Lower bounds for multicolor Ramsey numbers

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Abstract

We give an exponential improvement to the lower bound on diagonal Ramsey numbers for any fixed number of colors greater than two.

1 Introduction

The Ramsey number \( r(t; \ell) \) is the smallest natural number \( n \) such that every \( \ell \)-coloring of the edges of the complete graph \( K_n \) contains a monochromatic \( K_t \). For \( \ell = 2 \), the problem of determining \( r(t) := r(t; 2) \) is arguably one of the most famous in combinatorics. The bounds

\[
\sqrt{2^t} < r(t) < 4^t
\]

have been known since the 1940s, but, despite considerable interest, only lower-order improvements [2, 7, 8] have been made to either bound. In particular, the lower bound \( r(t) > (1 + o(1)) \frac{t}{\sqrt{2^t}} \), proved by Erdős [3] as one of the earliest applications of the probabilistic method, has only been improved [8] by a factor of 2 in the intervening 70 years.

If we ignore lower-order terms, the best known upper bound for \( \ell \geq 3 \) is \( r(t; \ell) < \ell^t \), proved through a simple modification of the Erdős–Szemerédi neighborhood-chasing argument [4] that yields \( r(t) < 4^t \). For \( \ell = 3 \), the best lower bound, \( r(t; 3) > \sqrt[3]{3^t} \), again comes from the probabilistic method. For higher \( \ell \), the best lower bounds come from the simple observation of Lefmann [5] that

\[
r(t; \ell_1 + \ell_2) - 1 \geq (r(t; \ell_1) - 1)(r(t; \ell_2) - 1).
\]

To see this, we blow up an \( \ell_1 \)-coloring of \( K_{r(t; \ell_1) - 1} \) with no monochromatic \( K_t \) so that each vertex set has order \( r(t; \ell_2) - 1 \) and then color each of these copies of \( K_{r(t; \ell_2) - 1} \) separately with the remaining \( \ell_2 \) colors so that there is again no monochromatic \( K_t \). By using the bounds \( r(t; 2) - 1 \geq 2^{t/2} \) and \( r(t; 3) - 1 \geq 3^{t/2} \), we can repeatedly apply this observation to conclude that

\[
r(t; 3k) > 3^{kt/2}, \quad r(t; 3k + 1) > 2^{3^{(k-1)t/2}}, \quad r(t; 3k + 2) > 2^{t/2 \cdot 3^{kt/2}}.
\]

Our main result is an exponential improvement to all these lower bounds for three or more colors.

Our principal contribution is the following theorem, proved via a construction which is partly deterministic and partly random. The deterministic part shares some characteristics with a construction of Alon and Krivelevich [1], in that we consider a graph whose vertices are vectors over a finite field where adjacency is determined by the value of their scalar product, while randomness comes in through both random coloring and random sampling.

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**Theorem 1.** For any prime $q$, $r(t; q + 1) > 2^{t/2} q^{3t/8 + o(t)}$.

In particular, the cases $q = 2$ and $q = 3$ yield exponential improvements over the previous bounds for $r(t; 3)$ and $r(t; 4)$, both of which came from the probabilistic method (in fact, Lefmann’s observation gives an additional polynomial factor in the four-color case, but this is of lower order than the exponential improvements that are our concern).

**Corollary 2.** $r(t; 3) > 2^{7t/8 + o(t)}$ and $r(t; 4) > 2^{7/2} 3^{t/8 + o(t)}$.

For the sake of comparison, we note that the improvement for three colors is from $1.732^t$ to $1.834^t$, while, for four colors, it is from $2^t$ to $2.135^t$. Improvements for all $\ell \geq 5$ now follow from repeated applications of Lefmann’s observation, yielding

$$r(t; 3k) > 2^{7kt/8 + o(t)}, \quad r(t; 3k + 1) > 2^{7(k-1)t/8 + t/2} 3^{t/8 + o(t)}, \quad r(t; 3k + 2) > 2^{7kt/8 + t/2 + o(t)},$$

where we used, for instance,

$$r(t; 3k + 1) - 1 \geq (r(t; 3(k-1)) - 1)(r(t; 4) - 1) \geq (r(t; 3) - 1)^{k-1}(r(t; 4) - 1).$$

# 2 Proof of Theorem 1

Let $q$ be a prime. Suppose $t \neq 0 \mod q$ and let $V \subseteq \mathbb{F}_q^t$ be the set consisting of all vectors $v \in \mathbb{F}_q^t$ for which $\sum_{i=1}^t v_i^2 = 0 \mod q$, noting that $q^{t-2} \leq |V| \leq q^t$. Here the lower bound follows from observing that we may pick $v_1, \ldots, v_{t-2}$ arbitrarily and, since every element in $\mathbb{F}_q$ can be written as the sum of two squares, there must then exist at least one choice of $v_{t-1}$ and $v_t$ such that $v_{t-1}^2 + v_t^2 = -\sum_{i=1}^{t-2} v_i^2$.

We will first color all the pairs $\binom{V}{2}$ and then define a coloring of $E(K_n)$ by restricting our attention to a random sample of $n$ vertices in $V$. Formally:

**Coloring all pairs in $\binom{V}{2}$.** For every pair $uv \in \binom{V}{2}$, we define its color $\chi(uv)$ according to the following rules:

- If $u \cdot v = i \mod q$ and $i \neq 0$, then set $\chi(uv) = i$.
- Otherwise, choose $\chi(uv) \in \{q, q + 1\}$ uniformly at random, independently of all other pairs.

**Mapping $[n]$ into $V$.** Take a random injective map $f : [n] \rightarrow V$ and define the color of every edge $ij$ as $\chi(f(i)f(j))$.

Our goal is to upper bound the orders of the cliques in each color class.

**Colors $1 \leq i \leq q-1$.** There are no $i$-monochromatic cliques of order larger than $t$ for any $1 \leq i \leq q-1$. Indeed, suppose that $v_1, \ldots, v_s$ form an $i$-monochromatic clique. We will try to show that they are linearly independent and, therefore, that there are at most $t$ of them. To this end, suppose that

$$u := \sum_{j=1}^s \alpha_j v_j = 0$$
and we wish to show that \( \alpha_j = 0 \mod q \) for all \( j \). Observe that since \( v_j \cdot v_j = 0 \mod q \) for all \( j \) (our ground set \( V \) consists only of such vectors) and \( v_k \cdot v_j = i \mod q \) for each \( k \neq j \), by considering all the products \( u \cdot v_j \), we obtain that the vector \( \bar{\alpha} = (\alpha_1, \ldots, \alpha_s) \) is a solution to

\[
M\bar{\alpha} = \bar{0}
\]

with \( M = iJ - iI \), where \( J \) is the \( s \times s \) all 1 matrix and \( I \) is the \( s \times s \) identity matrix. In particular, we obtain that the eigenvalues of \( M \) (over \( \mathbb{Z} \)) are \( is - i \) with multiplicity 1 and \( -i \) with multiplicity \( s - 1 \). Therefore, if \( s \neq 1 \mod q \), the matrix is also non-singular over \( \mathbb{Z}_q \), implying that \( \bar{\alpha} = 0 \), as required. On the other hand, if \( s = 1 \mod q \), we can apply the same argument with \( v_1, \ldots, v_{s-1} \) to conclude that \( s - 1 \leq t \). But, we cannot have \( s - 1 = t \), since this would imply that \( t = 0 \mod q \), contradicting our assumption. Therefore, we may also conclude that \( s \leq t \) in this case.

**Colors \( q \) and \( q + 1 \).** We call a subset \( X \subseteq V \) a potential clique if \( |X| = t \) and \( u \cdot v = 0 \mod q \) for all \( u, v \in X \). Given a potential clique \( X \), we let \( M_X \) be the \( t \times t \) matrix whose rows consist of all the vectors in \( X \). Observe that \( M_X \cdot M_X^T = 0 \), where we use the fact that each vector is self-orthogonal. First we wish to count the number of potential cliques and later we will calculate the expected number of cliques that survive after we color randomly and restrict to a random subset of order \( n \).

Suppose that \( X \) is a potential clique and let \( r := \text{rank}(X) \) be the rank of the vectors in this clique, noting that \( r \leq t/2 \), since the dimension of any isotropic subspace of \( F_q^t \) is at most \( t/2 \). By assuming that the first \( r \) elements are linearly independent, the number of ways to build a potential clique \( X \) of rank \( r \) is upper bounded by

\[
\left( \prod_{i=0}^{r-1} q^{t-i} \right) \cdot q^{(t-r)r} = q^{(t-r)q} = q^{2tr - \frac{3r^2}{2} + \frac{r}{2}}.
\]

Indeed, suppose that we have already chosen the vectors \( v_1, \ldots, v_s \in X \) for some \( s < r \). Then, letting \( M_s \) be the \( s \times t \) matrix with the \( v_i \) as its rows, we need to choose \( v_{s+1} \) such that \( M_s \cdot v_{s+1} = 0 \). Since the rank of \( M_s \) is assumed to be \( s \), there are exactly \( q^{t-s} \) choices for \( v_{s+1} + \mathbb{F}_q^t \) and, therefore, at most that many choices for \( v_{s+1} \in V \). If, instead, \( s \geq r \), then we need to choose a vector \( v_{s+1} \in \text{span}\{v_1, \ldots, v_r\} \) and there are at most \( q^r \) such choices in \( V \).

Now observe that the function \( 2tr - \frac{3r^2}{2} + \frac{r}{2} \) appearing in the exponent of the expression above is increasing up to \( r = \frac{2t}{3} + \frac{1}{6} \), so the maximum occurs at \( t/2 \). Therefore, by plugging this into our estimate and summing over all possible ranks, we see that the number \( N_t \) of potential cliques in \( V \) is upper bounded by \( q^{\frac{3t^2}{2} + o(t^2)} \).

The probability that a potential clique becomes monochromatic after the random coloring is \( 2^{1-\left(\frac{t}{2}\right)} \). Suppose now that \( p \) is such that \( p|\text{V} = 2n \) and observe that \( p = nq^{-t + O(1)} \). If we choose a random subset of \( V \) by picking each \( v \in V \) independently with probability \( p \), the expected number of monochromatic potential cliques in this subset is, for \( n = 2^{t/2}q^{3t/8 + o(t)} \),

\[
p^t 2^{1-\left(\frac{t}{2}\right)} N_t \leq q^{-t^2 + o(t^2)} n t^2 q^{-\frac{3t^2}{2} + o(t^2)} q^{\frac{3t^2}{2} + o(t^2)} = \left( 2^{-\frac{t}{2} q^{-\frac{3t^2}{2} + o(t^2)} n} \right)^t < 1/2.
\]

Since our random subset will also contain more than \( n \) elements with probability at least 1/2, there exists a choice of coloring and a choice of subset of order \( n \) such that there is no monochromatic potential clique in this subset. This completes the proof.
**Remark.** Our method also gives a construction which matches Erdős’ bound $r(t) > \sqrt{2^t}$ up to lower-order terms. To see this, we set $V = \mathbb{F}_2^t$ and color edges red or blue depending on whether $u \cdot v = 0$ or $1 \mod 2$. If we then sample $2^{t/2+o(t)}$ vertices of $V$ at random, we can show that w.h.p. the resulting set does not contain a monochromatic clique of order $t$. We believed this to be new, but, after the first version of this article was made public, we learned that such a construction was already discovered by Pudlák, Rödl and Savický [6] in 1988. It was also pointed out to us by Jacob Fox that one can achieve the same end by starting with any pseudorandom graph on $n$ vertices for which the count of cliques and independent sets of order $2c \log_2 n$ is approximately the same as in $G(n, 1/2)$ and sampling $n^c$ vertices. This can be applied, for instance, with the Paley graph.

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**References**

[1] N. Alon and M. Krivelevich, Constructive bounds for a Ramsey-type problem, *Graphs Combin.* **13** (1997), 217–225.

[2] D. Conlon, A new upper bound for diagonal Ramsey numbers, *Ann. of Math.* **170** (2009), 941–960.

[3] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

[4] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compos. Math.* **2** (1935), 463–470.

[5] H. Lefmann, A note on Ramsey numbers, *Studia Sci. Math. Hungar.* **22** (1987), 445–446.

[6] P. Pudlák, V. Rödl and P. Savický, Graph complexity, *Acta Inform.* **25** (1988), 515–535.

[7] A. Sah, Diagonal Ramsey via effective quasirandomness, preprint available at arXiv:2005.09251 [math.CO].

[8] J. Spencer, Ramsey’s theorem — a new lower bound, *J. Combin. Theory Ser. A* **18** (1975), 108–115.