The phase structure of a chirally invariant Higgs-Yukawa model

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Organization of the talk

- 1. Introduction and motivation
- 2. Phase structure at weak Yukawa coupling
  → Analytical large $N_f$-limit vs. Numerical results
- 3. Phase structure at strong Yukawa coupling
  → Analytical large $N_f$-limit vs. Numerical results
- 4. Preliminary results on upper Higgs mass bound
- 5. Outlook
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Results published in

| arXiv: 0705:2539 | See also Lattice-Talks by |
|------------------|--------------------------|
| arXiv: 0707:3849 | Julius Kuti |
|                  | Kieran Holland |
|                  | Daniel Nogradi |
1.1 Introduction and motivation

- LHC will explore Higgs sector soon.
  → Theoretical predictions on Higgs properties are of particular interest now.
- Available theo. Higgs mass bounds depend strongly on perturbation theory.
  → Concerns that at least lower bound, based on vacuum instability, is fake (Kuti, Holland)
- Non-perturbative determination of Higgs mass bounds desired.
  → Study pure Top-Higgs sector with Higgs-Yukawa models.
- Earlier Higgs-Yukawa models explicitly broke chiral symmetry.
  ▶ In continuum limit chiral symmetry restauration and lifting of fermion doublers could not be achieved simultaneously.

Study Higgs-Yukawa model with built-in chiral symmetry.
1.2 The model

- A chirally invariant $SU(2)_L \times SU(2)_R$ Higgs-Yukawa model can be constructed using the Neuberger overlap operator $D^{(N)}$ (Lüscher).

- The model, we consider, is discretized on a four-dimensional lattice with $L$ sites per dimension (volume $V = L^4$).

- It contains one four-component, real Higgs field $\Phi$, and $N_f$ fermion doublets $\psi^{(i)}$, but no gauge fields:

$$Z = \int D\Phi \prod_{i=1}^{N_f} \left[ D\psi^{(i)} D\bar{\psi}^{(i)} \right] \exp \left( -S_{F}^{kin} - S_Y - S_\Phi \right)$$

with kinetic fermion action $S_{F}^{kin}$, Yukawa coupling term $S_Y$, and Higgs action $S_\Phi$. 
1.2 The model

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with kinetic fermion action $S^{kin}_F$, Yukawa coupling term $S_Y$, and Higgs action $S_\Phi$.

- Kinetic fermion action:

$$S^{kin}_F = \sum_{i=1}^{N_f} \bar{\psi}^{(i)} D^{(N)} \psi^{(i)}$$
1.3 The model

- The Yukawa coupling term is given as

$$S_Y = y_N \sum_{i=1}^{N_f} \bar{\psi}^{(i)} B \cdot \left[ \mathbb{1} - \frac{1}{2\rho} \mathcal{D}(N) \right] \psi^{(i)}$$

$$B_{x,y} = \mathbb{1}_{x,y} \frac{1 - \gamma_5}{2} \phi_x + \mathbb{1}_{x,y} \frac{1 + \gamma_5}{2} \phi^+_x$$

where the Higgs field $\Phi_x$ is written as quaternion $\phi_x$ acting on flavor index

$$\phi_x = \Phi_x^0 \mathbb{1} - i(\Phi_x^1 \tau_1 + \Phi_x^2 \tau_2 + \Phi_x^3 \tau_3), \quad \tau_i : \text{Pauli-matrices.}$$
1.3 The model

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\]

\[
B_{x,y} = 1_{x,y} \frac{(1 - \gamma_5)}{2} \phi_x + 1_{x,y} \frac{(1 + \gamma_5)}{2} \phi_x^\dagger
\]

where the Higgs field \( \Phi_x \) is written as quaternion \( \phi_x \) acting on flavor index

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\phi_x = \Phi_x^0 1 - i(\Phi_x^1 \tau_1 + \Phi_x^2 \tau_2 + \Phi_x^3 \tau_3), \quad \tau_i : \text{Pauli-matrices.}
\]

• The Higgs action \( S_\Phi \) in lattice notation is

\[
S_\Phi = -\kappa_N \sum_{x,\mu} \Phi_x^\dagger \left[ \Phi_{x+\mu} + \Phi_{x-\mu} \right] + \sum_x \Phi_x^\dagger \Phi_x + \lambda_N \sum_x \left( \Phi_x^\dagger \Phi_x - N_f \right)^2
\]

related to usual notation by transforming couplings \( (\kappa, \lambda) \leftrightarrow (\kappa_N, \lambda_N) \).
2.1 Phase structure at small $y_N$

- We consider the limit $N_f \to \infty$ where the couplings scale according to

$$y_N = \frac{\tilde{y}_N}{\sqrt{N_f}}, \quad \tilde{y}_N = \text{const} \quad \lambda_N = \frac{\tilde{\lambda}_N}{N_f}, \quad \tilde{\lambda}_N = \text{const} \quad \kappa_N = \tilde{\kappa}_N, \quad \tilde{\kappa}_N = \text{const}$$

- The effective action

$$S_{\text{eff}}[\Phi] = S_{\Phi}[\Phi] - N_f \cdot \log \left[ \det \left( y_N B D^{(N)} - 2 \rho D^{(N)} - 2 \rho B \right) \right]$$

can be evaluated at least for the constant and staggered modes of $\Phi$.

- For the Higgs field we take a magnetization $(m)$ and a staggered magnetization $(s)$ into account by the ansatz

$$\Phi(x) = \hat{\Phi} \cdot \sqrt{N_f} \cdot \left( m + s \cdot (-1)^{\sum_{\mu=0}^{3} x_{\mu}} \right)$$

where $\hat{\Phi} \in \mathbb{R}^4$, $|\hat{\Phi}| = 1$ is a constant unit vector and $m, s \in \mathbb{R}$. 
2.2 Effective Potential

- At tree-level one finally finds for the effective potential $V(m, s)$

$$
\frac{V(m, s)}{L^4 N_f} = m^2 + s^2 - 8\tilde{\kappa}_N \left(m^2 - s^2\right) + \tilde{\lambda}_N \left(m^4 + s^4 + 6m^2 s^2 - 2 \left(m^2 + s^2\right)\right)
$$

$$
- \frac{1}{L^4} \sum_{p \in \mathcal{P}} \log \left[\left(\left|\nu^+(p)\right| \left|\nu^+(\varphi)\right| + \frac{\tilde{y}_N^2}{4\rho^2} (m^2 - s^2) \left|\nu^+(p) - 2\rho\right| \left|\nu^+(\varphi) - 2\rho\right\right)^2\right]
$$

$$
+ m^2 \frac{\tilde{y}_N^2}{4\rho^2} \left(\left|\nu^+(p) - 2\rho\right| \left|\nu^+(\varphi)\right| - \left|\nu^+(\varphi) - 2\rho\right| \left|\nu^+(p)\right|\right)^2
$$

with

$$
p = (p_0, p_1, p_2, p_3) \in \mathcal{P} : \text{allowed lattice momenta}
$$

$$
\nu(p) : \text{eigenvalues of Neuberger Dirac operator } \hat{D}^{(N)}
$$

$$
\varphi_\mu = p_\mu + \pi
$$

- The phase diagram can be explored by numerically searching for the absolute minima of the effective action with respect to $m$ and $s$. 

The phase structure of a chirally invariant Higgs-Yukawa model – p.7/26
2.3 Analytical phase diagrams

In general, four different phases can be obtained in this ansatz:

- **SYM**: $m = 0, \ s = 0$
- **FM**: $m \neq 0, \ s = 0$
- **AFM**: $m = 0, \ s \neq 0$
- **FI**: $m \neq 0, \ s \neq 0$
2.4 MC-simulations: Definitions

- We have implemented an HMC-algorithm for even values of $N_f$
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- As observables we consider the (staggered) magnetization $m (s)$

$$m = \left[ \sum_{i=0}^{3} \frac{1}{L^4} \sum_n \Phi^i_n \right]^2 \frac{1}{2}, \quad s = \left[ \sum_{i=0}^{3} \frac{1}{L^4} \sum_n (-1)^{\mu} n_\mu \cdot \Phi^i_n \right]^2 \frac{1}{2}$$

and the corresponding (staggered) susceptibility $\chi_m (\chi_s)$

$$\chi_m = L^4 \cdot [\langle m^2 \rangle - \langle m \rangle^2], \quad \chi_s = L^4 \cdot [\langle s^2 \rangle - \langle s \rangle^2],$$

where $\langle \ldots \rangle$ denotes the average over the generated $\Phi$-field configurations.
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and the corresponding (staggered) susceptibility $\chi_m$ ($\chi_s$)

\[
\chi_m = L^4 \cdot \left[ \langle m^2 \rangle - \langle m \rangle^2 \right], \quad \chi_s = L^4 \cdot \left[ \langle s^2 \rangle - \langle s \rangle^2 \right],
\]

where $\langle \ldots \rangle$ denotes the average over the generated $\Phi$-field configurations.
- Determine phase transition point by fit of $\chi_{m,s}$ to finite-size-scaling ansatz

\[
\chi_{m,s} = A_{1}^{m,s} \cdot \left( \frac{1}{L^{-2/\nu} + A_{2,3}^{m,s} (\kappa_N - \kappa_{m,s}^{\text{crit}})^2} \right)^{\gamma/2},
\]

with fitting parameters $\kappa_{\text{crit}}^{m,s}, A_{1}^{m,s}, A_{2}^{m,s}, A_{3}^{m,s}$. 

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2.5 Phase structure overview

- Numerically we find the expected phases at the predicted locations.
- Qualitatively, the phase diagram is in very good agreement with the large $N_f$ analysis.
2.6 Finite Size Effects

- Phase transition lines strongly shifted by finite size effects.
- We isolate finite size effects from $1/N_f$-corrections by choice $N_f = 50$.
- We compare $L = 4$ and $L = 8$ results with analytical finite size expectations.
2.7 $1/N_f$ corrections

- We demonstrate strength of $1/N_f$-corrections by determining phase transition points $\kappa_{m,s}^{\text{crit}}$ for several values of $N_f$.
- To isolate $1/N_f$-corrections from finite size effects we compare with analytical, finite size expectations.

\[ \tilde{y}_N = 1.0, \tilde{\lambda}_N = 0.1 \]

\[ \tilde{y}_N = 2.0, \tilde{\lambda}_N = 0.1 \]

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3.1 Phase structure at large $y_N$

- Idea: Divide out $y_N B (D^{(N)} - 2\rho)$ and develop logarithm into power series

$$S_{eff}[\Phi] = S_\Phi - N_f \cdot \log \left[ \det \left( y_N B D^{(N)} - 2\rho D^{(N)} - 2\rho B \right) \right]$$
3.1 Phase structure at large $y_N$

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\[
S_{eff}[\Phi] = S_\Phi - N_f \cdot \log \left[ \det \left( y_N B D^{(N)} - 2\rho D^{(N)} - 2\rho B \right) \right] \\
\rightarrow S_\Phi - N_f \cdot \log \left[ \det \left( 1 - \frac{2\rho}{y_N} D^{(N)} \left[ D^{(N)} - 2\rho \right]^{-1} B^{-1} \right) \right] \\
\rightarrow S_\Phi - N_f \sum_x \log(|\Phi_x|^8) - N_f \frac{(4\rho)^2}{y_N^2} \sum_{x,y} \frac{\Phi_x^\dagger K_{x,y} \Phi_y}{|\Phi_x|^2 \cdot |\Phi_y|^2}
\]

where the coupling matrix $K_{x,y}$ is explicitly known.
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$$\rightarrow S_\Phi - N_f \sum_x \log(|\Phi_x|^8) - N_f \frac{(4\rho)^2}{y_N^2} \sum_{x,y} \frac{\Phi_x^\dagger K_{x,y} \Phi_y}{|\Phi_x|^2 \cdot |\Phi_y|^2}$$

where the coupling matrix $K_{x,y}$ is explicitly known.

- In large $N_f$-limit amplitude $|\Phi_x|$ becomes fixed.

The model becomes an $O(4)$-symmetric sigma-model leading to the existence of a symmetric phase at strong Yukawa couplings.
3.1 Phase structure at large $y_N$

• Idea: Divide out $y_N B (D^{(N)} - 2\rho)$ and develop logarithm into power series

$$S_{e f f}[\Phi] = S_\Phi - N_f \cdot \log \left[ \det \left( y_N B D^{(N)} - 2\rho D^{(N)} - 2\rho B \right) \right]$$

$$\rightarrow S_\Phi - N_f \cdot \log \left[ \det \left( 1 - \frac{2\rho}{y_N} D^{(N)} \left[ D^{(N)} - 2\rho \right]^{-1} B^{-1} \right) \right]$$

$$\rightarrow S_\Phi - N_f \sum_x \log(|\Phi_x|^8) - N_f \frac{(4\rho)^2}{y_N^2} \sum_{x,y} \frac{\Phi_x^\dagger K_{x,y} \Phi_y}{|\Phi_x|^2 \cdot |\Phi_y|^2}$$

where the coupling matrix $K_{x,y}$ is explicitly known.

• Caution: $D^{(N)} - 2\rho$ has zero modes: More careful calculation yields

$$S_{e f f}[\Phi] \rightarrow S_\Phi - N_f \sum_x \log(|\Phi_x|^8) - N_f \frac{(4\rho)^2}{y_N^2} \sum_{x,y} \frac{\Phi_x^\dagger K_{x,y} \Phi_y}{|\Phi_x|^2 \cdot |\Phi_y|^2}$$

$$- N_f \cdot \log \det^* \left( B^{-1} \right) - N_f \cdot \log \det^* \left( 1 + \frac{2\rho}{y_N} F[\Phi] \right)$$

where $\det^*$ is the determinant over zero-modes (120 modes) and $F[\Phi]$ can be explicitly given.

The phase structure of a chirally invariant Higgs-Yukawa model – p.13/26
3.2 Analytical Phase Diagrams

- Neglecting the finite-volume terms, the phase structure can be derived in the large $N_f$-limit applying the ansatz

\[ y_N = \tilde{y}_N, \quad \tilde{y}_N = \text{const}, \quad \lambda_N = \frac{\tilde{\lambda}_N}{N_f}, \quad \tilde{\lambda}_N = \text{const}, \quad \kappa_N = \frac{\tilde{\kappa}_N}{N_f}, \quad \tilde{\kappa}_N = \text{const}, \]

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3.3 MC-results: Magnetizations

- We show the (staggered) magnetization for varying $\kappa_N$ at $\tilde{\lambda}_N = 0.1$, $\tilde{y}_N = 30$ for different lattice sizes.

- Strong finite volume effects prevent emergence of the symmetric phase on too small lattices and cause asymmetry in $m$ and $s$.
3.3 MC-results: Magnetizations

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- Strong finite volume effects prevent emergence of the symmetric phase on too small lattices and cause asymmetry in $m$ and $s$.

- Strongest finite-volume contribution is $\log \det^* (B^{-1})$.
  It can be written in terms of $m$ and $s$, explaining the observations

\[-N_f \log \det^* (B^{-1}) = \text{Const} - 8N_f \log |m| + 64N_f \log |m^2 - s^2|\]
3.4 MC-results: Susceptibilities

- We show the magnetic susceptibilities for varying $\kappa_N$ at $\tilde{\lambda}_N = 0.1$, $\tilde{y}_N = 30$ for different lattice sizes.
- On small lattices ($V = 4^4$) the maximum is at $\kappa_N = 0$, caused by the finite-volume effects.
- On larger lattices ($V = 8^4$) a second peak develops at $\kappa_N = 0.04$, which describes the location of the physical phase transition.
3.5 Phase Diagram

- We compare numerical and analytical results for the SYM-FM transition line.
- Good agreement is found even at $N_f = 2$.
- The SYM-AFM phase transition line was numerically too demanding for our HMC-algorithm.

![Graph showing phase structure with $\tilde{\lambda}_N = 0.1$, $N_f = 2$]
4.1 Towards Upper Mass Bounds:

- VERY PRELIMINARY
- To access physical setting $N_f = 1$, we implemented a PHMC-algorithm.
- We fixed physical scale by phenomenological value $vev = 246$ GeV.
- We searched for the physical region in the phase diagram by fixing the top quark mass to $m_{top} = 175$ GeV.
- To obtain upper mass bound, we went to strong quartic couplings $\lambda_N$.
- As a first step, we simulated the model on a $16^3 \times 32$ lattice close to the phase transition in the FM-phase.
- To account for the 3 Goldstone-modes we split the 4-component Higgs field into its radial $\phi$ and tangential components $\vec{\pi}$. 
4.2 Goldstone - Propagator:

- Obtain Goldstone renormalization factor $Z_G$ from inverse propagator of massless Goldstone-modes

$$G_{\pi}^{-1}(\hat{p}^2) = \frac{\hat{p}^2}{Z_G}$$

$$Z_G = 0.9683 \pm 0.0002$$
4.3 Higgs - Propagator:

- Obtain Higgs propagator-mass $m_{H,prop}$ from propagator of Higgs-mode

$$G_\phi^{-1}(\hat{p}^2) = \frac{\hat{p}^2 + m_{H,prop}^2}{Z_H}$$

$m_{H,prop} = 0.384 \pm 0.009$
4.4 Fermion correlator:

- Obtain top quark mass $m_{top}$ from fermion correlator $\langle \psi_{t_1} \bar{\psi}_{t_2} \rangle$

$$m_{top} = 0.0686 \pm 0.0021$$
4.5 Higgs correlator:

- Obtain Higgs mass $m_H$ from Higgs correlator $\langle \phi_{t_1} \phi_{t_2} \rangle$

$$m_H = 0.286 \pm 0.011$$
4.5 Summary of results:

- Cut-off $\Lambda$ $(2591 \pm 58)$ GeV
- Top mass $m_{\text{top}}$ $(178.8 \pm 6.8)$ GeV
- Higgs mass $m_H$ $(741 \pm 29)$ GeV
- Higgs prop. mass $m_{H,\text{prop}}$ $(994 \pm 22)$ GeV

- Bare Lambda $\lambda_0$ 4.4
- Ren. Lambda $\lambda_{\text{ren}}$ $4.53 \pm 0.18$
- Ren. $y \ y_{\text{ren}}$ $0.723 \pm 0.027$
Summary and Outlook

- Large $N_f$ analysis gives good understanding of qualitative phase structure.
- Finite size effects can be quantitatively described in large $N_f$-limit.
- A symmetric phase exists also at strong Yukawa coupling.
- First results on upper Higgs mass bound will become available soon.
- We will investigate the model at larger cut-offs and on larger lattices.
Evidence for FI-phase

- Also numerical evidence for the predicted FI-phase with
  \[ \langle m \rangle > 0 \text{ and } \langle s \rangle > 0 \]
  deeply inside the anti-ferromagnetic phase can be found.

- The plots were made for \( \tilde{\lambda}_N = 0.1, N_f = 10, L = 6. \)
Finite Size Effects

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