Equivariant $K$-theory of regular compactifications: further developments

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Abstract. We describe the $\tilde{G} \times G$-equivariant $K$-ring of $X$, where $\tilde{G}$ is a factorial covering of a connected complex reductive algebraic group $G$, and $X$ is a regular compactification of $G$. Furthermore, using the description of $K_{\tilde{G} \times G}(X)$, we describe the ordinary $K$-ring $K(X)$ as a free module (whose rank is equal to the cardinality of the Weyl group) over the $K$-ring of a toric bundle over $G/B$ whose fibre is equal to the toric variety $\mathcal{T}^+$ associated with a smooth subdivision of the positive Weyl chamber. This generalizes our previous work on the wonderful compactification (see [1]). We also give an explicit presentation of $K_{\tilde{G} \times G}(X)$ and $K(X)$ as algebras over $K_{\tilde{G} \times G}(G_{ad})$ and $K(G_{ad})$ respectively, where $G_{ad}$ is the wonderful compactification of the adjoint semisimple group $G_{ad}$. In the case when $X$ is a regular compactification of $G_{ad}$, we give a geometric interpretation of these presentations in terms of the equivariant and ordinary Grothendieck rings of a canonical toric bundle over $\tilde{G}_{ad}$.

Keywords: equivariant $K$-theory, regular compactification, wonderful compactification, toric bundle.

Introduction

Let $G$ be a connected complex reductive algebraic group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus of dimension $l$. Let $C$ be the centre of $G$, and let $G_{ad} := G/C$ be the corresponding semisimple adjoint group. We denote the Weyl group of $(G,T)$ by $W$.

A $G$-variety is a complex algebraic variety equipped with an algebraic action of $G$.

We now recall the definition of a regular $G$-variety given by Bifet, De Concini and Procesi (see [2], §3).

A $G$-variety $X$ is said to be regular if it satisfies the following conditions.

(i) $X$ is smooth and contains a dense $G$-orbit $X^0_G$ whose complement is a union of irreducible smooth divisors with normal crossings (the boundary divisors).

(ii) The closure in $X$ of every $G$-orbit is the transversal intersection of the boundary divisors which contain it.

(iii) For every $x \in X$, the normal space $T_xX/T_x(Gx)$ contains a dense orbit of the isotropy group $G_x$.

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A normal complete variety $X$ is called an *equivariant compactification* of $G$ if $X$ contains $G$ as an open subvariety and the action of $G \times G$ on $G$ by left and right multiplication extends to $X$. We say that $X$ is a *regular compactification* of $G$ if $X$ is an equivariant compactification of $G$ which is regular as a $G \times G$-variety in the sense of [2], §3 (see above).

Smooth complete toric varieties are exactly the regular compactifications of the torus. For the adjoint group $G_{\text{ad}}$, the wonderful compactification $\overline{G_{\text{ad}}}$ constructed by De Concini and Procesi [3] is the unique regular compactification of $G_{\text{ad}}$ with a unique closed $G_{\text{ad}} \times G_{\text{ad}}$-orbit.

We now recall some facts and notation from [4], §3.1.

Let $\overline{T}$ denote the closure of $T$ in $X$. Under the left action of $T$ (that is, the action of $T \times \{1\}$), $\overline{T}$ is a smooth complete toric variety. Let $\mathcal{F}$ be the fan associated with $\overline{T}$ in $X_*(T) \otimes \mathbb{R}$. Then $\mathcal{F}$ is a smooth subdivision of the fan associated with the Weyl chambers in $X_*(T) \otimes \mathbb{R}$. Moreover, $\mathcal{F} = WF_+$, where $\mathcal{F}_+$ is the subdivision of the positive Weyl chamber formed by the cones in $\mathcal{F}$ contained in this chamber. Let $T^+$ denote the smooth toric variety associated with the fan $\mathcal{F}_+$ (see [4], Propositions A1 and A2, for details).

Recall that there is an exact sequence

$$1 \to Z \to \tilde{G} := \tilde{C} \times G^{\text{ss}} \xrightarrow{\pi} G \to 1,$$  

where $Z$ is a finite central subgroup, $\tilde{C}$ is a torus and $G^{\text{ss}}$ is semisimple and simply connected. In particular, $\tilde{G}$ is *factorial* and $\tilde{B} := \pi^{-1}(B)$ and $\tilde{T} := \pi^{-1}(T)$ are respectively a Borel subgroup and a maximal torus of $\tilde{G}$ (see [5], Corollary 3.7, and [6]).

In this article we consider the $\tilde{G} \times \tilde{G}$-equivariant $K$-ring of $X$. We take the natural action of $\tilde{G} \times \tilde{G}$ on $X$ by means of the canonical surjection onto $G \times G$. Note that the main purpose of passing to the factorial covering $\tilde{G}$ of $G$ is to apply the description of the $\tilde{G} \times \tilde{G}$-equivariant $K$-ring to describe the ordinary $K$-ring of $X$. Indeed, the results of [6] show that the relation of $K_{\tilde{G} \times \tilde{G}}(X)$ to the ordinary $K$-theory is simpler than that of $K_{G \times G}(X)$.

We recall that in [1], §3, taking $G$ to be the simply connected covering of $G_{\text{ad}}$ and $T$ a maximal torus of $G$, we described the $G \times G$-equivariant $K$-ring of $\overline{G_{\text{ad}}}$. More precisely, we endowed $K_{G \times G}(\overline{G_{\text{ad}}})$ with the structure of an $R(T) \otimes R(G)$-algebra and determined the multiplication rule explicitly (see [1], Theorem 3.8). Then we used this structure to describe the ordinary Grothendieck ring $K(X)$ as a $K(G/B)$-algebra (see [1], Theorem 3.12).

Our primary aim in this article is to generalize these results on the wonderful compactification to any regular compactification $X$. With this aim in view, we first give in §2 a description of $K_{\tilde{G} \times \tilde{G}}(X)$ as a free module of rank $|W|$ over a canonical subring isomorphic to $K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{G})$ (see Theorem 2.2). Here $K_{\tilde{T}}(\tilde{T}^+)$ is the $\tilde{T}$-equivariant $K$-ring of the toric variety $\tilde{T}^+$, which is then identified with the subring of $K_{\tilde{G} \times \tilde{G}}(X)$ generated as an $R(\tilde{G}) \otimes R(\tilde{G})$-algebra by $\text{Pic}_{\tilde{G} \times \tilde{G}}(X)$. Furthermore we determine a canonical basis for $K_{\tilde{G} \times \tilde{G}}(X)$ as a $K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{G})$-module and describe the multiplication rule of the basis elements (see Theorem 2.2). In particular, when $X$ is the wonderful compactification of $G_{\text{ad}}$, the toric variety $\overline{T}_{\text{ad}}^+$
is the affine space $\mathbb{A}^r$, so that $K_{\overline{T}}(\overline{T}_{ad}^+) \subseteq K_{\overline{T}}(\overline{T}^+)$ can be identified with $R(\overline{T})$ and we get back to Theorem 3.8 of [1] in this case. Note that in the current notation $G$ stands for the simply connected covering of $G_{ad}$, and $\overline{T} \subseteq G$ is a maximal torus.

We mention here that [1], §2 contains general results on the $G \times G$-equivariant $K$-ring of a regular embedding which are closely related to our results in §2. In particular, Proposition 2.1 below is an extension of Proposition 2.5 of [1] to the setting of $G \times G$-equivariant $K$-theory (see [1], Remark 2.7). Moreover, in Proposition 2.1 we in fact establish an inclusion $K_{\tilde{G} \times \tilde{G}}(X) \subseteq K_{\overline{T}}(\overline{T}^+) \otimes R(\overline{T})$ of $R(\overline{T}) \otimes R(\tilde{G})$-algebras. This is sharper than the analogous inclusion $K_{\tilde{G} \times \tilde{G}}(X) \subseteq R(\overline{T})^{\mathbb{F}_p(l)} \otimes R(\overline{T})$ in Proposition 2.5 of [1]. The inclusion

$$K_{\tilde{G} \times \tilde{G}}(X) \subseteq K_{\overline{T}}(\overline{T}^+) \otimes R(\overline{T})$$

plays a key role in the proof of Theorem 2.2, which generalizes to any regular compactification Lemma 3.2 and Theorem 3.3 of [1] on the wonderful compactification.

In §3 we describe the ordinary $K$-ring of $X$ using the above structure of $K_{\tilde{G} \times \tilde{G}}(X)$ and applying Theorem 4.2 in [6]. More precisely, we prove that the subring generated by Pic$(X)$ in $K(X)$ is canonically isomorphic to $\mathcal{R}(\overline{T}^+) := \mathbb{Z} \otimes_{R(\tilde{G})} K_{\overline{T}}(\overline{T}^+)$. Further, in Theorem 3.1, we show that $K(X)$ is a free module of rank $|W|$ over $\mathcal{R}(\overline{T}^+)$. We finally construct an explicit basis of $K(X)$ over $\mathcal{R}(\overline{T}^+)$ and determine the multiplicative structure constants with respect to this basis. It is worth noticing here that

$$\mathcal{R}(\overline{T}^+) = \mathbb{Z} \otimes_{R(\tilde{G})} K_{\overline{T}}(\tilde{G} \times \tilde{T}^+) = K(\tilde{G} \times \tilde{T}^+) = K(G \times B \tilde{T}^+),$$

where $B$ acts on $\overline{T}^+$ by means of its quotient $T$. In other words, $\mathcal{R}(\overline{T}^+)$ is the Grothendieck ring of the toric bundle over $G/B$ associated with $\overline{T}^+$. In particular, when $X$ is the wonderful compactification of $G_{ad}$, one can identify $\mathcal{R}(\overline{T}_{ad}^+)$ with $K(G/B)$. Thus we get back to Theorem 3.12 in [1].

In §4 we use the canonical surjective morphism $f: X \to \overline{G}_{ad}$ to compare the equivariant and ordinary $K$-rings of any regular embedding with those of the wonderful compactification. We prove that $K_{\tilde{G} \times \tilde{G}}(X)$ is the tensor product of $K_{\tilde{G} \times \tilde{G}}(\overline{G}_{ad})$ (where $\tilde{G}$ acts on $\overline{G}_{ad}$ by means of its quotient $G_{ss}$) and the subalgebra generated by Pic$_{\tilde{G} \times \tilde{G}}(X)$, over the subalgebra of $K_{\tilde{G} \times \tilde{G}}(\overline{G}_{ad})$ generated by Pic$_{\tilde{G} \times \tilde{G}}(\overline{G}_{ad})$. A similar assertion holds for the ordinary $K$-ring. We actually give explicit presentations for $K_{\tilde{G} \times \tilde{G}}(X)$ as a $K_{\tilde{G} \times \tilde{G}}(\overline{G}_{ad})$-algebra and $K(X)$ as a $K(\overline{G}_{ad})$-algebra (see Theorems 4.1, 4.2).

In §4.2, for any regular embedding $X$ of the adjoint semisimple group $G_{ad}$ we give a geometric interpretation of the presentations of $K_{G_{ss} \times G_{ss}}(\overline{G}_{ad})$ as a $K_{G_{ss} \times G_{ss}}(\overline{G}_{ad})$-algebra and $K(X)$ as a $K(\overline{G}_{ad})$-algebra obtained in Theorems 4.1, 4.2. More precisely, in Corollary 4.1 we prove that $K_{G_{ss} \times G_{ss}}(X)$ as a $K_{G_{ss} \times G_{ss}}(\overline{G}_{ad})$-algebra and $K(X)$ as a $K(\overline{G}_{ad})$-algebra are respectively isomorphic to the $G_{ss} \times G_{ss}$-equivariant and ordinary Grothendieck rings of a canonical toric bundle over $\overline{G}_{ad}$. 


An appendix (§5) is devoted to the equivariant $K$-theory of toric bundles with fibre a smooth semi-projective toric variety. We take the left action of a connected complex reductive algebraic group $G$ on the total space and the base and assume that the bundle projection is $G$-invariant. In Theorem 5.1 we give a presentation of the $G$-equivariant Grothendieck ring of the toric bundle as an algebra over the $G$-equivariant Grothendieck ring of the base. This theorem plays a key role in the proof of Corollary 4.1.

Note that most of the results in §2 are essentially minor refinements of the analogous results in [1], §2, where the technique for studying the equivariant $K$-theory of regular compactifications was originally developed.

Our new results are firstly Theorems 2.2, 3.1, which generalize the explicit description of the multiplicative structure of the equivariant and ordinary $K$-rings (given in [1], §3, in the case of the wonderful compactification) to all regular compactifications.

In the main new results (Theorems 4.1, 4.2) we describe the structure of the equivariant and ordinary $K$-rings of any regular compactification as a quotient of a Stanley–Reisner algebra over the corresponding rings of the wonderful compactification $\overline{G_{ad}}$. For regular compactifications of $G_{ad}$ we also give a geometric interpretation of these structures in terms of toric bundles (Corollary 4.1).

§ 1. Preliminaries

Let $X$ be a smooth complex projective $G$-variety. We write $K_G(X)$ (resp. $K_T(X)$) for the Grothendieck group of $G$-equivariant (resp. $T$-equivariant) coherent sheaves on $X$. Recall that $K_T(\text{pt}) = R(T)$ and $K_G(\text{pt}) = R(G)$, where $R(T)$ and $R(G)$ are the Grothendieck groups of complex representations of $T$ and $G$ respectively. The Grothendieck group of equivariant coherent sheaves can be identified with the Grothendieck ring of equivariant vector bundles on $X$. Furthermore, the structure morphism $X \to \text{Spec} \mathbb{C}$ induces the canonical structures of an $R(G)$-module and an $R(T)$-module on $K_G(X)$ and $K_T(X)$ respectively (see [6], Example 2.1).

Let $W$ denote the Weyl group and $\Phi$ the root system of $(G,T)$. The set $\Phi^+$ of all positive roots with respect to $B$ contains the subset $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots, where $r$ is the semisimple rank of $G$. For every $\alpha \in \Delta$ we denote the corresponding simple reflection by $s_\alpha$. Given any subset $I \subset \Delta$, we write $W_I$ for the subgroup of $W$ generated by all $s_\alpha$, $\alpha \in I$. At the extremes we have $W_{\emptyset} = \{1\}$ and $W_\Delta = W$.

Put $\Lambda := X^*(T)$. Then $R(T)$ (the representation ring of the torus $T$) is isomorphic to the group algebra $\mathbb{Z}[\Lambda]$. We write $e^\lambda$ for the element of $\mathbb{Z}[\Lambda] = R(T)$ corresponding to a weight $\lambda \in \Lambda$. Then $(e^\lambda)_{\lambda \in \Lambda}$ is a basis of the $\mathbb{Z}$-module $\mathbb{Z}[\Lambda]$. The action of $W$ on $X^*(T)$ induces a natural action of $W$ on $\mathbb{Z}[\Lambda]$ by the formula $w(e^\lambda) = e^{w(\lambda)}$ for all $w \in W$ and $\lambda \in \Lambda$. We recall that $R(G)$ can be identified with $R(T)^W$ by means of restriction to $T$, where $R(T)^W$ stands for the subring of $R(T)$ invariant under the action of $W$ (see [6], Example 1.19).

It follows from (0.1) that $\tilde{B} := \pi^{-1}(B)$ and $\tilde{T} := \pi^{-1}(T)$ are respectively a Borel subgroup and a maximal torus of $G$. Restricting the map $\pi$ to $\tilde{T}$, we obtain the
exact sequence
\[ 1 \to \mathcal{Z} \to \tilde{T} \to T \to 1. \] (1.1)

Then it again follows from (0.1) that \( W \) and \( \Phi \) can be identified with the Weyl group and the root system of \((G, \tilde{T})\) respectively. Furthermore,
\[ R(\tilde{G}) = R(\tilde{C}) \otimes R(G_{ss}), \] (1.2)
\[ R(\tilde{T}) \simeq R(\tilde{C}) \otimes R(T_{ss}), \] (1.3)
where \( T_{ss} \) is the maximal torus \( \tilde{T} \cap G_{ss} \) of \( G_{ss} \).

We recall that \( R(\tilde{G}) \) can be identified with \( R(\tilde{T})^W \) by means of restriction to \( \tilde{T} \) and, moreover, \( R(\tilde{T}) \) is a free \( R(\tilde{G}) \)-module of rank \( |W| \) (see [7], Theorem 2.2). Since \( G_{ss} \) is semisimple and simply connected, we see that \( R(G_{ss}) \simeq \mathbb{Z}[x_1, \ldots, x_r] \) is a polynomial ring on the fundamental representations (see [6], Example 1.20). Hence \( R(\tilde{G}) = R(\tilde{C}) \otimes R(G_{ss}) \) is the tensor product of a polynomial ring and a Laurent polynomial ring, and hence is a regular ring of dimension \( r + \text{dim}(\tilde{C}) = \text{rank}(G) \), where \( r \) is the rank of \( G_{ss} \).

We shall consider the \( \tilde{T} \)- and \( \tilde{G} \)-equivariant \( K \)-theory of \( X \) for the natural actions of \( \tilde{T} \) and \( \tilde{G} \) on \( X \) by means of the canonical surjections to \( T \) and \( G \).

We regard \( \mathcal{Z} \) as an \( R(\tilde{G}) \)-module by means of the augmentation map \( \varepsilon : R(\tilde{G}) \to \mathcal{Z} \) sending every \( \tilde{G} \)-representation \( V \to \text{dim}(V) \). Moreover, there are natural restriction homomorphisms
\[ K_{\tilde{G}}(X) \to K_{\tilde{T}}(X), \quad K_{\tilde{G}}(X) \to K(X), \]
where \( K(X) \) is the ordinary Grothendieck ring of algebraic vector bundles over \( X \). Then we have the following isomorphisms (see [6], Proposition 4.1 and Theorem 4.2):
\[ R(\tilde{T}) \otimes_{R(\tilde{G})} K_{\tilde{G}}(X) \simeq K_{\tilde{T}}(X), \] (1.4)
\[ K_{\tilde{G}}(X) \simeq K_{\tilde{T}}(X)^W, \] (1.5)
\[ \mathcal{Z} \otimes_{R(\tilde{G})} K_{\tilde{G}}(X) \simeq K(X). \] (1.6)

Let \( R(\tilde{T})^{W_I} \) denote the invariant subring of the ring \( R(\tilde{T}) \) under the action of the subgroup \( W_I \) of \( W \) for every \( I \subseteq \Delta \). In particular, \( R(\tilde{T})^W = R(\tilde{G}) \) and \( R(\tilde{T})^{\{1\}} = R(\tilde{T}) \). Furthermore, for every \( I \subseteq \Delta \) the set \( R(\tilde{T})^{W_I} \) is a free module of rank \( |W/W_I| \) over \( R(\tilde{G}) = R(\tilde{T})^W \) (see [7], Theorem 2.2). We note that Theorem 2.2 in [7] actually holds only for \( R(T_{ss}) \). However, since the action of \( W \) on the central torus \( \tilde{C} \) (and hence on \( R(\tilde{C}) \)) is trivial, we have
\[ R(\tilde{T})^{W_I} = R(\tilde{C}) \otimes R(T_{ss})^{W_I} \] (1.7)
for every \( I \subseteq \Delta \), whence we obtain the analogous statement for \( R(\tilde{T}) \).

Let \( W^I \) denote the set of minimal-length coset representatives of the parabolic subgroup \( W_I \) for every \( I \subseteq \Delta \). Then
\[ W^I := \{ w \in W \mid l(wv) = l(w) + l(v) \ \forall \ v \in W_I \} = \{ w \in W \mid w(\Phi^+_I) \subseteq \Phi^+ \}, \]
where $\Phi_I$ is the root system associated with $W_I$ and $I$ is the set of simple roots. We recall (see [8], p. 19) that

$$W^I = \{ w \in W \mid l(ws) > l(w) \forall s \in I \}.$$ 

Note that $J \subseteq I$ implies that $W^{\Delta \setminus J} \subseteq W^{\Delta \setminus I}$. Put

$$C^I := W^{\Delta \setminus I} \setminus \left( \bigcup_{J \subseteq I} W^{\Delta \setminus J} \right). \quad (1.8)$$

Let $\alpha_1, \ldots, \alpha_r$ be an ordering of the set $\Delta$ of simple roots, and let $\omega_1, \ldots, \omega_r$ be the corresponding fundamental weights for the root system of $(G^{ss}, T^{ss})$. Since $G^{ss}$ is simply connected, the fundamental weights form a basis of $X^*(T^{ss})$. Hence, for every $\lambda \in X^*(T^{ss})$, $e^\lambda \in R(T^{ss})$ is a Laurent monomial in the elements $e^{\omega_i}$, $1 \leqslant i \leqslant r$.

Steinberg ([7], Theorem 2.2) defined a basis $\{f^I_v : v \in W^I\}$ of $R(T^{ss})^W$ as an $R(T^{ss})^W$-module. We recall this definition here. For $v \in W^I$ put

$$p_v := \prod_{\alpha_i < 0} e^{\omega_i} \in R(\tilde{T}). \quad (1.9)$$

Then

$$f^I_v := \sum_{x \in W_I(v) \setminus W_I} x^{-1}v^{-1}p_v, \quad (1.10)$$

where $W_I(v)$ is the stabilizer of $v^{-1}p_v$ in $W_I$.

We also write $\{f^I_v : v \in W^I\}$ for the corresponding basis of $R(\tilde{T})^W$ as an $R(\tilde{T})^W$-module, where it is understood that

$$f^I_v := 1 \otimes f^I_v \in R(\tilde{C}) \otimes R(T^{ss})^W. \quad (1.11)$$

**Notation 1.1.** Whenever $v \in C^I$, we denote $f_v^{\Delta \setminus I}$ simply by $f_v$. We can drop the superscript in the notation without any ambiguity since the sets $\{C^I : I \subseteq \Delta\}$ are pairwise disjoint. In this modified notation, Lemma 1.10 in [1] implies that the set $\{f_v : v \in W^{\Delta \setminus I} = \bigsqcup_{J \subseteq I} C^J\}$ is an $R(\tilde{T})^W$-basis of $R(\tilde{T})^{W^{\Delta \setminus I}}$ for every $I \subseteq \Delta$. We also put

$$R(\tilde{T})_I := \bigoplus_{v \in C^I} R(\tilde{T})^W \cdot f_v. \quad (1.12)$$

In $R(T)$ we write

$$f_v \cdot f_{v'} = \sum_{J \subseteq (I \cup I')} \sum_{w \in C^J} a^w_{v,v'} \cdot f_w \quad (1.13)$$

for some elements $a^w_{v,v'} \in R(G) = R(T)^W$, where $v \in C^I$, $v' \in C^I'$ and $w \in C^J$, $J \subseteq (I \cup I')$. 

§ 2. Equivariant K-theory of regular compactifications

In this section let $X$ be a projective regular compactification of $G$. We use the notation in § 1 along with the following.

Denote the set of maximal cones in the fan $\mathcal{F}_+$ by $\mathcal{F}_+(l)$. We know that $\mathcal{F}_+(l)$ parametrizes the closed $G \times G$-orbits on $X$. By Proposition A1 in [4], $X^{T \times T}$ is contained in the union $X_\tau$ of all closed $G \times G$-orbits on $X$. Moreover, all such orbits are isomorphic to $G/B^- \times G/B$. Hence $X^{T \times T}$ is parametrized by $\mathcal{F}_+(l) \times W \times W$. For $\sigma \in \mathcal{F}_+(l)$ we denote by $Z_\sigma \cong G/B^- \times G/B$ the corresponding closed orbit with base point $z_\sigma \in X^{T \times T}$. Moreover, $z_\sigma$ is also the $T \times \{1\}$-fixed point in $T^+$ corresponding to $\sigma \in \mathcal{F}_+(l)$.

Note that $\mathcal{F}_+$ is combinatorially complete (see [9], Definition 6.3) since $T$ acts on $T^+$ with enough limits (see [1], Remark 2.4). Thus it follows from Theorem 6.7 in [9] that $K_{\tilde{T}}(T^+)$ is a projective $R(\tilde{T})$-module of rank $|\mathcal{F}_+(l)|$. This further implies by Theorem 1.1 in [10] that $K_{\tilde{T}}(T^+)$ is a free module of rank $|\mathcal{F}_+(l)|$ over $R(\tilde{T})$. Moreover, since $R(\tilde{T})$ is a free module of rank $|W|$ over $R(\tilde{G})$, it follows that $K_{\tilde{T}}(X)$ is a free module of rank $|W| \cdot |\mathcal{F}_+(l)|$ over $R(\tilde{G})$.

Let $T_\tau \subseteq T$ denote the stabilizer along the orbit $O_\tau$ in $T^+$ for every $\tau \in \mathcal{F}_+$, and let $\tilde{T}_\tau \subseteq \tilde{T}$ be the inverse image of $T_\tau$ under the map (1.1). In particular, $\tilde{T}_\sigma = \tilde{T}$ is the stabilizer at $z_\sigma$ for $\sigma \in \mathcal{F}_+(l)$. Although $R(\tilde{T}_\sigma) = K_{\tilde{T}}(z_\sigma) = R(\tilde{T})$, we shall retain the subscript $\sigma$ for clarity, following [9], § 6.

**Proposition 2.1.** There is a chain of injective morphisms of $R(\tilde{G}) \otimes R(\tilde{G})$-algebras:

$$K_{\tilde{T}}(T^+) \otimes R(\tilde{G}) \subseteq K_{\tilde{G} \times \tilde{G}}(X) \subseteq K_{\tilde{T}}(T^+) \otimes R(\tilde{T}) \subseteq \prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma) \otimes R(\tilde{T}).$$

Moreover, the $R(\tilde{T})$-algebra structure on $K_{\tilde{T}}(T^+)$ induces an $R(\tilde{T}) \otimes R(\tilde{G})$-algebra structure on $K_{\tilde{G} \times \tilde{G}}(X)$.

**Proof.** Note that we have a split exact sequence

$$1 \to \text{diag}(\tilde{T}) \to \tilde{T} \times \tilde{T} \to \tilde{T} \to 1,$$

where the second map sends $(t_1, t_2)$ to $t_1 t_2^{-1}$, and the splitting is given by $t \mapsto (t, 1)$.

We recall from Theorem 2.1 in [1] that under restrictions to the $\tilde{T} \times \tilde{T}$-fixed points $(w, w') \cdot z_\sigma$ for $w, w' \in W$ the set $K_{\tilde{T} \times \tilde{T}}(X)$ consists of all families $(f_\sigma (\sigma \in \mathcal{F}_+))$ in $R(\tilde{T}) \times R(\tilde{T})$ such that

(i) $f_{\sigma,w \cdot \alpha, w' \cdot \alpha} \equiv f_{\sigma,w,w'} \text{ (mod}(1 - e^{-w(\alpha)} \otimes e^{-w'(\alpha)}))$ whenever $\alpha \in \Delta$ and the cone $\sigma \in \mathcal{F}_+(l)$ has a facet orthogonal to $\alpha$,

(ii) $f_{\sigma,w,w'} \equiv 0 \text{ (mod}(1 - e^{-\chi}))$ whenever $\chi \in X^*(\tilde{T})$ and the cones $\sigma$ and $\sigma'$ in $\mathcal{F}_+(l)$ have a common facet orthogonal to $\chi$.

(In (ii) $\chi$ is viewed as a character of $T \times T$ which is trivial on $\text{diag}(T)$ and hence is a character of $T$.)

Furthermore, taking $W \times W$-invariants and using Corollary 2.2 in [1] and (2.2), we see that $K_{\tilde{G} \times \tilde{G}}(X)$ consists of all families $(f_\sigma) (\sigma \in \mathcal{F}_+(l))$ of elements of $R(\tilde{T} \times \{1\}) \otimes R(\text{diag}(\tilde{T}))$ such that
(i) \((1, s_{\alpha}) f_\sigma(u, v) \equiv f_\sigma(u, v) \pmod{(1 - e^{-\alpha(u)})}\) whenever \(\alpha \in \Delta\) and the cone \(\sigma \in \mathcal{F}_+(l)\) has a facet orthogonal to \(\alpha\),

(ii) \(f_\sigma \equiv f_{\sigma'} \pmod{(1 - e^{-\chi(u)})}\) whenever \(\chi \in X^*(\tilde{T})\) and the cones \(\sigma\) and \(\sigma'\) in \(\mathcal{F}_+(l)\) have a common facet orthogonal to \(\chi\),

where the variables \(u\) and \(v\) correspond to \(R(\tilde{T} \times \{1\})\) and \(R(\text{diag}(\tilde{T}))\) respectively. Recall from Remark 2.4 in [1] that there is a canonical restriction homomorphism

\[
K_{\tilde{T}}(T^+) \to K_{\tilde{T}}((T^+)^T) = \prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma).
\]

(2.3)

The homomorphism (2.3) is injective and the image of \(K_{\tilde{T}}(T^+)\) can be identified with the subset of \(\prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma)\) consisting of the elements \((a_\sigma)\) such that \(a_\sigma \equiv a_{\sigma'} \pmod{(1 - e^{-\chi})}\) whenever \(\chi \in X^*(T)\) and the cones \(\sigma, \sigma' \in \mathcal{F}_+(l)\) have a common facet orthogonal to \(\chi\). Furthermore, (2.3) is compatible with the canonical \(R(\tilde{T})\)-algebra structure on \(K_{\tilde{T}}(T^+)\) and the \(R(\tilde{T})\)-algebra structure on \(\prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma)\) given by the diagonal map (see also the proof of Proposition 2.5 in [1]).

Thus \(K_{\tilde{T}}(T^+) \otimes R(\tilde{T})\) can be identified with the \(R(\tilde{T}) \otimes R(\tilde{G})\)-subalgebra of \(\prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma) \otimes R(\tilde{T})\) generated by the elements \((a_\sigma) \otimes b\) such that \(a_\sigma - a_{\sigma'} \equiv 0 \pmod{(1 - e^{-\chi})}\) whenever \(\sigma\) and \(\sigma'\) have a common facet orthogonal to \(\chi \in X^*(T)\).

Since \(R(\tilde{G}) = R(\tilde{T})^W\), it follows from (i) and (ii) that \(K_{\tilde{T}}(T^+) \otimes R(\tilde{G})\) is a subring of \(K_{\tilde{G} \times \tilde{G}}(X)\). In particular, \(K_{\tilde{G} \times \tilde{G}}(X)\) is an algebra over \(K_{\tilde{T}}(T^+) \otimes R(\tilde{G})\) and hence over \(R(\tilde{T}) \otimes R(\tilde{G})\).

Furthermore, by (ii), every element

\[
f(u, v) \in \prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma) \otimes R(\tilde{T})
\]

belonging to \(K_{\tilde{G} \times \tilde{G}}(X)\) must actually lie in \(K_{\tilde{T}}(T^+) \otimes R(\tilde{T})\). This yields the inclusion \(K_{\tilde{G} \times \tilde{G}}(X) \subseteq K_{\tilde{T}}(T^+) \otimes R(\tilde{T})\) of \(R(\tilde{T}) \otimes R(\tilde{G})\)-algebras. \(\Box\)

Let \(PL(\mathcal{F}_+)\) be the set of those piecewise-constant functions on \(C^+\) that are linear on the subdivision \(\mathcal{F}_+\). More precisely,

\[
PL(\mathcal{F}_+) := \{ h : C^+ \to \mathbb{R} \mid \forall \sigma \in \mathcal{F}_+(l) \text{ and } v \in \sigma, \ h(v) = \langle h_\sigma, v \rangle \text{ for some } h_\sigma \in X^*(T) \}.
\]

By the parametrization of line bundles on spherical varieties (see [11], §2.2) we know that the group of isomorphism classes of \(\tilde{G} \times \tilde{G}\)-linearized line bundles on \(X\) is isomorphic to \(PL(\mathcal{F}_+)\). Let \(L_h\) be the line bundle on \(X\) corresponding to an element \(h \in PL(\mathcal{F}_+)\). Then \(L_h\) is \(\tilde{G} \times \tilde{G}\)-linearized and \(\tilde{B}^- \times \tilde{B}\) acts on the fibre \(L_h|_{z_\sigma}\) by the character \((h_\sigma, -h_\sigma)\). It follows that \(L_h := L_h|_{T^+}\) is a \(\tilde{T} \times \tilde{T}\)-linearized line bundle on the toric variety \(\tilde{T}^+\) corresponding to the piecewise-linear function \(h \in PL(\mathcal{F}_+)\). In particular, \(\tilde{T}^- \times \tilde{T}\) acts on the fibre \(L_h|_{z_\sigma}\) by the character \(h_\sigma\).
Theorem 2.1. The ring \(K_{\tilde{G} \times \tilde{G}}(X)\) has the following direct sum decomposition as a \(K_{\tilde{T}}(T^+) \otimes R(\tilde{G})\)-module:

\[
K_{\tilde{G} \times \tilde{G}}(X) = \bigoplus_{I \subseteq \Delta} \prod_{\alpha \in I} (1 - e^{\alpha(u)}) \cdot K_{\tilde{T}}(T^+) \otimes R(\tilde{T})_I. \tag{2.4}
\]

This direct sum is a free \(K_{\tilde{T}}(T^+) \otimes R(\tilde{G})\)-module of rank \(|W|\) with basis

\[
\left\{ \prod_{\alpha \in I} (1 - e^{\alpha(u)}) \otimes f_v : v \in C^I \text{ and } I \subseteq \Delta \right\},
\]

where \(C^I\) is defined in (1.8) and \(\{f_v\}\) is defined above. Moreover, the component \(K_{\tilde{T}}(T^+) \otimes 1 \subseteq K_{\tilde{T}}(T^+) \otimes R(\tilde{T})^W\) of the direct sum can be identified with the subring of \(K_{\tilde{G} \times \tilde{G}}(X)\) generated by Pic\(\tilde{G} \times \tilde{G}(X)\).

Proof. By Notation 1.1 we have the following decompositions of \(R(\tilde{T})^W\)-modules:

\[
R(\tilde{T}) = \bigoplus_I R(\tilde{T})_I, \tag{2.5}
\]

\[
R(\tilde{T})^{W \setminus I} = \bigoplus_{J \subseteq I} R(\tilde{T})_J. \tag{2.6}
\]

Put

\[
L := \bigoplus_{I \subseteq \Delta} \prod_{\alpha \in I} (1 - e^{\alpha(u)}) K_{\tilde{T}}(T^+) \otimes R(\tilde{T})_I. \tag{2.7}
\]

When \(I \subseteq \Delta\) the piece \(\prod_{\alpha \in I} (1 - e^{\alpha(u)}) K_{\tilde{T}}(T^+) \otimes R(\tilde{T})_I\) of this direct sum decomposition is a free \(K_{\tilde{T}}(T^+) \otimes R(\tilde{G})\)-module with basis \(\{1 \otimes f_v : v \in C^I\}\), where \(f_v^I\) is as in (1.10). In particular, \(L\) is a free \(K_{\tilde{T}}(T^+) \otimes R(\tilde{G})\)-module of rank \(|W|\) and, therefore, a free module of rank \(|W| \cdot |\mathcal{F}_+(l)|\) over \(R(\tilde{T}) \otimes R(\tilde{G})\).

We observe that each element in \(\prod_{\alpha \in I} (1 - e^{\alpha(u)}) K_{\tilde{T}}(T^+) \otimes R(\tilde{T})_I\) clearly satisfies the conditions (i) and (ii), which determine \(K_{\tilde{G} \times \tilde{G}}(X)\) as a subalgebra of \(\prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma) \otimes R(\tilde{T})\). We therefore have an inclusion \(L \subseteq K\) of \(K_{\tilde{T}}(T^+) \otimes R(\tilde{G})\)-modules, where \(K := K_{\tilde{G} \times \tilde{G}}(X)\). Since \(K_{\tilde{T}}(T^+)\) is an \(R(\tilde{T})\)-algebra, we see that \(L \subseteq K\) is also an inclusion of \(R(\tilde{T}) \otimes R(\tilde{G})\)-modules.

It now follows from Lemma 1.6 in [1] that \(K\) is a free module of rank \(|W|^2 \cdot |\mathcal{F}_+(l)|\) over \(R(\tilde{G}) \otimes R(\tilde{G})\). Moreover, since \(K\) is an \(R(\tilde{T}) \otimes R(\tilde{G})\)-algebra by Proposition 2.1, it follows that \(K\) is a projective module over \(R(\tilde{T}) \otimes R(\tilde{G})\). (In view of Theorem 1.1 in [10] and, since \(R(\tilde{T}) \otimes R(\tilde{G})\) is a free module of rank \(|W|\) over \(R(\tilde{G}) \otimes R(\tilde{G})\), this further implies that \(K\) is a free module of rank \(|W| \cdot |\mathcal{F}_+(l)|\) over \(R(\tilde{T}) \otimes R(\tilde{G})\).)

Thus, \(L \hookrightarrow K \twoheadrightarrow K/L \twoheadrightarrow 0\) is a short exact sequence of \(K_{\tilde{T}}(T^+) \otimes R(\tilde{G})\)-modules. Since \(K\) and \(L\) are projective as \(R(\tilde{T}) \otimes R(\tilde{G})\)-modules, it follows that \(K/L\) is of projective dimension 1 as a module over \(R(\tilde{T}) \otimes R(\tilde{G})\).

We must now prove that \(L \cong K\). This is done in the following lemma using the inclusion \(L \subseteq K\) of \(R(\tilde{T}) \otimes R(\tilde{G})\)-modules.
Lemma 2.1. Put

\[ t_\alpha := \prod_{\beta \neq \alpha} (1 - e^{-\alpha(u)}) \in R(\tilde{T}) \otimes R(\tilde{G}) \]

for all \( \alpha \in \Delta \). Then the localization \((K/L)_{t_\alpha}\) is equal to 0 for every \( \alpha \in \Delta \).

Proof. Put

\[ M_\alpha := K_T(\tilde{T}^+) \otimes R(T)^{s_\alpha} + (1 - e^{-\alpha(u)})K_{\tilde{T}}(\tilde{T}^+ \otimes e^{\omega_\alpha(u)}R(\tilde{T})^{s_\alpha}). \quad (2.8) \]

Further, \( R(\tilde{T}) = R(\tilde{T})^{s_\alpha} \oplus e^{\omega_\alpha}R(\tilde{T})^{s_\alpha} \) for every \( \alpha \in \Delta \), where \( \omega_\alpha \) is the fundamental weight corresponding to \( \alpha \in \Delta \). Hence we have the following equality of \( K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})^W \)-modules:

\[ K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T}) = K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})^{s_\alpha} \oplus K_{\tilde{T}}(\tilde{T}^+) \otimes e^{\omega_\alpha(u)}R(\tilde{T})^{s_\alpha}. \quad (2.9) \]

Note that after localizing at \( t_\alpha = \prod_{\beta \neq \alpha} (1 - e^{-\beta(u)}) \), the only conditions defining \( K_{\tilde{G} \times \tilde{G}}(X) \) in \( \prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma) \otimes R(\tilde{T}) \) are the one corresponding to \( \alpha \) and the condition (ii). This together with (2.9) implies that \( K_{t_\alpha} \subseteq (M_\alpha)_{t_\alpha} \).

Using the equalities

\[ R(\tilde{T})^{s_\alpha} = \bigoplus_{\alpha \notin I} R(\tilde{T})_I, \quad (2.10) \]

\[ e^{\omega_\alpha} \cdot R(\tilde{T})^{s_\alpha} = \bigoplus_{\alpha \in I} R(\tilde{T})_I \quad (2.11) \]

and the definition of \( L \), we now deduce that \( L_{t_\alpha} = (M_\alpha)_{t_\alpha} \). Since \( L_{t_\alpha} \subseteq K_{t_\alpha} \subseteq (M_\alpha)_{t_\alpha} \), it follows that \((K/L)_{t_\alpha} = 0\) for every \( \alpha \in \Delta \). \( \square \)

Since the projective dimension of \((K/L)\) is equal to 1, the Auslander–Buchsbaum formula yields that \( \text{Supp}(K/L) \) is of pure codimension 1 in \( \text{Spec}(R(\tilde{T}) \otimes R(\tilde{G})) \). Hence \( \text{Supp}(K/L) \) must contain a prime ideal \( \mathfrak{p} \) of height 1 in \( R(\tilde{T}) \otimes R(\tilde{G}) \). Since \( R(\tilde{T}) \otimes R(\tilde{G}) \) is a unique factorization domain, \( \mathfrak{p} = (a) \) for some \( a \in R(\tilde{T}) \otimes R(\tilde{G}) \) and, by Lemma 2.1, we see that \( \mathfrak{p} \) contains \( 1 - e^{-\alpha(u)} \) and \( 1 - e^{-\beta(u)} \) for \( \alpha \neq \beta \in \Delta \).

It follows that \( a \) divides \( 1 - e^{-\alpha(u)} \) and \( 1 - e^{-\beta(u)} \) for distinct \( \alpha \) and \( \beta \). This is a contradiction since \( 1 - e^{-\alpha(u)} \) and \( 1 - e^{-\beta(u)} \) are relatively prime in the unique factorization domain \( R(\tilde{T}) \otimes R(\tilde{G}) \) (see [12], Lemma 1).

The resulting contradiction shows that \( K/L = 0 \) and, therefore, \( K = L \). This proves (2.4). We now prove the other assertions in Theorem 2.1.

It suffices to establish that the piece \( K_{\tilde{T}}(\tilde{T}^+) \otimes 1 \) in (2.4) is the subring of \( K_{\tilde{G} \times \tilde{G}}(X) \) generated by \( \{[\mathcal{L}_h]: h \in PL(\mathcal{F}_+)\} \), that is, the subring generated by \( \text{Pic}_{\tilde{G} \times \tilde{G}}(X) \).

We recall that the canonical embedding

\[ K_{\tilde{G} \times \tilde{G}}(X) \hookrightarrow \prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma) \otimes R(\tilde{T}), \]
which is obtained by restriction to \( z_{\sigma}, \sigma \in \mathcal{F}_+(l) \), sends \([\mathcal{L}_h]\) to \((e^{h_{\sigma}} \otimes 1)_{\sigma \in \mathcal{F}_+(1)}\).

Here we use the identification

\[ R(\tilde{T} \times \{1\}) \otimes R(\text{diag}(\tilde{T})) \simeq R(\tilde{T}) \otimes R(\tilde{T}), \]

which comes from (2.2).

Also note that under the restriction to \( z_{\sigma}, \sigma \in \mathcal{F}_+(l) \), the image of \([L_h] \in \text{Pic} \tilde{T}(\tilde{T}^+)\) in \( \prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma) \) is equal to \((e^{h_{\sigma}})_{\sigma \in \mathcal{F}_+(l)}\). Moreover, it is known that \( \text{Pic} \tilde{T}(\tilde{T}^+) \) generates \( K_{\tilde{T}}(\tilde{T}^+) \) as a ring (see [9], §6.2, where the piecewise-linear functions \( u_{\rho}(\sigma), \rho \in \Delta(1), \) on the fan \( \Delta \) which generate \( K_{\tilde{T}}(X(\Delta)) \) as a ring, correspond to the classes of the \( T \)-equivariant line bundles \( L_{u_{\rho}} \). Thus it follows that the image of \( K_{\tilde{T}}(\tilde{T}^+) \otimes 1 \) under the restriction to \( \prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma) \otimes R(\tilde{T}) \) coincides with the image of the subring generated by the elements \([\mathcal{L}_h], h \in PL(\mathcal{F}_+)\). □

**Theorem 2.2.** We have the following isomorphism of \( K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T}) \)-submodules of \( \prod_{\sigma \in \mathcal{F}_+(l)} R(\tilde{T}_\sigma) \otimes R(\tilde{T}) \):

\[
K_{G \times G}(X) = \bigoplus_{I \subseteq \Delta} \prod_{\alpha \in I} (1 - e^{\alpha(u)}) \cdot K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I \simeq \bigoplus_{I \subseteq \Delta} K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I.
\]

More explicitly, this isomorphism maps an arbitrary element

\[
a \otimes b \in K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I
\]

to the element

\[
\prod_{\alpha \in I} (1 - e^{\alpha(u)}) \cdot a \otimes b.
\]

In particular, the basis element

\[
1 \otimes f_v \in K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I
\]

is mapped to

\[
\prod_{\alpha \in I} (1 - e^{\alpha(u)}) \otimes f_v \in K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I,
\]

where \( v \in C^I \) for any fixed \( I \subseteq \Delta \).

Moreover, in \( \bigoplus_{I \subseteq \Delta} K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I \) any two basis elements \( 1 \otimes f_v, 1 \otimes f_{v'} \) with \( v \in C^I, v' \in C^{I'} \) \((I, I' \subseteq \Delta)\) are multiplied as follows:

\[
(1 \otimes f_v) \cdot (1 \otimes f_{v'})
\]

\[
= \sum_{J \subseteq (I \cup I')} \sum_{w \in C^J} \left( \prod_{\alpha \in I \cap I'} (1 - e^{\alpha(u)}) \cdot \prod_{\alpha \in (I \cup I') \setminus J} (1 - e^{\alpha(u)}) \right) \otimes a_{v, v'}^w \cdot (1 \otimes f_w).
\]

(2.12)

**Proof.** Note that \( \prod_{\alpha \in I} (1 - e^{\alpha(u)}) \) is not a zero divisor in the integral domain \( R(\tilde{T}) \) and, therefore, in \( K_{\tilde{T}}(\tilde{T}^+) \), which is a free module over \( R(\tilde{T}) \). Thus each piece
$K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I$, $I \subseteq \Delta$, is isomorphic to $\prod_{\alpha \in I} (1 - e^{\alpha(u)}) \cdot K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I$ as a $K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{G})$-submodule of $\prod_{\sigma \in \mathcal{F}(I)} R(\tilde{\sigma})_\sigma \otimes R(\tilde{T})$. The isomorphism sends an element

$$a \otimes b \in K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I$$

to the element

$$\prod_{\alpha \in I} (1 - e^{\alpha(u)}) \cdot a \otimes b \in K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I.$$  

This extends additively to the following isomorphism of $K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{G})$-submodules of $K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})$:

$$\bigoplus_{I \subseteq \Delta} \prod_{\alpha \in I} (1 - e^{\alpha(u)}) \cdot K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I \simeq \bigoplus_{I \subseteq \Delta} K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I. \tag{2.13}$$

Furthermore, in terms of the direct sum decomposition

$$K_{\tilde{G} \times \tilde{G}}(X) = \bigoplus_{I \subseteq \Delta} \prod_{\alpha \in I} (1 - e^{\alpha(u)}) \cdot K_{\tilde{T}}(\tilde{T}^+) \otimes R(T)_I$$

the product of the basis elements $\prod_{\alpha \in I} (1 - e^{\alpha(u)}) \otimes f_v$ and $\prod_{\alpha \in I'} (1 - e^{\alpha(u)}) \otimes f_{v'}$, where $v \in C^I$ and $v' \in C^{I'}$, is equal to

$$\prod_{\alpha \in I \cap I'} (1 - e^{\alpha(u)}) \cdot \prod_{\alpha \in I \cup I'} (1 - e^{\alpha(u)}) \otimes f_v \cdot f_{v'}. \tag{2.14}$$

Note that the element (2.14) belongs to the space

$$\prod_{\alpha \in I \cup I'} (1 - e^{\alpha(u)}) \cdot K_{\tilde{T}}(\tilde{T}^+) \otimes R(T)^{W_{\Delta \setminus (I \cup I')}} \subseteq \bigoplus_{J \subseteq (I \cup I')} \prod_{\alpha \in J} (1 - e^{\alpha(u)}) \cdot K_{\tilde{T}}(\tilde{T}^+) \otimes R(T)_J.$$

Therefore, using (1.13), we can rewrite (2.14) as

$$\sum_{J \subseteq (I \cup I')} \sum_{w \in C^J} \left( \prod_{\alpha \in I \cap I'} (1 - e^{\alpha(u)}) \prod_{\alpha \in I \cup I'} (1 - e^{\alpha(u)}) \otimes a_{v,v'}^w \right) \cdot \prod_{\alpha \in J} (1 - e^{\alpha(u)}) \otimes f_w. \tag{2.15}$$

Using the isomorphism (2.13), we now conclude that in $\bigoplus_{I \subseteq \Delta} K_{\tilde{T}}(\tilde{T}^+) \otimes R(T)_I$, the product of the basis elements $(1 \otimes f_v)$ and $(1 \otimes f_{v'})$, $v \in C^I$, $v' \in C^{I'}$, is of the form (2.12). □

§ 3. The ordinary $K$-ring of a regular compactification

In this section we use the results in § 2 to describe the ordinary $K$-ring of $X$.

For $u \in X^*(\tilde{T})$ we put $L(u) := (\tilde{G} \times \mathbb{C}_u)/\tilde{B}$, where $\tilde{B}$ acts diagonally, and the $\tilde{B}$-action on the one-dimensional vector space $\mathbb{C}_u$ is given by the surjection $\tilde{B} \to \tilde{T}$ followed by $u$. Then $L(u)$ is a $\tilde{G}$-linearized line bundle on $\tilde{G}/\tilde{B} = G/B$ associated with $u$, where the $\tilde{G}$-linearization comes from the left $\tilde{G}$-action.
Let $c_K: R(\tilde{T}) = \tilde{K}_G(G/B) \to K(G/B)$ be the characteristic homomorphism which sends $e^u$ to $[\mathcal{L}(u)]$ for all $u \in X^*(\tilde{T})$.

Following (1.3), we write $\overline{X}_I \in K(G/B)$ for the image of $1 \otimes \prod_{\alpha \in I} (1 - e^{-\alpha}) \in R(\tilde{T})$ for every $I \subseteq \Delta$. Put $\overline{f}_v := c_K(f_v)$, and let $c_{v,v'}^w \in \mathbb{Z}$ be the image of $a_{v,v'}^w \in R(\tilde{G})$ under the map $c_K|_{R(\tilde{G})}$, where $a_{v,v'}^w$ is as in (1.13).

Furthermore, if $K(G/B)_I := \bigoplus_{v \in C_I} \mathbb{Z}[\overline{f}_v]$, then we have $K(G/B) = \bigoplus_{I \subseteq \Delta} K(G/B)_I$. (3.1)

Put $R(\tilde{T}^+) := \mathbb{Z} \otimes_{R(\tilde{G})} K_{\tilde{T}}(\tilde{T}^+)$, (3.3)

where $\mathbb{Z}$ is an $R(\tilde{G})$-module under the augmentation $\varepsilon = c_K|_{R(\tilde{G})}: R(\tilde{G}) \to \mathbb{Z}$.

**Theorem 3.1.** We have a canonical $R(\tilde{T}^+)$-module structure on $K(X)$ induced from the $K_{\tilde{T}}(\tilde{T}^+) \otimes 1$-module structure on $K_{\tilde{G} \times \tilde{G}}(X)$ given in Theorem 2.2. Moreover, $K(X)$ is a free module of rank $|W|$ over $R(\tilde{T}^+)$, where $R(\tilde{T}^+)$ is identified with the subring of $K(X)$ generated by $\text{Pic}(X)$. More explicitly, let

$$\gamma_v := 1 \otimes [\overline{f}_v] \in R(\tilde{T}^+) \otimes K(G/B)_I$$

for $v \in C_I$ for every $I \subseteq \Delta$. Then we have

$$K(X) \simeq \bigoplus_{v \in W} R(\tilde{T}^+) \cdot \gamma_v.$$ (3.5)

This isomorphism is a ring isomorphism, where the multiplication of any basis elements $\gamma_v$ and $\gamma_{v'}$ is defined by the formula

$$\gamma_v \cdot \gamma_{v'} := \sum_{J \subseteq (I \cup I')} \sum_{w \in C_J} (\overline{X}_{I \cap I'} \cdot \overline{X}_{(I \cup I') \setminus J}) \cdot c_{v,v'}^w \cdot \gamma_w.$$ (3.6)

**Proof.** By Theorem 2.1 we have the following direct sum decomposition of $K_{\tilde{G} \times \tilde{G}}(X)$ as an $R(\tilde{T}) \otimes R(\tilde{G})$-module:

$$K_{\tilde{G} \times \tilde{G}}(X) \simeq \bigoplus_{I \subseteq \Delta} K_{\tilde{T}}(\tilde{T}^+) \otimes R(\tilde{T})_I.$$ (3.7)

Now, using the isomorphism

$$K(X) \simeq K_{\tilde{G} \times \tilde{G}}(X) \otimes_{R(\tilde{G}) \otimes R(\tilde{G})} \mathbb{Z}$$

(3.8)
(see [6]) and (2.13), we obtain that

$$K(X) \simeq \bigoplus_{I \subseteq \Delta} \mathcal{R}(T^+) \otimes K(G/B)_I.$$  \hspace{1cm} (3.9)

The canonical restriction homomorphism

$$K_{\tilde{G} \times \tilde{G}}(X) \to K(X)$$

maps the subring generated by Pic$\tilde{G} \times \tilde{G}(X)$ in $K_{\tilde{G} \times \tilde{G}}(X)$ surjectively onto the subring generated by Pic(X) in K(X). By Theorem 2.1 it follows that in (3.9) the subring generated by Pic(X) in K(X) is mapped isomorphically onto the piece $\mathcal{R}(T^+) \otimes 1 \subseteq \mathcal{R}(T^+) \otimes K_{\tilde{T}}(G/B)$.

We write $\gamma_v \in \mathcal{R}(\tilde{T}^+) \otimes K_{\tilde{T}}(G/B)_I$ for the element $1 \otimes \tilde{f}_v$, where $v \in C^I$ for some $I \subseteq \Delta$. Then it follows from Theorem 2.2 and (3.9) that $K(X)$ is a free module of rank $|W|$ over the ring $\mathcal{R}(\tilde{T}^+) \otimes K_{\tilde{T}}(G/B)$.

We recall from Theorem 2.2 that the multiplication of the basis elements $1 \otimes f_v$ and $1 \otimes f_{v'}$ of $K_{\tilde{G} \times \tilde{G}}(X) \simeq \bigoplus_{I \subseteq \Delta} \mathcal{R}(\tilde{T}) \otimes \mathcal{R}(\tilde{T})_I$ is given by the formula

$$\begin{align*}
(1 \otimes f_v) \cdot (1 \otimes f_{v'}) &= \sum_{J \subseteq (I \cup I')} \sum_{w \in C^J} \prod_{\alpha \in I \cap I'} \left(1 - e^{\alpha(u)}\right) \cdot \prod_{\alpha \in (I \cup I') \setminus J} \left(1 - e^{\alpha(u)}\right) \otimes a^{w}_{v,v'} \cdot (1 \otimes f_w).
\end{align*}$$

Thus their images $\gamma_v$ and $\gamma_{v'}$ in $\bigoplus_{I \subseteq \Delta} \mathcal{R}(\tilde{T}^+) \otimes K(G/B)_I$ are multiplied as follows:

$$\gamma_v \cdot \gamma_{v'} := \sum_{J \subseteq (I \cup I')} \sum_{w \in C^J} (\lambda_{I \cap I'} \cdot \lambda_{(I \cup I') \setminus J}) c^{w}_{v,v'} \cdot \gamma_w.$$  \hspace{1cm} (3.10)

We therefore conclude that the isomorphism

$$K(X) \simeq \bigoplus_{v \in W} \mathcal{R}(\tilde{T}^+) \cdot \gamma_v$$  \hspace{1cm} (3.11)

is in fact a ring isomorphism, where the multiplication of any basis elements $\gamma_v$ and $\gamma_{v'}$ for $v \in C^I$, $v' \in C^{I'}$ and $I, I' \subseteq \Delta$ is defined as in (3.10). □

§ 4. Relation to the equivariant and ordinary

K-rings of wonderful compactifications

We recall from [4], Proposition A2, that there is a canonical surjective morphism $f: X \to \overline{G_{ad}}$ from the regular compactification of $G$ to the wonderful compactification of $G_{ad}$, which further restricts to a proper surjective morphism $g: \overline{T^+} \to \overline{T^+}_{ad}$ of toric varieties. Furthermore, $f$ is equivariant with respect to the action of $\tilde{G} \times \tilde{G}$, where $\tilde{G} \times \tilde{G}$ acts on $\overline{G_{ad}}$ by means of the quotient $G^{ss} \times G^{ss}$. 
Lemma 4.1. The ring $K_{\tilde{T}}(\tilde{T}^+)$ has the following presentation as an $R(\tilde{T})$-algebra:

$$K_{\tilde{T}}(\tilde{T}^+) \simeq \frac{R(\tilde{T})[X_j^{\pm 1}; \rho_j \in \mathcal{F}_+(1)]}{J},$$

where $J$ is the ideal in $R(\tilde{T})[X_j^{\pm 1}; \rho_j \in \mathcal{F}_+(1)]$ generated by the elements $X_F$ for $F \notin \mathcal{F}_+$ and $\prod_{\rho_j \in \mathcal{F}_+(1)} X_j^{(u,v_j)} - e^u$ for $u \in X^*(T)$, and $v_j$ stands for the primitive vector along the edge $\rho_j$. Furthermore, the ring $\mathcal{R}(\tilde{T}^+)$ has the following presentation as a $K(G/B)$-algebra:

$$\mathcal{R}(\tilde{T}^+) \simeq \frac{K(G/B)[X_j^{\pm 1}; \rho_j \in \mathcal{F}_+(1)]}{\mathfrak{J}},$$

where $\mathfrak{J}$ is the ideal of $K(G/B)[X_j^{\pm 1}; \rho_j \in \mathcal{F}_+(1)]$ generated by the elements $X_F$ for $F \notin \mathcal{F}_+$ and $\prod_{\rho_j \in \mathcal{F}_+(1)} X_j^{(u,v_j)} - [L(u)]$ for $u \in X^*(T)$. (Here $u \in X^*(T)$ can be viewed as an element of $X^*(\tilde{T})$ by composing with the canonical surjection $\tilde{T} \to T$ involved in (1.1).)

Proof. Since $X$ is a projective regular embedding of $G$ and $\tilde{T}^+$ is the inverse image of $\mathbb{A}^r$ under the canonical morphism $f: X \to G_{\text{ad}}$, we see that the restriction $g: \tilde{T}^+ \to \mathbb{A}^r$ of the projective morphism $f$ is a projective morphism of toric varieties. Note that

$$K_{\tilde{T}}(\tilde{T}^+) = K_G(\tilde{G} \times_\tilde{T} \tilde{T}^+) = K_G(G \times_B \tilde{T}^+),$$

$$\mathcal{R}(\tilde{T}^+) = \mathbb{Z} \otimes_{R(\tilde{G})} K_G(\tilde{G} \times_\tilde{T} \tilde{T}^+) = K_G(G \times_\tilde{T} \tilde{T}^+) = K(G \times_B \tilde{T}^+),$$

where $B$ acts on $\tilde{T}^+$ by means of its quotient $T$. In other words, $K_{\tilde{T}}(\tilde{T}^+)$ (resp. $\mathcal{R}(\tilde{T}^+)$) is the $\tilde{G}$-equivariant (resp. ordinary) Grothendieck ring of the toric bundle $G \times_B \tilde{T}^+$ over $G/B$ with fibre $\tilde{T}^+$. Moreover, $e^u \in R(\tilde{T})$ corresponds to the class $[L(u)]_\tilde{G} \in K_G(G/B)$. Thus we get the presentations (4.1) and (4.2) by applying Theorem 5.1. (Although in §5 we consider toric bundles associated with principal $T$-bundles, Theorem 5.1 can be seen to hold for the toric bundles associated with the principal $B$-bundle $G \to G/B$, where $B$ acts on a semi-projective $T$-toric variety by means of its quotient $T$.) \square

In this section we use $f$ to relate the $\tilde{G} \times \tilde{G}$-equivariant (resp. ordinary) $K$-ring of the regular compactification $X$ with the equivariant (resp. ordinary) $K$-ring of the wonderful compactification $G_{\text{ad}}$.

It follows from (0.1) that $G^{\text{ss}}$ is the universal covering of $G_{\text{ad}}$, and $T_{\text{ad}} := T^{\text{ss}}/(C \cap G^{\text{ss}})$ is a maximal torus of $G_{\text{ad}}$. We recall that the number

$$\text{rank}(G_{\text{ad}}) = \text{rank}(G^{\text{ss}}) = r$$

coincides with the semisimple rank of $G$.

We also recall from [3] and [11], §2.2, that the isomorphism classes of line bundles on $G_{\text{ad}}$ correspond to the elements $\lambda \in X^*(T^{\text{ss}})$. Indeed, the line bundle $L_\lambda$ on $X$
(associated with λ) admits a unique $G^{ss} \times G^{ss}$-linearization, so that $(B^{ss})^{-} \times B^{ss}$ acts on the fibre $\mathcal{L}_{\lambda}|_{z}$ by the character $(\lambda, -\lambda)$, where $z$ stands for the base point of the unique closed orbit $G/B \times G/B$.

Note that $\tilde{G} \times \tilde{G}$ acts on $\overline{G}_{\text{ad}}$ by means of the quotient $G^{ss} \times G^{ss}$, so that $\tilde{C} \times \tilde{C}$ acts trivially on $\overline{G}_{\text{ad}}$. Since $\tilde{G} = G^{ss} \times \tilde{C}$, we see that every $\tilde{G} \times \tilde{G}$-equivariant line bundle on $\overline{G}_{\text{ad}}$ is isomorphic to the tensor product of a certain $G^{ss} \times G^{ss}$-equivariant line bundle and the trivial $\tilde{C} \times \tilde{C}$-equivariant line bundle corresponding to a character $u' \in X^*(\tilde{C})$. Moreover, since $\tilde{T} = T^{ss} \times \tilde{C}$, every element $u \in X^*(\tilde{T})$ can be expressed uniquely as $\lambda + u'$, where $\lambda \in \Lambda$ and $u' \in X^* (\tilde{C})$. For every $u \in X^*(\tilde{T})$ we denote the $\tilde{G} \times \tilde{G}$-equivariant line bundle $\mathcal{L}_{\lambda} \otimes e^{u'}$ on $\overline{G}_{\text{ad}}$ by $\mathcal{L}_{u}$. In particular, $\tilde{B}^{-} \times \tilde{B}$ acts on the fibre $\mathcal{L}_{u}|_{z}$ by the character $(u, -u)$. We also have the following isomorphism of $R(\tilde{G}) \otimes R(\tilde{G})$-algebras (see [13], (5.2.4)):

$$K_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}}) \simeq K_{G^{ss} \times G^{ss}}(\overline{G}_{\text{ad}}) \otimes R(\tilde{C} \times \tilde{C}).$$  

(4.3)

The ring $K_{\tilde{G} \times \tilde{G}}(X)$ acquires the structure of an algebra over the ring $K_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}})$ by pulling back equivariant vector bundles along $f$. The following theorem describes this structure explicitly.

**Theorem 4.1.** The ring $K_{\tilde{G} \times \tilde{G}}(X)$ can be described as a $K_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}})$-algebra as follows:

$$K_{\tilde{G} \times \tilde{G}}(X) = K_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}})[X_{j}^{\pm 1} : \rho_{j} \in \mathcal{F}_{+}(1)],$$  

(4.4)

where $\mathfrak{J}$ is the ideal in $K_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}})[X_{j}^{\pm 1} : \rho_{j} \in \mathcal{F}_{+}(1)]$ generated by the elements $X_{F}$ for $F \notin \mathcal{F}_{+}$ and $\prod_{\rho_{j} \in \mathcal{F}_{+}(1)} X_{j}^{(u,\nu_{j})} - [\mathcal{L}_{u}]_{\tilde{G} \times \tilde{G}}$ for $u \in X^*(T)$.

**Proof.** Using the equalities (1.2), (1.3), (1.7), (1.12) and (4.3), we see from Theorem 3.3 in [1] that the $\tilde{G} \times \tilde{G}$-equivariant $K$-ring of $\overline{G}_{\text{ad}}$ has the following description as an $R(\tilde{T}) \otimes R(\tilde{T})$-module:

$$K_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}}) = \bigoplus_{I \subseteq \Delta} \prod_{a \in I} (1 - e^{-\alpha(u)}) R(\tilde{T}) \otimes R(\tilde{T})_{I}.$$  

(4.5)

(Note that in [1], §3, G and T stand for $G^{ss}$ and $T^{ss}$ respectively and X stands for $\overline{G}_{\text{ad}}$.) Combining (2.4) and (4.5), we obtain the following isomorphism of $K_{\tilde{T}}(\overline{T}^{+}) \otimes 1$-algebras:

$$K_{\tilde{G} \times \tilde{G}}(X) = K_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}}) \otimes R(\tilde{T}) \otimes 1 (K_{\tilde{T}}(\overline{T}^{+}) \otimes 1),$$  

(4.6)

where the structure of an $R(\tilde{T}) \otimes 1$-algebra on $K_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}})$ is from the first factor in the direct sum decomposition (4.5). Recall that there is a canonical inclusion $K_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}}) \hookrightarrow R(\tilde{T}) \times R(\tilde{T})$ obtained by restriction to the base point $z$ of the unique closed orbit $G/B^{-} \times G/B$ (see [1], Lemma 3.2). By definition, this inclusion maps $\mathcal{L}_{u}$ to $e^{u} \otimes 1$ after identifying $R(\tilde{T}) \otimes R(\tilde{T})$ with $R(\tilde{T} \times 1) \otimes R(\text{diag}(\tilde{T}))$ following (2.2). Since $e^{u} \otimes 1$ generates $R(\tilde{T}) \otimes 1$, we see that $R(\tilde{T}) \otimes 1$ can further be identified with the subring of $K_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}})$ generated by $\text{Pic}_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}})$. The theorem follows since $K_{\tilde{T}}(\overline{T}^{+})$ has the presentation (4.1) as an $R(\tilde{T})$-algebra. □
4.1. Relation to the ordinary $K$-ring of the wonderful compactification.
The ring $K(X)$ acquires the stricture of an algebra over the ring $K(\overline{G}_{\text{ad}})$ by pulling back vector bundles along $f$. The following theorem explicitly describes this structure.

**Theorem 4.2.** The ring $K(X)$ has the following presentation as a $K(\overline{G}_{\text{ad}})$-algebra:

$$K(X) = \frac{K(\overline{G}_{\text{ad}})[X_j^{\pm 1} : \rho_j \in \mathcal{F}_+(1)]}{\mathfrak{I}},$$

where $\mathfrak{I}$ is the ideal in $K(\overline{G}_{\text{ad}})[X_j^{\pm 1} : \rho_j \in \mathcal{F}_+(1)]$ generated by the elements $X_F$ for $F \notin \mathcal{F}_+$ and $\prod_{\rho_j \in \mathcal{F}_+(1)} X_j^{(u,v_j)} - [\mathcal{L}_u]$ for $u \in X^*(T)$. Here $[\mathcal{L}_u]$ stands for the class of the line bundle $\mathcal{L}_u$ in $K(\overline{G}_{\text{ad}})$.

**Proof.** Since the action of $\tilde{C} \times \tilde{C}$ on $\overline{G}_{\text{ad}}$ is trivial, we see from (1.6) that

$$K_{G^n \times G^n}(\overline{G}_{\text{ad}}) \otimes_{R(G^n \times G^n)} \mathbb{Z} = K_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}}) \otimes_{R(\tilde{G} \times \tilde{G})} \mathbb{Z} = K(\overline{G}_{\text{ad}}).$$

Again by (1.6),

$$K_{\tilde{G} \times \tilde{G}}(X) \otimes_{R(\tilde{G} \times \tilde{G})} \mathbb{Z} = K(X).$$

Note that $[\mathcal{L}_u]_{\tilde{G} \times \tilde{G}}$ restricts to $[\mathcal{L}_u]$ under the canonical forgetful homomorphism from $\text{Pic}_{\tilde{G} \times \tilde{G}}(\overline{G}_{\text{ad}})$ to $\text{Pic}(\overline{G}_{\text{ad}})$. The equality (4.7) now follows from (4.4), (4.8) and (4.9). $\square$

4.2. Geometric interpretation. In this subsection we understand $X$ as a projective regular compactification of the adjoint semisimple group $G_{\text{ad}}$.

Recall that the $\mathcal{L}_{\alpha_i}$ are $G^n \times G^n$-linearized line bundles on $\overline{G}_{\text{ad}}$ such that $(B^n)^- \times B^n$ operates on $\mathcal{L}_{\alpha_i}|_z$ by the character $(\alpha_i, -\alpha_i)$, where $1 \leq i \leq r$. Here $z$ is the base point of the closed $G_{\text{ad}} \times G_{\text{ad}}$-orbit in $\overline{G}_{\text{ad}}$. Furthermore, since the centre $C \cap G^n$ of $G^n$ acts trivially on $G_{\text{ad}}$ and hence acts on the fibre by the character $(\alpha_i, -\alpha_i)$, the bundle $\mathcal{L}_{\alpha_i}$ is actually $G_{\text{ad}} \times G_{\text{ad}}$-linearized. Moreover, $\mathcal{L}_{\alpha_i}$ admits a $G_{\text{ad}} \times G_{\text{ad}}$-invariant section $s_i$ whose zero locus is the boundary divisor $D_i$ for $1 \leq i \leq r$.

We recall the following construction from [14], §§ 3, 5, and [2], §10.

The bundle $\mathcal{V} := \bigoplus_{1 \leq i \leq r} \mathcal{L}_{\alpha_i}$, being a direct sum of line bundles on $\overline{G}_{\text{ad}}$, admits a natural action of the $r$-dimensional torus $G^n_m$. Put

$$P := \mathcal{V} \setminus \bigcup_{i=1}^r \mathcal{L}_{\alpha_1} \oplus \cdots \oplus \mathcal{L}_{\alpha_i} \oplus \cdots \oplus \mathcal{L}_{\alpha_r}.$$  \hspace{1cm} (4.10)

Then $P$ is the principal $T_{\text{ad}} = G^n_m$-bundle associated with $\mathcal{V}$ over $\overline{G}_{\text{ad}}$. The section $s = s_1 \oplus s_2 \oplus \cdots \oplus s_r$ of $\mathcal{V}$ clearly maps $G_{\text{ad}}$ to $P$. Therefore, using $s$, we can embed $\overline{G}_{\text{ad}}$ in the bundle $P \times_{T_{\text{ad}}} \mathbb{A}^r$, which is nothing but $\mathcal{V}$. Since the bundles $\mathcal{L}_{\alpha_i}$ are $G^n \times G^n$-linearized, it follows that $P$ is a left $G^n \times G^n$-space whose bundle map $\pi: P \to \overline{G}_{\text{ad}}$ is $G^n \times G^n$-equivariant for the canonical $G^n \times G^n$-action on $\overline{G}_{\text{ad}}$. Moreover, the right $T_{\text{ad}}$-action is compatible with the left $G^n \times G^n$-action on $P$.  

If \( X \) is a regular compactification of \( G_{\text{ad}} \), then the associated toric variety \( Z \) corresponds to the fan \( (\mathcal{F}_{\text{ad}})^{+} \) in \( X_{\ast}(T_{\text{ad}}) \otimes \mathbb{R} \) associated with a smooth subdivision of the positive Weyl chamber \( \mathcal{C}^{+} \). In particular, when \( X = \overline{G_{\text{ad}}} \), the edges of the fan \( (\mathcal{F}_{\text{ad}})^{+} = \mathcal{C}^{+} \) are the fundamental coweights \( \omega_{1}', \ldots, \omega_{r}' \) dual to the simple roots \( \alpha_{1}, \ldots, \alpha_{r} \). In particular, \( \overline{T_{\text{ad}}^{\mathcal{C}^{+}}} \cong \mathbb{A}^{r} \), where \( T_{\text{ad}} \) acts on \( \mathbb{A}^{r} \) by the embedding \( t \mapsto (t^{\alpha_{1}}, \ldots, t^{\alpha_{r}}) \).

Consider the proper morphism of toric varieties \( g: Z \to \mathbb{A}^{r} \) corresponding to a smooth subdivision of the positive Weyl chamber. Since \( Z \) is a \( T_{\text{ad}} \)-toric variety, we can form the associated toric bundle \( P \times_{T_{\text{ad}}} Z \). The morphism \( g \) further induces a canonical morphism \( \overline{g}: P \times_{T_{\text{ad}}} Z \to P \times_{T_{\text{ad}}} \mathbb{A}^{r} \). Then the variety \( X_{Z} := \overline{g}^{-1}(s(\overline{G_{\text{ad}}})) \) is the fibre product of the diagram

\[
\begin{array}{ccc}
P \times_{T_{\text{ad}}} Z & \xrightarrow{\overline{g}} & P \times_{T_{\text{ad}}} \mathbb{A}^{r} \\
\overline{G_{\text{ad}}} & \xrightarrow{s} & P \times_{T_{\text{ad}}} \mathbb{A}^{r}
\end{array}
\]

Furthermore, \( X_{Z} \) is the closure of \( G_{\text{ad}} \) in \( P \times_{T_{\text{ad}}} Z \) and is a regular compactification of \( G_{\text{ad}} \) lying over \( \overline{G_{\text{ad}}} \). By [1], §5.2, every regular compactification \( X \) of \( \overline{G_{\text{ad}}} \) can be realized as \( X_{Z} \), where \( Z \) is the inverse image of \( \mathbb{A}^{r} \) under \( f \).

**Corollary 4.1.** The ring \( K_{G^{ss} \times G^{ss}}(X) \) as a \( K_{G^{ss} \times G^{ss}}(\overline{G_{\text{ad}}}) \)-algebra and \( K(X) \) as a \( K(\overline{G_{\text{ad}}}) \)-algebra are respectively isomorphic to the \( G^{ss} \times G^{ss} \)-equivariant and ordinary Grothendieck rings of the above-defined toric variety \( P \times_{T_{\text{ad}}} Z \) over \( \overline{G_{\text{ad}}} \).

**Proof.** Since \( X \) is a projective regular compactification of \( G_{\text{ad}} \), we see that \( Z \) is a semi-projective \( T_{\text{ad}} \)-toric variety as in the proof of Lemma 4.1.

Hence it follows from Theorem 5.1 that the right-hand sides of the formulae (4.4) and (4.7) in Theorems 4.1 and 4.2 are respectively isomorphic to the \( G^{ss} \times G^{ss} \)-equivariant and ordinary Grothendieck rings of the toric bundle \( P \times_{T_{\text{ad}}} Z \) over \( \overline{G_{\text{ad}}} \). □

**Remark 4.1.** Using techniques similar to those in [15], §4, we can alternatively describe \( K(X)_{\mathbb{Q}} \) as

\[
K(X)_{\mathbb{Q}} \cong \bigoplus_{v \in W} R(\overline{T}^{+}) \cdot \gamma_{v},
\]

where \( \gamma_{v} = [O_{X_{v}}] \in K(G/B)_{\mathbb{Q}} \) for \( v \in C^{I} \). Moreover, the multiplication of \( \gamma_{v} \) and \( \gamma_{v'} \) for \( v, v' \in C^{I} \), \( v' \in C^{I'} \) and \( I, I' \subseteq \Delta \) is as defined in (3.10), where the \( c_{v, v'}^{w} \) are now the multiplicative structure constants of the opposite Schubert classes described, for example, in [15], (3.82). Since this description involves Schubert classes, it is more geometric in nature. However, it requires rational coefficients since we are using the lifts of the Schubert classes in \( R(\overline{T}) \) instead of the Steinberg basis, and these lifts form a basis for \( R(\overline{G}) \) only after localizing at the augmentation ideal \( I(\overline{G}) = \{ a - \varepsilon(a) : a \in R(\overline{G}) \} \).
§ 5. Appendix

5.1. Semi-projective toric varieties. In this section we recall the geometric definition and some essential properties of semi-projective toric varieties. Our brief description fixes the notation and conventions which are necessary in order to state and prove the main result (Theorem 5.1) in the next subsection (for details we refer to [16], § 2).

5.1.1. The geometric definition. For every scheme $X$ there is a canonical morphism $\pi_X : X \to X_0$ to the affine scheme $X_0 := \text{Spec}(H^0(X, \mathcal{O}_X))$ of regular functions on $X$. A toric variety $X$ is said to be semi-projective if $X$ has at least one torus-fixed point and the morphism $\pi_X$ is projective (see [16], p. 501).

There are equivalent characterizations of a semi-projective toric variety in terms of the combinatorics of its defining fan as well as in terms of geometric invariant theory (see [16], Theorem 2.6 and Corollary 2.7).

5.1.2. Combinatorial characterization. Here we describe the fan associated with a smooth semi-projective toric variety (see [16], p. 499).

Let $B = \{v_1, \ldots, v_d\}$ be a configuration of vectors in the lattice $N \simeq \mathbb{Z}^n$ and let $\text{pos}(B)$ be the convex polyhedral cone in $N_\mathbb{R} := N \otimes \mathbb{Z} \mathbb{R}$ spanned by the vectors in $B$. A triangulation of $B$ is a simplicial fan $\Delta$ in $N$ whose rays lie in $B$ and whose support is equal to $\text{pos}(B)$. A triangulation is said to be unimodular if every maximal cone of $\Delta$ is spanned by a basis of $N$. This is equivalent to the smoothness of the toric variety $X(\Delta)$.

A $T$-Cartier divisor on a fan $\Delta$ is a continuous function $\Psi : \text{pos}(B) \to \mathbb{R}$ which is linear on each cone of $\Delta$ and takes integer values on $N \cap \text{pos}(B)$. Such a divisor is said to be ample if the function $\Psi : \text{pos}(B) \to \mathbb{R}$ is convex and restricts to a different linear function on each maximal cone of $\Delta$. Actually $\Psi$ is the piecewise-linear support function corresponding to an ample line bundle on $X(\Delta)$.

We say that the triangulation $\Delta$ of $B$ is regular if there is an ample $T$-Cartier divisor on $\Delta$.

We recall from the characterization of smooth semi-projective toric varieties in [16], Corollary 2.7, that the fan $\Delta$ associated with such a variety is a regular unimodular triangulation of a configuration of vectors $B$ which spans the lattice $N$.

5.1.3. Characterization in terms of geometric invariant theory. As above, let $\Delta$ be a regular unimodular triangulation of a configuration $B$ such that $X(\Delta)$ is a smooth semi-projective toric variety.

We first outline a construction realizing $X(\Delta)$ as a projective quotient (in the sense of geometric invariant theory) of the complex affine space $\mathbb{C}^d$ by a $(d-n)$-dimensional subtorus $(\mathbb{C}^*)^{d-n}$ of $(\mathbb{C}^*)^d$ (see [16], Theorem 2.4).

Let $M := \mathcal{X}^*(T) = \text{Hom}(N, \mathbb{Z})$ be the dual lattice of characters of $T$. We put $M_\mathbb{R} := M \otimes \mathbb{R}$. There is an integer-valued $d \times n$ matrix

$$Q := [v_1, \ldots, v_d]^t. \quad (5.1)$$

Consider the map $M \to \mathbb{Z}^d$ induced by the matrix $Q$. Since $\Delta$ is smooth and has at least one cone of maximal dimension, it follows that the cokernel of this map...
can be identified with $\mathbb{Z}^{d-n}$ and is isomorphic to the Picard group of $X(\Delta)$ (see the proposition on p. 63 of [17]). Therefore we have an exact sequence

$$0 \rightarrow M \rightarrow \mathbb{Z}^d \rightarrow \text{Pic}(X) \rightarrow 0. \quad (5.2)$$

Applying $\text{Hom}(\cdot, \mathbb{C}^*)$ to (5.2), we obtain an exact sequence

$$1 \rightarrow G := (\mathbb{C}^*)^{d-n} \rightarrow (\mathbb{C}^*)^d \rightarrow (\mathbb{C}^*)^n \rightarrow 1. \quad (5.3)$$

Thus the group $G$ is embedded in $(\mathbb{C}^*)^d$ as a $(d-n)$-dimensional subtorus and, therefore, acts on the affine space $\mathbb{C}^d$. Let $\theta := [L_\Psi]$ be the class in $\text{Pic}(X)$ of the ample line bundle $L_\Psi$ corresponding to the support function $\Psi$.

We assume that the action of $G$ on $\mathbb{C}^d$ is linearized on the trivial bundle by means of an ample character $\theta$ of $G$. Then the geometric quotient of the semistable locus of $\mathbb{C}^d$ modulo $G$ coincides with the toric variety $X(\Delta)$ (see [16], §2, or [18], §§ 7.2, 14.2, for details).

Conversely, every projective quotient (in the sense of geometric invariant theory) of the affine space $\mathbb{C}^d$ modulo a subtorus $G$ of $(\mathbb{C}^*)^d$, with respect to a $G$-linearization on the trivial bundle on $\mathbb{C}^d$ corresponding to a character $\theta$ of $G$, gives a semi-projective toric variety.

Indeed, applying $\text{Hom}(\cdot, \mathbb{C}^*)$ to the inclusion $G \hookrightarrow (\mathbb{C}^*)^d$ and using the fact that $\mathbb{C}^*$ is divisible, we get the following surjective map of Abelian groups: $\mathbb{Z}^d \rightarrow \mathbb{Z}^{d-n} = \text{Hom}(G, \mathbb{C}^*)$. Let $a_i$ be the image in $\mathbb{Z}^{d-n}$ of the standard basis element $e_i$ of $\mathbb{Z}^d$ under this surjection. Then we have a vector configuration $A := \{a_1, \ldots, a_d\}$ in $\mathbb{Z}^{d-n}$.

Let $S = \mathbb{C}[x_1, x_2, \ldots, x_d]$ be the ring of polynomial functions on $\mathbb{C}^d$. Then $S$ admits a grading by the semigroup $\mathbb{N}A \subseteq \mathbb{Z}^{d-n}$, where $\deg(x_i) = a_i$. There is a natural $G$-action on $S$ by the formula $g \cdot x_i := a_i(g) \cdot x_i$ for $g \in G$, $1 \leq i \leq d$. A polynomial in $S$ is homogeneous if and only if it is a $G$-eigenvector. We denote the set of homogeneous polynomials of degree $\theta$ by $S_\theta$. Then $S_\theta$ is a module over the subalgebra $S_0$ of polynomials of degree 0. The ring $S_{(\theta)} := \bigoplus_{r=0}^{\infty} S_{r\theta}$ is a finitely generated $S_0$-algebra. Then the projective quotient $\mathbb{C}^d/\!/G = \text{Proj}(S_{(\theta)})$ (in the sense of geometric invariant theory) is a toric variety which is projective over the affine toric variety $\mathbb{C}^d/\!/G = \text{Spec}(S_0)$ (see [18], §14.2, [16], Definition 2.2).

**5.2. Equivariant $K$-theory of semi-projective toric varieties.** Here we show that Theorem 1.2(iv) in [19] on the structure of the Grothendieck ring of a toric bundle associated with a smooth projective toric variety holds for toric bundles whose fibre is a smooth semi-projective toric variety.

Consider the following setting which is more general than the one considered in [19].

Let $\pi: E \rightarrow B$ be an algebraic principal $T$-bundle, where $B$ is an irreducible non-singular Noetherian scheme over $\mathbb{C}$. Let $G$ be a connected complex reductive algebraic group. We assume that $E$ and $B$ are algebraic left $G$-spaces and $\pi$ is $G$-invariant, so that the left $G$-action on $E$ commutes with the right $T \simeq (\mathbb{C}^*)^l$-action.

Let $X$ be a smooth semi-projective $T$-toric variety and let $\Delta$ be the associated fan in $N$. There is no loss of generality in assuming for simplicity of notation that
\{v_1, \ldots, v_d\} is the set of primitive vectors along the set \( \Delta(1) := \{\rho_1, \ldots, \rho_d\} \) of edges of \( \Delta \).

We write \( V(\gamma) \) for the orbit closure of the \( T \)-orbit \( O_\gamma \) of the toric variety \( X \) corresponding to a cone \( \gamma \in \Delta \). Let \( U_\sigma \) be the \( T \)-stable open affine subvariety corresponding to a cone \( \sigma \in \Delta \).

For every edge \( \rho_j \) there is a canonical \( T \)-equivariant line bundle \( L_j \) on \( X \). Moreover, there are trivial line bundles \( L_u = X \times \mathbb{C}_u \) on \( X \) corresponding to the characters \( u \in X^*(T) \). The bundle projection \( L_u \overset{h_u}{\to} X \) is \( T \)-equivariant, where \( T \) acts on the fibre \( \mathbb{C}_u \) by means of the character \( \chi_u : T \to \mathbb{C}^* \) (see [17], Ch. 3).

Consider the associated toric bundle \( E(X) := E \times_T X \), where \( X := X(\Delta) \) is the smooth semi-projective \( T \)-toric variety associated with the fan \( \Delta \).

We define the line bundle \( E(L_j) := E \times_T L_j \) on \( E(X) \) associated with \( L_j \), \( 1 \leq j \leq d \), and let \( E(L_u) = E \times_T L_u \) be the line bundle on \( E(X) \) associated with \( L_u \) and having the projection \( [e, v] \mapsto [e, h_u(v)] \). Then \( E(L_j) \) and \( E(L_u) \) are line bundles on \( B \) with the projection \( [e, v] \mapsto \pi(e) \). Both are \( G \)-linearized and their \( G \)-linearization comes from the left \( G \)-action on \( E \).

With every \( T \)-stable subvariety \( V(\gamma) \) corresponding to a cone \( \gamma \) in \( \Delta \), we associate the subvariety \( E(V(\gamma)) := E \times_T V(\gamma) \) of \( E(X) \).

Let \( x \) be a \( T \)-fixed point in \( X \). Then the projection \( p : E(X) \to B \) has a canonical section \( s : B \to E(X) \) given by the formula \( s(b) = [e, x] \), where \( e \in \pi^{-1}(b) \), so that \( p \circ s = \text{id}_B \). Thus the corresponding induced maps \( s^* : K_G(E(X)) \to K_G(B) \) and \( p^* : K_G(B) \to K_G(E(X)) \) between the \( G \)-equivariant Grothendieck rings satisfy \( s^* \circ p^* = \text{id}_B^* \). In particular, \( p^* \) is injective and determines the structure of a \( K_G(B) \)-module on \( K_G(E(X)) \).

We have the following theorem on the \( G \)-equivariant Grothendieck ring of the toric bundle \( E(X) \).

**Theorem 5.1.** The \( G \)-equivariant \( K \)-ring of \( E(X) \) has the following presentation as a \( K_G(B) \)-algebra:

\[
K_G(E(X)) = \frac{K_G(B)[X_j^{\pm 1}]}{\langle X_F : F \notin \Delta ; \prod_{\rho_j \in \Delta(1)}(1 - X_j)^{(u,v_j)} - [E(L_u)]_G, \forall u \in M \rangle}, \tag{5.4}
\]

where \( X_F = \prod_{\rho_j \in F}(1 - X_j) \). (Since \( E(L_u) \to B \) is a \( G \)-linearized line bundle on \( B \), as observed above, we can consider its class \( [E(L_u)]_G \in K_G(B) \).)

Before proving the theorem, we introduce some notation and recall necessary details on the structure of \( X \) and \( \Delta \).

The \( T \)-fixed locus in \( X \) coincides with the set of \( T \)-fixed points

\[
\{x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_m}\} \tag{5.5}
\]

which corresponds to the set of cones of maximal dimension

\[
\Delta(n) := \{\sigma_1, \sigma_2, \ldots, \sigma_m\}. \tag{5.6}
\]

We choose a generic one-parameter subgroup \( \lambda_v \in X_*(T) \) whose vector \( v \in N \) lies outside all hyperplanes spanned by the \( (n-1) \)-dimensional cones, so that the set of fixed points of \( \lambda_v \) is equal to \( \{x_{\sigma_1}, \ldots, x_{\sigma_m}\} \) (see [20], § 3.1).
For every $x_{\sigma_i}$ we define the plus strata by putting

$$Y_i = \{ x \in X \mid \lim_{t \to 0} \lambda_v(t)x \text{ exists and is equal to } x_{\sigma_i} \}. \quad (5.7)$$

Then

$$Y_i = \bigcup_{\tau_i \subseteq \gamma \subseteq \sigma_i} O_\gamma. \quad (5.8)$$

Here the union is taken over the set of all faces $\gamma$ of the cone $\sigma_i$ such that the image of $v$ in $N_R/\mathbb{R}\gamma$ lies in the relative interior of $\sigma_i/\mathbb{R}\gamma$ (see [16], Lemma 2.10). Since the set of such faces is closed under intersections, we can choose a minimal such face of $\sigma_i$. We denote it by $\tau_i$.

If we choose $v \in |\Delta|$, then we have $X = \bigcup_{i=1}^m Y_i$ (the Bialynicki–Birula decomposition of the toric variety $X$ with respect to the one-parameter subgroup $\lambda_v$).

It follows from (5.8) that

$$Y_i = V(\tau_i) \cap U_{\sigma_i} \simeq \mathbb{C}^{n-k_i}, \quad (5.9)$$

where $k_i = \dim(\tau_i)$, $1 \leq i \leq m$. Indeed, $Y_i$ is an affine open subset of $V(\tau_i)$ corresponding to a maximal $(n-k_i)$-dimensional cone in $N/\mathbb{R}\tau_i$ (see [17], Ch.3).

To an ample $T$-Cartier divisor $\Psi$ on $\Delta$ there corresponds a moment map $\mu: X \to \mathbb{R}^n$ (see [16], p. 503). The image of $X$ under $\mu$ is identified with a convex polyhedron

$$Q_\Psi := \{ v \in M_R : Qv \geq \psi \}$$

(see [16], p. 500), where $\psi := (\Psi(v_1), \ldots, \Psi(v_d)) \in \mathbb{Z}^d$ and $Q$ is as in (5.1).

One can also define the moment map $\mu_v$ for the circle action on $X$ induced by a generic one-parameter subgroup $\lambda_v$ by the formula $\mu_v(x) := \langle v, \mu(x) \rangle$. Using $\mu_v$, we can define an ordering

$$\sigma_1, \sigma_2, \ldots, \sigma_m$$

on the set of cones of maximal dimension in such a way that

$$\mu_v(x_{\sigma_i}) \leq \mu_v(x_{\sigma_j}) \quad \text{implies that} \quad \sigma_i \leq \sigma_j. \quad (5.11)$$

Then the distinguished faces $\tau_i \subseteq \sigma_i$ possess the following property (*) with respect to this ordering:

$$\tau_i \subseteq \sigma_j \quad \text{implies that} \quad i \leq j. \quad (5.12)$$

This is equivalent to

$$\overline{Y_i} \subseteq \bigcup_{j \geq i} Y_j \quad (5.13)$$

for every $i, 1 \leq i \leq m$. Hence the Bialynicki–Birula decomposition of $X$ is filtrable in the sense of [20], §3.2. We can order the cells $Y_1, Y_2, \ldots, Y_m$ in such a way that (5.13) holds or, equivalently, each $Y_i$ is closed in $Y_1 \cup Y_2 \cup \cdots \cup Y_i$.

Put

$$Z_i := \bigcup_{j \geq i} Y_j.$$
We see from (5.13) that the sets \( Z_i \) are closed subvarieties and form a chain
\[
X = Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_m = \{x_{\sigma_m}\},
\]  
(5.14)
where \( Z_i \setminus Z_{i+1} = Y_i \simeq \mathbb{C}^{n-k_i} \). This defines an algebraic cell decomposition of \( X \). In other words, \( X \) is paved by the affine spaces \( Y_i, 1 \leq i \leq m \).

Let \( V_i := V(\tau_i) \) be the toric varieties corresponding to the distinguished faces \( \tau_i \) of the maximal cones \( \sigma_i, 1 \leq i \leq m \). We put \( E(V_i) := E \times_T V_i \) for \( 1 \leq i \leq m \), and also \( E(Y_i) := E \times_T Y_i \) and \( E(Z_i) := E \times_T Z_i \).

We now prove the main theorem.

**Proof.** Let \( \mathcal{R} = \mathcal{R}(K_0(G(B), \Delta)) \) denote the ring defined in [19], Definition 1.1, where \( r_i = 1 - [E(L_{u_i})]_G \) in \( K_0(G(B)) \) corresponds to the basis of characters \( u_i \in X^*(T) \), \( 1 \leq i \leq n \), dual to a basis \( v_1, \ldots, v_n \) of \( N \) (we can renumber the edges so that the primitive vectors of the first \( n \) edges span a maximal cone and hence form a basis of \( N \)).

We have a canonical homomorphism of \( K_0(G(B)) \)-algebras \( \psi: \mathcal{R} \to K_0(G(E(X))) \) by the formula \( x_j \to 1 - [E(L_j)]_G \), \( 1 \leq j \leq d \). It is known from Lemma 2.2(iv) in [19] that the \( x(\tau_i), 1 \leq i \leq m \), span \( \mathcal{R} \) as a \( K_0(G(B)) \)-module. Thus if we show that \( K_0(G(E(X))) \) is a free \( K_0(G(B)) \)-module of rank \( m \) with basis \( [\mathcal{O}_{E(V_i)}], 1 \leq i \leq m \) (consisting of the images of \( x(\tau_i), 1 \leq i \leq m \), under \( \psi \)), then \( \psi \) will be an isomorphism and Theorem 5.1 will be proved.

**Claim.** The classes \( [\mathcal{O}_{E(V_i)}], 1 \leq i \leq m \), form a basis for the \( K_0(G(B)) \)-module \( K_0(G(E(X))) \).

**Proof.** Firstly, it follows from Corollary 6.10(i) of [9] and (4.1) that \( K_0(X) \) is isomorphic to the ring \( \mathcal{R}(Z, \Delta) \), where \( r_i = 1 \) for \( 1 \leq i \leq n \). Moreover, this isomorphism sends \( x(\tau_i) \) to \( [\mathcal{O}_{V_i}] \) for \( 1 \leq i \leq m \). It is known from [18], Lemma 2.2(iv) that the elements \( x(\tau_i), 1 \leq i \leq m \), span \( \mathcal{R}(Z, \Delta) \) as a \( \mathbb{Z} \)-module. Thus \( K_0(X) \) is spanned as a \( \mathbb{Z} \)-module by the classes \( [\mathcal{O}_{V_i}], 1 \leq i \leq m \). However, we also know from [9], Corollary 6.10(ii), that \( K_0(X) \) is a free Abelian group of rank \( m \), where \( m \) is equal to the number of maximal cones in \( \Delta \). (This also follows from the lemma on cellular fibration, Lemma 5.5.1 in [13], since \( m \) is the number of cells in the algebraic cell decomposition (5.14) of \( X \).) Hence the classes \( [\mathcal{O}_{V_i}], 1 \leq i \leq m \), form a basis for \( K_0(X) \) as a free \( \mathbb{Z} \)-module.

We now define a map \( \phi: K_0(G(B)) \otimes K_0(X) \to K_0(G(E(X))) \) by the formula
\[
\sum_{1 \leq i \leq m} b_i \otimes [\mathcal{O}_{V_i}] \mapsto \sum_{1 \leq i \leq m} p^*(b_i)[\mathcal{O}_{E(V_i)}].
\]  
(5.15)
It remains to show that the \( K_0(G(B)) \)-module homomorphism \( \phi \) is injective and surjective. This will be done below.

Note that there is a filtration
\[
E(Z_1) = E(X) \supseteq E(Z_2) \supseteq \cdots \supseteq E(Z_m) \simeq B,
\]  
(5.16)
where \( E(Z_i) \setminus E(Z_{i+1}) = E(Y_i) \) is an affine bundle on \( B \). Indeed, the restrictions of \( p: E(X) \to B \) to \( E(Z_i) \) and \( E(Y_i) \) are \( G \)-equivariant with respect to the left
$G$-action on $E$ and $B$. Then by the lemma on cellular filtration ([13], Lemma 5.5.1) we have the following short exact sequence for every $i$, $1 \leq i \leq m$:

$$0 \to K_G(E(Z_{i+1})) \to K_G(E(Z_i)) \to K_G(E(Y_i)) \to 0. \quad (5.17)$$

We claim that the restriction of $\phi$ determines isomorphisms $\phi_i : K_G(B) \otimes K(Z_i) \to K_G(E(Z_i))$ for all $i$, $1 \leq i \leq m$. This is proved by downward induction on $i$. The assertion holds trivially when $i = m$ since $E(Z_m) \simeq B$. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
K_G(B) & \otimes & K(Z_{i+1}) & \longrightarrow & K_G(B) & \otimes & K(Z_i) & \longrightarrow & K_G(B) & \otimes & K(Y_i) & \longrightarrow & 0 \\
\phi_{i+1} & & \downarrow & & \phi_i & & \downarrow & & \phi_i & & \cdots & & 0 \\
0 & \longrightarrow & K_G(E(Z_{i+1})) & \longrightarrow & K_G(E(Z_i)) & \longrightarrow & K_G(E(Y_i)) & \longrightarrow & 0
\end{array}
$$

where the bottom row is part of (5.17) and the top row is obtained as the tensor product of the exact sequence

$$0 \to K(Z_{i+1}) \to K(Z_i) \to K(Y_i) \to 0 \quad (5.19)$$

with $K_G(B)$. The exact sequence (5.19) is a consequence of the lemma on cellular fibration in the case when the base $B$ is a point. The Thom isomorphism ([13], Theorem 5.4.11) implies that $p^* : K_G(B) \simeq K_G(E(Y_i))$ is an isomorphism. Since $K(Y_i) \simeq Z$ (see, for example, [10]), it follows that the homomorphism $K_G(B) \otimes K(Y_i) \to K_G(E(Y_i))$ in (5.18) is an isomorphism. Therefore if we assume that $\phi_{i+1}$ is surjective, then it follows by diagram chasing that $\phi_i$ is surjective. In a similar vein, if we assume that $\phi_{i+1}$ is injective, then a diagram chase yields that $\phi_i$ is injective since the bottom horizontal row is left exact by (5.17). This proves the claim. □

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