MANIN’S CONJECTURE AND THE FUJITA INVARIANT OF FINITE COVERS

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ABSTRACT. We prove a conjecture of Lehmann-Tanimoto about the behaviour of the Fujita invariant (or $a$-constant appearing in Manin’s conjecture) under pull-back to generically finite covers. As a consequence we obtain results about geometric consistency of Manin’s conjecture.

1. INTRODUCTION

Let $X$ be a smooth projective variety over a field of $k$ of characteristic 0 and $L$ a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. Let $\Lambda_{\text{eff}}(X) \subset \text{NS}(X)_{\mathbb{R}}$ be the cone of pseudo-effective divisors. The Fujita invariant or the $a$-constant is defined as

$$a(X,L) = \min \{ t \in \mathbb{R} \mid [K_X] + t[L] \in \Lambda_{\text{eff}}(X) \}$$

The invariant $\kappa_L(X, L) = -a(X, L)$ was introduced and studied by Fujita under the name Kodaira energy in [Fuj87], [Fuj92], [Fuj96] and [Fuj97]. By [BDPP13], we know that $a(X,L) > 0$ if and only if $X$ is uniruled.

The $a$-constant was introduced in the context of Manin’s conjecture in [FMT89] and [BM90]. Manin’s conjecture predicts that the asymptotic behaviour of the number of rational points on Fano varieties over number fields is governed by certain geometric invariants (the $a$ and $b$-constants). In [LTT16], motivated by Manin’s conjecture, the authors studied the behaviour of the $a$-constant under restriction to subvarieties. We know (by [LTT16], [HJ16]) that if $X$ is uniruled and $L$ is a big and nef line bundle then there exists a proper closed subscheme $V \subset X$ such that any subvariety $Z \subset X$ satisfying $a(Z, L|_Z) > a(X,L)$ is contained in $V$. In [LT17], a similar finiteness statement was conjectured about the behaviour of the $a$-constant under pull-back to generically finite covers and in this paper we confirm the conjecture. In particular we prove the following:

**Theorem 1.1** (see [LT17, Conj. 1.7]) Let $X$ be a smooth projective uniruled variety and $L$ a big and nef $\mathbb{Q}$-divisor. Then, up to birational equivalence, there are only finitely many generically finite covers $f : Y \longrightarrow X$ such that $a(Y,f^*L) = a(X,L)$ and $\kappa(K_Y + a(Y,f^*L)f^*L) = 0$. 

The conjecture was proved in the case of \( \dim(X) = 2 \) in [LT17]. The authors also showed that the conjecture holds for Fano threefolds \( X \) and \( L = -K_X \) if \( \text{index}(X) \geq 2 \) or if \( \rho(X) = 1 \), \( \text{index}(X) = 1 \) and \( X \) is general in its moduli. Their idea was to reduce the statement to the finiteness of the étale fundamental groups of log Fano varieties. In this paper we take a different approach to prove the conjecture in general. It follows from the boundedness results in [Bir16] that the degree of morphisms \( f : Y \to X \) satisfying the hypothesis of Theorem 1.1 is bounded. Therefore it is enough to show that the branch divisors of all such morphisms are contained in a fixed proper closed subscheme \( V \subset X \). We show that if \( B \subset X \) is component of the branch divisor and \( B \not\subset B_+(L) \), then \( a(B, L|_B) > a(X, L) \). Here \( B_+(L) \) is a closed subset of \( X \) such that \( L|_B \) is big for any subvariety \( B \not\subset B_+(L) \) ([Laz04]). Therefore, by [HJ16], there is a fixed (depending only on \( X \) and \( L \)) closed subscheme \( V \subset X \) such that \( B \subset V \cup B_+(L) \).

We note that the converse of the statement above is not true. In particular, if \( B \) is a subvariety of \( X \) with \( a(B, L|_B) > a(X, L) \) then there might not exist a generically finite cover satisfying the hypothesis of Theorem 1.1 such that \( B \) is contained in the branch divisor. For example, let \( X = \text{Bl}_p \mathbb{P}^2 \) be the blow-up of \( \mathbb{P}^2 \) at a point \( p \) and \( B = E \) the exceptional divisor. Let \( L = -K_X \). Then we have \( a(X, L) = 1 \) and \( a(B, L) = 2 \). By [LT17, Theorem 6.2], \( X \) does not admit any generically finite cover \( f : Y \to X \) satisfying \( a(Y, f^*L) = a(X, L) \) and \( \kappa(K_Y + a(Y, f^*L)f^*L) = 0 \).

As a consequence of Theorem 1.1, we can obtain a statement about geometric compatibility of Manin’s conjecture. Let us recall the \( b \)-constant, which is the other geometric invariant involved in Manin’s conjecture. The \( b \)-constant is defined as (cf. [FMT89], [BM90])

\[
b(X, L) = \text{codim of minimal supported face of } \Lambda_{\text{eff}}(X) \\
\text{containing the class of } K_X + a(X, L)L
\]

In [FMT89] and [BM90], it was conjectured that the \( a \) and \( b \)-constants control the count of rational points on Fano varieties over a number field. The following version was suggested by Peyre in [Pey03] and later stated in [Rud14], [BL15].

**Manin’s conjecture:** Let \( X \) be a Fano variety defined over a number field \( F \) and \( L = (L, ||, ||) \) a big and nef adelically metrized line bundle on \( X \) with associated height function \( H_L \). Then there exists a thin set \( Z \subset \text{Pic}(X) \) such that

\[
\# \{ x \in \text{Pic}(X, F) - Z | H_L(x) \leq B \} \sim c(F, X(F) \setminus Z, L) B^{a(X, L)} (\log B)^{b(X, L) - 1}
\]
as $B \rightarrow \infty$.

Recall that a thin subset of $X(F)$ is a finite union $\bigcup \pi_i(Y_i(F))$ where $\pi_i : Y_i \rightarrow X$ is a morphism generically finite onto its image and admits no rational section. Initially Manin’s conjecture was stated for closed subsets instead of thin subsets (see [BM90], [Pey95], [BT98]). But it turned out that the closed set version is false (see [BT96],[Rud14]). The counterexamples arise from the existence of generically finite morphisms $f : Y \rightarrow X$ such that

$$(a(Y, f^*L), b(Y, f^*L)) > (a(X, L), b(X, L))$$

in the lexicographic order.

In [LT17], it was conjectured that such geometric incompatibilities cannot obstruct the thin set version of Manin’s conjecture.

**Conjecture 1.2** (see [LT17, Conj1.1]) Let $X$ be a smooth projective uniruled variety over a number field $F$. Consider all $F$-morphisms $f : Y \rightarrow X$ where $Y$ is a smooth projective variety and $f$ is generically finite. Then, as we vary over all such morphisms $f$ such that $f^*L$ is not big or

$$(a(Y, f^*L), b(Y, f^*L)) > (a(X, L), b(X, L))$$

in the lexicographic order, the points in $\bigcup f f(Y(F))$ are contained in a thin subset $Z \subset X$.

In [LT17], it was shown that the thinness statement of Conjecture 1.2 holds, when $f$ is varied over the inclusion morphism of subvarieties $Y \hookrightarrow X$ and $\rho(X) = \rho(X)$. However the finiteness statement of Theorem 1.1 does not hold over number fields due to the presence of twists. But it was proved in [LT17] that, for a fixed generically finite $F$-cover $f : Y \rightarrow X$ satisfying $\kappa(K_Y + a(X, L)L) = 0$, if we vary $f^\sigma : Y^\sigma \rightarrow X$ over all the twists of $f$, then the rational points, contributed by the twists satisfying the hypothesis of Conjecture 1.1, are contained in a thin set. Therefore as a consequence of Theorem 1.1, we obtain the following partial result towards Conjecture 1.2:

**Corollary 1.3** Let $X$ be a smooth uniruled variety over a number field $F$ such that $\rho(X) = \rho(X)$. Let $L$ be a big and nef $Q$-divisor on $X$ with $\kappa(K_X + a(X, L)L) = 0$. If we vary over all generically finite $F$-covers $f : Y \rightarrow X$ with $\kappa(K_Y + a(Y, f^*L)f^*L) = 0$ such that

$$(a(Y, f^*L), b(Y, f^*L)) > (a(X, L), b(X, L))$$

then the set of rational points $\bigcup f f(Y(F))$ are contained in a thin subset $Z \subset X$. 
The outline of the paper is as follows. In Section 2.1, we prove the key statements about the $a$-constant. In Section 2.2, we prove the boundedness of the degree of the morphisms in Theorem 1.1. In Section 2.3, we recall the facts about twists of morphisms over a number field. Finally, in section 3, we prove Theorem 1.1 and Corollary 1.3.

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2. Preliminaries

2.1. Geometric invariants. Let $X$ be a smooth projective variety over a field $k$. The Néron-Severi group $\text{NS}(X)$ is defined as the quotient of the group of Weil divisors, $\text{Wdiv}(X)$, modulo algebraic equivalence. The pseudo-effective cone $\Lambda_{\text{eff}}(X)$ is the closure of the cone of effective divisor classes in $\text{NS}(X)_{\mathbb{R}}$. The interior of $\Lambda_{\text{eff}}(X)$ is the cone of big divisors $\text{Big}^1(X)_{\mathbb{R}}$.

Definition 2.1 Let $L$ be a big Cartier $\mathbb{Q}$-divisor on $X$. The $a$-constant is defined as

$$a(X, L) = \min \{ t \in \mathbb{R} | [K_X + tL] \in \Lambda_{\text{eff}}(X) \}$$

If $L$ is not big, we formally set $a(X, L) = \infty$. For a singular projective variety we define $a(X, L) := a(\tilde{X}, \pi^*L)$ where $\pi: \tilde{X} \to X$ is a resolution of $X$. It is invariant under pull-back by a birational morphism of smooth varieties and hence independent of the choice of the resolution. By [BDPP13] we know that $a(X, L) > 0$ if and only if $X$ is uniruled. We note that the $a$-constant is independent of base change to another field. It was shown in [BCHM10] that, if $X$ is uniruled and has klt singularities, then for an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $L$, the Fujita invariant $a(X, L)$ is a rational number. If $L$ is big and not ample, then $a(X, L)$ can be irrational (see [HTT15, Example 6]).

Definition 2.2 Let $X$ be a smooth projective variety over $k$ and $L$ a big Cartier $\mathbb{Q}$-divisor. The $b$-constant is defined as

$$b(k, X, L) = \text{codim of minimal supported face of } \Lambda_{\text{eff}}(X)$$

containing the class of $K_X + a(X, L)L$. 

It is invariant under pullback by a birational morphism of smooth varieties ([HTT15]). For a singular variety $X$ we define $b(k, X, L) := b(k, \tilde{X}, \pi^* L)$, by pulling back to a resolution. In general the $b$-constant depends on the base field $k$. It is invariant under base change of algebraically closed fields.

For the rest of this section we work over an algebraically closed field $k$.

We have the following result about the behaviour of the $a$-constant under restriction to subvarieties.

**Theorem 2.3** (see [HJ16, Theorem 1.1], [LTT16, Theorem 4.8]) Let $X$ be a smooth uniruled projective variety and $L$ a big and nef $\mathbb{Q}$-divisor. Then there is a proper closed subset $V \subset X$ such that any subvariety $Y$ satisfying $a(Y, L|_Y) > a(X, L)$ is contained in $V$.

The above result was proved in [HJ16] when $L$ is big and semi-ample. In [LTT16], it was proved assuming the weak BAB-conjecture. By [Bir16], now we know that the BAB-conjecture holds. Therefore the above result works for $L$ big and nef.

The behaviour of the $a$-constant under pull-back to a generically finite cover is depicted in the following inequality.

**Lemma 2.4** Let $f : Y \longrightarrow X$ be a generically finite surjective morphism of varieties and $L$ a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. Then

$$a(Y, f^* L) \leq a(X, L).$$

**Proof.** Since the $a$-constant is computed on a resolution of singularities. We may assume $X$ and $Y$ are smooth. As $f : Y \longrightarrow X$ is generically finite, we may write

$$K_Y = f^* K_X + R$$

for some effective divisor $R$. Let $a(X, L) = a$. Then we have

$$K_Y + a f^* L = f^* (K_X + aL) + R.$$

Since $R \geq 0$ and $K_X + aL$ is pseudo-effective, we see that $K_Y + a f^* L$ is also pseudo-effective. Hence $a(Y, f^* L) \leq a(X, L) = a$.

Since the $a$-constant is not necessarily rational, we need to work with $\mathbb{R}$-divisors. We recall the definition of Iitaka dimension for $\mathbb{R}$-divisors.
Definition 2.5 Let $X$ be a smooth projective variety and $D$ be an $\mathbb{R}$-divisor on $X$. We define
\[
\kappa(X, D) = \limsup_{m \to \infty} \frac{\log h^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{\log m}
\]
If $X$ is a normal projective variety and $D$ an $\mathbb{R}$-Cartier divisor, then we define
\[
\kappa(X, D) = \kappa(\tilde{X}, \pi^*D)
\]
for a resolution of singularities $\pi : \tilde{X} \to X$. It is easy to see that the definition is independent of the choice of the resolution. Note that it is not necessarily true that if $D \sim_\mathbb{R} D'$ then $\kappa(X, D) = \kappa(X, D')$.

We form the following definition for convenience.

Definition 2.6 (cf. [LT17], Section 4.1) Let $X$ be a smooth uniruled variety and $L$ a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We say that a morphism $f : Y \to X$ is an adjoint-rigid cover preserving the $a$-constant if,

1. $f : Y \to X$ is a generically finite surjective morphism from a normal variety $Y$,
2. $a(Y, f^*L) = a(X, L)$,
3. $\kappa(K_Y + a(Y, f^*L)f^*L) = 0$.

Note that the conditions (1)-(3) are preserved under taking a resolution of singularities.

If $X$ is a smooth surface and $E$ is a curve contracted by the $K_X$-MMP, then $K_E$ is not pseudo-effective. The following proposition is a generalization of this fact and it is a key step for proving Theorem 1.1. For a smooth projective uniruled variety $X$ and a big and nef divisor $L$, this proposition enables us to compare $a(X, L)$ with the $a$-constants of $L$ under restriction to the exceptional divisors contracted by a $K_X + a(X, L)L$-MMP.

Proposition 2.7 Let $X$ be a normal variety with canonical singularities and $\Delta$ an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor which is nef. Suppose $\psi : X \to X'$ is a minimal model for $(X, \Delta)$ obtained by a running a $K_X + \Delta$-MMP. Let $E$ be an exceptional divisor contracted by $\psi$. Let $\pi : \tilde{E} \to E$ be a resolution of singularities. Then $K_{\tilde{E}} + \pi^*(\Delta|_E)$ is not pseudo-effective.

Proof. Note that it enough to prove the statement for one resolution of singularities of $E$. In particular, let $\pi : \tilde{X} \to X$ be a log resolution of $(X, E + \Delta)$.
such that we have a morphism $\phi = \psi \circ \pi : \widetilde{X} \to X'$. Let $\widetilde{E} = \pi_*^{-1}E$ be the strict transform. We reduce to the case when $X$ and $E$ are smooth and $\Delta$ is ample as follows. Let $H$ be a general ample divisor on $X$. Then $\psi : X \to X'$ is also a $K_X + \Delta + \epsilon H$-MMP for $\epsilon > 0$ sufficiently small. Hence, by replacing $\Delta$ with $\Delta + \epsilon H$, we may assume that $\Delta$ is ample. Note that, since $X$ has canonical singularities, $\phi : \widetilde{X} \to X'$ is a $K_{\widetilde{X}} + \pi^*\Delta$-minimal model. As $\pi^*\Delta$ is big and nef, we may choose $\Delta \sim_{\mathbb{R}} \pi^*\Delta$ such that $(\widetilde{X}, \Delta)$ is klt ([Xu15, Proposition 2.3]). Now $\phi : \widetilde{X} \to X'$ is a minimal model for $(\tilde{X}, \Delta)$ and since minimal models of klt pairs are isomorphic in codimension one ([BCHM10, Corollary 1.1.3]), we know that $\widetilde{E}$ will be contracted by any $K_{\widetilde{X}} + \Delta$-MMP. Therefore we may assume that $X$ and $E$ are smooth and $\Delta$ is ample. We need to show that $K_E + \Delta|_E$ is not pseudo-effective.

Since $\Delta$ is ample, we may choose $\Delta_0 \sim_{\mathbb{R}} \Delta$ such that $(X, E + \Delta_0)$ is simple normal crossing and divisorially log terminal, $(X, \Delta_0)$ is Kawamata log terminal and $(E, \Delta_0|_E)$ is canonical, by using the Bertini theorem ([Xu15, Lemma 2.2]).

As $(X, E + \Delta_0)$ is dlt and $\Delta_0$ is ample, by [BCHM10], we may run a $K_X + E + \Delta_0$-MMP. Since we know that the ACC holds for log canonical thresholds ([HMX14]) and special termination holds for dlt flips ([BCHM, Lemma 5.1]), the $K_X + E + \Delta_0$-MMP terminates with a minimal model $\theta : X \to X_m$ by [Bir07, Theorem 1.2]. Since $E$ is contained in the negative part of the Zariski decomposition of $K_X + E + \Delta_0$, the MMP given by $\theta$ contracts $E$. Let $\theta_k : X_k \to X_{k+1}$ be the divisorial contraction step of the $K_X + E + \Delta_0$-MMP that contracts the push-forward of $E$ on $X_k$. Let $\Theta_k = X \to X_k$ be the composition of the steps of the $K_X + E + \Delta_0$-MMP. We denote $\Delta_k = \Theta_k_\ast \Delta_0$ and $E_k = \Theta_k_\ast E$. Note that $E_k$ is normal ([KM98, Proposition 5.51]). By [AK17, Theorem 7], the restriction map $\Theta_k|_E : (E, \text{Diff}_E \Delta_0) \to (E_k, \text{Diff}_{E_k} \Delta_k)$ is a composition of steps of a $K_E + \text{Diff}_E \Delta_0$-MMP. As $\text{Diff}_E \Delta_0 = \Delta_0|_E$, it is enough to show that $K_{E_k} + \text{Diff}_{E_k} \Delta_k$ is not pseudo-effective. This follows from the fact that $E_k$ is covered by curves $C$ such that $(K_{E_k} + \text{Diff}_{E_k} \Delta_k) \cdot C = (K_{X_k} + E_k + \Delta_k) \cdot C < 0$.

Note that the assumption about $\Delta$ being nef is necessary. For example, let $Y$ be a minimal surface and $X = \text{Bl}_4(\text{Bl}_y Y)$ be the blow-up of $\text{Bl}_y Y$ at four distinct points $y_i \in E$, $1 \leq i \leq 4$, where $E \subset \text{Bl}_y Y$ is the exceptional curve corresponding to $y \in Y$. Let $E_i \subset X$ be the exceptional curve corresponding to $y_i$ for $1 \leq i \leq 4$ and $E_0$ be the strict transform of $E$ on $X$. Let $\Delta = \frac{1}{2}E_1 + \frac{1}{2}E_2 + \frac{1}{2}E_3 + \frac{1}{2}E_4$. Note
that $\Delta$ is not nef as $\Delta \cdot E_4 = -\frac{1}{2}$. Now $E_0$ is contracted by the $K_X + \Delta$-MMP but $\deg(K_{E_0} + \Delta|_{E_0}) = 0$.

**Corollary 2.8** Let $X$ be a smooth projective uniruled variety and $L$ a big and nef $\mathbb{Q}$-divisor. Let $f : Y \to X$ be a generically finite cover with $Y$ smooth and $a(Y, f^*L) = a(X, L)$. Let $R \subset Y$ be a component of the ramification divisor of $f$ (i.e. the strict transform of a component of the ramification divisor for the Stein factorization of $f$) and $B$ be the component of the branch divisor on $X$ which is the image of $R$. If $R$ is contracted by a $K_Y + a(X, L)f^*L$-MMP, then

$$a(B, L|_B) > a(X, L).$$

**Proof.** We may assume that $L|_B$ is big. We have a generically finite surjective map $f|_R : R \to B$. Therefore, by Lemma 2.4, we have $a(R, f^*L|_R) \leq a(B, L|_B)$. Now Proposition 2.7 implies that $a(R, a(X, L)f^*L) > 1$ and hence $a(R, f^*L|_R) > a(X, L)$.

2.2. **Boundedness statements.** Let $X$ be a normal projective variety of dimension $n$ and $D$ an $\mathbb{R}$-divisor. The volume of $D$ is defined by

$$\text{vol}(X, D) = \lim_{m \to \infty} \frac{n! h^0(X, O_X(\lceil mD \rceil))}{m^n}$$

If $D$ is nef then $\text{vol}(X, D) = D^n$. Also $D$ is big iff $\text{vol}(X, D) > 0$. The volume depends only on the numerical class $[D] \in N^1(X)$.

Let $L$ be a pseudoeffective $\mathbb{Q}$-Cartier divisor on $X$. The stable base locus of $L$ ([Laz04]) is defined as

$$B(L) := \cap_{m \in \mathbb{N}} \text{Bs}(mL)$$

where the intersection is taken over $m$ such that $mL$ is Cartier. The augmented base locus of $L$ is defined as

$$B_+(L) := \cap_A B(L - A)$$

where the intersection is over all ample $\mathbb{Q}$-Cartier divisors $A$. It is known that $B_+(L)$ is a closed subset of $X$. If $L$ is big, then $L|_Z$ is big for any subvariety $Z \not\subset B_+(L)$. We recall the following well-known result.

**Lemma 2.9** Let $f : Y \to X$ and $g : X \to W$ be a birational morphism of normal projective varieties and $D$ an $\mathbb{R}$-divisor on $X$.

1. $\text{vol}(W, g_*D) \geq \text{vol}(X, D)$,
(2) If $D$ is $\mathbb{R}$-Cartier and $E_i$ are $f$-exceptional, then
\[ \text{vol}(f^*D + \sum a_i E_i) = \text{vol}(X, D) \]
for $a_i > 0$.

**Definition 2.10** Let $\psi : X \dashrightarrow X'$ be a proper birational contraction (i.e. $\psi^{-1}$ does not contract any divisors) of normal quasi-projective varieties. Let $D$ be a $\mathbb{R}$-Cartier divisor such that $D' = \psi_* D$ is also $\mathbb{R}$-Cartier. We say that $\psi$ is $D$-negative if for some common resolution $p : W \to X$ and $q : W \to Y$, we may write
\[ p^* D = q^* D' + E \]
where $E \geq 0$ is $q$-exceptional and the support of $E$ contains the strict transform of the $\psi$-exceptional divisors.

Recall that if $\psi : X \dashrightarrow X'$ is a $K_X + \Delta$-minimal model, then it is $K_X + \Delta$-negative by definition. Further, if $(X, \Delta)$ is terminal and $p : W \to X$, $q : W \to X'$ is a common resolution, then $q : W \to X'$ is $K_W + p^* \Delta$-negative.

In general, if we have a $K_X + \Delta$-negative contraction $\psi : X \dashrightarrow X'$ of a terminal pair $(X, \Delta)$, then the pushforward $(X', \psi_* \Delta)$ might not be terminal since $\Delta$ might contain the $\psi$-exceptional divisors as components. If $\Delta$ is big and nef, then the following lemma shows that we can achieve the desired conclusion by passing to a resolution. This result will be used in Proposition 2.15 for proving the boundedness of degrees of adjoint-rigid covers preserving the $a$-constant.

**Lemma 2.11** Let $X$ be a normal variety with terminal singularities and $D$ a big and nef $\mathbb{R}$-Cartier divisor. Let $\psi : X \dashrightarrow X'$ be a $K_X + \Delta$-negative contraction. Then we may choose a common resolution $p : W \to X$ and $q : W \to X'$ with $\hat{\Delta} \sim_{\mathbb{R}} p^* D$ such that $(W, \hat{\Delta})$ and $(X', \Delta' = \mu_* \hat{\Delta})$ are both terminal.

**Proof.** Since $D$ is big and nef, we can find $\Delta \sim_{\mathbb{R}} D$ such that $(X, \Delta)$ is terminal (see the proof of [LTT16, Theorem 2.3]). Let $p : W \to X$ and $q : W \to X'$ be a common log resolution

\[
\begin{array}{c}
W \\
p \quad q \\
X \\
\quad \psi \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad X'
\end{array}
\]

Let $E_j$ be the $\psi$-exceptional divisors and $F_i$ the $p$-exceptional divisors. Note that the $q$-exceptional divisors are $F_i$ and the strict transforms $\tilde{E}_j$ of the $\psi$-exceptional
divisors. We have
\[ K_W + p_*^{-1} \Delta = p^*(K_X + \Delta) + \sum_i a_i F_i \]
where \( a_i > 0 \). We may add \( p \)-exceptional divisors to obtain
\[ K_W + p^* \Delta = p^*(K_X + \Delta) + \sum_i b_i F_i \]
with \( b_i > 0 \). Since \( \psi \) is \( K_X + \Delta \)-negative, we may write
\[ p^*(K_X + \Delta) = q^*(K_{X'} + \Delta') + \sum_j c_j \tilde{E}_j + \sum_i d_i F_i \]
where \( \Delta' = \psi^* \Delta \) and \( c_j > 0, d_i \geq 0 \). Therefore we have
\[ K_W + q_*^{-1} q_* \tilde{\Delta} = q^*(K_{X'} + q_* \tilde{\Delta}) + \sum_j \alpha_j \tilde{E}_j + \sum_i \beta_i F_i \]
with \( \alpha_j, \beta_i > 0 \). As \( p^* \Delta \) is big and nef, we may choose \( \tilde{\Delta} \sim_{\mathbb{R}} p^* \Delta \) with the coefficients of \( q \)-exceptional divisors in \( \tilde{\Delta} \) sufficiently small such that \((W, \tilde{\Delta})\) is a simple normal crossing terminal pair and
\[ K_W + q_*^{-1} q_* \tilde{\Delta} = q^*(K_{X'} + q_* \tilde{\Delta}) + \sum_j \alpha'_j \tilde{E}_j + \sum_i \beta'_i F_i \]
with \( \alpha'_j, \beta'_i > 0 \). Therefore \((X', q_* \tilde{\Delta})\) is also terminal.

We recall the definitions and results related to the BAB-conjecture.

**Definition 2.12** Let \( X \) be a normal projective variety and \( \Delta \) an effective boundary \( \mathbb{R} \)-divisor such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. We say the \((X, \Delta)\) is \( \epsilon \)-log canonical (resp. \( \epsilon \)-klt) if for a resolution of singularities \( \pi : X \to X \) with exceptional divisors \( E_i \), we have \( a(E_i, X, \Delta) \geq -1 + \epsilon \) (resp. \( a(E_i, X, \Delta) > -1 + \epsilon \)) where the discrepancies \( a(E_i, X, \Delta) \) are defined by the equation
\[ K_X + \pi_*^{-1} \Delta = \pi^*(K_X + \Delta) + a(E_i, X, \Delta) E_i. \]

The following is the BAB-conjecture proved by Birkar.

**Theorem 2.13** (see [Bir16, Theorem 1.1]) Let \( n \) be a natural number and \( \epsilon > 0 \) a real number. Then the set of projective varieties \( X \) such that

1. \( X \) is of dimension \( n \) with a boundary divisor \( \Delta \) such that \((X, \Delta)\) is \( \epsilon \)-log canonical
2. \(- (K_X + \Delta) \) is big and nef,

form a bounded family.

A consequence of the above theorem is the boundedness of anticanonical volumes.
Corollary 2.14 (Weak BAB-conjecture) Let $n$ be a natural number and $\epsilon > 0$ a real number. There exists a constant $M(n, \epsilon)$ such that, for any normal projective variety $X$ satisfying

1. $X$ is of dimension $n$ such that there is a boundary divisor $\Delta$ with $(X, \Delta)$ is $\epsilon$-klt and $K_X$ is $\mathbb{Q}$-Cartier.
2. $-(K_X + \Delta)$ is ample,

we have

$$\text{vol}(-K_X) < M(n, \epsilon).$$

As a consequence of the Weak BAB-conjecture we obtain the following result. It shows that the degrees of all adjoint-rigid covers preserving the $a$-constant are bounded by a constant.

Proposition 2.15 Let $X$ be a smooth uniruled variety and $L$ a big and nef $\mathbb{Q}$-divisor. Then there exists a constant $M > 0$ such that, if $f : Y \to X$ is an adjoint-rigid cover preserving the $a$-constant, then $\deg(f) < M$.

Proof. By Lemma 2.11, we may replace $Y$ by a resolution to assume that there exists $\Delta \sim af^*L$ with $(Y, \Delta)$ terminal and we have a morphism $\psi : Y \to Y'$ to a minimal model $(Y', \Delta')$ with $\mathbb{Q}$-factorial terminal singularities. Now $\kappa(K_Y + \Delta) = 0$ implies that $\kappa(K_{Y'} + \Delta') = 0$. As $K_{Y'} + \Delta'$ is semi-ample ([BCHM, Corollary 3.9.2]), we have $K_{Y'} + \Delta' \equiv 0$. Since $\Delta'$ is big, we can write $\Delta' \equiv A + E$ where $A$ is ample and $E$ is effective. Now for $0 < t \ll 1$, $(Y', (1-t)\Delta' + tE)$ is terminal and

$$K_{Y'} + (1-t)\Delta' + tE \equiv -tA$$

is anti-ample. Therefore $(Y', (1-t)\Delta' + tE)$ is terminal log Fano. In particular, it is $\epsilon$-klt for $\epsilon = \frac{1}{2}$. Therefore by the Weak BAB conjecture (Corollary 2.14), there exists $M > 0$ such that

$$\text{vol}(\Delta') = \text{vol}(-K_{Y'}) < M.$$ 

Since $f : Y \to X$ is generically finite, we have $\text{vol}(af^*L) = a^n\deg(f)\text{vol}(L)$. Now we have the following inequality

$$a^n\deg(f)\text{vol}(L) = \text{vol}(\Delta) \leq \text{vol}(\Delta') < M.$$ 

Therefore the degree of $f$ is bounded. \hfill \Box

2.3. Twists. Let us assume that the ground field is a number field $F$. Let $X$ be a smooth projective variety over $F$ and $L$ a big and nef $\mathbb{Q}$-divisor on $X$.

Let $f : Y \to X$ be a generically finite cover defined over $F$. A twist of $f : Y \to X$ is a generically finite cover $f' : Y' \to X$ such that, after base
change to the algebraic closure $\overline{F}$, we have an isomorphism $g : Y \sim \rightarrow Y'$ with $\overline{Y} = \overline{Y'} \circ g$.

All the twists of $f : Y \rightarrow X$ is parametrized by the Galois cohomology of $\text{Aut}(Y/X)$. Precisely, there is a bijection between the set of isomorphism classes of twists of $f$ and the Galois cohomology group $H^1(\text{Gal}(\overline{F}), \text{Aut}(Y/X))$. In view of Conjecture 1.2, even if we know the finiteness of adjoint-rigid $a$-covers over $\overline{F}$, the corresponding finiteness statement might not hold over $F$ itself due to the presence of twists. However the following result shows that, the rational points contributed by all twists of $f$, satisfying the hypothesis of Conjecture 1.2, are contained in a thin subset.

**Theorem 2.16** (see [LT17, Theorem 1.10]) Let $X$ be a smooth projective variety over a number field $F$ satisfying $\rho(X) = \rho(\overline{X})$ and let $L$ be a nef and big $\mathbb{Q}$-divisor on $X$. Suppose $f : Y \rightarrow X$ is a generically finite $F$-cover from a smooth projective variety $Y$, satisfying $\kappa(K_Y + a(X,L)f^*L) = 0$. If we vary $\sigma \in H^1(\text{Gal}(\overline{F}), \text{Aut}(Y/X))$ such that the corresponding twist $f^{\sigma} : Y^\sigma \rightarrow X$ satisfies

$$(a(Y, f^*L), b(F, Y^\sigma, (f^{\sigma})^*L)) > (a(X, L), b(F, X, L))$$

in the lexicographic order, then the set

$$\cup_{\sigma} f^{\sigma}(Y^\sigma(F)) \subset X(F)$$

is contained in a thin subset of $X(F)$.

### 3. Finiteness and Thinness

In this section we prove the main results.

**Proof of Theorem 1.1.** Let $X$ be a smooth uniruled variety of dimension $n$ and $L$ a big and nef $\mathbb{Q}$-divisor on $X$. Suppose $a(X, L) = a$. We need to show that, upto birational equivalence, there exist finitely many varieties $Y$ that admit a morphism $f : Y \rightarrow X$ which is an adjoint-rigid cover preserving the $a$-constant. The statement is obvious if $X$ is a curve. We assume that $n \geq 2$. By passing to a resolution it is enough to show that, upto birational equivalence, there exist finitely many smooth varieties $Y$ with a morphism $f : Y \rightarrow X$ which is an adjoint-rigid cover preserving the $a$-constant. By Proposition 2.15, we know that
there exists a constant $M > 0$ such that $\deg(f) < M$ for any adjoint-rigid cover preserving the $a$-constant $f : Y \to X$. Now, for an open $U \subset X$, there are finitely many étale covers (upto isomorphism) of $U$ of a given degree $d$. Hence it is enough to show that there is a proper closed subset $V \subsetneq X$, such that if $f : Y \to X$ is an adjoint-rigid cover preserving the $a$-constant and $Y$ is smooth, the branch locus of $f$ is contained in $V$.

Suppose $f : Y \to X$ is an adjoint-rigid cover preserving the $a$-constant and $Y$ is smooth. Let $Y \xrightarrow{\pi} \overline{Y} \xrightarrow{\pi} X$ be the Stein factorization of $f$. Let $B \subset X$ be a component of the branch divisor of $\overline{f}$. Note that, by the Zariski-Nagata purity theorem, the branch locus is a divisor. Let $\sum_j r_j R_j \subset \overline{Y}$ be the ramification divisor, i.e. $K_{\overline{Y}} = f^* K_X + \sum_j r_j R_j$. Let $R \subset \overline{Y}$ be a component of the ramification divisor mapping to $B$ and $R \subset Y$ be the strict transform $\pi^{-1}_*(R)$.

We have the following equation

$$K_Y + af^* L \equiv f^*(K_X + aL) + \pi_*^{-1}(\sum_i r_i R_i) + \sum_i a_i E_i$$

where $a_i > 0$. Note that, as $K_X + aL$ is pseudo-effective and $L$ is big and nef, by non-vanishing ([BCHM10, Theorem D]) we have

$$K_X + aL \sim D \geq 0.$$ 

Therefore we have

$$K_Y + af^* L \equiv \sum_j c_j F_j \geq 0$$

Now, we may find a $\Delta \equiv af^* L$ such that, $(Y, \Delta)$ is terminal. As $K_Y + \Delta$ is pseudo-effective, we can run a $K_Y + \Delta$-MMP

$$\psi : (Y, \Delta) \dasharrow (Y_1, \Delta_1) \dasharrow \cdots \dasharrow (Y_m, \Delta_m) = (Y', \Delta')$$

to obtain a $K_Y + \Delta$-minimal model $(Y', \Delta')$. Since $\kappa(K_Y + \Delta) = 0$, we have $K_{Y'} + \Delta' \equiv 0$. Hence the $K_Y + \Delta$-MMP contracts all components of the divisor $\sum_j c_j F_j$. As $R = F_j$ for some $j$, Corollary 2.8 implies that $a(R, af^* L|_R) > 1$ and hence

$$a(R, L|_R) > a = a(X, L).$$

Therefore, by Theorem 2.3, there exists a proper closed subset $V \subsetneq X$ such that $B \subset V' = V \cup B_+(L)$. Then, for any adjoint-rigid cover preserving the $a$-constant $f : Y \to X$ with $Y$ smooth, the branch locus of $f$ is contained in $V'$. Therefore we have the desired conclusion.

$\square$
**Proof of Corollary 1.3.** We have a smooth uniruled variety $X$ over a number field $F$ such that $\rho(X) = \rho(\overline{X})$ and $\kappa(K_X + a(X,L)L) = 0$ and $L$ is a big and nef $\mathbb{Q}$-divisor on $X$. Let $f : Y \to X$ be a generically finite $F$-cover such that $\kappa(K_Y + a(Y,f^*L)f^*L) = 0$ and

$$(a(Y,f^*L), b(F, Y, f^*L)) > (a(X, L), b(F, X, L)).$$

It is enough to show that the rational points contributed by the Stein factorization of $f : Y \to X$ are contained in a fixed thin set. Note that, by Lemma 2.4, $a(Y, f^*L) = a(X, L)$. Therefore, the morphism $\overline{f} : \overline{Y} \to \overline{X}$, obtained by base change to the algebraic closure of $F$, is an adjoint-rigid cover preserving the $a$-constant. Hence, by Theorem 1.1, the Stein factorizations of all such morphisms $\overline{f} : \overline{Y} \to \overline{X}$ vary in a finite set $S$. Hence we need to consider rational points contributed by twists of finitely many such Stein factorizations. So we may replace $Y$ by its Stein factorization. Further, by passing to a resolution of singularities, we may assume that $\text{Bir}(\overline{Y}/\overline{X}) = \text{Aut}(\overline{Y}/\overline{X})$ (see the proof of [LT17, Theorem 1.10]). By applying Theorem 2.16 to each $f \in S$, there is a thin subset $Z_f \subset X$ such that $\bigcup \sigma f^\sigma(Y^\sigma(F)) \subset Z_f$, where $\sigma$ varies over all the twists of $f$. Therefore we have

$$\bigcup f(Y(F)) = \bigcup_{f \in S} f(Y(F)) \subset \bigcup_{f \in S} Z_f = Z$$

where $Z \subset X$ is a thin subset and the union is taken over all the morphisms $f : Y \to X$ satisfying the hypothesis of Corollary 1.3.

\[\square\]

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