Oscillations and stability in the coupled mechanical system

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Abstract. Conservative mechanical systems are considered. Each system is supposed to admit a family of non-degenerate symmetric periodic motions. For the entire set of systems, they form a family of conditionally periodic oscillations, which is assumed to contain a symmetrical periodic motion. Universal couplings for mechanical systems are proposed such that they play the role of controls that ensure the existence, stability, and stabilization of oscillations at the same time for the whole system. Those couplings provide the natural, i.e. without involving any additional controls, stabilization of the oscillation. In addition, mechanical systems perform the synchronization in frequency and phase.

1. Introduction
We consider the autonomous model containing coupled subsystems (autonomous MCCS). The following problems are solved during the studying of oscillations of such systems: the existence of oscillations, the stability of oscillations, and stabilization of cycles. The synchronization of subsystems’ oscillations in frequency and phase is also of interest. The problem to find couplings that solve simultaneously all problems above is stated in [1]. Being found, the couplings-controls ensure the natural stabilization of the coupled system, so that additional control is not needed for the stabilization. This technique applied to the system that consists of the only subsystem results in the closed-loop control system. It is the case for the van der Pol system, where the orbitally asymptotically stable cycle arises as a result of the small control acting on the linear oscillator.

It is known that the conservative model does not allow cycles, so we are to consider perturbed systems. L. Pontryagin constructed a limit cycle for the perturbed Hamiltonian system in the plane [2]; van der Pol used the perturbed oscillator to study relaxational oscillations. Both constructed controlled systems. A similar controlled system is constructed for an arbitrary mechanical system in [3, 4], where the small control is taken in the form of the van der Pol-like nonlinear dissipation (linear in velocity), which is effective at the current point of the trajectory. This article extends this approach to weakly coupled mechanical systems.

The approach of [1] is applied to the MCCS that consists of mechanical subsystems. Each individual subsystem is supposed to admit a family $\sigma_s$ of non-degenerate symmetrical periodic motions (SPM). An SPM is in fact a single-frequency oscillation. The family of conditionally periodic oscillations $\sigma = \bigcup \sigma_s$ is assumed to contain an SPM for the whole system without couplings. Universal coupling controls are proposed for mechanical systems such that the coupled
model ensures the existence of the oscillation, its stability and stabilization simultaneously. So the natural stabilization without additional controls is realized. In addition, the oscillations of mechanical systems become synchronized in frequency and phase.

Coupled systems are intensively studied in relation to mathematical models used in various fields of knowledge. Appropriate examples can be found in [5–10]. In mechanics, the sympathetic Sommerfeld pendulums represent the classical example of the coupled system.

2. Oscillations of the MCCS and the synchronization of the oscillations in the subsystems

The \( \sigma_s \) family in a multi-degree-of-freedom mechanical system usually fills a two-dimensional central manifold \( \hat{\sigma}_s \), which is formed by non-degenerate SPM. Therefore, at first, we need to analyze the conservative single-degree-of-freedom systems.

Consider an MCCS of the type

\[
\ddot{q}_s + f_s(q_s) = \varepsilon u_s(q, \dot{q}), \quad s = 1, \ldots, n. \tag{1}
\]

Here \( n \) single-degree-of-freedom subsystems constitute the whole system. The numerical parameter \( \varepsilon \geq 0 \) sets up the magnitude of couplings between the subsystems. Given \( \varepsilon = 0 \), (1) becomes a set of \( n \) independent single-degree-of-freedom conservative systems. When \( \varepsilon \) is close to zero (1) is considered as weakly coupled. The time-invariant function \( u(q, \dot{q}) = (u_1(q, \dot{q}), \ldots, u_n(q, \dot{q})) \) is assumed to be smooth in its variables.

The \( s \)-th system has the energy integral

\[
E_s = \frac{\dot{q}_s^2}{2} + \int f(q_s) dq_s = h_s = \text{const}, \quad s = 1, \ldots, n.
\]

Suppose that the \( s \)-th system admits the family of periodic motions

\[
q_s = \varphi_s(h_s, t + \gamma_s), \quad s = 1, \ldots, n \tag{2}
\]

with the parameters \( h_s \) and \( \gamma_s \). Since the phase portrait is symmetrical with respect to the abscissa axis, the trajectories of (2) intersect the abscissa axis in two different points, and, therefore, (2) are SPM. The SPM of the whole uncoupled system corresponds to the zero time shifts along the trajectories: \( \gamma_s = 0, \quad s = 1, \ldots, n \).

**Definition 1** The family of SPM with the parameter \( h \) is called non-degenerate if the derivative of the period \( T(h) \) with respect to \( h \) is nonzero. An SPM of the non-degenerate family is called non-degenerate.

According to Definition 1 the oscillations of the linear oscillator are degenerate, while the oscillations of the mathematical pendulum are non-degenerate.

We consider conservative systems with non-degenerate families of SPM \( \sigma_s \). Necessary and sufficient conditions for the \( T^* \)-periodic solution of (1) to exist is written in the first approximation in \( \varepsilon \) in the form of the amplitude equation

\[
I(h, \gamma) \equiv \int_{0}^{T^*} \sum_{s=1}^{n} u_s(\varphi(h, t + \gamma), \dot{\varphi}(h, t + \gamma)) \psi_s dt = 0,
\]

\[
\varphi = (\varphi_1, \ldots, \varphi_n), \quad h = (h_1, \ldots, h_n), \quad \gamma = (\gamma_1, \ldots, \gamma_n).
\]

In (3) \( \psi_s \) is the solution of the conjugate linear system; \( \psi_s = \dot{q}_s \) for a conservative single-degree-of-freedom system [1].

It is clear that (3) is satisfied only if each subsystem admits a \( T^* \)-periodic SPM, i.e. if there exists \( h^* = (h^*_1, \ldots, h^*_n) \) such that \( T^* = T_s(h^*_s) \) for all \( s = 1, \ldots, n \). If, in addition,
then, according to [11], coupled system (1) admits a \((n - 1)\)-parametric family of \(T^*\)-periodic oscillations with respect to shifts \(\gamma_s\).

For subsystems’ oscillations to be synchronized in phase it is necessary that the shift along the trajectory be the same for all subsystems, i.e. \(\gamma_1 = \ldots = \gamma_n = \hat{\gamma}\). Then the condition

\[
\text{rank} \left\| \frac{\partial I(h^*, \gamma)}{\partial \gamma} \right\|_{\gamma = \hat{\gamma}} = n - 1
\]  

(5)

will be the sufficient condition of the synchronization at an appropriate point \(\hat{\gamma}\) (we can take \(\hat{\gamma} = 0\) without loss of generality). Condition (5) is satisfied for a cycle [11].

Thus, the synchronization problem is solved by using the coupling controls according to the following theorem.

**Theorem 1** If amplitude equation (3) for MCCS (1) admits the root \((h^*, \hat{\gamma})\) such that \(dI(h^*, \hat{\gamma})/dh \neq 0\) and (5) is satisfied, then the oscillations of the subsystems are synchronized in frequency and phase.

**Remark 1** According to Theorem 1, the synchronization of the subsystems’ oscillations in frequency and phase is performed in the cycle mode of MCCS.

**Remark 2** For MCCS (1) conditions (3) do not depend on whether the function of period \(T_s(h_s)\) is decreasing or increasing on the family \(\sigma_s\).

We find later universal couplings that satisfy Theorem 1 and ensure the orbital asymptotical stability of the MCCS’s cycle.

### 3. Universal coupling controls. Cycle of the MCCS

Consider the van der Pol equation

\[
\ddot{x} + x = \varepsilon (1 - x^2) \dot{x}.
\]

The right-hand side is considered as control and satisfies conditions (3), (4). Indeed, if we put \(\varepsilon = 0\), then the symmetrical solution to the linear oscillator is given by \(x = A \cos t\). From (3) we obtain the amplitude of the nonzero generating solution: \(A = 2\). Condition (4) is satisfied at \(A = 2\) since the derivative is negative. The last condition is sufficient for an attracting cycle to exist.

The linear oscillator has an isochronous family of periodic solutions. However, this fact does not prevent from extending van der Pol’s result to the case of the family of oscillations with the period depending monotonically on the family parameter [1, 3, 4]. In this case, a nonlinear van der Pol-type dissipation, which is linear in velocity and acting at the current point of the trajectory, is used.

We further assume that all subsystems have the same period behavior on SPM families: either increasing or decreasing. Then the union of uncoupled systems constitute a single conservative system such that any \(T\)-periodic solution implies a family \(\Sigma(h)\), where \(h\) is the constant energy [5]. So we have \(h_s = h_s(h), \ s = 1, \ldots, n\,\) on \(\Sigma(h)\), and \(\gamma_1 = \ldots = \gamma_n = \hat{\gamma}\) (we can put \(\hat{\gamma} = 0\) without loss of generality). The existence of \(\Sigma(h)\) allows to extend the approach of [3, 4] to MCCS by using the controls found there.

As a result, we apply the following coupling controls for mechanical system (1)

\[
\dot{u}_s = [1 - K(h)q^2] \dot{q}_s, \quad q^2 = \sum_{s=1}^{n} q_s^2, \quad s = 1, \ldots, n.
\]

(6)
Here the positive function $K(h)$ becomes the characteristic of the generating family $\Sigma$. The following identity is valid for $K(h)$

$$
T(h)
\int_0^T \left[ 1 - K(h)\varphi^2(h,t) \right] \varphi^2(h,t)dt = 0,
$$

(7)

so that the amplitude equation holds for all $h$. Hence, taking into account that $\dot{\varphi}$ takes the same value at the ends of the integration segment, we calculate

$$
K(h) = \frac{T(h)}{\int_0^{T(h)} \varphi^2(h,t)dt}.
$$

(8)

Thus, the amplitude equation for the SPM with energy $h = h^*$ has always the root $h = h^*$ if we put $K(h) = K(h^*)$.

**Lemma** Coupling controls (6) with characteristic of the SPM family (8) ensure the existence of the root of amplitude equation (3).

According to Lemma, coupling controls (6) do not depend on which specific subsystems are considered in MCCS (1). In this sense, coupling controls (6) are universal. It turns out that (6) almost always ensure the existence of a cycle. Indeed, amplitude equation (3) takes the following form for the SPM with the energy $h = h^*$

$$
I(h) \equiv \int_0^{T^*} \left[ 1 - K(h^*)\varphi^2(h,t) \right] \varphi^2(h,t)dt = 0.
$$

(9)

From (7), taking into account that $\dot{\varphi}(h,t)$ is odd, we calculate the derivative in (4) at $h = h^*$ as follows

$$
\frac{dI(h^*)}{dh} = \chi \nu, \quad \nu = \int_0^{T^*} \varphi^2(h^*,t)\dot{\varphi}^2(h^*,t)dt > 0.
$$

(10)

Hence, (4) is not satisfied only when $\chi = 0$.

**Definition 2** The point of the SPM family at which $\chi = 0$ is called the critical point of the SPM family with respect to function $K(h)$.

Basing on Lemma let us state the sufficient conditions for a cycle of the coupled system to exist in terms of Definition 2 with the aid of (10).

**Theorem 2** Let each $s$-th subsystem of (1) admit a family $\sigma_s$ of non-degenerate SPM. Let all periods decrease (increase) simultaneously. If (1) admits a periodic motion at $\varepsilon = 0$ then (1) with controls (6) admits a cycle at any non-critical point ($\chi \neq 0$).

Therefore, universal couplings (6) ensure the existence of a cycle in the MCCS at non-critical points of the characteristics $K(h)$.

4. Stabilization of the cycle of the MCCS

Recall that the cycle is the isolated solution of the autonomous system. This solution is defined to within the shift along the trajectory, therefore the single zero characteristic exponent (CE)
corresponds to the cycle. A corresponding real CE results from the decay of the zero CE Jordan cell and is obtained from the amplitude equation (9). Indeed, (1) admits the integral of total mechanical energy
\[ E = \sum_{s=1}^{n} E_s = h \quad \text{at} \quad \varepsilon = 0. \]
The law of change of the total mechanical energy at \( \varepsilon > 0 \) is written as follows
\[ \frac{dE}{dt} = \varepsilon \left[ 1 - K(h^*) \sum_{j=1}^{n} q_j^2 \right] \sum_{s=1}^{n} \dot{q}_s^2. \quad (11) \]
Amplitude equation (9) implies that the increment of the total mechanical energy \( \Delta E \) along the cycle on the segment \([0, T^*] \) is equal to zero. Therefore, taking into account (10), we can calculate the increment of the total energy for another trajectory close to the cycle as follows
\[ \Delta E(h) = \varepsilon \chi \nu (h - h^*) + o(\Delta h). \quad (12) \]
When \( \chi < 0 \) the sign of \( \Delta E \) on the segment \([0, t], t \leq T^* \), is opposite to the sign of \( \Delta h = h - h^* \). So, the trajectory of (1), (6) asymptotically approaches the surface of energy \( h = h^* \), which corresponds to the cycle.

In the subsystems, the laws of energy change are similar to (11). Hence, the following equalities hold
\[ q_s^2 \frac{dE}{dt} = \sum_{j=1}^{n} q_j^2 dE_j, \quad s = 1, \ldots, n. \]
This equation means that the change of energy in a subsystem has the same behavior as the change of energy in the entire system, i.e. the increments \( \Delta E_s \) on \([0, T(h^*)] \) satisfy the same law as (12). Therefore, when \( \chi < 0 \), the trajectory approach the cycle by each pair of variables \((q_s, \dot{q}_s)\). The implementation of the mapping \( G : 0 \rightarrow T^* \) of the phase space \((q, \dot{q})\) to itself completes the proof of the orbital asymptotic stability of the cycle.

The above reasoning are summarized as follows.

**Theorem 3** Let the point \( h = h^* \) be non-critical with respect to \( K(h) \) for SPM (1), (6). If \( \chi(h^*) < 0 \) then the appropriate cycle is orbitally asymptotically stable.

**Remark 3** The cycle has a single zero CE, while all other CE are in the left complex half-plane.

**Remark 4** Universal couplings ensure the natural stabilization, i.e. no additional control is needed to stabilize the MCCS’s cycle.

5. Generalization to multi-variable systems
For multi-degree-of-freedom mechanical systems the non-degenerate family of SPM fills the two-dimensional manifold [3]. Suppose that \( s \)-th subsystem admits the family \( \sigma_s \) of non-degenerate SPM with the corresponding two-dimensional manifold \( \hat{\sigma}_s \). Let us choose the generalized coordinates \( q_{s1}, \ldots, q_{sn_s} \) such that \( q_{s2} = \ldots = q_{sn_s} = 0 \) on \( \hat{\sigma}_s \). Then family \( \sigma_s \) is described by coordinate \( q_{s1} \). The coupled system that contains \( n_s \)-degree-of-freedom mechanical systems as subsystems admits a family \( \sigma = \bigcup \sigma_s \) of conditionally-periodic motions with the corresponding manifold \( \hat{\sigma} \). If \( \sigma \) comprises a \( T \)-periodic SPM, we can repeat all reasoning of Sections 1–3 for \( \hat{\sigma}_s \) to draw the same conclusions. In particular, an orbitally asymptotically stable cycle is realized on \( \hat{\sigma}_s \).

Evidently, the stability of the cycle on \( \hat{\sigma}_s \) is conditional. For the cycle to be unconditionally stable, the trajectory must be attracted by \( \hat{\sigma}_s \). This attraction can be achieved by implementing small additional controls in the form of linear dissipation. It is clear that this technique works only if all CEs are in the imaginary axis.
Theorem 4 For a coupled mechanical system containing multi-degree-of-freedom conservative systems with nondegenerate SPM families, the problem of the existence of an orbitally asymptotically stable cycle is solved by Theorem 3, provided that all CEs of subsystem’s SPM are in the imaginary axis.

6. Example
Consider two identical pendulums. For a single pendulum $\ddot{\varphi} + \sin \varphi = 0$, the function $K(h)$ is given in [12]. In view of (8), since the own frequencies of both pendulums are equal, we take $K(h)$ of [12] divided by 2 as the characteristic function of the coupled system.

We apply Theorem 3 to obtain an orbitally asymptotically stable cycle, which is close to the oscillations the pendulums as a whole.

7. Conclusion
Models containing coupled subsystems are actual in various fields of knowledge. A coupled mechanical system contains mechanical subsystems controlled by couplings. The problems of the existence of single-frequency oscillations, their stability, stabilization, and the problems of synchronization of the oscillations of subsystems in frequency and phase can be solved simultaneously by finding a single universal set of coupling controls. Such a set is proposed for subsystems that admit families of nondegenerate oscillations. These coupling controls are suitable for a coupled mechanical system containing subsystems with arbitrary degrees of freedom.

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