A Diagrammatic Approach for the Coefficients of the Characteristic Polynomial

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Abstract

In this work we provide a novel approach for computing the coefficients of the characteristic polynomial of a square matrix. We demonstrate that each coefficient can be efficiently represented by a set of circle graphs. Thus, one can employ a diagrammatic approach to determine the coefficients of the characteristic polynomial.

1 Introduction

A variety of different branches in Mathematical Physics boil down to calculating the eigenvalues of an $n \times n$ matrix $A$ over some field $K$ usually taken to be the real line or the complex plane. When we are interested in all the latent roots of the characteristic polynomial of a matrix and not in its latent vectors we usually expand out the expression

$$ f(x) = \det(A - xI) = \prod_{i=1}^{n}(x - \lambda_i) $$

$$ = x^n + (-1)^1e_1(\lambda)x^{n-1} + (-1)^2e_2(\lambda)x^{n-2} + (-1)^3e_3(\lambda)x^{n-3} + \cdots + (-1)^ne_n(\lambda) \tag{1} $$

where $e_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$ are the elementary symmetric polynomials in $k$ eigenvalues $\lambda_{i_s}$, $s = 1, \cdots, k$. There are different methods to determine explicitly $e_k(\lambda)$ in terms of a sum of trace products $\prod_{r=1}^{l}(tr(A^r))^{m_r}$, over all possible partitions $(1^{m_1}, \cdots, l^{m_l})$ of the positive integer $k$. Undoubtedly, all methods for high order matrices become very tedious.

One method for an $n \times n$ matrix is based upon the multinomial formula

$$ (\lambda_1 + \cdots + \lambda_n)^m = \sum_{k_1, \cdots, k_n} \binom{m}{k_1, \cdots, k_n} \lambda_1^{k_1} \cdots \lambda_n^{k_n} \tag{2} $$

where the summation is taken over all sequences of nonnegative integer indices $k_1, \cdots, k_n$ such that $\sum_{i=1}^{n} k_i = m$ and the coefficients are given by

$$ \binom{m}{k_1, \cdots, k_n} = \frac{m!}{k_1! \cdots k_n!} \tag{3} $$

$^1$We adopt the following notation: $n = (1^{m_1}, 2^{m_2}, \cdots, r^{m_r}, \cdots)$ where $m_i$ counts the number of parts of $n$ which are equal to $i$. It is called the multiplicity of $i$ in $n$. 


For \( n = 2 \) relation (2) gives

\[
\left( \sum_{i_1=1}^{n} \lambda_{i_1} \right)^2 = \sum_{i_1=1}^{n} \lambda_{i_1}^2 + 2 \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1} \lambda_{i_2} \tag{4}
\]

from which we find

\[
e_2(\lambda) = -\frac{1}{2} \left( (tr(A^2) - (trA)^2) \right). \tag{5}
\]

Similarly for \( e_3(\lambda) \) we have

\[
e_3(\lambda) = \frac{1}{6} \left( \left( \sum_{i_1=1}^{n} \lambda_{i_1} \right)^3 - 2 \sum_{i_1=1}^{n} \lambda_{i_1}^3 - 3 \sum_{i_1 \neq i_2=1}^{n} \lambda_{i_1}^2 \lambda_{i_2} \right)
= -\frac{1}{6} \left( 2tr(A^3) + 3tr(A^2)trA - (trA)^3 \right). \tag{6}
\]

Note that the three terms in (6) correspond to the partitions \( 3^1 = (1^1, 2^1) = 1^3 \). In the same vein we can compute any \( e_k(\lambda) \).

A second method is to use Newton–Girard's formulas which give the connection between the coefficients \( p_r(\lambda) = (-1)^{-r}e_r(\lambda) \) and the power sums

\[
S_r = \sum_{i_1=1}^{k} \lambda_{i_1}^r,
S_r + S_{r-1}p_1 + \cdots + S_1p_{r-1} + rp_r = 0, \quad r \leq n. \tag{7}
\]

From (7) it follows that

\[
p_1 = -S_1
p_2 = -\frac{1}{2}(S_1p_1 + S_2)
\cdots
p_r = -\frac{1}{r}(S_1p_{r-1} + \cdots + S_{r-1}p_1 + S_r). \tag{8}
\]

Relations (8) determine the coefficients \( p_r \) and the process is called Le Verrier's method.

## 2 The New Method

The method we propose for the evaluation of the coefficients of the characteristic polynomial is based upon the knowledge of the number of partitions \( q(k) \) for the index \( k \) which specifies the elementary symmetric polynomial \( e_k(\lambda) \). The generating function of \( q(k) \) is given by Euler's formula [2]

\[
F_{tot}(x) = \frac{1}{\prod_{k=1}^{\infty} (1 - x^k)} = \sum_{k=0}^{\infty} q(k)x^k. \tag{9}
\]

Other useful generating functions are

\[
F_{even}(x) = \frac{1}{\prod_{k=1}^{\infty} (1 - x^{2k})}, \quad F_{odd}(x) = \frac{1}{\prod_{k=0}^{\infty} (1 - x^{2k+1})}, \quad F_{uneq}(x) = \prod_{k=1}^{\infty} (1 + x^k). \tag{10}
\]
where \( F_{\text{even}}, F_{\text{odd}} \) and \( F_{\text{uneq}} \) enumerate partitions of \( k \) into \( \text{even}, \text{odd} \) and \( \text{unequal} \) parts respectively.

Having the partitions at hand we proceed by imposing certain rules for the construction of diagrams:

(1) Each matrix \((A_{ij})\) is represented by a line segment with indices \(i, j\) attached to the endpoints. The trace of \( A \) is graphically formed by gluing the two endpoints and thus resulting in a circle graph.

\[
(A_{ij}) \rightarrow \quad i \quad \longrightarrow \quad j
\]

\[
\text{tr}(A) \rightarrow \quad \bigcirc
\]

(2) For every positive integer \( k \) we associate a single circle graph with \( k \) points (we call it a \( k \)-circle from now on) and contributing a factor \(-\text{tr}(A^k)\).

(3) A partition of \( k = (1^{m_1}, 2^{m_2}, \ldots, r^{m_r}, \ldots) \) is represented by a set of circle graphs which can be constructed by a cutting and sewing procedure from the \( k \)-circle

\[
\bigcirc \quad \longrightarrow \left( \bigcirc \right)^{m_1} + \left( \bigcirc \right)^{m_2} + \cdots + \left( \bigcirc \right)^{m_r} + \cdots
\]

\( k \)-circle

where the powers on the right handside stand for the multiplicity of each graph.

(4) If \( r^{m_r} \) is a single partition of \( k \) then the coefficient is given by

\[
(-1)^{m_r}\text{tr}(A^r)\frac{((r-1)!)^{m_r}}{(m_r)!} \prod_{l=0}^{m_r-1} \binom{k-lr}{r}
\]

The interpretation of each factor in (11) is as follows:

\(\alpha\) The number of circular permutations of \( r \) distinct points on the circle is \((r-1)!\).

\(\beta\) The number of permutations of \( r \)-circles \( m_r \) times is \((m_r)!\).

\(\gamma\) The ways of extracting \( r \) points each time from a \( k \)-circle, \( m_r \) times, without replacement and disregarding order is given by the product.

Note that if we sum up the absolute values of factors (11) for all possible partitions of \( k \) then we recover \( k! \).

A more involved case study will be the partition \( k = (r^{m_r}, s^{m_s}) \). The combinatorial factor now reads

\[
(-1)^{m_r+m_s}\text{tr}(A^r)\text{tr}(A^s)\frac{((r-1)!)^{m_r}((s-1)!)^{m_s}}{(m_r)!(m_s)!} \prod_{l=0}^{m_r-1} \binom{k-lr}{r} \prod_{\rho=0}^{m_s-1} \binom{k-rm_r-\rho s}{s}
\]

As an application consider the case of \( k = 6 \). The total number of partitions is \( q(6) = 11 \) which splits into \( q_{\text{even}}(6) = 3 \) and \( q_{\text{odd}}(6) = q_{\text{uneq}}(6) = 4 \) parts. Adopting the convention \( T^k_l = (\text{tr}(A^l))^k \) a list of the contributions of all partitions of 6 is given in the following table.
Table 1: The summands of $e_6$ applying the diagrammatic approach.

| Partitions | Coefficients |
|------------|--------------|
| (6)        | $-120 T_6$   |
| (5,1)      | $144 T_5 T_1$ |
| (4,2)      | $90 T_4 T_2$ |
| (3,3)      | $40 T_3^2$   |
| (4,1,1)    | $-90 T_4 T_1^2$ |
| (3,2,1)    | $-120 T_3 T_2 T_1$ |
| (2,2,1)    | $-15 T_2^3$  |
| (3,1,1)    | $40 T_3 T_1^3$ |
| (2,1,1)    | $45 T_2^2 T_1^2$ |
| (1,1,1,1)  | $-15 T_2 T_1^4$ |

The connection of these coefficients with the elementary symmetric polynomial $e_6(\lambda)$ is

$$p_6(\lambda) = (-1)^6 e_6(\lambda) = \sum_{1 \leq i_1 < \ldots < i_6 \leq n} \lambda_{i_1} \cdots \lambda_{i_6}$$

$$= -\frac{1}{6!} \left( 120 T_6 - 144 T_5 T_1 - 90 T_4 T_2 - 40 T_3^2 + 90 T_4 T_1^2 + 120 T_3 T_2 T_1 + 15 T_2^3 
- 40 T_3 T_1^3 - 45 T_2^2 T_1^2 + 15 T_2 T_1^4 - T_1^6 \right)$$

(13)

3 Conclusions

In this letter we present explicitly a diagrammatic way to calculate the coefficients of the characteristic polynomial of a square matrix. All information is encoded in combinatorial form into the sets of circle graphs constructed for all partitions of the index associated with the corresponding elementary symmetric polynomial.

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References

[1] B. L. van Warden, *Algebra*, 5th ed. (Springer New York 1991), Vol. I, p. 99.

[2] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed. (Oxford University Press 2003), p. 276.