Topological optics with 3D-tangle laser solitons

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Abstract. We analyse, theoretically and numerically, stable localized 3D-structures with radiation wavefront singularities in lasers or laser media with fast saturable absorption; radiation has a fixed polarization state. These laser solitons’ internal structure is characterized by their skeletons, which consist of vortex lines where the field vanish and radiation phase changes by $2\pi m$ with integer $m$ – topological charge – as a result of a round-trip around the line. The vortex lines are oriented according the direction of radiation phase increase; high-order vortex lines ($m > 1$) are unstable and split into $m$ lines with $m = 1$. Complexes of fundamental (without vortices when isolated) laser solitons have unclosed vortex lines. Presented are topological laser solitons with vortex lines, which are unclosed and closed, unknotted and knotted, unlinked and linked. Various types of the solitons are stable in different, but overlapping domains of the scheme parameters; a slow variation of such parameters leads to hysteretic variations of solitons’ characteristics. Such hysteresis is reversible if soliton’s topology does not change during the parameters’ variation, and irreversible in the opposite case. The increased stability of dissipative solitons and the preservation of the topological characteristics of structures under perturbations make topological laser solitons promising for information applications.

1. Introduction and the model

Dissipative optical solitons, the localization of the field in which is caused by the balance of the inflow and outflow of energy, have an increased stability, being attractors, and can be implemented in schemes with any geometric dimensionality $D = 1, 2, 3$ [1]. On the other hand, the preservation, even with not small perturbations, of the topological characteristics of the structures, makes the search for topological dissipative solitons with a nontrivial internal structure attractive. Here we present the results of theoretical and numerical research of bright topological three-dimensional (3D) laser solitons considering both those found in [2-4] and some new types of such solitons.

The scheme considering is a large size homogeneous medium with saturation of gain and absorption or a ring laser with long cavity length filled in such a medium. The radiation is close to a plane monochromatic wave with a fixed polarization propagating along the Cartesian axis $z$. Then the governing equation for slowly varying electric field envelope $E$ has the following dimensionless form

$$\frac{\partial E}{\partial z} - \sum_{n=1}^{3} c_n \frac{\partial^2 E}{\partial x_n^2} = f(\|E\|^2)E.$$  (1)

Here $x_1 = x$ and $x_2 = y$ with the transverse Cartesian coordinates $x$ and $y$, $x_3 = \tau = t - \frac{z}{v_g}$ is the running time in the framework moving along $z$ with group velocity $v_g$, and $t$ is time. The coefficients $c_n$ are complex, $\Re c_n \geq 0$. The terms in the sum in equation (1) describe diffraction, frequency dispersion and dichroism. Finally, the complex, in general case, function $f$ of intensity $I = \|E\|^2$
represents non-resonance losses and nonlinear amplification, absorption and refractive index. We will consider here only the case of nearly coinciding transition frequencies of amplifying and absorbing centres (zero frequency detunings). Then function \( f \) is real and can be written in the form

\[
f(I) = -1 - \frac{a_0}{1 + I} + \frac{g_0}{1 + I / \beta}.
\]  

(2)

In the right-hand side of equation (2), \( a_0 \) is the small-signal resonance absorption, \( g_0 \) is small-signal gain, and \( \beta \) is the ratio of saturation intensities for gain and absorption. For large intensities, the non-resonance absorption dominates, \( f < 0 \). To ensure stability of a bright soliton tails we assume

\[
f(0) = -1 - a_0 + g_0 < 0.
\]  

(3)

In the special case of equal coefficients \( c_n = i + d \), governing equation (1) takes more symmetric form that is the main used here:

\[
\frac{\partial E}{\partial z} - (i + d) \nabla^2 E = f(|E|^2) E.
\]  

(4)

For fixed \( z \), the electromagnetic energy flow in the paraxial approximation is \( S = \text{Im}(E^{*} \nabla E) = i \nabla \Phi \) with radiation phase \( \Phi = \text{arg} E \). At vortex lines, the field vanish, \( \text{Re} E = 0, \text{Im} E = 0 \), and radiation phase changes by \( 2\pi m \) with integer \( m \) – topological charge – as a result of a round-trip around the line. The vortex lines of laser solitons are oriented according the direction of radiation phase increase; in our problem, high-order vortex lines \( (m > 1) \) are unstable and split into \( m \) lines with unit topological charges. Longitudinal (along the vortex line) component of the energy flow near the lines has the same sign along the whole length of non-alternating vortex lines and changes the sign for alternating lines. The unclosed vortex lines should be alternating, because the energy flow is directed out of the soliton centre at its periphery. As is presented below, closed vortex lines can be both alternating and non-alternating.

2. Vortex lines for a complex of weakly coupled fundamental solitons

Although the fundamental solitons do not contain vortex lines, the latter can appear for their complexes due to interference effects. It can be shown that for complexes of weakly coupled fundamental solitons, the asymptotics of the vortex lines are straight lines.

In Figure 1 we show a pyramidal complex of 4 fundamental solitons 3 of which are in-phase and one is out of phase. There are 3 hyperbolic-like unclosed, alternating vortex lines. At the complex periphery, the radial component of electromagnetic energy flow is directed out of the complex centre vanishing at the vortex lines.

![Figure 1](image_url)

*Figure 1.* Isointensity surfaces for a pyramidal complex of three in-phase fundamental solitons (red spheres) and one antiphase fundamental soliton (yellow sphere) and three hyperbolic-like vortex lines. Arrows show direction of increase of radiation phase. (a) Front view, (b) top view.
3. Topological tangle laser solitons
In Figure 2, we present a collection of topological laser solitons illustrated by the isointensity surfaces characterizing their sizes, and skeletons consisting of a number of unclosed, alternating, and closed, non-alternating vortex lines. The simplest of them, so-called “precession”, has asymmetric solid-like intensity distribution with its rotation and precession for some parameters, and only one unclosed vortex line. Other solitons have both closed and unclosed vortex lines and therefore are tangles [5].

![Figure 2](image-url)

Figure 2. The top and bottom rows of the figure depict the isosurfaces of the intensity of various solitons, and in the two middle rows, their skeletons: (a) precession with a single unclosed vortex line; (b) – (e) apple solitons one unclosed and one closed vortex lines; (f) two-ring soliton with three unclosed and two closed unlinked vortex lines; (g) single unknot closed line and three unclosed vortex lines; (h) Hopf soliton – two linked closed lines and three unclosed vortex lines; (i) trefoil knotted closed vortex line and three unclosed vortex lines; (j) Solomon soliton – two doubly linked vortex lines and two unclosed vortex lines. Colour indicates the radiation phase.

4. Topological reactions and hysteresis with regular variation of a control parameter
Different types of topological laser solitons are stable in different domains of the scheme parameters. Correspondingly, a slow variation of such parameters should result in changes of the solitons’ internal structure and of their characteristics. These changes are in form of cascades of the two main topological reactions with the solitons’ vortex lines (see Figure 3). First, two vortex lines approach and become anti-parallel near the point of tangency. In doing so, they exchange their branches, as in Figure 3(a). Second, after a strong curvature of the “parental” vortex line, a closed vortex loop, which is alternating, can break off from it, see Figure 3(b). These loops are metastable and their size monotonically decrease up to sizes of the computational mesh.

Due to overlapping of domains of stability of different types of topological solitons, hysteretic phenomena takes place with slow variations of the scheme parameters. We will take the small-signal gain coefficient $g_0$ as such a control parameter. A hysteretic cycle corresponds to slow increase, stabilization, and then slow decrease of $g_0$, see Figure 4(b). The hysteresis type depends on whether the topology of the soliton structure is preserved or changed. In Figure 4(a) we demonstrate that the hysteresis between “apple” solitons of types shown in Figure 2 (c) and (d), i.e., with the same topology, is reversible. Here $J_{\text{max}}$ is the absolute value if the main eigenvalue of inertia tensor.
\[ J_y = \oint (\delta_y x_j^2 - x_j y_j) \, d\mathbf{r}, \]  

(5)

where \( \delta_y \) is the Kronecker symbol. Contrary, in Figure 4 (b) we indicate that the initial trefoil soliton transforms after the hysteresis cycle into an “apple” soliton. The hysteresis is irreversible, it is accompanied with simplification of the topology (decrease of the topological characteristics) and decrease of electromagnetic energy. Similar phenomena take place for other topological solitons.

Figure 3. The main topological reactions with vortex lines: (a) – their reconnection, (b) – separation of closed loops from the parent vortex line. The vortex lines are given for fixed values of the evolution variable \( z \) indicated near figures. The reactions are fast as compared with a slow variation of \( z \).

Figure 4. (a): Reversible hysteresis for two types of “apple” solitons with increase and subsequent decrease of small-signal gain coefficient \( g_0 \) (arrows); \( J_{\text{max}} \) is the maximum moment of inertia. (b): Trapezoidal variation of gain \( g_0 \) and irreversible hysteresis for “trefoil” soliton (increase of \( g_0 \)) and “apple” solitons (decrease of \( g_0 \)). Insets: isointensity surfaces of solitons [(a), upper and lower figures, and (b), upper figures] and vortex lines [(b), lower figures], colour indicates the field phase.

5. References
[1] Rosanov N N 2011 Dissipative Optical Solitons (Fizmatlit, in Russian, Moscow, Russia)
[2] Veretenov N A, Rosanov N N and Fedorov S V 2015 Phys. Rev. Lett. 117, 183901
[3] Veretenov N A, Fedorov S V and Rosanov N N 2017 Phys. Rev. Lett. 119, 263901
[4] Fedorov S V, Veretenov N A and Rosanov N N 2019 Phys. Rev. Lett. 122, 023903
[5] Kawauchi A 1996 A survey of knot theory (Birkhauser, Basel, Switzerland)

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