) - F(Du(t_m)) \|_2^2 \, ds.
\]

Next, we split \( B_1 \) as follows, by adding and subtracting \( P_h \int_{I_m} u(s) \, ds = \int_{I_m} P_h u(s) \, ds \), again in the second entry,

\[
B_1 = \int_{I_m} \left( S(Du_h^n) - S(Du(t_m)), Du(s) - DP_h u(s) \right) \, ds \\
+ \int_{I_m} \left( S(Du_h^n) - S(Du(t_m)), DP_h u(s) - DP_h u(t_m) \right) \, ds =: C_1 + C_2.
\]

The term \( C_2 \) can be estimated by using Proposition 2.8 (i) and Young’s inequality (2.6) as

\[
C_2 \leq c \int_{I_m} \int_{\Omega} \varphi(\| Du(t_m) \|) \left( \| Du_h^n - Du(t_m) \| \right) \left( \| Du_h^n \| - Du(t_m) \| \right) \, dx \, ds \\
\leq \varepsilon \int_{I_m} \int_{\Omega} \varphi(\| Du(t_m) \|) \left( \| Du_h^n - Du(t_m) \| \right) \, dx \, ds \\
+ c \varepsilon \int_{I_m} \int_{\Omega} \varphi(\| Du(t_m) \|) \left( \| Du_h^n \| - Du(t_m) \| \right) \, dx \, ds \\
\leq \varepsilon \| F(Du_h^n) - F(Du(t_m)) \|_2^2 + c \varepsilon \int_{I_m} \int_{\Omega} \varphi(\| Du(t_m) \|) \left( \| Du_h^n \| - Du(t_m) \| \right) \, dx \, ds.
\]

The latter term from the above inequality can be estimated by a shift change, see Proposition 2.8 (ii)

\[
\int_{\Omega} \varphi(\| Du(t_m) \|) \left( \| Du(h) - Du(t_m) \| \right) \, dx \\
\leq c \| F(Du(t_m)) - F(Du(s)) \|_2^2 + c \varepsilon \int_{\Omega} \varphi(\| Du(s) \|) \left( \| Du(h) - Du(t_m) \| \right) \, dx.
\]

For the last term we use Proposition 2.16 (iii) and obtain

\[
|C_2| \leq \varepsilon \| F(Du_h^n) - F(Du(t_m)) \|_2^2 + c \varepsilon \int_{I_m} \| F(Du(t_m)) - F(Du(s)) \|_2^2 \, ds \\
+ c \varepsilon h^2 \int_{I_m} \| \nabla F(Du(s)) \|_2^2 \, ds.
\]
We estimate now $C_1$ by adding and subtracting $S(Du(s))$ in the first entry and get

$$C_1 = \int_{I_m} (S(Du(s)) - S(Du(t_m))), Du(s) - DP_h u(s)) \, ds$$

$$+ \int_{I_m} (S(Du^m) - S(Du(s))), Du(s) - DP_h u(s)) \, ds =: D_1 + D_2.$$  

Then, by Proposition 2.8 (ii) and Proposition 2.16 (i) it follows

$$|D_1| \leq c \int_{I_m} \|F(Du(s)) - F(Du(t_m))\|^2_2 \, ds + c \int_{I_m} \|F(DP_h u(s)) - F(Du(s))\|^2_2 \, ds$$

$$\leq c \int_{I_m} \|F(Du(s)) - F(Du(t_m))\|^2_2 \, ds + c h^2 \int_{I_m} \|\nabla F(Du(s))\|^2_2 \, ds.$$

The other term $D_2$ is estimated in the following manner, by Proposition 2.8 (ii), by adding and subtracting $F(Du(t_m)), \text{ and Proposition 1.26 (i)}$

$$|D_2| \leq \varepsilon \int_{I_m} \|F(Du(s)) - F(Du^m)\|^2_2 \, ds + c_2 \int_{I_m} \|F(DP_h u(s)) - F(Du(s))\|^2_2 \, ds$$

$$\leq \varepsilon \|F(Du^m) - F(Du(t_m))\|^2_2 + \varepsilon \int_{I_m} \|F(Du(s)) - F(Du(t_m))\|^2_2 \, ds$$

$$+ c_2 h^2 \int_{I_m} \|\nabla F(Du(s))\|^2_2 \, ds.$$

**Estimate of $A_2$:** We now estimate the term $A_2$, first by adding and subtracting $DP_h u(s)$ in the second entry to get

$$A_2 = \int_{I_m} (S(Du(t_m)) - S(Du(s))), DP_h u(s) - DP_h u(t_m)) \, ds$$

$$+ \int_{I_m} (S(Du(t_m)) - S(Du(s)), Du^m - DP_h u(s)) \, ds =: E_1 + E_2.$$  

The term $E_1$ is estimated using Young’s inequality, Proposition 2.8 and Proposition 2.16 (iii)

$$|E_1| \leq c \int_{I_m} \int_{\Omega} \varphi'(Du(s)) \left( |Du(t_m) - Du(s)| \right) |DP_h u(s) - DP_h u(t_m)| \, dx \, ds$$

$$\leq c \int_{I_m} \int_{\Omega} \varphi(Du(s)) (|Du(t_m) - Du(s)|) \, dx \, ds$$

$$+ c \int_{I_m} \int_{\Omega} \varphi(Du(s)) \left( |DP_h u(s) - DP_h u(t_m)| \right) \, dx \, ds$$

$$\leq c \int_{I_m} \|F(Du(t_m)) - F(Du(s))\|^2_2 \, ds + c h^2 \int_{I_m} \|\nabla F(Du(s))\|^2_2 \, ds.$$
The term $E_2$ is estimated by adding and subtracting $D\mathbf{u}(s)$ and $D\mathbf{u}(t_m)$ in the second entry to get
\[E_2 = \int_{I_m} (S(D\mathbf{u}(t_m)) - S(D\mathbf{u}(s)), D\mathbf{u}(s) - DP_h\mathbf{u}(s)) \, ds\]
\[+ \int_{I_m} (S(D\mathbf{u}(t_m)) - S(D\mathbf{u}(s)), D\mathbf{u}(t_m) - D\mathbf{u}(s)) \, ds\]
\[+ \int_{I_m} (S(D\mathbf{u}(t_m)) - S(D\mathbf{u}(s)), D\mathbf{u}_h - D\mathbf{u}(t_m)) \, ds =: E_{2,1} + E_{2,2} + E_{2,3}.

Then, Proposition 2.8 and Proposition 2.16 (i) yield
\[|E_{2,1}| \leq c \int_{I_m} \|F(D\mathbf{u}(s)) - F(D\mathbf{u}(t_m))\|^2 ds + c \int_{I_m} \|F(D\mathbf{u}(s)) - F(DP_h\mathbf{u}(s))\|^2 ds\]
\[\leq c \int_{I_m} \|F(D\mathbf{u}(s)) - F(D\mathbf{u}(t_m))\|^2 ds + c h^2 \int_{I_m} \|\nabla F(D\mathbf{u}(s))\|^2 ds,
\]
as well as
\[E_{2,2} \leq c \int_{I_m} \|F(D\mathbf{u}(s)) - F(D\mathbf{u}(t_m))\|^2 ds,
\]
and
\[|E_{2,3}| \leq \varepsilon \int_{I_m} \|F(D\mathbf{u}_h(t_m)) - F(D\mathbf{u}(t_m))\|^2 ds.
\]

Putting all these estimates together and choosing $\varepsilon$ small enough, we arrive at the estimate in Lemma 3.9. \qed

**Lemma 3.10.** It holds that
\[\left| \int_{I_m} (f(t_m) - f(s), u_h - P_h u(t_m)) \, ds \right|\]
\[\leq c \left\| f(t_m) - \int_{I_m} f \, ds \right\|_2^2 + c \left\| u_h - u(t_m) \right\|_2^2\]
\[+ c h^{2+4/p'} \int_{I_m} \|\nabla^2 u(s)\|_{2/p'}^2 ds + c h^{4/p'} \left\| u(t_m) - \int_{I_m} u(s) \, ds \right\|_1^{2/p}.
\]

**Proof.** Using the splitting (3.8) and Young’s inequality, we get
\[\left| \int_{I_m} (f(t_m) - f(s), u_h - P_h u(t_m)) \, ds \right|\]
\[\leq c \left\| f(t_m) - \int_{I_m} f \, ds \right\|_2^2 + c \left\| u_h - u(t_m) \right\|_2^2 + c \left\| u(t_m) - P_h u(t_m) \right\|_2^2.
\]
The last term was already treated in the proof of Lemma 3.7. There we proved
\[\left\| u(t_m) - P_h u(t_m) \right\|_2^2\]
\[\leq c h^{2+4/p'} \int_{I_m} \|\nabla^2 u(s)\|_{2/p'}^2 ds + c h^{4/p'} \left\| u(t_m) - \int_{I_m} u(s) \, ds \right\|_1^{2/p}.
\]
which yields the assertion. \qed
Proof of Proposition 3.5. The assertion follows from Lemma 3.7, Lemma 3.9 and Lemma (3.10). □

Proof of Theorem 2.17. We now prove the main result. Multiplying (3.6) by $\kappa$ and summing over $m = 1, \ldots, N$, for $N \leq M$, we get

$$
\|u_h^N - u(t_N)\|^2 + \sum_{m=1}^{N} \|F(Du_h^m) - F(Du(t_m))\|^2_2
$$

$$
\leq c h^2 \sum_{m=1}^{N} \int_{I_m} \|\nabla F(Du(s))\|^2_2 \, ds + c \kappa \sum_{m=1}^{N} \int_{I_m} \|F(Du(s)) - F(Du(t_m))\|^2_2 \, ds
$$

$$
+ c \frac{h^{2+4/p'}}{\kappa} \sum_{m=1}^{N} \int_{I_m} \|\nabla^2 u(s)\|^2_{2p'} \, ds + c h^{4/p'} \sum_{m=1}^{N} \|u(t_m) - \int_{I_m} u(s) \, ds\|^2_2 + c \kappa \sum_{m=1}^{N} \|u_h^m - u(t_m)\|^2_2 + c \|u_h^m - u_0\|^2_2.
$$

First we observe that by condition (2.19)

$$
\frac{h^{2+4/p'}}{\kappa} \sum_{m=1}^{N} \int_{I_m} \|\nabla^2 u(s)\|^2_{2p'} \, ds \leq \frac{h^{4/p'}}{\kappa} h^2 \int_{0}^{T} \|\nabla^2 u(s)\|^2_{2p'} \, ds \leq c h^2 \int_{0}^{T} \|\nabla^2 u(s)\|^2_{2p'} \, ds.
$$

Next, by using Lemma 3.1 we have

$$
\kappa \sum_{m=1}^{N} \int_{I_m} \|F(Du(s)) - F(Du(t_m))\|^2_2 \, ds \leq k^2 \int_{0}^{T} \left\|\frac{\partial F(Du)}{\partial t}(s)\right\|^2_2 \, ds,
$$

and also, by using again (2.19),

$$
\frac{h^{4/p'}}{\kappa} \sum_{m=1}^{N} \|u(t_m) - \int_{I_m} u(s) \, ds\|^2_2 \leq \frac{h^{4/p'}}{\kappa} \kappa^2 \int_{0}^{T} \|\nabla u(s)\|^2_{2p'} \, ds \leq c \kappa^2 \int_{0}^{T} \|\nabla u(s)\|^2_{2p'} \, ds.
$$

Moreover, Lemma 3.1 also yields

$$
\kappa \sum_{m=1}^{N} \|f(t_m) - \int_{I_m} f(s) \, ds\|^2_2 \leq c \kappa \int_{0}^{T} \left\|\frac{\partial f}{\partial t}(s)\right\|^2_2 \, ds.
$$

Hence we have, by using (2.24) and the fact that $u_h^m$ is the $L^2$-projection of $u_0$

$$
\|u_h^N - u(t_N)\|^2 + \sum_{m=1}^{N} \|F(Du_h^m) - F(Du(t_m))\|^2_2
$$

$$
\leq c (h^2 + \kappa^2) \int_{0}^{T} \|\nabla F(Du(s))\|^2_2 + \left\|\frac{\partial F(Du)}{\partial t}(s)\right\|^2_2 + \left\|\frac{\partial f}{\partial t}(s)\right\|^2_2 + \left\|\frac{\partial u}{\partial t}(s)\right\|^2_{2p'} + \|\nabla^2 u(s)\|^2_{2p'} \, ds
$$

$$
+ c \kappa \sum_{m=1}^{N} \|u_h^m - u(t_m)\|^2_2 + c h^2 \|\nabla u_0\|^2_2.
$$
Then, if $\kappa > 0$ is small enough such that $c\kappa < 1$, we can absorb the last addendum in the sum from the right-hand side and obtain (using the regularity of $u$ to bound the time integrals in the above formula)

$$
\|u_N^h - u(t_N)\|_2^2 + \sum_{m=1}^{N} \|F(Du_m^h) - F(Du(t_m))\|_2^2 \\
\leq c(h^2 + \kappa^2) + c\kappa \sum_{m=1}^{N-1} \|u_m^h - u(t_m)\|_2^2.
$$

The discrete Gronwall lemma yields the assertion. □

4. On the existence and uniqueness of regular solutions

In this section we prove Theorem 2.21, i.e., the existence and uniqueness of regular solutions of (parabolic), solely based on appropriate assumptions on the data. To this end we proceed as in [8] and treat a perturbed problem, obtained by adding to the tensor field $S$ with $(p, \delta)$-structure a linear perturbation. We use this approximation to justify the computations that follow and to avoid some technical problems related with the case $p \in (1, 6/5)$ and the lack of an evolution triple in this range. From now on we restrict ourselves to the case that $S$ has $(p, \delta)$-structure for some $p \in (1, 2]$, $\delta \in [0, \infty)$. Let $f \in L^p(I \times \Omega)$ and $u_0 \in L^2(\Omega)$ be given.

4.1. The perturbed problem and some global regularity in the time variable. We have the following result on existence and uniqueness of time-regular solutions of the perturbed problem.

**Proposition 4.1.** Let the tensor field $S$ have $(p, \delta)$-structure for some $p \in (1, 2]$, and $\delta \in [0, \infty)$ and let $f \in L^p(I \times L^p(\Omega)) \cap W^{1,2}(I; L^2(\Omega))$ and $u_0 \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)$ with $\text{div}(S^\varepsilon(Du_0)) \in L^2(\Omega)$ be given. Then, the perturbed problem

$$
\begin{align*}
\frac{\partial u}{\partial t} - \text{div} S^\varepsilon(Du) &= f & \text{in } I \times \Omega, \\
u_e &= 0 & \text{on } I \times \partial \Omega, \\
u_e(0) &= u_0 & \text{in } \Omega,
\end{align*}
$$

where

$$
S^\varepsilon(Q) := \varepsilon Q + S(Q), \quad \text{with } \varepsilon > 0,
$$

possesses a unique time-regular solution $u_e$, i.e., $u_e \in W^{1,\infty}(I; L^2(\Omega)) \cap W^{1,2}(I; W^{1,2}_0(\Omega))$ with $F(Du_e) \in W^{1,2}(I; L^2(\Omega))$ satisfies for all $\psi \in C_0^\infty(I)$ and all $v \in W^{1,2}_0(\Omega)$

$$
\int_0^T \left( \frac{\partial u}{\partial t}(t), v \right) \psi(t) \, dt + \int_0^T (S^\varepsilon(Du(t)), Dv) \psi(t) \, dt = \int_0^T (f(t), v) \psi(t) \, dt,
$$

(4.3)
and \( \mathbf{u}_\varepsilon(0) = \mathbf{u}_0 \) in \( W^{1,2}_0(\Omega) \). In addition, the solution \( \mathbf{u}_\varepsilon \) satisfies for a.e. \( t \in I \) the estimates\(^3\)

\[
\frac{1}{2} \| \mathbf{u}_\varepsilon(t) \|_2^2 + \int_0^t \int \varepsilon \| \nabla \mathbf{u}_\varepsilon \|_2^2 + \varphi(\| \mathbf{D} \mathbf{u}_\varepsilon \|) \, dx \, ds \\
\leq \frac{1}{2} \| \mathbf{u}_0 \|_2^2 + c(\delta, \Omega) \int_0^T \| \mathbf{f}(s) \|_{L^p}^p \, ds,
\]

\[\frac{\| \partial_t \mathbf{u}_\varepsilon(t) \|_2^2}{2} + \int_0^t \varepsilon \| \partial_t \mathbf{D} \mathbf{u}_\varepsilon(s) \|_2^2 + \| \partial_t \mathbf{F} \mathbf{D} \mathbf{u}_\varepsilon(s) \|_2^2 \, ds \\
\leq C(\delta, \Omega) \left( \| \mathbf{u}_0 \|_{2,2,2} + \| \operatorname{div}(S \mathbf{D} \mathbf{u}_0) \|_2^2 + \int_0^T \| \mathbf{f}(s) \|_2^2 + \| \partial_t \mathbf{D} \mathbf{u}_\varepsilon(s) \|_2^2 \right) \, ds \right).
\]

The estimate (4.4) and (4.5) imply that \( \mathbf{u}_\varepsilon \in L^\infty(I; L^2(\Omega)) \), \( \mathbf{F}(\mathbf{D} \mathbf{u}_\varepsilon) \in L^2(I \times \Omega) \); and \( \mathbf{u}_\varepsilon \in W^{1,\infty}(I; L^2(\Omega)) \), \( \mathbf{F}(\mathbf{D} \mathbf{u}_\varepsilon) \in W^{1,4}(I; L^2(\Omega)) \), resp., with bounds independent of \( \varepsilon > 0 \).

\textbf{Proof.} The proof is based on a standard Galerkin approximation. The existence of the Galerkin approximations follows from the standard theory of systems of ordinary differential equations. Estimate (4.4) is proved on the Galerkin level by testing with the Galerkin approximations. Estimate (4.5) is obtained by differentiating the Galerkin equations with respect to time and testing with the time derivative of the Galerkin approximation. We refer to [5, 19] for more details. Note that the regularity is enough to justify all calculations and to employ the Gronwall lemma to prove uniqueness. \[\square\]

\textbf{Remark 4.6.} Note that by the fundamental theorem on the calculus of variations the weak formulation (4.3) is equivalent to

\[
\left( \frac{\partial}{\partial t} \mathbf{u}(t), \mathbf{v} \right) + \left( S^\varepsilon(\mathbf{D} \mathbf{u}(t)), \mathbf{D} \mathbf{v} \right) = (\mathbf{f}(t), \mathbf{v}) \quad \text{a.e. } t \in I, \ \forall \mathbf{v} \in W^{1,2}_0(\Omega).
\]

In order to prove existence and uniqueness of regular solutions to (parabolic\(_c\)), by taking the limit \( \varepsilon \to 0^+ \), we need to prove further regularity for the solution \( \mathbf{u}_\varepsilon \), namely on the second order spatial derivatives. The regularity in the spatial variables requires an ad hoc treatment (localization) for the Dirichlet boundary value problem. To do this we adapt the argument in [8] for the steady problem, to handle the parabolic problem. We sketch the relevant steps, pointing out the main new aspects which are present in the time-dependent case.

\textbf{Remark 4.8.} In the space periodic case the requested regularity for the spatial derivatives can be obtained simply by testing (again the Galerkin approximations) with \( -\Delta \mathbf{u} \), as in [5] to prove for a.e. \( t \in I \) the inequality

\[
\frac{1}{2} \| \nabla \mathbf{u}_\varepsilon(t) \|_2^2 + \int_0^t \varepsilon \| \Delta \mathbf{u}_\varepsilon(s) \|_2^2 + \| \nabla \mathbf{F} \mathbf{D} \mathbf{u}_\varepsilon(s) \|_2^2 \, ds \\
\leq \frac{1}{2} \| \nabla \mathbf{u}_0 \|_2^2 + c \int_0^T \| \mathbf{f}(s) \|_{L^p}^p \, ds,
\]

with \( c \) depending only on \( \delta \) and \( \Omega \).

\(^3\)Note that \( c(\delta) \) only indicates that the constant \( c \) depends on \( \delta \) and will satisfy \( c(\delta) \leq c(\delta_0) \) for all \( \delta \leq \delta_0 \).
4.2. Description and properties of the boundary. We assume that the boundary $\partial \Omega$ is of class $C^{2,1}$, that is for each point $P \in \partial \Omega$ there are local coordinates such that in these coordinates we have $P = 0$ and $\partial \Omega$ is locally described by a $C^{2,1}$-function, i.e., there exist $R_P$, $R'_P \in (0, \infty)$, $r_P \in (0, 1)$ and a $C^{2,1}$-function $a_P : B^3_{R_P}(0) \to B^3_{R'_P}(0)$ such that

(b1) \( x \in \partial \Omega \cap (B^2_{R_P}(0) \times B^2_{R'_P}(0)) \iff x_3 = a_P(x_1, x_2) \),

(b2) \( \Omega_P := \{(x, x_3) \mid x = (x_1, x_2)^\top \in B^2_{R_P}(0), a_P(x) < x_3 < a_P(x) + R'_P \} \subset \Omega \),

(b3) \( \nabla a_P(0) = 0 \), and \( \forall x = (x_1, x_2)^\top \in B^2_{R_P}(0) \mid \nabla a_P(x) \mid < r_P \),

where $B^k(0)$ denotes the $k$-dimensional open ball with center 0 and radius $r > 0$. Note that $r_P$ can be made arbitrarily small if we make $R_P$ small enough. In the sequel we will also use, for $0 < \lambda < 1$, the scaled open sets $\lambda \Omega_P \subset \Omega_P$, defined as follows

$$\lambda \Omega_P := \{(x, x_3) \mid x = (x_1, x_2)^\top \in B^3_{\lambda R_P}(0), a_P(x) < x_3 < a_P(x) + \lambda R'_P \}.$$  

To localize near $\partial \Omega \cap \partial \Omega_P$, for $P \in \partial \Omega$, we fix smooth functions $\xi_P : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\begin{align*}
(1) \quad & \chi_{\Omega_P}(x) \leq \xi_P(x) \leq \chi_{\Omega_P}(x),
\end{align*}$$

where $\chi_A(x)$ is the indicator function of the measurable set $A$. For the remaining interior estimate we localize by a smooth function $0 \leq \chi_{00} \leq 1$ with $\text{spt} \xi_{00} \subset \partial \Omega$, where $\Omega_{00} \subset \Omega$ is an open set such that $\text{dist}(\partial \Omega_{00}, \partial \Omega) > 0$. Since the boundary $\partial \Omega$ is compact, we can use an appropriate finite sub-covering which, together with the interior estimate, yields the global estimate.

Let us introduce the tangential derivatives near the boundary. To simplify the notation we fix $P \in \partial \Omega$, $h \in (0, \frac{R_P}{10})$, and simply write $\xi := \xi_P$, $a := a_P$. We use the standard notation $x = (x', x_3)^\top$ and denote by $e_i$, $i = 1, 2, 3$ the canonical orthonormal basis in $\mathbb{R}^3$. In the following lower-case Greek letters take values 1, 2. For a function $g$ with $\text{spt} g \subset \text{spt} \xi$ we define for $\alpha = 1, 2$ tangential translations:

$$g_\tau(x', x_3) = g_{\tau_\alpha}(x', x_3) := g(x' + h e^\alpha, x_3 + a(x' + h e^\alpha) - a(x')), $$

tangential differences $\Delta^\alpha g := g_\tau - g$, and tangential divided differences $d^\alpha g := h^{-1} \Delta^\alpha g$. It holds that, if $g \in W^{1,1}(\Omega)$, then we have for $\alpha = 1, 2$

$$d^\alpha g \to \partial_\tau g = \partial_{\tau_\alpha} g := \partial_\tau g + \partial_\alpha a \partial_\tau g \quad \text{as } h \to 0,$$

almost everywhere in $\text{spt} \xi$, (cf. [25, Sec. 3]). Moreover, uniform $L^q$-bounds for $d^\alpha g$ imply that $\partial_\tau g$ belongs to $L^q(\text{spt} \xi)$. More precisely, if we define, for $0 < h < R_P$

$$\Omega_{P,h} = \{x \in \Omega_P \mid x' \in B^2_{R_P-h}(0)\},$$

and if $f \in W^{1,q}_{\text{loc}}(\mathbb{R}^3)$, then

$$\int_{\Omega_{P,h}} |d^\alpha f|^q \, dx \leq c \int_{\Omega_P} |\partial_\tau f|^q \, dx.$$  

Moreover, if $d^\alpha f \in L^q(\Omega_{P, h_0})$, for all $0 < h_0 < R_P$ and all $0 < h_0$ and if

$$\exists c_1 > 0 : \int_{\Omega_{P, h_0}} |d^\alpha f|^q \, dx \leq c_1 \quad \forall h_0 \in (0, R_P) \quad \text{and} \quad \forall h \in (0, h_0),$$

then $\partial_\tau f \in L^q(\Omega_P)$ and

$$\int_{\Omega_P} |\partial_\tau f|^q \, dx \leq c_1.$$  

For simplicity we denote $\nabla a := (\partial_1 a, \partial_2 a, 0)^\top$. The following variant of formula of integration by parts will be often used.
Lemma 4.10. Let \( \text{spt } g \cup \text{spt } f \subset \text{spt } \xi = \text{spt } \xi_P \) and \( 0 < h < \frac{R_P}{16} \). Then

\[
\int_{\Omega} fg_{-\tau} \, dx = \int_{\Omega} f g \, dx.
\]

Consequently, \( \int_{\Omega} f d^+ g \, dx = \int_{\Omega} (d^- f) g \, dx \). Moreover, if in addition \( f \) and \( g \) are smooth enough and at least one vanishes on \( \partial \Omega \), then

\[
\int_{\Omega} f \partial_\tau g \, dx = - \int_{\Omega} (\partial_\tau f) g \, dx.
\]

4.3. A first regularity result in space. We start proving spatial regularity for the perturbed problem in the non-degenerate case \( \delta > 0 \). The estimates proved in this intermediate step are uniform with respect to \( \varepsilon > 0 \) and \( \delta > 0 \) in: a) the interior and b) in the case of tangential derivatives; estimates depend on \( \varepsilon, \delta > 0 \) in the normal direction. Nevertheless, this allows later on to use the equations pointwise to prove in a different way estimates independent of \( \varepsilon, \delta > 0 \) even near the boundary. Thus, we can pass to the limit with \( \varepsilon \to 0 \) to treat the original problem in the non-degenerate case. Finally, the degenerate case is treated by a suitable approximation using that the estimates are independent of \( \delta > 0 \).

We observe that by using a translation method, the result is proved directly for solutions and not anymore for the Galerkin approximations.

Proposition 4.11. Let the tensor field \( S \) in (4.2) have \((p, \delta)\)-structure for some \( p \in [1, 2] \) and \( \delta \in (0, \infty) \), and let \( F \) be the associated tensor field to \( S \). Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with \( C^{2,1} \) boundary and let \( u_0 \in W_0^{1,2}(\Omega) \) and \( f \in L^p(I \times \Omega) \). Then, the unique time-regular solution \( u_c \) of the approximate problem (4.2) satisfies for a.e. \( t \in I \)

\[
\left\| \xi_0^2 \nabla u_c(t) \right\|_2^2 + \int_0^t \int_\Omega \xi_0^2 |\nabla^2 u_c|^2 + \xi_0^2 |\nabla F(Du_c)|^2 \, dx \, ds \\
\leq c(\|u_0\|_{1,2}, \|f\|_{L^p(I \times \Omega)}, \|\xi_0\|_{2,\infty}, \delta),
\]

(4.12)

\[
\left\| \xi_0^2 \partial_t u_c(t) \right\|_2^2 + \int_0^t \int_\Omega \xi_0^2 |\partial_t Du_c|^2 + \xi_0^2 |\partial_t F(Du_c)|^2 \, dx \, ds \\
\leq c(\|u_0\|_{1,2}, \|f\|_{L^p(I \times \Omega)}, \|\xi_0\|_{2,\infty}, \|\partial_P\|_{C^{2,1}, \delta}).
\]

Here \( \xi_0(x) \) is a cut-off function with support in the interior of \( \Omega \) and, for arbitrary \( P \in \partial \Omega \), the tangential derivative is defined locally in \( \Omega_P \) by (4.9).

Moreover, there exists a constant \( C_1 > 0 \) such that \(^4\) for a.e. \( t \in I \)

\[
\left\| \xi_0^2 \partial_\tau u_c(t) \right\|_2^2 + \int_0^t \int_\Omega \xi_0^2 |\partial_\tau Du_c|^2 + \xi_0^2 |\partial_\tau F(Du_c)|^2 \, dx \, ds \\
\leq c(\|u_0\|_{1,2}, \|f\|_{L^p(I \times \Omega)}, \|\xi_0\|_{2,\infty}, \|\partial_P\|_{C^{2,1}, \delta^{-1}, \varepsilon^{-1}, C_1}),
\]

(4.13)

provided that in the local description of the boundary there holds \( r_P < C_1 \) in (b3), where \( \xi_P(x) \) is a cut-off function with support in \( \Omega_P \).

Remark 4.14. Proposition 4.11 and Proposition 4.1 imply that \( u_c(t) \in W^{2,2}(\Omega) \) and \( \frac{\partial u_c}{\partial n}(t) \in L^2(\Omega) \) for a.e. \( t \in I \). Hence, equations (4.2) hold pointwise a.e. in \( I \).

\(^4\)Recall that \( c(\delta^{-1}) \) indicates a possibly critical dependence on \( \delta \) as \( \delta \to 0 \).
Proof of Proposition 4.11. Fix $P \in \partial \Omega$ and use in $\Omega_P$

$$\mathbf{v} = d^- (\xi^2 d^+ (u_\xi|_{\Omega_P})), $$

where $\xi := \xi_P$, $a := a_P$, and $h \in (0, \frac{\rho}{\kappa})$, as a test function in (4.3). This yields, after integration by parts over $\Omega$, for a.e. $t \in I$

$$\int_{\Omega} \xi^2 d^+ \frac{\partial u_\xi}{\partial t}(t) \cdot d^+ u_\xi(t) \, dx + \int_{\Omega} \xi^2 d^+ S^s(Du_\xi(t)) \cdot d^+ Du_\xi(t) \, dx $$

$$= - \int_{\Omega} S^s(Du_\xi(t)) \cdot (\xi^2 d^+ \partial_3 u_\xi(t) - (\xi d^- \xi + \xi d^- \xi) \partial_3 u_\xi(t)) \otimes d^- \nabla a \, dx $$

$$- \int_{\Omega} S^s(Du_\xi(t)) \cdot \xi^2(\partial_3 u_\xi(t))_{\xi} \otimes d^- \nabla a - S^s(Du_\xi(t)) \cdot d^- (2\xi \nabla \xi \xi \otimes d^+ u_\xi(t)) \, dx $$

$$+ \int_{\Omega} S^s((Du_\xi(t))_{\xi}) \cdot (2\xi \partial_3 \xi d^+ u_\xi(t) + \xi^2 d^+ \partial_3 u_\xi(t)) \otimes d^+ \nabla a \, dx $$

$$+ \int_{\Omega} f(t) \cdot d^- (\xi^2 d^+ u_\xi(t)) \, dx =: \sum_{j=1}^8 I_j.$$

Hence, by using the estimates for $I_j$ as in [8, Proposition 3.1] (see also [7, Proposition 4.4]) and by observing that $d^+ \frac{\partial u_\xi}{\partial t} = \frac{\partial d^+ u_\xi}{\partial t}$, one gets

$$\frac{d}{dt} \frac{1}{2} \int_{\Omega} \xi^2 |d^+ u_\xi(t)|^2 \, dx + \varepsilon \int_{\Omega} \xi^2 |d^+ \nabla u_\xi(t)|^2 + \xi^2 |\nabla d^+ u_\xi(t)|^2 \, dx $$

$$+ \int_{\Omega} \xi^2 |d^+ F(Du_\xi(t))|^2 + \varphi(\xi |d^+ \nabla u_\xi(t)|) + \varphi(\xi |\nabla d^+ u_\xi(t)|) \, dx $$

$$\leq c(||\xi||_{2, \infty}, ||a||_{C^{2,1}}, \delta) \int_{\Omega} |f(t)|^\rho + \varepsilon |\nabla u_\xi(t)|^2 + \varphi(|Du_\xi(t)|) \, dx,$$

and, after integration over $[0, t] \subseteq I$ and the use of the a priori estimate (4.4), we get

$$\frac{1}{2} \int_{\Omega} \xi^2 |d^+ u_\xi(t)|^2 \, dx + \varepsilon \int_0^t \int_{\Omega} \xi^2 |d^+ \nabla u_\xi(s)|^2 + \xi^2 |\nabla d^+ u_\xi(s)|^2 \, dx \, ds $$

$$+ \int_0^t \int_{\Omega} \xi^2 |d^+ F(Du_\xi(s))|^2 + \varphi(\xi |d^+ \nabla u_\xi(s)|) + \varphi(\xi |\nabla d^+ u_\xi(s)|) \, dx \, ds $$

$$\leq \frac{1}{2} \int_{\Omega} |d^+ u_\xi_0|^2 \, dx + \frac{1}{2} ||u_\xi_0||^2 + c(||\xi||_{2, \infty}, ||a||_{C^{2,1}}, \delta) \int_0^T ||f(t)||^p_{p'} \, dt $$

$$\leq \frac{1}{2} ||u_\xi_0||^2_{1,2} + c(||\xi||_{2, \infty}, ||a||_{C^{2,1}}, \delta) \int_0^T ||f(t)||^p_{p'} \, dt,$$

from which (4.12) follows by standard arguments, using that the estimates are independent of $h > 0$.

The same argument used with a test function $\xi_{\varepsilon_0}$ with compact support in $\Omega$, and standard finite differences can be used to prove (4.13): this implies that

$$u_\xi \in L^\infty(I; W^{1,2}_{loc}(\Omega)) \cap L^2(I; W^{2,2}_{loc}(\Omega)) \quad \text{and} \quad F(Du_\xi) \in L^2(I; W^{1,2}_{loc}(\Omega)).$$
Coupling the latter with the time regularity from Proposition 4.1 one obtains that the equations (4.2) hold pointwise a.e. in $I \times \Omega$.

We prove now the result on the regularity in the normal direction from (4.13). We re-write the equations in (4.2) as follows

$$-\frac{\partial u_i^t}{\partial t} + \sum_{k=1}^{3} \partial_{k3}S_{i3}^c(Du_k)\partial_{3}D_{33}u_3 + \partial_{3\alpha}S_{i3}^c(Du_\alpha)\partial_{3}D_{3\alpha}u_\alpha = f^i \quad \text{a.e. in } I \times \Omega,$$

where $f^i := -f^i - \partial_{i\sigma}S_{i3}^c(Du_\sigma)\partial_{3}D_{33}u_3 - \sum_{k,l=1}^{3} \partial_{k3}S_{i3}^c(Du_k)\partial_{l3}D_{3\alpha}u_\alpha$, $i = 1, 2, 3$.

We now proceed as in [8, Eq. (3.3)] and we multiply these equations by $\partial_{3}D_{3\alpha}u_\alpha$, where $\tilde{D}_{\alpha\beta}u_\epsilon = 0$, for $\alpha, \beta = 1, 2$, $\tilde{D}_{\alpha3}u_\epsilon = \tilde{D}_{3\alpha}u_\epsilon = 2D_{\alpha3}u_\epsilon$ for $\alpha = 1, 2$, and sum over $i = 1, 2, 3$. In this way we get a lower bound on the nonlinear term from the left-hand-side in such a way that

$$-\sum_{i=1}^{3} \frac{\partial u_i^t}{\partial t} \partial_{3}D_{3\alpha}u_\alpha + (\varepsilon + \varphi''(|Du_\alpha|))|\tilde{b}|^2 \leq c|f||\tilde{b}| \quad \text{a.e. in } I \times \Omega,$$

where $\tilde{b}_i := \partial_{3\alpha}D_{3\alpha}u_\alpha$.

By straightforward manipulations (cf. [8, Sections 3.2 and 4.2]) we have that for a.e. $(t, x)$ in $I \times \Omega_P$

$$|f| \leq c (|f| + (\varepsilon + \varphi''(|Du_\alpha|)) (|\partial_{\alpha}u_\alpha| + \|\nabla a\|_\infty|\nabla^2 u_\alpha|)),$$

$$|\tilde{b}| \geq 2|\tilde{b}| - |\partial_{\alpha}u_\alpha| - \|\nabla a\|_\infty|\nabla^2 u_\alpha|,$$

for $\tilde{b}_i := \partial_{3\alpha}u_\alpha$, $i = 1, 2, 3$. These results imply that a.e. in $I \times \Omega_P$

$$-\sum_{i=1}^{3} \frac{\partial u_i^t}{\partial t} \partial_{3}D_{33}u_3 + (\varepsilon + \varphi''(|Du_\alpha|))|\tilde{b}|^2$$

$$\leq c (|f| + (\varepsilon + \varphi''(|Du_\alpha|)) (|\partial_{\alpha3}u_\alpha| + \|\nabla a\|_\infty|\nabla^2 u_\alpha|))|\tilde{b}|.$$

We then add on both sides, for $\alpha = 1, 2$ and $i, k = 1, 2, 3$ the term

$$(\varepsilon + \varphi''(|Du_\alpha|)) |\partial_{\alpha3}u_\alpha|^2,$$

and estimate $\tilde{b}$ with all second order spatial derivatives obtaining

$$-\sum_{i=1}^{3} \frac{\partial u_i^t}{\partial t} \partial_{3i}D_{33}u_3 + (\varepsilon + \varphi''(|Du_\alpha|))|\nabla^2 u_\alpha|^2$$

$$\leq \frac{\varepsilon}{4} |\nabla^2 u_\alpha|^2 + \frac{1}{4} (\varepsilon + \varphi''(|Du_\alpha|))|\nabla^2 u_\alpha|^2$$

$$+ c (|f|^2 + (\varepsilon + \varphi''(|Du_\alpha|)) (|\partial_{\alpha3}u_\alpha|^2 + \|\nabla a\|_\infty|\nabla^2 u_\alpha|)),$$

where in the right-hand side we used also the definition of the tangential derivative (cf. (4.9)). Next, we choose the open sets $\Omega_P$ in such a way that $\|\nabla a\|_\infty = \|\nabla u_P(x_1, x_2)\|_\infty, \Omega_P$ is small enough, so that we can absorb the last term from the right-hand side. We finally arrive at the following pointwise inequality

$$-\sum_{i=1}^{3} \frac{\partial u_i^t}{\partial t} \partial_{3i}D_{33}u_3 + (\varepsilon + \varphi''(|Du_\alpha|))|\nabla^2 u_\alpha|^2$$

$$\leq c (|f|^2 + (\varepsilon + \varphi''(|Du_\alpha|)) (|\partial_{\alpha3}u_\alpha|^2)) \quad \text{a.e. in } I \times \Omega_P.$$

We neglect $\varphi''(|Du_\alpha|)$ (which is non-negative) from the left-hand side, multiply by $\xi^2$, and integrate in the spatial variable over the whole domain $\Omega$. In particular,
since \(u_\varepsilon\) and \(\frac{\partial u_\varepsilon}{\partial t}\) both vanish on \(I \times \partial \Omega\), the first term coming from the left-hand side of (4.15) can be written as follows by performing some integration by parts:

\[
\int_{\Omega} - \sum_{i=1}^{3} \xi^2 \frac{d}{dt} \frac{\partial u_i^\varepsilon}{\partial t} \partial_3 D_{13} u_\varepsilon \, d\mathbf{x}
\]

\[
= \int_{\Omega} \frac{\xi^2}{2} \frac{d}{dt} |\partial_3 u_\varepsilon|^2 - \sum_{\alpha=1}^{2} \xi^2 \frac{\partial u_\varepsilon^\alpha}{\partial t} \partial_3 u_\varepsilon^\alpha + \sum_{i=1}^{3} \xi \partial_3 \xi \frac{\partial u_i^\varepsilon}{\partial t} \partial_3 u_i^\varepsilon \, d\mathbf{x}.
\]

After a further integration over \([0, t] \subseteq I\) we obtain from (4.15), the tangential regularity already proved in (4.12) and Korn’s inequality the following inequality

\[
\|\xi^2 \partial_3 u_\varepsilon(t)\|_2^2 + \varepsilon \int_0^t \int_{\Omega} |\nabla^2 u_\varepsilon|^2 \, d\mathbf{x} \, ds \\
\leq \|\xi^2 \partial_3 u_\varepsilon\|_2^2 + c(\varepsilon, \|\xi\|_{2, \infty}, \|a\|_{C^{2, 1}}, \delta^{-1}) \int_0^T \|f(s)\|_{L^p}^p + \left\|\frac{\partial u_\varepsilon}{\partial t}(s)\right\|_2^2 \, ds,
\]

hence the boundedness of the right-hand side, by using Proposition 4.1.

With this estimate and recalling the properties of the covering we finish the proof. \(\square\)

4.4. Uniform estimates for the second order spatial derivatives. We now improve the estimate in the normal direction in the sense that we will show that they are bounded uniformly with respect to \(\varepsilon, \delta \geq 0\). The used method is an adaption to the time evolution problem of the treatment in [8], which is based on previous results from [29].

**Proposition 4.16.** Let the same hypotheses as in Theorem 2.21 be satisfied with \(\delta > 0\) and let the local description \(a_P\) of the boundary and the localization function \(\xi_P\) satisfy (b1)-(b3) and (f1) (cf. Section 4.2). Then, there exists a constant \(C_2 > 0\) such that the time-regular solution \(u_\varepsilon \in L^\infty(I; W^{2, 2}_0(\Omega)) \cap L^2(I; W^{2, 2}(\Omega))\) of the approximate problem (4.2) satisfies\(^5\) for every \(P \in \partial \Omega\) and for a.e. \(t \in I\)

\[
\int_{\Omega} \xi_P^2 |\partial_3 u_\varepsilon(t)|^2 \, d\mathbf{x} + \int_{\Omega} |\xi_P^2 |\partial_3 D u_\varepsilon|^2 | + \xi_P^2 |\partial_3 F(D u_\varepsilon)|^2 \, d\mathbf{x} \, ds \leq c,
\]

provided \(r_P < C_2\) in (b3), with \(c\) depending on \(\|u_\varepsilon\|_{2, 2}, \|\text{div} S(D u_\varepsilon)\|_2, \|f\|_{L^p(I \times \Omega)}, \|f\|_{L^2(I \times \Omega)}, \|\frac{\partial f}{\partial t}\|_{L^2(I \times \Omega)}, \|\xi_P\|_{2, \infty}, \|a_P\|_{C^{2, 1}}, \delta, C_2\).

**Proof.** We adapt the strategy as in [8, Proposition 3.2] to the time-dependent problem. Fix an arbitrary point \(P \in \partial \Omega\) and a local description \(a = a_P\) of the boundary and the localization function \(\xi = \xi_P\) satisfying (b1)-(b3) and (f1). In the following we denote by \(C\) constants that depend only on the characteristics of \(S\). First we observe that, by the results of Proposition 2.9 there exists a constant \(C_0\), depending only on the characteristics of \(S\), such that\(^6\)

\[
\frac{1}{C_0} |\partial_3 F(D u_\varepsilon)|^2 \leq P_3(D u_\varepsilon) \quad \text{a.e. in } I \times \Omega.
\]

\(^5\)Recall that \(c(\delta)\) only indicates that the constant \(c\) depends on \(\delta\) and will satisfy \(c(\delta) \leq c(\delta_0)\) for all \(\delta \leq \delta_0\).

\(^6\)In this section we do not write explicitly the dependence on space and time variables, since the reader at this point will be acquainted enough with the matter to avoid heavy notation.
Thus, we get, using also the symmetry of both $Du$ and $S$,

$$
\int_\Omega \varepsilon \xi |\partial_\beta Du_x|^2 + \frac{1}{C_0} \xi^2 |\partial_3 F(Du_x)|^2 \, dx \\
\leq \int_\Omega \xi^2 (\varepsilon \partial_3 D_{\alpha\beta} u_x + \partial_3 S_{\alpha\beta}(Du_x)) \partial_3 D_{\alpha\beta} u_x \, dx \\
+ \int_\Omega \xi^2 (\varepsilon \partial_3 D_{\alpha\beta} u_x + \partial_3 S_{\alpha\beta}(Du_x)) \partial_\alpha D_{3\beta} u_x \, dx \\
+ \int_\sum_j \xi^2 \partial_3 (\varepsilon D_{j\beta} u_x + S_{j\beta}(Du_x)) \partial_3 u_x^j \, dx =: J_1 + J_2 + J_3.
$$

The terms $J_1$ and $J_2$ can be estimated exactly as in [8] to prove, for $\lambda > 0$, that

$$
|J_1| + |J_2| \leq \lambda \int_\Omega \xi^2 |\partial_\beta F(Du_x)|^2 + \varepsilon \xi^2 |\partial_3 Du_x|^2 \, dx \\
+ c_{\lambda^{-1}} (1 + \|\nabla a\|_\infty^2) \sum_{\beta=1}^2 \int_\Omega \xi^2 |\partial_\beta F(Du_x)|^2 + \varepsilon \xi^2 |\partial_3 Du_x|^2 \, dx \\
+ c_{\lambda^{-1}} \sum_{\beta=1}^2 \int_\Omega \xi^2 |\partial_\beta F(Du_x)|^2 + \varepsilon \xi^2 |\partial_3 Du_x|^2 \, dx \\
+ c_{\lambda^{-1}} (1 + \|\nabla \xi\|_\infty^2 + \|\nabla a\|_\infty^2) \int_\Omega |\partial_\beta F(Du_x)| + \varepsilon |Du_x|^2 \, dx,
$$

for some constant $c_{\lambda^{-1}}$ depending only on $\lambda^{-1}$. The term $J_3$ can be estimated by observing that we can re-write the equations (4.2) as follows

$$
\partial_3 (\varepsilon D_{j\beta} u_x + S_{j\beta}(Du_x)) = \frac{\partial u_x^j}{\partial t} - \partial_\beta - \partial_3 (\varepsilon D_{j\beta} u_x + S_{j\beta}(Du_x)) \quad \text{a.e. in } I \times \Omega.
$$

Hence, we can multiply by $u_x$ and integrate by parts in space, since $u_x = \frac{\partial u}{\partial t} = 0$ on $I \times \partial \Omega$. We treat the terms without time derivative as $I_3$ in [8, p. 186] and integrate by parts the one involving $\partial_3 u_x$ to get the following

$$
J_3 = \\
= -\frac{1}{2} \int_\Omega \xi^2 |\partial_3 u_x|^2 \, dx - 2 \sum_{j=1}^3 \int_\Omega \xi \partial_\alpha \xi \frac{\partial u_x^j}{\partial t} \partial_3 u_x^j \, dx \\
- \lambda \int_\Omega \xi^2 |\partial_3 u_x|^2 \, dx + \lambda C \int_\Omega \xi^2 |\partial_\beta F(Du_x)|^2 \, dx + c_{\lambda^{-1}} \sum_{\beta=1}^2 \int_\Omega \xi^2 |\partial_\beta F(Du_x)|^2 \, dx \\
+ \lambda \int_\Omega \varepsilon \xi^2 |\partial_3 Du_x|^2 \, dx + c_{\lambda^{-1}} \sum_{\beta=1}^2 \int_\Omega \varepsilon \xi^2 |\partial_\beta Du_x|^2 \, dx + c \int_\Omega \xi^2 |\partial_3 u_x|^2 \, dx \\
+ c \|\nabla \xi\|_\infty^2 \left\|\frac{\partial u_x}{\partial t}\right\|_2^2 + c_{\lambda^{-1}} \left(\|f\|_{p'}^2 + \|Du_x\|_{p'}^p + \delta\right).
$$
In these estimates we use for the terms with $\partial_3 F(Du_{\varepsilon})$ and $\partial_{33} Du_{\varepsilon}$ the definition of the tangential derivative in (4.9) to get
\[
\int_{\Omega} \xi^2 |\partial_3 F(Du_{\varepsilon})|^2 + \varepsilon \xi^2 |\partial_{33} Du_{\varepsilon}|^2 \, dx
\] 
\[
\leq \int_{\Omega} \xi^2 |\partial_3 F(Du_{\varepsilon})|^2 + \varepsilon \xi^2 |\partial_{33} Du_{\varepsilon}|^2 \, dx + \|\nabla u\|^2_{\infty} \int_{\Omega} \xi^2 |\partial_3 F(Du_{\varepsilon})|^2 + \varepsilon \xi^2 |\partial_{33} Du_{\varepsilon}|^2 \, dx .
\]
Note that such terms already are present in the estimates for $\{J_i\}_{i=1,2,3}$. Now we choose the covering such that $\|\nabla a\|_{\infty}$ is small enough and only at this point we fix $\lambda > 0$ small enough (in order to absorb in the left-hand side terms involving $\partial_3 Du_{\varepsilon}$ and $\partial_{33} F(Du_{\varepsilon})$). We then obtain after integration in time over $[0,T]$ the following estimate
\[
\int_{\Omega} \xi^3 |\partial_3 u_{\varepsilon}(t)|^2 \, dx + \int_{0}^{T} \int_{\Omega} \xi^3 |\partial_3 Du_{\varepsilon}|^2 + \frac{1}{C_0} \xi^2 |\partial_3 F(Du_{\varepsilon})|^2 \, dx \, ds 
\] 
\[
\leq \int_{\Omega} \xi^3 |\partial_3 u_0|^2 \, dx + c \sum_{\beta=1}^{2} \int_{0}^{T} \int_{\Omega} \xi^2 |\partial_{\beta \alpha} F(u_{\varepsilon})|^2 + \varepsilon \xi^2 |\partial_{\beta \alpha} Du_{\varepsilon}|^2 \, dx \, ds 
\] 
\[
+ c \int_{0}^{T} \int_{\Omega} |f|^2 + \varphi(|Du_{\varepsilon}|) + \varphi(\delta) + \varepsilon |Du_{\varepsilon}|^2 + \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^2 \, dx \, ds + \int_{0}^{T} \int_{\Omega} \xi^3 |\partial_3 u_{\varepsilon}|^2 \, dx \, ds.
\]
Using the uniform estimates (4.4), (4.5) and (4.12) we can apply Gronwall’s inequality to prove the estimate (4.17). □

Choosing now an appropriate finite covering of the boundary (for the details see also [7]), Propositions 4.11, 4.16 yield the following result:

**Proposition 4.18.** Let the same hypotheses as in Theorem 2.21 with $\delta > 0$ be satisfied. Then, it holds\(^7\) for all $t \in I$
\[
\|\nabla u_{\varepsilon}(t)\|^2_{2} + \int_{0}^{t} \varepsilon \|\nabla Du_{\varepsilon}(s)\|^2_{2} + \|\nabla F(Du_{\varepsilon}(s))\|^2_{2} \, ds \leq C
\]
with $C$ depending on $\|u_0\|_{2,2}$, $\|\text{div} S(Du_0)\|_{2}$, $\|f\|_{L^p(I \times \Omega)}$, $\|\partial f\|_{L^p(I \times \Omega)}$, $\|\frac{\partial F}{\partial u}\|_{L^3(I \times \Omega)}$, $\delta$, $\partial \Omega$ and the characteristics of $S$.

4.5. **Passage to the limit.** Since the estimates in Propositions 4.1, 4.18 are uniform with respect to $\varepsilon > 0$, they are inherited by $u = \lim_{\varepsilon \to 0} u_{\varepsilon}$. The function $u$ is the unique regular solution to the initial boundary value problem (parabolic). We can now prove the existence result for regular solutions.

**Proof (of Theorem 2.21).** First, let us assume that $\delta > 0$. From Proposition 4.1, Proposition 2.8, and Proposition 4.18 we know that $F(Du_{\varepsilon})$ is uniformly bounded with respect to $\varepsilon$ in $W^{1,2}(I \times \Omega)$. This also implies (cf. [5, Lemma 4.4]) that $u_{\varepsilon}$ is uniformly bounded with respect to $\varepsilon$ in $L^p(I;W^{2,p}(\Omega)) \cap W^{3,p}(I;W^{1,p}(\Omega))$. The properties of $S$ and Proposition 4.1 also yield that $S(Du_{\varepsilon})$ is uniformly bounded

\(^7\)Recall that $c(\delta)$ only indicates that the constant $c$ depends on $\delta$ and will satisfy $c(\delta) \leq c(\delta_0)$ for all $\delta \leq \delta_0$. 
with respect to $\varepsilon$ in $L^p(I \times \Omega)$. Thus, there exists a sub-sequence $\{\varepsilon_n\}$ (which converges to 0 as $n \to +\infty$), $u \in L^p(I; W^{2,2}(\Omega)) \cap W^{1,2}(I; W_0^{1,2}(\Omega)) \cap W^{1,\infty}(I; L^2(\Omega))$, $F^* \in W^{1,2}(I \times \Omega)$, and $S^* \in L^p(I \times \Omega)$ such that

$$u_{\varepsilon_n} \to u \quad \text{in } L^p(I; W^{2,2}(\Omega)) \cap W^{1,2}(I; W_0^{1,2}(\Omega)) \cap W^{1,\infty}(I; L^2(\Omega)),$$

$$u_{\varepsilon_n} \rightharpoonup^* u \quad \text{in } W^{1,\infty}(I; L^2(\Omega)),$$

$$D(u_{\varepsilon_n}) \to D^* \quad \text{a.e. in } I \times \Omega,$$

$$F(D(u_{\varepsilon_n})) \to F^* \quad \text{in } W^{1,2}(I \times \Omega),$$

$$S(D(u_{\varepsilon_n})) \to S^* \quad \text{in } L^p(I \times \Omega).$$

The continuity of $S$ and $F$ and the classical result stating that the weak limit and the a.e. limit in Lebesgue spaces coincide (cf. [23]) implies that

$$F^* = F(Du) \quad \text{and} \quad S^* = S(Du).$$

These results enable us to pass to the limit in the weak formulation (4.3) of the perturbed problem (4.2), which yields for all $\psi \in C_0^\infty(I)$ and all $v \in V$

$$\int_0^T \left( \frac{\partial u}{\partial t}(t), v \right) \psi(t) \, dt + \int_0^T (S(Du(t)), Dv) \psi(t) \, dt = \int_0^T (f(t), v) \psi(t) \, dt, \quad (4.19)$$

since $\lim_{\varepsilon_n \to 0} \int_0^T \int_\Omega \varepsilon_n D(u_{\varepsilon_n}(t)) \cdot Dv \psi(t) \, dx \, dt = 0$. The weak lower semi-continuity of the norm implies that

$$\|F(Du)\|_{W^{1,2}(I \times \Omega)} \leq \liminf_{\varepsilon_n \to 0} \|F(D(u_{\varepsilon_n}))\|_{W^{1,2}(I \times \Omega)},$$

$$\|u\|_{W^{1,\infty}(I; L^2(\Omega))} \leq \liminf_{\varepsilon_n \to 0} \|u_{\varepsilon_n}\|_{W^{1,\infty}(I; L^2(\Omega))}.$$ 

By density and the strict monotonicity of $S$ we thus know that $u$ is the unique regular solution of problem (parabolic$_p$). This proves Theorem 2.21 in the case $\delta > 0$, since the weak formulation (1.1) follows immediately from (4.19).

Let us consider now the case $\delta = 0$. Proposition 4.11 and Proposition 4.16 are valid only for $\delta > 0$ and thus cannot be used directly for the case that $S$ has $(p, \delta)$-structure with $\delta = 0$. However, it is proved in [5, Section 3.1] that for any stress tensor with $(p, 0)$-structure $S$, there exist\footnote{The special case $S^*(D) = |D|^{p-2}D$ could be approximated by $S^*(D) := (\kappa + |D|)^{p-2}D$ as $\kappa \to 0$. However, for a general extra stress tensor $S$ having only $(p, \delta)$-structure this is not possible.} a stress tensors $S^{\kappa}$, having $(p, \kappa)$-structure with $\kappa > 0$ approximating $S$ in an appropriate way. Thus we approximate (parabolic$_p$) by the system

$$\frac{\partial u_{\varepsilon,\kappa}}{\partial t} - \text{div} S^{r,\kappa}(D(u_{\varepsilon,\kappa})) = f \quad \text{in } I \times \Omega,$$

$$u_{\varepsilon,\kappa} = 0 \quad \text{on } I \times \partial \Omega,$$

$$u_{\varepsilon,\kappa}(0) = u_0 \quad \text{in } \Omega,$$

where

$$S^{r,\kappa}(Q) := \varepsilon Q + S^*(Q), \quad \text{with } \varepsilon > 0, \kappa \in (0, 1).$$

For fixed $\kappa > 0$ we can use the above theory and use that fact that the estimates are uniform in $\varepsilon$ to pass to the limit as $\varepsilon \to 0$. Thus, we obtain that for all $\kappa \in (0, 1)$ there exists a unique $u_{\kappa} \in L^p(I; W_0^{1,2}(\Omega))$ fulfilling

$$\|u_{\kappa}\|_{W^{1,\infty}(I; L^2(\Omega))} + \|F(D(u_{\kappa}))\|_{W^{1,2}(I \times \Omega)} \leq c_0(f, u_0, \partial \Omega),$$

subject to the appropriate initial and boundary conditions.

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satisfying for all $\psi \in C_0^\infty(I)$ and all $v \in W_0^{1,2}(\Omega)$

$$\int_0^T \left( \frac{du_k}{dt}(t), v \right) \psi(t) \, dt + \int_0^T (S^c(Du_k(t)), Dv) \psi(t) \, dt = \int_0^T (f(t), v) \psi(t) \, dt.$$ 

The constant $c_0$ is independent of $\kappa \in (0,1)$ and $F^c : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}_{\text{sym}}$ is defined through

$$F^c(P) := (\kappa + |P|_{\text{sym}})^{\frac{p-2}{2}} P_{\text{sym}}.$$

Now we can proceed as in [5]. Indeed, it follows that $F^c(Du_k)$ is uniformly bounded in $W^{1,2}(I \times \Omega)$, that $u_k$ is uniformly bounded in $W^{1,p}(I \times \Omega)$ and that $S^c(Du_k)$ is uniformly bounded in $L^p(I \times \Omega)$. Thus, there exist $F^* \in W^{1,2}(I \times \Omega)$, $u \in L^p(I; W_0^{1,p}(\Omega))$, $S^* \in L^p(I \times \Omega)$, and a sub-sequence $\{\kappa_n\}$, with $\kappa_n \to 0$, such that

$$F^{c_n}(Du_{\kappa_n}) \to F^* \quad \text{in } W^{1,2}(I \times \Omega),$$

$$F^{c_n}(Du_{\kappa_n}) \to F^* \quad \text{in } L^2(I \times \Omega) \text{ and a.e. in } I \times \Omega,$$

$$u_{\kappa_n} \to u \quad \text{in } L^p(I; W_0^{1,p}(\Omega)),$$

$$S^c(Du_k) \to S^* \quad \text{in } L^p(I \times \Omega).$$

Setting $B := (F^0)^{-1}(F^*)$, it follows from [5, Lemma 3.23] that

$$Du_{\kappa_n} = (F^{c_n})^{-1}(F^{c_n}(Du_{\kappa_n})) \to (F^0)^{-1}(F^*) = B \quad \text{a.e. in } I \times \Omega.$$

Since weak and a.e. limit coincide we obtain that

$$Du_{\kappa_n} \to Du = B \quad \text{a.e. in } I \times \Omega.$$ 

From [5, Lemma 3.16] and [5, Corollary 3.22] it now follows that

$$F^{c_n}(Du_{\kappa_n}) \to F^0(Du) \quad \text{in } W^{1,2}(I \times \Omega),$$

$$S^{c_n}(Du_{\kappa_n}) \to S(Du) \quad \text{a.e. in } I \times \Omega.$$ 

Since weak and a.e. limit coincide we obtain that

$$F^* = F^0(Du) \quad \text{and } S^* = S(Du) \quad \text{a.e. in } I \times \Omega.$$ 

Now we can finish the proof in the same way as in the case $\delta > 0$. 

\section*{Appendix A. On the interpolation operator.}

We will deduce some results on interpolation operators which satisfy rather general assumptions. They are satisfied, e.g., by the Scott-Zhang operator. We work now in a general $d$-dimensional setting, i.e., we assume that $\Omega \subset \mathbb{R}^d$ is a polyhedral domain with Lipschitz continuous boundary. Let $T_h$ denote a family of shape-regular triangulations, consisting of $d$-dimensional simplices $K$. We assume that $T_h$ is non-degenerate, i.e., $\max_{K \in T_h} \frac{h_K}{\|K^{1/2}} \leq \gamma_0$. The global mesh-size $h$ is defined by $h := \max_{K \in T_h} h_K$. By the assumptions we obtain that $|S_K| \sim |K|$ and that the number of patches $S_K$ to which a simplex belongs are both bounded uniformly in $h$ and $K$. The finite element space $X_h$ is given by

$$X_h := \{ v \in L^1_{\text{loc}}(\Omega) \mid v \in P_K(T_h) \},$$

where $P_{r_0}(T_h) \subset P_K(T_h) \subset P_{r_1}(T_h)$ for some $r_0 \leq r_1 \in \mathbb{N}_0$.

We assume that the interpolation operator $P_h$ is $W^{\ell,1}$-stable.

\textbf{Assumption A.1.} Let $\ell_0 \in \mathbb{N}_0$ and let $P_h : (W^{\ell_0,1}(\Omega))^d \to (X_h)^d$.

(a) For some $\ell \geq \ell_0$ and $m \in \mathbb{N}_0$ holds uniformly in $K \in T_h$ and $v \in (W^{1,1}(\Omega))^d$

$$\sum_{j=0}^m h_K^j \int_K |\nabla^j P_h v| \, dx \leq c(m, \ell) \sum_{k=0}^\ell h_K^k \int_{S_K} |\nabla^k v| \, dx,$$
For all $v \in (P_{r_0})^d(\Omega)$ holds

$$P_h v = v.$$  

Note that we have to choose $\ell_0 \geq 1$, if the operator $P_h$ is preserving the boundary values, i.e., $P_h : (W^{\ell_0,1}(\Omega))^d \rightarrow (X_h \cap W^{\ell_0,1}(\Omega))^d$. Otherwise we allow $\ell_0 = 0$.

The properties of the interpolation operator $P_h$ are discussed in detail in [20, Sec. 4], [3, Sec. 3.2]. Let us now prove the two additional features formulated in Proposition 2.16 (ii), (iii). We start with the following non-homogeneous approximation property of $P_h$ (Note that Proposition 2.16 (ii) is a special case of the result below).

**Proposition A.2.** Let $P_h$ satisfy Assumption A.1 with $\ell \leq r_0 + 1$ and let $r, q \in [1, \infty)$ be such that $W^{\ell,q}(\Omega) \hookrightarrow W^{m,r}(\Omega)$. Moreover, assume that $h \sim h_K$ uniformly in $T_h$. Then, there exists a constant $c = c(\ell, m, q, r, r_0, r_1, \gamma_0)$ such that

$$\sum_{h=0}^{m} h^j \|\nabla^j (v - P_h v)\|_r \leq C \sum_{k=0}^{\ell} h^{\ell + d \min(0, \frac{1}{r} - \frac{1}{r})} \|\nabla^k v\|_q. \quad (A.3)$$

To prove Proposition A.2 we start by deriving from Assumption A.1 the non-homogeneous Sobolev stability adapting the approach in the case of Orlicz stability from [20] (cf. [28, Thm. 3.1] for the classical approach).

**Lemma A.4.** Let $P_h$ satisfy Assumption A.1 and let $r, q \in [1, \infty)$ be given. Then there exists $c = c(\ell, m, r_0, r_1)$ such that for all $K \in T_h$

$$\sum_{j=0}^{m} h^j \left( \int_K |\nabla^j P_h v|^r \, dx \right)^{\frac{1}{r}} \leq c \sum_{k=0}^{\ell} h^k \left( \int_{S_K} |\nabla^k v|^q \, dx \right)^{\frac{1}{q}},$$

or, in a non-averaged way, this can be formulated as follows

$$\sum_{j=0}^{m} h^j \|\nabla^j P_h v\|_{r,K} \leq c h^d \left( \frac{1}{r} - \frac{1}{q} \right) \sum_{k=0}^{\ell} h^k \|\nabla^k v\|_{q, S_K}.$$

**Proof.** We can write, using (2.13), (A.3), and Hölder’s inequality

$$\sum_{j=0}^{m} h^j \left( \int_K |\nabla^j P_h v|^r \, dx \right)^{\frac{1}{r}} \leq \sum_{j=0}^{m} h^j \|\nabla^j P_h v\|_{\infty,K} \leq c \sum_{j=0}^{m} h^j \int_K |\nabla^j P_h v| \, dx \leq c \sum_{k=0}^{\ell} h^k \int_{S_K} |\nabla^k v| \, dx \leq c \sum_{k=0}^{\ell} h^k \left( \int_{S_K} |\nabla^k v|^q \, dx \right)^{\frac{1}{q}}.$$  

Next, we prove a generalized Poincaré-Sobolev-Wirtinger inequality

**Lemma A.5.** Let $\ell \in \mathbb{N}$ and $q, r \in [1, \infty)$ be such that $W^{\ell,q}(K) \hookrightarrow W^{m,r}(K)$. Then, there exists a constant $c = c(\ell, m, q_0, r)$ such that for all $v \in W^{\ell,q}(K)$ with $\int_K \nabla^k v \, dx = 0$ for $k = 0, \ldots, \ell - 1$ it holds

$$\sum_{j=0}^{m} h^j \|\nabla^j v\|_{r,K} \leq c h^{\ell + d \left( \frac{1}{r} - \frac{1}{q} \right)} \|\nabla^\ell v\|_{q,K}.$$
Proof. Let us first show that for every \( j = 0, \ldots, m \) there exists \( c_j > 0 \) (depending on \( K \)) such that there holds \( \| \nabla^j v \|_{r,K} \leq c_j \| \nabla v \|_{q,K} \). Fix \( j \) and assume per absurdum that there exists \( \left\{ \tilde{v}_n \right\} \subset W^{\ell,q}(K) \) such that

\[
\| \nabla^j \tilde{v}_n \|_{r,K} > n \| \nabla^j \tilde{v}_n \|_{q,K}.
\]

Setting \( v_n := \frac{\tilde{v}_n}{\| \nabla^j \tilde{v}_n \|_{r,K}} \), we get

\[
\| \nabla^j v_n \|_{r,K} = 1 \quad \text{and} \quad \| \nabla^j v_n \|_{q,K} < \frac{1}{n}.
\]

Note that \( \| w \|_{q,K} + \| \nabla^j w \|_{q,K} \) is an equivalent norm on \( W^{\ell-j,q}(K) \) (cf. [31, p. 179]). We have to distinguish the cases \( r \geq q \) and \( r < q \).

**Case 1:** \( r \geq q \). The sequence \( \left\{ \nabla^j v_n \right\} \) is bounded in \( W^{\ell-j,q}(K) \), hence there exists a sub-sequence (relabelled as \( \left\{ \nabla^j v_n \right\} \)) such that \( \nabla^j v_n \to \nabla v \) strongly in \( L^r(K) \) and \( \| \nabla v \|_{r,K} = 1 \). This and (A.6) imply that \( \left\{ \nabla^j v_n \right\} \) is a Cauchy sequence in \( W^{\ell-j,q}(K) \), hence \( \nabla^j v_n \to \nabla v \) in \( W^{\ell-j,q}(K) \). Uniqueness of the limit implies that \( W = \nabla v \). This proves that \( \nabla^j v_n \to \nabla v \) in \( W^{\ell-j,q}(K) \). Moreover, (A.6) implies \( \| \nabla^{\ell-j} v \|_{q,K} = 0 \), hence that \( \nabla v \in P_{\ell-j-1} \). Next, the convergence in \( W^{\ell-j,q}(K) \) implies that also the averages converge. Hence

\[
0 = \int_K \nabla^k \nabla^j v_n \, dx \to \int_K \nabla^k \nabla v \, dx \quad \text{for} \ k = 0, \ldots, \ell-j-1,
\]

but as \( \nabla v \) is polynomial of degree less or equal than \( \ell - j \), this implies that \( \nabla v = 0 \). Thus, \( \| \nabla v \|_{r,K} = 0 \), contradicting the fact that \( \| \nabla v \|_{r,K} = 1 \).

**Case 2:** \( r < q \). In this case the same argument as in the previous case shows that

\[
\| \nabla^j v \|_{q,K} \leq c_j \| \nabla^j v \|_{q,K},
\]

and then by Hölder’s inequality

\[
\| \nabla^j v \|_{r,K} \leq c \| \nabla^j v \|_{q,K} \leq c c_j \| \nabla^j v \|_{q,K},
\]

with \( c = c(K) \).

To prove how the constants \( c_j \) depend on \( K \) we proceed as follows: We pass from a generic simplex \( K \) to the reference simplex \( \hat{K} \), use the previous inequalities in the reference domain with constants depending only \( \hat{K} \), and then we come back to the original simplex \( K \). This shows for every \( j = 0, \ldots, m \)

\[
\| \nabla^j v \|_{r,K} \leq h_K^j \| \nabla^j v \|_{r,\hat{K}}.
\]

Hence, we get

\[
h_K^j \| \nabla^j v \|_{r,K} \leq c_j(\hat{K}) h_K^{j+d(\frac{d}{2} - \frac{d}{4})} \| \nabla^j v \|_{q,K} ,
\]

which implies the assertion with \( c = \sum_{j=0}^m c_j(\hat{K}) \).

We recall now a Poincaré-Wirtinger type inequality, where it is possible to replace the average over the whole domain \( G \) with that one over a sub-domain \( A \subset G \) (cf. [15, Cor. 8.2.6], [24, Ch. 7.8]), provided that \( G \) is an \( \alpha \)-John domain and \( |A| \simeq |G| \).

Note that, due to our assumptions on the triangulation, we have that \( S_K \) are \( \alpha \)-John domains, where \( \alpha \) depends only on \( \gamma_0 \), and that \( |K| \simeq |S_K| \) for all \( K \in D_0 \).

**Lemma A.7.** There exists a constant \( c = c(d, \gamma_0) \) such that

\[
\| v - \langle v \rangle_K \|_{q,S_K} \leq c h_K \| \nabla v \|_{q,S_K} \quad \forall \ v \in W^{1,q}(S_K).
\]

This enables us to prove a local variant of Proposition A.2.
Lemma A.8. Let \( P_h \) satisfy Assumption A.1 with \( \ell \leq \ell_0 + 1 \) and let \( r, q \in [1, \infty) \) be such that \( W^{\ell,q}(\Omega) \hookrightarrow W^{r,q}(\Omega) \). Then there exists \( c = c(\ell, m, \ell_0, r_1, \gamma, r, q, d) \) such that for all \( \mathbf{v} \in W^{\ell,q}(\Omega) \) and all \( K \in \mathcal{T}_h \)
\[
\sum_{j=0}^{m} h_K^j \| \nabla^j \mathbf{v} - \nabla^j P_h \mathbf{v} \|_{r,K} \leq c h_K^{\ell + d(\frac{1}{r} - \frac{1}{q})} \| \nabla^\ell \mathbf{v} \|_{q,K}.
\]

Proof. We split the interpolation error by adding and subtracting a polynomial \( p \) of degree less than \( \ell \) and use Assumption A.1 (b) and Lemma A.4 to get for all \( j = 0, \ldots, m \)
\[
\sum_{j=0}^{m} h_K^j \| \nabla^j \mathbf{v} - \nabla^j P_h \mathbf{v} \|_{r,K} \leq \sum_{j=0}^{m} h_K^j \| \nabla^j \mathbf{v} - \nabla^j p \|_{r,K} + \sum_{j=0}^{m} h_K^j \| \nabla^j P_h (\mathbf{v} - p) \|_{r,K} \leq \sum_{j=0}^{m} h_K^j \| \nabla^j \mathbf{v} - \nabla^j p \|_{r,K} + c h_K^{\ell + d(\frac{1}{r} - \frac{1}{q})} \sum_{k=0}^{\ell} h_K^k \| \nabla^k (\mathbf{v} - p) \|_{r,S_K}.
\]

Since \( l \leq r_0 + 1 \) we can use Lemma A.5 to infer that for all polynomials \( p \) such that \( \int_K \nabla^k \mathbf{v} \, dx = \int_K \nabla^k p \, dx \), for \( k = 0, \ldots, \ell - 1 \), we have
\[
\sum_{j=0}^{m} h_K^j \| \nabla^j \mathbf{v} - \nabla^j p \|_{r,K} \leq c h_K^{\ell + d(\frac{1}{r} - \frac{1}{q})} \| \nabla^\ell \mathbf{v} \|_{q,K} \leq c h_K^{\ell + d(\frac{1}{r} - \frac{1}{q})} \| \nabla^\ell \mathbf{v} \|_{q,S_K}.
\]

For the same polynomials we have \( \int_K \nabla^k (\mathbf{v} - p) \, dx = 0 \), \( k = 0, \ldots, \ell - 1 \), and thus, Lemma A.7 yields for \( k = 0, \ldots, \ell - 1 \)
\[
\| \nabla^k (\mathbf{v} - p) \|_{q,S_K} \leq c h_K^{\ell - k} \| \nabla^\ell (\mathbf{v} - p) \|_{q,S_K} = c h_K^{\ell - k} \| \nabla^\ell \mathbf{v} \|_{q,S_K}.
\]

The last three inequalities prove the assertion. \( \square \)

We now have all results to prove Proposition A.2.

Proof of Proposition A.2. We split the integration over \( \Omega \) into a sum over \( K \), and then use Lemma A.8 to get for each \( j \in \{0, \ldots, m\} \)
\[
\| \nabla^j \mathbf{v} - \nabla^j P_h \mathbf{v} \|_{r,K} = \sum_{K \in \mathcal{T}_h} \| \nabla^j \mathbf{v} - \nabla^j P_h \mathbf{v} \|_{r,K} \leq c \sum_{K \in \mathcal{T}_h} h_K^{\ell - jr + d(\frac{1}{r} - \frac{1}{q})} \| \nabla^\ell \mathbf{v} \|_{q,S_K}.
\]

We set now \( \alpha_K := \| \nabla^\ell \mathbf{v} \|_{q,S_K} \) and observe that \( \nabla^\ell \mathbf{v} \in L^q(\Omega) \) is equivalent to \( \alpha_K \in \ell^q(\mathbb{N}) = \ell^1. \) We use Hölder inequality in the \( \ell^q \) spaces to estimate the right-hand side. We distinguish again the two cases \( q \leq r \) and \( q > r \).

Case 1: \( q \leq r \). In this case, since \( \frac{1}{q} - 1 \geq 0 \) and since for \( \{a_n\} \subset \ell^1 \) it holds \( \|a_n\|_{\ell^q} \leq \|a_n\|_{\ell^1} \), we can write
\[
\sum_{K \in \mathcal{T}_h} \| \nabla^\ell \mathbf{v} \|_{q,S_K} = \sum_{K \in \mathcal{T}_h} \alpha_K^\frac{q}{r} = \sum_{K \in \mathcal{T}_h} \alpha_K \alpha_K^{\frac{q}{r} - 1} \leq \sum_{K \in \mathcal{T}_h} \alpha_K \| \nabla^\ell \mathbf{v} \|_{\ell^q} \leq \| \alpha_K \|_{\ell^q} \| \nabla^\ell \mathbf{v} \|_{\ell^1} \leq \| \alpha_K \|_{\ell^1}.\]
Case 2: $q > r$. In this case

$$\sum_{K \in T_h} \alpha_K \leq \left( \sum_{K \in T_h} \alpha_K \right)^{\frac{1}{r}} \left( \sum_{K \in T_h} 1 \right)^{\frac{1}{q}} = \left( \sum_{K \in T_h} \alpha_K \right)^{\frac{1}{r}} \left( \sum_{K \in T_h} \alpha_K \right)^{\frac{1}{q}} = \left( \sum_{K \in T_h} \alpha_K \right)^{\frac{1}{r}} \left( \# K \right)^{\frac{1}{q}}.$$

Since $h \sim h_K$ uniformly in $T_h$ we get $|\Omega| \simeq \# K h^d$ and thus $\# K \sim h^{-d}$. Hence we obtain

$$\sum_{K \in T_h} \alpha_K \leq c h^{-d(r^{\frac{1}{q}} - \frac{1}{r})} \| \alpha_K \|_{\ell^q}^{\frac{1}{r}}.$$

Putting the two cases together, using $h_K \leq h$ and $W^{r,q}(\Omega) \hookrightarrow W^{m,r}(\Omega)$, we proved for each $j \in \{0, \ldots, m\}$

$$\| \nabla^j v - \nabla^j P_h v \|_{r,\Omega} \leq c h^{r - d(r^{\frac{1}{q}} - \frac{1}{r})} \left( \sum_{K \in T_h} \| \nabla^j v \|_{q,SK}^q \right)^{\frac{1}{r}}.$$

Taking the $r$-th root, multiplying by $h^j$ and summing up over $j = 0, \ldots, m$ proves the assertion, since $\left( \sum_{K \in T_h} \| \nabla^j v \|_{q,SK}^q \right)^{\frac{1}{r}} \leq c \| \nabla^j v \|_{q,\Omega}$.

Finally, we prove the following version of the continuity of the interpolation operator $P_h$ in Orlicz spaces.

**Lemma A.9.** Let $F(Dv), F(Dw) \in W^{1,2}(\Omega)$. Then there exists a constant $c = c(p, r_1, \gamma_0)$ such that

$$\int_{\Omega} |v| |DP_h v - DP_h w| \, dx \leq c h^2 \| Dv \|_{2}^2 + c \| F(Dv) - F(Dw) \|_{2}^2,$$

where the constants depend only on $\gamma_0$ and $p$.

**Proof.** Using again $\int_{\Omega} f \, dx = \sum_{K \in T_h} \int_K f \, dx$, it suffices to treat one simplex $K$. We obtain, thanks to back and forth shift changes (cf. Proposition 2.8 (ii)), the properties of the interpolation operator, Korn’s inequality (cf. [21, Thm. 6.13]) and Proposition 2.8 (i), Poincaré’s inequality applied to $F(Du)$ in $L^2(S_K)$ and Proposition 2.8 (iii), the properties of the triangulation, the following chain of inequalities

$$\int_{K} \varphi_{[Dv]}(|DP_h v - DP_h w|) \, dx$$

$$\leq c_{\delta} \int_{K} \varphi_{[Dv]}(|DP_h v - DP_h w|) \, dx + \varphi_{[Dv]}(|DP_h v - DP_h w|) \, dx$$

$$\leq c_{\delta} \int_{S_K} \varphi_{[Dv]}(|Dv - (Dv)_{SK}|) \, dx + \varphi_{[Dv]}(|Dv - (Dv)_{SK}|) \, dx$$

$$\leq c_{\delta} \int_{S_K} \varphi_{[Dv]}(|Dv - (Dv)|) \, dx + \varphi_{[Dv]}(|Dv - (Dv)|) \, dx$$

$$\leq c_{\delta} \int_{S_K} \varphi_{[Dv]}(|Dv - (Dv)|) \, dx + \varphi_{[Dv]}(|Dv - (Dv)|) \, dx$$

$$\leq c_{\delta} \int_{S_K} \varphi_{[Dv]}(|Dv - (Dv)|) \, dx + \varphi_{[Dv]}(|Dv - (Dv)|) \, dx$$

$$\leq c_{\delta} \int_{S_K} \varphi_{[Dv]}(|Dv - (Dv)|) \, dx + \varphi_{[Dv]}(|Dv - (Dv)|) \, dx$$

$$\leq c_{\delta} \| F(Dw) - F(Dv) \|_{2,SK}^2 + \varphi_{[Dv]}(|Dv - (Dv)|) \, dx$$
\[ \leq c_3 \| F(Dw) - F(Dv) \|_{L^2(S_K)}^2 + c_4 h^2 \int_{S_K} |\nabla F(Dv)|^2 \, dx. \]

This yields the assertion. \(\square\)

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**References**

[1] H. Beirão da Veiga, P. Kaplický, and M. Růžička. Boundary regularity of shear-thickening flows. *J. Math. Fluid Mech.*, 13:387–404, 2011.

[2] S. Bartels and M. Růžička. Convergence of fully discrete implicit and semi-implicit approximations of nonlinear parabolic equations, Tech. Report 1902.08122, arXiv, 2019, accepted in SINUM.

[3] L. Bělenský, L. C. Berselli, L. Diening, and M. Růžička. On the Finite Element approximation of p-Stokes systems. *SIAM J. Numer. Anal.*, 50(2):373–397, 2012.

[4] L. C. Berselli, L. Diening, and M. Růžička. Optimal error estimates for a semi implicit Euler scheme for incompressible fluids with shear dependent viscosities. *SIAM J. Numer. Anal.*, 47(4):2177–2202, 2009.

[5] L. C. Berselli, L. Diening, and M. Růžička. Existence of strong solutions for incompressible fluids with shear dependent viscosities. *J. Math. Fluid Mech.*, 12(1):101–132, 2010.

[6] L. C. Berselli, L. Diening, and M. Růžička. Optimal error estimates for semi-implicit space-time discretization for the equations describing incompressible generalized Newtonian fluids. *IMA J. Num. Anal.*, 25(2):680–697, 2015.

[7] L. C. Berselli and M. Růžička. Global regularity properties of steady shear thinning flows. *J. Math. Anal. Appl.*, 450(2):839–871, 2017.

[8] L. C. Berselli and M. Růžička. Global regularity for systems with p-structure depending on the symmetric gradient. *Adv. Nonlinear Anal.*, 9(1):176–192, 2020.

[9] D. Bothe and J. Prüss. \(L_p\)-theory for a class of non-Newtonian fluids. *SIAM J. Math. Anal.*, 39(2):379–421, 2007.

[10] D. Breit and P.R. Mensah. Space-time approximation of parabolic systems with variable growth *IMA J. Num. Anal.*, 2019 Online first: drz039

[11] F. Crispo and C.R. Grisanti and P. Maremonti. Singular \(p\)-Laplacian parabolic system in exterior domains: higher regularity of solutions and related properties of extinction and asymptotic behavior in time. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 19(3):913–949, 2019.

[12] F. Crispo and P. Maremonti. Higher regularity of solutions to the singular \(p\)-Laplacian parabolic system. *Adv. Differential Equations*, 8(9):849–894, 2013.

[13] L. Diening, C. Ebmeyer, and M. Růžička. Optimal convergence for the implicit space-time discretization of parabolic systems with \(p\)-structure. *SIAM J. Numer. Anal.*, 45:477–482, 2007.

[14] L. Diening and Ch. Kreuzer. Linear convergence of an adaptive finite element method for the \(p\)-Laplacian equation. *SIAM J. Numer. Anal.*, 46:614–638, 2008.

[15] L. Diening, A. Prohl, and M. Růžička. On time-discretizations for generalized Newtonian fluids. *SIAM J. Numer. Anal.*, 42:1172–1190, 2006.

[16] L. Diening and M. Růžička. Strong solutions for generalized Newtonian fluids. *J. Math. Fluid Mech.*, 7:413–450, 2005.

[17] L. Diening and M. Růžička. Interpolation operators in Orlicz–Sobolev spaces. *Num. Math.*, 107:107–129, 2007.
[21] L. Diening, M. Růžička, and K. Schumacher. A decomposition technique for John domains. *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, 35(1):87–114, 2010.

[22] S. Eckstein and M. Růžička. On the full space–time discretization of the generalized Stokes equations: The Dirichlet case. *SIAM J. Numer. Anal.*, 56(4):2234–2261, 2018.

[23] H. Gajewski, K. Gröger, and K. Zacharias. *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*. Akademie-Verlag, Berlin, 1974.

[24] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*. Springer, Berlin, 2001. Reprint of the 1998 edition.

[25] J. Málek, J. Nečas, and M. Růžička. On weak solutions to a class of non–Newtonian incompressible fluids in bounded three–dimensional domains. the case $p \geq 2$. *Adv. Differential Equations*, 6:257–302, 2001.

[26] A. Prohl and M. Růžička. On fully implicit space-time discretization for motions of incompressible fluids with shear dependent viscosities: The case $p \leq 2$. *SIAM J. Num. Anal.*, 39:214–249, 2001.

[27] M. Růžička and L. Diening. Non–Newtonian fluids and function spaces. In *Nonlinear Analysis, Function Spaces and Applications, Proceedings of NAFSA 2006 Prague*, volume 8, pages 95–144, 2007.

[28] L.R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[29] G. A. Seregin and T. N. Shilkin. Regularity of minimizers of some variational problems in plasticity theory. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 243(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funktsii. 28):270–298, 342–343, 1997.

[30] E. Süli and T. Tcherpel. Fully discrete finite element approximation of unsteady flows of implicitly constituted incompressible fluids. *IMA Journal of Numerical Analysis*, 2019. DOI 10.1093/imanum/dry097.

[31] H. Triebel. *Interpolation theory, function spaces, differential operators*. North-Holland Publishing Co., Amsterdam, 1978.

[32] T. Tcherpel. Finite element approximation for the unsteady flow of implicitly constituted incompressible fluids, 2018. PhD Thesis, University of Oxford.

[33] K. Yosida. *Functional analysis*. Springer, Berlin, 1980.

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