Applications of rational difference equations to spectral graph theory

Elismar R. Oliveira \( ^a \) and Vilmar Trevisan \( ^{a,b} \)

\( ^a \)Instituto de Matemática e Estatística, UFRGS – Universidade Federal do Rio Grande do Sul, Porto Alegre, Brazil; \( ^b \)Department of Mathematics and Applications, University of Naples Federico II, Napoli, Italy

ABSTRACT
We study a general class of recurrence relations that appear in the application of a matrix diagonalization procedure. We find a general closed formula and determine the analytical properties of the solutions. We finally apply these findings in several problems involving eigenvalues of graphs.

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1. Introduction

The main goal of this paper is to consider a general class of rational type recurrences appearing in graph applications, mainly in eigenvalue location, as a unified elementary form,

\[ x_{j+1} = \varphi(x_j), \quad j \geq 1 \tag{1} \]

where \( \varphi(t) = \alpha + \frac{\gamma}{t} \), for \( t \neq 0, \alpha, \gamma \in \mathbb{R} \) are fixed numbers \( (\gamma \neq 0) \) and \( x_1 \) is a given initial condition. The explicit solution is a function \( j \rightarrow f(j) \) such \( x_j = f(j) \) for \( j \geq 1 \).

Perhaps this study is interesting per se, but we explain now the motivation behind these recurrences that may be used for diagonalizing certain symmetric matrices and their important applications in graph theory.

For a real \( n \times n \) symmetric matrix \( M = [m_{ij}] \), we consider the graph of \( M \), as the graph with \( n \) vertices \( 1, 2, \ldots, n \) where an edge between \( i \) and \( j \) exists if and only if \( m_{ij} \neq 0 \). We may see \( m_{ij} \) as the weight of the edge \( ij \), whereas \( m_{ii} \), the diagonal element, as the weight of the vertex \( i \). Diagonalizing the matrix \( M \) has important applications in numerical linear algebra in general. Our focus here are applications we will describe later on in spectral graph theory. In particular, if the graph of \( M \) is a tree \( T \), the diagonalization algorithm of
Jacobs-Trevisan [8] (J-T algorithm, for short) provides an efficient and ingenious procedure that can be executed on the tree itself. We provide the J-T algorithm in Figure 1 for easy reference.

If we start the algorithm with a value $\alpha$, the algorithm computes a diagonal matrix $D_\alpha$ that is congruent to $M - \alpha I$, where the diagonal values are stored as the weight of the vertices. By the Silvester Law of Inertia, the inertia of $D_\alpha$ and $M - \alpha I$ are the same. This tells us that the number of (positive/negative/zero) final values of the vertices indicates the number of eigenvalue of $M$ that are (greater than/smaller than/equal to) $\alpha$. We call this an eigenvalue location algorithm.

The J-T algorithm works as follows. For a given $\alpha \in \mathbb{R}$, we initialize the weights of the vertices as $a(v_i) := m_{i,i} - \alpha$. For a child $v_j$ of $v_i$, if the value of the diagonal entry is $a(v_j) \neq 0$ and the correspondent value in the positions $(i,j)$ and $(j,i)$ is $w$ then we can annihilate the nondiagonal elements of row and column $j$ using $a(v_j)$ while the value of the diagonal entry $a(v_i)$ is replaced by

$$a(v_i) = \alpha_i - \frac{w^2}{a(v_j)}$$

where $\alpha_i = m_{i,i}$ is the diagonal value in the position $(i,i)$. When the value $a(v_j) = 0$, the algorithm does some other procedure (see Figure 1).

The information provided by the signs of the final values turns out to be of paramount importance for several applications in spectral graph theory. To be more concrete, and understand why the sequences appear, we consider three cases that will be prototypical to our studies.

- Index or spectral radius of a tree $T$. This is the largest eigenvalue of the adjacency matrix of $T$, let us denote it by $\lambda$. We consider applying the J-T algorithm in a pendant path, $M$ is the adjacency matrix of $T$ and we initialize the diagonal with $0 - \lambda$ ($M_{i,i} = 0$ and

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Input: scalar $\alpha$; matrix $M$ of a tree $T$ with vertices $v_1, v_2, \ldots, v_n$, in postorder
Output: diagonal matrix $\Lambda$ congruent to $M + \alpha I$

Initalize $a(v_i) := m_{i,i} - \alpha$, for all $v_i$ of $T$.
for $k = n$ to $1$ do
  if $v_k$ is not a leaf, then
    1. if $a(v_i) \neq 0$ for all children $v_i$ of $v_k$, then
       $a(v_k) \leftarrow a(v_k) - \sum_{v_i} \frac{(m_{ik})^2}{a(v_i)}$.
    2. if $a(v_i) = 0$ for some children $v_i$ of $v_k$, then
       choose a vertex $v_j$ that $a(v_j) = 0$;
       $a(v_k) \leftarrow -\frac{(m_{jk})^2}{2}; \ a(v_j) \leftarrow 2;
       $ if $v_k$ has a parent $v_\ell$, then remove the edge $\{v_k, v_\ell\}$.
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Figure 1. The J-T algorithm, Diagonalize$(T, \alpha)$.
\( \alpha = \lambda \). The nonzero entries are equal to 1. Starting in the end vertex of the path one obtains \( z_1 = -\lambda, z_2 = -\lambda - \frac{1}{z_1} = -\lambda - \frac{1}{z_1} \) and so on. Therefore, we have a recursion

\[
\begin{aligned}
z_1 &= -\lambda \\
z_{j+1} &= -\lambda - \frac{1}{z_j}, \quad j \geq 1.
\end{aligned}
\] (2)

A careful analysis of this recurrence relation (with different initial conditions), has been used to compare indices in classes of trees, enabling one to solve combinatorial/algebraic problems where traditional techniques had failed (see, for example [2,11,12]).

- **Average of the Laplacian eigenvalues of a tree** \( T \). We initialize the diagonal with degree \((v_i) - d\) where \( d = 2 - \frac{2}{n} \) (\( M_{i,i} = \text{degree}(v_i) \) and \( \alpha = d \) is the average of the Laplacian eigenvalues). The correspondent entries are equal to -1. Starting in the end vertex of a path one obtains \( a_1 = 1 - d = -1 + \frac{2}{n}, a_2 = 2 - d - \frac{(-1)^2}{a_1} = \frac{2}{n} - \frac{1}{a_1} \) and so on. Therefore, we have a recursion

\[
\begin{aligned}
a_1 &= -1 + \frac{2}{n} \\
a_{j+1} &= \frac{2}{n} - \frac{1}{a_j}, \quad j \geq 1.
\end{aligned}
\] (3)

This recurrence relation, used in an ingenious way, was the main tool to prove that at least half of the Laplacian eigenvalues of a tree are smaller than its average [9].

- **An eigenvalue** \( \lambda \in [0,2] \) of the normalized Laplacian of a tree \( T \). We initialize the diagonal with \( \frac{1}{\sqrt{\deg(v_i) \deg(v_j)}} \) which is always equal to \( \frac{1}{\sqrt{\deg(v)}} \) in the end of a path and \( \frac{1}{\sqrt{2}} = -\frac{1}{2} \) for the next vertices. Starting in the end vertex of a path one obtain \( x_0 = 1 - \lambda, x_1 = 1 - \lambda - \frac{(\frac{1}{x_0})^2}{x_0} = 1 - \lambda - \frac{1}{2}, x_2 = 1 - \lambda - \frac{(\frac{1}{x_1})^2}{x_1} = 1 - \lambda - \frac{1}{4}, x_3 = 1 - \lambda - \frac{(\frac{1}{x_2})^2}{x_2} = 1 - \lambda - \frac{1}{4} \) and so on. Therefore, we have a recursion

\[
\begin{aligned}
x_0 &= 1 - \lambda \\
x_1 &= 1 - \lambda - \frac{1}{2(1 - \lambda)} \\
x_{j+1} &= 1 - \lambda - \frac{1}{4x_j}, \quad j \geq 1.
\end{aligned}
\] (4)

As these examples show, applying the J-T algorithm on a path of a tree, produces certain numerical rational sequences. They are all of the general form given by Equation (1).

The main purpose of this paper is to understand the analytical behaviour of these sequences \( x_j \) when \( j \) is seen as a continuous variable, which allows one to obtain information about the sign on the discrete values of \( j \). The passage from natural numbers, representing the vertices of a graph, to arbitrary real numbers allows one to find solutions of analytical equations which will define intervals of indices where a certain property is true. By applying the general results to particular cases, we also demonstrate the potential of this technique in applications to spectral graph theory. For example, we determine limit points of spectral radius of certain trees and study and analyze the dependence of the solutions and the initial
conditions on a given parameter, applying to study the number of eigenvalues smaller than
its average.
In some cases, the applications we made require to study the dependence of the solution
with respect to a given parameter $a$ that may appear both in the initial condition and/or in
the recurrence formula.
Another aspect that we investigate and use in applications, is the extension properties.
Here we mean to extend the integer variable $j$ to values in $\mathbb{R}$ and to use the structure of the
resulting function to derive properties of the original sequence.
In several cases, the dynamical behaviour of the function $\varphi(t) = \alpha + \frac{\gamma}{t}$ brings us a lot of
information about the correspondent sequence. The knowledge of these properties allows
one to study some important problems in spectral graph theory regarding eigenvalue
location.
The paper is organized as follows. Next section presents a reduction method allowing
us to transform the general recurrence given by (1) into a second order linear recurrence,
which in turn, leads to a closed formula for the solution. In Section 3, we present general
properties of solutions given in Section 2. The next two sections present new applications in
spectral graph theory. Section 4 deals with limit points of spectral radius of graphs and uses
the solutions to prove two known results. The rational is to show the potential use of these
techniques in this area. Section 5 studies the dependence of the solutions and the initial
conditions on a given parameter. We also use a particular example - study the number of
eigenvalues smaller than its average – to show how an analytical approach may be used to
obtain very precise results. We finalize the paper with some concluding remarks and open
problems.

2. The reduction method
Before reducing equation $x_{j+1} = \varphi(x_j)$, where $\varphi(t) = \alpha + \frac{\gamma}{t}$, $\alpha, \gamma \in \mathbb{R}$, into a second-order linear recursion, we will discuss briefly two general properties.

2.1. Zeroes of $\varphi$
The first problem we face is to find the zeroes of $\varphi$. If $x_j = 0$ for some $j$, then we can not compute $x_{j+1}$. We observe that in the applications to the J-T algorithm, this means that the initial value $\alpha$ is in the spectrum, while there is a procedure to choose the next term. Therefore, the infinite orbit $(x_j)_{j \geq 1}$ are the complement of the pre-images of zero by $\varphi$. Hence, $(x_j)$ is finite if, and only if, $x_1 = \varphi^{-n}(0)$ for some $n \in \mathbb{N}$. We recall that $\psi(t) = \frac{\gamma}{t-\alpha}$, for $t \neq \alpha$, is the inverse of $\varphi(t) = \alpha + \frac{\gamma}{t}$. Therefore the initial conditions with finite orbit are the points $\{\psi^n(0) \mid n \geq 1\}$.
Evaluating $\psi(t)$ we obtain
\[
\left\{ \psi(0) = -\frac{\gamma}{\alpha}, \, \psi^2(0) = -\frac{\gamma}{\alpha+\alpha}, \ldots \right\}.
\]

Example 2.1: Considering $\alpha = 2$ and $\gamma = -1$, we have $\varphi(t) = 2 - \frac{1}{t}$ and $\psi(t) = \frac{1}{2-t}$. It is easy to see that the sequence $\{\psi(0), \psi^2(0), \ldots\}$ is the sequence $\left\{\frac{n-1}{n}\right\}_{n \geq 1}$. Thus, if we take $x_1 = 3/4$ (that is, $n = 4$) we get $x_2 = 2/3, x_3 = 1/2$ and $x_4 = 0$, so we can not compute
This is a powerful information: as \( \frac{n-1}{n} \) \( \rightarrow \) 1 increasingly, for any initial condition \( x_1 \) outside of a neighbourhood of 1 we can iterate \( \varphi \) to any order, except for a finite set of numbers of the form \( x_1 = \frac{n-1}{n}, n \leq n_0 \).

### 2.2. Fixed points of \( \varphi \)

Regarding the infinite orbits \((x_j)_{j \geq 1}\) the relevant question are about the accumulation points, that is, the asymptotic behaviour of \( \varphi \). As \( \varphi \) is piecewise monotonous, we notice that such a limit, if it exists, must be some fixed point of \( \varphi \).

Evaluating \( \varphi(t) = \alpha + \frac{\gamma}{t} = t \) we see that the only possible zeroes are the solutions of

\[
    t^2 - \alpha t - \gamma = 0. \tag{5}
\]

which is the characteristic equation of the auxiliary recursion \( c_j \) (see (8) below).

Therefore the solution is

\[
    t = \frac{\alpha}{2} \pm \frac{1}{2} \sqrt{\alpha^2 + 4\gamma}. \tag{6}
\]

In the analysis of the equivalent linear second order recurrence we make below, where the characteristic polynomial appears, the fixed point are important.

### 2.3. Reduction

Dealing with the non-linear recursion (1) one can apply a reduction method to transform \( x_{j+1} = \varphi(x_j) \) into a linear equation. Given the nature of the rational functions, we propose to consider an auxiliary sequence \( c_j \neq 0 \) and try to find \( x_j \) in the form

\[
    x_j = \frac{c_{j+1}}{c_j}, \quad j \geq 1. \tag{7}
\]

Substituting (7) in (1) we obtain

\[
    c_{j+2} - \alpha c_{j+1} - \gamma c_j = 0. \tag{8}
\]

### 2.4. Analysis of a second-order linear recursion

For basic results on difference equations we refer the book [4], but we recall here some facts about a second-order linear recursion such as (8),

\[
    c_{j+2} + Uc_{j+1} + Vc_j = 0 \tag{9}
\]

where \( U, V \in \mathbb{R} \) are fixed numbers. Additionally, we assume that (9) is irreducible, that is, \( V \neq 0 \). We assign to that the characteristic equation \( \theta^2 + U\theta + V = 0 \) which has three possible solutions:

(a) Only one real root \( \theta \). In this case, the solution of (general second-order linear recursion) is

\[
    c_j = (A + jB)\theta^j \tag{10}
\]
(b) Two real roots $\theta, \theta'$. In this case, the solution of (general second-order linear recursion) is

$$c_j = A\theta^j + B(\theta')^j$$  \hspace{1cm} (11)

(c) Two conjugated complex roots $\rho e^{\pm i\phi}$. In this case the solution of (general second-order linear recursion) is

$$c_j = \rho^j(A \cos(j\phi) + B \sin(j\phi))$$  \hspace{1cm} (12)

Denoting $\Delta = \alpha^2 + 4\gamma$ we can find an explicit formula in each case.

Case 1: $\Delta = 0$

In this case the only root is $\theta = \frac{\alpha}{2}$. Using (10) we obtain $c_j = (A + jB)\theta^j$. If $B = 0$ then $c_j = A\theta^j$ and $x_j = \frac{c_{j+1}}{c_j} = \theta$ is the trivial solution, which is obvious because $\varphi(\theta) = \theta$. Thus we can suppose $B \neq 0$ and obtain

$$x_j = \frac{c_{j+1}}{c_j} = \frac{(A + (j + 1)B)\theta^{j+1}}{(A + jB)\theta^j} = \theta \frac{(A + jB) + B}{(A + jB)} = \theta \left(1 + \frac{1}{(A/B + j)}\right).$$

As $B \neq 0$ we obtain the formula for non trivial solutions

$$x_j = \theta \left(1 + \frac{1}{(\beta + j)}\right)$$  \hspace{1cm} (13)

where $\beta := (\frac{2\theta - x_1}{x_1 - \theta}) \in \mathbb{R}$ is defined by the initial point $x_1$.

Case 2: $\Delta > 0$

In this case the two real roots are $\theta = \frac{\alpha}{2} - \frac{1}{2}\sqrt{\alpha^2 + 4\gamma}$ and $\theta' = \frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 + 4\gamma}$. Using (11) we obtain $c_j = A\theta^j + B(\theta')^j$ and

$$x_j = \frac{c_{j+1}}{c_j} = \frac{A\theta^{j+1} + B(\theta')^{j+1}}{A\theta^j + B(\theta')^j}.$$

As $\gamma \neq 0$ we know that $\theta \neq 0$ and $\theta' \neq 0$. Also, $A$ and $B$ can not be simultaneously zero. If $B = 0$ then $c_j = A\theta^j$ and $x_j = \frac{c_{j+1}}{c_j} = \theta$ is the trivial solution, which is obvious because $\varphi(\theta) = \theta$. Thus we can suppose $B \neq 0$ and divide the formula by $B(\theta')^j$ obtaining

$$x_j = \frac{A/B \theta \left(\frac{\theta'}{\varphi}\right)^j + \theta'}{A/B \left(\frac{\theta'}{\varphi}\right)^j + 1} = \frac{\theta \beta \left(\frac{\theta'}{\varphi}\right)^j + \theta'}{\beta \left(\frac{\theta'}{\varphi}\right)^j + 1} = \theta \left(\beta \left(\frac{\theta'}{\varphi}\right)^j + 1 - 1\right) + \theta'.$$

Therefore the explicit equation for other than the trivial solution $x_j = \theta$ is

$$x_j = \theta + \frac{\theta' - \theta}{\beta \left(\frac{\theta'}{\varphi}\right)^j + 1}$$  \hspace{1cm} (14)

where $\beta := (\frac{\theta'}{\varphi}) \left(\frac{\theta' - x_1}{x_1 - \theta}\right) \in \mathbb{R}$ is well defined by the initial point $x_1 \neq \theta$. Notice that the other trivial solution $x_j = \theta'$ is obtained from $\beta = 0$ (or equivalently $A = 0$).

Case 3: $\Delta < 0$
Since $\alpha^2 + 4\gamma < 0$ we obtain $\gamma < -\frac{1}{4}\alpha^2 \leq 0$ thus $\varphi(\theta) = \theta$ has no solution. In the case $\alpha \neq 0$ we have two complex roots are $Z = \frac{\alpha}{2} + i\frac{1}{2}\sqrt{-\alpha^2 - 4\gamma}$ and $\bar{Z} = \frac{\alpha}{2} - i\frac{1}{2}\sqrt{-\alpha^2 - 4\gamma}$. Using (12) we obtain $c_j = \rho^j (A \cos(j\phi) + B \sin(j\phi))$ where

$$\rho = \sqrt{\left(\frac{\alpha}{2}\right)^2 + \left(\frac{1}{2}\sqrt{-\alpha^2 - 4\gamma}\right)^2} = \sqrt{-\gamma},$$

$$\phi = \arctan\left(\frac{\sqrt{-\alpha^2 - 4\gamma}}{\alpha}\right) \text{ if } \alpha > 0,$$

$$\phi = \arctan\left(\frac{\sqrt{-\alpha^2 - 4\gamma}}{\alpha}\right) + \pi \text{ if } \alpha < 0$$

considering the branch $(-\frac{\pi}{2}, \frac{\pi}{2})$ of the function $\tan$, and

$$x_j = \frac{c_{j+1}}{c_j} = \frac{\rho^{j+1}(A \cos((j + 1)\phi) + B \sin((j + 1)\phi))}{\rho^j(A \cos(j\phi) + B \sin(j\phi))}$$

$$= \rho \frac{A}{\sqrt{A^2 + B^2}} \cos((j + 1)\phi) + \frac{B}{\sqrt{A^2 + B^2}} \sin((j + 1)\phi)$$

$$= \rho \frac{\sqrt{A^2 + B^2}}{\sqrt{A^2 + B^2}} \cos(j\phi) + \frac{B}{\sqrt{A^2 + B^2}} \sin(j\phi).$$

We denote $\omega \in [0, 2\pi)$ the angle such that $(\frac{A}{\sqrt{A^2 + B^2}}, -\frac{B}{\sqrt{A^2 + B^2}}) = (\cos(\omega), \sin(\omega))$. Using the addition formula we get

$$\frac{A}{\sqrt{A^2 + B^2}} \cos((j + 1)\phi) + \frac{B}{\sqrt{A^2 + B^2}} \sin((j + 1)\phi) = \cos((j + 1)\phi + \omega)$$

and

$$\frac{A}{\sqrt{A^2 + B^2}} \cos(j\phi) + \frac{B}{\sqrt{A^2 + B^2}} \sin(j\phi) = \cos(j\phi + \omega)$$

thus

$$x_j = \rho \frac{\cos((j + 1)\phi + \omega)}{\cos(j\phi + \omega)} = \rho \frac{\cos((j\phi + \omega) + \phi)}{\cos(j\phi + \omega)}$$

$$= \rho \frac{\cos(j\phi + \omega) \cos(\phi) - \sin(\phi) \sin(j\phi + \omega)}{\cos(j\phi + \omega)}.$$

Therefore the explicit solution for $\alpha \neq 0$ is

$$x_j = \rho \left( \cos(\phi) - \sin(\phi) \tan(j\phi + \omega) \right)$$

(15)

where $\omega := -\phi + \arctan(\frac{\cos(\phi) - (x_1/\rho)}{\sin(\phi)}) \in [0, 2\pi)$ is defined by the initial point $x_1$. The case $\alpha = 0$ does not appear in the graph applications and it is, in a certain way, trivial because $c_{j+2} - \alpha c_{j+1} - \gamma c_j = 0$ became $c_{j+2} = \gamma c_j$, a uncoupled equation. Taking $c_1 = 1$
and \( c_2 = x_1 \) \((x_1 = \frac{c_2}{c_1})\) we get \( c_{2k} = \gamma^{k-1} c_2 = \gamma^{k-1} x_1 \) and \( c_{2k+1} = \gamma^k c_1 = \gamma^k \) then

\[
x_j = \frac{c_{j+1}}{c_j} = \begin{cases} \frac{\gamma}{x_1}, & j = 2k \\ \frac{x_1}{\gamma}, & j = 2k + 1 \end{cases}, \quad j \geq 1.
\]

This is obviously the solution of \( x_{j+1} = \frac{\gamma}{x_j} \) for a given \( x_1 \neq 0 \).

Summarizing, we have proved the theorem below, which is well known in a general setting but is useful to have the explicit computations of the general formulas, given a prescribed initial condition, as we will see on the applications.

**Theorem 2.1:** Consider the rational first order difference equation \( x_{j+1} = \varphi(x_j), j \geq 1 \), where \( \varphi(t) = \alpha + \frac{\gamma}{t} \), for \( t \neq 0 \), \( \alpha, \gamma \in \mathbb{R} \) are fixed numbers \((\gamma \neq 0)\) and \( x_1 \) is a given initial condition. Then the general solution is one of three possibilities according the sign of \( \Delta = \alpha^2 + 4\gamma \):

**Type 1:** For \( \Delta = 0 \) the solution is

\[
x_j = \theta \left( 1 + \frac{1}{\beta + j} \right),
\]

where \( \theta = \frac{\alpha}{2} \) and \( \beta \in \mathbb{R} \) is defined by the initial point \( x_1 \neq \theta \) by the formula \( \beta = -1 + \frac{\theta}{x_1 - \theta} \). If \( x_1 = \theta \) then \( x_j = \theta, \forall j \geq 1 \) is the solution.

**Type 2:** For \( \Delta > 0 \) the solution is

\[
x_j = \theta + \frac{\theta' - \theta}{\beta \left( \frac{\theta'}{\theta} \right)^j + 1},
\]

where \( \theta = \frac{\alpha}{2} - \frac{1}{2} \sqrt{\alpha^2 + 4\gamma} \), \( \theta' = \frac{\alpha}{2} + \frac{1}{2} \sqrt{\alpha^2 + 4\gamma} \) and \( \beta \in \mathbb{R} \) is defined by the initial point \( x_1 \neq \theta \) by the formula \( \beta = \frac{\theta'}{\theta} \left( \frac{\theta'}{\theta} - 1 \right) \). If \( x_1 = \theta \) then \( x_j = \theta, \forall j \geq 1 \) is the solution.

**Type 3:** For \( \Delta < 0 \) the solution is

\[
x_j = \rho \left( \cos(\phi) - \sin(\phi) \tan(j\phi + \omega) \right),
\]

where \( \rho = \sqrt{-\gamma}, \phi = \arctan(\sqrt{-\frac{\alpha^2 + 4\gamma}{\alpha}}) \) if \( \alpha > 0, \phi = \arctan(\sqrt{-\frac{\alpha^2 + 4\gamma}{\alpha}}) + \pi \) if \( \alpha < 0 \) and \( \omega \in [0, 2\pi) \) is defined by the initial point \( x_1 \) by the formula \( \omega = -\phi + \arctan(\cot(\phi) - \frac{\Delta}{\rho} \csc(\phi)) \). If \( \alpha = 0 \) then

\[
x_j = \begin{cases} \frac{\gamma}{x_1}, & j = 2k \\ \frac{x_1}{\gamma}, & j = 2k + 1 \end{cases}, \quad j \geq 1
\]

is the solution of \( x_{j+1} = \frac{\gamma}{x_j} \) for a given \( x_1 \neq 0 \).

We analyze now a few properties of the solutions obtained that are useful for the applications we have in mind.
3. General properties of the solutions

By (1) we know that the rational recursion \( x_{j+1} = \varphi(x_j), \ j \geq 1 \) is defined by the rational map \( \varphi(t) = \alpha + \frac{\gamma t}{t-\alpha} \), for \( t \neq 0 \). We assumed that \( \gamma \neq 0 \) and the initial point must be \( x_1 \neq 0 \). We now give some properties of the solutions, some of them analytical.

3.1. Reversing initial conditions

In some cases we do not know what is \( x_1 \) but we know the value \( x_r \) for some \( r > 1 \). Usually we cannot reverse the role of the recursion \( x_{j+1} = \varphi(x_j) \) but it is not the case for \( \varphi \).

Lemma 3.1: Let \( \psi \) be the inverse function of \( \varphi \) and \( x_r \) a value of the solution for some \( r > 1 \). Then there exists a unique \( x_1 \) where \( y_j = \psi(x_j) \) and \( y_1 = x_r \).

Proof: We recall that \( \varphi(t) = \alpha + \frac{\gamma t}{t-\alpha} \), for \( t \neq \alpha \). As \( x_r \) is a value of the solution for some \( r > 1 \) we see that \( x_j \neq 0 \) for \( j = 1, \ldots, r - 1 \). Therefore, we can reverse the recursion

\[
x_r = \varphi(x_{r-1}) \Rightarrow x_{r-1} = \psi(x_r) = \psi(y_1),
\]

which means that \( x_{r-1} = y_2 \). Proceeding in this same way we obtain \( x_1 = y_r \). ■

As a consequence we obtain that if \( \alpha \) and \( \gamma \) are integers or rational numbers then \( x_j \neq 0 \) for all \( j \) provided that \( x_1 \) is an irrational number because the orbit of \( y_1 = 0 \) by \( \psi \) is a sequence of rational numbers \( \{ \psi(0), \psi^2(0) = \psi(\psi(0)), \psi^3(0) = \psi(\psi(\psi(0))) \ldots \} \).

Although, in applications when we consider a graph with \( n \) vertices, we just need to avoid \( \{ \psi(0), \psi^2(0), \ldots, \psi^n(0) \} \) as initial condition.

Example 3.1: Let \( \alpha = -\lambda \) and \( \gamma = -1 \) be fixed numbers such that \( \lambda > 2 \). We consider the associated recursion

\[
b_{j+1} = \varphi(b_j) = \alpha + \frac{\gamma}{b_j} = -\lambda - \frac{1}{b_j}
\]

and \( b_{r+2} = -\frac{2}{\lambda} < 0 \) for some fixed \( r \geq 4 \).

In this case, \( \Delta = \alpha^2 + 4\gamma = \lambda^2 - 4 > 0 \), therefore we have a Type 2 recursion, whose solution is

\[
b_j = \theta + \frac{\theta - \theta}{\beta (\frac{\theta}{\beta})^j + 1},
\]

where \( \theta = \frac{-\lambda}{2} - \frac{1}{2} \sqrt{\lambda^2 - 4} \) and \( \theta' = \frac{-\lambda}{2} + \frac{1}{2} \sqrt{\lambda^2 - 4} \). It is easy to see that \( \theta' + \theta = -\lambda \), \( \theta' - \theta = \sqrt{\lambda^2 - 4} \) and \( \theta \theta' = 1 \) thus

\[
b_j = \theta + \frac{\theta^{-1} - \theta}{\beta (\theta^2)^j + 1}.
\]

(16)
As \( b_{r+2} = -\frac{2}{r} < 0 \) we do not have \( b_1 \) available. Even in this case, substituting \( b_{r+2} = -\frac{2}{r} \) in (16), by Lemma 3.1, we can deduce that \( \beta = (\theta^2)^{-r-3} \) therefore

\[
b_j = \theta + \frac{\theta^{-1} - \theta}{(\theta^2)^{-r-3} + 1}
\]

is the explicit solution. An easy computation shows that \( b_1 = \theta + \frac{\theta^{-1} - \theta}{(\theta^2)^{-r-2} + 1} \).

### 3.2. Extended solutions

In our applications, the main concern is to predict when \( x_j \) is positive or negative. Eventually this will tell us the number of eigenvalues below or above the quantity in question. These values compose a discrete set, as \( j \) is a natural value. However, as Theorem 2.1 provides an explicit solution of \( x_{j+1} = \varphi(x_j) \), \( j \geq 1 \) that is defined for any \( j \in \mathbb{R} \) by \( x_j = f(j) \) for \( j \geq 1 \), we can take advantage of that. Except for vertical asymptotes the sign changes occurs when \( f(j) = 0 \). This property has been used in the proof of [9, Lemma 4.2]. We give an example that is more illustrative.

**Example 3.2:** When \( \alpha = 1 \) and \( \gamma = -\frac{1}{4} \) we obtain \( \Delta = 0 \) which is a Type 1 recursion, whose solution is

\[
x_j = f(j) = \theta \left( 1 + \frac{1}{(\beta + j)} \right),
\]

where \( \theta = \frac{\gamma}{2} = \frac{1}{2} \) and \( \beta \in \mathbb{R} \) is defined by the initial point \( x_1 \). Choosing \( x_1 = \frac{1}{2}(1 + \frac{1}{-5+\sqrt{2}}) \approx 0.36 \) in such way that \( \beta = -6 + \sqrt{2} \), we can have change of signal when \( \beta + j = 0 \) producing \( j = -\beta = 6 - \sqrt{2} \approx 4.58 \), a vertical asymptote of \( f \), or when \( 1 + \frac{1}{(\beta + j)} = 0 \) producing \( j = -\beta - 1 = 5 - \sqrt{2} \approx 3.58 \), a zero of the function \( f \). Neither of this values is an integer thus \( x_j \neq 0 \) for all \( j \).

From the general formula \( x_j = \frac{1}{2}(1 + \frac{1}{(\beta + j)} \). We know that \( \lim_{j \to \infty} x_j = \frac{1}{2} > 0 \).

By continuity of \( f \) we know that \( x_j > 0 \) for \( j \geq 5 \). \( \lim_{j \to -\infty} x_j = \frac{1}{2} > 0 \). By continuity of \( f \) we know that \( x_j > 0 \) for \( j \leq 3 \). Thus, the only possible negative term is \( x_4 \approx -0.35 \) because \( j = 4 \) is the only integer between 3.58 and 4.58.

### 3.3. Attracting and repelling sets

Given the sequence \( x_{j+1} = \varphi(x_j) \), \( j \geq 1 \) and a fixed point \( \theta \), that is, \( \varphi(\theta) = \theta \), the iteration of an initial condition contained in some interval \( I \) will be repelled from \( \theta \), resp. attracted to \( \theta \), according to whether \( |\varphi'| > 1 \) resp. \( |\varphi'| < 1 \), in \( I \).

\[
|x_{j+1} - \theta| = |\varphi(x_j) - \varphi(\theta)| = |\varphi'(\xi)| |x_j - \theta|,
\]

for some \( \xi \) between \( x_j \) and \( \theta \). Therefore

\[
\begin{align*}
|\varphi'(\xi)| &< 1, & \text{if } |x_{j+1} - \theta| < |x_j - \theta|, \\
|\varphi'(\xi)| &> 1, & \text{if } |x_{j+1} - \theta| > |x_j - \theta|.
\end{align*}
\]
As \( \varphi'(t) = -\frac{\gamma}{t^2} \) and \( \gamma \neq 0 \) we have two points \( \xi = \pm \sqrt{|\gamma|} \) where \( |\varphi'(\xi)| = 1 \). On the other points \( |\varphi'(\xi)| < 1 \) (\( t > \sqrt{|\gamma|} \)) or \( |\varphi'(\xi)| > 1 \) (\( -\sqrt{|\gamma|} < t < \sqrt{|\gamma|} \)).

The iterates \( x_{j+1} = \varphi(x_j), \ j \geq 1 \) not only are attracted to or are repelled from the fixed points but they do it in a monotone way because the sign of \( \varphi'(t) = -\frac{\gamma}{t^2} \) depends only on the sign of \( \gamma \).

**Example 3.3 ([2]):** Let \( \alpha = -\lambda \) and \( \gamma = -1 \) be fixed numbers such that \( \lambda > 2 \). We consider the associated recursion

\[
x_{j+1} = \varphi(x_j) = \alpha + \frac{\gamma}{x_j} = -\lambda - \frac{1}{x_j}
\]

and \( x_1 = -\lambda < 0 \).

In this case, \( \Delta = \alpha^2 + 4\gamma = \lambda^2 - 4 > 0 \), therefore we have a Type 2 equation whose solution is

\[
z_j = \theta + \frac{\theta' - \theta}{\beta (\theta')^j + 1},
\]

where \( \theta = -\frac{1}{2} - \frac{1}{2} \sqrt{\lambda^2 - 4} \) and \( \theta' = -\frac{1}{2} + \frac{1}{2} \sqrt{\lambda^2 - 4} \). It is easy to see that \( \theta' + \theta = -\lambda \), \( \theta' - \theta = \sqrt{\lambda^2 - 4} \) and \( \theta \theta' = 1 \) thus

\[
z_j = \theta + \frac{\theta^{-1} - \theta}{\beta (\theta^2)^j + 1}.
\]

As \( z_1 = -\lambda \) we can deduce that \( \beta = -1 \) therefore

\[
z_j = \theta - \frac{\theta^{-1} - \theta}{(\theta^2)^j - 1} \tag{19}
\]

is the explicit solution (Figure 2).

A quick examination shows that \( \theta^{-1} - \theta > 0 \) and \( \theta^2 > 1 \) because \( \lambda > 2 \). Thus \( z_j < \theta \) and \( z_j \to \theta \) increasingly when \( j \to \infty \) if \( z_1 < \theta \) which is the case for \( z_1 = -\lambda \).

**Remark 3.1:** We can summarize all the other possibilities for different values of \( z_1 \), when \( \alpha = -\lambda \) and \( \gamma = -1 \) be fixed numbers such that \( \lambda > 2 \), as follows. If \( \theta < z_1 < \theta' \) then \( z_j \to \theta \) decreasingly when \( j \to \infty \) by (18). If \( \theta' < z_1 < 0 \) then \( z_j \) is increasing by (18) and, in some point, \( z_j > 0 \). Finally, if \( z_1 > 0 \) then \( z_2 = -\lambda - \frac{1}{z_1} < -\lambda \) therefore \( z_j \to \theta \) increasingly when \( j \to \infty \).

### 3.4. Periodic behaviour

Type 3 solutions for \( \Delta < 0 \) are

\[
x_j = \rho \left( \cos(\phi) - \sin(\phi) \tan(j \phi + \omega) \right),
\]

where \( \rho = \sqrt{-\gamma}, \phi = \arctan\left(\frac{\sqrt{-\alpha^2 - 4\gamma}}{\alpha}\right) \) and \( \omega \in [0, 2\pi) \) is defined by the initial point \( x_1 \). We know that the tangent function is periodic with period \( \pi \) but \( \omega \) may be not 0. In this case we have a phase translation and the period is \( P = \frac{\pi}{\phi} \).
Figure 2. Function \( f(j) = \theta - \frac{\theta^{j-1}}{(\theta^2 - 1)^j} \) for \( \lambda = \sqrt{2 + \sqrt{5}} > 2 \). In this case, \( \theta \approx -1.27202, \theta' \approx -0.786145, \beta = -1 \). \( z_j = f(j) \to \theta \) increasingly when \( j \to \infty \).

Example 3.4: Let \( \alpha = \frac{2}{n}, n \geq 3, \) and \( \gamma = -1 \) be fixed numbers. We consider the associated recursion

\[
x_{j+1} = \varphi(x_j) = \alpha + \frac{\gamma}{x_j} = \frac{2}{n} - \frac{1}{x_j}
\]

and \( x_1 = 1 - (2 - \frac{2}{n}) = -1 + \frac{2}{n} < 0 \). Computing \( \Delta = \alpha^2 + 4\gamma = \frac{2^2}{n} - 4 = \frac{4}{n^2} - 4 < 0 \). Therefore, we have a Type 3 recursion where

- \( \rho = \sqrt{-\gamma} = 1 \);
- \( \phi = \arctan(\frac{\sqrt{n^2 - 4\gamma}}{\alpha}) = \arctan(\frac{\sqrt{\frac{4}{n^2} - 1}}{\frac{2}{n}}) = \arctan(\sqrt{n^2 - 1}) \in (0, \frac{\pi}{2}) \);
- \( \omega = -\phi + \arctan(\cot(\phi) - \frac{x_1}{\rho} \csc(\phi)) = -\arctan(\sqrt{n^2 - 1}) + \arctan(\cot(\phi) - \frac{-1 + \frac{n}{n^2}}{1}) \csc(\phi)). \)

From the identity \( \tan(\phi) = \sqrt{n^2 - 1} \) we obtain \( \csc(\phi) = \frac{n}{\sqrt{n^2 - 1}} \) and \( \cot(\phi) = \frac{1}{\sqrt{n^2 - 1}} \) thus

\[
\omega = -\arctan(\sqrt{n^2 - 1}) + \arctan\left(\frac{1}{\sqrt{n^2 - 1}} - \frac{-1 + \frac{2}{n}}{1} \frac{n}{\sqrt{n^2 - 1}}\right)
\]

\[
= -\arctan(\sqrt{n^2 - 1}) + \arctan\left(\frac{n - 1}{\sqrt{n^2 - 1}}\right)
\]

Using the formula for the sum \( \tan(a + b) \) we conclude that \( \omega = -\arctan(\frac{n-1}{\sqrt{n^2 - 1}}) \) and substituting in the previous formula we obtain \( \omega = -\frac{\phi}{2} \).

- The period is \( P = \frac{\pi}{\phi} = \frac{\pi}{\arctan(\sqrt{n^2 - 1})} \). Notice that, \( \arctan(\sqrt{n^2 - 1}) \in (0, \frac{\pi}{2}) \). Therefore, \( P > 2 \).
Figure 3. Function $f(j) = \left(\frac{1}{7} - \frac{\sqrt{17}}{7} \tan((j - 1/2)\phi)\right)$ for $n = 7$, $\phi = \arctan(4\sqrt{3}) \approx 1.42745$ and the period is $P \approx 2.20084$.

Thus, the general formula is

$$x_j = \left(\frac{1}{n} - \frac{\sqrt{n^2 - 1}}{n} \tan((j - 1/2)\phi)\right)$$

because $\csc(\phi) = \frac{1}{n}$ and $\sin(\phi) = \frac{\sqrt{n^2 - 1}}{n}$.

3.5. Dependence on a parameter

The explicit solution of $x_{j+1} = \varphi(x_j)$, $j \geq 1$ which is defined for any $j \in \mathbb{R}$ by $x_j = f(j)$ for $j \geq 1$, provided by Theorem 2.1, may have same dependence on a given parameter. In the Example 3.4 we have that

$$\begin{cases} x_1 = -1 + \frac{2}{n} \\ x_{j+1} = \frac{2}{n} - \frac{1}{x_j} \end{cases} \tag{20}$$

is depending on the parameter $n \geq 3$, both in the initial point and in the equation. Although, for each value of $n$ the solution presents a similar behaviour as we see in the Figure 3.

For instance, the period $P(n) = \frac{\pi}{\arctan(\sqrt{n^2 - 1})}$ depends analytically on $n$. A direct computation shows that $\frac{dP}{dn} < 0$ and $\lim_{n \to \infty} P = 2$. Therefore, the period is decreasing monotonously to 2. Varying $n \in [3, \infty)$ one can deduce several properties of the class of recursions (20).
4. Applications on limit points

A real number $r$ is said to be a limit point of the spectral radii of graphs if there exists a sequence $\{G_k\}$ of graphs such that

$$\rho(G_i) \neq \rho(G_j), \ i \neq j \quad \text{and} \quad \lim_{k \to \infty} \rho(G_k) = r,$$

where $\rho(G)$ is the spectral radius (or index) of the graph $G$.

A landmark paper in this area is due to J. Shearer [13] who proved that any real number $\lambda \geq \sqrt{2 + \sqrt{5}}$ is a limit point of the spectral radii of graphs. Precisely, there exists an infinite sequence of graphs $G_k$, $k = 1, 2, \ldots$, whose spectral radius $\rho(G_1) < \cdots < \rho(G_k) < \lambda$ is an increasing sequence and $\lim_{k \to \infty} \rho(G_k) = \lambda$.

In general, the techniques used to prove that a real number is a limit point are intricate. By way of an example, we show here that our results may be potentially applicable for finding limit points of spectral radii of graphs.

Consider the starlike tree $T_{l,m,n}$ illustrated in Figure 4 composed by paths $P_l, P_m$ and $P_n$ having one end point each joined by an edge to an isolated vertex.

**Theorem 4.1 ([3]):**

$$\lim_{n \to \infty} \rho(T_{1,n,n}) = \sqrt{2 + \sqrt{5}}.$$

**Proof:** Let $T_{1,n,n}$ for $n \geq 1$. After choosing the root as the unique vertex of degree three and enumerating the vertices from the leafs to the root, we see that applying the J-T algorithm, we have the recurrences given by $z_1 = -\lambda_n$ and $z_i = -\lambda_n - 1/z_{i-1}$, which is the sequence of Example 3.3. For a fixed value of $n$, Figure 5 illustrates the value in each vertex.

Here we consider that $\lambda_n = \rho(T_{1,n,n})$ is the spectral radius of $T_{1,n,n}$. Hence all the values $z_i$ are negative (as predicted by the theory), for $n \geq 3$. The value $a(v) = -\lambda_n - 1/z_i - 2/z_n = 0$, because $\lambda_n$ is an eigenvalue.

We notice that $\lambda_n$ is an increasing sequence as $T_{1,n,n}$ is a proper subgraph of $T_{1,n+1,n+1}$. Moreover $\lambda_n < \frac{3}{\sqrt{3-1}} \approx 2.12$ (see, for example, [10]). Hence $\lambda_n$ converges to some $\lambda_0$. We recall that $z_j = \theta_n - \frac{\theta_n^{-1} - \theta_n}{\theta_n^{j-1}}, \ j \geq 1$ and $\theta_n = -\frac{\lambda_n}{\sqrt{\lambda_n^2 - 4}}$ thus $z_n = \theta_n - \frac{\theta_n^{-1} - \theta_n}{\theta_n^{n-1}}$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tree.png}
\caption{Trees $T_{l,m,n}$.}
\end{figure}
Figure 5. Trees $T_{1,n,n}$ with labels.

Now, if $(\theta_n^2)^n \to \infty$, then $z_n \to \theta_0 = -\frac{\lambda_0 - \sqrt{\lambda_0^2 - 4}}{2}$. In order to see that $(\theta_n^2)^n \to \infty$, we recall that $\theta = \theta(\lambda)$ is a decreasing function for $\lambda \geq 2 = \lambda_2 = \rho(T_{1,2,2})$ meaning that, uniformly for $n \geq 3$, $(\theta_n^2)^n \to \infty$ because $\theta_3 < -1 = \theta_2$.

We therefore have $\lambda_n \to \lambda_0$, meaning that $\lambda_0 - \frac{1}{z_1} - \frac{2}{\theta_0} = 0$, which is equivalent to

$$-\frac{\lambda_0^2 \sqrt{\lambda_0^2 - 4} + \lambda_0^3 - \sqrt{\lambda_0^2 - 4} - 5 \lambda_0}{\lambda_0 (\lambda_0 + \sqrt{\lambda_0^2 - 4})} = 0.$$ 

Solving this gives $\lambda_0 = \sqrt{2 + \sqrt{5}}$. ■

The concept of limit point applies to the Laplacian matrix as well. The Laplacian spectral radius of the tree $T_{1,n,n}$ is denoted by $\mu(T_{1,n,n})$

**Theorem 4.2 ([7]):**

$$\lim_{n \to \infty} \mu(T_{1,n,n}) = 2 + \epsilon \approx 4.382975767,$$

where $\epsilon$ is the real root of $x^3 - 4x - 4$.

**Proof:** We denote by $\lambda_n = \mu(T_{1,n,n})$ for a given fixed $n$. We first notice that

$$4 < \lambda_n \leq 5.$$ 

The lower bound follows from [1] where it is shown that the Laplacian spectral radius is bounded by $\max\{d(v) + d(u) | u, v \text{ are adjacent}\}$, where $d(v)$ is the degree of the vertex $v$. The upper bound follows from [6], where it is shown that the Laplacian spectral radius is greater than or equal the largest degree + 1, with equality if and only if there is a universal vertex.

It follows that the sequence $\lambda_n$ converges to $\lambda_0$, since is limited and is monotonic increasing, as $T_{1,n,n}$ is a proper subgraph of $T_{1,n+1,n+1}$.

The same application of the J-T algorithm to $T_{1,n,n}$ with $\lambda_n = \mu(T_{1,n,n})$, gives the sequence $d_1 = 1 - \lambda_n$ and $d_i = 2 - \lambda_n - \frac{1}{d_{i-1}}$, for $i > 1$. In the language of Theorem 2.1, we have $\alpha = 2 - \lambda_n$ and $\gamma = -1$, leading to $\Delta = \alpha^2 + 4\gamma = \lambda_n^2 - 4\lambda_n > 0$, as $\lambda_n > 4$.

The fixed points are $\theta_n = \frac{2 - \lambda_n - \sqrt{\lambda_n^2 - 4\lambda_n}}{2}$ and $\frac{1}{\theta_n}$. Now, since $\lambda_n > 4$, we have that $\lambda_n < \theta_n$, and
then we know that \( d_i \to \theta_n \), increasingly. The value \( a(v) \) is now \( a(v) = 3 - \lambda_n - 1/d_1 - 2/d_n = 0 \), since \( \lambda_n \) is an eigenvalue.

We have \( \mu(T_{1,n,n}) \to \lambda_0 \) and, by a similar argument given in Theorem 4.1, \( \theta_n \to \theta_0 \). Therefore \( \mu(T_{1,n,n}) \to \lambda_0 \), with \( \lambda_0 = 3 - \lambda_0 - \frac{1}{d_1} - \frac{2}{\theta_0} = 3 - \lambda_0 - \frac{1}{\lambda_0} - \frac{2}{\theta_0} \) and \( \theta_0 = \frac{2 - \lambda_0 - \sqrt{\lambda_0^2 - 4\lambda_0}}{2} \), implying \( -\lambda_0 - \frac{1}{\lambda_0} - \frac{2}{\theta_0} = 0 \), which is equivalent to

\[
\lambda_0^3 + (\sqrt{\lambda_0 (\lambda_0 - 4)} - 6) \lambda_0^2 + (-4\sqrt{\lambda_0 (\lambda_0 - 4)} + 6) \lambda_0 + 2\sqrt{\lambda_0 (\lambda_0 - 4)} = 0.
\]

Solving this equation gives

\[
\frac{\sqrt[3]{54 + 6\sqrt{33}}}{3} + \frac{4}{\sqrt[6]{54 + 6\sqrt{33}}} + 2.
\]

To finish the proof, we let \( \epsilon = \frac{\sqrt[3]{54 + 6\sqrt{33}}}{3} + \frac{4}{\sqrt[6]{54 + 6\sqrt{33}}} \approx 2.38297567 \). It is easy to check that \( \epsilon \) is the only real root of \( x^3 - 4x - 4 \).

**5. A further look on parameter dependence**

In this section we emphasize the parameter dependence of the recurrences and apply to some classes of problems. The general idea is to study properties of a family of recursions

\[
\begin{align*}
x_1 &= h(s) \\
x_{j+1} &= \alpha(s) + \frac{\gamma(s)}{x_j}
\end{align*}
\]

(21)

where \( s \) is a real parameter varying in the set \( I \subset \mathbb{R} \) and \( h(s), \alpha(s), \gamma(s) \) are continuous functions of \( s \) (in general we assume to have good differential properties).

We think the dependence on the parameter in the following way: we fix a parameter \( s \) and solve (21) with respect to \( j \) obtaining \( x_j \). As the solution depends on the chosen \( s \), as \( j \) iterates, \( x_j \) is actually a function of \( s \), denoted by \( x_j(s) \). Now we want to understand the behaviour of \( x_j(s) \) for all \( s \in I \). We represent this dependence in the next table, where \( I = \mathbb{N} \),

| \( s \backslash j \) | 1     | 2     | 3     | 4     | \cdots |
|-------------------|-------|-------|-------|-------|--------|
| 1                 | \( x_1(1) \) | \( x_2(1) \) | \( x_3(1) \) | \( x_4(1) \) | \cdots |
| 2                 | \( x_1(2) \) | \( x_2(2) \) | \( x_3(2) \) | \( x_4(2) \) | \cdots |
| 3                 | \( x_1(3) \) | \( x_2(3) \) | \( x_3(3) \) | \( x_4(3) \) | \cdots |
| 4                 | \( x_1(4) \) | \( x_2(4) \) | \( x_3(4) \) | \( x_4(4) \) | \cdots |
| \vdots            | \vdots | \vdots | \vdots | \vdots | \ddots |

which is a pictorial representation of \( x_j(s) \). Suppose we are interested in understanding the behaviour of \( x_j(s) \), when \( j \) is fixed. This means we look downwards in the column \( j \). Perhaps a more standard technique would be, for a fixed value of \( s \), study the function \( x_j(s) \) as \( j \) varies. This means we look at the row \( j \) from left to right. We recall here that in general we are interested in the variation of \( x_j(s) \) when \( j \) is a natural number, but our formulas hold
Figure 6. White dots in the vertices means Diagonalize \( T, -2 + \frac{2}{n} \) produces a **negative** value, while black vertices means a positive value and light gray where we do not know the precise sign (e.g. \( b_3(r), b_4(r) \), etc).

for any real \( j \) and hence we may take advantage of the analytical properties of \( x_j(s) \) (see a simple illustration in Example 3.2).

By Theorem 2.1 all the parameters involved in the solutions are elementary functions (rational, trigonometric, inverse trigonometric, square, etc) of the parameters \( x_1, \alpha \) and \( \gamma \). Therefore, we can assume that all the iterates \( x_j(s) \) are smooth (say \( C^\infty \)) functions of \( s \), in its respective domains with vertical asymptotes in some points, provided that \( h(s), \alpha(s) \) and \( \gamma(s) \) have this property.

There are infinitely many possibilities of formulations of this problem. We will consider a specific example that may be seen as an application to important problems in spectral graph theory.

**5.1. Laplacian eigenvalues smaller than the average**

In a recent publication [9], it is shown that the number of eigenvalues of any tree (with \( n \) vertices) smaller than the average degree \( d = 2 - 2/n \) is at least \( \lceil \frac{n}{2} \rceil \). The main technique used was the analysis of the recurrence associated to the application of J-T algorithm to a path having a number \( r \) of pendant \( P_2 \) (see Figure 6).

Applying the J-T algorithm to the Laplacian matrix to locate \( \alpha = d \) we obtain, in each extremal vertex of the pendant paths \( P_{x_1x_2} \) the value

\[
x_1 = 1 - d = -1 + \frac{2}{n} < 0
\]

and the next value is

\[
x_2 = 2 - d - \frac{1}{x_1} = 2 - \frac{1}{n} - \frac{1}{x_1} > 1.
\]
From this values we follow the processing to a root obtaining
\[ b_1 = r + 1 - d - \frac{r}{x_2} = x_1 + r \left( 1 - \frac{1}{x_2} \right), \]
and the rest of the values are given by the recursion
\[
\begin{cases}
  b_1 = x_1 + r \left( 1 - \frac{1}{x_2} \right) \\
  b_{j+1} = \frac{2}{n} - \frac{1}{b_j}
\end{cases} \tag{22}
\]
for \( n \in [3, \infty) \).

In what follows, we obtain properties of the signs at the vertices of the paths that were not presented in [9].

5.1.1. Initial value

Our first concern is the dependence of the initial condition with respect to \( r \). The next lemma may be found in [9]. For completeness, we present a proof.

**Lemma 5.1:** Let \( n \geq 8 \). If \( 0 \leq r \leq \lfloor \frac{n}{4} \rfloor \) then \( b_1 (r) < 0 \).

**Proof:** Assume \( n \geq 8 \). Recall that \( b_1 (r) = x_1 + r (1 - \frac{1}{x_2}) \). On the non-negative reals, define the linear function into \( \mathbb{R} \)
\[ g(r) = x_1 + r \left( 1 - \frac{1}{x_2} \right). \]
Then \( g(0) = x_1 < 0 \). Since \( g'(r) = (1 - \frac{1}{x_2}) > 0 \), it is increasing. By continuity there is a unique point \( g(r_0) = 0 \). Solving for \( r_0 \) we have \( r_0 = \frac{-x_1}{1 - \frac{1}{x_2}} > 0 \). Since \( x_1 = -\frac{n-2}{n} \) and \( x_2 = \frac{n^2+2n-4}{n^2-2n} \), \( r_0 \) depends rationally on \( n \)
\[ r_0 = \frac{(n-2) \left( n^2 + 2n - 4 \right)}{4n(n-1)}. \]
Therefore \( b_1 (r) < 0 \) if and only if \( r \leq [r_0] \). We claim that \( \frac{n}{4} < r_0 \) for \( n \geq 7 \). Indeed, the inequality \( \frac{n}{4} < r_0 \) can be simplified to \( 0 < n^2 - 8n + 8 \) whose largest root is \( 4 + 2\sqrt{2} \approx 6.8 \).

5.1.2. General formula

The recursion (22) is the same we used in the Example 3.4 hence we have a Type 3 recursion, \( b_j = \rho (\cos(\phi) - \sin(\phi) \tan(j\phi + \omega)) \) where
\begin{itemize}
  \item \( \rho = 1 \);
  \item \( \phi = \arctan(\sqrt{n^2 - 1}) \in (0, \frac{\pi}{2}) \);
  \item The period is \( P = \frac{\pi}{\arctan(\sqrt{n^2 - 1})} > 2 \) because \( \arctan(\sqrt{n^2 - 1}) \in (0, \frac{\pi}{2}) \).
\end{itemize}
For the phase translation \( \omega \) the situation is different because it depends on \( r \). Denote \( \omega_r \) the phase translation associated with the initial condition \( b_1 = x_1 + r(1 - \frac{1}{x_2}) \) then

\[
\omega_r = -\phi + \arctan \left( \cot(\phi) - \frac{b_1}{\rho} \csc(\phi) \right).
\]

From the identity \( \tan(\phi) = \sqrt{n^2 - 1} \) we obtain \( \csc(\phi) = \frac{n}{\sqrt{n^2 - 1}} \) and \( \cot(\phi) = \frac{1}{\sqrt{n^2 - 1}} \) thus

\[
\omega_r = -\arctan \left( \sqrt{n^2 - 1} \right) + \arctan \left( \frac{1}{\sqrt{n^2 - 1}} - b_1 \frac{n}{\sqrt{n^2 - 1}} \right) = \arctan \left( \frac{1 - nb_1}{\sqrt{n^2 - 1}} \right) - \arctan \left( \sqrt{n^2 - 1} \right). \tag{23}
\]

For each fixed \( n \) the phase translation \( \omega_r \) depends differentially on \( r \). A straightforward computation shows that

\[
\frac{d\omega_r}{dr} = \frac{1}{1 + \left( \frac{1-nb_1}{\sqrt{n^2-1}} \right)^2} \frac{-n \left( 1 - \frac{1}{x_2} \right)}{\sqrt{n^2-1}} \]

\[
= \frac{-2\sqrt{n^2-1} \left( n^2 + 2n - 4 \right)}{8n(n-1)r^2 - 4(n-1)(n^2 + 2n - 4) r + (n^2 + 2n - 4)^2} < 0,
\]

therefore \( \omega_r \) is a strictly decreasing function of \( r \). Additionally \( \lim_{n \to \infty} \omega_r = -\frac{\pi}{2} + \arctan(\lim_{n \to \infty} b_1) = -\frac{\pi}{4} \), because \( \lim_{n \to \infty} b_1(r) = -1 \). Thus

\[-\frac{\pi}{2} < \omega_r < -\frac{\pi}{4}. \tag{24}\]

### 5.1.3. Periodicity and sign change

As we have observed before, the function \( j \mapsto f(j) := b_j = (\cos(\phi) - \sin(\phi) \tan(j\phi + \omega_r)) \) is actually defined for \( j \in \mathbb{R} \), except for vertical asymptotes. It is periodic with period \( P > 2 \) and piecewise decreasing in each interval of the period. As \( \lim_{n \to \infty} P = 2 \), the behaviour of \( b_j \) is very similar to a \( \mod 2 \)-periodic function.

From Lemma 5.1 we know that \( b_1 < 0 \), provided that \( 1 \leq r \leq \lfloor \frac{n}{4} \rfloor \). Using the recurrence formula, we have \( b_2 = \frac{2}{n} - \frac{1}{b_1} > 0 \). The main goal of this section is to discover how large is this alternating sign pattern, that is how large is \( k \) so that \( b_{2k+1} < 0 \) and consequently \( b_{2k+2} > 0 \). This change in the pattern will happen when we find a \( k_0 > 0 \) such that \( b_{2k_0+1} > 0 \) (notice that for \( k_0 = 0 \) we obtain \( b_{2k_0+1} = b_1 < 0 \)).

More formally we define

\[
k_0 := \begin{cases} 
\max\{k \mid b_{2k+1} < 0\}, & \text{if such } k \text{ exists} \\
\infty, & \text{otherwise}.
\end{cases}
\]

Obviously \( k_0 \geq 1 \). A remarkable fact is that, if \( k_0 < \infty \) then \( b_{2k_0+2} > 0 \) and \( b_{2k_0+3} > 0 \) is the first consecutive pair of positive numbers in the sequence \( b_j = b_j(r) \).
Theorem 5.1: Let \( n \geq 8 \) and \( 1 \leq r \leq \lfloor \frac{n}{2} \rfloor \). Then, \( k_0 < \infty \) and it is given by

\[
k_0 = \left\lfloor \frac{1}{P - 2} + \frac{\omega_r - \arctan(\cot(\phi))}{\phi(P - 2)} \right\rfloor \tag{25}\]

Proof: Our approach is to use the fact that \( b_j \) has a period close to 2 in order to compare different odd indices with \( b_1 \) which we know to be negative. For \( b_{2k+1} \) we compare

\[
b_{2k+1} = b_{1+2k-kP+kP} = b_{1-(P-2)k+kP} = b_{1-(P-2)k}.
\]

Since \( b_1 = b_{1+kP} < 0 \), \( b_j \) is decreasing and \(-(P-2)k < 0\), we conclude that the correspondence \( k \to b_{2k+1} \) is strictly increasing while \( 1 - (P-2)k > j^* \) where \( j^* \) is the first positive zero (see Figure 7). Stronger than that is the fact that if \( k \) is big enough so that \( 1 - (P-2)k < j^* \) then we have \( b_{2k+1} > 0 \). This shows that \( k_0 = \lfloor \hat{k} \rfloor \), where \( \hat{k} := \frac{1-j^*}{P-2} \), in particular \( k_0 < \infty \). In order to estimate \( k_0 \) we are going to introduce a new function

\[
G(k) := f(j^*) = f(1 - (P-2)k) \quad k \geq 0,
\]

where, \( f(j) = (\cos(\phi) - \sin(\phi) \tan(j\phi + \omega_r)) \) is the explicit solution of the recurrence. Obviously, \( \hat{k} \) is the first positive root of \( G \). A simple calculation shows that \( G(\hat{k}) = 0 \) is equivalent to \( \cos(\phi) - \sin(\phi) \tan((1 - (P-2)\hat{k})\phi + \omega_r) = 0 \). Now a (tedious) computation shows that it is equivalent to the existence of some \( m \in \mathbb{Z} \) such that

\[
\hat{k} = \frac{1}{P - 2} + \frac{\omega_r - \arctan(\cot(\phi))}{\phi(P - 2)} - m\frac{P}{P - 2}.
\]

We claim that \( m = 0 \) is the value that recovers the first positive root. By properties of periodic functions we just need to examine the values \( m \in \{-1, 0, 1\} \). To do that we introduce a new function \( H : \mathcal{A} \to \mathbb{R} \) given by

\[
H(n, r, m) = \frac{1}{P - 2} + \frac{\omega_r - \arctan(\cot(\phi))}{\phi(P - 2)} - m\frac{P}{P - 2} \tag{27}\]

where \( m \) is a fixed parameter and \( \mathcal{A} := \{(n, r) | n \geq 8, \ 1 \leq r \leq \lfloor \frac{n}{2} \rfloor \} \). The graphs of Figure 8 below confirm the correct choice \( m = 0 \) because decreasing or increasing \( m \) will generate negative or positive roots, respectively, showing that we reach the first positive root in \( m = 0 \), which concludes our proof.

5.1.4. The maximum length of alternating signs (mlas)

First we observe that there is no point in defining a minimum length of alternating signs in the sequence \( b_j(r) \) because it is always 2, since from Lemma 5.1 we know that \( b_1 < 0 \) and \( b_2 > 0 \), provided that \( n \geq 8 \) and \( 1 \leq r \leq \lfloor \frac{n}{2} \rfloor \). Our main concern is to predict the maximum length where the sign alternance happens. We introduce the definition of the maximum length of alternating signs or mlas, for short.

Definition 5.1: Given \( n \geq 8 \) and \( 1 \leq r \leq \lfloor \frac{n}{2} \rfloor \), consider the sequence \( \{b_j(r), \ j \geq 1\} \) given by Equation (22), then we define the maximum length of alternating signs of \( b_j(r) \) as being

\[
\text{mlas}_r(n) = 2k_r + 2, \quad \text{if } b_{2k+1} < 0, \text{ for } 0 \leq k \leq k_r \text{ and } b_{2(k_r+1)+1} > 0. \tag{28}\]
Figure 7. Extended solution $j \rightarrow f(j) = b_j$ for $r = 2$ and $n = 19$. In this case $P = 2.069368956$. The first positive root is $j^* = 0.6867$.

Figure 8. From left to right, the 3d plot of $H(n, r, m)$ for $m \in \{-1, 0, 1\}$: $H(n, r, -1) > 0$, $H(n, r, 0) > 0$, and $H(n, r, 1) < 0$.

We notice that, from Theorem 5.1, $\text{mlas}_r(n) = 2k_0 + 2 < \infty$ is well defined and, consequently, $b_{2k+1} < 0, b_{2k+2} > 0$ for $0 \leq k \leq k_r$, represents exactly the maximum index $j$ having only pairs $-, +$ in the sequence $b_j$.

**Example 5.1:** Consider the example given by Figure 7, $r = 2$ and $n = 19$. In this case $P \approx 2.06$, the first positive root is $j^* \approx 0.68$, $\phi \approx 1.51$, $\omega \approx -0.98$ and $k_0 = \lceil \frac{1}{P-2} + \frac{\omega}{\phi(P-2)} \rceil = \lceil 4.51 \rceil = 4$. This means that we must have alternance $(-, +)$ in the sequence elements until $2k_0 + 2 = 10$ and $b_{11} \approx 0.05 > 0$ should be the first odd index which is positive. Indeed, computing the entire sequence we verify that

$$\{b_j(2)\} = \{-0.53, 1.99, -0.39, 2.62, -0.27, 3.73, -0.16, 6.25, -0.05, 18.39, 0.05, \ldots\},$$

showing the power of the formula given by Theorem 5.1. In this example $\text{mlas}_2(19) = 2k_0 + 2 = 10$. 
Figure 9. Comparing $\text{mlas}_r(n) = 2k_0 + 2$ and $F_2(n, r)$ for $8 \leq n \leq 100$ (left) and a zoom for $8 \leq n \leq 10$ (right), showing the inequality from a different angle.

Despite the fact that Theorem 5.1 provides an exact formula for $\text{mlas}_r(n)$, we would like to have a lower bound for the $\text{mlas}_r(n)$ which allows one to have some idea of the size of this number in a general situation without computing $k_0$. The next result is remarkable and is obtained from estimates of the trigonometric functions involved in the formula for $k_0$.

**Theorem 5.2:** Consider $n \geq 8$ and $1 \leq r \leq \lfloor \frac{n}{4} \rfloor$, then a lower bound for the $\text{mlas}_r(n)$ is

$$\text{mlas}_r(n) \geq \max \left\{ 2 \left\lfloor \frac{\pi}{8} (n - 2) \right\rfloor - 4(r - 1), 2 \right\}.$$ 

**Proof:** We recall that, $\text{mlas}_r(n) = 2k_0 + 2 \geq 2$ and $k_0 = \lfloor \frac{1}{n - 2} + \frac{\omega_r - \arctan(\cot(\phi))}{\phi(P - 2)} \rfloor$. Plotting the functions $F_1 : (n, r) \rightarrow 2\left( \frac{1}{n - 2} + \frac{\omega_r - \arctan(\cot(\phi))}{\phi(P - 2)} \right) + 2$ (in red) and $F_2 : (n, r) \rightarrow 2\left( \frac{\pi}{8} (n - 2) \right) - 4(r - 1)$ (blue) in the domain $\mathcal{A} := \{(n, r) \mid n \geq 8, 1 \leq r \leq \lfloor \frac{n}{4} \rfloor \}$ one may see that $F_1(n, r) > F_2(n, r)$. We remark that the approximation is much sharper when $r \rightarrow 1$ (Figure 9).

5.1.5. **Estimating mlas$_1(n)$**

A very special case is when $r = 1$ because we recover information on the original sequence $x_j$ obtained by using the J-T algorithm applied to an actual pendant path of a given tree. Recall that, for a fixed $r$ we have $b_1(r) = x_1 + r(1 - \frac{1}{x_2})$ and $b_{j+1} = \frac{2}{n} - \frac{1}{b_j}$ where $x_1 = 1 - d = -1 + \frac{2}{n} < 0$ and $x_2 = 2 - d - \frac{1}{x_1} = \frac{2}{n} - \frac{1}{x_1} > 1$. Hence $b_1(1) = x_1 + (1 - \frac{1}{x_2}) = x_3$, analogously, $b_{j+1} = x_{j+2}$, $j \geq 1$. As the sequence $x_j$ starts with alternate signs (see Figure 10), we conclude that the maximum length of alternating signs for $x_j$, denoted $\text{mlas}(n)$ satisfy

$$\text{mlas}(n) := \text{mlas}_1(n) + 2.$$ (29)
We notice that, from the proof of Theorem 5.2, the estimate for $r = 1$ is quite sharp. In particular, for $r = 1$ we have, for $x_j$ that

$$\text{mlas}_1(n) \geq 2 \left\lfloor \frac{\pi}{8} (n - 2) \right\rfloor + 2 \approx \frac{3}{4} (n - 1).$$

Roughly speaking, if the length of a path is smaller than 75% of the graph order, we always alternate -,+ which allows us to conclude that 50% of the eigenvalues associated with the vertices in this path are above/under the average of the Laplacian eigenvalues!

Remark 5.1: We would like to conclude this section by pointing out the role of the discounting term $-4(r - 1)$ in Theorem 5.2. As the table below illustrates, we can see that:

- $\text{mlas}_1(n)$ is the largest possible interval of alternating signs;
- $\text{mlas}_r(n)$ actually decreases as a function of $r$;
- The lower bound given by Theorem 5.2 is sharp for $r = 1$ and looses accuracy as $r$ increases. In fact, each time $r$ increases by 1, the estimated value decreases by approximately 4, while the actual values may decrease by 2. It can even be negative, and in such case it is meaningless, because we always have $\text{mlas}_r(n) \geq 2$.

In this table we consider $n = 183$, a fairly large number, and compare $\text{mlas}_r(183)$ for increasing values of $r$ exhibiting the lower bound and the elements where the sign change. We recall that we must consider only $r \leq \lfloor \frac{183}{4} \rfloor = 45$.

As predicted $\text{mlas}_r(183)$ decreases when $r$ increases, sometimes by 2 and, in the worst case 4. When the pendant star becomes heavier, we loose the sign alternance sooner.
5.2. Proper transformations and the proof of a conjecture

In 2011 [14] it has been conjectured that the number of eigenvalues of any tree (with \( n \) vertices) smaller than the average degree \( d = 2 - \frac{2}{n} \) is at least \( \lceil \frac{n}{2} \rceil \). The key element of its recent proof [9] is the concept of proper transformation of a graph. Namely, a proper transformation changes a graph \( T \), producing a new graph \( T' \), preserving the number \( n \) of vertices in such way that the number of eigenvalues above the average degree, denoted by \( \sigma(T) \), does not decrease (but could stay the same), that is, \( \sigma(T) \leq \sigma(T') \). The main idea in the paper is that, given a tree \( T \) we can find a sequence of proper transformations changing \( T \) in to a new tree satisfying the conjecture, and consequently, \( T \) will satisfy the conjecture as well. The success of this strategy relies on a careful choice of a few proper transformations that change a given tree \( T \) by a finite number of steps into one of four prototype trees \( T_0, T_1, T_2 \) and \( T_3 \) given in Figure 11. It is not a coincidence the fact that the number of prototypes is four. Indeed, it is a fortuitous consequence of Lemma 5.1, which states that for \( n \geq 8 \), if \( 0 \leq r \leq \lfloor \frac{n}{4} \rfloor \) then \( b_1(r) < 0 \), and consequently \( b_2(r) > 0 \). As such, a generalized pendant path, i.e. \( r \) paths \( P_2 \) (2r vertices) attached to a path \( P_2 \) (as Figure 6 illustrates), has exactly half of the Laplacian eigenvalues below the average degree (negative vertices) and the other half of the Laplacian eigenvalues above the average degree (positive vertices). Eventually, we arrive at a tree having two generalized pendant paths with the largest possible \( r \) which is \( r = \lfloor \frac{n}{4} \rfloor \) or \( r = \lfloor \frac{n}{4} \rfloor - 1 \) we obtain the control of exactly \( 4r = 4\lfloor \frac{n}{4} \rfloor \) vertices of \( n \). The number of remaining vertices to be analyzed are 0,1,2 or 3, according to the congruence of \( n \mod 4 \). Those are exactly the prototypes trees.

**Example 5.2:** In order to obtain one of the prototype trees, one of the goals is to increase the number of \( P_2 \)'s attached to a vertex. Several of these transformations are described and proved to be proper in [9]. As an example we recall the Star-up transform, in which a path is shorten by two vertices, increasing the number of \( P_2 \)'s by one.

**Proposition 5.1 ([9], Star-up transform):** Let \( u \) be a vertex that is not a leaf of a tree \( T \) with \( n \geq 8 \) vertices. If \( u \) has a path \( P_q \), \( q \geq 2 \) connecting \( u \) to a starlike vertex that has exactly \( 0 \leq r \leq \lfloor \frac{n}{4} \rfloor - 1 \) pendant \( P_2 \), and no other pendant path, then the transformation in Figure 12 is proper.

**Corollary 5.1:** Consider \( T \) a double broom as in the picture below.

If \( 2q \leq \max\{2\lfloor \frac{n}{8}(n - 2) \rfloor - 4(r - 1), 2\} \), \( 2p \leq \max\{2\lfloor \frac{n}{8}(n - 2) \rfloor - 4(R - 1), 2\} \), \( r, R < \lfloor \frac{n-1}{4} \rfloor \) then \( \lfloor \frac{n}{2} \rfloor = \sigma(T) \) or \( \lfloor \frac{n}{2} \rfloor = \sigma(T) + 1 \).
Figure 12. Star-up transform increasing the number of $P_2$ attached to $u$. Black vertices are positive and white vertices are negative.

Figure 13. $T$.

**Proof:** It is a direct application of Theorem 5.2. From $2q \leq \max\{2\lfloor \frac{n}{8} (n-2) \rfloor - 4(r-1), 2\}$, $2p \leq \max\{2\lfloor \frac{n}{8} (n-2) \rfloor - 4(R-1), 2\}$ we conclude that we have exactly $p + q$ positive vertices and $p + q$ negative vertices. The same is true for each broom producing exactly $r + R$ positive vertices and $r + R$ negative vertices. Choosing the root as in the Figure 13 and applying the J-T algorithm to $T$ we conclude that the only unknown values is the root.

As $n = 2r + 2R + 2q + 2p + 1$ we obtain $\lfloor \frac{n}{2} \rfloor = r + R + q + p = \sigma(T)$ or $\sigma(T) + 1$, according to the signal of $\frac{2}{n} - \frac{1}{b_2q(r)} - \frac{1}{b_2p(R)}$ is negative or positive.

**Example 5.3:** We consider a tree $T$ with $n = 19$ vertices given in the Figure 14. Notice that the maximum number of $P_2$’s admitted by our results is $r = \lfloor \frac{19}{2} \rfloor = 4$. Computing the mlas$_r(19)$, through the exact formula (25), we obtain mlas$_1(19) = 12$, mlas$_2(19) = 10$, mlas$_3(19) = 8$ and mlas$_4(19) = 4$. Since the left side of $T$ has $3P_2$ and is connected to $u$ by $4 \leq$ mlas$_3(19) = 8$ vertices and, the right side of $T$ has $2P_2$ and is connected to $u$ by $4 \leq$ mlas$_2(19) = 10$ vertices, we can use the Corollary 5.1, concluding that the sign of each vertex is how we depicted in the figure, that is, 9 negative (eigenvalues below the average degree), 9 positive (eigenvalues above the average degree) and $u$ in red is unknown. If the sign of $u$ were positive we will have $\sigma(T) = 10 > \lfloor \frac{9}{2} \rfloor = 9$, as prescribed in the conjecture proof from [9].
In order to decide the sign of the remaining vertex, we emulate the argument in [9]. Since $19 \equiv 3 \mod 4$, according to [9], we must be able to transform it, properly, into the prototype $T_3$. We will do it using only star-up transform from Proposition 5.1:

- $T \xrightarrow{\text{star-up}} T'$, in the right side, using the root in $v$;
- $T' \xrightarrow{\text{star-up}} T''$, in the left side, using the root in $v$;
- $T'' \xrightarrow{\text{star-up}} T'''$, in the right side, using the root in $u$.

We observe that $T''' = T_3$ as we claimed.

Finally, we observe that as the transformations we used are proper we obtain $9 \leq \sigma(T) \leq \sigma(T') \leq \sigma(T'') \leq \sigma(T''') = \sigma(T_3) = 9$ concluding that sign of the red vertex, in $T$, is negative. Thus the number of Laplacian eigenvalues below the average degree is $\lceil \frac{9}{2} \rceil = 10$, as expected.

6. Concluding remarks

We have studied in this paper a class of recurrence relations given by

$$x_{j+1} = \varphi(x_j), j \geq 1,$$

where $\varphi(t) = \alpha + \frac{\gamma t}{1}$, for $t \neq 0$, $\alpha, \gamma \in \mathbb{R}$ are fixed numbers ($\gamma \neq 0$) and $x_1$ is a given initial condition. These recurrences appear in the process of diagonalizing certain matrices related to trees. We proposed in this paper an analytical approach to study the behaviour of the solutions of these relations recurrences. This study generalizes some particular relations that were studied in [2,9,11,12]. The central point about this work is to exhibit a powerful technique which is the passage from natural numbers, representing the vertices of a graph, to arbitrary real numbers, allowing to find solutions of analytical equations which will define intervals of indices where a certain property is true. Through this, we add a new layer of understanding of fine properties of graphs. What makes it possible, is the J-T algorithm, producing rational recurrence formulas which by in turn can be extended to real numbers, whenever we are able to find explicit expressions for them. Our goal is to
offer a broad and systematic study of the technical issues involved and show how it works
in many important problems of spectral graph theory. One may argue that trees constitute
a particular and small class of graphs. Nevertheless, the number of applications involving
trees is sufficient to grant importance to this study. Additionally, it seems that this novel
technique has been successful to tackle hard problems where traditional methods have not
being able to solve.

It is worth pointing out that the J-T algorithm has been generalized for locating eigen-
values of matrices of a graph based on the parameter treewidth in [5]. To be more precise,
if an order \( n \) graph \( G \) is given, together with a tree decomposition of width \( k \), then it is
possible to locate the eigenvalues of \( G \) in time \( O(k^2n) \). We anticipate that applications of
our technique using this algorithm to graphs with small treewidth may be possible.

We would like to finish the paper by suggesting a few problems where this technique
may be a tool.

- For a given \( n \), characterize the trees with \( n \) vertices having exactly \( \lceil \frac{n}{2} \rceil \) Laplacian
eigenvalues smaller than the average degree or, equivalently,

\[
\sigma(T) = \left\lfloor \frac{n}{2} \right\rfloor.
\]

- Prove that \( P_n \), the path with \( n \) vertices, is the only tree minimizing the Laplacian energy
among all trees with \( n \) vertices.
- Find limit points for eigenvalues of graphs.

**Note**

1. It is also possible to consider \( r = 0 \), in this case \( b_j(0) = x_j \), \( j \geq 1 \) because \( b_1(0) = x_1 + 0(1 - \frac{1}{x_2}) = x_1 \) and the recurrence is the same

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**ORCID**

Elismar R. Oliveira  http://orcid.org/0000-0003-2611-0489
Vilmar Trevisan  http://orcid.org/0000-0002-7053-8530
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