A COMPACTNESS THEOREM FOR HYPERKÄHLER 4-MANIFOLDS WITH BOUNDARY

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ABSTRACT. In this paper, we study the compactness of a boundary value problem for hyperkähler 4-manifolds. We show that under certain topological conditions and the positive mean curvature condition on the boundary, a sequence of hyperkähler triples converges smoothly up to diffeomorphisms if and only if their restrictions to the boundary converge smoothly up to diffeomorphisms. We also generalize this result to torsion-free hypersymplectic triples.

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1. INTRODUCTION

A Riemannian metric $g$ on a 4-manifold is called hyperkähler if its holonomy group $\text{Hol}(g)$ is contained in $Sp(1) = SU(2)$. A closed hyperkähler 4-manifold is diffeomorphic to either a torus or the K3 manifold, and the moduli space of all hyperkähler metrics are described by Torelli theorems. There have been extensive
recent studies on the Gromov-Hausdorff compactification of these moduli spaces, see for example [29] [32].

Hyperkähler metrics in dimension 4 are the simplest models for Riemannian metrics with special holonomy. Little general existence theory is developed for the latter in dimensions greater than 4, except for Calabi-Yau manifolds. Recently Donaldson [12] proposes to study special holonomy metrics on manifolds with boundary and set up suitable elliptic boundary value problems. To make further progress in this direction, it is clear that we need a compactness theory.

In this paper, we study the boundary value problem for hyperkähler 4-manifolds, which serves as the first step towards Donaldson’s program. We follow the general set-up by Fine-Lotay-Singer [16] in terms of hyperkähler triples. A hyperkähler triple on an oriented smooth 4-manifold $X$ is a triple of symplectic forms $\omega = (\omega_1, \omega_2, \omega_3)$ satisfying the following pointwise condition

$$\omega_i \wedge \omega_j = \frac{1}{3} \delta_{ij} (\omega_1^2 + \omega_2^2 + \omega_3^2).$$

It is well-known that a hyperkähler triple $\omega$ uniquely determines a compatible hyperkähler metric $g_\omega$ such that for each $i$, $\omega_i^2 = 2\text{vol}_{g_\omega}$ and $\omega_i$ is parallel with respect to the Levi-Civita connection. Conversely, given a hyperkähler metric $g$ on $X$, one can choose an orientation and find a compatible hyperkähler triple $\omega$, which is unique up to a $SO(3)$ rotation.

Now let $X$ be a compact oriented smooth 4-manifold with boundary $\partial X$. Note $\partial X$ has an induced orientation defined by contracting a volume form of $X$ with an outward vector field. If $\omega$ is a hyperkähler triple on $X$, then its restriction to $\partial X$ is a closed framing $\gamma$ on $\partial X$. The following is a natural filling problem, proposed by [16].

**Question 1.1.** Which closed framing $\gamma$ extends to a hyperkähler triple on $X$?

Notice a framing $\gamma$ defines a Riemannian metric $g_\gamma$ on $\partial X$ as follows: first, there exists a unique dual coframe $\eta = (\eta_1, \eta_2, \eta_3)$ such that $\gamma_i = \frac{1}{2} \delta^{jk} \eta_j \wedge \eta_k$ and such that $\eta_1 \wedge \eta_2 \wedge \eta_3$ is compatible with the orientation of $\partial X$; then the Riemannian metric $g_\gamma$ is defined by setting $\eta$ to be orthonormal. When there is no ambiguity, we always use $\eta$ to denote the dual coframe of $\gamma$ defined in this way and denote the Hodge star operator of the Riemannian metric by $*\gamma = *\eta$. It is well-known that if $\omega$ is a hyperkähler triple, then $g_\omega|\partial X = g_\gamma$; more importantly that the second fundamental form of $\partial X$ is determined intrinsically by $\gamma$ via the matrix $*\eta(\eta_i \wedge d\eta_j)$. In particular, the mean curvature $H_\gamma$ is given by one half of the trace of this matrix, i.e., $H_\gamma = \frac{1}{2} *\eta(\eta \wedge d\eta)^\gamma$.

There are some previous works on Question 1.1. Bryant [9] studied the local “thickening” problem and obtained both positive and negative results. It was shown that any real analytic closed framing on a closed oriented 3-manifold $Y$ can be extended to a hyperkähler triple on $Y \times (-\epsilon, \epsilon)$ for some $\epsilon > 0$, and the extension is essentially unique. On the other hand, there exists a smooth closed framing on an open ball $B^3 \subset \mathbb{R}^3$ that cannot be extended to a hyperkähler triple on $B^3 \times (-\epsilon, \epsilon)$ for any $\epsilon > 0$. Fine-Lotay-Singer [16] studied the local deformation theory for Question 1.1 and showed that the boundary framings must deform in certain directions. Roughly speaking, let $X = B^4$ for simplicity, suppose $\omega$ is a hyperkähler triple such that $\partial X$ has positive mean curvature, and $\omega'$ is a nearby hyperkähler triple, then after moduling out diffeomorphisms of $\partial X$, the dual coframe of $\omega'|\partial X$
must be a small perturbation of that of $\omega|_{\partial X}$ in the direction of negative frequency of the boundary Dirac operator defined by $g_{\omega|_{\partial X}}$.

A sequence of pairs of smooth covariant tensors $(T^1_i, \ldots, T^m_i)$ on a compact manifold $M$ with empty or nonempty boundary is said to converge in Cheeger-Gromov sense to $(T^1, \ldots, T^m)$ on $M$, if there exist diffeomorphisms $f_i: M \to M$ such that $f_i^*T^1_i \to T^1, \ldots, f_i^*T^m_i \to T^m$ smoothly on $M$.

Our main result is the following closedness result for Question 1.1:

**Theorem 1.2.** Let $X$ be a compact oriented smooth 4-manifold with boundary, such that there does not exist $C \in H_2(X,\mathbb{Z})$ with self intersection $C^2 = -2$. Let $\omega_i$ be a sequence of smooth hyperkähler triples on $X$. Suppose $\omega_i|_{\partial X}$ converges in Cheeger-Gromov sense to a closed framing $\gamma$ on $\partial X$ such that $H_3 > 0$, then there exists a smooth hyperkähler triple $\omega$ on $X$ with $\omega|_{\partial X} = \gamma$ and $\omega_i$ converges in Cheeger-Gromov sense to $\omega$ on $X$.

The proof here includes two parts: the compactness and uniqueness. The former is the main story of this paper, and the latter is a consequence of $[8]$ or $[6]$ on unique continuation of Einstein metrics with prescribed boundary metric and second fundamental form. It is worth noting that for the compactness part, no general Riemannian convergence theory can be applied directly. The difficulty here is that we only have data on the boundary, and a priori we do not know anything near the boundary or in the interior. Specifically, we worry about the following three bad geometric behaviours: curvature blow up, volume collapsing and boundary touching. These things are entangled, making it difficult to rule out any of them. However, we are able to separate these bad behaviours and rule them out. We will also give examples to demonstrate that the assumptions in Theorem 1.2 are essential, see Remark 4.9 and 4.10.

Such $C$ in the assumption of Theorem 1.2 is usually called a “$-2$ curve” in $X$, which appears in Kronheimer’s classification of hyperkähler ALE spaces [26] [27]. They appear in bubble limits of volume-noncollapsed hyperkähler manifolds. From this, one can replace the “no $-2$ curve” condition by an assumption on enhancements of $\omega_i|_{\partial X}$. Let $\gamma$ be a closed framing on $\partial X$. Following [14] [12], an enhancement of $\gamma$ is an equivalent class in the set of triples of closed 2-forms on $X$ whose restrictions to $\partial X$ are equal to $\gamma$. The equivalence relation is defined by $\theta \sim \theta + da$, where $a$ is a triple of smooth 1-forms on $X$ vanishing on $\partial X$. From the de Rham cohomology exact sequence of the pair $(X,\partial X)$,

$$H^2(X,\partial X) \to H^2(X) \to H^2(\partial X) \to H^1(X,\partial X),$$

we know $\gamma$ has at least one enhancement if and only if each $\gamma_i$ lies in the kernel of $H^2(\partial X) \to H^1(X,\partial X)$, and we know the set of all enhancements of $\gamma$ is an affine space over $H^2(X,\partial X) \otimes \mathbb{R}^3$. Choose an enhancement of $\gamma$ and denote it by $\bar{\gamma}$. Given a 2-cycle $\Sigma \in H_2(X,\mathbb{Z})$, then for any triple of closed 2-forms $\theta \in \bar{\gamma}$, $\int_{\Sigma} \theta$ does not depend on the choice of $\theta$ and we denote this invariant by $c_{\gamma,\Sigma} \in \mathbb{R}^3$.

The proof of Theorem 1.2 easily adapts to

**Theorem 1.3.** Let $X$ be a compact oriented smooth 4-manifold with boundary. Let $\omega_i$ be a sequence of smooth hyperkähler triples on $X$, and $\bar{\gamma}_i$ be the enhancement of $\gamma_i = \omega_i|_{\partial X}$ where $\omega_i$ lie in. Let $a > 0$ be a positive number. Suppose for any $C \in H_2(X,\mathbb{Z})$ with self intersection $C^2 = -2$, $|c_{\bar{\gamma}_i,C}| > a$ and $\omega_i|_{\partial X}$ converges in Cheeger-Gromov sense to a closed framing $\gamma$ on $\partial X$ such that $H_3 > 0$. Then there
exists a smooth hyperkähler triple $\omega$ on $X$ with $\omega|_{\partial X} = \gamma$, and $\omega_i$ converges in Cheeger-Gromov sense to $\omega$ on $X$.

It is worth noting that Question 1.1 is not an elliptic boundary value problem, observed by [16]. This can also be seen from the uniqueness result of [8] or [6]: the restriction of $\omega$ to any open boundary portion determines $g|_{\partial X}$ in the whole interior up to local isometries. So, it is natural to enlarge the class of closed triples of 2-forms on $X$ to obtain an elliptic boundary value problem. In [12], Donaldson studied the deformation theory of torsion-free $G_2$ structures on a compact oriented 7-manifold with boundary $M^7$ as follows. Suppose $\phi_0$ is a smooth torsion-free $G_2$ structure, $\rho_0 = \phi_0|_{\partial M^7}$, and denote the enhancement (defined in an analogous way as before) of $\rho_0$ where $\phi_0$ lies in by $\hat{\rho}_0$. Donaldson set up an elliptic boundary value problem for the torsion-free equation. So in particular, if the kernel space at $\phi_0$ is trivial, then for any small closed 3-form $\theta$ on $X$, $\hat{\rho}_0 + \theta|_{\partial X}$ contains a unique torsion-free $G_2$ structure that is close to $\phi_0 + \theta$ after gauge fixing. This phenomenon is quite different from the hyperkähler case, since in that case we cannot deform the boundary framing arbitrarily to extend it to a hyperkähler triple, as we discussed before.

It is well-known that $G_2$ structures have a reduction to dimension 4. Consider $X^4 \times T^3$. A triple of two forms $\omega = (\omega_1, \omega_2, \omega_3)$ on $X^4$ defines a 3-form on $X^4 \times T^3$ by

$$\phi = dt^1 \wedge dt^2 \wedge dt^3 - \omega_1 \wedge dt^1 - \omega_2 \wedge dt^2 - \omega_3 \wedge dt^3.$$  

The triple $\omega$ is called torsion-free hypersymplectic if $\phi$ is a torsion-free $G_2$ structure. Locally, this is a weaker condition than being hyperkähler, see examples in [14] or [18]. Donaldson observed in [14] that the boundary value problem for torsion-free $G_2$ structures can also be reduced to dimension 4. So, a compactness result is helpful to solve this dimension reduced boundary value problem.

Similar to the hyperkähler case, a torsion-free hypersymplectic triple $\omega$ defines a Riemannian metric $g|_{\omega}$ and a positive definite $SL(3, \mathbb{R})$-valued function $Q = (Q_{ij})$ such that

$$\omega_i \wedge \omega_j = 2Q_{ij} \text{dvol}_{g|_{\omega}}.$$  

We denote $Q'$ the restriction of $Q$ to $\partial X$. When there is ambiguity, we use notations $Q_{\omega}, Q'_{\omega}$ to denote their dependence on $\omega$. One can show that the mean curvature of $\partial X$ has an explicit expression in terms of $\gamma, Q'$, and we denote this explicit expression by $H_{\gamma, Q'}$. Note that on $\partial X$, $\gamma, Q'$ are subject to the constraints $d\gamma = 0, d(\gamma(Q')^{-1}) = 0$.

We have the following analogue of Theorem 1.3:

**Theorem 1.4.** Let $X$ be a compact oriented smooth 4-manifold with boundary. Let $\omega_i$ be a sequence of smooth torsion-free hypersymplectic triples on $X$, and $\hat{\gamma}_i$ be the enhancement of $\gamma_i = \omega_i|_{\partial X}$ where $\omega_i$ lie in. Let $a > 0$ be a positive number. Suppose for any $C \in H_2(X, \mathbb{Z})$ with self intersection $C^2 = -2$, $|c_{\gamma}, C| \geq a$, and $(\hat{\gamma}_i, Q'_i)$ converges in Cheeger-Gromov sense to some pair $(\gamma, Q')$ on $\partial X$, such that $\gamma$ is a framing, $Q'$ is positive definite and $H_{\gamma, Q'} > 0$. Then there exists a smooth torsion-free hypersymplectic triple $\omega$ on $X$ with $\omega|_{\partial X} = \gamma$, $Q_{\omega}' = Q'$, and $\omega_i$ converges in Cheeger-Gromov sense to $\omega$.

Note that Theorem 1.4 includes the previous two versions.
This paper is organized as follows: In Section 2, we discuss basics of hyperkähler triples on manifolds with boundary. In Section 3, we discuss Riemannian geometry for manifolds with boundary and Riemannian convergence theory. In Section 4, we prove Theorem 1.2 and Theorem 1.3 and give some remarks about the proofs. In Section 5, we discuss some basics for torsion-free hypersymplectic triples and prove Theorem 1.4.

Notations.

\[ R^n_+ = \{ x \in \mathbb{R}^n : x^n \geq 0 \}, \]
\[ B_r = \{ x \in \mathbb{R}^n : |x| < r \}, \]
\[ B_r^+ = B_r \cap \mathbb{R}^n_+, \]
\[ \partial B_r = \{ x \in \mathbb{R}^n : |x| = r, x^n = 0 \}, \]
\[ \partial^* B_r^+ = \{ x \in \mathbb{R}^n : |x| = r, x^n > 0 \}. \]

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2. Hyperkähler triples and closed framings

The discussions in this section are well-known.

2.1. Pointwise theory. Let \( V \) be an oriented 4-dimensional vector space, and \( \omega = (\omega_1, \omega_2, \omega_3) \in \Lambda^2(V^*) \otimes \mathbb{R}^3 \). Suppose \( \omega \) is a definite triple, i.e., \( \omega_i \) spans a maximum positive subspace of \( \Lambda^2(V^*) \) with respect to the wedge product, then \( \omega_i \) defines a unique conformal structure on \( V \) by making each \( \omega_i \) self dual. Fix a volume form \( \mu_0 \) on \( V \) that defines the orientation of \( V \), write \( \omega_i \wedge \omega_j = 2Q_{ij}\mu_0 \), we define a matrix \( Q \) associated to the definite triple \( \omega \) by \( Q_{ij} = \frac{q_{ij}}{\det(q_{ij})^{\frac{1}{3}}} \), which does not depend on the choice of \( \mu_0 \). We use \( Q^{-1} = (Q^{ij}) \) to denote the inverse matrix of \( Q \). If we write

\[ \omega_i \wedge \omega_j = 2Q_{ij}\mu, \]

then \( \mu \) is a volume form intrinsically defined by \( \omega \). We define a unique metric \( \langle,\rangle_\omega \) on \( V \) in the conformal structure by making \( \mu \) the volume form. Explicitly,

\[ \langle u, v \rangle_\omega = \frac{1}{6} \sum_{i,j,k=1}^3 \delta^{ijk} \frac{t_u \omega_i \wedge t_v \omega_j \wedge \omega_k}{\mu}. \]

So

\[ \langle u, u \rangle_\omega = \frac{t_u \omega_1 \wedge t_u \omega_2 \wedge \omega_3}{\mu}. \]

Denote \( *_\omega \) the Hodge star operator defined by this metric.

Let \( W \) be an orientated 3-dimensional vector space, and \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) be a framing on \( W \), i.e., a basis for \( \Lambda^3W^* \). Then by elementary linear algebra, there exists a coframe \( \eta = (\eta_1, \eta_2, \eta_3) \) such that

\[ \gamma_i = \frac{1}{2}\delta^{ijk}\eta_j \wedge \eta_k. \]
Such \( \eta \) is uniquely determined up to a sign and we choose \( \eta \) such that \( \eta_1 \land \eta_2 \land \eta_3 \)
defines the orientation of \( W \) and we denote this volume form by \( \text{vol}_\eta \). There is
a unique metric on \( W \), denoted by \( \langle \cdot \rangle_\eta \), that makes \( \eta \) an orthonormal coframe.
Denote \( *_\eta = *_\eta \) by the Hodge star operator of \( \langle \cdot \rangle_\eta \). Let \( e_i \in W \) be the dual
vector of \( \eta_i \), so \( \eta(e_j) = \delta_{ij} \). Then \( e = (e_1, e_2, e_3) \) is a frame of \( W \). Conversely,
given a coframe \( \eta = (\eta_1, \eta_2, \eta_3) \) on \( W \) compatible with the orientation, one can
define a framing \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) via (2), and a volume form \( \text{vol}_\eta = \text{vol}_\gamma \), a metric
\( \langle \cdot \rangle_\eta = \langle \cdot \rangle_\gamma \), a Hodge star operator \( *_\eta = *_\gamma \).

Now if \( W \subset V \) is a 3-dimensional subspace, \( \omega = (\omega_1, \omega_2, \omega_3) \) is a definite triple
on \( V \), and \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) is the restriction of \( \omega \) to \( W \). Let \( \langle \cdot \rangle_W \) be the restriction of the metric \( \langle \cdot \rangle_\omega \) on \( W \), which defines a volume form \( \text{vol}_W \), and a Hodge star operator \( *_W \)
compatible with the orientation of \( W \). Since \( \omega_i \) are self-dual, we can write
\[
\omega_i = \nu^* \land *_W \gamma_i + \gamma_i,
\]
where \( \nu^* = \omega \text{vol}_W \), then we have
\[
\omega_i \land \omega_j = 2\nu^* \land *_W \gamma_i \land \gamma_j = 2\langle \gamma_i, \gamma_j \rangle_W \nu^* \land \text{vol}_W = 2\langle \gamma_i, \gamma_j \rangle_W \mu.
\]

Hence on \( W \),
\[
Q_{ij} = \langle \gamma_i, \gamma_j \rangle_W,
\]
Since \( \text{det}(Q) = 1 \), we have \( \text{vol}_\gamma = \text{vol}_W \). If furthermore we assume \( Q_{ij} = \delta_{ij} \),
then \( \langle \gamma_i, \gamma_j \rangle_W = \delta_{ij} = \langle \gamma_i, \gamma_j \rangle_\gamma \). In this case, \( \langle \cdot \rangle_\gamma = \langle \cdot \rangle_W \) as an inner product on \( W = \Lambda^1 W \), so in particular \( *_W = *_\gamma \).

2.2. Local theory. Now we move our pointwise discussions to manifolds. Let \( X \) be an oriented 4-manifold with boundary, \( \omega = (\omega_1, \omega_2, \omega_3) \) be a smooth section in \( \Gamma(X, \Lambda^2 T^* X \otimes \mathbb{R}^3) \) such that it is a definite triple pointwise, and \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) be its restriction to \( \partial X \). By the discussions above, \( \omega \) defines a matrix valued function \( Q = (Q_{ij}) \), a volume form \( \mu \), and a Riemannian metric \( g_\omega \) which equals to \( \langle \cdot \rangle_\omega \) on the tangent spaces at each point. Similarly, \( \gamma \) defines \( \eta = (\eta_1, \eta_2, \eta_3) \) \( \in \Omega^1(\partial X) \otimes \mathbb{R}^3 \), \( e = (e_1, e_2, e_3) \in \Gamma(\partial X, T\partial X) \otimes \mathbb{R}^3 \). Denote \( \nabla \) the Levi-Civita connection of \( g_\omega \).

**Definition 2.1.** \( \omega \) is a hypersymplectic triple if \( d\omega = 0 \). A hypersymplectic triple \( \omega \) is called torsion-free if \( d(Q^{ij} \omega_j) = 0 \), and is called hyperkähler if \( Q_{ij} = \delta_{ij} \).

Note that the torsion-free definition coincides with the one defined in the introduction by direct calculation.

Let \( \nu \) be the outward unit normal vector field of \( \partial X \). We are going to calculate the second fundamental form of \( \partial X \), say \( II(v, w) = \langle \nabla_\nu v, w \rangle_{\partial X} \). Let \( S \in \Gamma(\partial X, \text{End} \ T\partial X) \) be the shape operator, i.e., \( \langle v, S(w) \rangle_{\partial X} = II(v, w) \). Denote \( \Gamma = (\Gamma_{ij}) \) the symmetric matrix
\[
\Gamma_{ij} = \frac{1}{2} \langle \gamma_i, d(*_\gamma \gamma_j) \rangle_\gamma + \frac{1}{2} \langle \gamma_j, d(*_\gamma \gamma_i) \rangle_\gamma = \frac{1}{2} *_\eta (\eta_i \land d\eta_j + \eta_j \land d\eta_i)
\]
which is completely determined by \( \gamma \). Denote the matrix \( II(e_i, e_j) \) by \( A \) and \( H = Tr S \).

**Lemma 2.2.** If \( \omega \) is a hyperkähler triple, then
\[
A = \frac{1}{2} (Tr \Gamma) I - \Gamma,
\]
In particular,
$$2H = s_\gamma (\eta \wedge d\eta^T) = \langle \gamma_1, d(*_\gamma \gamma_1) \rangle_\gamma + \langle \gamma_2, d(*_\gamma \gamma_2) \rangle_\gamma + \langle \gamma_3, d(*_\gamma \gamma_3) \rangle_\gamma,$$
so
$$|S|^2 = Tr\Gamma^2 - H^2.$$

Proof. Fix a point \( p \in \partial X \), choose a semi-geodesic coordinate system centered at \( p \), say \((x^1, x^2, x^3, t)\), such that a neighborhood of \( p \) is identified with \( \{t \geq 0\} \) and its intersection with \( \partial X \) is identified with \( \{t = 0\} \). The hyperkähler triple can be written as
$$\omega = -dt \wedge *_\gamma \gamma_t + \gamma_t,$$
where \( \gamma_t \) is a smooth family of closed framings on \( \partial X \) such that \( \gamma_0 = \gamma \) and
$$\frac{\partial \gamma_t}{\partial t} = -d(*_\gamma \gamma_t).$$

We can further choose \((x^1, x^2, x^3)\) to be a normal coordinate system for \( \partial X \) at \( p \) of the metric \( g_{\partial X} \) such that \( e_i = \partial_{x_i} \) at \( p \). Write \( \partial_{x_i} = \delta_i^k(x, t) e_k(x, t) \), where \( e_k(x, t) = (e_1(x, t), e_2(x, t), e_3(x, t)) \) is the dual frame of \( \eta_i = *_\gamma \gamma_t \). Then at \( p \),
$$II(e_i, e_j) = II(\partial_{x_i}, \partial_{x_j}) = -\frac{1}{2} \partial_t g_{ij} = -\frac{1}{2} \partial_t a^k_i a_j^k = -\frac{1}{2} (\partial_t a^k_i + \partial_t a_j^k).$$

Note that at \( p \),
$$\partial_t \eta_p = \partial_t a^k_i dx^k = \partial_t a^k_i \eta_m,$$
then we have
$$\langle \gamma_i, d(*_\gamma \gamma_j) \rangle_\gamma = -\langle \gamma_i, \partial_t \gamma_j \rangle_\gamma$$
$$= -\frac{1}{2} \delta_{ij} \partial_t a^k_i \eta_m \eta_p \eta_{\delta k} \eta_{\delta i} \eta_{\delta j}$$
$$= -\frac{1}{2} \delta_{ij} \partial_t a^k_i \eta_{\delta k} \eta_m \eta_p \eta_{\delta i} \eta_{\delta j}$$
$$= -\frac{1}{2} \delta_{ij} \partial_t a^k_i \eta_{\delta k} \eta_m \eta_p \eta_{\delta i} \eta_{\delta j}$$
$$= -\frac{1}{2} \delta_{ij} \partial_t a^k_i \eta_{\delta k} \eta_m \eta_p \eta_{\delta i} \eta_{\delta j}$$
$$= -\frac{1}{2} \delta_{ij} \partial_t a^k_i \partial_t a_j^k.$$

Combining (3) and (4), we have
$$\Gamma = (TrA)I - A,$$
so
$$Tr\Gamma = 2TrA, A = \frac{1}{2}(Tr\Gamma)I - \Gamma.$$

\[\square\]

3. General Riemannian geometry

3.1. Evolution equations of hypersurfaces. We refer to [30] Section 3.2 and [25] for discussions in this section.

Let \( M \) be a Riemannian manifold, \( f \) be a smooth distance function on \( M \), i.e.,
$$|\nabla f| = 1, \text{ so } \nabla \nabla f \nabla f = 0.$$ The \((1,1)\) tensor corresponds to \( \text{Hess} f \) is
$$S(X) = \nabla X \nabla f,$$
and its trace is \( H = \Delta f \in C^\infty(M) \). When \( \Sigma \subset M \) is a hypersurface defined by a level set of \( f \), the restriction of \( S \) to \( T \Sigma \) has image in \( T \Sigma \), which is the shape operator of \( \Sigma \). The second fundamental form of \( \Sigma \) with respect to \( \nabla f \) is
$$II(X,Y) := \langle S(X), Y \rangle = \text{Hess} f(X,Y),$$
so \( H \) is the mean curvature and \( \overline{H} = \)
$-H\nabla f$ is the mean curvature vector. By tensor calculations, we have evolution equations of second fundamental forms

\begin{align}
L_{\nabla f}S + S^2 &= -R(\cdot, \nabla f)\nabla f, \tag{6} \\
L_{\nabla f}\text{Hess} f - \text{Hess}^2 f &= -Rm(\cdot, \nabla f, \cdot, \nabla f), \tag{7}
\end{align}

where

$$\text{Hess}^2 f(X,Y) = \langle S^2(X), Y \rangle = \langle S(X), S(Y) \rangle,$$

and one has the equality

$$L_{\nabla f}S = \nabla_{\nabla f}S.$$ \tag{8}

Take the trace of (6), we have

$$L_{\nabla f}H = -|S|^2 - \text{Ric}(\nabla f, \nabla f).$$ \tag{9}

where $|S|^2 := \text{Tr}(S^2)$ is the norm square of the shape operator.

Besides the evolution equations, we have Gauss equations on $\Sigma$

$$Rm_M(X,W,Y,Z) - Rm_{\Sigma}(X,W,Y,Z) + II(X,Z)II(W,Y)$$ \tag{10}

$$- II(X,Y)II(W,Z).$$

Take the trace with respect to $W,Z$, we have

$$\text{Ric}_M = \text{Ric}_{\Sigma} + \text{Hess}^2 f - H \cdot \text{Hess} f + Rm_M(\cdot, \nabla f, \cdot, \nabla f),$$

take the trace again, we have

$$R_M = R_{\Sigma} + |S|^2 - H^2 + 2\text{Ric}_M(\nabla f, \nabla f),$$ \tag{12}

where $R$ denote scalar curvatures. Use equations (7) and (11) to cancel the curvature term involving $\nabla f$, we get

$$L_{\nabla f}\text{Hess} f = \text{Ric}_{\Sigma} - \text{Ric}_M + 2\text{Hess}^2 f - H \cdot \text{Hess} f.$$ \tag{13}

### 3.2. The boundary exponential map.

Let $(M,g)$ be a complete Riemannian manifold with boundary, which means the induced metric space is complete. Denote $T^\perp \partial M$ the normal line bundle of $\partial M$, which is a trivialized by the inward unit normal vector field $N$. We identify $T^\perp \partial M$ with $\partial M \times \mathbb{R}$ via this trivialization. For $p \in \partial M$, denote $\gamma_p(t)$ the geodesic such that $\gamma_p(0) = p, \gamma'_p(0) = N_p$. Denote

$$D(p) = \inf\{t > 0|\gamma_p(t) \in \partial M\} \in (0, \infty],$$

$$\tau(p) = \sup\{t > 0|d(\gamma_p(t), \partial M) = t\} \in (0, \infty].$$

We have a subset of $U_{\partial M} \subset T^\perp \partial M$ defined by

$$U_{\partial M} = \{(p,tN_p) \in T^\perp \partial M|0 \leq t < D(p)\},$$

which is the domain of the boundary exponential map

$$\exp_\perp : U_{\partial M} \to M, (p,s) \mapsto \gamma_p(s).$$ \tag{14}

and define

$$V_{\partial M} = \{(p,tN_p) \in T^\perp \partial M|0 \leq t < \tau(p)\} \subset U_{\partial M}.$$ 

There are some definitions, notations and terminologies related to the boundary exponential map that appear in this paper:

- The boundary injectivity radius $i_b$ is defined to be the supremum of $s \geq 0$ such that $\exp_\perp|_{\partial M \times [0,s]}$ is a diffeomorphism onto its image.
• A focal point \( q \) of \( \partial M \) is a critical value of the boundary exponential map (14). If \( q \) lies in \( \gamma_p \) for some \( p \in \partial M \), we say \( q \) is a focal point along \( \gamma_p \).
• A foot point of \( q \in M \) is a point \( p \in \partial M \) such that \( d(q, p) = d(q, \partial M) \).
• A cut point of \( \partial M \) is a point \( q \in M \) such that there exists a foot point \( p \) of \( q \) such that \( d(q, p) = \tau(p) \). We also say \( q \) is a cut point of \( p \).
• When we say a covariant tensor on \( M \) is written in geodesic gauge, we mean the pull back of this tensor via \( \exp_{\perp} \).
• For a subset \( B \subset \partial M \), we use the notation \( C(B, t_1, t_2) = \exp_{\perp}(B \times [t_1, t_2]) \) to denote a metric cylinder with base \( B \).
• \( N_r(\partial M, g) = \{ x \in M | d(x, \partial M) \leq r \} \).
• The \((1, 1)\) tensor \( S \) in (5) is defined with respect to \( f = -d(\cdot, \partial M) \) near \( \partial M \), so \( \nabla f = -N \) on \( \partial M \), and \( II \geq 0 \) if \( \partial M \) is convex.

Here are some remarks about some of these definitions:

• By definition, \( \gamma_p(s) \) is a focal point of along \( \gamma_p \), if and only if there exists a non-zero \( \partial M \)-Jacobi field \( V \) along \( \gamma_p \) (a Jacobi field with \( V(0) \in T_p \partial M, V'(0) + S(V(0)) \in T_{\perp p} \partial M \)) such that \( V(s) = 0 \). If there is no focal point along \( \gamma_p | [0, l] \), then

\[
I_0(W, W) = \int_0^l \langle W', W' \rangle - \langle R(W, \gamma_p')\gamma_p', W \rangle dt - \langle S(W), W \rangle(0) \geq 0
\]

for any piecewise smooth vector field \( W \) along \( \gamma \) with \( W(0) \in T_{\gamma(0)} \partial M \).
• By Lemma 3.2 in [31], \( \tau \) defines a continuous map from \( \partial M \) to \((0, \infty] \). It is well-known that by a second variation argument, the first focal point along \( \gamma_p \) appears no later than \( \tau(p) \), and moreover one can argue by contradiction to get (See Lemma 3.6 in [31])

**Proposition 3.1.** \( q \in M \) is a cut point of \( p \) if and only if at least one of the following holds:

• \( q \) is the first focal point of \( \gamma_p \);
• \( q \) has at least two foot points.

From this, we have \( \exp_{\perp} |_{V_{\partial M}} \) is a diffeomorphism and

\[
i_b = \inf_{p \in \partial M} \tau(p).
\]

It is worth noting that if \( M \) is embedded in some complete Riemannian manifold \( M' \) of the same dimension and view \( \partial M \) as an embedded hypersurface of \( M' \), then it may happen that a “focal point” of \( \partial M \) in \( M' \) lies outside \( M \) if we define the boundary exponential map in the whole normal bundle \( \nabla^2 \partial M \). However, our definition is intrinsic for \( M \).

### 3.3. Manifolds with mean convex boundary.
Now we focus on manifolds with mean convex boundary. We will summarize some results and discuss a new result.

The following results are well-known, see for example [28] and [14].

**Proposition 3.2.** Let \((M, g)\) be a compact, connected Riemannian manifold with boundary, \( \text{Ric}_M \geq 0 \). Suppose \( \partial M \) has mean curvature \( H \geq H_0 > 0 \), then \( \partial M \) is connected,
\( \pi_1(M, \partial M) = 0, \)
\[ \sup_{q \in M} d(q, \partial M) \leq (n-1)H_0^{-1}, \]
\[ \text{vol}(M) \leq C(n)H_0^{-1} \text{vol}(\partial M). \]

Proof. If \( \pi_1(M, \partial M) \neq 0 \) or \( \pi_0(\partial M) \neq 0 \), then every non-trivial class contains a non-trivial unit speed geodesic \( \gamma : [0, l] \to M \) which minimize the length of all curves in its class. From the first variation formula, \( \gamma \) intersects boundary perpendicularly at both end points. Pick an orthonormal basis \( \gamma \) and parallel them transport along \( \gamma \).

\[ \text{Proposition 3.1} \]
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It is worth noting that in this proof, Kodani used the first order variation of $h_1$ on $\Sigma$ to lead a contradiction. We can also investigate the second order variation of $h_1$ on $\Sigma$ and prove the following:

**Proposition 3.5.** Let $M$ be a compact Riemannian manifold with mean convex boundary, $\text{Ric}_M \geq 0$, then there exists a focal point of $\partial M$ whose distance to $\partial M$ is equal to $i_b$.

**Proof.** Suppose not, by the previous proposition, we have a smooth geodesic of length $2i_b$ which is perpendicular to $\partial M$ at both end points. We use notations $p_1, p_2, q, h_1, h_2, \Sigma$ as in the previous proposition. We claim $\Delta_\Sigma h_1(q) < 0$, so we get another point $q'' \in \Sigma$ near $p$ with $h_1(q'') < h_1(q) = i_b$ and get a contradiction. Denote $N_0 = \nabla h_1(q) = -\nabla h_2(q)$, $\Sigma_1 = h_1^{-1}(i_b), \Sigma_2 = h_2^{-1}(i_b)$, then $N_0$ is a common unit normal vector for $\Sigma_1, \Sigma_2$ at $q$. Let $II_\Sigma, II_{\Sigma_1}, II_{\Sigma_2}$ be second fundamental forms with respect to $N_0$ at $q$. Then at $q$,

$$II_\Sigma = \frac{1}{|\nabla(h_1 - h_2)|} \text{Hess}(h_1 - h_2) = \frac{1}{2} \text{Hess}(h_1 - h_2) = \frac{1}{2}(II_{\Sigma_1} + II_{\Sigma_2}),$$

hence

$$H_\Sigma = \frac{1}{2}(H_{\Sigma_1} + H_{\Sigma_2}), \quad \vec{H}_\Sigma = \frac{1}{2}(\vec{H}_{\Sigma_1} + \vec{H}_{\Sigma_2}).$$

From the formula of Laplace operator on a hypersurface, we know that at $q$,

$$\Delta_\Sigma h_1 = \Delta h_1 - \text{Hess} h_1(N_0, N_0) + \langle \nabla h_1, \vec{H}_\Sigma \rangle,$$

$$\Delta_{\Sigma_i} h_1 = \Delta h_1 - \text{Hess} h_1(N_0, N_0) + \langle \nabla h_1, \vec{H}_{\Sigma_i} \rangle.$$

Since $h_1$ is a constant on $\Sigma_1, \Sigma_2$, $h_1 = 0$, hence

$$\Delta_\Sigma h_1 = \Delta_{\Sigma_1} h_1 - \Delta_{\Sigma_2} h_1 = \langle \nabla h_1, \vec{H}_{\Sigma_1} - \vec{H}_{\Sigma_2} \rangle = \langle \nabla h_1, \frac{1}{2}(\vec{H}_{\Sigma_1} - \vec{H}_{\Sigma_2}) \rangle$$

(15)

$$= -\frac{1}{2}(H_{\Sigma_2} - H_{\Sigma_1}).$$

Since $\partial M$ is mean convex, $\text{Ric}_M \geq 0$, by the evolution equation of mean curvature (9), we have $-H_{\Sigma_1}(q) > H_{\partial M}(p_1) > 0, H_{\Sigma_2}(q) > H_{\partial M}(p_2) > 0$. Hence $\Delta_\Sigma h_1(p) < 0$, which completes the proof. 

\[\square\]

A moment thought about the arguments in the end of the previous proof yields that $\text{Ric}_M \geq 0$ is not so necessary, since we can make use of the evolution equation (9) to get an ordinary differential inequality for the mean curvature.

**Proposition 3.6.** If in the previous proposition we assume instead $\text{Ric}_M \geq -(n-1)c$ for some $c > 0$, and $H \geq H_0 > 0$. If $i_b < -\frac{1}{n}(n-1) \ln \left| \frac{h_0 - (n-1)\sqrt{c}}{h_0 + (n-1)\sqrt{c}} \right|$, then there exists a focal point of $\partial M$ whose distance to $\partial M$ is equal to $i_b$.

**Proof.** Suppose the conclusion is not true, follow the arguments as before except for second last sentence. Let $S_i(X) = -\nabla_X \nabla h_i, H_i = Tr S_i = -\Delta h_i$, and identify a neighborhood of $\gamma_{p_i}([0, i_b])$ with a subset of $\partial M \times \mathbb{R}$ via $\exp^\perp$. Then by (9),

$$\partial_t H_i = |S_i|^2 + \text{Ric}(\nabla h_i, \nabla h_i) \geq \frac{1}{n-1} H_i^2 - (n-1)c$$

and $H(p_i, 0) \geq H_0$. Let $f$ solves the ODE on $[0, i_b]$

$$f' = \frac{1}{n-1} f^2 - (n-1)c$$
and \( f(0) = H_0 \), then we have \( H_i(p_t, t) \geq f(t) \). In particular, \( H_i(p_t, i_b) \geq f(i_b) > 0 \), which leads to a contradiction as before.

Proposition 3.5 implies

**Corollary 3.7.** Let \((M, g)\) be a compact Riemannian manifold with boundary, \(K > 0, \lambda > 0\) are constants. Suppose \( \sec \leq K, S \leq \lambda, H \geq H_0 > 0\), then \( i_b \geq \frac{1}{\sqrt{K}} \arccot \frac{\lambda}{\sqrt{K}} \).

*Proof.* By Proposition 3.5, there exists \( p \in \partial M \) such that \( \gamma_p(i_b) \) is a focal point along \( \gamma_p \). If \( i_b < \frac{1}{\sqrt{K}} \arccot \frac{\lambda}{\sqrt{K}} \), from comparison theorem for Jacobi fields, we know \( \gamma_p(i_b) \) cannot be a focal point along \( \gamma_p \), which is a contradiction. \( \square \)

Similarly, Proposition 3.6 implies

**Corollary 3.8.** Let \((M, g)\) be a compact Riemannian manifold with boundary. Suppose \( |\text{Rm}| \leq C, |S| \leq C, H \geq H_0 > 0 \), then we can find \( i_0 \) depending explicitly on \( C, H_0 \) such that \( i_b \geq i_0 \).

**Remark 3.9.** In the previous two corollaries, if the sectional curvature and Ricci curvature bounds only holds for \( N_i(\partial M, g) \), then we also have a \( i_b \) lower bound. In the case of Corollary 3.7, we have \( i_b \geq \min\left\{ \frac{1}{\sqrt{K}}, \arccot \frac{\lambda}{\sqrt{K}}, 1 \right\} \).

**Remark 3.10.** In the same setting as the previous two corollaries, [22] Lemma 2.2 claimed to prove a lower bound for \( i_b \), using a similar method as [5] Lemma 2.4. In both papers, there is a logic problem that they get a contradiction with an unjustified statement: Let \( M \) be a Riemannian manifold with boundary, \( \gamma : [0, l] \to M \) be a geodesic that is perpendicular to the boundary at both end points, and suppose there is no focal point along \( \gamma \) for both boundary portions, then \( I_1(V, V) \geq 0 \) for any smooth vector field along \( \gamma \) with \( V(0), V(l) \in T\partial M \). Here

\[
I_1(V, V) = \int_0^l \langle V', V' \rangle - \langle R(V, \gamma')\gamma', V \rangle dt - \langle S(V(0)), V(0) \rangle - \langle S(V(l)), V(l) \rangle.
\]

In fact, this unjustified statement is not true, and one can easily think of an example: let

\[
\Sigma_1 = \{(x', x^n) \in \mathbb{R}^n | |x'|^2 + (1 - x^n)^2 = R_1^2 \},
\]

\[
\Sigma_2 = \{(x', x^n) \in \mathbb{R}^n | |x'|^2 + (1 + x^n)^2 = R_2^2 \},
\]

\( R_1, R_2 > 2 \) and \( \gamma(t) = (0, 1 - t), 0 \leq t \leq 2 \), then \( I_1(V, V) = -\frac{1}{R_1^4} - \frac{1}{R_2^4} < 0 \) for any unit-norm parallel vector field along \( \gamma \) with \( V(0) \in T_{\gamma(0)}\Sigma_1 \). In this case, there exist no focal points on \( \gamma \) for both \( \Sigma_1, \Sigma_2 \).

In fact, focal points give crucial information for index form defined by one submanifold and one point. However, as seen from the example, they do not fit well with the index form defined for two submanifolds. Indeed, there is some notion of “conjugate point” defined for two submanifolds, see [1].

It is easy to see focal points can “pass to the limit”, since they arise from kernels of the differential of exponential maps. Though one can use Corollary 3.8 directly in many situations, we point out this fact here, which may help in contradiction arguments.
Proposition 3.11. Let $M$ be a manifold with boundary, $g_i$ be a sequence of Riemannian metrics on $M$, and $p_i \in \partial M$. Suppose $g_i$ converges to a Riemannian metric $g_\infty$ smoothly and $p_i$ converges to $p_\infty \in \partial M$, $\gamma_{p_i}$ is defined on $[0, b]$ and $\gamma_{p_i}(t_i)$ is a focal point along $\gamma_{p_i}$ with $0 < a \leq t_i \leq b$. Then for a subsequence, $\gamma_{p_i}$ converges smoothly to $\gamma_{p_\infty}$, $t_i \to t_\infty$ and $\gamma_{p_\infty}(t_\infty)$ is a focal point along $\gamma_{p_\infty}$.

Proof. $\gamma'_{p_i}(0) \in TM$ is a bounded sequence, hence subconverges to some $v \in TM$, which must equal to $\gamma'_{p_\infty}(0)$. Hence by ODE theories, $\gamma_{p_i}$ converges smoothly to $\gamma_{p_\infty}$.

Suppose for a subsequence $t_i \to t_\infty$. Let $J_i : [0, t_i] \to TM$ be $\partial M$-Jacobi fields along $\gamma_i$ with $J_i(t_i) = 0$, $|J'_i(t_i)| = \frac{1}{\sqrt{n}}$. Normalize these geodesics and Jacobi-fields by $\bar{\gamma}_i(t) = \gamma_i(\frac{1}{t_i}t)$, $\bar{J}_i(t) = \bar{J}_i(\frac{1}{t_i}t)$, so $\bar{\gamma}_i, \bar{J}_i$ are defined on the same interval $[0, t_\infty]$, and

$$\begin{align*}
\bar{J}_i''(t) + Rm_{g_i}(\bar{J}_i(t), \bar{\gamma}'_i(t))\bar{\gamma}'_i(t) &= 0,
\bar{J}_i(t_\infty) &= 0, |\bar{J}_i(t_\infty)| = 1.
\end{align*}$$

(16)

For a subsequence $\bar{J}_i'(t_\infty) \to w$ with $|w| = 1$. Let $J_\infty$ be the non-trivial Jacobi-field along $\gamma_{p_\infty}$ with $J_\infty(t_\infty) = 0$, $J'_\infty(t_\infty) = w$, then $J_i$ converges smoothly as maps $[0, t_\infty] \to TM$ to $J_\infty$ by ODE theories. Hence $J_\infty$ is a $\partial M$-Jacobi field, which implies $\gamma_{p_\infty}(t_\infty)$ is a focal point along $\gamma_{p_\infty}$. \hfill \Box

3.4. Volume estimates near boundary. In this subsection, we show some volume lower bounds near boundary under some geometric control. These estimates can be found in [2] and [22].

Proposition 3.12. Let $(M, g)$ is a Riemannian manifold with boundary, suppose $\text{Ric} \geq -(n-1)c$ for some $c \geq 0$, and $\exp^+$ is an diffeomorphism in $B \times [0, T)$ for some open subset $B$ of $\partial M$, then

$$\begin{align*}
\sup_{B \times \{t\}} H &\leq \max\{(n-1)\sqrt{2c}, 4(n-1)^{-1}T^{-1}\}, \\
\text{vol}(C(B, t_1, t_2)) &\geq C(n, T, c)\text{vol}_{\partial M}(B)|t_2 - t_1|,
\end{align*}$$

where $0 \leq t, t_1, t_2 \leq \frac{1}{2}T$.

Proof. By the evolution equation (9),

$$\begin{align*}
\partial_t H &= |S|^2 + \text{Ric}(N_t, N_t),
\end{align*}$$

hence

$$\begin{align*}
\partial_t H &\geq \frac{1}{n-1}H^2 - (n-1)c.
\end{align*}$$

(18)

If for some $t_0 \in [0, \frac{1}{2}T]$, $z_0 \in B$, we have $H(z_0, t_0) \geq \delta$ and $\delta \geq (n-1)\sqrt{2c}$, then $\partial_t H(z_0, t) \geq 0$ and $H(z_0, t) \geq (n-1)\sqrt{2c}$ for $t \in [t_0, T)$. Hence

$$\begin{align*}
\partial_t H(z_0, t) &\geq \frac{1}{2}(n-1)^{-1}H(z_0, t)^2,
H^{-1}(z_0, t) &\leq H^{-1}(z_0, t_0) \left(\frac{1}{2}(n-1)(t - t_0)\right) \leq \delta^{-1} - \frac{1}{2}(n-1)(t - \frac{1}{2}T).
\end{align*}$$

Then the continuity of $H(z_0, \cdot)$ in $[0, T)$ forces $\delta \leq 4(n-1)^{-1}T^{-1}$, which proves the mean curvature estimate.

For $t \in [0, \frac{1}{2}T]$, let $B_t = \exp^+(B \times \{t\})$, then

$$\begin{align*}
\frac{d}{dt} H^{n-1}(B_t) &= -\int_{B_t} HdH^{n-1}(B_t) \geq -C_1(n, T, c)H^{n-1}(B_t),
\end{align*}$$

where $C_1(n, T, c) = C(n, T, c)|\partial M|(B_T)$. Therefore

$$\begin{align*}
H^{n-1}(B_t) \leq C_2(n, T, c)|\partial M|(B_T).
\end{align*}$$

(20)

Thus we have the required volume estimates near boundary.
where $\mathcal{H}^{n-1}$ denotes the \((n-1)\)-dimensional Hausdorff measure of $M$. Hence
\[
\mathcal{H}^{n-1}(B_t) \geq e^{-C_1(n,T,c)t} \text{vol}_{\partial M}(B) \geq e^{-\frac{1}{4}C_1(n,T,c)} \text{vol}_{\partial M}(B)
\]
and when $0 \leq t_1 < t_2 \leq \frac{1}{2}T$,
\[
\text{vol}(C(B,t_1,t_2)) = \int_{t_1}^{t_2} \mathcal{H}^{n-1}(B_t) dt \geq C(n, T, c) \text{vol}_{\partial M}(B)(t_2 - t_1).
\]

Proposition 3.13. Let $M$ be a Riemannian manifold with boundary, $p \in \partial M$. Suppose $B(p, 2r_0)$ has compact closure,
\[
\sup_{B(p, 2r_0)} |Rm| \leq C,
\]
\[
\text{vol}_{\partial M}(B_{\partial M}(p, r_0)) \geq v_0,
\]
exp is an diffeomorphism in $B_{\partial M}(p, r_0) \times [0, r_0)$, and on $B_{\partial M}(p, r_0)$
\[
\text{Ric}_{\partial M} \geq -(n-2)c_0, \quad |S| \leq C,
\]
then there exists $r_1 > 0, v_1 > 0$ depending on $n, C, c_0, r_0, v_0$, such that for $q = \exp_{\partial M}(p, 2r_1)$, we have
\[
\text{vol}(B(q, r_1)) \geq v_1.
\]

Proof. By definition $C(B_{\partial M}(p, r_0), 0, r_0) \subset B(p, 2r_0)$. We claim that there exists $r_2 > 0, C_1 > 0$ such that for any $p_1, p_2 \in B_{\partial M}(p, r_0)$,
\[
d_{\Sigma_t}(\exp_{\partial M}(p_1, t), \exp_{\partial M}(p_2, t)) \leq C_1 d_{\partial M}(p_1, p_2)
\]
when $0 \leq t \leq r_2$, where $\Sigma_t$ is the image of $B_{\partial M}(p, 2r_0)$ under $\exp_{\partial M}(\cdot, t)$. In fact, by (6)(8),
\[
\nabla_{\nabla t} S = S^2 + R(\cdot, \nabla t) \nabla t,
\]
hence
\[
\frac{d}{dt} |S| \leq |S|^2 + C.
\]
Integrate the inequality, we have
\[
\arctan\left(\frac{|S|}{\sqrt{C}}\right)(x, t) - \arctan\left(\frac{|S|}{\sqrt{C}}\right)(x, 0) \leq \sqrt{C}t.
\]
Hence there exist $r_2 > 0, C_2 > 1$ such that $|S| \leq \log C_2$ when $0 \leq t \leq r_2$. Now let $\gamma_0(s)$ be a smooth curve in $B_{\partial M}(p, 2r_0)$ that connects $p_1, p_2$, and let $\gamma_t(s) = \exp_{\partial M}(\gamma_0(s), t) \in \Sigma_t$, then we have
\[
\frac{d}{dt} \log |\gamma_t'(s)| = -\frac{\langle S_{\Sigma_t}(\gamma_t'(s)), \gamma_t'(s) \rangle}{\langle \gamma_t'(s), \gamma_t'(s) \rangle} \leq |S_{\Sigma_t}(\gamma_t(s))| \leq \log C_2.
\]
It follows that $|\gamma_t'(s)| \leq C_1 |\gamma_0'(s)|$ with $C_1 = C_2^2$ and the claim follows from integration. Now take
\[
r_1 = \min\{2C_1 r_0, \frac{1}{4} r_2\},
\]
we have
\[
C(B_{\partial M}(p, \frac{r_1}{2C_1}), \frac{3r_1}{2}, \frac{5r_1}{2}) \subset B(q, r_1),
\]
then we apply Proposition 3.12 and Bishop-Gromov volume comparison on $\partial M$ to get the desired conclusion. \qed
Remark 3.14. It is easy to give a quantitative version of the lemma from the proof. However, to the author’s knowledge, we cannot prove the last inclusion in the proof without a control of curvature. It may be possible that a metric ball of the boundary becomes “long and thin” under the flow of $\nabla d(\cdot, \partial M)$, while maintains an area lower bound.

3.5. Harmonic radius and convergence theory. Convergence theory of Riemannian manifolds is a powerful tool to prove conclusions in Riemannian geometry through contradiction arguments when explicit bounds is not required. In this section, we will restate some results of [2], follow the proof there, and discuss some direct corollaries.

Let $(M, g)$ be a Riemannian manifold with boundary, $m \in \mathbb{N}$, $0 < \alpha < 1$, $Q > 1$. For $p \in M$, define $r_{h}^{m,\alpha}(p, g, Q)$ to be the supremum of $\rho > 0$ such that if $d(p, \partial M) > \rho$, then there exists a neighborhood $U$ of $p$ in $M$ and an interior coordinate chart $\varphi : B_{\rho}^{+} \to U$, $\varphi(0) = p$, and if $d(p, \partial M) \leq \rho$, then there exists a neighborhood $U$ of $p$ in $M$ and a boundary coordinate chart $\varphi : B_{\rho}^{+} \to U$, $\varphi((0, d(p, \partial M))) = p$, $\varphi(\partial B_{\rho}^{+}) = U \cap \partial M$, and in either $B_{\rho}^{+}$ or $B_{\rho}^{+}$, we have

$$\Delta_{M} \varphi^{-1} = 0,$$

$$Q^{-2}(\delta_{ij}) \leq (g_{ij}) \leq Q^{2}(\delta_{ij}),$$

$$\rho^{m+\alpha} \sum_{|\beta|=m} |\partial_{\beta}g_{ij}(x) - \partial_{\beta}g_{ij}(y)| \leq (Q - 1)|x - y|^\alpha$$

We call such a coordinate chart a $(\rho, Q, m, \alpha)$-harmonic coordinate chart centered at $p$. Note that the second condition implies there exists $r_{1}, r_{2}$, depending on $\rho, Q$, such that $B(p, r_{1}) \subset U \subset B(p, r_{2})$.

Definition 3.15. Fix an integer $m \geq 0$, and $0 < \alpha < 1$. We say a sequence of Riemannian manifold with boundary $(M_{i}, g_{i}, p_{i})$ converges in pointed $C^{m,\alpha}$ to $(M_{\infty}, g_{\infty}, p_{\infty})$ if there exists precompact open subsets $\Omega_{i}$ of $M_{i}$ and $\Omega_{\infty,i}$ of $M_{\infty}$, and $\sigma_{i} > 0 \to \infty$ such that $B(p_{i}, \sigma_{i}) \subset \Omega_{i} \subset B(p_{i}, \sigma_{i})$, $B(p_{\infty}, \sigma_{i}) \subset \Omega_{i} \subset B(p_{\infty}, \sigma_{i})$ and there exists diffeomorphisms $F_{i} : \Omega_{\infty,i} \to \Omega_{i}$, $F_{i} : \Omega_{\infty,i} \cap \partial M_{i} \to \Omega_{i} \cap \partial M_{\infty}$ such that $F_{i}^{*}g_{i} \to g$ in $C^{m,\alpha}$ topology, and $F_{i}^{-1}(p_{i}) \to p_{\infty}$. If we replace $C^{m,\alpha}$ by $C^{\infty}$, we say the convergence is in pointed Cheeger-Gromov sense.

Remark 3.16. $(M_{\infty}, g_{\infty})$ is automatically a complete $C^{m,\alpha}$ or $C^{\infty}$ Riemannian manifold with boundary from the definitions. Sometimes we only need that one metric ball converges, so one can modify the definitions above: suppose $B(p_{i}, r) \subset \Omega_{i}$ for some precompact open set $\Omega_{i} \subset M_{i}$ and there exists a Riemannian manifold with boundary $(\Omega_{\infty}, g_{\infty})$, a point $p_{\infty} \in \Omega_{\infty}$, and diffeomorphisms $F_{i} : \Omega_{\infty} \to \Omega_{i}$ mapping $\partial \Omega_{\infty}$ onto $\Omega_{i} \cap \partial M_{i}$ such that $F_{i}^{\ast}g_{i} \to g_{\infty}$ in $C^{m,\alpha}$ or $C^{\infty}$ topology and $F_{i}^{-1}(p_{\infty}) \to p_{i}$, we say $B(p_{i}, r)$ converges in $C^{m,\alpha}$ or Cheeger-Gromov sense to $B(p_{\infty}, r)$.

The following theorem is well-known and is a fundamental theorem of Riemannian convergence theory.

Proposition 3.17. Let $(M_{i}, g_{i})$ be a sequence of complete Riemannian manifold with boundary, $p_{i} \in M_{i}$. Suppose there exists some $Q > 1$, and a positive function $r : (0, \infty) \to (0, \infty)$, such that $r^{m,\alpha}_{h}(p, g_{i}, Q) \geq r(R)$ for any $p \in B(p_{i}, R)$, then for a subsequence, $(M_{i}, g_{i}, p_{i})$ converges in pointed $C^{m,\beta}$ sense to $(M_{\infty}, g_{\infty}, p_{\infty})$. 
for any $0 < \beta < \alpha$. If the above assumption holds for only one $R$, then $B(p_1, R)$ converges in $C^{m,\beta}$ sense to $B(p_\infty, R)$.

Next, we discuss under what geometric control we can get a harmonic radius lower bound. We state and prove the following local version of Theorem 3.2.1 in [2], with simplified arguments in some parts.

**Theorem 3.18.** Fix $m \geq 1$. Let $(M, g)$ be a Riemannian manifold with boundary, and $\Sigma \subset \partial M$ be a boundary metric ball with compact closure, nonempty boundary. Suppose $\exp^+$ maps $\Sigma \times [0, i_0)$ diffeomorphically onto its image $\Omega$,

$$\text{inj}_\Omega \geq i_0, \text{ inj}_\Sigma \geq i_0,$$

in $\Omega$, (19)

$$|\nabla^l \text{Ric}_M| \leq \Lambda, 0 \leq l \leq m,$$

on $\Sigma$, (20)

$$|\nabla^l \text{Ric}_M| \leq \Lambda, |\nabla^{l+1} H| \leq \Lambda, 0 \leq l \leq m.$$ (21)

Then for any $Q > 1$, $\alpha \in (0, 1)$, $p \in \Omega$,

$$r_{h_p}^{m+1,\alpha}(p, g, Q) \geq r_0(i_0, \Lambda, m, \alpha, Q) d(p, \partial^+ \Omega),$$ (22)

where $\partial^+ \Omega = \Omega \setminus (\Omega \cup \partial M)$.

**Remark 3.19.** One should understand (19) as follows: for an open set $U$ inside a Riemannian manifold with boundary, $\text{inj}_U \geq i_0$ means for each $p \in U$, $\exp_p$ maps $B_0(0) \subset T_p M$ diffeomorphically onto its image if $d(p, U^c) \geq i_0$, and maps $B_0(D(p, U^c)) \subset T_p M$ diffeomorphically onto its image if $d(p, U^c) \leq i_0$.

Proof. If not, we have a sequence $(M_k, \bar{g}_k)$ and $\Sigma_k, \Omega_k$ that satisfies the conditions, but there exists $p_k \in \Omega_k$ with

$$\frac{r_{h_p}^{m+1,\alpha}(p_k, \bar{g}_k, Q)}{d_{\bar{g}_k}(p_k, \partial^+ \Omega_k)} = \inf_{p \in \Omega_k} \frac{r_{h_p}^{m+1,\alpha}(p_k, \bar{g}_k, Q)}{d_{\bar{g}_k}(p, \partial^+ \Omega_k)} \to 0.$$

Rescale the metric $g_k = (r_{h_p}^{m+1,\alpha}(p_k, \bar{g}_k, Q))^{-2} \bar{g}_k$, so $r_{h_p}^{m+1,\alpha}(p_k, g_k, Q) = 1$, then $d_{g_k}(p_k, \partial^+ \Omega_k) \to \infty$, and $r_{h_p}^{m+1,\alpha}(p_k, g_k, Q) \geq \frac{1}{2}$ if $d_{g_k}(p, p_k) \leq R$, $k \geq k(R)$. Fix any $\beta \in (0, \alpha)$. Then there are two cases:

**Case 1** $d_{g_k}(p_k, \Sigma_k) \to \infty$ for some subsequence.

Then a subsequence $(M_k, g_k, p_k)$ converges in pointed $C^{m+1,\beta}$ sense to a complete Riemannian manifold $(M_\infty, g_\infty, p_\infty)$. So $\text{Ric}_M \to 0$, $i_{\text{inj} M_\infty} = \infty$. By Cheeger-Gromoll splitting theorem, $(M_\infty, g_\infty)$ is isometric to $(\mathbb{R}^n, g_{\text{flat}})$.

Hence for any $L > 0$, there exist a coordinate $\varphi_{0,k} : B_{L+5} \to U_k \subset M_k, \varphi_{0,k}(0) = p_k, \text{ such that}$

$$\|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\beta}(B_{L+5})} \to 0, 1 \leq i, j \leq n.$$ (23)

We solve the Dirichlet problem for functions $u_\nu^\nu$, $1 \leq \nu \leq n$:

$$\Delta_{M_k} u_\nu^\nu = 0 \text{ in } B_{L+5}, u_\nu^\nu|_{\partial B_{L+5}} = x_\nu^\nu.$$ (24)

Recall the formula

$$\Delta_g = g^{ij} \partial_i \partial_j + \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} |g|^{ij}) \partial_j, |g| = \det(g_{ij}).$$
Then we have
\[ \|u_k^\nu - x^\nu\|_{C^{m+2,\alpha}(B_{L+1})} \leq C\|\Delta_{M_k}(u_k^\nu - x^\nu)\|_{C^{m,\beta}(B_{L+1})} \to 0. \]

Hence, we get a new coordinate system \((u_1^\nu, \ldots, u_n^\nu)\) and we discard the original coordinate system, and we use the same notation for tensors written in the new coordinate system, so in the new coordinate system we have
\[ \|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\alpha}(B_{L+1})} \to 0. \]

Now we want to improve the convergence of \(g_{k,ij}\) from elliptic equations. We have a system of equations
\[ \Delta_{M_k}g_{k,ij} + B_{ij}(g_k, \partial g_k) = -2\text{Ric}_{M_k,ij}, \]
where \(B_{ij}(g, \partial g)\) are polynomials of \(g, \partial g\) and are quadratic in \(\partial g\). From \(W^{m+2,p}\) estimates, Morrey embeddings, and
\[ |\nabla^l\text{Ric}_{M_k}| \to 0, \text{ for } 0 \leq l \leq m, \]
we have for 1 \(\leq i, j \leq n\)
\[ \|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\alpha}(B_{L+1})} \leq C(\|\Delta_{M_k}(g_{k,ij} - \delta_{ij})\|_{C^{m,\beta}(B_{L+1})} + \|g_{k,ij} - \delta_{ij}\|_{L^\infty(B_{L+1})}) \to 0. \]

Hence we get a \((2(L+2), Q, m+1, \alpha)\) harmonic coordinate chart centered at \(p^*_k\), with \(d_{g_k}(p^*_k, p_k) \to 0\), then \(r^m,\alpha(p_k, g_k, Q) \geq 2(L+1)\) for large \(k\), which is a contradiction.

**Case 2** \(d_{g_k}(p_k, \Sigma_k) \leq K\).

A subsequence \((M_k, g_k, p_k)\) converges in pointed \(C^{m+1,\beta}\) sense to a complete Riemannian manifold with boundary \((M_{\infty}, g_{\infty}, p_{\infty})\) and \((\partial M_k, g_k, q_k)\) converges in \(C^{m+1,\beta}\) sense to \((\partial M_{\infty}, g_{\infty}\partial M_{\infty}, q_{\infty})\), where \(q_k \in \Sigma_k\) is the unique foot point of \(p_k\) in \(\Sigma_k\). Then \(\text{Ric}_{M_{\infty}} = 0, \text{Ric}_{\partial M_{\infty}} = 0, H_{\infty} = 0, i_{\partial M_{\infty}} = \infty, i_{b,M_{\infty}} = \infty\). Hence \((\partial M_{\infty}, g_{\infty}\partial M_{\infty})\) is isometric to \((\mathbb{R}^{n-1}, g_{\text{flat}})\). By (12), we have \(S_{\infty} = 0\). Then by Lemma 4.5, \((M_{\infty}, g_{\infty})\) is a smooth Riemannian manifold with boundary and \(Rm_{M_{\infty}} = 0\). Since also \(i_{b,M_{\infty}} = \infty, (M_{\infty}, g_{\infty})\) is an isometric to \((\mathbb{R}^n, g_{\text{flat}})\).

Hence for any \(L > 2K + 10\), there exist a coordinate \(\varphi_{0,k} : B_{L+5}^+ \to U_k \subset M_k, \varphi_{0,k}(0) = q_k, \varphi_{0,k}(B_{L+5}) = U_k \cap \partial M_k\) such that
\[ \|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\beta}(B_{L+1})} \to 0, 1 \leq i, j \leq n. \]

First, we solve for functions \(v^\nu_{k'}\), \(1 \leq \nu \leq n-1\),
\[ \Delta_{\partial M_k}v^\nu_{k'} = 0 \text{ in } B_{L+5}, v^\nu_{k'}|_{\partial B_{L+5}} = x^\nu, \]
Then we have
\[ \|v^\nu_{k'} - x^\nu\|_{C^{m+2,\beta}(\overline{B_{L+5}})} \leq C\|\Delta_{\partial M_k}(v^\nu_{k'} - x^\nu)\|_{C^{m,\beta}(\overline{B_{L+5}})} \to 0. \]

Next, we solve for \(1 \leq \nu \leq n-1\),
\[ \Delta_{M_k}u^\nu_{k'} = 0 \text{ in } B_{L+5}^+, u^\nu_{k'}|_{\partial B_{L+5}} = v^\nu_{k'}|_{\partial B_{L+5}}, u^\nu_{k'}|_{\partial B_{L+5}^+} = x^\nu. \]

Note that \(\partial B_{L+5}^+\) is not a \(C^1\)-boundary, but it satisfies exterior sphere condition, so we can solve the equations by Perron’s method to get a unique solution \(u^\nu_{k'} \in C^\infty((B_{L+5}^+) \cap C^0(B_{L+5}^+))\). From definitions and the estimates above, we have
\[ \|\Delta_{M_k}(u^\nu_{k'} - x^\nu)\|_{C^{m,\beta}(B_{L+1})} \to 0, \]
but it satisfies exterior sphere condition, so we can solve the equations by Perron’s method to get a unique solution \(u^\nu_{k'} \in C^\infty((B_{L+5}^+) \cap C^0(B_{L+5}^+))\). From definitions and the estimates above, we have
\[ \|\Delta_{M_k}(u^\nu_{k'} - x^\nu)\|_{C^{m,\beta}(B_{L+1})} \to 0, \]
\[
\|u_k^\nu - x^\nu\|_{C^{m+2,\beta}(\bar{B}_{L+5})} \to 0,
\]
\[
\|u_k^\nu - x^\nu\|_{L^\infty(\partial B_{L+5}^+)} \to 0,
\]
then by maximum principle, we have
\[
\|u_k^\nu - x^\nu\|_{L^\infty(B_{L+5}^+)} \to 0,
\]
and by Schauder estimates
\[
\|u_k^\nu - x^\nu\|_{C^{m+2,\beta}(B_{L+5}^+)} \leq C(\|\Delta_{M_k}(u_k^\nu - x^\nu)\|_{C^{m,\beta}(B_{L+5}^+)} + \|u_k^\nu - x^\nu\|_{L^\infty(B_{L+5}^+)} + \|u_k^\nu - x^\nu\|_{C^{m+2,\beta}(\bar{B}_{L+5})}) \to 0.
\]
Next, we construct \(u_k^n\) by solving
\[
\Delta_{M_k} u_k^n = 0 \text{ in } B_{L+5}^+, u_k^n|_{\partial B_{L+5}^+} = x^n.
\]
We have
\[
\|u_k^n - x^n\|_{C^{m+2,\beta}(B_{L+5}^+)} \leq C(\|\Delta_{M_k}(u_k^n - x^n)\|_{C^{m,\beta}(B_{L+5}^+)} + \|u_k^n - x^n\|_{L^\infty(B_{L+5}^+)}) \to 0.
\]
Hence we get a new coordinate system \((u_1^n, \cdots, u_n^n)\) and we discard the original coordinate system, and we use the same notation for tensors written in both coordinate systems, so in the new coordinate system we have
\[
\|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\beta}(B_{L+5}^+)} \to 0.
\]
Now we want to improve the convergence of \(g_{k,ij}\) from elliptic equations with Neumann boundary conditions. We have equations
\[
\Delta_{\partial M_k} g_{k,ij} + \hat{B}_{ij}(g_k, \partial g_k) = -2\text{Ric}_{\partial M_k, ij}
\]
\[
\Delta_{M_k} g_{k,ij} + B_{ij}(g_k, \partial g_k) = -2\text{Ric}_{M_k, ij}
\]
Fix \(\theta \in (\beta, 1), p = \frac{m}{1+\theta}\), from \(W^{m+2,p}\) estimates, Morrey embeddings, and
\[
|\nabla_{L^\infty}^{l} g_{k,ij} \partial g_{k,ij}| \to 0, 0 \leq l \leq m,
\]
we have for \(1 \leq i, j \leq n - 1\),
\[
\|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\theta}(\bar{B}_{L+1.5}^+)} \leq C(\|\Delta_{M_k}(g_{k,ij} - \delta_{ij})\|_{C^{m}(\bar{B}_{L+1.5}^+)} + \|g_{k,ij} - \delta_{ij}\|_{L^\infty(\bar{B}_{L+1.5}^+)}) \to 0.
\]
By Theorem 8.33 in [19],
\[
\|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\theta}(\bar{B}_{L+1.5}^+)} \leq C(\|\Delta_{M_k}(g_{k,ij} - \delta_{ij})\|_{C^{m}(\bar{B}_{L+1.5}^+)} + \|g_{k,ij} - \delta_{ij}\|_{L^\infty(\bar{B}_{L+1.5}^+)} + \|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\theta}(\bar{B}_{L+1.5}^+)} \to 0.
\]
Note that
\[
N_k g_{k,nn} = -2(n-1)H_k g_{k,nn}^n,
\]
\[
N_k g_{k,nn} = -(n-1)H_k g_{k,nn} + \frac{1}{2g_{k,nn}} g_{k,ij} \partial_j g_{k,nn}^n,
\]
where $N_k = g^{i\bar{n}}_{\bar{j}} \partial_j$ is the unit normal vector of $\partial M_k$, $1 \leq i \leq n - 1$ and $j$ sums from 1 to $n$, then we have Neumann boundary conditions for (31). For simplicity, assume for a while $m = 0$. Since

$$\|g_{k,ij} - \delta_{ij}\|_{C^{1,\theta}(B^+_L)} \to 0,$$

$$|\text{Ric}_k|_{C^{0}(B^+_L)} \to 0, 1 \leq i, j \leq n,$$

$$|H_k|_{C^{1}(\partial B^+_L)} \to 0,$$

we have

$$\|\Delta M_k (g_{k}^{nn} - \delta^{nn})\|_{C^{0}(B^+_L)} \to 0, \|N_k g_{k}^{nn}\|_{C^{1}(\partial B^+_L)} \to 0,$$

then by Morrey embeddings (together with extensions), and $W^{2,p}$ estimates for Neumann boundary problems (for example, see a priori estimate 2.3.1.1 in [20]),

$$\|g_{k}^{nn} - \delta^{nn}\|_{C^{1,\theta}(B^+_L)} \leq C\|g_{k}^{nn} - \delta^{nn}\|_{W^{2,p}(B^+_L)}$$

$$\leq C(\|g_{k}^{nn} - \delta^{nn}\|_{L^p(B^+_L)} + \|\Delta M_k (g_{k}^{nn} - \delta^{nn})\|_{L^p(B^+_L)})$$

$$+ \|N_k g_{k}^{nn}\|_{W^{1-\frac{1}{p},p}(\partial B^+_L)} \to 0.$$

Now for $1 \leq l \leq n - 1$, since

$$\|\Delta M_k (g_{k}^{ln} - \delta^{ln})\|_{C^{0}(B^+_L)} \to 0,$$

and

$$\|N_k g_{k}^{ln}\|_{W^{1-\frac{1}{p},p}(B^+_L)} \leq C(\|g_{k}^{nn} - \delta^{nn}\|_{W^{2,p}(B^+_L)} + \|H_k\|_{W^{1-\frac{1}{p},p}(B^+_L)})$$

$$\leq C(\|g_{k}^{nn} - \delta^{nn}\|_{W^{2,p}(B^+_L)} + \|H_k\|_{C^{1}(\partial B^+_L)}) \to 0.$$

Then

$$\|\delta_{k}^{ln} - \delta^{ln}\|_{C^{1,\theta}(B^+_L)} \leq C(\|g_{k}^{nn} - \delta^{nn}\|_{W^{2,p}(B^+_L)}$$

$$+ \|\Delta M_k (g_{k}^{nn} - \delta^{nn})\|_{L^p(B^+_L)} + \|N_k g_{k}^{nn}\|_{W^{1-\frac{1}{p},p}(B^+_L)}) \to 0.$$

Hence

$$\|g_{k,ij} - \delta_{ij}\|_{C^{1,\theta}(B^+_L)} \to 0, 1 \leq i, j \leq n.$$

For general $m \geq 1$, take $m$-th derivatives of (31) and the Neumann boundary conditions (34)(35), and note that

$$[\partial_i, N_k] = (\frac{\partial_i g_{k}^{nn}}{\sqrt{g_{k}^{nn}}} - \frac{g_{k}^{nn} \partial_i g_{k}^{nn}}{2 \sqrt{g_{k}^{nn}}})\theta_j,$$

so we get a system of second order elliptic equations with Neumann boundary conditions in $\partial_i g_{k}^{nn}$ and $\partial_j g_{k}^{nn}$, $|\gamma| = m, 1 \leq l \leq m - 1$, with other terms freezeed. Apply the previous estimates in the case $m = 0$ and use (29)(32), we get

$$\|g_{k}^{nn} - \delta^{nn}\|_{C^{m+1,\theta}(B^+_L)} \to 0,$$
and then for $1 \leq l \leq n - 1$,
\[ \|g_{k,l} - \delta_{l}\|_{C^{m+1,\theta}(B_{k+1}^{+})} \to 0, \]

hence
\[ \|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\theta}(B_{k+1}^{+})} \to 0, \quad 1 \leq i, j \leq n \]

In particular, take $\theta = \alpha$, one can we get a $(\frac{L+1}{4}, Q, m + 1, \alpha)$ harmonic coordinate chart centered at $p_k'$, with $d_{g_k}(p_k', p_k) \to 0$. Then $r_{h}^{m+1,\alpha}(p_k, Q) \geq \frac{r}{2}$ for large $k$, which is a contradiction.

\[ \square \]

**Remark 3.20.** Note that the case $m = 0$ is also true, and one should be a little careful with the geometric arguments in the proof. Actually, the arguments in [2] prove a $C^{m+2}_{c}$ harmonic radius lower bound.

**Remark 3.21.** The proof also shows that if $M$ is complete, $i_b \geq i_0$, $inj_M \geq i_0$, $inj_{\partial M} \geq i_0$ and (20)(21) hold, then for any $p \in M$
\[ r_{h}^{m+1,\alpha}(p, Q) \geq r_0(i_0, \Lambda, m, \alpha, Q). \]

The following corollary is a version we will use often.

**Corollary 3.22.** Let $(M_i, g_i)$ be a sequence of complete Einstein manifold with boundary. Suppose $i_b \geq i_0$, $inj_{\partial M} \geq i_0$, $|Rm| \leq C$, $|S| \leq C$, $|\nabla_{\partial M}^k Rm| \leq C_k, |\nabla_{\partial M}^{k+1} H| \leq C_k, k \geq 0$, then for any $p_i \in M_i$, there exists some subsequence such that $(M_i, g_i, p_i)$ converges in pointed Cheeger-Gromov sense.

**Proof.** By Proposition 3.13, $|Rm| \leq C$, $|S| \leq C$, $i_b \geq i_0$, together imply volume lower bounds of interiors balls of some fixed radius near boundary, hence also gives an interior injectivity radius lower bound from the following lemma. Then use Remark 3.21 and Proposition 3.17.

\[ \square \]

The following lemma is well-known, which is a qualitative version of Theorem 4.3 in [10] and can also be easily proved by contradiction arguments.

**Lemma 3.23.** Let $(M, g)$ be a Riemannian manifold, and $B(p, r)$ be a metric ball that has compact closure. Suppose
\[ \sup_{B(p, r)} |Rm| \leq C, vol(B(p, r)) \geq v, \]

then there exists $r_0 > 0$ depending on $n, C, v, r$ such that $\exp_q : B_{r_0}(0) \subset T_qM \to B(q, r_0) \subset M$ is a diffeomorphism for any $q \in B(p, \frac{r}{2})$.

4. Convergence of hyperkähler manifolds

4.1. Curvature estimates near the boundary. This section serves as a first step for the proof of our main theorem. For an Einstein manifold with boundary, if the boundary intrinsic and extrinsic geometry are controlled well and $i_b$ is bounded from below, we hope to control the interior geometry within $i_b$. To the author’s knowledge, we do not know any general statement. We will first state and prove a version we need, and then discuss some lemmas needed in the proof.
Theorem 4.1. Let \((M, g)\) be a complete hyperkähler 4-manifold with compact boundary. Suppose \(|S| \leq C, \left|\nabla^j_\partial M Rm_{\partial M}\right| \leq C_j, |\nabla^{j+1} H| \leq C_j, j = 0, 1, \ldots, \) \(\int_{\partial M} |\nabla_{\partial M} H| \leq i_0, i_b \geq i_0, \int_M |Rm|^2 \leq C\). Then for any \(r_1 < i_0\), there exists \(C' > 0\), depending on \(C, C_j, i_0, r_1\), such that \(\sup_{N_{r_1}(\partial M, g)} |Rm| \leq C'\).

Proof. Without loss of generality, assume \(i_0 = 1\). Denote \(\alpha = r_1, \beta = \frac{1}{4}(1 - \alpha)\). Suppose the conclusion is not true, we have a sequence \((M_i, g_i)\) satisfying the conditions, but
\[
\sup_{N_{r_1}(\partial M_i, g_i)} |Rm_{g_i}| \to \infty.
\]

Let \(p_i \in N_{r_1}(\partial M_i, g_i)\) achieves this supremum.

Claim 1 There exists a subsequence such that
\[
d_{g_i}(p_i, \partial M_i)^2 |Rm_{g_i}(p_i)| \to \infty.
\]
If this is not true, we have \(\sup_{N_{r_1}(\partial M_i, g_i)} d_{g_i}(p_i, \partial M_i)^2 |Rm_{g_i}(p_i)| < \infty\). Rescale \(\tilde{g}_i = |Rm_{g_i}(p_i)|^{-1} g_i\), then \(\sup_{N_{r_1}(\partial M_i, \tilde{g}_i)} |Rm_{\tilde{g}_i}(p_i)| = 1\), and \(\sup_{N_{r_1}(\partial M_i, \tilde{g}_i)} |Rm_{\tilde{g}_i}| \leq 1\) in \(N_{\alpha(Rm_{g_i})}^{\frac{1}{2}} (\partial M_i, \tilde{g}_i)\) and \(N_{r_1}(\partial M_i, g_i)\) for \(i_0, \tilde{g}_i \geq |Rm_{\tilde{g}_i}(p_i)|^{\frac{1}{2}}\) for all \(i\). Hence by Corollary 3.22, \((M_i, g_i, p_i)\) subconverges in pointed Cheeger-Gromov sense to \((M_\infty, \tilde{g}_\infty, p_\infty)\), which is a complete Ricci-flat 4-manifold with flat, totally geodesic boundary, hence must be flat by Lemma 4.5. This contradicts that \(|Rm_{\tilde{g}_\infty}(p_\infty)| \leq 1\) and proves Claim 1.

Now rescale \(g_i\) in another way, let \(g'_i = d_{g_i}(p_i, \partial M_i)^{-2} g_i\), so \(d_{g_i}(p_i, \partial M_i) = 1\). Since \(d_{g_i}(p_i, \partial M_i) \leq \alpha\), the rescaled metric \(g'_i\) satisfies \(i_b g'_i \geq \alpha^{-1}\) as well as all other conditions of the assumptions of the theorem, but with different bounds, regardless of whether \(d_{g_i}(p_i, \partial M_i)\) is uniformly bounded from below or not. Moreover, (40) is equivalent to \(|Rm_{g'_i}(p_i)| \to \infty\).

By the \(\epsilon\)-regularity Theorem 4.3, there exists a universal constant \(\epsilon_0\) such that for sufficiently large \(i\),
\[
\int_{B_{g'_i}(p_i, \beta)} |Rm_{g'_i}|^2 \geq \epsilon_0.
\]

Claim 2 There exists a subsequence such that
\[
\sup_{N_{\alpha}(\partial M_i, g'_i)} |Rm_{g'_i}| \to \infty,
\]
If not, we have \(\sup_{N_{\alpha}(\partial M_i, g'_i)} |Rm_{g'_i}| \leq C\). By Lemma 3.13 and Bishop-Gromov volume comparison, \(\text{vol}(B_{g'_i}(p_i, \beta)) \geq v\). Since \(B_{g'_i}(p_i, \beta) \subset N_{\alpha^{-1}(\alpha + \beta)}(\partial M_i, g'_i) \subset N_{\alpha + \beta}(\partial M_i, g_i)\), and the last one is diffeomorphic to \(\partial M_i \times [0, \alpha + \beta]\), we conclude that there is no \(-2\) curve in \(B_{g'_i}(p_i, \beta)\). By Proposition 4.4, \(|Rm_{g'_i}(p_i)|\) is bounded, which is a contradiction to Claim 1 and finishes the proof of Claim 2.

Now Claim 2 enables us to get by induction, for each fixed positive integer \(N\), \(N\) sequences of metrics \(g_i^{(0)} = g_i, g_i^{(1)} = g'_i, \ldots, g_i^{(N)}\), and points \(p_i^{(0)} \in N_{\alpha}(\partial M_i, g_i^{(j)})\) for \(0 \leq j \leq N - 1\), \(p_i^{(0)} = p_i\), such that for \(0 \leq j \leq N - 1\), \(p_i^{(j)}\) achieves the supremum of \(|Rm_{g_i^{(j)}}|\) in \(N_{\alpha}(\partial M_i, g_i^{(j)})\), and
\[
|Rm_{g_i^{(j)}}(p_i)| \to \infty,
\]
\[
d_{g_i^{(j+1)}}(p_i^{(j)}, \partial M_i) = 1,
\]
\[
\int_{B_{g_{t+1}}(p_i^{(j)}, \beta)} |Rm_{g_{t+1}}|^2 \geq \epsilon_0,
\]
\[
B_{g_{t+1}}(p_i^{(j)}, \beta) \subset N_{\alpha - 1(a+\beta)}(\partial M, g_i^{(j+1)}) \subset N_{\alpha+\beta}(\partial M, g_i^{(j)}),
\]
\[
B_{g_{t+1}}(p_i^{(j)}, \beta) \cap N_{\alpha+\beta}(\partial M, g_i^{(j+1)}) = \emptyset.
\]

It follows that for each fixed \( i \), \( B_{g_{t+1}}(p_i^{(j)}, \beta) \) does not intersect each other for different \( j \). Since \( \int_{M_t} |Rm_{g_0}|^2 \leq C \), we have \( N_{\epsilon_0} \leq C \). This is a contradiction, since \( N \) can be any positive integer. \( \square \)

**Remark 4.2.** This theorem is purely local. In fact, by slightly modifying the proof, we see that if the bounds in the assumptions hold in a metric cylinder \( C(B_\partial M(p, r_0), 0, r_1) \) such that \( \exp^\perp \) maps \( B_\partial M(p, r_0) \times [0, r_1) \) diffeomorphically onto it, and such that \( B_\partial M(p, r_0) \) has compact closure, then we have curvature bounds in any interior metric cylinder \( C(B_\partial M(p, r_0'), 0, r_1') \) with fixed \( r_0' < r_0, r_1' < r_1 \).

The following collapsing \( \epsilon \)-regularity theorem is originally due to Cheeger-Tian in [11]. Recently, in the hyperkähler case, [32] gives a simple proof by a blow-up argument and studying complete collapsing limits of hyperkähler manifolds with bounded curvature.

**Proposition 4.3.** There exists \( \epsilon, c \) such that the following holds: Let \((M^4, g)\) be an Einstein 4-manifold, \(|\text{Ric}| \leq 3\), \( r \leq 1 \), and \( B(p, r) \) is a metric ball that has compact closure. If

\[
\int_{B(p, r)} |Rm| \leq \epsilon,
\]

then

\[
\sup_{B(p, \frac{r}{2})} |Rm| \leq cr^{-2}.
\]

The following proposition plays an important role. To avoid redundancy, we state it here without proving, but we will prove a more general version later, see Proposition 4.14.

**Proposition 4.4.** Let \((M, g)\) be an hyperkähler 4-manifold. Suppose \( B(p, 5) \) has compact closure, \( B(p, 3) \) contains no \(-2\) curve,

\[
\text{vol}(B(p, 1)) \geq v,
\]

\[
\int_{B(p, 3)} |Rm|^2 \leq C.
\]

Then there exists \( C' > 0 \), depending on \( v, C \) such that

\[
\sup_{B(p, 1)} |Rm| \leq C'.
\]

The following lemma originally dates back to Koiso in [25], and is an incredibly special case of the result in [8] [6]. Since it plays an important role throughout the paper, we provide a detailed proof here following [25].

**Lemma 4.5.** Let \((M, g)\) be a connected \( C^2 \) Riemannian manifold with boundary. Suppose \( \text{Ric}_M = 0 \) and for some open boundary portion \( T \), \( S|_T = 0 \) and \( \text{Rm}_{\partial M}|_T = 0 \), then \( g \) is smooth and \( \text{Rm}_M = 0 \).
Proof. Fix any point \( p \) in \( T \). First, we show \( g \) can be extended across the boundary near \( p \). Choose a semi-geodesic coordinate system \((x^1, \ldots, x^n)\) near \( p \), with \( \partial M \)
identified with \( \{x^n = 0\} \), and interior identified with \( \{x^n > 0\} \), \( \nabla x^n = \partial x^n \), and \( g_{ij}(x', 0) = \delta_{ij}, 1 \leq i, j \leq n - 1 \), where \( x' = (x^1, \ldots, x^{n-1}) \). We extend the metric tensor \( g \) by reflection across \( \{x^n = 0\} \), i.e., set \( g_{ij}(x', x^n) = g_{ij}(x', -x^n), 1 \leq i, j \leq n \). Then \( g \in C^0(B) \cap C^2(B^+) \), where \( B^+ \) is the boundary coordinate ball and \( B \) is its extension after reflection. We need to show \( g \in C^2(B) \). In fact, \( S = 0 \) is equivalent to \( \frac{\partial g_{ij}}{\partial x^k}(x', 0) = 0, 1 \leq i, j \leq n - 1 \), then we also have \( \frac{\partial g_{ij}}{\partial x^n}(x', 0) = -\frac{\partial g_{ij}}{\partial x^k}(x', 0) = 0 \), hence \( \frac{\partial g_{ij}}{\partial x^n}(x', 0) = 0 \) and \( g_{ij} \in C^1(B) \). Since \( g \in C^2(B^+) \), we have for \( 1 \leq l \leq n - 1 \),
\[
\frac{\partial}{\partial x^l} \frac{\partial g_{ij}}{\partial x^l}(x', 0) = \frac{\partial}{\partial x^l} \frac{\partial g_{ij}}{\partial x^l}(x', 0) = \frac{\partial}{\partial x^l} 0 = 0,
\]
\[
\frac{\partial}{\partial x^n} \frac{\partial g_{ij}}{\partial x^l}(x', 0) = \frac{\partial}{\partial x^n} \frac{\partial g_{ij}}{\partial x^l}(x', 0) = 0.
\]
(41)
Hence \( g_{ij} \in C^2(B) \). Finally, we have \( g_{nn} = 1, g_{ln} = g_{nl} = 0, 1 \leq l \leq n - 1 \), hence \( g \in C^2(B) \).

By elliptic regularity, all harmonic coordinate charts in \( B \) give rise to a real analytic structure in \( B \) such that \( g \) is real analytic. Hence if \( t \) is a distance function such that \( \partial M \) is defined by \( t^{-1}(0) \) near \( p \), then \( t \) is real analytic near \( p \). Choose a real analytic coordinate \((z, t)\) near \( p \). Since \( t^{-1}(0) \) is totally geodesic, \( \frac{\partial g}{\partial t}(z, 0) = 0 \).

The evolution equation (13) is equivalent to the second order PDE
\[
\frac{\partial^2 g}{\partial t^2} = 2\text{ric } g - \frac{1}{2} \text{tr} g \left( \frac{\partial g}{\partial t} \frac{\partial g}{\partial t} \right) + \left( \frac{\partial g}{\partial t} \right)^2,
\]
where \( \text{ric } g \) is the Ricci tensor of level sets of \( t \). By the uniqueness part of Cauchy-Kovalevskaya theorem, we know \( g(z, t) = g(z, 0) \). Hence \( Rm_M = 0 \) near \( p \). Since \( Rm_M \) is real analytic in the interior of \( M \), \( Rm_M = 0 \) in \( M \).

\[\square\]

4.2. Convergence of hyperkähler metrics.

**Theorem 4.6.** Let \((X_i, g_i)\) be a sequence of compact, connected hyperkähler 4-manifold with boundary, suppose on \( \partial X_i \), we have
\[
H_i \geq H_0 > 0, |S_i| \leq C, |\nabla^{j+1} H_i| \leq C_j,
\]
\[
inj_{\partial X_i} \geq i_0, \diam_{g_i, |\partial X_i|}(\partial X_i) \leq C, |\nabla^j_{\partial X_i} Rm_{\partial X_i}| \leq C_j, \forall j \geq 0,
\]
and \( \chi(X_i) \leq C \). Assuming there exists no \(-2\) curve on \( X_i \), then there exists a subsequence such that \((X_i, g_i)\) converges in Cheeger-Gromov sense to a compact, connected hyperkähler 4-manifold with boundary \((X_\infty, g_\infty)\).

**Remark 4.7.** By Chern-Gauss-Bonnet formula, our assumptions imply
\[
\int_{X_i} |Rm|^2 \leq C.
\]
Note that for a compact connected Einstein 4-manifold $(M, g)$ with boundary, the Chern-Gauss-Bonnet formula says

$$\frac{1}{8\pi^2} \int_M |Rm|^2 = \chi(M) - \frac{1}{2\pi^2} \int_{\partial M} \prod_{i=1}^3 \lambda_i - \frac{1}{8\pi^2} \int_{\partial M} \sum_{\sigma \in S_3} K_{\sigma_1 \sigma_2 \lambda \sigma_3}. $$

Here $\lambda_1$, $\lambda_2$, $\lambda_3$ are eigenvalues of the shape operator $S$ of $\partial M$. Let $e_i$ be eigenvectors of eigenvalue $\lambda_i$ such that $\{e_1, e_2, e_3\}$ is an orthonormal basis, then $K_{ij} = \sec(e_i, e_j)$. See for example (1.16) in [4].

**Remark 4.8.** If we drop the condition $\text{diam}_{g|_{\partial X_i}}(\partial X_i) \leq C$, and replace $H_i \geq H_0 > 0$ by $H_i > 0$, $\chi(X_i) \leq C$ by $\int_{X_i} |Rm|_g|^2 \leq C$, then for any point $p_i \in X_i$, a subsequence of $(X_i, g_i, p_i)$ converges in pointed Cheeger-Gromov sense to a complete, connected hyperkähler 4-manifolds with nonempty or empty boundary $(X_\infty, g_\infty, p_\infty)$, depending on whether the distance of $p_i$ to boundary is bounded or not.

**Remark 4.9.** The positive mean curvature condition is necessary. The following counterexample is natural and was observed by Donaldson in [13]. Consider the standard unit ball $B^4$ inside Euclidean $\mathbb{R}^4$, “squeeze” the ball such that the north pole and the south pole of the boundary $S^3$ comes together, so we get a sequence of embedded $B^4$ in $\mathbb{R}^4$ converging in Hausdorff sense to a limit homeomorphic to the wedge sum of two $B^4$, whose boundary is an immersed $S^3$ intersecting itself at one point. For this sequence, all other assumptions are satisfied. Slightly modifying the process, one can also have a sequence of $B^4$ of dumbbell shape such that the middle cylinder $B^3 \times [0, 1]$ collapses to $[0, 1]$, then they have a Hausdorff limit which is homeomorphic to two $B^4$ joint by a line segment.

In these types of examples, the curvatures are uniformly bounded, and the global volume are non-collapsing before taking the limit.

**Remark 4.10.** If we allow $-2$ curves and do not assume the positive mean curvature condition, something worse will happen: consider the Kummer construction. Let $T^4/\mathbb{Z}_2$ be the flat 4-orbifold with 16 singularities, remove small neighborhoods of the 16 singularities, and glue in 16 copies of $T^4 \times S^2$. By varying the sizes of these glue-in regions and perturbing the metrics, we get a sequence of hyperkähler 4-manifolds $(M_i, h_i)$, each of which is diffeomorphic to a K3 surface, such that $(M_i, h_i)$ converges in Gromov-Hausdorff sense to $T^4/\mathbb{Z}_2$, and converges in Cheeger-Gromov sense away from these 16 singularities. Now let $T^3/\mathbb{Z}_2 \subset T^4/\mathbb{Z}_2$ be the flat 3-orbifold, such that the last coordinate equal to 0. Let $\tilde{X}_i \subset T^4/\mathbb{Z}_2$ be the closure of the tubular neighborhood of $T^3/\mathbb{Z}_2$ of width $i^{-1}$, then $\partial \tilde{X}_i$ is connected, totally geodesic, isometric to the same flat $T^3$. Now for each $i$, choose $n(i)$ large enough such that one can find $X_i \subset M_{n(i)}$ which are compact domains with smooth boundary, $d_{GH}(X_i, \tilde{X}_i) \to 0$, $|\nabla_{\partial \tilde{X}_i} S_{\partial \tilde{X}_i}| \to 0$, $\forall j \geq 0$, $\partial X_i$ converges in Cheeger-Gromov sense to $\partial \tilde{X}_i$. Let $g_i = h_{n(i)}|_{X_i}$, then $(X_i, g_i)$ converges in Gromov-Hausdorff sense to flat $T^3/\mathbb{Z}_2$. In this case, $\text{vol}(X_i) \to 0$, $\sup_{X_i} |Rm_{g_i}| \to \infty$.

Note that in this example $\partial \tilde{X}_i$ cannot be perturbed in flat $T^4/\mathbb{Z}_2$ to have positive mean curvature. In fact, take a small tubular neighborhood of $\partial \tilde{X}_i$ of width less than $i^{-1}$, whose boundary has two totally geodesic connected components $T_1, T_2$. Suppose $\partial \tilde{X}_i$ can be perturbed to $T'$ such that its mean curvature has a strict sign,
say that its mean curvature vector points towards $T_1$. Then $T', T'_1$ bound a region $W$. By [13] Proposition 7, $T'$ and $T_1$ are isometric, so $T'$ is totally geodesic, which is a contradiction.

The major step of the proof in 4.6 is to show that these Riemannian manifolds have a nice neighborhood of definite size. We show that the boundary injectivity radius has a lower bound, so that the interior geometry within the boundary injectivity radius is nicely controlled by Theorem 4.1.

**Proposition 4.11.** There exists $i_1 > 0$, depending on the constants in Theorem 4.6 such that $i_b, g_i \geq i_1$.

**Proof.** Suppose not, we have a subsequence of hyperkähler metrics $g_i$, with $i_b, g_i \to 0$. Rescale the metric $\tilde{g}_i = \frac{-2}{i_b, g_i}$, then $i_b, \tilde{g}_i = 1$. For any point $p_i \in \partial X_i$, the restriction metric $(\partial X_i, \tilde{g}_i|_{\partial X_i}, p_i)$ converges in pointed Cheeger-Gromov sense to flat $\mathbb{R}^3$, $|S_{\tilde{g}_i}| \to 0$ and $|\nabla^{i+1}_X H_i| \to 0$ uniformly on $\partial X_i$. Consider $\sup_{B_{\tilde{g}_i}(p, 3)} |Rm_{\tilde{g}_i}|$.

We have two cases

**Case 1** $\sup_{B_{\tilde{g}_i}(p, 4)} |Rm_{\tilde{g}_i}| \leq C$.

We have a subsequence $(B_{\tilde{g}_i}(p_i, 3), \tilde{g}_i)$ converges in Cheeger-Gromov sense to a Riemannian manifold with boundary $(B_\infty, g_\infty)$, so $(B_\infty, g_\infty)$ is Ricci flat, and all boundary components are flat, totally geodesic. By Lemma 4.5, $(B_\infty, g_\infty)$ is flat. We need to choose good points $p_i$ to lead to a contradiction. In fact, by Proposition 3.5, there exists $p_i \in \partial X_i$ such that $\gamma_{p_i}(1)$ is a focal point along $\gamma_{p_i}$. Let $p_\infty$ be the limit of $p_i$. By Proposition 3.11, we get a limit geodesic $\gamma_{p_\infty} : [0, 1] \to B_\infty$, such that $\gamma_{p_\infty}(1)$ is a focal point along $\gamma_{p_\infty}$, contradiction.

**Case 2** For some subsequence we have $\sup_{B_{\tilde{g}_i}(p, 4)} |Rm_{\tilde{g}_i}| \to \infty$.

Then we can find points $q_i \in B_{\tilde{g}_i}(p, 4)$ such that $|Rm_{\tilde{g}_i}(q_i)| \to \infty$. By Theorem 4.1, we have $d_{\tilde{g}_i}(q_i, \partial X_i) \geq \frac{1}{2}$ for large $i$. By Theorem 4.1, Proposition 3.13, and the fact $d_{\tilde{g}_i}(q_i, \partial X_i) \leq 4$, we have $\text{vol}_{\tilde{g}_i}(B_{\tilde{g}_i}(q_i, \frac{5}{2})) \geq v_0$ for some $r_0, v_0$. Since there is no $-2$ curve in $X_i$, then by Proposition 4.4, $\sup_{B_{\tilde{g}_i}(q, \frac{5}{2})} |Rm_{\tilde{g}_i}| \leq C$, which is a contradiction.

Then we can finish the proof of Theorem 4.6 as follows: by Theorem 4.1, $|Rm_{g_i}|$ is uniformly bounded in $N_{\frac{1}{2}}(\partial X_i, g_i)$. By Proposition 3.13, there exist $r_1 < \frac{i_1}{10}, v_1$, such that $\text{vol}_{g_i}(B_{g_i}(p, r_1)) \geq v_1$ for any $p$ with $d_{g_i}(p, \partial X_i) = 2r_1$. By Proposition 3.2, $\sup_{g_i} d_{g_i}(q, \partial X_i) \leq 3H^{-1}_0$, hence $\text{diam}(X_i, g_i) \leq C$. Then from Bishop-Gromov volume comparison, we know $\text{vol}_{g_i}(B_{g_i}(p, r_1')) \geq v_2$ for any $p$ with $d_{g_i}(p, \partial X_i) \geq 5r_1$. By Proposition 4.4, for these $p$, $|Rm_{g_i}(p)|$ is uniformly bounded, with the bound independent of $i$ and $p$. Hence $\sup_X |Rm_{g_i}|$ is uniformly bounded. Then by Corollary 3.22, a subsequence $(X_i, g_i)$ converges in Cheeger-Gromov sense to a smooth Riemannian manifold with boundary $(X, g)$, so $g$ is a hyperkähler metric.

**Remark 4.12.** One can also directly prove Theorem 4.6 by rescaling the maximum curvature norm to be 1. Suppose it is achieved at $p_i$. Then from Corollary 3.7, we know $i_b \geq i_0$ for the rescaled metrics. If now $d(p_i, \partial X_i)$ is bounded, then the pointed Riemannian manifolds converge in pointed Cheeger-Gromov sense to a Ricci-flat manifold with flat and totally geodesic boundary, hence the limit is
flat, contradiction. Otherwise, for a subsequence \(d(p_i, \partial X_i) \to \infty\), then we rescale the metrics again such that this distance is 1. However, it is unknown whether the curvature is bounded in a fixed size neighborhood of the boundary in this scale. Using the idea of Theorem 4.1, we can keep rescaling such that this will happen at some stage, for possibly different points \(p'_i\), then get a contradiction. The contradiction is the same as in Case 2 in Proposition 4.11, since volume lower bounds can be passed within finite distance, so do the curvature bounds by Proposition 4.4. Alternatively, one can prove Theorem 4.6 by rescaling the harmonic radius, and all these methods turn out to be equivalent eventually.

4.3. Convergence of hyperkähler triples. Now we turn to the setting of 1.2. \(X\) will denote a compact oriented 4-manifold with boundary \(\partial X = Y\). Let \(\mathcal{N}^+\) be the set of closed framings \(\gamma = (\gamma_1, \gamma_2, \gamma_3)\) on \(Y\) that satisfies the “positive mean curvature” condition

\[
\sum_{i=1}^{3} (\gamma_i, d(\ast \gamma_i))_{\gamma} > 0.
\]

Let \(\mathcal{M}^+\) be the set of smooth hyperkähler triples \(\omega = (\omega_1, \omega_2, \omega_3)\) whose restriction to the boundary lies in \(\mathcal{N}^+\), so we have a restriction map \(p_0 : \mathcal{M}^+ \to \mathcal{N}^+\), which induce a map

\[
p : \mathcal{M}^+ / G_X \to \mathcal{N}^+ / G_Y.
\]

Here \(\mathcal{M}^+, \mathcal{N}^+\) are equipped with Fréchet topology defined by smooth convergence, \(G_X, G_Y\) are orientation preserving diffeomorphism groups of \(X, Y\), respectively. It is obvious that \(p_0, p\) are continuous. Then the compactness part of Theorem 1.2 is equivalent to:

**Proposition 4.13.** When there is no \(-2\) curve in \(X\), the map \(p : \mathcal{M}^+ / G_X \to \mathcal{N}^+ / G_Y\) is proper.

**Proof.** Suppose for some \(\phi_i \in G_Y\), and \(\gamma_i \in \mathcal{N}^+\), we have \(\phi_i \gamma_i \to \gamma \in \mathcal{N}^+\) and there exist \(\omega_i \in \mathcal{M}^+\) with \(\omega_i | Y = \gamma_i\), we want to show there exists \(\psi_i \in G_X\) and \(\omega \in \mathcal{M}^+\) such that \(\psi_i \ast \omega_i \to \omega\). Let \(g_i\) be the Riemannian metric defined by \(\omega_i\). By the assumptions, we have a uniform positive lower bound for the mean curvature \(H_i\) of \(Y\) of the metric \(g_i = g_{\omega_i}\), and bounds for \(|\nabla Y^l S_i|\) for all \(l \geq 0\), and \((Y, g_i| Y)\) converges in Cheeger-Gromov sense. Hence \((X, g_i)\) satisfies all conditions in Theorem 4.6. Then for a subsequence, there exists diffeomorphism \(\psi_i : X \to X\) such that \(\psi_i \ast g_i \to g\) smoothly as tensors. One can assume \(\psi_i\) is orientation preserving, otherwise, for a subsequence, compose them with a fixed orientation reversing diffeomorphism of \(X\). Since \(|\psi_i \ast \omega_i|^2 = 3\), \(\psi_i \ast \omega_i\) are parallel, we conclude that for some subsequence \(\psi_i \ast \omega_i \to \omega\) smoothly, and \(\omega \in \mathcal{M}^+\).

\[\square\]

4.4. Enhancements. For the proof of the compactness part of Theorem 1.3, one only needs the following proposition in place of Proposition 4.4. Then we argue in the same way as the proof of Theorem 1.2.

**Proposition 4.14.** Let \((M, g, \omega)\) be a hyperkähler 4-manifold. Suppose \(B(p, 5)\) has compact closure, and for any \(-2\) curve \(\Sigma\) in \(B(p, 3)\),

\[
\left| \int_\Sigma \omega \right| \geq a > 0,
\]
\[ \text{vol}(B(p, 1)) \geq v, \]
\[ \int_{B(p, 3)} |Rm|^2 \leq C. \]

Then there exists \( C' > 0 \), depending on \( a, v, C \) such that
\[ \sup_{B(p,1)} |Rm| \leq C'. \]

**Proof.** For all \( q \in B(p, 2) \), by volume comparison, we have
\[ \text{vol}(B(q, 1)) \geq 3^{-4} \text{vol}(B(q, 3)) \geq 3^{-4} \text{vol}(B(p, 1)) \geq 3^{-4} v. \]
Suppose the conclusion is not true, then we have a sequence \((M_i, g_i, p_i)\) satisfies the conditions, but there exists \( q'_i \in B(p_i, 1) \) with \( |Rm_{g_i}(q'_i)| \to \infty \). By the following Lemma 4.15, we can find points \( q_i \in B(p_i, 2) \) such that \( |Rm_{g_i}(q_i)| \geq |Rm_{g_i}(q'_i)| \), and
\[ \sup_{B_{\tilde{q}_i}(q_i, |Rm_{g_i}(q'_i)|^{\frac{1}{4}} |Rm_{g_i}(q_i)|^{-\frac{1}{4}})} |Rm_{g_i}| \leq 4 |Rm_{g_i}(q_i)|. \]

Rescale the metric \( \tilde{g}_i = |Rm_{g_i}(q_i)| g_i \), \( \tilde{\omega}_i = |Rm_{g_i}(q_i)| \omega_i \), so \( \tilde{\omega}_i \) defines \( \tilde{g}_i \). Then we have
\[ \sup_{B_{\tilde{q}_i}(q_i, |Rm_{g_i}(q'_i)|^{\frac{1}{4}} |Rm_{g_i}(q_i)|^{-\frac{1}{4}})} |Rm_{\tilde{g}_i}| \leq 4, \]
\[ |Rm_{\tilde{g}_i}(q_i)| = 1, \]
\[ \text{vol}_{\tilde{g}_i}(B_{\tilde{g}_i}(q_i, r)) \geq 3^{-4} vr^n, \forall r \leq |Rm_{g_i}(q_i)|^{\frac{1}{2}}, \]
and
\[ \int_{B_{\tilde{q}_i}(q_i, |Rm_{g_i}(q'_i)|^{\frac{1}{4}} |Rm_{g_i}(q_i)|^{-\frac{1}{4}})} |Rm_{\tilde{g}_i}|^2 \leq \int_{B_{\tilde{q}_i}(p_i, 3|Rm_{g_i}(q_i)|^{\frac{1}{4}} |Rm_{g_i}(q_i)|^{-\frac{1}{4}})} |Rm_{\tilde{g}_i}|^2 \leq C. \]

Hence, for a subsequence, \((M_i, \tilde{g}_i, q_i)\) converges in Cheeger-Gromov topology to a complete non-flat hyperkähler 4-manifold \((M_\infty, g_\infty, q_\infty)\) with maximum volume growth and
\[ \int_{M_\infty} |Rm_{g_\infty}|^2 \leq C. \]

Since \( \tilde{\omega}_i \) are parallel, and \( |\tilde{\omega}_i|^2 = 6 \), \( \tilde{\omega}_i \) also subconverges to a hyperkähler triple \( \omega_\infty \) that defines \( g_\infty \). By [7], \((M_\infty, g_\infty)\) is a hyperkähler ALE space of order 4. Hence, from Kronheimer’s classification [26] [27], there exists a \(-2\) curve \( C_\infty \) in \( B_{g_\infty}(q_\infty, R) \) for some \( R > 0 \) such that \( |\int_{C_\infty} \omega_\infty| \neq 0 \). Let \( \phi_i : B_{g_\infty}(q_\infty, R) \to V_i \) be diffeomorphisms, such that \( \phi_i^* \tilde{\omega}_i \to \omega_\infty \) smoothly. Let \( C_i = (\phi_i)_* C_\infty \), then \( C_i \) is a \(-2\) curve in \( B_{g_i}(p_i, 3) \) for large \( i \) and
\[ \int_{C_i} \omega_i = |Rm_{g_i}(q_i)|^{-1} \int_{C_i} \tilde{\omega}_i = |Rm_{g_i}(q_i)|^{-1} \int_{C_\infty} \phi_i^* \tilde{\omega}_i \to 0, \]
which contradicts our assumption. \( \square \)

The following point selection lemma is well-known and elementary.

**Lemma 4.15.** Let \((M, g)\) be a Riemannian manifold. Suppose \( \sup_{B(p,2)} |Rm| < \infty \), \( |Rm(p)| \neq 0 \). Then there exists a point \( q \in B(p, 2) \) such that \( |Rm(q)| \geq |Rm(p)| \), and
\[ \sup_{B(q, |Rm(p)|^{\frac{1}{4}} |Rm(q)|^{-\frac{1}{4}})} |Rm| \leq 4 |Rm(q)|. \]
Proof. If this is not true, let $A = |Rm(p)|^\frac{1}{2}$, then there exist $q_1 \in B(p, 2)$ such that $d(q_1, p) < 1$ and $|Rm(q_1)| > 4|Rm(p)| = 4A^2$. By induction, we can find a sequence of points $q_0 = p, q_1, q_2, \cdots$ with $d(q_{j+1}, q_j) < |Rm(q_j)|^{-\frac{1}{2}}A, d(q_{j+1}, p) < 2-2^{-j}$ and $|Rm(q_{j+1})| > 4^{j+1}A^2$. This is an obvious contradiction since $\sup_{B(p, 2)} |Rm| < \infty$. \hfill $\square$

4.5. Uniqueness. Given the compactness results, it is natural to ask whether a subsequential limit $\omega$ is unique (up to a diffeomorphism) in Theorem 1.2, Theorem 1.3. The answer is yes. In fact, both [8] and [6] proved a unique continuation theorem for Einstein metrics with prescribed boundary metric and second fundamental form, which implies the following:

**Proposition 4.16.** Let $X$ be a connected oriented 4-manifold with boundary. Suppose $\omega_1, \omega_2$ are two smooth hyperkähler triples on $X$. Suppose $\omega_1|_{\partial X} = \omega_2|_{\partial X}$, then in geodesic gauges of $g_{\omega_1}, g_{\omega_2}$, we have $\omega_1 = \omega_2$ near $\partial X$.

Proof. We have $g_{\omega_1}|_{\partial X} = g_{\omega_2}|_{\partial X}$ and $II_{\omega_1} = II_{\omega_2}$ on $\partial X$. By [8] Theorem 4, in geodesic gauges, we have $g_{\omega_1} = g_{\omega_2}$. Since $\nabla g_{\omega_1}|_{\partial X} = \nabla g_{\omega_2}|_{\partial X}$, we have $\omega_1 = \omega_2$ at one point, we have $\omega_1 = \omega_2$ everywhere near $\partial X$. \hfill $\square$

Now the following global uniqueness result follows from an analytic continuation argument (See [23] Chapter VI, Section 6):

**Theorem 4.17.** Let $X$ be a connected oriented 4-manifold with boundary, $\pi_1(X, \partial X) = 0$. Suppose $\omega_1, \omega_2$ are two smooth hyperkähler triples on $X$, and $\phi_0 : \partial X \rightarrow \partial X$ is a diffeomorphism, such that $\omega_1|_{\partial X} = \phi_0^*(\omega_2|_{\partial X})$, then there exists a diffeomorphism $\phi : X \rightarrow X, \phi|_{\partial X} = \phi_0$ such that $\omega_1 = \phi^*\omega_2$ on $X$.

**Remark 4.18.** This theorem implies that the map $p : \mathcal{M}^+/\mathcal{G}_X \rightarrow \mathcal{N}^+/\mathcal{G}_Y$ defined in subsection 4.3 is injective, provided that $\mathcal{M}^+$ is nonempty.

Proof. By the previous proposition, there exists a collar neighborhood $U$ of $\partial X$ and a diffeomorphism $\phi_1 : U \rightarrow V \subset X$ such that $g_{\omega_1} = \phi_1^*g_{\omega_2}, \phi_1|_{\partial X} = \phi_0$. Since $g_{\omega_1}, g_{\omega_2}$ are real analytic, $\phi_1$ is real analytic. Fix $p_0 \in U$ and a small neighborhood $U_0$ of $p_0$ in $X$. For any $p \in X \backslash \partial X$, choose a path $x(t), 0 \leq t \leq 1$ such that $x(0) = p_0, x(1) = p, x(t) \in X \backslash \partial X$, then an analytic continuation of the isometry $\phi_1|_{U_0}$ along $x(t)$ gives rise to an isometry defined near $p$. We claim that if we have two paths and two analytic continuations, then they define the same germ at $p$. In fact, one only needs to show that for the closed path $y_0(t)$ formed by concatenating these two paths, the isometry near $p_0$ given by an analytic continuation of $\phi_1|_{U_0}$ along $y_0(t)$ has the same germ as $\phi_1|_{U_0}$ at $p_0$. Since $\pi_1(X, \partial X) = 0, y_0(t)$ can be homotoped to a path $y_1(t)$ contained in $U$ via paths $y_s(t) \in X \backslash \partial X$, such that $y_s(0) = y_s(1) = p_0$. Since $\phi_1 : U \rightarrow V$ is a globally defined isometry, by uniqueness of analytic continuation, we know that any analytic continuation of $\phi_1|_{U_0}$ along $y_1(t)$ must coincide with $\phi_1$, which finishes the proof of the claim by invariance of analytic continuation via homotopy. This shows that $\phi_1 : U \rightarrow V$ can be extended to a global isometry on $X$. Hence $\omega_1 = \phi^*\omega_2$ by the same argument as before. \hfill $\square$

Given the compactness result, Theorem 4.17, together with the theorem of Ebin-Palais on properness of diffeomorphism group action on the space of Riemannian metrics on a closed manifold (See [15]), one can answer the question raised at the beginning of this subsection, thus finish the proof of Theorem 1.2, Theorem 1.3:
Suppose we have two sequences of diffeomorphism \( \phi_i, \psi_i \) of \( X \) such that \( \phi_i^* \omega_i \to \omega, \psi_i^* \omega_i \to \omega' \), then \( (\phi_i|_{\partial X})^* \gamma_i \to \omega|_{\partial X}, (\psi_i|_{\partial X})^* \gamma_i \to \omega'|_{\partial X} \), where \( \gamma_i = \omega_i|_{\partial X} \). Since \( \gamma_i \) converges to \( \gamma \) in Cheeger-Gromov sense, there exists diffeomorphisms \( u_i : \partial X \to \partial X \) such that \( u_i^* \gamma_i \to \gamma \). By the theorem of Ebin-Palais, we have for a subsequence \( (\phi_i|_{\partial X})^{-1} \circ u_i, (\psi_i|_{\partial X})^{-1} \circ u_i \) converge to some diffeomorphisms \( u, u' \) on \( \partial X \), respectively (because their inverses converge). Hence we also have \( u_i^* \gamma_i \to u^* \omega|_{\partial X}, u_i^* \gamma_i \to (u')^* \omega'|_{\partial X} \), so \( \omega|_{\partial X} = (u' \circ u^{-1})^* \omega'|_{\partial X} \). Note that the positive mean curvature condition implies that \( \pi_1(X, \partial X) = 0 \) (See Proposition 3.2), then by Theorem 4.17, there exists a diffeomorphism \( \varphi \) on \( X \) with \( \omega' = \varphi^* \omega \).

4.6. Some discussions. Suppose \( X \) is a connected complete metric space, and there exists a finite set \( \Sigma = \{p_1, \cdots, p_m\}, m \geq 0 \) such that \( X \setminus \Sigma \) is a smooth flat hyperkähler 4-manifold with nonempty boundary, and each boundary component \( Y_1, \cdots, Y_n, n \geq 1 \) is isometric to flat \( \mathbb{R}^4 \), \( X \setminus \cup_{i=1}^n Y_i \) is a flat hyperkähler 4-orbifold and \( \Sigma \) is the set of all orbifold points. Our goal is to classify all such \( X \).

The motivation of this problem is the following:

**Proposition 4.19.** Let \( (X_i, g_i) \) be a sequence of compact hyperkähler 4-manifolds with boundary, such that

\[
i_{b,g_i} \geq i_0, \text{inj}_{\partial X_i} \to \infty,
\]

\[
|S_i| \to 0, |\nabla^j_{\partial X_i} H_i| \to 0, |\nabla^j_{\partial X_i} Rm_{\partial X_i}| \to 0
\]
uniformly on \( \partial X_i \) for all \( j \geq 0 \), and

\[
\int_{X_i} |Rm_{g_i}|^2 \leq C.
\]

Then for any \( p_i \in X_i, d_{g_i}(p_i, \partial X_i) \leq K \), a subsequence of \( (X_i, g_i, p_i) \) converges in pointed Gromov Hausdorff sense to a complete metric space \( (X_\infty, d_\infty, p_\infty) \). The limit \( (X_\infty, d_\infty) \) satisfies all properties of \( X \) above.

By Theorem 4.1, the geometry near boundary is nicely controlled, so this theorem is a consequence of [3] or [7] etc.

**Remark 4.20.** In Theorem 4.6, under the mean positive condition, if we allow \( -2 \) curves in \( X_i \), then we are unable to show that \( i_{b,g_i} \) has a lower bound. In fact, the bad case is that focal points are exactly those curvature blow-up points, and they approach the boundary in a moderate rate. However, we can say something within the scale of the boundary injectivity radius. Suppose \( i_{b,g_i} \to 0 \), then we rescale the metric such that they are equal to 1, then the rescaled metric satisfies the assumption of Proposition 4.19.

**Theorem 4.21.** \( X \) is isometric to one of the following:

- \( (m=0, n=1) \mathbb{R}^4_+; \)
- \( (m=0, n=2) \) the region in \( \mathbb{R}^4 \) bounded by two parallel hyperplanes;
- \( (m=1, n=1) \) the connected component of 0 in \( (\mathbb{R}^4 \setminus H)/\mathbb{Z}_2 \), where \( H \) is a hyperplane such that \( 0 \not\in H \).

**Proof.** Consider another copy of \( X \), glue them together along \( Y_1, \cdots, Y_n \), we get a complete flat hyperkähler orbifold \( \tilde{X} = X \sqcup_{id} X, id: \cup_{i=1}^n Y_i \to \cup_{i=1}^n Y_i \subset X \), which contains \( Y_1, \cdots, Y_n \) as smooth hypersurfaces. Then \( \tilde{X} \) is a \( (SU(2), \mathbb{R}^4) \) orbifold in the sense of [33]. Let \( \tilde{X} \) be the universal covering orbifold of \( \tilde{X} \), then we have a developing map \( D : \tilde{X} \to \mathbb{R}^4 \), and since \( \tilde{X} \) is a complete orbifold, \( D \) is a covering
map. Hence $\hat{X}$ is homeomorphic to $\mathbb{R}^4$, and $\hat{X}$ is isometric to $\mathbb{R}^4/\Gamma$, where $\Gamma$ is a discrete subgroup of $\mathbb{R}^4 \times SU(2)$. Let $r$ be the projection map to the second factor, then $r(\Gamma)$ is a finite subgroup of $SU(2)$, and we have a short exact sequence

$$0 \to \Gamma \cap \mathbb{R}^4 \to \Gamma \to r(\Gamma) \to 0.$$ 

Then $\text{rank}(\Gamma \cap \mathbb{R}^4) \leq 1$, since otherwise $\hat{X}$ is a quotient of $\mathbb{R}^2 \times T^2$ and cannot contain a flat $\mathbb{R}^3$ as a Riemannian submanifold.

If $\Gamma \cap \mathbb{R}^4 = 0$, then $\hat{X} \cong \mathbb{R}^4/r(\Gamma)$ and hence $r(\Gamma) = \{1\}$, since otherwise $\hat{X}$ has exactly one orbifold point, contradiction. Hence, $m = 0$, $n = 1$ and $X$ is isometric to $\mathbb{R}^4$.

If $\Gamma \cap \mathbb{R}^4 = \mathbb{Z}_a$ for some $0 \neq a \in \mathbb{R}^4$, let $\pi : \mathbb{R}^4 \to \mathbb{R}^4/\Gamma$ be the covering map, then $\pi^{-1}(Y_1)$ is a disjoint union of parallel hyperplanes. Pick one of them $Z$, then $\pi^{-1}(Y_1)$ is a disjoint union of $\gamma Z$ for $\gamma \in \Gamma$. Suppose $Z$ is defined by $b^T x + c = 0$, then $\gamma^{-1} Z$ is defined by $b^T r(\gamma)x + c' = 0$. Since they are parallel to each other, $b^T = \pm b^T r(\gamma)$, and hence $\pm 1$ is an eigenvalue of $r(\gamma)$, which forces $r(\gamma) = \pm 1$. Hence $r(\Gamma) = \{1\}$ or $r(\Gamma) = \mathbb{Z}_2$. If $r(\Gamma) = \{1\}$, then $\Gamma$ is generated by $x \mapsto x + a$, so $m = 0$, $n = 2$ and $X$ is isometric to the region in $\mathbb{R}^4$ bounded by two parallel hyperplanes; If $r(\Gamma) = \mathbb{Z}_2$, then $\Gamma$ is generated by $x \mapsto -x + d$ and $x \mapsto x + a$ for some $d \in \mathbb{R}$. Hence $m = 1$ and $\mathbb{R}^4/\Gamma \setminus \{Y_1, Y_2, \ldots, Y_n\}$ has $n + 1$ connected components. But from the gluing construction and that $X$ is connected, we know $\hat{X} \setminus \{Y_1, Y_2, \ldots, Y_n\}$ has exactly two connected components. Hence $n + 1 = 2$, $n = 1$, and $X$ is isometric to the last case in the conclusion.

\[\square\]

5. Torsion-free hypersymplectic manifolds with boundary

5.1. Preliminaries. Recall that a hypersymplectic triple $\omega = (\omega_1, \omega_2, \omega_3)$ on an oriented 4-manifold $X$ with boundary is a definite triple of symplectic forms. Write

$$\omega_i \wedge \omega_j = 2Q_{ij} \mu$$

Denote $Q = (Q_{ij})$, $Q^{-1} = (Q_{ij})$, and $g = g_\omega$ the Riemannian metric. Recall that $\omega$ is called torsion-free if for each $i$,

$$d(Q^{ij} \omega_j) = 0,$$

which is equivalent to

$$dQ^{ij} \wedge \omega_j = 0.$$

Let us begin with some arguments and results in [18]. Let $\mathcal{P}$ denotes the set of symmetric positive-definite 3 by 3 matrices. There are two Riemannian metrics on $\mathcal{P}$: the first one is the Euclidean metric

$$\langle A, B \rangle = \text{Tr}(AB),$$

and the second one is the symmetric space metric, which has non-positive sectional curvature

$$\langle A, B \rangle_Q = \text{Tr}(Q^{-1}AQ^{-1}B)$$

at each point $Q \in \mathcal{P}$.
Then $Q$ can be regarded as a map $Q : X \to \mathcal{P}$. Let $\Delta Q, \hat{\Delta} Q$ denote the harmonic Laplacian of this map with respect to (47), (48), respectively. Explicitly, their components are related by

$$\hat{\Delta} Q_{ij} = \Delta Q_{ij} - Q^{km}(dQ_{ik}, dQ_{mj}),$$

For a hypersymplectic triple $\omega$, the calculations in [17] showed that the torsion-free condition is equivalent to

$$\hat{\Delta} Q = 0, \text{Ric} = \frac{1}{4} \langle dQ \otimes dQ \rangle_Q.$$ 

Hence if $\omega$ is torsion-free, then $Q$ is a harmonic map with respect to (48) and $\text{Ric} \geq 0$. Then the scalar curvature $R$ of $g$ is

$$R = \frac{1}{4} |dQ|^2_Q \geq 0,$$

which is a multiple of the energy density of the harmonic map $Q$. Take the trace of (49), we get

$$\Delta \text{Tr} Q = Q^{pq} \langle dQ_{kp}, dQ_{qk} \rangle \geq 0.$$ 

Moreover, [18] showed that the function $R$ satisfies the inequality

$$R \Delta R \geq \frac{1}{2} |\nabla R|^2 + \frac{1}{2} R^3,$$

hence a contradiction argument implies that everywhere

$$\Delta R \geq 0.$$ 

Then they used standard geometric analysis arguments for inequality (52) and the fact $\text{Ric} \geq 0$ to conclude

**Proposition 5.1.** Suppose $B(p, r) \subset X$ has compact closure, and $\partial B(p, r) \neq \emptyset$, then $R(p) \leq \frac{32}{r^2}$. In particular, a complete torsion-free hypersymplectic 4-manifold is hyperkähler.

Note that in the latter case, there exists a constant matrix $B$ in $SL(3, \mathbb{R})$ such that $\omega B$ is a hyperkähler triple, and the Riemannian metric defined by $\omega$ is the same as the one defined by $\omega B$.

**5.2. Compactness for the boundary value problem.** Now suppose $X$ is an oriented 4-manifold with compact boundary $Y = \partial X$, then through the boundary exponential map, a neighborhood $U$ of $Y$ is diffeomorphic to $Y \times [0, a)$. Let $t$ denote the distance function $d(\cdot, Y)$, i.e., the projection $Y \times [0, a) \to [0, a)$, then in $U$, $\omega$ can be written as

$$\omega = -dt \wedge *_{Y_t} \gamma_t + \gamma_t,$$

where $Y_t = Y \times \{t\}$ and $*_{Y_t}$ is the Hodge star operator of $g|_{Y_t}$. $\omega$ being closed is equivalent to

$$d_{Y_t} \gamma_t = 0,$$

$$\frac{\partial \gamma_t}{\partial t} = -d_{Y_t}(*_{Y_t} \gamma_t).$$

(46) is equivalent to

$$d_{Y_t} Q^{ij} \wedge *_{Y_t} \gamma_{t,j} + \frac{\partial Q^{ij}}{\partial t} \gamma_{t,j} = 0,$$
\[ (57) \quad dY_i Q^i_j \wedge \gamma_{t,j} = 0. \]

Note that (56) is equivalent to
\[ (58) \quad \frac{\partial Q^i_j}{\partial t} = \frac{dY_i Q^i_k \wedge \eta_{t,j} \wedge *Y_i \gamma_{k}}{\text{vol}_t}, \]
where \( \text{vol}_t = \eta_{t,1} \wedge \eta_{t,2} \wedge \eta_{t,3} \) and equals to the Riemannian volume form of \( g|_Y \).

From calculations in Lemma 2.2 and (55), (58), it is easy to see that the second fundamental form \( II(e_i, e_j) \) is in algebraic terms of \( \eta, d\alpha_X \eta, Q, d\alpha_X Q \).

Now let us try to prove Theorem 1.4, starting with basic observations. In Theorem 1.4, suppose \( \omega|_{\partial X}, Q \) converge in Cheeger-Gromov sense to the limit, then we have \( \text{diam}(\partial X, g_1|_{\partial X}) \leq C, \text{vol}_g(\partial X) \leq C, \text{inj}_{\partial X, g_1|_{\partial X}} \geq i_0, |\nabla^j_{\partial X} Rm_{\alpha X, g_1}| \leq C_j, |\nabla_{\partial X} S_i| \leq C_j \). By (53) and maximum principle, \( R_t \) is uniformly bounded on \( X \), so \( |\text{Ric}_{g_t}| \) is uniformly bounded on \( X \) since \( \text{Ric}_{g_t} \) is non-negative and its trace is uniformly bounded. Due to the uniform mean positive curvature condition, and \( \text{Ric}_{g_t} \geq 0 \), we have an upper bound of \( \text{sup}_{p \in X} d_{g_t}(p, \partial X), \text{vol}_{g_t}(X) \) by Proposition 3.2, and in particular an upper bound of \( \text{diam}(X, g_t) \). The Chern-Gauss-Bonnet formula thus gives an upper bound of \( \int_X |Rm_{g_t}|^2 \). Also, by (51) and maximum principle, \( \text{Tr}Q_t \) is uniformly bounded on \( X \), so \( Q_t \) is uniformly bounded on \( X \) and then \( |\text{d}Q_t| \) is uniformly bounded on \( X \).

Given these conclusions, we need to verify that all propositions that were used to prove Theorem 1.3 adapt to the torsion-free hypersymplectic setting. Firstly, we digress to discuss elliptic regularity for torsion-free equations (50). In harmonic coordinates in \( B_2 \) or \( B_2^T \), view \( g \) as a 4 by 4 matrix of functions, then we have a system of PDEs in \( g, Q \):
\[ (59) \quad \Delta_g Q_{kl} = Q^{pq}(dQ_{kp}, dQ_{ql})_g = 0, \]
\[ (60) \quad \Delta_g g_{ij} + B_{ij}(g, \partial g) = -\frac{1}{2} Q^{ab} Q^{cd} \partial_i Q_{bc} \partial_j Q_{da}. \]

From this, by a bootstrapping argument, one sees interior regularity: fix \( \beta \in (0, 1) \). If \( Q \) is \( C^1 \) bounded, \( g \) is \( C^{1,\beta} \) bounded, and they are uniform positive on \( B_2 \), then all derivatives of \( Q \) and \( g \) are bounded. For boundary regularity, there is no technical difficulty to get the following version from Neumann boundary conditions (34),(35): if all tangential derivatives of \( Q, g, H \) are bounded on \( \partial B_2 \), and \( Q \) is \( C^1 \) bounded, \( g \) is \( C^{1,\beta} \) bounded, and they are uniformly positive on \( B_2^T \), then all derivatives of \( Q \) and \( g \) are bounded in \( B_2^T \). If in both cases, we also assume \( g \) is \( C^{k,\beta} \) close to identity in \( B_2 \) or \( B_2^T \), then similarly, \( g \) is \( C^{k,\alpha} \) close to identity for any \( \alpha \in (\beta, 1) \).

Following the proof of Theorem 3.18, we get the following two propositions.

**Proposition 5.2.** Let \( (X, g, \omega) \) be a torsion-free hypersymplectic manifold and \( B(p, r) \) is a metric ball that has compact closure, \( \partial B(p, r) \neq \emptyset \). Suppose for any \( q \in B(p, r) \),
\[ \text{inj}_{q} \geq c(d(q, \partial B(p, r))), \]
\[ \text{Tr}Q, R \leq C. \]

Fix \( \Lambda > 1, 0 < \alpha < 1 \), then for any \( k \geq 0, q \in B(p, r) \),
\[ r^{k,\alpha}_h(q, g, \Lambda) \geq C'_k d(q, \partial B(p, r)). \]

In particular, \( |\nabla^k Q| \leq C'_k \) in \( B(p, \frac{r}{2}) \).
Proposition 5.3. Let \((X, g, \omega)\) be a compact torsion-free hypersymplectic manifold with boundary. Suppose \(i_b \geq i_0, \text{inj}_X \geq i_0, \text{inj}_{\partial X} \geq i_0, \text{Tr}Q, R \leq C\) on \(\partial X\) and \(|\nabla_{\partial X} Rm_{\partial X}| \leq C_j, |\nabla_{\partial X}^j S| \leq C_j, |\nabla_{\partial X}^j Q| \leq C_j\) on \(\partial X\), \(\forall k \geq 0\). Fix \(\Lambda > 1, 0 < \alpha < 1\), then for any \(k \geq 0, q \in X\),
\[
r_k^{k, \alpha}(q, g, \Lambda) \geq C''',
\]
Then we have

Proposition 5.4. Proposition 4.14 holds for torsion-free hypersymplectic manifolds \((X, g, \omega)\), provided an upper bound of \(\text{Tr}Q\) in \(B(p, 5)\).

Proof. The proof is almost the same as there. Let us list the ingredients here:

- We have Bishop-Gromov volume comparison, since \(\text{Ric}_g \geq 0\).
- Before rescaling, \(R_i, |\text{Ric}_g|\) are automatically bounded by Proposition 5.1.
- For the rescaled metric \(\check{g}_i\), the curvature bound and the volume non-collapsing condition imply injectivity radius lower bound on compact sets, hence by Proposition 5.2, we have harmonic radius lower bounds as well as bounds for derivatives of \(Q_i\) on compact sets, so we have pointed Cheeger-Gromov convergence of a subsequence \((M_i, g_i, \omega_i, q_i)\).
- The limit \(\omega_\infty\) is a hyperkähler triple up to a \(\text{SL}(3, \mathbb{R})\) rotation, because \(g_\infty\) is scalar flat, or because of Proposition 5.1.

So, we get the contradiction in the same way. \(\square\)

With the above three Propositions as tools and \(\epsilon\)-regularity, argue the same way as in Theorem 4.1, one gets an analogous version of Theorem 4.1, i.e., curvature control within \(i_b\), assuming \(i_b \geq i_0\).

Remark 5.5. We make a remark about the proof of \(\epsilon\)-regularity for torsion-free hypersymplectic manifolds here. Firstly, the proof in [32] Theorem 3.21 directly applies to this case by using Proposition 5.1. Alternatively, we can apply Remark 8.22 in [11] to conclude that \(g\) has \(C^{1, \alpha}\) bounded covering geometry when curvature \(L^2\) norm is small. By (50), we have \(|\nabla \text{Ric}| \leq C\), hence \(g\) has \(C^{2, \alpha}\) bounded covering geometry and \(|\text{Rm}|\) is bounded.

Now one can finish the proof of compactness part of Theorem 1.4 by the same arguments in Section 4. Note that \(\text{Ric} \geq 0, H > 0\) is enough for the focal point argument.

5.3. Uniqueness. Finally, we prove the uniqueness part of Theorem 1.4.

Proposition 5.6. Let \(\omega_1, \omega_2\) be two torsion-free hypersymplectic triples on an oriented 4-manifold \(X\) with compact boundary \(Y = \partial X\). Suppose \(\gamma_1 = \gamma_2, Q_1 = Q_2\) on \(\partial X\), then \(\omega_1 = \omega_2\) in geodesic gauges of \(g_\omega\) near \(\partial X\).

Proof. \(\omega_i\) defines a torsion-free \(G_2\) structure \(\phi_i\) on \(X \times T^3\) via (1), which defines a warped product metric
\[(61) \quad g_{\phi_i} = g_\omega + Q_{ij} dt^i dt^j.\]
In the geodesic gauge of \(g_{\omega_i}\), write
\[
\omega_i = -dt \wedge *_{Y_i} \gamma_t + \gamma_t,
\]
where \(t\) is the distance function \(d_{g_{\omega_i}}(\cdot, \partial X)\). Hence
\[(62) \quad \phi_i = -dt \wedge \theta_{1,i} + \rho_{t,i},\]
where
\[ \rho_{t,i} = dt^1 \wedge dt^2 \wedge dt^3 - \gamma_t^1 \wedge dt^1 - \gamma_t^2 \wedge dt^2 - \gamma_t^3 \wedge dt^3, \]
\[ \theta_{t,i} = - *_{Y_t} \gamma_t^{1,i} \wedge dt^1 - *_{Y_t} \gamma_t^{2,i} \wedge dt^2 - *_{Y_t} \gamma_t^{3,i} \wedge dt^3 \]
By (61), \( t \) can also be viewed as \( dz_i \times (, \partial X \times T^3) \), so (62) is written in the geodesic gauge of \( g_{\phi_i} \). By the calculations in [13] Section 2.2, for \( i = 1, 2 \), both \( g_{\phi_i}|_{\partial X \times T^3} \) and the second fundamental forms of \( \partial X \times T^3 \) are equal to each other, since they are explicitly in terms of \( \theta_{0,i}, \rho_{0,i} \), which are in terms of \( \gamma_i, Q_i \). Since \( g_{\phi_i} \) are Ricci-flat, by [8] Theorem 4, \( g_{\phi_1} = g_{\phi_2} \). Since \( \nabla_{g_{\phi_1}} |\phi_1 - \phi_2|^2 = 0 \) and \( \phi_1 - \phi_2 = 0 \) at one point, we have \( \phi_1 = \phi_2, \omega_1 = \omega_2 \).

Note that for a torsion-free hypersymplectic triple \( \omega \), the metric \( g_\omega \) is real analytic with respect to the analytic structure defined by harmonic coordinates, due to elliptic regularity of (59)(60), so the arguments in subsection 4.5 shows global uniqueness:

**Theorem 5.7.** Let \( X \) be a connected 4-manifold with boundary, \( \pi_1(X, \partial X) = 0 \). Suppose \( \omega_1, \omega_2 \) are two smooth torsion-free hypersymplectic triples on \( X \), and \( \varphi_0 : \partial X \to \partial X \) is a diffeomorphism, such that \( \omega_1|_{\partial X} = \varphi_0^*(\omega_2|_{\partial X}), Q_1|_{\partial X} = \varphi_0^*Q_2|_{\partial X} \), then there exists a diffeomorphism \( \varphi : X \to X, \varphi|_{\partial X} = \varphi_0 \), such that \( \omega_1 = \varphi^*\omega_2 \) on \( X \).

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