Chiral solitons from dimensional reduction of Chern-Simons gauged coupled non-linear Schrödinger model

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Abstract

The soliton structure of a gauge theory proposed to describe chiral excitations in the multi-Layer Fractional Quantum Hall Effect is investigated. A new type of derivative multi-component nonlinear Schrödinger equation emerges as effective description of the system that supports novel chiral solitons. We discuss the classical properties of the solutions and study relations to integrable systems.
1 Introduction

Study of two-dimensional electron system in a strong magnetic field is one of the most interesting in the condensed matter physics. In the fractional quantum hall effect, early work by Halperin anticipated novel fractional quantum hall effects due to inter-layer correlations in multi-layer systems. Recent technological progress has made it possible to produce double-layer two dimensional electron gas systems of extremely high mobility in which these effects can be observed.

One of the most fascinating aspects of classical field theory is the existence of traveling localized solutions of the nonlinear equations describing physical systems. From the quantum field theory point of view, they are believed to carry information about the non-perturbative structure of the quantum field theory. Their particle-like properties have an intuitive and reasonable interpretation as bound states of the elementary excitations of the corresponding quantum field theory.

Solitons made their appearance in a completely different context some years ago, in low energy phenomenological applications to physical systems confined to a plane. An interesting class of gauge theoretical models, describing matter coupled to a Chern-Simons gauge field was introduced to obtain a simple realization of interacting anyons. The physical reason lies that some characteristic of the model, in particular its fractional statistics, can be maintained by the related one-dimensional excitations and that, by a suitable modification, chiral behavior can be induced. This fact is relevant in the phenomenological description of the edge states in the quantum hall effect. As first observed in a novel and interesting soliton structure is present there, finding its origin in the gauge coupling and in the chiral modification. Our investigation is a extension of the work begun in.

We study a family of 1+1 dimensional theories that describes non relativistic bosons interacting with a gauge potential in a multi-component model. We show the chiral excitations which can be seen in one component case. This fact is relevant to describe the edge states in the multi-Layer quantum hall effect. In the multi-component case the theory contains self and inter-component interactions. We show the explicit soliton solutions which describe the soliton-soliton scattering for the inter-component interactions. In the one component case if the theory is modified by adding the suitable potential, the model becomes an integrable derivative nonlinear Schrödinger equation. The solitons are no longer chiral. In the multi-component case even if we add potential, we can not make this model integrable.

This letter is organized as follows. We start in section 2 by describing the extension of Jackiw-Pi model. We show that the system is equivalent to a general family of coupled non-linear Schrödinger equation that does not possess Galileian invariance. In the section
we discuss the conserved charges and derive some general properties of the localized classical solution. In the section 4 we compute the soliton solutions and discuss their properties. The last section is devoted to the concluding remarks.

2 Nonlinear Derivative Schrödinger Equation from a Gauge Theory

A non-relativistic gauge field theory that leads to planar anyons in the coupled nonlinear Schrödinger equation, gauged by a Chern-Simons field and governed by the Lagrange density

$$L^{(2+1)} = \frac{1}{4\kappa} \epsilon^{\alpha\beta\gamma} \hat{A}_\alpha \hat{F}_{\beta\gamma} + \frac{i\hbar}{2m} \sum_{k=1}^{N} \Psi_k^* (\partial_t + i\hat{A}_0) \Psi_k - \frac{\hbar^2}{2m} \sum_{k=1}^{N} \sum_{i=1}^{2} |(\partial_i + i\hat{A}_i) \Psi_k|^2 - V. \quad (2.1)$$

Here $\Psi_k$ is the Schrödinger quantum field in the $k$-th component, giving rise to charged bosonic particles after second quantization. $\hat{A}_\mu$ possesses no propagating degrees of freedom; it can be eliminated, leaving a statistical Aharonov-Bohm interaction between the particles. $V$ describes possible nonlinear self-interactions and interactions between components. It can be a general polynomial in the density $\Psi_k^* \Psi_k$, $k = 1, 2, \cdots, N$. $N$ is the number of components. We notice that the above Lagrangian is invariant under Galilean transformations, due to the topological nature of the Chern-Simons action. When analyzing the lineal problem [7], it is natural to consider a dimensional reduction of (2.1), by suppressing dependence on the second spatial coordinate, renaming $\hat{A}_2$ as $(mc/\hbar^2) B$ and redefining the gauge field as $\hat{A}_x = A_x$ and $\hat{A}_0 = A_0 + mc^2/2\hbar B^2$. In this way one is led to a $B$-$F$ gauge theory coupled to a non-relativistic bosonic fields in 1 + 1 dimensions:

$$L^{(1+1)} = \frac{1}{2\kappa} B \epsilon^{\mu\nu} F_{\mu\nu} + \frac{i\hbar}{2m} \sum_{k=1}^{N} \Psi_k^* (\partial_t + iA_0) \Psi_k - \frac{\hbar^2}{2m} \sum_{k=1}^{N} |(\partial_x + iA_1) \Psi_k|^2 - V, \quad (2.2)$$

where $\kappa \equiv (\hbar^2/mc) \tilde{\kappa}$ is dimensionless and we have neglected $\partial_x (B^3/3h\kappa)$ since it is a total spatial derivative. In the following we prefer a simpler expression that describes “chiral” Bose fields, propagating only in one direction we choose the $B$-kinetic Lagrange density to be

$$L_B = \frac{\lambda}{2\kappa^2 \hbar} \dot{B} B', \quad (2.3)$$

and the total Lagrange density is $L = L_B + L^{(1+1)}$. [8] The equations of motion are

$$i\hbar (\partial_t + iA_0) \Psi_i + \frac{\hbar^2}{2m} (\partial_x + iA_1)^2 \Psi_i - V' \Psi_i = 0, \quad (2.4a)$$

$$F_{01} - \frac{\lambda}{\kappa \hbar} \dot{B}' = 0, \quad (2.4b)$$
\[ B' - \hbar \kappa \sum_{k=1}^{N} \Psi^*_k \Psi_k = 0, \quad (2.4c) \]

\[ \dot{B} + \hbar \kappa \sum_{k=1}^{N} \dot{J}_k = 0, \quad (2.4d) \]

with \( \dot{J}_k = \frac{\hbar}{2im} (\Psi^*_k (\partial_x + iA_1) \Psi_k - \Psi_k (\partial_x - iA_1) \Psi^*_k) \). Here dot/prime indicate differentiation with respect to time/space. The integrability condition for the last two equation leads to the usual continuity equation

\[ \partial_t (\sum_{k=1}^{N} \Psi^*_k \Psi_k) + \partial_z (\sum_{k=1}^{N} \dot{J}_k) = 0. \quad (2.5) \]

Now in terms of field \( \hat{\phi}_i = \exp \left( i \int_{x_0}^{x} dy A_1(y, t) + i \int_{t_0}^{t} dt' A_0(x^0, t') - i \frac{\lambda}{\kappa \hbar} B(x^0, t) \right) \Psi_i \) (2.4a) and (2.4b) become respectively

\[ i\hbar \partial_t \hat{\phi}_i + \frac{\hbar^2}{2m} \partial^2_x \hat{\phi}_i - \hbar \lambda (\sum_{k=1}^{N} \hat{J}_k) \hat{\phi}_i - V' \hat{\phi}_i = 0, \quad (2.6) \]

\[ F_{01} = \lambda \partial_t (\sum_{k=1}^{N} \hat{\phi}^*_k \hat{\phi}_k). \quad (2.7) \]

The latter gives the electro-magnetic field as a function of the density \( \sum_{k=1}^{N} \hat{\phi}^*_k \hat{\phi}_k \) while the former encodes all the dynamical contents of the system. In the following we shall be mainly interested in the case in which the potential \( V \) is absent: this corresponds to a coupled Schrödinger equation with a current \( (\sum_{k=1}^{N} \hat{J}_k) \) nonlinearity,

\[ j_k = \frac{\hbar}{2im} \left( \hat{\phi}^*_k \partial_x \hat{\phi}_k - \hat{\phi}_k \partial_x \hat{\phi}^*_k \right). \quad (2.8) \]

Note that (2.6) does not possess a local Lagrangian formulation directly in terms of the field \( \hat{\phi}_i \) as the one component case.

Consider the action

\[ S = \int dt \ dx \ L = \int dt \ dx \ \left[ \frac{i\hbar}{2} \sum_{i=1}^{N} (\dot{\phi}^*_i \partial_t \phi_i - \phi_i \partial_t \dot{\phi}_i^*) - \frac{\hbar^2}{2m} \sum_{i=1}^{N} \left( \partial_x + i \frac{\lambda}{2 \rho^2} \right) \phi_i \right]^2 - V, \quad (2.9) \]

where \( \rho^2 = \sum_{k=1}^{N} \hat{\phi}_k \hat{\phi}^*_k \). The equation of motion is

\[ i\hbar \partial_t \phi_i = - \frac{\hbar^2}{2m} \left( \partial_x + i \frac{\lambda}{2 \rho^2} \right)^2 \phi_i + \frac{\lambda \hbar}{2} (\sum_{k=1}^{N} \hat{J}_k) \phi_i + V' \phi_i, \quad (2.10) \]

The relation between (2.6) and (2.10) is the gauge equivalent

\[ \phi_i = \exp \left[ i \frac{\lambda}{2} \int_{x_0}^{x} dy (\sum_{k=1}^{N} \hat{\phi}_k^* \hat{\phi}_k) (y, t) \right] \hat{\phi}_i. \quad (2.11) \]
The current corresponds to
\[ J_k = \frac{\hbar}{2im} \left( J_k^* \left( \partial_x + i\frac{\lambda}{2}\rho^2 \right) J_k - J_k \left( \partial_x - i\frac{\lambda}{2}\rho^2 \right) J_k^* \right), \quad (2.12) \]

### 3 Constants of Motion and Symmetries

We look for the conservation laws:
\[ \frac{\partial D_j}{\partial t} + \frac{\partial F_j}{\partial x} = 0, \quad (3.1) \]
where \( D_j \) and \( F_j \) are respectively conserved density and flux. There is the conserved quantities like amplitude,
\[ D_{ij}^1 = \phi_i^* \phi_j, \quad F_{ij}^1 = \frac{\hbar}{2im} \left( \phi_i^* \partial_x \phi_j - \hat{\phi}_j (\partial_x \phi_i)^* \right), \quad (3.2) \]
where \( \partial_x \) stands for the “covariant” derivative \( \partial_x + i\lambda\rho^2/2 \). In the case \( i = j \) conserved density and the flux become the amplitude and the current respectively. The conserved density in each line and inter-component are only \( D_{ij}^1 \). The other conserved densities are sum over all components. It is also true in the integrable model as \( (4.1) \) which has infinite number of conserved quantities.

Space translation invariance ensures momentum conservation, and the momentum density reads
\[ D_2 = \mathcal{P} = m \sum_{k=1}^{N} J_k - \hbar \frac{\lambda}{2} \rho^4. \quad (3.3) \]
With the momentum density \( D_2 \) we obtain the momentum flux
\[ F_2 = \frac{\hbar^2}{m} \sum_{k=1}^{N} |D_x \phi_k|^2 - \frac{\hbar^2}{4m} \partial_x^2 \rho^2 + V' \rho^2. \quad (3.4) \]
Here we define \( \mathcal{P}_i \)
\[ \mathcal{P}_i = m J_i - \hbar \frac{\lambda}{2} \rho^2 \phi_i \phi_i^*. \quad (3.5) \]
Note that \( \mathcal{P}_i \) is not conserved quantity and the sum of \( \mathcal{P}_i \) over all components is the momentum density.

Energy is conserved as a consequence of time transition invariance. Evidently the Hamiltonian density is
\[ D_3 = \mathcal{H} = \left[ \frac{\hbar^2}{2m} \sum_{k=1}^{N} |D_x \phi_k|^2 + V \right]. \quad (3.6) \]
With the energy density \( D_3 \) there is associated the energy flux
\[ F_3 = -\frac{\hbar^2}{2m} \sum_{k=1}^{N} [D_x \phi_k \partial_t \phi_k^* + (D_x \phi_k)^* \partial_t \phi_k]. \quad (3.7) \]
Here $D_2$ is not proportional to the current $\sum_{k=1}^{N} J_k$. Then we present the usual Galileo generator

$$G = t \int dx \mathcal{P} - m \int dx x^2. \quad (3.8)$$

We find

$$\frac{dG}{dt} = \int dx (\mathcal{P} - m \sum_{k} J_k) = -\hbar \frac{\lambda}{2} \int dx \rho^4,$$  \quad (3.9)

namely $G$, depending on the sign of the coupling constant, always increases or decreases in time. Here we introduce generators

$$G_i = t \int dx \mathcal{P}_i - m \int dx x^2 \rho_i^2. \quad (3.10)$$

In the same way we can obtain

$$\frac{dG_i}{dt} = \int dx (\mathcal{P}_i - m J_i) = -\hbar \frac{\lambda}{2} \int dx x^2 \rho_i^2,$$  \quad (3.11)

where $\rho_i^2$ is density in $i$-th component $\phi_i \phi_i^*$. Depending on the sign of the coupling constant, $G_i$ always increases or decreases in time, too.

The additional symmetry is dilaton invariance which can be seen in the no coupled case. In fact the action (2.9) is unchanged under a dilation, $t \rightarrow a^2 t$, $x \rightarrow ax$, and $\phi(x, t) \rightarrow a^{\frac{1}{2}} \phi(a^2 t, ax)$.

The generator $D$ of the scale symmetry takes the form

$$D = \int dx \mathcal{D} = t \int dx \mathcal{H} - \frac{1}{2} \int dx x \mathcal{P}, \quad (3.12)$$

where the density $\mathcal{D} = tD_3 - \frac{1}{2} x D_2$ obeys the continuity equation

$$\partial_t \mathcal{D} + \partial_x \left( t D_3 - \frac{1}{2} x D_2 - \frac{\hbar^2}{8m} \partial_x \rho^2 \right) = 0. \quad (3.13)$$

We can remove the last term in (3.13) proportional to the derivative of $\rho^2$ by adding a super potential to the energy-momentum tensor. In fact if we define an improved $\hat{D}$ and $\hat{F}$

$$\hat{D}_3 = D_3 - \frac{\hbar^2}{8m} \partial_x^2 \rho^2, \quad \hat{F}_3 = F_3 - \frac{\hbar^2}{8m} \partial_x^2 \left( \sum_{k=1}^{N} J_k \right), \quad \hat{D}_2 = D_2, \quad \hat{F}_2 = F_2,$$  \quad (3.14)

we obtain

$$\partial_t \mathcal{D} + \partial_x \left( t \hat{D}_3 - \frac{1}{2} x \hat{F}_2 \right) = 0. \quad (3.15)$$

The new energy momentum tensor satisfies $2\hat{D}_3 = \hat{F}_2$.

We look for solutions that possess particular symmetries or whose specific functional dependence simplified the structure of the original equation. In our case a simple ansatz is to assume that the density in $i$-th component is a function only of $x - v_i t$. We expect that
this choice will allow us to explore the presence of the soliton like solution. In the following we shall be conserved with solutions that approach the vacuum at spatial infinity. This class of solutions is strongly constrained by the symmetry of the problem. Substituting the ansatz into the continuity equation, yields

\[ \partial_x (-v_i \rho_i^2 (x - v_i t) + J_i(x, t)) = 0, \tag{3.16} \]

and hence

\[ J_i(x, t) = v_i \rho_i^2 (x - v_i t) + J_i^\infty(t). \tag{3.17} \]

From the ansatz at spatial infinity we consider the case \( J_i^\infty(t) = 0 \). (3.16) implies that \( P_i \) is a function of \( x - v_i t \). Sum \( P_i \) over all components is the momentum density \( \mathcal{P} \). The dilation charge takes the form

\[ D = \frac{t}{2} \sum_{i=1}^{N} \int_{-\infty}^{\infty} dx (x - v_i t) \rho_i^2(x - v_i t) - \frac{N}{2} v_i t \int_{-\infty}^{\infty} dx \mathcal{P}_i(x - v_i t) = t(H - \sum_{i=1}^{N} \frac{v_i}{2} P_i) - \frac{1}{2} D_0, \tag{3.18} \]

where \( D_0 = \sum_{i}^{N} \int_{-\infty}^{\infty} dx \mathcal{P}_i(x) \). Here we write a conserved momentum \( P = \int dx \mathcal{P} \) and energy \( H = \int dx \mathcal{H} \).

Since \( D \) is conserved and consequently time-independent we obtain

\[ H = \sum_{i=1}^{N} \frac{v_i}{2} P_i, \tag{3.19} \]

where \( P_i = \int dx \mathcal{P}_i \).

From the ansatz \( v_i P_i/2 \) is a function of \( x - v_i t \). Here we define the energy density \( \mathcal{H}_i \) in the \( i \)-th component

\[ \mathcal{H}_i = \frac{\hbar^2}{2m} |D_x \phi_i|^2. \tag{3.20} \]

Note that \( \mathcal{H}_i \) is not conserved quantity and the sum of \( \mathcal{H}_i \) over all components is the energy density. From the ansatz \( H_i \) is a function of \( x - v_i t \). Then we can obtain the relation

\[ H_i = \frac{v_i}{2} P_i, \tag{3.21} \]

where \( H_i = \int dx \mathcal{H}_i \).

To study further properties, we introduce the “center of mass” coordinate

\[ x_i^{\text{CM}}(t) = \frac{\int_{-\infty}^{\infty} dx \ x \rho_i^2(x, t)}{\int_{-\infty}^{\infty} dx \ \rho_i^2(x, t)}. \tag{3.22} \]

This name is easily understood if we think of \( \rho_i^2 \) as the mass density in the \( i \)-th component. Its velocity will be

\[ v_i^{\text{CM}} = \frac{\dot{x}_i^{\text{CM}}(t)}{N_i}, \quad \text{with} \ N_i = \int_{-\infty}^{\infty} dx \ \rho_i^2(x, t). \tag{3.23} \]
Here we have used the continuity equation for the current to eliminate the time derivative of the density.

From (3.9) we can obtain a suggestive form,

$$
\lambda (P_i - m N_i v_{CM}^i(t)) = -\hbar \frac{\lambda^2}{2} \int_{-\infty}^{\infty} dx \rho^2(x, t) \rho_i^2 \leq 0.
$$  (3.24)

Being valid for all \( t \), this implies

$$
\lambda P_i \leq m N_i \lambda \min_{t \in \mathbb{R}} \{ v_{CM}^i(t) \}. \tag{3.25}
$$

One can show an analogous inequality for the energy. In fact let us consider the following inequality

$$
\int_{-\infty}^{\infty} dx \left| \phi_i + w \frac{\hbar}{2m_1} D_x \phi_i \right|^2 \geq 0, \tag{3.26}
$$

where \( w \) is an arbitrary parameter. In terms of the physical quantities

$$
N_i + w N_i v_{CM}^i(t) + \frac{w^2 E_i}{2m} \geq 0, \tag{3.27}
$$

with \( E_i = \int_{-\infty}^{\infty} dx H_i \). The fact that the previous equation holds for all \( w \) entails

$$
E_i \geq \frac{m N_i (v_{CM}^i)^2}{2}. \tag{3.28}
$$

Note that here we consider the case the potential is absent. From the ansatz we can obtain that \( v_{CM}^i(t) = v_i \). Thus (3.24) can be written as

$$
\lambda v_i \left( H_i - \frac{m N_i v_i^2}{2} \right) \leq 0, \tag{3.29}
$$

where we used (3.21).

It implies \( \lambda v_i < 0 \), i.e. the soliton is “chiral”. If we set \( \lambda > 0 \), it shows that all the traveling waves move to the left.

### 4 Explicit solution

The coupled nonlinear Schrödinger (coupled NLS) equation [4] is

$$
\text{i} \partial_T \hat{\phi}_i + \partial_X^2 \hat{\phi}_i + 2 \left( \sum_{k=1}^{N} \kappa_k \rho_k \right) \hat{\phi}_i = 0, \tag{4.1}
$$

where use the normalized time and space variables \( t = \hbar T \) and \( x = \sqrt{\hbar^2/2mX} \). This equation has bright solutions only in all \( \kappa_i > 0 \). If the \( \kappa_i < 0 \) then the soliton solution in the \( i \)-th component becomes dark.[10] It possesses the famous soliton solution of the form

$$
\hat{\phi}_i = \frac{g_i}{T}, \quad \hat{\phi}_i^* = \frac{g_i}{T}, \tag{4.2}
$$
for \(i = 1, 2, \cdots, N\). \(f\) is a real function. Here we consider the simple case \(N = 2\). The solution which has one soliton for each component is expressed by

\[
g_1 = \exp[\eta_1] + \alpha_{12}\alpha_{22}\alpha_{21}\beta_{22} \exp[\eta_1 + \eta_2 + \eta_2], \]

\[
g_\bar{1} = \exp[\eta_\bar{1}] + \alpha_{1\bar{1}}\alpha_{\bar{2}2}\alpha_{\bar{1}2}\beta_{\bar{1}2} \exp[\eta_1 + \eta_2 + \eta_\bar{2}], \]

\[
g_2 = \exp[\eta_2] + \alpha_{21}\alpha_{11}\alpha_{12}\beta_{11} \exp[\eta_1 + \eta_1 + \eta_1], \]

\[
g_\bar{2} = \exp[\eta_\bar{2}] + \alpha_{\bar{2}1}\alpha_{\bar{1}1}\alpha_{\bar{2}1}\beta_{\bar{2}1} \exp[\eta_1 + \eta_1 + \eta_\bar{1}], \]

\[
f = 1 + \alpha_{1\bar{1}}\beta_{12} \exp[\eta_1 + \eta_1 + \eta_\bar{1} + \eta_\bar{1}], \] (4.3)

with

\[
\eta_\mu = p_\mu X + ip_\mu^2 T + \eta_\mu^0, \]

\[
\alpha_{\mu\nu} = \frac{p_\mu - p_\nu}{p_\mu + p_\nu}, \]

\[
\beta_{\mu\bar{\nu}} = \frac{\kappa_{\mu}}{p_\mu^2 - p_\bar{\nu}^2}. \] (4.4)

\(p_\mu\) and \(\eta_\mu\) are complex constant parameters related to the amplitudes and position of solitons respectively for \(\mu = 1, \bar{1}, 2, \bar{2}\). \(p_\mu, p_\bar{\nu}\) and \(\eta_\mu^0, \eta_\mu^0\) are the complex conjugate. Note that \(\kappa_\mu = \kappa_\mu\). Here after we choose the parameter

\[
\eta_\mu^0 = \log \frac{\kappa_\mu}{4a_\mu^2}. \]

If we set \(p_\mu = a_\mu + ib_\mu\), the speed of soliton \(V_\mu\) is presented as

\[
\frac{V_\mu}{2} = b_\mu. \] (4.5)

To find the solution of (2.10) we take the form (4.2) so that

\[
j_i = \frac{h}{2im} \frac{g_i'\bar{g}_i - \bar{g}_i'g_i}{f^2} = \frac{hb_i\bar{g}_i g_i}{mf^2}. \] (4.6)

where ‘ means differentiation with space and our equation (2.10) becomes identical (1.1) with

\[
\kappa_i = -\frac{h^2 \lambda V_i}{2}. \] (4.7)

The last equation can be obtained using (4.5). So that (1.3) with the condition (1.7) becomes the soliton solution of (2.10). Note that \(V_i\) is a speed in the coordinate \(X\) and \(T\). The soliton solution exists provided \(\kappa_i > 0\); this requires

\[
V_i < 0 \quad (\text{for all } i \text{ if } \lambda > 0). \] (4.8)

The soliton in all components can only move to the left. Had we taken \(\lambda < 0\), then the solitons could move only to the right.
We now discuss the phase shifts. We assume \( V_1 > V_2 > 0 \). In the limit \( t \to -\infty \) with \( \eta_1 \) and \( \bar{\eta}_1 \) fixed and \( \eta_2 \) and \( \bar{\eta}_2 \to -\infty \), the soliton solution approaches to
\[
\phi_1 = \frac{\exp[\eta_1]}{1 + \alpha_{11}\beta_{11}\exp[\eta_1 + \eta_1]} = a_1 \text{sech}_1(X - V_1T) \exp[i\frac{V_1}{2}X - i\frac{V_1^2}{4} - a_1T], \quad (4.9)
\]
which is the 1-soliton solution moving with the velocity \( V_1 \) in the one component case.

Also the other limit \( t \to \infty \) with \( \eta_1 \) and \( \bar{\eta}_1 \) fixed and \( \eta_2 \) and \( \bar{\eta}_2 \to \infty \), the result is
\[
\phi_1 = a_1 \text{sech}_1(X - V_1T + \text{Re}\Delta_1) \exp[i\frac{V_1}{2}X - i(V_1^2 - a_1)T + i\text{Im}\Delta_1], \quad (4.10)
\]
where \( \Delta_1 \) is the phase shift which is the effort of the inter-component interactions. One can repeat the same limiting procedure but with \( \eta_2 \) and \( \bar{\eta}_2 \) fixed and obtain the other soliton sector moving with velocity \( V_2 \). This shows that the solution describes the scattering of the chiral solitons.

From the solutions we can obtain the phase shifts
\[
\Delta_1 = \log \alpha_{12}\alpha_{21}, \quad \Delta_1 = \log \alpha_{12}\alpha_{21}, \\
\Delta_2 = \log \alpha_{12}\alpha_{21}, \quad \Delta_2 = \log \alpha_{12}\alpha_{21}, \quad (4.11)
\]
where \( \Delta_i \) is the phase shift of the \( i \)-th component.

Here we change the “charge density” \( \rho^2 \)
\[
\rho^2 = \rho_1^2 - \rho_2^2. \quad (4.12)
\]
The equation of motion becomes
\[
i\hbar \partial_t \hat{\phi}_i + \frac{\hbar^2}{2m} \partial_x^2 \hat{\phi}_i - \hbar\lambda(j_1 - j_2) \hat{\phi}_i = 0. \quad (4.13)
\]
To find the solution of (4.13) we take also the form (4.2). Using (4.6) we can obtain the relations
\[
\kappa_1 = -\frac{\hbar^2\lambda V_1}{2}, \quad \kappa_2 = \frac{\hbar^2\lambda V_2}{2}. \quad (4.14)
\]
We can obtain the explicit soliton solution which can move left in the component 1 and right in the component 2, in the case \( \lambda > 0 \).

### 5 Integrable Coupled Derivative Nonlinear Schrödinger equation

It is well known that the coupled hybrid nonlinear Schrödinger equation[1][2]
\[
i\hat{Q}_T + \hat{Q}_{XX} + \sum_k^N \beta|\hat{Q}|^2\hat{Q} + i\alpha \left( \sum_k^N |\hat{Q}|^{2k}\hat{Q} \right)_X = 0, \quad (5.1)
\]
is integrable. These equations are a hybrid of the coupled NLS \[^4\] and coupled derivative NLS equation. \[^3\] If we set
\[ Q^i = \hat{Q}^i \exp \left( -2i\delta \sum_k^N \int X |\hat{Q}^k|^2 dX \right), \]
then (5.1) is gauge-equivalent \[^4\]\[^3\] to
\[ iQ^i_T + Q^i_{XX} + \beta \rho^2 Q^i - 2i\delta A Q^i + i(2\delta + \alpha) B Q^i + i(4\delta + \alpha) \rho^2 Q^i_X + \delta (4\delta + \alpha) \rho^4 Q^i = 0, \] (5.3)
where
\[ A = \sum_k^N (Q^k X Q^{k*} - Q^{k*} X Q^k), \]
\[ B = \sum_k^N (Q^k X Q^{k*} + Q^{k*} X Q^k), \]
\[ \rho^2_Q = \sum_k^N |Q^k|^2, \] (5.4)
(5.3) is the coupled version of the generalized derivative NLS equation. \[^3\]\[^6\]
If we set $4\delta + \alpha = 0$, we can obtain the coupled version of Chen-Lee-Liu type equation\[^3\]
\[ iQ^i_T + Q^i_{XX} + \beta \rho^2 Q^i - 4i\delta (\sum_k^N (Q^k X Q^{k*})) Q^i = 0. \] (5.5)
In one component case there is the relation
\[ -A Q = B Q - 2\rho^2 Q_X, \] (5.6)
so that by adding an attractive potential with a fixed coefficient (in the case $3\delta + 8\alpha = 0$) the chiral soliton model can be made integrable.\[^5\]. On the other hand in multi-component case we can not use (5.6), then only adding potential we can not make the model integrable.

6  Concluding Remarks

We have studied a family of 1+1 dimensional theories that describes non relativistic bosons interacting with a gauge potential in a multi-component model. This form was suggested from the dimensional reduction of Chern-Simons theory coupled to non-relativistic matter and it represents a simple way to introduce chiral excitations as the one component case. It can be exactly reduced, solving for $A_\mu$ and $B$, to a bosonic theory with self and inter-component interactions. For this model a local Lagrangian formulation is possible. The soliton structure of the theory has been examined, and it exhibits an interesting chiral
behavior. In the multi-component case we can see the soliton-soliton scattering for the inter-component interactions. If we change the density, we can choose the chirality of the soliton solution in each component. Only adding potential we can not make the model integrable.

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