Forces and momenta caused by electromagnetic waves in magnetolectric media

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Abstract

We analyse the propagation of electromagnetic waves in magnetoelectric media. Recently, Feigel has predicted that such a medium may “extract momentum from vacuum” in the sense that the total momentum of the virtual waves (vacuum fluctuations of the electromagnetic field) is nontrivial. Our aim is to check the feasibility of this effect. The crucial point in our study is an assumption of the finite size of the magnetoelectric sample, which allows us to reduce the calculation of the momenta and forces of the electromagnetic waves acting on the sample to the vacuum region outside of the medium. In this framework, we demonstrate that, in contrast to Feigel, the total force caused by the virtual is zero, with an appropriate count of the modes that should be taken into account in this effect. Furthermore, we find that the two irreducible parts of the magnetoelectric matrix behave differently in the possible Feigel effect. Going beyond the original scheme of the virtual electromagnetic waves, we propose an experimental scheme which is suitable for the measurement of the magnetoelectric susceptibilities of the medium with the help of real electromagnetic waves.

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I. INTRODUCTION

Phenomenological macroscopic electrodynamics is based on the well known experimental observations that an external electric field can induce a polarization of a medium, whereas an external magnetic field can induce a magnetization of matter. As a result, the electric and magnetic excitations \( (D, H) \) (comprising a 2-form \( H \)) are functions of the electric and magnetic field strengths \( (E, B) \) (collected in a 2-form \( F \)) and of the permittivity \( \varepsilon \) and the permeability \( \mu \) of the medium. In the simplest case of an isotropic medium at rest, the constitutive relations read \( D = \varepsilon \varepsilon_0 E \) and \( H = (\mu \mu_0)^{-1} B \). Here \( \varepsilon_0 \) and \( \mu_0 \) are the electric and magnetic constants (permittivity and permeability of the vacuum). However, in the 1960s it was theoretically predicted \cite{1} and experimentally confirmed \cite{2, 3} that certain media become electrically polarized when placed into a magnetic field or are magnetized when put into an electric field. Since then, such a magnetoelectric effect was observed for many substances and studied in great detail both theoretically and experimentally, see for the reviews \cite{4, 5}, for example. The constitutive relation of a medium at rest is then modified to \( D = \varepsilon \varepsilon_0 E + \beta \cdot B \) and \( H = (\mu \mu_0)^{-1} B - \beta^T \cdot E \), where the traceless \( 3 \times 3 \) matrix \( \beta \) describes the magnetoelectric properties (\( ^T \) denotes the transposed matrix). The magnetoelectric effect can be observed in a certain class of media, but it also can be induced in media which are put into external electric and magnetic fields.

Recently, attention to magnetoelectric media was attracted in connection with an interesting new effect predicted by Feigel \cite{6}. He noticed that the propagation of electromagnetic waves in a magnetoelectric medium is essentially asymmetric in the sense that the waves moving in opposite directions carry different momenta. Then, calculating the total momentum of the virtual waves (or “vacuum fluctuations” of the electromagnetic field) which are present in the sample, he concluded that this quantity is nontrivial. In this sense, the Feigel effect predicts the extraction of momentum from vacuum. This possibility was discussed in \cite{7, 8, 9, 10, 11, 12}. In particular, in \cite{12} a certain similarity of the Feigel effect with the Casimir effect was noticed, and the computation of the total momentum transferred by the virtual waves was attacked by means of the Green’s function method. In this paper, we analyse the Feigel effect in a sample of finite size which was somehow neglected in the previous works.

It seems clear that the discussion of the Feigel effect is related to the definition of the
energy and momentum of the electromagnetic field in a medium, and moreover, in a moving medium (since a nontrivial velocity of a sample is predicted [6]). However, as it is well known, the issue of the energy and the momentum of the electromagnetic field (and of waves, in particular) in dielectric and magnetic media has a long and controversial history. The discussion of this issue began with the investigations of Minkowski [13], Abraham [14], and Einstein and Laub [15]. The problem is reviewed in [16, 17, 18, 19], and most recently, in [20]. However, till now the problem was not settled neither theoretically, nor experimentally (see a discussion in [20, 21, 22, 23], e.g.). In the recent paper [24] we have proposed a consistent definition of the electromagnetic energy-momentum in media, and furthermore, a variational approach was developed in [25] for the moving magnetoelectric medium specifically for the study of the Feigel effect. An interesting technical observation is then that the magnetoelectric matrix $\beta$ induces a term in the energy-momentum tensor which describes an additional flux of energy and momentum. Such an additional term is present even in the medium at rest, and it vanishes when the magnetoelectric properties are absent. This fact lends some support to the feasibility of the Feigel effect, at least on a qualitative level.

Here, however, we will analyse the possibility of the Feigel effect avoiding the problem of the electromagnetic energy-momentum in the medium. The crucial point is the assumption of a finite size of the magnetoelectric sample. Then we notice that the virtual waves, which are excited in the medium, are not confined to the sample. Since the boundaries are not impenetrable, the vacuum fluctuations exist everywhere, inside the medium as well as outside of it. Since the virtual electromagnetic waves in these regions of space are related by the jump conditions across the boundaries, we can eventually replace the evaluation of the electromagnetic momentum and forces inside the medium by the computation of these quantities in free space. In vacuum the energy-momentum is uniquely defined, and this simplifies the analysis of the possible Feigel effect to a considerable extent.

The structure of the paper is as follows. Sec. II presents some general material, introduces the basic notions and fixes the notation. In Sec. III we analyse the constitutive relation of the magnetoelectric medium. In particular, we provide an irreducible decomposition of the magnetoelectric matrix, which proves to be convenient for the subsequent theoretical analysis. Sec. IV demonstrates a general feature of the wave propagation in magnetoelectric media, namely the birefringence. We demonstrate that for a medium characterized by the second irreducible piece, the generic Fresnel covector surface factorizes into a product of
two light cones. We find the two corresponding optical metrics. In Sec. V the propagation of plane electromagnetic waves is studied in a magnetoelectric medium of the Feigel type. From the jump conditions on the boundaries between the vacuum regions and the medium, we derive relations between the transmitted and the reflected waves. These relations are then used in Sec. VI for the calculation of the electromagnetic energy, the momentum and the force acting on the magnetoelectric sample.

II. SOME BACKGROUND MATERIAL

If the 4-dimensional (4D) electromagnetic excitation 2-form is denoted by

$$H = -\mathcal{H} \wedge d\sigma + \mathcal{D} = \frac{1}{2} H_{ij} dx^i \wedge dx^j$$  \hspace{1cm} (2.1)

and the 4D electromagnetic field strength 2-form by

$$F = E \wedge d\sigma + B = \frac{1}{2} F_{ij} dx^i \wedge dx^j,$$  \hspace{1cm} (2.2)

then the Maxwell equations read

$$dH = J , \quad dF = 0.$$  \hspace{1cm} (2.3)

Here $\sigma$ is a time parameter and $J = -j \wedge d\sigma + \rho$ the 4D electric current. The 3D magnetic excitation $\mathcal{H} = \mathcal{H}_a dx^a$ is a 1-form, the electric excitation $\mathcal{D} = \frac{1}{2} D_{ab} dx^a \wedge dx^b$ a 2-form. Analogously, we have the 3D electric field strength $E = E_a dx^a$ and the 3D magnetic field strength $B = \frac{1}{2} B_{ab} dx^a \wedge dx^b$. We also work with the vector densities $\mathcal{D}^a = \frac{1}{2} \epsilon^{abc} D_{ab}$ and $B^a = \frac{1}{2} \epsilon^{abc} B_{ab}$.

We substitute (2.1) and (2.2) into (2.3), we find the 1+3 dimensional form of the Maxwell equations:

$$dD = \rho , \quad d\mathcal{H} - \dot{\mathcal{D}} = j ;$$  \hspace{1cm} (2.4)

$$dB = 0 , \quad dE + \dot{B} = 0.$$  \hspace{1cm} (2.5)

The 3D exterior derivative is denoted by $d$ and the time derivative by a dot. This is the premetric (i.e., metric independent) framework of electrodynamics which summarizes the Maxwell equations and their physical interpretation.
The properties of the medium under consideration are expressed by means of the constitutive law. With the assumptions of locality and linearity, we have

\[ H_{ij} = \frac{1}{2} \kappa_{ij}^{kl} F_{kl} = \frac{1}{4} \hat{\epsilon}_{ijmn} \chi^{mnkl} F_{kl}. \]  

(2.6)

It is convenient to introduce, besides the constitutive tensor density \( \kappa_{ij}^{kl} \), the tensor \( \chi^{ijkl} \), since the latter is used conventionally in electrodynamics, see [26]. Its symmetries are \( \chi^{ijkl} = -\chi^{jikl} = -\chi^{ijlk} \), i.e., it has 36 independent components.

The constitutive law can be put into different forms in order to customize it for different applications. If we put it into a \( 6 \times 6 \) form

\[
\begin{pmatrix}
\mathcal{H}_a \\
\mathcal{D}^a
\end{pmatrix} = \begin{pmatrix}
\mathcal{C}^b_a & \mathcal{B}_{ba} \\
\mathcal{A}^{ba} & \mathcal{D}_b^a
\end{pmatrix} \begin{pmatrix}
-E_b \\
B^b
\end{pmatrix},
\]

(2.7)

then the \( 3 \times 3 \) matrices \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \) can be related to the 4-dimensional constitutive tensor density \( \chi^{ijkl} \) by

\[
\mathcal{A}^{ba} = \chi^{0a0b}, \quad \mathcal{B}_{ba} = \frac{1}{4} \hat{\epsilon}_{acd} \hat{\epsilon}_{bef} \chi^{cdef}, \\
\mathcal{C}^a_b = \frac{1}{2} \hat{\epsilon}_{bcd} \chi^{cd0a}, \quad \mathcal{D}^b_a = \frac{1}{2} \hat{\epsilon}_{acd} \chi^{0bcd}.
\]

(2.8)

(2.9)

Then \( 6 \times 6 \) form of \( \chi^{ijkl} \) can be written as

\[
\chi^{IK} = \begin{pmatrix}
\mathcal{B}_{ab} & \mathcal{D}_a^b \\
\mathcal{C}^a_b & \mathcal{A}^{ab}
\end{pmatrix},
\]

(2.10)

with \( I, K, ... = 01, 02, 03, 23, 31, 12. \)

We can decompose \( \chi^{ijkl} \) irreducibly under the linear group. As we have shown elsewhere [28], we find

\[
\chi^{ijkl} \sim \begin{pmatrix}
(1) & (1) \\
(2) & (2) \\
(3) & (3)
\end{pmatrix} \chi^{ijkl}.
\]

(2.11)

The irreducible pieces carry the additional symmetries

\[
(1) \chi^{ijkl} = (1) \chi^{klij}, \quad (2) \chi^{ijkl} = -(2) \chi^{klij}, \quad (3) \chi^{ijkl} = (3) \chi^{[ijkl]}.
\]

(2.12)

The principal part with its 20 independent components is the only one discussed conventionally. The skewon part with its 15 components vanishes if one assumes the existence of a
Lagrangian 4-form from which the constitutive law can be derived completely. Finally, the axion piece with only 1 independent component is totally antisymmetric:

\[ \chi^{ijkl} = \alpha \epsilon^{ijkl}. \] (2.13)

The \( \alpha \) is a 4D pseudoscalar.

We take care of the irreducible decomposition and evaluate the matrix elements of (2.7):

\[ \mathcal{H}_a = (\mu_{ab}^{-1} - \hat{\epsilon}_{abc} m^c) B^b + (\beta^b_a + s_a^b - \delta_a^b s_c^c) E_b - \alpha E_a, \] (2.14)

\[ \mathcal{D}^a = (\varepsilon^{ab} - \epsilon^{abc} n_c) E_b + (\beta^a_b + s_a^b - \delta_a^b s_c^c) B^b + \alpha B^a. \] (2.15)

We have \( \varepsilon^{ab} = \varepsilon^{ba}, \mu_{ab}^{-1} = \mu_{ba}^{-1}, \) and \( \beta^c_c = 0. \) Thus we have the independent components of \( \varepsilon^{ab} (6), \mu_{ab}^{-1} (6), \beta^a_b (8), m^c (3), n_c (3), s_a^b (9), \) and \( \alpha (1). \) This adds up, as it is required, to 36.

In this paper, we assume that there exists a Lagrangian from which the constitutive law can be derived. Thus, \( (2) \chi^{ijkl} = 0. \) Moreover, we assume a vanishing axion part \( \alpha = 0. \) Still, in certain substances an axion part can be present, as we discussed recently, see [34, 35]. After these “amputations”, the constitutive law to be investigated reads

\[ \mathcal{H}_a = \mu_{ab}^{-1} B^b - \beta^b_a E_b, \] (2.16)

\[ \mathcal{D}^a = \varepsilon^{ab} E_b + \beta^a_b B^b. \] (2.17)

Recall that \( \beta^c_c = 0. \)

III. CONSTITUTIVE RELATION FOR MAGNETOELECTRIC MEDIA

Throughout the paper we will use the exterior calculus which proves to be very effective and convenient both in describing the general formalism and in the specific computations. Here we put the constitutive relation for the magnetoelectric medium into a simple and transparent form by using the language of exterior calculus.

As is worked out in [28], magnetoelectric properties of the medium are described by a tracefree \( 3 \times 3 \) matrix \( \beta^a_b \) in the rest frame. The corresponding spacetime foliation is called the laboratory foliation, with the coordinate time variable \( \sigma \) labeling the slices of this foliation. The spacetime metric \( g \) introduces a scalar product in the tangent space and defines the line element which reads \( (a, b, ... = 1, 2, 3) \)

\[ ds^2 = N^2 d\sigma^2 + g_{ab} \frac{dx^a}{dx^b} = N^2 d\sigma^2 - (3) g_{ab} \frac{dx^a}{dx^b}. \] (3.1)
Here $N^2 = g(n, n)$ is the length square of the foliation vector field $n$, and $dx^a = dx^a - n^a \, d\sigma$ is the transversal 3-covector basis, in accordance with the definitions above. The 3-metric $(3)^{g_{ab}}$ is the positive definite Riemannian metric on the spatial 3-dimensional slices corresponding to fixed values of the time $\sigma$. This metric defines a 3-dimensional Hodge duality operator $\star$. Modern discussion of the classical electrodynamics, in particular, using the exterior calculus, can be found in [26, 27, 28, 29, 30].

Following Feigel, we do not consider the effects of gravity. Accordingly, we are in the Minkowski spacetime and a convenient choice of the laboratory foliation is $\sigma = t$ and $n^a = 0$ (hence $dx^a = dx^a$). The spatial metric is Euclidean, $(3)^{g_{ab}} = \text{diag}(1, 1, 1)$, and $N = c$.

We introduce the 1-form $\beta^a = \beta^a_b dx^b$. Then the constitutive relation for the magneto-electric medium, provided we assume isotropic permittivity and permeability, reads

\begin{align*}
\mathcal{D} &= \varepsilon \varepsilon_0 \star E - \beta^a \wedge e_a \wedge B, \quad (3.2) \\
\mathcal{H} &= \frac{1}{\mu \mu_0} \star B - \beta^a \wedge e_a \wedge E. \quad (3.3)
\end{align*}

We can decompose the tracefree $\beta^a$ into two irreducible parts (symmetric, and antisymmetric):

$$\beta^a = (1)^{\beta^a} + (2)^{\beta^a} = (1)^{\beta^a} + \star (\tilde{\beta} \wedge dx^a).$$

(3.4)

The antisymmetric (pseudotrace) part is defined by $(2)^{\beta^a} := \star (\tilde{\beta} \wedge dx^a)$ with a 1-form $\tilde{\beta} := \frac{1}{3}(dx_a \wedge \beta^a)$. Obviously, $e_a (2)^{\beta^a} = 0$. Finally, the first irreducible part is trace- and pseudotrace-free, i.e., $e_a (1)^{\beta^a} = 0$ and $dx_a \wedge (1)^{\beta^a} = 0$.

The first term in (3.4) describes the tracefree symmetric part of the matrix $\beta^a_b$, whereas the second term is its antisymmetric part. When the first irreducible piece vanishes, the constitutive relation (3.2)-(3.3) becomes much simpler:

\begin{align*}
\mathcal{D} &= \varepsilon \varepsilon_0 \star E + \star B \wedge \tilde{\beta}, \quad (3.5) \\
\mathcal{H} &= \frac{1}{\mu \mu_0} \star B + \star (E \wedge \tilde{\beta}). \quad (3.6)
\end{align*}

In this paper, we will mainly analyse the general case (3.2)-(3.3), with special attention to the medium studied by Feigel. The latter is characterized by the magnetoelectric matrix

$$\beta^a_b = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \beta^2_3 \\
0 & \beta^3_2 & 0
\end{pmatrix}. \quad (3.7)$$
Such magnetoelectric susceptibilities can be induced in an ordinary medium (characterized by the permittivity $\varepsilon$ and permeability $\mu$) under the action of external electric and magnetic fields applied along the second and third axes. In spite of the simple form of (3.7), there are still two nontrivial irreducible parts. When $\beta_{23}^2 = \beta_{32}^3$, the pseudotrace $\tilde{\beta}$ vanishes, whereas for $\beta_{23}^2 = -\beta_{32}^3$ the first irreducible piece disappears.

Our subsequent analysis reveals that different irreducible parts of the magnetoelectric matrix are differently involved into the possible Feigel effect.

The magnetoelectric matrix has the dimension $[\beta_{ab}^a] = [\sqrt{\varepsilon_0/\mu_0}]$. Accordingly, it is convenient to introduce a dimensionless object $\tilde{\beta}_b^a := \beta_{ab}^a/\lambda$ where $\lambda = \sqrt{\varepsilon \varepsilon_0/\mu \mu_0}$. We will use the same “overlined” notation also for the various exterior forms constructed from the magnetoelectric matrix.

IV. BIREFRINGENCE IN MAGNETOELECTRIC MEDIA

The Fresnel approach (geometric optics) to the wave propagation in media and in space-time with the general linear constitutive law (2.6) gives rise to the extended covariant Fresnel equation [28] for the wave covector $q_i$:

$$G^{ijkl}(\chi) q_i q_j q_k q_l = 0. \quad (4.1)$$

Here the fourth order Tamm-Rubilar (TR) tensor density of weight +1 is constructed from the constitutive tensor $\chi^{ijkl}$ as a cubic contraction with the Levi-Civita densities:

$$G^{ijkl}(\chi) := \frac{1}{4!} \varepsilon_{mnps} \varepsilon_{rstu} \chi^{mnr(i} \chi^{j|ps|k} \chi^{l)tu}. \quad (4.2)$$
Using the \((1 + 3)\)-decomposed representation in terms of the \(3 \times 3\) matrices \((2.10)\), the independent components of the TR-tensor \((4.2)\) read explicitly as follows:

\[
M := G^{0000} = \det \mathcal{A}, \quad (4.3)
\]

\[
M^a := 4 \, G^{000a} = - \varepsilon b_{cd} \left( A^{ba} A^{ce} D^d_e + A^{ab} A^{ce} D^d_e \right), \quad (4.4)
\]

\[
M^{ab} := 6 \, G^{00ab} = \frac{1}{2} A^{(ab)} \left[ (C^d_e)^2 + (D^c_d)^2 - (C^c_d + D^c_d)(C^d_e + D^d_e) \right] + (C^d_e + D^c_d)(A^{c(a} C^{b)}_d + D^d_e A^{(a} C^{b)}_c) - C^d_e A^{c(a} C^{b)}_c
\]

\[- D^c_e (a A^{b)} D^d_d - \mathcal{A}^{dc} C^{(a} D^{b)}_d + (A^{ab} A^{de} - \mathcal{A}^{d(a} A^{b)}_c) B_{dc}, \quad (4.5)\]

\[
M^{abc} := 4 \, G^{abc} = \varepsilon^{de(c)} \left[ B_{df}(A^{ab} D^e_f - D^e_a A^{b)}_f \right) + B_{fd}(A^{ab} C^e_f - \mathcal{A}^{f(a} C^{e)}_c) + C^a_f D^e_b D^f_d + D^f_a C^{b)}_c c^f_d \right], \quad (4.6)\]

\[
M^{abcd} := G^{abcd} = \varepsilon^{ef(c|gh|d} B_{hbf} \left[ \frac{1}{2} A^{ab} B_{ge} - C^a_e \mathcal{A}^{b)}_g \right]. \quad (4.7)\]

Then, in \((1 + 3)\)-decomposed form, the extended Fresnel equation \((4.1)\) reads

\[
q^a_0 M + q^3_0 q_a M^a + q^2_0 q_a q_b M^{ab} + q_0 q_a q_b q_c M^{abc} + q_a q_b q_c q_d M^{abcd} = 0. \quad (4.8)\]

In Minkowski spacetime, for the general linear constitutive relation \((3.2)-(3.3)\) we have explicitly

\[
A^{ab} = - \varepsilon_0 \delta^{ab}, \quad B_{ab} = \frac{1}{\mu \mu_0} \delta_{ab}, \quad C^a_b = D^a_b = \beta^a_b. \quad (4.9)\]

In the special case \((3.5)-(3.6)\) when only the irreducible antisymmetric (pseudotrace) part of the magnetoelectric matrix is nontrivial, we analysis of the Fresnel equation \((4.8)\) reveals the clear birefringence effect. Indeed, using the constitutive relation \((3.5)-(3.6)\) in \((4.3)-(4.7)\), we find explicitly (with \(n = \sqrt{\varepsilon_0 \mu}\) as the usual index of refraction, and \(c = 1/\sqrt{\varepsilon_0 \mu_0}\) the vacuum speed of light):

\[
M = - \left( \frac{n}{c} \right)^3, \quad M^a = 4 \left( \frac{n}{c} \right)^2 \beta^a, \quad (4.10)\]

\[
M^{ab} = \frac{n}{c} \left[ \delta^{ab}(2 + \beta^2) - 5 \beta^a \beta^b \right], \quad (4.11)\]

\[
M^{abc} = - 2 \delta^{(ab}\beta^c) (2 + \beta^2) + 2 \beta^a \beta^b \beta^c, \quad (4.12)\]

\[
M^{abcd} = \frac{c}{n} \left[ - \delta^{(ab} \delta^{c)} (1 + \beta^2) + \delta^{(ab} \beta^c \beta^d \right]. \quad (4.13)\]

Here from the magnetoelectric 1-form \(\mathcal{A}\) we extract the dimensionless covector with the components \(\beta_a := e_a / \beta^a / \lambda\). Further, we denote \(\beta^a := \delta^{ab} \beta_b\) and \(\beta^2 := \beta^a \beta_a\).
Substituting this into the Fresnel equation (4.1), (4.8), we find birefringence: the quartic wave surface is factorized into the product of the two light cones,

\[ G^{ijkl}(\chi) q_i q_j q_k q_l = - (g_1^{ij} q_i q_j) (g_2^{kl} q_k q_l) = 0. \] (4.14)

The two optical metrics \[ g_1^{ij} \] depend explicitly on the magneto-electric properties according to

\[ g_1^{ij} = \begin{pmatrix} \frac{n^2}{c^2} & -\frac{n}{c} \beta^b \\ \frac{n}{c} \beta^a & -\delta^{ab} \end{pmatrix} \] (4.15)

and

\[ g_2^{ij} = \begin{pmatrix} \frac{n^2}{c^2} & -\frac{n}{c} \beta^b \\ \frac{n}{c} \beta^a & -\delta^{ab}(1 + \beta^2) + \beta^a \beta^b \end{pmatrix}, \] (4.16)

respectively. It is interesting that the magneto-electric covector manifests itself as an effective “rotation” of the spacetime related to the off-diagonal components of the optical metric.

In the general linear case of the constitutive relation (3.2)-(3.3), the quartic Fresnel surface does not factorize into the product of the two light cones, in general. The birefringence effect is much more nontrivial in this case. We, however, will eventually specialize to the medium (3.7) studied by Feigel, and in this case we can analyse the propagation of the planes waves to the end.

V. PLANE WAVES IN A MAGNETOELECTRIC MEDIUM

A general discussion of the propagation of electromagnetic waves in media with a local and linear constitutive relation can be found in [4, 28, 32, 33]. Various special cases of the constitutive tensor were analysed, including also the magneto-electric cases. Here we will confine our attention to the specific form of the magneto-electric matrix (3.7).

In order to clarify the possible Feigel effect, we need to analyse not only the waves inside the sample (as was done originally in [6]) but also the waves in the outside vacuum space. The appropriate qualitative picture is as follows (see Fig. 1): Let us put the magneto-electric matter between the two parallel planes \( S_1 = \{ x = -\ell \} \) and \( S_2 = \{ x = +\ell \} \). Feigel considers the case when the magneto-electric properties are induced by external electric and magnetic fields. When these external crossed electric and magnetic fields are applied parallelly to the boundaries \( S_1 \) and \( S_2 \), the magneto-electric matrix will have the form (3.7). The idea of
FIG. 1: An infinitely extended (in $y$ and $z$ directions) slab of a magnetoelectric medium is located between the plane boundaries $S_1$ and $S_2$. We study the propagation of electromagnetic waves in $x$-direction inside and outside the magnetoelectric medium.

Feigel is that the virtual electromagnetic waves (vacuum fluctuations of the electromagnetic field) propagating in such a magnetoelectric medium could produce a nontrivial momentum along the distinguished axis $x$. However, the presence of similar virtual waves outside of the medium was not taken into account (despite the fact that the magnetoelectric sample was assumed to have finite size). Here we reanalyse carefully the picture, taking into account all
TABLE I: Amplitudes in the different regions and direction of the corresponding wave. The polarization index 1 and 2 describes a wave with the electric field in y- and in z-direction, respectively.

| region 1 | region 3 | region 2 |
|----------|----------|----------|
| $a_1, a_2$ | $p_1, p_2$ | $c_1, c_2$ |
| $\rightarrow$ | $\rightarrow$ | $\rightarrow$ |
| $b_1, b_2$ | $q_1, q_2$ | $d_1, d_2$ |
| $\leftarrow$ | $\leftarrow$ | $\leftarrow$ |

the virtual electromagnetic waves inside and outside of the medium alike.

We begin by noticing that outside of the matter, we have the “bath” of the virtual photons some of which will penetrate the interior of the sample, reflecting and refracting at its boundaries. Obviously, the largest contribution to the possible effect should come from the electromagnetic waves which travel along the $x$-axis, i.e., with the wave vectors normal to the boundaries. Clearly, for each right-moving wave falling on the left boundary $S_1$, there exists an equal but opposite left-moving wave falling on the right boundary $S_2$. The contributions of these *ingoing* waves to the momentum density of the electromagnetic field are equal with opposite sign, thus providing a *balance* of the light pressures in the left and in the right vacuum regions. However, we have to find the contributions of the *outgoing* waves. If they turn out to be different in the left and in the right vacuum regions, this would seemingly yield a violation of the momentum balance and would encompass a nontrivial Feigel effect.

There are three regions (see Fig. 1 and Table I): 1) the left vacuum space (for $x < -\ell$), 2) the right vacuum space (for $x > \ell$), and 3) the interior region filled with magnetoelectric matter (for $-\ell < x < \ell$). The configurations of the electromagnetic field in these three domains read, respectively, as follows, where we use the complex representation to simplify the formulas:
1) In the first region \((x < -\ell)\):

\[
E = e^{-i\omega t} \left[ (a_1 e^{ikx} + b_1 e^{-ikx}) dy + (a_2 e^{ikx} + b_2 e^{-ikx}) dz \right],
\]

\[
B = \frac{k}{\omega} e^{-i\omega t} dx \land \left[ (a_1 e^{ikx} - b_1 e^{-ikx}) dy + (a_2 e^{ikx} - b_2 e^{-ikx}) dz \right],
\]

\[
\mathcal{D} = \varepsilon_0 e^{-i\omega t} dx \land \left[ -(a_1 e^{ikx} + b_1 e^{-ikx}) dz + (a_2 e^{ikx} + b_2 e^{-ikx}) dy \right],
\]

\[
\mathcal{H} = \frac{k}{\mu_0 \omega} e^{-i\omega t} \left[ (a_1 e^{ikx} - b_1 e^{-ikx}) dz - (a_2 e^{ikx} - b_2 e^{-ikx}) dy \right].
\]

Here the complex amplitudes \(a_1\) and \(a_2\) describe the right-moving waves with the two different polarizations (the subscript 1 refers to the first polarization with the electric field along the \(y\)-axis, whereas the subscript 2 denotes a second independent polarization with the electric field along the \(z\)-axis). The complex amplitudes \(b_1\) and \(b_2\) describe the corresponding left-moving waves. With \(k = \omega/c\) one can straightforwardly check that this configuration is a solution of the Maxwell equations.

2) Similarly, in the second region \((x > \ell)\):

\[
E = e^{-i\omega t} \left[ (c_1 e^{ikx} + d_1 e^{-ikx}) dy + (c_2 e^{ikx} + d_2 e^{-ikx}) dz \right],
\]

\[
B = \frac{k}{\omega} e^{-i\omega t} dx \land \left[ (c_1 e^{ikx} - d_1 e^{-ikx}) dy + (c_2 e^{ikx} - d_2 e^{-ikx}) dz \right],
\]

\[
\mathcal{D} = \varepsilon_0 e^{-i\omega t} dx \land \left[ -(c_1 e^{ikx} + d_1 e^{-ikx}) dz + (c_2 e^{ikx} + d_2 e^{-ikx}) dy \right],
\]

\[
\mathcal{H} = \frac{k}{\mu_0 \omega} e^{-i\omega t} \left[ (c_1 e^{ikx} - d_1 e^{-ikx}) dz - (c_2 e^{ikx} - d_2 e^{-ikx}) dy \right].
\]

Now the amplitudes \(c_1, c_2\) and \(d_1, d_2\) describe the right- and the left-moving waves, respectively, with the two polarizations, and again \(k = \omega/c\).

3) In the third (interior) region (with \(-\ell < x < \ell\)), the field configurations look somewhat more nontrivial:

\[
E = e^{-i\omega t} \left[ \left( p_1 e^{ik_1^+x} + q_1 e^{-ik_1^-x} \right) dy + \left( p_2 e^{ik_2^+x} + q_2 e^{-ik_2^-x} \right) dz \right],
\]

\[
B = \frac{e^{-i\omega t}}{\omega} dx \land \left[ \left( k_1^+ p_1 e^{ik_1^+x} - k_1^- q_1 e^{-ik_1^-x} \right) dy + \left( k_2^+ p_2 e^{ik_2^+x} - k_2^- q_2 e^{-ik_2^-x} \right) dz \right],
\]

\[
\mathcal{D} = e^{-i\omega t} dx \land \left[ -\left( \varepsilon \varepsilon_0 + \frac{k_1^+}{\omega} \beta_3 \right) p_1 e^{ik_1^+x} dz + \left( \varepsilon \varepsilon_0 - \frac{k_1^-}{\omega} \beta_3 \right) p_2 e^{ik_2^+x} dy 
\]

\[
- \left( \varepsilon \varepsilon_0 - \frac{k_1^-}{\omega} \beta_3 \right) q_1 e^{-ik_1^-x} dz + \left( \varepsilon \varepsilon_0 + \frac{k_1^+}{\omega} \beta_3 \right) q_2 e^{-ik_2^-x} dy \right],
\]

\[
\mathcal{H} = e^{-i\omega t} \left[ \left( \frac{k_1^+}{\mu_0 \omega} - \beta_3 \right) p_1 e^{ik_1^+x} dz - \left( \frac{k_2^+}{\mu_0 \omega} + \beta_3 \right) p_2 e^{ik_2^+x} dy 
\]

\[
- \left( \frac{k_1^-}{\mu_0 \omega} + \beta_3 \right) q_1 e^{-ik_1^-x} dz + \left( \frac{k_2^-}{\mu_0 \omega} - \beta_3 \right) q_2 e^{-ik_2^-x} dy \right].
\]
The matter is characterized by the permittivity $\varepsilon$, the permeability $\mu$, and the magneto-electric matrix $\beta_{ab}$ (the latter is given by 3.7). The right- and the left-movers are now described by complex amplitudes $p_1, p_2, q_1, q_2$, respectively. The birefringence, however, is manifest in the inequality of $k_1^\pm \neq k_2^\pm$: the waves with different polarization have different propagation vectors. Explicitly, we find

$$k_1^\pm = \frac{n\omega}{c} \left( \sqrt{(\beta_{23}^2)^2 + 1 \pm \beta_{3}^2} \right),$$
$$k_2^\pm = \frac{n\omega}{c} \left( \sqrt{(\beta_{23}^2)^2 + 1 \mp \beta_{3}^2} \right),$$

As before, we use here the dimensionless magnetoelectric variable $\beta_{ab} = \beta_{ab}/\lambda$.

The twelve amplitude coefficients $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, p_1, p_2, q_1, q_2$ are not arbitrary but are related among themselves via the jump conditions for the electromagnetic field strength and the excitations at the boundaries $S_1$ and $S_2$. There are, as usual, twelve jump conditions – six for every boundary surface. They read [28]:

$$(D_1 - D_3)|_{S_1} \land \nu = 0, \quad \tau_A (H_1 - H_3)|_{S_1} = 0,$$  
$$(B_1 - B_3)|_{S_1} \land \nu = 0, \quad \tau_A (E_1 - E_3)|_{S_1} = 0,$$  
$$(D_3 - D_2)|_{S_2} \land \nu = 0, \quad \tau_A (H_3 - H_2)|_{S_2} = 0,$$  
$$(B_3 - B_2)|_{S_2} \land \nu = 0, \quad \tau_A (E_3 - E_2)|_{S_2} = 0.$$  

Here $\nu = dx$ is the 1-form density normal to the surfaces and $\tau_1 = \partial_y, \tau_2 = \partial_z$ ($A = 1, 2$) are the two vectors tangential to the boundaries.

Substituting (5.1)–(5.12), we find that some of the jump conditions are trivially satisfied since $B \land \nu = 0$ and $D \land \nu = 0$ in all the three regions. Accordingly, we are left with only eight conditions which result from the continuity of $\tau_A |H$ and $\tau_A |E$ at the two boundaries. After some algebra, noting in particular that

$$\varepsilon\varepsilon_0 \pm \frac{k_1^\pm}{\mu_0} \beta_{3}^2 = \alpha_1 \sqrt{\varepsilon_0 \mu_0 \frac{k_1^\pm}{\omega}}, \quad \varepsilon\varepsilon_0 \pm \frac{k_2^\pm}{\mu_0} \beta_{2}^3 = \alpha_2 \sqrt{\varepsilon_0 \mu_0 \frac{k_2^\pm}{\omega}},$$

with the abbreviations

$$\alpha_1 := \sqrt{\frac{\varepsilon}{\mu} [1 + (\beta_{23}^2)^2]}, \quad \alpha_2 := \sqrt{\frac{\varepsilon}{\mu} [1 + (\beta_{22}^3)^2]},$$

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we can bring this system into the form of the _eight_ equations:

\[
e^{-ik\ell}a_1 - e^{ik\ell}b_1 - \alpha_1 e^{-ik_1^+\ell}p_1 + \alpha_1 e^{ik_1^-\ell}q_1 = 0,
\]

\[
e^{-ik\ell}a_1 + e^{ik\ell}b_1 - e^{-ik_1^+\ell}p_1 - e^{ik_1^-\ell}q_1 = 0,
\]

\[
e^{ik\ell}c_1 - e^{-ik\ell}d_1 - \alpha_1 e^{-ik_1^+\ell}p_1 + \alpha_1 e^{-ik_1^-\ell}q_1 = 0,
\]

\[
e^{ik\ell}c_1 + e^{-ik\ell}d_1 - e^{ik_1^+\ell}p_1 - e^{-ik_1^-\ell}q_1 = 0,
\]

\[
e^{-ik\ell}a_2 - e^{ik\ell}b_2 - \alpha_2 e^{-ik_2^+\ell}p_2 + \alpha_2 e^{ik_2^-\ell}q_2 = 0,
\]

\[
e^{-ik\ell}a_2 + e^{ik\ell}b_2 - e^{-ik_2^+\ell}p_2 - e^{ik_2^-\ell}q_2 = 0,
\]

\[
e^{ik\ell}c_2 - e^{-ik\ell}d_2 - \alpha_2 e^{-ik_2^+\ell}p_2 + \alpha_2 e^{-ik_2^-\ell}q_2 = 0,
\]

\[
e^{ik\ell}c_2 + e^{-ik\ell}d_2 - e^{ik_2^+\ell}p_2 - e^{-ik_2^-\ell}q_2 = 0.
\]

Thus, we can always choose 2 waves (inside or outside) the medium as independent and find all the other waves in all three regions of space from the system \((5.22)-(5.29)\). Obviously, the waves with a particular polarization are only related to the waves of the same polarization. They do not mix with the modes of a different polarization.

The following three cases exhaust all possible situations: (i) we choose as primary the waves inside the medium, i.e., the amplitudes \(p_A, q_A\) (with \(A = 1, 2\)) are independent, and the amplitudes \(a_A, b_A\) and \(c_A, d_A\) outside the matter are obtained from them as secondary, (ii) the waves in one vacuum region (for example, in the first one) are primary, then \(a_A, b_A\) are independent and the waves in the matter \(p_A, q_A\) and in the second vacuum region \(c_A, d_A\) are derived from them, (iii) the ingoing waves in the vacuum regions, \(a_A, d_A\), are independent, then the waves in the matter \(p_A, q_A\) and the outgoing waves \(b_A, c_A\) are derived. The system \((5.22)-(5.29)\) can be straightforwardly solved for all these cases.

(i) Assuming \(p_A, q_A\) to be the independent variables, see Table II we find from \((5.22)-(5.29)\) the amplitudes of the waves in the two vacuum regions:

\[
a_A = \frac{e^{ik\ell}}{2} \left[ (1 + \alpha_A)e^{-ik\lambda^+\ell}p_A + (1 - \alpha_A)e^{ik\lambda^-\ell}q_A \right], \tag{5.30}
\]

\[
b_A = \frac{e^{-ik\ell}}{2} \left[ (1 - \alpha_A)e^{-ik\lambda^+\ell}p_A + (1 + \alpha_A)e^{ik\lambda^-\ell}q_A \right], \tag{5.31}
\]

\[
c_A = \frac{e^{-ik\ell}}{2} \left[ (1 + \alpha_A)e^{ik\lambda^+\ell}p_A + (1 - \alpha_A)e^{-ik\lambda^-\ell}q_A \right], \tag{5.32}
\]

\[
d_A = \frac{e^{ik\ell}}{2} \left[ (1 - \alpha_A)e^{ik\lambda^+\ell}p_A + (1 + \alpha_A)e^{-ik\lambda^-\ell}q_A \right]. \tag{5.33}
\]
TABLE II: Case 1. Given are the amplitudes in region 3, the rest is computed

| region 1 | region 3 | region 2 |
|----------|----------|----------|
| compute  | $p_1, p_2$ | compute |
| $→$      | $→$      | $→$      |
| compute  | $q_1, q_2$ | compute |
| $←$      | $←$      | $←$      |

TABLE III: Case 2. Given are the amplitudes in region 1. The rest is computed. There is also an equivalent case if only the amplitudes in region 2 are specified

| region 1 | region 3 | region 2 |
|----------|----------|----------|
| $a_1, a_2$ | compute | compute |
| $→$      | $→$      | $→$      |
| $b_1, b_2$ | compute | compute |
| $←$      | $←$      | $←$      |

(ii) Assume now that the waves in the left vacuum region are primary, see Table III. Then $a_A, b_A$ are independent variables, and we find for the amplitudes in the medium and

TABLE IV: Case 3: The amplitudes of the ingoing waves (with respect to region 3) are specified, the rest is computed

| region 1 | region 3 | region 2 |
|----------|----------|----------|
| $a_1, a_2$ | compute | compute |
| $→$      | $→$      | $→$      |
| compute  | compute  | $d_1, d_2$ |
| $←$      | $←$      | $←$      |
in the second vacuum region:

\[ p_A = \frac{e^{ik_A^+\ell}}{2\alpha_A} \left[ (1 + \alpha_A)e^{-ik\ell}a_A - (1 - \alpha_A)e^{ik\ell}b_A \right], \]  

(5.34)

\[ q_A = \frac{e^{-ik_A^+\ell}}{2\alpha_A} \left[ -(1 - \alpha_A)e^{-ik\ell}a_A + (1 + \alpha_A)e^{ik\ell}b_A \right], \]  

(5.35)

\[ c_A = \frac{e^{i(k_A^+ - k_A^-)\ell}}{2\alpha_A} \left[ K_Ae^{-2ik\ell}a_A - i(1 - \alpha_A^2)\sin(k_A^+ + k_A^-)\ell b_A \right], \]  

(5.36)

\[ d_A = \frac{e^{i(k_A^+ - k_A^-)\ell}}{2\alpha_A} \left[ i(1 - \alpha_A^2)\sin(k_A^+ + k_A^-)\ell e^{-2ik\ell}a_A + K_A^*b_A \right]. \]  

(5.37)

Here we denoted

\[ K_A := 2\alpha_A \cos(k_A^+ + k_A^-)\ell + i(1 + \alpha_A^2)\sin(k_A^+ + k_A^-)\ell, \]  

(5.38)

\[ \Delta_A := K_A K_A^* = 4\alpha_A^2 + (1 - \alpha_A^2)^2 \sin^2(k_A^+ + k_A^-)\ell. \]  

(5.39)

The second quantity will be needed below. The star denotes complex conjugation as usual.

(iii) Finally, if we assume, in accordance with Table II, that the ingoing waves in the two vacuum regions, namely \( a_A, d_A \), are independent, the amplitudes of the waves in matter \( p_A, q_A \) and of the outgoing waves \( b_A, c_A \) read:

\[ p_A = \frac{K_Ae^{-ik\ell}}{\Delta_A} \left[ (1 + \alpha_A)e^{-ik\ell}a_A - (1 - \alpha_A)e^{ik\ell}d_A \right], \]  

(5.40)

\[ q_A = \frac{K_Ae^{-ik\ell}}{\Delta_A} \left[ (1 + \alpha_A)e^{-ik\ell}a_A - (1 - \alpha_A)e^{ik\ell}d_A \right], \]  

(5.41)

\[ b_A = \frac{K_Ae^{-2ik\ell}}{\Delta_A} \left[ -i(1 - \alpha_A^2)\sin(k_A^+ + k_A^-)\ell a_A + 2\alpha_A e^{i(k_A^+ - k_A^-)\ell}d_A \right], \]  

(5.42)

\[ c_A = \frac{K_Ae^{i(k_A^+ - k_A^-)\ell}}{\Delta_A} \left[ 2\alpha_A e^{-i(k_A^+ - k_A^-)\ell}a_A - i(1 - \alpha_A^2)\sin(k_A^+ + k_A^-)\ell d_A \right]. \]  

(5.43)

Both quantities (5.38) and (5.39) depend on the sums

\[ k_A^+ + k_A^- = \frac{2n\omega}{c} \sqrt{\beta_A^2 + 1}, \]  

(5.44)

with \( \beta_1 = \beta_3^2, \beta_2 = -\beta_2^3 \). In practice, the magnetoelectric parameters are rather small (typically of order of \( 10^{-4} - 10^{-6} \)), so with very high accuracy, we have \( k_A^+ + k_A^- = \frac{2n\omega}{c} = 2k \).

However, since the difference

\[ k_A^+ - k_A^- = \frac{2n\omega}{c} \beta_A \]  

(5.45)

for the magnetoelectric matter is nontrivial, the amplitudes of the left-moving waves in matter are in general distinct from that of the right-moving waves. Similarly, the amplitudes
of the outgoing waves in the two vacuum regions are different, in general. We have to check now if such a difference can be manifest in different field momentum densities and forces in the two vacuum regions.

VI. ENERGY, MOMENTUM, AND FORCES

In vacuum, the energy-momentum 3-form of the electromagnetic field reads \[ \Sigma_{\alpha} = \frac{1}{2} [F \wedge (e_{\alpha} H) - (e_{\alpha} F) \wedge H]. \] (6.1)

Using the (1 + 3)-decomposition, with \( F = E \wedge dt + B \) and \( H = -\mathcal{H} \wedge dt + \mathcal{D} \), we find explicitly for the temporal and spatial parts:

\[
\Sigma_0 = u - dt \wedge s, \\
\Sigma_a = -p_a - dt \wedge S_a.
\] (6.2, 6.3)

As we see, the energy-momentum generically decomposes into the four pieces: The energy density 3-form

\[
u := \frac{1}{2} (E \wedge \mathcal{D} + B \wedge \mathcal{H}),
\] (6.4)

the energy flux density (or Poynting) 2-form

\[
s := E \wedge \mathcal{H},
\] (6.5)

the momentum density 3-form

\[
p_a := -B \wedge (e_a \mathcal{D}),
\] (6.6)

and the Maxwell stress (or momentum flux density) 2-form of the electromagnetic field

\[
S_a := \frac{1}{2} [(e_a] E) \wedge \mathcal{D} - (e_a] \mathcal{D} \wedge E \\
+ (e_a] \mathcal{H} \wedge B - (e_a] B) \wedge \mathcal{H}].
\] (6.7)

We will evaluate these expressions for the plane wave configurations discussed in the previous section. Only the mean averaged (over a time period) quantities have a direct physical meaning. Using (5.1)-(5.4), we then find for the averaged quantities in the first
region:

\[ \langle u \rangle = \frac{\varepsilon_0}{2} \sum_{A=1}^{2} (|a_A|^2 + |b_A|^2) \, dx \wedge dy \wedge dz, \quad (6.8) \]

\[ \langle s \rangle = \frac{1}{2\mu_0 c} \sum_{A=1}^{2} (|a_A|^2 - |b_A|^2) \, dy \wedge dz, \quad (6.9) \]

\[ \langle p_a \rangle = \frac{\varepsilon_0}{2c} \sum_{A=1}^{2} \begin{pmatrix} (|a_A|^2 - |b_A|^2) \, dx \wedge dy \wedge dz \\ 0 \\ 0 \end{pmatrix}, \quad (6.10) \]

\[ \langle S_a \rangle = -\frac{\varepsilon_0}{2} \sum_{A=1}^{2} \begin{pmatrix} (|a_A|^2 + |b_A|^2) \, dy \wedge dz \\ 0 \\ 0 \end{pmatrix}. \quad (6.11) \]

The form of corresponding quantities in the second region is the same, with the amplitudes \( a_A \) replaced with \( c_A \) and \( b_A \) replaced with \( d_A \).

Consequently, it remains to calculate the moduli of the amplitudes of the incoming and outgoing waves, and compare the resulting quantities in the first and in the second regions. We begin with the case (i) which directly corresponds to the computations of Feigel. We take the virtual waves in the medium as primary fields, assuming the equal amplitudes \( p_A = q_A \) for the left- and right-movers, and then find from (5.30)-(5.33) the waves in the two vacuum regions:

\[ |a_A|^2 = |b_A|^2 = |c_A|^2 = |d_A|^2 = \frac{|p_A|^2}{2} \left[ 1 + \alpha_A^2 + (1 - \alpha_A^2) \cos(k_A^+ + k_A^-)\ell \right]. \quad (6.12) \]

Accordingly, we find that the total momentum and the total energy flux are vanishing in both vacuum regions, see (6.9) and (6.10), whereas the energy density and the stress densities are the same, see (6.8) and (6.11). Recalling that the force acting on the boundary can be calculated as an integral \( F_x = -\int <S_x> \), we then conclude that the resulting force acting on the medium is zero, since on the left boundary the normal vector points in the negative \( x \)-direction, and on the right boundary in the positive one.

The same conclusion is derived when we analyse the case (ii) with the waves of the equal amplitude travelling in the first or in the second vacuum regions. When we evaluate the combined effect of their contributions, we again find the zero resulting force acting on the medium.
Somewhat different is the case (iii) when the ingoing waves (the right-moving $a_A$ in the first (left) region and the left-moving wave in the second region with the same amplitude $d_A = a_A$) are considered as primary. Then for the outgoing waves we find from (5.42)-(5.43)

$$|b_A|^2 = |a_A|^2 \left[ 1 + \frac{4\alpha_A}{\Delta_A} (1 - \alpha_A^2) \sin(k_A^+ + k_A^-) \ell \sin(k_A^+ - k_A^-) \ell \right], \quad (6.13)$$

$$|c_A|^2 = |a_A|^2 \left[ 1 - \frac{4\alpha_A}{\Delta_A} (1 - \alpha_A^2) \sin(k_A^+ + k_A^-) \ell \sin(k_A^+ - k_A^-) \ell \right]. \quad (6.14)$$

As we can see, the contributions of the outgoing waves to the field momentum are clearly different in the two vacuum regions. The difference reads explicitly

$$|b_A|^2 - |c_A|^2 = 8 |a_A|^2 \frac{\alpha_A}{\Delta_A} (1 - \alpha_A^2) \sin(k_A^+ + k_A^-) \ell \sin(k_A^+ - k_A^-) \ell. \quad (6.15)$$

When $\beta_{ab} = 0$, in view of (5.45) the “bath” of virtual waves around the sample is in equilibrium since then (6.15) vanishes. Under this assumption, the total momentum of the waves in both vacuum regions is obviously equal to zero. However, for magnetoelectric matter, the mentioned “bath” is still balanced in the sense that the field momentum carried by the waves in the left vacuum region is the same as that of the waves in the right vacuum region. Namely, substituting (6.13) and (6.14) into (6.10), we find that the momentum densities of the electromagnetic field in both regions are equal

$$< p_x > = - \sum_{A=1}^2 \frac{2\varepsilon_0 |a_A|^2 \alpha_A}{c \Delta_A} \frac{1 - \alpha_A^2}{\Delta_A} \sin(k_A^+ + k_A^-) \ell \sin(k_A^+ - k_A^-) \ell \ dx \wedge dy \wedge dz. \quad (6.16)$$

At the same time, the stress is different in the two regions. When we compute the corresponding force $F_x = - \int < S_x >$, acting on the boundary of the sample, the resulting expressions will read for the first (left) and for the second (right) surfaces, respectively:

$$F_x^{\text{left}} = \sum_{A=1}^2 \varepsilon_0 |a_A|^2 A \left[ 1 + \frac{2\alpha_A}{\Delta_A} (1 - \alpha_A^2) \sin(k_A^+ + k_A^-) \ell \sin(k_A^+ - k_A^-) \ell \right], \quad (6.17)$$

$$F_x^{\text{right}} = - \sum_{A=1}^2 \varepsilon_0 |a_A|^2 A \left[ 1 - \frac{2\alpha_A}{\Delta_A} (1 - \alpha_A^2) \sin(k_A^+ + k_A^-) \ell \sin(k_A^+ - k_A^-) \ell \right]. \quad (6.18)$$

Here $A$ is the area of the boundary surface (we assume the left and right surfaces to be equal). Thus, there will be a nontrivial resulting force acting on the sample in the direction of the magnetoelectric vector:

$$F_x^{\text{left}} + F_x^{\text{right}} = \sum_{A=1}^2 \frac{4\varepsilon_0 |a_A|^2 A \alpha_A}{\Delta_A} (1 - \alpha_A^2) \sin(k_A^+ + k_A^-) \ell \sin(k_A^+ - k_A^-) \ell. \quad (6.19)$$
At first sight, the results obtained, namely (6.16) and (6.19), provide a theoretical support for the possible Feigel effect. For completeness, however, it is necessary to analyse also the situation when the directions of all waves are reversed, i.e., instead of assuming equal incoming waves, we should also consider the case of equal outgoing waves. Fortunately, it is not necessary to perform a new computation. All we need is to put $k \rightarrow -k$ in (5.1)-(5.12) and then note that in the jump equations (5.22)-(5.29) we have to change the sign of $\alpha_A \rightarrow -\alpha_A$. Then repeating the computations of the amplitudes $p_A, q_A, b_A, c_A$, we arrive again to the solution (5.40)-(5.43) with the replacements $k \rightarrow -k$ and $\alpha_A \rightarrow -\alpha_A$. As a result, the total momentum density turns out to be again (6.16). [It is important to note that here we do not have to replace $\alpha_A \rightarrow -\alpha_A$, since $(|a_A|^2 - |b_A|^2)$ is changed to $(|b_A|^2 - |a_A|^2)$ in (6.10), and analogously, $(|c_A|^2 - |d_A|^2)$ is changed to $(|d_A|^2 - |c_A|^2)$ in the similar formula in the right region]. However, the resulting force computed from the stress on the left and right boundary surfaces will have the opposite sign (and equal magnitude) to that of (6.19). Correspondingly, when we consider both contributions together, the total force will be found to be equal to zero. In other words, in contrast to Feigel’s result, the magnetoelectric body will not move, despite the presence of a certain asymmetry between the left- and right-moving waves in the matter.

Several remarks are in order. The above conclusions are based on the evident symmetry that characterizes the “bath” of the virtual photons (“vacuum fluctuations”) in the regions 1 and 2: for each left-moving virtual photon there is an equal right-moving virtual photon. Furthermore, our observations demonstrate the difference between the two irreducible parts of the magnetoelectric matrix $\beta_{ab}$. Namely, assuming that only first (symmetric) irreducible part is nontrivial, we have $\beta^3_3 = \beta^3_2$, and then the waves of the different polarization contribute to the above formulas with terms of the opposite sign since then $\sin(k_1^+ - k_1^-) \ell = -\sin(k_2^+ - k_2^-) \ell$. The natural assumption that the virtual waves of any polarization are produced with equal probability then leads to the conclusion that the net result will be zero due to the mutual compensation of the contributions of the waves of different polarizations. In other words, we find that the first irreducible part of the magnetoelectric matrix is irrelevant for the possible Feigel effect. However, for the second (antisymmetric) part the situation is different. Then $\beta^2_3 = -\beta^2_2$, and the waves of both polarizations now contribute with terms of the same sign. This was also observed in the original computation with the result obtained there which is proportional to the skew-symmetric combination
\( \beta^2_{3} - \beta^3_{2} \). This shows that the only the second (antisymmetric or pseudotrace) irreducible part of the magnetoelectric matrix (i.e., the 1-form \( \tilde{\beta} \)) is responsible for the possible Feigel effect.

Finally, we can make some predictions going beyond the scheme of the virtual waves of the original Feigel picture. Let us assume that real electromagnetic waves are directed on the magnetoelectric sample from the two sides. This can be easily achieved if we split the initial beam into the two beams which are then, with a simple system of mirrors, directed on the two opposite sides of the magnetoelectric sample. Then we are exactly in the situation described as the case (iii) above. The outcome which we derived in this case is that the sample will be affected by a nontrivial force \( (6.19) \) from the electromagnetic fields falling on it. The sign and the magnitude of this force is determined by the polarization of the infalling waves and by the magnetoelectric parameters \( \beta^2_{3}, \beta^3_{2} \) of the medium. This effect can be directly verified experimentally. Moreover, one can use this real Feigel effect for the measurement of the magnetoelectric parameters \( \beta^2_{3}, \beta^3_{2} \). This method is drastically different and much simpler in its practical realization than the usual measurements of the magnetoelectric parameters from the observation of magnetization of a medium in an external electric field (resp., electric polarization of a medium in an external magnetic field).

VII. DISCUSSION AND CONCLUSION

In this paper we analysed the propagation of electromagnetic waves in a magnetoelectric medium. Our aim was to use this analysis for the discussion of the possibility of the Feigel effect [6] that was predicted recently.

We derived a natural decomposition of the magnetoelectric matrix \( \beta^a_{\ b} \) into the two irreducible parts: the first (tracefree symmetric), and the second (antisymmetric, or pseudotrace). The magnetoelectric matrix is tracefree. However, the axion also describes the magnetoelectric effect, see (2.14) and (2.15). Being a 4D pseudoscalar, the axion can be formally treated as a trace of generalized magnetoelectric \( 3 \times 3 \) matrix. We discussed the physical properties of the axion piece elsewhere [34, 35] and here we assume (like in [6] and in the previous work [25]) that the axion piece is absent. Ultimately we find that only the second part (i.e., the magnetoelectric pseudotrace or antisymmetric part) can be responsible for the possible Feigel effect. Physically this result appears to be quite natural because
the magnetoelectric pseudotrace 1-form $\hat{\beta}$ selects a distinguished direction in space. Along this direction, we then subsequently find the nontrivial forces and electromagnetic momenta acting on the magnetoelectric medium.

We show that taking into account the finite size of a magnetoelectric sample, one can reduce the problem of computation of the electromagnetic energy, the momentum and the forces to the vacuum regions just outside the sample. We then demonstrate that the net force on the sample turns out to be zero for the careful count of the contributions from all the virtual electromagnetic waves excited in the medium.

At the same time, we discover that for real electromagnetic waves, that fall along the direction of the pseudotrace $\hat{\beta}$ on a sample from the two opposite sides, is nontrivial. Accordingly, it would make the magnetoelectric sample move along $\hat{\beta}$, indeed, as was predicted in the original work \[6\]. We propose to use this observation for the measurement of the magnetoelectric susceptibilities. Such an method could provide a useful alternative scheme for the measurements of the magnetoelectric parameters, along with the traditional methods based on the measurements of polarization induced in an external magnetic field (or of magnetization induced by an external electric field).

In the current paper, we have confined our attention (like also Feigel in \[6\]) to the waves induced along $\hat{\beta}$. Strictly speaking, the complete analysis requires also to take into account the waves that move along other directions. The extension of the above results to such waves is straightforward, although the formulas (especially those that relate the waves in the media and in vacuum at the boundary surfaces) turn out to be appreciable longer. The corresponding analysis, however, shows that the main contribution to the possible Feigel effect comes from the waves moving along $\hat{\beta}$, so the additional modes do not change the conclusions qualitatively.

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[36] Latin indices are coordinate indices. In 4D, they run over $i, j, \ldots = 0, 1, 2, 3$ and in 3D over $a, b, \ldots = 1, 2, 3$. The totally antisymmetric Levi-Civita symbol in 4D is denoted by $\hat{\epsilon}^{ijkl} = 0, 1, -1$ and in 3D by $\hat{\epsilon}_{abc} = 0, 1, -1$. 

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