TWO RESULTS ON HOMOGENEOUS HESSIAN NILPOTENT POLYNOMIALS

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Abstract. Let \( z = (z_1, \cdots, z_n) \) and \( \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2} \) the Laplace operator. A formal power series \( P(z) \) is said to be Hessian Nilpotent (HN) if its Hessian matrix \( \text{Hes} \ P(z) = (\frac{\partial^2 P}{\partial z_i \partial z_j}) \) is nilpotent. In recent developments in [BE1], [M] and [Z], the Jacobian conjecture has been reduced to the following so-called vanishing conjecture (VC) of HN polynomials: for any homogeneous HN polynomial \( P(z) \) (of degree \( d = 4 \)), we have \( \Delta^m P^{m+1}(z) = 0 \) for any \( m >> 0 \).

In this paper, we first show that, the VC holds for any homogeneous HN polynomial \( P(z) \) provided that the projective subvarieties \( \mathbb{Z}_P \) and \( \mathbb{Z}_{\sigma_2} \) of \( \mathbb{C}P^{n-1} \) determined by the principal ideals generated by \( P(z) \) and \( \sigma_2(z) := \sum_{i=1}^{n} z_i^2 \), respectively, intersect only at regular points of \( \mathbb{Z}_P \). Consequently, the Jacobian conjecture holds for the symmetric polynomial maps \( F = z - \nabla P \) with \( P(z) \) HN if \( F \) has no non-zero fixed point \( w \in \mathbb{C}^n \) with \( \sum_{i=1}^{n} w_i^2 = 0 \). Secondly, we show that the VC holds for a HN formal power series \( P(z) \) if and only if, for any polynomial \( f(z) \), \( \Delta^m (f(z) P(z)^m) = 0 \) when \( m >> 0 \).

1. Introduction and Main Results

Let \( z = (z_1, z_2, \cdots, z_n) \) be commutative free variables. Recall that the well-known Jacobian conjecture claims that: any polynomial map \( F(z) : \mathbb{C}^n \to \mathbb{C}^n \) with the Jacobian \( j(F)(z) \equiv 1 \) is an automorphism of \( \mathbb{C}^n \) and its inverse map must also be a polynomial map. Despite intense study from mathematicians in more than sixty years, the conjecture is still open even for the case \( n = 2 \). In 1998, S. Smale [S] included the Jacobian conjecture in his list of 18 important mathematical problems for 21st century. For more history and known results on the Jacobian conjecture, see [BCW], [E] and references there.

Recently, M. de Bondt and the first author [BE1] and G. Meng [M] independently made the following remarkable breakthrough on the Jacobian conjecture. Namely, they reduced the Jacobian conjecture to

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the so-called symmetric polynomial maps, i.e. the polynomial maps of the form \( F = z - \nabla P \), where \( \nabla P := \left( \frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2}, \cdots, \frac{\partial P}{\partial z_n} \right) \), i.e. \( \nabla P(z) \) is the gradient of \( P(z) \in \mathbb{C}[z] \).

For more recent developments on the Jacobian conjecture for symmetric polynomial maps, see [BE1]–[BE4].

Based on the symmetric reduction above and also the classical homogeneous reduction in [BCW] and [Y], the second author in [Z] further reduced the Jacobian conjecture to the following so-called vanishing conjecture.

Let \( \Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2} \) the Laplace operator and call a formal power series \( P(z) \) Hessian nilpotent (HN) if its Hessian matrix \( \text{Hes} P(z) := \left( \frac{\partial^2 P}{\partial z_i \partial z_j} \right) \) is nilpotent. It has been shown in [Z] that the Jacobian conjecture is equivalent to

**Conjecture 1.1. (Vanishing Conjecture of HN Polynomials)**

For any homogeneous HN polynomial \( P(z) \) (of degree \( d = 4 \)), we have \( \Delta^m P^{m+1} = 0 \) when \( m >> 0 \).

Note that, it has also been shown in [Z] that \( P(z) \) is HN if and only if \( \Delta^m P^m = 0 \) for \( m \geq 1 \).

In this paper, we will prove the following two results on HN polynomials.

Let \( P(z) \) be a homogeneous HN polynomial of degree \( d \geq 3 \) and \( \sigma_2(z) := \sum_{i=1}^{n} z_i^2 \). We denote by \( \mathbb{Z}_P \) and \( \mathbb{Z}_{\sigma_2} \) the projective subvarieties of \( \mathbb{C}P^{n-1} \) determined by the principal ideals generated by \( P(z) \) and \( \sigma_2(z) \), respectively. The first main result of this paper is the following theorem.

**Theorem 1.2.** Let \( P(z) \) be a homogeneous HN polynomial of degree \( d \geq 4 \). Assume that \( \mathbb{Z}_P \) intersects with \( \mathbb{Z}_{\sigma_2} \) only at regular points of \( \mathbb{Z}_P \), then the vanishing conjecture holds for \( P(z) \). In particular, the vanishing conjecture holds if the projective variety \( \mathbb{Z}_P \) is regular.

**Remark 1.3.** Note that, when \( \deg P(z) = d = 2 \) or \( 3 \), the Jacobian conjecture holds for the symmetric polynomial map \( F = z - \nabla P \). This is because, when \( d = 2 \), \( F \) is a linear map with \( j(F) \equiv 1 \). Hence \( F \) is an automorphism of \( \mathbb{C}^n \); while when \( d = 3 \), we have \( \deg F = 2 \). By Wang’s theorem [W], the Jacobian conjecture holds for \( F \) again. Then, by the equivalence of the vanishing conjecture for the homogeneous HN polynomial \( P(z) \) and the Jacobian conjecture for the symmetric map \( F = z - \nabla P \) established in [Z], we see that, when \( \deg P(z) = d = 2 \) or \( 3 \), Theorem 1.2 actually also holds even without the condition on the projective variety \( \mathbb{Z}_P \).
For any non-zero \( z \in \mathbb{C}^n \), denote by \([z]\) its image in the projective space \( \mathbb{C}P^{n-1} \). Set
\[
\tilde{Z}_{\sigma_2} := \{ z \in \mathbb{C}^n | z \neq 0; [z] \in Z_{\sigma_2} \}.
\] (1.1)

In other words, \( \tilde{Z}_{\sigma_2} \) is the set of non-zero \( z \in \mathbb{C}^n \) such that \( \sum_{i=1}^{n} z_i^2 = 0 \).

Note that, for any homogeneous polynomial \( P(z) \) of degree \( d \), it follows from the Euler’s formula \( dP = \sum_{i=1}^{n} z_i \frac{dP}{dz_i} \), that any non-zero \( w \in \mathbb{C}^n, [w] \in \mathbb{C}P^{n-1} \) is a singular point of \( Z_P \) if and only if \( w \) is a fixed point of the symmetric map \( F = z - \nabla P \). Furthermore, it is also well-known that, \( j(F) \equiv 1 \) if and only if \( P(z) \) is HN.

By the observations above and Theorem 1.2, it is easy to see that we have the following corollary on symmetric polynomial maps.

**Corollary 1.4.** Let \( F = z - \nabla P \) with \( P \) homogeneous and \( j(F) \equiv 1 \) (or equivalently, \( P \) is HN). Assume that \( F \) does not fix any \( w \in \tilde{Z}_{\sigma_2} \). Then the Jacobian holds for \( F(z) \). In particular, if \( F \) has no non-zero fixed point, the Jacobian conjecture holds for \( F \).

Our second main result is following theorem which says that the vanishing conjecture is actually equivalent to a formally much stronger statement.

**Theorem 1.5.** For any HN polynomial \( P(z) \), the vanishing conjecture holds for \( P(z) \) if and only if, for any polynomial \( f(z) \in \mathbb{C}[z], \Delta^m(f(z)P(z)^m) = 0 \) when \( m \gg 0 \).

2. **Proof of the Main Results**

Let us first fix the following notation. Let \( z = (z_1, z_2, \cdots, z_n) \) be free complex variables and \( \mathbb{C}[z] \) (resp. \( \mathbb{C}[[z]] \)) the algebra of polynomials (resp. formal power series) in \( z \). For any \( d \geq 0 \), we denote by \( V_d \) the vector space of homogeneous polynomials in \( z \) of degree \( d \).

For any \( 1 \leq i \leq n \), we set \( D_i = \frac{\partial}{\partial z_i} \) and \( D = (D_1, D_2, \cdots, D_n) \). We define a \( \mathbb{C} \)-bilinear map \( \{\cdot, \cdot\} : \mathbb{C}[z] \times \mathbb{C}[z] \rightarrow \mathbb{C}[z] \) by setting
\[
\{f, g\} := f(D)g(z)
\]
for any \( f(z), g(z) \in \mathbb{C}[z] \).

Note that, for any \( m \geq 0 \), the restriction of \( \{\cdot, \cdot\} \) on \( V_m \times V_m \) gives a \( \mathbb{C} \)-bilinear form of the vector subspace \( V_m \), which we will denote by \( B_m(\cdot, \cdot) \). It is easy to check that, for any \( m \geq 1 \), \( B_m(\cdot, \cdot) \) is symmetric and non-singular.

The following lemma will play a crucial role in our proof of the first main result.
Lemma 2.1. For any homogeneous polynomials \( g_i(z) \) (1 \( \leq \) \( i \) \( \leq \) \( k \)) of degree \( d_i \geq 1 \), let \( S \) be the vector space of polynomial solutions of the following system of PDEs:

\[
\begin{cases}
g_1(D) u(z) = 0, \\
g_2(D) u(z) = 0, \\
\hspace{0.5cm} \cdots \\
g_k(D) u(z) = 0.
\end{cases}
\]

(2.2)

Then, \( \dim S < +\infty \) if and only if \( g_i(z) \) (1 \( \leq \) \( i \) \( \leq \) \( k \)) have no non-zero common zeroes.

Proof: Let \( I \) the homogeneous ideal of \( \mathbb{C}[z] \) generated by \( \{ g_i(z) | 1 \leq i \leq k \} \). Since all \( g_i(z) \)'s are homogeneous, \( S \) is a homogeneous vector subspace \( S \) of \( \mathbb{C}[z] \).

Write

\[
S = \bigoplus_{m=0}^{\infty} S_m,
\]

(2.3)

\[
I = \bigoplus_{m=0}^{\infty} I_m.
\]

(2.4)

where \( I_m := I \cap V_m \) and \( S_m := I \cap V_m \) for any \( m \geq 0 \).

Claim: For any \( m \geq 1 \) and \( u(z) \in V_m, u(z) \in S_m \) if and only if \( \{ u, I_m \} = 0 \), or in other words, \( S_m = I_m^\perp \) with respect to the \( \mathbb{C} \)-bilinear form \( B_m(\cdot, \cdot) \) of \( V_m \).

Proof of the Claim: First, by the definitions of \( I \) and \( S \), we have \( \{ I_m, S_m \} = 0 \) for any \( m \geq 1 \), hence \( S_m \subseteq I_m^\perp \). Therefore, we need only show that, for any \( u(z) \in I_m^\perp \subseteq V_m \), \( g_i(D)u(z) = 0 \) for any \( 1 \leq i \leq n \).

We first fix any \( 1 \leq i \leq n \). If \( m < d_i \), there is nothing to prove. If \( m = d_i \), then \( g_i(z) \in I_m \), hence \( \{ g_i, u \} = g_i(D)u = 0 \). Now suppose \( m > d_i \). Note that, for any \( v(z) \in V_{m-d_i} \), \( v(z)g_i(z) \in I_m \). Hence we have

\[
0 = \{ v(z)g_i(z), u(z) \} \\
= v(D)g_i(D)u(z) \\
= v(D) (g_i(D)u) (z) \\
= \{ v(z), (g_i(D)u) (z) \}.
\]

Therefore, we have

\[
B_{m-d_i} ((g_i(D)u)(z), V_{m-d_i}) = 0.
\]
Since $B_{m-d}(\cdot, \cdot)$ is a non-singular $\mathbb{C}$-bilinear form of $V_{m-d}$, we have $g_i(D)u = 0$. Hence, the Claim holds. 

By a well-known fact in Algebraic Geometry (see Exercise 2.2 in [H], for example), we know that the homogeneous polynomials $g_i(z)$ $(1 \leq i \leq k)$ have no non-zero common zeroes if and only if $I_m = V_m$ when $m >> 0$. While, by the Claim above, we know that, $I_m = V_m$ when $m >> 0$ if and only if $S_m = 0$ when $m >> 0$, and if and only if the solution space $S$ of the system (2.2) is finite dimensional. Hence, the lemma follows. 

Now we are ready to prove our first main result, Theorem 1.2.

**Proof of Theorem 1.2**: Let $P(z)$ be a homogeneous HN polynomial of degree $d \geq 4$ and $S$ the vector space of polynomial solutions of the following system of PDEs:

\[
\begin{align*}
\frac{\partial P}{\partial z_1}(D) u(z) &= 0, \\
\frac{\partial P}{\partial z_2}(D) u(z) &= 0, \\
&\quad\vdots \\
\frac{\partial P}{\partial z_n}(D) u(z) &= 0, \\
\Delta u(z) &= 0.
\end{align*}
\]

(2.5)

First, note that the projective subvariety $Z_P$ intersects with $Z_{\sigma_2}$ only at regular points of $Z_P$ if and only if $\frac{\partial P}{\partial z_i}(z)$ $(1 \leq i \leq n)$ and $\sigma_2 = \sum_{i=1}^n z_i^2$ have no non-zero common zeros (again use Euler’s formula). Then, by Lemma 2.1, we have $\dim S < +\infty$.

On the other hand, by Theorem 6.3 in [Z], we know that $\Delta^m P^{m+1} \in S$ for any $m \geq 0$. Note that $\deg \Delta^m P^{m+1} = (d-2)m + d$ for any $m \geq 0$. So $\deg \Delta^m P^{m+1} > \deg \Delta^k P^{k+1}$ for any $m > k$. Since $\dim S < +\infty$ (from above), we have $\Delta^m P^{m+1} = 0$ when $m >> 0$, i.e. the vanishing conjecture holds for $P(z)$. 

Next, we give a proof for our second main result, Theorem 1.5.

**Proof of Theorem 1.5**: The $(\Rightarrow)$ part follows directly by choosing $f(z)$ to be $P(z)$ itself.

To show $(\Leftarrow)$ part, let $d = \deg f(z)$. If $d = 0$, $f$ is a constant. Then, $\Delta^m (f(z)P(z)^m) = f(z)\Delta^m P^m = 0$ for any $m \geq 1$.

So we assume $d \geq 1$. By Theorem 6.2 in [Z], we know that, if the vanishing conjecture holds for $P(z)$, then, for any fixed $a \geq 1$, $\Delta^m P^{m+a} = 0$ when $m >> 0$. Therefore there exists $N > 0$ such that, for any $0 \leq b \leq d$ and any $m > N$, we have $\Delta^m P^{m+b} = 0$. 


By Lemma 6.5 in [Z], for any $m \geq 1$, we have

\[
\Delta^m(f(z)P(z)^m) = \sum_{k_1+k_2+k_3=m} 2^{k_3} \binom{m}{k_1, k_2, k_3} \sum_{s \in \mathbb{N}^n \mid s} \binom{k_2}{s} \frac{\partial^{k_3} \Delta^{k_1} f(z)}{\partial z^s} \frac{\partial^{k_2} \Delta^{k_3} P^m(z)}{\partial z^s},
\]

where $\binom{m}{k_1, k_2, k_3}$ and $\binom{k_2}{s}$ denote the usual binomials.

Note first that, the general term in the sum above is non-zero only if $2k_1 + k_2 \leq d$. But on the other hand, since

\[
0 \leq k_1 + k_2 \leq 2k_1 + k_2 \leq d,
\]

by the choice of $N \geq 1$, we have $\Delta^{k_3} P^m(z) = \Delta^{k_3} P^{k_3+(k_1+k_2)}(z)$ is non-zero only if

\[
k_3 \leq N.
\]

From the observations above and Eqs. (2.6), (2.7), (2.8) it is easy to see that, $\Delta^m(f(z)P(z)^m) \neq 0$ only if $m = k_1 + k_2 + k_3 \leq d + N$. In other words, $\Delta^m(f(z)P(z)^m) = 0$ for any $m > d + N$. Hence Theorem 1.5 holds. \[\square\]

Note that all results used in the proof above for the ($\Leftarrow$) part of the theorem also hold for all HN formal power series. Therefore we have the following corollary.

**Corollary 2.2.** Let $P(z)$ be a HN formal power series such that the vanishing conjecture holds for $P(z)$. Then, for any polynomial $f(z)$, we have $\Delta^m(f(z)P(z)^m) = 0$ when $m >> 0$.

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