MAXIMAL NON-COMPACTNESS OF SOBOLEV EMBEDDINGS

JAN LANG¹, VÍT MUSIL²,³, MIROSLAV OLŠÁK⁴, AND LUBOŠ PICK⁵

ABSTRACT. It has been known that sharp Sobolev embeddings into weak Lebesgue spaces are non-compact but the question of whether the measure of non-compactness of such an embedding equals to its operator norm constituted a well-known open problem. The existing theory suggested an argument that would possibly solve the problem should the target norms be disjointly superadditive, but the question of disjoint superadditivity of spaces $L^{p,\infty}$ has been open, too. In this paper, we solve both these problems. We first show that weak Lebesgue spaces are never disjointly superadditive, so the suggested technique is ruled out. But then we show that, perhaps somewhat surprisingly, the measure of non-compactness of a sharp Sobolev embedding coincides with the embedding norm nevertheless, at least as long as $p < \infty$. Finally, we show that if the target space is $L^{\infty}$ (which formally is also a weak Lebesgue space with $p = \infty$), then the things are essentially different. To give a comprehensive answer including this case, too, we develop a new method based on a rather unexpected combinatorial argument and prove thereby a general principle, whose special case implies that the measure of non-compactness, in this case, is strictly less than its norm. We develop a technique that enables us to evaluate this measure of non-compactness exactly.

HOW TO CITE THIS PAPER

This paper has been accepted for publication in The Journal of Geometric Analysis and is available on https://doi.org/10.1007/s12220-020-00522-y.

Should you wish to cite this paper, the authors would like to cordially ask you to cite it appropriately.

1. INTRODUCTION

Given a linear mapping acting between two (quasi)normed linear spaces, one of the most important questions is whether it is compact. Compactness is often desired or even indispensable for specific applications in different areas of mathematics. It plays an important role in theoretical parts of functional analysis such as, for instance, in the proof of the Schauder fixed point theorem, and also in the most customary applications of functional analysis such as proving existence, uniqueness, and regularity of solutions to partial differential equations via compact embeddings of Sobolev-type spaces into various other function spaces. However, more often than not, the mapping in question is not compact.

For a non-compact mapping, more subtle techniques have to be developed. For example, in 1955, G. Darbo [11] extended the Schauder theorem to certain types of non-compact operators. The main tool she used was the measure of non-compactness, which had been introduced earlier by Kuratowski in [20] in connection with different problems in general topology. An important generalization was later added by Sadovskii [28]. Later still, the notion of the measure of non-compactness proved to be very useful for various applications, and a formidable theory was developed. See e.g. [27, 33] for a general survey and more references.

The concept of measure of non-compactness of a mapping is a good device for quantifying how bad the non-compactness is, or, perhaps, how far from the class of compact maps the given operator lies. Let us recall its definition here. Throughout the paper, $B_X$ denotes the open unit ball in $X$ centered at the origin.

2020 Mathematics Subject Classification. 46E35, 47B06.

Key words and phrases. Ball measure of non-compactness, maximal non-compactness, Sobolev embedding, Weak Lebesgue spaces.
Definition. Let \( X \) and \( Y \) be (quasi)normed linear spaces and let \( T \) be a bounded mapping defined on \( X \) and taking values in \( Y \), a fact we will denote by \( T : X \to Y \). The \textit{ball measure of non-compactness} \( \alpha(T) \) of \( T \) is defined as the infimum of radii \( \rho > 0 \) for which there exists a finite set of balls in \( Y \) of radii \( \rho \) that covers \( T(B_X) \).

There are other examples of measures of non-compactness, for instance, the Kuratowski measure of non-compactness, which is defined analogously but with balls replaced by arbitrary sets of diameter not exceeding \( \rho \). In our analysis, we focus on the ball measure of non-compactness even if we sometimes avoid the adjective “ball”.

The measure of non-compactness is an important geometric feature of images of bounded sets under an operator, see e.g. [3]. It is intimately connected for instance to the classical entropy numbers or certain types of the so-called \( s \)-numbers, see e.g. [12, 14, 27]. Its importance stems among other reasons from Carl’s inequality [6, 7] which establishes its relationship to eigenvalues of an operator. The measure of non-compactness is also related to the essential spectrum of a bounded map [12].

From the definition of the measure of non-compactness, we easily observe that

\[
0 \leq \alpha(T) \leq \|T\|
\]

where \( \|T\| \) denotes the norm of the operator \( T \) considered as a map from \( X \) to \( Y \). Then \( T \) is compact if and only if \( \alpha(T) = 0 \), and it is as non-compact as possible if \( \alpha(T) = \|T\| \). In the latter case, we say that \( T \) is \textit{maximally non-compact}.

Example. A simple example of a maximally non-compact operator is the embedding of sequence spaces

\[
I : \ell^p \to \ell^q \quad \text{for } 1 \leq p \leq q < \infty,
\]

where \( I \) is the identity (or the \textit{embedding operator}). Indeed, we obviously have \( \|I\| = 1 \). Suppose thus that \( \alpha(I) < 1 \) and fix \( \rho \) such that \( \alpha(I) < \rho < 1 \). Then there is some \( m \in \mathbb{N} \) and elements \( y^1, \ldots, y^m \) of \( \ell^q \) such that

\[
B_{\ell^p} \subset \bigcup_{k=1}^{m} (y^k + \rho B_{\ell^p}).
\]

Since all \( y^k \)'s belong to \( \ell^q \), they are also elements of \( c_0 \). Hence there is a \( j \in \mathbb{N} \) such that \( (y^k)_j < 1 - \rho \) for each \( k = 1, \ldots, m \). But then the vector \( \ell^j = (0, \ldots, 0, 1, 0, \ldots) \) with one on the \( j \)-th position belongs to \( B_{\ell^p} \) and does not belong to any of the balls \( y^k + \rho B_{\ell^p} \), which is a contradiction. Consequently, \( \alpha(I) = 1 \) and \( I \) is maximally non-compact.

Interestingly, the situation is dramatically different when the target space is \( \ell^\infty \). Then, for any fixed \( p \in [1, \infty) \), the norm of \( I : \ell^p \to \ell^\infty \) is again equal to 1, but \( I \) is no longer maximally non-compact. More precisely, we will demonstrate that \( \alpha(I) \leq 2^{-1/p} \). To this end, denote \( \sigma = 2^{(1-1/p)} \), fix \( \rho > \sigma /2 \), and consider \( m \in \mathbb{N} \) such that

\[
(1 + \frac{1}{m})^{\frac{\sigma}{2}} < \rho.
\]

Define \( \lambda_k = \frac{\sigma k}{2m} \) for \( k = -m, \ldots, m \) and let \( y^k \) be the constant sequence defined by \( (y^k)_j = \lambda_k \) for every \( j \in \mathbb{N} \) and \( k = -m, \ldots, m \). We show that

\[
B_{\ell^p} \subset \bigcup_{k=-m}^{m} (y^k + \rho B_{\ell^\infty}),
\]

proving \( \alpha(I) \leq \rho \). Assume that \( y \in B_{\ell^p} \). Then \( y \in B_{\ell^\infty} \) and \( |y_j| \leq 1 \leq \sigma \) for every \( j \in \mathbb{N} \). We claim that

\[
\sup y - \inf y \leq \sigma.
\]

Indeed, given \( \varepsilon > 0 \), we find \( s, i \in \mathbb{N} \) such that \( y_s > \sup y - \varepsilon \) and \( y_i < \inf y + \varepsilon \). We get

\[
1 \geq \|y\|_p \geq (|y_s|^p + |y_i|^p)^{\frac{1}{p}} \geq 2^{\frac{1}{p} - 1} (|y_s| + |y_i|) > \frac{1}{\sigma} (\sup y - \inf y - 2\varepsilon),
\]
and the claim follows on sending $\varepsilon \to 0_+$. Now, if $\inf y = -\sigma$, then $y_j \in [\sigma, 0]$ for each $j \in \mathbb{N}$ and $y \in y^{-m} + \sigma B_{\infty}$. If $\inf y \in (-\sigma, 0]$ instead, then there is a unique $k \in \{-m+1, \ldots, m\}$ such that $\inf y + \sigma/2 \in (\lambda_{k-1}, \lambda_k]$. Then, by the choice of $\sigma$ and inequality (1.3),

$$\lambda_k + \sigma > \lambda_k + \frac{\sigma}{2} \geq \inf y + \sigma \geq \sup y.$$ 

On the other hand, using the definition of $\lambda_k$ and (1.1),

$$\inf y > \lambda_{k-1} - \frac{\sigma}{2} = \lambda_k - \frac{\sigma}{2} \left(1 + \frac{1}{m}\right) > \lambda_k - \sigma.$$

Altogether, $y \in y^k + \sigma B_{\infty}$, and (1.2) follows.

We showed that $\alpha(I) \leq \sigma/2 = 2^{-1/p} < 1$, whence $I$ is not maximally non-compact. It is not difficult to prove that $\alpha(I)$ is actually equal to $2^{-1/p}$ (using similar reasoning as in the proof of Theorem 5.5), but that is beside the point here. This example is simple (and likely to be known, although we did not find it in the literature in this exact form), but it is a good illustration of much more involved attractions below.

An important operator to which the theory is often applied is the identity acting from a Sobolev space into another function space. Such an identity is also called a Sobolev embedding. Let $n \in \mathbb{N}$ and let $\Omega$ be an open, bounded and nonempty set in $\mathbb{R}^n$. Recall that the Sobolev space $V^k_0 X(\Omega)$ for $k \in \mathbb{N}$ is defined as a collection of all measurable functions $u: \Omega \to \mathbb{R}$ whose extension by zero outside $\Omega$ is $k$-times weakly differentiable and $|\nabla^k u| \in X(\Omega)$. Here $\nabla^k u = (D^\beta u)_{|\beta|=k}$ is the vector of all the derivatives of $u$ of order $k$, where, for an $n$-dimensional multiindex $\beta$, $D^\beta$ denotes $\partial^\beta/\partial x^\beta$. Once equipped with the norm

$$\|u\|_{V^k_0 X(\Omega)} = \sum_{|\beta|=k} \|D^\beta u\|_{X(\Omega)},$$

the Sobolev space $V^k_0 X(\Omega)$ forms a Banach space. If $X$ represents a classical Lebesgue space $L^p$ for some $p \in [1, \infty]$, we simply write $V^{k,p}_0(\Omega)$.

The compactness of a Sobolev embedding can constitute a crucial step in many applications in partial differential equations, probability theory, calculus of variations, mathematical physics and other disciplines and therefore it has been widely studied alongside with quantification of its absence when appropriate. To name just one of many interesting connections, let us recall that spectral properties of the Laplacian are governed by the measure of non-compactness of a Sobolev embedding [12, 15].

The variational approach to partial differential equations with singular coefficients often requires the use of an embedding of a Sobolev space into a two-parameter Lorentz space. Compactness properties of such embeddings are crucial under various circumstances [17, 21–24, 30, 32]. Of particular interest is an embedding into a Lorentz space whose second index is equal to infinity, equivalent to a weak Lebesgue space, see [8]. However, it is a rule of thumb that if a Sobolev embedding is sharp in the sense of function spaces, then it is never compact [9, 19, 29, 30]. It is thus of interest to study how bad is its non-compactness.

Our main goal in this paper is to study maximal non-compactness of general Sobolev embeddings with emphasis on embeddings involving Lorentz–Sobolev spaces. It is worth noticing that classical Sobolev embeddings built upon Lebesgue spaces are included as particular instances.

The classical Sobolev embedding theorem (cf. e.g. [1, 2, 25]) asserts that if $n, k \in \mathbb{N}$, $k < n$, $\Omega$ is an open set in $\mathbb{R}^n$, $p \in [1, \frac{n}{k})$ and $p^* = \frac{np}{n-kp}$, then one has

$$I : V^{k,p}_0(\Omega) \to L^{p^*}(\Omega),$$

where $I$ is the identity operator. The Sobolev space on the left-hand side, $V^{k,p}_0(\Omega)$, consists of functions of highest regularity ($k$) and relatively small integrability ($p$) while, on the right-most side, the degree of integrability is increased (note that $p^* > p$), balancing the loss of regularity. This example explains the great importance of Sobolev embeddings: it is an appropriate tool for “trading regularity for integrability”. 
Like the Lebesgue $L^p$ norm, the two-parameter Lorentz $L^{p,q}$ norm (or quasinorm, in general) captures the $p$-integrability of a function whereas the parameter $q$ measures how "spread out" the mass of the function is. This extra index $q$ thus provides a fine-tuning of the $L^p$ space with the property that $L^{p,q_1} \subseteq L^{p,q_2}$ if $q_1 \leq q_2$. Setting $q = p$, we recover the Lebesgue space $L^{p,p} = L^p$ to reveal that the Lorentz spaces refine the Lebesgue scale. For these and other details concerning Lorentz spaces, see e.g. [4, 26].

Having Lorentz spaces at hand, the classical Sobolev embedding (1.4) can be enhanced to

\[(1.5) \quad I : V_0^{k,p}(\Omega) \to L^{p,s}(\Omega),\]

in which the Lebesgue target space $L^s(\Omega)$ is replaced with the essentially smaller Lorentz space $L^{p,s}(\Omega)$ making thus the embedding stronger. The latter embedding is known to be optimal in the sense that the target space cannot be replaced by any smaller rearrangement-invariant Banach function space [13, 18]. However, none of the embeddings (1.4), (1.5) is compact, and, as recent advances show [19, 29], even when the target space is replaced by a considerably larger space $L^{s,\infty}(\Omega)$, the resulting embedding

\[(1.6) \quad I : V_0^{k,p}(\Omega) \to L^{s,\infty}(\Omega)\]

is still not compact. In [16] it is shown that the Sobolev embedding (1.4) is maximally non-compact. In [5], this result is extended to more general Sobolev embeddings of the form

\[I : V_0^{k,p}(\Omega) \to L^{s,\infty}(\Omega) \quad \text{for every } 1 \leq q \leq s < \infty.\]

These results leave open the case when $s = \infty$. Thus a natural question arises.

**Question 1.** Given $k, n \in \mathbb{N}$, $k < n$, and $p \in [1, \frac{n}{k}]$, is (1.6) maximally non-compact?

The key feature of the approach of both [16] and [5] is the fact that any Lebesgue space $L^p(\Omega)$ for $p \in [1, \infty)$, as well as any Lorentz space $L^{p,q}(\Omega)$ for $p, q \in [1, \infty)$, is disjointly superadditive.

**Definition.** We say that a (quasi)normed linear space $X(\Omega)$ containing functions defined on $\Omega$ is disjointly superadditive if there exist $\gamma > 0$ and $C > 0$ such that for every $m \in \mathbb{N}$ and every finite sequence of functions $\{f_k\}_{k=1}^m$ with pairwise disjoint supports in $\Omega$ one has

\[\sum_{k=1}^m \|f_k\|_{X(\Omega)} \leq C\left\|\sum_{k=1}^m f_k\right\|_{X(\Omega)}^\gamma.\]

In order to answer Question 1, one should first investigate the following closely related mystery.

**Question 2.** Is the space $L^{s,\infty}(\Omega)$ disjointly superadditive?

The reason for considering Question 2 is that if the answer to it was positive, then it would be very likely that using some not-so-difficult modification of techniques of [16] and [5] one should be able to prove that the answer to Question 1 is affirmative, too. However, it turns out (Theorem 2.1) that the answer to Question 2 is negative, that is, the space $L^{s,\infty}(\Omega)$ is not disjointly superadditive.

This result is interesting on its own, but it leaves us shorthanded as far as Question 1 is concerned. So, in order to answer it we have to develop new techniques. In Section 4 we prove that the answer to Question 1 is positive, even though the methods of [16] and [5] do not apply. More precisely, we in fact show that a slightly more general embedding than (1.6), namely

\[I : V_0^{k,p}(\Omega) \to L^{s,\infty}(\Omega),\]

is maximally non-compact for every $k, n \in \mathbb{N}$, $k < n$, $p \in [1, \frac{n}{k}]$ and $q \in [1, \infty]$. We obtain this result (Theorem 4.2) as a consequence of a fairly comprehensive principle (Theorem 4.1) which postulates maximal non-compactness of embeddings into weak Lebesgue spaces provided that the underlying identity operator has certain shrinking property, which roughly states that its norm over an open set $\Omega$ is attained at functions having their supports restricted to any (arbitrarily small) open subset of $\Omega$. In order to be able to apply this theory to our purposes, we need to know that the identity operator in (1.6) has shrinking property. We establish this fact in Proposition 3.1.
The techniques of Section 4 do not work for the case when the target space is $L^\infty(\Omega)$ although it is also a weak Lebesgue space. It was shown in [31] that a proper domain partner for a Sobolev embedding into $L^\infty(\Omega)$ is the Lorentz space $L^{\frac{n}{k},1}(\Omega)$. More precisely, we have

$$I: V^k_0 L^{\frac{n}{k},1}(\Omega) \to L^\infty(\Omega).$$

There is thus one more natural, and still unanswered, problem.

**Question 3.** Given $k, n \in \mathbb{N}, k \leq n$, is the Sobolev embedding (1.7) maximally non-compact?

Let us first recall that in the very special (one-dimensional) case when $n = k = 1$, the answer is known. In the one-dimensional setting, $\Omega$ is replaced by a compact interval $[a, b]$ and $L^\infty(\Omega)$ by $C([a, b])$, the space of all continuous functions on $[a, b]$ endowed with the $L^\infty$-norm. It is shown in [16] and [5] that (1.7) is not maximally non-compact in this case. However, when $n = k = 1$, the Lorentz space $L^{\frac{n}{k},1}(\Omega)$ on the domain position collapses to the Lebesgue space $L^1(\Omega)$ and things get much simpler.

The question of extending this result to the higher-dimensional and higher-order case has been one of the notoriously difficult open problems in the theory. In Section 5, we solve this problem with the help of a new method which we develop for this purpose. At the end, we show that the answer to Question 3 is negative (Theorem 5.5), and we obtain this fact as a consequence of a generic quantitative statement (Theorem 5.2) which works for a wide variety of operators.

The results contained in Theorems 5.2 and 5.5 are interesting for at least two reasons. First, it is striking that the situation is so drastically different from any other embedding into a weak Lebesgue space though it accords with the example mentioned in the introduction. The second is the innovative method of proof of Theorem 5.2 based on a combinatorial argument involving a coloring-type problem, see Figure 1. Such line of argumentation is rarely seen in the area of mathematical analysis.

![Figure 1](https://via.placeholder.com/150)

(a) An example of a valid coloring with $m$ colors. Triangle of side length $2^m - 1$ consists of two smaller triangles of side length $2^{m-1} - 1$ and of a square of length $2^{m-1}$. Pick a new color for the square and proceed inductively on smaller triangles.

(b) Sets of colors of $i$-th and $j$-th row ($C_i$ and $C_j$, resp.) differ. The color $c \in C_i$ of the element $w_{i,j}$ cannot appear in $C_j$, since otherwise $j$-th row and $j$-th column would share this color. To achieve $2^m - 1$ distinct (nonempty) sets, at least $m$ are needed.

Figure 1. Triangle coloring problem: Color a grid triangle of side length $2^m - 1$ provided that $j$-th line and $j$-th column do not share a color for every $j = 1, \ldots, 2^m - 1$. Show that at least $m$ colors are needed.

2. **Lack of disjoint superadditivity of weak Lebesgue spaces**

This section is devoted to the analysis of Question 2. We recall some definitions and fix the notation first.
Rearrangements. For a measurable function $u: \Omega \to \mathbb{R}$, its nonincreasing rearrangement, $u^*: [0, \infty) \to [0, \infty]$, is defined by

$$u^*(t) = \inf \{ \lambda > 0 : |\{x \in \Omega : |u(x)| > \lambda\}| \leq t \} \quad \text{for } t \in [0, \infty).$$

The absolute values $|\cdot|$ denote the Lebesgue measure. The maximal nonincreasing rearrangement of $u$, namely the function $u^{**}: (0, \infty) \to [0, \infty]$, is defined by

$$u^{**}(t) = \frac{1}{t} \int_0^t u^*(s) \, ds \quad \text{for } t \in (0, \infty).$$

An alternative formula for $u^{**}$ reads as

$$(2.1) \quad u^{**}(t) = \frac{1}{t} \sup_{E \subseteq \Omega} \int_E |u(x)| \, dx \quad \text{for } t \in (0, |\Omega|),$$

where the supremum is taken over all measurable sets $E \subseteq \Omega$ such that $|E| = t$, see e.g. [26, Proposition 7.4.5]. Note that the maximal nonincreasing rearrangement is subadditive, that is, given measurable functions $u, v: \Omega \to \mathbb{R}$, we have

$$(2.2) \quad (u + v)^{**}(t) \leq u^{**}(t) + v^{**}(t) \quad \text{for } t \in (0, \infty),$$

while for the nonincreasing rearrangement we have only

$$(2.3) \quad (u + v)^*(s + t) \leq u^*(s) + v^*(t) \quad \text{for } s, t \in (0, \infty).$$

Lorentz spaces. Given $0 < p, q \leq \infty$, the functional $\| \cdot \|_{L^{p,q}(\Omega)}$ is defined by

$$(2.4) \quad \|u\|_{L^{p,q}(\Omega)} = \left\| s^{\frac{1}{q} - \frac{1}{p}} u^*(s) \right\|_{L^q(0, |\Omega|)}$$

for a measurable function $u: \Omega \to \mathbb{R}$. We adopt the notation that $1/\infty = 0$. If either $1 < p < \infty$ and $1 \leq q \leq \infty$, or $p = q = 1$, or $p = q = \infty$, then $\| \cdot \|_{L^{p,q}(\Omega)}$ is equivalent to a norm (cf. e.g. [4] for details), in other cases it is a quasinorm. We further define the functional $\| \cdot \|_{L^{p,q}(\Omega)}$ as

$$(2.5) \quad \|u\|_{L^{p,q}(\Omega)} = \left\| s^{\frac{1}{q} - \frac{1}{p}} u^{**}(s) \right\|_{L^q(0, |\Omega|)}$$

for a measurable function $u: \Omega \to \mathbb{R}$. If either $0 < p < \infty$ and $1 \leq q \leq \infty$, or $p = q = \infty$, then $\| \cdot \|_{L^{p,q}(\Omega)}$ is a norm, in other cases it is a quasinorm. The functionals $\| \cdot \|_{L^{p,q}(\Omega)}$ and $\| \cdot \|_{L^{p,q}(\Omega)}$ are called Lorentz (quasi)norms, and the corresponding spaces $L^{p,q}(\Omega)$ and $L^{p,q}(\Omega)$, defined as collections of all measurable functions $u: \Omega \to \mathbb{R}$ such that $\|u\|_{L^{p,q}(\Omega)} < \infty$ or $\|u\|_{L^{p,q}(\Omega)} < \infty$, respectively, are called Lorentz spaces. Besides the relations mentioned in the introduction, it always holds that $L^{p,q}(\Omega) \to L^{p,q}(\Omega)$ and if either $p \in (1, \infty)$ and $q \in [1, \infty)$, or $p = q = \infty$, then $L^{p,q}(\Omega) = L^{p,q}(\Omega)$. By equality of spaces we mean the equality of the sets and equivalence of their norms.

Weak Lebesgue spaces. Recall that weak Lebesgue spaces coincide with Lorentz spaces when the second index equals to infinity, namely with $L^{p,\infty}(\Omega)$ and $L^{p,\infty}(\Omega)$ and their norms are given by

$$(2.6) \quad \|u\|_{L^{p,\infty}(\Omega)} = \sup_{t \in (0, |\Omega|)} t^{\frac{1}{p}} u^*(t) \quad \text{and} \quad \|u\|_{L^{p,\infty}(\Omega)} = \sup_{t \in (0, |\Omega|)} t^{\frac{1}{p}} u^{**}(t),$$

respectively. It follows from (2.2) that $\| \cdot \|_{L^{p,\infty}(\Omega)}$ is a norm for any $p > 0$. On the other hand, $\| \cdot \|_{L^{p,\infty}(\Omega)}$ is in general only a quasinorm. Indeed, by (2.3), we have for $u, v: \Omega \to \mathbb{R}$ measurable,

$$(2.7) \quad \|u + v\|_{L^{p,\infty}(\Omega)} \leq \sup_{t \in (0, |\Omega|/2)} (2t)^{\frac{1}{p}} (u^*(t) + v^*(t)) \leq 2^{\frac{1}{p}} \left( \|u\|_{L^{p,\infty}(\Omega)} + \|v\|_{L^{p,\infty}(\Omega)} \right).$$
The functional $\| \cdot \|_{L^{p,\infty}(\Omega)}$ is a norm only if $p = \infty$ and it is equivalent to a norm if $1 < p < \infty$ (see e.g. [26, Theorem 8.2.2 and Corollary 8.2.4]). Important particular instances of Lorentz spaces are the cornerstone spaces $L^1(\Omega)$ and $L^\infty(\Omega)$. Indeed, for $p \in (0, 1]$, one has by monotonicity

$$\|u\|_{L^{p,\infty}(\Omega)} = \sup_{t \in (0,|\Omega|)} t^{\frac{1}{p} - 1} \int_0^t u^*(s)ds = |\Omega|^{\frac{1}{p} - 1}\|u\|_{L^1(\Omega)}$$

for every measurable $u$ on $\Omega$, hence $L^1(\Omega) = L^{1,1}(\Omega) = L^{(p,\infty)}(\Omega)$. Moreover, obviously, $L^\infty(\Omega) = L^{\infty,\infty}(\Omega) = L^{(\infty,\infty)}(\Omega)$.

Our first result reads as follows.

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a nonempty set of positive measure and let $r \in (0, \infty]$. Then the space $L^{r,\infty}(\Omega)$ is not disjointly superadditive.

**Proof.** Let $m \in \mathbb{N}$ be given. There are pairwise disjoint measurable subsets $E_k$, $k = 1, \ldots, m$, of $\Omega$ such that their measures $s_k = |E_k|$ satisfy

$$s_{k+1} \leq \frac{s_k}{2} \quad \text{for each} \quad k = 1, \ldots, m - 1.$$  \hspace{1cm} (2.9)

We define the functions $u_k = s_k^{-\frac{1}{r}} \chi_{E_k}$ for $k = 1, \ldots, m$. Then

$$u_k^* = s_k^{-\frac{1}{r}} \chi_{(0,s_k)}, \quad \text{for} \quad k = 1, \ldots, m$$

and, consequently,

$$\|u_k\|_{L^{r,\infty}(\Omega)} = \sup_{t \in (0,|\Omega|)} t^{\frac{1}{r}} u_k^*(t) = 1.$$  \hspace{1cm} (2.10)

Let us denote

$$u = \sum_{k=1}^m u_k.$$  \hspace{1cm} (2.11)

Since the functions $u_k$ have pairwise disjoint supports, we have

$$u^* = \sum_{k=1}^m s_k^{-\frac{1}{r}} \chi_{(a_k,a_{k-1})},$$

where $a_k = s_{k+1} + \cdots + s_m$ for $k = 0, \ldots, m - 1$ and $a_m = 0$ (see Figure 2). Consequently,

$$\|u\|_{L^{r,\infty}(\Omega)} = \sup_{t \in (0,|\Omega|)} t^{\frac{1}{r}} u^*(t) = \max_{j \in \{1,\ldots,m\}} \sup_{t \in (a_j,a_{j-1})} \sum_{k=1}^m t^{\frac{1}{r}} s_k^{-\frac{1}{r}} \chi_{(a_k,a_{k-1})}(t)$$

$$= \max_{j \in \{1,\ldots,m\}} a_j^{-\frac{1}{r}} s_j^{-\frac{1}{r}} \leq \max_{j \in \{1,\ldots,m\}} (2s_j)^{\frac{1}{r}} s_j^{-\frac{1}{r}} = 2\frac{1}{r},$$

where we have used property (2.9) to show

$$a_{j-1} = s_j + \cdots + s_m \leq 2s_j \quad \text{for} \quad j = 1, \ldots, m.$$  \hspace{1cm} (2.12)

![Figure 2. Rearrangement of the sum $\sum u_k$.](image-url)
Now suppose that the space $L^{r,\infty}(\Omega)$ has the disjoint superadditivity property. Then there exists $\gamma > 0$ and $C > 0$ such that

$$
\sum_{k=1}^{m} \left\| u_{k} \right\|_{L^{r,\infty}(\Omega)}^\gamma \leq C \| u \|_{L^{r,\infty}(\Omega)}^\gamma,
$$

that is, by (2.11) and (2.14), $m \leq C 2^{\gamma/r}$, which is clearly absurd because $m$ was selected arbitrarily at the beginning.

If we consider the functional (2.5) instead of (2.4), then we obtain a slightly different result.

**Theorem 2.2.** Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a nonempty set of positive measure. Then the space $L^{r,\infty}(\Omega)$ is disjointly superadditive if and only if $r \leq (\gamma n)^{1/n} C^2 R^{1/r}$. Assume that $r \in (0,1]$. Then, by (2.8), $L^{r,\infty}(\Omega) = L^1(\Omega)$ with equivalent norms, hence $L^{r,\infty}(\Omega)$ is disjointly superadditive.

Assume that $r \in (1,\infty]$. Let $m \in \mathbb{N}$ be given and suppose that the sets $E_k$, $k = 1,\ldots,m$, are chosen as in the proof of Theorem 2.1. Let the functions $u_k$ for $k = 1,\ldots,m$, the numbers $s_k$ for $k = 1,\ldots,m$, and the numbers $a_j$ for $j = 0,\ldots,m$, have the same meaning as in the proof of Theorem 2.1. By (2.10),

$$
u_k(t) = s_k^{-\frac{1}{r}} x_{(0,s_k)}(t) + \frac{s_k^{-\frac{1}{r}}}{t} x_{[s_k,|\Omega|)}(t) \quad \text{for } t \in (0,|\Omega|) \text{ and } k = 1,\ldots,m,$$

and therefore

$$
\| u_k \|_{L^{r,\infty}(\Omega)} = \sup_{t \in (0,|\Omega|)} t^\frac{1}{r} \nu_k(t) = \max \left\{ \frac{1}{r}, \sup_{t \in (0,s_k)} t^\frac{1}{r}, \sup_{t \in (s_k,|\Omega|)} t^\frac{1}{r-1} \right\} = 1
$$

for every $k = 1,\ldots,m$. Denote by $u$ the sum of all $u_k$'s as in (2.12), fix $j \in \{1,\ldots,m-1\}$ and $t \in (a_j,a_{j-1}]$. Then, using (2.13),

$$
\int_0^t u^*(s) \, ds = \int_0^{a_j} u^*(s) \, ds + \int_{a_j}^t u^*(s) \, ds = \sum_{k=j+1}^{m} s_k^{-\frac{1}{r}} + (t-a_j)s_j^{-\frac{1}{r}},
$$

and therefore

$$
\sup_{t \in (a_j,a_{j-1}]} t^\frac{1}{r} u^*(t) \leq \sup_{t \in (a_j,a_{j-1}]} t^\frac{1}{r-1} \left( \sum_{k=j+1}^{m} s_k^{-\frac{1}{r}} + ts_j^{-\frac{1}{r}} \right) \leq a_j^{\frac{1}{r-1}} \sum_{k=j+1}^{m} s_k^{-\frac{1}{r}} + a_j^{-\frac{1}{r}} s_j^{-\frac{1}{r}}.
$$

Due to (2.15), we have $a_j^{-\frac{1}{r}} s_j^{-\frac{1}{r}} \leq 2^\frac{1}{r}$. By the definition of $a_j$, it is $a_j \geq s_j+1$ and, using (2.9), we have

$$
a_j^{-\frac{1}{r}} \sum_{k=j+1}^{m} s_k^{-\frac{1}{r}} \leq a_j^{-\frac{1}{r}} s_j^{-\frac{1}{r}} \sum_{k=j+1}^{m} (2^{1-r})^{j+1-k} \leq 2^{1-\frac{1}{r}}.
$$

Altogether, (2.16) yields

$$
\sup_{t \in (a_j,a_{j-1}]} t^\frac{1}{r} u^*(t) \leq 2^\frac{1}{r} + 2^{1-\frac{1}{r}} \leq 4 \quad \text{for each } j = 1,\ldots,m-1.
$$

It remains to consider the case when $t \in (0,a_{m-1}]$. But, for such $t$,

$$
\int_0^t u^*(s) \, ds = ts_m^{-\frac{1}{r}},
$$

and therefore

$$
\sup_{t \in (0,a_{m-1}]} t^\frac{1}{r} u^*(t) = \sup_{t \in (0,a_{m-1}]} t^\frac{1}{r} s_m^{-\frac{1}{r}} = a_m^{-\frac{1}{r}} s_m^{-\frac{1}{r}} = 1.
$$

Estimates (2.17) and (2.18) combined give $\| u \|_{L^{r,\infty}(\Omega)} \leq 4$. Now, assuming that $L^{r,\infty}(\Omega)$ has the disjoint superadditivity property for some $\gamma > 0$ and $C > 0$, we conclude that $m \leq C 4^\gamma$, which is impossible, as $m$ was arbitrary. 

\[ \square \]
Assume that
\[ \text{Proposition 3.1. Let } \| (3.3) \| \sup \text{concentric balls } B \text{ such that every } u \in \text{shrinking property.} \]

Clearly \( u \) is a Sobolev function on \( \Omega \). First, observe that
\[ (u_\kappa)^* (t) = u^* (\kappa^n t) \]
(3.4)
and
\[ (u_\kappa)^{**} (t) = u^{**} (\kappa^n t) \]
(3.5)
for \( t > 0 \). Consequently,
\[ (D^3 u_\kappa)^* (t) = \kappa |\beta| (D^3 u)^* (\kappa^n t) \]
(3.6)
and
\[ (D^3 u_\kappa)^{**} (t) = \kappa |\beta| (D^3 u)^{**} (\kappa^n t) \]
(3.7)
for \( t \in (0, |B_1|) \) and for each multiindex \( \beta \). Now, assume that \( q = \infty \) and \( X = L^{p, \infty} \). We have
\[
\| u_\kappa \|_{V^k_0 \hat{L}^{p, \infty}(B_1)} = \sum_{|\beta| = k} \| D^\beta u_\kappa \|_{L^{p, \infty}(B_1)} = \sum_{|\beta| = k} \sup_{t \in (0, |B_1|)} t^{\frac{1}{p}} (D^\beta u_\kappa)(t)
\]
\[
= \sum_{|\beta| = k} \kappa^{|\beta|} \sup_{t \in (0, |B_1|)} t^{\frac{1}{p}} (D^\beta u_\kappa)(\kappa^nt) = \kappa^{k - \frac{n}{p}} \sum_{|\beta| = k} \sup_{s \in (0, |B_2|)} s^{\frac{1}{p}} (D^\beta u_\kappa)(s)
\]
where we used (3.6) and the change of variables \( s = \kappa^nt \). If \( X = L^{(p, \infty)} \), we use (3.7) instead. For \( q < \infty \), analogous computations are shown in [5, Theorem 1.2]. Thus,
\[
(3.8) \quad \| u_\kappa \|_{V^k_0 X(B_1)} = \kappa^{k - \frac{n}{p}} \| u \|_{V^k_0 X(B_2)}.
\]
Similarly, if \( Y = L^{p^{*}, \infty} \), we have
\[
\| u_\kappa \|_{L^{p^{*}, \infty}(B_1)} = \sup_{t \in (0, |B_1|)} t^{\frac{1}{p^{*}}} (u_\kappa)(t) = \sup_{t \in (0, |B_1|)} t^{\frac{1}{p^{*}}} u^{*}(\kappa^nt)
\]
\[
= \kappa^{n - \frac{p}{p^{*}}} \sup_{s \in (0, |B_2|)} s^{\frac{p}{p^{*}}} u^{*}(s) = \kappa^{k - \frac{n}{p}} \| u \|_{L^{p^{*}, \infty}(B_2)},
\]
where we substituted \( s = \kappa^nt \) and used the relation
\[
- \frac{n}{p^{*}} = k - \frac{n}{p}.
\]
For \( Y = L^{(p^{*}, \infty)} \), we use (3.5) instead of (3.4). Altogether,
\[
(3.9) \quad \| u_\kappa \|_{Y(B_1)} = \kappa^{k - \frac{n}{p}} \| u \|_{Y(B_2)}
\]
and (3.3) follows by (3.8) and (3.9).

4. Maximal non-compactness of a Sobolev embedding into weak Lebesgue space

In this section, we focus on Question 1.

**Theorem 4.1.** Let \( \Omega \subset \mathbb{R}^n \), \( n \in \mathbb{N} \), be an open nonempty set of positive measure and let \( r \in (0, \infty) \). Assume that \( X(\Omega) \) is a quasinormed linear space containing measurable functions and \( Y(\Omega) \) is either \( L^{r, \infty}(\Omega) \) or \( L^{(r, \infty)}(\Omega) \). If \( I : X(\Omega) \to Y(\Omega) \) has shrinking property, then \( I \) is maximally non-compact.

**Proof.** Assume the contrary, that is \( \alpha(I) < \| I \| \), and set \( q \) and \( \varepsilon > 0 \) such that \( \alpha(I) < q < q + 2\varepsilon < \| I \| \). There exist \( j \in \mathbb{N} \) and functions \( v_1, \ldots, v_j \) in \( Y(\Omega) \) such that
\[
(4.1) \quad B_{X(\Omega)} \subset \bigcup_{k=1}^{j} (v_k + qB_{Y(\Omega)}) .
\]
We may assume that
\[
(4.2) \quad \| v_k \|_{Y(\Omega)} \leq 2^{j+\frac{1}{r}} \| I \| \quad \text{for each } k = 1, \ldots, j.
\]
Indeed, suppose that \( \| v_k \|_{Y(\Omega)} > 2^{j+\frac{1}{r}} \| I \| \) for some \( k \). Recall that \( \| \cdot \|_{L^{r, \infty}} \) is a norm while, by (2.7), \( \| \cdot \|_{L^{r, \infty}} \) is a quasinorm with the constant \( 2^{1/\ell} \). In any case, every \( v \in B_{X(\Omega)} \) obeys \( \| v \|_{Y(\Omega)} \leq \| I \| \) and
\[
\| v_k - v \|_{Y(\Omega)} \geq 2^{-\frac{1}{r}} \| v_k \|_{Y(\Omega)} - \| v \|_{Y(\Omega)} > 2\| I \| - \| I \| > \varrho.
\]
In other words,
\[
B_{X(\Omega)} \cap (v_k + qB_{Y(\Omega)}) = \emptyset,
\]
and so the function \( v_k \) can be excluded from the collection on the right-hand side of (4.1).

If \( Y = L^{r, \infty} \), we choose \( \eta \in (0, 1) \) such that
\[
(4.3) \quad 0 \geq \varrho \left( 1 - \eta^{-\frac{1}{r}} \right) \geq -\varepsilon
\]
and set $\eta = 0$ if $Y = L^{r, \infty}$. Fix $\ell \in \mathbb{N}$ such that

$$
(\ell(1 - \eta))^{\frac{1}{r}} \geq \frac{2^{1 + \frac{1}{r}}}{\varepsilon} \|I\|
$$

and set $m = j\ell$ and $\omega = |\Omega|/(\ell(1 - \eta))$. Denote by $B_1, \ldots, B_m$ pairwise disjoint balls of the same radii, centered at $x_1, \ldots, x_m$, respectively, and all contained in $\Omega$. We may without loss of generality assume that $|B_k| < \omega$ for all $k = 1, \ldots, m$. By the shrinking property, there exists $u_1 \in X(\Omega)$ such that $\|u_1\|_{X(\Omega)} = 1$, supp $u_1 \subset B_1$ and $\|u_1\|_{Y(\Omega)} > \varrho + 2\varepsilon$. For each $k = 2, \ldots, m$, we define $u_k : \Omega \to \mathbb{R}$ by

$$
u_k(x) = u_1(x + x_1 - x_k)\chi_{B_k}(x) \quad \text{for } x \in \Omega.
$$

Then all the functions $u_k$ are equimeasurable, i.e.

$$
u_k^* = \nu_i^* \quad \text{for every } k, i = 1, \ldots, m,
$$

one has $u_k \in X(\Omega)$, $\|u_k\|_{X(\Omega)} = 1$, supp $u_k \subset B_k$, and

$$\|u_k\|_{Y(\Omega)} > \varrho + 2\varepsilon \quad \text{for each } k = 1, \ldots, m.
$$

Due to (4.1), it holds that

$$\{u_1, \ldots, u_m\} \subset B_{X(\Omega)} \subset \bigcup_{k=1}^{j} (v_k + \varrho B_{Y(\Omega)}).
$$

By the Pigeonhole principle, at least one of the balls in the union on the rightmost side of (4.7) must contain at least $m/j = \ell$ functions from $\{u_1, \ldots, u_m\}$. More precisely, there exist $i \in \{1, \ldots, j\}$ and distinct functions $\{u^i_1, \ldots, u^i_\ell\} \subset \{u_1, \ldots, u_m\}$ such that $u^i_k \in v_i + \varrho B_{Y(\Omega)}$ for $k = 1, \ldots, \ell$. That is,

$$\|v_i - u^i_k\|_{Y(\Omega)} < \varrho \quad \text{for every } k = 1, \ldots, \ell.
$$

Let us denote

$$v^k = v_i \chi_{\text{supp } u^k} \quad \text{for } k = 1, \ldots, \ell
$$

and $v = \sum_{k=1}^{\ell} v^k$. Next, define

$$w^k(x) = \begin{cases} v^k(x) & \text{if } |u^k(x)| \geq |v^k(x)| \\ u^k(x) & \text{if } |u^k(x)| < |v^k(x)| \end{cases} \quad \text{for } x \in \Omega \text{ and } k = 1, \ldots, \ell
$$

and also $w = \sum_{k=1}^{\ell} w^k$. Then $|w| \leq |v|$, whence, thanks to the well-known lattice property of the weak Lebesgue functionals (2.6), by the definition of $v$, and by (4.2), we get

$$\|w\|_{Y(\Omega)} \leq \|v\|_{Y(\Omega)} \leq \|v_i\|_{Y(\Omega)} \leq 2^{1 + \frac{1}{r}} \|I\|.
$$

Since $w^k - u^k$ equals either zero or $v^k - u^k$, one clearly has $|w^k - u^k| \leq |v^k - u^k|$. Furthermore, by the definition of $v^k$, $v^k - u^k$ equals either zero or $v_i - u^k$, hence $|v^k - u^k| \leq |v_i - u^k|$. Altogether, calling into play again the lattice property of $\|\cdot\|_{Y(\Omega)}$ and finally using (4.8), we obtain

$$\|w^k - u^k\|_{Y(\Omega)} \leq \|v^k - u^k\|_{Y(\Omega)} \leq \|v_i - u^k\|_{Y(\Omega)} < \varrho
$$

for every $k = 1, \ldots, \ell$.

Now, assume that $Y = L^{r, \infty}$. By (4.6), there is some $t_0 \in (0, \omega)$ such that

$$\langle u^k \rangle_\ast(t_0) > (\varrho + 2\varepsilon) t_0^{-\frac{1}{r}} \quad \text{for every } k = 1, \ldots, \ell.
$$

It is important to notice that, thanks to (4.5), $t_0$ is independent of $k$. Since the measure is non-atomic, for each $k = 1, \ldots, \ell$ there exists $E_k \subset \text{supp } u^k$ such that $|E_k| = t_0$ and

$$\frac{1}{t_0} \int_{E_k} |u^k| > (\varrho + 2\varepsilon) t_0^{-\frac{1}{r}}$$
due to (2.1). By the Hardy–Littlewood inequality [4, Chapter 2, Theorem 2.2] and (4.10),
\[ \frac{1}{t_0} \int_{E^k} |u^k - w^k| \leq (u^k - w^k)^* (t_0) \leq \theta t_0^{-\frac{1}{p'}} \quad \text{for every } k = 1, \ldots, \ell. \]
Altogether, we have
\[ \frac{1}{t_0} \int_{E^k} |w^k| = \frac{1}{t_0} \int_{E^k} |u^k - (u^k - w^k)| \geq \frac{1}{t_0} \int_{E^k} |u^k| - \frac{1}{t_0} \int_{E^k} |u^k - w^k| \]
\[ > (\rho + 2\varepsilon) t_0^{\frac{1}{p'}} - \theta t_0^{-\frac{1}{p'}} = 2\varepsilon t_0^{-\frac{1}{p'}} \quad \text{for every } k = 1, \ldots, \ell. \]
Denote \( E = \bigcup_{k=1}^{\ell} E^k \). Since the sets \( E^k \) are pairwise disjoint, we get \( |E| = t_0 \) and
\[ w^* (t_0) \geq \frac{1}{t_0} \int_{E} |w| = \frac{1}{t_0} \sum_{k=1}^{\ell} \int_{E^k} |w^k| > 2\varepsilon t_0^{-\frac{1}{p'}}, \]
which gives
\[ \|w\|_{L^{(r,\infty)} (\Omega)} \geq (t_0)^{\frac{1}{p'}} w^* (t_0) > 2\varepsilon t_0^{\frac{1}{p'}} > 2^{1+\frac{1}{p'}} \|I\|, \]
thanks to (4.4). Recall that \( \eta = 0 \) in this case. Estimate (4.11) now contradicts (4.9).
Let \( Y = L^{r,\infty} \) instead. By (4.6), there is \( t_0 \in (0, \omega) \) such that
\[ (u^k)^* (t_0) > (\rho + 2\varepsilon) t_0^{-\frac{1}{p'}} \quad \text{for every } k = 1, \ldots, \ell. \]
Using (2.3), we have for all \( k = 1, \ldots, \ell \)
\[ (w^k)^* (t_0 - \eta t_0) \geq (u^k)^* (t_0) - (u^k - w^k)^* (\eta t_0) \geq (\rho + 2\varepsilon) t_0^{\frac{1}{p'}} - \rho (\eta t_0)^{-\frac{1}{p'}} \]
\[ = \rho \left( 1 - \eta^{-\frac{1}{p'}} \right) t_0^{\frac{1}{p'}} + 2\varepsilon t_0^{-\frac{1}{p'}} \geq \varepsilon t_0^{-\frac{1}{p'}}, \]
where we have used (4.10), (4.12) and (4.3). Now, since \( w^k \)'s have pairwise disjoint supports, one has
\[ w^* (\ell(1 - \eta) t_0) \geq \varepsilon t_0^{-\frac{1}{p'}}, \]
whence
\[ \|w\|_{L^{r,\infty} (\Omega)} \geq (\ell(1 - \eta) t_0)^{\frac{1}{p'}} w^* (\ell(1 - \eta) t_0) > \varepsilon (\ell(1 - \eta))^{\frac{1}{p'}} > 2^{1+\frac{1}{p'}} \|I\|, \]
where the last inequality is due to (4.4). Finally, (4.13) contradicts (4.9). \( \square \)

Theorem 4.2 combined with Proposition 3.1 leads to the following result, whose special case immediately answers Question 1.

**Theorem 4.2.** Let \( \Omega \subset \mathbb{R}^n \), \( n \in \mathbb{N} \), be open bounded and nonempty set and let \( k \in \mathbb{N} \), \( k < n \). Suppose that \( X(\Omega) \) is either \( L^{p,q}(\Omega) \) or \( L^{(p,q)}(\Omega) \) in which \( p \in [1, \frac{1}{2}) \), \( q \in [1, \infty) \), and \( Y(\Omega) \) is either \( L^{p',\infty}(\Omega) \) or \( L^{(p',\infty)}(\Omega) \), where \( p^* = \frac{np}{n-p} \). Then the Sobolev embedding (3.1) is maximally non-compact.

5. **Embeddings into the space of essentially bounded functions**

In this section, we shall exhibit that unlimited supply of disjointly supported functions of the same norm (or, in particular, the shrinking property) on its own is not enough to guarantee maximal non-compactness of an embedding. To this end, we shall investigate embeddings into \( L^\infty(\Omega) \) here. We begin by introducing a new quantity assigned to such an embedding, which will prove of substantial use later. Let \( X(\Omega) \) be a quasinormed linear space of measurable functions defined on \( \Omega \) and consider the identity operator
\[ I : X(\Omega) \to L^\infty(\Omega). \]
We define the span of \( I \) by
\[ \sigma (I) = \sup \{ \text{ess sup } u - \text{ess inf } u : u \in X(\Omega), \|u\|_{X(\Omega)} \leq 1 \}. \]
Trivial inspection shows that
\[ (5.2) \quad \sigma(I) \leq 2\|I\|. \]
We shall show that if inequality (5.2) is strict and the domain space has the unlimited supply (shrinking) property, then \( I \) is not maximally non-compact. The principal idea in the background of this result is rather neatly illustrated with the example which was mentioned in the introductory section. A key step to the result is the following general assertion. It requires a simple assumption on embedding (5.1), which reads
\[ (5.3) \quad \|I\| \leq \sigma(I). \]
This hypothesis prevents the space \( X(\Omega) \) from being too “poor”, like constant functions on \( \Omega \), for instance. Observe that shrinking property is a sufficient condition for \( X(\Omega) \) to obey (5.3).

**Proposition 5.1.** Let \( \Omega \subset \mathbb{R}^n, n \in \mathbb{N} \), be a nonempty set of positive measure. Assume that the embedding \( I \) in (5.1) obeys (5.3). The measure of non-compactness of \( I \) satisfies
\[ (5.4) \quad \alpha(I) \leq \frac{\sigma(I)}{2}. \]

**Proof.** Suppose that \( \varrho > \sigma(I)/2 \) and let \( m \in \mathbb{N} \) be such that
\[ \frac{\sigma(I)}{m} < \varrho - \frac{\sigma(I)}{2}. \]
Set
\[ \lambda_k = \frac{k\sigma(I)}{2m} \quad \text{for } k = -m, \ldots, m. \]
Observe that
\[ (5.5) \quad \lambda_k - \lambda_{k-1} < \varrho - \frac{\sigma(I)}{2} \quad \text{for } k = -m + 1, \ldots, m. \]
Define
\[ v_k(x) = \lambda_k \quad \text{for } x \in \Omega \text{ and } k = -m, \ldots, m. \]
Then of course each \( v_k \) belongs to \( L^\infty(\Omega) \). Now let \( u \in B_{X(\Omega)} \). Then, by (5.3), we have
\[ (5.6) \quad \|u\|_{L^\infty(\Omega)} \leq \|I\| \leq \sigma(I) \]
and, by the definition of \( \sigma(I) \),
\[ (5.7) \quad \text{ess sup } u - \text{ess inf } u \leq \sigma(I). \]
If \( \text{ess inf } u = -\sigma(I) \), then, by (5.7), \( u \) essentially ranges between \(-\sigma(I)\) and 0, whence \( u \in v_{-m} + \varrho B_{L^\infty(\Omega)} \). Suppose that \( \text{ess inf } u \in (-\sigma(I), 0] \). We find \( k \in \{-m + 1, \ldots, m\} \) such that
\[ (5.8) \quad \text{ess inf } u + \frac{\sigma(I)}{2} \in (\lambda_{k-1}, \lambda_k]. \]
Then, by (5.8) and (5.7) again,
\[ \lambda_k + \varrho > \lambda_k + \frac{\sigma(I)}{2} \geq \text{ess inf } u + \sigma(I) \geq \text{ess sup } u. \]
On the other hand, by (5.5),
\[ \text{ess inf } u \geq \lambda_{k-1} - \frac{\sigma(I)}{2} > \lambda_k - \varrho. \]
Therefore,
\[ \lambda_k - \varrho < \text{ess inf } u \leq \text{ess sup } u < \lambda_k + \varrho, \]
which means that \( u \in v_k + \varrho B_{L^\infty(\Omega)} \). Finally, if \( \text{ess inf } u \in (0, \sigma(I)] \), then, due to (5.6), \( u \) essentially takes values between 0 and \( \sigma(I) \), hence \( u \in v_m + \varrho L^\infty(\Omega) \). This shows that \( \alpha(I) < \varrho \). Since \( \varrho > \sigma(I)/2 \) was arbitrary, we get (5.4). \( \square \)
We shall now show that in the case when \( X(\Omega) \) has the unlimited supply property, the converse inequality to (5.4) holds as well. The principal task in such a case is, given \( \varrho > \alpha(I) \), to construct a large enough family of disjointly supported functions such that the span of the difference of each two of them exceeds \( 2\varrho \).

**Theorem 5.2.** Let \( \Omega \subset \mathbb{R}^n \), \( n \in \mathbb{N} \), be nonempty set of positive measure. Suppose that \( \varrho > 0 \) and assume that, for any \( \ell \in \mathbb{N} \), there exist pairwise disjointly supported functions \( u_k \in X(\Omega) \), \( k = 1, \ldots, \ell \), satisfying
\[
\|u_j - u_k\|_{X(\Omega)} \leq 1 \quad \text{for distinct} \; j, k = 1, \ldots, \ell
\]
and
\[
2\varrho > \text{ess sup}_{\Omega} u_k > \varrho \quad \text{for} \; k = 1, \ldots, \ell.
\]

Then, the measure of non-compactness of embedding \( I \) in (5.1) obeys
\[
\alpha(I) \geq \varrho.
\]

**Proof.** Assume that (5.11) is not satisfied, i.e. that \( \alpha(I) < \varrho \). This means that there exist \( m \in \mathbb{N} \) and a collection \( \{v_1, \ldots, v_{m-1}\} \subset L^\infty(\Omega) \) such that
\[
B_{X(\Omega)} \subset \bigcup_{k=1}^{m-1} \left(v_k + \varrho B_{L^\infty(\Omega)}\right).
\]
Set \( \ell = 2^m \) and let \( u_1, \ldots, u_{\ell} \) be the sequence guaranteed by the assumption of the theorem. Define
\[w_{i,j} = u_i - u_j \quad \text{for} \; 1 \leq i < j \leq \ell.\]
It follows from (5.9) that \( w_{i,j} \in B_X(\Omega) \) and, due to (5.10),
\[
\|w_{i,j} - w_{i',j'}\|_{L^\infty(\Omega)} > 2\varrho \quad \text{if and only if} \; i = j' \; \text{or} \; j = i'.
\]
for admissible indices. Now, let \( W_1, \ldots, W_{m-1} \) be arbitrary pairwise disjoint partitioning of the set \( \{w_{i,j} : 1 \leq i < j \leq \ell\} \) satisfying
\[w_{i,j} \in W_k \quad \text{if} \; w_{i,j} \in v_k + \varrho B_{L^\infty(\Omega)} \quad \text{for} \; k = 1, \ldots, m - 1,
\]
which is possible due to (5.12) and the fact that each \( w_{i,j} \) belongs to \( B_X(\Omega) \). Observe that if two functions \( w \) and \( \tilde{w} \) share the same class, then there exists \( k \in \{1, \ldots, m - 1\} \) such that both \( w \) and \( \tilde{w} \) belong to the ball \( v_k + \varrho B_{L^\infty(\Omega)} \), whence
\[
\|w - \tilde{w}\|_{L^\infty(\Omega)} \leq 2\varrho.
\]
Our goal now is to show that such a partitioning is impossible with less than \( m \) classes, which would lead to a contradiction.

To have a better understanding of this setup, imagine that every \( w_{i,j} \) is represented by a field \((i, j)\) in a grid \( 2^m \times 2^m \). Thanks to the constraint \( 1 \leq i < j \leq 2^m \), we deal just with the lower triangle under the diagonal (see Figure 1a). A membership of \( w_{i,j} \) to the class \( W_c \) may be represented as a coloring of the corresponding field \((i, j)\) by the color \( c \). The partitioning condition together with (5.13) and (5.14) translates to a simple constraint: \( i \)-th line cannot share any color with \( i \)-th column for any \( i = 1, 2, \ldots, 2^m - 1 \). The task is in showing that at least \( m \) colors are needed.

To this end, for each row \( i \in \{1, \ldots, 2^m - 1\} \), consider the sets
\[C_i = \{c : w_{i,j} \in W_c \text{ for some } i < j \leq 2^m\},\]
that is, the colors contained in the \( i \)-th row. We show that these sets need to be distinct across rows. Let \( 1 \leq i < j \leq 2^m \) be given and let \( c \) be the class index such that \( w_{i,j} \in W_c \). Obviously \( c \in C_i \). We claim that \( c \notin C_j \). Indeed, assume that there is some \( j' \) such that \( w_{i,j'} \in W_c \). Then both \( w_{i,j} \) and \( w_{i,j'} \) share the same class (see Figure 1b) which means that \( \|w_{i,j} - w_{i,j'}\|_{L^\infty(\Omega)} \leq 2\varrho \), due to (5.14). This however contradicts (5.13) and therefore \( C_i \neq C_j \). Since \( i \) and \( j \) were chosen arbitrarily, this shows that the family \( \{C_1, \ldots, C_{2^m-1}\} \) constitutes a collection of pairwise distinct nonempty subsets of the set of all classes \( \{1, \ldots, m - 1\} \), which is not possible. \( \square \)
In order to apply Theorem 5.2 to the Sobolev embedding (1.7), we need to show that the domain Sobolev space has the unlimited supply of functions satisfying (5.9) and (5.10). To this end, we shall employ the shrinking property.

**Proposition 5.3.** Let \( n \in \mathbb{N}, k \in \mathbb{N}, k \leq n, \) and let \( \Omega \subset \mathbb{R}^n \) be open bounded and nonempty set. Then the embedding from (1.7) has shrinking property.

**Proof.** Let \( G \subset \Omega \) be nonempty open set. Suppose that \( B_1 \) and \( B_2 \) are concentric balls such that \( B_1 \subset G \subset \Omega \subset B_2 \). By the translation invariance, we may assume that both the balls are centered at the origin. Let \( r_1 \) and \( r_2 \) denote their radii and set \( \kappa = r_2/r_1 \). Given \( u \in V_0^{k} L^{n/k, 1}(B_2) \), define \( u_\kappa(x) = u(\kappa x) \) for \( x \in B_1 \). Then we have

\[
\| u_\kappa \|_{L^\infty(B_1)} = \| u \|_{L^\infty(B_2)}
\]

and, by a simple computation,

\[
\| u_\kappa \|_{V_0^{k} L^{\frac{n}{k}, 1}(B_1)} = \| u \|_{V_0^{k} L^{\frac{n}{k}, 1}(B_2)}.
\]

The assertion then follows by the same argument as in Proposition 3.1. \( \square \)

Our next step will be exact evaluation of the span of the embedding operator \( I \) from (1.7).

**Proposition 5.4.** Let \( n, k \in \mathbb{N}, k \leq n, \) let \( \Omega \subset \mathbb{R}^n \) be open bounded and nonempty set and let \( I \) denote the embedding from (1.7). Then

\[
\sigma(I) = 2^{1 - \frac{k}{n}} \| I \|.
\]

**Proof.** Let us show the inequality “\( \leq \)”. Let \( u \in V_0^{k} L^{n/k, 1}(\Omega) \). Then \( u = u^+ - u^- \) where \( u^+ \) and \( u^- \) are the positive and the negative part of \( u \), respectively. Fix \( \varepsilon > 0 \). Let \( \psi_\varepsilon \) be the standard mollification kernel supported in an open set \( B_\varepsilon \) and let \( u_\varepsilon = \psi_\varepsilon * u \), \( u_\varepsilon^+ = \psi_\varepsilon * u^+ \) and \( u_\varepsilon^- = \psi_\varepsilon * u^- \). Denote \( \Omega_\varepsilon = \Omega + B_\varepsilon \). Then both \( u_\varepsilon^+ \) and \( u_\varepsilon^- \) are supported in \( \Omega_\varepsilon \) and belong to \( V_0^{k} L^{n/k, 1}(\Omega_\varepsilon) \). If we denote

\[
I_\varepsilon : V_0^{k} L^{\frac{n}{k}, 1}(\Omega_\varepsilon) \to L^\infty(\Omega_\varepsilon),
\]

then, by shrinking property of (5.16) ensured by Proposition 5.3

\[
\| I_\varepsilon \| = \| I \|,
\]

as \( \Omega \) is an open subset of \( \Omega_\varepsilon \). Also, by linearity, \( u_\varepsilon = u_\varepsilon^+ - u_\varepsilon^- \). Note that, since \( u \) vanishes at the boundary of \( \Omega \),

\[
\text{ess inf } u_\varepsilon \leq 0 \leq \text{ess sup } u_\varepsilon.
\]

Moreover, one has

\[
\text{ess sup } u_\varepsilon = \| u_\varepsilon^+ \|_{L^\infty(\Omega)} \leq \| I_\varepsilon \| \| u_\varepsilon^+ \|_{V_0^{k} L^{\frac{n}{k}, 1}(\Omega_\varepsilon)} \leq \| I \| \| u_\varepsilon^+ \|_{V_0^{k} L^{\frac{n}{k}, 1}(\Omega_\varepsilon)}
\]

and

\[
\text{ess inf } u_\varepsilon = \| u_\varepsilon^- \|_{L^\infty(\Omega)} \leq \| I_\varepsilon \| \| u_\varepsilon^- \|_{V_0^{k} L^{\frac{n}{k}, 1}(\Omega_\varepsilon)} \leq \| I \| \| u_\varepsilon^- \|_{V_0^{k} L^{\frac{n}{k}, 1}(\Omega_\varepsilon)},
\]

where we used (5.17). Summing both inequalities up, we get

\[
\text{ess sup } u_\varepsilon - \text{ess inf } u_\varepsilon \leq \| I \| \sum_{|\beta| = k} \left( \| D^\beta u_\varepsilon^+ \|_{L^{\frac{n}{k}, 1}(\Omega_\varepsilon)} + \| D^\beta u_\varepsilon^- \|_{L^{\frac{n}{k}, 1}(\Omega_\varepsilon)} \right),
\]

as the definition of the Sobolev norm yields. Next, since \( n/k \geq 1 \), the elementary inequality

\[
a + b \leq 2^{1 - \frac{k}{n}} (a^{\frac{n}{k}} + b^{\frac{n}{k}})^{\frac{k}{n}}
\]

holds for any non-negative \( a \) and \( b \). Therefore, applying (5.19) to

\[
a = \| D^\beta u_\varepsilon^+ \|_{L^{\frac{n}{k}, 1}(\Omega_\varepsilon)} \quad \text{and} \quad b = \| D^\beta u_\varepsilon^- \|_{L^{\frac{n}{k}, 1}(\Omega_\varepsilon)},
\]


we obtain from (5.18)
\begin{equation}
\text{ess sup } u_\varepsilon - \text{ess inf } u_\varepsilon \leq 2^{1 - \frac{n}{k}} ||I|| \sum_{|\beta| = k} \left( ||D^\beta u_\varepsilon^+||_{L^{\frac{n}{n-1}}(\Omega)} + ||D^\beta u_\varepsilon^-||_{L^{\frac{n}{n-1}}(\Omega)} \right)^{\frac{k}{n}}.
\end{equation}

By a particular case of [5, Proposition 2.5], the Lorentz space \( L^{\frac{n}{n-1}}(\Omega) \) is disjointly superadditive with power \( n/k \), whence we get, for each multiindex \( \beta \),
\begin{equation}
||D^\beta u_\varepsilon^+||_{L^{\frac{n}{n-1}}(\Omega)} + ||D^\beta u_\varepsilon^-||_{L^{\frac{n}{n-1}}(\Omega)} \leq ||D^\beta u_\varepsilon||_{L^{\frac{n}{n-1}}(\Omega)}.
\end{equation}

Altogether, (5.20) and (5.21) yield
\begin{equation}
\text{ess sup } u_\varepsilon - \text{ess inf } u_\varepsilon \leq 2^{1 - \frac{n}{k}} ||I|| \|u_\varepsilon\|_{V^k_0 L^{\frac{n}{n-1}}(\Omega)}.
\end{equation}

The desired inequality now follows on letting \( \varepsilon \rightarrow 0_+ \).

To show the converse inequality, let \( \varepsilon > 0 \) be given and suppose that \( B_1 \) and \( B_2 \) are two disjoint balls of the same measure contained in \( \Omega \) and let \( u_1 \) be a function supported in \( B_1 \) such that \( \|u_1\|_{L^\infty(\Omega)} \geq ||I|| - \varepsilon \) and \( \|u_1\|_{V^k_0 L^{n/k}(\Omega)} = 1 \). The existence of such a function is guaranteed by shrinking property of (1.7) due to Proposition 5.3. Denote by \( u_2 \) the shift of \( u_1 \) onto the domain \( B_2 \) and define \( v : \Omega \rightarrow \mathbb{R} \) by
\begin{equation}
v = u_1 \chi_{B_1} - u_2 \chi_{B_2}.
\end{equation}

Then \( v \in V^k_0 L^{n/k}(\Omega) \) and
\begin{equation}
\text{ess sup } v - \text{ess inf } v \geq 2(||I|| - \varepsilon).
\end{equation}

Next, observe that
\[
|D^\beta v|^*(t) = \begin{cases} 
|D^\beta u_1|^*(\frac{t}{2}) & \text{if } t \in (0, 2|B_1|) \\
0 & \text{if } t \in (2|B_1|, |\Omega|)
\end{cases}
\]
for every multiindex \( \beta \). This implies
\begin{equation}
\|v\|_{V^k_0 L^{n/k}(\Omega)} = \sum_{|\beta| = k} \int_0^{2|B_1|} |D^\beta v|^*(t) t^{\frac{k}{n} - 1} dt = \sum_{|\beta| = k} \int_0^{2|B_1|} |D^\beta u_1|^*(\frac{t}{2}) t^{\frac{k}{n} - 1} dt
\end{equation}
\begin{equation}
= 2^{\frac{k}{n}} \sum_{|\beta| = k} \int_0^{2|B_1|} |D^\beta u_1|^*(s) s^{\frac{k}{n} - 1} ds = 2^{\frac{k}{n}} \|u_1\|_{V^k_0 L^{n/k}(\Omega)} = 2^{\frac{k}{n}}
\end{equation}
and (5.23) together with (5.24) gives
\begin{equation}
\text{ess sup } v - \text{ess inf } v \geq 2^{1 - \frac{k}{n}} (||I|| - \varepsilon) \|v\|_{V^k_0 L^{n/k}(\Omega)}.
\end{equation}

The desired lower bound for \( \sigma(I) \) then follows by sending \( \varepsilon \rightarrow 0_+ \).

Finally, we will show that, for (1.7), there is equality in (5.4), that is,
\[
\alpha(I) = \frac{\sigma(I)}{2}.
\]

Combined with (5.15), this provides us with an exact evaluation of the measure of non-compactness of \( I \) in (1.7). In particular, this yields a negative answer to Question 3.

**Theorem 5.5.** Let \( n, k \in \mathbb{N}, k \leq n \), let \( \Omega \subset \mathbb{R}^n \) be open bounded and nonempty set and let \( I \) denote the embedding from (1.7). Then
\[
\alpha(I) = 2^{-\frac{k}{n}} ||I||.
\]

In particular, \( I \) is not maximally non-compact.
Proof. Due to Propositions 5.1 and 5.4, we only need to prove that
\begin{equation}
\alpha(I) \geq 2^{-\frac{k}{n}}\|I\|.
\end{equation}
Let \( \varrho > 0 \) obey \( 2^{-\frac{k}{n}}\|I\| > \varrho \). We show that then necessarily \( \alpha(I) \geq \varrho \) and (5.25) follows. Thanks to Theorem 5.2, it suffices to find the set of eligible functions satisfying assumptions (5.9) and (5.10). Let \( \ell \in \mathbb{N} \) be given. Denote by \( B_1, B_2, \ldots, B_\ell \) pairwise disjoint balls of the same volume and all contained in \( \Omega \). By the shrinking property of embedding (1.7) ensured by Proposition 5.3, there is a function \( v_1 : B_1 \to \mathbb{R} \) supported in \( B_1 \) such that \( \|v_1\|_{W_k^0 \mathcal{L}_n(\Omega)} = 1 \) and \( \|v_1\|_{L_\infty(\Omega)} > \varrho 2^{\frac{k}{n}} \), since \( \|I\| > \varrho 2^{\frac{k}{n}} \). For each \( j = 2, \ldots, \ell \), let \( v_j \) denote a shifted copy of \( v_1 \) supported on \( B_j \). Then, due to (5.22) and (5.24), we have \( \|v_i - v_j\| = 2^{\frac{k}{n}} \) for distinct \( i, j = 1, \ldots, \ell \), whence the functions \( u_j = 2^{-\frac{k}{n}}v_j \), \( j = 1, \ldots, \ell \), have the required properties. \( \square \)

In the case \( n = k = 1 \) one has \( \|I\| = \sigma(I) = 1/2 \). This is easy to observe using the fundamental theorem of calculus, see e.g. [5]. Note that this observation is consistent with (5.15). For \( n \geq 2 \) and \( k = 1 \), the inequality “\( \leq \)” in (5.15) was shown in [10, Theorem 3.5(ii)].

Acknowledgment. We would like to thank David E. Edmunds and Jan Malý for stimulating discussions and useful ideas. We are grateful to the referee for their careful reading of the manuscript and their valuable comments.

Funding. This research was partly funded by Czech Science Foundation grant P201-18-00580S.

References

[1] R. A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, second edition, 2003.

[3] J. Banas and K. Goebel. *Measures of noncompactness in Banach spaces*, volume 60 of Lecture Notes in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1980.

[4] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of Pure and Applied Mathematics. Academic Press, Inc., Boston, MA, 1988.

[5] O. Bouchala. Measures of Non-Compactness and Sobolev–Lorentz Spaces. *Z. Anal. Anwend.*, 39(1):27–40, 2020. doi:10.4171/zaa/1649.

[6] B. Carl. Entropy numbers, s-numbers, and eigenvalue problems. *J. Funct. Anal.*, 41(3):290–306, 1981. doi:10.1016/0022-1236(81)90076-8.

[7] B. Carl and H. Triebel. Inequalities between eigenvalues, entropy numbers, and related quantities of compact operators in Banach spaces. *Math. Ann.*, 251(2):129–133, 1980. doi:10.1007/BF01536180.

[8] D. Cassani, B. Ruf, and C. Tarsi. Optimal Sobolev type inequalities in Lorentz spaces. *Potential Anal.*, 39(3):265–285, 2013. doi:10.1007/s11118-012-9329-2.

[9] P. Cavaliere and Z. Mihula. Compactness for Sobolev-type trace operators. *Nonlinear Anal.*, 183:42–69, 2019. doi:10.1016/j.na.2019.01.013.

[10] A. Cianchi and L. Pick. Sobolev embeddings into BMO, VMO, and \( L_\infty \). *Ark. Mat.*, 36(2):317–340, 1998. doi:10.1007/BF02384772.

[11] G. Darbo. Punti uniti in trasformazioni a codominio non compatto. *Rend. Sem. Mat. Univ. Padova*, 24:84–92, 1955, http://www.numdam.org/item?id=RSMUP_1955__24__84_0.

[12] D. E. Edmunds and W. D. Evans. *Spectral theory and differential operators*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2018. doi:10.1093/acprof:oso/9780198812050.001.0001.

[13] D. E. Edmunds, R. Kerman, and L. Pick. Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms. *J. Funct. Anal.*, 170(2):307–355, 2000. doi:10.1006/jfan.1999.3508.

[14] D. E. Edmunds and H. Triebel. *Function spaces, entropy numbers, differential operators*, volume 120 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. doi:10.1017/CBO9780511662201.
[15] W. D. Evans and D. J. Harris. Sobolev embeddings for generalized ridged domains. *Proc. London Math. Soc. (3)*, 54(1):141–175, 1987. doi:10.1112/plms/s3-54.1.141.

[16] S. Hencl. Measures of non-compactness of classical embeddings of Sobolev spaces. *Math. Nachr.*, 258:28–43, 2003. doi:10.1002/mana.200310085.

[17] E. Jannelli and S. Solimini. Concentration estimates for critical problems. *Ricerche Mat.*, 48(suppl.):233–257, 1999. Papers in memory of Ennio De Giorgi (Italian).

[18] R. Kerman and L. Pick. Optimal Sobolev imbeddings. *Forum Math.*, 18(4):535–570, 2006. doi:10.1515/FORM.2006.028.

[19] R. Kerman and L. Pick. Compactness of Sobolev imbeddings involving rearrangement-invariant norms. *Studia Math.*, 186(2):127–160, 2008. doi:10.4064/sm186-2-2.

[21] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 1984, http://www.numdam.org/item?id=AIHPC_1984__1_2_109_0.

[22] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(4):223–283, 1984, http://www.numdam.org/item?id=AIHPC_1984__1_4_223_0.

[23] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985. doi:10.4171/RMI/6.

[24] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. II. *Rev. Mat. Iberoamericana*, 1(2):45–121, 1985. doi:10.4171/RMI/12.

[25] V. G. Maz’ja. *Sobolev spaces*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. doi:10.1007/978-3-662-09922-3. Translated from the Russian by T. O. Shaposhnikova.

[26] L. Pick, A. Kufner, O. John, and S. Fučík. *Function spaces. Vol. 1*, volume 14 of *De Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, extended edition, 2013.

[27] A. Pietsch. *History of Banach spaces and linear operators*. Birkhäuser Boston, Inc., Boston, MA, 2007.

[31] E. M. Stein. Editor’s note: the differentiability of functions in $\mathbb{R}^n$. *Ann. of Math. (2)*, 113(2):383–385, 1981, http://www.jstor.org/stable/2006989.

[32] M. Struwe. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.*, 187(4):511–517, 1984. doi:10.1007/BF01174186.

[33] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1978.