A Polynomial Time Graph Isomorphism Algorithm For Graphs That Are Not Locally Triangle-Free

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Abstract
In this paper, we show the existence of a polynomial time graph isomorphism algorithm for all graphs excluding graphs that are \textit{locally triangle-free}. This particular class of graphs allows to divide the graph into \textit{neighbourhood} sub-graph where each of induced sub-graph (neighbourhood) has at least 2 vertices. We construct all possible permutations for each induced sub-graph using a search tree. We construct automorphisms of subgraphs based on these permutations. Finally, we decide isomorphism through automorphisms.

\textbf{Keywords:} Graph, Isomorphism, Individualization Refinement, Search Tree.

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1 Introduction

Given two graphs $G$ and $H$, the Graph Isomorphism problem (GI) asks whether there exists a bijection from the vertices of $G$ to the vertices of $H$ that preserves adjacency. The graph isomorphism problem has a long history in the fields of mathematics, chemistry, and computing science. The problem is known to be in NP, but is not known to be in \textbf{P} or \textbf{NP-complete}. The best current theoretical algorithm is due to Babai and Luks (1983)[2]. The algorithm relies on the classification of finite simple groups. In 2015, Laszlo Babai claimed that the Graph Isomorphism problem can be solved in quasipolynomial time.

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1.1 Notations and Definitions

Let $G$ and $H$ be two graphs. Each graph has $n$ vertices. The cardinality of a set $B$ is the number of elements in it, denoted by $|B|$. For example, $|G| = |H| = n$.

The automorphism group of graph $G_1$ will be denoted by $\text{Aut}(G_1)$. The neighbourhood of a vertex $v$ in a graph $G$ is the induced subgraph of $G$ consisting of all vertices adjacent to $v$.

The neighbourhood is denoted $N_G(v)$ or (when the graph is unambiguous) $N(v)$. If the neighbourhood does not include $v$ itself then it is open neighbourhood of $v$; it is also possible to define a neighbourhood in which $v$ itself is included, called the closed neighbourhood and denoted by $N_G[v]$.

If all vertices in $G$ have neighbourhoods that are isomorphic to the same graph $G_1$, then $G$ is said to be locally $G_1$, and if all vertices in $G$ have neighbourhoods that belong to some graph family $\mathcal{F}$, $G$ is said to be locally $\mathcal{F}$ (Hell 1978, Sedlacek 1983).

A permutation of a vertex set $G$ is a bijection from $G$ to itself. For example, if $\pi = (1\ 2\ 3\ldots\ n\ 5\ 2\ 7\ldots\ 11)$, then the first vertex of $G$ moves to fifth position in the $G^\pi$. $\text{Sym}(G)$ or $S_n$ denotes the set of all permutations of $G$.

If $G$ is Isomorphic to $H$, then $\exists P \in S_n$ (Symmetric Group of $n$ vertices) such that $H^P = G$ (notation by Wielandt). We write $G \simeq H$ when $G$ is Isomorphic to $H$.

A tuple is a finite ordered list of vertices.

A search tree is an undirected graph in which any two vertices are connected by exactly one path. A rooted search tree is a tree in which one vertex has been designated the root. The tree elements are called nodes. Each of the nodes that is one graph-edge further away from a given node is called a child, i.e. the vertices adjacent to the root vertex are called its children. A rooted tree naturally imparts a notion of Levels (distance from the root), thus for every node a notion of children may be defined as the nodes connected to it a level below. Nodes without children are called leaf nodes, end-Nodes, leaves.

A walk on a graph is an alternating series of vertices and edges beginning and ending with a vertex in which each edge is incident with the vertex immediately preceding it and the vertex immediately following it. A trail is a walk in which all edges are distinct. A path is a trail in which all vertices are distinct. A path has a Sequences of Vertices. In a search tree, a Discrete Partition or Individualization-Refinement Path is a path which starts at the root and ends in a leaf [1].

A subset $B$ of a group is called generating set, if the smallest subgroup containing the subset is the group $S_n$ itself. We write, $\langle B \rangle = S_n$. A generating set is called minimal generating, if the set does not properly contain any generating set.
2 Overview

We define graphs \( G, H \) as regular, connected, and not locally triangle free. If \( G \cong H \) then \( H \) must have the same structure like \( G \). We will rearrange \( G \) according to 2.1. This rearrangement will split \( G \) in to vertex set \( G_1, G_2, \ldots, G_k \). In this paper, Graphs are not locally triangle-free. We will construct a search tree \( T_k \) for \( H_k \). The search tree \( T_k \) will provide a set of permutations which will create the set \( \beta_k \). We will construct generating set of automorphism from \( \beta_k \). The algorithm uses the same technique to find automorphism as [6].

2.1 Rearrangement of a Graph:

Consider any vertex of \( G \), say \( v \). We label \( v \) with the integer \( n \) (put \( n \) on it, as label). We use subscript \( n \) to denote the \( n \) labeled vertex and write \( v_n \). Now, construct neighbourhood induced sub-graph \( N_G[v_n] \) (closed neighbourhood). We rewrite \( N_G[v_n] \) as \( G_1 \) i.e. \( G_1 = N_G[v_n] \). Now, select any unlabeled vertex from \( G_1 \) and label it with \( (n - 1) \). The vertex with label \( (n - 1) \) would be denoted as \( v_{(n-1)} \). Now, construct induced sub-graph \( N_{G_1}(v_{(n-1)}) \) and label any vertex of it as \( v_{(n-2)} \).

We repeat the above procedure again. Label an unlabeled vertex (i.e. an unlabeled vertex of \( G \setminus G_1 \)) as \( v_{(n-3)} \) and obtain \( G_2 \) based on adjacency (neighbourhood) of vertex \( v_{(n-3)} \). Using the same procedure, we would be able to obtain \( G_3, G_4, \ldots, G_{(n/3)} \) subgraphs and to label all vertices of \( G \). Thus \( G \) could be split into \((n/3)\) subgraphs.

There is a bijection from the vertex set of \( G \) to a set \( L = \{1, 2, \ldots, n\} \) of labels. Now, let us define, a \( n \)-tuple (a sequence or ordered list of \( n \) vertices) \( w_G \) where vertices are ordered according to their labels (e.g. vertex labelled with label 1 is in \( 1^{st} \) position in \( w_G \)).

\( w_G \) is our desired arrangement (ordered) of graph \( G \). For some \( k \), \( G_k \) has vertices from \( i_k \) to \( j_k \) of \( w_G \) where \( (i_k + 2) = j_k \). Here, \( i_k \) is the starting position of \( k^{th} \) subgraph \( G_k \) in \( w_G \) and \( j_k \) is the ending position of \( k^{th} \) subgraph \( G_k \) in \( w_G \). Subscript \( k \) is used to distinguish \( i, j \) for the \( k^{th} \) subgraph \( G_k \) from other subgraphs. For example, if \( k = 1 \), then \( G_1 \) has vertices from \( (n - 2) \) to \( n \) in \( w_G \), here \( i_1 = (n - 2) \) and \( j_1 = n \).

2.2 Construction of Search Tree for Generating Set:

We have rearranged \( G \) (one of the two given graphs \( G \) and \( H \)). Now, we will construct permutations for \( H \) with respect to \( G \). Conversely, permutations can be generated for \( G \) with respect to \( H \) using procedure described in this subsection.

We label each vertex of \( H \) uniquely with elements from the same set \( L = \{1, 2, \ldots, n\} \). This labeling procedure is random. It must make sure that there is a bijection from the vertex set of \( H \) to the label set \( L \).

\( w_H \) is a \( n \)-tuple (a sequence of \( n \) vertices) where vertices are ordered according to their labels. The definition of \( w_H \) is similar to \( w_G \), except it is defined...
for graph $H$.

In $w_G$, position starts from left to right, so the subgraph $G_k$ starts at $i_k$ position and ends at $j_k$ where $n > j_k > i_k$.

We define the subgraph $H_k$ of $H$, which has consecutive vertices from $i_k$ position to $j_k$ position in $w_H$. Here, $i_k$ and $j_k$ have the same value as they had in $G_k$. So, $H_k$ has 3 vertices too.

If $G_k \simeq H_k$ then $\exists \pi_k$ such that $H_k^{\pi_k} = G_k$. Permutation $\pi_k$ moves vertices of $H$ to the interval between $i_k$ position and $j_k$ position, so that $H_k^{\pi_k} = G_k$. So, we construct a search tree $T_k$ for constructing all possible permutations which could be $\pi_k$. We will follow the construction method of $G_k$ described in 2.1, when we construct all possible permutations that could be $\pi_k$.

Let us define a rooted tree $T_k$ (for $H_k$), its nodes are labelled vertices of $H$. The children of root will be all possible candidates (vertices) for $j_k$ position of $H$ which could be $v_{j_k}$ where $v_{j_k}$ is the $j_k$th vertex of $w_G$. Since $\pi_k$ moves (or fixes) a vertex of $H$ to the $j_k$th position of $H$ such that $H_k^{\pi_k} = G_k$, consider all possible vertices that could be the vertex $v_{j_k}$ of $G_k \subset G$. All $n$ vertices of $H$ could be the $j_k$th vertex of $G$. So, at level 1, the children of root would be all $n$ vertices of $H$. It means, each node of $1^{st}$ level, has a unique label in $T_k$. Thus, we have a bijection form $H$ to the nodes of $1^{st}$ level of $T_k$. Let, $t_{k_1}$ is a node of level 1 of $T_k$.

Each node of $t_{k_1} \in T_k$ is related to a vertex, say $u_l$ in $H$ $(1 \leq l \leq n)$. All vertices that are adjacent to $u_l$ make a subgraph, say $A_H$.

The children nodes of $t_{k_1} \in T_k$ will be all vertices of subgraph $A_H$. Repeat previous procedure $\forall t_{k_1} \in T_k$. Thus we obtain all nodes of the $2^{nd}$ level of $T_k$.

We repeat the procedure for all $H_k$ graphs until we find all possible leaf node of $T_k$. Thus, we construct the search tree $T_k$. Note that, each level represents a position in $w_H$ of $H$, for example, in $T_k$ of $H_k$, $1^{st}$ level represents the $j_k$th position. We construct $x$ such trees.

The height of the $T_k$ is 2 . Note that an individualization-refinement path or discrete partition of $T_k$ is a permutation of $H_k \subset H$ (concept of [1]). All such paths, i.e. permutations create the set $\beta_k$. There will be total $x$ number of $\beta_k$.

For example, if a path is 5, 8, 9 (where 5, 8, 9 represent the labeled vertices of $H$) then, $\pi_k = \left( \frac{j_k}{5} \quad \frac{(j_k - 1)}{8} \quad \frac{(j_k - 2)}{9} \quad i_k \right)$.

It means, $\pi_k$ is a permutation that moves $5^{th}$ vertex of $w_H$ to $j_k^{th}$ vertex, $8^{th}$ vertex of $w_H$ to $(j_k - 1)^{th}$ vertex, and $9^{th}$ vertex of $w_H$ to $(j_k - 2)^{th}$ vertex.

## 3 Propositions

If $G \simeq H$, then $\exists \pi_k \in \beta_k$ such that $H_k^{\pi_k} = G_k$. For each subgraph $H_k$, we have found a set of permutations $\beta_k$(from 2.2). We would be able to construct the direct product $P$ such that $H^p = G$ if and only if $H \simeq G$. If we fail to construct such $P$, it implies that $H \not\simeq G$.

**Proposition 3.1**: $|\beta_k| < n^3$. 
Proposition 3.2: Given two graphs $G,H$ with $n$ vertices each, deciding whether they are isomorphic is polynomial time equivalent to determining generating sets of automorphism group of graphs $G,H$.

Proof: See [3].

So, to decide graph isomorphism of $G,H$, it is sufficient to construct generating sets of automorphism group of graphs $G,H$.

Proposition 3.3: Let $S_n$ be the finite group of order $n!$, There is a subset of elements of $S_n$ of size at most $\log_2(n!)$ which generates $S_n$.

Proof: The proof is similar to the Lemma 1 of [4] on page 3.

From now on, $G,H$ are adjacency matrices of graphs $G,H$ respectively. $H_k, G_k$ are blocks or sub-matrices of matrix $H,G$ respectively. The adjacency matrix of graph $H_k \cup H_e$ is $M_{(k,e)}$ where $M_{(k,e)} = (H_e R_{k,e} H_k)$, where, $R_{k,e}$ is the non symmetric sub-matrix of adjacency matrix $H$. Here, $R_{k,e}$ represents edges between $H_k, H_e$. Similarly, $S_{k,e}$ represents edges between $G_k, G_e$.

$$H = \begin{bmatrix} H(x) & R(x,x-1) & R(x,x-2) & \cdots & \cdots & R(x,1) \\ R(x,x-1) & H(x-1) & R(x-1,x-2) & \cdots & \cdots & R(x-1,1) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ R(x,1) & R(x-1,1) & R(x-2,1) & \cdots & H_1 \end{bmatrix}$$

Proposition 3.4: Generating set of automorphism group of graph $H$ can be constructed in polynomial time if $H$ is not locally triangle-free as defined above.

Proof: An algorithmic proof is presented here. At 1st iteration -

Step 1. Construct all possible direct product $(\pi_1 \times \pi_2)$ where $\pi_1 \in \beta_1$ and $\pi_2 \in \beta_2$.

There are $|\beta_1| \times |\beta_2| < n^9$ direct products (permutations). All these permutations (direct products) form set $\gamma_1$. Each element of $\gamma_1$ is a permutation that acts on graph $H_1 \cup H_2$.

Step 2. Construct/find -

$$\alpha_1 = \{ \pi \in \gamma_1 | (M_{(1,2)}^{(1,2)} = M_{(1,2)}) \land (R_{1,2} = S_{1,2}) \land (H_1^\pi = G_1) \land (H_2^\pi = G_2) \}$$

$\alpha_1$ is the set of automorphisms of matrix $M_{(1,2)}$. $|\alpha_1| < n^3$. There are two possible cases-

Case 1: If $|\alpha_1| = 1$, then for each $\pi_1 \in \beta_1$, there is only one permutation $\pi_2 \in \beta_2$. So, there could be maximum $n^2$ permutations in $\gamma_1$ but only one permutation could be included in $\alpha_1$.

Case 2: If $|\alpha_1| > 1$, we would be able to construct a generating set $S_1$ of an automorphism group of $Aut(M_{(1,2)})$. Note, that if $\pi_\alpha \in Aut(H)$ such that it acts on vertices of $H_1 \cup H_2$, then $\pi_\alpha \in \langle S_1 \rangle = Aut(M_{(1,2)})$. So, when we construct direct product of $S_1$ and another set, $\pi_\alpha$ can be found in the resulting generating set. This concept is similar to extending an automorphism described in [6]. Also, see Theorem 7, on page 31 of [5]. The theorem showed how to obtain the automorphism group of an arbitrary graph from the intersection of a specific permutation group with a direct product of symmetric groups.
Step 3. Now, we construct the generating set $S_1$ from $\alpha_1$. This construction of generating set can be done in polynomial time (see [5], page 40, theorem 9). From proposition 3.3, we find that $|S_1| \leq \log(n!)$. $S_1$ is the generating set of automorphism of $M_{(1,2)}$.

Step 4. We start 2nd iteration, for $\beta_3, S_1$ (instead of $\beta_2$), $M_{(2,3)}$ where $M_{(2,3)} = \begin{pmatrix} H_3 & R_{2,3} \\ R_{2,3}^T & H_2 \end{pmatrix}$. We find $\gamma_2, \alpha_2$ repeating steps 1, 2 and construct $S_2$ (repeating step 3) which is the generating set of automorphism of $M_{(2,3)}$, i.e. graph $H_1 \cup H_2 \cup H_3$. Note that, $|S_2| \leq \log(n!)$. 

Step 5. We keep repeating above four processes, until we find the set $S_{(x-1)}$ which is the generating set of automorphism of graph $H_1 \cup H_2 \cup H_3 \cdots \cup H_x = H$. Note that, $|S_{(x-1)}| \leq \log(n!)$, since $\langle S_{(x-1)} \rangle = \text{Aut}(H) \leq S_n$. ■

4 Conclusion

We repeat the process of construction of $S_{(x-1)}$ for graph $G$ and obtain set $\mathcal{R}_{(x-1)}$. Once we generate generating sets of $G, H$, we can decide isomorphism between them (3.2). The algorithm does not solve graph isomorphism problem in polynomial time if graphs are locally triangle-free. If the sub-matrix of edges is not a zero matrix then the problem reduces down to Bipartite Graph Isomorphism Problem. We can use the same approach there as above. This should lead to a practical solution which is the core idea of practical graph isomorphism [1].

References

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