CONSTANT \( Q \)-CURVATURE METRICS WITH DELAUNAY ENDS: THE NONDEGENERATE CASE

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Abstract. We construct a one-parameter family of solutions to the positive singular \( Q \)-curvature problem on compact nondegenerate manifolds of dimension bigger than four with finitely many punctures. If the dimension is at least eight we assume that the Weyl tensor vanishes to sufficiently high order at the singular points. On a technical level, we use perturbation methods and gluing techniques based on the mapping properties of the linearized operator both in a small ball around each singular point and in its exterior. Main difficulties in our construction include controlling the convergence rate of the Paneitz operator to the flat bi-Laplacian in conformal normal coordinates and matching the Cauchy data of the interior and exterior solutions; the latter difficulty arises from the lack of geometric Jacobi fields in the kernel of the linearized operator. We overcome both these difficulties by constructing suitable auxiliary functions.

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2020 Mathematics Subject Classification. 35J60, 35B09, 35J30, 35B40.

Key words and phrases. Paneitz operator, \( Q \)-curvature equation, Gluing construction, Prescribed asymptotics, Isolated singularities, Geometric equation.

This work was partially supported by São Paulo Research Foundation (FAPESP) \#2020/07566-3 and \#2021/15139-0, Paraiba State Research Foundation (FAPESQ) \#3034/2021, and National Council for Scientific and Technological Development (CNPq) \#312340/2021-4, \#403349/2021-4, and \#429285/2016-7.

arXiv:2110.05234v2 [math.DG]  4 Oct 2022
1. Introduction

In the last several decades, much of the geometric analysis community has explored curvature problems similar to and extending the classical Yamabe problem regarding scalar curvature. The present paper constructs several new metrics on a punctured manifold with constant (fourth order) \( Q \)-curvature. Let \((M^n, g)\) be a closed (compact and without boundary) Riemannian manifold of dimension \( n \geq 5 \) and

\[
Q_g = \frac{1}{2(n-1)} \Delta_g R_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2 - \frac{2}{(n-2)^2} |\text{Ric}_g|^2, \tag{1.1}
\]

where \( R_g \) is the scalar curvature, \( \text{Ric}_g \) is the Ricci curvature, and \( \Delta_g \) is the Laplace-Beltrami operator. It is well-known that \( Q_g \) transforms under a conformal change of metric, denoted by \( \tilde{g} = u^{4/(n-4)} g \) with \( u \in C^\infty(M) \) and \( u > 0 \), according to the rule

\[
Q_{\tilde{g}} = \frac{2}{n-4} u^{-\frac{n+4}{n-4}} P_g u, \tag{1.2}
\]

where the Paneitz operator \( P_g \) is given by

\[
P_g u = \Delta^2_g u + \text{div}_g \left( \frac{4}{n-2} \text{Ric}_g(\nabla u, \cdot) - \frac{(n-2)^2 + 4}{2(n-1)(n-2)} \text{R} \langle \nabla u, \cdot \rangle \right) + \frac{n-4}{2} Q_g u. \tag{1.3}
\]

We refer the interested reader to [8–10, 16, 22, 45] for a thorough background on the \( Q \)-curvature and on the Paneitz operator.

We are interested in the problem of prescribing the \( Q \)-curvature of a conformal metric. For this, we adopt the normalization \( Q_{\tilde{g}} = n(n^2 - 4)/8 \), which makes this curvature equals that of the sphere \((S^n, g_0)\) with its standard round metric. With this normalization, we define the \( Q \)-curvature operator \( H_g : C^{4,\alpha}(M) \rightarrow C^{0,\alpha}(M) \), for some \( \alpha \in (0,1) \), by

\[
H_g(u) = P_g u - \frac{n(n-4)(n^2 - 4)}{16} |u|^{-\frac{8}{n-4}} u, \tag{1.4}
\]

so that (1.2) can be reformulated as \( H_g(u) = 0 \). The operator \( H_g \) is conformally covariant, in that

\[
H_g(\phi) = u^{-\frac{n+4}{n-4}} H_g(u\phi) \quad \text{for all} \quad \phi \in C^\infty(M). \tag{1.5}
\]

For a given finite set of points \( \Lambda \), our main goal in this paper is to find positive smooth solutions to the singular problem

\[
\begin{cases}
H_g(u) = 0 & \text{on} \quad M \setminus \Lambda \\
\liminf_{x \to p} u(x) = \infty & \text{for each} \quad p \in \Lambda.
\end{cases} \tag{\text{Q}}
\]

When \((M^n, g)\) is conformally equivalent to the round sphere \((S^n, g_0)\) and \( \Lambda \) is a single point there exists no solution to \((\text{Q})\), as it is was proved by C. S. Lin [36] and J. Wei and X. Xu [52]. The first result proving existence of solutions to the positive singular \( Q \)-curvature problem is due to S. Baraket and S. Rebhi [4]. They provided a partial answer to the existence problem as follows

**Theorem A** ([4]). For each natural number \( k \) there exists a finite set \( \Lambda \) with cardinality \( 2k \) such that \( S^n \setminus \Lambda \) carries a complete, constant \( Q \)-curvature metric conformal to the round metric.

Using variational bifurcation theory and topological methods R. G. Bettiol, P. Piccione and Y. Sire [6] obtained the existence of infinitely many complete metrics with constant positive \( Q \)-curvature on \( S^n \setminus S^k \) with \( 0 \leq k \leq (n - 4)/2 \), conformal to the round metric. Recently, for arbitrary closed Riemannian manifolds, A. Hyder and Y. Sire [27] studied the positive singular \( Q \)-curvature problem when \( \Lambda \) is a smooth submanifold of \( M \) with Hausdorff dimension \( \dim_H(\Lambda) \) strictly between 0 and \((n - 4)/2 \). Motivated by the results in [40], they established the following

...
Theorem B (\cite{27}). Let \((M^n, g)\) be a closed Riemannian manifold with \(n \geq 5\) with semi-positive \(Q\)-curvature and nonnegative scalar curvature. Let \(\Lambda\) be a connected smooth closed submanifold of \(M\) with \(0 < \text{dim}_H(\Lambda) < (n-4)/2\). Then, there exists an infinite-dimensional family of complete metrics on \(M \setminus \Lambda\) with positive constant \(Q\)-curvature.

Let \((M^n, g)\) be a closed Riemannian manifold with dimension \(n \geq 5\) and let \(\Lambda\) be a finite set of points. Under two natural conditions on the metric, namely, nondegeneracy and vanishing of the Weyl tensor \(W_g\) up to some suitable order, we prove existence of a one-parameter family of solutions to \((\mathcal{Q})\). We say that \(g\) is nondegenerate if the linearized operator \(L_g : C^{4,\alpha}(M) \rightarrow C^{0,\alpha}(M)\) is invertible for some \(\alpha \in (0,1)\). Here

\[ L_gu = Pu - \frac{n(n^2 - 4)(n + 4)}{16}u, \]

that is, \(L_g\) is the linearization of \(H_g\) at the function 1. Let \(\mathcal{M}\) be the space of smooth metrics in \(M\). For \(5 \leq n \leq 7\), we define \(\mathcal{W}_\Lambda^d = \{g \in \mathcal{M} : g\) is nondegenerate\}. For \(n \geq 8\), and \(d \in \mathbb{N}\), we set

\[ \mathcal{W}_\Lambda^d = \left\{ g \in \mathcal{M} : \nabla_g^j W_g(p) = 0 \text{ for all } p \in \Lambda \text{ and } j = 0, \ldots, d, \right\}. \]

and \(g\) is nondegenerate.

Our main theorem extends the results in \([13,50]\) to the context of constant \(Q\)-curvature metrics

Theorem 1.1. Let \((M^n, g)\) be a closed Riemannian manifold with dimension \(n \geq 5\) and let \(\Lambda\) be a finite set of points. Assume that \(Q_g = n(n^2 - 4)/8\) and \(g \in \mathcal{W}_\Lambda^d\) for \(d = [(n-8)/2]\). Then, there exist a constant \(\varepsilon_0 > 0\) and a one-parameter family of complete metrics \(\{\tilde{g}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}\) on \(M \setminus \Lambda\), such that each \(\tilde{g}_\varepsilon\) is conformal to \(g\) with \(Q\)-curvature equal to \(n(n^2 - 4)/8\). Near each singular point \(p \in \Lambda\) the metric \(\tilde{g}_\varepsilon\) is asymptotic to one of the Delaunay metrics described in Section 2. Moreover, \(\tilde{g}_\varepsilon \rightarrow g\) uniformly on compact sets of \(M \setminus \Lambda\) as \(\varepsilon \rightarrow 0\).

In contrast with Theorem B, the assumption that the scalar curvature is nonnegative is not necessary. This set of hypotheses allows one to recover a maximum principle for the Paneitz operator, and it was first introduced by M. Gursky and A. Malchiodi \([21]\). Our techniques are point-fixed based and do not rely on such a principle. This condition was also used in \([3]\) to prove compactness of solutions to \((\mathcal{Q})\) in the spherical setting.

The vanishing condition on the Weyl tensor also appears in the scalar curvature setting. Indeed, in \([50]\), a similar hypothesis was used to prove existence of solutions to the Yamabe problem with finitely many singularities, which generalizes the results due to A. Byde \([13]\) in the conformally flat setting, at least around the singularities (see also \([48]\)). Both results were extended for the case of fully nonlinear equations and strongly coupled systems \([14,51]\). The vanishing condition is also one of the essential pieces in the program proposed by R. M. Schoen to establish compactness results for the Yamabe equation \([30,49]\), and it comes naturally from the Weyl vanishing conjecture \([49]\); this was proved to be true for \(3 \leq n \leq 24\) \([30,34,35,38]\), and disproved otherwise \([39]\).

The study of compactness for solutions to the \(Q\)-curvature equation is still under development. Recently, G. Li \([32]\) obtained compactness of solutions for \(5 \leq n \leq 7\) assuming the conditions in \([21]\) and without any hypothesis on the Weyl tensor. Independently, this was also proved by Y. Y. Li and J. Xiong \([33]\) for \(5 \leq n \leq 9\) assuming the Weyl tensor to vanish at singular points of a sequence of blowing-up solutions and the weaker hypothesis in \([23]\). In addition, since the Weyl tensor and its covariant derivatives appear in the expansion of the Green function for Paneitz operator, they observed that for \(n \geq 8\) its vanishing at the singular points is required. In our situation, a similar phenomenon happens. When \(5 \leq n \leq 7\), we do not need any assumption on the Weyl tensor in Theorem 1.1. Nevertheless, for \(n \geq 8\) we assume that it vanishes up to order
First, in the exterior analysis, we use the nondegeneracy condition to study the Weyl tensor up to some orders. To this end, the hypothesis on vanishing the Weyl tensor is constructed a solution inside this ball being uniformly bounded and satisfying proper estimates around each isolated singularity. The interior analysis then consists of constructing auxiliary functions and insert the missing variables into the problem, making it solvable.

We encounter several difficulties in carrying out the strategy outlined in the previous paragraph. First, one needs to control the convergence rate of the Paneitz operator to the flat bi-Laplacian in conformal normal coordinates. In dimension $n = 5, 6, 7$, this convergence comes naturally from local expansions in conformal normal coordinates, but in dimension $n \geq 8$, we only obtain reasonable convergence rates after constructing an auxiliary function. Gluing the interior and exterior solutions on the boundary up to third order is also problematic because of the lack of geometric Jacobi fields on the kernel of the linearized operator in the low-frequency mode, making the system involving the gluing variables under-determined. To circumvent this problem, we construct auxiliary functions and insert the missing variables into the problem, making it solvable.
For the sake of clarity, we provide an intuitive picture of our method. Figure 1 shows summands for our gluing construction before any modifications, namely the manifold minus a small geodesic ball and a punctured ball with the flat metric.

\[ M \setminus B_r(p) + B_r(p) \setminus \{p\} \]

**Figure 1.** This figure shows the summands in their original states.

Figure 2 shows the summands after we have conformally deformed each of them. We use the Green’s function of the flat bi-Laplacian on the left and the Delaunay metric on the right.

**Figure 2.** This figure shows the summands after conformal modification.

Now we explain the summands in our gluing procedure more precisely. The first summand is the underlying nondegenerate manifold with a point removed \( M \setminus B_r(p) \), where \( p \in \Lambda \) satisfies the Weyl vanishing hypotheses \( \mathcal{W}_3 \). The second summand is one end of a Delaunay metric. The Delaunay metric are all possible constant \( Q \)-curvature metrics on the cylinder \( \mathbb{R} \times S^{n-1} \) (or, equivalently, on \( S^n \setminus \{p_1, p_2\} \)) and we describe them in detail in Section 2. Most importantly, one can characterize a Delaunay metric by choosing a necksize parameter \( \varepsilon \) and when this necksize is small the conformal factor of the Delaunay metric is close to the Green’s function for the flat bi-Laplacian. Thus, after conformally modifying the given metric \( g \) on \( M \) by a multiple of the Green’s function whose pole is the gluing base point \( p \), we see that the two summands are geometrically sufficiently close, making it possible for us to glue them together. After describing the Delaunay metrics, we introduce some appropriate function spaces in Section 3. Next, we construct and study our model operators in Section 4. Then, we construct our solution operators for the geometric problem as perturbations of these model operators. In this same section, we also prove that the associated Navier-to-Neumann operator is an isomorphism. In Section 5 we complete our interior analysis, using a fixed-point argument and the hypothesis on the Weyl tensor to obtain a solution to \((Q)\) in the interior of a ball of small radius around a singular point with fine estimates. In Section 6 we use the nondegeneracy hypothesis to find a solution to \((Q)\) in the exterior of this ball. In Section 7 we use the Navier-to-Neumann operator to match the interior and exterior solutions on the boundary of the small ball around a unique singular point. At the beginning of this section we explain why matching the interior and exterior solutions to third order across their common boundaries suffices to give us a
smooth, global solution. In Section 8, we discuss the necessary modifications to extend the gluing
construction of the last section for the case of multiple points; this yields the proof of Theorem 1.1.

Acknowledgements: We would like to thank the referees for a close read of and valuable
comments regarding an earlier version of this manuscript. Additionally the first four authors
would like to thank the last author for performing above and beyond the call of duty.

2. Delaunay-type solutions

We introduce the (logarithmic) cylindrical coordinates. We present the Delaunay-type solutions
together with a classification result and some estimates. Finally, we consider deformations of the
Delaunay-type solutions. These solutions will be used as approximating solutions to the interior
problem. In what follows we will use dots to denote derivatives with respect to \( t \).

If \( g = u^{n−4} \delta \) is a complete metric in \((\mathbb{R}^n \setminus \{0\}, \delta)\) with constant \( Q \)-curvature \( Q_g = n(n^2 − 4)/8 \),
where \( \delta \) is the Euclidean metric in \( \mathbb{R}^n \), then the function \( u \) satisfies the following equation
\[
\Delta^2 u = \frac{n(n^2 − 4)(n − 4)}{16} u^{\frac{n+4}{n-4}}.
\] (2.1)

It was proved by C.-S. Lin [36] that solutions to (2.1) in \( \mathbb{R}^n \setminus \{0\} \) with a nonremovable singularity
at the origin are radially symmetric. If we consider the conformal diffeomorphism
\[
\Phi : (\mathbb{R} \times S^{n−1}, g_{cyl} := \text{dt}^2 + \text{d}\theta^2) \rightarrow (\mathbb{R}^n \setminus \{0\}, \delta),
\] (2.2)
given by \( \Phi(t, \theta) = e^{-t} \theta \), we find \( \Phi^* \delta = e^{-2t} g_{cyl} \). Also, if \( g = u^{n−4} \delta \), then \( \Phi^* g = v^{n−4} g_{cyl} \), where
\[
v(t) = e^{\frac{4-n}{2} t} u(e^{-t} \theta) = |x|^{\frac{n-4}{2}} u(x).
\]

In this logarithm cylindrical coordinates (also known as Endem-Fowler coordinates), (2.1) is
equivalent to
\[
P_{cyl} v = \frac{n(n-4)(n^2-4)}{16} v^{\frac{n+4}{n-4}},
\]
where
\[
P_{cyl} = \partial_t^4 + \Delta_{S^{n-1}}^2 + 2\Delta_{S^{n-1}}^2 \partial_t^2 - \frac{n^2-4n+8}{2} \partial_t^2 - \frac{n(n-4)}{2} \Delta_{S^{n-1}} + \frac{n^2(n^2-4)^2}{16}.
\]
Restricting to radial functions reduces the last PDE to the ODE below
\[
\ddot{v} - \frac{n^2-4n+8}{2} \dot{v} + \frac{n^2(n-4)^2}{16} v - \frac{n(n-4)(n^2-4)}{16} \frac{v^{\frac{n+4}{n-4}}}{v^{\frac{n+4}{n-4}}} = 0.
\] (2.3)

Following the notation of [18], we write this ODE as
\[
\ddot{v} - A \dot{v} = f(v),
\]
with \( A^2 = 4B + 4(n-2)^2 > 4B \) and
\[
f(v) = Cv^{\frac{n+4}{n-4}} - Bv = \frac{n(n-4)}{16} v \left((n^2-4)v^{\frac{8}{n-4}} - n(n-4)\right),
\] (2.4)
where
\[
A = \frac{n(n-4)+8}{2}, \quad B = \frac{n^2(n-4)^2}{16} \quad \text{and} \quad C = \frac{n(n-4)(n^2-4)}{16}.
\] (2.5)
2.1. Classification results and some estimates. Now we will present some recent results concerning the classification of solutions to (2.3). Notice that one can find a first integral for this ODE defined as

\[ H(x, y, z, w) = -xz + \frac{1}{2} y^2 + \frac{n^2 - 4n + 8}{4} z^2 - n^2(n - 4)^2 \frac{w^2}{32} + \frac{(n - 4)^2(n^2 - 4)}{32} w n - 4. \]  

This Hamiltonian energy is constant along solutions to (2.3), \( H_\varepsilon = H(\dot{v}_\varepsilon, \ddot{v}_\varepsilon, \dot{v}_\varepsilon, v_\varepsilon) = \text{constant}. \) Let

\[ v_{\text{sph}}(t) = (\cosh t)^{\frac{4-n}{2}} \quad \text{and} \quad v_{\text{cyl}}(t) = \left( \frac{n(n-4)}{n^2 - 4} \right)^{\frac{n}{4}} \]

be the spherical and cylindrical solutions, whose energies are

\[ H_{\text{sph}} = 0 \quad \text{and} \quad H_{\text{cyl}} = -\frac{(n - 4)(n^2 - 4)}{8} \left( \frac{n(n-4)}{n^2 - 4} \right)^{\frac{n}{4}} < 0, \]

respectively. It was observed in [47] (see also [46]) that \( H_\varepsilon < 0 \) and it is strictly decreasing function of \( \varepsilon < 0. \)

In a recent paper, R. Frank and T. König [18] provided a full characterization to the solutions to the ODE (2.3), as stated below

**Theorem C** ([18]). Let \( v \in C^4(\mathbb{R}) \) be a solution to (2.3). Then \( \inf |v| \leq v_{\text{cyl}}, \) with equality if and only if \( v \) is a non-zero constant. Moreover, one of the following three alternatives holds:

(a) \( v \equiv \pm v_{\text{cyl}} \) or \( v \equiv 0. \)

(b) \( v(t) = \cosh^{\frac{4-n}{2}}(t-T) \) for some \( T \in \mathbb{R}. \)

(c) Let \( \varepsilon \in (0, v_{\text{cyl}}). \) Then there is a unique (up to translations) bounded solution \( v_\varepsilon \in C^4(\mathbb{R}) \) of (2.3) with minimal value \( \varepsilon. \) This solution is periodic, has a unique local maximum and minimum per period and is symmetric with respect to its local extrema.

We call \( v_\varepsilon \) the Delaunay-type solution with necksize \( \varepsilon, \) and denote its minimal period by \( T_\varepsilon. \) As \( \varepsilon \to 0 \) one has \( T_\varepsilon \to \infty \) and \( v_\varepsilon(t + T_\varepsilon/2) \to \cosh^{\frac{4-n}{2}}(t) \) uniformly on compact sets. Also, for all \( \varepsilon \in (0, v_{\text{cyl}}), \) we have that \( 0 < v_\varepsilon < 1, \) see [47] and [2]. We observe that \( \varepsilon = v_\varepsilon(0) = \min v_\varepsilon, \) \( \dot{v}_\varepsilon(0) = 0 \) and \( \ddot{v}_\varepsilon(0) \geq 0. \) We write the corresponding solution to (2.1) as \( u_\varepsilon(x) = |x|^{\frac{4-n}{2}} v_\varepsilon(- \log |x|). \)

Let then \( \lambda, \mu \) and \( \omega \) be real numbers such that \( \lambda + \mu = A, \lambda \mu = \omega \) and \( A^2 \geq 4\omega. \) We see that \( \lambda \) and \( \mu \) satisfy \( \lambda^2 - A\lambda + \omega = 0. \) Thus

\[ \lambda = \frac{A}{2} - \frac{1}{2} \sqrt{A^2 - 4\omega} \quad \text{and} \quad \mu = \frac{A}{2} + \frac{1}{2} \sqrt{A^2 - 4\omega}, \]

implying that the equation (2.3) is equivalent to

\[
\begin{aligned}
\ddot{v} - \lambda v &= w \\
\dot{w} - \mu w &= f(v) + \omega v,
\end{aligned}
\]

where \( f \) is defined in (2.4). We can choose \( \omega \) such that

\[ B - \frac{n(n+4)(n^2 - 4)}{16} \varepsilon^{\frac{8}{n-4}} < \omega < \frac{A^2}{4} \]

and this implies that \( f'(v) + \omega > 0. \)

The proof of the next lemma is similar of [7, Lemma 5] and we include its proof for completeness.

**Lemma 2.1.** Let \( \lambda \leq \gamma \leq \mu. \) If \( v \) is a positive solution to (2.3), then

\[ \text{sign}(\ddot{v}(t) - \gamma \dot{v}(t)) = -\text{sign}(\dot{v}(t)) \]

for all \( t \in \mathbb{R}, \) where \( \text{sign}(0) = 0. \)
Proof. Let \( t_0 \in \mathbb{R} \). If \( \dot{v}(t_0) = 0 \), it follows by [18, Corollary 5] that \( v(t_0 + t) = v(t_0 - t) \) for all \( t \in \mathbb{R} \). Thus \( \dot{v}(t_0) = 0 \) the result follows. Then assume that \( \dot{v}(t_0) > 0 \). In this case, since \( v \) is bounded there exist real numbers \( t_1 \) and \( t_2 \) such that \( \dot{v}(t_1) = \dot{v}(t_2) = 0 \) and \( \dot{v}(t) > 0 \) for all \( t \in (t_1, t_2) \). Again we have \( \dot{v}(t_1) = \dot{v}(t_2) = 0 \). Define \( y = \dot{v} - \gamma \dot{v} \). Thus, using (2.8) we get
\[
\begin{align*}
\ddot{y} &= \dot{v} - \gamma \dot{v} = \ddot{v} - \lambda \dot{v} + (\lambda - \gamma) \dot{v} = \ddot{w} + (\lambda - \gamma) \dot{v} \\
&= \mu \dot{w} + f(v) + \omega \nu (\lambda - \gamma) \dot{v}
\end{align*}
\]
and
\[
\begin{align*}
\ddot{y} &= \mu \dot{w} + \dot{v}(f'(v) + \omega) + \lambda - \gamma) \dot{v} \\
&= \mu (\ddot{v} - \lambda \dot{v}) + \dot{v}(f'(v) + \omega) + \lambda - \gamma) \dot{v} \\
&= \mu (y - \gamma \dot{v} - \lambda \dot{v}) + \dot{v}(f'(v) + \omega) + \lambda - \gamma) (y + \gamma \dot{v}) \\
&= (A - \gamma) y + \dot{v}(f'(v) + \omega + (\gamma - \lambda)(\mu - \gamma)).
\end{align*}
\]
This implies that
\[
\begin{align*}
\{ \ddot{y} - (A - \gamma) y &= \ddot{v}(f'(v) + \omega + (\gamma - \lambda)(\mu - \gamma)) \geq 0 \\
y(t_1) &= y(t_2) = 0.
\end{align*}
\]
Therefore, by the strong maximum principle, \( y < 0 \) in \( (t_1, t_2) \). In particular, \( y(t_0) < 0 \).

If \( \dot{v}(t_0) < 0 \), the proof is similar. \( \square \)

Using the Lemma 2.1 and that the energy (2.6) is negative along the Delaunay solution we get that
\[
\left( \frac{A}{2} - \lambda \right) v^2 + \frac{1}{2} v^2 \leq \dot{v}(\ddot{v} - \lambda \dot{v}) + \frac{B}{2} v^2 - \frac{n - 4}{2n} C v_{\frac{2n}{n-4}} \leq \frac{B}{2} v^2, \tag{2.9}
\]
with \( A/2 - \lambda > 0 \), see (2.7).

**Proposition 2.2.** For any \( \varepsilon \in (0, v_{\text{cyl}}) \) and for all \( t \geq 0 \) the Delaunay-type solution \( v_\varepsilon \) satisfies the estimates
\[
\left| v_\varepsilon(t) - \alpha_\varepsilon \cosh \left( \frac{n - 4}{2} t \right) - \beta_\varepsilon \cosh \left( \frac{n}{2} t \right) \right| \leq c_n \varepsilon \frac{n + 4}{n + 4} \frac{n + 4}{2},
\]
\[
\left| \dot{v}_\varepsilon(t) - \frac{n - 4}{2} \alpha_\varepsilon \sinh \left( \frac{n - 4}{2} t \right) - \frac{n}{2} \beta_\varepsilon \sinh \left( \frac{n}{2} t \right) \right| \leq c_n \varepsilon \frac{n + 4}{n + 4} \frac{n + 4}{2},
\]
\[
\left| \ddot{v}_\varepsilon(t) - \left( \frac{n - 4}{2} \right)^2 \alpha_\varepsilon \cosh \left( \frac{n - 4}{2} t \right) - \left( \frac{n}{2} \right)^2 \beta_\varepsilon \cosh \left( \frac{n}{2} t \right) \right| \leq c_n \varepsilon \frac{n + 4}{n + 4} \frac{n + 4}{2},
\]
\[
\left| \dddot{v}_\varepsilon(t) - \left( \frac{n - 4}{2} \right)^3 \alpha_\varepsilon \sinh \left( \frac{n - 4}{2} t \right) - \left( \frac{n}{2} \right)^3 \beta_\varepsilon \sinh \left( \frac{n}{2} t \right) \right| \leq c_n \varepsilon \frac{n + 4}{n + 4} \frac{n + 4}{2},
\]
\[
\left| \ddddot{v}_\varepsilon(t) - \left( \frac{n - 4}{2} \right)^4 \alpha_\varepsilon \cosh \left( \frac{n - 4}{2} t \right) - \left( \frac{n}{2} \right)^4 \beta_\varepsilon \cosh \left( \frac{n}{2} t \right) \right| \leq c_n \varepsilon \frac{n + 4}{n + 4} \frac{n + 4}{2},
\]
for all \( t \geq 0 \), where
\[
\frac{n}{2(n - 2)} \varepsilon < \alpha_\varepsilon := \frac{1}{2(n - 2)} \left( \frac{n^2}{4} \varepsilon - \dddot{v}_\varepsilon(0) \right) < \frac{n}{4} \varepsilon \tag{2.10}
\]
\[
- \frac{n - 4}{4} \varepsilon < \beta_\varepsilon := \frac{1}{2(n - 2)} \left( \dddot{v}_\varepsilon(0) - \frac{(n - 4)^2}{4} \varepsilon \right) < - \frac{(n - 4)(n + 2)}{8n} \frac{\varepsilon}{n - 4}, \tag{2.11}
\]
for some positive constants \( C_n \) and \( c_n \) which depends only on \( n \).
Proof. First, note that by (2.9) we have
\[ |\tilde{v}_\varepsilon(0)| < \frac{n(n-4)}{4}\varepsilon \]  \hspace{1cm} (2.12)
and then we obtain (2.10). First, note that if \( v_\varepsilon \) is any solution to (2.3), then
\[ e^{\frac{n}{2}t} \left( e^{-nt} \left( e^{(n-4)t} \left( e^{4-n\varepsilon} v_\varepsilon \right) \right) \right)' = Cv_\varepsilon^{\frac{n+4}{4}}. \]
Thus
\[ v_\varepsilon(t) = \alpha_\varepsilon \cosh \frac{n-4}{2} t + \beta_\varepsilon \cosh \frac{n}{2} t \]
\[ + \frac{n(n-4)(n^2-4)}{16} e^{\frac{n-4}{2} t} \int_0^t e^{(4-n)s} \int_0^s e^{-2\varepsilon \int_0^s e^{ny} \int_0^y e^{-\frac{n}{4} r} v_\varepsilon^{\frac{n+4}{4}} dr dy dx ds}. \]
Using that \( v_\varepsilon \geq \varepsilon \) we obtain
\[ v_\varepsilon(t) \geq \alpha_\varepsilon \cosh \frac{n-4}{2} t + \beta_\varepsilon \cosh \frac{n}{2} t \]
\[ + \frac{n(n-4)(n^2-4)}{16} \varepsilon^{\frac{n+4}{4}} (\cosh \frac{n}{2} t - \frac{2}{n^2(n-2)} \cosh \frac{n-4}{2} t + \frac{16}{n^2(n-4)^2}), \]
for all \( t \geq 0 \). Since \( \alpha_\varepsilon > 0 \) we conclude the upper bound in (2.10) and (2.11). Using (2.12) we obtain the lower bound of (2.11). Also, using (2.13) we get
\[ 0 < v_\varepsilon(t) < \varepsilon \cosh \left( \frac{n-4}{2} t \right) \leq \varepsilon e^{\frac{n-4}{2} |t|} \]  \hspace{1cm} (2.14)
This and (2.13) implies the estimates. \( \square \)

For the next result we will write \( f = O^{(l)}(K|x|^{k}) \) to mean \( |\partial^i f| \leq c_n K|x|^{k-i} \), for all \( i \in \{1, \ldots, l\} \) and for some positive constant \( c_n \) which depends only on \( n \). Here \( K > 0 \) is a constant.

**Corollary 2.3.** For any \( \varepsilon \in (0, v_{cyl}) \) and any \( x \in \mathbb{R}^n \setminus \{0\} \) with \( |x| \leq 1 \), the Delaunay-type solution \( u_\varepsilon(x) \) satisfies the estimates
\[ u_\varepsilon(x) = \frac{\alpha_\varepsilon}{2} (1 + |x|^{4-n}) + \frac{\beta_\varepsilon}{2} (|x|^2 + |x|^{2-n}) + O^{(4)}(\varepsilon^{\frac{n+4}{n-4}} |x|^{-n}), \]
where \( \alpha_\varepsilon \) and \( \beta_\varepsilon \) are given by (2.10).

**Proof.** Remember that \( u_\varepsilon(x) = |x|^{\frac{4-n}{2}} v_\varepsilon(-\log |x|) \). Besides, since \( t = -\log |x| \), we get
\[ |x|^{\frac{4-n}{2}} \cosh \left( \frac{n-4}{2} t \right) = \frac{1}{2} (1 + |x|^{4-n}), \]
\[ |x|^{\frac{4-n}{2}} \cosh \left( \frac{n}{2} t \right) = \frac{1}{2} (|x|^2 + |x|^{2-n}). \]
From this we get the expansion for \( u_\varepsilon(x) \), since \( 0 < |x| \leq 1 \) implies that \( t = -\log |x| > 0 \). Also we have
\[ |x|^{\frac{4-n}{2}} \sinh \left( \frac{n-4}{2} t \right) = \frac{1}{2} (|x|^{4-n} - 1) \]
and
\[ |x| \partial_t u_\varepsilon(x) = \frac{4-n}{2} u_\varepsilon(x) - |x|^{\frac{4-n}{2}} \tilde{v}_\varepsilon(-\log |x|). \]  \hspace{1cm} (2.15)
Using the estimates of the derivative of $v_\varepsilon$ given by Proposition 2.2 we obtain the estimates for the radial derivatives of $u_\varepsilon$.

\section*{2.2. Variations of Delaunay-type solutions}

Applying rigid motions to the solutions to the PDE (2.1), we can obtain some important families of solutions. We focus in two types of variations: translations along the Delaunay axis and translations at infinity.

We can obtain the first family of solutions by simply noting that if $u$ is a solution to (2.1), then $R^{\frac{4-n}{n}} u(R^{-1}x)$, for $R > 0$, also solves this equation. Applying this transformation to a Delaunay-type solution yields the family

$$\mathbb{R}^+ \ni R \mapsto |x|^{\frac{4-n}{2}} v_\varepsilon(-\log |x| + \log R).$$

To construct the second family, define the standard inversion $I : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ given by $I(x) = x|x|^{-2}$. It is well known that $I^\ast \delta = |x|^{-4}\delta$. Hence, if $g = u^{\frac{4-n}{n}}\delta$, then $I^\ast g = (|x|^{4-n} u \circ I)\frac{n}{n-4}\delta$. Recall that the Kelvin transform $K$ is given by

$$K(u)(x) = |x|^{4-n}u(x|x|^{-2}).$$

By (1.2), we have

$$K(u) - \frac{n+4}{n-4} P_\delta(K(u)) = (u \circ I)^{-\frac{n+4}{n-4}} P_\delta u \circ I,$$

$$\Delta^2(K(u)) = K(|x|^8 \Delta^2 u).$$

Now suppose that $u$ is a solution to (2.1), then

$$\Delta^2 K(u)(x) = K(|x|^8 \Delta^2 u)(x) = K(c(n)|x|^8 u^{\frac{n+4}{n-4}})$$

$$= c(n)|x|^{-4-n}(u \circ I(x))^\frac{n+4}{n-4} = c(n)(K(u)(x))^\frac{n+4}{n-4}.$$ 

Therefore, $K(u)$ still is a solution to (2.1). The function $u(x-a)$ also solves (2.1), but with a singularity in $a$ in $\mathbb{R}^n$ instead of the origin.

For the purpose of this paper, we consider the family of solutions

$$u_{\varepsilon,R,a}(x) = K(K(u_\varepsilon)(-a))(x) = |x-a|x^2|\varepsilon|^{\frac{4-n}{2}} v_\varepsilon \left( -\log |x| + \log \frac{x}{|x|} - a|x| \right) + \log R. \quad (2.16)$$

This function has a singular point when $x = 0$ and $x = a/|a|$. In the particular case when $a = 0$, as a direct consequence of Corollary 2.3 the following asymptotic expansion

$$u_{\varepsilon,R}(x) = \frac{\alpha_\varepsilon}{2} \left( R^{\frac{4-n}{2}} + R^{\frac{n+4}{2}} |x|^{4-n} \right) + \frac{\beta_\varepsilon}{2} (R^{\frac{n}{2}} |x|^2 + R^{\frac{n+4}{2}} |x|^{2-n}) + O(R^{\frac{n+4}{2}} |x|^{-n}), \quad (2.17)$$

where $\alpha_\varepsilon$ and $\beta_\varepsilon$ are given by (2.10).

\begin{proposition}
There exists a constant $r_0 \in (0, 1)$, such that for any $x$ and $a$ in $\mathbb{R}^n$ with $|x| \leq 1$, $|a|/|x| < r_0$ and $\varepsilon \in (0, v_\varepsilon)$ the solution $u_{\varepsilon,R,a}$ satisfies the estimate

$$u_{\varepsilon,R,a}(x) = u_{\varepsilon,R}(x) + ((n-4)u_{\varepsilon,R}(x) + |x| \partial_x u_{\varepsilon}(x)) \langle a, x \rangle + O(|a|^2 |x|^{\frac{4-n}{2}}), \quad (2.18)$$

and if $R \leq |x|$ the estimate

$$u_{\varepsilon,R,a}(x) = u_{\varepsilon,R}(x) + ((n-4)u_{\varepsilon,R}(x) + |x| \partial_x u_{\varepsilon}(x)) \langle a, x \rangle + O(|a|^2 R^{\frac{4-n}{2}} |x|^2). \quad (2.19)$$

\end{proposition}
Proof. First, using the Taylor expansion we get
\[ |x - a|^\frac{4-n}{2} = |x|^{\frac{4-n}{2}} + \frac{n-4}{2} \langle a, x \rangle |x|^{\frac{4-n}{2}} + O(|a|^2 |x|^{\frac{8-n}{2}}) \] (2.20)
and
\[ \log \left| \frac{x}{|x|} - a \right| = -\langle a, x \rangle + O(|a|^2 |x|^2), \] (2.21)
for $|a||x| < r_0$ and some $r_0 \in (0, 1)$. Also
\[
v_\varepsilon \left( -\log |x| + \log \left| \frac{x}{|x|} - a \right| + \log R \right) = v_\varepsilon (-\log |x| + \log R)
+ \dot{v}_\varepsilon (-\log |x| + \log R) \log \left| \frac{x}{|x|} - a \right| + \ddot{v}_\varepsilon (-\log |x| + \log R + t_{a,x}) \left( \log \left| \frac{x}{|x|} - a \right| \right)^2
= v_\varepsilon (-\log |x| + \log R) - \dot{v}_\varepsilon (-\log |x| + \log R) \langle a, x \rangle + \ddot{v}_\varepsilon (-\log |x| + \log R) O(|a|^2 |x|^2)
+ \dddot{v}_\varepsilon (-\log |x| + \log R + t_{a,x}) O(|a|^2 |x|^2),
\]
for some $t_{a,x} \in \mathbb{R}$ with $0 < |t_{a,x}| < |\log \left| \frac{x}{|x|} - a \right| |$. Note that $t_{a,x} \to 0$ when $|a||x| \to 0$.

Note that, using (2.9) and (2.12) it follows that
\[ |\dot{v}_\varepsilon| \leq c_n v_\varepsilon \quad \text{and} \quad |\dddot{v}_\varepsilon| \leq c_n v_\varepsilon \] (2.22)
for some positive constant $c_n$ that depends only on $n$. Thus, by (2.16) and (2.20) we obtain
\[
u_{\varepsilon,R,a}(x) = u_{\varepsilon,R}(x) + \left( \frac{n-4}{2} u_{\varepsilon,R}(x) - |x|^{\frac{4-n}{2}} v_\varepsilon (-\log |x| + \log R) \right) \langle a, x \rangle + O(|a|^2 |x|^{\frac{8-n}{2}}).
\]
By (2.15) this implies (2.18). Moreover, by (2.9) and (2.14), since $-\log |x| + \log R \leq 0$ for $R \leq |x|$, then we get that $v_\varepsilon (-\log |x| + \log R)$ and $v_\varepsilon (-\log |x| + \log R + t_{a,x})$ are bounded by $c_n \varepsilon R^{\frac{4-n}{2}} |x|^{\frac{n-4}{2}}$, for some positive constant $c_n$ which depends only on $n$. Therefore, we get (2.19). \qed

3. Function spaces

In this section, we define some function spaces that will be useful in this work. The first one is the weighted H"older spaces in the punctured ball. It is well-established in the literature that these spaces are the most convenient spaces to define the linearized operator. The second one appears so naturally in our results that it is more helpful to put its definition here. Finally, the third one is the weighted H"older spaces in which the exterior analysis will be carried out. These are the same weighted spaces as in [29,41].

**Definition 3.1.** For each $k \in \mathbb{N}$, $r > 0$, $0 < \alpha < 1$ and $\sigma \in (0,r/2)$, let $u \in C^k(B_r(0)\setminus\{0\})$, set
\[
\|u\|_{(k,\alpha),[\sigma,2\sigma]} = \sup_{|x| \in [\sigma,2\sigma]} \left( \sum_{j=0}^{k} \sigma^j |\nabla^j u(x)| + \sigma^{k+\alpha} \sup_{|x|,|y| \in [\sigma,2\sigma]} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x - y|^\alpha} \right).
\]
Then, for any $\mu \in \mathbb{R}$, the space $C^{k,\alpha}_\mu(B_r(0)\setminus\{0\})$ is the collection of functions $u$ that are locally in $C^{k,\alpha}(B_r(0)\setminus\{0\})$ and for which the norm
\[
\|u\|_{(k,\alpha),\mu,r} = \sup_{0 < \sigma \leq r^\frac{1}{2}} \sigma^{-\mu} \|u\|_{(k,\alpha),[\sigma,2\sigma]}
\]
is finite.
The one result about these that we shall use frequently, and without comment, is that to check if a function $u$ is an element of some $C^{0,\alpha}_{\mu}$, say, it is sufficient to check that $|u(x)| \leq C|x|^\mu$ and $|\nabla u(x)| \leq C|x|^\mu$. In particular, the function $|x|^\mu$ is in $C^{k,\alpha}_{\mu}$ for any $k, \alpha$, or $\mu$.

Notice that $C^{k,\alpha}_{\mu} \subseteq C^{l,\alpha}_{\delta}$ if $\mu \geq \delta$ and $k \geq l$, and $\|u\|_{l,\delta} \leq C\|u\|_{k,\alpha,\mu}$ for all $u \in C^{k,\alpha}_{\mu}$.

**Definition 3.2.** For each $k \in \mathbb{N}$, $0 < \alpha < 1$ and $r > 0$, let $\phi \in C^k(S^n_{r^2})$, set

$$\|\phi\|_{(k),r} := \|\phi(r)\|_{C^k(S^n_{r^2})}.$$  

Then, the space $C^k(S^n_{r^2})$ is the collection of functions $\phi \in C^k(S^n_{r^2})$ for which the norm $\|\phi\|_{(k),r}$ is finite.

Next, consider an $n$-dimensional compact Riemannian manifold $(M^n, g)$ and a coordinate system $\Psi : B_{r_1}(0) \to M$ on $M$ centered at a point $p = \Psi(0) \in M$, where $B_{r_1}(0) \subset \mathbb{R}^n$ is the ball of radius $r_1$. For $0 < r < s \leq r_1$ define $M_r := M \setminus \Psi(B_r(0))$ and $\Omega_{r,s} := \Psi(A_{r,s})$, where $A_{r,s} := \{x \in \mathbb{R}^n; r \leq |x| \leq s\}$.

**Definition 3.3.** For all $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, the space $C^k_{\nu}(M \setminus \{p\})$ is the space of functions $v \in C^k_{\text{loc}}(M \setminus \{p\})$ for which the following norm is finite

$$\|v\|_{C^k_{\nu}(M \setminus \{p\})} := \|v\|_{C^k_{\nu}(M_{r_1/2})} + \|v \circ \Psi\|_{(k),\nu,r_1},$$

where the norm $\|\cdot\|_{(k),\nu,r_1}$ is the one defined in Definition 3.1.

For all $0 < r < s \leq r_1$, we can also define the spaces $C^k_{\mu}(\Omega_{r,s})$ and $C^k_{\mu}(M_r)$ to be the space of restriction of elements of $C^k_{\mu}(M \setminus \{p\})$ to $M_r$ and $\Omega_{r,s}$, respectively. These spaces is endowed with the following norm

$$\|f\|_{C^k_{\mu}(\Omega_{r,s})} := \sup_{r \leq \sigma \leq \frac{r_1}{2}} \sigma^{-\mu} \|f \circ \Psi\|_{(k),\sigma,2\sigma},$$

and

$$\|h\|_{C^k_{\mu}(M_r)} := \|h\|_{C^k_{\mu}(M_{r_1/2})} + \|h\|_{C^k_{\mu}(\Omega_{r_1})}.$$  

Notice that these norms are independent of the extension of the functions $f$ and $h$ to $M_r$.

4. Model operators

In this section, we follow [28, Sections 11 and 13] to construct a Poisson operator for the bi-Laplacian in the Euclidean space, which will be crucial to perform the gluing construction in Section 7.

Let us set some notation. We introduce $(e_j, \lambda_j)$, the eigendata of the Laplacian on $S^{n-1}$. That is, $\Delta_{S^{n-1}} e_j + \lambda_j e_j = 0$. We assume that the eigenvalues are counted with multiplicity and that the eigenfunctions are normalized so that their $L^2$-norm is equal to 1. It is well known that $(e_j, -\lambda_j^2)$ are the eigendata of the bi-Laplacian on $S^{n-1}$, that is, $\Delta_{S^{n-1}}^2 e_j = \lambda_j^2 e_j$. Thus, for $\phi \in L^2(S^{n-1})$ we use the following decomposition

$$\phi(\theta) = \sum_{j=0}^{\infty} \phi_j e_j(\theta).$$

Define the orthogonal projections

$$\pi', \pi'' : L^2(S^{n-1}) \to L^2(S^{n-1})$$
Moreover, if
\[ \phi \]

Proposition 4.1. Let
\[ \pi''(C^{4,\alpha}(\mathbb{S}^{n-1})) := \pi''(C^{4,\alpha}(\mathbb{S}^{n-1}))) \]

4.1. Interior Poisson operator. First, we can prove the existence of the Poisson operator for
the interior of the punctured unit ball.

Proposition 4.1. Let
\[ 0 < \alpha < 1 \]

Moreover, if \( \pi''(\phi_2) = 0 \), then
\[ \mathcal{P}_1(0, \phi_2) = |x|^2 \left( \frac{\phi_2(0)}{2n} + \frac{1}{6n-4} \sum_{j=1}^{n} (\phi_2)_j x_j \right). \]

Proof. By linearity, we reduce the proof to two cases.

Case 1: \( \phi_2 \equiv 0 \).

By [28, Proposition 11.25] and [44, Lemma 6.2], there exists a smooth function
\( v_{\phi_0} \in C^{2,\alpha}_2(B_1(0)\setminus\{0\}) \) satisfying the following
\[
\begin{cases}
\Delta v_{\phi_0} = 0 & \text{in } B_1(0)\setminus\{0\} \\
\pi''(v_{\phi_0}) = 0 & \text{on } \partial B_1.
\end{cases}
\]

Thus, we define \( \mathcal{P}_1(\phi_0, 0) := v_{\phi_0} \).

Case 2: \( \phi_0 \equiv 0 \). This case is divided into two steps.

Step 1: \( \pi''(\phi_2) = 0 \).

By a simple calculation we see that the function in (4.3) satisfies (4.2) and
\[ \sup_{B_1\setminus\{0\}} |x|^{-2} \mathcal{P}_1(0, \phi_2) \leq c \|\phi_2\|_{C^{4,\alpha}(\mathbb{S}^{n-1})} \]
for some constant \( c > 0 \) independent of \( \phi_2 \).

Step 2: \( \pi'(\phi_2) = 0 \).

Initially, by linearity, we may assume that \( \|\phi_2\|_{C^{4,\alpha}(\mathbb{S}^{n-1})} = 1 \). Hence the solution to the problem
(4.2), denoted by \( v_0 \), can be obtained as the limit \( \rho \to 0 \) of solutions \( v_\rho \) to
\[
\begin{cases}
\Delta^2 v_\rho = 0 & \text{in } B_1 \setminus B_\rho \\
v_\rho = 0 & \text{on } \partial B_1 \\
\Delta v_\rho = 0 & \text{on } \partial B_1 \\
\Delta v_\rho = v_\rho = 0 & \text{on } \partial B_\rho.
\end{cases}
\]

\[ \pi'(\phi) = \phi - \pi'(\phi). \]
Claim 1: There exists a constant $c > 0$, independent of $\varphi$ and $\rho$, such that
\[
\sup_{B_1 \setminus B_\rho} |x|^{-2} |v_\rho| \leq c.
\]

Arguing by contradiction, there would exist a sequence of numbers $\{\rho_i\}$ and functions $v_{\rho_i}$ solving (4.4) for $\rho_i$ with $\sup_{B_1 \setminus B_{\rho_i}} |x|^{-2} |v_{\rho_i}| \to \infty$ as $i \to \infty$. For each $r \in (\rho_i, 1)$ the function $v_{\rho_i}(r \cdot)$ is $L^2$-orthogonal to $e_0, \ldots, e_n$ on $\mathbb{S}^{n-1}$. Now choose $(r_i, \theta_i) \in (\rho_i, 1) \times \mathbb{S}^{n-1}$ such that
\[
A_i := \sup_{(\rho_i, 1) \times \mathbb{S}^{n-1}} r^{-2} |v_{\rho_i}(r \theta)| = r_i^{-2} |v_{\rho_i}(r_i \theta_i)| \to \infty \quad \text{as} \quad i \to \infty.
\]
For $|x| \in [\rho_i r_i^{-1}, r_i^{-1}]$, define
\[
\tilde{v}_{\rho_i}(x) = r_i^{-2} A_i^{-1} v_{\rho_i}(r_i x)
\]
which satisfies
\[
\sup_{(\rho_i r_i^{-1}, r_i^{-1}) \times \mathbb{S}^{n-1}} r^{-2} |\tilde{v}_{\rho_i}(r \theta)| = |\tilde{v}_{\rho_i}(\theta_i)| = 1. \tag{4.5}
\]

Arguing as in the proof of Lemma 5.8, in page 23, we conclude that the sequences $\rho_i r_i^{-1}$ and $r_i^{-1}$ remain bounded away from 1. Up to a subsequence, we can assume that the sequence $\rho_i r_i^{-1} \to \tau_1 \in [0, 1)$ and $r_i^{-1} \to \tau_2 \in (1, +\infty]$. Using (4.5) and Schauder estimates like in [19], by Arzelà–Ascoli Theorem we can suppose that $\tilde{v}_{\rho_i}$ converges to some biharmonic function $v_{\infty} \in C^4(B_{\tau_2} \setminus B_{\tau_1})$ satisfying $v_{\infty}(\theta_{\infty}) = 1$, for some $\theta_{\infty}$. If $\tau_1 = 0$ we consider $B_0 = \{0\}$. Moreover, $v_{\infty} = \Delta v_{\infty} = 0$ on the boundary and by (4.5) we conclude that $v_{\infty}$ is not identically zero and $|v_{\infty}(x)| \leq |x|^2$. Using an analogous argument as in the end of the proof of the Lemma 5.8, we obtain a contradiction.

The estimates for the derivatives follow from Schauder’s regularity (see [19, 43]). This finishes the proof of the proposition. \hfill \Box

Using a simple scaling argument, we can also define the Poisson operator in a more general context.

Corollary 4.2. Let $0 < \alpha < 1$, $\mu \leq 2$ and $r > 0$ be a fixed constants. There exists a bounded linear operator $P_r : \pi''(C^{4,\alpha}((\mathbb{S}_r^{n-1}))) \times C^{4,\alpha}(\mathbb{S}_r^{n-1}) \to C^{4,\alpha}(B_r \setminus \{0\})$ satisfying that for each $\phi_0 \in \pi''(C^{4,\alpha}((\mathbb{S}_r^{n-1})))$ and $\phi_2 \in C^{4,\alpha}(\mathbb{S}_r^{n-1})$
\[
\begin{cases}
\Delta^2 P_r(\phi_0, \phi_2) = 0 \quad \text{in} \quad B_r(0) \setminus \{0\}, \\
\pi''(P_r(\phi_0, \phi_2)) = \phi_0 \quad \text{on} \quad \partial B_r, \\
\Delta P_r(\phi_0, \phi_2) = r^{-2} \phi_2 \quad \text{on} \quad \partial B_r.
\end{cases}
\]
Moreover, there exists a constant $C > 0$ independently of $r$ such that
\[
\|P_r(\phi_0, \phi_2)\|_{(4,\alpha),\mu,r} \leq C r^{-2} \|\phi_0, \phi_2\|_{(4,\alpha),r},
\]
and if $\pi''(\phi_2) = 0$, then
\[
P_r(0, \phi_2) = r^{-2} |x|^2 \left( \frac{(\phi_2)_0}{2n} + \frac{1}{6n-4} \sum_{j=1}^n (\phi_0)_j x_j \right).
\]

Proof. Let us consider
\[
P_r(\phi_0, \phi_2)(x) = P_1(\phi_0, \phi_2)(r^{-1} x),
\]
where $\phi_i^r(\theta) := \phi_i(r \theta)$ for $i = 0, 2$. This operator is obviously bounded and satisfies
\[
\|P_r(\phi_0^r, \phi_2^r)\|_{(4,\alpha),\mu,r} = r^{-\mu} \|P_1(\phi_0^r, \phi_2^r)\|_{(4,\alpha),\mu,1},
\]
which concludes the proof. \hfill \Box
4.2. Exterior Poisson operator. Similar to the last proposition, we prove the existence of a Poisson operator for the exterior problem. This time we denote our boundary data as \( \psi_0, \psi_2 \in C^{4,\alpha}(\mathbb{S}^{n-1}) \).

Proposition 4.3. For each \( 0 < \alpha < 1 \) there exists a bounded linear operator \( \mathcal{Q}_1 : C^{4,\alpha}(\mathbb{S}^{n-1}, \mathbb{R}^2) \to C^{4,\alpha}(\mathbb{R}^n \setminus B_1) \) such that for all \( (\psi_0, \psi_2) \in C^{4,\alpha}(\mathbb{S}^{n-1}, \mathbb{R}^2) \)

\[
\begin{align*}
\Delta^2 \mathcal{Q}_1(\psi_0, \psi_2) &= 0 \quad \text{in} \quad \mathbb{R}^n \setminus B_1 \\
\mathcal{Q}_1(\psi_0, \psi_2) &= \psi_0 \quad \text{on} \quad \partial B_1 \\
\Delta \mathcal{Q}_1(\psi_0, \psi_2) &= \psi_2 \quad \text{on} \quad \partial B_1.
\end{align*}
\]

More specifically, we have that

\[
\mathcal{Q}_1(\psi_0, \psi_2) = |x|^{2-n} \sum_{j=0}^{\infty} |x|^{-j} \left( (\psi_0)_j + \frac{(\psi_2)_j}{c_2(n, j)}(|x|^2 - 1) \right),
\]

where

\[
c_2(n, j) = \left( 4 - n + \frac{n - \sqrt{n^2 + 8 + 4\lambda_j}}{2} \right) \left( 2 + \frac{n - \sqrt{n^2 + 8 + 4\lambda_j}}{2} \right)
\]

and \( \lambda_j \) is the \( j \)th eigenvalue of \( \Delta_{\mathbb{S}^{n-1}} \).

Remark 4.4. Recall that the eigenvalues of \( \Delta_{\mathbb{S}^{n-1}} \) all have the form \( \lambda_j = l(n - 2 + l) \) for some \( l = 0, 1, 2, 3, \ldots \). In this form, the expression for \( c_2(n, j) \) simplifies to \( (4 - n - l)(2 - l) \).

Proof. Since the expression (4.7) satisfies (4.6), we need only to prove that

\[
\| \mathcal{Q}_1(\psi_0, \psi_2) \|_{4,\alpha, 4-n} \leq c \| (\psi_0, \psi_2) \|_{4,\alpha}.
\]

To prove this, we consider two cases as follows.

Case 1: \( \psi_2 \equiv 0 \).

This case is similar to the Case 1 of Proposition 4.1. See [28, Proposition 13.25].

Case 2: \( \psi_0 \equiv 0 \).

We verify directly that there exists \( c > 0 \) such that

\[
\sup_{\mathbb{R}^n \setminus B_1} |x|^{n-4} |(|x|^{4-n} - |x|^{2-n}) (\psi_2)_0| \leq c \| \psi_2 \|_{L^\infty(\mathbb{S}^{n-1})}.
\]

Notice that since \( \mathbb{S}^{n-1} \) is compact, using Morrey’s inequality, we have \( H^s(\mathbb{S}^{n-1}) \hookrightarrow C^{0,\alpha}(\mathbb{S}^{n-1}) \hookrightarrow L^\infty(\mathbb{S}^{n-1}) \), for \( s > n/2 \) and \( \alpha = 1 - \frac{n}{2s} \). Considering \( (\psi_2)_0 = 0 \), by the definition of Sobolev fractional norm on the sphere (see, for instance, [12, page 406]), there exists \( c > 0 \) independently of \( r > 0 \), such that

\[
\sup_{\mathbb{S}^{n-1}} r^{n-4} |\mathcal{Q}_1(0, \psi_2)(r \cdot)| \leq c r^{n-4} \| \mathcal{Q}_1(0, \psi_2)(r \cdot) \|_{H^s(\mathbb{S}^{n-1})} \leq c \left[ \sum_{j=1}^{\infty} (1 + \lambda_j^s) |(\psi_2)_j|^2 r^{-j} \right]^{1/2}.
\]

Also, since \( j > 0 \), observe that

\[
\sup_{r \geq 2} \sup_{j \geq 1} (1 + \lambda_j^s) r^{-j} < \infty.
\]

Thus, we have

\[
\sup_{\mathbb{R}^n \setminus B_2} |x|^{n-4} |\mathcal{Q}_1(0, \psi_2)| \leq c \| \psi_2 \|_{L^2(\mathbb{S}^{n-1})} \leq c \| \psi_2 \|_{L^\infty(\mathbb{S}^{n-1})}.
\]
The maximum principle in [17], then implies that
\[ \sup_{\mathbb{R}^n \setminus B_1} |x|^{n-4}|Q_1(0, \psi_2)| \leq c\|\psi_2\|_{L^\infty(S^{n-1})}. \]

Finally, using the last estimate, we can use standard (rescaled) elliptic estimates to prove the estimates for the derivatives, which completes the proof of the proposition. \( \square \)

**Corollary 4.5.** Let \( 0 < \alpha < 1 \) and \( r > 0 \). There exists a bounded linear operator
\[ Q_r : C^{4,\alpha}(S^{n-1}, \mathbb{R}^2) \to C^{4,\alpha}_{4-n}(\mathbb{R}^n \setminus B_r) \]
satisfying that for each \( (\psi_0, \psi_2) \in C^{4,\alpha}(S^{n-1}, \mathbb{R}^2) \)
\[
\begin{align*}
\Delta^2 Q_r(\psi_0, \psi_2) &= 0 \quad \text{in} \quad \mathbb{R}^n \setminus B_r, \\
Q_r(\psi_0, \psi_2) &= \psi_0 \quad \text{on} \quad \partial B_r, \\
\Delta Q_r(\psi_0, \psi_2) &= r^{-2}\psi_2 \quad \text{on} \quad \partial B_r.
\end{align*}
\]
Moreover, there exists a constant \( C > 0 \) such that
\[ \|Q_r(\psi_0, \psi_2)\|_{C^{4,\alpha}_{4-n}(\mathbb{R}^n \setminus B_1)} \leq C r^{n-4}\|Q(\psi_0, \psi_2)\|_{C^{4,\alpha}(S^{n-1}, \mathbb{R}^2)}. \]

**Proof.** As before, let us consider
\[ Q_r(\psi_0^r, \psi_2^r)(x) = Q_1(\psi_0, \psi_2)\left(r^{-1}x\right), \]
where \( \psi_i^r(\theta) := \psi_i(r\theta) \) for \( i = 0, 2 \). The proof now follows the same lines as in the interior case. \( \square \)

### 4.3. Navier-to-Neumann operator

Finally we prove the main result of this section, which is the construction of an isomorphism we call the Navier-to-Neumann operator. To this end, we combine a Liouville-type result for bounded entire biharmonic functions and a standard result from the theory of pseudodifferential operators.

**Proposition 4.6.** The operator \( Z_1 : \pi''(C^{4,\alpha}(S^{n-1}, \mathbb{R}^2)) \to \pi''(C^{1,\alpha}(S^{n-1}, \mathbb{R}^2)) \) given by
\[ Z_1(\phi_0, \phi_2) = (\partial_r (P_1(\phi_0, \phi_2) - Q_1(\phi_0, \phi_2)), \partial_r (\Delta P_1(\phi_0, \phi_2) - \Delta Q_1(\phi_0, \phi_2))) \]
is an isomorphism.

**Proof.** Notice that \( Z_1 \) is a linear third order elliptic self-adjoint pseudodifferential operator with principal symbol \( \sigma_{Z_1}(\xi) = -2(|\xi|, |\xi|^2) \). Hence, since \( \sigma_{Z_1}(\xi) \not= 0 \) whenever \( \xi \not= 0 \), \( Z_1 \) is surjective by a classical result from the theory of linear operators.

Now it is sufficient to prove that this operator is also injective. Let us take \( (\phi_0, \phi_2) \in C^{4,\alpha}(S^n, \mathbb{R}^2) \) satisfying \( Z_1(\phi_0, \phi_2) = 0 \) and define \( w(\phi_0, \phi_2) \in C^{1,\alpha}(\mathbb{R}^n) \) as
\[ w(\phi_0, \phi_2)(x) = \begin{cases} w(\phi_0, \phi_2), & \text{if } x \in B_1 \\ \bar{w}(\phi_0, \phi_2), & \text{if } x \in \mathbb{R}^n \setminus B_1, \end{cases} \]
where
\[ \bar{w}(\phi_0, \phi_2) := r^{1-n} \sum_{j=2}^{\infty} \left( \phi_j e_j + \frac{\phi_j}{c_2(n, j)} (r^2 - 1) \right). \]
and \( \bar{w}(\phi_0, \phi_2) \) is the bounded entire biharmonic function defined in the exterior region \( \mathbb{R}^n \setminus B_1 \). Using the Liouville theorem in [26], it follows that \( w \equiv 0 \) in \( C^{1,\alpha}(\mathbb{R}^n) \), and so \( (\phi_0, \phi_2) \equiv 0 \) in \( C^{4,\alpha}(S^{n-1}, \mathbb{R}^2) \), which proves the proposition. \( \square \)
As in the proofs of Corollaries 4.2 and 4.5 we obtain
\[ r \partial_r (P_r(\phi_0^r, \phi_1^r) - Q_r(\phi_0^r, \phi_1^r))(r \theta) = \partial_r (P_1(\phi_0, \phi_2) - Q_1(\phi_0, \phi_2))(\theta) \]
and
\[ r^3 \partial_r \Delta (P_r(\phi_0^r, \phi_1^r) - Q_r(\phi_0^r, \phi_1^r))(r \theta) = \partial_r (\Delta P_1(\phi_0, \phi_2) - \Delta \Delta Q_1(\phi_0, \phi_2))(\theta). \]
Therefore, we obtain

**Corollary 4.7.** The operator \( Z_r : \pi''(C^4,\alpha(S^n_{r-1}, \mathbb{R}^2)) \to \pi''(C^1,\alpha(S^n_{r-1}, \mathbb{R}^2)) \) given by
\[ Z_r(\phi_0, \phi_2) = (r \partial_r (P_r(\phi_0, \phi_2) - Q_r(\phi_0, \phi_2)), r^3 \partial_r \Delta (P_r(\phi_0, \phi_2) - \Delta Q_r(\phi_0, \phi_2))) \]
is an isomorphism, with norm bounded independent of \( r \).

## 5. Interior Problem

Following the strategy described in the introduction, our goal in this section is to construct a solution in a small punctured ball centered at the singular point. Due to the conformal invariance of the \( Q \)-curvature equation, we will use conformal normal coordinates centered at the singular point. In these coordinates, we show the metric is sufficiently close to the Euclidean metric using the vanishing assumption on the Weyl tensor. We use these coordinates to show that the first terms in the expansion of \( P_0(u_{c, R}) \) are \( L^2 \)-orthogonal to the low Fourier modes on the sphere, see Lemma 5.6. This allows us to use an auxiliary function which will be important to define the fixed point problem. By the expansion of the metric, it should be possible to perturb a Delaunay-type solution to an exact solution to equation (\( Q \)).

### 5.1. Conformal normal coordinates

Let \((M, g_0)\) be a smooth Riemannian manifold of dimension \( n \geq 5 \). Given a point \( p \in M \), there exist \( r > 0 \) and a smooth function \( \mathcal{F} \) on \( M \) such that the conformal metric \( g = \mathcal{F} \frac{1}{r^2} g_0 \) satisfies
\[
\det g = 1, \quad \text{in } B_r(p) \quad (5.1)
\]
in \( g \)-normal coordinates. In these coordinates \( \mathcal{F} = 1 + O(|x|^2) \) and \( g = \delta + O(|x|^2) \). Additionally,
\[
R_{ij}(0) = 0, \quad R_g = O(|x|^2) \quad \text{and} \quad \Delta_g R_g(0) = -\frac{1}{6} |W_g(0)|^2, \quad (5.2)
\]
where \( W_g \) is the Weyl tensor of the metric \( g \). J. M. Lee and T. H. Parker [31] first proved the existence of such coordinates, and later J. G. Cao [15] and M. Günther [20] derived refined expansions of various curvature terms in conformal normal coordinates.

**Remark 5.1.** For the remainder of the paper we let \( d = \left[ \frac{n - 8}{2} \right] \) for \( n \geq 8 \) and \( d = -1 \) for \( 5 \leq n \leq 7 \).

Using the results in [24] and [25] we obtain improved estimates on curvature terms if we assume that the Weyl tensor vanishes to sufficiently high order. If the Weyl tensor satisfies
\[
\nabla^l W_g(p) = 0, \quad \text{for } l = 0, 1, \ldots, d, \quad (5.3)
\]
then
\[
g = \delta + O(|x|^{d+3}). \quad (5.4)
\]
This implies that \( R_g = O(|x|^{\max\{2, d+1\}}) \) and \( \text{Ric}_g = O(|x|^{\max\{1, d+1\}}) \). Therefore, by (1.1), we have \( Q_g = O(|x|^{\max\{0, d-1\}}) \).

Note that the Weyl tensor is conformally invariant, which implies that the condition (5.3) is satisfied for any metric in the conformal class of \( g \). Also, for dimensions \( 5 \leq n \leq 7 \), the condition on \( W_g \) is vacuous.
In conformal normal coordinates, we will always write
\[ g = \exp(h), \]
where \( h \) is a two-tensor on \( \mathbb{R}^n \) satisfying \( h_{ij}(x)x_j = 0 \) and \( tr(h(x)) = 0 \) (See [30]). Note that by (5.4), we have \( h = O(|x|^{d+3}) \).

Motivated by results in [1, 11] we prove the following lemma.

**Lemma 5.2.** We have
\[ R_{jk} = \frac{1}{2} (\partial_i \partial_j h_{ki} + \partial_i \partial_k h_{ij} - \partial_i \partial_l h_{jk}) + O(|\partial h|^2) + O(|h||\partial^2 h|) \] (5.5)
and
\[ R_g = \partial_i \partial_j h_{ij} + O(|\partial h|^2) + O(|h||\partial^2 h|) \] (5.6)

**Proof.** First we observe that \( g^{-1} = \exp(-h) \). The Ricci tensor is given by
\[ R_{jk} = \partial_i \Gamma^i_{jk} - \partial_j \Gamma^i_{ki} + \Gamma^p_{jk} \Gamma^i_{ip} - \Gamma^p_{ik} \Gamma^i_{jp}. \]
Note that \( \Gamma^i_{jk} = \frac{1}{2} g^{il} \partial_k g_{lj} = \frac{1}{2} \partial_k \log \det g = 0 \), by (5.1). Therefore, \( R_{jk} = \partial_i \Gamma^i_{jk} - \Gamma^p_{ik} \Gamma^i_{jp} \), where the Christoffel symbols are given by \( \Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}) \). Thus, \( \Gamma^k_{ij} = O(|\partial h|) \) and
\[ \partial_i \Gamma^i_{jk} = \frac{1}{2} (\partial_i \partial_j h_{ki} + \partial_i \partial_k h_{ij} - \partial_i \partial_l h_{jk}) + O(|\partial h|^2) + O(|h||\partial^2 h|). \]
This implies (5.5). Since \( R_g = g^{ik} R_{jk} \) and \( tr(h(x)) = 0 \) we obtain the result. \( \square \)

**Lemma 5.3.** Let \( \{x_1\} \) be conformal normal coordinates centered at \( p \). If \( u \) is radial with respect to these coordinates then in a small geodesic ball centered at \( p \) it holds
\[ \nabla_i \nabla_j u = \frac{x_i x_j}{r^2} u'' - \frac{x_i x_j}{r^3} u' + \frac{\delta_{ij}}{r} u' + O(|\partial h||u'|), \] (5.7)
and
\[ \Delta_g u = \Delta u = u'' + \frac{n-1}{r} u', \]
where \( \Delta \) is the Euclidean laplacian and we use \( ' \) to denote differentiation with respect to \( r \).

The equation (5.7) follows from (5.1) and [21, Lemma 2.6].

**Lemma 5.4.** We have \( x_j x_i \partial_i \partial_j h_{ki} = x_j x_k \Delta h_{jk} = x_i x_j \partial_i \partial_k h_{jk} = 0 \).

**Proof.** Recall that \( h_{ki} x_k = 0 \) and \( tr h = 0 \). Taking a derivative, we get \( \partial_j h_{ki} x_k + h_{ij} = 0 \) and
\[ x_k \partial_i h_{ki} = 0, \] (5.8)
which implies that \( \partial_j h_{ki} x_k x_j = 0 \). Differentiating again we see \( \partial_i \partial_j h_{ki} x_k x_j + \partial_j h_{ij} x_j + \partial_i h_{ki} x_k = 0 \), and so \( x_j x_k \partial_i \partial_j h_{ki} = 0 \). Thus \( \partial_j h_{ij} x_j + h_{ik} = 0 \), which implies that \( \partial_i h_{jh} x_j x_k = 0 \). Finally, \( \partial_i \partial_j h_{jk} x_k x_k + \partial_i h_{ji} x_j + \partial_j h_{ij} x_j = 0 \). Therefore \( x_j x_k \Delta h_{jk} = 0 \). \( \square \)

**Lemma 5.5.** The functions \( \partial_i \partial_j h_{ij}, \Delta \partial_i \partial_j h_{ij} \) and \( x_k \partial_i \partial_j h_{ij} \) are \( L^2 \)-orthogonal to the functions \( \{1, x_1, \ldots, x_n\} \) in \( \mathbb{S}_r^{n-1} \).

**Proof.** First observe that \( \partial_i \partial_j h_{ij} = \text{div}(\partial_i \partial_j e_i) \) and \( \partial_j h_{kj} = \text{div}(h_{kj} e_j) \). Using integration by parts and (5.8), we get
\[ \int_{B_r} \partial_i \partial_j h_{ij} = \frac{1}{r} \int_{\mathbb{S}_r^{n-1}} x_i \partial_j h_{ij} = 0 \] (5.9)
and
\[ \int_{B_r} x_k \partial_i \partial_j h_{ij} = - \int_{B_r} \partial_j h_{kj} + \frac{1}{r} \int_{\mathbb{S}_r^{n-1}} x_i \partial_j h_{ij} = - \frac{1}{r} \int_{\mathbb{S}_r^{n-1}} h_{kj} x_j = 0. \]
Since $\partial_k \partial_l h_{ij} = \operatorname{div}(\partial_k \partial_l h_{ij} e_i)$ and $\partial_k \partial_j h_{ij} = \operatorname{div}(\partial_k h_{ij} e_j)$, using Lemma 5.4, (5.8) and (5.9), we obtain

$$\int_{B_r} x_k \partial_k \partial_j h_{ij} = - \int_{B_r} \partial_k \partial_j h_{kj} + \frac{1}{r} \int_{S^m_{r-1}} x_k \partial_k \partial_j h_{ij} = 0$$

and

$$\int_{B_r} x_l x_k \partial_k \partial_j h_{ij} = - \int_{B_r} (x_l \partial_k \partial_j h_{kj} + x_k \partial_k \partial_j h_{lj}) + \frac{1}{r} \int_{S^m_{r-1}} x_l x_k \partial_k \partial_j h_{ij}$$

$$= - \int_{B_r} x_k \partial_k \partial_j h_{ij} = \int_{B_r} \partial_k h_{lk} - \frac{1}{2} \int_{S^m_{r-1}} x_l x_k \partial_k h_{ij} = 0.$$

By the previous results we get

$$\int_{B_r} \Delta \partial_l \partial_j h_{ij} = \frac{1}{r} \int_{S^m_{r-1}} x_k \partial_k \partial_l \partial_j h_{ij} = 0$$

and

$$\int_{B_r} x_l \Delta \partial_l \partial_j h_{ij} = - \int_{B_r} \partial_l \partial_l \partial_j h_{ij} = - \frac{1}{r} \int_{S^m_{r-1}} x_j \partial_l \partial_j h_{ij} = \frac{1}{r} \int_{S^m_{r-1}} \partial_l h_{ll} = 0.$$

□

**Lemma 5.6.** In conformal normal coordinates at $p$, for sufficiently small $r = |x|$, one has

$$P_g u_{\varepsilon,R} = \Delta^2 u_{\varepsilon,R} + \mathcal{P}(u_{\varepsilon,R}) + \mathcal{A}$$

where $\Delta$ is the Euclidean laplacian and

$$\mathcal{P}(u_{\varepsilon,R}) = \frac{4}{n-2} \left( \frac{u'_{\varepsilon,R}}{r} - \frac{(n-2)^2 + 4}{8(n-1)} \Delta u_{\varepsilon,R} \right) \partial_l \partial_j h_{ij} + \frac{6-n}{2(n-1)} x_l \partial_l \partial_j h_{jk} \frac{u'_{\varepsilon,R}}{r}$$

$$- \frac{n-4}{4(n-1)} u_{\varepsilon,R} \Delta \partial_l \partial_j h_{ij},$$

is $L^2$-orthogonal to the functions $\{1, x_1, \ldots, x_n\}$ and $\mathcal{A} = O(r^{d+4-\frac{d}{2}+\max\{d,0\}})$.

**Proof.** By (1.3)

$$P_g u_{\varepsilon,R} = \Delta^2 u_{\varepsilon,R} + \frac{4}{n-2} \langle \operatorname{Ric}_g, \nabla^2 u_{\varepsilon,R} \rangle - \frac{(n-2)^2 + 4}{2(n-1)(n-2)} R_g \Delta g u_{\varepsilon,R}$$

$$+ \frac{6-n}{2(n-1)} \langle \nabla R_g, \nabla u_{\varepsilon,R} \rangle + \frac{n-4}{2} Q_g u_{\varepsilon,R}.$$  \hspace{1cm} (5.10)

Now by (5.5) and (5.7), we obtain

$$\langle \operatorname{Ric}_g, \nabla^2 u_{\varepsilon,R} \rangle = g^{ij} g^{kl} R_{jk} \nabla_i \nabla_l u_{\varepsilon,R}$$

$$= g^{ij} g^{kl} R_{jk} \left( \frac{x_i x_l}{r^2} u''_{\varepsilon,R} - \frac{x_i x_l}{r^3} u'_{\varepsilon,R} + \frac{\delta_{il}}{r} u'_{\varepsilon,R} + O(|\partial h||u'_{\varepsilon,R}|) \right)$$

$$= R_g \frac{u'_{\varepsilon,R}}{r} + R_{jk} x_j x_k \left( \frac{u''_{\varepsilon,R}}{r^2} - \frac{u'_{\varepsilon,R}}{r^3} \right) + O(|h||\operatorname{Ric}|) \left( \frac{|u'_{\varepsilon,R}|}{r} + |u''_{\varepsilon,R}| \right)$$

$$+ O(|\operatorname{Ric}||\partial h||u'_{\varepsilon,R}|).$$
Using that \( h = O(r^{d+3}) \), \( \text{Ric} = O(r^{\max\{1,d+1\}}) \) and \( u_{\varepsilon, R} = O(r^{\frac{4-n}{2}}) \), as well (5.5) and (5.6) we obtain
\[
\langle \text{Ric}_g, \nabla^2 u_{\varepsilon, R} \rangle = \partial_i \partial_j h_{ij} \frac{u_{\varepsilon, R}'}{r} + \left( \partial_i \partial_j h_{kijx} x_k - \frac{1}{2} \partial_i \partial_j h_{ijk} x_j x_k \right) \left( \frac{u_{\varepsilon, R}''}{r^2} - \frac{u_{\varepsilon, R}'}{r^3} \right) + O(r^{d+4-\frac{n}{2}+\max\{d,0\}})
\]
\[
= \partial_i \partial_j h_{ij} \frac{u_{\varepsilon, R}'}{r} + O(r^{d+4-\frac{n}{2}+\max\{d,0\}}),
\]
where in the last equality we used (5.8) and (5.9). Now observe that
\[
\langle \nabla R_g, \nabla u_{\varepsilon, R} \rangle = g^{ij} \nabla_i R_g \nabla_j u_{\varepsilon, R} = \partial_i \partial_j h_{ij} \frac{u_{\varepsilon, R}'}{r} + O(r^{2d+4-\frac{n}{2}}).
\]
Using (1.1) and (5.6) we get
\[
R_g \Delta_g u_{\varepsilon, R} = \partial_i \partial_j h_{ij} \Delta u_{\varepsilon, R} + O(r^{2d+4-\frac{n}{2}})
\]
and
\[
Q_g = -\frac{1}{2(n-1)} \Delta \partial_i \partial_j h_{ij} + O(r^{2d+2}).
\]
This finishes the proof.

**Lemma 5.7.** In conformal normal coordinates one has
\[
(\Delta^2 - \Delta_g^2) u_{\varepsilon, R, a} = O(|x|^{d+2-\frac{n}{2}}),
\]
where \( \Delta \) is the Euclidean Laplacian.

**Proof.** First we write
\[
(\Delta^2 - \Delta_g^2) u_{\varepsilon, R, a} = (\Delta^2 - \Delta_g^2)(u_{\varepsilon, R, a} - u_{\varepsilon, R}) + (\Delta^2 - \Delta_g^2) u_{\varepsilon, R}.
\]
Since \( u_{\varepsilon, R} \) is a radial function, using (5.1) and Lemma 2.6 in [21] we have \( \Delta^2 u_{\varepsilon, R} = \Delta^2 u_{\varepsilon, R} \).
Now, note that by Lemma 2.4 we have \( v = u_{\varepsilon, R, a}(x) - u_{\varepsilon, R}(x) = O(|x|^{\frac{4-n}{2}}) \). Using the Laplacian in coordinates and (5.4) we obtain
\[
\Delta_g v = \Delta v + \partial_i g^{ij} \partial_j v + (g^{ij} - \delta^{ij}) \partial_i \partial_j v,
\]
which implies
\[
\Delta_g^2 v = \Delta^2 v + \partial_i g^{ij} \partial_j \Delta v + (g^{ij} - \delta^{ij}) \partial_i \partial_j \Delta v + \Delta_g (\partial_i g^{ij} \partial_j v + (g^{ij} - \delta^{ij}) \partial_i \partial_j v).
\]
This implies the result. \( \square \)

### 5.2. Linear analysis
In order to solve the singular Q-curvature problem we seek functions \( u \) that are close to a function \( u_0 \) such that \( (u + u_0)(x) \to +\infty \) as \( x \to p \), \( u + u_0 \) is a positive function, and \( H_g(u_0 + u) = 0 \). This is done by considering the linearization about \( u_0 \) of the operator \( H_g \), defined in (1.4), which is given by
\[
L_g^{u_0}(u) = \frac{\partial}{\partial t} \bigg|_{t=0} H_g(u_0 + tu) = P_g u - \frac{n(n^2 - 4)(n + 4)}{16} u_0 \frac{8}{n-4} u.
\]
Thus we can write
\[
H_g(u_0 + u) = H_g(u_0) + L_g^{u_0}(u) + \mathcal{R}^{u_0}(u),
\]
where the nonlinear term \( \mathcal{R}^{u_0}(u) \) is independent of the metric and given by
\[
\mathcal{R}^{u_0}(u) = -\frac{n(n^2 - 4)(n - 4)}{16} \left[ |u_0 + u|^{\frac{8}{n-4}} (u_0 + u) - u_0^{\frac{n+4}{n-4}} - \frac{n + 4}{n - 4} u_0^{\frac{8}{n-4}} u \right].
\]
This section is devoted to functional properties of the linearized operator around a Delaunay solution in $\mathbb{R}^n$. In this case the $Q$-curvature operator (1.4) is given by

$$ H_0(u) = \Delta^2 u - \frac{n(n^2 - 4)(n - 4)}{16} |u|^{\frac{8}{n-4}} u, $$

and its linearization (5.14) around a solution $u_{\varepsilon,R,a}$ (see (2.16)), denoted by $L_{\varepsilon,R,a}$, is given by

$$ L_{\varepsilon,R,a}(v) = \Delta^2 v - \frac{n(n^2 - 4)(n + 4)}{16} u_{\varepsilon,R,a}^\varepsilon v. \quad (5.16) $$

We abbreviate $L_{\varepsilon,R,0} = L_{\varepsilon,R}$ and $L_{\varepsilon,1,0} = L_{\varepsilon}$.

Remember that if $g = u^{\frac{4}{n-4}} \delta$, then $\Phi^* g = v^{\frac{4}{n-4}} g_{\text{cyl}}$, where $v(t) = e^{\frac{4}{n-4} t} u(e^{-t} \theta) = |x|^{\frac{4}{n-4}} u(x)$, i.e. $u(x) = |x|^{\frac{4-n}{2}} v(- \log |x|)$, where $\Phi$ is defined in (2.2). Hence, using (1.5) we see

$$ L_{\varepsilon,R}(u) = |x|^{-\frac{n+4}{2}} \mathcal{L}_{\varepsilon,R}(e^{\frac{n-4}{2} t} u \circ \Phi) \circ \Phi^{-1}, $$

where

$$ \mathcal{L}_{\varepsilon,R}(v) = P_{g_{\text{cyl}}}(v) - \frac{n(n + 4)(n^2 - 4)}{16} v_{\varepsilon,R}^2 v $$. 

and

$$ P_{g_{\text{cyl}}}(v) = \Delta^2_{g_{\text{cyl}}} v - \frac{n(n - 4)}{2} \Delta_{g_{\text{cyl}}} v - 4 \partial_t^2 v + \frac{n^2(n - 4)^2}{16} v. $$

Using $\Delta_{g_{\text{cyl}}} v = \partial_t^2 v + \Delta_{S^{n-1}} v$ we obtain

$$ \mathcal{L}_{\varepsilon,R}(v) = \partial_t^4 v + \frac{n^2(n - 4)^2}{16} v - \frac{n(n - 4)}{2} \Delta_{S^{n-1}} v - \frac{n(n - 4) + 8}{2} \partial_t^2 v + \frac{n(n + 4)(n^2 - 4)}{16} v_{\varepsilon,R}^2 v. \quad (5.17) $$

Now we focus on solving the following linear problem with Navier boundary conditions

$$ \left\{ \begin{array}{ll}
\mathcal{L}_{\varepsilon}(w) = f & \text{in } D_R \\
\Delta w = w = 0 & \text{on } \partial D_R,
\end{array} \right. \quad (5.18) $$

where $D_R := (-\log R, \infty) \times S^{n-1}$ and $\mathcal{L}_{\varepsilon}$ is given by (5.17) with $R = 1$.

We will follow closely the program due to R. Mazzeo and F. Pacard [41]. Due to the geometry of the domain, it is natural to approach our problem by means of a classical separation of variables, decomposing both $w$ and $f$ in Fourier series and using the projections (4.1).

Projecting the linear problem (5.18) along (the space generated by) each eigenfunction $e_j$ we obtain an infinite dimensional system of ODE's

$$ \left\{ \begin{array}{ll}
\mathcal{L}_j^\varepsilon(w_j) = f_j & \text{in } (-\log R, \infty) \\
\Delta w_j(-\log R) = w_j(-\log R) = 0,
\end{array} \right. $$

where the ordinary differential operator $\mathcal{L}_j^\varepsilon$ is given by

$$ \mathcal{L}_j^\varepsilon(w_j) = \ddot{w}_j - \left(2\lambda_j + \frac{n(n - 4) + 8}{2}\right) \dot{w}_j \\
+ \left(\lambda_j^2 + \frac{n(n - 4)}{2} \lambda_j + \frac{n^2(n - 4)^2}{16} - \frac{n(n + 4)(n^2 - 4)}{16} v_{\varepsilon,R}^2\right) w_j. \quad (5.19) $$

The problem will be divided in three cases: $j > n$, $j = 0$ and $j = 1, \ldots, n$. 
5.2.1. Case \( j > n \). We consider the projections \( w'' = \pi''(w) \) and \( f'' = \pi''(f) \) of \( w \) and \( f \) to the high Fourier modes

\[
\begin{aligned}
\{ \quad & \mathcal{L}_\varepsilon(w'') = f'' \quad \text{in} \quad D_R \\
& \Delta w'' = w'' = 0 \quad \text{on} \quad \partial D_R. 
\end{aligned}
\] (5.20)

First we solve the problem

\[
\begin{aligned}
\{ \quad & \mathcal{L}_\varepsilon(w_T)(t, \theta) = \overline{f} \quad \text{in} \quad D_R^T \\
& \Delta w_T = w_T = 0 \quad \text{on} \quad \partial D_R^T,
\end{aligned}
\] (5.21)

where \( D_R^T := (-\log R, T) \times S^{n-1} \). This solution can be obtained variationally. Consider the energy function \( \mathcal{E}_T \) given by

\[
\mathcal{E}_T(w) = \langle P_{g_{cyl}}w, w \rangle - \int_{D_R^T} \left( \frac{(n+4)(n^2-4)}{16} v_\varepsilon^\frac{s}{n-4} w^2 + f w \right) d\theta dt,
\]

where

\[
\langle P_{g_{cyl}}w, w \rangle = \int_{D_R^T} \left( \Delta_{g_{cyl}} w \Delta_{g_{cyl}} u - 4A_{g_{cyl}}(\nabla w, \nabla u) \right) d\theta dt = \int_{D_R^T} \left( \Delta_{g_{cyl}} w \right)^2 + \frac{n(n-4)}{2} |\nabla_{g_{cyl}} w|^2 + 4w^2 + \frac{n^2(n-4)^2}{16} w^2 \right) d\theta dt.
\]

It is easy to see that critical points of the functional \( \mathcal{E}_T \) are weak solutions to (5.21). Observe that

\[
\mathcal{E}_T(w) \geq \int_{D_R^T} \left( \lambda^2 w^2 + w^2 - 2\bar{w}w\lambda + \frac{n(n-4)}{2} \lambda w^2 + \frac{n(n-4)+8}{2} \bar{w}^2 \\
+ \frac{n^2(n-4)^2}{16} w^2 - \frac{n(n+4)(n^2-4)}{16} v_\varepsilon^\frac{s}{n-4} w^2 - f w \right) d\theta dt \\
\geq \int_{D_R^T} \left( \bar{w}^2 + \left( 2\lambda + \frac{n(n-4)+8}{2} \right) \bar{w}^2 \\
+ \left( \lambda^2 + \frac{n(n-4)}{2} \lambda + \frac{n^2(n-4)^2}{16} - \frac{n(n+4)(n^2-4)}{16} v_\varepsilon^\frac{s}{n-4} \right) w^2 - f w \right) d\theta dt.
\]

When \( \lambda \geq 2n \) we have

\[
\lambda^2 + \frac{n(n-4)}{2} \lambda + \frac{n^2(n-4)^2}{16} - \frac{n(n+4)(n^2-4)}{16} > 0. \quad (5.22)
\]

This implies that \( \mathcal{E}_T \) is coercive and convex in the space

\[
H_0^2(D_R^T) := \left\{ u \in H^2(D_R^T); \Delta u = u = 0 \text{ in } \partial D_R^T \text{ and } \int_{S^{n-1}} u(\cdot, \theta) e_j(\theta) d\theta = 0, \ j = 0, \ldots, n \right\}.
\]

The existence of a unique minimizer for \( \mathcal{E}_T \) is immediate. The standard elliptic theory yields the expected regularity issues for \( w_T \) in terms of the regularity of \( f \).

**Lemma 5.8.** For \( 2 - n < \mu < 2 \), let \( \delta = \frac{n-4}{T} + \mu \). Then there exist constants \( \varepsilon_0 > 0 \) and \( C > 0 \) independent of \( T \) and \( R \) such that for all \( \varepsilon \in (0, \varepsilon_0) \), we have

\[
\sup_{(t, \theta) \in D_R^T} e^{\delta t} |w_T(t, \theta)| \leq C \sup_{(t, \theta) \in D_R^T} e^{\delta t} |f(t, \theta)| \quad (5.23)
\]

**Proof.** The proof is by contradiction. Suppose that the estimate (5.23) does not hold. This implies that we can find a sequence \( (\varepsilon_i, T_i, R_i, w_{T_i}, f_i) \) such that

1. \( \varepsilon_i \to 0 \) as \( i \to \infty \).
where

Additionally

and

Claim 1: Either of the functions $v_n$ and the right hand side converges to $0$ as $i \to \infty$. Therefore, $v_i \to v_\infty$ in $C^1_{\text{loc}}(\mathbb{R})$. So that there is a subsequence, still denoted by $v_i$, and a function $v_\infty \in C^3_{\text{loc}}(\mathbb{R})$ such that $v_i \to v_\infty$ in $C^3_{\text{loc}}(\mathbb{R})$.

Since for each $i \in \mathbb{N}$ the function $v_i$ satisfies the ODE

and the right hand side converges to $\frac{n^2-4n+8}{16}v_\infty - \frac{n^2(n-4)^2}{16}v_\infty + \frac{n(n-4)(n^2-4)}{16}v_i^{\frac{n+4}{n-4}}$ in $C^0_{\text{loc}}(\mathbb{R})$, we conclude that $v_i \to v_\infty$ in $C^4_{\text{loc}}(\mathbb{R})$. Therefore, $v_\infty$ satisfies the ODE (2.3). Now, the claim follows by Theorem C, since $v_i \to v_\infty$ in $C^4_{\text{loc}}(\mathbb{R})$ as $\varepsilon_i := \min v_i \to 0$.

Combining (5.24), (5.25), and Schauder estimates (see [19]) the Arzela-Ascoli theorem gives us a subsequence, which we still denote by $w_i$, that converges to a function $w_\infty \in C^2_{\text{loc}}((\tau_1, \tau_2) \times S^{n-1})$. 
satisfying the equation
\[ \mathcal{L}_\infty(w_\infty) = 0 \quad \text{in} \quad (\tau_1, \tau_2) \times S^{n-1} \] (5.26)
in the sense of distributions, where
\[
\mathcal{L}_\infty = \Delta^2_{g_{S^1}} - \frac{n(n-4)}{2} \Delta_{g_{S^1}} - 4\partial_t^2 + \frac{n^2(n-4)^2}{16} - \frac{n(n+4)(n^2-4)}{16} \frac{w_\infty}{v_\infty^4}
= \partial_t^2 + \Delta^2_{S^{n-1}} + 2\Delta_{S^{n-1}} \partial_t^2 - \frac{n(n-4)}{2} \Delta_{S^{n-1}} - \frac{n(n-4) + 8}{2} \partial_t^2
+ \frac{n^2(n-4)^2}{16} - \frac{n(n+4)(n^2-4)}{16} \frac{w_\infty^2}{v_\infty^2}.
\]

If \( \tau_i \) is a real number, then the boundary condition becomes \( \Delta w_\infty(\tau_i) = w_\infty(\tau_i) = 0 \). It is important to point out that by (5.24) we have that \( \sup_{\theta \in S^{n-1}} |w_\infty(0, \theta)| = 1 \). Thus, \( w_\infty \) is nontrivial.

Decompose \( w_\infty \) as
\[
w_\infty(t, \theta) = \sum_{j \geq n+1} w^j_\infty(t) e_j(\theta).
\]
Projecting (5.26) along the eigenfunctions \( e_j \), for \( j \geq n+1 \), we obtain
\[
\ddot{w}^j_\infty - \left(2\lambda_j + \frac{n(n-4) + 8}{2}\right) \dot{w}^j_\infty
+ \left(\lambda_j^2 + \frac{n(n-4) + 8}{2}\lambda_j + \frac{n^2(n-4)^2}{16} - \frac{n(n+4)(n^2-4)}{16} \frac{w_\infty^2}{v_\infty^2}\right) w^j_\infty = 0.
\]

After multiplying this equation by \( w^j_\infty \), we use integration by parts and (5.22) to deduce that \( w^j_\infty \equiv 0 \), which is a contradiction. To justify the integration by parts, note that if both \( \tau_1 \) and \( \tau_2 \) are finite, we use that \( w^j_\infty(\tau_1) = 0 \). Otherwise, since \( \lim_{r \to \pm \infty} v_\infty = 0 \) in case \( 0 < v_\infty < 1 \), we deduce that
\[
w_\infty^j(t) \sim e^{\mu_j t}, \quad \text{as} \quad |t| \to +\infty,
\]
where
\[
\mu_j^\pm := \sqrt{\frac{n(n-4) + 8 + 4\lambda_j \pm 4\sqrt{(n-2)^2 + 4\lambda_j}}{2}},
\]
and \( \lambda_j \geq 2n \). We remark that \( \mu_j^\pm \geq \frac{n}{2} \). On the other hand, the condition \( |w_\infty| \leq e^{-\delta t} \), together with the fact that \( |\delta| < \frac{n}{2} \), implies that \( |w_\infty^j(t)| \leq C e^{-|t|\frac{\delta}{2}} \), as \( |t| \to +\infty \). Therefore, in this case we can also integrate by parts. This finishes the proof of the lemma.

Using a scaling argument as in [42, Proposition 6.2] we can see that if \( f'' \in C^{0,\alpha}_\delta(D_R) \), then \( w_T \in C^{4,\alpha}_\delta(D_T^R) \). If \( \gamma \in (-n/2, n/2) \) then there exist \( \varepsilon_0 > 0 \) and a positive constant \( C \), which does not depend on \( \varepsilon, R, T, \) such that for all \( \varepsilon \in (0, \varepsilon_0] \) we have
\[
\|w_T\|_{C^{4,\alpha}_\delta(D_T^R)} \leq C \|f''\|_{C^{0,\alpha}_\delta(D_R)}.
\]

Therefore, letting \( T \to +\infty \), we obtain the existence of a solution \( \tilde{w} \) to (5.20) fulfilling the estimate
\[
\|w''\|_{C^{4,\alpha}_\delta(D)} \leq C \|f''\|_{C^{0,\alpha}_\delta(D_R)},
\]
with \( C > 0 \) independent of \( R \) and \( \varepsilon \).
5.2.2. Case $j = 0$. We start by considering the projection of the problem (5.18) along the eigenfunction $\varphi_0$. We get
\[
\begin{aligned}
\mathcal{L}_0^\varepsilon(w_0) &= f_0 & \text{in } (-\log R, \infty) \\
\dot{w}_0(-\log R) &= 0 \\
w_0(-\log R) &= 0,
\end{aligned}
\]
where
\[
\mathcal{L}_0^\varepsilon(w_0) = \ddot{w}_0 - \frac{n(n - 4)}{2} \dot{w}_0 + \left( \frac{n^2(n - 4)^2}{16} - \frac{n(n + 4)(n^2 - 4)}{16} \frac{s}{v_\varepsilon^4} \right) w_0.
\]
Since we no longer have (5.22), it is necessary to use a different approach. Choose an extension of $f_0$ to $\mathbb{R}$, still denoted by $f_0$. For $T > -\log R$ consider the unique solution $w_T$ of the problem
\[
\mathcal{L}_0^\varepsilon(w_T) = f_0 \text{ in } (-\infty, T)
\]
with $\ddot{w}_T(T) = \dot{w}_T(T) = w_T(T) = 0$.

**Lemma 5.9.** For $\mu > 0$, let $\delta = \frac{n-4}{2} + \mu$. Then there are constants $\varepsilon_0 > 0$ and $C > 0$ independent of $T$ and $R > 0$, such that, for $\varepsilon \in (0, \varepsilon_0]$, we have
\[
\sup_{t \in (-\log R, T)} e^{\delta t} |w_T(t)| \leq C \sup_{t \in (-\log R, T)} e^{\delta t} |f_0(t)|. \tag{5.27}
\]

**Proof.** The proof is similar to that of Lemma 5.8. Suppose that (5.27) does not hold. Then there is a sequence $(\varepsilon_i, T_i, R_i, w_{T_i}, f_{0,i})$ such that
\begin{enumerate}
\item $\varepsilon_i \to 0$ as $i \to \infty$;
\item $\mathcal{L}_0^\varepsilon(w_{T_i}) = f_{0,i}$ in $(-\infty, T_i)$, with $\ddot{w}_{T_i}(T_i) = \dot{w}_{T_i}(T_i) = w_{T_i}(T_i) = 0$, for all $i \in \mathbb{N}$;
\item $\sup_{t \in (-\log R_i, T_i)} e^{\delta t} |f_0(t, \theta)| = 1$, for every $i \in \mathbb{N}$;
\item $\sup_{t \in (-\log R_i, T_i)} e^{\delta t} |w_{T_i}(t, \theta)| \to \infty$ as $i \to \infty$.
\end{enumerate}
There exists $t_i \in [-\log R_i, T_i]$ such that
\[
A_i := \sup_{t \in [-\log R_i, T_i]} e^{\delta t} |w_{T_i}(t)| = e^{\delta t_i} |w_{T_i}(t_i)|.
\]
Now set $w_i(t) = A_i^{-1} e^{\delta t_i} w_{T_i}(t + t_i)$ and $f_i(t) = A_i^{-1} e^{\delta t_i} f_{0,i}(t + t_i)$, for $t \in [-\log R_i - t_i, T_i - t_i]$ and every $i \in \mathbb{N}$. Note that $|w_i(t)|$ and $|f_i(t)|$ are bounded by a positive constant times $e^{-\delta t}$. Besides
\[
\mathcal{L}_i^0(w_i) = f_i \text{ in } (-\log R_i - t_i, T_i - t_i) \tag{5.28}
\]
with $\ddot{w}_i(T_i - t_i) = \dot{w}_i(T_i - t_i) = \dot{w}_i(T_i - t_i) = 0$, as well as
\[
\sup_{t \in [-\log R_i - t_i, T_i - t_i]} e^{\delta t} |w_i(t)| = |w_i(0)| = 1
\]
and
\[
\sup_{t \in [-\log R_i - t_i, T_i - t_i]} e^{\delta t} |f_i(t)| \to 0, \quad \text{as } i \to \infty,
\]
where
\[
\mathcal{L}_i^0(w_i) = \ddot{w}_i - \frac{n(n - 4)}{2} \dot{w}_i + \left( \frac{n^2(n - 4)^2}{16} - \frac{n(n + 4)(n^2 - 4)}{16} \frac{s}{v_\varepsilon^{n-4}} \right) w_i
\]
and $v_i(t) = v_{\varepsilon_i}(t + t_i)$. Up to a subsequence, $-\log R_i - t_i \to \tau_1 \in [-\infty, 0]$ and $T_i - t_i \to \tau_2 \in [0, \infty]$. 

**Claim:** $\tau_2 > 0$. 

If $\tau_2 = 0$, since $|w_i(0)| = 1$ and $w_i(T_i - t_i) = 0$, then $\dot{w}_i$ must blow up in $[T_i - t_i - 1, T_i - t_i]$, as $i \to \infty$. Using that $|w_i(t)|, |f_i(t)| \leq Ce^{-\delta(T_i-t_i)}$ in this interval and the boundary condition in (5.28), we integrate the ODE to obtain that

$$|\ddot{w}_i(t)| \leq Ce^{-\delta(T_i-t_i)},$$

in the same interval. Thus, since $T_i - t_i \to 0$, we obtain a contradiction. The claim follows.

The claim implies that the interval $(\tau_1, \tau_2)$ is not empty.

As in the proof of Lemma 5.8 we can suppose that $v_i \to v_{\infty}$ in $C_{\text{loc}}^4(\mathbb{R})$, where $v_{\infty}$ satisfies (2.3) and is such that either $v_{\infty} \equiv 0$ or $0 < v_{\infty} < 1$ with $\lim_{t \to \pm \infty} v_{\infty}(t) = 0$. Passing to a subsequence if necessary, there exists a function $w_{\infty}$ such that $w_i \to w_{\infty}$ in $C_{\text{loc}}^3(\tau_1, \tau_2)$ and satisfies the equation

$$\mathcal{L}_\infty^j(w_{\infty}) = \ddot{w}_{\infty} - \frac{n(n-4) + 8}{2} \dot{w}_{\infty} + \left(\frac{n^2(n-4)^2}{16} - \frac{n(n+4)(n^2-4)}{16} \frac{\delta}{v_{\infty}^3}\right)\dot{v}_{\infty} = 0$$

in $(\tau_1, \tau_2)$. By the normalization $|w_i(0)| = 1$ we have $|w_{\infty}(0)| = 1$, and so $w_{\infty}$ is not trivial.

If $\tau_2 < \infty$, then $\ddot{w}_{\infty}(\tau_2) = \ddot{w}_{\infty}(\tau_2) = w_{\infty}(\tau_2) = 0$ implies that $w_{\infty} \equiv 0$, which is a contradiction. If $\tau_2 = \infty$, we use the fact that $v_{\infty} \to 0$ as $|t| \to \infty$ to conclude that

$$w_{\infty}(t) \sim e^{\pm \mu t} \quad \text{as} \quad t \to +\infty,$$

where $\mu = \frac{n}{2}$ or $\frac{n-4}{2}$. Using that $|w_{\infty}(t)| \leq e^{-\delta t}$ with $\delta > \frac{n-4}{2}$, we get that $w_{\infty} \equiv 0$, which again is a contradiction. \qed

Since the estimate (5.27) is independent of the parameters $T$, $R$ and $\varepsilon$, we let $T \to +\infty$ to obtain a function $w_0$ which satisfies

$$\mathcal{L}_\infty^j(w_0) = f_0$$

in $\mathbb{R}$ with the estimate

$$\sup_{t \in (-\log R, +\infty)} e^{\delta t} |w_0(t)| \leq C \sup_{t \in (-\log R, +\infty)} e^{\delta t} |f_0(t)|,$$

where $\delta > \frac{n-4}{2}$ and the positive constant $C$ does not depend on $\varepsilon \in (0, \varepsilon_0)$ and $R$. If $f_0$ belongs to $C^0_{\delta, \alpha}(-\log R, +\infty)$, then $w_0$ belongs to $C^4_{\delta, \alpha}(-\log R, +\infty)$ and there exists a positive constant $C$ which does not depend on $\varepsilon$ and $R$ such that

$$\|w_0\|_{C^4_{\delta, \alpha}(-\log R, +\infty)} \leq C \|f_0\|_{C^0_{\delta, \alpha}(-\log R, +\infty)}.$$

There is no reason why the function $w_0$ satisfies the boundary condition $\ddot{w}_0 = w_0 = 0$ in $t = -\log R$. Although it would be nice to find a right inverse with these conditions, we can not control the boundary condition in the right function space.

5.2.3. Case $j = 1, \ldots, n$. Projecting (5.18) along the eigenfunction $\varphi_j$, with $j = 1, \ldots, n$, we get

$$\begin{cases}
\mathcal{L}_\varepsilon^j(w_j) = f_j & \text{in} \quad (-\log R, \infty) \\
\ddot{w}_j(-\log R) = 0 \\
\dot{w}_j(-\log R) = 0,
\end{cases}$$

where $\mathcal{L}_\varepsilon^j$ is given by (5.19). The proof is similar to the case $j = 0$. First we consider an extension of $f_j$, still denoted by $f_j$, to $\mathbb{R}$ and find the unique solution $w_T$ of

$$\mathcal{L}_\varepsilon^j(w_T) = f_j \quad \text{in} \quad (-\infty, T),$$

with $\ddot{w}_T(T) = \dot{w}_T(T) = \dot{w}_T(T) = w_T(T) = 0$. In a similar manner of the case $j = 0$ we can prove the following lemma.
Lemma 5.10. For $\mu > 1$, let $\delta = \frac{n-4}{2} + \mu$. Then there are constants $\varepsilon_0 > 0$ and $C > 0$ independent of $T$ and $R > 0$, such that, for all $\varepsilon \in (0, \varepsilon_0]$, we have

$$\sup_{t \in (-\log R, T)} e^{\delta |w_T(t, \theta)|} \leq C \sup_{t \in (-\log R, T)} e^{\delta |f_0(t, \theta)|}.$$ 

The proof of this lemma is similar to that of the Lemma 5.9, so we will not include it here. The fact that $\lambda_j = n - 1$ for each $j = 1, 2, \ldots, n$ forces us to choose a weight $\delta > \frac{n-2}{2}$. As in the case of $j = 0$, as $T \to +\infty$ we get, for every $j = 1, \ldots, n$ a solution $w_j$ to the equation

$$\mathcal{L}_j^j(w_j) = f_j$$

in the whole $\mathbb{R}$. Moreover, if $f_j \in C^{0, \alpha}_\delta(-\log R, +\infty)$ with $\delta > \frac{n-2}{2}$, then $w_j \in C^{4, \alpha}_\delta(-\log R, +\infty)$ and there exists a positive constant $C$, which does not depend on $R$ and $\varepsilon$, such that

$$\|w_j\|_{C^{4, \alpha}_\delta(-\log R, +\infty)} \leq C\|f_j\|_{C^{0, \alpha}_\delta(-\log R, +\infty)}.$$

Combining the previous three lemmas we obtain the following proposition,

Proposition 5.11. Let $R > 0$, $\alpha \in (0, 1)$ and $\mu \in (1, 2)$. Then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, there is an operator

$$G_{\varepsilon, R} : C^{0, \alpha}_{\mu-4}(B_1(0) \setminus \{0\}) \to C^{4, \alpha}_{\mu}(B_1(0) \setminus \{0\})$$

with the norm bounded independently of $\varepsilon$ and $R$, such that for all $f \in C^{0, \alpha}_{\mu-4}(B_1(0) \setminus \{0\})$, the function $w := G_{\varepsilon, R}(f)$ solves the equation

$$\begin{cases}
L_{\varepsilon, R}(w) = f & \text{in } B_1(0) \setminus \{0\} \\
\pi''(w|_{S^{n-1}}) = 0 & \text{on } \partial B_1(0) \\
\pi''(\Delta w|_{S^{n-1}}) = 0 & \text{on } \partial B_1(0).
\end{cases}$$

(5.29)

Moreover, if $f \in C^{n''}(C^{0, \alpha}_{\mu-4}(B_1(0) \setminus \{0\}))$, then $w \in C^{n''}(C^{4, \alpha}_{\mu}(B_1(0) \setminus \{0\}))$ and we may take $\mu \in (2 - n, 2)$.

We will work with the solution $u_{\varepsilon, R, a}$ given in (2.16) and with functions defined in $B_{r}(0) \setminus \{0\}$ with $0 < r < 1$. Then it is convenient to study the operator $L_{\varepsilon, R, a}$ in weighted function spaces defined in $B_{r}(0) \setminus \{0\}$.

Our construction requires a right-inverse for the linearized operator in a ball of radius $r > 0$, which will be very small. We obtain the solution operator in $B_{r}(0) \setminus \{0\}$ from the solution operator in the punctured unit ball by rescaling.

First, we observe that if $f \in C^{0, \alpha}_{\mu-4}(B_1(0) \setminus \{0\})$ and $w \in C^{4, \alpha}_{\mu}(B_1(0) \setminus \{0\})$ satisfy (5.29), then the function $f_r(x) = r^{-4} f(r^{-1} x)$ and $w_r(x) = w(r^{-1} x)$ satisfy

$$\begin{cases}
L_{\varepsilon, R}(w_r) = f_r & \text{in } B_r(0) \setminus \{0\} \\
\pi''(w_r|_{S^{n-1}}) = 0 & \text{on } \partial B_r(0) \\
\pi''(\Delta w_r|_{S^{n-1}}) = 0 & \text{on } \partial B_r(0).
\end{cases}$$

(5.29)

Also, it is not difficulty to see that

$$\|w_r\|_{C^{(4, \alpha)}_{\mu-4}(B_{r}(0) \setminus \{0\})} \leq c\|f_r\|_{C^{(0, \alpha)}_{\mu-4}(B_{r}(0) \setminus \{0\})},$$

where $c > 0$ is a constant that does not depend on $\varepsilon$, $R$, $a$ and $r$.

Finally using a perturbation argument we obtain the following two propositions.

Proposition 5.12. Let $R > 0$, $\alpha \in (0, 1)$ and $\mu \in (1, 2)$. Then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, $a \in \mathbb{R}^n$ and $0 < r \leq 1$ with $|a|r \leq r_0$ for some $r_0 \in (0, 1)$, there is an operator

$$G_{\varepsilon, R, a, r} : C^{0, \alpha}_{\mu-4}(B_{r}(0) \setminus \{0\}) \to C^{4, \alpha}_{\mu}(B_{r}(0) \setminus \{0\})$$
with the norm bounded independently of \(\varepsilon, R, a\) and \(r\), such that for all \(f \in C^\alpha_{\mu - 4}(B_r(0)\{0}\})\), the function \(w := G_{\varepsilon,R,a}(f)\) solves the equation

\[
\begin{align*}
L_{\varepsilon,R,a}(w) &= f \quad \text{in} \quad B_r(0)\{0}\}, \\
\pi''(w|_{\overline{S}^{n-1}}) &= 0 \quad \text{on} \quad \partial B_r(0) \\
\pi''(\Delta w|_{\overline{S}^{n-1}}) &= 0 \quad \text{on} \quad \partial B_r(0).
\end{align*}
\]

Moreover, if \(f \in \pi''(C^\alpha_{\mu - 4}(B_r(0)\{0}\})\), then \(w \in \pi''(C^\alpha_{\mu - 4}(B_r(0)\{0}\})\) and we may take \(\mu \in (2 - n, 2)\).

**Proof.** Observe that

\[
(L_{\varepsilon,R,a} - L_{\varepsilon,R})v = c(n)(u_{\varepsilon,R,a}^{\frac{8}{n-4}} - u_{\varepsilon,R}^{\frac{8}{n-4}})v.
\]

By (2.16) we have

\[
u_{\varepsilon,R,a}^{\frac{8}{n-4}}(x) = \frac{|x - a|}{|x|}^{n-4} \left( - \log |x| + \log \frac{x}{|x|} + \log R \right).
\]

However, \(|x - a| = |x|^4 + O(|a||x|^{-3})\), \(|x/|x| - a| = O(|a||x|)\) and

\[
v_{\varepsilon}^{\frac{8}{n-4}}(x) = \frac{8}{n-4} \frac{\log |x| - \log |a|}{|x| - |a|} + \frac{8}{n-2} \int_0^{\log \frac{|x|}{|a|}} \left( \frac{12 - n}{8} \right) (- \log |x| + \log R + t) dt.
\]

Therefore, using (2.22) and fact that \(0 < v_\varepsilon < 1\) we get

\[
|(u_{\varepsilon,R,a}^{\frac{8}{n-4}} - u_{\varepsilon,R}^{\frac{8}{n-4}})(x)| \leq c_n |x|^{-4} \int_0^{O(|a||x|)} \frac{8}{v_{\varepsilon}^{\frac{8}{n-4}}} (- \log |x| + \log R + t) dt + O(|a||x|^{-3})
\]

(5.30)

This implies that

\[
\|u_{\varepsilon,R,a}^{\frac{8}{n-4}} - u_{\varepsilon,R}^{\frac{8}{n-4}}(x)\|_{(0,\alpha),|x|,2\varepsilon} \leq c_n |a|^{-3},
\]

which implies that

\[
\|(L_{\varepsilon,R,a} - L_{\varepsilon,R})v\|_{(0,\alpha),\mu - 4,r} \leq c_n |a||v|_{(4,\alpha),\mu,r}.
\]

Therefore, by an perturbation argument we obtain the result. \(\square\)

### 5.3. Nonlinear analysis.

The main goal of this section is to construct constant \(Q\)-curvature metrics in the punctured ball \(B_r(p)\) that are close to the deformed Delaunay metrics \(g_{\varepsilon,R,a} = u_{\varepsilon,R,a}^{\frac{4}{n-4}}\) with partially prescribed boundary Navier boundary data. More precisely, given \(\phi_0, \phi_2 \in C^4(\overline{S}^{n-1})\) we want to find a complete metric \(g_{\text{int}}\) in the punctured ball \(B_r(p)\{p\}\) with \(Q\)-curvature equal to \(\frac{n(n^2-4)}{8}\) where

\[
g_{\text{int}} = (u_{\varepsilon,R,a} + \Upsilon + v_{\phi_0,\phi_2} + w_{\varepsilon,R} + v)^{\frac{4}{n-4}} g.
\]

Here

\[
\Upsilon := -\frac{\beta_\varepsilon}{2} R^{-\frac{2}{n}} |x|^2,
\]

(5.31)

where \(\beta_\varepsilon\) is defined in (2.11), \(v_{\phi_0,\phi_2} = \mathcal{P}_r(\phi_0, \phi_2)\), where \(\mathcal{P}_r\) is the Poisson operator we constructed in Corollary 4.1, and

\[
w_{\varepsilon,R} = \begin{cases}
0, & 5 \leq n \leq 9 \\
-G_{\varepsilon,R,a}(\mathcal{P}(u_{\varepsilon,R})), & n \geq 10,
\end{cases}
\]

where \(G_{\varepsilon,R,a}\) is given in Proposition 5.12 and \(\mathcal{P}\) is given in Lemma 5.6. Note that \(\Upsilon\) is biharmonic.
Following (1.4) we arrive at the boundary value problem

\[ H_{g}(u_{\varepsilon,R,a}+\Upsilon+v_{\phi_0,\phi_2}+w_{\varepsilon,R}+v) = 0, \quad \pi''(v|_{\partial B_{r_0}}) = 0, \quad \pi''((\Delta v)|_{\partial B_{r_0}}) = 0. \]  

(5.32)

for some 0 < r ≤ r_1. We can expand (5.32) as

\[ L_{\varepsilon,R,a}(v) = (\Delta^2 - P_{g})(u_{\varepsilon,R,a}+\Upsilon+v_{\phi_0,\phi_2}+w_{\varepsilon,R}+v) - \mathcal{R}_{\varepsilon,R,a}(\Upsilon+v_{\phi_0,\phi_2}+w_{\varepsilon,R}+v) \]

\[ - L_{\varepsilon,R,a}(w_{\varepsilon,R}) \left[ \frac{n(n^2-4)(n+4)}{16} u_{\varepsilon,R,a}(\Upsilon+v_{\phi_0,\phi_2}+w_{\varepsilon,R}) =: \mathcal{M}_{\varepsilon,R,a}(v), \right. \]

where \( L_{\varepsilon,R,a} \) is defined in (5.16), \( P_{g} \) is the Paneitz-Branson operator (1.3), \( \mathcal{R}_{\varepsilon,R,a} := \mathcal{R}^{u_{\varepsilon,R,a}} \) is defined in (5.15).

We first show that for a suitable choice of parameters the right hand side \( \mathcal{M}_{\varepsilon,R,a}(v) \) lies in the space \( C_{\mu-4}^{0,\alpha}(B_{r_0}(0)\setminus\{0\}) \), which is the domain of the right inverse \( G_{\varepsilon,R,a} \) constructed in Proposition 5.12. This will allow us to consider the map defined in (5.15).

Remark 5.13. Throughout this work let \( \delta_0 > 0 \) and \( m > \delta_2 > \delta_1 > 0 \) be sufficiently small, but fixed, constants. We fix \( s = \frac{2}{n-4} - \delta_0 \) and \( r_\varepsilon = \alpha_\varepsilon^s \) and choose \( a \in \mathbb{R}^n \) such that \(|a| r_\varepsilon \leq 1\). Finally we let \( b \in \mathbb{R} \) satisfy \(|b| \leq 1/2\) and choose \( R = \left( \frac{\alpha_\varepsilon}{2+2b} \right)^\frac{n}{n-4} \).

From the parameter choices in Remark 5.13 and (2.16) it follows that there are positive constants \( c_1 \) and \( c_2 \) independent of \( R \) and \( a \) such that

\[ c_1 |x|^\frac{4-n}{2} \leq u_{\varepsilon,R,a}(x) \leq c_2 |x|^\frac{4-n}{2}, \]

for each \( x \in B_{r_\varepsilon}(0)\setminus\{0\} \). Furthermore, since \( s > 8/(n(n-4)) \) when \( \delta_0 \) is sufficiently small, there exists a constant \( c_3 \) such that

\[ c_1 |x|^\frac{4-n}{2} \leq u_{\varepsilon,R,a} + \Upsilon \leq c_3 |x|^\frac{4-n}{2}. \]

(5.35)

Lemma 5.14. Let \( \mu \in (1,3/2) \). There exist \( \varepsilon_0 > 0 \) and \( m > \delta_1 > 0 \), such that for each \( \varepsilon \in (0,\varepsilon_0) \), \( a \in \mathbb{R}^n \) with \(|a| r_\varepsilon \leq 1\), and for all \( v_i \in C_{\mu}^{4,\alpha}(B_{r_\varepsilon}(0)\setminus\{0\}) \), \( i = 0,1 \), and \( w \in C_{5+d-\frac{\mu}{2}}^{4,\alpha}(B_{r_\varepsilon}(0)\setminus\{0\}) \) with \( \|w\|_{(4,\alpha),5+d-\frac{\mu}{2},r_\varepsilon} \leq c \) and \( \|v_i\|_{(4,\alpha),\mu,r_\varepsilon} \leq c r_\varepsilon^{m-\mu-\delta_1} \) there exists a constant \( C > 0 \) independent of \( \varepsilon, R \) and \( a \) such that \( \mathcal{R}_{\varepsilon,R,a} \) satisfies

\[ \|\mathcal{R}_{\varepsilon,R,a}(\Upsilon+w+v_1) - \mathcal{R}_{\varepsilon,R,a}(\Upsilon+w+v_0)\|_{(0,\alpha),\mu-4,r_\varepsilon} \leq C \varepsilon^{\delta'} \|v_1 - v_0\|_{(4,\alpha),\mu,r_\varepsilon} \]

and

\[ \|\mathcal{R}_{\varepsilon,R,a}(\Upsilon+v_1 + w)\|_{(0,\alpha),\mu-4,r_\varepsilon} \leq C \varepsilon^{\delta'} r_\varepsilon^{m-\mu}, \]

for some \( \delta' > 0 \).
Proof. First, by (5.15), we can write
\[ R_{\varepsilon,R,a}(Y + w + v_1) - R_{\varepsilon,R,a}(Y + w + v_0) = -\frac{n(n + 4)^2}{2}(v_1 - v_0) \int_0^1 \int_0^1 (u_{\varepsilon,R,a} + szt) \frac{12-s}{n-s} z_t dt ds \]
where \( z_t = Y + w + tv_1 + (1 - t)v_0 \). Combining this last displayed equation with (5.35) implies
\[ c|x|^{\frac{4-n}{2}} \leq u_{\varepsilon,R,a}(x) + Y(x) + v_i(x) + w(x) \leq C|x|^{\frac{4-n}{2}} \]
for sufficiently small \( \varepsilon > 0 \). Thus
\[ \max_{0 \leq s,t \leq 1} \|(u_{\varepsilon,R,a} + szt)\|_{(0,\alpha),[\sigma,2\sigma]} \leq C\varepsilon^{\lambda_n} \sigma^{\frac{n-12}{n-4}}, \]
where \( \lambda_n = 0 \) for \( 5 \leq n \leq 12 \) and \( \lambda_n = \frac{12-n}{n-4} \) for \( n \geq 13 \), which in turn yields
\[
\sigma^{4-\mu}\|R_{\varepsilon,R,a}(Y + w + v_1) - R_{\varepsilon,R,a}(Y + w + v_0)\|_{(0,\alpha),[\sigma,2\sigma]} \leq \nonumber \\
C\varepsilon^{\lambda_n} \sigma^{\frac{n-4}{n-2}}\|v_1 - v_0\|_{(4,\alpha),\mu,r_\varepsilon} (\|w\|_{(0,\alpha),[\sigma,2\sigma]} + \|v_1\|_{(0,\alpha),[\sigma,2\sigma]} + \|v_2\|_{(0,\alpha),[\sigma,2\sigma]} + \|Y\|_{(0,\alpha),[\sigma,2\sigma]}) \\
\leq C\varepsilon^{\lambda_n - \frac{4-n}{n-4} + \frac{n-4}{2} s} \|v_1 - v_0\|_{(4,\alpha),\mu,r_\varepsilon} \]
and
\[
\sigma^{4-\mu}\|R_{\varepsilon,R,a}(Y + v_1 + w)\|_{(0,\alpha),[\sigma,2\sigma]} \leq \nonumber \\
C\varepsilon^{\lambda_n} \sigma^{\frac{n-4}{n-2} - \mu}(\|v_1\|_{(0,\alpha),[\sigma,2\sigma]} + \|w\|_{(0,\alpha),[\sigma,2\sigma]} + \|Y\|_{(0,\alpha),[\sigma,2\sigma]}) \\
\leq C\varepsilon^{\lambda_n - \frac{4-n}{n-4} + \frac{n-4}{2} s + \frac{n-4}{2} r_\varepsilon^{-\mu}}, \]
with \( \lambda_n - \frac{4-n}{n-4} + \frac{n-4}{2} s > 0 \) and \( \lambda_n - \frac{8}{n-4} + \frac{n-4}{2} > 0 \). From these inequalities we obtain the result. \( \square \)

Lemma 5.15. Let \( 5 \leq n \leq 9 \), let \( \mu \in (1,3/2) \), and let \( c \) and \( \kappa \) be fixed, positive constants. There exists \( \varepsilon_0 > 0 \) such that for each \( \varepsilon \in (0,\varepsilon_0) \), for all \( v \in C^{\mu,\alpha}_H(B_{r_\varepsilon}(0)\{0\}) \), \( \phi_0 \in C^{\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon}) \) and \( \phi_2 \in \pi''(C^{\mu,\alpha}(\mathbb{S}^{n-1}_{r_\varepsilon})) \), with \( \|v\|_{(4,\alpha),\mu,r_\varepsilon} \leq cr_\varepsilon^{-\mu-\delta_1} \) and \( \|\phi_0,\phi_2\|_{(4,\alpha),r_\varepsilon} \leq \kappa r_\varepsilon^{-\mu-\delta_1} \), the right-hand side of (5.33) belongs to \( C^{\mu,\alpha}_H(B_{r_\varepsilon}(0)\{0\}) \).

Proof. By Corollary 4.2, the norm of \( v_{\phi_0,\phi_2} + v \) is bounded by a positive constant times \( r_\varepsilon^{-\delta_1} \), so Lemma 5.14 implies the term \( R_{\varepsilon,R,a}(Y + v_{\phi_0,\phi_2} + v) \) belongs to \( C^{\mu,\alpha}_H(B_{r_\varepsilon}(0)\{0\}) \). We also have
\[
\frac{1}{\varepsilon^{n-4}} v_{\varepsilon,R,a}v_{\phi_0,\phi_2} = O(\|x\|^{-2}) = O(\|x\|^{-\delta_1}). \]

To prove the remaining term of (5.33) has the right decay, we write
\[
(\Delta^2 - P_g)(u_{\varepsilon,R,a} + v_{\phi_0,\phi_2} + v) = (\Delta^2 - \Delta^2_g) u_{\varepsilon,R,a} + (\Delta^2 - P_g)(v_{\phi_0,\phi_2} + v) + (\Delta^2_g - P_g)u_{\varepsilon,R,a} \]
and estimate each of these terms separately. Observe that \( v_{\phi_0,\phi_2} = O(\|x\|^2) \) and \( v = O(\|x\|^\mu) \), and so the second terms decays like \( O(\|x\|^{-\delta_1}) \). By Lemma 5.7 we also have the estimate \( (\Delta^2 - \Delta^2_g) u_{\varepsilon,R,a} = O(\|x\|^{-\mu-\delta_4}) \), and so it only remains to estimate \( (\Delta^2_g - P_g)u_{\varepsilon,R,a} \).

Observe that (5.2) gives us \( R_g = O(\|x\|^2) \), \( \text{Ric}_g = O(\|x\|) \) and \( Q_g = O(1) \), and so from (5.10) we obtain directly that \( (\Delta^2_g - P_g)u_{\varepsilon,R,a} = O(\|x\|^{1-\frac{\mu}{2}}) \) for \( 5 \leq n \leq 7 \). For dimension 8 and 9, (5.11) yields
\[
\langle \text{Ric}_g, \nabla^2 u_{\varepsilon,R} \rangle = R_g \frac{u'}{r} + O(\|x\|^{4-\frac{\mu}{2}}). \]

Thus, using that \( R_g = O(\|x\|^2) \), \( Q_g = O(1) \) and (5.10) we obtain \( (\Delta^2_g - P_g)u_{\varepsilon,R} = O(\|x\|^{2-\frac{\mu}{2}}) \). Now, using (2.18) and (5.10) we get \( (\Delta^2_g - P_g)(\varepsilon,R,a) = O(\|x\|^{2-\frac{\mu}{2}}) \). Therefore, \( (\Delta^2_g - P_g)u_{\varepsilon,R,a} = \)
$O(|x|^{-\frac{n}{2}})$ for $n = 8$ or 9. In either case we have $(\Delta_g^2 - P_g)u_{\epsilon,R,a} = O(|x|^{\mu - 4})$. This completes the proof, since $\Upsilon = O(|x|^2)$.

Let us now consider the case $n \geq 10$. Since $(P_g - \Delta_g^2)u_{\epsilon,R,a} = O(|x|^{d + 1 - \frac{n}{2}})$, unfortunately $(P_g - \Delta_g^2)u_{\epsilon,R,a} \not\subset C_{\mu-4}^{0,\alpha}(B_r(0) \setminus \{0\})$ for any $\mu > 1$, and so we cannot use the right inverse of $L_{\epsilon,R,a}$ directly. To overcome this difficulty we will consider the expansion (2.18) and the fact that, by Lemma 5.6, $\mathcal{R}(u_{\epsilon,R})$ is orthogonal to $\{1, x_1, \ldots, x_n\}$. If follows from this fact and Proposition 5.12 that there exists $w_{\epsilon,R} \in \pi''(C_{5+d-\frac{n}{2}}^{1,\alpha}(B_r(0) \setminus \{0\}))$ such that

$$L_{\epsilon,R,a}(w_{\epsilon,R}) = -\mathcal{R}(u_{\epsilon,R}).$$

(5.36) This auxiliary function $w_{\epsilon,R}$ will eliminate the terms that were preventing us from applying the right inverse to the right hand-side of (5.33). From this, the expansion (2.18) and (5.33) we obtain

$$L_{\epsilon,R,a}(v) = \big[(\Delta^2 - P_g)(u_{\epsilon,R,a} - u_{\epsilon,R} + \Upsilon + v_{\phi_0,\phi_2} + w_{\epsilon,R} + v) + \frac{1}{(n+4)(n^2 - 4)} u_{\epsilon,R,a}(\Upsilon + v_{\phi_0,\phi_2}) + \mathcal{R}_{\epsilon,R,a}(\Upsilon + v_{\phi_0,\phi_2} + w_{\epsilon,R} + v) - \mathcal{R}\big],$$

(5.37) where Lemma 5.6 implies $\mathcal{R} = O(|x|^{2d+4-\frac{n}{2}})$.

**Lemma 5.16.** Suppose $n \geq 10$, $\mu \in (1, 3/2)$, $\kappa > 0$ and $c > 0$ be fixed constants. Under the same assumption of Lemma 5.15 the right-hand side of (5.37) belongs to $C_{\mu-4}^{0,\alpha}(B_r(0) \setminus \{0\})$.

**Proof.** The terms $(\Delta^2 - P_g)(v_{\phi_0,\phi_2} + v)$, $u_{\epsilon,R,a}v_{\phi_0,\phi_2}$ and $\mathcal{R}_{\epsilon,R,a}(\Upsilon + v_{\phi_0,\phi_2} + w_{\epsilon,R} + v)$ are similar to the proof in Lemma 5.15, while the term $\mathcal{R} \in C_{\mu-4}^{0,\alpha}(B_r(0) \setminus \{0\})$ because $2d + 4 - \frac{n}{2} > \mu - 4$.

Since $w_{\epsilon,R} = O(|x|^{5+d-\frac{n}{2}})$, (5.4) and (5.13) imply $(\Delta^2 - \Delta_g^2)w_{\epsilon,R} = O(|x|^{2d+4-\frac{n}{2}})$, with $2 + 2d - n/2 > \mu - 4$. Observe that $\Delta_g - P_g$ is a second order operator and $3 + d - n/2 > \mu - 4$. Thus $(\Delta^2 - P_g)w_{\epsilon,R} = O(|x|^{\mu - 4})$. In an analogous way, by (2.18), we obtain $(\Delta^2 - P_g)(u_{\epsilon,R,a} - u_{\epsilon,R}) = O(|x|^{2d+4-\frac{n}{2}})$ with $2 + d - \frac{n}{2} > \mu - 4$. This completes the proof, since $\Upsilon = O(|x|^2)$.

**Remark 5.17.** The vanishing of the Weyl tensor up to order $d = \left\lceil \frac{n-8}{2} \right\rceil$ is sharp in the following sense: If the Weyl tensor vanishes up to order $d - 1$, then for $n \geq 10$, $g_{ij} = \delta_{ij} + O(|x|^{d+2})$ and

$$(\Delta^2 - \Delta_g^2)(u_{\epsilon,R,a} - u_{\epsilon,R}) = O(|x|^{1+d-\frac{n}{2}}),$$

which implies that $(\Delta^2 - \Delta_g^2)(u_{\epsilon,R,a} - u_{\epsilon,R}) \notin C_{\mu-4}^{0,\alpha}(B_r(0) \setminus \{0\})$.

### 5.3.1. Fixed point argument

In this section we prove the map

$$\mathcal{N}_{\epsilon}(R,a,\phi_0,\phi_2,v) = G_{\epsilon,R,a,r}(\mathcal{M}_{\epsilon,R,a}(v)),$$

defined in (5.34) has a fixed point, where $\mathcal{M}_{\epsilon,R,a}(v)$ is the right hand side of (5.33) for $5 \leq n \leq 9$, and the right hand side of (5.37) for $n \geq 10$.

**Lemma 5.18.** For all $\mu \in \mathbb{R}$ and $v \in C_{\mu}^{4,\alpha}(B_r(0) \setminus \{0\})$ there exists a constant $c > 0$ that does not depend on $r$ and $\mu$ such that

$$\| (P_g - \Delta_g^2)v \|_{(0,\alpha),\mu - 4,r} \leq cr^{d+3}\| v \|_{(4,\alpha),\mu,r}$$

**Proof.** First we use (5.4) and (5.13) to obtain

$$\sigma^{4-\mu}\| (\Delta^2 - \Delta_g^2)v \|_{(0,\alpha),\sigma,2\sigma,\alpha} \leq C\sigma^{d+3-\mu}\| v \|_{(4,\alpha),\sigma,2\sigma,\alpha}.$$

Also, by (1.3) we get

$$\sigma^{4-\mu}\| (P_g - \Delta_g^2)v \|_{(0,\alpha),\sigma,2\sigma,\alpha} \leq C\sigma^{\max\{3,d+3\}-\mu}\| v \|_{(4,\alpha),\sigma,2\sigma,\alpha}.$$

Combining these two inequalities implies the result.

$\square$
Lemma 5.19. Given \( a \in \mathbb{R}^n \) with \( |a|r_\varepsilon^{1-\delta_2} \leq 1 \) and \( \|(\phi_0, \phi_2)\|_{(4, \alpha), r_\varepsilon} \leq \kappa r_\varepsilon^{m-\delta_1} \), if \( \varepsilon > 0 \) is sufficiently small there exists a constant \( \beta > 0 \) such that
\[
\left\| u_{\varepsilon, R, a}^{n, \frac{\pi}{2}} (\Upsilon + v_{\phi_0, \phi_2}) \right\|_{(0, \alpha), \mu - 4, r_\varepsilon} \leq c \varepsilon^\beta r_\varepsilon^{m-\mu}
\]
where \( m \) is defined in Remark 5.13.

Proof. First we note that by Remark 5.13 we obtain \( \log R = \frac{2}{n-4} \log \alpha_\varepsilon + \frac{2}{4-n} \log (2 + 2b) \). By (2.20) and (2.21) we have
\[
\left| x \right| \left| \frac{1}{n} \frac{x}{|x|} - a \right| = \left| x \right| \left| \frac{1}{n} \frac{1}{1 + O(r_\varepsilon^{\delta_2})} \right| \quad \text{and} \quad \log \left| \frac{x}{|x|} - a \right| = O(r_\varepsilon^{\delta_2})
\]
respectively. This implies that if \( r_\varepsilon^{1+\ell} \leq |x| \leq r_\varepsilon \) and \( \ell \geq 0 \) is small enough, we have
\[
\left( \frac{2}{n-4} - s \right) \log \alpha_\varepsilon - C \leq - \log |x| + \log \left| \frac{x}{|x|} - a \right| + \log R \leq \left( \frac{2}{n-4} - s(1 + \ell) \right) \log \alpha_\varepsilon + C.
\]
Observe \( 2/(n-4) - s > 0 \) by Remark 5.13. Thus, for \( 0 < \ell < \frac{2}{s(n-4)} - 1 \) and \( \varepsilon > 0 \) small enough (2.14) implies
\[
v_\varepsilon \left( - \log |x| + \log \left| \frac{x}{|x|} - a \right| + \log R \right) \leq c_n r_\varepsilon^{\frac{n-4}{2}}.
\]
By (2.16) and (5.38) we have \( u_{\varepsilon, R, a}(x) \leq C|x|^{-\frac{n+4}{2}}, \) and so
\[
u_{\varepsilon, R, a}(x) = c_n |x|^{-4 r_\varepsilon^{1+\ell}}
\]
for all \( r_\varepsilon^{1+\ell} \leq |x| \leq r_\varepsilon \). Then, for \( \frac{1}{4} r_\varepsilon^{1+\ell} \leq \sigma \leq \frac{1}{2} r_\varepsilon \), we have
\[
\sigma^4 \mu \left\| u_{\varepsilon, R}^{n, \frac{\pi}{2}} (\Upsilon + v_{\phi_0, \phi_2}) \right\|_{(0, \alpha), |\sigma, 2\sigma|} \leq c \sigma^\mu r_\varepsilon^{4 \left\| v_{\phi_0, \phi_2} \right\|_{(0, \alpha), |\sigma, 2\sigma|} + \| \Upsilon \|_{(0, \alpha), |\sigma, 2\sigma|}} \leq c (r_\varepsilon^{m-\mu+4-\delta_1} + r_\varepsilon^{6-\mu} \alpha_\varepsilon^{-\frac{4}{n-4}}) \leq cr_\varepsilon^{\beta m-\mu},
\]
with \( 6s - 4/(n-4) > 0 \) by Remark 5.13. For \( 0 \leq \sigma \leq 4r_\varepsilon^{1+\ell} \), we use \( (2-\mu) \ell - \delta_1 > 0 \) and \( (2+\mu(2-\mu))s - \frac{4}{n-4} > 0 \) to see
\[
\sigma^4 \mu \left\| u_{\varepsilon, R}^{n, \frac{\pi}{2}} (\Upsilon + v_{\phi_0, \phi_2}) \right\|_{(0, \alpha), |\sigma, 2\sigma|} \leq c \sigma^\mu \left\| \right\| v_{\phi_0, \phi_2} \right\|_{(0, \alpha), |\sigma, 2\sigma|} + \| \Upsilon \|_{(0, \alpha), |\sigma, 2\sigma|}} \leq cr_\varepsilon^{(2-\mu) \ell + \delta_1 r_\varepsilon^{m-\mu} + r_\varepsilon^{2-\mu} (2-\mu) \ell \alpha_\varepsilon^{-\frac{4}{n-4}} \leq r_\varepsilon^{\beta m-\mu},
\]
which completes our proof. \( \square \)

Proposition 5.20. Let \( \mu \in (1, 3/2), \tau > 0 \) and \( \kappa > 0 \) be fixed constants. There exists \( \varepsilon_0 > 0 \) such that for each \( \varepsilon \in (0, \varepsilon_0), a \in \mathbb{R}^n \) with \( |a|r_\varepsilon^{1-\delta_2} \leq 1 \), \( \phi_0 \in C^{4, \alpha}(S^{n-1}) \) and \( \phi_2 \in \pi''(C^{4, \alpha}(S^{n-1})) \) with \( \|(\phi_0, \phi_2)\|_{(4, \alpha), r_\varepsilon} \leq \kappa r_\varepsilon^{m-\delta_1} \), there exists a fixed point of the map \( N_\varepsilon(R, a, \phi_0, \phi_2, \cdot) \) in the ball of radius \( \tau r_\varepsilon^{m-\mu} \).

Proof. We will prove that the map \( N_\varepsilon(R, a, \phi_0, \phi_2, \cdot) \) is a contraction for \( \varepsilon > 0 \) small enough by proving the following two inequalities
\[
\left\| N_\varepsilon(R, a, \phi_0, \phi_2, 0) \right\|_{(4, \alpha), \mu, r_\varepsilon} < \frac{1}{2} \tau r_\varepsilon^{m-\mu}
\]
and
\[
\left\| N_\varepsilon(R, a, \phi_0, \phi_2, v_1) - N_\varepsilon(R, a, \phi_0, \phi_2, v_2) \right\|_{(4, \alpha), \mu, r_\varepsilon} < \frac{1}{2} \left\| v_1 - v_2 \right\|_{(4, \alpha), \mu, r_\varepsilon}
\]
for \( v_i \in C^{4, \alpha}(B_{r_\varepsilon}(0) \setminus \{0\}), i = 1, 2 \) with \( \left\| v_i \right\|_{(4, \alpha), \mu} \leq \tau r_\varepsilon^{m-\mu} \).
Case $5 \leq n \leq 9$: Since the right inverse $G_{\varepsilon,R,a,r}$ given by Proposition 5.12 is bounded independently of $\varepsilon$, $R$, $a$ and $r$, the inequality (5.39) follows once we estimate the $C_{\mu-4}^{0,\alpha}$-norm of the right hand side of (5.33) when $v = 0$. Using (2.18), Lemma 5.3 and the estimates in the proof of Lemma 5.7 we get

$$
\| (\Delta^2 - \Delta_g^2) u_{\varepsilon,R,a} \|_{(0,\alpha),\mu-4,r} \leq \| (\Delta^2 - \Delta_g^2) (u_{\varepsilon,R,a} - u_{\varepsilon,R}) \|_{(0,\alpha),\mu-4,r} \\
\leq c|a|^6 + d + \frac{5}{2} - \mu \leq c\varepsilon^\delta_2 r_{\varepsilon}^{m-\mu}.
$$

Here we have used $|a|r_{\varepsilon}^{1-\delta_2} \leq 1$, with $\delta_2 > 0$. By the proof of Lemma 5.15 we have $(\Delta_g^2 - P_g)u_{\varepsilon,R,a} = O(|x|^{2+d-\frac{2}{2}})$, which implies that

$$
\| (\Delta_g^2 - P_g)u_{\varepsilon,R,a} \|_{(0,\alpha),\mu-4,r} \leq c\varepsilon^\delta_2 r_{\varepsilon}^{m-\mu}.
$$

(5.41)

By Corollary 4.2 and Lemma 5.18 we have

$$
\| (\Delta^2 - P_g) v_{\phi_0,\phi_2} \|_{(0,\alpha),\mu-4,r} \leq c\varepsilon^{d+3} \| v_{\phi_0,\phi_2} \|_{(4,\alpha),\mu,r} \leq c\varepsilon^{d+3} - \mu \| (\phi_0, \phi_2) \|_{(4,\alpha),r}
$$

(5.42)

and

$$
\| (\Delta^2 - P_g) \Upsilon \|_{(0,\alpha),\mu-4,r} \leq \varepsilon^{-\frac{1}{n-4} + (d+5-m)} r_{\varepsilon}^{m-\mu},
$$

(5.43)

since $\Upsilon = O(\varepsilon^{-\frac{d}{4}}|x|^2)$. We also have

$$
\| Q_g v_{\phi_0,\phi_2} \|_{(0,\alpha),\mu-4,r} \leq c\varepsilon^{4-\mu} \| (\phi_0, \phi_2) \|_{(4,\alpha),r}.
$$

(5.44)

Using Lemma 5.14, one has

$$
\| R_{\varepsilon,R,a} (\Upsilon + v_{\phi_0,\phi_2}) \|_{(0,\alpha),\mu-4,r} \leq c\varepsilon^{\delta'} r_{\varepsilon}^{m-\mu},
$$

(5.45)

where $\delta' > 0$. Therefore, by (5.41), (5.42), (5.43), (5.44), (5.45) and Lemma 5.19 we obtain (5.39) for $\varepsilon > 0$ small enough. Now, using (5.34) with $\mathcal{M}_{\varepsilon,R,a}(v)$ as the right hand side of (5.33) we obtain

$$
\| N(\varepsilon, R, a, \phi_0, \phi_2, v_1) - N(\varepsilon, R, a, \phi_0, \phi_2, v_2) \|_{(4,\alpha),\mu,r} \leq C \left( \| (\Delta^2 - P_g) (v_1 - v_2) \|_{(0,\alpha),\mu-4,r} \\
+ \| R_{\varepsilon,R,a} (\Upsilon + v_{\phi_0,\phi_2} + v_1) - R_{\varepsilon,R,a} (\Upsilon + v_{\phi_0,\phi_2} + v_2) \|_{(0,\alpha),\mu-4,r} \right).
$$

By Lemmas 5.18 and 5.14 we obtain directly (5.40) for $\varepsilon > 0$ small enough.

Case $n \geq 10$: Now we use (5.34) with $\mathcal{M}_{\varepsilon,R,a}$ as the right hand side of (5.37). As in the previous case, to prove (5.39) it is enough to estimate the $C_{\mu-4}^{0,\alpha}$-norm of the right hand side of (5.37) when $v = 0$. Similar to the proof of Lemma 5.18, we obtain

$$
\| (\Delta^2 - P_g) w_{\varepsilon,R} \|_{(0,\alpha),\mu-4,r} \leq c\varepsilon^{8+2d-\frac{6}{2}} \| w_{\varepsilon,R} \|_{(4,\alpha),5+d-\frac{2}{2},r}. 
$$

(5.46)

By Lemma 5.14 we know that

$$
\| R_{\varepsilon,R,a} (\Upsilon + v_{\phi_0,\phi_2} + w_{\varepsilon,R}) \|_{(0,\alpha),\mu-4,r} \leq c\varepsilon^{\delta'} r_{\varepsilon}^{m-\mu},
$$

(5.47)

where $\delta' > 0$. Therefore, using $\mathcal{R} = O(\varepsilon^{2d+4-\frac{5}{2}})$, Lemma 5.19, (5.42), (5.46), (5.47) and a similar calculation to that of (5.41) we conclude that (5.39) holds. Moreover,

$$
\| N(\varepsilon, R, a, \phi_0, \phi_2, v_1) - N(\varepsilon, R, a, \phi_0, \phi_2, v_2) \|_{(4,\alpha),\mu,r} \leq C \left( \| (\Delta^2 - P_g) (v_1 - v_2) \|_{(0,\alpha),\mu-4,r} \\
+ \| R_{\varepsilon,R,a} (\Upsilon + v_{\phi_0,\phi_2} + w_{\varepsilon,R} + v_1) - R_{\varepsilon,R,a} (\Upsilon + v_{\phi_0,\phi_2} + w_{\varepsilon,R} + v_2) \|_{(0,\alpha),\mu-4,r} \right).
$$

By Lemma 5.14

$$
\| R_{\varepsilon,R,a} (\Upsilon + v_{\phi_0,\phi_2} + w_{\varepsilon,R} + v_1) - R_{\varepsilon,R,a} (\Upsilon + v_{\phi_0,\phi_2} + w_{\varepsilon,R} + v_2) \|_{(0,\alpha),\mu-4,r} \leq c\varepsilon^{\delta'} \| v_1 - v_2 \|_{(4,\alpha),\mu,r}.
$$

with $\delta' > 0$. From this and Lemma 5.18, we obtain (5.40) for $\varepsilon$ sufficiently small. □
Theorem 5.21. Let $\mu \in (1, 3/2)$, $\tau > 0$, $\kappa > 0$ and $\delta_2 > \delta_1 > 0$ be fixed constants. There exists $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, $|b| \leq 1/2$, $a \in \mathbb{R}^n$, $\phi_2 \in C^{4, \alpha}(\mathbb{S}^{n-1})$ and $\phi_0 \in \pi''(C^{4, \alpha}(\mathbb{S}^{n-1}))$, with $|a| r^{1-\delta_2} \leq 1$, $\|\phi_0, \phi_2\|_{(4, \alpha), r_\varepsilon} \leq kr^{m-\delta_1}$, there exists $U_{\varepsilon, R, a, \phi_0, \phi_2} \in C^{4, \alpha}(B_{r_\varepsilon}(0)\setminus\{0\})$ satisfying
\[
\begin{align*}
H_0(u_{\varepsilon, R, a} + \varphi + w_{\varepsilon, R} + v_{\phi_0, \phi_2} + U_{\varepsilon, R, a, \phi_0, \phi_2}) &= 0 \quad \text{in} \quad B_{r_\varepsilon}(0)\setminus\{0\} \\
\pi''(\Delta U_{\varepsilon, R, a, \phi_0, \phi_2}) &= 0 \quad \text{on} \quad \partial B_{r_\varepsilon}(0)
\end{align*}
\]
where $w_{\varepsilon, R} \equiv 0$ when $5 \leq n \leq 9$, $w_{\varepsilon, R} \in \pi''(C^{4, \alpha}(\mathbb{S}^{n-1})(B_{r_\varepsilon}(0)\setminus\{0\}))$ is the solution of (5.36) when $n \geq 10$. Moreover,
\[
\|U_{\varepsilon, R, a, \phi_0, \phi_2}\|_{(4, \alpha), \mu, r_\varepsilon} \leq \tau r^{m-\mu}
\]
and
\[
\|U_{\varepsilon, R, a, \phi_0, \phi_2} - U_{\varepsilon, R, a, \tilde{\phi}_0, \tilde{\phi}_2}\|_{(4, \alpha), \mu, r_\varepsilon} \leq cr^{3-\mu}\|\phi_0, \phi_2\|_{(4, \alpha), r_\varepsilon},
\]
for some constants $c > 0$ and $\delta_3 > 0$ that do not depend on $\varepsilon$, $R$, $a$ and $(\phi_0, \phi_2)$, $(\tilde{\phi}_0, \tilde{\phi}_2)$.

Proof. The solution $U_{\varepsilon, R, a, \phi_0, \phi_2}$ is the fixed point of the map $N_\varepsilon(R, a, \phi_0, \phi_2, \cdot)$ given by Proposition 5.20 with the estimate (5.48). Using this fact and (5.40), we obtain that
\[
\begin{align*}
\|U_{\varepsilon, R, a, \phi_0, \phi_2} - U_{\varepsilon, R, a, \tilde{\phi}_0, \tilde{\phi}_2}\|_{(4, \alpha), \mu, r_\varepsilon} &\leq 2\|N_\varepsilon(R, a, \phi_0, \phi_2, U_{\varepsilon, R, a, \phi_0, \phi_2}) - N_\varepsilon(R, a, \tilde{\phi}_0, \tilde{\phi}_2, U_{\varepsilon, R, a, \tilde{\phi}_0, \tilde{\phi}_2})\|_{(4, \alpha), \mu, r_\varepsilon}.
\end{align*}
\]

The estimate (5.49) now follows from the definition of $N_\varepsilon$ together with the estimates obtained in Proposition 5.20.

---

6. Exterior Problem

In the previous section we exploited the conformal structure of the equation to perturb a Delaunay-type solution to a exact solution to the Q-curvature problem in a small neighborhood of the singularity. To find a solution on the complement of this neighborhood whose boundary data matches the solution constructed in the previous section, we need to choose the approximate solution that will be perturbed on the exterior domain carefully. Such approximate solution must be close to the Delaunay-type solution $u_{\varepsilon, R, a}$ near to the boundary. Our asymptotic expansion in Corollary 2.3 shows that as $\varepsilon \to 0$ the function $u_{\varepsilon, R, a}$ is close to the Green’s function of the flat bi-Laplacian with a pole on the singularity, at least in a sufficiently small annulus. Consequently, since our manifold has constant $Q$-curvature (the constant function 1 is a solution to $H_{g_0}(1) = 0$), the natural approximate solution on the exterior domain will be the constant 1 plus the Green function with a pole at $p$.

Let $r_1 \in (0, 1)$ and $\Psi : B_{r_1}(0) \to M$ be a normal coordinate system with respect to $g = F^{1/4}g_0$ on $M$ centered in $p$, where $F$ is defined in Section 5.1. In these coordinates we have $g_{ij} = \delta_{ij} + O(|x|^2)$ and $F = 1 + O(|x|^2)$, which implies that $(g_0)_{ij} = \delta_{ij} + O(|x|^2)$. Remember that $g_0$ has constant $Q$-curvature equal to $n(n^2 - 8)/8$. Denote by $G_p(x)$ the Green’s function with pole at $p$ for $L_{g_0} = P_{g_0} - \frac{n(n-4)(n+4)}{16}$, the linearization of $H_{g_0}$ at the constant function 1. The Green’s function $G_p$ exists by our hypothesis that $g_0$ is nondegenerate. By a similar argument to the one in Proposition 2.7 of [23] we can normalize $G_p$ such that $\lim_{x \to 0} |x|^{n-4}G_p(x) = 1$. This implies that $|G_p(x)| \leq C|x|^{4-n}$, for all $x \in B_{r_1}(0)$. Our main goal in this section is to solve the following problem
\[
H_{g_0}(1 + \lambda G_p + u) = 0 \quad \text{on} \quad M \setminus B_r(p),
\]
with $\lambda \in \mathbb{R}$, $r \in (0, r_1)$ and prescribed boundary data. Similarly to the strategy of the previous section, expanding this equation we obtain
\[
H_{g_0}(1 + \lambda G_p + u) = L_{g_0}(u) + R(\lambda G_p + u) = 0,
\]
since \( H_{g_0}(1) = 0 \) and \( L_{g_0}(G_p) = 0 \). Here \( \mathcal{R} \) is defined in (5.15) with \( u_0 = 1 \). In order to apply a fixed point theorem in this case, it is necessary to show that the linearized operator in \( M \setminus B_r(p) \) has a right inverse. At this point, the nondegeneracy of the metric will be necessary.

### 6.1. Linear analysis.

The goal of this subsection is to find a right inverse of the operator \( L_{g_0} \). To this aim, given \( 0 < r < s \) consider in \( \mathbb{R}^n \) the annulus \( \Omega_{r,s} := B_s(0) \setminus B_r(0) \). We start proving the following result.

#### Proposition 6.1.

Let \( \nu \in (4 - n, 5 - n) \) fixed and \( 0 < 2r < s \). There exists an operator \( \mathcal{G}_{r,s} : C^{0, \alpha}_{\nu - 4}(\Omega_{r,s}) \to C^{4, \alpha}_{\nu - 4}(\Omega_{r,s}) \) such that, for all \( f \in C^{0, \alpha}_{\nu - 4}(\Omega_{r,s}) \), the function \( w = \mathcal{G}_{r,s}(f) \) satisfies

\[
\begin{align*}
\Delta^2 w &= f & \text{in} & \Omega_{r,s} \\
\Delta w &= 0 & \text{in} & \partial B_s.
\end{align*}
\]

Moreover, there exists a positive constant \( C \) that does not depend on \( s \) and \( r \) such that

\[
\| \mathcal{G}_{r,s}(f) \|_{C^{4, \alpha}_{\nu - 4}(\Omega_{r,s})} \leq C \| f \|_{C^{0, \alpha}_{\nu - 4}(\Omega_{r,s})}. \]

**Proof.** The existence of \( w \) satisfying (6.2) follows by standard elliptic theory. Now we need to prove the existence of a constant \( c > 0 \) which does not depend on \( r \) and \( s \) such that

\[
\sup_{\Omega_{r,s}} \| r^{-\nu} w \| \leq c \sup_{\Omega_{r,s}} \| r^{4-\nu} |f| \|,
\]

Suppose by contradiction that the estimate does not hold. Then there exist a sequence of numbers \( 0 < r_i < s_i \), a sequence of functions \( f_i \) satisfying \( \sup_{\Omega_{r_i,s_i}} r^{4-\nu} |f_i| = 1 \), and a sequence of solutions \( w_i \) to the problem

\[
\begin{align*}
\Delta^2 w_i &= f_i & \text{in} & \Omega_{r_i,s_i} \\
\Delta w_i &= 0 & \text{in} & \partial \Omega_{r_i,s_i}
\end{align*}
\]

such that \( A_i := \sup_{\Omega_{r_i,s_i}} r^{-\nu} |w_i| \to +\infty \). Consider \( (t_i, \theta_i) \in (r_i, s_i) \times S^{n-1} \), a point where this supremum \( A_i \) is attained. Up to a subsequence, we can assume that \( t_i / \tau_i \to \tau_1 \in [0, 1] \) and \( s_i / t_i \to \tau_2 \in [1, +\infty) \). With a similar argument applied in page 23 we can prove that \( \tau_1 \) and \( \tau_2 \) is not equal to 1.

Define the sequence of rescaled functions

\[
\hat{w}_i(r\theta) := \frac{t_i^{-\nu}}{A_i} w_i(t_i \tau \theta) \quad \text{and} \quad \hat{f}_i(r\theta) := \frac{t_i^{2-\nu}}{A_i} f_i(t_i \tau \theta).
\]

Note that \( 1 \in (\tau_1, \tau_2) \), \( \hat{w}_i(\tau_i) = 1 \) and \( |\hat{w}_i(\tau \theta)| \leq r^\nu \). By Arzelà-Ascoli’s Theorem we can assume, up to a subsequence, the sequence of functions \( \hat{w}_i \) converges uniformly on compact subsets of \( (\tau_1, \tau_2) \times S^{n-1} \) to a biharmonic function \( \hat{w}_\infty \) which is not identically zero. Moreover, \( \hat{w}_\infty \) is bounded by \( r^\nu \) on \( (\tau_1, \tau_2) \times S^{n-1} \), and \( \Delta \hat{w}_\infty = \hat{w}_\infty = 0 \) on the boundary if \( \tau_1 > 0 \) or if \( \tau_2 < \infty \).

Writing

\[
\hat{w}_\infty(r, \theta) = \sum_{j=0}^{\infty} \hat{w}_j(r)e_j(\theta),
\]

where \( \hat{w}_j \) is given explicitly by

\[
\hat{w}_j := a_j r^j + a_j^+ r^{2-n-j} + b_j^+ r^{2+j} + b_j^+ r^{4-n-j}.
\]

We examine the various possibilities of the values for \( \tau_1 \) and \( \tau_2 \) case by case.

**Case 1:** \( \tau_1 > 0 \) and \( \tau_2 < +\infty \). In this case, \( \hat{w}_\infty \) is biharmonic with zero Navier boundary data, which implies that \( \hat{w}_\infty \equiv 0 \).
Case 2: \( \tau_1 = 0 \) and \( \tau_2 = +\infty \). Since \( \hat{w}_\infty \) is bounded by \( r^\nu \), using that \( \tau_2 = +\infty \) and \( \nu < 5 - n \leq 0 \), we find that \( a_j^+ = b_j^- = 0 \), and using that \( \tau_1 = 0 \) and \( \nu > 4 - n \) we find that \( a_j^+ = b_j^- = 0 \).

Case 3: \( \tau_1 = 0 \) and \( \tau_2 < +\infty \). As in the case 2 we find that \( a_j^+ = b_j^- = 0 \). Now, using that 
\[
\hat{w}_\infty = \Delta \hat{w}_\infty = 0 \quad \text{on } \partial B_{r_2}
\]
we find that \( a_j^+ = b_j^- = 0 \).

Case 4: \( \tau_1 > 0 \) and \( \tau_2 = +\infty \). Similarly to the previous case we conclude that \( \hat{w}_\infty \equiv 0 \).

In each of these cases above, we obtain the zero function in the limit, contradicting the normalization \( \hat{w}_i(\theta_i) = 1 \). Consequently, we proved the existence of a function satisfying (6.2) and the estimate (6.3). The estimates for the derivatives follow from standard elliptic estimates. \( \square \)

**Corollary 6.2.** Let \( \nu \in (4 - n, 5 - n) \). There exists \( s_0 > 0 \) such that for all \( 0 < 2r < s < s_0 \) there is an operator

\[ \mathcal{H}_{r,s} : C^{0,\alpha}_{\nu-4}(\Omega_{r,s}) \rightarrow C^{0,\alpha}_{\nu}(\Omega_{r,s}) \]

with the function \( w = \mathcal{H}_{r,s}(f) \) satisfying

\[
\begin{align*}
L_{g_0}(w) &= f & \text{in} & & \Omega_{r,s} \\
\Delta w &= w = 0 & \text{on} & & \partial B_r.
\end{align*}
\]

Moreover, there exists a positive constant \( C \) that does not depend on \( s \) and \( r \) such that

\[
\||\mathcal{H}_{r,s}(f)||_{C^{0,\alpha}_{\nu}(\Omega_{r,s})} \leq C \||f||_{C^{0,\alpha}_{\nu-4}(\Omega_{r,s})}.
\]

**Proof.** Using (5.13) and the fact that in normal coordinates one has \( g_0 = \delta + O(|x|^2) \), we find that
\[
\|(\Delta g_0 - \Delta^2)v\|_{C^{0,\alpha}_{\nu-4}(\Omega_{r,s})} \leq s^2 \|v\|_{C^{0,\alpha}_{\nu-2}(\Omega_{r,s})}.
\]
From this and (1.3) we obtain
\[
\|(L_{g_0} - \Delta^2)v\|_{C^{0,\alpha}_{\nu-4}(\Omega_{r,s})} \leq s^2 \|v\|_{C^{0,\alpha}_{\nu-2}(\Omega_{r,s})}.
\]
Finally, the result follows by a perturbation argument and Proposition 6.1. \( \square \)

For the remainder of the construction we let \( M_r = M \setminus B_r(p) \).

**Proposition 6.3.** Let \( \nu \in (4 - n, 5 - n) \). There exists \( r_2 < r_1 \) such that for all \( r \in (0, r_2) \) we can define an operator \( G_{r,g_0} : C^{0,\alpha}_{\nu-4}(M_r) \rightarrow C^{0,\alpha}_{\nu}(M_r) \) where the function \( w = G_{r,g_0}(f) \) satisfies

\[
L_{g_0}(w) = f, \quad w|_{\partial B_r} = 0, \quad (\Delta w)|_{\partial B_r} = 0
\]
in \( M_r \). In addition, there exists a constant \( C > 0 \) independent of \( r \) such that

\[
\||G_{r,g_0}(f)||_{C^{0,\alpha}_{\nu}(M_r)} \leq C \||f||_{C^{0,\alpha}_{\nu-4}(M_r)}.
\]

**Proof.** The proof follows the ideas of the proof of Proposition 13.28 in [28] (see also [50]). Given \( f \in C^{0,\alpha}_{\nu-4}(M_r) \) define a function \( w_0 \in C^{0,\alpha}_{\nu}(M_r) \) by \( w_0 := \eta \mathcal{H}_{r_1}(f|_{\Omega_{r_1}}) \), where \( \eta \) is a smooth radial function equal to 1 in \( B_{1/2r_1}(p) \) and equal to zero in \( M_{r_1} \). There is a constant \( C > 0 \) independent of \( r \) and \( r_1 \) so that

\[
\|w_0\|_{C^{0,\alpha}(M_r)} \leq C \|f\|_{C^{0,\alpha}_{\nu-4}(M_r)}. \tag{6.4}
\]
Note that the function \( h := f - L_{g_0}(w_0) \) is supported in \( M_{1/2r_1} \), and so it follows that

\[
\|h\|_{C^{0,\alpha}(M)} \leq \|f\|_{C^{0,\alpha}(M)} + \|L_{g_0}(w_0)\|_{C^{0,\alpha}_{\nu-4}(M_r)} \leq C \|f\|_{C^{0,\alpha}_{\nu-4}(M_r)};
\]
for some constant \( C \) independent of \( r \). By the non degeneracy of the metric, we can define \( w_1 := \chi L_{g_0}^{-1}(h) \), where \( \chi \) is a smooth radial function equal to 1 in \( M_{2r_2} \) and equal to zero in \( B_{r_2}(p) \) with \( 4r_2 < r_1 \). Hence, there exists a constant \( C > 0 \) independent of \( r \) and \( r_2 \) such that

\[
\|w_1\|_{C^{0,\alpha}_{\nu}(M_r)} \leq C \|f\|_{C^{0,\alpha}_{\nu-4}(M_r)}. \tag{6.5}
\]
This implies that there exists $C > 0$ independent of $r$ and $r_2$ so that
\[ \|L_{\delta_0}(w_0 + w_1) - f\|_{C^{0,\alpha}_{\nu-4}(M_r)} \leq C r_2^{-1-\nu} \|f\|_{C^{0,\alpha}_{\nu-4}(M_r)}. \]
From this, by (6.4), (6.5) and a perturbation argument we obtain the result. \qed

6.2. Nonlinear analysis. Now that we have shown the linearized operator about the constant function 1 has a right-inverse on $M_r$ for $r$ sufficiently small, we use a fixed point argument to show the existence of a function satisfying (6.1). However, because we must prescribe the Navier boundary data appropriately, in addition to the constant 1 and the Green function, we add the exterior Poisson operator (constructed in section 4.2) in the expansion of equation (6.1).

Recall that $\Psi$ parameterizes $B_1(p)$ in conformal normal coordinates. We let
\[ \eta : \mathbb{R}^n \to \mathbb{R}, \quad \eta(x) = \begin{cases} 1, & |x| < r_1/2 \\ 0, & |x| > r_1 \end{cases} \]
be a smooth, radial cut-off function satisfying $|\partial_r^{(k)} \eta| \leq c|x|^{-k}$ on $B_1(0)$ for each $k = 0, 1, \ldots, 4$. Consequently, we have $\|\eta\|_{(4,\alpha),[\sigma,2\sigma]} \leq c$ for every $r \leq \sigma \leq \frac{1}{2} r_1$. For each $\psi_0, \psi_2 \in C^{4,\alpha}(\mathbb{S}_r^{n-1})$ we define $u_{\psi_0,\psi_2}$ by
\[ u_{\psi_0,\psi_2} \equiv 0 \text{ on } M_{r_1}, \quad u_{\psi_0,\psi_2} \circ \Psi = \eta Q_r(\psi_0, \psi_2) \text{ on } B_{r_1} \setminus B_r, \]
where $Q_r$ is the Poisson operator on the exterior domain constructed in Corollary 4.5. By Corollary 4.5 we obtain
\[ \|u_{\psi_0,\psi_2}\|_{C^{4,\alpha}_{\nu}(M_r)} \leq c r^{-\nu} \|\psi_0, \psi_2\|_{(4,\alpha),r} \] (6.6)
for $\nu \geq 4 - n$.

Finally, we seek a solution to (6.1) of the form $u_{\psi_0,\psi_2} + v$, which is equivalent to finding a fixed point of the mapping
\[ \mathcal{M}_r(\lambda, \psi_0, \psi_2, \cdot) : C^{4,\alpha}_{\nu}(M_r) \to C^{4,\alpha}_{\nu}(M_r) \]
defined by
\[ \mathcal{M}_r(\lambda, \psi_0, \psi_2, v) := -G_{r,\delta_0}(\mathcal{R}(\lambda G_p + u_{\psi_0,\psi_2} + v) + L_{\delta_0}(u_{\psi_0,\psi_2})). \]

Proposition 6.4. Let $\nu \in (9/2 - n, 5 - n)$, $\delta_4 \in (0,1/2)$ and $\beta, \gamma > 0$ fixed constants. There exists $r_2 > 0$, such that for all $r \in (0, r_2)$, $\lambda \in \mathbb{R}$, $\psi_0, \psi_2 \in C^{4,\alpha}(\mathbb{S}_r^{n-1})$ satisfying $|\lambda| \leq r^{n+\delta_4}$, and $\|\psi_0, \psi_2\|_{(4,\alpha),r} \leq \beta r^{m-\delta_4}$, there exists a fixed point to the map $\mathcal{M}_r(\lambda, \psi_0, \psi_2, \cdot)$ in the ball of radius $\gamma r^{m-\nu}$, where $m$ is defined in Remark 5.13.

Proof. We follow the same strategy as in the proof of Proposition 5.20, that is, showing that $\mathcal{M}_r(\lambda, \psi_0, \psi_2, \cdot)$ is a contraction in the ball of radius $\gamma r^{m-\nu}$. First, since $u_{\psi_0,\psi_2} \equiv 0$ in $M_{r_1}$, we have
\[ \|\mathcal{R}(\lambda G_p + u_{\psi_0,\psi_2})\|_{C^{0,\alpha}_{\nu-4}(M_r)} = \|\mathcal{R}(\lambda G_p)\|_{C^{0,\alpha}_{\nu-4}(M_{r_1})} + \|\mathcal{R}(\lambda G_p + u_{\psi_0,\psi_2})\|_{C^{0,\alpha}_{\nu-4}(\Omega_{r,1})} \]
(6.7)
Note that $1 + \lambda G_p > 0$ in $M_{r_1}$ for $r$ small enough. Additionally, the $C^{0,\alpha}(M_{r_1})$-norm of $(1 + st\lambda G_p)^{\frac{12-n}{n-4}}$ and $G_p$ are bounded independently of $r$. Thus, it follows from (5.15) that
\[ \|\mathcal{R}(\lambda G_p)\|_{C^{0,\alpha}(M_{r_1})} \leq C |\lambda|^2 \leq C r^{\delta'} r^{m-\nu}, \]
(6.8)
where the constant $C > 0$ does not depend on $r$. The fact that $\nu > 9/2 - n$ implies $\delta' > 0$. Since $r < \sigma$ and $19/2 - 2n - \nu < 0$, similarly to (6.6) we can prove that
\[ \sigma^{4-\nu} \|u_{\psi_0,\psi_2}\|^2_{(0,\alpha),[\sigma,2\sigma]} \leq C r^{\frac{4}{2} - \delta_4 + m - \nu}. \]
By (6.6) we have $|u_{\psi_0,\psi_2}(x)| \leq cr^{m-\nu-\delta_4}$, for all $x \in M_r$ with $m - \nu - \delta_4 > 0$, and we have $|\lambda G_p(x)| \leq cr^{m/2}$, where the exponent is positive. This implies that
$$0 < c < 1 + t(\lambda G_p + u_{\psi_0,\psi_2}) < C$$
for every $t \in [0, 1]$, and so we obtain that the Hölder norm of $(1 + t(\lambda G_p + u_{\psi_0,\psi_2}))^{12-n}r^{-\delta_4}$ is bounded independently of $r$. Thus, since $19/2 - 2n - \nu < 0$, by (5.14) we obtain
$$\|\mathcal{R}(\lambda G_p + u_{\psi_0,\psi_2})\|_{C^{0,\alpha}_{\nu-4}(\Omega_r, r)} \leq C r^{\frac{1}{2}-\delta_4} r^{m-\nu}. \quad (6.9)$$

From (6.8), (6.9), it follows that the right hand side of (6.7) is bounded by $cr^{m-\nu-\delta_4}$, for some constants $\delta_4 > 0$ and $c > 0$ independent of $r$. Using that $u_{\psi_0,\psi_2} = Q(\psi_0, \psi_2)$, the fact that the derivative of $\eta$ is zero in $B_{\frac{r}{2}} \setminus B_r$, and (5.13) we obtain
$$\|\Delta^2_0 u_{\psi_0,\psi_2}\|_{(0,\alpha),\sigma,2\sigma} \leq C_{r_1}\|Q(\psi_0, \psi_2)\|_{(4,\alpha),\sigma,2\sigma},$$
for some constant $C_{r_1}$ independently of $r$. By (1.3) and (5.14) the operator $L_{g_0} - \Delta^2_0$ is second order, and so for some $C > 0$ independent of $r$ we have
$$\|L_{g_0}(u_{\psi_0,\psi_2})\|_{(0,\alpha),\sigma,2\sigma} \leq C\sigma^{-2}\|Q_r(\psi_0, \psi_2)\|_{(4,\alpha),\sigma,2\sigma}.$$  

Note that $6 - n - \nu > 0$ and $n - 4 + \nu - \delta_4 > 0$, and so
$$\|L_{g_0}(u_{\psi_0,\psi_2})\|_{C^{0,\alpha}_{\nu-4}(\Omega_r, r)} \lesssim C r^{n-4+\nu-\delta_4} r^{m-\nu}. \quad (6.10)$$

Using (6.7), (6.8), (6.9) and (6.10), and the fact that the right inverse of $G_{r,g_0}$ constructed in Proposition 6.3 is bounded independently of $r$ together with $u_{\psi_0,\psi_2} \equiv 0$ in $M_{r_1}$ for $r$ small enough, we have
$$\|M_r(\lambda, \psi_0, \psi_2, 0)\|_{C^{4,\alpha}_{\nu-4}(M_r)} \lesssim \frac{1}{2} \gamma r^{m-\nu}. \quad (6.11)$$

Now, note that for $r > 0$ small enough we have $0 < c < z_t = 1 + \lambda G_p + u_{\psi_0,\psi_2} + \psi_2 + t(\psi_2 - v_1) < C$, for all $t \in [0, 1]$. Thus, the $C^{0,\alpha}$-norms of $(1 + sz_t)^{12-n}r^{-\delta_4}$ and $(1 + sz_t)^{12-n}r^{-\delta_4}$ are bounded independently of $r$. Using an analogous equality as in the beginning of the proof of Lemma 5.14, we obtain
$$\|\mathcal{R}(\lambda G_p + v_1) - \mathcal{R}(\lambda G_p + v_2)\|_{C^{0,\alpha}(M_{r_1})} \leq C (r^{n-4+\frac{1}{2}} + r^{m-\nu}) \|v_1 - v_2\|_{C^{4,\alpha}_{\nu}(M_r)}$$
and
$$\|\mathcal{R}^1(\lambda G_p + u_{\psi_0,\psi_2} + v_1) - \mathcal{R}^1(\lambda G_p + u_{\psi_0,\psi_2} + v_2)\|_{C^{0,\alpha}_{\nu-4}(\Omega_{r_2}, r_2)} \leq C (r^{\frac{1}{2}} + r^{m-\delta_4}) \|v_1 - v_2\|_{C^{4,\alpha}(M_r)}.$$

Using again the fact that the right inverse $G_{r,g_0}$ is bounded independently of $r$, $u_{\psi_0,\psi_2} \equiv 0$ in $M_{r_1}$, and the two previous estimates, for $r > 0$ small enough, we get that
$$\|M_r(\lambda, \psi_0, \psi_2, v_1) - M_r(\lambda, \psi_0, \psi_2, v_2)\|_{C^{4,\alpha}(M_r)} \leq \frac{1}{2} \|v_1 - v_2\|_{C^{4,\alpha}(M_r)}. \quad (6.12)$$

Therefore, by (6.11) and (6.12) the proof of the proposition is complete. \qed

**Theorem 6.5.** Let $\nu \in (9/2 - n, 5 - n)$, $\delta_4 \in (0, 1/2)$ and $\beta, \gamma > 0$ fixed constants. There exists $r_2 > 0$, such that for all $r \in (0, r_2)$, $\lambda \in \mathbb{R}$ with $|\lambda| \leq r^{n-4+\frac{1}{2}}$ and $\psi_0, \psi_2 \in C^{4,\alpha}(\mathbb{S}^{n-1})$ satisfying $\|\psi_0, \psi_2\|_{(4,\alpha), r} \leq \beta r^{m-\delta}$, there exists a solution $V_{\lambda, \psi_0, \psi_2} \in C^{4,\alpha}_{\nu}(M_r)$ of
$$H_{g_0}(1 + \lambda G_p + u_{\psi_0,\psi_2} + V_{\lambda, \psi_0, \psi_2}) = 0, \quad \text{in } M_r.$$
Moreover, $\|V_{\lambda, \psi_0, \psi_2}\|_{C^{4,\alpha}(M_r)} \leq \gamma r^{m-\nu}$ and
$$\|V_{\lambda, \psi_0, \psi_2} - V_{\lambda, \psi_0, \psi_2}\|_{C^{4,\alpha}(M_r)} \leq cr^{\delta_5-\nu}\|\psi_0 - \tilde{\psi}_0, \tilde{\psi}_2\| \quad (6.13)$$
for some sufficiently small constant $\delta_5 > 0$ independent of $r$.

Proof. After the previous proposition, it remains only to show (6.13), which we prove using the same method as in the proof of inequality (5.49).

7. ONE-POINT GLUING PROCEDURE

In this section, we complete our gluing construction in the case of a single puncture by matching the boundary data of our interior and exterior solutions.

We begin our discussion by explaining why matching the boundary data of our interior and exterior solutions across their common boundary gives us a smooth, global solution. The weak form of (Q) states

$$0 = \int_{M \setminus \{p\}} \left( \Delta_g \phi \Delta_g u - \frac{4}{n-2} \text{Ric}_g(\nabla \phi, \nabla u) + \frac{(n-2)^2 + 4}{2(n-1)(n-2)} R_g(\nabla \phi, \nabla u) \right)$$

$$+ \frac{n-4}{2} Q_g u \phi - \frac{n(n-4)(n^2-4)}{16} n \phi \right) \, d\mu_g$$

for every $\phi \in C^0(M \setminus \{p\})$. Now let $A$ denote the interior solution, defined on $\overline{B_r(p)} \setminus \{p\}$ for a sufficiently small $r$, and let $B$ denote the exterior solution defined on $M \setminus B_r(p)$. Integrating by parts in (7.1) we obtain

$$0 = \int_{M \setminus \{p\}} \left( \Delta_g \phi \Delta_g u - \frac{4}{n-2} \text{Ric}_g(\nabla \phi, \nabla u) + \frac{(n-2)^2 + 4}{2(n-1)(n-2)} R_g(\nabla \phi, \nabla u) \right)$$

$$+ \frac{n-4}{2} Q_g u \phi - \frac{n(n-4)(n^2-4)}{16} n \phi \right) \, d\mu_g$$

$$= \int_{\partial B_r(p)} \left( \frac{\partial \phi}{\partial r} (\Delta_g B - \Delta_g A) \right) + \phi \left( \frac{\partial \Delta_g A}{\partial r} - \frac{\partial \Delta_g B}{\partial r} \right) \, d\mu_g$$

$$+ \int_{\partial B_r(p)} \left( -\frac{4}{n-2} \text{Ric}_g(\partial_r, \nabla A - \nabla B) + \frac{(n-2)^2 + 4}{2(n-1)(n-2)} R_g(\phi(\partial_r A - \partial_r B) \right) \, d\mu_g.$$ 

Thus we see that, provided $A$ and $B$ satisfy the compatibility equations

$$A = B, \quad \partial_r A = \partial_r B, \quad \Delta_g A = \Delta_g B, \quad \partial_r \Delta_g A = \partial_r \Delta_g B$$

along the boundary $\partial B_r(p)$ we obtain a weak solution to (Q) which is smooth, and therefore also a strong solution.

By Theorem 5.21, for each sufficiently small $\varepsilon > 0$ there exists a mapping $A_\varepsilon$ such that the metric defined in $\overline{B_{r_\varepsilon}(p)} \setminus \{p\}$ by

$$\tilde{g} = (A_\varepsilon(R, a, \phi_0, \phi_2))^{\frac{4}{n-4}} g$$

has $Q$-curvature equal to $\frac{n(n^2-4)}{8}$. Also, by Theorem 6.5, for each sufficiently small $\varepsilon > 0$ there exists a mapping $B_\varepsilon$ such that the metric defined in $M_{r_\varepsilon}$ by

$$\tilde{g} = (B_\varepsilon(\lambda, \psi_0, \psi_2))^{\frac{4}{n-4}} g$$

has $Q$-curvature equal to $\frac{n(n^2-4)}{8}$, where $f - 1 = O(|x|^2)$ in conformal normal coordinates.
We would like now to define our metric to be \( \hat{g} = A^4 \xi g \) in the punctured ball \( B_{\varepsilon}(p) \setminus \{p\} \) and \( \bar{g} = B^4 \xi g \) in \( M \setminus B_{\varepsilon}(p) \). However, this is not \textit{a priori} a smooth (or even continuous!) metric. To complete our gluing construction, we show that one can choose geometric parameters \( a, R, \lambda, \phi_0 \) and \( \phi_2 \) such that

\[
\begin{align*}
A_{\varepsilon} &= B_{\varepsilon}, \\
\partial_{\varepsilon}A_{\varepsilon} &= \partial_{\varepsilon}B_{\varepsilon}, \\
\Delta_{\varepsilon}A_{\varepsilon} &= \Delta_{\varepsilon}B_{\varepsilon}, \\
\partial_{\varepsilon}\Delta_{\varepsilon}A_{\varepsilon} &= \partial_{\varepsilon}\Delta_{\varepsilon}B_{\varepsilon}.
\end{align*}
\]

(7.4)

To solve this system of equations, we once again decompose our boundary data into high and low Fourier modes, writing

\[
\phi_i(\theta) = \sum_{j=0}^{\infty} (\phi_i)_j e_j(\theta), \quad \pi'(\phi_i) = \sum_{j=0}^{n} (\phi_i)_j e_j, \quad \pi''(\phi_i) = \sum_{j=n+1}^{\infty} (\phi_i)_j e_j,
\]

where \( e_j \) is the \( j \)-th eigenfunction of \( \Delta_{\varepsilon^{-1}} \), counted with multiplicity and normalized to have \( L^2 \)-norm equal to 1. We will sometimes abbreviate \( \pi'(\phi_i) = \phi_i' \) and \( \pi''(\phi_i) = \phi_i'' \). We similarly decompose \( \psi_i = \psi_i' + \psi_i'' \) into low and high Fourier modes, respectively.

### 7.1. High Fourier modes

In this subsection use the model Navier-to-Neumann operator we constructed in Corollary 4.7 to show that we can match the high Fourier modes of our boundary data. Observe that \( \Upsilon, \) defined in (5.31), is radial in the ball \( B_{\varepsilon}(p) \subset M \), which implies that it will only appear in the constant term of the Fourier expansions. In particular,

\[
\partial_{\varepsilon}^k \pi''(\Upsilon|_{\partial B_{\varepsilon}(p)}) = 0, \quad k = 0, 1, 2, 3,
\]

and so we do not encounter \( \Upsilon \) in matching the high Fourier modes of the boundary data of the interior and exterior solutions.

Before we do this, however, we must first ensure that

\[
\pi''(A_{\varepsilon}|_{\partial B_{\varepsilon}(p)}) = \pi''(B_{\varepsilon}|_{\partial B_{\varepsilon}(p)}), \quad \pi''(\Delta_{\varepsilon}A_{\varepsilon}|_{\partial B_{\varepsilon}(p)}) = \pi''(\Delta_{\varepsilon}B_{\varepsilon}|_{\partial B_{\varepsilon}(p)}).
\]

Using

\[
\pi''(v_{\phi_0,\phi_2}|_{\partial B_{\varepsilon}(p)}) = \phi_0'', \quad \pi''(w_{\varepsilon,R}|_{\partial B_{\varepsilon}(p)}) = 0, \quad \pi''(U_{\varepsilon,R,a,\phi_0,\phi_2}|_{\partial B_{\varepsilon}(p)}) = 0
\]

and

\[
\pi''(f|_{\partial B_{\varepsilon}(p)}) = f'', \quad \pi''(u_{\psi_0,\psi_2}|_{\partial B_{\varepsilon}(p)}) = \psi_0'', \quad \pi''(\psi_0|_{\partial B_{\varepsilon}(p)}) = 0,
\]

we see from (7.2) and (7.3) that

\[
\begin{align*}
\pi''(A_{\varepsilon}|_{\partial B_{\varepsilon}(p)}) &= \pi''(u_{\varepsilon,R,a}|_{\partial B_{\varepsilon}(p)}) + \phi_0'' \\
\pi''(B_{\varepsilon}|_{\partial B_{\varepsilon}(p)}) &= f'' + \lambda \pi''(f G_{|\partial B_{\varepsilon}(p)}) + \pi''((f - 1)\psi_0) + \psi_0''.
\end{align*}
\]

Thus taking

\[
\phi_0'' = f'' + \lambda \pi''(f G_{|\partial B_{\varepsilon}(p)}) + \pi''((f - 1)\psi_0) + \psi_0'' - \pi''(u_{\varepsilon,R,a}|_{\partial B_{\varepsilon}(p)})
\]

(7.5)

gives us the first identity

\[
\pi''(A_{\varepsilon}|_{\partial B_{\varepsilon}(p)}) = \pi''(B_{\varepsilon}|_{\partial B_{\varepsilon}(p)}).
\]

Next we use the fact that

\[
\pi''(\Delta v_{\phi_0,\phi_2}|_{\partial B_{\varepsilon}(p)}) = \phi_2'', \quad \pi''(\Delta w_{\varepsilon,R}|_{\partial B_{\varepsilon}(p)}) = 0, \quad \pi''(\Delta U_{\varepsilon,R,a,\phi_0,\phi_2}|_{\partial B_{\varepsilon}(p)}) = 0
\]

and

\[
\begin{align*}
\pi''(\Delta u_{\psi_0,\psi_2}|_{\partial B_{\varepsilon}(p)}) &= \psi_2'', \quad \Delta V_{\lambda,\psi_0,\psi_2}|_{\partial B_{\varepsilon}(p)} = 0,
\end{align*}
\]

(7.6)
to see
\[\pi''(\Delta_g A|_{\partial B_{r\varepsilon}(p)}) = \pi''((\Delta_g - \Delta)(w_\varepsilon, R + v_{\phi_0, \phi_2} + U_{\varepsilon, R, a, \phi_0, \phi_2})|_{\partial B_{r\varepsilon}(p)}) + \phi_2''
\]
\[+ \pi''((\Delta_g(u_{\varepsilon, R, a} + \Upsilon))|_{\partial B_{r\varepsilon}(p)})
\]
\[\pi''(\Delta_g B|_{\partial B_{r\varepsilon}(p)}) = \pi''((\Delta_g(1 + u_{\psi_0, \psi_2} + V_{\lambda, \psi_0, \psi_2} + \lambda f G_p)|_{\partial B_{r\varepsilon}(p)})
\]
\[+ \pi''((\Delta_g - \Delta)u_{\psi_0, \psi_2}|_{\partial B_{r\varepsilon}(p)}) + \psi_2''.
\]
Thus we need to choose \(\phi_2''\) satisfying
\[
\phi_2'' = \pi''((\Delta_g((f - 1)(1 + u_{\psi_0, \psi_2} + V_{\lambda, \psi_0, \psi_2} + \lambda f G_p)|_{\partial B_{r\varepsilon}(p)}) + \psi_2''
\]
\[+ \pi''((\Delta_g - \Delta)u_{\psi_0, \psi_2}|_{\partial B_{r\varepsilon}(p)}) - \pi''((\Delta_g - \Delta)(w_\varepsilon, R + v_{\phi_0, \phi_2} + U_{\varepsilon, R, a, \phi_0, \phi_2})|_{\partial B_{r\varepsilon}(p)})
\]
\[= \mathcal{X}_{\varepsilon, R, a, \lambda, \psi_0, \psi_2}(\phi_2').
\]
To solve (7.6) it suffices to find a fixed point of the mapping
\[\phi_2'' \mapsto \mathcal{X}_{\varepsilon, R, a, \lambda, \psi_0, \psi_2}(\phi_2'')\]
for a given choice of parameters \(\varepsilon, R, a, \lambda, \psi_0, \psi_2\). We have \((f - 1)|_{\partial B_{r\varepsilon}(p)} = O(r_{\varepsilon}^2)\) and similarly all the coefficients of the second order operator \(\Delta_g - \Delta\) are \(O(r_{\varepsilon}^2)\). Therefore, following the same arguments as in the proof of Proposition 5.20 we find that \(\mathcal{X}\) is a contraction provided we choose all the parameters to be sufficiently small. We conclude that (7.6) admits a solution \(\phi_2''\) depending continuously on the parameters \(\varepsilon, R, a, \lambda, \psi_0, \psi_2\).

For the remainder of this construction, for given boundary functions \(\psi_0\) and \(\psi_2\), we choose \(\phi_0\) to satisfy (7.5) and choose \(\phi_2\) to satisfy (7.6). In this fashion we may regard \(\phi_0\) and \(\phi_2\) as functions of the parameters \(\psi_0\) and \(\psi_2\).

It remains to show that
\[\pi''((\partial_r A)|_{\partial B_{r\varepsilon}}) = \pi''((\partial_r B)|_{\partial B_{r\varepsilon}}), \quad \pi''((\partial_r \Delta_g A)|_{\partial B_{r\varepsilon}}) = \pi''((\partial_r \Delta_g B)|_{\partial B_{r\varepsilon}}).
\]
We see from (7.2) and (7.3) that
\[\pi''((\partial_r A)|_{\partial B_{r\varepsilon}}) = \pi''(\partial_r(\varepsilon, R, a + \Upsilon) + w_\varepsilon, R + U_{\varepsilon, R, a, \phi_0, \phi_2})|_{\partial B_{r\varepsilon}}
\]
\[+ \partial_r v_{\phi_0, \phi_2} - v_{\psi_0, \psi_2} - v_{\phi_0, \phi_2}|_{\partial B_{r\varepsilon}}.
\]
\[\pi''((\partial_r B)|_{\partial B_{r\varepsilon}}) = \pi''(\partial_r(f(1 + \lambda G_p + V_{\lambda, \psi_0, \psi_2}) + (f - 1)u_{\psi_0, \psi_2})|_{\partial B_{r\varepsilon}}
\]
\[+ \partial_r \varepsilon, R, a, \lambda, \psi_0, \psi_2|_{\partial B_{r\varepsilon}}.
\]
\[\pi''((\partial_r \Delta_g A)|_{\partial B_{r\varepsilon}}) = \pi''(\partial_r((\Delta_g(1 + \lambda G_p + V_{\lambda, \psi_0, \psi_2}) + (f - 1)u_{\psi_0, \psi_2})|_{\partial B_{r\varepsilon}}
\]
\[+ \partial_r \Delta_g v_{\phi_0, \phi_2} - v_{\psi_0, \psi_2} - v_{\phi_0, \phi_2}|_{\partial B_{r\varepsilon}}
\]
\[\pi''((\partial_r \Delta_g B)|_{\partial B_{r\varepsilon}}) = \pi''(\partial_r((\Delta_g(f(1 + \lambda G_p + V_{\lambda, \psi_0, \psi_2}) + (f - 1)u_{\psi_0, \psi_2})|_{\partial B_{r\varepsilon}}
\]
\[+ \partial_r \Delta_g v_{\phi_0, \phi_2} - v_{\psi_0, \psi_2} - v_{\phi_0, \phi_2}|_{\partial B_{r\varepsilon}}.
\]
Multiplying the first two equations by \(r_{\varepsilon}\) and the other two by \(r_{\varepsilon}^3\) we obtain the system
\[
\begin{align*}
S_{\varepsilon}(\psi_0', \psi_2') &= r_{\varepsilon} \partial_r(v_{\phi_0, \phi_2}' - u_{\psi_0, \psi_2}')|_{\partial B_{r\varepsilon}} \\
T_{\varepsilon}(\psi_0', \psi_2') &= r_{\varepsilon}^3 \partial_r \Delta_g(v_{\phi_0, \phi_2}' - u_{\psi_0, \psi_2}')|_{\partial B_{r\varepsilon}},
\end{align*}
\]
where
\[
S_\varepsilon(\psi_0', \psi_2'') = \pi'' \left( \partial_r (f(1 + \lambda G_p + V_\lambda, \psi_0, \psi_2) + (f - 1)u_{\psi_0, \psi_2})|_{\partial B_\varepsilon} \right)
- \pi'' \left( (\partial_r (u_{\varepsilon,R, a,Y}) + w_{\varepsilon,R} + U_{\varepsilon,R,a, \phi_0, \phi_0})|_{\partial B_\varepsilon} \right) - \partial_r v_{\phi_0, \phi_2} - v_{\phi_0, \phi_2} - \partial_r \psi_2'\bigg|_{\partial B_\varepsilon}
\]
and
\[
T_\varepsilon(\psi_0', \psi_2'') = \pi'' \left( \partial_r \Delta_g (f(1 + \lambda G_p + V_\lambda, \psi_0, \psi_2) + (f - 1)u_{\psi_0, \psi_2})|_{\partial B_\varepsilon} \right)
+ \pi'' \left( \partial_r (\Delta_g - \Delta)u_{\psi_0, \psi_2}|_{\partial B_\varepsilon} \right) - \partial_r \Delta v_{\phi_0, \phi_2} - v_{\psi_0, \phi_2} - \partial_r \psi_2'\bigg|_{\partial B_\varepsilon}
- \pi'' \left( \partial_r (\Delta_g (u_{\varepsilon,R, a,Y} + w_{\varepsilon,R} + U_{\varepsilon,R,a, \phi_0, \phi_0}) + (\Delta_g - \Delta)v_{\phi_0, \phi_2})|_{\partial B_\varepsilon} \right).
\]

(Recall that \(\psi_0\) and \(\psi_2\) determine \(\phi_0\) and \(\phi_2\) in the formulas above.)

Recalling the definition of the isomorphism \(Z_{r_\varepsilon}\) we constructed in Corollary 4.7, we see that solving (7.7) is equivalent to
\[
Z_{r_\varepsilon}(\psi_0', \psi_2'') = (S_\varepsilon(\psi_0', \psi_2''), T_\varepsilon(\psi_0', \psi_2''))
\]
and so it suffices to find a fixed point of the mapping
\[
\mathcal{H}_\varepsilon : D_\varepsilon \to \pi''(C^{4,\alpha}(S^{n-1}; \mathbb{R}^2)), \quad \mathcal{H}_\varepsilon(\psi_0', \psi_2'') = Z_{r_\varepsilon}^{-1}(S_\varepsilon(\psi_0', \psi_2''), T_\varepsilon(\psi_0, \psi_2)),
\]
where
\[
D_\varepsilon := \{ (\psi_0', \psi_2') \in \pi''(C^{4,\alpha}(S^{n-1}; \mathbb{R}^2)) : \| (\psi_0', \psi_2'') \|_{(4,\alpha), 1} \leq r_\varepsilon^{m-\delta_1} \}.
\]

**Lemma 7.1.** Let \(a \in \mathbb{R}^n\), \(b, \lambda \in \mathbb{R}\) be constants with \(|a|^2 \leq r_\varepsilon^{m-2}, \ |b| \leq 1/2\) and \(|\lambda| \leq r_\varepsilon^{-4+\frac{m}{2}}\). Then there exists \(\varepsilon_0 > 0\) such that for \(\varepsilon \in (0, \varepsilon_0)\), the map \(\mathcal{H}_\varepsilon\) has a fixed point. Moreover, this fixed point depends continuously on the parameters.

*Proof.* The proof of this lemma is based in the same ideas of the previous results where we have used the fixed point argument. We have to prove that the map \(\mathcal{H}_\varepsilon\) is a contraction with the norm of \(\mathcal{H}_\varepsilon(0,0)\) bounded by \(\frac{1}{2}r_\varepsilon^{m-\delta_1}\). To this end we use that the map \(Z_{r_\varepsilon}\) has norm bounded independently of \(r_\varepsilon\) and the estimates obtained in Propositions 5.20 and 6.4, as well (5.12), (5.49) and (6.13). This is a long and technical computation, which we be omitted. □

### 7.2. Constant functions.
We denote the fixed point of the mapping \(\mathcal{H}_\varepsilon\) by \((\psi_0', \psi_2'')\) and, abusing notation slightly, we denote the associated boundary functions for \(A_\varepsilon\) as \((\phi_0', \phi_2'')\).

By Remark 5.13, (2.17) and (3.31) we have
\[
u_{\varepsilon,R}(x) + Y = 1 + b + \frac{\alpha_\varepsilon^2}{4(1+b)} |x|^{4-n} + \frac{\beta_\varepsilon}{2} \left( \frac{\alpha_\varepsilon}{2(1+b)} \right)^{\frac{n-4}{n-2}} |x|^{2-n} + O(\varepsilon) \epsilon^{2n+4 \frac{n}{n-4}} |x|^{-n}.
\]

Note that
\[
\frac{\beta_\varepsilon}{2} \left( \frac{\alpha_\varepsilon}{2(1+b)} \right)^{\frac{n-4}{n-2}} \varepsilon^{2n-4} = O(\varepsilon^{\frac{2(n-2)}{n-4} - s(n-2)}) \quad \text{and} \quad \varepsilon^{\frac{2n+4}{n-4} - sn} = \varepsilon^{\frac{2n+4}{n-4} - sn},
\]
with \(\frac{2(n-2)}{n-4} - s(n-2) > 0\) and \(\frac{2n+4}{n-4} - sn > 0\). Assuming the hypotheses of Lemma 7.1, combining (2.19) and (4.6) with the bound \(|a|^2 \leq r_\varepsilon^{m-2}\) gives us
\[
A_\varepsilon(r_\varepsilon \theta) = 1 + b + \frac{\alpha_\varepsilon^2}{4(1+b)} r_\varepsilon^{4-n} + ((n-4)u_{\varepsilon,R}(r_\varepsilon \theta) + r_\varepsilon^2 \partial_r u_{\varepsilon,R}(r_\varepsilon \theta)) r_\varepsilon a \cdot \theta + w_{\varepsilon,R}(r_\varepsilon \theta) + v_{\phi_0, \phi_2}(r_\varepsilon \theta) + U_{\varepsilon,R,a, \phi_0, \phi_2}(r_\varepsilon \theta) + O(r_\varepsilon^m) + O(r_\varepsilon^d) \quad (7.8)
\]
with the last term independent of $\theta$. We also have
\[
B_\varepsilon(r_\varepsilon \theta) = 1 + \lambda r_\varepsilon^{-n} + u_{\psi_0, \psi_2}(r_\varepsilon \theta) + (f - 1)(r_\varepsilon \theta) + (f - 1)u_{\psi_0, \psi_2}(r_\varepsilon \theta) + (fV_{\lambda, \psi_0, \psi_2})(r_\varepsilon \theta) + O(|\lambda| r_\varepsilon^{5-n}).
\]

Now we project the system (7.4) onto the space generated by the constant functions. Let us define $\xi_0$, $\xi_2$ as the zero Fourier modes of $\phi_0$ and $\phi_2$, respectively. Using Corollary 4.2 and 4.5, we obtain
\[
\begin{cases}
\frac{b - \left(\frac{\alpha_2^2}{4(1+b)} - \lambda\right)}{(4-n)} r_\varepsilon^{4-n} + \frac{1}{2n} \xi_0 = \mathcal{H}_0^0 \\
(n-1) \left(\frac{\alpha_2^2}{4(1+b)} - \lambda\right) r_\varepsilon^{4-n} + \frac{1}{2n} \xi_0 = r_\varepsilon \partial_r \mathcal{H}_0^0 \\
2(n-1) \left(\frac{\alpha_2^2}{4(1+b)} - \lambda\right) r_\varepsilon^{4-n} + \xi_0 = r_\varepsilon^2 \Delta_g \mathcal{H}_0^0 \\
\end{cases}
\]  
(7.10)

where $\mathcal{H}_0^0$, $r_\varepsilon \partial_r \mathcal{H}_0^0$, $r_\varepsilon^2 \Delta_g \mathcal{H}_0^0$ and $r_\varepsilon^3 \partial_r \Delta_g \mathcal{H}_0^0$ depend continuously on the parameters $b$, $\lambda$, $\xi_0$ and $\xi_2$. Also, by (7.8), (7.9) and the estimates obtained in Sections 5.3.1 and 6.2 we obtain that $\mathcal{H}_0^0$, $r_\varepsilon \partial_r \mathcal{H}_0^0$, $r_\varepsilon^2 \Delta_g \mathcal{H}_0^0$ and $r_\varepsilon^3 \partial_r \Delta_g \mathcal{H}_0^0$ have order $r_\varepsilon^m$.

**Lemma 7.2.** Let $a \in \mathbb{R}^n$ with $|a|^2 \leq r_\varepsilon^{m-2}$ and $\omega_i \in C^{4,\alpha}(\mathbb{S}^{n-1})$ belonging to the space spanned by the coordinate functions and with norm bounded by $r_\varepsilon^{m-\delta}$. There exists a constant $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$ the system (7.10) has a solution $(b, \lambda, \xi_0, \xi_2)$ with $|b| \leq 1/2$, $|\lambda| \leq r_\varepsilon^{n-4+\frac{m}{2}}$ and $|\xi_i| \leq r_\varepsilon^{m-\delta}$.

**Proof.** Define a continuous map $G_{\varepsilon, 0} : \mathcal{D}_{\varepsilon, 0} \to \mathbb{R}^4$ as
\[
G_{\varepsilon, 0}(b, \lambda, \xi_0, \xi_2) = \left(\mathcal{H}_\varepsilon + \frac{r_\varepsilon}{n-2} \partial_r \mathcal{H}_\varepsilon - \frac{r_\varepsilon^2}{2(n-2)} \Delta_g \mathcal{H}_\varepsilon - \frac{r_\varepsilon^3}{2(n-4)(n-2)} \partial_r \Delta_g \mathcal{H}_\varepsilon, \right)
\]
where $\mathcal{D}_{\varepsilon, 0} := \{ (b, \lambda, \xi_0, \xi_2) \in \mathbb{R}^4 : |b| \leq 1/2, |\lambda| \leq r_\varepsilon^{n-4+\frac{m}{2}}$ and $|\xi_i| \leq r_\varepsilon^{m-\delta} \}$. Using the fact that that $\mathcal{H}_\varepsilon^0$, $r_\varepsilon \partial_r \mathcal{H}_\varepsilon^0$, $r_\varepsilon^2 \Delta_g \mathcal{H}_\varepsilon^0$ and $r_\varepsilon^3 \partial_r \Delta_g \mathcal{H}_\varepsilon^0$ are all $O(r_\varepsilon^m)$, we can show that $G_{\varepsilon, 0}(\mathcal{D}_{\varepsilon, 0}) \subset \mathcal{D}_{\varepsilon, 0}$, and so it follows from Brouwer’s fixed point theorem that $G_{\varepsilon, 0}$ has a fixed point. In fact, we can prove that $G_{\varepsilon, 0}$ is a contraction, hence the fixed point depends continuously on the parameters. It is not difficult to see that this fixed point is a solution to the system (7.10).

**7.3. Coordinate functions.** Finally, we consider the solutions given by Lemmas 7.1 and 7.2, and project the system (7.4) in the space spanned by the coordinate functions. Let us consider that the component of the functions $\phi_0$, and $\phi_2$ in the direction of the coordinates functions are given by
\[
\sum_{j=1}^{n} r_j e_j, \sum_{j=1}^{n} \xi_j e_j \text{ and } \sum_{j=1}^{n} \partial_j e_j,
\]
Using Corollary 4.2 and 4.5, projecting the system (7.4) in direction of $e_j$ we obtain

\[
\begin{align*}
F(\varepsilon_\tau)e_j + \frac{1}{2n+4}\tau_j - g_j &= H_{\varepsilon_j} \\
G(\varepsilon_\tau)e_j + \frac{3}{2n+4}\tau_j - \frac{2}{3n-1}\varepsilon_j - (1-n)g_j &= r_\varepsilon \partial_r H_{\varepsilon_j} \\
M(\varepsilon_\tau)e_j + \tau_j - g_j &= r_\varepsilon^2 \Delta H_{\varepsilon_j} \\
N(\varepsilon_\tau)e_j + \tau_j - (1-n)\varepsilon_j &= r_\varepsilon^2 \partial_r \Delta H_{\varepsilon_j},
\end{align*}
\]  
(7.11)

where

\[
\begin{align*}
F(\varepsilon_\tau) &= ((n-4)u_\varepsilon, r_\varepsilon \partial_r u_\varepsilon) (\varepsilon_\tau), \\
G(\varepsilon_\tau) &= ((n-4)u_\varepsilon, r_\varepsilon \partial_r u_\varepsilon, r_\varepsilon^2 \varepsilon^2 u_\varepsilon) (\varepsilon_\tau), \\
M(\varepsilon_\tau) &= ((n-3)(n+1)r_\varepsilon \partial_r u_\varepsilon, r_\varepsilon^2 \varepsilon^2 u_\varepsilon, r_\varepsilon^3 \varepsilon^3 u_\varepsilon) (\varepsilon_\tau), \\
N(\varepsilon_\tau) &= ((n^2-4)r_\varepsilon^2 \varepsilon^2 u_\varepsilon, r_\varepsilon^2 \varepsilon^2 u_\varepsilon, r_\varepsilon^3 \varepsilon^3 u_\varepsilon) (\varepsilon_\tau),
\end{align*}
\]

and $H_{\varepsilon_j}$, $r_\varepsilon \partial_r H_{\varepsilon_j}$, $r_\varepsilon^2 \Delta g H_{\varepsilon_j}$ and $r_\varepsilon^2 \partial_r \Delta g H_{\varepsilon_j}$ depends continuously on the parameters $a_j$, $\tau_j$, $\varepsilon_j$ and $g_j$. Also, by (7.8), (7.9) and the estimates obtained in Sections 5.3.1 and 6.2 we obtain that $H_{\varepsilon_j}$, $r_\varepsilon \partial_r H_{\varepsilon_j}$, $r_\varepsilon^2 \Delta g H_{\varepsilon_j}$ and $r_\varepsilon^2 \partial_r \Delta g H_{\varepsilon_j}$ have order $r_\varepsilon^m$.

**Lemma 7.3.** There exists a constant $\varepsilon_2 > 0$ such that if $\varepsilon \in (0, \varepsilon_2)$, then the system (7.11) has a solution $(a_j, \tau_j, \varepsilon_j, g_j)$.

**Proof.** The proof of this lemma is similar to the previous one, that is, by a fixed point argument. First, by (2.17) and Remark 5.13 we have

\[
\mathcal{T} = F + \frac{G}{n-1} + c_1 M + c_2 N = \frac{n(n-4)}{n-1}(1+b) + O(\varepsilon^2),
\]

for some $\gamma > 0$, where $c_i$, $i = 1, 2$, are constants which depends only on $n$. Then, we can reduce the system (7.11) to the system

\[
\begin{align*}
a_j &= \mathcal{T}^{-1} r_\varepsilon^{-1}(H_{\varepsilon_j} + c_3 r_\varepsilon \partial_r H_{\varepsilon_j} + c_4 r_\varepsilon^2 \Delta g H_{\varepsilon_j} + c_5 r_\varepsilon^3 \partial_r \Delta g H_{\varepsilon_j}) \\
\tau_j &= c_6 r_\varepsilon^2 H_{\varepsilon_j} + c_7 r_\varepsilon^3 \partial_r H_{\varepsilon_j} + ((1-n)M - N)n^{-1} r_\varepsilon a_j \\
\varepsilon_j &= c_8 r_\varepsilon^3 \Delta g H_{\varepsilon_j} + c_9 r_\varepsilon^4 \partial_r \Delta g H_{\varepsilon_j} + (M - N)n^{-1} r_\varepsilon a_j \\
g_j &= c_{10} r_\varepsilon \partial_r H_{\varepsilon_j} + c_{11} r_\varepsilon^2 \Delta g H_{\varepsilon_j} + c_{12} r_\varepsilon^3 \partial_r \Delta g H_{\varepsilon_j} + c_{13} G + c_{14} M + c_{15} N) r_\varepsilon a_j
\end{align*}
\]

where $c_i$ are constants which depends only on $n$. Now, using that $H_{\varepsilon_j}$, $r_\varepsilon \partial_r H_{\varepsilon_j}$, $r_\varepsilon^2 \Delta g H_{\varepsilon_j}$ and $r_\varepsilon^2 \partial_r \Delta g H_{\varepsilon_j}$ are bounded by $r_\varepsilon^m$, we obtain a solution to this system as a fixed point of a map $K_j(a_j, \tau_j, \varepsilon_j, g_j) = (a_j, \tau_j, \varepsilon_j, g_j)$. \hfill \Box

We summarize this construction with the following theorem.

**Theorem 7.4.** Let $(M^n, g_0)$ be a nondegenerate compact Riemannian manifold of dimension $n \geq 5$, with constant Q-curvature equal to $n(n^2 - 4)/8$. For $n \geq 8$, suppose there exists a point $p \in M$ with the Weyl tensor satisfying

\[
\nabla^k W_{g_0}(p) = 0, \quad \text{for} \quad k = 0, \ldots, \left[\frac{n-8}{2}\right].
\]

Then there exists $\varepsilon_0 > 0$ and a one-parameter family of complete metrics $g_\varepsilon$ on $M \setminus \{p\}$, for $\varepsilon \in (0, \varepsilon_0)$, such that each $g_\varepsilon$ is conformal to $g_0$ and has constant Q-curvature equal to $n(n^2 - 4)/8$, $g_\varepsilon$ is asymptotically Delaunay and $g_\varepsilon$ converges to $g_0$ uniformly on compact sets of $M \setminus \{p\}$ as $\varepsilon \to 0$. 


Proof. We keep the previous notations. Consider the metric \( g = F_0^{4,4} g_0 \) given in Section 5. For suitable parameters \( R, a \) and \( \phi_0, \phi_2 \in C^{4,\alpha}(\Sigma_{r_0}) \) with \( \phi_0 = \xi_0 e_0 + \sum_{j=1}^n \tau_j e_j + \phi''_0, \pi''(\phi''_0) = \phi'_0 \) and \( \pi''(\phi_2) = \phi_2 \), by Theorem 5.21 there exists a family of constant \( Q \)-curvature complete metrics in \( B_{r_0}(p) \setminus \{p\} \subset M_\varepsilon \), for \( \varepsilon > 0 \) small enough, given by \( \hat{g} = A_\varepsilon^{4,4} g \).

Also, for suitable parameters \( \lambda \) and \( \psi_0, \psi_2 \in C^{4,\alpha}(\Sigma_{r_0}) \) with \( \psi_0 = \xi_2 e_0 + \sum_{j=1}^n \zeta_j e_j + \psi''_0, \psi_2 = \sum_{j=1}^n \eta_j e_j + \psi'_2 \), by Theorem 6.5 there exists a family of constant \( Q \)-curvature metrics in \( M \setminus B_{r_0}(p) \), for \( \varepsilon > 0 \) small enough, given by \( \tilde{g} = B_\varepsilon^{4,4} g\).

From Lemmas 7.1, 7.2 and 7.3 there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) there are parameters for which the functions \( A_\varepsilon \) and \( B_\varepsilon \) satisfy the system 7.4. Hence using elliptic regularity we can show that the function \( W_\varepsilon \), defined by \( W_\varepsilon := A_\varepsilon \) in \( B_{r_0}(p) \setminus \{p\} \) and \( W_\varepsilon := B_\varepsilon \) in \( M \setminus B_{r_0}(p) \), is a positive smooth function in \( M \setminus \{p\} \). Moreover, \( W_\varepsilon \) is asymptotically to some Delaunay-type solution \( u_{\varepsilon, R} \). This implies that the metric \( g_\varepsilon := W_\varepsilon^{4,4} g \) is the one that proves the theorem. \( \square \)

8. PROOF OF THE MAIN RESULT

Finally we are ready to prove the main result of this work, particularly adapting our construction in the case of a single puncture to handle finitely many punctures.

Recall that the singular set is given by \( \Lambda = \{p_1, \ldots, p_k\} \) and at each point it satisfies \( \nabla^2 W_{g_0}(p_j) = 0 \), for \( j = 0, \ldots, \lfloor \frac{n-8}{2} \rfloor \) and \( i = 1, \ldots, k \).

Proof of Theorem 1.1. As before, we also have three steps: we construct an interior solution, construct an exterior solution, and finally match boundary data across an interface to obtain a globally smooth solution. Fix a positive number \( \delta \) and fix \( t_\varepsilon \in (\delta, \delta^{-1}) \). For each \( i = 1, \ldots, k \) let \( \varepsilon_i = \varepsilon t_\varepsilon \). Our interior domain is \( \cup_{i=1}^k B_{r_{\varepsilon_i}}(p_i) \), where we choose \( \varepsilon \) sufficiently small such that these balls are pairwise disjoint. Thus one can perform the same interior analysis on each of these balls individually, as we have already done in Section 5. Given geometric parameters \( R_i > 0 \) and \( a_i \in \mathbb{R}^n \) and boundary functions \( \phi_i^0 \in \pi''(C^{4,\alpha}(\Sigma_{r_{\varepsilon_i}})) \) and \( \phi_i^2 \in C^{4,\alpha}(\Sigma_{r_{\varepsilon_i}}) \) we obtain a constant \( Q \)-curvature metric of the form

\[
\check{g} = (u_{\varepsilon_i, R_i, a_i} + Y_i + v_0, \phi_i^0, \phi_i^2 + w_{\varepsilon_i, R_i, a_i} + U_{\varepsilon_i, R_i, a_i, \phi_i^0, \phi_i^2})^{4,4} g_0
\]

on the union \( \bigcup_{i=1}^k B_{r_{\varepsilon_i}}(p_i) \setminus \{p_j\} \).

On the other hand, some adjustments are necessary in order to construct a family of metrics as in Section 6. Let \( \Psi_i : B_{2r_0}(0) \to M \) be a normal coordinate system with respect to \( g_i = F_i^{4,4} g_0 \) on \( M \) centered at \( p_i \in \Lambda \). Here, \( F_i \) is defined in Section 6 for all \( i = 1, \ldots, k \). Hence, each metric \( g_i \) yields conformal normal coordinates centered at \( p_i \). Recall that \( F_i = 1 + O(|x|^2) \) in the coordinate system \( \Psi_i \). Denote by \( G_{p_i} \) the Green’s function of \( L_{g_0} \) with pole at \( p_i \) and assume that \( \lim_{x \to 0} |x|^{n-4} G_{p_i}(x) = 1 \) in the coordinate system \( \Psi_i \) for all \( i = 1, \ldots, k \). Define \( G_{p_1, \ldots, p_k} \in C^\infty(M \setminus \Lambda) \) to be

\[
G_{p_1, \ldots, p_k} = \sum_{i=1}^k \lambda_i G_{p_i},
\]

where \( \lambda_i \in \mathbb{R} \). Let \( r = (r_{\varepsilon_1}, \ldots, r_{\varepsilon_k}) \) and denote by \( M_r := M \setminus \bigcup_{i=1}^k \Psi_i(B_{r_{\varepsilon_i}}(0)) \). Define the space \( C^{4,\alpha}(M \setminus \Lambda) \) as in Definition 3.3 with the following norm

\[
\|v\|_{C^{4,\alpha}(M \setminus \Lambda)} := \|v\|_{C^{4,\alpha}(M_{\frac{1}{2}r_0})} + \sum_{i=1}^k \|v \circ \Psi_i\|_{1, \alpha, \nu, r_0}.
\]
The space $C^{l,\alpha}_{\nu}(M_r)$ is defined similarly.

We now outline how to prove the analog of Proposition 6.3. Choose a radial cutoff function
\[ \eta : \mathbb{R}^n \to [0,\infty), \quad \eta(x) = \begin{cases} 
1, & |x| < r_1/2 \\
0, & |x| > r_1 
\end{cases} \]
and let
\[ w_0 = \sum_{i=1}^k \eta \mathcal{K}_{r,r_1}^i(f|_{\Omega_{r,r_1}^i}). \]
Here $\Omega_{r,r_1}$ is the annulus centred at $p_i$ with inner radius $r$ and outer radius $r_1$, with respect to our chosen conformal normal coordinates described above. Letting
\[ h = f - Lg_0(w_0) \]
we see as in the proof of Proposition 6.3 that there exists $C_1 > 0$ such that
\[ \|h\|_{C^{0,\alpha}(M)} \leq C_1 \|f\|_{C^{0,\alpha}_{\nu-4}(M_r)}. \]

Now choose
\[ \chi : M \to [0,\infty), \quad \chi(x) = \begin{cases} 
0, & x \in \bigcup_{i=1}^k B_{r_2}(p_i) \\
1, & x \in M \setminus \left( \bigcup_{i=1}^k B_{2r_2}(p_i) \right) 
\end{cases} \]
where $4r_2 < r_1$ and let
\[ w_1 = \chi Lg_0^{-1}(h). \]
The same estimates as before tell us
\[ \|w_1\|_{C^{4,\alpha}(M)} \leq C_2 \|f\|_{C^{0,\alpha}_{\nu-4}(M_r)} \]
for some constant $C_2$ independent of $r$. Finally, as in the previous proof, we see
\[ \|Lg_0(w_0 + w_1) - f\|_{C^{0,\alpha}_{\nu-4}(M_r)} \leq C_3 r_2^{-1-\nu} \|f\|_{C^{0,\alpha}_{\nu-4}(M_r)}, \]
and so we may find our right-inverse for the exterior problem by perturbations, such that the resulting function $w$ is constant on any component of $\partial M_r$.

Choosing boundary functions $\psi_0^i, \psi_2^i \in C^{4,\alpha}(S^{n-1})$ we define $u_{\psi_0^i,\psi_2^i}$ by
\[ u_{\psi_0^i,\psi_2^i} \circ \Psi_i = \eta Q_{r_{r_1}}(\psi_0^i, \psi_2^i) \quad \text{in } B_{2r_0}(p_i) \]
and letting $u_{\psi_0^i,\psi_2^i}$ be zero otherwise. The same fixed-point argument as before will give us $V_{\lambda_i,\psi_0^i,\psi_2^i}$ such that
\[ Hg_0(1 + G_{p_1,\ldots,p_k} + u_{\psi_0^i,\psi_2^i} + V_{\lambda_i,\psi_0^i,\psi_2^i}) = 0, \]
which gives us our exterior solution.

We complete the gluing construction by once more using fixed-point theorems to show that one can choose the boundary date and parameters such that the interior and exterior solutions match to third order across the interface. This part of the proof is the same as in the one-point gluing construction. \[ \square \]
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