An elliptic analogue of Fukuhara’s trigonometeric identities

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MSC classes: 11F11, 11F20, 11M36, 33E05

Abstract

We obtain new elliptic function identities, which are an elliptic analogue of Fukuhara’s trigonometric identities. We show that the coefficients of Laurent expansions at $z = 0$ of our elliptic identities give rise to some reciprocity laws for elliptic Dedekind sums.

1 Introduction

Our starting point is the following identities. Let $a$ and $b$ be relatively prime positive integers and $z \in \mathbb{C}$.

(0) ((1.1) in [2]) For any complex number $z$,

$$\frac{1}{a} \sum_{\nu=1}^{a-1} \cot \left( \frac{\pi b \nu}{a} \right) \cot \left( \pi \left( z + \frac{\nu}{a} \right) \right) + \frac{1}{b} \sum_{\nu=1}^{b-1} \cot \left( \frac{\pi a \nu}{b} \right) \cot \left( \pi \left( z + \frac{\nu}{b} \right) \right)$$

$$= - \cot (\pi az) \cot (\pi bz) + \frac{1}{ab} \csc(\pi z)^2 - 1.$$ (1.1)

(1) ((1.2) in [2]) If $a$ is even, then

$$\frac{1}{a} \sum_{\nu=1}^{a-1} (-1)^\nu \cot \left( \frac{\pi b \nu}{a} \right) \cot \left( \pi \left( z + \frac{\nu}{a} \right) \right) + \frac{1}{b} \sum_{\nu=1}^{b-1} \csc \left( \frac{\pi a \nu}{b} \right) \cot \left( \pi \left( z + \frac{\nu}{b} \right) \right)$$

$$= - \csc (\pi az) \cot (\pi bz) + \frac{1}{ab} \csc(\pi z)^2.$$ (1.2)

(2) ((1.4) in [2]) If $a$ is odd, then

$$\frac{1}{a} \sum_{\nu=1}^{a-1} (-1)^\nu \cot \left( \frac{\pi b \nu}{a} \right) \csc \left( \pi \left( z + \frac{\nu}{a} \right) \right) + \frac{1}{b} \sum_{\nu=1}^{b-1} \csc \left( \frac{\pi a \nu}{b} \right) \csc \left( \pi \left( z + \frac{\nu}{b} \right) \right)$$

$$= - \csc (\pi az) \cot (\pi bz) + \frac{1}{ab} \csc(\pi z) \cot(\pi z).$$ (1.3)
If $a + b$ is even, then

\[ \frac{1}{a} \sum_{\nu=1}^{a-1} (-1)^\nu \csc \left( \frac{\pi b\nu}{a} \right) \cot \left( \pi \left( z + \frac{\nu}{a} \right) \right) + \frac{1}{b} \sum_{\nu=1}^{b-1} (-1)^\nu \csc \left( \frac{\pi a\nu}{b} \right) \cot \left( \pi \left( z + \frac{\nu}{b} \right) \right) \]

\[ = - \csc (\pi az) \csc (\pi bz) + \frac{1}{ab} \csc(\pi z)^2. \]  

(1.4)

If $a + b$ is odd, then

\[ \frac{1}{a} \sum_{\nu=1}^{a-1} (-1)^\nu \csc \left( \frac{\pi b\nu}{a} \right) \csc \left( \pi \left( z + \frac{\nu}{a} \right) \right) + \frac{1}{b} \sum_{\nu=1}^{b-1} (-1)^\nu \csc \left( \frac{\pi a\nu}{b} \right) \csc \left( \pi \left( z + \frac{\nu}{b} \right) \right) \]

\[ = - \csc (\pi az) \csc (\pi bz) + \frac{1}{ab} \csc(\pi z) \cot(\pi z). \]  

(1.5)

The formula (1.1) was given by Fukuhara [2]. Precisely, in [2], Fukuhara pointed out that (1.1) is derived from specialization of Dieter’s formula (Theorem 2.4 of [1]) and proved (1.2) - (1.5). Further, he compared the coefficients of Laurent expansions at $z = 0$ of identities (1.1) - (1.5) and obtained reciprocity laws for Dedekind-Apostol sums. The most simplest case of his reciprocity laws are the following.

(0) ((1.11) in [2]) If $a$ and $b$ are relatively prime positive integers, then

\[ \frac{1}{a} \sum_{\nu=1}^{a-1} \cot \left( \frac{\pi b\nu}{a} \right) \cot \left( \frac{\pi \nu}{a} \right) + \frac{1}{b} \sum_{\nu=1}^{b-1} \cot \left( \frac{\pi a\nu}{b} \right) \cot \left( \frac{\pi \nu}{b} \right) \]

\[ = \frac{a^2 + b^2 + 1 - 3ab}{3ab}. \]  

(1.6)

(1) ((1.12) in [2]) If $a$ is even, then

\[ \frac{1}{a} \sum_{\nu=1}^{a-1} (-1)^\nu \cot \left( \frac{\pi b\nu}{a} \right) \cot \left( \frac{\pi \nu}{a} \right) + \frac{1}{b} \sum_{\nu=1}^{b-1} \csc \left( \frac{\pi a\nu}{b} \right) \cot \left( \frac{\pi \nu}{b} \right) \]

\[ = \frac{-a^2 + 2b^2 + 2}{6ab}. \]  

(1.7)

(2) ((1.14) in [2]) If $a$ is odd, then

\[ \frac{1}{a} \sum_{\nu=1}^{a-1} (-1)^\nu \cot \left( \frac{\pi b\nu}{a} \right) \csc \left( \frac{\pi \nu}{a} \right) + \frac{1}{b} \sum_{\nu=1}^{b-1} \csc \left( \frac{\pi a\nu}{b} \right) \csc \left( \frac{\pi \nu}{b} \right) \]

\[ = \frac{-a^2 + 2b^2 - 1}{6ab}. \]  

(1.8)

(3) ((1.13) in [2]) If $a + b$ is even, then

\[ \frac{1}{a} \sum_{\nu=1}^{a-1} (-1)^\nu \csc \left( \frac{\pi b\nu}{a} \right) \cot \left( \frac{\pi \nu}{a} \right) + \frac{1}{b} \sum_{\nu=1}^{b-1} (-1)^\nu \csc \left( \frac{\pi a\nu}{b} \right) \cot \left( \frac{\pi \nu}{b} \right) \]

\[ = \frac{-a^2 - b^2 + 2}{6ab}. \]  

(1.9)
If $a + b$ is odd, then
\[
\frac{1}{a} \sum_{\nu=1}^{a-1} (-1)^\nu \csc \left( \frac{\pi b \nu}{a} \right) \csc \left( \frac{\pi \nu}{a} \right) + \frac{1}{b} \sum_{\nu=1}^{b-1} (-1)^\nu \csc \left( \frac{\pi a \nu}{b} \right) \csc \left( \frac{\pi \nu}{b} \right) = -\frac{a^2 - b^2 - 1}{6ab}.
\]

(1.10)

On the other hand, Fukuhara and Yui obtained the following elliptic function identity
(Theorem 2.1 in [3]) which is regarded as an elliptic analogue of the trigonometric identity (1.1). If $a + b$ is odd, then
\[
\frac{1}{a} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{a-1} (-1)^\mu \cs \left( 2K b^{\mu \tau + \nu} \right) \cs \left( \frac{z + \mu \tau + \nu}{a} \right), \kappa + \frac{1}{b} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{b-1} (-1)^\mu \cs \left( 2K a^{\mu \tau + \nu} \right) \cs \left( \frac{z + \mu \tau + \nu}{b} \right), \kappa = -\cs \left( 2K a z, \kappa \right) \cs \left( 2K b z, \kappa \right) + \frac{1}{ab} \dt \left( 2K z, \kappa \right) \nt \left( 2K z, \kappa \right),
\]

(1.11)

where $\cs(z, \kappa)$ is the Jacobi elliptic function (see Section 2). From this elliptic function identity, Fukuhara and Yui also gave reciprocity laws (Theorem 2.2 in [3]) for the elliptic Dedekind-Apostol sums
\[
\frac{1}{b} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{b-1} (-1)^\mu \cs \left( 2K a^{\mu \tau + \nu} \right) \cs \left( \frac{z + \mu \tau + \nu}{b} \right), \kappa \]

where $\cs^{(N)}(z)$ is the $N$-th derivative of the $\cs(z)$. The simplest case of Fukuhara-Yui’s reciprocity is the following (Lemma 3.1 in [3])
\[
\frac{1}{a} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{a-1} (-1)^\mu \cs \left( 2K b^{\mu \tau + \nu} \right) \cs \left( \frac{z + \mu \tau + \nu}{a} \right), \kappa + \frac{1}{b} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{b-1} (-1)^\mu \cs \left( 2K a^{\mu \tau + \nu} \right) \cs \left( \frac{z + \mu \tau + \nu}{b} \right), \kappa = \frac{a^2 + b^2 + 1}{6ab} (2 - \lambda(\tau)),
\]

(1.12)

which is an elliptic analogue of the reciprocity law (1.6) for the classical Dedekind sum
\[
s(a; b) := \frac{1}{4} \sum_{\nu=1}^{b-1} \cot \left( \frac{\pi a \nu}{b} \right) \cot \left( \frac{\pi \nu}{b} \right).
\]
In this article, we give an elliptic analogue of (1.2) - (1.5) and (1.7) - (1.10). The content of this paper is as follows. In Section 2, we introduce the elliptic functions cs(z, k), ds(z, k), ns(z, k), and list their fundamental properties. Section 3 is the main part of this article and we prove our main results (Theorem 3, Theorem 4 and Corollary 5). In Section 4, we give all the examples of Theorem 3 and Corollary 5.

2 Preliminaries

Throughout the paper, we denote the ring of rational integers by \( \mathbb{Z} \), the field of real numbers by \( \mathbb{R} \), the field of complex numbers by \( \mathbb{C} \) and the upper half plane \( \mathbb{H} := \{ z \in \mathbb{C} \mid \text{Im} \, z > 0 \} \). For \( \tau \in \mathbb{H} \), we put

\[ e(x) := e^{2\pi \sqrt{-1} x}, \quad q := e(\tau). \]

First, we recall the Jacobi theta functions

\[ \theta_1(z, \tau) := 2 \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} \sin ((2n+1)\pi z) = 2q^{\frac{1}{8}} \sin \pi z \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e(z))(1 - q^n e(-z)), \]

\[ \theta_2(z, \tau) := 2 \sum_{n=0}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} \cos ((2n+1)\pi z) = 2q^{\frac{1}{8}} \cos \pi z \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n e(z))(1 + q^n e(-z)), \]

\[ \theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^{\frac{1}{2}n^2} \cos (2n\pi z) = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-\frac{1}{2}} e(z))(1 + q^{n-\frac{1}{2}} e(-z)), \]

\[ \theta_4(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{\frac{1}{2}n^2} \cos (2n\pi z) = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-\frac{1}{2}} e(z))(1 - q^{n-\frac{1}{2}} e(-z)). \]

Further we put

\[ k = k(\tau) := \frac{\theta_2(0, \tau)^2}{\theta_3(0, \tau)^2}, \quad \lambda = \lambda(\tau) := k(\tau)^2, \quad K = K(\tau) := \frac{\pi}{2} \theta_3(0, \tau)^2. \]
and introduce the Jacobi elliptic functions

\[
\begin{align*}
\text{sn} (2Kz, k) & := \frac{\theta_3(0, \tau) \theta_1(z, \tau)}{\theta_2(0, \tau) \theta_4(z, \tau)}, \\
\text{cn} (2Kz, k) & := \frac{\theta_4(0, \tau) \theta_2(z, \tau)}{\theta_2(0, \tau) \theta_4(z, \tau)}, \\
\text{dn} (2Kz, k) & := \frac{\theta_4(0, \tau) \theta_3(z, \tau)}{\theta_3(0, \tau) \theta_4(z, \tau)}.
\end{align*}
\]

As is well known, the Jacobi elliptic functions sn (2Kz, k), cn (2Kz, k) and dn (2Kz, k) only depend on \( \lambda(\tau) = k(\tau)^2 \) (elliptic lambda function) that is a modular function of the modular subgroup

\[
\Gamma(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \left| \begin{array}{c}
 a \equiv d \equiv 1 \pmod{2}, \\
 b \equiv c \equiv 0 \pmod{2}
\end{array} \right. \right\}.
\]

Therefore under the following we restrict \( \tau \) to the fundamental domain of \( \Gamma(2) \)

\[
\Gamma(2) \setminus \mathcal{H} \simeq \left\{ \tau \in \mathcal{H} \left| \begin{array}{c}
 |\text{Re} \tau| \leq 1, \\
 |\tau \pm \frac{1}{2}| \geq \frac{1}{2}
\end{array} \right. \right\}.
\]

The elliptic functions cs (2Kz, k), ds (2Kz, k) and ns (2Kz, k) are defined by

\[
\begin{align*}
\text{cs} (2Kz, k) & := \frac{\text{cn} (2Kz, k)}{\text{sn} (2Kz, k)}, \\
\text{ds} (2Kz, k) & := \frac{\text{dn} (2Kz, k)}{\text{sn} (2Kz, k)}, \\
\text{ns} (2Kz, k) & := \frac{1}{\text{sn} (2Kz, k)}.
\end{align*}
\]

The elliptic function cs (2Kz, k) is regarded as an elliptic analogue of \( \cot (\pi z) = \frac{\cos (\pi z)}{\sin (\pi z)} \). Similarly, ds (2Kz, k) and ns (2Kz, k) are regarded as an elliptic analogue of \( \csc (\pi z) = \frac{1}{\sin (\pi z)} \). According to the wolfram functions site [6] and [5], we list fundamental properties of cs (2Kz, k), ds (2Kz, k) and ns (2Kz, k).

**Lemma 1.** (1) (parity)

\[
\begin{align*}
\text{cs} (-2Kz, k) & = -\text{cs} (2Kz, k), \\
\text{ds} (-2Kz, k) & = -\text{ds} (2Kz, k), \\
\text{ns} (-2Kz, k) & = -\text{ns} (2Kz, k).
\end{align*}
\]

[6] http://functions.wolfram.com/EllipticFunctions/JacobiCS/04/02/01/
[5] http://functions.wolfram.com/EllipticFunctions/JacobiDS/04/02/01/
(2) (periodicity) For any $\mu, \nu \in \mathbb{Z}$,

\[
\begin{align*}
\text{cs} (2K(z + \mu \tau + \nu), k) &= (-1)^\mu \text{cs} (2Kz, k), \\
\text{ds} (2K(z + \mu \tau + \nu), k) &= (-1)^{\mu+\nu} \text{ds} (2Kz, k), \\
\text{ns} (2K(z + \mu \tau + \nu), k) &= (-1)^\nu \text{ns} (2Kz, k).
\end{align*}
\]

(3) (Laurent expansions at $z = 0$)

\[
\begin{align*}
2K\text{cs} (2Kz, k) &= \frac{1}{z} + \left(-\frac{1}{3} + \frac{1}{6} \lambda\right) (2K)^2 z + \left(-\frac{1}{45} + \frac{1}{45} \lambda + \frac{7}{360} \lambda^2\right) (2K)^4 z^3 + \cdots, \\
2K\text{ds} (2Kz, k) &= \frac{1}{z} + \left(\frac{1}{6} - \frac{1}{3} \lambda\right) (2K)^2 z + \left(\frac{7}{360} + \frac{1}{45} \lambda - \frac{1}{45} \lambda^2\right) (2K)^4 z^3 + \cdots, \\
2K\text{ns} (2Kz, k) &= \frac{1}{z} + \left(\frac{1}{6} + \frac{1}{6} \lambda\right) (2K)^2 z + \left(\frac{7}{360} - \frac{11}{180} \lambda + \frac{7}{360} \lambda^2\right) (2K)^4 z^3 + \cdots.
\end{align*}
\]

(4) (Partial fraction expansions) (5.1) in [3]

\[
\begin{align*}
2K\text{cs} (2Kz, k) &= \sum_{m \in \mathbb{Z}} e \sum_{n \in \mathbb{Z}} e \frac{(-1)^m}{m \tau + n + z}, \\
2K\text{ds} (2Kz, k) &= \sum_{m \in \mathbb{Z}} e \sum_{n \in \mathbb{Z}} e \frac{(-1)^{m+n}}{m \tau + n + z}, \\
2K\text{ns} (2Kz, k) &= \sum_{m \in \mathbb{Z}} e \sum_{n \in \mathbb{Z}} e \frac{(-1)^n}{m \tau + n + z},
\end{align*}
\]

where $\sum_{n \in \mathbb{Z}} e$ is the Eisenstein convention

\[
\sum_{n \in \mathbb{Z}}^e f(n) := f(0) + \sum_{n=1}^{\infty} \{f(n) + f(-n)\}.
\]

In particular, for any non zero constant $A$, we have

\[
\begin{align*}
\lim_{z \to \frac{\mu \tau + \nu}{A}} \left(z + \frac{\mu \tau + \nu}{A}\right) 2K\text{cs} (2KAz, k) &= \frac{1}{A} (-1)^\mu, \\
\lim_{z \to \frac{\mu \tau + \nu}{A}} \left(z + \frac{\mu \tau + \nu}{A}\right) 2K\text{ds} (2KAz, k) &= \frac{1}{A} (-1)^{\mu+\nu}, \\
\lim_{z \to \frac{\mu \tau + \nu}{A}} \left(z + \frac{\mu \tau + \nu}{A}\right) 2K\text{ns} (2KAz, k) &= \frac{1}{A} (-1)^\nu.
\end{align*}
\]
(6) (Fourier expansions) p107 in [5]

\[ 2K \text{cs} (2Kz, k) = \pi \cot (\pi z) + \sum_{m=1}^{\infty} \frac{(-1)^m \pi \sin (2\pi z)}{\sin (\pi (z + m\tau)) \sin (\pi (z - m\tau))}, \quad (2.16) \]

\[ 2K \text{ds} (2Kz, k) = \sum_{m \in \mathbb{Z}} \frac{\pi}{\sin (\pi (z + m\tau))}, \quad (2.17) \]

\[ 2K \text{ns} (2Kz, k) = \sum_{m \in \mathbb{Z}} \frac{(-1)^m \pi}{\sin (\pi (z + m\tau))}. \quad (2.18) \]

(4) (Derivations)

\[ \text{cs}'(z, k) = -\text{ds} (z, k) \text{ns} (z, k), \quad (2.19) \]

\[ \text{ds}'(z, k) = -\text{cs} (z, k) \text{ns} (z, k), \quad (2.20) \]

\[ \text{ns}'(z, k) = -\text{cs} (z, k) \text{ds} (z, k). \quad (2.21) \]

http://functions.wolfram.com/EllipticFunctions/JacobiCS/20/01/01/
http://functions.wolfram.com/EllipticFunctions/JacobiDS/20/01/01/
http://functions.wolfram.com/EllipticFunctions/JacobiNS/20/01/01/

(5) (Relations between the Weierstrass \( \wp \) function)

\[ (2K \text{cs} (2Kz, k))^2 = \wp (z, \tau) - \wp \left( \frac{1}{2}, \tau \right), \quad (2.22) \]

\[ (2K \text{ds} (2Kz, k))^2 = \wp (z, \tau) - \wp \left( \frac{1 + \tau}{2}, \tau \right), \quad (2.23) \]

\[ (2K \text{ns} (2Kz, k))^2 = \wp (z, \tau) - \wp \left( \frac{\tau}{2}, \tau \right), \quad (2.24) \]

\[ \wp'(z, \tau)^2 = 4(2K \text{cs} (2Kz, k))(2K \text{ds} (2Kz, k))(2K \text{ns} (2Kz, k)). \quad (2.25) \]

Here, \( \wp (z, \tau) \) is the Weierstrass \( \wp \) function defined by

\[ \wp (z, \tau) := \frac{1}{z^2} + \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} \left\{ \frac{1}{(m\tau + n + z)^2} - \frac{1}{(m\tau + n)^2} \right\}. \]
Here, $\lfloor w \rfloor$ denotes the greatest integer not exceeding $w$.

For convenience, we put

\[
f_{1,0}(z, \tau) := 2K \cos(2Kz, k), \quad C_{1,0}(\tau) := \left( -\frac{1}{3} + \frac{1}{6} \lambda \right) (2K)^2,
\]

\[
f_{1,1}(z, \tau) := 2K \sinh(2Kz, k), \quad C_{1,1}(\tau) := \left( \frac{1}{6} - \frac{1}{3} \lambda \right) (2K)^2,
\]

\[
f_{0,1}(z, \tau) := 2K \cosh(2Kz, k), \quad C_{0,1}(\tau) := \left( \frac{1}{6} + \frac{1}{6} \lambda \right) (2K)^2.
\]

According to these notations, we have the following expressions of parity (2.1) - (2.3), periodicity (2.4) - (2.6), Laurent expansions at $z = 0$ (2.7) - (2.9), partial fraction expansions (2.10) - (2.12), residues at simple poles (2.13) - (2.15), derivations (2.19) - (2.21) and relations between the Weierstrass $\wp$ function (2.22) - (2.24) respectively.

\[
f_{i,j}(-z, \tau) = -f_{i,j}(z, \tau), \tag{2.31}
\]

\[
f_{i,j}(z + \mu \tau + \nu, \tau) = (-1)^{i\mu+j\nu} f_{i,j}(z, \tau), \tag{2.32}
\]

\[
f_{i,j}(z, \tau) = \frac{1}{z} + C_{i,j}(\tau) z + O(z^3), \tag{2.33}
\]

\[
f_{i,j}(z, \tau) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{(-1)^{im+jn}}{m\tau + n + z}, \tag{2.34}
\]

\[
\lim_{z \to -\frac{\mu \tau + \nu}{A}} f_{i,j}(Az, \tau) = \frac{1}{A} (-1)^{i\mu+j\nu}. \tag{2.35}
\]

\[
f'_{i,j}(z, \tau) = -f_{i+1,j}(z, \tau)f_{i,j+1}(z, \tau), \tag{2.36}
\]

\[
f_{i,j}(z, \tau)^2 = \wp(z, \tau) - \wp \left( \frac{i + j\tau}{2}, \tau \right). \tag{2.37}
\]

Here indices of $f_{i,j}(z, \tau)$ are regarded as elements in $\mathbb{Z}/2\mathbb{Z}$. 

Here, $[w]$ denotes the greatest integer not exceeding $w$.

For convenience, we put

\[
f_{1,0}(z, \tau) := 2K \cos(2Kz, k), \quad C_{1,0}(\tau) := \left( -\frac{1}{3} + \frac{1}{6} \lambda \right) (2K)^2,
\]

\[
f_{1,1}(z, \tau) := 2K \sinh(2Kz, k), \quad C_{1,1}(\tau) := \left( \frac{1}{6} - \frac{1}{3} \lambda \right) (2K)^2,
\]

\[
f_{0,1}(z, \tau) := 2K \cosh(2Kz, k), \quad C_{0,1}(\tau) := \left( \frac{1}{6} + \frac{1}{6} \lambda \right) (2K)^2.
\]
Remark 2. (1) Fukuhara-Yui use
\[ \varphi(\tau, z) := \sqrt{\wp(z, \tau) - \wp \left( \frac{1}{2}, \tau \right)} = \frac{1}{z} + O(z) \quad (z \to 0) \]
instead of \(2Kcs(2Kz, k)\). However, Fukuhara-Yui did not mention that \(\varphi(\tau, z)\) is the Jacobi elliptic function \(2Kcs(2Kz, k)\) exactly.

(2) If we use Mumford’s notations \([4]\)
\[\theta_{0,0}(z, \tau) := \theta_3(z, \tau), \quad \theta_{1,0}(z, \tau) := \theta_2(z, \tau), \quad \theta_{0,1}(z, \tau) := \theta_4(z, \tau), \quad \theta_{1,1}(z, \tau) := -\theta_1(z, \tau),\]
then our \(f_{i,j}(z, \tau)\) is written by
\[ f_{i,j}(z, \tau) = -\frac{\pi}{2} \theta_{i,0}(0, \tau) \theta_{0,j}(0, \tau) \theta_{j+1,i+1}(z, \tau). \]

3 Main results

Under the following we assume \(a\) and \(b\) are relatively prime positive numbers and \(i, j, m, n \in \{0, 1\}\). We mention and prove the main theorem.

**Theorem 3.** If \(ia + mb\) or \(ja + nb\) is odd, then

\[
\begin{align*}
&\frac{1}{a} \sum_{\substack{\mu, \nu = 0 \\
(\mu, \nu) \neq (0,0)}}^{a-1} (-1)^{\mu+j\nu} f_{m,n} \left( b \frac{\mu \tau + \nu}{a}, \tau \right) f_{ia+mb,ja+nb} \left( z + \frac{\mu \tau + \nu}{a}, \tau \right) \\
&\quad + \frac{1}{b} \sum_{\substack{\mu, \nu = 0 \\
(\mu, \nu) \neq (0,0)}}^{b-1} (-1)^{m+\nu} f_{i,j} \left( a \frac{\mu \tau + \nu}{b}, \tau \right) f_{ia+mb,ja+nb} \left( z + \frac{\mu \tau + \nu}{b}, \tau \right) \\
&= -f_{i,j}(az, \tau) f_{m,n}(bz, \tau) + \frac{1}{ab} f_{ia+mb+1,1}(z, \tau) f_{1,ja+nb+1}(z, \tau). \quad (3.1)
\end{align*}
\]

**Proof.** We put

\[
\Phi_{(i,j),(m,n)}((a,b), z, \tau) := \frac{1}{a} \sum_{\substack{\mu, \nu = 0 \\
(\mu, \nu) \neq (0,0)}}^{a-1} (-1)^{\mu+j\nu} f_{m,n} \left( b \frac{\mu \tau + \nu}{a}, \tau \right) f_{ia+mb,ja+nb} \left( z + \frac{\mu \tau + \nu}{a}, \tau \right)
\]

\[
+ \frac{1}{b} \sum_{\substack{\mu, \nu = 0 \\
(\mu, \nu) \neq (0,0)}}^{b-1} (-1)^{m+\nu} f_{i,j} \left( a \frac{\mu \tau + \nu}{b}, \tau \right) f_{ia+mb,ja+nb} \left( z + \frac{\mu \tau + \nu}{b}, \tau \right),
\]

\[
\Psi_{(i,j),(m,n)}((a,b), z, \tau) := -f_{i,j}(az, \tau) f_{m,n}(bz, \tau) + \frac{1}{ab} f_{ia+mb+1,1}(z, \tau) f_{1,ja+nb+1}(z, \tau)
\]

\[
= -f_{i,j}(az, \tau) f_{m,n}(bz, \tau) - \frac{1}{ab} f'_{ia+mb,ja+nb}(z, \tau).
\]
and
\[
U_{(i,j),(m,n)}((a,b),z,\tau) := \Phi_{(i,j),(m,n)}((a,b),z,\tau) - \Psi_{(i,j),(m,n)}((a,b),z,\tau)
\]
Under the condition \(2 \nmid ia + mb\) or \(2 \nmid ja + nb\), we claim that
\[
U_{(i,j),(m,n)}((a,b),z,\tau) \equiv 0.
\]
First we show that \(\Phi_{(i,j),(m,n)}((a,b),z,\tau)\) and \(\Psi_{(i,j),(m,n)}((a,b),z,\tau)\) have same periodicity. From periodicity \((2.32)\), for any integers \(M\) and \(N\) we have
\[
\Psi_{(i,j),(m,n)}((a,b),z + M\tau + N,\tau) = -f_{i,j}(az + M\tau + N, k) f_{m,n}(bz + M\tau + N, k)
\]
\[+ \frac{1}{ab} f_{ia+mb+1,1}(z + M\tau + N, \tau) f_{1,ja+nb+1}(z + M\tau + N, \tau)
\]
\[= -(-1)^{iaM+jaN} f_{i,j}(az, k) (-1)^{mbM+nbN} f_{m,n}(bz, k)
\]
\[+ \frac{1}{ab} (-1)^{(ia+mb+1)M+N} f_{ia+mb+1,1}(z, \tau) (-1)^{(ja+nb+1)N} f_{1,ja+nb+1}(z, \tau)
\]
\[= (-1)^{(ia+mb)M+(ja+nb)N} \Psi_{(i,j),(m,n)}((a,b),z,\tau).
\]
Similarly, for \(\Phi_{(i,j),(m,n)}((a,b),z,\tau)\) we have
\[
\Phi_{(i,j),(m,n)}((a,b),z + M\tau + N,\tau) = (-1)^{(ia+mb)M+(ja+nb)N} \Phi_{(i,j),(m,n)}((a,b),z,\tau).
\]
Thus we obtain double periodicity of \(U_{(i,j),(m,n)}((a,b),z,\tau)\)
\[
U_{(i,j),(m,n)}((a,b),z + M\tau + N,\tau) = (-1)^{(ia+mb)M+(ja+nb)N} U_{(i,j),(m,n)}((a,b),z,\tau). \quad (3.2)
\]
Next we consider all the poles of \(U_{(i,j),(m,n)}((a,b),z,\tau)\) and their Laurent expansions. We remark that \(\Phi_{(i,j),(m,n)}((a,b),z,\tau)\), \(\Psi_{(i,j),(m,n)}((a,b),z,\tau)\) and \(U_{(i,j),(m,n)}((a,b),z,\tau)\) are holomorphic at \(z = 0\). Actually, since \(a\) and \(b\) are relatively prime positive integers, the Laurent expansions of \(\Phi_{(i,j),(m,n)}((a,b),z,\tau)\) at \(z = 0\) is the following.
\[
\Phi_{(i,j),(m,n)}((a,b),z,\tau) = \frac{1}{a} \sum_{i,j=0}^{a-1} (-1)^{ia+j1} f_{m,n} \left( \frac{b \mu + \nu}{a}, \tau \right) f_{ia+mb,jb+nb} \left( \frac{\mu \tau + \nu}{a}, \tau \right)
\]
\[+ \frac{1}{b} \sum_{i,j=0}^{b-1} (-1)^{mb+nv} f_{i,j} \left( \frac{a \mu + \nu}{b}, \tau \right) f_{ia+mb,jb+nb} \left( \frac{\mu \tau + \nu}{b}, \tau \right) + O(z).
\]
On the other hand, from \((2.36)\) and \((2.33)\), we have
\[
\Psi_{(i,j),(m,n)}((a,b),z,\tau) = -f_{i,j}(az, \tau) f_{m,n}(bz, \tau) - \frac{1}{ab} f_{ia+mb,jb+nb}(z, \tau)
\]
\[= -\left( \frac{1}{az} + C_{i,j}(\tau) az + O(z^3) \right) \left( \frac{1}{bz} + C_{m,n}(\tau) bz + O(z^3) \right)
\]
\[= -\left( \frac{1}{az} + C_{i,j}(\tau) az + O(z^2) \right) \left( \frac{1}{bz} + C_{m,n}(\tau) bz + O(z^2) \right)
\]
\[= -b C_{m,n}(\tau) - \frac{a}{b} C_{i,j}(\tau) - \frac{1}{ab} C_{ia+mb,jb+nb}(\tau) + O(z^2).
\]
Then we obtain the Laurent expansion of $U_{i,j,(m,n)}((a, b), z, \tau)$ at $z = 0$

$$U_{i,j,(m,n)}((a, b), z, \tau) = -\frac{b}{a} C_{i,n}(\tau) - \frac{a}{b} C_{i,j}(\tau) - \frac{1}{ab} C_{ia+mb,ja+nb}(\tau)$$

$$- \frac{1}{a} \sum_{\substack{\mu, \nu = 0 \to \infty \atop (\mu, \nu) \neq (0, 0)}} (-1)^{\mu + \nu} f_{m,n} \left( b\frac{\mu \tau + \nu}{a}, \tau \right) f_{i+mb,ja+nb} \left( \frac{\mu \tau + \nu}{a}, \tau \right)$$

$$- \frac{1}{b} \sum_{\substack{\mu, \nu = 0 \to \infty \atop (\mu, \nu) \neq (0, 0)}} (-1)^{\mu + \nu} f_{i,j} \left( a\frac{\mu \tau + \nu}{b}, \tau \right) f_{ia+mb,ja+nb} \left( \frac{\mu \tau + \nu}{b}, \tau \right) + O(z).$$

Hence we investigate other poles. By the definition or partial fractional expansion (2.34) of $f_{i,j}(z, \tau)$, all other poles of $\Phi_{i,j,(m,n)}((a, b), z, \tau)$ and $\Psi_{i,j,(m,n)}((a, b), z, \tau)$ are

$$-\frac{\mu \tau + \nu}{a} + M \tau + N, \quad \mu, \nu = 0, 1, \ldots, a - 1, \quad (\mu, \nu) \neq (0, 0), \quad M, N \in \mathbb{Z}$$

or

$$-\frac{\mu \tau + \nu}{b} + M \tau + N, \quad \mu, \nu = 0, 1, \ldots, b - 1, \quad (\mu, \nu) \neq (0, 0), \quad M, N \in \mathbb{Z}.$$
Thus for \( M, N \in \mathbb{Z}, \mu, \nu \in \{0, 1, \ldots, a-1\} \) and \((\mu, \nu) \neq (0,0)\),
\[
\lim_{z \to -a^{-1} M - N} \left( z + M \tau + N + \frac{\mu \tau + \nu}{a} \right) U_{(i,j),(m,n)}((a,b), z, \tau) = 0.
\]

Similarly, for \( M, N \in \mathbb{Z}, \mu, \nu \in \{0, 1, \ldots, b-1\} \) and \((\mu, \nu) \neq (0,0)\) we have
\[
\lim_{z \to -b^{-1} M - N} \left( z + M \tau + N + \frac{\mu \tau + \nu}{b} \right) U_{(i,j),(m,n)}((a,b), z, \tau) = 0.
\]

Therefore \( U_{(i,j),(m,n)}((a,b), z, \tau) \) is an entire function.

Summarizing the above discussion, \( U_{(i,j),(m,n)}((a,b), z, \tau) \) is a doubly periodic entire function on \( \mathbb{C} \). Then by the well-known Liouville theorem, there exists a constant \( c_{(i,j),(m,n)}((a,b), \tau) \) such that
\[
U_{(i,j),(m,n)}((a,b), z, \tau) = c_{(i,j),(m,n)}((a,b), \tau).
\]

If \( ia + mb \) is odd, changing the variable from \( z \) to \( z + \tau \), we have
\[
c_{(i,j),(m,n)}((a,b), \tau) = U_{(i,j),(m,n)}((a,b), z + \tau, \tau)
= (-1)^{ia + mb} U_{(i,j),(m,n)}((a,b), z, \tau)
= -U_{(i,j),(m,n)}((a,b), z, \tau)
= -c_{(i,j),(m,n)}((a,b), \tau).
\]

Here the second equality follows from double periodicity (3.2). If \( ia + mb \) is even, from the assumption of theorem, then \( ja + nb \) is odd. Hence, changing the variable from \( z \) to \( z + 1 \), we have
\[
c_{(i,j),(m,n)}((a,b), \tau) = U_{(i,j),(m,n)}((a,b), z + 1, \tau)
= (-1)^{ja + nb} U_{(i,j),(m,n)}((a,b), z, \tau)
= -U_{(i,j),(m,n)}((a,b), z, \tau)
= -c_{(i,j),(m,n)}((a,b), \tau).
\]

Therefore \( c_{(i,j),(m,n)}((a,b), \tau) \equiv 0 \) in any cases and we obtain the conclusion. \( \square \)

Expanding both side of (3.1) and comparing coefficients of \( z^{2N} \), we obtain reciprocity laws for elliptic Dedekind sums, which is a natural generalization of Fukuhara-Yui’s main result (Theorem 2.2 (1) in [3]).
**Theorem 4** (Reciprocity laws for elliptic Dedekind sums). If \(ia + mb\) or \(ja + nb\) is odd, then

\[
\frac{1}{(2N)!} \frac{1}{a} \sum_{\mu,\nu=0}^{a-1} (-1)^{\mu+j\nu} f_{m,n} \left( b \frac{\mu \tau + \nu}{a}, \tau \right) f_{ia+mb,ja+nb} \left( \frac{\mu \tau + \nu}{a}, \tau \right)
\]

\[
+ \frac{1}{(2N)!} \frac{1}{b} \sum_{\mu,\nu=0}^{b-1} (-1)^{m\mu+n\nu} f_{i,j} \left( a \frac{\mu \tau + \nu}{b}, \tau \right) f_{ia+mb,ja+nb} \left( \frac{\mu \tau + \nu}{b}, \tau \right)
\]

\[
= - \frac{N}{s=0} C_{i,j}(s, \tau) C_{m,n}(N-s, \tau) a^{2s-1} b^{2N-2s-1} - \frac{1}{ab} (2N+1) C_{ia+mb,ja+nb}(N, \tau),
\]

(3.4)

where \(f_{ia+mb,ja+nb}(z)\) is the \(2N\)-th derivative of the \(f_{ia+mb,ja+nb}(z)\), and \(C_{i,j}(s, \tau)\) is the coefficients of the Laurent expansions of \(f_{i,j}(z, \tau)\) at \(z = 0\)

\[
f_{ij}(z, \tau) = \frac{1}{z} \sum_{s=0}^{\infty} C_{ij}(s, \tau) z^{2s}, \quad C_{ij}(0, \tau) = 1, \quad C_{ij}(1, \tau) := C_{ij}(\tau).
\]

In particular, considering the case of \(N = 0\) of (3.4) or taking the limit \(z \to 0\) in (3.3), we obtain reciprocity laws for elliptic Dedekind sums.

**Corollary 5.** If \(ia + mb\) or \(ja + nb\) is odd, then

\[
\frac{1}{a} \sum_{\mu,\nu=0}^{a-1} (-1)^{\mu+j\nu} f_{m,n} \left( b \frac{\mu \tau + \nu}{a}, \tau \right) f_{ia+mb,ja+nb} \left( \frac{\mu \tau + \nu}{a}, \tau \right)
\]

\[
+ \frac{1}{b} \sum_{\mu,\nu=0}^{b-1} (-1)^{m\mu+n\nu} f_{i,j} \left( a \frac{\mu \tau + \nu}{b}, \tau \right) f_{ia+mb,ja+nb} \left( \frac{\mu \tau + \nu}{b}, \tau \right)
\]

\[
= - \frac{b}{a} C_{m,n}(\tau) - \frac{a}{b} C_{i,j}(\tau) - \frac{1}{ab} C_{ia+mb,ja+nb}(\tau).
\]

(3.5)

### 4 All the examples of (3.1) and (3.5)

In this section, we give all the examples of (3.1) and (3.5) up to the constant factor \((2K)^2\) explicitly.

#### 4.1 \((i, j) = (1, 0), (m, n) = (1, 0)\)

**4.1.1** \(2 \nmid a + b\)

In this case, (3.1) and (3.5) are Fukuhara-Yui’s results (1.11) and (1.12) respectively.
4.2 $(i,j) = (1,1), \ (m,n) = (1,0)$

4.2.1 $2 \not| a + b, \ 2 \not| a$

\[
\frac{1}{a} \sum_{\mu,\nu=0}^{a-1} (-1)^{\mu+\nu} \text{cs} \left( 2Kb\frac{\mu \tau + \nu}{a},k \right) \text{cs} \left( 2K \left( z + \frac{\mu \tau + \nu}{a} \right),k \right) + \frac{1}{b} \sum_{\mu,\nu=0}^{b-1} (-1)^{\mu} \text{ds} \left( 2Ka\frac{\mu \tau + \nu}{b},k \right) \text{ds} \left( 2K \left( z + \frac{\mu \tau + \nu}{b} \right),k \right)
\]

\[= -\text{ds} (2Kaz,k) \text{cs} (2Kbz,k) + \frac{1}{ab} \text{ns} (2Kz,k) \text{cs} (2Kz,k), \quad (4.1)\]

\[
\frac{1}{a} \sum_{\mu,\nu=0}^{a-1} (-1)^{\mu+\nu} \text{cs} \left( 2Kb\frac{\mu \tau + \nu}{a},k \right) \text{cs} \left( 2K \left( z + \frac{\mu \tau + \nu}{a} \right),k \right) + \frac{1}{b} \sum_{\mu,\nu=0}^{b-1} (-1)^{\mu} \text{ds} \left( 2Ka\frac{\mu \tau + \nu}{b},k \right) \text{ds} \left( 2K \left( z + \frac{\mu \tau + \nu}{b} \right),k \right)
\]

\[= -a^2 + 2b^2 - 1 + \frac{2a^2 - b^2 + 2}{6ab} \lambda(\tau). \quad (4.2)\]

By taking the limit $\tau \to \sqrt{-1}\infty$ and (2.26) - (2.30), (4.1) and (4.2) degenerate to (1.3) and (1.8) respectively.

4.2.2 $2 \not| a + b, \ 2 \mid a$

\[
\frac{1}{a} \sum_{\mu,\nu=0}^{a-1} (-1)^{\mu+\nu} \text{cs} \left( 2Kb\frac{\mu \tau + \nu}{a},k \right) \text{cs} \left( 2K \left( z + \frac{\mu \tau + \nu}{a} \right),k \right) + \frac{1}{b} \sum_{\mu,\nu=0}^{b-1} (-1)^{\mu} \text{ds} \left( 2Ka\frac{\mu \tau + \nu}{b},k \right) \text{ds} \left( 2K \left( z + \frac{\mu \tau + \nu}{b} \right),k \right)
\]

\[= -\text{ds} (2Kaz,k) \text{cs} (2Kbz,k) + \frac{1}{ab} \text{ns} (2Kz,k) \text{cs} (2Kz,k), \quad (4.3)\]
By taking the limit $\tau \to \sqrt{-1}\infty$ and (2.26) - (2.30), we have

\[
\lim_{\tau \to \sqrt{-1}\infty} \Phi_{(1,1),(1,0)}((a, b), z, \tau) = -\csc \left( \pi a z \right) \cot \left( \pi b z \right) + \frac{1}{ab} \csc(\pi z)^2.
\]

Since $a$ is even and the third term in (4.5) vanishes, (4.3) and (4.4) degenerate to (1.2) and (1.7) respectively.
4.2.3 2 | a + b, 2 \nparallel a

\[ \frac{1}{a} \sum_{\mu, \nu = 0}^{a-1} (-1)^{\mu+\nu} \text{cs} \left( 2Kb\frac{\mu+\nu}{a}, k \right) \text{ns} \left( 2K \left( z + \frac{\mu+\nu}{a} \right), k \right) \]

\[ + \frac{1}{b} \sum_{\mu, \nu = 0}^{b-1} (-1)^{\mu} \text{ds} \left( 2Ka\frac{\mu+\nu}{b}, k \right) \text{ns} \left( 2K \left( z + \frac{\mu+\nu}{b} \right), k \right) \]

\[ = -\text{ds} (2Kaz, k) \text{cs} (2Kbz, k) + \frac{1}{ab} \text{ds} (2Kz, k) \text{cs} (2Kz, k), \quad (4.6) \]

\[ \frac{1}{a} \sum_{\mu, \nu = 0}^{a-1} (-1)^{\mu+\nu} \text{cs} \left( 2Kb\frac{\mu+\nu}{a}, k \right) \text{ds} \left( 2K \left( z + \frac{\mu+\nu}{a} \right), k \right) \]

\[ + \frac{1}{b} \sum_{\mu, \nu = 0}^{b-1} (-1)^{\mu} \text{ds} \left( 2Ka\frac{\mu+\nu}{b}, k \right) \text{ds} \left( 2K \left( z + \frac{\mu+\nu}{b} \right), k \right) \]

\[ = -a^2 + 2b^2 - 1 \frac{1}{6ab} + 2a^2 - b^2 - 1 \frac{1}{6ab} \lambda(\tau). \quad (4.7) \]

Taking the limit \( \tau \to \sqrt{-1}\infty \), (4.6) and (4.7) degenerate to (1.3) and (1.8) respectively.

4.3 \( (i, j) = (0, 1), (m, n) = (1, 0) \)

4.3.1 2 \nparallel b, 2 \nparallel a

\[ \frac{1}{a} \sum_{\mu, \nu = 0}^{a-1} (-1)^{\nu} \text{cs} \left( 2Kb\frac{\mu+\nu}{a}, k \right) \text{ds} \left( 2K \left( z + \frac{\mu+\nu}{a} \right), k \right) \]

\[ + \frac{1}{b} \sum_{\mu, \nu = 0}^{b-1} (-1)^{\mu} \text{ns} \left( 2Ka\frac{\mu+\nu}{b}, k \right) \text{ds} \left( 2K \left( z + \frac{\mu+\nu}{b} \right), k \right) \]

\[ = -\text{ns} (2Kaz, k) \text{cs} (2Kbz, k) + \frac{1}{ab} \text{ns} (2Kz, k) \text{cs} (z, k), \quad (4.8) \]
\[
\begin{align*}
\frac{1}{a} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0, 0)}^{a-1} (-1)^\nu \text{cs} \left( 2Kb \frac{\mu \tau + \nu}{a}, k \right) \text{cs} \left( 2K \frac{\mu \tau + \nu}{a}, k \right) & \\
+ \frac{1}{b} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0, 0)}^{b-1} (-1)^\mu \text{ns} \left( 2Ka \frac{\mu \tau + \nu}{b}, k \right) \text{cs} \left( 2K \frac{\mu \tau + \nu}{b}, k \right) & \\
= -a^2 + 2b^2 - 1 & \quad + -a^2 - b^2 + 2 \quad 6ab \quad \lambda(\tau).
\end{align*}
\] (4.9)

Taking the limit \( \tau \to \sqrt{-1} \infty \), (4.8) and (4.9) degenerate to (1.3) and (1.8) respectively.

4.3.2 \( 2 \parallel b, 2 \parallel a \)

\[
\begin{align*}
\frac{1}{a} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0, 0)}^{a-1} (-1)^\nu \text{cs} \left( 2Kb \frac{\mu \tau + \nu}{a}, k \right) \text{cs} \left( 2K \left( z + \frac{\mu \tau + \nu}{a} \right), k \right) & \\
+ \frac{1}{b} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0, 0)}^{b-1} (-1)^\mu \text{ns} \left( 2Ka \frac{\mu \tau + \nu}{b}, k \right) \text{cs} \left( 2K \left( z + \frac{\mu \tau + \nu}{b} \right), k \right) & \\
= -\text{ns} \left( 2Kaz, k \right) \text{cs} \left( 2Kbz, k \right) + \frac{1}{ab} \text{ns} \left( 2Kz, k \right) \text{ds} \left( 2Kz, k \right),
\end{align*}
\] (4.10)

\[
\begin{align*}
\frac{1}{a} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0, 0)}^{a-1} (-1)^\nu \text{cs} \left( 2Kb \frac{\mu \tau + \nu}{a}, k \right) \text{cs} \left( 2K \frac{\mu \tau + \nu}{a}, k \right) & \\
+ \frac{1}{b} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0, 0)}^{b-1} (-1)^\mu \text{ns} \left( 2Ka \frac{\mu \tau + \nu}{b}, k \right) \text{cs} \left( 2K \frac{\mu \tau + \nu}{b}, k \right) & \\
= -a^2 + 2b^2 + 2 & \quad + -a^2 - b^2 - 1 \quad 6ab \quad \lambda(\tau).
\end{align*}
\] (4.11)

Taking the limit \( \tau \to \sqrt{-1} \infty \), (4.10) and (4.11) degenerate to (1.2) and (1.7) respectively.
4.3.3 2 | b, 2 \not| a

\[
\frac{1}{a} \sum_{\mu,\nu=0}^{a-1} (-1)^\nu \cos \left( 2K b \frac{\mu \tau + \nu}{a}, k \right) \text{ns} \left( 2K \left( z + \frac{\mu \tau + \nu}{a} \right), k \right)
\]
\[+ \frac{1}{b} \sum_{\mu,\nu=0}^{b-1} (-1)^\mu \text{ns} \left( 2K a \frac{\mu \tau + \nu}{b}, k \right) \text{ns} \left( 2K \left( z + \frac{\mu \tau + \nu}{b} \right), k \right)
\]
\[= -\text{ns} (2Kaz, k) \cos (2Kbz, k) + \frac{1}{ab} \text{ds} (2Kz, k) \cos (2Kz, k),
\]
(4.12)

\[
\frac{1}{a} \sum_{\mu,\nu=0}^{a-1} (-1)^\nu \cos \left( 2K b \frac{\mu \tau + \nu}{a}, k \right) \text{ns} \left( 2K \frac{\mu \tau + \nu}{a}, k \right)
\]
\[+ \frac{1}{b} \sum_{\mu,\nu=0}^{b-1} (-1)^\mu \text{ns} \left( 2K a \frac{\mu \tau + \nu}{b}, k \right) \text{ns} \left( 2K \frac{\mu \tau + \nu}{b}, k \right)
\]
\[= -\frac{a^2 + 2b^2 - 1}{6ab} + \frac{-a^2 - b^2 - 1}{6ab} \lambda(\tau).
\]
(4.13)

Taking the limit \( \tau \to \sqrt{-1}\infty \), (4.12) and (4.13) degenerate to (1.3) and (1.8) respectively.

4.4 \( (i, j) = (1, 1), (m, n) = (1, 1) \)

4.4.1 2 \not| a + b

\[
\frac{1}{a} \sum_{\mu,\nu=0}^{a-1} (-1)^{\mu+\nu} \text{ds} \left( 2K b \frac{\mu \tau + \nu}{a}, k \right) \text{ds} \left( 2K \left( z + \frac{\mu \tau + \nu}{a} \right), k \right)
\]
\[+ \frac{1}{b} \sum_{\mu,\nu=0}^{b-1} (-1)^{\mu+\nu} \text{ds} \left( 2K a \frac{\mu \tau + \nu}{b}, k \right) \text{ds} \left( 2K \left( z + \frac{\mu \tau + \nu}{b} \right), k \right)
\]
\[= -\text{ds} (2Kaz, k) \text{ds} (2Kbz, k) + \frac{1}{ab} \text{ns} (2Kz, k) \cos (2Kz, k),
\]
(4.14)
\[
\frac{1}{a} \sum_{\substack{\mu, \nu = 0 \\
(\mu, \nu) \neq (0,0)}}^{a-1} (-1)^{\mu+\nu} ds \left( 2K b^{\frac{\mu \tau + \nu}{a}, k} \right) ds \left( 2K^{\frac{\mu \tau + \nu}{a}, k} \right) \\
+ \frac{1}{b} \sum_{\substack{\mu, \nu = 0 \\
(\mu, \nu) \neq (0,0)}}^{b-1} (-1)^{\mu+\nu} ds \left( 2K a^{\frac{\mu \tau + \nu}{b}, k} \right) ds \left( 2K^{\frac{\mu \tau + \nu}{b}, k} \right) \\
= -\frac{a^2 + b^2 + 1}{6ab} (1 - 2\lambda(\tau)).
\]

(4.15)

Taking the limit \( \tau \to \sqrt{-1}\infty \), (4.14) and (4.15) degenerate to (1.5) and (1.10) respectively.

4.5 \((i, j) = (0, 1), (m, n) = (1, 1)\)

4.5.1 \(2 \not\parallel b, 2 \not\parallel a + b\)

\[
\frac{1}{a} \sum_{\substack{\mu, \nu = 0 \\
(\mu, \nu) \neq (0,0)}}^{a-1} (-1)^{\nu} ds \left( 2K b^{\frac{\mu \tau + \nu}{a}, k} \right) ds \left( 2K^{\left( z + \frac{\mu \tau + \nu}{a} \right), k} \right) \\
+ \frac{1}{b} \sum_{\mu, \nu = 0 \atop (\mu, \nu) \neq (0,0)}^{b-1} (-1)^{\mu+\nu} ns \left( 2K a^{\frac{\mu \tau + \nu}{b}, k} \right) ds \left( 2K^{\left( z + \frac{\mu \tau + \nu}{b} \right), k} \right) \\
= -ns (2Kaz, k) ds (2Kbz, k) + \frac{1}{ab} ns (2Kz, k) cs (2Kz, k),
\]

(4.16)

\[
\frac{1}{a} \sum_{\substack{\mu, \nu = 0 \\
(\mu, \nu) \neq (0,0)}}^{a-1} (-1)^{\nu} ds \left( 2K b^{\frac{\mu \tau + \nu}{a}, k} \right) ds \left( 2K^{\mu \tau + \nu, k} \right) \\
+ \frac{1}{b} \sum_{\mu, \nu = 0 \atop (\mu, \nu) \neq (0,0)}^{b-1} (-1)^{\mu+\nu} ns \left( 2K a^{\frac{\mu \tau + \nu}{b}, k} \right) ds \left( 2K^{\mu \tau + \nu, k} \right) \\
= \frac{-a^2 - b^2 - 1}{6ab} + \frac{-a^2 + 2b^2 + 2}{6ab} \lambda(\tau).
\]

(4.17)

Taking the limit \( \tau \to \sqrt{-1}\infty \), (4.16) and (4.17) degenerate to (1.5) and (1.10) respectively.
4.5.2 \(2 \not| b, 2 | a + b\)

\[
\frac{1}{a} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{a-1} (-1)^{\nu} \text{ds} \left( 2K \frac{\mu \tau + \nu}{a}, k \right) \text{cs} \left( 2K \left( z + \frac{\mu \tau + \nu}{a} \right), k \right) \\
+ \frac{1}{b} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{b-1} (-1)^{\mu+\nu} \text{ns} \left( 2K \frac{\mu \tau + \nu}{b}, k \right) \text{cs} \left( 2K \left( z + \frac{\mu \tau + \nu}{b} \right), k \right)
\]

\[
= -\text{ns} (2K a z, k) \text{ds} (2K b z, k) + \frac{1}{ab} \text{ns} (2K z, k) \text{ds} (2K z, k), \quad (4.18)
\]

\[
\frac{1}{a} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{a-1} (-1)^{\nu} \text{ds} \left( 2K \frac{\mu \tau + \nu}{a}, k \right) \text{ns} \left( 2K \left( z + \frac{\mu \tau + \nu}{a} \right), k \right) \\
+ \frac{1}{b} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{b-1} (-1)^{\mu+\nu} \text{ns} \left( 2K \frac{\mu \tau + \nu}{b}, k \right) \text{ns} \left( 2K \left( z + \frac{\mu \tau + \nu}{b} \right), k \right)
\]

\[
= -a^2 - b^2 + 2 + \frac{a^2 + 2b^2 - 1}{6ab} \lambda(\tau). \quad (4.19)
\]

Taking the limit \(\tau \to \sqrt{-1}\infty\), (4.18) and (4.19) degenerate to (1.4) and (1.9) respectively.

4.5.3 \(2 \parallel b, 2 \not| a + b\)

\[
\frac{1}{a} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{a-1} (-1)^{\nu} \text{ds} \left( 2K \frac{\mu \tau + \nu}{a}, k \right) \text{ns} \left( 2K \left( z + \frac{\mu \tau + \nu}{a} \right), k \right) \\
+ \frac{1}{b} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{b-1} (-1)^{\mu+\nu} \text{ns} \left( 2K \frac{\mu \tau + \nu}{b}, k \right) \text{ns} \left( 2K \left( z + \frac{\mu \tau + \nu}{b} \right), k \right)
\]

\[
= -\text{ns} (2K a z, k) \text{ds} (2K b z, k) + \frac{1}{ab} \text{ds} (2K z, k) \text{cs} (2K z, k), \quad (4.20)
\]

\[
\frac{1}{a} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{a-1} (-1)^{\nu} \text{ds} \left( 2K \frac{\mu \tau + \nu}{a}, k \right) \text{ns} \left( 2K \left( z + \frac{\mu \tau + \nu}{a} \right), k \right) \\
+ \frac{1}{b} \sum_{\mu, \nu=0 \atop (\mu, \nu) \neq (0,0)}^{b-1} (-1)^{\mu+\nu} \text{ns} \left( 2K \frac{\mu \tau + \nu}{b}, k \right) \text{ns} \left( 2K \left( z + \frac{\mu \tau + \nu}{b} \right), k \right)
\]

\[
= -a^2 - b^2 + 2 + \frac{a^2 + 2b^2 - 1}{6ab} \lambda(\tau). \quad (4.21)
\]
Taking the limit $\tau \to \sqrt{-1}\infty$, (4.20) and (4.21) degenerate to (1.5) and (1.10) respectively.

4.6 $(i, j) = (0, 1)$, $(m, n) = (0, 1)$

4.6.1 $2 \not| a + b$

\[
\begin{align*}
\frac{1}{a} \sum_{\mu, \nu = 0}^{a-1} \frac{(-1)^\nu}{(\mu, \nu) \neq (0, 0)} 
& \quad \left(2Kb\frac{\mu\tau + \nu}{a}, k\right) \text{ns} \left(2K \left(z + \frac{\mu\tau + \nu}{a}\right), k\right) \\
& + \frac{b}{b} \sum_{\mu, \nu = 0}^{b-1} \frac{(-1)^\nu}{(\mu, \nu) \neq (0, 0)} 
& \quad \left(2Ka\frac{\mu\tau + \nu}{b}, k\right) \text{ns} \left(2K \left(z + \frac{\mu\tau + \nu}{b}\right), k\right) \\
& = -\text{ns} (2Kaz, k)\text{ns} (2Kbz, k) + \frac{1}{ab} \text{ds} (2Kz, k)\text{cs} (2Kz, k),
\end{align*}
\]

(4.22)

\[
\begin{align*}
\frac{1}{a} \sum_{\mu, \nu = 0}^{a-1} \frac{(-1)^\nu}{(\mu, \nu) \neq (0, 0)} 
& \quad \left(2Kb\frac{\mu\tau + \nu}{a}, k\right) \text{ns} \left(2K \left(z + \frac{\mu\tau + \nu}{a}\right), k\right) \\
& + \frac{b}{b} \sum_{\mu, \nu = 0}^{b-1} \frac{(-1)^\nu}{(\mu, \nu) \neq (0, 0)} 
& \quad \left(2Ka\frac{\mu\tau + \nu}{b}, k\right) \text{ns} \left(2K \left(z + \frac{\mu\tau + \nu}{b}\right), k\right) \\
& = -\frac{a^2 + b^2 + 1}{6ab} (1 + \lambda(\tau)).
\end{align*}
\]

(4.23)

Taking the limit $\tau \to \sqrt{-1}\infty$, (4.22) and (4.23) degenerate to (1.5) and (1.10) respectively.

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