Dynamics of Information Entropies in Nonextensive Systems

Hideo Hasegawa

Department of Physics, Tokyo Gakugei University, Koganei, Tokyo 184-8501, Japan

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Abstract

We have discussed dynamical properties of the Tsallis entropy and the generalized Fisher information in nonextensive systems described by the Langevin model subjected to additive and multiplicative noise. Analytical expressions for the time-dependent Tsallis entropy and generalized Fisher information have been obtained with the use of the $q$-moment approach to the Fokker-Planck equation developed in a previous study [H. Hasegawa, Phys. Rev. E 77, 031133 (2008)]. Model calculations of the information entropies in response to an applied pulse and sinusoidal inputs have been presented.

Keywords: Fisher information; nonextensive systems; Fokker-Planck equation

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\[^{1}\text{E-mail address: hideohasegawa@goo.jp}\]
1 Introduction

Information entropies in nonextensive systems have been extensively investigated since Tsallis [1, 2] proposed the generalized entropy (called Tsallis entropy hereafter) defined by

\[ S_q = \frac{1}{(q-1)} \left[ 1 - \int p(x,t)^q \, dx \right], \]

where \( q \) is the entropic parameter and \( p(x,t) \) denotes the probability distribution of a state variable \( x \) at time \( t \). The Tsallis entropy is not extensive for \( q \neq 1.0 \) in the sense that the total entropy of a given system is not proportional to the number of constituent elements. In the limit of \( q = 1.0, S_q \) reduces to the Boltzmann-Gibbs-Shannon entropy.

The generalized Fisher information, which is derived for the generalized Kullback-Leibler divergence in conformity with the Tsallis entropy, is expressed by [3]-[15]

\[ g_{ij} = q E \left[ \left( \frac{\partial \ln p(x,t)}{\partial \theta_i} \right) \left( \frac{\partial \ln p(x,t)}{\partial \theta_j} \right) \right], \]

where \( E[\cdot] \) denotes the average over the probability \( p(x,t) \) characterized by a set of parameters of \( \{ \theta_i \} \). The Tsallis entropy and the generalized Fisher information provide the important measure of the information in nonextensive systems. The Boltzman-Gibbs-Shannon-Tsallis entropy represents a global measure of ignorance while the Fisher information expresses a local measure of positive amount of information [16]. In particular, the Fisher information signifies the distance between the neighboring points in the Riemann space spanned by probability distributions in the information geometry. Its inverse expresses the lower bound of the decoding error for unbiased estimator in the Cramér-Rao inequality.

In a previous paper [15] (referred to as I hereafter), we calculated the Tsallis entropy and generalized Fisher information in a microscopic nonextensive system described by the Langevin model subjected to additive and multiplicative noise. Employing an analytic time-dependent solution of the relevant Fokker-Planck equation (FPE), we discussed the stationary and dynamical properties of the Tsallis entropy \( S_q \) and the generalized Fisher information \( g_{xx} \) given by

\[ g_{xx} = q E \left[ \left( \frac{\partial \ln p(x,t)}{\partial x} \right)^2 \right], \]

instead of \( g_{ij} \) given by Eq. (2). Recently Konno and Watanabe [17] have made a detailed study of the Fisher information \( g_{ij} \) in the stationary state of the Langevin model.
subjected to cross-correlated additive and multiplicative noise, related discussion being given in Sec. 4. It is worthwhile to investigate the dynamics of various elements of the generalized Fisher information $g_{ij}$ besides $g_{xx}$. The purpose of the present paper is to perform detailed calculations of the time-dependent Tsallis entropy and generalized Fisher information with the use of the analytic $q$-moment approach to the FPE which was previously developed in I.

The paper is organized as follows. In Section 2, we discuss the adopted Langevin model and $q$-moment method to obtain the analytic solution of the FPE. With the use of the calculated time-dependent distributions, analytical expressions for the Tsallis entropy and generalized Fisher information are derived in Section 3. In Section 4, we present some model calculations of noise-intensity dependences of the generalized Fisher information in the stationary state as well as the dynamical response of Tsallis entropy and generalized Fisher information to an applied pulse and sinusoidal inputs. Section 5 is devoted to conclusion and discussion on relevant previous studies.

2 Adopted model and method

2.1 The Fokker-Planck equation

We have adopted the Langevin model subjected to additive ($\xi$) and multiplicative noise ($\eta$) given by

$$\frac{dx}{dt} = F(x) + G(x)\eta(t) + \xi(t) + I(t), \quad (4)$$

where $F(x)$ and $G(x)$ are arbitrary functions of $x$, $I(t)$ stands for an external input, and $\eta(t)$ and $\xi(t)$ express zero-mean Gaussian white noises with correlations given by

$$\langle \eta(t) \eta(t') \rangle = \alpha^2 \delta(t - t'), \quad (5)$$

$$\langle \xi(t) \xi(t') \rangle = \beta^2 \delta(t - t'), \quad (6)$$

$$\langle \eta(t) \xi(t') \rangle = 0 \quad (7)$$

$\alpha$ and $\beta$ denoting the strengths of multiplicative and additive noises, respectively.

The FPE in the Stratonovich representation is expressed by

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x} \left\{ [F(x) + I(t)] p(x,t) \right\} + \left( \frac{\beta^2}{2} \right) \frac{\partial^2}{\partial x^2} p(x,t)$$

$$+ \left( \frac{\alpha^2}{2} \right) \frac{\partial}{\partial x} \left[ G(x) \frac{\partial}{\partial x} \{G(x)p(x,t)\} \right]. \quad (8)$$

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Although we have adopted the single-variable Langevin model in this study, it is straightforward to extend it to the coupled Langevin model with the use of the mean-field approximation \[15\].

2.2 Stationary distribution

For the linear Langevin model given by

\[
\begin{align*}
F(x) &= -\lambda x, \quad \text{(9)} \\
G(x) &= x, \quad \text{(10)}
\end{align*}
\]

where \(\lambda\) denotes the relaxation rate, the stationary probability distribution \(p(x)\) is expressed by \[15\]

\[
p(x) = \frac{1}{Z} \left[ 1 - \frac{(1 - q)}{2\phi^2} x^2 \right]^{1/(1-q)} \exp[Y(x)], \quad \text{(11)}
\]

with

\[
q - 1 = \frac{2\alpha^2}{(2\lambda + \alpha^2)}, \quad \text{(12)}
\]
\[
\phi^2 = \frac{\beta^2}{(2\lambda + \alpha^2)}, \quad \text{(13)}
\]
\[
Y(x) = 2c \tan^{-1} \left( \sqrt{\frac{(q - 1)}{(2\phi^2)}} x \right), \quad \text{(14)}
\]
\[
c = \frac{I}{\alpha\beta}, \quad \text{(15)}
\]
\[
Z = \left[ \frac{2\phi^2}{(q - 1)} \right]^{1/2} \frac{\sqrt{\pi} \Gamma(\frac{1}{q-1}) \Gamma(\frac{1}{q-1} - \frac{1}{2})}{|\Gamma(\frac{1}{q-1} + ic)|^2}, \quad \text{(16)}
\]

where \(\Gamma(z)\) denotes the gamma function. In deriving Eq. \[16\], we have employed the following formula \[17\]:

\[
\int_0^{\pi/2} \cos^u(y) \cosh(vy) \, dy = \frac{\pi \Gamma(u + 1)}{2^{u+1} \left| \Gamma\left(\frac{u + 2 + iv}{2}\right) \right|^2}, \quad \text{(17)}
\]
\[
\Gamma(2z) = \frac{2^{2z}}{2\sqrt{\pi}} \Gamma(z) \Gamma\left( z + \frac{1}{2} \right). \quad \text{(18)}
\]

To make the expression of \(p(x)\) compact, we have introduced new parameters \(a\) and \(b\), defined by

\[
a = \left[ \frac{(q - 1)}{2\phi^2} \right]^{1/2} = \frac{\alpha}{\beta}, \quad \text{(19)}
\]
\[
b = \frac{1}{(q - 1)} = \frac{(2\lambda + \alpha^2)}{2\alpha^2}, \quad \text{(20)}
\]
which lead to

\[ p(x) = \left( \frac{1}{Z} \right) \frac{\exp \left[ 2c \tan^{-1}(ax) \right]}{(1 + a^2 x^2)^b}, \]  

(21)

with

\[ Z = \frac{\sqrt{\pi} \Gamma(b) \Gamma(b - \frac{1}{2})}{a \left| \Gamma(b + ic) \right|^2}. \]  

(22)

We define the \(n\)th \(q\)-moment defined by

\[ E_q[x^n] = \int P_q(x) x^n \, dx, \]  

(23)

where \(E_q[\cdot]\) expresses the average over the escort distribution \(P_q(x)\) given by

\[ P_q(x) = \frac{p(x)^q}{c_q}, \]  

(24)

\[ c_q = \int p(x)^q \, dx, \]  

(25)

while \(E[\cdot]\) denotes the average over \(p(x)\) \([e.g. \, Eq. (2)]. By using Eqs. (23)-(25) for the stationary distribution given by Eq. (21), we may obtain the variance \(\mu_q (= E_q[x])\) and covariance \(\sigma_q^2 (= E_q[x^2] - E_q[x]^2)\) in the stationary state given by

\[ \mu_q = \frac{b(b + 1)}{ab^2} = \frac{2(2\lambda + 3\alpha^2)I}{(2\lambda + \alpha^2)^2} \simeq \frac{I}{\lambda}, \]  

(26)

\[ \sigma_q^2 = \frac{[b^2 + (qc)^2]}{a^2b^2(2b - 1)} = \frac{(\alpha^2 \mu_q^2 + \beta^2)}{2\lambda}. \]  

(27)

Depending on the model parameters, the stationary distribution given by Eq. (21) or (24) may reproduce various distributions such as the Gaussian \((q = 1.0)\), \(q\)-Gaussian \((q \neq 1.0, c = 0.0)\), Cauchy \((q = 2.0, c \neq 0.0)\) and inverse-gamma distributions \((\beta = 0.0, c \neq 0.0)\) [15].

### 2.3 Dynamical distribution

In order to obtain the dynamical solution of the FPE given by Eq. (8), we have adopted the following \(q\)-moment approach [15].

(1) From an equation of motion for the \(n\)th \(q\)-moment of \(E_q[x^n]\) given by Eq. (23),

\[ \frac{dE_q[x^n]}{dt} = \frac{d}{dt} \int P_q(x, t) x^n \, dx, \]  

(28)

\[ = \frac{q}{c_q} \int \left( \frac{\partial p(x, t)}{\partial t} \right) p(x, t)^{q-1} x^n \, dx - \frac{1}{c_q} \left( \frac{dc_q}{dt} \right) E_q[x^n], \]  

(29)

\[ \frac{dc_q}{dt} = q \int \left( \frac{\partial p(x, t)}{\partial t} \right) p(x, t)^{q-1} \, dx, \]  

(30)
we have derived equations of motion for $\mu_q(t)$ and $\sigma_q(t)^2$ given by [15]

$$
\frac{d\mu_q(t)}{dt} \simeq -\lambda \mu_q(t) + I(t),
$$

(31)

$$
\frac{d\sigma_q(t)^2}{dt} \simeq -2\lambda \sigma_q(t)^2 + \alpha^2 \mu_q(t)^2 + \beta^2.
$$

(32)

(2) We have assumed that the dynamical distribution has the same functional form as that of the stationary one,

$$
p(x,t) = \frac{1}{Z(t)} \exp\left[2c(t) \tan^{-1} a(t)x\right]
$$

(33)

with the time-independent $b \equiv 1/(q-1)$ and the time-dependent $a(t)$ and $c(t)$ given by [15]

$$
a(t) = \frac{1}{[(2b-1)\sigma_q(t)^2 - \mu_q(t)^2]^{1/2}},
$$

(34)

$$
c(t) = \frac{a(2b-1)\mu_q(t)}{2},
$$

(35)

$$
Z(t) = \frac{\sqrt{\pi} \Gamma(b)\Gamma\left(b - \frac{1}{2}\right)}{a(t) \Gamma[b + ic(t)]^{1/2}}.
$$

(36)

where $\mu_q(t)$ and $\sigma_q(t)^2$ obey equations of motion given by Eqs. (31) and (32). Equations (34) and (35) are derived from the relations for the stationary state given by Eqs. (26) and (27) after I (see Eqs. (86)-(90) in Ref. [15]).

With the assumption (2), we once tried to derive equations of motion for $a(t)$, $b(t)$, $c(t)$ and $Z(t)$ from the FPE, but failed: they are not determined in an appropriate way [15]. Then we have decided to express the parameters of $a(t)$ and $c(t)$ in terms of $\mu_q(t)$ and $\sigma_q(t)^2$, as given by Eqs. (34) and (35). The time-dependent distributions calculated by Eqs. (31)-(36) are in good agreement with those obtained by the partial difference equation method (PDE), as was discussed in I (see Fig. 7 of Ref. [15]) and will be shown shortly (Fig. 2).

## 3 Information entropies

### 3.1 Tsallis Entropy

In a previous study [15], we numerically calculated the Tsallis entropy $S_q$ given by Eq. (1). By using the time-dependent distribution $p(x,t)$ given by Eq. (33), we may obtain
analytical expression for $S_q$ given by

$$S_q = \frac{1}{2} [1 + \ln(2\pi \sigma_q^2)] \quad \text{(for } q = 1),$$

$$= \left(\frac{1 - c_q}{q - 1}\right), \quad \text{(for } q \neq 1),$$

with

$$c_q = Z^{1-q} \left(\frac{b(2b - 1)}{2[b^2 + (qc)^2]}\right) \frac{\left|\Gamma(b + ic)^2\right|}{\left|\Gamma(b + iqc)^2\right|},$$

where $Z$ is given by Eq. (36).

### 3.2 Generalized Fisher information

In order to derive the analytic expression for the generalized Fisher information, we first calculate the derivatives of $\ln p(x,t)$ with respect to $\theta = (a,b,c)$, as given by (the argument $t$ in $a$ and $b$ being suppressed),

$$\frac{\partial \ln p(x,t)}{\partial a} = 2c \left(\frac{x}{U}\right) + \frac{2b}{a} \left(\frac{1}{U}\right) - \frac{(2b - 1)}{a},$$

$$\frac{\partial \ln p(x,t)}{\partial b} = -\ln U + A,$$

$$\frac{\partial \ln p(x,t)}{\partial c} = 2 \tan^{-1}(ax) - B,$$

with

$$U = 1 + ax^2,$$

$$A = 2 \Re \psi(b + ic) - \psi(b) - \psi(b - 1/2),$$

$$B = 2 \Im \psi(b + ic),$$

where the relation given by $x^2/U = (1 - 1/U)/a^2$ is employed. Substituting Eqs. (40)-(42) to Eq. (2) and evaluating various averages such as $E[1/U]$ and $E[\ln U]$ after the method mentioned in the Appendix A (and the Table 1), we have obtained the generalized Fisher information matrix $G$, whose elements $g_{ij} = (G)_{ij}$ are expressed by

$$g_{aa} = \frac{q(2b - 1)(b + 1 + c^2)}{a^2[(b + 1)^2 + c^2]},$$

$$g_{bb} = q \left[\psi'(b) + \psi'(b - 1/2) - 2 \Re \psi'(b + ic)\right],$$

$$g_{cc} = 2q \Re \psi'(b + ic),$$

$$g_{ab} = g_{ba} = \frac{q(b + 2c^2)}{a(b^2 + c^2)},$$
\[ g_{ac} = g_{ca} = -q(c(2b - 1)) \quad \text{(50)} \]
\[ g_{bc} = g_{cb} = 2q \text{Im} \psi'(b + ic). \quad \text{(51)} \]

In the limit of \( c = 0 \), Eqs. (46)-(51) reduce to

\[ g_{aa} = \frac{q(2b - 1)}{a^2(b + 1)}, \quad \text{(52)} \]
\[ g_{bb} = q[\psi'(b - 1/2) - \psi'(b)], \quad \text{(53)} \]
\[ g_{cc} = 2q \psi'(b), \quad \text{(54)} \]
\[ g_{ab} = g_{ba} = \frac{q}{ab}, \quad \text{(55)} \]

and \( g_{ij} = 0 \) otherwise.

4 Model calculations

4.1 Stationary properties

By using Eqs. (46)-(51), we have performed some model calculations of the generalized Fisher information matrix \( G \), whose dependences on \( \alpha, \beta \) and \( I \) are plotted in Fig. 1(a)-(f); relevant calculations of the Tsallis entropy were presented in I (see Figs. 2, 3 and 4 in Ref. [15]).

(a) \( \alpha \) dependence

First we show in Figs. 1(a) and (b), the diagonal and off-diagonal elements of \( G \) as a function of \( \alpha^2 \) for \( \beta = 0.5 \) and \( I = 1.0 \). We note that with increasing \( \alpha \), \( g_{aa} \) is significantly decreased while \( g_{bb} \) and \( g_{cc} \) are increased. \(|g_{bc}|\) is gradually increased with a negative sign whereas the saturation behavior is realized in \( g_{ab} \) and \( g_{ac} \) for a large \( \alpha \).

(b) \( \beta \) dependence

Figures 1(c) and (d) show the \( \beta \) dependence of the diagonal and off-diagonal elements, respectively, of \( G \) for \( \alpha = 0.5 \) and \( I = 1.0 \). With increasing \( \beta \), \( g_{aa} \) and \( g_{cc} \) are increased whereas \( g_{bb} \) is decreased. Off-diagonal elements are small at \( \beta = 0.0 \). With increasing \( \beta \) from zero value, \( g_{ab} \) is increased with a positive sign while \( |g_{bc}| \) and \( |g_{ac}| \) are increased with negative signs.

(c) \( I \) dependence

The \( I \) dependences of diagonal and off-diagonal elements of \( G \) are plotted in Figs. 1(e) and (f), respectively. With increasing \( I \), both \( g_{aa} \) and \( g_{bb} \) are increased whereas \( g_{cc} \) is decreased. Although off-diagonal elements are vanishing at \( I = 0.0 \) except for \( g_{ab} \) as
Eq. (55) shows, a positive $g_{ab}$ is increased while $|g_{bc}|$ and $|g_{ac}|$ are increased with negative signs when $I$ is increased.

### 4.2 Dynamical properties

We will discuss the response of the system to applied external signals in this subsection. In order to examine the validity of the analytic dynamical approach, we have employed the PDE derived from Eq. (8), as given by

$$
p(x,t+u) = p(x,t) + \left(-F'(x) + \frac{\alpha^2}{2} [G'(x)^2 + G(x)G^{(2)}(x)]\right) v p(x,t)
+ \left[-F(x) - I(t) + \frac{3\alpha^2}{2} G(x)G'(x)\right] \left(\frac{v}{2u}\right) [p(x+u) - p(x-u)]
+ \left(\frac{\alpha^2}{2} G(x)^2 + \frac{\beta^2}{2}\right) \left(\frac{v}{u^2}\right) [p(x+u,t) + p(x-u,t) - 2p(x,t)],
$$

(56)

where $u$ and $v$ denote incremental steps of $x$ and $t$, respectively. We impose the boundary condition:

$$p(x,t) = 0, \quad \text{for } |x| \geq x_m$$

(57)

with $x_m = 5$, and the initial condition of $p(x,0) = p_0(x)$ where $p_0(x)$ is the stationary distribution given by Eqs. (21) and (22). We have chosen parameters of $u = 0.05$ and $v = 0.0001$ such as to satisfy the condition: $(\alpha^2 x_m v / 2 u^2) < 1/2$, which is required for stable, convergent solutions of the PDE [15].

(a) Pulse Input

We first apply a pulse input given by

$$I(t) = \Delta I [\Theta(t-2)\Theta(6-t) - \Theta(t-10)\Theta(14-t)],$$

(58)

to a system with $\lambda = 1.0$, $\alpha = 0.5$ and $\beta = 0.5$, where $\Delta I = 1.0$ and $\Theta(x)$ denotes the Heaviside function: $\Theta(x) = 1$ for $x \geq 1$ and zero for $x < 0$. Positive and negative pulses with the magnitude of $|\Delta I|$ are applied at $2.0 \leq t < 6.0$ and $10.0 \leq t < 14.0$, respectively. The time-dependent distributions of $p(x,t)$ at $0 \leq t < 10.0$ are plotted in Fig. 2 where solid (dashed) curves express the results of the $q$-moment (PDE) method [15]. By a positive pulse applied at $2.0 \leq t < 6.0$, $p(x,t)$ moves rightward with a slight modification of its shape. In contrast, when a negative pulse is applied at $10.0 \leq t < 14.0$, $p(x,t)$ moves leftward again with a slight modification of its shape. We note that the results

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calculated by the $q$-moment method are in good agreement with those obtained by the PDE method.

Figure 3(a) shows the time dependence of $\mu_q(t)$ and $\sigma_q(t)^2$ in response to the applied pulse input. We note that $\mu_q$ is increased (decreased) by an applied positive (negative) pulse whereas $\sigma_q^2$ is increased for both positive and negative inputs. The result calculated with equations of motion given by Eqs. (31) and (32) shown by solid curves well agrees with those obtained by the PDE method shown by dashed curves.

The Tsallis entropy $S_q(t)$ in response to an applied pulse is plotted in Fig. 3(b), where the solid and dashed curves express the results calculated by Eqs. (38) and (39) and by the PDE method, respectively. It is shown that $S_q(t)$ is increased for both positive and negative input pulses.

The time dependences of the diagonal and off-diagonal elements of the generalized Fisher information are plotted in Figs. 4(a) and 4(b), respectively. When a positive input signal is applied at $2.0 \leq t < 6.0$, $g_{aa}$, $g_{bb}$ and $g_{ab}$ are increased whereas $g_{cc}$, $g_{bc}$ and $g_{ac}$ are decreased. When a negative input signal is applied at $10.0 \leq t < 14.0$, all the elements except for $g_{ac}$ show the same behaviors as those for a positive input: only $g_{ac}$ is expressed by an odd function of $c$ in Eqs. (46)-(51).

Figure 5 shows the time dependence of inverse of the generalized Fisher information matrix, $h_{ij} [= (G^{-1})_{ij}]$. With an applied pulse, $h_{cc}$ is increased while $h_{aa}$ and $h_{bb}$ are decreased. It implies from the Cramér-Rao theorem that when an external signal is applied, the lower bound in estimating the parameter $c$ is increased while those in estimating $a$ and $b$ are decreased.

(b) Sinusoidal Input

Next we apply a sinusoidal input given by

$$ I(t) = \Delta I \sin \left( \frac{2\pi (t - 2)}{T_p} \right) \Theta(t - 2) \Theta(T_p + 2 - t), \quad (59) $$

where $\Delta I = 1.0$ and $T_p = 10.0$. The time dependences of $\mu_q(t)$ and $\sigma_q(t)^2$ in response to the applied sinusoidal input are plotted in Figure 6(a), where bold and thin solid curves express the results calculated by Eqs. (31) and (32), and where dashed curves show those obtained by the FPE method. The response of $\mu_q(t)$ lags behind the input $I(t)$ by about 0.85. The time dependence of $S_q(t)$ is plotted in Fig. 6(b), where solid and dashed curves denote the results of the $q$-moment and PDE methods, respectively. Figure 7(a) and (b) show the time dependences of diagonal and off-diagonal elements, respectively, of the
generalized Fisher information. A comparison of Figs. 6 and 7 to Figs. 3 and 4 shows that
time dependences of $\mu(t)$, $\sigma_q(t)^2$, $S_q(t)$ and $g_{ij}(t)$ for a sinusoidal input are not dissimilar
to those for a pulse input. This is true also for the time dependence of the inversed Fisher
information (relevant results not shown).

5 Discussion and conclusion

In a previous study [15], we numerically calculated the generalized Fisher information $g_{xx}$ given by Eq. (3). By using the dynamical distribution given by Eq. (33), we may analytically calculate $g_{xx}$ given by

$$g_{xx} = \frac{q a^2 b (b + 1)(2b - 1)}{[(b + 1)^2 + c^2]}, \quad (60)$$

$$= \frac{q a^2 b (2b - 1)}{(b + 1)} = \frac{(3 - q)}{2c^2} = \frac{1}{\sigma_q^2}. \quad (for \ c = 0.0). \quad (61)$$

The time dependence of $g_{xx}$ when a pulse input given by Eq. (58) is applied, is plotted by the chain curve in Fig. 8(a).

For a comparison, we have calculated the generalized Fisher information for $\theta_i = \theta_j = \mu_q$ and $\theta_i = \theta_j = \sigma_q^2 \equiv \gamma$ in Eq. (2), by using the relations given by

$$g_{\mu\mu} = \left( \frac{\partial a}{\partial \mu_q} \right)^2 g_{aa} + \left( \frac{\partial c}{\partial \mu_q} \right)^2 g_{cc} + 2 \left( \frac{\partial a}{\partial \mu_q} \right) \left( \frac{\partial c}{\partial \mu_q} \right) g_{ac}, \quad (62)$$

$$g_{\gamma\gamma} = \left( \frac{\partial a}{\partial \sigma_q^2} \right)^2 g_{aa}, \quad (63)$$

where $\partial a/\partial \mu_q$ et. al. may be calculated from Eqs. (34) and (35). Solid curves in Figs. 8(a) and (b) show the time dependences of $g_{\mu\mu}$ and $g_{\gamma\gamma}$, respectively, in response of a pulse input given by Eq. (58). When an input pulse is applied, $g_{\mu\mu}$ is decreased while $g_{\gamma\gamma}$ is increased. We note that $g_{xx}$ and $g_{\mu\mu}$ are almost the same in the stationary state and that their responses to an input are similar apart from their magnitudes.

It is well known that the relation given by

$$g_{\mu\mu} = g_{xx}, \quad (64)$$

holds for $q = 1.0 \ (\alpha = 0.0)$ because the time-dependent Gaussian distribution is given by

$$p(x, t) = \frac{1}{\sqrt{2\pi \sigma(t)^2}} e^{-[x - \mu(t)]^2/2\sigma(t)^2}, \quad (65)$$
with $\mu(t)$ and $\sigma(t)^2$ satisfying equations of motion given by
\[
\frac{d\mu(t)}{dt} = -\lambda \mu(t) + I(t), \quad (66)
\]
\[
\frac{d\sigma(t)^2}{dt} = -2\lambda \sigma(t)^2 + \beta^2. \quad (67)
\]

The relation given by Eq. (64) is obtainable also for the $q$-Gaussian distribution given by
\[
p(x,t) = \frac{1}{Z_q} \left[ 1 - \frac{(1-q)}{(3-q)\sigma(t)^2} \{x - \mu_q(t)\}^2 \right]^{1/(1-q)}, \quad (68)
\]
which is derived from the maximum-entropy method for given $\mu_q$ and $\sigma_q^2$. It is natural that Figs. 8(a) shows $g_{\mu\mu} \neq g_{xx}$ for $I(t) \neq 0.0$ because the distribution given by Eq. (68) is rather different from the $q$-Gaussian given by Eq. (68) except for $c = \mu_q = 0.0$ (see Figs. 10 and 11 of Ref. [15]).

Although additive and multiplicative noise is assumed to be uncorrelated in Sec. 2, we may take into account effects of the cross-correlation between the two noise, introducing the degree of the cross-correlation $\epsilon$ with $\langle \eta(t) \xi(t') \rangle = \epsilon \alpha \beta \delta(t-t')$ in place of Eq. (7).

The stationary distribution is given by (for details, see the Appendix B)
\[
p(x) = \left( \frac{1}{Z} \right) \frac{\exp \left[ 2c \tan^{-1} a(x+f) \right]}{[1 + a^2(x+f)^2]^b}, \quad (69)
\]
with
\[
a = \frac{\alpha}{\beta \sqrt{1-\epsilon^2}}, \quad (70)
\]
\[
b = \frac{2\lambda + \alpha^2}{2\alpha^2}, \quad (71)
\]
\[
c = \frac{(I + \lambda f)}{\beta \sqrt{1-\epsilon^2}}, \quad (72)
\]
\[
f = \frac{\epsilon \beta}{\alpha}, \quad (73)
\]
\[
Z = \frac{\sqrt{\pi} \Gamma(b) \Gamma(b-1/2)}{a \mid \Gamma(b+ic) \mid^2}. \quad (74)
\]

The distribution given by Eq. (69), which is shifted by an amount of $f$ by an introduced cross-correlation, has the same dependence on the parameters $a$, $b$ and $c$ as that of Eq. (21). The Langevin model with the cross-correlated noise ($\epsilon \neq 0.0$) but no external inputs ($I = 0.0$) was discussed by Konno and Watanabe, who derived useful expressions for the generalized Fisher information in its stationary state \[17\].

\[^2\] The factor corresponding to $\epsilon \alpha \beta G'(x)$ in the first term of Eq. (32) is missing in Eq. (23) of Konno and Watanabe \[17\]: $K(x) = (\alpha - D_p)x$ and $f = 2(b-1)c/A$ in Eqs. (23) and (25) of Ref. [17] should be replaced by $K(x) = (\alpha - D_p)x - D_{ap}$ and $f = (2b-1)c/A$, respectively. The probability distribution given by Eq. (27) of Ref. \[17\] is valid with these replacements.
Our result presented in this study may be applied to another type of the Langevin model, for example, with $\epsilon = 0.0$, and $F(x)$ and $G(x)$ given by

\begin{align}
F(x) &= -\lambda (x + s), \quad (75) \\
G(x) &= \sqrt{r^2 + 2sx + x^2}, \quad (76)
\end{align}

where $\lambda$ expresses the relaxation rate, and $r$ and $s$ are additional parameters. In the case of $[\beta^2 + \alpha^2(r^2 - s^2)] > 0$, the stationary distribution is given by

\begin{equation}
p(x) = \left(\frac{1}{Z}\right) \exp \left[2c \tan^{-1} a(x + s) \right] \left[1 + a^2(x + s)^2\right]^b, \quad (77)
\end{equation}

with

\begin{align}
a &= \frac{\alpha}{\sqrt{\beta^2 + \alpha^2(r^2 - s^2)}}, \quad (78) \\
b &= \frac{(2\lambda + \alpha^2)}{2\alpha^2}, \quad (79) \\
c &= \frac{I}{\alpha\sqrt{\beta^2 + \alpha^2(r^2 - s^2)}}, \quad (80) \\
Z &= \frac{\sqrt{\pi} \Gamma(b) \Gamma(b - \frac{1}{2})}{a \Gamma(b + ic)} \left|\Gamma(b + ic)\right|^2. \quad (81)
\end{align}

Equation (77) is equivalent to Eqs. (21) and (69). Discussion on the information entropies presented in Sec. 3 may be applied also to this model.

Quite recently we have investigated the effect of spatially correlated variability on the information entropies in nonextensive systems, by using the maximum-entropy method [18]. It has been shown that effects of the spatially correlated variability on the generalize Fisher information are different from those on the Tsallis entropy: the generalized Fisher information is increased (decreased) by a positive (negative) spatial correlation, whereas the Tsallis entropy is decreased with increasing an absolute magnitude of the correlation independently of its sign. This fact arises from the difference in their characteristics [16].

In Ref. [18], we obtained analytic expressions for the Tsallis entropy and the generalized Fisher entropy in spatially correlated nonextensive systems from the $q$-Gaussian-type distribution derived by the maximum-entropy method [18]. Unfortunately, we cannot obtain analytic distributions in the Langevin model subjected to spatially correlated additive and multiplicative noise, because we have no analytic approaches to solve the relevant FPE, even for $N = 2$ [19]. It is necessary to develop an appropriate analytic method to solve the FPE including the spatial correlation in additive and multiplicative noise.
To summarize, we have discussed the stationary and dynamical properties of Tsallis entropy and generalized Fisher entropy in the Langevin model subjected to additive and multiplicative noise, which is a typical microscopic nonextensive model. By employing the dynamical solution of the FPE, we have derived analytical expressions for the time-dependent information entropies, with which model calculations of their responses to applied pulse and sinusoidal inputs have been performed. The analytic $q$-moment approach to the FPE developed in I is useful and applicable to the Langevin model which includes the time-dependent model parameters [15]. It is possible to generalize the moment method to the FPE for various types of Langevin models, which will be reported in a separate paper [20].

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**Appendix A: Evaluations of averaged values**

Various quantities averaged over the distribution $p(x)$, which appear in the Fisher information, may be evaluated with the use of $K_m$ defined by

$$K_m \equiv K_m(a, b, c) = \int_{-\infty}^{\infty} \frac{e^{2c \tan^{-1}(ax)}}{U^{b+m}} \, dx,$$

$$= \frac{\sqrt{\pi} \Gamma(b+m) \Gamma(b+m-1/2)}{a \Gamma(b+m+i c)^2},$$

where $U = 1 + ax^2$ and $K_0 = Z$. Taking derivatives of Eq. (A1) with respect to parameters of $a$, $b$ and $c$, we obtain

$$\frac{K_m}{K_0} = E \left[ \frac{1}{U^m} \right],$$

$$\frac{1}{K_0} \frac{\partial K_m}{\partial a} = -2a(b+m) E \left[ \frac{x^2}{U^{m+1}} \right] + 2c E \left[ \frac{x}{U^{m+1}} \right],$$

$$\frac{1}{K_0} \frac{\partial K_m}{\partial b} = -E \left[ \ln U \right] \frac{1}{U^m},$$

$$\frac{1}{K_0} \frac{\partial K_m}{\partial c} = 2E \left[ \frac{\tan^{-1}(ax)}{U^m} \right],$$

$$\frac{1}{K_0} \frac{\partial^2 K_m}{\partial a^2} = 4a^2(b+m)(b+m+1)E \left[ \frac{x^4}{U^{m+2}} \right] - 4ac(b+m+1)E \left[ \frac{x^3}{U^{m+2}} \right].$$
\[-4ac(b + m)E \left[ \frac{x^3}{U^{m+1}} \right] + 4c^2 E \left[ \frac{x^2}{U^{m+2}} \right] - 2(b + m)E \left[ \frac{x^2}{U^{m+1}} \right], \quad (A7)\]
\[
\frac{1}{K_0} \frac{\partial^2 K_m}{\partial b^2} = E \left[ \frac{(\ln U)^2}{U^m} \right], \quad (A8)\]
\[
\frac{1}{K_0} \frac{\partial^2 K_m}{\partial c^2} = 4E \left[ \frac{[\tan^{-1}(ax)]^2}{U^m} \right], \quad (A9)\]
\[
-2aE \left[ \frac{x^2}{U^{m+1}} \right] + 2a(b + m)E \left[ \frac{x^2 \ln U}{U^{m+1}} \right] - 2cE \left[ \frac{x \ln U}{U^{m+1}} \right], \quad (A10)\]
\[
\frac{1}{K_0} \frac{\partial^2 K_m}{\partial a \partial b} = 2 \frac{\partial^2 K_m}{\partial a \partial c} = 0. \quad (A20)\]

From derivatives of Eq. (A2) with respect to \(a\), \(b\) and \(c\), we obtain
\[
\frac{\partial \ln K_m}{\partial a} = -\frac{1}{a}, \quad (A13)\]
\[
\frac{\partial \ln K_m}{\partial b} = \psi(b + m) + \psi(b + m - 1/2) - 2 \text{Re} \, \psi(b + m + ic), \quad (A14)\]
\[
\frac{\partial \ln K_m}{\partial c} = 2 \text{Im} \, \psi(b + m + ic), \quad (A15)\]
\[
\frac{\partial^2 \ln K_m}{\partial a^2} = \frac{1}{a^2}, \quad (A16)\]
\[
\frac{\partial^2 \ln K_m}{\partial b^2} = \psi'(b + m) + \psi'(b + m - 1/2) - 2 \text{Re} \, \psi'(b + m + ic), \quad (A17)\]
\[
\frac{\partial^2 \ln K_m}{\partial c^2} = 2 \text{Re} \, \psi'(b + m + ic), \quad (A18)\]
\[
\frac{\partial^2 \ln K_m}{\partial b \partial c} = 2 \text{Im} \, \psi'(b + m + ic), \quad (A19)\]
\[
\frac{\partial^2 \ln K_m}{\partial a \partial b} = \frac{\partial^2 K_m}{\partial a \partial c} = 0. \quad (A20)\]

By using Eqs. (A3)-(A20) and the relations given by
\[
\frac{x^2}{U} = \frac{1}{a^2} \left( 1 - \frac{1}{U} \right), \quad (A21)\]
\[
\frac{1}{K_0} \frac{\partial^2 K_m}{\partial a \partial b} = \frac{K_m}{K_0} \left[ \left( \frac{\partial \ln K_m}{\partial a} \right) \left( \frac{\partial \ln K_m}{\partial b} \right) + \left( \frac{\partial^2 \ln K_m}{\partial a \partial b} \right) \right], \quad (A22)\]

we may calculate averaged values of various quantities, whose results are summarized in the Table 1.

Appendix B: The Langevin model with cross-correlated noise
We assume the Langevin model subjected to cross-correlated additive and multiplicative noise, which is given by Eqs. (1)–(7) but Eq. (7) is replaced by

$$\langle \eta(t) \xi(t') \rangle = \epsilon \alpha \beta \delta(t - t'),$$

(B1)

with $\epsilon$ expressing the degree of the cross-correlation.

The FPE for the Langevin model is given by [21, 22, 23]

$$\frac{\partial}{\partial t} p(x, t) = - \frac{\partial}{\partial x} \left( \left[ F(x) + I + \left( \frac{\phi}{2} \right) [\alpha^2 G'(x) G(x) + \epsilon \alpha \beta G'(x)] \right] p(x, t) \right)$$
$$+ \left( \frac{1}{2} \right) \frac{\partial^2}{\partial x^2} \left\{ [\alpha^2 G(x)^2 + 2\epsilon \alpha \beta G(x) + \beta^2] p(x, t) \right\},$$

(B2)

where $\phi = 1$ and 0 in the Stratonovich and Ito representations, respectively.

The stationary probability distribution $p(x)$ is expressed by

$$\ln p(x) = 2 \int \left[ \frac{F(x) + I}{\alpha^2 G(x)^2 + 2\epsilon \alpha \beta G(x) + \beta^2} \right] dx$$
$$- \left( 1 - \frac{\phi}{2} \right) \ln \left( \frac{1}{2} [\alpha^2 G(x)^2 + 2\epsilon \alpha \beta G(x) + \beta^2] \right).$$

(B3)

For the linear Langevin model given by $F(x) = -\lambda x$ and $G(x) = x$, Eq. (B3) in the Stratonovich representation ($\phi = 1$) yields the stationary distribution expressed by Eqs. (69)–(74).
**Table 1** Averaged values of various quantities: \( E[Q(x)] = \int p(x) Q(x) \, dx \)

| \( Q(x) \) | \( E[Q(x)] \) |
|---|---|
| \( x \) | \( c/a(b - 1) \) |
| \( x^2 \) | \( (b - 1 + 2c^2)/a^2(b - 1)(2b - 3) \) |
| \( x^2 - (E[x])^2 \) | \( [(b - 1)^2 + c^2]/a^2(b - 1)^2(2b - 3) \) |
| \( 1/U \) | \( b(2b - 1)/2(b^2 + c^2) \) |
| \( 1/U^2 \) | \( b(b + 1)(2b + 1)(2b - 1)/4(b^2 + c^2)[(b + 1)^2 + c^2] \) |
| \( x/U \) | \( c(2b - 1)/2a(b^2 + c^2) \) |
| \( x/U^2 \) | \( bc(2b + 1)(2b - 1)/4a(b^2 + c^2)[(b + 1)^2 + c^2] \) |
| \( \ln U \) | \( -[\psi(b) + \psi(b - 1/2) - 2\Re \psi(b + ic)] \) |
| \( \tan^{-1}(ax) \) | \( \psi'(b) + \psi'(b - 1/2) - 2\Re \psi'(b + ic) \) |
| \( [\tan^{-1}(ax)]^2 - (E[\tan^{-1}(ax)])^2 \) | \( \Im \psi(b + ic) \) |
| \( (U = 1 + a^2x^2) \) | \( \Re \psi'(b + ic)/2 \) |
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Figure 1: The $\alpha$ dependence of (a) diagonal and (b) off-diagonal elements of the generalized Fisher information matrix $G$ with $\beta = 0.5$ and $I = 1.0$. The $\beta$ dependence of (c) diagonal and (d) off-diagonal elements of $G$ with $\alpha = 0.5$ and $I = 1.0$. The $I$ dependence of (e) diagonal and (f) off-diagonal elements of $G$ with $\alpha = 0.5$ and $\beta = 0.5$. In (a), (c) and (e), solid, dashed and chain curves denote $g_{aa}/5$, $g_{bb}$ and $g_{cc}$, respectively: in (b), (d) and (f), solid, dashed and chain curves denote $g_{ab}$, $g_{bc}$ and $g_{ac}$, respectively.

Figure 2: The time-dependent distribution $p(x, t)$ in response to an applied pulse input given by Eq. (58): solid and dashed curves express the results calculated by the $q$-moment approach and the PDE method, respectively ($\lambda = 1.0$, $\alpha = 0.5$ and $\beta = 0.5$). Curves are consecutively shifted downward by 0.25 for a clarity of the figure.

Figure 3: (a) The time dependence of $\mu_q$ (the bold solid curve) and $\sigma_q^2$ (the thin solid curve) calculated by Eqs. (31) and (32) for an applied pulse input $I$ (the chain curve): dashed curves denote the results of the PDE method: $\sigma_q(t)^2$ is multiplied by a factor of five. (b) The time dependence of $S_q$ calculated by Eqs. (38) and (39) (the solid curve) and by the PDE method (the dashed curve) ($\lambda = 1.0$, $\alpha = 0.5$ and $\beta = 0.5$).

Figure 4: (a) The time dependence of diagonal elements of the generalized Fisher information matrix $G$ in response to a pulse input given by Eq. (58): $g_{aa}$ (the solid curve), $g_{bb}$ (the dashed curve) and $g_{cc}$ (the chain curve), $g_{aa}$ being divided by a factor of five. (b) The time dependence of off-diagonal elements of $G$: $g_{ab}$ (the solid curve), $g_{bc}$ (the dashed curve) and $g_{ac}$ (the chain curve) ($\lambda = 1.0$, $\alpha = 0.5$ and $\beta = 0.5$).

Figure 5: The time dependence of the inverse of the generalized Fisher information matrix, $h_{ij} = (G^{-1})_{ij}$, in response to a pulse input given by Eq. (58): $h_{aa}$ (the solid curve), $h_{bb}$ (the dashed curve) and $h_{cc}$ (the chain curve), $h_{bb}$ being divided by a factor of twenty ($\lambda = 1.0$, $\alpha = 0.5$ and $\beta = 0.5$).

Figure 6: (a) The time dependence of $\mu_q$ (the bold solid curve) and $\sigma_q^2$ (the thin solid curve) calculated by Eqs. (31) and (32) for an applied sinusoidal input $I$ (the chain curve): dashed curves denote the results of the PDE method: $\sigma_q(t)^2$ is multiplied by a factor of five. (b) The time dependence of $S_q$ calculated by Eqs. (38) and (39) (the solid curve) and by the PDE method (the dashed curve) ($\lambda = 1.0$, $\alpha = 0.5$ and $\beta = 0.5$).
Figure 7: (a) The time dependence of the diagonal elements of the generalized Fisher information matrix $G$ in response to a sinusoidal input given by Eq. (59): $g_{aa}$ (the solid curve), $g_{bb}$ (the dashed curve) and $g_{cc}$ (the chain curve), $g_{aa}$ being divided by a factor of five. (b) The time dependence of the off-diagonal elements of $G$: $g_{ab}$ (the solid curve), $g_{bc}$ (the dashed curve) and $g_{ac}$ (the chain curve) ($\lambda = 1.0$, $\alpha = 0.5$ and $\beta = 0.5$).

Figure 8: The time dependence of the generalized Fisher information matrices of (a) $g_{xx}$ (the chain curve) and $g_{\mu\mu}$ (the solid curve), and (b) $g_{\gamma\gamma}$ (the solid curve) in response to a pulse input given by Eq. (58) ($\lambda = 1.0$, $\alpha = 0.5$ and $\beta = 0.5$) (see text).
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