Understanding the Complex Position in a $\mathcal{PT}$-symmetric Oscillator

Jin-Ho Cho

Department of Physics & Research Institute for Natural Sciences,
Hanyang University, Haengdang-dong, Seongdong-gu, Seoul 133-791, Korea
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We study how to understand the complex coordinates involved in the non-Hermitian but $\mathcal{PT}$-symmetric systems. We explore a $\mathcal{PT}$-symmetric oscillator model to show that the entire information on the complex position is attainable. Its real part is from the observation while its imaginary part is from the non-Hermiticity parameter. We also propose a new complex extension of $\mathcal{P}$-transformation and $\mathcal{T}$-transformation (the ‘parity’ and ‘time reflection’ respectively). Particularly, the $\mathcal{T}$-transformation realizes the left-right reflection in the complex plane.

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Solving the eigenvalue problem of a system associated with ‘non-Hermitian’ potential, we generically meet the complex coordinates. For example, in the non-Hermitian potential like $V(x) \sim x^2 (ix)^\nu$ proposed in Ref. [1], the turning points, which are relevant in finding the connection formula in WKB approximation, are determined by the relation $E = V(x)$. The points are definitely in the complex plane.

It is confusing that we have to work in complex coordinates finding the connection formula in WKB method. We already know that the position operator $\hat{x}$ is Hermitian with respect to the $L^2$-inner product (the conventional one we use in quantum mechanics), thus has real eigenvalues.

The aim of this letter is to clarify this confusion by inspecting a ‘non-Hermitian’ but $\mathcal{PT}$-symmetric harmonic oscillator. (We will be more specific about these jargons later.) As a warmup for the non-Hermitian system, let us consider a massive charged particle, living in one-dimension under an external potential $V(x)$, and being coupled with an external constant ‘gauge’ $A$:

$$\hat{H}\psi(x) = \left\{ \frac{1}{2} \left( \hat{p} - A \right)^2 + V(\hat{x}) \right\} \psi(x).$$

(1)

Since no electric field or magnetic field is involved, one can neglect the trivial gauge coupling by redefining the wave function as $\psi(x) = e^{iA^* x} \phi(x)$.

What if the constant ‘gauge’ $A$ is pure imaginary? Indeed, one occasionally encounters such a case in the solid state physics system exhibiting the delocalization phenomena [2] or in the optics system like the channel wave guide involving the surface wave in the cladding layers [3]. The above method of redefining the wave function is no help if $A$ is pure imaginary. Though the newly defined wave function $\phi(x)$ is normalizable, the function $\psi(x)$ diverges in the asymptotic region. The difficulty with the case of the imaginary $A$ is originating from the non-Hermiticity of the Hamiltonian resulting in non-unitary time evolution.

A class of non-Hermitian Hamiltonians can possess a real spectrum. Behind those systems lies $\mathcal{PT}$-symmetry [1], or pseudo-Hermiticity [4]. In a broader sense, $\mathcal{P}$- and $\mathcal{T}$-transformation are a linear and an anti-linear involution on the phase space preserving the commutation relation $[\hat{x}, \hat{p}] = i\hat{I}$ [5]. Under the combined action of $\mathcal{P}$ ($\hat{x} \rightarrow -\hat{x}$, $\hat{p} \rightarrow -\hat{p}$, $i\hat{I} \rightarrow i\hat{I}$) and $\mathcal{T}$ ($\hat{x} \rightarrow \hat{x}$, $\hat{p} \rightarrow -\hat{p}$, $i\hat{I} \rightarrow -i\hat{I}$), the above Hamiltonian (with the imaginary $A$) will be invariant if $V^{\mathcal{PT}}(\hat{x}) = \hat{V}(\hat{x})$.

A concrete model in the class was first provided and pioneered in Refs. [1] [6] [7], where $\hat{A} = 0$ and $\hat{V}(\hat{x}) = \hat{x}^2 (i\hat{x})^\nu$, thus is $\mathcal{PT}$-symmetric. It was shown by numerical method or WKB approximation that the system with $\nu \geq 0$ has real eigenvalues when $\mathcal{PT}$-symmetry is exact, that is, when $\mathcal{PT}$ shares its eigenstates with the Hamiltonian. Experimental observation on the symmetry has been put forward in optics [8], especially in the wave guide physics. (See Ref. [9] and references therein.)

In this letter, we consider a model with

$$\hat{A} = iz^* \hat{I}, \quad \hat{V}(\hat{x}) = \frac{1}{2} (\hat{x} - z^* \hat{I})^2,$$

(2)

where $\hat{I}$ is the identity operator and $z^* = \rho e^{i\lambda}$ is a complex number. Unless $z$ is pure imaginary ($\lambda = \pm \pi/2$), $\mathcal{PT}$ symmetry is not clear. At the final stage, we will see the system has a real spectrum for arbitrary value of $z$ and how $\mathcal{PT}$ symmetry reconcile with the general case.

The model has the virtue of exact solvability. Moreover, it has not only the coordinate space representation but also the spectral representation, which allows a clear connection between the $\mathcal{PT}$ symmetry (studied in Refs. [1] [6]) and the pseudo-Hermiticity (discussed in Refs. [4] [10]).

Regarding the spectral representation, it is more instructive to write the Hamiltonian in terms of the ladder operators:

$$\hat{a} = (\hat{x} + i\hat{p}) / \sqrt{2}, \quad \hat{a}^\dagger = (\hat{x} - i\hat{p}) / \sqrt{2}$$

(3)

(Note that we have set $m = \hbar = \omega = e = 1$ for simplicity.)
The non-Hermiticity is clear in this representation,

$$\hat{H} = \left(\hat{a}^\dagger - z^* \sqrt{2} \hat{I}\right) \hat{a} + \frac{1}{2}. \quad (4)$$

We hope it to be pseudo-Hermitian instead, because it will have a real spectrum then. (See Ref. [4] for the rigorous mathematics behind the pseudo-Hermitian system.) The form suggests that the ladder operators be redefined as

$$\hat{b}^\dagger := e^{-i\theta} (\hat{a}^\dagger - z^* \sqrt{2} \hat{I}), \quad \hat{b} := \hat{a} e^{i\theta}. \quad (5)$$

Indeed, they compose ladder operators satisfying $[\hat{b}, \hat{b}^\dagger] = \hat{I}$. If $\hat{b}^\dagger$ is the pseudo-Hermitian conjugate of $\hat{b}$, more specifically if $\hat{b}^\dagger = \eta^{-1} \hat{b} \eta$ for some Hermitian operator $\eta$, the Hamiltonian will be pseudo-Hermitian, that is, $H = \eta^{-1} \hat{H}^\dagger \eta = \hat{H}^\dagger$. Inspired by the coherent state computation, one can determine it as $\eta = e^{z^2 \sqrt{2} \hat{a} + z \sqrt{2} \hat{a}^\dagger}$. It is not only Hermitian but also positive in the sense of the quasi-Hermitian Hamiltonian is proportional to the number operator $\hat{c}_n = \hat{c}_n^\dagger$.

Two different Fock spaces are involved in the system. On the common vacuum $|0\rangle_a = |0\rangle_b := |0\rangle$ ( annihilated by $\hat{b} = \hat{a} e^{i\theta}$), one can successively apply either $\hat{a}^\dagger$ or $\hat{b}^\dagger$ to construct the Fock space $\{|n\rangle_a\}$ or $\{|n\rangle_b\}$. Their relation is

$$|n\rangle_a = e^{-i n \theta} \prod_{l=0}^{n-1} \sqrt{C_l} \left(-z^* \sqrt{2}\right)^l \left(\frac{1}{\sqrt{n!}}\right) |n-l\rangle_a. \quad (6)$$

Only the states $\{|n\rangle_b\}$ form the eigenstates because the hamiltonian is proportional to the number operator $\hat{N}_b = \hat{b}^\dagger \hat{b}$ associated with the set $\{\hat{b}, \hat{b}^\dagger\}$.

The eigenstates $|n\rangle_b$ of the quasi-Hermitian Hamiltonian $\hat{H}$ are not orthonormal with respect to $L^2$-inner product. Indeed, an explicit computation shows that

$$\langle m|n\rangle_b = N_{mn} \sum_{k=0}^{\min\{m,n\}} C_k n P_k \left(\frac{z^m \left(z^*\right)^n}{|z|^2}\right). \quad (7)$$

where the numerical factor is determined as $N_{mn} = e^{i(m-n)\theta} (-\sqrt{2})^{m-n} / \sqrt{m! n!}$.

The ‘non-Hermit’ Hamiltonian [4] is Hermitian with respect to a new-inner product $\langle \cdot | \cdot \rangle_{\eta} := \langle \cdot | \eta \cdot \rangle$ (hereafter called as $\eta$-inner product). Indeed $\langle \phi | \hat{H} | \psi \rangle_{\eta} = \langle \phi | \eta \hat{H} | \psi \rangle = \langle \hat{H} \phi | \psi \rangle_{\eta} = \langle \hat{H} \phi | \psi \rangle_{\eta}$ for arbitrary states $\phi$ and $|\psi\rangle$. Specifically for the choice of two eigenstates $|m\rangle_b$ and $|n\rangle_b$, it implies that the eigenvalues are real, that is, $E_m = E_n^a$, and the eigenstates can be orthonormal, $\langle m|n\rangle_b = \delta_{mn}$, with respect the $\eta$-inner product.

The Hamiltonian has not only the real spectrum but also the real expectation values. According to Dirac-von Neumann axiom of quantum mechanics, the expectation value rather than the spectrum concerns the observation of $\eta$-inner product. An observable corresponds to an operator in the Hilbert space and it is Hermitian if and only if it has real expectation values [10]. The Hamiltonian [4] is not Hermitian with respect to $L^2$-inner product, can possess real expectation values as far as $\eta$-inner product is used. The Hamiltonian [4] can delineate a physical system if $\eta$-inner product is adopted.

The problem with using $L^2$-inner product for the system [4] is that the energy expectation value for a non-eigenstate depends on time, and what is worse, it develops imaginary value as time flows, thus is unphysical in the sense of Dirac-von Neumann axiom. Indeed for a specific state $|\psi\rangle = |1\rangle_a = z^* \sqrt{2} |0\rangle_b + e^{i\theta} |1\rangle_b$, the energy expectation value with respect to $L^2$-inner product is given by

$$\langle \psi|\hat{H}|\psi\rangle = 1 + \frac{1 + 4i |z|^2 \sin t}{2 + 8 |z|^2 (1 - \cos t)}, \quad (8)$$

whereas with respect to $\eta$-inner product it is a real constant;

$$\langle \psi|\hat{H}_\eta|\psi\rangle = \frac{3 + 2 |z|^2}{2 + 4 |z|^2}. \quad (9)$$

The results are illustrated in Fig. 1. (The figures are generated by using Mathematica [13].)

The orthonormality condition $\langle m| \eta | n \rangle_b = \delta_{mn}$ suggests that the states $|m\rangle_{\eta} = \eta |n\rangle_b$ are dual to the states $|m\rangle_b$. They are the eigensates of the operator $\hat{H}_\eta = \eta^{-1} \hat{b} \hat{b}^\dagger + 1/2$ satisfying $\hat{H}_\eta^\dagger = E^a_n |n\rangle_{\eta}$, and compose the biorthonormal set of eigenstates along with the states $|m\rangle_b$. Note that the operators $\hat{b}^\dagger$ and $\hat{b} := \hat{b}^\dagger \eta^{-1}$ form the dual set of ladder operators satisfying $[\hat{b}, \hat{b}^\dagger] = \hat{I}^\dagger$.

The biorthonormal set of eigenstates allows the spectral representations for various operators. Some of them are $\hat{H} = \sum_{n=0}^\infty \hat{E}_n |n\rangle_b \langle n|$ and $\eta = \sum_{n=0}^\infty |n\rangle_{\eta} \langle n|$.

It is easy to see that they satisfy the pseudo-Hermiticity condition $\hat{H} = \hat{H}_\eta^\dagger$.

Let us find the anti-linear involution symmetry associated with the system for arbitrary value of $z$. A pseudo-Hermitian Hamiltonian admits an anti-linear involution operator as a symmetry [3]. As was mentioned earlier, $PT$ symmetry looks clear when $z = \pm i \rho$. A systematic way of constructing the generalized $P$, $T$, and $C$-operator has been developed in Ref. [3], which we will employ here dealing with the case of arbitrary complex $z$.

Associated with the metric operator $\eta$, there are canoni- cal pseudo-metric operators and the anti-linear operators

$$\eta = \sum_{n=0}^\infty \sigma_n |n\rangle_{\eta} \langle n|, \quad \sigma' = \sum_{n=0}^\infty \sigma_n'$

Here, the series elements $\sigma_n$ and $\sigma_n'$ take values in $\{-1, 1\}$ and the operation $\star$ is the complex conjugation on the
expression appearing on its right. With proper normalization of the metric operator as $\eta \to \tilde{\eta} = e^{-2|z|^2}\eta$, one can show that the operator $P_\sigma$ with $\sigma_\rho = (-1)\rho$ is an involution, that is, it is squared to the identity. The operator $T_\sigma$, can be seen to be involutory too. The sigma factor is identified as $\sigma'_\rho = (-1)^\rho$ when $\theta - \lambda = 0, \pi, 2\pi, \ldots$, and while $\sigma'_\rho = 1$ if $\theta - \lambda = \pi/2, 3\pi/2, 5\pi/2, \ldots$. This means that for arbitrary complex value of $z^* = re^{i\lambda}$, one can always adjust the phase $\theta$ in $\tilde{b} = \tilde{a}e^{i\theta}$ so that $P_\sigma$ and $T_\sigma'$ be involutions.

From the above involutory operators, one can construct involutory symmetry generators as $C_\sigma := \tilde{\eta}^{-1}\tilde{\eta}_\sigma$, and $X_{\sigma\sigma'} := P_\sigma^{-1}T_\sigma = P_\sigma T_\sigma'$. They commute with the Hamiltonian because they satisfy $\tilde{H}^\dagger = \tilde{\eta}\tilde{H}\tilde{\eta}^{-1} = P_\sigma H P_\sigma^{-1} = T_\sigma H T_\sigma^{-1}$. Their spectral representations are $C_\sigma = \sum_{n=0}^\infty \sigma_n |n\rangle_b \sigma'_n \langle n|_b$, and $X_{\sigma\sigma'} = \sum_{n=0}^\infty \sigma_n \sigma'_n |n\rangle_b \sigma'_n \langle n|$.

The spectral representation,

$$b = \sum_{n=0}^\infty \sqrt{n + 1} |n\rangle_b \nu' \langle n + 1|,$$

$$b^\dagger = \sum_{n=0}^\infty \sqrt{n + 1} |n + 1\rangle_b \nu' \langle n|$$

enables us to compute the following transformations explicitly. Under the pseudo-metric operator $P_\sigma$, $b \to -\tilde{b}$, $b^\dagger \to -\tilde{b}^\dagger$, and $i\tilde{I} \to i\tilde{I}'$. Here $\tilde{I}' = \tilde{I}^\dagger$ represents the identity operator on $b'$-Fock space. Under the anti-linear operator $T_\sigma$, $\tilde{b} \to \mp b'$, $\tilde{b}^\dagger \to \mp b^\dagger$, and $i\tilde{I} \to -i\tilde{I}'$. The upper/lower sign is for the case when $\theta - \lambda$ is an integer/0 half-integer multiple of $\pi$, respectively. (The same correspondence will be applied upon the appearance of the multiple signs hereafter.) Lastly under the generalized charge operator $C_\sigma$, $\tilde{b} \to -\tilde{b}$, $\tilde{b}^\dagger \to -\tilde{b}^\dagger$, and $i\tilde{I} \to -i\tilde{I}$.

The phase space variables $\tilde{x}$ and $\tilde{p}$ transform nontrivially under the above involutions. Exploiting the relations (9) and (5), we obtain

$$P_\sigma : \tilde{x} \to -\tilde{x} + (z + z^*)\tilde{I}'\, , \quad \tilde{p} \to -\tilde{p} - i(z - z^*)\tilde{I}'\, ,$$

$$T_\sigma : \tilde{x} \to \mp(\tilde{x}\cos 2\theta - \tilde{p}\sin 2\theta) + (z + z^*)\tilde{I}'\, ,$$

$$\tilde{p} \to \pm(\tilde{x}\cos 2\theta + \tilde{p}\cos 2\theta) - i(z - z^*)\tilde{I}'\, ,$$

$$C_\sigma : \tilde{x} \to -\tilde{x} + 2z^*\tilde{I}'\, , \quad \tilde{p} \to -\tilde{p} + 2iz^*\tilde{I}.'$$

They look entirely different from the ones given in Ref. [1], or those mentioned in the earlier part of this letter. Under the combination of $P_\sigma T_\sigma'$, the operators transform homogeneously as $\tilde{q} \to \pm(\tilde{x}\cos 2\theta - \tilde{p}\sin 2\theta)$, $\tilde{p} \to \mp(\tilde{x}\sin 2\theta + \tilde{p}\cos 2\theta)$. Only with $z$ pure imaginary, thus $\cos 2\theta = \mp 1$, the result accords with those in Ref. [1].

It is possible to define the pseudo-Hermitian phase variables. By construction, the operators $\tilde{X} := (\tilde{b} + \tilde{b}^\dagger)/\sqrt{2}$ and $\tilde{P} := -i(\tilde{b} - \tilde{b}^\dagger)/\sqrt{2}$ are pseudo-Hermitian possessing real spectra. They are physical in the sense that their expectation values are real. They transform homogeneously under the involutions:

$$P_\sigma \tilde{X} \to -\tilde{X}^\dagger\, , \quad \tilde{P} \to -\tilde{P}^\dagger\, , \quad i\tilde{I} \to i\tilde{I}'\, ,$$

$$T_\sigma \tilde{X} \to \mp\tilde{X}^\dagger\, , \quad \tilde{P} \to \pm\tilde{P}^\dagger\, , \quad i\tilde{I} \to -i\tilde{I}'\, ,$$

$$C_\sigma \tilde{X} \to -\tilde{X}\, , \quad \tilde{P} \to -\tilde{P}\, , \quad i\tilde{I} \to -i\tilde{I}.'$$

When $\theta - \lambda$ is a half-integer multiple of $\pi$, $\sigma'_\rho = 1$ and the transformations (with lower sign) look analogous to the one given earlier. However, the transformations relate the operators $\tilde{X}, \tilde{P}$ with their Hermitian conjugates $\tilde{X}^\dagger, \tilde{P}^\dagger$. They amount to the complex extension of $P$- and $T$-transformation. A different complex extension of $P$- and $T$-transformation is given in Ref. [14], but we emphasize that the above transformation is perfectly allowable too, even with the other choice ($\theta - \lambda$ is an integer multiple of $\pi$) though the meaning of ‘time reflection’ become obscure then. Moreover, the classical counterpart of the ‘parity’ operation on the complex $X$-plane
\[ P : X \rightarrow -X^* \] naturally realizes the ‘left-right’ reflection mentioned in Ref. [2].

The eigenstates of \( \hat{H} = (\hat{p}^2 + \hat{X}^2)/2 \) can be represented in \( X \)-space. Being pseudo-Hermitian, \( \hat{X}^2 = \hat{X} \) has real spectrum and its eigenstates \( |X \rangle \) are ‘orthonormal’ with respect to \( \eta \)-inner product, that is, \( \langle X|\eta|X' \rangle = \delta(X - X') \). This result leads to the completion relation; 
\[
\int dx |X\rangle \langle X| \eta = I. \]
In \( X \)-space representation, \( b\)-vacuum \( |0\rangle_b \), being annihilated by \( \hat{b} = (\hat{X} + i\hat{P})/\sqrt{2} \), satisfies
\[
\langle X|\eta|0\rangle_b = \frac{1}{\sqrt{2}} (X + \partial_X) \langle X|\eta|0\rangle_b = 0. \tag{12}
\]
Therefore, the eigenfunctions \( \langle X|\eta|n\rangle_b \) are given in terms of Hermite polynomial as
\[
\langle X|\eta|n\rangle_b = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2^n n! \sqrt{\pi}}} H_n(X). \tag{13}
\]
Regarding the probability density function, defined to be real \textit{ab initio}, one can use two different well-defined representations. The relations
\[
\int dx \langle \psi|x\rangle \langle x|\psi \rangle = \int \langle \psi|X\rangle \langle X|\eta|\psi \rangle = 1 \tag{14}
\]
suggest that the integrands (including the denominator) be the probability density functions in \( x \)-space and in \( X \)-space, respectively.

It is the expectation value of the position operator \( \hat{x} \) that is the complex coordinate we meet in the non-Hermitian system. Exploiting Eqs. (3) and (5), one can obtain the operator relations,
\[
\hat{x} = \hat{X} \cos \theta + \hat{P} \sin \theta + z^* \hat{I},
\hat{p} = -\hat{X} \sin \theta + \hat{P} \cos \theta + iz^* \hat{I}. \tag{15}
\]
The operators \( \hat{x} \) and \( \hat{p} \), though Hermitian with respect to \( L^2 \)-inner product, have complex expectation values for any state \( |\psi\rangle \) if we use \( \eta \)-inner product. For example in the expectation value \( \langle \psi|\eta \hat{x}|\psi \rangle = \langle \psi|\eta \hat{X} \cos \theta + \langle \psi|\eta \hat{P} |\psi \rangle + z^* \), the parts \( \langle \psi|\eta \hat{X}|\psi \rangle \) and \( \langle \psi|\eta \hat{P}|\psi \rangle \) are real with \( \hat{X} \) and \( \hat{P} \) pseudo-Hermitian, but \( z^* \) is complex valued in general. The imaginary part \( \Im((\psi|\eta \hat{x}|\psi)) = \Im(z^*) \) looks like an ‘order parameter’ signaling the non-Hermiticity. Employing \( L^2 \)-inner product, we can make the value \( \langle \psi|\hat{x}|\psi \rangle \) real, but then the Hamiltonian will be invalidated, being complex valued.

As regards the measurement, a proper choice of the inner product should be prior to determining the ‘physical’ observables. Dirac-von Neumann axiom of measurement underpins quantum mechanics. In the conventional Hermitian system, we need not worry much about the inner-product despite that it is an essential component of Hilbert space. In the non-Hermitian system, we have to devise a new inner product that renders the Hamiltonian Hermitian. However, the operators \( \hat{x} \) and \( \hat{p} \) will be non-Hermitian in the new inner product, making them ‘unphysical’ in the Dirac-von Neumann sense.

In conclusion, the entire information on the ‘complex position’ \( \hat{x} \) for a state is attainable, even though the operator is not pseudo-Hermitian. In this letter we constructed the ‘physical’ position \( X \) and ‘physical’ momentum \( P \) explicitly and found their relations with the ‘unphysical’ counterparts, \( \hat{x} \) and \( \hat{p} \). From the relations we note that \( \hat{x} = -\Re(z^*)\hat{I} \) and \( \hat{p} = -i\Im(z^*)\hat{I} \) are pseudo-Hermitian, thus ‘physical’. One can always prepare for a state in an eigenstate of these pseudo-Hermitian operators. Therefore the pseudo-Hermitian part of the position operator \( \hat{x} \) is definitely observable. Its anti-pseudo Hermitian part is unobservable in the experiment, but it just reads the non-Hermiticity parameter \( \Im(z^*) \). This part remains the same under the time flow.

It is interesting to see that the operator \( \hat{x} - \langle \hat{x} \rangle \hat{I} \) is pseudo-Hermitian, thus, observable. The same is true for \( \hat{p} \). This implies that albeit \( \hat{x} \) and \( \hat{p} \) are unobservable, their corresponding uncertainties are observable. Notably, they satisfy the uncertainty principle.

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\[ jinhocho@hanyang.ac.kr \]

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[1] & C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
[2] & N. Hatano and D. R. Nelson, Phys. Rev. Lett. 77, 570 (1996).
[3] & K. Kawano and T. Kiotoh, Introduction to Optical Waveguide Analysis, (John Wiley & Sons, INC., 2001).
[4] & A. Mostafazadeh, J. Math. Phys. 43, 205 (2002).
[5] & A. Mostafazadeh, J. Math. Phys. 44, 974 (2003).
[6] & C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. 89, 270401 (2002); \textit{ibid.} 92, 119902 (2004).
[7] & C. M. Bender, Rept. Prog. Phys. 70, 947 (2007).
[8] & C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. 6, 192 (2010).
[9] & Y. Choi, J.-K. Hong, J.-H. Cho, K.-G. Lee, J. W. Yoon, and S. H. Song, Opt. Express 23 (2015).
[10] & A. Mostafazadeh, Int. J. Geom. Math. Mod. Phys. 7, 1191 (2010).
[11] & P.A.M. Dirac, \textit{The Principles of Quantum Mechanics}, 4th ed., (Oxford University Press, 1958).
[12] & J. von Neumann, \textit{Mathematical Foundations of Quantum Mechanics}, transl. by R.T. Beyer, 2nd ed., (Princeton University Press, 1955).
[13] & Wolfram Research, Inc., Mathematica, Version 10.2, (Champaign, IL, 2015).
[14] & C. M. Bender, S. Boettcher and P. Meisinger, J. Math. Phys. 40, 2201 (1999).
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