AREA-MINIMIZING SURFACES IN ASYMPTOTICALLY FLAT THREE-MANIFOLDS

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Abstract. We show that an asymptotically flat Riemannian three-manifold with non-negative scalar curvature is isometric to flat $\mathbb{R}^3$ if it admits an unbounded area-minimizing surface. This answers a question of R. Schoen.

1. Introduction

A Riemannian three-manifold $(M, g)$ is asymptotically flat if it is connected and there is a coordinate chart at infinity

$$M \setminus U \cong \{ x \in \mathbb{R}^3 : |x| > 1 \}$$

where $U \subset M$ is bounded and open such that

$$g_{ij} = \delta_{ij} + O(|x|^{-p})$$

and

$$\partial_\ell g_{ij} = O(|x|^{-p-1}) \quad \text{and} \quad \partial_\ell \partial_k g_{ij} = O(|x|^{-p-2})$$

for some $p > 1/2$. We also require that the scalar curvature of $(M, g)$ is integrable and that the boundary of $M$ is a minimal surface in $(M, g)$. Given $\rho > 1$, we denote by $S_\rho$ the surface in $M$ corresponding to the coordinate sphere $\{ x \in \mathbb{R}^3 : |x| = \rho \}$, and by $B_\rho$ the bounded open subset of $M$ that is enclosed by $S_\rho$. The ADM-mass (after R. Arnowitt, S. Deser, and C. Misner [3]) of such $(M, g)$ is defined by

$$m_{\text{ADM}} = \lim_{\rho \to \infty} \frac{1}{16\pi} \int_{\{ |x| = \rho \}} \sum_{i,j=1}^3 (\partial_\ell g_{ij} - \partial_j g_{i\ell}) \frac{x^j}{\rho}$$

where the integral is with respect to the Euclidean metric. It was shown by R. Bartnik [4] that this definition is independent of the particular choice of the chart at infinity.

It would be hard to overstate the significance of the positive mass theorem as well as that of its proof using minimal surface theory due to R. Schoen and S.-T. Yau.

Theorem 1.1 (Positive mass theorem [25]). Let $(M, g)$ be an asymptotically flat Riemannian three-manifold with non-negative scalar curvature. Then $m_{\text{ADM}} \geq 0$. Equality holds if and only if $(M, g)$ is isometric to $\mathbb{R}^3$ with the flat metric.

The proof that $m_{\text{ADM}} \geq 0$ in [25] is based on the insight that asymptotically flat Riemannian three-manifolds with positive scalar curvature do not admit planar two-sided stable minimal surfaces with quadratic area growth. A review of the full argument that is tailored to our discussion here can be found in the introduction of [11].
The proof of the positive mass theorem suggests the following conjecture of R. Schoen: An asymptotically flat Riemannian manifold with non-negative scalar curvature that contains an unbounded area-minimizing surface is isometric to Euclidean space. Cf. [23, p. 48]. The goal of this paper is to prove this conjecture:

**Theorem 1.2.** The only asymptotically flat Riemannian three-manifold with non-negative scalar curvature that admits a non-compact area-minimizing boundary is flat $\mathbb{R}^3$.

We make precise our meaning of area-minimizing boundaries in Appendix A. The following surprising result of A. Carlotto and R. Schoen shows that the condition that $\Sigma \subset M$ be area-minimizing is sharp.

**Theorem 1.3** (A. Carlotto and R. Schoen [12]). There exists an asymptotically flat Riemannian metric $g = g_{ij} dx^i \otimes dx^j$ with non-negative scalar curvature and positive mass on $\mathbb{R}^3$ such that $g_{ij} = \delta_{ij}$ on $\mathbb{R}^2 \times (0, \infty)$.

In particular, the coordinate planes $\mathbb{R}^2 \times \{z\}$ with $z > 0$ in Theorem 1.3 are two-sided stable minimal surfaces. These planes are not area-minimizing by Theorem 1.2.

On the other hand, when $(M, g)$ is asymptotic to Schwarzschild with mass $m > 0$ in that instead of (1) we require that

$$g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + o(|x|^{-1})$$

with corresponding estimates for the derivatives, then Theorem 1.2 follows from the work of A. Carlotto [10]. We also point out the following rigidity result for two-sided stable minimal immersions.

**Theorem 1.4** (A. Carlotto, O. Chodosh, and M. Eichmair [11]). Let $(M, g)$ be a Riemannian three-manifold with non-negative scalar curvature that is asymptotic to Schwarzschild with $m > 0$. There is no unbounded complete two-sided stable minimal immersion $\varphi : \Sigma \to M$ that does not cross itself.

We refer the reader to the introduction of our paper [11] with A. Carlotto for further remarks and a survey of related results. Our proof of Theorem 1.2 is of a very different flavor.

Theorem 1.2 has an important consequence for the study of large isoperimetric regions. In [11], we observe that the ideas of Y. Shi [26] imply the following existence result: given an asymptotically flat Riemannian three-manifold $(M, g)$ with non-negative scalar curvature and $V > 0$, among all smooth compact regions $\Omega \subset M$ with $\partial M \subset \Omega$ and $\text{vol}(\Omega) = V$, there is one whose boundary has least area. Let $\Omega_V$ denote one such isoperimetric region of volume $V$. The boundary of $\Omega_V$ is a closed stable constant mean curvature surface. J. Metzger and the second named author have shown in [14] that $\Omega_V$ is unique when $(M, g)$ is asymptotic to Schwarzschild with mass $m > 0$ and $V > 0$ is sufficiently large. Moreover, its “outer boundary” is close to $S_\rho$ where $\rho > 1$ is such that $\text{vol}(B_\rho) = V$. The isoperimetric regions of the exact Schwarzschild geometry

$$g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} \quad \text{on} \quad \{x \in \mathbb{R}^3 : |x| \geq m/2\}$$
where $m > 0$ had been characterized in the fundamental work of H. Bray [6]. We also mention the seminal uniqueness results of G. Huisken and S.-T. Yau [16] and of J. Qing and G. Tian [22] for large two-sided stable constant mean curvature spheres in Riemannian three-manifolds that are asymptotic to Schwarzschild with mass $m > 0$, as well as extensions of these results obtained in [13, 11, 8].

It is natural to wonder about the behavior of large isoperimetric regions when $(M, g)$ has general asymptotics. For simplicity of exposition, let us assume that the boundary of $M$ is empty. Let $\Sigma_i = \partial \Omega_i$ be a sequence of isoperimetric surfaces with $0 < V_i \to \infty$. It has been shown in [14] that these surfaces either diverge to infinity as $i \to \infty$, or that a subsequence of these surfaces converges geometrically to a non-compact area-minimizing boundary $\Sigma \subset M$. In view of Theorem 1.2, the latter is impossible unless $(M, g)$ is flat $\mathbb{R}^3$. In conclusion, we arrive at the dichotomy that large isoperimetric regions in $(M, g)$ are either drawn far into the asymptotically flat end, or they contain the center of the manifold. When the scalar curvature of $(M, g)$ is everywhere positive, this was observed in Corollary 6.2 of [14].

**Corollary 1.5.** Let $(M, g)$ be an asymptotically flat Riemannian three-manifold with non-negative scalar curvature and positive mass. We also assume that the components of the boundary of $M$ are the only closed minimal surfaces in $(M, g)$. Let $U \subset M$ be a bounded open subset that contains the boundary of $M$. There is $V_0 > 0$ so that for every isoperimetric region $\Omega \subset M$ of volume $V \geq V_0$, either $U \subset \Omega$ or $U \cap \Omega$ is a thin smooth region that is bounded by the components of $\partial M$ and nearby two-sided stable constant mean curvature surfaces.

Note that the conclusion of the corollary is wrong for flat $\mathbb{R}^3$. The role of Theorem 1.2 in the proof of this corollary is similar to that of Theorem 1.4 in the analysis [11] of large stable constant mean curvature surfaces in three-manifolds that are asymptotic to Schwarzschild with mass $m > 0$.

Finally, we mention several other results that are based on scalar curvature and the existence of two-sided stable minimal surfaces of a certain topological type.

**Theorem 1.6** (R. Schoen and S.-T. Yau [24]). A Riemannian three-manifold with positive scalar curvature does not admit a two-sided stable minimal immersion of positive genus.

**Theorem 1.7** (M. Cai and G. Galloway [9]). Let $(M, g)$ be a closed Riemannian three-manifold with non-negative scalar curvature. If $(M, g)$ contains an embedded area-minimizing two-torus, then $(M, g)$ is flat.

**Theorem 1.8** (H. Bray, S. Brendle, M. Eichmair, and A. Neves [5]). Let $(M, g)$ be a closed Riemannian three-manifold that contains an embedded projective plane. We have that

$$\inf\{\text{area}(\Sigma) : \Sigma \subset M \text{ is an embedded projective plane}\} \cdot \inf_M R \leq 12\pi$$

where $R$ denotes the scalar curvature of $(M, g)$. Equality holds if and only if $(M, g)$ is isometric to real projective space $\mathbb{RP}^3$ with the round metric, up to scaling.

There have recently been some important developments in this area. We refer the reader to [7, 18, 19, 21, 11, 20] and the references therein.
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2. Tools

The following result is a technical variation of a crucial insight in the proof of the positive mass theorem due to R. Schoen and S.-T. Yau [25].

Lemma 2.1 (Section 6 in [14]). Let \((M, g)\) be an asymptotically flat Riemannian three-manifold. Assume that \(\Sigma \subset M\) is the unbounded component of an area-minimizing boundary in \((M, g)\), and that the scalar curvature of \((M, g)\) is non-negative along \(\Sigma\). Then \(\Sigma \subset M\) is totally geodesic and the scalar curvature of \((M, g)\) vanishes along this surface. Moreover, for all \(\rho > 1\) sufficiently large, \(\Sigma\) intersects \(S_\rho\) transversely in a nearly equatorial circle.

We also need the following rigidity result for asymptotically flat slabs with minimal boundary from our paper [11]. The proof adapts to the setting of non-negative scalar curvature a method that was developed by M. Anderson and L. Rodríguez [2] and crucially refined by G. Liu [17] to characterize complete manifolds with non-negative Ricci curvature.

Lemma 2.2 (Theorem 1.5 in [11]). Let \((M, g)\) be an asymptotically flat Riemannian three-manifold with non-negative scalar curvature. Any two disjoint unbounded connected properly embedded minimal surfaces in \((M, g)\) bound a region that is isometric to a Euclidean slab \(\mathbb{R}^2 \times [a, b]\).

3. Proof of Theorem 1.2

We first deal with the case where the boundary of \(M\) is empty.

Let \(r_0 > 0\) be as in Appendix B. Let \(\rho_0 > 1\) be such that \(S_\rho\) is convex for all \(\rho \geq \rho_0\). Every closed minimal surface of \((M, g)\) is contained in \(B_\rho_0\).

Let \(\Sigma \subset M\) be a connected unbounded properly embedded and separating surface that is area-minimizing with respect to \(g\). Fix a component \(M_+\) of the complement of \(\Sigma\) in \(M\) and choose a point \(p \in M_+\) to the following specifications:

- \(r = \frac{\text{dist}_g(x, p)}{2} < r_0\);
- \(p \notin B_{\rho_0 + 4r_0}\);
- \(\Sigma\) intersects \(\{x \in M : \text{dist}_g(x, p) < 4r\}\) in a single component, and this component is transverse to \(\nabla_g \text{dist}_g(\cdot, p)\).

In Appendix B we construct a family of conformal Riemannian metrics \(\{g(t)\}_{t \in [0, \epsilon)}\) on \(M\) with the following properties (see also Figure 1):

1. This result is stated in [11] with the additional assumption that the components of the boundary of \(M\) are the only closed minimal surfaces in \((M, g)\). By choosing the curve \(\gamma\) in the proof of Theorem 1.5 in [11] to lie sufficiently far out, we ensure that every complete minimal surface that intersects \(\gamma\) is unbounded. A similar argument is made in the choice of the point \(p\) in Section 3. The rest of the proof in [11] goes through verbatim.
(iv) the scalar curvature of $g(t)$ is positive on $\{x \in M : r < \text{dist}_g(x, p) < 3r\}$;
(v) the region $M_+$ is weakly mean-convex with respect to $g(t)$.

By taking $\epsilon > 0$ smaller if necessary, we may assume that all closed minimal surfaces of $(M, g(t))$ are contained in $B_{3r}$.

According to Lemma 2.1 for all $\rho \geq \rho_0$ sufficiently large, the intersection of $\Sigma$ with $S_\rho$ is transverse in a nearly equatorial circle. We denote this circle by $\Gamma_\rho = \Sigma \cap S_\rho$. Consider all properly embedded surfaces in $M$ that have boundary $\Gamma_\rho$ and which together with $\Sigma \cap B_\rho$ bound an open subset of $M_+ \cap B_\rho$. Using (v) and standard existence results from geometric measure theory, we see that among all these surfaces there is one — call it $\Sigma_\rho(t)$ — that has least area with respect to $g\big(\Sigma_\rho(t)\big)$. This surface is disjoint from $M_+ \cap S_\rho$ by convexity. It has one component with boundary $\Gamma_\rho$. Its other components are closed minimal surfaces in $(M, g(t))$. In particular, they are disjoint from $\{x \in M : \text{dist}_g(x, p) < 3r\}$.

Immediately, $\Sigma_\rho(t)$ does intersect $\{x \in M : \text{dist}_g(x, p) < 3r\}$ since otherwise,

\begin{equation}
\text{area}_g(\Sigma_\rho(t)) = \text{area}_{g(t)}(\Sigma_\rho(t)) \leq \text{area}_{g(t)}(\Sigma \cap B_\rho) < \text{area}_g(\Sigma \cap B_\rho).
\end{equation}

The strict inequality holds on account of (iii) and because $\Sigma$ intersects $\{x \in M : \text{dist}_g(x, p) < 3r\}$. Observe that (2) violates the area-minimizing property of $\Sigma$ with respect to $g$.

Using standard convergence results from geometric measure theory, we now find a connected unbounded properly embedded separating surface $\Sigma(t) \subset M$ as a subsequential geometric limit of $\Sigma_\rho(t)$ as $\rho \to \infty$. By construction, $\Sigma(t)$ is contained in $M_+ \cup \Sigma$ where it is area-minimizing with respect to $g(t)$. Moreover, $\Sigma(t)$ intersects $\{x \in M : \text{dist}_g(x, p) \leq 3r\}$. If $\Sigma(t)$ intersects $\{x \in M : \text{dist}_g(x, p) < 3r\}$, then it also intersects $\{x \in M : \text{dist}_g(x, p) \leq r\}$ because of (iv) and Lemma 2.1. Passing to a subsequential geometric limit as $t \to 0$, we obtain a connected unbounded properly embedded separating surface $\Sigma_+ \subset M$ that is contained in $M_+ \cup \Sigma$ where it is area-minimizing with respect to $g(t)$. Using now the area-minimizing property of $\Sigma$, we see that $\Sigma_+$ is in fact area-minimizing in all of $M$. Note that $\Sigma$ intersects $\{x \in M : \text{dist}_g(x, p) < 3r\}$ while it is disjoint from $\{x \in M : \text{dist}_g(x, p) \leq r\}$. It follows from the maximum principle that $\Sigma$ and $\Sigma_+$ are disjoint. The region in $(M, g)$ that is bounded by $\Sigma$ and $\Sigma_+$ is isometric to a Euclidean slab by Lemma 2.2.
Consider now the set of all $h > 0$ so that $(M, g)$ contains a Euclidean slab of the form $\mathbb{R}^2 \times [0, h]$ with $\mathbb{R}^2 \times \{0\}$ corresponding to $\Sigma$ and $\mathbb{R}^2 \times \{h\}$ corresponding to another area-minimizing surface in $(M, g)$. We have seen that this set is non-empty and open. It is clearly closed. It follows that $(M, g)$ is isometric to a Euclidean half-space to one side (namely $M_-$) of $\Sigma$. Since the choice of $M_+$ as one of the two components of the complement of $\Sigma$ in $M$ was arbitrary, we conclude that $(M, g)$ is indeed isometric to flat $\mathbb{R}^3$.

We now turn to the general case where $M$ has boundary. Consider $\Omega \in \mathcal{F}$ with non-compact area-minimizing boundary $\Sigma \subset M$. The unique non-compact component $\Sigma_0 \subset M$ is a separating surface. Let $M_-$ and $M_+$ denote the two components of its complement in $M$. Note that the interior of $\Omega \cap M$ agrees with either $M_-$ (Case 1) or $M_+$ (Case 2) outside of $B_{\rho_0}$. The proof that $g$ is flat in $M_+$ proceeds exactly as above, except for the following change. In Case 1, we let $\Sigma_\rho(t)$ have least area among properly embedded surfaces with boundary $\Gamma_\rho$ that bound together with $\Sigma_0 \cap B_\rho$ in $M_+ \cap B_\rho$ and relative to $M_+ \cap \partial M$. In Case 2, we let $\Sigma_\rho(t)$ have least area among properly embedded surfaces with boundary $\Gamma_\rho$ that bound together with $M_+ \cap S_\rho$ in $M_+ \cap B_\rho$ and relative to $M_+ \cap \partial M$. Theorem 1.2 follows upon switching the roles of $M_-$ and $M_+$.

Remark 3.1. The use of the conformal change of metric in this proof is inspired by an idea of G. Liu in his classification of complete non-compact Riemannian three-manifolds with non-negative Ricci curvature [17]. This idea has partially been adapted to non-negative scalar curvature in our proof of Lemma 2.2. The observation [2] in the proof of Theorem 1.2 is crucial: we use it to conclude that the surfaces $\Sigma_\rho(t)$ cannot run off as $\rho \to \infty$. At a related point in the work of M. Anderson and L. Rodríguez [2], the assumption of non-negative Ricci curvature is used tacitly in their delicate estimation of comparison surfaces [2, (1.5)].

APPENDIX A.

Let $(M, g)$ be an asymptotically flat Riemannian three-manifold. Extend $M$ inwards across each of its minimal boundary components by thin open collar neighborhoods to a new manifold $\hat{M}$. Denote the union of these finitely many collar neighborhoods by $C$. We consider the collection $\mathcal{F}$ of all proper three-dimensional submanifolds with boundary $\Omega \subset \hat{M}$ with $C \subset \Omega$. A surface $\Sigma \subset M$ bounds in $M$ relative to $\partial M$ if it arises as the boundary of such a smooth region. Note that $\partial M$ bounds in this sense.

We say that the boundary of $\Omega \in \mathcal{F}$ is area-minimizing if for all $\rho > 1$ and $\tilde{\Omega} \in \mathcal{F}$ with $\Omega \setminus B_\rho = \tilde{\Omega} \setminus B_\rho$ we have that

$$\text{area}(B_{2\rho} \cap \partial \Omega) \leq \text{area}(B_{2\rho} \cap \partial \tilde{\Omega}).$$

The components of the boundary $\Sigma \subset M$ of such $\Omega \in \mathcal{F}$ are two-sided stable minimal surfaces in $(M, g)$. It follows from the arguments in Section 6 of [14] that $\Sigma$ has at most one unbounded component $\Sigma_0$. More precisely, if we consider the homothetic blow-downs of $\Omega$ in the chart at infinity

$$M \setminus U \cong \{x \in \mathbb{R}^3 : |x| > 1\}$$

by a sequence $\lambda_i \to \infty$, then we can pass to a geometric subsequential limit in $\mathbb{R}^3 \setminus \{0\}$. This limit is either a half-space through the origin, or empty in the case where $\Omega$ is bounded.
We mention that when $M$ is orientable and when there are no closed minimal surfaces in $(M, g)$ other than the components of the boundary of $M$, then $M$ is diffeomorphic to the complement in $\mathbb{R}^3$ of the union of finitely many open balls whose closures are disjoint; see Section 4 in [15] and the references therein.

Appendix B.

Let $f \in C^\infty(\mathbb{R})$ be a non-positive function with support in the interval $[0, 3]$ such that

$$f(s) = -\exp(18/(s - 3))$$

when $s \in (1, 3)$. This definition is made so that

$$0 < f'(s) \quad \text{and} \quad sf''(s) + 3f'(s) < 0$$

for all $s \in (1, 3)$.

Let $(M, g)$ be a homogeneously regular Riemannian three-manifold. Choose $0 < r_0 < \text{inj}(M, g)/4$ so that

$$\Delta \text{dist}_g(\cdot, p)^2 \leq 8 \quad \text{on} \quad \{x \in M : \text{dist}_g(x, p) \leq 3r_0\}$$

for all $p \in M$. Fix $p \in M$ and $0 < r \leq r_0$. Consider the function $v : M \to \mathbb{R}$ given by

$$x \mapsto f(\text{dist}_g(x, p)/r).$$

Note that $v$ is smooth, non-positive, and supported in $\{x \in M : \text{dist}_g(x, p) \leq 3r\}$. Moreover,

$$v < 0, \quad g(\nabla_g v, \nabla_g \text{dist}_g(\cdot, p)) > 0, \quad \Delta_g v < 0$$

on $\{x \in M : r < \text{dist}_g(x, p) < 3r\}$. For $\epsilon > 0$ sufficiently small, a smooth family of conformal metrics $\{g(t)\}_{t \in [0, \epsilon)}$ with the properties needed in the proof of Theorem 1.2 is given by

$$g(t) = (1 + tv)^4g.$$
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