The category of 3-computads is not cartesian closed

Mihaly Makkai and Marek Zawadowski
Department of Mathematics and Statistics, McGill University,
805 Sherbrooke St., Montréal, PQ, H3A 2K6, Canada
Instytut Matematyki, Uniwersytet Warszawski
ul. S.Banacha 2, 00-913 Warszawa, Poland

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Abstract
We show, using [CJ] and Eckmann-Hilton argument, that the category of
3-computads is not cartesian closed. As a corollary we get that neither the
category of all computads nor the category of $n$-computads, for $n > 2$, do form
locally cartesian closed categories, and hence elementary toposes.

1 Introduction

S.H. Schanuel (unpublished) made an observation, c.f. [CJ], that the category of
2-computads $\text{Comp}_2$ is a presheaf category. We show below that neither the category
of computads nor the categories $n$-computads, for $n > 2$, are locally cartesian closed.
This is in contrast with a remark in [CJ] on page 453, and an explicit statement in
[B] claiming that these categories are presheaves categories. Note that some inter-
esting subcategories of computads, like many-to-one computads, do form presheaf
categories, c.f. [HMP], [HMZ].

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2 Computads

Computads were introduced by R.Street in [S], see also [B]. Recall that a computad
is an $\omega$-category that is levelwise free. Below we recall one of the definitions.

Let $\text{nCat}$ be the category of $n$-categories and $n$-functors between them, $\omega\text{Cat}$
be the category of $\omega$-categories and $\omega$-functors between them. We have the obvious
truncation functors

$$tr_{n-1} : \text{nCat} \rightarrow (n-1)\text{Cat}$$

By $\text{Comp}_n$ we denote the category of $n$-computads, a non-full subcategory of the
category $\text{nCat}$. By $\text{CCat}_n$ we denote the non-full subcategory of $\text{nCat}$, whose
objects are 'computads up to the level $n - 1$', i.e. an $n$-functor $f : A \rightarrow B$ is a
morphism in $\text{CCat}_n$ if and only if $tr_{n-1}(f) : tr_{n-1}(A) \rightarrow tr_{n-1}(B)$ is a morphism
in $\text{Comp}_{n-1}$. Clearly $\text{CCat}_n$ is defined as soon as $\text{Comp}_{n-1}$ is defined. The
categories $\text{Comp}_n$ and $n$-comma category $\text{Com}_n$ are defined below.

The categories $\text{Comp}_0$, $\text{CCat}_0$ and $\text{Com}_0$ are equal to $\text{Set}$, the category of
sets. We have an adjunction
with both functors being the identity on $\text{Set}$, $F_0 \dashv U_0$. $\text{Comp}_0$ is the image of $\text{Com}_0$ under $F_0$.

$\text{Com}_1$ is the category of graphs, i.e. an object of $\text{Com}_1$ is a pair of sets and a pair of functions between them $\langle d, c : E \to V \rangle$. $\text{CCat}_1$ is simply $\text{Cat}$, the category of all small categories. The forgetful functor $U_1$ (forgetting compositions and identities) has a left adjoint $F_1$ 'the free category (over a graph)' functor

We have a diagram

where three triangles commute, moreover the left triangle and the outer square commute up to an isomorphism. $tr_1$ and $tr'_1$ are the obvious truncation morphisms. Then we define the category of 1-computads $\text{Comp}_1$ as the essential (non-full) image of the functor $F_1$ in $\text{CCat}_1$, i.e. 1-computads are the free categories over graphs and computad maps between them are functors sending indets (=indeterminates=generators) to indets.

Now suppose that we have an adjunction $U_n \dashv F_n$

and $\text{Comp}_n$ is defined as the the essential (non-full) image of the functor $F_n$ in $\text{CCat}_n$. We define the $n$-parallel pair functor

such that

for any $n$-computad $A$. The $(n + 1)$-comma category $\text{Comp}_{n+1}$ is the category $\text{Set} \downarrow \Pi_n$. Thus an object in $\text{Com}_{n+1}$ is a pair $(A, \langle d, c \rangle : X \to \Pi_n(A))$, such that $A$ is an $n$-computad $X$ is a set of $(n + 1)$-indets and $\langle d, c \rangle$ is a function associating $n$-domains and $n$-codomains. The forgetful functor $U_{n+1} : \text{CCat}_{n+1} \to \text{Com}_{n+1}$ (forgetting compositions and identities at the level $n + 1$) creates limits and satisfies the solution set condition. Thus it has a left adjoint $F_{n+1}$. We get a diagram
where three triangles commute, moreover the left triangle and the outer square commute up to an isomorphism. \( tr_n \) are the obvious truncation functors and \( tr'_n \) is a truncation functor that at the level \( n \) leaves the indets only. Then we define the category of \((n+1)\)-computads \( \text{Comp}_{n+1} \) as the essential (non-full) image of the functor \( F_{n+1} \) in \( \text{CCat}_{n+1} \), i.e. \((n+1)\)-computads are the free \((n+1)\)-categories over \((n+1)\)-comma categories and \((n+1)\)-computad maps between them are \((n+1)\)-functors sending indets to indets. The category of computads \( \text{Comp} \) is a (non-full) subcategory of the category of \( \omega \)-categories and \( \omega \)-functors \( \omega \text{Cat} \) such, that for each \( n \), the truncation of objects and morphisms to \( n \text{Cat} \) is in \( \text{Comp}_n \). As \( F_n : \text{Comp}_n \rightarrow \text{CCat}_n \) is faithful and full on isomorphisms, after restricting the codomain we get an equivalence of categories \( F_n : \text{Comp}_n \rightarrow \text{Comp}_n \).

\textbf{Notation.} If \( A \) is a computad then \( A_n \) denotes the set of \( n \)-cells of \( A \) and \( |A|_n \) denotes the set of \( n \)-indets of \( A \).

The truncation functor \( tr_n : \text{Comp}_{n+1} \rightarrow \text{Comp}_n \) has both adjoints \( i_n + tr_n + f_n \)

\[
\text{Comp}_{n+1} \xrightarrow{\text{tr}_n} \text{Comp}_n \xleftarrow{\text{f}_n} \text{Comp}_{n+1}
\]

where

\[
i_n(A) = F_{n+1}(A, \emptyset) \rightarrow \Pi_n(A)
\]

and

\[
f_n(A) = F_{n+1}(A, id_{\Pi_n(A)} : \Pi_n(A) \rightarrow \Pi_n(A))
\]

for \( A \) in \( \text{Comp}_n \). This shows that \( tr_n \) preserves limits and colimits. The colimits in \( \text{Comp}_{n+1} \) are calculated in \((n+1)\text{Cat}\) but the limits in \( \text{Comp}_{n+1} \) are more involved. It is more convenient to describe them in \( \text{Comp}_{n+1} \) and then apply the functor \( F_{n+1} \). If \( H : J \rightarrow \text{Comp}_{n+1} \) is a functor and \( P \) is the limit of its truncation \( tr_n \circ H \) to \( \text{Comp}_n \) then \( \text{Lim} \, H \), the limit of \( H \), truncated to \( \text{Comp}_n \) is \( P \) and the \((n+1)\)-indets \( |\text{Lim} \, H|_{n+1} \) of \( \text{Lim} \, H \) are as follows

\[
|\text{Lim} \, H|_{n+1} = \{ \langle a_i \rangle_{i \in J} \mid a_i \in |H(i)|_{n+1}, (d(a_i))_{i \in J}, (c(a_i))_{i \in J} \in P_n \}
\]

The terminal object \( 1_n \) in \( \text{Comp}_n \) is quite complicated, for \( n \geq 2 \). However the \( \text{Comp}_2 \) part of \( 1_2 \) is still easy to describe. \( 1_2 \) has one 0-indet \( x \) and one 1-indet \( \xi : x \rightarrow x \). Thus the 1-cells can be identified with finite (possibly empty) strings of of arrows:

\[
x, \quad x \xrightarrow{\xi} x \xrightarrow{\xi} x \quad \cdots \quad x \xrightarrow{\xi} x
\]

or simply with elements of \( \omega \). The set \( |1_2|_2 \) of 2-indets in \( 1_2 \) contains exactly one indet for every pair of strings. The first element of such a pair is the domain of the indet and the second element of the pair is the codomain of the indet. Thus \( |1_2|_2 \) can be identified with the set \( \omega \times \omega \). In particular \( \langle 0,0 \rangle \) correspond to the only indet from \( id_x \) to \( id_x \) (\( id_x \) is the identity on \( x \)). The description of all 2-cells in \( 1_2 \) is more involved but we don’t need it here.
3 The counterexample

Lemma 3.1 Comp_3 is not cartesian closed.

Proof. As it was noted in Lemma 4.2 [CJ], the functor $\Pi_2$ factorizes as

$$
\begin{array}{ccc}
\text{Comp}_2 & \xrightarrow{\Pi_2} & \text{Set} \downarrow \Pi_2(1_2) \\
\downarrow & & \downarrow \Sigma \\
\text{Set} & & 
\end{array}
$$

where $\Pi_2(A) = \Pi_2(!: A \to 1_2)$, and $\Sigma(b: B \to \Pi_2(1_2)) = B$, for $A$ in $\text{Comp}_2$ and $b$ in $\text{Set} \downarrow \Pi_2(1_2)$. Moreover, the category $\text{Set} \downarrow \Pi_2(1_2)$, which is equivalent to $\text{Comp}_3$, is also equivalent to $(\text{Set} \downarrow \Pi_2(1_2)) \downarrow \Pi_2$. Now, as $\text{Comp}_2$ and $\text{Set} \downarrow \Pi_2(1_2)$ are cartesian closed categories with initial objects (in fact both categories are presheaf toposes) and $\Pi_2$ preserves the terminal object, by Theorem 4.1 of [CJ], $\text{Comp}_3$ is a cartesian closed category if and only if $\Pi_2$ preserves binary products. We finish the proof by showing that $\Pi_2$ does not preserves the binary products.

Let $A$ be a 2-computad with one 0-cell $x$, one 1-cell $id_x$ the identity on $x$ (no 1-indets). Moreover $A$ has as 2-cells all cells generated by the two indeterminate 2-cells $a_1, a_2 : id_x \to id_x$. Thus, by Eckmann-Hilton argument, any 2-cell in $A$ is of form $a_1^m \circ a_2^n$, for $m, n \in \omega$ (if $m = n = 0$ then $a_1^m \circ a_2^n = id_{id_x}$). Let $B$ be a 2-computad isomorphic to $A$ with indeterminate 2-cells $b_1, b_2$. Let $x$ be the unique 0-cell in $1_2$, $c$ be the only indeterminate 2-cell in $1_2$ that has $id_x$ as its domain and codomain and $A$ a subcomputad of $1_2$ generated by $c$. The unique maps of 2-computads $!: A \to 1_2$ and $!: B \to 1_2$ sends $a_i$ and $b_i$ to $c$, for $i = 1, 2$. Thus they factor through $C$ as $\alpha: A \to C$ and $\beta: B \to C$, respectively. The 2-computad $C$ does not play a crucial role in the counterexample but it makes the explanations simpler.

Let us describe the product $A \times B$ in $\text{Comp}_2$. The 0-cell and 1-cells are as in $A$, $B$ and $C$. As there is only one 1-cell $id_x$ in $A \times B$, the compatibility condition for domain and codomains of 2-indets is trivially satisfied, and the set 2-indets of $A \times B$ is just the product of 2-indets of $A$ and $B$, i.e.

$$
|A \times B|_2 = \{(a_i, b_j) | i, j = 1, 2\}
$$

and the set of all 2-cells of $A \times B$ is

$$
(A \times B)_2 = \{(a_1, b_1)^{n_1} \circ (a_1, b_2)^{n_2} \circ (a_2, b_1)^{n_3} \circ (a_2, b_2)^{n_4} | n_1, n_2, n_3, n_4 \in \omega\}
$$

The projections

$$
A \xrightarrow{\pi_1} A \times B \xrightarrow{\pi_B} B
$$

are defined as the only 2-functors such that $\pi_A(a_i, b_j) = a_i$ and $\pi_A(a_i, b_j) = b_j$, for $i, j = 1, 2$. Thus we have a commuting square

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\pi_A} & A \\
\downarrow & & \downarrow \alpha \\
A & \xrightarrow{\pi_B} & B \\
\downarrow m & & \downarrow \beta \\
1_2 & \xrightarrow{!} & C \\
\end{array}
$$

(*)
As $C$ is a subobject of the terminal object $A \times B$ is $A \times_C B$ and $A \times_{1_2} B$, i.e. both inner and outer squares in the above diagram are pullbacks.

Since all the 2-cells in $A$, $B$, $C$ and $A \times B$ are parallel we have

$$\Pi_2(A) = A_2 \times A_2, \quad \Pi_2(B) = B_2 \times B_2, \quad \Pi_2(C) = C_2 \times C_2,$$

and

$$\Pi_2(A \times B) = (A \times B)_2 \times (A \times B)_2.$$ 

$\tilde{\Pi}_2$ preserves the product of $A$ and $B$ if in the diagram (**) below, which is the application of $\Pi_2$ to the diagram (*) above, the outer square is a pullback in $\text{Set}$

\[
\begin{array}{ccc}
(A \times B)_2 \times (A \times B)_2 & \rightarrow & (A \times B)_2 \\
\Pi_2(\pi_A) & \downarrow & \Pi_2(\pi_B) \\
A_2 \times A_2 & \rightarrow & B_2 \times B_2 \\
\Pi_2(\alpha) & \downarrow & \Pi_2(\beta) \\
C_2 \times C_2 & \rightarrow & \Pi_2(!) \\
\Pi_2(\alpha) & \downarrow & \Pi_2(\beta) \\
\Pi_2(1_2) & \rightarrow & \Pi_2(1_2) \\
\end{array}
\]

As $\Pi_2(m)$ is mono, the outer square in (**) is a pullback in $\text{Set}$ if and only if the inner square in (**) is a pullback in $\text{Set}$. We have

$$\Pi_2(\pi_A) = (\pi_A)_2 \times (\pi_A)_2, \quad \Pi_2(\pi_B) = (\pi_B)_2 \times (\pi_B)_2,$$

$$\Pi_2(\alpha) = \alpha_2 \times \alpha_2, \quad \text{and} \quad \Pi_2(\beta) = \beta_2 \times \beta_2.$$ 

Hence the inner square in (**) is a pullback if and only if the square (***) below

\[
\begin{array}{ccc}
(A \times B)_2 & \rightarrow & (A \times B)_2 \\
(\pi_A)_2 & \downarrow & (\pi_B)_2 \\
A_2 & \rightarrow & B_2 \\
\alpha_2 & \downarrow & \beta_2 \\
(C)_2 & \rightarrow & (C)_2 \\
\end{array}
\]

is a pullback. But (***) is not a pullback in $\text{Set}$. The two 2-cells

$$\langle a_1, b_1 \rangle \circ \langle a_2, b_2 \rangle, \quad \text{and} \quad (a_1, b_2) \circ (a_2, b_1)$$

in $A \times B$ are different since they are compositions of different indets. On the other hand

$$(\pi_A)_2((a_1, b_1) \circ (a_2, b_2)) = a_1 \circ a_2 = (\pi_A)_2((a_1, b_2) \circ (a_2, b_1))$$

and

$$(\pi_B)_2((a_1, b_1) \circ (a_2, b_2)) = b_1 \circ b_2 = b_2 \circ b_1 = (\pi_B)_2((a_1, b_2) \circ (a_2, b_1))$$

i.e. they agree on both projections and hence (***) is not a pullback. Thus $\tilde{\Pi}_2$ does not preserve binary products, as required. □

**Theorem 3.2** The category of computads $\text{Comp}$ and the categories of $n$-computads $\text{Comp}_n$, for $n > 2$, are not locally cartesian closed.
Proof. The slice categories $\text{Comp} \downarrow 1_3$, as well as $\text{Comp}_n \downarrow 1_3$, for $n > 2$, are equivalent to $\text{Comp}_3$, where $1_3$ is the terminal object in $\text{Comp}_3$ lifted (by adding suitable identities) to the category of appropriate computads. As, by Lemma 3.1, $\text{Comp}_n \downarrow 1_3$ is not cartesian closed we get the theorem. □

Remark. In particular the categories mentioned in the above theorem are not presheaf (or even elementary) toposes.

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