CHARACTERIZATION OF ISOMETRIC EMBEDDINGS OF GRASSMANN GRAPHS

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Abstract. Let $V$ be an $n$-dimensional left vector space over a division ring $R$. We write $G_k(V)$ for the Grassmannian formed by $k$-dimensional subspaces of $V$ and denote by $\Gamma_k(V)$ the associated Grassmann graph. Let also $V'$ be an $n'$-dimensional left vector space over a division ring $R'$. Isometric embeddings of $\Gamma_k(V)$ in $\Gamma_k(V')$ are classified in [13]. A classification of $J(n,k)$-subsets in $G_k(V')$, i.e. the images of isometric embeddings of the Johnson graph $J(n,k)$ in $\Gamma_k(V')$, is presented in [12]. We characterize isometric embeddings of $\Gamma_k(V)$ in $\Gamma_k(V')$ as mappings which transfer apartments of $G_k(V)$ to $J(n,k)$-subsets of $G_k(V')$. This is a generalization of the earlier result concerning apartments preserving mappings [11, Theorem 3.10].

1. Introduction

Let $V$ be an $n$-dimensional left vector space over a division ring $R$ and let $G_k(V)$ be the Grassmannian formed by $k$-dimensional subspaces of $V$. The associated Grassmann graph will be denoted by $\Gamma_k(V)$. By classical Chow’s theorem [2], every automorphism of $\Gamma_k(V)$ with $1 < k < n - 1$ is induced by a semilinear automorphism of $V$ or a semilinear isomorphism of $V$ to the dual vector space $V^*$. The second possibility can be realized only in the case when $n = 2k$. The statement fails for $k = 1, n - 1$. In this case, any two distinct vertices of $\Gamma_k(V)$ are adjacent and any bijective transformation of $G_k(V)$ is an automorphism of $\Gamma_k(V)$.

Results closely related to Chow’s theorem can be found in [1, 3, 4, 6, 7, 9, 10], see also [11, Section 3.2].

One of recent generalizations of Chow’s theorem is the classification of isometric embeddings of $\Gamma_k(V)$ in $\Gamma_k(V')$, where $V'$ is an $n'$-dimensional left vector space over a division ring $R'$. The existence of such embeddings implies that

$$\min\{k, n-k\} \leq \min\{k', n'-k'\},$$

i.e. the diameter of $\Gamma_k(V)$ is not greater than the diameter of $\Gamma_k(V')$. The case $k = 1, n - 1$ is trivial: every isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(V')$ is a bijection to a clique of $\Gamma_k(V')$. If $1 < k < n - 1$ then isometric embeddings of $\Gamma_k(V)$ in $\Gamma_k(V')$ are defined by semilinear $(2k)$-embeddings, i.e. semilinear injections which transfer any $2k$ linearly independent vectors to linearly independent vectors.

A result of similar nature is obtained in [12]. This is the classification of the images of isometric embeddings of the Johnson graph $J(n,k)$ in the Grassmann graph $\Gamma_k(V')$. As above, we need (1.1) which guarantees that the diameter of $J(n,k)$ is not greater than the diameter of $\Gamma_k(V')$. The images of isometric embeddings of $J(n,k)$ in $\Gamma_k(V')$ will be called $J(n,k)$-subsets of $G_k(V')$.

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Suppose that $1 < k < n - 1$ (the case $k = 1, n - 1$ is trivial). If $n = 2k$ then every $J(n, k)$-subset is an apartment in a parabolic subspace of $G_k(V')$ and we get an apartment of $G_k(V')$ if $n = n'$ and $k' = k, n - k$. In the case when $n \neq 2k$, there are two distinct types of $J(n, k)$-subsets.

If $n = n'$ then every apartments preserving mapping of $G_k(V)$ to $G_k(V')$ with $1 < k < n - 1$ is induced by a semilinear embedding of $V$ in $V'$ or a semilinear embedding of $V$ in $V''$ and the second possibility can be realized only in the case when $n = 2k$ [11, Theorem 3.10]. For $k = 1, n - 1$ this fails. By [5], there are apartments preserving mappings of $G_1(V)$ to itself which can not be defined by semilinear mappings.

Our main result (Theorem 3.1) characterizes isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ as mappings which transfer apartments of $G_k(V)$ to $J(n, k)$-subsets of $G_{k'}(V')$. As a consequence, we get a generalization of the above mentioned result on apartments preserving mappings.

2. Grassmann graph and Johnson graph

2.1. Graph theory. In this subsection we recall some concepts of the general graph theory.

A subset in the vertex set of a graph is called a clique if any two distinct vertices in this subset are adjacent (connected by an edge). Every clique is contained in a maximal clique (this is trivial if the vertex set is finite and we use Zorn lemma in the infinite case).

The distance between two vertices in a connected graph $\Gamma$ is defined as the smallest number $i$ such that there is a path consisting of $i$ edges and connecting these vertices. The diameter of $\Gamma$ is the greatest distance between two vertices.

An embedding of a graph $\Gamma$ in a graph $\Gamma'$ is an injection of the vertex set of $\Gamma$ to the vertex set of $\Gamma'$ such that adjacent vertices go to adjacent vertices and non-adjacent vertices go to non-adjacent vertices. Every surjective embedding is an isomorphism. An embedding is said to be isometric if it preserves the distance between any two vertices. Every embedding preserves the distances 1 and 2. Thus any embedding of a graph with diameter 2 is isometric.

2.2. Grassmann graph. Let $V$ be an $n$-dimensional left vector space over a division ring $R$. For every $k \in \{0, \ldots, n\}$ we denote by $G_k(V)$ the Grassmannian formed by $k$-dimensional subspaces of $V$. Then $G_0(V) = \{0\}$ and $G_n(V) = \{V\}$. In the case when $1 \leq k \leq n - 1$, two elements of $G_k(V)$ are said to be adjacent if their intersection is $(k - 1)$-dimensional (this is equivalent to the fact that their sum is $(k + 1)$-dimensional).

The Grassmann graph $\Gamma_k(V)$ is the graph whose vertex set is $G_k(V)$ and whose edges are pairs of adjacent $k$-dimensional subspaces. The graph $\Gamma_k(V)$ is connected, the distance $d(S, U)$ between two vertices $S, U \in G_k(V)$ is equal to $k - \dim(S \cap U) = \dim(S + U) - k$ and the diameter of $\Gamma_k$ is equal to $\min\{k, n - k\}$.

Let $V^*$ be the dual vector space. This is an $n$-dimensional left vector space over the opposite division ring $R^*$ (the division rings $R$ and $R^*$ have the same set of elements and the same additive operation, the multiplicative operation $*$ on $R^*$ is defined by the formula $a * b := ba$ for all $a, b \in R$). The second dual space $V^{**}$ is canonically isomorphic to $V$. 
For a subset $X \subset V$ the subspace
\[ X^0 := \{ x^* \in V^* : x^*(x) = 0 \ \forall x \in X \} \]
is called the annihilator of $X$. The dimension of $X^0$ is equal to the codimension of $\langle X \rangle$. The annihilator mapping of the set of all subspaces of $V$ to the set of all subspaces of $V^*$ is bijective and reverses the inclusion relation, i.e.
\[ S \subset U \iff U^0 \subset S^0 \]
for any subspaces $S, U \subset V$. Since $S^{00} = S$ for every subspace $S \subset V$, the inverse bijection is also the annihilator mapping. The restriction of the annihilator mapping to each $\mathcal{G}_k(V)$ is an isomorphism of $\Gamma_k(V)$ to $\Gamma_{n-k}(V^*)$.

**Lemma 2.1.** If $S_1, \ldots, S_m$ are subspaces of $V$ then
\[
(S_1 + \cdots + S_m)^0 = (S_1)^0 \cap \cdots \cap (S_m)^0,
\]
\[
(S_1 \cap \cdots \cap S_m)^0 = (S_1)^0 + \cdots + (S_m)^0.
\]

Consider incident subspaces $S \in \mathcal{G}_s(V)$ and $U \in \mathcal{G}_u(V)$ such that $s < k < u$. We define
\[ [S, U]_k := \{ P \in \mathcal{G}_k(V) : S \subset P \subset U \}. \]
In the case when $U = V$ or $S = 0$, this subset will be denoted by $[S]_k$ or $[U]_k$, respectively. Subsets of such type are called parabolic subspaces of $\mathcal{G}_k(V)$, see [11, Section 3.1].

There is the natural isometric embedding $\Phi^U_S$ of $\Gamma_{k-s}(U/S)$ in $\Gamma_k(V)$ which sends every $(k-s)$-dimensional subspace of $U/S$ to the corresponding $k$-dimensional subspace of $V$. In the case when $U = V$ or $S = 0$, this embedding will be denoted by $\Phi_S$ or $\Phi^U$, respectively. The image of $\Phi^U_S$ is the parabolic subspace $[S, U]_k$.

If $k = 1, n - 1$ then any two distinct vertices of $\Gamma_k(V)$ are adjacent. In the case when $1 < k < n - 1$, there are precisely the following two types of maximal cliques of $\Gamma_k(V)$:

- the star $[S]_k$, $S \in \mathcal{G}_{k-1}(V)$,
- the top $[U]_k$, $U \in \mathcal{G}_{k+1}(V)$.

The annihilator mapping transfers every parabolic subspace $[S, U]_k$ to the parabolic subspace $[U^0, S^0]_{n-k}$; in particular, it sends stars to tops and tops to stars.

**2.3. Johnson graph.** The Johnson graph $J(n, k)$ is the graph whose vertices are $k$-element subsets of $\{1, \ldots, n\}$ and whose edges are pairs of $k$-element subsets with $(k-1)$-element intersections. The graph $J(n, k)$ is connected, the distance $d(X, Y)$ between two vertices $X, Y$ is equal to
\[ k - |X \cap Y| = |X \cup Y| - k \]
and the diameter of $J(n, k)$ is equal to $\min\{k, n - k\}$. The mapping
\[ X \rightarrow X^c := \{1, \ldots, n\} \setminus X \]
is an isomorphism between $J(n, k)$ and $J(n, n - k)$.

If $k = 1, n - 1$ then any two distinct vertices of $J(n, k)$ are adjacent. In the case when $1 < k < n - 1$, there are precisely the following two types of maximal cliques of $J(n, k)$:

- the star which consists of all vertices containing a certain $(k - 1)$-element subset,
• the top which consists of all vertices contained in a certain \((k + 1)\)-element subset.

The stars and tops of \(J(n,k)\) consist of \(n - k + 1\) and \(k + 1\) vertices, respectively. The isomorphism \(X \rightarrow X^*\) transfers stars to tops and tops to stars.

Let \(B\) be a base of \(V\). The associated apartment of \(G_k(V)\) consists of all \(k\)-
dimensional subspaces spanned by subsets of \(B\). This is the image of an isometric embedding of \(J(n,k)\) in \(\Gamma_k(V)\). We will use the following facts:

• for any two \(k\)-dimensional subspaces of \(V\) there is an apartment of \(G_k(V)\) containing both of them;
• the annihilator mapping of \(G_k(V)\) to \(G_{n-k}(V^*)\) transfers apartments to apartments.

Let \([S,U]_k\) be a parabolic subspace of \(G_k(V)\). Let also \(B\) be a base of \(V\) such that \(S\) and \(U\) are spanned by subsets of \(B\). The intersection of the corresponding apartment of \(G_k(V)\) with \([S,U]_k\) is said to be an apartment in the parabolic sub-

2.4. Isometric embeddings of Johnson graphs in Grassmann graphs. Let \(V'\) be an \(n'\)-dimensional left vector space over a division ring \(R'\). Isometric embed-
dings of \(J(n,k)\) in \(\Gamma_{k'}(V')\) are classified in [12]. The existence of such embeddings implies that the diameter of \(J(n,k)\) is not greater than the diameter of \(\Gamma_{k'}(V')\), i.e.

\[
\min\{k, n-k\} \leq \min\{k', n'-k'\}.
\]

Since \(J(n,k)\) and \(J(n,n-k)\) are isomorphic, we can suppose that \(k \leq n-k\). Then

\[
k = \min\{k', n-k, n'-k'\}.
\]

The case \(k = 1\) is trivial: any two distinct vertices of \(J(n,1)\) are adjacent and every isometric embedding of \(J(n,1)\) in \(\Gamma_{k'}(V')\) is a bijection to a clique of \(\Gamma_{k'}(V')\).

We say that a subset \(X \subset V\) is \(m\)-independent if every \(m\)-element subset of \(X\) is independent. If \(x_1, \ldots, x_m\) are linearly independent vectors of \(V\) and

\[
x_{m+1} = a_1x_1 + \cdots + a_mx_m,
\]

where each \(a_i\) is non-zero, then \(x_1, \ldots, x_{m+1}\) form an \(m\)-independent subset. Every \(n\)-independent subset of \(V\) consisting of \(n\) vectors is a base of \(V\). By [12] Proposition

1], if the division ring \(R\) is infinite then for every natural integer \(l \geq n\) there is an \(n\)-independent subset of \(V\) consisting of \(l\) vectors.

Suppose that \(k < n-k \) and \(X\) is a \((2k)\)-independent subset of \(V\) consisting of \(l\) vectors. Every \(k\)-element subset of \(X\) spans a \(k\)-dimensional subspace and we denote by \(J_k(X)\) the set formed by all such subspaces. This is the image of an isometric embedding of \(J(l,k)\) in \(\Gamma_k(V)\). We will write \(J_k^*(X)\) for the subset of \(G_{n-k}(V^*)\) consisting of the annihilators of elements from \(J_k(X)\). If \(X\) is a base of \(V\) then \(J_k(X)\) and \(J_k^*(X)\) are apartments of \(G_k(V)\) and \(G_{n-k}(V^*)\), respectively.

The images of isometric embeddings of \(J(n,k)\) in \(\Gamma_{k'}(V')\) are called \(J(n,k)\)-

subsets of \(G_k(V')\).

Theorem 2.1 ([12]). Let \(J\) be a \((n,k)\)-subset of \(G_k(V')\) and \(1 < k \leq n - k\). In the case when \(n = 2k\), there exist \(S \in G_{k'-k}(V')\) and \(U \in G_{k'+k}(V')\) such that \(J\) is
an apartment in the parabolic subspace $[S,U]_{k'}$, i.e.
\[ J = \Phi_S^U(A), \]
where $A$ is an apartment of $G_k(U/S)$. If $k < n - k$ then one of the following possibilities is realized:

1. there exist $S \in G_{k'}(V')$ and a $(2k)$-independent $n$-element subset $X \subset V'/S$ such that \[ J = \Phi_S(J_k(X)); \]
2. there exist $U \in G_{k'+1}(V')$ and a $(2k)$-independent $n$-element subset $Y \subset U^*$ such that \[ J = \Phi_U(J_{k+1}(V')). \]

Remark 2.1. Suppose that $1 < k < n - k$ and $J \subset G_{k'}(V')$ is a $J(n,k)$-subset of second type. Let $U$ and $Y$ be as in Theorem 2.1. The annihilators of vectors belonging to $Y$ form an $n$-element subset $\mathcal{Y} \subset G_{k'+1}(U)$. Every element of $\mathcal{J}$ can be presented as the intersection of $k$ distinct elements of $\mathcal{Y}$.

Let $C$ be a maximal clique of $\Gamma_{k'}(V')$ (a star or a top). As above, we suppose that $J$ is a $J(n,k)$-subset of $G_{k'}(V')$ and $1 < k \leq n - k$. If $J \cap C$ contains more than one element than it is a maximal clique of the restriction of $\Gamma_{k'}(V')$ to $J$ (this restriction is isomorphic to $J(n,k)$). In this case, we say that $J \cap C$ is a star or a top of $J$ if $C$ is a star or a top, respectively.

Lemma 2.2. Suppose that $1 < k < n - k$. If $J$ is a $J(n,k)$-subset of first type then the stars and tops of $J$ consist of $n - k + 1$ and $k + 1$ vertices, respectively. In the case when $J$ is a $J(n,k)$-subset of second type, the stars and tops of $J$ consist of $k + 1$ and $n - k + 1$ vertices, respectively.

Proof. Easy verification. \[ \square \]

Lemma 2.2 shows that the two above determined classes of $J(n,k)$-subsets are disjoint.

2.5. Isometric embeddings of Grassmann graphs. Isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ are classified in [13]. As in the previous subsection, we have (2.1) which implies that the diameter of $\Gamma_k(V)$ is not greater than the diameter of $\Gamma_{k'}(V')$.

A mapping $l : V \rightarrow V'$ is called semilinear if
\[ l(x + y) = l(x) + l(y) \]
for all $x, y \in V$ and there is a homomorphism $\sigma : R \rightarrow R'$ such that
\[ l(ax) = \sigma(a)l(x) \]
for all $a \in R$ and $x \in V$. If $l$ is non-zero then there is only one homomorphism $\sigma$ satisfying this condition. Every non-zero homomorphism of $R$ to $R'$ is injective.

A semilinear injection of $V$ to $V'$ is said to be a semilinear $m$-embedding if it transfers any $m$ linearly independent vectors to linearly independent vectors. The existence of such mappings implies that $m \leq n'$. A semilinear $n$-embedding of $V$ in $V'$ will be called a semilinear embedding. It maps every independent subset to an
Every isometric embedding of $\Gamma_k$ in $\Gamma_k$ is canonical isomorphic, every isometric embedding of $\Gamma_k$ is considered as an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(V')$ and $\Gamma_{n-k}(V')$, respectively.

**Theorem 3.1.** Let $f$ be an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ and $1 < k \leq n - k$. Then one of the following possibilities is realized:

1. there exist $S \in \mathcal{G}_{k-r}(V')$ and a semilinear $2k$-embedding $l : V \to V'/S$ such that $f = \Phi_S \circ (l)_k$;
2. there exist $U \in \mathcal{G}_{k^*+1}(V')$ and a semilinear $2k$-embedding $s : V \to U^*$ such that $f = \Phi_U \circ (s)_k^*$.

In particular, if $n = 2k$ then there exist incident $S \in \mathcal{G}_{k-r}(V')$ and $U \in \mathcal{G}_{k^*+1}(V')$ such that $f$ is induced by a semilinear embedding $l : V \to U/S$ or a semilinear embedding $s : V \to (U/S)^*$, i.e.

$$f = \Phi_S^U \circ (l)_k \quad \text{or} \quad f = \Phi_S^U \circ (s)_k^*.$$

The case $k = 1, n - 1$ is trivial. The case when $1 < k \leq n - k$ is considered in Theorem 2.2. Suppose that $n - k < k < n - 1$. Since $\Gamma_k(V)$ and $\Gamma_{n-k}(V^*)$ are canonically isomorphic, every isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ can be considered as an isometric embedding of $\Gamma_{n-k}(V^*)$ in $\Gamma_{k'}(V')$. The latter embedding is one of the mappings described in Theorem 2.2. In contrast to the case when $1 < k \leq n - k$, we can not show that isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ are defined by semilinear mappings of $V$.

### 3. Main result

Let $f$ be a mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_{k'}(V')$. If the restriction of $f$ to every apartment of $\mathcal{G}_k(V)$ is an isometric embedding of $J(n, k)$ in $\Gamma_{k'}(V')$ then $f$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. This follows from the fact that for any two elements of $\mathcal{G}_k(V)$ there is an apartment containing both of them.

We say that $f$ is a $J$-mapping if it sends every apartment of $\mathcal{G}_k(V)$ to a $J(n, k)$-subset. Every isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ satisfies this condition. Our main result states that this property characterizes isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$.

**Theorem 3.1.** Every $J$-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_{k'}(V')$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$.

Some corollaries of Theorem 3.1 will be given in Section 6.
4. Intersections of $J(n,k)$-subsets

4.1. Special subsets. Let $X = \{x_1, \ldots, x_n\}$ be a $(2k)$-independent subset of a vector space $W$ (the dimension of $W$ is assumed to be not less than $2k$ and $n \geq 2k$) and let $k \geq 2$. Consider the set $\mathcal{J} = \mathcal{J}_k(X)$ formed by all $k$-dimensional subspaces spanned by subsets of $X$. For every $i \in \{1, \ldots, n\}$ we denote by $\mathcal{J}(+i)$ and $\mathcal{J}(-i)$ the sets consisting of all elements of $\mathcal{J}$ which contain $x_i$ and do not contain $x_i$, respectively. Also, we write $\mathcal{J}(+i, +j)$ for the intersection of $\mathcal{J}(+i)$ and $\mathcal{J}(+j)$.

Every

$$\mathcal{J}(+i, +j) \cup \mathcal{J}(-i), \quad i \neq j$$

is said to be a special subset of $\mathcal{J}$.

We say that a subset $\mathcal{X} \subset \mathcal{J}$ is inexact if there is a $(2k)$-independent $n$-element subset $Y \subset W$ such that $\mathcal{J}_k(Y) \neq \mathcal{J}$ (at least one of the vectors belonging to $Y$ is not a scalar multiple of a vector from $X$) and $\mathcal{X} \subset \mathcal{J}_k(Y)$.

**Lemma 4.1.** Every inexact subset is contained in a special subset.

**Proof.** Let $\mathcal{X}$ be an inexact subset. Denote by $S_i$ the intersection of all elements of $\mathcal{X}$ containing $x_i$ and set $S_i = 0$ if there are no elements of $\mathcal{X}$ containing $x_i$. There is at least one $i$ such that $S_i \neq \{x_i\}$ (otherwise, $\mathcal{X}$ is not inexact). Then $S_i = 0$ or $\dim S_i \geq 2$. In the first case, $\mathcal{X}$ is contained in $\mathcal{J}(-i)$ which gives the claim. If $\dim S_i \geq 2$ then the inclusion

$$\mathcal{X} \subset \mathcal{J}(+i, +j) \cup \mathcal{J}(-i)$$

holds for any $j \neq i$ such that $x_j \in S_i$. \qed

**Lemma 4.2.** If $X$ is independent then the class of maximal inexact subsets coincides with the class of special subsets.

**Proof.** By Lemma 4.1, it sufficient to show that every special subset is inexact. Since $X$ is independent,

$$Y := (X \setminus \{x_i\}) \cup \{x_i + x_j\}$$

is independent and $\mathcal{J}_k(Y)$ contains the special subset $\mathcal{J}(+i, +j) \cup \mathcal{J}(-i)$. \qed

**Remark 4.1.** Suppose that $R = \mathbb{Z}_2$ and $X = \{x_1, \ldots, x_5\}$, where $x_1, \ldots, x_4$ are linearly independent vectors and

$$x_5 = x_1 + \cdots + x_4.$$ 

Then $k = 2$ and $X$ is 4-independent. The vectors $x_1 + x_2, x_3, x_4, x_5$ are not linearly independent and $x_1$ can not be replaced by $x_1 + x_2$ as in the proof of Lemma 4.2. The subspace $(x_1, x_2)$ contains only three non-zero vectors — $x_1, x_2, x_1 + x_2$. This means that $\mathcal{J}(+1, +2) \cup \mathcal{J}(-1)$ can not be inexact. The same arguments show that every special subset is not inexact.

**Remark 4.2.** It is not difficult to prove that all special subsets are inexact if $R$ is infinite, but we do not need this fact.

The subsets $\mathcal{J}(+i, +j)$ and $\mathcal{J}(-i)$ are disjoint. This means that every special subset contains precisely

$$a(n,k) := |\mathcal{J}(+i, +j)| + |\mathcal{J}(-i)| = \binom{n-2}{k-2} + \binom{n-1}{k}$$

elements. Lemma 4.1 implies the following.
Lemma 4.3. If an inexact subset consists of \( a(n, k) \) elements then it is a special subset.

A subset \( X \subset \mathcal{J} \) is said to be complement if \( \mathcal{J} \setminus X \) is special, i.e.
\[
\mathcal{J} \setminus X = \mathcal{J}(+i, +j) \cup \mathcal{J}(-i)
\]
for some distinct \( i, j \). Then
\[
X = \mathcal{J}(+i) \cap \mathcal{J}(-j).
\]
This complement subset will be denoted by \( \mathcal{J}(+i, -j) \).

Lemma 4.4. Let \( P, Q \in \mathcal{J} \). Then \( d(P, Q) = m \) if and only if there are precisely
\[
(k-m)(n-k-m)
\]
distinct complement subsets of \( \mathcal{J} \) containing both \( P \) and \( Q \).

Proof. The equality \( d(P, Q) = m \) implies that
\[
\dim(P \cap Q) = k - m \quad \text{and} \quad \dim(P + Q) = k + m.
\]
The complement subset \( \mathcal{J}(+i, -j) \) contains both \( P \) and \( Q \) if and only if
\[
x_i \in P \cap Q \quad \text{and} \quad x_j \notin P + Q.
\]
So, there are precisely \( k - m \) possibilities for \( i \) and precisely \( n - k - m \) possibilities for \( j \). \( \square \)

4.2. Connectedness of the apartment graph. Suppose that \( 1 < k \leq n - k \). If \( X \) is a base of \( V \) then \( \mathcal{J}_k(X) \) is an apartment of \( \mathcal{G}_k(V) \) and, by Lemma 4.2, the class of maximal inexact subsets coincides with the class of special subsets. Two apartments of \( \mathcal{G}_k(V) \) are said to be adjacent if their intersection is a maximal inexact subset. Consider the graph \( A_k \) whose vertices are apartments of \( \mathcal{G}_k(V) \) and whose edges are pairs of adjacent apartments.

Proposition 4.1. The graph \( A_k \) is connected.

Proof. Let \( B \) and \( B' \) be bases of \( V \). The associated apartments of \( \mathcal{G}_k(V) \) will be denoted by \( A \) and \( A' \), respectively. Suppose that \( A \neq A' \) and show that these apartments can be connected in \( A_k \).

First we consider the case when \( |B \cap B'| = n - 1 \). Let
\[
B = \{x_1, \ldots, x_{n-1}, x_n\} \quad \text{and} \quad B' = \{x_1, \ldots, x_{n-1}, x'_n\}.
\]
Since \( A \neq A' \), the vector \( x'_n \) is a linear combination of \( x_n \) and some others \( x_{i_1}, \ldots, x_{i_m} \). Clearly, we can suppose that
\[
x'_n = ax_n + \sum_{i=1}^{m} a_i x_i \quad \text{with} \quad m \leq n - 1.
\]
We prove the statement induction on \( m \). If \( m = 1 \) then
\[
A \cap A' = \mathcal{J}(+n, +1) \cup \mathcal{J}(-n)
\]
is a maximal inexact subset and \( A, A' \) are adjacent. Let \( m \geq 2 \). Denote by \( A'' \) the apartment of \( \mathcal{G}_k(V) \) associated with the base \( x_1, \ldots, x_{n-1}, x''_n \), where
\[
x''_n := ax_n + \sum_{i=1}^{m-1} a_i x_i.
\]
By inductive hypothesis, $A$ and $A''$ can be connected in $A_k$. The equality

$$x_n' = x_n'' + a_m x_m$$

guarantees that $A''$ and $A'$ are adjacent. This implies the existence of a path connecting $A$ with $A'$.

Now consider the case when $|B \cap B'| = m < n - 1$ (possible $m = 0$). Suppose that

$$B \setminus B' = \{x_1, \ldots, x_{n-m}\} \text{ and } x' \in B' \setminus B.$$  

For every $i \in \{1, \ldots, n-m\}$ we define

$$S_i := \langle B \setminus \{x_i\} \rangle.$$  

Since the intersection of all $S_i$ coincides with $\langle B \cap B' \rangle$ and $x'$ does not belong to $\langle B \cap B' \rangle$, there is at least one $S_i$ which does not contain $x'$. Then

$$B_1 := (B \setminus \{x_i\}) \cup \{x'\}$$  

is a base of $V$. Denote by $A_1$ the associated apartment of $\mathcal{G}_k(V)$. It is clear that

$$|B \cap B_1| = n - 1 \quad \text{and} \quad |B_1 \cap B'| = m + 1.$$  

The apartment $A_1$ coincides with $A$ (if $x'$ is a scalar multiple of $x_i$) or $A$ and $A_1$ are connected in $A_k$. Step by step we construct a sequence of bases

$$B = B_0, B_1, \ldots, B_{n-m} = B'$$  

such that $|B_{i-1} \cap B_i| = n - 1$ for every $i \in \{1, \ldots, n-m\}$. Let $A_i$ be the apartment of $\mathcal{G}_k(V)$ associated with $B_i$. Then for every $i \in \{1, \ldots, n-m\}$ we have $A_{i-1} = A_i$ or $A_{i-1}$ and $A_i$ are connected in $A_k$. This means that $A = A_0$ and $A' = A_{n-m}$ are connected in $A_k$.  

\[\square\]

4.3. Intersections of $J(n,k)$-subsets of different types. In this subsection we suppose that $W$ is a $(2k)$-dimensional vector space and $k \geq 2$. Let

$$X = \{x_1, \ldots, x_n\} \quad \text{and} \quad Y = \{y_1^*, \ldots, y_n^*\}, \quad n > 2k$$  

be $(2k)$-independent subsets of $W$ and $W^*$, respectively. Denote by $U_i$ the annihilator of $y_i^*$. This is a $(2k-1)$-dimensional subspace of $W$. Suppose that the following conditions hold:

- every $U_i$ is spanned by a subset of $X$,
- every $\langle x_i \rangle$ is the intersection of some $U_j$.

Since $X$ is a $(2k)$-independent subset, every $U_i$ is spanned by a $(2k-1)$-element subset $X_i \subset X$ and it does not contain any vector of $X \setminus X_i$. Similarly, $Y$ is $(2k)$-independent and every $x_i$ is contained in precisely $2k - 1$ distinct $U_j$ whose intersection coincides with $\langle x_i \rangle$.

We will investigate the intersection

$$Z := J_k(X) \cap J_k^*(Y).$$  

It is formed by all elements of $\mathcal{G}_k(W)$ which are spanned by subsets of $X$ and can be presented as the intersections of $k$ distinct $U_j$.

We define

$$b(n,k) := \frac{(2k-1)n}{k}.$$  

Note that this integer is not necessarily natural.

**Lemma 4.5.** $|Z| \leq b(n,k)$.  


Proof. Denote by $Z_t$ the set of all elements of $Z$ containing $x_t$. There are precisely $2k - 1$ distinct $U_j$ containing $x_t$ and every element of $Z$ is the intersection of $k$ distinct $U_j$. This means that $Z_t$ contains not greater than $\binom{2k-1}{k}$ elements. Since every element of $Z$ belongs to $k$ distinct $Z_i$, we have

$$|Z| = \frac{|Z_1| + \cdots + |Z_n|}{k}$$

which implies the required inequality. \hfill \square

Lemma 4.6. $a(n,k) > b(n,k)$ except the case when $n = 5$ and $k = 2$.

Proof. We have

$$a(n,2) = 1 + \frac{(n-1)(n-2)}{2} = \frac{n^2 - 3n + 4}{2} \quad \text{and} \quad b(n,2) = \frac{3n}{2}.$$ 

An easy verification shows that the equality $a(n,2) > b(n,2)$ does not hold only for $n = 5$.

From this moment we suppose that $k \geq 3$. Then

$$a(n,k) = \binom{n-2}{k-2} + \binom{n-1}{k} = \frac{(n-2)!}{(k-2)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} =$$

$$= \prod_{j=2}^{k} \binom{n-j}{k-j} + \frac{(n-1)\cdot \ldots \cdot (n-k+1)}{k!} =$$

$$= [k(k-1) + (n-1)(n-k)]\frac{(n-2)\cdot \ldots \cdot (n-k+1)}{k!}$$

and

$$b(n,k) = \binom{2k-1}{k} = \frac{n(2k-1)!}{k!k!} = \frac{n(2k-1)\cdot \ldots \cdot (k+1)}{k!} =$$

$$= [n(n+1)]\frac{(2k-1)\cdot \ldots \cdot (k+2)}{k!}.$$ 

Since $n \geq 2k + 1$ and $k \geq 3$,

$$n - 1 + k - 1 = (n-1)(n-k) + (k+1)(k-1) - (k-1) \geq$$

$$\geq (n-1)(k+1) + (k+1)(k-1) - (k-1) = (n+k-2)(k+1) - (k-1) \geq$$

$$\geq (n+1)(k+1) - (k-1) = n(k+1) + 2 > n(k+1).$$

So,

$$\text{(4.1)} \quad k(k-1) + (n-1)(n-k) > n(k+1).$$

Also, $n \geq 2k + 1$ implies that

$$n - 2 \geq 2k - 1, \ldots, n - k + 1 \geq k + 2$$

and we have

$$\text{(4.2)} \quad (n-2)\cdot \ldots \cdot (n-k+1) \geq (2k-1)\cdot \ldots \cdot (k+2).$$

The inequality

$$a(n,k) = [k(k-1) + (n-1)(n-k)]\frac{(n-2)\cdot \ldots \cdot (n-k+1)}{k!} >$$

$$> [n(n+1)]\frac{(2k-1)\cdot \ldots \cdot (k+2)}{k!} = b(n,k)$$

follows from (4.1) and (4.2). \hfill \square

Lemma 4.7. If $n = 5$ and $k = 2$ then $|Z| \leq 5 < 7 = a(5,2)$. 
Lemma 5.1. The mapping

Proof. In the present case, $U_1, \ldots, U_5$ are 3-dimensional, each $x_i$ is contained in precisely 3 distinct $U_i$ and every element of $Z$ is the intersection of 2 distinct $U_j$. If every $U_i$ contains not greater than 2 elements of $Z$ then $|Z| \leq \frac{35}{2} = 5$ (since every element of $Z$ is contained in 2 distinct $U_j$).

Suppose that $U_1$ is spanned by $x_1, x_2, x_3$ and contains 3 elements of $Z$. These are $\langle x_1, x_2 \rangle, \langle x_1, x_3 \rangle, \langle x_2, x_3 \rangle$. Suppose that these subspaces are the intersections of $U_1$ with $U_2, U_3, U_4$. Then each $x_i, i \in \{1, 2, 3\}$ is contained in 3 distinct $U_i, j \in \{1, 2, 3, 4\}$. The subspace $U_5$ contains at least one of $x_i, i \in \{1, 2, 3\}$ and this $x_i$ is contained in 4 distinct $U_j$, a contradiction.

The same arguments show that every $U_i$ contains not greater than 2 elements of $Z$ and we get the claim.

Joining all results of this subsection, we get the following.

Lemma 4.8. $|Z| < a(n, k)$.

5. Proof of Theorem 3.1

Let $f$ be a $J$-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_{k'}(V')$.

Lemma 5.1. The mapping $f$ is injective.

Proof. Let $P, Q$ be distinct elements of $\mathcal{G}_k(V)$. We take an apartment $A \subset \mathcal{G}_k(V)$ containing $P$ and $Q$. Since $f(A)$ is a $J(n, k)$-subset, $A$ and $f(A)$ have the same number of elements which implies that $f(P) \neq f(Q)$.

Consider the mapping $f_*$ which transfers every $P \in \mathcal{G}_{n-k}(V^*)$ to $f(P^0)$. This is a $J$-mapping of $\mathcal{G}_{n-k}(V^*)$ to $\mathcal{G}_{k'}(V')$. It is clear that $f$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(V')$ if and only if $f_*$ is an isometric embedding of $\Gamma_{n-k}(V^*)$ in $\Gamma_{k'}(V')$. Therefore, it sufficient to prove Theorem 3.1 only in the case when $k \leq n - k$.

Suppose that $k = 1$, i.e. $f$ is a $J$-mapping of $\mathcal{G}_1(V)$ to $\mathcal{G}_{k'}(V')$. Any distinct $P, Q \in \mathcal{G}_1(V)$ are adjacent and there is an apartment $A \subset \mathcal{G}_1(V)$ containing $P, Q$. Since $f(A)$ is a $J(n, 1)$-subset, $f(P)$ and $f(Q)$ are adjacent vertices of $\Gamma_{k'}(V')$.

Thus $f$ is an isometric embedding of $\Gamma_1(V)$ in $\Gamma_{k'}(V')$.

From this moment we suppose that $2 \leq k \leq n - k$. By Subsection 2.4, we have

$$k \leq \min\{k', n - k, n' - k'\}.$$

Lemma 5.2. If $n = 2k$ then there exists $S \in \mathcal{G}_{k'-k}(V')$ such that the image of $f$ is contained in $|S|_{k'}$.

Proof. Let $A$ and $A'$ be distinct apartments of $\mathcal{G}_k(V)$. Then $f(A)$ and $f(A')$ are $J(n, k)$-subsets and, since $n = 2k$, Theorem 2.1 implies that

$$f(A) = \Phi_S(J_k(X)) \quad \text{and} \quad f(A') = \Phi_{S'}(J_k(X')),$$

where $S, S' \in \mathcal{G}_{k'-k}(V')$ and $X, X'$ are independent $(2k)$-element subsets of $V'/S$ and $V'/S'$, respectively. We need to show that $S = S'$.

By Proposition 4.1, it is sufficient to consider the case when $A$ and $A'$ are adjacent. Then

$$|f(A) \cap f(A')| = |A \cap A'| = a(2k, k)$$

and

$$X := (\Phi_S)^{-1}(f(A) \cap f(A')).$$
is a subset of $\mathcal{J}_k(X)$ consisting of $a(2k, k)$ elements. Since $S + S'$ is contained in all elements of $f(A) \cap f(A')$, every element of $\mathcal{X}$ contains $T := (S + S')/S$. If $S \neq S'$ then $t = \dim T \geq 1$ and

$$|\mathcal{X}| \leq \binom{2k-t}{k-t}$$

which implies that

$$|\mathcal{X}| \leq \binom{2k-1}{k-1} = \frac{(2k-1)!}{(k-1)!k!} = \binom{2k-1}{k} + \binom{2k-2}{k-2} = a(2k, k),$$

a contradiction. Thus $S = S'$. □

**Lemma 5.3.** Suppose that $k < n - k$. If $f$ transfers an apartment $A \subset G_k(V)$ to a $J(n, k)$-subset of first type then the images of all apartments of $G_k(V)$ are $J(n, k)$-subsets of first type and there exists $S \in G_{k'+k}(V')$ such that the image of $f$ is contained in $[S]_{k'}$.

**Proof.** By our hypothesis,

$$f(A) = \Phi_S(\mathcal{J}_k(X)),$$

where $S \in G_{k'+k}(V')$ and $X$ is a $(2k)$-independent subset of $V'/S$ consisting of $n$ vectors

$$\tau_1 = x_1 + S, \ldots, \tau_n = x_n + S.$$  

Denote by $S_i$ the $(k'-k+1)$-dimensional subspace of $V'$ corresponding to $\tau_i$. Every element of $f(A)$ is the sum of $k$ distinct $S_j$.

Let $A'$ be an apartment of $G_k(V)$ distinct from $A$. We need to show that $f(A')$ is a $J(n, k)$-subset of first type and is contained in $[S]_{k'}$. By Proposition 4.1 it is sufficient to consider the case when $A$ and $A'$ are adjacent. As in the proof of the previous lemma,

$$\mathcal{X} := (\Phi_S)^{-1}(f(A) \cap f(A'))$$

is a subset of $\mathcal{J}_k(X)$ consisting of $a(n, k)$ elements. There are the following possibilities:

(1) $\mathcal{X}$ is contained in a special subset of $\mathcal{J}_k(X)$,

(2) there is no special subset of $\mathcal{J}_k(X)$ containing $\mathcal{X}$.

**Case (1).** Every special subset of $\mathcal{J}_k(X)$ consists of $a(n, k) = |\mathcal{X}|$ elements. This implies that $\mathcal{X}$ is a special subset of $\mathcal{J}_k(X)$. Suppose that

$$\mathcal{X} = \mathcal{J}(+i, +j) \cup \mathcal{J}(-i)$$

(see Subsection 4.1 for the notation). We take any $(k'-1)$-dimensional subspace $T \subset V'/S$ spanned by a subset of $X$ containing $\tau_j$. Then

$$S := \mathcal{J}_k(X) \cap |T|_k$$

is a star of $\mathcal{J}_k(X)$ contained in $\mathcal{X}$ (if $P \in S$ contains $\tau_i$ then it belongs to $\mathcal{J}(+i, +j)$ and $P \in S$ is an element of $\mathcal{J}(-i)$ if it does not contain $\tau_i$).

Consider $\Phi_S(S)$. This is a star of $f(A)$. By Lemma 2.2, this star consists of $n - k + 1$ vertices (since $f(A)$ is a $J(n, k)$-subset of first type). Also, it is contained in $\Phi_S(\mathcal{X}) \subset f(A')$ and Lemma 2.2 guarantees that $f(A')$ is a $J(n, k)$-subset of first type.

We take $P, Q \in \mathcal{X}$ such that $P \cap Q = 0$. The intersection of $\Phi_S(P)$ and $\Phi_S(Q)$ coincides with $S$. Since $\Phi_S(P)$ and $\Phi_S(Q)$ both belong to $f(A')$ and $f(A')$ is
a $J(n,k)$-subset of first type, the associated $(k' - k)$-dimensional subspace of $V'$ coincides with $S$ and $f(A')$ is contained in $|S|_{k'}$.

Case (2). For every $i \in \{1, \ldots, n\}$ the intersection of all elements of $\mathcal{X}$ containing $\mathcal{X}_i$ coincides with $\langle y^*_i \rangle$ (otherwise, as in the proof of Lemma 4.1 we show that $\mathcal{X}$ is contained in a special subset of $\mathcal{J}_k(X)$ which is impossible). Then the intersection of all elements of

$$\Phi_S(\mathcal{X}) = f(A) \cap f(A')$$

containing $S_i$ coincides with $S_i$. This implies that the intersection of all elements of $f(A) \cap f(A')$ is $S$.

Therefore, if $f(A')$ is a $J(n,k)$-subset of first type then the associated $(k' - k)$-dimensional subspace of $V'$ coincides with $S$, i.e. $f(A')$ is contained in $|S|_{k'}$. Then $\mathcal{X}$ is an inexact subset of $\mathcal{J}_k(X)$. By Lemma 1.3 $\mathcal{X}$ is a special subset of $\mathcal{J}_k(X)$ which is impossible.

So, $f(A')$ is a $J(n,k)$-subset of second type. Then

$$f(A') = \Phi^U(\mathcal{J}^*_S(Y)),$$

where $U \in G_{k'+k}(V')$ and $Y$ is a $(2k)$-independent subset of $U^*$ consisting of $n$ vectors $y^*_1, \ldots, y^*_n$. Denote by $U_i$ the annihilator of $y^*_i$ (in $U$). By Remark 2.1 every element of $f(A')$ is the intersection of $k$ distinct $U_j$.

The set

$$(5.1) \quad (\Phi^U)^{-1}(f(A) \cap f(A'))$$

is contained in $\mathcal{J}^*_k(Y)$. Denote by $\mathcal{Y}$ the subset of $\mathcal{J}_k(Y)$ formed by the annihilators of all elements of (5.1). It consists of a $(n,k)$ elements. If $\mathcal{Y}$ is contained in a special subset of $\mathcal{J}_k(Y)$ then it coincides with this special subset. In this case, there is a star $S \subset \mathcal{J}_k(Y)$ contained in $\mathcal{Y}$. Let $S^0$ be the subset of $\mathcal{J}^*_k(Y)$ consisting of the annihilators of all elements of $S$. Then $\Phi^U(S^0)$ is a top of $f(A')$ contained in $f(A) \cap f(A')$. This contradicts Lemma 2.2 since $f(A)$ and $f(A')$ are $J(n,k)$-subsets of different types.

Thus there is no special subset of $\mathcal{J}_k(Y)$ containing $\mathcal{Y}$. This means that for every $i \in \{1, \ldots, n\}$ the intersection of all elements of $\mathcal{Y}$ containing $y^*_i$ coincides with $\langle y^*_i \rangle$.

By Lemma 2.1 $U_i$ is the sum of the annihilators (in $U$) of these elements; hence it is the sum of some elements of $f(A) \cap f(A')$. Since every element of $f(A)$ is the sum of $k$ distinct $S_j$,

(*) every $U_i$ is the sum of some $S_j$.

This implies that every $U_i$ contains $S$ (since $S$ is contained in all $S_i$) and $f(A')$ is a subset of $|S|_{k'}$ (every element of $f(A')$ is the intersection of $k$ distinct $U_j$).

Since the intersection of all elements of $f(A) \cap f(A')$ containing $S_i$ coincides with $S_i$ and every element of $f(A')$ is the intersection of $k$ distinct $U_j$,

(**) every $S_i$ is the intersection of some $U_j$.

Then every $S_i$ is contained in $U$ and $f(A)$ is a subset of $|U|_{k'}$ (since every element of $f(A)$ is the sum of $k$ distinct $S_j$).

So, $f(A)$ and $f(A')$ both are contained in $|S,U|_{k'}$. The vector space $W := U/S$ is $2k$-dimensional. It is clear that

$$f(A) = \Phi^U_S(\mathcal{J}_k(X)) \text{ and } f(A') = \Phi^U_S(\mathcal{J}^*_k(Y')).$$
where $Y'$ is the $(2k)$-independent $n$-element subset of $W^*$ induced by $Y$. The annihilators of the vectors belonging to $Y'$ are $U_i/S$, $i \in \{1, \ldots, n\}$. The facts (*) and (***) guarantee that $X$ and $Y'$ satisfy the conditions of Subsection 4.3:

- the annihilator of every element of $Y'$ is spanned by a subset of $X$,
- every $(\overline{x}_i)$ is the intersection of the annihilators of some elements from $Y'$.

By Subsection 4.3,
\[ Z := J_k(X) \cap J_k^*(Y') \]
contains less than $a(n, k)$ elements. This contradicts the fact that
\[ \Phi^U_S(Z) = f(A) \cap f(A') \]
consists of $a(n, k)$ elements. So, the case (2) is impossible. \( \square \)

By Lemma 5.3 if $k < n - k$ then the images of all apartments of $G_k(V)$ are $J(n, k)$-subsets of the same type.

Suppose that one of the following possibilities is realized:

- $n = 2k$,
- $k < n - k$ and the images of all apartments of $G_k(V)$ are $J(n, k)$-subsets of first type.

By Lemmas 5.2 and 5.3 the image of $f$ is contained in $[S]_{k'}$ with $S \in G_{k'-k}(V')$. This implies the existence of a mapping
\[ g : G_k(V) \to G_k(V'/S) \]
such that $f = \Phi_S \circ g$. This is a $J$-mapping which transfers every apartment of $G_k(V)$ to a certain $J_k(X)$, where $X$ is a $(2k)$-independent $n$-element subset of $V'/S$. Using results of Subsection 4.1, we prove the following.

**Lemma 5.4.** The mapping $g$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(V'/S)$.

**Proof.** Let $P, Q \in G_k(V)$ and let $A$ be an apartment of $G_k(V)$ containing $P$ and $Q$. If $\mathcal{X}$ is a special subset of $A$ then $\mathcal{X} = A \cap A'$, where $A'$ is an apartment of $G_k(V)$ adjacent with $A$. By Subsection 4.1,
\[ g(\mathcal{X}) = g(A) \cap g(A') \]
is an inexact subset of $g(A)$. It consists of $a(n, k)$ elements and Lemma 5.3 implies that $g(\mathcal{X})$ is a special subset of $g(A)$. Since $A$ and $g(A)$ have the same number of special subsets, a subset of $A$ is special if and only if its image is a special subset of $g(A)$. Then $\mathcal{X}$ is a complement subset of $A$ if and only if $g(\mathcal{X})$ is a complement subset of $g(A)$. Lemma 4.3 implies that
\[ d(P, Q) = d(g(P), g(Q)) \]
and we get the claim. \( \square \)

Since $\Phi_S$ is an isometric embedding of $\Gamma_k(V'/S)$ in $\Gamma_{k'}(V')$, Lemma 5.4 guarantees that $f = \Phi_S \circ g$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$.

Now suppose that $k < n - k$ and the images of all apartments of $G_k(V)$ are $J(n, k)$-subsets of second type. Consider the mapping $f^\ast$ which sends every $P \in G_k(V)$ to $f(P)^0$. This is a $J$-mapping of $G_k(V)$ to $G_{n'-k}(V'^\ast)$. It transfers every apartment of $G_k(V)$ to a $J(n, k)$-subset of first type. Then $f^\ast$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{n'-k}(V'^\ast)$ which means that $f$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. 

6. Strong J-mappings

A J-mapping of $G_k(V)$ to $G_{k'}(V')$ is said to be strong if there is an apartment of $G_k(V)$ whose image is an apartment in a parabolic subspace of $G_{k'}(V')$. The apartments preserving mappings considered in [11] Section 3.4 are strong J-mappings.

If $n = 2k \geq 4$ then every J-mapping of $G_k(V)$ to $G_{k'}(V')$ is strong (Theorem 2.1) and, by Theorems 2.2 and 3.1, it is induced by a semilinear embedding of $V$ in $U/S$ or a semilinear embedding of $V$ in $(U/S)^*$, where
\[ S \in G_{k'-k}(V') \quad \text{and} \quad U \in G_{k'+k}(V'). \]

In this section, we show that all strong J-mappings of $G_k(V)$ to $G_{k'}(V')$ are induced by semilinear embeddings if $1 < k < n - 1$. For $k = 1, n - 1$ this fails [5].

First we prove the following generalization of [11] Theorem 3.10.

**Corollary 6.1.** If $n = n'$ and $1 < k < n - 1$ then every strong J-mapping of $G_k(V)$ to $G_{k'}(V')$ is induced by a semilinear embedding of $V$ in $V'$ or a semilinear embedding of $V$ in $V'^*$ and the second possibility can be realized only in the case when $n = 2k$.

**Proof.** Let $f$ be a strong J-mapping of $G_k(V)$ to $G_{k'}(V')$. By Theorem 3.1, $f$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. We suppose that $n = n'$ and $1 < k < n - 1$. Then there is an apartment $A \subset G_k(V)$ such that $f(A)$ is an apartment of $G_{k'}(V')$.

In the case when $n = 2k$, the required statement follows from Theorem 3.1. If $k < n - k$ then, by Theorem 2.2, we have the following possibilities:

- $f = (l)_k$, where $l : V \to V'$ is a semilinear $(2k)$-embedding;
- $f = (s)_k^*$, where $s : V \to U^*$ is a semilinear $(2k)$-embedding and $U$ is a $(2k)$-dimensional subspace of $V'$.

In the second case, the image of $f$ is contained in $\langle U \rangle_k$. Since $2k < n$, $\langle U \rangle_k$ does not contain any apartment of $G_{k'}(V')$. This is impossible and $f = (l)_k$. Then $l$ transfers any base of $V$ associated with $A$ to a base of $V'$. This implies that $l$ is a semilinear embedding.

Let $k > n - k$. Consider the mapping which transfers every $P \in G_{n-k}(V^*)$ to $f(P)^0\circ$. This is a strong J-mapping of $G_{n-k}(V^*)$ to $G_{k-k}(V'^*)$. By the arguments given above, it is induced by a semilinear embedding $s : V^* \to V'^*$. Denote by $g$ the mapping of the set of all subspaces of $V$ to the set of all subspaces of $V'$ which sends every $P$ to $s(P)^0$. By [11] Subsection 3.4.3, it is induced by a semilinear embedding $l : V \to V'$, i.e.
\[ g(P) = \langle l(P) \rangle \]
for every subspace $P \subset V$. Since the restriction of $g$ to $G_k(V)$ coincides with $f$, we have $f = (l)_k$. □

**Corollary 6.2.** Suppose that $1 < k < n - 1$ and $n \neq 2k$. Then for every strong J-mapping $f : G_k(V) \to G_{k'}(V')$ one of the following possibilities is realized:

1. there exist $S \in G_{k'-k}(V')$ and $U \in G_{n+k-k}(V')$ such that $f = \Phi_S \circ (l)_k$, where $l : V \to U/S$ is a semilinear embedding;
2. there exist $S' \in G_{n-k'}-k(V'^*)$ and $U' \in G_{n+n'-k-k}(V'^*)$ such that $f = A \circ \Phi_{S'} \circ (l)_k$, where $l : V \to U'/S'$ is a semilinear embedding and $A$ is the annihilator mapping of $G_{n-k}(V'^*)$ to $G_{k'}(V')$.

**Proof.** By Theorem 3.1, $f$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. Suppose that $k < n - k$. Theorem 2.2 states that one of the following possibilities is realized:
\[ f = \Phi_S \circ (l)_k, \text{ where } S \in \mathcal{G}_{k'-k}(V') \text{ and } l : V \to V'/S \text{ is semilinear (2k)-embedding}; \]

\[ f = \Phi_U \circ (s)_k, \text{ where } U \in \mathcal{G}_{k'+k}(V') \text{ and } s : V \to U^* \text{ is a semilinear (2k)-embedding}. \]

As in Corollary 6.1 we establish that \( l \) and \( s \) both are semilinear embeddings.

We get a mapping of type (1) in the first case.

In the second case, the image of \( f \) is contained in \([T, U]_{k'}\), where \( T \in \mathcal{G}_{k+k'-n}(V') \) is the annihilator of \( s(V) \) in \( U \). Consider the mapping \( f^* \) sending every \( P \in \mathcal{G}_k(V) \) to \( f(P)^0 \). The image of this mapping is contained in \([S', U']_{n'-k'}\) with

\[ S' := U^0 \in \mathcal{G}_{n'-k'}(V'^*), \quad U' := T^0 \in \mathcal{G}_{n+n'-k'-k}(V'^*). \]

Then \( f^* = \Phi_{U'^*} \circ g \), where \( g \) is a \( J \)-mapping of \( \mathcal{G}_k(V) \) to \( \mathcal{G}_k(U'/S') \). This \( J \)-mapping is strong (since \( f \) and \( f^* \) are strong \( J \)-mappings). The dimension of \( U'/S' \) is equal to \( n \) and Corollary 6.1 implies that \( g \) is induced by a semilinear embedding of \( V \) in \( U'/S' \). Thus \( f \) is a mapping of type (2).

Now suppose that \( k > n - k \). The image of \( f \) coincides with the image of the mapping \( f_* \) which transfers every \( P \in \mathcal{G}_{n-k}(V^*) \) to \( f(P)^0 \). This image is contained in

\[ [N, M]_{k'}, \quad N \in \mathcal{G}_{k'-n+k}(V'), \quad M \in \mathcal{G}_{k'+k}(V') \]

(\( f_* \) is a mapping of type (1)) or it is a subset of

\[ [S, U]_{k'}, \quad S \in \mathcal{G}_{k'-k}(V'), \quad U \in \mathcal{G}_{k'+n-k}(V') \]

(\( f_* \) is a mapping of type (2)).

In the second case, \( f = \Phi_{U'} \circ g \), where \( g \) is a strong \( J \)-mapping of \( \mathcal{G}_k(V) \) to \( \mathcal{G}_k(U/S) \). Since \( U/S \) is \( n \)-dimensional, Corollary 6.1 implies that \( g \) is induced by a semilinear embedding of \( V \) in \( U/S \) and \( f \) is a mapping of type (1).

Suppose that the image of \( f \) is contained in \([N, M]_{k'}\). As above, we consider the mapping \( f^* \) which sends every \( P \in \mathcal{G}_k(V) \) to \( f(P)^0 \). Its image is a subset of \([S', U']_{n'-k'}\) with

\[ S' := M^0 \in \mathcal{G}_{n'-k'}(V'^*), \quad U' := N^0 \in \mathcal{G}_{n+n'-k'-k}(V'^*). \]

Then \( f^* = \Phi_{S'^*} \circ g \), where \( g \) is a strong \( J \)-mapping of \( \mathcal{G}_k(V) \) to \( \mathcal{G}_k(U'/S') \). The standard arguments show that \( f \) is a mapping of type (2). \( \square \)

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