Intermittency in the $q$-state Potts model

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Abstract

We define a block observable for the $q$-state Potts model which exhibits an intermittent behaviour at the critical point. We express the intermittency indices of the normalised moments in terms of the magnetic critical exponent $\beta/\nu$ of the model. We confirm this relation by a numerical simulation of the $q = 2$ (Ising) and $q = 3$ two-dimensional Potts model.

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1 Introduction

Recently, the concept of intermittency has been introduced in the field of equilibrium critical phenomena as a tool for studying local fluctuations of a system undergoing a second order phase transition [1, 2]. For example, in the Ising model, as the the magnetisation fluctuates without limit at the critical point, one expects that the normalised moments of this observable, measured on sub-blocks of the system, scale with the size of the block, with exponents that may depend on the order of the moment, the so-called intermittency indices. On the basis of real space renormalisation group arguments, Satz [3] established that, for the Ising model, the intermittency indices are directly connected to the magnetic critical exponent. However, the normalised moments of the magnetisation used in Satz’s argument are not suitable for a numerical simulation, as the net magnetisation, which enters in the denominator of the moments, is almost zero on a (large) finite lattice. In subsequent numerical simulations, Bambah et. al. [4] and Gupta et. al. [5], using a new observable which respects the \( \mathbb{Z}_2 \) symmetry and remains finite on finite lattices, present clear evidence for an intermittent behaviour in the two-dimensional Ising model. However, they do not observe the expected relation between the intermittency exponents and the critical ones. Burda et. al. [6] suggest that this is a consequence of their choice for the fluctuating observables, among which the moments have only a subdominant scaling behaviour in the critical regime.

Therefore, the problem of finding convenient block observables which have well-defined moments on finite lattices and which exhibit the expected scaling law, remains open. In this paper we address this question in the more general framework of the \( q \)-state Potts model.

In section 2, we define the fluctuating observable for which we derive the intermittency indices in terms of the magnetic exponent by generalising the Satz’s argument [3]. In section 3 we give the details of the numerical simulation and discuss the results. Conclusions are drawn in the last section.

2 Definition of the observables

The most convenient definition of the moments for studying critical spin systems has been thoroughly discussed in ref [1, 2]. Following the authors
of ref. \[5\] we use the so-called standard block moments of order \(p\) defined as

\[ F_p(L; \ell) = \frac{1}{M} \sum_{\alpha} \frac{< k_\alpha^p >}{< k_\alpha >^p} \]  

(1)

where \(L\) is the lattice size, \(\ell\) the cell size, \(M = (L/\ell)^d\) the number of cells and \(k_\alpha\) the block observable defined on the \(\alpha^{th}\) cell. The symbols \(<>\) stands for the thermodynamical average which will be taken at the bulk critical temperature \(T_c\) defined by \(K_c = J/k_b T_c = \ln[1 + \sqrt{q}]\) for the \(q\)-state Potts model. We recall that the intermittency indices \(\lambda_p\) are defined according to the behaviour of the moments with respect to the block size \(\ell\):

\[ F_p(L; \ell) \sim \left( \frac{\ell}{L} \right)^{-\lambda_p} \quad \text{for} \quad \frac{\ell}{L} \to 0 \]

Let us now define the block observable \(k_\alpha\).

We consider a system of Potts spins \(s_i = 0, 1, \ldots, q - 1\), interacting with their nearest neighbours via the hamiltonian

\[ -\beta H = K \sum_{<i,j>} \delta_{s_i s_j} \]

on a two-dimensional square lattice of linear size \(L\). For a particular configuration, we denote by \(Q\) the spin value which occurs the most frequently and to each site \(i\) we assign the variable \(n_i = \delta_{s_i Q}\). We now define

\[ k_i = \frac{qn_i - 1}{q - 1} \]

which is such that

\[ \frac{1}{L^d} < \sum_i k_i > = < k_i > = < m_q > \]

is the order parameter for the Potts model \[7\]. For the Ising model \((q = 2)\) this definition reduces to

\[ k_i = \text{sgn}(m_I)\sigma_i \]

where \(m_I\) is the Ising magnetisation of the configuration of Ising spins \(\{\sigma_i = \pm 1\}\).

We then define the block observable to be

\[ k_\alpha = \frac{1}{\ell^d} \sum_{i \in \alpha} k_i = \frac{qn^{(\alpha)}_Q - 1}{q - 1} \]  

(2)
where \( n_Q^{(\alpha)} \) is the fraction of spins in the cell \( \alpha \) having the value \( Q \).

Unlike the original definition of ref [3], on a finite lattice, \( \langle k_\alpha \rangle \) is non-zero at the critical point, \( \langle k_\alpha \rangle = \langle |m_f| \rangle \) for the Ising model, allowing the moments of eq. (2.1) to be measured in a numerical simulation. Moreover, one can extend to this observable the renormalisation argument which predicts the intermittent behaviour of the moments.

Let us associate to the blocking procedure the following renormalisation prescription: we define a block spin \( s_\alpha \) according to the majority rule in the \( \alpha \)th cell, associate the corresponding renormalised site variable \( \tilde{n}_Q^{(\alpha)} = \delta_{s_\alpha Q} \) and then the renormalised quantity \( \tilde{k}_\alpha \)

\[
\tilde{k}_\alpha = \frac{q \tilde{n}_Q^{(\alpha)} - 1}{q - 1}
\]

Since \( k_\alpha \) is related to the magnetic scaling operator, we expect the following renormalisation relation

\[
k_\alpha = Z(\ell) \tilde{k}_\alpha
\]

where \( Z(\ell) \) is the renormalisation factor depending only on the block size \( \ell \). Therefore

\[
\langle k^p_\alpha \rangle = [Z(\ell)]^p < \left( \frac{q \tilde{n}_Q^{(\alpha)} - 1}{q - 1} \right)^p \\
= [Z(\ell)]^p \left[ \tilde{n}_Q^{(\alpha)} > \left( \frac{1}{1 + \frac{(-1)^{(p+1)}(q-1)^p}{(q - 1)^p}} + \frac{(-1)^p}{(q - 1)^p} \right) \right]
\]

(4)

where we have used \( (\tilde{n}_Q^{(\alpha)})^p = \tilde{n}_Q^{(\alpha)} \). Reintroducing \( \tilde{k}_\alpha \) in the left-hand side of eq. (2.4), we get the relation

\[
\langle k^p_\alpha \rangle = [Z(\ell)]^p \left( A_p \langle \tilde{k}_\alpha \rangle + B_p \right)
\]

(5)

where the coefficients \( A_p \) and \( B_p \) are given exactly by

\[
A_p = \frac{q - 1}{q} \left( 1 + \frac{(-1)^{(p+1)}}{(q - 1)^p} \right)
\]

(6)

\[
B_p = \frac{1}{q} \left( 1 + \frac{(-1)^{(p+1)}}{(q - 1)^p} \right) + \frac{(-1)^p}{(q - 1)^p}
\]

(7)

\(^1\)For a given configuration, the value of \( Q \) is preserved by the blocking procedure.
According to the standard renormalisation analysis, we expect \(Z(\ell) \sim \ell^{-\frac{1}{2}(d-2-\eta)} = \ell^{-\beta/\nu}\). Moreover, \(<k_\alpha>\) and \(<\tilde{k}_\alpha>\) are the order parameter of the system of size \(L\) and of the rescaled system of size \(L/\ell\), respectively. At the bulk critical temperature, these quantities behave according to the finite size scaling law

\[
<k_\alpha> \sim L^{-\beta/\nu} \quad (8)
\]

\[
<\tilde{k}_\alpha> \sim (L/\ell)^{-\beta/\nu} \quad (9)
\]

Thus, for \(\ell \gg 1\) (in lattice units) in order to sum up enough degrees of freedom for the renormalisation arguments to be valid and \(\ell \ll L\) to avoid finite size effects, we expect the moments \(F_p\) to behave like

\[
F_p(L; \ell) \sim \left(\frac{\ell}{L}\right)^{-p\frac{\beta}{\nu}} A'_p \left(\frac{\ell}{L}\right)^w + B'_p \quad (10)
\]

where the coefficients \(A'_p\) and \(B'_p\) are proportional to the coefficients \(A_p\) and \(B_p\) defined above.

For the Ising model \((q = 2)\), eqs (2.6) and (2.7) give \(A_p \equiv 0\) for \(p\) even and \(B_p \equiv 0\) for \(p\) odd, leading to the scaling law, derived previously by Satz [3]

\[
F_p(L; \ell) \sim \left(\frac{\ell}{L}\right)^{-p\frac{\beta}{\nu}} \quad \text{for } p \text{ even} \quad (11)
\]

\[
\sim \left(\frac{\ell}{L}\right)^{-(p-1)\frac{\beta}{\nu}} \quad \text{for } p \text{ odd} \quad (12)
\]

For the three-state Potts model such a cancellation does not occur and the leading behaviour \((L/\ell)^{-p\beta/\nu}\) is affected by a corrective term as shown in eq. (2.10). However, similar corrections also appear in the scaling laws of eqs (2.9), which, when taken into account, modify the exact exponent \(\beta/\nu\) in eq.(2.10). Therefore, for the three-state Potts model, we expect a more general form

\[
F_p(L; \ell) \sim \left(\frac{\ell}{L}\right)^{-p\frac{\beta}{\nu}} \left[A''_p \left(\frac{\ell}{L}\right)^w + B''_p\right] \quad (13)
\]

where \(A''_p\), \(B''_p\) and \(w\) are unknown parameters.
From this behaviour we deduce the intermittency indices:

For the Ising model

\[ \lambda_p = \begin{cases} \frac{1}{8} p & \text{for } p \text{ even} \\ \frac{1}{8} (p - 1) & \text{for } p \text{ odd} \end{cases} \]  

\[ (14) \]

For the three-state Potts model

\[ \lambda_p = \frac{2}{15} p \]

3 Numerical tests

We have performed a simulation of the \( q = 2 \) and \( q = 3 \) Potts models at the bulk critical temperature, using the Swendsen Wang dynamics, on two-dimensional lattices of size up to \( 256 \times 256 \). For the largest lattice sizes, the data were taken on four independent runs of \( 2 \times 10^4 \) Monte-Carlo lattice updates, with a spacing between two consecutive measurements of five MC lattice updates. The error analysis is realized on this total sample of \( 1.6 \times 10^4 \) independent measurements, leading to quite small error bars (of the order of the width of the data points). In order to check the reliability of our simulation, we have tested the scaling law for the order parameter, \( < m_q > \) (eq. 2.8), for lattice sizes ranging from \( L = 16 \) up to \( L = 256 \). The results are displayed in figure 1, with a linear fit in log scale, giving \( \beta/\nu = 0.124(5) \) for the Ising model (the exact value is \( 1/8=0.125 \)) and \( \beta/\nu = 0.131(7) \) for the three-state Potts model (the exact value is \( 2/15=0.133 \)).

We then determine the intermittency indices \( \lambda_p \). We measure the moments for \( p = 2 \) to \( p = 5 \) and for \( \ell = 2, 4, 8, \ldots, L \). Notice that the values of the moments for \( \ell = L \) approach, as \( L \to \infty \), the ratio of universal critical amplitudes. For instance, the ratio \( [F_2(L, \ell = L)]^2 / F_4(L, \ell = L) \) has been thoroughly studied [8] and found to have the value 0.85622 in the Ising model [9]. Our result is 0.856(4) providing a further check of our data.

In figure 2 we show the moments for the Ising model on a \( 256 \times 256 \) periodic lattice. The behaviour of \( F_p \) depending on the parity of \( p \) depicted by eqs (2.11-12) is quite visible on the data. According to these equations,
we have performed a power law fit of the moments for $4 \leq \ell \leq L/2$, shown as the straight lines in figure 2. We observe that the odd moments are remarkably well fitted by the power law, whereas the even moments exhibit deviations from this leading behaviour. Therefore, we proceed to a double determination of the exponents: first, from the exact power law of eqs.(2.11-12) and second, from a corrected parametrisation of the form of eq.(2.13). The results of both fitting methods for the largest lattice size $L = 256$ are shown in the last two lines of table I. The difference between the values corresponding to the same moment give an estimate of the systematic error on the exponents. In addition, we give the result of the power law fit for the $L = 128$ lattice, which shows the stability of this determination.

| $L$ | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|
| 128 | 0.111 | 0.124 | 0.117 | 0.123 |
| 256 | 0.114 | 0.124 | 0.119 | 0.123 |
| 128 | 0.135 | 0.126 | 0.129 | 0.122 |

Table I  The exponents $\lambda_p/p$ ($p$ even) and $\lambda_p/(p - 1)$ ($p$ odd) for the Ising model on lattices of size $L = 128, 256$ and for the moments of order $p = 2, 3, 4, 5$. The exponents given in the bottom line correspond to a fit of the $L = 256$ moments with a corrected scaling law. The expected exact value is $\beta/\nu = 0.125$.

The agreement of these results with the prediction of eqs.(2.14) $\beta/\nu = 0.125$, is qualitatively good for the even moments and excellent for the odd ones.

We have repeated the same analysis for the three-state Potts model. The moments are shown in figure 3 for the largest lattice size $L = 256$. The fit corresponds to the corrected power law fit of eq.(2.13)

$$F_p(L; \ell) = \left(\frac{\ell}{L}\right)^{-px} \left[ a \left(\frac{\ell}{L}\right)^w + b \right]$$

where $x, w, a, b$ are free parameters. The results are shown in table II for the lattice size $L = 256$. 

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Table II  The exponents $x = \lambda_p/p$ for the moments of order $p = 2, 3, 4, 5$ for the three-state Potts model on the $L = 256$ lattice. The expected exact value is $\beta/\nu = 0.133$. We show the fitted values of the other variables entering the parametrisation of the moments $F_p(z) = z^{-p}(az^w + b)$ where $z = \ell/L$. $x$ values from the $L = 128$ data are reported at the bottom line.

| $L$ | $p$ | 2     | 3     | 4     | 5     |
|-----|-----|-------|-------|-------|-------|
|     | $x$ | 0.141 | 0.124 | 0.131 | 0.125 |
| 256 | $\pm$ | 0.015 | $\pm$ | 0.009 | $\pm$ | 0.009 | $\pm$ | 0.011 |
|     | $w$ | 0.359 | 0.270 | 0.307 | 0.235 |
|     | $a$ | 0.652 | 0.772 |
|     | $b$ | 0.516 | 0.503 | 0.521 | 0.560 |
| 128 | $x$ | 0.144 | 0.119 | 0.128 | 0.124 |
|     | $\pm$ | 0.022 | $\pm$ | 0.017 | $\pm$ | 0.020 | $\pm$ | 0.011 |

The errors quoted are the statistical ones corresponding to 95% confidence level. They are quite large due to the freedom allowed by our parametrisation. Actually, we observe that, if we set the exponents to the exact value, $x = 0.133$, the minimum $\chi^2$ does not change significantly with respect to its best-fit value. We give the other variables of our parametrisation in Table II. They are only weakly dependent on the order of the moment, but the exponents $w$ differ significantly from $x$, indicating, as expected, a strong contamination of the behaviour of eq.(2.10) by corrections to the scaling law of eq. (2.9). The exponents obtained from the fit of the $L = 128$ data are reported at the bottom line of table II, showing again the stability of their determination.

Nevertheless, the agreement with the expected intermittency indices is again quite good.

4 Conclusions

We have defined a block observable for the $q$-state Potts model which exhibits an intermittent behaviour at the critical point of the model. The moments of this observable are measurable on finite lattices and have well-defined scaling behaviour in terms of the block-size. By a numerical simula-
tion, we confirm the previously observed intermittent behaviour, and we find good agreement between the measured exponents and the magnetic exponent $\beta/\nu$, as predicted by renormalisation group arguments.

Although a critical spin system is presumably not multifractal, a complete description of its scaling properties requires several exponents. Besides the thermal and magnetic critical indices which drive the fluctuations of energy and of magnetisation, the geometrical aspects of the critical clusters of spins are characterised by other exponents, not related to the thermodynamics ones \cite{11, 12, 13}. For this reason, the intermittency phenomenon in a critical system may assume several aspects depending on the definition of the block observable. This may explain the results of ref \cite{5}, where the measured intermittency indices coincide with the fractal dimension of the clusters of spins. However, further theoretical and numerical studies are needed in order to investigate the geometrical origin of intermittency.

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References

[1] R. Peschanski, *Int. J. Mod. Phys.* **A6** (1991) 3681 and references therein.

[2] J. Wosiek, *Acta Phys. Pol.* **B19** (1988) 863

[3] H. Satz, *Nucl. Phys.* **B326** (1989) 613

[4] B. Bambah, J. Fingberg, H. Satz *Nucl. Phys.* **B332** (1990) 629

[5] S. Gupta, P. Lacock, H. Satz *Nucl. Phys.* **B362** (1991) 583

[6] Z. Burda, J. Wosiek, K. Zalewski *Phys. Lett.* **B266** (1991) 439

[7] F.Y. Wu *Rev. Mod. Phys.* **54** (1982) 235

[8] This is an explicit realisation of the formal definition given by D. Hajduković, CERN preprint CERN-TH.6440/92

[9] K. Binder, *Z. Phys.* **B43** (1981) 119

[10] G. Kamieniarz, HW Blöte *J. Phys.* **A26** (1993) 201

[11] A. Coniglio, C. Nappi, F. Peruggi and L. Russo *J. Phys.* **A10** (1977) 205

[12] C. Vanderzande and A. Stella, *J. Phys.* **A22** (1989) L445, C. Vanderzande, *J. Phys.* **A25** (1991) L75

[13] H.W. Blöte, Y.M.M. Knops and B. Nienhuis *Phys. Rev. Let.* **68** (1992) 3440
Figure Captions

Figure 1 The order parameter for the $q = 2$ (Ising) and $q = 3$ Potts model, measured at the bulk critical temperature as a function of the lattice size. The lines result from a power fit, giving the exponent $\beta/\nu = 0.124(5)$ for the Ising model (exact value 0.125) and $\beta/\nu = 0.131(7)$ for the three-state Potts model (exact value 0.133).

Figure 2 The moments of order $p = 2, 3, 4, 5$ for the Ising model as a function of the block size $\ell$. The straight lines result from a power law fit.

Figure 3 The moments of order $p = 2, 3, 4, 5$ for the three-state Potts model as a function of the block size $\ell$. The curves result from a corrected power law fit.
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