Amenability of groups and $G$-sets

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Abstract. This text surveys classical and recent results in the field of amenability of groups, from a combinatorial standpoint. It has served as the support of courses at the University of Göttingen and the École Normale Supérieure. The goals of the text are (1) to be as self-contained as possible, so as to serve as a good introduction for newcomers to the field; (2) to stress the use of combinatorial tools, in collaboration with functional analysis, probability etc., with discrete groups in focus; (3) to consider from the beginning the more general notion of amenable actions; (4) to describe recent classes of examples, and in particular groups acting on Cantor sets and topological full groups.

1 Introduction

In 1929, John von Neumann introduced in [103] the notion of amenability of $G$-spaces. Fundamentally, it considers the following property of a group $G$ acting on a set $X$: The right $G$-set $X$ is amenable if there exists a $G$-invariant mean on the power set of $X$, namely a function $m: \{\text{subsets of } X\} \to [0,1]$ satisfying $m(A \sqcup B) = m(A) + m(B)$ and $m(X) = 1$ and $m(Ag) = m(A)$ for all $A, B \subseteq X$ and all $g \in G$.

Amenability may be thought of as a finiteness condition, since non-empty finite $G$-sets are amenable with $m(A) = \#A/\#X$; it may also be thought of as a fixed-point property: on a general $G$-set there exists a $G$-invariant mean; on a compact $G$-set there exists a $G$-invariant measure, and on a convex compact $G$-set there exists a $G$-fixed point, see §6; on a $G$-measure space there exists a $G$-invariant measurable family of means on the orbits, see §6.2.

Amenability may be defined for other objects such as graphs and random walks on sets. If $X$ is a $G$-set and $G$ is finitely generated, then $X$ naturally has the structure of a graph, with one edge from $x$ to $xs$ for every $x \in X, s \in S$. Amenability means, in the context of graphs, that there are finite subsets of $X$ with arbitrarily small boundary with respect to their size. In terms of random walks, it means that there are finite subsets with arbitrarily small connectivity between the set and its complement, and equivalently that the return probability of the random walk decreases subexponentially in time; see §8.

The definition may also be modified in another direction: rather than considering group actions, we may consider equivalence relations, or more generally groupoids. The case we concentrate on is an equivalence relation with countable leaves on a standard measure space. The orbits of a countable group acting measurably naturally give rise to such an equivalence relation. This point of view is actually very valuable:

\[\text{\footnotesize \cite{arXiv:1705.04091v1 [math.GR]} 11 May 2017}}\]
quite different groups (e.g. one with free subgroups, one without) may generate the same equivalence relation; see §6.2.

One of the virtues of the notion of amenability of $G$-sets is that there is a wealth of equivalent definitions; depending on context, one definition may be easier than another to check, and another may be more useful. In summary, the following will be shown, in the text, to be equivalent for a $G$-set $X$:

- $X$ is amenable; i.e. there is a $G$-invariant mean on subsets of $X$;
- There is a $G$-invariant normalized positive functional in $\ell^\infty(X)^*$, see Corollary 2.25;
- For every bounded functions $h_i$ on $X$ and $g_i \in G$ the function $\sum_i (1 - g_i)$ is non-negative somewhere on $X$, see Theorem 2.29;
- For every finite subset $S \subseteq G$ and every $\varepsilon > 0$ there exists a finite subset $F \subseteq X$ with $\#(FS \setminus F) < \varepsilon \#F$, see Theorem 3.23(5);
- For every finite subset $S \subseteq G$, every $\varepsilon > 0$ and every $p \in [1, \infty)$ there exists a positive function $\phi \in \ell^p(X)$ with $\|\phi s - \phi\| < \varepsilon \|\phi\|$ for all $s \in S$, see Theorem 3.23(4);
- There does not exist a “paradoxical decomposition” of $X$, namely $X = Z_1 \sqcup \cdots \sqcup Z_m = Z_{m+1} \sqcup \cdots \sqcup Z_{m+n} = Z_1 g_1 \sqcup \cdots \sqcup Z_{m+n} g_{m+n}$ for some $Z_i \subseteq X$ and $g_i \in G$, see Theorem 5.14(2);
- There does not exist a map $\phi : X \rightarrow Y$ and a finite subset $S \subseteq G$ with $\#\phi^{-1}(x) = 2$ and $\phi(x) \in xS$ for all $x \in X$, see Theorem 5.14(4);
- There does not exist a free action of a non-amenable group $H$ on $X$ with the property that for every $h \in H$ there is a finite subset $S \subseteq G$ with $xh \in xS$ for all $x \in X$, see Theorem 5.15(3);
- Every convex compact set equipped with a $G$-equivariant map from $X$ admits a fixed point, see Theorem 6.4;
- Every compact set equipped with a $G$-equivariant map from $X$ admits an invariant measure, see Theorem 6.7;
- The isoperimetric constant (Definition 8.2) of every non-degenerate $G$-driven random walk on $X$ vanishes, see Theorem 8.4(2);
- The spectral radius (Definition 8.2) of every non-degenerate $G$-driven random walk on $X$ is equal to 1, see Theorem 8.4(3).

Amenability has been given particular attention for groups themselves, seen as $G$-sets under right multiplication; see the next section. We stress that many results that exclusively concern groups (e.g., the recent proofs that topological full groups are amenable) are actually proven using amenable $G$-sets in a fundamental manner. The reason is that a group is amenable if and only if it acts on an amenable $G$-set with amenable point stabilizers, see Proposition 2.26.

Quotients of amenable $G$-sets are again amenable; but sub-$G$-sets of amenable $G$-sets need not be amenable. A stronger notion will be developed in §9 that of extensively amenable $G$-sets. It has the fundamental property that, if $\pi : X \rightarrow Y$ is a $G$-equivariant map between $G$-sets, then $X$ is extensively amenable if and only if both $Y$ and all $\pi^{-1}(y)$ are extensively amenable, the latter for the action of the stabilizer $G_y$. 
We detail slightly Reiter's characterization of amenability given above: the space $\ell^1(G)$ of summable functions on $G$ is a Banach algebra under convolution, and $\ell^1(X)$ is a Banach $\ell^1(G)$-module. We denote by $\mathfrak{I}(\ell^1 G)$ and $\mathfrak{I}(\ell^1 X)$ respectively the ideal and submodule of functions with 0 sum, and by $\ell^1_+(G)$ and $\ell^1_+(X)$ the cones of positive elements.

Then $X_G$ is amenable if and only if for every $\varepsilon > 0$ and every $g \in \mathfrak{I}(\ell^1 G)$ there exists $f \in \ell^1_+(X)$ with $\|fg\| < \varepsilon \|f\|$, see Proposition 3.25.

The quantifiers may be exchanged; we call $X_G$ laminable if for every $\varepsilon > 0$ and every $f \in \mathfrak{I}(\ell^1 X)$ there exists $g \in \ell^1_+(G)$ with $\|fg\| < \varepsilon \|g\|$, see Theorem 8.20. It has the consequence that there exists a measure $\mu$ on $G$ such that every $\mu$-harmonic function on $X$ is constant.

In case $X = G_G$, these definitions are equivalent, but for $G$-sets the properties of being amenable or Liouville are in general position.

1.1 Amenability of groups

John von Neumann's purpose, in introducing amenability of $G$-spaces, was to understand better the group-theoretical nature of the Hausdorff-Banach-Tarski paradox. This paradox, due to Banach and Tarski [4] and based on Hausdorff's work [62], states that a solid ball can be decomposed into five pieces, which when appropriately rotated and translated can be reassembled in two balls of same size as the
original one. It could have been felt as a death blow to measure theory; it is now resolved by saying that the pieces are not measurable.

A group is called amenable if all non-empty $G$-sets are amenable; and it suffices to check that the regular $G$-set $G$ is amenable, see Corollary 2.11.

Using the “paradoxical decompositions” criterion, it is easy to see that the free group $F_2 = \langle a, b \rangle$ is not amenable: we exhibit a partition $F_2 = G_1 \sqcup \cdots \sqcup G_m \sqcup H_1 \sqcup \cdots \sqcup H_n$ and elements $g_1, \ldots, g_m, h_1, \ldots, h_n$ with $F_2 = G_1 g_1 \sqcup \cdots \sqcup G_m g_m = H_1 h_1 \sqcup \cdots \sqcup H_n h_n$ as follows. Set

- $G_1 = \{\text{words whose reduced form ends in } a\} \cup \{1, a^{-1}, a^{-2}, \ldots\}$,
- $G_2 = \{\text{words whose reduced form ends in } a^{-1}\} \setminus \{a^{-1}, a^{-2}, \ldots\}$,
- $H_1 = \{\text{words whose reduced form ends in } b\}$,
- $H_2 = \{\text{words whose reduced form ends in } b^{-1}\}$;

then $F_2 = G_1 \sqcup G_2 \sqcup H_1 \sqcup H_2 = G_1 \sqcup G_2 a = H_1 \sqcup H_2 b$.

The group of rotations $SO_3(\mathbb{R})$ contains a free subgroup $F_2$, and even one that acts freely on the sphere $S^2$; so its orbits are all isomorphic to $F_2$. Choose a transversal: a subset $T \subset S^2$ intersecting every $F_2$-orbit in exactly one point. Consider then the sphere partition $S^2 = TG_1 \sqcup TG_2 \sqcup TH_1 \sqcup TH_2 = TG_1 \sqcup TG_1 a = TH_1 \sqcup TH_2 b$; this is the basis for the paradoxical Hausdorff-Banach-Tarski decomposition.

John von Neumann also noted that the class of amenable groups is closed under the following operations ($*$): subgroups, quotients, extensions, and directed unions. It contains all finite and abelian groups. More generally, a criterion due to Følner, Theorem 3.23(5), shows that all groups in which every finite subset generates a group of subexponential word growth is amenable. One may therefore define the following classes:

- $EG$ = the smallest class containing finite and abelian groups and closed under ($*$),
- $SG$ = the smallest class containing groups of subexponential growth and closed under ($*$),
- $AG$ = the class of amenable groups,
- $NF$ = the class of groups with no free subgroups;

and concrete examples show that all inclusions

$$EG \subsetneq SG \subsetneq AG \subsetneq NF$$

are strict: the “Grigorchuk group” $G$ for the first inclusion, see [4.2], the group of “bounded tree automorphisms” for the second inclusion, see [7.2] and the “Frankenstein group” for the last one, see [1.3].

This text puts a strong emphasis on examples; they are essential to obtain a (however coarse) picture of the universe of discrete groups, see Figure 1. A fairly general framework contains a large number of important constructions: groups acting on

\[ ^2 \text{Namely, in which the number of elements expressible as a product of at most } n \text{ generators grows subexponentially in } n. \]
Cantor sets. On the one hand, if we choose \( X = \mathcal{A}^\mathbb{N} \) as model for the Cantor set, we have examples of groups defined by automatic transformations of \( X \), namely by actions of invertible transducers. On the other hand, we may fix a “manageable” group \( H \) acting on \( X \), and consider the group of self-homeomorphisms of \( X \) that are piecewise \( H \).

Examples of the first kind may be constructed via their recursively-defined actions on \( X \). The Grigorchuk group \( G \) is the group acting on \{0,1\}^\mathbb{N} and generated by four elements \( a, b, c, d \) defined by

\[
\begin{align*}
  a(x_0x_1\ldots) &= (1-x_0)x_1\ldots, & b(x_0x_1\ldots) &= \begin{cases} x_0a(x_1\ldots) & \text{if } x_0 = 0, \\ x_0c(x_1\ldots) & \text{if } x_0 = 1, \end{cases} \\
  c(x_0x_1\ldots) &= \begin{cases} x_0a(x_1\ldots) & \text{if } x_0 = 0, \\ x_0d(x_1\ldots) & \text{if } x_0 = 1, \end{cases} & d(x_0x_1\ldots) &= \begin{cases} x_0x_1\ldots & \text{if } x_0 = 0, \\ x_0b(x_1\ldots) & \text{if } x_0 = 1. \end{cases}
\end{align*}
\]

The Grigorchuk group gained prominence in group theory for being a finitely generated infinite torsion group, and for having intermediate word-growth between polynomial and exponential, see [43]. An amenable group that does not belong to the class \( SG \) is the “Basilica group” \( B \), generated by two elements \( a, b \) acting recursively on \{0,1\}^\mathbb{N} by

\[
\begin{align*}
  a(x_1x_2\ldots) &= \begin{cases} 1x_2\ldots & \text{if } x_1 = 0, \\ 0b(x_2\ldots) & \text{if } x_1 = 1, \end{cases} & b(x_1x_2\ldots) &= \begin{cases} 0x_2\ldots & \text{if } x_1 = 0, \\ 1a(x_2\ldots) & \text{if } x_1 = 1. \end{cases}
\end{align*}
\]

The Basilica group is a subgroup of the group of bounded tree automorphisms, whose amenability will be proven in [7.2].

These groups are residually finite: the action on \{0,1\}^\mathbb{N} is the limit of actions on the finite sets \{0,1\}^n as \( n \to \infty \), so that the groups may be arbitrarily well approximated by their finite quotients. More conceptually, the actions of \( G \) and \( B \) on \{0,1\}^\mathbb{N} induce actions on the clopens of \{0,1\}^\infty, and every clopen has a finite orbit, giving rise to a finite quotient acting by permutation on the orbit.

Examples of the second kind include the “Frankenstein” group mentioned above, which is a non- amenable group acting on the circle by piecewise projective transformations, and “topological full groups” of a minimal action of \( H = \mathbb{Z} \) on a Cantor set; for example, let \( \sigma : 0 \mapsto 01, 1 \mapsto 0 \) be the Fibonacci substitution, and consider \( H = \langle S \rangle \) the two-sided shift on the subset \( X = \Sigma^n(\sigma^{-1}(0)) \subset \{0,1\}^\mathbb{Z} \). Let \( G \) be the group of piecewise-\( H \) homeomorphisms of \( X \). Then \( G' \) is an example of a simple, infinite, finitely generated, amenable group.

These groups’ actions on the Cantor set exhibit behaviours at the exact opposite of \( G \) and \( B \): the actions are expansive: the orbit of a clopen may be used to separate points in \( X \). Topological full groups shall be used to produce examples of finitely generated, infinite, amenable simple groups.

Finally, we consider in [10] the adaptation of amenability to a linear setting: on the one hand, a natural notion of amenability of \( \mathcal{A} \)-modules for an associative algebra \( \mathcal{A} \); and, on the other hand, a characterization of amenability by cellular automata.
1.2 Why this text?

After John von Neumann’s initial work in the late 1920’s, amenability of groups has developed at great speed in the 1960’s, and then remained mostly dormant till the late 2000’s, when a variety of new techniques and examples appeared. It seems now to be a good time to reread and rewrite the fundamentals of the field with these developments in mind.

I have done my best to include all the material I found digestible, and to express it in the “best” generality, namely the maximum generality that does not come at the price of arcane definitions or notation. Whenever possible, I included complete proofs of the results, so that the text may be used for a course as well as for a reference.

I have also striven to follow von Neumann’s use of $G$-sets rather than groups; it seems to me that clarity is gained by separating the set $X$ (with a right $G$-action) from the group $G$.

I have also, consciously, avoided any mention of amenability for topological groups. This notion is well developed for second-countable locally compact groups, see e.g. [14,15], so I should justify its exclusion. I have felt that either the results stated for discrete groups extend more-or-less obviously to topological groups (and then there is no point in loading the notation with topology), or they don’t extend, and then the additional effort would be a distraction from the main topic.

I have also devoted a fairly large part of the text to examples; and, in particular, to groups defined by their action on a Cantor set, see the previous section. I have included exercises, with ranking *=just check the definitions, **=requires some thought, ***=probably very difficult. Problems are like ***-exercises, but are questions rather than statements.

I have consulted a large number of sources, and did my best to attribute to their original authors all results and fragments of proof that I have used. Apart from articles, these sources include notes from a course given by Nicolas Monod at EPFL in 2007 and from a course given by Anna Erschler and myself at ENS in 2016, and books in preparation by Kate Juschenko and Gábor Pete. I have also made abundant use of [52], [24], and [14, Chapter 5 and Appendix G].

I benefited from useful conversations with and remarks from Yves de Cornulier, Anna Erschler, Vadim Kaimanovich, Peter Kropholler, Yash Lodha, Nicolas Matte Bon, Nicolas Monod, Volodya Nekrashevych and Romain Tessera. I thank all of them heartily.

1.3 Why not this text?

For lack of space, I have left out much material that I wanted to include. First and foremost, I have not touched at all at the boundary initiated by Furstenberg; the “size” of its boundary is an indication of the non-amenability of a $G$-set.
I have also left out much material related to quantitative invariants — drift, entropy, on- and off-diagonal probabilities of return of random walks, and their relation to other invariants such as growth and best-case distortion of embeddings in convex metric spaces such as Hilbert space. This topic is evolving rapidly, and I fear that my rendition would be immediately obsolete.

I would have preferred to write §6.2 in terms of groupoids, especially since groupoids appear anyways in §9.2. In the end, I have opted for directness at the cost of generality.

Finally, I put as much effort as I could into including applications and examples in the text; but I omitted the most important ones, e.g. Margulis’s work on lattices in semisimple Lie groups and percolation on graphs, feeling they would take us too far adrift.

1.4 Notation

We mainly use standard mathematical notation. We try to keep Latin capitals for sets, Latin lowercase for elements, and Greek for maps. A subset inclusion \( A \subset B \) is strict, while \( A \subseteq B \) means that \( A \) could equal \( B \). The difference and symmetric difference of two sets \( A, B \) are respectively written \( A \setminus B \) and \( A \triangle B \). We denote by \( \mathcal{P}(X) \) the power set of \( X \), and by \( \mathcal{P}_f(X) \) the collection of finite subsets of \( X \). Since it appears quite often in the context of amenability, we use \( A \preccurlyeq B \) (“compactly contained”) to mean that \( A \) is a finite subset of \( B \).

We denote by \( A^X \) the set of maps \( X \to A \), and by \( A^{(X)} \) or by \( \prod_X A \) the restricted product of \( A \), namely the set of finitely-supported maps \( X \to A \). Under the operation of symmetric difference, \( \mathcal{P}(X) \) and \( \mathcal{P}_f(X) \) are respectively isomorphic to \((\mathbb{Z}/2)^X\) and \((\mathbb{Z}/2)^{(X)}\).

We denote by \( \text{Sym}(X) \) the group of finitely-supported permutations of a set \( X \), and abbreviate \( \text{Sym}(n) = \text{Sym}(\{1, \ldots, n\}) \). Groups and permutations always act on the right, and we denote by \( X \twoheadrightarrow G \) a set \( X \) equipped with a right \( G \)-action.

We denote by \( 1_A \) the characteristic function of a set \( A \), and also by \( 1_{\mathcal{P}} \) the function that takes value 1 when property \( \mathcal{P} \) holds and 0 otherwise.

Finally, we write \( x \downarrow S \) for various kinds of restriction of the object \( x \) to a set \( S \).
2 Means and amenability

Definition 2.1. Let $X$ be a set. A mean on $X$ is a function $m : \mathcal{P}(X) \to [0, 1]$ satisfying

$$m(X) = 1, \\
m(A \cup B) = m(A) + m(B)$$

for all disjoint $A, B \subseteq X$.

(This last property is often called finite additivity, as opposed to the $\sigma$-additivity property enjoyed by measures, in which countable unions are allowed).

It easily follows from the definition that $m(\emptyset) = 0$; that $m(A) \leq m(B)$ if $A \subseteq B$; and that $m(A_1 \cup \cdots \cup A_k) = m(A_1) + \cdots + m(A_k)$ for pairwise disjoint $A_1, \ldots, A_k$.

We denote by $\mathcal{M}(X)$ the set of means on $X$, with the usual topology on a set of functions; namely, a sequence $m_n \in \mathcal{M}(X)$ converges to $m$ precisely if for every $\varepsilon > 0$ and every finite collection $A_1, \ldots, A_k \subseteq X$ we have $|m_n(A_i) - m(A_i)| < \varepsilon$ for all $i \in \{1, \ldots, k\}$ and all $n$ large enough.

Observe that $\mathcal{M}$ is a covariant functor: if $f : X \to Y$, then we have a natural map $f_* : \mathcal{M}(X) \to \mathcal{M}(Y)$ given by

$$f_*(m) : B \mapsto m(f^{-1}(B))$$

for all $B \subseteq Y$.

In particular, if a group $G$ acts on $X$, then it also acts on $\mathcal{M}(X)$. For a right action $\cdot : X \times G \to X$, we have a right action on $\mathcal{M}(X)$ given by $(m \cdot g)(A) = m(A \cdot g^{-1})$ for all $A \subseteq X$.

Definition 2.2 (von Neumann [103]). Let $G$ be a group and let $X \hookrightarrow G$ be a set on which $G$ acts. The $G$-set $X$ is amenable if there is a $G$-fixed element in $\mathcal{M}(X)$.

A group $G$ is amenable if all non-empty right $G$-sets are amenable.

In other words, the $G$-set $X$ is amenable if $\mathcal{M}(X)^G \neq \emptyset$, namely if there exists a mean $m$ on $X$ such that $m(Ag) = m(A)$ for all $g \in G$ and all $A \subseteq X$.

2.1 First examples

Proposition 2.3. Every finite, non-empty $G$-set is amenable. More generally, every $G$-set with a finite orbit is amenable.

Note that, trivially, the empty set is never amenable since a mean requires $m(\emptyset) = 0 \neq 1 = m(X)$.

Proof. Let $xG$ be a finite $G$-orbit in the $G$-set $X$. Then $m(A) := \#(A \cap xG)/\#(xG)$ defines a $G$-invariant mean on $X$. \qed

3 By $\mathcal{P}(X)$ we denote the power set of $X$, namely the set of its subsets.
In particular, finite groups are amenable. We shall now see that, although amenable groups abound, extra logical tools are necessary to provide more examples.

**Proposition 2.4.** The infinite cyclic group \( \mathbb{Z} \) is amenable.

**False proof.** Define \( m \in \mathcal{M}(\mathbb{Z}) \) by

\[
m(A) = \lim_{n \to \infty} \frac{\#(A \cap \{1, 2, \ldots, n\})}{n}.
\]

It is clear that \( m(A) \) is contained in \([0, 1]\), and the axioms of a mean are likewise easy to check. Finally, if \( g \) denote the positive generator of \( \mathbb{Z} \),

\[
|m(Ag) - m(A)| = \lim_{n \to \infty} \frac{|\#(Ag \cap \{1, 2, \ldots, n\}) - \#(A \cap \{1, 2, \ldots, n\})|}{n} = \lim_{n \to \infty} \frac{|\#(A \cap \{0, 1, \ldots, n-1\}) - \#(A \cap \{1, 2, \ldots, n\})|}{n} = \lim_{n \to \infty} \frac{\#(A \cap \{0, n\})}{n} = 0.
\]

The problem in this proof, of course, is that the limit need not exist. Consider typically

\[
A = \bigcup_{k \geq 0} \{2^k + 1, 2^k + 2, \ldots, 2^k + 2^{k-1}\} = \{2, 3, 5, 6, 9, 10, 11, 12, 17, \ldots\}.
\]

The arguments of the “limit” above oscillate between 2/3 and 1/2. To correct this proof, we make use of a logical axiom:

**Definition 2.5.** Let \( X \) be a set. A filter is a family \( \mathcal{F} \) of subsets of \( X \), such that

1. \( X \in \mathcal{F} \) and \( \emptyset \notin \mathcal{F} \);
2. if \( A \in \mathcal{F} \) and \( B \supseteq A \) then \( B \in \mathcal{F} \);
3. if \( A, B \in \mathcal{F} \) then \( A \cap B \in \mathcal{F} \).

An ultrafilter is a maximal filter (under inclusion). It therefore satisfies the extra condition

4. if \( A \subseteq X \), then either \( A \in \mathcal{F} \) or \( X \setminus A \in \mathcal{F} \).

For every \( x \in X \), there is a principal ultrafilter \( \mathcal{F}_x = \{ A \subseteq X \mid x \in A \} \).

The set of ultrafilters on \( X \) is called its Stone-Čech compactification and is written \( \beta X \). Its topology is defined by declaring open, for every \( Y \subseteq X \), the collection \( \{ \mathcal{F} \in \beta X \mid Y \in \mathcal{F} \} \cong \beta Y \).

Elements of a filter are thought of as “large”. As a standard example, consider the “cofinite filter” on \( \mathbb{N} \),

\[
\mathcal{F}_c = \{ A \subseteq \mathbb{N} \mid \mathbb{N} \setminus A \text{ is finite} \}.
\]
Using this notion, the standard definition of convergence in analysis can be phrased as follows: “a sequence \((x_n)\) converges to \(x\) if for every \(\varepsilon > 0\) we have \(\{n \in \mathbb{N} \mid \varepsilon > |x_n - x|\} \in \mathcal{F}_c\).” More generally, for a filter \(\mathcal{F}\) on \(\mathbb{N}\) we define convergence with respect to \(\mathcal{F}\) by

\[
\lim_{\mathcal{F}} x_n = x \quad \text{if and only if} \quad \forall \varepsilon > 0 : \{n \in \mathbb{N} \mid \varepsilon > |x_n - x|\} \in \mathcal{F}.
\]

A standard axiom asserts the existence of non-principal ultrafilters on every infinite set. In fact, Zorn’s lemma implies that the cofinite filter \(\mathcal{F}_c\) is contained in an ultrafilter \(\mathcal{F}\). Using this axiom, \(\beta X\) is compact, and in fact is universal in the sense that every map \(X \to K\) with \(K\) compact Hausdorff factors uniquely through \(\beta X\). We state this universal property in the following useful form sometimes called “stone duality’’:

**Lemma 2.6.** Let \(X\) be a set. The map \(f \mapsto (\mathcal{F} \mapsto \lim_{\mathcal{F}} f)\) is an isometry between the spaces \(\ell^\infty(X)\) of bounded functions on \(X\) and \(C(\beta X)\) of continuous functions on \(\beta X\) with supremum norm.

In particular, if \(\mathcal{F}\) is an ultrafilter on \(\mathbb{N}\) then every bounded sequence converges with respect to \(\mathcal{F}\).

**Proof.** We first prove that if \(f : X \to \mathbb{C}\) is bounded and \(\mathcal{F}\) is an ultrafilter then it has a well-defined limit with respect to \(\mathcal{F}\). Assume \(f(x) \in [L_0, U_0]\) for all \(x \in X\). For \(i = 0, 1, \ldots\) repeat the following.

1. Set \(M_i = (L_i + U_i)/2\).
2. Define \(A_i = \{x \in X \mid f(x) \in [L_i, M_i]\}\) and \(B_i = \{x \in X \mid f(x) \in [M_i, U_i]\}\).
3. By induction, \(A_i \cup B_i \in \mathcal{F}\); so either \(A_i \in \mathcal{F}\) or \(B_i \in \mathcal{F}\). In the former case, set \((L_{i+1}, U_{i+1}) = (L_i, M_i)\) while in the latter case set \((L_{i+1}, U_{i+1}) = (M_i, U_i)\).

Then \((L_i)\) is an increasing sequence, \((U_i)\) is a decreasing sequence, and they both have the same limit; call that limit \(f(\mathcal{F})\).

We have extended \(f\) to \(\beta X\). Let us show that this extension is continuous at every \(\mathcal{F}\): keeping the notation from the previous paragraph, for every \(\varepsilon > 0\) there is some \(i\) with \(U_i - L_i < \varepsilon\); so \(\{x \in X \mid |f(x) - f(\mathcal{F})| < \varepsilon\} \supseteq A_i \cup B_i \in \mathcal{F}\) and therefore \(f(x) \to f(\mathcal{F})\) when \(x \to \mathcal{F}\).

Finally the inverse map \(C(\beta X) \to \ell^\infty(X)\) is simply given by restriction to the discrete subspace \(X \subseteq \beta X\). \(\square\)

**Exercise 2.7 (★).** Prove that the Stone-Čech compactification \(\beta X\) is homeomorphic to the set of continuous algebra homomorphisms \(\ell^\infty(X) \to \mathbb{C}\), with the induced topology of \(\ell^\infty(X)^*\).

**Exercise 2.8 (★).** Let \(\mathcal{F}\) be an ultrafilter on \(\mathbb{N}\). Prove \(\lim_{\mathcal{F}} (x_n + y_n) = \lim_{\mathcal{F}} x_n + \lim_{\mathcal{F}} y_n\) when these last two limits exist.

Using a non-principal ultrafilter \(\mathcal{F}\) on \(\mathbb{N}\), we may correct the “proof” that \(\mathbb{Z}\) is amenable, by replacing ‘lim’ by ‘\(\lim_{\mathcal{F}}\)’; but in some sense we have done nothing except shuffling axioms around. Indeed, an ultrafilter \(\mathcal{F}\) on \(X\) is precisely the same
thing as a \{0, 1\}-valued mean on \(X\): given an ultrafilter \(\mathcal{F}\), we define a mean \(m\) on \(X\) by
\[
m(A) = \begin{cases} 
0 & \text{if } A \not\in \mathcal{F}, \\
1 & \text{if } A \in \mathcal{F},
\end{cases}
\]
and given a mean \(m\) taking \{0, 1\} values we define a filter \(\mathcal{F} = \{A \subseteq X | m(A) = 1\}\); so the construction of complicated means is as hard as the construction of complicated filters.

**Proposition 2.9.** The free group \(F_k\) is not amenable if \(k \geq 2\).

**Proof.** We reason by contradiction, assuming that the regular right \(F_k\)-set \(F_k \rightharpoonup F_k\) is amenable. Assume that there were an invariant mean \(m: \mathcal{P}(F_k) \to [0, 1]\). In \(F_k = \langle x_1, \ldots, x_k \rangle\), let \(A\) denote those elements whose reduced form ends by a non-trivial (positive or negative) power of \(x_1\). Then clearly \(F_k = A \cup Ax_1\), so
\[
1 = m(F_k) \leq m(A) + m(Ax_1) = 2m(A).
\]
On the other hand, \(F_k \supseteq Ax_2^{-1} \cup A \cup Ax_2\), so
\[
1 = m(F_k) \geq m(Ax_2^{-1}) + m(A) + m(Ax_2) = 3m(A).
\]
These statements imply \(1/2 \leq m(A) \leq 1/3\), a contradiction. \(\square\)

### 2.2 Elementary properties

**Proposition 2.10.** Let \(G, H\) be groups, let \(X \rightharpoonup G\) and \(Y \rightharpoonup H\) be respectively a \(G\)-set and an \(H\)-set, let \(\phi: G \to H\) be a surjective homomorphism, and let \(f: X \to Y\) be an equivariant map, namely satisfying \(f(xg) = f(x)\phi(g)\) for all \(x \in X, g \in G\). If \(X\) is amenable, then \(Y\) is amenable.

**Proof.** If \(\mathcal{M}(X)^G \neq \emptyset\), then \(f_* (\mathcal{M}(X)^G) = \mathcal{M}(X)^{\phi(G)} \subseteq \mathcal{M}(Y)^H\) so \(\mathcal{M}(Y)^H \neq \emptyset\). \(\square\)

**Corollary 2.11.** (Corollary 3.2). Let \(G\) be a group. Then \(G\) is amenable if and only if the right \(G\)-set \(G_G\) is amenable.

**Proof.** Assume the right \(G\)-set \(G \rightharpoonup G\) is amenable. For every non-empty \(G\)-set \(X\), choose \(x \in X\); then \(g \mapsto xg\) is a \(G\)-equivariant map \(G \to X\), so \(X\) is amenable by Proposition 2.10. The converse is obvious. \(\square\)

Thus amenability of a group is equivalent to amenability of the right-regular action, and also to amenability of all actions. We give another characterization:

**Proposition 2.12.** Let \(G\) be a group. Then the following are equivalent:

1. \(G\) is amenable;

2. every non-empty G-set is amenable;
3. G admits an amenable free action.

Proof. In view of the previous corollary, it suffices to prove (3) \(\Rightarrow\) (1). Let \(X\) be a free G-set, and choose a G-isomorphism \(X \cong T \times G\). Let \(m: \mathcal{P}(X) \to [0,1]\) be a G-invariant mean. Define a mean \(m'\) on G by \(m'(A) = m(T \times A)\), and check that \(m'\) is G-invariant. \(\square\)

Exercise 2.13 (*). Let \(X, Y\) be G-sets. Then

1. \(X \sqcup Y\) is amenable if and only if \(X\) or \(Y\) is amenable;
2. \(X \times Y\) is amenable if and only if \(X\) and \(Y\) are amenable.

Proposition 2.10 says that quotients of amenable G-sets are amenable. Note however that subsets of amenable G-sets need not be amenable; the empty set being the extreme example. See [9] for a notion of amenability better suited to subsets and extensions of G-sets.

Definition 2.14 (Wreath product). We introduce a construction of groups that serve as important examples. Let \(A, G\) be groups and let \(X\) be a G-set. Their (restricted) wreath product is

\[ A \wr_X G := A^X \rtimes G, \]

the semidirect product of the group of finitely-supported maps \(X \to A\) with \(G\), under the action of \(G\) at the source. Elements of \(A \wr_X G\) may be written as \((f, g)\) with \(f: X \to A\) and \(g \in G\); they multiply by \((f, g) \cdot (f', g') = (f' \cdot (f'g^{-1}), gg')\) with \((f'g^{-1})(x) = f'(xg)\).

In case \(G\) acts faithfully on \(X\), elements of \(A \wr_X G\) may be thought of as “decorated permutations”: permutations, say \(\sigma\) represented by a diagram with vertex set \(X\) and an arrow from \(x\) to \(\sigma(x)\), and with a label in \(A\) on each arrow in such a manner that only finitely many labels are non-trivial. Decorated permutations are composed by concatenating their arrows and multiplying their labels.

The wreath product is associative, in the sense that if \(A, G, H\) are groups, \(X\) is a \(G\)-set and \(Y\) is an \(H\)-set, then \(G \wr_Y H\) naturally acts on \(X \times Y\) and \(A \wr_{X \times Y} (G \wr_Y H) = (A \wr_X G) \wr_Y H\).

On the other hand, for groups \(A, G\) we write \(\mathcal{A} \wr G\) for the wreath product \(A \wr G\) with regular right action of \(G\) on itself, and that operation is not associative.

Definition 2.15 (Tree automorphisms). For a finite set \(\mathcal{A}\), consider the set \(X := \mathcal{A}^*\) of words over \(\mathcal{A}\). This set is naturally the vertex set of a rooted tree \(\mathcal{T}\); the root is the empty word, and there is an edge between \(x_1 \cdots x_n\) and \(x_1 \cdots x_n x_{n+1}\) for all \(x_i \in \mathcal{A}\). The space \(\mathcal{A}^{\mathcal{A}}\) corresponds to infinite paths in \(\mathcal{T}\), and thus naturally describes the boundary of \(\mathcal{T}\).

Let \(G\) be the group of graph automorphisms of \(\mathcal{T}\): maps \(\mathcal{A}^* \to \mathcal{A}^*\) that preserve the edge set. Then there is a natural map \(\pi: G \to \text{Sym}(\mathcal{A})\) defined by restricting the action of \(G\) to the neighbours of the root; and \(\ker(\pi)\) acts on the \(#\mathcal{A}\) disjoint
trees hanging from the root, so is isomorphic to $G_{\mathcal{A}}$. We therefore have a natural isomorphism

$$\Phi : G \to G_{\mathcal{A}} \ Sym(\mathcal{A}).$$

(2)

A subgroup $H \leq G$ is called 

self-similar

if the isomorphism (2) restricts to a homomorphism $\Phi : H \to H_{\mathcal{A}} \ Sym(\mathcal{A})$. In that case, elements of $H$ may be defined recursively in terms of their image under $\Phi$, and conversely such a recursive description defines uniquely an action on $\mathcal{T}$.

The Grigorchuk group $G$ (see §4.3 or the Introduction) acts faithfully on the binary rooted tree $T_2$, and as such is a subgroup of the automorphism group of $T_2$. It is self-similar, and the generators $\{a,b,c,d\}$ of $G$ may be written using decorated permutations as follows:

$$a \mapsto \begin{cases} \ , \\ a \end{cases}, \quad b \mapsto \begin{cases} a \ , \\ a \end{cases}, \quad c \mapsto \begin{cases} a \ , \\ d \end{cases}, \quad d \mapsto \begin{cases} b \ , \\ a \end{cases}.$$

Example 2.16 (The “lamplighter group”). Consider $G = \mathbb{Z}$ acting on itself by translation, and $A = \mathbb{Z}/2$. The wreath product $W = A \wr G$ is called the “lamplighter group”. The terminology is justified as follows: consider an bi-infinite street with a lamp at each integer location. The group $G$ consists of invertible instructions for a person, the “lamplighter”: either move up or down the street, or toggle the state of a lamp before him/her.

If we denote by $a$ the operation of toggling the lamp at position 0 and by $t$ the movement of the lamplighter one step up the street, then $G$ is generated by $\{a,t\}$; and it admits as presentation

$$G = \langle a,t \mid [a,a^k] \text{ for all } k \in \mathbb{N} \rangle.$$

(3)

Exercise 2.17 (**). Let $A$ be a simple group and let $H$ be perfect. Let $G := H \wr X A$ be their wreath product. Then $G$ is perfect, and all normal subgroups of $G$ are $G$ or of the form $N \wr X$ for a normal subgroup $N \triangleleft H$.

Example 2.18 (Monod-Popa [95]). There are groups $K \triangleleft H \triangleleft G$ such that the $G$-sets $K \setminus G$ and $H \setminus G$ are amenable but the $H$-set $K \setminus H$ is not.

Choose indeed any non-amenable group $Q$, and set $G := Q \wr Z$ and $H = \prod_Q Z$ and $K = \prod_Q Q$.

The $G$-set $H \setminus G$ is clearly amenable, since the action of $G$ factors through an action of $Z$. To prove that $K \setminus G$ is amenable, it therefore suffices to find an $H$-invariant mean on $\ell^\infty(K \setminus G)$, and then apply Proposition 2.26. Let $t$ denote the positive generator of $Z$. For every $k \in \mathbb{N}$, define a mean $m_k$ by $m_k(f) = f(Kt^k)$ for $f \in \ell^\infty(K \setminus G)$. This mean is invariant by the group $Kt^k$. Since $H = \bigcup_{k \in \mathbb{N}} Kt^k$, any weak limit of the $m_k$ is an $H$-invariant mean.

On the other hand, $K \setminus H$ is just a restricted direct product of $Q$’s, so is not amenable by Proposition 2.12.

Exercise 2.19 (**). Give an amenable $G$-set such that none of its orbits are amenable.
Hint: Consider the “lamplighter group” $G = \langle a, t \rangle$, see Example 2.16, and the groups $G_n = \langle a, t \mid [a, a^k t] \rangle$ for all $k = 1, \ldots, n$. Consider the natural action of $F_2 = \langle a, t \mid \rangle$ on $X = \bigsqcup_{n \geq 0} G_n$, and show that (i) each $G_n$ is non-amenable (ii) the group $G$ is amenable (iii) the action on $X$ approximates arbitrarily well the action on $G$.

We return to the definition of means we started with; we shall see more criteria for amenability. Recall that $\mathcal{M}(X)$ denotes the set of means on $X$.

Lemma 2.20. $\mathcal{M}(X)$ is compact.

Proof. Since $\mathcal{M}(X)$ is a subset of $[0, 1]^{\mathcal{P}(X)}$ which is compact by Tychonoff’s theorem, it suffices to show that $\mathcal{M}(X)$ is closed.

Now each of the conditions defining a mean, namely $m(X) = 0$ and $m(A) = m(B) = 0$, defines a closed subspace of $[0, 1]^{\mathcal{P}(X)}$ because it is the zero set of a continuous map. The intersection of these closed subspaces is $\mathcal{M}(X)$ which is therefore closed.

Here are simple examples of means. For $x \in X$, define $\delta_x \in \mathcal{M}(X)$ by

$$\delta_x(A) = \begin{cases} 0 & \text{if } x \notin A, \\ 1 & \text{if } x \in A. \end{cases}$$

It is easy to see that the axioms of a mean are satisfied. We have thus obtained a map $\delta : X \to \mathcal{M}(X)$, which is clearly injective.

Lemma 2.21. $\delta(X)$ is discrete in $\mathcal{M}(X)$.

Proof. Given $x \in X$, set $\mathcal{U} = \{m \in \mathcal{M}(X) \mid m(\{x\}) > 0\}$.

Corollary 2.22. If $X$ is infinite, then $\delta(X)$ is not closed.

Proof. Indeed, if $\delta(X)$ is closed in $\mathcal{M}(X)$, then it is compact; being furthermore discrete, it is finite; $\delta$ being injective, $X$ itself is finite.

Recall that a subset $K$ of a topological vector space is convex if for all $x, y \in K$ the segment $\{(1 - t)x + ty \mid t \in [0, 1]\}$ is contained in $K$; see [6] for more on convex sets. The convex hull of a subset $S$ of a topological vector space is the intersection $\hat{S}$ of all the closed convex subspaces containing $S$.

Lemma 2.23. $\mathcal{M}(X)$ is convex.

Proof. Consider means $m_i$ and positive numbers $t_i$ such that $\sum t_i = 1$. Then $\sum t_i m_i$ clearly satisfies the axioms of a mean.

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4 We are using here, and throughout this chapter, the Axiom of Choice; see [81].

5 Recall that $D$ is discrete in a topological space $X$ if for every $x \in D$ there is an open set $\mathcal{U}$ $x$ with $D \cap \mathcal{U} = \{x\}$. 

For a set $X$ and $p \in [1, \infty)$ we denote by $\ell^p(X)$ the Banach space of functions $\phi : X \to \mathbb{R}$ satisfying $\|\phi\|^p_p := \sum |\phi(x)|^p < \infty$, and by $\ell^\infty(X)$ the space of bounded functions with supremum norm. For $p \in [1, \infty]$ the space $\ell^p(X)$ carries a natural isometric $G$-action by $(\phi g)(x) = \phi(xg^{-1})$. Of particular interest is the space $\ell^1(X)$, and its subset

$$\mathcal{P}(X) = \{ \mu \in \ell^1(X) \mid \mu \geq 0, \sum_{x \in X} \mu(x) = 1 \},$$  \hspace{1cm} (4)

the space of probability measures on $X$. It is a convex subspace of $\ell^1(X)$, compact for the weak*-topology, and (for infinite $X$ strictly) contained in $\mathcal{M}(X)$:

**Proposition 2.24.** For a set $X$, consider the following subset of $\ell^\infty(X)^*$:

$$\mathcal{B}(X) := \{ m \in \ell^\infty(X)^* \mid m(f) \geq 0 \text{ whenever } f \geq 0, m(1) = 1 \}.$$

Then the map $\tilde{f} : \mathcal{B}(X) \to \mathcal{M}(X)$ defined by

$$(\tilde{f} m)(A) := m(1_A) \text{ with } 1_A \text{ the characteristic function of } A$$

is a homeomorphism, functorial in $X$.

The subspace $\ell^1(X) \cap \mathcal{B}(X) \subset \ell^\infty(X)^*$ corresponds via $\tilde{f}$ to the convex hull $\hat{\delta}(X)$ of $\delta(X)$.

We recall that there is a natural non-degenerate pairing $\ell^1(X) \times \ell^\infty(X) \to \mathbb{C}$, given by $(f, g) \mapsto \sum f(x)g(x)$. For that pairing, $(\ell^1(X))^* = \ell^\infty(X)$; but $(\ell^\infty(X))^*$ is much bigger than $\ell^1(X)$, as is clear from the proposition. In fact, $\ell^\infty(X)$ is in isometric bijection with the space of continuous functions on the Stone-Čech compactification $\beta X$ of $X$, see Lemma[2.6] so

$$(\ell^\infty(X))^* = L^1(\beta X) \text{ the set of Borel measures on } \beta X.$$  \hspace{1cm} (5)

**Proof of Proposition 2.24** Let $\mathscr{S}$ be the set of simple functions on $X$, namely the functions that take only finitely many values. Consider first $m \in \ell^\infty(X)^*$ with $m(1_A) = 0$ for all $A \subseteq X$. Then $m$ vanishes on $\mathscr{S}$ by linearity; and $\mathscr{S}$ is dense in $\ell^\infty(X)$, so $m = 0$. This proves that $\tilde{f}$ is injective.

On the other hand, let $m : \Psi(X) \to [0, 1]$ be a mean. For $f \in \mathscr{S}$, we have

$$(\tilde{f} m)(f) = \sum_{v \in f(X)} m(f^{-1}(v)).$$

We check that $\tilde{f} m$ is a continuous function $\mathscr{S} \to \mathbb{C}$ for the $\ell^\infty$ norm on $\mathscr{S}$; indeed, for $f, g$ simple functions on $X$,
Therefore, $\int m$ extends to a continuous function $\ell^\infty(X) \to \mathbb{C}$, which clearly belongs to $\mathscr{B}(X)$. Since $\mathcal{F}$ is dense, this extension is unique.

Finally recall that $\ell^1(X)$ embeds in $\ell^\infty(X)^*$ by $f \mapsto (f' \mapsto \sum_x f(x) f'(x))$. The element $f \in \ell^1(X)$ therefore corresponds to the affine combination $\sum f(x) \delta_x$ of Dirac means.

\[ m(\bigcup A_i) = \sum m(A_i). \]

Consider now an invariant mean $m$ on $\mathbb{Z}$, as given by Proposition 2.24. Assume for contradiction that $m$ were $\sigma$-additive. Then either $m(\{0\}) = 0$, so $m(\{n\}) = 0$ for all $n \in \mathbb{Z}$ by $\mathbb{Z}$-invariance and $m(\mathbb{Z}) = 0$ by $\sigma$-additivity; or $m(\{0\}) = \varepsilon > 0$ and $m(\{0, 1, \ldots, n\}) > 1$ as soon as $n > 1/\varepsilon$. In all cases we have reached a contradiction.

**Corollary 2.25.** Let $X$ be a $G$-set. Then $X$ is amenable if and only if there exists a $G$-invariant positive functional in $\ell^\infty(X)^*$.

The fact that the “Dirac” means $\hat{\delta}(X)$ constitute a small subset of $\mathscr{M}(X)$ may be confirmed as follows. Every mean $m \in \hat{\delta}(X)$ enjoys an additional property, namely $\sigma$-additivity: for disjoint $A_1, A_2, \ldots$ we have

\[ m(\bigcup A_i) = \sum m(A_i). \]

**Proposition 2.26.** Let $X$ be an amenable $G$-set such that all point stabilizers $G_x$ are amenable. Then $G$ itself is amenable.

**Proof.** Thanks to Proposition 2.24 for all $Y$ we view $\mathscr{M}(Y)$ as the set of normalized positive functionals $m : \ell^\infty(Y) \to \mathbb{R}$. Let us first define a map $\Phi : X \to \mathscr{M}(G)$.

Since every $G_x$ is amenable, there exists for all $x \in X$ an invariant mean $m_x \in \mathscr{M}(G_x)^{G_x}$, which we extend via the inclusion $G_x \hookrightarrow G$ to mean still written $m_x \in \mathscr{M}(G)^{G_x}$. Choose for every $G$-orbit in $X$ a point $x$, and set $\Phi(xg) = m_x g$ on that orbit. This is well-defined: if $xg = xh$, then $hg^{-1} \in G_x$ so $m_x h = m_x h g^{-1} g = m_x g$. It follows automatically that $\Phi$ is $G$-equivariant.

By functoriality, $\Phi$ induces a $G$-equivariant map $\Phi_* : \mathscr{M}(X) \to \mathscr{M}(\mathscr{M}(G))$.

Now there is, for all $Y$, a functorial map $\beta : \mathscr{M}(\mathscr{M}(Y)) \to \mathscr{M}(Y)$ called the barycentre: it is given by

\[ \beta(m)(f) = m(n \mapsto n(f)) \text{ for } m \in \mathscr{M}(\mathscr{M}(Y)), f \in \ell^\infty(Y), n \in \mathscr{M}(Y). \]
Composing, we get a map \( \Upsilon \circ \Phi : \mathcal{M}(X) \to \mathcal{M}(G) \), which is still \( G \)-equivariant. Now since \( X \) is amenable \( \mathcal{M}(X)^G \) is non-empty, so \( \mathcal{M}(G)^G \) is also non-empty.

**Corollary 2.27.** Let \( 1 \to N \to G \to Q \to 1 \) be an exact sequence of groups. Then \( G \) is amenable if and only if both \( N \) and \( Q \) are amenable.

**Proof.** If \( G \) is amenable, then its quotient \( Q \) is amenable by Proposition 2.10 and its subgroup \( H \) is amenable by Proposition 2.12 since it acts freely on the amenable \( G \)-set \( G \).

Conversely, if \( N \) and \( Q \) are amenable, then the natural action of \( G \) on \( Q \) satisfies the hypotheses of Proposition 2.26.

**Exercise 2.28 (\(*\).** Let \( G \) be a group. We might have called \( G \) left-amenable if there exists a left-invariant mean on \( G \), namely a mean \( m \in \mathcal{M}(G) \) with \( m(gA) = m(A) \) for all \( g \in G, A \subseteq G \); and have called \( G \) bi-amenable if there exists a mean \( m \in \mathcal{M}(G) \) with \( m(gAh) = m(A) \) for all \( g,h \in G, A \subseteq G \).

Prove that in fact \( G \) is amenable if and only if it is left-amenable, if and only if it is bi-amenable.

We conclude with yet another criterion, attributed to Dixmier:

**Theorem 2.29 (Følner \([38, \text{Theorem 4}], \text{Dixmier \([34, \text{Théorème 1}]\); see \([50, \text{Theorem 4.2}]\)).** Let \( X \) be a \( G \)-set. Then \( X \) is amenable if and only if for any \( h_1, \ldots, h_n \in \ell^\infty(X) \) and any \( g_1, \ldots, g_n \in G \) the function

\[
H := \sum_{i=1}^{n} (h_i - h_ig_i)
\]

satisfies \( \sup_{x \in X} H(x) \geq 0 \).

**Proof.** If \( X \) is amenable then there is an invariant positive mean \( m \in \ell^\infty(X)^\ast \); then for every function \( H \) as above \( m(H) = 0 \) by invariance while \( m(H) \leq \sup H \) by positivity.

On the other hand, if \( \sup H \geq 0 \) for all \( H \) as above, then an invariant mean may be constructed as follows: set

\[
\tilde{m}(f) = \inf_{H \text{ as above}} \sup_X (f + H).
\]

Clearly \( \tilde{m} \) satisfies \( \tilde{m}(\lambda f) = \lambda \tilde{m}(f) \) for \( \lambda \geq 0 \) and \( \tilde{m}(fg) = \tilde{m}(f) \) for \( g \in G \) and \( \tilde{m}(1) = 1 \) and \( \tilde{m}(f) \geq 0 \) if \( f \geq 0 \); and \( \tilde{m}(f + g) \leq \tilde{m}(f) + \tilde{m}(g) \) because if \( \tilde{m}(f) \geq \sup_X (f + H) - \varepsilon \) and \( \tilde{m}(g) \geq \sup_X (g + K) - \varepsilon \) then \( \tilde{m}(f + g) \leq \sup_X (f + g + H + K) \leq \sup_X (f + H) + \sup_X (g + K) \leq \tilde{m}(f) + \tilde{m}(g) - 2\varepsilon \). The Hahn-Banach theorem (see e.g. \([116, \text{Theorem 3.12}]\)) implies the existence of a linear functional \( m \) with the same properties.

\( ^{6} \) Erroreously!
3 Følner and Reiter’s criteria

The following combinatorial criterion will be shown equivalent to amenability; it is sometimes the easiest path to prove a group’s amenability. It was introduced by Erling Følner [39], though the idea of averaging over larger and larger finite sets to construct invariant means can be traced back at least to Ahlfors [2, Chapter III.25].

Definition 3.1. Let $X$ be a $G$-set. We say that $X$ satisfies Følner’s condition if for all finite $S \subseteq G$ and all $\varepsilon > 0$, there is a finite subset $F \subseteq X$ with

$$\#(FS \setminus F) < \varepsilon \#F.$$  

When we say that a group $G$ satisfies Følner’s condition, we mean it for the right $G$-set $X = G \looparrowright G$.

For example, $\mathbb{Z}$ satisfies Følner’s condition: given $\varepsilon > 0$ and $S \subseteq \mathbb{Z}$ finite, find $k$ such that $S \subseteq \{-k, \ldots, k\}$. Let $\ell \in \mathbb{N}$ be such that $\ell > 2k/\varepsilon$, and set $F = \{1, 2, \ldots, \ell\}$. Then $FS \setminus F \subseteq \{1-k, \ldots, 0, \ell+1, \ldots, \ell+k\}$ has size at most $2k$, so $\#(FS \setminus F) < \varepsilon \#F$.

Actually, the definition makes sense in a much more general context, that of graphs:

Definition 3.2. A directed graph (digraph) is a pair of sets $G = (V, E)$ called vertices and edges, with maps $\pm : E \to V$ giving for each edge $e \in E$ its head $e^+ \in V$ and tail $e^- \in V$.

A graph $G = (V, E)$ has bounded valency if there is a bound $K \in \mathbb{N}$ such that at every vertex $v \in V$ there are at most $K$ incoming and outgoing edges, namely if

$$\#\{e \in E \mid v = e^+\} \leq K \quad \text{and} \quad \#\{e \in E \mid v = e^-\} \leq K.$$  

Consider a $G$-set $X$ and a finite set $S \subseteq G$. The Schreier graph of $X$ with respect to $S$ is the graph with vertex set $V = X$ and edge set $E = X \times S$, with $(x, s)^- = x$ and $(x, s)^+ = xs$. In other words, there is an edge from $x$ to $xs^+$ for all $x \in X, s \in S$. If $X = G \looparrowright G$, then the Schreier graph is usually called the Cayley graph of $G$.

Let $(V, E)$ be a graph. For a subset $F \subseteq V$, its boundary is the set of edges connecting $F$ to its complement, in formulae

$$\partial F = \{e \in E \mid e^- \in F, e^+ \notin F\}.$$  

Definition 3.3. A graph $G = (V, E)$ satisfies Følner’s condition if for all $\varepsilon > 0$ there is a finite subset $F \subseteq V$ with $\#\partial F < \varepsilon \#F$.

Thus Følner’s criterion asks for the existence of subgraphs of $X$ with an arbitrarily small relative outer boundary. It is clear that a $G$-set $X$ satisfies Følner’s condition if and only if its Schreier graphs satisfy it for all choices of $S \subseteq G$.

Lemma 3.4. Let $X$ be a $G$-set. Følner’s condition is equivalent to: for all finite subsets $S \subseteq G$ and all $\varepsilon > 0$, there is a finite subset $F \subseteq X$ with

$$\#(Fs \setminus F) < \varepsilon \#F \text{ for all } s \in S.$$
Proof. If \( \#(FS \setminus F) < \varepsilon \#F \) then in particular \( \#(Fs \setminus F) < \varepsilon \#F \) for all \( s \in S \). Conversely, if \( \#(Fs \setminus F) < \varepsilon \#F / \#S \) for all \( s \in S \) then \( \#(FS \setminus F) < \varepsilon \#F \).

Recall that a directed set is a partially ordered set \( (\mathcal{N}, \leq) \) with finite upper bounds, i.e., for every \( m, n \in \mathcal{N} \) there exists an element \( \max\{m, n\} \in \mathcal{N} \) with \( m, n \leq \max\{m, n\} \). A net is a sequence indexed by a directed set. For \( (x_n)_{n \in \mathcal{N}} \) a real-valued net, we write

\[
\lim_{n \to \infty} x_n = x \quad \text{to mean} \quad \forall \varepsilon > 0 : \exists n_0 \in \mathcal{N} : \forall n \geq n_0 : |x_n - x| < \varepsilon,
\]

as in usual calculus.

**Exercise 3.5** (*). Let \( \mathcal{N} \) be a non-empty net. Then \( \{ F \subseteq \mathcal{N} \mid \exists n_0 \in \mathcal{N} : n \geq n_0 \Rightarrow n \in F \} \) is a filter on \( \mathcal{N} \), and the notions of convergence in (7) and in the filter coincide.

We have the following alternative definition of Følner’s condition:

**Lemma 3.6.** Let \( G \) be a group and let \( X \) be a \( G \)-set. Then \( X \) satisfies Følner’s condition if and only if there exists a net \( (F_n)_{n \in \mathcal{N}} \) of finite subsets of \( X \) with

\[
\lim_{n \to \infty} \frac{\#(F_n g \setminus F_n)}{\#F_n} = 0 \quad \text{for all} \quad g \in G.
\]

**Proof.** Assume (8), and let \( S \subseteq G, \varepsilon > 0 \) be given. For each \( s \in S \), let \( n(s) \in \mathcal{N} \) be such that \( \#(F_n s \setminus F_n) < \varepsilon \#F_n / \#S \) for all \( n \geq n(s) \), and set \( F = F_{\max\{n(s)\}} \); then \( \#(FS \setminus F) \leq \sum_{s \in S} \#(Fs \setminus F) < \varepsilon \#F \), so Følner’s condition is satisfied.

Conversely, define \( \mathcal{N} = \{ (S, \varepsilon) \mid S \subseteq G, \varepsilon > 0 \} \), ordered as follows: \( (S, \varepsilon) \leq (T, \delta) \) if \( S \subseteq T \) and \( \varepsilon > \delta \); so \( \max\{(S, \varepsilon), (T, \delta)\} = (S \cup T, \min\{\varepsilon, \delta\}) \). For each \( n = (S, \varepsilon) \in \mathcal{N} \), choose a finite set \( F_n \subseteq X \) with \( \#(FS \setminus F) < \varepsilon \#F \). These satisfy (8).

In case \( G \) is finitely generated, we also have the following alternative definition:

**Lemma 3.7.** Let \( G \) be finitely generated, say by a finite set \( S \) containing 1, and let \( X \) be a \( G \)-set. Then \( X \) satisfies Følner’s condition if and only if for all \( \varepsilon > 0 \) there is a finite subset \( F \subseteq X \) with

\[
\#(FS \setminus F) < \varepsilon \#F.
\]

**Proof.** One direction is obvious. In the other direction, let \( S' \subseteq G \) and \( \varepsilon' > 0 \) be given. Since \( S \) generates \( G \), there exists \( k \in \mathbb{N} \) with \( S' \subseteq S^k \). Set \( \varepsilon = \varepsilon' / k \), and let \( F \subseteq X \) satisfy \( \#(Fs \setminus F) < \varepsilon \#F \) for all \( s \in S \).

Consider \( g \in S' \), and write it as \( g = s_1 \ldots s_k \) with \( s_1, \ldots, s_k \in S \). Then

\[
Fg \setminus F = \bigsqcup_{j=1}^k Fs_j \ldots s_k \setminus Fs_{j+1} \ldots s_k,
\]

so
\[
(Fg \setminus F) = \sum (Fs_j \setminus Fs_{j+1} \cdots s_k) \quad = \sum (Fs_j \setminus F)s_{j+1} \cdots s_k < k \epsilon \#F = e' \#F.
\]

We are done by Lemma 3.4.

We shall see in Theorem 3.23 that a $G$-space $X$ satisfies Følner’s criterion if and only if it is amenable. This can be used to prove (non-)amenability in numerous cases; for example,

**Proposition 3.8.** A $G$-set $X \rightarrow G$ is amenable if and only if for every finitely generated subgroup $H \leq G$ the $H$-set $X \rightarrow H$ is amenable.

**Proof.** ($\Leftarrow$) Given $S \subset G$ and $\epsilon > 0$, consider $H = \langle S \rangle$ and apply Følner’s criterion.

($\Rightarrow$) Every $G$-invariant mean is also $H$-invariant. □

Thus for instance the action of $\mathbb{Q}$ on $\mathbb{Q}/\mathbb{Z}$ is amenable, because every finitely generated subgroup of $\mathbb{Q}$ has a finite orbit on $\mathbb{Q}/\mathbb{Z}$. (We shall later see that all actions of $\mathbb{Q}$ are amenable.)

**Example 3.9.** The group of permutations $\text{Sym}(\mathbb{N})$ of $\mathbb{N}$ with finite support is amenable; indeed every finite subset generates a finite group.

**Example 3.10.** The group of “bounded-displacement permutations of $\mathbb{Z}$”

\[G = W(\mathbb{Z}) = \{ \tau : \mathbb{Z} \ni \sup_{n \in \mathbb{Z}} |\tau(n) - n| < \infty \}\]

acts amenably on $\mathbb{Z}$. Indeed given $S \subset G$ finite and $\epsilon > 0$, the maximum displacement of elements of $S$ is bounded, say $\leq k$; and then $\mathbb{Z} \rightarrow G$ satisfies Følner’s condition with $F = \{ 0, \ldots, \lfloor k/\epsilon \rfloor \}$.

**Example 3.11.** The “lamplighter group” $G$ from Example 2.16 is amenable. Indeed elements of $G$ may be written as pairs $(f, m)$ with $f : \mathbb{Z} \rightarrow \mathbb{Z}/2$ and $m \in \mathbb{Z}$, and one may consider as Følner sets

\[F_n = \{(f, m) : \text{support}(f) \subseteq [-n, n] \text{ and } m \in [-n, n]\}.
\]

**Example 3.12.** The free group $F_k = \langle x_1, \ldots, x_k \rangle$ is amenable if and only if $k \leq 1$, see Proposition 2.9. Indeed if $k \leq 1$ then $F_k$ is $\{ 1 \}$ or $\mathbb{Z}$; while in general, choose $S = \{ x_1^{\pm 1}, \ldots, x_k^{\pm 1} \}$ and consider $F \subset X$. In the Cayley graph of $F_k$, which is a $2k$-regular tree (see Figure 2 left), consider the subgraph spanned by $F$. It suffices to consider connected components of the graph once at a time; each connected component is a tree, with say $v$ vertices and therefore $v - 1$ edges. The sum of the vertex degrees within that tree is therefore $2v - 2$, so the total number of edges pointing out of the component is at least $2kv - (2v - 2) \geq (2k - 2)v$; these edges point to distinct elements in $SF \setminus F$. Therefore, Følner’s criterion is not satisfied as soon as $\epsilon < 2n - 2$. 

Amenability of groups and $G$-sets

There are plenty of non-amenable groups with amenable actions, and even faithful amenable actions; here is one.

**Example 3.13.** Consider $F_2 = \langle a, b \rangle$ and its subgroup $H = \langle a^n : n \leq 0 \rangle$. Then $F_2$ acts naturally on the coset space $X := H \backslash F_2$, see Figure 2 right, and this action is amenable. Indeed with $S = \{a^{\pm 1}, b^{\pm 1}\}$ and $\varepsilon > 0$ given, consider the set $F = \{H, Hb, \ldots, Hb^n\}$ for $n > \varepsilon^{-1}$. It satisfies Følner’s criterion. Note that the action of $F_2$ on $X$ is not free, but it is nevertheless faithful.

### 3.1 Growth of sets

Let $X \curvearrowright G$ be a $G$-set, and consider $S \subseteq G$ and $x_0 \in X$. The orbit growth of $X$ is the function $v_{X,x_0,S} : \mathbb{N} \to \mathbb{N}$ given by

$$v_{X,x_0,S}(n) = \#\{x \in X \mid x = x_0s_1 \cdots s_m \text{ for some } s_i \in S, m \leq n\}.$$ 

If $G$ is finitely generated, then the orbit growth depends only mildly on the choice of $S$ as soon as it generates $G$: if $S'$ be another generating set of $G$, then there exists a constant $C > 0$ with $v_{X,x_0,S'}(n) \leq v_{X,x_0,S}(Cn)$ and $v_{X,x_0,S'}(n) \leq v_{X,x_0,S}(Cn)$. Similarly, if $x_0, x'_0 \in X$ belong to the the same $G$-orbit, then there exists a constant $C \in \mathbb{N}$ with $v_{X,x_0,S}(n) \leq v_{X,x'_0,S}(n + C)$ and $v_{X,x'_0,S}(n) \leq v_{X,x_0,S}(n + C)$. Therefore, the equivalence class of $v_{X,x_0,S}$ under linear transformations of its argument is independent of the choice of $S$ if $S$ generates $G$, and of $x_0$ if $X$ is transitive; it is denoted simply $v_{X,S}$, $v_X$, and $v_{x_0}$ respectively.

As usual, we consider $G$ as a $G$-set under right translation, and denote by $v_{G,S}$ and $v_G$ its growth function. We also write $B_{G,S}(n)$ for the ball of radius $n$ in $G$, and more generally $B_{X,x_0,S}(n)$ for the ball of radius $n$ in $X$ around $x_0$.
Proposition 3.14. Let $X$ be a $G$-set and let $x_0 \in X$ be such that $v_{x_0,S}$ grows subexponentially for all $S \subseteq G$. Then $X$ satisfies Følner’s condition.

Proof. Let a finite subset $S \subseteq G$ and $\varepsilon > 0$ be given. Since $X$ has subexponential growth, we have $\lim \sqrt[n]{v_{x_0,S}(n)} = 1$; therefore $\liminf v_{x_0,S(n+1)} = 1$, so for some $n$ we have $v_{x_0,S}(n+1) < (1 + \varepsilon)v_{x_0,S}(n)$. Set $F = B_{x_0,S}(n)$. We have $\#(FS) < (1 + \varepsilon)\#F$, so $X$ satisfies Følner’s condition by Lemma 3.7.

Note that it is unknown whether in every finitely generated group $G$ of subexponential growth we have $\lim v_{G,S}^{(n+1)} = 1$; only the ‘liminf’ is known to equal 1.

One may study more quantitatively the Følner condition as follows: let $X$ be a $G$-set and let $S$ be a generating set for $G$. Define $\text{Føl}: \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by

$$\text{Føl}(n) = \inf \{\#F \mid F \subseteq X, \#(F \Delta FS) < \#F/n \text{ for all } s \in S\}. \quad (9)$$

Then $\text{Føl}(n) < \infty$ for all $n$ precisely if $X$ is amenable. A similar definition may be given for graphs, which we leave to the reader. Groups admit the following lower bound on $\text{Føl}$:

Proposition 3.15 (Coulhon-(Saloff-Coste) [30]). Let $G = \langle S \rangle$ be a finitely generated group, with growth function $v_{G,S}(n)$. Then

$$\text{Føl}(n) \geq \frac{1}{2}v_{G,S}(n) \text{ for all } n \in \mathbb{N}.$$ 

Proof. We shall prove the following equivalent form: given $F \subseteq G$, choose $n \in \mathbb{N}$ such that $v_{G,S}(n) \geq 2\#F$. We are required to find $s \in S$ with $\#(F \Delta FS) \geq \#F/n$.

First, for all $x \in F$ we have $\#(xB_{G,S}(n) \setminus F) \geq \#F \geq (xB_{G,S}(n) \cap F)$, so

$$\sum_{g \in B_{G,S}(n)} 1_{xg \not\in F} \geq v(n)/2 \geq \sum_{g \in B_{G,S}(n)} 1_{xg \in F},$$

so for some $g \in B_{G,S}(n)$ we have $\sum_{x \in F} 1_{xg \not\in F} \geq \#F/2$, namely $\#(F \Delta FG) \geq \#F$.

Write now $g = s_1 \cdots s_n$; then $F \Delta FG = (F \Delta Fs_1) \Delta \cdots \Delta (F \Delta Fs_2 \cdots s_{n-1} \Delta Fg) \subseteq (F \Delta Fs_n) \cup \cdots \cup (F \Delta Fs_1 \Delta Fs_2 \cdots s_n)$. It follows that there exists some $k \in \{1, \ldots, n\}$ with $\#(F \Delta Fs_k) \geq \#F/n$.

On the other hand, we have an upper bound on $\text{Føl}$ coming from balls: with $F = B_{G,S}(n)$ we have $F \Delta FS \subseteq B_{G,S}(n+1)$ so $\#(F \Delta FS) \leq (v(n + 1) - v(n)) \leq v(n)$. Assuming that $v$ is the restriction to $\mathbb{N}$ of a differentiable function, we may seek a function $f$ satisfying $f(1/\log(v')) = v$ to obtain an upper bound $\text{Føl}(n) \leq f(n)$. For example, if $v(n) \propto n^d$ then $f(n) \propto n^{d'}$ and therefore the estimate given by Proposition 3.15 is at worst a constant off. The “1/2” in Proposition 3.15 cannot easily be eliminated: in a finite group, we shouldn’t expect any good estimates for sets larger than half of the group.
Note also that we have Føl(n) > n as soon as X is infinite, since then #(F \triangle F_s) \geq 1 for all F \in X. No analogue of Proposition 3.15 may hold for G-sets in general:

**Exercise 3.16 (**)». Let X be a G-set for a finitely generated group G. Prove that Føl(n) is linear (i.e. Føl(n) \leq Cn for some constant C) if and only if the Schreier graph of X has bounded cutsets, namely there is a bound C′ such that every finite set of vertices can be separated by removing at most C′ vertices.

**Exercise 3.17 (**)». We saw in Exercise 2.28 that a group is “left-amenable” if and only if it is amenable. First prove directly that if a group admits sets that are almost invariant under right translation, then it admits sets that are almost invariant under left translation.

Next, prove that the infinite dihedral group \(D_\infty = \langle a, b \mid a^2, b^2 \rangle\) admits finite subsets that are almost right-invariant but far from left-invariant, namely subsets \(F_n \in D_\infty\) with \(\#(F_n \triangle F_n^{-1})/\#F_n \to 0\) for all \(g \in D_\infty\) but \(\#(F_n \triangle gF_n)/\#F_n \not\to 0\).

Give on the other hand a family of sets \(F_n \in D_\infty\) with \(\#(F_n \triangle gF_n)/\#F_n \to 0\) for all \(g, h \in D_\infty\).

We return to Definition 3.3. A connected graph \(\mathcal{G} = (V, E)\) endows its set of vertices with the structure of a metric space still written \(\mathcal{G}\): the distance between two vertices is the minimal length of a path connecting them. Given two metric spaces (e.g. connected graphs) \(X, Y\), a map \(f: X \to Y\) is quasi-Lipschitz if there is a constant \(C\) with

\[d(f(x), f(y)) \leq Cd(x, y) + C,\]

and \(f\) is a quasi-isometry if there is a quasi-Lipschitz map \(g: Y \to X\) with \(\sup_{x \in X} d(x, g(f(x))) < \infty\) and \(\sup_{y \in Y} d(y, g(f(y))) < \infty\).

**Exercise 3.18 (**)». Let \(\mathcal{G} = (V, E)\) be a graph, and let \(\mathcal{G}' = (V', E')\) be its barycentric subdivision: \(V' = V \sqcup E\) and \(E' = E \times \{+, -\}\) with \((e, \pm)\pm = e\pm\) and \((e, \pm)\mp = e\mp\). Prove that \(\mathcal{G}\) and \(\mathcal{G}'\) are quasi-isometric.

**Exercise 3.19 (**)». Let G be a finitely generated group. Prove that all Cayley graphs of G with respect to finite generating sets are quasi-isometric; that all finite-index subgroups of G are have quasi-isometric Cayley graphs; and that all quotients of G by finite subgroups have quasi-isometric Schreier graphs.

**Proposition 3.20.** Let \(\mathcal{G} = (V, E)\) and \(\mathcal{G}' = (V', E')\) be bounded-degree graphs, and let \(f: \mathcal{G} \to \mathcal{G}'\) be quasi-Lipschitz with \(\sup_{x \in X} d(y, f(V)) < \infty\). If \(\mathcal{G}\) is amenable then \(\mathcal{G}'\) is amenable.

In particular, if \(\mathcal{G}, \mathcal{G}'\) are quasi-isometric then \(\mathcal{G}\) is amenable if and only if \(\mathcal{G}'\) is amenable.

**Proof.** Let \(\mathcal{G}'' = (V'', E'')\) be a graph. For \(F \subseteq V''\) and \(k \in \mathbb{N}\), define

\[\partial^k(F) = \{ (e_1, \ldots, e_k) \mid e_i \in E'', e_i^+ = e_{i+1}, e_i^- \in E, e_i^+, e_{i+1}^- \not\in F \}.\]

Recall that \(\mathcal{G}\) is amenable if and only if \(\inf_{F \subseteq V} \#\partial F / \#F = 0\). Equivalently, \(\inf_{F \subseteq V} \#\{ e^+ \mid e \in \partial F \} / \#F = 0\). There exists a constant \(D\) such that, for every \(F \subseteq V\), we have
\{ f(e^+) \mid e \in \partial F \} \subseteq \{ e^*_D \mid (e_1, \ldots, e_D) \in \partial^D(f(F)) \}. Therefore, \inf_{F \in V} \#\partial^D(f(F))/\#f(F) = 0, and therefore \inf_{F' \in V} \#\partial(F')/\#f(F') = 0. \quad \Box

**Exercise 3.21 (\text{*}).** Prove that if \( G, G' \) are quasi-isometric graphs then their Følner functions \( \mathcal{F} \) are equivalent in the sense that \( \mathcal{F} \mathcal{F}_{G}(n) \leq C \mathcal{F} \mathcal{F}_{G'}(Cn + C) + C \) and conversely, for some constant \( C \).

There are quasi-invariant groups with quite distinct algebraic properties; e.g., \( A \rtimes Z \) and \( B \rtimes Z \) are quasi-isometric for all finite groups \( A, B \) of same cardinality. If \( A \) is Abelian but \( B \) is simple, then \( A \rtimes Z \) is metabelian and residually finite but \( B \rtimes Z \) is neither. However, these groups are quasi-isometric (and both amenable).

### 3.2 Reiter’s criterion

Følner sets — finite subsets \( F \subseteq X \) that are almost invariant under translation — may be thought of as almost-invariant characteristic functions.

**Definition 3.22 (see [115, page 168]).** Let \( X \) be a \( G \)-set. It satisfies Reiter’s condition for \( p \geq 1 \) if for every finite subset \( S \subseteq G \) and every \( \varepsilon > 0 \) there exists a positive function \( \phi \in \ell^p(X) \) with \( \|\phi s - \phi\| < \varepsilon \|\phi\| \) for all \( s \in S \).

**Theorem 3.23.** Let \( X \) be a \( G \)-set. The following are equivalent:

1. \( X \) is amenable;
2. \( X \) satisfies Reiter’s condition for \( p = 1 \);
3. \( X \) satisfies Reiter’s condition for some \( p \in [1, \infty) \);
4. \( X \) satisfies Reiter’s condition for all \( p \in [1, \infty) \);
5. \( X \) satisfies Følner’s condition.

**Proof.** \((1) \Rightarrow (2)\) Given \( S \subseteq G \) and \( \varepsilon > 0 \), consider the subset

\[
K = \{ \bigoplus_{s \in S} (\mu s - \mu) \mid \mu \in \mathcal{P}(X) \} \subseteq \ell^1(X)^S.
\]

Since \( X \) is amenable, there exists a \( G \)-invariant functional \( m \in \ell^m(X)^* \) by Corollary 2.25. Since \( \ell^1(X) \) is weak*-dense in \( \ell^m(X)^* \), there exists a net \( (\mu_n)_{n \in \mathcal{N}} \) in \( \ell^1(X) \) with \( \mu_n \to \mu \) in the weak*-topology, so \( \bigoplus_{s \in S} (\mu_n s - \mu_n) \in K \) converges to 0 in the weak*-topology on \( \ell^1(X)^S \), so \( K \) weak*-contains 0. Since \( K \) is convex, its norm closure \( \overline{K} \) also contains 0, by the Hahn-Banach theorem (see e.g. [116] Theorem 3.12); so there exists \( \mu \in \mathcal{P}(X) \) with \( \|\mu s - \mu\| < \varepsilon \) for all \( s \in S \).

\((2) \Rightarrow (4)\) Let \( \psi \in \ell^1(X) \) satisfy \( \|\psi s - \psi\| < \varepsilon \|\psi\| \) for all \( s \in S \). Define \( \phi(x) := \psi(x)^{1/p} \); then \( \phi \in \ell^p(X) \) with \( \|\phi\| = \|\psi\|^{1/p} \), and
$$
\|\phi_s - \phi\|_p^p = \sum_{x \in X} |\phi(xs^{-1}) - \phi(x)|^p = \sum_{x \in X} |\psi(xs^{-1})^{1/p} - \psi(x)^{1/p}|^p \\
\leq \sum_{x \in X} |\psi(xs^{-1}) - \psi(x)| \text{ because } |A^{1/p} - B^{1/p}| \leq |A - B|^{1/p} \text{ for all } A, B \\
= \|\psi_s - \psi\| < \varepsilon \|\psi\| = \varepsilon \|\phi\|_p^p.
$$

(4) \Rightarrow (3) is obvious, and so is (2) \Rightarrow (3).

(3) \Rightarrow (2) Let \( \psi \in \ell^p(X) \) satisfy \( \|\psi_s - \psi\| < \varepsilon \|\psi\| \) for all \( s \in S \). Define \( \phi(x) := \psi(x)^p \); then \( \phi \in \ell^1(X) \) with \( \|\phi\|_1 = \|\psi\|^p \), and

$$
\|\phi_s - \phi\| = \sum_{x \in X} |\phi(xs^{-1}) - \phi(x)| = \sum_{x \in X} |\psi(xs^{-1})^p - \psi(x)^p| \\
\leq \sum_{x \in X} p|\psi(xs^{-1}) - \psi(x)| \max\{\psi(xs^{-1}), \psi(x)\}^{p-1} \text{ because } |X^p - Y^p| \leq p|X - Y| \max\{X, Y\}^{p-1} \\
\leq p\left(\sum_{x \in X} |\psi(xs^{-1}) - \psi(x)|^p\right)^{1/p} \left(\sum_{x \in X} |\psi(xs^{-1}) + \psi(x)|^p\right)^{1-1/p} \text{ by Hölder's inequality} \\
= p\|\psi_s - \psi\|_p \|\psi_s + \psi\|_p^{p-1} < p\varepsilon \|\psi\|^p_2 \|\psi\|^{p-1}_p = p2^{p-1}\varepsilon \|\phi\|.
$$

(2) \Rightarrow (5) Given \( S \subseteq G \) and \( \varepsilon > 0 \), let \( \phi \in \ell^1(X) \) be positive and satisfy \( \|\phi_s - \phi\| < \varepsilon \|\phi\| \) for all \( s \in S \). For all \( r \in \mathbb{R}_+ \), consider the set \( F_r = \{x \in X | \phi(x) \geq r\} \).

Then \( \phi = \int \mathbb{1}_{F_r} dr \) and \( \phi_s = \int \mathbb{1}_{F_r} dr \), so

$$
\int (\#F_s \triangle F_r) dr = \|\phi_s - \phi\| < \varepsilon \|\phi\| = \varepsilon \int \#F_r dr,
$$

therefore, there exists \( r \in \mathbb{R}_+ \) with \( \#(F_s \triangle F_r) < \varepsilon \#F_r \), and \( X \) satisfies Følner's criterion by Lemma 3.4.

(5) \Rightarrow (1) By Lemma 3.6 there exists a net \( (F_n)_{n \in \mathcal{N}} \) with \( \lim_{n \to \infty} \#(F_n \setminus F_n) / \#F_n \to 0 \) for all \( g \in G \).

For each \( n \in \mathcal{N} \), consider the “discrete” mean \( \mu_n \in M(X) \) defined by

$$
\mu_n(A) = \frac{\#(A \cap F_n)}{\#F_n}.
$$

Since \( M(X) \) is compact, the net \( (\mu_n)_{n \in \mathcal{N}} \) has an accumulation point, say \( \mu \). We will show that \( \mu \) is a \( G \)-invariant mean by a standard “\( \delta/3 \)” argument.

Given \( g \in G \) and \( A \subseteq X \), we show \( |\mu(A) - \mu(Ag)| < \delta \) for any \( \delta > 0 \). There is \( n \in \mathcal{N} \) with

$$
n > (\{g, g^{-1}\}, \delta/3), \quad |\mu_n(A) - \mu(A)| < \delta/3, \quad |\mu_n(Ag) - \mu(A)| < \delta/3,
$$

because the \( \mu_n \) converge pointwise to \( \mu \). Then
We have sequences \((\mathcal{F}_n)_{n \in \mathcal{N}}\) so that
\[
\#(A \cap F_n) - \#(Ag \cap F_n) = \#(A \cap F_n) - \#(A \cap F_n g^{-1}) \leq \max \{\#(F_n \setminus F_n g^{-1}), \#(F_n g^{-1} \setminus F_n)\} \leq \#(F_n \setminus g, g^{-1}) \setminus F_n) < \varepsilon \#F_n < \delta/3 \#F_n,
\]
so \(|\mu_n(A) - \mu_n(Ag)| < \delta/3\) and
\[
|\mu(A) - \mu(Ag)| \leq |\mu_n(A) - \mu(A)| + |\mu_n(A) - \mu_n(Ag)| + |\mu_n(Ag) - \mu(Ag)| < \delta.
\]
Since this holds for all \(\delta > 0\), we get \(\mu(A) = \mu(Ag)\). \(\square\)

In fact, the \('#(F_s \setminus F)/\#F \to 0'\) in Følner’s condition can be substantially weakened:

**Proposition 3.24 (Gournay).** Let \(X\) be a \(G\)-set. Then \(X\) is amenable if and only if there is a constant \(c < 1\) with the following property: for every finite subset \(S \subseteq X\) there is a finite subset \(F \subseteq X\) with \('#(F_s \setminus F)/\#F \leq c#F\) for all \(s \in S\).

**Proof.** \((\Rightarrow)\) is obvious, by Lemma 3.4 and Theorem 3.23.

\((\Leftarrow)\) by the condition of the proposition, there exists a net \((F_n)_{n \in \mathcal{N}}\) of finite subsets of \(X\) (say indexed by \(\mathcal{F}_f(G)\)) with \(\limsup_{n} \#(F_n \setminus gF_n)/\#F_n \leq c\) for all \(g \in G\). Set \(\xi_n := 1_{F_n}/\sqrt{\#F_n} \in \ell^2(X)\) be the normalized characteristic function of \(F_n\). We have
\[
2 - 2\langle \xi_n, \xi_n g \rangle = ||\xi_n g - \xi_n||^2 = ||1_{F_n g} - 1_{F_n}||_1/\#F_n = 2\#(F_n g \setminus F_n)/\#F_n,
\]
so \(\langle \xi_n, \xi_n g \rangle \geq 1 - c\) for all \(n \gg 1\).

Choose now a non-principal ultrafilter \(\mathcal{F}\) on \(\mathcal{N}\), and consider the ultraproduct space \(\mathcal{H} := \ell^2(X)^\mathcal{F}\); it is a Hilbert space, whose elements are equivalence classes of sequences \((\eta_n)_{n \in \mathcal{N}}\) with \(\eta_n \in \ell^2(X)\) for all \(n\) and \(\sum_{n \in \mathcal{N}} ||\eta_n||^2 < \infty\), under the relation \((\eta_n) \sim (\eta'_n)\) if \(\lim_{\mathcal{F}} ||\eta_n - \eta'_n|| = 0\).

Write \(\hat{\xi} = (\xi_n) \in \mathcal{H}\), and let \(K\) denote the convex hull in \(\mathcal{H}\) of \(\{\xi g \mid g \in G\}\). We have \(\langle \xi g, \hat{\xi} \rangle \geq 1 - c\) for all \(g \in G\), so \(\langle \xi, \hat{\eta} \rangle \geq 1 - c > 0\) for all \(\eta \in K\), and in particular \(0 \notin K\). Let \(\xi'\) be the element of \(K\) of minimal norm, and set \(\xi = \xi' ||\xi'||\), represented by a sequence \((\xi_n)_{n \in \mathcal{N}}\) with \(\xi_n \in \ell^2(X)\) of norm 1. Since \(\xi g = \xi\) by unicity of the element of minimal norm in \(K\), we have ||\(\xi_n - \xi_n g|| \to 0\) for all \(g \in G\), so \(X\) is amenable by Theorem 3.23(3). \(\square\)

We finally present a result that puts as much symmetry between \(X\) and \(G\) as possible, with an eye towards the corresponding notion with the roles of \(G\) and \(X\) interchanged, see Theorem 8.20.

**Proposition 3.25.** Let \(X\) be a \(G\)-set. Then \(X\) is amenable if and only if for every \(\varepsilon > 0\) and every \(g \in G\) \(\in \ell^1(\mathcal{F})\) there exists a positive function \(f \in \ell^1(X)\) with \(||fg|| < \varepsilon \|f\||\).

**Proof.** \((\Rightarrow)\) Given \(\varepsilon > 0\) and \(g = \sum_{x \in G} \xi_x x \in \ell^1 G\) with \(\sum \xi_x = 0\), let \(S \subseteq G\) be such that \(g' := g - \sum_{s \in S} \xi_s (s - 1)\) satisfies ||\(g'|| < \varepsilon/2\. Since \(X\) is amenable, there exists \(F \subseteq X\) with \('#(F_s \setminus F)/\#F < \varepsilon/4\|g\|\#F\) for all \(s \in S\). Set \(f := 1_F\).
\[ \|fg\| \leq \|fg'\| + \sum_{s \in S} \|g_s(fs-f)\| < \#F\|g'\| + 2\#(FS \setminus F)\|g\| < \varepsilon\#F = \varepsilon\|f\|. \]

\((\Leftarrow)\) Given \(\varepsilon > 0\) and \(S \subseteq G\), set \(g = \sum_{s \in S} s - 1\), and let \(f \in \ell^1(G)\) be a positive function satisfying \(\|fg\| < \varepsilon\|f\|/2\). Then

\[ \varepsilon\|f\| > 2\|fg\| = 2\left(\sum_{s \in S} \max (fs-f,0)\right) = 2\sum_{s \in S} \max (fs-f,0) \geq \sum_{s \in S} \|fs-f\|. \]

\[\Box\]

### 3.3 Non-amenability

It may be interesting to consider weaker versions of amenability for groups; for instance, to consider groups admitting faithful amenable actions.

Osin considers in [110] a class of “weakly amenable groups”, which in our context are groups \(G\) admitting an amenable action \(X\) such that, for every finite \(F \subset G\), there exists \(x \in X\) with \(\#(xF) = \#F\); namely, the orbit map \(f \mapsto x \cdot f\) is injective on \(F\). An example of a weakly amenable, non-amenable group is the Baumslag-Solitar group \(\langle a, t | a^mt = ta^n \rangle\), for \(m > n \geq 2\).

If a group \(G\) is not amenable, but all its proper subgroups are amenable, then \(G\) does not have any “interesting” amenable actions: by Proposition 2.26, every amenable action of \(G\) has a fixed point. This applies in particular to Tarski monsters [106], which are non-amenable torsion groups in which every proper subgroup is cyclic.

**Definition 3.26 (Kazhdan, see [79] or [14]).** A group \(G\) has property (T) if every unitary representation \(G \rightarrow U(\mathcal{H})\) in a Hilbert space \(\mathcal{H}\) with almost invariant vectors (in the sense that for every \(\varepsilon > 0\) and every finite \(S \subseteq G\) there exists non-trivial \(x \in \mathcal{H}\) with \(\|x - xs\| < \varepsilon\) for all \(s \in S\)) has a non-trivial fixed vector.

If \(G\) is infinite, then Kazhdan’s property (T) restricted to the unitary representation on \(\ell^2(G)\) is thus precisely the negation of amenability: there are invariant vectors in \(\ell^2(G)\) if and only if \(G\) is finite, and the existence of almost-invariant vectors is Reiter’s condition for \(p = 2\).

Thus an amenable group with property (T) restricted to the unitary representation on \(\ell^2(G)\) has a finite orbit.

Glasner and Monod consider in [44] another group property, which they call property (F): “every amenable action has a fixed point”. They show that a free product of groups always has a faithful, transitive, amenable action unless one factor is (F) and the other is virtually (F). Thus for example \(G \ast \mathbb{Z}\) is not amenable if \(G \neq 1\), yet admits a faithful, transitive, amenable action.

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7 This was exploited in a fundamental manner by Margulis in [89] to prove his “normal subgroup theorem”.
4 Growth of groups

We cover here some classical material on asymptotic growth of groups. Recall from [3.1] that $v_{G,S}(n)$ denotes the number of elements in a group $G$ that are expressible as products of at most $n$ elements of $S$. The group $G$ has exponential growth if $v_{G,S}(n) \geq B^n$ for some $B > 1$, and subexponential growth otherwise; it has polynomial growth if $v_{G,S}(n) \leq p(n)$ for some polynomial $p$; and it has intermediate growth if its growth is neither polynomial nor exponential. These properties are easily seen to be independent of the choice of generating set $S$.

4.1 Groups of polynomial growth

Groups of polynomial growth admit an elegant algebraic characterization. The “if” part is due to Hyman Bass [12] and independently Yves Guivarc’h [58], with an explicit computation of the growth degree of $G$, which is always an integer; the harder, “only if” part is due to Misha Gromov.

We recall some basic group theoretical terminology. For $P$ a property of groups (abelian, ...), a group $G$ is called virtually $P$ if $G$ admits a finite-index subgroup satisfying $P$.

A group $G$ is nilpotent if there exists a constant $c$ such that every $(c+1)$-fold iterated commutator $[[g_0,[g_1,[g_2,\ldots,g_{c-1},g_c],\ldots]]]$ vanishes in $G$; the minimal $c$ is called the nilpotency class of $G$. A group $G$ is polycyclic if it admits a sequence of subgroups $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = 1$ with $G_k/G_{k+1}$ cyclic for all $k$. Finitely generated nilpotent groups are polycyclic.

Theorem 4.1 (Gromov [54]). Let $G$ be a finitely generated group. Then $G$ has polynomial growth if and only if $G$ is virtually nilpotent, namely $G$ has a finite-index nilpotent subgroup.

Proof of Theorem 4.1 “if” direction. Let $G_0$ be a finite-index nilpotent subgroup of $G$. It suffices to prove that $G_0$ has polynomial growth, since then $G$ will have polynomial growth of same degree as $G_0$. Denote by $c$ the nilpotency class of $G_0$, so all $(c+1)$-fold iterated commutators vanish.

Let $(G_k)_{0 \leq k \leq \ell}$ be a composition series for $G$, namely a series of subgroups such that $G_k/G_{k+1}$ is cyclic for all $k$; and for each $k$ let $x_k \in G_k$ be a lift of a generator of $G_k/G_{k+1}$ so that $G_0 = \langle x_0, x_1, \ldots, x_{\ell-1} \rangle$.

We reason by induction on $\ell$. If $\ell = 0$, or if $G/G_1$ is finite, we are done. Assume then $G_0/G_1 \cong \mathbb{Z}$, and by induction that the growth of $G_1$ is bounded by a polynomial, say of degree $d$.

Consider $x \in G_0$, of the form $x = x_{i_1}^{\pm 1} \cdots x_{i_n}^{\pm 1}$. Write it in the form $x_0^e z$, with $e \in \mathbb{Z}$ and $z \in G_1$. This requires us to exchange past each other some letters $x_0$ and $x_i$.

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8 Much to the annoyance of finite group theorists, some people call finite groups “virtually trivial”.
producing subexpressions \([x_i, x_0, \ldots, x_0]\) along the process: indeed one has \(Wx_0 = x_0W[W, x_0]\) for any expression \(W\).

There are at most \(n\) letters \(x_0\) in \(x\); each of them must be brought past at most \(n\) other letters, producing at most \(n^2\) expressions \([x_i, x_0]\); each of these produces in turn at most \(n^3\) expressions \([x_i, x_0, x_0]\); etc. We take as generating set \(S\) for \(G\) all expressions of the form \([x_i, x_0, \ldots, x_0]\) with \(i \geq 1\) and length \(\in \{1, \ldots, c\}\). We have then expressed \(x\) by an integer \(e \in \{-n, \ldots, n\}\) and a word \(z\) of length at most \(n + \cdots + n^e\) in these generators; so

\[
v_{G_0, S \cup \{x_0\}}(n) \leq (2n + 1)v_{G_1, S}(n + \cdots + n^e)
\]

is bounded by a polynomial of degree \(\leq cd + 1\).

We shall give at the end of §8.2 a sketch of the “only if” direction, via slowly growing harmonic functions.

**Corollary 4.2.** Let \(G\) be a virtually nilpotent group. Then \(G\) is amenable.

**Proof.** If \(G\) is virtually nilpotent, then every finitely generated subgroup of \(G\) is also virtually nilpotent, so by Theorem [4.1] has polynomial growth, so is amenable by Proposition [3.14].

### 4.2 Groups of exponential growth

At the other end of the growth spectrum, we find groups of exponential growth. In fact, as soon as a group has a non-abelian free subgroup, it has exponential growth; so a large class of groups, including all non-elementary hyperbolic groups [42], have exponential growth.

In the class of soluble groups, the growth of a group is either polynomial or exponential, as we shall see below. Recall that the derived series of a group \(G\) is the series of normal subgroups defined by \(G^{(0)} = G\) and \(G^{(i+1)} = [G^{(i)}, G^{(i)}]\), and that \(G\) is soluble if \(G^{(n)} = 1\) for some \(n\). The minimal such \(n\) is called the derived length of \(G\).

**Proposition 4.3.** Let \(G\) be a finitely generated group of subexponential growth, and let \(N \triangleleft G\) be a normal subgroup with \(G/N \cong \mathbb{Z}\). Then \(N\) is also finitely generated.

**Proof.** Let \(S = \{x_1, \ldots, x_d\}\) generate \(G\), and let \(x \in G\) generate \(G/N\). Write each \(x_i = x^\ell y_j\), with \(y_j \in N\); so \(G = \langle x, y_1, \ldots, y_d \rangle\), and \(N = \langle y_1, \ldots, y_d \rangle^G\).

Consider further \(N_i = \langle y_i^n \mid n \in \mathbb{Z} \rangle\), so that \(N = \langle N_1, \ldots, N_d \rangle\). It is sufficient to show that each \(N_i\) is finitely generated.

Write then \(y = y_i\), and consider all expressions \(x^{-1}y \epsilon_1 x^{-1}y \epsilon_2 \cdots x^{-1}y \epsilon_m x^n\), with all \(\epsilon_j \in \{0, 1\}\). There are \(2^n\) such expressions, and their length is linear in \(n\), so two must be equal in \(G\) because \(G\) has subexponential growth. Let

\[
y_1 \epsilon_1 \cdots y_m \epsilon_m \cdots y_1 \epsilon_1 \cdots y_m \epsilon_m = y_1 y_2 \cdots y_m y_1 \cdots y_m
\]
be such an equality in $G$, without loss of generality with $1 = e_m \neq f_m = 0$. It follows that $y^m$ is in the group generated by $\{y^x, \ldots, y^{x^{m-1}}\}$, so that $N_i = \langle y_i^n \mid n < m \rangle$. Now a similar argument, replacing $x$ by $x^{-1}$ in (10), shows that $N_i$ is finitely generated. \hfill $\square$

**Corollary 4.4.** Let $G$ be a finitely generated group of subexponential growth, and let $N \triangleleft G$ be a normal subgroup such that $G/N$ is virtually polycyclic. Then $N$ is finitely generated. \hfill $\square$

**Corollary 4.5 (Milnor).** Let $G$ be a finitely generated soluble group of subexponential growth. Then $G$ is polycyclic.

**Proof.** Consider the derived series $G^{(i)}$ of $G$; by assumption, $G^{(s+1)} = 1$ for some minimal $s \in \mathbb{N}$. Set $A = G^{(s)}$. We may assume, by induction, that $G/A$ is polycyclic. By Corollary 4.4, the subgroup $A$ is finitely generated and abelian, so is polycyclic too. It follows that $G$ is polycyclic. \hfill $\square$

**Lemma 4.6.** Let $G$ be a finitely generated group that is an extension $N \cdot Q$ of finitely generated virtually nilpotent groups. Then $G$ is virtually soluble.

**Proof.** Assume first that $N$ is finite; we then claim that $G$ is virtually nilpotent. Indeed the centralizer $Z_G(n)$ has finite index in $G$, so $Z = \bigcap_{n>0} Z_G(n)$ has finite index in $G$. Then $Z$ is a central extension of $Z \cap N$ by $Z/(Z \cap N)$, so is virtually nilpotent; and then so is $G$.

We turn to the general case. Let $N_0$ be a nilpotent subgroup of finite index in $N$. Up to replacing $N_0$ by $\bigcap_{[N:M]=[N_0:M]} M$, we may assume $N_0$ is characteristic in $N$, and therefore normal in $G$. By the first paragraph, $G/N_0$ is virtually nilpotent, so $G$ is virtually soluble. \hfill $\square$

We recall that a group is **noetherian** if all its subgroups are finitely generated; in other words, if every chain $H_1 < H_2 < \cdots$ of subgroups of $G$ is finite.

**Lemma 4.7.** A group $G$ is polycyclic if and only if it is both soluble and noetherian.

**Proof.** Note first that an abelian group is noetherian if and only if it is finitely generated; if finitely generated, it is of the form $\mathbb{Z}^d \times F$ for a finite abelian group $F$, and is clearly noetherian.

If $G$ is soluble and noetherian, then all quotients $G^{(i)} / G^{(i+1)}$ along its derived series are also noetherian, so finitely generated; the derived series may then be refined into a polycyclic series.

Conversely, an extension of noetherian groups is noetherian, so if $G$ is polycyclic, then it is noetherian by induction. \hfill $\square$

This reduction to polycyclic groups brings us closer to groups of polynomial growth; the next step is the

**Theorem 4.8 (Wolf).** Let $G$ be a polycyclic group of subexponential growth. Then $G$ is virtually nilpotent.
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Proof. Let $G = G_0 > G_1 > \cdots$ be a polycyclic series of minimal length. If $[G : G_1] < \infty$, proceed inductively with $G_1$. Assume therefore that $G/G_1 \cong \mathbb{Z} = \langle x \rangle$. By induction, there is a nilpotent subgroup $N \leq G_1$ of finite index. Furthermore, since $G_1$ is finitely generated by Proposition 4.3, we may suppose that $N$ is characteristic in $G_1$, at the cost of intersecting it with its finitely many images under automorphisms of $G_1$; so we may assume $N \triangleleft G$. We have $N\langle x \rangle \leq G$ of finite index, and we replace $G$ by $N\langle x \rangle$, to simplify notation.

We now seek a central series $(N_k)$ in $N$, i.e. a series with $N_0 = N$, all $N_k$ normal in $G$, and $N_k/N_{k+1} \leq \mathbb{Z}(N/N_{k+1})$; and we require that some non-zero power $x^n$ centralizes $N_k/N_{k+1}$ for all $k$. Then $(N,x^n)$ will be the finite-index nilpotent subgroup of $G$ we are after.

Among central series, choose one maximizing the number of $k$ such that $N_k/N_{k+1}$ is infinite; it exists because the number of factors is bounded by the Hirsch length of $G$. The torsion subgroup of $N_k/N_{k+1}$ is characteristic, so insert it in the series between $N_k$ and $N_{k+1}$. The resulting series is such that each quotient $N_k/N_{k+1}$ is either finite or free abelian; and, in the latter case, if $M \triangleleft G$ and $N_{k+1} \leq M \leq N_k$, then either $N_{k+1} = M$ or $N_k/M$ is finite.

If $N_k/N_{k+1}$ is finite, then certainly some non-zero power of $x$ will act trivially on it. We therefore consider $N_k/N_{k+1} \cong \mathbb{Z}^m$, and we study the $\mathbb{Q}[x]$-module $V := N_k/N_{k+1} \otimes \mathbb{Q} \cong \mathbb{Q}^m$.

The module $V$ is irreducible; indeed, otherwise there would exist a proper, non-trivial invariant subspace $W < V$; then $M := \{x \in N_k \mid xN_{k+1} \in W\}$ is a normal subgroup of $G$, of infinite index in $N_k$, contradicting the maximality of the number of infinite factors in $(N_k)$. We then use the

Lemma 4.9 (Schur). Let $V$ be an irreducible module. Then $\text{End}(V)$ is a division ring.

Proof. Let $\alpha \neq 0 \in \text{End}(V)$ be an endomorphism; then $\ker(\alpha)$ and $\alpha(V)$ are invariant subspaces, so $\ker(\alpha) = 0$ and $\alpha(V) = V$; so $\alpha$ is invertible. □

We see $x \in G$ as an endomorphism of $V$; by Lemma 4.9, the ring $\text{End}(V)$ does not contain nilpotent elements, so $x$ generates a field $\mathbb{Q}(x)$ within $\text{End}(V)$. Since $\text{End}(V)$ is finite-dimensional, $x$ is algebraic. Since $x$ preserves the lattice $N_k/N_{k+1} \subset V$, it is an algebraic integer. We now recall the classical

Lemma 4.10 (Kronecker). Let $\tau$ be an algebraic number, all of whose conjugates have norm 1. Then $\tau$ is a root of unity.

Proof. Let $\tau$ be algebraic of degree $n$, and consider some power $\sigma = \tau^n$. Then $\sigma \in \mathbb{Q}(\tau)$, and all conjugates of $\sigma$ have norm 1, so the coefficients of the minimal polynomial of $\sigma$, which are symmetric functions of the conjugates of $\sigma$, have norm at most $2^n$. It follows that there are finitely many such minimal polynomials, so $\sigma^n = \sigma^M$ for some $M > N$. □

We are now ready to finish the proof. Either all conjugates of $x$ (seen now as an algebraic number) have norm $\leq 1$; and then $x$ is a root of unity by Lemma 4.10 so
$x^n$ acts trivially for some $n > 0$; or there exists an embedding of $\mathbb{Q}(x)$ in $\mathbb{C}$ such that $|x| > 1$.

In that last case, we may replace $x$ by a power of itself so that $|x| > 2$. Choose $y \in N_k \setminus N_{k+1}$, seen as a vector $v \neq 0 \in V$. Consider as in the proof of Proposition 4.3 all expressions $x^{-1}y^{e_1}x^{-1}y^{e_2} \ldots x^{-1}y^{e_n}x^n$, with all $e_j \in \{0, 1\}$. There are $2^n$ such expressions, and their length is linear in $n$, so two must be equal in $G$ because $G$ has subexponential growth. This leads in $V$ to the relation

$$(e_1 - f_1)x(v) + \cdots + (e_{n-1} - f_{n-1})x^{n-1}(v) + x^n(v) = 0,$$

so $(e_1 - f_1)x + \cdots + (e_{n-1} - f_{n-1})x^{n-1} + x^n = 0$, because only 0 is non-invertible in $\text{End}(V)$. Now taking norms we get

$$|x|^n \leq (e_1 - f_1)|x| + \cdots + (e_{n-1} + f_{n-1})|x|^{n-1} \leq |x| |x|^{n-1} - 1 \leq |x|^n$$

using $|x| > 2$, a contradiction. \hfill $\square$

**Corollary 4.11.** Let $G$ be a virtually soluble finitely generated group. Then $G$ has either polynomial or exponential growth, and has polynomial growth precisely when it is virtually nilpotent. \hfill $\square$

### 4.3 Groups of intermediate growth

The previous sections were aimed at showing that “most” groups have polynomial or exponential growth; John Milnor asked in 1968 whether there existed any groups of intermediate growth [92]. There can be no such examples among virtually soluble groups, as we saw above; nor among linear groups (subgroups of matrix groups over fields), by Tits’ alternative [122].

Milnor’s question has, however, a positive answer, which was given in the early 1980’s by Slava Grigorchuk. We give here his example.

Set $\mathcal{A} = \{0, 1\}$, and consider the following group $G$ acting recursively on the set $X := \mathcal{A}^\mathbb{N}$ of infinite sequences over $\mathcal{A}$. It is generated by four elements $a, b, c, d$ defined by

$$(x_0x_1 \cdots)a = (1 - x_0)x_1 \cdots,$$

$$(x_0x_1 \cdots)b = \begin{cases} 
  x_0 \cdots (1 - x_n)x_{n+1} \cdots & \text{if } x_0 = \cdots = x_{n-2} = 0 \neq x_{n-1}, n \neq 0 \pmod{3} \\
  x_0x_1 \cdots & \text{else},
\end{cases}$$

$$(x_0x_1 \cdots)c = \begin{cases} 
  x_0 \cdots (1 - x_n)x_{n+1} \cdots & \text{if } x_0 = \cdots = x_{n-2} = 0 \neq x_{n-1}, n \neq 2 \pmod{3} \\
  x_0x_1 \cdots & \text{else},
\end{cases}$$

$$(x_0x_1 \cdots)d = \begin{cases} 
  x_0 \cdots (1 - x_n)x_{n+1} \cdots & \text{if } x_0 = \cdots = x_{n-2} = 0 \neq x_{n-1}, n \neq 1 \pmod{3} \\
  x_0x_1 \cdots & \text{else}.
\end{cases}$$
This action is the limit of an action on finite sequences \( A^* \), which is the vertex set of the binary rooted tree, and \( G \) is self-similar, see Definition 2.15.

**Theorem 4.12 (Grigorchuk).** The group \( G \) has intermediate growth. More precisely, let \( \eta \approx 0.811 \) be the positive root of \( X^3 + X^2 + X - 2 = 0 \); then

\[
\exp(n^{1/2}) \lesssim v_{G,S}(n) \lesssim \exp(n^{\log(2)/(\log(2) - \log(\eta))}).
\]

We begin by a series of exercises deriving useful properties of \( G \). Details may be found e.g. in [61, Chapter 8]. The self-similar structure of \( G \) is at the heart of all arguments; let us describe it again, starting from the action above.

There is an injective group homomorphism \( \Phi : \{1, X, 0, 1\} \to \langle (G \times G) \rangle \), written \( g \mapsto \langle g, 0, g \rangle \), and defined as follows. If \( g \) permutes \( 0X \) and \( 1X \) then \( \pi_g = \varepsilon \neq 1 \) while if \( g \) preserves \( 0X \) and \( 1X \), and for \( i = 0,1 \) define a permutation \( g_i \) of \( X \) by \( (x_0 x_1 \ldots) g = (x_0 \pi_{g_i})(x_1 \ldots) g_{s_0} \). To see that the \( g_i \) belong to \( G \), note that \( \Phi \) is given on the generators by

\[
\Phi : \begin{cases} 
 a \mapsto \langle 1, 1 \rangle \varepsilon, \\
 b \mapsto \langle a, c \rangle, \\
 c \mapsto \langle a, d \rangle, \\
 d \mapsto \langle 1, b \rangle.
\end{cases}
\]

**Exercise 4.13 (**). Check in \( G \) the relations \( a^2 = b^2 = c^2 = d^2 = bcd = (ad)^4 = 1 \).

We fix once and for all the generating set \( S = \{a, b, c, d\} \) of \( G \). It follows from the exercise that every element of \( G \) may be written as a word of minimal length in the form \( s_0 a s_1 \ldots s_{n-1} a s_n \) for some \( s_0, s_n \in \{1, b, c, d\} \) and other \( s_i \in \{b, c, d\} \).

We let \( \eta \approx 0.811 \) be the real root of \( X^3 + X^2 + X - 2 = 0 \), and define a metric on \( G \) by setting

\[
\|a\| = 1 - \eta^3, \quad \|b\| = \eta^3, \quad \|c\| = 1 - \eta^2, \quad \|d\| = 1 - \eta
\]

and extending the metric to \( G \) by the triangle inequality: \( \|g\| = \min\{\|s_1\| + \cdots + \|s_n\| : g = s_1 \cdots s_n\} \).

**Lemma 4.14.** If \( \Phi(g) = \langle g_0, g_1 \rangle \pi \), then \( \|g_0\| + \|g_1\| \leq \eta(\|g\| + \|a\|) \).

**Proof.** Consider \( g \in G \). Since \( \|c\| + \|d\| \geq \|b\| \) etc., \( g \) may be written as a word of minimal norm in the form \( s_0 a s_1 \cdots s_{n-1} a s_n \) for some \( s_0, s_n \in \{1, b, c, d\} \) and other \( s_i \in \{b, c, d\} \), using Exercise 4.13. Now among the \( s_i \), each ‘\( b \)’ after it, contributes \( \|b\| + \|a\| = 1 \) to \( \|g\| \), and contributes at most \( \|a\| + \|c\| = \eta \) to \( \|g_0\| + \|g_1\| \) because \( \Phi(b) = \langle a, c \rangle \). Similarly, each ‘\( c \)’ or ‘\( a \)’ contributes \( \eta \) to \( \|g_0\| + \|g_1\| \), and at most \( \eta^2 \) to \( \|g_0\| + \|g_1\| \), and each ‘\( d \)’ or ‘\( a \)’ contributes \( \eta^2 \) to \( \|g_0\| + \|g_1\| \). Only the last \( s_n \) may not have an ‘\( a \)’ after it. Summing all these inequalities proves the lemma.

**Exercise 4.15 (**). Define \( \sigma : G \to G \) by
\[ \sigma : a \mapsto c^a, \quad b \mapsto d, \quad d \mapsto c, \quad c \mapsto b, \]

extended multiplicatively. Prove \( \Phi(\sigma(g)) = \{\theta(g), g\} \) for all \( g \in G \), where \( \theta(a) = d, \theta(b) = 1, \theta(c) = \theta(d) = a \) is a homomorphism to the finite group \( \langle a, d \rangle \cong D_4 \).

Deduce that \( \sigma \) is well-defined, and is an injective endomorphism of \( G \). For the usual word metric, prove that \( |\sigma(g)| \leq 2|g| + 1 \) for all \( g \in G \).

**Proof of Theorem 4.12** see [5]. For the lower bound, consider the map (not homomorphism!)

\[ F : G \times G \to G, \quad (g_0, g_1) \mapsto \sigma(g_0)^a \sigma(g_1). \]

By the exercise, we have \( \Phi(F(g_0, g_1)) = \{g_0 \theta(g_1), \theta(g_0) g_1\} \). Since \( \#G = 8 \) and \( \Phi \) is injective, we have \( \#\Phi^{-1}(g) = 8 \) for all \( g \in G \). Also, \( |\sigma(g)| \leq 2|g| + 1 \) for the usual word metric, so \( |\Phi(g_0, g_1)| \leq 2|g_0| + 2|g_1| + 4 \). Denoting by \( B(n) \) the ball of radius \( n \) in \( G \) for the word metric, we have \( F(B(n) \times B(n)) \subseteq B(4n + 4) \), so the growth function \( v(n) \) of \( G \) satisfies \( 8v(n - 2)^2 \leq v(4n - 2) + 4 \leq v(4n - 2) \). Iterating, we have \( v(4n - 2) \geq 8^{2^{n-2}} \), so \( v(n) \geq 8^{\sqrt{n^2} - 1} \).

For the upper bound, we make use of the norm \( \|\cdot\| \), and represent every \( g \in G \) by a finite rooted tree \( R(g) \). Fix any constant \( K > \|a\|/\eta - 1 \). Given \( g \in G \), construct \( R(g) \) as follows. If \( \|g\| \leq K \), let \( R(g) \) be the one-vertex tree with label \( g \) written at the root, which is also a leaf of the tree.

If \( \|g\| > K \), compute \( \Phi(g) = \{g_0, g_1\} \). Note \( \|g_0\|, \|g_1\| < \|g\| \), and construct \( R(g_0), R(g_1) \) recursively. Let then \( R(g) \) be the tree with a root labeled \( \pi \) connected by two edges leading to the roots of \( R(g_0) \) and \( R(g_1) \) respectively. Since \( \Phi \) is injective, the map \( R \) is injective, and it remains to count the number of trees of given size.

Up to replacing \( \|g\| \) by \( \max\{1, \|g\| - K\} \), we may assume that, in Lemma 4.14, we have \( \|g_0\| + \|g_1\| \leq \eta \|g\| \) as soon as \( \|g\| \) is large enough.

Let us denote by \( \#R(g) \) the number of leaves of \( R(g) \), and set \( \alpha = \log 2/(\log 2 - \log \eta) \). We claim that there is a constant \( D \) such that \( \#R(g) \leq D\|g\|^\alpha \) for all \( g \in G \). This is certainly true if \( \|g\| \) is small enough. For \( \|g\| > K \), we proceed by induction:

\[
\#R(g) = \#R(g_0) + \#R(g_1) \leq D(\|g_0\|^\alpha + \|g_1\|^\alpha) \\
\leq 2D\left( \frac{\|g_0\| + \|g_1\|}{2} \right)^\alpha \quad \text{by convexity of } X^\alpha \\
\leq 2D\|g\|^\alpha (\frac{\eta}{2})^\alpha = D\|g\|^\alpha.
\]

We finally count the number of trees with \( n \) leaves. There are Catalann\( n \) such tree shapes; each of the \( n - 1 \) non-leaf vertices has a label in \( \{1, e\} \), and each of the \( n \) leaf vertices has a label in \( B(K) \). It follows that there are Catalann\( n \)2\( n-1 \)b\( K^n \leq E^n \) trees with at most \( n \) leaves, for some constant \( E \); and then \( v(n) \leq E^n \).

**Exercise 4.16 (**) Prove that \( G \) is a torsion group.

*Hint:* Use Exercise 4.13, Lemma 4.14 and induction.
5 Paradoxical decompositions

We consider again the general case of a group $G$ acting on a set $X$, and shall derive other characterizations of amenability, based on finite partitions of $X$.

**Definition 5.1.** A $G$-set $X$ is **paradoxical** if there are partitions
\[ X = Y_1 \sqcup \cdots \sqcup Y_m = Z_1 \sqcup \cdots \sqcup Z_n, \]
and $g_1, \ldots, g_m, h_1, \ldots, h_n \in G$, such that
\[ X = Y_1 g_1 \sqcup \cdots \sqcup Y_m g_m \sqcup Z_1 h_1 \sqcup \cdots \sqcup Z_n h_n. \]

As a naive example, relax the condition that $G$ be a group, and consider the monoid of affine transformations of $\mathbb{N}$. Then $\mathbb{N} = \mathbb{N} g_1 \sqcup \mathbb{N} h_1$ for $g_1(n) = 2n$ and $h_1(n) = 2n + 1$ defines a paradoxical decomposition.

**Example 5.2.** We return to Proposition 2.9. More precisely, now, consider $X = G = \langle x_1, x_2 \rangle$ a free group of rank 2; and
\[ Y_1 = \{ \text{reduced words ending in } x_1 \}, \quad Y_2 = G \setminus Y_1, \]
\[ Z_1 = \{ \text{reduced words ending in } x_2 \} \cup \{ 1, x_2^{-1}, x_2^{-2}, \ldots \}, \quad Z_2 = G \setminus Z_1; \]
then $G = Y_1 \sqcup Y_2 = Z_1 \sqcup Z_2 = Y_1 \sqcup Y_2 x_1^{-1} \sqcup Z_1 \sqcup Z_2 x_2^{-1}$.

### 5.1 Hausdorff’s Paradox

John von Neumann had noted already in [103] that non-amenability of $F_2$ was at the heart of the Hausdorff-Banach-Tarski paradox. We first show:

**Proposition 5.3.** The group $SO_3(\mathbb{R})$ of rotations of the sphere contains a non-abelian free subgroup.

**Proof.** There are many classical proofs of this fact. Consider for example the matrices
\[ U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \]
in $SO_3(\mathbb{R})$. They satisfy the relations $U^2 = V^3 = 1$, but no other, since in a product $W = U^{\varepsilon_1} V^{\varepsilon_2} U^{\varepsilon_3} \cdots V^{\varepsilon_n} U^{\varepsilon_{n+1}}$ with $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{0, 1\}$ and $n$ letters $V^{\pm 1}$ we have

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9 This should not come as a surprise, since $\{g_1, h_1\}$ generate a free monoid.
with $a_{ij} \in \mathbb{Z}$ and $a_{33}$ odd, as can be seen from computing $2^n W \mod 2$; so $W \neq 1$. Then $\langle [U, V], [U, V^{-1}] \rangle$ is a free group of rank 2.

Here is another proof: $SO_3(\mathbb{R})$ is the group of quaternions of norm 1. Let $p$ be a prime $\equiv 1 \pmod{4}$, and set

$$S = \{(a + bi + cj + dk)/\sqrt{p} \mid a \in 2\mathbb{N} + 1, b, c, d \in 2\mathbb{Z}, a^2 + b^2 + c^2 + d^2 = p\}.$$ 

It follows from Lagrange’s Theorem on sums of four squares that $\#S = p + 1$, and from the unique factorization of quaternions that $S$ generates a free group of rank $(p + 1)/2$. See [66] for proofs of these facts.

The following paradox follows:

**Theorem 5.4 (Hausdorff [62]).** There exists a partition of the sphere $S^2$, or of the ball $B^3$, into two pieces; and a further partition of each of these into respectively two and three pieces, in such a manner that these be reassembled, using only isometries of $\mathbb{R}^3$, into two spheres or balls respectively.

**Proof.** We first show the following: there is a countable subset $D \subset S^2$ such that one can decompose $S^2 \setminus D = P \sqcup Q$, and further decompose $P = P_1 \sqcup \cdots \sqcup P_m$ and $Q = Q_1 \sqcup \cdots \sqcup Q_n$, so that $S^2 \setminus D = P_1 g_1 \sqcup \cdots \sqcup P_m g_m = Q_1 h_1 \sqcup \cdots \sqcup Q_n h_n$.

Indeed, by Proposition 5.3 there is a free subgroup $G$ of $SO_3(\mathbb{R})$, acting on the sphere. Every non-trivial element of $G$ acts as a rotation, and therefore has two fixed points. Let $D$ denote the collection of all fixed points of all non-trivial elements of $G$; clearly $D$ is countable. The group $G$ acts freely on $S^2 \setminus D$: let $T$ be a choice of one point per orbit. Let $(Y_i, Z_j, g_i, h_j)$ be a paradoxical decomposition of $G$ as in Definition 5.1. Set then $P_i = TY_i g_i^{-1}$ and $Q_j = T Z_j h_j^{-1}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Keeping the same notation, we now show that $S^2$ can be cut as $S^2 = U \sqcup V$, such that for an appropriate rotation $\rho$ we have $\rho(U) \sqcup V = S^2 \setminus D$. Since $D$ is countable, there is a direction $\mathbb{R}v \subset \mathbb{R}^3$ that does not intersect $D$. There are continuously many rotations $\rho$ with axis $\mathbb{R}v$, and only countably many that satisfy $D \cap \rho^n(D) \neq \emptyset$ for some $n \neq 0$; let $\rho$ be any other rotation. Set $U = \bigcup_{n \geq 0} \rho^n(D)$ and $V = S^2 \setminus U$; then $\rho(U) = U \setminus D$ and we are done.

These paradoxical decompositions can be combined (see Corollary 5.8 below for details), proving the statement for $S^2$.

The same argument works for all concentric spheres simultaneously, and therefore for $B^3 \setminus \{0\}$. It remains to show that $B^3$ and $B^3 \setminus \{0\}$ can respectively be cut into isometric pieces. Let $\rho$ be a rotation about $(1/2, 0, \mathbb{R})$ with angle 1 (in radians), and set $W = \{\rho^n(0) \mid n \in \mathbb{N}\}$. Then $\rho(W) = W \setminus \{0\}$, so $B^3 = W \cup (B^3 \setminus W)$ and $B^3 \setminus \{0\} = \rho(W) \cup (B^3 \setminus W)$.  

---

[10] The Axiom of Choice is required here.
5.2 Doubling conditions

Let us restate paradoxical decompositions in a more sophisticated way.

**Definition 5.5.** Let a group $G$ act on a set $X$. A $G$-wobble is a map $\phi : Y \to Z$ for two subsets $Y, Z \subseteq X$, such that there exists a finite decomposition $Y = Y_1 \sqcup \cdots \sqcup Y_n$ and elements $g_1, \ldots, g_n \in G$ with $\phi(y) = yg_i$ whenever $y \in Y_i$.

We define a preorder on subsets of $X$ by $Y \not\leq Z$ if there exists an injective $G$-wobble $Y \to Z$; and an equivalence relation $Y \sim Z$ if there exists a bijective $G$-wobble $Y \to Z$; in that case, we say that $Y$ and $Z$ are **equidecomposable**.

Using that terminology, the $G$-set $X$ is paradoxical if one may decompose $X = Y \sqcup Z$ with $Y \sim Z$.

**Lemma 5.6.** The map $\phi : Y \to Z$ is a $G$-wobble if and only if there exists a finite subset $S \subseteq G$ such that $\phi(y) \in yS$ for all $y \in Y$.

**Proof.** If $\phi$ is a $G$-wobble, set $S = \{g_1, \ldots, g_n\}$, and note $\phi(y) \in yS$ for all $y \in Y$.

Conversely, if $\phi(y) \in yS$ for all $y \in Y$, write $S = \{g_1, \ldots, g_n\}$, and set

$$Y_n = \{y \in Y \mid \phi(y) = yg_n$ and $\phi(y) \neq yg_m$ for all $m < n\}. \quad \square$$

**Corollary 5.7.** The composition of $G$-wobbles is again a $G$-wobble, and the inverse of a bijective $G$-wobble is also a $G$-wobble.

It follows that the set of invertible $G$-wobbles is actually a group. If the space $X$ is assumed compact and the pieces in the decomposition are open, then this group is known as the “topological full group” of $G$, see [9.2].

**Corollary 5.8.** The relation $\not\leq$ is a preorder, and $\sim$ is an equivalence relation.

**Proof.** Consider injective $G$-wobbles $\phi : Y \to Z$ and $\psi : W \to Y$. By Lemma 5.6 there are $S, T \subseteq G$ such that $\phi(y) \in yS$ and $\phi(w) \in wT$ for all $y \in Y, w \in W$. Then $\phi \psi(w) \in wTS$ for all $w \in W$, so $\phi \psi : W \to Z$ is an injective $G$-wobble, again by Lemma 5.6. \square

**Theorem 5.9 (Cantor-Schröder-Bernstein [23]).** Let $Y, Z$ be sets. If there exists an injection $\alpha : Y \to Z$ and an injection $\beta : Z \to Y$, then there exists a bijection $\gamma : Y \to Z$.

Furthermore, $\gamma$ may be chosen so that $\gamma(y) \in \{\alpha(y), \beta^{-1}(y)\}$ for all $y \in Y$.

**Proof.** Let $\alpha : Y \to Z$ and $\beta : Z \to Y$ be injective maps. Set $Y_0 = Y$ and $Z_0 = Z$; and, for $n \geq 1$, set $Y_n = \beta(Z_{n-1})$ and $Z_n = \alpha(Y_{n-1})$. Partition $Y$ as follows:

$$U = \bigsqcup_{n \in \mathbb{N}} Y_{2n} \setminus Y_{2n+1}, \quad V = \bigsqcup_{n \in \mathbb{N}} Y_{2n+1} \setminus Y_{2n+2}, \quad W = \bigcap_{n \in \mathbb{N}} Y_n.$$ 

---

I.e. a transitive, reflexive relation.
Define then \( \gamma : Y \to Z \) as follows:

\[
\gamma(y) = \begin{cases} 
\alpha(y) & \text{if } y \in U; \\
\beta^{-1}(y) & \text{if } y \in V \cup W.
\end{cases}
\]

Therefore \( \gamma \) sends \( Y_n \setminus Y_{n-1} \to Z_{2n+1} \setminus Z_{2n} \) and \( Y_{2n+1} \setminus Y_{2n+2} \to Z_{2n} \setminus Z_{2n-1} \); while sending \( \bigcap Y_n \to \bigcap Z_n \). It follows that \( \gamma \) is a bijection. \( \square \)

**Corollary 5.10.** If \( Y \preceq Z \) and \( Z \preceq Y \), then \( Y \sim Z \).

**Proof.** Consider injective \( G \)-wobbles \( \alpha : Y \to Z \) and \( \beta : Z \to Y \). By Lemma \[5.6\] there are finite sets \( S, T \subseteq G \) such that \( \alpha(y) \in yS \) and \( \beta(z) \in zT \) for all \( y \in Y, z \in Z \).

Let \( \gamma : Y \to Z \) be the bijection given by Theorem \[5.9\] with \( \gamma(y) \in y(S \cup T^{-1}) \). Then \( \gamma \) is a bijective \( G \)-wobble, again by Lemma \[5.6\]. \( \square \)

We also need a little more terminology, coming from graph theory and following Definition \[3.2\].

**Definition 5.11.** A digraph \((V, E)\) is **bipartite** if there is a decomposition \( V = V^+ \cup V^- \) such that \( e^+ \in V^+ \) and \( e^- \in V^- \) for every edge.

If \( V^+ \) and \( V^- \) are \( G \)-sets and are identified, the graph \((V, E)\) is **bounded** if there exists a finite subset \( S \subseteq G \) with \( e^+ \in e^-S \) for all \( e \in E \).

An \( m : n \) matching in \((V, E)\) is a subgraph \((V, M)\) with \( M \subseteq E \), such that for each \( v \in V^+ \) there are precisely \( n \) edges \( e \in M \) with \( e^+ = v \), and for each \( v \in V^- \) there are precisely \( m \) edges \( e \in M \) with \( e^- = v \). We define similarly \( m : (\leq n) \) and \( m : (\geq n) \) matchings.

If \( X \) is a \( G \)-set, a **bounded matching on \( X \)** is a matching in a bounded graph with vertex set \( X \cup X \).

In particular, a \( 1 : 1 \) matching is nothing but a bijection \( V^- \to V^+ \); and a bounded \( 1 : 1 \) matching is a bijective \( G \)-wobble. A \( 1 : (\leq 1) \) matching is an injective map, and a \( 1 : (\geq 0) \) matching is just a map.

**Theorem 5.12 (Hall [60]-Hall-Rado [14]).** Let \( V, W \) be sets, and for each \( v \in V \), let \( E_v \subseteq W \) be a finite set. Assume that, for every finite subset \( F \subseteq V \),

\[
\text{the set } E_F := \bigcup_{v \in F} E_v \text{ contains at least } \#F \text{ elements.} \tag{11}
\]

Then there exists an injection \( e : V \to W \) with \( e(v) \in E_v \) for all \( v \in V \).

**Proof.** Assume first that \( (E_v) \) satisfies \[11\], and that \( \#E_v \geq 2 \) for some \( v \in V \). We show that we may replace \( E_v \) by \( E_v \setminus \{w\} \) for some \( w \in E_v \) and still satisfy \[11\].

Indeed, consider \( w_0 \neq w_1 \in E_v \), and assume that neither \( w_0 \) nor \( w_1 \) may be removed from \( E_v \). Then there are \( F_0, F_1 \subseteq V \) and \( N_i = E_{F_i} \cup (E_v \setminus \{w_i\}) \subseteq W \) for \( i = 0, 1 \) such that \( \#N_i < \#(F_i \cup \{v\}) \); i.e. \( \#N_i \leq \#F_i \). Then
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\[ \#F_0 + \#F_1 \geq \#N_0 + \#N_1 = \#(N_0 \cup N_1) + \#(N_0 \cap N_1) \]
\[ \geq \#(E_{F_0 \cup F_1}) + \#(E_{F_0 \cap F_1}) \]
\[ \geq \#(F_0 \cup F_1) + 1 + \#(F_0 \cap F_1) = \#F_0 + \#F_1 + 1, \]

a contradiction. Then, inductively, we may suppose \( \#E_v = 1 \) for any given \( v \in V \).

If \( V \) is finite, we are done by repeatedly replacing each \( E_v \) by a singleton; the injection is \( v \mapsto w \) for the unique \( w \in E_v \).

If \( V \) is countable, we may write \( V = \{ v_1, v_2, \ldots \} \) and define recursively \( E_v^0 = E_v \) for all \( v \in V \), and, for \( i, j > 0 \),

\[ E_v^j = \begin{cases} E_v^{j-1} & \text{if } j \neq i, \\ \text{the singleton coming from the above operation} & \text{if } j = i; \end{cases} \]

then the required injection is \( v_i \mapsto w \) for the unique \( w \in E_v^i \).

For general \( V \), we need the help of an axiom. Order all systems \( (E_v^i) \) satisfying (11) by \( (E_v^i) \leq (E_v^j) \) if \( E_v^i \subseteq E_v^j \) for all \( v \in V \). By Zorn’s lemma, \( \{(E_v^i) \leq (E_v^j)\} \) admits a minimal element \( (E_v^i) \). If \#\( E_v^i \) \( \geq 2 \) for some \( v \in V \), then by the above it could be made strictly smaller; therefore \#\( E_v^i \) \( = 1 \) for all \( v \in V \) and we again have an injection \( V \to W \).

Note that, if one drops the assumption that \( E_v \) is finite for all \( v \), then there are counterexamples to the theorem, e.g. \( V = W = \mathbb{N} \), \( E_0 = \mathbb{N} \) and \( E_{n+1} = \{ n \} \) for all \( n \in \mathbb{N} \). For more details see [93].

**Corollary 5.13.** Let \( (V, E) \) be a bipartite graph, and assume that for all \( \varepsilon \in \{ \pm 1 \} \) and all finite subsets \( F \subset V^\varepsilon \) the set

\[ \{ v \in V^{-\varepsilon} \mid e^{-\varepsilon} = v, e^\varepsilon \in F \text{ for some } e \in E \} \]

is finite and contains at least \#\( F \) elements. Then there exists a 1 : 1 matching in \( (V, E) \).

**Proof.** By Theorem 5.12 there exists a subgraph of \( (V, E) \) defining an injection \( V^- \to V^+ \); and symmetrically there exists a subgraph of \( (V, E) \) defining an injection \( V^+ \to V^- \). Applying Theorem 5.9 there exists a subgraph of \( (V, E) \) defining a bijection \( V^- \to V^+ \). \( \square \)

We are ready to prove the equivalence of our new notions:

**Theorem 5.14.** Let \( X \) be a \( G \)-set. The following are equivalent:

1. \( X \) is paradoxical;
2. \( X \) is not amenable;
3. For any \( m > n > 0 \) there exists a bounded \( m : n \) matching on \( X \);
4. There exists a \( G \)-wobble \( \phi : X \to X \) with \#\( \phi^{-1}\{ x \} \) \( = 2 \) for all \( x \in X \);
5. There exists a \( G \)-wobble \( \phi : X \to X \) with \#\( \phi^{-1}\{ x \} \) \( \geq 2 \) for all \( x \in X \).
and consider a finite subset \( F \) of \( G \) which we project to a bounded subset \( V \) of \( X \) and all these vertices are reached from \( X \) by edges in \( (V, E) \). Conversely, fix \( g \in S \), and consider a finite subset \( F \subset V^+ \). Because \( m > n \), every \((x, i) \in F\) is connected by an edge to \((xg^{-1}, i) \in V^- \). Therefore, every finite \( F \subset V^\pm \) has at least \#F neighbours in \( V^\pm \).

We now invoke the Hall-Rado theorem \ref{Hall-Rado} to obtain a 1 : 1 matching \((V, \mathcal{M})\); which we project to a bounded \( m : n \) matching \((X \cup X, \mathcal{M})\) by setting \( e^\pm = x \) whenever we had \( e^\pm = (x, *) \) in \((V, \mathcal{M})\).

(3) \( \Rightarrow \) (4) Let \( \mathcal{M} \) be a bounded 2 : 1 matching on \( X \). Given \( x \in X \), there is a unique \( e \in \mathcal{M} \) with \( e^- = x \); set \( \phi(x) = e^+ \). This defines a \( G \)-wobble \( \phi : X \rightarrow X \) with \( \#\phi^{-1}(y) = 2 \).

(4) \( \Rightarrow \) (5) is obvious.

(5) \( \Rightarrow \) (1) For each \( x \in X \) choose \( y_x \in X \) with \( \phi(y_x) = y \); this is possible using the Axiom of Choice. Set \( Y = \{ y_x \mid x \in X \} \), and \( Z = X \setminus Y \). We have \( X = Y \cup Z \), and \( \phi \) restricts to bijective \( G \)-wobbles \( Y \rightarrow X \) and \( Z \rightarrow X \), so \( Y \sim X \sim Z \).

(5) \( \Rightarrow \) (2) Let \( S \subset G \) satisfy \( \phi(x)S \ni x \) for all \( x \in X \). Then, for any finite \( F \subset X \), we have \( \phi^{-1}(F) \subset FS \) so \( \#(FS) \geq 2\#F \).

If a group \( G \) contains a non-abelian free subgroup, then \( G \) is not amenable. The converse is not true, as we shall see in \[7.3\]. However, the following weaker form of the converse holds:

**Theorem 5.15 (see [127]).** Let \( X \) be a \( G \)-set. The following are equivalent:

1. \( X \) is not amenable;
2. There is a free action of the free group \( F_2 \) on \( X \) by bijective \( G \)-wobbles;
3. There is a free action of a non-amenable group on \( X \) by bijective \( G \)-wobbles.

**Proof.** (1) \( \Rightarrow \) (2) Assume that \( X \) is non-amenable, so by Theorem 5.14 there exists a \( G \)-wobble \( \phi : X \rightarrow X \) with \( \#\phi^{-1}(x) = 2 \) for all \( x \in X \). Let \( S \subset G \) satisfy \( \phi(x) \in xS \) for all \( x \in X \).
View $X$ as a directed graph $T$, with an edge from $x$ to $\phi(x)$ for all $x$; and let $U$ be the corresponding undirected graph. These graphs are 3-regular: in $T$ every vertex has one outgoing and two incoming edges. Assume that there is a cycle in $U$. This cycle is necessarily oriented, for otherwise there would be two outgoing edges at a vertex. Furthermore, there cannot be two cycles in the same connected component of $U$: if there were two such cycles, consider a minimal path $p$ joining them. At least one of $p$'s extremities would be oriented away from its end, and again there would be two outgoing edges at a vertex.

It follows that all connected components of $U$ are either 3-regular trees, or cycles with 3-regular trees attached to them. Remove an edge from each cycle, creating in this manner either two vertices of degree 2 or one of degree 1. In all cases, at each vertex $v$ of degree $< 3$ choose a ray $\rho_v$ going to infinity consisting entirely of degree-3 vertices, and shift the edges attached to $\rho_v$ towards $v$ along $\rho_v$ so as to increase the degree of $v$. In this manner, we obtain a 3-regular forest $U$ with vertex set $X$, with the following property: there exists a finite subset $S' \subseteq G$ such that every edge of $U$, joining say $x$ to $y$, satisfies $y \in xS'$. In fact, $S' = S \cup S^{-1}S^2$ will do.

Now label all edges of $U$ with $\{a, b, c\}$ in such a manner that at every vertex all three colours appear exactly once on the incident edges. This is easy to do: on each connected component label arbitrarily an edge; then at each extremity label the two other incident edges by the two remaining symbols, and continue.

In this manner, every connected component of $U$ becomes the Cayley graph of $H := \langle a, b, c \mid a^2, b^2, c^2 \rangle$. In effect, we have defined an action of $H$ on $X$ by $G$-wobbles: the image of $x$ under $a, b, c$ respectively is the other extremity of the edge starting at $x$ and labeled $a, b, c$ respectively. The group $H$ contains a free subgroup of rank 2, namely $\langle ab, bc \rangle$. (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (1) Assume that $X$ admits a free action of a non-amenable group $H$ by bijective $G$-wobbles; without loss of generality, $H$ is finitely generated, say by a set $T$. Since $H$ is not amenable and acts freely, it does not satisfy Følner’s condition by Proposition 2.12, so there exists $\delta > 0$ such that $\#(FT) \geq \delta \#F$ for all $F \subseteq X$.

Let $S \subseteq G$ satisfy $xT \subseteq xS$ for all $x \in X$. In particular, $\#(FS) \geq \delta \#F$, so $X$ does not satisfy Følner’s condition. □

Note that the proof becomes trivial in case $X = G \leftrightarrow G$ and $G$ contains a non-abelian free subgroup; indeed the action of $G$ itself is by $G$-wobbles.

It is possible to modify slightly this construction to make $F_2$ act transitively by $G$-wobbles, see [117].
6 Convex sets and fixed points

We consider an abstract version of convex sets, introduced by Stone in [119] as sets with barycentric coordinates:

**Definition 6.1.** A convex space is a set $K$ with an operation $[0, 1] \times K \times K \to K$ of taking convex combinations, written $(t, x, y) \mapsto t(x, y)$, satisfying the axioms

\[
\begin{align*}
0(x, y) &= x = t(x, x), \\
t(x, y) &= (1 - t)(y, x), \quad \text{for all } x, y, z \in K \text{ and } 0 \leq u \leq t \leq 1. \\
t(x, t(y, z)) &= \frac{t - u}{1 - u} (x, u(y, z)),
\end{align*}
\]

It is called cancellative if it furthermore satisfies the axiom

\[t(x, y) = t(x, z), t > 0 \Rightarrow y = z.\]

An affine map is a map $f : K \to L$ between convex spaces satisfying $t(f(x), f(y)) = t(x, y)$ for all $t \in [0, 1]$ and all $x, y \in K$.

Usual convex subsets of vector spaces are typical examples; if $K \subseteq V$ is convex, then $t(x, y) := (1 - t)x + ty$ gives $K$ the structure of a convex space. There are other examples: for any set $X$ one may take $K = P(X)$ with $t(x, y) = x \cup y$ whenever $t \in (0, 1)$.

As another example, trees (and more generally $\mathbb{R}$-trees: geodesic metric spaces in which every triangle is isometric to a tripod) are convex spaces: for $x, y$ in a tree, there is a unique geodesic from $x$ to $y$, and $t(x, y)$ is defined as the point at distance $t d(x, y)$ from $x$ along this geodesic. Unless the tree is a line segment, this convex space is not cancellative.

The set of closed balls in an ultrametric space with Hausdorff distance, is also an example of a convex space; it is actually isomorphic to the convex space associated with an $\mathbb{R}$-tree, see [65].

It turns out [119 Theorem 2] that those convex spaces that are embeddable in real vector spaces as convex subsets are precisely the cancellative ones.

A topological convex space is a convex space $K$ with the structure of a topological space, such that the structure map $[0, 1] \times K \times K \to K$ is continuous. A convex $G$-space is a convex space on which a group $G$ acts by affine maps. The convex hull of a subset $X \subseteq K$ of a convex space is the intersection $\hat{X}$ of all convex subspaces of $K$ containing $X$.

**Exercise 6.2 (*).** Convex spaces form a variety. Prove that the free convex space on $n + 1$ generators is isomorphic to the standard $n$-simplex $\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_i \geq 0, \sum x_i = 1\}$, and also to the convex hull of the basis vectors in $\mathbb{R}^{n+1}$. In particular, it is cancellative.

\[\text{12 Namely, a metric space in which the ultratriangle inequality } d(x, z) \leq \max\{d(x, y), d(y, z)\} \text{ holds.}\]
Definition 6.3. Let $X, Y$ be $G$-sets. We say that $Y$ is $X$-markable if there exists an equivariant $G$-map $X \to Y$.

Theorem 6.4. Let $X$ be a $G$-set. The following are equivalent:

1. $X$ is amenable;
2. Every compact $X$-markable convex space admits a fixed point;
3. Every compact $X$-markable convex subset of a locally compact topological vector space admits a fixed point.

Proof. (1) $\Rightarrow$ (2) By Lemma 3.6, there exists a net $(F_n)_{n \in \mathcal{N}}$ of Følner sets in $X$. Let $K$ be a compact $X$-markable convex space, and let $\pi : X \to K$ be a $G$-equivariant map. For each $n \in \mathcal{N}$, set

$$k_n := \sum_{x \in F_n} \frac{1}{|F_n|} \pi(x) \in K.$$

Then $(k_n)$ is a net in $K$, so by compactness admits a cluster point, say $k$. The $k_n g$ have the same limit, so $k$ is a fixed point.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Take $K = \mathcal{M}(X)$; it is compact by Lemma 2.20, $X$-marked by $\delta$, convex by Lemma 2.23 and contained in the topological vector space $\ell^1(X)^*$ which is locally compact by the Banach-Alaoglu theorem [116, Theorem 3.15]. A fixed point is an invariant mean on $X$. $\square$

In particular, a group $G$ is amenable if and only if every compact non-empty convex $G$-space admits a fixed point. We may thus show that amenability of $G$-sets is stable under amenable extensions:

Proposition 6.5. Let $X$ be a $G$-set, and let $N \normal G$ be a normal subgroup with $G/N$ amenable. Then $X \\leftrightarrow G$ is amenable if and only if $X \\leftrightarrow N$ is amenable.

Proof. Let $K$ be an $X$-markable convex compact space. The “if” direction is obvious, since every $G$-fixed point in $K$ is $N$-fixed. Conversely, if $K^N \neq \emptyset$, then $K^N$ is a non-empty convex compact space on which $G/N$ acts, and has a fixed point because $G/N$ is amenable. Clearly $(K^N)^{G/N} = K^G$, so $X \\leftrightarrow G$ is amenable. $\square$

6.1 Measures

Consider a topological space $X$. We recall that $\mathcal{C}(X)$ denotes the space of continuous functions $X \to \mathbb{R}$, and that probability measures on $X$ are identified with functionals $\lambda \in \mathcal{C}(X)^*$ such that $\lambda(1) = 1$ and $\lambda(\phi) \geq 0$ if $\phi \geq 0$. One sometimes writes $\lambda(\phi) = \int \phi d\lambda$.

An important property of measures on subsets of vector spaces is that they have barycentres:
Lemma 6.6. Let $K$ be a non-empty convex compact subset of a locally compact topological vector space, and let $\mu \in C(K)^+$ be a probability measure. Then there exists a unique $b \in K$ such that $\mu(\phi) = \phi(b)$ for all affine maps $\phi \in C(K)$. We write $b = \int td\mu(t)$ and call it the barycentre of $\mu$.

Proof. For any affine function $\phi : K \to \mathbb{R}$, set
$$K_\phi := \{ x \in K \mid \mu(\phi) = \phi(x) \}.$$ It is clear that $K_\phi$ is convex and compact. Furthermore, it is non-empty; more generally, we will show that $K_{\phi_1} \cap \cdots \cap K_{\phi_n} \neq \emptyset$ for all affine functions $\phi_1, \ldots, \phi_n : K \to \mathbb{R}$.

Write $\phi = (\phi_1, \ldots, \phi_n) : K \to \mathbb{R}^n$. Define $L = \{ \phi(x) : x \in K \}$; this is a convex compact in $\mathbb{R}^n$. Define $p \in \mathbb{R}^n$ by $p_i = \mu(\phi_i) = \int_K \phi_i d\mu$. We claim that $p$ belongs to $L$; once this is shown, every $x \in K$ with $\phi(x) = p$ belongs to $K_{\phi_1} \cap \cdots \cap K_{\phi_n}$, so the intersection is non-empty.

We now show that, for any $q \in L$, we have $p \neq q$. There exists then a hyperplane that separates $q$ from $L$, namely the nullspace of any affine map $\tau : \mathbb{R}^n \to \mathbb{R}$ with $\tau(q) < 0$ and $\tau(L) > 0$. In particular $\tau(\phi(x)) > 0$ for all $x \in K$, so by integrating $\tau(p) > 0$, and therefore $p \neq q$.

Set now $B = \bigcap_{\phi \text{ affine}} K_\phi$. It is non-empty by compactness of $K$, because any finite sub-intersection is non-empty.

Affine functions separate points in $K$, so $B$ contains a single point $b$. $\square$

Theorem 6.7. Let $X$ be a $G$-set. The following are equivalent:

1. $X$ is amenable;
2. Every compact $X$-markable set admits an invariant probability measure.

Proof. (1) $\Rightarrow$ (2) Let $K$ be a compact $G$-set and let $\pi : X \to K$ be a $G$-equivariant map. Let $m \in \ell^\infty(X)^+$ be a $G$-invariant positive functional; then $m \circ \pi^* : \ell^\infty(K) \to \ell^\infty(X) \to \mathbb{R}$ is a $G$-invariant, positive functional on $K$, and its restriction to $\ell^\infty(K)$ is an invariant probability measure on $K$.

(2) $\Rightarrow$ (1) Let $K$ be a compact $X$-markable convex subset of a locally compact topological vector space, and let $\lambda$ be an invariant probability measure on $K$. Then $\lambda$’s barycentre, which exists by Lemma 6.6, is a fixed point in $K$, so $X$ is amenable by Theorem 6.4 (3) $\Rightarrow$ (1). $\square$

Exercise 6.8 (*). Reprove that the free group $F_2$ is not amenable as follows: write $F_2 = \langle a, b \rangle$, and make it act on the circle $X = [0,1] / (0 \sim 1)$ by $xa = x^2$ and $xb = (x + 1/2) \mod 1$ for all $x \in [0,1]$. Show that the only $a$-invariant measure on $X$ is $\delta_0$, and that it is not $b$-invariant.

We proved in Corollary 4.2 that abelian groups are amenable. We may reprove it as follows:

Proposition 6.9 (Kakutani [76]-Markov [90]). Let $G$ be an abelian group. Then $G$ is amenable.

13 Note that we use here the Hahn-Banach theorem, which requires certain logical axioms.
Amenability of groups and $G$-sets

Proof. Let $G$ act affinely on a convex compact $K$. For every $g \in G$ and every $n \geq 1$ define a continuous transformation $A_{n,g}: K \to K$ by

$$A_{n,g}(x) = \frac{1}{n} \sum_{i=0}^{n-1} xg^i.$$ 

Let $\mathcal{S}$ denote the monoid generated by $\{A_{n,g} \mid g \in G, n \geq 1\}$. We show that $\bigcap_{s \in \mathcal{S}} s(K)$ is not empty. Since $K$ is compact, it suffices to show that every finite intersection $s_1(K) \cap \cdots \cap s_k(K)$ is non empty. To that end, set $t = s_1 \cdots s_k$. We have

$$s_i(K) \subseteq s_i s_1 \cdots \hat{s}_i \cdots s_k(K) = t(K),$$

because $\mathcal{S}$ is commutative. Therefore $s_1(K) \cap \cdots s_k(K)$ contains $t(K)$ so is not empty.

Pick now $x \in \bigcap_{s \in \mathcal{S}} s(K)$. To show that $x$ is $G$-fixed, choose any affine function $\phi : K \to \mathbb{R}$, and any $g \in G$. For all $n$, write $x = A_{n,g}(y)$, and compute

$$\phi(x) - \phi(xg) = \frac{1}{n} (\phi(y) - \phi(yg^n)) \leq \frac{2}{n} \|\phi\|_{\infty};$$

Since $\phi, g$ are fixed and $n$ is arbitrary, we have $\phi(x) = \phi(xg)$ for all affine $\phi : K \to \mathbb{R}$, from which $x = xg$. 

Furstenberg studied in [40] a condition at the exact opposite of amenability: a boundary for a group $G$ is a compact $G$-space $K$ which is minimal and such that every probability measure on $K$ admits point measures in the closure of its $G$-orbit. By Theorem 6.7, if $G$ is amenable then its only boundary is the point. See §11.1 for more details.

6.2 Amenability of equivalence relations

In the previous section, we gave conditions on a compact $G$-set to admit an invariant measure. Here, we assume that we are given a measure space on which a group acts measurably.

In the abstract setting, we are given a set $X$, a $\sigma$-algebra $\mathcal{M}$ of subsets of $X$, and a map $\lambda : \mathcal{M} \to \mathbb{R}$.

To simplify the presentation, and focus on the interesting cases, we assume that $(X, \lambda)$ is $\sigma$-finite, namely $X$ is the countable union of subsets of finite measure. In this case, it costs nothing to assume that $\lambda$ is a probability measure, namely $\lambda(X) = 1$. (Indeed, if $X = \bigsqcup_{n \in \mathbb{N}} X_n$ with $\lambda(X_n) < \infty$, define a new measure $\lambda'(A) = \sum_{n \in \mathbb{N}} 2^{-n} \lambda(A \cap X_n)/\lambda(X_n)$.) We will even assume that $(X, \lambda)$ is a standard probability space [104], such as $([0,1], \text{Lebesgue})$ or $([0,1]^{\mathbb{N}}, \text{Bernoulli})$; these spaces are isomorphic as measure spaces.
Let $G$ be a group, and assume that $G$ acts measurably on $(X, \lambda)$. Recall that this means that $G$ acts on $\lambda$-null sets: if $A \subset X$ satisfies $\lambda(A) = 0$, then $(\lambda g)(A) = \lambda(Ag^{-1}) = 0$ for all $g \in G$. In other words, the measures $\lambda$ and $\lambda g$ are absolutely continuous with respect to each other, and the Radon-Nikodym theorem \cite{105} implies that there is an essentially unique measurable function $\partial(\lambda g)/\partial \lambda : X \to \mathbb{R}$ satisfying

$$
\int_X f(xg)d\lambda(x) = \int_X f(x) \frac{\partial(\lambda g)}{\partial \lambda} d\lambda(x) \text{ for all } f \in L^1(X, \lambda).
$$

If $(X, \lambda) = ([0,1], \text{Lebesgue})$ and $g : X \to X$ is differentiable, then $\partial(\lambda g)/\partial \lambda = dg/dx$, the usual derivative. The chain rule gives a “cocycle” identity

$$
\frac{\partial(\lambda gh)}{\partial \lambda} = \frac{\partial(\lambda g)}{\partial \lambda} \cdot \left( \frac{\partial(\lambda h)}{\partial \lambda} \cdot g \right).
$$

In the extreme case (which is not the typical case we are interested in), the measure $\lambda$ might be $G$-invariant: $\lambda(A) = \lambda(Ag)$ for all $A \subseteq X, g \in G$, and then the Radon-Nikodym derivative is constant $\equiv 1$.

To simplify the presentation and concentrate on the useful cases, we also restrict ourselves to a countable group $G$. Recall that an action is essentially free if $\lambda$-almost every point has a trivial stabilizer, namely $\lambda(\{x \in X \mid G_x \neq 1\}) = 0$. More generally, everything is considered “up to measure 0”: a group action, isomorphisms between measured actions etc. only need to be defined on sets of full measure.
It will be useful to forget much about the group action, and only remember its orbits. This is captured in the following definitions:

**Definition 6.10.** A countable (respectively finite) measurable equivalence relation on \((X, \lambda)\) is an equivalence relation \(R \subseteq X \times X\) that is measurable qua subset of \(X \times X\), such that for every \(x \in X\) the equivalence class \(xR := \{ y \in X \mid (x, y) \in R \}\) is countable (respectively finite) and such that for every measurable \(A \subseteq X\) with \(\lambda(A) = 0\) one has \(\lambda(A R) = 0\).

The set \(R\) itself is treated as a measure space, with the counting measure on each equivalence class: \(d\mu(x, y) = d\lambda(x)\).

A fundamental example is given by a measurable action of a countable group \(G\), as above: one sets \(R_G = \{(x, y) \in X^2 \mid \exists g \in G\ \text{with}\ xg = y\}\).

**Definition 6.11.** A countable measurable equivalence relation \(R\) on \((X, \lambda)\) is amenable if there is a measurable invariant mean \(m: X \to \mathcal{M}(R)\), written \(x \mapsto m_x\), with \(m_x \in \mathcal{M}(xR)\) for all \(x \in X\). Here “measurable” means that for every \(F \in \mathcal{L}^\infty(X, \lambda)\) the map \(x \mapsto m_x(F)\) is measurable, and “invariant” means that \(m_x = m_y\) almost whenever \((x, y) \in R\).

By \([29]\), a countable measurable equivalence \(R\) relation is amenable if and only if it is hyperfinite: \(R\) is the increasing union of countably many finite measurable equivalence relations, if and only if it is given by an action of \(\mathbb{Z}\).

The following lemma rephrases amenability of equivalence relations as an analogue of Reiter’s criterion; we omit the proof which essentially follows that of Theorem 3.23; see \([73]\):

**Lemma 6.12.** The equivalence relation \(R\) on \((X, \lambda)\) is amenable if and only if there exists a system \((\phi_{x,n})_{x \in X, n \in \mathbb{N}}\) of measures, with \(\phi_{x,n} \in \ell^1(xR)\), which is

- measurable: for all \(n \in \mathbb{N}\) the function \((x, y) \mapsto \phi_{x,n}(y)\) is measurable on \(R\),
- asymptotically invariant: \(\|\phi_{x,n} - \phi_{x,m}\| \to 0\) for almost all \((x, y) \in R\). \(\Box\)

**Proposition 6.13.** If \(G\) is amenable and acts measurably on \((X, \lambda)\), then \(G\) generates an amenable equivalence relation.

**Proof.** Since \(G\) is amenable, there exists a sequence of almost invariant measures \(\phi_n \in \ell^1(G)\), in the sense that \(\|\phi_n - \phi_g\| \to 0\) for all \(g \in G\). Let \(R_G\) be the equivalence relation generated by \(G\) on \(X\). For \(x \in X\), set \(\phi_{x,n} := x \cdot \phi_n\), the push-forward of \(\phi_n\) along the orbit of \(x\). Clearly \((\phi_{x,n})_{x \in X, n \in \mathbb{N}}\) is an asymptotically invariant system, and it is measurable since for all \(n \in \mathbb{N}\) the level sets \(\{(x, y) \in R \mid \phi_{x,n}(y) > a\}\) are the unions of the graphs of finitely many elements of \(G\). \(\Box\)

Note that the proposition does not admit a converse: for instance, if \(G\) is a discrete subgroup of a Lie group \(L\) and \(P \trianglelefteq L\) is soluble, then the action of \(G\) on \(P \setminus L\) is amenable. Indeed the action of \(G\) on \(L\) is amenable: letting \(T\) be a measurable transversal of \(G\) in \(L\), choose arbitrarily a measurable assignment \(m: T \to \mathcal{M}(R_G)\) on the transversal, and extend it to \(L\) by translation. The map \(m\) may easily be required to be \(P\)-invariant, so passes to the quotient \(P \setminus L\).
Proposition 6.14. If $G$ acts essentially freely by measure-preserving transformations on the probability space $(X, \lambda)$, and the generated equivalence relation $R$ is amenable, then $G$ is amenable.

Proof. Given $f \in \ell^\infty(G)$, set

$$m(f) = \int_X m_x(xg \mapsto f(x))d\lambda(x).$$

It is possible for a non-amenable group to act essentially freely on a probability space:

Example 6.15. Let $F_k = \langle x_1, \ldots, x_k \rangle$ be a free group of rank $k$, and consider its boundary $\partial F_k$: it is the space of infinite reduced words over the generators of $F_k$.

$$\partial F_k = \{a_0a_1 \cdots \in \{x_1^\pm, \ldots, x_k^\pm\}^\infty \mid a_i a_{i+1} \neq 1 \text{ for all } i \in \mathbb{N}\}.$$ 

The measure is equidistributed on cylinders: $\lambda(a_0a_1 \cdots a_n(x_1^\pm, \ldots, x_k^\pm)^n) = (2k)^{-1}(2k-1)^{1-n}$. The action of $F_k$ on $\partial F_k$ is by pre-catenation:

$$(a_0a_1 \cdots) \cdot x_i = \begin{cases} x_i a_0 a_1 \cdots & \text{if } x_i a_0 \neq 1, \\ a_1 \cdots & \text{if } x_i a_0 = 1. \end{cases}$$

Then the action of $F_k$ on $\partial F_k$ is essentially free and amenable, although $F_k$ is not amenable.

Proof. For $1 \neq g = a_1 \ldots a_n \in F_k$, its only fixed points in $\partial F_k$ are $g^x$ and $g^{-x}$; since $F_k$ is countable and $\partial F_k$ has the cardinality of the continuum, the action of $F_k$ is free almost everywhere in $\partial F_k$.

For all $x = a_0 a_1 \cdots \in \partial F_k$, define probability measures $\mu_{x,n}$ on the orbit of $x$ by

$$\mu_{x,n} = \frac{1}{n} (\delta_{x} + \delta_{a_0} + \cdots + \delta_{a_{n-1}}).$$

These measures converge weakly to a mean $m_x$ on the orbit of $x$, and clearly $m_x$ and $m_{xg}$ have the same limit, since the sums defining $\mu_{x,n}$ and $\mu_{xg,n}$ agree on all but at most $|g|$ terms. Therefore, $m: X \to R_{F_k}$ is invariant, so $R_{F_k}$ is amenable.

Consider a non-amenable group acting on $(X, \lambda)$. So as to guarantee that the equivalence relation $R_G$ be non-amenable, we may relax somewhat the condition that $G$ preserve $\lambda$. We also assume that $X$ is a compact topological space on which $G$ acts by homeomorphisms. In fact, this is not a strong restriction: given a measurable action of $G$ on $(X, \lambda)$, we may always construct a compact topological $G$-space $Y$, with a measure $\mu$ on its Borel subsets, such that $(X, \lambda)$ and $(Y, \mu)$ are isomorphic as $G$-measure spaces; see [15] Theorem 5.2.1.

We will call the action of $G$ indiscrete if for every $\varepsilon > 0$ and every neighbourhood $\mathcal{U}$ of the diagonal in $X \times X$ there exists $g \neq 1 \in G$ with $\{(x, xg) \mid x \in X\} \subseteq \mathcal{U}$ and $\partial(\lambda g)/\partial \lambda \in (1 - \varepsilon, 1 + \varepsilon)$ almost everywhere.
The measurable action of $G$ on $X$ induces an action of $G$ by isometries on the Banach space $L^1(X, \lambda)$ of integrable functions on $X$, by

$$(fg)(x) = \left( \frac{\partial(\lambda g)}{\partial \lambda} f \right)(xg^{-1})$$ for $f \in L^1(X, \lambda)$.

**Lemma 6.16.** If we give $G$ the topology of uniform convergence in its action on $X$, then the action of $G$ on $L^1(X, \lambda)$ is continuous.

**Proof.** Consider $f \in L^1(X, \lambda)$; we wish to show $fg \to f$ whenever $g \to 1$.

The closure of $G$ in the homeomorphism group of $X$ is second-countable locally compact; it therefore admits a Haar measure $\eta$. Let $K \subseteq \overline{G}$ be a compact with $\eta(K) = 1$, and let $V$ be a compact neighbourhood of $1$ in $G$. Since the Haar measure is invariant, we have

$$\|fg - f\| = \int_K \|fg - fh\| d\eta.$$

since $f$ is measurable, there is for all $\varepsilon > 0$ a continuous function $f^\prime: VK \to \mathbb{C}$ with $\int_V \|fh - f^\prime h\| d\eta < \varepsilon$, and there is also a neighbourhood $W$ of $1$ in $V$ such that $\|f^\prime g - f^\prime h\| < \varepsilon$ for all $h \in K, g \in W$. Then $\|fg - f\| < 3\varepsilon$ as soon as $g \in W$ by a standard ‘3δ’ argument.

**Proposition 6.17 (Monod).** Let $G$ contain an indiscrete non-abelian free group acting essentially freely on a measure space $(X, \lambda)$. Then $G$ generates a non-amenable equivalence relation.

**Proof.** It suffices to prove the claim with $G = \langle a, b \rangle$ itself free. Let $A \subseteq G$ denote those elements whose reduced form starts with a non-trivial power of $a$, and define similarly $B$ using $b$; so $G = A \cup B \cup \{1\}$.

Assume for contradiction that $R_G$ is amenable, and let $m: X \to \mathcal{M}(R_G)$ be an invariant mean. Define measurable maps $u, v: X \to [0, 1]$ by

$$u(x) = m_a(xA), \quad v(x) = m_x(xB).$$

Then $u + v = 1$ almost everywhere, and $0 \leq \sum_{n \in \mathbb{Z}} u(xb^n) \leq 1$ and $0 \leq \sum_{n \in \mathbb{Z}} v(xa^n) \leq 1$ almost everywhere, because the sets $b^nA$ are all disjoint. In particular, if $v(x) > \frac{1}{2}$ then $v(xa^n) < \frac{1}{2}$ for all $n \neq 0$, so if $u(x) < \frac{1}{2}$ then $u(xb^n) > \frac{1}{2}$ for all $n \neq 0$. Define

$$P = \{ x \in X \mid u(x) < \frac{1}{2} \}, \quad Q = \{ x \in X \mid u(x) > \frac{1}{2} \}.$$

Denote furthermore by $A' \subseteq A$ those elements of $G$ that start and end with a non-trivial power of $a$, and by $B' \subseteq B$ those elements of $G$ that start and end with a non-trivial power of $b$. Then $PA' \subseteq Q$, and $QB' \subseteq P$.

Since $G$ is indiscrete, there exist $g_n \in G \setminus \{1\}$ with $g_n \to 1$ and $\partial(\lambda g_n)/\partial \lambda \to 1$ uniformly. Up to taking a subsequence, we may assume all $g_n$ have the same first letter and the same last letter, and have increasing lengths. Up to switching the roles.
of $a$ and $b$, we may assume they all start with $a^\pm 1$. Up to replacing $g_n$ by $g_n g_{n-1}^{-1}$, we may assume they all belong to $A'$.

Since $P$ is measurable, its characteristic function $1_P$ is measurable and $\lambda(P) = \int_X 1_P d\lambda$. Then $\lambda(P \triangle P g_n) = \int_X |1_P - 1_{P g_n}| d\lambda$; now $1_{P g_n} = \partial(\lambda g_n) / \partial \lambda$ with $\partial(\lambda g_n) / \partial \lambda \to 1$, and by Lemma 6.16, $1_{P g_n} \to 1_P$, so $\lambda(P \triangle P g_n) \to 0$ as $n \to \infty$.

However, $P g_n \subseteq Q \subseteq X \setminus P$, so $\lambda(P \triangle P g_n) = 2\lambda(P)$; so $\lambda(P) = 0$. Next $\lambda(Q B') \leq \lambda(P) = 0$ so $\lambda(Q) = 0$. It follows that $u = \frac{1}{2}$ almost everywhere, but this contradicts $0 \leq \sum_{n \in \mathbb{Z}} u(xb^n) \leq 1$. \hfill \square

**Example 6.18.** Let $G$ be a countable indiscrete, non-soluble subgroup of $\text{PSL}_2(\mathbb{R})$. Then $G$ contains a non-discrete free group acting essentially freely on $X = \mathbb{P}^1(\mathbb{R})$. It follows that $G$ generates a non-amenable equivalence relation on $X$.

Indeed, $G$ contains an elliptic element of infinite order, namely an element with $|\text{trace}(g)| \in [-2, 2] \setminus 2 \cos(\pi \mathbb{Q})$, see [68]. The group generated by some power of $g$ and of a hyperbolic element not fixing $g$'s fixed points is a non-discrete Schottky group.

Note that groups and equivalence relations are two special cases of groupoids, see Definition 9.17. There is a well-developed theory of amenability for groupoids with a measure on their space of units, see [73], and [3] for a full treatise.
7 Elementary operations

We turn to a more systematic study of the class $AG$ of amenable groups. John von Neumann already noted in [103] that $AG$ is closed under the following operations:

**Proposition 7.1.** Let $G$ be a group.

1. Let $N \triangleleft G$ be a normal subgroup. If $G$ is amenable, then $G/N$ is amenable.
2. Let $H < G$ be a subgroup. If $G$ is amenable, then $H$ is amenable.
3. Let $N \triangleleft G$ be a normal subgroup. If $N$ and $G/N$ are amenable, then $G$ is amenable.
4. Let $(G_n)_{n \in \mathcal{N}}$ be a directed family of groups: $\mathcal{N}$ is a directed set, and for all $m < n$ there is a homomorphism $f_{mn}: G_m \to G_n$ with $f_{mn}f_{np} = f_{mp}$ whenever $m < n < p$. If $G_n$ is amenable for all $n$, then $\lim_{n \to \infty} G_n$ is amenable.

In particular, if the $G_n$ form a nested sequence of amenable groups, i.e. $G_m \leq G_n$ for $m < n$, then $\bigcup_{n \in \mathcal{N}} G_n$ is amenable.

It is an amusing exercise to prove the proposition using a specific definition of amenability. Below we prove it using the fixed point property of convex compact $G$-sets, and give references to previous statements where other proofs were given.

**Proof.**

1. **Proposition 2.10**
   For another proof, let $G/N$ act on a non-empty convex compact $K$. Then in particular $G$ acts on $K$, and since $G$ is amenable we have $K^G \neq \emptyset$ by Theorem 6.4.

2. **Proposition 2.12**
   For another proof, let $H$ act on a non-empty convex compact $K$, and define $K^{G/H} = \{f: G \to K \mid f(xh) = f(x)h \text{ for all } x \in G, h \in H\}$.

   Then $K^{G/H}$ is a convex compact $G$-set under the action $(f \cdot g)(x) = f(gx)$, so admits a fixed point. This fixed point is a constant function, whose value is an $H$-fixed point in $K$.

3. **Proposition 2.26**
   For another proof, let $G$ act on a non-empty convex compact $K$. Since $N$ is amenable, $K^N \neq \emptyset$. Since $N$ is normal, $G/N$ acts on $K^N$, and since $G/N$ is amenable, $(K^N)^{G/N} \neq \emptyset$. But this last set is nothing but $K^G$.

4. **Proposition 3.8**
   For another proof, write $G = \lim_{n \to \infty} G_n$, with natural homomorphisms $f_n: G_n \to G$ such that $f_m = f_{mn}f_n$ for all $m < n$. Let $G$ act on a non-empty convex compact $K$. Then each $G_n$ acts on $K$ via $f_n$, and $K^{G_n}$ is non-empty because $G_n$ is amenable. Furthermore the $K^{G_n}$ form a directed sequence of closed subsets of $K$: given $I \subseteq \mathcal{N}$ finite, there is $n \in \mathcal{N}$ greater than $I$, so $\bigcap_{i \in I} K^{G_i} \supseteq K^{G_n}$ is not empty. By compactness, $\bigcap_{n \in \mathcal{N}} K^{G_n} = K^G \neq \emptyset$. \hfill $\Box$

We deduce immediately
Corollary 7.2. A group $G$ is amenable if and only if all its finitely generated subgroups are amenable.

Indeed one direction follows from (2), the other from (4) with $\mathcal{N}$ the family of finite subsets of $G$, ordered by inclusion, and $G_n = \langle n \rangle$.

7.1 Elementary amenable groups

Finite groups are amenable; we saw in Corollary 4.2 and Proposition 6.9 that abelian groups are amenable; and saw in Proposition 7.1 that the class of amenable groups is closed under extensions and colimits. Following Mahlon Day [32], let us define the class of elementary amenable groups, $E_G$. This is the smallest class of groups that contains finite and abelian groups, and is closed under the four operations of Proposition 7.1: quotients, subgroups, extensions, and directed unions.

Example 7.3. Virtually soluble groups are in $E_G$.

Indeed, they are obtained by a finite number of extensions using finite and abelian groups.

Example 7.4. For a set $X$, the group $\text{Sym}(X)$ of finitely-supported permutations is in $E_G$.

Indeed, $X$ is the union of its finite subsets, so $\text{Sym}(X)$ is the directed limit of finite symmetric groups.

Example 7.5. Consider

$$G = \langle \ldots, x_{-1}, x_0, x_1, \ldots | \langle x_i, \ldots, x_{i+k} \rangle^{(k)} \text{ for all } i \in \mathbb{Z}, k \in \mathbb{N} \rangle,$$

where $F^{(k)}$ denotes the $k$th term of the derived series of $F$. Then $G$ is in $E_G$.

Obviously the map $x_i \mapsto x_{i+1}$ extends to an automorphism of $G$; let $\hat{G}$ denote the extension $G \rtimes \mathbb{Z}$ using this automorphism. Then $\hat{G}$ also is in $E_G$.

Indeed, $G = \bigcup_{k \in \mathbb{N}} \langle x_{-k}, \ldots, x_k \rangle$, where each term is soluble. However, $G$ itself is not soluble.

Example 7.6. This example is similar to 7.5 but more concrete. Consider formal symbols $e_{mn}$ for all $m < n \in \mathbb{Z}$. The group $M$ is the set of formal expressions $1 + \sum_{m<n} \alpha_{mn} e_{mn}$, with $\alpha_{mn} \in \mathbb{Z}$ and almost all 0; multiplication is defined by the formulas $e_{mn} e_{np} = e_{mp}$, all other products being 0. Then $M$ is locally nilpotent, so is in $E_G$.

Extend then $M$ by the automorphism $\sigma: e_{mn} \mapsto e_{m+1,n+1}$; the resulting group $\hat{M} = M \rtimes \mathbb{Z}$ is again in $E_G$, and is finitely generated, by $1 + e_{12}$ and $\sigma$. 
The class \( EG \) may be refined using transfinite induction. Let \( EG_0 \) denote the class of finite or abelian groups. For an ordinal \( \alpha \), let \( EG_{\alpha+1} \) denote the class of extensions or directed unions of groups in \( EG_\alpha \); and for a limit ordinal \( \alpha \) set \( EG_\alpha = \bigcup_{\beta < \alpha} EG_\beta \).

**Lemma 7.7.** A group is elementary amenable if and only if it belongs to \( EG_\alpha \) for some ordinal \( \alpha \).

**Proof.** It suffices to see that the classes \( EG_\alpha \) are closed under subgroups and quotients. This is clear for \( EG_0 \). If \( \alpha \) is a successor, consider a subgroup \( H \leq G \in EG_\alpha \). Either \( G = N.\hat{Q} \) is an extension of groups in \( EG_{\alpha-1} \); and then \( H = (N\cap H).(H/N\cap H) \) with \( H/N\cap H \leq Q \); or \( G = \bigcup G_i \), in which case \( H = \bigcup (H\cap G_i) \); in both cases, \( H \in EG_\alpha \) by induction. Consider next a quotient \( \pi : G \twoheadrightarrow H \). Either \( G = N.\hat{Q} \), and \( H = \pi(N).(H/\pi(N)) \) with \( Q \twoheadrightarrow H/\pi(N) \), or \( G = \bigcup G_i \), in which case \( H = \bigcup \pi(G_i) \); in both cases, \( H \in EG_\alpha \) by induction.

If \( \alpha \) is a limit ordinal, then each \( G \in EG_\alpha \) actually belongs to \( EG_\beta \) for some \( \beta < \alpha \) and there is nothing to do. \( \Box \)

**Example 7.8.** Continuing Example 7.4 consider \( H = \text{Sym}(\mathbb{Z}) \rtimes \mathbb{Z} \), with \( \mathbb{Z} \) acting on functions in \( \text{Sym}(\mathbb{Z}) \) by shifting: \((n \cdot p)(x) = p(x-n) \). Then \( H \) is 2-generated, for example by \((1,2) \in \text{Sym}(\mathbb{Z}) \) and a generator of \( \mathbb{Z} \).

Since \( \text{Sym}(\mathbb{Z}) \) is a union of finite groups but is neither finite nor abelian, \( \text{Sym}(\mathbb{Z}) \in EG_1 \setminus EG_0 \). Likewise, \( H \in EG_2 \setminus EG_1 \).

Example 7.5 is a bit more complicated. \( F_k/F_k^{(k)} \) is soluble of class precisely \( k \); so it belongs to \( EG_{k-1} \setminus EG_{k-2} \). Therefore, \( G \in EG_\omega \), but \( G \not\in EG_n \) for finite \( n \). Similarly, \( \hat{G} \in EG_{\omega+1} \). The same holds for \( M \) and \( \hat{M} \) from Example 7.6.

Note also in Example 7.4 that the group of all permutations of \( \mathbb{Z} \) is not amenable. Indeed it contains every countable group (seen as acting on itself); so if it were amenable then by Proposition 7.1 any countable group would be amenable.

Recall that \( AG \) denotes the class of amenable groups. In [12], Mahlon Day asks whether the inclusion \( EG \subseteq AG \) is strict; in other words, is there an amenable group that may not be obtained by repeated application of Proposition 7.1 starting with finite or abelian groups?

**Theorem 7.9 (Chou [26 Theorems 2.3 and 3.2]).** Finitely generated torsion groups in \( EG \) are finite.

No finitely generated group in \( EG \) has intermediate word-growth.

The inequality \( EG \neq AG \) follows, since there exist finitely generated infinite torsion groups (see [46] or Exercise 4.16) and groups of intermediate word growth, see Theorem 4.12.

**Proof.** The two statements are proven in the same manner, by transfinite induction. We only prove the second, and leave the (easier) first one as an exercise. Let us show that, if \( G \in EG \) has subexponential word-growth, then \( G \) is virtually nilpotent. Groups in \( EG_0 \) have polynomial growth, and are therefore virtually nilpotent.
by Theorem 4.1. Consider next $\alpha$ a limit ordinal, and $G \in EG_\alpha$ a finitely generated group. We may assume that $\alpha$ is minimal, so in particular $\alpha$ is not a limit ordinal. Since $G$ is finitely generated, we have $G = N \cdot Q$ for $N, Q \in EG_{\alpha - 1}$. By induction $Q$ is virtually nilpotent, so in particular is virtually polycyclic. By Corollary 4.4 the subgroup $N$ is finitely generated, so is virtually nilpotent by induction. By Lemma 4.5 the group $G$ is virtually soluble, and by Corollary 4.11 it has either polynomial or exponential growth. □

7.2 Subexponentially amenable groups

In [24, §14], Tullio Ceccherini-Silberstein, Pierre de la Harpe and Slava Grigorchuk consider the class $SG$ of subexponentially amenable groups as the smallest class containing groups of subexponential growth and closed under taking subgroups, quotients, extensions, and direct limits. We then have $EG \subsetneq SG \subseteq AG$, and we shall see promptly that the last inclusion is also strict.

We introduce a general construction of groups: let $H$ be a permutation group acting on a set $\mathcal{A}$. We assume that the action is transitive, and choose a point $0 \in \mathcal{A}$. Let us construct a self-similar group $\mathcal{M}(H)$ acting on the rooted tree $\mathcal{A}^*$, see Definition 2.15.

The group $\mathcal{M}(H)$ is generated by two subgroups, written $H$ and $K$ and isomorphic respectively to $H$ and to $H/H_0 = H/\mathcal{A}^\mathcal{A} \times H_0$. We first define the actions of $H$ and $K$ on the boundary $\mathcal{A}^N$ of the tree. The action of $h \in H$ is on the first letter:

$$(a_0a_1\ldots)h = (a_0h)a_1\ldots.$$

The action of $(f,h) \in K$, with $f : \mathcal{A} \setminus \{a\} \rightarrow H$ finitely supported, fixes $a^N$ and is as follows on its complement:

$$(a_0a_1\ldots)(f,h) = 0\ldots0(a_nh)(a_{n+1}f(a_n))a_{n+2}\ldots\text{ with }n\text{ minimal such that }a_n \neq 0.$$

The self-similarity of $\mathcal{M}(H)$ is encoded by an injective homomorphism $\Phi : \mathcal{M}(H) \rightarrow \mathcal{M}(H)\backslash_{\mathcal{A}^*} H = \mathcal{M}(H)^{\mathcal{A}} \times H$, written $g \mapsto \{g_a | a \in \mathcal{A}\}\pi$ and defined as follows. Given $g \in \mathcal{M}(H)$, its image $\pi$ in $H$ is the natural action of $g$ on $\{a^N | a \in \mathcal{A}\} \cong \mathcal{A}$. The permutation $g_a$ of $\mathcal{A}^N$ is the composition $\mathcal{A}^N \rightarrow a^N \rightarrow (a\pi)^N \rightarrow \mathcal{A}^N$ of the maps $(w \mapsto aw)$, $g$ and $((a\pi)w \mapsto w)$ respectively. On the generators of $\mathcal{M}(H)$, we have

$$\Phi(h) = \{1 | a \in \mathcal{A}\}h, \quad \Phi((f,h)) = \{f(a) | a \in \mathcal{A}\}h.$$

Proposition 7.10. If $H$ is perfect and 2-transitive, then $\Phi$ is an isomorphism.

Proof. First, if $H$ is 2-transitive, then $\mathcal{M}(H)$ is generated by three subgroups $H,H_0,\overline{H}$. Fix a letter $1 \in \mathcal{A}$; then $H_0$ consists of those $(1,h) \in K$, and $\overline{H}$ consists of those $(f,1)$ where $f(a) = 1$ for all $a \neq 1$. To avoid confusions between these subgroups, we write $h,h_0,\overline{h}$ for respective elements of $H,H_0,\overline{H}$.
To prove that $\Phi$ is an isomorphism, it suffices to prove that $\{h,1,\ldots,1\}, \{h_0,1,\ldots,1\}$ and $\{\tilde{h},1,\ldots,1\}$ belong to $\Phi(\mathscr{M}(H))$ for all $h \in H, h_0 \in H_0, \tilde{h} \in \tilde{H}$.

First, choose $k \in H_0$ with $1k \neq 1$. For all $\tilde{h}, \tilde{h}' \in \tilde{H}$ we have $\Phi([\tilde{h},(\tilde{h})^k]) = \langle [\tilde{h},1],\ldots,1 \rangle$; and since $\tilde{H} \times 1 \cdots \times 1$ contains $H \times 1 \cdots \times 1$. Consider next $h_0 \in H_0$; then $\Phi(h_0h^{-1}) = \langle h_0,1,\ldots,1 \rangle$. Finally, $\Phi(\tilde{h}) = \langle h,1,\ldots,1 \rangle$ and $\langle \tilde{h},1,\ldots,1 \rangle$ belongs to the image of $\Phi$, so $\langle 1,1,\ldots,1 \rangle$ also belongs to its image. Conjugating by an appropriate element of $H$, we see that $\langle h,1,\ldots,1 \rangle$ belongs to the image of $\Phi$. □

**Theorem 7.11** ([10]; see [20] for the proof). If $H$ is finite, then the group $\mathscr{M}(H)$ is amenable.

**Proof.** If $H \leq \tilde{H}$ as permutation groups then $\mathscr{M}(H) \leq \mathscr{M}(\tilde{H})$. It therefore does not reduce generality, in proving that $\mathscr{M}(H)$ is amenable, to consider $H$ perfect and 2-transitive.

We consider $S = H \cup K$ as generating set for $\mathscr{M}(H)$. Let us define finite subsets $I_k \subseteq L_k$ of $\mathscr{M}(H)$ inductively as follows:

$I_0 = K, \quad L_0 = I_0H,$

$I_k = H \cdot \Phi^{-1}(I_{k-1} \times L_{k-1}^{\mathscr{A} \setminus \{0\}}),$

$L_k = H \cdot \Phi^{-1}(L_{k-1}^{\mathscr{A}} \setminus (L_{k-1} \setminus I_{k-1})^{\mathscr{A}}).$

**Lemma 7.12.** For all $k \in \mathbb{N}$ we have $I_kK = I_k$ and $I_kH = L_kH = L_k$; therefore, $I_kS = L_k$.

**Proof.** The claims are clear for $k = 0$. Also, $L_kH = L_k$ for all $k$. Consider $g \in I_k$ and $f \in K$, and write them $g = h\langle g_a | a \in \mathscr{A} \rangle$ and $f = \langle f_a | a \in \mathscr{A} \rangle h'$. Note $gf = a\langle g_a f_a | a \in \mathscr{A} \rangle h'$. We have $f_a \in H$ for all $a \neq 0$, so $g_a f_a \in L_{k-1}$ for all $a \neq 0$; and $f_0 \in K$ so go$f_0 \in I_{k-1}$. □

**Lemma 7.13.** Setting $\rho_k = \#I_k / \#L_k$, we have

$$\rho_k = \frac{\rho_{k-1}}{1 - (1 - \rho_{k-1})^{\#\mathscr{A}}}.$$

**Proof.** Set $d = \#\mathscr{A}$. From the definition, we get $\#L_k = \#L_k^{d-1} \#H(1 - (1 - \rho_{k-1})^d)$ and $\#I_k = \#I_{k-1} \#L_{k-1}^{d-1} \#H$, so

$$\rho_k = \frac{\#I_k}{\#L_k} = \frac{\#I_{k-1}}{\#L_{k-1}(1 - (1 - \rho_{k-1})^d)}.$$ □

We are ready to prove that the sequence $(I_k)$ is a Følner sequence. In view of Lemma 7.12 it suffices to prove $\rho_k \to 1$. Note $0 < \rho_{k-1} < \rho_k < 1$, so the sequence $(\rho_k)$ has a limit, $\rho$. Then $\rho$ satisfies $\rho = \rho(1 - (1 - \rho)^d)$, so $\rho = 1$. □

To prove that $\mathscr{M}(H)$ has exponential growth, we use a straightforward criterion:
**Proposition 7.14.** Let a left-cancellative monoid $G = \langle S \rangle_+$ act on a set $X$; let there be a point $x \in X$ and disjoint subsets $Y_s \subseteq X \setminus \{x\}$ satisfying $xs \in Y_s$ and $Y_s \subseteq Y_{u_s}$ for all $s \in S$. Then $G$ is free on $S$, namely $G \cong S^\ast$.

**Proof.** Consider distinct words $u = u_1 \ldots u_m, v = v_1 \ldots v_n \in S^\ast$; we are to prove that they have distinct images in $G$. Since $G$ is left-cancellative, we may assume either $m = 0$ or $u_1 \neq v_1$. In the first case $xu = x \neq xv \in Y_{v_1}$, and in the second case $Y_{u_1} \ni xu \neq xv \in Y_{v_1}$.

The proposition implies that $\mathcal{M}(H)$ has exponential growth for almost all $H$; it seems difficult to formulate a general result, so we content ourselves with an example:

**Example 7.15.** The group $\mathcal{M}(S_3)$ has exponential growth.

**Proof.** Write $\mathcal{A} = \{0, 1, 2\}$ and $S_3 = \{(0, 1), (0, 2)\}$. In our notation, consider the elements $s = (0, 1) (0, 1)$ and $t = (0, 2)^{(1, 2, 0)} (0, 2)$. A quick calculation gives

$$\Phi(s) = \{s(0, 1), (0, 1), 1\} (0, 1), \quad \Phi(t) = \{t(0, 2), 1, (0, 2)\} (0, 2),$$

Proposition 7.14 applies with $G = \langle s, t \rangle_+$ and $X = \mathcal{A}^{N^\ast}$ and $x = 0^{N^\ast}$ and $Y_s = \mathcal{A}^{x10^{N^\ast}}$ and $Y_t = \mathcal{A}^{x20^{N^\ast}}$.

The first construction of an amenable, not subexponentially amenable group appears in [7], with an explicit subgroup of (what was later defined to be) $\mathcal{M}(D_4)$.

**Example 7.16.** The group $\mathcal{M}(A_5)$ belongs to $AG \setminus SG$.

**Proof.** The group $G := \mathcal{M}(A_5)$ is amenable by Theorem 7.11. It contains $\mathcal{M}(S_3)$, e.g. because the permutations $(0, 1)(3, 4)$ and $(0, 2)(3, 4)$ generate a copy of $S_3$ in $A_5$, so $G$ has exponential growth by Example 7.15.

It remains to prove that $G$ does not belong to $SG$, and we do this by transfinite induction, defining (just as we did for $EG$) the class $SG_\beta$ of groups of subexponential growth and for an ordinal $\alpha$ by letting $SG_\alpha$ denote those extensions and directed unions of groups in $SG_\beta$ for $\beta < \alpha$.

By way of contradiction, let $\alpha$ be the minimal ordinal such that $G$ belongs to $SG_\alpha$. Since $G$ is finitely generated, it is an extension of groups in $SG_\beta$ for some $\beta < \alpha$. Now the only normal subgroups of $G$ are 1 and the groups $G_n$ in the series defined by $G_0 = G$ and $G_{n+1} = \Phi^{-1}(G_n^{\mathcal{A}} \times H)$; the argument is similar to that used to show that $G$ is not in $EG$, see Exercise 2.17. In particular, every non-trivial normal subgroup of $G$ maps onto $G$, so cannot belong to $SG_\beta$ for some $\beta < \alpha$.  

\[\square\]
7.3 Free group free groups

For levity, in this section by “free group” we always mean “non-abelian free group”. It follows from Proposition 7.1 that every group containing a free subgroup is itself not amenable; this covers surface groups, or more generally word-hyperbolic groups; free products of a group of size at least 2 with a group of size at least 3; and \(SO_3(\mathbb{R})\); that last example is important in relation to the Banach-Tarski paradox, see §5.1.

Let us denote by \(NF\) the class of groups with no free subgroup. In [32], Mahlon Day asks whether the inclusion \(AG \subseteq NF\) is an equality; in other words, does every non-amenable group contains a free subgroup?

This was made into a conjecture by Frederick Greenleaf [52, Page 9], attributed to von Neumann. Ching Chou [26] proved \(EG \neq NF\); while Alexander Ol’shanski˘ı [107] proved \(AG \neq NF\), see also Sergei Adyan [1]. Indeed, they proved the much stronger result that the free Burnside groups \(B(n,m) = \langle x_1, \ldots, x_n | w^n \text{ for all words } w \in x_1^{\pm 1}, \ldots, x_n^{\pm 1} \rangle \) (12) are non-amenable as soon as \(n \geq 2\) and \(m \geq 665\) is odd. These groups, of course, do not contain any non-trivial free subgroup.

The following examples of groups are called “Frankenstein groups”, since (as their namesake) they have rather different properties than the groups they are built of:

**Theorem 7.17 (Monod [94]).** Let \(A\) be a countable subring of \(\mathbb{R}\) properly containing \(\mathbb{Z}\); let \(P_A \subseteq P^1(\mathbb{R})\) be the set of fixed points of hyperbolic elements in \(PSL_2(A)\), and let \(H(A)\) be the group of self-homeomorphisms of \(P^1(\mathbb{R})\) that fix \(\infty\) and are piecewise elements of \(PSL_2(A)\) with breakpoints in \(P_A\). Then \(H(A)\) is a nonamenable free group.

**Proof.** Since \(A\) properly contains \(\mathbb{Z}\), it is dense in \(\mathbb{R}\), so \(PSL_2(A)\) is a countable dense subgroup of \(PSL_2(\mathbb{R})\). It therefore generates a non-amenable equivalence relation on \(P^1(\mathbb{R})\), by Example 6.18.

**Lemma 7.18 ([94, Proposition 9]).** For all \(p \in P^1(\mathbb{R}) \setminus \{\infty\}\) we have

\[ p \cdot PSL_2(A) \subseteq \{\infty\} \cup p \cdot H(A). \]

**Proof.** Given \(g \in PSL_2(A)\) with \(pg \neq \infty\), we seek \(h \in H(A)\) with \(ph = pg\). It will be made of two pieces, \(g\) near \(p\) and \(z \mapsto z + r\) near \(\infty\) for a suitable choice of \(r \in A\). Consider the quotient \(q := g \cdot (z \mapsto z - r) \in PSL_2(A)\); if \(q\) is hyperbolic, say with fixed points \(\xi_{\pm}\), and \(\{\xi_{\pm}\}\) separates \(p\) from \(\infty\), then we may define \(h\) as \(g\) on the component of \(P^1(\mathbb{R}) \setminus \{\xi_{\pm}\}\) containing \(p\) and as \(z \mapsto z + r\) on its complement. Now an easy calculation shows that \(q\) is hyperbolic for all \(|r|\) large enough, and as \(|r| \to \pm\infty\) one of the fixed points of \(q\) approaches \(\infty\) and the other approaches \(\infty\)g,

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and as the sign of \( r \) changes the approach to \( \infty \) is from opposite sides; so in all cases it is easy to find a suitable \( r \).

Therefore, the equivalence relation generated by \( H(\mathbb{A}) \) is non-amenable, so \( H(\mathbb{A}) \) is itself non-amenable.

On the other hand, consider \( f, g \in H(\mathbb{A}) \). We claim that they do not generate a free group, and more precisely either that \( \langle f, g \rangle \) either is metabelian or contains a subgroup isomorphic to \( \mathbb{Z}^2 \).

Let \( 1 \neq h \in \langle f, g \rangle^\mathbb{N} \) belong to the second derived subgroup, and intersect as few connected components of \( \text{support}(f) \cup \text{support}(g) \) as possible — if no such \( h \) exists, we are already done. For every endpoint \( p \in \partial (\text{support}(f) \cup \text{support}(g)) \), the element \( h \) acts trivially in a neighbourhood of \( p \), because both \( f \) and \( g \) act as affine maps in a neighbourhood of \( p \); so the support of \( h \) is strictly contained in \( \text{support}(f) \cup \text{support}(g) \). Since the dynamics of \( \langle f, g \rangle \) has attracting elements in the neighbourhood of \( p \), there exists \( k \in \langle f, g \rangle \) such that \( \text{support}(h) \) and \( \text{support}(h)k \) are disjoint; then \( \langle h, h^k \rangle \cong \mathbb{Z}^2 \).

**Exercise 7.19 (***).** Since \( H(\mathbb{A}) \) is not amenable, there is a free action of \( \text{PSL}_2(\mathbb{A}) \) on \( \mathbb{R} \) by \( H(\mathbb{A}) \)-wobbles. Construct explicitly such an action.

**Hint:** This is essentially what [87] does in computing the minimal number of pieces in a paradoxical decomposition of \( H(\mathbb{A}) \), but it’s still highly non-explicit.

Thus we have \( \text{EG} \subset \mathbb{Z} \subset \text{SG} \subset \text{AG} \subset \mathbb{NF} \). The last inequality also holds for finitely generated groups — any finitely generated nonamenable subgroup of \( H(\mathbb{A}) \) will do. Lodha and Moore construct finitely presented examples in [86].

**Problem 7.20.** Is the group \( H(\mathbb{Z}) \) amenable?

The group \( H(\mathbb{Z}) \) is related to a famous group acting on the real line, consider Thompson’s group \( F \) (see Problem 7.13), which we describe here.

**Example 7.21.** Let \( F \) be the group of self-homeomorphisms of \( [0,1] \) that are piecewise affine with slopes in \( 2\mathbb{Z} \) and breakpoints in \( \mathbb{Z} \frac{1}{2} \).

Conjugating \( F \) by Minkowski’s “\( \ast \)” map, defined by \( ?(x) = \sum_{n>0} (-1)^n 2^{-a_0 - \cdots - a_n} \) if \( x \)'s continued fraction expansion is \( [a_0, a_1, \ldots] \), one obtains a group of piecewise- \( \text{PSL}_2(\mathbb{Z}) \) homeomorphisms of the real line with rational breakpoints; it is easy to see that having rational breakpoints is equivalent to the maps being \( \text{diffeomorphisms} \).

The same argument as that given in the proof of Theorem 7.17 shows that \( F \) is a free group free group.

The difference with \( H(\mathbb{Z}) \) is that breakpoints of maps in \( H(\mathbb{Z}) \) are in \( \mathbb{P}_Z \), which is disjoint from \( \mathbb{Q} \). There are embeddings of \( F \) in \( H(\mathbb{Z}) \), so amenability of \( H(\mathbb{Z}) \) would imply that of \( F \).

Yet another description of \( F \) is by an action on the Cantor set. For this, break the interval \( [0,1] \) open at every dyadic rational; one obtains in this manner a Cantor set, modeled on \( \{0,1\}^\mathbb{N} \) by the usual binary expansion of real numbers, except that one does not identify \( a_1 \ldots a_n 01^\infty \) with \( a_1 \ldots a_n 10^\infty \). The action of \( F \) is then by lexicographical order-preserving maps that are piecewise of the form \( a_1 \ldots a_n v \mapsto b_1 \ldots b_k v \).
for a collection of words \((a_1 \ldots a_n, b_1 \ldots b_k)\) and every \(v \in \{0, 1\}^N\). The group \(F\) is finitely generated, by the elements \(x_0: 00v \mapsto 0v, 01v \mapsto 10v, 1v \mapsto 11v\) and \(x_1: 0v \mapsto 0v, 1v \mapsto 1x_0(v)\), and is even finitely presented. See [22] for a detailed survey of \(F\).
8 Random walks

We now turn to other criteria for amenability, expressed in terms of random walks. For a thorough treatment of random walks consult the book [131]; we content ourselves with the subset most relevant to amenability. One is given a space $X$, and a random walker $W$ moving at random in $X$. There is thus a random process $W \in X \rightarrow S(W) \in X$, describing a single step of the random walk. One asks for the distribution $W_n$ of the random walker after a large number $n$ of iterations of $S$.

More formally, we are given one-step transition probabilities $p_1(x,y) = \mathbb{P}(W_n = x \mid W_{n-1} = y)$ of moving to $x$ for a particle lying at $y$; they satisfy $p_1(x,y) \geq 0$ and $\sum_{x \in X} p_1(x,y) = 1$ for all $y \in X$. We define iteratively $p_n(x,y) = \sum_{z \in X} p_{n-1}(x,z)p_1(z,y)$, and then ask for asymptotic properties of $p_n$.

Here are two fundamental examples. First, if $X$ is a graph with finite degree, set $p_1(x,y) = \frac{1}{\text{deg}(y)}$ if $x$, $y$ are neighbours, and $p_1(x,y) = 0$ otherwise. This is called the simple random walk (SRW) on the graph $X$.

Another fundamental example is given by a group $G$, a right $G$-set $X$, and a probability measure $\mu$ on $G$, namely a map $\mu : G \rightarrow [0,1]$ with $\sum_{g \in G} \mu(g) = 1$ as in (4). The random walk is then defined by

$$p_1(x,y) = \sum_{g \in G, x = yg} \mu(g).$$

(13)

It is called the random walk driven by $\mu$. The measure $\mu$ is called symmetric if $\mu(g) = \mu(g^{-1})$ for all $g \in G$, and is called non-degenerate if its support generates $G$ qua semigroup.

These two examples coincide in case $G = \langle S \rangle$ is finitely generated and the driving measure $\mu$ is equidistributed on $S$, one considers then SRW on the Schreier graph of the action of $G$ on $X$.

A random walk $p$ on a set $X$ is reversible if there exists a function $s : X \rightarrow (0,\infty)$ satisfying $s(x)p_1(x,y) = s(y)p_1(y,x)$ for all $x, y \in X$. SRW is reversible on undirected graphs, with $s(x) = \text{deg}(x)$, and if $\mu$ is symmetric then the random walk driven by $\mu$ is reversible with $s(x) \equiv 1$. We shall always assume that the random walks we consider are reversible, and to lighten notation actually assume that they are symmetric: $s(x) \equiv 1$ so $p_1(x,y) = p_1(y,x)$.

8.1 Spectral radius

We shall prove a criterion, due to Harry Kesten, relating the spectral radius of the linear operator associated with $p$ to amenability. It first appeared in [82]. Let $p$ be a reversible random walk on a set $X$, assumed symmetric for simplicity. Set $E = \{(x,y) \in X^2 \mid p_1(x,y) > 0\}$. We introduce two Hilbert spaces:
Lemma 8.1. \( T \) is a self-adjoint operator on \( \ell^2 \) of norm at most 1. The operator \( d^* \) is the adjoint of \( d \), and \( T = 1 - d^* d \).

Proof. The first claim follows from the second. For \( f \in \ell^2 \) and \( g \in \ell^2 \), we compute

\[
(d f, g) = \frac{1}{2} \sum_{(x,y) \in E} p_1(x,y) (\overline{f(x)} - \overline{f(y)}) g(x,y)
\]

\[
= \frac{1}{2} \sum_{x \in X} \overline{f(x)} \sum_{y \in X} p_1(x,y) (g(x,y) - g(y,x))
\]

\[
= \sum_{x \in X} \overline{f(x)} (d^* g)(x) = (f, d^* g),
\]

and

\[
(1 - d^* d) f = f(x) - \sum_{y \in X} p_1(x,y) (f(x) - f(y)) = \sum_{y \in X} p_1(x,y) f(y). \square
\]

The following definitions are more commonly given in the context of graphs; our more general setting coincides with it if \( p \) is the simple random walk:

Definition 8.2. Let \( p \) be a random walk on a set \( X \). The isoperimetric constant of \( p \) is

\[
\tau(p) = \inf_{F \subset X} \frac{p_1(F, X \setminus F)}{\#F} = \inf_{F \subset X} \frac{\sum_{f \in X \setminus F} p_1(x,y)}{\#F}.
\]

The spectral radius of \( p \) is the spectral radius — or, equivalently, the norm — of the operator \( T \).

The following inequalities relating spectral radius and isoperimetric constant appear, with different notation and normalization, in [18]:

\[
\ell_0^2 = \{ f : X \to \mathbb{C} \mid \langle f, f \rangle < \infty \},
\]

\[
\ell_2^2 = \{ g : E \to \mathbb{C} \mid g(x,y) = -g(y,x), \langle g, g \rangle < \infty \}
\]

with scalar products \( \langle f, f' \rangle = \sum_{x \in X} \overline{f(x)} f'(x) \) and \( \langle g, g' \rangle = \frac{1}{2} \sum_{x,y \in X} p_1(x,y) \overline{g(x,y)} g'(x,y) \). Elements of \( \ell_1^2 \) are naturally extended to functions on \( X^2 \) which vanish on \( X^2 \setminus E \).

One step of the random walk \( p \) induces a linear operator \( T \) on \( \ell_0^2 \) given by

\[
(Tf)(x) = \sum_{y \in X} p_1(x,y) f(y).
\]

Writing \( \delta_x \) for the function taking value 1 at \( x \in X \) and 0 elsewhere, we then have \( p_n(x,y) = (T^n \delta_x)(x) \). We also define operators \( d, d^* \) between \( \ell_0^2 \) and \( \ell_1^2 \) by

\[
d : \ell_0^2 \to \ell_1^2, \quad (df)(x,y) = f(x) - f(y),
\]

\[
d^* : \ell_1^2 \to \ell_0^2, \quad (d^* g)(x) = \sum_{y \in X} p_1(x,y) g(x,y).
\]
Proposition 8.3. Let \( p \) be a reversible random walk on a set \( X \). Then the isoperimetric constant \( \iota \) and spectral radius \( \rho \) of \( p \) are related by

\[
\iota^2 + \rho^2 \leq 1 \leq 1 + \rho.
\]

Proof. We begin by the second inequality. For \( \varepsilon > 0 \), let \( F \subseteq X \) satisfy \( p_1(F, X \setminus F)/|F| < 1 + \varepsilon \). Let \( \phi \in \ell^2_0 \) denote the characteristic function of \( F \). Then \( \|\phi\|^2 = |F| \), and

\[
\|d\phi\|^2 = \frac{1}{2} \sum_{(x,y) \in E} p_1(x,y)(\phi(x) - \phi(y))^2 = \sum_{x \in F, y \in X \setminus F} p_1(x,y) < (1 + \varepsilon)\|\phi\|^2;
\]

then \( (\rho + 1 + \varepsilon)\|\phi\|^2 > \langle \phi, T\phi \rangle + \|d\phi\|^2 = \langle \phi, (1 - d'd)\phi \rangle + \|d\phi\|^2 = \|\phi\|^2 \). The conclusion \( \rho + 1 \geq 1 \) follows under \( \varepsilon \to 0 \).

In the other direction, consider for finite \( F \subseteq X \) the projection \( \pi_F : \ell^2 \to \ell^2 \subseteq \ell^2 \) defined by \( (\pi_F f)(x) = f(x) \) if \( x \in F \) and 0 otherwise, and set \( T_F := \pi_F T \pi_F \). The operator \( T_F \) is self-adjoint, and converges strongly to \( T \) as \( F \) increases, so the spectral radius of \( T_F \) converges to \( \rho \). For \( \varepsilon > 0 \), let \( F \) be such that the spectral radius \( \rho_F \) of \( T_F \) is larger than \( \rho - \varepsilon \). Since \( T_F \) has non-negative entries, its eigenvalue \( \rho_F \) is simple and has a non-negative eigenvector \( \phi \), by the Perron-Frobenius theorem. We extend \( \phi \) by 0 into an element of \( \ell^2_0 \), and normalize it so that \( \|\phi\| = 1 \). Set then

\[
A := \frac{1}{2} \sum_{(x,y) \in E} p_1(x,y)|\phi(x) - \phi(y)|^2,
\]

and compute

\[
A^2 \geq \left( \frac{1}{2} \sum_{(x,y) \in E} p_1(x,y)(\phi(x) + \phi(y)) \cdot |\phi(x) - \phi(y)| \right)^2
\]

\[
\leq \frac{1}{2} \sum_{(x,y) \in E} p_1(x,y)(\phi(x) + \phi(y))^2 \cdot \frac{1}{2} \sum_{(x,y) \in E} p_1(x,y)(\phi(x) - \phi(y))^2
\]

\[
= (\|\phi\|^2 + \langle \phi, T_F\phi \rangle)(\|\phi\|^2 - \langle \phi, T_F\phi \rangle) = (1 + \rho_F)(1 - \rho_F),
\]

because \( \sum_{(x,y) \in E} p_1(x,y)\phi(x)\phi(y) = \sum_{x \in F} \phi(x) \sum_{y \in X} p_1(x,y)\phi(y) = \langle \phi, T_F\phi \rangle \).

On the other hand, let \( 0 < s_1 < s_2 < \cdots < s_n \) denote the finitely many values that \( \phi \) takes, and define, for \( k = 1, \ldots, n \),

\[
F_k = \{ x \in X \mid \phi(x) \geq s_k \},
\]

with the additional conventions \( s_0 = 0 \) and \( F_{k+1} = \emptyset \). Then
A = \frac{1}{2} \sum_{(x,y) \in E} p_1(x,y)\phi(x)^2 - \phi(y)^2 = \sum_{k=1}^{n} \sum_{x \in F_k, y \notin F_k} p_1(x,y)(s_k^2 - s_{k-1}^2) \\
\geq \sum_{k=1}^{n} \epsilon(F_k)(s_k^2 - s_{k-1}^2) = t \sum_{k=1}^{n} (\#F_k - \#F_{k+1})s_k^2 = t\|\phi\| = t.

Combining, we get

\( 1 - (\rho - \epsilon)^2 \geq 1 - \rho^2 \geq A^2 \geq t^2; \)

and the conclusion \( \rho^2 + t^2 \leq 1 \) follows under \( \epsilon \to 0. \)

This section’s main result is the following characterization of amenable \( G \)-sets:

**Theorem 8.4.** Let \( \mu \) be a symmetric, non-degenerate probability measure on a group \( G \), let \( X \) be a \( G \)-set, and let \( p \) be the random walk on \( X \) driven by \( \mu \). Then the following are equivalent:

1. \( X \) is amenable;
2. \( t(p) = 0; \)
3. \( \rho(p) = 1. \)

**Proof.** \((1) \Rightarrow (2)\) Assume first that \( X \) is amenable, and let \( \epsilon > 0 \) be given. Let \( S \in G \) satisfy \( \mu(S) > 1 - \epsilon/2. \) Let \( F \in X \) satisfy \( \#(FS \setminus F) < \epsilon\#F/2. \) Then

\[ \sum_{x \in F, y \notin F} p_1(x,y) \leq \sum_{x \in F, y \in FS \setminus F} \mu\{s \in S \mid y = xs\} + \sum_{x \in F, y \in G \setminus S} \mu(g) \leq \epsilon\#F, \]

so \( t(p) \leq \epsilon \) for all \( \epsilon > 0. \) (Note that we have not used the assumption that \( \mu \) is non-degenerate here.)

\((2) \Rightarrow (1)\) Let \( \epsilon > 0 \) and a finite subset \( S \) of \( G \) be given. By assumption, there exists \( n \in \mathbb{N} \) and \( \delta > 0 \) such that \( \mu^n(s) \geq \delta \) for all \( s \in S. \) Let \( F \) be a finite subset of \( X \) such that \( \sum_{x \in F, y \notin F} p_n(x,y) < \delta \epsilon\#F. \) Then \( \#(FS \setminus F) < \epsilon\#F, \) so \( X \) is amenable by Følner’s criterion, Theorem 3.23, \( \Rightarrow (1). \)

The equivalence \((2) \Leftrightarrow (3)\) is given by Proposition 8.3.

The spectral radius of the random walk has a direct interpretation in terms of probabilities of return of the random walk, at least when we restrict to transitive random walks: random walks with the property that, for any two \( x, y \in X \) there exists \( n \in \mathbb{N} \) such that \( p_n(x,y) > 0 \) (not to be confused with random walks invariant under a transitive group action!). Let us make the following temporary

**Definition 8.5.** The spectral radius of the random walk \( p \) based at \( x \) is

\[ \rho(p,x) := \limsup_{n \to \infty} \sqrt[n]{p_n(x,x)}. \]

**Lemma 8.6 (Fekete).** Let \( N \in \mathbb{N} \) be given, and let \( \alpha: \{N,N+1,\ldots\} \to \mathbb{R} \) be a subadditive function, i.e. a function satisfying \( \alpha(m+n) \leq \alpha(m) + \alpha(n). \) Then
\[\lim_{n \to \infty} \frac{\alpha(n)}{n} = \inf_{n \geq 4} \frac{\alpha(n)}{n};\]

in particular \(\alpha(n)/n\) either converges, or diverges to \(-\infty\).

**Proof.** Consider any \(a \geq N\), and write every \(k \geq N\) as \(k = qa + r\) with \(q \in \mathbb{N}\) and \(r \in \{N, N+1, \ldots, N+a-1\}\). Then, for \(k \geq N\),

\[
\frac{\alpha(k)}{k} \leq \frac{qa\alpha(a) + \alpha(r)}{qa + r} \leq \frac{\alpha(a)}{a} + \frac{\alpha(r)}{k},
\]

letting \(k \to \infty\), we get \(\limsup_{n \to \infty} \alpha(k)/k \leq \alpha(a)/a\) for every \(a \geq N\); so \(\limsup_{n \to \infty} \alpha(k)/k = \inf_{a \geq N} \alpha(a)/a\) converges or diverges to \(-\infty\).

The “limsup” in the definition of the spectral radius is in fact a limit, and is independent of the starting and endpoints:

**Proposition 8.7.** Assume \(p\) is transitive. Then

\[\rho(p, z) = \limsup_{n \to \infty} \sqrt{n} p_n(x, y) = \lim_{n \to \infty} \sqrt{n} p_{2n}(x, y)\]

for all \(x, y, z \in X\).

**Proof.** For the first claim, consider more generally \(w, x, y, z \in X\). There are \(\ell \in \mathbb{N}\) such that \(p_{\ell}(x, w) > 0\); and \(m \in \mathbb{N}\) such that \(p_{m}(z, y) > 0\). Since \(p_{n+\ell+m}(x, y) \geq p_{\ell}(x, w)p_{m}(w, z)p_{m}(z, y)\) for all \(n \in \mathbb{N}\), we have

\[\limsup_{n \to \infty} \sqrt{n} p_{n}(x, y) \geq \limsup_{n \to \infty} \sqrt{n} p_{\ell}(x, w)p_{m}(z, y)\]

\[\sqrt{n} p_{n}(w, z) = \limsup_{n \to \infty} \sqrt{n} p_{n}(w, z).\]

Applying it to \((w, x, y, z) = (z, x, y, z)\) and \((x, z, y, z)\) respectively gives the claim.

It is then clear that \(\limsup_{n \to \infty} \sqrt{n} p_{2n}(x, y) \leq \rho(p, x)\); but conversely \(p_{2n}(x, x) \leq p_{n}(x, x)^2\), so \(\limsup_{n \to \infty} \sqrt{n} p_{2n}(x, y) \geq \limsup_{n \to \infty} \sqrt{n} p_{n}(x, y) = \rho(p, x)\).

Now \(p_{2r+2s}(x, x) \geq p_{2r}(x, x)p_{2s}(x, x)\) for all \(r, s \in \mathbb{N}\). Setting \(\alpha(r) = -\log p_{2r}(x, x)\), we get \(\alpha(r + s) \leq \alpha(r) + \alpha(s)\); furthermore, because \(p\) is transitive, \(\alpha(r)\) is defined for all \(r\) large enough, and \(\alpha(r) \geq 0\) because \(p_{2r}(x, x) \leq 1\). By Lemma 8.6, \(\alpha(r)/r\) converges, whence \(\sqrt{n} p_{2n}(x, x)\) converges.

**Proposition 8.8.** Let \(p\) be symmetric and transitive. Then the spectral radius of \(p\) is equal to the norm of \(T\) acting on \(\ell^2(X)\).

**Proof.** Let us write \(\|T\|\) the operator norm of \(T\) on \(\ell^2(X)\). First, by Proposition 8.7

\[\rho(p, x) = \lim_{n \to \infty} \sqrt{n} p_{2n}(x, x) = \lim_{n \to \infty} \sqrt{n} \langle T^{2n} \delta_x, \delta_x \rangle \leq \sqrt{n} \|T^{2n}\| \leq \|T\|\]

Next, consider \(f \in \mathbb{C}X\). By Cauchy-Schwartz’s inequality, for all \(m \in \mathbb{N}\) we have

\[\langle T^{m+1} f, T^{m+1} f \rangle = \langle T^m f, T^{m+2} f \rangle \leq \|T^m f\| \cdot \|T^{m+2} f\|;\]

so \(\|T^{m+1} f\| / \|T^m f\|\) is increasing, with limit \(\lim_{n \to \infty} \sqrt{n} \|T^n f\|\). Now
Example 8.9. Let us look first at an amenable example: consider the free group \(F\) on \(2\) generators; at time \(t\), one chooses \(+1\) or \(-1\) with equal probabilities; at time \(n\) we then made \(2^n\) choices. If \((n + x - y)/2\) of these are \(+1\) and \((n + x + y)/2\) are \(-1\), then we end up at \(x + (n + x - y)/2 - (n + x + y)/2 = y\).

In particular, if \(n\) is even, we have \(p_n(x, x) = 2^{-n} (\frac{n}{2})\), so by Stirling’s formula \(n! \approx \sqrt{2\pi n} (n/e)^n\) we get
\[
p_n(x, x) \approx \sqrt{\frac{2}{\pi n}}.
\]

Example 8.10. Consider the free group \(F_d\), whose Cayley graph is a \(2d\)-regular tree \(\mathcal{T}\). Fix an edge of this tree, e.g. between 1 and \(x_1\), let \(A_n\) denote the number of closed paths in \(\mathcal{T}\) based at 1, and by \(B_n\) the number of closed paths in \(\mathcal{T}\), based at 1, that do not cross the fixed edge. Consider the generating series \(A(z) = \sum A_n z^n\) and \(B(z) = \sum B_n z^n\). Then \(A(z) = 1/(1 - 2d z B(z))\), because every closed path factors uniquely as a product of closed paths that reach 1 only at their endpoints; and \(B(z) = 1/(1 - (2d - 1)/z^2 B(z))\) for the same reason; so
\[
A(z) = \frac{1 - d + d \sqrt{1 - 4(2d - 1)z^2}}{1 - 4d^2 z^2}
\]
and \(A_n \approx (8d - 4)^{n/2}\) and \(p_n(1, 1) \approx (8d - 4)^{n/2}/(2d)^n\). Therefore, SRW on \(F_d\) has spectral radius \(\rho(p) = \sqrt{2d - 1}/d\).

The isoperimetric constant of SRW may also easily be computed. A connected, finite subset \(F\) of \(\mathcal{T}\) has \(\#F\) vertices and is connected to \(2\#F\) edges, of which \(2(\#F - 1)\) point back to \(F\), so \(\sum_{x \in F, y \notin F} p_1(x, y) = ((2n - 2)\#F + 2)/2n\). The isoperimetric constant is therefore \(\iota(p) = 1 - 1/n\).
Exercise 8.11 (**). Compute the isoperimetric constant of SRW on the surface group \( \Sigma_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle \).

Hint: its Cayley graph is a tiling of hyperbolic plane by \( 4g \)-gons, meeting \( 4g \) per vertex. Use Euler characteristic.

Note that is is substantially harder to compute the spectral radius of SRW; only estimates are known, proportional to \( \sqrt{g} \); see [48] for the best bounds.

It is sometimes easier to count reduced paths in graphs, rather than general paths. Formally, this may be expressed as follows: let \( G = \langle S \cup S^{-1} \rangle \) be a finitely generated group, and write \( S^\pm = S \cup S^{-1} \). There is a natural map \( \pi: F_S \to G \) induced by the inclusion \( S \hookrightarrow G \). The spectral radius of SRW on \( G \) is

\[
\rho = \lim_{n \to \infty} \sqrt{\# \{ w \in (S^\pm)^n \mid w = G^1 \}} / \#(S \cup S^{-1}) \in [0, 1].
\]

The cogrowth of \( G \) is

\[
\gamma = \lim_{n \to \infty} \sqrt{n \in F_S \mid \pi(w) = 1 \#(S \cup S^{-1}) \in [1, \#S^\pm - 1].}
\]

Theorem 8.12 ([53]; see also [6, 28, 120, 130]). The parameters \( \gamma, \rho \) are related by the equation

\[
\rho = \frac{\gamma + (\#S^\pm - 1) / \gamma}{\#S^\pm} \quad \text{if } \gamma > 1.
\]

In particular, \( G \) is amenable if and only if \( \gamma = \#S^\pm - 1 \).

Proof. The most direct proof is combinatorial. Define formal matrices \( B, C \) indexed by \( G \) with power series coefficients by

\[
B(z)_{g,h} = \sum_{w \in F_S : \pi(w) = h} z^{\omega}, \quad C(z)_{g,h} = \sum_{w \in (S^\pm)^*: \pi(w) = h} z^{\omega}.
\]

Set for convenience \( q := \#S^\pm - 1 \). We shall prove the formal relationship

\[
\frac{B(z)}{1 - z^2} = \frac{C(z/(1 + qz^2))}{1 + qz^2}, \quad (14)
\]

from which the claim of the theorem follows. Define the adjacency matrix

\[
A_{g,h} = \sum_{s \in S^\pm : gs = h} 1;
\]

then \( C(z) = 1/(1 - zA) \). If for all \( s \in S^\pm \) we define \( B_s(z)_{g,h} = \sum_{w \in F_S \setminus \{1\} : s \pi(w) = h} z^{\omega} \)

then

\[
B(z) = 1 + \sum_{s \in S^\pm} B_s(z), \quad B_s(z) = sz(B(z) - B_{s^{-1}}(z))
\]

which solve to \( B_s(z) = (1 - z^2)^{-1} (sz - z^2)B(z) \) and therefore to
\[
\frac{1 + qz^2}{1 - z^2} B(z) = 1 + \sum_{s \in S^\pm} \frac{z}{1 - z^2} s B(z) = 1 + \frac{z}{1 - z^2} AB(z);
\]
so \((1 + qz^2)/(1 - z^2) \cdot B(z) = 1/(1 - z/(1 + qz^2)) A\), which is equivalent to \((14)\). \(\square\)

It is also known that \(\rho \geq \sqrt{|S^\pm| - 1}\), with equality if and only if \(G \cong F_S\), see \([111]\).

\section*{8.2 Harmonic functions}

We shall obtain, in this subsection, yet another characterization of amenability in terms of bounded harmonic functions.

\textbf{Definition 8.13.} Let \(p\) be a random walk on a set \(X\). A \textit{harmonic function} is a function \(f : X \to \mathbb{R}\) satisfying

\[
f(x) = \sum_{y \in X} p_1(y, x) f(y).
\]

In other words, \(f\) is a martingale: along a trajectory \((W_n)\) of a random walk, the expectation of \(f(W_n)\) given \(W_0, \ldots, W_{n-1}\) is \(f(W_{n-1})\).

A random walk is called Liouville if the only bounded harmonic functions are the constants.

If \(X\) is a \(G\)-set and \(p\) is the random walk driven by a measure \(\mu\) on the group \(G\), we say that \((X, \mu)\) is Liouville when the corresponding random walk is Liouville.

Bounded harmonic functions are fundamental in understanding long-term behaviour of random walks. The space of trajectories of a random walk on \(X\) is \((X^\mathbb{N}, \nu)\), in which the trajectory \((W_0, W_1, \ldots)\) has probability \(\nu(W_0, W_1, \ldots) = \prod_{n \geq 0} \mu(\{g \in G \mid W_n g = W_{n+1}\})\). An asymptotic event on \((X^\mathbb{N}, \nu)\) is a measurable subset of \(X^\mathbb{N}\) that is invariant under the shift map of \(X^\mathbb{N}\). Given an asymptotic event \(E\), we define a bounded function \(f(x) = \nu(E \cap \{W_0 = x\})\) and check that it is harmonic by conditioning on the first step of the random walk; conversely, given a bounded harmonic function \(f\) the limit \(f(W_n)\) almost surely exists along trajectories, by Doob’s martingale convergence theorem, so \(E_{[a, b]} = \{(W_0, W_1, \ldots) \mid \lim f(W_n) \in [a, b]\}\) is a asymptotic event. In summary, a random walk is Liouville if and only if there are no non-trivial asymptotic events.

Let us continue with the example of SRW on \(\mathbb{Z}\): a harmonic function satisfies \(f(x-1) + f(x+1) = 2f(x)\), so \(f\) is affine. In particular, SRW on \(\mathbb{Z}\) is Liouville.

Let us consider next the example of SRW on the Cayley graph of \(F_2 = \langle a, b \rangle\), which is a tree. The random walk \((W_n)\) escapes at speed \(1/2\) towards the boundary of the tree, since at every position except the origin it has three ways of moving one step farther and one way of moving one step closer; so in particular almost surely \(W_n \neq 1\) for all \(n\) large enough. Let \(A \subset F_2\) denote those elements whose reduced form starts with \(a\), and define
\[ f(g) = \mathbb{P}(W_n \in g^{-1}A \text{ for all } n \text{ large enough}). \]

In words, \( f(g) \) is the probability that a random walk started at \( g \) escapes to the boundary of the tree within \( A \). It is clear that \( f \) is bounded, and it is seen to be harmonic by conditioning on the first step of the random walk. More succinctly, “the random walk eventually escapes in \( A \)” is a non-trivial asymptotic event. Therefore, SRW on a regular tree is not Liouville.

**Exercise 8.14 (**) Let \((X, \mu)\) and \((Y, \nu)\) be Liouville random walks. Prove that \((X \times Y, \mu \times \nu)\) is Liouville.

Let us recall some properties of measures and random walks. The set \( \ell^1(G) \) of summable functions on \( G \) is a Banach *-algebra, for the convolution product

\[ (\mu \nu)(g) = \sum_{g=hk} \mu(h) \nu(k) \text{ for } \mu, \nu \in \ell^1(G). \]  

(15)

We denote by \( \tilde{\mu} \) the adjoint of \( \mu \), defined by \( \tilde{\mu}(g) = \mu(g^{-1}) \). If \( X \) is a \( G \)-set, then \( \ell^p(X) \) is an \( \ell^1(G) \)-module for all \( p \in [1, \infty] \), under

\[ (f \mu)(x) = \sum_{x=yh} f(y) \mu(h) \text{ for } f \in \ell^p(X), \mu \in \ell^1(G). \]

If a random walk is driven by a measure \( \mu \), then from (13) we get \( T f = f \mu \). With our notation, a function \( f \in \ell^p(X) \) is harmonic for the random walk driven by a measure \( \mu \) if and only if \( f \tilde{\mu} = f \).

The Liouville property is fundamentally associated with a measure, or a random walk. It has a counterpart which solely depends on the space, and is a variant of amenability with switched quantifiers (see Proposition 3.25):

**Definition 8.15.** A \( G \)-set \( X \) is called **laminable** if for every \( \varepsilon > 0 \) and every \( f \in \sigma(\ell^1X) \) there exists a positive function \( g \in \ell^1(G) \) with \( \|fg\| < \varepsilon \|g\| \).

**Proposition 8.16.** Let \( G \) be a group, viewed as a right \( G \)-set \( G_G \). Then \( G_G \) is amenable if and only if \( G_G \) is laminable.

Let \( G \) be an amenable group, and let \( X \rightarrow G \) be a \( G \)-set. Then \( X \) is laminable if and only if it is transitive or empty.

**Proof.** By Proposition 3.25, \( G_G \) is amenable if and only if for every \( \varepsilon > 0 \) and every \( \neq g \in \sigma(\ell^1G) \) there exists a positive function \( g \in \ell^1(G) \) with \( \|fg\| < \varepsilon \|f\| \|g\| \); equivalently, \( \|fg\| < \varepsilon \|g\| \|f\| \), which is the definition of laminability of \( G_G \).

For the second statement: if there is more than one \( G \)-orbit on \( X \), choose \( x, y \) in different orbits; then \( \|(\delta_x - \delta_y)g\| = 2\|g\| \) for all positive \( g \in \ell^1(G) \). Conversely, given \( \varepsilon > 0 \) and \( f \in \sigma(\ell^1X) \), choose \( x \in X \) and \( h \in \sigma(\ell^1G) \) with \( f = \delta_x h \). Since \( G \) is amenable, there is a positive function \( g \in \ell^1(G) \) with \( \|g\| < \varepsilon \|g\| \); so \( \|fg\| = \|xg\| = \|hg\| < \varepsilon \|g\| \), and \( X \) is laminable. \( \square \)

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15 This is a contraction of “Liouville” and “amenable”.
The following easy proposition is an analogue to Proposition 2.10:

**Proposition 8.17.** Let $G, H$ be groups, let $X \xrightarrow{\phi} G$ and $Y \xrightarrow{\psi} H$ be respectively a $G$-set and an $H$-set, let $\phi : G \rightarrow H$ be a homomorphism, and let $f : X \rightarrow Y$ be a surjective equivariant map, namely satisfying $f(xg) = f(x)\psi(g)$ for all $x \in X, g \in G$. If $X$ is laminable, then $Y$ is laminable.

**Proof.** Given $\varepsilon > 0$ and $e \in \sigma(\ell^1 Y)$, there is $e' \in \sigma(\ell^1 X)$ with $e' = e \circ f$, because $f$ is surjective; then there is a positive function $g \in \ell^1(G)$ with $\|e'g\| < \varepsilon \|g\|$, because $X$ is laminable; then $\|e\phi(g)\| \leq \|e'g\| < \varepsilon \|g\| = \varepsilon \phi(g)$, so $Y$ is laminable. \qed

**Corollary 8.18.** Let $X \xrightarrow{} G$ be a $G$-set and let $H \leq G$ be a subgroup. If $H$ is amenable and transitive, then $X$ is laminable. \qed

**Lemma 8.19.** If $X \xrightarrow{} G$ is laminable, then for every $x \in X$, every finite subset $S \subset X$ and every $\varepsilon > 0$ there exists a positive function $g \in \ell^1(G)$ with

$$\|\delta_s g - \delta_x g\| < \varepsilon \|g\| \text{ for all } s \in S.$$

Furthermore $g$ may be supposed to be of finite support.

**Proof.** Consider $f = \sum_{s \in S} \delta_s - \#S \delta_x$. Since $X$ is laminable, there is for every $\varepsilon > 0$ a positive function $g \in \ell^1(G)$ with $\|fg\| < \varepsilon \|g\|$. Then

$$\varepsilon \|g\| > 2 \|fg\| \geq 2 \left\| \sum_{s \in S} \max(\delta_s g - \delta_x g, 0) \right\| = \sum_{s \in S} 2 \max(\delta_s g - \delta_x g, 0) \geq \sum_{s \in S} \|\delta_s g - \delta_x g\|.$$

Using density of finitely-supported functions in $\ell^1(G)$ gives the last claim. \qed

The main result of this section is:

**Theorem 8.20.** Let $X$ be a $G$-set. The following are equivalent:

1. $X$ is laminable;
2. There exists a symmetric measure $\mu$ with support equal to $G$ such that $(X, \mu)$ is Liouville;
3. There exists a measure $\mu$ on $G$ such that $(X, \mu)$ is Liouville.

**Corollary 8.21 (Kaimanovich-Vershik [74]).** Let $G$ be a group. Then $G$ is amenable if and only if there exists a measure $\mu$ (ad lib. symmetric, with full support) such that $(G, \mu)$ is Liouville. \qed

Note that there exist amenable non-laminable $G$-sets, such as Example 3.13, and non-amenable graphs for which SRW is Liouville, see [15] or [16, Chapter 13]. At the extreme, note that the empty set is laminable but not amenable, and the disjoint union of two points is amenable but not laminable. Here is a slightly less contrived example:
Example 8.22 (Kaimanovich). Consider the binary rooted tree with vertex set $\{0, 1\}^*$ and an edge between $a_1 \ldots a_n$ and $a_1 \ldots a_{n+1}$ for all $a_i \in \{0, 1\}$. Fix a function $f: \mathbb{N} \to \mathbb{N}$ satisfying $f(n) < n$ for all $n \in \mathbb{N}$, and put also an edge between $a_1 \ldots a_n$ and $a_1 \ldots a_{f(n)} \ldots a_n$ for all $a_i \in \{0, 1\}$. Finally add some loops at the root so as to make the graph 6-regular; we have constructed a graph $\mathcal{G}$, with a natural action of $F_6$ once the edges are appropriately labeled. We consider SRW on $\mathcal{G}$.

On the one hand, $\mathcal{G}$ is not amenable; for example, because SRW drifts away from the root at speed $(4 - 2)/6 = 1/3$, or because the isoperimetric inequality in $\mathcal{G}$ is at least as bad as in a binary tree.

On the other hand, if $f$ grows slowly enough then SRW on $\mathcal{G}$ is Liouville; indeed SRW converges to the boundary of the binary tree, represented by binary sequences $\{0, 1\}^\mathbb{N}$, and it suffices to show that there are no asymptotic events on this boundary. If $f$ is such that $f^{-1}(n)$ is infinite for all $n \in \mathbb{N}$, then each coordinate in $\{0, 1\}^\mathbb{N}$ is randomized infinitely often by the walk when it follows the $f$-edges, so there is no non-constant measurable function on the space of trajectories.

For the remainder of the section, we assume the hypotheses of the theorem: a countable group $G$ and a transitive $G$-set $X$ are fixed. We also assume that all measures $\mu$ under consideration satisfy $\mu(1) > 0$, and call such $\mu$ aperiodic. This is harmless: a function $f$ is harmonic for $\mu$ if and only if it is harmonic for $q\mu + (1 - q)\delta_1$ whenever $q \in (0, 1]$.

**Lemma 8.23.** Let $\mu$ be an aperiodic measure on $G$. Then there exists a sequence $(\epsilon_n) \to 0$, depending only on $\mu(1)$, such that, for all $f \in L^\infty(X)$,

$$\|f\mu^n - f\mu^{n+1}\| \leq \epsilon_n \|f\|.$$  

**Proof.** Since $\|f\mu^n - f\mu^{n+1}\|_{\infty} \leq \|f\|_{\infty} \|\mu^n - \mu^{n+1}\|_1$, it suffices to prove $\|\mu^n - \mu^{n+1}\|_1 \to 0$. Set $q = \mu(1)$; we assume $q \in (0, 1)$. Define a measure $\lambda$ on $\mathbb{N}$ by $\lambda(0) = q, \lambda(1) = p = 1 - q$, and let $\nu$ be the probability measure on $G$ such that $\mu = q\delta_1 + p\nu$. Then

$$\mu^n(g) = (q\delta_1 + p\nu)^n(g) = \sum_{i=0}^n \lambda^n(i)\nu^i(g),$$

so $\|\mu^n - \mu^{n+1}\| = \left\| \sum_{i=0}^{n+1} (\lambda^n(i) - \lambda^{n+1}(i))\nu^i \right\|$. Since $\lambda^n(i) - \lambda^{n+1}(i) = \lambda^n(i) - q\lambda^n(i) - p\lambda^n(i-1) = p(\lambda^n(i) - \lambda^n(i-1))$, it suffices to prove $\sum_{i=0}^{n+1} |\lambda^n(i) - \lambda^n(i-1)| \to 0$. Remembering $\lambda^n(i) = \binom{n}{i} p^i q^{n-i}$, the argument of the absolute value is positive for $i < pn$ and negative for $i > pn$, so $\sum_{i=0}^{n+1} |\lambda^n(i) - \lambda^n(i-1)| \leq \lambda^n([pn]) + \lambda^n([pn]) \to 0$. \hfill \Box

**Corollary 8.24.** For every bounded sequence of functions $(F_n)$ in $L^\infty(X)$, every pointwise accumulation point of the sequence $(f_n\hat{\lambda}^n)$ is harmonic. \hfill \Box

**Proposition 8.25.** Let $\mu$ be aperiodic and non-degenerate. Then $(X, \mu)$ is Liouville if and only if
for all \( x, y \in X \): \( \| \delta_i \mu^n - \delta_j \mu^n \|_1 \to 0 \) as \( n \to \infty \).

**Proof.** Let first \( f \in \ell^\infty(X) \) be harmonic. Then for all \( n \in \mathbb{N} \)

\[
|f(x) - f(y)| = \left| \sum_{z \in X} f(z) \left( \sum_{z=xh} \mu^n(h) - \sum_{z=yh} \mu^n(h) \right) \right|
\leq \|f\|_\infty \cdot \sum_{z \in X} |(\delta_i \mu^n)(z) - (\delta_j \mu^n)(z)| = \|f\|_\infty \cdot \| \delta_i \mu^n - \delta_j \mu^n \| \to 0,
\]

so \( f \) is constant.

Conversely, assume that there exist \( x, y \in X \) and a sequence \( (n_i) \) such that \( \| \delta_i \mu^{n_i} - \delta_j \mu^{n_i} \| \geq 4c > 0 \) for all \( i \). Set \( V_i := \{ z \in X \mid (\delta_i \mu^{n_i})(z) > (\delta_j \mu^{n_i})(z) \} \); then

\[
\sum_{z \in V_i} (\delta_i \mu^{n_i})(z) - (\delta_j \mu^{n_i})(z) \geq 2c.
\]

Set then \( W_i := \{ z \in X \mid (\delta_i \mu^{n_i})(z) \geq (1 + c)(\delta_j \mu^{n_i})(z) \} \); then \( (\delta_i \mu^{n_i})(W_i) \geq c \), for otherwise one would have \( \sum_{z \in V_i \setminus W_i} (\delta_i \mu^{n_i})(z) - (\delta_j \mu^{n_i})(z) = \sum_{z \in W_i} (\delta_i \mu^{n_i})(z) - (\delta_j \mu^{n_i})(z) < c + c = 2c \). Set finally \( f_i = \mathbb{1}_{W_i} \mu^{n_i} \). Note then

\[
f_i(y) = \sum_{z = yh} \mathbb{1}_{W_i}(z) \mu^n(h) = \sum_{z \in W_i} (\delta_i \mu^{n_i})(z) \leq \sum_{z \in W_i} (\delta_j \mu^{n_i})(z)/(1 + c) = f_i(x)/(1 + c),
\]

and similarly \( f_i(x) = (\delta_i \mu^{n_i})(W_i) \geq c \), so any accumulation point of the \( f_i \) is harmonic by Corollary 8.24 bounded and non-constant.

**Proof of Theorem 8.20** (1) \( \Rightarrow \) (2) We assume throughout that \( X \) is transitive and therefore countable. Fix a basepoint \( x_0 \in X \), and let \( \{x_0, x_1, \ldots\} \) be an enumeration of \( X \). Choose two sequences \( (t_i)_{i \in \mathbb{N}} \) and \( (\epsilon_i)_{i \in \mathbb{N}} \) of positive real numbers with \( \sum t_i = 1 \) and \( \lim \epsilon_i = 0 \). Let \( (n_i) \) be a sequence of integers with \( (t_1 + \cdots + t_{i-1})^{n_i} < \epsilon_i \) for all \( i \). Since \( X \) is laminable, by Lemma 8.19 there exists for every \( i \) a positive function \( \alpha_i \in \ell^1(G) \), normalized by \( \| \alpha_i \| = 1 \) and supported on a finite set (say \( F_i \)), with

\[
\| \delta_i - \delta_{n_0} \alpha_i \| < \epsilon_i \text{ for all } s \in \{x_0, \ldots, x_i\} \cdot \{\{1\} \cup F_1 \cup \cdots \cup F_{i-1}\}^{n_i}.
\]

Let us set \( \mu = \sum_{i \in \mathbb{N}} t_i \alpha_i \). To prove that \( (G, \mu) \) is Liouville, it suffices, by Proposition 8.25, to prove that \( \| \delta_i \mu^n - \delta_{n_0} \mu^n \| \to 0 \) for all \( x \in X \). Say \( x = x_0 \); we claim that

\[
\| \delta_i \mu^n - \delta_{n_0} \mu^n \| < 4 \epsilon_i,
\]

and this is sufficient to conclude the proof. For convenience let us write \( n = n' = n \), and expand

\[
\mu^n = \sum_{k_1, \ldots, k_n} t_{k_1} \cdots t_{k_n} \alpha_{k_1} \cdots \alpha_{k_n}.
\]

We subdivide the sum (16) into two summands, \( v_1 \) on which all \( k_j < \ell \) and \( v_2 = \mu^n - v_1 \). First, \( \|v_1\| = \sum_{k < \ell} t_{k_1} \cdots t_{k_n} = (t_1 + \cdots + t_{\ell-1})^{n_0} < \epsilon_i \), so \( \| \delta_i v_1 - \delta_{n_0} v_1 \| < 2 \epsilon_i \).

Secondly, consider a summand \( \theta = \alpha_{k_1} \cdots \alpha_{k_n} \) appearing in \( v_2 \); by hypothesis \( k_1 \geq \ell \) for some \( i \), which we choose minimal. The summand then has the form \( \theta_1 \alpha_{k_{\ell+1}} \cdots \alpha_{k_n} \). The supports of \( \delta_i \theta_1 \) and of \( \delta_{n_0} \theta_1 \) are by hypothesis contained in \( \{x, x_0\} \cdot \{\{1\} \cup F_1 \cup \cdots \cup \)
\[ F_{\ell-1}^n, \text{ so } \| \delta_1 \alpha_k - \delta_0 \alpha_k \| < \epsilon \ell \text{ and } \| \delta_0 \theta \alpha_k - \delta_0 \alpha_k \| < \epsilon \ell. \] Consequently, \[ \| \delta_0 \theta - \delta_0 \theta_1 \| < 2\epsilon \ell, \text{ so } \| \delta_2 \alpha_k \| - \| \delta_0 \alpha_k \| < 2\epsilon \ell \] and finally \[ \| \delta_\nu \alpha_k - \delta_0 \alpha_k \| < 4\epsilon \ell \text{ as required.} \]

\( (2) \Rightarrow (3) \) is obvious.

\( (3) \Rightarrow (1) \) By Proposition [\textbf{8.25}], the sequence \( (\mu_n)_{n \in \mathbb{N}} \) is asymptotically invariant. Consider \( \epsilon > 0 \) and \( f \in \mathcal{O}(\ell^1 X) \). There is then a subset \( S \subseteq X \times X \) such that \( f = f' + \sum_{(x,y) \in S} \delta_x - \delta_y \) with \( \| f' \| < \epsilon / 2 \); we have \( \| f \mu_n \| \leq \sum_{(x,y) \in S} \| \delta_x \mu_n - \delta_y \mu_n \| + \| f' \mu_n \| \), and for \( n \) large enough each \( \delta_x \mu_n - \delta_y \mu_n \) has norm at most \( \epsilon / 2 \# S \), from which \( \| f \mu_n \| < \epsilon = \epsilon \| \mu_n \| \). and \( X \) is laminable.

\( \square \)

**Exercise 8.26** (*). Let \( G \) be a group and let \( \mu \) be a probability measure on \( G \). Prove that \( (G, \mu) \) is Liouville if and only if \( (G, \tilde{\mu}) \) is Liouville.
9 Extensive amenability

We introduce now a property stronger than amenability for $G$-sets, a property that behaves better with respect to extensions of $G$-sets (whence the name). This section is based on [70].

**Definition 9.1.** Let $X$ be a set; recall that $\mathcal{P}_f(X)$ denotes the collection of finite subsets of $X$. An **ideal** in $\mathcal{P}_f(X)$ is a subset, for some $x \in X$, of the form $\{E \subseteq X \mid x \in E\}$.

Let $X$ be a $G$-set. It is **extensively amenable** if there exists a $G$-invariant mean $m$ on $\mathcal{P}_f(X)$ giving weight 1 to every ideal.

It follows immediately from the definition that $m(\{\emptyset\}) = 0$ if $X \neq \emptyset$, and that for every $E \in X$ we have $m(\{F \subseteq X \mid E \subseteq F\}) = 1$.

Recall that $\mathcal{P}_f(X)$ is an abelian group under symmetric difference $\triangle$, and is naturally isomorphic to $\prod_X \mathbb{Z}/2$ under the map $E \mapsto 1_E$. Recall also from (1) that the wreath product $\mathbb{Z}/2 \wr X G$ is the semidirect product $G \ltimes (\prod_X \mathbb{Z}/2)$, with $G$ acting on $\prod_X \mathbb{Z}/2$ by permuting its factors.

**Lemma 9.2.** If $G$ is amenable, then all $G$-sets are extensively amenable. Every extensively amenable non-empty $G$-set is amenable.

**Proof.** Let $G$ be amenable and let $X$ be a $G$-set. Consider the set $K$ of means on $\mathcal{P}_f(X)$ giving full weight to every ideal. Clearly $K$ is a convex compact subset of $\ell_\infty(\mathcal{P}_f(X))^*$, and is non-empty because it contains any cluster point of $(\delta_E)_{E \in X}$. Since $G$ is amenable, there exists a fixed point in $K$, so $X$ is extensively amenable.

Let next $X \xrightarrow{\phi} G$ be extensively amenable, and let $m$ be an invariant mean in $\ell_\infty(\mathcal{P}_f(X) \setminus \{\emptyset\})^*$. Define a mean on $X$ by

$$
\ell_\infty(X) \ni f \mapsto m \left( E \mapsto \frac{1}{\#E} \sum_{x \in E} f(x) \right),
$$

and note that it is $G$-invariant because $m$ is. \hfill \Box

**Lemma 9.3.** Let $X$ be a $G$-set. Then the following are equivalent:

1. $X$ is extensively amenable;
2. For every finitely generated subgroup $H$ of $G$ and every $H$-orbit $Y \subseteq X$, the $H$-set $Y$ is extensively amenable;
3. For every finitely generated subgroup $H$ of $G$ and every $x_0 \in X$, there is an $H$-invariant mean on $\mathcal{P}_f(x_0H)$ that gives non-zero weight to $\{E \subseteq x_0H \mid x_0 \in E\}$;
4. There is a $G$-invariant mean on $\mathcal{P}_f(X)$ that gives non-zero weight to $\{E \subseteq X \mid x_0 \in E\}$ for all $x_0 \in X$.

\[\text{It is really the ideal generated by } \{x\} \text{ in the semigroup } (\mathcal{P}_f(X), \cup).\]
Proof. (1) ⇒ (4) by definition.

(4) ⇒ (3) There is a natural map \( \ell^\infty(\mathcal{P}_f(x_0H)) \to \ell^\infty(\mathcal{P}_f(X)) \) given by \( f \mapsto f(- \cap x_0H) \), inducing an \( H \)-equivariant map \( \mathcal{M}(\mathcal{P}_f(X)) \to \mathcal{M}(\mathcal{P}_f(x_0H)) \).

(3) ⇒ (2) Let \( Y = x_0H \) be an \( H \)-orbit, and let \( m_0 \) be an \( H \)-invariant mean on \( \mathcal{P}_f(Y) \) that gives positive weight to \( \{ A \subseteq Y \mid x_0 \in A \} \). As in Theorem 5.23, the mean \( m_0 \) may be approximated by a net \( p_n \) of probability measures on \( \mathcal{P}_f(Y) \): these are maps \( \mathcal{P}_f(Y) \to [0,1] \) with total mass 1. Define now for every \( k \in \mathbb{N} \) new probability measures on \( \mathcal{P}_f(Y) \) by

\[
p_{n,k}(E) = \sum_{E_1 \cup \ldots \cup E_k = E} p_n(E_1) \cdots p_n(E_k).
\]

Let \( m \) be an cluster point of the \( p_{n,k} \) as \( n,k \to \infty \); then \( m \) is an \( H \)-invariant mean on \( \mathcal{P}_f(Y) \), and we check that it gives mass 1 to the ideal \( S := \{ E \subseteq Y \mid x_0 \in E \} \), and therefore also to every ideal because \( H \) acts transitively on \( Y \) and \( m \) is \( H \)-invariant: since \( m_0(S) > 0 \), there exists \( \delta < 1 \) such that \( p_n(S) > 1 - \delta \) for all \( n \) large enough, and then \( p_{n,k}(S) = 1 - \delta^k \) so at the cluster point \( m(S) = 1 \).

(2) ⇒ (1) For every finitely generated subgroup \( H \) of \( G \) and every finite union \( Y = Y_1 \cup \ldots \cup Y_n \) of \( H \)-orbits, choose for \( i = 1,\ldots,n \) an \( H \)-invariant mean \( m_i \) on \( \mathcal{P}_f(Y_i) \), and construct a mean \( m_{H,Y} \) on \( \mathcal{P}_f(X) \) by \( m_{H,Y}(S) = m_1(\{ E \cap Y_1 \mid E \subseteq S \}) \cdots m_n(\{ E \cap Y_n \mid E \subseteq S \}) \). Clearly \( m_{H,Y} \) is \( H \)-invariant and gives full weight to ideals in \( \mathcal{P}_f(Y) \). Order the pairs \( (H,Y) \) by inclusion, and consider a cluster point of the net \( (m_{H,Y}) \). It is \( G \)-invariant, and gives full weight to ideals in \( \mathcal{P}_f(X) \). \( \square \)

Note that Lemma 9.3.2) implies in particular that extensively amenable sets are hereditarily amenable: every subgroup acting on every orbit is amenable. We obtain in this manner an abundance of amenable actions that are not extensively amenable. For instance, consider Example 3.13 of amenable action of \( F_2 = \langle a, b \rangle \), and the subgroup \( K = \langle a^{b^{-1}}, a^{b^{-2}} \rangle \). Then \( K \) is a free group of rank 2, and the \( K \)-orbit \( Y \) of 1 in \( X \) is free, so \( Y \) is not an amenable \( K \)-set, and therefore \( X \) is not extensively amenable. We shall see in Example 9.20 a hereditarily amenable \( G \)-set that is not extensively amenable.

We come to the justification of the terminology “extensive amenability”: the analogue of Corollary 2.7 for \( G \)-sets.

**Proposition 9.4.** Let \( G \) be a group acting on two sets \( X,Y \), and let \( q : X \to Y \) be \( G \)-equivariant. If \( Y \) is extensively amenable and if for every \( y \in Y \) the \( G_y \)-set \( q^{-1}(y) \) is an extensively amenable, then \( X \) is extensively amenable. The converse holds if \( q \) is onto.

**Proof.** The proof follows closely that of Proposition 2.26 see [70] Proposition 2.4 for details. Assume that \( q^{-1}(y) \) is extensively amenable for all \( y \in Y \), and let \( m_y \) be a \( G_y \)-invariant mean giving full weight to ideals. By making one choice per \( G \)-orbit, we may also assume that \( m_y \) is the push-forward by \( g \) of \( m_y \) whenever \( y' = yg \). Extend every \( m_y \) to a mean on \( \mathcal{P}_f(X) \); then \( (m_y) \) is a \( G \)-equivariant map \( Y \to \mathcal{M}(\mathcal{P}_f(X)) \).
For every \( F = \{ y_1, \ldots, y_n \} \subseteq Y \), we set \( m_F(S) = m_{y_1}(\{ E \cap q^{-1}(\{ y_1 \}) \mid E \in S \}) \cdots m_{y_n}(E \cap q^{-1}(y_n)) \), and note that \( m_F \) gives full weight to every ideal of the form \( \{ E \in X \mid x \in E \} \) for some \( x \in q^{-1}(F) \). The map \( F \mapsto m_F \) defines a \( G \)-equivariant map \( \Psi^F(Y) \to \mathcal{M}(\Psi^F(X)) \). Composing with the barycentre \( Y \) as in \([5]\), we obtain a \( G \)-equivariant map \( m_*: \mathcal{M}(\Psi^F(Y)) \to \mathcal{M}(\Psi^F(X)) \).

Assume now that \( Y \) is extensively amenable, and let \( n \) be a \( G \)-invariant mean on \( \Psi^F(Y) \) giving full weight to ideals. Set \( m := m_*(n) \); then \( m \) is a \( G \)-invariant mean on \( \Psi^F(X) \) giving full weight to ideals, so \( X \) is extensively amenable.

Assume finally that \( q \) is onto and that \( X \) is extensively amenable. By Lemma 9.3 the \( G \)-subset \( q^{-1}(y) \) of \( X \) is extensively amenable for all \( y \in Y \). Let \( m \) be a mean on \( \Psi^F(X) \) giving full weight to ideals, and define a mean \( n \) on \( \Psi^F(Y) \) by \( n(S) = m(\{ E \in X \mid q(E) \in S \}) \). Given \( y \in Y \), choose \( x \in q^{-1}(y) \), and note

\[
n(\{ F \subseteq Y \mid y \in F \}) = m(\{ E \subseteq X \mid y \in q(E) \}) \geq m(\{ E \subseteq X \mid x \in E \}) = 1. \quad \Box
\]

In particular, let \( K \leq H \leq G \) be groups. Then \( K \setminus G \) is an extensively amenable \( G \)-set if and only if both \( K \setminus H \) and \( H \setminus G \) are extensively amenable. This is in contrast with Example 2.18 where the corresponding property is shown not to hold for amenable sets.

The following proposition relates Definition 9.1 to the original definition; we begin by introducing some vocabulary. Let \( \mathbf{A} \) denote the category of group actions: its objects are pairs \( X \rightharpoonup G \) of a set \( X \) and an action of \( G \) on \( X \), and a morphism \( (X \rightharpoonup G) \to (Y \rightharpoonup H) \) is a pair of maps \( (f: X \to Y, \phi: G \to H) \) intertwining the actions on \( X \) and \( Y \), namely satisfying \( f(x)\phi(g) = f(xg) \) for all \( x \in X, g \in G \). We denote by \( \mathbf{AA} \) and \( \mathbf{EA} \) the subcategories of amenable, respectively extensively amenable actions.

We are interested in functors \( F: \{ \text{finite sets, injections} \} \to \mathbf{AA} \), written \( F(X) = F_0(X) \rightharpoonup F_1(X) \) for a group \( F_1(X) \) and an \( F_1(X) \)-set \( F_0(X) \). Since amenable actions are closed under directed unions, and every set is the directed union of its finite subsets, we get by continuity a functor still written \( F: \{ \text{sets, injections} \} \to \mathbf{AA} \), called an amenable functor. If furthermore \( F \) takes values in \( \mathbf{EA} \) then we call it an extensively amenable functor. We call the functor \( F \) tight if the map \( F_0(X \setminus \{ x \}) \to F_0(X) \) is never onto.

We already saw some examples of tight functors: for any amenable group \( A \), the functor \( X \mapsto A^X \rightharpoonup A^X \) since \( A^X \) is the directed union of its amenable subgroups \( A^E \) over all \( E \subseteq X \); the functor \( X \mapsto \text{Sym}(X) \rightharpoonup \text{Sym}(X) \), by the same reasoning (see Example 7.4); and the functor \( X \mapsto X \rightharpoonup \text{Sym}(X) \). Note that if \( X \) is a \( G \)-set then \( F_0(X) \) and \( F_1(X) \) inherit \( G \)-actions by functoriality.

**Proposition 9.5 ([70], Theorem 3.14]).** Let \( F \) be a functor as above, and let \( X \) be a \( G \)-set. If \( X \) is extensively amenable and \( F \) is amenable then \( F_0(X) \rightharpoonup (G \rtimes F_1(X)) \) is amenable, and if furthermore \( F \) is extensively amenable then \( F_0(X) \rightharpoonup (G \rtimes F_1(X)) \) is extensively amenable.

Conversely, if \( F \) is tight and \( F_0(X) \rightharpoonup (G \rtimes F_1(X)) \) is amenable then \( X \) is extensively amenable.

**Proof.** Assume first that \( F \) is amenable. For every \( E \subseteq X \) let \( m_E \in \mathcal{M}(F_0(E))^{F_1(E)} \) be an invariant mean, and extend it functorially to a mean still written \( m_E \in
\(\mathcal{M}(F_0(X))^{F_1(E)}\). By choosing once \(m_E\) per cardinality class of subsets of \(X\), we may ensure that we have \(f_*(m_E) = m_{E'}\) for every bijection \(f : E \to E'\). We obtain in this manner a \(G\)-equivariant map \(\mathcal{P}_f(X) \to \mathcal{M}(F_0(X))\), and therefore, composing with the barycentre \(Y\) as in \([9]\), a map \(\mathcal{M}(\mathcal{P}_f(X)) \to \mathcal{M}(F_0(X))\).

By assumption, there exists \(m_0 \in \mathcal{M}(\mathcal{P}_f(X))\) giving full mass to ideals; let \(m\) be the image of \(m_0\) under the above map. Clearly \(m\) is a \(G\)-invariant mean on \(F_0(X)\). It is also \(F(A)\)-invariant for every \(A \subseteq X\): one may restrict \(m_0\) to \(\{E \subseteq X \mid A \subseteq E\}\) and still obtain a mean. Every \(m_E\) is \(F_1(E)\)-invariant, so is in particular \(F(A)\)-invariant, and therefore \(m\) is also \(F(A)\)-invariant. In summary, \(m\) is \(G \ltimes F_1(X)\)-invariant, so \(F_0(X)\) is an amenable \(G \ltimes F_1(X)\)-set.

For the converse, define a \(G\)-equivariant map \(\text{support} : F_0(X) \to \mathcal{P}_f(X)\) by

\[
\text{support}(x) = \bigcap \{E \subseteq X \mid x \in \text{image}(F_0(E) \to F_0(X))\}.
\]

Assume that \(F_0(X)\) is an amenable \(G \ltimes F_1(X)\)-set, and let \(m_0\) be a \(G\)-invariant mean on \(F_0(X)\). Let \(m\) the push-forward of \(m_0\) via \(\text{support}\); it is a \(G\)-invariant mean on \(\mathcal{P}_f(X)\). Choose \(x_0 \in X\). By definition, \(m(\{E \subseteq X \mid x_0 \in E\}) = m_0(S)\) for the ideal

\[
S = \{x \in F_0(X) \mid x_0 \in \text{support}(x)\}
\]

\[
= \bigcap_{x_0 \notin E \in X} (F_0(X) \setminus \text{image}(F_0(E) \to F_0(X)))
\]

\[
= F_0(X) \setminus F_0(X \setminus \{x_0\}).
\]

Since \(F\) is tight, \(S \neq \emptyset\). Furthermore, \(m_0\) is \(F_1(X)\)-invariant, so \(m_0(S) > 0\). We conclude by Lemma \([9.3]\) that \(X\) is extensively amenable.

Finally, to prove that \(F_0(X)\) is an extensively amenable \(G \ltimes F_1(X)\)-set whenever \(F\) is an extensively amenable functor, we apply the converse just proven to the functor \(H(X) = (\mathbb{Z}/2)_{\langle X \rangle} \to (\mathbb{Z}/2)^{\langle X \rangle}\). For every \(X\), we know from the first part of the proof that \(H_0(F_0(X))\) is an amenable \((G \ltimes F_1(X)) \ltimes H_1(F_0(X))\)-set, since we assumed \(F_1(X) \leftrightarrow G \ltimes F_1(X)\) is extensively amenable. Therefore, the functor \(X \mapsto H_0(F_0(X)) \leftrightarrow (F_1(X) \ltimes H_1(F_0(X)))\) is amenable, and yet again the second part of the proof allows us to deduce that \(F_0(X) \leftrightarrow G \ltimes F_1(X)\) is extensively amenable. \(\square\)

A fundamental application of Proposition \([9.5]\) is the following

**Corollary 9.6.** Let \(H\) be a subgroup of \(G \ltimes F(X)\) for some extensively amenable \(G\)-set \(X\). If \(H \cap (G \times 1)\) is amenable, then \(H\) is amenable too.

**Proof.** By Proposition \([9.5]\) \(F(X)\) is extensively amenable, so by Lemma \([9.3]\) the \(H\)-orbit \(1 \cdot H \subseteq X\) is an extensively amenable \(H\)-set, and is therefore amenable by Lemma \([9.2]\). The stabilizers in this action are conjugate to \(H \cap (G \times 1)\), which is amenable by assumption, so \(H\) is amenable by Proposition \([2.26]\). \(\square\)

There is also a connection between extensive amenability and laminability, see Definition \([8.15]\) by Corollary \([8.18]\) if \(X \leftrightarrow G\) is extensively amenable then \(F(X) \leftrightarrow G \ltimes F(X)\) is laminable.
In the next section, we shall see a sufficient condition for an action to be extensively amenable, and in Example 9.15 an application to interval exchange transformations.

We finish this section by a very brief summary of the “only if” part of a proof of Theorem 4.1 due to Kleiner [83] and simplified by Tao; we include it here because it combines amenability and the study of (now unbounded) harmonic functions.

Let $G$ be a group of polynomial growth; we are to show that $G$ has a nilpotent subgroup of finite index. We may of course assume that $G$ is infinite, and by induction on the growth degree it suffices to show that $G$ has a finite-index subgroup mapping onto $\mathbb{Z}$. For that purpose, it suffices to show that $G$ has an infinite image in some virtually soluble group. By [118], every amenable finitely generated subgroup of $\text{GL}_n(\mathbb{C})$ is virtually soluble, and $G$ is amenable by Proposition 3.14, so it suffices to construct a representation $G \to \text{GL}_n(\mathbb{C})$ with infinite image. The proof uses the following arguments:

**Lemma 9.7.** Let $G$ be a countably infinite amenable group. Then there exists an action of $G$ on a Hilbert space $\mathcal{H}$ with no fixed points.

**Proof.** Consider $\mathcal{H} = \ell^2(\mathbb{N} \times G)$, the space of square-summable functions $(f_1, f_2, \ldots)$ in $\ell^2(G)$. There is a natural, diagonal action of $G$ on $\mathcal{H}$ by right-translation. This action has a fixed point 0, but we can construct an affine action without fixed point as follows.

Let $(F_n)_{n \in \mathbb{N}}$ be a Følner sequence in $G$, and define $h = (\frac{n_{F_n}}{\sqrt{|F_n|}})_{n \in \mathbb{N}}$. Then $h \notin \mathcal{H}$, but $h - hg \in \mathcal{H}$ for all $g \in G$, using the almost-invariance of $(F_n)$. We let $G$ act on $\mathcal{H}$ by $f \cdot g = fg + h - hg$, namely we move the fixed point to $h$. \qed

The main result, whose proof we omit, is the following control on the growth of harmonic functions. It follows easily from Gromov’s theorem, but Kleiner gave a direct and elementary proof of it:

**Lemma 9.8.** Let $G$ be a group of polynomial growth, and let $\mu$ be a measure on $G$. Then for every $d \in \mathbb{N}$ the vector space of harmonic maps $u : G \to \mathbb{C}$ of growth degree at most $d$ (namely for which there is a constant $C$ with $|u(g)| \leq C|g|^d$ for all $g \in G$) is finite-dimensional.

The proof of Theorem 4.1 is then finished: a group $G$ of polynomial growth is amenable, so by Lemma 9.7 it has an affine, fixed-point-free action on a Hilbert space $\mathcal{H}$. Let $\mu$ be SRW on $G$, and define

$$E : \mathcal{H} \to \mathbb{R}_+, \quad v \mapsto \frac{1}{2} \sum_{s \in S} \mu(s) \|vs - v\|^2.$$  

Since $\mathcal{H}$ has no fixed point, $E(v) > 0$ for all $v \in \mathcal{H}$. Let us assume that $E(v)$ attains its minimum — this can be achieved by considering a sequence of better and better approximations to a minimum in an ultrapower of $\mathcal{H}$ — and call its minimum $h$. One directly sees from $\frac{\partial E(v)}{\partial v} |_{\mathcal{H}} = 0$ that $h$ is $\mu$-harmonic, and it is not constant. Then $V := \{ (h|v) \mid v \in \mathcal{H} \}$ is a vector space of Lipschitz harmonic maps, so is
finite-dimensional by Lemma 9.8, and $G$’s action on $V$ has infinite image because $V$ has no non-zero $G$-fixed point.

### 9.1 Recurrent actions

We saw in Proposition 3.14 that actions on subexponentially-growing spaces are amenable; and in Theorem 8.4 that random walks on graphs in which the probability of return to the origin decays subexponentially give amenable actions. We see here that stronger conditions — quadratic growth, recurrent random walks — produce extensively amenable actions.

Let $p_1 : X \times X \to [0, 1]$ be a random walk on a set $X$. It is recurrent at $x \in X$ if $\sum_{n \geq 0} p_n(x, x) = \infty$, namely if a random walk started at $x$ is expected to return infinitely often to $x$, and equivalently if it is certain to return to $x$. It is transient if it is not recurrent.

We computed in Example 8.9 that the probability of return in to the origin in $n$ steps of SRW on $Z$ is $\approx n^{-1/2}$; so the probability of return to the origin on $Z^d$ is $\approx n^{-d/2}$. It follows that SRW on $Z^d$ is recurrent precisely for $d \leq 2$.

**Lemma 9.9.** The random walk $p$ is recurrent if and only if for every $x \in X$ there exists a sequence of functions $(a_n)$ in $\ell^2(X)$ with $a_n(x) = 1$ and $\|a_n - Ta_n\| \to 0$, for $T$ the associated random walk operator.

**Proof.** For a function $\phi \in \ell^2(X)$, define its Dirichlet norm as $D(\phi) = \|d\phi\|^2 = \frac{1}{2} \sum_{y \in X} (f(x) - f(y))^2 p(x, y)$. The claim is equivalent to requiring the existence of functions $a_n \in \ell^2(X)$ with $a_n(x) = 1$ and arbitrarily small Dirichlet norm. If $X$ is finite, there is nothing to do, as the functions $a_n \equiv 1$ have $D(a_n) = 0$.

Choose $x \in X$. Assume first that $p$ is transient, so that $G(y) := \sum_{n \geq 0} p_n(x, y)$ is well-defined. Then for all $\phi \in \ell^2(X)$ we have

$$\langle d\phi, dG \rangle = \langle \phi, d^* dG \rangle = \phi(x),$$

and $\|d\phi, dG\|^2 \leq D(\phi)D(g)$, so $D(\phi) \geq \phi(1)/D(g)$ is bounded away from 0.

Assume next that $p$ is recurrent. For every $n \in \mathbb{N}$, set $G_n(y) = \sum_{m=0}^n p_m(x, y)$ and $a_n(y) = G_n(y)/G_n(x)$. Since by assumption $G_n(x) \to \infty$, the functions $a_n(y)$ satisfy the requirement. \(\square\)

For random walks with finite range, the following criterion due to Nash-Williams is very useful. Let $p$ be a transitive random walk on a set $X$, and let $x \in X$ be a basepoint. A slow constriction of $X$ is a family $\{x\} = V_0 \subset V_1 \subset \cdots$ of finite subsets of $X$, such that $\bigcup V_n = X$ and $p_1(V_m, V_n) = 0$ whenever $|m - n| \geq 2$ and $\sum_{n \geq 0} p_1(V_n, V_{n+1})^{-1} = \infty$. A refinement of $p$ is the random walk on a set obtained by subdividing arbitrarily each transition $p_1(x, y)$ by inserting midpoints along it.

**Theorem 9.10 (Nash-Williams [100]).** Let $p$ be a transitive random walk on a set $X$. Then $p$ is recurrent if and only if it has a refinement admitting a slow constriction.
The result applies to \( \mathbb{Z}^d \) for \( d \leq 2 \): the sets \( V_n \) may be chosen as \( \{-n, \ldots, n\}^d \).

We only prove the “only if” direction, which is the important direction for us.

**First proof of Theorem 9.10 “only if” direction.** Given a constriction \((V_n)\), set \( c_n = 1/p_1(V_n, V_{n+1}) \), and define an associated random walk \( q \) on \( \mathbb{N} \) by \( q_1(n, n + 1) = c_n/(c_n + c_{n–1}) \) and \( q_1(n, n - 1) = c_{n-1}/(c_n + c_{n–1}) \). It is easy to check \( \sum q_n(1, 1) = \infty \) if the constriction is slow. \( \square \)

We shall give another proof of the “only if” direction, based on Lemma 9.9; we begin by a simple

**Lemma 9.11.** Let \( \sum v_i \) be a positive, divergent series. Then there exist \( \lambda_{i,n} \geq 0 \) such that \( \sum \lambda_{i,n} v_i = 1 \) for all \( n \) and \( \sum \lambda_{i,n}^2 v_i \to 0 \) as \( n \to \infty \).

**Proof.** Let \( \alpha_n = 1 + 1/n \) be a decreasing sequence converging to 1. Group the terms in \( \sum v_i \) into blocks \( w_1 + w_2 + \cdots \) such that \( w_i \geq 1 \) for all \( i \). Set

\[
\lambda_{i,n} = \frac{\alpha_n - 1}{w_i \alpha_n^{k–1}} \quad \text{if } v_i \text{ belongs to the block } w_k. \quad \square
\]

**Second proof of Theorem 9.10 “only if” direction.** Let \((V_i)_{i \geq 1}\) be a slow constriction of \( X \) with basepoint \( x \), and set \( v_i := 1/p_1(V_i, V_{i+1}) \). Apply the lemma to the divergent series \( \sum v_i \), and define maps \( a_n : X \to [0, 1] \) by

\[
a_n(y) = 1 - \sum \lambda_{i,n} v_i 1_{y \in V_i}. \]

Then \( a_n \) has finite support so in particular belongs to \( \ell^2(X) \); and \( a_n(x) = 1 \) because \( x \in V_i \) for all \( i \); and \( ||a_n - a_n g||_2 \to 0 \) for all \( g \in G \) because \( \sum \lambda_{i,n}^2 v_i \to 0 \). \( \square \)

The main result of this section is the following. We will prove it in two different manners, and in fact in this manner recover the “if” direction of Theorem 9.10

**Theorem 9.12.** If \( X \) is a \( G \)-set with a non-degenerate recurrent random walk, then \( X \) is extensively amenable.

We begin with some preparation for the proof. Let \( \mu \) be a symmetric, non-degenerate measure on a group \( G \), and let \( X \) be a \( G \)-set. For a basepoint \( x \in X \) and a trajectory \( x, xg_1, xg_1g_2, \ldots \) of the random walk on \( X \), the corresponding length-\( n \) inverted orbit is the random subset

\[
O_n = \{x, xg_n, xg_n^{-1}g_n, \ldots, xg_1 \cdots g_n\}. 
\]

If \( X \) is transitive, then \( \# O_n \) depends only mildly on the choice of \( x \).

**Proposition 9.13.** Let \( X \) be a transitive \( G \)-set and let \( \mu \) be a symmetric, non-degenerate probability measure on \( G \). Then \( X \) is extensively amenable if and only if

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}(2^{-\# O_n}) = 0. \quad (17)
\]
Proof. Thanks to Proposition 9.5, it is enough to prove that (17) is equivalent to
the amenability of the group \( G \). Choose a basepoint \( x \in X \), and consider on \( G \) the probability distribution \( v := \frac{1}{2} (1 + \delta_x) * \mu * \frac{1}{2} (1 + \delta_x) \), called the “switch-walk-switch” measure: in the action on \( (\mathbb{Z}/2)^X \), it amounts to randomizing the current copy of \( \mathbb{Z}/2 \), moving to another position in \( X \), and randomizing the new copy of \( \mathbb{Z}/2 \). By Kesten’s Theorem 8.4, amenability of the action on \( (\mathbb{Z}/2)^X \) is equivalent to subexponential decay of return probabilities of a random walk \( (f_0 = 1, f_1, f_2, \ldots) \) on \( (\mathbb{Z}/2)^X \), namely to \( \lim_{n \to \infty} \frac{1}{n} \log P(f_1 \cdot \cdots \cdot f_n = 1) = 0 \). Now the support of \( f_1 \cdot \cdots \cdot f_n \) is contained in \( O_n \); writing each \( f_i = \delta_i^g \cdot \eta_i^g \) with \( g_i \in G \), we get \( f_1 \cdot \cdots \cdot f_n = g_1 \cdots \cdot g_n \delta_1^{g_1 \cdots \cdot g_n} \cdots \cdot \delta_n^{g_1 \cdots \cdot g_n} \); and \( f_n \) randomizes every copy of \( \mathbb{Z}/2 \) indexed by \( O_n \), so \( P(f_n = 1) = \mathbb{E}(2^{-\#O_n}) \). \( \square \)

**Lemma 9.14.** Let \( p \) be a transitive random walk on a \( G \)-set \( X \) driven by a symmetric probability measure \( \mu \). Then \( X \) is recurrent if and only if \( \lim \frac{1}{n} \mathbb{E}(\#O_n) = 0 \).

**Proof.** Choose a basepoint \( x \in X \) for the random walk \( (x = x_0, x_1, \ldots) \), and define the random variable \( \Theta = \min \{ n \geq 1 \mid x_n = x \} \). Then

\[
\mathbb{E}(\#O_{n+1} - \#O_n) = \mathbb{P}(x_{g_{n+1}} \notin O_n) = \mathbb{P}(x_{g_{n+1}} \notin \{ x, x_{g_n}, x_{g_n+1}, \ldots, x_{g_1} \}) = \mathbb{P}(\{ x_{g_{n+1}+1}, x_{g_{n+1}+1}, x_{g_n+1}, \ldots, x_{g_1} \} \notin x) = \mathbb{P}(\Theta > n + 1),
\]

because the random walk with increments \( g_{n+1}, g_n^{-1}, \ldots, g_1^{-1} \) has the same law as \( \mu^n \). Therefore, \( \mathbb{E}(\#O_n)/n \to \mathbb{P}(\Theta = \infty) \), which vanishes if and only if \( X \) is recurrent. \( \square \)

**First proof of Theorem 9.12.** We may assume, by Lemma 9.3, that \( X \) is transitive. Let \( p \) be a non-degenerate, transitive, recurrent random walk on \( X \). By Lemma 9.14, we have \( \frac{1}{n} \mathbb{E}(\#O_n) \to 0 \), so by convexity

\[
-\frac{1}{n} \log \mathbb{E}(2^{-\#O_n}) \leq \frac{1}{n} \mathbb{E}(\#O_n) \log 2 \to 0,
\]

so \( X \) is extensively amenable by Proposition 9.13. \( \square \)

**Second proof of Theorem 9.12.** Let \( x \in X \) be arbitrary. We start, using Lemma 9.9, with a sequence of functions \( (a_n) \in \ell^2(X) \) satisfying \( a_n(x) = 1 \) and \( \lim \| a_n - a_{n+1} \| = 0 \) for all \( g \in G \). (This is also the outcome of the second proof of Theorem 9.10.) We construct then maps \( b_n : \mathcal{P}(X) \to [0, 1] \) by

\[
b_n(E) = \prod_{y \in E} a_n(y).
\]

They are finitely supported, and therefore may be viewed in \( \ell^2(\mathcal{P}(X)) \). It remains to check that they are almost invariant under the action of \( \mathbb{Z}/2 \) on \( G \). Assuming that \( X \) is transitive, this last group is generated by \( \delta_2 : X \to \mathbb{Z}/2 \) and \( G \). We have \( b_n \delta_2 = b_n \), because \( b_n(E) = b_n(E \triangle \{ x \}) \).
The name “interval exchange” comes from opening up the circle into an interval \([0, 1]\); the rotation on the circle may be viewed as an exchange of two intervals \([0, 1 - \alpha] \mapsto [\alpha, 1], [1 - \alpha, 1] \mapsto [0, \alpha]\).

The spaces \(\ell^2(\mathcal{Q}_f(X))\) and \(\bigotimes_X \ell^2(C^2)\) are isometric; the isometry is the obvious one mapping \(\delta_E\) to \(\bigotimes_{x \in X} \delta_x \in E\), if we take \(\{\delta_{\text{false}}, \delta_{\text{true}}\}\) as basis of \(\ell^2(C^2)\). We compute

\[
||b_n||^2 = \langle b_n, b_n \rangle = \prod_{y \in X} (1 + a_n(y))^2,
\]

and for \(g \in G\) we similarly have \(\langle b_n, b_ng^{-1} \rangle = \prod_{y \in X} (1 + a_n(y)a_n(yg))\), so

\[
\left(\frac{\langle b_n, b_n \rangle}{\langle b_n, b_ng^{-1} \rangle}\right)^2 = \prod_{y \in X} \frac{(1 + a_n(y)^2)(1 + a_n(yg)^2)}{(1 + a_n(y)a_n(yg))^2}.
\]

Taking logarithms, and using the approximation \(\log(t) \leq t - 1\),

\[
0 \leq 2 \log(A) \leq \sum_{y \in X} \log(B) \leq \sum_{y \in X} \frac{(a_n(y) - a_n(yg))^2}{(1 + a_n(ya_n(yg)))^2} \leq ||a_n - a_ng^{-1}||^2 \to 0,
\]

so \(\langle b_n, b_n \rangle/\langle b_n, b_ng^{-1} \rangle \to 1\) and therefore \(||b_n - b_ng|| \to 0\). \(\square\)

Example 9.15. An interval exchange is a piecewise-translation self-map of the circle. More precisely, it is a right-continuous map \(g : \mathbb{R}/\mathbb{Z} \rightharpoonup \mathbb{R}/\mathbb{Z}\) such that \(\langle g \rangle := \{g(x) - x \mid x \in \mathbb{R}/\mathbb{Z}\}\) is finite.

The rotation \(x \mapsto x + \alpha\) is an extreme example of interval exchange. The interval exchange transformations naturally form a group \(\text{IET}\) acting on \(\mathbb{R}/\mathbb{Z}\); and every countable subgroup \(G \leq \text{IET}\) can be made to act on the Cantor set by letting \(\mathcal{D}\) be the union of the \(G\)-orbits of discontinuity points of \(G\) (or of 0 if all elements of \(G\) are rotations) and replacing \(\mathbb{R}/\mathbb{Z}\) by

\[
X := (\mathbb{R}/\mathbb{Z} \setminus \mathcal{D}) \cup (\mathcal{D} \times \{+, -\}),
\]

namely by opening up the circle at every point of \(\mathcal{D}\); see [80 §5].

Little is known on the group \(\text{IET}\); in particular, it is not known whether it contains non-abelian free groups, or whether it is amenable. We prove:

Theorem 9.16 ([70] Theorem 5.1]. Let \(\Lambda \leq \mathbb{R}/\mathbb{Z}\) be a finitely generated subgroup with free rank at most 2, namely \(\dim(\Lambda \otimes \mathbb{Q}) \leq 2\). Then

\[
\text{IET}(\Lambda) := \{g \in \text{IET} \mid \langle g \rangle \subseteq \Lambda\}
\]

is an amenable subgroup of \(\text{IET}\).

Proof. We first prove that the action of \(\text{IET}(\Lambda)\) on \(\mathbb{R}/\mathbb{Z}\) is extensively amenable. Choose a finite generating set for \(\Lambda\); then the Cayley graph of \(\Lambda\) is quasi-isometric.
to $\mathbb{Z}^d$ for $d \leq 2$, and in particular is recurrent. Let $G = \langle S \rangle$ be a finitely generated subgroup of $\text{IET}(\Lambda)$. For $x \in \mathbb{R}/\mathbb{Z}$, the orbit $xG$ injects into $\Lambda$ under the map $y \mapsto y - x$, and this map is Lipschitz with Lipschitz constant $\max_{\lambda \in \Lambda} \max_{\lambda \in \langle S \rangle} \| \lambda \|$, so the Schreier graph of $xG$ is recurrent. Theorem 9.12 implies that $xG$ is an extensively amenable $G$-set, so $\mathbb{R}/\mathbb{Z}$ is an extensively amenable $\text{IET}(\Lambda)$-set by Lemma 9.3. 

We wish to apply Corollary 9.6 to $X = \mathbb{R}/\mathbb{Z}$ and $G = \text{IET}(\Lambda)$ and $F(X) = X \leftrightharpoons \text{Sym}(X)$. Given an interval exchange map $g \in \text{IET}$, let $\bar{g}$ be the unique left-continuous self-map of $\mathbb{R}/\mathbb{Z}$ that coincides with $g$ except at its discontinuity points, and let $\tau_g = g^{-1} \bar{g} \in \text{Sym}(\mathbb{R}/\mathbb{Z})$ be the corresponding permutation of the discontinuity points of $g^{-1}$. We have a cocycle identity $\tau_{gh} = \tau_g \tau_h$, so the map

$$t : \text{IET} \to \text{IET} \ltimes \text{Sym}(\mathbb{R}/\mathbb{Z}) \quad \text{given by} \quad (g, \tau_g) \mapsto (tg)$$

is an embedding. Observe that $\tau_g = 1$ if and only if $g$ is continuous, namely is a rotation. Therefore, $t(\text{IET}(\Lambda)) \cap (\text{IET}(\Lambda) \times 1) = \Lambda$ consists of rotations, so is amenable. We deduce by Corollary 9.6 that $\text{IET}(\Lambda)$ is amenable. 

\section{9.2 Topological full groups}

We apply the results from the previous sections to exhibit a wide variety of amenable groups.

We begin by a fundamental construction. Let $G$ be group acting on a compact set $X$. The associated \textit{topological full group} is the group $[\langle G, X \rangle]$ of piecewise-$G$ homeomorphisms of $X$:

$$[\langle G, X \rangle] = \{ \phi : X \hookrightarrow \exists v : X \to G \text{ continuous with } \phi(x) = vx(x) \text{ for all } x \}.$$

Note that $v$ takes finitely many values since it is a map from a compact set to a discrete set. If we suppose $X$ discrete rather than compact, then $[\langle G, X \rangle]$ becomes the group of bijective $G$-wobbles of $X$ that we saw in \S 5.2. The connection is even more direct: let $x \in X$ be such that its orbit $xG$ is dense in $X$. Then $[\langle G, X \rangle]$ acts faithfully on the orbit $xG$ by $G$-wobbles.

The natural setting for the definition of the topological full group is that of groupoids of germs. We recall the basic notions:

\textbf{Definition 9.17.} A \textit{groupoid} is a set $\mathcal{G}$ with source and range maps $s, r : \mathcal{G} \rightrightarrows \mathcal{G}$, with an associative multiplication $\gamma \gamma_2$ defined whenever $r(\gamma_1) = s(\gamma_2)$, and with an everywhere-defined inverse satisfying $\gamma \gamma^{-1} = s(\gamma) = r(\gamma^{-1})$. Its \textit{set of units} is the subset $\mathcal{G}_0$ of elements of the form $\gamma \gamma^{-1}$. The groupoid $\mathcal{G}$ is called \textit{topological} if $\mathcal{G}$ is a topological space and the multiplication and inverse maps are continuous. Note that for every $x \in \mathcal{G}_0$ the subset $\mathcal{G}_x := \{ \gamma \in \mathcal{G} \mid s(\gamma) = r(\gamma) = x \}$ is a group, called the \textit{isotropy group} of $\mathcal{G}$ at $x$. 


A fundamental example is given by a $G$-set $X$: the associated groupoid is $X \times G$ as a set, with $s(x, g) = x$ and $r(x, g) = xg$ and $(x, g)(xg, h) = (x, gh)$ and $(x, g)^{-1} = (xg, g^{-1})$. One writes this groupoid as $X \rtimes G$ and calls it the action groupoid of $X \rtimes G$.

Another example is given by the groupoid of germs, see [71, Theorem 11]. Let $X \rtimes G$ be an action groupoid, and declare $(x, g) \sim (y, h)$ when $x = y$ and there exists an open neighbourhood of $x$ on which $g$ and $h$ agree. The set of equivalence classes $\mathcal{G}$ is called the groupoid of germs of $X \rtimes G$.

**Definition 9.18.** Let $\mathcal{G}$ be a groupoid of germs, and let $\mathcal{G}_0$ be its space of units. A bisection is a subset $F$ of $\mathcal{G}$ such that $s, r : F \to \mathcal{G}_0$ are homeomorphisms. Note in particular that bisections are open and closed. Bisections may be composed and inverted, qua subsets of $\mathcal{G}$. The full group $[[\mathcal{G}]]$ of a groupoid $\mathcal{G}$ is the group of its bisections.

Note that the topological full group of the groupoid of germs of the action of a group $G$ coincides with the earlier definition of topological full group. It is more convenient to consider the full group of a groupoid of germs, because it is defined only in terms of local homeomorphisms, and not of the global action of a group.

**Theorem 9.19 (see [71] Theorem 11).** Let $X$ be a $G$-topological space, let $\mathcal{G}$ denote the groupoid of germs of $X$, and let $\mathcal{H}$ be a groupoid of germs of homeomorphisms of $X$. Assume that

1. $G$ is finitely generated;
2. At every $x \in X$ the group of germs $\mathcal{G}_x$ is amenable;
3. For every $g \in G$, there are only finitely many $x \in X$ such that $(x, g) \notin \mathcal{H}$, and then for each of these $x$ the action of $G$ on $xG$ is extensively amenable;
4. The topological full group $[[\mathcal{H}]]$ is amenable.

Then $G$ is amenable, and if $X$ is compact then $[[\mathcal{G}]]$ is amenable too.

**Proof.** Let $P$ be the space of “finitely-supported sections of $\mathcal{H} \setminus \mathcal{G}$”: the quotient $\mathcal{H} \setminus \mathcal{G}$ is the set of equivalence classes in $\mathcal{G}$ under $\gamma \sim \delta \gamma$ for all $\gamma \in \mathcal{G}, \delta \in \mathcal{H}$, and

$$P = \{ \phi : X \to \mathcal{H} \setminus \mathcal{G} \text{ finitely supported} \mid s(\phi(x)) \in x\mathcal{H} \text{ for all } x \in X \}.$$  

There is a natural action of $G$ on $P$, by $(\phi g)(x) = \phi(x) \cdot (t(\phi(x)), g)$.

We claim that $P$ is an amenable $G$-set. For this, note first that there are only finitely many $G$-orbits in $X$ at which at element of $P$ can possibly be non-trivial: let $S$ be a finite generating set for $G$; then for every $s \in S$ there is a finite subset $\Sigma_s \subseteq X$ at which $(x, x) \notin \mathcal{H}$, so if $(x, g) \notin \mathcal{H}$ for some $g = s_1 \ldots s_n$ then $(xs_1 \ldots s_{i-1}, s_i) \notin \mathcal{H}$ for some $i$ and therefore $x \in \Sigma_0 G$ for some $i$.

The $G$-set $P$ is naturally the direct product, with diagonal action, of its restrictions to the finitely many $G$-orbits in $X$ at which $P$ can possibly be non-trivial. We therefore restrict ourselves to a single $G$-orbit $Y \subseteq X$, and the corresponding image $P_Y = \{ \phi : Y \to \mathcal{H} \setminus \mathcal{G} \}$ of $P$. 


Let us choose, for every \( y, z \in Y \), an element \( f_{y,z} \in \mathcal{G} \) with \( s(f_{y,z}) = y \) and \( r(f_{y,z}) = z \), taking \( f_{z,y} = f_{y,z}^{-1} \) and \( f_{y,y} = 1 \). Choose also a basepoint \( x \in Y \). We have a “twisted” embedding \( \tau : G \to \mathcal{G}_{x} \), given by \( g \mapsto ((y \mapsto f_{x,y}(y,g), f_{y,x}(y,g)) \). Note that \( P_{Y} \) is isomorphic, qua \( G \)-set, to \( \bigsquare_{x} \mathcal{G}_{x} \) with natural action of \( \tau(G) \).

Now since \( \mathcal{G}_{x} \) is amenable, we have a functor \( \{ \text{finite sets} \} \to \{ \text{amenable groups} \} \) given by \( E \to \mathcal{G}(E) \); since \( X \) and therefore \( Y \) are extensively amenable, Proposition \( 9.5 \) implies that \( \bigsquare_{x} \mathcal{G}_{x} \) is an amenable \( G \)-set, and \textit{a fortiori} so is its quotient \( P \).

We next prove that the stabilizer \( G_{\phi} \) of every \( \phi \in P \) is amenable. Let \( \{ v_{1}, \ldots, v_{n} \} \) be the support of \( \phi \), and set \( K = G_{\phi} \cap G_{v_{1}} \cap \cdots \cap G_{v_{n}} \). We have a natural homomorphism \( K \to \mathcal{G}_{v_{1}} \times \cdots \times \mathcal{G}_{v_{n}} \) to an amenable group, whose kernel is contained in \( \bigsquare_{y} \mathcal{G}_{y} \); so \( K \) is amenable. Iteratively applying Proposition \( 2.26 \) proves that \( G_{\phi} \cap G_{v_{1}} \cap \cdots \cap G_{v_{i}} \) is amenable for all \( i = n, n - 1, \ldots, 0 \).

We apply once more Proposition \( 2.26 \) to deduce that \( G \) is amenable. Finally, the full group \( \bigsquare_{y} \mathcal{G}_{y} \) is the union of groups generated by finite sets of bisections, to which the theorem applies, so \( \bigsquare_{y} \mathcal{G}_{y} \) itself is amenable. \( \square \)

\textit{Example 9.20} (\cite{70}, Theorem 6.1). Consider the “Frankenstein group” \( H(\mathbb{A}) \) from Theorem \( 7.17 \). Then the action of \( H(\mathbb{A}) \) on \( \mathbb{R} \) is hereditarily amenable, but not extensively amenable.

Indeed, consider first \( H \leq H(\mathbb{A}) \) and any \( x \in \mathbb{R} \), and set \( m := \inf(xH) \in \mathbb{R} \cup \{ \infty \} \), as at the end of the proof of Theorem \( 7.17 \). Every element of \( H'' \) acts trivially in a neighbourhood of \( m \). Consider a sequence \( (x_{n}) \) in \( \mathbb{R} \) converging to \( m \); then any cluster point of the sequence of measures \( (\delta_{x_{n}}) \) is an \( H'' \)-invariant mean on \( xH \). Since \( H/H'' \) is amenable, there is also an \( H \)-invariant mean on \( xH \).

On the other hand, since \( H(\mathbb{A}) \) is not amenable there exists a non-amenable finitely generated subgroup \( G \leq H(\mathbb{A}) \), and Theorem \( 9.19 \) should not apply to \( G \) with \( \mathcal{G} \) the groupoid of germs of the action of \( \text{PSL}_{2}(\mathbb{R}) \) on \( \mathbb{R} \cup \{ \infty \} \). However, the first condition is satisfied by assumption, the second one is satisfied because the group of germs at \( x \in \mathbb{R} \) is at most \( \text{Affine}(\mathbb{R}) \times \text{Affine}(\mathbb{R}) \), and the fourth one is satisfied because projective transformations are analytic, so their germs coincide with point stabilizers, namely with \( \text{Affine}(\mathbb{R}) \). Therefore, the third condition fails, so there exists \( x \in \mathbb{R} \) such that the action of \( G \) on \( xG \) is not extensively amenable.

We now specialize the results to \( X \) a Cantor set, and more precisely the Cantor set of paths in a specific kind of graph:

\textbf{Definition 9.21} (\cite{19}; see \cite{35}). A \textit{Bratteli diagram} is a directed graph \( \mathcal{D} = (V, E) \) along with decompositions \( V = \bigcup_{i \geq 0} V_{i} \) and \( E = \bigcup_{i \geq 1} E_{i} \) in non-empty finite subsets, such that \( e^{-1} \in V_{i-1} \) and \( e \subseteq V_{i} \) for all \( e \in E_{i} \). For \( v \in V \) we denote by \( X_{v} \) the set of paths starting at \( V_{0} \) and ending at \( v \); by \( X_{n} = \bigcup_{v \in V_{n}} X_{v} \) the set of paths of length \( n \) starting at \( V_{0} \); and by \( X \) the set of infinite paths starting at \( V_{0} \).

If for any \( n \gg m \) there exists a path from every vertex in \( V_{m} \) to every vertex in \( V_{n} \), the diagram is called \textit{simple}.

For \( e = (e_{1}, \ldots, e_{n}) \in X_{n} \), we denote by \( eX \) the set of paths beginning with \( e \); it is a basic open set for the topology on \( X \), which turns \( X \) into a compact, totally disconnected space. If \( \mathcal{D} \) is simple then \( X \) has no isolated points, so is a Cantor set.
For two paths \( e, f \in X_v \) for some \( v \in V_n \) we define a homeomorphism \( T_{e,f} : eX \to fX \) by
\[
T_{e,f}(e, e_{n+1}, \ldots) = (f, e_{n+1}, \ldots) \quad \text{for all } e_i \in E_i.
\]

Denote by \( \mathfrak{T} \) the groupoid of germs of all homeomorphisms of \( T_{e,f} \). It coincides with the tail equivalence groupoid of \( \mathcal{D} \):
\[
\mathfrak{T} = \{(e, f) \in X \times X \mid e = (e_i)_{i \geq 1}, f = (f_i)_{i \geq 1}, \text{ and } e_i = f_i \text{ for all } i \text{ large enough}\},
\]
with the obvious groupoid structure \( s(e, f) = e, r(e, f) = f, \) and \( (e, f) \cdot (f, g) = (e, g) \). The topology on \( \mathfrak{T} \) has as basic open sets \{germs of \( T_{e,f} \}\}.

Let us describe the topological full group \([\mathfrak{T}]\). Every \( g \in [\mathfrak{T}] \) acts locally like \( T_{e,f} \) for some \( v \in X_n \) and some \( e, f \in X_v \); since \( X \) is compact, there exists a common \( n \in \mathbb{N} \), assumed minimal, for all these local actions. Write \([\mathfrak{T}]_n = \{g \in [\mathfrak{T}] \mid n \leq n\}; then \([\mathfrak{T}]_n \) is a group, and is in fact isomorphic to \( \prod_{v \in V_n} \text{Sym}(X_v) \leq \text{Sym}(X_n) \), since every \( g \in [\mathfrak{T}]_n \) is uniquely determined by the rule \( (e, e_{n+1}, \ldots)^g = (e^g, e_{n+1}, \ldots) \). It follows that \([\mathfrak{T}] = \bigcup_{n \geq 0} [\mathfrak{T}]_n\) is a locally finite group.

**Definition 9.22 ([71]).** Consider a homeomorphism \( a : X \to \mathbb{S} \). For \( v \in V_n \) denote by \( \alpha_a(v) \) the number of paths \( e \in X_v \) such that \( a\mid eX \) does not coincide with a transformation of the form \( T_{e,f} \) for some \( f \in X_v \). The homeomorphism \( a \) is called of bounded type if \( \|a\| = \sup_{v \in V_n} \alpha_a(v) \) is finite and there are only finitely many points \( x \in X \) at which the germ \( (a, x) \) does not belong to \( \mathfrak{T} \).

It is easy to see that the set of bounded-type self-homeomorphisms of \( X \) forms a group. The following result produces a wide variety of amenable groups.

**Theorem 9.23 ([71], Theorem 16]).** Let \( \mathcal{D} \) be a Bratteli diagram, and let \( G \) be a group of homeomorphisms of bounded type of \( X \). If the groupoid of germs of \( G \) has amenable isotropy groups, then \( G \) is amenable.

**Proof.** We may assume without loss of generality that \( G \) is finitely generated. We apply Theorem 9.19 with \( \mathcal{D} = [\mathfrak{T}] \); since \([\mathfrak{T}]\) is locally finite, it is amenable. The only condition to check is that the action of \( G \) on \( X \) is extensively amenable; we prove that it is recurrent and apply Theorem 9.12.

Consider therefore an orbit \( xG \) of \( G \), and a finite generating set \( S \) of \( G \). We will in fact prove that the simple random walk on \( xG \) admits a slow constriction, and apply Theorem 9.10.

The Schreier graph of the orbit \( xG \subset X \) is an \( S \)-labelled graph. In it, remove all edges \( y \to ys \) such that the germ \( (y, s) \) does not belong to \( \mathfrak{T} \). By assumption, only finitely many edges were removed, so the resulting graph has finitely many connected components; let \( P \subseteq xG \) be a choice of one point per connected component. We have covered \( xG \) by finitely many \( \mathfrak{T} \)-orbits. For \( e = (e_i)_{i \geq 1} \in S \) consider
\[
F_{n,e} = \{(a_1, a_2, \ldots, a_n, e_{n+1}, \ldots) \in xG \mid a_1 \in E_1, \ldots, a_n \in E_n\},
\]
and set \( F_n = \bigcup_{e \in P} F_{n,e} \). The \( F_n \) are finite subsets of \( xG \), and \( xG = \bigcup F_n \). For \( e \in P, s \in S \), there are at most \( \alpha_e(e_i^s) \) paths \( f \in F_{n,e} \) with \( fs \not\in F_{n,e} \); so \( #(F_n S \setminus F_n) \leq \)
Definition 9.24 ([35] Definition 6.3.2]). A Bratteli-Vershik diagram is a Bratteli diagram \( \mathcal{D} = (V, E) \) together with a partial order \( \leq \) on \( E \) such that \( e, f \) are comparable if and only if \( e^+ = f^+ \). For every \( v \in V \) there is an induced linear order on \( X_v \): if \( e = (e_1, \ldots, e_n), f = (f_1, \ldots, f_n) \in X_v \) then \( e \leq f \) if and only if \( e_i \leq f_i, e_{i+1} = f_{i+1}, \ldots, e_n = f_n \) for some \( i \in \{1, \ldots, n\} \). We let \( X^{\max} \) denote those \( e = (e_1, \ldots) \in X \) such that \( (e_1, \ldots, e_n) \) is maximal for all \( n \in \mathbb{N} \), define \( X^{\min} \) similarly, and say \( \mathcal{D} \) is properly ordered if \( #X^{\max} = #X^{\min} = 1 \).

The adic transformation of a properly-ordered Bratteli-Vershik diagram \( (\mathcal{D}, \leq) \) is the self-homeomorphism \( a : X \rightrightarrows \) defined as follows. If \( e = (e_1, \ldots) \in X \) is such that \( (e_1, \ldots, e_n) \) is not maximal in \( X^{\max}_+ \) for some \( n \in \mathbb{N} \), then \( e^a := (f_1, \ldots, f_n, e_{n+1}, \ldots) \). Otherwise, \( e \) is the unique maximal path in \( X \), and \( e^a \) is defined to be the unique minimal path in \( X \).

If \( \mathcal{D} \) is simple, then \( a \) is a minimal transformation of \( X \). Bratteli-Vershik diagrams encode all minimal homeomorphisms of Cantor sets:

Theorem 9.25 ([64], see [35] Theorem 6.4.6]). Every minimal homeomorphism of the Cantor set is topologically conjugate to the adic transformation of a properly ordered simple Bratteli-Vershik diagram. \( \square \)

(The idea of the proof is to choose a decreasing sequence \( (C_n)_{n \geq 0} \) of clopen sets, shrinking down to a base point \( \{x\} \), and to consider the associated “Kakutani-Rokhlin tower”: the largest collection of iterated images of \( C_n \) under the homeomorphism that are disjoint. These translates of \( C_n \) make up the \( n \)th level of the Bratteli-Vershik diagram.)

Corollary 9.26 ([69]). Let \( a \) be a minimal homeomorphism of a Cantor set \( X \). Then the topological full group \( \mathbb{F}(a, X) \) is amenable.

Proof. Using Theorem 9.25 we may assume \( a \) is the adic transformation of a Bratteli-Vershik diagram. It follows directly that \( \alpha_a(v) = 1 \) for every \( v \in V \), and that the germs of \( a \) belong to \( \mathcal{T} \) for all points \( x \in X \setminus X^{\max} \). No power of \( a \) has fixed points so their germs are all trivial. \( \square \)

Here are some typical examples of minimal \( \mathbb{Z} \)-actions on a Cantor set, to which Corollary 9.26 applies to produce amenable groups:

Example 9.27. Consider an irrational \( \alpha \in (0, 1) \), and the transformation \( x \mapsto x + \alpha \) on \( \mathbb{R}/\mathbb{Z} \). It is minimal, since \( \mathbb{Z} + \mathbb{Z} \alpha \) is dense in \( \mathbb{R} \). We can replace \( \mathbb{R}/\mathbb{Z} \) by a Cantor set as follows: set

\[
X_\alpha := (\mathbb{R} \setminus \mathbb{Z} \alpha / \{(\mathbb{Z} \alpha \times \{+,-\})/\mathbb{Z}\}),
\]

namely replace every point \( x \in \mathbb{Z} \alpha \subset \mathbb{R} / \mathbb{Z} \) by a pair \( x^\pm \). Give \( X_\alpha \) the cyclic order induced from the circle and \( x^- < x^+ \), and its associated topology. Then \( X_\alpha \) is a Cantor set, and \( x \mapsto x + \alpha \) is a minimal transformation of \( X_\alpha \); see Example 9.15.
As another example, consider the substitution \( a \mapsto ab, b \mapsto a \) on \( \{a,b\}^\mathbb{Z} \) and let \( x \in \{a,b\}^\mathbb{Z} \) denote a fixed point of the substitution; for example, with ‘a’ denoting the position of the 0th letter, \( x = \lim(ab,abaab,abaababa,\ldots) \). Set \( X = \overline{x\mathbb{Z}} \). Then the action of \( \mathbb{Z} \) by shift on \( X \) is minimal.

In fact, this example coincides with the first one if one takes \( \alpha = (\sqrt{5} - 1)/2 \) the golden ratio and \( x = 0^+ \), decomposes \( X_\alpha = [0^+, \alpha^-] \cup [\alpha^+, 1^-] \), defines \( \pi : X_\alpha \to \{a,b\} \) by \( \pi(x) = a \) if \( x \in [0^+, \alpha^-] \) and \( \pi(x) = b \) if \( x \in [\alpha^+, 1^-] \), and puts \( X_\alpha \) in bijection with \( X \) via the map \( x \mapsto (n \mapsto \pi(x+n)) \).

The encoding of this example as a Bratteli diagram \( \mathcal{D} \) is as follows:

\[
\begin{array}{c}
\vdots \\
\bullet \\
\downarrow \\
\bullet \\
\downarrow \\
\bullet \\
\downarrow \\
\bullet \\
\vdots \\
\end{array}
\]

\( a \) \quad \bullet \quad \ldots \quad \bullet \quad \ldots \quad \bullet \quad \ldots \\
\downarrow \quad b \quad a \quad \downarrow \quad b \quad a \\
\bullet \quad \bullet \quad \bullet \\
\downarrow \quad \downarrow \\
\bullet \quad \bullet \\
\downarrow \quad \downarrow \\
V_0 \quad V_1 \quad V_2 \quad V_3
\]

where now a point \( x \in X_\alpha \) is encoded by the path in \( \mathcal{D} \) with labels \( (\pi(\hat{s}/\alpha^n))_{n \geq 1} \) for the unique representative \( \hat{s} \) of \( x \) in \([0,1]\).

We next quote some results from [102] to exhibit some properties of the topological full groups \([[G,X]]\) constructed above.

**Definition 9.28.** Let \( \mathcal{G} \) be a groupoid. A multisection of degree \( d \) is a collection \( M \) of \( d^2 \) non-empty, disjoint bisections \( \{F_{i,j}\}_{i,j=1,\ldots,d} \) of \( \mathcal{G} \) such that \( F_{i,j} \subset \mathcal{G}_0 \) and \( F_{i,j}F_{i,k} = F_{j,k} \) for all \( i, j, k \in \{1,\ldots,d\} \).

For \( \pi \in \text{Sym}(d) \), we denote by \( M_\pi \) the element of \([[\mathcal{G}]]\) that maps \( x \) to \( xF_{i,\pi} \) if \( x \in F_{i,\pi} \) and fixes \( \mathcal{G}_0 \setminus \bigcup_{i=1}^d F_{i,j} \), and by \( \text{Alt}(M) \) the subgroup \( \{M_\pi \mid \pi \in \text{Alt}(d)\} \) of \([[\mathcal{G}]]\). Finally, we denote by \( \text{Alt}(\mathcal{G}) \) the subgroup of \([[\mathcal{G}]]\) generated by \( \text{Alt}(M) \) for all multisections \( M \) of \( \mathcal{G} \).

**Proposition 9.29 ([102], Theorem 4.1).** Let \( \mathcal{G} \) be a minimal groupoid of germs. Then every non-trivial subgroup of \([[\mathcal{G}]]\) normalized by \( \text{Alt}(\mathcal{G}) \) contains \( \text{Alt}(\mathcal{G}) \). In particular, \( \text{Alt}(\mathcal{G}) \) is simple and is contained in every non-trivial normal subgroup of \([[\mathcal{G}]]\).

(Note that the minimality assumption is always necessary: if \( \mathcal{G} \) does not act minimally, then let \( Y \neq \mathcal{G}_0 \) be a closure of an orbit; then there is a natural quotient map \([[\mathcal{G}]] \to [[\mathcal{G}]/Y]]\), proving that \([[\mathcal{G}]]\) is not simple.)

We call a groupoid \( \mathcal{G} \) compactly generated if there exists a compact subset \( S \) of \( \mathcal{G} \) that generates it. This is for example the case if \( \mathcal{G} \) is the action groupoid of a finitely generated group \( G \) acting on a compact set (in which case one bisection per generator of \( G \) suffices to generate \( \mathcal{G} \)).
Let $\mathcal{G}$ be a compactly generated groupoid, say by $S \subseteq \mathcal{G}$. We call $\mathcal{G}$ expansive if there exists a finite cover $\mathcal{S}$ of $\mathcal{G}$ by bisections such that $\bigcup_{n \geq 0} \mathcal{S}^n$ generates the topology on $\mathcal{G}$; so in particular for every $x \neq y \in \mathcal{G}_0$ there exists a bisection $F \in \mathcal{S}^n$ with $x \in s(F) \neq y$.

**Proposition 9.30** ([122, Theorem 5.6]). If $\mathcal{G}$ is compactly generated and expansive then $\text{Alt}(\mathcal{G})$ is finitely generated. \[\square\]

**Example 9.31.** Consider the $\mathbb{Z}$-action from Example [9.27] for $\alpha$ the golden ratio. We claim that the group $G = [[\mathbb{Z},X]]'$ is infinite, amenable, finitely generated and simple.

Amenability of $G$ was proven in Corollary 9.26. Let $\mathcal{G}$ be the groupoid of the action of $\mathbb{Z} = \{a\}$ on $X$. It is minimal, so $\text{Alt}(\mathcal{G})$ is simple by Proposition 9.29 and it is easy to check $\text{Alt}(\mathcal{G}) = [[\mathcal{G}]]'$. The groupoid $\mathcal{G}$ is compactly generated, say by $S = X \cup \{(x,xa^{\pm 1}) \mid x \in X\}$. Finally $X \subset \{0,1\}^\mathbb{Z}$ is a subshift, so $\mathcal{G}$ is expansive: the cover of $S$ by $X$ by $\{\{x \in X \mid x_0 = 0\} \cup \{x \in X \mid x_0 = 1\} \cup \{(x,xa) \mid x \in X\} \cup \{(x,xa^{-1}) \mid x \in X\}\}$ generates the topology on $X$ and therefore on $\mathcal{G}$.

Finally, we end with examples of topological full groups of non-minimal $\mathbb{Z}$-actions and of minimal $\mathbb{Z}^2$-actions which are not amenable, showing that Corollary 9.26 does not generalize without extra conditions:

**Example 9.32 (Geodesic flow).** Consider a free group $F_k$, and the space $X$ of geodesic maps $a : \mathbb{Z} \to F_k$ into the Cayley graph of $F_k$, namely of bi-infinite geodesic rays. The $\mathbb{Z}$-action is by shifting: $\sigma(a) = (i \mapsto a_{i+1})$. The space $X$ is a Cantor set, and may be identified with $\{a \in \{x_1^\pm,\ldots,x_k^\pm\}^\mathbb{Z} \mid a_{i+1} a_i^{-1} \neq 1 \text{ for all } i \in \mathbb{Z}\}$. For $a \in X$ and $j \in \{1,\ldots,k\}$, define

$$a \cdot x_j = \begin{cases} \sigma(a) & \text{if } a_0 = x_j, \\ \sigma^{-1}(a) & \text{if } a_{-1} = x_j^{-1}, \\ a & \text{otherwise.} \end{cases}$$

This defines a piecewise-$\mathbb{Z}$ action of $F_k$ on $X$, which is easily seen to be faithful: for $w \in F_k$ a non-trivial reduced word, extend $w$ arbitrarily but non-periodically to a bi-infinite geodesic $a$ containing $w$ at positions $\{0,\ldots,|w|−1\}$; then $a \cdot w = \sigma^{|w|}(a) \neq a$.

We may modify the example above by letting $C_2*C_2*C_2$ rather than $F_k$ act on the space of geodesics of its Cayley graph, and then embed that system into a minimal $\mathbb{Z}^2$-action, as follows:

**Example 9.33 ([36]).** Consider the space $X$ of proper colourings of the edges of the standard two-dimensional grid by $\mathcal{A} = \{A,B,C,D,E,F\}$. There is a natural action of $\mathbb{Z}^2$ on $X$ by translations.

To each $a \in \mathcal{A}$ corresponds a continuous involution $a : X \to X$, defined as follows. For $\sigma \in X$, if there is an edge between $(0,0)$ and one of its neighbours $v$ with colour $a$, then $\sigma \cdot a := \sigma \cdot v$; otherwise $\sigma \cdot a := \sigma$. These involutions clearly belong to $[[\mathbb{Z}^2,X]]$. 

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L. Bartholdi
We shall exhibit a minimal non-empty closed $\mathbb{Z}^2$-invariant subset $Y$ of $X$ on which $\mathbb{Z}^2$ acts freely and $H := \langle A, B, C \mid A^2, B^2, C^2 \rangle$ acts faithfully as subgroup of $[[\mathbb{Z}^2, X]]$; since $H$ contains free subgroups, we will have proved that $[[\mathbb{Z}^2, Y]]$ may contain free subgroups (and therefore be non-amenable) for minimal, free $\mathbb{Z}^2$-spaces $Y$.

We create a specific colouring of the grid, namely an element $\sigma \in X$, as follows: first, colour every horizontal line of the grid alternately with $E$ and $F$. Enumerate $H = \{w_0, w_1, \ldots \}$. For all $x \in \mathbb{N}$, write $x = 2^i x'$ with $x'$ odd, and colour the vertical lines $\{x\} \times \mathbb{R}$ and $\{-x\} \times \mathbb{R}$ by the infinite word $(w_i D)^\infty$. Set $Y = \sigma \mathbb{Z}^2$.

Every finite patch of $\sigma \downharpoonright S$ repeats infinitely, and moreover there exists $n(S)$ such that every ball of radius $n(S)$ in the grid contains a copy of $\sigma \downharpoonright S$. It follows (see [47]) that $Y$ is minimal, that $\mathbb{Z}^2$ acts freely on $Y$ because $\sigma$ is aperiodic, and that every $\tau \in Y$ also uniformly contains copies of every patch.

Consider now $w \neq 1 \in \langle A, B, C \rangle$, and let $\tau$ be a translate of $\sigma$ in which $wD$ reads vertically at the origin. Then $\tau w$ reads $D$ vertically at the origin, so $\tau w \neq \tau$, and therefore $w$ acts non-trivially.
10 Cellular automata and amenable algebras

Von Neumann defined cellular automata as creatures built out of infinitely many finite-state devices arranged on the nodes of $\mathbb{Z}^2$ or $\mathbb{Z}^3$, each device being capable of interaction with its immediate neighbours. Algebraically, we consider the natural generalization to creatures living on the vertices of a Cayley graph. We shall see that some fundamental properties of the automaton are characterized by amenability of the underlying graph.

**Definition 10.1.** Let $G$ be a group. A finite cellular automaton on $G$ is a $G$-equivariant continuous map $\Theta : \mathcal{A}^G \rightarrow \mathcal{A}$, where $\mathcal{A}$, the state set, is a finite set, and $G$ acts on $\mathcal{A}^G$ by left-translation: $(xg)(h) = x(gh)$ for $x \in \mathcal{A}^G$ and $g, h \in G$. Elements of $\mathcal{A}^G$ are called configurations.

A linear cellular automaton is defined similarly, except that $\mathcal{A}$ is rather required to be a finite-dimensional vector space, and $\Theta$ is required to be linear.

Note that usually $G$ is infinite; much of the theory holds trivially if $G$ is finite. The map $\Theta$ computes the 1-step evolution of the automaton; its continuity implies that the evolution of a site depends only on a finite neighbourhood, and its $G$-equivariance implies that all sites evolve with the same rule.

**Lemma 10.2 (Lyndon-Curtis-Hedlund).** A map $\Theta : \mathcal{A}^G \rightarrow \mathcal{A}$ is a cellular automaton if and only if there exists a finite subset $S \subset G$ and a map $\theta : \mathcal{A}^S \rightarrow \mathcal{A}$ such that

$$\Theta(x)(g) = \theta(x \mapsto x(gs))$$

for all $x \in \mathcal{A}^G$. The minimal such $S$ is called the memory set of $\Theta$.

**Proof.** Such a map $\Theta$ is continuous in the product topology if and only if $\Theta(x)(1)$ depends only on the restriction of $x$ to $S$ for some finite $S$. \qed

A classical example of cellular automaton is Conway’s Game of Life. It is defined by $G = \mathbb{Z}^2$ and $\mathcal{A} = \{\text{alive, dead}\}$, and by the following local rule $\theta$ as in Lemma 10.2: $S = \{-1, 0, 1\} \times \{-1, 0, 1\}$, and $\theta(x)$ depends only on $x(0, 0)$ and on the number of alive cells among its eight neighbours:

$$\theta(x)(0, 0) = \begin{cases} \text{alive} & \text{if } x(0, 0) \text{ is alive and two or three of its neighbours are alive,} \\ \text{alive} & \text{if } x(0, 0) \text{ is dead and exactly three of its neighbours are alive,} \\ \text{dead} & \text{in all other cases, from loneliness or overpopulation.} \end{cases}$$

For example, here is the evolution of a piece of the plane; we represent alive in black and dead in white:

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18 It seems that von Neumann never published his work on cellular automata — see [21] for history of the subject.
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Note that the last configuration is the first one, transformed by $(x, y) \mapsto (1 - y, -x)$, so the pattern moves by a sliding reflection along the $x + y = 0$ direction.

Some properties have been singled out in attempts to understand the global, long-term behaviour of cellular automata: a cellular automaton $\Theta$ may have “Gardens of Eden” (GOE) if the map $\Theta$ is not surjective, the biblical metaphor expressing the notion of paradise lost forever. Note that $\Theta(A^G)$ is compact, hence closed in $A^G$, so if $\Theta$ is not surjective then there exists a finite subset $F \subseteq G$ such that the projection of $\Theta(A^G)$ to $A^F$ is not onto; have “Mutually Erasable Patterns” (MEP) if $\Theta$ fails in a strong way to be injective: there are configurations $x \neq y$ which nevertheless agree at all but finitely many places, and such that $\Theta(x) = \Theta(y)$. The opposite is sometimes called pre-injectivity; preserve the Bernoulli measure; open sets of the form $O_{g,q} = \{x \in A^G \mid x(g) = q\}$ are declared to have measure $\beta(O_{g,q}) = 1/\#A^G$, and one may ask whether $\beta(M) = \beta(\Theta^{-1}(M))$ for every measurable $M \subseteq A^G$.

For example, it is clear that the Game of Life has Mutually Erasable Patterns, because of the “loneliness” clause:

but it is less clear that there are also Gardens of Eden (there are some; the smallest known one is specified by $\#F = 92$ cells).

Before addressing the question of relating the GOE and MEP properties, we introduce one more tool: entropy. Assume that the group $G$ is amenable, and let $(F_n)$ be a Følner net in $G$, which exists by Lemma 3.6 and Theorem 3.23. For subsets $X \subseteq A^G$ and $S \subseteq G$, we let $X|S$ denote the projection of $X$ to $A^S$. We set

$$h(X) = \liminf_n \frac{\log(\#X|F_n)}{\#F_n}. \quad (18)$$

If $X$ is $G$-invariant, then the liminf in (18) is a limit and is independent of the choice of Følner net. This follows from the following more general statement (in-
dependence of the Følner net follows from interleaving two Følner nets), which we quote without proof:

**Lemma 10.3 (Ornstein-Weiss, see [55] §1.3.1 and [84]).** Let \( h : \mathcal{P}_f(G) \to \mathbb{R} \) be subadditive: \( h(A \cup B) \le h(A) + h(B) \), and \( G \)-invariant: \( h(Ag) = h(A) \). Then the limit \( \lim_{n \to \infty} h(F_n)/\#F_n \) exists for every Følner net \((F_n)_{n \in \mathbb{N}}\).

The following is the “Second Principle of Thermodynamics”:

**Lemma 10.4.** For every cellular automaton \( \Theta \) and every \( G \)-invariant \( X \subseteq \mathcal{A}^G \), we have \( h(\Theta(X)) \le h(X) \).

**Proof.** Let \( S \subseteq G \) be a memory set for \( G \). For every finite \( F \subseteq G \), consider \( F \), such that \( ES \subseteq F \); then \( \Theta(x)|E \) depends only on \( x|F \). Therefore, \( \sum(\Theta(x)|E) \le \#(X|F) \), so \( \#(\Theta(X)|F) \le \#(X|F)\#\mathcal{A}^{#F-E} \). Take now \( F = F_nS \) and \( E = F_n \) for a net of Følner sets, and apply the definition from (18). \[ \Box \]

Finally, given a measure \( \nu \) on \( \mathcal{A}^G \), we may define a measured entropy as follows: for \( S \subseteq G \) and \( y \in \mathcal{A}^S \), denote by \( O(y) \) the open set \( \{ x \in \mathcal{A}^G | x|S = y|S \} \); and for \( X \subseteq \mathcal{A}^G \) set

\[
h_\nu(X) = \liminf_{n \to \infty} \frac{-\sum_{y \in X \cap F_n} \nu(O(y)) \log \nu(O(y))}{\#F_n}.
\]

Note that \( \beta(O(y)) = 1/\#\mathcal{A}^S \) if \( y \in \mathcal{A}^S \), so the measured entropy coincides with (18) if \( \nu = \beta \).

We are ready to state the main result, called the “Gardens of Eden theorem”. It was first proven for \( G = \mathbb{Z}^d \) by Moore [96, the \( (1) \Rightarrow (2) \) direction], Myhill [98, the \( (2) \Rightarrow (1) \) direction], and Hedlund [63, the \( (1) \Leftrightarrow (3) \) equivalence]:

**Theorem 10.5 ([25, 91]).** Let \( G \) be an amenable group, and let \( \Theta \) be a cellular automaton. Then the following are equivalent:

1. \( \Theta \) has Gardens of Eden;
2. \( \Theta \) has Mutually Erasable Patterns;
3. \( \Theta \) does not preserve Bernoulli measure \( \beta \);
4. \( h(\Theta(\mathcal{A}^G)) < \log \#\mathcal{A} \).

**Remark 10.6.** The same theorem holds for linear cellular automata (except that I do not know an analogue of Bernoulli measure), with the entropy replaced in the last statement by mean dimension:

\[
\text{mdim}(X) = \liminf_n \frac{\dim(#X \cap F_n)}{\#F_n}.
\]

**Proof.** Throughout the proof, we let \( S \) denote the memory set of \( \Theta \).

(1) \( \Rightarrow \) (4) If there exists a GOE, then there exists \( F \subseteq G \) with \( \Theta(\mathcal{A}^G) \cap F \neq \emptyset \), so

\[
h(\Theta(\mathcal{A}^F)) \le \log \#\mathcal{A}^{#F} < \log \#\mathcal{A}.
\]
(4) \implies (1) If \( h(\Theta(\mathcal{A}^G)) < \log \# \mathcal{A} \), then there exists \( F \subseteq G \) with \( \Theta(\mathcal{A}^G)|F \neq \mathcal{A}^F \), and a GOE exists in \( \mathcal{A}^F \setminus \Theta(\mathcal{A}^G)|F \).

(2) \implies (4) If \( y \neq z \) are MEP, which differ on \( F \) and agree elsewhere, set \( E = FS \) and let \( T \subseteq G \) be maximal such that \( Et_1 \cap Et_2 = \emptyset \) for all \( t_1 \neq t_2 \in T \); note that \( T \) intersects every translate of \( E^{-1}E \). Define

\[
Z = \{ x \in \mathcal{A}^G \mid x|Et \neq y|Et \text{ for all } t \in T \},
\]

and compute \( h(\Theta(\mathcal{A}^G)) = h(\Theta(Z)) \leq h(Z) < \log \# \mathcal{A} \); the first equality follows since given in \( x \in \Theta(\mathcal{A}^G) \), say \( x = \Theta(w) \), one may replace in \( w \) every occurrence of \( y|Et \) by \( z|Et \) so as to obtain a \( y|Et \)-free configuration, which therefore belongs to \( Z \), and has the same image as \( x \) under \( \Theta \); the second inequality follows from Lemma 10.4 and the last inequality because there are forbidden patterns \( y|Et \) in \( Z \), with “density” at least \( 1/\#(E^{-1}E) \).

(4) \implies (2) If \( h(\Theta(\mathcal{A}^G)) < \log \# \mathcal{A} \), there exists \( F_n \) with \( \log \#(\Theta(\mathcal{A}^G)|F_n) < \log \# A \), because \( \#F_n \) may be made arbitrarily close to \( \#A \) for \( n \) large enough. Therefore, by the pigeonhole principle, there exist \( y \neq z \in \mathcal{A}^G \) with \( y|G \setminus F_n = z|G \setminus F_n \) and \( \Theta(y) = \Theta(z) \).

(1) \implies (3) This is always true: if \( \Theta \) has GOE, then there exists a non-empty open set \( \mathcal{U} \) in \( \mathcal{A}^G \setminus \Theta(\mathcal{A}^G) \); then \( \beta(\mathcal{U}) \neq 0 \) while \( \beta(\Theta^{-1}(\mathcal{U})) = 0 \).

(3) \implies (1) Define

\[
K = \{ \nu \text{ probability measure on } \mathcal{A}^G \mid \beta = \Theta_\nu \}.
\]

Note that \( K \) is convex and compact, no admits a \( G \)-fixed point because \( G \) is amenable. Consider \( \nu \in K \). Then \( \phi: (\mathcal{A}^G, \nu) \to (\mathcal{A}^G, \beta) \) is a factor map because \( \Theta \) is onto, so \( h_\nu(\mathcal{A}^G) \geq h_\beta(\mathcal{A}^G) \). However, \( \beta \) is the unique measure of maximal entropy, so \( \nu = \beta \) and therefore \( \beta = \Theta_\nu \).

It turns out that Theorem 10.5 is essentially optimal, and yields characterizations of amenable groups:

**Theorem 10.7 (9,11).** Let \( G \) be a non-amenable group. Then there exist

1. cellular automata (ad lib linear) that admit Mutually Erasable Patterns but no Gardens of Eden;
2. cellular automata (ad lib linear) that admit Gardens of Eden but no Mutually Erasable Patterns;
3. cellular automata that do not preserve Bernoulli measure but have no Gardens of Eden.

In fact, we shall prove Theorem 10.7 for finite fields, answering at the same time the classical and linear questions. Let \( \Theta \) be a linear cellular automaton; then \( \mathcal{A} = k^n \) for some field \( k \) and some integer \( n \), and there exists an \( n \times n \) matrix \( M \) over \( kG \) such that \( \Theta(x) = xM \) for all \( x \in \mathcal{A}^G \). Conversely, every such matrix defines a linear cellular automaton.

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19 One says that the \( G \)-action is intrinsically ergodic.
The ring \( kG \) admits an anti-involution \( * \), defined on its basis \( G \) by \( g^* = g^{-1} \) and extended by linearity. This involution extends to an anti-involution on square matrices by \( (M^*)_{i,j} = (M_{j,i})^* \), and \( M^* \) is called the adjoint of \( M \).

We put on \( \mathcal{A} \) the natural scalar product \( \langle x,y \rangle = \sum_{g \in G} x(g)y(g) \). Consider the vector space \( \mathcal{A} \) of \( \sum_{g \in G} x(g)y(g) \) for \( x \in \mathcal{A}G \) and \( y \in \mathcal{A}^G \).

**Exercise 10.8 (*)**. Prove that \( M^* \) is the adjoint with respect to this pairing; namely

\[ \langle xM, y \rangle = \langle x, yM^* \rangle \quad \text{for all} \quad x \in \mathcal{A}G, y \in \mathcal{A}^G. \]

We put a topology on \( \mathcal{A}^G \) by declaring that, for every finite \( S \subseteq G \) and every vector space \( V \leq \mathcal{A}^S \), the subset \( \{ x \in \mathcal{A}^G \mid x|S \in V \} \) is closed. With this topology, \( \mathcal{A}^G \) is compact (but not Hausdorff). Nevertheless,

**Lemma 10.9.** If \( \Theta \) is a cellular automaton then \( \Theta(\mathcal{A}^G) \) is closed.

**Proof.** Let \( S \) be the memory of \( \Theta \). Consider \( y \) in the closure of \( \Theta(\mathcal{A}^G) \). Then for every \( F \subseteq G \) the affine space \( L_F = \{ x \in \mathcal{A}^F \mid \Theta(x)|F = y|F \} \) is finite-dimensional and non-empty, and if \( F \subseteq F' \) then \( L_F|FS \subseteq L_{F'} \); so \( \{ L_F|FS \mid F \subseteq F' \} \) is a nested sequence of non-empty affine spaces, and in particular stabilizes at a non-empty affine space \( J_F \). We still have restriction maps \( J_{F'} \to J_F \) for all \( F \subseteq F' \), which are easily seen to be surjective. Then \( \lim_{F \subseteq F' \subseteq G} J_F \) is non-empty and contains all preimages of \( y \).

The following proposition extends to the infinite-dimensional setting the classical statement that the image of a matrix is the orthogonal of the nullspace of its transpose:

**Proposition 10.10 ([123]).** Let \( M \) be an \( n \times n \) matrix over \( kG \), let \( M^* \) be its adjoint, and set \( \mathcal{A} = k^n \). Then

\[ \ker(M) \cap \mathcal{A}G = \operatorname{image}(M^*)^\perp = \{ x \in \mathcal{A}G \mid \langle x, \mathcal{A}^G M^* \rangle = 0 \}. \]

Equivalently, right-multiplication by \( M \) is injective on \( \mathcal{A}G \) if and only if right-multiplication by \( M^* \) is surjective on \( \mathcal{A}^G \).

**Proof.** Assume first that right-multiplication by \( M \) is not injective, and consider a non-trivial element \( c \in \mathcal{A}G \) with \( cM = 0 \). We claim that for every \( y \in (\mathcal{A}^G)M^* \) we have \( \langle c, y \rangle = 0 \). Say \( y = zM^* \); then the claim follows from the computation

\[ \langle c, y \rangle = \langle c, zM^* \rangle = \langle cM, z \rangle = \langle 0, z \rangle = 0. \]

Since \( \langle -,- \rangle \) is non-degenerate, this implies that \( y \) cannot range over all of \( \mathcal{A}^G \), so right-multiplication by \( M^* \) is not surjective.

Conversely, suppose that right-multiplication by \( M \) is not surjective. Since \( \mathcal{A}^G M \) is closed, there exists an open set in its complement; so there exists a finite subset \( S \subseteq G \) and a proper subspace \( V \leq \mathcal{A}S \) such that, for every \( c \in \mathcal{A}^G M \), its projection \( c|S \) belongs to \( V \). Since \( \mathcal{A}^S \) is finite-dimensional, there exists a linear form \( y \) on
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that vanishes on $V$. Note that $y$, qua element of $(\mathcal{A}^S)^*$, is canonically identified with an element of $\mathcal{A}^S$, and therefore with an element of $\mathcal{A}^G$. We claim $yM^* = 0$, proving that right-multiplication by $M^*$ is not injective. This follows from the following computation: consider an arbitrary $c \in \mathcal{A}^G$. Then

$$\langle yM^*, c \rangle = \langle y, cM \rangle = 0.$$ 

Since $\langle -, - \rangle$ is non-degenerate and $c \in V^G$ is arbitrary, this forces $yM^* = 0$. \hfill $\square$

Before embarking in the main step of the proof of Theorem 10.7, we give a simple example of a cellular automaton that is pre-injective but not surjective:

**Example 10.11 (Muller, see [88, page 55]).** Consider the free product of cyclic groups $G = \langle a, b, c | a^2, b^2, c^2 \rangle$. Fix a field $k$, and set $A := k^2$. Define the linear cellular automaton $\Theta : \mathcal{A}^G \rightarrow \mathcal{A}^G$ by $\Theta(x) = x \cdot \begin{pmatrix} a + b & 0 \\ b & c \end{pmatrix}$.

It is obvious that $\Theta$ is not surjective: its image is $(k \times 0)^G$. To show that it is pre-injective, consider $x$ a non-zero configuration with finite support, and let $F \subseteq G$ denote its support. Let $f \in F$ be an element of maximal length; then at least two among $fa, fb, fc$ will be reached precisely once as products of the form $F \cdot \{a, b, c\}$. Write $x(f) = (\alpha, \beta) \neq (0, 0)$; then at least two among the equations

$$\Theta(x)(fa) = \alpha, \quad \Theta(x)(fb) = \alpha + \beta, \quad \Theta(x)(fc) = \beta$$

hold, and this is enough to force $\Theta(x) \neq 0$.

In the general case of a non-amenable group $G = \langle S \rangle$, we may not claim that there exist two elements reached exactly once from an arbitrary finite set $F$ under right $S$-multiplication; but we shall see that there exists “many” elements reached “not too many” times, in the sense that there exists $f \in F$ with $\sum_{s \in S} 1/\#(\{t \in S \mid fs \in Ft\}) > 1$; and this will suffice to construct a pre-injective, non-surjective cellular automaton. We begin by a combinatorial lemma:

**Lemma 10.12.** Let $n$ be an integer. Then there exists a set $Y$ and a family of subsets $X_1, \ldots, X_n$ of $Y$ such that, for all $I \subseteq \{1, \ldots, n\}$ and all $i \in I$, we have

$$\#(X_i \setminus \bigcup_{j \in F \setminus \{i\}} X_j) \geq \frac{\#Y}{(1 + \log n)\#I}. \quad (19)$$

Furthermore, if $n \geq 2$ then we may require $X_1 \cup \cdots \cup X_n \neq Y$.

**Proof.** We denote by $\text{Sym}(n)$ the symmetric group on $n$ letters. Define

$$Y := \{(1, \ldots, n) \times \text{Sym}(n) \mid (i, \sigma) \sim (j, \sigma) \text{ if } i \text{ and } j \text{ belong to the same cycle of } \sigma\};$$
in other words, $Y$ is the set of cycles of elements of $\text{Sym}(n)$. Let $X_i$ be the natural image of \{$i\}$ \times $\text{Sym}(n)$ in the quotient $Y$.

First, there are $(i-1)!$ cycles of length $i$ in $\text{Sym}(i)$, given by all cyclic orderings of $\{1, \ldots, i\}$; so there are \binom{n}{i}(i-1)! cycles of length $i$ in $\text{Sym}(n)$, and they can be completed in $(n-i)!$ ways to a permutation of $\text{Sym}(n)$; so

$$#Y = \sum_{i=1}^{n} \binom{n}{i} (i-1)! (n-i)! = \sum_{i=1}^{n} \frac{n!}{i!} \leq (1 + \log n)n!$$  \hspace{1cm} (20)

since $1 + 1/2 + \cdots + 1/n \leq 1 + \log n$ for all $n$.

Next, consider $I \subseteq \{1, \ldots, n\}$ and $i \in I$, and set $X_{i,I} := X_i \setminus \bigcup_{j \in \mathbb{P}_{i}(I)} X_{j}$. Then $X_{i,I} = \{(i, \sigma) : (i, \sigma) \sim (j, \sigma) \text{ for all } j \in I \setminus \{i\}\}$. Summing over all possibilities for the length-$(j+1)$ cycle $(i, t_1, \ldots, t_j)$ of $\sigma$ intersecting $I$ in $\{i\}$, we get

$$#X_{i,I} = \sum_{j=0}^{n} \binom{n-\#I}{j} j!(n-j-1)!$$

$$= \sum_{k=0}^{n} (n-\#I)! ((I-1)! \binom{k-1}{k-\#I})$$

$$= (n-\#I)! ((I-1)! \binom{n}{n-\#I} = \frac{n!}{\#I}$$  \hspace{1cm} (21)

Combining (20) and (21), we get

$$#X_{i,I} = \frac{n!}{\#I} \leq \frac{(1 + \log n)n!}{(1 + \log n)!} \leq #Y \leq \frac{#Y}{(1 + \log n)!}.$$

Finally, if $n \geq 2$ then (19) may be improved to $#Y \leq (0.9 + \log n)n!$; for even larger $n$ one could get to $#Y \leq (0.57721 \cdots + \log n)n!$. Since clearly $#Y/(0.9 + \log n) \geq (#Y + 1)/((1 + \log n)$, one may simply replace $Y$ by $Y \cup \{\cdot\}$. \hfill \Box

**Proposition 10.13.** Let $k$ be a field, and let $G$ be a non-amenable group. Then there exists a finite extension $K$ of $k$ and an $n \times (n-1)$ matrix $M$ over $\mathbb{K}G$ such that multiplication by $M$ is an injective map $(\mathbb{K}G)^n \to (\mathbb{K}G)^{n-1}$.

**Proof.** Since $G$ is non-amenable, there exists by Theorem 3.23 a finite subset $S_0 \subset G$ and $\varepsilon > 0$ with $#(FS_0) \geq (1 + \varepsilon)#F$ for all finite $F \subset G$. We then have $#(FS_0^k) \geq (1 + \varepsilon)^k#F$ for all $k \in \mathbb{N}$. Let $k$ be large enough so that $(1 + \varepsilon)^k > 1 + k \log #S_0$, and set $S := S_0^k$ and $n := #S$. We will seek $M$ supported in $\mathbb{K}S$. We have

$$#(FS) \geq (1 + \varepsilon)^k#F > (1 + k \log #S_0)#F$$

$$\geq (1 + \log n)#F \text{ for all finite } F \subset G.$$  \hspace{1cm} (22)

Apply Lemma 10.12 to this $n$, and identify $\{1, \ldots, n\}$ with $S$ to obtain a set $Y$ and subsets $X_s$ for all $s \in S$. We have $\bigcup_{s \in S} X_s \subseteq Y$ and
We shall specify soon how large the finite extension \( K \) of \( k \) should be. Under that future assumption, set \( \mathcal{A} := \mathbb{A} Y \). For each \( s \in S \), we shall construct a linear map \( \alpha_s : \mathcal{A} \to \mathbb{K} X_s \subset \mathcal{A} \); for this, we introduce the following notation: for \( T \supseteq s \) denote by \( \alpha_s : \mathcal{A} \to \mathbb{K} X_s, T \) the composition of \( \alpha_s \) with the coordinate projection \( \mathcal{A} \to \mathbb{K} X_s, T \). We wish to impose the condition that, whenever \( \{ T_s : s \in S \} \) is a family of subsets of \( S \) with \( \sum_{s \in S} \# X_s, T_s \geq \# Y \), we have

\[
\bigcap_{s \in S} \ker(\alpha_s, T_s) = 0. \tag{23}
\]

As a first step, we treat each \( \alpha_s \) as a \( \# X_s \times \# Y \) matrix with variables as coefficients, by considering only its rows indexed by \( X_s \subset Y \); and we treat each \( \alpha_s, T \) as a \( \# X_s, T \times \# Y \) submatrix of \( \alpha_s \). The space of all \( (\alpha_s)_{s \in S} \) therefore consists of \( N := \# Y \sum_{s \in S} \# X_s \) variables, so is an affine space of dimension \( N \).

Equations (23) amounts to the condition, on these variables, that all matrices obtained by stacking vertically a collection of \( \alpha_s, T_s \)’s have full rank as soon as \( \sum_{s \in S} \# X_s, T_s \geq \# Y \). The complement of these conditions is an algebraic subvariety of \( \mathbb{K}^N \), given by a finite union of hypersurfaces of the form ‘\( \det(\cdots) = 0 \)’. Crucially, the equations of these hypersurfaces are defined over \( Z \), and in particular are independent of the field \( K \). Therefore, as soon as \( K \) is large enough, there exist points that belong to none of these hypersurfaces; and any such point gives a solution to (23).

Define now the matrix \( M \) with coefficients in \( \mathbb{K} G \) by

\[
M = \sum_{s \in S} \alpha_s. \tag{24}
\]

It maps \( \mathbb{K}^n G \) to \( \mathbb{K}^{n-1} G \) as required, since \( \bigcup_{s \in S} X_s \varsubsetneq Y \). To show that \( M \) is injective, consider \( x \in \mathbb{K}^n G \) non-trivial, and let \( \emptyset \neq F \subseteq G \) denote its support. Define \( \rho : FS \to (0,1] \) by \( \rho(g) := 1/\#\{ s \in S : g \in F_s \} \). Now

\[
\sum_{f \in F} \left( \sum_{s \in S} \rho(fs) \right) = \sum_{g \in FS} \sum_{s \in S} \rho(g) = \sum_{g \in FS} 1 = \#(FS),
\]

so there exists \( f \in F \) with \( \sum_{s \in S} \rho(fs) \geq \#(FS)/\# F \geq 1 + \log n \) by (22). For every \( s \in S \), set \( T_s := \{ t \in S : f \in F_t \} \), so \( \# T_s = 1/\rho(fs) \). We obtain

\[
\sum_{s \in S} \# X_s, T_s \geq \sum_{s \in S} \frac{\# Y}{(1 + \log n) \# T_s} \text{ by Lemma 10.12} \]

\[
= \sum_{s \in S} \frac{\# Y \rho(fs)}{1 + \log n} \geq \# Y,
\]

so by (23) the map \( \mathcal{A} \ni a \mapsto (\alpha_s, T_s(a))_{s \in S} \) is injective. Set \( y := xM \). Since by assumption \( x(f) \neq 0 \), we get \( (\alpha_s, T_s(x(f)))_{s \in S} \neq 0 \), namely there exists \( s \in S \) with
\( \alpha_{s,t}(x(f)) \neq 0 \). Now \( y(f,s)|_{X_{s,t}} = \alpha_{s,t}(x(f)) \) by (24), so \( y \neq 0 \) and we have proven that \( \mathbf{M} \) is injective. \( \square \)

**Proof of Theorem 10.7.** We start by (2). Apply Proposition 10.13 to \( k = F_2 \), and let \( K = F_{2^n} \) and \( \mathbf{M} \) be the \( n \times (n - 1) \) resulting matrix over \( K G \). Set \( \mathcal{A}^G \approx K^n \), and extend \( \mathbf{M} \) to an \( n \times n \) matrix by adding a column on 0’s to its right. Then \( \Theta : \mathcal{A}^G \rightarrow \mathcal{A}^G \) given by \( \Theta(x) = x \mathbf{M} \) is a \( G \)-equivariant endomorphism of \( \mathcal{A}^G \), is pre-injective because \( \mathbf{M} \) is injective on \( \mathcal{A} \), and is not surjective because no configuration in its image has a non-trivial last coordinate.

Right-multiplication by \( \mathbf{M}^* \) on \( \mathcal{A}^G \) is surjective and not pre-injective by Proposition 10.10, so this answers (1).

Finally, let \( y \in \mathcal{A}^S \), for some \( S \subset G \), be such that \( O_y \) is a Garden of Eden for \( \mathbf{M} \). Then \( O_y \mathbf{M}^* = 0 \), so \( \mathbf{M}^* \) does not preserve Bernoulli measure, answering (3). \( \square \)

### 10.1 Goldie rings

We saw in the last section that linear cellular automata are closely related to group rings. We give now a characterization of amenability of groups in terms of ring theory. We recommend [112] as a reference for group rings.

**Definition 10.14.** Let \( R \) be a ring. It is **semiprime** if \( aRa \neq 0 \) whenever \( a \in R \setminus \{ 0 \} \). An element \( a \in R \) is **regular** if \( xay \neq 0 \) whenever \( x,y \in R \setminus \{ 0 \} \), and the ring \( R \) is a **domain** if \( xy \neq 0 \) whenever \( x,y \in R \setminus \{ 0 \} \). The **right annihilator** of \( a \in R \) is \( \{ x \in R \mid ax = 0 \} \) and is a right ideal in \( R \).

The ring \( R \) is **Goldie** if (1) there is no infinite ascending chain of right annihilators in \( R \) and (2) there is no infinite direct sum of nonzero right ideals in \( R \).

Clearly \( R \) is a domain if and only if all its non-zero elements are regular; annihilators of regular elements are trivial; and all domains are semiprime.

These definitions may be difficult to digest, but they have strong consequences for the structure of \( R \), see [31] and Goldie’s theorem below. In terms of their ideal structure, the simplest rings are **skew fields**, in which all non-zero elements are invertible. Next best are **Artinian rings**, which do not admit infinite descending chains of ideals. Finitely generated modules over Artinian rings have a well-defined notion of dimension, namely the maximal length of a composition series.

Ore studied in [108] when a ring \( R \) may be imbedded in a ring in which all regular elements of \( R \) become invertible. Let us denote by \( R^* \) the set of regular elements in \( R \). A naive attempt is to consider expressions of the form \( as^{-1} \) with \( a,s \in R \) and \( s \) regular; then to multiply them one must rewrite \( as^{-1}bt^{-1} = ab'(s')^{-1}t^{-1} = (ab')(ts')^{-1} \), and to add them one must rewrite \( as^{-1} + bt^{-1} = (at' + bs')(st')^{-1} \). In all cases, it is sufficient that \( R \) satisfy the following property, called **Ore’s condition:**

for all \( a,s \in R \) with \( s \) regular there exist \( b,t \in R \) with \( t \) regular and \( sb = at \),
namely every pair of elements \(a, s\) admits a common “right multiple” \(at = sb\). The ring

\[
R(R^e)^{-1} := \{as^{-1} \mid a \in R, s \in R^e\}/\langle as^{-1} = at(st)^{-1} \text{ for all } a \in R, s, t \in R^e \rangle
\]

is called \(R\)'s classical ring of fractions. It naturally contains \(R\) as the subring \(\{a1^{-1}\}\).

If \(R\) is a domain, then \(R(R^e)^{-1}\) is a skew field.

**Theorem 10.15 (Goldie [45]).** Let \(R\) be a semiprime Goldie ring. Then \(R\) satisfies Ore’s condition, and its classical ring of fractions is Artinian. \(\square\)

Let \(R \subseteq S\) be a subring of a ring. The ring \(S\) is called flat over \(R\) if for every exact sequence \(0 \to A \to B \to C \to 0\) of \(R\)-modules the corresponding sequence \(0 \to A \otimes_R S \to B \otimes_R S \to C \otimes_R S \to 0\) of \(S\)-modules is exact.

**Exercise 10.16 (**)**. For \(R\) a domain, show that \(S := R(R^e)^{-1}\) is flat.

*Hint:* there is an equational criterion for flatness: \(S\) is flat if and only if every \(R\)-linear relation \(\sum r_i x_i = 0\), with \(r_i \in R\) and \(x_i \in S\), “follows from linear relations in \(R^e\), in the following sense: the equation in matrix form \(r^T x = 0\), with \(r \in R^e\) and \(x \in S^n\), implies equations \(r^T B = 0\) and \(x = By\) for some \(n \times m\) matrix \(B\) over \(R\) and some \(y \in S^m\); see [85] 4.24(2)].

Using Ore’s condition, apply this criterion by expressing in a \(R\)-linear relation \(\sum r_i x_i = 0\) every \(x_i = a_i s^{-1}\) for \(a_i \in R\) and a common denominator \(s \in R^e\).

Let now \(G\) be a group, let \(k\) be a field, and consider the group ring \(kG\). It is the \(k\)-vector space with basis \(G\), and multiplication extended multilinearly from the multiplication in \(G\). Is is well understood when the group ring \(kG\) is semiprime:

**Theorem 10.17 (Passman, see [112] Theorems 2.12 and 2.13).** If \(k\) has characteristic \(0\), then \(kG\) is semiprime for all \(G\). If \(k\) has characteristic \(p > 0\), then \(kG\) is semiprime if and only if \(G\) has no finite normal subgroup of order divisible by \(p\). \(\square\)

**Exercise 10.18 (*).** If \(G\) is non-amenable, then it has a non-amenable quotient \(\overline{G}\) whose group ring \(k\overline{G}\) is semiprime for all \(k\).

**Theorem 10.19 (Tamari [121], Kielak [111], Kropholler).** Let \(k\) be a field and let \(G\) be group such that \(kG\) is Goldie and semiprime. Then \(G\) is amenable.

Furthermore, if \(kG\) is a domain \(^{20}\) then \(kG\) satisfies Ore’s condition if and only if \(G\) is amenable.

**Proof.** Assume first that \(G\) is amenable and that \(kG\) is a domain, and let \(a, s \in kG\) be given. Let \(S \subseteq G\) contain the supports of \(a\) and \(s\). Since \(G\) is amenable, there exists \(F \subseteq G\) with \(#(FS) < 2#F\), by Følner’s Theorem [3.23]. Consider \(b, t \in kG\) as unknowns in \(kF\). The equation \(st = at\) which they must satisfy is linear in their coefficients, and there are more variables \(2#F\) than constraints \(#(FS)\), so there exists a non-trivial solution, in which \(t \neq 0\) if \(s \neq 0\); so Ore’s condition is satisfied.

\(^{20}\) Conjecturally (see [77] and [78] Problem 6), \(kG\) is a domain if and only if \(G\) is torsion-free.
Assume next that $G$ is not amenable. By Proposition 10.13 there exists a finite field extension $k$ of $k$ and an $n \times (n - 1)$ matrix $M$ over $kG$ such that multiplication by $M$ is an injective map $(kG)^n \to (kG)^{n-1}$. Restricting scalars, namely writing $k = k^d$ qua $k$-vector space, we obtain an exact sequence of free $kG$-modules

$$0 \longrightarrow (kG)^{dn} \longrightarrow (kG)^{d(n-1)}.$$  

(25)

Suppose now for contradiction that $kG$ is a semiprime Goldie ring, and let $S$ be its classical ring of fractions, which exists and is Artinian by Theorem 10.15. By Exercise 10.16 the ring $S$ is flat over $k$, so tensoring (25) with $S$ we obtain an exact sequence

$$0 \longrightarrow S^{dn} \longrightarrow S^{d(n-1)}$$

which is impossible for reasons of composition length. \hfill \Box

### 10.2 Amenable Banach algebras

We concentrated, in this text, on amenability of groups. The topic of amenability of associative algebras has been developed in various directions; although the different definitions are in general inequivalent, we stress here the connections between amenability of a group (or a set) and that of an associated algebra (or module).

Let $A$ be a Banach algebra, and let $V$ be a Banach bimodule: a Banach space $V$ endowed with commuting actions $V \hat{\otimes} A \to V$ and $A \hat{\otimes} V \to V$. Recall that a derivation is a map $\delta: A \to V$ satisfying $\delta(ab) = a\delta(b) + \delta(a)b$, and a derivation $\delta$ is inner if it is of the form $\delta(a) = av - va$ for some $v \in V$. The dual $V^*$ of a Banach bimodule is again a Banach bimodule, for the adjoint actions $(g \cdot \phi \cdot h)(x) = \phi(h^{-1}xg^{-1})$.

**Definition 10.20.** The Banach $A$-module $V$ is amenable if all bounded derivations of $A$ into $V$ are inner. More pedantically: the Hochschild cohomology group $H^1(A, V)$ is trivial.

The algebra $A$ itself is called amenable if all $H^1(A, V^*) = 0$ for all Banach bimodules $V$.

**Exercise 10.21 (***, see Johnson [67, Proposition 5.1]).** Prove that the tensor product of amenable Banach algebras is amenable.

This definition seems quite distinct from everything we have seen in the context of groups and $G$-sets; yet it applies to the Banach algebra $\ell^1(G)$ introduced in (15). For a set $X$, denote by $\ell^\infty(X)^*_0$ those functionals $\Phi: \ell^\infty(X) \to \mathbb{C}$ such that $\Phi(1_X) = 0$.

**Theorem 10.22 ([67] Theorem 2.5]).** Let $G$ be a group. Then the following are equivalent:

1. $G$ is amenable;
2. $\ell^1(G)$ is amenable;
3. the Banach $\ell^1(G)$-module $\ell^\infty(G)_0^*$ is amenable.

Proof. We begin by remarking that the bimodule structure on $V$ can be modified into a right module structure: let $\overline{V}$ be $V$ qua Banach space, with actions $g \cdot v \cdot h = h^{-1}vh$ for $g, h \in G$; in other words, the left action becomes trivial while the right action is by conjugation. A derivation $\delta: \ell^1(G) \to V$ gives rise to a “crossed homomorphism” $\eta: \ell^1(G) \to \overline{V}$, defined by $\eta(g) = g^{-1}\delta(g)$. It satisfies $\eta(gh) = \eta(g)h + \eta(h)$. Inner derivations give rise to crossed homomorphisms of the form $\eta(g) = v - vg$ for some $v \in V$. For the rest of the proof, we replace $V$ by $\overline{V}$.

(1) $\Rightarrow$ (2) Let $m: \ell^\infty(G) \to C$ be a mean on $G$. Given a Banach module $V$ and a crossed homomorphism $\eta: \ell^1(G) \to V^*$, define $v \in V^*$ by

$$v(f) = m(g \mapsto \eta(g)(f)) \text{ for all } f \in V.$$ Compute then, for $h \in G$,

$$(vh)(f) = v(fh^{-1}) = m(g \mapsto \eta(g)(fh^{-1})) = m(g \mapsto (\eta(g)h)(f)) = m(g \mapsto (\eta(gh) - \eta(h))(f)) = (v - \eta(h))(f),$$

so $\eta(h) = v - vh$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) More generally, if $X$ is a $G$-set and $\ell^\infty(X)_0^*$ is amenable then $X$ is amenable: choose $\Phi \in \ell^\infty(X)^*$ with $\Phi(1_X) = 1$, and set $\eta(g) := \Phi - \Phi g$. Then $\eta: \ell^1(G) \to \ell^\infty(X)_0^*$ is a crossed homomorphism, so since $\ell^\infty(X)_0^*$ is amenable there exists $\Psi \in \ell^\infty(X)_0^*$ with $\Psi - \Psi g = \Phi - \Phi g$, namely $(\Phi - \Psi)g = \Phi - \Psi$. Then $\Phi - \Psi: \ell^\infty(X) \to \mathbb{C}$ is a $G$-invariant functional on $X$.

Furthermore, using (3), $\Phi - \Psi$ may be viewed as a measure on the Stone-Čech compactification $\beta X$; its normalized absolute value is a positive measure, and therefore a $G$-invariant mean on $X$. $\square$

As a corollary, we may deduce that $\ell^1(G)$ is amenable if and only if its augmentation ideal has approximate identities; though we prefer to give a direct proof. Recall that an approximate identity in a Banach algebra $\mathcal{A}$ is a bounded net $(e_n)$ in $\mathcal{A}$ with $e_n a \to a$ for all $a \in \mathcal{A}$, and that the augmentation ideal $\mathcal{I}(\ell^1G)$ is $\{f \in \ell^1(G) \mid \sum_{g \in G} f(g) = 0\}$.

Lemma 10.23. Let $\mathcal{A}$ be a Banach algebra with approximate identities, and let $f_1, \ldots, f_N \in \mathcal{A}$ and $\varepsilon > 0$ be given. Then there exists $e \in \mathcal{A}$ with $\|f_i - ef_i\| < \varepsilon$ for all $i = 1, \ldots, N$.

Proof. Let $K = \sup \|e_n\|$ be a bound on the norms of approximate identities in $\mathcal{A}$. For $N = 0$ there is nothing to do. If $N \geq 1$, find by induction $e' \in \mathcal{A}$ satisfying $\|f_i - ef_i\| < \varepsilon/(1 + K)$ for all $i < N$, and let $e'' \in \mathcal{A}$ satisfy $\|(f_N - e'f_N) - e''(f_N - e'f_N)\| < \varepsilon$. Set $e := e' + e'' - e''e'$, and check. $\square$
Theorem 10.24. Let $G$ be a group. Then $G$ is amenable if and only if $\mathfrak{A}(\ell^1 G)$ has approximate identities.

Proof. $(\Rightarrow)$ Given $f \in \mathfrak{A}(\ell^1 G)$ and $\varepsilon > 0$, let $S \subseteq G$ be such that $\sum_{g \in G} |f(g)| < \varepsilon / 2$. Since $G$ is amenable, there exists $h \in \ell^1(G)$ with $h \geq 0$ and $\|h - hs\| < \varepsilon / 2$ for all $s \in S$; so $\|hf\| < \varepsilon$. Set $e := 1 - h$; then $\|e\| \leq 2$, and $\|f - ef\| = \|hf\| < \varepsilon$.

$(\Leftarrow)$ Let $S = \{s_1, \ldots, s_n\} \subseteq G$ and $\varepsilon > 0$ be given, and apply Lemma 10.23 with $f_i = 1 - s_i$ to obtain $e \in \mathfrak{A}$ satisfying $\|1 - e(1 - s)\| < \varepsilon$ for all $s \in S$; set $g := 1 - e$ to rewrite this as $\|g - gs\| < \varepsilon$. Finally set $h(x) = |g(x)|/\|g\|$ for all $x \in G$; we have obtained $h \geq 0$ and $\|h\| = 1$ and $\|h - hs\| < \varepsilon$, so $G$ is amenable by Theorem 3.23. □

We recall without proof Cohen’s factorization theorem:

Lemma 10.25 (Cohen [27]). Let $\mathfrak{A}$ be a Banach algebra with approximate identities, and consider $z \in \mathfrak{A}$. Then for every $\varepsilon > 0$ there exists $x, y \in \mathfrak{A}$ with $z = xy$ and $\|z - y\| < \varepsilon$. □

For instance, it follows that if $G$ is an amenable group then $\mathfrak{A}(\ell^1 G)^2 = \mathfrak{A}(\ell^1 G)$. Amenity, and the Liouville property, are tightly related to the ideal structure of $\ell^1(G)$. The following is in fact a reformulation of Theorem 8.21.

Theorem 10.26 (Willis [128]). Let $G$ be a group and let $X$ be a $G$-set. For a probability measure $\mu$ on $G$, let

$$\ell^1_\mu(X) := \{f - f\mu \mid f \in \ell^1(X)\}$$

denote the closed submodule of $\ell^1(X)$ generated by $1 - \mu$, and write $\mathfrak{A}(\ell^1 X) = \{f \in \ell^1(X) \mid \sum_{g \in G} f(g) = 0\}$. Then $(X, \mu)$ is Liouville if and only if $\ell^1_\mu(X) = \mathfrak{A}(\ell^1 X)$.

In particular, $G$ is amenable if and only if $\{\ell^1_\mu(G) \mid \mu \in \mathcal{P}(G)\}$ has a unique maximal element, which is $\mathfrak{A}(\ell^1 G)$.

Proof. Assume first that $(X, \mu)$ is Liouville, and consider an arbitrary $f \in \mathfrak{A}(\ell^1 X)$. By Proposition 8.25 we have $\|f\mu^n\| \to 0$, so $f - f\mu\to f$, and $f - f\mu^n = f(1 + \mu + \cdots + \mu^{n-1})(1 - \mu) \in \ell^1_\mu(X)$, so $f \in \ell^1_\mu(X)$.

Conversely, if $\mu$ is such that $\ell^1_\mu(X) = \mathfrak{A}(\ell^1 X)$, then given $f \in \mathfrak{A}(\ell^1 X)$ we may for every $\varepsilon > 0$ find $g \in \ell^1(X)$ with $\|f - g(1 - \mu)\| < \varepsilon$; then $\|f - \frac{1}{\varepsilon} \sum_{i=0}^{n-1} \mu f\| \approx \|g(1 - \mu^n)/\|g\| \to 0, \text{ so } f\mu^n \to 0$. By Proposition 8.25 the random walk $(X, \mu)$ is Liouville.

By Theorem 8.21 $G$ is amenable if and only if there exists a Liouville measure on $G$.

It remains to prove that if $\ell^1_\mu(X)$ is the unique maximal element in $\{\ell^1_\mu(X) \mid \nu \in \mathcal{P}(G)\}$ then $\ell^1_\mu(X) = \mathfrak{A}(\ell^1 X)$. For this, $f$ belong to $\mathfrak{A}(\ell^1 X)$ and write $f = g + ih$ with $g, h$ real. Furthermore, write $g = g^+ - g^-$ and $h = h^+ - h^-$ for positive $g^+, h^+$, and set $c = \sum_{x \in X} g^+(x) = \sum_{x \in X} g^-(x)$ and $d = \sum_{x \in X} h^+(x) = \sum_{x \in X} h^-(x)$. Then
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$$f = c(1 - g^+/c) + (-c)(1 - g^-/c) + (id)(1 - h^+/d) + (-id)/(1 - h^-/d),$$

and each term belongs to some $\ell^1_\mu(X)$ and therefore to $\ell^1_\mu(X)$ because $\ell^1_\mu(X)$ is maximal; so $f \in \ell^1_\mu(X)$.

\[\Box\]

**Exercise 10.27 (**, see Johnson [67, Proposition 5.1]). Let $\mathcal{A}$ be an amenable algebra, and let $J \triangleleft \mathcal{A}$ be a closed ideal. Prove that if $J$ and $\mathcal{A}/J$ are amenable, then $\mathcal{A}$ is amenable. Conversely, if $\mathcal{A}$ is amenable then $\mathcal{A}/J$ is amenable, and if $J$ has approximate identities then it is amenable.

### 10.3 Amenable algebras

We now turn to the group algebra $kG$ for a field $k$. Note that we do not make any assumption on the field, which could be finite.

**Definition 10.28.** Let $\mathcal{A}$ be an associative algebra, and let $V$ be an $\mathcal{A}$-module. We call $V$ amenable if for every finite-dimensional subspace $S \leq \mathcal{A}$ and every $\varepsilon > 0$ there exists a finite-dimensional subspace $F \leq V$ with

$$\dim(FS) < (1 + \varepsilon)\dim(F).$$

The algebra $\mathcal{A}$ itself is called amenable if all non-zero $\mathcal{A}$-modules are amenable.\(^{21}\)

We note in passing that if $\mathcal{A}$ is finitely generated, then the ‘$S$’ in Definition 10.28 may be fixed once and for all to be a generating subspace of $\mathcal{A}$.

**Theorem 10.29 ([8]).** Let $G$ be a group and let $X$ be a $G$-set. Then $kX$ is an amenable $kG$-module if and only if $X$ is amenable.

Proof, after [57, §3.6]. $(\Rightarrow)$ Consider the set $O(X)$ of orders on $X$; it is a closed subspace of $\{0,1\}^{X \times X}$, so is compact. It is also the inverse limit of $O(F)$ over all $F \subseteq X$.

Let $\Pi$ denote the group of all bijections of $X$. There exists a unique $\Pi$-invariant probability measure on $O(X)$, which may be defined as the inverse limit of the uniform probability measures on $O(F)$ over $F \subseteq X$. For an order $\leq \in O(X)$, consider

$$\Phi^\leq : \begin{cases} \text{finite-dim}'1 \text{subspaces of } kX \to \text{finite subsets of } X \\
W \mapsto \{ \min^\leq (\text{support}(w)) \mid w \in W \setminus \{0\} \} ,\end{cases}$$

and let $m^\leq_w := \mathbb{1}_{\Phi^\leq(W)}$ be the corresponding characteristic function in $\ell^1(X)$. We clearly have

$$||m^\leq_w|| = \dim W, \quad W_1 \leq W_2 \Rightarrow m^\leq_{w_1} m^\leq_{w_2} \text{ pointwise. \quad (26)}$$

\(^{21}\) Some people defined amenability of algebras — erroneously, in my opinion — as mere amenability of the regular right module.
Define then $m_w := [\varphi|\mathcal{M}]m_W^\mathcal{M}d\lambda(\leq)$, and observe that (26) still holds for $m_w$. Furthermore, the map $W \mapsto m_w$ is $\Pi$-equivariant, so in particular is $G$-equivariant; and (26) further implies $\|m_w - m_w\| = \dim W_2 - \dim W_1$ whenever $W_1 \leq W_2$.

Now given $S \subseteq G$ finite and $\varepsilon > 0$, there exists $W \leq kX$ with $\dim(W + Ws) < (1 + \varepsilon)\dim W$ for all $s \in S$, because $kX$ is amenable. Thus $\|m_{W + Ws} - m_w\| < \varepsilon\dim W$, and similarly $\|m_{W + Ws} - m_w\| < \varepsilon\dim W$, so

$$\|m_w - m_{W,w}\| = \|m_w - m_{W,w}\| < 2\varepsilon\|m_w\|,$$

and $G$ is amenable by Theorem 3.23(2).

$(\Leftarrow)$ Let a finite-dimensional subspace $S$ of $kG$ and $\varepsilon > 0$ be given. There is a finite subset $T \subseteq G$ with $S \leq kT$, so because $X$ is amenable there is $F \subseteq X$ with $\#(FT) < (1 + \varepsilon)\#F$. Set $E := kF$; then

$$\dim(ES) \leq \dim((kF)(kT)) \leq \#(FT) < (1 + \varepsilon)\#F = (1 + \varepsilon)\dim E. \quad \Box$$

Note that, although $kG_G$ is amenable if and only if $kG_{kG}$ is amenable, the growth of almost-invariant subsets and subspaces may behave quite differently. In Example 3.11 we saw Følner sets $F_n$ for the “lamplighter group” $G$, and we may convince ourselves that they are optimal, so $G$’s Følner function, see (9), satisfies $\text{Fol}(n) = n^{2n}$. On the other hand,

$$W_n = k\left\{ \sum_{\text{support}(f) \subseteq [-n,n]} (f, m) \mid m \in [-n,n] \right\}$$

are subspaces of $kG$ of dimension $2n + 1$ with $\dim(W_n + W_n)/\dim W_n = \#(F_n \cup F_n)/\#F_n$, so the “linear Følner function” of $G$ grows linearly.

The following is an analogue, for linear spaces, of the space $\ell^1$ of summable functions on a set. Let $V$ be a vector space. Consider the free $\mathbb{Z}$-module with basis $\{ [A] \mid A \subseteq V \text{ a finite-dimensional subspace} \}$, and let $\ell^1(V, \mathbb{Z})$ be its quotient under the relations $[A] + [B] = [A \cap B] + [A + B]$ for all $A, B \subseteq V$. Note that every $x \in \ell^1(V, \mathbb{Z})$ may be represented as $x = \sum_i [X_i^+] - \sum_j [X_j^-]$. Define a metric on $\ell^1(V, \mathbb{Z})$ by

$$d(x, y) = \|x - y\|, \quad \|x\| = \inf\{ \sum_i \dim(X_i) + \sum_j \dim(X_j^-) \mid x = \sum_i [X_i^+] - \sum_j [X_j^-] \}.$$

**Lemma 10.30.** Let $\mathcal{A}$ be an algebra generated by a set $B$ of invertible elements, and let $V$ be an $\mathcal{A}$-module. Then $V$ is amenable if and only if for every $S \subseteq B$ and every $\varepsilon > 0$ there exists $f \in \ell^1(V, \mathbb{B})$ with $\|f - fs\| < \varepsilon\|f\|$ for all $s \in S$.

**Proof.** If $V$ is amenable, then for every $S \subseteq B$ and every $\varepsilon > 0$ there exists $F \subseteq V$ finite-dimensional with $\dim(F + FS) < (1 + \varepsilon)\dim F$; so in particular $\dim(F + FS) < (1 + \varepsilon)\dim F$ for all $s \in S$; since $\dim(Fs) = \dim F$ because $s$ is invertible, we get $\dim(F \cap Fs) > (1 - \varepsilon)\dim F$ so $f := [F]$ satisfies $\|f - fs\| < 2\varepsilon\|f\|$.

Conversely, given $S \subseteq B$ and $f \in \ell^1(V, \mathbb{B})$ with $\|f - fs\| < \varepsilon\|f\|$ for all $s \in S$, we have $\sum_{s \in S} \|f - fs\| < \varepsilon\|s\|\|f\|$. There is a unique expression $f = [X_0] + \cdots + [X_n]$.
with $X_0 \leq \cdots \leq X_n \leq V$; so there exists $i \in \{0, \ldots, n\}$ with $\sum_{s \in S} \|X_i - [X_i]s\| < \varepsilon \#S \|X_i\|$, and therefore $\sum_{s \in S} \dim(X_i + X_is) < (1 + \varepsilon \#S) \dim X_i$, so $\dim(X_i + X_is) < (1 + \varepsilon \#S) \dim X_i$. We are done since $S \subseteq B$ was arbitrary and $B$ generates $A$. □

**Corollary 10.31.** Let $\mathcal{A}$ be a group ring. Then $\mathcal{A}$ is amenable if and only if the regular right module $A \rightarrow \mathcal{A}$ is amenable.

**Proof.** Consider $\mathcal{A} = \mathbb{Z}G$ a group ring. If $\mathcal{A}$ is amenable, then obviously the regular module $\mathcal{A} \rightarrow \mathcal{A}$ is amenable.

Conversely, if $\mathcal{A} \rightarrow \mathcal{A}$ is amenable, then $G_G$ is amenable by Theorem 10.29. Let $V$ be a non-zero $\mathcal{A}$-module, and consider $\nu \in V \setminus \{0\}$. By Theorem 3.23(5) for every $S \subseteq G$ and every $\varepsilon > 0$ there exists a subset $F \subseteq G$ with $\#(F \triangle Fs) < \varepsilon \#F$. Consider $x := \sum_{s \in S} [\nu f] \in \ell^1(V, \mathbb{N})$, and note $\|x - xs\| < \varepsilon \|x\|$. Thus $\mathcal{A}$ is amenable by Lemma 10.30. □

**Problem 10.32 (Gromov).** Let $G$ be a group. If the $\mathbb{R}G$-module

$$\mathcal{C}_0(G) = \{f \colon G \to \mathbb{R} \mid \inf_{F \subseteq G} \sup_{f \in G} (f|G \setminus F) = 0\}$$

is amenable, does it follow that $G$ is amenable?
11 Further work and open problems

For lack of space, some important and interesting topics have been omitted from this text. Here are a few of the most significant ones, with very brief descriptions.

11.1 Boundary theory

Furstenberg initiated a deep theory of “boundaries” for random walks. Given a random walk on a set \( X \), say driven by a measure \( \mu \) on a group \( G \), a boundary is a measure space \((Y, \nu)\) with a measurable map from the orbit space \((X^\mathbb{N}, \mu^\mathbb{N}) \to Y\) that quotients through asymptotic equivalence, namely if \( (x_0, x_1, \ldots) \) and \( (x'_0, x'_1, \ldots) \) differ in only finitely many positions then their images are the same in \( Y \).

There is a universal such space, written \( \partial(X, \mu) \) and called the Poisson boundary of \((X, \mu)\), such that all boundaries are quotients of \( \partial(X, \mu) \). This space, as a measure space, may be characterized by the identity

\[
L^1(\partial(X, \mu), \nu) = \ell^1(X)/\ell^1_\mu(X),
\]

see Theorem 10.26. The Poisson boundary is reduced to a point if and only if \((X, \mu)\) is Liouville.

In fact, it is better to view \( \partial(X, \mu) \) as a measure space with a family of measures \( \nu_x \), one for each \( x \in X \), satisfying \( \nu_x g = \nu_x \) for all \( g \in G \). One then has a “Poisson formula” for harmonic functions on \( X \): if \( f \in \ell^\infty(X) \) is harmonic, then there exists an integrable function \( \hat{f} \) on \( \partial(X, \mu) \) such that

\[
f(x) = \int_{\partial(X, \mu)} \hat{f}(\xi) d\nu_x(\xi).
\]

There is another construction of \( \partial(X, \mu) \) based on \( \ell^\infty(X) \) rather than \( \ell^1(X) \): the subspace \( h^\infty(X) \leq \ell^\infty(X) \) of harmonic functions is a commutative Banach algebra, under the product

\[
(f_1 \cdot f_2)(x) = \lim_{n \to \infty} \sum_{g \in G} f_1(xg) f_2(xg) \mu^n(g).
\]

The spectrum of \( h^\infty(X) \), namely the set of algebra homomorphisms \( h^\infty(X) \to \mathbb{C} \), is naturally a measure space and is isomorphic to \( \partial(X, \mu) \). The function \( \hat{f} \) is the Gelfand transform of \( f \), given by \( \hat{f}(\xi) = \xi(f) \).

The Poisson boundary is naturally defined as a measure space, and is directly connected to the space of bounded harmonic functions; but other notions of boundary have been considered, for example the space of positive harmonic functions, leading to the Martin boundary which is a well-defined topological space; for a natural measure, it becomes measure-isomorphic to the Poisson boundary.
Glasner considers in [43] “strongly amenable” groups: they are groups all of whose proximal actions on a compact space has a fixed point; see the comments at the end of §6.1. Recall that an action of \( G \) on a compact Hausdorff space \( X \) is \textit{proximal} if for every \( x, y \in X \) there exists a net \( (g_n) \) of elements of \( G \) such that \( \lim_n x g_n = \lim_n y g_n \).

For details, we refer to the original articles [40, 41], the classical [74], and the survey [37].

### 11.2 Consequences

Little has been said about the uses of amenability. On the one hand, it plays a major role in the study of Lie groups and their lattices; for example, Margulis’s “normal subgroup theorem” states that a normal subgroup of a lattice in a higher-rank semisimple Lie group is either finite or finite-index [89]. Ruling out finite-index subgroups, the strategy is to show that such a group is amenable and has property (T).

Witte-Morris uses amenability, and Poincaré’s recurrence theorem, to prove in [129] that all finitely generated amenable groups that act on the real line have homomorphisms onto \( \mathbb{Z} \).

Benjamini and Schramm consider in [17] percolation on Cayley graphs. One fixes \( p \in (0, 1) \) and a finitely generated group \( G = \langle S \rangle \); call \( S \) the corresponding Cayley graph. Then every vertex \( v \in S \) is made independently at random “open” with probability \( p \) (and “closed” with probability \( 1 - p \)). “Open clusters” are connected components of the subgraph of \( S \) spanned by open vertices. We define \textit{critical probabilities}

\[
\begin{align*}
  p_c &= \sup \{ p \in (0, 1) \mid \text{the open cluster containing 1 is almost surely finite} \}, \\
  p_a &= \inf \{ p \in (0, 1) \mid \text{there is almost surely a single infinite open cluster} \}.
\end{align*}
\]

They conjecture that \( p_c < 1 \) for all \( G \) which are not virtually cyclic; this is known for all groups of polynomial or exponential growth, and for all groups containing subgroups of the form \( \Lambda \times B \) with \( \Lambda, B \) infinite, finitely generated groups.

They also conjecture that \( p_c < p_a \) holds precisely when \( G \) is not amenable; see [59] for a survey of known results.

### 11.3 Ergodic theory

One of the standard tools of ergodic theory is the “Rokhlin-Kakutani lemma”: let \( T : X \to X \) be an invertible, measure-preserving transformation of a measure space \((X, \mu)\) that is aperiodic in the sense that almost all points have infinite orbits. Then for every \( n \in \mathbb{N} \) and every \( \varepsilon > 0 \) there exists a measurable subset \( E \subseteq X \) such that \( E, T(E), \ldots, T^{n-1}(E) \) are all disjoint with \( \mu(E \sqcup \cdots \sqcup T^{n-1}(E)) > 1 - \varepsilon \).
It may be understood as the following statement. Given $S, T : X \to X$, define their distance as $d(S, T) = \mu(\{x \in X \mid S(x) \neq T(x)\})$. Then for every $n \in \mathbb{N}, \varepsilon > 0$ there exists $S$ of period $n$ with $d(S, T) < \varepsilon$. In other words, $\mathbb{Z}$ may be approximated arbitrarily closely by $\mathbb{Z}/n$.

The Rokhlin lemma is essential in reducing ergodic theory problems to combinatorial ones. For example, it serves to prove that two Bernoulli shifts (the shift on $\mathcal{A}^\mathbb{Z}$ for a given probability measure on $\mathcal{A}$) are isomorphic if and only if they have the same entropy.

Ornstein and Weiss generalize in [109] the Rokhlin lemma to some amenable groups; see also [124]. Let $G$ be a group; we say that a subset $F \subseteq G$ tiles $G$ if $G$ is a disjoint union of translates of $F$; namely, if there exists a subset $C \subseteq G$ with $G = \bigcup_{c \in C}Fc$. They prove:

**Theorem 11.1.** Let $G$ be amenable, and let $F \subseteq G$ be a finite subset. Then $F$ tiles $G$ if and only if for every free measure-preserving action of $G$ on a probability space $(X, \mu)$ and every $\varepsilon > 0$ there is a measurable subset $E \subseteq X$ such that $\{Ef \mid f \in F\}$ are all disjoint and $\mu(EF) > 1 - \varepsilon$.

In [126], Weiss calls $G$ monotileable if it admits arbitrarily large tiles. He proves that amenable, residually finite are monotileable; more precisely, in Følner’s definition of amenability it may be assumed that the Følner sets tile $G$. For example, $\mathbb{Z}$ is tiled by sets of the form $\{-n, \ldots, n\}$ which form an exhausting sequence of Følner sets are are also transversals for the subgroups $(2n+1)\mathbb{Z}$.

Let us denote by $MG$ the class of monotileable groups; then $MG$ contains all residually amenable groups, and is closed under taking extensions, quotients, subgroups and directed unions [26 §4].

It is at the present (2017) unknown whether every group is monotileable, and whether $AG \subseteq MG$. It is also unknown whether, if a group $G$ belongs to $MG \cap AG$, then $G$ may be tiled by Følner sets.

### 11.4 Numerical invariants

Recall that the *entropy* of a probability measure $\mu$ on a countable set $X$ is defined as

$$H(\mu) = -\sum_{x \in X} \mu(x) \log \mu(x),$$

where as usual $0 \log(0) = 0$.

The Liouville property can, in some favourable cases, be detected by a single numerical invariant, its *entropy* or its *drift*. Given a random walk $p$ on a set $X$, starting at $x \in X$, its *entropy growth* is the function $h(n) := H(p_n(x, -))$ computing the entropy of distribution of the random walker after $n$ steps. If furthermore $X$ is a metric space, the *drift growth* of $p$ is the function $\ell(n) := \sum_{y \in X} p_n(x,y)d(x,y)$ estimating the expected distance from the random walker to the origin after $n$ steps.

A celebrated criterion by Derriennic ([33]; see also [74]) shows that, if $H(\mu) < \infty$, then $(X, \mu)$ is Liouville if and only if $h$ is sublinear. Moreover, the volume, en-
tropy and drift growth are related by the inequality
\[
\lim_{n \to \infty} \frac{\log v(n)}{n} \leq \lim_{n \to \infty} \frac{\ell(n)}{n} \leq \lim_{n \to \infty} \frac{h(n)}{n}.
\]

Finer estimates relate these functions \(\log v, \ell, h\), in particular if all are sublinear; additionally, the probability of return \(p(n) = -\log p_n(x, x)\) and the \(\ell^p\) distortion
\[
d_p(n) = \sup_{\Phi: G \to \ell^p 1\text{-Lipschitz}} \inf \{\|\Phi(g) - \Phi(h)\| \mid d(g, h) \geq n\}
\]
are all related by various inequalities; see [49, 99, 113].

### 11.5 Sofic groups

The class of sofic groups is a common extension of amenable and residually-finite groups. We refer to [56, 125] for its introduction. The definition may be seen as a variant of Følner’s criterion:

**Definition 11.2.** Let \(G\) be a group. It is **sofic** if for every finite subset \(S \subseteq G\) and every \(\varepsilon > 0\) there exists a finite set \(F\) and a mapping \(\pi: S \to \text{Sym}(F)\) such that

- if \(s, t, st \in S\) then \#\{\(f \in F \mid f \pi(s)\pi(t) \neq f \pi(st)\}\} < \varepsilon\#F,
- if \(s \neq t \in S\) then \#\{\(f \in F \mid f \pi(s) = f \pi(t)\}\} < \varepsilon\#F.

Two cases are clear: if \(G\) is residually finite, then for every \(S \subseteq G\) there exists a homomorphism \(\rho: G \to F\) to a finite group that is injective on \(S\); define then \(f \pi(s) = f \rho(s)\) for all \(s \in S, f \in F\), showing that \(G\) is sofic. If on the other hand \(G\) is amenable, then for every \(S \subseteq G\) and every \(\varepsilon > 0\) there exists \(F \subseteq G\) with \#\((FS \setminus F)\) < \(\varepsilon\#F\); define then \(f \pi(s) = fs\) if \(fs \in F\), and extend the partial map \(\pi(s): F \to F\) arbitrarily into a permutation, showing that \(G\) is sofic.

Remarkably, there is at the present time (2017) no known example of a non-sofic group.

### 11.6 Is this group amenable?

We list here some examples of groups for which it is not known whether they are amenable or not. These problems are probably very hard.

**Problem 11.3 (Geoghegan).** Is Thompson’s group \(F\) amenable?

Recall that \(F\) is the group of piecewise-linear homeomorphisms of \([0, 1]\), with slopes in \(2\mathbb{Z}\) and breakpoints in \(\mathbb{Z}\frac{1}{2}\); see [22] and Example [7,21]
There have been numerous attempts at answering Problem 11.3 too many to cite them all; a promising direction appears in [97]. Kaimanovich proves in [75] that, for every finitely supported measure $\mu$ on $F$, the orbit $(\frac{1}{2}F, \mu)$ is not Liouville; however, Juschenko and Zhang prove in [72] that $\frac{1}{2}F$ is laminable.

There is a group that is related to $F$, and acts on the circle $[0,1]/(0 \sim 1)$: it satisfies the same definition as $F$, namely the group $T$ of piecewise-linear self-homeomorphisms with slopes in $2\mathbb{Z}$ and breakpoints in $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$. Its amenable subgroup $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$ acts transitively on the orbit $0T$, so $0T \leftrightarrow T$ is laminable by Corollary 8.18.

**Problem 11.4 (Nekrashevych).** Are all contracting self-similar groups amenable?

Recall that a self-similar group is a group $G$ generated invertible transducers; it acts on $\mathcal{A}^\mathbb{N}$, and may be given by a map $\phi: G \to G \wr \text{Sym}(\mathcal{A})$, as in (2). It is contracting if there is a proper metric on $G$ and constants $\lambda < 1, C$ such that whenever $\phi(g) = \langle g_1, \ldots, g_{\#A} \rangle \pi$ we have $\|g_i\| < \lambda \|g\| + C$. See [101].

**Problem 11.5 (Folklore, often attributed to Katok).** Is the group of interval exchange transformations amenable? Does it contain non-abelian free subgroups?

A partial, positive result appears in Example 9.15. It would suffice, following the strategy in that example (see [70, Proposition 5.3]), to prove that the group of $\mathbb{Z}^d$-wobbles $W(\mathbb{Z}^d)$ acts extensively amenably on $\mathbb{Z}^d$ for all $d \in \mathbb{N}$; at present (2017), this is known only for $d \leq 2$, see Theorem 9.16.
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