Borromean binding

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Abstract
A review is first presented of the Hall–Post inequalities relating \( N \)-body to \( (N-1) \)-body energies of quantum bound states. These inequalities are then applied to delimit, in the space of coupling constants, the domain of Borromean binding where a composite system is bound while smaller subsystems are unbound.

I. INTRODUCTION

There are many examples, at various scales, of composite systems at the edge between binding and non-binding. In nuclear physics, a proton–proton or neutron–neutron pair misses binding by a small margin, while a proton and a neutron form a rather weakly bound deuteron. The existence of a near-threshold state can induce dramatic consequences, for instance on fusion probabilities [1]. A pair of charmed mesons near-threshold state can induce dramatic consequences, for instance on fission probabilities [1]. Atoms such as \( ^3\text{He} \) were for a long time believed to be unable to merge into a molecule. Recent studies indicates a tiny binding of the order of 1 mK for \( ^4\text{He}_2 \).

An intriguing question is whether it is easier to bind three or more components than to form a mere two-body bound state. An answer is provided by the study of halo nuclei, which contain peripheral neutrons. Consider for instance the \( ^6\text{He} \) nucleus. It is stable against any dissociation, while the lighter \( ^5\text{He} \) spontaneously decays into a neutron and a \( ^4\text{He} \). In the (reasonable) approximation where the structure of the core is neglected, this means that the \((\alpha, n, n)\) three-body system is bound, while neither \((\alpha, n)\) nor \((n, n)\) have a discrete spectrum.

This property of 3-body binding without 2-body binding was astutely named Borromean [3], after the Borromean rings, which are interlaced in a subtle topological way (see Fig. 1) such that if any one of them is removed, the two other become unlocked. The adjective Borromean is nowadays broadly accepted in the field of quantum few-body systems.

Borromean binding is intimately related to two other fascinating properties of few-body quantum systems. The Efimov effect [6] indicates that when the two-body energy vanishes (e.g., by tuning the strength of the potential), a myriad of weakly-bound states show up in the three-body spectrum. This implies that the three-body ground-state already exists at this point. Slightly above the onset of two-body binding, the ratio \( E_2/E_3 \) of two-body to three-body binding energies is very small. By rescaling, one can reach a situation with a finite 2-body energy, and a 3-body energy that becomes infinite when the range of the potential is made shorter and shorter: this is the Thomas collapse [7].

This review is organised as follows. In Sec. II the Hall–Post inequalities are briefly recalled. They are applied in Sec. III to constraint the domain of coupling constants leading to Borromean binding for bosons interacting through short-range forces. The difficulties arising in the case of fermions are described in Sec. IV. Borromean binding with Coulomb forces is the subject of Sec. V before the conclusions.

II. HALL–POST INEQUALITIES

A number of inequalities can be written down for binding energies in quantum mechanics if one splits the Hamiltonian into pieces (each piece being hermitian). Thus, for example,

\[
H = A + B + \cdots \Rightarrow E(H) \geq E(A) + E(B) + \cdots , \quad (1)
\]

in an obvious notation where \( E(H) \) is the ground-state energy of \( H \). Saturation is obtained if \( A, B, \) etc., reach their minimum simultaneously. If, for instance, \( H = p^2 - 1/r + r^2/2 \) describes the motion of a particle feeling both a Coulomb and an harmonic potential, then \( E(H) \geq (-1/2) + (3/2), \) corresponding to an equal share of the kinetic energy. A slight improvement is obtained by writing \( H = [\alpha p^2 - 1/r] + [(1-\alpha)r^2 + r^2/2], \) and optimising \( \alpha \).

The reasoning can be applied to obtain a lower bound on 3-body energies in terms of 2-body energies. This has been discovered independently by several authors working on the stability of matter [8] or baryon spectroscopy in simple quark

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* Dedicated to my colleague and friend Vladimir Belyaev at the occasion of his 70th birthday
models [9]. Let
\[ H_N(m, g) = \sum_{i=1}^{N} \frac{p_i^2}{2m} + g \sum_{i<j} V(r_{ij}), \]  
(2)
be the Hamiltonian describing a system of \( N \) identical particles interacting with pairwise forces, and \( E_N(m, g) \) be its ground state energy. From the identity
\[ H_N(m, g) = \sum_i H_{N-1}^{(i)} \left( (N-1)m, \frac{g}{N-2} \right), \]  
(3)
where \( H_{N-1}^{(i)} \) does not include particle \( i \), one derives
\[ E_N(m, g) \geq N E_{N-1} \left( (N-1)m, \frac{g}{N-2} \right) \]
\[ \geq \frac{N(N-1)}{2} E_2(m(N-1), g), \]  
(4)
which can supplement any upper bound provided by a variational method, to frame the exact energy. The generalisation to unequal masses or different potentials among the pairs is straightforward. For instance,
\[ H_3(\{m_i\}; \{g_{ij}\}) = \left[ \frac{p_1^2}{4m_1} + \frac{p_2^2}{4m_2} + g_{12} V(r_{12}) \right] + \cdots \]  
(5)
Numerical investigations show that the lower bound [4] is not very accurate and never reaches saturation. The relative wave function of particles 1 and 2 is of course modified by the presence of the third particle, and thus the three-body wave function does not describe optimally the (1,2) subsystem. However, this is a small effect which even disappears for the harmonic oscillator. The main source of inaccuracy in this decomposition is that the contribution of the (1,2) subsystem is replaced by the rest energy of an isolated (1,2) pair, although the (1,2) pair is not at rest in the whole system.

To overcome this difficulty, a better decomposition was written down by Hall and Post [10], and rediscovered in Ref. [11]. It involves the translation-invariant part \( \tilde{H}_N \)
\[ \tilde{H}_N = H_N - \frac{(p_1 + p_2 + \cdots)^2}{2Nm}, \]  
(6)
of each Hamiltonian \( H_N \) and reads
\[ \tilde{H}_N(m, g) = \sum_{i<j} \tilde{H}_{N-1}^{(ij)} \left( m, \frac{N(N-2)}{N-1}, \frac{g}{N-2} \right). \]  
(7)
This leads to the new inequality
\[ E_N(m, g) \geq E_{N-1} \left( m, \frac{N(N-2)}{N-1}, \frac{g}{N-2} \right) \]
\[ \geq \frac{N(N-1)}{2} E_2 \left( \frac{mN}{2}, g \right), \]  
(8)
which is necessarily better than [4] since, \( mN/2 < m(N-1) \) for \( N \geq 3 \), and for a given potential, the binding energy is a decreasing function of the constituent mass. This inequality is optimal in the sense that it can be saturated: this occurs for the harmonic oscillator and only in this case. The inequality [8] also holds for the variational approximation \( E_N^{opt} \) to the binding energy \( E_N \), provided the sets of trial functions \( \psi \) for \( N = 2 \) and \( N = 3 \) are consistent. In particular, one gets saturation, \( E_N^{opt}(m, v) = 3E_2^{opt}(3m/2, v) \) if each variational energy is calculated with a single harmonic-oscillator function \( \psi_N \propto \exp(-\alpha_N \sum r_{ij}^2) \). This is a rather frequent pattern that (with minimal restrictions) variational solutions share the properties of the exact ones. For the virial theorem this was underlined by Fock and rediscovered by many authors [12].

The case of particles with unequal masses has revealed some surprises. The simple extension (here for \( N = 3 \)) of the decomposition [7]
\[ \tilde{H}_3(m_i; V_{ij}, \ldots) = \alpha_{12} \left( \frac{m_1 p_2 - m_2 p_1}{m_1 + m_2} \right)^2 + V_{12} + \cdots, \]  
(9)
does not always give saturation for the harmonic oscillator. The remedy was found in Ref. [13] for \( N = 3 \) particles and extended in Ref. [14] for \( N = 4 \) : one should introduce more freedom in the decomposition of the Hamiltonian and find the optimal choice by suitable adjustment.

For \( N = 3 \), the decomposition is written as
\[ H_3 = B \cdot P + \alpha_{12} \left( \frac{p_1 - x_3 p_2}{1 + x_3} \right) + gV_{12} + \cdots, \]  
(10)
where \( P = \sum p_i \). The identification gives the irrelevant vector \( B \) and the inverse mass \( \alpha_{ij} \) in terms of the free parameters \( x_i \). The lower bound on \( E_3 \) is expressed as a sum of 2-body energies. It depends on these \( x_i \) and is maximised by varying these parameters, for each given set of potentials \( V_{ij} \). In the case of the harmonic oscillator with unequal masses and even unequal strength factors, saturation is reached, i.e., the exact energy always coincides with the lower bound.

For \( N = 4 \), one should go a step further, and write the decomposition as
\[ H_4 = B \cdot P + \alpha_{12} \left( \sum_i x_{12,i} p_i \right)^2 + gV_{12} + \cdots \]  
(11)
The momentum \( \sum_i x_{12,i} p_i \) is constrained by translation invariance, and \( x_{12,2} = x_{12,1} = 1 \), to make it conjugate to \( r_2 - r_1 \). This leaves two free parameters for each pair, which are adjusted to optimise the lower bound.

Refs. [11, 13, 14] contain several examples, where the three- or four-body energy is computed and compared to the above lower bounds. For confining potentials, one is not too far from the limiting case of the harmonic oscillator for which the inequality is saturated, and the inequalities [8] provide a very good approximation to the exact energy. The quality of the approximation deteriorates for short-range or Coulomb forces.
III. BORROMEO BINDING OF BOSONS

The Hall–Post decomposition was used in the previous section to obtain inequalities on energies at given coupling constant $g$. They can also be used to provide relations between coupling constants at given energy $E$. In particular for $E = 0$, one can derive inequalities on the critical couplings $g_N$, where $g_N$ is the minimal strength required to achieve $N$-body binding.

In one- or two-dimensional quantum mechanics, any attractive potential supports at least one bound state [15]. More precisely, $g_2 = 0$ if the integral $\int d^nxV(x)$ is negative. For $n = 3$ dimensions, the problem is more subtle [16]. An attractive potential with short range, say $gV(r)$, requires a minimal strength $g > g_2$ to achieve binding of two constituents of unit mass, and $mg > g_2$ for constitutents with mass $m$. A classic paper by Blatt and Jackson [17] gives estimates of $g_2$ for simple potentials such as Yukawa.

In this context, the phenomenon of Borromean binding is expressed by the property that the critical coupling $g_3$ required to bind three bosons is smaller than $g_2$. More generally, $g_N \leq g_{N-1}$. Several questions can be raised: what is the typical range of values for $g_3/g_2$, $g_4/g_3$, etc.? Are there rigorous constraints on these ratios? Are these constraints saturated for particular potentials? What can be said in the limit of large $N$?

Some results are given in Refs. [18, 19, 20], which contain references to earlier papers. In particular, the decomposition

$$H_N \propto \sum H_{N-1}$$

implies that $H_N$ hardly explores the domain of binding with negative expectation values, if all $H_{N-1}$ remain positive. Thus

$$Ng_N \geq (N-1)g_{N-1} \geq \ldots g_2.$$  \hspace{1cm} (12)

For simple monotonic potentials, one gets $g_3/g_2 \approx 0.80$, well above the rigorous bound $g_3/g_2 \geq 2/3$. Still, there is a remarkable window of about 20% in the coupling constant, where three-body systems are bound without two-body binding. One also gets $g_4/g_2 \approx 0.67$, typically, which, when compared to $g_3/g_2 \approx 0.80$, reveals a window of about 13% for four-body binding without three-body binding.

For bosons, the situation is rather simple. If a potential does not succeed in binding a pair of bosons, but is predominately attractive, that is to say has a negative scattering length, it will bind a sufficiently large number of bosons.

We have seen that Hall–Post inequalities are saturated for harmonic potentials. A pure $V(r) \propto r^2$ does not give Borromean binding, as $gV$ would confine even for $g \rightarrow 0+$. But a potential like $V \propto r^2 - C$ in the domain of the relevant wave functions, and vanishing at very large distances $r$, would give $g_3/g_2 \approx 2/3$, i.e., nearly saturate the bound [20]. This corresponds to an attractive potential with an external barrier.

More important for physical applications are potentials with an internal repulsive core. Their wave functions significantly depart from those of the harmonic oscillator and the ratio $g_3/g_2$ become closer to 1. Examples are given in [20].

IV. THE CASE OF FERMIONS

In general, an inequality such as (8), though remaining valid, is of little interest for fermions. The ground state $E_2$ corresponds to a symmetric orbital wave function, with, for instance, the spins arranged in a singlet state to fulfill the antisymmetry requirement. On the other hand, three or more electrons can never be in a fully-symmetric orbital wave function. Thus the actual $E_N$, $N \geq 3$, will be an excited level of the $E_N$ entering the inequality (8).

A minimal knowledge of the structure of the wave function is thus needed to write down an useful inequality. Consider for instance three spin 1/2 fermions in a symmetric spin (or spin–isospin, or spin–isospin–colour) state. Then the space wave function of the ground state should be a $L^P = 1^+$ antisymmetric wave function, a prototype being the harmonic oscillator state $\rho \times \lambda \exp[-\alpha (\rho^2 + \lambda^2)]$ in terms of Jacobi coordinates $\rho$ and $\lambda$. The two-body subsystems are in an antisymmetric state with orbital momentum $\ell = 1$. Then $E_3(m, g) \geq 3E_2(3m/2, g)$ results from the identity $E_3(m, g) = \sum E_2(3m/2, g)$ applied in the subspace of antisymmetric wave functions.

For more than $N = 3$ fermions, or other spin states of $N = 3$, the problem is more complicated but still, some rigorous results can be obtained.

The basis is the decomposition of $N$-particle representations of the permutation group in terms of $N-1$ states [21]. For instance, in a spin $S = 1/2$ state of three electrons a given pair is half of the time in a singlet state, and half of time in a triplet state. This means a proper bound on $E_3$ involves, for $E_2$, and average over the $\ell = 0$ and the $\ell = 1$ ground state. For other cases, one needs some group theoretical calculations [21]. This gives nice local constraints, i.e., links from $E_N$ to $E_{N-1}$ (see also the improvements proposed in Ref. [22]). But this approach fails when extrapolated from $E_2$ to the large-$N$ limit of $E_N$. This means that much progress remains to be made. Checking the constraints with exactly solvable potentials [23] gives valuable insight into the problem.

Note that the question of writing inequalities for fermion energies is rather old. In the course of the pioneering papers on the stability of matter, a very clever decomposition was written down

$$H_{N+1} = \sum_i h_N^{(i)} ,$$

$$h_N^{(i)} = \sum_{\alpha} \frac{p^{(i)}_{\alpha}^2}{2 N m} + \frac{g}{N-1} \sum_{\alpha < \beta} \epsilon_{\alpha \beta} ,$$  \hspace{1cm} (13)

where the superscript in $\sum^{(i)}$ indicates that particle $i$ is omitted. Each $h_N^{(i)}$ is an independent-particle Hamiltonian with ground-state energy $\sum_{\epsilon_j}$, in terms of one-particle energies $\epsilon_j$, properly piled up as explained in textbooks of elementary chemistry (2 particles in 1s, 6 in 2p, etc.), to ensure antisymmetry. However, the recoil of, e.g., the $\{2, 3, \ldots, N\}$ subsystem against the first particle is not taken into account. As a consequence, the inequality is never saturated.

Significant progress was achieved by Basdevant and Martin [24], who used sophisticated convexity inequalities to derive a
bound for power-law potentials, that becomes saturated in the limit of harmonic forces.

V. BORROMEAN MOLECULES

Borromean binding as described in Sec. III seems at first hardly conceivable for a Coulomb potential, since changing the strength by an overall factor $g$ simply results in a mere rescaling by $g^2$ of all binding energies $E_N$. However, if the stability of ions and molecules does not depend on the overall strength of the interaction, it relies on an adequate balance between attraction and repulsion, and, for a given set of charges, on the ratios of the constituent masses which are involved.

Consider for instance a set of three masses $\{m_i\}$ carrying charges $\{q_i\} = \{+1, -1, -1\}$, (times an overall factor). Systems with identical or nearly identical inverse masses $m_2^{-1}$ and $m_3^{-1}$, such as $H_2^+ (e^-, p, p)$, $Ps^- (e^+, e^-, e^-)$ or $H^2_+ (p, e^-, e^-)$, or neighbouring configurations, are stable, while less symmetric systems such as $(e^-, p, e^+)$ or $(p, \bar{p}, e^-)$ spontaneously decay into an atom and an isolated charge 24.

Note that the stable systems with three unit charges are not Borromean, since there are always two stable atoms among the subsystems, $\{1, 2\}$ and $\{1, 3\}$ in our notation. This means that one can construct each stable system step by step: two charges form a stable atom, which attracts the third charge.

For $N = 4$ unit charges $\{q_i\} = +1, +1, -1, -1$, there are new possibilities. The most familiar cases are the following:

- unstable configurations, such as $(p, e^+, \bar{p}, e^-)$ 32], which spontaneously splits into a protonium and a positronium,
- stable systems, the simplest being the $Ps_2$ molecule $(e^+, e^+, e^-, e^-)$, whose all three-body subsystems $\pm (e^+, e^-, e^-)$ are stable,
- less stable systems, such as the positronium hydride $(p, e^+, e^-, e^-)$, which can be viewed as a stable $H^+$ fixing a positron, or a stable $Ps^-$ attached to a proton, or, more interestingly, as an unstable $(e^-, p, e^+)$ stabilised by the addition of a second electron.

More recently, a new category of four unit-charge systems was revealed 24. They are stable, but all three-body subsystems are unstable. Hence, they are Borromean, in the sense that they cannot be built by adding the constituents one by one. An example is $(p, d, \bar{p}, d)$, with a proton, a deuteron and their antiparticles. Indeed, a study by Mitroy 27, who uses Varga’s stochastic variational method 28, indicates that asymmetric ions $(m^+, M^-, m^-)$ are stable in the range

$$0.70 \lesssim \frac{M}{m} \lesssim 1.69$$

while other studies, by Bressanini et al. 29, who use a diffusion Monte-Carlo approach, or Varga 30 with a stochastic search of the range coefficients of a Gaussian parametrisation, have established that $(M^+, m^+, M^-, m^-)$ remain stable for

$$\frac{1}{2.2} \lesssim \frac{M}{m} \lesssim 2.2$$

Then, for $M/m \sim 2$, and for neighbouring configurations $(M^+, m^+, M^-, m^-)$, the four-body system is stable while none of its three-body subsystems is stable.

VI. OUTLOOK

In molecular, nuclear, or hadronic physics, there are fragile and subtle quantum states, whose stability relies on the joint efforts of many subsystems, which by themselves do not contain enough attraction to be stable. It is of course tempting to extend to political or sociological situations: l’union fait la force, as it is said in a popular French proverb.

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lation, strong interaction, etc., are neglected.