Pins, Stakes, Anchors and Gaussian Triangles

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Abstract. Among the topics we discuss are certain joint densities (for sides and for angles), acuteness probabilities and bivariate Rice moments.

A planar triangle is pinned Gaussian if one vertex is fixed at the origin and the other two have \( x, y \) coordinates that are independent normally distributed random variables with mean 0 and variance 1. Let \( a, b \) denote the sides opposite the random vertices; let \( c \) denote the side opposite \((0,0)\). A Jacobian determinant calculation similar to [1] yields

\[
\begin{cases}
\frac{2}{\pi} \frac{x y z}{\sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}} \exp \left( -\frac{1}{2} (x^2 + y^2) \right) \\
\text{if } |x-y| < z < x+y,
\end{cases}
\]

as the trivariate density \( f(x,y,z) \) for \( a, b, c \). The condition \(|x-y| < z < x+y\) is equivalent to \(|x-z| < y < x+z\) and to \(|y-z| < x < y+z\) via the Law of Cosines. As a consequence, the univariate density for \( a \) (or \( b \)) is

\[
x \exp \left( -\frac{1}{2} x^2 \right), \quad x > 0
\]

and the univariate density for \( c \) is

\[
\frac{x}{2} \exp \left( -\frac{1}{4} x^2 \right), \quad x > 0
\]

which are Rayleigh with means \( \sqrt{\pi/2}, \sqrt{\pi} \) and mean squares 2, 4 respectively. Side \( c \) is distributed the same as that for “pure” Gaussian triangles. Also \( E(ab) = \pi/2 \) and \( E(ac) = \sqrt{2\pi} \).

From this, another calculation like [1] gives

\[
\begin{cases}
\frac{2 \sin(x) \sin(y) \sin(x+y)}{\pi (\sin(x)^2 + \sin(y)^2)^2} & \text{if } 0 < x < \pi, 0 < y < \pi \text{ and } x + y < \pi, \\
0 & \text{otherwise}
\end{cases}
\]

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as the bivariate density for angles $\alpha$, $\beta$ opposite sides $a, b$. It is clear that $\gamma = \pi - \alpha - \beta$ is Uniform$[0, \pi]$ by construction. Let $\varphi$ denote the probability that a pinned Gaussian triangle is obtuse. The univariate density for $\alpha$ (or $\beta$) is obtained via integration:

\[
\frac{2}{\pi} \int_{0}^{\pi-x} \sin(x) \sin(y) \sin(x+y) \, dy
\]

\[
= \frac{2}{\pi} \int_{0}^{\pi-x} \frac{\cos(x) \sin(x)}{2(2-\cos(x)^2)(\sin(x)^2+\sin(y)^2)} \, dy
\]

\[
+ \frac{2}{\pi} \int_{0}^{\pi-x} \left( \frac{\sin(x) \sin(y) \sin(x+y)}{\sin(x)^2+y^2} - \frac{\cos(x) \sin(x)}{2(2-\cos(x)^2)(\sin(x)^2+\sin(y)^2)} \right) \, dy
\]

\[
= \frac{1}{\pi} \frac{\cos(x)}{(2-\cos(x)^2)^{3/2}} \left( \frac{\pi}{2} + \arcsin \left( \frac{\cos(x)}{\sqrt{2}} \right) \right) + \frac{1}{\pi} \frac{1}{2-\cos(x)^2}.
\]

Call this latter expression $g(x)$. Several results follow:

\[E(\alpha) = \frac{\pi}{4}, \quad E(\gamma) = \frac{\pi}{2},\]

\[E(\alpha^2) = \frac{5}{48} \pi^2 + \frac{1}{4} \ln(2)^2, \quad E(\gamma^2) = \frac{\pi^2}{3},\]

\[E(\alpha \beta) = \frac{1}{16} \pi^2 - \frac{1}{4} \ln(2)^2, \quad E(\alpha \gamma) = \frac{\pi^2}{12}.\]

The cross-correlation coefficient

\[\rho(\alpha, \beta) = \frac{\text{Cov}(\alpha, \beta)}{\sqrt{\text{Var}(\alpha) \text{Var}(\beta)}} = \frac{E(\alpha \beta) - \pi^2/16}{E(\alpha^2) - \pi^2/16} \approx -0.226\]

indicates weak negative dependency; replacing $\beta$ by $\gamma$, such association becomes stronger since $\rho(\alpha, \gamma) \approx -0.622$. Finally,

\[G(x) = \int_{0}^{x} g(\xi) \, d\xi = \frac{1}{\pi} \frac{\sin(x)}{(2-\cos(x)^2)^{1/2}} \left( \frac{\pi}{2} + \arcsin \left( \frac{\cos(x)}{\sqrt{2}} \right) \right) + \frac{1}{\pi} x\]

which implies that

\[\varphi = P(\alpha > \pi/2) + P(\beta > \pi/2) + P(\gamma > \pi/2)
= 2 \left( 1 - G(\pi/2) \right) + 1/2
= 3/2 - 1/\sqrt{2} \approx 0.793\]
because a triangle can have at most one obtuse angle. See [2, 3] for alternative approaches for computing $\wp$.

We conjecture that the bivariate density for $\alpha$, $\beta$ for pinned Gaussian triangles in $n$-dimensional space is

$$
\begin{cases}
C_n \frac{\sin(x)^{n-1} \sin(y)^{n-1} \sin(x+y)^{n-1}}{(\sin(x)^2 + \sin(y)^2)^n} & \text{if } 0 < x < \pi, 0 < y < \pi \text{ and } 0 < x + y < \pi, \\
0 & \text{otherwise}
\end{cases}
$$

where $C_n = (n - 1)^2 n^{-1} / \pi$. Values of $\wp$ for all $n$ are known [2] and are consistent with predictions based on our conjecture:

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|
| $\wp_n$ | $1 - \frac{1}{3}$ | $\frac{3}{2} - \frac{3}{4\sqrt{2}}$ | $1 - \frac{1}{3\pi}$ | $\frac{3}{2} - \frac{43}{32\sqrt{2}}$ | $1 - \frac{2\pi}{15\pi}$ | $\frac{3}{2} - \frac{11\pi}{128\sqrt{2}}$ |

but the subtle dependency between angles remains open.

Henceforth let $c > 0$ be constant. A planar triangle is **staked Gaussian** if one vertex is fixed at $(0, 0)$, another vertex is fixed at $(c, 0)$ and the third has $x$, $y$ coordinates that are independent Normal$(0,1)$. The term *stake* (as in “staking a tent”) is new in geometric probability, as far as is known. For definiteness, define the vertices $A = (c, 0)$, $B = (0, 0)$ and $C = (u, v)$. We shall enter into more details than previously, as the territory is uncharted. Clearly

$$
\tan(\alpha) = \frac{v}{c-u}, \quad \tan(\beta) = \frac{v}{u}.
$$

To compute the Jacobian of the transformation $(u, v) \mapsto (\alpha, \beta)$, note that

$$
\sec(\alpha)^2 \frac{\partial \alpha}{\partial u} = \frac{\partial v}{\partial u} = \frac{v}{(c-u)^2}, \quad \sec(\alpha)^2 \frac{\partial \alpha}{\partial v} = \frac{\partial v}{\partial v} = \frac{1}{c-u},
$$

$$
\sec(\beta)^2 \frac{\partial \beta}{\partial u} = \frac{\partial v}{\partial u} = -\frac{v}{u^2}, \quad \sec(\beta)^2 \frac{\partial \beta}{\partial v} = \frac{\partial v}{\partial v} = \frac{1}{u}
$$

and

$$
\cos(\alpha)^2 = \frac{(c-u)^2}{(c-u)^2 + v^2}, \quad \cos(\beta)^2 = \frac{u^2}{u^2 + v^2}
$$

hence the determinant is

$$
|J| = \begin{vmatrix}
\frac{v}{(c-u)^2 + v^2} & \frac{c-u}{(c-u)^2 + v^2} \\
\frac{-v}{u^2 + v^2} & \frac{u}{u^2 + v^2}
\end{vmatrix} = \frac{cv}{(u^2 + v^2) [(c-u)^2 + v^2]}.
$$
Solving for \( u, v \) in terms of \( \alpha, \beta \), we obtain

\[
\begin{align*}
u &= \frac{c \tan(\alpha)}{\tan(\alpha) + \tan(\beta)}, \\
v &= \frac{c \tan(\alpha) \tan(\beta)}{\tan(\alpha) + \tan(\beta)}.
\end{align*}
\]

Substituting these expressions into the standard bivariate normal density

\[
\frac{1}{2\pi} \exp \left[ -\frac{1}{2} \{u^2 + v^2\} \right]
\]

and dividing by \( |J| \) yields

\[
\frac{c^2}{2\pi} \exp \left[ -\frac{c^2}{2} \frac{\sin^2(\alpha)}{\sin(\alpha + \beta)^2} \right] \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^3}.
\]

Multiplying by 2 gives the correct normalization. We have not attempted to find any univariate densities or moments here. For simplicity, set \( c = 1 \). The acuteness probability can be found numerically:

\[
1 - \varphi = \frac{1}{\pi} \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \exp \left[ -\frac{1}{2} \frac{\sin^2(\alpha)}{\sin(\alpha + \beta)^2} \right] \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^3} d\beta d\alpha = 0.23685...
\]

and symbolically:

\[
1 - \varphi = \frac{1}{2} \left[ -1 + \text{erf} \left( \frac{1}{\sqrt{2}} \right) + \frac{I_0(1/4)}{\exp(1/4)} \right]
\]

where \( \text{erf} \) is the error function and \( I_m \) is the \( m \)th modified Bessel function of the first kind. The exact expression comes not from evaluating the double integral in some unforeseen manner, but via geometry as follows.

Shift the Gaussian mean to the left by \( 1/2 \). Likewise, translate the triangle \( ABC \) to \( A'B'C' \), where \( A' = (1/2, 0) \), \( B' = (-1/2, 0) \) and \( C' = (u - 1/2, v) \). The set of points \( C \) where \( \alpha + \beta = \pi/2 \) is satisfied corresponds precisely to the points \( C' = (x, y) \) on the circle given by

\[
x^2 + y^2 = 1/4 \quad \text{or} \quad r = 1/2
\]

(the latter is in polar coordinates). Acute triangles \( ABC \) correspond to points \( C' \) outside the circle but inside the strip \( |x| = 1/2 \). The probability that \( C' \) falls inside the strip is

\[
\frac{1}{2\pi} \int_{-1/2}^{1/2} \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{2} \left\{ \left( x + \frac{1}{2} \right)^2 + y^2 \right\} \right] dy dx = \frac{1}{2} \text{erf} \left( \frac{1}{\sqrt{2}} \right)
\]
and the probability that $C'$ falls inside the circle is

$$\frac{1}{2\pi} \int_{-1/2}^{1/2} \int_{-\sqrt{1/4-x^2}}^{\sqrt{1/4-x^2}} \exp \left[ -\frac{1}{2} \left( \left( x + \frac{1}{2} \right)^2 + y^2 \right) \right] dy \, dx$$

$$= \frac{1}{2\pi} \int_0^{1/2} \int_0^{1/2} \exp \left[ -\frac{1}{2} \left( r^2 + r \cos(\theta) + \frac{1}{4} \right) \right] r \, dr \, d\theta$$

$$= \exp(-1/8) \int_0^{1/8} \exp \left( -\frac{1}{2} r^2 \right) I_0 \left( \frac{r}{2} \right) r \, dr$$

$$= \exp(-1/8) \int_0^{1/8} \exp (-s) I_0 \left( 2\sqrt{s/8} \right) ds$$

$$= 1 - \frac{1}{2} \left[ 1 + \exp(-1/4)I_0(1/4) \right] = \frac{1}{2} \left[ 1 - \frac{I_0(1/4)}{\exp(1/4)} \right]$$

using a formula for what is called $J(1/8, 1/8)$ in [4, 5]. This completes the calculation of $1 - \wp$.

It is shown in the Appendix that

$$\frac{2}{\pi} \frac{x \, y}{\sqrt{(x + y + c)(-x + y + c)(x - y + c)(x + y - c)}} \exp \left( -\frac{1}{2} x^2 \right)$$

$$\begin{cases} \frac{x \, y}{\sqrt{(x + y + c)(-x + y + c)(x - y + c)(x + y - c)}} \exp \left( -\frac{1}{2} x^2 \right) & \text{if } |x - y| < c < x + y, \\ 0 & \text{otherwise} \end{cases}$$

is the bivariate density for sides $a, b$. The asymmetry is unsurprising. Side $a$ is distributed the same as that for pinned Gaussian triangles: Rayleigh with mean $\sqrt{\pi/2}$ and mean square 2. For simplicity, set $c = 1$. The univariate density for $b$ is

$$x \exp \left( -\frac{1}{2} (x^2 + 1) \right) I_0(x), \quad x > 0$$

which is Rice with mean

$$\frac{\sqrt{2\pi}}{4e^{1/4}} \left( 3I_0 \left( \frac{1}{4} \right) + I_1 \left( \frac{1}{4} \right) \right) = 1.5485724605511453806302363...$$
and mean square 3. Also,
\[ E(a, b) = \frac{2}{\pi} \int_0^\infty x^2(x + 1) \exp \left( -\frac{1}{2} x^2 \right) E \left( \frac{2\sqrt{x}}{x + 1} \right) \, dx \]
\[ = 2.2627965282687383013183035... \]
where
\[ E(\xi) = \int_0^{\pi/2} \sqrt{1 - \xi^2 \sin^2(\theta)} \, d\theta = \int_0^1 \sqrt{\frac{1 - \xi^2 t^2}{1 - t^2}} \, dt \]
is the complete elliptic integral of the second kind. An expression for \( E(a, b) \) in terms of Meijer \( G \)-function values is possible [6].

A planar triangle is **anchored Gaussian** if one vertex is fixed at \((-c/2, 0)\), another vertex is fixed at \((c/2, 0)\) and the third has \(x, y\) coordinates that are independent Normal(0,1). The term anchoring (as in “anchoring a ship”) is again new in this context. For definiteness, define the vertices \( A = (c/2, 0), B = (-c/2, 0) \) and \( C = (u, v) \). As before, here are details:

\[ \tan(\alpha) = \frac{v}{\frac{c}{2} - u} = \frac{2v}{c - 2u}, \quad \tan(\beta) = \frac{v}{\frac{c}{2} + u} = \frac{2v}{c + 2u}; \]
\[ \frac{\sec(\alpha)^2}{\partial u} = \frac{\partial}{\partial u} \frac{2v}{c - 2u} = \frac{4v}{(c - 2u)^2}, \quad \frac{\sec(\beta)^2}{\partial u} = \frac{\partial}{\partial u} \frac{2v}{c + 2u} = \frac{4v}{(c + 2u)^2}; \]
\[ \cos(\alpha)^2 = \frac{(c - 2u)^2}{(c - 2u)^2 + 4v^2}, \quad \cos(\beta)^2 = \frac{(c + 2u)^2}{(c + 2u)^2 + 4v^2} \]
hence the Jacobian determinant is
\[ |J| = \begin{vmatrix} \frac{4v}{(c - 2u)^2 + 4v^2} & \frac{2(c - 2u)}{(c - 2u)^2 + 4v^2} \\ \frac{2(c + 2u)}{4v} & \frac{2(c + 2u)}{(c + 2u)^2 + 4v^2} \end{vmatrix} = \frac{16cv}{[(c - 2u)^2 + 4v^2][(c + 2u)^2 + 4v^2]}. \]
Solving for \( u, v \) in terms of \( \alpha, \beta \), we obtain
\[ u = c \tan(\alpha) - \tan(\beta), \quad v = c \frac{\tan(\alpha) \tan(\beta)}{\tan(\alpha) + \tan(\beta)}. \]
Substituting these expressions into the standard bivariate normal density and dividing by $|J|$ yields
\[
\frac{c^2}{2\pi} \exp \left[ -\frac{c^2 \sin(\alpha - \beta)^2 + 4 \sin(\alpha)^2 \sin(\beta)^2}{8} \sin(\alpha + \beta)^2 \right] \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^3}.
\]

Multiplying by 2 gives the correct normalization. We have not attempted to find any univariate densities or moments here. For simplicity, set $c = 1$. The acuteness probability can be found numerically:
\[
1 - \varphi = \frac{1}{\pi} \int_{0}^{\pi} \int_{0}^{\pi} \exp \left[ -\frac{1}{8} \sin(\alpha - \beta)^2 + 4 \sin(\alpha)^2 \sin(\beta)^2 \right] \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^3} d\beta d\alpha = 0.26542...
\]

and symbolically:
\[
1 - \varphi = -1 + \frac{1}{\exp(1/8)} + \text{erf} \left( \frac{1}{2\sqrt{2}} \right).
\]

This value is slightly larger for anchored triangles than for staked triangles. Proving this formula is similar to before, with $x + 1/2$ replaced by $x$ in both of the double integrals, which simplifies matters.

It is shown in the Appendix that
\[
\begin{cases}
\frac{2e^{c^2/8}}{\pi} \frac{xy}{\sqrt{(x + y + c)(-x + y + c)(x - y + c)(x + y - c)}} \exp \left( -\frac{1}{4} (x^2 + y^2) \right) \\
0 & \text{if } |x - y| < c < x + y,
\end{cases}
\]
is the bivariate density for sides $a, b$. The symmetry is very helpful. For simplicity, set $c = 1$. The univariate density for $a$ (or $b$) is
\[
x \exp \left( -\frac{1}{2} \left( x^2 + \frac{1}{4} \right) \right) I_0 \left( \frac{x}{2} \right), \quad x > 0
\]
which is Rice with mean
\[
\sqrt{\frac{2\pi}{16e^{1/16}}} \left( 9I_0 \left( \frac{1}{16} \right) + I_1 \left( \frac{1}{16} \right) \right) = 1.3304473406107031708025583...
\]
and mean square $9/4$. Also,
\[
E(ab) = \frac{1}{64} (I_0 \left( \frac{1}{16} \right) K_0 \left( \frac{1}{16} \right) + 8I_0 \left( \frac{1}{16} \right) K_1 \left( \frac{1}{16} \right) - 8I_1 \left( \frac{1}{16} \right) K_0 \left( \frac{1}{16} \right) + I_1 \left( \frac{1}{16} \right) K_1 \left( \frac{1}{16} \right))
\]
\[
= 2.0303939030262620132069685...
\]
where $K_m$ is the $m^{\text{th}}$ modified Bessel function of the second kind.

The bivariate density for $\alpha, \beta / a, b$ for staked/anchored Gaussian triangles in $n$-dimensional space is also of interest, as well as acuteness probabilities and cross-covariances.

In closing, let us return to “pure” Gaussian triangles in $n$-dimensional space [1][2].

We conjecture that the bivariate density for $\alpha, \beta$ is

$$
\begin{cases}
\tilde{C}_n \frac{\sin(x)^{n-1} \sin(y)^{n-1} \sin(x+y)^{n-1}}{(\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)^n} & \text{if } 0 < x < \pi, 0 < y < \pi \text{ and } x + y < \pi,
0 & \text{otherwise}
\end{cases}
$$

where $\tilde{C}_n = (n-1)2^{n-1}3^{n/2}/\pi$. Values of $\tilde{\varphi}$ for all $n$ are known [2] and are consistent with predictions based on our conjecture:

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|
| $\tilde{\varphi}_n$ | $\frac{1}{4}$ | $1 - \frac{3\sqrt{3}}{4\pi}$ | $\frac{13}{32}$ | $1 - \frac{9\sqrt{3}}{8\pi}$ | $\frac{353}{312}$ | $1 - \frac{27\sqrt{3}}{20\pi}$ | $\frac{867}{4096}$ |

Surely someone else has examined this issue! More relevant information would be appreciated.

1. Related Work

Let $(X, Z)$ be bivariate normally distributed random variables with mean 0, variance 1 and cross-correlation coefficient $\rho$. Let $(Y, W)$ be likewise and independent of $(X, Z)$. The density of

$$(A, B) = \left(\sqrt{(X - 1/2)^2 + Y^2}, \sqrt{(Z - 1/2)^2 + W^2}\right)$$

is [7][8]

$$
\Omega a b \exp \left(-\frac{a^2 + b^2}{2(1 - \rho^2)}\right) \sum_{k=0}^{\infty} \varepsilon_k I_k \left(\frac{a b \rho}{1 - \rho^2}\right) I_k \left(\frac{a}{2(1 + \rho)}\right) I_k \left(\frac{b}{2(1 + \rho)}\right)
$$

where $\varepsilon_0 = 1$, $\varepsilon_k = 2$ for $k > 0$ and the normalizing constant is

$$
\Omega = \frac{1}{1 - \rho^2} \exp \left(-\frac{1}{4(1 + \rho)}\right).
$$

This result concerns the joint distribution of correlated distances from a point $(1/2, 0)$, which is not what we truly seek. Using an argument in [8] applied instead to a more general theorem in [9], we evaluate that

$$(\bar{A}, \bar{B}) = \left(\sqrt{(X + 1/2)^2 + Y^2}, \sqrt{(Z - 1/2)^2 + W^2}\right)$$
has density
\[
\tilde{\Omega} \, a \, b \exp \left( -\frac{a^2 + b^2}{2(1 - \rho^2)} \right) \sum_{k=0}^{\infty} (-1)^k \varepsilon_k I_k \left( \frac{a \rho}{1 - \rho^2} \right) I_k \left( \frac{b}{2(1 - \rho)} \right)
\]
with normalizing constant
\[
\tilde{\Omega} = \frac{1}{1 - \rho^2} \exp \left( -\frac{1}{4(1 - \rho)} \right).
\]
The changes from former to latter are slight. Our conjecture would be that this joint distribution approaches that for anchored triangle sides (with \( c = 1 \)) as \( \rho \to 1^- \). In particular, \( E(\bar{A} \, \bar{B}) \) would approach \( 2.03039... \) as \( \rho \) increases and \( E(\bar{A} \, \bar{B}) \) can be rewritten as a doubly infinite series as in [10], avoiding integration entirely. Numerical convergence issues associated with the doubly infinite series for large \( \rho \), however, make us hesitate to commit further.

2. Acknowledgement
I am thankful to M. Larry Glasser [6] for discovering the Meijer G-function expression
\[
E(a \, b) = \frac{1}{4} \left[ G_{2,3}^{2,1} \left( \frac{1}{2} \left| \begin{array}{c} -\frac{3}{2}, -\frac{1}{2} \\ -2, 0, -2 \end{array} \right. \right) + G_{2,3}^{2,1} \left( \frac{1}{2} \left| \begin{array}{c} -\frac{1}{2}, -\frac{1}{2} \\ -1, 0, -1 \end{array} \right. \right) + G_{2,3}^{2,1} \left( \frac{1}{2} \left| \begin{array}{c} -\frac{1}{2}, -\frac{1}{2} \\ 0, 0, -2 \end{array} \right. \right) \right]
\]
corresponding to staked triangle sides. Much more relevant material can be found at [11], including experimental computer runs that aided theoretical discussion here.

3. Appendix
Let \( \Delta = (a + b + c)(-a + b + c)(a - b + c)(a + b - c) \). The natural transformation \((\alpha, \beta, c) \mapsto (a, b, c)\) appearing in [1] has Jacobian determinant \( a \, b \). Using the identities
\[
a = \frac{\sin(\alpha)}{\sin(\alpha + \beta)}, \quad b = \frac{\sin(\beta)}{\sin(\alpha + \beta)}, \quad \sqrt{\Delta} = \frac{\sin(\alpha) \sin(\beta)}{2c^2}
\]
we have
\[
\sin(\alpha + \beta) = \frac{cc\sqrt{\Delta}}{a \, b \, 2c^2} = \frac{\sqrt{\Delta}}{2a \, b}
\]
thus the bivariate staked angle density can be rewritten as
\[
\frac{c^2}{\pi} \exp \left[ -\frac{c^2}{2} \sin(\alpha)^2 \right] \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^2} \frac{1}{ab}
\]

\[
= \frac{c^2}{\pi} \exp \left[ -\frac{c^2}{2} \frac{a^2}{c^2} \right] \frac{ab}{cc\sqrt{\Delta}} \frac{1}{ab}
\]

\[
= \frac{2}{\pi} \frac{ab}{\sqrt{\Delta}} \exp \left( -\frac{1}{2}a^2 \right).
\]
Also, we have
\[
\frac{\sin(\alpha - \beta)}{\sin(\alpha + \beta)} = \frac{\sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta)}{\sin(\alpha + \beta)}
\]
\[
= \frac{a}{2a} \frac{-b^2 + a^2 + c^2}{c} \frac{b}{2b} \frac{-a^2 + b^2 + c^2}{c}
\]
\[
= \frac{-b^2 + a^2 + c^2}{2c^2} - \frac{-a^2 + b^2 + c^2}{2c^2}
\]
\[
= \frac{a^2 - b^2}{c^2}
\]

thus the bivariate anchored angle density can be rewritten as
\[
\frac{\frac{e^{\frac{c^2}{8}}}{\pi} \exp \left[ \frac{\frac{c^2 \sin(\alpha - \beta)^2 + 4 \sin(\alpha)^2 \sin(\beta)^2}{8 \sin(\alpha + \beta)^2}}{\sin(\alpha + \beta)^3 \frac{1}{a b}} \frac{\sin(\alpha) \sin(\beta)}{\sin(\alpha + \beta)^3 \frac{1}{a b}} \right] \sin(\alpha) \sin(\beta) \frac{1}{a b}}{\pi} \exp \left[ \frac{-\frac{c^2 (a^2 - b^2)^2 + \Delta}{8 \frac{1}{c^4}}}{\frac{ab}{2a} \frac{1}{ab}} \frac{ab}{c} \frac{\sqrt{\Delta}}{ab} \right]
\]
\[
= \frac{2a b}{\pi \sqrt{\Delta}} \exp \left[ \frac{-\frac{c^2 (a^2 - b^2)^2 + (a + b + c)(-a + b + c)(a - b + c)(a + b - c)}{8 \frac{1}{c^4}}}{\frac{1}{c^4}} \right]
\]
\[
= \frac{e^{\frac{c^2}{8}} \frac{2a b}{\pi \sqrt{\Delta}}}{\frac{1}{4} \left( a^2 + b^2 \right)}
\]

Let \(c = 1\) from now on and let \(S\) denote the infinite strip \(\{(x, y) : |x - y| < 1 < x + y\}\).

The asymmetry within
\[
E(ab) = \frac{2}{\pi} \int_S \frac{x^2 y^2}{\sqrt{(x + y + 1)(-x + y + 1)(x + y - 1)(x + y - 1)}} \exp \left( -\frac{1}{2} x^2 \right) dy dx
\]

for staked triangles is actually advantageous since
\[
\int_{|x-1|}^{x+1} \frac{y^2}{\sqrt{(x + y + 1)(-x + y + 1)(x + y - 1)(x + y - 1)}} dy = (x + 1)E \left( \frac{2\sqrt{x}}{x + 1} \right)
\]

therefore a single (but difficult) integral remains. The symmetry within
\[
\frac{2e^{\frac{c^2}{8}}}{\pi} \int_S \frac{x^2 y^2}{\sqrt{(x + y + 1)(-x + y + 1)(x + y + 1)(x + y - 1)}} \exp \left( -\frac{1}{4} \left( x^2 + y^2 \right) \right) dy dx
\]
for anchored triangles seems problematic at first, until we set

\[ u = x + y, \quad v = -x + y; \quad \text{equivalently,} \quad 2x = u - v, \quad 2y = u + v \]

and obtain

\[ 2(x^2 + y^2) = u^2 + v^2, \quad 4xy = u^2 - v^2 \]

therefore

\[
E(ab) = \frac{1}{16} \frac{2e^{1/8}}{\pi} \int_1^\infty \frac{1}{1} \int_{-1}^1 \frac{(u^2 - v^2)^2}{\sqrt{(u^2 - 1)(1 - v^2)}} \exp \left( -\frac{1}{8} (u^2 + v^2) \right) \frac{1}{2} dv \ du \\
= \frac{e^{1/8}}{16\pi} \int_1^\infty \frac{1}{\sqrt{u^2 - 1}} v^4 \exp \left( -\frac{1}{8} u^2 \right) du \cdot \int_{-1}^1 \frac{1}{\sqrt{1 - v^2}} \exp \left( -\frac{1}{8} v^2 \right) dv \\
= \frac{e^{1/8}}{16\pi} \int_1^\infty \frac{1}{\sqrt{u^2 - 1}} \exp \left( -\frac{1}{8} u^2 \right) du \cdot \int_{-1}^1 \frac{v^4}{\sqrt{1 - v^2}} \exp \left( -\frac{1}{8} v^2 \right) dv \\
= \frac{1}{64} \left[ K_0 \left( \frac{1}{16} \right) + 9K_1 \left( \frac{1}{16} \right) \right] I_0 \left( \frac{1}{16} \right) - \\
\frac{1}{64} \left[ K_0 \left( \frac{1}{16} \right) + K_1 \left( \frac{1}{16} \right) \right] \left[ I_0 \left( \frac{1}{16} \right) - I_1 \left( \frac{1}{16} \right) \right] + \\
\frac{1}{64} K_0 \left( \frac{1}{16} \right) \left[ I_0 \left( \frac{1}{16} \right) - 9I_1 \left( \frac{1}{16} \right) \right],
\]

as was to be shown.

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