Holographic temperature bound in the slow-roll inflation

Yun Soo Myung

Relativity Research Center and School of Computer Aided Science, Inje University
Gimhae 621-749, Korea

Abstract

We investigate the relationship between the holographic temperature bound and the slow-roll inflation. For this purpose we introduce the holographic temperature bound for a radiation matter: \( T \geq T_H \). Here \( T_H \) is the Hubble temperature which arises from the cosmological holographic description of a radiation-dominated universe. For the quasi-de Sitter phase of slow-roll inflation, we find that the holographic temperature bound of \( T_{GH} \geq T_H \) is guaranteed with the Gibbons-Hawking temperature \( T_{GH} \). When \( T_{GH} = T_H \), inflation ends.

*e-mail address: ysmyling@physics.inje.ac.kr
The inflation turned out to be a successful tool to resolve the problems of the hot big bang model [1]. Thanks to the recent observations of the cosmic microwave background anisotropies and large scale structure galaxy surveys, it has become widely accepted by the cosmology community [2]. The idea of inflation is based on the very early universe dominance of vacuum energy density of a hypothetical scalar field, the inflaton. This produces the quasi-de Sitter spacetime [3] and during the slow-roll period, the equation of state can be approximated by the vacuum state as $p \approx -\rho$. After that there must exist a strong non-adiabatic and out-of-equilibrium phase called reheating to produce a large increase of the entropy. However we don’t know exactly how inflation started.

On the other hand, the implications of the holographic principle for cosmology have been investigated in the literature [4, 5, 6, 7, 8]. Verlinde proposed the cosmological holographic bound Eq.(7) in a radiation-dominated phase by introducing three entropies [9]. As an example, such a radiation-dominated phase is provided by a conformal field theory (CFT) with a large central charge which is dual to the AdS-black hole [10]. In this case it appeared an interesting relationship between the Friedmann equation governing the cosmological evolution and the square root form of entropy-energy relation, called Cardy-Verlinde formula [11]. Although the Friedmann equation has the geometric origin and the Cardy-Verlinde formula is designed only for the matter content, it suggested that both may arise from a single fundamental theory.

In this work we will explore the implications of the holographic principle for describing the slow-roll inflation. It is not easy to obtain the holographic bounds for this period, compared with a radiation-dominated universe [12]. However, considering a quasi-de Sitter phase of slow-roll inflation leads to the holographic temperature bound. This work will provide a solution to the question of how the holographic principle is useful to describe inflation.

Let us start an $(n + 1)$-dimensional Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -d\tau^2 + R(\tau)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega_{n-1}^2 \right],$$

where $R$ is the scale factor of the universe and $d\Omega_{n-1}^2$ denotes the line element of an $(n - 1)$-dimensional unit sphere. Here $k = -1, 0, 1$ represent that the universe is open, flat, closed, respectively. A cosmological evolution is determined by the two Friedmann equations

$$H^2 = \frac{16\pi G_{n+1} E}{n(n-1) V} - \frac{k}{R^2},$$

$$\dot{H} = -\frac{8\pi G_{n+1}}{n-1} \left( \frac{E}{V} + p \right) + \frac{k}{R^2},$$

(2)
where $H$ represents the Hubble parameter with the definition $H = \dot{R}/R$ and the overdot stands for derivative with respect to the cosmic time $\tau$, $E$ is the total energy of matter filling the universe, and $p$ is its pressure. $V$ is the volume of the universe, $V = R^n \Sigma^n_k$ with $\Sigma^n_k$ being the volume of an $n$-dimensional space with $k$, and $G_{n+1}$ is the Newton constant in $(n + 1)$ dimensions. Here we assume the equation of state: $\rho = \omega p$, $\rho = E/V$. Before we proceed, we introduce three entropies for a holographic description of a radiation-dominated universe [9]:

\begin{align*}
\text{Bekenstein – Verlinde entropy} : & \quad S_{BV} = \frac{2\pi}{n} ER, \\
\text{Bekenstein – Hawking entropy} : & \quad S_{BH} = (n - 1) \frac{V}{4G_{n+1} R}, \\
\text{Hubble entropy} : & \quad S_H = (n - 1) \frac{HV}{4G_{n+1}}. \quad (3)
\end{align*}

We define a quantity $E_{BH}$ which corresponds to energy needed to form a universe-size black hole: $S_{BH} = (n - 1)V/4G_{n+1} R \equiv 2\pi E_{BH} R/n$. The Friedmann equations (2) can be further cast to the cosmological Cardy-Verlinde formula and cosmological Smarr formula respectively

\begin{align*}
S_H &= \frac{2\pi R}{n} \sqrt{E_{BH}(2E - kE_{BH})}, \\
kE_{BH} &= n(E + pV - T_H S_H), \quad (4)
\end{align*}

where the Hubble temperature ($T_H$) is given by

\begin{equation}
T_H = -\frac{\dot{H}}{2\pi H} \quad (5)
\end{equation}

as the minimum temperature during the strongly gravitating phase of $HR \geq \sqrt{2 - k}$.

Eq.(4) corresponds to another representation of the Friedmann equations expressed in terms of holographic quantities.

On the matter-side, the entropy of radiation and its Casimir energy can be described by the Cardy-Verlinde formula and the Smarr formula, respectively

\begin{align*}
S &= \frac{2\pi R}{n} \sqrt{E_c(2E - E_c)}, \\
E_c &= n(E + pV - TS). \quad (6)
\end{align*}

The first denotes the entropy-energy relation, where $S$ is the entropy of a CFT-like radiation living on an $n$-dimensional sphere with radius $R$ and $E$ is the total energy of the CFT. The second represents the relation between a non-extensive part of the total energy
(Casimir energy) and thermodynamic quantities. Here $E_c$ and $T$ stand for the Casimir energy of the system and the temperature of radiation with $\omega = 1/n$. We note again that the above equations correspond to thermodynamic relations for the CFT-matter which are originally independent of the geometric Friedmann equations. Suppose that the entropy of radiation in the FRW universe can be described by the Cardy-Verlinde formula. Then comparing (4) with (6), one finds that if $E_{BH} = E_c$, then $S_H = S$ and $T_H = T$. For a $k = 1$ closed radiation-dominated universe, a bound on the Casimir energy ($E_c \leq E_{BH}$) leads to the Hubble bounds for entropy and temperature [9]

$$S \leq S_H, \quad T \geq T_H, \quad \text{for } HR \geq 1$$

which shows inequalities between geometric and matter quantities. The Hubble entropy bound can be saturated by the entropy of a radiation-matter filling the universe when its Casimir energy $E_c$ is enough to form a universe-size black hole. If this happens, equations (4) and (5) coincide. This implies that the first Friedmann equation somehow knows the entropy formula for a radiation-matter filling the universe. As an example, we consider a moving brane universe in the background of the 5D Schwarzschild-AdS black hole. Savonije and Verlinde [10] found that when this brane crosses the black hole horizon, the Hubble entropy bound is saturated by the entropy of black hole (= the entropy of the CFT-radiation). At this moment $(T_H, E_{BH})$ are identical with $(T, E_c)$ of the CFT-matter dual to the AdS black hole respectively. For a radiation-dominated universe with a positive cosmological constant, the holographic bound was discussed in [13].

For arbitrary $k$, the Hubble bounds for a radiation-dominated universe are still valid [14]

$$S \leq S_H, \quad T \geq T_H, \quad \text{for } HR \geq \sqrt{2 - k}.$$  

(8)

On the other hand, for the general equation of state of $p = \omega \rho$, the entropy-energy relation no longer coincide with the first Friedmann equation and the conjectured bound on the Casimir energy does not leads to the Hubble entropy bound. But it was argued that even for $\omega \neq 1/n$ including $\omega = -1$ for a cosmological constant $\Lambda$, the Hubble temperature bound $(T \geq T_H)$ is still satisfied [15]. We wish to test here whether or not this argument is correct for the inflation.

For this purpose, we adopt a model of primordial inflation based on the quasi-de Sitter space and FRW space [3]. In what follows we work with the (3+1)-dimensional flat FRW slicing of de Sitter spacetime, because this maps directly onto the FRW spacetime of the post inflationary universe. The line element which covers half of the full de Sitter solution is given by

$$ds_{FRW-dS}^2 = -d\tau^2 + \exp[2H\tau](dr^2 + r^2 d\Omega_2^2).$$  

(9)
Another slicing of de Sitter spacetime is given by the static coordinates\(^1\)

\[
ds^2_{s-deS} = -(1 - H^2\tilde{r}^2)dt^2 + \left(1 - H^2\tilde{r}^2\right)^{-1}d\tilde{r}^2 + \tilde{r}^2d\Omega_2^2,
\]

where \(H^{-1}\) is the size of the cosmological horizon. The cosmological horizon is similar to the event horizon of the black hole. Accordingly the Gibbons-Hawking temperature is defined by \(T_{GH} = H/2\pi\) \([17]\) and the area of the horizon is \(A = 4\pi/H^2\). A role of the Gibbons-Hawking temperature in inflation was discussed in \([18]\). In order to see a route of information flow from inflation to observable anisotropy, see the Penrose diagram in Fig.1.

\[\text{Figure 1: Penrose diagram of a cosmology with inflation based on quasi-de Sitter space and Friedmann-Robertson-Walker space.}\]

\(^1\)The coordinates of \(\tau, r\) and \(t, \tilde{r}\) are related by the transformations \(r = e^{H\tau\tilde{r}/\sqrt{1 - H^2\tilde{r}^2}}, \tau = t + \frac{1}{2H} \ln[1 - H^2\tilde{r}^2]\) \([10]\).
of an observer at the origin. Others are boundaries at infinity. The lower half stands for a quasi-de Sitter space (QdS) and the upper half for a FRW spacetime. The join between them is the epoch of reheating (RH) and shaded regions of each show the regions within the apparent horizon of an observer at the origin: one is an apparent cosmological horizon for QdS and the other is an apparent particle horizon for FRW. REC and BBN represent spacelike hypersurfaces for the recombination epoch and big bang nucleosynthesis epoch. Two regions in QdS are necessary, one appropriate for matching onto FRW and the other for holographic analysis. Actually perturbations may be imprinted by fluctuating quantum field (QF) on the scale of the apparent cosmological horizon during the slow-roll period of inflation (SR). The apparent horizon grows slightly during SR, as is shown by two closely parallel null lines. This happens because the spacetime becomes asymptotically de Sitter space due to the increase of entropy during SR. The intersection of our past light cone (null line) with REC is the two-sphere of the last scattering surface (LS) for the cosmic background radiation. A particular timelike trajectory of a comoving sphere (CMS) is shown. The radiation-dominated era (RD) is from the end of RH to the time of REC and the matter-dominated era (MD) is extended from REC to the present: US, NOW. Finally a high frequency gravitational wave background (GWR) can reach US via direct null trajectories.

In order to describe the inflation, we introduce a scalar field \( \phi \equiv \phi(\tau) \): inflaton). This gives us the energy density and pressure

\[
\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \quad p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi).
\]

(11)

Note that although the scalar field acts as a matter, it does not possess an exact equation of state like \( p_\phi = \omega_\phi \rho_\phi \). Assuming a spatially flat universe, we obtain

\[
H^2 = \frac{1}{3 M_p^2} \left[ V(\phi) + \frac{\dot{\phi}^2}{2} \right]
\]

(12)

and

\[
\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0
\]

(13)

where the Planck mass is given by \( M_p = 1/\sqrt{8\pi G_4} \) in the units of \( c = \hbar = 1 \). \( V'(\phi) \) denotes the differentiation with respect to its argument. The first equation is obtained from Eqs. (2) and (11), whereas the second from the conservation law of \( \dot{\rho} + 3H(\rho + p) = 0 \). Inflation occurs when the potential energy of the scalar is dominant in Eq. (12). This situation is approximated by the slow-roll period of inflation which is formally defined by

\[
\epsilon(\phi) = \frac{M_p^2}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 \ll 1, \quad |\eta(\phi)| \ll 1 \text{ with } \eta = M_p^2 \frac{V''(\phi)}{V(\phi)}.
\]

(14)
Then we obtain two equations (12) and (13) in the slow-roll approximation
\[ H^2 \simeq \frac{V(\phi)}{3M_p^2}, \quad 3H \frac{\dot{\phi}}{M_p} \simeq -V'(\phi), \] (15)
where \(\simeq\) indicates that the quantities are equal with the slow-roll approximation. From Eq. (2) one finds a relation
\[ \dot{H} = -\frac{\dot{\phi}^2}{2M_p^2}. \] (16)
In the slow-roll approximation the potential can be taken to be a nearly constant. Hence this can be approximated by a quasi-de Sitter phase with its temperature \(T_{GH}\).

The slow-roll approximation is sufficient condition for inflation. To see this, let us rewrite the condition for inflation as
\[ \frac{\ddot{R}}{R} = \dot{H} + H^2 > 0. \] (17)
This is obviously satisfied if \(\dot{H}\) is positive. Otherwise, we require
\[ \epsilon_{HJ} \equiv -\frac{\dot{H}}{H^2} < 1 \] (18)
where \(\epsilon_{HJ}\) is the slow-roll parameter in the Hamilton-Jacobi formalism. \(\epsilon_{HJ}\) leads to
\[ \epsilon_{HJ} \simeq \frac{3}{2} \frac{\dot{\phi}^2}{V(\phi)} \simeq \frac{M_p^2}{2} \left(\frac{V'(\phi)}{V(\phi)}\right)^2 = \epsilon. \] (19)
Hence, if the slow-roll approximation is valid \((\epsilon \ll 1)\), then inflation \((\epsilon_{HJ} < 1)\) is guaranteed.

On the other hand, when expressing \(\epsilon_{HJ}\) in terms of the Hubble temperature \(T_H = -\frac{\dot{H}}{2\pi H}\) and the Gibbons-Hawking temperature \(T_{GH} = \frac{H}{2\pi}\), one finds the relation
\[ \epsilon_{HJ} = \frac{T_H}{T_{GH}} \simeq \epsilon. \] (20)
Similarly, if the slow-roll approximation is valid \((\epsilon \ll 1)\), an inequality of \(T_{GH} > T_H\) is guaranteed. From Eq. (19) this inequality is another representation for existing inflation. Here the matter-temperature \(T\) is replaced by \(T_{GH}\) because in the slow-roll period of quasi-de Sitter space, the matter-distribution is approximately given by the positive cosmological constant \((\Lambda \simeq V(\phi))\). From the above, we arrive at the holographic temperature bound
\[ T_{GH} \geq T_H \] (21)
which is valid for the period of inflation. This is the main result of our work. Eq. (21) has the same form as in the Hubble temperature bound \( T \geq T_H \) for a radiation-dominated universe except replacing \( T \) by \( T_{GH} \). Here \( T_H \) corresponds to the minimum temperature in the period of inflation. When the Gibbons-Hawking temperature is equal to the Hubble temperature \( (T_{GH} = T_H) \), inflation comes to an end \( (\epsilon = 1) \).

Although two temperature of \( T_{GH} \) and \( T_H \) have the geometric origin, our new interpretation for the inflationary period are helpful to understand when the inflation ended. That is, when \( T_{GH} = T_H \), inflation ended. However, concerning the question of when the inflation started \[20\], we have still no information.

**Acknowledgment**

This work was supported in part by KOSEF, Project Number: R02-2002-000-00028-0.

**References**

[1] A. H. Guth, Phys. Rev. D 23, 347 (1981); A. D. Linde, Phys. Lett. B 108, 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).

[2] J. Garcia-Bellido, Nucl. Phys. Proc. Suppl. 114, 13 (2003) [arXiv:hep-ph/0210050].

[3] C. J. Hogan, Phys. Rev. D 66, 023521 (2002) [arXiv:astro-ph/0201020].

[4] G. ’t Hooft, [arXiv:gr-qc/9310026](http://arxiv.org/abs/gr-qc/9310026) L. Susskind, J. Math. Phys. 36, 6377 (1995) [arXiv:hep-th/9409089](http://arxiv.org/abs/hep-th/9409089).

[5] J. D. Bekenstein, Phys. Rev. D 23, 287 (1981).

[6] W. Fischler and L. Susskind, [arXiv:hep-th/9806039](http://arxiv.org/abs/hep-th/9806039).

[7] R. Easther and D. A. Lowe, Phys. Rev. Lett. 82, 4967 (1999) [arXiv:hep-th/9902088](http://arxiv.org/abs/hep-th/9902088); G. Veneziano, Phys. Lett. B 454, 22 (1999) [arXiv:hep-th/9902126](http://arxiv.org/abs/hep-th/9902126); G. Veneziano, arXiv:hep-th/9907012; R. Brustein and G. Veneziano, Phys. Rev. Lett. 84, 5695 (2000) [arXiv:hep-th/9912055](http://arxiv.org/abs/hep-th/9912055); D. Bak and S. J. Rey, Class. Quant. Grav. 17, L83 (2000) [arXiv:hep-th/9902173](http://arxiv.org/abs/hep-th/9902173); N. Kaloper and A. D. Linde, Phys. Rev. D 60, 103509 (1999) [arXiv:hep-th/9904120](http://arxiv.org/abs/hep-th/9904120).

[8] R. Bousso, JHEP 9907, 004 (1999) [arXiv:hep-th/9905177](http://arxiv.org/abs/hep-th/9905177); R. Bousso, JHEP 9906, 028 (1999) [arXiv:hep-th/9906022](http://arxiv.org/abs/hep-th/9906022).
[9] E. Verlinde, arXiv:hep-th/0008140.

[10] I. Savonije and E. Verlinde, Phys. Lett. B 507, 305 (2001) arXiv:hep-th/0102042.

[11] J. L. Cardy, Nucl. Phys. B 270, 186 (1986).

[12] Y. S. Myung, arXiv:hep-th/0301073.

[13] R. G. Cai and Y. S. Myung, Phys. Rev. D 67, 124021 (2003) arXiv:hep-th/0210272; Y. S. Myung, arXiv:hep-th/0306180.

[14] D. Youm, Phys. Lett. B 515, 170 (2001) arXiv:hep-th/0105093.

[15] D. Youm, Phys. Lett. B 531, 276 (2002) arXiv:hep-th/0201268.

[16] A. Frolov and L. Kofman, arXiv:hep-th/0212327.

[17] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15, 2738 (1977).

[18] A. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic Publishers, 1990).

[19] A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large-Scale Structure* (Cambridge Univ. Press, 2000).

[20] A. Borde, A. H. Guth, and A. Vilenkin, Phys. Rev. Lett. 90, 151301 (2003) arXiv:gr-qc/0110012.