N–transitivity of Certain Diffeomorphism Groups

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n-TRANSITIVITY OF CERTAIN DIFFEOMORPHISM GROUPS

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ABSTRACT. It is shown that some groups of diffeomorphisms of a manifold act n-transitively for each finite n.

In this paper we show that the following groups of diffeomorphism of a manifold $M$ act transitively of order $n$ for each finite $n$: All diffeomorphisms with compact support (this is folklore, the first trace is in [8]) and all real analytic diffeomorphisms (from [7]). Furthermore all real analytic diffeomorphisms, or smooth ones with compact support, which preserve either a volume form, or a symplectic form, or are contact diffeomorphisms. The symplectic ones can also be chosen ‘globally hamiltonian’. For the smooth cases 1-transitivity is due to [3], $n$-transitivity to [1].

1. Proposition. Let $M$ be a connected smooth manifold of dimension $\dim M \geq 2$. Then the group $\text{Diff}_c(M)$ of all smooth diffeomorphisms with compact support acts $n$-transitively on $M$, for each finite $n$. Thus for any two ordered sets of $n$ different points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in $M$ there is a smooth diffeomorphism $f$ with compact support such that $f(x_i) = y_i$ for each $i$.

This result is folklore. In order to be complete and since we shall need an argument later on we include a short proof.

Proof. Let us first choose a finite $n \in \mathbb{N}$. Let $M^{(n)}$ denote the open submanifold of all $n$-tuples $(x_1, \ldots, x_n) \in M^n$ of pairwise distinct points. $\text{Diff}_c(M)$ acts on $M^{(n)}$ by the diagonal action, and we have to show, that this action is transitive.

Let us first assume that $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are pairwise disjoint. For some $\varepsilon > 0$ let $c_i : (-\varepsilon, 1+\varepsilon) \to M$ be smooth curves with $c_i(0) = x_i$ and $c_i(1) = y_i$, which are embeddings and do not intersect each other. From a drawing it can be seen that this exists if $\dim M \geq 2$, since $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are disjoint.

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We choose pairwise disjoint tubular neighborhoods \( U_i \) of \( c_i(-\varepsilon, 1 + \varepsilon) \), extend the velocity vector fields of the curves to them, and use a smooth bump function to obtain vector fields \( X_i \) with compact support in \( U_i \) which coincides with the velocity vector field \( c_i^* \circ c_i^{-1} \) along each curve \( c_i \). Then the vector field \( X = X_1 + \ldots + X_n \) on \( M \) with compact support coincides with the velocity vector field \( c^* \circ c^{-1} \) along each curve \( c_i \) and the flow mapping \( \Phi_t^X \) maps each \( x_i \) to \( y_i \).

This argument shows that each \( \text{Diff}_e(M) \) -orbit in \( M^{(n)} \) is dense. We may replace in the argument the points \( y_i \) by points \( z_i \) in small open pairwise disjoint neighborhoods \( U_i \) of \( y_i \), not meeting \( \{x_1, \ldots, x_n\} \). Then the argument shows that each orbit contains an open set in \( M^{(n)} \), thus is open. Since the dimension of \( M \) is at least 2, \( M^{(n)} \) is connected, so there is only one orbit and the result on \( n \)-transitivity follows. \( \square \)

2. Lemma. Let \( M \) be a real analytic manifold. Then the group \( \text{Diff}^\omega(M) \) of all real analytic diffeomorphisms is dense in the group \( \text{Diff}^\infty(M) \) of smooth diffeomorphisms, in the Whitney \( C^\infty \)-topology.

Proof. By [2], theorem 3, there is a real analytic embedding \( i : M \to \mathbb{R}^k \) on a closed submanifold, for some \( k \). We use the constant standard inner product on \( \mathbb{R}^k \) to obtain a real analytic tubular neighborhood \( U \) of \( i(M) \) with projection \( p : U \to i(M) \). By [2], proposition 8, applied to each coordinate of \( \mathbb{R}^k \) the space \( C^\omega(M, \mathbb{R}^k) \) of real analytic \( \mathbb{R}^k \)-valued functions is dense in the space \( C^\infty(M, \mathbb{R}^k) \) of smooth functions, in the Whitney \( C^\infty \)-topology. If \( f : M \to M \) is a smooth diffeomorphism we may approximate \( i \circ f \) by real analytic mappings \( g \) in \( C^\omega(M, U) \), then \( p \circ g \) is real analytic \( M \to i(M) \) and approximates \( i \circ f \). Since the set of diffeomorphisms is open in the Whitney topology, this approximation becomes eventually a diffeomorphism. \( \square \)

3. Lemma. Let \( M \) be a real analytic manifold. Then for any real analytic vector bundle \( E \to M \) the space \( C^\omega(E) \) of real analytic sections of \( E \) is dense in the space \( C^\infty(E) \) of smooth sections. In particular the space \( \mathfrak{X}^\omega(M) \) of real analytic vector fields is dense in the space \( \mathfrak{X}(M) \) of smooth vector fields, in the Whitney \( C^\infty \)-topology.

Proof. Either repeat the proof of lemma 2 with some changes or use [6], 7.5. \( \square \)

4. Theorem. Let \( M \) be a connected real analytic manifold of dimension \( m \geq 2 \). Then the group \( \text{Diff}^\omega(M) \) acts \( n \)-transitively on \( M \), for each finite \( n \).

Proof. Let us fix a natural number \( n \). The group \( \text{Diff}^\omega(M) \) acts on the open submanifold \( M^{(n)} \) of all \( n \)-tuples \( (x_1, \ldots, x_n) \in M^n \) of pairwise distinct points by the diagonal action. Again we have to show, that this action is transitive.

First we show that each \( \text{Diff}^\omega(M) \) -orbit in \( M^{(n)} \) is dense. Let \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_n) \) be in \( M^{(n)} \) and consider an open neighborhood of \( (y_1, \ldots, y_n) \) in \( M^{(n)} \), which we may suppose to be of the form \( \prod_i U_i \), where \( U_i \) is a neighborhood of \( y_i \) in \( M \) for each \( i \). Then by proposition 1 there is a smooth diffeomorphism \( f : M \to M \) with \( f(x_i) = y_i \) for all \( i \), and by lemma 2 there exists a real analytic diffeomorphism \( g \in \text{Diff}^\omega(M) \) with \( g(x_i) \in U_i \) for each \( i \). So \( g : (x_1, \ldots, x_n) \in \prod_i U_i \).

Next we show that the orbit through \( (x_1, \ldots, x_n) \in M^{(n)} \) in \( M^{(n)} \) contains an open neighborhood of \( (x_1, \ldots, x_n) \). This will finish the proof: Since each orbit is dense, each orbit meets this nonempty open subset, so all orbits coincide.

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We choose again a complete Riemannian metric \( g \) on \( M \). Then we let \( (Y_{ij})_{j=1}^m \) be an orthonormal basis of \( T_x M \) with respect to \( g \), for all \( i \). Then we choose real analytic vector fields \( X_k \) for \( 1 \leq k \leq N = nm \) which satisfy the following conditions:

\[
|X_k(x_i) - Y_{ij}|_g < \varepsilon \quad \text{for } k = (i-1)m + j, \\
|X_k(x_i)|_g < \varepsilon \quad \text{for all } k \notin [(i-1)m+1, im], \\
|X_k(x)|_g < 2 \quad \text{for all } x \in M \text{ and all } k.
\]

These fields exist by lemma 3. Since the fields are bounded with respect to a complete Riemannian metric, they have complete real analytic flows, see e.g. [4]. We consider the real analytic mapping

\[
f : \mathbb{R}^N \to M^{(n)}
\]

\[
f(t_1, \ldots, t_N) := \left( (F_{t_1} X_1 \circ \cdots \circ F_{t_N} X_n)(x_1) \right) \\
\cdots \\
\left( (F_{t_1} X_1 \circ \cdots \circ F_{t_N} X_n)(x_n) \right)
\]

which has values in the \( \text{Diff}^\omega(M) \)-orbit through \( (x_1, \ldots, x_n) \). To get the tangent mapping at 0 of \( f \) we consider the partial derivatives

\[
\frac{\partial}{\partial t_k} |f(0, \ldots, 0, t_k, 0, \ldots, 0) = (X_k(x_1), \ldots, X_k(x_n)).
\]

If \( \varepsilon > 0 \) is small enough, this is near an orthonormal basis of \( T_{(x_1, \ldots, x_n)} M^{(n)} \) with respect to the product metric \( g \times \ldots \times g \). So \( T_0 f \) is invertible and the image of \( f \) contains thus an open subset. \( \square \)

5. Lemma. Let \( c : (-\varepsilon, 1 + \varepsilon) \to M^m \) be a smooth embedding. Then every 1-form (respectively \((m-1)\)-form) along \( c([0,1]) \) can be extended to an exact 1-form (respectively \((m-1)\)-form) on \( M \) with compact support in a tubular neighborhood of the image of \( c \).

Proof. There exists a tubular neighborhood of \( c(-\varepsilon, 1+\varepsilon) \), i.e. a diffeomorphism from \((-\varepsilon, 1+\varepsilon) \times \mathbb{R}^{m-1}\) to an open neighborhood \( U \) of the image of \( c \) in \( M \) which coincides with \( c \) on \((-\varepsilon, 1+\varepsilon) \times \{0\} \), and whose inverse \( u : U \to (-\varepsilon, 1+\varepsilon) \times \mathbb{R}^{m-1} \) we may use as a chart with \( u(c(t)) = (t,0) \).

(i) The case of a 1-form.

A 1-form along \( c \) is given by \( \omega(t) = \sum_{i=1}^m a_i(t) du^i |_{c(t)} \) for \( t \in [0, 1] \), where \( a_i : [0, 1] \to \mathbb{R} \) are smooth and we may extend them smoothly to \( a_i : (-\varepsilon, 1+\varepsilon) \to \mathbb{R} \).

Consider the function \( f : U \to \mathbb{R} \), given by

\[
f = A_1(u^1) + u^2 a_2(u^1) + \cdots + u^m a_m(u^1),
\]

where \( A_1(t) = \int_0^t a_1(s) ds \). Then \( df(c(t)) = \omega(t) \). Let \( h, k : \mathbb{R} \to \mathbb{R} \) be smooth bump functions such that \( \text{supp} h \subset (-\delta, \delta) \), \( \text{supp} k \subset (-\varepsilon, 1+\varepsilon) \), \( h = 1 \) in a neighborhood of 0, and \( k = 1 \) in a neighborhood of \([0, 1]\). Then

\[
F := k(u^1) h(u^2) \cdots h(u^m) f
\]

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Theorem. Let \((M, \sigma)\) be a connected symplectic smooth manifold of dimension \(m \geq 2\). Then the group \(\text{Diff}_c(M, \sigma)\) of all smooth diffeomorphisms with compact support which preserve the symplectic form \(\sigma\) acts \(n\)-transitively on \(M\), for each finite \(n\).

If \(M\) is a real analytic manifold with a real analytic symplectic form \(\sigma\), then also the group \(\text{Diff}^\omega(M, \sigma)\) of real analytic symplectomorphisms acts \(n\)-transitively on \(M\), for each finite \(n\).

The \(n\)-transitivity of the group of smooth symplectomorphisms is due to [1], with essentially the same method. The proof will also show that the Lie subgroup of \(\text{Diff}_c(M, \sigma)\) whose Lie algebra is the Lie algebra of compactly supported globally Hamiltonian vector fields acts \(n\)-transitively on \(M\). This group has been identified as a Lie group in [11], for compact \(M\). Also in the real analytic case the subgroup of globally Hamiltonian real analytic symplectomorphisms act \(n\)-transitively.

Proof. First the smooth case. By the argument used at the end of the proof of proposition 1 it suffices to show, that there exists \(\varphi \in \text{Diff}_c(M, \sigma)\) with \(\varphi h(x_i) = y_i\), for any \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) in \(M^{(n)}\) which are pairwise disjoint sets in \(M\). We take again smooth curves \(c_i : (-\epsilon, 1+\epsilon) \to M\) with \(c_i(0) = x_i\) and \(c_i(1) = y_i\) which are embeddings and do not intersect. Let \(U_i\) be pairwise disjoint tubular neighborhoods of \(c_i((-\epsilon, 1+\epsilon))\).

The velocity field of the curve \(c_i\) defines the 1-form \(\alpha_i = i_{c_i^*}\sigma\) along the curve \(c_i\). Using lemma 6 we extend this form to an exact 1-form \(df_i\) on \(M\) with \(\text{supp} f_i \subset M\).
Let $f := f_1 + \cdots + f_n$ and consider the (globally) Hamiltonian vector field $\text{grad}^\sigma(f) = -\sigma^{-1}df$ with compact support corresponding to $f$. It coincides with the velocity field $c_1 \circ c^{-1} \circ [0,1])$. Hence the flow $F_t^{\text{grad}^\sigma(f)} \in \text{Diff}_c(M,\sigma)$ and $F_t^{\text{grad}^\sigma(f)}(x_i) = y_i$.

If $M$ and $\sigma$ are real analytic, we may approximate the smooth function $f$ from above by a real analytic function $g$ in the Whitney $C^1$-topology in such a way that:

1. The Hamiltonian vector field $\text{grad}^\sigma(g)$ is bounded with respect to some complete Riemannian metric and thus has a global real analytic flow $F_t^{\text{grad}^\sigma(g)} \in \text{Diff}_c(M,\sigma)$.

2. $F_t^{\text{grad}^\sigma(g)}(x_i)$ is near $y_i$ for all $i$.

Thus it follows that each $\text{Diff}_c(M,\sigma)$-orbit in $M^{(n)}$ is dense. Similarly as in the proof of theorem 4 we will show that the orbit through $(x_1,\ldots,x_n) \in M^{(n)}$ is open, which finishes the proof of $n$-transitivity.

We choose again a complete Riemannian metric $g$ on $M$. Then we let $(Y_{ij})_{j=1}^m$ be an orthonormal basis of $T_xM$ with respect to $g$, for all $i$. Then we choose real analytic functions $f_k$ for $1 \leq k \leq N = nm$ whose Hamiltonian vector fields satisfy the following conditions:

- $|\text{grad}^\sigma(f_k)(x_i) - Y_{ij}| < \varepsilon$ for $k = (i-1)m + j$.
- $|\text{grad}^\sigma(f_k)(x_i)| < \varepsilon$ for all $k \notin [(i-1)m + 1, im]$.
- $|\text{grad}^\sigma(f_k)(x)| < 2$ for all $x \in M$ and all $k$.

Since these conditions describe Whitney $C^1$ open subsets, such functions exist by [2], proposition 8. Now we may finish the proof as at the end of theorem 4. \qed

7. Contact manifolds. Let $M$ be a smooth manifold of dimension $m = 2n + 1 \geq 3$. A contact form on $M$ is a 1-form $\alpha \in \Omega^1(M)$ such that $\alpha \wedge (da)^n \in \Omega^{2n+1}(M)$ is nowhere zero. This is sometimes called an exact contact structure. The pair $(M,\alpha)$ is called a contact manifold (see [5]). The contact vector field $X_\alpha \in \mathfrak{X}(M)$ is the unique vector field satisfying $i_{X_\alpha} \alpha = 1$ and $i_{X_\alpha} da = 0$.

A diffeomorphism $f \in \text{Diff}(M)$ with $f^*\alpha = \lambda_f \alpha$ for a nowhere vanishing function $\lambda_f \in C^\infty(M,\mathbb{R} \setminus 0)$ is called a contact diffeomorphism. Note that then $\lambda_f = i_{X_\alpha}(\lambda_f) = i_{X_\alpha} f^*\alpha = f^*(i_{f^{-1}X_\alpha}) = f^*(i_{f^{-1}X_\alpha})$. The group of all contact diffeomorphisms will be denoted by $\text{Diff}(M,\alpha)$.

A vector field $X \in \mathfrak{X}(M)$ is called a contact vector field if $\mathcal{L}_X \alpha = \mu_X \alpha$ for a smooth function $\mu_X \in C^\infty(M,\mathbb{R})$. The linear space of all contact vector fields will be denoted by $\mathfrak{X}_\alpha(M)$ and it is clearly a Lie algebra. Contraction with $\alpha$ is a linear mapping again denoted by $\alpha : \mathfrak{X}_\alpha(M) \to C^\infty(M,\mathbb{R})$. It is bijective since we may apply $i_{X_\alpha}$ to the equation $\mathcal{L}_X \alpha = i_X da + da(X) = \mu_X \alpha$ and get $0 + i_{X_\alpha} da(X) = \mu_X X$; but the equation uniquely determines $X$ from $\alpha(X)$ and $\mu_X$. The inverse $f \mapsto \text{grad}^\sigma(f)$ of $\alpha : \mathfrak{X}_\alpha(M) \to C^\infty(M,\mathbb{R})$ is a linear differential operator of order 1.

**Theorem.** Let $M$ be a connected smooth manifold of dimension $m \geq 2$, and let $\alpha$ be a contact form on $M$. Then the group $\text{Diff}_c(M,\alpha)$ of contact diffeomorphisms with compact support acts $n$-transitively on $M$ for all finite $n$.

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If $M$ and $\alpha$ are real analytic then also the group $\text{Diff}^\omega(M, \alpha)$ of real analytic contact diffeomorphisms acts $n$-transitively on $M$ for each finite $n$.

The $n$-transitivity of $\text{Diff}_c(M, \alpha)$ is due to [1].

Proof. By the argument used at the end of the proof of proposition 1 it suffices to show, that there exists $\varphi \in \text{Diff}_c(M, \mu)$ with \( \varphi(x_i) = y_i \), for any \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) in $M^{(n)}$ which are pairwise disjoint sets in $M$. For $\varepsilon > 0$ let again $c_i : (-\varepsilon, 1 + \varepsilon) \to M$ be smooth embeddings with $c_i(0) = x_i$, $c_i(1) = y_i$ which do not intersect. We choose pairwise disjoint tubular neighborhoods $U_i$ of $c_i(-\varepsilon, 1 + \varepsilon)$. Let $f_i : M \to \mathbb{R}$ be a smooth extension of $\alpha(c_i' \circ c_i^{-1}) : c_i([0,1]) \to \mathbb{R}$ with support in $U_i$ and \( f := \sum_{i=1}^n f_i \in C_c^\infty(M, \mathbb{R}) \). Then the contact vector field $\text{grad}^\alpha(f) \in \mathfrak{X}_\alpha(M)$ coincides with the velocity field $c_i' \circ c_i^{-1}$ on $c_i([0,1])$ for each $i$.

Hence $\text{Fl}^{\omega}_1 \in \text{Diff}_c(M, \alpha)$ and $\text{Fl}^{\omega}_1(x_i) = y_i$ for $i = 1, \ldots, n$.

If $M$ and $\alpha$ are real analytic, we may approximate the smooth function $f$ from above by a real analytic function $g$ in the Whitney $C^1$-topology in such a way that:

1. The contact vector field $\text{grad}^\alpha(g)$ is bounded with respect to a complete Riemannian metric and so has a global real analytic flow $\text{Fl}^{\text{grad}^\alpha(g)}_1 \in \text{Diff}(M, \alpha)$, see [4].
2. $\text{Fl}^{\text{grad}^\alpha(g)}_1(x_i)$ is near $y_i$ for all $i$.

Thus it follows that each $\text{Diff}^\omega(M, \alpha)$-orbit in $M^{(n)}$ is dense. Similarly as in the proof of theorem 4 we will show that the orbit through \((x_1, \ldots, x_n)\) in $M^{(n)}$ is open, which finishes the proof.

We choose again a complete Riemannian metric $g$ on $M$. Then we let $\left(Y_{ij}\right)_{i,j=1}^n$ be an orthonormal basis of $T_{x_i}M$ with respect to $g$, for all $i$. Then we choose real analytic functions $f_k$ for $1 \leq k \leq N = nm$ which satisfy the following conditions:

\[
\begin{align*}
|\text{grad}^\alpha(f_k)(x_i) - Y_{ij}|_g &< \varepsilon & \text{for } k = (i-1)m + j, \\
|\text{grad}^\alpha(f_k)(x_i)|_g &< \varepsilon & \text{for all } k \notin [(i-1)m + 1, im], \\
|\text{grad}^\alpha(f_k)(x_i)|_g &< 2 & \text{for all } x \in M \text{ and all } k.
\end{align*}
\]

Since these conditions describe Whitney $C^1$ open subsets, such functions exist by [2], proposition 8. Now we may finish the proof as at the end of theorem 4. \( \square \)

8. Theorem. Let $(M, \mu)$ be a connected smooth manifold of dimension $m \geq 2$ with a positive volume density. Then the group $\text{Diff}_c(M, \mu)$ of all smooth volume preserving diffeomorphisms of $M$ with compact support acts $n$-transitively on $M$, for each finite $n$.

If $M$ and $\mu$ are real analytic then also the group $\text{Diff}^\omega(M, \mu)$ of real analytic volume preserving diffeomorphisms acts $n$-transitively on $M$, for each finite $n$.

Proof. First the smooth case. By the argument used at the end of the proof of proposition 1 it suffices to show, that there exists $f \in \text{Diff}_c(M, \mu)$ with $f(x_i) = y_i$, for any $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in $M^{(n)}$ which are pairwise disjoint sets in $M$.

Having fixed the points, we may find an orientable connected open subset $U$ of $M$ containing all points. Since we are going to construct a volume preserving diffeomorphism with support in $U$, for the smooth case we can replace $M$ by $U$ and...
without loss assume that $M$ is orientable. But we shall need the setting $U \subset M$ later.

For some $\varepsilon > 0$ let $c_i : (-\varepsilon, 1 + \varepsilon) \to M$, $i = 1, \ldots, n$ be smooth embeddings with $c_i(0) = x_i$, $c_i(1) = y_i$ which do not intersect. We choose pairwise disjoint tubular neighborhoods $U_i$ of $c_i(-\varepsilon, 1 + \varepsilon)$, $i = 1, \ldots, n$.

We can find a Riemannian metric $g$ on $M$ whose volume form is $\mu$. Then the divergence of a vector field $X \in \text{Vect}(M)$ is $\text{div} X = \ast d \ast X^g$, where $X^g = g(X) \in \Omega^1(M)$ (here we view $g : TM \to T^*M$) and $\ast$ is the Hodge star operator. The velocity field of the curve $c_i$ defines an $(m-1)$-form $((c'_i \circ c_i^{-1})^g)$ along $c_i([0, 1])$. Using lemma 8 we extend it to an exact $(m-1)$-form $d\alpha_i$ on $M$ with $\alpha_i \subset U_i$, and we put $\alpha = \sum_{i=1}^n \alpha_i \in \Omega^{m-2}(M)$. We consider the vector field

\begin{equation}
X_\alpha = (-1)^{m+1}(d\alpha)^2 = (-1)^{m+1}g^{-1} \ast d\alpha,
\end{equation}

i.e. by the relation $d\alpha = \ast X^g_\alpha$. Then $X_\alpha$ is divergence free, $\text{div} X_\alpha = \ast d \ast X^g = \ast d \ast X^g = 0$, and has compact support in the union of all $U_i$. It also coincides on $c_i([0, 1])$ with the velocity field of the curve $c_i$. Hence $\text{Fl}^{X_\alpha}_1 \in \text{Diff}_e(M, \mu)$ with $\text{Fl}^{X_\alpha}_1(x_i) = y_i$.

We treat now the real analytic case. The Riemannian metric $g$ with volume form $\mu$ can be chosen real analytic. We also choose a complete Riemannian metric $\gamma$.

First we assume that $M$ is orientable. We approximate the smooth $(m-2)$-form $\alpha$ from above by real analytic $(m-2)$-forms $\beta$ in such a way that:

1. The real analytic vector field $X_\beta = (-1)^{m+1}g^{-1} \ast d\beta$ is bounded with respect to the complete Riemannian metric $\gamma$ and thus has a global real analytic flow $\text{Fl}^{X_\beta}_1 \in \text{Diff}_e(M, \mu)$.

2. $\text{Fl}^{X_\beta}_1(x_i)$ is near $y_i = \text{Fl}^{X_\gamma}_1(x_i)$ for all $i$.

Since these conditions describe a Whitney $C^1$-open set, such real analytic forms $\beta$ exist by lemma 3. Thus it follows that each $\text{Diff}_e(M, \mu)$-orbit in $M^{(n)}$ is dense. Similarly as in the proof of theorem 4 we will show that the orbit through $(x_1, \ldots, x_n) \in M^{(n)}$ is open, which finishes the proof.

We let $(Y_{ij})_{i=1}^m$ be an orthonormal basis of $T_{x_i}M$ with respect to the complete Riemannian metric $\gamma$, for all $i$. Then we choose real analytic $(m-2)$-forms $\beta_k$ for $1 \leq k \leq N = nm$ whose vector fields $X_{\beta_k} = (-1)^{m+1}g^{-1} \ast d\beta_k$ satisfy the following conditions:

\begin{equation}
\begin{aligned}
|X_{\beta_k}(x_i) - Y_{ij}|_\gamma &< \varepsilon \quad \text{for } k = (i-1)m + j, \\
|X_{\beta_k}(x_i)|_\gamma &< \varepsilon \quad \text{for all } k \notin [(i-1)m+1, im], \\
|X_{\beta_k}(x_i)|_\gamma &< 2 \quad \text{for all } x \in M \text{ and all } k.
\end{aligned}
\end{equation}

Since these conditions describe Whitney $C^1$ open subsets, such $(m-2)$-forms exist by lemma 3. Now we may finish the proof as at the end of theorem 4.

Now we treat the case of non-orientable $M$. Let $\pi : \tilde{M} \to M$ be the real analytic connected oriented double cover of $M$, and let $\varphi : \tilde{M} \to M$ be the real analytic involutive covering map. Recall the orientable connected open subset $U \subset M$ containing all points $x_i$ and $y_i$ from above. The form $\alpha$ from above had compact

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support in \( U \). The inverse image \( \pi^{-1}(U) \subset \tilde{M} \) is the disjoint union of two connected open subsets \( W_1 \) and \( W_2 \) such that \( \pi|_{W_p} : W_p \to U \) is a diffeomorphism for both \( p = 1, 2 \). We let \( x^p_i = (\pi|_{W_p})^{-1}(x_i) \) and \( y^p_i = (\pi|_{W_p})^{-1}(y_i) \), and we pull back both metrics to \( \tilde{M} \), so \( \tilde{g} := \pi^*g \) and \( \tilde{\gamma} := \pi^*\gamma \).

We approximate the smooth \((m - 2)\)-form \( \tilde{\alpha} := \pi^*\alpha \) by real analytic \((m - 2)\)-forms \( \beta \in \Omega^{m-2}(\tilde{M}) \) in such a way that the conditions \((2)\) and \((3)\) from above are satisfied now on \( M \) for \( x^p_i \) and \( y^p_i \).

\( \tilde{X}_\beta := (-1)^{m+1} \tilde{g}^{-1} \ast d\beta \) is bounded with respect to the complete Riemannian metric \( \tilde{g} \) and thus has a global real analytic flow \( \text{Fl}_{\tilde{X}\beta} \in \text{Diff}^\omega(\tilde{M}, \pi^*\mu) \).

\( \text{Fl}_{\tilde{X}\beta}^X(x^p_i) \) is near \( y^p_i = \text{Fl}_{\tilde{X}\beta}^X(x^p_i) \) for all \( i \), and for \( p = 1, 2 \).

Since these conditions describe a Whitney \( C^1 \)-open set, such real analytic forms \( \beta \) exist by lemma 3. Since \( \tilde{\alpha} = \pi^*\alpha \) is invariant under \( \varphi^* \), the real analytic vector field \( \frac{1}{2}(X_\beta + \varphi_*X_\beta) \) still satisfies both \((1)\) and \((2)\), is divergence free, invariant under the covering transformation \( \varphi \), thus it induces a real analytic vector field \( Z_\beta \in \mathfrak{X}(M) \) which is bounded with respect to \( \gamma \), such that \( \text{Fl}_{Z\beta}^X(x_i) \) is near \( y_i \) for each \( i \), and \( Z_\beta \) is now divergence free in the sense that \( \mathcal{L}_{Z_\beta} \mu = 0 \). Thus it follows that each \( \text{Diff}^\omega(M, \mu) \)-orbit in \( M^{(n)} \) is dense.

Next we will show that the orbit through \((x_1, \ldots, x_n) \in M^{(n)} \) is open, which finishes the proof. We choose real analytic \((m - 2)\)-forms \( \beta_k \in \Omega^{m-2}(M) \) for \( 1 \leq k \leq N = nm \) whose vector fields \( X_{\beta_k} = (-1)^{m+1} \tilde{g}^{-1} \ast d\beta_k \) satisfy the following conditions, where we put \( Y^p_{ij} := T_{x^p_i} \pi^{-1} Y_{ij} \) for \( p = 1, 2 \):

\[ |X_{\beta_k}(x^p_i) - Y^p_{ij}|_\tilde{g} < \varepsilon \quad \text{for } k = (i-1)m + j, p = 1, 2, \]

\[ |X_{\beta_k}(x^p_i)|_\tilde{g} < \varepsilon \quad \text{for all } k \notin [(i-1)m + 1, im], p = 1, 2, \]

\[ |X_{\beta_k}|_\tilde{g} < 2 \quad \text{for all } x \in \tilde{M} \text{ and all } k. \]

Since these conditions describe Whitney \( C^1 \) open subsets, such \((m - 2)\)-forms exist by lemma 3. Then the vector fields \[ \frac{1}{2}(X_{\beta_k} + \varphi_*X_{\beta_k}) \] still satisfy the conditions \((4)\), are still divergence free and induce divergence free vector fields \( Z_{\beta_k} \in \mathfrak{X}(M) \) which satisfy the conditions \((4)\) on \( M \) as in the oriented case, and we may finish the proof as above. \( \square \)

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