State diagrams of functional programs

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Abstract
In the paper we introduce graphical objects (called state diagrams) related to functional programs. It is shown that state diagrams of functional programs can be used to solve problems of verification of functional programs. The proposed approach is illustrated by an example of verification of a sorting program.

1 Introduction
The problem of program verification consists of proving statements that analyzed programs have specified properties. This problem is one of the main problems of theoretical computer science.

For various classes of programs there are used various verification methods. For example, for a verification of sequential programs there are used Floyd’s inductive assertions method [2], Hoare’s logic [3], etc., are used. For verification of parallel and distributed programs there are used methods based on Milner’s calculus of communicating systems (CCS) and π-calculus [4] [5], Hoare’s theory of communicating sequential processes (CSP) and its generalizations [6], [7], temporal logic and model checking [8], process algebra [9], Petri nets [10], etc. are used. Main methods of verification of functional programs (FPs) are computational induction and structural induction [1]. Disadvantages of these methods are related to difficulties to construct formal proofs of program correctness. Among other methods of verification of FPs it should be noted a method based on reasoning with datatypes and abstract interpretation through type inference [12], a model checking method to verify FPs [13], [14], methods based on flow analysis [11], methods based on the concept of a multiparametric tree transducer [15].

In this article we consider FPs as systems of algebraic equations over strings. We introduce a concept of a state diagram for such FPs and present the verification method based on state diagrams. The main advantage of our approach in comparison with all the above approaches to verification of FPs is that our approach allows to present proofs of correctness of FPs in the form of simple properties of their state diagrams.

The basic idea of our approach is the following:
we assume that a specification of properties of FP Σ under verification is expressed by another FP Σ′, whose input is equal to the output of FP Σ,

we say that a FP Σ is correct with respect to the specification Σ′, iff the composition \( f_{Σ′}(f_Σ) \) of input-output maps corresponded to FPs Σ and Σ′ has an output value 1 on all its input values, we denote this statement by the notation

\[ f_{Σ′}(f_Σ) = 1 \]  \hspace{1cm} (1)

we reduce the problem of a proving statement (1) to the problem of an analysis of a state diagram for the FP Σ′(Σ), whose input-output map \( f_{Σ′(Σ)} \) is equal to the composition \( f_{Σ′}(f_Σ) \).

The proposed method of verification of FPs is illustrated by an example of verification of a sorting FP:

• at first, we present a proof of correctness of this FP by structural induction,
• at second, we present a correctness proof of the FP by the method based on constructing of state diagrams, the proof by the second method can be generated automatically.

2 Main concepts

2.1 Terms

We assume that there is given the set \( D \) of values, and each element of \( D \) has one of the following types: C, S or B. The sets of values of the types C, S and B are denoted by \( D_C \), \( D_S \) and \( D_B \), respectively, and

• values of the type C are called symbols,
• values of the type S are called symbolic strings (or briefly strings), each string is a finite (maybe empty) sequence of symbols,
• values of the type B are called boolean values, there are two boolean values: \( \top \) (true) and \( \bot \) (false).

We assume also that there are sets

• \( X \) of data variables (or briefly variables),
• \( C \) of constants,
• \( F \) functional symbols (FSs), and
• \( Φ \) of functional variables.

Each element \( x \) of any of the above sets is associated with a type of this element, denoted by the notation \( τ(x) \), and

• if \( x \in X \) or \( x \in C \), then \( τ(x) \in \{C, S, B\} \), and
• if \( x \in F \) or \( x \in Φ \), then \( τ(x) \) is a notation of the form \( (t_1, \ldots, t_n) \to t \), where \( t_1, \ldots, t_n, t \in \{C, S, B\} \).
Each constant \( c \in C \) corresponds to an element of the set \( \mathcal{D}_{\tau(c)} \), called a value of this constant. The notation \( \varepsilon \) denotes a constant of the type \( S \), whose value is an empty string. There are constants of the type \( B \) which correspond to the values \( \top \) and \( \bot \), these constants are denoted by \( \top \) and \( \bot \) respectively.

Each FS \( f \in F \) corresponds to a partial function, which is denoted by the same symbol \( f \), and has the form

\[
f : \mathcal{D}_{t_1} \times \ldots \times \mathcal{D}_{t_n} \to \mathcal{D}_t,
\]

where \( \tau(f) = (t_1, \ldots, t_n) \to t \).

Below we list some of the FSs which belong to \( F \), beside each FS we point out (with a colon) its type.

1. \( \text{head} : S \to C \). The function \( \text{head} \) is defined for non-empty strings, it maps each non-empty string to its first element (i.e. if a string \( u \) has the form \( a_1 \ldots a_n \), then \( \text{head}(u) = a_1 \)).

2. \( \text{tail} : S \to S \). The function \( \text{tail} \) is defined for non-empty strings, it maps each non-empty string \( u \) to a string (called a tail of the string \( u \)), derived from \( u \) by removal of its first element (i.e. if a string \( u \) has the form \( a_1 a_2 \ldots a_n \), then \( \text{tail}(u) = a_2 \ldots a_n \)).

3. \( \text{conc} : (C, S) \to S \). For each pair \( (a, u) \in \mathcal{D}_C \times \mathcal{D}_S \) the string \( \text{conc}(a, u) \) is derived by a writing the symbol \( a \) before \( u \).

4. \( = : (t, t) \to B \), where \( t \in \{C, S, B\} \), i.e. the symbol \( = \) denotes three FSs. A value of the function \( = \) on the pair \( (x, y) \) is \( \top \), if \( x \) and \( y \) are equal, and \( \bot \), otherwise.

5. \( \leq : (C, C) \to B \). We assume that \( \mathcal{D}_C \) is a linearly ordered set, and the value of the function \( \leq \) on the pair \( (a, b) \) is \( \top \), if \( a \leq b \), and \( \bot \), otherwise.

6. Boolean FSs:
   \( \neg : B \to B \), \( \land : (B, B) \to B \), etc.,
   the corresponding functions are standard boolean functions on the arguments \( \top \) and \( \bot \) (i.e. \( \neg(\top) = \bot \), etc.).

7. \( \text{if\_then\_else} : (B, t, t) \to t \), where \( t \in \{C, S, B\} \), i.e. the notation \( \text{if\_then\_else} \) denotes three FSs. Functions corresponding to these FSs are defined as follows:

\[
\text{if\_then\_else} (a, x, y) \overset{\text{def}}{=} \begin{cases}
  x, & \text{if } a = \top, \\
  y, & \text{if } a = \bot.
\end{cases}
\]

A concept of a term is defined inductively. Each term \( e \) is associated with a type \( \tau(e) \in \{C, S, B\} \). A definition of a term has the following form:

- each data variable and each constant is a term, its type is equal to the type of this variable or constant,
- if \( f \) is a FS or a functional variable, \( e_1, \ldots, e_n \) are terms, and

\[
\tau(f) = (\tau(e_1), \ldots, \tau(e_n)) \to t,
\]

then \( f(e_1, \ldots, e_n) \) is a term of the type \( t \).
We shall use the following concepts and notations.

- A set of all terms is denoted by the symbol $\mathcal{E}$.
- Terms of the type $\mathbf{B}$ are called formulas.

- $\forall e, e' \in \mathcal{E}, e'$ is a subterm of $e$, if either $e' = e$, or $e = f(e_1, \ldots, e_n)$, and
  $\exists i \in \{1, \ldots, n\} : e'$ is a subterm of $e_i$.
- $\forall e \in \mathcal{E}$, $X_e$ and $\Phi_e$ are sets of data variables and functional variables respectively, occurred in $e$.
- $\forall X \subseteq X \subseteq \mathcal{E}$, $\mathcal{E}_X$ is denoted by the symbols $\{e \in \mathcal{E} | X_e \subseteq X\}$.
- The terms
  
  \begin{align*}
  \text{head}(e), \text{tail}(e), \text{conc}(e, e'), & = (e, e'), \leq (e, e'), \text{if then else} (e, e', e'')
  \end{align*}

are denoted by $e_h, e_i, e', e = e', [e] e': e''$, respectively.

- A term $e \in \mathcal{E}$ is said to be simple, if $e = e_1 \ldots e_n$, where each term from the list $e_1, \ldots, e_n$ is a data variable or a constant.

- Terms containing boolean FSs will be denoted as in mathematical texts (i.e. in the form $e \land e'$, etc.), terms of the form $e_1 \land \ldots \land e_n$ can also be denoted by the notation $\{e_1, \ldots, e_n\}$.

- $\forall e \in \mathcal{E}$, the notation $\mathcal{D}_e$ denotes the set $\mathcal{D}_{\tau(e)}$.

- Lists of terms are denoted by the notations of the form $\bar{e}$.

- If $\bar{e}$ is a list of terms of the form $(e_1, \ldots, e_n)$, then
  - $\tau(\bar{e})$ denotes the list $(\tau(e_1), \ldots, \tau(e_n))$.
  - $X_e, \Phi_e$ denote the sets $\bigcup_{i=1}^n X_{e_i}, \bigcup_{i=1}^n \Phi_{e_i}$ respectively,
  - $\mathcal{D}_e$ denotes the set $\mathcal{D}_{e_1} \times \ldots \times \mathcal{D}_{e_n}$.

- If $\bar{e}' = (e'_1, \ldots, e'_n), \bar{e}'' = (e''_1, \ldots, e''_n)$ are lists of terms, $\tau(\bar{e}') = \tau(\bar{e}'')$, then
  - the notation $\bar{e}' = \bar{e}''$ denotes the term $(e'_1 = e''_1) \land \ldots \land (e'_n = e''_n)$,
  - if in addition it is assumed that for each pair $i, j$ of different indices from $\{1, \ldots, n\}$ the term $e'_i$ is a subterm of $e''_j$, then
    * $\forall e \in \mathcal{E}$ the notation
      \begin{equation}
      e[e''_1/e'_1, \ldots, e''_n/e'_n]
      \end{equation}
    denoting a term derived from $e$ by replacing $\forall i = 1, \ldots, n$ each subterm of $e$, which is equal to $e'_i$, on the term $e''_i$, term (2) is denoted also by the notation $e[\bar{e}'', \bar{e}']$,
    * for each list of terms $\bar{e} = (e_1, \ldots, e_m)$ the notation $\bar{e}[\bar{e}'', \bar{e}']$ denotes
      the term $\left(e_1[e''_1/e']', \ldots, e_m[e''_m/e']\right)$.
• A clarification is a notation $\theta$ of the form

$$e_1/x_1, \ldots, e_n/x_n,$$

where $x_1, \ldots, x_n$ are different variables, $e_1, \ldots, e_n$ are simple terms, such that $\forall i = 1, \ldots, n \quad \tau(x_i) = \tau(e_i)$. $\forall e \in E$ the notation $e[\theta]$ denotes the term $e[e_1/x_1, \ldots, e_n/x_n]$ (similar notations are used when a list of terms is considered instead of the a term $e$).

(3) is called a renaming, if $e_1, \ldots, e_n$ are different variables.

2.2 A concept of a functional program

In this article, a functional program (FP) refers to a finite set $\Sigma$ of equalities of the form

$$\begin{align*}
\varphi_1(x_{11}, \ldots, x_{1n_1}) & = e_1 \\
\vdots \end{align*}$$

where

• $\varphi_1, \ldots, \varphi_m$ are different functional variables, and
• $\forall i = 1, \ldots, m \quad \varphi_i(x_{i1}, \ldots, x_{im_i})$ and $e_i$ are terms of the same type, and

$$X_{e_i} = \{x_{i1}, \ldots, x_{im_i}\}, \quad \Phi_{e_i} \subseteq \{\varphi_1, \ldots, \varphi_m\}.$$

A main term of FP (4) is the left side of first equality in (4) (i.e. the term $\varphi_1(x_{11}, \ldots, x_{1n_1})$).

The set of equalities in FP (4) can be considered as a system of functional equations for functional variables $\varphi_1, \ldots, \varphi_m$. This system defines a list

$$(f_{\varphi_1}, \ldots, f_{\varphi_m})$$

of partial functions corresponding to $\varphi_1, \ldots, \varphi_m$, which is the least (in the sense of the order on lists of partial functions described in [1]) a solution of system of functional equations (4). List (5) is called a least fixpoint (LFP) of FP (4).

All details related to the concept of a LFP of a FP, can be found in chapter 5 of the book [1]. The first function in the list (5) (i.e. $f_{\varphi_1}$) is denoted by $f_{\Sigma}$, and is called a function defined by the FP $\Sigma$.

Let $\Sigma$ be a FP. The notation $E_{\Sigma}$ denotes the set of all terms, such that all functional variables occurred in them, are occurred in $\Sigma$.

FPs $\Sigma$ and $\Sigma'$ are considered as equal, if $\Sigma'$ is derived from $\Sigma$ by renaming of data variables and functional variables, i.e. if $X\Phi_\Sigma$ and $X\Phi_{\Sigma'}$ are sets of data variables and functional variables occurred in $\Sigma$ and $\Sigma'$ respectively, then there is a one-to-one correspondence $f : X\Phi_\Sigma \rightarrow X\Phi_{\Sigma'}$, such that $\Sigma'$ is derived from $\Sigma$ by replacing each variable $v \in X\Phi_\Sigma$ on $f(v)$.
3 An example of specification and verification of a functional program

3.1 An example of a functional program

Consider the following FP:

\[
\begin{align*}
\text{sort}(x) &= [x = \varepsilon] \varepsilon : \text{insert}(x_h, \text{sort}(x_t)) \\
\text{insert}(a, y) &= [y = \varepsilon] a\varepsilon : \begin{cases} 
[a \leq y_h] ay : y_h \text{ insert}(a, y_t)
\end{cases}
\end{align*}
\]  

This FP defines a sorting function on strings. The FP consists of two equations that define the following functions:

- \textbf{sort} : S \rightarrow S is a main function, and
- \textbf{insert} : (C, S) \rightarrow S is an auxiliary function, this function maps a pair \((a, y) \in \mathcal{D}_C \times \mathcal{D}_S\) to the string derived by inserting a character \(a\) to the string \(y\), such that the following condition holds: if the string \(y\) is ordered, then the string \(\text{insert}(a, y)\) also is ordered (a string is ordered, if its components form a non-decreasing sequence).

3.2 An example of a specification of a functional program

One of the properties of correctness of FP (6) has the form: \(\forall x \in \mathcal{D}_S\) the string \(\text{sort}(x)\) is ordered. This property can be described formally as follows. Consider a FP defining a function \textbf{ord} of string ordering checking:

\[
\text{ord}(x) = [x = \varepsilon] 1 : [x_t = \varepsilon] 1 : [x_h \leq (x_t)_h] \text{ ord}(x_t) : 0
\]  

(7)

The function \textbf{ord} allows to describe the above property of correctness as the following statement:

\(\forall x \in \mathcal{D}_S \ \text{ord}(\text{sort}(x)) = 1\).  

(8)

3.3 An example of verification of a functional program

The problem of verification of the correctness property of FP (8) of FP (6) consists of a formal proof of proposition (8). This proposition can be proved like an ordinary mathematical theorem, for example using the method of mathematical induction. A proof of this proposition can be the following.

If \(x = \varepsilon\), then, according to first equation of system (6), the equality \(\text{sort}(x) = \varepsilon\) holds, and therefore

\(\text{ord}(\text{sort}(x)) = \text{ord}(\varepsilon) = 1\).

Let \(x \neq \varepsilon\). We prove (8) for this case by induction. Assume that for each string \(y\), which is shorter than \(x\), the equality

\(\text{ord}(\text{sort}(y)) = 1\)


holds. Prove that this implies the equality
\[
\text{ord}(\text{sort}(x)) = 1. \tag{9}
\]

(9) is equivalent to the equality
\[
\text{ord}(\text{insert}(x_h, \text{sort}(x_t))) = 1. \tag{10}
\]

By the induction hypothesis, the equality
\[
\text{ord}(\text{sort}(x_t)) = 1,
\]
holds, and this implies (10) on the reason of the following lemma.

**Lemma.**

The following implication holds:
\[
\text{ord}(y) = 1 \implies \text{ord}(\text{insert}(a, y)) = 1 \tag{11}
\]

**Proof.**

We prove the lemma by induction on the length of \( y \). If \( y = \varepsilon \), then the right side of (11) has the form
\[
\text{ord}(a\varepsilon) = 1,
\]
which is true by definition of \( \text{ord} \).

Let \( y \neq \varepsilon \), and for each string \( z \), which is shorter than \( y \), the following implication holds:
\[
\text{ord}(z) = 1 \implies \text{ord}(\text{insert}(a, z)) = 1 \tag{12}
\]

Let \( c \overset{\text{def}}{=} y_h, d \overset{\text{def}}{=} y_t \). Then (11) has the form
\[
\text{ord}(cd) = 1 \implies \text{ord}(\text{insert}(a, cd)) = 1 \tag{13}
\]

To prove the implication (13) it is necessary to prove that if \( \text{ord}(cd) = 1 \), then the following implications hold:

(a) \( a \leq c \implies \text{ord}(a(cd)) = 1 \),
(b) \( c < a \implies \text{ord}(c \text{ insert}(a, d)) = 1 \).

(a) holds because \( a \leq c \) implies
\[
\text{ord}(a(cd)) = \text{ord}(cd) = 1.
\]

Let us prove (b).

- \( d = \varepsilon \). In this case, right side of (b) has the form
\[
\text{ord}(c(a\varepsilon)) = 1 \tag{14}
\]

(14) follows from \( c < a \).
\[ d \neq \varepsilon. \] Let \( p \overset{\text{def}}{=} d_h, \ q \overset{\text{def}}{=} d_t. \]

In this case, it is necessary to prove that if \( c < a \), then

\[ \text{ord}(c \text{ insert}(a, pq)) = 1 \quad (15) \]

1. if \( a \leq p \), then (15) has the form

\[ \text{ord}(c(a(pq))) = 1 \quad (16) \]

Since \( c < a \leq p \), then (16) follows from the equalities

\[ \text{ord}(c(a(pq))) = \text{ord}(a(pq)) = \text{ord}(pq) = \text{ord}(c(pq)) = \text{ord}(cd) = 1 \]

2. if \( p < a \), then (15) has the form

\[ \text{ord}(c(p \text{ insert}(a, q))) = 1 \quad (17) \]

Since, by assumption,

\[ \text{ord}(cd) = \text{ord}(c(pq)) = 1 \]

then \( c \leq p \), and therefore (17) can be rewritten as

\[ \text{ord}(p \text{ insert}(a, q)) = 1 \quad (18) \]

If \( p < a \), then

\[ \text{insert}(a, d) = \text{insert}(a, pq) = p \text{ insert}(a, q) \]

therefore (18) can be rewritten as

\[ \text{ord}(\text{insert}(a, d)) = 1 \quad (19) \]

(19) follows from the induction hypothesis for the Lemma (i.e., from the implication (12), where \( z \overset{\text{def}}{=} d \)) and from the equality

\[ \text{ord}(d) = 1 \]

which is justified by the chain of equalities

\[
1 = \text{ord}(cd) = \text{ord}(c(pq)) = \text{ord}(pq) = \text{ord}(d). \]

From the above example we see that even for the simplest FP, which consists of several lines,

- a proof of its correctness is not trivial mathematical reasoning,
- it is difficult to check this proof, and
- it is much more difficult to construct this proof.

Below we present a radically different method for verification of FPs based on a construction of state diagrams for FPs. We illustrate our approach by a proof of the proposition (8) on the base of the proposed method. This proof can be generated automatically, that is an evidence of an advantage of the method for verification of FPs based on state diagrams.
4 States of functional programs

4.1 A concept of a state of a functional program

Let $\Sigma$ be a FP. A state of $\Sigma$ is a notation $s$ of the form $\{\beta\}_y^{x_1,\ldots,x_n}$, where

- $\beta \in E_\Sigma$ is a formula, called a condition of the state $s$,
- $y$ is a simple term, called an output term of the state $s$, and
- $x_1,\ldots,x_n$ is a list of simple terms, called input terms of the state $s$.

We shall use the following notations:

- the set of all states of FP $\Sigma$ is denoted by $S_\Sigma$,
- $\forall s \in S_\Sigma$ the set of all data variables occurred in $s$ is denoted by $X_s$,
- if the state $s$ has the form $\{\beta\}_y^{x_1,\ldots,x_n}$, then the terms $\beta$, $y$ and the list $x_1,\ldots,x_n$ can be denoted by $\beta_s$, $y_s$, and $\bar{x}_s$, respectively,
- if $\beta_s$ has the form $e_1 \land \ldots \land e_n$, then $s$ can be denoted by $\{e_1^{\bar{x}_i}\}_y^{\bar{x}_s}$,
- if $\beta_s = \top$, then $s$ can be denoted by $\{\}_y^{\bar{x}_s}$.

A state $s \in S_\Sigma$ is said to be an initial state of FP $\Sigma$ (and is denoted by $s^0_\Sigma$), if it has the form $\{y = \varphi(\bar{x})\}_y^{\bar{x}_s}$ where

- $\varphi$ is a functional variable occurred in main term of FP $\Sigma$,
- $\bar{x}$ is a list of different variables,
- $y$ is a variable which is not occurred in $\bar{x}$, and
- $\tau(\varphi) = \tau(\bar{x}) \rightarrow \tau(y)$.

A state $s \in S_\Sigma$ is said to be terminal, if $\Phi_s = \emptyset$.

If $s \in S_\Sigma$ and $\theta$ is a clarification, then

$$s[\theta] \overset{\text{def}}{=} \{\beta_s[\theta]\}_y^{\bar{x}_s[\theta]}.$$  

If $\theta$ is a renaming, then we say that $s[\theta]$ is derived from $s$ by a renaming.

4.2 Equality of terms and states

Let $X$ be a subset of $\mathcal{X}$. An evaluation of variables from $X$ is a function $\xi : X \rightarrow D$, which maps each variable $x \in X$ to a value $\xi(x)$ of the type $\tau(x)$.

A set of all evaluations of variables from $X$ $X$ is denoted by $X^\bullet$. For

- each evaluation $\xi \in X^\bullet$, and
- each term $e$, such that $X_e \subseteq X$ and $\Phi_e = \emptyset$,

$e^\xi$ denoted an object which either is a value from $D$, or is not defined, and is defined, and is computed recursively:

- if $e = x \in X$, then $e^\xi = \xi(x)$,
• if $e = c \in C$, then $e^\xi$ is a value of the constant $c$,
• if $e = f(e_1, \ldots, e_n)$, where $f \in F$, then
  – $e^\xi$ is equal to the value $f(e_1^\xi, \ldots, e_n^\xi)$, if this value is defined,
  – $e^\xi$ is not defined, otherwise.

Let $\Sigma$ be a FP. $\forall e \in E_\Sigma, \forall \xi \in X^\bullet : X \supseteq X_e$, the notation $e^\xi_\Sigma$ denotes a value of $e$ on $\xi$ with respect to $\Sigma$, which is defined as above, but with the following difference: functional variables from $\Phi_e$ are considered as FSs, associated with partial functions, which are corresponding components of a least fixpoint of $\Sigma$.

Terms $e_1$ and $e_2 \in E_\Sigma$ are considered as equal (with respect to the FP $\Sigma$), if $\forall \xi \in (X_{e_1} \cup X_{e_2})^\bullet$ the objects $e_1^\xi_\Sigma$ and $e_2^\xi_\Sigma$ are both
• either not defined,
• or defined and equal.

Examples of pairs of equal terms:

$e_1 e_1' = e_2 e_2'$ and $(e_1 = e_2) \land (e_1' = e_2')$,
$e e' = e$ and $\bot$,
$[T] e : e'$ and $e$,
$[\bot] e : e'$ and $e'$,
$e = T$ and $e$,
$e = \bot$ and $\neg e$,
$e \land (e' = e''$) and $e[e'/e'] \land (e' = e'')$.

Let $\Sigma$ be a FP. States $s, s' \in S_\Sigma$ are considered as equal, if one of the following conditions hold:
• $s'$ can be derived from $s$ by a renaming, and $\bar{x}_s = \bar{x}_{s'}$,
• $\beta_{s'} = \beta \land (x = e)$, where $x \in X, e \notin X_e, e \in E$, $\beta$ can be derived from $\beta_s$ by a replacement of some occurrences of $e$ on $x$, $y_{s'} = y_s, \bar{x}_{s'} = \bar{x}_s$,
• $\beta_s = \beta \land (x = e)$, where $x \in X, e$ is a simple term, $x \notin X_e, s' = s[e/x]$.

4.3 Transitions of functional programs

In this section we define a concept of a transition of a FP. A transition of a FP
• represents a relation between states of the FP, and has a label which is
  – either a functional variable, or
  – or a formula, called a condition of this transition.

A transition of FP $\Sigma$ is a triple $r = (s, s', l)$, where $s, s'$ are states from $S_\Sigma$, called a start and an end of the transition $r$, respectively, and $l$ is a label of transition $r$. A transition $(s, s', l)$ is called a transition from $s$ to $s'$. It can be denoted by $s \xrightarrow{l} s'$.

Let $s = \{\beta\}_X^e \in S_\Sigma$. There are the following transitions starting from $s$: 
1. if $\beta$ contains a subterm of the form $\varphi(\bar{e})$, and one of equations in $\Sigma$ has the form $\varphi(\bar{x}) = e$, then there is a transition (called an expansion)

$$s \xrightarrow{\varphi} \{\beta[e/\bar{x}]/[\varphi(\bar{e})]\}_y^y,$$

2. if $\beta$ contains a subformula $e$, then there is a pair of transitions

$$s \xrightarrow{e} \{\beta \land e\}_2^y, \quad s \xrightarrow{-e} \{\beta \land \neg e\}_2^y, \quad (20)$$

3. if $\beta$ contains a subterm $e$ of the type $S$, then there is a pair of transitions

$$s \xrightarrow{e=x} \{\beta \land (e = \varepsilon)\}_2^y, \quad s \xrightarrow{e=x'} \{\beta \land (e = xx')\}_2^y, \quad (21)$$

where $x, x'$ are fresh variables (which are not occurred in $X_s$).

Any transition, occurred in a pair of the form (20) or (21), is said to be complementary to another transition from this pair.

A set of all transitions of FP $\Sigma$ is denoted by $R_\Sigma$.

5 Neighborhoods of states of functional programs

5.1 Unfoldings of states

Let $\Sigma$ be a FP. An unfolding of a state $s \in S_\Sigma$ is a finite tree $V$,

- each node $v$ of which is associated with a state $s_v \in S_\Sigma$,
- a root of this tree is associated with the state $s$, and
- each edge $r$ of which is associated with a transition from $R_\Sigma$ of the form $s_v \xrightarrow{l} s_{v'}$, where $v$ and $v'$ are a start and an end of the edge $r$, and a label $l$ of this transition is also a label of the edge $r$.

Nodes and edges of $V$ will be identified with those states and transitions respectively, which are associated with them.

5.2 A concept of a neighborhood of a state

Let $\Sigma$ be a FP. Each state $s \in S_\Sigma$ is associated with a set $U_s$ of neighborhoods of the state $s$. Each neighborhood $U \in U_s$ is a tree,

- nodes of which are associated with states from $S_\Sigma$; and
- edges of which are labeled by lists of labels used in unfoldings of states.

The set $U_s$ is defined as follows.

1. Each unfolding $V$ of $s$, such that $\forall v \in V$ the set of edges outgoing from $v$
   - either is empty (in this case $v$ is said to be a leaf),
or consists of only edge, which is labeled by an expansion,
or consists of two complementary edges,

belongs to the set $U_s$.

2. Let $U \in U_s$, and $s'$ is a node of $U$, which is not a root or leaf, then if

- an edge ended in $s'$, has the form $s_0 \xrightarrow{l} s'$, and
- edges started in $s'$, have the form $s' \xrightarrow{l_1} s_1$, ..., $s' \xrightarrow{l_n} s_n$,

then $U_s$ has a tree $U'$, derived from by $U$

- a removing of the node $s'$ and edges related to this node, and
- adding edges $s_0 \xrightarrow{l_1} s_1$, ..., $s_0 \xrightarrow{l_n} s_n$, where $\forall i = 1, \ldots, n$ $l_i$ is a concatenation of lists $l$ and $l_i$.

3. Let $U \in U_s$, and $s'$ is a contradictory node $U$ (i.e. $\beta_{s'} = \bot$), which is not a root, then $U_s$ has a tree $U''$, derived from $U$ by a removing of nodes reachable from $s'$ (i.e. such that there are paths from $s'$ to these nodes), and edges related to these nodes.

A neighborhood $U''$, derived from $U$ according to items 2 and 3 of this definition, is said to be a reduction of the neighborhood $U$.

It is not so difficult to prove that

- a node $s$ of some neighborhood is contradictory iff ends of all edges outgoing from $s$ are contradictory, and
- a state is contradictory iff all leaves of some its neighborhood are contradictory.

If $U$ is a neighborhood of some state, then $\forall v \in U$ there is a unique path from a root of $U$ to the node $v$. All nodes of $U$, lying on this path and not coinciding with $v$, are said to be ancestors of $v$.

We shall use the following agreement in graphical representation of neighborhoods: if $U$ is a neighborhood of some state, then in a graphical representation of the neighborhood $U$

- nodes of $U$ are represented by ovals,
- a root of $U$ is represented by a double oval,
- contradictory nodes can be represented by black boxes ($\blacksquare$),
- $\forall s \in U$ an oval $O_s$, representing $s$, has the following form:
  - conjunctive terms occurred in $\beta_s$, are displayed in a column inside $O_s$ (if $\beta_s = \top$, then nothing is drawn inside $O_s$),
  - the list $\bar{x}_s$ of input terms and output term $y_s$ of the state $s$ are displayed to the right of $O_s$ from the bottom and from the top, respectively, and
  - an identifier of the state $s$ is displayed at the top from the left of $O_s$.  

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• edges occurred in $U$ are represented by arrows connecting ovals: if $U$
contains the edge $s \rightarrow s'$, then
  – then this edge is represented by an arrow from $O_s$ to $O_{s'}$, and
  – near this arrow the components of the label $l$ may be depicted.

5.3 Examples of neighborhoods

In this section, we give examples of neighborhoods of states for a FP of sorting
(6) and for FP of checking the ordering of strings (7).

5.3.1 Examples of neighborhoods for the program of sorting

We rewrite the FP of sorting (6), using shorter notation for the function variables
occurred in it (we denote terms of the form $\text{sort}(x)$ and $\text{insert}(a, y)$ by the
notations $\varphi(x)$ and $a \rightarrow y$ respectively):

\begin{equation}
\begin{aligned}
\{ & \varphi(x) = \{ [x = \epsilon] \in : x_h \rightarrow \varphi(x_t) \} \\
& a \rightarrow y = \{ [y = \epsilon]a\epsilon : \{ a \leq y_h \} ay : y_h(a \rightarrow y_t) \}
\end{aligned}
\end{equation}

One of unfoldings of the state $s_0 \overset{\varphi}{\rightarrow} \{ y = \varphi(x) \}$ consists of the following
states and edges:

\begin{align*}
s_0 & \xrightarrow{\varphi} \{ y = \varphi(x) \} \\
s & \xrightarrow{x = \epsilon} \{ y = \varphi(x) \} \\
s & \xrightarrow{x = ab} \{ y = \varphi(x) \}
\end{align*}

One of neighborhoods, corresponded to this unfolding, has the form

\begin{equation}
\begin{aligned}
&s_0 \xrightarrow{\varphi} \{ y = \varphi(x) \} \\
&s_1 \xrightarrow{\varphi} \{ y = \varphi(x) \}
\end{aligned}
\end{equation}
One of neighborhoods of state $s_2$ in (23) has the form

$$s_2 \xrightarrow{y = a \rightarrow p} p = \varepsilon \xrightarrow{\varepsilon = \varphi(b)} b = \varepsilon \xrightarrow{\varepsilon = \varphi(ij)} y \xrightarrow{y = \varphi(ij)} s_2 \xrightarrow{a\varepsilon} \cdots \xrightarrow{ab^{s_3} a\varepsilon}$$

One of neighborhoods of the state $s^*$ in (24) has the form

$$s^* \xrightarrow{\varepsilon = \varphi(ij)} \cdots \xrightarrow{ab^{s_3} a\varepsilon}$$

All leaves of the last neighborhood are contradictory, since among the conjunctive terms occurred in their conditions, there are equalities of the form $\varepsilon = uv$, which are equal to the term $\bot$. Therefore, as it was said above, $s^*$ is contradictory.

Bringing together neighborhoods (23) and (24), in view of the foregoing, we conclude that one of the neighborhoods of state $s_0 = \{y = \varphi(x)\}_x$ has the form

$$s_0 \xrightarrow{y = \varphi(x)} x \xrightarrow{a\varepsilon} \cdots \xrightarrow{ab^{s_3} a\varepsilon}$$

Another example of a neighborhood is related to the state $s_5$. One of neigh-
This neighborhood can be reduced, and we can get the neighborhood

5.3.2 Examples of neighborhoods for string ordering checking program

Other examples of neighborhoods of states are related to the FP (17) of string ordering checking. We rewrite this FP using a shorter notation for the function variable occurred in it:

\[
o(x) = [x = \epsilon]1 : ([x_t = \epsilon]1 : [x_h \leq (x_t)_h]o(x_t) : 0)\]
One of neighborhoods of the state \( \sigma_0 \overset{\text{def}}{=} \{ z = o(h) \} \) of this FP has the form

\[
\begin{align*}
\sigma_0 \quad z = o(h) \\
h = \varepsilon \\
h = u.f \\
z = o(u.f) \\
f = \varepsilon \\
f = v.f \\
(28)
\end{align*}
\]

Using the definition of the concept of a neighborhood of a state of a FP, (28) can be transformed to the neighborhood

\[
\begin{align*}
\sigma_1 \quad u \leq v \\
z = o(vw) \\
(29)
\end{align*}
\]

\[
\begin{align*}
\sigma_0 \quad z = o(h) \\
h = \varepsilon \\
h = u.f \\
z = o(u.f) \\
f = \varepsilon \\
f = v.f \\
(28)
\end{align*}
\]

Using the definition of the concept of a neighborhood of a state of a FP, (28) can be transformed to the neighborhood

\[
\begin{align*}
\sigma_0 \quad z = o(h) \\
h = \varepsilon \\
h = u.f \\
z = o(u.f) \\
f = \varepsilon \\
f = v.f \\
(28)
\end{align*}
\]

\[
\begin{align*}
\sigma_1 \quad u \leq v \\
z = o(vw) \\
(29)
\end{align*}
\]

6 Embeddings of states of functional programs

6.1 Explicit, conditional and justified embeddings

Let \( s, s' \) be states from \( S_\Sigma \).

- An explicit embedding \( s \) in \( s' \) is a notation of the form

\[
\theta : s \leftrightarrow s',
\]

where \( \theta \) is a clarification, and \( \beta_s = \beta_v[\theta] \land \beta, \) and \( \Phi_\beta = \emptyset. \)

- A conditional embedding \( s \) in \( s' \) is a notation of the form

\[
\left\{ \begin{array}{l}
\eta : r \leftrightarrow r' \\
u[\theta] \leftrightarrow u'[\theta']
\end{array} \right\} : s \leftrightarrow s',
\]

where \( \eta : r \leftrightarrow r' \) is an explicit embedding, \( u, u' \in S_\Sigma; \) \( \theta \) and \( \theta' \) are clarifications, and

\[
\beta_s = \beta_u[\theta] \land \beta_r, \quad \beta_{s'} = \beta_{u'}[\theta'] \land \beta_{r'}.
\]

A premise of the conditional embedding (30) is a notation \( u \leftrightarrow u' \), where \( u \) and \( u' \) are corresponding states occurred in (30).
A justified embedding of $s$ in $s'$ is a notation of the form

$$s^1 \rightarrow s'$$

(31)

if $\exists U \in \mathcal{U}_s, \exists U' \in \mathcal{U}_{s'}$: for each non-terminal leaf $r \in U$

- either there is an explicit embedding $r$ in some $r' \in U'$,
- or there is a conditional embedding $r$ in some $r' \in U'$, and its premise has the form $s \rightarrow s'$, where $s$ and $s'$ are states from $\mathcal{U}$.

A state $s$ is said to be embedded in $s'$, if there is

- either explicit embedding $s$ in $s'$,
- or conditional embedding $s$ in $s'$ with a justified premise.

The notation $s \subseteq s'$ means that $s$ is embedded in $s'$.

Note that each justified embedding can be considered as a conditional embedding with a justified premise (an “explicit embedding” component in this conditional embedding is trivial).

6.2 Examples of embeddings of states

6.2.1 Examples of explicit embeddings of states

1. For the states $s_4 = \{ a \leq c \}
\text{ and } \{ cd = \varphi(b) \}$, occurred in neighborhood (25), there is an explicit embedding $\text{[cd/y,b/x]} : s_4 \rightarrow s_0$. (32)

2. For the states $s_7 = \{ c < a, c \leq i \}
\text{ and } \{ q = a \rightarrow ij \}
\text{ and } \{ ij = \varphi(g) \}$, occurred in neighborhoods (26) and (23) respectively, there is an explicit embedding $\text{[q/y,ij/p,g/b]} : s_7 \rightarrow s_2$. (33)

3. For the states $\sigma_1 = \{ r \leq v \}
\text{ and } \{ z = o(vw) \}$, occurred in neighborhood (29), there is an explicit embedding $\text{[vw/h]} : \sigma_1 \rightarrow \sigma_0$. (34)

6.2.2 An example of conditional embedding

For the states $s_8 = \{ c < a, c < r \}
\text{ and } \{ q = a \rightarrow d \}
\text{ and } \{ d = r \rightarrow j \}
\text{ and } \{ cj = \varphi(g) \}$, occurred in neighborhoods (26) and (23) respectively, there is a conditional embedding
with the premise $s_5 \leadsto s_0$:

\[
\begin{align*}
[q/y, d/p] : & \left\{ \begin{array}{c}
c < a \\
q = a \rightarrow d \\
cq \end{array} \right\}_{a+d} \leadsto \left\{ y = a \rightarrow p \right\}_{d+p}^y \\
s_5[\theta] = & \left\{ \begin{array}{c}
c < r \\
d = r \rightarrow j \\
\varphi = \varphi(\varphi) \\
cj \end{array} \right\}_{r} \leadsto s_0[\theta'] = \left\{ p = \varphi(b) \right\}_b^p
\end{align*}
\]

\[ s_8 \leadsto s_2, \quad (35) \]

where $\theta = [r/a, d/q, j/d, g/b]$, $\theta' = [p/y, b/x]$.

### 6.2.3 An example of a justified embedding

An example of a justified embedding is $s_5 \leadsto s_0$, where $s_5$ and $s_0$ are states from $(25)$. In this case $U = (26)$ and $U' = (23)$.

In the neighborhood $(26)$

- state $s_6$ is terminal,
- there is explicit embedding $(33)$ of state $s_7$ in state $s_2$, and
- there is a conditional embedding $(35)$ of state $s_8$ in state $s_2$ with the premise $s_5 \leadsto s_0$.

### 7 State diagrams

#### 7.1 A concept of a state diagram

Let $\Sigma$ be a FP. A **state diagram** (SD) of $\Sigma$ is a triple

\[ D = (U, N, I), \]

whose components have the following meaning:

- $U$ is a neighborhood of the initial state $s_0^0$,
- $N$ is a set of all non-terminal leaves of $U$, and
- $I$ is a set of pairs of the form $(s, s')$, where $s \in N$, $s'$ is an ancestor of $s$, $s \subseteq s'$, and $\forall s \in N \; \exists s' : (s, s') \in I$.

In the graphic representation of SD $(36)$ we will denote pairs from $I$ by labelled arrows on the neighborhood $U$: a pair $(s, s') \in I$ will be represented by an arrow starting from $s$, ending to $s'$ and labelled by $\subseteq$.

#### 7.2 Examples of state diagrams

**7.2.1 State diagram for a sorting program**

SD for FP $(22)$ is based on neighborhood $(25)$ and has the form
7.2.2 State diagram for the program of string ordering checking

SD for FP (27) is built on the base of neighborhood (29) and has the form

In this SD, an edge labeled by $\subset$ corresponds to explicit embedding (34).

8 Verification of functional programs based on the concept of a state diagram

8.1 Composition of functional programs

Let $\Sigma$ and $\Sigma'$ be FPs, and main terms in $\Sigma$ and $\Sigma'$ have the form $\varphi(\bar{x})$ and $\varphi'(u)$ respectively, where $\tau(\varphi(\bar{x})) = \tau(u)$, and $X\Phi_\Sigma \cap X\Phi_{\Sigma'} = \emptyset$.

In this case, it can be defined a FP $\Sigma'(\Sigma)$, called a composition of FPs $\Sigma$ and $\Sigma'$, and is a set of equalities,

- the first of which has the form $\psi(\bar{x}) = \varphi'(\varphi(\bar{x}))$, where $\psi$ is a fresh functional a variable of the appropriate type, and
- other equalities are all equalities, occurred in $\Sigma$ and $\Sigma'$.

It is easy to see that $\forall \bar{d} \in D_\Sigma \ f_{\Sigma'(\Sigma)}(\bar{d}) = f_{\Sigma'}(f_{\Sigma}(\bar{d}))$. 

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8.1.2 Neighborhoods of an initial state of a composition of functional programs

Let

- Σ and Σ’ be FPs satisfying conditions at the beginning of item 8.1.1 and
- U ∈ U_Σ, U’ ∈ U_Σ’ be neighborhoods such that

∀ s ∈ U, ∀ s’ ∈ U’ \ X_s \cap X_{s'} = \emptyset.

∀ s ∈ U, ∀ s' ∈ U' it will be denoted by ss' a state FP Σ'(Σ), defined as follows: let s = {β}^{y}_{x}, s' = {β'}^{y'}_{x'}, then

\[ ss' \overset{\text{def}}{=} \{ β \land β' \land (y = y') \}^{z}_{x}. \]

It is easy to see that if \( s_0^{0}_{\Sigma} = \{ y = \varphi(\bar{x}) \}^{y}_{x} \) and \( s_0^{0}_{\Sigma'} = \{ z = \varphi'(y') \}^{z}_{x'} \), then

\[ s_0^{0}_{\Sigma'}(\Sigma) = \{ z = \psi(\bar{x}) \}^{z}_{x} = \{ z = \varphi'(\varphi(\bar{x})) \}^{z}_{x} = s_0^{0}_{\Sigma} s_0^{0}_{\Sigma'}. \]

Let \( UU' \) be a tree,

- nodes of which have labels of the form ss', where \( s \in U, s' \in U' \), and
- which is defined by an a non-deterministic algorithm for its construction.

The algorithm of construction of the tree \( UU' \) consists of several stages. A tree built at each of these stages is denoted by the same notation \( UU' \).

- At the first stage, \( UU' \) is defined as a tree from one node that has the label \( s_0^{0}_{\Sigma} s_0^{0}_{\Sigma'} \).

- Each subsequent step is that if the tree \( UU' \) constructed so far contains a leaf \( v \) labeled by ss', where either \( s \) is not a leaf in \( U \), or \( s' \) is not a leaf in \( U' \) then one of the following two operations is performed:

  - if \( s \) is not a leaf in \( U \), and the list of its followers is of the form \( s_1, \ldots, s_n \), then the followers of the node \( v \) with the labels \( s_1 s', \ldots, s_n s' \), are added to the constructed tree \( UU' \).

  - if \( s' \) is not a leaf in \( U' \), then then instead of the previous operation a similar operation can be performed for followers of \( s' \).

Theorem 1
The above tree \( UU' \) is a neighborhood of an initial state of FP Σ'(Σ).

8.1.3 A state diagram of a composition of functional programs

Theorem 2
Let FPs Σ and Σ’ have SDs, and the composition Σ'(Σ) is defined.
Then FP Σ'(Σ) also has SD.
8.2 The problem of verification of functional programs

The problem of verification of a FP $\Sigma$ is in constructing the proof of the statement that FP $\Sigma$ satisfies property expressed by some formal specification $Spec$.

Below we shall use the following agreement: the notation of the form $f = 1$, where $f$ is a function, denotes the following statement:

the function $f$ has a value 1 on all its arguments.

In some cases

- formal specification $Spec$ is expressed by another FP $\Sigma'$, and
- correctness $\Sigma$ with respect to $Spec$ is represented by the statement

$$f_{\Sigma'}(\Sigma) = 1. \quad (39)$$

For example, one of the correctness properties of sorting FP in section 3.1 is expressed by a statement of the form (39) (namely, by the statement (8)).

Theorem 3

Let FP $\Sigma$ has a SD, in which for each terminal state $s$ the term $y_s$ is a constant 1. Then $f_\Sigma = 1$.

The above theorems are the theoretical basis of the FP verification method based on the construction of SD

- for the analyzed FP $\Sigma$, and
- for FP $\Sigma'$, representing the property being checked.

If these FPs have SDs, then, according to the theorem 2, $\Sigma'(\Sigma)$ also has a SD. If this SD has the property described in theorem 3, then (39) holds.

The following section provides an example of the application of this method.

8.3 An example of verification of a sorting functional program using a state diagram

In this section, we illustrate the verification method described above with an example of the proof of the statement (8) for FP defined in section 3.1.

To prove equality (39), where $\Sigma = (22)$ and $\Sigma' = (27)$, we construct a neighborhood of the initial state $s_0^\Sigma$ as a neighborhood if the form $UU'$, according to the algorithm at (8.1.2) where

- $U$ is the neighborhood (25) of the state $s_0^\Sigma$ and
- $U'$ is the neighborhood (29) of the state $s_0^{\Sigma'}$. 

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Some states of the resulting neighborhood will be contradictory. After removing them, we get the following neighborhood:

Neighborhood (40) has 5 leaves. It is easy to see that

- output term of two of these leaves ($\sigma_3s_1$ and $\sigma_4s_3$) is equal to 1,
- these is an explicit embedding of the leaf $\sigma_1s_4$ to the state $\sigma_0s_0$:
  $$[b/x, cd/y] : \sigma_1s_4 \hookrightarrow \sigma_0s_0,$$
- there is a conditional embedding $\sigma_1s_5$ to $\sigma_0s_0$ with a justified premise $s_5 \hookrightarrow s_0$:
  $$\begin{align*}
  &\left\{ [vw/y] : \begin{cases} 
  c \leq v \\
  z = o(vw)
  \end{cases} \right\} : \sigma_1s_5 \hookrightarrow \sigma_0s_0.
  
  \right\}
\end{align*}$$

Let is construct a neighborhood of the leaf $\sigma_2s_5$. Consider followers of the state $\sigma_2s_5$, corresponding to followers $s_6, s_7, s_8$ of the state $s_5$.

- $\sigma_2s_6 = \{ v < u, c < a, uvw = caε \}_{acε}^0$, this state is contradictory.
- $\sigma_2s_7 = \{ v < c < a, c \leq i \}_{acg}^0$. There are two complementary transitions from this state to states,
  - one of which has a conjunctive term $v = a$ in its condition, and
  - another state has a conjunctive term $v = i$ in its condition.

It is easy to see that both of these states are contradictory.
\[ \sigma_{2s_{8}} = \begin{cases} v < c < a, c < r \\ vw = a \rightarrow d \\ d = r \rightarrow j \\ cj = \varphi(g) \end{cases} \] . There are two complementary transitions from this state to states \( s, s' \), where

\( \beta_{s} \) has a conjunctive term \( d = \varepsilon \), thus \( \beta_{s} \) has a conjunctive term \( v = a \), where it is easy to get a conjunctive term \( a < c < a \) in \( \beta_{s} \), i.e. \( s \) is contradictory,

\( \beta_{s}' \) has a conjunctive term \( d = pq \), where \( p, q \) are fresh variables. There are two complementary transitions from \( s' \) to the states \( \tilde{s}, \tilde{s}' \), where

\( \star \beta_{s} \) has a conjunctive term \( a \leq p \), whence it follows that there is a conjunctive term \( v = a \) in \( \beta_{s} \), where it is easy to prove that \( \tilde{s} \) is contradictory, and

\( \star \) there is a conjunctive term \( p < a \) in \( \beta_{s}' \), and

\[ \tilde{s}' = \begin{cases} v < c < a, c < r \\ w = a \rightarrow q \\ vq = r \rightarrow j \\ cj = \varphi(g) \end{cases} \] \( \sigma_{2s_{5}} \). Thus, one of neighborhoods of \( \sigma_{2s_{5}} \) has the form

\[ \sigma_{2s_{5}} \rightarrow \tilde{s}' . \] (41)

There is an explicit embedding

\[ [q/w, r/a, j/d, g/b] : \tilde{s}' \hookrightarrow \sigma_{2s_{5}}. \]

A union of (40) and (41) is a neighborhood with five leaves, such that

\( \bullet \) two of there leaves are terminal, and their output term is equal to 1, and

\( \bullet \) other leaves are non-terminal, and each of them is included in some its ancestor.

Thus, the union of neighborhoods (40) and (41), with the above embeddings of non-terminal leaves, is a SD of FP \( \Sigma^{'},(\Sigma) \). On the reason of theorem 3 we conclude that equality (39) holds.

9 Conclusion

In the article, we have introduced the concept of a state diagram of a functional program and have proposed a verification method based on the concept of a state diagram.

The main advantage of the proposed verification method is the possibility of its full automation: a construction of a state diagram for a functional program can be performed automatically using a fairly simple algorithm.

One of the problems for further research related to the concept of a state diagram is the following: to find a sufficient condition (possibly the strongest) for a functional program, such that if a functional program satisfies this condition, then it has a state diagram.
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