SCHRÖDER COMBINATORICS AND \( \nu \)-ASSOCIAHEDRA

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Abstract. We study \( \nu \)-Schröder paths, which are Schröder paths which stay weakly above a given lattice path \( \nu \). Some classical bijective and enumerative results are extended to the \( \nu \)-setting, including the relationship between small and large Schröder paths. We introduce two posets of \( \nu \)-Schröder objects, namely \( \nu \)-Schröder paths and trees, and show that they are isomorphic to the face poset of the \( \nu \)-associahedron \( A_\nu \) introduced by Ceballos, Padrol and Sarmiento. A consequence of our results is that the \( i \)-dimensional faces of \( A_\nu \) are indexed by \( \nu \)-Schröder paths with \( i \) diagonal steps, and we obtain a closed-form expression for these Schröder numbers in the special case when \( \nu \) is a ‘rational’ lattice path. Using our new description of the face poset of \( A_\nu \), we apply discrete Morse theory to show that \( A_\nu \) is contractible. This yields one of two proofs presented for the fact that the Euler characteristic of \( A_\nu \) is one. A second proof of this is obtained via a formula for the \( \nu \)-Narayana polynomial in terms of \( \nu \)-Schröder numbers.

Keywords: Schröder paths, Schröder trees, \( \nu \)-associahedron, face poset, Morse matching.

1. Introduction

The \( n \)-dimensional associahedron is a simple polytope whose face poset is isomorphic to the poset of diagonal dissections of a convex \((n + 3)\)-gon ordered by coarsening, so that the minimal elements of the poset are the triangulations of the \((n + 3)\)-gon and the maximal element is the empty dissection. A well-known equivalent statement is that the face poset of the \( n \)-associahedron is isomorphic to the poset of Schröder paths from \((0, 0)\) to \((n + 1, n + 1)\). The number of elements in the poset of Schröder paths is known as a Schröder–Hipparchus number or a small Schröder number. This poset is graded by the number of diagonal steps in a Schröder path, so the number of Schröder paths with \( i \) diagonal steps is the number of faces of the associahedron of dimension \( i \). In particular, the vertices of the \( n \)-associahedron correspond to the Schröder paths with no diagonal steps, which are better known as Dyck paths. The Tamari lattice is a partial order on the set of Dyck paths, and a notable result \([10, 12]\) is that its Hasse diagram is realizable as the 1-skeleton of the \( n \)-associahedron.

With the viewpoint that the set of Dyck paths is the set of lattice paths that lie weakly above the staircase path \((NE)^{n+1}\) from \((0, 0)\) to \((n+1, n+1)\), Préville-Ratelle and Viennot \([13]\) extended the notion of the Tamari lattice to the \( \nu \)-Tamari lattice, a partial order on the set of \( \nu \)-Dyck paths, which are lattice paths that lie weakly above a fixed lattice path \( \nu \) from \((0,0)\) to \((b,a)\). A striking result of Ceballos, Padrol and Sarmiento \([5, \text{Theorem 5.2}]\) is that the Hasse diagram of the \( \nu \)-Tamari lattice is realizable as the 1-skeleton of a polyhedral complex induced by an arrangement of tropical hyperplanes. This polyhedral complex is the \( \nu \)-associahedron \( A_\nu \).

Date: June 18, 2020.
Inspired by the connection between Schröder paths and faces of the associahedron, we define \( \nu \)-Schröder paths (see Definition 2.1) and make a similar connection to the faces of the \( \nu \)-associahedron. In this article we also consider another family of Schröder objects: \( \nu \)-Schröder trees (see Definition 3.1) are a generalization of the \( \nu \)-trees of Ceballos, Padrol and Sarmiento [6].

The face poset of \( A_\nu \) is defined in [5] to be the poset on covering \((I, \overline{J})\)-forests, which may be considered as a \( \nu \)-analogue of non-crossing partitions. A central result of this article is the following pair of alternative descriptions of the face poset of \( A_\nu \).

**Theorem 4.7.** The face poset of the \( \nu \)-associahedron is isomorphic to the poset on \( \nu \)-Schröder paths, and to the poset on \( \nu \)-Schröder trees.

**Corollary 4.8.** The number of \( i \)-dimensional faces of the \( \nu \)-associahedron is the number of \( \nu \)-Schröder paths with \( i \) diagonal steps.

This is the outcome of combining bijections between covering \((I, \overline{J})\)-trees, \( \nu \)-Schröder trees, and \( \nu \)-Schröder paths from Theorem 4.2 and Theorem 3.13 and showing how the cover relation on the covering \((I, \overline{J})\)-trees is translated to cover relations for the other \( \nu \)-Schröder objects. Figures 10 and 11 show an example of a \( \nu \)-associahedron and its face poset in terms of \( \nu \)-Schröder paths.

Combining Theorem 4.7 with the fact that \( A_\nu \) is a polyhedral complex, we conclude in Theorem 4.9 that if \( P_\nu \) is the contraction poset of \( \nu \)-Schröder paths or the poset of \( \nu \)-Schröder trees, then every interval in \( P_\nu \) is an Eulerian lattice. As a consequence of results by Prévile-Ratelle and Viennot [13] we conclude in Corollary 4.10 that for non-classical \( \nu \), adjoining a minimal and a maximal element to \( P_\nu \) yields a sublattice in the face lattice of a classical associahedron.

From Theorem 4.7 it also follows that the Euler characteristic of the \( \nu \)-associahedron is 
\[
\chi(A_\nu) = \sum_{i \geq 0} (-1)^i \text{sch}_\nu (i),
\]
where \( \text{sch}_\nu (i) \) is the number of \( \nu \)-Schröder paths with \( i \) diagonal steps. We present two proofs of the fact that \( \chi(A_\nu) = 1 \); one enumerative and one topological. The enumerative proof relies on the \( \nu \)-Narayana polynomial \( N_\nu(x) \) (see Definition 2.5), which is a generating function for \( \nu \)-Dyck paths with respect to their valleys. In Proposition 2.6 we show that 
\[
N_\nu(x + 1) = \sum_{i \geq 0} \text{sch}_\nu (i) x^i,
\]
from which it follows that \( \chi(A_\nu) = N_\nu(0) = 1 \), as there is a unique \( \nu \)-Dyck path with zero valleys.

From Proposition 2.6 it can also be deduced that the number of large \( \nu \)-Schröder paths is twice the number of (small) \( \nu \)-Schröder paths. A bijective proof of this fact is also given. This generalizes results of Aguiar and Moreira [1] and Gessel [8].

The topological proof that \( \chi(A_\nu) = 1 \) employs discrete Morse theory. We show in Theorem 4.13 that the contraction poset of \( \nu \)-Schröder paths has an acyclic matching with a unique critical element, thereby showing that the \( \nu \)-associahedron is contractible. The matching has a simple description in terms of \( \nu \)-Schröder paths, highlighting a benefit of this viewpoint.

Another avenue for generalizing the theory of Schröder paths is to define \((q, t)\)-analogues. Haglund [9, Section 4] developed the theory of the \((q, t)\)-Schröder polynomial in connection with the theory of Macdonald polynomials and diagonal harmonics. Song [14], and Aval and Bergeron [3] further extended this \((q, t)\) generalization to the case of \((a, ma + 1)\) and \((a, b)\)-Schröder paths for any positive integers \( a, b \). The idea of Schröder parking functions
was also explored in [3]. We anticipate that the \((q,t)\)-analogue can be further extended to the case of \(\nu\)-Schröder paths.

This article is organized as follows. In Section 2 a number of existing bijective and enumerative results on Schröder paths are extended to the case of \(\nu\)-Schröder paths. Closed-form expressions are obtained for Schröder numbers with respect to the number of diagonal step in the special case when \(\nu\) is a ‘rational’ lattice path. In Section 3 a bijection between \(\nu\)-Schröder trees and paths is given. Furthermore, a poset structure on the set of \(\nu\)-Schröder trees given by contraction operations is defined, and it is shown that this induces a poset structure on the set of \(\nu\)-Schröder paths. In Section 4 the face poset of the \(\nu\)-associahedron is shown to have alternative descriptions in terms of \(\nu\)-Schröder trees and paths. An acyclic partial matching on the Hasse diagram of the contraction poset of \(\nu\)-Schröder paths is exhibited, giving a proof that the \(\nu\)-associahedron is contractible.

Acknowledgments. We thank Richard Ehrenborg for suggesting the use of Discrete Morse Theory to show the contractibility of the \(\nu\)-associahedron. MY was partially supported by the Simons Collaboration Grant 429920.

2. Small and large Schröder paths

We begin by presenting some preliminary definitions of the various kinds of lattice paths that we will consider.

Definition 2.1. A lattice path in the rectangle defined by \((0,0)\) and \((b,a)\) in \(\mathbb{Z}_{\geq 0}^2\) is a sequence of \(a\) north steps \(N = (0,1)\) and \(b\) east steps \(E = (1,0)\).

Let \(\nu\) be a lattice path in the rectangle defined by \((0,0)\) and \((b,a)\). A \(\nu\)-Dyck path is a lattice path from \((0,0)\) to \((b,a)\) which stays weakly above the path \(\nu\). A peak of \(\nu\) is a consecutive \(NE\) pair in \(\nu\), and a high peak is a peak that occurs strictly above the path \(\nu\). A valley of \(\nu\) is a consecutive \(EN\) pair in \(\nu\). The \(\nu\)-diagonal is defined to be the set of squares immediately below the peaks of \(\nu\). Let \(\mu\) denote the path obtained from \(\nu\) by replacing each of its peaks with a diagonal \(D = (1,1)\) step. A (small) \(\nu\)-Schröder path is a Schröder path from \((0,0)\) to \((b,a)\) which stays weakly above the path \(\nu\). A large \(\nu\)-Schröder path is a a Schröder path from \((0,0)\) to \((b,a)\) which stays weakly above the path \(\mu\). Let \(P_\nu\) and \(LP_\nu\) denote the set of small and large \(\nu\)-Schröder paths respectively. Figure 1 provides some examples of both small and large \(\nu\)-Schröder paths. For a small \(\nu\)-Schröder path \(\pi\) we define its area, denoted by area\((\pi)\), to be the area of the region between \(\pi\) and \(\nu\). For example, the small \(\nu\)-Schröder path on the right in Figure 1 has area 1.5.

**Figure 1.** From left to right: a large \(\nu\)-Schröder path with \(\nu = (NEE)^3\), a large \(\nu\)-Schröder path with \(\nu = ENEENNNEE\), and a (small) \((4,5)\)-Schröder path. The region below \(\nu\) is shaded in gray, with the \(\nu\)-diagonal in a darker gray.
Remark 2.2. We point out two special cases; the rational \((a, b)\) case, and the classical case.

First, the line segment from \((0, 0)\) to \((b, a)\) determines a unique lowest lattice path that stays weakly above \(a\), that is, the unique lattice path \(\nu = \nu(a, b)\) with valleys at the lattice points \(\{(k, \frac{ka}{b}) \mid \frac{ka}{b} \neq \frac{(k+1)a}{b}, 1 \leq k \leq b - 1\}\). When \(a\) and \(b\) are coprime with \(a < b\), the set of \(\nu\)-Dyck paths is the set of ‘rational’ \((a, b)\)-Dyck paths defined in by Armstrong, Rhoades and Williams \([2]\). The lattice path on the right in Figure 1 is an example of a rational \((4, 5)\)-Schröder path where \(\nu = \nu(4, 5)\) is determined by the white dotted line segment from \((0, 0)\) to \((5, 4)\). We point out that the lattice path \(\nu\) on the left in Figure 1 is also determined by the line segment from \((0, 0)\) to \((b, a)\), but we do not consider this to be a rational case as \(a = 3\) and \(b = 6\) are not coprime.

Furthermore, in the case \(a = n\) and \(b = n + 1\) for some positive integer \(n\), the path \(\nu(n, n + 1) = (NE)^nE\), and the set of \(\nu\)-Dyck paths is equivalent to the set of ‘classical’ Dyck paths, which are often defined as lattice paths from \((0, 0)\) to \((n, n)\) that do not fall below the line \(y = x\).

Aguiar and Moreira \([1]\, Proposition 3.1\] showed that the set of classical large Schröder paths can be partitioned into two halves where one half consists of paths that do not contain \(D\) steps on the diagonal, and the other half consists of paths that contain at least one \(D\) step on the diagonal. Gessel \([8]\] showed that the same result holds in the more general rational \((a, b)\)-case. We further generalize Gessel’s argument to the setting of \(\nu\)-Schröder paths.

**Theorem 2.3.** Let \(\nu\) be a lattice path. Then \(|\mathcal{LP}_\nu| = 2|\mathcal{P}_\nu|\) if and only if \(\nu\) begins with a north step and ends with an east step.

**Proof.** Suppose \(\nu\) begins with a north step and ends with an east step. Let \(\mathcal{A} = \mathcal{LP}_\nu \setminus \mathcal{P}_\nu\) denote the set of \(\nu\)-Schröder paths with at least one \(D\) step on the \(\nu\)-diagonal. Define a map \(f : \mathcal{P}_\nu \to \mathcal{A}\) as follows: A path \(\mu \in \mathcal{P}_\nu\) can be partitioned as \(N\mu_1E\mu_2\), where \(E\) is the first \(D\) step on the \(\nu\)-diagonal. The existence of such an \(E\) step is guaranteed by the fact that \(\nu\) ends in an \(E\) step, so there is a \(\nu\)-diagonal square in the top row. Let \(f(\mu)\) be the path \(\mu_1D\mu_2\). We claim that \(f\) is a bijection.

To see that \(f(\mu) \in \mathcal{A}\), note that \(f\) shifts the steps in \(\mu_1\) down by one unit, while the steps in \(\mu_2\) remain fixed. Thus the \(D\) step of \(f(\mu)\) which is between \(\mu_1\) and \(\mu_2\) occurs on the \(\nu\)-diagonal, since it replaced the \(E\) step of \(\mu\) which preceeded \(\mu_2\).

A step in \(N\mu_1\) can only intersect a horizontal run in \(\nu\) at the leftmost lattice point of the horizontal run, since otherwise the first \(E\) step of the horizontal run is an \(E\) step of \(\mu\) on the \(\nu\)-diagonal. Therefore, only \(N\) steps and \(D\) steps which intersect only the leftmost lattice points of horizontal runs can occur in \(N\mu_1\), both of which remain weakly above \(\nu\) after shifting down by one unit. Thus \(f(\mu) \in \mathcal{P}_\nu\), and so \(f\) is well-defined.

The inverse map \(f^{-1} : \mathcal{A} \to \mathcal{P}_\nu\) is defined as follows: For \(\pi \in \mathcal{A}\), partition \(\pi\) into \(\pi_1D\pi_2\) where \(D\) is the last \(D\) step on the \(\nu\)-diagonal. Then \(f^{-1}\) is given by \(\pi_1D\pi_2 \mapsto N\pi_1E\pi_2\), with \(f(f^{-1}(\pi)) = \pi\) and \(f^{-1}(f(\mu)) = \mu\). Hence \(f\) is a bijection, and \(|\mathcal{LP}_\nu| = 2|\mathcal{P}_\nu|\).

Conversely, suppose \(\nu\) does not begin with a north step. The map \(f^{-1} : \mathcal{A} \to \mathcal{P}_\nu\) is injective, but for any path \(\rho \in \mathcal{P}_\nu\) that begins with a \(D\) (or \(E\)) step there is no path \(\sigma \in \mathcal{A}\) such that \(f^{-1}(\rho) = \sigma\). Hence \(|\mathcal{A}| < |\mathcal{P}_\nu|\) and so \(2|\mathcal{P}_\nu| \neq |\mathcal{LP}_\nu|\). The case when \(\nu\) does not end with an \(E\) step can be argued similarly. \(\square\)
Recall that a high peak of a lattice path $\nu$ is a peak that occurs strictly above $\nu$. A $\nu$-Dyck path is completely determined by its high peaks. It is also completely determined by its valleys. See Figure 2 for an example.

Figure 2. The possible high peaks and valleys for $\nu = NEENEENEE$ (top row), and a $\nu$-Dyck path determined by a pair of high peaks along with the $\nu$-Dyck path determined by the corresponding pair of valleys.

The proof of the next result is a direct generalization of the arguments in Deutsch [7] and Gessel [8] to the $\nu$-setting.

**Lemma 2.4.** Let $\nu$ be a lattice path that begins with a north step and ends with an east step. The set of $\nu$-Dyck paths with $i$ high peaks is in bijection with the set of $\nu$-Dyck paths with $i + 1$ peaks.

**Proof.** Since $\nu$ is a lattice path that begins with a north step and ends with an east step, then each $\nu$-Dyck path with $i + 1$ peaks is determined by its $i$ valleys, and it suffices to show that there is a bijection between the set of $\nu$-Dyck paths with $i$ high peaks and the set of $\nu$-Dyck paths with $i$ valleys. A bijection is given by mapping a $\nu$-Dyck path with high peaks at the lattice points $(p_1, q_1), \ldots, (p_i, q_i)$ to the $\nu$-Dyck path with valleys at the lattice points $(p_1 + 1, q_1 - 1), \ldots, (p_i + 1, q_i - 1)$, and mapping the unique $\nu$-Dyck path with no valleys to the unique $\nu$-Dyck path with no high peaks (which is $\nu$ itself). This map is well-defined because high peaks are strictly above the path $\nu$. The inverse map sends a $\nu$-Dyck path with $i$ valleys at the lattice points $(p_1, q_1), \ldots, (p_i, q_i)$ to the $\nu$-Dyck path with $i$ high peaks at $(p_1 - 1, q_1 + 1), \ldots, (p_i - 1, q_i + 1)$, so the map is a bijection. \hfill $\square$

**Definition 2.5.** The $i$-th $\nu$-Narayana number $\text{Nar}_\nu(i)$ is the number of $\nu$-Dyck paths with exactly $i$ valleys. The $\nu$-Narayana polynomial is

$$N_\nu(x) = \sum_{i \geq 0} \text{Nar}_\nu(i)x^i.$$  

This generalization of the Narayana numbers was introduced by Ceballos, Padrol and Sarmiento [5] as the $h$-vector of the $\nu$-Tamari complex. The rational $(a, b)$ case also appears
in the work of Armstrong, Rhoades and Williams [2] as the \( h \)-vector of their rational associahedron. Bonin, Shapiro and Simion [4] considered the Narayana polynomial for the dual associahedron.

**Proposition 2.6.** Let \( \text{sch}_\nu(i) \) denote the number of \( \nu \)-Schröder paths with \( i \) diagonal steps. Then

\[
N_\nu(x + 1) = \sum_{i \geq 0} \text{sch}_\nu(i)x^i.
\]

**Proof.** Note that \( |\mathcal{P}_\nu| = |\mathcal{P}_{N\nu E}| \), that is, appending an \( N \) step to the beginning of \( \nu \) and an \( E \) step to the end of \( \nu \) does not change the number of \( \nu \)-Schröder paths. Hence we can assume without loss of generality that \( \nu \) begins with an \( N \) step and ends with an \( E \) step. By Lemma [2.3] \( \text{Nar}_\nu(i) \) is also the number of \( \nu \)-Dyck paths with exactly \( i \) high peaks. The result then follows from the computation

\[
\sum_{j \geq 0} \text{Nar}_\nu(j)(x + 1)^j = \sum_{i \geq 0} \sum_{j \geq 0} \text{Nar}_\nu(j) \binom{j}{i} x^i = \sum_{i \geq 0} \text{sch}_\nu(i)x^i,
\]

where the last equality follows from the observation that for each \( \nu \)-Dyck path with \( j \) high peaks there are exactly \( \binom{j}{i} \) ways to choose which \( i \) of the high peaks to replace with a \( D \) step.

**Corollary 2.7.** The number of \( \nu \)-Schröder paths is given by specializing \( N_\nu(x) \) at \( x = 2 \).

**Proof.** The claim follows by noting that \( |\mathcal{P}_\nu| = \sum_{i \geq 0} \text{sch}_\nu(i) = N_\nu(2) \). An alternative way to see this is to note that \( \text{Nar}_\nu(i) \) is the number \( \nu \)-Dyck paths with \( i \) high peaks. For each of the high peaks, there are two choices; keep the peak or replace it with a \( D \) step. Thus the total number of \( \nu \)-Schröder paths is

\[
|\mathcal{P}_\nu| = \sum_{i \geq 0} \text{Nar}_\nu(i)2^i = N_\nu(2).
\]

**Corollary 2.8.** Let \( \text{sch}_\nu(i) \) denote the number of \( \nu \)-Schröder paths with \( i \) diagonal steps. Then

\[
\sum_{i \geq 0} (-1)^i \text{sch}_\nu(i) = 1.
\]

**Proof.** This follows from the fact that \( \sum_{i \geq 0} (-1)^i \text{sch}_\nu(i) = N_\nu(0) \), and there is a unique \( \nu \)-Dyck path with no valleys.

**Remark 2.9.** Corollary 2.8 can be obtained topologically from the results in Section 4 since \( \sum_{i \geq 0} (-1)^i \text{sch}_\nu(i) \) is the Euler characteristic of the contractible polyhedral complex known as the \( \nu \)-associahedron. See Theorem 4.13.

**Remark 2.10.** Theorem 2.3 can be deduced from Corollary 2.7 since \( \text{Nar}_\nu(i) \) is the number \( \nu \)-Dyck paths with \( i + 1 \) peaks if and only if \( \nu \) begins with a \( N \)-step and ends with an \( E \)-step, in which case

\[
|\mathcal{LP}_\nu| = \sum_{i \geq 0} \#(\nu \text{-Dyck paths with } i + 1 \text{ peaks}) \cdot 2^{i+1} = 2 \sum_{i \geq 0} \text{Nar}_\nu(i)2^i = 2 |\mathcal{P}_\nu|.
\]
Recall that we refer to the special case when $\nu = \nu(a, b)$ is the lattice path with valleys at $\{(k, \lfloor ka/b \rfloor) \mid \lfloor ka/b \rfloor \neq \lfloor (k+1)a/b \rfloor\}$ as the ‘rational’ case. We end this section with some enumerative results for the rational $(a, b)$-Schröder paths, but we first recall some results on the rational $(a, b)$-Dyck paths.

For coprime positive integers $a, b$ the rational $(a, b)$-Catalan number $\text{Cat}(a, b)$ is the number of $(a, b)$-Dyck paths, and the rational $(a, b)$-Narayana number $\text{Nar}(a, b, i)$ is the number of $(a, b)$-Dyck paths with $i$ peaks. Armstrong, Rhoades and Williams [2] showed that

$$\text{Cat}(a, b) = \frac{1}{a+b} \binom{a+b}{a} = \frac{1}{a} \binom{a+b-1}{b} = \frac{1}{b} \binom{a+b-1}{a},$$

and for $i = 0, \ldots, a,$

$$\text{Nar}(a, b, i) = \frac{1}{a} \binom{a}{i} \binom{b-1}{b-i}.$$

We now enumerate $(a, b)$-Schröder paths with respect to the number of diagonal steps.

**Definition 2.11.** For coprime positive integers $a, b$, and $i = 0, \ldots, a$, let $\text{sch}(a, b, i)$ denote the number of (small) $(a, b)$-Schröder paths with $i$ diagonal steps and let $\text{Sch}(a, b, i)$ denote the number of large $(a, b)$-Schröder paths with $i$ diagonal steps.

We can give an explicit formula for the numbers $\text{Sch}(a, b, i)$. The proof of the following result closely mirrors the one given by Song [14, Theorem 2.1], who studied Schröder paths from $(0, 0)$ to $(kn, n)$, which is equivalent to the rational case when $a = n$ and $b = kn+1$.

**Proposition 2.12.** For coprime positive integers $a, b$, and $i = 0, \ldots, a$,

$$\text{Sch}(a, b, i) = \frac{1}{a} \binom{a}{i} \binom{a+b-1-i}{b-i} = \frac{1}{b} \binom{b}{i} \binom{a+b-1-i}{a-i}.$$

**Proof.** The crucial observation is that the set of large $(a, b)$-Schröder paths with $i$ diagonal steps can be generated by taking the set of $(a, b)$-Dyck paths with at least $i$ peaks, and replacing $i$ of the peaks with diagonal steps. Each large $(a, b)$-Schröder path is obtained in a unique way in this construction, thus

$$\text{Sch}(a, b, i) = \sum_{p \geq i} \binom{p}{i} \text{Nar}(a, b, p) = \sum_{p \geq i} \binom{p}{i} \frac{1}{a} \binom{a}{p} \binom{b-1}{b-p} = \frac{1}{a} \binom{a}{i} \sum_{p \geq i} \binom{a-i}{p-i} \binom{b-1}{b-p} = \frac{1}{a} \binom{a}{i} \binom{a+b-1-i}{a-i}.$$

\qed

Following directly from the bijection $f$ constructed in Theorem 2.3, we have the next result which relates $\text{Sch}(a, b, i)$ and $\text{sch}(a, b, i)$.

**Corollary 2.13.** For coprime positive integers $a, b$, and $i = 0, \ldots, a$,

$$\text{Sch}(a, b, i) = \text{sch}(a, b, i) + \text{sch}(a, b, i-1),$$

with the understanding that $\text{sch}(a, b, -1) = 0$.

**Proof.** From this Corollary, we can deduce an explicit formula for the numbers $\text{sch}(a, b, i)$.
Proposition 2.14. For coprime positive integers \( a, b \) and \( i = 0, \ldots, a - 1 \),

\[
\text{sch}(a, b, i) = \frac{1}{a} \binom{b-1}{i} \binom{a+b-1-i}{b} = \frac{1}{b} \binom{a-1}{i} \binom{a+b-1-i}{a}.
\]

Proof. Induct on \( i \). By definition, \( \text{sch}(a, b, 0) = \text{Sch}(a, b, 0) = \text{Cat}(a, b) \), and one can check via a direct computation that \( \text{Sch}(a, b, i) - \text{sch}(a, b, i - 1) = \text{sch}(a, b, i) \). \( \square \)

3. \( \nu \)-Schröder Trees

In this section we introduce \( \nu \)-Schröder trees, which generalize the \( \nu \)-trees of Ceballos, Padrol and Sarmiento [6]. They showed that the rotation lattice of \( \nu \)-trees is an alternative description of the \( \nu \)-Tamari lattice, which is the 1-skeleton of the \( \nu \)-associahedron. Using the structural insight gained from the \( \nu \)-tree perspective, they showed that the \( \nu \)-Tamari lattice is isomorphic to the increasing flip poset of a suitably chosen subword complex, and solve a special case of Rubey’s Lattice Conjecture.

We define poset structures on \( \nu \)-Schröder trees and \( \nu \)-Schröder paths, and show that these posets are isomorphic. In Section 4, we show that these posets are an alternative description for the face poset of the \( \nu \)-associahedron.

Let \( \nu \) be a lattice path from \((0,0)\) to \((b,a)\). Let \( R_\nu \) denote the region of the plane which lies weakly above \( \nu \) inside the rectangle defined by \((0,0)\) and \((b,a)\). In Figure 3, \( R_\nu \) is represented by the unshaded region in the rectangular grid. Two lattice points \( p \) and \( q \) in \( R_\nu \) are \( \nu \)-incompatible if and only if \( p \) is southwest or northeast of \( q \), and the smallest rectangle containing \( p \) and \( q \) is contained in \( R_\nu \). We say that \( p \) and \( q \) are \( \nu \)-compatible if they are not \( \nu \)-incompatible.

**Definition 3.1.** A \( \nu \)-Schröder tree is a set of \( \nu \)-compatible points in \( R_\nu \) which includes the point \((0,b)\), such that each row and each column contains at least one point. The point \((0,b)\) in a \( \nu \)-Schröder tree is the root, and the other points will be called nodes. A maximal collection of pairwise \( \nu \)-compatible lattice points in \( R_\nu \) will be referred to as a \( \nu \)-binary tree. Let \( T_\nu \) denote the set of \( \nu \)-Schröder trees.

Note that \( \nu \)-binary trees are equal to the \( \nu \)-trees of [6]. We use the term \( \nu \)-binary tree to emphasize their binary nature. This way, the \( \nu \)-Schröder trees generalize \( \nu \)-binary trees just as Schröder trees generalize binary trees in the classical sense.

It may seem peculiar that a collection of points is called a ‘tree’, but this is justified as we may associate a non-crossing plane tree embedded in \( R_\nu \) to each \( \nu \)-Schröder tree \( T \) as follows. If a non-root node \( p \) of \( T \) in \( R_\nu \) has a node above it in the same column or a node to the left of it in the same row, we connect them by an edge. Note that the \( \nu \)-compatibility of the nodes guarantees that it does not have both. However, it could have neither, in which case we consider the smallest rectangular box containing \( p \) and exactly one other point \( q \) of \( T \). The point \( q \) must be the northwest corner of such a box. The root guarantees the existence of such a box, and uniqueness follows from the fact that the northwest corners of two such hypothetical boxes would be \( \nu \)-incompatible. We then connect \( p \) and \( q \) by an edge. The resulting tree is guaranteed to be non-crossing, as otherwise the parent nodes of the two crossing edges would be \( \nu \)-incompatible.
Example 3.2. Letting $\nu = \nu(3,5)$, Figure 3 provides two examples of $\nu$-Schröder trees. The region $R_\nu$ is the unshaded region weakly above $\nu$. The root is the node at (0,3). Note that although the node (2,3) is northeast of the node at (0,1) in the left tree, they are $\nu$-compatible since the rectangle determined by them is not contained in $R_\nu$. No more nodes can be added to the left tree in Figure 3 without introducing a pair of $\nu$-incompatible nodes, hence it is a $\nu$-binary tree.

![Figure 3. A $\nu$-binary tree (left) and a $\nu$-Schröder tree (right), where $\nu = \nu(3,5)$.](image)

Definition 3.3. Let $p$, $q$ and $r$ be nodes in a $\nu$-Schröder tree $S$ such that either $p$ is the first node above $q$ and $r$ is the first node to the right of $q$, or $p$ is the first node to the left of $q$ and $r$ is the first node below $q$. We define a contraction of $S$ at node $q$ as the $\nu$-Schröder tree resulting from removing the node $q$ from $S$. When $p$ is above $q$ we call it a right contraction, when $q$ is above $r$ we call it a left contraction. Define a rotation at $q$ by removing the point $q$ and placing it in the other corner of the box determined by $p$ and $r$. If $p$ is above $q$, we call it a right rotation, and if $q$ is above $r$, we call it a left rotation. There is a third contraction possible, namely when $r$ is southeast of a non-leaf node $p$, with neither corner of the box determined by $p$ and $r$ containing a node of $S$. If removing the node $p$ yields a $\nu$-Schröder tree, the removal of $p$ will be called a diagonal contraction.

Figures 4 and 5 give diagrammatic illustrations of these definitions.

![Figure 4. A right and left contraction as intermediate steps in a right and left rotation, respectively.](image)

![Figure 5. A diagonal contraction at the node (1,2).](image)
Remark 3.4. The term contraction comes from noticing that removing the node \( q \) is equivalent to contracting the edge between \( q \) and its neighbor closest to the root. The tree on the right in Figure 3 is formed from the tree on the left by contracting at the points \((0, 1)\) and \((2, 2)\). Performing a contraction on a \( \nu \)-binary tree \( T \) can be thought of as an intermediate step in a left or right rotation of \( \nu \)-binary trees as defined in [6]. See Figure 4.

Proposition 3.5. The set of \( \nu \)-Schröder trees is the set of trees obtained from contracting \( \nu \)-binary trees.

Proof. Since contraction always leaves at least one node in every row and column, performing a sequence of contractions on a \( \nu \)-binary tree results in a \( \nu \)-Schröder tree. Conversely, given a \( \nu \)-Schröder tree \( T \), it is contained in a maximal set of \( \nu \)-compatible nodes, that is, a \( \nu \)-binary tree \( T' \). Contracting \( T' \) at the nodes not appearing in \( T \) in any order yields \( T \). □

Remark 3.6. Since the set of \( \nu \)-binary trees determine a set of binary trees with labels left and right [6, Lemma 2.4], we can define the set of \( \nu \)-Schröder trees as the set of labeled trees resulting from contracting internal edges in the corresponding set of binary trees. When contracting at a node \( p \) labeled left or right, assign the label middle to all children with a different label than \( p \). If \( p \) has label middle, assign the label middle to all of its children. Relabel the left and right children \( E \) and \( N \) respectively. In a contraction at \( q \), each child of \( q \) receives the label \( D \).

Next, we show that the leaves of a \( \nu \)-Schröder tree determine the path \( \nu \), and vice versa. As a result, the path \( \nu \) can be read from any \( \nu \)-Schröder tree.

Lemma 3.7 ([6, Lemma 2.2]). A non-root node in a \( \nu \)-binary tree has a node above it in the same column or to its left in the same row.

Proposition 3.8. A node in a \( \nu \)-Schröder tree is a leaf if and only if it is the starting point of a vertical run or an end point of a horizontal run in \( \nu \).

Proof. Note that contraction does not change the number of leaves, thus by Lemma 3.7 it suffices to only consider \( \nu \)-binary trees. If a node in a \( \nu \)-binary tree \( T \) occurs at the starting point of a vertical run of \( \nu \) or at the end point of a horizontal run, then it must be a leaf as it cannot have any nodes to its south or east.

Conversely let \( p \) be a leaf in \( T \). Suppose toward a contradiction that \( p \) is not the starting point of a vertical run or the end point of a horizontal run in \( \nu \). Thus there is a lattice point \( s \in R_\nu \) either below or to the right of \( p \). Assume without loss of generality that \( s \) is to the right of \( p \). Then since \( T \) is maximal, \( s \) is not a node of \( T \), and so it is \( \nu \)-incompatible with some \( q \in T \). Since \( q \) is \( \nu \)-compatible with \( p \), it must be south or southeast of \( p \). If it is south of \( p \), then \( s \) and \( p \) are connected by an edge, and \( p \) is not a leaf. If \( s \) is southeast of \( p \), then by Lemma 3.7 there is a node \( t \in T \) that is either west of \( s \) or north of \( s \). By \( \nu \)-compatibility with \( p \), the node \( t \) cannot be southwest or northeast of \( p \), thus \( t \) is on the boundary of the rectangular box determined by \( p \) and \( s \). The nodes \( s \) and \( t \) are connected by an edge in \( T \). Iterating the argument using \( p \) and the point inside the box generates a path from \( s \) to \( p \), hence \( p \) is not a leaf. It follows that \( p \) is the starting point of a vertical run or the end point of a horizontal run in \( \nu \). □

Since a \( \nu \)-Dyck path is determined by its horizontal and vertical runs, we have the following corollary.
Corollary 3.9. The path $\nu$ is determined by a $\nu$-Schröder tree. \hfill $\square$

Remark 3.10. When $\nu = (NE)^n$ we recover the classical Schröder trees, that is, trees with $n + 1$ leaves where each non-leaf node has at least two children.

3.1. The bijection between $\nu$-Schröder trees and $\nu$-Schröder paths. The bijection $\varphi: T_\nu \rightarrow P_\nu$ given here between $\nu$-Schröder trees and $\nu$-Schröder paths is a generalization of the bijection between $\nu$-binary trees and $\nu$-Dyck paths given by Ceballos, Padrol and Sarmiento [6, Theorem 3.3]. Given a $\nu$-Schröder tree $T$, we assign labels $N$, $E$ and $D$ to its non-root nodes as follows: if its parent node is in the same column then label it $N$, if its parent node is in the same row then label it $E$, and if its parent node is in neither then label it $D$.

First define a right-flushing map $\mathcal{R}$, which takes a $\nu$-Schröder path $\mu$ and maps it to a $\nu$-Schröder tree $T = \mathcal{R}(\mu)$ by right-flushing the lattice points of $\mu$ as follows. Begin by labeling the points in $\mu$ in the order they appear on the path, as it is traversed from the origin to $(b, a)$. Starting from the bottom row in $R_\nu$ and proceeding upward, place the points in the same row of $R_\nu$ from right to left as far right as possible, while avoiding $x$-coordinates forbidden by previously right-flushed rows. An $x$-coordinate is forbidden if it corresponds to the initial point of an $E$ or $D$ step in $\mu$. We claim that the lattice points obtained by right-flushing all the lattice points in $\mu$ are the nodes of a $\nu$-Schröder tree $T$. See the top of Figure 5 for an example of the right-flushing map $\mathcal{R}$.

We first check that $\mathcal{R}$ is well-defined. It is not immediately clear that right-flushing is always possible on a row, that is, that there is always an $x$-coordinate available in a row for the placement of a node. To verify that placing a node is always possible, suppose that we are right-flushing a point $p$ in the $\nu$-Schröder path $\mu$. Let $\overline{p}$ denote the node to which $p$ is right-flushed. We need the number of lattice points in the row in $R_\nu$ on which $\overline{p}$ lies to be greater than the number of forbidden $x$-coordinates before $\overline{p}$. The latter is equal to the number of $E$ and $D$ steps before $p$. Let $\text{horiz}_\nu(p)$ denote the maximal number of east steps that can be placed starting at $p$ before crossing $\nu$ (while remaining in the smallest rectangle containing $\nu$). For example, in Figure 5 $\text{horiz}_\nu(4) = 2$ and $\text{horiz}_\nu(9) = 3$. The difference between the number of lattice points in the row with $p$ and the number of $E$ and $D$ steps before $p$ is equal to $\text{horiz}_\nu(p) + 1$, and since this quantity is greater than or equal to one, there is a free column for the placement of $\overline{p}$.

Next, we verify that $T = \mathcal{R}(\mu)$ is in fact a $\nu$-Schröder tree. The construction guarantees the $\nu$-compatibility of the nodes, so it remains to verify the existence of the root, and that every row and column has a node. It is clear that every row has a node, as there is a lattice point of $\mu$ in every row. The total number of forbidden $x$-coordinates is the number of $E$ and $D$ steps in $\mu$, which is $b$, thus when flushing the last point of $\mu$, we must have $b$ forbidden $x$-coordinates, or in other words, nodes in $b$ columns. Note that the first column cannot be forbidden by any previous node, as such a forbidding node would correspond to a $E$ or $D$ step crossing $\nu$. Thus the last node must be placed in $(0, a)$, and so we have a node in each column, and a root.

Now that $\mathcal{R}$ is well-defined, we define its inverse known as the left-flushing map $\mathcal{L}$, which left-flushes the nodes in a $\nu$-Schröder tree $T$ to form a $\nu$-Schröder path $\mu = \mathcal{L}(T)$ as follows. First order the nodes in $T$ from bottom to top and right to left. Starting from the bottom row in $R_\nu$ and proceeding upward, place the nodes from left to right in the same row as
far left as possible, while avoiding x-coordinates forbidden by previously left-flushed rows. The forbidden x-coordinates of a row are the x-coordinates of lattice points corresponding to nodes labeled E or D in T. We claim that the resulting collection of lattice points is a ν-Schröder path μ. Note that by construction μ is the same ν-Schröder path as the one obtained by reading the labels in a post-order traversal of T. See the bottom of Figure 6 for an example of the left-flushing map L.

We verify that L is well-defined. First we check that left-flushing a node p in a ν-tree is always possible, that is, that there is always an available lattice point of Rν in the row of p in which to place p. We need more lattice points of Rν on the row of p than the number of x-coordinates forbidden prior to p. Let hrootν(p) denote the number of nodes labeled E or D in the unique path from p to the root. The difference between the number of lattice points of Rν on the row of p and the number of x-coordinates forbidden prior to p is hrootν(p) + 1. Since this quantity is greater than or equal to one, there is an available x-coordinate in the row of p in which to place p.

It remains to check that μ = L(T) is a ν-Schröder path. It is clear from the construction that μ is a lattice path with N, E and D steps. For any p ∈ T the quantity hrootν(p) is one less than the difference between the number of lattice points of Rν on the row of p and the number of E and D nodes read before p, which is precisely horizν(p). Thus we have hrootν(p) = horizν(p) ≥ 0 for any p, that is, μ lies weakly above ν.

![Figure 6. The right-flushing map R (top) and the left-flushing map L (bottom). The action of L is equivalent to reading the labels of the ν-Schröder tree in post-order traversal starting at the root and going counter-clockwise. The zigzag lines indicate the forbidden x-coordinates.](image)

Finally, we check that the right and left flushing maps R and L are inverses. Any ν-Schröder path μ is uniquely determined by its lattice points. The x-coordinate of a point p in μ is determined by the number of E and D steps before p, which is precisely the number
of forbidden $x$-coordinates before $\overline{p}$ in $R(\mu)$. Therefore the $x$-coordinate of $L(\overline{p})$ is the same as that of $p$, and since $R$ and $L$ do not alter the $y$-coordinates, we have $L(R(p)) = p$. Note that $R$ is injective, as two different $\nu$-Schröder paths have at least one row with a different number of lattice points, and so the corresponding $\nu$-Schröder trees differ on that row.

The next theorem now follows.

**Theorem 3.11.** The map $\varphi : T_\nu \rightarrow P_\nu$ is a bijection between the set of $\nu$-Schröder trees and the set of $\nu$-Schröder paths. \hfill $\square$

### 3.2. The posets of $\nu$-Schröder trees and paths.

The set of $\nu$-Schröder trees satisfy a partial order induced by the covering relation $T < T'$ if and only if $T'$ is a contraction of $T$. We call this the **poset of $\nu$-Schröder trees**.

To define a poset on $\nu$-Schröder paths, we translate contractions of $\nu$-Schröder trees to $\nu$-Schröder paths. The right, left and diagonal contractions are considered separately, as they correspond to different contraction moves on $\nu$-Schröder paths.

Let $T$ be a $\nu$-Schröder tree. First we consider a right contraction of $T$ at a node $\overline{q}$ with parent node $\overline{p}$ above $\overline{q}$ and with a child node $\overline{r}$ to the right of $\overline{q}$. The labels of the nodes $\overline{q}$ and $\overline{r}$ are $N$ and $E$ respectively. Contracting at $\overline{q}$ removes the node $\overline{q}$ and the label on the node $\overline{r}$ becomes $D$. This corresponds to replacing an $E$ step and a $N$ step in $\varphi(T)$ with a $D$ step. In the counterclockwise post-order traversal of $T$, the $E$ and $N$ steps are consecutive, and so correspond to a valley in $\varphi(T)$. Thus a right contraction in $T$ corresponds to replacing a valley in $\varphi(T)$ with a $D$ step.

Next, consider a left contraction in $T$ at a node $\overline{q}$ with parent node $\overline{p}$ to the left of $\overline{q}$ and with a child node $\overline{r}$ below $\overline{q}$. As in the case above, contracting at $\overline{q}$ replaces an $E$ step and $N$ step with a $D$ step at $\overline{r}$. However, this time the $N$ and $E$ steps are not necessarily consecutive in $\varphi(T)$, as $\overline{q}$ may have other children which are read before $\overline{q}$ in the post-order traversal of $T$. The node $\overline{r}$ is the previous node in the post order traversal of $T$ such that $\text{hroot}_\nu(\overline{r}) = \text{hroot}_\nu(\overline{q})$. Recall from Section 3.1 that $\text{hroot}_\nu(\overline{x}) = \text{horiz}_\nu(x)$. Therefore, $r$ is the previous lattice point on $\varphi(T)$ such that $r$ is the initial point of an $N$ step and $\text{horiz}_\nu(r) = \text{horiz}_\nu(q)$. Left contraction deletes this pair of $E$ and $N$ steps, and places a $D$ step at $r$. See Figure 7 for an example.

Lastly, consider a diagonal contraction in $T$ at a node $\overline{q}$ with parent node $\overline{p}$. Note that $\overline{r}$ must have a left child $\overline{s}$ and a right child $\overline{t}$, as otherwise contracting at $\overline{q}$ would not yield a $\nu$-Schröder tree (either the row or column of $\overline{q}$ would not contain a node). The labels of the nodes $\overline{q}$, $\overline{r}$, and $\overline{t}$ are $D$, $N$, and $E$ respectively. Contracting at $\overline{q}$ changes the labels of both $\overline{s}$ and $\overline{t}$ to $D$. In the post-order traversal of the tree, this contraction corresponds to replacing the label $N$ at $\overline{s}$ with $D$, replacing the label $E$ at $\overline{t}$ with $D$, and removing the point $\overline{q}$ labeled $D$. Note that $\overline{s}$ is the first point before $\overline{t}$ in the post-order traversal satisfying $\text{hroot}_\nu(\overline{s}) = \text{hroot}_\nu(\overline{r}) = \text{hroot}_\nu(\overline{t}) - 1$. Therefore, $s$ is the previous point on $\varphi(T)$ such that $\text{horiz}_\nu(s) = \text{horiz}_\nu(r) = \text{horiz}_\nu(t) - 1$. Diagonal contraction thus deletes the step $E$ with end point $r$ and the step $N$ with initial point $s$, and places a $D$ step at $s$. See Figure 8 for an example.

The set of $\nu$-Schröder paths then form a poset with the cover relation inherited from the poset of $\nu$-Schröder trees.
$\phi$

**Figure 7.** A right and left contraction of a pair of (3,5)-Schröder trees, and the corresponding contractions in the associated (3,5)-Schröder paths.

$\phi$

**Figure 8.** A diagonal contraction of a (3,5)-Schröder tree and the corresponding diagonal contraction in the associated (3,5)-Schröder path.

**Definition 3.12.** The (contraction) poset $P_\nu$ of $\nu$-Schröder paths is the set of $\nu$-Schröder paths with cover relation $\mu \prec \lambda$ if and only if $\lambda$ is formed from $\mu$ by a contraction. The contraction moves are the following:

1. **Right Contraction:** Replace a consecutive $EN$ pair with $D$.
2. **Left Contraction:** Delete an $E$ step with initial point $q$, along with the preceding $N$ step with initial point $r$ satisfying $\text{horiz}_\nu(r) = \text{horiz}_\nu(q)$. Shift the subpath between the deleted steps one unit to the right, and place a $D$ step at $r$.
3. **Diagonal Contraction:** Delete an $E$ step ending at a point $r$, which is the initial point of a $D$ step, along with the preceding $N$ step with initial point $s$ satisfying $\text{horiz}_\nu(s) = \text{horiz}_\nu(r)$. Shift the subpath between the deleted steps one unit to the right, and place a $D$ step at $s$. 
See Figure 10 for an example of the poset of $\nu$-Schröder paths for the rational $\nu = \nu(3, 5)$.

By the bijection in Theorem 3.11 and the translation between contractions of $\nu$-Schröder trees and contractions of $\nu$-Schröder paths above, the next theorem now follows.

**Theorem 3.13.** The poset of $\nu$-Schröder trees is isomorphic to the poset of $\nu$-Schröder paths. $\square$

### 4. The face poset of the $\nu$-associahedron

The $\nu$-associahedron $A_\nu$ is a polyhedral complex which generalizes the classical associahedron. It was introduced by Ceballos, Padrol and Sarmiento [5], and they gave a geometric realization of $A_\nu$ via tropical hyperplane arrangements. $A_\nu$ also has a combinatorial definition [5, Theorem 5.2] as a polyhedral complex whose face poset is determined by objects known as covering $(I, \overline{J})$-forests.

In this section, we show that the face poset of the $\nu$-associahedron has alternative descriptions as a poset on $\nu$-Schröder trees and as a poset of $\nu$-Schröder paths by showing that these posets are isomorphic to the poset of covering $(I, \overline{J})$-forests. We begin by recalling the definition of the covering $(I, \overline{J})$-forests of [5], and for our purposes it suffices to restrict the definition slightly to set partitions of $[n]$.

**Definition 4.1.** Let $I \sqcup \overline{J}$ be a partition of $[n]$ such that $1 \in I$ and $n \in \overline{J}$. An $(I, \overline{J})$-forest is a subgraph of the complete bipartite graph $K_{|I|,|\overline{J}|}$ that is

1. **Increasing:** each arc $(i, \overline{j})$ fulfills $i < \overline{j}$; and
2. **Non-crossing:** it does not contain two arcs $(i, \overline{j})$ and $(i', \overline{j}')$ satisfying $i < i' < j < \overline{j}'$.

An $(I, \overline{J})$-tree is a maximal $(I, \overline{J})$-forest. A covering $(I, \overline{J})$-forest is an $(I, \overline{J})$-forest with the arc $(1, n)$ and no isolated nodes.

To a set of covering $(I, \overline{J})$-forests we can associate a unique path $\nu$ as follows. Assign the label $E_{i-1}$ to the $i$-th element in $I$, and assign the label $N_{i-1}$ to the $i$-th element in $\overline{J}$. Reading the labels of the nodes $k = 2, \ldots, n - 1$ in increasing order yields a lattice path $\nu$ from $(0, 0)$ to $(|I| - 1, |\overline{J}| - 1)$. See Figure 9 for an illustration.

**Figure 9.** On the left is a covering $(I, \overline{J})$-forest $F$ for $I = \{1, 3, 5, 6, 8, 9\}$ and $\overline{J} = \{2, 4, 7, 10\}$. The associated path $\nu$ is read from the red labels below the covering $(I, \overline{J})$-forest. On the right is the $\nu$-Schröder tree that corresponds to $F$ under the bijection of Theorem 4.2.
Theorem 4.2. Covering \((I, J)\)-forests are in bijection with \(\nu\)-Schröder trees.

Proof. Given a covering \((I, J)\)-forest \(F\), the arcs of \(F\) can be identified with the labels at their end points, that is, pairs of the form \((E_i, N_j)\). For each such arc, insert a node at the coordinate \((i, j)\) of the grid from \((0,0)\) to \((|I| - 1, |J| - 1)\), and call the resulting configuration of nodes in the grid \(T\). The fact that \(F\) has no isolated nodes guarantees that each row and column of the grid contains a node of \(T\). The increasing condition guarantees that the nodes are in \(R_\nu\), and the non-crossing condition guarantees that the nodes in \(T\) are \(\nu\)-compatible. Thus \(T\) is a \(\nu\)-Schröder tree. This construction is readily invertible. □

Corollary 4.3. Covering \((I, J)\)-forests are in bijection with \(\nu\)-Schröder paths. □

Remark 4.4. The \(\nu\)-Schröder trees are thus grid representations of covering \((I, J)\)-forests, just as \(\nu\)-binary trees are grid representations of \((I, J)\)-trees in [6, Remark 3.7].

Definition 4.5. The poset of covering \((I, J)\)-forests is the set of covering \((I, J)\)-forests equipped with the partial order \(T \leq T'\) if and only if the arcs of \(T'\) are a subset of the arcs of \(T\). Note that \(T\) is covered by \(T'\) if \(T'\) has all but one of the arcs of \(T\).

Recall that for a polyhedral complex \(C\), the face poset of \(C\) is the poset of non-empty faces of \(C\) with partial order \(F_1 \leq F_2\) if and only if \(F_1 \subseteq F_2\). The combinatorial definition of the \(\nu\)-associahedron is then given as follows.

Definition 4.6. Let \(\nu\) be the lattice path associated with the set of the covering \((I, J)\)-forests. The \(\nu\)-associahedron is the polyhedral complex whose face poset is the poset of covering \((I, J)\)-forests.

We can combinatorially describe the \(\nu\)-associahedron in terms of \(\nu\)-Schröder objects.

Theorem 4.7. The following posets are isomorphic:

1. The face poset of the \(\nu\)-associahedron.
2. The poset of \(\nu\)-Schröder trees.
3. The poset of \(\nu\)-Schröder paths.

Proof. Posets 1 and 2 are seen to be isomorphic since the cover relation in the poset of covering \((I, J)\)-forests is equivalent to contracting the corresponding node in the \(\nu\)-Schröder tree. The isomorphism between posets 2 and 3 was shown in Theorem 3.13. □

Corollary 4.8. The number of \(i\)-dimensional faces of the \(\nu\)-associahedron is the number of \(\nu\)-Schröder paths with \(i\) diagonal steps, and therefore, the \(\nu\)-Schröder numbers \(\text{sch}_\nu(i)\) enumerate the faces of \(\nu\)-associahedra. □

A lattice is Eulerian if every nontrivial interval has an equal number of elements in the even ranks versus the odd ranks.

Theorem 4.9. Let \(\hat{P}\) denote the poset \(P\) with an adjoined minimal element \(\hat{0}\) and maximal element \(\hat{1}\). If \(P_\nu\) is the poset of \(\nu\)-Schröder paths or \(\nu\)-Schröder trees, then \(\hat{P}_\nu\) is a lattice. Furthermore, every interval \([x, y]\) in \(\hat{P}_\nu\setminus\{\hat{1}\}\) is an Eulerian lattice.
Proof. Since $A_\nu$ is a polytopal complex, $P \cup \{\hat{0}\}$ is a meet semilattice, with the meet of two faces being their (possibly empty) intersection. Since $z \wedge P \{\hat{1}\} = z$, $\hat{P}$ is a meet semilattice.

Let $x, y \in \hat{P}$. If there exists upper bounds $z, w \in \hat{P}$ of both $x$ and $y$, that is, $z$ and $w$ satisfy $x < P z$, $y < P z$, $x < P w$ and $y < P w$. Then the unique face at the intersection of the faces $w$ and $z$ is the unique join $x \lor P y$. If there is no face containing $x$ and $y$ as subfaces in $A_\nu$, then $x \lor P y = \hat{1}$. Every interval $[\hat{0}, y] \in \hat{P} \setminus \{\hat{1}\}$ corresponds to a convex polytope in $A_\nu$, and hence is Eulerian. Therefore every subinterval $[x, y] \subseteq [\hat{0}, y]$ in $\hat{P} \setminus \{\hat{1}\}$ is an Eulerian lattice. $\square$

Let $\nu$ be a lattice path with $n$ steps. Préville-Ratelle and Viennot [13, Theorem 3] showed that the $\nu$-Tamari lattice is isomorphic to an interval in the classical $(NE)^{n+1}$ Tamari lattice. Extending this isomorphism gives that the $\nu$-associahedron is isomorphic to a connected subcomplex of the boundary complex of the $n$-associahedron. As a result, we have the following corollary.

**Corollary 4.10.** If $\nu = (EN)^{n+1}$, then $P_\nu \cup \{\hat{0}\}$ is isomorphic to the face lattice of the $n$-associahedron. For general $\nu$, $\hat{P}_\nu$ is isomorphic to a sublattice of the face lattice of the $m$-associahedron, where $m$ is the number of steps in $\nu$. $\square$

Since the classical Tamari lattice can be partitioned into disjoint intervals of $\nu$-Tamari lattices [13, Theorem 3], another consequence is that

$$\bigcup_{\nu \text{ path of length } n} \hat{P}_\nu \cong F$$

where $F$ is a sublattice of the face lattice of the $n$-associahedron.

For our last result, we apply discrete Morse theory to the contraction poset of $\nu$-Schröder paths to show that the $\nu$-associahedron $A_\nu$ is contractible. See [11] for background on discrete Morse theory.

**Definition 4.11.** Given a poset $P$, a partial matching in $P$ is a matching in the underlying graph of the Hasse diagram of $P$. That is, a subset $M \subseteq P \times P$, such that

- $(a, b) \in M$ implies $a \prec b$;
- each element $a \in P$ belongs to at most one element of $M$.

When $(a, b) \in M$, we write $a = d(b)$ and $b = u(a)$. A partial matching is acyclic if there does not exist a cycle

$$b_1 \succ d(b_1) \prec b_2 \succ d(b_2) \prec \cdots \prec b_n \succ d(b_n) \prec b_1$$

where $n \geq 2$ and the $b_i \in P$ are distinct. Any elements of $P$ not in an element of $M$ are called critical elements.

The main theorem of discrete Morse theory for complexes is the following.

**Theorem 4.12** ([11, Theorem 11.13]). Let $C$ be a polyhedral complex with face poset $F$. Let $M$ be an acyclic matching on $F$, and let $c_i$ denote the number of critical elements in $F$ corresponding to $i$-dimensional faces of $C$. Then $C$ is homotopy equivalent to a subcomplex of $C$ consisting of $c_i$ faces of dimension $i$. 
Theorem 4.13. The $\nu$-associahedron $A_\nu$ is contractible.

Proof. Let $A_\nu$ be the $\nu$-associahedron with face poset $P_\nu$. By Theorems \[4.7\] and \[4.12\] it suffices to find an acyclic matching on $P_\nu$ with a single critical element corresponding to a vertex in $A_\nu$.

Let $M$ be the set of edges $(\pi, \sigma)$ where $\pi$ is formed from $\sigma$ by replacing with $EN$ the first $D$ step not preceded by any valley. We claim that $M$ is the desired acyclic partial matching. See Figure 10 for an example.

First we check that $M$ is in fact a partial matching. If $(\pi, \sigma) \in M$, then $\sigma$ is formed by a contraction of $\pi$, so $\pi \prec \sigma$ in $P_\nu$. Next we show that a path $\pi$ cannot be in more than one element of $M$. First, there cannot be a pair of elements $(\tau, \pi)$ and $(\pi, \sigma)$ in $M$ because all $D$ steps in $\pi = d(\sigma)$ are preceded by the added valley and so $\tau = d(\pi)$ cannot exist. Second, since $d(\pi)$ is unique by construction, it follows that there cannot be two pairs $(\tau, \pi)$ and $(\tau', \pi)$ in $M$ where $\tau \neq \tau'$. It remains to check that there are no two pairs $(\pi, \sigma)$ and $(\pi, \rho)$ in $M$ with $\sigma \neq \rho$. Suppose the contrary; then $\sigma$ and $\rho$ can be partitioned into $\sigma = \sigma_1D_\sigma\sigma_2$ and $\rho = \rho_1D_\rho\rho_2$, where the $D_\sigma$ and $D_\rho$ steps are the first $D$ steps not preceded by a valley in the respective paths $\sigma$ and $\rho$. If $\sigma_1$ and $\rho_1$ have the same number of steps, then it follows from $\pi = \sigma_1EN\sigma_2 = \rho_1EN\rho_2$ that $\sigma_1 = \rho_1$. However, we cannot have $\sigma_1 = \rho_1$, because then we would also have $\sigma_2 = \rho_2$, from which it would follow that $\sigma = \sigma_1D_\sigma\sigma_2 = \rho_1D_\rho\rho_2 = \rho$. Thus either $\sigma_1$ has fewer steps than $\rho_1$ or vice versa. If $\sigma_1$ has fewer steps, then $\pi$ can be partitioned as $\pi = \pi_1EN\pi_2EN\pi_2$, where $\pi_1EN\pi_2 = \rho_1$. However, this means $\rho = \pi_1EN\pi_2D_\sigma\rho_2$ has a valley before $D_\rho$, which contradicts the fact that $(\pi, \rho)$ is in $M$. Similarly $\rho_1$ cannot have fewer steps. We conclude that $M$ is a partial matching.

Next, we check that $M$ is acyclic. Suppose to the contrary that there exists a cycle

$\pi_1 \succ d(\pi_1) \prec \pi_2 \succ d(\pi_2) \prec \cdots \prec \pi_n \succ d(\pi_n) \prec \pi_1$

with $n \geq 2$. Note that any pair $(d(\pi_i), \pi_i)$ satisfies area$(d(\pi_i)) = \text{area}(\pi_i) - 1/2$. Every pair $d(\pi_i) \prec \pi_j$ in the cycle is related by a contraction of $d(\pi_i)$, and each contraction move either decreases the area of the path, or adds exactly half a unit of area. Since area$(\pi_1)$ at the beginning and the end of the cycle must be equal, each contraction between $d(\pi_i)$ and $\pi_j$ must increase the area by exactly one half, and must therefore be a right contraction. The first valley in $d(\pi_1)$ is the one added to $\pi_1$. Since $\pi_2$ must have a $D$ step not preceded by a valley, it must be a result of a right contraction at the first $EN$ pair in $d(\pi_1)$, which means $\pi_1 = \pi_2$. Therefore $n < 2$, giving the desired contradiction, and so $M$ is acyclic.

Finally, we check that the only critical element in $P_\nu$ is the path $N^aE^b$. Any other path $\pi$ will have either a first $D$ step not preceded by a valley, or not. If it does, then $(d(\pi), \pi) \in M$. If it does not have such a $D$, step, then it must have a first valley. Letting $\sigma$ be the path $\pi$ but with the first valley replaced with a $D$ step gives an element $(\pi, \sigma) \in M$.

Remark 4.14. The acyclic matching $M$ is more difficult to describe in the setting of $\nu$-Schröder trees or of $(I, J)$-trees, thus highlighting a benefit of the $\nu$-Schröder path perspective. The utility of paths is the clear linear order on the steps, making it easy to check if valleys occur before a $D$ step.
Figure 10. The contraction poset of $(3,5)$-Schröder paths, which is the face poset of the $(3,5)$-associahedron of Figure 11. The blue edges denote the acyclic partial matching $M$ described in the proof of Theorem 4.13. The path $N^3E^5$ is the unique critical element in this matching.

Figure 11. The $(3,5)$-associahedron with its faces indexed by $(3,5)$-Schröder paths.

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