Solution Independent Analysis of Black Hole Entropy in Brick Wall Model

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Abstract
Using the brick wall regularization of ’t Hooft, the entropy of non-extreme and extreme black holes is investigated in a general static, spherically symmetric spacetime. We classify the singularity in the entropy by introducing a new index $\delta$ with respect to the brick wall cut-off $\epsilon$. The leading contribution to entropy for non-extreme case ($\delta \neq 0$) is shown to satisfy the area law with quadratic divergence with respect to the invariant cut-off $\epsilon_{\text{inv}}$ while the extreme case ($\delta = 0$) exhibits logarithmic divergence or constant value with respect to $\epsilon$. The general formula is applied to Reissner-Nordström, dilaton and brane-world black holes and we obtain consistent results.

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1 Introduction

An important part of black hole physics is concerned with its dynamical behavior which includes studying its thermal property [1], information content [2], not to mention the possible laboratory simulation of the behavior. A basic ingredient in black hole thermodynamics is the notion of entropy. The entropy $S$ of a standard non-extreme black hole obeys the well known Bekenstein-Hawking area law, $S = A/(4G)$, where $A$ is the area of the horizon [3, 4, 5]. This result can be derived by the action integral method for the Kerr-Newman solutions and de Sitter space [6] and for dilaton black holes [7]. The contribution to the entropy of quantum fields in black hole backgrounds was studied using the brick wall model [8], the WKB approximation [9] and the path integral method [10]. However, the entropy due to quantum fields in the black hole background introduces divergences which are interpreted as renormalizations of the gravitational coupling constant $G$.

In the extreme case, the situation is different, that is, the area law does not hold and the classical entropy becomes zero by the topological argument [11] and by the Gauss-Bonnet theorem [12]. It has been argued that extreme and non-extreme black holes should be regarded as qualitatively different objects due to discontinuity in the Euclidean topology. In this context, one would recall that the extreme dilaton black holes have zero area and hence $S = 0$ [13]. However, its temperature could be zero or infinite (even arbitrary) as the Euclidean section is smooth without identification [14]. However, $S$ does not remain zero when quantum field contributions are taken into account: the linearly divergent contribution vanishes but the logarithmic divergence persists [15]. The situation is somewhat similar to the case for the Reissner-Nordström black holes [11, 16].

Additionally, given the current widespread interest in the brane theory, it would be important to assess similar quantum field contributions to entropy in the background of brane-world black holes. The brane theory is described by the Randall-Sundrum framework [17] and the black hole solutions are obtained by solving the Shiromizu-Maeda-Sasaki equations [18]. The general class of four dimensional black hole solutions has been recently obtained by Bronnikov, Melnikov and Dehnen [19]. Typically, black holes on the brane are characterized by induced tidal charge [20], negative energy [21], non-singular nature [22] and so on. In these respects, they are different from the Reissner-Nordström and dilaton black holes.

The purpose of this paper is to systematically study the quantum scalar field
contributions to the entropy using the general form of static, spherically symmetric background spacetime. The regularization is done using the standard ’t Hooft brick wall model in evaluating the scalar field contribution to entropy [8]. Now, in the literature, the black hole radiation is assumed to emerge from the inner boundary wall itself and the Hawking temperature in the observed power spectrum followed quite beautifully [23]. In a similar methodological spirit, we define the temperature at the brick wall position as the local temperature that leads to the correct Hawking temperature in the horizon limit. With this regularization scheme for temperature and entropy together, we want to examine the relation between non-extreme and extreme black holes in detail. It might be parenthetically noted that there are several other regularization schemes too. For instance, Pauli-Villars regulator was applied to study black hole entropy by Demers, Lafrance and Myers [24]. Also, the concept of local temperature was applied to extreme black holes by Wang, Su and Abdalla [25] that yielded consistent results. On the problem of black hole entropy, see also [26].

The plan of the paper is as follows: Formulation of the scalar field entropy in the brick wall model will be done in section 2. We apply the general expression for the entropy to Reissner-Nordström, dilaton and brane-world black holes in section 3. Special interest is in the relation between extreme, non-extreme black holes. Summary and discussion will appear in section 4. The rationale and validity of our regularization scheme are elucidated in the Appendix.

2 Entropy of scalar field in the gravitational background

In order to study the black hole entropy for a scalar field in the brick wall model of ’t Hooft [8], we start with the general diagonal expression for the static background metric using units such that \( \hbar = c = G = 1 \) unless otherwise specified.

2.1 The brick wall model

General arguments imply that the metric functions depend only on radial coordinate \( r \) in (3+1) space-time dimension:

\[
d s^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{\theta\theta}(r)d\Omega^2,
\]

(1)
where $d\Omega_2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element on a unit sphere. The matter action for the scalar field with mass $m$ is

$$I_M(\phi) = -\frac{1}{2} \int d^4x \sqrt{-g} \left(g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi + \mu^2 \phi^2\right).$$

(2)

The field equation for the scalar field is obtained from this action as

$$\frac{1}{\sqrt{-g}} \partial_\rho \left( \sqrt{-g} g^{\rho\sigma} \partial_\sigma \phi \right) - \mu^2 \phi = 0.$$  

(3)

We may take the ansatz for the scalar field $\phi$ in the form

$$\phi(t, r, \theta, \varphi) = \exp(-iEt) Y_{\ell m}(\theta, \varphi) f_{E,\ell}(r),$$

(4)

and obtain the radial equation as

$$\left( -\frac{E^2}{g_{tt}} + \frac{1}{g_{\theta\theta}} \partial_r g_{\theta\theta} \sqrt{-g} g_{t\theta} \partial_r - \frac{\ell(\ell + 1)}{g_{\theta\theta}} - \mu^2 \right) f_{E,\ell}(r) = 0.$$  

(5)

Here we write the radial function as

$$f_{E,\ell}(r) \sim \exp \left( \pm i \int_r^r dr k_{E,\ell}(r) \right),$$

(6)

where the radial momentum $k_{E,\ell}$ in eq.(6) is obtained from eq.(5) in the semi-classical approximation as

$$k_{E,\ell} = \sqrt{g_{rr} \left( -\frac{E^2}{g_{tt}} - \frac{\ell(\ell + 1)}{g_{\theta\theta}} - \mu^2 \right)}.$$  

(7)

The non-negative integer number of radial modes $n_{E,\ell}$ using the semi-classical quantization condition in the brick wall model is given by

$$\pi n_{E,\ell} = \int_{r_h+\epsilon}^{r_h+L} dr k_{E,\ell}(r),$$

(8)

where the brick wall distance $\epsilon$ denotes an ultraviolet cutoff from $r_h$ and radius $L$ denotes an infrared cutoff measured from $r_h$. The total number of states with energy $E$ is given by

$$g(E) = \sum_{\ell, \ell_z} n_{E,\ell} \simeq \int_0^{\ell_{max}} d\ell (2\ell + 1) \frac{1}{\pi} \int_{r_h+\epsilon}^{r_h+L} dr k_{E,\ell}(r).$$

(9)
To study the thermodynamics of black holes, we make the Wick rotation from Minkowski time $t$ to Euclidian time $\tau = it$ so that the metric eq.(1) becomes

$$ds^2 = g_{\tau\tau}(r)d\tau^2 + g_{rr}(r)dr^2 + g_{\theta\theta}(r)d\Omega_2,$$

(10)

where $g_{\tau\tau} = -g_{tt}$ is the Euclidean time component of the metric. The free energy $F(\beta)$ at the inverse temperature $\beta = 1/T$ is obtained by the partition function $Z_M(\beta)$ for the matter action in eq.(2) as

$$Z_M(\beta) = \int [d\phi] e^{-I_M(\phi)},$$

$$F(\beta) = -\frac{1}{\beta} \ln Z_M(\beta) = -\int_0^\infty dE \frac{g(E)}{e^{\beta E} - 1}. \quad (11)$$

The integration in eq.(11) with respect to angular momentum $\ell$ can be performed using eq.(9) and the free energy is expressed in the form

$$F(\beta) = -\frac{1}{\pi} \int_0^\infty dE \frac{1}{e^{\beta E} - 1} \int_{r_h+\epsilon}^{r_h+L} dr \frac{2}{3}(g_{rr})^{1/2} g_{\theta\theta} \left( \frac{E^2}{g_{\tau\tau}} - \mu^2 \right)^{3/2}. \quad (12)$$

If the mass of the scalar field is zero, then the expression of the free energy becomes simpler, viz.,

$$F(\beta) = -\frac{2}{45\pi} \left( \frac{\pi}{\beta} \right)^4 \frac{V}{4\pi}, \quad (13)$$

where $V$ denotes the volume of optical space defined as

$$V := 4\pi \int_{r_h+\epsilon}^{r_h+L} dr \left( g_{\tau\tau} \right)^{-3/2} (g_{rr})^{1/2} g_{\theta\theta}. \quad (14)$$

Then the entropy of the black holes is obtained in a compact form

$$S := \beta^2 \frac{\partial F(\beta)}{\partial \beta} = \frac{1}{45} \left( \frac{2\pi}{\beta} \right)^3 \frac{V}{4\pi}. \quad (15)$$

This result coincides with that obtained by the path integral method by de Alwis and Ohta [10]. We need to evaluate the temperature on the horizon in order to obtain the value of entropy in eq.(15). The temperature is defined by the condition that no conical singularity is imposed on the Euclidean Rindler space.
While this prescription consistently yields a unique temperature of the horizon for non-extreme black holes, it fails for the extreme case. Nevertheless, we introduce the ultraviolet regularization $\epsilon$ and uniformly define the temperature that should be valid for extreme as well as the non-extreme case, that is, we define the local temperature [25]:

$$\frac{2\pi}{\beta} = \frac{\partial_r g_{rr}}{2\sqrt{g_{rr}g_{rr}}} \bigg|_{r=r_h+\epsilon}.$$  

(16)

Note that the regularization position is not just anywhere off the horizon but at the boundary position of the brick wall infinitesimally close to the horizon. The background (Hawking) temperature and gravitational entropy of the horizon (at $r = r_h$) are consistently determined from the above equation (with $\epsilon \to 0$) both for non-extreme and extreme black holes. The meaning of this temperature regularization is further illustrated in the Appendix in the extreme Reissner-Nordström case.

2.2 Laurent expansion around the horizon

In order to evaluate the entropy formula of eq.(15) independently of any explicit form of black hole solutions, we make Laurent expansion of the metric around the horizon $r \sim r_h$ as

$$g_{rr}(r) = (r - r_h)^a \sum_{i=0}^{\infty} a^{(i)}(r_h)(r - r_h)^i,$$

$$g_{rr}(r) = (r - r_h)^b \sum_{i=0}^{\infty} b^{(i)}(r_h)(r - r_h)^i,$$

$$g_{\theta\theta}(r) = (r - r_h)^c \sum_{i=0}^{\infty} c^{(i)}(r_h)(r - r_h)^i,$$

(17)

where the coefficients $a^{(0)}, b^{(0)}, c^{(0)}$, respectively, are assumed not to be zero and $a, b, c$ denote the leading exponents of Laurent expansion for each metric function. In the following, we keep only the ultraviolet singularity on the horizon, which is the main contribution to entropy.

A. Leading contribution

The volume of optical space in eq.(14) is evaluated using the expansion in
eq.(17), and the leading term of the contribution is obtained as
\[
\frac{V'(0)}{4\pi} = \begin{cases} 
(a^{(0)})^{-3/2}(b^{(0)})^{1/2}c^{(0)} \log \left( \frac{L}{\epsilon} \right) & \text{if } \gamma = 0 \\
(a^{(0)})^{-3/2}(b^{(0)})^{1/2}c^{(0)} \gamma^{-1} e^{-\gamma} & \text{otherwise ,}
\end{cases}
\] (18)
where we introduce \( \gamma \) as the index of exponent of the optical volume
\[
\gamma := \frac{3a}{2} - \frac{b}{2} - c - 1 .
\] (19)
The leading contribution to the local temperature \( T = 1/\beta \) at the brick wall distance \( \epsilon \) is obtained from eq.(16) as
\[
\frac{2\pi}{\beta(0)} = \frac{a}{2} (a^{(0)})^{1/2} (b^{(0)})^{-1/2} e^{a/2-b/2-1} .
\] (20)
The leading contribution to the entropy in eq.(15) is obtained by combining the expressions in eqs.(18) and (20)
\[
S^{(0)} = \begin{cases} 
\frac{1}{45} (a) \cdot (b^{(0)})^{-1} c^{(0)} \log \left( \frac{L}{\epsilon} \right) \epsilon^{-\delta} & \text{if } \gamma = 0 \\
\frac{1}{45} (a) \cdot (b^{(0)})^{-1} c^{(0)} \gamma^{-1} \epsilon^{-\delta} & \text{otherwise ,}
\end{cases}
\] (21)
where we introduce \( \delta \) as a new index of exponent of entropy
\[
\delta := b - c + 2 .
\] (22)

B. Area law
By defining the area of the brick wall surface
\[
A = \int d\Omega_2 g_{\theta\theta} \bigg|_{r=r_h+\epsilon} ,
\] (23)
and the invariant distance to the brick wall
\[
\epsilon_{\text{inv}} := \int_{r_h}^{r_h+\epsilon} dr (g_{rr})^{1/2} ,
\] (24)
which holds for \( b+2 > 0 \), we obtain the area law for the leading contribution to the entropy from eq.(21)
\[
S^{(0)} = \frac{1}{45} \left( \frac{a}{2} \right)^{3} \left( \frac{2}{b+2} \right)^{2} \gamma^{-1} (\epsilon_{\text{inv}})^{-2} \frac{4A}{4\pi} \frac{1}{4\pi} \text{ if } 0 < \delta \text{ and } \gamma \neq 0 .
\] (25)
The leading contribution to entropy for $\delta > 0$ is proportional to the area but diverges in $\epsilon$ (quadratically divergent in $\epsilon_{\text{inv}}$). This term is considered as the renormalization effect to the gravitational constant $G$ in the Bekenstein-Hawking entropy $S_{BH} = A/(4G)$, and is the same as the one suggested by Susskind and Uglum [9]. Note that the leading contribution to entropy for $\delta < 0$ is zero in the limit $\epsilon \to 0$.

C. Extreme case

We define the extreme case as $\delta = 0$ in which case the leading contribution to the entropy doesn’t show the area law. This contribution can directly be read off from eq.(21)

$$S^{(0)} = \begin{cases} 
\frac{1}{45} \left( \frac{a}{2} \right)^{3} (b^{(0)})^{-1} c^{(0)} \log \left( \frac{L}{\epsilon} \right) & \text{if } \delta = 0 \text{ and } \gamma = 0 \\
\frac{1}{45} \left( \frac{a}{2} \right)^{3} (b^{(0)})^{-1} c^{(0)} \gamma^{-1} & \text{if } \delta = 0 \text{ and } \gamma \neq 0 .
\end{cases}$$

(26)

The equation (26) shows that the leading contribution to entropy for $\delta = 0$ is constant or logarithmically divergent in $\epsilon$ and is not proportional to the area on the horizon. The coefficient term $(b^{(0)})^{-1} c^{(0)}$ is shown to be a numerical factor from the dimensional consideration. This term is considered as the renormalization of the quadratic-curvature coupling constant in the one-loop effective gravitational action suggested by Demers, Lafrance and Myers [24].

D. Next leading contributions

The next leading contribution to the black hole entropy $S^{(1)}$ is from the optical volume $V^{(1)}$ and the temperature $1/\beta^{(1)}$. They are calculated to be

$$\frac{V^{(1)}}{4\pi} = \begin{cases} 
(a^{(0)})^{-3/2} (b^{(0)})^{1/2} c^{(0)} X \log \left( \frac{L}{\epsilon} \right) & \text{if } \gamma = 1 \\
(a^{(0)})^{-3/2} (b^{(0)})^{1/2} c^{(0)} X \frac{\epsilon^{-\gamma+1}}{\gamma-1} & \text{if } \gamma \neq 1 ,
\end{cases}$$

(27)

with

$$X := -\frac{3a^{(1)}}{2a^{(0)}} + \frac{b^{(1)}}{2b^{(0)}} + \frac{c^{(1)}}{c^{(0)}} .$$

(28)

Similarly,

$$\frac{2\pi}{\beta^{(1)}} = \frac{a}{2} (a^{(0)})^{1/2} (b^{(0)})^{-1/2} Y e^{a/2 - b/2} ,$$

(29)
with

$$Y := \left( \frac{1}{2} + \frac{1}{a} \frac{a^{(1)}}{a^{(0)}} - \frac{1}{2} \frac{b^{(1)}}{b^{(0)}} \right).$$ \hfill (30)

Combining them we obtain the next leading contribution to the entropy as

$$S^{(1)} = \frac{2\pi^2}{45} \left( (\beta^{(0)})^{-3} V^{(1)} + 3(\beta^{(0)})^{-2}(\beta^{(1)})^{-1} V^{(0)} \right)$$

$$= \frac{1}{45} \left( \frac{a}{2} \right)^3 b^{(0)} \log (\epsilon) \epsilon^{-\delta + 1} Z,$$ \hfill (31)

where the function $Z$, given below, shows logarithmic divergence or constant value depending on the value of index $\gamma$:

$$Z := \begin{cases} 
-X + 3Y \log \frac{L}{\epsilon} & \text{if } \gamma = 0 \\
X \log \frac{L}{\epsilon} + 3Y & \text{if } \gamma = 1 \\
X \frac{1}{\gamma - 1} + 3Y \frac{1}{\gamma} & \text{if } \gamma \neq 0, 1.
\end{cases} \hfill (32)$$

We can immediately derive the conclusion that the next leading contribution to entropy $S^{(1)}$ is zero if $\delta < 1$, logarithmically divergent or constant if $\delta = 1$ and shows other type of divergence if $\delta > 1$, which are derived from eq.(31).

### 2.3 Classification of scalar field entropy in terms of indices

Combining the leading and next leading contributions from the ultraviolet region of the horizon, we can classify the scalar field entropy in the black hole background from eqs.(21) and (31) according to the two indices $\gamma$ in eq.(19) and $\delta$ in eq.(22). This is summarized in Table 1.

**Table 1.** Classification of the black hole entropy in terms of the indices
We remark that the leading contribution to the entropy $S^{(0)}$ in Table 1 satisfies the area law if $\delta > 0$ and $\gamma \neq 0$ as shown in eq.(25).

### 3 Examples

In this section we apply our model independent analysis of the black hole entropy for a scalar field in the brick wall model to some special black hole solutions. Examples that we consider are standard Schwarzschild, Reissner-Nordström, dilaton black hole and a typical example of brane-world black hole solutions, the corresponding metrics being listed in Table 2.

**Table 2.** Metrics of black hole solutions

| Black hole solution | Metric |
|---------------------|--------|
| $g_{\tau \tau}$   | $g_{rr}$ | $g_{\theta \theta}$ |
| Schwarzschild       | $1 - 2M/r$ | $1$ | $r^2$ |
| Reissner-Nordström  | $1 - 2M/r + Q^2/r^2$ | $1 - 2M/r + Q^2/r^2$ | $r^2$ |
| dilaton             | $1 - 2M/r$ | $1 - 2M/r$ | $r(r - p_0)$ |
| brane world         | $1 - 2M/r$ | $1 - 3M/2r$ | $r^2$ |

In Table 2, $M$ and $Q$ denote the black hole mass and charge respectively, and
$p_0$ and $q_0$ are parameters. We discuss these examples in the non-extreme and extreme cases separately.

### 3.1 Non-extreme black holes

Non-extreme black hole is defined as $M \neq Q$ for Reissner-Nordström, $p_0 \neq 2M$ for dilaton and $q_0 \neq 2M$ for the brane-world black holes. In all cases, the leading exponents of metrics defined in eq.(17) are of the same value $a = 1, b = -1$ and $c = 0$. This result leads the indices defined in eqs.(19) and (22) to have values $\gamma = 1$ and $\delta = 1$, which case we called the Schwarzschild-type in Table 1. The reason is that, for all non-extreme black hole solution of this type, the entropy behaves like that for the Schwarzschild solution with respect to the ultra violet cut-off $\epsilon$ although the coefficient functions depend on the example chosen, but the area law holds.

The explicit corrections to entropy in the Schwarzschild-like black hole backgrounds are obtained by using the coefficient functions from the general Laurent expansion in eq.(17) for each black hole solution listed in Table 2. The leading and next leading contributions to entropy in each non-extreme background solution are given in the following.

**E1. Non-extreme Reissner-Nordström black hole**

\[
S^{(0)\text{RN}} = \frac{r_+ - r_-}{360 \epsilon}, \\
S^{(1)\text{RN}} = \frac{1}{360 r_+} \left\{ 2(2r_+ - 3r_-) \log \frac{L}{\epsilon} - 6(r_+ - 2r_-) \right\},
\]

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$ are the outer and inner horizons.

**E2. Non-extreme dilaton black hole**

\[
S^{(0)\text{dilaton}} = \frac{r_h - p_0}{360 \epsilon}, \\
S^{(1)\text{dilaton}} = \frac{1}{360 r_h} \left\{ (4r_h - 3p_0) \log \frac{L}{\epsilon} - 6(r_h - p_0) \right\},
\]

where $r_h = 2M$ and $p_0$ are the horizon and parameter of dilaton black hole respectively.
E3. Non-extreme brane-world black hole

\[ S^{(0)\text{BWBH}} = \frac{r_h - q_0}{90\epsilon}, \]
\[ S^{(1)\text{BWBH}} = \frac{1}{180r_h} \left\{ (11r_h - 12q_0) \log \frac{L}{\epsilon} - 3(7r_h - 8q_0) \right\}, \quad (35) \]

where \( r_h = 2M \) is the horizon and \( q_0 \) is the parameter of brane world black hole respectively.

Clearly, the linear and logarithmic divergences in each of the above cases are similar. Each Schwarzschild-like black hole tends to Schwarzschild black hole if \( Q = 0 \) for Reissner-Nordström, \( p_0 = 0 \) for dilaton and \( q_0 = 3M/2 \) for the brane-world black hole respectively.

### 3.2 Extreme black holes

Next we consider the extreme cases, which are \( M = Q \) for Reissner-Nordström, \( p_0 = 2M \) for dilaton and \( q_0 = 2M \) for brane-world black holes. For each solution the corresponding leading exponent of the Laurent expansion and the indices deriving from them are listed in the Table 3.

**Table 3.** Leading exponents in eq.(17) and index of the optical volume in eq.(19) and that of the entropy in eq.(22) for the extreme black holes

| Extreme black hole     | Exponent | Index |
|-------------------------|----------|-------|
|                         | \( a \)  | \( b \) | \( c \) | \( \gamma \) | \( \delta \) |
| Reissner-Nordström \( M = Q \) | 2        | -2    | 0      | 3        | 0          |
| dilaton \( p_0 = 2M \)     | 1        | -1    | 1      | 0        | 0          |
| brane-world \( q_0 = 2M \) | 1        | -2    | 0      | 3/2      | 0          |

For all extreme cases, the values of the index \( \delta \) in Table 3 show zero. The corresponding leading contributions to entropy are either logarithmically divergent or constant depending on the value of another index \( \gamma \), which can be read off from eq.(26).

The explicit expressions obtained from the metric form in Table 2. are

\[ S^{(0)\text{Ext–RN}} = S^{(0)\text{Ext–BWBH}} = \frac{1}{135} \quad (36) \]
for the extreme Reissner-Nordström and brane-world black holes, and

\[
S^{(0)\text{Ext-dilaton}} = \frac{1}{360} \log \frac{L}{\epsilon}
\]

for the extreme dilaton black hole. The optical volumes for Reissner-Nordström and brane-world black holes diverge like \(\epsilon^{-3}\) and \(\epsilon^{-3/2}\) but their temperatures tend to zero as \(\epsilon\) and \(\epsilon^{1/2}\) respectively, and combining them the entropies become finite as in eq.(36). For dilaton black hole, the optical volume diverges logarithmically so that the entropy also becomes logarithmically divergent as in eq.(37). Note that the next leading term to entropy from the ultraviolet contribution vanishes for the extreme black holes, that is, \(S^{(1)} = 0\).

Earlier calculations for the extreme Reissner-Nordström case [16] have shown that \(S^{(0)\text{Ext-RN}} = 8\pi^3 M^6/(135\beta^3 \epsilon^3)\) is either zero if \(\beta \to \infty\) or cubically divergent in the cut-off (as \(\epsilon^{-3}\)) if the period \(\beta\) is left arbitrary. Clearly, there is no unique answer here. Therefore, one good way to dismiss the ambiguity due to \(\beta\) is to regularize both the temperature and the optical volume with the same regularization parameter \(\epsilon\). The brick wall regularization gives \(\beta \propto \epsilon^{-1}\) which is what we have been actually doing (see the Appendix for details). The constant value thereby obtained for \(S^{(0)\text{Ext-RN}}\) in eq.(36) is perfectly consistent with the cited expression from [16] in the square bracket above under necessary dimensional readjustments. Similar considerations apply for \(S^{(0)\text{Ext-BWBH}}\).

3.3 Extreme black holes vs extremal limit of non-extreme black holes

As mentioned earlier, extremal black holes (EBH) are topologically dissimilar to their non-extreme counterparts. It is not possible to obtain properties of EBH by continuously extremalizing their non-extreme partner. These are the well known Hawking-Horowitz-Ross EBH [11] for which the gravitational entropy is zero. See also [12]. (There is a second kind of EBH due to Zaslavskii [27] in which the topology is assumed to belong to the non-extreme sector and as a result the entropy satisfies the Bekenstein-Hawking area law. A consistent grand canonical ensemble approach [28] however does not lead to the area law [29].)

We have been concerned here with the contribution to gravitational entropy due to a scalar field in a fixed gravitational background. We state here how these corrections in the background of EBH contrast with those in the extreme limit of the non-extreme black hole background. The result is solution dependent.
The limits of entropy for the non-extreme Reissner-Nordström and brane-world black holes in eqs.(33) and (35) are logarithmically divergent and give different results from those for the extreme black holes in eq.(36). The limit of entropy for the non-extreme dilaton black hole in eq.(34) gives the same value as in the extreme dilaton black hole in eq.(37). The reason is that additional poles appear in Laurent expansion in eq.(17) for RN and BWBH but gives additional zero for the dilaton black hole in the limiting procedure.

The leading contributions to the entropy which are either logarithmically divergent or constant can be absorbed in the renormalization of the quadratic-curvature coupling constant in the one-loop effective gravitational action [24].

4 Summary and discussion

The merits of the present paper lie in the (a) general expression for entropy of the scalar field, eq.(25), applicable to a wide variety of spherically symmetric fixed backgrounds, (b) introduction of a new parameter \( \delta \), eq.(22), leading to the classifications and (c) in the easiness in which the higher order corrections, eqs.(31), (32), can be computed. We have also exemplified the validity of our method with some known solutions, the brane-world case being a new addition. The fact that \( S^{(1)} = 0 \) in the regular, intermediate and extreme cases (Table 1) is an interesting result.

Entropy is composed of the optical volume \( V \) and the (local) temperature \( 1/\beta \), both of which are evaluated on the surface at the brick wall distance \( \epsilon \) from the horizon \( r_h \). Physically, the local temperature here is analogous to the one originally devised by Pretorius, Vollick and Israel [30] and later used by others, for instance, [25]. There the temperature is defined on a reversible contracting thin shell gradually zeroing on the horizon while being in thermal equilibrium with acceleration radiation. Mathematically, the local temperature on the wall surface approaching the Hawking temperature on the horizon can be likened to test functions approaching a generalized function (Gaussian distributions tending to Dirac delta function, for instance). This underlines the spirit in the definition of a local temperature as well as in the overall approach.

This kind of approach has allowed us to classify the singularity in entropy by an index \( \delta \) with respect to \( \epsilon \). The leading contribution to entropy for non-extreme case (\( \delta \neq 0 \)) is shown to satisfy the area law with quadratic divergence with respect to the invariant cut-off \( \epsilon_{\text{inv}} \). For the extreme case (\( \delta = 0 \)), the
contribution is either logarithmically divergent or constant with respect to $\epsilon$. The results are displayed in Tables 1 and 3.

The general result is applied to Reissner-Nordström, brane-world and dilatonic black holes. We have compared the extreme black holes with the extremal limit of non-extreme black holes. The limits of entropy for the non-extreme Reissner-Nordström and brane-world black holes show logarithmic divergence while the corresponding extreme cases produce a constant value. These results are consistent given the fact that the Euclidean topologies in the two situations are different. Note that the behavior of the two solutions, viz., RN and brane-world black hole, in this respect is very similar. This is probably due to the fact that both the solutions satisfy the same field equation $R = 0$ where $R$ is the Ricci scalar.

On the other hand, the limit of entropy for the non-extreme dilatonic black hole gives the same result as for the extreme dilatonic black hole, both being logarithmically divergent (see [15]) unlike in the other solutions. Does it imply that the background dilatonic spacetime is intrinsically different from those of other black holes? We do not know, but would point to a certain difference: In the extreme dilatonic case, the background gravitational entropy is zero due to the fact that the area is zero. In the other two extreme cases, the background entropy is also zero but the area is nonzero. It would be of interest to examine this issue in more detail and we leave it for future investigations.

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**Appendix**

A Regularization method for the temperature

Why do we need to consider this temperature regularization at all? The reason that has prompted us is this: For EBH, Hawking temperature is completely arbitrary because the period can be identified with any value. It is assumed that the black hole entropy $S = MdS/dM - I$ can not depend on arbitrary
temperature, and the only conclusion is to set the classical action $I \propto 1/T$ to zero [12]. This leads to $S = kM$, where $k$ is an arbitrary constant. There is indeed no clear cut reason as yet to set $k$ to zero [16], but if we still do it, we get $S = 0$ for the background EBH entropy. With this scenario in view, and fixing $S$ to its classical zero value, we attempted to try out an alternative method for semiclassical corrections to entropy involving a well defined temperature.

Regularization method for the temperature involves defining the (local) temperature at a surface, infinitely close to the horizon, where it has a definite value and after the calculation of entropy, the limit is taken to the original place, the horizon. Now, the brick wall boundary provides a natural surface close to the horizon. The temperature so defined is the same as the temperature defined by the condition that no conical singularity appears in the Euclidean Rindler space. We illustrate this by the extreme RN case in detail.

The Euclideanized extreme RN metric is

$$ds^2 = (1 - M/r)^2 d\tau^2 + \frac{1}{(1 - M/r)^2} dr^2 + \text{(angle term)}. \quad (A.1)$$

Let us define a new radial variable $R$ as

$$R := \int_{r_h}^r \frac{1}{1 - M/r} \, dr. \quad (A.2)$$

However the integration diverges at the horizon $r_h = M$. To avoid this, the regularization is taken at the boundary position of quantum scalar field $r_h + \epsilon$ as

$$R := \int_{r_h+\epsilon}^{r'} \frac{1}{1 - M/r} \, dr \simeq \frac{M}{r_h + \epsilon - M} \int_{r_h+\epsilon}^{r'} dr \simeq \frac{M}{\epsilon} (r - M). \quad (A.3)$$

Note that $R$ tends to zero as $r \to r_h$. The original radial coordinate $r$ is expressed inversely as

$$r - M \simeq \epsilon \frac{R}{M}. \quad (A.4)$$

Then the Euclidean component of metric is expressed by the new variable

$$g_{\tau\tau} = (1 - M/r)^2 \simeq \frac{(r - M)^2}{M^2} \simeq \frac{\epsilon^2 R^2}{M^4}. \quad (A.5)$$

The Rindler form of the metric is therefore

$$ds^2 = R^2 \left( \frac{\epsilon d\tau}{M^2} \right)^2 + dR^2 + \text{(angle term)}. \quad (A.6)$$
No conical singularity condition is imposed to this Rindler metric when the Euclidean time varies \([0, \beta = 1/T]\), and obtain
\[ \frac{2\pi}{\beta} = \frac{\epsilon}{M^2}. \]  
\(\text{(A.7)}\)

The temperature of extreme RN black holes is zero as the regularization parameter \(\epsilon\) tends to zero.

It is worthwhile noting that when the general expression of temperature in eq. (16) is applied to the extreme RN black hole, we obtain
\[ \frac{2\pi}{\beta} \bigg|^{\text{Ext-RN}} = \frac{\partial_r g_{rr}}{2 \sqrt{g_{rr} g_{rr}}} \bigg|^{\text{Ext-RN}}_{r_h + \epsilon} = \frac{\epsilon}{M^2}, \]  
\(\text{(A.8)}\)

which coincides the explicit calculation of eq.(A.7). We have applied this prescription to define the temperature at the boundary position (brick wall) of the scalar field. This value of \(2\pi/\beta\) when combined with the corresponding optical volume yields nontrivial results in the limit \(\epsilon \to 0\), as we have seen in the text.

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