The Critical Ising Model on a Möbius Strip

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Abstract

We study the two-dimensional critical Ising model on a Möbius strip based on a duality relation between conformally invariant boundary conditions. By using a Majorana fermion field theory, we obtain explicit representations of crosscap states corresponding to the boundary states. We also discuss the duality structure of the partition functions.
The continuum limit of the two-dimensional critical Ising model with boundaries is one of the simplest systems realized by field theories with boundaries. The presence of boundaries gives rise to new effects which depend on boundary conditions. Simple but important boundary conditions for Ising spins are those of the fixed and free ones. These boundary conditions are conformal invariant and are related by duality. A model with such boundary conditions has been analyzed in Ref. [1] using a boundary conformal field theory with the central charge \( c = \frac{1}{2} \). However, only a model defined on a cylinder has been considered to this time.

The purpose of this paper is to study the critical Ising model on a Möbius strip based on the duality relation between conformally invariant boundary conditions. We use a free massless Majorana fermion field theory as used in Ref. [4] to compute loop amplitudes of superstring theories with open strings. We obtain explicit representations of crosscap states corresponding to the boundary states for conformally invariant boundary conditions and discuss the duality structure of these crosscap states. We also show that the periodic Möbius partition function with the free boundary condition is constructed from those of the periodic cylinder and antiperiodic cylinder with free boundary conditions.

We now consider the continuum limit of the critical Ising model on a Möbius strip of width \( L \) and boundary length \( R \). We define coordinates \( x, y \) as \( 0 \leq x \leq L \) and \( 0 \leq y \leq R \). When we take the coordinate \( x \) to be Euclidean time, the system is described by the Hamiltonian \( H_R \) for the periodic space \( 0 \leq y \leq R \). The partition function is then given by

\[
Z = \langle B | e^{-\frac{L}{2} H_R} | C \rangle ,
\]

where \( |B\rangle \) is a boundary state placed at \( x = 0 \) and \( |C\rangle \) is a crosscap state placed at \( x = \frac{L}{2} \).

In the form (1) the partition function is a function of the modular parameter \( \tilde{\tau} = i\frac{R}{L} + \frac{1}{2} \). To represent the system concretely, we introduce free massless Majorana fermion fields \( \psi(x, y) \) and \( \overline{\psi}(x, y) \). In the picture (1) the fields have standard oscillator modes \( \psi_n \) and \( \overline{\psi}_n \) for the periodic space \( 0 \leq y \leq R \). In terms of these modes the Hamiltonian is

\[
H_R = \frac{2\pi}{R} \sum_{n>0} n \left( \psi_{-n} \psi_n + \overline{\psi}_{-n} \overline{\psi}_n \right) + E_0 ,
\]

where \( E_0 = -\frac{\pi}{12R} \) in the NS-sector (\( n \in \mathbb{Z}_+ + \frac{1}{2} \)) and \( E_0 = \frac{\pi}{6R} \) in the R-sector (\( n \in \mathbb{Z}_+ \)).

The explicit form of the boundary state in Eq. (1) depends on the boundary conditions of the model. Here we consider fixed and free boundary conditions for Ising spins, which are related by duality. For the fixed boundary condition, there are two possibilities: Ising spins on the boundary are all fixed in the \( s = +1 \) state (fixed(+) or in the \( s = -1 \) state (fixed(-)). Boundary states corresponding to the above three different boundary conditions
were obtained in Ref. [1]. In terms of the above modes they are

\begin{align}
|B_{\text{fixed}(\pm)}\rangle &= \frac{1}{\sqrt{2}} |B, -\rangle_{\text{NS}} \pm \frac{1}{\sqrt{2}} |B, -\rangle_R, \\
|B_{\text{free}}\rangle &= |B, +\rangle_{\text{NS}} ^{\text{(dual)}} = \frac{1}{\sqrt{2}} \left(|B_{\text{fixed}(+)}\rangle + |B_{\text{fixed}(-)}\rangle\right) \quad (3)
\end{align}

with

\begin{align}
|B, \pm\rangle_{\text{NS}} &= e^{\pm i \sum_{n \geq 1} \frac{1}{2} \psi_{-n} \bar{\psi}_{-n}} |0\rangle, \\
|B, \pm\rangle_R &= e^{\pm i \sum_{n \geq 1} \psi_{-n} \bar{\psi}_{-n}} |\pm\rangle. \quad (5)
\end{align}

The second equality of Eq. (4) implies that the free boundary condition is obtained from the fixed boundary conditions by duality \(\psi \rightarrow \psi, \bar{\psi} \rightarrow -\bar{\psi}\). Note that the free boundary state defined by the duality relation (4) describes periodic boundary Ising spins around the boundary since the fixed boundary states have periodic boundary spins. On the other hand, the crosscap state in Eq. (1) is generally defined by requiring the condition

\[\left[T\left(\frac{L}{2}, y\right) - T\left(\frac{L}{2}, y + \frac{R}{2}\right)\right] |C\rangle = 0, \quad (6)\]

where \(T(x, y)\) and \(\overline{T}(x, y)\) are the left moving and right moving components of the energy momentum tensor. In terms of the modes, solutions of Eq. (3) are given by

\begin{align}
|C, \pm\rangle_{\text{NS}} &= e^{\pm i \sum_{n \geq 1} \frac{1}{2} e^{\pi i n} \psi_{-n} \bar{\psi}_{-n}} |0\rangle, \\
|C, \pm\rangle_R &= e^{\pm i \sum_{n \geq 1} e^{\pi i n} \psi_{-n} \bar{\psi}_{-n}} |\pm\rangle. \quad (7)
\end{align}

To determine the crosscap states corresponding to the boundary states (3) and (4) in Eq. (1), we must evaluate these partition functions. They can be obtained from cylinder partition functions as follows. When we regard the coordinate \(y\) as Euclidean time, the partition function \(Z\) has the form

\[Z = \text{tr} f e^{-\frac{R}{2} H_L}, \quad (8)\]

where \(f\) is a flip operator and \(H_L\) is the Hamiltonian for the space \(0 \leq x \leq L\). The partition function without the flip operator \(Z^{\text{cyl}} = \text{tr} e^{-\frac{R}{2} H_L}\) is that of the critical Ising model defined on a cylinder of length \(L\) and circumference \(\frac{R}{2}\). The Möbius partition functions with fixed or free boundary conditions can be constructed from the cylinder partition functions with the same boundary conditions at the ends of the cylinder. There are three possible constructions:

\begin{align}
Z^{\text{cyl}(P)}_{\text{fixed}(\pm), \text{fixed}(\pm)} &\rightarrow Z^{\text{fixed}(\pm)}_{\text{fixed}(\pm)} , & Z^{\text{cyl}(P)}_{\text{free}, \text{free}} &\rightarrow Z^{(1)}_{\text{free}}, & Z^{\text{cyl}(A)}_{\text{free}, \text{free}} &\rightarrow Z^{(2)}_{\text{free}}, \quad (9)
\end{align}
where the lower two indices in the cylinder partition functions are the boundary conditions at \( x = 0 \) and \( x = L \). (P) and (A) denote periodic and antiperiodic boundary conditions of Ising spins around the cylinder, and (A) is only allowed for the free sector. These two types of boundary conditions come from traces in the lattice transfer matrix formulation.\(^{1}\)

We note that \( Z_{\text{cyl}}^{(P)} \) free/free is obtained from \( Z_{\text{cyl}}^{(P)} \) fixed(\( \pm \)) fixed(\( \mp \)) + \( Z_{\text{cyl}}^{(P)} \) fixed(\( \mp \)) fixed(\( \pm \)) by duality \((4)\), but \( Z_{\text{cyl}}^{(P)} \) fixed(\( \pm \)) fixed(\( \pm \)) does not appear in Eq. \((9)\), and that \( Z_{\text{cyl}}^{(P)} \) free/free cannot be expressed by using the boundary state \((4)\).

Let us now obtain explicit forms of Möbius partition functions in Eq. \((9)\) following the above considerations. For the same spin boundary conditions at \( x = 0 \) and \( x = L \), the fields \( \psi(x,y) \) and \( \overline{\psi}(x,y) \) are not independent and are expanded as\(^{2}\)

\[
e^{\frac{\pi}{4}i} \psi(x,y) = \sqrt{\pi L} \sum_{n \in \mathbb{Z} + \frac{1}{2}} b_n e^{\frac{\pi}{L} n(y-ix)},
\]

\[
e^{-\frac{\pi}{4}i} \overline{\psi}(x,y) = \sqrt{\pi L} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \overline{b}_n e^{\frac{\pi}{L} n(y+ix)},
\]

\((10)\)

where \( b_n = b_n \) for the fixed boundary condition and \( \overline{b}_n = -b_n \) for the free boundary condition. The mode \( b_n \) satisfies \( \{b_n, b_m\} = \delta_{n+m,0} \). On the other hand, the cylinder partition functions in Eq. \((9)\) were obtained in terms of the \( c = \frac{1}{2} \) Virasoro characters. Their explicit forms can be found in Ref. \((1)\). Therefore, by using the mode \( b_n \) representations of these cylinder partition functions, we can evaluate them in which the flip operator \( f = (-1)^{\sum_{n \geq \frac{1}{2}} nb_n} \) is inserted. In this way, we obtain

\[
Z_{\text{fixed}(\pm)}(\tau) = \frac{1}{2} \left( Z_{\text{free}}^{(1)}(\tau) + Z_{\text{free}}^{(2)}(\tau) \right),
\]

\[
Z_{\text{free}}^{(1)}(\tau) = e^{\frac{\pi}{4}i} \sqrt{\frac{\eta(\tau)}{\vartheta_3(0|\tau)}} \vartheta_3(0|\tau),
\]

\[
Z_{\text{free}}^{(2)}(\tau) = e^{\frac{\pi}{4}i} \sqrt{\frac{\eta(\tau)}{\vartheta_4(0|\tau)}} \vartheta_4(0|\tau),
\]

\((11), (12)\)

where we have introduced Jacobi theta functions \( \vartheta_i(z|\tau) \) and the Dedekind eta function \( \eta(\tau) \) with modular parameter \( \tau = \frac{iB}{4M} + \frac{1}{2} \). We use the notation of Ref. \((3)\).

To compare the above partition functions with those in the form \((4)\), we change the variable \( \tau \) to \( \bar{\tau} \) in Eqs. \((11)\) and \((12)\). The transformation properties of \( \vartheta \) and \( \eta \) functions under the modular transformation \( \tau \to \bar{\tau} = \frac{-1}{2\tau - 1} \) can be obtained by using the Poisson resummation formula. The results are

\[
\vartheta_3(0|\tau) = e^{\frac{\pi}{4}i} (-2\bar{\tau} + 1)^{\frac{1}{2}} \vartheta_4(0|\bar{\tau}),
\]

\[
\vartheta_4(0|\tau) = (-2\bar{\tau} + 1)^{\frac{1}{2}} \vartheta_3(0|\bar{\tau}),
\]

\[
\eta(\tau) = e^{\frac{\pi}{4}i} (-2\bar{\tau} + 1)^{\frac{1}{2}} \eta(\bar{\tau}).
\]

\((13)\)
Substituting Eq. (13) into Eqs. (12) and (11), we obtain the partition functions as functions of $\tilde{\tau}$. On the other hand, by using Eqs. (2)~(4) and (7), these partition functions can be uniquely expressed as

\[
\begin{align*}
Z_{\text{fixed}}^{(\pm)}(\tilde{\tau}) &= \langle B_{\text{fixed}}^{(\pm)} | e^{-\frac{L}{2}H_R} | C_{\text{(fixed)}} \rangle, \\
Z_{\text{free}}^{(1)}(\tilde{\tau}) &= \langle B_{\text{free}}^{(1)} | e^{-\frac{L}{2}H_R} | C_{\text{(free)}}^{(1)} \rangle, \\
Z_{\text{free}}^{(2)}(\tilde{\tau}) &= \langle B_{\text{free}}^{(2)} | e^{-\frac{L}{2}H_R} | C_{\text{(free)}}^{(2)} \rangle,
\end{align*}
\]

where

\[
\begin{align*}
|C_{\text{(fixed)}}\rangle &= \frac{1}{\sqrt{2}} \left( e^{+\frac{\pi}{8}i} |C, +\rangle_{\text{NS}} + e^{-\frac{\pi}{8}i} |C, -\rangle_{\text{NS}} \right), \\
|C_{\text{(free)}}^{(1)}\rangle &= e^{+\frac{\pi}{8}i} |C, -\rangle_{\text{NS}}, \\
|C_{\text{(free)}}^{(2)}\rangle &= e^{-\frac{\pi}{8}i} |C, +\rangle_{\text{NS}}.
\end{align*}
\]

Thus we have obtained the crosscap states corresponding to the boundary states (3) and (4).

Let us discuss the above results. We first find that the crosscap state $|C_{\text{(fixed)}}\rangle$ constructed by only the NS-sectors corresponds to two different fixed boundary states $|B_{\text{fixed}}^{(+)\rangle}$ and $|B_{\text{fixed}}^{(-)}\rangle$. This implies that the R-sector in the fixed boundary states (3) makes no contribution to the partition function $Z_{\text{fixed}}^{(\pm)}$. Next, we remark that $Z_{\text{free}}^{(1)}$ and $Z_{\text{free}}^{(2)}$ are described by the same boundary state $|B_{\text{free}}\rangle$, and therefore not only $Z_{\text{free}}^{(1)}$ but also $Z_{\text{free}}^{(2)}$ have periodic boundary Ising spins around the boundary. This can be understood as follows. If we define a crosscap state $|C_{\text{(free)}}\rangle$ by

\[
|C_{\text{(free)}}\rangle = \frac{1}{\sqrt{2}} \left( |C_{\text{(free)}}^{(1)}\rangle + |C_{\text{(free)}}^{(2)}\rangle \right),
\]

it satisfies $|C_{\text{(free)}}\rangle^{(\text{dual})} \equiv |C_{\text{(fixed)}}\rangle$. Then, from Eqs. (4), (14) and (16) we obtain a duality relation between fixed and free boundary conditions in the form (4) as

\[
\langle B_{\text{free}} | e^{-\frac{L}{2}H_R} | C_{\text{(free)}} \rangle^{(\text{dual})} = \frac{1}{\sqrt{2}} \left[ \left( \langle B_{\text{fixed}}^{(+)\rangle} + \langle B_{\text{fixed}}^{(-)}\rangle \right) e^{-\frac{L}{2}H_R} | C_{\text{(fixed)}} \rangle \right].
\]

Therefore, from Eqs. (16) and (17) we finally obtain the following identification in Eq. (11)

\[
Z_{\text{free}} \equiv \frac{1}{2} \left( Z_{\text{free}}^{(1)} + Z_{\text{free}}^{(2)} \right).
\]

We see that $Z_{\text{free}}$ defined by Eq. (18) is just the Möbius partition function with the free boundary condition. It is identical to $Z_{\text{fixed}}^{(\pm)}$ in Eq. (11), showing the duality relation between fixed and free boundary conditions corresponding to the form (17).

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