ON EIGENFUNCTION EXPANSIONS OF FIRST-ORDER SYMMETRIC SYSTEMS AND ORDINARY DIFFERENTIAL OPERATORS OF AN ODD ORDER

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Abstract. We study general (not necessarily Hamiltonian) first-order symmetric systems
\[ Jy' - B(t)y = \Delta(t)f(t), \] on an interval \( I = [a, b] \) with the regular endpoint \( a \). It is assumed that the deficiency indices \( n_{\pm}(T_{\text{min}}) \) of the minimal relation \( T_{\text{min}} \) satisfy
\[ n_+(T_{\text{min}}) < n_-(T_{\text{min}}). \] We define \( \lambda \)-depending boundary conditions which are analogs of separated self-adjoint boundary conditions for Hamiltonian systems. With a boundary value problem involving such conditions we associate an exit space self-adjoint extension \( \tilde{T} \) of \( T_{\text{min}} \) and the \( m \)-function \( m(\cdot) \), which is an analog of the Titchmarsh-Weyl coefficient for the Hamiltonian system. By using \( m \)-function we obtain the eigenfunction expansion with the spectral function \( \Sigma(\cdot) \) of the minimally possible dimension and characterize the case when spectrum of \( \tilde{T} \) is defined by \( \Sigma(\cdot) \). Moreover, we parametrize all spectral functions in terms of a Nevanlinna type boundary parameter. Application of these results to ordinary differential operators of an odd order enables us to complete the results by Everitt and Krishna Kumar on the Titchmarsh-Weyl theory of such operators.

1. Introduction

Let \( H \) and \( \hat{H} \) be finite dimensional Hilbert spaces and let
\[
H_0 := H \oplus \hat{H}, \quad \mathbb{H} := H_0 \oplus H = H \oplus \hat{H} \oplus H.
\]
In the paper we study first-order symmetric systems of differential equations defined on an interval \( I = [a, b] \), \( -\infty < a < b \leq \infty \), with the regular endpoint \( a \) and regular or singular endpoint \( b \). Such a system is of the form [3, 12]
\[
Jy' - B(t)y = \Delta(t)f(t), \quad t \in I,
\]
where \( B(t) = B^*(t) \) and \( \Delta(t) \geq 0 \) are the \([\mathbb{H}]\)-valued functions on \( I \) and
\[
J = \begin{pmatrix} 0 & 0 & -I_H \\ 0 & iI_{\hat{H}} & 0 \\ I_H & 0 & 0 \end{pmatrix} : H \oplus \hat{H} \oplus H \to H \oplus \hat{H} \oplus H.
\]
System (1.2) is called a Hamiltonian system if \( \hat{H} = \{0\} \).
Throughout the paper we assume that system (1.2) is definite. The latter means that for any \( \lambda \in \mathbb{C} \) each solution \( y(\cdot) \) of the equation
\[
Jy' - B(t)y = \lambda \Delta(t)y
\]
satisfying \( \Delta(t)y(t) = 0 \) (a.e. on \( I \)) is trivial, i.e., \( y(t) = 0, \ t \in I \).

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In what follows we denote by $\mathcal{H} := L_2^\Delta(\mathcal{I})$ the Hilbert space of $\mathbb{H}$-valued Borel functions $f(\cdot)$ on $\mathcal{I}$ (in fact, equivalence classes) satisfying $||f||_\mathcal{H}^2 := \int_I (\Delta(t) f(t), f(t))_\mathbb{H} \, dt < \infty$.

Studying of symmetric systems is basically motivated by the fact that system (1.4) is a more general object than a formally self-adjoint differential equation of an arbitrary order with matrix coefficients. In fact, such equation is reduced to the system (1.4) of a special form with $J$ given by (1.3); moreover, this system is Hamiltonian precisely in the case when the differential equation is of an even order (we will concern this question below).

As is known, the extension theory of symmetric linear relations gives a natural framework for investigation of the boundary value problems for symmetric systems (see [4, 7, 20, 28, 36] and references therein). According to [20, 28, 36] the system (1.2) generates the minimal linear relation $T_{\min}$ and the maximal linear relation $T_{\max}$ in $\mathcal{H}$. It turns out that $T_{\min}$ is a closed symmetric relation with not necessarily equal deficiency indices $n_{\pm}(T_{\min})$ and $T_{\max} = T_{\min}^*$. Since system (1.2) is definite, $n_{\pm}(T_{\min})$ can be defined as a number of $L^2_{\Delta}$-solutions of (1.4) for $\lambda \in \mathbb{C}_\pm$.

A description of various classes of extensions of $T_{\min}$ (self-adjoint, $m$-dissipative, etc.) in terms of boundary conditions is an important problem in the spectral theory of symmetric systems. For Hamiltonian system (1.2) self-adjoint separated boundary conditions were described in [16]. Moreover, the Titchmarsh–Weyl coefficient $M_{TW}(\lambda)$ of the boundary value problem for Hamiltonian system with self-adjoint separated boundary conditions was defined by various methods in [16, 25, 23]. Using $M_{TW}(\cdot)$ one obtains the Fourier transform with the spectral function $\Sigma(\cdot)$ of the minimally possible dimension $N_\Sigma = \dim H$ (see [7, 18, 20]). At the same time according to [34] non-Hamiltonian system (1.2) does not admit self-adjoint separated boundary conditions. Moreover, the inequality $n_{+}(T_{\min}) \neq n_{-}(T_{\min})$, and hence absence of self-adjoint boundary conditions is a typical situation for such systems. Therefore the following problems are of certain interest:

- To find (may be $\lambda$-depending) analogs of self-adjoint separated boundary conditions for general (not necessarily Hamiltonian) systems (1.2) and describe such type conditions;
- To describe in terms of boundary conditions all spectral matrix functions that have the minimally possible dimension and investigate the corresponding Fourier transforms.

In the paper [2] these problems were considered for symmetric systems (1.2) satisfying $n_{-}(T_{\min}) \leq n_{+}(T_{\min})$. In the present paper we solve these problems in the opposite case $n_{+}(T_{\min}) < n_{-}(T_{\min})$. It turns out that this case requires a somewhat another approach in comparison with [2], although the ideas of both the papers are similar. Moreover, we show that in the case $n_{+}(T_{\min}) < n_{-}(T_{\min})$ there is a class of exit space self-adjoint extensions of $T_{\min}$ with special spectral properties. We apply also the obtained results to ordinary differential operators of an odd order with matrix valued coefficients and arbitrary deficiency indices.

Let $\nu_{b+}$ and $\nu_{b-}$ be indices of inertia of the skew-Hermitian bilinear form $[y, z]_b$ defined on $\text{dom} \ T_{\max}$ by

$$[y, z]_b = \lim_{t \to b} (J y(t), z(t)), \quad y, z \in \text{dom} \ T_{\max}.$$

It turns out that system (1.2) with $n_{+}(T_{\min}) < n_{-}(T_{\min})$ has different properties depending on the sign of $\nu_{b+} - \nu_{b-}$. Within this section we present the results of the paper assuming a simpler case $\nu_{b+} - \nu_{b-} \leq 0$.

Let a function $y \in \text{dom} \ T_{\max}$ be represented as $y(t) = \{y_0(t), \hat{y}(t), y_1(t)\} \in H \oplus \hat{H} \oplus H$. A crucial role in our considerations is played by a symmetric linear relation $T(\cdot \ T_{\min})$ in $\mathcal{H}$. 

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given by means of boundary conditions as follows:
\[ T = \{ (y, f) \in T_{\max} : y_1(a) = 0, \hat{y}(a) = 0, \Gamma_{0b}y = \Gamma_{1b}y = 0 \} \]
Here \( \Gamma_{0b} : \text{dom} \ T_{\max} \to \mathcal{H}_b \) and \( \Gamma_{1b} : \text{dom} \ T_{\max} \to \mathcal{H}_b \) are linear mappings with special properties, \( \mathcal{H}_b \) and \( \mathcal{H}_b(\subset \mathcal{H}_b) \) are auxiliary finite dimensional Hilbert spaces. In fact \( \Gamma_{0b}y \) and \( \Gamma_{1b}y \) are singular boundary values of a function \( y \in \text{dom} \ T_{\max} \) at the endpoint \( b \).

Recall that a linear relation \( \mathcal{T} = \mathcal{T}^* \) in a wider Hilbert space \( \mathfrak{H} \supset \mathfrak{H} \) satisfying \( \mathcal{T} \subset \mathfrak{H} \) is called an exit space self-adjoint extension of \( \mathcal{T} \). Moreover, a generalized resolvent \( R(\cdot) \) and a spectral function \( F(\cdot) \) of \( \mathcal{T} \) are defined by
\[ R(\lambda) = P_\mathfrak{H}(\mathfrak{T} - \lambda)^{-1} \upharpoonright \mathfrak{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad \text{and} \quad F(t) = P_\mathfrak{H}E(t) \upharpoonright \mathfrak{H}, \quad t \in \mathbb{R}, \]
where \( E(\cdot) \) is the orthogonal spectral function (resolution of identity) of \( \mathfrak{T} \). As is known [1] exit space self-adjoint extensions exist for symmetric linear relations with arbitrary (possibly unequal) deficiency indices.

To describe the set of all generalized resolvents of \( T \) we use the Nevanlinna type class \( \mathcal{R}_- (\mathcal{H} \oplus \mathcal{H}_b, \mathcal{H}_b) \) of holomorphic operator pairs \( \tau = \{ D_0(\lambda), D_1(\lambda) \} \). Such a pair is formed by defined on \( \mathbb{C}_- \) holomorphic operator functions
\[ D_0(\lambda) = (\hat{D}_0(\lambda), \hat{D}_{0b}(\lambda)) : \hat{\mathcal{H}} \oplus \mathcal{H}_b \to (\hat{\mathcal{H}} \oplus \mathcal{H}_b) \quad \text{and} \quad D_1(\lambda)(\in [\mathcal{H}_b, \hat{\mathcal{H}} \oplus \mathcal{H}_b]) \]
with special properties (see [35]). We show that each generalized resolvent \( y = R(\lambda)f, f \in \mathfrak{H} \), is given as the \( L^2_\mathcal{A} \)-solution of the following boundary-value problem with \( \lambda \)-depending boundary conditions:
\begin{align*}
Jy' - B(t)y &= \lambda \Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I}, \\
y_1(a) &= 0, \quad i\hat{D}_0(\lambda)\hat{y}(a) + \hat{D}_{0b}(\lambda)\Gamma_{0b}y + D_1(\lambda)\Gamma_{1b}y = 0, \quad \lambda \in \mathbb{C}_-.
\end{align*}
Here \( (\hat{D}_0(\lambda), \hat{D}_{0b}(\lambda)) =: D_0(\lambda) \) and \( D_1(\lambda) \) are components of a pair \( \tau = \{ D_0(\lambda), D_1(\lambda) \} \in \mathcal{R}_- (\hat{\mathcal{H}} \oplus \mathcal{H}_b, \mathcal{H}_b) \) (see (1.5)), so that the second equality in (1.7) is a Nevanlinna type boundary condition involving boundary values of a function \( y \) at both endpoints \( a \) and \( b \). Thus, investigation of boundary value problems for the system (1.2) in the case \( n_+(T_{\min}) < n_-(T_{\min}) \) require use of boundary conditions of another class in comparison with the case \( n_-(T_{\min}) \leq n_+(T_{\min}) \) (cf. [2]). One may consider a pair \( \tau \) as a boundary parameter, since \( R(\lambda) \) runs over the set of all generalized resolvents of \( T \) when \( \tau \) runs over the set of all holomorphic operator pairs of the class \( \mathcal{R}_- (\hat{\mathcal{H}} \oplus \mathcal{H}_b, \mathcal{H}_b) \). To indicate this fact explicitly we write \( R(\lambda) = R_\tau(\lambda) \) and \( F(t) = F_\tau(t) \) for the generalized resolvents and spectral functions of \( T \) respectively. Moreover, we denote by \( \mathcal{T} = \mathcal{T}_\tau \) the exit space self-adjoint extension of \( T \) generating \( R_\tau(\cdot) \) and \( F_\tau(\cdot) \).

Next assume that \( \varphi(\cdot, \lambda) \) and \( \psi(\cdot, \lambda) \) are \( [H_0, \mathbb{H}^\lambda] \)-valued operator solutions of the equation (1.4) satisfying the initial conditions
\[ \varphi(a, \lambda) = \left( \begin{array}{c} I_{H_0} \\ 0 \end{array} \right) (\in [H_0, H_0 \oplus H]), \quad \psi(a, \lambda) = \left( \begin{array}{c} -\frac{i}{\lambda}P_{\hat{H}} \\ -P_H \end{array} \right) (\in [H_0, H_0 \oplus H]). \]
(here \( P_H \) and \( P_{\hat{H}} \) are the orthoprojectors in \( H_0 \) onto \( H \) and \( \hat{H} \) respectively). We show that, for each Nevanlinna boundary parameter \( \tau = \{ D_0(\lambda), D_1(\lambda) \} \) of the form (1.5), there exists a unique operator function \( m_\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [H_0] \) such that the operator solution
\[ v_\tau(t, \lambda) := \varphi(t, \lambda)m_\tau(\lambda) + \psi(t, \lambda) \]
of Eq. (1.4) has the following property: for every $h_0 \in H_0$ the function $y = v_\tau(t, \lambda)h_0$ belongs to $L_2^\Delta(I)$ and satisfies the boundary condition
\[
\hat{D}_0(\lambda)(i\hat{g}(a) - P_Rh_0) + \hat{D}_0(\lambda)\hat{\Gamma}_0y + D_1(\lambda)\hat{\Gamma}_1y = 0, \quad \lambda \in \mathbb{C}_-.
\]
We call $m_\tau(\cdot)$ the $m$-function corresponding to the boundary value problem (1.6), (1.7); really, $m_\tau(\cdot)$ is an analog of the Titchmarsh-Weyl coefficient $M_{TW}(\lambda)$ for Hamiltonian systems. It turns out that $m_\tau(\cdot)$ is a Nevanlinna operator function satisfying the inequality
\[
(\text{Im}\lambda)^{-1} \cdot \text{Im} m_\tau(\lambda) \geq \int_I v_\tau^*(t, \lambda)\Delta(t)v_\tau(t, \lambda) \, dt, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.
\]
Next we study eigenfunction expansions of the boundary value problems for symmetric systems. Namely, let $\tau$ be a boundary parameter and let $F_\tau(\cdot)$ be the spectral function of $T$ generated by the boundary value problem (1.6), (1.7). A distribution operator-valued function $\Sigma_\tau(\cdot) : \mathbb{R} \to [H_0]$ is called a spectral function of this problem if, for each function $f \in \mathcal{F}$ with compact support, the Fourier transform
\[
\hat{f}(s) = \int_I \varphi^*(t, s)\Delta(t)f(t) \, dt
\]
satisfies
\[
((F_\tau(\beta) - F_\tau(\alpha))f, f)_{\mathcal{F}} = \int_{[\alpha, \beta]} (d\Sigma_\tau(s)\hat{f}(s), \hat{f}(s))
\]
for any compact interval $[\alpha, \beta) \subset \mathbb{R}$. We show that for each boundary parameter $\tau$ there exists a unique spectral function $\Sigma_\tau(\cdot)$ and it is recovered from the $m$-function $m_\tau(\cdot)$ by means of the Stieltjes inversion formula
\[
\Sigma_\tau(s) = -\lim_{\delta \to +0} \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{s-\delta}^{s+\delta} \text{Im} m_\tau(\sigma - i\varepsilon) \, d\sigma.
\]
Below (within this section) we assume for simplicity that $T$ is a (not necessarily densely defined) operator, i.e., $\text{mul} T = \{0\}$.

It follows from (1.8) that the mapping $Vf = \hat{f}$ (the Fourier transform) admits a continuous extension to a contractive linear mapping $V : \mathcal{F} \to L^2(\Sigma; H_0)$ (for the strict definition of the Hilbert space $L^2(\Sigma; H_0)$ see e.g. [8, Ch.13.5]. As in [2] one proves that $V$ is an isometry (that is, the Parseval equality $||\hat{f}||_{L^2(\Sigma; H_0)} = ||f||_{\mathcal{F}}$ holds for every $f \in \mathcal{F}$) if and only if the exit space extension $\overline{T_\tau}$ in $\tilde{\mathcal{F}} \supset \mathcal{F}$ is an operator, i.e., $\text{mul} \overline{T_\tau} = \{0\}$. In this case one may define the inverse Fourier transform in the explicit form (see (5.30)). Moreover, if $V$ is an isometry, then there exists a unitary extension $\overline{V} \in [\tilde{\mathcal{F}}, L^2(\Sigma; H_0)]$ of $V$ such that the operator $\overline{T_\tau}$ and the multiplication operator $\Lambda$ in $L^2(\Sigma; H_0)$ are unitarily equivalent by means of $\overline{V}$. Hence, the operators $T^\tau$ and $\Lambda$ have the same spectral properties; for instance, multiplicity of the spectrum of $\overline{T_\tau}$ does not exceed $\dim H_0 = \dim H + \dim \overline{H}$.

Now assume that the boundary parameter $\tau = \{D_0(\lambda), D_1(\lambda)\}$ is (cf. (1.5))
\[
\begin{align*}
D_0(\lambda) &= \text{diag}(I_{\overline{H}}, \overline{\mathcal{T}}_0(\lambda))(\in [\overline{H} \oplus \overline{H}_b], \quad D_1(\lambda) = (0, \mathcal{T}_1(\lambda))^\top(\in [H_b, \overline{H} \oplus \overline{H}_b]).
\end{align*}
\]
We show that in this case the corresponding $m$-function $m_\tau(\cdot)$ is of the triangular form
\[
m_\tau(\lambda) = \begin{pmatrix}
m_{\tau,1}(\lambda) & m_{\tau,2}(\lambda) \\
0 & -\frac{1}{2}I_{\overline{H}}
\end{pmatrix} : H \oplus \overline{H} \to H \oplus \overline{H}, \quad \lambda \in \mathbb{C}_-,
\]
so that the spectral function $\Sigma_{\tau}(\cdot)$ has the block matrix representation

$$
(1.11) \quad \Sigma_{\tau}(s) = \begin{pmatrix}
\Sigma_{\tau,1}(s) & \Sigma_{\tau,2}(s) \\
\Sigma_{\tau,3}(s) & \Sigma_{\tau,4}(s)
\end{pmatrix}_{\tau \in \pi} : H \oplus \tilde{H} \to H \oplus \tilde{H}, \quad s \in \mathbb{R}.
$$

Here $\Sigma_{\tau,1}(\cdot)$ is an $[H]$-valued distribution function, which can be defined by means of the Stieltjes inversion formula for $m_{\tau,1}(\lambda)$. It follows from (1.11), that in the case when a boundary parameter $\tau = \{D_0(\lambda), D_1(\lambda)\}$ is of the form (1.10) and the Fourier transform is an isometry, the corresponding exit space self-adjoint extension $\tilde{T^\tau}$ of $T$ has the following spectral properties (for more details see Theorem 5.14):

(S1) $\sigma_{ac}(\tilde{T^\tau}) = \mathbb{R}$, where $\sigma_{ac}(\tilde{T^\tau})$ is the absolutely continuous spectrum of $\tilde{T^\tau}$.

(S2) $\sigma_s(\tilde{T^\tau}) = Ss(\Sigma_{\tau,1})$, where $\sigma_s(\tilde{T^\tau})$ is the singular spectrum of $\tilde{T^\tau}$ and $Ss(\Sigma_{\tau,1})$ is a closed support of the measure generated by the singular component of $\Sigma_{\tau,1}$. Hence the multiplicity of the singular spectrum of $\tilde{T^\tau}$ does not exceed $\dim H$, which yields the same estimate for multiplicity of each eigenvalue $\lambda_0$ of $\tilde{T^\tau}$.

Next, we show that all spectral functions $\Sigma_{\tau}(\cdot)$ can be parametrized immediately in terms of the boundary parameter $\tau$. More precisely, we show that there exists an operator function $m_{\tau}(\cdot)$ is given by

$$
(1.12) \quad X_-(\lambda) = \begin{pmatrix}
m_0(\lambda) & \Phi_-(\lambda) \\
\Psi_-(\lambda) & M_-(\lambda)
\end{pmatrix}_{\psi} : H_0 \oplus (\tilde{H} \oplus \tilde{H}_b) \to H_0 \oplus H_b, \quad \lambda \in \mathbb{C}_-,
$$

such that for each boundary parameter $\tau = \{D_0(\lambda), D_1(\lambda)\}$ the corresponding $m$-function $m_{\tau}(\cdot)$ is given by

$$
(1.13) \quad m_{\tau}(\lambda) = m_0(\lambda) + \Phi_-(\lambda)(D_0(\lambda) - D_1(\lambda)M_-(\lambda))^{-1}D_1(\lambda)\Psi_-(\lambda), \quad \lambda \in \mathbb{C}_-.
$$

Thus, formula (1.13) together with the Stieltjes inversion formula (1.9) defines a (unique) spectral function $\Sigma_{\tau}(\cdot)$ of the boundary value problem (1.6), (1.7). Note that entries of the matrix (1.12) are defined in terms of the boundary values of respective operator solutions of Eq.(1.4). We also describe boundary parameters $\tau$ for which the Fourier transform $V$ is an isometry and characterize the case when $V$ is an isometry for every boundary parameter $\tau$ (see Theorem 5.15). Note that a description of spectral functions for various classes of boundary problems in the form close to (1.13), (1.9) can be found in [11, 13, 15, 22, 21, 33].

Clearly, all the foregoing results can be reformulated (with obvious simplifications) for Hamiltonian systems (1.2).

We suppose that the above results about exit space extensions $\tilde{T^\tau}$ of $T$ may be useful in the case when $\tilde{T^\tau}$ is a self-adjoint relation in $\tilde{D} = L^2_{\Delta}(\mathbb{R})$ induced by the symmetric system on the whole line $\mathbb{R}$. More precisely, we assume that spectral properties of such $\tilde{T^\tau}$ may be characterized in terms of the objects ($m$-function, boundary parameter etc.) associated with the restriction of this system onto the semi-axis $I = [0, \infty)$. These problems will be considered elsewhere.

If $n_+(T_{\min})$ takes on the minimally possible value $n_+(T_{\min}) = \dim H$, then $n_+(T_{\min}) \leq n_-(T_{\min})$ and the above results can be rather simplified. Namely, in this case $T$ is a maximal symmetric relation and hence there exists a unique generalized resolvent of $T$. Therefore there is a unique $m$-function $m(\cdot)$ which has the triangular form

$$
(1.14) \quad m(\lambda) = \begin{pmatrix}
M(\lambda) & N_-(\lambda) \\
0 & -\frac{1}{2}I_{\tilde{H}}
\end{pmatrix}_{\psi} : H \oplus \tilde{H} \to H \oplus \tilde{H}, \quad \lambda \in \mathbb{C}_-.
$$
Moreover, the spectral function \( \Sigma(\cdot) \) of the corresponding boundary value problem is of the form (1.11) and the Fourier transform \( V \) is an isometry. Therefore a (unique) exit space self-adjoint extension \( T_0 \) of \( T \) has the spectral properties (S1) and (S2).

Note that systems (1.2) with \( \hat{H} \neq \{0\} \) and both minimal deficiency indices \( n_+(T_{\text{min}}) = \dim H \) and \( n_-(T_{\text{min}}) = \dim H + \dim \hat{H} \) were studied in the paper by Hinton and Schneider [17], where the concept of the “rectangular” Titchmarsh-Weyl coefficient \( M_{TW}(\lambda) \in [H, H \oplus \hat{H}], \lambda \in \mathbb{C}_+, \) was introduced. One can easily show that in fact \( M_{TW}(\lambda) = (M^*(\overline{\lambda}), N^*(\overline{\lambda}))^T \), where \( M(\lambda) \) and \( N_-(\lambda) \) are taken from (1.14).

In the final part of the paper we consider the operators generated by a differential expression \( l[y] \) of an odd order \( 2m+1 \) with \( [H] \)-valued coefficients (\( H \) is a Hilbert space with \( r := \dim H < \infty \)) defined on an interval \( I = [a, b] \) (see (6.1)). In particular, case of scalar coefficients such differential operators have been investigated in the papers by Everitt and Krishna Kumar [9, 10, 26], where the limiting process from the compact intervals \([a, \beta]\) \( \subset I \) was used for construction of the Titchmarsh-Weyl matrix \( M_{TW}(\lambda) = (m_{jk}(\lambda))^{m+1}_{j,k=1} \). Note that the results of these papers can not be considered to be completed; in particular, an attempt to define self-adjoint boundary conditions in [10] gave rise to hardly verifiable assumptions even in the case of minimally possible equal deficiency indices \( n_{\pm}(L_{\text{min}}) = m+1 \) of the minimal operator \( L_{\text{min}} \).

Our approach is based on the known fact [24] that the equation \( l[y] = \lambda y \) is equivalent to a special symmetric non-Hamiltonian system (1.4). This enables us to extend the obtained results concerning symmetric systems to the expression \( l[y] \) with arbitrary (possibly unequal) deficiency indices of \( L_{\text{min}} \). In particular, we describe self-adjoint and \( \lambda \)-depending boundary conditions for \( l[y] \), which are analogs of self-adjoint separated boundary conditions for differential expressions of an even order. This makes it possible to construct eigenfunction expansion with spectral matrix function \( \Sigma(\cdot) \) of the dimension \((m+1)r \times (m+1)r \) and to describe all \( \Sigma(\cdot) \) immediately in terms of boundary conditions (for operators of an even order and separated boundary conditions such a description was obtained in [33]).

In conclusion note that the above results are obtained with the aid of the method of boundary triplets and the corresponding Weyl functions in the extension theory of symmetric linear relations (see [6, 14, 29, 31, 32] and references therein).

2. Preliminaries

2.1. Notations. The following notations will be used throughout the paper: \( \mathcal{H} \), \( \mathcal{H} \) denote Hilbert spaces: \([\mathcal{H}_1, \mathcal{H}_2]\) is the set of all bounded linear operators defined on the Hilbert space \( \mathcal{H}_1 \) with values in the Hilbert space \( \mathcal{H}_2 \); \([\mathcal{H}] := [\mathcal{H}, \mathcal{H}]; A \mid \mathcal{L} \) is the restriction of an operator \( A \) onto the linear manifold \( \mathcal{L} \); \( P_{\mathcal{L}} \) is the orthogonal projector in \( \mathcal{H} \) onto the subspace \( \mathcal{L} \subset \mathcal{H} \); \( \mathbb{C}_+ \) (\( \mathbb{C}_- \)) is the upper (lower) half-plane of the complex plane.

Recall that a closed linear relation from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \) is a closed linear subspace in \( \mathcal{H}_0 \oplus \mathcal{H}_1 \). The set of all closed linear relations from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \) (in \( \mathcal{H} \)) will be denoted by \( \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1) \) (\( \tilde{\mathcal{C}}(\mathcal{H}) \)). A closed linear operator \( T \) from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \) is identified with its graph \( \text{gr} \ T \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1) \).

For a linear relation \( T \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1) \) we denote by \( \text{dom} \ T, \text{ran} \ T, \text{ker} \ T \) and \( \text{mul} \ T \) the domain, range, kernel and the multivalued part of \( T \) respectively. Recall also that the inverse and adjoint linear relations of \( T \) are the relations \( T^{-1} \in \tilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0) \) and \( T^* \in \tilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0) \).
defined by
\[ T^{-1} = \{ \{ h_1, h_0 \} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : \{ h_0, h_1 \} \in T \}, \quad T^* = \{ \{ k_1, k_0 \} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : (k_0, h_0) - (k_1, h_1) = 0, \{ h_0, h_1 \} \in T \}. \] (2.1)

In the case \( T \in \tilde{C}(\mathcal{H}_0, \mathcal{H}_1) \) we write \( 0 \in \rho(T) \) if ker \( T = \{ 0 \} \) and ran \( T = \mathcal{H}_1 \), or equivalently if \( T^{-1} \in [\mathcal{H}_1, \mathcal{H}_0] ; 0 \in \overline{\rho(T)} \) if ker \( T = \{ 0 \} \) and ran \( T \) is a closed subspace in \( \mathcal{H}_1 \). For a linear relation \( T \in \tilde{C}(\mathcal{H}) \) we denote by \( \rho(T) := \{ \lambda \in \mathbb{C} : 0 \in \rho(T - \lambda) \} \) and \( \overline{\rho(T)} = \{ \lambda \in \mathbb{C} : 0 \in \overline{\rho(T - \lambda)} \} \) the resolvent set and the set of regular type points of \( T \) respectively.

Recall also the following definition.

**Definition 2.1.** A holomorphic operator function \( \Phi(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\mathcal{H}] \) is called a Nevanlinna function if \( \text{Im} \lambda \cdot \text{Im} \Phi(\lambda) \geq 0 \) and \( \Phi^*(\lambda) = \Phi(\overline{\lambda}) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

**2.2. Symmetric and self-adjoint linear relations.** A linear relation \( A \in \tilde{C}(\mathfrak{H}) \) is called symmetric (self-adjoint) if \( A \subseteq A^* \) (resp. \( A = A^* \)). For each symmetric relations \( A \in \tilde{C}(\mathfrak{H}) \) the following decompositions hold
\[ \mathfrak{H} = \mathfrak{H}_0 \oplus \text{mul} A, \quad A = \text{gr} A' \oplus \text{mul} A, \]
where \( \text{mul} A = \{ 0 \} \oplus \text{mul} A \) and \( A' \) is a closed symmetric (not necessarily densely defined) operator in \( \mathfrak{H}_0 \) (the operator part of \( A \)); moreover, \( A = A^* \) if and only if \( A' = (A')^* \).

A spectral function \( E(\cdot) : \mathbb{R} \rightarrow [\mathfrak{H}] \) of the relation \( A = A^* \in \tilde{C}(\mathfrak{H}) \) is defined as \( E(t) = E'(t)P_{\mathfrak{H}_0}, \) where \( E'(\cdot) : \mathbb{R} \rightarrow [\mathfrak{H}_0] \) is the orthogonal spectral function of \( A' \).

For an operator \( A = A^* \) we denote by \( \sigma(A) \), \( \sigma_{ac}(A) \) and \( \sigma_s(A) \) the spectrum, the absolutely continuous spectrum and the singular spectrum of \( A \) respectively [19, Section 10.1].

Next assume that \( \mathcal{H} \) is a finite dimensional Hilbert space. A non-decreasing operator function \( \Sigma(\cdot) : \mathbb{R} \rightarrow [\mathcal{H}] \) is called a distribution function if it is left continuous and satisfies \( \Sigma(0) = 0 \). For each distribution function \( \Sigma(\cdot) \) there is a unique pair of distribution functions \( \Sigma_{ac}(\cdot) \) and \( \Sigma_s(\cdot) \) such that \( \Sigma_{ac}(\cdot) \) is absolutely continuous, \( \Sigma_s(\cdot) \) is singular and \( \Sigma(t) = \Sigma_{ac}(t) + \Sigma_s(t) \) (the Lebesgue decomposition of \( \Sigma \)). For a distribution function \( \Sigma(\cdot) \) we denote by \( S(\Sigma) \) the set of all \( t \in \mathbb{R} \) such that \( \Sigma(t - \delta) \neq \Sigma(t + \delta) \) for any \( \delta > 0 \). Moreover, we let \( S_{ac}(\Sigma) = S(\Sigma_{ac}) \) and \( S_s(\Sigma) = S(\Sigma_s) \).

With each distribution function \( \Sigma(\cdot) \) one associates the Hilbert space \( L^2(\Sigma; \mathcal{H}) \) of all functions \( f(\cdot) : \mathbb{R} \rightarrow \mathcal{H} \) such that \( \int d\Sigma(t)f(t), f(t) < \infty \) (for more details see e.g. [8, Section 13.5]). The following theorem is well known.

**Theorem 2.2.** Let \( \Sigma(\cdot) \) be a \([\mathcal{H}]\)-valued distribution function. Then the relations
\[ \text{dom} \Lambda_\Sigma = \{ f \in L^2(\Sigma; \mathcal{H}) : t f(t) \in L^2(\Sigma; \mathcal{H}) \}, \quad (\Lambda_\Sigma f)(t) = t f(t), \quad f \in \text{dom} \Lambda_\Sigma \]
define a self-adjoint operator \( \Lambda = \Lambda_\Sigma \) in \( L^2(\Sigma; \mathcal{H}) \) (the multiplication operator) and
\[ \sigma(\Lambda_\Sigma) = S(\Sigma), \quad \sigma_{ac}(\Lambda_\Sigma) = S_{ac}(\Sigma), \quad \sigma_s(\Lambda_\Sigma) = S_s(\Sigma). \]

**2.3. The class \( \tilde{C}_-(\mathcal{H}_0, \mathcal{H}_1) \).** Let \( \mathcal{H}_0 \) be a Hilbert space, let \( \mathcal{H}_1 \) be a subspace in \( \mathcal{H}_0 \) and let \( \tau = \{ \tau_+, \tau_- \} \) be a collection of holomorphic functions \( \tau_{\pm}(\cdot) : \mathbb{C}_\pm \rightarrow \tilde{C}(\mathcal{H}_0, \mathcal{H}_1) \). In the paper we systematically deal with collections \( \tau = \{ \tau_+, \tau_- \} \) of the special class \( \tilde{C}_-(\mathcal{H}_0, \mathcal{H}_1) \) introduced in [35]. In the case \( \mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H} \) this class turns into the well known class
$\tilde{R}(\mathcal{H})$ of Nevanlinna functions $\tau(\cdot): \mathbb{C} \setminus \mathbb{R} \to \tilde{C}(\mathcal{H})$ (see for instance [5]). If $\dim \mathcal{H}_0 < \infty$, then according to [35] the collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}^{-} (\mathcal{H}_0, \mathcal{H}_1)$ admits the representation

\begin{equation}
\tau_+(\lambda) = \{(C_0(\lambda), C_1(\lambda)) : \mathcal{H}_1\}, \quad \lambda \in \mathbb{C}_+;
\end{equation}

\begin{equation}
\tau_-(\lambda) = \{(D_0(\lambda), D_1(\lambda)) : \mathcal{H}_0\}, \quad \lambda \in \mathbb{C}_-.
\end{equation}

by means of holomorphic operator pairs

\begin{equation}
(C_0(\lambda), C_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_1, \quad \lambda \in \mathbb{C}_+;
\end{equation}

\begin{equation}
(D_0(\lambda), D_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0, \quad \lambda \in \mathbb{C}_-.
\end{equation}

In [35] the class $\tilde{R}^{-} (\mathcal{H}_0, \mathcal{H}_1)$ is characterized both in terms of $\tilde{C}(\mathcal{H}_0, \mathcal{H}_1)$-valued functions $\tau_\pm(\cdot)$ and in terms of operator functions $C_j(\cdot)$ and $D_j(\cdot)$, $j \in \{0, 1\}$, from (2.3).

2.4. Boundary triplets and Weyl functions. Here we recall some definitions and results from our paper [35].

Let $A$ be a closed symmetric linear relation in the Hilbert space $\mathcal{H}$, let $\mathfrak{N}_A(\lambda) = \ker (A^* - \lambda)$ ($\lambda \in \rho(A)$) be a defect subspace of $A$, let $\widehat{\mathfrak{N}}_A(\lambda) = \{f, \lambda f : f \in \mathfrak{N}_A(\lambda)\}$ and let $n_\pm(\lambda) := \dim \mathfrak{N}_A(\lambda) \leq \infty$, $\lambda \in \mathbb{C}_\pm$, be deficiency indices of $A$. Denote by $\text{Ext}_A$ the set of all proper extensions of $A$, i.e., the set of all relations $\tilde{A} \in \tilde{C}(\mathcal{H})$ such that $A \subset \tilde{A} \subset A^*$.

Next assume that $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$ and $\mathcal{H}_2 := \mathcal{H}_0 \oplus \mathcal{H}_1$, so that $\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$. Denote by $P_j$ the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j \in \{1, 2\}$.

**Definition 2.3.** A collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where $\Gamma_j : A^* \to \mathcal{H}_j$, $j \in \{0, 1\}$, are linear mappings, is called a boundary triplet for $A^*$, if the mapping $\Gamma: f \to (\Gamma_0 f, \Gamma_1 f)$, $f \in A^*$, from $A^*$ into $\mathcal{H}_0 \oplus \mathcal{H}_1$ is surjective and the following Green’s identity

\begin{equation}
(f', g) - (f, g') = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_{\mathcal{H}_0} - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_{\mathcal{H}_0} - i(P_2 \Gamma_0 \hat{f}, P_2 \Gamma_0 \hat{g})_{\mathcal{H}_2}
\end{equation}

holds for all $\hat{f} = \{f, f'\}$, $\hat{g} = \{g, g'\} \in A^*$.

**Proposition 2.4.** Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $A^*$. Then:

1. $\dim \mathcal{H}_1 = n_+(A) \leq n_-(A) = \dim \mathcal{H}_0$.
2. $\ker \Gamma_0 \cap \ker \Gamma_1 = \mathbb{C}$ and $\Gamma_j$ is a bounded operator from $A^*$ into $\mathcal{H}_j$, $j \in \{0, 1\}$.
3. The equality

\begin{equation}
A_0 := \ker \Gamma_0 = \{\hat{f} \in A^* : \Gamma_0 \hat{f} = 0\}
\end{equation}

defines the maximal symmetric extension $A_0 \in \text{Ext}_A$ such that $\mathbb{C}_- \subset \rho(A_0)$.

**Proposition 2.5.** Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $A^*$ (so that in view of Proposition 2.4, (1) $n_+(A) \leq n_-(A)$). Denote also by $\pi_1$ the orthoprojector in $\mathcal{H} \oplus \mathcal{H}_0$ onto $\mathcal{H}_0 \oplus \mathcal{H}_1 \subset \mathcal{H}$. Then:

1. The operators $P_1 \Gamma_0 \mid \widehat{\mathfrak{N}}_A(\lambda)$, $\lambda \in \mathbb{C}_+$, and $\Gamma_0 \mid \widehat{\mathfrak{N}}_z(A)$, $z \in \mathbb{C}_-$, isomorphically map $\widehat{\mathfrak{N}}_A(\lambda)$ onto $\mathcal{H}_1$ and $\widehat{\mathfrak{N}}_z(A)$ onto $\mathcal{H}_0$ respectively. Therefore the equalities

\begin{equation}
\gamma_+(\lambda) = \pi_1(P_1 \Gamma_0 \mid \widehat{\mathfrak{N}}_A(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_+;
\end{equation}

\begin{equation}
\gamma_-(z) = \pi_1(\Gamma_0 \mid \widehat{\mathfrak{N}}_z(A))^{-1}, \quad z \in \mathbb{C}_-;
\end{equation}

\begin{equation}
M_+(\lambda) h_1 = (\Gamma_1 - iP_2 \Gamma_0) \{\gamma_+(\lambda) h_1, \lambda \gamma_+(\lambda) h_1\}, \quad h_1 \in \mathcal{H}_1, \quad \lambda \in \mathbb{C}_+
\end{equation}

\begin{equation}
M_-(z) h_0 = \Gamma_1 \{\gamma_-(z) h_0, z \gamma_-(z) h_0\}, \quad h_0 \in \mathcal{H}_0, \quad z \in \mathbb{C}_-
\end{equation}
correctly define the operator functions \( \gamma_+(: \mathbb{C} \to [\mathcal{H}_1, \mathfrak{F}], \gamma_-(:: \mathbb{C} \to [\mathcal{H}_0, \mathfrak{F}] \) and \( M_+: \mathbb{C} \to [\mathcal{H}_1, \mathcal{H}_0], \quad M_-(:: \mathbb{C} \to [\mathcal{H}_0, \mathcal{H}_1], \) which are holomorphic on their domains. Moreover, the equality \( M_+^{*}(\lambda) = M_-(\lambda), \lambda \in \mathbb{C}, \) is valid.

(2) Assume that

\[
M_+ (\lambda) = (M(\lambda), N_+ (\lambda)) : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}^+
\]

\[
M_-(z) = (M(z), N_- (z)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1, \quad z \in \mathbb{C}^-
\]

are the block representations of \( M_+ (\lambda) \) and \( M_- (z) \) respectively and let

\[
\mathcal{M}(\lambda) = \begin{pmatrix}
M(\lambda) & 0 \\
N_+ (\lambda) & \frac{1}{2} I_{\mathcal{H}_2}
\end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}^+
\]

\[
\mathcal{M}(\lambda) = \begin{pmatrix}
M(\lambda) & 0 \\
N_- (\lambda) & -\frac{1}{2} I_{\mathcal{H}_2}
\end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}^-.
\]

Then \( \mathcal{M}(\cdot) \) is a Nevanlinna operator function satisfying the identity

\[
\mathcal{M}(\mu) - \mathcal{M}^*(\lambda) = (\mu - \lambda) \gamma_+ (\lambda) \gamma_- (\mu), \quad \mu, \lambda \in \mathbb{C}.
\]

**Definition 2.6.** [35] The operator functions \( \gamma_+ (\cdot) \) and \( M_+ (\cdot) \) are called the \( \gamma \)-fields and the Weyl functions, respectively, corresponding to the boundary triplet \( \Pi_- \).

It follows from (2.7) that for each \( h_1 \in \mathcal{H}_1 \) and \( h_0 \in \mathcal{H}_0 \) the following equalities hold

\[
P_1 \Gamma_0 \gamma_+(\lambda) h_1, \lambda \gamma_+(\lambda) h_1 \} = h_1, \quad \Gamma_0 \{ \gamma_-(z) h_0, z \gamma_-(z) h_0 \} = h_0.
\]

**Proposition 2.7.** Let \( \Pi_- = \{ \mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( A^* \) and let \( \gamma_+ (\cdot) \) and \( M_+ (\cdot) \) be the corresponding \( \gamma \)-fields and Weyl functions respectively. Moreover, let the spaces \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) be decomposed as

\[
\mathcal{H}_1 = \mathcal{H} \oplus \tilde{\mathcal{H}}_1, \quad \mathcal{H}_0 = \mathcal{H} \oplus \tilde{\mathcal{H}}_0
\]

(so that \( \mathcal{H}_0 = \tilde{\mathcal{H}}_1 \oplus \mathcal{H}_2 \) and let

\[
\Gamma_0 = (\tilde{\Gamma}_0, \tilde{\Gamma}_0)^T : A^* \to \mathcal{H} \oplus \mathcal{H}_0, \quad \Gamma_1 = (\tilde{\Gamma}_1, \tilde{\Gamma}_1)^T : A^* \to \mathcal{H} \oplus \mathcal{H}_1
\]

be the block representations of the operators \( \Gamma_0 \) and \( \Gamma_1 \). Then:

1. The equalities
   \[
   \tilde{A} = \{ \tilde{f} \in A^* : \tilde{\Gamma}_0 \tilde{f} = \tilde{\Gamma}_1 \tilde{f} = 0 \}, \quad \tilde{A}^* = \{ \tilde{f} \in A^*: \tilde{\Gamma}_0 \tilde{f} = 0 \}
   \]
   define a closed symmetric extension \( \tilde{A} \in Ext \mathcal{A} \) and its adjoint \( \tilde{A}^* \).

2. The closed collection \( \Pi_- = \{ \mathcal{H}_0 \oplus \mathcal{H}_1, \tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{A}^* \} \) is a boundary triplet for \( \tilde{A}^* \).

3. The \( \gamma \)-fields \( \gamma_\pm (\cdot) \) and the Weyl functions \( M_\pm (\cdot) \) corresponding to \( \Pi_- \) are given by
   \[
   \gamma_+(\lambda) = \gamma_+(\lambda) \upharpoonright \mathcal{H}_1, \quad M_+(\lambda) = P_{\mathcal{H}_0} M_+(\lambda) \upharpoonright \mathcal{H}_1, \quad \lambda \in \mathbb{C}^+
   \]
   \[
   \gamma_-(\lambda) = \gamma_-(\lambda) \upharpoonright \mathcal{H}_0, \quad M_-(\lambda) = P_{\mathcal{H}_1} M_-(\lambda) \upharpoonright \mathcal{H}_0, \quad \lambda \in \mathbb{C}^-
   \]

The proof of Proposition 2.7 is similar to that of Proposition 4.1 in [5].

**Remark 2.8.** If \( \mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H} \), then the boundary triplet in the sense of Definition 2.3 turns into the boundary triplet \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) for \( A^* \) in the sense of [14, 29]. In this case \( n_+ (A) = n_-(A) = \dim \mathcal{H} \) and the \( \gamma \)-fields \( \gamma_\pm (\cdot) \) and Weyl functions \( M_\pm (\cdot) \) turn into the \( \gamma \)-field \( \gamma (\cdot) \) and Weyl function \( M (\cdot) \) respectively introduced in [6, 29]. Observe also that along with \( \Pi_- \) we define in [32, 35] a boundary triplet \( \Pi_+ = \{ \mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1 \} \) for \( A^* \). Such a triplet is applicable to symmetric relations \( A \) with \( n_-(A) \leq n_+(A) \).
2.5. Generalized resolvents and exit space extensions. The following definitions are well known.

**Definition 2.9.** Let \( \tilde{\mathcal{H}} \) be a Hilbert space and let \( \mathcal{H} \) be a subspace in \( \tilde{\mathcal{H}} \). A relation \( \tilde{A} = \tilde{A}^* \in \mathcal{C}(\tilde{\mathcal{H}}) \) is called \( \mathcal{H} \)-minimal if \( \text{span}\{\mathcal{H}, (\tilde{A} - \lambda)^{-1}\tilde{\mathcal{H}} : \lambda \in \mathbb{C} \setminus \mathbb{R}\} = \tilde{\mathcal{H}} \).

**Definition 2.10.** The relations \( T_j \in \mathcal{C}(\mathcal{H}_j) \), \( j \in \{1, 2\} \), are said to be unitarily equivalent (by means of a unitary operator \( U \in [\mathcal{H}_1, \mathcal{H}_2] \)) if \( T_2 = UT_1 \) with \( U = U \oplus U \in [\mathcal{H}_1^2, \mathcal{H}_2^2] \).

**Definition 2.11.** Let \( A \) be a symmetric relation in a Hilbert space \( \mathcal{H} \). The operator functions \( R(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathcal{H} \) and \( F(\cdot) : \mathbb{R} \to \mathcal{H} \) are called the generalized resolvent and the spectral function of \( A \) respectively if there exist a Hilbert space \( \tilde{\mathcal{H}} \supset \mathcal{H} \) and a self-adjoint relation \( \tilde{A} \in \mathcal{C}(\tilde{\mathcal{H}}) \) such that \( A \subset \tilde{A} \) and the following equalities hold:

\[
R(\lambda) = P_A(\tilde{A} - \lambda)^{-1} \upharpoonright \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \\
F(t) = P_A E(t) \upharpoonright \mathcal{H}, \quad t \in \mathbb{R}
\]

(in formula (2.15) \( E(\cdot) \) is the spectral function of \( \tilde{A} \)).

The relation \( \tilde{A} \) in (2.14) is called an exit space extension of \( A \).

According to [27] each generalized resolvent of \( A \) is generated by some \( \mathcal{H} \)-minimal exit space extension \( \tilde{A} \) of \( A \). Moreover, if the \( \mathcal{H} \)-minimal exit space extensions \( \tilde{A}_1 \in \mathcal{C}(\tilde{\mathcal{H}}_1) \) and \( \tilde{A}_2 \in \mathcal{C}(\tilde{\mathcal{H}}_2) \) of \( A \) induce the same generalized resolvent \( R(\lambda) \), then there exists a unitary operator \( V' \in [\tilde{\mathcal{H}}_1 \ominus \mathcal{H}, \tilde{\mathcal{H}}_2 \ominus \mathcal{H}] \) such that \( \tilde{A}_1 \) and \( \tilde{A}_2 \) are unitarily equivalent by means of \( \tilde{V} = I_{\tilde{\mathcal{H}}} \oplus V' \). By using these facts we suppose in the following that the exit space extension \( \tilde{A} \) in (2.14) is \( \mathcal{H} \)-minimal, so that \( \tilde{A} \) is defined by \( R(\cdot) \) uniquely up to the unitary equivalence. Note that in this case the equality \( \text{mul} \tilde{A} = \text{mul} A \) holds for any exit space extension \( \tilde{A} = \tilde{A}^* \) of \( A \) if and only if \( \text{mul} A = \text{mul} A^* \) or, equivalently, if and only if the operator \( A' \) (the operator part of \( A \)) is densely defined.

### 3. Boundary triplets for symmetric systems

3.1. **Notations.** Let \( I = [a, b] \) \((-\infty < a < b \leq \infty\) be an interval of the real line (the symbol \( \cdot \)) means that the endpoint \( b < \infty \) might be either included to \( I \) or not). For a given finite-dimensional Hilbert space \( \mathbb{H} \) denote by \( AC(I; \mathbb{H}) \) the set of functions \( f(\cdot) : I \to \mathbb{H} \) which are absolutely continuous on each segment \([a, \beta] \subset I\) and let \( AC(I) := AC(I; \mathbb{C}) \).

Next assume that \( \Delta(\cdot) \) is a \( [\mathbb{H}] \)-valued Borel functions on \( I \) integrable on each compact interval \([a, \beta] \subset I \) and such that \( \Delta(t) \geq 0 \). Denote by \( L^2_\Delta(I) \) the semi-Hilbert space of Borel functions \( f(\cdot) : I \to \mathbb{H} \) satisfying \( \|f\|_{L^2_\Delta} := \int_I \|\Delta(t)f(t)\|_{\mathbb{H}} dt < \infty \) (see e.g. [8, Chapter 13.5]). The semi-definite inner product \( (\cdot, \cdot)_\Delta \) in \( L^2_\Delta(I) \) is defined by \( (f, g)_\Delta = \int_I (\Delta(t)f(t), g(t))_{\mathbb{H}} dt, \ f, g \in L^2_\Delta(I) \). Moreover, let \( L^\Delta(I) \) be the Hilbert space of the equivalence classes in \( L^2_\Delta(I) \) with respect to the semi-norm \( \| \cdot \|_\Delta \) and let \( \pi \) be the quotient map from \( L^2_\Delta(I) \) onto \( L^\Delta(I) \).

For a given finite-dimensional Hilbert space \( \mathbb{K} \) denote by \( L^2_\Delta[\mathbb{K}, \mathbb{H}] \) the set of all Borel operator-functions \( F(\cdot) : I \to [\mathbb{K}, \mathbb{H}] \) such that \( F(t)h \in L^2_\Delta(I) \) for each \( h \in \mathbb{K} \). It is clear that the latter condition is equivalent to \( \int_I \|F(t)h\|^2 dt < \infty \).
3.2. Symmetric systems. In this subsection we provide some known results on symmetric systems of differential equations.

Let $H$ and $\hat{H}$ be finite-dimensional Hilbert spaces and let

$$H_0 = H \oplus \hat{H}, \quad \mathbb{H} = H_0 \oplus H = H \oplus \hat{H} \oplus H.$$  

In the following we put

$$\nu_+ := \dim H, \quad \hat{\nu} := \dim \hat{H}, \quad \nu_- := \dim H_0 = \nu_+ + \hat{\nu}, \quad n := \dim \mathbb{H} = \nu_+ + \nu_-.$$  

Let as above $\mathcal{I} = [a, b]$ ($-\infty < a < b \leq \infty$) be an interval in $\mathbb{R}$. Moreover, let $B(\cdot)$ and $\Delta(\cdot)$ be $[\mathbb{H}]$-valued Borel functions on $\mathcal{I}$ integrable on each compact interval $[a, \beta] \subset \mathcal{I}$ and satisfying $B(t) = B^*(t)$ and $\Delta(t) \geq 0$ a.e. on $\mathcal{I}$ and let $J \in [\mathbb{H}]$ be operator (1.3).

A first-order symmetric system on an interval $\mathcal{I}$ (with the regular endpoint $a$) is a system of differential equations of the form

$$Jy'(t) - B(t)y(t) = \Delta(t)f(t), \quad t \in \mathcal{I},$$  

where $f(\cdot) \in \mathcal{L}_A^2(\mathcal{I})$. Together with (3.3) we consider also the homogeneous system

$$Jy'(t) - B(t)y(t) = \lambda \Delta(t)y(t), \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}.$$  

A function $y \in AC(\mathcal{I}; \mathbb{H})$ is a solution of (3.3) (resp. (3.4)) if equality (3.3) (resp. (3.4)) holds a.e. on $\mathcal{I}$. Moreover, a function $Y(\cdot, \lambda): \mathcal{I} \to [\mathbb{K}, \mathbb{H}]$ is an operator solution of equation (3.4) if $y(t) = Y(t, \lambda)b$ is a (vector) solution of this equation for every $h \in \mathbb{K}$ (here $\mathbb{K}$ is a Hilbert space with $\dim \mathbb{K} < \infty$).

In what follows we always assume that system (3.3) is definite in the sense of the following definition.

Definition 3.1. [12, 24] Symmetric system (3.3) is called definite if for each $\lambda \in \mathbb{C}$ and each solution $y$ of (3.4) the equality $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$) implies $y(t) = 0$, $t \in \mathcal{I}$.

As it is known [36, 20, 28] symmetric system (3.3) gives rise to the maximal linear relations $T_{\max}$ and $T_{\max}$ in $\mathcal{L}_A^2(\mathcal{I})$ and $\mathcal{L}_A^2(\mathcal{I})$, respectively. They are given by

$$T_{\max} = \{ \{y, f\} \in (\mathcal{L}_A^2(\mathcal{I}))^2 : y \in AC(\mathcal{I}; \mathbb{H}) \}$$

and $T_{\max} = \{ \{\pi y, \pi f\} : \{y, f\} \in T_{\max}\}$. Moreover the Lagrange’s identity

$$(f, z)_\Delta - (y, g)_\Delta = [y, z]_b - (Jy(a), z(a)), \quad \{y, f\}, \{z, g\} \in T_{\max}.$$  

holds with

$$[y, z]_b := \lim_{t \to b} (Jy(t), z(t)), \quad y, z \in \text{dom } T_{\max}.$$  

Formula (3.7) defines the skew-Hermitian bilinear form $[\cdot, \cdot]_b$ on $\text{dom } T_{\max}$, which plays a crucial role in our considerations. By using this form we define the minimal relations $T_{\min}$ in $\mathcal{L}_A^2(\mathcal{I})$ and $T_{\min}$ in $\mathcal{L}_A^2(\mathcal{I})$ via

$$T_{\min} = \{ \{y, f\} \in T_{\max} : y(a) = 0 \text{ and } [y, z]_b = 0 \text{ for each } z \in \text{dom } T_{\max}\}.$$  

and $T_{\min} = \{ \{\pi y, \pi f\} : \{y, f\} \in T_{\min}\}$. According to [36, 20, 28] $T_{\min}$ is a closed symmetric linear relation in $\mathcal{L}_A^2(\mathcal{I})$ and $T_{\min}^* = T_{\max}$. 
Remark 3.2. It is known (see e.g. [28]) that the maximal relation $T_{\text{max}}$ induced by the definite symmetric system (3.3) possesses the following property: for any $\{\tilde{y}, \tilde{f}\} \in T_{\text{max}}$ there exists a unique operator solution $y \in AC(I; \mathbb{H}) \cap L^2_{\Delta}(I)$ such that $y \in \tilde{y}$ and $\{y, f\} \in T_{\text{max}}$ for any $f \in \tilde{f}$. Below we associate such a function $y \in AC(I; \mathbb{H}) \cap L^2_{\Delta}(I)$ with each pair $\{\tilde{y}, \tilde{f}\} \in T_{\text{max}}$.

For any $\lambda \in \mathbb{C}$ denote by $N_\lambda$ the linear space of solutions of the homogeneous system (3.4) belonging to $L^2_{\Delta}(I)$. Definition (3.5) of $T_{\text{max}}$ implies

$$N_\lambda = \ker(T_{\text{max}} - \lambda) = \{y \in L^2_{\Delta}(I) : \{y, \lambda y\} \in T_{\text{max}}\}, \quad \lambda \in \mathbb{C},$$

and hence $N_\lambda \subset \text{dom} T_{\text{max}}$. As usual, denote by $n_\pm(T_{\text{min}}) := \dim N_{\lambda}(T_{\text{min}})$, $\lambda \in \mathbb{C}_\pm$, the deficiency indices of $T_{\text{min}}$. Since the system (3.3) is definite, $\pi N_{\lambda} = N_{\lambda}(T_{\text{min}})$ and $\ker(\pi \upharpoonright N_{\lambda}) = \{0\}$, $\lambda \in \mathbb{C}$. This implies that $\dim N_{\lambda} = n_\pm(T_{\text{min}})$, $\lambda \in \mathbb{C}_\pm$.

The following lemma is obvious.

Lemma 3.3. If $Y(\cdot, \lambda) \in L^2_{\Delta}(K, \mathbb{H})$ is an operator solution of Eq. (3.4), then the relation

$$K \ni h \to Y(\lambda)h(t) = Y(t, \lambda)h \in N_{\lambda},$$

defines the linear mapping $Y(\lambda) : K \to N_{\lambda}$ and, conversely, for each such a mapping $Y(\lambda)$ there exists a unique operator solution $Y(\cdot, \lambda) \in L^2_{\Delta}(K, \mathbb{H})$ of Eq. (3.4) such that (3.8) holds.

Next assume that

$$U = \begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \end{pmatrix} : H \oplus \tilde{H} \oplus H \to \tilde{H} \oplus H$$

is the operator satisfying the relations

$$\text{ran} U = \tilde{H} \oplus H$$

$$iu_2u_2^* - u_1u_3^* + u_3u_2^* = iI_{\tilde{H}}, \quad iu_5u_2^* - u_4u_3^* + u_6u_1^* = 0$$

$$iu_5u_2^* + u_6u_1^* - u_4u_3^* = 0$$

One can prove that the operator (3.9) admits an extension to the $J$-unitary operator

$$\tilde{U} = \begin{pmatrix} u_7 & u_8 & u_9 \\ u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \end{pmatrix} : H \oplus \tilde{H} \oplus H \to \tilde{H} \oplus H,$$

i.e. the operator satisfying $\tilde{U}^* J \tilde{U} = J$.

In view of (3.1) each function $y \in AC(I; \mathbb{H})$ admits the representation

$$y(t) = \{y_0(t), \tilde{y}(t), y_1(t)\} \in H \oplus \tilde{H} \oplus H, \quad t \in I.$$

Using (3.13) and the representation (3.14) of $y$ we introduce the linear mappings $\Gamma_{ja} : AC(I; \mathbb{H}) \to H$, $j \in \{0, 1\}$, and $\tilde{\Gamma}_a : AC(I; \mathbb{H}) \to \tilde{H}$ by setting

$$\Gamma_{0a} y = u_7y_0(a) + u_8\tilde{y}(a) + u_9y_1(a), \quad y \in AC(I; \mathbb{H})$$

$$\tilde{\Gamma}_a y = u_1y_0(a) + u_2\tilde{y}(a) + u_3y_1(a), \quad \Gamma_{1a} y = u_4y_0(a) + u_5\tilde{y}(a) + u_6y_1(a).$$

Clearly, the mappings $\tilde{\Gamma}_a$ and $\Gamma_{1a}$ are determined by the operator $U$, while $\Gamma_{0a}$ is determined by the extension $\tilde{U}$. Moreover, the mapping

$$\Gamma_a := (\Gamma_{0a}, \tilde{\Gamma}_a, \Gamma_{1a})^\top : AC(I; \mathbb{H}) \to H \oplus \tilde{H} \oplus H$$
satisfies $\Gamma_ay = \tilde{U}y(a)$, $y \in AC(I;\mathbb{H})$. Hence $\Gamma_a$ is surjective and

$$\text{(3.18)} \quad (Jy(a), z(a)) = -(\Gamma_{a1}y, \Gamma_{a2}z) + (\Gamma_{ao}y, \Gamma_{az}) + i(\tilde{\Gamma}_{a1}y, \tilde{\Gamma}_{az}), \quad y, z \in AC(I;\mathbb{H}).$$

In what follows we associate with each operator $U$ (see (3.9)) the operator solution $\varphi(\cdot, \lambda) = \varphi_U(\cdot, \lambda)(\in [H_0, \mathbb{H}])$, $\lambda \in \mathbb{C}$, of Eq. (3.4) with the initial data

$$\text{(3.19)} \quad \varphi_U(a, \lambda) = \begin{pmatrix} u_0^* & iu_2^* \\ -iu_5^* & u_2^* \\ -u_4^* & -iu_1^* \end{pmatrix} : H \oplus H \to H \oplus H.$$

One can easily verify that for each $J$-unitary extension $\tilde{U}$ of $U$ one has

$$\text{(3.20)} \quad \tilde{U}\varphi_U(a, \lambda) = \begin{pmatrix} I_{H_0} \\ 0 \end{pmatrix} : H_0 \to H_0 \oplus H.$$

The particular case of the operator $U$ and its $J$-unitary extension $\tilde{U}$ is (cf. [17])

$$\text{(3.21)} \quad U = \begin{pmatrix} 0 & I_{\tilde{H}} \\ \cos B & 0 \end{pmatrix}, \quad \tilde{U} = \begin{pmatrix} \sin B & 0 & -\cos B \\ 0 & I_{\tilde{H}} & 0 \\ \cos B & 0 & \sin B \end{pmatrix},$$

where $B = B^* \in [H]$. For such $U$ the solution $\varphi_U(\cdot, \lambda)$ is defined by the initial data

$$\varphi_U(a, \lambda) = \begin{pmatrix} \sin B & 0 \\ 0 & I_{\tilde{H}} \\ -\cos B & 0 \end{pmatrix} : H \oplus \tilde{H} \to H \oplus \tilde{H} \oplus H.$$

3.3. Decomposing boundary triplets. According to [4, 34] the skew-Hermitian bilinear form (3.7) has finite indices of inertia $v_{b+}$ and $v_{b-}$ and

$$\text{(3.22)} \quad n_+(T_{\text{min}}) = v_{b+} + v_{b+}, \quad n_-(T_{\text{min}}) = v_{b-} + v_{b-} + \tilde{\nu} + v_{b-}$$

(for $\nu_{\pm}$ see (3.2)). Moreover, the following lemma is immediate from [34, Lemma 5.1].

**Lemma 3.4.** For any pair of finite-dimensional Hilbert spaces $\mathcal{H}_b$ and $\tilde{\mathcal{H}}_b$ with $\dim \mathcal{H}_b = \min \{v_{b+}, v_{b-}\}$ and $\dim \tilde{\mathcal{H}}_b = |v_{b+} - v_{b-}|$ there exists a surjective linear mapping

$$\text{(3.23)} \quad \Gamma_b = (\Gamma_{0b}, \tilde{\Gamma}_b, \Gamma_{1b})^\top : \text{dom } T_{\text{max}} \to \mathcal{H}_b \oplus \tilde{\mathcal{H}}_b \oplus \mathcal{H}_b$$

such that for all $y, z \in \text{dom } T_{\text{max}}$ the following equality is valid

$$\text{(3.24)} \quad [y, z]_b = i \cdot \text{sign}(v_{b+} - v_{b-})(\tilde{\Gamma}_{b1}y, \tilde{\Gamma}_{b1}z) - (\Gamma_{1b}y, \Gamma_{0b}z) + (\Gamma_{0b}y, \Gamma_{1b}z).$$

Note that $\Gamma_by$ is in fact a singular boundary value of a function $y \in \text{dom } T_{\text{max}}$ in the sense of [8, Chapter 13.2] (for more details see Remark 3.5 in [2]).

Assume that $n_+(T_{\text{min}}) > n_-(T_{\text{min}})$. Then by (3.22) and (3.2) $\tilde{\nu} > v_{b+} - v_{b-}$ and hence the following two alternative cases may hold:

**Case 1.** $\tilde{\nu} > v_{b+} - v_{b-} > 0$.

**Case 2.** $\tilde{\nu} \geq 0 \geq v_{b+} - v_{b-}$ and $\tilde{\nu} \neq v_{b+} - v_{b-}(\neq 0)$.

Below we construct a boundary triplet for $T_{\text{max}}$ separately in each of these cases.

In Case 1 one has $\dim \tilde{\mathcal{H}}(=\tilde{\nu}) > v_{b+} - v_{b-} > 0$. Let $\tilde{H}_1$ be a subspace in $\tilde{H}$ with $\dim \tilde{H}_1 = v_{b+} - v_{b-}$ and let $\mathcal{H}_b$ be a Hilbert space with $\dim \mathcal{H}_b = v_{b-}$. Then by Lemma 3.4 there exists a surjective linear mapping

$$\text{(3.25)} \quad \Gamma_b = (\Gamma_{0b}, \tilde{\Gamma}_b, \Gamma_{1b})^\top : \text{dom } T_{\text{max}} \to \mathcal{H}_b \oplus \tilde{H}_1 \oplus \mathcal{H}_b.$$
such that (3.24) holds for all \( y, z \in \text{dom} \, T_{\text{max}} \). Let \( \hat{H} \) be decomposed as \( \hat{H} = \hat{H}_1 \oplus \hat{H}_2 \) with \( \hat{H}_2 := \hat{H} \ominus \hat{H}_1 \) and let

\[
\hat{\Gamma}_a = (\hat{\Gamma}_{a1}, \hat{\Gamma}_{a2})^T : AC(I; H) \to \hat{H}_1 \oplus \hat{H}_2
\]

be the block representation of the mapping \( \hat{\Gamma}_a \) (see (3.26)). Moreover, let \( H'_0 := H \oplus \hat{H}_1 \), so that in view of (3.1) \( H_0 \) admits the representation

\[
H_0 = H_0' \oplus \hat{H}_1 \oplus \hat{H}_2 = H'_0 \oplus \hat{H}_2
\]

In Case 1 we let

\[
H_0 = H'_0 \oplus \hat{H}_2 \oplus H_b, \quad H_1 = H'_0 \oplus H_b,
\]

\[
\Gamma_0 \{ \bar{y}, \bar{f} \} = \{ -\Gamma_{1a}y + i(\hat{\Gamma}_{a1} - \hat{\Gamma}_b)y, i\hat{\Gamma}_{a2}y, \Gamma_{0b}y \} (\in H'_0 \oplus \hat{H}_2 \oplus H_b),
\]

\[
\Gamma_1 \{ \bar{y}, \bar{f} \} = \{ \Gamma_{0a}y + \frac{1}{2}(\hat{\Gamma}_{a1} + \hat{\Gamma}_b)y, -\Gamma_{1b}y \} (\in H'_0 \oplus H_b), \quad \{ \bar{y}, \bar{f} \} \in T_{\text{max}}.
\]

Now assume that Case 2 holds. Let \( H_b \) and \( \hat{H}_b \) be Hilbert spaces with \( \text{dim} \, H_b = \nu_b^+ \) and \( \text{dim} \, \hat{H}_b = \nu_b^- \), then by Lemma 3.4 there is a surjective linear mapping (3.23) satisfying (3.24). Let \( \hat{H}_b := H_b \oplus \hat{H}_b \) (so that \( H_b \subset \hat{H}_b \)) and let \( \Gamma_{0b} : \text{dom} \, T_{\text{max}} \to \hat{H}_b \) be the linear mapping given by

\[
\Gamma_{0b} = \Gamma_{0b} + \hat{\Gamma}_b.
\]

In Case 2 we put

\[
H_0 = H \oplus \hat{H} \oplus \hat{H}_b, \quad H_1 = H \oplus H_b
\]

\[
\Gamma_0 \{ \bar{y}, \bar{f} \} = \{ -\Gamma_{1a}y, i\hat{\Gamma}_{a}y, \Gamma_{0b}y \} (\in H \oplus \hat{H} \oplus \hat{H}_b)
\]

\[
\Gamma_1 \{ \bar{y}, \bar{f} \} = \{ \Gamma_{0a}y, -\Gamma_{1b}y \} (\in H \oplus H_b), \quad \{ \bar{y}, \bar{f} \} \in T_{\text{max}}.
\]

Note that in both Cases 1 and 2 \( H_j \) is a subspace in \( H_0 \) and \( \Gamma_j \) is an operator from \( T_{\text{max}} \) to \( H_j \), \( j \in \{0, 1\} \). Moreover, \( H_2 = (H_0 \oplus H_1) = \hat{H}_2 \) in Case 1 and \( H_2 = \hat{H} \oplus \hat{H}_b \) in Case 2.

**Proposition 3.5.** Assume that \( \hat{U} \) is \( J \)-unitary operator (3.13) and \( \Gamma_{0a}, \Gamma_{1a} \) and \( \hat{\Gamma}_a \) are linear mappings (3.15), (3.16). Moreover, let \( \Gamma_b \) be surjective linear mapping given either by (3.25) (in Case 1) or (3.23) (in Case 2) and satisfying (3.24). Then a collection \( \Pi_- = \{ H_0 \oplus H_1, \Gamma_0, \Gamma_1 \} \) defined either by (3.28)–(3.30) (in Case 1) or (3.32)–(3.34) (in Case 2) is a boundary triplet for \( T_{\text{max}} \).

**Proof.** The immediate calculation with taking (3.18) and (3.24) into account gives

\[
(\Gamma_1 \{ \bar{y}, \bar{f} \}, \Gamma_0 \{ \bar{z}, \bar{g} \}) - (\Gamma_0 \{ \bar{y}, \bar{f} \}, \Gamma_1 \{ \bar{z}, \bar{g} \}) - i(\gamma_2 \Gamma_0 \{ \bar{y}, \bar{f} \}, \gamma_2 \Gamma_0 \{ \bar{z}, \bar{g} \}) =
\]

\[
= [z, z]_0 - (Jy(a), z(a)), \quad \{ \bar{y}, \bar{f} \}, \{ \bar{z}, \bar{g} \} \in T_{\text{max}}.
\]

This and the Lagrange’s identity (3.6) yield identity (2.5) for \( \Gamma_0 \) and \( \Gamma_1 \). Moreover, the mapping \( \Gamma = (\Gamma_0, \Gamma_1)^T \) is surjective, because so are \( \Gamma_a \) (see (3.17)) and \( \Gamma_b \). \( \square \)

**Definition 3.6.** The boundary triplet \( \Pi_- = \{ H_0 \oplus H_1, \Gamma_0, \Gamma_1 \} \) constructed in Proposition 3.5 is called a decomposing boundary triplet for \( T_{\text{max}} \).
Proposition 3.7. Let in Case 1 $U$ be operator (3.9) and let $\hat{\Gamma}_a$ and $\Gamma_{1a}$ be linear mappings (3.16). Moreover, let $\hat{\mathcal{H}}_1$ be a subspace in $\mathcal{H}$, let $\hat{\mathcal{H}}_2 = \hat{\mathcal{H}} \ominus \hat{\mathcal{H}}_1$, let $\hat{\Gamma}_{aj}$, $j \in \{1, 2\}$, be defined by (3.26) and let $\Gamma_b$ be surjective linear mapping (3.25) satisfying (3.24). Then:

(1) The equalities

\begin{equation}
T = \{\{\tilde{y}, \tilde{f}\} \in T_{\text{max}} : \Gamma_{1a}y = 0, \ \hat{\Gamma}_{a1}y = \hat{\Gamma}_b y, \ \hat{\Gamma}_{a2}y = 0, \ \Gamma_{0b}y = \Gamma_{1b}y = 0\}
\end{equation}

\begin{equation}
T^* = \{\{\tilde{y}, \tilde{f}\} \in T_{\text{max}} : \Gamma_{1a}y = 0, \ \hat{\Gamma}_a y = \hat{\Gamma}_b y\}
\end{equation}

define a symmetric extension $T$ of $T_{\text{min}}$ and its adjoint $T^*$. Moreover, the deficiency indices of $T$ are $n_+(T) = \nu_b^-$ and $n_-(T) = \nu_b^+ + 2\nu_{b^-} - \nu_{b^+}$.

(2) The collection $\hat{\Pi}_- = \{\mathcal{H}_0 \oplus \mathcal{H}_b, \Gamma_0, \hat{\Gamma}_1\}$ with $\mathcal{H}_0 = \hat{\mathcal{H}}_2 \oplus \mathcal{H}_b$ and the operators

\begin{equation}
\hat{\Gamma}_0 \{\tilde{y}, \tilde{f}\} = \{i\hat{\Gamma}_{a2}y, \Gamma_{0b}y\} \in \mathcal{H}_2 \oplus \mathcal{H}_b, \quad \hat{\Gamma}_1 \{\tilde{y}, \tilde{f}\} = -\Gamma_{1b}y, \quad \{\tilde{y}, \tilde{f}\} \in T^*,
\end{equation}

is a boundary triplet for $T^*$ and the (maximal symmetric) relation $A_0(= \ker \hat{\Gamma}_0)$ is

\begin{equation}
A_0 = \{\{\tilde{y}, \tilde{f}\} \in T_{\text{max}} : \Gamma_{1a}y = 0, \ \hat{\Gamma}_{a1}y = \hat{\Gamma}_b y, \ \hat{\Gamma}_{a2}y = 0, \ \Gamma_{0b}y = 0\}.
\end{equation}

Proof. Let $\tilde{U}$ be $J$-unitary extension (3.13) of $U$, let $\Gamma_{0a}$ be operator (3.15) and let $\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \hat{\Gamma}_1\}$ be the decomposing boundary triplet (3.28)–(3.30) for $T_{\text{max}}$. Applying to this triplet Proposition 2.7 one obtains the desired statements. \qed

Proposition 3.8. Let in Case 2 $U$ be operator (3.9) and let $\hat{\Gamma}_a$ and $\Gamma_{1a}$ be linear mappings (3.16). Moreover, let $\Gamma_b$ be surjective linear mapping (3.23) satisfying (3.24), let $\hat{\mathcal{H}}_b = \mathcal{H}_b \oplus \hat{\mathcal{H}}_0$ and let $\hat{\Gamma}_{0b}$ be given by (3.31). Then:

(1) The equalities

\begin{equation}
T = \{\{\tilde{y}, \tilde{f}\} \in T_{\text{max}} : \Gamma_{1a}y = 0, \ \hat{\Gamma}_a y = 0, \ \hat{\Gamma}_{0b}y = \Gamma_{1b}y = 0\}
\end{equation}

\begin{equation}
T^* = \{\{\tilde{y}, \tilde{f}\} \in T_{\text{max}} : \Gamma_{1a}y = 0\}
\end{equation}

define a symmetric extension $T$ of $T_{\text{min}}$ and its adjoint $T^*$. Moreover, $n_+(T) = \nu_{b^+}$ and $n_-(T) = \nu_{b^-} + \nu_{b^+}$.

(2) The collection $\hat{\Pi}_- = \{\mathcal{H}_0 \oplus \mathcal{H}_b, \Gamma_0, \hat{\Gamma}_1\}$ with $\mathcal{H}_0 = \hat{\mathcal{H}}_2 \oplus \mathcal{H}_b$ and the operators

\begin{equation}
\hat{\Gamma}_0 \{\tilde{y}, \tilde{f}\} = \{i\hat{\Gamma}_{a2}y, \Gamma_{0b}y\} \in \mathcal{H} \oplus \mathcal{H}_b, \quad \hat{\Gamma}_1 \{\tilde{y}, \tilde{f}\} = -\Gamma_{1b}y, \quad \{\tilde{y}, \tilde{f}\} \in T^*,
\end{equation}

is a boundary triplet for $T^*$ and the (maximal symmetric) relation $A_0(= \ker \hat{\Gamma}_0)$ is

\begin{equation}
A_0 = \{\{\tilde{y}, \tilde{f}\} \in T_{\text{max}} : \Gamma_{1a}y = 0, \ \hat{\Gamma}_a y = 0, \ \hat{\Gamma}_{0b}y = 0\}.
\end{equation}

We omit the proof of this proposition, because it is similar to that of Proposition 3.7.

4. $L^2_\Lambda$-Solutions of Boundary Value Problems

4.1. Basic assumption. In what follows we suppose (unless otherwise stated) that system (3.3) satisfies $n_-(T_{\text{min}}) < n_+(T_{\text{min}})$ and the following assumptions are fulfilled:

(A1) $U$ is the operator (3.9) satisfying (3.10) - (3.12) and $\hat{\Gamma}_a$ and $\Gamma_{1a}$ are the linear mappings (3.16).

(A2) In Case 1 $\hat{\mathcal{H}}$ is decomposed as $\hat{\mathcal{H}} = \hat{\mathcal{H}}_1 \oplus \hat{\mathcal{H}}_2$, $\hat{\Gamma}_{aj} : AC(I; \mathcal{H}) \rightarrow \hat{\mathcal{H}}_j, j \in \{1, 2\}$, are the operators given by (3.26), $\mathcal{H}_b$ is a finite-dimensional Hilbert space and $\Gamma_b$ is a surjective operator (3.25) satisfying (3.24).
(A3) In Case 2 $H_b$ and $\hat{H}_b$ are finite-dimensional Hilbert spaces, $\Gamma_b$ is a surjective operator (3.23) satisfying (3.24), $H_b = H_b \oplus \hat{H}_b$ and $\Gamma_{0b} : \text{dom} \, T_{\text{max}} \to H_b$ is the operator (3.31).

4.2. Case 1.

Definition 4.1. In Case 1 a boundary parameter is a collection $\tau = \{\tau_+, \tau_-\} \in \mathcal{R} - (\hat{H}_2 \oplus H_b, H_b)$. A truncated boundary parameter is a boundary parameter $\tau = \{\tau_+, \tau_-\}$ satisfying $\tau_- (\lambda) \subset \{\{0\} \oplus H_b\} \oplus H_b$, $\lambda \in \mathbb{C}_-$. According to Subsection 2.3 a boundary parameter $\tau = \{\tau_+, \tau_-\}$ admits the representation (4.1)

$$\tau_+ (\lambda) = \{(C_0 (\lambda), C_1 (\lambda)); H_b\}, \; \lambda \in \mathbb{C}_+; \quad \tau_- (\lambda) = \{(D_0 (\lambda), D_1 (\lambda)); \hat{H}_2 \oplus H_b\}, \; \lambda \in \mathbb{C}_-$$

with holomorphic operator functions $C_0 (\lambda) (\in [\hat{H}_2 \oplus H_b, H_b]), \; C_1 (\lambda) (\in [H_b])$ and $D_0 (\lambda) (\in [\hat{H}_2 \oplus H_b], \; D_1 (\lambda) (\in [H_b, \hat{H}_2 \oplus H_b])$. Moreover, a truncated boundary parameter $\tau = \{\tau_+, \tau_-\}$ admits the representation (4.1) with

$$D_0 (\lambda) = \left( \begin{array}{c} I_{\hat{H}_2} \\ 0 \end{array} \right) : \hat{H}_2 \oplus H_b \to \hat{H}_2 \oplus H_b, \quad D_1 (\lambda) = \left( \begin{array}{c} 0 \\ \mathcal{T}_1 (\lambda) \end{array} \right) : H_b \to \hat{H}_2 \oplus H_b.$$

Let $\tau = \{\tau_+, \tau_-\}$ be a boundary parameter (4.1) and let

$$C_0 (\lambda) = (\hat{C}_0 (\lambda), C_0 b (\lambda)): \hat{H}_2 \oplus H_b \to H_b, \quad D_0 (\lambda) = (D_0 (\lambda), D_0 b (\lambda)) : \hat{H}_2 \oplus H_b \to (\hat{H}_2 \oplus H_b)$$

be the block representations of $C_0 (\lambda)$ and $D_0 (\lambda)$. For a given function $f \in L^2 (\mathcal{I})$ consider the following boundary value problem:

$$Jy' - B(t)y = \lambda \Delta(t)y + \Delta (t)f(t), \quad t \in \mathcal{I},$$

$$\Gamma_{1a} y = 0, \quad \hat{\Gamma}_{a2} y = \hat{\Gamma}_b y, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

$$i\hat{C}_0 (\lambda) \hat{\Gamma}_{a2} y + C_0 b (\lambda) \Gamma_{0b} y + C_1 (\lambda) \Gamma_{1b} y = 0, \quad \lambda \in \mathbb{C}_+$$

$$i\hat{D}_0 (\lambda) \hat{\Gamma}_{a2} y + D_0 b (\lambda) \Gamma_{0b} y + D_1 (\lambda) \Gamma_{1b} y = 0, \quad \lambda \in \mathbb{C}_-.$$  

A function $y(\cdot, \cdot) : \mathcal{I} \times (\mathbb{C} \setminus \mathbb{R}) \to \mathbb{H}$ is called a solution of this problem if for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the function $y(\cdot, \lambda)$ belongs to $AC(\mathcal{I}; \mathbb{H}) \cap L^2 (\mathcal{I})$ and satisfies the equation (4.4) a.e. on $\mathcal{I}$ (so that $y \in \text{dom} \, T_{\text{max}}$) and the boundary conditions (4.5) – (4.7).

Theorem 4.2. Let in Case 1 $T$ be a symmetric relation in $L^2 (\mathcal{I})$ defined by (3.35). If $\tau = \{\tau_+, \tau_-\}$ is a boundary parameter (4.1), then for every $f \in L^2 (\mathcal{I})$ the boundary problem (4.4) - (4.7) has a unique solution $y(t, \lambda) = y_f (t, \lambda)$ and the equality

$$R (\lambda) \hat{f} = \pi (y_f (\cdot, \cdot), \lambda), \quad \hat{f} \in L^2 (\mathcal{I}), \quad f \in \hat{f}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

defines a generalized resolvent $R (\lambda) = R_T (\lambda)$ of $T$. Conversely, for each generalized resolvent $R (\lambda)$ of $T$ there exists a unique boundary parameter $\tau$ such that $R (\lambda) = R_T (\lambda)$.

Proof. Let $\hat{\Pi}_-$ be the boundary triplet for $T^*$ defined in Proposition 3.7. Applying to this triplet [35, Theorem 3.11] we obtain the required statements. \qed

Remark 4.3. Let $\tau_0 = \{\tau_+, \tau_-\}$ be a boundary parameter (4.1) with

$$C_0 (\lambda) \equiv P_{H_b}, \quad C_1 (\lambda) \equiv 0, \quad D_0 (\lambda) \equiv I_{\hat{H}_2 \oplus H_b}, \quad D_1 (\lambda) \equiv 0.$$
and let $A_0$ be a symmetric relation (3.38). Then
\begin{equation}
R_{\tau_0}(\lambda) = (A_0^* - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_+ \quad \text{and} \quad R_{\tau_0}(\lambda) = (A_0 - \lambda)^{-1}, \quad \lambda \in \mathbb{C}_-.
\end{equation}

**Proposition 4.4.** Let in Case 1 $P_H$, $P_{\tilde{H}_1}$ and $P_{\tilde{H}_2}$ be the orthoprojectors in $H_0$ onto $H$, $\tilde{H}_1$ and $\tilde{H}_2$ respectively (see (3.27)) and let $P_{\hat{H}_2}$ ($P_{\hat{H}_b}$) be the orthoprojector in $\hat{H}_2 \oplus \mathcal{H}_b$ onto $\hat{H}_2$ (resp. $\mathcal{H}_b$). Then:

1. For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists a unique operator solution $v_0(\cdot, \lambda) \in \mathcal{L}_2^2[\mathcal{H}_0, \mathbb{H}]$ of Eq. (3.4) satisfying
\begin{equation}
\Gamma_{1a}v_0(\lambda) = -P_H, \quad i(\hat{\Gamma}_{a1} - \hat{\Gamma}_b)v_0(\lambda) = P_{\hat{H}_1}, \quad \Gamma_{0b}v_0(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\end{equation}
\begin{equation}
\Gamma_{0b}v_0(\lambda) = P_{\hat{H}_b}, \quad \lambda \in \mathbb{C}_-
\end{equation}

2. For any $\lambda \in \mathbb{C}_+ \quad (\lambda \in \mathbb{C}_-)$ there exists a unique operator solution $u_+(\cdot, \lambda) \in \mathcal{L}_2^2[\mathcal{H}_b, \mathbb{H}]$ (resp. $u_-(\cdot, \lambda) \in \mathcal{L}_2^2[\hat{H}_2 \oplus \mathcal{H}_b, \mathbb{H}]$) of Eq. (3.4) such that
\begin{equation}
\Gamma_{1a}u_\pm(\lambda) = 0, \quad i(\hat{\Gamma}_{a1} - \hat{\Gamma}_b)u_\pm(\lambda) = 0, \quad \lambda \in \mathbb{C}_\pm
\end{equation}
\begin{equation}
\Gamma_{0b}u_+(\lambda) = I_{\mathcal{H}_b}, \quad \lambda \in \mathbb{C}_+
\end{equation}
\begin{equation}
\Gamma_{0b}u_-(\lambda) = P_{\hat{H}_b}, \quad i\hat{\Gamma}_{a2}u_-(\lambda) = P_{\hat{H}_2}, \quad \lambda \in \mathbb{C}_-.
\end{equation}

In formulas (4.10)–(4.14) $v_0(\lambda)$ and $u_\pm(\lambda)$ are linear mappings from Lemma 3.3 corresponding to the solutions $v_0(\cdot, \lambda)$ and $u_\pm(\cdot, \lambda)$ respectively.

**Proof.** Let $\tilde{U}$ be the $J$-unitary extension (3.13) of $U$, let $\Gamma_{0a}$ be the operator (3.15) and let $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be the decomposing boundary triplet (3.28)-(3.30) for $T_{\max}$. Assume also that $\gamma_{\pm}(\cdot)$ are the $\gamma$-fields of $\Pi_-$. Since the quotient mapping $\pi$ isomorphically maps $\mathcal{N}_\lambda$ onto $\mathcal{N}_\lambda(T_{\min})$, it follows that for every $\lambda \in \mathbb{C}_+ \quad (\lambda \in \mathbb{C}_-)$ there exists an isomorphism $Z_+(\lambda) : \mathcal{H}_1 \rightarrow \mathcal{N}_\lambda$ (resp. $Z_-(\lambda) : \mathcal{H}_0 \rightarrow \mathcal{N}_\lambda$) such that
\begin{equation}
\gamma_+(\lambda) = \pi Z_+(\lambda), \quad \lambda \in \mathbb{C}_+; \quad \gamma_-(\lambda) = \pi Z_-\lambda), \quad \lambda \in \mathbb{C}_-.
\end{equation}
Let $\Gamma_0$ and $\Gamma_1'$ be the linear mappings given by
\begin{equation}
\Gamma_0 = \begin{pmatrix}
-Gamma_{1a} + i(\hat{\Gamma}_{a1} - \hat{\Gamma}_b) \\
\Gamma_{0b}
\end{pmatrix} : \text{dom} \ T_{\max} \rightarrow \mathcal{H}_0 \oplus \mathcal{H}_2 \oplus \mathcal{H}_b,
\end{equation}
\begin{equation}
\Gamma_1' = \begin{pmatrix}
\Gamma_{0a} + \frac{1}{2}(\hat{\Gamma}_{a1} + \hat{\Gamma}_b) \\
-Gamma_{1b}
\end{pmatrix} : \text{dom} \ T_{\max} \rightarrow \mathcal{H}_0 \oplus \mathcal{H}_b.
\end{equation}

Then by (3.29) and (3.30) one has $\Gamma_j\{\pi y, \lambda \pi y\} = \Gamma'_j y$, $y \in \mathcal{N}_\lambda$, $j \in \{0, 1\}$. Combining of this equality with (4.15) and (2.13) gives
\begin{equation}
P_{H_0^C \mathcal{H}_b} \Gamma_0 Z_+(\lambda) = I_{\mathcal{H}_1}, \quad \lambda \in \mathbb{C}_+; \quad \Gamma_0' Z_-\lambda) = I_{\mathcal{H}_0}, \quad \lambda \in \mathbb{C}_-,
\end{equation}

which in view of (4.16) can be written as
\begin{equation}
\begin{pmatrix}
-Gamma_{1a} + i(\hat{\Gamma}_{a1} - \hat{\Gamma}_b) \\
\Gamma_{0b}
\end{pmatrix} Z_+(\lambda) = \begin{pmatrix}
I_{H_0^C} & 0 \\
0 & I_{H_2}
\end{pmatrix}, \quad \lambda \in \mathbb{C}_+
\end{equation}
\begin{equation}
\begin{pmatrix}
-Gamma_{1a} + i(\hat{\Gamma}_{a1} - \hat{\Gamma}_b) \\
\Gamma_{0b}
\end{pmatrix} Z_-\lambda) = \begin{pmatrix}
0 & I_{H_2} \\
0 & 0
\end{pmatrix}, \quad \lambda \in \mathbb{C}_-
\end{equation}
It follows from (4.18) and (4.19) that

\begin{align}
(4.20) & \quad \Gamma_{1a}Z_+ (\lambda) = (- P_H, 0), \quad \frac{i}{2} (\tilde{\Gamma}_{a1} - \tilde{\Gamma}_b) Z_+ (\lambda) = (- \frac{i}{2} P_{\tilde{H}_1}, 0), \quad \Gamma_{0b} Z_+ (\lambda) = (0, I_{\mathcal{H}_b}) \\
(4.21) & \quad \Gamma_{1a}Z_- (\lambda) = (- P_H, 0, 0), \quad \frac{i}{2} (\tilde{\Gamma}_{a1} - \tilde{\Gamma}_b) Z_- (\lambda) = (- \frac{i}{2} P_{\tilde{H}_1}, 0, 0) \\
(4.22) & \quad \tilde{\Gamma}_{a2} Z_- (\lambda) = (0, -i \tilde{I}_{\tilde{H}_2}, 0), \quad \Gamma_{0b} Z_- (\lambda) = (0, 0, I_{\mathcal{H}_b})
\end{align}

Next assume that

\begin{align}
(4.23) & \quad Z_+ (\lambda) = (r (\lambda), u_+ (\lambda)) : H'_0 \oplus \mathcal{H}_b \to \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_+ \\
(4.24) & \quad Z_- (\lambda) = (r (\lambda), \omega_- (\lambda), \tilde{u}_- (\lambda)) : H'_0 \oplus \tilde{H}_2 \oplus \mathcal{H}_b \to \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_-
\end{align}

are the block representations of \( Z_\pm (\lambda) \) and let

\begin{align}
(4.25) & \quad v_0 (\lambda) := (r (\lambda), 0) : H'_0 \oplus \tilde{H}_2 \to \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_+ \\
(4.26) & \quad v_0 (\lambda) := (r (\lambda), \omega_- (\lambda)) : H'_0 \oplus \tilde{H}_2 \to \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_-
\end{align}

Then in view of (3.27) the equalities (4.25) and (4.26) define the operator \( v_0 (\lambda) \in [H_0, \mathcal{N}_\lambda], \lambda \in \mathbb{C} \setminus \mathbb{R} \). Moreover, (4.23) induces the operator \( u_+ (\lambda) \in [\mathcal{H}_b, \mathcal{N}_\lambda] \). Introduce also the operator

\begin{equation}
(4.27) \quad u_- (\lambda) = (\omega_- (\lambda), \tilde{u}_- (\lambda)) : \tilde{H}_2 \oplus \mathcal{H}_b \to \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_-.
\end{equation}

Now assume that \( v_0 (\cdot, \lambda) \in \mathcal{L}^2_\Delta [H_0, \mathbb{H}], \quad u_+ (\cdot, \lambda) \in \mathcal{L}^2_\Delta [\mathcal{H}_b, \mathbb{H}] \) and \( u_- (\cdot, \lambda) \in \mathcal{L}^2_\Delta [\tilde{H}_2 \oplus \mathcal{H}_b, \mathbb{H}] \) are the operator solutions of Eq. (3.4) corresponding to \( v_0 (\lambda), u_+ (\lambda) \) and \( u_- (\lambda) \) in accordance with Lemma 3.3. Then combining of (4.23)–(4.27) with (4.20)–(4.22) yields the relations (4.10)–(4.14) for \( v_0 (\cdot, \lambda) \) and \( u_\pm (\cdot, \lambda) \). Finally, by using uniqueness of the solution of the boundary value problem (4.4)–(4.7) (with \( f = 0 \)) one proves uniqueness of \( v_0 (\cdot, \lambda) \) and \( u_\pm (\cdot, \lambda) \) in the same way as in [2, Proposition 4.3]. □

Let \( \tilde{U} \) be a \( J \)-unitary extension (3.13) of \( U \) and let \( \Gamma_{0a} \) be the mapping (3.15). By using the solutions \( v_0 (\cdot, \lambda) \) and \( u_\pm (\cdot, \lambda) \) we define the operator functions

\begin{align}
(4.28) & \quad X_+ (\lambda) = \begin{pmatrix} m_0 (\lambda) \\ \Phi_+ (\lambda) \\ \Psi_+ (\lambda) \\ M_+ (\lambda) \end{pmatrix} : H_0 \oplus \mathcal{H}_b \to H_0 \oplus (\tilde{H}_2 \oplus \mathcal{H}_b), \quad \lambda \in \mathbb{C}_+ \\
(4.29) & \quad X_- (\lambda) = \begin{pmatrix} m_0 (\lambda) \\ \Phi_- (\lambda) \\ \Psi_- (\lambda) \\ M_- (\lambda) \end{pmatrix} : H_0 \oplus (\tilde{H}_2 \oplus \mathcal{H}_b) \to H_0 \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C}_-, 
\end{align}

where

\begin{align}
(4.30) & \quad m_0 (\lambda) = (\Gamma_{0a} + \tilde{\Gamma}_a) v_0 (\lambda) + \frac{i}{2} P_{\tilde{H}_1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \\
(4.31) & \quad \Phi_\pm (\lambda) = (\Gamma_{0a} + \tilde{\Gamma}_a) u_\pm (\lambda), \quad \lambda \in \mathbb{C}_\pm \\
(4.32) & \quad \Psi_+ (\lambda) = (\tilde{\Gamma}_{a2} - \Gamma_{1b}) v_0 (\lambda) + iP_{\tilde{H}_1}, \quad M_+ (\lambda) = (\tilde{\Gamma}_{a2} - \Gamma_{1b}) u_+ (\lambda), \quad \lambda \in \mathbb{C}_+ \\
(4.33) & \quad \Psi_- (\lambda) = - \Gamma_{1b} v_0 (\lambda), \quad M_- (\lambda) = - \Gamma_{1b} u_- (\lambda), \quad \lambda \in \mathbb{C}_-
\end{align}

In the following proposition we specify a connection between the operator functions \( X_\pm (\cdot) \) and the Weyl functions \( M_\pm (\cdot) \) of the decomposing boundary triplet \( \Pi_- \) for \( T_{\text{max}} \).
Proposition 4.5. Let \( \Pi_- = \{ \mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1 \} \) be the decomposing boundary triplet (3.28)–(3.30) for \( T_{\text{max}} \) and let

\[
M_+(\lambda) = \begin{pmatrix} M_1(\lambda) & M_2(\lambda) \\ N_1(\lambda) & N_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix} : H'_0 \oplus H_b \to H'_0 \oplus \tilde{H}_2 \oplus H_b, \quad \lambda \in \mathbb{C}_+
\]

\[
M_-(\lambda) = \begin{pmatrix} M_1(\lambda) & N_1(-\lambda) \\ N_2(-\lambda) & M_2(-\lambda) \\ M_3(-\lambda) & M_4(-\lambda) \end{pmatrix} : H'_0 \oplus \tilde{H}_2 \oplus H_b \to H'_0 \oplus H_b, \quad \lambda \in \mathbb{C}_-
\]

be the block representations of the corresponding Weyl functions \( M_\pm(\cdot) \). Then the entries of the operator matrices (4.28) and (4.29) are holomorphic on their domains and have the following block matrix representations:

\[
m_0(\lambda) = \begin{pmatrix} M_1(\lambda) & 0 \\ N_1(\lambda) & -i\Gamma_2 \end{pmatrix} : H'_0 \oplus H_2 \to H'_0 \oplus \tilde{H}_2, \quad \lambda \in \mathbb{C}_+
\]

\[
m_0(\lambda) = \begin{pmatrix} M_1(\lambda) & 0 \\ N_1(-\lambda) & -i\Gamma_2 \end{pmatrix} : H'_0 \oplus H_2 \to H'_0 \oplus \tilde{H}_2, \quad \lambda \in \mathbb{C}_-
\]

\[
\Phi_+(\lambda) = \begin{pmatrix} M_2(\lambda) \\ N_2(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_+; \quad \Phi_-(\lambda) = \begin{pmatrix} 0 \\ -iI \end{pmatrix}, \quad \lambda \in \mathbb{C}_-
\]

\[
\Psi_+(\lambda) = \begin{pmatrix} N_1(\lambda) \\ M_3(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_+; \quad \Psi_-(\lambda) = (M_3(-\lambda), N_2(-\lambda)), \quad \lambda \in \mathbb{C}_-
\]

Moreover, the following equalities hold

\[
m_0^0(\lambda) = m_0(\lambda), \quad \Phi_+^0(\lambda) = \Phi_-(\lambda), \quad \Psi_+^0(\lambda) = \Psi_-(\lambda), \quad M_+^0(\lambda) = M_-(\lambda), \quad \lambda \in \mathbb{C}_-.
\]

Proof. Let \( \Gamma'_0 \) and \( \Gamma'_1 \) be given by (4.16) and let \( Z_\pm(\cdot) \) be the same as in Proposition 4.4. Then by (4.15) and (2.8), (2.9) one has

\[
(\Gamma'_1 - iP_H \Gamma'_0)Z_+(\lambda) = M_+(\lambda), \quad \lambda \in \mathbb{C}_+; \quad \Gamma'_1Z_-(\lambda) = M_-(\lambda), \quad \lambda \in \mathbb{C}_-,
\]

which in view of (4.34) and (4.35) can be represented as

\[
\begin{pmatrix} \Gamma_0 + \frac{1}{2}(\tilde{\Gamma}_a + \tilde{\Gamma}_b) \\ \Gamma_0a_2 \end{pmatrix}Z_+(\lambda) = \begin{pmatrix} M_1(\lambda) & M_2(\lambda) \\ N_1(\lambda) & N_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_+
\]

\[
\begin{pmatrix} \Gamma_0 + \frac{1}{2}(\tilde{\Gamma}_a + \tilde{\Gamma}_b) \\ -\Gamma_0b_1 \end{pmatrix}Z_-(\lambda) = \begin{pmatrix} M_1(\lambda) & N_1(-\lambda) & M_2(-\lambda) \\ M_3(-\lambda) & N_2(-\lambda) & M_4(-\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_-.
\]

This implies that

\[
\Gamma_0Z_+(\lambda) = P_H(M_1(\lambda), M_2(\lambda)), \quad \tilde{\Gamma}_aZ_+(\lambda) = P_{\tilde{H}_1}(M_1(\lambda), M_2(\lambda))
\]

\[
\Gamma_0Z_-(\lambda) = P_H(M_1(\lambda), N_1(\lambda), M_2(-\lambda))
\]

\[
\Gamma_0a_2Z_+(\lambda) = (N_1(\lambda), N_2(\lambda)), \quad \Gamma_0b_1Z_+(\lambda) = (-M_3(\lambda), -M_4(\lambda))
\]

\[
\Gamma_0Z_-(\lambda) = P_H(M_1(\lambda), N_1(\lambda), M_2(-\lambda))
\]

\[
\tilde{\Gamma}_aZ_-(\lambda) = P_{\tilde{H}_1}(M_1(\lambda), N_1(\lambda), M_2(-\lambda))
\]
Summing up the second equality in (4.20) with the equalities (4.42) and the first equality in (4.43) one obtains

\[(4.47) \quad (\Gamma_{0a} + \tilde{\Gamma}_a)Z_+ (\lambda) = (M_1 (\lambda) + N_{1+} (\lambda) - \frac{i}{2} P_{\tilde{H}_1}, M_{2+} (\lambda) + N_{2+} (\lambda)) : H_0' \oplus H_b \to H_0.\]

Similarly, summing up the second equality in (4.21), the first equality in (4.22) and the equalities (4.44) and (4.45) one gets

\[(4.48) \quad (\Gamma_{0a} + \tilde{\Gamma}_a)Z_- (\lambda) = (M_1 (\lambda) - \frac{i}{2} P_{\tilde{H}_1}, N_{1-} (\lambda) - i I_{\tilde{H}_2}, M_{2-} (\lambda)).\]

Combining (4.47) and (4.43) with the block representation (4.23) of \(Z_+ (\lambda)\) and taking definition (4.25) of \(v_0 (\lambda)\) into account we obtain

\[(4.49) \quad (\Gamma_{0a} + \tilde{\Gamma}_a)v_0 (\lambda) = (M_1 (\lambda) + N_{1+} (\lambda) - \frac{i}{2} P_{\tilde{H}_1}) P_{H_0'}, \quad \lambda \in \mathbb{C}_+\]

\[(4.50) \quad (\Gamma_{0a} + \tilde{\Gamma}_a)u_+ (\lambda) = M_{2+} (\lambda) + N_{2+} (\lambda), \quad \lambda \in \mathbb{C}_+\]

\[(4.51) \quad \tilde{\Gamma}_{a2} v_0 (\lambda) = N_{1+} (\lambda) P_{H_0'}, \quad \tilde{\Gamma}_{a2} u_+ (\lambda) = N_{2+} (\lambda), \quad \lambda \in \mathbb{C}_+\]

\[(4.52) \quad \Gamma_{1b} v_0 (\lambda) = -M_{3+} (\lambda) P_{H_0'}, \quad \Gamma_{1b} u_+ (\lambda) = -M_{4+} (\lambda), \quad \lambda \in \mathbb{C}_+.\]

Moreover, (4.48) and (4.46) together with (4.24), (4.26) and (4.27) give

\[(4.53) \quad (\Gamma_{0a} + \tilde{\Gamma}_a)v_0 (\lambda) = (M_1 (\lambda) - \frac{i}{2} P_{\tilde{H}_1}) P_{H_0'} + (N_{1-} (\lambda) - i I_{\tilde{H}_2}) P_{H_2}, \quad \lambda \in \mathbb{C}_-\]

\[(4.54) \quad (\Gamma_{0a} + \tilde{\Gamma}_a)u_- (\lambda) = (N_{1-} (\lambda) - i I_{\tilde{H}_2}, M_{2-} (\lambda)) : \tilde{H}_2 \oplus H_b \to H_0, \quad \lambda \in \mathbb{C}_-\]

\[(4.55) \quad \Gamma_{1b} v_0 (\lambda) = -M_3 (\lambda), \quad \Gamma_{1b} u_- (\lambda) = -(N_{2-} (\lambda), M_{4-} (\lambda)), \quad \lambda \in \mathbb{C}_-.\]

Combining (4.30) with (4.49) and (4.53) yields

\[m_0 (\lambda) = (M_1 (\lambda) + N_{1+} (\lambda) - \frac{i}{2} P_{\tilde{H}_1}) P_{H_0'} + \frac{i}{2} P_{\tilde{H}_1} = (M_1 (\lambda) + N_{1+} (\lambda)) P_{H_0'}, \quad \lambda \in \mathbb{C}_+\]

\[+ \frac{i}{2} (P_{\tilde{H}_1} + P_{\tilde{H}_2}) = (M_1 (\lambda) + N_{1+} (\lambda)) P_{H_0'} + \frac{i}{2} P_{\tilde{H}_2}, \quad \lambda \in \mathbb{C}_+\]

\[m_0 (\lambda) = (M_1 (\lambda) - \frac{i}{2} P_{\tilde{H}_1}) P_{H_0'} + (N_{1-} (\lambda) - i I_{\tilde{H}_2}) P_{H_2} + \frac{i}{2} P_{H_2} = M_1 (\lambda) P_{H_0'} + N_{1-} (\lambda) P_{\tilde{H}_2} - \frac{i}{2} P_{\tilde{H}_1} - i P_{\tilde{H}_2} + \frac{i}{2} (P_{\tilde{H}_1} + P_{\tilde{H}_2}) = M_1 (\lambda) P_{H_0'} + N_{1-} (\lambda) P_{\tilde{H}_2} - \frac{i}{2} P_{\tilde{H}_1} + \frac{i}{2} (P_{\tilde{H}_1} + P_{\tilde{H}_2}), \quad \lambda \in \mathbb{C}_-\]

which is equivalent to the block representations (4.36) and (4.37) of \(m_0 (\lambda)\). Moreover, (4.31) together with (4.50) and (4.54) gives the block representations (4.38) of \(\Phi_+ (\lambda)\) and \(\Phi_- (\lambda)\). Combining (4.32) and (4.33) with (4.51), (4.52) and (4.55) we arrive at the block representations (4.39) of \(\Psi_+ (\lambda)\) and (4.40) of \(M_\pm (\lambda)\). Finally, holomorphy of the operator functions \(m_0 (\cdot), \Phi_\pm (\cdot), \Psi_\pm (\cdot)\) and \(M_\pm (\cdot)\) and the equalities (4.41) are implied by (4.34)–(4.40) and the equality \(M_\pm (\lambda) = M_- (\lambda), \lambda \in \mathbb{C}_-\).

**Theorem 4.6.** Let in Case 1 \(\tau = \{\tau_+, \tau_-\}\) be a boundary parameter (4.1), (4.3). Then:

(1) For each \(\lambda \in \mathbb{C} \setminus \mathbb{R}\) there exists a unique operator solution \(v_\tau (\cdot, \lambda) \in \mathcal{L}_2^2 [H_0, H]\) of Eq. (3.4) satisfying the boundary conditions

\[(4.56) \quad \Gamma_{1a} v_\tau (\lambda) = -P_H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}\]

\[(4.57) \quad i(\tilde{\Gamma}_{a1} - \tilde{\Gamma}_b) v_\tau (\lambda) = P_{\tilde{H}_1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}\]

\[(4.58) \quad \tilde{C}_{02} (\lambda) (i\tilde{\Gamma}_{a2} v_\tau (\lambda) - P_{\tilde{H}_2}) + C_{10} (\lambda) \Gamma_{0b} v_\tau (\lambda) + C_1 (\lambda) \Gamma_{1b} v_\tau (\lambda) = 0, \quad \lambda \in \mathbb{C}_+\]

\[(4.59) \quad \tilde{D}_{02} (\lambda) (i\tilde{\Gamma}_{a2} v_\tau (\lambda) - P_{\tilde{H}_2}) + D_{0b} (\lambda) \Gamma_{0b} v_\tau (\lambda) + D_1 (\lambda) \Gamma_{1b} v_\tau (\lambda) = 0, \quad \lambda \in \mathbb{C}_-\]

(here \(v_\tau (\cdot, \lambda)\) is the linear map from Lemma 3.3 corresponding to the solution \(v_\tau (\cdot, \lambda)\)).
(2) \( v_\tau(\cdot, \lambda) \) is connected with the solutions \( v_0(\cdot, \lambda) \) and \( u_\pm(\cdot, \lambda) \) from Proposition 4.4 by
\[
(4.60) \quad v_\tau(t, \lambda) = v_0(t, \lambda) - u_+(t, \lambda)(\tau_+^{\ast}(\lambda) + \hat{M}_+(\lambda))^{-1}\Psi_+(\lambda), \quad \lambda \in \mathbb{C}_+,
\]
\[
(4.61) \quad v_\tau(t, \lambda) = v_0(t, \lambda) - u_-(t, \lambda)(\tau_-(\lambda) + \hat{M}_-(\lambda))^{-1}\Psi_-(\lambda), \quad \lambda \in \mathbb{C}_-,
\]
where \( \Psi_\pm(\lambda) \) and \( \hat{M}_\pm(\lambda) \) are the operator functions defined in (4.32) and (4.33).

Proof. It follows from (4.40) and (4.35) that \( \hat{M}_-(\lambda) = P_{H_b}M_-(\lambda) \mid \tilde{H}_2 \oplus H_b, \lambda \in \mathbb{C}_- \).
Therefore by Proposition 2.7, (3) \( \hat{M}_-(\lambda) \) is the Weyl function of the boundary triplet \( \tilde{H}_- \) for \( T^* \) and in view of [35, Theorem 3.11] one has \( 0 \in \rho(\tau_-(\lambda) + \hat{M}_-(\lambda)), \lambda \in \mathbb{C}_- \). Hence for each \( \lambda \in \mathbb{C}_- \) the equalities (4.60) and (4.61) correctly define the solution \( v_\tau(\cdot, \lambda) \in L^2([H_0, \hat{H}]) \) of Eq. (3.4) and to prove the theorem it is sufficient to show that such \( v_\tau(\cdot, \lambda) \) is a unique solution of (3.4) belonging to \( L^2([H_0, \hat{H}]) \) and satisfying (4.56)–(4.59).

Combining (4.60) and (4.61) with (4.10) and (4.12) one gets the equalities (4.56) and (4.57). To prove (4.58) and (4.59) we let
\[
(4.62) \quad T_+(\lambda) = (\tau_+^{\ast}(\lambda) + M_+(\lambda))^{-1}, \lambda \in \mathbb{C}_+, \quad T_-(\lambda) = (\tau_-(\lambda) + \hat{M}_-(\lambda))^{-1}, \lambda \in \mathbb{C}_-,
\]
so that \( \tau_+^{\ast}(\lambda) \) and \( \tau_-(\lambda) \) can be written as
\[
(4.63) \quad \tau_+^{\ast}(\lambda) = \{(T_+(\lambda)h, (I - \hat{M}_+(\lambda)T_+(\lambda))h) : h \in \tilde{H}_2 \oplus H_b\},
\]
\[
(4.64) \quad \tau_-(\lambda) = \{(T_-(\lambda)h, (I - \hat{M}_-(\lambda)T_-(\lambda))h) : h \in H_b\}.
\]
By using definition of the class \( \tilde{R}_-(\mathcal{H}_0, \mathcal{H}_1) \) in [35] one can easily show that
\[
\tau_+(\lambda) = \{-h_1 + iP_0h_0, -P_1h_0\} : h_1, h_0 \in \tau_+^{\ast}(\lambda)\}
\]
This and (4.63) yield
\[
(4.65) \quad \tau_+(\lambda) = \{(\lambda + \tilde{P}_{\tilde{H}_2} - i\tilde{P}_{\tilde{H}_2}\hat{M}_+(\lambda)T_+(\lambda))\hat{P}_0, \hat{P}_1h_0 : h_1, h_0 \in \tau_+^{\ast}(\lambda)\},
\]
where \( \tilde{P}_{\tilde{H}_2} \) and \( P_{\tilde{H}_b} \) are the same as in Proposition 4.4 and \( h \) runs over \( \tilde{H}_2 \oplus H_b \).

It follows from (4.32) that
\[
(4.66) \quad \hat{\Gamma}_{a_2}v_0(\lambda) = \tilde{P}_{\tilde{H}_2}\Psi_+(\lambda) - iP_{\tilde{H}_2}, \quad \Gamma_{b_2}v_0(\lambda) = -P_{\tilde{H}_2}\Psi_+(\lambda), \quad \lambda \in \mathbb{C}_+
\]
\[
(4.67) \quad \hat{\Gamma}_{a_2}u_+(\lambda) = \tilde{P}_{\tilde{H}_2}\hat{M}_+(\lambda), \quad \Gamma_{b_2}u_+(\lambda) = -P_{\tilde{H}_2}\hat{M}_+(\lambda), \quad \lambda \in \mathbb{C}_+.
\]
Combining (4.60) with (4.66), (4.67), (4.13) and the last equality in (4.10) one gets
\[
(4.68) \quad \Gamma_{b_2}v_\tau(\lambda) = (-P_{\tilde{H}_2} + \hat{P}_{\tilde{H}_2}\hat{M}_+(\lambda)\hat{T}_+(\lambda))\Psi_+(\lambda), \quad \lambda \in \mathbb{C}_+.
\]
Moreover, (4.61) together with (4.11), (4.14), (4.33) and the last equality in (4.10) yields
\[
(4.69) \quad \Gamma_{b_2}v_\tau(\lambda) = -(\hat{P}_{\tilde{H}_2}\hat{T}_-(\lambda)\Psi_-(\lambda) - \hat{P}_{\tilde{H}_2}\hat{T}_-(\lambda)\Psi_-(\lambda) = -T_-(\lambda)\Psi_-(\lambda), \quad \lambda \in \mathbb{C}_-.
\]
Hence by (4.64) and (4.65) one has
\[
\{(i\hat{\Gamma}_{a_2}v_\tau(\lambda) - P_{\tilde{H}_2})h_0 + \Gamma_{b_2}v_\tau(\lambda)h_0, \Gamma_{b_2}v_\tau(\lambda)h_0\} \in \tau_\pm(\lambda), \quad h_0 \in H_0, \quad \lambda \in \mathbb{C}_\pm,
\]
which in view of (2.4) and the block representations (4.3) yields (4.58) and (4.59). Finally, uniqueness of \( v_\tau(\cdot, \lambda) \) is implied by uniqueness of the solution of the boundary value problem (4.4)–(4.7) (see Theorem 4.2). \( \square \)
4.3. Case 2.

Definition 4.7. A boundary parameter in Case 2 is a collection \( \tau = \{ \tau^+, \tau^- \} \in \tilde{R} - (\tilde{H} \oplus \tilde{H}_b, \tilde{H}_b) \). A truncated boundary parameter is a boundary parameter \( \tau = \{ \tau^+, \tau^- \} \) satisfying \( \tau^-(\lambda) \subset \{ \{0\} \oplus \tilde{H}_b \} \oplus \tilde{H}_b, \lambda \in \mathbb{C}_- \).

According to Subsection 2.3 a boundary parameter \( \tau = \{ \tau^+, \tau^- \} \) admits the representation (4.70)
\[
\tau^+(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \tilde{H}_b\}, \lambda \in \mathbb{C}_+; \tau^-(\lambda) = \{(D_0(\lambda), D_1(\lambda)); \tilde{H} \oplus \tilde{H}_b\}, \lambda \in \mathbb{C}_-
\]

with holomorphic operator functions \( C_0(\lambda) \in [\tilde{H} \oplus \tilde{H}_b, \tilde{H}_b] \), \( C_1(\lambda) \in [\tilde{H}_b] \) and \( D_0(\lambda) \in [\tilde{H} \oplus \tilde{H}_b] \), \( D_1(\lambda) \in [\tilde{H}_b, \tilde{H} \oplus \tilde{H}_b] \). Moreover, a truncated boundary parameter \( \tau = \{ \tau^+, \tau^- \} \) admits the representation (4.70) with
\[
D_0(\lambda) = \begin{pmatrix} I_{\tilde{H}} & 0 \\ 0 & \mathcal{T}_{D_0(\lambda)} \end{pmatrix} : \tilde{H} \oplus \tilde{H}_b \to \tilde{H} \oplus \tilde{H}_b, \quad D_1(\lambda) = \begin{pmatrix} 0 \\ \mathcal{T}_{D_1(\lambda)} \end{pmatrix} : \tilde{H}_b \to \tilde{H} \oplus \tilde{H}_b.
\]

Let \( \tau = \{ \tau^+, \tau^- \} \) be a boundary parameter (4.70) and let
\[
C_0(\lambda) = (\tilde{C}_0(\lambda), \tilde{C}_0b(\lambda)) : \tilde{H} \oplus \tilde{H}_b \to \tilde{H}_b, \quad D_0(\lambda) = (\tilde{D}_0(\lambda), \tilde{D}_0b(\lambda)) : \tilde{H} \oplus \tilde{H}_b \to (\tilde{H} \oplus \tilde{H}_b)
\]
be the block representations of \( C_0(\lambda) \) and \( D_0(\lambda) \). For a given function \( f \in L^2(\mathcal{I}) \) consider the following boundary value problem:
\[
(4.73) \quad Jy' - B(t)y = \lambda \Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I},
\]
\[
(4.74) \quad \Gamma_{1a}y = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
\[
(4.75) \quad i\tilde{C}_0(\lambda)\tilde{\Gamma}_ay + \tilde{C}_0b(\lambda)\tilde{\Gamma}_b y + C_1(\lambda)\Gamma_{1b} y = 0, \quad \lambda \in \mathbb{C}_+,
\]
\[
(4.76) \quad i\tilde{D}_0(\lambda)\tilde{\Gamma}_ay + \tilde{D}_0b(\lambda)\tilde{\Gamma}_b y + D_1(\lambda)\Gamma_{1b} y = 0, \quad \lambda \in \mathbb{C}_-.
\]

One proves the following theorem in the same way as Theorem 4.2.

Theorem 4.8. Let in Case 2 \( T \) be a symmetric relation in \( L^2(\mathcal{I}) \) defined by (3.39). Then statements of Theorem 4.2 hold with a boundary parameter \( \tau = \{ \tau^+, \tau^- \} \) of the form (4.70) and the boundary value problem (4.73) - (4.76) in place of (4.4) - (4.7).

Remark 4.9. Let \( \tau_0 = \{ \tau^+, \tau^- \} \) be a boundary parameter (4.70) with
\[
(4.77) \quad C_0(\lambda) \equiv P_{\tilde{H}_b}(\tilde{H} \oplus \tilde{H}_b), \quad C_1(\lambda) \equiv 0, \quad D_0(\lambda) \equiv I_{\tilde{H} \oplus \tilde{H}_b}, \quad D_1(\lambda) \equiv 0
\]
and let \( A_0 \) be the symmetric relation (3.42). Then \( R_{\tau_0}(\lambda) \) is of the form (4.9).

Proposition 4.10. Let in Case 2 \( P_H \) (resp. \( P_{\tilde{H}} \)) be the orthoprojector in \( H_0 \) onto \( H \) (resp. \( \tilde{H} \)) and let \( \tilde{P}_{\tilde{H}} \) (resp. \( P_{\tilde{H}_b} \)) be the orthoprojector in \( \tilde{H} \oplus \tilde{H}_b \) onto \( \tilde{H} \) (resp. \( \tilde{H}_b \)). Then:
\( 1 \) For any \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) there exists a unique operator solution \( v_0(\cdot, \lambda) \in L^2[H_0, \mathbb{H}] \) of Eq. (3.4) satisfying
\[
(4.78) \quad \Gamma_{1a}v_0(\lambda) = -P_H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\]
\[
(4.79) \quad \Gamma_{1b}v_0(\lambda) = 0, \quad \lambda \in \mathbb{C}_+; \quad i\tilde{\Gamma}_av_0(\lambda) = P_{\tilde{H}}, \quad \tilde{\Gamma}_bv_0(\lambda) = 0, \quad \lambda \in \mathbb{C}_-
\]
(2) For any \( \lambda \in \mathbb{C}_+ (\lambda \in \mathbb{C}_-) \) there exists a unique operator solution \( u_+ (\cdot, \lambda) \in \mathcal{L}_2^2 [\mathcal{H}_b, \mathbb{H}] \) (resp. \( u_- (\cdot, \lambda) \in \mathcal{L}_2^2 [\mathcal{H} \oplus \mathcal{H}_b, \mathbb{H}] \)) of Eq. (3.4) satisfying

\[
\Gamma_{1a} u_\pm (\lambda) = 0, \quad \lambda \in \mathbb{C}_\pm
\]

(4.80)

\[
\Gamma_{0b} u_+ (\lambda) = I_{\mathcal{H}_b}, \quad \lambda \in \mathbb{C}_+; \quad \tilde{\Gamma}_{0a} u_- (\lambda) = \tilde{P}_{\mathcal{H}}, \quad \Gamma_{0b} u_- (\lambda) = P_{\mathcal{H}_b}, \quad \lambda \in \mathbb{C}_-.
\]

(4.81)

Proof. Let \( \Pi_\pm \) be the decomposing boundary triplet for \( T_{\text{max}} \) constructed with the aid of some \( J \)-unitary extension \( \tilde{U} \) of \( U \) (see Proposition 3.5). By using \( \gamma \)-fields of this triplet one proves in the same way as in Proposition 4.4 the existence of isomorphisms \( Z_+ (\lambda) : \mathcal{H}_1 \to \mathcal{N}_\lambda, \lambda \in \mathbb{C}_+, \) and \( Z_- (\lambda) : \mathcal{H}_0 \to \mathcal{N}_\lambda, \lambda \in \mathbb{C}_- \), such that

\[
\begin{pmatrix}
-\Gamma_{1a} \\
\Gamma_{0b}
\end{pmatrix}
\begin{pmatrix}
Z_+ (\lambda) \\
\lambda \in \mathbb{C}_+
\end{pmatrix} = 
\begin{pmatrix}
I_H & 0 \\
0 & I_{\mathcal{H}_b}
\end{pmatrix},
\]

(4.82)

\[
\begin{pmatrix}
-\Gamma_{1a} \\
\Gamma_{0b}
\end{pmatrix}
\begin{pmatrix}
Z_- (\lambda) \\
\lambda \in \mathbb{C}_-
\end{pmatrix} = 
\begin{pmatrix}
I_H & 0 \\
0 & I_{\mathcal{H}_b}
\end{pmatrix},
\]

(4.83)

It follows from (4.82) and (4.83) that

\[
\Gamma_{1a} Z_+ (\lambda) = (-I_H, 0), \quad \Gamma_{0b} Z_+ (\lambda) = (0, I_{\mathcal{H}_b})
\]

(4.84)

\[
\Gamma_{1a} Z_- (\lambda) = (-I_H, 0, 0), \quad \tilde{\Gamma}_{0a} Z_- (\lambda) = (0, I_\mathcal{H}), \quad \Gamma_{0b} Z_- (\lambda) = (0, 0, I_{\mathcal{H}_b}).
\]

(4.85)

Assume that

\[
Z_+ (\lambda) = (r (\lambda), u_+ (\lambda)) : H \oplus \mathcal{H}_b \to \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_+
\]

(4.86)

\[
Z_- (\lambda) = (r (\lambda), \omega_- (\lambda), \bar{u}_- (\lambda)) : H \oplus \mathcal{H} \oplus \mathcal{H}_b \to \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_-
\]

(4.87)

are the block representations of \( Z_\pm (\lambda) \) (see (3.32)) and let \( v_0 (\lambda) : H_0 \to \mathcal{N}_\lambda, \lambda \in \mathbb{C} \setminus \mathbb{R} \), be the operator given by

\[
v_0 (\lambda) := (r (\lambda), 0) : H \oplus \mathcal{H} \to \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_+
\]

(4.88)

\[
v_0 (\lambda) := (r (\lambda), \omega_- (\lambda)) : H \oplus \mathcal{H} \to \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_-
\]

(4.89)

It is clear that (4.86) induces the operator \( u_+ (\lambda) \in [\mathcal{H}_b, \mathcal{N}_\lambda] \). Introduce also the operator

\[
u_- (\lambda) = (\omega_- (\lambda), \bar{u}_- (\lambda)) : \mathcal{H} \oplus \mathcal{H}_b \to \mathcal{N}_\lambda, \quad \lambda \in \mathbb{C}_-
\]

(4.90)

Let \( v_0 (\cdot, \lambda) \in \mathcal{L}_2^2 [H_0, \mathbb{H}], u_+ (\cdot, \lambda) \in \mathcal{L}_2^2 [\mathcal{H}_b, \mathbb{H}] \) and \( u_- (\cdot, \lambda) \in \mathcal{L}_2^2 [\mathcal{H} \oplus \mathcal{H}_b, \mathbb{H}] \) be the operator solutions of Eq. (3.4) corresponding to \( v_0 (\lambda), u_+ (\lambda) \) and \( u_- (\lambda) \) respectively (see Lemma 3.3). Then combining of (4.86)–(4.90) with (4.84) and (4.85) gives the relations (4.78)–(4.81). Finally, uniqueness of \( v_0 (\cdot, \lambda) \) and \( u_\pm (\cdot, \lambda) \) follows from uniqueness of the solution of the boundary value problem (4.73)–(4.76).

Let \( \tilde{U} \) be a \( J \)-unitary extension (3.13) of \( U \) and let \( \Gamma_{0a} \) be the mapping (3.15). Introduce the operator functions

\[
X_+ (\lambda) = \begin{pmatrix}
\Phi_0 (\lambda) \\
\Psi_0 (\lambda)
\end{pmatrix} : H_0 \oplus \mathcal{H}_b \to H_0 \oplus (\mathcal{H} \oplus \mathcal{H}_b), \quad \lambda \in \mathbb{C}_+
\]

(4.91)

\[
X_- (\lambda) = \begin{pmatrix}
\Phi_0 (\lambda) \\
\Psi_0 (\lambda)
\end{pmatrix} : H_0 \oplus (\mathcal{H} \oplus \mathcal{H}_b) \to H_0 \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C}_-
\]

(4.92)
with the entries defined in terms of the solutions $v_0(\cdot, \lambda)$ and $u_\pm(\cdot, \lambda)$ as follows: $m_0(\lambda)$, $\Phi_\pm(\lambda)$, $\Psi_-(\lambda)$ and $M_-(\lambda)$ are given by (4.30), (4.31) and (4.33), while

\begin{equation}
(4.93) \quad \Psi_+(\lambda) = (\widehat{\Gamma}_a - \Gamma_{1b} - i\widehat{\Gamma}_b)v_0(\lambda) + i\widehat{P}_H, \quad M_+(\lambda) = (\widehat{\Gamma}_a - \Gamma_{1b} - i\widehat{\Gamma}_b)u_+(\lambda), \quad \lambda \in \mathbb{C}_+.
\end{equation}

**Proposition 4.11.** Let $\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be the decomposing boundary triplet (3.32)–(3.34) for $T_{\text{max}}$ and let

\begin{equation}
(4.94) \quad M_+(\lambda) = \begin{pmatrix} M_1(\lambda) & M_2(\lambda) \\ N_1(\lambda) & N_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix} : H \oplus \mathcal{H}_b \rightarrow H \oplus \widehat{H} \oplus \widehat{\mathcal{H}}_b, \quad \lambda \in \mathbb{C}_+
\end{equation}

\begin{equation}
(4.95) \quad M_-(\lambda) = \begin{pmatrix} M_1(\lambda) & N_1(\lambda) & M_2(\lambda) \\ M_3(\lambda) & N_2(\lambda) & M_4(\lambda) \end{pmatrix} : H \oplus \widehat{H} \oplus \widehat{\mathcal{H}}_b \rightarrow H \oplus \mathcal{H}_b, \quad \lambda \in \mathbb{C}_-
\end{equation}

be the block matrix representations of the corresponding Weyl functions. Then the entries of the operator matrices (4.91) and (4.92) are holomorphic on their domains and satisfy

\begin{equation}
(4.96) \quad m_0(\lambda) = \begin{pmatrix} M_1(\lambda) \\ N_1(\lambda) \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{i}I_{\widehat{H}} \\ \frac{1}{i}I_{\widehat{H}} \\ H_0 \end{pmatrix} : H \oplus \widehat{H} \rightarrow H \oplus \widehat{H}, \quad \lambda \in \mathbb{C}_+
\end{equation}

\begin{equation}
(4.97) \quad m_0(\lambda) = \begin{pmatrix} M_1(\lambda) \\ N_1(\lambda) \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{i}I_{\widehat{H}} \\ \frac{1}{i}I_{\widehat{H}} \\ H_0 \end{pmatrix} : \mathcal{H}_b \oplus \widehat{H} \rightarrow \mathcal{H}_b \oplus \widehat{H}, \quad \lambda \in \mathbb{C}_-
\end{equation}

and (4.38)–(4.40). Moreover, the equalities (4.41) are valid.

**Proof.** Let $Z_\pm(\cdot)$ be the same as in Proposition 4.10. By using the reasonings similar to those in the proof of Proposition 4.5 one proves the equalities

\begin{equation}
(4.98) \quad \begin{pmatrix} \Gamma_{0a} \\ \widehat{\Gamma}_a \\ -\Gamma_{1b} - i\widehat{\Gamma}_b \end{pmatrix} Z_+(\lambda) = \begin{pmatrix} M_1(\lambda) & M_2(\lambda) \\ N_1(\lambda) & N_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_+,
\end{equation}

\begin{equation}
(4.99) \quad \begin{pmatrix} \Gamma_{0a} \\ -\Gamma_{1b} \end{pmatrix} Z_-(\lambda) = \begin{pmatrix} M_1(\lambda) & N_1(\lambda) & M_2(\lambda) \\ M_3(\lambda) & N_2(\lambda) & M_4(\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}_-.
\end{equation}

It follows from (4.98) that

\begin{align*}
(\Gamma_{0a} + \widehat{\Gamma}_a)Z_+(\lambda) &= (M_1(\lambda) + N_1(\lambda), M_2(\lambda) + N_2(\lambda)), \\
(\widehat{\Gamma}_a - \Gamma_{1b} - i\widehat{\Gamma}_b)Z_+(\lambda) &= (N_1(\lambda) + M_3(\lambda), N_2(\lambda) + M_4(\lambda)), \quad \lambda \in \mathbb{C}_+.
\end{align*}

Moreover, (4.99) and the second equality in (4.85) yield

\begin{align*}
(\Gamma_{0a} + \widehat{\Gamma}_a)Z_-(\lambda) &= (M_1(\lambda), N_1(\lambda) - iI_{\widehat{H}}, M_2(\lambda)), \\
-\Gamma_{1b}Z_-(\lambda) &= (M_3(\lambda), N_2(\lambda), M_4(\lambda)), \quad \lambda \in \mathbb{C}_-.
\end{align*}
Combining these equalities with the block representations (4.86) and (4.87) of \( Z_\pm(\lambda) \) and taking (4.88)–(4.90) into account we obtain

\[
(G_0 + \tilde{a})v_0(\lambda) = (M_1(\lambda) + N_1(\lambda))P_H, \quad (G_0 + \tilde{a})u_+(\lambda) = M_2(\lambda) + N_2(\lambda), \quad \lambda \in \mathbb{C}_+
\]

\[
(\tilde{a} - \tilde{\Gamma}_b)v_0(\lambda) = (N_1(\lambda) + M_3(\lambda))P_H, \quad \lambda \in \mathbb{C}_+
\]

\[
(\tilde{a} - \tilde{\Gamma}_b)u_+(\lambda) = N_2(\lambda) + M_4(\lambda), \quad \lambda \in \mathbb{C}_+
\]

\[
(G_0 + \tilde{a})v_0(\lambda) = M_1(\lambda)P_H + N_1(\lambda)P_{\tilde{H}} - iP_{\tilde{H}}, \quad \lambda \in \mathbb{C}_-
\]

\[
(G_0 + \tilde{a})u_-(\lambda) = (N_1(\lambda) - i\tilde{\Gamma}_b, M_2(\lambda)) : \tilde{H} \oplus \tilde{\mathcal{H}}_b \to H_0, \quad \lambda \in \mathbb{C}_-
\]

\[-\Gamma_0v_0(\lambda) = (N_3(\lambda), N_2(-\lambda)), \quad -\Gamma_1u_-(\lambda) = (N_2(-\lambda), M_4(-\lambda)), \quad \lambda \in \mathbb{C}_-
\]

This and definitions (4.30), (4.31), (4.33) and (4.93) of \( m_0(\cdot), \Phi_\pm(\cdot), \Psi_\pm(\cdot) \) and \( M_\pm(\cdot) \) yield the equalities (4.96), (4.97) and (4.38)–(4.40). Finally one proves (4.41) in the same way as in Proposition 4.5. \( \square \)

**Theorem 4.12.** Let in Case 2 \( \tau = \{\tau_+, \tau_-\} \) be a boundary parameter defined by (4.70) and (4.72). Then:

1. Statement (1) of Theorem 4.6 holds with the boundary condition (4.56) and the following boundary conditions in place of (4.57)–(4.59):

\[
(4.100) \quad \tilde{C}(\lambda)(i\tilde{\Gamma}_b v_\tau(\lambda) - P_{\tilde{H}}) + \tilde{a}v_\tau(\lambda) + C(\lambda)\Gamma_{1b}v_\tau(\lambda) = 0, \quad \lambda \in \mathbb{C}_+
\]

\[
(4.101) \quad \tilde{D}(\lambda)(i\tilde{\Gamma}_b v_\tau(\lambda) - P_{\tilde{H}}) + \tilde{a}v_\tau(\lambda) + D(\lambda)\Gamma_{1b}v_\tau(\lambda) = 0, \quad \lambda \in \mathbb{C}_-
\]

2. The solution \( v_\tau(\cdot, \lambda) \) is of the form (4.60), (4.61), where \( v_0(\cdot, \lambda) \) and \( u_\pm(\cdot, \lambda) \) are defined in Proposition 4.10 and \( \Psi_\pm(\lambda) \) and \( M_\pm(\lambda) \) are given by (4.33) and (4.93).

**Proof.** As in Theorem 4.6 one proves that (4.60) and (4.61) correctly define the solution \( v_\tau(\cdot, \lambda) \in \mathcal{C}_2^2[H_0, \mathbb{H}] \) of Eq. (3.4). Therefore it remains to show that such \( v_\tau(\cdot, \lambda) \) is a unique solution of (3.4) belonging to \( \mathcal{C}_2^2[H_0, \mathbb{H}] \) and satisfying (4.56), (4.100) and (4.101).

First observe that (4.78) and (4.80) yield (4.56). Next assume that \( T_\pm(\lambda) \) are defined by (4.62). Then in the same way as in Theorem 4.6 one gets (4.64) and the equality

\[
(4.102) \quad \tau_+(\lambda) = \{(-T_+(\lambda) + iP_{\tilde{H}}\tilde{H}) - iP_{\tilde{H}}\tilde{H}, -P_{\tilde{H}}\tilde{H}M_+(-\lambda)T_+(\lambda)h, \quad -P_{\tilde{H}}\tilde{H}M_+(-\lambda)T_+(\lambda)h) : h \in \tilde{H} \oplus \tilde{\mathcal{H}}_b
\]

where \( P_{\tilde{H}}\tilde{H} \) is the orthoprojector in \( \tilde{H} \oplus \tilde{\mathcal{H}}_b \) onto \( \tilde{H} \oplus \tilde{\mathcal{H}}_b(= \tilde{H} \oplus \tilde{\mathcal{H}}_b) \).

It follows from (4.93) that

\[
(4.103) \quad \tilde{a}v_0(\lambda) = \tilde{P}_{\tilde{H}}\tilde{H}_b + iP_{\tilde{H}}\tilde{H}_b, \quad \Gamma_{1b}v_0(\lambda) = -P_{\tilde{H}}\tilde{H}_b\Psi_\pm(\lambda), \quad \lambda \in \mathbb{C}_+
\]

\[
(4.104) \quad \tilde{a}u_+(\lambda) = \tilde{P}_{\tilde{H}}\tilde{H}_b M_+(-\lambda), \quad \Gamma_{1b}u_+(\lambda) = -P_{\tilde{H}}\tilde{H}_b M_+(-\lambda), \quad \lambda \in \mathbb{C}_+
\]

\[
(4.105) \quad \Gamma_{1b}u_+(\lambda) = \tilde{P}_{\tilde{H}}\tilde{H}_b M_+(-\lambda), \quad \Gamma_{1b}u_+(\lambda) = -P_{\tilde{H}}\tilde{H}_b M_+(-\lambda), \quad \lambda \in \mathbb{C}_+
\]

where \( \tilde{P}_{\tilde{H}}\tilde{H}_b \) and \( P_{\tilde{H}}\tilde{H}_b \) are the orthoprojectors in \( \tilde{H} \oplus \tilde{\mathcal{H}}_b \) onto \( \tilde{H}, \tilde{\mathcal{H}}_b \) and \( \tilde{\mathcal{H}}_b \) respectively. Moreover, (4.104), the last equality in (4.105) and the first equalities in (4.79) and (4.81) give

\[
(4.106) \quad \tilde{a}v_0(\lambda) = iP_{\tilde{H}}\tilde{H}_b\Psi_\pm(\lambda), \quad \Gamma_{1b}u_+(\lambda) = iP_{\tilde{H}}\tilde{H}_b M_+(-\lambda), \quad \lambda \in \mathbb{C}_+
\]

Now combining (4.60) with (4.103)–(4.106) one gets (4.68) and the equality

\[
(i\tilde{a}v_\tau(\lambda) - P_{\tilde{H}}) + \tilde{a}v_\tau(\lambda) = (-T_+(\lambda) + iP_{\tilde{H}}\tilde{H}_b - iP_{\tilde{H}}\tilde{H}_b M_+(-\lambda)T_+(\lambda))\Psi_\pm(\lambda), \quad \lambda \in \mathbb{C}_+
\]
Moreover, (4.61) together with (4.79), (4.81) and (4.33) leads to (4.69) and the equality
\[(i\hat{\Gamma}_av_\tau(\lambda)-P_H)+\hat{\Gamma}_{06}v_\tau(\lambda)=-\hat{P}_HT_-(\lambda)\Psi_-(\lambda)-P_{\hat{H}a}T_-(\lambda)\Psi_-(\lambda), \ \lambda \in \mathbb{C}_-\]
Therefore in view of (4.64) and (4.102) one has
\[\{(i\hat{\Gamma}_av_\tau(\lambda)-P_H)h_0+\hat{\Gamma}_{06}v_\tau(\lambda)h_0,\Gamma_{16}v_\tau(\lambda)h_0\} \in \tau(\lambda), \ h_0 \in H_0, \ \lambda \in \mathbb{C}_\pm,\]
which implies (4.100) and (4.101). Finally, uniqueness of \(v_\tau(\cdot, \lambda)\) follows from uniqueness of the solution of the boundary value problem (4.73)–(4.76).

5. m-FUNCTIONS AND EIGENFUNCTION EXPANSIONS

5.1. m-functions. In this subsection we assume that \(\tilde{U}\) is a J-unitary extension (3.13) of \(U\) and \(\Gamma_{0a}\) is the mapping (3.15).

Let \(\tau\) be a boundary parameter and let \(v_\tau(\cdot, \lambda) \in \mathcal{L}_2^2[H_0, \mathbb{H}]\) be the operator solution of Eq. (3.4) defined in Theorems 4.6 and 4.12. Similarly to the case \(n_-(T_{\min}) \leq n_+(T_{\min})\) (see [2]) we introduce the following definition.

**Definition 5.1.** The operator function \(m_\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [H_0]\) defined by
\[(5.1) \quad m_\tau(\lambda) = (\Gamma_{0a} + \hat{\Gamma}_av_\tau(\lambda) + i\hat{P}_H), \ \lambda \in \mathbb{C} \setminus \mathbb{R},\]
is called the \(m\)-function corresponding to the boundary parameter \(\tau\) or, equivalently, to the boundary value problem (4.4)–(4.7) (in Case 1) or (4.73)–(4.76) (in Case 2).

From (4.56) it follows from that \(m_\tau(\cdot)\) satisfies the equality
\[(5.2) \quad \tilde{U}v_\tau(a, \lambda) = \begin{pmatrix} \Gamma_{0a} + \hat{\Gamma}_a \\ \Gamma_{1a} \end{pmatrix} v_\tau(\lambda) = \begin{pmatrix} m_\tau(\lambda) - i\hat{P}_H \\ -\hat{P}_H \end{pmatrix} : H_0 \rightarrow H_0 \oplus H, \ \lambda \in \mathbb{C} \setminus \mathbb{R}.\]

It is easily seen that Proposition 5.3 in [2] remains valid in the case \(n_+(T_{\min}) < n_-(T_{\min})\). This means that for a given operator \(U\) (see (3.9)) and a boundary parameter \(\tau\) the \(m\)-functions \(m^{(1)}(\cdot)\) and \(m^{(2)}(\cdot)\) corresponding to J-unitary extensions \(\tilde{U}_1\) and \(\tilde{U}_2\) of \(U\) are connected by \(m^{(2)}(\lambda) = m^{(1)}(\lambda) + B, \ \lambda \in \mathbb{C} \setminus \mathbb{R},\) with some \(B = B^* \in [H_0].\)

A definition of the \(m\)-function \(m_\tau\) in somewhat other terms is given in the following proposition, which directly follows from Theorems 4.6 and 4.12.

**Proposition 5.2.** Let \(\tau = \{\tau_+, \tau_-\}\) be a boundary parameter (4.1), (4.3) in Case 1 (resp. (4.70), (4.72) in Case 2), let \(\varphi_U(\cdot, \lambda)(\in [H_0, \mathbb{H}]), \ \lambda \in \mathbb{C},\) be the operator solution of Eq. (3.4) defined by (3.19) and let \(\psi(\cdot, \lambda)(\in [H_0, \mathbb{H}]), \ \lambda \in \mathbb{C},\) be the operator solutions of Eq. (3.4) with
\[(5.3) \quad \tilde{U}_\psi(a, \lambda) = \begin{pmatrix} -i\hat{P}_H \\ -\hat{P}_H \end{pmatrix} : H_0 \rightarrow H_0 \oplus H.\]

Then there exists a unique operator function \(m(\cdot) = m_\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [H_0]\) such that for any \(\lambda \in \mathbb{C} \setminus \mathbb{R}\) the operator solution \(v(\cdot, \lambda) = v_\tau(\cdot, \lambda)\) of Eq. (3.4) given by
\[(5.3) \quad v(t, \lambda) := \varphi_U(t, \lambda)m(\lambda) + \psi(t, \lambda)\]
belaongs to \(\mathcal{L}_2^2[H_0, \mathbb{H}]\) and satisfies the boundary conditions (4.57)–(4.59) in Case 1 and (4.100), (4.101) in Case 2.

In the following theorem we give a description of all \(m\)-functions immediately in terms of the boundary parameter \(\tau\).
Theorem 5.3. Assume the following hypothesis:

(i) \( \tau_0 \) is a boundary parameter from Remark 4.3 in Case 1 and Remark 4.9 in Case 2.
(ii) \( m_0(\cdot), \Phi_-(\cdot), \Psi_-(\cdot) \) and \( M_-(\cdot) \) are the operator functions given by (4.30), (4.31) and (4.33) (this functions form the operator matrix \( X_-(\cdot) \) given by (4.29) in Case 1 and (4.92) in Case 2).
(iii) \( M_1(\cdot) \) and \( M_2(\cdot), j \in \{2, 3, 4\} \) are the operator functions defined by the block representations (4.35) in Case 1 and (4.95)(in Case 2) of the Weyl function \( M_-(\cdot) \) (according to Propositions 4.5 and 4.11 these functions can be also defined as the entries of the block matrix representations (4.36)–(4.40) of \( m_0(\cdot), \Phi_-(\cdot), \Psi_-(\cdot) \) and \( M_-(\cdot) \)).

Then: (i) \( m_0(\lambda) = m_{\tau_0}(\lambda) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), and for any boundary parameter \( \tau = \{\tau_+, \tau_-\} \) defined by (4.1) in Case 1 and (4.70) in Case 2 the corresponding \( m \)-function \( m_+(\cdot) \) is
\[
(5.4) \quad m_+(\lambda) = m_0(\lambda) + \Phi_-(\lambda)(D_0(\lambda) - D_1(\lambda)\hat{M}_-(\lambda))^{-1}D_1(\lambda)\Psi_-(\lambda), \quad \lambda \in \mathbb{C}.
\]
(ii) For each truncated boundary parameter \( \tau = \{\tau_+, \tau_-\} \) defined by (4.1), (4.2) in Case 1 and (4.70), (4.71) in Case 2 the corresponding \( m \)-function \( m_-(\cdot) \) has the triangular block matrix representation
\[
(5.5) \quad m_-(\lambda) = \begin{pmatrix} m_{-1}(\lambda) & m_{-2}(\lambda) \\ 0 & \frac{1}{2}I \end{pmatrix}, \quad \lambda \in \mathbb{C},
\]
with respect to the decomposition \( H_0 = H'_0 \oplus \tilde{H}_2 \) in Case 1 (see (3.27)) and \( H_0 = H \oplus \tilde{H} \) in Case 2. Moreover, the operator function \( m_{\tau,1}(\cdot) \) in (5.5) is
\[
(5.6) \quad m_{\tau,1}(\lambda) = M_1(\lambda) + M_2(-\lambda)(\overline{D}_0(\lambda) - \overline{D}_1(\lambda)M_4(-\lambda))^{-1}\overline{D}_1(\lambda)M_3(-\lambda), \quad \lambda \in \mathbb{C}.
\]

Proof. We prove the theorem only for Case 1, because in Case 2 the proof is similar.

(1) It is easily seen that \( v_0(t, \lambda) = v_{\tau_0}(t, \lambda) \), where \( v_{\tau_0}(\cdot, \lambda) \) is defined in Proposition 4.4. Hence by (4.30) one has \( m_0(\lambda) = m_{\tau_0}(\lambda) \). Next, applying the operator \( \Gamma_0 + \tilde{\Gamma}_a \) to the equalities (4.60) and (4.61) with taking (4.30) and (4.31) into account one gets
\[
(5.7) \quad m_+(\lambda) = m_0(\lambda) - \Phi_+(\lambda)(\tau^*_+(\lambda) + \hat{M}_+(\lambda))^{-1}\Psi_+(\lambda), \quad \lambda \in \mathbb{C},
\]
\[
(5.8) \quad m_-(\lambda) = m_0(\lambda) - \Phi_-(\lambda)(\tau_-(\lambda) + \hat{M}_-(\lambda))^{-1}S_-(\lambda), \quad \lambda \in \mathbb{C}.
\]
Moreover, according to [30, Lemma 2.1] \( 0 \in \rho(D_0(\lambda) - D_1(\lambda)\hat{M}_-(\lambda)) \) and
\[
-(\tau_-(\lambda) + \hat{M}_-(\lambda))^{-1} = (D_0(\lambda) - D_1(\lambda)\hat{M}_-(\lambda))^{-1}D_1(\lambda), \quad \lambda \in \mathbb{C},
\]
which together with (5.8) yields (5.4).

(2) It follows from (4.2) and the second equality in (4.40) that \( (D_0(\lambda) - D_1(\lambda)\hat{M}_-(\lambda))^{-1} =
\[
\begin{pmatrix} I & 0 \\ -\overline{D}_1(\lambda)N_2(-\lambda) & \overline{D}_0(\lambda) - \overline{D}_1(\lambda)M_4(-\lambda) \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ \star & (\overline{D}_0(\lambda) - \overline{D}_1(\lambda)M_4(-\lambda))^{-1} \end{pmatrix}
\]
\)
(the entry \( \star \) does not matter). Combining this equality with (5.4) and taking (4.37)–(4.39) into account one gets the equalities (5.5) and (5.6).

By using the reasonings similar to those in the proof of Proposition 5.7 in [2] one proves the following proposition.

Proposition 5.4. The \( m \)-function \( m_+(\cdot) \) is a Nevanlinna operator function satisfying
\[
(\text{Im} \lambda)^{-1} \cdot \text{Im} m_+(\lambda) \geq \int_I v^*_+(t, \lambda)\Delta(t)v_+(t, \lambda) \, dt, \quad \lambda \in \mathbb{C}.
\]
5.2. **Green’s function.** In the sequel we put $\mathcal{H} := L^2(\mathcal{I})$ and denote by $\mathcal{H}_0$ the set of all $\tilde{f} \in \mathcal{H}$ with the following property: there exists $\beta_{\tilde{f}} \in \mathcal{I}$ such that for some (and hence for all) function $f \in \tilde{f}$ the equality $\Delta(t)f(t) = 0$ holds a.e. on $(\beta_{\tilde{f}}, b)$.

Let $\varphi_U(\cdot, \cdot) \in L^2(\mathcal{H})$ be the operator-valued solution (3.19), let $\tau$ be a boundary parameter and let $v_{\tau}(\cdot, \lambda) \in L^2(\mathcal{H}_0, \mathbb{H})$ be the operator-valued solution of Eq. (3.4) defined in Theorems 4.6 and 4.12.

**Definition 5.5.** The operator function $G_{\tau}(\cdot, \cdot, \lambda) : \mathcal{I} \times \mathcal{I} \to [\mathbb{H}]$ given by

\[
G_{\tau}(x, t, \lambda) = \begin{cases} 
v_{\tau}(x, \lambda) \varphi_U(t, \lambda), & x > t \\
\varphi_U(x, \lambda) v_{\tau}(t, \lambda), & x < t 
\end{cases}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\]

will be called the Green’s function corresponding to the boundary parameter $\tau$.

**Theorem 5.6.** Let $\tau$ be a boundary parameter and let $R_{\tau}(\cdot)$ be the corresponding generalized resolvent of the relation $T$ (see Theorems 4.2 and 4.8). Then

\[
R_{\tau}(\lambda)\tilde{f} = \pi \left( \int_{\mathcal{I}} G_{\tau}(\cdot, t, \lambda)\Delta(t)f(t) \, dt \right), \quad \tilde{f} \in \mathcal{H}, \quad f \in \tilde{f}.
\]

**Proof.** As in [2, Theorem 6.2] one proves that for each $f, \tilde{f} \in L^2(\mathcal{I})$ the inequality

\[
\int_{\mathcal{I}} ||G_{\tau}(x, t, \lambda)\Delta(t)f(t)|| \, dt < \infty
\]

correctly defines the function $y_{\tau}(\cdot, \cdot, \lambda) : \mathcal{I} \times \mathcal{I} \to \mathbb{H}$. Therefore (5.10) is equivalent to the following statement: for each $\tilde{f} \in \mathcal{H}_0$

\[
y_{\tau}(\cdot, \lambda) \in L^2(\mathcal{I}) \quad \text{and} \quad R_{\tau}(\lambda)\tilde{f} = \pi(y_{\tau}(\cdot, \cdot, \lambda)), \quad f \in \tilde{f}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\]

To prove (5.12) first assume that $\tilde{f} \in \mathcal{H}_0$. We show that the function $y_{\tau}(\cdot, \lambda)$ given by (5.11) is a solution of the boundary problem (4.4)–(4.7) in Case 1 (resp. (4.73)–(4.76) in Case 2). As in [2, Theorem 6.2] one proves that $y_{\tau}(\cdot, \lambda)$ belongs to $AC(\mathcal{I}; \mathbb{H}) \cap L^2(\mathcal{I})$ and satisfies (4.4) a.e. on $\mathcal{I}$. Now it remains to show that $y_{\tau}$ satisfies the boundary conditions (4.5)–(4.7) in Case 1 (resp. (4.74)–(4.76) in Case 2).

It follows from (5.11) and (5.9) that

\[
y_{\tau}(a, \lambda) = \varphi_U(a, \lambda) \int_{\mathcal{I}} v_{\tau}(x, \lambda)\Delta(t)f(t) \, dt,
\]

\[
y_{\tau}(x, \lambda) = v_{\tau}(x, \lambda) \int_{\mathcal{I}} \varphi_U(t, \lambda)\Delta(t)f(t) \, dt, \quad x \in (\beta_{\tilde{f}}, b).
\]

Let $\tilde{U}$ be a $J$-unitary extension (3.13) of $U$ and let $\Gamma_a$ be the operator (3.17). Then $\Gamma_a y_{\tau} = \tilde{U} y_{\tau}(a, \lambda)$ and in view of (5.13) one has

\[
\Gamma_a y_{\tau} = \tilde{U} \varphi_U(a, \lambda) \int_{\mathcal{I}} v_{\tau}(x, \lambda)\Delta(t)f(t) \, dt.
\]

This and (3.20) imply

\[
\Gamma_a y_{\tau} = P_{\tilde{f}} \int_{\mathcal{I}} v_{\tau}(x, \lambda)\Delta(t)f(t) \, dt, \quad \Gamma_1 y_{\tau} = 0.
\]
Next assume that
\[
\tau := \int \varphi^*_U(t, \bar{\lambda}) \Delta(t) f(t) \, dt \in H_0
\]
Then in view of (5.14) one has (see [2, Remark 3.5 (1)])
\[
\hat{\Gamma}_0 y_f = \Gamma_0 v_r(\lambda) h_f, \quad \Gamma_0 y_f = \Gamma_0 v_r(\lambda) h_f, \quad \Gamma_1 y_f = \Gamma_1 v_r(\lambda) h_f.
\]
Let us also prove the equality
\[
\hat{\Gamma}_a y_f = (\hat{\Gamma}_a v_r(\lambda) + i P_{\hat{H}}) h_f.
\]
It follows from (5.2) that
\[
\hat{U} v_r(a, \lambda) | \hat{H} = \left( (m_r^{+}(\lambda) - \frac{1}{2} I_{H_0}) | \hat{H} \right) : \hat{H} \to H_0 \oplus H, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
Therefore by (3.20) one has $v_r(t, \lambda) | \hat{H} = \varphi^U(t, \lambda)(m_r^{+}(\lambda) - \frac{1}{2} I_{H_0}) | \hat{H}$. Moreover, according to Proposition 5.4 $m_r^{+} (\bar{\lambda}) = m_r(\lambda)$. This implies that
\[
P_{\hat{H}} v^r_r(t, \bar{\lambda}) = P_{\hat{H}} m_r(\lambda) + \frac{1}{2} I_{H_0}) \varphi^U(t, \bar{\lambda}) = (P_{\hat{H}} m_r(\lambda) + \frac{1}{2} P_{\hat{H}}) \varphi^U(t, \bar{\lambda})
\]
and (5.15), (5.16) yield
\[
\hat{\Gamma}_a y_f = (\hat{\Gamma}_a v_r(\lambda) + \frac{1}{2} P_{\hat{H}}) \int \varphi^U(t, \bar{\lambda}) \Delta(t) f(t) \, dt = (P_{\hat{H}} m_r(\lambda) + \frac{1}{2} P_{\hat{H}}) h_f.
\]
Since in view of (5.1) $P_{\hat{H}} m_r(\lambda) = \hat{\Gamma}_a v_r(\lambda) + \frac{1}{2} P_{\hat{H}}$, the equality (5.19) gives (5.18).

The second equality in (5.15) gives the first condition in (4.5) and the condition (4.74).

Next assume Case 1 and the hypothesis (A2). Then by (3.26) and (5.18)
\[
\hat{\Gamma}_a y_f = (\hat{\Gamma}_a v_r(\lambda) + i P_{\hat{H}}) h_f, \quad \hat{\Gamma}_a y_f = (\hat{\Gamma}_a v_r(\lambda) + i P_{\hat{H}}) h_f
\]
and combining of (5.20) and (5.17) with (4.57)–(4.59) shows that the second condition in (4.5) and the conditions (4.6) and (4.7) are fulfilled for $y_f$.

Now assume Case 2 and the hypothesis (A3). Then by (3.31) and (5.17) one has $\hat{\Gamma}_0 y_f = \Gamma_1 y_f = \Gamma_1 v_r(\lambda) h_f$. Combining of this equality and (5.17), (5.18) with (4.100) and (4.101) implies fulfillment of the conditions (4.75) and (4.76) for $y_f$.

Thus $y_f(\cdot, \lambda)$ is a solution of the boundary value problem (4.4)–(4.7) in Case 1 (resp. (4.73)–(4.76) in Case 2) and by Theorems 4.2 and 4.8 relations (5.12) hold (for $f \in \mathcal{H}_0$).

Finally, one proves (5.12) for arbitrary $f \in \mathcal{H}$ in the same way as in [2, Theorem 6.2].

Remark 5.7. Theorem 5.6 is a generalization of similar results obtained in [7, 16, 25] for Hamiltonian systems (i.e., for systems (3.3) with $\hat{H} = \{0\}$). Moreover, for non-Hamiltonian systems (3.3) in the case of minimally possible deficiency indices $n_{\pm}(T_{min}) = \nu_{\pm}$ formulas (5.9) and (5.10) were proved in [17].

5.3. Spectral functions and the Fourier transform. Let $T$ be a symmetric relation in $\mathcal{H}$ defined by (3.35) in Case 1 and (3.39) in Case 2 and let $\tau = \{\tau_+, \tau_\pm\}$ be a boundary parameter given by (4.1) in Case 1 and (4.70) in Case 2. According to Theorems 4.2 and 4.8 the boundary value problems (4.4)–(4.7) and (4.73)–(4.76) establish a bijective correspondence between boundary parameters $\tau$ and generalized resolvents $R(\cdot) = R_\tau(\cdot)$ of
T. Denote by $\tilde{T}^\tau$ the (f-minimal) exit space self-adjoint extension of $T$ in the Hilbert space $\tilde{H}_0 \supset \tilde{H}$ generating $R_\tau(\cdot)$:

$$R_\tau(\lambda) = P_\delta(\tilde{T}^\tau - \lambda)^{-1} |_{\tilde{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (5.21)$$

Clearly, formula (5.21) gives a parametrization of all such extensions $\tilde{T} = \tilde{T}^\tau$ by means of a boundary parameter $\tau$. Denote by $F_\tau(\cdot)$ the spectral function of $T$ corresponding to $\tilde{T}^\tau$.

In the following we assume that a certain $J$-unitary extension $\tilde{U}$ of $U$ is fixed and hence the $m$-function $m_\tau(\cdot)$ is defined (see (5.1)). Note that in view of the assertion just after (5.2) a choice of $U$ does not matter in our further considerations.

For each $f \in \tilde{H}_0$ introduce the Fourier transform $f(\cdot) : \mathbb{R} \to H_0$ by setting

$$\hat{f}(s) = \int_{\mathbb{R}} \varphi^*_\nu(t, s) \Delta(t) f(t) \, dt, \quad f \in \tilde{f}. \quad (5.22)$$

**Definition 5.8.** A distribution function $\Sigma_\tau(\cdot) : \mathbb{R} \to [H_0]$ is called a spectral function of the boundary value problem (4.4)–(4.7) in Case 1 (resp. (4.73)–(4.76) in Case 2) if, for each $f \in \tilde{H}_0$ and for each finite interval $[\alpha, \beta) \subset \mathbb{R}$, the Fourier transform (5.22) satisfies the equality

$$((F_\tau(\beta) - F_\tau(\alpha)) \hat{f}, \hat{f})_R = \int_{[\alpha, \beta)} (d\Sigma_\tau(s) \hat{f}(s), \hat{f}(s)). \quad (5.23)$$

In the following "the boundary value problem" means either the boundary value problem (4.4)–(4.7) (in Case 1) or the boundary value problem (4.73)–(4.76) (in Case 2).

Let $\Sigma_\tau(\cdot)$ be the spectral function of the boundary value problem. Then in view of (5.23) $\hat{f} \in L^2(\Sigma_\tau; H_0)$ and the same reasonings as in [2] give the Bessel inequality $‖\hat{f}‖_{L^2(\Sigma_\tau; H_0)} \leq ‖\hat{f}‖_{\tilde{H}}$ (for the Hilbert space $L^2(\Sigma; \mathcal{H})$ see Subsection 2.2). Therefore for each $\tilde{f} \in \tilde{H}_0$ there exists a function $\tilde{f} \in L^2(\Sigma_\tau; H_0)$ (the Fourier transform of $\tilde{f}$) such that

$$\lim_{\beta \to 0} ‖\hat{f} - \int_{\alpha}^{\beta} \varphi^*_\nu(t, \cdot) \Delta(t) f(t) \, dt‖_{L^2(\Sigma_\tau; H_0)} = 0, \quad f \in \tilde{f},$$

and the equality $Vf = \hat{f}, \quad \tilde{f} \in \tilde{H}_0$ defines the contraction $V : \tilde{H}_0 \to L^2(\Sigma_\tau; H_0)$.

**Theorem 5.9.** For each boundary parameter $\tau$ there exists a unique spectral function $\Sigma_\tau(\cdot)$ of the boundary value problem. This function is defined by the Stieltjes inversion formula

$$\Sigma_\tau(s) = - \lim_{\delta \to +0} \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \text{Im} m_\tau(\sigma - i\varepsilon) \, d\sigma. \quad (5.24)$$

If in addition $\tau$ is a truncated boundary parameter, then the corresponding spectral function $\Sigma_\tau(\cdot)$ has the block matrix representation

$$\Sigma_\tau(s) = \begin{pmatrix} \Sigma_{\tau,1}(s) & \Sigma_{\tau,2}(s) \\ \Sigma_{\tau,3}(s) & \frac{s}{\pi} s I \end{pmatrix}, \quad s \in \mathbb{R} \quad (5.25)$$

with respect to the decomposition $H_0 = H_0^1 \oplus \tilde{H}_2$ in Case 1 (resp. $H_0 = H \oplus \tilde{H}$ in Case 2).

In (5.25) $\Sigma_{\tau,1}(\cdot)$ is an $[H_0^1]$-valued (resp. $[H]$-valued) distribution function defined by

$$\Sigma_{\tau,1}(s) = - \lim_{\delta \to +0} \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \text{Im} m_{\tau,1}(\sigma - i\varepsilon) \, d\sigma \quad (5.26)$$

with $m_{\tau,1}(\cdot)$ taken from (5.5).
Proof. In the case of an arbitrary boundary parameter \( \tau \) one proves the required statement by using Theorem 5.6 and the Stieltjes-Livščic inversion formula in the same way as in [2, Theorem 6.5]. The statements of the theorem for a truncated boundary parameter \( \tau \) are implied by (5.24) and the block matrix representation (5.5) of \( m_\tau(\cdot) \).

Let \( \mathcal{H} \) be decomposed as

\[
\mathcal{H} = \mathcal{H}_0 \oplus \text{mul } T
\]

and let \( T' \) be the operator part of \( T \). Moreover, let \( \tau \) be a boundary parameter, let \( \tilde{T}^\tau = (\tilde{T}^\tau)^* \) be an exit space extension of \( T \) in the Hilbert space \( \tilde{\mathcal{H}} \supset \mathcal{H} \), let \( \mathcal{H} \) be decomposed as

\[
\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 \oplus \text{mul } \tilde{T}^\tau
\]

and let \( T^\tau \) be the operator part of \( \tilde{T}^\tau \) (so that \( T^\tau \) is a self-adjoint operator in \( \tilde{\mathcal{H}}_0 \)). Assume also that \( \Sigma_\tau(\cdot) \) is a spectral function of the boundary value problem and \( V \in [\mathcal{H}, L^2(\Sigma_\tau; H_0)] \) is the Fourier transform. As in [2] one proves that \( V \upharpoonright \text{mul } T = 0 \). Therefore by (5.27) \( \mathcal{H}_0 \) is the maximally possible subspace of \( \mathcal{H} \) on which \( V \) may be isometric.

**Definition 5.10.** A spectral function \( \Sigma_\tau(\cdot) \) is referred to the class \( SF_0 \) if the operator \( V_0 := V \upharpoonright \mathcal{H}_0 \) is a unitary isometry from \( \mathcal{H}_0 \) to \( L^2(\Sigma_\tau; H_0) \).

Similarly to [2] the following equivalence is valid:

\[
\Sigma_\tau(\cdot) \in SF_0 \iff \text{mul } \tilde{T}^\tau = \text{mul } T.
\]

Hence all spectral functions \( \Sigma_\tau(\cdot) \) of the boundary value problem belong to \( SF_0 \) if and only if \( \text{mul } T = \text{mul } T^* \) or, equivalently, if and only if the operator \( T^\tau \) is densely defined.

As in [2] one proves the following statement: if \( \Sigma_\tau(\cdot) \in SF_0 \), then for each \( \bar{f} \in \tilde{\mathcal{H}}_0 \) the inverse Fourier transform is

\[
\bar{f} = \pi \left( \int_\mathbb{R} \varphi_U(\cdot, s) \, d\Sigma_\tau(s) \bar{f}(s) \right),
\]

where the integral converges in the semi-norm of \( L^2_{\Delta}(\mathcal{I}) \).

The following theorem can be proved in the same way as Theorem 6.9 in [2].

**Theorem 5.11.** Let \( \tau \) be a boundary parameter such that \( \Sigma_\tau(\cdot) \in SF_0 \) and let \( V \) be the corresponding Fourier transform. Then \( \mathcal{H}_0 \subset \tilde{\mathcal{H}}_0 \) and there exists a unitary extension \( V \in [\tilde{\mathcal{H}}, L^2(\Sigma_\tau; H_0)] \) of the isometry \( V_0 := V \upharpoonright \mathcal{H}_0 \) such that the operator \( T^\tau \) and the multiplication operator \( \Lambda = \Lambda_{\Sigma_\tau} \) in \( L^2(\Sigma_\tau; H_0) \) are unitarily equivalent by means of \( \tilde{V} \).

Moreover, if \( \text{mul } T = \text{mul } T^* \), then the statements of the theorem hold for arbitrary boundary parameter \( \tau \) (that is, for any spectral function \( \Sigma_\tau(\cdot) \)).

Since in the case of a densely defined \( T \) one has \( \text{mul } T = \text{mul } T^* = \{0\} \), the following theorem is implied by Theorem 5.11.

**Theorem 5.12.** If \( T \) is a densely defined operator, then for each boundary parameter \( \tau \) and the corresponding spectral function \( \Sigma_\tau(\cdot) \) the following hold: (i) \( \tilde{T}^\tau \) is an operator, that is, \( \tilde{T}^\tau = T^\tau \); (ii) the Fourier transform \( V \) is an isometry; (iii) there exists a unitary extension \( V \in [\tilde{\mathcal{H}}, L^2(\Sigma_\tau; H_0)] \) of \( V \) such that \( \tilde{T}^\tau \) and \( T_\Delta \) are unitarily equivalent by means of \( \tilde{V} \).

Since \( n_+(T_{\min}) \neq n_-(T_{\min}) \), the equality \( \sigma(T^\tau) = \mathbb{R} \) holds for any boundary parameter \( \tau \). Moreover, Theorem 5.11 yields the following corollary.
Corollary 5.13. (1) If \( \tau \) is a boundary parameter such that \( \Sigma_\tau(\cdot) \in SF_0 \), then the spectral multiplicity of the operator \( T^* \) does not exceed \( \mu = \dim H_0 \).

(2) If \( \text{mul} \, T = \text{mul} \, T^* \), then the above statement about the spectral multiplicity of \( T^* \) holds for any boundary parameter \( \tau \).

In the case of the truncated boundary parameter \( \tau \) there is a somewhat more information about the spectrum of \( T^* \). Namely, the following theorem is valid.

Theorem 5.14. (1) Let \( \tau \) be a truncated boundary parameter such that \( \Sigma_\tau(\cdot) \in SF_0 \) and let \( \Sigma_{\tau,1}(\cdot) \) be a distribution function given by the block matrix representation (5.25) of \( \Sigma_\tau(\cdot) \). Moreover, let \( E(\cdot) \) be the orthogonal spectral measure of the operator \( T^* \) and let \( E_s(\cdot) \) be the singular part of \( E(\cdot) \). Then

\[
\sigma_{ac}(T^*) = \mathbb{R}, \quad \sigma_s(T^*) = S_s(\Sigma_{\tau,1})
\]

and the multiplicity of the orthogonal spectral measure \( E_s(\cdot) \) does not exceed \( \nu_+ + \nu_{b+} - \nu_{b-} = \dim H_0 \) in Case 1 and \( \nu_+ = \dim H \) in Case 2 (this implies that the same estimates are valid for multiplicity of each eigenvalue \( \lambda_0 \) of \( T^* \)).

(2) If \( \text{mul} \, T = \text{mul} \, T^* \), then the above statements about the spectrum of \( T^* \) hold for each truncated boundary parameter \( \tau \).

Proof. (1) let \( \Sigma_{\tau,ac}(\cdot), \Sigma_{\tau,s}(\cdot) \) and \( \Sigma_{\tau,1,ac}(\cdot), \Sigma_{\tau,1,s}(\cdot) \) be absolutely continuous and singular parts of \( \Sigma_\tau(\cdot) \) and \( \Sigma_{\tau,1}(\cdot) \) respectively. Then in view of (5.25) the block matrix representations

\[
\Sigma_{\tau,ac}(s) = \begin{pmatrix} \Sigma_{\tau,1,ac}(s) & \Sigma_{\tau,2}(s) \\ \Sigma_{\tau,3}(s) & D \end{pmatrix}, \quad \Sigma_{\tau,s}(s) = \begin{pmatrix} \Sigma_{\tau,1,s}(s) & 0 \\ 0 & 0 \end{pmatrix}
\]

hold with respect to the same decompositions of \( H_0 \) as in Theorem 5.9. This and Theorems 5.11 and 2.2 give the required statements about the spectrum of \( T^* \).

The statement (2) of the theorem follows from the fact that in the case \( \text{mul} \, T = \text{mul} \, T^* \) the inclusion \( \Sigma_\tau(\cdot) \in SF_0 \) holds for any boundary parameter \( \tau \) and hence for any truncated boundary parameter \( \tau \).

In the next theorem we give a parametrization of all spectral functions \( \Sigma_\tau(\cdot) \) immediately in terms of the boundary parameter \( \tau \).

Theorem 5.15. Let \( X_- (\cdot) \) be the operator matrix (4.29) in Case 1 (resp. (4.92) in Case 2). Then for each boundary parameter \( \tau = \{ \tau_+, \tau_- \} \) of the form (4.1) in Case 1 (resp. (4.70) in Case 2) the equality

\[
m_\tau(\lambda) = m_0(\lambda) + \Phi_-(\lambda)(D_0(\lambda) - D_1(\lambda)\hat{M}_-(\lambda))^{-1}D_1(\lambda)\Psi_-(\lambda), \quad \lambda \in \mathbb{C}_-
\]

together with (5.24) defines a (unique) spectral function \( \Sigma_\tau(\cdot) \) of the boundary value problem. If in addition \( \tau = \{ \tau_+, \tau_- \} \) is a truncated boundary parameter defined by (4.1), (4.2) in Case 1 and (4.71), (4.70) in Case 2, then the distribution function \( F_\tau(\cdot) \) in (5.25) is defined by (5.6) and (5.26). Moreover, the following statements are valid:

(1) Let in Case 1 \( \tilde{C}_{02}(\lambda), \tilde{C}_{0b}(\lambda) \) and \( M_{4+}(\lambda), N_{2+}(\lambda) \) be defined by (4.3) and (4.40) and let \( P_{\mathcal{H}_b} \) be the orthoprojector in \( \tilde{H}_2 \oplus \mathcal{H}_b \) onto \( \mathcal{H}_b \). Then there exist the limits

\[
\mathcal{B}_\tau := \lim_{y \to +\infty} \frac{1}{iy} (C_{0b}(iy) - C_1(iy)M_{4+}(iy) + i\tilde{C}_{02}(iy)N_{2+}(iy))^{-1}C_1(iy) = \\
= \lim_{y \to +\infty} \frac{1}{iy} P_{\mathcal{H}_b}(D_0(iy) - D_1(iy)\hat{M}_-(iy))^{-1}D_1(iy)
\]

\[
\tilde{\mathcal{B}}_\tau := \lim_{y \to +\infty} \frac{1}{iy} M_{4+}(iy)(C_{0b}(iy) - C_1(iy)M_{4+}(iy) + i\tilde{C}_{02}(iy)N_{2+}(iy))^{-1}C_0(iy)
\]
and the following equivalence holds:

\[(5.32)\quad \Sigma_\tau(\cdot) \in SF_0 \iff \mathcal{B}_\tau = \tilde{\mathcal{B}}_\tau = 0\]

(2) Let in Case 2 \(C_0(\lambda)\) and \(M_+(\lambda)\) has the block representations (cf. \(4.72\) and \(4.40\))

\[
C_0(\lambda) = (C_{01}(\lambda), C_{02}(\lambda)) : (\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}_b) \oplus \mathcal{H}_b \to \mathcal{H}_b, \quad \lambda \in \mathbb{C}_+ \\
M_+(\lambda) = (N_{2+}(\lambda), N_{4+}(\lambda), M_4(\lambda)) : \mathcal{H}_b \to \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}_b, \quad \lambda \in \mathbb{C}_+
\]

and let \(N_+(\lambda) = (N_{2+}(\lambda), N_{4+}(\lambda)) \in [\mathcal{H}_b, \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}_b], \lambda \in \mathbb{C}_+ \). Then there exist limits

\[
\mathcal{B}_\tau := \lim_{y \to +\infty} \frac{1}{iy} (C_{02}(iy) - C_1(iy)M_4(iy) + iC_{02}(iy)N_+(iy))^{-1}C_1(iy) = \\
= \lim_{y \to -\infty} \frac{1}{iy} P_{\mathcal{H}_b}(D_0(iy) - D_1(iy)M_-(iy))^{-1}D_1(iy) \\
\tilde{\mathcal{B}}_\tau := \lim_{y \to +\infty} \frac{1}{iy} M_4(iy)(C_{02}(iy) - C_1(iy)M_4(iy) + iC_{02}(iy)N_+(iy))^{-1}C_{02}(iy)
\]

and the equivalence \((5.32)\) holds.

(3) Each spectral function \(\Sigma_\tau(\cdot)\) belongs to \(SF_0\) if and only if \(\lim_{y \to +\infty} \frac{1}{iy} M_4(iy) = 0\) in Case 1 (resp. \(\lim_{y \to +\infty} \frac{1}{iy} M_4(iy) = 0\) in Case 2) and

\[
\lim_{y \to -\infty} y(\text{Im}(M_-(iy))b, h) - \frac{1}{2}||P_2h||^2) = +\infty, \quad h \neq 0,
\]

where \(h \in \tilde{\mathcal{H}}_2 \oplus \mathcal{H}_b\) and \(P_2\) is the orthoprojector in \(\tilde{\mathcal{H}}_2 \oplus \mathcal{H}_b\) onto \(\tilde{\mathcal{H}}_2\) in Case 1 (resp. \(h \in H_2 \oplus \tilde{\mathcal{H}}_b\) and \(P_2\) is the orthoprojector in \(H_2 \oplus \tilde{\mathcal{H}}_b\) onto \(H_2 \oplus \tilde{\mathcal{H}}_b\) in Case 2).

**Proof.** The main statement of the theorem directly follows from Theorems 5.3 and 5.9. Next, consider the boundary triplet \(\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_b, \tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_1\}\) for \(T^*\) defined in Propositions 3.7 and 3.8. Since the Weyl functions \(M_\pm(\cdot)\) of the decomposing boundary triplet \(\Pi_-\) for \(T_{\text{max}}\) have the block representations \((4.34), (4.35)\) in Case 1 and \((4.94), (4.95)\) in Case 2, it follows from \((4.40)\) and Proposition 2.7, (3) that the Weyl functions of the triplet \(\Pi_-\) are \(\tilde{M}_+(\lambda)\) and \(\tilde{M}_-(\lambda)\). Now applying to the boundary triplet \(\Pi_-\). Theorems 4.12 and 4.13 from [35] and taking equivalence \((5.29)\) into account one obtains statements (1)–(3) of the theorem.

5.4. **The case of minimal deficiency indices.** It follows from \((3.22)\) that for a given system \((3.3)\) minimally possible deficiency indices of the linear relation \(T_{\text{min}}\) are

\[(5.33)\quad n_+(T_{\text{min}}) = \nu_+, \quad n_-(T_{\text{min}}) = \nu_-
\]

and the first (second) equality in \((5.33)\) holds if and only if \(\nu_{b+} = 0\) (resp. \(\nu_{b-} = 0\)). If \(n_+(T_{\text{min}}) = \nu_+\), then by \((3.22)\) one has \(n_+(T_{\text{min}}) \leq n_-(T_{\text{min}})\); moreover, \(n_+(T_{\text{min}}) < n_-(T_{\text{min}})\) if in addition \(\nu > 0\) (\(\Leftrightarrow \tilde{H} \neq \{0\}\)).

In the case of the minimal deficiency index \(n_+(T_{\text{min}})\) the above results can be somewhat simplified. Namely, assume the first equality in \((5.33)\) and let \(n_+(T_{\text{min}}) < n_-(T_{\text{min}})\). Then the equality \(\nu_{b+} = 0\) implies Case 2. Moreover, Lemma 3.4 gives \(\mathcal{H}_b = \{0\}\) and hence \(\tilde{\Gamma}_b = \Gamma_{1b} = 0\). Therefore \(\tilde{\mathcal{H}}_b = \tilde{\mathcal{H}}_b\) and \((3.31)\) yields \(\tilde{\mathcal{H}}_b = \tilde{\mathcal{H}}_b\).

If the assumption \((A1)\) from Subsection 4.1 is satisfied, then by Proposition 3.8 the equality

\[(5.34)\quad T = \{[\tilde{y}, 0] \in T_{\text{max}} : \Gamma_{1a}y = 0, \tilde{\Gamma}_ay = 0, \tilde{\Gamma}_by = 0\}.\]
defines a maximal symmetric relation $T$ in $\mathcal{D}(= L^2_\Delta(I))$ with the deficiency indices $n_+(T) = 0$ and $n_-(T) = \bar{\nu} + \bar{\nu}_b$. Moreover, in this case $T$ coincides with the relation $A_0$ defined by (3.42). Hence $T$ has a unique generalized resolvent $R(\lambda)$ of the form (4.9), which in view of Theorem 4.8 and Remark 4.9 is given by the boundary value problem

\begin{equation}
(5.35) \quad Jy' - B(t)y = \lambda \Delta(t)y + \Delta(t)f(t), \quad t \in I,
\end{equation}

\begin{equation}
(5.36) \quad \Gamma_1 a y = 0, \quad \lambda \in \mathbb{C}^+; \quad \Gamma_1 a y = 0, \quad \hat{\Gamma}_a y = 0, \quad \lambda \in \mathbb{C}^-.
\end{equation}

If $\hat{U}$ is a $J$-unitary extension (3.13) of $U$, then according to Proposition 5.2 the $m$-function $m(\cdot)$ of the problem (5.35), (5.36) is given by the relations

\begin{equation}
v(t, \lambda) := \varphi_U(t, \lambda)m(\lambda) + \psi(t, \lambda) \in L^2_\Delta[H_0, \mathbb{H}], \quad \lambda \in \mathbb{C} \setminus \mathbb{R}
\end{equation}

and $\hat{\Gamma}_a v(\lambda) = P_{\hat{H}}$, $\hat{\Gamma}_b v(\lambda) = 0$, $\lambda \in \mathbb{C}^-$. 

Since $\mathcal{H}_b = \{0\}$, the decomposing boundary triplet (3.32)–(3.34) for $T_{\text{max}}$ takes the form $\Pi_- = \{H_0 \oplus H, \Gamma_0, \Gamma_1\}$, where

\begin{equation}
\mathcal{H}_0 = H \oplus \hat{H} \oplus \hat{\mathcal{H}}_b = H_0 \oplus \hat{\mathcal{H}}_b,
\end{equation}

\begin{equation}
\Gamma_0 \{\vec{y}, \vec{f}\} = \{-\Gamma_1 a y, \hat{\Gamma}_a y, \hat{\Gamma}_b y\} \in H \oplus \hat{H} \oplus \hat{\mathcal{H}}_b, \quad \Gamma_1 \{\vec{y}, \vec{f}\} = \Gamma_0 a y, \quad \{\vec{y}, \vec{f}\} \in T_{\text{max}}.
\end{equation}

Assume that the Weyl function $M_- (\cdot)$ of $\Pi_-$ has the block matrix representation (cf. (4.95))

\begin{equation}
(5.37) \quad M_- (\lambda) = (M(\lambda), N_- (\lambda), M_- (\lambda)) : H \oplus \hat{H} \oplus \hat{\mathcal{H}}_b \to H, \quad \lambda \in \mathbb{C}^-.
\end{equation}

According to Theorem 5.3 $m(\lambda) = m_0 (\lambda)$ and by (4.97) one has

\begin{equation}
(5.38) \quad m(\lambda) = \begin{pmatrix} M(\lambda) & N_- (\lambda) \\ 0 & -\frac{1}{2} I_{\hat{H}} \end{pmatrix} : H \oplus \hat{H} \Rightarrow H \oplus \hat{H}, \quad \lambda \in \mathbb{C}^-.
\end{equation}

Let $\mathcal{M}(\cdot)$ be the operator function (2.10), (2.11) corresponding to the decomposing boundary triplet $\Pi_-$. Then by (5.37) and (5.38)

\begin{equation}
(5.39) \quad \mathcal{M}(\lambda) = \begin{pmatrix} m(\lambda) & \mathcal{M}_1 (\lambda) \\ 0 & -\frac{1}{2} I_{\hat{H}} \end{pmatrix} : H_0 \oplus \hat{\mathcal{H}}_b \Rightarrow H_0 \oplus \hat{\mathcal{H}}_b, \quad \lambda \in \mathbb{C}^-,
\end{equation}

where $\mathcal{M}_1 (\lambda) = (M_- (\lambda), 0)^T (\in [\hat{\mathcal{H}}_b, H \oplus \hat{H}])$. Using (5.39) and (2.12) one can show that

\begin{equation}
m(\mu) - m^*(\lambda) = (\mu - \lambda) \int_I v^*(t, \lambda) \Delta(t)v(t, \mu) dt, \quad \mu, \lambda \in \mathbb{C}^-.
\end{equation}

The spectral function $\Sigma(\cdot)$ of the problem (5.35), (5.36) has the block matrix representation

\begin{equation}
\Sigma(s) = \begin{pmatrix} F(s) & \Sigma_4 (s) \\ \Sigma_2 (s) & \Sigma_3 (s) \end{pmatrix} : H \oplus \hat{H} \Rightarrow H \oplus \hat{H}, \quad s \in \mathbb{R},
\end{equation}

where $F(s)$ is an $[H]$-valued distribution function defined by the Stieltjes formula (5.26) with $M(\lambda)$ in place of $m_{\tau, 1}(\lambda)$. Moreover, since $T$ is maximal symmetric, it follows that $\text{mul} T = \text{mul} T^*$ and, consequently, $\Sigma(\cdot) \in SF_0$. Hence the operator part $T_0^* \equiv (\text{the (unique) exit space self-adjoint extension}) T_0$ of $T$ satisfies statements of Theorem 5.14 with $T_0$ and $F(\cdot)$ in place of $T^*$ and $F(\cdot)$ respectively.

If in addition to the above assumptions $n_-(T_{\text{min}}) = \nu_-$ (i.e., both the deficiency indices $n_+ (T_{\text{min}})$ are minimal), then $\hat{\mathcal{H}}_b = \{0\}$ and $\hat{\Gamma}_b = 0$. This implies the corresponding modification of the results of this subsection. In particular, formula (5.39) takes the form $m(\lambda) = \mathcal{M}(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}$.
6. Differential operators of an odd order

In this section we apply the above results to ordinary differential operators of an odd order on an interval \( I = [a, b] \) \((-\infty < a < b \leq \infty\) with the regular endpoint \( a \).

Assume that \( H \) is a finite dimensional Hilbert space and

\[
l[y] = \sum_{k=0}^{m} (-1)^k \left( \frac{i}{2}[(q_n-k y^{(k)})^{(k+1)} + (q_n-k y^{(k+1)})^{(k)}] + (p_n-k y^{(k)})^{(k)} \right)
\]

is a differential expression of an odd order \( 2m+1 \) with sufficiently smooth operator valued coefficients \( p_k(\cdot), q_k(\cdot) : I \to [H] \) such that \( p_k(t) = p^*_k(t) \), \( q_k(t) = q^*_k(t) \) and \( 0 \in \rho(q_0(t)) \).

Denote by \( y^{[k]}(\cdot), k \in \{0, \ldots, 2m+1\} \), the quasi-derivatives of \( y \in AC(I; H) \) and let \( dom l \) be the set of all functions \( y \in AC(I; H) \) for which \( l[y] = y^{[2m+1]} \) makes sense [37, 24].

As is known \( H = H^+_t \oplus H^-_t \), where \( H^+_t \) (\( H^-_t \)) is an invariant subspace of the operator \( q_0(t) \), on which \( q_0(t) \) is strictly positive (resp. negative). We put

\[
\nu_{0+} = \dim H^+_t, \quad \nu_{0-} = \dim H^-_t
\]

(this numbers does not depend on \( t \); moreover, we assume for definiteness that \( \nu_{0-} \leq \nu_{0+} \)).

By using formula (1.27) in [24] one can easily show that there exist finite dimensional Hilbert spaces \( H' \) and \( \tilde{H} \) and an absolutely continuous operator function

\[
Q(t) = (Q_1(t), \tilde{Q}(t), Q_2(t))^T : H \to H' \oplus \tilde{H} \oplus H', \quad t \in I,
\]

such that \( 0 \in \rho(Q(t)) \) and the following holds:

\[
iq(t) = -Q_1(t)Q_2(t) + Q_2(t)Q_1(t) + i\tilde{Q}(t)\tilde{Q}(t), \quad t \in I
\]

\[
\dim H' = \nu_{0-}, \quad \dim \tilde{H} = \nu_{0+} - \nu_{0-}.
\]

Introduce the finite dimensional Hilbert spaces (cf. (3.1))

\[
H_0 = H \oplus \tilde{H} = H^m \oplus H' \oplus \tilde{H}, \quad \mathbb{H} = H_0 \oplus H = H \oplus \tilde{H} \oplus H
\]

Clearly, the space \( \mathbb{H} \) admits the representation

\[
\mathbb{H} = \underbrace{H \oplus \cdots \oplus H}_m \oplus \underbrace{H' \oplus \tilde{H} \oplus H}_m \oplus \underbrace{H \oplus \cdots \oplus H}_m
\]

For each function \( y \in \text{dom } l \) we let

\[
y_0(t) = \{y(t), \ldots, y^{[m-1]}(t), Q_1(t)y^{(m)}(t)\} \in H
\]

\[
y_1(t) = \{y^{[2m]}(t), \ldots, y^{[m+1]}(t), Q_2(t)y^{(m)}(t)\} \in H
\]

\[
y(t) = \{y_0(t), \tilde{Q}(t)y^{(m)}(t), y_1(t)\} \in H \oplus \tilde{H} \oplus H = \mathbb{H}.
\]

Let \( \mathcal{K} \) be a finite dimensional Hilbert space. For an operator valued solution \( Y(\cdot) : I \to [\mathcal{K}, H] \) of the equation

\[
l[y] = \lambda y \quad (\lambda \in \mathbb{C})
\]

we define the operator function \( Y(\cdot) : I \to [\mathcal{K}, \mathbb{H}] \) by the following relations: if \( h \in \mathcal{K} \) and \( Y(t)h = y(t) \), then \( Y(t)h = y(t) \).
Next assume that $\mathcal{S} := L^2(\mathcal{I})$ is the Hilbert space of all Borel $H$-valued functions $f(\cdot)$ on $\mathcal{I}$ satisfying $\int \|f(t)\|^2 dt < \infty$. Denote also by $L^2[\mathcal{K}, H]$ the set of all operator functions $Y(\cdot) : \mathcal{I} \to [\mathcal{K}, H]$ such that $Y(t)h \in \mathcal{S}$, $h \in \mathcal{K}$. Moreover, by using (6.4) we associate with a function $f(\cdot) \in \mathcal{S}$ the $\tilde{H}$-valued functions $\hat{f}(\cdot)$ on $\mathcal{I}$ given by $\hat{f}(t) = \{f(t), 0, \ldots, 0\}, t \in \mathcal{I}$.

According to $[37]$ expression (6.1) induces in $\mathcal{S}$ the maximal operator $L_{\text{max}}$ and the minimal operator $L_{\text{min}}$. Moreover, $L_{\text{min}}$ is a closed densely defined symmetric operator and $L_{\text{min}}^* = L_{\text{max}}$.

It turns out that the expression $[\gamma]$ is equivalent in fact to a certain symmetric system. More precisely, the following proposition is implied by the results of $[24]$.

**Proposition 6.1.** Let $[\gamma]$ be the expression (6.1) and let

$$J = \begin{pmatrix} 0 & 0 & -\hat{I}_H \\ 0 & i\hat{I}_H & 0 \\ \hat{I}_H & 0 & 0 \end{pmatrix} \in [H \oplus \hat{H} \oplus H], \quad \Delta(t) = \begin{pmatrix} P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in [H \oplus \hat{H} \oplus H],$$

where $P$ is the orthoprojector in $H$ onto the first term in the right hand side of (6.3). Then there exists a continuous operator function $B(t) = B^*(t)([H])$, $t \in \mathcal{I}$, (defined in terms of $p_j$ and $q_j$) such that the first-order symmetric system

$$Jy'(t) - B(t)y(t) = \Delta(t)f(t), \quad t \in \mathcal{I}$$

and the corresponding homogeneous system

$$(6.10) \quad Jy'(t) - B(t)y(t) = \lambda \Delta(t)y(t), \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}$$

possess the following properties:

1. There is a bijective correspondence $Y(\cdot, \lambda) \leftrightarrow \hat{Y}(\cdot, \lambda) = Y(\cdot, \lambda)$ between all $[\mathcal{K}, H]$-valued operator solutions $Y(\cdot, \lambda)$ of Eq. (6.8) and all $[\mathcal{K}, \hat{H}]$-valued operator solutions $\hat{Y}(\cdot, \lambda)$ of the system (6.10). Moreover, $Y(\cdot, \lambda) \in \mathcal{L}^2[\mathcal{K}, H]$ if and only if $Y(\cdot, \lambda) \in \mathcal{L}^2[\mathcal{K}, \hat{H}]$.

2. Let $T_{\text{max}}$ be the maximal linear relation in $L^2(\mathcal{I})$ induced by the system (6.9). Then the equality $(V_1y)(t) = y(t)$, $y(\cdot) \in \text{dom} L_{\text{max}}$, defines a bijective linear mapping $V_1$ from $\text{dom} L_{\text{max}}$ onto $\text{dom} T_{\text{max}}$ satisfying $(V_1y, V_1z)_{\Delta} = (y, z)_{\Delta'}$, $y, z \in \text{dom} L_{\text{max}}$.

3. Let $T_{\text{min}}$ and $T_{\text{max}}$ be minimal and maximal relations in $\mathcal{S} := L^2(\mathcal{I})$ induced by system (6.9). Then $T_{\text{min}}$ is a densely defined operator and the equality $V_2 f = \pi(\hat{f}(\cdot))$, $f = f(\cdot) \in \mathcal{S}$, defines a unitary operator $V_2 : \text{dom} T_{\text{min}} \to \text{dom} T_{\text{max}}$ such that $(V_2 \oplus V_2)(\text{gr} T_{\text{min}}) = \text{gr} T_{\text{max}}$ and $(V_2 \oplus V_2)(\text{gr} T_{\text{max}}) = \text{gr} T_{\text{min}}$ (i.e., the operators $L_{\text{min}}$ and $T_{\text{min}}$ as well as $L_{\text{max}}$ and $T_{\text{max}}$ are unitarily equivalent by means of $V_2$).

By using Proposition 6.1 one can easily translate all the results of [2] and the present paper to the expression (6.1). Below we specify only the basic points in this direction.

Let $U \in [\hat{H}]$ be a $J$-unitary operator given by (3.13) with $H$ and $\hat{H}$ in place of $H$ and $\hat{H}$ respectively. Using (6.5)–(6.7) introduce the linear mappings $\Gamma_{ja} : \text{dom} l \to H$, $j \in \{0, 1\}$, and $\tilde{\Gamma}_a : \text{dom} l \to \hat{H}$ by setting

$$(6.11) \quad \Gamma_{0a} y = u_7 y_0(a) + u_8 \hat{Q}(a)y^{(m)}(a) + u_9 y_1(a), \quad y \in \text{dom} l$$

$$(6.12) \quad \tilde{\Gamma}_a y = u_4 y_0(a) + u_5 \hat{Q}(a)y^{(m)}(a) + u_6 y_1(a),$$

Next, assume that $\nu_+$ and $\nu_-$ are indices of inertia of the bilinear form

$$[y, z]_b := \lim_{t \uparrow b}(Jy(t), z(t)), \quad y, z \in \text{dom} L_{\text{max}}.$$
Combining (3.22) with (6.3) and (6.2) and taking Proposition 6.1, (3) into account one gets the following equality for deficiency indices \( d_+ = n_+(L_{\text{min}}) \) of the operator \( L_{\text{min}} \):

\[
(6.13) \quad d_+ = m \cdot \dim H + \nu_{0-} + \nu_{b+}, \quad d_- = m \cdot \dim H + \nu_{0+} + \nu_{b-}.
\]

Therefore \( d_+ < d_- \) if and only if one of the following two alternative cases holds:

- **Case 1.** \( \nu_{0+} - \nu_{0-} < \nu_{b+} - \nu_{b-} \).
- **Case 2.** \( \nu_{0+} - \nu_{0-} \geq 0 \geq \nu_{b+} - \nu_{b-} \) and \( \nu_{0+} - \nu_{0-} \neq \nu_{b+} - \nu_{b-} (\neq 0) \).

Proposition 6.1, (2) enables one to identify \( \text{dom} L_{\text{max}} \) and \( \text{dom} \Gamma_{\text{max}} \). Therefore in the case \( d_- \leq d_+ \) we may assume that the linear mapping \( \Gamma_b \) in [2, Lemma 3.4] is of the form

\[
(6.14) \quad \Gamma_b = (\Gamma_{0b}, \tilde{\Gamma}_b, \Gamma_{1b})^\top : \text{dom} L_{\text{max}} \rightarrow \mathcal{H}_{0b} \oplus \tilde{\mathcal{H}} \oplus \mathcal{H}_{1b},
\]

where \( \mathcal{H}_{0b} \) is a finite dimensional Hilbert space and \( \mathcal{H}_{1b} \) is a subspace in \( \mathcal{H}_{0b} \). Similarly in the case \( d_+ < d_- \) we define on \( \text{dom} L_{\text{max}} \) the linear mappings \( \Gamma_{0b}, \Gamma_{1b}, \tilde{\Gamma}_b \) (Case 1) and \( \Gamma_{0b}, \Gamma_{1b} \) (Case 2) in the same way as in Subsection 3.3.

In the following we suppose that: (i) \( U \) is the operator (3.9) (with \( H \) and \( \tilde{H} \) in place of \( H \) and \( H \) satisfying (3.10)-(3.12); (ii) \( \tilde{\Gamma}_a \) and \( \Gamma_{1a} \) are the mappings (6.12). Then the same boundary conditions as in the right hand sides of (3.35), (3.39) and [2, (3.40)] give a symmetric operator \( T(\ominus L_{\text{min}}) \) in \( \mathcal{S}' \). Moreover, we define a boundary parameter \( \tau \) in the same way as for symmetric systems in [2, Definition 5.1] (the case \( d_- \leq d_+ \) and Definitions 4.1 and 4.7 (the case \( d_+ < d_- \)). A boundary parameter \( \tau \) gives a parametrization of all generalized resolvents \( R_\tau(\cdot) \) and, consequently, all spectral functions \( F_\tau(\cdot) \) of \( T \). Such a parametrization is generated by means of a boundary value problem involving the equation

\[
(6.15) \quad l[y] - \lambda y = f(t), \quad t \in \mathcal{I}
\]

and the boundary conditions [2, (4.2) and (4.3)] \( (d_- \leq d_+), (4.5)-(4.7) \) \( (d_+ < d_-), \text{Case 1} \) or \( (4.74)-(4.76) \) \( (d_+ < d_-), \text{Case 2} \). In the following "the boundary value problem for \( l[y] \)" means one of the listed above boundary value problems for the expression \( l[y] \).

If \( d_+ = d_- \), then in (6.14) \( \mathcal{H}_{0b} = \mathcal{H}_{1b} =: \mathcal{H}_b \) and \( R_\tau(\cdot) \) is a canonical resolvent of \( T \) if and only if \( \tau = \{(C_0, C_1); \mathcal{H}_b\} \) is a self-adjoint operator pair (this means that \( C_0, C_1 \subseteq \mathcal{H}_b \), \( \text{Im} C_1 C_0^* = 0 \) and \( 0 \in \rho(C_0 \pm i C_1) \)). In this case the corresponding boundary problem involves equation (6.15) and self-adjoint boundary conditions

\[
(6.16) \quad \Gamma_{1a} y = 0, \quad \tilde{\Gamma}_a y = \tilde{\Gamma}_b y, \quad C_0 \Gamma_{0b} y + C_1 \Gamma_{1b} y = 0,
\]

where \( \Gamma_{0b}, \Gamma_{1b} \) and \( \tilde{\Gamma}_b \) are taken from (6.14). Moreover, \( R_\tau(\lambda) = (T^\tau - \lambda)^{-1} \), where \( T^\tau = L_{\text{max}} \upharpoonright \text{dom} T^\tau \) is a self-adjoint extension of \( L_{\text{min}} \) with the domain

\[
(6.17) \quad \text{dom} T^\tau = \{ y \in \text{dom} L_{\text{max}} : \Gamma_{1a} y = 0, \tilde{\Gamma}_a y = \tilde{\Gamma}_b y, C_0 \Gamma_{0b} y + C_1 \Gamma_{1b} y = 0 \}.
\]

Next assume that \( \tilde{U} \in [H \oplus \tilde{H} \oplus H] \) is a \( J \)-unitary extension (3.13) of \( U \) and \( \Gamma_{0a} \) is the mapping (6.11).

**Definition 6.2.** The \( m \)-function of the expression \( l[y] \) corresponding to the boundary parameter \( \tau \) is the \( m \)-function \( m_\tau(\cdot) \) of the equivalent system (6.9).

Definition 6.2 means that \( m_\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow [\mathcal{H}_0] \) is a unique operator function such that for any \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the operator solution \( v_\tau(\cdot, \lambda) \) of Eq. (6.8) given by

\[
(6.18) \quad v_\tau(t, \lambda) := \varphi_U(t, \lambda)m_\tau(\lambda) + \psi(t, \lambda)
\]

belongs to \( L^2[\mathcal{H}_0, H] \) and satisfies the boundary conditions [2, (4.41)-(4.43)] \( (d_- \leq d_+), (4.57)-(4.59) \) \( (d_+ < d_-), \text{Case 1} \) or (4.100) and (4.101) \( (d_+ < d_-), \text{Case 2} \).
\[ \varphi_U(\cdot, \lambda) \in [H_0, H] \] and \( \psi(\cdot, \lambda) \in [H_0, H] \) are the operator solutions of Eq. (6.8) with the initial data

\[ \tilde{U}\varphi_U(a, \lambda) = \begin{pmatrix} I_{H_0} \\ 0 \end{pmatrix} : H_0 \to H_0 \oplus H, \quad \tilde{U}\psi(a, \lambda) = \begin{pmatrix} -P_H \Psi \Psi \psi(\cdot, \lambda) \\ -P_H \psi(\cdot, \lambda) \end{pmatrix} : H_0 \to H_0 \oplus H \]

(note that \( \varphi_U(\cdot, \lambda) \) does not depend on a choice of the extension \( \tilde{U} \supset U \)).

Let \( \mathcal{S}_0^\prime \) be the set of all functions \( f(\cdot) \in \mathcal{S}^\prime \) such that \( f(t) = 0 \) on some interval \( (\beta, b) \) (depending on \( f \)). For a function \( f(\cdot) \in \mathcal{S}_0^\prime \) the Fourier transform \( \tilde{f}(\cdot) : \mathbb{R} \to H_0 \) is

\[ \tilde{f}(s) = \int_I \varphi_U(t, s) f(t) \, dt. \quad (6.19) \]

**Definition 6.3.** Let \( \tau \) be a boundary parameter. A distribution function \( \Sigma_\tau(\cdot) : \mathbb{R} \to [H_0] \) is called a spectral function of the boundary value problem for \( l[y] \) if for each \( f(\cdot) \in \mathcal{S}_0^\prime \) the Fourier transform \( \tilde{f}(\cdot) : \mathbb{R} \to H_0 \) is

\[ \tilde{f}(s) = \int_I \varphi_U(t, s) f(t) \, dt. \]

Since \( L_{\min} \) is densely defined, formula (5.23) yields the Parseval equality \( ||\tilde{f}||_{L^2(\Sigma_\tau; H_0)} = ||f||_{\mathcal{S}^\prime} \). Therefore for each \( f \in \mathcal{S}^\prime \) there exists the Fourier transform (6.19) (the integral converges in the norm of \( L^2(\Sigma_\tau; H_0) \)) and the equality \( Vf = \tilde{f}, f \in \mathcal{S}^\prime \), define an isometry \( V : \mathcal{S}^\prime \to L^2(\Sigma_\tau; H_0) \).

In view of Proposition 6.1 each spectral function \( \Sigma_\tau(\cdot) \) of the expression \( l[y] \) is a spectral function of the equivalent symmetric system (6.9) and vice versa. Hence by [2, Theorem 6.5] and Theorem 5.9 for each boundary parameter \( \tau \) there exists a unique spectral function \( \Sigma_\tau(\cdot) \) of the boundary value problem for \( l[y] \) and this function is defined by the Stieltjes inversion formula (5.24). Moreover, for each \( f \in \mathcal{S}^\prime \) the inverse Fourier transform is

\[ f(t) = \int_I \frac{d}{d\tau}(\varphi_U(t, s) d\Sigma_s(s) \tilde{f}(s). \]

For a given boundary parameter \( \tau \) denote by \( T^\tau \) the \( (\mathcal{S}) \)-minimal exit space self-adjoint extension of \( T \) generating \( R_\tau(\lambda) \). Since \( T \) is densely defined, \( T^\tau \) is a self-adjoint operator in the Hilbert space \( \mathcal{S}^\prime \).

The following theorem is implied by Theorem 5.12 and [2, Theorem 6.11].

**Theorem 6.4.** Let \( \tau \) be a boundary parameter and let \( \Sigma_\tau(\cdot) \) and \( V \) be the corresponding spectral function and Fourier transform respectively. Then there exists a unitary extension \( V \in [\mathcal{S}, L^2(\Sigma_\tau; H_0)] \) of \( V \) such that \( T^\tau \) and the multiplication operator \( \Lambda \) in \( L^2(\Sigma_\tau; H_0) \) are unitarily equivalent by means of \( V \). Moreover, the following statements are equivalent:

1. \( d_+ = d_- \) and \( \tau = \{(C_0, C_1); H_0\} \) is a self-adjoint operator pair, so that \( T^\tau \) is the canonical self-adjoint extension (6.17) of \( T \);
2. \( V\mathcal{S}^\prime = L^2(\Sigma_\tau; H_0) \), that is the fourier transform \( V \) is a unitary operator.

If the statement (1) (and hence (2)) is valid, then the operators \( T^\tau \) and \( \Lambda \) are unitarily equivalent by means of \( V \).

Theorem 6.4 immediately implies that the spectral multiplicity of \( T^\tau \) does not exceed \( m \cdot \dim H + \nu_{0+} \). Similarly one can translate to the expression \( l[y] \) Theorem 5.14 and parametrization of all spectral functions \( \Sigma_\tau(\cdot) \) by means of (5.31), [35, (5.22)] and the Stieltjes inversion formula. It follows from (6.13) that for a given expression (6.1) minimally possible deficiency indices of the operator \( L_{\min} \) are

\[ n_+(L_{\min}) = m \cdot \dim H + \nu_{0-}, \quad n_-(L_{\min}) = m \cdot \dim H + \nu_{0+}. \]
The routine reformulation of the results of Subsection 5.4 to the case of the expression \( l[y] \) with minimal deficiency index \( n_{+}(L_{\text{min}}) \) of the operator \( L_{\text{min}} \) is left to the reader.

**Remark 6.5.** Let \( H = \mathbb{C} \), so that \( l[y] \) is a scalar expression with real valued coefficients \( p_{k}(\cdot) \) and \( q_{k}(\cdot) \). Then \( q_{0}(t) > 0 \), \( \hat{Q}(t) = q_{0}^{-1}(t) \), \( \hat{H} = \mathbb{C} \) and the equalities (6.3)–(6.7) take the form

\[
\begin{align*}
\mathbf{H} &= \mathbb{C}^{m}, \\
\mathbf{H}_{0} &= \mathbb{C}^{m} \oplus \mathbb{C} = \mathbb{C}^{m+1}, \\
\mathbf{H} &= \mathbb{C}^{m+1} \oplus \mathbb{C}^{m} = \mathbb{C}^{2m+1}.
\end{align*}
\]

\[ y_{0}(t) = \{y(t), \ldots, y^{[m-1]}(t)\}(\in \mathbb{C}^{m}), \quad y_{1}(t) = \{y^{[2m]}(t), \ldots, y^{[m+1]}(t)\}(\in \mathbb{C}^{m}), \]

\[ y(t) = \{y_{0}(t), \hat{q}_{0}^{-1}(t)\gamma^{(m)}(t), y_{1}(t)\}(\in \mathbb{C}^{2m+1}). \]

In this case \( \varphi_{U}(\cdot, \lambda) \) is the \((m + 1)\)-component operator solution

\[ \varphi_{U}(t, \lambda) = (\varphi_{1}(t, \lambda), \varphi_{2}(t, \lambda), \ldots, \varphi_{m}(t, \lambda), \varphi_{m+1}(t, \lambda)) : \mathbb{C}^{m+1} \to \mathbb{C} \]

of Eq. (6.8) with the initial data

\[ \hat{U}\varphi_{U}(a, \lambda) = \begin{pmatrix} I_{m+1} \\ 0 \end{pmatrix} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1} \oplus \mathbb{C}^{m} \]

and (6.19) defines the Fourier transform \( \hat{f}(\cdot) : \mathbb{R} \to \mathbb{C}^{m+1} \). Moreover, in a fixed basis of \( \mathbb{C}^{m+1} \) the \( m \)-function \( m_{\tau}(\cdot) \) and spectral function \( \Sigma_{\tau}(\cdot) \) can be represented as \((m + 1) \times (m + 1)\)-matrix valued functions \( m_{\tau}(\lambda) = (m_{ij}(\lambda))^{m+1}_{i,j=1} \) and \( \Sigma_{\tau}(\lambda) = (\Sigma_{ij}(\lambda))^{m+1}_{i,j=1} \) respectively.

In conclusion note that for a scalar expression (6.1) one has \( \nu_{0+} = 1 \) and \( \nu_{0-} = 0 \). Therefore for a scalar expression \( l[y] \) with \( n_{+}(L_{\text{min}}) < n_{-}(L_{\text{min}}) \) Case 1 is impossible. In other words, for such an expression either \( n_{-}(L_{\text{min}}) \leq n_{+}(L_{\text{min}}) \) or \( n_{+}(L_{\text{min}}) < n_{-}(L_{\text{min}}) \) and Case 2 holds.

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