ON THE HASSE PRINCIPLE FOR CONIC BUNDLES OVER EVEN DEGREE EXTENSIONS

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ABSTRACT. Let $k$ be a number field and let $\pi: X \to \mathbb{P}^1_k$ be a smooth conic bundle. We show that if $X/k$ has four geometric singular fibers with $X(\mathbb{A}_k) \neq \emptyset$ or non-trivial Brauer group, then $X$ satisfies the Hasse principle over any even degree extension $L/k$. Furthermore, for arbitrary $X$ we show that, conditional on Schinzel’s hypothesis, $X$ satisfies the Hasse principle over all but finitely many quadratic extensions of $k$. We prove these results by showing the Brauer-Manin obstruction vanishes and then apply fibration method results of Colliot-Thélène, following Colliot-Thélène and Sansuc.

1. INTRODUCTION

Let $X \to \mathbb{P}^1_k$ be a smooth conic bundle over a global field $k$ of characteristic not equal to 2. Over local and global fields, the arithmetic of such conic bundles has been well studied. By the Hasse-Minkowski theorem, conic bundles over $\mathbb{Q}$ with two geometric singular fibers always satisfy the Hasse principle and work of Iskovskikh shows that conic bundles over $\mathbb{Q}$ with three geometric singular fibers always have rational points [Isk96]. However, there exist conic bundles with 4 geometric singular fibers, specifically Châtelet surfaces, that fail the Hasse principle [Isk71,Poo09].

Since conics have numerous quadratic points, for any conic bundle there are numerous quadratic extensions over which they satisfy the Hasse principle. However, we show that a much stronger statement holds, namely that for general conic bundles with 4 geometric singular fibers, the Hasse principle holds over every even degree extension.

Theorem 1.1. Let $k$ be a number field, let $X \to \mathbb{P}^1_k$ be a conic bundle with four geometric singular fibers, and assume that one of the following holds:

1. $\text{Br}_X/\text{Br}_k \neq 0$
2. $X(\mathbb{A}_k) \neq \emptyset$
3. If the singular fibers lie over an $A_4$ or $S_4$ extension of $k$

Then, if $L/k$ is an even degree extension, we have

$$X(L) \neq \emptyset \iff X(\mathbb{A}_L) \neq \emptyset. \quad (1.1)$$

In 2009, Pete Clark coined the notion of a potential Hasse principle failure for smooth, geometrically irreducible varieties over number fields. In particular, given such a variety $V/k$, we say that it is a potential Hasse principle failure (PHPF) if there exists an extension $L/k$ such that $V$ fails the Hasse principle over $L$. Clark showed the existence of infinitely many curves which were PHPFs. It was also conjectured that for any positive genus curve $C/k$ with no $k$-rational points, $C$ is a PHPF, see [Cla09, Theorem 1, Conjecture 4]. However, despite the fact that most conic bundles with four bad fibers are not PHPFs (Theorem 1.1),
there do exist some cases where new Brauer classes can give a Brauer-Manin obstruction over $L$.

If the conditions of Theorem 1.1 fail, then one can prove that the singular fibers of $X \to \mathbb{P}^1_k$ lie over a single closed point of $\mathbb{P}^1_k$ and there exists a quadratic extension $L/k$ such that the singular fibers of $X_L \to \mathbb{P}^1_L$ lie over two closed points of degree 2. For these quadratic extensions, the Hasse principle can fail.

**Theorem 1.2.** The Châtelet surface $X/\mathbb{Q}$ given by
\[
y^2 - 5z^2 = \frac{3}{5}(5t^4 + 7t^2 + 1)
\]
has no $\mathbb{A}_\mathbb{Q}$-points and fails the Hasse principle over $L = \mathbb{Q}(\sqrt{29})$, i.e. $X$ is a potential Hasse principle failure.

Indeed, we show that these are the only even degree extensions over which the Hasse principle can fail (see Corollary 4.2). In addition, we trace the failure of the Hasse principle to a parity condition on the number of ramification places of a conic that splits in a fixed quadratic extension (see Theorem 6.1).

For arbitrary conic bundles we do not have such unconditional results, however, we can extend our prior results to conic bundles with more geometric singular fibers at the expense of restricting to quadratic extensions.

**Theorem 1.3.** Let $k$ be a number field, let $X \to \mathbb{P}^1_k$ be a conic bundle, let $S$ denote the set of closed points on $\mathbb{P}^1_k$ corresponding to the singular fibers, and for $P \in S$, let $k(P)$ denote its residue field. Assume Schinzel’s hypothesis. If $L/k$ is a quadratic extension linearly disjoint from $k(P)$ for all $P \in S$, then
\[
X(L) \neq \emptyset \iff X(\mathbb{A}_L) \neq \emptyset.
\]

**1.1. Outline of the proof of Theorems 1.1 and 1.3.** The statements of both Theorems follow from results about Brauer-Manin obstructions. We prove these results by first showing (in section 4) that
\[
X(\mathbb{A}_L)^{\text{Res}_{L/k}(\text{Br}(X_k))} \neq \emptyset \iff X(\mathbb{A}_L) \neq \emptyset. \tag{1.2}
\]

In using this result to deduce the main theorems, a careful examination of the map $\text{Res}_{L/k} : \text{Br}(X_k) \to \text{Br}(X_L)$ is needed. We then show that the same assumptions on $X$ imply
\[
X(\mathbb{A}_L)^{\text{Br}(X_L)} \neq \emptyset \iff X(\mathbb{A}_L) \neq \emptyset. \tag{1.3}
\]

A case by case analysis shows that when $\text{Res}_{L/k}$ is not surjective, statement (1.3) still holds for all even degree extensions $L/k$ except (at most) three.

When $X \to \mathbb{P}^1_k$ is a Châtelet surface, Theorem 1.1 follows from statement (1.3) by a landmark result of Colliot-Thélène, Sansuc, and Swinnerton-Dyer [CTSSD87] which states that for Châtelet surfaces, the Brauer-Manin obstruction to the Hasse principle is the only
one. More generally, Theorem 1.1 follows from (1.3) by work of Colliot-Thélène and Coray combined with results of Colliot-Thélène and Swinnerton-Dyer. Colliot-Thélène and Coray show that if $X/k$ is a conic bundle with five or fewer geometric singular fibers, then $X(k) \neq \emptyset$ if and only if there exists a zero cycle of degree one [CTC79]. Furthermore, Colliot-Thélène and Swinnerton-Dyer have results which, when applied to conic bundles, state that if there is no Brauer-Manin obstruction to the existence of a 0-cycle of degree one then there exists a 0-cycle of degree one [CTSD94, Theorem 5.1]. In particular, no Brauer-Manin obstruction to the existence of rational points implies no Brauer-Manin obstruction to the existence of a zero cycle of degree one, hence we can conclude Theorem 1.1.

When $X$ has arbitrarily many geometric singular fibers, Theorem 1.3 follows from statement (1.3) by different means. In 1982, Colliot-Thélène and Sansuc pioneered the method of using Schnizel’s hypothesis and the fibration method to prove that certain varieties have rational points [CTS82]. In particular, a theorem of Colliot-Thélène and Swinnerton-Dyer shows that if $k$ is a number field, $X$ is a conic bundle, and one assumes Schinzel’s hypothesis, then $X(k) \neq \emptyset$ if and only if $X(A_k) \neq \emptyset$ [CTSD94]. Consequently, the conditional statement of Theorem 1.3 follows from this work.

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3. The Brauer group of conics and conic bundles

In this section, we collect some general results concerning Brauer groups of conic bundles. For a background on the Brauer-Manin obstruction see [Poo17, §8.2]. We note that many of these results are well known but we include their proofs for completeness.

Lemma 3.1. Let $k$ be a local field and let $C/k$ be a smooth conic. If $L/k$ is an even degree extension then $C(L) \neq \emptyset$.

Proof. From local class field theory, the restriction map factors as

$$\text{Res}_{L/k} : \text{Br } k \cong \mathbb{Q}/\mathbb{Z} \xrightarrow{[L:k]} \mathbb{Q}/\mathbb{Z} \cong \text{Br } L$$

Using the correspondence between smooth conics and quaternion algebras, we can apply this to any element of $\text{Br } k[2]$ and obtain the result. □

3.1. The Brauer group of a conic bundle. Throughout, let $k$ denote an infinite field of characteristic not equal to 2 and let $X \to \mathbb{P}^1_k$ be a conic bundle. After a suitable change of coordinates on $\mathbb{P}^1_k$, we can and will assume that the fiber over the point at infinity, $X_\infty$, is a
smooth conic. Let \( t \) be the coordinate on \( \mathbb{A}^1_k = \mathbb{P}^1_k \setminus \{ \infty \} \) and let \( S \) denote the finite set of closed points on \( \mathbb{A}^1_k \) with geometric singular fiber. Let \( |S| \) denote the number of such closed points, not counting their degree, i.e. if \( X \) has four geometric singular fibers and \( |S| = 1 \) then \( S \) is irreducible and \( \deg(S) = 4 \). For any such point \( P \in \mathbb{A}^1_k \), let \( k(P) \) denote its residue field. The fiber \( X_P \) degenerates to the union of two lines, \( \ell_P \) and \( \ell'_P \), defined over \( k(P)(\sqrt{a_P}) \) for some \( a_P \in k(P)^* \). Let \( \tau_P \in k(P) \) be the image of \( t \) in \( k(P) \) and let \( \varepsilon = (\varepsilon_P) \in \mathbb{F}^{|S|}_2 \) be any vector satisfying

\[
\prod_{P \in S} N_{k(P)/k}(a_P)^{\varepsilon_P} \in k^{\times 2}
\]

where \( N_{k(P)/k} \) denotes the usual norm map. With this setup, we obtain an especially useful generating set for \( \frac{\text{Br} X}{\text{Br} k} \) which comes from the following fact.

**Corollary 3.4.** Let \( \pi: X \to \mathbb{P}^1_k \) be a conic bundle with four geometric singular fibers. Then

\[
\text{Corollary 3.4.} \quad \text{Let } k \text{ be a global field of characteristic not equal to } 2 \text{ and let } X/k \text{ be a conic bundle with four geometric singular fibers. Then}
\]

**The subgroup \( \ker(\text{Br} X[2] \to \text{Br} X_\infty) \) generates \( \frac{\text{Br} X}{\text{Br} k} \)**.

**Proof.** By the description given in equation \((3.1)\), we can see that \( \mathcal{A}_P(\infty) = (1, a_P) = 0 \in \text{Br} k(P) \). Since \( \text{Cor}_{k(P)/k} \) is a group homomorphism, \( A_{\varepsilon} \in \ker(\text{Br} k(\mathbb{P}^1)[2] \xrightarrow{\text{er} \ast} \text{Br} k[2]) \) and so \( \pi^*(A_{\varepsilon}) \in \ker(\text{Br} X[2] \to \text{Br} X_\infty) \).

From Lemma 3.2, we can deduce the following.

**Corollary 3.3.** The subgroup \( \ker(\text{Br} X[2] \to \text{Br} X_\infty) \) generates \( \frac{\text{Br} X}{\text{Br} k} \).

**Proof.** By the description given in equation \((3.1)\), we can see that \( \mathcal{A}_P(\infty) = (1, a_P) = 0 \in \text{Br} k(P) \). Since \( \text{Cor}_{k(P)/k} \) is a group homomorphism, \( A_{\varepsilon} \in \ker(\text{Br} k(\mathbb{P}^1)[2] \xrightarrow{\text{er} \ast} \text{Br} k[2]) \) and so \( \pi^*(A_{\varepsilon}) \in \ker(\text{Br} X[2] \to \text{Br} X_\infty) \).

Moreover, if \( X(k) = \emptyset \) and \( \frac{\text{Br} X}{\text{Br} k} \neq 0 \) then there exists a Galois-invariant decomposition \( S = S_1 \cup S_2 \) with both \( S_i \) irreducible of degree 2.

**Proof.** The computations for \( \frac{\text{Br} X}{\text{Br} k} \) follow from a case-by-case analysis, see [Sko15, §2.2]. In particular, following the notation of Lemma 3.2, any Brauer class of the form \( A_{\varepsilon} \) for \( \varepsilon = (1, 1, \ldots, 1) \in \mathbb{F}_2^{|S|} \) is trivial, hence \( |S| = 1 \) implies that \( \frac{\text{Br} X}{\text{Br} k} = 0 \).

If any of the singular fibers of \( X \) lie over a degree 1 point \( P \in \mathbb{A}^1_k \), then we can obtain a \( k \)-rational point on \( X \) by taking the intersection point of the lines \( \ell_P \) and \( \ell'_P \) for such \( P \in S \). Therefore, if \( X(k) = \emptyset \) and \( \frac{\text{Br} X}{\text{Br} k} \neq 0 \), it must be the case that \( |S| = 2 \) and the singular fibers both lie over degree 2 points of \( k \), hence we obtain the Galois-invariant decomposition \( S_1 \cup S_2 \).
Corollary 3.5. Let $k$ be a global field of characteristic not equal to 2 and let $X \to \mathbb{P}^1_k$ be a conic bundle with four geometric singular fibers. If there exists an even degree extension $L/k$ such that $X(L) = \emptyset$ and $\text{Res}_{L/k}: \Br X_L \to \Br X_L$ is not surjective, then $S$ is irreducible over $k$ and the singular fibers of $X_L \to \mathbb{P}^1_L$ lie over two degree 2 closed points that are interchanged by $\text{Gal}(L/k)$.

Proof. From Corollary 3.4, we have that $\text{Res}_{L/k}$ is not surjective when $\Br X_{L/k} = \{0\}$ and $\Br X_{L/k} \neq \{0\}$ or $\Br X_{L/k} = \mathbb{Z}/2\mathbb{Z}$ and $\Br X_{L/k} \neq \{\mathbb{Z}/2\mathbb{Z}\}$. If $\Br X_{L/k} = \mathbb{Z}/2\mathbb{Z}$ and $\Br X_{L/k} = (\mathbb{Z}/2\mathbb{Z})^2$, then $X_L$ will always have an $L$-rational point. If $\Br X_{L/k} = \{0\}$ and $\Br X_{L/k} \neq \{0\}$, then the only case where $X_L$ may not have an $L$-rational point is when all geometric singular fibers of $X/k$ lie over a single closed point of $\mathbb{A}^1_k$ and over $L$, there exists a Galois-invariant decomposition $S = S_1 \cup S_2$ with both $S_i$ of degree 2. \hfill \Box

Remark 3.6. Corollary 3.5 implies that the degree 4 closed point of $\mathbb{A}^1_k$ must correspond to a polynomial of the form $N_{k_0/k}(g(t))$, where $g(t) \in k_0[t]$ is an irreducible quadratic, and $k_0/k$ is a quadratic extension. In fact, finite extensions such as these are the only possible extensions of $k$ over which the Hasse principle may fail.

4. Brauer-Manin Obstructions over Extensions

In this section, we prove some general results relating the Brauer-Manin obstruction on a conic bundle to the Brauer-Manin obstruction over certain extensions. These results will play a critical role in the proofs of the main theorems.

Theorem 4.1. Let $k$ be a global field of characteristic not equal to 2 and let $X \to \mathbb{P}^1_k$ be a conic bundle with four geometric singular fibers, then for any even degree extension $L/k$

$$X(\mathbb{A}_L)^{\text{Res}_{L/k}(\Br X_k)} \neq \emptyset \iff X(\mathbb{A}_L) \neq \emptyset.$$  

In particular, if $\text{Res}_{L/k}: \Br X_{L/k} \to \Br X_{L/k}$ is surjective, then $X(\mathbb{A}_L)^{\Br(X_L)} \neq \emptyset \iff X(\mathbb{A}_L) \neq \emptyset$.

Proof. We begin by observing that one direction is immediate since $X(\mathbb{A}_L) = \emptyset$ implies that $X(\mathbb{A}_L)^{\text{Res}_{L/k}(\Br X_k)} = \emptyset$. Now assume that $X(\mathbb{A}_L) \neq \emptyset$ and let $\Omega_k$ denote the set of places of $k$. We will show that for all $v \in \Omega_k$ and all $w|v$, there exists a point $(P_w) \in X(L \otimes_k k_v)$ such that $\sum_{w|v} \text{inv}_w(\text{ev}_A(P_w)) = 0$ for all $A \in \ker(\Br X[2] \to \Br X_{\infty})$, and so

$$\sum_{v \in \Omega_k} \sum_{w|v} \text{inv}_w(\text{ev}_A(P_w)) = 0.$$  

Since by Corollary 3.3, $\ker(\Br X[2] \to \Br X_{\infty})$ generates $\Br X_{Br k}$, this will imply that $X(\mathbb{A}_L)^{\text{Res}_{L/k}(\Br X_k)} \neq \emptyset$.

We first consider the case when $X(k_v) \neq \emptyset$. Choose a point $P_v \in X(k_v)$, and for all $w|v$, set $P_w = P_v$. Then

$$\sum_{w|v} \text{inv}_w(\text{ev}_A(P_w)) = \sum_{w|v} \text{inv}_w(\text{Res}_{L_w/k_v}(\text{ev}_A(P_v))) = \sum_{w|v} [L_w : k_v] \text{inv}_v(\text{ev}_A(P_v))$$  

$$= \text{inv}_v(\text{ev}_A(P_v)) \sum_{w|v} [L_w : k_v] = \text{inv}_v(\text{ev}_A(P_v)) \cdot [L : k] = 0 \in \mathbb{Q}/\mathbb{Z}.$$  

5
Now consider the case where $X(k_v) = \emptyset$. By [CTC79], $X(k_v) \neq \emptyset$ if and only if there exists a zero cycle of degree 1, hence $X(k_v) = \emptyset$ implies that $X(L_w) = \emptyset$ for all odd degree extensions $L_w/k_v$. Since $X(L \otimes_k k_v) \neq \emptyset$, $L_w/k_v$ is an even degree extension for all $w|v$. Thus, by Lemma 3.1, there exists a point $P_w \in X_\infty(L_w)$ satisfying $\text{inv}_w(\text{ev}_A(P_w)) = 0$. □

**Corollary 4.2.** Let $k$ be a global field of characteristic not equal to 2, let $X \to \mathbb{P}^1_k$ be a conic bundle with four geometric singular fibers. There exist at most three quadratic extensions $k_i/k$ such that for all even degree extensions $L/k$ that do not contain any $k_i$, we have

$$X(\mathbb{A}_L)^{\text{Br}(X_L)} \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset.$$  

**Proof.** If $X(k) \neq \emptyset$ then $X(\mathbb{A}_L)^{\text{Br}(X_L)}$ and $X(\mathbb{A}_L)$ are non-empty for all extensions $L/k$, hence we assume that $X(k) = \emptyset$. By Theorem 4.1, it remains to show that there exist quadratic extensions $k_i \supseteq k$ such that $\text{Res}_{L/k}: \text{Br}_L \to \text{Br}_L$ is surjective whenever $k_i \not\subseteq L$ for any $i$. If $\text{Res}_{L/k}$ is not surjective for all $L/k$, then by Corollary 3.5, it remains to consider the case when $|S| = 1$. Here, the problematic extensions $k_i/k$ are precisely the extensions for which the singular fibers of $X_{k_i} \to \mathbb{P}^1_{k_i}$ lie over two degree 2 closed points that are interchanged by $\text{Gal}(k_i/k)$.

By Remark 3.6, we have at most three such extensions, arising from the case where $S$ corresponds to a polynomial of the form $N_{F/k}(\ell(t))$, where $F$ is a bi-quadratic extension of $k$ and $\ell(t) \in k[t]$ is a linear polynomial. In this case, the extensions $k_i/k$ correspond to the three quadratic subextensions of $F/k$. If $L$ does not contain any such $k_i$, then this case never occurs, and the result follows. □

We can now extend Theorem 4.1 to arbitrary conic bundles at the expense of restricting to quadratic extensions.

**Theorem 4.3.** Let $k$ be a global field of characteristic not equal to 2 and let $X \to \mathbb{P}^1_k$ be a conic bundle, then for any quadratic extension $L/k$

$$X(\mathbb{A}_L)^{\text{Res}_{L/k}(\text{Br}_L)} \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset.$$  

**Proof.** Observe that one direction is again immediate, hence we assume that $X(\mathbb{A}_L) \neq \emptyset$. In a similar fashion to the proof of Theorem 4.1, we will show that for all $v \in \Omega_k$ there exists a point $(P_w) \in X(L \otimes_k k_v)$ such that $\sum_{w|v} \text{inv}_w(\text{ev}_A(P_w)) = 0$ for all $A \in \ker(\text{Br}_L[2] \to \text{Br}_X)$, and thus $X(\mathbb{A}_L)^{\text{Res}_{L/k}(\text{Br}_L)} \neq \emptyset$.

If $X(k_v) \neq \emptyset$ then by an identical argument to the proof of Theorem 4.1, there exists a point $(P_w) \in X(\mathbb{A}_L)$ such that $\sum_{w|v} \text{inv}_w(\text{ev}_A(P_w)) = 0$ for all $A \in \ker(\text{Br}_L[2] \to \text{Br}_X)$.

If $X(k_v) = \emptyset$ and $X(L \otimes_k k_v) \neq \emptyset$, then $v$ is non-split and so $L_w/k_v$ is quadratic. Thus, Lemma 3.1 implies the existence of a point $P_w \in X_\infty(L_w)$ which satisfies $\text{inv}_w(\text{ev}_A(P_w)) = 0$ for all $A \in \ker(\text{Br}_L[2] \to \text{Br}_X)$. □

When considering extensions analogous to the extensions $k_i$ of Corollary 4.2 for arbitrary conic bundles there are more issues that can arise, but nonetheless, there are still only finitely many. These problematic extensions arise in a similar manner to those of Corollary 4.2. Namely, if $S \subseteq \mathbb{P}^1_k$ is the finite set of closed points with geometric singular fiber, then the problematic extensions, $k_i/k$, are those which coincide with residue fields, $k(P)$, for $P \in S$.  

6
Over these extensions, \( \text{Res}_{k//k} : \text{Br}_{k} \to \text{Br}_{k} \) could fail to be surjective and new Brauer classes in \( \text{Br}_{k} \) could give a Brauer-Manin obstruction to \( X_k \). For all extensions \( L/k \) over which the points in \( S \) remain unchanged, \( \text{Res}_{L/k} \) is surjective and indeed \( X(A_{L})^{\text{Br}(X_L)} \neq \emptyset \Leftrightarrow X(A_{L}) \neq \emptyset \). Since there are only finitely many geometric singular fibers, there are only finitely many problematic extensions of this form, hence we have the following corollary.

**Corollary 4.4.** Let \( k \) be a global field of characteristic not equal to 2, let \( X \to \mathbb{P}^1_k \) be a conic bundle, and let \( L/k \) be a quadratic extension. If \( L \not
\subseteq k(P) \) for all \( P \in S \), then

\[
X(A_{L})^{\text{Br}(X_L)} \neq \emptyset \Leftrightarrow X(A_{L}) \neq \emptyset.
\]

**Remark 4.5.** Quite generally, if \( k \) is a number field and \( X/k \) is a smooth, projective, and geometrically rational variety, then \( \text{Pic}(X) \) is a Galois lattice split by a finite Galois extension \( K/k \). If \( E/k \) is a finite field extension which is linearly disjoint from \( K/k \), then the map \( \text{Res}_{E/k} : \text{Br}_{E} \to \text{Br}_{K} \) is an isomorphism, and in particular, is surjective. Corollary 4.4 is a slight refinement of this result for the case of conic bundles.

### 5. Proofs Of The Main Theorems

#### 5.1. Proof of Theorem 1.1.

If \( \text{Br}_{k} \) \( X(k) \neq \emptyset \) or there exists a Galois-invariant decomposition \( S = S_1 \cup S_2 \). The result trivially follows if \( X(k) \neq \emptyset \). If the latter case holds, then \( \text{Br}_{X} \cong \mathbb{Z}/2\mathbb{Z} \) and for any even degree extension \( L/k \) we have

\[
\text{Br}_{X_L} \cong \mathbb{Z}/2\mathbb{Z} \quad \text{or} \quad \text{Br}_{X_L} \cong (\mathbb{Z}/2\mathbb{Z})^2.
\]

If \( \text{Br}_{X_L} \cong \mathbb{Z}/2\mathbb{Z} \), then \( \text{Res}_{L/k} \) is surjective and the result follows from Theorem 4.1. If \( \text{Br}_{X_L} \cong (\mathbb{Z}/2\mathbb{Z})^2 \), then \( X(L) \neq \emptyset \) by Corollary 3.5.

It remains to prove the theorem when \( \text{Br}_{X} = 0 \) and \( X(A_k) \neq \emptyset \). If \( \text{Br}_{X} = 0 \) then

\[
X(A_k)^{\text{Br}(X_k)} = X(A_k) \neq \emptyset.
\]

By [CTSD94, Theorem 5.1], \( X(A_k)^{\text{Br}(X_k)} \neq \emptyset \Leftrightarrow X(k) \neq \emptyset \) hence

\[
X(A_k)^{\text{Br}(X_k)} \neq \emptyset \implies X(k) \neq \emptyset \implies X(L) \neq \emptyset
\]

completing the proof. \( \square \)

#### 5.2. Proof of Theorem 1.2.

Let \( X/\mathbb{Q} \) be the Châtelet surface given by

\[
y^2 - 5z^2 = \frac{3}{5}(5t^4 + 7t^2 + 1).
\]

One can check that \( 5t^4 + 7t^2 + 1 \) is irreducible hence \( \text{Br}_{X/\mathbb{Q}} = 0 \) by Corollary 3.4.

We will begin by showing \( X(Q_p) \neq \emptyset \) for all \( p \neq 3 \) and that \( X(Q_3) = \emptyset \). First, observe that for all \( p \neq 3, 5 \) we have \( X_\infty (Q_p) \neq \emptyset \) because \( X_\infty \) is the conic \( y^2 - 5z^2 = 3w^2 \), which has \( Q_p \) points for all \( p \neq 3, 5 \). Furthermore, the fiber \( X_2 \) is the conic \( y^2 - 5z^2 = \frac{1239}{5}w^2 \) which has \( Q_5 \) points since \( 1239 \in Q_5 \times Q \). For the case when \( p = 3 \), observe that \( 5 \notin Q_3 \times Q \) which implies that \( y^2 - 5z^2 \) is a norm from the unramified extension \( Q_3(\sqrt{5}) \). It remains to see that \( 5t^4 + 7t^2 + 1 \) always has even 3-adic valuation and this follows from the fact that


\(5t^4 + 7t^2 + 1\) is irreducible over the residue field \(\mathbb{F}_3\). We can now conclude that \(\frac{1}{5}(5t^4 + 7t^2 + 1)\) always has odd valuation hence is never of the form \(y^2 - 5z^2\), so \(X(\mathbb{Q}_3) = \emptyset\). This shows that \(X(A_{\mathbb{Q}}) = \emptyset\).

Now, consider the quadratic extension \(L = \mathbb{Q}(\sqrt{29})\). One can see that \(|\mathbb{Q}_3(\sqrt{29}) : \mathbb{Q}_3| = 2\), hence \(X(\mathbb{Q}_3(\sqrt{29})) \neq \emptyset\) from Lemma 3.1. This shows that \(X(A_L) \neq \emptyset\). Furthermore, a computation shows that \(X/\mathbb{Q}(\sqrt{29})\) is the Châtelet surface given by

\[
y^2 - 5z^2 = 3\left(t^2 + \frac{1}{10}(7 + \sqrt{29})\right)\left(t^2 + \frac{1}{10}(7 - \sqrt{29})\right)
\]

hence by Corollary 3.4 we have

\[
\frac{Br_{X_{\mathbb{Q}(\sqrt{29})}}}{Br_{\mathbb{Q}(\sqrt{29})}} \cong \mathbb{Z}/2\mathbb{Z} = \langle A \rangle
\]

where \(A\) denotes the quaternion algebra \((5, t^2 + \frac{1}{10}(7 + \sqrt{29}))\).

It remains to show that \(X(A_{\mathbb{Q}(\sqrt{29})})^{Br_{(X_{\mathbb{Q}(\sqrt{29})})}} = \emptyset\) and we do so by first showing that \(ev_A : X(L_w) \to Br L_w\) is identically zero for all \(w \not\equiv 5\). Begin by observing that for all primes \(w|p\) where \(p \neq 3, 5\), we know that \(X_\infty(L_w) \neq \emptyset\), hence \(ev_A : X(L_w) \to Br L_w\) takes the value 0 at such primes. Since the evaluation map is constant at all primes \(w\) of good reduction \([\text{CTS}13, \text{Theorem 3.1}]\), it remains to check \(ev_A(X(L_w ))\) for primes \(w\) lying over \(p = 2, 3, 5, \) and 29. Now observe that, for \(w|2, 3, 29, 5 \in L_w^x\) hence the algebra \(A = (5, t^2 + \frac{1}{10}(7 + \sqrt{29})\) is identically zero and \(ev_A(X(L_w )) = 0\). Thus for any \((P_w) \in X(A_L)\) we have

\[
\sum_{w \in \Omega_L} \text{inv}_w ev_A(P_w) = \sum_{w|5} \text{inv}_w ev_A(P_w).
\]

Let \(w_1\) and \(w_2\) denote the places lying over 5 corresponding to the embeddings in which \(\sqrt{29} \equiv 2 \pmod{5}\) and \(\sqrt{29} \equiv 3 \pmod{5}\) respectively and let \(P_i = (t_i, y_i, z_i) \in X(L_{w_i})\). We now show that

\[
\text{inv}_{w_i}(A(P_i)) = \begin{cases} 
\frac{1}{2} & \text{if } i = 1 \\
0 & \text{if } i = 2
\end{cases}
\]

Let \(\alpha = \frac{1}{10}(7 + \sqrt{29})\) and \(\overline{\alpha} = \frac{1}{10}(7 - \sqrt{29})\). Over \(\mathbb{Q}_5\), we have

\[
X/\mathbb{Q}_5 : y^2 - 5z^2 = 3(t^2 + \alpha)(t^2 + \overline{\alpha})
\]

hence \(\frac{Br_{X_{\mathbb{Q}_5}}}{Br_{\mathbb{Q}_5}}\) is generated by the quaternion algebra \(A = (5, t^2 + \alpha)\). Note that in \(Br X\), the quaternion algebra \(A\) is equivalent to the algebra \(B = (5, 3(t^2 + \overline{\alpha}))\). This will be an important tool in the final part of our proof.

First we consider the place \(w_1\) and observe that since \(\sqrt{29} \equiv 2 \pmod{5}\), we have \(w_1(\alpha) = -1\). Moreover, since \(\alpha \overline{\alpha} = \frac{1}{5}\), it follows that \(w_1(\overline{\alpha}) = 0\). Take \(P_1 = (t_1, y_1, z_1)\) and assume \(w_1(t_1) \geq 0\). Then, by the strong triangle inequality, \(w_1(t_1^2 + \alpha) = w_1(\alpha)\) and since \(5\alpha \equiv \frac{1}{5}(7 + \sqrt{29}) \equiv 2 \pmod{5}\), it follows that \(10(t_1^2 + \alpha) \in \mathbb{Q}_5^x\) meaning \(ev_A(P_1) = (5, 10)\) which is a non-split quaternion algebra in \(Br Q_5\). We can now see that \(\text{inv}_{w_1}(ev_A(P_1)) = \frac{1}{2}\).

Further, assume that \(w_1(t_1) < 0\) and consider the polynomial \(P(t) = \frac{2}{5}(5t^4 + 7t^2 + 1)\). We have that \(5^{-4w_1(t_1)}P(t) \equiv 3 \pmod{5}\) hence \(P(t)\) is never a norm from \(\mathbb{Q}_5(\sqrt{5})\) so all \(P_1 \in X(L_{w_1})\) must satisfy \(w_1(t_1) \geq 0\). Therefore, for all \(P_1 \in X(L_{w_1})\) we have \(\text{inv}_{w_1}(ev_A(P_1)) = \frac{1}{2}\).
Lastly, we consider the case of $w_2$ and recall that in this case, $\sqrt{29} \equiv 3 \pmod{5}$, hence $w_2(\alpha) = 0$ and $w_2(\overline{\alpha}) = -1$. Take $P_2 = (t_2, y_2, z_2)$ and observe that just as in the previous case, if $P_2 \in X(L_{w_2})$ we must have $w_2(t_2) \geq 0$. If $w_2(t_2) > 0$ then by the same argument for when $w_1(t_1) \geq 0$, we have that $t_2^2 + \alpha$ is always a square in $\mathbb{Q}_5$, hence $ev_\mathcal{A}(P_2) = 0 \in Br \mathbb{Q}_5$.

It remains to consider the case when $w_2(t_2) = 0$ and for this, we consider the algebra $\mathcal{B} = (5, 3(t^2 + \overline{\alpha}))$ and show that $ev_\mathcal{B}(P_2)$ is identically zero in $Br \mathbb{Q}_5$. For such values of $t_2$ we have $w_2(3(t^2 + \overline{\alpha})) = w_2(\overline{\alpha})$ hence $5 \cdot 3(t^2 + \overline{\alpha}) \equiv 3\overline{\alpha} \pmod{5}$, which by Hensel’s lemma, is a square in $\mathbb{Q}_5$. It now follows that for all $P_2 \in X(L_{w_2})$ we have $inv_{w_2}(ev_\mathcal{A}(P_2)) = 0$.

We can now conclude that for any $P_w \in X(\mathbb{A}_L)$ we have

$$\sum_{w \in \Omega_L} inv_w(ev_\mathcal{A}(P_w)) = \frac{1}{2}$$

and $X$ fails the Hasse principle over $L$. $\square$

**5.3. Proof of Theorem 1.3.** This result follows immediately from Corollary 4.4. $\square$

6. A Partial Converse to Theorem 1.2

In our consideration of even degree extensions over which $X$ may fail the Hasse principle, we saw that the extensions which intersect with $k(S)$ in a quadratic extension $F/k$ open the possibility for a potential Hasse principle failure. In the case of Châtelet surfaces, this can occur for surfaces $X/k$ of the form

$$y^2 - az^2 = cN_{F/k}(g(t))$$

where $a, c \notin k^{\times 2}$, $g(t) \in k[t]$ is a monic, irreducible polynomial, and $N_{F/k}$ denotes the usual norm map. In contrast to Theorem 1.2 not all such extensions $F/k$ are guaranteed to produce a Brauer-Manin obstruction over $F$. In order for such an obstruction to exist, the fiber $X_\infty$, the places for which it has no local points, and the places of bad reduction, must satisfy several necessary conditions.

**Theorem 6.1.** Let $k$ be a number field, let $F/k$ be a quadratic extension, and let $X/k$ be the Châtelet surface given by

$$y^2 - az^2 = cN_{F/k}(g(t))$$

where $a, c \notin k^{\times 2}$ and $g(t) \in k[t]$ is monic and irreducible. Let $\Omega_k$ denote the set of places of $k$, let $\Omega_F$ denote the set of places of $F$, and assume $X(\mathbb{A}_k) = \emptyset$.

If

$$\sum_{\nu \in \Omega_k \text{ s.t. } \nu \text{ splits in } F} inv_{\nu}(a, c) = 0 \in \mathbb{Q}/\mathbb{Z}$$

then for all even degree extensions $L/k$, we have $X(L) \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset$.

**Proof.** Let $v \in \Omega_k$, let $w \in \Omega_F$ such that $w|v$. Let $\mathcal{A} = (a, g(t))$ be a generator of $Br_{X_F}$ and let $\sigma$ denote the generator of $Gal \left( F/k \right)$. If $[F_w : k_v] = 2$ then there exists a point $P_w \in X_\infty(F_w)$ such that $inv_w(\mathcal{A}(P_w)) = 0$ by Lemma 3.1. Now assume that $F_w = k_v$, then
there exists a unique place \( w' \) such that \( w' \neq w \) and \( w'|v \), hence \( F_w = F_{w'} = k_v \). Take \( P_v \in X(k_v) \) and set \( P_v = P_w = P_{w'} = (t_v, y_v, z_v) \). We then have

\[
\text{inv}_w(\mathcal{A}(P_w)) + \text{inv}_{w'}(\mathcal{A}(P_{w'})) = \text{inv}_w\left((a, g(t_v)) + (a, \sigma(g)(t_v))\right) = \text{inv}_w\left(a, g(t_v)\sigma(g)(t_v)\right) = \text{inv}_v((a, c))
\]

Now, picking \((P_w) \in X(\mathbb{A}_F)\) as above, we have

\[
\sum_{w \in \Omega_F} \text{inv}_w(\mathcal{A}(P_w)) = \sum_{[F_w : k_v] = 2} 0 + \sum_{w \in \Omega_F} \text{inv}_w(\mathcal{A}(P_w)) = \sum_{v \in \Omega_F} \text{inv}_v(a, c)
\]

Now, if

\[
\sum_{v \in \Omega_F} \text{inv}_v(a, c) = 0 \in \mathbb{Q}/\mathbb{Z}
\]

then \( X(\mathbb{A}_F)^{Br(X_F)} \neq \emptyset \) and it follows that \( X(F) \neq \emptyset \). Since \( F/k \) is the only even degree extension over which the set \( S \) admits a Galois-invariant decomposition \( S = S_1 \cup S_2 \), it follows from Corollary 3.4 and Corollary 3.5 that if \( L/k \) is any even degree extension, then \( X(L) \neq \emptyset \iff X(\mathbb{A}_L) \neq \emptyset \).

\[
\square
\]

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