Several Turán-Type Inequalities for the Generalized Mittag-Leffler Function

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Research Article

1. Introduction

The Mittag-Leffler function was first defined by Mittag-Leffler in 1903 [1]. In this paper, he defined the function by

\[ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha + n + 1)}, \quad z, \alpha \in \mathbb{C}, \text{Re} (\alpha) > 0, \quad (1) \]

where \( \Gamma(\cdot) \) is a classical gamma function. In 1905, Wiman [2] generalized \( E_\alpha(z) \) as

\[ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha + n + \beta)}, \quad z, \alpha, \beta \in \mathbb{C}, \text{Re} (\alpha) > 0, \text{Re} (\beta) > 0. \quad (2) \]

In 1971, Prabhakar [3] introduced the function

\[ E_{\alpha,\beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha + n + \beta) n!}, \quad z, \alpha, \beta, \gamma \in \mathbb{C}, \text{Re} (\alpha) > 0, \text{Re} (\beta) > 0. \quad (3) \]

Later, Dorrego and Cerutti [4] introduced the \( k \)-Mittag-Leffler function

\[ E_{\kappa,\alpha,\beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk} z^n}{\Gamma_k(\alpha + n + \beta) n!}, \quad z, \alpha, \beta, \gamma, k \in \mathbb{R}, \text{Re} (\alpha) > 0, \text{Re} (\beta) > 0. \quad (4) \]

where \( (\gamma)_{nk} \) is the \( k \)-Pochhammer symbol defined by \( (\gamma)_{nk} = (\Gamma_k(\gamma + nk))/\Gamma_k(\gamma) \). Recently, a generalization of the \( k \)-Mittag-Leffler function

\[ E_{\kappa,\alpha,\beta}^{\gamma,\tau}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk} z^n}{\Gamma_k(\alpha + n + \beta) n!}, \quad z, \alpha, \beta, \gamma, \tau \in \mathbb{C}, k \in \mathbb{R}, \text{Re} (\alpha) > 0, \text{Re} (\beta) > 0. \quad (5) \]

was introduced and studied in [5].

The Mittag-Leffler function plays an important role in various branches of applied mathematics and engineering sciences, such as chemistry, biology, statistics, thermodynamics,
mechanics, quantum physics, informatics, and signal processing. In 1930, the most known result in this field is an explicit formula for the resolvent of Riemann-Liouville fractional integral proved by E. Hille and J. Tamarkin. Based on these important formulas, many results are based still for solving fractional integral and differential equations. More properties and numerous applications of the Mittag-Leffler function to fractional calculus are collected, for instance, in References [6, 7]. In particular, we also refer to References [3, 8–10]. On the recent introduction of the Mittag-Leffler function and its generalizations, the reader may see [11, 12]. There are further related generalizations of the Mittag-Leffler function.

Recently, Mehrez and Sitnik ([13, 14]) obtained some Turán-type inequalities for the Mittag-Leffler function by considering monotonicity for the special ratio of sections for series of the Mittag-Leffler function. In the course of their research, they used a new method. We call this the Mehrez-Sitnik method (the reader can refer to [15–17]). And then they applied this method to the Fox-Wright function and got a lot of interesting new results.

Turán-type inequalities which initiated a new field of research on inequalities for special functions were proved by Paul Turán, it states

\[ |P_n(x)|^2 - P_{n+1}(x)P_{n-1}(x) \geq 0, \]

where \(-1 < x < 1, n \in \mathbb{N}, \) and \( P_n(x) \) stands for the classical Legendre polynomial.

In this paper, we mainly consider a more general generalization

\[ E_{k,a,b,\delta}^{\tau}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \cdot z^n}{\Gamma((an+\beta)/(\delta))}, \]

where \( \delta \neq 0, -1, -2, \cdots \) and \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0. \) It is clear that \( E_{k,a,b,\delta}^{\tau}(z) = E_{k,a,b,\delta}^{\tau}(z) \) and \( E_{k,a,b,\delta}^{\tau}(z) = E_{k,a,b,\delta}^{\tau}(z). \) In the following, we mainly prove the monotonicity of ratios for sections of series of generalized Mittag-Leffler functions; the result is also closely connected with Turán-type inequalities.

2. Definition of the k-Gamma Function and Lemmas

In 2007, Díaz and Pariguan [18] defined the \( k \)-analogue of the gamma function for \( k > 0 \) and \( x > 0 \) as

\[ \Gamma_k(x) = \int_0^\infty t^{x-1}e^{-tk}dt = \lim_{n \to \infty} n!k^x/(nk)^{x/k-1}, \]

where \( \lim_{k \to \infty} \Gamma_k(x) = \Gamma(x). \) Similarly, we may define the \( k \)-analogue of the digamma and polygamma functions as

\[ \psi_k(x) = \frac{d}{dx} \ln \Gamma_k(x), \]

\[ \psi_k^{(m)}(x) = \frac{d^m}{dx^m} \psi_k(x). \]

It is well known that the \( k \)-analogues of the digamma and polygamma functions satisfy the following recursive formula and series identities (see [18]):

\[ \Gamma_k(x+k) = x\Gamma_k(x), \quad x > 0, \]

\[ \psi_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(nk+x)}, \]

\[ \psi_k^{(m)}(x) = (-1)^{m+1}m! \sum_{n=0}^{\infty} \frac{1}{(nk+x)^{m+1}} \]

\[ \quad = (-1)^{m+1} \int_0^\infty \frac{1}{1-e^{-tk}} t^m e^{-tk}dt. \]

For more properties of these functions, the reader may see Reference [19].

**Lemma 1** (see [16]). Let \( \{a_n\} \) and \( \{b_n\} (n = 0, 1, 2, \cdots) \) be real numbers such that \( b_n > 0 \) and \( \{a_n/b_n\}_{n \geq 0} \) is increasing (decreasing); then, \( \{(a_n + a_{n+1} + \cdots + a_n)/(b_n + b_{n+1} + \cdots + b_n)\} \) is increasing (decreasing).

**Lemma 2** (see [20]). Let \( \{a_n\} \) and \( \{b_n\} (n = 0, 1, 2, \cdots) \) be real numbers and let the power series \( A(x) = \sum_{n=0}^{\infty} a_n x^n \) and \( B(x) = \sum_{n=0}^{\infty} b_n x^n \) be convergent if \( |x| < r. \) If \( b_n > 0 (n = 0, 1, 2, \cdots) \) and the sequence \( \{a_n/b_n\}_{n \geq 0} \) is (strictly) increasing (decreasing), then the function \( A(x)/B(x) \) is also (strictly) increasing (decreasing) on \( [0, r). \)

3. Main Results

Our results read as follows.

**Theorem 3.** Let \( n \in \mathbb{N}, \alpha > 0, \beta > 0, \gamma > 0, k > 0, \tau \in (0, 1) \cup \mathbb{N}, \) and \( \delta \in (0, 1] \). We define the function \( E_{k,a,b,\delta}^{\tau}(z) \) on \( (0, \infty) \) by

\[ E_{k,a,b,\delta}^{\tau}(z) = E_{k,a,b,\delta}^{\tau}(z) - \sum_{m=0}^{n} \frac{(\gamma)_{m,k}^{\tau}}{\Gamma((am+\beta)/(\delta))} \]

\[ = \sum_{m=0}^{n} \frac{(\gamma)_{m,k}^{\tau}}{\Gamma((am+\beta)/(\delta))}. \]

Then, the Turán-type inequality

\[ E_{k,a,b,\delta}^{\tau}(z) \cdot E_{k,a,b,\delta}^{\tau+2}(z) \leq \left( E_{k,a,b,\delta}^{\tau+1}(z) \right)^2, \]

holds true.
Proof. Using the formulas

\[
E_{k,a,b,\delta}(z) = E_{k,a,b,\delta}^{\gamma,n+1}(z) + \frac{(\gamma)_{(n+1)\cdot k} \cdot z^{n+1}}{\Gamma_k(a(n+1) + \beta)},
\]

we have

\[
E_{k,a,b,\delta}^{\gamma,n+2}(z) = E_{k,a,b,\delta}^{\gamma,n+1}(z) - \frac{(\gamma)_{(n+1)\cdot k} \cdot z^{n+1}}{\Gamma_k(a(n+1) + \beta)}.
\]

\[
E_{k,a,b,\delta}(z) \cdot E_{k,a,b,\delta}^{\gamma,n+2}(z) - \left( E_{k,a,b,\delta}^{\gamma,n+1}(z) \right)^2
\]

\[
eq \frac{(\gamma)_{(n+1)\cdot k} \cdot z^{n+1}}{\Gamma_k(a(n+1) + \beta)} - \frac{(\gamma)_{(n+2)\cdot k} \cdot z^{n+2}}{\Gamma_k(a(n+2) + \beta)} - \frac{(\gamma)_{(n+1)\cdot k} \cdot (\gamma)_{(n+2)\cdot k} \cdot z^{n+3}}{\Gamma_k(a(n+1) + \beta) \cdot \Gamma_k(a(n+2) + \beta)}
\]

where

\[
A_m^{\gamma}(k, a, \beta, \delta) = (\delta + n + 1)(\gamma)_{(n+1)\cdot k}(\gamma)_{mr,k} \Gamma_k(a(n+2) + \beta) \Gamma_k(a(m-1) + \beta) - (\delta + m - 1)
\]

\[
\cdot (\gamma)_{(n+2)\cdot k}(\gamma)_{(m-1)\cdot k} \Gamma_k(a(n+1) + \beta) \Gamma_k(a(m-1) + \beta).
\]

On the other hand, we have

\[
A_m^{\gamma}(k, a, \beta, \delta) = (\delta + n + 1)(\gamma)_{(n+1)\cdot k}(\gamma)_{mr,k}
\]

\[
\cdot \left( \Gamma_k(a(n+2) + \beta) \Gamma_k(a(m-1) + \beta) - (\delta + m - 1)(\gamma)_{(n+2)\cdot k}(\gamma)_{(m-1)\cdot k} \right)
\]

\[
\cdot \Gamma_k(a(n+1) + \beta) \Gamma_k(a(m-1) + \beta)
\]

\[
= (\delta + n + 1)(\gamma)_{(n+1)\cdot k}(\gamma)_{mr,k} \left( \Gamma_k(a(n+2) + \beta) \right)
\]

\[
\cdot \left( \Gamma_k(a(m-1) + \beta) - (\delta + m - 1) \right)
\]

\[
\cdot \Gamma_k(y + (n+1) rk) \Gamma_k(y + (m-1) rk)
\]

\[
\frac{1}{\Gamma_k(y + (n+1) rk) \Gamma_k(y + (m-1) rk)}
\]

Taking into account the inequality [21],

\[
\frac{\Gamma_k(a(n+1) + \beta) \Gamma_k(a(n+2))}{\Gamma_k(a(m+1) + \beta)} \leq \frac{\Gamma_k(a(m-1) + \beta) \Gamma_k(a(m+1) + \beta)}{\Gamma_k(a(m-1) + \beta)},
\]

which holds for all \(a > 0, \beta > 0, \) and \(m \geq 3, \) and clearly, we have \(A_m^{\gamma}(k, a, \beta, \delta) \leq 0.\) This in turn implies that inequality (13) holds.

**Corollary 4.** Let \(n \in \mathbb{N}, a > 0, \beta > 0, y > 0, k > 0, \) and \(q \in (0, 1) \cup \mathbb{N}.\) Then,

\[
E_{1,a,b,\delta}^{\gamma,n+2}(z) \cdot E_{1,a,b,\delta}^{\gamma,n+1}(z) \leq \left( E_{1,a,b,\delta}^{\gamma,n+1}(z) \right)^2.
\]

Proof. By taking \(\delta = k = 1\) and \(q = \tau\) in Theorem 3, we easily obtain the above Turán-type inequality. The proof is complete.

**Remark 5.** It is worth noting that Mehrez and Sitnik posed an open problem 1 in [13]: “find the generalization of the inequality in the following inequality..."
\[ E_{a,\beta}^{\tau, n}(z) \cdot E_{a,\beta}^{\tau, n+2}(z) \leq \left( E_{a,\beta}^{\tau, n+1}(z) \right)^2, \] 
(21)

where \( \alpha > 0, \beta > 0, \gamma > 0, q \in (0, 1) \cup \mathbb{N}, \) and \( z > 0. \)

Clearly, Corollary 4 gives an affirmative answer to problem 1 in [13].

**Theorem 6.** Let \( n \in \mathbb{N}, \alpha > 0, \beta > 0, \gamma > 0, k > 0, \tau \in (0, 1) \cup \mathbb{N}, \) and \( \delta \in (0, 1]. \) We define the function \( H_{k,\alpha,\beta,\delta}^{\tau, n}(z) \) on \((0, \infty)\) by
\[
H_{k,\alpha,\beta,\delta}^{\tau, n}(z) = \frac{E_{k,\alpha,\beta,\delta}^{\tau, n}(z) \cdot E_{k,\alpha,\beta,\delta}^{\tau, n+2}(z)}{E_{k,\alpha,\beta,\delta}^{\tau, n+1}(z)}, \quad z > 0.
\]
(22)

Then, the function \( z \mapsto H_{k,\alpha,\beta,\delta}^{\tau, n}(z) \) is increasing on \((0, \infty).\)

So the Turán-type inequality
\[
\frac{(\delta + n)(\gamma + (n + 1)\tau k)}{\Gamma_k(\alpha(n + 1) + \beta) \cdot \Gamma_k(\alpha(n + 2 + i - j) + \beta)} \leq E_{k,\alpha,\beta,\delta}^{\tau, n+2}(z) \cdot E_{k,\alpha,\beta,\delta}^{\tau, n+1}(z),
\]
(23)

is valid for \( n \in \mathbb{N}, \alpha > 0, \beta > 0, \gamma > 0, \tau \in \mathbb{C}, \) and \( \delta \in (0, 1]. \) The constant on the left-hand side of inequality is sharp.

**Proof.** By using the Cauchy product, we have
\[
H_{k,\alpha,\beta,\delta}^{\tau, n}(z) = \frac{E_{k,\alpha,\beta,\delta}^{\tau, n}(z) \cdot E_{k,\alpha,\beta,\delta}^{\tau, n+2}(z)}{E_{k,\alpha,\beta,\delta}^{\tau, n+1}(z)}.
\]
(24)

We define the sequence \( V_j(k, \alpha, \beta, \delta, \gamma) \) by
\[
V_j(k, \alpha, \beta, \delta, \gamma) = \frac{\delta \cdot \Gamma_k(\alpha(n + 1) + \beta) \cdot \Gamma_k(\alpha(n + 2 + i - j) + \beta)}{\delta \cdot \Gamma_k(\alpha(n + 1 + j) + \beta) \cdot \Gamma_k(\alpha(n + 1 + i - j) + \beta)} \cdot U_j(k, \alpha, \beta),
\]
(25)

where the sequence \( U_j(k, \alpha, \beta) \) is defined by
\[
U_j(k, \alpha, \beta) = \frac{\Gamma_k(\alpha(n + 1) + \beta) \cdot \Gamma_k(\alpha(n + 1 + i - j) + \beta)}{\Gamma_k(\alpha(n + j) + \beta) \cdot \Gamma_k(\alpha(n + 2 + i - j) + \beta)}.
\]
(26)

In [21], the authors proved that the sequence \( U_j(k, \alpha, \beta) \) is increasing for all \( j = 0, 1, \ldots. \) Thus,
\[
\frac{V_{j+1}(k, \alpha, \beta)}{V_j(k, \alpha, \beta)} = K_{i,j}(k, \alpha, \beta) \geq K_{i,j}(k, \alpha, \beta).
\]
(27)

By using the inequality
\[
\frac{(\delta + n + 1)(\delta + n + 1 + i - j)}{(\delta + n + j)(\delta + n + i - j)} \geq 1,
\]
(28)

we have
\[
K_{i,j}(k, \alpha, \beta) = \left[ \frac{(\gamma + (n + j)\tau k)(\gamma + (n + i - j)\tau k)}{(\gamma + (n + 1 + j)\tau k)(\gamma + (n + 1 + i - j)\tau k)} \cdot \frac{(\delta + n + 1 + j)(\delta + n + 1 + i - j)}{(\delta + n + j)(\delta + n + i - j)} \right] \geq 1.
\]
(29)

So the sequence \( V_j(k, \alpha, \beta, \delta, \gamma) \) is increasing for \( j = 0, 1, \ldots, \) and \( n \in \mathbb{N}, \alpha > 0, \beta > 0, \gamma > 0, \tau \in \mathbb{C}, \) and \( \delta \in (0, 1]. \) This implies that the ratios
\[
H_{k,\alpha,\beta,\delta}^{\tau, n}(z) = \frac{\sum_{j=0}^\infty \sum_{i=0}^j (\gamma(n+j)\tau k)(\gamma(n+i-j)\tau k) \cdot \Gamma_k(\alpha(n+j) + \beta) \cdot \Gamma_k(\alpha(n+2+i-j) + \beta) \cdot (\delta)_{n+j} \cdot (\delta)_{n+2+i-j}}{\sum_{j=0}^\infty \sum_{i=0}^j (\gamma(n+j)\tau k)(\gamma(n+i-j)\tau k) \cdot \Gamma_k(\alpha(n+j) + \beta) \cdot \Gamma_k(\alpha(n+1+i-j) + \beta) \cdot (\delta)_{n+1+j} \cdot (\delta)_{n+1+i-j}},
\]
(30)
increase by Lemma 1. So the function \( z \mapsto H^{|z|}_{\kappa,z} \) is increasing on \((0, \infty)\) by Lemma 2. Finally, it is easy to see that

\[
\lim_{z \to \infty} H^{|z|}_{\kappa,z} = \frac{\delta + n}{\delta + n + 1}(y + (n + 1)rk) = \binom{\delta + n + 1}{\delta + n + 1}(y + ntk).
\]

So the constant \((\delta + n/y + (n + 1)rk)/(\delta + n + 1) = (\delta + n + 1)\) is the best possible for which the inequality holds for all \( n \in \mathbb{N}, \alpha > 0, \beta > 0, \gamma > 0, \kappa > 0, \) and \( q \in (0, 1) \cap \mathbb{N}, \delta > 0, \) \( \tau \in (0, 1) \cup \mathbb{N}, \delta \)

**Corollary 7.** For \( n \in \mathbb{N}, \alpha > 0, \beta > 0, \gamma > 0, k > 0, \) and \( q \in (0, 1) \cup \mathbb{N}, \) the function \( z \mapsto H^{|z|}_{\kappa,z} \) is increasing on \((0, \infty)\).

**Proof.** By putting \( \delta = k = 1 \) and \( q = \tau \), we can get the above inequality. The proof is complete.

**Remark.** In [13], Mehrez and Sitnik also posed another open problem: “for \( z \in (0, \infty) \), find the monotonicity of the function

\[
E^{|z|}_{u_\beta} \left( \frac{E^{2|z|+1}_{u_\beta}}{E_u^{2|z|}} \right),
\]

where \( n \in \mathbb{N}, \alpha > 0, \beta > 0, \gamma > 0, q \in (0, 1) \cup \mathbb{N}, \) and \( z > 0 \). Here, Corollary 7 gives an affirmative answer to problem 2 in [13].

**Theorem 9.** For \( n \in \mathbb{N}, \alpha > 0, \beta > 0, \gamma > 0, k > 0, \tau \in (0, 1) \cup \mathbb{N}, \delta \in (0, 1), \) and fixed \( z > 0 \), the function \( f : \beta \mapsto \Gamma_\kappa(\beta) E^{|z|}_{\kappa,\tau,\delta,\gamma} \) is strictly log-convex on \((0, \infty)\). As a result, we have the following inequality:

\[
\frac{E^{(\cdot)3}_{u_\beta}}{E^{(\cdot)3}_{u_\beta}} = \frac{E^{(\cdot)3}_{u_\beta}}{E^{(\cdot)3}_{u_\beta}}.
\]

**Proof.** Simple computation yields

\[
\frac{\partial}{\partial \beta} \left( \log \frac{\Gamma_\kappa(\beta)}{\Gamma_\kappa(\beta + \delta)} \right) = \psi_\kappa(\beta) - \psi_\kappa(ak + \beta),
\]

where we apply that the function \( \psi_\kappa(x) \) is concave on \( \mathbb{R} \). Therefore, we get that the function \( \beta \mapsto \Gamma_\kappa(\beta)/\Gamma_\kappa(ak + \beta) \) is strictly log-convex on \((0, \infty)\). Using the fact that the sum of the log-convex functions is also log-convex, we obtain that the function \( f \) is strictly log-convex on \((0, \infty)\).

Due to inequality (33), we easily know

\[
\frac{\beta + 2}{\beta + 2} < \frac{\beta + 2}{\beta + 2}.
\]

That is,

\[
\left( \frac{E^{|z|}_{u_\beta}}{E^{|z|}_{u_\beta}}(z) \right)^2 < \frac{\Gamma_\kappa(\beta + 2k)}{[\Gamma(\beta + k)]^2} E^{|z|}_{u_\beta}(z).
\]

Using the definition of \( \Gamma_\kappa(x) \), we easily obtain

\[
\frac{\Gamma_\kappa(\beta + 2k)}{[\Gamma(\beta + k)]^2} = \frac{(\beta + k)}{\beta}.
\]

The proof is complete.

**Corollary 10.** For \( \alpha > 0, \beta > 0, \gamma > 0, k > 0, \tau \in (0, 1) \cup \mathbb{N}, \delta \in (0, 1), \beta_1 > \beta_2, \) and fixed \( z > 0 \), we have

\[
\frac{E^{(\cdot)3}_{u_\beta}}{E^{(\cdot)3}_{u_\beta}} < \frac{\beta_1}{\beta_1} E^{(\cdot)3}_{u_\beta}(z).
\]

**Proof.** Since the function \( f(\beta) \) is strictly log-convex, we obtain that the function

\[
\frac{f(\beta + k)}{f(\beta)} = \frac{\Gamma_\kappa(\beta + k)}{\Gamma_\kappa(\beta)} E^{|z|}_{u_\beta}(z),
\]

is strictly increasing on \((0, \infty)\). Taking \( 0 < \beta_1 < \beta_2 \), we have

\[
\frac{\Gamma_\kappa(\beta + k)}{\Gamma_\kappa(\beta)} E^{|z|}_{u_\beta}(z) < \frac{\Gamma_\kappa(\beta + k)}{\Gamma_\kappa(\beta)} E^{|z|}_{u_\beta}(z).
\]

This completes the proof.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interests.

**Authors’ Contributions**

All authors contributed equally to the manuscript and read and approved the final manuscript.

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