Choice number and energy of graphs

Saieed Akbari∗, Ebrahim Ghorbani

Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11365-9415, Tehran, Iran
Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran

Received 14 August 2007; accepted 30 November 2007
Available online 25 January 2008
Submitted by R.A. Brualdi

Abstract

The energy of a graph $G$, denoted by $E(G)$, is defined as the sum of the absolute values of all eigenvalues of the adjacency matrix of $G$. It is proved that $E(G) \geq 2(n - \chi(G)) \geq 2(\text{ch}(G) - 1)$ for every graph $G$ of order $n$, and that $E(G) \geq 2\text{ch}(G)$ for all graphs $G$ except for those in a few specified families, where $G$, $\chi(G)$, and $\text{ch}(G)$ are the complement, the chromatic number, and the choice number of $G$, respectively.

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AMS classification: 05C15; 05C50; 15A03

Keywords: Energy; Choice number

1. Introduction

All the graphs that we consider in this paper are finite, simple and undirected. Let $G$ be a graph. Throughout this paper the order of $G$ is the number of vertices of $G$. If $\{v_1, \ldots, v_n\}$ is the set of vertices of $G$, then the adjacency matrix of $G$, $A = [a_{ij}]$, is an $n \times n$ matrix where $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and $a_{ij} = 0$ otherwise. Thus $A$ is a symmetric matrix with zeros on the diagonal, and all the eigenvalues of $A$ are real and are denoted by $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$. By the eigenvalues of $G$ we mean those of its adjacency matrix. The energy $E(G)$ of a graph $G$ is defined as the sum of the absolute values of all eigenvalues of $G$, which is twice the sum of the positive eigenvalues since the sum of all the eigenvalues is zero. For a survey on the energy of graphs, see [7].

∗ Corresponding author. Address: Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11365-9415, Tehran, Iran.

E-mail addresses: s_akbari@sharif.edu (S. Akbari), e_ghorbani@math.sharif.edu (E. Ghorbani).

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doi:10.1016/j.laa.2007.11.028
For a graph $G$, the chromatic number of $G$, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices of $G$ so that no two adjacent vertices have the same color. Suppose that to each vertex $v$ of a graph $G$ we assigned a set $L_v$ of $k$ distinct elements. If for any such assignment of sets $L_v$ it is possible, for each $v \in V(G)$, to choose $\ell_v \in L_v$ so that $\ell_u \neq \ell_v$ if $u$ and $v$ are adjacent, then $G$ is said to be $k$-choosable. The choice number $ch(G)$ of $G$ is the smallest $k$ such that $G$ is $k$-choosable.

We denote by $A_{n,t}$, $1 \leq t \leq n - 1$, the graph obtained by joining a new vertex to $t$ vertices of the complete graph $K_n$. If we add two pendant vertices to a vertex of $K_n$, the resulting graph has order $n + 2$ and we denote it by $B_n$.

In [1], it is proved that apart from a few families of graphs, $E(G) \geq 2 \max(\chi(G), n - \chi(G))$ (see the following theorem). Our goal in this paper is to extend this result to the choice number of graphs.

**Theorem A.** Let $G$ be a graph. Then $E(G) < 2\chi(G)$ if and only if $G$ is a union of some isolated vertices and one of the following graphs:

(i) the complete graph $K_n$;
(ii) the graph $B_n$;
(iii) the graph $A_{n,t}$ for $n \leq 7$, except when $(n, t) = (7, 4)$, and also for $n \geq 8$ and $t \in \{1, 2, n - 1\}$;
(iv) a triangle with two pendant vertices adjacent to different vertices.

The following is our main result.

**Theorem 1.** Let $G$ be a graph. Then $E(G) < 2ch(G)$ if and only if $G$ is a union of some isolated vertices and one of the following graphs:

(i)–(iv) as in Theorem A;
(v) the complete bipartite graph $K_{2,4}$.

2. Proofs

In this section, we present a proof for Theorem 1. To do so we need some preliminaries.

A well-known theorem of Nordhaus and Gaddum [8] states that for every graph $G$ of order $n$, $\chi(G) + \chi(G) \leq n + 1$. This inequality can be extended to the choice number. The graphs attaining equality are characterized in [3]. It is proved that there are exactly three types of such graphs defined as follows:

- A graph $G$ is of type $F_1$ if its vertex set can be partitioned into three sets $S_1, T, S_2$ (possibly, $S_2 = \emptyset$) such that $S_1 \cup S_2$ is an independent set of $G$, every vertex of $S_1$ is adjacent to every vertex of $T$, every vertex of $S_2$ has at least one non-neighbor in $T$, and $|S_1|$ is sufficiently large that the choice number of the induced subgraph on $T \cup S_1$ is equal to $|T| + 1$. This implies that $ch(G) = |T| + 1$ also. Theorem 1 of [6] states that if $T$ does not induce a complete graph, then $|S_1| \geq |T|^2$; we will use this result later.
- A graph is of type $F_2$ if it is the complement of a graph of type $F_1$.
- A graph is of type $F_3$ if its vertex set can be partitioned into a clique $K$, an independent set $S$, and a five-cycle $C$ such that every vertex of $C$ is adjacent to every vertex of $K$ and to no vertex of $S$.
Theorem B. (a) \([4]\) \(\text{ch}(G) + \text{ch}(\overline{G}) \leq n + 1\) for every graph \(G\) of order \(n\).

(b) \([3]\) Equality holds in (a) if and only if \(G\) is of type \(F_1, \overline{F}_1\) or \(F_2\).

Lemma 1. For every graph \(G\) of order \(n\),
\[
E(G) \geq 2(n - \chi(\overline{G})) \geq 2(n - \text{ch}(\overline{G})) \geq 2\text{ch}(G) - 1.
\]

Proof. As remarked in \([1]\), the first inequality follows from Theorem 2.30 of \([5]\), which states that \(n - \chi(\overline{G}) \leq \lambda_1(G) + \cdots + \lambda_{\chi(\overline{G})}(G)\). The second inequality holds because \(\text{ch}(G) \geq \chi(G)\) for every graph \(G\), and the third inequality holds by Theorem B(a). \(\square\)

Lemma 2. For every graph \(G\), \(\text{ch}(G) \leq \lambda_1(G) + 1\).

Proof. Wilf (\([9]\), see also \([2, p. 90]\)) proved that every graph \(G\) has a vertex with degree at most \(\lambda_1(G)\), and so does every induced subgraph of \(G\). He deduced from this that \(\chi(G) \leq \lambda_1(G) + 1\), and the same argument also proves that \(\text{ch}(G) \leq \lambda_1(G) + 1\). \(\square\)

Lemma 3. Suppose \(G\) has \(2K_2\) as an induced subgraph. Then \(E(G) \geq 2\text{ch}(G)\).

Proof. By the Interlacing Theorem (Theorem 0.10 of \([2]\)), \(\lambda_2(G) \geq \lambda_2(2K_2) = 1\), and so \(E(G) \geq 2(\lambda_1(G) + \lambda_2(2K_2)) \geq 2(\lambda_1(G) + 1) \geq 2\text{ch}(G)\) by Lemma 2. \(\square\)

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let \(G\) be a graph such that \(E(G) < 2\text{ch}(G)\). We may assume that \(G\) has at least one edge, since otherwise \(G\) is the union of some isolated vertices and \(K_1\), which is permitted by (i) of Theorem 1. Since removing isolated vertices does not change the value of \(E(G)\) or \(\text{ch}(G)\), we may assume that \(G\) has no isolated vertices. If \(\text{ch}(G) + \text{ch}(\overline{G}) \leq n\), then \(E(G) \geq 2\text{ch}(G)\) by Lemma 1; this contradiction shows that \(\text{ch}(G) + \text{ch}(\overline{G}) = n + 1\), which means that \(G\) has one of the types \(F_1, \overline{F}_1\) and \(F_2\) by Theorem B(b). We consider these three cases separately.

Case 1. \(G\) has type \(F_1\). Then \(G\) has \(G[T] \vee K_k\) as an induced subgraph, where \(G[T]\) is the subgraph induced by \(G\) on \(T, k = |S_1|\), and \(\vee\) denotes ‘join’. Let \(|T| = t\), so that \(\text{ch}(G) = t + 1\). If \(G[T]\) is a complete graph, then \(\chi(G) = t + 1 = \text{ch}(G)\), so that \(E(G) < 2\chi(G)\) and \(G\) is one of the graphs listed in Theorem A. So we may assume that \(G[T]\) is not a complete graph. In this case, as remarked after the definition of type \(F_1\), \(k = |S_1| \geq |T|^2 \geq t^2\). Thus
\[
\lambda_1(G[T] \vee K_k) \geq \lambda_1(K_{t,t^2}) = t\sqrt{t} \geq t + 1,
\]
provided \(t \geq 3\); since \(\text{ch}(G) = t + 1\), we have \(E(G) \geq 2\text{ch}(G)\). So we may assume that \(t \leq 2\), when \(G[T] = K_2\) and \(k \geq t^2 = 4\). For \(k \geq 5\), we have \(\lambda_1(K_{2,k}) \geq \sqrt{\frac{k}{10}} > 3 = \text{ch}(K_{2,k})\), thus \(E(G) \geq 2\text{ch}(G)\). So we may assume that \(k = 4\). If \(G \neq K_{2,4}\), then either \(|S_1| \geq 5\) or \(|S_2| > 0\); thus \(G\) has either \(K_{2,5}\) or \(H\) as an induced subgraph, where \(H\) is formed from \(K_{2,4}\) by adding an extra vertex joined to one of the vertices of degree 4. We have \(E(K_{2,5}) = 2\sqrt{10} > 6\). The graph \(H\) has a \(P_4\) as an induced subgraph so \(\lambda_2(H) > \lambda_2(P_4) > 0.6\). On the other hand \(\lambda_1(H) \geq \lambda_1(K_{2,4}) = 2\sqrt{2}\). Therefore, \(E(H) > 2(2\sqrt{2} + 0.6) > 6\). Hence \(E(G) > 6 = 2\text{ch}(G)\) if \(G \neq K_{2,4}\). Therefore, \(G = K_{2,4}\).

Case 2. \(G\) has type \(\overline{F}_1\). So \(\overline{G}\) is of type \(F_1\) with the associated partition \(\{S_1, T, S_2\}\). Let \(t = |T|\) and \(k = |S_1|\). If \(G[T]\) is not a complete graph, then \(k \geq t^2 > 1\) as in Case 1; hence \(G\) has \(2K_2\) as an induced subgraph, which gives a contradiction by Lemma 3. So \(G[T]\) is a complete graph.
Let $J$ be the set of those vertices of $T$ that are adjacent to all vertices of $S_2$ in $G$. Let $v$ be a vertex of $S_1$. Then $G$ is a graph of type $F_1$ with the associated partition $\{S_1', T', S_2'\}$, in which

$$
S_1' = \{v\}, \quad T' = S_2 \cup (S_1 \setminus \{v\}), \quad S_2' = T, \quad \text{if } k \geq 2;
$$

$$
S_1' = J \cup \{v\}, \quad T' = S_2, \quad S_2' = T \setminus J, \quad \text{if } k = 1.
$$

Therefore, the result follows by Case 1.

**Case 3.** $G$ has type $F_2$. Thus $G$ has a 5-cycle as an induced subgraph. So $\lambda_2(G) + \lambda_3(G) \geq \lambda_2(C_5) + \lambda_3(C_5) > 1$. Hence, by Lemma 2, we obtain

$$
E(G) \geq 2(\lambda_1 + \lambda_2 + \lambda_3) > 2(1 + \lambda_1) \geq 2\text{ch}(G). \quad \square
$$

**Acknowledgments**

The authors are indebted to the Institute for Studies in Theoretical Physics and Mathematics (IPM) for support; the research of the first author was in part supported by a grant from IPM (No. 86050212). They are also grateful to the referee for her/his helpful suggestions.

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