Sloshing of Melts in a Bath Smelting Process Having an Elliptic Cross Section

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Recently, the use of the elliptic cross section cylinder for bath smelting operations was adopted by the AusIron process. The high gas flow rates used have been found to generate wave motion in the bath that can enhance refractory wear. The standing wave modes were found for the bath. Two distinct sets of eigenvalues were obtained, one in the major axis and the other in the minor axis. The two sets converged when the ellipticity of the cylinder was zero, which is the shape of a circle. The calculated wave frequency was found to agree well with experimental data. A complete set of eigenvalues have been calculated for elliptic cylinders and fitted to Chebyshev polynomials, enabling quick estimation of the standing wave frequencies. The study showed that the ellipticity of the vessel for the AusIron process needs to be carefully chosen as the standing wave modes that can be generated may result in beat frequencies that can interact with the natural frequencies of the vessel support and auxiliary equipment.

KEY WORDS: sloshing; bath smelting; Mathieu functions; eigenvalues; AusIron process; elliptic cross section cylinder.

1. Introduction and Previous Work

The use of elliptic cross section vessels in the metallurgical industry is seldom encountered. The drum converters studied by Kootz and Gille were rotated from the vertical to give an elliptic free surface shape in the belief that tilting might reduce slopping for a given gas flow rate when compared to a vertical converter. This resulted in a vessel where the liquid depth varied with position and with gas being injected from the side. Analysis of the vessel behaviour was difficult. Wear was found to occur more at the free surface level and above the tuyere where the splash impacted.

Elliptic cross section basins have been studied earlier this century. Jeffreys studied lakes with small and great eccentricity and calculated the period for a lake where the minor to major axes were in the ratio 3:5. He found that the slowest period differed by only 2% from a circular cross section. However, the solution in Mathieu functions do not converge rapidly and higher order modes were not calculated. Jeffreys also calculated the modes for an elliptical canal. Goldstein calculated the first five modes for an elliptical canal to a much higher accuracy.

Goldsbrough showed that for an elliptic basin with a depth that varied as \( h = h_0(1 - \cos^2 \beta)(1 - \cos^2 \eta) \), the solution can be obtained in closed form in terms of the Gauss hypergeometric function. Chu compared two elliptic cylinders with 1 and 5% out-of-roundness from a circular cylinder and showed that the resonant frequencies were higher for the minor axis and lower for the major axis when compared to the circular cylinder of equal cross-sectional area. The difference increases with increasing out-of-roundness. New modes and frequencies were introduced that have no counterpart in the perfectly circular tank.

The use of cylindrical vessels with a single lance at the centre is not suitable when very large diameters are required as the influence of the lance diminishes with distance from the centre. Large diameter cylindrical vessels set up higher order modes whereby an inner annulus of melt is thoroughly mixed by the gas injected through the lance. The outer annulus is not affected, resulting in little or no mixing, and much calmer than the central portion. To overcome this limitation, the AusIron process uses multiple lances situated in an elliptical cross section vessel. Hitherto, due to the difficulty in solving the Mathieu functions, there are no complete calculations spanning the range of ellipticity to enable prediction of the period of the standing waves present in a cylinder of elliptic cross section.

In the modelling of standing waves, the Reynolds number of the flow can be determined by \( \text{Re} = cL/(2\pi v) \). For pig iron \( (\nu = 1 \times 10^{-6} \text{ m}^2/\text{s}) \) in a pilot plant of diameter 1 m, bath depth of 0.5 m \( (c^2 = gh) \), a wave speed of 3.1 m/s is obtained giving a Reynolds number of \( 5 \times 10^3 \). For a bath of slag with a kinematic viscosity about 100 times of pig iron, the Reynolds number is still quite large. The Navier–Stokes equation reduces to the Euler inviscid flow equation for high Reynolds numbers. A potential flow approach is therefore appropriate for the study of the sloshing behaviour.

2. Governing Equations

An elliptical cross section cylinder in the \( x \) and \( y \) plane, with the vertical component in the \( z \) direction can be trans-
formed into elliptical coordinates by the use of
\[ x = a \cosh u \cos v, \quad y = a \sinh u \sin v. \quad \cdots \cdots (1) \]
where 0 ≤ u < ∞; 0 ≤ v < 2π. The coordinate system and lines of constant u and v are shown in Fig. 1. The distance between the two foci is 2a. The eccentricity e is defined as
\[ \cosh u = 1/e, \quad \cdots \cdots \cdots (2) \]
where e → 1 as u → 0 and e → 0 as u → ∞; the shapes being a ribbon and circle respectively. The ratios of the minor (B) to major axis (A) is related to e by B/A = \( \sqrt{1-e^2} \) and A is given by a/e. Note that for B/A = 0, e = 1 and for B/A = 1, e = 0. Assuming an inviscid fluid, we required the Laplacian of the liquid potential to satisfy
\[ \nabla^2 \phi = 0, \quad \cdots \cdots \cdots \cdots \cdots (3) \]
within the cylinder. In elliptical coordinates, this is
\[ \frac{2}{a^2 (\cosh 2u - \cos 2v)} \left( \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} \right) + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad \cdots \cdots (4) \]
The boundary conditions are:
\[ \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -h; \quad \cdots \cdots (5) \]
\[ \frac{\partial \phi}{\partial u} = 0 \quad \text{at} \quad u = 0, u_0; \quad \cdots \cdots (6) \]
\[ \frac{\partial \phi}{\partial v} = 0 \quad \text{at} \quad v = 0, 2\pi; \quad \cdots \cdots (7) \]
\[ \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = 0. \quad \cdots \cdots (8) \]
The solution is obtained from the separation of variable by substituting \( \phi = e^{iu} U(u) V(v) \zeta(z) \) into Eq. (4) to give
\[ \frac{2}{a^2 (\cosh 2u - \cos 2v)} \left( \frac{U''}{U} + \frac{V''}{V} \right) + \frac{Z''}{Z} = 0 \quad \cdots \cdots (9) \]
Let \( Z''/Z = e^{2z} \) and a solution for the depth with boundary condition (5) gives
\[ \frac{Z}{\cosh \xi h} = C_1 \cosh (\xi (z + h)) / \cosh \xi h \quad \cdots \cdots (10) \]
Substituting \( \xi^2 \) into Eq. (9) and separating \( U \) and \( V \) gives two equations that are independent of each other. They are therefore equal to a constant k.
\[ \frac{U''}{U} + \frac{(\xi a)^2}{2} \cosh 2u = - \frac{V''}{V} + \frac{(\xi a)^2}{2} \cos 2v = k \quad \cdots \cdots (11) \]
The two resultant differential equations are the Mathieu (12) and modified Mathieu (13) equations respectively.
\[ V'' + (k - 2q \cos 2v) V = 0, \quad \cdots \cdots (12) \]
\[ U'' - (k - 2q \cosh 2u) U = 0, \quad \cdots \cdots (13) \]
where \( q = (\xi a/2)^2 \). The solutions are given (McLachlan) by
\[ \phi(u, v, z, t) = e^{iut} \left( \sum_{n=0}^{\infty} A_n C_e(u, q) C_e(v, q) \right. \]
\[ \left. + \sum_{n=0}^{\infty} S_n S_e(u, q) S_e(v, q) \right) C_1 \cosh (\xi (z + h)) / \cosh \xi h \quad \cdots \cdots (14) \]
\[ ce, se, Ce, Se \] are the cosine elliptic, sine elliptic, cosh elliptic and sinh elliptic functions respectively, where
\[ cen(z, q) = \sum_{r=0}^{\infty} \phi_{en} \cos 2rz, \]

Fig. 1. The elliptic cylinder coordinate system. For higher values of u, the lines of constant u approach a circle.
ce_{2n+1}(z, q) = \sum_{r=0}^{\infty} \mathcal{B}_r q^{r+1} \cos(2r+1)z,
se_{2n+1}(z, q) = \sum_{r=0}^{\infty} \mathcal{B}_r q^{r+1} \sin(2r+1)z,
se_{2n+2}(z, q) = \sum_{r=0}^{\infty} \mathcal{B}_r q^{r+2} \sin(2r+2)z,

C_{2n}(z, q) = c c_{2n}(iz, q) = \sum_{r=0}^{\infty} \mathcal{B}_r q^{r+1} \cosh 2rz,

C_{2n+1}(z, q) = c c_{2n+1}(iz, q) = \sum_{r=0}^{\infty} \mathcal{B}_r q^{r+1} \cosh(2r+1)z,

Se_{2n+1}(z, q) = -i s e_{2n+1}(iz, q) = \sum_{r=0}^{\infty} \mathcal{B}_r q^{r+1} \sinh(2r+1)z,

Se_{2n+2}(z, q) = -i s e_{2n+2}(iz, q) = \sum_{r=0}^{\infty} \mathcal{B}_r q^{r+2} \sinh(2r+2)z,

where \mathcal{B}_r's and \mathcal{B}_r's here are arbitrary constants. The boundary conditions (6) and (7) give

\begin{align}
C_{2n}(u_0, q) &= 0, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldot

3. Eigenvalues of the Slosh Modes

For the first eigenvalue, the series solution was used. The eigenvalues for the modes in the major axis are calculated from Eq. (15). The accuracy of the eigenvalues improves as more terms are used in the expansion. For $B/A = 0$, an elliptic canal, the eigenvalue was calculated to 29 decimal places as 1.88660541060568590263369945233 and was used as a check with the numerical approach for obtaining the eigenvalues. Moreover, for the elliptic canal, there is only one root to each $m$ as there are no nodal ellipses associated with the modes of oscillation of an elliptic canal. Figure 2 shows the error in the eigenvalues as the number of terms in the series is increased. For $B/A$ approaching 1, the value of $q$ decreases rapidly and the series expansion can be used. However, a singularity exists for $B/A = 1$ whereupon the eigenvalue is the root of the derivative of the Bessel function. Figure 3 shows the eigenvalues for the first five modes for $0 \leq \epsilon \leq 1$. The eigenvalues do not vary greatly over the range while the value of $q$ changes quite substantially. Tables 1 and 2 list the Chebyshev approximations to the eigenvalues for the major ($c_1$–$c_5$ in Fig. 3) and minor ($s_1$–$s_5$ in Fig. 3) axes respectively. In Table 1, the eigenvalue is bounded and the approximation is fitted for $0 \leq \epsilon \leq 0.95$ as the eigenvalues approach infinity as the ellipse approaches a ribbon shape (or elliptic canal). The inclusion of points between 0.95 and 1 severely
reduced the accuracy of the approximation and have been excluded since such shapes are not commonly used. The evaluation of the Chebyshev approximation for Table 1 is given by

\[ A_x(x) = \sum_{k=0}^{N} c_k T_k(2x-1) - c_1/2, \ldots \ldots \ldots (18) \]

and for Table 2 is

\[ 1/(A_x(x)) = \sum_{k=0}^{N} c_k T_k((40x/19)-1) - c_1/2, \ldots \ldots \ldots (19) \]

The theory and computation of Chebyshev approximations are described in Press et al. The accuracy of the approximation is given by the value of the \( c_p \), the truncation error.

### 3.1. The Shapes of the Standing Waves

The shape of the wave of an elliptic canal, along the major axis, is shown in Fig. 4. The shape does not vary significantly as the ellipticity of the vessel is varied. However, the distance from the crest to crest do not form a smooth cosine or sine shape, but varies with the wave mode. For the even values of \( m \), the wave is symmetric about the major and minor axes, while for odd values of \( m \), the wave is symmetric about the major axis and anti-symmetric about the minor axis.

The AusIron furnace is elongated in shape, of which the precise dimensions are not available. A value of \( B/A = 0.6 \) falls within the range of ellipses considered for

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**Table 1.** Chebyshev coefficients for the eigenvalue of the standing waves in the major axis elliptic cross-section cylinder geometry.

| \# | 1st asymmetric | 1st symmetric | 2nd asymmetric | 2nd symmetric | 3rd asymmetric |
|---|---|---|---|---|---|
| \( c_1 \) | 3.7147025779 | 6.488191160 | 9.127813348 | 1.170239630E1 | 1.423721423E1 |
| \( c_2 \) | 2.214725084E-2 | 2.292887042E-1 | 4.617195610E-1 | 7.014226050E-1 | 9.436006283E-1 |
| \( c_3 \) | 6.445043561E-3 | 2.398717990E-2 | 7.588758683E-2 | 1.464784160E-1 | 2.275010606E-1 |
| \( c_4 \) | 5.578689878E-4 | -1.457534380E-2 | -3.606987963E-2 | -3.743972550E-2 | -3.524734806E-2 |
| \( c_5 \) | 9.697169155E-5 | 2.431498382E-3 | -6.173265403E-3 | -1.92542121E-2 | -3.138035040E-2 |
| \( c_6 \) | 9.615534292E-6 | 1.623324408E-3 | 3.304550873E-3 | -7.269445969E-4 | -8.750005962E-3 |
| \( c_7 \) | 1.281363989E-6 | -1.175117797E-4 | 2.292602037E-3 | 4.294235886E-3 | 3.278296174E-3 |
| \( c_9 \) | 1.154892099E-7 | -1.857781510E-4 | 1.087077090E-4 | 2.294875393E-3 | 4.455313103E-3 |
| \( c_{10} \) | 1.059326404E-8 | 1.518329213E-5 | -3.929162415E-4 | 1.173305088E-4 | 1.923636430E-3 |

**Table 2.** Chebyshev coefficients for the eigenvalue of the standing waves in the minor axis elliptic cross-section cylinder geometry.

| \# | 1st asymmetric | 1st symmetric | 2nd asymmetric | 2nd symmetric | 3rd asymmetric |
|---|---|---|---|---|---|
| \( c_1 \) | 8.551447347E-1 | 5.564089254E-1 | 4.158192588E-1 | 3.328898647E-1 | 2.77895672E-1 |
| \( c_2 \) | -1.683518090E-1 | -7.351771045E-2 | -4.453099099E-2 | -3.140708527E-2 | -2.422911117E-2 |
| \( c_3 \) | -5.761019456E-2 | -3.203518059E-2 | -1.913108421E-2 | -1.270951804E-2 | -9.201833848E-3 |
| \( c_4 \) | -1.315871037E-2 | -1.102742145E-3 | -7.167652144E-3 | -4.586245843E-3 | -3.032425848E-3 |
| \( c_5 \) | -5.444426060E-3 | -3.478487312E-3 | -3.630972089E-3 | -2.572911643E-3 | -1.789883610E-3 |
| \( c_6 \) | -2.402619479E-3 | -2.115208287E-3 | -1.755520139E-3 | -1.369281130E-3 | -1.026470505E-3 |
| \( c_7 \) | -1.163345144E-3 | -1.019730300E-4 | -8.584445447E-4 | -7.108744010E-4 | -5.672758661E-4 |
| \( c_8 \) | -5.869829249E-4 | -5.155277542E-4 | -4.360105682E-4 | -3.712221434E-4 | -3.095021158E-4 |
| \( c_9 \) | -3.040076512E-4 | -2.704792740E-4 | -2.344199183E-4 | -1.987003880E-4 | -1.659218890E-4 |

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Fig. 3. The eigenvalues for the first five modes in the major and minor axis as a function of the ellipticity of the container. For the \( \Phi_R(x, q) = 0 \) solution, the eigenvalue approaches infinity as the ellipticity approaches 1.

Fig. 4. The shapes of the standing waves along the major axis where the amplitude is normalised to the maximum value. The intersections of the standing waves with the quiescent surface in the normalised x direction are: 1st symmetric, 0.44092197; 2nd asymmetric, 0.61167593; 2nd symmetric, 0.23494167 and 0.70241105; 3rd asymmetric, 0.39048359 and 0.7587375.
the Ausiron furnace\textsuperscript{13} which allows the positioning of two top submerged lances with a slight overlap of splashing during operation. It is also a common value chosen for the design of road oil tankers from the viewpoint of structural and economical optimisation, and those factors are applicable to the AusIron furnace design. Jeffreys\textsuperscript{2} also provides some data for comparison for $B/A=0.6$ and verification of the solutions. The results and analysis based on this single $B/A$ value can however be used to understand the behaviour in vessels of different ellipticity.

The shapes of the different wave modes governed by the minor axis are shown in the three dimensional contour plot in Fig. 5 for $B/A=0.6$. For the even values of $m$, the wave is anti-symmetric about the major and minor axes, while for odd values of $m$, the wave is symmetric about the minor axis and anti-symmetric about the major axis. $se_1$ is an asymmetric wave slopping from side to side in the minor axis direction. $se_2$ is anti-symmetric, where the tank is compartmented into quarters and symmetric on a plane 45° to the axes. The higher values of $m$ result in fairly complicated wave shapes.

Figure 6 shows the shapes of standing waves for the sum of the first three modes of the major axis and the first two modes of the minor axis. The standing wave shape is generally not only asymmetric but there is no plane of reflection or symmetry except when the $m$ value of the major and minor axis modes have the same integer. The standing wave shapes are much more complex and when they rotate around the vessel boundary, will result in shapes that show a number of maxima and minima for each revolution around the vessel. Higher wave modes can be plotted but become increasingly difficult to visualise.

4. Application

In Table 3, the periods of the different wave modes have been calculated for $B/A=0.5887$. A videotape of the standing wave formed in an elliptic cross section vessel with two swirled lances injecting air was provided by Ausmelt Limited through Swinburne University of Technology. This was used to compare with the theoretical predictions. The experimental elliptic vessel had $B/A=0.5887$ with $2A=0.535$ m. The water depth was 120 mm, lance submergence of 50 mm, and each lance positioned 110 mm from the centre of the major axis. The position of the lances could be varied from 110 to 150 mm from the centre of the major axis. The foci is 214 mm from the centre of the major axis. The annular swirled lances have an annulus with inner and outer diameters of 13 and 18 mm respectively. The annulus holds a four start swirler with a pitch angle of 57.5°. The gas flow rate in each lance was varied between 1 to 2 l/s and kept identical.

The video footage was captured at 50 Hz and deinterlaced to estimate the wave period. The wave period was found to be independent of the position in the bath chosen to estimate the wave period. Theoretical values of the different wave modes are given in Table 3 and calculated from the eigenvalues by

$$T = \frac{2\pi}{\sqrt{\alpha^2}} \quad \text{(20)}$$

where $\tanh(\xi h)=1$ as the bath depth was at least half or more than the wavelength dimension.

Figure 7 shows the standing waves formed with and without gas injection. The number of video frames measured between crest to crest was consistent, whether gas was injected or not, giving a value between 27 to 28 ($T=0.54-0.56$ s). This corresponds to the first symmetric standing wave as seen in Fig. 6. It is not possible to distin-
guish between the major or minor axis mode as the periods differ by only a fraction of a percent. During gas injection, the wave oscillated along the major axis and was sustained by the rising gas plume. The amplitude is higher when the wave converges to the centre as the gas plumes reinforce each other. When the standing wave moves to the vessel boundary, the gas plumes do not reinforce each other. Instead, spouts are formed individually at each lance resulting in the amplitude of the standing wave being smaller at the boundary. Nevertheless, the first symmetric mode will result in substantial refractory wear being localised to the opposite ends of the major axis but due to symmetry, the forces on the vessel wall are balanced out. This is in contrast to top submerged lancing in a circular cylinder (Ausmelt Limited’s top submerged gas injection process) where the first asymmetric standing wave is more likely to be formed resulting in force imbalance on the vessel.

When gas injection is stopped, the standing wave takes a minute or more to dissipate while rotating around the vessel boundary. Figures 7C–7F shows the maximum amplitudes observed and in between those times, the surface is much flatter but it is not completely symmetric or asymmetric. This suggest that the standing waves of close frequencies in the major and minor axes are interacting (see Fig. 6) but the variation in amplitude is not large enough to allow good visualisation of the interaction. The rotating standing wave in Figs. 7D–7F is similar in shape to Fig. 6D and the small out of symmetry may be due to the presence of more than two waves modes being present.

From Table 3, as one goes to higher wave modes, the periods between the standing waves in the major and minor axis become closer. The largest difference occur with the lowest mode in the minor axis pulsates 27% faster than that in the major axis. This falls to 11 and 5% for next two modes. If a large vessel is required for bath smelting, two lances may not be able to provide sufficient agitation and bath mixing for the reactions. In such a case, rows of multiple lances may be required. Under such circumstances, both the major and minor axis standing waves may be excited and coupling between wave modes will occur predominantly with the first two wave modes. Since they differ in frequencies, beat frequencies may occur. For example, the coupling of the first asymmetric standing wave based on the

Table 3. Theoretical periods corresponding to the experimental vessel, $B/A=0.5887$.  

| Wave mode | Eigenvalue, $\xi$ | Wave period (s) |
|-----------|------------------|-----------------|
| Major axis |                  |                 |
| 1st asymmetric | 1.86888 | 0.759 |
| 1st symmetric  | 3.39034 | 0.564 |
| 2nd asymmetric | 4.86171 | 0.470 |
| 3rd asymmetric | 7.71634 | 0.374 |
| Minor axis |                  |                 |
| 1st asymmetric  | 3.03742 | 0.595 |
| 1st symmetric   | 4.15792 | 0.509 |
| 2nd asymmetric  | 5.36704 | 0.448 |
| 3rd symmetric   | 6.62756 | 0.403 |

Fig. 6. The shapes of the standing waves due to the addition of wave modes in the major and minor axes, the amplitude is arbitrarily scaled. A. $ce_1 + se_1$, B. $ce_1 + se_2$, C. $ce_2 + se_1$, D. $ce_2 + se_2$, E. $ce_3 + se_1$, F. $ce_3 + se_2$.  

(Ausmelt Limited’s top submerged gas injection process)
values in Table 3 gives beat frequencies of 3 and 0.4 Hz, compared to 1.3 and 1.7 Hz for the standing waves.

As the lances are positioned far from the foci, the results indicate that the standing wave periods are governed by the bath shape and size rather than the position of the lances. Trials with different lance submergences have shown that the wave periods are not affected except that higher air flow rates are needed at lower lance submergences to initiate standing waves. Varying the position of the lance along the major axis does not affect the wave periods, but it is limited to 110 to 150 mm from the centerline in this experimental setup. The wave period increases as the wave amplitude increases but this was not pursued as the time resolution using a video camera is inadequate to resolve the increase in period accurately.

5. Conclusion

The analytical solution of the standing wave modes in an elliptical cross section cylindrical vessel was presented. The eigenvalues of the first five modes in the major and minor axes were obtained and fitted to Chebyshev polynomials to allow easy calculation for engineering purposes. The results obtained were compared to a set of experimental results with good agreement. The analysis showed that the wear will occur at the ends of the major axis when a standing wave is formed in a bath with two top submerged lances. The use of larger vessels and more tuyeres must be carefully considered as the elliptical cylinder has multiple wave modes that may be excited in the major and minor axes. Coupling of these wave modes results in beat frequencies, higher and lower than that of the standing wave frequencies and care must be taken to ensure that these frequencies do not resonate with the frequencies of other structural components in the vessel.

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Nomenclature

\[ a : \text{ Foci distance (m)} \]
\[ A : \text{ Major axis distance (m)} \]
\[ A_k : \text{ Arbitrary constant} \]
\[ B : \text{ Minor axis distance (m)} \]
\[ B : \text{ Arbitrary constant} \]
\[ c : \text{ Wave velocity} \]
\[ c_k : \text{ Chebyshev coefficients} \]
\[ C_m : \text{ Arbitrary constants} \]
\[ \cos(z, q) : \text{ Cosine elliptic function} \]
\[ \cosh(z, q) : \text{ Cosh elliptic function} \]
\[ \sin(z, q) : \text{ Sine elliptic function} \]
\[ \sinh(z, q) : \text{ Sinh elliptic function} \]
\[ t : \text{ Time (s)} \]
\[ U(u) : \text{ Function of } u \text{ only} \]
\[ V(v) : \text{ Function of } v \text{ only} \]
Greek symbols

\( \delta \): Surface height (m)
\( \nu \): Kinematic viscosity \((\text{m}^2/\text{s})\)
\( \pi \): Liquid potential \((\text{m}^2 \cdot \text{s}^{-1})\)
\( \xi \): Characteristic value
\( \omega_{mn} \): Oscillation frequency corresponding to \( q_{mn} \)

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