Central limit theorems for belief measures

Xiaomin Shi

Abstract. Recently a new type of central limit theorem for belief functions was given in Epstein and Seo \[13\]. In this paper, we generalize the result in Epstein and Seo \[13\] from Bernoulli random variables to general bounded random variables. These results are natural extension of the classical central limit theory for additive probability measures.

Key words. central limit theorem, belief measure, non-additive measure

1 Introduction

It is well known the classical laws of large numbers and central limit theorems as fundamental limit theorems play an important role in the development of probability theory and its applications. The key in the proof of these theorems lies in the additivity of probabilities and the widely use of characteristic functions which have a one to one correspondence to distribution functions. However, such additivity assumption is not reasonable in many areas of applications because many uncertain phenomena can not easily be modeled using additive probabilities. Moreover, economists have found that additive probabilities result in the Allais paradox and the Ellsberg paradox, see Allais \[1\] and Ellsberg \[11\]. So a lot of papers have used non-additive probabilities to describe and interpret the phenomena(see for example \[3, 8, 15, 16, 17, 22, 23, 29, 30, 31, 32\]).

Given a sequence \(\{X_i\}_{i=1}^\infty\) of independent and identically distributed (i.i.d. for short) random variables for non-additive probabilities, there have been a number of papers related to strong laws of large numbers. We refer to \([4, 5, 10, 12, 19, 20, 31]\). But to the best of our knowledge, only a few papers have studied the central limit theorems for non-additive probabilities. Motivated by measuring risk and other financial problems with uncertainty, Peng \[24, 25, 26, 27\] put forward the notion of i.i.d. random variables in the sublinear expectations space, which is a generalization of a probability space. He proved a law of large numbers and a central limit theorem under this sub-linear framework. However, all the functions considered in his papers are continuous functions which are not suitable for us. Recently Epstein and Seo \[13\] gave a central limit theorem for belief functions which is now contained as an independent theorem in Epstein et al. \[14\]. The unilateral CLT was relatively easily characterized by a probability measure, but as they say "A CLT for two-sided intervals is less trivial because minimizing measures are not easily identified". At the same time, they argued that the central limit theory for empirical frequency of the Bernoulli trials can be depicted by a bivariate normal distribution, in the two-sided interval case.

Thanks to a theorem in the seminar paper of Choquet \[6\] and the recent work of Epstein and Seo \[13\], we are able to obtain a central limit theorem for belief measures. Different from Bernoulli random variables

*Qilu Institute of Finance, Shandong University, Jinan, Shandong 250100, PR China. shixm@mail.sdu.edu.cn.
considered in Epstein and Seo \cite{13}, we are mainly interested in general bounded random variables. Our results are extension of the classical central limit theory for additive probability measures.

The paper is organized as follows. In section 2, we give some preliminaries about belief measures. In section 3, we give a unidirectional central limit theorem under belief measures and a central limit theorem for two-sided intervals is given in section 4.

2 Preliminaries

Let $\Omega$ be a Polish space and $\mathcal{B}(\Omega)$ be the Borel $\sigma$-algebra on $\Omega$. We denote by $\mathcal{K}(\Omega)$ the compact subsets of $\Omega$ and $\Delta(\Omega)$ the space of all probability measures on $\Omega$. The set $\mathcal{K}(\Omega)$ will be endowed with the Hausdorff topology generated by the topology of $\Omega$.

Definition 2.1 A belief measure on $(\Omega, \mathcal{B}(\Omega))$ is most commonly defined as a set function $\nu : \mathcal{B}(\Omega) \to [0, 1]$ satisfying:

(i) $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$;

(ii) $\nu(A) \leq \nu(B)$ for all Borel sets $A \subset B$;

(iii) $\nu(B_n \downarrow \nu(B))$ for all sequences of Borel sets $B_n \downarrow B$;

(iv) $\nu(G) = \sup\{\nu(K) : K \subset G, K \text{ is compact}\}$, for all open set $G$;

(v) $\nu$ is totally monotone (or $\infty$-monotone): for all Borel sets $B_1, \ldots, B_n$,

$$\nu(\bigcup_{j=1}^{n} B_j) \geq \sum_{\emptyset \neq J \subseteq \{1, \ldots, n\}} (-1)^{|J|+1} \nu(\bigcap_{j \in J} B_j)$$

where $|J|$ is the number of elements in $J$.

By Phillipe et al.\cite{28}, for all $A \in \mathcal{B}(\Omega)$, $\{K \in \mathcal{K}(\Omega) : K \subset A\}$ is universally measurable. Denote by $\mathcal{B}_u(\mathcal{K}(\Omega))$ the $\sigma$-algebra of all subsets of $\mathcal{K}(\Omega)$ which are universally measurable. The following theorem belongs to Choquet \cite{6}. We also refer to Phillipe et al. \cite{28}.

Theorem 2.2 The set function $\nu : \mathcal{B}(\Omega) \to [0, 1]$ is a belief measure if and only if there exists a probability measure $P_\nu$ on $(\mathcal{K}(\Omega), \mathcal{B}(\mathcal{K}(\Omega)))$ such that

$$\nu(A) = P_\nu(\{K \in \mathcal{K}(\Omega) : K \subset A\}), \quad \forall A \in \mathcal{B}(\Omega).$$

Moreover, there exists a unique extension of $P_\nu$ to $(\mathcal{K}(\Omega), \mathcal{B}_u(\mathcal{K}(\Omega)))$. In the following, we still denote the extension by $P_\nu$.

For any $A \in \mathcal{B}(\Omega^\infty)$, we define

$$\nu^\infty(A) = P_\nu^\infty(\{\tilde{K} = K_1 \times K_2 \times \ldots \in (\mathcal{K}(\Omega))^\infty : \tilde{K} \subset A\}),$$

where $P_\nu^\infty \in \Delta((\mathcal{K}(\Omega))^\infty)$ is the i.i.d. product probability measure. According to Epstein and Seo \cite{13} (Lemma A.2.), $\nu^\infty$ is the unique belief measure on $(\Omega^\infty, \mathcal{B}(\Omega^\infty))$ corresponding to $P_\nu^\infty$. 

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**Definition 2.3** The elements of a sequence \( \{ \xi_i \}_{i=1}^{\infty} \) of random variables on \((\Omega, B(\Omega))\) are called identically distributed w.r.t. \( \nu \), denoted by \( \xi_i \overset{d}{=} \xi_j \), if for each \( i, j \geq 1 \) and for all intervals \( G \) of \( \mathbb{R} \),

\[
\nu(\xi_i \in G) = \nu(\xi_j \in G).
\]

In this paper, we are mainly interested in bounded random variables. The following assumption will be in force throughout this paper.

**Assumption 2.4** There exists a constant \( M > 0 \) such that, for any random variable \( X \) considered in this paper, we have \( |X| \leq M \).

### 3 Unidirectional central limit theorem

Our main result in this section is the following central limit theorem.

**Theorem 3.1** Suppose that Assumption 2.4 holds. Let \( Y_i \) be a random variable on \((\Omega, B(\Omega))\), \( Y_i \overset{d}{=} Y_1, i \geq 1 \). We denote \( X_i(\omega_1, \omega_2, ...) := Y_i(\omega_i), i \geq 1 \). Then we have

\[
\lim_{n \to \infty} \nu^\infty \left( \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma}} \geq \alpha \right) = 1 - N(\alpha),
\]

and

\[
\lim_{n \to \infty} \nu^\infty \left( \frac{\sum_{i=1}^{n} X_i - n\bar{\mu}}{\sqrt{n\bar{\sigma}}} < \alpha \right) = N(\alpha), \quad \forall \alpha \in \mathbb{R},
\]

where \( \mu = \mathbb{E}_{\nu^\infty}(X_i), \bar{\mu} = \mathbb{E}_{V^\infty}(X_i), \)

\[
\bar{\sigma} = \sqrt{\int_{0}^{\infty} 2t\nu^\infty(X_i \geq t)dt + \int_{-\infty}^{0} 2|\nu^\infty(X_i \geq t) - 1|dt - (\mu)^2},
\]

\[
\bar{\sigma} = \sqrt{\int_{0}^{\infty} 2tV^\infty(X_i \geq t)dt + \int_{-\infty}^{0} 2|V^\infty(X_i \geq t) - 1|dt - (\bar{\mu})^2}, \quad i \geq 1.
\]

**Proof.** Step 1: Let

\[
Z_i(K_1 \times K_2 \times ...) = \inf_{\omega_i \in K_i} X_i(\omega_1, \omega_2, ...),
\]

\[
\bar{Z}_i(K_1 \times K_2 \times ...) = \sup_{\omega_i \in K_i} X_i(\omega_1, \omega_2, ...), K_i \in \mathcal{K}(\Omega), \quad i \geq 1.
\]

We claim that \( \{Z_i\}_{i=1}^{\infty} \) and \( \{\bar{Z}_i\}_{i=1}^{\infty} \) are two sequences of i.i.d. random variables defined on \( (\mathcal{K}(\Omega))^{\infty}, B_{\infty}((\mathcal{K}(\Omega))^{\infty}) \) w.r.t. \( P_{\nu^\infty} \). Moreover,

\[
E_{\nu^\infty}[Z_i] = \mu, \quad V_{\nu^\infty}[Z_i] = \sigma^2, \quad i \geq 1,
\]

\[
E_{\nu^\infty}[\bar{Z}_i] = \bar{\mu}, \quad V_{\nu^\infty}[\bar{Z}_i] = \bar{\sigma}^2, \quad i \geq 1.
\]

We only prove the assertions for \( \{Z_i\}_{i=1}^{\infty} \). The proofs for \( \{\bar{Z}_i\}_{i=1}^{\infty} \) are parallel.
For each $i, j \geq 1, \forall t, t_i, t_j \in \mathbb{R}$,

\[
P^\infty_\nu(Z_i \geq t) = P^\infty_\nu(\{\tilde{K} = K_1 \times K_2 \times \ldots \in (\mathcal{K}(\Omega))^{\infty} : Z_i(\tilde{K}) \geq t\})
\]

\[
= P^\infty_\nu(\{\tilde{K} = K_1 \times K_2 \times \ldots \in (\mathcal{K}(\Omega))^{\infty} : \inf_{\omega_j \in K_i} X_i(\omega_1, \omega_2, \ldots) \geq t\})
\]

\[
= P^\infty_\nu(\{\tilde{K} = K_1 \times K_2 \times \ldots \in (\mathcal{K}(\Omega))^{\infty} : \tilde{K} \subset \{(\omega_1, \omega_2, \ldots) : X_i(\omega_1, \omega_2, \ldots) \geq t\}\})
\]

\[
= \nu^\infty(X_i \geq t)
\]

\[
= \nu^\infty(X_j \geq t)
\]

\[
= P^\infty_\nu(Z_j \geq t)
\]

and

\[
P^\infty_\nu(Z_i \geq t_i, Z_j \geq t_j)
\]

\[
= P^\infty_\nu(\{\tilde{K} = K_1 \times K_2 \times \ldots \in (\mathcal{K}(\Omega))^{\infty} : Z_i(\tilde{K}) \geq t_i, Z_j(\tilde{K}) \geq t_j\})
\]

\[
= P^\infty_\nu(\{\tilde{K} = K_1 \times K_2 \times \ldots \in (\mathcal{K}(\Omega))^{\infty} : \inf_{\omega_j \in K_i} X_i(\omega_1, \omega_2, \ldots) \geq t_i, \inf_{\omega_j \in K_j} Y_j(\omega_1, \omega_2, \ldots) \geq t_j\})
\]

\[
= P^\infty_\nu(\{\tilde{K} = K_1 \times K_2 \times \ldots \in (\mathcal{K}(\Omega))^{\infty} : \inf_{\omega_j \in K_i} X_i(\omega_1, \omega_2, \ldots) \geq t_i\}) \times
\]

\[
P^\infty_\nu(\{\tilde{K} = K_1 \times K_2 \times \ldots \in (\mathcal{K}(\Omega))^{\infty} : \inf_{\omega_j \in K_j} Y_j(\omega_1, \omega_2, \ldots) \geq t_j\})
\]

\[
= P^\infty_\nu(Z_i \geq t_i) P^\infty_\nu(Z_j \geq t_j).
\]

Thus, $\{Z_i\}_{i=1}^\infty$ is a sequence of pairwise independent and identically distributed random variables w.r.t. $P^\infty_\nu$.

Moreover,

\[
E^{P^\infty_\nu}(Z_i) = \int_0^M P^\infty_\nu(Z_i \geq t)dt + \int_{-M}^0 [P^\infty_\nu(Z_i \geq t) - 1]dt
\]

\[
= \int_0^M \nu^\infty(X_i \geq t)dt + \int_{-M}^0 [\nu^\infty(X_i \geq t) - 1]dt
\]

\[
= \mu^\infty_i, i \geq 1.
\]

By Assumption 2.4 we have

\[
Var^{P^\infty_\nu}(Z_i) = E^{P^\infty_\nu}(Z_i^2) - (E^{P^\infty_\nu}(Z_i))^2
\]

\[
= \int_0^M P^\infty_\nu(Z_i^2 \geq t)dt - \mu^2
\]

\[
= \int_0^M P^\infty_\nu(Z_i \geq \sqrt{t})dt + \int_0^M P^\infty_\nu(Z_i \leq -\sqrt{t})dt - \mu^2
\]

\[
= \int_0^M \nu^\infty(X_i \geq \sqrt{t})dt + \int_0^M [1 - \nu^\infty(X_i \geq -\sqrt{t})]dt - \mu^2
\]

\[
= \int_0^M 2t \nu^\infty(X_i \geq t)dt + \int_{-M}^0 2t[\nu^\infty(X_i \geq t) - 1]dt - \mu^2
\]

\[
= t^2, \ i \geq 1.
\]
Step 2: \( \nu^\infty \left( \sum_{i=1}^{n} X_i \geq \alpha \right) = P^\nu \left( \sum_{i=1}^{n} Z_i \geq \alpha \right), \quad \forall \alpha \in \mathbb{R} \).

Actually, by Theorem 2.2, we have
\[
\nu^\infty \left( \sum_{i=1}^{n} X_i \geq \alpha \right) = P^\nu \left( \{ \tilde{K} = K_1 \times K_2 \times \cdots \in (\mathcal{K}(\Omega))^\infty : \tilde{K} \in \{ (\omega_1, \omega_2, \ldots) : \sum_{i=1}^{n} Y_i(\omega_i) \geq \alpha \} \} \right)
\]
\[
= P^\nu \left( \{ \tilde{K} = K_1 \times K_2 \times \cdots \in (\mathcal{K}(\Omega))^\infty : \inf_{(\omega_1, \omega_2, \ldots) \in K_1 \times K_2 \times \cdots} \sum_{i=1}^{n} Y_i(\omega_i) \geq \alpha \} \right)
\]
\[
= P^\nu \left( \{ \tilde{K} = K_1 \times K_2 \times \cdots \in (\mathcal{K}(\Omega))^\infty : \sum_{i=1}^{n} Z_i(1, K_2 \times \cdots) \geq \alpha \} \right)
\]
\[
= P^\nu \left( \sum_{i=1}^{n} Z_i \geq \alpha \right), \quad \forall \alpha \in \mathbb{R}.
\]

Step 3: From step 2, we have
\[
\nu^\infty \left( \sum_{i=1}^{n} X_i \geq \alpha \right) = P^\nu \left( \sum_{i=1}^{n} Z_i \geq \alpha \right), \quad \forall \alpha \in \mathbb{R}.
\] (3.5)

Applying the classical central limit theorem to \( \{ Z_i \}_{i=1}^{\infty} \) deliver the corresponding limit theorem (3.1).

As for the proof of limit theorem (3.2), we just reverse the inequality sign and replace \( \{ Z_i \}_{i=1}^{\infty} \) by \( \{ \bar{Z}_i \}_{i=1}^{\infty} \) in the preceding proof. \( \square \)

4 Central limit theorem for two-sided intervals

Let \( N_2(\cdot, \cdot; \rho) \) be the cumulate distribution function for the bivariate normal with zero means, unit variances and correlation coefficient \( \rho \), i.e.,
\[
N_2(\alpha_1, \alpha_2; \rho) = \Pr(\tilde{Z}_1 \leq \alpha_1, \tilde{Z}_2 \leq \alpha_2),
\]
where \( (\tilde{Z}_1, \tilde{Z}_2) \) is bivariate normal with the indicated moments.

In this section we will prove the following CLT for two-sided intervals using the strategy similar to Epstein and Seo [13].

**Theorem 4.1** Suppose the conditions of Theorem 3.1 and Assumptions 2.4 hold. Then there is a constant \( K \) which does not dependent on \( \alpha_1, \alpha_2 \) or \( n \), such that,
\[
\left| \nu^\infty (\alpha_1 \sqrt{n} \sigma + n \mu - \sum_{i=1}^{n} X_i \leq \alpha_2 \sqrt{n} \sigma + n \bar{\mu}) - N_2(-\alpha_1, \alpha_2; -\rho) \right| \leq \frac{K}{\sqrt{n}}.
\]
where, \( \rho = \frac{M^2 - M\mu + M\nu - \mu \bar{\nu}}{2\sigma} \), \( \rho' = \int_{-M}^{M} \int_{-M}^{t_2} \nu(t_1 \leq Y_i \leq t_2) dt_1 dt_2 \). Moreover, the similar result holds if \( \alpha_1 \) and \( \alpha_2 \) depend on \( n \).

**Proof:** Let \( \{Z_i\}^\infty_{i=1} \) and \( \{\tilde{Z}_i\}^\infty_{i=1} \) be the same as in Theorem 3.1. For each \( i \geq 1 \), we first compute the correlation of \( Z_i \) and \( \tilde{Z}_i \).

\[
E_{\tilde{P}_\nu}^{(\infty)}(Z_i, \tilde{Z}_i)
\]

\[
= \int_{-M}^{M} \int_{-M}^{M} xydP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y)
\]

\[
= \int_{-M}^{0} \int_{-M}^{0} xydP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y) + \int_{-M}^{0} \int_{0}^{M} xydP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y)
\]

\[
+ \int_{0}^{M} \int_{-M}^{0} xydP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y) + \int_{0}^{M} \int_{0}^{M} xydP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y)
\]

\[
= \int_{-M}^{0} \int_{-M}^{0} \int_{x}^{0} \int_{y}^{0} dt_1 dt_2 dP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y) - \int_{-M}^{0} \int_{0}^{M} \int_{x}^{0} \int_{y}^{0} dt_1 dt_2 dP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y)
\]

\[
- \int_{0}^{M} \int_{-M}^{0} \int_{x}^{0} \int_{y}^{0} dt_1 dt_2 dP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y) + \int_{0}^{M} \int_{0}^{M} \int_{x}^{0} \int_{y}^{0} dt_1 dt_2 dP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y)
\]

\[
= \int_{-M}^{0} \int_{-M}^{0} \int_{x}^{0} \int_{y}^{0} dP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y) dt_1 dt_2 - \int_{-M}^{0} \int_{0}^{M} \int_{x}^{0} \int_{y}^{0} dP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y) dt_1 dt_2
\]

\[
- \int_{0}^{M} \int_{-M}^{0} \int_{x}^{0} \int_{y}^{0} dP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y) dt_1 dt_2 + \int_{0}^{M} \int_{0}^{M} \int_{x}^{0} \int_{y}^{0} dP_\nu^\infty(Z_i \leq x, \tilde{Z}_i \leq y) dt_1 dt_2
\]

\[
= \int_{-M}^{0} \int_{-M}^{0} P_\nu^\infty(Z_i \leq t_1, \tilde{Z}_i \leq t_2) dt_1 dt_2 - \int_{-M}^{0} \int_{0}^{M} [P_\nu^\infty(Z_i \leq t_1) - P_\nu^\infty(Z_i \leq t_1, \tilde{Z}_i \leq t_2)] dt_1 dt_2
\]

\[
- \int_{0}^{M} \int_{-M}^{0} [P_\nu^\infty(\tilde{Z}_i \leq t_2) - P_\nu^\infty(Z_i \leq t_1, \tilde{Z}_i \leq t_2)] dt_1 dt_2
\]

\[
+ \int_{0}^{M} \int_{0}^{M} [1 - P_\nu^\infty(Z_i \leq t_1) - P_\nu^\infty(\tilde{Z}_i \leq t_2) + P_\nu^\infty(Z_i \leq t_1, \tilde{Z}_i \leq t_2)] dt_1 dt_2
\]

\[
= \int_{-M}^{0} \int_{-M}^{0} P_\nu^\infty(Z_i \leq t_1, \tilde{Z}_i \leq t_2) dt_1 dt_2 - \int_{-M}^{0} \int_{0}^{M} P_\nu^\infty(\tilde{Z}_i \leq t_2) dt_1 dt_2 + \int_{0}^{M} \int_{-M}^{0} P_\nu^\infty(Z_i \leq t_2) dt_1 dt_2
\]

\[
- \int_{-M}^{0} \int_{-M}^{0} P_\nu^\infty(Z_i \leq t_1) dt_1 dt_2 - \int_{0}^{M} \int_{0}^{M} P_\nu^\infty(\tilde{Z}_i \leq t_2) dt_1 dt_2
\]

\[
+ \int_{0}^{M} \int_{0}^{M} [1 - P_\nu^\infty(Z_i \leq t_1) - P_\nu^\infty(\tilde{Z}_i \leq t_2)] dt_1 dt_2
\]

\[
= - \int_{-M}^{0} \int_{-M}^{0} P_\nu^\infty(t_1 \leq Z_i \leq \tilde{Z}_i \leq t_2) dt_1 dt_2 + M \int_{-M}^{M} P_\nu^\infty(\tilde{Z}_i \leq t_2) dt_2 - M \int_{-M}^{M} P_\nu^\infty(\tilde{Z}_i \leq t_1) dt_1 + M^2
\]

\[
= - \int_{-M}^{0} \int_{-M}^{0} \nu(\tilde{Z}_i \leq t_1 \leq t_2) dt_1 dt_2 + M \int_{-M}^{M} P_\nu^\infty(Z_i \leq t_2) dt_2 - M \int_{-M}^{M} P_\nu^\infty(Z_i \leq t_1) dt_1 + M^2
\]

\[
= - \int_{-M}^{0} \int_{-M}^{0} \nu(t_1 \leq Y_i \leq t_2) dt_1 dt_2 + M(M - \bar{\mu} - M(M - \bar{\mu}) + M^2
\]

\[
= M^2 - M\bar{\mu} - \int_{-M}^{M} \int_{-M}^{t_2} \nu(t_1 \leq Y_i \leq t_2) dt_1 dt_2
\]

\[
= M^2 - M\bar{\mu} + M\bar{\mu} - \rho'.
\]
It yields that
\[ \text{cov}^{P_{\nu}}(Z_i, \tilde{Z}_i) = E^{P_{\nu}}(Z_i \tilde{Z}_i) - E^{P_{\nu}}(Z_i)E^{P_{\nu}}(\tilde{Z}_i) = M^2 - M\bar{\mu} + \bar{M}\rho' - \bar{\mu} \]
and
\[
E^{P_{\nu}} \left( \frac{Z_i - \mu}{\sigma} \right) = 0, \quad \text{Var}^{P_{\nu}} \left( \frac{Z_i - \mu}{\sigma} \right) = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},
\]
where \( \rho = \text{corr}^{P_{\nu}}(Z_i, \tilde{Z}_i) = \frac{\text{cov}^{P_{\nu}}(Z_i, \tilde{Z}_i)}{\sqrt{\text{Var}^{P_{\nu}}(Z_i)} \sqrt{\text{Var}^{P_{\nu}}(\tilde{Z}_i)}} = \frac{M^2 - M\bar{\mu} + \bar{M}\rho' - \bar{\mu}}{\sigma}. \)

By Theorem 2.2,
\[
\nu^\infty(\alpha_1 \sqrt{n\sigma} + n\mu) \leq \sum_{i=1}^{n} X_i \leq \alpha_2 \sqrt{n\sigma} + n\mu
\]
\[= P_{\nu} \{ \{ \tilde{K} = K_1 \times K_2 \times \ldots \in (K(\Omega))^\infty : \tilde{K} \subset \{ (\omega_1, \omega_2, \ldots) : \alpha_1 \sqrt{n\sigma} + n\mu \leq \sum_{i=1}^{n} X_i \leq \alpha_2 \sqrt{n\sigma} + n\mu \} \} \}
\]
\[= P_{\nu} \{ \{ \tilde{K} = K_1 \times K_2 \times \ldots \in (K(\Omega))^\infty : \tilde{K} \subset \{ (\omega_1, \omega_2, \ldots) : \alpha_1 \sqrt{n\sigma} + n\mu \leq \sum_{i=1}^{n} Y_i(\omega_i) \leq \alpha_2 \sqrt{n\sigma} + n\mu \} \} \}
\]
\[= P_{\nu} \{ \{ \tilde{K} = K_1 \times K_2 \times \ldots \in (K(\Omega))^\infty : \alpha_1 \sqrt{n\sigma} + n\mu \leq \inf_{(\omega_1, \omega_2, \ldots) \in K_1 \times K_2 \times \ldots} \sum_{i=1}^{n} Y_i(\omega_i) \} \}
\]
\[\leq \sup_{(\omega_1, \omega_2, \ldots) \in K_1 \times K_2 \times \ldots} \sum_{i=1}^{n} Y_i(\omega_i) \leq \alpha_2 \sqrt{n\sigma} + n\mu \}
\]
\[= P_{\nu} \{ \{ \tilde{K} = K_1 \times K_2 \times \ldots \in (K(\Omega))^\infty : \alpha_1 \sqrt{n\sigma} + n\mu \leq \sum_{i=1}^{n} \inf_{\omega_i \in K_i} Y_i(\omega_i) \leq \sum_{i=1}^{n} \sup_{\omega_i \in K_i} Y_i(\omega_i) \leq \alpha_2 \sqrt{n\sigma} + n\mu \} \}
\]
\[= P_{\nu} \{ \{ \tilde{K} = K_1 \times K_2 \times \ldots \in (K(\Omega))^\infty : \alpha_1 \sqrt{n\sigma} + n\mu \leq \sum_{i=1}^{n} Z_i(\tilde{K}) \leq \sum_{i=1}^{n} \tilde{Z}_i(\tilde{K}) \leq \alpha_2 \sqrt{n\sigma} + n\mu \} \}
\]
\[= P_{\nu}(\alpha_1 \sqrt{n\sigma} + n\mu \leq \sum_{i=1}^{n} Z_i(\tilde{K}) \leq \sum_{i=1}^{n} \tilde{Z}_i(\tilde{K}) \leq \alpha_2 \sqrt{n\sigma} + n\mu)
\]
\[= P_{\nu}(\alpha_1 \leq \frac{\sum_{i=1}^{n} Z_i - n\mu}{\sqrt{n\sigma}}, \frac{\sum_{i=1}^{n} \tilde{Z}_i - n\bar{\mu}}{\sqrt{n\sigma}} \leq \alpha_2).
\]

Thus, the classical central limit theorem is applicable.

Let \( T = \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix} \). Then
\[
E^{P_{\nu}} \left( T \left( \frac{Z_i - \mu}{\sigma} \right) \right) = 0, \quad \text{Var}^{P_{\nu}} \left( T \left( \frac{Z_i - \mu}{\sigma} \right) \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

and
\[ \alpha_1 \leq \frac{\sum_{i=1}^{n} Z_i - n\mu}{\sqrt{n\sigma}}, \quad \frac{\sum_{i=1}^{n} \tilde{Z}_i - n\tilde{\mu}}{\sqrt{n\tilde{\sigma}}} \leq \alpha_2 \Leftrightarrow T \left( \frac{\sum_{i=1}^{n} Z_i - n\mu}{\sqrt{n\sigma}} \right) \in C \] (4.2)

for some convex \( C \subset \mathbb{R}^2 \).

According to the multidimensional Berry-Esseen Theorem, we get
\[ \left| P_{\nu} \left( T \left( \frac{\sum_{i=1}^{n} Z_i - n\mu}{\sqrt{n\sigma}} \right) \in C \right) - \text{Pr} \left( \left( \frac{\tilde{Z}}{\tilde{\sigma}} \right) \in C \right) \right| \leq \frac{K}{\sqrt{n}} \] (4.3)

for some constant \( K \), where \( \left( \frac{\tilde{Z}}{\tilde{\sigma}} \right) \) is the standard bivariate normal.

Define
\[ \left( \frac{\tilde{Z}}{\tilde{\sigma}} \right) = T^{-1} \left( \frac{\tilde{Z}}{\tilde{\sigma}} \right). \]

By (4.2), we have
\[ \text{Pr} \left( \left( \frac{\tilde{Z}}{\tilde{\sigma}} \right) \in C \right) = \text{Pr}(\alpha_1 \leq \tilde{Z}, \tilde{Z}' \leq \alpha_2) = \text{Pr}(\tilde{Z} \leq -\alpha_1, \tilde{Z}' \leq \alpha_2) = N_2(-\alpha_1, \alpha_2; -\rho). \] (4.4)

Then Theorem 4.1 follows from (4.1), (4.2), (4.3), (4.4). This completes the proof. \( \square \)

**Remark 4.2** We claim that \( \rho \) does not depend on the value of \( M \) (the upper bound of \( \{X_i\}_{i=1}^{\infty} \)). For simplicity, we suppose \( 0 \leq X_i \leq M, \ i \geq 1 \). In this case, \( E_{\nu}^{\inf}(Z_i, \tilde{Z}_i) = M\mu - \rho' \), where \( \rho' = \int_0^M \int_0^t \nu(t_1 \leq \tilde{Z} \leq t_2) \, dt_1 \, dt_2 \). The result follows by noticing that
\[
(M + 1)\mu - \int_0^M \int_0^{t_2} \nu(t_1 \leq Y_1 \leq t_2) \, dt_1 \, dt_2
\]
\[= (M + 1)\mu - \int_0^M \int_0^{t_2} \nu(t_1 \leq Y_1 \leq t_2) \, dt_1 \, dt_2 - \int_M^{M+1} \int_0^{t_2} \nu(t_1 \leq Y_1 \leq t_2) \, dt_1 \, dt_2
\]
\[= (M + 1)\mu - \int_0^M \int_0^{t_2} \nu(t_1 \leq Y_1 \leq t_2) \, dt_1 \, dt_2 - \int_M^{M+1} \int_0^{t_2} \nu(t_1 \leq Y_1) \, dt_1 \, dt_2
\]
\[= (M + 1)\mu - \int_0^M \int_0^{t_2} \nu(t_1 \leq Y_1 \leq t_2) \, dt_1 \, dt_2 - \mu
\]
\[= M\mu - \int_0^M \int_0^{t_2} \nu(t_1 \leq Y_1 \leq t_2) \, dt_1 \, dt_2.
\]

**Remark 4.3** When \( \nu \) is additive, Theorem 4.1 degenerates into the classical central limit theorem,
\[ \lim_{n \to \infty} \nu^\infty(\alpha_1 < \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma}} \leq \alpha_2) = \text{Pr}(\alpha_1 < \tilde{Z} \leq \alpha_2), \]

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where \( \tilde{Z} \) is the standard normal. Note that in this case, \( \mu = \tilde{\mu}, \sigma = \tilde{\sigma} \) and \( \rho = 1 \).

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