Locally Rotationally Symmetric Vacuum Solutions in f(R) Gravity

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Abstract

This paper is devoted to find the Locally Rotationally Symmetric (LRS) vacuum solutions in the context of $f(R)$ theory of gravity. Actually, we have considered the three metrics representing the whole family of LRS spacetimes and solved the field equations by using metric approach as well as the assumption of constant scalar curvature. It is mention here that $R$ may be zero or non-zero. In all we found 10 different solutions.

Keywords: Locally rotationally symmetric, f(R) gravity.

1 Introduction

The rapid growth of the universe is one of the undo crisis in the cosmology [1]. Some of researchers consider this rushing growth of universe is may be due to some unknown energy momentum component. The equation of state of energy momentum component is $P = \omega \rho$. A number of theoretical models such as quintessential scenarios [2], which generalize the cosmological constant approach [3], higher dimensional scenarios [4,5] or alternative to cosmological fluids with exotic equation of state [6] have been given to solve a problem which untilled seems to be still unresolved.

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Since General Relativity (GR) fails to explain the acceleration of the universe. Thus, we are compelled to introduced some kind of dark matter (DM) or dark energy (DE), which are responsible for the rapid growth of the universe [7]. DE and DM models have been inspected in relative to their ability of explaining the acceleration of the universe, however until now there are no experimental indications for the existence of the predicted amount of dark energy in the universe.

So, there is a need of modified or generalized theories to solve the above mention problem. One of these modified theories that arose a lot of stimulation is called \( f(R) \) theory of gravity which is obtained by simply replacing \( R \) with the general function \( f(R) \) in the Einstein-Hilbert Lagrangian of GR. The resultant field equations become more complicated as well as of higher order due to the use of general function \( f(R) \) in the action. Thus, more exact solutions are expected in this theory than GR, due to higher order derivative. The Ricci scalar \( R \) and trace of the energy-momentum tensor \( T \) have a differential relationship in this theory whereas, in GR, they are algebraically related, i.e., \( R = -\kappa T \). Moreover, in this theory Birkhoff’s theorem does not hold [8] and also \( T = 0 \) not imply \( R = 0 \) as it is true in GR.

Weyl [9] and Eddington [10] were the first who studied the action in the context of \( f(R) \) theory of gravity. Jakubiec and Kijowski [11] investigated theories of gravitation with non-linear Lagrangian. There is quite enough material in literature in which different issues have been explored in \( f(R) \) theories of gravity. Multamaki and Vilja [12] worked on spherically symmetric vacuum solutions in \( f(R) \) theory. The same authors [13] also studied the perfect fluid solutions and they found that pressure and density did not uniquely determine the function \( f(R) \).

In 2008, the static cylindrically symmetric vacuum solutions in metric \( f(R) \) theory of gravity have been investigated by Azadi and his coworkers [14]. Momeni [15] extended this work to the non-static cylindrically symmetric solutions. Sharif and Rizwana [16] explored the non-vacuum solution of Bianchi type \( V I_0 \) universe in \( f(R) \) gravity. Capozziello et al. [17] investigated spherically symmetric solutions of \( f(R) \) theories of gravity by the Noether symmetry approach. Rebovocas and Santos [18] analyzed Gödel-type universes in \( f(R) \) gravity. Sharif and Shamir [19] studied static plane symmetric vacuum solutions in \( f(R) \) gravity.

Many authors studied Bianchi type spacetimes in different frameworks. Kumar and Sing [20] investigated solutions of the field equations in presence of perfect fluid using Bianchi type \( I \) spacetime in GR. Lorenz-Petzold [21]
explored exact Bianchi type \textit{III} solutions in the presence of the electromagnetic field. Xing-Xiang [22] investigated Bianchi type \textit{III} string cosmology with bulk viscosity. Wang [23] studied strings cosmological models with bulk viscosity in Kantowski-sachs spacetime.

Shri and Singh [24] explored the analytical solutions of the Einstein-Maxwell field equations for cosmological models of LRS Bianchi type-II, VIII and IX. Pant and Oli [25] investigated two fluid Bianchi type-II cosmological models. Singh and Kumar [26] studied the Bianchi type-II cosmological model by means of constant deceleration parameter. The massive cosmic string in extent of BII model was discussed by Belinchon [27]-[28]. Pradhan et. al. [29], Kumar [30] and Yadav et. al. [31]-[32] inspected the LRS BII cosmological models in existence of massive cosmic string and varying cosmological constant.

The study of Bianchi type models in alternative or modified theories of gravity is also an fascinating discussion. Kumar and Sing studied perfect fluid solutions using Bianchi type \textit{I} spacetimes in scalar tensor theory. Sing and his coworkers [33] investigated some Bianchi type \textit{III} cosmological models in frame work of scalar tensor theory. Sharif and Shamir [34]-[35] investigated the solutions of Bianchi type \textit{I} and \textit{V} spacetimes in framework of \( f(R) \) gravity. Paul and collaborators [36] studied FRW cosmologies in \( f(R) \) gravity. Recently, Jamil and Saima [?] explored the spatially homogeneous rotating solution in \( f(R) \) gravity and Its energy contents.

In this paper, we explore the vacuum solutions of the whole family of LRS spacetimes in metric \( f(R) \) gravity. The field equations are solved by taking the assumption of constant scalar curvature. The paper is organized as follows: In section 2 a brief introduction of the field equations in metric version of \( f(R) \) gravity is given. Section 3 contains LRS solutions in \( f(R) \) theory of gravity, especially, solutions with constant scalar curvature. In the last section, we summarize the results.

\section{Field Equations in Metric \( f(R) \) Gravity}

The three main approaches in \( f(R) \) theory of gravity are ”Metric Approach”, ”Palatini formalism” and ” affine \( f(R) \) gravity”. In metric approach, the connection is the Levi-Civita connection and variation of the action is done with respect to the metric tensor. While, in Palatini formalism, the metric and the connection are independent of each other and variation is done for the
two mentioned parameters independently. In metric-affine $f(R)$ gravity, both
the metric tensor and connection are treating independently and assuming
the matter action depends on the connection as well. In our work, we use
the metric approach only. The action for $f(R)$ theory of gravity is given by

$$S = \int \sqrt{-g} \left( \frac{1}{16\pi G} f(R) + L_m \right) d^4x$$  \hspace{1cm} (1)

Here, $f(R)$ is the general function of the Ricci scalar and $L_m$ is the matter
Lagrangian. To derive the field equations for the metric $f(R)$ gravity, we vary
the action given by Eq. (1) with respect to the metric tensor $g_{\mu\nu}$ and arrive
at

$$F(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \Box F(R) = \kappa T_{\mu\nu},$$  \hspace{1cm} (2)

where $F(R) \equiv \frac{df(R)}{dR}$ and $\nabla_\mu \nabla_\nu$ represents the covariant derivative
and $T_{\mu\nu}$ is the standard matter energy-momentum tensor which is derived
from the Lagrangian $L_m$. These are fourth order partial differential equations
in the metric tensor due to the last two terms on the left hand side of the
equation. These equations can be reduced to the field equations of GR, i.e.,
EFEs by replacing $f(R)$ with $R$. To contract the field equations (2), we
multiply it by the suitable components of the inverse metric functions and
obtain

$$F(R)R - 2f(R) + 3\Box F(R) = \kappa T.$$  \hspace{1cm} (3)

In vacuum, i.e., for $T = 0$, the last equation takes the form as

$$F(R)R - 2f(R) = 0.$$  \hspace{1cm} (4)

This is an important equation as it will be helpful to find the general function
$f(R)$ and also will be used in simplifying the field equations. It is clear from
Eq. (1) that any metric with constant curvature, say $R = R_0$, is a solution of
this equation if the following condition holds

$$F(R_0)R_0 - 2f(R_0) = 0.$$  \hspace{1cm} (5)

This is known as constant scalar curvature condition in vacuum. While, for
non-vacuum case, the constant scalar curvature condition can be obtained
from Eq. (3) and is given by

$$F(R_0)R_0 - 2f(R_0) = \kappa T.$$  \hspace{1cm} (6)
The conditions (5) and (6) are very important to check the stability conditions of the $f(R)$ models. If we differentiate the Eq.(3) w.r.t. $R$, we obtain

$$F'(R)R - RF(R) + 3(\Box F(R))' = 0, \tag{7}$$

where prime represents derivative with respect to $R$. This equation gives the consistency relation for $F(R)$.

## 3 Locally Rotationally Symmetric Solutions in f(R) Gravity

The LRS spacetimes which contain well-known exact solutions of the EFEs are widely studied by many authors [37]-[41]. It has been shown that they admit a group of motions $G_4$ acting multiply transitively on three dimensional non-null orbits space-like ($S_3$) or time-like ($T_3$) and the isotropy group is a spatial rotation. The whole family of LRS spacetimes is represented by the following three metrics:

$$ds^2 = \epsilon [-dt^2 + A^2(t)dx^2] - B^2(t)dy^2 - B^2(t)\Sigma^2(y, k)dz^2, \tag{8}$$

$$ds^2 = \epsilon [-dt^2 + A^2(t)(dx - \Lambda(y, k)dz)^2] - B^2(t)dy^2$$

$$- B^2(t)\Sigma^2(y, k)dz^2, \tag{9}$$

$$ds^2 = \epsilon [-dt^2 + A^2(t)dx^2] - e^{2x}B^2(t)(dy^2 + dz^2), \tag{10}$$

where $k = -1, 0, 1$, $\epsilon = \pm 1$. Here $\Sigma$ and $\Lambda$ are the multivalued functions depending upon the value of $k$ and are defined as

$$\Sigma = \begin{cases} 
\sin y, & k = 1, \\
y, & k = 0, \\
\sinh y, & k = -1, 
\end{cases}$$

and

$$\Lambda = \begin{cases} 
\cos y, & k = 1, \\
y^2, & k = 0, \\
\cosh y, & k = -1. 
\end{cases}$$

Further, it is mentioned here that corresponding to $\epsilon = 1$ and $\epsilon = -1$ we obtain the static and non-static LRS solutions respectively. In this paper,
we will discuss only the non-static case as the results for the static case can be obtained consequently. For $\epsilon = -1$, the above equations take the form

\[
\begin{align*}
\text{Metric-I} & : ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - B^2(t)\Sigma^2(y,k)dz^2, \\
\text{Metric-II} & : ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)e^{2x}dy^2 - B^2(t)e^{2x}dz^2, \\
\text{Metric-III} & : ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - \{A^2(t)\Lambda^2(y,k) \\
& \quad + B^2(t)\Sigma^2(y,k)\}dz^2 + 2A^2(t)\Lambda(y,k)dx dz.
\end{align*}
\]

It has been shown that, for $k = 0$, the metric-I may reduce to Bianchi types $I(BI)$ or $VII_0,(BVII_0)$, for $k = -1$, Bianchi type $III,(BIII)$ and Kantowski-Sachs (KS) for $k = +1$. The metric-II represents Bianchi type $V(BV)$ or $VII_h,(BVII_h)$ metric while the metric-III reduces to Bianchi types $II(BII)$ for $k = 0$, $VIII(BVIII)$ or $III(BIII)$ for $k = -1$ and $IX(BIX)$ for $k = +1$.

### 3.1 Solution of the Metric-I

Now, we shall solve the metric-I by considering the following three cases:

a). When $k = 0$ \quad b). When $k = +1$ \quad c). When $k = -1$

#### 3.1.1 Case a:

For $k = 0$, the metric-I takes the form

\[
ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)(d^2y + y^2d^2z).
\]

After evaluating the components of Ricci tensor, the Ricci scalar turns out to be

\[
R = -2\frac{\ddot{A}}{A} - 4\frac{\ddot{B}}{B} - 4\frac{\dot{A}\dot{B}}{AB} - 2\frac{\dot{B}^2}{B^2},
\]

where dot represents derivative with respect to time. Writing the Eq. (11) in the form

\[
f(R) = \frac{3\Box F(R) + F(R)R}{2}.
\]

Using this value of $f(R)$ in the field equations (2) and setting $T_{\mu\nu} = 0$ (for vacuum solutions), we have

\[
\frac{F(R)R_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}F(R)}{g_{\mu\nu}} = \frac{F(R)R - \Box F(R)}{4}.
\]
It is clear from equation (12) that the Ricci scalar depends only on $t$ and hence $F(R)$ will be a function of $t$ only, that is, $F \equiv F(t)$. So, Eq.(14) becomes the set of ordinary differential equations involving $F(t), A(t)$ and $B(t)$. Now, one can straightforward write from the L.H.S. of Eq.(14) that

$$E_\mu \equiv \frac{F(R)R_{\mu\mu} - \nabla_\mu \nabla_{\mu} F(R)}{g_{\mu\mu}},$$

which is independent of the index $\mu$ and hence $E_\mu - E_\nu = 0$ for all $\mu$ and $\nu$. Thus, $E_0 - E_1 = 0$ gives

$$-2 \ddot{B}B + 2\dot{A}\dot{B}AB + \dot{A}\dot{F}AF - \ddot{F}F = 0.$$  

Similarly, $E_0 - E_2 = 0$ yields

$$-\ddot{A}A - \ddot{B}B + \dot{B}^2B^2 + \dot{A}\dot{B}AB + \dot{B}\dot{F}BF - \ddot{F}F = 0.$$  

while the remaining cases, obtained by varying $\mu$ and $\nu$, yield the equations which are scalar multiple of the last two equations. As we obtain two nonlinear differential equations and three unknowns, i.e., $A, B$ and $F$ so we have to impose an other condition to find the solution of these equations. We shall use assumption of constant curvature and find the solution of these equations as follows:

**Constant Curvature Solution**

For constant curvature solution, i.e., $R = R_0$, we have

$$\dot{F}(R_0) = \ddot{F}(R_0) = 0.$$  

Making use of this condition of constant curvature, the Eqs.(16) and (17) reduce to

$$\frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} = 0,$$

and

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{\dot{A}\dot{B}}{AB} = 0$$

respectively. Further, we will solve the last two equations by using following two assumptions:
Case aI:

Here we assume that $A \propto t^m$ and $B \propto t^n$, where $m$ and $n$ are any real numbers. In other words, we substitute $A = K_1 t^m$ and $B = K_2 t^n$ in Eqs.(19) and (20), where $K_1$ and $K_2$ are constants of proportionality, to obtain

\begin{align*}
  n(n - 1 - m) &= 0, \\  m^2 - m - n - mn &= 0.
\end{align*}

The simultaneous solution of these equations gives two possibilities for the values of $m$ and $n$

i). $m = -\frac{1}{3}$, $n = \frac{2}{3}$  
ii). $m = 1$, $n = 0$

Case aI(i):

In this case, we have $A = K_1 t^{-\frac{1}{3}}$ and $B = K_2 t^{\frac{2}{3}}$. Then the corresponding solution turns out to be

\[ ds^2 = dt^2 - (K_1)^2 t^{-\frac{2}{3}} dx^2 - (K_2)^2 t^{\frac{4}{3}} (dy^2 + y^2 dz^2). \] (23)

It is mentioned here that the Ricci scalar becomes zero for this case. This solution corresponds to the Kinematics self-similar solution of the second kind for the tilted dust case given in table 1 of the [41].

Case aI(ii):

In this case, we obtain $A = K_1 t$ and $B = K_2$ and arrive at the solution

\[ ds^2 = dt^2 - (K_1)^2 t^2 dx^2 - (K_2)^2 dy^2 + y^2 dz^2. \] (24)

In this case, again the Ricci scalar vanishes. The solution given in Eq.(24) coincides to the Kinematics self-similar solution of the first kind for the tilted dust case given in table 1 of the [41].

Case aII:

In this case, we assume that $A(t) = e^{2\mu(t)}$ and $B(t) = e^{2\lambda(t)}$ so that metric (11) becomes

\[ ds^2 = dt^2 - e^{4\mu(t)} dx^2 - e^{4\lambda(t)} (dy^2 + y^2 dz^2). \] (25)

The corresponding Ricci scalar turns out to be

\[ R = -8\ddot{\mu} - 4\dot{\mu} - 24\ddot{\lambda} - 8\dot{\lambda} - 16\dot{\mu}\dot{\lambda}. \] (26)
Now, by substituting \(A(t) = e^{2\mu(t)}\) and \(B(t) = e^{2\lambda(t)}\) in Eqs.(19) and (20), we have
\[
\ddot{\lambda} + 2(\dot{\lambda})^2 - 2\dot{\mu}\dot{\lambda} = 0
\] (27)
and
\[
\ddot{\mu} + \ddot{\lambda} + 2\dot{\mu}^2 - 2\dot{\mu}\dot{\lambda} = 0
\] (28)
respectively. Eq.(27) can be written as
\[
\dot{\lambda}(\ddot{\lambda} + 2\dot{\lambda} - 2\dot{\mu}) = 0,
\] (29)
which yields the following two cases:

i) \(\dot{\lambda} = 0\),

ii) \(\ddot{\lambda} + 2\dot{\lambda} - 2\dot{\mu} = 0\)

We solve the Eqs.(19) and (20) for these two cases as:

Case aII(i):
This case implies that
\[
\lambda = a,
\] (30)
where \(a\) is constant of integration. By putting this value of \(\lambda\) in Eq.(28), we obtain
\[
\ddot{\mu} + 2\dot{\mu}^2 = 0.
\] (31)
On integration, the last equation implies that
\[
\mu = \ln(b\sqrt{t - c}),
\] (32)
where \(b\) and \(c\) are integration constants. Thus, the metric (25) takes the form
\[
\begin{align*}
\text{ds}^2 &= dt^2 - (b^2t - b^2c)^2dx^2 - e^{4a}(dy^2 + y^2dz^2) \\
&= dt^2 - (b^2t - b^2c)^2dx^2 - e^{4a}(dy^2 + y^2dz^2).
\end{align*}
\] (33)

Case aII(ii):
In this case, the constraint equation
\[
\frac{\ddot{\lambda}}{\dot{\lambda}} + 2\dot{\lambda} - 2\dot{\mu} = 0
\] (34)
yields the value of \(\mu\) as
\[
\mu = \lambda + \frac{1}{2}\ln\dot{\lambda} + d.
\] (35)
Substituting this value in Eq. (26), we get the scalar curvature as

\[ R = -2 \frac{\lambda^{\cdots}}{\lambda} - 28\dddot{\lambda} - 48\dddot{\lambda}^2. \]  

(36)

The last equation implies that, for constant scalar curvature, we have to assume

\[ -2 \frac{\lambda^{\cdots}}{\lambda} - 28\dddot{\lambda} - 48\dddot{\lambda}^2 = \text{constant}. \]  

(37)

This is a third order linear ordinary differential equation which cannot be analytically solved easily. So, we assume that \( \lambda(t) \) is a linear function of \( t \), i.e., \( \lambda(t) = ft + g \), where \( f \) and \( g \) are any arbitrary constants. Then Eq. (35) gives \( \mu = ft + \bar{g} \), where \( \bar{g} = g + \frac{1}{2} \ln f + d \). Consequently, the metric (25) becomes

\[ ds^2 = dt^2 - e^{4(f t + \bar{g})} dx^2 - e^{4(f t + g)} [dy^2 + y^2 dz^2]. \]  

(38)

The corresponding Ricci scalar takes the form

\[ R = -48f^2. \]  

(39)

We can write the metric (38) as

\[ ds^2 = dt^2 - a^2 e^{2t} dx^2 - b^2 e^{2t} [dy^2 + y^2 dz^2], \]  

(40)

where \( a = e^{\bar{g}/2f} \) and \( b = e^{g/2f} \). Hence, this solution corresponds to the Kinematics self-similar solution of the zeroth kind for the parallel vector field case [41]. It is mentioned here that the trivial solution of the Eq. (11) coincides with the Kinematics self-similar solution of the infinite kind for the parallel dust case given in Eq. (39) of [41].

### 3.1.2 Case b:

For \( k = 1 \), the Metric-I takes the form

\[ ds^2 = dt^2 - A^2(t) dx^2 - B^2(t) [dy^2 + \sin^2 y dz^2]. \]  

(41)

With the help of the components of Ricci tensor, the Ricci scalar has been evaluated as

\[ R = \frac{-2}{AB^2} [\dot{A}B^2 + 2\dot{B}AB + 2A\dot{B}B + A(\ddot{B})^2 + A]. \]  

(42)
Eq. (15) gives the following three independent equations for the corresponding values of $\mu$ and $\nu$:

$$E_0 - E_1 = 0 \text{ yields } -\frac{2\ddot{B}}{B} + \frac{2\dot{A}\dot{B}}{AB} + \frac{\ddot{A}\dot{F}}{AF} - \frac{\dddot{F}}{F} = 0; \quad (43)$$

$$E_0 - E_2 = 0 \text{ gives } -\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{1}{B^2} + \frac{\dot{B}\dot{F}}{BF} - \frac{\dddot{F}}{F} = 0 \quad (44)$$

and $E_1 - E_2 = 0$ implies that

$$-\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} + \frac{(\dot{B})^2}{B^2} + \frac{1}{B^2} + \frac{\dot{B}\dot{F}}{BF} - \frac{\dddot{F}}{AF} = 0. \quad (45)$$

We will also use the constant curvature assumption to solve this system of three equations

**Constant Curvature Solution:**

Using condition (18), Eqs. (43)–(45) reduce to

$$\frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} \quad (46)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} - \frac{(\dot{B})^2}{B^2} - \frac{\dot{A}\dot{B}}{AB} - \frac{1}{B^2} \quad (47)$$

and

$$-\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} + \frac{(\dot{B})^2}{B^2} + \frac{1}{B^2} = 0. \quad (48)$$

We solve the equation (46) by using power law assumption and substitute $A = K_1 t^m$ and $B = K_2 t^n$, where $K_1$ and $K_2$ are constants of proportionality. Consequently, we have

$$n(m - n + 1) = 0. \quad (49)$$

Here only the case $n = 0$ gives constant curvature so we leave the case for which $m - n + 1 = 0$. Substituting $n = 0$, i.e., $B = K_2$ and $A = K_1 t^m$ in Eqs. (47) and (48), we obtain the following single equation

$$m(m - 1) - \frac{t^2}{K_2} = 0. \quad (50)$$
Comparing the co-efficient of $t^0$, we have $m(m-1) = 0$, i.e., $m = 0$ or $m = 1$, which yields the only non-trivial solution when $A = k_1 t$ and $B = k_2$. Hence, the metric (41) takes the form

$$ds^2 = dt^2 - (K_1 t)^2 dx^2 - (K_2)^2 [dy^2 + \sin^2 y dz^2].$$  (51)

The corresponding Ricci scalar is given by $R = -2/k_2^2$. The trivial solution of the Eq.(41) is exactly the same as the Kinematics self-similar solution of the infinite kind given in Eq.(32) of [41].

### 3.1.3 Case c:

Now, we discuss the case when $k = -1$, for which the metric-I takes the form

$$ds^2 = dt^2 - A^2(t) dx^2 - B^2(t) [dy^2 + \sinh^2 y dz^2].$$  (52)

With the help of the components of Ricci tensor, the Ricci scalar becomes

$$R = \frac{-2}{AB^2} [\ddot{A}B^2 + 2\dot{A}\dot{B}AB + 2\dot{A}\dot{B}B + A(\dot{B})^2 + A].$$  (53)

Eq.(15) gives the following three independent equations for the corresponding values of $\mu$ and $\nu$:

$E_0 - E_1 = 0$ yields

$$-\frac{2\dot{B}}{B} + \frac{2\dot{A}}{AB} + \frac{\dot{A}}{AF} - \frac{\ddot{F}}{F} = 0;$$  (54)

$E_0 - E_2 = 0$ gives

$$-\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} + \frac{\dot{A}}{AB} - \frac{1}{B^2} + \frac{\dot{B}}{BF} - \frac{\ddot{F}}{F} = 0$$  (55)

and $E_1 - E_2 = 0$ implies that

$$-\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} - \frac{\dot{A}}{AB} + \frac{(\dot{B})^2}{B^2} - \frac{1}{B^2} + \frac{\dot{B}}{BF} - \frac{\ddot{F}}{AF} = 0.$$(56)

Again we use the constant curvature assumption to solve these equations

**Constant Curvature Solution:**
Using the condition (18), Eqs. (54)-(56) reduce to
\[ \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} = 0, \quad (57) \]
\[ \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} - \frac{(\dot{B})^2}{B^2} - \frac{\dot{A}\dot{B}}{AB} + \frac{1}{B^2} = 0, \quad (58) \]
\[ \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} + \frac{(\dot{B})^2}{B^2} - \frac{1}{B^2} = 0. \quad (59) \]

We solve these equations by using the power law assumption and substitute \( A = K_1 t^m \) and \( B = K_2 t^n \), where \( K_1 \) and \( K_2 \) are constants of proportionality. Only the case when \( n = 0 \) and \( m = 1 \) gives constant curvature solution. Then, the metric (52) takes the form
\[ ds^2 = dt^2 - (K_1 t) dx^2 - (K_2)^2 [dy^2 + \sinh^2 y dz^2]. \quad (60) \]

In this case, the Ricci scalar is given by \( R = 2/k_2^2 \).

The trivial solution of the Eq. (52) is exactly the same as the Kinematics self-similar solution of the infinite kind given in Eq. (33) of [41].

### 3.2 Solution of Metric-II

With the help of the components of Ricci tensor, Ricci scalar for the Metric-II turns out to be
\[ R = -2\left[ \frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} - \frac{3}{A^2} \right]. \quad (61) \]

From Eq. (15), \( E_0 - E_1 = 0 \) gives
\[ (-2\frac{\ddot{B}}{B} + 2\frac{\dot{A}\dot{B}}{AB} - \frac{2}{A^2}) F(R) + \left( \frac{\dot{A}}{A} \dot{F} - \ddot{F} \right) = 0; \quad (62) \]

\( E_0 - E_2 = 0 \) yields
\[ (\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} + \frac{\dot{A}\dot{B}}{AB} - \frac{2}{A^2}) F(R) + \left( \frac{\dot{B}}{B} \dot{F} - \ddot{F} \right) = 0 \quad (63) \]

and \( E_1 - E_2 = 0 \) implies that
\[ (-\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} - \frac{\dot{A}\dot{B}}{AB}) F(R) + \left( \frac{\dot{B}}{B} \dot{F} - \frac{\dot{A}}{A} \dot{F} \right) = 0, \quad (64) \]
which are the only independent equations for all possible cases.

**Constant Curvature Solution:**

Now, for constant curvature solution, the Eqs.\((62)-(64)\) reduce to

\[
-2\frac{\ddot{B}}{B} + 2\frac{\dot{A}\dot{B}}{AB} - \frac{2}{A^2} = 0, \tag{65}
\]

\[
-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} + \frac{\dot{A}\dot{B}}{AB} - \frac{2}{A^2} = 0, \tag{66}
\]

\[
-\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} - \frac{\dot{A}\dot{B}}{AB} = 0. \tag{67}
\]

By subtracting Eq.\((66)\) form Eq.\((67)\) we get the only independent Eq.\((65)\). Using Eq.\((65)\) in Eq.\((61)\), the corresponding Ricci scalar becomes

\[
R = -2\frac{\ddot{\lambda}}{\lambda} - 48\dot{\lambda}^2. \tag{68}
\]

Now, we assume that \(A(t) = e^{2\alpha}\) and \(B(t) = e^{2\lambda(t)}\) so that Eq.\((68)\) takes the form

\[
R = -28\ddot{\lambda} - 48\dot{\lambda}^2. \tag{69}
\]

But according to our assumption, the scalar curvature must be constant, i.e.,

\[
\text{cons tan } t = -28\ddot{\lambda} - 48\dot{\lambda}^2 \tag{70}
\]

which implies that \(\lambda(t)\) must be a linear function of \(t\), i.e., \(\lambda(t) = ft + g\), where \(f\) and \(g\) are any arbitrary constants. Consequently, the metric takes the form

\[
ds^2 = dt^2 - e^{4\alpha}dx^2 - e^{2(f t + g)} e^{2\dot{\lambda}} dy^2 + dz^2. \tag{71}
\]

The corresponding Ricci scalar becomes \(R = -24f^2 + 6e^{-4\alpha}\).

It is mentioned here that, for the trivial case of the metric-II, i.e., when \(A(t)\) and \(B(t)\) are both taken to be constant, the solution corresponds to the Kinematics self-similar solution of the infinite kind for the parallel vector field given in the Eq.(50) of [41].

### 3.3 Solution of Metric-III

In this section, we will solve the Metric-III by considering the following three cases:

\(\alpha\). When \(k = 0\) \hspace{1cm} \(\beta\). When \(k = +1\) \hspace{1cm} \(\gamma\). When \(k = -1\)
3.3.1 Case $\alpha$

For $k = 0$, the metric-III can be written as

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - \left\{ A^2(t)\frac{y^4}{4} + B^2(t)y^2 \right\}dz^2 + 2A^2(t)dxdz.$$  \hfill (72)

The corresponding Ricci scalar has been evaluated, by using the components of the Ricci tensor, as

$$R = \frac{-2}{AB^4}[\ddot{A}B^4 + 2\dot{B}AB^3 + 2\dot{A}\dot{B}B^3 + A(\dot{B})^2B^2 - \frac{A^3}{4}].$$  \hfill (73)

The Eq.(15) gives the following independent three equations for the possibilities, $E_0 - E_1 = 0$, $E_0 - E_2 = 0$ and $E_1 - E_2 = 0$

$$-\frac{2\ddot{B}}{B} + \frac{2\dot{A}\dot{B}}{AB} + \frac{A^2}{2B^4} + \frac{\dot{A}\dot{F}}{AF} - \frac{\ddot{F}}{F} = 0,$$  \hfill (74)

$$-\frac{\ddot{A}}{A} - \frac{\dot{B}}{B} + \frac{(\dot{B})^2}{AB} - \frac{\dot{A}\dot{B}}{AB} - \frac{A^2}{2B^4} + \frac{\dot{B}\dot{F}}{BF} - \frac{\ddot{F}}{F} = 0,$$  \hfill (75)

$$-\frac{\ddot{A}}{A} + \frac{\dot{B}}{B} - \frac{A^2}{2B^4} - \frac{\dot{B}\dot{F}}{BF} + \frac{\dot{A}\dot{F}}{AF} = 0$$  \hfill (76)

respectively.

**Constant curvature Solution:**

Using the condition of constant curvature given by equation (18) in the Eqs.(74)-(76), we obtain

$$-\frac{2\ddot{B}}{B} + \frac{2\dot{A}\dot{B}}{AB} + \frac{A^2}{2B^4} = 0,$$  \hfill (77)

$$-\frac{\ddot{A}}{A} - \frac{\dot{B}}{B} + \frac{(\dot{B})^2}{AB} - \frac{\dot{A}\dot{B}}{AB} = 0,$$  \hfill (78)

$$-\frac{\ddot{A}}{A} + \frac{\dot{B}}{B} - \frac{A^2}{2B^4} = 0.$$  \hfill (79)

It is noticed that when we subtract Eq.(79) from Eq.(78), it yields the Eq.(77). Now, we solve Eq.(77) by assuming $A(t) = B^2(t)$ and arrive at

$$\frac{\ddot{B}}{B} = \frac{1}{4} + \frac{2(\dot{B})^2}{B^2}.$$
We further assume that $B(t) = e^{2\lambda(t)}$ so that the corresponding Ricci scalar turns out to be
\[ R = -120\dot{\lambda}^2 - \frac{3}{2}. \] (80)

Now, to attain the assumption of scalar constant curvature, we must put the following constraint
\[ \cos t \tan t = -120\dot{\lambda}^2 - \frac{3}{2}, \] (81)
which implies that $\lambda(t)$ must be a linear function of $t$, that is, $\lambda(t) = ft + g$, where $f$ and $g$ are any arbitrary constants. Consequently, we get the following solution
\[ ds^2 = dt^2 - e^{8(ft+g)} dx^2 - e^{4(ft+g)} dy^2 - e^{4(ft+g)} \left\{ \frac{y^4 e^{4(ft+g)}}{4} + y^2 \right\} dz^2 + 2e^{8(ft+g)} dx dz. \] (82)

In this case, the corresponding Ricci scalar becomes $R = -120f^2 - \frac{3}{2}$.

3.3.2 Case $\beta$: 
In this case, i.e., for $k = 1$, the Metric-III takes the form
\[ ds^2 = dt^2 - A^2(t) dx^2 - B^2(t) dy^2 - \left\{ A^2(t) \cos^2 y + B^2(t) \sin^2 y \right\} dz^2 + 2 \cos y A^2(t) dx dz. \] (83)

The corresponding Ricci scalar, evaluated with the help of the components of Ricci tensor, turns out to be
\[ R = \frac{-1}{2AB^4} [4\dot{A}\dot{B}^4 + 8\dot{B}AB^3 + 8\dot{A}\dot{B}B^3 + 4A(\dot{B})^2B^2 + 4AB^2 - A^3]. \] (84)

The Eq.(15) yields the only two independent equation, given below: $E_0 - E_1 = 0$ yields
\[ -\frac{2\ddot{B}}{B} + \frac{2\dot{A}\ddot{B}}{AB} + \frac{A^2}{2B^4} + \frac{\dot{A}\dot{F}}{AF} - \frac{\ddot{F}}{F} = 0 \] (85)
and $E_0 - E_2 = 0$ gives
\[ -\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} + \frac{\dot{A}\dot{B}}{AB} - \frac{A^2}{2B^4} + \frac{1}{B^2} + \frac{\dot{B}\dot{F}}{BF} - \frac{\ddot{F}}{F} = 0. \] (86)
Constant Curvature Solution:

Utilizing the constant curvature condition (18), the Eqs.(85) and (86) take the form

\[ -\frac{2\ddot{B}}{B} + \frac{2\dot{A}\dot{B}}{AB} + \frac{A^2}{2B^4} = 0, \]  
(87)

\[ -\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{1}{B^2} - \frac{A^2}{2B^4} = 0. \]  
(88)

To solve these equations, we put \( A(t) = 2B(t) \) so that Eqs.(87) and (88) reduce to

\[ -\frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} + \frac{1}{B^2} = 0, \]  
(89)

\[ -\frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} - \frac{1}{B^2} = 0. \]  
(90)

After addition the last two equations yield the following single equation

\[ -\frac{\ddot{B}}{B} + \frac{(\dot{B})^2}{B^2} = 0, \]  
(91)

which can be easily solved by making the assumption, \( B(t) = e^\lambda(t) \) and yields

\[ \ddot{\lambda} = 0. \]

The solution of the last equation is obvious that \( \lambda(t) \) must be a linear function of \( t \), i.e., \( \lambda(t) = ft + g \), where \( f \) and \( g \) are any arbitrary constants. Consequently, we obtain the solution given by

\[ ds^2 = dt^2 - 4e^{2(f \cdot t + g)}dx^2 - e^{2(f \cdot t + g)}\{dy^2 + (4\cos^2 y + \sin^2 y)dz^2\} + 2\sin y e^{2(f \cdot t + g)}dxdz. \]  
(92)

In this case, the corresponding Ricci scalar turns out to be constant, i.e., \( R = -12f^2 \).

3.3.3 Case \( \gamma \):

For this case, the Metric-III takes the following form

\[ ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - \{A^2(t)\cosh^2 y + B^2(t)\sinh^2 y\}dz^2 + 2\sinh yA^2(t)dxdz. \]  
(93)
The corresponding Ricci scalar has been evaluated, using the components of Ricci tensor, as

$$ R = \frac{-1}{2AB^4}[4\ddot{A}B^4 + 8\dot{B}AB^3 + 8\dddot{A}\dot{B}B^3 + 4A(\dot{B})^2B^2 - 4AB^2 - A^3].$$  \hfill (94)

From Eq. (15), \( E_0 - E_1 = 0 \) yields

$$ -\frac{2\dddot{B}}{B} + \frac{2\dot{A}\dot{B}}{AB} + \frac{A^2}{2B^4} + \frac{\dot{A}\dot{F}}{AF} - \frac{\ddot{F}}{F} = 0; \hfill (95)$$

\( E_0 - E_2 = 0 \) yields

$$ -\frac{\ddot{A}}{A} + \frac{\dddot{B}}{B} + \frac{(\dot{B})^2}{B^2} + \frac{\dot{A}\dot{B}}{AB} - \frac{A^2}{2B^4} - \frac{1}{B^2} + \frac{\dot{B}\ddot{F}}{BF} - \frac{\ddot{F}}{F} = 0 \hfill (96)$$

and \( E_1 - E_2 = 0 \) implies that

$$ -\frac{\ddot{A}}{A} + \frac{\dddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{(\dot{B})^2}{B^2} - \frac{A^2}{2B^4} - \frac{1}{B^2} + \frac{\dot{B}\ddot{F}}{BF} - \frac{\ddot{F}}{F} = 0. \hfill (97)$$

**Constant Curvature Solution:**

By making use of the constant curvature condition (18), the Eqs. (95)-(97) reduce to

$$ -\frac{2\dddot{B}}{B} + \frac{2\dot{A}\dot{B}}{AB} + \frac{A^2}{2B^4} = 0. \hfill (98)$$

$$ -\frac{\ddot{A}}{A} - \frac{\dddot{B}}{B} + \frac{(\dot{B})^2}{B^2} + \frac{\dot{A}\dot{B}}{AB} - \frac{A^2}{2B^4} = 0. \hfill (99)$$

$$ -\frac{\ddot{A}}{A} + \frac{\dddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} + \frac{(\dot{B})^2}{B^2} - \frac{1}{B^2} - \frac{A^2}{2B^4} = 0. \hfill (100)$$

Subtracting Eq. (98) from Eq. (99) we obtain the Eq. (100), which yields a trivial constant curvature solution for \( A = a \) and \( B = b \) and \( b = \pm a \). Consequently, the solution is given by

$$ ds^2 = dt^2 - a^2dx^2 - a^2dy^2 - \{a^2\cosh^2 y + a^2\sinh^2 y\} dz^2 + 2\sinh ya^2 dx dz. \hfill (101)$$

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4 Summary and Discussion

The study of $f(R)$ models has been carried out by many researchers during the last two decades. They discussed classification and comparison of the different approaches of $f(R)$ theories of gravity. Also, different $f(R)$ models have been introduced to evaluate the energy density in $f(R)$ theory of gravity. In this paper, we have solved the field equations of $f(R)$ theory of gravity for LRS spacetimes by using metric $f(R)$ gravity approach.

As the field equations of $f(R)$ theory of gravity are highly non-linear and complicated to be solved analytically due to the arbitrary function $F$, so we use the assumption of constant scalar curvature. This assumption is found to be the most appropriate and make the field equations solvable in some cases. However, it is not always be assured that the constant scalar curvature assumption yield a solution.

The whole family of LRS spacetimes is represented by three metrics, i.e., Metric-I, Metric-II and Metric-III. The Metric-I yields further three cases. The Case(a) yields four non-trivial solutions as given by Eqs.(23), (24), (33) and (40). While the Case(b) and Case(c) give the nontrivial solutions given by Eqs.(51) and (60) respectively. The Metric-II yields only one non-trivial solution given by Eq.(71). There arise three cases for the Metric-III, i.e., Case($\alpha$), Case($\beta$) and the Case($\gamma$). We obtain only one nontrivial solution for the each Cases ($\alpha$) and ($\beta$) given by Eqs.(82) and (92) respectively. While the Case($\gamma$) has been solved for trivial case only as given by Eq.(101).

The above mentioned solutions of LRS space-times in $f(R)$ theory of gravity are given in the form of tables below:

Table 1. Solution of Metric-I

| Case     | Solution                                                                                           |
|----------|----------------------------------------------------------------------------------------------------|
| Case aI(i)| $ds^2 = dt^2 - (K_1)^2 t^{-\frac{4}{3}} dx^2 - (K_2)^2 t^\frac{2}{3} (dy^2 + y^2 dz^2)$.          |
| Case aI(ii)| $ds^2 = dt^2 - (K_1)^2 t^2 dx^2 - (K_2)^2 (dy^2 + y^2 dz^2)$.                                       |
| Case aII(i)| $ds^2 = dt^2 - (b^2 t - b^2 c) dy^2 - (e^2 a) (dy^2 + y^2 dz^2)$.                                |
| Case aII(ii)| $ds^2 = dt^2 - (e^4 ft + g) dy^2 - e^{4 ft + g} [dy^2 + y^2 dz^2]$.                              |
| Case b    | $ds^2 = dt^2 - (K_1 t)^2 dx^2 - (K_2)^2 [dy^2 + \sin^2 y dz^2]$                                  |
| Case c    | $ds^2 = dt^2 - (K_1 t)^2 dx^2 - (K_2)^2 [dy^2 + \sinh^2 y dz^2]$                                 |

Table 2. Solution of Metric-II
Table 3. Solution of Metric-III

| Case | Solution |
|------|----------|
| Case \(\alpha\) | \(ds^2 = dt^2 - e^{4(f t + g)}(e^{4(f t + g)} dx^2 - dy^2 + 2e^{4(f t + g)} dx dy) dz^2 - \left\{e^{4(f t + g)} \frac{x^2}{4} + y^2\right\} dz^2\). |
| Case \(\beta\) | \(ds^2 = dt^2 - e^{2(f t + g)}(4 dx^2 - dy^2 - \left\{4 \cos^2 y + \sin^2 y\right\} dz^2 + 8 \cos y dx dz\). |
| Case \(\gamma\) | \(ds^2 = dt^2 - a^2 dx^2 - a^2 dy^2 - \left\{a^2 \cosh^2 y + a^2 \sinh^2 y\right\} dz^2 + 2 \sinh ya^2 dx dz\). |

For the solution given in case aI(i), case aI(ii) and case aII(ii) corresponds to the solution given in table 1 and the Eq.(39) of the [41] respectively. It is mentioned here that the trivial solution of case b, case c and the metric-II corresponds to the solution given in Eq.(32), Eq.(33) and Eq.(50) of the [41] respectively.

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