REGULAR ALGEBRAS OF GLOBAL DIMENSION TWO

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Abstract. Let $V$ be a finite-dimensional positively-graded vector space. Let $b \in V \otimes V$ be an element whose rank is $\dim(V)$. Let $A = TV/(b)$, the quotient of the tensor algebra $TV$ modulo the 2-sided ideal generated by $b$. Let $\text{gr}(A)$ be the category of finitely presented graded left $A$-modules and $\text{fdim}(A)$ its full subcategory of finite dimensional modules. Let $q\text{gr}(A)$ be the quotient category $\text{gr}(A)/\text{fdim}(A)$. We compute the Grothendieck group $K_0(q\text{gr}(A))$. In particular, if the reciprocal of the Hilbert series of $A$, which is a polynomial, is irreducible, then $K_0(q\text{gr}(A)) \cong \mathbb{Z}[\theta] \subset \mathbb{R}$ as ordered abelian groups where $\theta$ is the smallest positive real root of that polynomial.

1. Introduction

1.1. Let $k$ be a field and $A = \bigoplus_{n \geq 0} A_n$ an $\mathbb{N}$-graded $k$-algebra such that $A_0 = k$. The left and right global dimensions of $A$ are the same and equal the projective dimension of the $A$-module $k := A/A_{\geq 1}$. We say $A$ is regular if it has finite global dimension, $n$ say, and

$$\text{Ext}_A^j(k, A) \cong \begin{cases} k & \text{if } j = n \\ 0 & \text{if } j \neq n. \end{cases}$$

Zhang [9, Theorem 0.1] proved that $A$ is regular of global dimension 2 if and only if it is isomorphic to some

$$A := \frac{k\langle x_1, \ldots, x_g \rangle}{(b)}$$

where $g \geq 2$, the $x_i$’s can be labelled so that $\deg(x_i) + \deg(x_{g+1-i}) =: d$ is the same for all $i$, and $\sigma$ is a graded $k$-algebra automorphism of the free algebra $k\langle x_1, \ldots, x_g \rangle$, and $b = \sum_{i=1}^{g} x_i \sigma(x_{g+1-i})$.

1.2. From now on $A$ denotes the algebra in (1-1) where the degrees of the generators and the relation $b$ have the properties stated after (1-1). We will also assume that $g \geq 3$; the case $g = 2$ is well-understood.

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1.3. Let $\text{Gr}(A)$ denote the category of graded left $A$-modules and $\text{gr}(A)$ its full subcategory of finitely presented modules. Because $A$ is defined by a single homogeneous relation $\text{gr}(A)$ is an abelian category [8, Theorem 1.2]. Let $\text{fdim}(A)$ be the full subcategory of finite dimensional graded left $A$-modules and write $\text{qgr}(A)$ for the quotient category $\text{gr}(A)/\text{fdim}(A)$, and $\pi^* : \text{gr}(A) \to \text{qgr}(A)$ for the quotient functor.

If $M$ is a graded $A$-module, $M(1)$ denotes the graded module that is $M$ as an abelian group with grading $M(1)_n = M_1 + n$ and the same action of $A$. Since $\text{fdim}(A)$ is stable under the functor $M \mapsto M(1)$ there is an induced functor $\pi^* M \mapsto \pi^*(M(1)) =: (\pi^* M)(1)$ on the quotient category $\text{qgr}(A)$.

1.3.1. We follow the convention in algebraic geometry where, for a smooth projective variety $X$, $K_0(\text{coh}(X))$ denotes the free abelian group generated by coherent $\mathcal{O}_X$-modules modulo the relations $[M_2] = [M_1] + [M_3]$ for every exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$. In this paper, $K_0(\text{qgr}(A))$ denotes the free abelian group generated by the objects in $\text{qgr}(A)$ modulo the “same” relations. The order structure on $K_0(\text{qgr}(A))$ is defined by $K_0(\text{qgr}(A)) \geq 0 := \{[M] \mid M \in \text{qgr}(A)\}$.

1.4. Because $A$ is regular of global dimension two the minimal projective resolution of $\mathcal{O}_A$ is

$$0 \to \bigoplus A(-d) \xrightarrow{\alpha} \bigoplus A(-\deg(x_i)) \xrightarrow{\beta} A \xrightarrow{\gamma} k \to 0$$

where $d = \deg(x_i) + \deg(x_{g+1-i})$, $\alpha$ is right multiplication by $(x_g, \ldots, x_1)$, and $\beta$ is right multiplication by $(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_g))^T$. The Hilbert series for $A$ is therefore

$$H_A(t) := \sum_{n=0}^{\infty} \dim_k(A_n)t^n = \frac{1}{f(t)}$$

where

$$f(t) := t^d - \sum_{i=1}^{g} t^{\deg(x_i)} + 1.$$  

(1-2)

1.5. The minimal projective resolution of a module $M \in \text{gr}(A)$ has the form

$$0 \to \bigoplus A(-i)^{e_i} \to \bigoplus A(-i)^{c_i} \to \bigoplus A(-i)^{b_i} \to M \to 0$$

where the sums are finite. Since $H_{A(-i)}(t) = t^i H_A(t)$, the Hilbert series for $M$, $H_M(t)$, is $H_A(t)q_M(t)$ where

$$q_M(t) := \sum_{i \geq 0} (b_i - c_i + e_i)t^i \in \mathbb{Z}[t^{\pm 1}].$$
1.6. The main result and remarks.

**Theorem 1.1.** Let $A$ be the algebra in (1-1) and assume that the greatest common divisor of the degrees of its generators $x_i$ is 1. Let $f(t)$ be the polynomial in (1-2) and let $\theta$ be its smallest positive real root. Let $Z[\theta] \subset \mathbb{R}$ be the $\mathbb{Z}$-subalgebra generated by $\theta$ viewed as an ordered abelian subgroup of $\mathbb{R}$. The Grothendieck group $K_0(\text{qgr}(A))$ is isomorphic as an ordered abelian group to $\mathbb{Z}[t, t^{-1}]$ via the map $\pi^* M \mapsto q\theta M(t)$. If $f$ is irreducible, $K_0(\text{qgr}(A))$ is isomorphic as an ordered abelian group to $\mathbb{Z}[\theta]$ via the map $\pi^* M \mapsto q\theta M(\theta)$.

Furthermore, under the isomorphism(s), the functor $M \mapsto M(1)$ on $\text{qgr}(A)$ corresponds to multiplication by $t^{-1}$ and multiplication by $\theta^{-1}$.

1.6.1. Descartes’ rule of signs implies that $f(t)$ has either 0 or 2 positive real roots. The hypothesis that $g \geq 3$ implies $f(1) < 0$. Since $f(0) > 0$, we conclude that $f(t)$ has two positive roots, $\theta^{-1} > 1$ and $\theta \in (0, 1)$, say.

1.6.2. We make $\mathbb{Z}[t^\pm]/(f)$ an ordered abelian group by defining $\left(\mathbb{Z}[t, t^{-1}]/(f)\right)_{\geq 0} := \{ p | p(\theta) > 0 \} \cup \{ 0 \}$ where $p$ denotes the image of the Laurent polynomial $p$ in $\mathbb{Z}[t^\pm]/(f)$.

1.6.3. In general, $f(t)$ need not be irreducible. If $A = k\langle x, y, z \rangle/(xz + zx + y^2)$ with $\text{deg}(x, y, z) = (5, 6, 7)$, then $f(t) = t^{12} - t^7 - t^6 - t^5 + 1 = (t^2 - t + 1)(t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1)$. There are reducible $f(t)$ of smaller degree but this particular example is interesting because the degree-10 factor is Lehmer’s polynomial \[4\]. Lehmer’s number is $\theta^{-1} \approx 1.17628$.

1.7. The proof of Theorem 1.1 uses the next result with $d_i = \text{deg}(x_i)$.

**Proposition 1.2.** Let $d_1, \ldots, d_g$ be non-negative integers. Suppose that $g \geq 3$, that $d_i + d_{g+1-i} = d$ for all $i$, and $\gcd\{d_1, \ldots, d_g\} = 1$. The polynomial

\[(1-3) f(t) := t^d - \sum_{i=1}^{g} t^{d_i} + 1\]

has a unique root of maximal modulus and that root is real.

We give two proofs of this result. That in \[3\] uses a directed graph associated to $d_1, \ldots, d_g$ and the Perron-Frobenius theorem. That in \[4\] uses elementary trigonometric identities, but only works when $g \geq 4$. 

2. Proof of Theorem 1.1 modulo Proposition 1.2

2.1.

Lemma 2.1. Let \( \sum_{n=0}^{\infty} c_n z^n \) be a power series in which \( c_n > 0 \) for all \( n \gg 0 \). Suppose

1. \( \sum_{n=0}^{\infty} c_n z^n \) has radius of convergence \( R > 0 \) and on the disk \( |z| < R \) it converges to a rational function \( s(z) \) that has a simple pole at \( z = R \);
2. all other poles of \( \sum_{n=0}^{\infty} c_n z^n \) have modulus \( > R \).

Then
\[
\lim_{n \to \infty} \frac{c_n}{c_{n+1}} = R.
\]

Proof. There are polynomials \( p(z) \) and \( q(z) \), neither divisible by \( R - z \), such that
\[
s(z) = \frac{p(z)}{(R - z)q(z)} = \frac{\alpha}{R - z} + \frac{r(z)}{q(z)}
\]
where \( \alpha \in \mathbb{C} \), \( r(z) \) is a polynomial, and \( r(z)/q(z) \) has a Taylor series expansion \( \sum_{n=0}^{\infty} b_n R^n \) with radius of convergence > \( R \) by (2). Since \( \sum_{n=0}^{\infty} b_n R^n \) converges, \( \lim_{n \to \infty} b_n R^n = 0 \).

Since
\[
s(z) = \frac{\alpha}{R} \sum_{n=0}^{\infty} \frac{z^n}{R^n} + \sum_{n=0}^{\infty} b_n z^n
\]
for \( |z| < R \),
\[
c_n = \frac{\alpha}{R} R^{n+1} + b_n.
\]

Therefore
\[
\lim_{n \to \infty} \left( \frac{c_n}{c_{n+1}} \right) = \lim_{n \to \infty} \left( \frac{\alpha R + b_n R^{n+2}}{\alpha + b_{n+1} R^{n+2}} \right) = \frac{\alpha R}{\alpha} = R,
\]
as claimed. \( \square \)

Let \( A \) be the algebra in (1-1). Let \( d_i := \deg(x_i) \) and \( d := \deg(b) \). We assume that \( g \geq 3 \), that \( \gcd\{d_1, \ldots, d_g\} = 1 \), and \( d = d_1 + d_{g+1-i} \) for all \( i \).

We suppose \( d_1 \leq d_2 \leq \cdots \leq d_g \) for simplicity. We write \( n_i \) for the number of \( x_j \)'s whose degree is \( i \). Thus
\[
f(t) = t^d - \sum_{i=1}^{d-1} n_i t^i + 1.
\]

Because \( A \) is a domain \( [9, \text{Thm. 0.2}] \) and 1 is the greatest common divisor of the degrees of its generators, \( A_n \neq 0 \) for all \( n \gg 0 \).

We write \( a_n := \dim_k(A_n) \).

Lemma 2.2. For all \( m \geq 1 \),
\[
\lim_{n \to \infty} \frac{a_n}{a_{n+m}} = \theta^m.
\]
Proof. Since
\[
\frac{a_n}{a_{n+m}} = \frac{a_n}{a_{n+1}} \frac{a_{n+1}}{a_{n+2}} \cdots \frac{a_{n+m-1}}{a_{n+m}}
\]
for \( n \gg 0 \), it suffices to prove the result for \( m = 1 \). Since \( H_A(t) = \sum_{i=0}^{\infty} a_i t^i \) satisfies the conditions of Lemma 2.1 for \( R = \theta \), the result follows from the conclusion of Lemma 2.1.

Proposition 2.3. If \( M \in \text{gr}(A) \), then \( q_M(\theta) \geq 0 \).

Proof. Write \( q_M(t) = \sum_{i=-s}^{s} p_i t^i \) and define \( e_i := \sum_{j=-s}^{s} p_j a_i - j \). Then
\[
H_M(t) = q_M(t) H_A(t) = \left( \sum_{i=-s}^{s} p_i t^i \right) \left( \sum_{i=0}^{\infty} a_i t^i \right) = \sum_{i=-s}^{\infty} e_i t^i.
\]
As \( m \to \infty \),
\[
\frac{e_m}{a_m} = \sum_{j=-s}^{s} \left( \frac{a_{m-j}}{a_m} \right) p_j \to \sum_{j=-s}^{s} p_j \theta^j = q_M(\theta).
\]
Since \( e_i = \dim(M_i) \), \( \{e_m/a_m\}_{m \gg 0} \) is a sequence of non-negative numbers its limit, \( q_M(\theta) \), is \( \geq 0 \).

Lemma 2.4. Let \( M \in \text{gr}(A) \). The following are equivalent:

1. \( M \in \text{fdim}(A) \);
2. \( f(t) \) divides \( q_M(t) \);
3. \( q_M(\theta) = 0 \).

Proof. (1) \( \Rightarrow \) (2) If \( \dim_k(M) < \infty \), then \( H_M(t) \in \mathbb{N}[t, t^{-1}] \) so \( q_M(t) \) is a multiple of \( f(t) \).

(2) \( \Rightarrow \) (3) If \( f(t) \) divides \( q_M(t) \) then \( q_M(\theta) = 0 \) since \( \theta \) is a root of \( f(t) \).

(3) \( \Rightarrow \) (1) Suppose \( q_M(\theta) = 0 \) but \( \dim_k(M) = \infty \). The power series \( H_M(t) \) has non-negative coefficients and a finite radius of convergence \( R \leq 1 \). Since \( H_M(t) = q_M(t) H_A(t) \), \( q_M(\theta) = 0 \) and \( \theta \) is a simple pole of \( H_A(t) \) and the only pole of \( H_A(t) \) in the interval \( [0, 1] \), \( H_M(t) \) has no poles in the interval \( [0, 1] \). This contradicts Pringsheim’s Theorem [2] Theorem IV.6] which says that \( H_M(t) \) has a pole at \( t = R \). We therefore conclude that \( \dim_k(M) < \infty \). □

2.2. The converse of Proposition 2.3 is not true: there are Laurent polynomials \( p \in \mathbb{Z}[t, t^{-1}] \) such that \( p(\theta) \geq 0 \) but \( p(t) \) is not equal to \( q_M(t) \) for any \( M \in \text{gr}(A) \). The following example illustrates this fact.

Example 2.5. Let \( A = k(x, y, z)/(xz + y^2 + zx) \) with \( \deg(x) = \deg(y) = \deg(z) = 1 \). In this case, \( f(t) = 1 - 3t + t^2 \) and \( \theta = \frac{1}{2}(3 - \sqrt{5}) \).

Let \( p(t) = -3 + 13t - 4t^2 \). Then \( p(\theta) = 1 + \theta > 0 \); but
\[
p(t) H_A(t) = -3 + 4t^2 + 11t^2 + \cdots
\]
has a negative coefficient so it is not the Hilbert series of any module.
However, if \( M = A \oplus A(-1) \), then \( q_M(t) = 1 + t \) and \( q_M(t) - p(t) = 4 - 12t + 4t^2 = 4f(t) \), so

\[
(q_M(t) - p(t)) H_A(t) = 4.
\]

In other words, \( H_M(t) \) and \( p(t) H_A(t) \) differ only in the first term. The following lemma is a generalization of this fact.

**Lemma 2.6.** Let \( p \in \mathbb{Z}[t, t^{-1}] \). If \( p(\theta) > 0 \), then there is an \( M \) in \( \text{gr}(A) \) such that \( q_M(t) - p(t) \in (f) \).

**Proof.** It suffices to show that \( t^s q_M(t) - t^s p(t) \in (f) \) for some integer \( s \).

Since \( q_M(t^{-s}) = t^s q_M(t) \), we can, and will, assume \( p(t) \in \mathbb{Z}[t] \).

Write \( p(t) = \sum_{i=0}^{s} p_i t^i \). Define integers \( b_j, j \geq 0 \), by the requirement that

\[
\sum_{j=0}^{\infty} b_j t^j := p(t) H_A(t).
\]

Therefore

\[
p(t) = f(t) \sum_{j=0}^{\infty} b_j t^j = \left(1 - \sum_{\ell=1}^{d-1} n_{\ell} t^\ell + t^d \right) \sum_{j=0}^{\infty} b_j t^j.
\]

Equating coefficients gives

\[
p_i = b_i + b_{i-d} - \sum_{\ell=1}^{d-1} n_{\ell} b_{i-\ell}
\]

for all \( i \geq 0 \) with the convention that \( p_i = 0 \) for \( i > s \) and \( b_j = 0 \) for \( j < 0 \).

Since \( a_j \neq 0 \) for \( j > 0 \),

\[
\lim_{j \to \infty} \left( \frac{b_j}{a_j} \right) = \lim_{j \to \infty} \left( \sum_{i=0}^{s} \frac{a_{j-i}}{a_j} p_i \right) = \sum_{i=0}^{s} p_i \theta^i = p(\theta) > 0.
\]

Therefore

\[
\lim_{j \to \infty} \left( \frac{b_j}{b_{j+1}} \right) = \left( \frac{b_j}{a_j + a_{j+1}} \right) = p(\theta) p(\theta)^{-1} \theta = \theta.
\]

There is therefore an integer \( m \geq s \) such that \( \{b_j\}_{j=m+1-d} \) is a strictly increasing sequence of positive integers. We fix such an \( m \).

We will complete the proof by showing that the Laurent polynomial

\[
q(t) := p(t) - \left( \sum_{i=0}^{m} b_i t^i \right) f(t)
\]

is \( q_M(t) \) for a suitable \( M \in \text{gr}(A) \). Before beginning the proof we define

\[
r_i := \sum_{\ell=i-m}^{d-1} n_{\ell} b_{i-\ell} - b_{i-d}
\]
for \( m + 1 \leq i \leq m + d \). To start the proof, we note that \( q(t) \) is equal to
\[
p(t) - \left( \sum_{i=0}^{m} b_i t^i \right) \left( 1 - \sum_{i=1}^{d-1} n_i t^i + t^d \right)
\]
which equals
\[
p(t) - \sum_{i=0}^{m} \left[ b_i - \sum_{\ell=1}^{d-1} n_{\ell} b_{i-\ell} + b_{i-d} \right] t^i + \sum_{i=m+1}^{m+d} \left[ \sum_{\ell=m+1}^{d-1} n_{\ell} b_{i-\ell} - b_{i-d} \right] t^i.
\]
By (2.2), the left-hand sum is \( p(t) \) so
\[
q(t) = \sum_{i=m+1}^{m+d} r_i t^i.
\]
Suppose \( \text{deg}(x_i) = 1 \) for all \( i = 1, \ldots, g \). Then
\[
q(t) = r_{m+1} t^{m+1} + r_{m+2} t^{m+2} = a t^{m+1} + b_m (1 - t) t^{m+1}
\]
where \( a = (g-1) b_m - b_{m-2} \geq 0 \). Thus, \( q(t) = q_M(t) \) where \( M = M'(-m-1) \) and

\[
M' = A^n \oplus \left( \frac{A}{x_1 A} \right)^{b_m}.
\]

Suppose \( \text{deg}(x_i) \neq 1 \) for some \( i \). Then \( d_1 \neq d_g \) and Lemmas 2.8 and 2.9 below show that \( q(t) = q_M(t) \) for some \( M \in \text{gr}(A) \).

2.3. Technical lemmas. The next three lemmas complete the proof of Lemma 2.6 when \( d_1 \neq d_g \) so are proved under that hypothesis.

**Lemma 2.7.** For each integer \( i \) between \( m + d_1 + 1 \) and \( m + d_g \),
\[
\sum_{\ell=i-m}^{d-1} n_{\ell} b_{i-\ell} \geq b_{i-d}.
\]

**Proof.** Since \( n_{\ell} \) is the number of the generators \( x_1, \ldots, x_g \) having degree \( \ell \), \( n_{\ell} \geq 0 \) for all \( \ell \) between \( i - m \) and \( d - 1 \). Since \( i - m \) is between \( d_1 + 1 \) and \( d_g \), the only \( \ell \)'s between \( i - m \) and \( d - 1 \) for which \( n_{\ell} \) is non-zero are \( d_2, \ldots, d_g \). If \( \ell = d_j \), then \( n_{\ell} b_{i-\ell} = n_{d_j} b_{i-d_j} \); but \( i - d_j \geq i - d \geq m + 1 - d \) so \( b_{i-d_j} \geq b_{i-d} \). The result follows.

**Lemma 2.8.** There is \( N \in \text{gr}(A) \) such that
\[
q_N(t) = \sum_{i=m+d_1+1}^{m+d_g} r_i t^i.
\]

**Proof.** By definition,
\[
r_i = -b_{i-d} + \sum_{\ell=i-m}^{d-1} n_{\ell} b_{i-\ell}.
\]
By Lemma 2.7, \( r_i \geq 0 \) for all \( i \) between \( m + d_1 + 1 \) and \( m + d_g \). The module

\[ N := \bigoplus_{i=m+d_1+1}^{m+d_g} A^r_i(-i), \]

satisfies the conclusion of the lemma. \( \square \)

**Lemma 2.9.** There is \( L \in \text{gr}(A) \) such that

\[ q_L(t) = \sum_{i=m+1}^{m+d_1} r_i t^i + \sum_{i=m+d_1+1}^{m+d} r_i t^i. \]

**Proof.** Because \( d_1 + d_g = d \),

\[ \sum_{i=m+1}^{m+d_1} r_i t^i + \sum_{i=m+d_1+1}^{m+d} r_i t^i = \sum_{i=m+1}^{m+d_1} (r_i + r_{i+d_g} t^{d_g}) t^i. \]

However, \( n_\ell = 0 \) for all \( \ell \geq d_g + 1 \) so, when \( m + 1 \leq i \leq m + d_1 

\[ r_i + r_{i+d_g} t^{d_g} = r_i - b_i - d_g + d_g = r_i - b_i - d_1 + b_i - d_1 (1 - t^{d_g}), \]

We must therefore show there is \( L \in \text{gr}(A) \) such that

\[ q_L(t) = \sum_{i=m+1}^{m+d_1} (r_i - b_i - d_1) t^i + \sum_{i=m+1}^{m+d_1} b_i - d_1 (1 - t^{d_g}) t^i \]

Since \( t^i = q_{A(-i)}(t) \) and \( (1 - t^{d_g}) t^i = q(A/x_g, A)(-i)(t), q(t) \) equals \( q_L(t) \) where

\[ L = \left( \bigoplus_{i=m+1}^{m+d_1} A^{r_i-b_i-d_1}(-i) \right) \oplus \left( \bigoplus_{i=m+1}^{m+d_1} \left( \frac{A}{x_g A} \right)^{b_i-d_1}(-i) \right) \]

provided the coefficients \( r_i - b_i - d_1 \) and \( b_i - d_1 \) are non-negative. Since \( i - d_1 \geq m + 1 - d_g, b_i - d_1 > 0. \)

If \( m + 1 \leq i \leq m + d_1 \), then

\[ r_i \geq n_{d_1} b_i - d_1 + n_{d_g} b_i - d_g - b_i - d \geq b_i - d_1 \]

so \( r_i - b_i - d_1 \geq 0. \) \( \square \)

2.4. The proof of Theorem 1.1 modulo Proposition 1.2 By localization and dévissage, the map

\[ (2-4) \quad K_0(q_{\text{gr}}(A)) \to \frac{\mathbb{Z}[t, t^{-1}]}{(f)}, \quad [\pi^* M] \mapsto [q_M(t)], \]

is an isomorphism of abelian groups and \([M(1)] = t^{-1} [M]\) under this isomorphism.

Under the isomorphism \((2-4)\), the positive cone in \( K_0(q_{\text{gr}}(A)) \) is mapped to \( \{ q_M(t) \mid M \in \text{gr}(A) \} \). To show that \((2-4)\) is an isomorphism of ordered abelian groups we must show that

\[ (2-5) \quad \{ \overline{p} \mid p(\theta) > 0 \} \cup \{ 0 \} = \{ q_M(t) \mid M \in \text{gr}(A) \}. \]
Let $M \in \text{gr}(A)$. By Proposition 2.3 $q_M(\theta) \geq 0$. If $q_M(\theta) > 0$, then $q_M(t)$ is in the left-hand side of (2-5). If $q_M(\theta) = 0$, then $f(t)$ divides $q_M(t)$ by Lemma 2.4 whence $q_M(t) = 0$. Thus, the right-hand side of (2-5) is contained in the left-hand side of (2-5).

If $p \in \mathbb{Z}[t \pm 1]$ and $p(\theta) > 0$, then $p = q_M$ for some $M \in \text{gr}(A)$ by Lemma 2.6 so $p$ is in the right-hand side of (2-5). It is clear that 0 is in the right-hand side of (2-5). Hence (2-4) is an isomorphism of ordered abelian groups.

Suppose $f$ is irreducible. The composition

$$K_0(\text{qgr}(A)) \to \frac{\mathbb{Z}[t, t^{-1}]}{(f)} \to \mathbb{Z}[\theta], \quad [\pi^* M] \mapsto q_M(\theta),$$

is certainly an isomorphism of abelian groups. By (2-5), the image of the positive cone in $K_0(\text{qgr}(A))$ under this composition is $\mathbb{R}_{\geq 0} \cap \mathbb{Z}[\theta]$, the positive cone in $\mathbb{Z}[\theta]$. Hence (2-5) is an isomorphism of ordered abelian groups and $[\mathcal{M}(1)] = \theta^{-1}[\mathcal{M}]$ under the isomorphism.

□

3. First proof of Proposition 1.2

3.1. We recall the statement of Proposition 1.2 and explain our strategy to prove it.

Let $d_1, \ldots, d_g$ be non-negative integers. Suppose $g \geq 3$, that $d_i + d_{g+1-i} = d$ for all $i$, and $\gcd\{d_1, \ldots, d_g\} = 1$. Let $n_j$ be the number of $d_i$ that equal $j$. We will show that the polynomial

$$f(t) = t^d - n_{d-1}t^{d-1} - \cdots - n_1 t + 1$$

has a unique root of maximal modulus, and that root is real.

We will associate to the data $d_1, \ldots, d_g$ a particular finite directed graph $G$. An incidence matrix for $G$ is a square matrix whose rows and columns are labelled by the vertices of $G$ and whose $uv$-entry is the number of arrows from $v$ to $u$. The characteristic polynomial of $G$ is

$$p_G(t) := \det(tI - M)$$

where $M$ is an incidence matrix for $G$. We will show that $p_G(t) = t^{\ell-d} f(t)$ where $\ell = d_1 + \cdots + d_g$. We also show that $M$ is primitive, i.e., all entries of $M^n$ are positive for $n \gg 0$. We then apply the Perron-Frobenius theorem which says that a primitive matrix has a positive real eigenvalue of multiplicity 1, $\rho$ say, with the property that $|\lambda| < \rho$ for all other eigenvalues $\lambda$. But the non-zero eigenvalues of $M$ are the roots of $f(t)$. Since we already know that $f(t)$ has only two positive real roots, $\theta < 1$ and $\theta^{-1} > 1$, $\rho = \theta^{-1}$. Since the coefficient of $t^\ell$ in $f(t)$ is the same as that of $t^{d-i}$, $f(t) = t^d f(t^{-1})$. Thus $f(\lambda) = 0$ if and only if $f(\lambda^{-1}) = 0$. Hence $\theta^{-1}$ is the unique root of $f(t)$ having largest modulus.
3.2. We will use Theorem 3.1 to compute the characteristic polynomial of \( G \). First we need some notation.

A simple cycle in \( G \) is a directed path that begins and ends at the same vertex and does not pass through any vertex more than once. We introduce the notation for an arbitrary directed graph \( G \):

1. \( v(G) := \) the number of vertices in \( G \);
2. \( c(G) := \) the number of connected components in \( G \);
3. \( Z(G) := \) \{simple cycles in \( G \}\};
4. \( \overline{Z}(G) := \) \{subgraphs of \( G \) that are a disjoint union of simple cycles\}.

**Theorem 3.1.** \([1, \text{Theorem 1.2}]\) Let \( G \) be a directed graph with \( \ell \) vertices. Then

\[
p_G(t) = t^{\ell} + c_1 t^{\ell-1} + \cdots + c_{\ell-1} t + c_{\ell}
\]

where

\[
c_i := \sum_{\substack{Q \in Z(G) \\
v(Q) = i}} (-1)^{c(Q)}.
\]

3.3. The \( x_i \)'s are labelled so that \( \deg(x_1) \leq \cdots \leq \deg(x_g) \).

The free algebra on \( k\langle x_1, \ldots, x_g \rangle \) is the path algebra of the quiver with one vertex \( \star \) and \( g \) loops from \( \star \) to \( \star \) labelled \( x_1, \ldots, x_g \). We replace each loop \( x_i \) by \( d'_i := \deg(x_i) - 1 = d_i - 1 \) vertices labelled \( x_{i1}, \ldots, x_{id'_i} \) and arrows

\[
\begin{array}{ccccccc}
\star & \to & x_{i1} & \to & x_{i2} & \cdots & \cdots & \to & x_{id'_i} & \to & \star
\end{array}
\]

The graph obtained by this procedure is the graph associated to \( k\langle x_1, \ldots, x_g \rangle \) in \([5]\).

3.3.1. **Example.** If \( A \) is generated by \( x_1, x_2, x_3 \) and \( \deg(x_i) = i \), the associated graph is

\[
\begin{array}{c}
x_{21} \\
\alpha_{21} \quad \alpha_{20} \quad \alpha_{30} \\
x_{31} \\
\alpha_{31} \\
\end{array}
\]

3.4. **The second graph associated to \( A \).** We now form a second directed graph, the vertices of which are the arrows in the previous graph. In the second graph there is an arrow from vertex \( u \) to vertex \( v \) if in the first graph the arrow \( u \) can be followed by the arrow \( v \), except we do not include an arrow \( \alpha_{id'_i} \to \alpha_{g0} \).

We write \( G \), or \( G(A) \), for the second graph associated to \( A \).
The second graph associated to Example 3.3.1 is

Note the absence of an arrow from $\alpha_{10}$ to $\alpha_{30}$.

**Proposition 3.2.** If $u$ and $v$ are vertices in $G$, there is a directed path starting at $u$ and ending at $v$.

**Proof.** There is a directed path $\alpha_{i0} \to \alpha_{i1} \to \cdots \to \alpha_{id} \to \alpha_{i0}$ so the result is true if $u = \alpha_{ij}$ and $v = \alpha_{ik}$. There are also arrows

\[ \alpha_{1d'}, \alpha_{2d'}, \ldots, \alpha_{g-1d'_{g-1}}, \alpha_{gd'}, \alpha_{10} \]

so the result is true if $u = \alpha_{i_{1}j_{1}}$ and $v = \alpha_{i_{2}j_{2}}$. □

**Proposition 3.3.** Let $M$ be an incidence matrix for $G$. Then every entry in $M^n$ is non-zero for $n \gg 0$.

**Proof.** In the language of [6, Defn. 4.2.2], Proposition 3.2 says that $M$ is irreducible.

The period of a vertex $v$ in $G$ is the greatest common divisor of the non-trivial directed paths that begin and end at $v$. The period of $G$ is the greatest common divisor of the periods of its vertices. Since there is a directed path of length $d_i = \deg(x_i)$ from $\alpha_{i0}$ to itself, the period of $G$ divides $\gcd\{d_1, \ldots, d_g\}$ which is 1. The period of $G$ is therefore 1. Thus, in the language of [6, Defn. 4.5.2], $M$ is aperiodic and therefore primitive [6, Defn. 4.5.7]. Hence [6, Thm. 4.5.8] applies to $M$, and gives the result claimed. □

The Perron-Frobenius theorem [3, Thm. 1, p.64] therefore applies to $M$ giving the following result.

**Corollary 3.4.** The characteristic polynomial for $G$ has a unique eigenvalue of maximal modulus and that eigenvalue is simple and real.

Our next goal, achieved in Proposition 3.9 is to show that $p_G(t) = t^{\ell-d}f(t)$ for a suitable $\ell$.

3.5. **Other graphs associated to $A$.** We now write $\mathcal{X} := \{x_1, \ldots, x_g\}$ and define the directed graph $\hat{\mathcal{X}}$ by declaring that its vertex set is $\mathcal{X}$ and there is an arrow $x_i \to x_j$ for all $(x_i, x_j) \in \mathcal{X}^2 - \{(x_1, x_g)\}$. For each non-empty subset $X \subset \mathcal{X}$ let $\hat{X}$ be the full subgraph of $\hat{\mathcal{X}}$ with vertex set $X$. 
3.5.1. If $g = 4$, then
\[
\{x_1, x_2, x_3\} = \begin{array}{c}
\vdots \\
\end{array}
\]
and
\[
\{x_1, x_2, x_4\} = \begin{array}{c}
\vdots \\
\end{array}
\]

Lemma 3.5. Let $X \subset \{x_1, \ldots, x_g\}$. The constant term in the characteristic polynomial for $\hat{X}$ is
\[
p_{\hat{X}}(0) = \begin{cases} 
1 & \text{if } X = \{x_1, x_g\} \\
-1 & \text{if } |X| = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. Let $M$ be an incidence matrix of $\hat{X}$. Then the constant term in the characteristic polynomial for $\hat{X}$ is $p_{\hat{X}}(0) = (-1)^{|X|} \det(M)$.

If $|X| = 1$, then $\hat{X}$ consists of one vertex with a single loop so $M = (1)$ whence $p_{\hat{X}}(0) = -1$.

If $X = \{x_1, x_g\}$, then $\hat{X}$ has vertices $x_1$ and $x_g$, an arrow from $x_g$ to $x_1$, and a loop at each vertex. Hence $\binom{1}{0,1}$ is an incidence matrix for $\hat{X}$ and the constant term is 1.

If $|X| = 2$ and $X \neq \{x_1, x_g\}$, then the incidence matrix for $\hat{X}$ is $\binom{1}{1,1}$ so the constant term is 0.

Suppose $|X| \geq 3$. If $\{x_1, x_g\} \subseteq X$, then $M$ has a single off-diagonal 0 and all its other entries are 1; in particular, $M$ is singular so the constant term is 0. If $\{x_1, x_g\} \not\subseteq X$, then every entry in $M$ is 1 so $M$ is singular and the constant term is 0. \qed

3.6. The paths $\beta_1, \ldots, \beta_g$ in $G$. For each $1 \leq i \leq g$, let $\beta_i$ be the path
\[
\alpha_{i0} \rightarrow \alpha_{i1} \rightarrow \cdots \rightarrow \alpha_{id_i}.
\]
In example 3.3.1, $\beta_1$ is the trivial path at vertex $\alpha_{10}$, $\beta_2$ is the arrow $\alpha_{20} \rightarrow \alpha_{21}$, and $\beta_3$ is the path $\alpha_{30} \rightarrow \alpha_{31} \rightarrow \alpha_{32}$.

Proposition 3.6. Let $i_1, \ldots, i_m$ be pairwise distinct elements of $\{1, \ldots, g\}$ such that $(1, g) \notin \{(i_m, i_1), (i_1, i_2), \ldots, (i_{m-1}, i_m)\}$. Then there is a simple cycle in $G$ of the form
\[
(3-2) \quad \beta_{i_1} \rightarrow \beta_{i_2} \rightarrow \cdots \rightarrow \beta_{i_m} \rightarrow \alpha_{i_1}
\]
and every simple cycle in $G$ is of this form, up to choice of starting point.
Proof. Let \( r, s \in \{1, \ldots, g\} \) and assume \( r \neq s \). If \((r, s) \neq (1, g)\), then there is an arrow from \( \alpha_{rd} \); the vertex at which \( \beta_r \) ends, to \( \alpha_{s0} \), the vertex at which \( \beta_s \) starts; hence there is a path “traverse \( \beta_r \) then traverse \( \beta_s \)”; we denote this path by \( \beta_r \rightarrow \beta_s \). It follows that there is a path of the form (3-2).

Let \( p \) be a simple cycle in \( G \). A simple cycle passes through a vertex \( \alpha_{ij} \) if and only if it passes through \( \alpha_{i0} \). Every simple cycle that passes through \( \alpha_{i0} \) contains \( \beta_i \) as a subpath because there is a unique arrow starting at \( \alpha_{ij} \) for all \( j = 0, \ldots, d_i' - 1 \). Hence \( p \) is of the form (3-2). \( \square \)

Lemma 3.7. There is a bijection \( \Phi : Z(\hat{X}) \to Z(G) \) defined by

\[
(3-3) \quad \Phi(x_{i1} \rightarrow \cdots \rightarrow x_{im} \rightarrow x_{i1}) := \beta_{i1} \rightarrow \cdots \rightarrow \beta_{im} \rightarrow \alpha_{i10}
\]

whose inverse is

\[
(3-4) \quad \Phi^{-1}(\beta_{i1} \rightarrow \cdots \rightarrow \beta_{im} \rightarrow \alpha_{i10}) := x_{i1} \rightarrow \cdots \rightarrow x_{im} \rightarrow x_{i1}.
\]

Proof. We need only check that \( \Phi \) and \( \Psi \) are well-defined. Because \( \hat{X} \) does not contain an arrow \( x_1 \rightarrow x_g \) and \( G \) does not contain an arrow \( \alpha_{1d_i'} \rightarrow \alpha_{g0} \), the right-hand sides of (3-3) and (3-4) are simple cycles. \( \square \)

The next result is obvious.

Proposition 3.8. The function \( \Phi \) extends to a bijection \( \Phi : Z(\hat{X}) \to Z(G) \) defined by

\[
\Phi(E_1 \sqcup \cdots \sqcup E_m) := \Phi(E_1) \sqcup \cdots \sqcup \Phi(E_m)
\]

for disjoint simple cycles \( E_1, \ldots, E_m \) in \( \hat{X} \). Furthermore, \( c(E) = c(\Phi(E)) \) for all \( E \in Z(\hat{X}) \).

3.7. The support of a subgraph \( Q \) of \( G \) is

\[
\text{Supp}(Q) := \{ x_i \mid \beta_i \text{ is a path in } Q \}.
\]

For each non-empty subset \( X \subset \{ x_1, \ldots, x_g \} \) let

\[
Z(G, X) := \{ Q \in Z(G) \mid \text{Supp}(Q) = X \}
\]

and let \( d(X) = \sum_{x \in X} \deg(x) \).

Proposition 3.9. Let \( \ell = \sum_{i=1}^g \deg(x_i) \). The characteristic polynomial of \( G \) is \( t^{\ell-d} f(t) \).

Proof. The characteristic polynomial of \( G \) is

\[
p_G(t) = t^\ell + c_1 t^{\ell-1} + \cdots + c_{\ell-1} t + c_\ell
\]

where \( \ell = v(G) = \sum_{i=1}^g d_i \) and

\[
(3-5) \quad c_i = \sum_{Q \in Z(G)} (\text{parity}(Q) = i) = \sum_{X \in \mathcal{X}} \left( \sum_{Q \in Z(G, X)} (\text{parity}(Q) = i) \right).
\]
Since $Z(G, X) = \{ \Phi(E) \mid E \in Z(\hat{X}) \& v(E) = d(X) \}$ we have

\begin{equation}
\sum_{Q \in Z(G, X)} (-1)^{c(Q)} = \sum_{E \in Z(\hat{X})} (-1)^{c(E)}.
\end{equation}

Since $v(\hat{X}) = |X|$, the right-hand side of (3-6) is $p_{\hat{X}}(0)$. Hence by Lemma 3.5,

\[ c_i = \begin{cases} 
1 & \text{if } i = d_1 + d_g = d, \\
-n_i & \text{if } 1 \leq i \leq d_g, \\
0 & \text{otherwise.}
\end{cases} \]

Thus $p_G(t) = t^\ell - n_1 t^{\ell-1} - \cdots - n_{d_g} t^{\ell-d_g} + t^{\ell-d} = t^{\ell-d} f(t)$, as claimed. \qed

As explained at the end of §3.1, Proposition 1.2 follows from Proposition 3.9 and Corollary 3.4.

3.7.1. Example. In order to clarify some of the technicalities in this section, we will compute the coefficient $c_5$ in $p_G(t) = t^9 + c_1 t^8 + \cdots + c_8 t + c_9$ where $G$ is the second graph associated to the algebra $A = k(x_1, x_2, x_3)/(b)$ where $\deg(x_i) = i + 1$. First, $G$ is

\[ Q_1 = \begin{cases} 
\alpha_{10} & \text{and } Q_2 = \begin{cases} 
\alpha_{10} & \text{and } Q_2 = \end{cases} \\
\alpha_{20} & \\
\alpha_{22}
\end{cases} \]

There are two subgraphs of $G$ that have exactly five vertices and are disjoint unions of simple cycles, namely

\[ Q_1 = \begin{cases} 
\alpha_{10} & \text{and } Q_2 = \begin{cases} 
\alpha_{10} & \text{and } Q_2 = \end{cases} \\
\alpha_{20} & \\
\alpha_{22}
\end{cases} \]
The only subset $X$ of $\mathcal{X} = \{x_1, x_2, x_3\}$ such that $d(X) = 5$ is $X = \{x_1, x_2\}$. The graph $\hat{X}$ is

$$
\begin{array}{c}
\circ x_1 \\
\circ x_2 \\
\circ x_3 \\
\end{array}
\begin{array}{c}
\downarrow \quad \downarrow \\
x_3 \quad x_2 \\
\end{array}
\begin{array}{c}
\uparrow \quad \uparrow \\
x_1 \quad x_3 \\
\end{array}
\begin{array}{c}
\downarrow \quad \downarrow \\
x_2 \quad x_1 \\
\end{array}
\begin{array}{c}
\uparrow \quad \uparrow \\
x_3 \quad x_2 \\
\end{array}
\begin{array}{c}
\downarrow \quad \downarrow \\
x_2 \quad x_3 \\
\end{array}
\begin{array}{c}
\uparrow \quad \uparrow \\
x_1 \quad x_2 \\
\end{array}
\begin{array}{c}
\end{array}
$$

Since $Q_1 = \Phi(E_1)$ and $Q_2 = \Phi(E_2)$ where

$$
E_1 = \begin{cases}
\{x_1\} \\
\{x_2\} \\
\{x_3\}
\end{cases}
$$

and

$$
E_2 = \begin{cases}
\{x_1\} \\
\{x_2\}
\end{cases}
$$

equations (3-5) and (3-6) give

$$
e_5 = (-1)^{c(Q_1)} + (-1)^{c(Q_2)}
= (-1)^{c(E_1)} + (-1)^{c(E_2)}
= 1 - 1
= 0.
$$

4. Second proof of Proposition 1.2

The following proof of Proposition 1.2 is made under the additional assumption that $g \geq 4$.

**Lemma 4.1.** Let $\eta, \phi, \psi \in \mathbb{R}$. The following inequalities hold:

1. $3 + \cos(2\eta) - \cos(\eta + \psi) - \cos(\eta - \psi) - \cos(\eta + \phi) - \cos(\eta - \phi) \geq 0$;
2. $3 + \cos(2\eta) - \cos(\eta + \psi) - 2\cos(\eta) - \cos(\eta - \psi) \geq 0$;
3. $3 + \cos(2\eta) - 2\cos(\eta + \psi) - 2\cos(\eta - \psi) \geq 0$.

In each case the inequality is an equality if and only if

1. $\cos(\eta) = \cos(\phi) = \cos(\psi) = 1$ or $\cos(\eta) = \cos(\phi) = \cos(\psi) = -1$;
2. $\cos(\eta) = \cos(\psi) = 1$;
3. $\cos(\eta) = \cos(\psi) = 1$ or $\cos(\eta) = \cos(\psi) = -1$.

**Proof.** The expression in (2) is obtained from that in (1) by taking $\phi = 0$. The expression in (3) is obtained from that in (1) by taking $\psi = \phi$. The statements about equality in (2) and (3) follow from that about equality in (1). It therefore suffices to prove (1).

Let $\lambda = \frac{1}{2}(\cos(\phi) + \cos(\psi))$. Since $-1 \leq \lambda \leq 1$,

$$
1 + (\cos(\eta) - \lambda)^2 - \lambda^2 \geq 0.
$$
Therefore
\[ 0 \leq 2 + 2(\cos(\eta) - \lambda)^2 - 2\lambda^2 \]
\[ = 2 + 2\cos^2(\eta) - 4\lambda \cos(\eta) \]
\[ = 2 + 2\cos^2(\eta) - 2\cos(\phi)\cos(\eta) - 2\cos(\psi)\cos(\eta) \]
\[ = 3 + \cos(2\eta) - \cos(\eta + \phi) - \cos(\eta - \phi) - \cos(\eta + \psi) - \cos(\eta - \psi). \]

This proves the inequality in (1). The inequality in (1) is an equality if and only if \( \lambda^2 = 1 \) and \( \cos(\eta) = \lambda \), i.e., if and only if either \( \cos(\eta) = \lambda = 1 \) or \( \cos(\eta) = \lambda = -1 \); i.e., if and only if \( \cos(\eta) = \cos(\phi) = \cos(\psi) = 1 \) or \( \cos(\eta) = \cos(\phi) = \cos(\psi) = -1 \).

As always, \( f(t) \) is the polynomial defined in (3-1) or, equivalently, the polynomial in (1-3).

**Proposition 4.2.** Suppose \( g \geq 4 \). If \( f(\lambda) = 0 \) and \( |\lambda| = \theta^{-1} \) then \( \lambda = \theta^{-1} \).

**Proof.** Suppose \( d = 2 \). Then \( f(t) = t^2 - gt + 1 \) with \( g \geq 4 \). The only roots of \( f \) are \( \theta \) and \( \theta^{-1} \).

Suppose \( d \geq 3 \). Let \( \omega = (\lambda\theta)^{-1} = \cos(\alpha) + i\sin(\alpha) \) and define
\[ (4-1) \quad Z := \cos(d\alpha) - 1 + \sum_{j=1}^{d-1} n_j \theta^{-j}(1 - \cos((d - j)\alpha)). \]

Since \( f(\theta^{-1}) = f(\omega^{-1}\theta^{-1}) = 0 \),
\[ 0 = \theta^{-d} + 1 - \sum_{j=1}^{d-1} n_j \theta^{-j} \]
\[ = (\omega\theta)^{-d} + 1 - \sum_{j=1}^{d-1} n_j (\omega\theta)^{-j} \]
\[ = \theta^{-d} + \omega^d - \sum_{j=1}^{d-1} n_j \theta^{-j}\omega^{d-j}. \]

Therefore
\[ (4-2) \quad 0 = \omega^d - 1 + \sum_{j=1}^{d-1} n_j \theta^{-j}(1 - \omega^{d-j}). \]

The real part of (4-2) is \( Z \), so \( Z = 0 \).

We will use the fact that \( Z = 0 \) to show that \( \cos(j\alpha) = 1 \) for all \( j \in D \); i.e., \( \omega^j = 1 \) for all \( j \in D \). But \( \gcd D = 1 \) so it will then follow that \( \omega = 1 \), whence \( \lambda = \theta^{-1} \). The proof of the proposition will then be complete.

Because \( g \geq 4 \), one of the following must be true:
(a) there is \( q \in D \) such that \( n_q \geq 2 \) and \( q < d/2 \);
(b) (a) does not occur, \( d \) is even, \( n_{d/2} \geq 2 \), and \( n_q \neq 0 \) for some \( q < d/2 \);
(c) neither (a) nor (b) occurs, and there are four distinct elements $q, r, d - q, d - r \in D$ such that $q, r < d/2$.

In each of these cases we make the following definitions:

(a) $C = \{q, d - q\}, \quad n'_d = n'_{d-q} = 2$, and 
$$X = 3 + \cos(d\alpha) - 2 \cos(q\alpha) - 2 \cos((d - q)\alpha)$$

(b) $C = \{q, d - q, d/2\}, \quad n'_q = n'_{d-q} = 1, n'_{d/2} = 2$, and 
$$X = 3 + \cos(d\alpha) - \cos(q\alpha) - 2 \cos(d\alpha/2) - \cos((d - q)\alpha)$$

(c) $C = \{q, r, d - q, d - r\}, \quad n'_q = n'_r = n'_{d-q} = n'_{d-r} = 1$, and 
$$X := 3 + \cos(d\alpha) - \cos(q\alpha) - \cos(r\alpha) - \cos((d - q)\alpha) - \cos((d - r)\alpha).$$

Let $q$ and $r$ be as in (a), (b), and (c). Define $\eta = d\alpha/2$, $\psi = (d/2 - q)\alpha$, and $\phi = (d/2 - r)\alpha$. In case (a), Lemma 4.1(3) implies $X \geq 0$. In case (b), Lemma 4.1(2) implies $X \geq 0$. In case (c), Lemma 4.1(1) implies $X \geq 0$.

In each of the three cases,
$$X = \cos(d\alpha) - 1 + \sum_{j \in C} n'_j (1 - \cos((d-j)\alpha)).$$

Therefore, $-X = Z - X = \sum_{j \in D - C} n_j \theta^{-j} (1 - \cos((d-j)\alpha)) + \sum_{j \in C} (n_j \theta^{-j} - n'_j) (1 - \cos((d-j)\alpha)).$

However,
- $n_j \theta^{-j} > 0$ for all $j \in D - C$,
- $n_j \theta^{-j} - n'_j > 0$ for all $j \in C$, and
- $1 - \cos((d-j)\alpha)$ is always $\geq 0$,

so $-X \geq 0$ and $-X = 0$ if and only if $\cos(j\alpha) = 1$ for all $j \in D$. Since both $-X$ and $X$ are $\geq 0$ we conclude that $X = 0$, whence $\cos(j\alpha) = 1$ for all $j \in D$. The proof is now complete. \hfill \square

**Lemma 4.3.** Suppose $g \geq 4$. If $\lambda$ is a root of $f$ other than $\theta$, then $\theta < |\lambda|$.

**Proof.** Let $\rho$ be a root of $f$ having minimal modulus. Because $f(t)$ is the reciprocal of a Hilbert series, the power series

$$\frac{1}{f(t)} = \sum_{i=0}^{\infty} a_i t^i$$

has non-negative coefficients and a finite radius of convergence $|\rho|$. By Pringsheim’s Theorem \cite{2} Theorem IV.6], $|\rho|$ is a singularity of $H_\lambda(t)$, i.e. a root of $f$. Since $\theta$ is the smallest positive root of $f$, $|\rho| = \theta$, so $\theta \leq |\lambda|$.

Suppose $\lambda = \theta$. Since $f$ is reciprocal, $\lambda^{-1}$ is a root of $f$, and $|\lambda^{-1}| = \theta^{-1}$. By Proposition 4.2 $\lambda^{-1} = \theta^{-1}$, so $\lambda = \theta$, a contradiction. The result follows. \hfill \square
5. Examples

5.1. When $A$ is generated by $g \geq 3$ elements of degree one, $f(t)$ is the irreducible polynomial $1 - gt + t^2$ so

$$ K_0(\text{qgr}(A)) \cong \mathbb{Z} \left[ \frac{g - \sqrt{g^2 - 4}}{2} \right] \subset \mathbb{R} $$

as ordered abelian groups.

5.2. Non-irreducible $f$. Suppose $g = 4$, $d_1 = d_2 = 1$ and $d_3 = d_4 = 2$. Then $f(t) = 1 - 2t - 2t^2 + t^3 = (1 + t)(1 - 3t + t^2)$ and $\theta = \frac{1}{2}(3 - \sqrt{5})$. The map

$$ \frac{\mathbb{Z}[t, t^{-1}]}{(f)} \to \mathbb{Z} \oplus \mathbb{Z}[\theta], \quad p \mapsto (p(-1), p(\theta)) $$

is an isomorphism of abelian groups. The image of the positive cone under that isomorphism $K_0(\text{qgr}(A)) \to \mathbb{Z} \oplus \mathbb{Z}[\theta]$ is $(\mathbb{Z} \oplus \mathbb{Z}[\theta]_{\geq 0}) \cup \{0\}$.

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