Coherence and finiteness spaces
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Abstract. This very short note presents an unsuspected relation between coherence spaces from [4] and finiteness spaces from [2] in the form of a linear functor from \textbf{Coh} to \textbf{Fin}. The main used in this analysis is the infinite Ramsey theorem.

0. Introduction. The category of coherence spaces was the first denotational model for linear logic (see [4]): the basic objects are reflexive, non oriented graphs; and we are more specifically interested by their cliques (complete subgraph). If \( C \) is such a graph, we write \( C(C) \) for the collection of its cliques.

Coherence spaces enjoy a very rich algebraic structure where the most important operations on them are:
- taking the (reflexive closure of the) complement (written \( C^\perp \));
- taking a cartesian product (written \( C_1 \otimes C_2 \));
- taking a disjoint union (written \( C_1 \oplus C_2 \)).

If one only looks at the vertices, the corresponding operations are simply the identity, the usual cartesian product “\( \times \)” and the disjoint union “\( \oplus \)”.

More recently, Thomas Ehrhard introduced the notion of finiteness spaces ([2]) to give a model to the differential \( \lambda \)-calculus ([3]), which can be seen as an enrichment of linear logic. The point that interests us most here is that the collection of finitary sets of a finiteness space are closed under finite sums (i.e. finite unions) to take into account a notion of “non-deterministic sum” of terms. (See also [5].) This is definitely not possible with the cliques of a coherence space.

Very briefly, a finiteness space is given by a set \( |F| \), called the web, and a collection \( F \) of subsets of \( |F| \) such that
\[
F^{\perp \perp} = F
\]
where
\[
D^\perp = \{ x \mid \forall y \in D, \#(x \cap y) < \omega \}.
\]

Constructions similar to the one above can be defined; and they are characterized by:
- the dual \( F^\perp \);
- \( F_1 \oplus F_2 = \{ x_1 \cup x_2 \mid x_1 \in F_1, x_2 \in F_2 \} \);
- \( F_1 \otimes F_2 = \{ r \mid \pi_1(r) \in F_1, \pi_2(r) \in F_2 \} \).

Here again, if one looks only at the web \( |F| \) of finiteness spaces, the corresponding operations are just the identity, the usual cartesian product and the disjoint union.

Remarks:
- the constructions on finiteness spaces are actually defined in a way that makes it clear that they yield finiteness spaces. They are latter proved to be equivalent what is given above. (See [2].)
- There is another presentation of coherent spaces that closely matches the definition of finiteness spaces: a coherent space is given by a collection \( C \) of subsets of \( |C| \) which satisfy \( C^{**} = C \), where \( D^* = \{ x \mid \forall y \in D, \#(x \cap y) \leq 1 \} \).

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Any operator of the form $\mathcal{X} \mapsto \mathcal{X}^\bullet = \{ y \mid \forall x \in \mathcal{X}, R(x, y) \}$ is contravariant and yields a closure operator when applied twice.

1. From “coherence” to finiteness. The idea is rather simple: we would like to close the collection of cliques of a coherence space under finite unions. Unfortunately (but unsurprisingly), the notion of “finite unions of cliques” is not very well behaved. We instead consider the following notion:

**Definition.** If $C$ is a coherent space, we call a subset of $|C|$ finitely incoherent if it doesn’t contain infinite anticliques. We write $\mathcal{F}(C)$ for the collection of all finitely incoherent subsets of $C$.

The following follows directly from the definition:

**Lemma.**
- any finite subset of $|C|$ is finitely incoherent;
- any clique is finitely incoherent;
- a subset of a finitely incoherent subset is finitely incoherent;
- finitely incoherent subsets are closed under finite unions.

Note however that a finitely incoherent set needs not be a finite union of cliques: take for example the graph composed of the disjoint union of all the complete graphs $K_n$ for $n \geq 1$. This graph doesn’t contain an infinite clique, but it is not a finite union of anticliques; so, its dual is finitely incoherent but is not a finite union of cliques.

The next lemma is more interesting as it implies that the collection of finitely incoherent subsets forms a finiteness space in the sense of [2]:

**Lemma.** If $C$ is a coherence space, we have:

$$\mathcal{C}(C)^\perp = \mathcal{F}(C^\perp).$$

**Proof:**

1. Let $x \in \mathcal{C}(C)^\perp$, and suppose, by contradiction, that $x$ is not in $\mathcal{F}(C^\perp)$, i.e. $x$ contains an infinite anticlique $y$ of $C^\perp$. This set $y$ is a clique in $C$, i.e. $y \in \mathcal{C}(C)$. Since $x \cap y = y$ is infinite, this contradicts the hypothesis that $x \in \mathcal{C}(C)^\perp$.

2. Let $x$ be finitely incoherent in $C^\perp$, i.e. $x$ doesn’t contain an infinite clique of $C$; let $y$ be in $\mathcal{C}(C)$. Since $x \cap y \in \mathcal{C}(C)$ and $x \cap y$ is contained in $x$, it cannot be infinite. This shows that $x \in \mathcal{C}(C)^\perp$.

We thus get the expected corollary:

**Corollary.** If $C$ is a coherent space, then $\mathcal{F}(C)$ is a finiteness space.

What was slightly unexpected was the following:

**Lemma.** If $C$ is a coherence space, then:

$$\mathcal{F}(C^\perp) = \mathcal{F}(C)^\perp.$$ 

**Proof:** because of the previous lemma, and because $\_\perp$ is contravariant, we only need to show that $\mathcal{C}(C)^\perp \subseteq \mathcal{F}(C)^\perp$. Suppose that $x \in \mathcal{C}(C)^\perp$, and let $y \in \mathcal{F}(C)$; we need to show that $x \cap y$ is finite.

- Since $x \cap y \subseteq y \in \mathcal{F}(C)$, $x \cap y$ cannot contain an infinite anticlique;
- since $x \cap y \subseteq x \in \mathcal{C}(C)^\perp$, $x \cap y$ cannot contain an infinite clique.

Those two points imply, by the infinite Ramsey theorem, that $x \cap y$ is finite.
The other linear connectives are similarly behaved with respect to the notion of finitely incoherent sets. We have:

**Lemma.** If $C_1$ and $C_2$ are coherent spaces, then we have both

$$
\mathcal{F}(C_1 \oplus C_2) = \mathcal{F}(C_1) \oplus \mathcal{F}(C_2),
$$

and

$$
\mathcal{F}(C_1 \otimes C_2) = \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)
$$

where the connectives on the left are the coherent spaces ones, and the connectives on the right are the finiteness ones.

**Proof:** the $\oplus$ part is direct; for the $\otimes$ part, recall that $r \in \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)$ is equivalent to $\pi_1(r) \in \mathcal{F}(C_1)$ and $\pi_2(r) \in \mathcal{F}(C_2)$.

1. Suppose $r$ doesn’t contain an infinite anticlique; neither $\pi_1(r)$ nor $\pi_2(r)$ can contain an infinite anticlique, as it would imply the existence of an infinite anticlique in $r$.

2. Suppose that $r \in \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)$ contains an infinite anticlique $r'$ of $C_1 \otimes C_2$. At least one of $\pi_1(r')$ or $\pi_2(r')$ must be infinite, otherwise, $r'$ itself would be finite. Suppose $\pi_1(r')$ is infinite; because $\pi_1(r') \subseteq \pi_1(r)$, it cannot contain an infinite anticlique. By the infinite Ramsey theorem, it thus contains an infinite clique $x$. For each $a \in x$, chose one element $b$ inside the “fiber” $r'(a) = \{b \mid (a, b) \in r'\}$. Two such $b$’s can’t be coherent as it would contradict the fact that $r'$ is an anticlique: we have constructed an infinite anticlique in $\pi_2(r')$. Contradiction!

Note that because the same is true of finiteness spaces, the previous corollary implies in particular that

$$
\mathcal{F}((C_1 \oplus C_2)^\perp) = \mathcal{F}(C_1^\perp \oplus C_2^\perp).
$$

The direct proof of this equality is also quite easy.

Both coherent spaces and finiteness spaces form categories, where:

- a morphism from $C$ to $D$ in $\text{Coh}$ is a clique in $(C \otimes D^\perp)^\perp$,
- a morphism from $\mathcal{F}$ to $\mathcal{G}$ in $\text{Fin}$ is a finitary set in $(\mathcal{F} \otimes \mathcal{G}^\perp)^\perp$.

In both cases, morphisms are special relations between the webs and composition is the usual composition of relations. From all the above, we can conclude that:

**Proposition.** $\mathcal{F}(\_)$ can be lifted to a functor from $\text{Coh}$ to $\text{Fin}$:

- it sends $C$ to $\mathcal{F}(C)$
- and $r \in \text{Coh}[C, D]$ to $r \in \text{Fin}[\mathcal{F}(C), \mathcal{F}(D)]$.

Moreover, this functor commutes with the logical connectives.

Note that this functor is faithful (but not full); and that it is not injective on objects: adding and removing any finite number of edges to a coherence space doesn’t change its image via $\mathcal{F}(\_)$.

In a sense, coherence spaces allow to define a collection of “simple” finiteness spaces. An informal argument regarding this “simplicity” can be found in the following remark: the logical complexity of the formula expressing $x \in \mathcal{A}^{\perp \perp \perp}$ for a lower-closed collection of subsets $\mathcal{A}$. In the general case (see [2]), we have

$$
x \in \mathcal{A}^{\perp \perp} \iff \forall y \subseteq x, \#(y) = \infty \exists z \subseteq y, \#(z) = \infty, z \in \mathcal{A}.
$$
Because \( \#(y) = \infty \) is a \( \Sigma^1_1 \)-formula (only existential second order quantifiers), this is a \( \Pi^1_3 \)-formula (second order quantifiers are \( \forall \exists \)). For the particular case when \( A \) is the set of cliques of a coherent spaces \( C \), we obtain

\[
x \in A^\perp \perp \iff \forall y \subseteq x, \#(y) = \infty \exists a, b \in y \ (a, b) \in C
\]

which is only a \( \Pi^1_3 \)-formula.

1\( ^\perp \). Cardinality of finiteness spaces. So, coherence spaces can be used to define a (non-full) subcategory of finiteness spaces, closed under the logical operations \( (\perp, \otimes, \oplus) \). It is natural to ask whether all finiteness spaces can be obtained in this way. The previous remark about the logical complexity of coherence versus finiteness points toward a negative answer. Here is a more formal proof, which also answers a question raised by T. Ehrhard:

**Proposition.** If \( A \) is infinite countable, the cardinality of finiteness spaces on \( A \) is exactly that of \( \mathcal{P}(\mathcal{P}(A)) \). The cardinality doesn’t change if we consider finiteness on \( A \) “up to permutations”.

Since the cardinality of coherence spaces on \( A \) is the same as that of \( \mathcal{P}(A \times A) \simeq \mathcal{P}(A) \), we have:

**Corollary.** If \( A \) is infinite countable, there are strictly more finiteness spaces on \( A \) than coherence spaces on \( A \).

**Proof of the proposition:** let \( A \) be infinite countable; “up to isomorphism”, we can assume that \( A = B^{<\omega} \), the set of finite sequences of bits. If \( x \) is an infinite sequence of bits (a non dyadic real), write \( x^\perp \) for the set of finite approximations of \( x \); and if \( X \) is a set of such “real numbers”, write \( X^\perp \) for the set \( \{ x^\perp \mid x \in X \} \). We have \( X^\perp \subseteq \mathcal{P}(A) \) for any such \( X \).

Suppose now that \( X \neq X' \) with, for example, \( x \in X \) but \( x \not\in X' \). Since \( x^\perp \) is infinite and \( x^\perp \in X^\perp \), we have \( x \not\in X'^{\perp} \). However since two different reals must differ on some finite approximation, we have that \( x^\perp \in X'^{\perp} \). Thus, the finiteness spaces \( (A, X^\perp) \) and \( (A, X'^{\perp}) \) differ.

Thus, finiteness spaces on \( A \) have the same cardinality as \( \mathcal{P}(\mathbb{R}) \simeq \mathcal{P}(\mathcal{P}(A)) \).

The cardinality of finiteness spaces on \( A \) modulo permutation on \( A \) is the same because equivalence classes are of cardinality at most \( \#(\mathcal{P}(A)) \) (there are exactly \( \#(\mathcal{P}(A)) \) permutations on \( A \)), and since \( \kappa \times \#(\mathcal{P}(A)) = \max (\kappa, \#(\mathcal{P}(A))) \), there must be at least \( \#(\mathcal{P}(\mathcal{P}(A))) \) such equivalence classes to cover the whole collection of finiteness spaces.

It is slightly interesting to note that the same reasoning doesn’t apply to higher cardinalities since \( \#(A^{<\omega}) \neq \#(A) \) if \( A \) is uncountable.

\( \omega \). Conclusion. The situation with respect to full linear logic isn’t totally clear. We have a base category \( \mathbf{F} \) with:

- coherence spaces as objects
- and finitely incoherent linear maps as morphisms: \( \mathbf{F}[C, D] = \mathcal{F} ((C \otimes D^{\perp})^{\perp}) \).

This category is a linear, full subcategory of the category \( \mathbf{Fin} \) of finiteness spaces.

Lifting the usual notion set-based of exponentials for coherence spaces to this category is impossible: because of uniformity (considering only the cliques of \( C \) as vertices of \( \mathcal{F}[C] \), this construction isn’t even functorial. Take for example \( K_n \) and \( K_n^\perp \): they are isomorphic in \( \mathbf{F} \) but \( !K_n \) and \( !(K_n^\perp) \) cannot be because their webs have different cardinalities, namely \( 2^n \) and \( n+1 \). A similar phenomenon is expected if we use the multiset-based notion of exponentials. Using non-uniform coherence spaces ([1]) isn’t a solution because Ramsey theorem cannot be used anymore. Finding an appropriate notion of exponential to extend this category to a model of the algebraic \( \lambda \)-calculus ([5]), or better yet, of the differential \( \lambda \)-calculus by is thus left open at this point.
References

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