A NEW CLASS OF EXCEPTIONAL ORTHOGONAL POLYNOMIALS: THE
TYPE III $X_m$-LAGUERRE POLYNOMIALS AND THE SPECTRAL ANALYSIS
OF THREE TYPES OF EXCEPTIONAL LAGUERRE POLYNOMIALS

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Abstract. The Bochner Classification Theorem (1929) characterizes the polynomial sequences
$\{p_n\}_{n=0}^{\infty}$, with $\deg p_n = n$ that simultaneously form a complete set of eigenstates for a second-
order differential operator and are orthogonal with respect to a positive Borel measure having
finite moments of all orders. Indeed, up to a complex linear change of variable, only the classical
Hermite, Laguerre, and Jacobi polynomials, with certain restrictions on the polynomial parameters,
satisfy these conditions. In 2009, Gómez-Ullate, Kamran, and Milson found that for sequences
$\{p_n\}_{n=1}^{\infty}$, $\deg p_n = n$ (without the constant polynomial), the only such sequences satisfying these
conditions are the exceptional $X_1$-Laguerre and $X_1$-Jacobi polynomials. Subsequently, during the
past five years, several mathematicians and physicists have discovered and studied other exceptional
orthogonal polynomials $\{p_n\}_{n \in \mathbb{N}_0 \setminus A}$, where $A$ is a finite subset of the non-negative integers $\mathbb{N}_0$ and
where $\deg p_n = n$ for all $n \in \mathbb{N}_0 \setminus A$. We call such a sequence an exceptional $X_{|A|}$ sequence, where
$|A|$ denotes the cardinality of $A$. All exceptional sequences found, to date, have the remarkable
feature that they form a complete orthogonal set in their natural Hilbert space setting.

Among the exceptional sets already known are two types of exceptional Laguerre polynomials,
called the Type I and Type II $X_m$-Laguerre polynomials, each omitting $m$ polynomials. In this paper,
we briefly discuss these polynomials and construct the self-adjoint operators generated by their
respective second-order differential expressions in the appropriate Hilbert spaces. In addition,
we present a new Type III family of $X_m$-Laguerre polynomials along with a detailed disquisition
of its properties. We include several representations of these polynomials, orthogonality, norms,
completeness, the location of their local extrema and roots, root asymptotics, as well as a complete
spectral study of the second-order Type III exceptional $X_m$-Laguerre differential expression.

1. Introduction

An exceptional orthogonal polynomial system is a sequence $\{p_n\}_{n \in \mathbb{N}_0 \setminus A}$ with the following character-
istic properties:

(a) $\deg(p_n) = n$ for $n \in \mathbb{N}_0 \setminus A$, where $A$ is a finite subset of $\mathbb{N}_0$;

(b) there exists an interval $I = (a, b)$ and a Lebesgue measurable weight $w \geq 0$ on $I$ such that

$$\int_I p_n p_m w = k_n \delta_{n,m} \quad (n, m \in \mathbb{N}_0 \setminus A)$$

for some $k_n > 0$; here $\delta_{n,m}$ denotes the Kronecker delta symbol;

(c) there exists a second-order differential expression

$$\ell[y](x) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x)$$

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and, for each \( n \in \mathbb{N}_0 \setminus A \), there exists a \( \lambda_n \in \mathbb{C} \) such that \( y = p_n(x) \) is a solution of
\[
\ell[y](x) = \lambda_n y(x) \quad (x \in I);
\]
(d) for \( n \in A \), there does not exist a polynomial \( p \) of degree \( n \) such that \( y = p(x) \) satisfies
\[
\ell[y] = \lambda y \quad \text{for any choice of } \lambda \in \mathbb{C};
\]
(e) all of the moments
\[
\int_I x^n w(x) dx \quad (n \in \mathbb{N}_0)
\]
of \( w \) exist and are finite.

If \( |A| \) denotes the cardinality of the set \( A \), we call a sequence \( \{p_n\}_{n \in \mathbb{N}_0 \setminus A} \) satisfying conditions (a)-(e) above an \textit{exceptional sequence of codimension} \( |A| \) or an \( X_{|A|} \) \textit{polynomial sequence}. The case \( A = \{0\} \) was treated in [9], where the authors classified \( X_1 \) exceptional orthogonal polynomials and introduced the \( X_1 \)-Laguerre and the \( X_1 \)-Jacobi polynomials, so named because of their similarity to their classical cousins. The fact that these sequences omit a constant polynomial distinguishes their characterization from the Bohner classification [3] characterizing the Jacobi, Laguerre, and Hermite polynomials; of course, the Bohner classification corresponds to \( A = \emptyset \).

Since 2009, several authors have generalized the results of Kamran, Milson and Gómez-Ullate in [9] by finding other sequences of exceptional polynomials \( \{p_n\}_{n \in \mathbb{N}_0 \setminus A} \), where \( A \) is a finite subset of \( \mathbb{N}_0 \), satisfying each of the conditions in (a)-(e).

In this paper, we discuss three families of exceptional Laguerre polynomials, each spanning a flag of codimension \( m \). Specifically, we deal with two such exceptional Laguerre sequences associated with
\[
(1.1) \quad A = \{0, 1, \ldots, m - 1\}
\]
and a new exceptional Laguerre sequence when
\[
(1.2) \quad A = \{1, 2, \ldots m\}.
\]
Associated with (1.1), the two exceptional Laguerre sequences are known as the Type I and Type II exceptional Laguerre polynomials, discovered and introduced into the literature by Odake and Sasaki in [22] and [23]; properties of these two sets have been studied at length and can be found in, among others, the contributions [11, 12, 16]. In this paper, after a brief review of their properties, we develop the spectral theory of the two second-order exceptional Laguerre differential equations having the Type I and Type II sequences as eigenfunctions. As mentioned above, we also present a new sequence of exceptional Laguerre polynomials, naturally named the Type III exceptional \( X_m \)-Laguerre polynomials; these polynomials are associated with the set \( A \) given in (1.2).

The discovery of the exceptional orthogonal polynomials is one of the most interesting, and intensive studies, over the past five years in the area of exactly solvable models in quantum mechanics. We refer the interested reader to the contributions [10, 11, 12, 13, 14, 15, 18, 19, 22, 23, 24, 25], each of which has been influential in our study of this developing subject. Exceptional orthogonal polynomials and their associated exactly solvable potentials have applications in a wide range of problems in mathematical physics, including mass-dependent potentials [20], supersymmetric quantum mechanics [7, 8], quasi-exact solvability [27], and Fokker-Planck and Dirac equations [17]. Both mathematicians and physicists have discovered some of these new orthogonal polynomials as a result of deforming the radial oscillator potential and the Darboux-Pöschl-Teller potential in terms of an eigenfunction which is a polynomial of degree \( m \). The lowest \( (m = 1) \) such examples, the exceptional \( X_1 \)-Laguerre and \( X_1 \)-Jacobi polynomials, are equivalent to those introduced by Kamran, Gómez-Ullate and Milson in their seminal paper [9]. Subsequently, their results were reformulated in the framework of quantum mechanics and shape-invariant potentials by Quesne (see [21, 25]).
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who first related these orthogonal polynomials to the Darboux transformation and classical orthogonal polynomials. The paper [22] first introduced higher order codimension exceptional orthogonal polynomials for arbitrary positive integers $m$. Besides the Darboux transformation being intimately connected with the derivation of such exceptional families so is the notion of bispectrality and other tools that appear in the theory of integrable systems. In mathematical physics, these functions allow us to write exact solutions to rational extensions of classical quantum potentials. From the point of view of special functions and orthogonal polynomials, these exceptional orthogonal polynomials are polynomial systems formed by solutions to Fuchsian linear equations that belong to the Heine-Stieltjes class.

From a mathematical point of view, there are several surprises above and beyond the implications of generalizing the Bochner classification theorem. Indeed, these new exceptional orthogonal polynomials are wonderful new examples to illustrate the classical Glazman-Krein-Naimark theory [21] of constructing self-adjoint operators from Lagrangian symmetric second-order differential expressions. Furthermore, the fact that each of the exceptional orthogonal polynomial sequences found, to date, are complete in their natural Hilbert space setting is nothing short of remarkable. In particular, it is remarkable that the three exceptional Laguerre sequences that we discuss in this paper are complete since each of them are missing $m$ polynomials. On the other hand, their completeness suggests that interesting Müntz-type theorems (see, for example, [4] and [6, Theorem 7.6]) in weighted $L^2$-spaces lie in waiting to be discovered.

The contents of this paper are as follows. Section 2 reviews some essential facts about the classical Laguerre expression and its solutions. We also give a brief synopsis of rational factorizations and the Darboux transformation as it relates to the Laguerre case in this section. In Section 3 we review the main properties of the Type I exceptional $X_m$-Laguerre polynomials and then develop the spectral theory of their associated second-order differential expression. In particular, we construct a self-adjoint operator, generated from the second-order Type I exceptional $X_m$-Laguerre differential expression, which has the Type I exceptional $X_m$-Laguerre polynomials as eigenfunctions (Theorem 3.5). We also discuss another interesting self-adjoint operator (Theorem 3.6), generated by the Type I exceptional $X_m$-Laguerre differential expression, which has a complete set of eigenfunctions involving the Type III exceptional $X_m$-Laguerre polynomials. Section 4 treats the Type II exceptional $X_m$-Laguerre polynomials in a similar fashion. In Section 5 we introduce the Type III exceptional $X_m$-Laguerre polynomials and we develop many of their properties in full detail. Among these properties, we will establish several explicit representations of these polynomials. In Section 5 we will also compute the norms of these polynomials (Theorem 5.3) in the appropriate Hilbert space $H$ and show that the sequence of Type III exceptional $X_m$-Laguerre polynomials, despite missing polynomials $p$ of degrees $1 \leq \deg(p) \leq m$, forms a complete orthogonal set of polynomials (Theorem 5.4) in $H$. We also determine the location of the roots of these polynomials (Theorem 5.5); more precisely, we show that the Type III exceptional $X_m$-Laguerre polynomial of degree $m + k$ has $k$ positive roots and $m$ negative roots and we give careful interlacing properties of these roots with the roots of the two classical Laguerre polynomials $L_{k+1}^{\alpha+1}(x)$ and $L_{m-\alpha-1}^{-\alpha-1}(-x)$. We also discuss the asymptotic behavior of the roots of these Type III exceptional $X_m$-Laguerre polynomials (Theorem 5.6). In Section 5 we develop spectral properties of the second-order Type III exceptional $X_m$-Laguerre differential expression $\ell_{m,\alpha}^{III}[-]$ and, in particular, determine the self-adjoint operator in $H$, generated by $\ell_{m,\alpha}^{III}[-]$, which has the Type III exceptional $X_m$-Laguerre polynomials as eigenfunctions (Theorem 5.8). Lastly, in the Appendix, we list some examples of Type III exceptional $X_m$-Laguerre polynomials.
**Notation:** Let \( \mathbb{N} \) denote the set of positive integers and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). The set \( \mathcal{P} \) will denote the vector space of all complex-valued polynomials \( p(x) \) in the real variable \( x \). For \( n \in \mathbb{N}_0 \), let \( \mathcal{P}_n \) denote the \((n+1)\)-dimensional vector space of all polynomials of degree \( \leq n \).

2. **The Classical Laguerre Differential Expression and Rational Factorizations**

Let
\[
\ell^\alpha[y] = -xy'' + (-\alpha - 1 + x)y'
\]
 denote the classical Laguerre differential expression. Of course, for each \( m \in \mathbb{N}_0 \), \( y = L_m^\alpha(x) \) is a solution of
\[
\ell^\alpha[y](x) = my(x).
\]

**Remark 2.1.** In the contributions [9, 10, 11, 12], the authors define the Laguerre expression as
\[
\bar{\ell}^\alpha[y] = xy'' + (\alpha + 1 - x)y';
\]
for operator-theoretic and spectral-analytic reasons, we elect to define the Laguerre expression as in (2.1).

A rational factorization of \(-\ell^\alpha[\cdot]\) is an identity of the form
\[
\ell^\alpha = BA - \lambda,
\]
where \( A \) and \( B \) are first-order linear differential expressions with rational coefficients. We call
\[
\hat{\ell}^\alpha := AB - \lambda
\]
the *partner operator* corresponding to the above rational factorization. Suppose \( \phi(x) \) is a quasi-rational solution (that is, \( \frac{\phi'(x)}{\phi(x)} \) is a rational function) of \( A[y] = 0 \). Notice, from (2.2), that
\[
\ell^\alpha[\phi](x) = \lambda \phi(x).
\]

The operators \( A \) and \( B \) are given by
\[
A[y](x) = b(x) \left( y'(x) - \frac{\phi'(x)}{\phi(x)} y(x) \right)
\]
and
\[
B[y](x) = \hat{b}(x) \left( y'(x) - \hat{w}(x) y(x) \right),
\]
where \( b(x) \) is a rational function, called the *factorization gauge*, and
\[
\hat{b}(x) = \frac{x}{b(x)}, \quad \hat{w}(x) = -\frac{\phi'(x)}{\phi(x)} + \frac{b'(x)}{b(x)} - \frac{1 + \alpha - x}{x}.
\]

In this case, the second-order partner operator is given by
\[
\hat{\ell}^\alpha[y](x) := xy''(x) + \hat{q}(x)y'(x) + \hat{r}(x)y(x),
\]
where
\[
\hat{q}(x) = 2 + \alpha - x - 2x \frac{b'(x)}{b(x)}
\]
and
\[
\hat{r}(x) = -x \left( (\hat{w}(x))' + (\hat{w}(x))^2 \right) - \hat{q}(x)\hat{w}(x) + \lambda.
\]

In choosing the factorization gauge \( b(x) \), we are guided by two principles: (a) we want polynomial eigenfunctions of the partner operator and (b) we do not want these polynomial eigenfunctions to
have a common factor. For further information on rational factorizations and the above transformation formulas, we refer the reader to [11] Section 3.

As discussed in [2], the quasi-rational solutions of the differential equation \( \ell^\alpha[y] = \lambda y \) are given by the following four families:

\[
\begin{align*}
\phi_0(x) &= L_0^\alpha(x) \quad \lambda = m \\
\phi_1(x) &= e^x L_0^\alpha(-x) \quad \lambda = -(\alpha + 1 + m) \\
\phi_2(x) &= x^{-\alpha} L_0^\alpha(x) \quad \lambda = m - \alpha \\
\phi_3(x) &= x^{-\alpha} e^x L_0^\alpha(-x) \quad \lambda = -(m + 1).
\end{align*}
\]

(2.4) (2.5) (2.6)

Our choice of labels differs from [2] to better conform to the Type I, II, III nomenclature for the exceptional Laguerre polynomials.

As described in [11], each of these quasi-rational solutions corresponds to a rational factorization of \( \ell^\alpha[\cdot] \) and, through the Darboux transform (2.3), lead to the Type I, Type II and, as we will see in this manuscript, the Type III exceptional \( X_m \)-Laguerre operators.

3. The Type I Exceptional \( X_m \)-Laguerre Polynomials

Unless otherwise indicated, for the Type I exceptional \( X_m \)-Laguerre polynomials, we assume that the parameter \( \alpha > 0 \). Explicit details of properties of these polynomials can be found in [11, 12, 16]. In this section, we will briefly discuss some properties of these polynomials and then study the spectral properties of the associated second-order Type I differential expression.

3.1. Properties of the Type I Exceptional \( X_m \)-Laguerre Polynomials.

For fixed \( m \in \mathbb{N} \), the classical Laguerre polynomial \( L_{m-1}^\alpha(-x) \) has no roots in \([0, \infty)\) when \( \alpha > 0 \). Taking \( \phi_1(x) \) in (2.4) as the quasi-rational solution and the classical Laguerre polynomial \( L_{m}^\alpha(-x) \) as the factorization gauge, it can be seen that the classical Laguerre expression \( \ell^\alpha[\cdot] \), given in (2.1), may be rewritten as

\[
-\ell^\alpha = B_{m}^1, \quad A_{m}^1[y] := L_{m}^\alpha(-x)y'(x) - L_{m}^{\alpha+1}(-x)y(x) \quad \text{and} \quad B_{m}^1[y] := \frac{xy'(x) + (1 + \alpha)y(x)}{L_{m}^\alpha(-x)}.
\]

With these definitions, the Type I exceptional \( X_m \)-Laguerre expression \( \ell_{1}^{\alpha}[\cdot] \) may be written as

\[
\ell_{1}^{\alpha}[y] := -\ell^\alpha[y](x) + 2(\log L_{m}^{\alpha-1}(-x))'(xy'(x) + \alpha y(x)) - my(x)
\]

(3.1)

Written out, this expression is given by

\[
= -xy''(x) + \left(x - \alpha - 1 + 2x(\frac{L_{m}^{\alpha-1}(-x))'}{L_{m}^{\alpha-1}(-x)} \right) y'(x)
\]

\[
+ \left(\frac{2\alpha(L_{m}^{\alpha-1}(-x))'}{L_{m}^{\alpha-1}(-x)} - m\right) y(x).
\]

The Type I exceptional \( X_m \)-Laguerre polynomial \( y = L_{m}^{\alpha}(x) \) \( (n \in \mathbb{N} \setminus \{0, 1, 2, \ldots, m - 1\}) \) satisfies the second-order differential equation

\[
\ell_{1}^{\alpha}[y] = \lambda_n y \quad (0 < x < \infty),
\]

where

\[
\lambda_n := n - m \quad (n = m, m + 1, m + 2, \ldots)
\]
so \( \{\lambda_n \mid n = m, m + 1, m + 2, \ldots\} = \mathbb{N}_0 \).

The \( n \)th degree Type I exceptional \( X_m \)-Laguerre polynomial can be expressed in terms of classical Laguerre polynomials through the formula
\[
L_{m,n}^\alpha(x) = L_m^\alpha(-x)L_{n-m}^{\alpha-1}(x) + L_m^{\alpha-1}(-x)L_{n-m-1}^\alpha(x) \quad (n \geq m).
\]

In fact, (3.2) follows by expanding the expression
\[
L_{m,n}^\alpha(x) := -A_m^{\alpha-1}[L_{n-m}^{\alpha-1}](x) \quad (n \geq m).
\]

The Type I exceptional \( X_m \)-Laguerre polynomials \( \{L_{m,n}^\alpha\}_{n=m}^\infty \) are orthogonal on \((0, \infty)\) with respect to the weight function
\[
W_m^\alpha(x) = \frac{x^\alpha e^{-x}}{(L_{m,n}^{\alpha-1}(-x))^2} \quad (0 < x < \infty).
\]

**Remark 3.1.** Notice that, since \( L_m^{\alpha-1}(-x) \) is positive and increasing on \((0, \infty)\), the function \(1/(L_m^{\alpha-1}(-x))^2\) is both bounded and bounded away from zero on \((0, \infty)\). Consequently, all moments of the weight function \( W_m^\alpha \) on the interval \((0, \infty)\) exist and are finite.

In [12], the Type I exceptional \( X_m \)-polynomials \( \{L_{m,n}^\alpha\}_{n=m}^\infty \) are shown to be complete in the Hilbert space \( L^2((0, \infty); W_m^\alpha) \). With \( \| \cdot \|_{m}^{\alpha} \) denoting the norm
\[
\|f\|_{m}^{\alpha} = \left( \int_0^\infty |f(x)|^2 W_m^\alpha(x)dx \right)^{1/2} \quad (f \in L^2((0, \infty); W_m^\alpha))
\]
in \( L^2((0, \infty); W_m^\alpha) \), derived from the inner product
\[
(f, g)_{m}^{\alpha} := \int_0^\infty f(x)\overline{g(x)}W_m^\alpha(x)dx \quad (f, g \in L^2((0, \infty); W_m^\alpha)),
\]
the explicit norms of the Type I exceptional \( X_m \)-Laguerre polynomials are given by
\[
\left( \|L_{m,n}^\alpha\|_m^{\alpha} \right)^2 = \frac{(\alpha + n)!}{(n - m)!} \frac{\Gamma(\alpha + n - m)}{\Gamma(\alpha + m)} \quad (n \geq m);
\]
see [11].

In [16], the authors prove the following two theorems concerning the zeros of \( \{L_{m,n}^\alpha\}_{n=m}^\infty \).

**Theorem 3.1.** [16, Proposition 3.2] For \( \alpha > 0 \), the Type I exceptional \( X_m \)-Laguerre polynomial \( L_{m,m+k}^\alpha(x) \) has \( k \) simple zeros in \((0, \infty)\) and \( m \) simple zeros in \((-\infty, 0)\). More specifically, the positive roots of \( L_{m,m+k}^\alpha(x) \) are located between consecutive roots of \( L_k^{\alpha}(x) \) and \( L_{k-1}^{\alpha}(x) \) with the smallest positive root of \( L_{m,m+k}^\alpha(x) \) located to the left of the smallest root of \( L_k^{\alpha}(x) \). The negative roots of \( L_{m,m+k}^\alpha(x) \) are located between the consecutive roots of \( L_{m-1}^{\alpha}(-x) \) and \( L_m^{\alpha}(-x) \).

**Theorem 3.2.** [16, Corollary 3.1 and Proposition 3.4] For \( k \geq 1 \), the following asymptotic results for the roots of \( L_{m,m+k}^\alpha(x) \) hold:

(a) Let \( \{j_{\alpha,i}\} \) denote the sequence of zeros of the Bessel function of the first kind \( J_\alpha(x) \) listed in increasing order and let \( \{x_{k,i}\}_{i=1}^k \) denote the positive zeros of \( L_{m,m+k}^\alpha(x) \) arranged in increasing order. Then
\[
\lim_{k \to \infty} kx_{k,i}^\alpha = \frac{j_{\alpha,i}^2}{4} \quad (i \in \mathbb{N}).
\]

(b) As \( k \to \infty \), the \( m \) negative roots of \( L_{m,m+k}^\alpha(x) \) converge to the \( m \) roots of \( L_m^{\alpha-1}(-x) \).
3.2. Type I Exceptional \( X_m \)-Laguerre Spectral Analysis. In Lagrangian symmetric form, the Type I exceptional \( X_m \)-Laguerre differential expression \((3.1)\) is given by

\[
\ell_{\alpha m}^f[y](x) = \frac{1}{W_{\alpha m}(x)} \left( -x^{\alpha+1}e^{-x} \left( x^{\alpha} \frac{y'(x)}{W_{\alpha m}(x)} \right)' + \left( 2\alpha x^{\alpha}e^{-x}(L_m^{\alpha-1}(-x))' - \frac{m\alpha x^{\alpha}e^{-x}}{(L_m^{\alpha-1}(-x))^2 - (L_m^{\alpha-1}(-x))^3} \right) \right) y(x). 
\]

When \( m = 1 \), the spectral analysis of \((3.4)\) in \( L^2((0, \infty); W_1^{1, \alpha}) \) was completed in \([1]\).

The maximal domain associated with \( \ell_{\alpha m}^f[\cdot] \) in the Hilbert space \( L^2((0, \infty); W_1^{1, \alpha}) \) is defined to be

\[
\Delta_{\alpha m}^{1, \alpha} := \{ f : (0, \infty) \to \mathbb{C} \mid f, f' \in AC_{loc}(0, \infty); f, \ell_{\alpha m}^f[f] \in L^2((0, \infty); W_1^{1, \alpha}) \}.
\]

The associated maximal operator

\[
T_{1, m}^{1, \alpha} : \mathcal{D}(T_{1, m}^{1, \alpha}) \subset L^2((0, \infty); W_1^{1, \alpha}) \to L^2((0, \infty); W_1^{1, \alpha}),
\]

is defined to be

\[
f \in \mathcal{D}(T_{1, m}^{1, \alpha}) := \Delta_{\alpha m}^{1, \alpha}.
\]

For \( f, g \in \Delta_{\alpha m}^{1, \alpha} \), Green’s formula is given by

\[
\int_0^\infty \ell_{\alpha m}^f[f](x)g(x)W_1^{1, \alpha}(x)dx = [f, g]_{\alpha m}^{1, \alpha}(x) \big|_{x=0}^\infty + \int_0^\infty f(x)\ell_{\alpha m}^g[g](x)W_1^{1, \alpha}(x)dx,
\]

where \([\cdot, \cdot]_{\alpha m}^{1, \alpha}(\cdot)\) is the sesquilinear form defined by

\[
[f, g]_{\alpha m}^{1, \alpha}(x) := \frac{x^{\alpha+1}e^{-x}}{(L_m^{\alpha-1}(-x))^2}(f(x)g'(x) - f'(x)g(x)) \quad (0 < x < \infty),
\]

and where

\[
[f, g]_{\alpha m}^{1, \alpha}(\infty) := [f, g]_{\alpha m}^{1, \alpha}(\infty) - [f, g]_{\alpha m}^{1, \alpha}(0).
\]

By Green’s formula and the definition of \( \Delta_{\alpha m}^{1, \alpha} \), both limits

\[
[f, g]_{\alpha m}^{1, \alpha}(\infty) := \lim_{x \to \infty} [f, g]_{\alpha m}^{1, \alpha}(x) \text{ and } [f, g]_{\alpha m}^{1, \alpha}(0) := \lim_{x \to 0^+} [f, g]_{\alpha m}^{1, \alpha}(x)
\]

exist and are finite for all \( f, g \in \Delta_{\alpha m}^{1, \alpha} \).

The adjoint of the maximal operator in \( L^2((0, \infty); W_1^{1, \alpha}) \) is the minimal operator

\[
T_{0, m}^{1, \alpha} : \mathcal{D}(T_{0, m}^{1, \alpha}) \subset L^2((0, \infty); W_1^{1, \alpha}) \to L^2((0, \infty); W_1^{1, \alpha}),
\]

defined by

\[
f \in \mathcal{D}(T_{0, m}^{1, \alpha}) := \{ f \in \Delta_{\alpha m}^{1, \alpha} \mid [f, g]_{\alpha m}^{1, \alpha} \big|_{x=0}^\infty = 0 \text{ for all } g \in \Delta_{\alpha m}^{1, \alpha} \}.
\]

We seek to find the self-adjoint extension \( T_{m}^{1, \alpha} \) in \( L^2((0, \infty); W_1^{1, \alpha}) \), generated by \( \ell_{\alpha m}^f[\cdot] \), which has the Type I exceptional \( X_m \)-Laguerre polynomials \( \{ L_{m, \alpha} \}_{n=\infty}^{\infty} \) as eigenfunctions. To do this, we first need to study the behavior of solutions near the singular endpoints \( x = 0 \) and \( x = \infty \) in order to determine the deficiency indices and to determine the appropriate boundary conditions (if any).
Theorem 3.3. For \( \alpha > 0 \), let \( L^{1,\alpha}_m[\cdot] \) be the Type I exceptional \( X_m \)-Laguerre differential expression (3.1) on the interval \((0, \infty)\).

(a) \( L^{1,\alpha}_m[\cdot] \) is in the limit-circle case at \( x = 0 \) when \( 0 < \alpha < 1 \) and is in the limit-point case at \( x = 0 \) when \( \alpha \geq 1 \).

(b) \( L^{1,\alpha}_m[\cdot] \) is in the limit-point case at \( x = \infty \) for any choice of \( \alpha > 0 \).

Proof. (a): The endpoint \( x = 0 \) is, in the sense of Frobenius, a regular singular endpoint of the Type I exceptional \( X_m \)-Laguerre expression \( L^{1,\alpha}_m[y] = 0 \). The Frobenius indicial equation at \( x = 0 \) is

\[
\rho(r + \alpha) = 0.
\]

Consequently, two linearly independent solutions of \( L^{1,\alpha}_m[y] = 0 \) on \((0, \infty)\) will behave asymptotically like

\[
z_1(x) = 1 \quad \text{and} \quad z_2(x) = x^{-\alpha}
\]

near \( x = 0 \). Now, for any \( \alpha > 0 \), we see from Remark 3.1 that

\[
\int_0^\infty |z_1(x)|^2 W^{1,\alpha}_m(x) \, dx = \int_0^\infty \frac{x^\alpha e^{-x}}{(L^{a-1}_m(-x))^2} \, dx < \infty.
\]

However, for any choice of \( x^* \in (0, \infty) \),

\[
\int_0^{x^*} |z_2(x)|^2 W^{1,\alpha}_m(x) \, dx = \int_0^{x^*} \frac{x^{-\alpha} e^{-x}}{(L^{a-1}_m(-x))^2} \, dx < \infty
\]

only when \( 0 < \alpha < 1 \). In the language of the Weyl limit-point/limit-circle theory, it follows that the Type I exceptional \( X_m \)-Laguerre differential expression is in the limit-circle case at \( x = 0 \) when \( 0 < \alpha < 1 \) and is in the limit-point case at \( x = 0 \) when \( \alpha \geq 1 \).

(b): Since \( x = \infty \) is an irregular singular endpoint of the Type I exceptional \( X_m \)-Laguerre differential expression, the above Frobenius method cannot be employed. Fortunately, we are able to explicitly solve the differential equation

\[
L^{1,\alpha}_m[y](x) = 0 \quad (0 < x < \infty)
\]

for a basis \( \{y_1(x), y_2(x)\} \) of solutions and, from this, we are able to determine the \( L^2 \) behavior of these solutions near \( x = \infty \). The function \( y_1(x) = L^{1,\alpha}_m(x) = L^\alpha_m(-x) \), the Type I exceptional \( X_m \)-Laguerre polynomial of degree \( m \), is one solution of \( L^{1,\alpha}_m[y](x) = 0 \) on \((0, \infty)\). Using the well-known reduction of order method, we obtain a second linearly independent solution \( y_2(x) \). Indeed,

\[
y_2(x) = L^\alpha_m(-x) \int_a^x \frac{e^t (L^{a-1}_m(-t))^2}{t^{a+1} (L^a_m(-t))^2} \, dt,
\]

where \( a \) is a fixed, arbitrary, positive constant. Clearly \( y_1 \in L^2((0, \infty); W^{1,\alpha}_m) \). However, as we now show, \( y_2 \notin L^2((0, \infty); W^{1,\alpha}_m) \). To see this, note that since

\[
\lim_{t \to \infty} \left( \frac{L^{a-1}_m(-t)}{L^a_m(-t)} \right)^2 = 1,
\]

there exists \( x_0 > 0 \) such that

\[
\left( \frac{L^{a-1}_m(-t)}{L^a_m(-t)} \right)^2 \geq \frac{1}{2} \quad (t \geq x_0).
\]
Theorem 3.4. Let \( y \)

\[
\int_{x_0}^{x} \frac{e^t}{t^{\alpha+1}} \left( \frac{L_{m}^{\alpha-1}(-t)}{L_{m}^{\alpha}(-t)} \right)^2 \, dt \geq \frac{1}{2} \int_{x_0}^{x} \frac{e^t}{t^{\alpha+1}} \, dt
\]

Moreover, for large enough \( x \),

\[
\int_{x_1}^{x} e^{t/2} \, dt \quad \text{where} \quad A = \frac{e^{\alpha+1}}{2^{\alpha+1}(\alpha+1)^{\alpha+1}} \quad (x \geq x_1).
\]

Hence, from (3.5) with the choice \( a = x_2 \), we see that

\[
|y_2(x)|^2 = (L_m^\alpha(-x))^2 \left( \int_{x_2}^{x} \frac{e^t}{t^{\alpha+1}} \left( \frac{L_{m}^{\alpha-1}(-t)}{L_{m}^{\alpha}(-t)} \right)^2 \, dt \right)^2
\]

\[
\geq \left( \frac{A}{2} \right)^2 (L_m^\alpha(-x))^2 e^x \quad (x \geq x_2).
\]

Mimicking the argument in (3.6), there exists an \( x_3 \geq x_2 \) such that

\[
\left( \frac{L_{m}^{\alpha}(-t)}{L_{m}^{\alpha-1}(-t)} \right)^2 \geq \frac{1}{2} \quad (t \geq x_3).
\]

Consequently, from (3.7), we see that for \( t \geq x_3 \),

\[
\int_{x_3}^{\infty} |y_2(t)|^2 W_m^{1,\alpha}(t) \, dt
\]

\[
\geq \left( \frac{A}{2} \right)^2 \int_{x_3}^{\infty} t^\alpha \left( \frac{L_{m}^{\alpha}(-t)}{L_{m}^{\alpha-1}(-t)} \right)^2 \, dt
\]

\[
\geq A^2 \int_{x_3}^{\infty} t^\alpha \, dt = \infty.
\]

Therefore, \( y_2 \notin L^2((x_3, \infty); W_m^{1,\alpha}) \). \( \square \)

As a result,

**Theorem 3.4.** Let \( T_{0,m}^{1,\alpha} \) be the minimal operator in \( L^2((0, \infty); W_m^{1,\alpha}) \) generated by the Type I exceptional \( X_m \)-Laguerre differential expression \( \ell_m^{1,\alpha} \).

(a) If \( 0 < \alpha < 1 \), the deficiency index of \( T_{0,m}^{1,\alpha} \) is \( (1,1) \);

(b) If \( \alpha \geq 1 \), the deficiency index of \( T_{0,m}^{1,\alpha} \) is \( (0,0) \).

If \( \alpha \geq 1 \), there is only one self-adjoint extension (restriction) of the minimal operator \( T_{0,m}^{1,\alpha} \) (maximal operator \( T_{1,m}^{1,\alpha} \)), namely \( T_m^{1,\alpha} := T_{0,m}^{1,\alpha} = T_{1,m}^{1,\alpha} \). However, when \( 0 < \alpha < 1 \), there are infinitely many self-adjoint extensions of \( T_{0,m}^{1,\alpha} \). Furthermore, when \( 0 < \alpha < 1 \), in order to obtain a self-adjoint extension of the minimal operator \( T_{0,m}^{1,\alpha} \) having the Type I exceptional \( X_m \)-Laguerre polynomials \( \{f_m^{1,\alpha}\}_m \) as eigenfunctions, the Glazman-Krein-Naimark theory (see [21]) requires that we impose one particular boundary condition of the form,

\[
[f,g_0]_m^{1,\alpha}(0) = 0 \quad (f \in \Delta_m^{1,\alpha}),
\]
where \( g_0 \in \Delta_0 \setminus \mathcal{D}(T_{0,m}^{I,m}) \). We claim \( g_0 \equiv 1 \) on \((0, \infty)\) is an appropriate choice.

Note that the function \( y(x) = x^{-\alpha} \in L^2((0, \infty); W_m^{I,m}) \) when \( 0 < \alpha < 1 \). Remarkably, it is the case that
\[
\ell_m^{I,m}[x^{-\alpha}] = (-m - \alpha)x^{-\alpha};
\]
hence, it follows that \( x^{-\alpha} \in \Delta_0^{I,m} \) for \( 0 < \alpha < 1 \). Additionally,
\[
[x^{-\alpha}, 1]^{I,m}_m(0) = \alpha \lim_{x \to 0^+} \frac{e^{-x}}{(I_m^{-1}(-x))^2} \neq 0
\]
so \( g_0 = 1 \in \Delta_0^{I,m} \setminus \mathcal{D}(T_{0,m}^{I,m}) \). Furthermore, a calculation shows that
\[
[f, 1]^{I,m}_m(0) = 0 \iff \lim_{x \to 0^+} x^{a+1} f'(x) = 0.
\]
Therefore, we obtain the following theorem.

**Theorem 3.5.** Let \( \alpha > 0 \).

(a) Suppose \( 0 < \alpha < 1 \). The operator
\[
T_m^{I,m} : \mathcal{D}(T_m^{I,m}) \subset L^2((0, \infty); W_m^{I,m}) \to L^2((0, \infty); W_m^{I,m})
\]
defined by
\[
T_m^{I,m} f = \ell_m^{I,m}[f]
\]
is self-adjoint in \( L^2((0, \infty); W_m^{I,m}) \) and has the Type I exceptional \( X_m \)-Laguerre polynomials \( \{L_m^{I,m}\} \) as eigenfunctions. Moreover, the spectrum of \( T_m^{I,m} \) consists only of eigenvalues and is given by
\[
\sigma(T_m^{I,m}) = \mathbb{N}_0.
\]

(b) Suppose \( \alpha \geq 1 \). The operator
\[
T_m^{I,m} : \mathcal{D}(T_m^{I,m}) \subset L^2((0, \infty); W_m^{I,m}) \to L^2((0, \infty); W_m^{I,m})
\]
defined by
\[
T_m^{I,m} f = \ell_m^{I,m}[f]
\]
is self-adjoint in \( L^2((0, \infty); W_m^{I,m}) \) and has the Type I exceptional \( X_m \)-Laguerre polynomials \( \{L_m^{I,m}\} \) as eigenfunctions. Moreover, the spectrum of \( T_m^{I,m} \) consists only of eigenvalues and is given by
\[
\sigma(T_m^{I,m}) = \mathbb{N}_0.
\]

3.3. Spectral Properties of Another Self-Adjoint Operator in \( L^2((0, \infty); W_m^{I,m}) \). From (3.8) we can interchange the roles of 1 and \( x^{-\alpha} \) to obtain another interesting self-adjoint operator \( S_m^{I,m} \), generated by \( \ell_m^{I,m}[\cdot] \), in \( L^2((0, \infty); W_m^{I,m}) \) with different boundary conditions. Observing that
\[
0 = [f, x^{-\alpha}]^{I,m}_m(0) \iff \lim_{x \to 0^+} (xf'(x) + \alpha f(x)) = 0,
\]
we are now in position to prove the following theorem regarding the self-adjoint operator \( S_m^{I,m} \), defined below in (3.10), in the space \( L^2((0, \infty); W_m^{I,m}) \). This operator \( S_m^{I,m} \) is only quasi-isospectral to the classical Laguerre operator in the sense that the commutation transformation that relates
the classical Laguerre operator to its Type III counterpart represents a state-adding transformation in the sense of Deift [5].

**Theorem 3.6.** Suppose $0 < \alpha < 1$. Define

$$S_m^{\alpha, \omega} : \mathcal{D}(S_m^{\alpha}) \subset L^2((0,\infty); W_m^{\alpha} \to L^2((0,\infty); W_m^{\alpha})$$

by

$$S_m^{\alpha} f = \ell_m^{\alpha}[f] \quad \mathcal{D}(S_m^{\alpha}) = \{ f \in \Delta_m^{\alpha} \mid \lim_{x \to 0^+} (xf'(x) + \alpha f(x)) = 0 \}$$

(3.10)

Then $S_m^{\alpha}$ is self-adjoint in $L^2((0,\infty); W_m^{\alpha})$. Furthermore,

$$\{ x^{-\alpha}L_{m,n}^{\alpha}(x) \mid n = 0, m + 1, m + 2, m + 3, \ldots \}$$

forms a complete set of (orthogonal) eigenfunctions of $S_m^{\alpha}$ in $L^2((0,\infty); W_m^{\alpha})$, where

$$\{ L_{m,n}^{\alpha} \mid n = 0, m + 1, m + 2, m + 3, \ldots \} \quad (-1 < \alpha < 0)$$

is the sequence of Type III exceptional $X_m$-Laguerre polynomials which we introduce below in Section 5. Finally,

$$\sigma(S_m^{\alpha}) = \sigma_p(S_m^{\alpha}) = \{ n - m - \alpha \mid n = 0, m + 1, m + 2, m + 3, \ldots \}.$$

**Proof.** The self-adjointness of $S_m^{\alpha}$ is clear from the Glazman-Krein-Naimark theory. We need to prove

(i) For $n = 0, m + 1, m + 2, m + 3, \ldots$, $y = x^{-\alpha}L_{m,n}^{\alpha}(x)$ is a solution of

$$\ell_m^{\alpha}[y](x) = (n - m - \alpha)y(x).$$

(3.11)

(ii) For $n = 0, m + 1, m + 2, m + 3, \ldots$,

$$x^{-\alpha}L_{m,n}^{\alpha} \in \mathcal{D}(S_m^{\alpha}).$$

(iii) $\{ x^{-\alpha}L_{m,n}^{\alpha} \mid n = 0, m + 1, m + 2, m + 3, \ldots \}$ is a complete (orthogonal) set of eigenfunctions of $S_m^{\alpha}$ in $L^2((0,\infty); W_m^{\alpha})$; equivalently,

$$\text{span} \{ x^{-\alpha}L_{m,n}^{\alpha} \mid n = 0, m + 1, m + 2, m + 3, \ldots \}$$

is dense in $L^2((0,\infty); W_m^{\alpha})$.

(iv) The spectrum of $S_m^{\alpha}$ is pure point spectrum and given by

$$\sigma(S_m^{\alpha}) = \{ n - m - \alpha \mid n = 0, m + 1, m + 2, m + 3, \ldots \}.$$

Appealing to (5.12), (5.13), and (5.14) in Section 5, we see from (5.1) and (5.2) that $y = x^{-\alpha}L_{m,n}^{\alpha}(x)$ satisfies

$$\ell_m^{\alpha}[y](x) = (n - m - \alpha)y(x) \quad (n = 0, m + 1, m + 2, m + 3, \ldots).$$

This establishes (i). From the proof of completeness in part (iii) below, part (iv) will follow.

It is clear, since $0 < \alpha < 1$, that $x^{-\alpha}L_{m,n}^{\alpha} \in \Delta_m^{\alpha}$. Moreover, for fixed $n \in \{0, m + 1, m + 2, \ldots \}$, let $h(x) = x^{-\alpha}L_{m,n}^{\alpha}(x)$; a calculation shows that

$$\lim_{x \to 0^+} xh'(x) + \alpha h(x) = \lim_{x \to 0^+} \left(-\alpha x^{-\alpha}L_{m,n}^{\alpha}(x) + x^{-\alpha+1} \left( L_{m,n}^{\alpha}(x) \right)' + \alpha x^{-\alpha}L_{m,n}^{\alpha}(x) \right)$$

$$= \lim_{x \to 0^+} x^{-\alpha+1} \left( L_{m,n}^{\alpha}(x) \right)' = 0.$$
Hence $h \in \mathcal{D}(S_m^{1,\alpha})$ and this proves (ii).

To prove (iii), let $f \in L^2((0, \infty); W_m^{1,\alpha})$ and $\varepsilon > 0$. A calculation shows that

$$
\|f\|_{L_m^{1,\alpha}} = \|\tilde{f}\|^{-\alpha},
$$

where $\tilde{f}(x) = \frac{f(x)}{x^{-\alpha} L_m^{-1}(x)}$, $\|\cdot\|_{L_m^{1,\alpha}}$ is the norm defined in (3.3) and $\|\cdot\|^{\alpha}$ ($\alpha > -1$) is the (usual) norm in $L^2((0, \infty); x^\alpha e^{-x})$

given by

$$
\|g\|^\alpha := \left( \int_0^\infty |g(x)|^2 x^\alpha e^{-x} \, dx \right)^{1/2} \quad (g \in L^2((0, \infty); x^\alpha e^{-x})).
$$

Let

$$
\eta(x) = L_m^{-1}(-x);
$$

it is clear that $\eta(x) \neq 0$ on $(0, \infty)$. Appealing to Lemma 5.5 in Section 5, it follows that

$$
\{ L_m^{-1}(-x)p \mid p \in \mathcal{P} \}
$$

is dense in $L^2((0, \infty); x^{-\alpha} e^{-x})$, where $\mathcal{P}$ is the space of all polynomials in the real variable $x$. Hence, from (3.12), there exists $p \in \mathcal{P}$, say $\deg(p) = n$, such that

$$
\varepsilon > \|\tilde{f} - L_m^{-1}(-x)p\|^{-\alpha} = \|f - x^{-\alpha} (L_m^{-1}(-x))^2 p\|_{L_m^{1,\alpha}}.
$$

Define

$$
\mathcal{E}_{n+2m} := \text{span} \left\{ x^{-\alpha} L_{m,j}^{\text{III},-\alpha} \mid j = 0, m + 1, m + 2, \ldots, n + 2m \right\}
$$

and

$$
\mathcal{F}_{n+2m} := \{ x^{-\alpha} q(x) \mid q \in \mathcal{P}_{n+2m}, q'(-x_j) = 0 \ (j = 1, 2, \ldots, m) \},
$$

where $\mathcal{P}_{n+2m} \subset \mathcal{P}$ is the space of all polynomials of degree $\leq n + 2m$ and $\{x_j\}_{j=1}^m \subset (0, \infty)$ are the simple, distinct roots of the Laguerre polynomial $L_m^{-1}(x)$. Note that both $\mathcal{E}_{n+2m}$ and $\mathcal{F}_{n+2m}$ are subspaces of $L^2((0, \infty); W_m^{1,\alpha})$ with

$$
\dim(\mathcal{F}_{n+2m}) = \dim(\mathcal{E}_{n+2m}) = n + m + 1.
$$

From part (i), we know that $\mathcal{E}_{n+2m}$ is an invariant subspace of the Type I exceptional $X_m$-Laguerre differential expression $\ell_m^{1,\alpha}[\cdot]$; that is to say,

$$
\ell_m^{1,\alpha}[\mathcal{E}_{n+2m}] \subset \mathcal{E}_{n+2m}.
$$

Furthermore, it is clear that

$$
x^{-\alpha} (L_m^{-1}(-x))^2 p \in \mathcal{F}_{n+2m}.
$$

In lieu of (3.13), in order to show that

$$
\text{span} \left\{ x^{-\alpha} L_{m,n}^{\text{III},-\alpha} \mid n = 0, m + 1, m + 2, m + 3, \ldots \right\}
$$

is dense in $L^2((0, \infty); W_m^{1,\alpha})$, we need to show that

$$
\|x^{-\alpha} (L_m^{-1}(-x))^2 p\|_{L_m^{1,\alpha}}.
$$
Claim: $\mathcal{E}_{n+2m} \subset \mathcal{F}_{n+2m}$

Let $g \in \mathcal{E}_{n+2m}$ so $g(x) = x^{-\alpha}q(x)$ where $q(x)$ is a linear combination of $L_{m,0}^{\alpha}(x)$, $L_{m,m+1}^{\alpha}(x)$, $L_{m,m+2}^{\alpha}(x)$, \ldots, $L_{m,n}^{\alpha}(x)$. A calculation shows

$$\ell_{m}^{\alpha}[g](x) = x^{-\alpha} \left((\alpha - 1)q'(x) - xq''(x) + xq'(x) - \alpha q(x) + 2x \frac{L_{m}^{\alpha-1}(-x)'}{L_{m}^{\alpha-1}(-x)} q'(x) - mq(x)\right).$$

Since $\ell_{m}^{\alpha}[g] \in \mathcal{E}_{n+2m}$ we see that, from the term

$$2x \frac{L_{m}^{\alpha-1}(-x)'}{L_{m}^{\alpha-1}(-x)} q'(x),$$

that, necessarily, $q \in \mathcal{P}_{n+2m}$ and $q'(-x_j) = 0$ for $j = 1, 2, \ldots, m$. This proves the claim. Notice also that this implies that the polynomial $q$ is either a constant or $\deg(q) \geq m + 1$.

By (3.14), we see that

$$\mathcal{E}_{n+2m} = \mathcal{F}_{n+2m}.$$

From (3.15), we obtain $x^{-\alpha} \left(L_{m}^{\alpha-1}(-x)\right)^2 \in \mathcal{E}_{n+2m}$, establishing (3.16) and this proves part (iii). Finally, part (iv) follows immediately from part (iii) and (3.11). This completes the proof of the theorem.

4. The Type II Exceptional $X_m$-Laguerre Polynomials

Throughout this section, we let $m \in \mathbb{N}_0$; allowing $m = 0$ reproduces the classical Laguerre polynomials. We also assume that

$$\alpha > m - 1$$

throughout this section. We refer the reader to the contributions [11, 12, 16] for full details of the Type II exceptional $X_m$-Laguerre polynomials. We briefly discuss some of their properties, and later, develop the spectral theory for the Type II exceptional $X_m$-Laguerre differential expression.

4.1. Properties of the Type II Exceptional $X_m$-Laguerre Polynomials. Choosing the factorization function $\phi_2(x)$, as given in (2.3), and letting $xL_m^{-\alpha}(x)$ be the factorization gauge, the classical Laguerre differential expression (2.1) may be written as

$$-\ell^{\alpha} = B_{m}^{\alpha} \circ A_{m}^{\alpha} + \alpha - m,$$

where

$$A_{m}^{\alpha}[y](x) = xL_{m}^{-\alpha}(x)y'(x) + (\alpha - m)L_{m}^{-\alpha-1}(x)y(x)$$

and

$$B_{m}^{\alpha}[y](x) = \frac{y'(x) - y(x)}{L_{m}^{-\alpha}(x)}.$$

Based on this factorization, we define the Type II exceptional $X_m$-Laguerre expression $\ell_{m}^{\alpha}[\cdot]$ by

$$\ell_{m}^{\alpha}[y] = -\left(A_{m}^{\alpha+1} \circ B_{m}^{\alpha+1}[y] - m + \alpha + 1\right)$$

$$= -\ell^{\alpha}[y](x) - 2x(\log L_{m}^{-\alpha-1}(x))'(y(x) - y'(x)) + my(x)$$

(4.2) $$= -xy''(x) + \left(-1 - \alpha + x + 2x \frac{L_{m}^{-\alpha-1}(x)'}{L_{m}^{-\alpha-1}(x)}\right)y'(x) + \left(m - 2x \frac{(L_{m}^{-\alpha-1}(x))'}{L_{m}^{-\alpha-1}(x)}\right)y(x).$$
The Type II exceptional $X_m$-Laguerre polynomial $y = L_{m,n}^{\alpha}(x)$, where $n \geq m$, satisfies the second-order differential equation
\[ \ell_m^{\alpha}[y] = \lambda_n y \quad (0 < x < \infty) \]
where
\[ \lambda_n = n - m \quad (n \geq m). \]
Note that $\{\lambda_n\}_{n=m}^{\infty} = \mathbb{N}_0$.

The $n^\text{th}$ degree Type II exceptional $X_m$-Laguerre polynomial is explicitly given by
\[ L_{m,n}^{\alpha}(x) = -A_{m,n}^{\alpha+1}[L_{n-m}^{\alpha+1}](x) \]
\[ = xL_{m}^{\alpha-1}(x)L_{n-m-1}^{\alpha+2}(x) + (m - \alpha - 1)L_{m}^{\alpha-2}(x)L_{n-m}^{\alpha+1}(x) \quad (n \geq m). \]

The sequence $\{L_{m,n}^{\alpha}\}_{n=m}^{\infty}$ of Type II exceptional $X_m$-Laguerre polynomials is orthogonal on $(0, \infty)$ with respect to the weight function
\[ W_{m}^{\alpha}(x) = \frac{x^\alpha e^{-x}}{(L_{m}^{\alpha-1}(x))^2} \quad (x \in (0, \infty)). \]

**Remark 4.1.** Requiring $\alpha > m - 1$ is equivalent to $L_{m,n}^{\alpha-1}(x)$ having no zeros in $[0, \infty);$ see Proposition 4.1. Notice that the function $1 / (L_{m-1}^{\alpha}(-x))^2$ is bounded and bounded away from zero on $(0, \infty);$ hence all moments for the weight function $W_{m}^{\alpha}$ on the interval $(0, \infty)$ exist and are finite.

In fact, in [12], the authors show that $\{L_{m,n}^{\alpha}\}_{n=m}^{\infty}$ forms a complete orthogonal set in the Hilbert space $L^2((0, \infty); W_{m}^{\alpha})$. With $\|\cdot\|_{m}^{\alpha}$ denoting the norm in $L^2((0, \infty); W_{m}^{\alpha})$ defined by
\[ \|f\|_{m}^{\alpha} = \left( \int_{0}^{\infty} |f(x)|^2 W_{m}^{\alpha}(x)dx \right)^{1/2} \quad (f \in L^2((0, \infty); W_{m}^{\alpha})) \]
in $L^2((0, \infty); W_{m}^{\alpha})$, derived from the inner product
\[ (f,g)_{m}^{\alpha} := \int_{0}^{\infty} f(x)\overline{g(x)}W_{m}^{\alpha}(x)dx \quad (f,g \in L^2((0, \infty); W_{m}^{\alpha})), \]
the norms of the Type II exceptional $X_m$-Laguerre polynomials are explicitly given by
\[ \|L_{m,n}^{\alpha}\|_{m}^{\alpha} = \frac{(\alpha + 1 + n - 2m)\Gamma(\alpha + 2 + n - m)}{(n-m)!} \quad (n \geq m); \]
see [11].

In [16], the authors establish the following two theorems concerning properties of the zeros of $\{L_{m,n}^{\alpha}\}_{n=m}^{\infty}$.

**Theorem 4.1.** [16, Propositions 4.3, 4.4, and 4.5] For $n \geq m$, the Type II exceptional $X_m$-Laguerre polynomial $L_{m,n}^{\alpha}(x)$ has $n - m$ simple, positive zeros in $(0, \infty)$. Moreover, $L_{m,n}^{\alpha}(x)$ has either 1 or 0 negative roots according to, respectively, whether $m$ is odd or even.

**Theorem 4.2.** [16, Corollary 4.1 and Proposition 4.8] Let $\{j_{\alpha,i}\}$ denote the sequence of positive zeros of the Bessel function of the first kind $J_{\alpha}(x)$ listed in increasing order and let $\{x_{n,i}^{\alpha}\}_{i=1}^{n-m}$ denote the positive zeros of $L_{m,n}^{\alpha}(x)$ arranged in increasing order. Then
\[ \lim_{n \to \infty} nx_{n,i}^{\alpha} = \frac{j_{\alpha,i}^2}{4} \quad (i \in \mathbb{N}). \]
Furthermore, as \( n \to \infty \), the negative and complex roots of \( L_{m,n}^{\alpha}(z) \) converge to the zeros of \( L_{m}^{-\alpha-1}(z) \).

4.2. Type II Exceptional \( X_m \)-Laguerre Spectral Analysis. In Lagrangian symmetric form, the Type II exceptional \( X_m \)-Laguerre differential expression (4.2) is given by

\[
\ell_m^{\alpha}[y](x) = \frac{1}{W_m^{\alpha}(x)} \left( -\frac{x^{\alpha+1}e^{-x}}{(L_m^{-\alpha-1}(x))^2} y'(x) \right) + \left( -\frac{m x^\alpha e^{-x}}{(L_m^{-\alpha-1}(x))^2} - \frac{2x^{\alpha+1}e^{-x} (L_m^{-\alpha-1}(x))'}{(L_m^{-\alpha-1}(x))^3} \right) y(x) .
\]

The maximal domain associated with \( \ell_m^{\alpha}[-] \) in the Hilbert space \( L^2((0, \infty); W_m^{\alpha}) \) is defined by

\[
\Delta_m^{\alpha} := \{ f : (0, \infty) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(0, \infty); f, \ell_m^{\alpha}[f] \in L^2((0, \infty); W_m^{\alpha}) \} .
\]

The associated maximal operator

\[
T_m^{\alpha} : D(T_m^{\alpha}) \subset L^2((0, \infty); W_m^{\alpha}) \to L^2((0, \infty); W_m^{\alpha}),
\]

is defined to be

\[
f \in D(T_m^{\alpha}) : = \Delta_m^{\alpha}.
\]

For \( f, g \in \Delta_m^{\alpha} \), Green’s formula is

\[
\int_0^\infty \ell_m^{\alpha}[f](x)\overline{g}(x)W_m^{\alpha}(x)dx = [f, g]_m^{\alpha}(x)|_{x=\infty} + \int_0^\infty f(x)\ell_m^{\alpha}[\overline{g}](x)W_m^{\alpha}(x)dx ,
\]

where \([\cdot, \cdot]_m^{\alpha}(\cdot)\) is the sesquilinear form defined by

\[
[f, g]_m^{\alpha}(x) := \frac{x^{\alpha+1}e^{-x}}{(L_m^{-\alpha-1}(x))^2}(f(x)g'(x) - f'(x)g(x)) \quad (0 < x < \infty),
\]

and where

\[
[f, g]_m^{\alpha}(\infty) := [f, g]_m^{\alpha}(0) = [f, g]_m^{\alpha}(0).
\]

By Green’s formula and the definition of \( \Delta_m^{\alpha} \), both limits

\[
[f, g]_m^{\alpha}(\infty) := \lim_{x \to \infty} [f, g]_m^{\alpha}(x) \quad \text{and} \quad [f, g]_m^{\alpha}(0) := \lim_{x \to 0^+} [f, g]_m^{\alpha}(x)
\]

exist and are finite for all \( f, g \in \Delta_m^{\alpha} \).

The adjoint of the maximal operator in \( L^2((0, \infty); W_m^{\alpha}) \) is the minimal operator \( T_{0,m}^{\alpha} \), defined in \( L^2((0, \infty); W_m^{\alpha}) \), by

\[
T_{0,m}^{\alpha} f = \ell_m^{\alpha}[f] \quad (f \in D(T_{0,m}^{\alpha}) : = \{ f \in \Delta_m^{\alpha} \mid [f, g]_m^{\alpha}|_{x=\infty} = 0 \text{ for all } g \in \Delta_m^{\alpha} \}).
\]

In the same manner as in the Type I exceptional \( X_m \)-Laguerre case, we seek to find the self-adjoint extension \( T_m^{\alpha} \) in \( L^2((0, \infty); W_m^{\alpha}) \), generated by \( \ell_m^{\alpha}[-] \), which has the Type II exceptional \( X_m \)-Laguerre polynomials \( \{ T_m^{\alpha,n}\}_{n=m}^{\infty} \) as eigenfunctions. As in the Type I case, we first need to determine the deficiency index of the minimal operator \( T_{0,m}^{\alpha} \) in \( L^2((0, \infty); W_m^{\alpha}) \). In turn, this requires a study of the behavior of solutions near the singular endpoints \( x = 0 \) and \( x = \infty \) of the
differential expression \(\ell_m^{II,\alpha}[\cdot]\). This analysis is similar to the Type I case in the previous section so we omit many of the details.

The point \(x = 0\) is a regular singular endpoint of \(\ell_m^{II,\alpha}[\cdot]\); the Frobenius indicial equation is \(r(r + \alpha) = 0\). Consequently, two linearly independent solutions of \(\ell_m^{II,\alpha}[y] = 0\) on \((0, \infty)\) behave asymptotically like \(z_1(x) = 1\) and \(z_2(x) = x^{-\alpha}\) near \(x = 0\). Clearly \(z_1 \in L^2((0, \infty); W_m^{II,\alpha})\) but a calculation shows that \(z_2 \in L^2((0, 1); W_m^{II,\alpha})\) only when \(\alpha < 1\). Consequently, \(\ell_m^{II,\alpha}[\cdot]\) is in the limit-point case at \(x = 0\) when \(\alpha \geq 1\) and is in the limit-circle case at \(x = 0\) when \(\alpha < 1\). More specifically, recalling \((4.1)\),

(i) if \(m = 0\) (the classical Laguerre case), \(\ell_m^{II,\alpha}[\cdot]\) is in the limit-circle case at \(x = 0\) when \(-1 < \alpha < 1\) and in the limit-point case when \(\alpha \geq 1\);

(ii) if \(m = 1\) (so, by \((4.1)\), \(\alpha > 0\)), \(\ell_m^{II,\alpha}[\cdot]\) is in the limit-circle case at \(x = 0\) when \(0 < \alpha < 1\) and in the limit-point case when \(\alpha \geq 1\);

(iii) if \(m \geq 2\), then \(\alpha > m - 1 \geq 1\) and thus \(\ell_m^{II,\alpha}[\cdot]\) is in the limit-point case at \(x = 0\).

The point \(x = \infty\) is an irregular singular endpoint of \(\ell_m^{II,\alpha}[\cdot]\). Again, we can explicitly solve \(\ell_m^{II,\alpha}[y] = 0\) on \((0, \infty)\) for a basis of solutions. One solution is \(y_1(x) = L_m^{II,\alpha}(x) := L_m^{-\alpha-2}(x)\) which clearly belongs to \(L^2((0, \infty); W_m^{II,\alpha})\). A second solution \(y_2(x)\) can be found by the reduction of order method; this method shows that

\[
y_2(x) = L_m^{-\alpha-2}(x) \int_a^x \frac{e^t}{t^{\alpha+1}} \left( \frac{L_m^{-\alpha-1}(t)}{L_m^{-\alpha-2}(t)} \right)^2 \, dt \quad (x > 0)
\]

where \(a > 0\) is fixed but otherwise arbitrary. An analysis similar to that given in part (b) of Theorem 3.3 shows that \(y_2 \notin L^2((x^*, \infty); W_m^{II,\alpha})\) for some \(x^* > 0\). Consequently, \(\ell_m^{II,\alpha}[\cdot]\) is in the limit-circle case at \(x = \infty\) for any choice of \(\alpha > m - 1\).

When \(\ell_m^{II,\alpha}[\cdot]\) is in the limit-circle case at \(x = 0\), the Glazman-Krein-Naimark theory requires that one appropriate boundary condition be imposed in order to generate a self-adjoint extension of the minimal operator \(T_{0,m}^{II,\alpha}\) in \(L^2((0, \infty); W_m^{II,\alpha})\). We are interested in a particular self-adjoint extension, namely that operator \(T_m^{II,\alpha}\) that has the Type II exceptional \(X_m\)-Laguerre polynomials \(\{L_{m,n}^{II,\alpha}\}_{n=m}^{\infty}\) as eigenfunctions. As in the case of the Type I exceptional \(X_m\)-Laguerre case, we can take this boundary condition to be

\[
[f, 1]_{m}^{II,\alpha}(0) = 0,
\]

where \([\cdot, \cdot]_{m}^{II,\alpha}\) is the sesquilinear form given in \((4.2)\). This boundary condition simplifies to

\[
\lim_{x \to 0^+} x^{\alpha+1}f'(x) = 0.
\]

We summarize this discussion in the following theorem.

**Theorem 4.3.** Let \(m \in \mathbb{N}_0\) and \(\alpha > m - 1\). Let \(T_{0,m}^{II,\alpha}\) be the minimal operator, defined in \((4.5)\) in \(L^2((0, \infty); W_m^{II,\alpha})\) generated by the Type II exceptional \(X_m\)-Laguerre differential expression \(\ell_m^{II,\alpha}[\cdot]\) given in \((4.2)\) or \((4.3)\).

(a) The deficiency index of \(T_{0,m}^{II,\alpha}\) is

(i) \((1, 1)\) when \(m = 0\) and \(-1 < \alpha < 1\), or when \(m = 1\) and \(0 < \alpha < 1\);

(ii) \((0, 0)\) when \(m = 0\) and \(\alpha \geq 1\), or when \(m = 1\) and \(\alpha \geq 1\), or when \(m \geq 2\).

(b) The operator

\[
T_{m}^{II,\alpha} : D(T_{m}^{II,\alpha}) \subset L^2((0, \infty); W_m^{II,\alpha}) \to L^2((0, \infty); W_m^{II,\alpha}),
\]
THREE TYPES OF EXCEPTIONAL X_\(m\)-LAGUERRE POLYNOMIALS

defined by

\[ T_{m}^{\Pi,\alpha} f = \ell_{m}^{\Pi,\alpha}[f] \]

\[ f \in \mathcal{D}(T_{m}^{\Pi,\alpha}), \]

is self-adjoint. The domain of \( T_{m}^{\Pi,\alpha} \) is given by

(i) \[
\mathcal{D}(T_{m}^{\Pi,\alpha}) := \{ f \in \Delta_{m}^{\Pi,\alpha} \mid \lim_{x \to 0^+} x^{\alpha+1} f'(x) = 0 \}
\]

when the deficiency index of \( T_{0,m}^{\Pi,\alpha} \) is \((1,1)\), or by

(ii) \[
\mathcal{D}(T_{m}^{\Pi,\alpha}) := \Delta_{m}^{\Pi,\alpha}
\]

when the deficiency index of \( T_{0,m}^{\Pi,\alpha} \) is \((0,0)\).

Moreover, in either case, \( T_{m}^{\Pi,\alpha} \) has the Type II exceptional \( X_{m}\)-Laguerre polynomials \( \{L_{m,n}^{\Pi,\alpha}\}_{n=m}^{\infty} \) as a complete set of eigenfunctions in \( L^2((0,\infty); W_{m}^{\Pi,\alpha}) \). Lastly, the spectrum of \( T_{m}^{\Pi,\alpha} \) consists only of eigenvalues and is given by

\[ \sigma(T_{m}^{\Pi,\alpha}) = \mathbb{N}_0. \]

5. A NEW SEQUENCE OF EXCEPTIONAL LAGUERRE POLYNOMIALS: THE TYPE III EXCEPTIONAL \( X_{m}\)-LAGUERRE POLYNOMIALS

The Type III exceptional \( X_{m}\)-Laguerre polynomials

\[ \{L_{m,n}^{\Pi,\alpha} \mid n = 0, m+1, m+2, m+3, \ldots\} \]

is a new class of exceptional \( X_{m}\)-Laguerre orthogonal polynomials for the parameter range \(-1 < \alpha < 0\). They can be derived from the quasi-rational eigenfunctions of the classical Laguerre differential expression (2.1) and they can also be obtained from a transformation of the Type I exceptional \( X_{m}\)-Laguerre differential expression (3.1). Both of these derivations will be developed in Section 5.1. In Section 5.2, we introduce the Type III exceptional \( X_{m}\)-Laguerre polynomials and derive several representations of them. Section 5.3 deals with the computation of the norms of these polynomials in \( L^2((0,\infty); W_{m}^{\Pi,\alpha}) \). In Section 5.4, we prove that the sequence of Type III exceptional \( X_{m}\)-Laguerre polynomials forms a complete set of functions in \( L^2((0,\infty); W_{m}^{\Pi,\alpha}) \). We will deal with a comprehensive study of the location of the roots and an asymptotic analysis of the roots for the Type III exceptional \( X_{m}\)-Laguerre polynomials in Section 5.5. Lastly, in Section 5.6, we construct a self-adjoint operator in \( L^2((0,\infty); W_{m}^{\Pi,\alpha}) \), generated by the second-order Type III exceptional \( X_{m}\)-Laguerre differential expression, having the sequence of Type III exceptional \( X_{m}\)-Laguerre polynomials as eigenfunctions.

5.1. TWO DERIVATIONS OF THE TYPE III EXCEPTIONAL \( X_{m}\)-LAGUERRE DIFFERENTIAL EXPRESSION

Consider the transformation arising from the quasi-rational solution \( \phi_2(x) \) in (2.5)

\[ z(x) = x^{-\alpha} y(x). \]

A calculation shows that

\[ \ell_{m}^{I,\alpha}[z](x) = x^{-\alpha} \ell_{m}^{\Pi,\alpha}[y](x), \]

where

\[ \ell_{m}^{\Pi,\alpha}[y](x) := -xy''(x) + \left( -1 - \alpha + x + 2x \frac{(L_m^{\alpha-1}(-x))'}{L_m^{\alpha-1}(-x)} \right) y'(x) + (-m + \alpha)y(x) \]
or, equivalently, with the notation \( M(g)(f(x)) := g(x)f(x) \),

\[(5.4) \quad M(x^{\alpha}) \circ \ell_m^{\alpha} \circ M(x^{-\alpha}) = \ell_m^{\alpha} \circ M(x) = \ell_m^{\alpha} \.
\]

With regards to this identity, we say that the Type I and Type III expressions are related by a gauge transformation.

We call (5.3) the Type III exceptional \( X_m \)-Laguerre differential expression. In Lagrangian symmetric form, this expression can be written as

\[(5.5) \quad \ell_m^{\alpha} y(x) = \frac{1}{W_m^{\alpha}(x)} \left( \left( -\frac{x^{\alpha+1}e^{-x}}{(L_m^{\alpha-1}(x))^2} y'(x) \right) + \frac{(-m + \alpha)x^{\alpha}e^{-x}}{(L_m^{\alpha-1}(x))^2} y(x) \right),
\]

where

\[ W_m^{\alpha}(x) = \frac{x^{\alpha}e^{-x}}{(L_m^{\alpha-1}(-x))^2} \quad (x \in (0, \infty)). \]

**Remark 5.1.** As we will see below, the parameter range for the identity (5.4) is \( 0 < \alpha < 1 \). In this regard, we remark that the Type III polynomials, which we show are solutions of

\[ \ell_m^{\alpha} y(x) = \lambda y(x), \]

for a certain sequence of the eigenvalue parameter \( \lambda \), are related to the \( L3 \) family of rational extensions of the isotonic oscillator which were investigated by Grandati in [8]. From the point of view of Schrödinger operators, the parameter range \( 0 < \alpha < 1 \) corresponds to a potential with a weakly attracting singularity at the origin. Qualitatively, this kind of singularity makes the physics of the system ambiguous and requires the imposition of a boundary condition at the origin for a well-defined eigenvalue problem; see Section 5.6.

We note that all of the moments of \( W_m^{\alpha} \) exist and are finite only when

\[ -1 < \alpha < 0. \]

At this point, it is unclear if the eigenvalue problem

\[(5.6) \quad \ell_m^{\alpha} y(x) = \lambda y(x) \]

produces polynomial solutions for certain values of \( \lambda \). In the next section, we will show that (5.6) has polynomial solutions of degrees \( n = 0 \) and all \( n \geq m + 1 \). We now argue that there cannot be polynomial solutions to (5.6) of degrees \( n = 1, 2, \ldots, m \) for any value of \( \lambda \in \mathbb{C} \). Indeed, suppose \( y = p(x) \) is a polynomial solution to (5.6) for some \( \lambda \in \mathbb{C} \). From (5.3), it follows that the term

\[ 2x \frac{L_m^{\alpha-1}(-x)}{L_m^{\alpha-1}(-x)} p'(x) \]

is a polynomial. However, since the roots of the Laguerre polynomial \( L_m^{\alpha-1}(-x) \) are simple and negative, we see in fact that \( p'(x)/L_m^{\alpha-1}(-x) \) is a polynomial. Consequently, either \( p \) is a constant or a polynomial of degree \( \geq m + 1 \). More specifically, it is the case, for some polynomial \( q \), that \( p'(x) = L_m^{\alpha-1}(-x)q(x) \); see Lemmas 5.2 and 5.3 below.

To see that (5.6) has orthogonal polynomial eigenfunctions, we turn to a special rational factorization of the classical Laguerre expression (2.1). Indeed, the rational factorization function in this case is \( \phi_3(x) \), where \( \phi_3 \) is defined in (2.6), and the corresponding gauge function is \( xL_m^{\alpha}(-x) \).
Define the first-order operators $A_m^\text{III},\alpha$ and $B_m^\text{III},\alpha$ by

$$
A_m^\text{III},\alpha[y](x) := xL_m^{-\alpha}(-x)y'(x) - (m + 1)L_{m+1}^{-\alpha}(-x)y(x)
$$

$$
B_m^\text{III},\alpha[y](x) := \frac{y'(x)}{L_m^{-\alpha}(-x)}.
$$

**Lemma 5.1.** The operators $A_m^\text{III},\alpha$ and $B_m^\text{III},\alpha$ satisfy the following factorization properties:

(a) $-\ell^\alpha = B_m^\text{III},\alpha \circ A_m^\text{III},\alpha + m + 1$, where $\ell^\alpha$ is the classical Laguerre second-order differential expression defined in (2.1);

(b) $-\ell_m^\text{III,}\alpha = A_m^\text{III,}\alpha+1 \circ B_m^\text{III,}\alpha+1 + m - \alpha$.

**Proof.** The proofs of these identities are similar so we give only the proof of part (b). Our proof will make repeated use of two facts:

(5.7) $$(L_n^\alpha(-x))' = L_{n+1}^\alpha(-x)$$ for any $\alpha > -1$ and $n \in \mathbb{N}_0$, and

(5.8) $$x(L_{m+1}^{-\alpha-2}(-x))'' + (x - \alpha - 1)(L_{m+1}^{-\alpha-2}(-x))' - (m + 1)L_{m+1}^{-\alpha-2}(-x) = 0;$$

see [26] Chapter V, (5.1.2) and (5.1.14). Now

$$
A_m^\text{III,}\alpha+1(B_m^\text{III,}\alpha+1[y]) = xL_m^{-\alpha-1}(-x)
$$

$$
\left(\frac{y'}{L_m^{-\alpha-1}(-x)}\right)' - (m + 1)L_{m+1}^{-\alpha-2}(-x)\left(\frac{y'}{L_m^{-\alpha-1}(-x)}\right) = xL_m^{-\alpha-1}(-x)
$$

$$
\left(\frac{L_m^{-\alpha-1}(-x)y'' - L_{m-1}^{-\alpha}(-x)y'}{(L_m^{-\alpha-1}(-x))^2}\right) - (m + 1)L_{m+1}^{-\alpha-2}(-x)
$$

$$
\frac{y'}{L_m^{-\alpha-1}(-x)} = xy'' + \left(\frac{-xL_{m+1}^{-\alpha-1}(-x) - (m + 1)L_{m+1}^{-\alpha-2}(-x)}{L_{m+1}^{-\alpha-1}(-x)}\right)y'.
$$

Moreover,

$$
\frac{-xL_{m+1}^{-\alpha-1}(-x) - (m + 1)L_{m+1}^{-\alpha-2}(-x)}{L_{m+1}^{-\alpha-1}(-x)}
$$

$$
= \frac{-x(L_{m+1}^{-\alpha-2}(-x))'' - (m + 1)L_{m+1}^{-\alpha-2}(-x)}{L_{m+1}^{-\alpha-1}(-x)} \quad \text{by (5.7)}
$$

$$
= \frac{(x - \alpha - 1)(L_{m+1}^{-\alpha-2}(-x))' - 2(m + 1)L_{m+1}^{-\alpha-2}(-x)}{L_{m+1}^{-\alpha-1}(-x)} \quad \text{from (5.8)}
$$

$$
= \frac{(x - \alpha - 1)L_{m+1}^{-\alpha-1}(-x) - 2(m + 1)L_{m+1}^{-\alpha-2}(-x)}{L_{m+1}^{-\alpha-1}(-x)} \quad \text{by (5.7)}
$$

$$
= \frac{(x - \alpha - 1)L_{m+1}^{-\alpha-1}(-x) - 2x(L_{m+1}^{-\alpha-2}(-x))'' + (2\alpha + 2 - 2x)(L_{m+1}^{-\alpha-2}(-x))'}{L_{m+1}^{-\alpha-1}(-x)} \quad \text{by (5.8)}
$$

$$
= \frac{(x - \alpha - 1)L_{m+1}^{-\alpha-1}(-x) - 2x(L_{m+1}^{-\alpha-1}(-x))' + (2\alpha + 2 - 2x)L_{m+1}^{-\alpha-1}(-x)}{L_{m+1}^{-\alpha-1}(-x)} \quad \text{from (5.7)}.
$$

(5.10) $$
1 + \alpha - x - 2x\left(\frac{L_{m+1}^{-\alpha-1}(-x)}{L_{m+1}^{-\alpha-1}(-x)}\right)'.
$$
Substitution of (5.10) into (5.9) yields

\[ A_m^{\text{III}, \alpha+1} \left( B_m^{\text{III}, \alpha+1}[y] \right) = xy'' + \left( 1 + \alpha - x - 2x \frac{L_m^{-\alpha-1}(-x)'}{L_m^{-\alpha-1}(-x)} \right) y'; \]

adding the term \((m - \alpha)y\) to both sides of this latter identity completes the proof. \(\square\)

**Remark 5.2.** With reference to (2.2) and (2.3), where the reader will notice that the parameters \(\lambda\) in both expressions are equal, we could define the Type III exceptional \(X_m\)-Laguerre differential expression by

\[ \ell_m^{\text{III}, \alpha}[y](x) := -\left( A_m^{\text{III}, \alpha+1} \circ B_m^{\text{III}, \alpha+1} + m + 1 \right)[y](x) = -xy''(x) + \left( -1 - \alpha + x + 2x \frac{L_m^{-\alpha-1}(-x)'}{L_m^{-\alpha-1}(-x)} \right) y'(x) + (m - 1)y(x). \]

In this case, we would not have the identity (5.4); however, by mimicking the proof of Theorem 5.7 below, the Type III exceptional \(X_m\)-Laguerre polynomial \(y = \hat{L}_m^{\text{III}, \alpha}(x)\) can be shown to be a solution of the eigenvalue equation

\[ \ell_m^{\text{III}, \alpha}[y](x) = (n - m - 1)y(x) \quad (n = 0, m + 1, m + 2, m + 3, \ldots). \]

### 5.2. The Type III Exceptional \(X_m\)-Laguerre Polynomials

For the remainder of this section, we assume that \(-1 < \alpha < 0\).

Similar to how we introduced the Type I and Type II exceptional \(X_m\)-Laguerre polynomials, we define the \(n^{th}\) degree Type III exceptional \(X_m\)-Laguerre polynomial by

\[ L_m^{\text{III}, \alpha}(x) := \begin{cases} -A_m^{\text{III}, \alpha+1}[L_{n-m-1}^{\alpha+1}(x)] & \text{if } n \geq m + 1 \\ 1 & \text{if } n = 0. \end{cases} \]

From the definition of \(A_m^{\text{III}, \alpha+1}\), a calculation shows that

\[ L_m^{\text{III}, \alpha}(x) = \begin{cases} xL_{n-m-2}^{\alpha+2}(x)L_m^{-\alpha-1}(-x) + (m + 1)L_{n-m-1}^{\alpha+1}(x)L_m^{-\alpha-2}(-x), & \text{if } n \geq m + 1 \\ 1 & \text{if } n = 0. \end{cases} \]

The following lemmas (Lemmas 5.2 and 5.3) are critical for several reasons. Indeed, they will ultimately help show that \(y = \hat{L}_m^{\text{III}, \alpha}(x)\) is a solution of the eigenvalue equation

\[ \ell_m^{\text{III}, \alpha}[y](x) = \lambda_n y(x), \]

where

\[ \lambda_n = n - m + \alpha \quad (n = 0, m + 1, m + 2, m + 3, \ldots). \]

In addition, both lemmas give new characterizations of the Type III exceptional \(X_m\)-Laguerre polynomials and lead to an additional representation (Theorem 5.2) of these polynomials. Lastly, these lemmas will be critically important in our analysis of the location of the roots (Lemma 5.4 and Theorem 5.5) of \(\{L_m^{\text{III}, \alpha}\}\) and to proving root asymptotic results (Theorem 5.6) of these roots.

**Lemma 5.2.** For \(k \in \mathbb{N}\),

\[ \left( \frac{L_m^{\text{III}, \alpha}(x)}{L_m^{-\alpha-1}(-x)} \right)' = -xL_k^{\alpha+3}(-x) + (\alpha + 2 - x)L_k^{\alpha+2}(-x) + (m + 1)L_k^{\alpha+1}(-x). \]
Therefore, in particular, for all $(5.20)$

we see that

For $(5.16)$ we have

Using the Laguerre identity

Hence $(5.18)$ becomes

Therefore,

Now, using $(5.16)$,

Since $y = L_{m+1}^{-\alpha-2}(x)$ satisfies $xy'' + (\alpha - 1 - x)y' + (m + 1)y = 0$, a simple calculation shows that $y = L_{m+1}^{-\alpha-2}(-x)$ satisfies

Hence $(5.18)$ becomes

since $(L_{m+1}^{-\alpha-2}(-x))' = L_{m}^{-\alpha-1}(-x)$. Substituting $(5.19)$ into $(5.17)$ establishes $(5.15)$. \qed

Lemma 5.3. For $k \in \mathbb{N}$,

In particular, for all $k \in \mathbb{N}$, the Type III exceptional $X_m$-Laguerre polynomial $L_{m,m+k}^{\alpha}(x)$ has a local extremum at each of the $m$ roots of $L_{m+1}^{-\alpha-1}(-x)$ and at each of the $k - 1$ roots of $L_{k-1}^{\alpha}(x)$. 

Proof. Recall the representation $(5.12)$:

$$L_{m,m+k}^{\alpha}(x) = xL_{k-2}^{\alpha+2}(x)L_{m-1}^{-\alpha-1}(x) + (m + 1)L_{k-1}^{\alpha+1}(x)L_{m+1}^{-\alpha-2}(x).$$
Proof. The Laguerre polynomial \( y = L_{k-1}^{\alpha+1}(x) \) is a solution of Laguerre’s equation

\[ xy'' + (\alpha + 2 - x)y' + (k - 1)y = 0. \]

Consequently, we see that the right-hand side of (5.15) simplifies to

\[
-xL_{m-3}^{\alpha+3}(x) + (\alpha + 2 - x)L_{m-2}^{\alpha+2}(x) + (m + 1)L_{m-1}^{\alpha+1}(x)
= -x(L_{m-1}^{\alpha+1}(x))'' - (\alpha + 2 - x)(L_{m-1}^{\alpha+1}(x))' + (m + 1)L_{m-1}^{\alpha+1}(x)
= (m + k)L_{m-1}^{\alpha+1}(x).
\]

The result now follows from this identity and (5.15).

A degree count implies that all roots of \( L_{m,m+k}^{\alpha}(x) \) must be simple. \( \square \)

We are now in position to prove the following theorem which will establish (5.13) and (5.14).

**Theorem 5.1.** For \( n = 0, m + 1, m + 2, m + 3, \ldots \), the function \( y = L_{m,n}^{\alpha}(x) \) is a solution of (5.13), where \( \lambda_n \) is given in (5.14).

Proof. The proof is straightforward when \( n = 0 \) so we assume \( n \geq m + 1 \). With \( y = L_{m,n}^{\alpha}(x) \), we see from Lemma 5.1 (b) that

\[
\ell_{m,n}^{\alpha}(y)(x) = -A_{m}^{\alpha+1}B_{m}^{\alpha+1}[L_{m,n}^{\alpha}](x) + (-m + \alpha)L_{m,n}^{\alpha}(x)
= -A_{m}^{\alpha+1}[(L_{m,n}^{\alpha}(x))'/L_{m}^{\alpha-1}(-x)] + (-m + \alpha)L_{m,n}^{\alpha}(x) \quad \text{by definition of } B_{m}^{\alpha+1}
= -A_{m}^{\alpha+1}[nL_{n-1}^{\alpha}(x) + (-m + \alpha)L_{m,n}^{\alpha}(x)] \quad \text{by Lemma 5.3}
= nL_{m,n}^{\alpha+1}(x) + (-m + \alpha)L_{m,n}^{\alpha}(x) \quad \text{by (5.11)}
= (n - m + \alpha)y(x).
\]

\( \square \)

The next two results give new representations of the Type III exceptional \( X_m \)-Laguerre polynomials.

**Theorem 5.2.** For \( k \in \mathbb{N} \),

\[
L_{m,m+k}^{\alpha}(x) = (m + k) \int_{0}^{x} L_{k-1}^{\alpha+1}(t)L_{m-1}^{\alpha-1}(-t)dt + (m + 1) \begin{pmatrix} k + \alpha \\ k \end{pmatrix} \begin{pmatrix} m - \alpha - 1 \\ m + 1 \end{pmatrix}.
\]

Proof. This follows immediately from (5.20) and the normalizations

\[ L_{k}^{\alpha}(0) = \begin{pmatrix} k + \alpha \\ k \end{pmatrix} \quad \text{and} \quad L_{m,m+k}^{\alpha}(0) = (m + 1)L_{k-1}^{\alpha+1}(0)L_{m+1}^{\alpha-2}(0). \]

\( \square \)

The following representation of the Type III exceptional \( X_m \)-Laguerre polynomials will be important for determining the location of their zeros.

**Lemma 5.4.** For \( m, k \in \mathbb{N} \) and \(-1 < \alpha < 0\), the Type III exceptional \( X_m \)-Laguerre polynomial \( L_{m,m+k}^{\alpha}(x) \) can be written as

\[
L_{m,m+k}^{\alpha}(x) = (k + \alpha)L_{k-2}^{\alpha+1}(x)L_{m-1}^{\alpha-1}(-x) + (m + 1)L_{k-1}^{\alpha+1}(x)L_{m+1}^{\alpha-1}(-x)
- (m + k)L_{k-1}^{\alpha+1}(x)L_{m}^{\alpha-1}(-x).
\]

\( \square \)
Proof. Recall (5.12):

\[ L_{m,m+k}^{\alpha}(x) = xL_{m-1}^{\alpha}(x) + (m+1)L_{m}^{\alpha+1}(x)L_{m+1}^{\alpha-2}(x). \]

From [26, p. 102, (5.1.14)],

\[ x(L_n^{\alpha}(x))' = -(n+\alpha)L_n^{\alpha-1}(x) + nL_n^{\alpha}(x), \]

we see that

\[ xL_{k-2}^{\alpha+2}(x) = -x(L_{k-1}^{\alpha+1}(x))' = (k+\alpha)L_{k-1}^{\alpha+1}(x) - (k-1)L_{k-1}^{\alpha+1}(x). \]

Likewise, from the identity (see [26, p. 102, (5.1.13)])

\[ L_n^{\alpha}(x) = L_n^{\alpha+1}(x) - L_n^{\alpha+1}(x), \]

we obtain

\[ L_{n+1}^{\alpha-2}(-x) = L_{n+1}^{\alpha-1}(-x) - L_{n+1}^{\alpha-1}(-x). \]

Substituting (5.23) and (5.24) into (5.12) yields (5.22). □

Remark 5.3. Our discussion to this point shows that if we take (5.22) (or (5.21)) as our definition of the Type III exceptional \(X_m\)-Laguerre polynomials, they are orthogonal polynomials for \(-1 < \alpha < 0\). Regardless of this parameter restriction, the polynomial defined in either (5.22) (or (5.21)) is of degree \(m+k\). A natural question to ask is whether or not these polynomials are orthogonal, in some sense, for values of \(\alpha \notin (-1, 0)\). It would be interesting to look into this question even when some of the associated moments do not exist.

We note that the Type III exceptional \(X_m\)-Laguerre polynomials are negative at the origin; that is to say, for \(-1 < \alpha < 0\) and \(k \in \mathbb{N}\), we have

\[ L_{m,m+k}^{\alpha}(0) < 0. \]

To see this, recall from Theorem 5.2 that

\[ L_{m,m+k}^{\alpha}(0) = (m+1)L_{k-1}^{\alpha+1}(0)L_{m+1}^{\alpha-2}(0). \]

Now, in general,

\[ L_n^{\alpha}(0) = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!} = \frac{(1+\alpha)(2+\alpha)\ldots(n+\alpha)}{n!}, \]

so \(L_{k-1}^{\alpha+1}(0) > 0\) when \(-1 < \alpha < 0\). However, this parameter range implies that \(-1 < -\alpha - 1 < 0\) while \(-\alpha + j > 0\) for \(j = 0, 1, \ldots, m-1\); thus

\[ L_{m+1}^{\alpha-2}(0) = \frac{(-\alpha-1)(-\alpha)(-\alpha+1)\ldots(-\alpha+m-1)}{(m+1)!} < 0. \]

From (5.26), the inequality in (5.25) now follows. The negativity of \(L_{m,m+k}^{\alpha}(0)\) turns out to be essential in our analysis (see Section 5.5) of determining the location of the roots of \(L_{m,m+k}^{\alpha}(x)\).
5.3. The Norms of the Type III Exceptional $X_m$-Laguerre Polynomials. We now compute the norms of these Type III polynomials.

**Theorem 5.3.** Suppose $-1 < \alpha < 0$. The Type III exceptional $X_m$-Laguerre polynomials
\[
\{ f_{m,n}^{\text{III}} : \alpha \in (0, \infty) \} \quad \text{are orthogonal in the Hilbert space } L^2((0, \infty); W_m^{\text{III}, \alpha}) \text{ and the norms are explicitly given by}
\]
\[
\| f_{m,n}^{\text{III}} \|_{m, \alpha}^2 = \int_0^\infty (L_m^\alpha(x))^2 W_m^{\text{III}, \alpha}(x) dx = \begin{cases} 
\frac{n \Gamma(n - m + \alpha + 1)}{(n - m - 1)!} & \text{if } n \geq m + 1 \\
\frac{\Gamma(\alpha + 1) \Gamma(-\alpha)m!}{\Gamma(m - \alpha)} & \text{if } n = 0.
\end{cases}
\]

**Proof.** We compute the norms in this proof; the orthogonality will follow directly from the self-adjointness of the operator $T_m^{\text{III}, \alpha}$ in Theorem 5.8 in Section 5.6 below. The proof, when $n \geq m + 1$, rests on the following adjoint relationship for the $A_m^{\text{III}, \alpha}$ and $B_m^{\text{III}, \alpha}$ operators
\[
B_m^{\text{III}, \alpha}[f](x) g(x) W_\alpha(x) + A_m^{\text{III}, \alpha}[g](x) f(x) W_m^{\text{III}, \alpha - 1}(x) = \frac{d}{dx} \left( \frac{W_\alpha(x)}{L_m^\alpha(-x)} f(x) g(x) \right),
\]
where we take $f = f(x), g = g(x)$ to be polynomials and where $W_\alpha(x) = x^\alpha e^{-x}$ is the classical Laguerre weight. To prove this, divide the left-hand side of (5.28) by
\[
\frac{W_\alpha(x)}{L_m^\alpha(-x)} f(x) g(x)
\]
to obtain
\[
\frac{L_m^\alpha(-x) B_m^{\text{III}, \alpha}[f](x)}{f(x)} + \frac{A_m^{\text{III}, \alpha}[g](x)}{x L_m^\alpha(-x) g(x)},
\]
on the other hand, a tedious calculation shows that
\[
\frac{d}{dx} \log \left( \frac{W_\alpha(x)}{L_m^\alpha(-x)} f(x) g(x) \right) = \frac{f'(x)}{f(x)} + \frac{x L_m^\alpha(-x) g'(x) + (\alpha L_m^\alpha(-x) - x L_m^\alpha(-x) - x(L_m^\alpha(-x))') g(x)}{x L_m^\alpha(-x) g(x)}
\]
\[
= \frac{f'(x)}{f(x)} + \frac{x L_m^\alpha(-x) g'(x) + (-x(L_m^{\alpha+1}(-x))' + (\alpha - x)(L_m^{\alpha+1}(-x))' g(x)}{x L_m^\alpha(-x) g(x)}
\]
\[
= \frac{f'(x)}{f(x)} + \frac{x L_m^\alpha(-x) g'(x) - (m + 1)L_m^{\alpha+1}(-x) g(x)}{x L_m^\alpha(-x) g(x)}
\]
\[
= \frac{L_m^{\alpha+1}(-x) B_m^{\text{III}, \alpha}[f](x)}{f(x)} + \frac{A_m^{\text{III}, \alpha}[g](x)}{x L_m^\alpha(-x) g(x)} \quad \text{by definition of } A_m^{\text{III}, \alpha} \text{ and } B_m^{\text{III}, \alpha}.
\]
This establishes (5.28). Setting $f = A_m^{\text{III}, \alpha}[L_m^n]$ and $g = L_m^n$ in (5.28), integrating and using Lemma 5.1 Part (a), and (5.11) gives
\[
-(n + m + 1) \int_0^\infty (L_n^\alpha(x))^2 W_\alpha(x) dx + \int_0^\infty (L_m^{\alpha-1}(x))^2 W_m^{\text{III}, \alpha - 1}(x) dx
\]
\[
= - \left( \frac{L_m^{\alpha-1}(x) L_n^\alpha(x)}{L_m^\alpha(-x)} W_\alpha(x) \right) \bigg|_{x=0}^{x=\infty}.
\]
In [5.29], we are assuming that $-1 < \alpha - 1 < 0$ so $0 < \alpha < 1$. It follows that
\[
\left( \frac{L_{m,m+n+1}^{\alpha-1}(x) L_n^{\alpha}(x)}{L_m^{\alpha}(-x)} W_\alpha(x) \right) \bigg|_{x=\infty} = \left( \frac{L_{m,m+n+1}^{\alpha-1}(x) L_n^{\alpha}(x)}{L_m^{\alpha}(-x)} x^\alpha e^{-x} \right) \bigg|_{x=\infty} = 0.
\]

Hence, we see that
\[
\int_0^\infty \left( \frac{L_{m,m+n+1}^{\alpha-1}(x)}{L_m^{\alpha}(-x)} \right)^2 W_m^{\alpha-1}(x) dx = (n + m + 1) \int_0^\infty (L_n^{\alpha}(x))^2 W_\alpha(x) dx.
\]

Since (see [26, Chapter V, (5.1.1)])
\[
\int_0^\infty (L_n^{\alpha}(x))^2 W_\alpha(x) dx = \frac{\Gamma(n + \alpha + 1)}{n!} (j \in \mathbb{N}_0),
\]
we see from [5.29] that
\[
\int_0^\infty \left( \frac{L_{m,m+n+1}^{\alpha-1}(x)}{L_m^{\alpha}(-x)} \right)^2 W_m^{\alpha-1}(x) dx = \frac{(n + m + 1) \Gamma(n + \alpha + 1)}{n!}.
\]

Replacing $\alpha$ by $\alpha + 1$ and $n + m + 1$ by $n$ yields
\[
\int_0^\infty (L_m^{\alpha}(x))^2 W_m^{\alpha}(x) dx = \frac{n \Gamma(n - m + \alpha + 1)}{(n - m - 1)!} \quad \text{for } n \geq m + 1, \text{ as required.}
\]

To prove the norm formula in [5.27] for $n = 0$, we first establish the following identity:
\[
(5.30) \quad \int W_m^{\alpha}(x) dx + m \int W_{m-1}^{\alpha-1}(x) dx = -\frac{x^\alpha e^{-x}}{L_m^{\alpha-1}(-x)L_{m-1}^{\alpha}(-x)}.
\]

Let $\psi(x) = L_m^{\alpha-1}(-x)$ so that $\psi'(x) = L_{m-1}^{\alpha}(-x)$ and $\psi''(x) = L_{m-2}^{\alpha+1}(-x)$. Now $\psi(x)$ is a solution of the Laguerre differential equation
\[
x \psi''(x) + (x - \alpha) \psi'(x) - m \psi(x) = 0.
\]
Divide this differential equation by $\psi(x)(\psi'(x))^2$ and rearrange to obtain
\[
\frac{m}{(\psi'(x))^2} + \frac{x}{\psi(x)} \left( \frac{1}{\psi'(x)} \right)' + \frac{\alpha - x}{\psi(x)\psi'(x)} = 0.
\]

Multiplying by $x^{\alpha-1} e^{-x}$ yields
\[
(5.31) \quad \frac{m x^{\alpha-1} e^{-x}}{(\psi'(x))^2} + \frac{x^{\alpha-1} e^{-x}}{\psi(x)} \left( \frac{1}{\psi'(x)} \right)' + \frac{(x^{\alpha-1} e^{-x})'}{\psi(x)\psi'(x)} = 0.
\]

Since
\[
\frac{x^\alpha e^{-x}}{(\psi(x))^2} + \frac{x^\alpha e^{-x}}{\psi'(x)} \left( \frac{1}{\psi(x)} \right)' \equiv 0,
\]
we see that [5.31] can be rewritten as
\[
(5.32) \quad \frac{x^\alpha e^{-x}}{(\psi(x))^2} + \frac{m x^{\alpha-1} e^{-x}}{(\psi'(x))^2} = -\left( \frac{x^\alpha e^{-x}}{\psi'(x)} \left( \frac{1}{\psi(x)} \right)' + \frac{x^\alpha e^{-x}}{\psi(x)} \left( \frac{1}{\psi'(x)} \right) + \frac{(x^\alpha e^{-x})'}{\psi(x)\psi'(x)} \right).
\]
From the product rule for derivatives, notice that
\begin{equation}
(5.33) \quad \frac{x^{\alpha}e^{-x}}{\psi'(x)} \left( \frac{1}{\psi(x)} \right)' + \frac{x^{\alpha}e^{-x}}{\psi(x)} \left( \frac{1}{\psi'(x)} \right)' + \frac{(x^{\alpha}e^{-x})'}{\psi(x)\psi'(x)} = \left( \frac{1}{\psi(x)} \right) \cdot \left( \frac{1}{\psi'(x)} \right) \cdot x^{\alpha}e^{-x}'
= \left( \frac{x^{\alpha}e^{-x}}{L_{m-\alpha}^{-1}(-x)L_{m-\alpha}^{-1}(-x)} \right)'.
\end{equation}

Substituting (5.33) into (5.32) and using the definition of $W_{m}^{\alpha}(x)$, we obtain
\begin{equation}
W_{m}^{\alpha}(x) + mW_{m-1}^{\alpha-1}(x) = -\left( \frac{x^{\alpha}e^{-x}}{L_{m-\alpha}^{-1}(-x)L_{m-\alpha}^{-1}(-x)} \right)'.
\end{equation}

Integrating this expression now yields (5.30). Applying this relation inductively yields
\begin{equation}
(5.34) \quad \int W_{m}^{\alpha}(x)dx = m!(-1)^{m} \int e^{-x}x^{\alpha-m}dx - \sum_{j=0}^{m-1}(-1)^{j} \binom{m}{j} j! \frac{x^{\alpha-j}e^{-x}}{L_{m-j-1}^{-\alpha}(-x)L_{m-j}^{-\alpha}(-x)}.
\end{equation}

For $r > 0$, let $C_{r} = C_{1} + C_{2} + C_{3}$ denote the contour given by the ray
\begin{equation}
C_{1} = \{ x - ir : 0 \leq x < \infty \}
\end{equation}
oriented from right to left, by the left-side semi-circle
\begin{equation}
C_{2} = \{ re^{it} : \frac{\pi}{2} \leq t \leq \frac{3\pi}{2} \}
\end{equation}
oriented clockwise, and by the ray
\begin{equation}
C_{3} = \{ x + ir : 0 \leq x < \infty \}
\end{equation}
oriented from left to right. The zeros of $L_{m}^{-\alpha-1}(-x)$ are all negative, and so by taking $r$ sufficiently small, the contour $C_{r}$ can be made to not include these negative roots. Observe that the integrand denominator is the square of a polynomial with simple roots. Hence the residues of the integrand at the roots of $L_{m}^{-\alpha-1}(-x)$ vanish, which means that it suffices to impose the condition that $r > 0$ and that $L_{m}^{-\alpha-1}(r) \neq 0$. With this assumption,
\begin{equation}
\int_{0}^{\infty} W_{m}^{\alpha}(x)dx = \frac{1}{1-e^{2\pi i a}} \int_{C_{r}} \frac{(-z)^{a}e^{\pi i a}e^{-z}}{(L_{m-\alpha}^{-1}(-z))^{2}}dz
\end{equation}
where $(-z)^{a}$ denotes the principal branch of the power function. By deforming $C_{r}$ we can rewrite the latter as a Mellin-Barnes integral, namely
\begin{equation}
(5.35) \quad \int_{0}^{\infty} W_{m}^{\alpha}(x)dx = \frac{i}{2\sin(\pi a)} \int_{r-i\infty}^{r+i\infty} \frac{(-z)^{a}e^{-z}}{(L_{m-\alpha}^{-1}(-z))^{2}}dz
\end{equation}
Applying the same procedure to the usual integral representation of the $\Gamma$-function gives
\begin{equation}
(5.36) \quad \frac{i}{2\sin(\pi a)} \int_{r-i\infty}^{r+i\infty} (-z)^{a}e^{-z}dz = \Gamma(1 + a)
\end{equation}
valid for all non-integral values of $a$ and all $r > 0$. Applying (5.35) and (5.36) with $a = \alpha - m$ to (5.34) gives
\begin{equation}
\int_{0}^{\infty} W_{m}^{\alpha}(x)dx = m!(-1)^{m}\Gamma(1 + \alpha - m) = \frac{\Gamma(\alpha + 1)\Gamma(-\alpha)m!}{\Gamma(m - \alpha)}.
\end{equation}
This completes the proof of the theorem. □
5.4. The Completeness of the Type III Exceptional $X_m$-Laguerre Polynomials. In preparation for the proof of completeness of $\{L_{m,n}^{\text{III},\alpha} \mid n = 0, m + 1, m + 2, m + 3, \ldots\}$, we remind the reader that the set $\mathcal{P}$ denotes the vector space of all polynomials with complex coefficients in the real variable $x$ and, for $n \in \mathbb{N}_0$, let $\mathcal{P}_n$ denote the vector space of all $p \in \mathcal{P}$ with degree $\leq n$. The following lemma is critical for our argument; a proof can be found in [12, Lemma 3, p. 416].

**Lemma 5.5.** Suppose $\eta(x)$ is a polynomial such that $\eta(x) \neq 0$ for all $x \geq 0$. Then, for $\alpha > -1$, the subspace

$$\eta \mathcal{P} := \{\eta(x)p(x) \mid p \in \mathcal{P}\}$$

is dense in $L^2((0, \infty); x^\alpha e^{-x})$.

We now prove the following completeness result.

**Theorem 5.4.** The set of Type III exceptional $X_m$-Laguerre polynomials

$$\{L_{m,n}^{\text{III},\alpha} \mid n = 0, m + 1, m + 2, m + 3, \ldots\}$$

forms a complete orthogonal set of polynomials in the Hilbert space $L^2((0, \infty); W_m^{\text{III},\alpha})$.

**Proof.** The proof that we give of completeness is similar to the proof that we give of completeness in Theorem 3.6, we give the full proof since some essential ingredients of the proof below are different. Let $\epsilon > 0$ and let $f \in L^2((0, \infty); W_m^{\text{III},\alpha})$. Define

$$\tilde{f}(x) := \frac{f(x)}{L_m^{-\alpha-1}(-x)};$$

clearly

$$\|f\|_{m,\alpha}^\alpha = \|	ilde{f}\|_{m}^\alpha,$$

where $\|\cdot\|_{\alpha}$ denotes the norm in $L^2((0, \infty); x^\alpha e^{-x})$. Hence $\tilde{f} \in L^2((0, \infty); x^\alpha e^{-x})$. From Lemma 5.5 with

$$\eta(x) = L_m^{-\alpha-1}(-x),$$

there exists $p \in \mathcal{P}$, say with $\deg(p) = n$, such that

$$\|\tilde{f}(x) - L_m^{-\alpha-1}(-x)p(x)\|_{m}^\alpha < \epsilon.$$

Hence it follows that

$$\epsilon^2 > \int_0^{\infty} \left| \frac{f(x)}{L_m^{-\alpha-1}(-x)} - L_m^{-\alpha-1}(-x)p(x) \right|^2 x^\alpha e^{-x} dx$$

$$= \int_0^{\infty} \left| \frac{f(x) - (L_m^{-\alpha-1}(-x))^2p(x)}{L_m^{-\alpha-1}(-x)} \right|^2 x^\alpha e^{-x} dx$$

$$= \int_0^{\infty} \left| \frac{f(x) - (L_m^{-\alpha-1}(-x))^2p(x)}{L_m^{-\alpha-1}(-x)} \right|^2 W_m^{\text{III},\alpha} dx$$

$$= \left( \|f(x) - (L_m^{-\alpha-1}(-x))^2p(x)\|_{m}^{\text{III},\alpha} \right)^2.$$

Notice that

$$L_m^{-\alpha-1}(-x))^2 p \in \mathcal{F}_{n+2m},$$

where

$$\mathcal{F}_{n+2m} := \{P \in \mathcal{P}_{n+2m} \mid P'(-x_j) = 0 \ (j = 1, 2, \ldots, m)\},$$

(5.37)
and where \( \{x_j\}_{j=1}^m \subset (0, \infty) \) are the simple roots of the Laguerre polynomial \( L_m^{-\alpha-1}(x) \). We now show that \((L_m^{-\alpha-1}(-x))^2 p(x) \in \mathcal{E}_{n+2m}, \) where

\[
\mathcal{E}_{n+2m} := \text{span} \left\{ L_{n,j}^{\text{III},\alpha} \mid j = 0, m+1, m+2, \ldots, n+2m \right\};
\]

this will complete the proof of the theorem. Note that

\[
(5.38) \quad \dim (\mathcal{E}_{n+2m}) = \dim (\mathcal{F}_{n+2m}) = m + n + 1.
\]

Since the span of eigenfunctions of an operator is an invariant subspace of that operator, we see that

\[
L_m^{\text{III},\alpha}[\mathcal{E}_{n+2m}] \subset \mathcal{E}_{n+2m}.
\]

In particular, if \( P \in \mathcal{E}_{n+2m}, \) then

\[
L_m^{\text{III},\alpha}[P](x) = -x P''(x) + \left( -1 - \alpha + x + 2x \frac{L_m^{-\alpha-1}(-x)'}{L_m^{-\alpha-1}(-x)} \right) P'(x) + (-m + \alpha) P(x)
\]

\[
\in \mathcal{P}.
\]

Consequently, the term

\[
2x \frac{L_m^{-\alpha-1}(-x)'}{L_m^{-\alpha-1}(-x)} P'(x)
\]

must be a polynomial. Since \( L_m^{-\alpha-1}(x) \) is a classical Laguerre polynomial, its roots \( \{x_j\}_{j=1}^m \subset (0, \infty) \) are simple so it follows that \( P'(-x_j) = 0 \) for \( j = 1, 2, \ldots, m. \) Thus,

\[
\mathcal{E}_{n+2m} \subset \mathcal{F}_{n+2m}.
\]

From \((5.38)\) we see, in fact, that

\[
\mathcal{E}_{n+2m} = \mathcal{F}_{n+2m}.
\]

From \((5.37)\), it follows that

\[
(L_m^{-\alpha-1}(-x))^2 p(x) \in \mathcal{E}_{n+2m};
\]

and this completes the proof. \( \square \)

5.5. Location of Roots and Root Asymptotics of the Type III Exceptional \( X_m \)-Laguerre Polynomials. The following theorem gives us exact location of the \( m+k \) (real) roots of the Type III exceptional \( X_m \)-Laguerre polynomial \( L_{m,m+k}^{\text{III},\alpha}(x) \).

**Theorem 5.5.** Suppose \( m, k \in \mathbb{N} \) and \(-1 < \alpha < 0\). The Type III exceptional \( X_m \)-Laguerre polynomial \( L_{m,m+k}^{\text{III},\alpha}(x) \) has \( k \) positive roots which interlace the roots of \( L_{k-1}^{\alpha+1}(x) \) and \( m \) negative roots which interlace the roots of \( L_{-\alpha}^{-1}(x) \). More precisely, let \( \{x_{k-1,i}^{\alpha+1}\}_{i=1}^{k-1} \subset (0, \infty) \) denote the (simple) roots of the Laguerre polynomial \( L_{k-1}^{\alpha+1}(x) \), and let \( \{z_{m,i}^{-\alpha-1}\}_{i=1}^{m} \subset (-\infty, 0) \) denote the (simple) roots of the Laguerre polynomial \( L_{m}^{-\alpha-1}(x) \), with both sets ordered as follows:

\[
z_{m,m}^{-\alpha-1} < z_{m,m-1}^{-\alpha-1} < \ldots < z_{m,1}^{-\alpha-1} < 0 < x_{k-1,1}^{\alpha+1} < x_{k-1,2}^{\alpha+1} < \ldots < x_{k-1,k-1}^{\alpha+1}.
\]

Then

(a) each of the \( k \) intervals

\[
(0, x_{k-1,1}^{\alpha+1}), (x_{k-1,1}^{\alpha+1}, x_{k-1,2}^{\alpha+1}), \ldots, (x_{k-1,k-2}^{\alpha+1}, x_{k-1,k-1}^{\alpha+1}), (x_{k-1,k-1}^{\alpha+1}, \infty)
\]

contains exactly one root of \( L_{m,m+k}^{\text{III},\alpha} \).
(b) each of the \(m\) intervals
\((-\infty, z_{m,m}^{-\alpha-1}), (z_{m,m}^{-\alpha-1}, z_{m,m-1}^{-\alpha-1}), \ldots, (z_{m,2}^{-\alpha-1}, z_{m,1}^{-\alpha-1})\)
contains exactly one root of \(L_{m,m+k}^{III,\alpha}\).

Proof. The key identity in establishing both (a) and (b) is the identity given in (5.22), namely
\[
L_{m,m+k}^{III,\alpha}(x) = (k + \alpha)L_{k-2}^{\alpha+1}(x)L_m^{-\alpha-1}(-x) + (m + 1)L_{k-1}^{\alpha+1}(x)L_{m+1}^{-\alpha-1}(-x)
- (m + k)L_{k-1}^{\alpha+1}(x)L_m^{-\alpha-1}(-x).
\]

We first prove part (a). Letting \(x = x_{k-1,i}^{\alpha+1} (i = 1, 2, \ldots, k - 1)\) yields
\[
L_{m,m+k}^{III,\alpha}(x_{k-1,i}^{\alpha+1}) = (k + \alpha)L_{k-2}^{\alpha+1}(x_{k-1,i}^{\alpha+1})L_m^{-\alpha-1}(-x_{k-1,i}^{\alpha+1}).
\]
Since \(L_m^{-\alpha-1}(-x)\) has no roots in \((0, \infty)\) and \(L_m^{-\alpha-1}(0) > 0\), we see that
\[
L_m^{-\alpha-1}(-x_{k-1,i}^{\alpha+1}) > 0 \quad (i = 1, 2, \ldots, k - 1).
\]
Furthermore, from the classical theory, the roots of \(L_{k-1}^{\alpha+1}(x)\) and \(L_{k-2}^{\alpha+1}(x)\) interlace and since \(L_{k-2}^{\alpha+1}(0) > 0\), we see that
\[
L_{k-2}^{\alpha+1}(x_{k-1,i}^{\alpha+1}) > 0.
\]
Hence, from (5.39), we deduce that
\[
\text{sgn}(L_{m,m+k}^{III,\alpha}(x_{k-1,i}^{\alpha+1})) = \text{sgn}(L_{k-2}^{\alpha+1}(x_{k-1,i}^{\alpha+1})) = (-1)^{i+1} \quad (i = 1, \ldots, k - 1).
\]
It follows that \(L_{m,m+k}^{III,\alpha}(x)\) has a root in each of the \(k - 2\) intervals
\((x_{k-1,1}^{\alpha+1}, x_{k-1,2}^{\alpha+1}), (x_{k-1,2}^{\alpha+1}, x_{k-1,3}^{\alpha+1}), \ldots, (x_{k-1,k-2}^{\alpha+1}, x_{k-1,k-1}^{\alpha+1})\).

From (5.25), \(L_{m,m+k}^{III,\alpha}(0) < 0\); hence, from (5.40) with \(i = 1\), we see that there is another root of \(L_{m,m+k}^{III,\alpha}(x)\) in the interval \((0, x_{k-1,1}^{\alpha+1})\). Lastly, from (5.20), we see that \(x = x_{k-1,k-1}^{\alpha+1}\) is the right-most extreme point of \(L_{m,m+k}^{III,\alpha}(x)\). Regardless of whether \(x = x_{k-1,k-1}^{\alpha+1}\) is a relative maximum or relative minimum point, the graph of \(y = L_{m,m+k}^{III,\alpha}(x)\) necessarily must cross the \(x\)-axis once more at a point \(x^* > x_{k-1,k-1}^{\alpha+1}\). Hence we see that \(L_{m,m+k}^{III,\alpha}(x)\) has a root in the interval \((x_{k-1,k-1}^{\alpha+1}, \infty)\). Summarizing, we have shown that \(L_{m,m+k}^{III,\alpha}(x)\) has \(k\) distinct, positive roots.

The proof of (b) is similar. In this case, from (5.22), we see that
\[
L_{m,m+k}^{III,\alpha}(z_{m,i}^{-\alpha-1}) = (m + 1)L_{k-1}^{\alpha+1}(z_{m,i}^{-\alpha-1})L_{m+1}^{-\alpha-1}(-z_{m,i}^{-\alpha-1}).
\]
Since \(L_{k-1}^{\alpha+1}(x)\) has \(k - 1\) positive roots and \(L_{k-1}^{\alpha+1}(0) > 0\), we see that
\[
L_{k-1}^{\alpha+1}(z_{m,i}^{-\alpha-1}) > 0 \quad (i = 1, 2, \ldots, m).
\]
Moreover, since \(L_m^{-\alpha-1}(0) > 0\), it follows from the interlacing property of the roots of \(L_m^{-\alpha-1}(x)\) and \(L_{m+1}^{-\alpha-1}(x)\) that
\[
L_m^{-\alpha-1}(-z_{m,i}^{-\alpha-1}) = (-1)^i \quad (i = 1, 2, \ldots, m).
\]
Hence, from (5.41) and (5.42), we see that
\[
\text{sgn}(L_{m,m+k}^{III,\alpha}(z_{m,i}^{-\alpha-1})) = \text{sgn}(L_{m+1}^{-\alpha-1}(-z_{m,i}^{-\alpha-1})) = (-1)^i \quad (i = 1, \ldots, m).
\]
This implies that each of the \( m - 1 \) intervals
\[
(z_{m,m}^{-\alpha-1}, z_{m,m-1}^{-\alpha-1}), \ldots, (z_{m,2}^{-\alpha-1}, z_{m,1}^{-\alpha-1})
\]
contains a root of \( L_{m,m+k}^{\alpha}(x) \). We claim that there is an additional root of \( L_{m,m+k}^{\alpha}(x) \) in the interval \((-\infty, z_{m,m}^{-\alpha-1})\). Indeed, from [5.20], \( x = z_{m,m}^{-\alpha-1} \) is the left-most extreme point of \( L_{m,m+k}^{\alpha}(x) \) so, as in part (a), there must be another root of \( L_{m,m+k}^{\alpha}(x) \) at a point \( z^* < z_{m,m}^{-\alpha-1} \). This completes the proof that \( L_{m,m+k}^{\alpha}(x) \) has \( m \) roots in \((-\infty, 0)\). Combining this fact with part (a), we have found all \( m + k \) roots of \( L_{m,m+k}^{\alpha}(x) \) and this completes the proof of the theorem. \( \square \)

**Remark 5.4.** When \( k = 1 \), \( L_{m,m+1}^{\alpha}(x) \) has one positive root; the exact location of this root cannot be specifically identified. When \( m = 1 \), the Laguerre polynomial \( L_1^{\alpha-1}(-x) \) has a unique root \( z_{1,1}^{-\alpha-1} < 0 \). In this case, the above theorem indicates there is a unique root of \( L_{1,k+1}^{\alpha}(x) \) in the interval \((-\infty, z_{1,1}^{-\alpha-1})\).

We call the \( m \) negative roots of \( L_{m,m+k}^{\alpha}(x) \) above the ‘exceptional’ roots of \( L_{m,m+k}^{\alpha}(x) \). We now discuss the asymptotic behavior of the roots as \( k \to \infty \).

**Theorem 5.6.** As \( k \to \infty \):

(a) The exceptional roots of \( L_{m,m+k}^{\alpha}(x) \) converge to the roots of \( L_m^{-\alpha-1}(-x) \).

(b) The first positive root of \( L_{m,m+k}^{\alpha}(x) \) tends to zero.

**Proof.** Recall, from (5.12), that
\[
L_{m,m+k}^{\alpha}(x) = xL_{k-2}^{\alpha+2}(x)L_m^{-\alpha-1}(-x) + (m + 1)L_{k-1}^{\alpha+1}(x)L_m^{-\alpha-2}(-x).
\]

Now, the outer ratio asymptotics for the classical Laguerre polynomials give
\[
\frac{L_{k-1}^{\alpha+1}(x)}{L_{k-2}^{\alpha+2}(x)} \sim \left(\frac{-x}{k}\right)^{1/2} + O\left(\frac{1}{k}\right), \quad k \to \infty
\]
with convergence uniform on compact sets that avoid the positive real axis. Therefore, dividing the identity in (5.43) by \( L_{k-2}^{\alpha+2}(x) \) and taking the limit as \( k \to \infty \), we obtain
\[
\frac{L_{m,m+k}^{\alpha}(x)}{L_{k-2}^{\alpha+2}(x)} \xrightarrow{k \to \infty} xL_m^{-\alpha-1}(-x).
\]

Part (a) follows by Hurwitz’s theorem ([26, Theorem 1.91.3]) and Theorem 5.5. The extra \( x \) factor implies Part (b) as follows. At \( x = 0 \) we have
\[
\frac{L_{k-1}^{\alpha+1}(0)}{L_{k-2}^{\alpha+1}(0)} = \frac{k + \alpha}{k - 1} \xrightarrow{k \to \infty} 1.
\]

So the ratio asymptotics hold at \( x = 0 \). And from (5.21), it follows that \( L_{m,m+k}^{\alpha}(0) \to 0 \) as \( k \to \infty \). \( \square \)
5.6. **Type III Exceptional Xₘ-Laguerre Spectral Analysis.** The maximal domain associated with the differential expression \( \ell_{\text{III}, \alpha} [\cdot] \), given in either [5.3] or [5.3], in the Hilbert space \( L^2((0, \infty); W_{m \cdot \cdot}^{\text{III}, \alpha}) \) is defined to be

\[
\Delta_{m \cdot \cdot}^{\text{III}, \alpha} := \{ f : (0, \infty) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(0, \infty); f, \ell_{m \cdot \cdot}^{\text{III}, \alpha} [f] \in L^2((0, \infty); W_m^{\text{III}, \alpha}) \}.
\]

The associated maximal operator

\[
T_{1, m}^{\text{III}, \alpha} : \mathcal{D}(T_{1, m}^{\text{III}, \alpha}) \subset L^2((0, \infty); W_m^{\text{III}, \alpha}) \to L^2((0, \infty); W_m^{\text{III}, \alpha}),
\]

is defined to be

\[
T_{1, m}^{\text{III}, \alpha} f = \ell_{m \cdot \cdot}^{\text{III}, \alpha} [f]
\]

\[
f \in \mathcal{D}(T_{1, m}^{\text{III}, \alpha}) : = \Delta_{m \cdot \cdot}^{\text{III}, \alpha}.
\]

For \( f, g \in \Delta_{m \cdot \cdot}^{\text{III}, \alpha} \), Green’s formula may be written as

\[
\int_{0}^{\infty} \ell_{m \cdot \cdot}^{\text{III}, \alpha} [f](x) \overline{g}(x) W_{m \cdot \cdot}^{\text{III}, \alpha}(x) dx = [f, g]_{m \cdot \cdot}^{\text{III}, \alpha}(x) \bigg|_{x=0}^{x=\infty} + \int_{0}^{\infty} f(x) \ell_{m \cdot \cdot}^{\text{III}, \alpha} [\overline{g}](x) W_{m \cdot \cdot}^{\text{III}, \alpha}(x) dx,
\]

where \([\cdot, \cdot]_{m \cdot \cdot}^{\text{III}, \alpha}(\cdot)\) is the sesquilinear form defined by

\[
[f, g]_{m \cdot \cdot}^{\text{III}, \alpha}(x) := \frac{x^{\alpha+1} e^{-x}}{(L_{m \cdot \cdot}^{\alpha+1}(-x))^2} (f(x) \overline{g}(x) - f'(x) \overline{g}(x)) \quad (0 < x < \infty)
\]

and where

\[
[f, g]_{m \cdot \cdot}^{\text{III}, \alpha}(x) \bigg|_{x=0}^{x=\infty} := [f, g]_{m \cdot \cdot}^{\text{III}, \alpha}(\infty) - [f, g]_{m \cdot \cdot}^{\text{III}, \alpha}(0) = \lim_{x \to \infty} [f, g]_{m \cdot \cdot}^{\text{III}, \alpha}(x) - \lim_{x \to 0^+} [f, g]_{m \cdot \cdot}^{\text{III}, \alpha}(x)
\]

The adjoint of the maximal operator in \( L^2((0, \infty); W_m^{\text{III}, \alpha}) \) is the minimal operator \( T_{0, m}^{\text{III}, \alpha} \), defined by

\[
T_{0, m}^{\text{III}, \alpha} f = \ell_{m \cdot \cdot}^{\text{III}, \alpha} [f]
\]

\[
f \in \mathcal{D}(T_{0, m}^{\text{III}, \alpha}) : = \{ f \in \Delta_{m \cdot \cdot}^{\text{III}, \alpha} \mid [f, g]_{m \cdot \cdot}^{\text{III}, \alpha} \bigg|_{x=0}^{x=\infty} = 0 \text{ for all } g \in \Delta_{m \cdot \cdot}^{\text{III}, \alpha} \}.
\]

Both endpoints \( x = 0 \) and \( x = \infty \) are singular points of \( \ell_{m \cdot \cdot}^{\text{III}, \alpha} [\cdot] \). In fact, \( x = 0 \) is a regular singular endpoint in the sense of Frobenius and \( x = \infty \) is an irregular singular endpoint. The associated Frobenius indicial equation at \( x = 0 \) is \( r(r + \alpha) = 0 \). Consequently, two linearly independent solutions of \( \ell_{m \cdot \cdot}^{\text{III}, \alpha} [y] = 0 \) will behave asymptotically like

\[
z_1(x) := 1 \quad \text{and} \quad z_2(x) := x^{-\alpha}
\]

near \( x = 0 \). Since \(-1 < \alpha < 0\), it is clear that both solutions are in \( L^2((0, \infty); W_m^{\text{III}, \alpha}) \); in other words, \( \ell_{m \cdot \cdot}^{\text{III}, \alpha} [\cdot] \) is in the limit-circle case at \( x = 0 \).

For the analysis at the irregular singular endpoint, \( x = \infty \), we obtain two linearly independent solutions using the standard reduction of order method. Solving the differential equation \( \ell_{m \cdot \cdot}^{\text{III}, \alpha} [y](x) = 0 \) we have a basis of solutions \( \{y_1(x), y_2(x)\} \), where

\[
y_1(x) = 1 \in L^2((0, \infty); W_m^{\text{III}, \alpha})
\]

and

\[
y_2(x) = \int_{a}^{x} e^{t} (L_{m \cdot \cdot}^{\alpha+1}(t))^{2} dt \quad (a > 0 \text{ is arbitrary}).
\]
Mimicking the proof of the Type I case in Section 3.2 we find that \( y_2 \notin L^2((1, \infty); W_{m, \alpha}^{III}) \). Consequently, we obtain the following result on the deficiency indices of \( T_{0,m}^{III,\alpha} \).

**Theorem 5.7.** Let \( T_{0,m}^{III,\alpha} \) be the minimal operator in \( L^2((0, \infty); W_{m, \alpha}^{III}) \) generated by the Type III exceptional \( X_m \)-Laguerre differential expression \( l_{m, \alpha}^{III} \). For \(-1 < \alpha < 0\), the deficiency index of \( T_{0,m}^{III,\alpha} \) is \((1,1)\).

For \(-1 < \alpha < 0\), we must impose one boundary condition at \( x = 0 \) in order to obtain a self-adjoint extension of the minimal operator \( T_{0,m}^{III,\alpha} \). We seek to find that self-adjoint operator \( T_{0,m}^{III,\alpha} \) which has the Type III polynomials \( \{ L_{m,n,\alpha}^{III} \mid n = 0, m + 1, m + 2, m + 3, \ldots \} \) as eigenfunctions.

Note that \( x^{-\alpha} \in L^2((0, \infty); W_{m, \alpha}^{III}) \) since \(-1 < \alpha < 0\). A calculation shows that

\[
l_{m, \alpha}^{III} x^{-\alpha} = \frac{-2\alpha x^{-\alpha} L_{m-1}^{-\alpha}(-x)}{L_{m-1}^{\alpha-1}(-x)} - mx^{-\alpha}.
\]

Since

\[
\int_0^\infty \left| \frac{x^{-\alpha} L_{m-1}^{-\alpha}(-x)}{L_{m-1}^{\alpha-1}(-x)} \right|^2 \frac{x^\alpha e^{-x}}{(L_{m-1}^{\alpha-1}(-x))^2} dx \\
\leq \frac{1}{(L_{m-1}^{\alpha-1}(0))^2} \int_0^\infty \left( L_{m-1}^{-\alpha}(-x) \right)^2 x^{-\alpha} e^{-x} dx < \infty,
\]

we see that \( l_{m, \alpha}^{III} x^{-\alpha} \in L^2((0, \infty); W_{m, \alpha}^{III}) \). Consequently, \( x^{-\alpha} \in \Delta_{m, \alpha}^{III} \) for \(-1 < \alpha < 0\). Moreover, the calculation

\[
[x^{-\alpha}, 1]_{m, \alpha}^{III}(0) = \alpha \lim_{x \to 0} \frac{e^{-x}}{(L_{m-1}^{\alpha-1}(-x))^2} \neq 0,
\]

proves that \( 1 \notin D(T_{0,m}^{III,\alpha}) \), the minimal domain, and thus we can use the function \( 1 \) as an appropriate Glazman boundary function. For \( f \in \Delta_{m, \alpha}^{III} \), further calculations show that

\[
0 = [f, 1]_{m, \alpha}^{III}(0) = \lim_{x \to 0^+} x^{\alpha+1} f'(x)
\]

and

\[
\lim_{x \to 0^+} x^{\alpha+1} (L_{m,n}^{III,\alpha}(x))' = 0.
\]

Summarizing, and using Theorem 5.4, we obtain the following theorem.

**Theorem 5.8.** Suppose \(-1 < \alpha < 0\). The operator

\[
T_{m, \alpha}^{III} : D(T_{m}^{III,\alpha}) \subseteq L^2((0, \infty); W_{m, \alpha}^{III}) \to L^2((0, \infty); W_{m, \alpha}^{III}),
\]

defined by

\[
T_{m, \alpha}^{III} f = l_{m, \alpha}^{III} f
\]

\( f \in D(T_{m, \alpha}^{III}, \alpha) : = \{ f \in \Delta_{m, \alpha}^{III} \mid \lim_{x \to 0^+} x^{\alpha+1} f'(x) = 0 \}, \)

is a self-adjoint extension of the minimal operator \( T_{0,m}^{III,\alpha} \) in \( L^2((0, \infty); W_{m, \alpha}^{III}) \) having the Type III exceptional \( X_m \)-Laguerre polynomials

\[
\{ L_{m,n}^{III,\alpha} \mid n = 0, m + 1, m + 2, m + 3, \ldots \}\]
as a complete set of eigenfunctions. Moreover, the spectrum of $T_{m}^{\alpha}_{III}$ consists only of eigenvalues and is given explicitly by

$$
\sigma(T_{m}^{\alpha}_{III}) = \{n - m + \alpha \mid n = 0, m + 1, m + 2, m + 3, \ldots\}.
$$

6. Appendix

The following is a list of a few Type III exceptional $X_{m}$-Laguerre polynomials $\{L_{m,n}^{III,\alpha}\}$ for various values of $m$ and $n$. A similar list of Type I and II exceptional $X_{m}$-Laguerre polynomials can be found in [19].

For $m = 1$ we have:

$$
L_{1,0}^{III,\alpha}(x) = 1
$$
$$
L_{1,1}^{III,\alpha}(x) = x^2 - 2\alpha x + \alpha(\alpha + 1)
$$
$$
L_{1,2}^{III,\alpha}(x) = -x^3 + 3(\alpha + 1)x^2 - 3\alpha(x + 2)x + \alpha(\alpha + 1)(\alpha + 2)
$$
$$
L_{1,3}^{III,\alpha}(x) = \frac{1}{2}x^4 - 2(\alpha + 2)x^3 + (\alpha + 3)(3\alpha + 2)x^2 - 2\alpha(\alpha + 2)(\alpha + 3)x
$$
$$
+ \frac{1}{2}\alpha(\alpha + 1)(\alpha + 2)(\alpha + 3)
$$
$$
L_{1,4}^{III,\alpha}(x) = -\frac{1}{6}x^5 + \frac{5}{6}(\alpha + 3)x^4 - \frac{5}{6}(\alpha + 4)(2\alpha + 3)x^3
$$
$$
+ \frac{5}{6}(\alpha + 3)(\alpha + 4)(2\alpha + 1)x^2 - \frac{5}{6}\alpha(\alpha + 2)(\alpha + 3)(\alpha + 4)x
$$
$$
+ \frac{1}{6}\alpha(\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4).
$$

For $m = 2$ we obtain:

$$
L_{2,0}^{III,\alpha}(x) = 1
$$
$$
L_{2,1}^{III,\alpha}(x) = \frac{1}{2}x^3 - \frac{3(\alpha - 1)}{2}x^2 + \frac{3\alpha(\alpha - 1)}{2}x - \frac{\alpha(\alpha - 1)(\alpha + 1)}{2}
$$
$$
L_{2,2}^{III,\alpha}(x) = -\frac{1}{2}x^4 + 2\alpha x^3 - (\alpha - 1)(3\alpha + 4)x^2 + 2\alpha(\alpha - 1)(\alpha + 2)x
$$
$$
- \frac{\alpha(\alpha - 1)(\alpha + 1)(\alpha + 2)}{2}
$$
$$
L_{2,3}^{III,\alpha}(x) = \frac{1}{4}x^5 - \frac{5(\alpha + 1)}{4}x^4 + \frac{5}{2}(\alpha^2 + 2\alpha - 1)x^3
$$
$$
- \frac{5(\alpha - 1)(\alpha + 1)(\alpha + 3)}{2}x^2 + \frac{5\alpha(\alpha - 1)(\alpha + 2)(\alpha + 3)}{4}x
$$
$$
- \frac{\alpha(\alpha - 1)(\alpha + 1)(\alpha + 2)(\alpha + 3)}{4}.$$
$$L_{2,6}^{\alpha,3}(x) = -\frac{1}{12}x^6 + \frac{(\alpha+2)}{2}x^5 - \frac{(5\alpha^2 + 19\alpha + 6)}{4}x^4$$
$$+ \frac{(\alpha + 2)(\alpha + 4)(5\alpha - 3)}{3}x^3 - \frac{(\alpha - 1)(\alpha + 3)(\alpha + 4)(5\alpha + 4)}{4}x^2$$
$$+ \frac{\alpha(\alpha - 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)}{2}x$$
$$- \frac{\alpha(\alpha - 1)(\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)}{12}.$$

For $m = 3$:

$$L_{3,0}^{\alpha,3}(x) = 1$$
$$L_{3,4}^{\alpha,3}(x) = \frac{1}{6}x^4 - \frac{2(\alpha - 2)}{3}x^3 + (\alpha - 1)(\alpha - 2)x^2$$
$$- \frac{2\alpha(\alpha - 1)(\alpha - 2)}{3}x + \frac{\alpha(\alpha - 2)(\alpha - 1)(\alpha + 1)}{6}$$
$$L_{3,5}^{\alpha,3}(x) = -\frac{1}{6}x^5 + \frac{5(\alpha - 1)}{6}x^4 - \frac{5(\alpha - 2)(2\alpha + 1)}{6}x^3$$
$$+ \frac{5(\alpha - 2)(\alpha - 1)(2\alpha + 3)}{6}x^2 - \frac{5\alpha(\alpha - 2)(\alpha - 1)(\alpha + 2)}{6}x$$
$$+ \frac{\alpha(\alpha - 2)(\alpha - 1)(\alpha + 1)(\alpha + 2)}{6}$$
$$L_{3,6}^{\alpha,3}(x) = \frac{1}{12}x^6 - \frac{\alpha}{2}x^5 + \frac{(5\alpha^2 + \alpha - 12)}{4}x^4$$
$$- \frac{\alpha(\alpha - 2)(5\alpha + 13)}{3}x^3$$
$$+ \frac{(\alpha - 2)(\alpha - 1)(\alpha + 3)(5\alpha + 6)}{5}x^2$$
$$- \frac{\alpha(\alpha - 2)(\alpha - 1)(\alpha + 2)(\alpha + 3)}{2}x$$
$$+ \frac{\alpha(\alpha - 2)(\alpha - 1)(\alpha + 1)(\alpha + 2)(\alpha + 3)}{12}$$
$$L_{3,7}^{\alpha,3}(x) = -\frac{1}{36}x^7 + \frac{7(\alpha + 1)}{36}x^6 - \frac{7(\alpha^2 + 2\alpha - 2)}{12}x^5$$
$$+ \frac{35(\alpha + 1)(\alpha^2 + 2\alpha - 6)}{36}x^4 - \frac{7(\alpha - 2)(\alpha + 4)(5\alpha^2 + 10\alpha - 3)}{36}x^3$$
$$+ \frac{7(\alpha - 2)(\alpha - 1)(\alpha + 1)(\alpha + 3)(\alpha + 4)}{12}x^2$$
$$- \frac{7\alpha(\alpha - 2)(\alpha - 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)}{36}x$$
$$+ \frac{\alpha(\alpha - 2)(\alpha - 1)(\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)}{36}.$$

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