Rigidity of spacelike translating solitons in pseudo-Euclidean space

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Abstract: In this paper, we investigate the parametric version and non-parametric version of rigidity theorem of spacelike translating solitons in pseudo-Euclidean space $\mathbb{R}^{m+n}$. Firstly, we classify $m$-dimensional complete spacelike translating solitons in $\mathbb{R}^{m+n}$ by affine technique and classical gradient estimates, and prove the only complete spacelike translating solitons in $\mathbb{R}^{m+n}$ are the spacelike $m$-planes. This result provides another proof of a nonexistence theorem for complete spacelike translating solitons in [8], and a simple proof of rigidity theorem in [33]. Secondly, we generalize the rigidity theorem of entire spacelike Lagrangian translating solitons in [34] to spacelike translating solitons with general codimensions. As a directly application of theorem, we obtain two interesting corollaries in terms of Gauss image.

Keywords: translating soliton; rigidity theorem; pseudo-Euclidean space.

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1 Introduction

The mean curvature flow (MCF) in Euclidean space (pseudo-Euclidean space resp.) is a one-parameter family of immersions

$$X_t = X(\cdot, t) : M^m \to \mathbb{R}^{m+n}(\mathbb{R}^{m+n}_n \text{ resp.})$$

with the corresponding image $M_t = X_t(M)$ such that

$$\begin{cases} \frac{d}{dt}X(x, t) = H(x, t), & x \in M, \\ X(x, 0) = X(x), \end{cases}$$

is satisfied, here $H(x, t)$ is the mean curvature vector of $M_t$ at $X(x, t)$ in $\mathbb{R}^{m+n}(\mathbb{R}^{m+n}_n \text{ resp.})$. The MCF in higher codimension has been studied extensively in the last few

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years (cf. [6, 24, 26, 28, 29]). Translating soliton and self-shrinker play an important role in analysis of singularities in MCF. A submanifold $M^m$ is said to be a translating soliton in $\mathbb{R}^{m+n}$ (spacelike translating soliton in $\mathbb{R}^{m+n}_n$ resp.) if the mean curvature vector $H$ satisfies

$$H = T^\perp,$$

(1.2)

where $T \in \mathbb{R}^{m+n}$ is a non-zero constant vector, which is called a translating vector, and $T^\perp$ denotes the orthogonal projection of $T$ onto the normal bundle of $M$. $M^m$ is said to be a self-shrinker in $\mathbb{R}^{m+n}$ (spacelike self-shrinker in $\mathbb{R}^{m+n}_n$ resp.) if it satisfies a quasi-linear elliptic system

$$H = -X^\perp,$$

(1.3)

where $X^\perp$ is the normal part of $X$, which is an important class of solutions to (1.1).

There is a plenty of works on the classification and uniqueness problem for translating soliton and self-shrinker in Euclidean space (cf. [2, 4, 9, 11, 14, 18, 21, 23, 25, 27]). On the other hand, there are many works on the rigidity problem for complete spacelike submanifolds in pseudo-Euclidean space. Calabi [3] proposed the Bernstein problem for spacelike extremal hypersurfaces in Minkowski space $\mathbb{R}^{m+1}_{1}$ and proved such hypersurfaces have to be hyperplanes when $m \leq 4$. In [10] Cheng-Yau solved the problem for all dimensions. Later, Jost-Xin [20] generalized the results to higher codimensions.

**Theorem 1 [20]:** Let $M^m$ be a spacelike extremal submanifold in $\mathbb{R}^{m+n}_{n}$. If $M^m$ is closed with respect to the Euclidean topology, then $M$ has to be a linear subspace.

Translating soliton and self-shrinker can be regarded as generalizations of extremal submanifold. Thus it is natural to study the corresponding rigidity problem for spacelike translating soliton and self-shrinker. Here we only investigate the rigidity of spacelike translating soliton in $\mathbb{R}^{m+n}_{n}$. For complete spacelike Lagrangian translating solitons in $\mathbb{R}^{2n}_{n}$, Xu-Huang [33] proved the following rigidity theorem.

**Theorem 2 [33]:** Let $f(x_1, \ldots, x_n)$ be a strictly convex $C^\infty$-function defined on a convex domain $\Omega \subset \mathbb{R}^n$. If the graph $M_\nabla f = \{(x, \nabla f(x))\}$ in $\mathbb{R}^{2n}_{n}$ is a complete spacelike translating soliton, then $f(x)$ is a quadratic polynomial and $M_\nabla f$ is an affine $n$-plane.

From Theorem 2 above, it is easy to see that the corresponding translating vector must be **spacelike**. In [9], Chen-Qiu proved a nonexistence theorem for complete spacelike translating solitons in $\mathbb{R}^{m+n}_{n}$ by establishing a very powerful generalized Omori-Yau maximum principle. They proved that there exists no complete $m$-dimensional spacelike translating soliton (with a timelike translating vector). In fact, such nonexistence conclusion still holds for the lightlike translating vector case. Motivating by papers [8, 33],
we classify $m$-dimensional complete spacelike translating solitons in $\mathbb{R}^{m+n}$ by affine technique and classical gradient estimates, and obtain the following Bernstein theorem.

**Theorem 3:** Let $M^m$ be a complete spacelike translating soliton in $\mathbb{R}^{m+n}$, then it is an affine $m$-plane.

A more precise statement of the assertion in Theorem 3 says that there exists an $m$-dimensional complete spacelike translating soliton in $\mathbb{R}^{m+n}$ only if the translating vector is spacelike. It provides another proof of the nonexistence theorem of translating soliton in \cite{8}, and a simple proof of the Bernstein theorem for translating soliton in \cite{33}. Here we mention that it is valid if one shall use the generalized Omori-Yau maximum principle in \cite{8} to prove Theorem 3 above.

For non-parametric Lagrangian solitons of mean curvature flow, there are several interesting rigidity theorems. As to entire self-shrinker for mean curvature flow in $\mathbb{R}^{2n}$ with the indefinite metric $\sum_{i=1}^{n} dx_i dy_i$, Huang-Wang \cite{16} and Chau-Chen-Yuan \cite{5} used different methods to prove the rigidity of entire self-shrinker under a decay condition on the induced metric $(D^2 f)$. Later, using an integral method, Ding and Xin \cite{13} removed the additional decay condition and proved any entire smooth convex self-shrinking solution for mean curvature flow in $\mathbb{R}^{2n}$ is a quadratic polynomial. In \cite{17}, Huang and Xu investigated the rigidity problem of entire spacelike translating soliton graph $(x, \nabla f)$ in $\mathbb{R}^{2n}$ with some symmetry conditions. Motivating by \cite{5,13,16,17}, Xu-Zhu \cite{34} proved a rigidity theorem of entire convex translating solutions for mean curvature flow in $\mathbb{R}^{2n}$ under a decay condition on the induced metric $(D^2 f)$ and provided a class of nontrivial entire spacelike Lagrangian translating solitons.

**Theorem 4** \cite{34}: Let $f(x)$ be an entire smooth strictly convex function on $\mathbb{R}^n$ ($n \geq 2$) and its graph $M_{\nabla f} = \{ (x, \nabla f(x)) \}$ be a translating soliton in $\mathbb{R}^{2n}$. If there exists a number $\epsilon > 0$ such that the induced metric $(D^2 f)$ satisfies

\[
(D^2 f) > \frac{\epsilon}{|x|^2} I, \quad \text{as} \quad |x| \to \infty, \tag{1.4}
\]

then $f(x)$ must be a quadratic polynomial and $M_{\nabla f}$ is an affine $n$-plane.

Here we shall use the idea in \cite{34} to generalize Theorem 4 to spacelike graphic translating solitons in pseudo-Euclidean space $\mathbb{R}^{m+n}$.

**Theorem 5:** Let $u^\alpha (1 \leq \alpha \leq n)$ be smooth functions defined everywhere in $\mathbb{R}^m$ and their graph $M = (x, u^1(x), u^2(x), \ldots, u^n(x))$ be a spacelike translator in $\mathbb{R}^{m+n}$. If there exists a number $\epsilon > 0$ such that the induced metric $(g_{ij})$ satisfies

\[
(g_{ij}) > \frac{\epsilon}{|x|} I, \quad \text{as} \quad |x| \to \infty, \tag{1.5}
\]
then $u^1(x), \cdots, u^n(x)$ are linear functions on $\mathbb{R}^m$, and $M$ is an affine $m$-plane in $\mathbb{R}^{m+n}_n$.

**Remark:** It is necessary that there is a restriction on the induced metric $(g_{ij})$ for the rigidity of translating solitons. If not, there exist nontrivial entire smooth spacelike translating solitons, which are not planes. For example, submanifold

$$(x, y, \ln(1 + \exp\{2x\}) - x, \mu y), \quad |\mu| < 1, \quad (x, y) \in \mathbb{R}^2$$

(1.6)
is an entire spacelike translator with the translating vector $(0, 1, 1, \mu)$ in $\mathbb{R}^4_2$, which graphic functions satisfy the PDE (2.11).

By studying the distribution of the Gauss map, they obtained Bernstein theorems of translating solitons in Euclidean space (see [2], [21] and [32]). Notice that the Gauss image of spacelike graphic submanifold $M^m$ in $\mathbb{R}^{m+n}_n$ is bounded if and only if the induced metric $g = \text{det}(g_{ij})$ is bounded (see [31]). Therefore, it is easy to see that example (1.6) above and example (1.7) in [34] have boundless Gauss images. As a directly application of theorem 5, we have

**Corollary 1:** Let $M^m$ be an entire spacelike graphic translator in $\mathbb{R}^{m+n}_n$ as defined in Theorem 5. If the Gauss image of $M$ is bounded, then $M$ is an affine $m$-plane.

By relaxing the bound of the Gauss image to controlled growth, we also get a more general corollary from theorem 5 as Prof. Dong generalized a rigidity theorem for spacelike graph with parallel mean curvature in [15].

**Corollary 2:** Let $M^m$ be an entire spacelike graphic translator in $\mathbb{R}^{m+n}_n$ as defined in Theorem 5. If there exists a number $\epsilon > 0$ such that the induced metric $(g_{ij})$ satisfies

$$\text{det}(g_{ij}) > \frac{\epsilon}{|x|}, \quad \text{as} \quad |x| \to \infty,$$

(1.7)
then $M$ is an affine $m$-plane.

# 2 Preliminaries

The pseudo-Euclidean space $\mathbb{R}^{m+n}_n$ is the linear space $\mathbb{R}^{m+n}$ endowed with the metric

$$ds^2 = \sum_{i=1}^{m} (dx^i)^2 - \sum_{\alpha=m+1}^{m+n} (dx^\alpha)^2.$$  

(2.1)

Let $X : M^m \to \mathbb{R}^{m+n}_n$ be a spacelike $m$-submanifold in $\mathbb{R}^{m+n}_n$ with the second fundamental form $B$ defined by

$$B_{YW} := (\nabla_Y W)^\perp$$

(2.2)
for $Y, W \in \Gamma(TM)$, where $\nabla$ denotes the connection on $\mathbb{R}^{m+n}_n$. Let $(\cdot)^\top$ and $(\cdot)^\perp$ denote the orthogonal projections into the tangent bundle $TM$ and the normal bundle.
NM, respectively. Let $\nabla$ and $\nabla^\perp$ be connections on the tangent bundle and the normal bundle of $M$, respectively. Choose a local Lorentzian frame field $\{e_i, e_\alpha\}(i = 1, \cdots, m; \alpha = m + 1, \cdots, m + n)$ such that $\{e_i\}$ are tangent vectors to $M$. The mean curvature vector of $M$ in $\mathbb{R}^{m+n}$ is defined by

$$H = \sum_\alpha H^\alpha e_\alpha = \sum_i B_{ii} = -\sum_{i,\alpha} h_{ii}^\alpha e_\alpha,$$

(2.3)

where $h_{ii}^\alpha = \langle B_{ii}, e_\alpha \rangle$. Here, $\langle \cdot, \cdot \rangle$ is the canonical inner product in $\mathbb{R}^{m+n}$. We have the Gauss equation

$$R_{ijkl} = -\sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),$$

(2.4)

and the Ricci curvature

$$R_{ij} = -\sum_{\alpha, k} (h_{kk}^\alpha h_{ij}^\alpha - h_{ki}^\alpha h_{kj}^\alpha).$$

(2.5)

For a spacelike translating soliton $M^m$, by definition we can decompose the translating vector $T$ into a tangential part $V$ and a normal part $H$ on $M^m$, namely $T = V + H$. Define

$$\|H\|^2 = -\langle H, H \rangle = -|H|^2,$$

where $\|H\|^2$ is absolute value of the norm square of the mean curvature. Similarly define

$$\|B\|^2 = -\langle B, B \rangle = -|B|^2.$$

From (2.3) and the inequality of Schwartz, we have

$$\|B\|^2 \geq \frac{1}{m} \|H\|^2.$$  

(2.6)

Note that when the spacelike manifold $M^m$ is a Lagrangian gradient graph $(x, \nabla f)$ in $\mathbb{R}^{2n}$ with the indefinite metric $\sum_{i=1}^n dx_i dy_i$, the functions $\|H\|^2$ and $\|B\|^2$ are the norm of Tchebychev vector field $\Phi$ and the Pick invariant $J$ in relative geometry respectively (see [33]), up to a constant. Therefore we can use some affine technique to estimate functions $\|H\|^2$ and $\|B\|^2$.

Let $z = \langle X, X \rangle$ be the pseudo-distance function on $M$. Then we have (see also [30])

$$z_i = e_i(z) = 2 \langle X, e_i \rangle,$$

(2.7)

$$z_{ij} = Hess(z)(e_i, e_j) = 2(\delta_{ij} - \langle X, h_{ij}^\alpha e_\alpha \rangle),$$

(2.8)

$$\Delta z = \sum z_{ii} = 2m + 2\langle X, H \rangle.$$

(2.9)

In the following we set up the basic notations and formula for an $m$-dimension spacelike graphic submanifold in $\mathbb{R}_n^{m+n}$. Let

$$M := \{(x_1, \cdots, x_m, u^1, \cdots, u^n); x_i \in \mathbb{R}, u^n(x) = u^n(x_1, \cdots, x_m)\},$$
where \( i = 1, \ldots, m \) and \( \alpha = 1, \ldots, n \). Denote
\[
x = (x_1, \ldots, x_m) \in \mathbb{R}^m; u = (u^1, \ldots, u^n) \in \mathbb{R}^n.
\]
Let \( E_A (A = 1, \ldots, m + n) \) be the canonical Lorentzian basis of \( \mathbb{R}^{m+n} \). Namely, every component of the vector \( E_A \) is 0, except that the \( A \)-th component is 1. Then
\[
e_i = E_i + \sum_{\alpha} u^\alpha_i E_{m+\alpha}, \quad i \in \{1, \ldots, m\}
\]
give a tangent frame on \( M \). Here, \( u^\alpha_i = \frac{\partial u^\alpha}{\partial x_i} \). In pseudo-Euclidean space with index \( n \), the induced metric on spacelike submanifold \( M \) is
\[
g_{ij} = \langle e_i, e_j \rangle = \delta_{ij} - \sum_{\alpha} u^\alpha_i u^\alpha_j.
\]
Then there are \( n \) linear independent unit normal vectors,
\[
e_\alpha = \frac{\sum_i u^\alpha_i E_i + E_{m+\alpha}}{(1 - |Du^\alpha|^2)^{\frac{1}{2}}}, \quad \alpha \in \{1, \ldots, n\},
\]
where \( Du^\alpha = (u^\alpha_1, \ldots, u^\alpha_m) \). Thus the Levi-Civita connection with respect to the induced metric has the Christoffel symbols
\[
\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right) = -g^{kl} u^\alpha_{ij} u^\alpha_l. \quad (2.10)
\]
Put a nonzero constant vector
\[
T := \sum a^i E_i + \sum b^\alpha E_{m+\alpha}.
\]
By the definition of the translator (1.2), for each \( e_\alpha \), there holds
\[
\langle T, e_\alpha \rangle = -H^\alpha.
\]
By calculation, we have
\[
\langle H, e_\alpha \rangle = g^{ij} \langle \nabla_{e_i} e_j, e_\alpha \rangle = g^{ij} \langle u^\beta_{ij} E_{m+\beta}, e_\alpha \rangle
\]
\[
= -g^{ij} u^\alpha_{ij} \frac{1}{(1 - |Du^\alpha|^2)^{\frac{1}{2}}},
\]
and
\[
\langle T, e_\alpha \rangle = \frac{a^i u^\alpha_i - b^\alpha}{(1 - |Du^\alpha|^2)^{\frac{1}{2}}}.
\]
Then (1.2) is equivalent to the following elliptic system
\[
g^{ij} u_{ij}^\alpha = -a^i u^\alpha_i + b^\alpha, \quad \alpha \in \{1, \ldots, n\}. \quad (2.11)
\]
3 Calculation of $\Delta \|H\|^2$ and $\Delta \|B\|^2$

Using the idea and technique in [8] and [30], we have the following propositions.

**Proposition 3.1:** For the spacelike translating soliton $M^m$ in $\mathbb{R}^{m+n}$, the following estimate holds

$$\Delta \|H\|^2 \geq \frac{2}{m} \|H\|^4 - \langle T, \nabla \|H\|^2 \rangle.$$

**Proof:** Let $\{e_i\}$ be a local orthonormal normal frame field at the considered point of $M$. From (1.2), we derive

$$\nabla_{e_j} H = \left( \nabla_{e_j} \left( T - \sum_k \langle T, e_k \rangle e_k \right) \right)^\perp = - \sum_k \langle T, e_k \rangle B_{jk}, \quad (3.1)$$

and

$$\nabla_{e_i} \nabla_{e_j} H = - \sum_k \langle T, e_k \rangle \nabla_{e_i} B_{jk} - \sum_k \langle H, B_{ik} \rangle B_{jk}. \quad (3.2)$$

Hence, using the Codazzi equation

$$\nabla_{e_i} B_{jk} = \nabla_{e_k} B_{ji},$$

we have

$$\begin{align*}
\Delta |H|^2 &= 2|\nabla_{\perp} H|^2 + 2\langle H, \Delta_{\perp} H \rangle \\
&= 2|\nabla_{\perp} H|^2 - 2\langle H, \nabla_{\perp} H \rangle - 2 \sum_{i,k} \langle H, B_{ik} \rangle^2. \quad (3.3)
\end{align*}$$

It follows that

$$\begin{align*}
\Delta \|H\|^2 &= 2\|\nabla_{\perp} H\|^2 + 2\langle H, \nabla_{\perp} H \rangle + 2 \sum_{i,k} \langle H, B_{ik} \rangle^2 \\
&\geq 2\langle H, \nabla_{\perp} H \rangle + \frac{2}{m} \|H\|^4. \quad (3.4)
\end{align*}$$

Note that

$$2\langle H, \nabla_{\perp} H \rangle = \nabla_{\perp} \langle H, H \rangle = - \nabla_{\perp} \|H\|^2 = - \langle T, \nabla \|H\|^2 \rangle. \quad (3.5)$$

Then (3.4) and (3.5) together give proposition 3.1. □

In order to prove the completeness of entire graph spacelike translator with respect to the induced metric, we need the following type estimate for $\Delta \|B\|^2$.

**Proposition 3.2:** For the spacelike translating soliton $M$ in $\mathbb{R}^{m+n}$, the Laplacian of the second fundamental form $B$ satisfies

$$\Delta \|B\|^2 \geq \frac{2}{n} \|B\|^4 - \langle T, \nabla \|B\|^2 \rangle.$$
Proof: Let \( \{e_i\} \) be a local tangent orthonormal frame field on \( M \) and \( \{e_\alpha\} \) a local normal orthonormal frame field on \( M \) such that \( \nabla e_\alpha = 0 \) at the considered point. From [30], we have

\[
\Delta \|B\|^2 = \Delta \sum_{\alpha,i,j} (h^{\alpha}_{ij})^2 = 2 \sum_{\alpha,i,j} (h^{\alpha}_{ij,k})^2 + h^{\alpha}_{ij}h^{\alpha}_{i,lj}h^{\alpha}_{j,k} + h^{\alpha}_{ij}h^{\alpha}_{i,jk}h^{\alpha}_{k,j} + \sum_{\alpha,\beta} (S_{\alpha \beta})^2 + N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}) \tag{3.6}
\]

Let

\[S_{\alpha \beta} = \sum_{i,j} h^{\alpha}_{ij}h^{\beta}_{ij}, \quad N(h^{\alpha}) = \sum_{i,j} (h^{\alpha}_{ij})^2,\]

then

\[\|B\|^2 = \sum_{\alpha} S_{\alpha \alpha}.
\]

So (3.6) becomes

\[
\Delta \|B\|^2 = 2 \sum_{\alpha,i,j} (h^{\alpha}_{ij,k})^2 + h^{\alpha}_{ij}h^{\alpha}_{i,lj}h^{\alpha}_{j,k} + \sum_{\alpha,\beta} (S_{\alpha \beta})^2 + N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}) \tag{3.7}
\]

Note that

\[
\sum_{\alpha,\beta} (S_{\alpha \beta})^2 \geq \frac{1}{n} \left( \sum_{\alpha} S_{\alpha \alpha} \right)^2 = \frac{1}{n} \|B\|^4,
\]

\[N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}) \geq 0.
\]

From (3.2) and Codazzi equation, we get

\[
\sum_{\alpha,i,j} h^{\alpha}_{ij}H^{\alpha}_{ij} = \sum_{i,j} \langle B_{ij}, \nabla_{e_i} \nabla_{e_j} H \rangle = -\sum_{i,j} \langle B_{ij}, \nabla_{e_i} B_{ij} \rangle - \sum_{i,j,k} \langle H, B_{ik} \rangle \langle B_{ij}, B_{jk} \rangle = -\frac{1}{2} \langle V, \nabla \|B\|^2 \rangle + \sum_{\alpha,i,j} h^{\alpha}_{ij}h^{\alpha}_{j,k}h^{\beta}_{ik}H^{\beta}.
\]

Then substituting these inequalities into (3.7) completes the proof of proposition 3.2.

\[\square\]

4 Proof of theorem 3

To gain Bernstein theorem, we are about to prove \( \|H\|^2 \equiv 0 \) on \( M \). If \( \|H\|^2 \neq 0 \), then there exists a point \( p_0 \in M \) such that \( \|H\|^2(p_0) > 0 \). Set \( \|H\|^2(p_0) = \lambda \). Denote by \( r(p_0,p) \) the geodesic distance function from \( p_0 \in M \) with respect to the induced metric \( g \). For any positive number \( a \), let \( B_a(p_0) := \{ p \in M \mid r(p_0,p) \leq a \} \). We consider the function

\[\Psi := (a^2 - r^2)^2 \|H\|^2 \tag{4.1}\]
defined on $B_a(p_0)$. Obviously, $\Psi$ attains its supremum at some interior point $p^*$. We may assume that $r^2$ is a $C^2$-function in a neighborhood of $p^*$. Choose an orthonormal frame field on $M$ around $p^*$ with respect to the metric $g$. Then, at $p^*$, we have

$$\nabla \Psi = 0, \quad \Delta \Psi \leq 0.$$}

Hence

$$- \frac{2(r^2)_{,i}}{a^2 - r^2} + \frac{\|H\|_{,i}^2}{\|H\|^2} = 0,$$

(4.2)

$$- \frac{2\Delta r^2}{a^2 - r^2} - \frac{2|\nabla r^2|^2}{(a^2 - r^2)^2} + \frac{\Delta\|H\|^2}{\|H\|^2} - \frac{(\nabla\|H\|^2)^2}{\|H\|^4} \leq 0,$$

(4.3)

where "\nabla" denotes the covariant derivative with respect to the metric $g$.

Inserting Proposition 3.1 into (4.3), we get

$$- \frac{2\Delta r^2}{a^2 - r^2} - \frac{2|\nabla r^2|^2}{(a^2 - r^2)^2} - \frac{\langle T, \nabla\|H\|^2 \rangle}{\|H\|^2} + \frac{2}{m}\|H\|^2 - \frac{(\nabla\|H\|^2)^2}{\|H\|^4} \leq 0.$$  

(4.4)

Combining (4.2) with (4.4), we have

$$0 \geq - \frac{2\Delta r^2}{a^2 - r^2} - \frac{24r^2}{(a^2 - r^2)^2} - \frac{\langle T, \nabla\|H\|^2 \rangle}{\|H\|^2} + \frac{2}{m}\|H\|^2.$$  

(4.5)

By (4.2) and the inequality of Schwarz, we obtain

$$\langle T, \nabla\|H\|^2 \rangle = \langle V, \nabla\|H\|^2 \rangle \leq \epsilon|V|^2 + \frac{4}{\epsilon}\frac{r^2}{(a^2 - r^2)^2}$$

(4.6)

$$= \epsilon(C_0 + \|H\|^2) + \frac{4}{\epsilon}\frac{r^2}{(a^2 - r^2)^2},$$

where $C_0 = \langle T, T \rangle$ and $\epsilon$ is a small positive constant to be determined later. Inserting (4.6) into (4.5), we have

$$0 \geq - \frac{2\Delta r^2}{a^2 - r^2} - (24 + \frac{4}{\epsilon}) \cdot \frac{r^2}{(a^2 - r^2)^2} + \frac{2}{m} - \epsilon C_0.$$  

(4.7)

Now we shall use the technique in [19] to calculate the term $\Delta r^2$. In the case $p_0 \neq p^*$, we denote $a^* = r(p_0, p^*) > 0$ and assume that

$$\max_{B_{a^*}(p_0)} \|H\|^2 = \|H\|^2(\tilde{p}).$$

By Proposition 3.1 we have

$$\max_{B_{a^*}(p_0)} \|H\|^2 = \max_{\partial B_{a^*}(p_0)} \|H\|^2.$$
Thus

\[(a^2 - r^2(p^*))^2 \|H\|^2(\tilde{p}) = (a^2 - r^2(\tilde{p}))^2 \|H\|^2(\tilde{p}) \leq (a^2 - r^2(p^*))^2 \|H\|^2(p^*). \quad (4.8)\]

For any \(p \in B_{a^*}(p_0)\), by the Gauss equation we know that the Ricci curvature \(\text{Ric}(M, g)\) is bounded from below by

\[R_{ii}(p) \geq -\frac{1}{4} \|H\|^2(\tilde{p}). \quad (4.9)\]

By the Laplacian comparison theorem, we get

\[\Delta r^2 \leq 2m + (m - 1)a \|H\|((\tilde{p})). \quad (4.10)\]

Substituting (4.10) into (4.7), we have

\[(\frac{2}{m} - \epsilon) \|H\|^2(p^*) \leq c(m, \epsilon) \frac{a^2}{(a^2 - r^2)^2} + \epsilon C_0 + 2(m - 1) \|H\|(\tilde{p}) \frac{a}{a^2 - r^2}. \quad (4.11)\]

where \(c(m, \epsilon)\) is a positive constant depending only on \(m\) and \(\epsilon\). Multiplying both sides of (4.11) by \((a^2 - r^2)^2(p^*)\), we have

\[(\frac{2}{m} - \epsilon) \Psi \leq c(m, \epsilon)a^2 + \epsilon C_1 a^4 + 2(m - 1)a(a^2 - r^2)(p^*) \|H\|(\tilde{p}), \quad (4.12)\]

where

\[C_1 = \begin{cases} C_0, & \text{if } C_0 \geq 0 \\ 0, & \text{if } C_0 < 0. \end{cases} \]

By (4.8) and the inequality of Schwarz, we get

\[\Psi \leq c(m, \epsilon)a^2 + \frac{mC_1 \epsilon}{2 - 2m \epsilon} a^4. \quad (4.13)\]

In the case \(p_0 = p^*\), we have \(r(p_0, p^*) = 0\). It is easy to get (4.13) from (4.7). Therefore (4.13) holds on \(B_a(p_0)\).

Then, at any interior point \(q \in B_a(p_0)\), we get

\[\|H\|^2(q) \leq c(m, \epsilon) \frac{a^2}{(a^2 - r^2)^2} + \frac{mC_1 \epsilon}{2 - 2m \epsilon} \frac{a^4}{(a^2 - r^2)^2}. \quad (4.14)\]

**Case 1.** If \(C_0 > 0\), we choose \(\epsilon\) sufficiently small, such that

\[\frac{mC_1 \epsilon}{2 - 2m \epsilon} \leq \frac{\lambda}{2}. \]

For \(a \to \infty\), at the point \(q\), we get

\[\|H\|^2 \leq \frac{2\lambda}{3}. \]
In particular,
\[ \|H\|^2(p_0) \leq \frac{2\lambda}{3}. \]
This contradicts to \( \|H\|^2 = \lambda \) at \( p_0 \), so we can conclude that \( \|H\|^2 \equiv 0 \) on \( M^n \). Therefore it is an affine \( m \)-plane by theorem 1.

**Case 2.** If \( C_0 = 0 \), let \( a \to \infty \) in (4.14), we get
\[ \|H\|^2 = 0. \]
On the other hand, it is easy to see that \( H \) is a zero vector, since \( M \) is an \( m \)-dimensional spacelike submanifold. Thus \( T \in TM \). This contradicts to the translating vector \( T \) is nonzero. Then there exists no complete \( m \)-dimensional spacelike translating soliton with a **lightlike** translating vector.

**Case 3.** If \( C_0 < 0 \), let \( a \to \infty \) in (4.14), we get
\[ \|H\|^2 \leq 0. \]
This is impossible. Then there exists no complete \( m \)-dimensional spacelike translating soliton with a **timelike** translating vector. This completes the proof of Theorem 3. \( \square \)

**Remark:** Here we mention that we can replace \( \|H\| \) with \( \|B\| \) in the proof of theorem 3 and use proposition 3.2 to prove the second fundamental form \( \|B\| \equiv 0 \). It directly prove \( M^m \) must be a plane.

### 5 Proof of theorem 5

To gain theorem 5, we will show that the graph spacelike translating soliton is complete with respect to the induced metric \( g \). Thus from theorem 3, we complete the proof of theorem 5. Using the similar calculation of section 5 in [34], we prove the completeness of the induced metric \( g \) if \( \|B\| \) has an upper bound. To the simplicity, here we shall use theorem 2.1 of Prof. Xin in [29] to obtain the completeness.

**Proposition 5.1:** Let \( u^\alpha \) be smooth functions defined everywhere in \( \mathbb{R}^m \). Suppose their graph \( M = (x, u(x)) \) is a spacelike translator in \( \mathbb{R}^{m+n} \). If the norm of the second fundamental form \( \|B\| \) has a bound, then \( M \) is complete with respect to the induced metric.

**Proof:** Without loss of generality, we assume that the origin \( 0 \in M \). From Proposition 3.1 of [20], we know that the pseudo-distance function \( z = \langle X, X \rangle \) on \( M \) is a non-negative proper function. By (2.6), we know that if \( \|B\| \) has an upper bound,
\[ \|H\| \text{ also has an upper bound. On the other hand, by (3.1) we get} \]
\[ \|\nabla^\perp H\|^2 = -\langle \nabla^\perp H, \nabla^\perp H \rangle = -\sum_j \sum_k \langle T, e_k \rangle B_{jk}, \sum_l \langle T, e_l \rangle B_{jl} \]
\[ \leq |V|^2 \|B\|^2 \leq (|T|^2 + \|H\|^2) \|B\|^2. \]

Then \( \|\nabla^\perp H\| \) also has a bound. By Theorem 2.1 in [29], we have for some \( k > 0 \), the set \( \{ z \leq k \} \) is compact, then there is a constant \( c \) depending only on the dimension \( m \) and the bounds of mean curvature and its covariant derivatives such that for all \( x \in M \) with \( z(x) \leq \frac{k}{2} \),
\[ |\nabla z| \leq c(z + 1). \quad (5.1) \]

Let \( \gamma : [0, r] \to M \) be a geodesic on \( M \) issuing from the origin 0. Integrating (5.1) gives
\[ z(\gamma(r)) + 1 \leq \exp(cr), \]
which forces \( M \) to be complete with respect to the induced metric. \( \Box \)

In the following we prove the bound of \( \|B\| \).

**Proposition 5.2:** Under the assumption in Theorem 5, the function \( \|B\| \) has an upper bound on \( \mathbb{R}^m \).

**Proof:** By assumption in theorem 5, there exists a large enough \( R_0 \) such that \( (g_{ij}) > \frac{\epsilon}{|x|} I \) in case \( |x| > R_0 \). Choose a positive number \( a > R_0 \). Let \( B_a(0) := \{ x \in \mathbb{R}^m \mid |x| \leq a \} \). Consider the function below
\[ F(x) := (a^2 - |x|^2)^2 \|B\|^2 \]
defined on \( B_a(0) \). Obviously, \( F \) attains its supremum at some interior point \( p^* \). We can assume that \( \|B\|(p^*) > 0 \). Then, at \( p^* \),
\[ (\log F)_i = 0, \quad (\log F)_{ij} \leq 0. \]

In the following we calculate the first equation explicitly, and contract the second term above with the positive definite matrix \( (g^{ij}) \).
\[ \frac{(\|B\|^2)_i}{\|B\|^2} - \frac{4x_i}{a^2 - |x|^2} = 0, \quad (5.2) \]
\[ \sum_{i,j=1}^n g^{ij} \left( \frac{(\|B\|^2)_{ij}}{\|B\|^2} - \frac{(\|B\|^2)_i(\|B\|^2)_j}{\|B\|^4} - \frac{8x_i x_j}{(a^2 - |x|^2)^2} - \frac{4\delta_{ij}}{a^2 - |x|^2} \right) \leq 0. \quad (5.3) \]
From Proposition 3.2 and the inequality of Schwarz, we get

\[
\sum_{i,j} g_{ij} \frac{\|B\|^2}{\|B\|^2} = \Delta \frac{\|B\|^2}{\|B\|^2} + \sum_{i,j} \sum_{k} g_{ij} \Gamma_{ij}^k \frac{\|B\|^2}{\|B\|^2} \geq \frac{2}{n} \|B\|^2 - \langle V, \nabla \frac{\|B\|^2}{\|B\|^2} \rangle + \sum_{i,j,k} g_{ij} \Gamma_{ij}^k \frac{\|B\|^2}{\|B\|^2} \geq \frac{2}{n} \|B\|^2 - \frac{1}{mn} |\bar{V}|^2 - mn \left( \sum_{i,j,k} \frac{\|B\|^2}{\|B\|^2} \right) \}
\]

(5.4)

In the following we shall estimate the term \( \sum_{i,j,k} g_{ij} \Gamma_{ij}^k \frac{\|B\|^2}{\|B\|^2} \). By (2.10) (2.11) and (5.2), we have

\[
\sum_{i,j,k} g_{ij} \Gamma_{ij}^k \frac{\|B\|^2}{\|B\|^2} = \sum_{\alpha} (a_i^\alpha - b^\alpha) g_{kl}^\alpha \frac{4x_k}{a^2 - |x|^2}.
\]

By the inequality of Cauchy, we get

\[
\sum_{\alpha} (a_i^\alpha - b^\alpha) g_{kl}^\alpha x_k \leq \left( \sum_{\alpha} (a_i^\alpha - b^\alpha)^2 \right)^{\frac{1}{2}} \left( \sum_{\alpha} (g_{kl}^\alpha x_k)^2 \right)^{\frac{1}{2}}.
\]

By the spacelike of \( M \), we have

\[
\sum_{\alpha} (u_i^\alpha)^2 < 1, \quad \forall \quad 1 \leq i \leq m.
\]

Then

\[
\left[ \sum_{\alpha} (a_i^\alpha - b^\alpha)^2 \right]^{\frac{1}{2}} \leq \left[ \sum_{\alpha} (b^\alpha)^2 \right]^{\frac{1}{2}} + \left[ \sum_{\alpha} \left( \sum_{i} a_i^\alpha \right)^2 \right]^{\frac{1}{2}}
\]

\[
\leq |\bar{b}| + \left[ |\bar{a}|^2 \left( \sum_{\alpha} \sum_{i} (u_i^\alpha)^2 \right) \right]^{\frac{1}{2}}
\]

\[
= |\bar{b}| + \left[ |\bar{a}| \left( \sum_{i} (\sum_{\alpha} u_i^\alpha)^2 \right) \right]^{\frac{1}{2}} \leq |\bar{b}| + \sqrt{m} |\bar{a}|,
\]

where \( \bar{a} = (a^1, ..., a^m) \), \( \bar{b} = (b^1, ..., b^n) \). On the other hand,
\[
\left[ \sum_{\alpha} \left( g^{kl \alpha} u^\alpha x_k \right)^2 \right]^{\frac{1}{2}} \leq \left[ \sum_{\alpha} \left( g^{kl \alpha} u^\alpha \right)^2 \right]^{\frac{1}{2}} (g^{kl} x_l x_k)^{\frac{1}{2}} \\
\leq \left[ (\sum_i g^{ii})(\sum_k (u_k^\alpha)^2) \right]^{\frac{1}{2}} \left[ (\sum_k g^{kk}) |x|^2 \right]^{\frac{1}{2}} \\
\leq \left[ (\sum_i g^{ii})(\sum_k (u_k^\alpha)^2) \right]^{\frac{1}{2}} (\sum_k g^{kk})^{\frac{1}{2}} |x| \\
\leq \sqrt{ma} (\sum_i g^{ii}).
\]

Therefore
\[
\sum_{i,j,k} g^{ij} \Gamma^k_{ij} \left( \frac{\|B\|^2}{\|B\|^2} \right)_k \leq \frac{C_1 a \sum g^{ii}}{a^2 - |x|^2}, \tag{5.5}
\]

where \( C_1 = 4m(|\vec{a}| + |\vec{b}|) \).

Thus, inserting (2.6), (5.2) and (5.5) into (5.4), we get
\[
\sum_{i,j} g^{ij} \frac{\|B\|^2}{\|B\|^2} \geq \frac{2}{n} \|B\|^2 - \frac{1}{mn} |V|^2 - mn \left( \frac{\nabla \|B\|^2}{\|B\|^4} \right) - \frac{C_1 a \sum g^{ii}}{a^2 - |x|^2} \\
\geq \frac{1}{n} \|B\|^2 - \frac{1}{mn} C_0 - 16mn \frac{a^2 \sum g^{ii}}{(a^2 - |x|^2)^2} - \frac{C_1 a \sum g^{ii}}{a^2 - |x|^2}, \tag{5.6}
\]

where \( C_0 = \langle T, T \rangle \).

Then, inserting (5.2) and (5.6) into (5.3), we have
\[
\|B\|^2 \leq \frac{1}{m} C_0 + (16mn^2 + 28n) \frac{a^2 \sum g^{ii}}{(a^2 - |x|^2)^2} + nC_1 a \frac{\sum g^{ii}}{a^2 - |x|^2}. \tag{5.7}
\]

Multiplying both sides of (5.7) by \((a^2 - |x|^2)^2(p^*)\), we have
\[
(a^2 - |x|^2)^2 \|B\|^2(p^*) \leq \frac{1}{m} C_0 a^4 + [(16mn^2 + 28n)a^2 + nC_1 a^3] \sum g^{ii}. \tag{5.8}
\]

If \( p^* \in B_{R_0}(0) \), then for any \( x \in B_a(0) \),
\[
F(x) \leq a^4 \max_{B_{R_0}(0)} \|B\|^2.
\]

If \( p^* \in B_a(0) \setminus B_{R_0}(0) \), by assumption
\[
\sum g^{ii} \leq m \frac{|x|}{\epsilon},
\]
we get
\[
[(a^2 - |x|^2)^2 \|B\|^2](p^*) \leq \frac{1}{m} C_0 a^4 + C_2 a^3 + \frac{mnC_1}{\epsilon} a^4. \tag{5.9}
\]
where $C_2$ is a positive constant depending only on $m$, $n$ and $\epsilon$. Thus, in both cases, we always have

\[ F(x) \leq [(a^2 - |x|^2)^2 \|B\|^2](p^*) \leq C_3 a^4 + C_2 a^3, \]  

(5.10)

where $C_3$ is a positive constant depending only on $m$, $n$, $\epsilon$, $C_1$, $C_0$ and $\max_{B_{\delta_0}(0)} \|B\|^2$. Then, at any interior point $q \in B_\delta(0)$, we obtain

\[ [(a^2 - |x|^2)^2 \|B\|^2](q) \leq [(a^2 - |x|^2)^2 \|B\|^2](p^*) \leq C_3 a^4 + C_2 a^3. \]  

(5.11)

Dividing both sides of (5.11) by $(a^2 - |x|^2)^2$, we obtain

\[ \|B\|^2(q) \leq C_3 \left( \frac{a^4}{(a^2 - |x|^2)^2} + \frac{a^3}{(a^2 - |x|^2)^2} \right). \]  

(5.12)

Let $a \to \infty$ in (5.12), we get

\[ \|B\|^2(q) \leq C_3. \]

This completes the proof of Proposition 5.2. □

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