Critical behavior of charged Gauss-Bonnet-AdS black holes in the grand canonical ensemble

De-Cheng Zou, Yunqi Liu and Bin Wang

Department of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China

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We study the thermodynamics in the grand canonical ensemble of D-dimensional charged Gauss-Bonnet-AdS black holes in the extended phase space. We find that the usual small-large black hole phase transition, which exhibits analogy with the Van de Waals liquid-gas system holds in five-dimensional spherical charged Gauss-Bonnet-AdS black holes when its potential is fixed within the range $0 < \Phi < \sqrt{3}\pi/4$. For the other higher dimensional and topological charged Gauss-Bonnet-AdS black holes, there is no such phase transition. In the limiting case, Reissner-Nordstrom-AdS black holes, with vanishing Gauss-Bonnet parameter, there is no critical behavior in the grand canonical ensemble. This result holds independent of the spacetime dimensions and topologies. We also examine the behavior of physical quantities in the vicinity of the critical point in the five-dimensional spherical charged Gauss-Bonnet-AdS black holes.

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I. INTRODUCTION

Black hole thermodynamics has been an intriguing subject of discussions for decades. In view of AdS/CFT correspondence, the black hole thermodynamics in the presence of a negative cosmological constant becomes more interesting nowadays. The thermodynamic properties of AdS black holes was initially studied in [1], where a Hawking-Page phase transition between the phase spaces of the Schwarzschild AdS black hole and pure AdS space was disclosed. Later, more interesting discoveries were obtained for the charged AdS black holes [2, 3], where it was found that a first order small black hole and big black hole phase transition is allowed in the canonical ensemble where the black hole charge is kept fixed. This phase transition is superficially analogous to a liquid-gas phase transition of the Van der Waals fluid. This superficial reminiscence was

*Electronic address: zoudecheng@sjtu.edu.cn
†Electronic address: liuyunqi@sjtu.edu.cn
‡Electronic address: wang_b@sjtu.edu.cn
also observed in other AdS backgrounds [4–18]. The understanding of the phase transition and the critical phenomena has been further extended to more complicated background, such as the charged Gauss-Bonnet(GB)-AdS black holes [6, 7]. In the canonical ensemble where the black hole charge is fixed, the phase transition between small and big black holes exits as well in the charged GB-AdS holes. The discussion has also been generalized to the grand canonical ensemble where the black hole is allowed to emit and absorb charged particles keeping the potential fixed till the thermal equilibrium is reached. For the spherical Reissner-Nordstrom(RN)-AdS black hole, there is no critical behavior observed in the grand canonical ensemble [2, 3]. While for the spherical charged GB-AdS black hole background in five-dimensions, the phase transition between the small and large black holes in the grand canonical ensemble can still happen [7, 8].

In the usual discussions of thermodynamical properties of black holes in (A)dS spaces, the cosmological constant is treated as a fixed parameter. Recently the study of thermodynamics in AdS black holes has been generalized to the extended phase space, where the cosmological constant is regarded as a variable and also identified with thermodynamic pressure [19, 20]

\[ P = -\frac{\Lambda}{8\pi} = \frac{(D - 1)(D - 2)}{16\pi l^2} \]  

(1)

in the geometric units \( G_N = h = c = k = 1 \). Here \( D \) stands for the number of spacetime dimensions and \( l \) denotes the AdS radius. Taking the cosmological constant as a thermodynamic pressure is nowadays a common practice, where such operations implicitly assume that gravitational theories including different values of the cosmological constants fall in the “same class”, with unified thermodynamic relations. The common excuse for doing this is that the classical theory of gravity may be an effective theory which follows from a yet unknown fundamental theory, in which all the presently “physical constants” are actually moduli parameters that can run from place to place in the moduli space of the fundamental theory. Since the fundamental theory is yet unknown, it is more reasonable to consider the extended thermodynamics of gravitational theories involving only a single action, and then all variables will appear in the thermodynamical relations. Similar situations the parameter \( b \) of Born-Infeld term [23] and coupling coefficient of GB term [36] can be evaluated in different gravitational theories. In addition, the variation of the cosmological constant is included in the first law of black hole thermodynamics, which ensures the consistency between the first law of black hole thermodynamics and the Smarr formula. Including the variation of the cosmological constant in the first law, the AdS black hole mass is identified with enthalpy and there exists a natural conjugate thermodynamic volume to the cosmological constant. In the extended phase space with cosmological constant and volume as thermodynamic variables, it was
Interestingly observed that the system admits a more direct and precise coincidence between the first order small-large black hole phase transition and the liquid-gas change of phase occurring in fluids [21]. More discussions on phase transitions in AdS black holes by treating the cosmological constant as a dynamical quantity can be found in [22, 43]. However, all these discussions were concentrated on the canonical ensemble by fixing the charge of the black hole in the extended phase space.

It is of great interest to generalize the discussion of the phase transitions of black holes in the extended phase space to the grand canonical ensemble where the background AdS black hole has a constant fixed potential. In the footnote of [21], it was briefly argued that in the extended phase space the criticality cannot happen in the grand canonical ensemble for the four-dimensional spherical RN-AdS black hole. It is interesting to ask whether in the extended phase space the critical behavior of the RN-AdS black hole will emerge for different numbers of spacetime dimensions or other topologies. Whether in the grand canonical ensemble, the critical behavior in the extended phase space can be in consistent with the result when the cosmological constant was fixed [2]. This is the first motivation of the present paper. Besides, we would like to generalize the exploration of the critical behaviors to a more complicated black hole background, the charged GB-AdS black hole, in the grand canonical ensemble. Until now, $P - V$ criticality of GB-AdS black hole has been discussed in the canonical ensemble [36], in which the coupling coefficient of GB term has been regarded as a variable appearing in the first law of black hole thermodynamics. On the other hand, in the GB gravity, the coupling coefficient of GB term should be also regarded as a variable in order to get a consistent Smarr relation for black hole thermodynamics, which has been verified in [44] by employing the Hamiltonian perturbation theory techniques. Moreover, similar situation occurred for the Born-Infeld black holes [23, 45, 46]. In this paper, we will concentrate on the extended phase space with fixed potential and examine whether the critical behavior in the charged GB-AdS black holes can happen once the black hole pressure and volume are identified. We will disclose how the criticality will be influenced by spacetime dimension and topology in the higher order derivative gravity.

This paper is organized as follows. In Sec. II we will firstly present the solutions of the $D$-dimensional charged GB-AdS black holes. Then we will examine the critical behaviors in the charged GB-AdS black holes in the grand canonical ensemble for fixed potential. In Sec. III we will discuss the influence of the critical behavior by topologies and spacetime dimensions. Later in Sec. IV we will study the critical point by using the Ehrenfest equations. Finally in Sec. V we will present our closing remarks.
II. CHARGED GAUSS-BONNET ADS BLACK HOLE BACKGROUND

The solution of $D$-dimensional charged GB-AdS black hole with negative cosmological constant is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2h_{ij}dx^idx^j,$$

$$f(r) = k + \frac{r^2}{2\tilde{\alpha}_{GB}}\left[1 - \sqrt{1 - \frac{64\pi\tilde{\alpha}_{GB}P}{(D-1)(D-2)} + \frac{64\tilde{\alpha}_{GB}\pi M}{(D-2)\Sigma_k r^{D-1}} - \frac{2\tilde{\alpha}_{GB}Q^2}{(D-2)(D-3)r^{2D-4}}}\right],$$

where $\Sigma_k$ is the volume of the $(D-2)$-dimensional unit hypersurface, $\tilde{\alpha}_{GB} = (D-3)(D-4)\alpha_{GB}$, $M$ is the black hole mass and $Q$ is related to the charge of the black hole. We will only consider the positive GB coefficient $\alpha_{GB}$ in the following discussion. The coordinates are labeled as $x^\mu = (t, r, x^i)$, $(i = 1, \cdots, D-2)$ and the metric function $h_{ij}$ is a function of the coordinates $x^i$, which span an $(D-2)$-dimensional hypersurface with constant scalar curvature $(D-2)(D-3)k$. The constant $k$ characterizes the geometric property of hypersurface, which takes values $k = 0$ for flat, $k = -1$ for negative curvature and $k = 1$ for positive curvature, respectively. If we take $\tilde{\alpha}_{GB} \to 0$, the solution $f(r)$ reduces to the RN-AdS case. In order to have a well-defined vacuum solution with $M = Q = 0$, the Gauss-Bonnet coefficient $\tilde{\alpha}_{GB}$ and pressure $P$ have to satisfy the following constraint

$$0 < \frac{64\pi\tilde{\alpha}_{GB}P}{(D-1)(D-2)} < 1.$$  (4)

In terms of the horizon radius $r_+$, the mass $M$, Hawking temperature $T$, entropy $S$ and electromagnetic potential $\Phi$ of charged GB-AdS black holes can be written as

$$M = \frac{(D-2)\Sigma_k r_+^{D-3}}{16\pi} \left[k + \frac{k^2\tilde{\alpha}_{GB}}{r_+^2} + \frac{16\pi Pr_+^2}{(D-1)(D-2)} + \frac{2Q^2}{(D-2)(D-3)r_+^{2D-6}}\right],$$

$$T = \frac{1}{4\pi r_+ (r_+^2 + 2k\tilde{\alpha}_{GB})} \left[\frac{16\pi Pr_+^4}{D-2} + (D-3)kr_+^2 + (D-5)k^2\tilde{\alpha}_{GB} - \frac{2Q^2}{(D-2)r_+^{2D-8}}\right],$$

$$S = \int_0^{r_+} T^{-1} \frac{dM}{dr_+} dr_+ = \frac{\Sigma_k r_+^{D-2}}{4} \left[1 + \frac{2k(D-2)\tilde{\alpha}_{GB}}{(D-4)r_+^2}\right], \quad \Phi = \frac{\Sigma_k Q}{4\pi(D-3)r_+^{D-3}}.$$  (7)

Note that the entropy $S$ will take negative values for sufficiently small black holes for hyperbolic charged GB-AdS black hole. However, an ambiguity can be added into the definition of the entropy which can be appropriately chosen to avoid the appearance of negative entropies [47, 48]. Here we do not rewrite the expression for entropy $S$ with $k = -1$, since the entropy of hyperbolic GB-AdS black holes will be not considered in the next sections.
The expressions for mass $M$ and Hawking temperature $T$ of charged GB-AdS black holes with fixed potential can be rewritten as

$$T = \frac{1}{4\pi r_+ (r_+^2 + 2k\tilde{\alpha}_{GB})} \left[ \frac{16\pi P r_+^4}{D-2} + (D-3)kr_+^2 + (D-5)k^2\tilde{\alpha}_{GB} - \frac{32\pi^2(D-3)^2\Phi^2 r_+^2}{(D-2)\Sigma_k^2} \right],$$

$$M = \frac{(D-2)\Sigma_k r_+^{D-3}}{16\pi} \left[ k + \frac{k^2\tilde{\alpha}_{GB}}{r_+^2} + \frac{16\pi P r_+^2}{(D-1)(D-2)} + \frac{32\pi^2(D-3)\Phi^2}{(D-2)\Sigma_k^2} \right].$$

The black hole mass $M$ is considered as the enthalpy rather than the internal energy of the gravitational system $\text{[22]}$. Moreover, the thermodynamic quantities satisfy the following differential relation

$$dM = TdS + Qd\Phi + VdP + \Omega d\tilde{\alpha}_{GB},$$

where $V$ denotes the thermodynamic volume with

$$V = \left( \frac{\partial M}{\partial P} \right)_{S,\Phi,\tilde{\alpha}_{GB}} = \frac{\Sigma_k r_+^{D-1}}{D-1}, \quad \Omega = \left( \frac{\partial M}{\partial \tilde{\alpha}_{GB}} \right)_{S,\Phi,P} = \frac{(D-2)k^2\Sigma_k r_+^{D-5}}{16\pi}.$$

By the scaling argument, we can obtain the generalized Smarr relation for the charged GB-AdS black holes in the extended phase space

$$M = \frac{D-2}{D-3}TS + \Phi Q - \frac{2}{D-3}VP + \frac{2}{D-3}\Omega\tilde{\alpha}_{GB}.$$  

**III. CRITICAL BEHAVIORS OF CHARGED GB-ADS BLACK HOLES IN THE GRAND CANONICAL ENSEMBLE**

From Eq. (8), the equation of state $P(V,T,\tilde{\alpha}_{GB},\Phi)$ can be expressed into

$$P = \frac{(D-2)T}{4r_+} \left( 1 + \frac{2k\tilde{\alpha}_{GB}}{r_+^2} \right) - \frac{(D-2)(D-3)k}{16\pi r_+^2} - \frac{(D-2)(D-5)k^2\tilde{\alpha}_{GB}}{16\pi r_+^4} + \frac{2\pi(D-3)^2\Phi^2}{\Sigma_k r_+^2}.$$  

To compare with the Van der Waals fluid equation in $D$-dimensions, we can translate the “geometric” equation of state to physical one by identifying the specific volume $v$ of the fluid with the horizon radius of the black hole as $v = \frac{4r_+}{D-2}$.

We know that the critical points occur when $P$ has an inflection point,

$$\left. \frac{\partial^2 P}{\partial r_+^2} \right|_{T=T_c,r_+=r_c} = 0.$$  

Then we can obtain the critical temperature

$$T_c = \frac{(D-3)kr_c}{2\pi (r_c^2 + 6k\tilde{\alpha}_{GB})} + \frac{(D-5)k^2\tilde{\alpha}_{GB}}{\pi (r_c^2 + 6k\tilde{\alpha}_{GB}) r_c} - \frac{16\pi(D-3)^2\Phi^2 r_c}{(D-2) (r_c^2 + 6k\tilde{\alpha}_{GB}) \Sigma_k^2}.$$  

**P.S.**
and the equation for the critical horizon radius $r_c$ (specific volume $v_c = \frac{4r_c}{D-2}$) is

$$\frac{8\pi(D-3)^2 \left( r_c^2 - 6k\tilde{\alpha}_{GB} \right) r_c \Phi^2}{(D-2) (r_c^2 + 6k\tilde{\alpha}_{GB}) (r_c^2 + 12k\tilde{\alpha}_{GB}) \Sigma_k^2} - \frac{(D-3)kr_c^4 - 12k^2\tilde{\alpha}_{GB}r_c^2 + 12k(D - 5)\tilde{\alpha}_{GB}^2}{4\pi r_c (r_c^2 + 6k\tilde{\alpha}_{GB}) (r_c^2 + 12k\tilde{\alpha}_{GB})} = 0, \quad (16)$$

where $r_c$ denotes the critical value of $r$.

Taking $\tilde{\alpha}_{GB} = 0$, one can reduce Eq. (16) to the RN-AdS case

$$\frac{1}{r_c} \left( \frac{8\pi(D-3)\Phi^2}{(D-2)\Sigma_k^2} - \frac{k}{4\pi} \right) = 0. \quad (17)$$

Different from the discussion in the canonical ensemble [23], $r_c$ maintains the same order in the two terms related to $\Phi$ and $k$ in Eq. (17). For $k = 0$, Eq. (17) cannot be satisfied for nonzero $\Phi$. For $k = -1$, the right hand side of Eq. (17) can never be zero. For $k = 1$, Eq. (17) can hold only when $\Phi = \frac{\Sigma_k}{4\pi} \sqrt{\frac{(D-2)}{2(D-3)}}$. However, when $k = 1$ and $\Phi = \frac{\Sigma_k}{4\pi} \sqrt{\frac{(D-2)}{2(D-3)}}$, both the critical pressure $P_c$ and the critical temperature $T_c$ vanish. This result tells us that in the grand canonical ensemble, there is no critical behavior between small-large black holes in the RN-AdS backgrounds in the extended phase space. This result holds independent of the spacetime dimensions and topology. This generalized the first observation in the four-dimensional spherical RN-AdS black holes with fixed potential in the extended phase space [21]. The obtained result in the extended phase space supports the arguments with fixed cosmological constant [2].

Now we would like to extend the discussion of the critical behavior to the charged GB-AdS black holes in the extended phase space with fixed potential. Looking at the critical temperature $T_c$ (Eq. (15)), we find that it is negative for $k = 0$, which indicates that no criticality can happen when $k = 0$. Thus we only need to focus on the cases with $k = -1$ and $k = 1$.

A. Five-dimensional Charged GB-AdS black holes with fixed potential

We first concentrate on the five dimensional spacetime, $D = 5$, so that Eq. (16) reduces to

$$\frac{64\pi^2\Phi^2 - 3k\Sigma_k^2 \left( r_c^2 - 6k\tilde{\alpha}_{GB} \right)}{6\pi\Sigma_k^2 (r_c^2 + 6k\tilde{\alpha}_{GB}) (r_c^2 + 12k\tilde{\alpha}_{GB})} r_c = 0. \quad (18)$$

For $k = -1$, it is impossible to have a physical solution of $r_c$ to make the above equation to be satisfied. This corresponds to say that there does not exist the Van der Waals like small-large black hole phase transition in the hyperbolic space.

For $k = 1$, $\Sigma_k$ equals to $2\pi^2$. When $3\pi^2 - 16\Phi^2 = 0$, namely $\Phi = \frac{\sqrt{7} \pi}{4}$, $r_c$ can take any positive values, while the critical temperature $T_c$ and $P_c$ both disappear from Eqs.(15)(17). In case of
Φ ≠ \frac{\sqrt{3}\pi}{4}$, the solution of Eq. (20) and corresponding critical values of temperature and pressure read

\[ r_c = \sqrt{6\bar{\alpha}_{GB}}, \quad T_c = \frac{3\pi^2 - 16\Phi^2}{6\sqrt{6\bar{\alpha}_{GB}\pi^3}}, \quad P_c = \frac{3\pi^2 - 16\Phi^2}{144\pi^3\bar{\alpha}_{GB}}. \tag{19} \]

It is easy to see that for the potential with fixed values in the range $0 < \Phi < \frac{\sqrt{3}\pi}{4}$, criticality can appear in this charged GB-AdS black hole background. This result holds for all positive GB coupling constants $\bar{\alpha}_{GB}$. Considering the specific volume $v_c = \frac{4r_c}{D-2}$, we can obtain an interesting relation $\frac{P_c v_c}{T_c} = \frac{1}{D-2} = \frac{1}{3}$ from Eq. (19), which is independent of $\bar{\alpha}_{GB}$ and potential $\Phi$. We plot the $P - r_+$ isotherm diagram with different values of $\Phi$ for $D = 5$ in Fig. 1. It shows that for $\Phi = 1 < \frac{\sqrt{3}\pi}{4}$, the dashed line corresponds to the “idea gas” phase behavior when $T > T_c$, and the Van der Waals like small-large black hole phase transition appears in the system when $T < T_c$.

![FIG. 1: The $P - r_+$ diagram of spherical GB-AdS black holes in the grand canonical ensemble for $\Phi = 1$, $\bar{\alpha}_{GB} = 0.01$ and $D = 5$.](image)

On the other hand, in the Van der Waals fluid we have the liquid-gas phase transition equation

\[ \left( P + \frac{a}{v^2} \right)(v - b) = KT, \tag{20} \]

where $v$ is the specific volume of the fluid, $P$ is the pressure, $T$ is the temperature, and $K$ is the Boltzmann constant. The constant $b > 0$ describes the molecules with nonzero size in the fluid, and the constant $a > 0$ is a measure of the attraction between them. Defining

\[ p = \frac{P}{P_c}, \quad \nu = \frac{v}{v_c}, \quad \tau = \frac{T}{T_c}, \tag{21} \]

the Van der Waals fluid equation can be rewritten in a dimensionless form

\[ 8\tau = (3\nu - 1) \left( p + \frac{3}{\nu^2} \right). \tag{22} \]
Here the compressibility factor $P_{c_v} = \frac{3}{8}$ is a universal number for all fluids. For the $D$-dimensional RN-AdS black hole, the analogy of the Van der Waals liquid-gas phase transition equation is described by

$$4(D - 2)\tau = (2D - 5)\nu \left(p + \frac{D - 2}{(D - 3)\nu^2}\right) - \frac{1}{(D - 3)\nu^{2D-5}}. \tag{23}$$

The ratio is $P_{c_v} = \frac{2D-5}{4D-8}$, which depends on the spacetime dimensions. For $D = 5$, Eq. (23) reduces to

$$12\tau = 5\nu \left(p + \frac{3}{2\nu^2}\right) - \frac{1}{2\nu^5} \tag{24}$$

with the ratio $P_{c_v} = 5/12$. For the five dimensional charged GB-AdS black hole with fixed potential, Eq. (13), the analogy of the Van der Waals liquid-gas phase transition equation has the form

$$\tau = \frac{3\nu (3 + p\nu^2)}{4 (2 + 3\nu^2)}. \tag{25}$$

One can see the Van der Waals like small-large black hole phase transition happen for the these black hole backgrounds. However, the analogy of the Van der Waals liquid-gas phase transition equations can take different forms, and the ratio $P_{c_v}$ arrives at different values for various black hole backgrounds.

The behavior of the Gibbs free energy $G$ is important to determine the thermodynamic phase transition. In the grand canonical ensemble with fixed potential, the Gibbs free energy $G$ obeys the following thermodynamic relation

$$G = M - TS - Q\Phi = \frac{1}{24\pi (r_+^2 + 2\tilde{\alpha}_{GB})} \left[3\pi^2 (r_+^4 - 3\tilde{\alpha}_{GB} r_+^2 + 6\tilde{\alpha}_{GB}^2) - 4\pi^3 r_+^4 (r_+^2 + 18\tilde{\alpha}_{GB}) P - 16r_+^2 (r_+^2 - 6\tilde{\alpha}_{GB}) \Phi^2 \right]. \tag{26}$$

Here $r_+$ is understood as a function of pressure and temperature, $r_+ = r_+(P,T)$, via equation of state Eq. (13). In Fig. 2, we see that the $G$ surface demonstrates the characteristic “swallow tail” behavior, which shows that there is a Van der Waals like first order phase transition in the system. We also plot the coexistence line in the $(P,T)$ plane by finding a curve where the Gibbs free energy and temperature coincide for small and large black holes as shown in Fig. 2. The coexistence line in the $(P,T)$ plane is very similar to that in the Van der Waals fluid. The critical point is shown by a small circle at the end of the coexistence line. The small-large black hole phase transition occurs for $T < T_c$. 
B. Higher dimensional charged GB-AdS black holes with Fixed potential

Defining $\tilde{\Phi} = \pi \Phi / \Sigma_k$, equation Eq. (16) can be rewritten as

$$\frac{1}{4(D - 2)\pi r_c(r_c^2 + 6k\tilde{\alpha}_GB)(r_c^2 + 12k\tilde{\alpha}_GB)} \left[ (D - 2) ((D - 3)kr_c^4 + 12kr_c^2\tilde{\alpha}_GB - 12k(D - 5)\tilde{\alpha}_GB^2) \right. \\
+ 32(D - 3)^2r_c^2(r_c^2 - 6k\tilde{\alpha}_GB)\tilde{\Phi}^2 \left. \right] = 0. \quad (27)$$

For $k = -1$, the right hand side of Eq. (27) can never disappear for any real value of $r_c$. This tells us that there does not exist any critical behavior in the hyperbolic charged GB-AdS black holes.

Now we turn our discussion to the spherical case with $k = 1$. The Eq. (27) reduces to

$$(D - 3) \left[ 32(D - 3)\tilde{\Phi}^2 - (D - 2) \right] r_c^4 - 12\tilde{\alpha}_GB \left[ 16(D - 3)^2\tilde{\Phi}^2 - (D - 2) \right] r_c^2 \\
- 12(D - 5)(D - 2)\tilde{\alpha}_GB^2 = 0. \quad (28)$$

For $D \geq 6$, we have the relation $\frac{(D-2)^2}{16(D-3)^2} < \frac{(D-2)^4}{48(D-3)^4} < \frac{(D-2)^2}{32(D-3)^2}$. When the potential is fixed with the value $\tilde{\Phi}^2 = \frac{(D-2)^2}{32(D-3)^2} > \frac{(D-2)^2}{16(D-3)^2}$, we cannot find any real solution of $r_c$ from Eq. (28). If the potential is fixed with other values, $\tilde{\Phi}^2 \neq \frac{(D-2)^2}{32(D-3)^2}$, we can obtain the solution of Eq. (28) in the form

$$r_c^2 = \frac{\tilde{\alpha}_GB \left[ 12 \left( 16(D - 3)^2\tilde{\Phi}^2 - (D - 2) \right) \pm \sqrt{\Delta} \right]}{2(D - 3) \left[ 32(D - 3)\tilde{\Phi}^2 - (D - 2) \right]}, \quad (29)$$

where $\Delta = 48 \left[ 768(D - 3)^4\tilde{\Phi}^4 + 32(D - 2)(D - 8)(D - 3)^2\tilde{\Phi}^2 - (D - 6)(D - 2)^3 \right]$. To keep the $\Delta$ non-negative, we need the potential satisfy $\tilde{\Phi}^2 \geq \frac{(D-2)^2}{48(D-3)^2}$. When the potential is fixed in the range $\frac{(D-2)^2}{48(D-3)^2} \leq \tilde{\Phi}^2 < \frac{(D-2)^2}{32(D-3)^2}$, we have $12 \left( 16(D - 3)^2\tilde{\Phi}^2 - (D - 2) \right) > \sqrt{\Delta} \geq 0$. The numerator of Eq. (29) is always positive independent of the sign we choose in front of the square root. However the
denominator of Eq. (29) is always negative in this fixed potential range. Thus in the above chosen range of the potential, there is no criticality to happen. We can also fix the potential in the range $\tilde{\Phi}^2 > \frac{(D-2)}{32(D-3)}$ to avoid the negative $\Delta$. Now the denominator in Eq. (29) can always remain positive. In the numerator when the potential is fixed in this range, $\sqrt{\Delta} > 12 \left(16(D-3)^2\tilde{\Phi}^2 - (D-2)\right) > 0$, we can only select the positive sign in front of the square root term. We can always find the physical solution of $r_c$ when the potential is fixed in the range $\tilde{\Phi}^2 > \frac{(D-2)}{32(D-3)}$.

Can the criticality appear for the higher dimensional spherical charged GB-AdS black holes? To answer this question, let us further examine the critical temperature $T_c$. From Eq. (15) we have

$$T_c = \frac{-(D-3)^{3/2} \left[32(D-3)\tilde{\Phi}^2 - (D-2)\right]^{3/2} \left(\kappa + \sqrt{\Delta}/4\right)}{6(D-2)\sqrt{2\kappa_{GB}\pi} \left[3 \left(16(D-3)^2\tilde{\Phi}^2 - (D-2)\right) + \sqrt{\Delta}/4\right]^{1/2} \left(\kappa + \sqrt{\Delta}/12\right)}, \quad (30)$$

where $\kappa = 48(D-3)^2\tilde{\Phi}^2 - (D-2)^2$. It is worth noting that $\kappa > 0$ when the potential is fixed in the range $\tilde{\Phi}^2 > \frac{(D-2)}{32(D-3)}$, so that the critical temperature $T_c$ is always negative in this potential range. Therefore the criticality cannot appear and there is no small-large black hole phase transition, in resemblance with the Van der Waals phase transition, for higher dimensional charged GB-AdS black holes in the grand canonical ensemble.

In summary, the Van der Waals like phase transition only happens in the five-dimensional spherical charged GB-AdS black hole in the grand canonical ensemble when the potential is fixed within the range $0 < \Phi < \sqrt{\frac{3\pi}{4}}$. The criticality cannot appear in the spacetime with other dimensions and topologies. To further examine the critical behavior in the five-dimensional spherical charged GB-AdS black holes, in the following we investigate the behaviors of thermodynamic quantities near the critical point of the phase transition.

C. Critical exponents near critical point

Now we turn to compute the critical exponents $\alpha$, $\beta$, $\gamma$, $\delta$ for the black hole system, which characterize the behaviors of physical quantities in the vicinity of the critical point ($r_+ = r_c$, $v = v_c$, $T = T_c$, $P = P_c$) for the five-dimensional spherical charged GB-AdS black hole in the grand
canonical ensemble. Near the critical point, critical exponents are defined as follows \[21\]

\[ C_v = T \frac{\partial S}{\partial T} \bigg|_v \propto \left( -\frac{T - T_c}{T_c} \right)^{-\alpha}, \]

\[ \eta = \frac{v_s - v_l}{v_c} \propto \left( -\frac{T - T_c}{T_c} \right)^{\beta}, \]

\[ \kappa_T = -\frac{1}{v} \frac{\partial v}{\partial P} \bigg|_T \propto \left( -\frac{T - T_c}{T_c} \right)^{-\gamma}, \]

\[ P - P_c \propto (v - v_c)^{\delta}, \quad (31) \]

where “c” denotes the quantity at the critical point of the system.

In order to compute the critical exponent \( \alpha \) for \( D = 5 \), we rewrite the entropy of black hole (Eq. (7)) as

\[ S = \frac{27\pi^2 v^3}{128}\left( 1 + \frac{32\alpha GB v^2}{3\pi^2} \right). \]

Obviously this entropy \( S \) is independent of \( T \) for the constant value of specific volume \( v \), so we conclude that the critical exponent \( \alpha = 0 \). To obtain the other exponents, we introduce the expansion parameters

\[ \tau = t + 1, \quad \nu = \omega + 1, \quad (32) \]

and expand this equation of state \(25\) near the critical point

\[ p = 1 + a_{10} t + a_{11} t \omega + a_{03} \omega^3 + \mathcal{O}(t^2, \epsilon^4). \quad (33) \]

During the phase transition, the pressure remains constant

\[ p = 1 + a_{10} t + a_{11} t \omega_s + a_{03} \omega_s^3 = 1 + a_{10} t + a_{11} t \omega_l + a_{03} \omega_l^3, \]

\[ \Rightarrow \quad a_{11} t (\omega_s - \omega_l) + a_{03} (\omega_s^3 - \omega_l^3) = 0, \quad (34) \]

where \( \omega_s \) and \( \omega_l \) denote the ‘volume’ of small and large black holes.

Using Maxwell’s area law, we obtain

\[ \int_{\omega_l}^{\omega_s} \omega \frac{dp}{d\omega} d\omega = 0 \Rightarrow a_{11} t (\omega_s^2 - \omega_l^2) + \frac{3}{2} a_{03} (\omega_s^4 - \omega_l^4) = 0. \quad (35) \]

With Eqs. (34) (35), the nontrivial solutions appear only when \( a_{11} a_{03} t < 0 \). Then we can get

\[ \omega_s = \frac{\sqrt{-a_{11} a_{03} t}}{3|a_{03}|} \approx 3.23\sqrt{-t}, \quad \omega_l = -\frac{\sqrt{-a_{11} a_{03} t}}{3|a_{03}|} \approx -3.23\sqrt{-t}. \quad (36) \]

Therefore, we have

\[ \eta = \omega_s - \omega_l = 2\omega_s = 6.46\sqrt{-t} \Rightarrow \beta = 1/2. \quad (37) \]

The isothermal compressibility can be computed as

\[ \kappa_T = -\frac{1}{v} \frac{\partial v}{\partial P} \bigg|_{v_c} \propto -\frac{1}{\frac{\partial v}{\partial \omega}} \bigg|_{\omega = 0} = \frac{2}{3t}. \quad (38) \]
which indicates that the critical exponent \( \gamma = 1 \). Moreover, the shape of the critical isotherm \( t = 0 \) is given by

\[
p - 1 = -\omega^3 \Rightarrow \delta = 3.
\]  

(39)

Evidently in the grand canonical ensemble, these critical exponents of the five-dimensional spherical charged GB-AdS black holes coincide with those of the Van der Waals liquid-gas system [21].

IV. PHASE TRANSITION AT THE CRITICAL POINT AND EHRENFEST’S EQUATIONS

For Van der Waals liquid-gas system, the liquid-gas structure does not change suddenly but undergoes the second order phase transition at the critical point \((V = V_c, T = T_c, P = P_c)\). This is described by the Ehrenfest’s description [50, 51]. In conventional thermodynamics, Ehrenfest’s description consists of the first and second Ehrenfest’s equations [52, 53]

\[
\frac{\partial P}{\partial T} \bigg|_S = \frac{C_{P2} - C_{P1}}{TV(\zeta_2 - \zeta_1)} = \frac{\Delta C_P}{TV\Delta \zeta},
\]

\[
\frac{\partial P}{\partial T} \bigg|_V = \frac{\zeta_2 - \zeta_1}{\kappa_{T2} - \kappa_{T1}} = \frac{\Delta \zeta}{\Delta \kappa_T}.
\]

(40)

(41)

For a genuine second order phase transition, both of these equations have to be satisfied simultaneously. Here \( \zeta \) and \( \kappa_T \) denote the volume expansion and isothermal compressibility coefficients of the system respectively

\[
\zeta = \frac{1}{V} \frac{\partial V}{\partial T} \bigg|_P, \quad \kappa_T = -\frac{1}{V} \frac{\partial V}{\partial P} \bigg|_T.
\]

(42)

Let us concentrate on the five-dimensional spherical charged GB-AdS black hole in the grand canonical ensemble. From Eq. (42), we obtain

\[
V \zeta = \frac{\partial V}{\partial T} \bigg|_P = \frac{\partial V}{\partial P} \bigg|_T \times \frac{\partial S}{\partial T} \bigg|_P = \frac{\partial V}{\partial S} \bigg|_P \times \frac{C_P}{T}.
\]

(43)

The right hand side of Eq. (40) can be expressed into

\[
\frac{\Delta C_P}{TV\Delta \zeta} = \left[ \frac{\partial S}{\partial V} \bigg|_P \right]_{r_+ = r_c} = \frac{3(r_c^2 + 2\tilde{\alpha}_{GB})}{4r_c^3},
\]

(44)

where the thermodynamic volume \( V \) is described in Eq. (11) and the subscript denotes the physical quantities at the critical point. From Eq. (13), the left hand side of Eq. (40) at the critical point can be got as

\[
\left[ \frac{\partial P}{\partial T} \bigg|_S \right]_{r_+ = r_c} = \frac{3(r_c^2 + 2\tilde{\alpha}_{GB})}{4r_c^3}.
\]

(45)
Therefore, the first of Ehrenfest’s equations can be satisfied at the critical point.

Now let’s examine the second of Ehrenfest’s equations. In order to compute $\kappa_T$, we make use
of the thermodynamic identity
\[
\frac{\partial V}{\partial P} \bigg|_T \times \frac{\partial P}{\partial T} \bigg|_V \times \frac{\partial T}{\partial V} \bigg|_P = -1. \tag{46}
\]
Considering Eq. (42), we can have
\[
\kappa_T V = \frac{\partial V}{\partial P} \bigg|_T = \frac{\partial T}{\partial P} \bigg|_V \times \frac{\partial V}{\partial T} \bigg|_P = \frac{\partial T}{\partial P} \bigg|_V \quad V \zeta. \tag{47}
\]
which reveals the validity of the second Ehrenfest equations at the critical point. Moreover, the
right hand side of Eq. (41) is given by
\[
\Delta \zeta \Delta \kappa_T = \left[ \frac{\partial P}{\partial T} \bigg|_V \right]_{r_+ = r_c} = \frac{3(r_c^2 + 2\delta_{GB})}{4r_c^3}. \tag{48}
\]
Using Eqs. (44) and (48), the Prigogine-Defay (PD) ratio (II) \[54] is
\[
\Pi = \frac{\Delta C_P \Delta \kappa_T}{T v (\Delta \zeta)^2} = 1. \tag{49}
\]
Hence in the grand canonical ensemble this phase transition at the critical point in the five-
dimensional charged GB-AdS black hole is of the second order. This result is consistent with the
nature of the liquid-gas phase transition at the critical point.

V. CLOSING REMARKS

In this paper we have studied the thermodynamic behaviors in the grand canonical ensemble by
fixing the potential of D-dimensional charged GB-AdS black holes in an extended phase space by
treating the cosmological constant and their conjugate quantity as thermodynamic variables. We
have written out the equations of state and examined the phase structures by using the standard
thermodynamic techniques. We have shown that only five-dimensional spherical GB-AdS black
holes admit a first order small-large black hole phase transition when its potential is fixed within
the range $0 < \Phi < \sqrt{\frac{\pi}{3}}$, which resembles the liquid-gas phase transition in fluids. For the other
higher dimensional and topological charged GB-AdS black holes, we have not found the phase
transition. When we take the limiting case with vanishing GB parameter, we conclude that in the
grand canonical ensemble, there is no criticality for the RN-AdS black holes in the extended phase
space. This result is independent of the spacetime dimensions and topologies.

We have also computed the critical exponents of the phase transition and found that in the fixed
potential ensemble the thermodynamic exponents associated with the five-dimensional spherical
charged GB-AdS black hole coincide with those of the Van der Waals fluid. We finally analyzed
the phase transition at the critical point by employing Ehrenfest’s scheme and verified that both
of the Ehrenfest’s equations can hold at the critical point. This shows that in resemblance with
the liquid-gas phase transition, at the critical point the phase transition of the five-dimensional
charged GB-AdS black hole in the fixed potential ensemble is of the second order.

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