Extended Graph of the Fuzzy Topographic Topological Mapping Model

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Abstract: Fuzzy topological topographic mapping (FTTM) is a mathematical model which consists of a set of homeomorphic topological spaces designed to solve the neuro magnetic inverse problem. A sequence of FTTM, FTTM₀ᵢ, is an extension of FTTM that is arranged in a symmetrical form. The special characteristic of FTTM, namely the homeomorphisms between its components, allows the generation of a set of homeomorphic topological spaces. The generated FTTM₁ pseudo degree zero with respect to n number of components and k number of versions. In this paper, the conjecture is proven analytically for the first time using a newly developed grid-based method. Some definitions and properties of the novel grid-based method are introduced and developed along the way. The developed definitions and properties of the method are then assembled to prove the conjecture. The grid-based technique is simple yet offers some visualization features of the conjecture.

Keywords: FTTM; graph; pseudo degree; sequence

1. Introduction

Fuzzy topographic topological mapping (FTTM) [1] was introduced to solve the neuro magnetic inverse problem, particularly with regards to the sources of electroencephalography (EEG) signals recorded from epileptic patients. Originally, the model was a 3-tuple of topological spaces and mappings. The topological spaces are the magnetic plane (MC), base magnetic plane (BM), fuzzy magnetic field (FM) and topographic magnetic field (TM). The third component of FTTM, FM, is a set of three tuples with the membership function of its potential reading obtained from a recorded EEG. FTTM is defined formally as follows (see Figure 1).

\[ MC = \{(x, y, 0), \beta_2 | x, y, \beta_2 \in \mathbb{R}\} \]
\[ = \{(x, y), \beta_2 \in \mathbb{R}\} \]
\[ TM = \{(x, y, z) | x, y \in \mathbb{R}, z \in (-h, 0)\} \]
\[ BM = \{(x, y, h), \beta_2 | x, y, \beta_2 \in \mathbb{R}\} \]
\[ = \{(x, y), x, y, \beta_2 \in \mathbb{R}\} \]
\[ FM = \{(x, y, h), \mu_\beta | x, y, h \in \mathbb{R}, \mu_\beta \in (0, 1)\} \]
\[ = \{(x, y), x, y, h, \mu_\beta \in (0, 1)\} \]

**Figure 1. The FTTM.**

**Definition 1.** Ref. [1] Let \( FTTM_i = (MC_i, BM_i, FM_i, TM_i) \) such that \( MC_i, BM_i, FM_i, TM_i \) are topological spaces with \( MC_i \cong BM_i \cong FM_i \cong TM_i \). Set of \( FTTM_i \) is denoted by...
FTTM = \{FTTM_i : i = 1, 2, 3, \ldots, n\}. Sequence of nFTTM_i of FTTM is FTTM_1, FTTM_2, FTTM_3, \ldots, FTTM_n such that MC_i = MC_{i+1}, BM_i = BM_{i+1}, FM_i = FM_{i+1} and TM_i = TM_{i+1}.

Furthermore, a sequence of FTTM, FTTM_n, is an extension of FTTM and illustrated in Figure 2. It is arranged in a symmetrical form, since the model can accommodate magnetoencephalography (MEG) signals as well as image data due to its homeomorphism.

![Figure 2. The sequence of FTTM_n.](image)

2. Generalized FTTM

Generally, the FTTM structure can also be expanded for any n number of components.

Definition 2. Ref. [2] A FTTM is defined as

\[
FTTM_n = \{\{A_1, A_2, \ldots, A_n\} : A_1 \cong A_2 \cong \ldots \cong A_n\} \tag{1}
\]

such that A_1, A_2, \ldots, A_n are the components of FTTM_n

The same generalization can be applied to any k number of FTTM versions as well, denoted as FTTM^k_n. Without the loss of generality, the collection of the k version of FTTM, in short FTTM^k_n, is now simply called as a sequence of FTTM unless otherwise stated.

Definition 3. Ref. [2] A sequence of k versions of FTTM_n denoted by *FTTM^k_n such that

\[
*FTTM^k_n = \{FTTM^1_n, FTTM^2_n, \ldots, FTTM^k_n\} \tag{2}
\]

where FTTM^1_n is the first version of FTTM_n, the FTTM^2_n is the second version of FTTM_n and so forth.

Obviously, a new FTTM can be generated from a combination of components from different versions of FTTM due to their homeomorphisms.

Definition 4. Ref. [2] A new FTTM generated from *FTTM^k_n is defined as

\[
F = \{A_1^{m_1}, A_2^{m_2}, \ldots, A_n^{m_n}\} \in FTTM \tag{3}
\]

where 0 \leq m_1, m_2, \ldots, m_n \leq k and m_i \neq m_j for at least one i, j.
A set of elements generated by $*\text{FTTM}_{n}^{k}$ is denoted by $G\left(*\text{FTTM}_{n}^{k}\right)$. Mukaram et al. [2] showed that the number of FTTM can be determined from $*\text{FTTM}_{4}^{k}$ using the geometrical features of its graph representation.

**Theorem 1.** Ref. [2] The number of generated FTTM that can be created from $*\text{FTTM}_{4}^{k}$ is

$$|G\left(*\text{FTTM}_{4}^{k}\right)| = k^4 - k. \quad (4)$$

Theorem 1 is then extended to include $n$ number of FTTM components.

**Theorem 2.** Ref. [2] The number of generated FTTM that can be created from $*\text{FTTM}_{n}^{k}$ is

$$|G\left(*\text{FTTM}_{n}^{k}\right)| = k^n - k. \quad (5)$$

The following example is presented to illustrate Theorem 2.

**Example 1.** Consider $*\text{FTTM}_{2}^{3}$, with $\text{FTTM}_{1}^{1} = \{A_{1}^{1}, A_{1}^{2}, A_{1}^{3}\}$ and $\text{FTTM}_{2}^{2} = \{A_{1}^{2}, A_{2}^{2}, A_{3}^{2}\}$, then $G\left(*\text{FTTM}_{2}^{3}\right) = \{\{A_{1}^{1}, A_{1}^{2}, A_{1}^{3}\}, \{A_{1}^{1}, A_{1}^{2}, A_{1}^{3}\}, \{A_{1}^{2}, A_{2}^{2}, A_{3}^{2}\}, \{A_{1}^{2}, A_{2}^{2}, A_{3}^{2}\}\}$ that is $|G\left(*\text{FTTM}_{2}^{3}\right)| = 2^3 - 2 = 6$ as given by Theorem 2.

3. Extended Generalization of FTTM

There are many studies on ordinary and fuzzy hypergraphs available in the literature such as [3,4]. However, $*\text{FTTM}_{n}^{k}$ is an extended generalization of FTTM that is represented by a graph of a sequence of $k$ number of polygons with $n$ sides or vertices. The polygon is arranged from back to front where the first polygon represents $\text{FTTM}_{n}^{1}$, the second polygon represents $\text{FTTM}_{n}^{2}$ and so forth. An edge is added to connect $\text{FTTM}_{n}^{1}$ to the $\text{FTTM}_{n}^{2}$ component wisely. A similar approach is taken for $\text{FTTM}_{n}^{2}$, $\text{FTTM}_{n}^{3}$ and the rest (Figure 3).

![Figure 3. Graph of $*\text{FTTM}_{n}^{k}$.](image)

When a new FTTM is obtained from $*\text{FTTM}_{n}^{k}$, it is then called a pseudo-graph of the generated FTTM and plotted on the skeleton of $*\text{FTTM}_{n}^{k}$. A generated element of
a pseudo-graph consists of vertices that signify the generated FTTM and edges which connect the incidence components. Two samples of pseudo-graphs are illustrated in Figure 4.

![Figure 4. Pseudo graph: (a) \{A_1^1, A_2^1, A_3^1\}; (b) \{A_1^1, A_2^2, A_3^3\} of *FTTM^3.](image)

Another concept related closely to the pseudo-graph is the pseudo degree. It is defined as the sum of the pseudo degree from each component of the FTTM. The pseudo degree of a component is the number of other components that are adjacent to that particular component.

**Definition 5.** Ref. [2] The \( \deg_p : \text{FTTM} \rightarrow \mathbb{Z} \) defines the pseudo degree of the FTTM component. It maps a component of \( F \in G\left(*\text{FTTM}^k_n\right) \) to an integer

\[
\deg_p(A_j^{m_j}) = \begin{cases} 
0; & m_{j-1} \neq m_j \neq m_{j+1} \\
1; & m_{j-1} = m_j = m_{j+1}, \quad m_{j-1} = m_j = m_{j+1} \\
2; & m_{j-1} = m_j = m_{j+1} 
\end{cases} \quad (6)
\]

for \( A_j^{m_j} \in \text{FTTM} \).

**Definition 6.** Ref. [2] The \( \deg_p G : G\left(*\text{FTTM}^k_n\right) \rightarrow \mathbb{Z} \) defines the pseudo degree of the FTTM graph. Let \( F \in \text{FTTM} \)

\[
\deg_p G(F) = \sum_{i=1}^{n} \deg_p A_i^{m_i} \quad (7)
\]

where \( F = \{A_1^{m_1}, A_2^{m_2}, \ldots, A_n^{m_n}\} \in G\left(*\text{FTTM}^k_n\right) \).

**Definition 7.** Ref. [2] The set of elements generated by \(*\text{FTTM}^k_n\) that have pseudo degree zero is

\[
G_0\left(*\text{FTTM}^k_n\right) = \left\{ F \in G\left(*\text{FTTM}^k_n\right) \left| \deg_p G(F) = 0 \right. \right\} \quad (8)
\]

From now on,

1. \( G_0\left(*\text{FTTM}^k_n\right) \) is simply denoted by \( G_0\left(\text{FTTM}^k_n\right) \).
2. \( \#G_0\left(\text{FTTM}^k_n\right) \) denotes the cardinality of the set \( G_0\left(\text{FTTM}^k_n\right) \).
Example 2. (See Figure 5).

\[
G_0(FTTM_4^3) = \{(A_1, A_2, A_3, A_4), (B_1, B_2, B_3, B_4), (C_1, C_2, C_3, C_4)\}
\]

\[
G_0(FTTM_4^3) = \{(A_1, B_2, A_3, C_4), (A_1, B_2, C_3, B_4), (A_1, C_2, A_3, B_4), (A_1, C_2, B_3, C_4),
(B_1, A_2, B_3, C_4), (B_1, A_2, C_3, A_4), (B_1, C_2, B_3, A_4), (B_1, C_2, A_3, C_4),
(C_1, B_2, C_3, A_4), (C_1, B_2, A_3, B_4), (C_1, A_2, C_3, B_4), (C_1, A_2, B_3, A_4)\}
\] (9)

\[
G_0(FTTM_4^3) = 12.
\]

Figure 5. FTTM_4^3.

Previously, Elsafi proposed a conjecture in [5] related to the graph of pseudo degree.

Conjecture 1. Ref. [5]

\[
|C_0^3(FTTM_n^3)| = \begin{cases} 4|G_0^3(FTTM_{n-2}^3)| + 12, & \text{when } n \text{ is even} \\ 4|G_0^3(FTTM_{n-2}^3)| + 6, & \text{when } n \text{ is odd} \end{cases}
\] (10)

In order to observe some patterns that may appear from the proposed conjecture, Mukaram et al. [2] have developed an algorithm to compute \(|G_0(FTTM_n^3)|\) in order to prove the conjecture analytically. A flowchart on \(|G_0(FTTM_n^3)|\) is sampled in Figure 6.
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The researchers generated all $FTTM$ combinations for $3 \leq k \leq 4$, $4 \leq n \leq 15$ and were able to isolate graphs with pseudo degree zero, which are listed below (Table 1).

**Table 1.** $|G_0(FTTM_n^k)|$ for $4 \leq n \leq 15$ and $k = 3, 4$.

| $n$ | $|G_0(FTTM_n^3)|$ | $|G_0(FTTM_n^4)|$ |
|-----|-------------------|-------------------|
| 4   | 12                | 24                |
| 5   | 30                | 120               |
| 6   | 60                | 480               |
| 7   | 126               | 1680              |
| 8   | 252               | 5544              |
| 9   | 510               | 17,640            |
| 10  | 1020              | 54,960            |
| 11  | 2046              | 168,960           |
| 12  | 4092              | 515,064           |
| 13  | 8190              | 1,561,560         |
| 14  | 16,380            | 4,717,440         |
| 15  | 32,766            | 14,217,840        |
The researchers then simulated $|G_0\left(FTTM_n^k\right)|$ for some values of $k$ as well [2]. The number of graphs of pseudo degree zero for $2 \leq k \leq 8$ and $2 \leq n \leq 10$ are listed in Table 2.

### Table 2. $|G_0\left(FTTM_n^k\right)|$ for $2 \leq k \leq 8$ and $2 \leq n \leq 10.$

| $k/n$ | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-------|----|----|----|----|----|----|----|----|----|
| 2     | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
| 3     | 0  | 6  | 12 | 30 | 60 | 126| 252| 510| 1020|
| 4     | 0  | 0  | 24 | 120| 480| 1680|5544|17,640|54,960|
| 5     | 0  | 0  | 0  | 120| 1080|6720|35,280|168,840|763,560|
| 6     | 0  | 0  | 0  | 0  | 720 |10,080|90,720|665,280|4,339,440|
| 7     | 0  | 0  | 0  | 0  | 0  | 5040|100,800|1,239,840|12,096,000|
| 8     | 0  | 0  | 0  | 0  | 0  | 0  | 40,320|1,088,640|17,539,200|

4. Grid of FTTM

An alternative presentation of a sequence of FTTM, called an FTTM grid, is briefly overviewed. It provides a different perspective of the structure of FTTM. Instead of a polygon representation for each version of FTTM, a straight line is now used. The components of $FTTM_n$ are arranged on a horizontal line of vertices and the lines represent the homeomorphisms between the components of $FTTM_n$. The only exception is the homeomorphism between the first and last components of $FTTM_n$, $A_1$ and $A_n$, respectively. Two open segments on the left of $A_1$ and on the right of $A_n$ are used to represent the homeomorphism between them. A vertical line is added to represent a homeomorphism between two components of different versions; hence, a grid is created (see Figure 7).

![Figure 7. A graph representation of $\ast FTTM_n^k$ as a grid.](image)

There are four advantages when FTTM is represented as a grid instead of a sequence of polygon.

- It is represented in two dimensions; therefore, it reduces the complexity of the structure.
- The process of adding a new component is easier than in a sequence of polygon.
- It can take any number of components by adding the number of vertices at the end of the grid.
- The homeomorphism between two components of the same version is presented as a horizontal edge, whereas the homeomorphism between two components of different versions is represented by a diagonal edge (see Figure 8). These arrangements are necessary to produce the graph of pseudo degree zero.
Furthermore, Zilullah et al. [2] introduced some operations and properties with respect to the FTTM grid. They are recalled, summarized and listed below for convenience. Then, we will move on to the next main section of the paper wherein Conjecture 1 is finally proven as a theorem.

**Definition 8.** Let \( F \in G \left( *FTTM_n^k \right) \) and \( F = \{ A_{m_1}^{p_1}, A_{m_2}^{p_2}, \ldots, A_{m_n}^{p_n} \} \). A block \( B \), where \( B \subseteq F \) is defined as

\[
B = \{ A_i^{m_{i+1}}, A_i^{m_{i+2}}, \ldots, A_i^{m_{i+j}} \}, \quad 1 \leq i < n, \quad 0 < j \leq n - 1
\]

This set \( B \left( *FTTM_n^k \right) \) is the set of FTTM blocks that can be generated from \( G \left( *FTTM_n^k \right) \).

**Definition 9.** The function \( C_i \) is defined as \( C : G \left( *FTTM_n^k \right) \to B \left( *FTTM_n^k \right) \) for \( F \in G \left( *FTTM_n^k \right) \),

\[
B = \{ A_i^{m_i}, A_i^{m_{i+1}}, A_i^{m_{i+2}}, \ldots, A_i^{m_{i+j}} \}, \quad 1 \leq i < n, \quad 0 < j \leq n - 1
\]

for \( 1 < i < j < n \), where \( F = \{ A_{m_1}^{p_1}, A_{m_2}^{p_2}, A_{m_3}^{p_3}, \ldots, A_{m_n}^{p_n} \} \).

**Definition 10.** The operation \( \oplus \) is a mapping \( \oplus : B \left( *FTTM_n^k \right) \times B \left( *FTTM_n^k \right) \to B \left( *FTTM_n^k \right) \) such that

\[
\{ A_i^{m_i}, A_i^{m_{i+1}}, \ldots, A_i^{m_k} \} \oplus \{ A_j^{m_p}, A_j^{m_{p+1}}, \ldots, A_j^{m_j} \} = \{ A_i^{m_i}, A_i^{m_{i+1}}, \ldots, A_j^{m_j} \}
\]

when \( k = p \) and \( m_k = m_p \), then \( B_3 = B_1 \oplus B_2 = \{ A_i^{m_i}, A_i^{m_{i+1}}, \ldots, A_j^{m_j} \} \).

**Definition 11.** An indexed FTTM \( G \left( *FTTM_n^k \right) \) is defined as

\[
G \left( *FTTM_n^k \right) = \{ F \in G \left( *FTTM_n^k \right) | A_j^{m_j} \in F, m_j = i \}
\]

**Figure 8.** Generated element \( \{ A_1^1, A_2^1, A_3^1 \} \) on \( *FTTM_3^3 \) grid.
A generated FTTM is then divided into blocks of three components. A set of blocks is defined as follows.

**Definition 12.** A set of blocks \( B_{ijk} \) is defined as

\[
B_{ijk} = \left\{ B \in G \left( \ast \text{FTTM}_n^k \right) \mid B = \{ A_{m_p}^i, A_{m_{p+1}^i}, A_{m_{p+2}^i} \}, m_p = i, m_{p+1} = j, m_{p+2} = k \right\} \tag{15}
\]

Since this study is concerned with graphs of pseudo degree zero, the sets that need to be taken into consideration are the ones with diagonal paths, namely, \( B_{121, 121, 123}, B_{132, 211, 212, 213}, B_{321, 321, 323}, B_{312, 321, 323} \) and \( B_{313} \).

**Lemma 1.** Let \( F \in \ast \text{FTTM}_n^k \) and \( F = \{ A_1^{m_1}, A_2^{m_2}, \ldots, A_n^{m_n} \} \). For any \( A_j^{m_j} \in F, 1 < j < n \), then \( \deg_p \left( A_j^{m_j} \right) = 0 \) if \( A_j^{m_j} \) is connected to \( A_{j-1}^{m_{j-1}} \) and \( A_{j+1}^{m_{j+1}} \) by a diagonal path.

**Theorem 3.** If \( F \in G_2 \left( \ast \text{FTTM}_n^k \right) \) where \( G_2 \left( \ast \text{FTTM}_n^k \right) \) is the set of generated FTTMs with a diagonal path, then \( \deg_p G(F) = 2 \) or 0.

**Corollary 1.** The element of \( G_0 \left( \ast \text{FTTM}_n^k \right) \) has a FTTM path with the following properties:
1. All the edges connecting the path are diagonal.
2. The starting and the end points of the path belong to different versions of FTTM.

**Theorem 4.** If \( x \in B \left( G_0 \left( \ast \text{FTTM}_n^k \right) \right) \), then all the paths for \( x \) are diagonals.

**Proposition 1.** If \( F \in G \left( \ast \text{FTTM}_n^k \right) \), then \( C_{1}^{n-2}(F) \in G \left( \ast \text{FTTM}_{n-2} \right) \).

**Lemma 2.** If \( F \in G \left( \ast \text{FTTM}_n^k \right) \), then \( \exists x, y \) such that \( x \in G \left( \ast \text{FTTM}_{n-2} \right), y \in C_{1}^{n-2} \left( G \left( \ast \text{FTTM}_n^k \right) \right) \) and \( F = x \oplus y \).

**Lemma 3.** If \( F \in G \left( \ast \text{FTTM}_n^k \right) \), then \( \exists \) unique tuple \( (x, y) \) such that \( x \in G \left( \ast \text{FTTM}_{n-2} \right), y \in C_{1}^{n-2} \left( G \left( \ast \text{FTTM}_n^k \right) \right) \) and \( F = x \oplus y \).

**Theorem 5.** If \( H \subseteq G \left( \ast \text{FTTM}_n^k \right) \) and \( K = \{ (x, y) \mid x \oplus y \in H \}, x \in G \left( \ast \text{FTTM}_{n-2} \right), y \in C_{1}^{n-2} \left( G \left( \ast \text{FTTM}_n^k \right) \right) \), then \( |K| = |C| \).

**Lemma 4.**

\[
\left( \ast \text{FTTM}_n^3 \right) = \bigcup_{m_{n-2} = 1} G \left( \ast \text{FTTM}_n^3 \right) \bigcup_{m_{n-2} = 2} G \left( \ast \text{FTTM}_n^3 \right) \bigcup_{m_{n-2} = 3} G \left( \ast \text{FTTM}_n^3 \right) \tag{16}
\]

**Lemma 5.**

\[
G \left( \ast \text{FTTM}_n^3 \right) \bigcap_{m_{n-2} = a} G \left( \ast \text{FTTM}_n^3 \right) = \emptyset \tag{17}
\]

for any \( a, b \in \mathbb{Z} \) and \( a \neq b \).

**Theorem 6.**

\[
\left| G \left( \ast \text{FTTM}_n^3 \right) \right| = \left| G \left( \ast \text{FTTM}_n^3 \right) \bigcap_{m_{n-2} = 1} \right| + \left| G \left( \ast \text{FTTM}_n^3 \right) \bigcap_{m_{n-2} = 2} \right| + \left| G \left( \ast \text{FTTM}_n^3 \right) \bigcap_{m_{n-2} = 3} \right| \tag{18}
\]
5. The Theorem

All the materials laid down in previous sections are assembled to produce the analytical proof of Conjecture 1. The first step is to find \(|G_d(\ast FTTM^3_n)|\) since \(G_0(\ast FTTM^3_n)\) is a subset of \(G_d(\ast FTTM^3_n)\) by Theorem 2.

**Theorem 7.**

\[
|G_d(\ast FTTM^3_n)| = \begin{cases} 
12 \cdot 4^{\frac{n-3}{2}}, & \text{if } n \text{ is odd, } n \geq 3 \\
6 \cdot 4^{\frac{n-2}{2}}, & \text{if } n \text{ is even, } n \geq 4.
\end{cases} 
\]  

**Proof of Theorem 7.** (By mathematical induction)

Let \(P(m) = |G_d(\ast FTTM^3_n)| = \begin{cases} 
12 \cdot 4^{\frac{n-3}{2}}, & \text{if } n \text{ is odd, } n \geq 3 \\
6 \cdot 4^{\frac{n-2}{2}}, & \text{if } n \text{ is even, } n \geq 4.
\end{cases} \)

For odd numbers, \(P(3) : n = 3,

\[
P(3) = |G_d(\ast FTTM^3_3)| = 12 \cdot 4^{\frac{3-3}{2}} = 12.
\]  

There are exactly 12 combinations, namely

\[
\{A_1^1, A_2^3, A_3^1\}, \{A_1^2, A_2^3, A_3^1\}, \{A_1^3, A_2^1, A_3^1\}, \{A_1^1, A_2^3, A_3^2\}, \{A_1^2, A_2^3, A_3^2\}, \{A_1^3, A_2^3, A_3^1\}, \{A_2^1, A_2^3, A_3^1\}, \{A_2^2, A_2^3, A_3^1\}, \{A_2^3, A_2^3, A_3^1\}, \{A_3^1, A_2^3, A_3^1\}, \{A_3^2, A_2^3, A_3^1\}, \{A_3^3, A_2^3, A_3^1\}
\]

Now assume \(P(m = 2k+1) : n = 2k+1\) is true with

\[
P(m) = |G_d(\ast FTTM^3_{2k+1})| = 12 \cdot 4\frac{2k+1-3}{2} = 12 \cdot 4^{k-1}
\]  

for \(P\left( m + 2 = 2k + 1 + 2 \right)\).

By using Theorem 4, \(P(m + 1) = |G_0(\ast FTTM^3_{2k+3})| = |K|\) such that

\[
K = \{(x, y) : x \oplus y \in H, x \in G(\ast FTTM^3_{2k+1}), y \in C_n C_{n-2}(G(\ast FTTM^3_{2k+3}))\}.
\]

By using Theorem 5,

\[
|P(m + 1)| = |G_d(\ast FTTM^3_{2k+3})| = |G_d(\ast FTTM^3_{2k+1})| + |G_d(\ast FTTM^3_{2k+3})| + |G_d(\ast FTTM^3_{2k+3})| = 4 |G_d(\ast FTTM^3_{2k+1})|.
\]

The set \(G_d(x, y)\) can be constructed from \((x, y)\) where \(x \in G_d(\ast FTTM^3_{2k+1})\) and \(y \in C_n C_{n-2}(G_d(\ast FTTM^3_{2k+3}))\). There are four options for \(y\), namely \(B_{121}, B_{123}, B_{131},\) and \(B_{132}\). Hence,

\[
|G_d(\ast FTTM^3_{2k+3})| = 4 |G_d(\ast FTTM^3_{2k+1})|.
\]

The same process can be applied to \(G_d(\ast FTTM^3_{2k+3})\) and \(G_d(\ast FTTM^3_{2k+3})\).

Thus,
Thus, from Theorem 3. Case \(m_{n-2} = 1\).

\[
|P(m+1)| = |G_0(\text{*FTTM}_{n-2}^3)|
= 4 \left| G_{d_{m_{n-2}=1}}(\text{*FTTM}_{n-2}^3) \right| + 4 \left| G_{d_{m_{n-2}=2}}(\text{*FTTM}_{n-2}^3) \right| + 4 \left| G_{d_{m_{n-2}=3}}(\text{*FTTM}_{n-2}^3) \right|
= 4 \left( G_{d_{m_{n-2}=1}}(\text{*FTTM}_{n-2}^3) \right) + G_{d_{m_{n-2}=2}}(\text{*FTTM}_{n-2}^3) + G_{d_{m_{n-2}=3}}(\text{*FTTM}_{n-2}^3)
= 4 G_{d_{m_{n-2}=3}}(\text{*FTTM}_{n-2}^3) = 4 \cdot 12 \cdot 4^{n-2} = 12 \cdot 4^n.
\]

Similarly, the same induction process can be used as proof for even parts. \(\Box\)

The set \(G_0(\text{*FTTM}_n^3)\) has only two possible subsets, namely \(G_0(\text{*FTTM}_n^3)\) and \(H_n = \left\{ x \in G_0(\text{*FTTM}_n^3) \right\}\). To find \(G_0(\text{*FTTM}_n^3)\), the relation between \(G_0(\text{*FTTM}_n^3)\), \(G_0(\text{*FTTM}_n^3)\) and \(H_n\) must be investigated.

**Lemma 6.** If \(H_n = \left\{ x \in G_0(\text{*FTTM}_n^3) \right\}\), then \(|H_n| = |G_0(\text{*FTTM}_n^3)| - |G_0(\text{*FTTM}_n^3)|\).

**Proof of Lemma 6.** Let \(x \in G_0(\text{*FTTM}_n^3)\), then \(\text{deg}_p(x) = 0\) or \(\text{deg}_p(x) = 2\) by Theorem 5. Thus, \(x \in G_0(\text{*FTTM}_n^3)\) or \(x \in H_n\), i.e., \(|G_0(\text{*FTTM}_n^3)| = |G_0(\text{*FTTM}_n^3)| + |H_n|\) or \(|H_n| = |G_0(\text{*FTTM}_n^3)| - |G_0(\text{*FTTM}_n^3)|\). \(\Box\)

Finally, \(G_0(\text{*FTTM}_n^3)\) is determined using Lemma 6 and Theorem 5.

**Theorem 8.**

\[
|G_0(\text{*FTTM}_n^3)| = 3 |G_0(\text{*FTTM}_{n-1}^3)| + 2|H_{n-2}|, n > 4
\]

**Proof of Theorem 8.** By Theorem 5, \(G_0(\text{*FTTM}_n^3)\) can be determined by the combination of \((x,y)\) where \(x \oplus y \in G_0(\text{*FTTM}_n^3), x \in G_0(\text{*FTTM}_{n-1}^3), y \in G_0(\text{*FTTM}_{n-2}^3)\), \(y \in C^m_{n-2} G_0(\text{*FTTM}_{n-2}^3)\). By Theorem 4, all \(x\) edges must be diagonal; hence, \(x \in G_0(\text{*FTTM}_{n-2}^3)\) or \(x \in H_{n-2}\), where \(H_{n-2} = \left\{ x \in G_d(\text{*FTTM}_{n-2}^3) \right\}\). From Theorem 3. Case \(i = 1\):

Let \(X_2 = \left\{ x \in G_0(\text{*FTTM}_{n-2}^3) \right\}\), \(X_3 = \left\{ x \in G_0(\text{*FTTM}_{n-2}^3) \right\}\), then for any \(x \in X_2\), then \(y \in B_{1,2}, B_{1,3}, B_{3,3}\) and also for any \(x \in X_3\), then \(y \in B_{1,2}, B_{1,3}, B_{3,3}\) by Corollary 1.

Thus, for \(x \in H_{n-2}\), there are \(3 |G_0(\text{*FTTM}_{n-2}^3)|\) combinations of tuple \((x,y)\).

If \(x \in H_{n-2}\), then \(A_{m_1}^1 \in x, m_1 = 1\) when \(x \in H_{n-2}\) and \(y \in B_{1,2}, B_{1,3}\) by Corollary 1. Thus, there are \(3 |H_{n-2}|\) combinations of tuple \((x,y)\). Hence, \(G_0(\text{*FTTM}_n^3) = 3|H_{n-2}|\).
3 \left| G_0 \left( \star \text{FTTM}_{n-2}^3 \right) \right| + 2|H_{n-2}|, \ n > 4. \text{ Using the same procedure as for } i = 1, \text{ the same result can be obtained for } i = 2, 3. \ □

Theorem 9.

\[ |G_0 \left( \star \text{FTTM}_{n}^3 \right)| = \begin{cases} |G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 3 \cdot 2^{n-2}, & \text{n is odd, } n > 3 \\ |G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 3 \cdot 2^{n}, & \text{n is even, } n > 4 \end{cases} \]  \hspace{1cm} (28)

where \( |G_0 \left( \star \text{FTTM}_{3}^3 \right)| = 6, |G_0 \left( \star \text{FTTM}_{4}^3 \right)| = 12. \)

Proof of Theorem 9. Using Theorem 6, \( |G_0 \left( \star \text{FTTM}_{n}^3 \right)| = \begin{cases} |G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 2 \left| G_{m-2} \left( \star \text{FTTM}_{n}^3 \right) \right| & \text{for } m = n-1 \\ |G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 2 \left| G_{m-2} \left( \star \text{FTTM}_{n}^3 \right) \right| & \text{for } m = n-2 \\ |G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 2 \left| G_{m-2} \left( \star \text{FTTM}_{n}^3 \right) \right| & \text{for } m = n-3 \end{cases} \)

\[ |G_0 \left( \star \text{FTTM}_{n}^3 \right)| = \begin{cases} |G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 2 \left| G_{m-2} \left( \star \text{FTTM}_{n}^3 \right) \right| & \text{for } m = n-1 \\ |G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 2 \left| G_{m-2} \left( \star \text{FTTM}_{n}^3 \right) \right| & \text{for } m = n-2 \\ |G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 2 \left| G_{m-2} \left( \star \text{FTTM}_{n}^3 \right) \right| & \text{for } m = n-3 \end{cases} \)  \hspace{1cm} (29)

Hence by Theorem 7,

\[ |G_0 \left( \star \text{FTTM}_{n}^3 \right)| = \begin{cases} |G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 3 \cdot 2^{n-2}, & \text{n is odd, } n > 3 \\ |G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 3 \cdot 2^{n}, & \text{n is even, } n > 4 \end{cases} \]  \hspace{1cm} (30)

such that \( |G_0 \left( \star \text{FTTM}_{3}^3 \right)| = 6, |G_0 \left( \star \text{FTTM}_{4}^3 \right)| = 12. \ □

Theorem 9 is another version of the earlier conjecture. A simple algebraic manipulation is needed to show their equivalence. We formally state and prove this as the final theorem.

Theorem 10.

\[ |G_0 \left( \text{FTTM}_{n}^3 \right)| = \begin{cases} 4|G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 12, & \text{n is even} \\ 4|G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 6, & \text{n is odd} \end{cases} \]  \hspace{1cm} (31)

where \( |G_0 \left( \star \text{FTTM}_{3}^3 \right)| = 6, |G_0 \left( \star \text{FTTM}_{4}^3 \right)| = 12. \)

Proof of Theorem 10. By Theorem 9,

\[ |G_0 \left( \star \text{FTTM}_{n}^3 \right)| = \begin{cases} 4|G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 12, & \text{n is even} \\ 4|G_0 \left( \star \text{FTTM}_{n-2}^3 \right)| + 6, & \text{n is odd} \end{cases} \]  \hspace{1cm} (32)

and \( |G_0 \left( \text{FTTM}_{3}^3 \right)| = 6, |G_0 \left( \text{FTTM}_{4}^3 \right)| = 12. \)
However, when \( n \) is odd,
\[
\begin{align*}
|G_0(FTTM_3^n)| &= 4 \cdot 6 + 6 \\
&= 4^1 \cdot 6 + 4^0 \cdot 6 \\
|G_0(FTTM_5^n)| &= 4(4 \cdot 6 + 6) + 6 \\
&= 4^2 \cdot 6 + 4^1 \cdot 6 + 4^0 \cdot 6 \\
|G_0(FTTM_7^n)| &= 4(4(4 \cdot 6 + 6) + 6) + 6 \\
&= 4^3 \cdot 6 + 4^2 \cdot 6 + 4^1 \cdot 6 + 4^0 \cdot 6 \\
|G_0(FTTM_9^n)| &= 4(4(4(4 \cdot 6 + 6) + 6) + 6) + 6 \\
&= 4^4 \cdot 6 + 4^3 \cdot 6 + 4^2 \cdot 6 + 4^1 \cdot 6 + 4^0 \cdot 6
\end{align*}
\] (33)
Thus, \( |G_0(FTTM_3^n)| = \sum_{k=0}^{n-3} 4^k \cdot 6 \).

Notice that
\[
|G_0(FTTM_3^n)| = \sum_{k=0}^{n-3} 4^k \cdot 6 \\
= 4^{n-3} \cdot 6 + \sum_{k=0}^{n-5} 4^k \cdot 6 \\
= 2^{n-3} \cdot 6 + |G_0(FTTM_3^{n-2})| \\
= 2^{n-2} \cdot 3 + |G_0(FTTM_3^{n-2})| \\
\] (34)

When \( n \) is even,
\[
\begin{align*}
|G_0(FTTM_6^n)| &= 4 \cdot 12 + 12 \\
&= 4^1 \cdot 12 + 4^0 \cdot 12 \\
|G_0(FTTM_8^n)| &= 4(4 \cdot 12 + 12) + 12 \\
&= 4^2 \cdot 12 + 4^1 \cdot 12 + 4^0 \cdot 12 \\
|G_0(FTTM_{10}^n)| &= 4(4(4 \cdot 12 + 12) + 12) + 12 \\
&= 4^3 \cdot 12 + 4^2 \cdot 12 + 4^1 \cdot 12 + 4^0 \cdot 12 \\
|G_0(FTTM_{12}^n)| &= 4(4(4(4 \cdot 12 + 12) + 12) + 12) + 12 \\
&= 4^4 \cdot 12 + 4^3 \cdot 12 + 4^2 \cdot 12 + 4^1 \cdot 12 + 4^0 \cdot 12
\end{align*}
\] (35)
Thus, \( |G_0(FTTM_3^n)| = \sum_{k=0}^{n-4} 4^k \cdot 12 \).

Notice that
\[
|G_0(FTTM_6^n)| = \sum_{k=0}^{n-4} 4^k \cdot 12 \\
= 4^{n-4} \cdot 12 + \sum_{k=0}^{n-6} 4^k \cdot 12 \\
= 2^{n-2} \cdot 3 + \sum_{k=0}^{n-6} 4^k \cdot 12 \\
= 2^{n-2} \cdot 3 + |G_0(FTTM_3^{n-2})| \\
\] (36)
It shows that the equation in Theorem 9 is exactly the statement of the conjecture. In other words, the conjecture is proven by construction. \(\square\)

The whole process of proving Conjecture 1 is summarized below in Figure 9.
Notice that, \[
|G_{\omega}(FTTM_{3})| = 4 \cdot 12
\]
It shows that the equation in Theorem 9 is exactly the statement of the conjecture. In other words, the conjecture is proven by construction. □

The whole process of proving Conjecture 1 is summarized below in Figure 9.

Figure 9. Outline of proving Conjecture 1 by construction.

6. Conclusions

The developed grid-based method of proof is new; some definitions and properties were introduced, whereas others were investigated along the way. The originality and advantages of this method can be summarized in the point forms below.

- It provides a different perspective to the structure of FTTM. Instead of a polygon representation for each version of FTTM, a straight line is now used. The components of FTTMn are arranged on a horizontal line of vertices and the lines represent the homeomorphisms between the components of FTTMn.
- A vertical line is added to represent a homeomorphism between two components of different versions; hence, a grid is created.
- It is represented in two dimensions; therefore, it reduces the complexity of the structure.
- The process of adding a new component is easier than in a sequence of polygon.
- It can take any number of components by adding the number of vertices at the end of the grid.
- The homeomorphism between two components of the same version is presented as a horizontal edge, whereas the homeomorphism between two components of two different versions is represented by a diagonal edge (see Figure 8).
- This grid-based technique offers an edge in proving the conjecture; in particular, it enables one to visualize a given problem in a 2-dimensional space.
- Finally, the conjecture that spells the number of the generated FTTM graph of pseudo degree zero with respect to n number of components and k number of versions is proven analytically for the first time using this method.

However, the lengthy computing time for simulation needs to be resolved for larger k and n, accordingly. This may be overcome by employing parallel computing, and the grid-based technique can be very handy for such enumerative combinatorics problems in the near future.

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**Abbreviations**

The following abbreviations are used in this manuscript.

| Abbreviation | Description                          |
|--------------|--------------------------------------|
| BM           | Base magnetic plane                  |
| EEG          | Electroencephalography               |
| FM           | Fuzzy magnetic field                 |
| FTTM         | Fuzzy topological topographic mapping|
| FTTM<sub>n</sub> | Sequence of FTTM              |
| MC           | Magnetic plane                       |
| MEG          | Magnetoencephalography               |
| TM           | Topographic magnetic field           |
| ∗FTTM<sub>n</sub> | Sequence of k versions of FTTM<sub>n</sub> |
|<sub>G<sub>0</sub>(∗FTTM<sub>k</sub>)</sub> | Set of elements generated by ∗FTTM<sub>k</sub> that have pseudo degree zero |
|<sub>G<sub>0</sub>(FTTM<sup>k</sup><sub>n</sub>)</sub> | Set of elements generated by ∗FTTM<sup>k</sup><sub>n</sub> that have pseudo degree zero |

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