Trois couleurs: A new non-equational theory

by

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Abstract. A first order theory is equational if every definable set is a Boolean combination of instances of equations, that is, of formulae such that the family of finite intersections of instances has the descending chain condition. Equationality is a strengthening of stability yet so far only two examples of non-equational stable theories are known. We construct non-equational \( \omega \)-stable theories by a suitable colouring of the free pseudospace, based on Hrushovski and Srour’s original example.

1. Introduction. Consider a first order complete theory \( T \). A formula \( \varphi(x; y) \) is an equation (for a given partition of the free variables into \( x \) and \( y \)) if, in every model of \( T \), the family of finite intersections of instances \( \varphi(x, a) \) has the descending chain condition. The theory \( T \) is equational if every formula \( \psi(x; y) \) is equivalent modulo \( T \) to a Boolean combination of equations \( \varphi(x; y) \).

Determining whether a particular stable theory is equational is not obvious. So far, the only known natural example of a stable non-equational theory is the free non-abelian group [12, 10], though the first example of a non-equational stable theory is of combinatorial nature and appeared in unpublished notes of Hrushovski and Srour [7]. They coloured the free pseudospace [4] with two colours in order to obtain two types \( r(x, y) \neq r'(x, y) \) which are not equationally separated, according to the terminology of [6, Section 2.1], that is, there are sequences \( (a_i, b_i)_{i \in \mathbb{N}} \) and \( (c_i, d_i)_{i \in \mathbb{N}} \), which can be taken indiscernible over \( \emptyset \), such that \( r(a_i, b_i) \) and \( r'(c_i, d_i) \) hold for all \( i \), but \( r'(a_i, b_j) \) and \( r(c_i, d_j) \) hold for \( i < j \). In an equational theory, any two distinct types are equationally separated.

In all previously known examples of non-equational theories the failure of equationality is due to the presence of two distinct non-equationally sep-
arated types \( r(x, y) \neq r'(x, y) \) such that the length of \( x \) is 1. In this note, we will build on Hrushovski–Srour’s example in order to construct new examples of non-equational theories, where all distinct real types \( p \neq q \) in finitely many variables are equationally separated.

### 2. Equations and indiscernibly closed sets.

Most of the results in this section come from [11, 9].

Consider a first order theory \( T \). A formula \( \varphi(x; y) \) is an equation (with respect to a given partition of the free variables into \( x \) and \( y \)) if, in every model of \( T \), the family of finite intersections of instances \( \varphi(x, b) \) has the descending chain condition. An easy compactness argument shows the following:

**Lemma 2.1.** The formula \( \varphi(x; y) \) is an equation if there is no sequence \( (a_i, b_i)_{i \in \mathbb{N}} \) in any model \( M \) such that \( M \models \varphi(a_i, b_j) \) and \( M \not\models \varphi(a_i, b_i) \) for all \( i < j \).

A Ramsey argument shows that, working in a sufficiently saturated model, the sequence \( (a_i, b_i) \) can be assumed to be indiscernible of any infinite order type. Thus, if \( \varphi(x; y) \) is an equation, then so are \( \varphi^{-1}(x; y) = \varphi(y, x) \) and \( \varphi(f(x); y) \), whenever \( f \) is a \( \emptyset \)-definable function which maps finite tuples to finite tuples. Finite conjunctions and disjunctions of equations are again equations. Note that equations are stable formulae.

In [9], an equivalent definition of equations was obtained in terms of indiscernibly closed sets: an element \( c \) lies in the indiscernible closure \( \text{icl}(X) \) of a set \( X \) if there is an indiscernible sequence \( (a_i)_{i \in \mathbb{N}} \) such that \( a_i \) lies in \( X \) for \( i > 0 \) and \( a_0 = c \). Note that \( X \subset \text{icl}(X) \). A set \( X \) is indiscernibly closed if \( X = \text{icl}(X) \).

**Lemma 2.2 ([9 Theorem 3.16]).** A formula \( \varphi(x; y) \) is an equation if and only if the set \( \varphi(M, b) \) is indiscernibly closed in every model \( M \) of \( T \).

**Proof.** Let us work inside a sufficiently saturated model \( M \). If \( \varphi(x; y) \) is not an equation, witnessed by an indiscernible sequence \( (a_i, b_i)_{i \in \mathbb{N}} \) as in Lemma 2.1, the set defined by \( \varphi(x, b_0) \) is not indiscernibly closed, for it contains all \( a_i \)'s with \( i < 0 \), but does not contain \( a_0 \). Conversely, if some instance \( \varphi(x, b) \) is not indiscernibly closed, then there is an indiscernible sequence \( (a_i)_{i \in \mathbb{Z}} \) such that \( M \models \varphi(a_i, b) \) for \( i < 0 \), but \( M \not\models \varphi(a_0, b) \). For every \( j \) in \( \mathbb{Z} \), there is an element \( b_j \) in \( M \) such that \( M \models \varphi(a_i, b_j) \) for \( i < j \), but \( M \not\models \varphi(a_j, b_j) \).

The theory \( T \) is **equational** if every formula \( \psi(x; y) \) is equivalent modulo \( T \) to a Boolean combination of equations \( \varphi(x; y) \). Since Boolean combinations of stable formulae are stable, equational theories are stable.
Typical examples of equational theories are the theory of an equivalence relation with infinitely many infinite classes, the theory of \( R \)-modules for some ring \( R \), or the theory of algebraically closed fields.

Equationality is preserved under bi-interpretability as well as addition of parameters \([8]\). It is unknown whether equationality holds if every formula \( \varphi(x; y) \), with \( x \) a single variable, is a Boolean combination of equations.

It is easy to see that \( T \) is equational if and only if all completions of \( T \) are equational. So for the rest of this section we assume that \( T \) is complete and work in a sufficiently saturated model \( U \).

Notice that a theory \( T \) is equational if and only if every type \( p \) over \( A \) is implied by its equational part \( \{ \varphi(x, a) \in p \mid \varphi(x; y) \text{ is an equation} \} \).

**Definition 2.3.** Given two types \( p(x, b) \) and \( q(x, b) \), define \( p(x, b) \rightarrow q(x, b) \) if \( q(x, b) \subseteq \text{cl}(p(x, b)) \), or equivalently, if there is an indiscernible sequence \((a_i)_{i \in \mathbb{N}} \) such that all \( \models p(a_i, b) \) for \( i > 0 \) and \( \models q(a_0, b) \). If \( p(x, y) \) and \( q(x, y) \) are the corresponding (complete) types over \( \emptyset \), we write

\[
p(x; y) \rightarrow q(x; y).
\]

A standard argument as in Lemma 2.2 with \( p \) instead of \( \varphi \) and \( q \) instead of \( \neg \varphi \) yields the following:

**Lemma 2.4.** We have \( p(x; y) \rightarrow q(x; y) \) if and only if there is a sequence \((a_i, b_i)_{i \in \mathbb{N}} \) such that \( \models p(a_i, b_j) \) for \( i < j \), and \( \models q(a_i, b_i) \) for all \( i \). Furthermore, we may assume that the sequence is indiscernible and of any given infinite order type.

The above characterisation provides an easy proof of the following remark:

**Remark 2.5.** Clearly \( p \rightarrow p \). If \( p(x; y) \rightarrow q(x; y) \), then \( p^{-1} \rightarrow q^{-1} \), where \( p^{-1}(x; y) = p(y; x) \).

Furthermore, if \( \text{tp}(a; b) \rightarrow \text{tp}(a'; b) \), then \( a \equiv_{\text{tp}} a' \). Thus, if \( p(x; y) \) implies that \( x \) (or \( y \)) is algebraic, then \( p \rightarrow q \) only when \( q = p \).

**Corollary 2.6.** Let \( f \) and \( g \) be \( \emptyset \)-definable functions and \( a, a', b, b' \) finite tuples with \( \text{tp}(a; b) \rightarrow \text{tp}(a'; b') \). Then \( \text{tp}(f(a); g(b)) \rightarrow \text{tp}(f(a'); g(b')) \).

**Corollary 2.7.** A formula \( \varphi(x; y) \) is an equation if and only if, whenever a type \( p(x, y) \) contains \( \varphi(x, y) \) and \( p(x; y) \rightarrow q(x; y) \), then \( \varphi(x, y) \) lies in \( q(x; y) \).

**Proof.** One direction follows clearly from Lemma 2.2. For the converse, assume that \( \varphi(x; y) \) is not an equation and choose an indiscernible sequence \((a_i, b_i)_{i \in \mathbb{N}} \) as in Lemma 2.1. Let \( p \) be the common type of the pairs \((a_i, b_j)\) with \( i < j \) and \( q \) be the common type of the pairs \((a_i, b_i)\). Then \( p \rightarrow q \) and \( \varphi \) belongs to \( p \), but not to \( q \). \( \blacksquare \)
Definition 2.8. A cycle of types is a sequence
\[ p_0(x; y) \to p_1(x; y) \to \cdots \to p_{n-1}(x; y) \to p_0(x; y), \]
The cycle is proper if all the \( p_i \)'s are different. The theory \( T \) is indiscernibly acyclic if there is no proper cycle of types of length \( n \geq 2 \).

Following the terminology of [9, Section 2.1], two distinct types \( p(x; y) \) and \( q(x; y) \) are not equationally separated if and only if \( p \to q \to p \).

Remark 2.9. Every indiscernibly acyclic theory is stable.

Proof. If there is a formula \( \varphi(x; y) \) in \( T \) with the order property, find an indiscernible sequence \( (a_i)_{i \in \mathbb{Z}} \) in \( U \) such that \( \models \varphi(a_i, a_j) \) if and only if \( i < j \). Set \( p = \text{tp}(a_1; a_0) \) and \( q = \text{tp}(a_{-1}; a_0) \). Then \( p \neq q \), and since \( (a_i)_{i \neq 0} \) is indiscernible, we have \( p \to q \to p \), so there is a proper cycle of types of length 2.

Remark 2.10. Every equational theory is indiscernibly acyclic.

Proof. Consider a cycle
\[ p_0 \to p_1 \to \cdots \to p_{n-1} \to p_0, \]
By Corollary 2.7 all the types \( p_i \) contain the same equations, so they all agree, by equationality of \( T \).

Definition 2.11. The theory \( T \) satisfies the MS-criterion if there is some formula \( \varphi(x; y) \) and a matrix \((a_{ij}, b_{ij})_{i,j \in \mathbb{N}}\) such that
(1) \( \models \varphi(a_{ij}, b_{il}) \) if and only if \( j = l \),
(2) \( a_{ij}, b_{ij} \equiv a_{ik}, b_{kl} \) whenever \( i < k \) and \( j < l \).

Lemma 2.12. If a theory \( T \) satisfies the MS-criterion, then there is a proper cycle of types \( p \to q \to p \). In particular, the theory is not equational (cf. [10, Proposition 2.6]).

Proof. We may assume that the matrix \((a_{ij}, b_{ij})_{i,j \in \mathbb{N}}\) is indiscernible, that is, the type \( \text{tp}(a_{ij}, b_{ij})_{i \in I, j \in J} \) only depends on \( |I| \) and \( |J| \). Set \( p = \text{tp}(a_{00}; b_{00}) \), \( q = \text{tp}(a_{00}; b_{01}) \) and \( r = \text{tp}(a_{00}; b_{11}) \). Since \((a_{0j}b_{0j})_{j \in \mathbb{N}}\) is indiscernible, we have \( q \to p \). Since \((a_{i0}b_{i1})_{i \in \mathbb{N}}\) is indiscernible, we have \( r \to q \).

Now, by Definition 2.11, the formula \( \varphi(x; y) \) belongs to \( p(x; y) \) but not to \( q(x; y) \), so \( p \neq q \). By Definition 2.11 we have \( p = r \), as desired.

Since \( p \) and \( q \) contain the same equations, it follows that the above formula \( \varphi \) cannot be a Boolean combination of equations (cf. [10, Proposition 2.6]).

Let us assume for the rest of this section that \( T \) is stable.
Lemma 2.13. Let \( p_0(x, b) \rightarrow \cdots \rightarrow p_{n-1}(x, b) \rightarrow p_0(x, b) \) be a proper cycle of types and \( b' \) be some tuple such that \( p_0(x, b) \) has only finitely many distinct non-forking extensions to \( bb' \). Then there is a proper cycle of types starting with some non-forking extension \( p'_0(x; b, b') \) of \( p_0(x, b) \) whose length is a multiple of \( n \).

Proof. First notice that if \( p(x, b) \rightarrow q(x, b) \) and \( q'(x, b, b') \) is a non-forking extension of \( q(x, b) \), then \( p(x, b) \) has a non-forking extension \( p'(x, b, b') \) with

\[
p'(x; b, b') \rightarrow q'(x; b, b')
\]

Indeed, consider an indiscernible sequence \( (a_i)_{i \in \mathbb{N}} \) such that \( \models p(a_i, b) \) for \( i > 0 \), and \( q(a_0, b) \). We may assume that \( a_0 \) realises \( q'(x, b, b') \) and that \( (a_i)_{i \in \mathbb{N}} \) is independent from \( b' \) over \( b \). By a Ramsey argument, we may assume that \( (a_i)_{i>0} \) is indiscernible over \( a_0 bb' \). Set now \( p'(x, b, b') \) to be the type of \( a_1 \) over \( bb' \), so \( p'(x, b, b') \rightarrow q'(x, b, b') \), as desired.

Let \( k \) now be the number of distinct non-forking extensions of \( p_0(x, b) \) to \( bb' \). Working backwards in the cycle of types, we deduce from the above that there is a sequence \( r_0(x; b, b') \rightarrow \cdots \rightarrow r_{n,k}(x; b, b') \), where \( r_{n,i+j}(x, b, b') \) is a non-forking extension of \( p_j(x, b) \) for each \( i \leq k \). Since \( p_0 \) has only finitely many distinct non-forking extensions to \( bb' \), there are indices \( i < i' \) such that \( r_{n,i}(x, b, b') = r_{n,i'}(x, b, b') \). Choose \( i \) and \( i' \) such that \( i' - i > 0 \) is least possible. Then

\[
r_{n,i}(x; b, b') \rightarrow \cdots \rightarrow r_{n,i'}(x; b, b')
\]

is a proper cycle of types.

Corollary 2.14. If \( T \) is totally transcendental, then it is indiscernibly acyclic if and only if so is \( T'^{eq} \).

Proof. We need only show that \( T'^{eq} \) is indiscernibly acyclic provided that \( T \) is indiscernibly acyclic. Assume first that the type \( p(x, e) \) starts a proper cycle of types, where \( e \) is an imaginary element. Choose a real tuple \( b \) such that \( \pi_E(b) = e \) for some 0-definable equivalence relation \( E \). Since \( T \) is totally transcendental, the type \( p(x, e) \) has only finitely many non-forking extensions to \( \{b, e\} \), so there is a proper cycle starting with some non-forking extension \( p'(x, b, e) \) of \( p(x, e) \), by Lemma 2.13. By Corollary 2.6, if we restrict the types in the cycle to \( b \), we have a cycle of types which must be proper, because \( e \) is definable from \( b \).

Since the relation \( \rightarrow \) is symmetric in \( x \) and \( y \), we can now replace \( x \) by some real tuple, so \( T \) is not indiscernibly acyclic.

Notation. Given stationary types \( p_1(x, b) \) and \( p_2(x', b) \), denote by \( p_1(x, b) \otimes p_2(x', b) \) the type of the pair \((a_1, a_2)\) over \( b \), where \( \models p_i(a_i, b) \) for \( i = 1,2 \), and \( a_1 \downarrow_b a_2 \). More generally, for a sequence of stationary types,
$p_0(x^0, b) \otimes \cdots \otimes p_n(x^n, b)$ is the type of a sequence $(a_0, \ldots, a_n)$ of realisations $a_i \models p_i(x^i, b)$ which is independent over $b$.

Observe that
$$p_0(x^0, b) \otimes \cdots \otimes p_n(x^n, b) = p_0(x^0, b) \otimes (p_1(x^1, b) \otimes \cdots \otimes p_n(x^n, b)).$$

**Lemma 2.15.** Given stationary types $p^j(x^j, y^j, c)$ and $q^j(x^j, y^j, c)$ over a tuple $c$ in $\text{acl}^{eq}(\emptyset)$ such that
$$p^j(x^j; y^j, c) \rightarrow q^j(x^j; y^j, c) \quad \text{for } j = 1, 2,$$
we have
$$p^1(x^1; y^1, c) \otimes p^2(x^2; y^2, c) \rightarrow q^1(x^1; y^1, c) \otimes q^2(x^2; y^2, c).$$

By the above, the lemma generalises to an arbitrary finite product of types.

**Proof.** For $j = 1, 2$, choose a tuple $b^j$ and an indiscernible sequence $(a_i^j)_{i \in \mathbb{N}}$ such that $\models p^j(a_i^j, b^j, c)$ for $i > 0$, and $\models q^j(a_i^j, b^j, c)$. We may assume that
$$b^1 \cup \{a_i^1\}_{i \in \mathbb{N}} \downarrow_c b^2 \cup \{a_i^2\}_{i \in \mathbb{N}}.$$
Since $c$ is algebraic over $\emptyset$, the sequences $\{a_i^1\}_{i \in \mathbb{N}}$ and $\{a_i^2\}_{i \in \mathbb{N}}$ are both indiscernible over $c$ and therefore mutually indiscernible, by stationarity of strong types, so $\{a_i^1, a_i^2\}_{i \in \mathbb{N}}$ is indiscernible. Notice that $(a_0^1, a_0^2)$ realises $p^1(x^1; b^1, c) \otimes p^2(x^2; b^2, c)$ for $i > 0$, and $(a_0^1, a_0^2)$ realises the type
$$q^1(x^1; b^1, c) \otimes q^2(x^2; b^2, c),$$
as desired. \hfill \blacksquare

**Proposition 2.16.** If $T$ is totally transcendental, then it is indiscernibly acyclic if and only if there is no proper cycle of types in $T^{eq}$ of length 2.

**Proof.** By Corollary 2.14, we need only prove one direction, so suppose
$$p_0(x, y) \rightarrow \cdots \rightarrow p_{n-1}(x, y) \rightarrow p_0(x, y)$$
is a proper cycle of types with real variables. Since $T$ is totally transcendental, there is a finite tuple $c$ in $\text{acl}^{eq}(\emptyset)$ such that all non-forking extensions of all $p_i$'s to $c$ are stationary. Lemma 2.13 gives a proper cycle of stationary types
$$p_0(x; y, c) \rightarrow \cdots \rightarrow p_{k-1}(x; y, c) \rightarrow p_0(x; y, c)$$
for some $k$ in $\mathbb{N}$.

Denote $\bar{x} = (x^0, \ldots, x^{k-2})$ and $\bar{y} = (y^0, \ldots, y^{k-2})$ and consider the types
$$r_1(\bar{x}; \bar{y}, c) = p_0(x^0, y^0, c) \otimes p_1(x^1, y^1, c) \otimes \cdots \otimes p_{k-2}(x^{k-2}, y^{k-2}, c),$$
$$r_2(\bar{x}; \bar{y}, c) = p_1(x^0, y^0, c) \otimes p_2(x^1, y^1, c) \otimes \cdots \otimes p_{k-1}(x^{k-2}, y^{k-2}, c),$$
$$r_3(\bar{x}; \bar{y}, c) = p_1(x^0, y^0, c) \otimes p_2(x^1, y^1, c) \otimes \cdots \otimes p_{k-2}(x^{k-3}, y^{k-3}, c),$$
$$\otimes p_0(x^{k-2}, y^{k-2}, c).$$
Lemma 2.15 yields the path
\[ r_1(\bar{x}; \bar{y}, c) \to r_2(\bar{x}; \bar{y}, c) \to r_3(\bar{x}; \bar{y}, c). \]

Given a realisation \((a^0, a^1, \ldots, a^{k-2}; b^0, b^1, \ldots, b^{k-2})\) of \(r_1(\bar{x}; \bar{y}, c)\), notice that
\[
(a^1, \ldots, a^{k-2}, a^0; b^1, \ldots, b^{k-2}, b^0)
\]
realises \(r_3(\bar{x}; \bar{y}, c)\). Corollary 2.6 applied to a function which maps a \(k-1\)-tuple \((c^1, \ldots, c^{k-1})\) to the imaginary coding the set \(\{c^1, \ldots, c^{k-1}\}\) yields a cycle of imaginary types
\[
\text{tp}(\{a^0, \ldots, a^{k-2}\}; b^0, \ldots, b^{k-2}), c) \to \text{tp}(\{a^1, \ldots, a^{k-1}\}; b^1, \ldots, b^{k-1}), c) \to \text{tp}(\{a^0, \ldots, a^{k-2}\}; b^0, \ldots, b^{k-2}), c).
\]

In order to conclude, we need only show that the types
\[
\text{tp}(\{a^1, \ldots, a^{k-1}\}; b^1, \ldots, b^{k-1}), c)
\]
and
\[
\text{tp}(\{a^0, \ldots, a^{k-2}\}; b^0, \ldots, b^{k-2}), c)
\]
are different. Assume not, and find an automorphism \(\rho\) of the sufficiently saturated model \(\mathbb{U}\) of \(T\) fixing \(c\) pointwise and mapping
\[
\{a^1, \ldots, a^{k-1}\} \text{ to } \{a^0, \ldots, a^{k-2}\}
\]
and
\[
\{b^1, \ldots, b^{k-1}\} \text{ to } \{b^0, \ldots, b^{k-2}\}.
\]
Observe that no two \(a^i\) and \(a^j\) with \(i \neq j\) can be equal since the independence \(a^i \downarrow_c a^j\) would imply that \(a_i\) is algebraic and thus \(p_i(x, y, c) = p_j(x, y, c)\), by Remark 2.5. Likewise, no two elements \(b^i\) and \(b^j\) can be equal for \(i \neq j\).

It follows that the automorphism \(\rho\) induces two bijections \(\sigma\) and \(\tau\) from \(\{1, \ldots, k-1\}\) to \(\{0, \ldots, k-2\}\) such that
\[
\rho(a^i) = a^{\sigma(i)} \quad \text{and} \quad \rho(b^j) = b^{\tau(j)}.
\]
If \(\sigma(k-1) = \tau(k-1) = j\), then \((a^j, b^j)\) realises both \(p_j(x; y, c)\) and \(p_{k-1}(x; y, c)\), which contradicts the cycle of types being proper. Hence, the values \(\sigma(k-1)\) and \(\tau(k-1)\) must be different. So there exists \(1 \leq i < k-1\) such that \(\sigma(i)\) and \(\tau(i)\) must be different. The independences
\[
a^{\sigma(i)} \downarrow_c b^{\tau(i)} \quad \text{and} \quad a^{\sigma(k-1)} \downarrow_c b^{\tau(k-1)}
\]
implicate (after composing with \(\rho^{-1}\)) that
\[
a^i \downarrow_c b^i \quad \text{and} \quad a^{k-1} \downarrow_c b^{k-1}.
\]
We conclude that \(p_i(x, y, c) = p_{k-1}(x, y, c)\), by Remark 2.5 and by stationarity of strong types, which gives the desired contradiction. 

We do not know whether Corollary 2.14 and Proposition 2.16 are true for arbitrary stable theories.
All known examples of non-equational stable theories have a proper cycle of real types of length 2. Indeed, in Hrushovski and Srour’s primordial example [7], the type of a white point and the type of a red point in a plane indiscernibly converge to each other, whereas the non-abelian free group satisfies the MS-criterion [10, Lemmata 3.4 & 3.6]. In this note, we will provide new examples of non-equational totally transcendental theories, one for each natural number \( k \), having proper cycles of length \( k \) but no proper cycles of real types of length strictly smaller than \( k \). We will do so by suitably colouring the free pseudospace, mimicking the construction of Hrushovski and Srour. The following question seems hence natural, though we do not have a solid guess what the answer will be.

**Question.** Is there a non-equational indiscernibly acyclic theory?

Related to the above, we wonder whether there is a local characterisation of equationality in terms of cycles of types:

**Question.** Is a formula \( \varphi(x, y) \) a Boolean combination of equations if and only if whenever

\[
\varphi \in p_0(x, y) \rightarrow p_1(x, y) \rightarrow \cdots \rightarrow p_{n-1}(x, y) \rightarrow p_0(x, y),
\]

then \( \varphi \) belongs to \( p_i \) for every \( i > 0 \)?

Do two types \( p \) and \( q \) contain the exact same equations if and only if \( p \) and \( q \) both occur in a (proper) cycle of types?

Observe that a positive answer to the second question would positively answer the first one.

### 3. Indiscernible kernels.

To our knowledge, the results in this section only appeared in print in Adler’s Master’s Thesis [1] (in German). Therefore, we will include their proofs, even if the results are most likely well-known among the community.

As before, work inside a sufficiently saturated model \( U \) of the complete theory \( T \).

**Notation.** Given two subsets \( I_0 \) and \( I_1 \) of a linearly ordered infinite index set with no endpoints, we write \( I_0 \ll I_1 \) if \( i_0 < i_1 \) for all \( i_0 \) in \( I_0 \) and \( i_1 \) in \( I_1 \). If \( (a_i)_{i \in I} \) is a sequence indexed by \( I \), set \( \text{acl}^\text{eq}(a_{I_0}) = \text{acl}^\text{eq}(\{a_i\}_{i \in I_0}). \)

**Definition 3.1.** The kernel of the indiscernible sequence \( (a_i)_{i \in I} \) is defined as

\[
\text{Ker}((a_i)_{i \in I}) = \bigcup_{I_0, I_1 \subseteq I \atop I_0 \ll I_1} \text{acl}^\text{eq}(a_{I_0}) \cap \text{acl}^\text{eq}(a_{I_1}).
\]

Note that we may assume that both \( I_0 \) and \( I_1 \) are finite subsets of \( I \). Furthermore, the set \( \text{acl}^\text{eq}(a_{I_0}) \cap \text{acl}^\text{eq}(a_{I_1}) \) only depends on \( |I_0| \) and \( |I_1| \) (possibly
after enlarging $I$), since $(a_i)_{i \in J}$ is indiscernible over $a_{I_0}$, whenever the segment $J \subset I \setminus I_0$ is either to the left of $I_0$ or to the right of $I_0$. If the sequence is indiscernible as a set (which is always the case in stable theories), then we may define the kernel by considering all the intersections given by pairs $(I_0, I_1)$ with $I_0 \cap I_1 = \emptyset$. Observe that (if $I$ is large enough)
\[
\text{Ker}((a_i)_{i \in I}) = \text{acl}^{\text{eq}}(a_{I_0}) \cap \text{acl}^{\text{eq}}(a_{I_1}) \quad \text{for any } I_0 \ll I_1 \text{ both infinite.}
\]

**Lemma 3.2.** The kernel $K$ of an indiscernible sequence $(a_i)_{i \in I}$ is the largest subset of $\text{acl}^{\text{eq}}((a_i)_{i \in I})$ over which the sequence is indiscernible.

**Proof.** We may assume that $I$ has no endpoints. Clearly, the sequence is indiscernible over $K$. Given a tuple $b$ in $\text{acl}^{\text{eq}}(a_{I_0})$, for $I_0 \subset I$ finite, such that the sequence is indiscernible over $b$, the tuple $b$ also lies in $\text{acl}^{\text{eq}}(a_{I_1})$ whenever the segment $I_1 \gg I_0$ has the same size as $I_0$, so $b$ lies in the kernel
\[
K = \bigcup_{J_0,J_1 \subset I, I_0 \ll I_1} \text{acl}^{\text{eq}}(a_{J_0}) \cap \text{acl}^{\text{eq}}(a_{J_1}).
\]

**Lemma 3.3.** If $T$ is stable, then the kernel $K$ of an indiscernible sequence $(a_i)_{i \in I}$ is the smallest algebraically closed subset (in $T^{\text{eq}}$) over which the sequence is independent.

**Proof.** Let $E$ be an algebraically closed subset (in $T^{\text{eq}}$) such that $(a_i)_{i \in I}$ is $E$-independent. In particular, for each $I_0 < I_1$, we have
\[
a_{I_0} \downarrow_E a_{I_1},
\]
so $K \subset E$. Let now $p = \text{Av}((a_i)_{i \in I})$ be the average type, that is,
\[
p = \{ \varphi(x,b) \text{ L-U-formula} \mid \varphi(a_i,b) \text{ for all but finitely many } i \in I \}.
\]
Since $p$ is invariant over every infinite subsequence of $(a_i)_{i \in I}$, its canonical base $C$ is contained in $K$. Thus, the sequence is $C$-indiscernible and $p$ is a non-forking extension of the stationary type $p|_K$.

It suffices to show that $a_i \models p|_{K \cup (a_j)_{j<i}}$, since any Morley sequence of $p|_K$ has this property and its type over $K$ is unique. Thus, let $\varphi(x,(a_j)_{j<i})$ be a formula in $p|_{K \cup (a_j)_{j<i}}$. We may clearly assume that $I$ does not have a last element. By definition of the average type, there is some $a_t \models p|_{K \cup (a_j)_{j<i}}$ with $t \geq i$. By indiscernibility,
\[
a_i \models p|_{K \cup (a_j)_{j<i}},
\]
as desired. $lacksquare$

**Corollary 3.4.** In a stable theory $T$, every indiscernible sequence is a Morley sequence over its kernel.

Using kernels, we can provide a different characterisation of the relation $\rightarrow$ in a stable theory.
Corollary 3.5. Given types $p(x; y)$ and $q(x; y)$ in a stable theory $T$, we have $p \rightarrow q$ if and only if there is a set $C$ and tuples $a$, $a'$ and $b$ such that

- $\models p(a, b)$ and $\models q(a', b)$,
- $a \equiv_C a'$,
- $a \downarrow_C b$.

In particular, given a cycle

$$p_0(x, y) \rightarrow p_1(x, y) \rightarrow \cdots \rightarrow p_{n-1}(x, y) \rightarrow p_0(x, y),$$

there are tuples $b$, $a_0, \ldots, a_n$ and subsets $C_0, \ldots, C_{n-1}$ such that

- $\models p_r(a_r, b)$ for $0 \leq r \leq n - 1$, and $\models p_0(a_n, b)$,
- $a_r \equiv_{C_r} a_{r+1}$ for $0 \leq r \leq n - 1$,
- $a_r \downarrow_{C_r} b$ for $0 \leq r \leq n - 1$.

Proof. If $p \rightarrow q$, choose some tuple $b$ and an indiscernible sequence $(a_i)_{i < |T|}$ such that $q(a_0, b)$ and $p(a_i, b)$ for each $i > 0$. Consider the kernel $K$ of the sequence, which is algebraically closed in $T^{eq}$, so $a_i \equiv_K a_j$ for all $i, j$. In particular, the subsequence $(a_i)_{0 < i < |T|}$ is a Morley sequence over $K$, so there is some $i_0 < |T|^+$ such that $a_{i_0} \downarrow_K b$. Set $C = K$, $a' = a_0$ and $a = a_{i_0}$.

For the other direction, set $a_0 = a'$ and choose for each $0 \neq i$ in $\mathbb{N}$ a realisation $a_i \equiv_C a$ such that $a_i \downarrow_C b, \{a_j\}_{j < i}$. Since strong types are stationary, we have $p(a_i, b)$ for $i \neq 0$. Furthermore, the sequence $\{a_i\}_{i \geq 0}$ is indiscernible over $C$, by construction.

The above provides a simpler characterisation of equations in stable theories (cf. [5, Remark 2.4]).

Remark 3.6. In a stable theory $T$, a formula $\varphi(x; y)$ is an equation if and only if for every set $C$ and tuples $a$, $a'$ and $b$ such that $\varphi(a, b)$ holds with $a \downarrow_C b$, so does $\varphi(a', b)$ whenever $a' \equiv_C a$.

Proof. Given $C$, $a$, $a'$ and $b$ as in the statement, Corollary 3.5 yields $tp(a, b) \rightarrow tp(a', b)$. As $\varphi$ belongs to $tp(a, b)$, it must lie in $tp(a', b)$, by Corollary 2.7.

For the other direction, it suffices to show that $\varphi$ lies in $q$ whenever $\varphi$ belongs to $p$ and $p \rightarrow q$, by Corollary 2.7. By Corollary 3.5 there are $C$, $a$, $a'$ and $b$ such that $p(a, b)$, $q(a', b)$, $a \downarrow_C b$ and $a' \equiv_C a$. Since $\varphi(a, b)$ holds, so does $\varphi(a', b)$, that is, the formula $\varphi$ belongs to $q$, as desired.

4. A blank pseudospace. Hrushovski and Srour [7] produced the first example of a non-equational stable theory by adding two colours to an underlying (2-dimensional) free pseudospace, a structure later studied by Baudisch and Pillay [4]. Subsequently, the free (n-dimensional) pseudospace has been
considered from different perspectives, either as a lattice \[13, 14\] or as a right-angled building \[2, 3\], in order to show that the ample hierarchy is strict. In this section, we will recall the basic properties of the free 2-dimensional pseudospace.

A geometry is a graph whose vertices have levels 0, 1 and 2. Vertices of level 0 are called points (usually denoted by the letter \(c\)), whereas vertices of level 1 are lines (denoted by \(b\)) and vertices of level 2 are planes (denoted by \(a\)). In an abuse of notation, we say that the point \(c\) lies in the plane \(a\) if there is a line \(b\) contained in \(a\) passing through \(c\), though there are no edges between points and planes. We refer to a subgraph of the form \(a - b - c\) as a flag.

A letter \(s\) is a non-empty subinterval of \([0, 2]\). Given a flag \(F\) in a geometry \(A\), a new geometry \(B\) is obtained from \(A, F\) via the operation \(s\) by freely adding a new flag \(G\) which agrees with \(F\) on the levels in \([0, 2] \setminus s\):

The free pseudospace \(M_\infty\) is obtained by successively applying countably many times all of the above operations starting from a flag. The geometry \(M_\infty\) is independent, up to isomorphism, of the order in which the operations are applied. It is denoted by \(M^2_\infty\) in \[2\] Definition 4.6]. Observe that the geometry obtained by only considering the operations 0, 1 and 2 is an elementary substructure (cf. \[2\] Corollary 4.31]) of \(M_\infty\) (namely, the prime model).

We will now exhibit the axioms for the theory \(PS\) of \(M_\infty\). Let us first
fix some notation. We will write the symbol $i$ for the letter $[i]$, that is, from now on, we will write 0 instead of $[0]$, etc. A \textit{word} is a sequence of letters. A \textit{permutation} of the word $u$ is obtained by successively replacing an occurrence of the subword $0 \cdot 2$ by the subword $2 \cdot 0$; similarly the subword $2 \cdot 0$ is permuted to $0 \cdot 2$ (note that $0 \cdot 2$ denotes the word $[0] \cdot [2]$, but we do not want to render the notation cumbersome with additional brackets). The word $u$ is \textit{reduced} if it does not contain, up to permutation, a subword of the form $s \cdot t$, where $s \subset t$ or $t \subset s$ (note that our notation $s \subset t$ does not imply $s \subset t$).

A \textit{flag path}

$$F_0 \xrightarrow{s_1} F_1 \to \cdots \to F_{n-1} \xrightarrow{s_n} F_n$$

with word $u = s_1 \cdots s_n$ is a sequence of flags such that, for each $1 \leq i \leq n$, the flag $F_i$ agrees with $F_{i-1}$ exactly in the levels in $[0,2] \setminus s_i$. The above flag path is \textit{reduced} if its word is reduced and for each $i$, the flags $F_{i-1}$ and $F_i$ cannot be connected by a \textit{splitting}, that is, a flag subpath whose word consists of proper subletters of $s_i$. It is not hard to show that any two flags are connected by a reduced path \cite{3} Corollary 3.13.

\textbf{Fact 4.1} (\cite{3} Theorem 4.12). \textit{The theory PS is axiomatised by the following properties:}

\begin{enumerate}
\item The universe is a geometry such that every vertex lies in a flag.
\item For every level $i$ in $[0,2]$ and every flag $F$, there are infinitely many flags $G$ with $F \xrightarrow{i} G$.
\item Every closed reduced flag path $F_0 \xrightarrow{s_1} F_1 \to \cdots \to F_{n-1} \xrightarrow{s_n} F_0$ has length $n = 0$.
\end{enumerate}

It was proven in \cite{3} Theorem 3.26 that property (3) can be expressed by a set of elementary sentences.

We will now describe types and the geometry of forking in the pseudospace. We refer the reader to \cite{3} Sections 3–7 for the corresponding proofs. Since there are no non-trivial reduced closed paths of flags, the word $u$ connecting two flags $F$ and $G$ by a reduced path $F \xrightarrow{u} G$ is unique, up to permutation, and will be denoted by $d(F,G)$. The flags $F$ and $G$ agree modulo a subset $S$ of $[0,2]$, that is, they have the same vertices in all levels off $S$, if and only if the letters in $d(F,G)$ are all contained in $S$. In particular, the collection of points and lines, resp. lines and planes, forms a pseudoplane, so any two lines intersect in at most one point, resp. lie in at most one plane. Furthermore, the intersection of two distinct planes is either empty, a unique point or a unique line \cite{4} Axiom $\Sigma 3$. Actually, the geometry forms a lattice, once a smallest element $0$ and a largest element $1$ are added \cite{13,14}. 
If \( u = d(F, G) = u_1 \cdot u_2 \), given two reduced flag paths

\[
\begin{array}{c}
F \\ \downarrow u_1 \\
H \\ \quad H_1 \\
\downarrow u_1 \\
G \\ \quad G_1 \\
\downarrow u_2 \\
\end{array}
\]

and a vertex \( p \) in \( H \) of level \( i \) which does not wobble, that is, such that the word \( u_1 \cdot i \) or the word \( i \cdot u_2 \) is reduced, then \( p \) is also a vertex of \( H_1 \). In particular, the vertex \( p \) is definable over \( F,G \).

A non-empty subset \( A \) of \( M_\infty \) is nice if

- every vertex in \( A \) lies in a flag fully contained in \( A \),
- any two flags in \( A \) are connected by a reduced path of flags in \( A \).

The algebraic closure and the definable closure of a set \( X \) agree [13, Proposition 2.29] and coincide with the intersection of all nice sets \( A \supset X \) [13, Corollary 2.27] (see also [14, Proposition 5.1 & Corollary 5.4]). If \( X \) is finite, then so is the algebraic closure. The quantifier-free type of a nice subset determines its type. More generally:

**FACT 4.2 (cf. [13, Remark 2.26] & [14, Corollary 3.12]).** The quantifier-free type of an algebraically closed subset determines its type in \( \text{PS} \).

Observe that if we apply one of the operations \([0], [1] \) or \([2]\) to a flag in a nice set \( A \), the resulting geometry is again nice.

Given a flag \( F \) and a nice subset \( A \), there is a flag \( G \) in \( A \) (called a base-point of \( F \) over \( A \)) such that, for any flag \( G' \) in \( A \), the word \( d(F,G') \) is the non-splitting reduction of \( d(F,G) \cdot d(G,G') \), that is, whenever a subword \( s \cdot t \) or \( t \cdot s \) occurs in a permutation of the product \( d(F,G) \cdot d(G,G') \), with \( s \subset t \), we cancel \( s \). If we consider a reduced flag path \( P \) connecting \( F \) to some base-point \( G \) over \( A \) with word \( d(F,G) \), the set \( A \cup P \) is again nice. Any flag occurring in the nice set \( P \) appears in a permutation of the path \( P \).

The theory \( \text{PS} \) of \( M_\infty \) is \( \omega \)-stable of rank \( \omega^2 \), equational with perfectly trivial forking and has weak elimination of imaginaries. Forking can be easily described: Given nice sets \( A \) and \( B \) containing a common algebraically closed subset \( C \), we have \( A \downarrow_C B \) if and only if for every nice set \( D \supset C \) and flags \( F \) in \( A \) and \( H \) in \( B \), \( d(F,H) \) is the non-splitting reduction of \( d(F,G) \cdot d(G,H) \), where \( G \) is a base-point of \( F \) over \( D \). In particular,

\[
F \downarrow_G D.
\]

**REMARK 4.3 ([13, Theorem 2.35] & [14, Proposition 4.3 & Theorem 4.13]).** Assume that \( A, B \) and \( C = A \cap B \) are algebraically closed and \( A \downarrow_C B \). Then

1. \( A \cup B \) is algebraically closed,
(2) if a vertex \( x \) in \( A \) is directly connected to a vertex \( y \) in \( B \), then \( x \) or \( y \) must lie in \( C \).
(3) if a point in \( A \) lies in a plane of \( B \), then there is a line in \( C \) connecting them,
(4) a point \( c \) which belongs to both a line in \( A \setminus C \) and a line in \( B \setminus C \), lies in \( C \).

Before introducing the \( k \)-coloured pseudospace in Section 5, we will prove several auxiliary results about the free pseudospace. We hope that this will allow the reader to become more familiar with the theory \( PS \).

**Lemma 4.4.** Let \( X \) and \( Y \) be algebraically closed sets independent over their common intersection \( Z \). If a point \( c \) not contained in \( Y \setminus Z \) lies on the line \( b \) of \( X \), then
\[
X \cup \{c\} \downarrow_{Z} Y.
\]

**Proof.** By the transitivity of non-forking, we may assume that \( Z = X \). If \( c \) belongs to \( X \), then there is nothing to prove. Otherwise, the type of \( c \) over \( X \) has Morley rank \( 1 \) (it is actually strongly minimal), by \([4\), Remark 6.2\]) (cf. \([2\), Corollary 7.13\]). Since the extension \( \text{tp}(c/Y) \) is not algebraic, it does not fork over \( X \).

**Lemma 4.5.** The type of a set \( X \) is determined by the collection of types \( \text{tp}(x,x') \), with \( x \) and \( x' \) in \( X \). In particular, if \( X \equiv_{Z} X' \) and \( X \equiv_{Y} X' \), then \( X \equiv_{YZ} X' \).

**Proof.** Choose an enumeration \( X = \{x_{\alpha}\}_{\alpha<\kappa} \) and flags \( F_{\alpha} \) containing \( x_{\alpha} \), for \( \alpha < \kappa \), such that \( F_{\alpha} \downarrow_{x_{\alpha}} X \cup \{F_{\beta}\}_{\beta<\alpha} \). In particular, for \( \alpha \neq \beta \),
\[
F_{\alpha} \downarrow_{x_{\alpha}} x_{\beta} \quad \text{and} \quad F_{\alpha} \downarrow_{x_{\alpha},x_{\beta}} F_{\beta}.
\]
Since the type of \( F_{\alpha} \) over \( x_{\alpha} \) is stationary, the type of the pair \( (x_{\alpha},x_{\beta}) \) determines the type of \( F_{\alpha}, F_{\beta} \). By \([3\), Theorem 7.24\], the type of \( (F_{\alpha})_{\alpha<\kappa} \), hence the type of \( X \), is uniquely determined by the collection of types \( \text{tp}(x_{\alpha},x_{\beta}) \) for \( \alpha, \beta < \kappa \).

5. A coloured pseudospace. Work inside a sufficiently saturated model \( U \) of the theory \( PS \) of the free pseudospace and consider a natural number \( k \geq 2 \). For \( 0 \leq i < k \), we use the notation \( i + 1 \) instead of \( i + 1 \mod k \), and likewise \( i - 1 \) for \( i - 1 \mod k \).

We colour the lines in \( U \), as well as the pairs \( (a,c) \), where the point \( c \) lies in the plane \( a \), with \( k \) colours. Formally, we partition the set of lines into subsets \( C_{0}, \ldots, C_{k-1} \), and the set of pairs \( (a,c) \), where \( c \) lies in the plane \( a \), into \( I_{0}, \ldots, I_{k} \). Given a plane \( a \) and an index \( 0 \leq i < k \), we define the section \( I_{i}(a) \) to be the collection of points \( c \) with \( I_{i}(a,c) \).
Consider the theory CPS$k$ of $k$-coloured pseudospaces with the following axioms:

• The axioms of PS.

**Universal Axioms**

• For each $0 \leq i < k$, given a line $b$ with colour $i$ in a plane $a$, all the points $c$ in $b$ lie in the section $I_i(a)$ except at most one point, which lies in $I_{i+1}(a)$ (if such a point exists, we call it the *exceptional* point of $b$ in $a$).

**Inductive Axioms**

• Every line $b$ in a plane $a$ contains an exceptional point, denoted by $ep(a, b)$.

• For each $0 \leq i < k$, given a point $c$ and a plane $a$ with $I_i(a, c)$, there are infinitely many lines in $a$ passing through $c$ with colour $i$.

• For each $0 \leq i < k$, given a point $c$ and a plane $a$ with $I_i(a, c)$, there are infinitely many lines in $a$ passing through $c$ with colour $i - 1$.

• For every point $c$ on a line $b$, there are infinitely many planes $a$ containing $b$ such that $c$ is exceptional for $b$ in $a$.

We can construct a model of CPS$k$ as follows: We start with a flag $A_0 = \{a - b - c\}$ with any colouring, e.g. $b \in C_0$ and $I_0(a, c)$ and construct an ascending sequence $A_0 \subset A_1 \subset \cdots$ of coloured geometries by applying one of the operations [0], [1] and [2] to the flag $a - b - c$ in $A_j$ to obtain $A_{j+1}$, extending the colouring to $A_{j+1}$ in an arbitrary way whilst preserving the Universal Axioms. For example, do as follows:

• Operation [0] adds a new point $c'$ to $b$. If $b$ has colour $i$, then for all $a''$ in $A_j$ containing $b$, paint the pair $(a'', c')$ with colour $i$, if $ep(a'', b)$ already exists in $A_j$. Otherwise, paint $(a'', c')$ with colour $i + 1$.

• Operation [1] adds a new line $b'$ between $a$ and $c$. If $(a, c)$ has colour $i$, then paint $b'$ with colour $i$ or colour $i - 1$, and see to it that each choice occurs infinitely often in the sequence.

• Operation [2] adds a new plane $a'$ which contains $b$. If $b$ has colour $i$, then for all $c''$ in $A_j$ which lie on $b$, we give the pair $(a', c'')$ one of the colours $i$ or $i - 1$. Each choice should occur infinitely often.

It is easy to see that the structure obtained in this fashion satisfies all axioms of CPS$k$, so the theory CPS$k$ is consistent.

**Notation.** Given a subset $X$ of a model of CPS$k$, we will denote by $\langle X \rangle$ the algebraic closure of $X$ in the reduct PS, and by $EP(X) = \{ep(a, b) \mid (a, b) \in X \times X\}$ the exceptional points of lines and planes from $X$.

**Remark 5.1.** If the point $c$ is directly connected to a line in $X$, then $\langle X, c \rangle = \langle X \rangle \cup \{c\}$. 
In particular, if \( X = \langle X \rangle \), given \( c \in \text{EP}(X) \), then \( X \cup \{c\} \) is algebraically closed in the reduct PS.

**Proof.** To show that \( \langle X, c \rangle = \langle X \rangle \cup \{c\} \), it suffices to consider the case when \( X \) is nice. The geometry \( X \cup \{c\} \) is either \( X \) or obtained from \( X \) by applying the operation \([0]\), so it is nice again, and thus algebraically closed. \( \blacksquare \)

Similar to [2, Proposition 4.26] or [14, Proposition 3.10], it is easy to see that the collection of partial isomorphisms between PS-algebraically closed finite sets closed under exceptional points inside two \( \aleph_0 \)-saturated models of CPS\(_k\) is non-empty and has the back-and-forth property, so we deduce

**Theorem 5.2.** The theory CPS\(_k\) is complete. Given a set \( X \) in a model of CPS\(_k\) with \( X = \langle X \rangle \) and \( \text{EP}(X) \subset X \), the quantifier-free type of \( X \) determines its type.

The back-and-forth system yields an explicit description of the algebraic closure, as well as showing that the theory CPS\(_k\) is \( \omega \)-stable, by a standard type-counting argument.

**Corollary 5.3.** The theory CPS\(_k\) is \( \omega \)-stable. The algebraic closure \( \text{acl}(X) \) of a set \( X \) is obtained by closing \( \langle X \rangle \) under exceptional points:

\[
\text{acl}(X) = \langle X \rangle \cup \text{EP}(\langle X \rangle).
\]

We deduce the following characterisation of forking over (coloured) algebraically closed sets.

**Corollary 5.4.** Let \( X \) and \( Y \) be supersets of an algebraically closed set \( Z = \text{acl}(Z) \) in CPS\(_k\). We have

\[
X \mathrel{\downarrow^{\text{CPS}_k} Z} Y
\]

if and only if

- \( X \mathrel{\downarrow^\text{PS} Z} Y \),
- \( \text{EP}(\langle X \rangle) \cap \text{EP}(\langle Y \rangle) \subset Z \).

Types over algebraically closed sets are stationary, that is, the theory CPS\(_k\) has weak elimination of imaginaries.

**Proof.** Since PS has weak elimination of imaginaries, non-forking in CPS\(_k\) implies non-forking in the reduct PS over algebraically closed sets, by [5, Lemme 2.1]. Clearly \( \text{EP}(\langle X \rangle) \cap \text{EP}(\langle Y \rangle) \subset Z \).

For the other direction, we may assume that \( X = \langle X \rangle \) and \( Y = \langle Y \rangle \). Lemma 4.4 yields

\[
X \cup \text{EP}(X) \mathrel{\downarrow^\text{PS} Z} Y \cup \text{EP}(Y).
\]

Since \( \text{acl}(X) = X \cup \text{EP}(X) \), Remark 4.3 implies that \( \text{acl}(X) \cup \text{acl}(Y) \) is algebraically closed in PS. We need only show that it contains all exceptional points, so it determines a unique type in the stable theory CPS\(_k\). If \( c \) is an
exceptional point of a plane \(a\) and a line \(b\) in \(\text{acl}(X) \cup \text{acl}(Y)\), we may assume that \(a\) lies in \(X\) and \(b\) lies in \(Y\). Since \(a\) and \(b\) are directly connected and \(X \downarrow^{PS} Z\), Remark 4.3 implies that \(a\) or \(b\) lies in \(Z\). Therefore \(c\) lies in \(\text{EP}(X) \cup \text{EP}(Y)\) and hence is contained in \(\text{acl}(X) \cup \text{acl}(Y)\), as desired. 

**Corollary 5.5.** Let \(X, Y\) and \(Z = \text{acl}(Z)\) be sets such that

\[
X \downarrow Z Y.
\]

Then \(\langle X, Y \rangle \cap \text{acl}(X, Z) = \langle X, Y \rangle \cap \langle X, Z \rangle\).

**Proof.** Let \(\xi\) be in \(\langle X, Y \rangle \cap \text{acl}(X, Z)\). The independence \(X \downarrow Z Y\) yields \(\xi, X \downarrow Z Y\). It follows from Corollary 5.4 that 

\[
\xi, X \downarrow^{PS} Z Y, \quad \text{and thus} \quad \xi \downarrow^{PS}_{X, Z} X, Y.
\]

Since \(\xi\) lies in \(\langle X, Y \rangle\), the above independence implies that \(\xi\) lies in \(\langle X, Z \rangle\), as desired.

**Proposition 5.6.** Let \(X = \langle X \rangle\) and \(Y = \langle Y \rangle\) be subsets of a model of \(\text{CPS}_k\). A map \(F : X \to Y\) is elementary with respect to the theory \(\text{CPS}_k\) if and only if it satisfies the following conditions:

1. The map \(F\) is a partial isomorphism with respect to the reduct \(\text{PS}\).
2. The function \(F\) preserves colours of lines and sections.
3. For all \(a, a'\) and \(b\) in \(X\), we have \(\text{ep}(a, b) = \text{ep}(a', b)\) if and only if \(\text{ep}(F(a), F(b)) = \text{ep}(F(a'), F(b))\).

**Proof.** We need only show that \(F\) is elementary if it satisfies all three conditions. By Theorem 5.2 it suffices to show that \(F\) extends to a partial isomorphism \(\tilde{F}\) preserving colours between \(\text{acl}(X) = X \cup \text{EP}(X)\) and \(\text{acl}(Y) = Y \cup \text{EP}(Y)\).

For each line \(b\) in \(X\) contained in a plane \(a\) of \(X\), set \(\tilde{F}(\text{ep}(a, b)) = \text{ep}(F(a), F(b))\). Let us first show that \(\tilde{F}\) is well-defined, which analogously implies that \(\tilde{F}\) is a bijection. Suppose that \(\text{ep}(a, b) = \text{ep}(a_1, b_1)\) for a line \(b_1\) contained in the plane \(a_1\), both in \(X\). If \(b \neq b_1\), then \(\text{ep}(a, b)\) is the unique intersection of \(b\) and \(b_1\), both lines in \(X\), so \(\text{ep}(a, b)\) lies in \(X\) and hence its image is determined by \(F\). Otherwise, we conclude that \(b = b_1\), and thus \(\tilde{F}\) is bijective, by \([3]\).

Similarly, the map \(\tilde{F}\) defined above is a partial isomorphism with respect to the reduct \(\text{PS}\). We need only show that \(\tilde{F}\) preserves the colours of sections. Choose a new point \(\text{ep}(a, b)\) not in \(X\) and an arbitrary plane \(a_1 \neq a\) in \(X\) containing \(\text{ep}(a, b)\). Since \(\text{ep}(a, b)\) does not lie in \(X\), the intersection of \(a\) and \(a_1\) cannot solely consist of the point \(\text{ep}(a, b)\). Hence, this intersection is given by a unique line \(b_1\), which lies in \(X\) and contains \(\text{ep}(a, b)\). We conclude as before that \(b = b_1\). The colour of \(\text{ep}(a, b)\) in \(a_1\) is uniquely determined.
according to whether $\text{ep}(a, b) = \text{ep}(a_1, b)$, and thus so is the colour of its image in $F(a_1)$ by $\tilde{F}$, by (3).

**6. Coloured paths.** We will now show that the theory $\text{CPS}_k$ is not indiscernibly acyclic, and hence it is not equational, yet every proper cycle of types has length at least $k$ (cf. Theorem 6.2), so we expect the complexity of these theories to increase as $k$ grows. However, we do not know whether two of these theories are bi-interpretable.

**Theorem 6.1.** In $\text{CPS}_k$ there is a proper cycle of types

$$p_0(x; y) \rightarrow p_1(x; y) \rightarrow \cdots \rightarrow p_{k-1}(x; y) \rightarrow p_0(x; y),$$

where both the variables $x$ and $y$ have length 1. In particular, the theory $\text{CPS}_k$ is not equational.

*Proof.* For each $0 \leq r < k$, a pair $(a, c)$ with colour $I_r$ has a unique type $p_r = \text{tp}(c, a)$, for the set $\{a, c\}$ is algebraically closed, since it is the intersection of all the flags containing $a$ and $c$, and it is closed under exceptional points, for it contains no line. Clearly $p_r \neq p_{r+1}$ for each $0 \leq r < k$.

It suffices to show that $p_r \rightarrow p_{r+1}$: Let $(a, c)$ with colour $I_r$, and choose a line $b$ connecting them with colour $r$. Let $c'$ be the exceptional point of $b$ in $a$, so $(c', a) \models p_{r+1}$. Now, the set $\{b\}$ is algebraically closed in $\text{CPS}_k$. By Corollary 5.4, the points $c$ and $c'$ have the same strong type over $b$, and

$$c \downarrow_b a.$$

Corollary 3.5 implies that $p_r = \text{tp}(c, a) \rightarrow \text{tp}(c', a) = p_{r+1}$, as desired. ■

**Theorem 6.2.** Let $x$ and $y$ be finite tuples of variables. In $\text{CPS}_k$, every proper cycle of types

$$p_0(x; y) \rightarrow p_1(x; y) \rightarrow \cdots \rightarrow p_{n-1}(x; y) \rightarrow p_0(x; y)$$

has length $n \geq k$.

*Proof.* A proper cycle of types $p_0(x; y) \rightarrow p_1(x; y) \rightarrow \cdots \rightarrow p_{n-1}(x; y) \rightarrow p_0(x; y)$ as above induces a cycle in the reduct $\text{PS}$, which is equational. Therefore, the colourless reducts of $p_r$ and $p_s$ agree for all $r, s$.

Corollary 3.5 implies that there are tuples $f, e_0, \ldots, e_n$ and algebraically closed subsets $Z_0, \ldots, Z_{n-1}$ such that

- $\models p_r(e_r, f)$ for $0 \leq r \leq n - 1$, and $\models p_0(e_n, f)$,
- $e_r \equiv_{Z_r} e_{r+1}$ for $0 \leq r \leq n - 1$,
- $e_r \downarrow_{Z_r} f$ for $0 \leq r \leq n - 1$.

Set $Y = \text{acl}(f)$ and $X_r = \text{acl}(e_r)$ for $0 \leq r \leq n$. Since the definable closure and the algebraic closure coincide, and the colourless reducts of all $p_r$ agree, all the types $\text{tp}_{\text{PS}}(X_rY)$ are equal. Denote $\langle X_rY \rangle$ by $P_r$. We find colourless
isomorphisms

\[ F_r : P_r \to P_{r+1} \]

which fix \( Y \) pointwise. Note that \( X_r \) and \( X_{r+1} \) have the same type over \( Z_r \) for \( r \leq n - 1 \). Lemma 4.3\[4.5\] yields \( \text{tp}^{PS}(X_r Y Z_r) = \text{tp}^{PS}(X_{r+1} Y Z_r) \) for \( r \leq n - 1 \). The above map \( F_r \) extends to a colourless isomorphism between \( \langle X_r Y Z_r \rangle \) and \( \langle X_{r+1} Y Z_r \rangle \), which is the identity on \( \langle Y Z_r \rangle \). We will still refer to this colourless isomorphism as \( F_r \), keeping in mind that it is elementary in the sense of \( \text{CPS}_k \) on \( \langle X_r Z_r \rangle \) and (clearly) on \( \langle Y Z_r \rangle \) separately. Observe that

\[ \langle X_r Y Z_r \rangle = \langle X_r Z_r \rangle \cup \langle Y Z_r \rangle, \]

by Remark 4.3\[4.3\](1).

If a set \( W \) is finite, so are the closures \( \langle W \rangle \) and \( \text{acl}(W) \). Define the defect of \( W \) as the natural number

\[ \text{defect}(W) = |\text{acl}(W) \setminus \langle W \rangle| = |\text{EP}(\langle W \rangle) \setminus \langle W \rangle|. \]

**Claim.** For each \( r \leq n - 1 \), we have \( \text{defect}(P_r) \geq \text{defect}(P_{r+1}) \).

**Proof of Claim.** If \( \text{ep}(a, b) = \text{ep}(a_1, b_1) \) for \( b \) and \( b_1 \) in \( P_r \), with \( b \neq b_1 \), then the point \( \text{ep}(a, b) \) lies in \( P_r \) by Remark 4.3\[4.3\](4). Thus, it suffices to show the following:

(1) If the line \( b \) in \( P_r \) lies in the plane \( a \) in \( P_r \) with \( \text{ep}(a, b) \) in \( P_r \), then \( \text{ep}(F_r(a), F_r(b)) \) lies in \( P_{r+1} \).

(2) If \( a, a_1 \) and \( b \) lie in \( P_r \) and \( \text{ep}(a, b) = \text{ep}(a_1, b) \), then

\[ \text{ep}(F_r(a), F_r(b)) = \text{ep}(F_r(a_1), F_r(b)). \]

For \[1\], since \( X_r \upharpoonright_{Z_r} Y \), the plane \( a \) and the line \( b \) must both lie in \( P_r \cap \langle X_r Z_r \rangle \) or both in \( P_r \cap \langle Y Z_r \rangle \), by the independence

\[ a \upharpoonright_{Z_r}^\text{PS} b \]

and Remark 4.3\[4.3\](2). For example, let \( a \) and \( b \) lie in \( P_r \cap \langle X_r Z_r \rangle \), so \( \text{ep}(a, b) \) lies in \( P_r \cap \text{acl}(X_r Z_r) = P_r \cap \langle X_r Z_r \rangle \), by Corollary 5.5. Since \( F_r \) is elementary on \( P_r \cap \langle X_r Z_r \rangle \), we see that \( \text{ep}(F_r(a), F_r(b)) = F_r(\text{ep}(a, b)) \) lies in \( P_{r+1} \), as desired. Observe that we have actually shown that

\[ \text{ep}(a, b) \in P_r \iff \text{ep}(F_r(a), F_r(b)) \in P_{r+1}. \]

For \[2\], we need only consider the case when \( a \neq a_1 \) and the exceptional point \( \text{ep}(a, b) = \text{ep}(a_1, b) \) does not lie in \( P \), by \[1\]. Again, if \( a \) and \( a_1 \) lie both in \( \langle X_r Z_r \rangle \) or both in \( \langle Y Z_r \rangle \), then so does \( b \), and we are done by Proposition 5.6, since \( F_r \) is elementary on each side. If this is not the case, and \( a \) lies in \( P_r \cap \langle X_r Z_r \rangle \) and \( a_1 \) in \( P_r \cap \langle Y Z_r \rangle \), then the line \( b \) lies in \( P_r \cap Z_r \), by Remark 4.3\[4.3\](3). Thus the point \( \text{ep}(a, b) = \text{ep}(a_1, b) \) lies in \( \text{acl}(X_r, Z_r) \cap \text{acl}(Y, Z_r) = Z_r \), so we conclude as before since \( F_r \) is elementary on each side separately. **Claim**
As $P_0$ and $P_n$ have the same type, their defect is the same, so $\text{defect}(P_r) = \text{defect}(P_{r+1})$ for all $0 \leq r \leq n - 1$. Hence, for all $a$, $a'$ and $b$ in $P_r$, we have $\text{ep}(a, b) = \text{ep}(a', b)$ if and only if $\text{ep}(F_r(a), F_r(b)) = \text{ep}(F_r(a'), F_r(b))$.

Since $P_r$ and $P_{r+1}$ are closed in the reduct $PS_k$, but $\text{tp}(P_r) \neq \text{tp}(P_{r+1})$, Proposition 5.6 implies that $F_r$ restricted to $P_r$ cannot preserve colours. As $F_r$ is elementary on each side separately, the colours of lines are preserved. Thus, there is a pair $(a, c)$ in $P_r$ whose colour $j$, with $0 \leq j < k$, is not preserved under $F_r$. We will show that the colour of $(F_r(a), F_r(c))$ is $j + 1$.

Since $F_r$ is elementary on $X_r$ and on $Y Z_r$ separately, neither $a$ nor $c$ lie in $Z_r$. The independence $X_r \upharpoonright PS Y$ and Remark 4.3(3) imply that there is a line $b$ in $Z_r$ connecting $a$ and $c$. The characterisation of the independence in Corollary 5.4 implies that $c \neq \text{ep}(a, b)$. Hence the line $b$ must have colour $j$. The map $F_r$ is the identity on $Z_r$, and the plane $F_r(a)$ is connected to the point $F_r(c)$ by $b = F_r(b)$, so the only possible colours for $(F_r(a), F_r(c))$ are $j$ or $j + 1$. As the colour of $(a, c)$ is not preserved, we deduce that $(F_r(a), F_r(c))$ has colour $j + 1$, as desired.

Let $F_n$ be the $\text{CPS}_k$-elementary map mapping $P_n$ to $P_0$ (as both $(e_0, f)$ and $(e_n, f)$ realise the type $p_0$) and write $F^r = F_r \circ \cdots \circ F_0$. Notice that $F^n$ is the identity of $P_0$. Let $(a, c)$ be one of the pairs in $P_0$ whose colour $j_0$ changes under $F_0$. The colours of the pairs

$$(a, c), F^0(a, c), \ldots, F^{n-1}(a, c)$$

change at each step by at most adding 1 (modulo $k$), so the colour of $F^{n-1}(a, c)$ equals $j_0 + m$ modulo $k$, for some $1 \leq m \leq n$. Since $F_n$ preserves colours and $F^n(a, c) = (a, c)$, we find that $m$ is divisible by $k$, and thus $k \leq m \leq n$. We conclude that the original cycle had length at least $k$.

**Remark 6.3.** Given a function $\pi : \{0, \ldots, k - 1\} \to \{0, \ldots, k - 1\}$ with no fixed points, we could similarly consider the theory $\text{CPS}_\pi$ of coloured pseudospaces such that given a line $b$ with colour $j$ inside a plane $a$, all points in $b$ lie in the section $I_i(a)$ except one unique exceptional point which lies in $I_{\pi(i)}(a)$.

The corresponding theory $\text{CPS}_\pi$ is not equational. Every closed path of real types has length at least the length of the shortest $\pi$-cycle.

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