SUBCRITICALITY, POSITIVITY, AND GAUGEABILITY
OF THE SCHRÖDINGER OPERATOR

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1. INTRODUCTION

We investigate properties of the Schrödinger operator
\[ H := -(\Delta/2) + V \geq 0 \]
in \( \mathbb{R}^d (d \geq 3) \) in the following three aspects:

(I) **Subcriticality**: Intuitively, the idea is that if \( H \geq 0 \) is subcritical, then it should be possible to perturb \( H \) by small perturbations and still keep its nonnegativity. More precisely, we have the following assertions:

(a) For any \( q \in B_c \) (\( B_c \) denotes the class of bounded Borel functions with compact support), there exists an \( \varepsilon > 0 \) such that \( -(\Delta/2) + V + \varepsilon q \geq 0 \).

(b) There exists a function \( q \in B_c \), \( q \leq 0 \) and \( q \not\equiv 0 \) a.e. such that \( -(\Delta/2) + V + q \geq 0 \).

There have been two other definitions of subcriticality:

(c) (B. Simon [7]) There exists \( \beta > 0 \) such that \( -(\Delta/2) + (1 + \beta)V \geq 0 \).

(d) (M. Murata [6]) There exists a positive Green function \( G^H(\cdot, \cdot) \) for \( H \).

(II) **Strong Positivity**:

(e) There exists a positive solution \( u > 0 \) of \( Hu = 0 \) with the limit: \( \lim_{|x| \to \infty} u(x) > 0 \).

(f) There exists a solution \( u \) of \( Hu = 0 \) with \( c' \geq u \geq c > 0 \).

(g) There exists a solution \( u \) of \( Hu = 0 \) with \( u \geq c > 0 \).

(III) **Gaugeability**: Let \( \{X_t : t \geq 0\} \) be the Brownian motion in \( \mathbb{R}^d \) and let \( E^x \) denote the expectation over the Brownian paths starting from \( x \in \mathbb{R}^d \). Put \( u_0(x) := E^x[\exp(-\int_0^\infty V(X_s) \, ds)] \).

(h) \( u_0(x) \neq \infty \) in \( \mathbb{R}^d \).

(i) \( u_0(x) \) is bounded in \( \mathbb{R}^d \).

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For any \( y \) in \( \mathbb{R}^d \), we define the \( y \)-conditional Brownian motion of Doob type (see [10]) and use \( E^x_y \) to denote the expectation over the \( y \)-conditional Brownian paths starting from \( x \). Put

\[
u_0(x, y) := E^x_y \left[ \exp \left(-\int_0^\xi V(X_s) \, ds \right) \right], \quad x, y \in \mathbb{R}^d,
\]

where \( \xi \) is the lifetime of the process.

(j) \( \nu_0(x, y) < \infty \) for some \( (x, y) \) in \( \mathbb{R}^d \times \mathbb{R}^d \), \( x \neq y \).

(k) \( \nu_0(x, y) \) is bounded in \( \mathbb{R}^d \times \mathbb{R}^d \).

Our main result is the equivalence of all the assertions (a) through (k) listed above for a large class of potentials \( V \) given below.

2. Restricted Kato class \( K_d^{\infty} \)

For a function \( V \) in \( K_d^{\text{loc}, \infty} \), \( d \geq 3 \) (see [8] for definition of the Kato classes \( K_d^{\text{loc}} \) and \( K_d^{\infty} \)), we add a similar Kato condition around the point at \( \infty \) and then form a new class \( K_d^{\infty} \) called the restricted Kato class:

\[
K_d^{\infty} := \left\{ V \in K_d^{\text{loc}, \infty} : \lim_{A \to \infty} \sup_{|y| \geq A} \int_{|y| \geq A} \frac{|V(y)|}{|y-x|^{d+2}} \, dy = 0 \right\}.
\]

It is easy to see that \( K_d^{\infty} \cap L^1(\mathbb{R}^d) \subseteq K_d^{\infty} \subseteq K_d^{\infty} \). It can be verified by Hölder’s inequality that \( K_d^{\infty} \) also contains the class of “short range potentials”:

\[
\{ V \in K_d^{\infty}, V(x) = O(|x|^{-\rho}) \text{ as } |x| \to \infty, \rho > 2 \}.
\]

We note that Murata [5] proved some part of the above-mentioned equivalences for subcriticality for potentials satisfying the condition in (2) with \( \rho > 4 \).

For \( V \in K_d^{\infty} \), put \( ||V|| := \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (|V(y)|/|x-y|^{d-2}) \, dy < \infty \). We add two more assertions to the list in (I):

(l) There exists an \( \varepsilon > 0 \) such that for any \( q \in K_d^{\infty} \) with \( ||q|| < \varepsilon \), \( -(\Delta/2) + V + q \geq 0 \).

(m) There exists a function \( q \in K_d^{\infty} \), \( q \leq 0 \) and \( q \neq 0 \) a.e. such that \( -(\Delta/2) + V + q \geq 0 \).
3. Main Theorem and Sketch of the Proof

**Theorem.** For any $V \in K^\infty_d (d \geq 3)$, the conditions (a) through (m) are equivalent.

**Sketch of the proof.** Since $V \in K^\infty_d$, there exists a $r > 0$ such that

\[
\sup_{|x| \geq r} \left[ C_d \int_{|y| \geq r} \frac{|V(y)|}{|x - y|^{d-2}} \, dy \right] < \frac{1}{2},
\]

where $C_d = \Gamma((d/2) - 1)/2\pi^{d/2}$. Let $D = \{x \in R^d : |x| > r\}$ and $B = \{x \in R^d : |x| < 2r\}$. Put $T := \tau_B + \tau_D \circ \theta_{\tau_B}$ (the shuttle time), where $\tau_U$ is the exit time from a domain $U$ and $\theta$ is the shift operator on paths. We define the shuttle operator $S_V$ in the Banach space $C(\partial D)$: for $f \in C(\partial D)$,

\[
S_V f(x) := \mathcal{E}^x \left[ T < \infty ; \exp \left( - \int_0^T V(X(s)) \, ds \right) f(X(T)) \right],
\]

$x \in \partial D$.

By Khasmin'skii's lemma together with (3) and the arguments similar to those in [10], we can prove $S_V$ is an integral operator with continuous kernel:

\[
S_V f(x) = \int_{\partial D} \Phi(x, y) f(y) \sigma(dy) \quad (\sigma \text{ is the area measure}),
\]

where

\[
\Phi(x, y) = 9(d - 2)^2 C^2 d r^2 \times \int_{\partial D} \frac{E_x^z \exp \left( - \int_0^{\tau_D} V(X(s)) \, ds \right) E_y^z \exp \left( - \int_0^{\tau_D} V(X(s)) \, ds \right)}{|x - z|^d |y - z|^d} \sigma(dz),
\]

$(x, y) \in \partial D \times \partial D$.

Put $\lambda(V) := \lim_{n \to \infty} \sqrt[n]{\|S_V^n\}$.

Introducing the shuttle operator $S_V$ and its spectral radius $\lambda(V)$ is the key idea in connecting the seemingly different assertions in the list (a) through (m). In fact, we add a new equivalent assertion as a linkage among the assertions (a) through (m):

\[
(n) \quad \lambda(V) < 1.
\]
\[ \lambda(V) \], as a function of \( V \), has the following properties:

**Lemma.** (L1) If \( ||V_n - V|| \to 0 \), then \( \lambda(V_n) \to \lambda(V) \).

(L2) If \( V_1 \leq V_2 \) and \( V_1 \neq V_2 \) a.e., then \( \lambda(V_1) > \lambda(V_2) \).

Both properties are based on the integral kernel representation (5) in terms of path integrals. We also need a characterization of nonnegativity of \( H \), which can be regarded as a higher dimensional version of a result by Chung and Varadhan [2].

**Proposition A.** For \( V \in K_{d}^\infty \), \( -(\Delta/2) + V \geq 0 \) if and only if \( \lambda(V) \leq 1 \).

We now sketch the proof of some nontrivial implications in connection with (n). (n) \( \Leftrightarrow \) (h): This equivalence is mainly given by the equality:

(6) \[ E^x \left[ \exp \left( - \int_0^\infty V(X_s) \, ds \right) \right] = \sum_{n=0}^\infty (S_{V})^n g(x), \quad x \in \partial D, \]

where \( g(x) := E^x[T = \infty ; \exp(- \int_0^T V(X_s) \, ds)] \). The idea behind the equality (6) is that almost every Brownian path in \( R^d (d \geq 3) \) will shuttle finitely many times between \( \partial B \) and \( \partial D \) before it goes off to \( \infty \).

(n) \( \Rightarrow \) (1): Suppose \( \lambda(V) < 1 \). By (L1), if \( ||q|| \) is small enough, then \( \lambda(V + q) < 1 \). Therefore \( -(\Delta/2) + V + q \geq 0 \) by Proposition A.

(m) \( \Rightarrow \) (n): By (L2) and Proposition A, we have \( \lambda(V) < \lambda(V + q) \leq 1 \).

(c) \( \Rightarrow \) (n): For each \( 0 \leq t \leq 1 + \beta \), put \( f(t) := \ln[\lambda(tV)] = \lim_{n \to \infty} (1/n) \ln ||S_{tV}||^n \).

Since for each \( n \), \( \ln ||S_{tV}||^n \) is a convex function of \( t \) by using the stopped path integral and the Cauchy-Schwarz inequality, so is the limit \( f(t) \). Since \( f(t) \leq 0 \) in \([0, 1 + \beta]\) by Proposition A and \( f(0) < 0 \) by the transient property of the Brownian motion in \( R^d (d \geq 3) \), we obtain \( f(1) < 0 \), i.e. \( \lambda(V) < 1 \).

Another key idea is the connection between the Green function \( G^H(x, y) \) and the conditional Feynman–Kac gauge (see Zhao [10]):

\[ G^H(x, y) = G^{A/2}(x, y)E^x_y \left[ \exp \left( - \int_0^\xi V(X_s) \, ds \right) \right]. \]

The proof of equivalences in the list (III) involves gauge and conditional gauge arguments similar to those in [1], [3] and [9].
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