Observables for spacetimes with two Killing field symmetries

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Abstract

The Einstein equations for spacetimes with two commuting spacelike Killing field symmetries are studied from a Hamiltonian point of view. The complexified Ashtekar canonical variables are used, and the symmetry reduction is performed directly in the Hamiltonian theory. The reduced system corresponds to the field equations of the SL(2,R) chiral model with additional constraints.

On the classical phase space, a method of obtaining an infinite number of constants of the motion, or observables, is given. The procedure involves writing the Hamiltonian evolution equations as a single 'zero curvature' equation, and then employing techniques used in the study of two dimensional integrable models. Two infinite sets of observables are obtained explicitly as functionals of the phase space variables. One set carries sl(2,R) Lie algebra indices and forms an infinite dimensional Poisson algebra, while the other is formed from traces of SL(2,R) holonomies that commute with one another. The restriction of the (complex) observables to the Euclidean and Lorentzian sectors is discussed.

It is also shown that the sl(2,R) observables can be associated with a solution generating technique which is linked to that given by Geroch.
In classical general relativity one of the important questions is that of finding exact solutions and extracting their properties. This is hindered by the complexity of Einstein’s equations, and the discovery of a new solution is rare.

It is therefore usual to simplify the problem by seeking solutions that have certain symmetries. These are normally specified by requiring the metric to have a number of Killing vector fields, which leads to a simplified set of equations to solve.

One such set of reduced equations is obtained by requiring the metric to have two commuting vector fields. This simplification leads to a two dimensional field theory, and has the advantage that it still leaves the gravitational field with two local degrees of freedom, (unlike, for example the minisuperspace reductions, where only a finite number of degrees of freedom remain). This symmetry reduction was first studied in detail by Geroch [1], who found that the resulting Einstein equations have an infinite dimensional ‘hidden’ symmetry. These symmetry transformations of the equations provide a solution generating technique, whereby, given one solution with two commuting Killing fields, a new family of solution can be generated. The solution generating technique was later presented from other points of view [2–4]. These equations have also been studied using the inverse scattering method [5] to obtain solitonic solutions.

The question of exact solutions is related to that of conserved quantities. It is expected, as for any dynamical system, that exact solutions will be labelled by values of the conserved quantities. In general relativity, for spacetimes with compact spacelike hyper surfaces, the latter are also referred to as observables. This is because if conserved quantities can be written explicitly as functionals of the phase space variables (which should always be possible), they would also be the fully gauge invariant variables.

It is useful to have phase space observables for the classical theory, in particular in attempts to prove integrability. For example, in all the known two dimensional integrable models such as the KdV and Sine-Gordon equations, an explicit generating procedure for
observables may be used to prove integrability [6,7].

Apart from the classical questions, in attempts to construct a canonical quantum theory starting from general relativity, a complete set of such classical variables is a prerequisite for certain quantization schemes, where the quantum theory is to be obtained as a representation of the Poisson algebra of observables [8–10]. This method has been under study for the nonperturbative approach to quantum gravity using the Ashtekar variables [9,11] and the related loop space representation [12]. It has been successful for the quantization of 2+1 gravity [13].

For the full Einstein equations, it is known that the only ‘hidden’ symmetries, apart from diffeomorphisms, are constant rescalings of the metric [14]. From this result it follows that no observables can be built as integrals of local functions of the initial data [15]. However, from the works mentioned above, the two Killing field reduced equations are known to have an associated infinite dimensional symmetry group. It is then natural to ask what are the conserved quantities associated with these symmetries, and in particular what they are as functionals of the phase space variables.

In some recent work [16], a procedure based on methods used for finding conservation laws for soliton equations has been applied to the two Killing field reduced Einstein equations. The starting point in this work was a particular form of the metric with two commuting spacelike Killing fields. The dynamical Einstein equations following from this were then studied using ideas from two dimensional integrable models. If these quantities can be written as phase space functionals, one would have an infinite number of observables for this sector of Einstein gravity. However it is not clear from this work how the conserved quantities can be rewritten in terms of the ADM phase space variables.

This paper addresses the question of obtaining observables for two Killing field reduced Einstein gravity. The main result presented below is an explicit construction of an infinite number of phase space observables for spacetimes with two commuting spacelike Killing fields, and with compact spatial hypersurfaces. The observables are obtained for complexified gravity (i.e. complex phase space variables on a real manifold). The reality conditions are
then discussed for the Euclidean and Lorentzian restrictions.

The natural starting point is the Hamiltonian form of the Einstein equations. The Ashtekar Hamiltonian formulation [9, 11] is used for this, and in the next section the two Killing field symmetry is imposed in these variables to obtain a reduced first class Hamiltonian system which still has two local degrees of freedom. This reduction corresponds to the Gowdy cosmological models [17], and has been studied earlier by the author and Smolin [18]. In the third section the reduced system is fully gauge fixed, with the gauge fixing conditions chosen to put the Hamiltonian evolution equations in a suggestive form. This is discussed further in the following section, where a zero curvature form of the evolution equations is given. The fifth section gives the procedure for obtaining the observables, and is based on methods used in two dimensional integrable models. There is also a discussion of the Poisson algebra of the observables. The sixth section describes a solution generating technique for this sector of the Einstein equations using the observables, and its connection with the Geroch method. The paper ends with a summary and outlook for the quantization of this sector of gravity.

II. TWO KILLING VECTOR FIELD REDUCTION

The Ashtekar Hamiltonian variables for complexified general relativity are the (complex) canonically conjugate pair \((A^i_a, \tilde{E}^{ai})\) where \(A^i_a\) is an \(so(3)\) connection and \(\tilde{E}^{ai}\) is a densitized dreibein. \(a, b, ..\) are three dimensional spatial indices and \(i, j, .. = 1, 2, 3\) are internal \(so(3)\) indices. The constraints of general relativity are

\[
\mathcal{G}^i := D_a \tilde{E}^{ai} = 0,
\]

\[
C^a := F^{ai}_{ab} \tilde{E}^{ai} = 0,
\]

\[
\mathcal{H} := \epsilon^{ijk} F^{ai}_{ab} \tilde{E}^{aj} \tilde{E}^{bk} = 0,
\]

where \(D_a \lambda^i = \partial_a \lambda^i + \epsilon^{ijk} A^j_a \lambda^k\) is the covariant derivative, and \(F^{ai}_{ab}\) is its curvature.

Since the phase space variables are complex, reality conditions need to be imposed to
obtain the Euclidean or Lorentzian sectors. These are $A^i_a = \bar{A}^i_a$, $E^{ai} = \bar{E}^{ai}$ for the former, and $A^i_a + \bar{A}^i_a = 2\Gamma^i_a(E)$, $E^{ai} = \bar{E}^{ai}$ for the latter. The $\Gamma^i_a(E)$ is the connection for spatial indices and the bar denotes complex conjugation.

We now review the two commuting spacelike Killing field reduction of these constraints which was first presented in [18]. Working in spatial coordinates $x, y$, such that the Killing vector fields are $(\partial/\partial x)^a$ and $(\partial/\partial y)^a$ implies that the phase space variables will depend on only one of the three spatial coordinates. Specifically, we assume that the spatial topology is that of a three torus so that the phase space variables depend on the time coordinate $t$ and one angular coordinate $\theta$. This situation corresponds to one of the Gowdy cosmological models [17]. (The other permitted spatial topologies for the Gowdy cosmologies are $S^1 \times S^2$ and $S^3$.)

In addition to these Killing field conditions, we set to zero some of the phase space variables as a part of the symmetry reduction:

$$\tilde{E}^{x3} = \tilde{E}^{\eta 3} = \tilde{E}^{\eta 1} = \tilde{E}^{\eta 2} = 0,$$
$$A^3_x = A^3_y = A^1_\theta = A^2_\theta = 0. \quad (2.4)$$

These conditions may be viewed as implementing a partial gauge fixing and solution to some of the constraints. The end result below (2.5-2.7) is a simplified set of first class constraints which describes a two dimensional field theory on $S^1 \times R$ with two local degrees of freedom.

Renaming the remaining variables $A := A^3_\theta$, $E := \tilde{E}^{\eta 3}$ and $A^I_\alpha$, $\tilde{E}^{\alpha I}$, where $\alpha, \beta, .. = x, y$ and $I, J, .. = 1, 2$, the reduced constraints are

$$G := \partial E + J = 0, \quad (2.5)$$
$$C := F^I_\alpha E^{\alpha I} = 0, \quad (2.6)$$
$$H := -2\epsilon^{IJ} F^I_{\alpha \beta} E^{\alpha J} E + F_{\alpha \beta} E^{\alpha I} E^{\beta J} \epsilon_{IJ}$$
$$= -2EE^{\alpha I} \epsilon^{IJ} \partial A^I_\alpha + 2AEK - K^\beta_\alpha K^\alpha_\beta + K^2 = 0, \quad (2.7)$$

where $\partial = (\partial/\partial \theta)$.
\[ K_\alpha^\beta : = A_\alpha^I E^{\beta I}, \quad K := K_\alpha^\alpha, \quad (2.8) \]
\[ J_\alpha^\beta : = \epsilon^{IJ} A_\alpha^I E^{\beta J} \quad J := J_\alpha^\alpha, \quad (2.9) \]

and \( \epsilon^{12} = 1 = -\epsilon^{21} \).

The SO(3) Gauss law has been reduced to U(1) and the spatial diffeomorphism constraint to Diff(S^1) as may be seen by calculating the Poisson algebra of the constraints smeared by functions \( \Lambda, V, \) and the lapse \( N \) (which is a density of weight -1):

\[ G(\Lambda) = \int_0^{2\pi} d\theta \ \Lambda G, \quad (2.10) \]
\[ C(V) = \int_0^{2\pi} d\theta \ V C, \quad (2.11) \]
\[ H(N) = \int_0^{2\pi} d\theta \ N H \quad (2.12) \]

\[ \{G(\Lambda), G(\Lambda')\} = \{G(\Lambda), H(N)\} = 0, \quad (2.13) \]
\[ \{C(V), C(V')\} = C(\mathcal{L}_V V'), \quad (2.14) \]
\[ \{H(N), H(N')\} = C(W) - G(AW), \quad (2.15) \]

where

\[ W \equiv E^2 (N \partial N' - N' \partial N). \quad (2.16) \]

This shows that \( C \) generates Diff(S^1). Also we note that this reduced system still describes a sector of general relativity due to the Poisson bracket \( \{H(N), H(N')\} \), which is the reduced version of that for full general relativity in the Ashtekar variables.

The variables \( K_\alpha^\beta \) and \( J_\alpha^\beta \) defined above will be used below in the discussion of observables. Here we note their properties. They are invariant under the reduced Gauss law (2.5), transform as densities of weight +1 under the Diff(S^1) generated by \( C \), and form the Poisson algebra

\[ \{K_\alpha^\beta, K_\gamma^\sigma\} = \delta_\alpha^\sigma K_\gamma^\beta - \delta_\beta^\sigma K_\alpha^\gamma, \quad (2.17) \]
\[ \{J_\alpha^\beta, J_\gamma^\sigma\} = -\delta_\alpha^\sigma K_\gamma^\beta + \delta_\beta^\sigma K_\alpha^\gamma, \quad (2.18) \]
\[ \{K_\alpha^\beta, J_\gamma^\sigma\} = \delta_\alpha^\sigma J_\gamma^\beta - \delta_\gamma^\sigma J_\alpha^\beta. \quad (2.19) \]
This shows that $K_{\alpha}^\beta$ form the gl(2) Lie algebra, and hence generate gl(2) rotations on variables with indices $\alpha, \beta, \ldots = x, y$.

The following linear combinations of $K_{\alpha}^\beta$ form the sl(2,R) subalgebra of gl(2,R):

$$L_1 = \frac{1}{2}(K_y^x + K_x^y) \quad L_2 = \frac{1}{2}(K_x^x - K_y^y) \quad L_3 = \frac{1}{2}(K_y^x - K_x^y)$$  \hspace{1cm} (2.20)

The Poisson bracket algebra of these is

$$\{L_i, L_j\} = C^{k}_{ij} L_k,$$

where $C_{12}^3 = -1, C_{23}^1 = 1, C_{31}^2 = 1$ are the sl(2,R) structure constants. The corresponding linear combinations of $J_{\alpha}^\beta$ are denoted by $J_i, i=1,2,3$. Their Poisson brackets are

$$\{L_i, J_j\} = C^{k}_{ij} J_k, \quad \{J_i, J_j\} = -C^{k}_{ij} L_k.$$  \hspace{1cm} (2.22)

Also

$$\{J, L_i\} = \{J, L_i\} = \{K, J_i\} = \{K, L_i\} = 0.$$  \hspace{1cm} (2.23)

For discussing observables, it will turn out to be very convenient to replace the eight canonical phase space variables $A_I^\alpha, \tilde{E}^{\alpha I}$ by the eight Gauss law invariant variables $K_{\alpha}^\beta, J_{\alpha}^\beta$.

**III. GAUGE FIXING AND THE METRIC**

The Dirac observables are defined as the phase space functionals $O[A, E]$ that have vanishing Poisson brackets with all the first class constraints of the theory. This is because the first class constraints generate local gauge transformations via Poisson brackets. The question of finding the observables can be equally well addressed prior to, or after, full gauge fixing of a first class system. Each will yield observables in terms of the phase space variables.

Assuming that variables $O[A, E]$ invariant under the kinematical Gauss law and spatial diffeomorphism invariant have already been determined, (which is relatively easy), the first case would correspond to solving for $O[A, E]$ the equation:
\{H(N), O\} \sim 0. \quad (3.1)

The second amounts to solving

\{\tilde{H}, O\} = 0 \quad (3.2)

where the last equality is strong, and \(\tilde{H}\) is a suitably gauge fixed Hamiltonian constraint.

The second procedure will be followed here since, with a particular gauge choice to be described in this section, the Hamiltonian evolution equations can be put in a very simple form.

Full gauge fixing using the Ashtekar variables requires a careful consideration of the reality conditions on the phase space coordinate \(A^I_a\). This is because the (complex) phase space variables depend on real coordinates. For conventional gauge fixing where some functions of the phase space variables are chosen as the coordinates, real functions must be chosen. But since the constraints themselves are complex, two real conditions must be imposed for complete gauge fixing. Here we gauge fix the complex theory by requiring that certain (complex) functions of the phase space variables vanish. This results in (complex) gauge fixed evolution equations and second class constraints. The reality conditions are discussed below, where the metric resulting from the gauge fixing is compared with the standard metric for this reduction, and in section V where the observables are obtained.

We start by fixing the Gauss law (2.5) by imposing the gauge fixing condition \(A = 0\). Solving this constraint gives

\[ E = c - \int^{\theta'} d\theta' J(\theta'), \quad (3.3) \]

where \(c\) is an arbitrary constant. The diffeomorphism and Hamiltonian constraints (2.6-2.7) in this gauge become

\[ H = -2(c - \int^{\theta'} d\theta' J(\theta'))E^\alpha I \epsilon^I J \partial A^J_\alpha - K^\beta K^\alpha + K^2 \quad (3.4) \]

\[ C = E^\alpha I \partial A^J_\alpha. \quad (3.5) \]

and are still first class. In particular (3.4) satisfies the Poisson bracket relation
\{H(N), H(N')\} = C(W), \quad (3.6)

with \(W\) given by (2.16), which is the usual Poisson bracket of the Hamiltonian constraint with itself. Thus (3.4-3.5) on the \(A^I, \tilde{E}^{\alpha I}\) phase space still describe general relativity with two local degrees of freedom.

We now work with the eight Gauss law invariant densities \(L_i, J_i\) and \(K, J\) introduced in the last section instead of the eight remaining phase space variables \(A^I, \tilde{E}^{\alpha I}\). The evolution equations \(\dot{F} = \{F, H(N)\}\) with \(H\) from (3.4) for these variables are

\[
\dot{L}_i = -2\partial [N(c - \int d\theta' J(\theta'))J_i], \quad (3.7)
\]

\[
\dot{J}_i = 2\partial [N(c - \int d\theta' J(\theta'))L_i] + 4NC_{ij} J_j L_k, \quad (3.8)
\]

and

\[
\dot{J} = 2\partial [N(c - \int d\theta' J(\theta'))K], \quad (3.9)
\]

\[
\dot{K} = -2\partial [N(c - \int d\theta' J(\theta'))J]. \quad (3.10)
\]

A natural and consistent gauge fixing for the remaining gauge freedom is achieved by the choice \(J = 0, K = \text{constant}\), so that the diffeomorphism and Hamiltonian constraints become strongly zero.

The condition \(J = 0\) is second class with the Hamiltonian constraint, and in this gauge the evolution equations (3.9-3.10) for \(J, K\) reduce to

\[
\dot{K} = 0, \quad \dot{J} = 0 = 2c\partial(NK). \quad (3.11)
\]

The first implies that \(K = K(\theta)\). The second fixes the lapse to be \(N(\theta) = a/K(\theta)\), where \(a\) is a complex constant. We may now fix the (real) \(\theta\) coordinate condition by setting \(ReK(\theta) = k \neq 0\) and \(ImK(\theta) = 0\), where \(k\) is a real constant density on the circle. Thus these gauge fixing conditions on the phase space variables imply that the lapse and shift functions are constants. If we choose \(a\) to be real, the lapse is real.

The evolution equations (3.7-3.8) for the remaining variables, the six \(L_i, J_i\), with the choice \(N = 1/4, c = 2\) become
\[ \dot{L}_i + J'_i = 0 \]  \hspace{1cm} (3.12)

\[ \dot{J}_i - L'_i + C_i^{jk} L_j J_k = 0. \]  \hspace{1cm} (3.13)

where \( \dot{} \equiv \partial/\partial \theta \). These, together with the strongly imposed Hamiltonian and diffeomorphism constraints

\[ \epsilon^{ij} \alpha_j \partial A^i + L_1^2 + L_2^2 - L_3^2 = 0 \]  \hspace{1cm} (3.14)

\[ E^{\alpha i} \partial A_{\alpha}^i = 0, \]  \hspace{1cm} (3.15)

form the fully gauge fixed set of two Killing field reduced complex Einstein equations. There are \( 6 - 2 = 4 \) local phase space degrees of freedom. We note that these are written in terms of the original phase space variables, so that Poisson brackets may still be calculated using the fundamental \( (A_i^I, E_{\alpha I}) \) bracket. Also, the gauge fixing has reduced the \( \text{gl}(2,\mathbb{R}) \) Casimir term in the Hamiltonian constraint to the \( \text{sl}(2,\mathbb{R}) \) Casimir \( (3.14) \).

We have used the gauge condition \( J = 0 \) which is not explicitly time dependent and is second class with the Hamiltonian constraint. Normally such a gauge condition for compact spatial hypersurfaces implies that the lapse function must be zero, which implies no evolution and a degenerate metric. But this is not the case for the two Killing field reduction with this gauge due to \( (3.9-3.10) \), which are consistent with constant non-zero lapse, and as we see below, lead to a metric of the standard type for two Killing field reductions. The two conditions \( J = 0, K = \text{constant} \) imply that the shift is also constant.

The main purpose of the gauge fixing was not to get an explicitly reduced Hamiltonian in terms of the two physical degrees of freedom, but to look at the full evolution equations \( (3.7-3.8) \) in a convenient gauge which is useful for obtaining the conserved quantities. The \( J = 0 \) gauge is very convenient for this. However one can obtain a non-vanishing reduced Hamiltonian as a function of \( L_i, J_i \) that leads to the evolution equations \( (3.12-3.13) \). It is that for the \( \text{Sl}(2,\mathbb{R}) \) chiral model.

Since the gauge fixed evolution equations \( (3.12-3.13) \) involve only \( J_i, L_i \), the conserved charges will depend only on these. It is therefore important to check that the charges
commute with the second class constraints (3.14-3.15). The commutation with the strong Hamiltonian constraint is guaranteed because the variables \( J_i, L_i \) commute with \( J, K \), (which are the variables fixed in the gauge choice):

\[
\dot{L}_i = \{ L_i, H(N) \} \bigg|_{J=0, K=\text{const.}} = \{ L_i, H(N) \} \bigg|_{J=0, K=\text{const.}},
\]

(3.16)

with the same equation holding for \( J_i \). This is another reason why it appears natural to separate the phase space variables into \( \text{sl}(2,\mathbb{R}) \) variables \( J_i, L_i \), with gauge conditions imposed on the traces \( J, K \). As we will see below, the charges also commute with the diffeomorphism constraint (3.13) by construction.

We will not solve the second class constraints explicitly, since the goal is only to obtain the conserved quantities. The second class constraints imply that there are two relations among the six \( J_i, L_i \). In principle these can be substituted into the conserved quantities to rewrite them in terms of four independent reduced variables.

For comparison with the usual metric variables, it is useful to see what form of the metric arises from the gauge choices made above. The general line element with two commuting spacelike Killing fields can be put in the form [17]

\[
ds^2 = e^{2F} (-dt^2 + d\theta^2) + g_{\alpha\beta} dx^\alpha dx^\beta
\]

(3.17)

where the four functions \( F, g_{\alpha\beta} \) are four functions of \( t, \theta \) only. On the other hand, the gauge choices made above lead to the line element

\[
ds^2 = -(\frac{1}{16} + \frac{1}{2}\sqrt{q} C^2) dt^2 + \frac{C}{\sqrt{q}} dt d\theta + \frac{1}{2}\sqrt{q} d\theta^2 + \frac{2}{\sqrt{q}} q_{\alpha\beta} dx^\alpha dx^\beta
\]

(3.18)

where \( q_{\alpha\beta} \) is the matrix inverse of \( \tilde{E}^\alpha_i \tilde{E}^\beta_i \), \( q = det q_{\alpha\beta} \), and \( C \) is a constant (the shift). By a suitable gauge condition which fixes \( F \) as a function of the other three metric variables \( g_{\alpha\beta} \), and a coordinate transformation, (3.17) can be brought into the form (3.18). In arriving at (3.18), \( E^a_i \) has been fixed to be real (reality condition), and the lapse and shift chosen to be real constants. Note that while the reality conditions on \( A^I_a \) have not been imposed, this does not affect the general form (3.18) of the Lorentzian metric that will result.
We now note an alternative natural gauge fixing which may also be useful for this system but will not be used in this paper. The Hamiltonian constraint (2.7) contains the product $AE$, and $E$ transforms like a scalar under the reduced diffeomorphism constraint (2.6). This suggests the (partial) gauge fixing $ReE = t$, $ImE = 0$, which gives $H_R := -A$ as the (complex) reduced Hamiltonian. Substituting this gauge condition into (2.3-2.7) gives the first class constraints

$$J = 0,\quad E^{\alpha I} \partial A^I_\alpha = 0,$$

and the time dependent reduced Hamiltonian

$$H_R = -\frac{1}{K} E^{\alpha J} \epsilon^{IJ} \partial A^I_\alpha + \frac{1}{2K^t}(K^2 - K^\alpha K^\beta).$$

The time dependence in $H_R$ is associated only with the ultralocal part, which is also the gl(2,R) Casimir invariant. This suggests that for small times the ultralocal piece dominates the dynamics and that a perturbation theory in $t$ may be possible. The reality conditions on the $A$’s still need to be applied.

**IV. EVOLUTION EQUATIONS AS A ZERO CURVATURE CONDITION**

The evolution equations (3.12-3.13) derived in the last section can be rewritten in a compact form using the sl(2,R) matrix generators

$$g_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which satisfy the relations $[g_i, g_j] = C_{ij}^k g_k$ and $g_i g_j = \frac{1}{2} C_{ij}^k g_k$. Defining the matrices

$$A_0 := L_i g_i, \quad A_1 := J_i g_i,$$

the evolution equations (3.12-3.13) become

$$\partial_0 A_0 + \partial_1 A_1 = 0.$$

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\[ \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = 0. \quad (4.4) \]

Equations (4.3-4.4) are the first order form of the SL(2,R) chiral model field equations.

The two evolution equations (4.3-4.4) may be rewritten as a single equation in the following way. Define for a real parameter \( \lambda \)

\[
a_0 := \frac{1}{1 + \lambda^2} (A_0 - \lambda A_1) \quad \text{and} \quad a_1 := \frac{1}{1 + \lambda^2} (\lambda A_0 + A_1) \quad (4.5)
\]

Then equations (4.3-4.4) follow from the single ‘zero curvature’ equation

\[
a_1' - a_0' + [a_0, a_1] = 0. \quad (4.6)
\]

This equation, together with the strong constraints (3.14-3.15) form the two spacelike commuting Killing field reduction. The dynamical equation (4.6) is used in the following section to obtain the conserved charges.

\[
V. \text{OBSERVABLES}
\]

The field equations for all the known two dimensional integrable models have zero curvature formulations analogous to that given in the last section. This is a direct consequence of the existence of two distinct symplectic forms on the phase spaces of the models \( \mathfrak{g} \), which is also the geometric way of viewing the Lax pair formulation. Another consequence of the zero curvature formulation is a procedure for generating an infinite number of conserved charges. We now apply this to the dynamical equation (4.6) arising from the two Killing field reduction. The resulting observables will be for complex gravity and the reality conditions on them will be discussed at the end of the section.

The transfer matrix used in the study of two dimensional models is analogous to the Wilson loop. For the present case, it is the path ordered exponential associated with the matrix \( a_1 \):

\[
U[A_0, A_1](0, \theta) := \lim_{N \to \infty} \prod_{\Delta \to 0} [1 + a_1(\theta) \Delta \theta] \equiv \exp \int_0^\theta a_1(A_0, A_1, \lambda) \, d\theta' \quad (5.1)
\]
The trace of the transfer matrix is preserved under time evolution as may be seen by deriving its equation of motion using equation (4.6). We note first that

\[ U'(0, \theta) = U(0, \theta) a_1(\theta), \quad U'(\theta, 2\pi) = -a_1(\theta) U(\theta, 2\pi). \]  

(5.2)

The time derivative of the first gives

\[ \dot{U}'(0, \theta) = \dot{U}(0, \theta) a_1 + U(0, \theta) \dot{a}_1 \]

\[ = \dot{U}(0, \theta) a_1 + U(0, \theta) (a_0' - [a_0, a_1]), \]

which may be rewritten as

\[ (\dot{U}(0, \theta) - U(0, \theta) a_0)' = (\dot{U}(0, \theta) - U(0, \theta) a_0) a_1. \]  

(5.3)

Thus, since \( \dot{U}(0, \theta) - U(0, \theta) a_0 \) satisfies the same equation as \( U(0, \theta) \), we get the equation of motion

\[ \dot{U}(0, \theta) = U(0, \theta) a_0(\theta) - a_0(0) U(0, \theta). \]  

(5.4)

From this it follows that

\[ M[A_0, A_1](\lambda) := \text{Tr} U(0, 2\pi) \]  

(5.5)

is conserved in time. The conservation of this trace follows in basically the same way as the conservation of the Wilson loop observable when there is a zero-curvature constraint on the phase space, such as in 2+1 gravity [13]. That \( M \) is spatial diffeomorphism invariant follows from noting that \( a_1 \) transforms like a density under the Diff(\( S^1 \)) generated by (3.5):

\[ \{C(V), a_1\} = -\partial(V a_1), \]  

(5.6)

from which it follows that \( \{C(V), M\} = 0 \).

Expanding \( M \) in a power series in \( \lambda \) gives explicitly the phase space observables, which are the coefficients of powers of \( \lambda \): The first three observables are

\[ Q^0 := M|_{\lambda=0} = \text{TrPexp}\left[ \int_0^{2\pi} d\theta \ A_1 \right] =: \text{Tr} V(0, 2\pi), \]  

(5.7)
\[ Q^1 := \frac{\partial M}{\partial \lambda} \big|_{\lambda=0} = \int_0^{2\pi} d\theta \; \text{Tr}[V(0, \theta)A_0(\theta)V(\theta, 2\pi)], \] \quad (5.8)

and

\[ \frac{\partial^2 M}{\partial \lambda^2} \big|_{\lambda=0} = -2 \int_0^{2\pi} d\theta \; \text{Tr}[V(0, \theta)A_1(\theta)V(\theta, 2\pi)] \\
+ \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \; \Theta(\theta_2 - \theta_1) \\
\text{Tr}[V(0, \theta_1)A_0(\theta_1)V(\theta_1, \theta_2)A_0(\theta_2)V(\theta_2, 2\pi)], \] \quad (5.9)

where \( \Theta(\theta - \theta') = 1, \theta \geq \theta' \) and zero otherwise. It is straightforward to verify directly the conservation of these functionals using the equations of motion (4.3-4.4).

The structure of the general observable can now be seen and we can write down the observable with \( n \) insertions of \( A_0 \) in the holonomies \( V \):

\[ Q^n := \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \Theta(\theta_n - \theta_{n-1}) \cdots \Theta(\theta_2 - \theta_1) \]
\[ \text{Tr}[V(0, \theta_1)A_0(\theta_1)V(\theta_1, \theta_2)A_0(\theta_2) \cdots A_0(\theta_n)V(\theta_n, 2\pi)]. \] \quad (5.10)

This has a remarkable resemblance to the \( T \) variables used in 3+1 gravity \([12]\),

\[ T^{a_1 \cdots a_n} [A_i^a, \tilde{E}^{ai}](x_1, \ldots x_n; \alpha) := \text{Tr}[U_\alpha(x_0, x_1)\tilde{E}^{a_1}(x_1)U_\alpha(x_1, x_2) \cdots \tilde{E}^{a_n}(x_n)U_\alpha(x_n, x_0)], \] \quad (5.11)

where the holonomies \( U_\alpha \) are based on the loop \( \alpha \) are made from Ashtekar’s connection \( A_i^a \), and the insertions in the product of holonomies are the conjugate momenta \( \tilde{E}^{ai} \) instead of \( A_0 \). The other difference is that in equation (5.11) there is an integration over all the point insertions of \( A_0 \), (which gives invariance under the remaining spatial diffeomorphisms \( \text{Diff}(S^1) \) in the present reduction).

Another set of observables is obtained by looking at the first term in (5.9) where there is an insertion of \( A_1 \) in the holonomies instead of \( A_0 \). The general observables of this type is similar to (5.10) but with \( n \) insertions of \( A_1 \):

\[ P^n := \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \Theta(\theta_n - \theta_{n-1}) \cdots \Theta(\theta_2 - \theta_1) \\
\text{Tr}[V(0, \theta_1)A_1(\theta_1)V(\theta_1, \theta_2)A_1(\theta_2) \cdots A_1(\theta_n)V(\theta_n, 2\pi)]. \] \quad (5.12)
The Poisson algebra of the observables (5.10-5.12) may be calculated using the trace identity

$$\text{Tr}[Xg^i]\text{Tr}[Yg^j] = C_{kj}^{ij}(\text{Tr}[Xg^kY] - \text{Tr}[Yg^kX]),$$

(5.13)

for SL(2,R) matrices $X, Y$ and generators $g^i$. We find

$$\{P^m, P^n\} = 0 \quad (5.14)$$

$$\{Q^m, Q^n\} \sim Q^{m+n-1} + Q^{m+n-1}, \quad m, n > 1 \quad (5.15)$$

$$\{Q^0, Q^m\} \sim Q^{m+1} \quad (5.16)$$

There is another method for generating conserved charges for two dimensional chiral models [19] which can be applied here to generate observables. This is useful for comparison with the above procedure. Also, as discussed below, the resulting observables for the Killing field reduction give a solution generating technique which may be viewed as the Hamiltonian analog of Geroch method [1]. This procedure for obtaining observables has also been applied to self-dual gravity [20, 21].

The starting point is the dynamical equations (4.3-4.4). We note that (4.4) is already like a conservation law and so the first conserved charge is

$$q^{(1)} = \int_0^{2\pi} d\theta \ A_0 = \int_0^{2\pi} d\theta \ L_i g^{i}, \quad (5.17)$$

which gives the three sl(2,R) charges

$$q^{(1)}_i = \int_0^{2\pi} d\theta \ L_i. \quad (5.18)$$

These observables were obtained earlier in [18].

The current $J^{(1)}_\mu := A_\mu (\mu, \nu, .. = 0, 1)$ is conserved so there exists a (matrix) function $f^{(1)}(t, \theta)$ such that

$$J^{(1)}_\mu = \epsilon^{\mu}_{\nu} \partial_\nu f^{(1)}. \quad (5.19)$$

We now define the second current by
\[ \mathcal{J}^{(2)}_\mu := D_\mu f^{(1)} \equiv \partial_\mu f^{(1)} + A_\mu f^{(1)} \]  

(5.20)

With this definition of a derivative operator, the equation of motion (1.4) may be rewritten as \([D_0, D_1] = 0\). The conservation of \(\mathcal{J}^{(2)}_\mu\) is easy to show:

\[ \delta^{\mu\nu} \partial_\nu \mathcal{J}^{(2)}_\mu = \delta^{\mu\nu} \partial_\nu D_\mu f^{(1)} = \delta^{\mu\nu} \epsilon^{\mu\alpha} D_\mu D_\alpha f^{(0)} = 0, \]  

(5.21)

where the last equality follows because \(\mathcal{J}^{(1)}_\mu = D_\mu f^{(0)} = A_\mu\), where \(f^{(0)}\) is the identity matrix, and \([D_0, D_1] = 0\) by the equation of motion (4.4). This procedure generalizes, and it is straightforward to give an inductive proof that \(\mathcal{J}^{(n+1)} := D_\mu f^{(n)}\) is conserved, assuming \(\mathcal{J}^{(n)}_\mu\) is conserved. The observables are

\[ q^{(n)} := \int_0^{2\pi} d\theta \ \mathcal{J}^{(n)}_0. \]  

(5.22)

The second conserved charge is

\[ q^{(2)} := \int_0^{2\pi} d\theta \ D_0 f^{(1)}(\theta, t) = \int_0^{2\pi} d\theta \ (-A_1(\theta, t) + A_0(\theta, t) \int^\theta d\theta' A_0(\theta', t)). \]  

(5.23)

In terms of the sl(2,R) phase space functions this is

\[ q_i^{(2)} = \int_0^{2\pi} d\theta \ (-J_i + \frac{1}{2} C_{ij}^k L_j \int^{\theta} d\theta' L_k). \]  

(5.24)

The conservation of this may be checked directly using (3.12-3.13):

\[ q_i^{(2)} = \int_0^{2\pi} d\theta \ [-L_i' + C_{ij}^k L_j J_k - \frac{1}{2} C_{ij}^k (J'_j \int^{\theta} d\theta' L_k + L_j \int^{\theta} d\theta' J'_k)] \]

\[ = \int_0^{2\pi} d\theta \ [C_{ij}^k L_j J_k + \frac{1}{2} C_{ij}^k (J_j L_k - L_j J_k)] = 0. \]  

(5.25)

The Poisson bracket of the first two charges is

\[ \{q_i^{(1)}, q_j^{(2)}\} = C_{ij}^k q_k^{(2)}. \]  

(5.26)

Since \(q_i^{(1)}\) form an \(\text{sl}(2,\mathbb{R})\sim\text{so}(2,1)\) Lie algebra it follows that all the observables \(q_i^{(n)}\) with \(\text{sl}(2,\mathbb{R})\) indices will have the Poisson algebra
The Poisson algebra of the higher observables \( q_i^{(n)} \) with themselves is more involved and there are in general non-linear combinations of observables on the right hand sides. We note that given the first two observables \( q_i^{(1)}, q_j^{(2)} \), the remaining may also be generated by taking Poisson brackets of these with themselves. Another feature of this set is that they are sl(2,R) Lie algebra valued whereas the first set obtained above, using \( M(5.5) \), are traces of SL(2,R) group elements.

In the steps above, we have obtained a gauge fixed version of complex two Killing field reduced gravity, and given two methods for obtaining observables. The observables are for the complexified theory and reality conditions must be imposed on them to obtain their restrictions on the Euclidean or Lorentzian sections.

The restriction to the Euclidean section involves just setting the \( L_i, J_i \) to be real. The Lorentzian restriction requires setting the triads to be real, and imposing \( A^I_{\alpha} - \bar{A}^{I}_{\alpha} = 2\Gamma^I_{\alpha}(E) \). This reality condition implies that the complex conjugate of the observables are also observables. Therefore when the triads are set to be real, if \( Q[A, E] \) is an observable, so is \( Q[\bar{A}, E] \). Thus \( Q[A, E] + Q[\bar{A}, E] \) is a real observable for the complex theory. The real observables for the Lorentzian section in terms of the original phase space variables may be obtained as

\[
(Q[A, E] + Q[\bar{A}, E])|_{\bar{A}=2\Gamma-A}.
\]

VI. SOLUTION GENERATING TECHNIQUE

In this section we discuss the relation between the second set of observables obtained above and the solution generating technique for spacetimes with two commuting Killing fields given by Geroch [1]. We note only the general features of the method, which are unchanged by the reality conditions.

A solution of the Einstein equation with two commuting spacelike Killing fields is a phase space trajectory labelled by values of the conserved quantities \( q_i^{(n)} \). A new solution can be
generated from a given one by considering the Hamiltonian flow of the phase space variables \( L_i, J_i \) generated by the observables \( q_i^{(n)} \). This flow may be parametrized by a parameter \( s \), and specified by giving three ‘shift’ functions \( F^i(s) \):

\[
\begin{align*}
\frac{dL_i(t, \theta; s)}{ds} &= \{ L_i(t, \theta; s), F^k(s)q_k^{(n)} \} \\
\frac{dJ_i(t, \theta; s)}{ds} &= \{ J_i(t, \theta; s), F^k(s)q_k^{(n)} \}.
\end{align*}
\tag{6.1}
\]

Integration of these equations with the initial condition that \( L_i(t, \theta; s = 0), J_i(t, \theta; s = 0) \) lie on the given solution, gives the values of these variables on the new solution at say, \( s = 1 \).

We therefore see that a new exact solution of the Einstein equations may be constructed from a given one by specifying a curve \( \gamma(s) \) \((0 \leq s \leq 1)\) in a three dimensional vector space with tangent vector \( F^i(s) \), and with \( \gamma(0) \) at the origin. But these are precisely the conditions given by Geroch for generating new solutions from a given one \([1]\). In particular, the intermediate equations (6.1) that need to be integrated as a part of the procedure are of exactly the same form as those present in ref. \([1]\). Thus the infinite number of \( \text{sl}(2,\mathbb{R}) \) observables obtained in the last section may be viewed as the phase space analogs of the generators of Geroch’s transformation.

VII. DISCUSSION

The main new result given in this paper is the explicit construction of an infinite number of phase space observables for spacetimes with two commuting spacelike Killing vector fields. The previous studies of this reduction of the Einstein equations, in particular Geroch’s work, provided strong indications of the existence of such observables.

Our approach involved rewriting the Hamiltonian evolution equations using the Ashtekar variables, and then choosing a particular gauge fixing which allowed these equations to be rewritten as those of the \( \text{SL}(2,\mathbb{R}) \) chiral model \((4.3-4.4)\). From this form of the equations, two known methods were used to obtain the observables. The first made use of the conservation of the trace of the monodromy matrix \( M \) \((5.3)\), which acts as the generating functional for
the observables. The second made use of a recursive procedure given by Brezin et. al. [19] to calculate non-local conserved charges in two dimensional models.

One set of observables obtained from the monodromy matrix have a structure similar to that of the loop observables that have been used to study the quantization of full 3+1 gravity [12]. This is interesting and suggests that it should be possible to obtain the quantized two-Killing field reduction directly from the full 3+1 observables.

The second set have an infinite dimensional algebra which doesn’t appear to have a simple form. However, as discussed in section VI these observables can be used to give a solution generating method for this sector of the Einstein equations. In particular, the solution generating procedure has exactly the same ingredients as Geroch’s one, which indicates that it is the phase space analog of it.

One of main reasons for addressing the observables problem is that it provides one way to address the quantization issue. For generally covariant theories the observables are also the fully gauge invariant phase space variables. A quantum theory may be constructed by finding a representation of the Poisson algebra of a complete set of classical observables. From the results given above, the second set of observables $q_i^{(n)}$ may be suitable for this provided their Poisson algebra can be put into a more manageable form. Previous work [4] on a simpler method of obtaining the Geroch procedure provides a hint that this Poisson algebra may actually be an SL(2,R) Kac-Moody (Affine) algebra. The task is then to see if the $q_i^{(n)}$ can be replaced by some functions of them such that the Poisson algebra simplifies to this. This is under investigation.

A further question regarding the observables that hasn’t been addressed is the question of completeness: Can any invariant phase space variable be expressed as a sum of products of the observables obtained here? In particular, is there any relation between the observables obtained using the two different methods? These questions are important for studying quantization, which has been previously studied in the loop space representation in ref. [18]. It was found that there are an infinite number of observables in the quantum theory that form a gl(2) loop algebra. However, surprisingly the classical counterparts of these
observables was not known. It is likely that the observables given here form a subset of these quantum observables, and the correspondence merits further study.

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