Modular invariance approach to masses and mixing of neutrino flavors

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Abstract. We discuss the phenomenological implications of the modular symmetry \( \Gamma(3) \simeq A_4 \) of lepton flavors. The mass matrices of neutrinos and charged leptons are given by fixing the modulus \( \tau \), which is the only source of the breaking of the modular invariance.

1. Introduction

One of the interesting approaches to flavors is to impose non-Abelian discrete symmetries for flavors. Many models have been proposed by using non-Abelian discrete groups, \( S_3, A_4, S_4, A_5 \), and other groups with larger orders \([1-5]\). Interestingly, the modular group includes \( S_3, A_4, S_4, \) and \( A_5 \) as its congruence subgroups, \( \Gamma(N) \). In the modular symmetry, coupling constants such as Yukawa couplings also transform non-trivially under the modular symmetry and are written as functions of the modulus called modular forms. In this aspect, an attractive ansatz was proposed in Ref. \([6]\), where \( \Gamma(3) \simeq A_4 \) was taken.

In this talk, we present a model of \( \Gamma(3) \simeq A_4 \), where the mass matrices of neutrinos and charged leptons are given by fixing the expectation value of the modulus \( \tau \), which is the only source of the breaking of the modular invariance.

2. Modular Symmetry

The torus compactification is the simplest compactification. The two-dimensional torus \( T^2 \) can be constructed as division of \( \mathbb{R}^2 \) by a two-dimensional lattice \( \Lambda \), i.e. \( T^2 = \mathbb{R}^2 / \Lambda \). Here, we use the complex coordinate on \( \mathbb{R}^2 \), and the lattice is spanned by two lattice vectors, \( \alpha_1 = 2\pi R \) and \( \alpha_2 = 2\pi R\tau \), where \( R \) is real and \( \tau \) is a complex modulus parameter. The same lattice can be spanned by the following basis vectors,

\[
\left( \begin{array}{c} \alpha'_1 \\ \alpha'_2 \end{array} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} \alpha_2 \\ \alpha_1 \end{array} \right),
\]

where \( a, b, c, d \) are integer with satisfying \( ad - bc = 1 \). That is the \( SL(2, \mathbb{Z}) \) transformation.

Under the above transformation, the modulus parameter transforms as

\[
\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d},
\]

and this modular transformation is generated by \( S \) and \( T \),

\[
S: \tau \rightarrow -\frac{1}{\tau}, \quad T: \tau \rightarrow \tau + 1.
\]
They satisfy the following algebraic relations,

$$S^2 = I, \quad (ST)^3 = I.$$  

This is the modular symmetry $\Gamma$. The congruence subgroups of level $N$ are defined as

$$\Gamma(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{PSL}(2, \mathbb{Z}), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \pmod{N} \right\}.$$  

Furthermore, the quotient subgroups $\Gamma_N$ are given as $\Gamma_N = \Gamma/\Gamma(N)$. These are finite groups for $N = 2, 3, 4, 5$, and isomorphic to $A_N$ or $S_N$: $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, $\Gamma_5 \simeq A_5$, where the algebraic relation $T^N = I$ is satisfied. Holomorphic functions which transform as

$$f(\tau) \to (c\tau + d)^k f(\tau),$$  

under the modular transformation (2) are called modular forms of weight $k$.

Superstring theory on the torus $T^2$ or orbifold $T^2/Z_N$ has the modular symmetry. Its low-energy effective field theory is described in terms of supergravity theory, and string-derived supergravity theory has also the modular symmetry. Under the modular transformation (2), chiral superfields $\phi^{(I)}$ transform as,

$$\phi^{(I)} \to (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \phi^{(I)},$$  

where $-k_I$ is the so-called modular weight and $\rho^{(I)}$ denotes a representation matrix. That is, the superpotential should have vanishing modular weight in global supersymmetric models, while the superpotential in supergravity should be invariant under the modular symmetry up to the Kähler transformation.

The Dedekind eta-function $\eta(\tau)$ is one of famous modular forms, which is written by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$  

where $q = e^{2\pi i \tau}$ and $\eta(\tau)^{24}$ is a modular form of weight 12. By use of $\eta(\tau)$ and its derivative, $A_4$ triplet modular forms $Y = (Y_1, Y_2, Y_3)^T$ of modular weight 2 are written by [6],

$$Y_1(\tau) = \frac{i}{2\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} - \frac{27q'(3\tau)}{\eta(3\tau)} \right),$$  

$$Y_2(\tau) = \frac{-i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2 \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right),$$  

$$Y_3(\tau) = \frac{-i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega^2 \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right),$$  

where $\omega = e^{2\pi i/3}$. The overall coefficient in (9) is one choice and cannot be determined essentially.

3. Models with modular symmetry

Let us consider the modular invariant flavor model with the $A_4$ symmetry for lepton mass matrices. We consider the type I seesaw model where neutrinos are Majorana particles [7]. The assignments of representations and modular weights to the MSSM fields as well as right-handed neutrino superfields are presented in Table 1.
For the charged leptons, we assign three right-handed charged leptons for three different singlets of \( A_4 \), \( (1, 1', 1'') \), respectively. Therefore, there are three independent couplings in the superpotential of the charged lepton sector. The modular invariant mass terms of the leptons are given by the following superpotentials:

\[
\begin{align*}
\mathcal{W}_e &= \alpha e_R H_d(LY) + \beta \mu_R H_d(LY) + \gamma \tau_R H_d(LY), \\
\mathcal{W}_D &= g(\nu_R H_u LY)_1, \\
\mathcal{W}_N &= \Lambda(\nu_R \nu_R Y)_1,
\end{align*}
\]

where the sums of the modular weights vanish. The parameters \( \alpha, \beta, \gamma, g, \) and \( \Lambda \) are constant coefficients. The functions \( Y_1(\tau) \) are \( A_4 \) triplet modular forms in Eq. (9).

\[ M_E = \text{diag}[\alpha, \beta, \gamma] \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix} \]

\[ M_D = v_u \begin{pmatrix} 2g_1 Y_1 & (-g_1 + g_2)Y_3 & (-g_1 - g_2)Y_2 \\ (-g_1 - g_2)Y_3 & 2g_1 Y_2 & (-g_1 + g_2)Y_1 \\ (-g_1 + g_2)Y_2 & (-g_1 - g_2)Y_1 & 2g_1 Y_3 \end{pmatrix}_{RL}. \]

\[ M_N = \Lambda \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \]

Finally, the effective neutrino mass matrix is obtained by the type I seesaw.

### 4. Numerical results

Our lepton mass matrices are given by modulus parameter \( \tau \). By fixing \( \tau \), the modular invariance is broken, and then the lepton mass matrices give the mass eigenvalues and flavor mixing. We consider NH of neutrino masses \( m_1 < m_2 < m_3 \). By inputting the data of \( \Delta m^2_{\text{atm}} \equiv m_2^2 - m_1^2 \), \( \Delta m^2_{\text{sol}} \equiv m_3^2 - m_2^2 \), and three mixing angles \( \theta_{23}, \theta_{12}, \) and \( \theta_{13} \) with \( 3\sigma \) error-bar, we fix the modulus \( \tau \) and the other parameters.
Now, we can predict the CP violating Dirac phases $\delta_{CP}$ and Majorana phases. The coefficients $\alpha/\gamma$ and $\beta/\gamma$ in the charged lepton mass matrix are given only in terms of $\tau$ by the input of the observed values $m_e/m_\tau$ and $m_\mu/m_\tau$. Then, we have two free parameters, $g_1/g_2$ and the modulus $\tau$ apart from the overall factors in the neutrino sector. Since these are complex, we set

$$\tau = \text{Re}[\tau] + i \text{Im}[\tau], \quad \frac{g_2}{g_1} = g e^{i\phi}.$$ (14)

At first, we present the prediction of the Dirac CP violating phase $\delta_{CP}$ versus $\sin^2 \theta_{23}$ for NH of neutrino masses in Fig.1. It is emphasized that $\sin^2 \theta_{23}$ is restricted to be larger than 0.54, and $\delta_{CP} = (50^\circ - 180^\circ)$. Since the correlation of $\sin^2 \theta_{23}$ and $\delta_{CP}$ is characteristic, this prediction is testable in the future experiments of neutrinos.

The effective mass $\langle m_{ee} \rangle$ which is the measure of the neutrinoless double beta decay is around 22meV while $m_1$ is 40meV. Note that the range of $m_1$ is restricted by the cosmological upper bound for the sum of neutrino masses.

5. Summary
We study the phenomenological implications of the modular symmetry $\Gamma(3) \simeq A_4$. The mass matrices of neutrinos and charged leptons are essentially given by fixing the expectation value of the modulus $\tau$, which is the only source of the breaking of the modular invariance.

For NH of neutrino masses, we have found that the seesaw model is available facing recent experimental data and the cosmological bound of the sum of neutrino masses. The predicted $\sin^2 \theta_{23}$ is restricted to be larger than 0.54 and $\delta_{CP} = (50^\circ - 180^\circ)$. The distinct correlation between $\sin^2 \theta_{23}$ and $\delta_{CP}$ is testable in the future experiments of the neutrino oscillations. It is remarkable that $\langle m_{ee} \rangle$ is around 22meV while the sum of neutrino masses is 145meV.

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