MAPPING THE DISCRETE LOGARITHM

DANIEL R. CLOUTIER AND JOSHUA HOLDEN

Abstract. The discrete logarithm is a problem that surfaces frequently in
the field of cryptography as a result of using the transformation $g^x \mod n$.
This paper focuses on a prime modulus, $p$, for which it is shown that the basic
structure of the functional graph is largely dependent on an interaction be-
tween $g$ and $p - 1$. In fact, there are precisely as many different functional
structures as there are divisors of $p - 1$. This paper extracts two of
these structures, permutations and binary functional graphs. Estimates exist
for the shape of a random permutation, but similar estimates must be created
for the binary functional graphs. Experimental data suggests that both the
permutations and binary functional graphs correspond well to the theoretical
data which provides motivation to extend this to larger divisors of $p - 1$ and
study the impact this forced structure has on the many cryptographic algo-
rithms that rely on the discrete logarithm for their security. This is especially
applicable to those algorithms that require a “safe” prime ($p = 2q + 1$, where
$q$ is prime) modulus since all non-trivial functional graphs generated using a
safe prime modulus can be analyzed by the framework presented here.

1. Introduction

Just a few decades ago, cryptography was considered a domain exclusive to
national governments and militaries. However, the computer explosion has changed
that. Every day, millions of people trust that their privacy will be protected as
they make online purchases or communicate privately with a friend. Many of the
cryptographic algorithms they will use are built upon a common transformation,

\[ g^x \equiv y \mod n. \]

For instance, Diffie-Hellman key exchange, RSA and the Blum-Micali pseudorand-
dom bit generator all use (1). This paper will examine some of the properties
exhibited by this sort of transformation and provide theoretical and experimen-
tal data describing how the interaction between $g$ and the modulus impacts the
behavior of this function.

2. Terminology and Background

In this paper, we will restrict the values of $n$ to primes and examine mappings
\[ f : S = \{1, 2, ..., p - 1\} \rightarrow S \]
of the form $x \mapsto g^x \mod p$, where $p$ is a prime modulus. In some instances, it
will prove to be useful to interpret the mappings as functional graphs. A functional
graph is a directed graph such that each vertex must have exactly one edge directed
out from it. The relationship between the mappings which interest us and functional
graphs is straightforward. Each element in \( S \) can be interpreted as a vertex. The edges are defined such that an edge \( (a, b) \) is in the graph if and only if \( f(a) = b \).

There are a number of statistics of interest derived from functional graphs. Following the convention of [4], which treats random mappings in detail, let \( f : S \to S \) be the transition function so that the edges in the functional graph can be expressed as the ordered pair \( \langle x, f(x) \rangle \) for \( x, f(x) \in S \). By applying the pigeonhole principle and noting that the cardinality of \( S \) is \( p - 1 \), we can say that by starting at any random point \( u_0 \) and following the sequence \( u_1 = f(u_0), u_2 = f(u_1), \ldots \), there must be a \( u_i = u_j \) after at most \( p \) iterations. Suppose \( u_i \) occurs before \( u_j \) in the sequence of nodes. In this case, the tail length is the number of iterations from \( u_0 \) to \( u_i \). The cycle length is the number of iterations from \( u_i \) to \( u_j \). In more natural graphical terms, the cycle length is the number of edges (or equivalently nodes) involved in the directed path from \( u_i \) to itself. The tail length is the number of edges from \( u_0 \) to \( u_i \). Additionally, a terminal node is one with no pre-image, or more formally, \( x \) is a terminal node if \( f^{-1}(x) = \emptyset \). A node is an image node if it is not a terminal node. Since each node has an out-degree of exactly one, each cycle with the trees grafted onto its nodes will form a connected component.

The value of \( g \) plays a major role in determining the basic structure of the graph. In fact, as Theorem 1 formalizes, the interaction among \( g \) and \( p - 1 \) will effectively fix the in-degrees of the nodes in the graph. First, though, define an \( m \)-ary functional graph to be a graph where each node has in-degree of exactly zero or \( m \). The proof of the following theorem is then straightforward.

**Theorem 1.** Let \( p \) be fixed and let \( m \) be any positive integer that divides \( p - 1 \). Then as \( g \) ranges over all integers, there are \( \phi(\frac{p-1}{m}) \) different functional graphs which are \( m \)-ary produced by maps of the form \( f : x \mapsto g^x \mod p \). Furthermore, if \( r \) is any primitive root modulo \( p \), and \( g \equiv r^a \mod p \), then the values of \( g \) that produce an \( m \)-ary graph are precisely those for which \( \gcd(a, p - 1) = m \).

Theorem 1 gives a strong indication that the graphs generated by (1) have to be considered separately for different values of \( m \).

It should be noted, though, that there are some values of \( m \) which lead to completely predicatable graphs. For instance, there is one \((p-1)\)-ary graph that corresponds to \( g \equiv 1 \mod p \). There is also one \((\frac{p-1}{2})\)-ary graph that corresponds to \( g \equiv -1 \mod p \). In general, however, an \( m \)-ary graph is not trivially predictable. This paper will restrict its focus to unary functional graphs (which will be referred to as permutations since they simply permute the numbers 1, \( \ldots \), \( p - 1 \)) and binary functional graphs. The values of \( g \) which produce a permutation are precisely those which are primitive roots modulo \( p \).

In cryptography, it is common to look for primes where \( p - 1 \) has at least one large prime factor. For instance, the pseudorandom bit generator described by Gennaro in [1] and mentioned in Section 1 requires the modulus to be of the form \( p = 2q + 1 \) where \( q \) is also prime. A prime of this form is known as a safe prime (\( q \) is also known as a Sophie Germain prime). These primes are of interest here not only because of their extensive use in cryptography, but also because \( p - 1 \) has only four divisors, namely 1, 2, \( q \) and 2\( q \). It can be quickly verified that there is only one \( q \)-ary \((g \equiv -1 \mod p)\) and one 2\( q \)-ary \((g \equiv 1 \mod p)\) graph generated. More importantly, there are \( \phi(q) \) permutations and \( \phi(q) \) binary functional graphs.

\footnote{Throughout this paper, \( \phi \) denotes the Euler phi function.}
which represent the remaining values of \( g \) (since \( \phi(q) = q - 1 \)). Thus, not only do safe primes provide large numbers of permutations and binary functional graphs, but every graph generated by a safe prime is either trivial (the graphs where \( g \) is either 1 or -1) or fits into the theoretical framework presented in Section 3.

3. Theoretical Results

In Theorem 1, it is shown that the in-degree of each node is dependent on the value of both \( g \) and \( p \). This is clearly imposing a structure on any functional graphs generated using (1). It seems reasonable, though, that a large collection of functional graphs generated by using (1) as the transition function would tend toward exhibiting behavior similar to that of a collection of random functional graphs. At a minimum, a factorization for \( p - 1 \) with many divisors would certainly seem to hide the structure imposed by Theorem 1 since the many divisors of \( p - 1 \) would each contribute some graphs. Section 4.1 will give evidence that this is not the case. However, the methods used to obtain the theoretical bounds for the random functional graphs can be extended to analyze \( m \)-ary graphs for specific \( m \).

While most of the parameters that are of interest depend on the exact graph generated, the number of image nodes can be computed directly from the values of \( g \) and \( p \). The proof is again straightforward.

**Theorem 2.** The number of image nodes in any \( m \)-ary graph is \( \frac{p-1}{m} \).

Theorem 2 helps to quantify the repercussions of Theorem 1 and the restrictions on in-degree in \( m \)-ary graphs. The number of image nodes is a direct function of \( m \) which can greatly limit the shapes each graph can take on. None of the other parameters appear to have a generalization as convenient as the image nodes and will be treated as specific parameters in permutations and binary functional graphs.

3.1. Random Functional Graphs. Flajolet and Odlyzko do a thorough analysis of functional graphs in [2]. While none of these results are original, Flajolet and Odlyzko demonstrate that all of these parameters can be estimated through a singularity analysis of generating functions. This appears to be the first method that can be applied to all of these parameters. Their methods can then be adapted for any fixed value of \( m \) to estimate the parameters of interest for an \( m \)-ary graph. Specifically, the methods will be used to confirm some permutation results and to develop all of the binary functional graph results. The results from [2] are summarized below in Theorem 3.
Theorem 3. The asymptotic values for the parameters of interest in a random functional graph of size $n$ are:

(i) **Number of components** \( \ln(2n) + \gamma \)
(ii) **Number of cyclic nodes** \( \sqrt{\pi n/2} - \frac{1}{3} \)
(iii) **Number of tail nodes** \( n - \sqrt{\pi n/2} + \frac{1}{3} \)
(iv) **Number of terminal nodes** \( e^{-1}n \)
(v) **Number of image nodes** \( (1 - e^{-1})n \)
(vi) **Average cycle length** \( \sqrt{\pi n/8} \)
(vii) **Average tail length** \( \sqrt{\pi n/8} \)
(viii) **Maximum cycle length** \( \sqrt{\frac{\pi n}{2}} \int_0^\infty \left[ 1 - \exp \left( -\int_v^\infty e^{-u}du \right) \right] du \approx 0.78248\sqrt{n} \)
(ix) **Maximum tail length** \( \sqrt{2\pi n \ln 2} \approx 1.73746\sqrt{n} \)

(2)

In part (i), \( \gamma \) refers to the Euler constant which is approximately 0.57721566. The second order terms for parts (i), (ii), and (iii) were not given in [4], but can be computed with a careful singularity analysis using precisely the same methods used there.

3.2. **Permutations.** Predicting the behavior of the permutations is, in many ways, much easier than other $m$-ary graphs. The most important reason for this is that there are no terminal nodes or tail nodes. This follows quickly from the definition of a permutation as a unary functional graph and the fact that the sum of the in-degrees must be the same as the sum of the out-degrees. Each node has an out-degree of exactly one, and if any node were to have an in-degree of zero, then, by the pigeon-hole principle, at least one node must have an in-degree of more than one. This is not allowed so each node must have in-degree of exactly one. Furthermore, since every tail must contain at least one terminal node, this also implies that every node is cyclic. The parameters that can then be determined from the definition of a permutation are given below.

- **Number of cyclic nodes** \( n \)
- **Number of tail nodes** \( 0 \)
- **Number of terminal nodes** \( 0 \)
- **Number of image nodes** \( n \)
- **Average tail length** \( 0 \)

There are three non-trivial parameters of interest. They are expressed in Theorem 4.

Theorem 4. The asymptotic values for the number of components, the average cycle length as seen from a random node and the maximum cycle length in a random
permutation of size $n$ have the following values:

(i) \[ \text{Number of components} \quad \sum_{i=1}^{n} \frac{1}{i} \]

(ii) \[ \text{Average cycle length} \quad \frac{n + 1}{2} \]

Maximum cycle length \[ n \int_{0}^{\infty} \left[ 1 - \exp \left( - \int_{v}^{\infty} e^{-u \frac{du}{u}} \right) \right] dv \]
\[ \approx 0.62432965n \]

Parts (i) and Part (ii) are fairly well known. Part (iii) seems to have first been solved by Shepp and Lloyd in 1966 \[7\]. An alternative solution and proof more similar to the methods used here is offered by Flajolet and Odlyzko in \[3\].

3.3. Binary Functional Graphs. While estimates for the parameters investigated here exist in literature for the random functional graphs and permutations, it does not appear similar estimates exist for binary functional graphs. However, the methods in \[4\] can be extended to develop these estimates. Imitating the methods of \[4\], we first need to convert our ideas of a binary functional graph into corresponding generating functions. The machinery is fairly straightforward once we define the following as in \[4\]:

\begin{align*}
\text{BinFunGraph} & = \text{set(Components)} \\
\text{Component} & = \text{cycle(Node*BinaryTree)} \\
\text{BinaryTree} & = \text{Node + Node*set(BinaryTree, cardinality = 2)} \\
\text{Node} & = \text{Atomic Unit}
\end{align*}

This implies that a binary functional graph is a set of components. Each component is a cycle of nodes with each node having an attached binary tree to bring its in-degree to two. A binary tree is either a node (terminal node) or a node with two binary trees attached. Finally, a node is simply an atomic unit. A moment’s reflection should indicate that this natural specification does, in fact, specify a binary functional graph. Imitating the transformations in \[4\] Section 2.1, the generating functions of interest are

\begin{align*}
\text{(3)} & \quad f(z) = e^{c(z)} = \frac{1}{1 - zb(z)} \\
\text{(4)} & \quad c(z) = \ln \frac{1}{1 - zb(z)} \\
\text{(5)} & \quad b(z) = z + \frac{1}{2} zb^2(z)
\end{align*}

Here $f$ generates the number of binary functional graphs, $c$ generates the number of components, and $b$ generates the number of binary trees of a given size. Solving the quadratic formula for \( b \), we can produce the following formulas for $f$ and $c$ which simplify some of the cases:

\begin{align*}
\text{(6)} & \quad f(z) = \frac{1}{\sqrt{1 - 2z^2}} \\
\text{(7)} & \quad c(z) = \ln \frac{1}{\sqrt{1 - 2z^2}}
\end{align*}
In order to compute asymptotic forms of any of the statistics of interest, we must first compute an asymptotic form for \( f \) to normalize results. The following derivations give only a highlight of the methods used by Flajolet and Odlyzko. The interested reader is encouraged to see \([3, 4]\) for detailed proofs.

From equation (6) it is clear that there is a singularity at \( z = \frac{1}{\sqrt{2}} \). Performing a singularity analysis\(^2\) as in \([4, \text{Section 2}]\), the asymptotic form for \( f \) falls out quickly as

\[
(f(z) \sim \frac{2^{n/2}}{\sqrt{\pi n/2}}. \tag{8}
\]

In at least one case, there are some important second-order interactions between the error terms of the number of graphs and the appropriate statistic. In these cases, a more exact form of (8) must be used. Expanding one more term in the expansion of \( f \) gives

\[
(f(z) \sim \frac{2^{n/2}}{\sqrt{\pi n/2}} - \frac{2^{n/2}}{4n\sqrt{\pi n/2}} = \frac{2^{n/2}(4n - 1)}{4n\sqrt{\pi n/2}} \tag{9}
\]

In most cases, using this more precise expansion of \( f \) is not necessary and does not change the results. Therefore, in all but the necessary cases, (8) will be used.

We begin by deriving the results for the most simple parameters.

**Theorem 5.** The asymptotic forms for the number of components, number of cyclic nodes, number of tail nodes, number of terminal nodes and number of image nodes in a random binary functional graph of size \( n \), as \( n \to \infty \) are

\[
\begin{align*}
\text{(i) Number of components} & \quad \ln(2n) + \gamma \\
\text{(ii) Number of cyclic nodes} & \quad \sqrt{\pi n/2} - 1 \\
\text{(iii) Number of tail nodes} & \quad n - \sqrt{\pi n/2} + 1 \\
\text{(iv) Number of terminal nodes} & \quad n/2 \\
\text{(v) Number of image nodes} & \quad n/2
\end{align*}
\]

In part (i), \( \gamma \) represents the Euler constant which is approximately 0.57721566. The highlights of the proofs as they differ from those in \([4]\) follow.

**Proof.** As in \([4]\), the following bivariate generating functions need to be defined with parameter \( u \) marking the elements of interest. The generating functions for the number of components, number of cyclic nodes and number of terminal nodes are respectively:

\[
\begin{align*}
\xi_1(u, z) &= \exp \left( u \ln \frac{1}{1 - zb(z)} \right) \tag{10} \\
\xi_2(u, z) &= \frac{1}{1 - ub(z)} \tag{11} \\
\xi_3(u, z) &= \frac{1}{\sqrt{1 - 2uz^2}} \tag{12}
\end{align*}
\]

\( ^2 \)The analyses in this paper have been performed using the computer algebra program Maple and the packages created as part of the Algorithms Project at INRIA, Rocquencourt, France. The packages can be found online at \( \text{http://pauillac.inria.fr/algo/libraries/software.html} \).
Imitating the methods in [4], the mean value generating function, $\Xi(z)$, is found by taking the partial derivative of $\xi(u, z)$ with respect to $u$ and evaluating at $u = 1$. This yields the following results

\begin{align*}
\Xi_1(z) &= \frac{1}{1 - zb(z)} \ln \left( \frac{1}{1 - zb(z)} \right) \\
\Xi_2(z) &= \frac{zb(z)}{(1 - zb(z))^2} \\
\Xi_3(z) &= \frac{z^2}{(1 - 2z^2)^{3/2}}.
\end{align*}

The forms in the statement of the theorem follow by expanding around the singularity $z = 1/\sqrt{2}$, applying singularity analysis as in [4], and normalizing parts (i) and (ii) by (8) and (iv) by (9). Parts (iii) and (v) follow from parts (ii) and (iv) respectively since the respective pairs must sum to $n$. Also note that part (iv) can also be derived in an elementary fashion from the definition of the binary functional graph.

The asymptotic values for the average length of cycles and tails as seen from a random point in the graph are also interesting. The asymptotic forms of these values are given in Theorem [6].

**Theorem 6.** The expected values for the cycle size and tail length as seen from a random node in a random binary functional graph of size $n$ are asymptotic to

\begin{align*}
\text{(i) Average cycle length} &\quad \sqrt{\pi n / 8} \\
\text{(ii) Average tail length} &\quad \sqrt{\pi n / 8}
\end{align*}

**Proof.** In order to calculate the average cycle length and average tail length, the generating functions must be manipulated to account for each node in the cycle or tail. This can be done by using the same methods as in the previous proof, but on the component function and taking an additional derivative with respect to $z$ to weight each cycle and tail by the nodes involved. Multiplying again by $z$ replaces the factor lost in the differentiation and by $1/(1 - b(z))$ cumulates over all of the components. This strategy is used to prove the result for average cycle size in [4]. More background on the method can be found there.

Marking the appropriate elements, performing a singularity analysis of the two generating functions and normalizing by $2^{n/2}/(n\sqrt{\pi n/2})$, as done in the previous theorems, leads to the statement of the theorem. The additional factor of $n$ in the denominator is needed to compensate for the fact that the parameters were estimated across all nodes in the graph and the goal is to determine them from any single random node in the graph.

The final parameters that needs to be calculated are the average maximum cycle length and the average maximum tail length.

**Theorem 7.** The asymptotic forms for the expected sizes of the largest cycle and the largest tail in a random binary functional graph of size $n$, as $n \to \infty$, are

\begin{align*}
\text{(i) Largest cycle} &\quad \sqrt{\frac{\pi n}{2}} \int_0^\infty \left[ 1 - \exp \left( - \int_0^\infty e^{-u} \frac{du}{u} \right) \right] dv \approx 0.78248 \sqrt{n} \\
\text{(ii) Largest tail} &\quad 2\pi n \ln 2 - 3 + 2 \ln 2 \approx 1.73746 \sqrt{n} - 1.61371
\end{align*}
Proof. The proof for part (i) result follows precisely the methods of [4] with substitution of the proper generating function \( f \), and is therefore omitted.

The proof for part (ii) follows a combination of [4, Theorem 6] and [2, Sections 3–5]. Let \( b^{[h]}(z) \) be the exponential generating function for the number of binary trees with height at most \( h \) and \( f^{[h]}(z) \) be the exponential generating function for the number of binary functional graphs with maximum tail length less than or equal to \( h \), so that (as in Equations (41) and (42) of [4])
\[
    f^{[h]}(z) = \frac{1}{1 - zb^{[h]}(z)}
\]
and
\[
b^{[h+1]}(z) = z + \frac{1}{2} z \left( b^{[h]}(z) \right)^2, \quad b^{[0]}(z) = z.
\]
Now, as in [2, Proposition 2], note that
\[
b(z) - b^{[h+1]} = \frac{1}{2} z \left( b(z) - b^{[h]}(z) \right) \left( b(z) + b^{[h]}(z) \right)
\]
so if we let
\[
e_h(z) = \frac{b(z) - b^{[h]}(z)}{2b(z)}.
\]
then
\[
e_{h+1}(z) = (1 - \sqrt{1 - 2z^2})e_h(z)(1 - e_h(z)).
\]
Now we want to approximate \( e_h(z) \) with a function of \( h \) and some \( \epsilon(z) \). If we let
\[
\epsilon = \sqrt{1 - 2z^2}
\]
then we have
\[
e_{j+1} = (1 - \epsilon)e_j(1 - e_j); \quad e_{-1} = 2.
\]
This is essentially the same recursion as in [2], and as in [2, Lemma 5], we can then “normalize” and “take inverses” to get the approximation
\[
e_h \approx \frac{(1 - \epsilon)^{h+1}\epsilon}{1 - (1 - \epsilon)^{h+1}}.
\]
(The details of the error bounds proceed as in [2]; we omit them here.)

The generating function associated to the average maximum tail length is (as in Equation (43) of [4])
\[
\Xi(z) = \sum_{h \geq 0} \left[ \frac{1}{1 - zb(z)} - \frac{1}{1 - zb^{[h]}(z)} \right]
\]
and we proceed as in Equation (51) of [4] to write
\[
\Xi(z) = \frac{2zb(z)}{1 - zb(z)} \sum_{h \geq 0} \frac{e_h(z)}{1 - zb(z) + 2e_h(z)zb(z)}.
\]
Putting this entirely in terms of \( \epsilon \) and \( h \), and shifting the index of summation for convenience, we can write
\[
\Xi(z) \approx \frac{2(1 - \epsilon)}{\epsilon} \sum_{h \geq 1} \frac{(1 - \epsilon)^h}{1 + (1 - 2\epsilon)(1 - \epsilon)^h}.
\]
We approximate the sum with an integral, using Euler-Maclaurin summation.
Taking the integral and noting that \( \ln(1 - \epsilon) \sim -\epsilon \) as \( \epsilon \to 0 \), we finally get:
\[
\Xi(z) \approx \frac{2(1 - \epsilon)}{\epsilon^2(1 - 2\epsilon)} \ln(2 - 3\epsilon + 2\epsilon^2).
\]
The next step is to substitute $\epsilon = \sqrt{1 - 2z^2}$ into (18) and do the singularity analysis, which gives us the statement of the theorem. □

4. Observed Results

In [6], heuristics and observed values for the number of small cycles (fixed points and two-cycles) in graphs of the type investigated here are given. Our methods build on this to generate experimental data for the parameters described by the theoretical predictions in Section 3. The method of data collection was straightforward. A prime was chosen as the modulus and then for each $g \in \{1, 2, 3, \ldots, p - 1\}$, the corresponding map or permutation was generated. The results were then computed as averages over all $p - 1$ graphs observed. The permutations and binary functional graphs were noted and their results were also tabulated separately. In this manner, the data can be examined in its complete form over all graphs and individually over the permutations and binary functional graphs. The generation and analysis of each of the graphs was handled by C++ code written by the first author.

The primes chosen for these calculations were

$$100043 = 2 \cdot 50021 + 1,$$
$$100057 = 2^3 \cdot 3 \cdot 11 \cdot 379 + 1,$$
$$106261 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 + 1.$$

The total number of graphs, permutations and binary functional graphs can be computed using Theorem 1 and are shown in Table 1. The combined results of

| Permutations | 100043 | 100057 | 106261 |
|--------------|--------|--------|--------|
| 50020        | 30240  | 21120  |
| Binary Functional Graphs | 50020 | 15120 | 10560 |
| 100042 | 100056 | 106260 |

Table 1. The number of permutations, binary functional graphs and total functional graphs associated with $p = 100043$, $p = 100057$, and $p = 106260$.

all functional graphs will be examined first in Section 4.1 where the observed results will be compared to the theoretical framework for random functional graphs given in Theorem 3. In Section 4.2, the observed results for the permutations will be compared to the theoretical results given in Theorem 4. Finally, the observed results for the binary functional graphs will be examined in Section 4.3. Theorems 5 through 7 will provide the theoretical predictions for these values. Since the terminal nodes and tail nodes can be directly computed from the image nodes and cyclic nodes, including them in the collected data does not add any insight. For this reason, they have both been excluded from the analysis conducted in the following sections. Appendix A gives some of the interesting extremal data such as the longest cycle observed for each prime.
4.1. Combined Results. It would seem that by combining better than one hundred thousand functional graphs generated by (1), the results would tend toward a random functional graph. Theorem 1 shows that the modular exponentiation function imposes some structure onto the functional graphs, but especially if \( p - 1 \) has a complex factorization, the large number of graphs might be thought to approach a lack of structure. However, as Table 2 clearly shows, these graphs are not tending toward a random functional graph.

| Components   | 100043 | 100057 | 106261 |
|--------------|--------|--------|--------|
| Observed     | 9.235  | 7.603  | 6.742  |
| Error        | 44.481%| 18.947%| 4.983% |
| Cyclic Nodes | 50271.600 | 33999.400 | 21268.600 |
| Error        | 12578.567% | 7574.478% | 5110.130% |
| Image Nodes  | 75029.000 | 47838.800 | 60435.300 |
| Error        | 18.644% | 24.363% | 3.374% |
| Avg Cycle    | 25088.934 | 15249.500 | 10629.500 |
| Error        | 12557.883% | 7593.148% | 5093.529% |
| Avg Tail     | 197.951  | 114.215 | 92.590  |
| Error        | 0.130%  | 42.380% | 54.674% |
| Max Cycle    | 31320.700 | 19027.821 | 13259.600 |
| Error        | 12555.466% | 7587.860% | 5098.564% |
| Max Tail     | 271.408  | 217.842 | 202.581 |
| Error        | 50.613% | 60.363% | 64.232% |

Table 2. The observed results for the three primes over all functional graphs generated and the corresponding percent errors.

4.2. Permutation Results. The results in Section 3 and Section 4.1 imply that the graphs should be split based on the value of \( m \), or the possible in-degrees of each node. The results of looking at only the values of \( g \) that were a primitive root modulo \( p \) (permutation graphs) can be found in Table 3.

| Components   | 100043 | 100057 | 106261 |
|--------------|--------|--------|--------|
| Observed     | 12.081 | 12.054 | 12.126 |
| Error        | 0.083% | 0.306% | 0.205% |
| Avg Cycle    | 49980.551 | 50191.352 | 53105.104 |
| Error        | 0.082% | 0.326% | 0.048% |
| Max Cycle    | 62395.488 | 62627.745 | 66245.807 |
| Error        | 0.102% | 0.256% | 0.144% |

Table 3. The observed results for the three primes over the permutations and the corresponding percent errors.

The percent error here is nearly zero in every instance. This seems to indicate that there are no obvious structural differences between a random permutation and a permutation generated by the process used here.

4.3. Binary Functional Graph Results. The binary functional graphs should prove more interesting than the permutations examined in the previous section. Unlike permutations, binary functional graphs do not appear to have been previously studied in detail. The statistics derived from the binary functional graphs and the error when compared to the results derived in Section 3.3 can be found in Table 4.
The observed results for the three primes over all binary functional graphs generated and the corresponding percent errors.

|                        | 100043   | 100057   | 106261   |
|------------------------|----------|----------|----------|
|                        | Observed | Error    | Observed | Error    | Observed | Error    |
| Components             | 6.389    | 0.047%   | 6.364    | 0.437%   | 6.370    | 0.810%   |
| Cyclic Nodes           | 395.303  | 0.029%   | 395.858  | 0.105%   | 408.433  | 0.217%   |
| Image Nodes            | 50021    | 6%       | 50028    | 0%       | 53130    | 0%       |
| Avg Cycle              | 198.319  | 0.056%   | 197.766  | 0.230%   | 202.651  | 0.795%   |
| Avg Tail               | 197.961  | 0.125%   | 197.550  | 0.339%   | 202.422  | 0.907%   |
| Max Cycle              | 247.261  | 0.094%   | 247.302  | 0.082%   | 256.986  | 0.754%   |
| Max Tail               | 541.827  | 1.115%   | 549.588  | 1.145%   | 566.370  | 1.744%   |

Table 4. The observed results for the three primes over all binary functional graphs generated and the corresponding percent errors.

The number of image nodes came out exactly as expected and predicted by Theorem 2. However, in many other cases the results were nearly as good. The relative size of the error follows the number of binary functional graphs for each prime. This is especially worth noting for \( p = 100043 \) which has over fifty thousand binary functional graphs while 100057 and 106261 have approximately fifteen thousand and ten thousand respectively. Since having more graphs appears to push the results closer to those derived in Section 3.3, this seems to further support the claim that the results hold for any binary functional graph produced by our mapping.

5. Conclusions and Future Work

The transformation used here to generate functional graphs and permutations is an exceedingly important transformation in cryptography. If the output of the function were to fall into a predictable pattern, it could be an exploitable flaw in many algorithms considered secure today. For instance, the average cycle length seems particularly important for pseudorandom bit generators since, in many cases, it relates directly to the predictability of the pseudorandom bit generator. As Theorem 1 demonstrates, the use of (1) repeatedly forces a non-trivial structure onto the graphs generated. This is certainly worth investigating as any imposed structure may be open to an exploit.

The advantage of using a safe prime is that every non-trivial graph can be analyzed by the theoretical framework laid out in this paper. Their use is also very prevalent in cryptographic applications. As mentioned above, the pseudorandom bit generator specified in [5] requires the use of a safe prime to defend against other attacks. However, the methods used for binary functional graphs in Section 3.3 can and should be extended to larger values of \( m \). In an ideal case, they should be extended in the general case for an \( m \)-ary graph that can be specified by

\[
\text{FunctionalGraph} = \text{set}(\text{Components}) \\
\text{Component} = \text{cycle}(\text{Node*Set}(\text{Tree}, \text{cardinality} = m - 1)) \\
\text{Tree} = \text{Node} + \text{Node*set}(\text{Tree}, \text{cardinality} = m) \\
\text{Node} = \text{Atomic Unit}
\]
The associated generating functions for these functional graphs would be
\[ f(z) = e^{c(z)} \]
\[ c(z) = \ln \left( 1 - \frac{z}{(m-1)!} t^{m-1}(z) \right)^{-1} \]
\[ t(z) = z + \frac{z}{m!} t^m(z) \]
where \( f(z) \) is the exponential generating function associated to the functional graphs, \( c(z) \) is the exponential generating function associated to the connected components and \( t(z) \) is associated to the trees. The methods in Section 3.3 could also be extended to obtain values for additional parameters such as the maximum tail length.

This paper has focused on the graphs generated when the modulus is prime. In practice, though, this is not always the case. For this reason, it could be worthwhile to attempt to extend the type of analysis done here to a composite modulus.

While the data generated for this project appears to confirm that the graphs do tend toward the shape and structure of a random graph of the appropriate type, no data was collected on the distribution of the different parameters. This data could help to give a clearer picture of how closely individual graphs may be expected to exhibit the characteristics of a random graph, especially given the observation that primes with a larger number of binary functional graphs seem to conform better to prediction on the average. The methods used in [1] would seem to be potentially helpful here.

**Appendix A. Extremal Data**

For \( p = 100043 \), the longest cycle observed was 100042 which occurred for two different values of \( g \). They were \( g = 20812 \) and \( g = 94034 \). The longest tail had a length of 1448 and was observed when \( g = 89339 \). There were five instances where the graphs contained no cycles longer than one which occurred for \( g = 1, 72116, 91980, 95997, \) and 100042.

The graphs generated by \( p = 100057 \) had an overall longest cycle of 100052 when \( g = 58303 \). The longest tail observed was 1589 when \( g = 18115 \). There were also 26 different values of \( g \) that produced a graph that did not have a cycle longer than one.

The largest cycle observed in graphs generated using \( p = 106261 \) was 106257 when \( g = 102141 \). The longest tail was 35822 when \( g = 1480 \). There were 92 different values of \( g \) that produced graphs with no cycles longer than a fixed point.

**References**

[1] Philippe Flajolet, Zhicheng Gao, Andrew Odlyzko, and Bruce Richmond. The distribution of heights of binary trees and other simple trees. *Combinatorics, Probability and Computing*, 2(2):145–156, 1993.
[2] Philippe Flajolet and Andrew Odlyzko. The average height of binary trees and other simple trees. *Journal of Computer and System Sciences*, 25:171–213, 1982.
[3] Philippe Flajolet and Andrew Odlyzko. Singularity analysis of generating functions. *SIAM J. Discrete Math.*, 3(2):216–240, 1990.
[4] Philippe Flajolet and Andrew M. Odlyzko. Random mapping statistics. In *Advances in cryptology—EUROCRYPT ’89 (Houthalen, 1989)*, volume 434 of *Lecture Notes in Comput. Sci.*, pages 329–354. Springer, Berlin, 1990.
[5] Rosario Gennaro. An improved pseudo-random generator based on the discrete logarithm problem. *Journal of Cryptology*, 18(2):91–110, 2005.

[6] Joshua Holden. Fixed points and two-cycles of the discrete logarithm. In *Algorithmic number theory (Sydney, 2002)*, volume 2369 of *Lecture Notes in Comput. Sci.*, pages 405–415. Springer, Berlin, 2002.

[7] L. A. Shepp and S. P. Lloyd. Ordered cycle lengths in a random permutation. *Trans. Amer. Math. Soc.*, 121:340–357, 1966.

CINCINNATI, OH, USA

*E-mail address*: Daniel.R.Cloutier@alumni.rose-hulman.edu

ROSE-HULMAN INSTITUTE OF TECHNOLOGY, TERRE HAUTE, IN 47803, USA

*E-mail address*: Joshua.Holden@rose-hulman.edu