Application of Cauchy-Schwarz inequality method for resolving constrained optimization problems at classroom with formal pre-operational phase of thinking

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Abstract. The optimization problem is a mathematical problem that is introduced at the junior high school level. Material related to optimization has even been introduced for elementary school levels in students with a level of concrete operations, especially for students who are prepared to take part in the Math Olympiad. The problem arises here, is that complicated methods for finding solutions to optimization problems, such as the simplex method, ordinary differential equations, partial differential equations, and the Lagrange multiplier were only introduced in high school or in college. To bridge this we need a special method that is easily understood by students participating in the pre-Olympic who are at the level of formal pre-operative cognition, one of these methods is called the Cauchy-Schwarz inequality method. The Cauchy-Schwarz inequality is a powerful tool for finding the maximum and minimum solutions for a target equation with one or two constraint equations. The interesting thing is that the prices of the optimum manufacturing variables are reached when the Cauchy-Schwarz equation is fulfilled.

1. Introduction
In order to obtain meaningful learning, the mathematics material is taught to follow and harmonize with the cognitive levels of students. According to Piaget the level of cognition or maturity of thinking, divided into four: (1) Sensory motor, (2) Pre-operational, (3) Concrete operations, and (4) Operations or formal reasoning[1]. Adapting Piaget's theory can be made in a hierarchical-sequential stage of the relationship between thinking maturity and the age of the child i.e sensory motor is in the age range of 0-3 years, and formal operations are in the range of 13 years and above. The age of concrete operational is usually students who are at the end of primary education, 6th grade and junior high school. In general, at the stage of concrete operations, students are able to think concretely, but do not close the possibility there are a number of students who can think abstractly. So, there are a number of children of new age at the stage of concrete operations, but the mindset has entered formal pre-surgery. This level is a transition from concrete post-surgery to formal preoperative. Usually these group students enter the superior class or the Olympic class. Thus difficult mathematical material based on a certain level of cognition, sometimes it has been taught to students below that level. For example the problem of optimization, determining the maximum and minimum values of the constrained function, has been included as children's learning material at elementary school age (grade 6) and junior high school, especially related to the Math Olympics.

In USAMO 1978 [2], optimization problems have arisen: "The sum of 5 real numbers is 8 and the sum of their squares is 16. What is the largest possible value for one of the numbers?" In the 2007 Mexico Mathematical Olympiad [3] questions were also raised about optimization problems: If $a_1,a_2,a_3$ are positive real numbers, where $a_3 + a_2 + a_1 = 1$, prove that the maximum value of $\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3} + \sqrt{a_1a_2a_3} = 2$.

The way to solve these problems is of course using calculus or advanced calculus (partial differential equations). This material is only obtained by students when they are in the formal post-operative level. To associate this we need a method concept. One of them is the Cauchy-Schwarz Words Concept. The inequality theorem is written below. If $n$ is a natural number and $a_1,a_2,...,a_n$ and $b_1,b_2,...,b_n$ are real numbers, then $(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2)$. In addition, if not all $b_i = 0$, then the similarity is fulfilled if and only if there is $r \in \mathbb{R}$ so that $b_1 = ra_1, b_2 = ra_2, ..., b_n = ra_n[4]$. In another representation: If $u = (a_1,a_2,...,a_n)$ and $v = (b_1,b_2,...,b_n)$ then $|u,v| \leq ||u|| ||v||$. If $v \neq 0$, then there is $k$ so that $v = ku[5]$.

Cauchy-Schwarz's inequality in optimization problems has been discussed but generally leads to formal post-operative cognitive levels and student levels at universities. This is shown in textbooks by[3, 6, 7, 8, 9, 10, 11] and in magazines or scientific journals by [12, 13, 14, 15, 16]. However, the results achieved do not discuss specifically in solving mathematical optimization problems at the elementary or secondary school level related to cognitive levels.

2. Result and discussion

The problems to be discussed are related to the shortest path, minimum circumference, maximum area and volume, economical cost or other general optimization problems.

2.1. Shortest path

Given the line $ax + by = c$ and the point $(p,q)$. The shortest path from point to line follows the following path. Referring to the right triangle rules $\sqrt{(x - p)^2 + (y - q)^2}$ is the shortest segment. Noted that

$$[a(x - p) + b(y - q)]^2 \leq [(x - p)^2 + (y - q)^2][a^2 + b^2]$$

This inequality leads to results.
\[ [c - (ap + bq)]^2 \leq d^2[a^2 + b^2] \]

or
\[ \frac{1}{\sqrt{a^2 + b^2}} |c - (ap + bq)| \leq d \]

so, the shortest path length of the point \((p, q)\) to the line \(ax + by = c\) is
\[ d = \frac{|c - (ap + bq)|}{\sqrt{a^2 + b^2}} \]

and this was achieved at the time
\[ \frac{x - p}{a} = \frac{y - q}{b} \]

This problem is extended to more general problems, \(d = (\sum_{i=1}^{n}(x_i - p_i)^2)^{1/2}\) with one obstacle problem \(\sum_{i=1}^{n} a_i x_i = c\) where \(a_i, x_i, p_i, c \in \mathbb{R}\), \(a_i, p_i\) and \(c\) constant, and there are \(k \in \mathbb{N}, 1 \leq k \leq n\) such that \(a_k \neq 0\).

From that
\[ \left( \sum_{i=1}^{n} a_i (x_i - p_i) \right)^2 \leq \sum_{i=1}^{n} (a_i)^2 \sum_{i=1}^{n} (x_i - p_i)^2 \]

obtained
\[ \left( c - \sum_{i=1}^{n} a_i p_i \right)^2 \leq \sum_{i=1}^{n} a_i^2 \cdot \frac{1}{d^2} \]

with result \(\frac{|c - \sum_{i=1}^{n} a_i p_i|}{\sum_{i=1}^{n} a_i^2}^{1/2}\) as the minimum value from \(d\). This value is reached at time \(\frac{x_i - p_i}{a_i} = \frac{x_j - p_j}{a_j}\) for each \(i, j\) with \(i \neq j\).

2.2. Maximum area
Related to this, several optimization issues are reviewed in rectangular flat shapes and conic sections. Let \(K = K(x, y)\) represent the circumference of a rectangle where \(x\) and \(y\) respectively indicate the length and width. The area of the rectangle is \(L = L(x, y) = xy\).

Because \(x + y = \frac{K}{2}\) so \((\sqrt{xy} + \sqrt{xy})^2 \leq (x + y) (y + x)\)

or
\[ L \leq \frac{K}{2\sqrt{2}} \]

The maximum value of the area of the rectangle, \(L \leq \frac{K}{2\sqrt{2}}\) reached at \(\frac{y}{x} = \frac{x}{y}\)
Now it continues with the calculation for the conic section (conic). Suppose that a flat shape has the formula \( ax^2 + by^2 = c \) with \( a, b, c > 0 \). We can calculate the maximum area of a rectangle that can be accommodated by \( ax^2 + by^2 = c \). In this case the target function is \( L = 4xy \) with the constraint equation \( ax^2 + by^2 = c \). Thus, using the Cauchy-Schwarz inequality was obtained

\[
(xy + xy)^2 \leq (ax^2 + by^2) \left( \frac{y^2}{a} + \frac{x^2}{b} \right) \leq \frac{c^2}{ab}
\]
or

\[ L \leq \frac{2c}{\sqrt{ab}} \]

So the square area that can be accommodated by the conic must not pass through the value \( \frac{2c}{\sqrt{ab}} \).

Specifically, for circles \( x^2 + y^2 = r^2 \), the maximum area of a rectangle is \( 2r \) reached when \( y = r / \sqrt{2} \).

2.3. Perimeter and Minimum Material

It is known that the perimeter of a rectangle is \( K = K(x, y) = 2x + 2y \). We can find the minimum perimeter length formed by taking into account the conditions of the constraint \( c = xy \). In this case the objective function is \( K = K(x, y) \). Starting with \( (\sqrt{xy} + \sqrt{xy})^2 \leq (x + y)(y + x) \) it will be obtained \( 4c \leq K^2 \) or \( 2\sqrt{2L} \leq K \). This value is reached when \( x = y = \sqrt{c} \).

The link between material and circumference is minimized, we find out the following optimization problem which was adapted from[17]. A breeder wants to build a 200 \( m^2 \) cage. The cage wants to be lined with two razor wire. One of the sides of the cage is limited by a wall. What is the minimum length of barbed wire required by the breeder?

Suppose \( L \) is the area and \( K \) is the length of the wire, we get the constraints equation: \( 200 = xy \) and the function \( K = 2x + 4y \). Cauchy-Schwarz’s inequality leads to a link

\[ K = (\sqrt{2x \cdot 4y} + \sqrt{2x \cdot 4y})^2 \leq (2x + 4y)(4y + 2x) \]
or

\[ 32xy \leq K^2 \]

This inequality implies that the minimum value of \( K \) is 80 meters, which occurs in the condition \( x : y = 2 : 1 \).

2.4. Maximum Volume

The discussion in this context is the maximum volume of a block and the constraint requirement for the length of all the ribs is constant. Then proceed with a more general case formula. For example \( x_1, x_2 \) and \( x_3 \) represent the edges of a block and \( V \) is the volume of the beam. Therefore we have

\[ 4\sum_{i=1}^{3} x_i = k \]

with positive constants, and \( V = \prod_{i=1}^{3} x_i \)

\[
\left( 3 \prod_{i \in I} x_i^{1/2} \right)^2 \leq \left( \sum_{i \in I} x_i \right) \left( \prod_{i \neq 1} x_i + \prod_{i \neq 2} x_i + \prod_{i \neq 3} x_i \right)
\]

where \( I = \{1, 2, 3\} \)
\[ V \leq \frac{k}{36} \left( \prod_{i \neq 1} x_i + \prod_{i \neq 3} x_i + \prod_{i \neq 2} x_i \right) \]

But then

\[ (\prod_{i \neq 1} x_i + \prod_{i \neq 3} x_i + \prod_{i \neq 2} x_i)^2 \leq (\sum_{i=1}^{3} x_i^2)^2 \]

or

\[ \prod_{i \neq 1} x_i + \prod_{i \neq 3} x_i + \prod_{i \neq 2} x_i \leq \sum_{i=1}^{3} x_i^2 \]

\[ \prod_{i \neq 1} x_i + \prod_{i \neq 3} x_i + \prod_{i \neq 2} x_i \leq (\sum_{i=1}^{3} x_i^2)^2 - 2 \left( \prod_{i \neq 1} x_i + \prod_{i \neq 3} x_i + \prod_{i \neq 2} x_i \right) \]

\[ \left( \prod_{i \neq 1} x_i + \prod_{i \neq 3} x_i + \prod_{i \neq 2} x_i \right) \leq \frac{k^2}{48} \]

Because of that

\[ V \leq \left( \frac{k}{12} \right)^3 \]

Maximum volume value \( (\frac{k}{12})^3 \) reached at

\[ \frac{\prod_{i \neq 1} x_i}{x_1} = \frac{\prod_{i \neq 3} x_i}{x_2} = \frac{\prod_{i \neq 2} x_i}{x_3} \]

This optimization problem can be extended to the case of constraints \( \sum_{i=1}^{n} x_i = c \) and objective function \( \rho = \rho(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i \).

\[ \left( \left( n \prod_{i \in I} x_i^{1/2} \right) \right)^2 \leq (\sum_{i=1}^{n} x_i) \left( \prod_{i \neq 1} x_i + \prod_{i \neq 2} x_i + \ldots + \prod_{i \neq n} x_i \right) \]

where \( I = \{1, 2, 3, \ldots, n \} \)

Therefore

\[ n^2 \rho \leq (\sum_{i \in I} x_i) \left( \prod_{i \neq 1} x_i + \prod_{i \neq 2} x_i + \ldots + \prod_{i \neq n} x_i \right) \]

\[ n^2 \rho \leq c \left( \prod_{i \neq 1} x_i + \prod_{i \neq 2} x_i + \ldots + \prod_{i \neq n} x_i \right) \]

If the process continues then

\[ \left( \prod_{i \neq 1} x_i + \prod_{i \neq 2} x_i + \ldots + \prod_{i \neq n} x_i \right) \leq \frac{n-1}{n^2} \]
so that

$$\rho \leq \frac{c^n}{n^n}$$

The maximum value for \(\rho\) is \(\frac{c^n}{n^n}\) and reached when \(x_1 = x_2 = \cdots = x_n\).

### 2.5. Optimization of Triangles and Tetrahedrons

For instance the right triangle of \(ABC\) at point \(C\), with the lengths of the sides is \(a, b\) and hypotenuse \(c\). We can define the minimum ratio of \(c/(a + b)\). In this case the constraint function is \(a^2 + b^2 = c^2\) and the objective function is \(f(a, b) = a + b\). Based on the inequality Cauchy-Schwarz

\[(a + b)^2 \leq 2(a^2 + b^2)\]

Therefore

$$\frac{c}{a + b} \geq \frac{c}{\sqrt{2(a^2 + b^2)}} = \frac{\sqrt{2}}{2}$$

Furthermore, the minimum value of comparison \(\frac{\sqrt{2}}{2}\) grasped when \(a = b\) when the area of the right triangle is maximum.

Assumed the \(T.\ ABC\) tetrahedron which is right at point \(C\). For example \(\alpha, \beta, y, \lambda\) successive states broadly \(\Delta ABC, \Delta TAC, \Delta TCB\) and \(\Delta TAB\). The maximum value of comparison \(\frac{\alpha + \beta + y}{\lambda}\) can be calculated using the Pythagorean theorem on three dimensions, \(\alpha^2 + \beta^2 + y^2 = \lambda^2\) and Cauchy-Schwarz inequality.

$$\frac{(\alpha + \beta + y)^2}{\lambda} \leq 3\frac{(\alpha^2 + \beta^2 + y^2)}{\lambda^2}$$

or

$$\frac{\alpha + \beta + y}{\lambda} \leq \sqrt{3}$$

### 2.6. Mathematical Competition Questions

Here we deliberate two math competition questions related to optimization problems. The expenditure process that using the Cauchy-Schwarz inequality.

Let

$$2x + 2y - 5z + 6w - 8u = 50$$

and

$$3x^2 + 3y^2 + 6xy + 64z^2 + 16w^2 + 144u^2 = 150.$$  

The maximum and minimum of \(x + y\) can be found by the formula

\[(-5z + 6w - 8u)^2 \leq (64z^2 + 16w^2 + 144u^2) \left(\frac{25}{64} + \frac{9}{4} + \frac{4}{9}\right)\]
Then

\[ [50 - 2r]^2 \leq [150 - 3r^2]c \]

where \( c = \frac{25}{64} + \frac{9}{4} + \frac{4}{9} \) and \( r = x + y \)

Attained inequality squared which has the formula of the solution and \( a \leq r \leq b \). Minimum values of \( a \) and maximum \( b \) that accomplished at \( \frac{64z}{5} = \frac{8w}{3} = \frac{18u}{4} \).

Furthermore, there is the optimization of problem: If \( u,v,w,x,y,z > 0 \) prove that \( \left( \frac{1}{u+v} + \frac{1}{v+w} + \frac{1}{w+x} + \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+u} \right)(u + v + w + x + y + z) \geq 18 \)

The disbursement process by using a trivial modification,

\[
(6)^2 = \left( \frac{\sqrt{u+v}}{\sqrt{u+v}} + \frac{\sqrt{v+w}}{\sqrt{v+w}} + \frac{\sqrt{w+x}}{\sqrt{w+x}} + \frac{\sqrt{x+y}}{\sqrt{x+y}} + \frac{\sqrt{y+z}}{\sqrt{y+z}} + \frac{\sqrt{z+u}}{\sqrt{z+u}} \right)^2 \\
\leq (u + v) + (v + w) + (w + x) + (x + y) + (y + z) + (z + u) \left( \frac{1}{u+v} + \frac{1}{v+w} + \frac{1}{w+x} + \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+u} \right) \\
\]

Then

\[ 18 \leq \left( \frac{1}{u+v} + \frac{1}{v+w} + \frac{1}{w+x} + \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+u} \right)(u + v + w + x + y + z) \]

3. Conclusions

Optimization problems can already be taught to students with a cognitive level of congressional operations or formal pre-surgery. One method that can be used is the Cauchy-Schwarz inequality. The characteristic of this method is that the optimum solution is reached when a similarity occurs. It requires the skill of changing (transmuting) a constraint equation with a faintly protracted work process, however the process of modification and workmanship is aligned with the level of adulthood of students’ formal pre-operative thinking in thoughtful it. Optimization problems that should be solved by using derivatives or partial differential equations can be overcome with the concept of Cauchy-Schwarz inequality. It is interesting that the critical (stationary) point in the goal equation is equivalent to the Cauchy-Schwarz similarity, and it is quite easy to find such a fact in the Cauchy-Schwarz inequality.

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