Analytical Study of Mode Coupling in Hybrid Inflation

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We provide an analytical study of the coupling of short and long wavelength fluctuation modes during the initial phase of reheating in two field models like hybrid inflation. In these models, there is - at linear order in perturbation theory - an instability in the entropy modes of cosmological perturbations which, if not cut off, could lead to curvature fluctuations which exceed the current observational values. Here, we demonstrate that the back-reaction of short wavelength fluctuations is too weak to lead to a truncation of the instability for the long wavelength modes on time scales comparable to the typical instability time scale of the long wavelength entropy modes. Hence, unless there are other mechanisms which truncate the instability, then in models such as hybrid inflation the curvature perturbations produced during reheating on scales of current observational interest may be very important.

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I. INTRODUCTION

The inflationary scenario [1] is the current paradigm of early universe cosmology. It addresses several conceptual problems which Standard Big Bang cosmology was unable to solve, and - perhaps more importantly - provides a causal mechanism to generate the primordial cosmological fluctuations which lead to the large-scale structure and cosmic microwave background anisotropies which are currently observed [2]. Reheating after the period of inflation is a key aspect of inflationary cosmology (see the recent review of [3]). During reheating the energy is transformed from the inflaton, the scalar matter field which is responsible for providing the inflationary expansion of space, to the matter we see today. Without reheating, the inflationary universe would leave behind a large universe empty of any matter, in obvious contradiction with observations. Reheating requires some coupling between the inflaton and regular matter.

The initial studies of reheating were based on first order perturbation theory applied to the inflaton decay [4]. This approach, however, fails to take into account the coherent nature of the inflaton field. At the end of the period of inflation, the inflaton begins coherent homogeneous oscillations about the ground state of its potential. In [5] (see also [6, 7]) it was realized that this gives rise to a potential parametric instability in any matter field which couples to the inflaton. This instability can take various forms depending on the nature of the coupling between the inflaton and the matter field. Rather generally, the process leads to a rapid energy transfer from the inflaton to the matter fields [4]. In a wide class of models, the resonance is of “broad resonance” [8] type and affects all fluctuations with wavenumbers smaller than a characteristic mass set by the particle physics Lagrangian. In some cases - in particular in the hybrid inflation model which we study in this paper - there is a tachyonic instability in the matter sector at the end of inflation, in which case the reheating instability is of “tachyonic resonance” [9] type and hence extremely efficient.

The inflaton field obviously couples to gravity, and therefore, as first conjectured in [10], it is possible that its oscillations will lead to a parametric resonance instability of the cosmological fluctuation modes. Due to the exponential growth of the causal horizon during the inflationary phase with constant Hubble radius, even super-Hubble modes can be causally affected by this instability [11]. This will inherently cause a long wavelength curvature mode to develop which potentially exhibits a tachyonic resonance. These large scale modes are of prime importance, since they correspond to scales observable today. If an inflationary model fails to control this possible instability of the curvature mode, the observed smallness of the magnitude of curvature perturbations today might rule out this model.

If matter consists of a single scalar field, then - as studied in [11] and [12] - there is no parametric instability of super-Hubble scale modes. However, if entropy fluctuations are present, then an instability of entropy modes can occur which will lead to rapid growth of the curvature fluctuations. For this instability to be effective, the entropy mode cannot have been exponentially redshifted during the period of inflation. Models which satisfy the conditions for parametric instability of the metric fluctuations were first discussed in [13] and [14].

If the duration of the parametric instability of long wavelength curvature fluctuations were long, then the curvature fluctuations induced by the resonant entropy modes could become larger than the primordial curvature perturbations and - in fact - larger than the observed amplitude of the inhomogeneities. Thus, one might potentially be able to use the parametric resonant instability of the entropy modes of the cosmological fluctuations to rule out large classes of multi-field inflationary models.

One class of inflationary models which at linear order in perturbation theory leads to an instability of the entropy fluctuations is hybrid inflation [15]. In this scenario, two scalar fields $\phi$ and $\psi$, are involved in the inflationary process, the slowly rolling inflaton field $\phi$ and the “waterfall” field $\psi$. The key point of this class of models is that while $\phi$ is slowly rolling during inflation, the
energy density of the Universe is dominated by the potential of \( \psi \). Over the past few years, a lot of interest has been devoted to these models, mainly since they provide a framework for the realization of inflation in the context of both supersymmetry \([16]\) and string theory (see \([17]\) for reviews and \([18]\) for an original reference). Among the promising approaches stand the D-brane / antibrane inflation models (e.g. the D3/D7 brane inflation model \([19]\) and the KKLMMT model \([20]\)). Though these models may resolve some of the conceptual problems from which simple single scalar-field driven inflation models suffer, the large number of light moduli fields present in string compactifications can give rise to entropy fluctuation modes, which could in turn enter a parametric resonance phase at early stages of reheating \([13, 14]\).

Hybrid inflation models are characterised by symmetry breaking along the \( \psi \) direction: The symmetry \( \psi \to -\psi \) is spontaneously broken for field values \( \phi \) smaller than a critical value \( \phi_c \). Inflation takes place while \( \phi \) is slowly rolling at field values \( \phi > \phi_c \). Once the inflaton crosses the critical value, the effective potential for \( \psi \) develops a tachyonic instability which triggers the rapid rolling of \( \psi \) towards one of the ground states of its potential. This causes inflation to end.

Reheating in hybrid inflation models on a fixed Friedmann background cosmology (i.e. no cosmological perturbations) was studied in detail in \([21, 22]\) (see also \([23]\)). By means of numerical simulations it was observed that within a very short time, non-linearities on a length scale given by the mass of the waterfall field develop and dominate the subsequent stages of the reheating process. It is important to study how large-scale metric perturbations evolve during reheating in these models. In previous work, linear evolution of the entropy and curvature modes was studied \([24, 25, 42]\) and it was shown that it is possible that an important entropy perturbation mode develops. In these works it was assumed that non-linear effects, e.g. the back-reaction of small-scale fluctuations, would unlikely have a dominant effect on large-scale fluctuation. Due to the large range of scales (the small wavelength modes we are interested in have wavelengths comparable to the inverse Hubble radius at the end of inflation whereas the wavelengths of modes we are interested in for current cosmological observations are of the order of 1mm, assuming that the scale of inflation is about \( 10^{15} \)GeV) numerical studies do not have the dynamical range to study this question. Instead, an analytical understanding of the dynamics is required.

Our work demonstrates, by means of an analytical analysis, that the back-reaction of short wavelength modes with random phases is too weak to truncate the instability of the long wavelength entropy modes. We have studied the effects of short wavelength fluctuations both on the background inflaton and on long wavelength modes of the inflaton and waterfall fields. Most importantly, we have shown that the back-reaction on large scale fluctuations of the waterfall field is too weak to shut off the resonance of this entropy mode one the relevant time scale of the instability. This implies that hybrid inflation models of the type analyzed in this paper may suffer from a potential “entropy mode problem”, unless there are other ways of truncating the resonance (e.g. the exponential decrease in the initial value of the fluctuations modes during the period of inflation - see e.g. \([29]\) for a discussion).

Our study of the back-reaction of fluctuations in hybrid inflation models is not the first analytical study. For an analytical study of the back-reaction effect of fluctuations on the background using very different techniques than the ones we use see \([31]\). For a study of the effects of metric fluctuations on the observable local expansion rate of the universe due to fluctuations see \([32]\).

The structure of this paper is as follows. In Section 2, we review the class of hybrid inflation models which are studied here, and introduce the perturbative approach employed to study non-linear interactions that arise during preheating. In Section 3, we solve the background theory, and in Section 4 we review the evolution of quantum fluctuations to linear order. In Section 5, we analyze the generation of long wavelength second order perturbation modes sourced by shorter wavelength first order fluctuations (summing over the contribution from all frequencies and assuming random phases), for both the inflaton and the waterfall field. This is the leading order back-reaction effect between short and long wavelength modes. We also study the back-reaction of the fluctuations on the background. We find that small-scale first-order modes with random phases have a contribution on the evolution of large-scale modes which does not start to dominate until long after the instability of the first order long wavelength modes has had a chance to develop. Hence, in spite of their large phase space, the non-linear effects of the short wavelength perturbations do not become dominant for the evolution of long wavelength modes during the early stages of preheating.

II. THE MODEL AND PERTURBATIVE APPROACH

We focus on the category of hybrid inflation models with Lagrangian density for matter given by:

\[
\mathcal{L}_m(\phi, \psi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda (\psi^2 - \nu^2)^2 - \frac{1}{2} g^2 \phi^2 \psi^2 + \mathcal{L}_{\text{matter}}
\]

The equations of motion form a coupled system of partial differential equations

\[
\ddot{\phi} + 3H \dot{\phi} - \frac{1}{a^2} \nabla^2 \phi = - (m^2 + g^2 \psi^2) \phi
\]

\[
\ddot{\psi} + 3H \dot{\psi} - \frac{1}{a^2} \nabla^2 \psi = - (\lambda (\psi^2 - \nu^2) + g^2 \phi^2) \psi.
\]
The Hubble parameter $H(t)$ at time $t$ is given by:

$$H(t)^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho,$$

(4)

where $\rho$ is the energy density, and is given by:

$$\rho = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \dot{\psi}^2 + \frac{1}{2} a^{-2} (\nabla \phi)^2 + \frac{1}{2} a^{-2} (\nabla \psi)^2 + V(\phi, \psi),$$

(5)

where the potential $V$ is given in the second line of (1).

The turnover value of $\phi$ at which $\psi$ develops a tachyonic instability will then be

$$\phi_c = \frac{v \chi^{1/2}}{g}.$$

(6)

To study this model we work in discrete Fourier space (discrete because of a finite volume cutoff) and expand to second order about a homogeneous and isotropic cosmological background. The expansion parameter $\varepsilon$ is the amplitude of the linear fluctuations. Our goal is to study how first order perturbations of high $k$ modes (with wavelengths comparable to the Hubble length at the end of inflation or to the wavelength associated with the mass of the waterfall field, whichever is smaller) influence the low $k$ modes (modes which affect cosmological observations today which correspond to a scale of roughly $l_0 \sim 1 \text{ mm}$ at the end of inflation) at second order. We want to estimate the time interval $\Delta t$ it will take before this back-reaction effect becomes dominant, and we also want to see whether back-reaction effects on modes with wavelength of the order $l_0$ decreases with the wavelength.

The expansion of the fields to second order in $\varepsilon$ is:

$$\phi(x, t) = \phi^{(0)}(t) + \varepsilon \delta \phi^{(1)}(x, t) + \varepsilon^2 \delta \phi^{(2)}(x, t),$$

(7)

$$\psi(x, t) = 0 + \varepsilon \delta \psi^{(1)}(x, t) + \varepsilon^2 \delta \psi^{(2)}(x, t).$$

(8)

Since we will eventually be evaluating mode sums numerically, it is useful to work with real Fourier modes. Hence, we can expand the first and second order field perturbations as follows:

$$\delta \phi^{(1)} = \sum_{n=0}^{\infty} \left[ \phi^{(1)^{(n)}}(x) \sin \left( \frac{n \pi x}{L} \right) + \phi^{(1)^{(c)}}(x) \cos \left( \frac{n \pi x}{L} \right) \right],$$

$$\delta \phi^{(2)} = \sum_{n=0}^{\infty} \left[ \phi^{(2)^{(n)}}(x) \sin \left( \frac{n \pi x}{L} \right) + \phi^{(2)^{(c)}}(x) \cos \left( \frac{n \pi x}{L} \right) \right],$$

$$\delta \psi^{(1)} = \sum_{n=0}^{\infty} \left[ \psi^{(1)^{(n)}}(x) \sin \left( \frac{n \pi x}{L} \right) + \psi^{(1)^{(c)}}(x) \cos \left( \frac{n \pi x}{L} \right) \right],$$

$$\delta \psi^{(2)} = \sum_{n=0}^{\infty} \left[ \psi^{(2)^{(n)}}(x) \sin \left( \frac{n \pi x}{L} \right) + \psi^{(2)^{(c)}}(x) \cos \left( \frac{n \pi x}{L} \right) \right].$$

Here, $2L$ is the size of the finite one-dimensional box inside of which we are performing the discrete Fourier expansion. Physically, we need to take $L$ larger (but not much larger) than the largest scale of the problem we study; hence we fix it to be $\sim 1 \text{ mm} = 6 \times 10^{31} \text{ l}_p$. This effectively imposes a cutoff on the largest scale studied. For clarity, we only write explicitly the one-dimensional expansions, but generalisation to three-dimensional Fourier series is straightforward and will not modify in a crucial way the form of the obtained solutions, unless otherwise mentioned.

### III. ZEROTH ORDER EXPANSION

Inserting the above ansatz into the equations of motion of the system and expanding to zeroth order in $\varepsilon$, (2) and (3) reduce to

$$\ddot{\phi}^{(0)}(t) + 3H \dot{\phi}^{(0)}(t) = -m^2 \phi^{(0)}(t)$$

(9)

$$\psi^{(0)}(t) = 0.$$  

(10)

For values of $|\phi|$ smaller than $\phi_c$, there is an instability of the background solution for $\psi$. Because of this instability, the $\psi$ field grows fast on the scale of a Hubble expansion time. Hence, we expect the Hubble damping term to be negligible, and thus we can set $H \approx 0$. This amounts to setting $a(t)$ to a constant (which we pick to be $1$) and $\dot{a}(t) = 0$. The linear equation for $\phi$ then becomes that of a harmonic oscillator and has the solution

$$\phi^{(0)}(t) = A^{(0)} \cos(mt) + B^{(0)} \sin(mt),$$

(11)

where $A^{(0)}$ and $B^{(0)}$ are constants depending on the initial conditions on $\phi^{(0)}(t = 0)$ and its derivative. Since in our case, we are interested in the end of the inflationary era, we want $\phi^{(0)}(t = 0) = \phi_c$ and we also want $\dot{\phi}$ to be initially slowly rolling, i.e. $|\dot{\phi}^{(0)}(t = 0)| \ll 1$. Thus, we set $B^{(0)}$ to zero and obtain:

$$\phi^{(0)}(t) = \phi_c \cos(mt)$$

(12)

$$\psi^{(0)}(t) = 0.$$  

(13)

### IV. FIRST ORDER EXPANSION

#### A. Equations

Now, going back to the system (2) and (3), we expand and keep terms of first order in $\varepsilon$:

$$\ddot{\phi}^{(1)}(x, t) + 3H \dot{\phi}^{(1)}(x, t) - \frac{1}{a^2} \nabla^2 \phi^{(1)}(x, t) = -m^2 \phi^{(1)}(x, t)$$

(14)

$$\ddot{\psi}^{(1)}(x, t) + 3H \dot{\psi}^{(1)}(x, t) - \frac{1}{a^2} \nabla^2 \psi^{(1)}(x, t) = \lambda v^2 \delta \psi^{(1)}(x, t) - g^2 \left( \phi^{(0)}(x, t) \right)^2 \delta \psi^{(1)}(x, t),$$

(15)
Inserting the explicit form of the first order perturbations, we make use of the orthogonality relations for trigonometric functions to convert \( \sin^2 \) and \( \cos^2 \) to discrete Fourier space.

Doing so, we obtain the following set of differential equations for the first order correction to each Fourier mode:

\[
\psi_{n}^{(1)s,c} + 3H\psi_{n}^{(1)s,c} + \left[ \left( \frac{n\pi}{aL} \right)^2 + m^2 \right] \phi_{n}^{(1)s,c} = 0, \tag{16}
\]

\[
\psi_{n}^{(1)s,c} + 3H\psi_{n}^{(1)s,c} + \left[ \left( \frac{n\pi}{aL} \right)^2 - \lambda t^2 + g^2 \left( \phi_{0} \right)^2 \right] \psi_{n}^{(1)s,c} = 0. \tag{17}
\]

Replacing the wavenumber \( n \) by its vectorial expression \( \mathbf{n} \) yields, without any other modification, the generalisation of the 1+1-dimensional equations to the 3+1-dimensional setting.

**B. Solutions**

Again setting \( H \approx 0 \), the first order \( \phi \) equations become that of harmonic oscillators, and so can be solved easily:

\[
\phi_{n}^{(1)s,c} = A_{n}^{(1)s,c} \cos \left( \sqrt{\left( \frac{n\pi}{aL} \right)^2 + m^2 t} \right) + B_{n}^{(1)s,c} \sin \left( \sqrt{\left( \frac{n\pi}{aL} \right)^2 + m^2 t} \right). \tag{18}
\]

Let us now have a closer look at the characteristic parameter values. We have in mind a hybrid inflation model stemming from string scale physics. Hence, the value of \( v \) will be taken to be \( v = 10^{-2} \) in Planck units. We choose coupling constants \( g = 10^{-2} \) and \( \lambda = 10^{-4} \), so that \( \phi_c = 10^{-2} \) (again in Planck units). We will consider the range \( m \in \left[ 10^{-7}, 10^{-5} \right] \) (in Planck units). For a value of \( m \) at the upper end of this interval, the hybrid inflation model will result in cosmological fluctuations of the observed order of magnitude (see e.g. [34] for a review of the theory of cosmological perturbations and [35] for an introductory overview). We explore a range of masses \( m \) in order to study how the strength of our back-reaction effect depends on the model parameters.

Moreover, we are obviously interested in modes whose wavelength is larger than \( m^{-1} \) and larger than the Hubble radius. For such modes \( \left( \frac{n\pi}{aL} \right) \ll m \). Therefore, (18) can be approximated by:

\[
\phi_{n}^{(1)s,c} = A_{n}^{(1)s,c} \cos \left( mt \right) + B_{n}^{(1)s,c} \sin \left( mt \right), \tag{19}
\]

which describes stable harmonic oscillation. No instability is manifest in the \( \phi \) field.

On the other hand, the first order \( \psi \) equations become:

\[
\psi_{n}^{(1)s,c}(t) + \left[ \left( \frac{n\pi}{aL} \right)^2 - \lambda t^2 + g^2 \phi_c^2 \cos^2 \left( mt \right) \right] \psi_{n}^{(1)s,c}(t) = 0. \tag{20}
\]

The equation can be reduced to the Mathieu equation by performing the transformation \( z = mt \) and using the identity \( \cos^2(z) = 1/2 (1 + \cos(2z)) \):

\[
\psi_{n}^{(1)s,c} + \left[ \left( \frac{n\pi}{amL} \right)^2 - \frac{\lambda t^2 + g^2 \phi_c^2}{2m^2} (1 + \cos(2z)) \right] \psi_{n}^{(1)s,c} = 0, \tag{21}
\]

where the prime refers to a derivation with respect to \( z \). Defining:

\[
q = \frac{g^2 \phi_c^2}{4m^2}, \tag{22}
\]

\[
\omega_n = \left( \frac{n\pi}{amL} \right)^2 - \frac{\lambda t^2}{m^2} + 2q, \tag{23}
\]

we indeed recover the canonical form of the Mathieu equation (see e.g. [22])

\[
\psi_{n}^{(1)s,c}(t) + [\omega_n - 2q \cos(2z)] \psi_{n}^{(1)s,c}(t) = 0. \tag{24}
\]

The parameter \( q \) is called the “Floquet exponent”, and we will call \( \omega_n \) the “square frequency”.

For the parameter values we are using, the value of \( q \) is much larger than 1. Hence, we are in the parameter region of either “tachyonic resonance” (if the tachyonic term in the expression for \( \omega_n \) dominates over the third term, or “broad parametric resonance” if the third term dominates (the first term in \( \omega_n \) is negligible for the modes we are interested in). In either case, all of the infrared modes which we study here will experience an exponential instability with a growth rate characterized by the Floquet exponent. To first order in perturbation theory, every \( \psi \) mode evolves in an independent way and there is no interaction between different modes. In particular, a mode that is not initially excited will not grow to first order at any later time (obviously, we expect quantum vacuum fluctuations on all scales to seed the instability). Inserting the expression (19) for \( \phi_c \), we find that for all values of \( n \) of interest to us

\[
\left( \frac{n\pi}{amL} \right)^2 \ll \left| \frac{\lambda t^2}{m^2} + \frac{g^2 \phi_c^2}{2m^2} \right| \leq \frac{g^2 \phi_c^2}{2m^2}
\]

(independently of the value of \( m \)); which means that \( \omega_n \) is always negative. Hence, we conclude that all modes we are interested in undergo tachyonic parametric resonance.

The solution to the second order differential equation (24) can be written in terms of two linearly independent solutions, the so-called Mathieu functions \( M_{ath}C \) and \( M_{ath}S \):

\[
\psi_{n}^{(1)s,c}(z) = C_{n}^{(1)s,c}M_{ath}C(\omega_n, q, z) + D_{n}^{(1)s,c}M_{ath}S(\omega_n, q, z), \tag{25}
\]

where the \( C \)'s and \( D \)'s are the coefficients.

Note that an important property of the solution to (24) for any choice of the parameters is the existence of
for some constant parametric resonance \( \phi \) (but slightly smaller) than 1.

\[ \mu \]

of discussed in the next subsection, is a good assumption an upper bound for a better approximation to the Mathieu function, and it is for certain ranges of the parameter value

\[ \text{FIG. 1: Analytical solution to (24) in red, } \cosh \left( \frac{\mu z}{m} (1 - \cos (ms)) \right) \text{ in green, and } \exp \left( \frac{\mu z}{m} (1 - \cos (ms)) \right) \text{ in yellow, all on a logarithmic scale, as a function of } z = ms; \text{ for } m = 10^{-6} \text{ and for the reference parameters cited in the text. The } \cosh \text{ function is a better approximation to the Mathieu function, and it is an upper bound for } z < \pi. \]

ses exponentially. In the cases of tachyonic and broad parametric resonance

\[ \phi_{n}(1)_{s,c}(t) \propto \exp(\mu z) \]  

(26)

for some constant \( \mu \) depending on \( q \). In the case of broad parametric resonance \( \tilde{\phi} \), the parameter \( \mu \) is of the order (but slightly smaller) than 1.

We can find an approximate solution of (24) valid over the first half of the period of \( \phi(0) \) under the assumption that the initial value is 1 (this is a normalization) and that the initial velocity of the mode vanishes (which, as discussed in the next subsection, is a good assumption for the modes we are interested in). Our approximate solution in fact gives an upper bound on the value of the mode function (which is within a factor two of the exact solution) and is given by

\[ M_{\text{ath}} C \left( -g^{2} \frac{\phi_{c}^{2}}{2m^{2}}, -g^{2} \frac{\phi_{c}^{2}}{4m^{2}}, z \right) \lesssim \cosh \left( \frac{\mu z}{m} (1 - \cos(z)) \right). \]

(27)

This result is found by approximating the equation of motion as

\[ \psi'' = \left( \frac{\phi_{c}}{m} \right)^{2} (1 - \cos^{2} z) \psi = \left( \frac{\phi_{c}}{m} \right)^{2} \sin^{2}(z) \psi, \]

(28)

and imposing the initial conditions mentioned above. Indeed, if we set

\[ \tilde{\psi} = \cosh \left( \frac{\mu z}{m} (1 - \cos z) \right), \]

(29)

we obtain

\[ \tilde{\psi}'' = \left( \frac{g \phi_{c}}{m} \right)^{2} \sin^{2} z \tilde{\psi} + g \frac{\phi_{c}}{m} \sin z \sinh \left( \frac{g \phi_{c}}{m^{2}} (1 - \cos z) \right). \]

But \( \frac{\mu z}{m} \sim 100 \) from our choice of parameters, and \( \sinh x < \cosh x \). So we have \( \tilde{\psi} > \psi \) at least from \( z = 0 \) to \( z = \pi \) (in which region \( \cos z > 0 \)).

C. Initial Conditions

Initial conditions on the first order fluctuation modes are given by the quantum fluctuations at the end of inflation. These modes begin on sub-Hubble scales at the beginning of the inflationary phase in their quantum vacuum state. As reviewed e.g. in [24, 32], the fluctuations freeze out and undergo squeezing once the wavelength exits the Hubble radius. The squeezing implies that the velocity of the mode functions will redshift. For purely notational simplicity we have chosen to excite only the \( \phi^{(1)c} \) and \( \psi^{(1)c} \) modes and to set the \( \phi^{(1)s}, \psi^{(1)s} \) modes to zero. This implies that we are taking correlated phases for the first order modes. At the end of the calculation we will restore the randomness of the phases and comment on the effect that this has on the strength of the back-reaction.

Because of the squeezing of the super-Hubble modes discussed above, we start \( \phi^{(1)c}(x) \) and \( \psi^{(1)c}(x) \) with zero velocity, that is to say, we set \( B_{n}(1)^{c} = D_{n}(1)^{c} = 0 \), while \( A_{n}(1)^{c} \) and \( C_{n}(1)^{c} \) are determined by the Bunch-Davies state.

When numerically computing mode sums later on in this article, it is important to know the initial amplitude of the mode functions in discrete momentum space. In continuous Fourier space, these amplitudes are given by (in \( d \) spatial dimensions):

\[ \tilde{\phi}^{(1)c}(x) = k^{-1/2}, \]

(30)

where we are using the Fourier decomposition in the form

\[ \delta \phi^{(1)}(x, 0) = \Re \left[ \int \frac{d^{d}k}{(2\pi)^{d}} e^{i\pi k} \phi_{C}^{(1)c}(x) \right] V^{1/2}, \]

(31)

where \( V \) is the spatial volume. We want to match this set of initial conditions with our discrete Fourier series:

\[ \delta \phi^{(1)}(x, 0) = \Re \left[ \sum_{n=0}^{\infty} e^{\frac{2n \pi}{L} L \phi^{(1)c}}(x) \right]. \]

(32)

We know \( 2 \pi L = \pi m \) and \( \Delta k L = \pi \), and thus want to relate \( \tilde{\phi}_{D}^{(1)c}(m) \) to \( \tilde{\phi}_{D}^{(1)c}(x) \) in terms of \( m \). To do so, we make use of the identity (recalling that \( V = (2L)^{d} \))

\[ \Re \left[ \frac{1}{V} \sum_{n=0}^{\infty} \right] \Re \left[ \sum_{n=0}^{\infty} \left( \frac{\Delta k}{2\pi} \right)^{d} \right] \to \Re \left[ \int \frac{d^{d}k}{(2\pi)^{d}} \right]. \]

(33)
in the continuum limit in \( d \) dimensions. In this limit, the discrete expansion (32) needs to converge to (31), i.e.:

\[
\text{Re} \left[ \sum_{n=0}^{\infty} \left( \frac{\Delta k}{2\pi} \right)^d e^{i\phi_D(1)^c(n)} \left( \frac{2\pi}{\Delta k} \right)^d \right] \\
\rightarrow \text{Re} \left[ \int \frac{d^d k}{(2\pi)^d} e^{i\phi_D(1)^c(\mathbf{k})} \left( \frac{2\pi}{\Delta k} \right)^d \right] \\
= \text{Re} \left[ \int \frac{d^d k}{(2\pi)^d} e^{i\phi_D(1)^c(\mathbf{k})} V^{1/2} \right],
\]

where the last step expresses our requirement of convergence.

Hence, for the initial values of the discrete Fourier modes we find the relation:

\[
\phi_D(1)^c(\mathbf{k}, 0) = \left( \frac{\Delta k}{2\pi} \right)^d \phi_D(1)^c(\mathbf{k}, 0) (2L)^{d/2} \\
= \left( \frac{1}{2L} \right)^{d-1} \frac{1}{(2\pi)^{1/2} m^{1/2}}.
\]  

V. SECOND ORDER EXPANSION

A. Equations

Going back to the system (2) and (3) and again inserting the ansatz (7) and (8), we expand and now keep terms of second order in \( \varepsilon \). We obtain

\[
\delta \phi^{(2)}(x, t) + 3H \delta \phi^{(2)}(x, t) - \frac{1}{a^2} \nabla^2 \delta \phi^{(2)}(x, t) = 0
\]

and

\[
\delta \psi^{(2)}(x, t) + 3H \delta \psi^{(2)}(x, t) - \frac{1}{a^2} \nabla^2 \delta \psi^{(2)}(x, t) = 0
\]

Inserting the explicit form of the first and second order perturbations, we make use of the orthogonality relations for trigonometric functions to convert (38) and (39) to discrete Fourier space. However, this time the process is slightly non-trivial due to the presence of the interaction terms at this order in perturbation theory which give rise to mode mixing. Indeed, the last terms in equations (38) and (39) describe how the growth of first order perturbations will source second order perturbations. They involve products of modes, which requires the use of trigonometric identities to split these terms in a way that allows the use of the canonical orthogonality conditions for sines and cosines. After some algebra, this yields the following set of differential equations for the second order correction to each Fourier mode. In one spatial dimension (recalling that \( \phi^{(1)s}(t) = \psi^{(1)s}(t) = 0 \) by definition, and \( \phi^{(1)c}(t) = \psi^{(1)c}(t) = 0 \) because these modes are part of the background) we obtain the following results:

For \( n \geq 1 \), the equations describing the back-reaction of the fluctuation modes on the perturbations themselves take the form

- \( \phi_{n}^{(2)s}(t) + 3H \phi_{n}^{(2)s}(t) + \left( \frac{n\pi}{aL} \right)^2 \phi_{n}^{(2)s}(t) = m^2 \phi_{n}^{(2)s}(t) - g^2 \left( \phi^{(0)}(t) \right)^2 \sum_{j=1}^{\infty} \left[ \psi_{j}^{(1)s}(t) \left( \psi_{j-n}^{(1)c}(t) - \psi_{j+n}^{(1)c}(t) \right) \right] \) (40)

- \( \phi_{n}^{(2)c}(t) + 3H \phi_{n}^{(2)c}(t) + \left( \frac{n\pi}{aL} \right)^2 \phi_{n}^{(2)c}(t) = m^2 \phi_{n}^{(2)c}(t) \)

- \( \psi_{n}^{(2)s}(t) + 3H \psi_{n}^{(2)s}(t) + \left( \frac{n\pi}{aL} \right)^2 \psi_{n}^{(2)s}(t) = \lambda v^2 \psi_{n}^{(2)s}(t) - g^2 \left( \phi^{(0)}(t) \right)^2 \psi_{n}^{(2)s}(t) \)
\[-g^2 \langle \phi^{(0)}(t) \rangle \sum_{k=1}^{\infty} \left[ \left( \phi^{(1)c}_{k-n}(t) - \phi^{(1)c}_{k+n}(t) \right) \psi^{(1)\ast}(t) + \left( \psi^{(1)c}_{k-n}(t) - \psi^{(1)c}_{k+n}(t) \right) \phi^{(1)\ast}(t) \right] \tag{42} \]

\[ \cdot \psi^{(2)c}_{n}(t) + 3H \psi^{(2)c}_{n}(t) + \left( \frac{2\pi}{aL} \right)^2 \psi^{(2)c}_{n}(t) \] 

\[-g^2 \langle \phi^{(0)}(t) \rangle \sum_{k=1}^{\infty} \left[ \psi^{(1)\ast}(t) \left( \phi^{(1)c}_{k-n} + \phi^{(1)c}_{k+n} \right) + \psi^{(1)c}(t) \left( \phi^{(1)c}_{k-n} + \phi^{(1)c}_{k+n} \right) \right], \tag{43} \]

while for \( n = 0 \), that is, for the back-reaction on the background fields, we have:

\[ \cdot \phi^{(2)c}_{0}(t) + 3H \phi^{(2)c}_{0}(t) = -m^2 \phi^{(2)c}_{0}(t) - g^2 \langle \phi^{(0)}(t) \rangle \left( \sum_{j=1}^{\infty} \left[ \frac{1}{2} \left( \psi^{(1)\ast}(t) \right)^2 + \frac{1}{2} \left( \psi^{(1)c}(t) \right)^2 \right] \right) \tag{44} \]

\[ \cdot \psi^{(2)c}(t) + 3H \psi^{(2)c}(t) = \lambda n^2 \psi^{(2)c}(t) - g^2 \langle \phi^{(0)}(t) \rangle \sum_{j=1}^{\infty} \left[ \phi^{(1)c}(t) \psi^{(1)c}(t) + \phi^{(1)c}(t) \psi^{(1)c}(t) \right]. \tag{45} \]

Note that the phases cancel out in the back-reaction on the inflaton field (in Eq. 43), but not in any of the other equations.

The physics which these equations describe is the following: Since the system initially has no second order perturbations, it is the interaction of two first order modes whose wavenumbers add up to \( k \) that will source fluctuations of wavenumber \( k \) at second order. The second order perturbation at wavenumber \( k \) is affected by all first order modes. Hence, even though the effect of each individual first order mode is of the order \( \varepsilon^2 \), the large phase space of modes which contribute can lead to a large back-reaction effect [44].

The above equations can luckily be reduced slightly. Since we have chosen not to excite the \( \phi^{(1)c} \) and \( \psi^{(1)c} \) modes, no second order sinusoidal fluctuations will arise, that is, \( \phi^{(2)c}_n = \psi^{(2)c}_n = 0 \) for all \( n \), and so there is no need to consider equations (40) and (42). Moreover, \( H \) can again be set to zero; and every term involving \( \phi^{(2)c}_n \) or \( \psi^{(2)c}_n \) in the interaction sum acting as a source in each equation can be set to zero. Also, as discussed above, the terms linear in the fields having as coefficient \( \left( \frac{2\pi}{aL} \right)^2 \) in equations (40) through (43) are negligible compared to the mass term of the fields, and thus can be dropped. However, recall that this conclusion was reached by imposing a cutoff equal to the Hubble radius on the smallest scale excited to first order. Consequently, all sums involving interactions of first order modes acting as source terms for the second order perturbation modes can be performed up to \( n = \frac{LH}{\sqrt{2}a^2} \sim 6 \times 10^{24} \).

Generalizing these equations from the 1+1-dimensional case to the higher-dimensional case this time is a bit more involved than simply replacing the wavenumber \( n \) by its vectorial expression \( \mathbf{n} \). In fact, the simple replacement works for every terms except for the interaction sum in each equation, which needs to be modified as follows:

\[ \sum_{j_1, ..., j_d = 0}^{\frac{LH}{2\sqrt{2}a^2}} \psi^{(1)c}_{j_1 \ldots j_d}(t) \phi^{(1)c}_{k_1 \ldots k_d}(t) \left[ \frac{1}{2} \left( \delta_{k_1 \ldots k_d, j_1 \ldots j_d} + \delta_{k_1 \ldots k_d, j_1 \ldots j_d} \right) \right] \ldots \left[ \frac{1}{2} \left( \delta_{k_d \ldots k_1, j_d \ldots j_1} + \delta_{k_d \ldots k_1, j_d \ldots j_1} \right) \right] \tag{46} \]

\[ \sum_{j_1, ..., j_d = 0}^{\frac{LH}{2\sqrt{2}a^2}} \psi^{(1)c}_{j_1 \ldots j_d}(t) \phi^{(1)c}_{k_1 \ldots k_d}(t) \left[ \frac{1}{2} \left( \delta_{k_1 \ldots k_d, j_1 \ldots j_d} + \delta_{k_1 \ldots k_d, j_1 \ldots j_d} \right) \right] \ldots \left[ \frac{1}{2} \left( \delta_{k_d \ldots k_1, j_d \ldots j_1} + \delta_{k_d \ldots k_1, j_d \ldots j_1} \right) \right] \tag{47} \]

\[ \sum_{j_1, ..., j_d = 0}^{\frac{LH}{2\sqrt{2}a^2}} \psi^{(1)c}_{j_1 \ldots j_d}(t) \phi^{(1)c}_{k_1 \ldots k_d}(t) \left[ \frac{1}{2} \left( \delta_{k_1, j_1} + \delta_{k_1, j_1} \right) \right] \ldots \left[ \frac{1}{2} \left( \delta_{k_d, j_d} \right) \right] \tag{48} \]
for the $\phi^{(2)c}$, $\psi^{(2)c}$, $\phi^{(2)c}_0$ and $\psi^{(2)c}_0$ equations, respectively, in the case of $d$ spatial dimensions. In particular, for $d = 3$, the sum for the $\phi^{(2)c}$ equation (49) can be rewritten as:

$$
g^2 \phi^{(0)} = \frac{\pi}{8} \sum_{i,j,k} \left[ \sum_{i,j,k=0}^{\pm 2} \psi^{(1)c}_{ijk} \psi^{(1)c}_{ijk} (k+n_z)(k+n_y)(k+n_x) + \sum_{i,j,k=0}^{\pm 2} \psi^{(1)c}_{ijk} \psi^{(1)c}_{ijk} (k+n_z)(k+n_y)(k+n_x) + \sum_{i,j,k=0}^{\pm 2} \psi^{(1)c}_{ijk} \psi^{(1)c}_{ijk} (k+n_z)(k+n_y)(k+n_x) \right]. \tag{50}
$$

Similarly, for the $\psi^{(2)c}$ equation (49):

$$
g^2 \psi^{(0)} = \frac{\pi}{8} \sum_{i,j,k} \left[ \sum_{i,j,k=0}^{\pm 2} \psi^{(1)c}_{ijk} \psi^{(1)c}_{ijk} (k+n_z)(k+n_y)(k+n_x) + \sum_{i,j,k=0}^{\pm 2} \psi^{(1)c}_{ijk} \psi^{(1)c}_{ijk} (k+n_z)(k+n_y)(k+n_x) + \sum_{i,j,k=0}^{\pm 2} \psi^{(1)c}_{ijk} \psi^{(1)c}_{ijk} (k+n_z)(k+n_y)(k+n_x) \right]. \tag{51}
$$

Finally, for the $\phi^{(2)c}$ and $\psi^{(2)c}$ equations (44) and (45), the sum at the end of each equation gets replaced by, respectively:

$$
g^2 \phi^{(0)} = \frac{\pi}{8} \sum_{i,j,k=1}^{\pm 2} \left( \psi^{(1)c}_{ijk} \right)^2 + 3 \sum_{i,j,k=1}^{\pm 2} \left( \psi^{(1)c}_{ijkl} \right)^2 + 3 \sum_{i,j,k=1}^{\pm 2} \left( \psi^{(1)c}_{ijklm} \right)^2 \tag{52}
$$

From the above discussion, we see that the differential equations for the second order perturbation modes reduce to that of driven harmonic oscillators, with a constant period in the case of $\phi^{(2)c}$, and with a time-dependent period in the case of $\psi^{(2)c}$. It is thus possible to solve them by making use of the Green’s function method.

For the $\phi^{(2)c}$ equations, we use the causal Green’s function for a simple harmonic oscillator:

$$
G(s,t) = \begin{cases} 
\frac{1}{m} \sin(m(t-s)), & \text{for } t \geq 0 \\
0, & \text{for } t < 0 
\end{cases}. \tag{53}
$$
while for the $\psi^{(2)c}$ equation, we make use of the function:

$$G(s,t) = \begin{cases} \frac{1}{m} M_{\text{reh}} S(\omega_0, q, m(t-s)) & , \text{for } t \geq 0 \\ 0 & , \text{for } t < 0 \end{cases}$$

where $\omega_0$ and $q$ are defined as above.

In the following, we first analyze the back-reaction effect on the background fields, and then move on to study the back-reaction of first order fluctuations on the perturbation modes themselves [43].

**B. Back-reaction on the $\phi$ Background**

We first tackle the description of the mode $\phi^{(2)c}_{\pi=0}$ whose evolution is dictated by equation (44) generalised to 3 spatial dimensions. This mode describes how second order perturbations of the $\phi$ field will modify how the background inflaton $\phi^{(0)}$ is oscillating around the minimum of its potential. From (44) and (42), we see that it is the growth of the amplitude of the first order perturbations in the $\psi$ field that will source this back-reaction.

More precisely, solving the $\phi^{(2)c}$ equation using the Green’s function method and making use of (53), we obtain the following solution:

$$
\phi^{(2)c}(t) = \int_0^t ds \frac{-1}{m} \sin(m(t-s)) g^2 \phi_c \cos(ms) \\
\times \left[ \frac{1}{8} \sum_{i,j,k=1}^{\frac{k_H}{2}} \left( \psi^{(1)c}_{ij,k} \right)^2 + \frac{3}{4} \sum_{j,k=1}^{\frac{k_H}{2}} \left( \psi^{(1)c}_{0jk} \right)^2 \\
+ \frac{1}{4} \sum_{k=1}^{\frac{k_H}{2}} \left( \psi^{(1)c}_{00k} \right)^2 \right].
$$

Note that the phases won’t cause any major cancellation of the terms in the summation, since all terms are positive. The expectation value of the sum can thus easily be taken, which will simply add an additional factor of $1/2$ to the solution (that is, the expectation value of $\cos^2 \theta$ over one period). Thus, we can write:

$$
\phi^{(2)c}(t) = -\frac{g^2 \phi_c}{4\pi m} \frac{1}{(2L)^2} \int_0^t ds \sin(m(t-s)) \cos(ms) \\
\times \cos^2 \left[ \frac{g \phi_c (1 - \cos ms)}{m} \right] \left[ \frac{1}{8} \sum_{i,j,k=1}^{\frac{k_H}{2}} \frac{1}{(i^2 + j^2 + k^2)^{1/2}} \\
+ \frac{3}{4} \sum_{j,k=1}^{\frac{k_H}{2}} \frac{1}{(j^2 + k^2)^{1/2}} + \frac{3}{2} \sum_{k=1}^{\frac{k_H}{2}} \frac{1}{k} \right].
$$

We see that the time-dependent part of the sums factors out, which means that the summations can be performed independently of the time integral. This process of summing is made much easier by replacing the sums with integrals, which is a reasonable approximation, since the range of the $i, j, k$ variables is quite wide. Moreover, instead of integrating over a cube of length size $\frac{k_H}{2}$ in momentum space, we integrate over one eighth of a sphere of radius $\frac{LH}{\pi}$, which yields the result:

$$
\sum_{i,j,k=1}^{\frac{k_H}{2}} \frac{1}{(i^2 + j^2 + k^2)^{1/2}} \approx \frac{1}{8} \int_1^{\frac{k_H}{4}} r dr d\Omega \frac{1}{r} \\
= \frac{(L\lambda^{1/2}m^2)^2}{48\pi}.
$$

Similarly,

$$
\sum_{j,k=1}^{\frac{k_H}{2}} \frac{1}{(j^2 + k^2)^{1/2}} \approx \frac{1}{4} \int_1^{\frac{k_H}{4}} r dr d\Omega \frac{1}{r} = \frac{L\lambda^{1/2}m^2}{4\sqrt{3}}
$$

These sums are dominated by the small wavelength first order modes since these modes have the largest volume in momentum space. Hence, since the momentum space is three-dimensional, it is clear that the first of the sums dominates over the two other. Indeed, for $m = 10^{-6}$ (in Planck mass units), the first term contributes $O(10^9)$, compared to $O(10^{25})$ and $O(10)$ for the two others. The prefactor of the integral (54) therefore becomes $\frac{g^2 \phi_c m^2}{2^{3} \pi^{3/2} m}$, independently of $L$, and the integral itself can be performed numerically.

As a stability and consistency check of our method, we study how the amplitude of the second order fluctuations in $\phi$ affects the background field as we vary the value of the mass of the inflaton field in Figure 2. The first striking feature of these graphs is that, for all considered values of $m$, the second order perturbation of the background mode becomes dominant before $\phi$ reaches the minimum of the potential, that is, within a short time relative to the time scale of the problem.

Moreover, as $m$ increases from $10^{-7}$ to $10^{-5.5}$, the fraction of the phase of $\phi^{(0)}$ needed before the back-reaction term starts to dominate decreases. Equivalently, the value of $mt$ when the amplitude of the second order perturbations becomes of similar order as the background field $\phi^{(0)}$ shrinks as the mass of the inflaton is increased.

However, in none of the cases studied is the time elapsed before the back-reaction effects on the background field becomes dominant shorter than

$$|m|^{-1} = \lambda^{-1/2}v^{-1}
$$

(which is $10^3$ Planck times for the parameter values we have chosen), the other relevant time scale of the problem under study which is the typical time scale of the motion in the tachyonic direction. Since the tachyonic field sources the instability at the end of inflation, its inverse mass will set the time scale dominating the growth
of second order perturbations. Varying $m$ should thus affect the growth of second order perturbations in a very smooth and mild way, which is what Figure 2 shows.

The $\phi^{(2)c}_0$ mode is found to grow as $\sim -e^{mt^2}$, an instability that rapidly outruns the amplitude corresponding to the first order perturbation modes, which we found above to oscillate as $\cos mt$. A fit of the function $-e^{at^2} + b$ with $a$ and $c$ as fitting parameters, presented in Figure 3 for the case $m = 10^{-6}$, is found to be in agreement with the analytical values over a time scale of $m^{-1}$. However, it is to be expected that during its very early evolution, the relative error between the fit and the numerical $\phi^{(2)c}_0$ will be large. Indeed, $\phi^{(2)c}_0(0) = 0$, while the exponential fitting function can never reach zero.

A plot of how the fitting parameters $a$ and $b$ vary with $m$ is displayed in the bottom part of Figure 3, confirming the stability and smoothness of the evolution of the $\phi^{(2)c}_0$ mode under variation of the inflaton’s mass. The parameter $a$ goes as $\sim m$, while the behavior of $b$ was found to match a $\sim 1/m$ function. Therefore, we find the following fitting function for the growth of the second order back-reaction on the background:

$$
\phi^{(2)c}_0 \approx -e^{\frac{8.071 e^{-5} m t^2}{m + 1.33 e^{-8}} - 35.60}
$$

(61)

C. Back-reaction on the $\psi$ Background

We now describe the mode $\psi^{(2)c}_0$, whose evolution is dictated by equation (45) generalised to 3 spatial dimensions. This mode describes how second order perturbations of the $\psi$ field will induce the growth of a spatially homogeneous contribution to the tachyonic field. At first order in perturbation theory the mode is obviously uncited, and so there is no non-trivial background to compare its growth against. Note that a second order $\psi$ background only develops since our choice of phases of the first order fluctuations breaks the $\psi \rightarrow -\psi$ symmetry of the Lagrangian. Averaged over the entire universe, we would expect the symmetry to be restored. However, in a finite volume there is no reason why the phases should obey the symmetry.

Combining the $\psi^{(2)c}_0$ equation and the Green’s function (51), we obtain the following solution:

$$
\psi^{(2)c}_0(t) = \int_0^t ds \frac{(-1)}{m} M_{\text{ath}}(\omega_0, q, m(t - s)) g^2 \phi_c \cos(ms)
$$

$$
\times \left[ \sum_{i, k=1}^{4H} \psi^{(1)c}_i \phi^{(1)c}_{0, k} + 6 \sum_{j=1}^{4H} \psi^{(1)c}_{0, j} \phi^{(1)c}_{0, k} + 12 \sum_{k=1}^{4H} \psi^{(1)c}_{0, k} \phi^{(1)c}_{0, 0k} \right] + 12 \sum_{k=1}^{4H} \psi^{(1)c}_{0, k} \phi^{(1)c}_{0, 0k}
$$

(62)

Substituting the solution for $\psi^{(1)c}_0$ and $\psi^{(1)c}_0$ obtained in (36) and (37), respectively, the sums to be performed turn out to be the same as above for $\phi^{(2)c}_0$. However, there is one important difference: in the case of the backreaction equation for the background $\phi$ field, the phases of the first order modes cancelled out. This will not be the case here. We will take this difference into account by performing the sums including random phases (see Appendix). Moreover, the $M_{\text{ath}}$ function appearing in the Green’s function can be dealt with by means of the same kind of approximation performed for the $M_{\text{ath}}C$ function in (27):

$$
M_{\text{ath}}(\frac{g^2 \phi_c^2}{2m^2}, -\frac{g^2 \phi_c^2}{4m^2}, z) \approx \sinh \left( \frac{g \phi_c}{m} \right) \left( 1 - \cos(z) \right)
$$

(63)
Proceeding with these substitutions and performing the sums as a random walk, we get:

$$
\psi_0^{(2)c}(t) = \frac{-g^2 \phi_0 \lambda^4}{3 \times 8 \pi^2 m L^2} \int_0^t ds \sinh \left[ \frac{g \phi_0 (1 - \cos(m(t - s)))}{m} \right] \cos^2(ms) \cosh \left[ \frac{g \phi_0 (1 - \cos ms)}{m} \right] \tag{64}
$$

which can now be integrated numerically. The result turns out to be quite similar to the one obtained for \( \psi_0^{(2)c} \): the second order background mode again grows as \( \sim -e^{t^2} \), which is much faster than the first order perturbation modes, which we found above to grow as \( \sim \cosh(1 - \cos mt) \). However, due to the cancellation of terms in the sum induced by the random walk, their overall amplitude is significantly smaller. In fact, even their rapid growth cannot compensate for this factor over the time scale set by the mass of \( \phi \).

A fitting function very similar to the one found above for \( |\phi_0^{(2)c}| \) is found to match the growth of \( |\psi_0^{(2)c}| \) as \( m \)

\[
\psi_0^{(2)c} \approx -\exp \left[ \frac{4.139 e^{-5 mt^2} - 5.336 e^{-6}}{m + 4.781 e^{-8}} - 111.6 \right] \tag{65}
\]

This relation is displayed in Figure 4 for the case \( m = 10^{-6} \), where it is found to be in agreement with the analytical values over a time scale of \( m^{-1} \), but as explained above, it deviates over smaller time scales.

D. Back-reaction on Long Wavelength \( \phi \) Modes

We now consider the case of main interest, i.e. the evolution of the large wavelength perturbation modes \( \phi_0^{(2)c} \) (with \( n \) small) at second order in perturbation theory. Their evolution is governed by equation (41), with the interaction term replaced by (50). However, once simplified, the homogeneous version of these equations become exactly the homogeneous version of the equation we had for \( \phi_0^{(2)c} \), so that they can be solved by the same Green’s function. Their solution thus reduces to the same integral as before, and only the prefactor will differ. The main difference is that the cancellation of the phases observed in the \( \phi_0^{(2)} \) case is not present anymore, and sums over random phases need to be taken into account. The
expectation value over phases must thus be performed as a random walk, as was done in the $\psi_0^{(2)}$ case. Terms with sums over 3 dimensions will again dominate over the ones with sums over only 1 or 2 dimensions. Thus, we are left with the task of evaluating:

$$
\frac{1}{8} \sum_{i,j,k=0}^{k_H} \left[ \frac{(i^2 + j^2 + k^2)^{-1/4}}{[(i-n_x)^2 + (j-n_y)^2 + (k-n_z)^2]^{1/4}} + \frac{(i^2 + j^2 + k^2)^{-1/4}}{[(i+n_x)^2 + (j+n_y)^2 + (k+n_z)^2]^{1/4}} + \frac{3(i^2 + j^2 + k^2)^{-1/4}}{[(i-n_x)^2 + (j-n_y)^2 + (k+n_z)^2]^{1/4}} + \frac{3(i^2 + j^2 + k^2)^{-1/4}}{[(i+n_x)^2 + (j+n_y)^2 + (k+n_z)^2]^{1/4}} \right] (66)
$$

where each term can be summed independently as a distinct sum (again with random phases).

To evaluate these, we assume, for the moment, that all modes are in phase. Under this assumption, we compute the first and second sums separately, and note that the third and fourth will be bounded by the values of the two firsts. Since they turn out to differ by a negligible amount, we will be able to set them all to be equal to each other. We then restore the randomness of the phases to compute the sums under the approximation that they are equal their monopole term in a multipole expansion.

Starting with the first sum, we note that it is bounded below by the sum (67), which is in fact the monopole contribution to its value in a multipole expansion. Since we are considering long wavelength modes ($n$ small), setting the sum equal to its monopole contribution constitutes a first approximation of its value. However, we wish to know how good this approximation will be as a function of $n$. To answer this question, we compute the difference between the sum and the monopole term (67), term by term:

$$
\frac{1}{2} \frac{(i^2 + j^2 + k^2)^{-1/2}}{[(i-n_x)^2 + (j-n_y)^2 + (k-n_z)^2]^{1/4}} - \frac{1}{2} \frac{(i^2 + j^2 + k^2)^{-1/2}}{[(i+n_x)^2 + (j+n_y)^2 + (k+n_z)^2]^{1/4}} \leq \frac{1}{2} \frac{(m^2)}{[(i^2 + j^2 + k^2)^{5/4}]^{1/4}} - \frac{1}{2} \frac{(m^2)}{[(i^2 + j^2 + k^2)^{5/4}]^{1/4}} (67)
$$

This expression puts an upper bound on the term by term difference between the sum we want to evaluate and its monopole approximation (67). This difference can then be summed over the three indices $i, j, k$, in order to obtain an upper bound on the total difference. To do so, we again use integrals to approximate the sums, and perform them numerically. We obtain, for $m = 10^{-6}$ in Planck mass units:

$$
\frac{n_x + n_y + n_z}{2} \int_0^{k_H} \frac{didjdk}{[i^2 + j^2 + k^2]^{5/4}} \approx \frac{n_x + n_y + n_z}{2} 1.8 \times 10^{15} (68)
$$

and

$$
- \frac{m^2}{4} \int_0^{k_H} \frac{didjdk}{[i^2 + j^2 + k^2]^{5/4}} \approx - \frac{m^2}{4} 2.1 \times 10^{12}. (69)
$$

First, note that once the value of $m$ is fixed, which mode exactly we choose to consider on the momentum shell of radius $|m|$ will make at most a factor of $\sqrt{3}$ difference in the result we just obtained. Hence we can take $m = n_x$ and $n_y = n_z = 0$ for simplicity.

Now, since the monopole term (67) is $O(10^{49})$ for the parameter values which we are considering, we can be confident that the higher multipole corrections to the first sum are negligible as long as we choose $O(m) < 10^{-12}$. Since we are interested in long wavelength modes, we can take the first sum in (66) to be equal to $\frac{(m^2)^2}{16\pi}$.

If we now tackle the evaluation of the second sum in (67), we find that (67) is an upper bound on its value. To compute the difference between this upper bound and the actual sum, we proceed by following the exact same approximation scheme as above. In doing so, we find that the difference is bounded by:

$$
\leq \sum_{i,j,k=0}^{k_H} \frac{1}{2} \frac{(n_x + n_y + n_z)}{[i^2 + j^2 + k^2]^{5/4}} + \frac{1}{2} \frac{(m^2)}{[i^2 + j^2 + k^2]^{5/4}}. (70)
$$

FIG. 5: Comparison on a log-log scale of the growth in amplitude of the $\phi$ back-reaction on large-scale fluctuations (blue) with the growth of the first order $\psi$ perturbations, for $n$ ranging from 1 to 10000 (red to yellow spectra), and for $m = 10^{-6}$. The evolution of the $\phi$ background is also shown (topmost green curve) to provide the overall time scale of the evolution. Back-reaction becomes dominant within less than $mt = 1$ for all long wavelength modes considered (about $5 \times 10^5$ Planck times).
which is once again negligible, provided that we choose $O(n) < 10^{-12}$.

Hence, we conclude that the four sums in (66) can be taken to be equal without making any significant error. Now, performing the sums including random phases within this approximation, we find that the evolution of the back-reaction term for long wavelength modes (with $O(n) < 10^{-12}$) is given by the solution:

$$
\psi_n^{(2)c}(t) = \frac{g^2 \phi_c v \lambda^{1/4}}{3^{1/4} 8\pi^2 m L^{3/2}} \int_0^t ds \sin(m(t-s)) \cos(ms) \times \cosh^2 \left[ \frac{g \phi_c (1 - \cos ms)}{m} \right].
$$

(71)

Since this integral form of the solution is the same as the one we obtained for $\phi_0^{(2)c}$, the growth and $m$ dependence of $\phi_0^{(2)c}$ (for long wavelength modes) will be exactly as studied above for the back-reaction on the background. The only difference comes in the overall normalisation of this integral, which is now greatly reduced by the random phase, instead of being proportional to the volume of first order excited modes in phase space. Our conclusions concerning the consistency and stability as a function of $m$ near the value $m = 10^{-6}$ (in Planck units) thus still hold. In particular, we obtain the following fitting function:

$$
\psi_n^{(2)c} \approx -\exp \left[ 8.071 e^{-5mt^2} + \frac{7.477 e^{-6}}{(m + 1.33 e^{-8})} - 146.4 \right].
$$

(72)

Note that for long wavelength modes, it is independent of $m$, and so the back-reaction term will have an identical growth for all modes with $n \sim O(10^{12})$.

Comparing the growth of the second order contributions to the amplitude of the long wavelength modes with the growth of the first order terms, we immediately see that, since the former grow as $\sim e^{c t}$, they will eventually come to dominate over the corresponding first order terms. Indeed, the latter exhibit no instability since they simply oscillate as $\sim \cos mt$. The precise time scale on which the second order terms become dominant can be found by comparing the the fitting function (72) to obtain a semi-analytical estimate.

Figure 6 shows the evolution of long wavelength fluctuations and the corresponding back-reaction term for various values of $n$, for $m = 10^{-6}$. For this value of $m$, it takes between $O(10^5)$ and $O(10^6)$ Planck times for the amplitude of the second order long wavelength mode to become equal to the first order fluctuation of that mode. This time scale is comparable to the longest typical time scales of the problem, $m^{-1}$, but is very long compared to the instability time scale in the tachyonic direction, $m \phi$. Since the only $n$ dependence of the intersection is through the normalization of the first order fluctuations modes in (66) (increasing by two orders of magnitude the momentum considered will reduce the amplitude of the first order perturbation by only one order of magnitude), the time when the second order effects start to dominate over the first order perturbations will depend only slightly on the value of $n$ considered, as long as only long wavelength modes are considered.

### E. Back-Reaction on Long Wavelength $\psi$ Modes

Repeating the above analysis to compute the back-reaction on the $\psi$ perturbations, i.e. on $\psi_n^{(2)c}$, for long wavelength modes we obtain (once again taking into account that the phases are random):

$$
\psi_n^{(2)c}(t) = \frac{g^2 \phi_c v \lambda^{1/4}}{3^{1/4} \pi^2 m L^{3/2}} \int_0^t ds \sinh \left[ \frac{g \phi_c (1 - \cos(m(t-s)))}{m} \right] 
\times \cos^2(ms) \cosh \left[ \frac{g \phi_c (1 - \cos ms)}{m} \right],
$$

(73)

which is simply $8\psi_0^{(2)c}$. We thus obtain the following fitting function:

$$
\psi_n^{(2)c} \approx -\exp \left[ 4.139 e^{-5mt^2} - \frac{5.336 e^{-6}}{(m + 4.781 e^{-8})} - 113.7 \right],
$$

(74)

which is still independent of $n$ for long wavelengths, i.e. for $n \sim O(10^{12})$.

Again, the growth of second order modes will be much faster than the growth of the corresponding first order modes, since their instability only grows as $\cosh(1 - \cos mt)$. However, the amplitude of the back-reaction term is suppressed by the random phases, and so the
time interval $\Delta t$ it takes before the second order term becomes more important than the first order term for long wavelength modes is much larger than the time interval for the onset of the instability of the first order fluctuations. To find the exact crossing time, we need to find the intersection of (73) and (37). The result, shown in Figure 6, is $O(10^9)$ Planck times for $m = 10^{-6}$. This is much longer than the instability time scale for fluctuations of the waterfall field. Since once again the only $n$ dependence of this crossing point is through the normalization of the first order modes, the chosen value of the long wavelength mode $n$ will only mildly reduce this result as $n$ is increased.

Since $\Delta t$ is much larger than the inverse mass of the waterfall field $m_\phi$, we conclude that $\Delta t$ is sufficiently large so that the back-reaction cannot shut off the growth of the entropy fluctuations before they have time to induce an appreciable contribution to the curvature perturbations during the early stages of the tachyonic instability.

VI. CONCLUSIONS

We have considered classical dynamics in a two field inflation model like hybrid inflation in which one field - $\phi$ - is slowly rolling during inflation and a second field, the so-called “waterfall field” $\psi$, develops a tachyonic instability once $\phi$ decreases below some critical value $\phi_c$. We focus on the dynamics right after the instability develops, which corresponds to the initial stages of preheating at the end of inflation.

We have provided a perturbative analytical analysis of the back-reaction effects of linear fluctuations on the background fields and on the perturbations themselves during the initial phase of the tachyonic instability which arises at the end of a period of hybrid inflation. We conclude that the tachyonic growth of the short wavelength fluctuation modes leads - at second order in perturbation theory, the lowest order at which mode mixing appears - to contributions which could in principle shut off the tachyonic growth of long wavelength entropy modes. However, in the case of random phases for the linear fluctuations considered here, we find that time scale when the back-reaction terms become important is much longer than the typical instability time scale of the tachyonic field. This process cannot shut off the instability of entropy modes before the latter have had time to become important. Note that there are other mechanisms which could dramatically weaken the strength of the instability of the long wavelength modes of the waterfall field, e.g. the decrease in amplitude of the linear fluctuations on super-Hubble scales during inflation.

It is important to stress the crucial role which the assumption of random phases has played in our analysis. Had we worked with correlated phases, then the back-reaction effects would have been larger by a factor comparable to $L^{3/2}$ (in Planck units), and back-reaction would then be able to cut off the tachyonic instability of modes of the waterfall field long before they have time to develop.

Our work complements the numerical studies of this model [21, 22]. Our analytical approach has as a big advantage that we can access the large dynamical range of scales which is required in order to draw conclusions for modes of cosmological interest today from dynamics taking place on microphysical scales at the end of inflation, the scales which are studied in the numerical works. Disadvantages of our approach based on a discrete Fourier space analysis are that it is purely perturbative and that we cannot study position space phenomena expected in the model such as the formation of topological defects.

The reader may worry that the back-reaction effect which we find on the adiabatic mode of the fluctuations, the $\phi$ fluctuations, might affect the predicted curvature fluctuations of the model. However, this is not the case. On scales larger than the Hubble radius, the curvature fluctuations are dominated by gravity. Microphysical effects which occur at the time of reheating cannot affect the adiabatic modes of the curvature perturbations. This follows in full generality from the Traschen integral constraints [31] which hold as long as we operate in the framework of General Relativity. In the case of scalar field models of inflation, it was shown in [11] that effects at preheating cannot affect the adiabatic mode of the curvature fluctuations.

The reader should also keep in mind that our analysis does not concern other sources of entropy fluctuations in models like the one considered here. Specifically, if topological defects are produced during preheating, then the active evolution of the defects between the time of formation and the present time will lead to a large contribution to the entropy fluctuations. In fact, the specific model considered in this paper, where both $\phi$ and $\psi$ are single component real fields, would lead to domain walls and would in fact be ruled out because of the domain wall problem [40], the over-abundance of energy in the walls.

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Appendix: Sum over Random Phases

In the evaluation of the sum of back-reaction terms, each term arises from an interaction between the two fields which each have a random phase and which, in general, do not cancel. We need to take the average over possible choices of phases not before studying the
effects on back-reaction, but only after. That is, the sum present in the expression for the background fields and their fluctuations must be performed for every possible choice of phases, and only then the expectation value of the obtained displacement distribution must be taken.

Since we take the phases to be random, this effect can be modeled by weighting randomly each amplitude, in the sum of the backreaction term, with 1 or -1. Calculating the expectation value of the absolute value of this sum is similar to calculating the average displacement of a random walk in one dimension, so we will call this quantity a random walk summation.

For a one dimensional random walk with \( N \) steps of size \( \delta \), the expectation value of the displacement \( (E[D]) \) is given by:

\[
\lim_{N \to \infty} \frac{E[D]}{\sqrt{N}} = \delta \sqrt{\frac{\pi}{2}} \tag{75}
\]

But in our case, this process is in fact slightly non-trivial, since the step size \( \delta \) is a function of the radius \( r \) in momentum space: \( \delta \to \delta(r) = \frac{1}{2\pi(2L)^2} \). For simplicity, we consider the random walk sum on the different shells of the sphere in momentum space independently, and we give each shell a thickness of 1. The number of steps is then \( \frac{1}{8} \) the volume of the shell, that is:

\[
\sqrt{\frac{4\pi r^2}{8}} \Delta r = 1 \tag{76}
\]

As the random walk summation of one shell, we thus obtain:

\[
\sqrt{\frac{4\pi r^2}{8}} \frac{1}{r} \frac{1}{2\pi(2L)^2} \sqrt{\frac{2}{\pi}} = \sqrt{\frac{2}{2\pi(2L)^2}} \frac{1}{\sqrt{8\pi^2 L^{3/2}}} \tag{77}
\]

for each shell. This result is now independent of the radius, which allows us to take a random walk over the shells, that is a random walk of \( L H / \pi \) steps with stepsize \( \frac{1}{2\pi(2L)^2} \) (note that just summing the above result over the shells is not the right way of proceeding, because the value calculated above could be + or − from one shell to the other, and doing so would thus greatly over-estimate the average result). We finally obtain:

\[
\sqrt{\frac{L H}{\pi}} \frac{1}{2\pi(2L)^2} \sqrt{\frac{2}{\pi}} = \sqrt{\frac{2H}{8\pi^2 L^{3/2}}} \tag{78}
\]

We also did a root mean square approximation to what the expectation value of the absolute value of the sum is, and we got the same expression within a numerical factor of order 1.

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Rapid meaning on a time scale less than the Hubble time at the end of the period of inflation.

Other work on fluctuations beyond linear analysis in hybrid inflation model focused on non-Gaussianities 26, 27.

There is another way to test these models observationally: hybrid inflation models typically produce topological defects such as cosmic strings. These strings, if stable on cosmological scales as they are in field theory models of hybrid inflation, lead to a scaling solution of strings at all late times (see e.g. 25 for reviews on cosmic strings and structure formation). The strings, in turn, produce line discontinuities in cosmic microwave temperature anisotropy maps 24 which can be searched for in observational maps using edge detection algorithms such as the Canny algorithm, as recently studied in 30.

Similarly in spirit, the large phase space of linear perturbation modes can lead to a large back-reaction effect of linear cosmological fluctuations on the background metric, an effect studied in 36 and reviewed in 37.

In the case of cosmological perturbations, the latter problem was studied in 38.