HECKE ALGEBRAS OF FINITE TYPE ARE CELLULAR

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Abstract. Let $H$ be the one-parameter Hecke algebra associated to a finite Weyl group $W$, defined over a ground ring in which “bad” primes for $W$ are invertible. Using deep properties of the Kazhdan–Lusztig basis of $H$ and Lusztig’s $a$-function, we show that $H$ has a natural cellular structure in the sense of Graham and Lehrer. Thus, we obtain a general theory of “Specht modules” for Hecke algebras of finite type. Previously, a general cellular structure was only known to exist in types $A_n$ and $B_n$.

1. Introduction

The concept of “cellular algebras” was introduced by Graham and Lehrer [16]. It provides a systematic framework for studying the representation theory of non-semisimple algebras which are deformations of semisimple ones. The original definition was modeled on properties of the Kazhdan–Lusztig basis in Hecke algebras of type $A_n$. There is now a significant literature on the subject, and many classes of algebras have been shown to admit a “cellular” structure, including Ariki-Koiki algebras, $q$-Schur algebras, Temperley–Lieb algebras, and a variety of other algebras with geometric connections; see, e.g., [16], [17], [3], [1] and the references there. However, the question of whether all Hecke algebras of finite type (i.e., the originally motivating examples) are cellular remained open. A positive answer to this question would provide a general theory of “Specht modules” which so far has only been established in types $A_n, B_n$ (see [2], [27], [4], [16]) and, with some restrictions on the ground ring, in type $D_n$ with $n$ odd (see [28], [20]). In these cases, the constructions heavily rely on the underlying combinatorics of Young tableaux.

The purpose of this paper is to solve this problem in general. We prove:

Theorem 1.1. Let $H$ be the one-parameter Hecke algebra associated with a finite Weyl group $W$, defined over an integral domain in which all “bad” primes for $W$ are invertible. Then $H$ admits a natural “cellular” structure, where the elements in the “cellular basis” are integral linear combinations of Kazhdan–Lusztig basis elements $C_w$ with constant value $a(w)$ (Lusztig’s $a$-function [24]).

If we apply this in type $A_n$, the linear combinations will have only one non-zero term and we recover the Kazhdan–Lusztig basis in this case. In type $B_n$, we obtain a cellular structure which is different from the one by Dipper–James–Murphy [4] or Graham–Lehrer [16]. Our construction works in the general setting of multi-parameter Hecke algebras, assuming that

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Lusztig’s conjectures on Hecke algebras with unequal parameters in [25, Chapter 14] hold. We can also formulate a similar result for Hecke algebras associated to non-crystallographic finite Coxeter groups \( W \); some additional care is needed since \( \mathbb{Q} \) is not a splitting field for such a group \( W \).

Thus, if “bad” primes are invertible in the ground ring, then all Hecke algebras of finite type are “cellular”. The general theory of cellular algebras then produces “cell representations” and a natural parametrisation of the irreducible representations for non-semisimple versions of \( \mathcal{H} \), in terms of properties of certain bilinear forms on the cell representations; see [16, §3]. We show that this parametrization is precisely given by the “canonical basic sets” introduced by Rouquier and the author [7], [15], [11].

This paper is organised as follows. In Section 2 we recall the fundamental facts from Kazhdan–Lusztig theory, including Lusztig’s construction of the “asymptotic ring” \( J \). We also establish a basic result about the representations of \( J \) in Proposition 2.6. In Section 3, we prove Theorem 1.1 and briefly discuss non-crystallographic Coxeter groups in Remark 3.3. The cellular basis that we construct turns out to be unique up to integral equivalence of the representations of \( J \). Finally, in Section 4, we consider examples and discuss the applications to non-semisimple specialisations. For types \( A_n \) and \( B_n \), we explain the relation of our cellular bases with the previously known ones.

2. KAZHDAN–LUZTIG THEORY AND THE ASYMPTOTIC RING

References for this section are the books [14] and [25]. Let \( W \) be a finite Weyl group, with generating set \( S \). Let \( L: W \to \mathbb{Z} \) be a weight function. Thus, we have \( L(ww') = L(w) + L(w') \) whenever \( l(ww') = l(w) + l(w') \) for \( w, w' \in W \); here \( l(w) \) denotes the length of \( w \in W \). Note that \( L(s) = L(t) \) whenever \( s, t \in S \) are conjugate in \( W \).

We will assume throughout that \( L(s) \geq 0 \) for all \( s \in S \).

Let \( R \subseteq \mathbb{C} \) be a subring and \( A = R[v, v^{-1}] \) be the ring of Laurent polynomials in an indeterminate \( v \). Let \( \mathcal{H} = \mathcal{H}_A(W, L) \) be the corresponding generic Iwahori–Hecke algebra. This is an associative algebra, free over \( A \) with a basis \( \{ T_w \mid w \in W \} \) and multiplication given as follows.

\[
T_s T_w = \begin{cases} 
T_{sw} & \text{if } l(sw) > l(w), \\
T_{sw} + (v^{L(s)} - v^{-L(s)}) T_w & \text{if } l(sw) < l(w),
\end{cases}
\]

where \( s \in S \) and \( w \in W \).

2.1. Irreducible representations of \( W \) and \( \mathcal{H} \). It is known that \( \mathbb{Q} \) is a splitting field for \( W \); see, for example, [14, 6.3.8]. We will write

\[
\text{Irr}(W) = \{ E^\lambda \mid \lambda \in \Lambda \}, \quad d_\lambda = \dim E^\lambda,
\]

for the set of irreducible representations of \( W \) (up to equivalence), where \( \Lambda \) is some finite indexing set.
Now let $K$ be the field of fractions of $A$. By extension of scalars, we obtain a $K$-algebra $\mathcal{H}_K = K \otimes_A \mathcal{H}$. This algebra is known to be split semisimple; see [14, 9.3.5]. (Note the form of the quadratic relations for $T_s$, $s \in S$.) Furthermore, by Tits’ Deformation Theorem, the irreducible representations of $\mathcal{H}_K$ (up to isomorphism) are in bijection with the irreducible representations of $W$; see [14, 8.1.7]. Thus, we can write

\[ \text{Irr}(\mathcal{H}_K) = \{ E^\lambda_v \mid \lambda \in \Lambda \}. \]

The correspondence $E^\lambda \leftrightarrow E^\lambda_v$ is uniquely determined by the following condition:

\[ \text{trace}(w, E^\lambda) = \text{trace}(T_w, E^\lambda_v) \bigg|_{v=1} \quad \text{for all } w \in W; \]

note that $\text{trace}(T_w, E^\lambda_v) \in A$ for all $w \in W$. See also [25, 20.2, 20.3] for a discussion of the above correspondence, but note that this relies on the validity of Lusztig’s conjectures (see §2.4 below).

2.2. The integers $a_\lambda$ and $f_\lambda$. The algebra $\mathcal{H}$ is symmetric, where $\{ T_w \mid w \in W \}$ and $\{ T_{w^{-1}} \mid w \in W \}$ form a pair of dual bases. Hence we have the following orthogonality relations for the irreducible representations of $\mathcal{H}_K$:

\[ \sum_{w \in W} \text{trace}(T_w, E^\lambda_v) \text{trace}(T_{w^{-1}}, E^\mu_v) = \begin{cases} d_\lambda c_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu; \end{cases} \]

see [14, 8.1.7]. Here, $0 \neq c_\lambda \in \mathbb{Z}[v, v^{-1}]$ and, following Lusztig, we can write

\[ c_\lambda = f_\lambda v^{-2a_\lambda} + \text{combination of strictly higher powers of } v, \]

where $a_\lambda, f_\lambda$ are integers such that $a_\lambda \geq 0$ and $f_\lambda > 0$; see [14, 9.4.7]. Thus, using $\mathcal{H}$, we have associated with each $E^\lambda \in \text{Irr}(W)$ two integers $a_\lambda$ and $f_\lambda$. These integers are explicitly known for all types of $W$; see Lusztig [25, Chap. 22].

Now let $p$ be a prime number. We say that $p$ is $L$-bad for $W$ if $p$ divides $f_\lambda$ for some $\lambda \in \Lambda$. Otherwise, $p$ is called $L$-good. If $L$ is a positive multiple of the length function, this corresponds to the familiar definition of “bad” primes; see Lusztig [23, Chap. 4]. Recall that, in this case, the conditions for being good the various irreducible types of $W$ are as follows:

- $A_n$: no condition,
- $B_n, C_n, D_n$: $p \neq 2$,
- $G_2, F_4, E_6, E_7$: $p \neq 2, 3$,
- $E_8$: $p \neq 2, 3, 5$.

See [11, Example 4.7] and [13, Def. 2.3] for a description of the $L$-bad primes when $L$ is not constant on $S$. 
2.3. The Kazhdan–Lusztig basis of $\mathcal{H}$. Let $\{c_w \mid w \in W\}$ be the “new” basis of $\mathcal{H}$ defined in [25, Theorem 5.2]. We have $c_w = T_w + \sum_y p_{y,w} T_y$ where $p_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}]$ and $p_{y,w} = 0$ unless $y < w$ in the Bruhat–Chevalley order. Given $x, y \in W$, we write

$$c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$$

where $h_{x,y,z} \in A$.

As in [25], we usually work with the elements $c_w^\dagger$ obtained by applying the unique $A$-algebra involution $H \rightarrow H, h \mapsto h^\dagger$ such that $T_s^\dagger = -T_s^{-1}$ for any $s \in S$; see [25, 3.5]. We refer to [25, Chap. 8] for the definition of the preorders $\leq_L, \leq_R, \leq_{LR}$ and the corresponding equivalence relations $\sim_L, \sim_R, \sim_{LR}$ on $W$, induced by $L$. The equivalence classes with respect to these relations are called left, right and two-sided cells of $W$, respectively. If $L(s) = 1$ for all $s \in S$, then $C_w = (-1)^{l(w)} c_w^\dagger$ where $\{C_w\}$ is the basis originally constructed in [22].

Let $z \in W$. Following Lusztig [25, 13.6], we define $a(z) \in \mathbb{Z}_{\geq 0}$ by the condition that

$$v^{a(z)} h_{x,y,z} \in \mathbb{Z}[v] \quad \text{for all } x, y \in W,$$

$$v^{a(z)-1} h_{x,y,z} \not\in \mathbb{Z}[v] \quad \text{for some } x, y \in W.$$

Furthermore, if $p_{1,z} \neq 0$, we define $\Delta(z) \in \mathbb{Z}_{\geq 0}$ and $n_z = 0$ if $n_z \in \mathbb{Z}$ by the condition that

$$p_{1,z} = n_z v^{-\Delta(z)} + \text{strictly smaller powers of } v; \quad \text{see [25, 14.1].}$$

Otherwise, we set $\Delta(z) = \infty$ and leave $n_z$ undefined. (This case can only occur when $L(s) = 0$ for some $s \in S$.) We set

$$D = \{z \in W \mid a(z) = \Delta(z)\}.$$

(In what follows, the coefficients $n_z$ will only play a role when $z \in D$.)

2.4. Lusztig’s conjectures and the asymptotic ring $J$. In the sequel, we assume that the following hypotheses are satisfied.

**Lusztig’s conjectures (P1)–(P15) in [25, 14.2] hold for $\mathcal{H}$.**

(Actually, for our purposes, instead of (P15) it is enough to require the somewhat weaker statement [25, 18.9(b)], which we called (P15’) in [11, §5].) By [25, Chap. 15], these conjectures are known to hold when $L$ is a positive multiple of the length function (“equal parameter case”), thanks to a deep geometric interpretation of the basis $\{c_w\}$. They are also known for a certain class of non-trivial weight functions in type $B_n$; see [12], [10].

Assuming the above hypotheses, we can perform the following constructions. Following Lusztig [25, Chap. 18], let $J$ be a free $\mathbb{Z}$-module with basis $\{t_w \mid w \in W\}$. We define a multiplication on $J$ by

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z}^{-1} t_z,$$

where $\gamma_{x,y,z}^{-1} \in \mathbb{Z}$ is the constant term of $v^{a(z)} h_{x,y,z} \in \mathbb{Z}[v]$. Then it turns out that this multiplication is associative and we have an identity element given by $1_J = \sum_{d \in D} n_d t_d$. 
Furthermore, we have a homomorphism of $A$-algebras $\phi: \mathcal{H} \to J_A = A \otimes_{\mathbb{Z}} J$ such that

$$\phi(c^i_w) = \sum_{z \in W, d \in D, a(z) = a(d)} h_{w, d, z} n_z t_z \quad (w \in W),$$

where $n_z$ is defined as follows. Given $z \in W$, let $d$ be the unique element of $D$ such that $d \sim_L z^{-1}$; then $n_z = n_d = \pm 1$. (See properties (P5), (P13) in Lusztig’s conjectures [25, 14.2].) Note that the function $z \mapsto n_z$ is constant on the right cells of $W$.

2.5. **Representations of $J$.** Until the end of this section, we will assume that $R = \mathbb{Q}$. Upon substituting $v \mapsto 1 \in \mathbb{Q}$, the algebra $\mathcal{H}$ specialises to $\mathbb{Q}[W]$. Hence, we obtain a homomorphism of $\mathbb{Q}$-algebras

$$\phi_1: \mathbb{Q}[W] \to J_Q, \quad \text{where } J_Q = \mathbb{Q} \otimes_{\mathbb{Z}} J.$$  

This is an isomorphism by the argument in [25, 20.1]. Since $\mathbb{Q}[W]$ is split semisimple, we can conclude that $J_Q$ also is split semisimple. Via the isomorphism $\phi_1: \mathbb{Q}[W] \to J_Q$, the set $\Lambda$ can be used to parametrize the irreducible representations of $J_Q$ (up to isomorphism). As in [25, 20.2], we write

$$\text{Irr}(J_Q) = \{ E_\lambda^\Lambda \mid \lambda \in \Lambda \},$$

where $E_\lambda^\Lambda$ coincides with $E^\Lambda$ as $\mathbb{Q}$-vector space and the action of $j \in J_Q$ on $E_\lambda^\Lambda$ is the same as the action of $\phi_1^{-1}(j)$ on $E^\Lambda$. Now, by [25, 20.1(b)], the algebra $J$ is symmetric where \{ $t_w \mid w \in W$ \} and \{ $t_{w^{-1}} \mid w \in W$ \} form a pair of dual bases. Thus, we have the following orthogonality relations:

$$\sum_{w \in W} \text{trace}(t_w, E_\lambda^\Lambda) \text{trace}(t_{w^{-1}}, E_\mu^\Lambda) = \begin{cases} d_\lambda f_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

(The fact that $f_\lambda$ appears on the right hand side is shown in [25, 20.11].) Choosing a vector space basis of $E^\Lambda$, we obtain a matrix representation

$$\rho^\lambda: J_Q \to M_{d_\lambda}(\mathbb{Q}), \quad t_w \mapsto \left( \rho^\lambda_{st}(t_w) \right)_{s, t \in M(\lambda)};$$

where $M(\lambda) = \{1, \ldots, d_\lambda\}$. Then the following *Schur relations* hold; see [14, §7.2]. For $\lambda, \mu \in \Lambda$, $s, t \in M(\lambda)$ and $u, v \in M(\mu)$, we have:

$$\sum_{w \in W} \rho^\lambda_{st}(t_w) \rho^\mu_{uv}(t_{w^{-1}}) = \begin{cases} f_\lambda & \text{if } (\lambda, s, t) = (\mu, v, u), \\ 0 & \text{otherwise}. \end{cases}$$

These equations can be inverted and this yields the “second” Schur relations:

$$\sum_{\lambda \in \Lambda} \sum_{s, t \in M(\lambda)} \frac{1}{f_\lambda} \rho^\lambda_{st}(t_x) \rho^\lambda_{tu}(t_{y^{-1}}) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise}. \end{cases}$$

By a general argument, every irreducible representation of $J_Q$ leaves a positive-definite quadratic form invariant. More precisely, we have the following result.

**Proposition 2.6.** Let $\lambda \in \Lambda$. Then a basis of $E^\Lambda$ can be chosen such that
(a) $\rho^\lambda_{st}(t_w) \in \mathbb{Z}$ for all $w \in W$ and $s, t \in M(\lambda)$.

Furthermore, there exists a symmetric, positive-definite matrix

$$B^\lambda = (\beta^\lambda_{st})_{s, t \in M(\lambda)} \quad \text{where} \quad \beta^\lambda_{st} \in \mathbb{Z} \quad \text{for all} \quad s, t \in M_d,$$

such that the following two conditions hold:

(b) $B^\lambda \cdot \rho^\lambda(t_{w^{-1}}) = \rho^\lambda(t_w)^{tr} \cdot B^\lambda$ for all $w \in W$;

(c) Any prime which divides $\det(B^\lambda) \neq 0$ is $L$-bad for $W$.

(For any matrix $M$, we denote by $M^{tr}$ the transpose of $M$.)

Proof. To simplify the notation, write $E = E^\lambda$, $d = d_\lambda$ and $\rho = \rho^\lambda$ so that we can omit a subscript or superscript $\lambda$ in the subsequent formulas. Since $J$ is defined over $\mathbb{Z}$, the statement in (a) follows by a general argument; see, for example, [14, 7.3.7]. Now let

$$B_1 = \sum_{y \in W} \rho(t_y)^{tr} \cdot \rho(t_y) \in M_d(\mathbb{Z}).$$

This matrix clearly is symmetric. Now let $0 \neq e = (e_1, \ldots, e_d) \in \mathbb{Z}^d$. Since $\rho$ is irreducible, there exists some $y \in W$ such that $e \rho(t_y)^{tr} \neq 0$ and, hence, the standard scalar product of this vector with itself will be strictly positive. Consequently, we have $e B_1 e^{tr} > 0$. Thus, $B_1$ is positive-definite and, in particular, $\det(B_1) \neq 0$. For any $x \in W$, we have

$$B_1 \cdot \rho(t_{x^{-1}}) = \sum_{y \in W} \rho(t_y)^{tr} \cdot \rho(t_y t_{x^{-1}}) = \sum_{y, z \in W} \gamma_{y, x^{-1}, z^{-1}} \rho(t_y)^{tr} \cdot \rho(t_z).$$

Now $\gamma_{y, x^{-1}, z^{-1}} = \gamma_{x^{-1}, z^{-1}, y}$ by (P7) in [25, 14.2], and $\gamma_{x^{-1}, z^{-1}, y} = \gamma_{z, x, y^{-1}}$ by [25 Prop. 13.9]. Hence, the right hand side of the above identity equals

$$\sum_{y, z \in W} \gamma_{z, x, y^{-1}} \rho(t_y)^{tr} \cdot \rho(t_z) = \sum_{z \in W} \rho(t_z t_x)^{tr} \cdot \rho(t_z) = \rho(t_x)^{tr} \cdot B_1.$$

Let $0 \neq n \in \mathbb{Z}$ be the greatest common divisor of all non-zero coefficients of $B_1$ and $B = n^{-1} B_1 \in M_d(\mathbb{Z})$. Then $B$ is a symmetric, positive-definite matrix such that (b) holds. It remains to prove (c). Let $p$ be a prime number and denote by $\bar{B}$ the matrix obtained by reducing all coefficients modulo $p$. By reduction modulo $p$, we also obtain an $\mathbb{F}_p$-algebra $J_p = \mathbb{F}_p \otimes_{\mathbb{Z}} J$ and a corresponding matrix representation $\bar{\rho}: J_p \to M_d(\mathbb{F}_p)$. Hence we have

$$\bar{B} \neq 0 \quad \text{and} \quad \bar{B} \cdot \bar{\rho}(t_w) = \bar{\rho}(t_w)^{tr} \cdot \bar{B} \quad \text{for all} \quad w \in W.$$

By [25, 13.9], the map $t_w \mapsto t_{w^{-1}}$ defines an involutory anti-automorphism of $J$. Hence the assignment $t_w \mapsto \bar{\rho}(t_w)^{tr}$ also defines a representation of $J_p$. The above identity now shows that $\bar{B} \neq 0$ is an “intertwining operator”. Hence, if we knew that $\bar{\rho}$ was irreducible, then Schur’s Lemma would imply that $\bar{B}$ were invertible and so $p$ could not divide $\det(B)$.

Thus, it remains to show that $\bar{\rho}$ is an irreducible representation of $J_p$ whenever $p$ is $L$-good. But this follows from a general argument about symmetric algebras. Indeed, as already noted
in §2.5, $J$ is symmetric and we have the Schur relations for the matrix coefficients of $\rho = \rho^\lambda$. Reducing these relations modulo $p$, we obtain:

$$\sum_{w \in W} \rho^\lambda_{st}(t_w) \rho^\lambda_{uw}(t_{w^{-1}}) = \begin{cases} f^\lambda \mod p & \text{if } s = v, u = t, \\ 0 & \text{otherwise}. \end{cases}$$

Since $f^\lambda \not\equiv 0 \mod p$, one easily deduces from this that $\bar{\rho}^\lambda$ is (absolutely) irreducible; see [14, Remark 7.2.3].

3. A cell datum for $\mathcal{H}$

We keep the notation of the previous sections. In order to show that $\mathcal{H}$ is “cellular” in the sense of Graham–Lehrer [16, Definition 1.1], we must specify a quadruple $(\Lambda, M, C, *)$ satisfying the following conditions.

(C1) $\Lambda$ is a partially ordered set, $\{M(\lambda) \mid \lambda \in \Lambda\}$ is a collection of finite sets and $C : \prod_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \to \mathcal{H}$ is an injective map whose image is an $A$-basis of $\mathcal{H}$;

(C2) If $\lambda \in \Lambda$ and $s, t \in M(\lambda)$, write $C^\lambda_{s,t} = C_{s,t}^\lambda \in \mathcal{H}$. Then $* : \mathcal{H} \to \mathcal{H}$ is an $A$-linear anti-involution such that $(C^\lambda_{s,t})^* = C^\lambda_{t,s}$.

(C3) If $\lambda \in \Lambda$ and $s, t \in M(\lambda)$, then for any element $h \in \mathcal{H}$ we have

$$hC^\lambda_{s,t} \equiv \sum_{s' \in M(\lambda)} r_h(s', s) C^\lambda_{s',t} \mod \mathcal{H}(<\lambda),$$

where $r_h(s', s) \in A$ is independent of $t$ and where $\mathcal{H}(<\lambda)$ is the $A$-submodule of $\mathcal{H}_n$ generated by $\{C^\mu_{s',t} \mid \mu < \lambda; s', t' \in M(\mu)\}$.

We now define a required quadruple $(\Lambda, M, C, *)$ as follows.

Let $\Lambda$ be an indexing set for the irreducible representations of $W$, as in §2.1. Using the $a$-invariants in §2.2, we define a partial order $\preceq$ on $\Lambda$ by

$$\lambda \preceq \mu \quad \iff \quad \lambda = \mu \quad \text{or} \quad a_\lambda > a_\mu.$$

Thus, $\Lambda$ is ordered according to decreasing $a$-value. Next, we define an $A$-linear anti-involution $* : \mathcal{H} \to \mathcal{H}$ by $T_w^* = T_{w^{-1}}$ for all $w \in W$. Thus, $T_w^* = T_w^0$ in the notation of [25, 3.4].

For $\lambda \in \Lambda$, we set $M(\lambda) = \{1, \ldots, d_\lambda\}$ as before. The trickiest part is, of course, the definition of the basis elements $C^\lambda_{s,t}$ for $s, t \in M(\lambda)$. We can now state the main result of this paper.

**Theorem 3.1.** Recall that Lusztig’s conjectures (P1)–(P15) are assumed to hold for $\mathcal{H}$. Let $R \subseteq \mathbb{C}$ be a subring such that all $L$-bad primes are invertible in $R$. Let $(\rho^\lambda_{st}(t_w))$ and
\((\beta_{w}^{\lambda})\) be as in Proposition 2.6. For any \(\lambda \in \Lambda\) and \(s, t \in M(\lambda)\), define
\[
C_{s,t}^{\lambda} = \sum_{w \in W} \sum_{u \in M(\lambda)} \hat{n}_w \hat{n}_{w^{-1}} \beta_{wu}^{\lambda} \rho_{us}^{\lambda} (t_{w^{-1}}) c_{w}^{t}.
\]

Then \(C_{s,t}^{\lambda}\) is a \(\mathbb{Z}\)-linear combination of Kazhdan–Lusztig basis elements \(c_{w}^{t}\) where \(a(w) = a_{\lambda}\). The quadruple \((\Lambda, M, C, \ast)\) is a “cell datum” in the sense of Graham–Lehrer [16].

Proof. First note that, by [25, Prop. 20.6], we have \(\rho^{\lambda}(t_{w^{-1}}) = 0\) unless \(a(w) = a_{\lambda}\). Thus, \(C_{s,t}^{\lambda}\) is an integral linear combination of elements \(c_{w}^{t}\) where \(a(w) = a_{\lambda}\). In what follows, it will be convenient to write the coefficients occurring in various sums as entries of matrices. Thus, for example, the defining formula for \(C_{s,t}^{\lambda}\) reads:
\[
C_{s,t}^{\lambda} = \sum_{w \in W} \hat{n}_w \hat{n}_{w^{-1}} \left( B^{\lambda} \cdot \rho^{\lambda}(t_{w^{-1}}) \right)_{t,s} c_{w}^{t}
\]
\[
= \sum_{w \in W} \hat{n}_w \hat{n}_{w^{-1}} \left( \rho^{\lambda}(t_{w})^{tr} \cdot B^{\lambda} \right)_{t,s} c_{w}^{t},
\]
where the second equality holds by Proposition 2.6. We now proceed in three steps.

Step 1. (C1) holds, that is, the elements \(\{C_{s,t}^{\lambda} \mid \lambda \in \Lambda \text{ and } s, t \in M(\lambda)\}\) form a basis of \(H\). This is proved as follows. By Wedderburn’s Theorem, \(\dim H_k = |W| = \sum_{\lambda \in \Lambda} |M(\lambda)|^2\). Hence the above set has the correct cardinality. It is now sufficient to show that the elements \(\{C_{s,t}^{\lambda}\}\) span \(H\) as an \(A\)-module.

Let us fix \(y \in W\). We consider the following \(R\)-linear combination:
\[
\sum_{\lambda \in \Lambda} \sum_{s, t \in M(\lambda)} \frac{1}{f_{\lambda}} \left( \rho^{\lambda}(t_{y}) \cdot (B^{\lambda})^{-1} \right)_{s,t} C_{s,t}^{\lambda}.
\]
Note that the coefficients lie in \(R\) since \(f_{\lambda}\) and \(\det(B^{\lambda})\) are invertible in \(R\). Inserting the first of the above-mentioned two expressions for \(C_{s,t}^{\lambda}\), we obtain:
\[
\sum_{w \in W} \hat{n}_w \hat{n}_{w^{-1}} \sum_{\lambda \in \Lambda} \sum_{s, t \in M(\lambda)} \frac{1}{f_{\lambda}} \left( \rho^{\lambda}(t_{y}) \cdot \rho^{\lambda}(t_{w^{-1}}) \right)_{s,s} c_{w}^{t}.
\]
Now, writing out the product \(\rho^{\lambda}(t_{y}) \cdot \rho^{\lambda}(t_{w^{-1}})\) and using the “second” Schur relations, we find that
\[
\sum_{\lambda \in \Lambda} \sum_{s, t \in M(\lambda)} \frac{1}{f_{\lambda}} \left( \rho^{\lambda}(t_{y}) \cdot \rho^{\lambda}(t_{w^{-1}}) \right)_{s,s} = \delta_{yw}.
\]
Hence our linear combination reduces to \(\hat{n}_y \hat{n}_{y^{-1}} c_{y}^{t} = \pm c_{y}^{t}\). Thus, \(c_{y}^{t}\) is an \(R\)-linear combination of the elements \(C_{s,t}^{\lambda}\), as required.

Step 2. (C2) holds, that is, we have \((C_{s,t}^{\lambda})^{\ast} = C_{t,s}^{\lambda}\) for all \(\lambda \in \Lambda\) and \(s, t \in M(\lambda)\). This is seen as follows. By [25, 4.9 and 5.6], we have \((c_{w}^{t})^{\ast} = (c_{w}^{s})^{\dagger} = c_{w^{-1}}^{t}\). Thus, using the above
two expressions for \( C_{s,t}^\lambda \), we obtain:

\[
(C_{s,t}^\lambda)^* = \sum_{w \in W} \hat{n}_w \hat{n}_{w^{-1}} \left( B^\lambda \cdot \rho^\lambda(t_{w^{-1}}) \right)_{t,s} c_{w^{-1}}^f
\]

\[
= \sum_{w \in W} \hat{n}_w \hat{n}_{w^{-1}} \left( \rho^\lambda(t_w)\cdot B^\lambda \right)_{t,s} c_{w^{-1}}^f
\]

\[
= \sum_{w \in W} \hat{n}_w \hat{n}_{w^{-1}} \left( B^\lambda \cdot \rho^\lambda(t_w) \right)_{s,t} c_{w^{-1}}^f = C_{s,t}^\lambda
\]

as required.

**Step 3.** Finally, we consider the multiplication rule (C3). By Lusztig [25, 18.10], there is a natural left \( J_A \)-module structure on \( \mathcal{H} \) given by the formula

\[
t_x * c^f_w = \sum_{z \in W} \gamma_{x,w,z^{-1}} \hat{n}_w \hat{n}_z c^f_z \quad (x, w \in W).
\]

We begin by studying the effect of the \( J_A \)-action on the element \( C_{s,t}^\lambda \). Let \( x \in W \). We claim that

\[
t_x * C_{s,t}^\lambda = \sum_{s' \in M(\lambda)} \rho^\lambda_{s's}(t_x) C_{s',t}^\lambda.
\]

Indeed, recalling the defining formula for \( C_{s,t}^\lambda \), we have:

\[
t_x * C_{s,t}^\lambda = \sum_{u \in W} \sum_{w \in M(\lambda)} \hat{n}_w \hat{n}_{w^{-1}} \beta^\lambda_{t u} \rho^\lambda_{u s}(t_{w^{-1}}) t_x * c^f_w
\]

\[
= \sum_{w, z \in W} \sum_{u \in M(\lambda)} \hat{n}_z \hat{n}_{w^{-1}} \beta^\lambda_{t u} \rho^\lambda_{u s}(t_{w^{-1}}) \gamma_{x,w,z^{-1}} c^f_z.
\]

Assume that the term in the sum corresponding to \( w, z \in W \) is non-zero. Then \( \gamma_{x,w,z^{-1}} \neq 0 \) and so \( w \sim_L z \), by property (P8) in [25, 14.2]. Hence \( w^{-1} \sim_R z^{-1} \) and so \( \hat{n}_w = \hat{n}_{z^{-1}} \). Furthermore, \( \gamma_{x,w,z^{-1}} = \gamma_{z^{-1},x,w} \) by property (P7) in [25, 14.2]. Thus, we obtain

\[
t_x * C_{s,t}^\lambda = \sum_{z \in W} \sum_{u \in M(\lambda)} \hat{n}_z \hat{n}_{z^{-1}} \beta^\lambda_{t u} \rho^\lambda_{u s}(t_{z^{-1},x,w} t_{w^{-1}}) c^f_z
\]

\[
= \sum_{z \in W} \sum_{u \in M(\lambda)} \hat{n}_z \hat{n}_{z^{-1}} \beta^\lambda_{t u} \rho^\lambda_{u s}(t_{z^{-1},t_x}) c^f_z
\]

\[
= \sum_{z \in W} \hat{n}_z \hat{n}_{z^{-1}} \left( B^\lambda \cdot \rho^\lambda(t_{z^{-1}}) \right)_{t,s} c^f_z
\]

\[
= \sum_{s' \in M(\lambda)} \rho^\lambda_{s's}(t_x) C_{s',t}^\lambda.
\]
as claimed. Now let \( h \in \mathcal{H} \) and write \( \phi(h) = \sum_{x \in W} a_h(x) t_x \) where \( a_h(x) \in A \). Then define
\[
\rho_h(s', s) = \sum_{x \in W} a_h(x) \rho^\lambda_{s',s}(t_x) \quad \text{for any } s', s \in M(\lambda).
\]
Note that, indeed, this coefficient lies in \( A \) and it only depends on \( s, s' \) and \( h \). Then the above computation shows that
\[
\phi(h) \ast C^\lambda_{s,1} = \sum_{s' \in M(\lambda)} r_h(s', s) C^\lambda_{s',1}.
\]
For any \( a \geq 0 \), we define \( \mathcal{H}^{\geq a} \) to be the \( A \)-span of all elements \( c^\dagger_y \) where \( y \in W \) is such that \( a(y) \geq a \). By [25, 18.10(a)], we have
\[
h c^\dagger_w = \phi(h) \ast c^\dagger_w \mod \mathcal{H}^{\geq a(w) + 1} \quad \text{for any } h \in \mathcal{H} \text{ and } w \in W.
\]
We have already noted in the beginning of the proof that \( C^\lambda_{s,1} \) is a linear combination of elements \( c^\dagger_x \) where \( a(z) = a_\lambda \). Hence the above relations imply that
\[
h C^\lambda_{s,1} \equiv \phi(h) \ast C^\lambda_{s,1} = \sum_{s' \in M(\lambda)} r_h(s', s) C^\lambda_{s',1} \mod \mathcal{H}^{\geq a_\lambda + 1}.
\]
The definition of the partial order \( \preceq \) on \( \Lambda \) now shows that (C3) holds.

**Corollary 3.2.** Let \( \theta : A \to k \) be a ring homomorphism into an integral domain \( k \) (i.e., a “specialisation”). By extension of scalars, we obtain a \( k \)-algebra \( \mathcal{H}_k = k \otimes_A \mathcal{H} \). Then the above ingredients define a “cell datum” for \( \mathcal{H}_k \), where
\[
C^\lambda_{s,1} = \sum_{w \in W} \sum_{\omega \in M(\lambda)} \hat{n}_w n_{w^{-1}} \theta(\beta^\lambda_{w,\omega}) \theta(\rho^\lambda_{w,\omega}(t_{w^{-1}})) (1 \otimes c^\dagger_w) \in \mathcal{H}_k,
\]
for \( \lambda \in \Lambda \) and \( s, t \in M(\lambda) \).

**Proof.** This immediately follows from Theorem \[3.1\] see the remarks in [16, (1.8)].

Thus, since Lusztig’s conjectures (P1)–(P15) hold in the “equal parameter” case, we have proved Theorem \[1.1\] as stated in the introduction.

**Remark 3.3.** Assume that \( W \) is a finite Coxeter group which is not a Weyl group, i.e., \( W \) is of type \( H_3, H_4 \) or \( I_2(m) \) where \( m = 5 \) or \( m \geq 7 \). In these cases, \( Q \) is no longer a splitting field for \( W \), and this leads to some technical complications. However, choosing \( R \subseteq \mathbb{C} \) appropriately, Theorem \[3.1\] remains valid in these cases as well.

First note that Lusztig’s properties (P15)–(P15) (see §2.4) are known to hold for types \( H_3, H_4 \) and \( I_2(m) \) (where \( L \) is a positive multiple of the length function), thanks to DuCloux [6]. (As far as we know, these properties do not seem to have been verified in type \( I_2(m) \) with unequal parameters.)

Now, the irreducible representations in type \( H_4, H_4 \) and \( I_2(m) \) (and much further information) are explicitly known; see [14, §8.1 and §11.2]. Using this explicit information, one can show that Theorem \[3.1\] holds if we require that the subring \( R \subseteq \mathbb{C} \) satisfies the following
two conditions: (1) $\mathcal{R}$ contains $\zeta + \zeta^{-1}$ where $\zeta$ is a root of unity of order 5 (in type $H_3$, $H_4$) or $m$ (in type $I_2(m)$); (2) the order of $W$ is invertible in $R$.

A more detailed discussion of these cases will appear elsewhere.

Jeong et al. [21] point out that Fakiolas [5] also produces a cellular basis in type $I_2(m)$ where $L$ is a positive multiple of the length function.

4. Examples and applications to modular representations

Throughout this section, we assume that Lusztig’s conjectures on Hecke algebras with unequal parameters hold; see [24]. (Recall that this is the case, for example, if the weight function $L$ is a positive multiple of the length function.) We now discuss examples and applications of Theorem 3.1.

Example 4.1. Assume that $L(s) > 0$ for all $s \in S$ and that $\lambda \in \Lambda$ is such that $\dim E^\lambda = d_\lambda = 1$. Then there exists a group homomorphism $\eta: W \to \{\pm 1\}$ and a weight function $m: W \to \mathbb{Z}$ such that $T_w$ acts on $E^\lambda_w$ via the $A$-algebra homomorphism

$$\mu^\lambda: \mathcal{H} \to A, \quad T_w \mapsto \eta(w)v^m(w).$$

By [25] Prop. 20.6], the corresponding representation of $J$ is given by

$$t_w \mapsto \begin{cases} (-1)^{l(w)} \hat{n}_w \hat{n}_{w^{-1}} \eta(w) & \text{if } a_\lambda + m(w) = 0, \\ 0 & \text{otherwise}. \end{cases}$$

(Note that a factor $\hat{n}_w \hat{n}_{w^{-1}}$ should be inserted into the formula in [25] 20.6.) Clearly, we can take $B^\lambda = (1)$ in this case. Hence, the unique element of the cellular basis corresponding to $\lambda$ is given by

$$C_{1,1}^\lambda = \sum_{\substack{w \in W \\ a_\lambda + m(w) = 0}} (-1)^{l(w)} \eta(w) c^w_{1,1}.$$ 

For example, if $E^\lambda$ is the unit representation, we have $\eta(w) = 1$ and $m(w) = L(w)$ for all $w \in W$; furthermore, $a_\lambda = 0$ and so $C_{1,1}^\lambda = c^w_{1,1}$ (since $L(w) > 0$ for all $w \neq 1$).

If $E^\lambda$ is the sign representation, we have $\eta(w) = (-1)^{l(w)}$ and $m(w) = -L(w)$ for all $w \in W$. Furthermore, $a_\lambda = L(w_0)$ where $w_0 \in W$ is the longest element; see [25] 20.18. Since $L(w) < L(w_0)$ for all $w \neq w_0$, we obtain $C_{1,1}^\lambda = c^w_{w_0}$ in this case.

Example 4.2. Assume that there are no $L$-bad primes for $W$, that is, we have $f_\lambda = 1$ for all $\lambda \in \Lambda$. Then we claim that there are signs $\delta_w = \pm 1$ ($w \in W$) such that

$$\{\delta_w c^w_{1,1} \mid w \in W\}$$

is a cellular basis.

This is seen as follows. Let $\Gamma$ be a left cell of $W$ and $[\Gamma] = \langle t_x \mid x \in \Gamma \rangle_Q \subseteq J_Q$. Then $[\Gamma]$ is a simple left ideal of $J_Q$; see [25] Chap. 21 and [8] Cor. 4.8. Hence $[\Gamma]$ affords an irreducible representation of $J_Q$, and all irreducible representations arise in this way. So, for any $\lambda \in \Lambda$, we can choose a left cell $\Gamma^\lambda$ such that $[\Gamma^\lambda] \cong E^\lambda$. Let us write

$$\Gamma^\lambda = \{s \in M(\lambda)\}$$

for $\lambda \in \Lambda$. 

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Then \( \{t_s \mid s \in M(\lambda)\} \) is a basis of \( [\Gamma^\lambda] \) and the corresponding matrix coefficients are

\[
\rho_{st}(t_w) = \gamma_{s,x_1,x_s^{-1}} \quad \text{for } w \in W \text{ and } s, t \in M(\lambda).
\]

By (P7) in [25, 14.2] and [25 Prop. 13.9], we have

\[
\gamma_{s,x_1,x_s^{-1}} = \gamma_{x_1^{-1},w^{-1},x_s} = \gamma_{w^{-1},x_s,x_1^{-1}}.
\]

This implies that \( \rho^\lambda(t_{w^{-1}}) = \rho^\lambda(t_w)^{\mu} \) for all \( w \in W \). Thus, if we take for \( B^\lambda \) the identity matrix of size \( d_\lambda \), then the conditions in Proposition [2.6] are satisfied. Furthermore, for fixed \( s, t \in M(\lambda) \), the Schur relations now read:

\[
\sum_{y \in W} (\gamma_{y,x_1,x_s^{-1}})^2 = \sum_{y \in W} \rho_{st}(t_y) \rho_{ts}(t_{y^{-1}}) = f_\lambda = 1.
\]

We deduce that there is a unique \( w = w_\lambda(s,t) \in W \) such that \( \gamma_{w,x_1,x_s^{-1}} \neq 0 \); in fact, we have \( \gamma_{w,x_1,x_s^{-1}} = \pm 1 \). Now the formula in Theorem [3.1] reads

\[
C_{s,t}^\lambda = \sum_{y \in W} \hat{n}_y \hat{n}_{y^{-1}} \gamma_{y^{-1},x_1,x_s^{-1}} c^\dagger_{y} = \sum_{y \in W} \hat{n}_y \hat{n}_{y^{-1}} \gamma_{w,x_1,x_s^{-1}} c^\dagger_{w} = \delta_w c^\dagger_{w},
\]

where \( w = w_\lambda(s,t) \) and \( \delta_w = \pm 1 \). Thus, \( \{\delta_w c^\dagger_{w} \mid w \in W\} \) is a cellular basis, as claimed.

The above assumptions are satisfied for \( W \) of type \( A_\infty \) where \( L \) is a positive multiple of the length function; see [25, 22.4]. Thanks to a geometric interpretation of the basis \( \{c^\dagger_w\} \), we know that \( \hat{n}_{x,y,z} \in \mathbb{Z}_{\geq 0}[v,v^{-1}] \) for all \( x, y, z \in W \); see [25 Chap. 15] and the references there. Hence we also have \( \hat{n}_x \geq 0 \) and \( \gamma_{x,y,z} \geq 0 \) for all \( x, y, z \in W \). So, in this case, \( \{c^\dagger_w \mid w \in W\} \) is a cellular basis of \( H \), as originally pointed out by Graham–Lehrer [16 Example 1.2]. Note that another cellular structure in type \( A_\infty \) was constructed by Murphy [26, 27], using purely combinatorial methods. The exact relation between the two cellular structures is determined in [9].

**Example 4.3.** Let \( W \) be the Weyl group of type \( B_2 \) where \( S = \{s_1, s_2\} \) and \((s_1 s_2)^4 = 1\). Let \( L: W \rightarrow \mathbb{Z} \) be any weight function such that \( L(s_i) > 0 \) for \( i = 1, 2 \). If \( L(s_1) \neq L(s_2) \), then there are no \( L \)-bad primes for \( W \). Furthermore, by [12, 10], Lusztig’s conjectures hold in this case. (More generally, they hold for a certain class of weight functions in type \( B_n \) for any \( n \geq 2 \).) Hence, in this case, \( \{\delta_w c^\dagger_w \mid w \in W\} \) is a cellular basis by Example [4.2]. By [10 Cor. 6.4], we actually have \( \delta_w = 1 \) for all \( w \in W \).

Now assume that \( L(s_1) = L(s_2) = 1 \) (“equal parameter case”). Then \( 2 \) is the only bad prime. We have \( \text{Irr}(W) = \{1, \varepsilon_1, \varepsilon_2, \varepsilon, r\} \) where \( 1 \) is the unit representation, \( \varepsilon \) is the sign representation, \( \varepsilon_1, \varepsilon_2 \) have dimension one, and \( r \) has dimension two. The \( a \)-invariants are
a_1 = 0, a_{c_1} = a_{c_2} = a_r = 1 and a_c = 4. A cellular basis as in Theorem 3.1 is given as follows:

\[ C_{1,1}^1 = c_1^+, \quad C_{1,1}^r = c_{s_1}^+ + c_{s_1 s_2 s_1}^+, \]
\[ C_{1,1}^{c_1} = c_{w_0}^+, \quad C_{1,1}^{c_2} = -2c_{s_1 s_2}^+, \]
\[ C_{1,1}^{c_1} = c_{s_2}^+ - c_{s_2 s_1 s_2}^+, \quad C_{1,1}^{c_2} = 2c_{s_2}^+ + 2c_{s_2 s_1 s_2}^+. \]

Indeed, by Example 4.1 we already know that \( c_1^+ \) and \( c_{w_0}^+ \) belong to the cellular basis. Now consider the two further one-dimensional representations. We fix the notation such that \( \varepsilon_1(T_{s_1}) = \varepsilon_2(T_{s_2}) = v \) and \( \varepsilon_1(T_{s_1}) = \varepsilon_2(T_{s_2}) = -v^{-1} \). Using the formula in Example 4.1 we obtain the above expressions for \( C_{1,1}^1 \) and \( C_{1,1}^{c_2} \). Finally, consider the two-dimensional representation \( r \). We have to determine a corresponding representation of \( J \) such that the conditions in Proposition 2.6 hold. Now, the basis elements \( \{c_w\} \) are explicitly determined in \([25, \text{Prop. 7.3}]\). From this, one easily deduces the left cells. They are given by

\[ \{1\}, \ \{s_1, s_2 s_1, s_1 s_2 s_1\}, \ \{s_2, s_1 s_2, s_2 s_1 s_2\}, \ \{w_0 = s_1 s_2 s_1 s_2\}; \]

see \([25, \text{8.7}]\). Furthermore, one finds the following relations in \( J \). We have \( t_1^2 = t_1 \) and \( t_1 t_x = 0 \) for \( x \neq 1 \); we have \( t_{w_0}^2 = t_{w_0} \) and \( t_{w_0} t_x = 0 \) for \( x \neq w_0 \). Furthermore,

\[ t_{s_1}^2 = t_{s_1}, \quad t_{s_1} t_{s_2} = t_{s_1 s_2}, \quad t_{s_1} t_{s_1 s_2 s_1} = t_{s_1 s_2 s_1}, \]
\[ t_{s_2}^2 = t_{s_2}, \quad t_{s_2} t_{s_2 s_1} = t_{s_2 s_1}, \quad t_{s_2} t_{s_2 s_1 s_2} = t_{s_2 s_1 s_2}, \]
\[ t_{s_1 s_2} t_{s_1 s_2 s_1} = t_{s_1} s_2, \quad t_{s_1 s_2} t_{s_2 s_1} = t_{s_1} s_2 s_1, \]
\[ t_{s_2 s_1} t_{s_1 s_2 s_1} = t_{s_2} s_1, \quad t_{s_2 s_1} t_{s_2 s_1 s_2} = t_{s_2 s_1 s_2}. \]

All other products \( t_x t_y \) are either zero or can be deduced from the above list using the formulas \( \gamma_{x,y,z} = \gamma_{y^{-1} x^{-1} z^{-1}} \) (see \([25, \text{Prop. 13.9}]\)) and \( \gamma_{x,y,z} = \gamma_{y,z,x} \) (see \([P7]\) in \([25, \text{14.2}]\)). Note also that, by \([P8]\) in \([25, \text{14.2}]\), we have \( \gamma_{x,y,z} = 0 \) unless \( x \sim_L y^{-1}, y \sim_L z^{-1} \) and \( z \sim_L x^{-1} \). Now one readily checks that the following assignments define an irreducible two-dimensional representation of \( J_Q \):

\[ t_1 \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad t_{s_1} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad t_{s_2 s_1} \mapsto \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad t_{s_1 s_2 s_1} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \]
\[ t_{s_2} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad t_{s_1 s_2} \mapsto \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}, \quad t_{s_2 s_1 s_2} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad t_{w_0} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

Finally, taking for \( B^\lambda \) the diagonal matrix with diagonal entries 1 and 2, we see that all the conditions in Proposition 2.6 are satisfied. This yields the expressions for the basis elements \( C_{s_1}^r \) where \( 1 \leq s, t \leq 2 \).
Example 4.4. Following Graham–Lehrer [16], we define cell representations of $H$ as follows. Fix $\lambda \in \Lambda$ and let $W(\lambda)$ be a free $A$-module with basis $\{C_s \mid s \in M(\lambda)\}$. Then $W(\lambda)$ is an $H$-module with action given by

$$h.C_s = \sum_{s' \in M(\lambda)} \tau_h(s', s) C_{s'}$$

for $h \in H$ and $s \in M(\lambda)$. There is a symmetric bilinear form $g^\lambda: W(\lambda) \times W(\lambda) \to A$ defined by

$$g^\lambda(C_s, C_t) = r_h(s, s)$$

where $s, t \in M(\lambda)$ and $h = C_{s,t}^\lambda$; see [16, Def. 2.3]. We have $g^\lambda(h.C_s, C_t) = g^\lambda(C_s, h^* C_t)$ for all $s, t \in M(\lambda)$ and $h \in H$; see [16, Prop. 2.4]. Using the formula for $r_h(s', s)$ in the proof of Theorem 3.1 and a computation analogous to that in the proof of [25, Prop. 20.6], it is straightforward to check that

$$v^a g^\lambda(C_s, C_t) \equiv f_\lambda \beta^\lambda_{st} \mod vR[v].$$

In particular, we can see from this expression that the determinant of the Gram matrix of $g^\lambda$ is non-zero. (This fact also follows from the general theory of cellular algebras, since $H_K$ is known to be split semisimple.)

Now let $K$ be the field of fractions of $A$; we write $H_K = K \otimes_A H$ and $W_K(\lambda) = K \otimes_A W(\lambda)$. Since $g^\lambda$ induces a non-degenerate form on $W_K(\lambda)$ for each $\lambda \in \Lambda$, we have

$$\text{Irr}(H_K) = \{W_K(\lambda) \mid \lambda \in \Lambda\};$$

see [16, Theorem 3.8]. In fact, by the formula for $r_h(s', s)$ in the proof of Theorem 3.1, we see that the action of $H$ on $W(\lambda)$ is obtained by pulling back the action of $J$ on $E^\lambda_A$ via Lusztig’s homomorphism $\phi: H \to J_A$. This shows that $W_K(\lambda) \cong E^\lambda_A$; see [25, 20.2, 20.3].

Now let $\theta: A \to k$ be any ring homomorphism where $k$ is a field. By extension of scalars, we obtain a $k$-algebra $H_k = k \otimes_A H$ and cell representations $W_k(\lambda) = k \otimes_A W(\lambda)$ ($\lambda \in \Lambda$), which may no longer be irreducible. Denoting by $g_k^\lambda$ the induced bilinear form on $W_k(\lambda)$, we set $L^\lambda = W_k(\lambda)/\text{rad}(g_k^\lambda)$. Then, by [16, Theorem 3.4], each $L^\lambda$ is either 0 or an absolutely irreducible representation of $H_k$, and we have

$$\text{Irr}(H_k) = \{L^\mu \mid \mu \in \Lambda^o\}$$

where $\Lambda^o = \{\lambda \in \Lambda \mid L^\lambda \neq 0\}$.

Denote by $[W_k(\lambda) : L^\mu]$ the multiplicity of $L^\mu$ as a composition factor of $W_k(\lambda)$. Then

$$(\Delta) \quad \begin{cases} [W_k(\mu) : L^\mu] = 1 & \text{for any } \mu \in \Lambda^o, \\ [W_k(\lambda) : L^\mu] = 0 & \text{unless } \lambda \preceq \mu \text{ (i.e., } \lambda = \mu \text{ or } a_\mu < a_\lambda)\end{cases}$$

see [16, Prop. 3.6]. Thus, the decomposition matrix

$$D = ([W_k(\lambda) : L^\mu])_{\lambda \in \Lambda, \mu \in \Lambda^o}$$

has a lower unitriangular shape, if the rows and columns are ordered according to increasing $a$-value.

The relations $(\Delta)$ show that the subset $\Lambda^o \subseteq \Lambda$ defines a “canonical basic set” in the sense of [11, Def. 4.13], which itself is an axiomatization of the results obtained earlier by Rouquier.
and the author [7], [15]. Note that the definition of “canonical basic sets” only requires the invariants $a_\lambda$ (with respect to $L$) and the general set-up of Brauer’s theory of decomposition numbers for associative algebras. Explicit descriptions of these “basic sets” are now known in all cases; see the survey [11] and the references there. In [19], [13], the existence of such basic sets for type $B_n$ and any $L$ (where $k$ has characteristic 0) has been established without assuming Lusztig’s conjectures on Hecke algebras with unequal parameters.

**Example 4.5.** Assume that $W$ is of type $B_n$ ($n \geq 2$) with diagram and weight function given by

$$
\begin{array}{c}
  B_n \\
  b \quad a \quad a \quad \ldots \quad a
\end{array}
$$

where $a, b$ are non-negative integers. Then we have two cellular structures on $\mathcal{H}$: the one given by Theorem 3.1 and the one given by Dipper–James–Murphy [4] (or, alternatively, by Graham–Lehrer [16, §5]). In both cases, $\Lambda$ is the set of all pairs of partitions $(\lambda, \mu)$ such that $|\lambda| + |\mu| = n$. The partial order on $\Lambda$ that we use here is defined in terms of the $a$-function; it heavily depends on the weight function $L$, i.e., on the integers $a$ and $b$. The partial order used by Dipper–James–Murphy (or Graham–Lehrer) is defined in purely combinatorial terms: it does not depend at all on the weight function $L$. Small examples show that the two partial orders indeed are different. Hence, in general, the cellular structure that we construct will be essentially different from the one constructed by Dipper–James–Murphy (or Graham–Lehrer). These differences can be seen explicitly in the descriptions of the sets $\Lambda^0 \subseteq \Lambda$ parametrizing the irreducible representations of non-semisimple specializations of $\mathcal{H}$, as discussed in [13]. By [13, Theorem 2.8], the two parametrizations coincide if $b > (n - 1)a > 0$. Thus, one may expect that the two cellular structures are equivalent under this condition; see Iancu–Pallikaros [18] for a further discussion of this question.

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