Generating All Partitions: A Comparison Of Two Encodings

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Abstract

Integer partitions may be encoded as either ascending or descending compositions for the purposes of systematic generation. Many algorithms exist to generate all descending compositions, yet none have previously been published to generate all ascending compositions. We develop three new algorithms to generate all ascending compositions and compare these with descending composition generators from the literature. We analyse the new algorithms and provide new and more precise analyses for the descending composition generators. In each case, the ascending composition generation algorithm is substantially more efficient than its descending composition counterpart. We develop a new formula for the partition function \( p(n) \) as part of our analysis of the lexicographic succession rule for ascending compositions.

1 Introduction

A partition of a positive integer \( n \) is an unordered collection of positive integers whose sum is \( n \). Partitions have been the subject of extensive study for many years and the theory of partitions is a large and diverse body of knowledge. Partitions are a fundamental mathematical concept and have connections with number theory [4], elliptic modular functions [48, p.224], Schur algebras and representation theory [33, p.13], derivatives [61], symmetric groups [10, 7], Gaussian polynomials [6, ch.7] and much else [2]. The theory of partitions also has many and varied applications [12, 41, 30, 59, 18, 1, 57].

Combinatorial generation algorithms allow us to systematically traverse all possibilities in some combinatorial universe, and have been the subject of sustained interest for many years [28]. Many algorithms are known to generate fundamental combinatorial objects; for example, in 1977 Sedgewick reviewed...
more than thirty permutation generation algorithms [49]. Many different orders have been proposed for generating combinatorial objects, the most common being lexicographic [23] and minimal-change order [46]. The choice of encoding, the representation of the objects we are interested in as simpler structures, is of critical importance to the efficiency of combinatorial generation.

In this paper we demonstrate that by changing the encoding for partitions from \textit{descending} compositions to \textit{ascending} compositions we obtain significantly more efficient generation algorithms. We develop three new algorithms under the most common generation idioms: recursion (Section 2), succession rules (Section 3), and efficient sequential generation (Section 4). In each case we rigorously analyse the new algorithm and use this analysis to compare with a commensurable algorithm from the literature, for which we provide a new and more precise analysis. These analyses are performed using a novel application of Kemp's abstraction of counting read and write operations [23] and this approach is validated in an empirical study (Section 4.3). In all three cases the new ascending composition generation algorithm is substantially more efficient than the algorithm from the literature. As part of our study of partition generation algorithms we provide a new proof of a partition identity in Section 4.1.1. We also develop a new formula for the partition function \( p(n) \), one of the most important functions in the theory of partitions [5], in Section 3.4.

1.1 Related Work

A composition of a positive integer \( n \) is an expression of \( n \) as an ordered sum of positive integers [52, p.14], and a composition \( a_1 + \cdots + a_k = n \) can be represented by the sequence \( a_1 \ldots a_k \). Since there is a unique way of expressing each partition of \( n \) as composition of \( n \) in either ascending or descending order\(^1\), we can generate either the set of ascending or descending compositions of \( n \) in order to obtain the set of partitions. More precisely, we can say that we are encoding partitions as either ascending or descending compositions for the purposes of systematic generation.

Although partitions are fundamentally unordered they have come to be defined in more and more concrete terms as descending compositions. This trend can be clearly seen in the works of Sylvester [56], MacMahon [32, Vol.II p.91] and finally Andrews [4, p.1]. Sylvester’s “constructive theory of partitions”, based on the idea of treating a partition as “definite thing” [56] (in contrast to Euler’s algebraical identities [20]), has been extremely successful [39]. As a result of this, partitions are now often defined as descending compositions [4, p.1]; thus, algorithms to generate all partitions have naturally followed the prevailing definition and generated descending compositions.

It is widely accepted that the most efficient means of generating descending compositions is in reverse lexicographic order: see Andrews [4, p.230], Knuth [27, p.1], Nijenhuis & Wilf [36, p.65–68], Page & Wilson [38, §5.5], Skiena [50, p.52],

\(^1\)For our purposes the terms ‘ascending’ and ‘descending’ are synonymous with ‘nondecreasing’ and ‘nonincreasing’, respectively.
Several different representations (concrete data structures) have been used for generating descending compositions: namely the sequence [34], multiplicity [36, ch.9] and part-count [55] representations. Although the lexicographic succession rules for descending compositions in the multiplicity or part-count representations can be implemented looplessly [14], they tend to be less efficient that their sequence representation counterparts [27, ex.5]. In an empirical analysis, Zoghbi & Stojmenović [62] demonstrated that their sequence representation algorithms are significantly more efficient than all known multiplicity and part-count representation algorithms.

Algorithms to generate descending compositions in lexicographic order have also been published. See Knuth [25, p.147] and Zoghbi & Stojmenović [62] for implementations using the sequence representation; Reingold, Nievergelt & Deo [42, p.193] and Fenner & Loizou [16] for implementations using the multiplicity representation; and Klimko [24] for an implementation using the part-count representation. Fenner & Loizou’s tree construction operations [15] can be used to generate descending compositions in several other orders.

Several algorithms are known to generate descending \( k \)-compositions in lexicographic [19, 58], reverse lexicographic [43], and minimal-change [45] order. Hindenburg’s eighteenth century algorithm [13, p.106] generates ascending \( k \)-compositions in lexicographic order and is regarded as the canonical method to generate partitions into a fixed number of parts: see Knuth [27, p.2], Andrews [4, p.232] or Reingold, Nievergelt & Deo [42, p.191]. Algorithms due to Stockmal [54], Lehmer [31, p.26], Narayana, Mathsen & Sarangi [35], and Boyer [9] also generate ascending \( k \)-compositions in lexicographic order. Algorithms to generate all ascending compositions, however, have not been considered.

1.2 Notation

In general we use the notation and conventions advocated by Knuth [26, p.1], using the term visit to refer to the process of making a complete object available to some consuming procedure. Thus, any combinatorial generation algorithm must visit every element in the combinatorial universe in question exactly once. In discussing the efficiency of combinatorial generation, we say that an algorithm is constant amortised time [44, §1.7] if the average amount of time required to generate an object is bounded, from above, by some constant.

Ordinarily, we denote a sequence of integers as \( a_1 \ldots a_k \), which denotes a sequence of \( k \) integers indexed \( a_1, a_2, \) etc. When referring to short specific sequences it is convenient to enclose each element using \( ( \) and \( ) \). Thus, if we let \( a_1 \ldots a_k = \langle 3 \rangle \langle 23 \rangle \), we have \( k = 2, a_1 = 3 \) and \( a_2 = 23 \). We will also use the idea of prepending a particular value to the head of a sequence: thus, the notation \( 3 \cdot \langle 23 \rangle \) is the same sequence as given in the preceding example.

**Definition 1.1.** A sequence of positive integers \( a_1 \ldots a_k \) is an ascending composition of the positive integer \( n \) if \( a_1 + \cdots + a_k = n \) and \( a_1 \leq \cdots \leq a_k \).
Definition 1.2. Let \( \mathcal{A}(n) \) be the set of all ascending compositions of \( n \) for some \( n \geq 1 \), and let \( \mathcal{A}(n, m) \subseteq \mathcal{A}(n) \) be defined for \( 1 \leq m \leq n \) as \( \mathcal{A}(n, m) = \{ a_1 \ldots a_k \mid a_1 \ldots a_k \in \mathcal{A}(n) \text{ and } a_1 \geq m \} \). Also, let \( A(n) = |\mathcal{A}(n)| \) and \( A(n, m) = |\mathcal{A}(n, m)| \).

Definition 1.3. A sequence of positive integers \( d_1 \ldots d_k \) is a descending composition of the positive integer \( n \) if \( d_1 + \cdots + d_k = n \) and \( d_1 \geq \cdots \geq d_k \).

Definition 1.4. Let \( \mathcal{D}(n) \) be the set of all descending compositions of \( n \) for some \( n \geq 1 \), and let \( \mathcal{D}^*(n, m) \subseteq \mathcal{D}(n) \) be defined for \( 1 \leq m \leq n \) as \( \mathcal{D}^*(n, m) = \{ d_1 \ldots d_k \mid d_1 \ldots d_k \in \mathcal{D}(n) \text{ and } d_1 = m \} \). Also, let \( D(n) = |\mathcal{D}(n)| \) and \( D^*(n) = |\mathcal{D}^*(n, m)| \).

There is an asymmetry between the function used to enumerate the ascending compositions and the descending compositions: \( \mathcal{A}(n, m) \) counts the ascending compositions of \( n \) where the first part is at least \( m \), whereas \( \mathcal{D}^*(n, m) \) counts the number of descending compositions of \( n \) where the first part is exactly \( m \). This asymmetry is necessary as we require \( \mathcal{A}(n, m) \) in our analysis of ascending composition generation algorithms and \( \mathcal{D}^*(n, m) \) is essential for the analysis of the recursive descending composition generation algorithm of Section 2.2.

2 Recursive Algorithms

In this section we examine recursive algorithms to generate ascending and descending compositions. Recursion is a popular technique in combinatorial generation as it leads to elegant and concise generation procedures [44]. In Section 2.1 we develop and analyse a simple constant amortised time recursive algorithm to generate all ascending compositions of \( n \). Then, in Section 2.2 we study Ruskey’s descending composition generator [44, §4.8], and provide a new analysis of this algorithm. We compare these algorithms in Section 2.3 in terms of the total number of recursive invocations required to generate all \( p(n) \) partitions of \( n \).

2.1 Ascending Compositions

The only recursive algorithm to generate all ascending compositions available in the literature is de Moivre’s method [11]. In de Moivre’s method we generate the ascending compositions of \( n \) by prepending \( m \) to the (previously listed) ascending compositions of \( n - m \), for \( m = 1, \ldots, n \) [28, p.20]. Our new recursive algorithm to generate all ascending compositions of \( n \) operates on a similar principle, but does not require us to have large sets of partitions in memory.

We first note that we can generate all ascending compositions of \( n \), with smallest part at least \( m \), by prepending \( m \) to all ascending compositions of \( n - m \). We then observe that \( m \) can range from 1 to \( \lfloor n/2 \rfloor \), since the smallest part in a partition of \( n \) (with more than one part) cannot be less than 1 or greater than \( \lfloor n/2 \rfloor \); and we complete the process by visiting the singleton composition

\[ \{ n \} \]

This requires \( (n-1) \) recursive invocations to generate all partitions of \( n \) with at least \( m \) parts.

\[ A(n, m) = \{ a_1 \ldots a_k \mid a_1 \ldots a_k \in \mathcal{A}(n) \text{ and } a_1 \geq m \} \]

\[ A(n, m) = |\mathcal{A}(n, m)| \]

\[ D^*(n, m) = \{ d_1 \ldots d_k \mid d_1 \ldots d_k \in \mathcal{D}(n) \text{ and } d_1 = m \} \]

\[ D^*(n, m) = |\mathcal{D}^*(n, m)| \]

\[ p(n) \]

\[ \{ n \} \]

\[ (n-1) \]

\[ A(n, m) \]

\[ A(n, m) = \{ a_1 \ldots a_k \mid a_1 \ldots a_k \in \mathcal{A}(n) \text{ and } a_1 \geq m \} \]

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\[ D^*(n, m) = |\mathcal{D}^*(n, m)| \]

\[ p(n) \]

\[ \{ n \} \]

\[ (n-1) \]

\[ A(n, m) \]

\[ A(n, m) = \{ a_1 \ldots a_k \mid a_1 \ldots a_k \in \mathcal{A}(n) \text{ and } a_1 \geq m \} \]

\[ A(n, m) = |\mathcal{A}(n, m)| \]

\[ D^*(n, m) = \{ d_1 \ldots d_k \mid d_1 \ldots d_k \in \mathcal{D}(n) \text{ and } d_1 = m \} \]

\[ D^*(n, m) = |\mathcal{D}^*(n, m)| \]

\[ p(n) \]

\[ \{ n \} \]

\[ (n-1) \]

\[ A(n, m) \]

\[ A(n, m) = \{ a_1 \ldots a_k \mid a_1 \ldots a_k \in \mathcal{A}(n) \text{ and } a_1 \geq m \} \]

\[ A(n, m) = |\mathcal{A}(n, m)| \]

\[ D^*(n, m) = \{ d_1 \ldots d_k \mid d_1 \ldots d_k \in \mathcal{D}(n) \text{ and } d_1 = m \} \]

\[ D^*(n, m) = |\mathcal{D}^*(n, m)| \]

\[ p(n) \]

\[ \{ n \} \]

\[ (n-1) \]

\[ A(n, m) \]

\[ A(n, m) = \{ a_1 \ldots a_k \mid a_1 \ldots a_k \in \mathcal{A}(n) \text{ and } a_1 \geq m \} \]

\[ A(n, m) = |\mathcal{A}(n, m)| \]

\[ D^*(n, m) = \{ d_1 \ldots d_k \mid d_1 \ldots d_k \in \mathcal{D}(n) \text{ and } d_1 = m \} \]

\[ D^*(n, m) = |\mathcal{D}^*(n, m)| \]

\[ p(n) \]

\[ \{ n \} \]

\[ (n-1) \]

\[ A(n, m) \]
Algorithm 2.1 RecAsc($n, m, k$)

\begin{algorithmic}[1]
  \Require $1 \leq m \leq n$
  \State $x \leftarrow m$
  \While{$2x \leq n$}
  \State $a_k \leftarrow x$
  \State RecAsc($n - x, x, k + 1$)
  \State $x \leftarrow x + 1$
  \EndWhile
  \State $a_k \leftarrow n$
  \State \textbf{end while}$\langle n \rangle$. This provides sufficient information for us to derive a recursive generation procedure, Algorithm 2.1, in the idiom of Page & Wilson [38]. This algorithm generates all ascending compositions of $n$ where the first part is at least $m$ in lexicographic order. See Kelleher [22, §5.2.1] for a complete discussion and proof of correctness of Algorithm 2.1.

Following the standard practise for the analysis of recursive generation algorithms, we count the number of recursive calls required to generate the set of combinatorial objects in question (e.g. Sawada [47]). By counting the total number of recursive invocations required, we obtain a bound on the total time required, as each invocation, discounting the time spent in recursive calls, requires constant time. To establish that Algorithm 2.1 generates the set $A(n)$ in constant amortised time we must count the total number of invocations, $I_{A2.1}(n)$, and show that this value is proportional to $p(n)$.

**Theorem 2.1.** For all positive integers $n$, $I_{A2.1}(n) = p(n)$.

**Proof.** Each invocation of Algorithm 2.1 visits exactly one composition (line 8). The invocation RecAsc($n, m, 1$) correctly visits all $p(n)$ ascending compositions of $n$ [22, p.78] and it immediately follows, therefore, that there must be $p(n)$ invocations. Hence, $I_{A2.1}(n) = p(n)$.

Theorem 2.1 gives us an asymptotic measure of the total computational effort required to generate all partitions of $n$ using Algorithm 2.1. It is also useful to know the average amount of effort that this total implies per partition. Therefore, we let $\bar{I}_{A2.1}(n)$ denote the average number of invocations of RecAsc required to generate an ascending composition of $n$. We then trivially get

$$\bar{I}_{A2.1}(n) = 1$$

from Theorem 2.1, and we can see that Algorithm 2.1 is obviously constant amortised time.

In this subsection we have developed a new algorithm to generate all ascending compositions of $n$. This algorithm, although concise and simple, can be easily shown to be constant amortised time. In the next subsection we examine the most efficient known algorithm to generate descending compositions,
### Algorithm 2.2 \texttt{RecDesc}(n, m, k)

**Require:** \(1 \leq m \leq n\) and \(d_j = 1\) for \(j > k\)

1: \(d_k \leftarrow m\)
2: if \(n = m\) or \(m = 1\) then
3: \(\text{visit } d_1 \ldots d_k+n-m\)
4: else
5: \(\text{for } x \leftarrow 1\) to \(\min(m, n - m)\) do
6: \(\text{RecDesc}(n - m, x, k + 1)\)
7: end for
8: \(d_k \leftarrow 1\)
9: end if

which we subsequently compare to the ascending composition generator of this subsection.

### 2.2 Descending Compositions

Two recursive algorithms are available to generate all descending compositions of \(n\): Page & Wilson’s [38, §5.5] generator (variants of which have appeared in several texts, including Kreher & Stinson [29, p.68], Skiena [50, p.51] and Pemmaraju & Skiena [40, p.136]) and Ruskey’s improvement thereof [44, §4.8]. Ruskey’s algorithm, given in Algorithm 2.2, generates all descending compositions of \(n\) in which the first (and largest) part is exactly \(m\); thus \texttt{RecDesc}(8, 4, 1) visits the compositions 41111, 4211, 422, 431, 44. \texttt{RecDesc} uses what Ruskey refers to as a ‘path elimination technique’ [44, §4.3] to attain constant amortised time performance.

A slight complication arises when we wish to use \texttt{RecDesc} to generate all descending compositions. As the algorithm generates all descending compositions where the first part is exactly \(m\), we must iterate through all \(j \in \{1, \ldots, n\}\) and invoke \texttt{RecDesc}(n, j, 1). Following Ruskey’s recommendations [44, §4.3], we consider instead the invocation \texttt{RecDesc}(2n, n, 1). This invocation will generate all descending compositions of \(2n\) where the first part is exactly \(n\); therefore the remaining parts will be a descending composition of \(n\). Thus, if we alter line 3 to ignore the first part in \(d\) (i.e. \texttt{visit } \(d_2 \ldots d_{k+n-m}\)), we will visit all descending compositions of \(n\) in lexicographic order.

Ruskey’s algorithm generates descending compositions where the largest part is exactly \(m\), and so we require a recurrence relation to count objects of this type. Ruskey [44, §4.8] provides a recurrence relation to compute \(D^*(n, m)\), which we shall use for our analysis. Thus, we define \(D^*(n, n) = D^*(n, 1) = 1\), and in general,

\[
D^*(n, m) = \min(m, n-m) \sum_{x=1}^{\min(m, n-m)} D^*(n-m, x).
\]

(2)

Recurrence (2) is useful here because it is the recurrence relation upon which \texttt{RecDesc} is based. Using this recurrence we can then easily count the number
of invocations of \texttt{RecDesc} required to generate the descending compositions of $n$. Let us define $I'_{A2,2}(n, m)$ as the number of invocation of \texttt{RecDesc} required to generate all descending compositions of $n$ where the first part is exactly $m$. Then, $I'_{A2,2}(n, n) = I'_{A2,2}(n, 1) = 1$, and

$$I'_{A2,2}(n, m) = 1 + \sum_{x=1}^{\min(m,n-m)} I'_{A2,2}(n - m, x). \tag{3}$$

Recurrence (3) computes the number of invocations of Algorithm 2.2 required to generate all descending compositions of $n$ with first part exactly $m$, but tells us little about the actual magnitude of this value. As a step towards solving this recurrence in terms of the partition function $p(n)$ we require the following lemma, in which we relate the $I'_{A2,2}(n, m)$ numbers to the $D^*(n, m)$ numbers.

\begin{lemma}
If $1 < m \leq n$ then $I'_{A2,2}(n, m) = D^*(n, m) + D^*(n - 1, m)$.
\end{lemma}

\textbf{Proof.} Proceed by strong induction on $n$.

\textbf{Base case:} $n = 2$ Suppose $1 < m \leq 2$; it follows immediately that $m = 2$. Thus, by recurrence (3) we compute $I'_{A2,2}(2, 2) = 1$ and by recurrence (2) compute $D^*(2, 2) = 1$ and $D^*(1, 2) = 0$. Therefore, $I'_{A2,2}(2, 2) = D^*(2, 2) + D^*(1, 2)$, and so the inductive basis holds.

\textbf{Induction step} Suppose, for some positive integer $n$, $I'_{A2,2}(n', m') = D^*(n', m') + D^*(n' - 1, m')$ for all positive integers $1 < m' \leq n' < n$. Then, suppose $m$ is an arbitrary positive integer such that $1 < m \leq n$. Now, suppose $m = n$. By (3) we know that $I'_{A2,2}(n, m) = 1$ since $m = m$. Also, $D^*(n, m) = 1$ as $m = n$, and $D^*(n - 1, m) = 0$ as $n - 1 \neq m$, $m \neq 1$ and $\min(m, n - m - 1) = -1$, ensuring that the sum in (2) is empty. Therefore, $I'_{A2,2}(n, m) = D^*(n, m) + D^*(n - 1, m)$.

Suppose, on the other hand, that $1 < m < n$. We can see immediately that $\min(m, n - m) \geq 1$, and so there must be at least one term in the sum of (3). Extracting this first term where $x = 1$ from (3) we get

$$I'_{A2,2}(n, m) = 1 + I'_{A2,2}(n - m, 1) + \sum_{x=2}^{\min(m,n-m)} I'_{A2,2}(n - m, x),$$

and furthermore, as $I'_{A2,2}(n, 1) = 1$, we obtain

$$I'_{A2,2}(n, m) = 2 + \sum_{x=2}^{\min(m,n-m)} I'_{A2,2}(n - m, x). \tag{4}$$

We are assured that $1 < x \leq n - m$ by the upper and lower bounds of the
summation in (4), and so we can apply the inductive hypothesis to get

\[
I_{A2.2}(n, m) = 2 + \sum_{x=2}^{\min(m, n-m)} (D^*(n - m, x) + D^*(n - m - 1, x)) \]

By the definition of \(D^*\) we know that \(D^*(n, 1) = 1\), and so \(D^*(n - m, 1) + D^*(n - m - 1, 1) = 2\). Replacing the leading 2 above with this expression, and inserting the terms \(D^*(n - m, 1)\) and \(D^*(n - m - 1, 1)\) into the appropriate summations we find that

\[
I_{A2.2}(n, m) = 2 + \sum_{x=2}^{\min(m, n-m)} D^*(n - m, x) + \min(m, n-m) \sum_{x=2}^{n-m} D^*(n - m - 1, x). \tag{5}
\]

By (2) we know that the first term of (5) is equal to the first term of \(I_{A2.2}(n, m) = D^*(n, m) + D^*(n - 1, m)\), it therefore remains to show that

\[
D^*(n - 1, m) = \sum_{x=1}^{\min(m, n-m)} D^*(n - m - 1, x),
\]

or equivalently, that

\[
\sum_{x=1}^{\min(m, n-m-1)} D^*(n - m - 1, x) = \sum_{x=1}^{\min(m, n-m)} D^*(n - m - 1, x). \tag{6}
\]

Suppose \(m \leq n - m - 1\). Then, \(\min(m, n - m - 1) = \min(m, n - m)\), and so the left and right-hand sides of (6) are equal. Suppose, alternatively, that \(m > n - m - 1\). Hence, \(\min(m, n - m - 1) = n - m - 1\) and \(\min(m, n - m) = n - m\) and so we get

\[
\sum_{x=1}^{n-m} D^*(n - m - 1, x) = \sum_{x=1}^{n-m-1} D^*(n - m - 1, x) + D^*(n - m - 1, n - m).
\]

Since \(n - m - 1 < n - m\) we know that \(D^*(n - m - 1, n - m) = 0\), and therefore (6) is verified.

Therefore, by (5) and (6) we know that \(I_{A2.2}(n, m) = D^*(n, m) + D^*(n - 1, 1)\), as required. \(\square\)

Lemma 2.1 is a crucial step in our analysis of Algorithm 2.2 as it relates the number of invocations required to generate a given set of descending compositions to the function \(D^*(n, m)\). Much is known about the \(D^*(n, m)\) numbers, as they count the partitions of \(n\) where the largest part is \(m\); thus, we can then relate the number of invocations required to the partition numbers, \(p(n)\). Therefore, let us formally define \(I_{A2.2}(n)\) to be number of invocations of Algorithm 2.2 required to generate all \(p(n)\) descending compositions of \(n\). We then get the following result.
Theorem 2.2. If \( n > 1 \) then \( I_{A2.2}(n) = p(n) + p(n - 1) \).

Proof. Suppose \( n > 1 \). To generate all descending compositions of \( n \) we invoke \( \text{RecDesc}(2n, n, 1) \) (see discussion above), and as \( n > 1 \) we can apply Lemma 2.1, to obtain \( I'_{A2.2}(2n, n) = D^*(2n, n) + D^*(2n - 1, n) \), and thus \( I_{A2.2}(n) = D^*(2n, n) + D^*(2n - 1, n) \). We know that \( D^*(2n, n) = p(n) \), as we can clearly obtain a descending composition of \( n \) from any descending composition of \( 2n \) where the first part is exactly \( n \) by removing that first part. Similarly, \( D^*(2n - 1, n) = p(n - 1) \), as we can remove the first part of size \( n \) from any descending composition of \( 2n - 1 \) with first part equal to \( n \), obtaining a descending composition of \( n - 1 \). Thus, the descending compositions counted by the functions \( D^*(2n, n) = p(n) \) and \( D^*(2n - 1, n) = p(n - 1) \). Hence, \( I_{A2.2}(n) = p(n) + p(n - 1) \), completing the proof. \( \square \)

Note that in Theorem 2.2, and in many of the following analyses, we restrict our attention to values \( n > 1 \). This is to avoid unnecessary complication of the relevant formulas in accounting for the case where \( n = 1 \). In the above, if we compute \( I_{A2.2}(n) = p(n) + p(n - 1) \) for \( n = 1 \), we arrive at the conclusion that the number of invocations required is 2, as \( p(0) = 1 \) by convention. In the interest of clarity we shall ignore such contingencies, as they do not affect the general conclusions we draw.

Using Theorem 2.2 it is now straightforward to show that \( \text{RecDesc} \) generates all descending compositions of \( n \) in constant amortised time. To show that the algorithm is constant amortised time we must demonstrate that the average number of invocations of the algorithm per object generated is bounded, from above, by some constant. To do this, let us formally define \( \bar{I}_{A2.2}(n) \) as the average number of invocations of \( \text{RecDesc} \) required to generate a descending composition of \( n \). Clearly, as the total number of invocations is \( I_{A2.2}(n) \) and the number of objects generated is \( p(n) \), we have \( \bar{I}_{A2.2}(n) = I_{A2.2}(n)/p(n) \).

Since \( I_{A2.2}(n) = p(n) + p(n - 1) \) by Theorem 2.2, we have \( \bar{I}_{A2.2}(n) = 1 + p(n)/p(n - 1) \). It is well known that \( p(n) > p(n - 1) \) for all \( n > 1 \), and therefore \( p(n - 1)/p(n) < 1 \). From this inequality we can then deduce that \( \bar{I}_{A2.2}(n) < 2 \), proving that Algorithm 2.2 is constant amortised time. It is useful to have a more precise asymptotic expression for the average number of invocations required to generate a descending composition using \( \text{RecDesc} \), \( \bar{I}_{A2.2}(n) \). By the asymptotic estimate for \( p(n - t)/p(n) \) [27, p.11] we then get \( \bar{I}_{A2.2}(n) = 1 + e^{-C/\sqrt{n}} \left( 1 + O(n^{-1/6}) \right) \), with \( C = \pi/\sqrt{6} \). Simplifying this expression we get

\[
\bar{I}_{A2.2}(n) = 1 + e^{-\pi/\sqrt{6n}} \left( 1 + O(n^{-1/6}) \right), \tag{7}
\]

In this subsection we have described and provided a new analysis for the most efficient known recursive descending composition generation algorithm, which is due to Ruskey [44, §4.8]. Ruskey demonstrates that \( \text{RecDesc} \) is constant amortised time by reasoning about the number of children each node in the computation tree has, but does not derive the precise number of invocations involved. In this section we rigorously counted the number of invocations required to generate all descending compositions of \( n \) using this algorithm, and
related the recurrence involved to the partition numbers. We then used an asymptotic formula for $p(n)$ to derive the number of invocations required to generate each partition, on average. In the next subsection we use this analysis to compare Ruskey’s descending composition generator with our new ascending composition generator.

## 2.3 Comparison

Performing the comparison between the recursive algorithms to generate all ascending compositions and to generate all descending compositions of $n$ is a simple procedure. **RECASC** requires $p(n)$ invocations to generate all $p(n)$ partitions of $n$ whereas **RecDesc** requires $p(n) + p(n - 1)$ invocations. The asymptotics of $p(n)$ show that, as $n$ becomes large, $p(n - 1)/p(n)$ approaches 1. Thus, we can reasonably expect the descending composition generator to require approximately twice as long as the ascending composition generator to generate all partitions of $n$.

In Table 1 we see a comparison of the actual time spent in generating partitions of $n$ using Ruskey’s algorithm, Algorithm 2.2, and our ascending composition generator, Algorithm 2.1. In this table we report the time spent by Algorithm 2.1 in generating all ascending compositions of $n$, divided by the time required by Ruskey’s algorithm (we report these ratios as the actual durations are of little interest). Several steps were taken in an effort to address Sedgewick’s concerns about the empirical comparisons of algorithms [49]. Direct and literal implementation of the algorithms concerned were written in the C and Java languages and compiled in the simplest possible manner (i.e., without the use of compiler ‘optimisations’). Execution times were measured as accurately as possible and the minimum value over five runs used. The C programs were compiled using GCC version 3.3.4 and the Java programs compiled and run on the Java 2 Standard Edition, version 1.4.2. All programs were executed on an Intel Pentium 4 processor running Linux kernel 2.6.8. See Kelleher [22, p.111–114] for a full discussion of the methodology adopted in making these observations.

The values of $n$ are selected such that $n$ is the smallest integer where $p(n) > 1 \times 10^6$ and $p(n) > 5 \times 10^6$ for $6 \leq x \leq 8$. Orders of magnitude larger than these values proved to be infeasible on the experimental platform; similarly, the time

### Table 1: A comparison of recursive partition generators. The ratio of the time required by our ascending composition generation algorithm and Ruskey’s algorithm in the Java and C languages is shown.

| n   | 61  | 72  | 77  | 90  | 95  | 109 |
|-----|-----|-----|-----|-----|-----|-----|
| $p(n)$ | 1.12×10^6 | 5.39×10^6 | 1.06×10^7 | 5.66×10^7 | 1.05×10^8 | 5.42×10^8 |
| Java | 0.56 | 0.56 | 0.56 | 0.55 | 0.55 | 0.55 |
| C    | 0.40 | 0.48 | 0.49 | 0.50 | 0.50 | 0.50 |
| Theoretical | 0.54 | 0.54 | 0.53 | 0.53 | 0.53 | 0.53 |
elapsed in generating fewer than a million partitions was too brief to measure accurately. Along with the observed ratios of the time required by RecAsc and RecDesc we also report the theoretically predicted ratio of the running times: \( p(n)/(p(n)+p(n-1)) \). We can see from Table 1 that these theoretically predicted ratios agree well with the empirical evidence. We can also see that as \( n \) becomes larger, Ruskey’s algorithm is tending towards taking twice as long as RecAsc to generate the same partitions.

3 Succession Rules

In this section we consider algorithms of the form studied by Kemp in his general treatment of the problem of generating combinatorial objects [23]. Kemp reduced the problem of generating combinatorial objects to the generation of all words in a formal language \( \mathcal{L} \), and developed powerful general techniques to analyse such algorithms. Specifically, Kemp studied “direct generation algorithms” that obey a simple two step procedure: (1) scan the current word from right-to-left until we find the end of the common prefix shared by the current word and its immediate successor; and (2) attach the new suffix to the end of this shared prefix. The cost of this process can be easily quantified by counting the number of ‘read’ operations required in step (1), and the number of ‘write’ operations in step (2). To determine the complexity of generating a given language, we can count the number of these operations incurred in the process of generating all words in the language.

The section proceeds as follows. In Section 3.1 we derive a new succession rule for ascending compositions. We then use this succession rule to develop a generation algorithm, which we subsequently analyse. Then, in Section 3.2 we examine the well-know succession rule for generating descending compositions in reverse lexicographic order, and analyse the resulting algorithm. Following this, Section 3.3 compares the two algorithms in terms of Kemp’s read and write operations. Finally, in Section 3.4 we develop a new formula for \( p(n) \) using our analysis the succession rule for ascending compositions.

3.1 Ascending Compositions

We are concerned here with developing a simple succession rule that will allow us to generate the lexicographic successor of a given ascending composition, and using this rule to develop a generation algorithm. To do this it is convenient to define the following notation.

**Definition 3.1** (Lexicographic Minimum). *For some positive integers \( m \leq n \), the function \( M_{A}(n, m) \) computes the lexicographically least element of the set \( A(n, m) \).*

**Definition 3.2** (Lexicographic Successor). *For any \( a_1 \ldots a_k \in A(n) \setminus \langle n \rangle \) the function \( S_{A}(a_1 \ldots a_k) \) computes the immediate lexicographic successor of \( a_1 \ldots a_k \).*
The succession rule for ascending compositions is then stated simply. We obtain the lexicographically least composition in $A(n, m)$ by appending $m$ to the lexicographically least composition in $A(n - m, m)$. If $2m > n$ then there are no compositions in $A(n, m)$ with more than one part, leading us to conclude that there is only one possible composition; and this must be the lexicographically least. This leads us to the following recurrence:

$$ M_{A}(n, m) = m \cdot M_{A}(n - m, m) \tag{8} $$

where $M_{A}(n, m) = \langle n \rangle$ if $2m > n$. See Kelleher [22, p.84] for a proof of (8). We can also derive a nonrecursive succession rule for $A(n, m)$. We develop in the following sequence of results.

Lemma 3.1. For all positive integers $m \leq n$, the lexicographically least element of the set $A(n, n)$ is given by

$$ M_{A}(n, m) = \underbrace{m \ldots m}_{\mu} \langle n - \mu m \rangle, \tag{9} $$

where $\mu = \lfloor n/m \rfloor - 1$.

Proof. Proceed by strong induction on $n$.

Base case: $n = 1$ Since $1 \leq m \leq n$ and $n = 1$, then $m = 1$, and so $2m > n$. Then, by (8), we know that $M_{A}(n, m) = \langle n \rangle$. Thus, as $\mu = 0$ when $n = 1$, (9) correctly computes $M_{A}(n, m)$ when $n = 1$.

Induction step Suppose, for some positive integer $n$ that (9) holds true for all positive integers $m \leq n$. Suppose $m$ is an arbitrary positive integer such that $m \leq n$. Suppose then that $2m > n$. By dividing both sides of this inequality by $m$, we see that $n/m < 2$, and so $\lfloor n/m \rfloor \leq 1$. Similarly, as $m \leq n$, it follows that $1 \leq n/m$, and so $1 \leq \lfloor n/m \rfloor$. Thus, $1 \leq n/m \leq 1$, and so $\lfloor n/m \rfloor = 1$; hence $\mu = 0$. By (8) $M_{A}(n, m) = \langle n \rangle$, and as $\mu = 0$, zero copies of $m$ are concatenated with $\langle n - \mu m \rangle$, and so (9) correctly computes $M_{A}(n, m)$.

Suppose then that $2m \leq n$. By the inductive hypothesis and (8) we have

$$ M_{A}(n, n) = m \cdot \underbrace{m \ldots m}_{\mu'} \langle n - \mu' m \rangle $$

Clearly, if $\mu = \mu' + 1$, then (9) correctly computes the lexicographically least element of $A(n, m)$. We know that $\mu' = \lfloor (n - m)/m \rfloor - 1$, which gives us $\mu' = \lfloor n/m - 1 \rfloor - 1$. It follows that $\mu' = \lfloor n/m \rfloor - 2$, and, as $\mu = \lfloor n/m \rfloor - 1$ from (9), we have $\mu = \mu' + 1$, completing the proof.

Theorem 3.1 (Lexicographic Successor). If $a_1 \ldots a_k \in A(n) \setminus \{ \langle n \rangle \}$ then

$$ S_{A}(a_1 \ldots a_k) = a_1 \ldots a_{k-2} \underbrace{m \ldots m}_{\mu} \langle n' - \mu m \rangle \tag{10} $$

where $m = a_{k-1} + 1$, $n' = a_{k-1} + a_k$, and $\mu = \lfloor n'/m \rfloor - 1$. 


Algorithm 3.1 RuleAsc($n$)

Require: $n > 0$

1: $k ← 2$
2: $a_1 ← 0$
3: $a_2 ← n$
4: while $k ≠ 1$ do
5:  $y ← a_k - 1$
6:  $k ← k - 1$
7:  $x ← a_k + 1$
8:  while $x ≤ y$ do
9:     $a_k ← x$
10:    $y ← y - x$
11:    $k ← k + 1$
12:  end while
13:  $a_k ← x + y$
14:  visit $a_1 \ldots a_k$
15: end while

Proof. Suppose $n$ is an arbitrary positive integer. Let $a_1 \ldots a_k$ be an arbitrary element of $\mathcal{A}(n) \setminus \{⟨n⟩\}$. Clearly, there is no positive integer $x$ such that $a_1 \ldots a_k-1⟨a_k+x⟩ ∈ \mathcal{A}(n)$. The initial part of $M_\mathcal{A}(a_k + a_{k-1}, a_{k-1} + 1)$ is the least possible value we can assign to $a_{k-1}$; and the remaining parts (if any) are the lexicographically least way to extend $a_1 \ldots a_{k-1}$ to a complete ascending composition of $n$. Therefore, $S_\mathcal{A}(a_1 \ldots a_k) = a_1 \ldots a_{k-2}M_\mathcal{A}(a_{k-1}+a_k, a_{k-1}+1)$. Then, using Lemma 3.1 we get (10) as required.

The succession rule (10) is implemented in RuleAsc (Algorithm 3.1). Each iteration of the main loop visits exactly one composition, and the internal loop generates any sequences of parts required to find the lexicographic successor. We concentrate here on analysis of this algorithm; see Kelleher [22, §5.3.1] for a full discussion and proof of correctness. The goal of our analysis is to derive a simple expression, in terms of the number of partitions of $n$, for the total number of read and write operations [23] made in the process of generating all ascending compositions of $n$. We do this by first determining the frequency of certain key instructions and using this information to determine the number of read and write operations involved.

Lemma 3.2. The number of times line 6 is executed during the execution of Algorithm 3.1 is given by $t_6(n) = p(n)$.

Proof. As Algorithm 3.1 correctly visits all $p(n)$ ascending compositions of $n$, we know that line 14 is executed exactly $p(n)$ times. Clearly line 6 is executed precisely the same number of times as line 14, and so we have $t_6(n) = p(n)$, as required.

Lemma 3.3. The number of times line 11 is executed during the execution of Algorithm 3.1 is given by $t_{11}(n) = p(n) - 1$.
Proof. The variable $k$ is used to control termination of the algorithm. From line 1 we know that $k$ is initially 2, and from line 4 we know that the algorithm terminates when $k = 1$. Furthermore, the value of $k$ is modified only on lines 6 and 11. By Lemma 3.2 we know that $k$ is decremented $p(n)$ times; it then follows immediately that $k$ must be incremented $p(n) - 1$ times, and so we have $t_{11}(n) = p(n) - 1$, as required. □

Theorem 3.2. Algorithm 3.1 requires $R_{A3.1}(n) = 2p(n)$ read operations to generate the set $\mathcal{A}(n)$.

Proof. Read operations are carried out on lines 7 and 5, which are executed $p(n)$ times each by Lemma 3.2. Thus, the total number of read operations is $R_{A3.1}(n) = 2p(n)$. □

Theorem 3.3. Algorithm 3.1 requires $W_{A3.1}(n) = 2p(n) - 1$ write operations to generate the set $\mathcal{A}(n)$, excluding initialisation.

Proof. After initialisation, write operations are carried out in Algorithm 3.1 only on lines 9 and 13. Line 13 is executed $p(n)$ times by Lemma 3.2. We can also see that line 9 is executed exactly as many times as line 11, and by Lemma 3.3 we know that this value is $p(n)$. Therefore, summing these contributions, we get $W_{A3.1}(n) = 2p(n) - 1$, completing the proof. □

From Theorem 3.3 and Theorem 3.2 it is easy to see that we require an average of two read and two write operations per partition generated, as we required $2p(n)$ of both operations to generate all $p(n)$ partitions of $n$. Thus, for any value of $n$ we are assured that the total time required to generate all partitions of $n$ will be proportional to the number of partitions generated, implying that the algorithm is constant amortised time.

3.2 Descending Compositions

Up to this point we have considered only algorithms that generate compositions in lexicographic order. The majority of descending composition generation algorithms, however, visit compositions in reverse lexicographic order (McKay [34] refers to it as the ‘natural order’ for partitions). There are many different presentations of the succession rule required to transform a descending composition from this list into its immediate successor: see Andrews [4, p.230], Knuth [27, p.1], Nijenhuis & Wilf [36, p.65–68], Page & Wilson [38, §5.5], Skiena [50, p.52], Stanton & White [53, p.13], Wells [60, p.150] or Zoghbi & Stojmenović [62]. No analysis of this succession rule in terms of the number of read and write operations [23] involved has been published, however, and in this section we analyse a basic implementation of the rule (we study more sophisticated techniques in Section 4.2).

If we formally define $S_D(d_1 \ldots d_k)$ to be the immediate lexicographic predecessor of a $d_1 \ldots d_k \in \mathcal{D}(n) \setminus 1 \ldots 1$, the succession rule can be formulated as
follows. Given a descending composition \(d_1 \ldots d_k\) where \(d_1 \neq 1\), we obtain the next composition in the ordering by applying the transformation

\[
S_D(d_1 \ldots d_k) = d_1 \ldots d_q m \ldots m^{\mu} \langle n' - \mu m \rangle
\]

where \(q\) is the rightmost non-1 value (i.e., \(d_j > 1\) for \(1 \leq j \leq q\) and \(d_j = 1\) for \(q < j \leq k\)), \(m = d_q - 1\), \(n' = d_q + k - q\) and \(\mu = \lfloor n'/m \rfloor - \lfloor n' \mod m = 0 \rfloor\). This presentation can readily be derived from the treatments cited in the previous paragraph.

The succession rule (11) is implemented in RuleDesc (Algorithm 3.2), where each iteration of the main loop implements a single application of the rule. The internal loop of lines 7–9 implements a right-to-left scan for the largest index \(q\) such that \(d_q > 1\), and the loop of lines 13–17 inserts \(\mu\) copies of \(m\) into the array. We analyse the algorithm by first determining the frequency of certain key statements, and using this information to derive the number of read and write operations needed to generate all descending compositions of \(n\).

**Lemma 3.4.** The number of times line 8 is executed during the execution of Algorithm 3.2 is given by \(t_8(n) = 1 - n + \sum_{x=1}^{n-1} p(x)\).

**Proof.** As exactly one descending composition is visited per iteration of the outer while loop, we know that upon reaching line 6 there is a complete descending
composition of \( n \) contained in \( d_1 \ldots d_k \). Furthermore, as \( d_1 \geq \cdots \geq d_k \), we know that all parts of size 1 are at the end of the composition, and so it is clear that line 7 will be executed exactly once for each part of size 1 in any given composition. As we visit the compositions at the end of the loop and we terminate when \( k = n \) we will not reach line 5 when the composition in question consists of \( n \) copies of 1 (as this is the lexicographically least, and hence the last descending composition in reverse lexicographic order). Thus, line 7 will be executed exactly as many times as there are parts of size 1 in all partitions of \( n \), minus the \( n \) 1s contained in the last composition. It is well known [21, p. 8] that the number of 1s in all partitions of \( n \) is \( 1 + p(1) + \cdots + p(n-1) \), and therefore we see that line 7 is executed exactly \( 1 - n + \sum_{x=1}^{n-1} p(x) \) times, as required.

Lemma 3.5. The number of times line 16 is executed during the execution of Algorithm 3.2 is given by \( t_{16}(n) = \sum_{x=1}^{n-1} p(x) \).

Proof. The variable \( k \) is used to control termination of Algorithm 3.2: the algorithm begins with \( k = 1 \) and terminates when \( k = n \). Examining Algorithm 3.2 we see that \( k \) is modified on only two lines: it is incremented on line 16 and decremented on line 8. Thus, we must have \( n - 1 \) more increment operations than decrements; by Lemma 3.4 there are exactly \( 1 - n + \sum_{x=1}^{n-1} p(x) \) decrement operations, and so we see that line 14 is executed \( \sum_{x=1}^{n-1} p(x) \) times, as required.

Theorem 3.4. Algorithm 3.2 requires \( R_{A3.2}(n) = \sum_{x=1}^{n} p(x) - n \) read operations to generate the set \( D(n) \).

Proof. Read operations are performed on lines 6 and 9 of Algorithm 3.2. By Lemma 3.4 we know that line 8 is executed \( 1 - n + \sum_{x=1}^{n-1} p(x) \) times, and so line 9 is executed an equal number of times. Clearly line 6 is executed \( p(n) - 1 \) times, and so we get a total of \( R_{A3.2}(n) = \sum_{x=1}^{n} p(x) - n \), as required.

Theorem 3.5. Algorithm 3.2 requires \( W_{A3.2}(n) = \sum_{x=1}^{n} p(x) - 1 \) write operations to generate the set \( D(n) \), excluding initialisation.

Proof. The only occasions in Algorithm 3.2 where a value is written to the array \( d \) are lines 14 and 18. By Lemma 3.5 we know that line 16 is executed exactly \( \sum_{x=1}^{n-1} p(x) \) times, and it is straightforward to see that line 14 is executed precisely the same number of times. As we visit exactly one composition per iteration of the outer while loop, and all descending compositions except the composition \( \langle n \rangle \) are visited with this loop, we then see that line 18 is executed \( p(n) - 1 \) times in all. Therefore, summing these contributions we get \( W_{A3.2}(n) = \sum_{x=1}^{n-1} p(x) + p(n) - 1 = \sum_{x=1}^{n} p(x) - 1 \) as required.

Theorems 3.4 and 3.5 derive the precise number of read and write operations required to generate all descending compositions of \( n \) using Algorithm 3.2, and this completes our analysis of the algorithm. We discuss the implications of these results in the next subsection, where we compare the total number of read and write operations required by RuleAsc\((n)\) and RuleDesc\((n)\).
3.3 Comparison

In this section we developed two algorithms. The first algorithm we considered, **RULEASC** (Algorithm 3.1), generates ascending compositions of \( n \); the second algorithm, **RULEDESC** (Algorithm 3.2), generates descending compositions of \( n \). We analysed the total number of read and write operations required by these algorithms to generate all partitions of \( n \) by iteratively applying the succession rule involved. The totals obtained, disregarding unimportant (i.e. \( O(1) \) or \( O(n) \)) trailing terms, for the ascending composition generator are summarised as follows.

\[
R_{A3.1}(n) \approx 2p(n) \quad \text{and} \quad W_{A3.1}(n) \approx 2p(n)
\]  

(12)

That is, we require approximately \( 2p(n) \) operations of the form \( x \leftarrow a_j \) and approximately \( 2p(n) \) operations of the form \( a_j \leftarrow x \) to generate all partitions of \( n \) using the ascending composition generator. Turning then to the descending composition generator, we obtained the following totals, again removing insignificant trailing terms.

\[
R_{A3.2}(n) \approx \sum_{x=1}^{n} p(x) \quad \text{and} \quad W_{A3.2}(n) \approx \sum_{x=1}^{n} p(x)
\]  

(13)

These totals would appear to indicate a large disparity between the algorithms, but we must examine the asymptotics of \( \sum_{x=1}^{n} p(x) \) to determine whether this is significant. We shall do this in terms of the average number of read and write operations per partition which is implied by these totals.

We know the total number of read and write operations required to generate all \( p(n) \) partitions of \( n \) using both algorithms. Thus, to determine the expected number of read and write operations required to transform the average partition into its immediate successor we must divide these totals by \( p(n) \). In the case of the ascending composition generation algorithms this is trivial, as both expressions are of the form \( 2p(n) \), and so dividing by \( p(n) \) plainly yields the value 2. Determining the average number of read and write operations using the succession rule for descending compositions is more difficult, however, as both expressions involve a factor of the form \( \sum_{x=1}^{n} p(x) \).

Using the asymptotic expressions for \( p(n) \) we can get a qualitative estimate of these functions. Odlyzko [37, p.1083] derived an estimate for the value of sums of partition numbers which can be stated as follows

\[
\sum_{x=1}^{n} p(x) = \frac{e^{\pi \sqrt{2n/3}}}{2\pi \sqrt{2n}} \left( 1 + O(n^{-1/6}) \right).
\]

Then, dividing this by the asymptotic expression for \( p(n) \) we get the following approximation

\[
\frac{1}{p(n)} \sum_{x=1}^{n} p(x) \approx 1 + \frac{\sqrt{6n}}{\pi},
\]  

(14)

which, although crude, is sufficient for our purposes. The key feature of (14) is that the value is not constant: it is \( O(\sqrt{n}) \). Using this approximation we obtain
the following values for the number of read and write operations expected to transform a random partition of \( n \) into its successor.

|             | Reads | Writes |
|-------------|-------|--------|
| Ascending   | 2     | 2      |
| Descending  | \( 1 + 0.78\sqrt{n} \) | \( 1 + 0.78\sqrt{n} \) |

We can see the qualitative difference between the algorithms by examining their read and write tapes in Figure 1. The tapes in question are generated by imagining that read and write heads mark a tape each time one of these operations is made. The horizontal position of each head is determined by the index of the array element involved. The tape is advanced one unit each time a composition is visited, and so we can see the number of read and write operations required for each individual partition generated. Regarding Figure 1 then, and examining the read tape for \( \text{RuleAsc} \), we can see that every partition requires exactly 2 reads; in contrast, the read tape for \( \text{RuleDesc} \) shows a maximum of \( n - 1 \) read operations per partition, and this oscillates rapidly as we move along the tape. Similarly, the write tape for \( \text{RuleAsc} \) shows that we sometimes need to make a long sequence of write operations to make the transition in question, but that these are compensated for — as our analysis has shown — by the occasions where we need only one write. The behaviour of the write head in \( \text{RuleDesc} \) is very similar to that of its read head, and we again see many transitions where a large number of writes are required.

The difference between \( \text{RuleAsc} \) and \( \text{RuleDesc} \) is not due to some algorithmic nuance; rather, it reflects of a structural property of the objects in question. The total suffix length [23] of descending compositions is much greater than that of ascending compositions, because in many descending composition the suffix consists of the sequence of 1s; and we known that the total number of 1s in all partitions of \( n \) is \( \sum_{x=1}^{n-1} p(x) \). In this well-defined way, it is more efficient to generate all ascending compositions than it is to generate all descending compositions.

### 3.4 A new formula for \( p(n) \)

Although not strictly relevant to our analyses of ascending and descending composition generation algorithms, another result follows directly from the analysis of Algorithm 3.1. If we compare the lexicographic succession rule (10) and Algorithm 3.1 carefully, we realise that the \( \mu \) copies of \( m \) must be inserted into the array within the inner loop of lines 8–12; and our analysis has given us the precise number of times that this happens. Therefore, we know that the sum of \( \mu \) values over all ascending compositions of \( n \) (except the last composition, \( \langle n \rangle \)), must equal the number of write operations made in the inner loop. Using this observation we then get the following theorem.
Figure 1: Read and write tapes for the direct implementations of succession rules to generate ascending and descending compositions. On the left we have the read and write tapes for the ascending composition generator, Algorithm 3.1; on the right, then, are the corresponding tapes for the descending composition generator, Algorithm 3.2. In both cases, the traces correspond to the read and write operations carried out in generating all partitions of 12.
**Theorem 3.6.** For all $n \geq 1$

\[
p(n) = \frac{1}{2} \left( 1 + n + \sum_{a_1 \ldots a_k \in A(n) \setminus \{n\}} \left\lfloor \frac{a_{k-1} + a_k}{a_{k-1} + 1} \right\rfloor \right).
\]  

(15)

**Proof.** We know from Lemma 3.3 that the total number of write operations made by Algorithm 3.1 in the inner loop of lines 8–12 is given by $p(n) - 1$. Algorithm 3.1 applies the lexicographic succession rule above to all elements of $A(n) \setminus \{n\}$, as well as one extra composition, which we refer to as the 'initialisation composition'. The initialisation composition is not in the set $A(n)$ as $a_1 = 0$, and so we must discount the number of writes incurred by applying the succession rule to this composition. The composition visited immediately after $0n$ is $1 \ldots 1$, and so $n - 1$ copies of 1 must have been inserted into the array in the inner loop during this transition. Therefore, the total number of writes made within the inner loop in applying the succession rule to all elements of $A(n) \setminus \{n\}$ is given by $p(n) - 1 - (n - 1) = p(n) - n$. Therefore, from this result and the succession rule of Theorem 3.1 we get

\[
p(n) - n = \sum_{a_1 \ldots a_k \in A(n) \setminus \{n\}} \left( \left\lfloor \frac{a_{k-1} + a_k}{a_{k-1} + 1} \right\rfloor - 1 \right),
\]

from which it is easy to derive (15), completing the proof. \qed

We can simplify (15) if we suppose that all $a_1 \ldots a_k \in A(n)$ are prefixed by a value 0. More formally, a direct consequence of Theorem 3.6 is that

\[
p(n) = \frac{1}{2} \left( 1 + \sum_{a_1 \ldots a_k \in A'(n)} \left\lfloor \frac{a_{k-1} + a_k}{a_{k-1} + 1} \right\rfloor \right),
\]

(16)

where $A'(n) = \{0 \cdot a_1 \ldots a_k \mid a_1 \ldots a_k \in A(n)\}$. Fundamentally, what Theorem 3.6 shows us is that if we let $y$ be the largest part and $x$ the second largest part in an arbitrary partition of $n$, we can count the partitions of $n$ by summing $\lfloor (x + y)/(x + 1) \rfloor$ over all partitions of $n$.

The partition function $p(n)$ is one of the most important functions in the theory of partitions and has been studied for several centuries [5]. The asymptotic [20] and arithmetic [2] properties of $p(n)$ have been very thoroughly examined. While (16) is clearly not an efficient means of computing $p(n)$, it may provide some new insight into this celebrated function.

### 4 Accelerated Algorithms

In this section we examine algorithms that use structural properties of the sets of ascending and descending compositions to reduce the number of read and write
operations required. The algorithms presented are the most efficient known examples of ascending and descending composition generators, ensuring that we have a fair comparison of the algorithms arising from the two candidate encodings for partitions. In Section 4.1 we develop a new ascending composition generator that requires fewer read operations than RuleAsc. Then, in Section 4.2 we study the most efficient known descending composition generation algorithm, due to Zoghbi & Stoimenović [62], which requires far fewer read and write operations than RuleDesc. In Section 4.3, we compare these two algorithms to determine which of the two is more efficient.

4.1 Ascending Compositions

In this subsection we improve on RuleAsc (Algorithm 3.1) by applying the theory of ‘terminal’ and ‘nonterminal’ compositions. To enable us to fully analyse the resulting algorithm we require an expression to enumerate terminal ascending compositions in terms of $p(n)$. In the opening part of this subsection we develop the theory of terminal and nonterminal compositions. A byproduct of this analysis is a new proof for a partition identity on the number of partitions where the largest part is less than twice the second largest part. After developing this necessary theory, we move on to the description of the algorithm itself, and its subsequent analysis.

4.1.1 Terminal and Nonterminal Compositions

The algorithm that we shall examine shortly uses some structure within the set of ascending compositions to make many transitions very efficient. This structure is based on the ideas of ‘terminal’ and ‘nonterminal’ compositions. We now define these concepts and derive some basic enumerative results to aid us in our analysis.

Definition 4.1 (Terminal Ascending Composition). For some positive integer $n$, an ascending composition $a_1 \ldots a_k \in A(n)$ is terminal if $k = 1$ or $2a_{k-1} \leq a_k$. Let $T_A(n, m)$ denote the set of terminal compositions in $A(n, m)$, and $T_A(n, m)$ denote the cardinality of this set (i.e. $T_A(n, m) = |T_A(n, m)|$).

Definition 4.2 (Nonterminal Ascending Composition). For some positive integer $n$, $a_1 \ldots a_k \in A(n)$ is nonterminal if $k > 1$ and $2a_{k-1} > a_k$. Let $N_A(n, m)$ denote the set of nonterminal compositions in $A(n, m)$, and let $N_A(n, m)$ denote the cardinality of this set (i.e. $N_A(n, m) = |N_A(n, m)|$).

If we let $A(n, m)$ denote the number of ascending compositions of $n$ where the initial part is at least $m$ it can be shown [22, ch.3] that

$$A(n, m) = 1 + \sum_{x=m}^{\lfloor n/2 \rfloor} A(n-x, x)$$

holds for all positive integers $m \leq n$. We require a similar recurrence to enumerate the terminal ascending compositions, and so we let $T_A(n, m)$ denote the
number of terminal compositions in the set $A(n, m)$. The terminal ascending compositions are a subset of the ascending compositions, and the construction rule implied is the same: the number of terminal ascending compositions of $n$ where the initial part is exactly $m$ is equal to the number of terminal compositions of $n - m$ with initial part at least $m$. The only difference, then, between the recurrences for ascending compositions and terminal ascending compositions occurs in the boundary conditions. The recurrence can be stated as follows: for all positive integers $m \leq n$, $T_A(n, m)$ satisfies

$$T_A(n, m) = 1 + \sum_{x=m}^{\lfloor n/3 \rfloor} T_A(n - x, x).$$

(18)

See Kelleher [22, p.160–161] for the proofs of recurrences (17) and (18).

Before we move onto the main result, where we prove that $T_A(n, m) = A(n, m) - A(n - 2, m)$, we require some auxiliary results which simplify the proof of this assertion. In Lemma 4.1 we prove an equivalence between logical statements of a particular form involving the floor function, which is useful in Lemma 4.2; the latter lemma then provides the main inductive step in our proof of the central theorem of this section. In the interest of brevity, we limit our proofs to values of $n > 3$, since $n \leq 3$ can be easily demonstrated and would unnecessarily complicate the proofs.

**Lemma 4.1.** If $x$, $m$ and $n$ are positive integers then $x \leq \lfloor (n - x)/m \rfloor \iff x \leq \lfloor n/(m + 1) \rfloor$.

**Proof.** Suppose $x$, $m$ and $n$ are positive integers. Suppose $x \leq \lfloor (n - x)/m \rfloor$. Thus, $x \leq (n - x)/m$, and so $x \leq n/(m + 1)$. Then, as $\lfloor n/(m + 1) \rfloor \leq n/(m + 1)$ and $x$ is an integer, we know that $x \leq \lfloor n/(m + 1) \rfloor$, and so $x \leq \lfloor (n - x)/m \rfloor \iff x \leq \lfloor n/(m + 1) \rfloor$.

Suppose that $x \leq \lfloor n/(m + 1) \rfloor$. Then, $x \leq n/(m + 1)$, and so $x \leq (n - x)/m$. Once again, as $x$ is an integer it is apparent that $x \leq \lfloor (n - x)/m \rfloor \leq (n - x)/m$, and so $x \leq \lfloor n/(m + 1) \rfloor \iff x \leq \lfloor (n - x)/m \rfloor$. Therefore, as $x \leq \lfloor (n - x)/m \rfloor \iff x \leq \lfloor n/(m + 1) \rfloor$ and $x \leq \lfloor n/(m + 1) \rfloor \iff x \leq \lfloor (n - x)/m \rfloor$ we see that $x \leq \lfloor (n - x)/m \rfloor \iff x \leq \lfloor n/(m + 1) \rfloor$, as required. \hfill $\Box$

**Lemma 4.2.** For all positive integers $n > 3$

$$\sum_{x=\lfloor n/3 \rfloor+1}^{\lfloor n/2 \rfloor} A(n - x, x) = 1 + \sum_{x=\lfloor n/3 \rfloor+1}^{\lfloor (n-2)/2 \rfloor} A(n - 2 - x, x).$$

(19)

**Proof.** Suppose $n > 3$ and $1 \leq m \leq n$, and consider the left-hand side of (19). We know that $A(n, m) = 1$ if $m > \lfloor n/2 \rfloor$, as the summation in recurrence (17) will be empty. By the contrapositive of Lemma 4.1 we know that $x > \lfloor (n - x)/2 \rfloor \iff x > \lfloor n/3 \rfloor$, and we therefore know that each term in the summation of the left-hand side of (19) is equal to 1. Thus, we see that

$$\sum_{x=\lfloor n/3 \rfloor+1}^{\lfloor n/2 \rfloor} A(n - x, x) = \lfloor n/2 \rfloor - \lfloor n/3 \rfloor - 1.$$

(20)

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Similarly, as \( x > \lfloor n/3 \rfloor \implies x > \lfloor (n-x)/2 \rfloor \), it clearly follows that 
\( x > \lfloor n/3 \rfloor \implies x > \lfloor (n-x)/2 \rfloor - 1 \), or \( x > \lfloor n/3 \rfloor \implies x > \lfloor (n-2-x)/2 \rfloor \).
Thus, each term in the summation on the right-hand side of (19) must also equal 1, and so we get

\[
1 + \sum_{x=\lfloor n/3 \rfloor + 1}^{\lfloor (n-2)/2 \rfloor} A(n - 2 - x, x) = 1 + \lfloor (n-2)/2 \rfloor - \lfloor n/3 \rfloor - 1
\]

\[= \lfloor n/2 \rfloor - \lfloor n/3 \rfloor - 1. \quad (21)\]

Therefore, as (20) and (21) show that the left-hand and right-hand side of (19) are equal, the proof is complete.

**Theorem 4.1.** If \( n \geq 3 \), then \( T_A(n, m) = A(n, m) - A(n-2, m) \) for all \( 1 \leq m \leq \lfloor n/2 \rfloor \).

**Proof.** Proceed by strong induction on \( n \).

**Base case:** \( n = 3 \). As \( 1 \leq m \leq \lfloor n/2 \rfloor \) and \( n = 3 \), we know that \( m = 1 \). Computing \( T_A(3, 1) \), we get \( 1 + T_A(2, 1) = 2 \). We also find \( A(3, 1) = 3 \) and \( A(1, 1) = 1 \), and so the base case of the induction holds.

**Induction step** Suppose \( T_A(n', m) = A(n', m) - A(n'-2, m) \) when \( 1 \leq m \leq \lfloor n'/2 \rfloor \), for all \( 3 < n' < n \), and some integer \( n \). Then, as \( x \leq \lfloor (n-x)/2 \rfloor \iff x \leq \lfloor n/3 \rfloor \), by Lemma 4.1, we can apply this inductive hypothesis to each term \( T_A(n-x, x) \) in (18), giving us

\[
T_A(n, m) = 1 + \sum_{x=m}^{\lfloor n/3 \rfloor} (A(n-x, x) - A(n-2-x, x))
\]

\[= 1 + \sum_{x=m}^{\lfloor n/3 \rfloor} A(n-x, x) - \sum_{x=m}^{\lfloor n/3 \rfloor} A(n-2-x, x). \quad (22)\]

By Lemma 4.2 we know that

\[
\sum_{x=\lfloor n/4 \rfloor + 1}^{\lfloor n/2 \rfloor} A(n-x, x) - \sum_{x=\lfloor n/3 \rfloor + 1}^{\lfloor (n-2)/2 \rfloor} A(n-2-x, x) - 1 = 0,
\]

and so we can add the left-hand side of this equation to the right-hand side of (22), to get

\[
T_A(n, m) = 1 + \sum_{x=m}^{\lfloor n/3 \rfloor} A(n-x, x) - \sum_{x=m}^{\lfloor n/3 \rfloor} A(n-2-x, x)
\]

\[+ \sum_{x=\lfloor n/3 \rfloor + 1}^{\lfloor n/2 \rfloor} A(n-x, x) - \sum_{x=\lfloor n/3 \rfloor + 1}^{\lfloor (n-2)/2 \rfloor} A(n-2-x, x) - 1.
\]

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Then, gathering the terms $A(n - x, x)$ and $A(n - 2 - x, x)$ into the appropriate summations we get

$$T_A(n, m) = 1 + \sum_{x=m}^{[n/2]} A(n - x, x) - 1 - \sum_{x=m}^{[(n-2)/2]} A(n - 2 - x, x),$$

which by (18) gives us $T_A(n, m) = A(n, m) - A(n - 2, m)$, as required.

For the purposes of our analysis it is useful to know the total number of terminal and nonterminal compositions of $n$, and it is worthwhile formalising the results here for reference. Therefore, letting $T_A(n) = T_A(n, 1)$ and $N_A(n) = N_A(n, 1)$, we get the following corollaries defined in terms of the partition function $p(n)$.

**Corollary 4.1.** For all positive integers $n$, $T_A(n) = p(n) - p(n - 2)$.

*Proof.* As $T_A(n) = T_A(n, 1)$ and $A(n, 1) = p(n)$, proof is immediate by Theorem 4.1 for all $n \geq 3$. Since $p(n) = 0$ for all $n < 0$ and $p(0) = 1$, we can readily verify that $T_A(2) = T_A(1) = 1$, as required.

**Corollary 4.2.** If $n$ is a positive integer then $N_A(n) = p(n - 2)$.

*Proof.* An ascending composition is either terminal or nonterminal. As the total number of ascending compositions of $n$ is given by $p(n)$, we get $N_A(n) = p(n) - (p(n) - p(n - 2)) = p(n - 2)$, as required.

Corollaries 4.1 and 4.2 prove a nontrivial structural property of the set of all ascending compositions, and can be phrased in more conventional partition theoretic language. Consider an arbitrary partition of $n$, and let $y$ be the largest part in this partition. We then let $x$ be the second largest part ($x \leq y$). Corollary 4.2 then shows that the number of partitions of $n$ where $2x > y$ is equal to the number of partitions of $n - 2$. This result is known, and has been reported by Adams-Watters [51, Seq.A027336]. The preceding treatment, however, would appear to be the first published proof of the identity.

### 4.1.2 Algorithm

Having derived some theoretical results about the terminal and nonterminal ascending compositions of $n$, we are now in a position to exploit those properties in a generation algorithm. In the direct implementation of the lexicographic succession rule for ascending compositions, RuleAsc, we generate the successor of $a_1 \ldots a_k$ by computing the lexicographically least element of the set $A(a_k + 1 + a_k, a_k + 1)$, and visit the resulting composition. The algorithm operates by implementing exactly one transition per iteration of the main loop. The accelerated algorithm, AcCELAsc, developed here operates on a slightly different principle: we compute the lexicographically least composition of $A(a_k + 1 + a_k, a_k + 1)$, as before, but now keep a watchful eye to see if the resulting composition is nonterminal. If it is, we can compute the lexicographic
Algorithm 4.1 \textsc{AccelAsc}(n)

\begin{algorithm}
\textbf{Require: } \(n \geq 1\)
1: \(k \leftarrow 2\)
2: \(a_1 \leftarrow 0\)
3: \(y \leftarrow n - 1\)
4: \textbf{while } \(k \neq 1\) \textbf{do}
5: \(k \leftarrow k - 1\)
6: \(x \leftarrow a_k + 1\)
7: \textbf{while } \(2x \leq y\) \textbf{do}
8: \(a_k \leftarrow x\)
9: \(y \leftarrow y - x\)
10: \(k \leftarrow k + 1\)
11: \textbf{end while}
12: \(\ell \leftarrow k + 1\)
13: \textbf{while } \(x \leq y\) \textbf{do}
14: \(a_k \leftarrow x\)
15: \(a_\ell \leftarrow y\)
16: \textbf{visit } \(a_1 \ldots a_\ell\)
17: \(x \leftarrow x + 1\)
18: \(y \leftarrow y - 1\)
19: \textbf{end while}
20: \(y \leftarrow y + x - 1\)
21: \(a_k \leftarrow y + 1\)
22: \textbf{visit } \(a_1 \ldots a_k\)
23: \textbf{end while}
\end{algorithm}

successor simply by incrementing \(a_{k-1}\) and decrementing \(a_k\). Otherwise, we revert to the standard means of computing the lexicographic successor. By analysing this algorithm, we shall see that this approach provides significant gains. We concentrate on the analysis of Algorithm 4.1 here — see Kelleher [22, §4.4.2] for further discussion and proof of correctness.

**Lemma 4.3.** The number of times line 16 is executed during the execution of Algorithm 4.1 is given by \(t_{16}(n) = p(n - 2)\).

**Proof.** Compositions visited on line 16 must be nonterminal because upon reaching line 12, the condition \(2x > y\) must hold. As \(x\) and \(y\) are the second-last and last parts, respectively, of the composition visited on line 16, then this composition must be nonterminal by definition. Subsequent operations on \(x\) and \(y\) within this loop do not alter the property that \(2x > y\), and so all compositions visited on line 16 must be nonterminal.

Furthermore, we also know that all compositions visited on line 22 must be terminal. To demonstrate this fact, we note that if \(a_1 \ldots a_k\) is the last composition visited before we arrive at line 20, the composition visited on line 22 must be \(a_1 \ldots a_{k-2}(a_{k-1} + a_k)\). Therefore, to demonstrate that this composition is terminal, we must show that \(2a_{k-2} \leq a_{k-1} + a_k\). We know that \(a_{k-2} \leq
$a_{k-1} \leq a_k$. It follows that $2a_{k-2} \leq 2a_{k-1}$, and also that $2a_{k-1} \leq a_{k-1} + a_k$. Combining these two inequalities, we see that $2a_{k-2} \leq 2a_{k-1} \leq a_{k-1} + a_k$, and so $2a_{k-2} \leq a_{k-1} + a_k$. Thus all compositions visited on line 22 must be terminal.

Then, as Algorithm 4.1 correctly visits all $p(n)$ ascending compositions of $n$ [22, p.105], since all compositions visited on line 22 are terminal and as all compositions visited on line 16 are nonterminal, we know that all nonterminal compositions of $n$ must be visited on line 16. By Corollary 4.2 there are $p(n-2)$ nonterminal compositions of $n$, and hence $t_{16} = p(n-2)$, as required.

**Lemma 4.4.** The number of times line 5 is executed during the execution of Algorithm 4.1 is given by $t_5(n) = p(n) - p(n-2)$.

**Proof.** By Lemma 4.3 we know that the visit statement on line 16 is executed $p(n-2)$ times. As Algorithm 4.1 correctly visits all $p(n)$ ascending compositions of $n$, then the remaining $p(n) - p(n-2)$ compositions must be visited on line 22. Clearly then, line 22 (and hence line 5) is executed $p(n) - p(n-2)$ times. Therefore, $t_5 = p(n) - p(n-2)$, as required.

**Lemma 4.5.** The number of times line 10 is executed during the execution of Algorithm 4.1 is given by $t_{10}(n) = p(n) - p(n-2) - 1$.

**Proof.** The variable $k$ is assigned the value 2 upon initialisation, and the algorithm terminates when $k = 1$. As the variable is only updated via increment (line 10) and decrement (line 5) operations, we know that there must be one more decrement operation than increments. By Lemma 4.4 we know that there are $p(n) - p(n-2)$ decrements, and so there must be $p(n) - p(n-2) - 1$ increments on the variable. Therefore, $t_{10} = p(n) - p(n-2) - 1$.

**Theorem 4.2.** Algorithm 4.1 requires $R_{A4.1}(n) = p(n) - p(n-2)$ read operations to generate the set $A(n)$.

**Proof.** Only one read operation occurs Algorithm 4.1, and this is done on line 6. By Lemma 4.4 we know that line 5 is executed $p(n) - p(n-2)$ times, and it immediately follows that line 6 is executed the same number of times. Therefore, $R_{A4.1}(n) = p(n) - p(n-2)$, as required.

**Theorem 4.3.** Algorithm 4.1 requires $W_{A4.1}(n) = 2p(n) - 1$ write operations to generate the set $A(n)$, excluding initialisation.

**Proof.** Write operations are performed on lines 8, 14, 15 and 21. Lemma 4.4 shows that line 21 is executed $p(n) - p(n-2)$ times. From Lemma 4.5 we know that line 8 is executed $p(n) - p(n-2) - 1$ times. Then, by Lemma 4.3 we know that lines 14 and 15 are executed $p(n-2)$ times each. Summing these contributions we get $W_{A4.1}(n) = p(n) - p(n-2) + p(n) - p(n-2) - 1 + 2p(n-2) = 2p(n) - 1$, as required.

Theorems 4.2 and 4.3 derive the precise number of read and write operations required to generate all partitions of $n$ using Algorithm 4.1. This algorithm is a considerable improvement over our basic implementation of the succession rule,
Algorithm 3.1, in two ways. Firstly, by keeping \( p(n - 2) \) of the visit operations within the loop of lines 13–19, we significantly reduce the average cost of a write operation. Thus, although we do not appreciably reduce the total number of write operations involved, we ensure that \( 2p(n - 2) \) of those writes are executed at the cost of an increment and decrement on a local variable and the cost of a \( \leq \) comparison of two local variables — in short, very cheaply.

The second improvement is that we dramatically reduce the total number of read operations involved. Recall that \textsc{RuleAsc} required \( 2p(n) \) read operations to generate all ascending compositions of \( n \); Theorem 4.2 shows that \textsc{AccelAsc} requires only \( p(n) - p(n - 2) \) read operations. We also reduced the number of read operations by a factor of 2 by maintaining the value of \( y \) between iterations of the main while loop, but this could equally be applied to \textsc{RuleAsc}, and is only a minor improvement at any rate. The real gain here is obtained from exploiting the block-based nature of the set of ascending compositions, as we do not need to perform any read operations once we have begun iterating through the nonterminal compositions within a block.

4.2 Descending Compositions

In Section 3.2 we derived a direct implementation of the succession rule for descending compositions. We then analysed the cost of using this direct implementation to generate all descending compositions of \( n \), and found that it implied an average of \( O(\sqrt{n}) \) read and write operations per partition. There are, however, several constant amortised time algorithms to generate descending compositions, and in this section we study the most efficient example.

There is one basic problem with the direct implementation of the succession rule for descending compositions (\textsc{RuleDesc}): most of the read and write operations it makes are redundant. To begin with, the read operations incurred by \textsc{RuleDesc} in scanning the current composition to find the rightmost non-1 value are unnecessary. As McKay [34] noted, we can easily keep track of the index of the largest non-1 value between iterations, and thereby eliminate the right-to-left scan altogether. The means by which we can avoid the majority of the write operations is a little more subtle, and was first noted by Zoghbi & Stojmenović [62]. For instance, consider the transition

\[ 3321111 \rightarrow 33111111. \]  

\textsc{RuleDesc} implements the transition from 3321111 to 33111111 by finding the prefix 33 and writing six copies of 1 after it, oblivious to the fact that 4 of the array indices already contain 1. Thus, a more reasonable approach is to make a special case in the succession rule so that if \( d_q = 2 \), we simply set \( d_q \leftarrow 1 \) and append 1 to the end of the composition. This observation proves to be sufficient to remove the worst excesses of \textsc{RuleDesc}, as 1s are by far the most numerous part in the partitions of \( n \).

Zoghbi & Stojmenović’s algorithm implements both of these ideas, and makes one further innovation to reduce the number of write operations required.
By initialising the array to hold \( n \) copies of 1, we know that any index \( k > n \) must contain the value 1, and so we can save another write operation in the special case of \( d_q = 2 \) outlined above. Thus, Zoghbi & Stojmenović’s algorithm is the most efficient example, and consequently it is the algorithm that we shall use for our comparative analysis. Knuth developed a similar algorithm [27, p.2]: he also noted the necessity of keeping track of the value of \( q \) between iterations, and also implemented the special case for \( d_q = 2 \) outlined above. Knuth’s algorithm, however, does not contain the further improvement included by Zoghbi & Stojmenović (i.e. initialising the array to 1...1 and avoiding the second write operation in the \( d_q = 2 \) special case), and therefore requires strictly more write operations than Zoghbi & Stojmenović’s. Zoghbi & Stojmenović’s algorithm also consistently outperforms Knuth’s algorithm in empirical tests.

Zoghbi & Stojmenović’s algorithm is presented in Algorithm 4.2, which we shall also refer to as AccelDesc. Each iteration of the main loop implements a single transition, and two cases are identified for performing the transition. In the conditional block of lines 8–10 we implement the special case for \( d_q = 2 \): we can see that the length of the composition is incremented, \( d_q \) is assigned to 1 and the value of \( q \) is updated to point to the new rightmost non-1 part. The general case is dealt with in the block of lines 11–29; the approach is much the same as that of RuleDesc, except in this case we have the additional complexity of maintaining the value of \( q \) between iterations.

**Lemma 4.6.** The number of times line 10 is executed during the execution of Algorithm 4.2 is given by \( t_{10}(n) = p(n - 2) \).

**Proof.** The variable \( q \) points to the smallest non-1 value in \( d_1 \ldots d_k \), and we have a complete descending composition in the array each time we reach line 7. Therefore, line 10 will be executed once for every descending composition of \( n \) which contains at least one 2; and it is well known that this is \( p(n - 2) \). Therefore, \( t_{10}(n) = p(n - 2) \), as required. \( \square \)

**Lemma 4.7.** The number of times line 16 is executed during the execution of Algorithm 4.2 is given by \( t_{16}(n) + t_{25}(n) = p(n - 2) - 1 \).

**Proof.** The variable \( q \) controls the termination of the algorithm. It is initialised to 1 on line 2, and the algorithm terminates when \( q = 0 \). We modify \( q \) via increment operations on lines 16 and 25, and decrement operations on line 10 only. Therefore, there must be one more decrement operation than increments on \( q \). By Lemma 4.6 there are \( p(n - 2) \) decrements performed on \( q \), and there must therefore be \( p(n - 2) - 1 \) increments. Therefore, \( t_{16}(n) + t_{25}(n) = p(n - 2) - 1 \), as required. \( \square \)

**Theorem 4.4.** Algorithm 4.2 requires \( R_{A4.2}(n) = 2p(n) - p(n - 2) - 2 \) read operations to generate the set \( D(n) \).

**Proof.** Read operations are performed on lines 7 and 12 of Algorithm 4.2. Clearly, as all but the composition \( \langle n \rangle \) are visited on line 30, line 7 is executed \( p(n) - 1 \) times. Then, as a consequence of Lemma 4.6, we know that
Algorithm 4.2 AccelDesc($n$)

Require: $n \geq 1$

1: $k \leftarrow 1$
2: $q \leftarrow 1$
3: $d_2 \ldots d_n \leftarrow 1 \ldots 1$
4: $d_1 \leftarrow n$
5: visit $d_1$
6: while $q \neq 0$ do
7:  if $d_q = 2$ then
8:      $k \leftarrow k + 1$
9:      $d_q \leftarrow 1$
10:     $q \leftarrow q - 1$
11:  else
12:      $m \leftarrow d_q - 1$
13:      $n' \leftarrow k - q + 1$
14:      $d_q \leftarrow m$
15:      while $n' \geq m$ do
16:          $q \leftarrow q + 1$
17:          $d_q \leftarrow m$
18:          $n' \leftarrow n' - m$
19:      end while
20:  if $n' = 0$ then
21:      $k = q$
22:  else
23:      $k \leftarrow q + 1$
24:      if $n' > 1$ then
25:          $q \leftarrow q + 1$
26:          $d_q \leftarrow n'$
27:      end if
28:  end if
29: end if
30: visit $d_1 \ldots d_k$
31: end while

line 12 is executed $p(n) - p(n - 2) - 1$ times. Therefore, the total number of read operations is given by $R_{A4.2}(n) = 2p(n) - p(n - 2) - 2$, as required.

Theorem 4.5. Algorithm 4.2 requires $W_{A4.2}(n) = p(n) + p(n - 2) - 2$ write operations to generate the set $D(n)$, excluding initialisation.

Proof. After initialisation, write operations are performed on lines 9, 14, 17 and 26 of Algorithm 4.2. Line 9 contributes $p(n - 2)$ writes by Lemma 4.6; and similarly, line 14 is executed $p(n) - p(n - 2) - 1$ times. By Lemma 4.7 we know that the total number of write operations incurred by lines 17 and 26 is $p(n - 2) - 1$. Therefore, summing these contributions we get $W_{A4.2}(n) = p(n) + p(n - 2) - 2$, as required.
Theorems 4.4 and 4.5 show that Zoghbi & Stojmenović’s algorithm is a vast improvement on RuleDesc. Recall that RuleDesc\(n\) requires roughly \(\sum_{x=1}^{n} p(x)\) read and \(\sum_{x=1}^{n} p(x)\) write operations; and we have seen that AccelDesc\(n\) requires only \(2p(n) - p(n - 2)\) read and \(p(n) + p(n - 2)\) write operations.

Zoghbi & Stojmenović [62] also provided an analysis of AccelDesc, and proved that it generates partitions in constant amortised time. We briefly summarise this analysis to provide some perspective on the approach we have taken. Zoghbi & Stojmenović begin their analysis by demonstrating that \(D(n, m) \geq \frac{n^2}{12}\) for all \(m > 2\), where \(D(n, m)\) enumerates the descending compositions of \(n\) in which the initial part is no more than \(m\). They use this result to reason that, for each \(d_q > 2\) encountered, the total number of iterations of the internal while loop is < \(2c\), for some constant \(c\). Thus, since the number of iterations of the internal loop is constant whenever \(d_q \geq 3\) (the case for \(d_q = 2\) obviously requires constant time), the algorithm generates descending compositions in constant amortised time.

The preceding paragraph is not a rigorous argument proving that AccelDesc is constant amortised time. It is intended only to illustrate the difference in the approach that we have taken in this section to Zoghbi & Stojmenović’s analysis, and perhaps highlight some of the advantages of using Kemp’s abstract model of counting read and write operations [23]. By using Kemp’s model we were able to ignore irrelevant details regarding the algorithm’s implementation, and concentrate instead on the algorithm’s effect: reading and writing parts in compositions.

4.3 Comparison

Considering AccelAsc (Algorithm 4.1) first, we derived the following numbers of read and write operations required to generate all ascending compositions of \(n\), ignoring inconsequential trailing terms.

\[
R_{A4.1}(n) \approx p(n) - p(n - 2) \quad \text{and} \quad W_{A4.1}(n) \approx 2p(n)
\] (24)

We can see that the total number of write operations is \(2p(n)\): i.e., the total number of write operations is twice the total number of partitions generated. On the other hand, the total number of read operations required is only \(p(n) - p(n - 2)\), which, as we shall see presently, is asymptotically negligible in comparison to \(p(n)\). The number of read operations is small because we only require one read operation per iteration of the outer loop. Once we have stored \(a_{k-1}\) in a local variable, we can then extend the composition as necessary and visit all of the following nonterminal compositions without needing to perform a read operation. Thus, it is the write operations that dominate the cost of generation with this algorithm and, as we noted earlier, the average cost of a write operation in this algorithm is quite small.

For the descending composition generator, AccelDesc (Algorithm 4.2), the following read and write totals were derived (we ignore the insignificant trailing
terms in both cases).

\[ R_{A4.2}(n) \approx 2p(n) - p(n - 2) \quad \text{and} \quad W_{A4.2}(n) \approx p(n) + p(n - 2) \quad (25) \]

The total number of write operations required by this algorithm to generate all partitions of \( n \) is \( p(n) + p(n - 2) \). Although this value is strictly less than the write total for AcclAsc, the difference is not asymptotically significant as \( p(n - 2)/p(n) \) tends towards 1 as \( n \) becomes large. Therefore, we should not expect any appreciable difference between the performances of the two algorithms in terms of the number of write operations involved. There is, however, an asymptotically significant difference in the number of read operations performed by the algorithms.

The total number of read operations required by AcclDesc is \( 2p(n) - p(n - 2) \). This expression is complicated by an algorithmic consideration, where it proved to be more efficient to perform \( p(n) - p(n - 2) \) extra read operations than to save the relevant value in a local variable. Essentially, AcclDesc needs to perform one read operation for every iteration of the external loop, to determine the value of \( d_q \). If \( d_q = 2 \) we execute the special case and quickly generate the next descending composition; otherwise, we apply the general case. We cannot keep the value of \( d_q \) locally because the value of \( q \) changes constantly, and so we do not spend significant periods of time operating on the same array indices, as we do in AcclAsc. Thus, we must read the value of \( d_q \) for every transition, and we can therefore simplify by saying that AcclDesc(\( n \)) requires \( p(n) \) read operations.

In the interest of the fairest possible comparison between ascending and descending compositions generation algorithms, let us therefore simplify, and assume that any descending composition generation algorithm utilising the same properties as AcclDesc requires \( p(n) \) read operations. We know from (24) that our ascending composition generation algorithm required only \( p(n) - p(n - 2) \) reads. We can therefore expect that an ascending composition generator will require \( p(n - 2) \) less read operations than a descending composition generator similar to AcclDesc. Other things being equal, we should expect a significant difference between the total time required to generate all partitions using an ascending composition generation algorithm and a commensurable descending composition generator.

We can gain a qualitative idea of the differences involved if we examine the average numbers of read and write operations using the asymptotic values of \( p(n) \). Again, to determine the average number of read and write operations required per partition generated we must divide the totals involved by \( p(n) \). We stated earlier that the value of \( p(n) - p(n - 2) \) is asymptotically negligible compared to \( p(n) \); we can quantify this statement using the asymptotic formulas for \( p(n) \). Knuth [27, p.11] provides an approximation of \( p(n - 2)/p(n) \), which can be expressed as follows:

\[ \frac{p(n - 2)}{p(n)} \approx \frac{1}{e^{2\pi/\sqrt{6n}}} \quad (26) \]
Using this approximation, we obtain the following estimates for the average number of read and write operations required to generate each ascending and descending composition of \( n \).

|        | Reads | Writes |
|--------|-------|--------|
| Ascending | \( 1 - e^{-2\pi/\sqrt{6n}} \) | \( \frac{2}{\sqrt{6n}} \) |
| Descending | \( 1 \) | \( 1 + e^{-2\pi/\sqrt{6n}} \) |

Suppose we wished to generate all partitions of 1000. Then, using the best known descending composition generation algorithm we would expect to make 1 read and 1.92 write operations per partition generated. On the other hand, if we used ACCELASC, we would expect to make only 0.08 read and 2 write operations per partition.

The qualitative behaviour of ACCELASC and ACCELDESC can be seen from their read and write tapes (Figure 2). Comparing the write tapes for the algorithms, we can see that the total number of write operations is roughly equal in both algorithms, although they follow an altogether different spatial pattern. The read tapes for the algorithms, however, demonstrate the essential difference between the algorithms: ACCELDESC makes one read operation for every partition generated, while the read operations for ACCELASC are sparsely distributed across the tape.

We have derived expressions to count the total number of read and write operations required to generate all partitions of \( n \) using ACCELASC and ACCELDESC. We can now use these expressions to make some quantitative predictions about the relative efficiencies of the algorithms. If we assume that the cost of read and write operations are equal, we can then derive a prediction for the ratio of the total time elapsed using both algorithms. Therefore, let \( E_{4.1}(n) \) be the expected total running time of ACCELASC\((n)\), and similarly define \( E_{4.2}(n) \) for ACCELDESC\((n)\). We can then predict that the ratio of the running times should be equal to the ratio of their total read and write counts. Thus, using the values of (24) and (25), we get

\[
\frac{E_{4.1}(n)}{E_{4.2}(n)} = \frac{3p(n) - p(n - 2)}{3p(n)}.
\]

Consequently, we expect that the total amount of time required to generate all ascending compositions of \( n \) should be a factor of \( p(n - 2)/3p(n) \) less than that required to generate all descending compositions of \( n \). To test this hypothesis we measured the total elapsed time required to generate all partitions of \( n \) using ACCELASC and ACCELDESC, using the methodology outlined in Section 2.3. We report the ratio of these times in Table 2, for both the C and Java implementations of the algorithms.

Table 2 supports our qualitative predictions well. The theoretical analysis of ascending and descending composition generation algorithms in this section suggests that the ascending composition generator should require significantly less time to generate all partitions of \( n \) than its descending composition counterpart; and the data of Table 2 supports this prediction. In the Java implementations,
Figure 2: Read and write tapes for the accelerated algorithms to generate ascending and descending compositions. On the left we have the read and write tapes for the ascending composition generator, Algorithm 4.1; on the right, then, are the corresponding tapes for the descending composition generator, Algorithm 4.2. In both cases, the traces correspond to the read and write operations carried out in generating all partitions of 12.
Table 2: Empirical analysis of accelerated ascending and descending composition generation algorithms. The ratio of the time required to generate all partitions of \( n \) using AccelAsc and AccelDesc is given: measured ratios for implementations in the Java and C languages as well as the theoretically predicted ratio are shown.

| \( n \)   | \( p(n) \)           | Java | C   | Theoretical |
|----------|----------------------|------|-----|-------------|
| 100      | \( 1.91 \times 10^8 \) | 0.85 | 0.77| 0.74        |
| 105      | \( 3.42 \times 10^8 \) | 0.85 | 0.77| 0.74        |
| 110      | \( 6.07 \times 10^8 \) | 0.84 | 0.75| 0.74        |
| 115      | \( 1.06 \times 10^9 \) | 0.84 | 0.75| 0.73        |
| 120      | \( 1.84 \times 10^9 \) | 0.83 | 0.75| 0.73        |
| 125      | \( 3.16 \times 10^9 \) | 0.83 | 0.74| 0.73        |
| 130      | \( 5.37 \times 10^9 \) | 0.83 | 0.74| 0.73        |
| 135      | \( 9.04 \times 10^9 \) | 0.82 | 0.74| 0.73        |

\( r_{\text{Java}} = 0.9891 \) \( r_{\text{C}} = 0.9321 \)

the ascending composition generator requires 15% less time to generate all partitions of 100 than the descending composition generation algorithm; in the C version, the difference is around 23%. These differences increase as the value of \( n \) increases: when \( n = 135 \), we see that AccelAsc requires 18% and 26% less time than AccelDesc in the C and Java implementations, respectively.

We also made a quantitative prediction about the ratio of the time required to generate all partitions of \( n \) using AccelAsc and AccelDesc. Using the theoretical analysis, where we counted the total number of read and write operations required by these algorithms, we can predict the expected ratio of the time required by both algorithms. This ratio is also reported in Table 2, and we can see that it is consistent with the measured ratios for the Java and C implementations of the algorithms. In the case of the Java implementation, the theoretically predicted ratios are too optimistic, suggesting that the model of counting only read and write operations is a little overly simplistic in this case. The correspondence between the measured and predicted ratios in the C implementation is much closer, as we can see from Table 2. In both cases there is a strong positive correlation between the predicted and measured ratios.

5 Conclusion

In this paper we have systematically compared algorithms to generate all ascending and descending compositions, two possible encodings for integer partitions. In Section 2 we compared two recursive algorithms: our new ascending composition generator, and Ruskey’s descending composition generator. By analysing these algorithms we were able to show that although both algorithms are constant amortised time, the descending composition generator requires approximately twice as long to generate all partitions of \( n \). In Section 3 we compared
two generators in Kemp’s idiom: succession rules that require no state to be maintained between transitions. We developed a new succession rule for ascending compositions in lexicographic order, and implemented the well known succession rule for descending compositions in reverse lexicographic order. The analyses of these algorithms showed that the ascending composition generator required constant time, on average, to make each transition; whereas the descending composition generator required $O(\sqrt{n})$ time. Section 4 then compared the most efficient known algorithms to generate all ascending and descending compositions. We developed a new generation algorithm for the ascending compositions by utilising structure within the set of ascending compositions. We also analysed Zoghbi & Stojmenović’s algorithm and compared these two algorithms theoretically and empirically. As a result of this analysis, we showed that the ascending composition generator requires roughly three quarters of the time required by the descending composition generator. These three comparisons of algorithms show that ascending compositions are a superior encoding for generating all partitions.

Generation efficiency is not the only advantage of encoding partitions as ascending compositions. As part of our analysis of the succession rule for ascending compositions in Section 3 we proved a new formula for computing the number of partitions of $n$ in terms of the largest and second largest parts. In Section 4.1 we developed a new proof for a combinatorial identity, showing that the number of partitions of $n$ where the largest part is less than twice the second largest part is equal to the number of partitions of $n - 2$. These mathematical results were motivated by studying algorithms to generate ascending compositions.

Another advantage of using ascending compositions to encode partitions, not mentioned here, is the possibility of developing algorithms to generate a very flexible class of restricted partition. By generalising the algorithms developed in this paper it is possible to generate (and enumerate) combinatorially important classes of partition such as the partitions into distinct parts [8, §2], Rogers-Ramanujan partitions [17] and Göllnitz-Gordon partitions [3]. The framework for describing these restrictions and developing generation and enumeration algorithms is described by Kelleher [22, ch.3–4].

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