Fractionalized Metals and Superconductors in Three Dimensions

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We study three-dimensional metals with nontrivial correlation functions and fractionalized excitations. We formulate for such states a gauge theory, which also naturally describes the fractional quantization of chiral anomaly. We also study fractional superconductors in this description. This formulation leads to the “three-dimensional chiral Luttinger liquids” and fractionalized Weyl semimetals, which can arise in both fermion and boson models. We also propose experiments to detect these fractionalized phases.

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Introduction.

Fractionalization of quantum numbers\textsuperscript{1,20} such as electrical charge is an intriguing phenomenon in condensed matter physics. Fractionalization in weakly interacting systems is usually associated with topological solitons, a simplest example being the Su-Schrieffer-Heeger soliton\textsuperscript{2–5} in polyacetylene. Fractionalized excitations can also arise in strongly correlated systems, in which strong interaction among electrons plays an essential role in fractionalization. In fact, one of the best known examples of charge fractionalization is the elementary excitations (quasi-particles or quasi-holes) carrying charge $\pm e/m$ in the $m$-th Laughlin state\textsuperscript{21,23}, which describes a fractional quantum Hall state, one of the prototypes of strongly correlated systems. The one-dimensional (1D) chiral Luttinger liquids\textsuperscript{24–27} (CLLs) at the edge of the Laughlin states are fractionalized metals, for which the electron Green’s function follows a nontrivial power-law\textsuperscript{24,27}

$$G(x,t) \propto \frac{1}{(x-v_F t)^\nu}$$

where $v_F$ is a velocity parameter. This correlation is in sharp contrast with expectation of Landau’s Fermi liquid theory.

Charge fractionalization is a much more subtle problem for strongly correlated systems in spatial dimensions higher than one ($d>1$), due to absence of powerful analytical tools such as bosonization. Describing fractionalized metals in $d>1$ analogous to the 1D chiral Luttinger liquids is one of the motivations of the present work. A deeper motivation, as we now explain, is to describe a class of quantum phenomena we dubbed “fractional anomalies”. This was the initial motivation of this work.

In quantum mechanics, an “anomaly” refers to the failure of a symmetry of the classical action to be a symmetry of the full quantum theory. The earliest example is the chiral anomaly\textsuperscript{28–30}, which implies chiral current nonconservation in the presence of nontrivial gauge field backgrounds. Remarkably, chiral anomaly also has significant implications in condensed matter systems\textsuperscript{31–37}, where it is deeply related to topological states such as topological insulators and topological superconductors\textsuperscript{31,32,57}. As a simplest example, the one-dimensional edge of 2D integer quantum Hall edge states has the chiral anomaly for electrical current\textsuperscript{103}

$$\partial_\mu j^{R} = n_R \frac{e^2}{4\pi} \epsilon^{\nu\rho\sigma} F_{\rho\sigma}$$

the superscript and subscript “$R$” appears because low energy fermions have definite chirality, which is taken to be right-handed, i.e., they all move towards the right direction. Here the integer $n_R$ is the number of these edge modes. According to the bulk-edge correspondence, $n_R$ is also equal to the Chern number of the two-dimensional bulk of quantum Hall insulators. In this example the origin of chiral anomaly is transparent: The transverse current (Hall current) in the bulk adds or removes charges at the edge, which is regarded as charge nonconservation by an edge observer.

On one hand, there have been various arguments pointing to the quantization (non-renormalization) of chiral anomaly. On the other hand, the existence of fractional quantum Hall effects has deepened our understanding of chiral anomaly. For the Laughlin state with filling factor $\nu = \frac{1}{3}$, the chiral edge state has the fractional chiral anomaly\textsuperscript{38–40}

$$\partial_\mu j^{R} = \frac{e}{\pi} \epsilon^{\nu\rho\sigma} F_{\rho\sigma},$$

which is consistent with the chiral Luttinger liquid theory\textsuperscript{24–27}.

In 3D, the integer chiral anomaly reads $\partial_\mu j^{R} = n_R \frac{e^2}{32\pi^2} \epsilon^{\nu\rho\sigma} F_{\rho\sigma}$ and $\partial_\mu j^{L} = -n_L \frac{e^2}{32\pi^2} \epsilon^{\nu\rho\sigma} F_{\rho\sigma}$, or more compactly\textsuperscript{28}

$$\partial_\mu j^{R,3D} = (n_R + n_L) \frac{e^3}{32\pi^2} \epsilon^{\nu\rho\sigma} F_{\rho\sigma}$$

where $n_R$ and $n_L$ is the number of modes of right-handed and left-handed chiral fermions, respectively, and the chiral current $j^{R,3D} \equiv j^{R} - j^{L}$. Note that in Eq.(3) we have the factor $\epsilon^3$ because $j^{R}$ refers to electrical current (for the current associated with particle number this factor should be $e^2$). The chiral anomaly have interesting physical implications for Weyl semimetals (and Weyl superconductors)\textsuperscript{32,37,59–79} (“Weyl fermions” is another name of chiral fermions).

By analogy with 1D fractional chiral anomaly, the “fractional chiral anomaly” for right-handed chiral fermions in 3D reads

$$\partial_\mu j^{R} = \nu \frac{e^3}{32\pi^2} \epsilon^{\nu\rho\sigma} F_{\rho\sigma}$$

The chiral anomaly have interesting physical implications for Weyl superconductors\textsuperscript{32,37,59–79} (“Weyl fermions” is another name of chiral fermions).
where \( K_{ijab'} = e^{i\epsilon_{ij}(N-1)/N} (1/N)!^2 \langle e_{ab...c}q_{ijb}...q_{iec} \rangle \), and similarly for \( K'_{ijab} \). In Eq. (7) we have added the Lagrange multiplier \( \lambda_{ij} \) for the physical Hilbert space, \( T \) being the generators of \( SU(N) \). Writing the mean-field Hamiltonian in Eq. (7) more compactly, we have

\[
H_{\text{mean}} = \sum_{ij} \sum_{ab} q_{ab}^a U_{ij} e^{i\epsilon_{ij}N} + \langle \lambda_{ij} \rangle \sum_{ij} \frac{A_0}{N} \delta_{ij} q_{ij} \]

where \( U_{ij} = \bar{U}_{ij} e^{\tilde{a}_{ij}} \) in which \( a_{ij} = a_i^T a_j \) is a \( N \times N \) Hermitian matrices describing the \( SU(N) \) phase fluctuations of \( U_{ij} \) around the mean field value \( \bar{U}_{ij} \), while the amplitude fluctuation of \( U_{ij} \) is ignored. Similarly, we can split the Lagrange multipliers as \( \lambda_{ij} = \bar{\lambda}_{ij} + a_{ij} \) and \( a_{ij} \) being the vacuum expectation value and the fluctuation of \( \lambda_{ij} \), respectively. We can see that \( a_{ij} \) can be regarded as the temporal component of an emergent \( SU(N) \) gauge field, whose spatial components are \( \bar{a}_{ij} \).

Now let us discuss the dynamics of the \( SU(N) \) gauge potentials \( (a_0^i, a_1^j) \) in the long wavelength limit, which are referred to as “\( a^\mu_0 \)” \( (\mu = 0, 1, 2, 3) \), or more compactly as “\( a^\mu \)” when there is no confusion. The most important problem now is whether the gauge bosons are massless or massive, namely, whether the effective action \( L_{\text{eff}}(U) \) contains mass terms for \( a^\mu \). In fact, the gauge bosons can become massive by the Higgs mechanism, the vacuum expectation value \( \bar{a}_{ij} \) playing the role of “Higgs fields”. To be precise, the gauge bosons \( a^\mu \) will generally be massive if there exists certain loop \( C_i \) given by \( i \rightarrow j \rightarrow k \rightarrow \ldots \rightarrow l \rightarrow i \) for which \( [T^T P(C_i)] \neq 0 \). Thus, the gauge bosons \( a^\mu \) are massive in this case. According to this general criterion, there are several scenarios associated with different types of \( (\bar{a}_{ij}, \bar{U}_{ij}) \):

1. “Generic flux”. For any \( SU(N) \) group generator \( T^\mu \), there exists certain loop \( C_i \) for which the commutator \([P(C_i), T^\mu] \neq 0 \). All gauge bosons are massive in this case.

2. Trivial flux. We have \( P(C_i) = 1 \) for an arbitrary loop \( C \), in other words, \([P(C_i), T^\mu] = 0 \) for all \( T^\mu \) and all \( C_i \). All gauge bosons are massless in this case.

3. Coplanar flux. For some of the \( SU(N) \) group generators, say \( T^m \), the relation \([P(C_i), T^m] = 0 \) (for any \( C_i \)) is satisfied; while for other group generators,
say $T^a$, there exists at least one choice of $C_i$ such that $\{P(C_i), T^a\} \neq 0$. The gauge bosons associated with the first class of group generators are massless, while those associated with the second class are massive.

Similar scenarios emerge in the $SU(2)$ gauge-field formulation of spin liquid\cite{58, 59, 60}. Without going into technical details, here we simply provide an intuitive argument for these scenarios (not a proof). In the continuum limit, the effective Lagrangian generally contain a term $\text{tr}((a_\mu, a_\nu)^2)$, where $a_\mu$ denotes the background field determined by $(\bar{U}_i(t), \bar{U}_j)$. When this term is non-vanishing, it can be regarded as a mass term for $a_\mu$, therefore, the Higgs mechanism of non-Abelian gauge field does not require matter fields (partons in our context); gauge fields themselves can trigger the Higgs mechanism because they carry gauge charge\cite{58} (Unlike the Abelian gauge field theory, in which the gauge boson is charge neutral).

Here let us focus on the first scenario, namely that all $SU(N)$ gauge bosons acquire an mass from the Higgs mechanism. An explicit ansatz realizing this scenario will be given shortly. In this case the gauge bosons can only mediate short-range interactions, thus they do not cause infrared divergences.

If the original electron system has $n$ bands, namely, there are $n$ microscopic electronic states (referred to as $\alpha, \beta, \ldots$) in each unit cell, then the mean-field Hamiltonian Eq.\,(8) for partons has $nN$ bands.

Suppose that the parton mean-field Hamiltonian $H_{\text{mean}}$ has $m$ valleys near $K_\sigma$ ($s = 1, 2, \ldots, m$), where the band structure is that of right-handed chiral fermions (Weyl fermions). Within the valley around $K_\sigma$, the two low energy bands are described approximately by the Hamiltonian $q^i(k)\bar{P}_i h_\sigma(k)\bar{P}_i q(k)$, where $q(k)$ is the abbreviation for $(q_1(k), q_2(k), \ldots, q_{\alpha N}(k))^T$, $\bar{P}_i$ is the projection operator to the two low energy bands, and

$$h_\sigma(k) = \sum_{i,j=1,2,3} v_{ij} \sigma_i(k - K_\sigma)_{ij} - \mu \quad (10)$$

where $v_{ij}$ ($i, j = 1, 2, 3$ or $x, y, z$) play the role of Fermi velocities, and the Pauli matrices $\sigma_i$ ($i = 1, 2, 3$) act on the two low energy bands. In the following we will take all $K_\sigma = 0$ and $v_{ij} = v_F \delta_{ij}$ ($v_F > 0$) without affecting main physical conclusions. At $\mu = 0$, the Fermi surface shrinks to a point. By dimensional analysis, various short range interactions such as $q^i q^j q_k$ (indices omitted) are irrelevant in the renormalization group sense. The electromagnetic interaction is marginally irrelevant\cite{10}. Therefore, at low energy or long distance the Green’s function of partons is just that of the free fermions. The elementary excitations are partons (and holes of partons) carrying charge $e\epsilon/N$, which is a direct manifestation of fractionalization.

In the momentum space, the parion Green’s function is denoted as $G_{\alpha \beta}(x, k)$, where $\alpha, \beta = 1, 2, \ldots, n$ and $a, b = 1, 2, \ldots, N$. In the low energy limit it is given by $g = \sum q h(k) q$, where $g_{\alpha}(k)$ being the contribution of the $s$-th valley. In our simplest ansatz it reads $g_{s}(\omega, k) = 1/(\omega + \epsilon^l \text{sgn}(\omega) - v_F \sigma \cdot k)$. Fourier-transformed to the real space, the parion Green’s function is

$$g_s(x, t) \propto \frac{v_F t + \sigma \cdot x}{(x^2 - v_F^2 t^2)^2} \quad (11)$$

where $x^2 \equiv x_1^2 + x_2^2 + x_3^2$. The electron Green’s function is

$$G_{ap}(x, t) = \det g_{a\beta}(x, t) \quad (12)$$

where the determinant is calculated regarding the color index $a (b )$ as the row (column) index of matrix $g_{a\beta}$, the indices $\alpha, \beta$ being fixed. Because $g(x, t) \propto 1/t^4$ in the long time limit, it readily follows that in this limit

$$G(0, t) \propto \frac{1}{t^{5N}} \quad (13)$$

Similarly, we have $G(x, 0) \propto 1/x^{3N}$ in the long distance limit. The power-law behavior of Green’s function, with the exponent quantized as an integer\cite{57}, is reminiscent of the chiral Luttinger liquids in 1D. As a comparison, the 1D chiral Luttinger liquid at the edge of $m$-th Laughlin state has\cite{58} $G(0, t) \propto t^{-m}$.

Since each parion carries electrical charge $e/N$, the coefficient of chiral anomaly in Eq.(4) is

$$\nu = (1/N)^3 \times m = m/N^3 \quad (14)$$

which can be readily obtained from the triangular diagram\cite{28}, with parion propagators extracted from Eq.(10). Due to the non-renormalization property of chiral anomaly, modest modifications of Hamiltonian cannot change this quantized $\nu$.

Sofar we have formulated a self-consistent theory of fractionalized metals in 3D, for which the Green’s function follows a nontrivial power-law, and the chiral anomaly is fractionally quantized. Now we would like to discuss the physical applications of this formulation. The most direct application can be found at the surface of 4D fractional quantum Hall effects\cite{88-91}. From our formulation it is clear that fractionalized metal (“3D chiral Luttinger liquid”) is a possible and consistent scenario in certain regime. Our formulation can also be applied to Weyl semimetals with equal number of modes of right-handed and left-handed chiral fermions. For Weyl semimetals the nontrivial exponents in Green’s function can be measured in tunneling experiments, which provide information of the electron density of states.

Now let us study two simple examples. We consider single-band lattice boson models, and write the (hard-core) boson operator $b_i$ as $b_i = q_1 q_2$. The mean-field ansatz reads $H_{\text{mean}} = \sum_{\epsilon} q^h(k) q$, where $q^h(k)$ is the shorthand notation for $(q_1(k), q_2(k))^T$.

The first example we consider is $h(k) = v_F \sigma \cdot k$, which gives nonzero masses to all three $SU(2)$ gauge bosons because none of $\sigma_i$ ($i = 1, 2, 3$) commutes with $v_F \sigma \cdot k$ for a generic $k$. According to Eq.(14), we have fractional anomaly

$$\nu = (1/N)^3 = 1/8 \quad (15)$$
Such a fractional anomaly can be realized at the boundary of a 4D (boson) quantum Hall insulators. The parton propagator for this model is exactly given by Eq. (11). The boson Green’s function can be obtained from Eq. (12), with the simplification in this special case that the indices $\alpha, \beta$ are absent. Explicitly, we have

$$G(x, t) = \det g \propto \frac{1}{(x^2 - v_F t^2)^3}$$

(16)

which implies $G(0, t) \propto 1/t^6$ in the $t \to \infty$ limit.

The second example of parton mean-field Hamiltonian is $h(k) = \{2t_x \sin k_x \sigma_x + 2t_y \sin k_y \sigma_y + 2t_z \sin k_z \sigma_z\}$, which is borrowed from Ref. [93]. It has two Weyl valleys near $K = \pm 0.0, 0$, where $h(p) \approx \mp 2t_x (\sin k_x) p_x \sigma_x + 2t_y p_y \sigma_y + 2t_z p_z \sigma_z$, with $p \equiv k \mp K$. The elementary excitations are chiral fermions carrying charge $\pm e/2$, and the fractional anomaly is given by Eq. (5) with $\nu = 1/8$. There is a similar equation for $j^L$ with $\nu = -1/8$. The boson Green’s function reads $G(x, t) = \det g$, which scales as $G(0, t) \propto 1/t^6$. This example can be realized in 3D lattice boson model (rather than just realized at the surface of 4D lattice models).

**One dimensional fractionalized chiral modes propagating along dislocations.**

One way to detect the charge fractionalization in the fractionalized Weyl semimetals is to induce an energy gap and then create dislocations.

For simplicity, let us suppose that the mean field Hamiltonian Eq. (8) has one valley (right-handed fermion) around $K_R$, where the low energy Hamiltonian is $h_R(k) = v_f \sigma \cdot (k - K_R)$, and another valley (left-handed fermion) at momentum $K_L$, described by $h_L(k) = -v_f \sigma \cdot (k - K_L)$. Now let us add an external magnetic field $B = B_z \hat{z}$. In the presence of this magnetic field, the energy spectra are $E_{\alpha}(p_z) = \pm v_F \sqrt{p_z^2 + 2eB|l|/N}$ with $l = 0, 1, 2, \ldots$, where $p_z = (k - K_R(l))$. For the $l = 0$ mode, the ± sign before $v_F$ is determined by the chirality (+ for right-handed, − for left-handed). Therefore, the effective Hamiltonian for the zeroth Landau level can be written compactly as $h_0(p_z) = v_F \tau_z p_z$, where the Pauli matrix $\tau_z$ refers to the two chiralities. Note that $h_0(p_z)$ is independent on $B$. Due to perfect nesting of Fermi surface, an infinitesimal interaction can dynamically generate a mass $m(r) = \langle q'(x)q(x) \rangle \propto e^{\theta(|Q + \delta Q|)}$, where $\tau_+ = \tau_3 + i\tau_1$, $Q = K_r - K_R$, and $\theta(x)$ is a slowly varying variable. The (charge or spin) density vary as $\cos(Q \cdot x + \theta(x))$, in which $\theta(x)$ determines the locations of peaks and troughs of the density wave.

Suppose that there is a line dislocation $l$, such that for a loop $C$ around $l$ we have $\int_C d\theta = 2\pi$, thus the peaks and troughs shift by one wavelength by making a circle around $l$ [see Fig. 1 in Ref. [75]]. In Ref. [75], it was shown for a similar problem (integer case) that there is a chiral mode along $l$, which is analogous to the edge state of integer quantum Hall effect with Hall conductance $e^2/h$. By similar calculation, it can be shown for the present problem that there is chiral modes along $l$ with “Hall conductance” $e^2/N^2h$. In this sense the fractionalized chiral mode along $l$ is analogous to the 1D chiral Luttinger liquid. Measuring this fractional “Hall conductance” will be one way to confirm the fractionalized metals in 3D.

**Fractional superconductors in 3D.**

In the previous sections we have focused on the case with parton chemical potential $\mu = 0$. When $\mu \neq 0$, an infinitesimal attractive interaction can induce Cooper instability, and the ground state is a superconductor. Let us consider a Weyl valley around $K_r$. For a short-range interaction $g\delta^3(x)$, the superconducting gap is

$$\Delta_{\text{wave}} = \langle q_R(x + p)q_R(x - p) \rangle \propto e^{-|p|/c}$$

(17)

where $\uparrow$ and $\downarrow$ refer to the two low energy degrees of freedom, $\rho$ is the density of states of partons at the Fermi level, and $c$ is a numerical coefficient of order of unity. More complicated scenarios such as color superconductors [95–99] can also arise in our description, which will be left for future studies. Here we would like to focus on model-independent physical consequences.

One of the physical predictions is about the Josephson effect [77, 98]. In fact, the parton Cooper pairs carry charge $2e/N$, therefore, we expect fractional Josephson effects. In the presence of a voltage $V_0$ between two fractional superconductors connected by a weak link, alternating tunneling current with frequency

$$\omega_0 = \frac{2eV_0}{Nh}$$

(18)

can be observed (In this formula we have restored the Planck constant $h$, which has been set to unity in our previous presentation). We would like to mention that the fractional Josephson effect has also been studied in 2D systems [99, 102] in different approaches.

**Conclusions.**

In the present paper we have studied fractionalized (semimetallic in 3D through an $SU(N)$ gauge theory. We have found for them power-law Green’s functions with quantized exponents. This formulation will be useful to 3D chiral Luttinger
liquids and fractionalized Weyl semimetals. This gauge-theoretical formulation resembles the standard model of particle physics, in which the chiral fermions are coupled to $U(1)$ and $SU(N)$ gauge fields with $N = 2, 3$, the $SU(2)$ being suppressed at low energy due to the Higgs mechanism. In the field of condensed matter, we believe that fractional anomalies will provide much information about fractional topological states of quantum matter. In addition to the significance of their own right, the fractionalized (semi-)metals can also be regarded as the “mother states” of gapped fractional topological states. In future it will be fruitful to establish explicit many-body Hamiltonians for the scenarios proposed in the present work, and to explore the implications of fractional anomalies in depth.

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