Integrability as Effective Principle of Nonperturbative Field and String Theories

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Abstract

One of the perspectives in modern quantum field and string theory is related with the attempts to go beyond the perturbation theory. It turns out that a key principle in the formulation of all known non-perturbative results is integrability, i.e. arising of the structures of completely integrable systems. I discuss several important steps in this direction and speculate on its further possible development.

Recent investigations in gauge and string theories showed that sometimes there exists a way to find explicitly the exact nonperturbative results (spectrum, correlation functions, effective actions) even in the quantum theories which can not be considered as quantum integrable models at least in conventional and naive sense. In contrast to "naive" quantum integrable models where usually the quantum (infinite-dimensional) symmetry algebra allows one to compute explicitly the spectrum and the correlation functions, there is no even to such extent "straightforward" way in the theories I am going to discuss. The intention to study these particular models is caused by the hope that they are not so far away from the realistic quantum gauge and string theories where the solution to the basic problems of confinement and quantum gravity is looked for, being on the other hand solvable at least in the sense to be discussed below.

The starting, and maybe the most pessimistic point is that already in this class of models there are still no direct ways of solution – like there are no such ways for in many other models of quantum field and string theory. However, this particular class is distinguished by the fact (indeed a strong hypothesis) that (generally some pieces of) the exact solutions do really exist. A key reason for that is that the topological excitations play an important role and determine the structure of the theory at very low distances when the other excitations cancel each other due to, say, (extended) supersymmetry. Amazingly enough, all existing examples subjected to this scheme of behaviour

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have almost identically the same effective formulation which can be presented in terms of effective completely integrable model forgetting about many properties of the bare theory – in particular there is no real distinction between the theories living in different space-time dimensions. The low-energy limit of such models is described by a sort of topological theory.

Below, I will not pay enough attention to the mechanism which allows us to distinguish this class of theories – just mentioning again that $N \geq 2$ supersymmetry (which is always possible to find for such models at least as a BRST-symmetry) plays an important role. Instead I will try to describe the basic features of the effective formulation, which attract a lot of interest themselves.

**Moduli of the theory.** Any exact nonperturbative solution should present the spectrum, correlation functions, effective actions as functions of the parameters of the theory – or its moduli, coming usually from the (gauge-invariant) low-energy values of the fields in the target-space. For example, in 4d supersymmetric gauge theories this is the v.e.v.'s of the Higgs fields $h_k$, in string theories these are the moduli of the target-space metric (e.g. Kähler or complex structures), gauge fields (e.g. moduli of flat connections or selfdual gauge fields) etc. The problem itself is to find the (exact nonperturbative) dependence of the physical objects on these moduli parameters.

**Spectral curve.** Next point is that the class of the theories we discuss is distinguished by the holomorphic dependence of the (complex) moduli parameters – i.e. there exists a complex structure on the moduli space and only functions with ”good” global behaviour enter the game. This goes back to the holomorphic structures arising in the instantonic calculus and to the Belavin-Knizhnik theorem in string theory which leads to the draustical simplifications removing the possible ambiguity in the form of the exact answer. In the known examples, this holomorphic structures arise moreover in a nice geometric way – the moduli of the theory appear to be (a subspace in) the moduli space of complex structures of the target-space spectral curves $\Sigma$. It turns out, that nonperturbatively the target-space spectral curve acquires a nontrivial topological structure (being just defined locally – as a sphere of the scale parameter in perturbation theory) and a complex structure on the spectral curve is parameterized by moduli of the theory. This additional complicated structure means that ”stringy” nature plays an important role in the nonperturbative formulation and strings (and $D$-branes) wrapping along topologically nontrivial directions produce important effects in the exact effective formulation while perturbatively the spectral curve can be tested only ”locally”.

**Curves and Integrable Systems.** Now let us discuss the fact of appearence of integrable systems of KP/Toda type in the framework of the quantum field and string theories. Indeed, since

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Of course, in addition to the parametric dependence of moduli themselves the physical quantities can depend on the topological (discrete) characteristics of moduli spaces, moreover in the simplest topological string models only this dependence is essential and the correlation functions can be just numbers.
the solutions are formulated in terms of periods of some differentials on a complex spectral curve – it means that an integrable system (moreover the particular class of systems of KP/Toda type – where the Liouville torus is restricted to be a real section of a Jacobian of a complex curve) appears more or less by definition by means of the Krichever construction. This very useful observation leads us to a possibility of applying rather simple technique of Lax pairs, spectral curves, symplectic forms and $\tau$-functions to naively infinite-dimensional quantum field and string theories.

**Symplectic geometry of integrable systems.** To go further we will start with the point – what should be added to a complex curve to have an integrable system. Indeed, the Liouville torus as a real section of a Jacobian is determined after one introduces following Dubrovin, Krichever and Novikov a meromorphic 1-form $dS$ whose derivatives $\frac{\partial dS}{\partial h_k} \equiv \omega_k$ give holomorphic differentials. This generating 1-form defines a completely integrable system on a symplectic manifold $\Omega = \delta dS$ which in all cases we discuss can be explicitly rewritten as

$$
\begin{align*}
    dS &= \lambda d\log w = \text{Tr}\mathcal{L} d\log T \\
    \Omega &= \delta \lambda \wedge \delta \log w = \text{Tr}\delta\mathcal{L} \wedge \delta \log T
\end{align*}
$$

(1)

**Symplectic form and duality.** The symplectic form (1) is defined by the eigenvalues of two operators playing the essential role in integrable systems. Quasiclassically their common spectrum defines the spectral curve. The symplectomorphisms of (1) can be considered as transformations between the dual integrable systems with the generating function $S = \sum_k \int^{\gamma_k} dS$.

Indeed, writing (1) more explicitly one finds that $\Omega = \sum_k \delta\lambda \wedge \delta \log w|_{\gamma_k} = \sum_k \delta h_k \wedge \delta \phi_k$ and the Hamiltonian flows provided by $h_k$ produce a completely integrable system on Jacobian with co-ordinates $\{\phi_k\}$. The symplectomorphisms of (1) can bring us to a dual system, for example when $h_k$ become the time-variables themselves. Then $dS$ (1) plays the role of the generating differential of the Whitham hierarchy describing the flows in moduli space around a point, corresponding to a finite-gap solution.

**$\mathcal{T}$-function and prepotential.** The most complete information about the integrable system is given by the object $\log \mathcal{T} = \log \mathcal{T}_0 + \log \mathcal{T}_\theta$ where the first part (a logarithm of a quasiclassical $\tau$-function) restricted to the dependence on moduli $\log \mathcal{T}_0|_{\text{moduli}} \equiv F$ is usually called a prepotential and is defined by

$$
\frac{\partial^2 \mathcal{F}}{\partial t_i \partial t_j} = T_{ij}(t)
$$

(2)

The sense of parameters $t_k$ and the r.h.s. $T_{ij}(t)$ is different for the different systems, but their geometrical meaning is always related to certain "periods" $t_k = \oint_{A_k} dS$ or $t_\alpha = \text{res}_{P_\alpha}(\lambda^{-\alpha} dS)$ and intersection form $T_{ik} = \int_{\Sigma} \omega_i \wedge \omega_j$ on $\Sigma$. 3
The prepotential $F = \log T_0$ in general satisfies the associativity equation having the form (for the matrices $F_{i,j,k} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} \equiv (F_i)_{jk}$)

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i \quad \forall i, j, k$$

(3)

The full generating (partition) function depends on the infinite amount of variables – related to all excitations of the effective theory (which should include in particular gravitational dressing). It is defined by a generalization of (3) – usually called a string equation. The string equation can be formulated in terms of quantization of the symplectic structure (1) corresponding in general to a sort of quantization of the spectral curves: where $\hat{L}$ and $\hat{Q} \equiv \hat{\log} T$ obey $[\hat{L}, \hat{Q}] = 1$.

"Toy string" solutions effective theories. In this simplest example the already known explicit solutions exist for the spherical spectral curve $\Sigma$ where only the times related to the residues are valid. $T_{ij}(t)$ is a linear function of the $t$-variables (giving rise to the prepotential $F = t_3 + \ldots$) and the eigenvalues of two operators in (1) are just polynomial functions. The topological correlators are numbers and count the intersection indices on moduli space – this is an example of topological gravity.

The case of physical ($c < 1$ or pq-) gravity is known much less explicitly and correspond already to nontrivial spectral curves $\Sigma_{g=(p-1)(q-1)}$. This is however the case where the exact form of the duality transformation – relating the partition functions in the dual points – is known exactly, having the form of a Fourier transform with the exponent $S = \int \lambda \, dS$.

The Seiberg-Witten effective theories. The higher genus complex curves arise also in 4d SUSY gauge theories where the nonperturbative exact solution is formally defined as a map

$$G, \tau, h_k \rightarrow T_{ij}, a_i, a_i^D$$

(4)

$G$ is gauge group, $\tau$ – the UV coupling constant, $h_k = \frac{1}{k} \langle \text{Tr} \Phi^k \rangle$ – the v.e.v.’s of the Higgs field) and an elegant description in terms of $\Sigma_{g=\text{rank} G}$ with $h_k$ parameterizing some of the "hyperelliptic" moduli of complex structures. The periods of meromorphic 1-form $a_i = \oint A_i$ and $a_i^D = \oint B_i$ determine the BPS massive spectrum, $a_i^D = \frac{\partial^2 F}{\partial a_i \partial a_j}$ the prepotential $F$ (giving the low-energy effective action) and, thus, the set of low-energy coupling constants $T_{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j} = \frac{\partial a_i^D}{\partial a_j}$. The curves $\Sigma_{p=\text{rank} G}$ are spectral curves of the nontrivial finite-gap solutions of the periodic Toda-chain problem and its natural deformations into Calogero-Moser and spin chains. The "period"-times $a_i, a_i^D$ are related to the action integrals ($\oint pdq$) of the system.

String Duality. The picture presented above should be actually considered as a simple version of a generic nonperturbative effective target-space formulation of string theory. String theory
possesses a huge amount of "hidden symmetries" which allow sometimes to determine the answer without a direct computation. The introduced objects have a direct generalization for the whole string theory picture where at the moment only some observations based on consistency requirements for the relations among dual theories are made. The difference of the presented above picture with generic conception of string duality is that the above construction is formulated in strict mathematical sense what still remains to be done for more "rich" string models.

A stringy generalization is straightforward and related first of all with the prepotentials arising in the study of realistic models related to the Calabi-Yau compactifications. All the steps described above can be in principle repeated leading finally to the integrable models based on the higher-dimensional complex manifolds (instead of 1C-dimensional Σ). Such integrable systems (in spirit of Hitchin-Donagi-Markman) are not investigated yet in such detail.

Another problem is that even for the simplest cases considered above the complete picture has a lot of holes. In particular the exact form of the generating function log T is not yet known even for the Seiberg-Witten effective theories.

In spite of all the problems it is easy to believe that for all the theories where it is possible to make any statement about the nonperturbative and exact quantities there exists something more than a summation of a perturbation theory. The main idea I tried to advocate above that this could be the principle of integrability, which has been checked already in several examples and based on general belief that the realistic theory should be a selfconsistent one and adjust automatically its properties not to be ill-defined both at large and small distances. It looks that an adequate language for the effective formulation of nonperturbative field and string theories obeying such property can be looked for among integrable systems.

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