MEAN OSCILLATION BOUNDS ON REARRANGEMENTS

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ABSTRACT. We use geometric arguments to prove explicit bounds on the mean oscillation for two important rearrangements on \( \mathbb{R}^n \). For the decreasing rearrangement \( f^* \) of a rearrangeable function \( f \) of bounded mean oscillation (BMO) on cubes, we improve a classical inequality of Bennett–DeVore–Sharpley, \( \| f^* \|_{\text{BMO}(\mathbb{R}^n)} \leq C_n \| f \|_{\text{BMO}(\mathbb{R}^n)} \), by showing the growth of \( C_n \) in the dimension \( n \) is not exponential but at most of the order of \( \sqrt{n} \). This is achieved by comparing cubes to a family of rectangles for which one can prove a dimension-free Calderón–Zygmund decomposition. By comparing cubes to a family of polar rectangles, we provide a first proof that an analogous inequality holds for the symmetric decreasing rearrangement, \( Sf \).

1. Introduction

Equimeasurable rearrangements are used in analysis and mathematical physics to reduce extremal problems involving functions on higher-dimensional spaces to problems for functions of a single variable. Here, we consider the action of two classical rearrangements on the space \( \text{BMO} \), consisting of functions with bounded mean oscillation. The rearrangements are symmetrization (symmetric decreasing rearrangement), which replaces a given function \( f \) on \( \mathbb{R}^n \) with a radially decreasing function \( Sf \), and the decreasing rearrangement, which replaces \( f \) with a decreasing function \( f^* \) on \( \mathbb{R}^+ \).

Symmetrization, when applicable, offers a direct path to geometric inequalities in functional analysis. To name a few examples, in the first computation of the sharp constants in the Sobolev inequality for \( \| \nabla f \|_p \), due to Talenti [26] and Aubin [1], symmetrization is used to reduce the problem to radial functions. A similar construction appears in the work of Lieb [7] on extremals of the Hardy–Littlewood–Sobolev inequality. Sharp rearrangement inequalities, which also characterize equality cases, are crucial for identifying extremals. They can also be used constructively, according to the Competing Symmetries principle of Carlen–Loss [9], to find sequences with rearrangements that converge strongly to a particular extremal.

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Many applications rely on classical symmetrization inequalities in Lebesgue and Sobolev spaces. Rearrangements generally preserve $L^p$-norms and contract $L^p$-distances for $1 \leq p \leq \infty$. Symmetrization also improves the modulus of continuity. By the Pólya–Szegő inequality, it decreases $\|\nabla f\|_p$ for $p \geq 1$, and by the Riesz–Sobolev inequality it increases norms in certain negative Sobolev spaces $\|f\|_{H^{-s}}$ for $0 < s < \frac{n}{2}$. These inequalities indicate that symmetrization reduces the overall oscillation of $f$. They contain the classical isoperimetric inequality as the limiting case where $f$ is the characteristic function of a set of finite volume.

The decreasing rearrangement, on the other hand, is determined by the distribution function, which contains no information about the shape of level sets. It can be defined on a general measure space and is widely applied, for example, on metric spaces endowed with a doubling measure. The decreasing rearrangement plays an important role in interpolation theory as many familiar function spaces are invariant under equimeasurable rearrangements, including the Lebesgue, Lorentz, and Orlicz spaces (see [4]).

We study the rearrangements of functions of bounded mean oscillation (BMO), a condition introduced by John and Nirenberg in [17]. By definition, a locally integrable function $f$ is in $\text{BMO}(\mathbb{R}^n)$ if it satisfies a uniform bound on its mean oscillation over cubes. Unlike $L^p$-spaces, BMO is not invariant under equimeasurable rearrangements (for instance, $f$ defined by $f(x) = (-\log |x|)_+$ lies in $\text{BMO}(\mathbb{R})$, but $g$ defined by $g(x) = f(\frac{1}{2}x)$ for $x > 0$ and $g(x) = 0$ for $x \leq 0$ does not.) Moreover, recent results [13] on the John–Nirenberg space $JN_p$, a variant of BMO introduced in [17], show that spaces defined by mean oscillation need not be preserved by the decreasing rearrangement. The work of Bennett–DeVore–Sharpley in [3] implies, as shown in [21], that the decreasing rearrangement of a function $f$ in $\text{BMO}(\mathbb{R}^n)$ belongs to $\text{BMO}(\mathbb{R}^n)$. Moreover, there are constants $C_n$ (depending only on dimension) such that

\begin{equation}
\|f^*\|_{\text{BMO}} \leq C_n \|f\|_{\text{BMO}}.
\end{equation}

The explicit bound on the constant arising from [3] is $C_n \leq 2^{n+5}$. When $n = 1$, the results of Klemes [18], along with subsequent steps taken by Korenovskii [19], imply that $C_1 = 1$. (We neglect, for the moment, the distinction between $\mathbb{R}^n$ and finite cubes, see Section 3.)

Our first main result is that the exponential dependence on the dimension in the bound of Bennett–DeVore–Sharpley can be eliminated.

**Theorem 1.1.** For rearrangeable $f \in \text{BMO}(\mathbb{R}^n)$, Eq. (1.1) holds with $C_n \leq 2(1 + 2\sqrt{n-1})$.

To achieve this, we exhibit a basis $\mathcal{W}$ of rectangles in $\mathbb{R}^n$, comparable with cubes, such that

\begin{equation}
\|f^*\|_{\text{BMO}} \leq 2\|f\|_{\text{BMO},\mathcal{W}}
\end{equation}

where $\|f\|_{\text{BMO},\mathcal{W}}$ denotes the supremum of the mean oscillation of $f$ over these rectangles. Moreover,

\begin{equation}
\|f\|_{\text{BMO}} \leq \|f\|_{\text{BMO},\mathcal{W}} \leq (1 + 2\sqrt{n-1})\|f\|_{\text{BMO}}.
\end{equation}
The family of rectangles $W$ consists of the false cubes introduced by Wik in his work on the constants in the John-Nirenberg inequality [27]. The collection of false cubes has the property that each false cube can be bisected into two smaller false cubes, and in Section 3 we prove Eq. (1.2) for any such bisection basis. The equivalence of the seminorms $\| \cdot \|_{BMO_W}$ and $\| \cdot \|_{BMO}$ up to a constant of order $O(\sqrt{n})$ was first proved by Wik (Theorem 2 in [27]), using a combinatorial argument and a slightly different definition of the norms. Here, we give a simple probabilistic proof for Eq. (1.3).

While the sharp constant $C_1 = 1$ of Klemes-Korenovskii holds in dimension $n = 1$, the question of the best constants $C_n$ in Eq. (1.1), and whether such constants can be dimension-free, remains open for $n > 1$. However, sharp results are known for other BMO-spaces. For the smaller space $BMO_R$, called anisotropic or strong BMO, consisting of functions of bounded mean oscillation on rectangles, a sharp constant of 1 was obtained by Korenovskii in [20] by using a higher-dimensional version of the Riesz Rising Sun lemma for rectangles - see also Korenovskii–Lerner–Stokolos [22].

Korenovskii shows that for a rectangle $R_0$,

$$\| f^* \|_{BMO(0,|R_0|)} \leq \| f \|_{BMO_R(R_0)}.$$  

The inequalities of Klemes and Korenovskii extend to the entire space $\mathbb{R}^n$ (using the techniques in Section 3.2 below). On the other hand, for the larger space dyadic BMO, working with the $L^2$-mean oscillation, Stolyarov, Vasyunin, and Zatitskiy [25] obtained a sharp dimension-dependence of $2^n/2$ in the boundedness of the decreasing rearrangement.

There has been recent interest in dimension-free bounds for averaging operators [5, 6], and dimension-free constants in the John–Nirenberg inequality [11]. Improved bounds on the decreasing rearrangement on BMO yield improved constants in the John–Nirenberg inequality, by reducing to the one-dimensional case: from Eq. (1.4), Korenovskii [20] obtained sharp constants for the John–Nirenberg inequality in anisotropic BMO on a rectangle.

It is apparent from the results discussed above that the geometry of the sets over which mean oscillation is measured plays a crucial role in the properties of the corresponding BMO-space. In previous work, two of the authors [12] introduce the space $BMO_\mathcal{S}$, consisting of functions of bounded mean oscillation on sets $S$, called shapes, forming a basis $\mathcal{S}$ — see Section 2 for the relevant definitions. The proof of Theorem 1.1 demonstrates that equivalent bases of shapes provide powerful tools for geometric analysis in BMO.

Comparison of shapes also plays a key role in the relation between $\| Sf \|_{BMO(\mathbb{R}^n)}$ and $\| f^* \|_{BMO(\mathbb{R}^n)}$, shown in Section 4. Geometrically, the symmetric decreasing rearrangement $Sf$ is linked with $f^*$ by means of the formula $Sf(x) = f^*(\omega_n|x|)$). We give the following bi-Lipschitz equivalence between $Sf$ and $f^*$.

**Theorem 1.2.** If $f_1, f_2$ are rearrangeable functions in $BMO(\mathbb{R}^n)$, then

$$2^{-2^n} \omega_n \| f_1^* - f_2^* \|_{BMO} \leq \| Sf_1 - Sf_2 \|_{BMO} \leq n^{\frac{2}{n}} \omega_n \| f_1^* - f_2^* \|_{BMO}.$$  

In the proof of Eq. (1.5), we compare cubes in $\mathbb{R}^n$ with an equivalent basis of shapes $\mathcal{S}$, comprised of certain annular sectors. The oscillation of $Sf$ over these sectors coincides with the mean
oscillation of $f^*$ over corresponding intervals. The dimension-dependent constants arise from volume factors associated when inscribing and circumscribing sectors in $\mathcal{A}$ with balls and cubes.

This theorem makes it possible to transfer results on the decreasing rearrangement $f^*$ to the symmetrization $Sf$. In particular, taking $f_1 = f$ and $f_2 = 0$ in Theorem 1.2, together with Theorem 1.1, we see that the symmetric decreasing rearrangement is bounded on $\text{BMO}(\mathbb{R}^n)$:

**Corollary.** There are constants $D_n$ (depending only on dimension) such that if $f \in \text{BMO}(\mathbb{R}^n)$ is rearrangeable, then $Sf \in \text{BMO}(\mathbb{R}^n)$ with

$$\|Sf\|_{\text{BMO}} \leq D_n \|f\|_{\text{BMO}}. \tag{1.6}$$

One can take $D_n = 2(1 + 2\sqrt{n - 1})n^2 \omega_n$. To our knowledge, there are no prior results on symmetrization in $\text{BMO}(\mathbb{R}^n)$ for dimensions $n > 1$. In one dimension, Klemes [18] proved that the symmetric decreasing rearrangement on the circle satisfies $\|Sf\|_{\text{BMO}} \leq 2\|f^*\|_{\text{BMO}}$, and gives an example for which $\|Sf\|_{\text{BMO}} > \|f^*\|_{\text{BMO}}$. The sharp constants $D_n$ are not known in any dimension.

2. Preliminaries

2.1. Rearrangements. Let $f$ be a measurable function on a domain $\Omega \subset \mathbb{R}^n$. The **distribution function** of $f$ is defined, for $\alpha \geq 0$, by

$$\mu_f(\alpha) = |\{x \in \Omega : |f(x)| > \alpha\}|.$$

Here $|\cdot|$ denotes Lebesgue measure. Note that $\mu_f : [0, \infty) \to [0, \infty]$ is decreasing (meaning, here and in the rest of the paper, nonincreasing) and right-continuous.

**Definition 2.1.** We say that a measurable function $f$ is **rearrangeable** if $\mu_f(\alpha) \to 0$ as $\alpha \to \infty$. The **decreasing rearrangement** of such a function is the function $f^* : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$f^*(s) = \inf \{\alpha \geq 0 : \mu_f(\alpha) \leq s\}.$$

In other words, the decreasing rearrangement of a function $f$ is the generalized inverse of its distribution function. In the case where $|\Omega| < \infty$, we consider $f^*$ as a function on $(0, |\Omega|)$, since $f^*(s) = 0$ for all $s \geq |\Omega|$.

The condition that $f$ is rearrangeable guarantees that the set $\{\alpha \geq 0 : \mu_f(\alpha) \leq s\}$ is nonempty for $s > 0$ and so $f^*$ is finite on its domain. The set $\{\alpha \geq 0 : \mu_f(\alpha) = 0\}$, however, can be empty. If $f$ is bounded, then $f^*$ tends to $\|f\|_{L_\infty}$ as $s \to 0^+$; otherwise, $f^*$ is unbounded at the origin. As is the case with the distribution function, $f^*$ is decreasing and right-continuous. Furthermore, $f$ and $f^*$ are equimeasurable in the sense that the distribution function of $f^*$ equals that of $f$ for all $\alpha \geq 0$.

We will need to use the following elementary form of the **Hardy-Littlewood inequality**: for any measurable set $A \subset \Omega$,

$$\int_A |f| \leq \int_0^{|A|} f^*. \tag{2.1}$$
Definition 2.2. Let $f$ be a rearrangeable function on $\mathbb{R}^n$. Its symmetric decreasing rearrangement $Sf$ is defined by

$$Sf(x) = f^*(\omega_n|x|^n), \quad x \in \mathbb{R}^n \setminus \{0\},$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.

The symmetric decreasing rearrangement defines a map from functions on $\mathbb{R}^n$ to functions on $\mathbb{R}^n$, and from $f^*$ it inherits equimeasurability with $f$. The reader is invited to see [24] for more details on the decreasing rearrangement, and [2] for the symmetric decreasing rearrangement.

2.2. Bounded mean oscillation. A shape is an open set $S \subset \mathbb{R}^n$ with $0 < |S| < \infty$. A basis of shapes in a domain $\Omega \subset \mathbb{R}^n$, then, is a collection $\mathcal{S}$ of shapes $S \subset \Omega$ forming a cover of $\Omega$. Common examples of bases are the collections of all open Euclidean, $\mathcal{B}$; all finite open cubes with sides parallel to the axes, $\mathcal{Q}$; and, all finite open rectangles with sides parallel to the axes, $\mathcal{R}$. In one dimension, these three choices coincide with the collection of all finite open intervals, $\mathcal{I}$. It is understood that when working on a domain $\Omega$, the notation for the bases above refers to those shapes contained in $\Omega$.

Functions on $\Omega$ are generally assumed to be real-valued and measurable, as well as integrable on every shape $S \subset \Omega$ coming from a basis $\mathcal{S}$; such functions are automatically locally integrable. The mean oscillation of a function $f$ on a shape $S \in \mathcal{S}$ is defined by

$$O(f, S) := \int_S |f - f_S|,$$

where $f_S = \frac{1}{|S|} \int_S f$ is the average of $f$ over $S$. It is immediate from the definition that $O(f + \alpha, S) = O(f, S)$ for any constant $\alpha$.

The following inequality allows for the comparison of mean oscillation over different shapes at the cost of the volume ratio: for any pair of shapes $S \subset \tilde{S}$,

$$O(f, S) \leq \frac{\tilde{S}}{|S|} O(f, \tilde{S}). \quad (2.2)$$

This follows from the fact that $O(f, S) = 2 \int_S (f - f_S)_+$, where $y_+ = \max(y, 0)$ (see [12]).

Definition 2.3. Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be two bases of shapes in $\Omega$. We say that $\mathcal{S}$ is equivalent to $\tilde{\mathcal{S}}$, written $\mathcal{S} \approx \tilde{\mathcal{S}}$, if there are constants $c, \tilde{c} > 0$ such that for every $S \in \mathcal{S}$ there exists $\tilde{S} \in \tilde{\mathcal{S}}$ with $S \subset \tilde{S}$ and $|\tilde{S}| \leq c|S|$, and for every $\tilde{S} \in \tilde{\mathcal{S}}$ there exists $S \in \mathcal{S}$ with $\tilde{S} \subset S$ and $|S| \leq \tilde{c}|\tilde{S}|$.

The most standard example of equivalent bases is that of balls and cubes in $\mathbb{R}^n$. We calculate the explicit constants $c, \tilde{c}$ in this equivalence, as they will be used in Section 4.

Lemma 2.4. The basis $\mathcal{B}$ is equivalent to the basis $\mathcal{Q}$.

Proof. Given a cube $Q$ of sidelength $\ell$, there is a ball $B$ of radius $\sqrt{n} \ell/2$ containing $Q$, so $|B| = 2^{-n} \frac{n}{n} \omega_n |Q|$. In the other direction, given a ball $B$ of radius $r$, there is a cube $Q$ of sidelength $2r$ containing $B$, so $|Q| = 2^n \frac{1}{\omega_n} |B|$.
**Definition 2.5.** We say a function $f$ has *bounded mean oscillation* with respect to a basis $\mathcal{S}$, denoted $f \in \text{BMO}_{\mathcal{S}}(\Omega)$, if $f \in L^1(S)$ for all $S \in \mathcal{S}$ and

$$\|f\|_{\text{BMO}_{\mathcal{S}}} := \sup_{S \in \mathcal{S}} O(f, S) < \infty.$$  

Since mean oscillation is translation invariant under the addition of constants, Eq. (2.3) defines a seminorm that vanishes on constant functions. If one considers $\text{BMO}_{\mathcal{S}}(\Omega)$ modulo constants, one obtains a Banach space, as was shown in [12]. However, as we will see below, for the purpose of rearrangement it is useful to just think of $\text{BMO}_{\mathcal{S}}(\Omega)$ as a linear space with a seminorm. The notation $\text{BMO}(\Omega)$ will be reserved for the case $\mathcal{S} = \mathcal{Q}$.

From Eq. (2.2), it follows that $f \in \text{BMO}_{\mathcal{S}}(\mathbb{R}^n)$ if and only if $f \in \text{BMO}_{\mathcal{S}}(\mathbb{R}^n)$ whenever $\mathcal{S} \approx \mathcal{T}$. More precisely, if the comparability constants are $c$ and $\tilde{c}$, then

$$c^{-1}\|f\|_{\text{BMO}_{\mathcal{S}}} \leq \|f\|_{\text{BMO}_{\mathcal{S}}} \leq \tilde{c}\|f\|_{\text{BMO}_{\mathcal{S}}}.$$  

We point out that a decreasing function $f$ in $\text{BMO}(\mathbb{R}^+)$ automatically satisfies a stronger condition, called *bounded lower oscillation*, denoted $f \in \text{BLO}(\mathbb{R}^+)$, introduced by Coifman–Rochberg [10]. $\text{BLO}(\Omega)$ is the class of $f \in L^1_{\text{loc}}(\Omega)$ such that

$$\sup_Q \frac{1}{|Q|} \int_Q \left( f - \text{ess inf}_Q f \right) < \infty,$$

where the supremum is taken over all cubes $Q \subset \Omega$. It is a strict subset of $\text{BMO}(\Omega)$ and is not closed under multiplication by negative scalars. For the decreasing rearrangement, any statement about $f^* \in \text{BMO}(\mathbb{R}^+)$ can be interpreted in the stronger sense that $f^* \in \text{BLO}(\mathbb{R}^+)$. For a reference on BMO functions, see [21].

### 2.3. Rearrangeability in BMO

In defining the decreasing rearrangement for functions in BMO, several issues arise. Since mean oscillation is invariant under the addition of constants, while rearrangement is not, the mapping from $f$ to $f^*$ is not a mapping between equivalence classes modulo constants, but between individual functions. This can be avoided, for BMO functions that are bounded below, by considering the map $f \rightarrow (f - \text{inf} f)^*$, which is well-defined modulo constants.

Moreover, functions in BMO need not be rearrangeable. One example is $-\log |x|$, the prototypical unbounded function in $\text{BMO}(\mathbb{R}^n)$. On the other hand, the positive part $(-\log |x|)_+$ is rearrangeable, as is any other BMO-function of compact support since such functions are integrable.

In this paper, we use Definition 2.1 according to which $f^* = |f|^*$. Since functions in BMO are locally integrable, hence finite almost everywhere, a function $f$ is rearrangeable provided that $\mu_f(\alpha) < \infty$ for some $\alpha \geq 0$. This property is preserved under the addition of constants.
3. Decreasing rearrangement

We first present a general result from [8] which guarantees the BMO-boundedness of the decreasing rearrangement under assumptions on the basis $\mathcal{S}$. We then proceed to derive from it bounds with improved constants for several well-known cases, culminating in the proof of Theorem 1.1.

3.1. General boundedness criterion. We start with the definition, in our general setting, of a version of the Calderón–Zygmund decomposition, one of most used tools in harmonic analysis.

**Definition 3.1.** Let $f$ be a nonnegative measurable function on $\Omega$ and $c_* \geq 1$. We say that $\mathcal{S}$ admits a $c_*$-Calderón–Zygmund decomposition for $f$ at a level $\gamma > 0$ if there exist a pairwise-disjoint sequence $\{S_i\} \subset \mathcal{S}$ and a corresponding sequence $\{\tilde{S}_i\} \subset \mathcal{S}$ such that

(i) for all $i$, $\tilde{S}_i \supset S_i$ and $|\tilde{S}_i| \leq c_*|S_i|$;

(ii) for all $i$, $f$ is integrable on $\tilde{S}_i$, with $\int_{\tilde{S}_i} f \leq \gamma \leq \int_{S_i} f$;

and

(iii) $f \leq \gamma$ almost everywhere on $\Omega \setminus \bigcup \tilde{S}_i$.

We now state the boundedness criterion from [8]. The proof given there owes much to the work of Klemes [18] in dimension one, though it does not yield a sharp constant since there is no Rising Sun lemma in this generality. Note that the constant for the bound equals the constant in the Calderón–Zygmund decomposition, clearly demonstrating the dependence on the geometry of the shapes.

**Lemma 3.2 ([8, Theorem 4.4]).** Let $c_* \geq 1$. Suppose that for every nonnegative $g \in L^\infty(\Omega)$ and each $t \in (0, |\Omega|)$, $\mathcal{S}$ admits a $c_*$-Calderón–Zygmund decomposition at level $\gamma = g^*_{(0,t)}$. Then for every rearrangeable function $f \in \text{BMO}_\mathcal{S}(\Omega)$, the decreasing rearrangement $f^*$ is locally integrable, and

$$\|f^*\|_{\text{BMO}} \leq c_* \|f\|_{\text{BMO}_\mathcal{S}}.$$ 

The boundedness of $g$ implies that $g^* \in L^\infty(\mathbb{R}_+) \subset L^1_{\text{loc}}(\mathbb{R}_+)$, so $\gamma$ as defined in this lemma is necessarily finite. For a general rearrangeable $f \in \text{BMO}_\mathcal{S}(\Omega)$, we do not know a priori that $f^*$ is locally integrable and $f^*_{(0,t)}$ is finite.

The following lemma will be used in the construction of the families $\{S_i\}$ and $\{\tilde{S}_i\}$ satisfying conditions (i), (ii), and (iii) of Definition 3.1.

**Lemma 3.3.** Let $0 \leq g \in L^\infty(\mathbb{R}^n)$, $t > 0$, and $\gamma = g^*_{(0,t)}$. Then $g_S \leq \gamma$ for any shape $S$ with $|S| \geq t$.

**Proof.** By the Hardy–Littlewood inequality Eq. (2.1),

$$g_S \leq g^*_{(0,|S|)} \leq g^*_{(0,t)} = \gamma,$$

where the second inequality holds since $g^*$ is decreasing. \qed
3.2. Families of rectangles. We briefly describe the construction of the Calderón–Zygmund decompositions required for certain bases of rectangles (Cartesian products of intervals) in order for Lemma 3.2 to hold.

We start with the collection of cubes, \( Q \), in \( \Omega = \mathbb{R}^n \). Given \( 0 \leq g \in L^\infty(\mathbb{R}^n) \) and \( t > 0 \), we choose a countable collection of cubes of measure \( t \) that partition \( \mathbb{R}^n \) up to a set of measure zero. By Lemma 3.3, the mean of \( g \) over each of these cubes is at most \( \gamma \). On each of these cubes, we follow the proof of the classical Calderón–Zygmund decomposition (see [23, Theorem 1.3.2]) to obtain a \( 2^n \)-Calderón–Zygmund decomposition of cubes for \( g \) at level \( \gamma = f^t \). Therefore, Lemma 3.2 implies that every rearrangeable \( f \in \text{BMO}(\mathbb{R}^n) \) has decreasing rearrangement \( f^* \in \text{BMO}(\mathbb{R}^n) \) with \( \|f^*\|_{\text{BMO}} \leq 2^n \|f\|_{\text{BMO}} \). This improves on the Bennett–DeVore–Sharpley bound in Eq. (1.1) by a constant factor, but falls far short of Theorem 1.1.

For the purpose of comparison, again on \( \Omega = \mathbb{R}^n \), consider the basis \( R \) comprised of rectangles of arbitrary proportions. Let \( 0 \leq g \in L^\infty(\mathbb{R}^n) \) and \( t > 0 \). Applying Lemma 3.3 and the multidimensional analogue of Riesz’ Rising Sun lemma of Korenovskii-Lerner-Stokolos [22], one obtains an extreme case of Definition 3.1, where \( c^* = 1 \) and one has a single countable family \( \{R_i\} = \{\tilde{R}_i\} \) that is both pairwise disjoint and on which the averages of \( g \) can be made equal to \( \gamma \). In this case, Lemma 3.2 implies that every rearrangeable \( f \in \text{BMO}(\mathbb{R}^n) \) has decreasing rearrangement \( f^* \in \text{BMO}(\mathbb{R}_+ \mathbb{R}^n) \), with \( \|f^*\|_{\text{BMO}} \leq \|f\|_{\text{BMO}} \). This inequality was proved on finite rectangles \( R_0 \) by Korenovskii [20].

We point out that when the domain \( \Omega \) is a cube \( Q_0 \) or a rectangle \( R_0 \), the bound on the rearrangement can be used to obtain a John-Nirenberg inequality on \( \Omega \) with constants depending on \( c^* \), in the same way as Korenovskii derived the sharp John–Nirenberg inequality on a rectangle. In such a case \( \Omega \) is itself a shape, and \( \mathcal{S} \) admits a \( c^* \)-Calderón–Zygmund decomposition for every \( g \in L^\infty(\Omega) \) at any level \( \gamma \leq |g|_{\Omega} \), which by Lemma 3.3 includes \( \gamma = g(t_0,t) \) for \( t < |\Omega| \).

Simple examples show that an analogue of Riesz’ Rising Sun lemma cannot exist for cubes [20]. Moreover, we cannot deduce from Korenovskii’s sharp inequality any information about bounds on the decreasing rearrangement on \( \text{BMO}(\mathbb{R}^n) \), because arbitrary rectangles are not comparable to cubes.

3.3. Bisection bases. As a compromise between the rigidity of cubes and the freedom of arbitrary rectangles, we consider particular families of rectangles of bounded eccentricity. If such a family \( \mathcal{S} \) satisfies the assumptions of Lemma 3.2 with constant \( c^* \), and moreover \( \|f\|_{\text{BMO},\mathcal{S}} \leq c \|f\|_{\text{BMO}} \) for some constant \( c \), then it follows that Eq. (1.1) holds with a constant \( C_n \leq c^* c \).

To this end, we introduce a class of bases \( \mathcal{F} \) of rectangles that will be shown to satisfy the assumptions of Lemma 3.2 with constant \( c^* = 2 \), regardless of the dimension.

**Definition 3.4.** We say that a basis \( \mathcal{F} \) in \( \mathbb{R}^n \) is a bisection basis, if for every \( S \in \mathcal{F} \) there is a bisection that splits \( S \) into two shapes in \( \mathcal{F} \).

For such bases, we obtain the following dimension-free bound.
Lemma 3.5. Let $F$ be a bisection basis of rectangles of uniformly bounded eccentricity in $\mathbb{R}^n$ that contains, for each $t > 0$, a partition of $\mathbb{R}^n$ (up to a set of measure zero) into rectangles of measure at least $t$. If $f \in \text{BMO}_F(\mathbb{R}^n)$ is rearrangeable, then $f^* \in \text{BMO}(\mathbb{R}^+)$ with

$$\|f^*\|_{\text{BMO}} \leq 2\|f\|_{\text{BMO}_F}.$$ 

Proof. We need to show that the basis $F$ satisfies the assumptions of Lemma 3.2. Let $0 \leq g \in L^\infty(\mathbb{R}^n)$, $t > 0$, and $\gamma = g^*_{(0,t)}$. Partition $\mathbb{R}^n$ into rectangles from $F$ with measure no smaller than $t$. By Lemma 3.3, the mean of $g$ is at most $\gamma$ for each of these rectangles. Bisect each of these rectangles, select any subrectangles satisfying $g_R \geq \gamma$, and continue to bisect those for which $g_R \leq \gamma$.

The result is a pairwise-disjoint collection of rectangles $R_i \in F$ satisfying $g_{R_i} \geq \gamma$, with their parent rectangles $\widetilde{R}_i \in F$ satisfying $g_{\widetilde{R}_i} \leq \gamma$. By construction $|\widetilde{R}_i| = 2|R_i|$, and the Lebesgue differentiation theorem implies that $g \leq \gamma$ a.e. on $\mathbb{R}^n \setminus \bigcup \widetilde{R}_i$. \hfill \Box

As pointed out earlier, a consequence of this result is the following: if $\|f\|_{\text{BMO}_F} \leq c\|f\|_{\text{BMO}}$ holds with some constant $c > 0$ for some bisection basis $F$, then Eq. (1.1) holds with $C_n \leq 2c$.

Consider the basis $P$ of rectangles with sidelengths given by a permutation of $\{s^{2^{-j/n}} : 0 \leq j \leq n - 1\}$ for some $s > 0$. For example, when $n = 2$, these correspond to rectangles whose sides have lengths $s$ and $s/\sqrt{2}$ (the dimensions of the ISO 216 standard paper size, which includes the commonly used A4 format). The basis $P$ is a bisection basis (with bisection along the longest side) of rectangles of constant eccentricity, and so $\|f^*\|_{\text{BMO}} \leq 2\|f\|_{\text{BMO}_P}$ holds for all rearrangeable $f \in \text{BMO}_P(\mathbb{R}^n)$ by Lemma 3.5. Moreover, these rectangles are comparable with cubes, resulting in a relationship $\|f\|_{\text{BMO}_P} \leq 2^{n+1} \|f\|_{\text{BMO}}$. The constant’s dependence on dimension is a result of the volume ratio obtained by circumscribing a rectangle in $P$ by a cube. This implies an improvement of the constants in Eq. (1.1) to $C_n \leq 2^{n+1}$, but one that does not break past exponential dependence on the dimension.

Consider now the basis $W$ of false cubes in $\mathbb{R}^n$. As defined in [27], a false cube is a rectangle in $\mathbb{R}^n$ with sidelengths $2s$ in the coordinate directions $i = 1, \ldots, m$ and side lengths $s$ in the remaining coordinate directions. Here, $s > 0$, and $1 \leq m \leq n$. This is also a bisection basis: for an arbitrary false cube, bisecting it along the $m$th coordinate results in two congruent rectangles in $W$. Moreover, rectangles in $W$ have bounded eccentricity and so Lemma 3.5 implies that

$$(3.1) \quad \|f^*\|_{\text{BMO}} \leq 2\|f\|_{\text{BMO}_W}$$

for all rearrangeable $f \in \text{BMO}_W(\mathbb{R}^n)$.

In order to obtain an improvement of the constants in Eq. (1.1), it remains to investigate the relationship between $\|\cdot\|_{\text{BMO}_W}$ and $\|\cdot\|_{\text{BMO}}$. As a first step, we estimate the oscillation of $f$ over a false cube in terms of its means over its subcubes.
Lemma 3.6. Let $R$ be a false cube, given (up to a set of measure zero) by a disjoint union $R = \bigcup_{\nu} Q(\nu)$ of $2^m$ cubes of equal size. Then
\[
\frac{1}{2^m} \sum_{\nu} |f_{Q(\nu)} - f_R| \leq O(f, R) \leq \|f\|_{\text{BMO}} + \frac{1}{2^m} \sum_{\nu} |f_{Q(\nu)} - f_R|
\]
for all $f \in \text{BMO}(\mathbb{R}^n)$.

Proof. For the lower bound, we write
\[
O(f, R) = \sum_{\nu} \frac{|Q(\nu)|}{|R|} \int_{Q(\nu)} |f - f_R| \geq \frac{1}{2^m} \sum_{\nu} |f_{Q(\nu)} - f_R| .
\]
For the upper bound, we use the triangle inequality to obtain
\[
O(f, R) \leq \frac{1}{2^m} \sum_{\nu} \int_{Q(\nu)} (|f - f_{Q(\nu)}| + |f_{Q(\nu)} - f_R|)
\]
\[
\leq \sup_{\nu} O(f, Q(\nu)) + \frac{1}{2^m} \sum_{\nu} |f_{Q(\nu)} - f_R| ,
\]
and bound the first term by $\|f\|_{\text{BMO}}$. □

The next lemma gives a dimension-free bound on the difference between the means of a function $f \in \text{BMO}(\mathbb{R}^n)$ on a pair of neighbouring cubes. In [27, Lemma 3], Wik obtains the analogous estimate for the medians, with the constant 6.

Lemma 3.7. Let $Q_1, Q_2$ be two cubes of equal size in $\mathbb{R}^n$ that share an $(n - 1)$-dimensional face. Then $|f_{Q_1} - f_{Q_2}| \leq 4\|f\|_{\text{BMO}}$.

Proof. Let $Q_0$ be the cube of the same size as $Q_1, Q_2$ that is bisected by the common interface. We will prove that
\[
|f_{Q_j} - f_{Q_0}| \leq 2\|f\|_{\text{BMO}}, \quad j = 1, 2,
\]
and then apply the triangle inequality.

Let $A_1 = Q_1 \setminus Q_0, A_0 = Q_0 \setminus Q_1, \text{ and } B_1 = Q_1 \cap Q_0$. By construction, $|A_1| = |A_0| = |B_1| = \frac{1}{2}|Q_1|$. By linearity of the mean,
\[
f_{Q_j} = \frac{1}{2}(f_{A_j} + f_{B_1}), \quad j = 0, 1.
\]
On the other hand, in the same way as in the proof of Lemma 3.6
\[
\frac{1}{2}(|f_{A_j} - f_{Q_j}| + |f_{B_1} - f_{Q_j}|) \leq O(f, Q_j) \leq \|f\|_{\text{BMO}}, \quad j = 0, 1.
\]
It follows that
\[
|f_{A_j} - f_{Q_j}| = |f_{B_1} - f_{Q_j}| \leq \|f\|_{\text{BMO}}, \quad j = 0, 1,
\]
and we conclude that
\[
|f_{Q_1} - f_{Q_0}| \leq |f_{Q_1} - f_{B_1}| + |f_{B_1} - f_{Q_0}| \leq 2\|f\|_{\text{BMO}}.
\]
By the same argument, $|f_{Q_2} - f_{Q_0}| \leq 2\|f\|_{\text{BMO}}$. □
We make use of the following concentration inequality.

**Lemma 3.8.** Let \( \{X_1, \ldots, X_m\} \) be \( m \) independent random variables on a probability space, and let \( g = g(X_1, \ldots, X_m) \) be a random variable of finite expectation. If there are constants \( a_1, \ldots, a_m \) such that

\[
|g(x_1, \ldots, x_i, \ldots, x_m) - g(x_1, \ldots, x'_i, \ldots, x_m)| \leq a_i
\]

for all \( x_i, x'_i \) and all \( 1 \leq i \leq m \), then

\[
\mathbb{E} |g - \mathbb{E} g| \leq \frac{1}{2} \|a\|_2,
\]

where \( \|a\|_2 = (\sum a_i^2)^{\frac{1}{2}} \) is the Euclidean length of the vector \((a_1, \ldots, a_m) \in \mathbb{R}^m\).

**Proof.** We estimate the variance of \( g(X_1, \ldots, X_m) \) using a martingale argument. Let \( \mu_i \) be the probability distribution of \( X_i \). The difference \( g - \mathbb{E} g \) can be written as a telescoping sum,

\[
g - \mathbb{E} g = \sum_{i=1}^m (g_i - g_{i-1}),
\]

where \( g_i(x_1, \ldots, x_i) := \mathbb{E} g(x_1, \ldots, x_i, X_{i+1}, \ldots, X_m) \). By construction, \( g_0 = \mathbb{E} g \) and \( g_m = g \).

Consider the differences \( Y_i := g_i(X_1, \ldots, X_i) - g_{i-1}(X_1, \ldots, X_{i-1}) \). By independence,

\[
\int g_i(x_1, \ldots, x_i) \, d\mu_i(x_i) = g_{i-1}(x_1, \ldots, x_{i-1}),
\]

which implies \( \mathbb{E} Y_i = 0 \). Moreover, for each pair \( i < j \), we have by a similar calculation

\[
\int (g_i(x_1, \ldots, x_i) - g_{i-1}(x_1, \ldots, x_{i-1}))(g_j(x_1, \ldots, x_j) - g_{j-1}(x_1, \ldots, x_{j-1})) \, d\mu_j(x_j) = 0,
\]

which implies that \( \text{Cov}(Y_i, Y_j) = \mathbb{E}(Y_i Y_j) = 0 \) for \( 1 \leq i < j \leq m \).

We next estimate the variance of \( Y_i \). By Eq. \((3.2)\), we have for any \( x_1, \ldots, x_i \),

\[
\int (g_i(x_1, \ldots, x_i) - g_{i-1}(x_1, \ldots, x_{i-1}))^2 \, d\mu_i(x_i) = \inf_{c \in \mathbb{R}} \int (g_i(x_1, \ldots, x_i) - c)^2 \, d\mu_i(x_i)
\]

\[
\leq \frac{1}{4} a_i^2.
\]

The first inequality holds by definition of \( g_{i-1} \) as a mean, and the second inequality follows by choosing \( c \) as the arithmetic mean of the extremal values of \( g(x_1, \ldots, x_{i-1}, \cdot) \). Integrating over the remaining variables, we obtain \( \text{Var} Y_i \leq \frac{1}{4} a_i^2 \).

We conclude, using that the random variables \( Y_i \) are uncorrelated, that

\[
\mathbb{E}(g - \mathbb{E} g)^2 = \text{Var} g(X_1, \ldots, X_m) = \sum_{i=1}^m \text{Var} Y_i \leq \frac{1}{4} \|a\|_2^2.
\]

By Schwarz’ inequality, this implies Eq. \((3.3)\). \( \square \)

The previous three lemmas combine to give a simple proof of Eq. \((1.3)\).

**Lemma 3.9.** Let \( f \in \text{BMO}(\mathbb{R}^n) \). Then \( \|f\|_{\text{BMO}} \leq (1 + 2\sqrt{n-1}) \|f\|_{\text{BMO}} \).
Proof. Let \( f \in \text{BMO}(\mathbb{R}^n) \) and \( R \) be an arbitrary false cube that has sidelength \( 2s \) in the first \( m \) directions, \( 1 \leq m \leq n \), and sidelength \( s \) in the remaining \( n - m \) dimensions. If \( m = n \), then \( R \) is a cube and there is nothing to show.

If \( 1 \leq m < n \), we write \( R \) as a union of \( 2^m \) subcubes \( Q(\nu) \) indexed by binary strings \( \nu \in \{0, 1\}^m \).

By Lemma 3.6,
\[
\int_R |f - f_R| \leq \|f\|_{\text{BMO}} + \frac{1}{2^m} \sum_{\nu \in \{0, 1\}^m} |f_{Q(\nu)} - f_R|.
\]

Set \( g(\nu) := f_{Q(\nu)} \) for \( \nu \in \{0, 1\}^m \). We consider \( g \) as a random variable on the space of binary strings \( \{0, 1\}^m \), equipped with the uniform measure. and verify the hypotheses of Lemma 3.8. Note that \( f_R = \mathbb{E}g \) by the linearity of expectation. The components \((X_1, \ldots, X_m)\) of a random binary string are Bernoulli random variables with bias \( p = \frac{1}{2} \). By Lemma 3.7,
\[
|f_Q - f_Q'| \leq 4\|f\|_{\text{BMO}}
\]
for any pair of adjacent cubes. Therefore, Eq. (3.2) holds with \( a_i = 4\|f\|_{\text{BMO}} \) for \( i = 1, \ldots, m \), and we obtain from Lemma 3.8 that \( \mathbb{E}|g - \mathbb{E}g| \leq 2\sqrt{m}\|f\|_{\text{BMO}} \); that is,
\[
\frac{1}{2^m} \sum_{\nu \in \{0, 1\}^m} |f_{Q(\nu)} - f_R| \leq 2\sqrt{m}\|f\|_{\text{BMO}}.
\]

Since \( m \leq n - 1 \), this concludes the proof. \( \square \)

We finally have all the necessary components to prove Theorem 1.1.

Proof of Theorem 1.1 This follows from a combination of Eq. (3.1) and Lemma 3.9. \( \square \)

4. Symmetric decreasing rearrangement

This section is dedicated to the proof of Theorem 1.2. We consider BMO with respect to a basis \( \mathscr{A} \) that is adapted to the radial symmetry of \( Sf \). This basis consists of all balls centred at the origin, along with annular sectors obtained by intersecting an annulus with a suitable spherical cone.

For \( x \in \mathbb{R}^n \setminus \{0\} \), \( 0 < \rho \leq |x| \), and \( \alpha \in (0, \frac{\pi}{2}] \), write
\[
A(x, \rho, \alpha) := \{ y \in \mathbb{R}^n : |x| - \rho < |y| < |x| + \rho, y \cdot x > |x||y| \cos \alpha \}.
\]
The set \( A(x, \rho, \alpha) \) is the sector of width \( 2\rho \) and aperture \( \alpha \) centred at \( x \). Then
\[
\mathscr{A} = \{ B(0, r) : r > 0 \} \cup \left\{ A(x, \rho, \alpha) : \rho = |x| \sin \alpha, \alpha \in \left(0, \frac{\pi}{2}\right) \right\}
\]
is a basis of shapes in \( \mathbb{R}^n \).

This basis captures those sets on which the symmetric decreasing rearrangement \( Sf \) can be compared directly with \( f^* \). More generally, \( \text{BMO}(\mathbb{R}^n) \) is isometric to the subspace of radial functions in \( \text{BMO}_{\mathscr{A}}(\mathbb{R}^n) \).
Lemma 4.1. Let \( f \) be a radial function on \( \mathbb{R}^n \), defined by \( f(x) = g(\omega_n|x|^n) \). Then, \( f \in \text{BMO}_\mathcal{A}(\mathbb{R}^n) \) if and only if \( g \in \text{BMO}(\mathbb{R}^+) \), with
\[
\|f\|_{\text{BMO}_\mathcal{A}} = \|g\|_{\text{BMO}}.
\]

Proof. Integration over any shape \( A \in \mathcal{A} \) can be represented by an integral in spherical coordinates over \( J \times S \), where \( J = (a,b) \subset \mathbb{R}^+ \) is an interval and \( S \) is either the entire sphere or a spherical cap. Let \( \nu \) be the measure on \( \mathbb{R}^+ \) with density \( n\omega_n r^{n-1} \). Since \( f \) is radial, two changes of variables give
\[
f_A = \int_A f = \int_J g(\omega_n r^n) d\nu(r) = \int_I g = g_I,
\]
where \( I = (\omega_n a^n, \omega_n b^n) \) (see \cite{16} Corollary 2.51). In the same way, \( \mathcal{O}(f, A) = \mathcal{O}(g, I) \), from where it follows that \( \|f\|_{\text{BMO}_\mathcal{A}} \leq \|g\|_{\text{BMO}} \).

Conversely, given an open interval \( I \subset (0, \infty) \), set \( J = \{ s > 0 : \omega_n s^n \in I \} \). We distinguish between two cases. If the left endpoint of \( J \) is at 0, we take \( A \) to be the ball of measure \( |I| \) centred at the origin. Otherwise, we choose \( A = A(Re_1, \rho, \alpha) \), where \( R \) is the centre of \( J \), \( \rho = \frac{1}{2}|J| < R \), \( \alpha = \arcsin(\rho/R) \), and \( e_1 \) is the standard unit vector \( e_1 = (1, 0, \ldots, 0) \). Reversing the calculation above, we see that \( \mathcal{O}(g, I) = \mathcal{O}(f, A) \), and hence \( \|g\|_{\text{BMO}} \leq \|f\|_{\text{BMO}_\mathcal{A}} \). \( \square \)

Consider two rearrangeable functions \( f_1, f_2 \). The function \( Sf_1 - Sf_2 \) is radial, and satisfies
\[
(Sf_1 - Sf_2)(x) = (f_1^* - f_2^*)(\omega_n|x|^n)
\]
by the definition of the symmetric decreasing rearrangement. Lemma 4.1 implies that
\[
\|Sf_1 - Sf_2\|_{\text{BMO}_\mathcal{A}} = \|f_1^* - f_2^*\|_{\text{BMO}}.
\]
We next compare \( \mathcal{A} \) with the basis of balls, \( \mathcal{B} \), and hence with the standard basis \( \mathcal{Q} \).

Lemma 4.2. The basis \( \mathcal{A} \) is equivalent to the basis \( \mathcal{B} \).

Proof. We first show that for each ball \( B \) there exist \( A \in \mathcal{A} \) and \( \tilde{B} \in \mathcal{B} \) such that
\[
B \subset A \subset \tilde{B} \quad \text{and} \quad |A| < |\tilde{B}| = 2^n|B|,
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Equivalence of the bases \( \mathcal{A} \) and \( \mathcal{B} \).}
\end{figure}
see Figure 1.

Let a ball \( B = B(x, r) \) be given. If \(|x| < r\), then \( B \) contains the origin. In this case, we choose \( A \) to be the ball of radius \(|x| + r\) centred at the origin and \( \tilde{B} \) to be the ball of radius \( 2r \) centred at the origin. It follows that \(|A| < |\tilde{B}| = 2^n|B|\), where the inequality is strict because \( A \) is a proper subset of \( \tilde{B} \). Since \( A \) exhausts \( \tilde{B} \) as \(|x|\) approaches \( r \) from below, the constant \( 2^n \) is sharp.

If \(|x| \geq r\), then the ball does not contain the origin. In that case, we take \( A = A(x, r, \alpha) \) where \( \alpha = \arcsin \left( \frac{r}{|x|} \right) \), and \( \tilde{B} = B(\tilde{x}, 2r) \), where \( \tilde{x} = (\cos \alpha) x \). Clearly, \( A \in \mathcal{A} \). We begin by showing that \( B \subset A \). To that end, fix \( y \in B \). The triangle inequality implies that

\[
|x| - r < |x| - |x - y| \leq |y| \leq |x| + |x - y| < |x| + r.
\]

Writing \( x = |x| \xi \) and \( y = |y| \eta \), where \( \eta \) and \( \xi \) are unit vectors, it follows that

\[
r^2 > |y - x|^2 = |x|^2 + |y|^2 - 2|x||y| \xi \cdot \eta \geq |x|^2 (1 - (\xi \cdot \eta)^2).
\]

Hence,

\[
(\xi \cdot \eta)^2 > 1 - \frac{r^2}{|x|^2} = 1 - \sin^2 \alpha = \cos^2 \alpha.
\]

Since \( B \) does not include the origin, the angle between any vector terminating in \( B \) and the vector terminating at the centre of \( B \) is at most \( \pi/2 \). Thus, \( x \cdot y > 0 \) and so \( \xi \cdot \eta > 0 \). It follows from this and Eq. (4.4) that \( \xi \cdot \eta > \cos \alpha \). Together with Eq. (4.3), this implies that \( y \in A \), demonstrating that \( B \subset A \).

To see that \( A \subset \tilde{B} \), we bound the distance to any point \( y = |y| \eta \in A \) from \( \tilde{x} = (\cos \alpha)|x| \xi \). Since both \( x \in B \subset A \) and \( y \in A = A(x, r, \alpha) \), we have \( \xi \cdot \eta > \cos \alpha \). Using that \(|\tilde{x}|^2 = |x|^2 - r^2\), we calculate

\[
|\tilde{x} - y|^2 < |x|^2 - r^2 + |y|^2 - 2|x||y| \cos^2 \alpha = (|x| - |y|)^2 - r^2 + 2|x||y| \sin^2 \alpha.
\]

Since \(|x| - r < |y| < |x| + r\) and \(|x| \sin \alpha = r\), it follows that

\[
(|x| - |y|)^2 - r^2 + 2|x||y| \sin^2 \alpha < 2|x|(|x| + r) \sin^2 \alpha < 4r^2.
\]

Therefore \( A \subset B(\tilde{x}, 2r) = \tilde{B} \). This proves Eq. (4.2).

In the other direction, given \( A \in \mathcal{A} \), we will construct \( B, \tilde{B} \in \mathcal{B} \) such that

\[
B \subset A \subset \tilde{B} \quad \text{and} \quad |\tilde{B}| = 2^n|B| \leq 2^n|A|.
\]

If \( A \) is a centred ball, we take \( B = A \) and \( \tilde{B} \) to be the centred ball of twice the radius. If \( A = A(x, \rho, \alpha) \) with \( \rho = |x| \sin \alpha \), we take \( B = B(x, \rho) \) and \( \tilde{B} = B(\tilde{x}, 2\rho) \), where \( \tilde{x} = (\cos \alpha) x \). The same calculations as above show that Eq. (4.5) holds for this choice of \( B \) and \( \tilde{B} \).

We can now establish the bi-Lipschitz equivalence between rearrangements.
Proof of Theorem 1.2. Let $f_1, f_2$ be rearrangeable functions. From Lemmas 2.4 and 4.2, along with Eq. (2.4), it follows that

$$2^{-n} \omega_n \|Sf_1 - Sf_2\|_{\text{BMO}_B} \leq \|Sf_1 - Sf_2\|_{\text{BMO}} \leq 2^{-n} n^{\frac{n}{2}} \omega_n \|Sf_1 - Sf_2\|_{\text{BMO}_B}$$

and

$$2^{-n} \|Sf_1 - Sf_2\|_{\text{BMO}_{A'}} \leq \|Sf_1 - Sf_2\|_{\text{BMO}_B} \leq 2^n \|Sf_1 - Sf_2\|_{\text{BMO}_{A'}}.$$ 

Combining these with Eq. (4.1) yields Eq. (1.5). \qed

We also have the following local version of Theorem 1.2.

Lemma 4.3. Let $R > 0$ and $Q \subset B(0, R)$ be a cube of diameter $d$, centred at a point $x$ with $|x| \leq R - d/2$. There is an interval $I \subset (0, \omega_n R^n)$ of length $|I| \leq n \omega_n R^{n-1} d$, such that if $f_1, f_2$ are rearrangeable, then

$$\mathcal{O}(Sf_1 - Sf_2, Q) \leq n^{\frac{n}{2}} \omega_n \mathcal{O}(f_1^* - f_2^*, I).$$

Proof. Let $Q$ be as in the statement of the lemma. Lemmas 2.4 and 4.2, along with Eq. (2.2), imply that

$$\mathcal{O}(Sf_1 - Sf_2, Q) \leq n^{\frac{n}{2}} \omega_n \mathcal{O}(Sf_1 - Sf_2, A),$$

where $A \in \mathcal{A}$ is constructed, as in the proof of Lemma 4.2, from the ball $B(x, d/2)$ and satisfies $Q \subset B \subset A \subset B(0, R)$ and $|A| \leq n^{\frac{n}{2}} \omega_n |Q| = \omega_n d^n$. As in the proof of Lemma 4.1, $A$ is represented in polar coordinates by $J \times S$, where $J \subset (0, R)$ is an interval of length at most $d$, and $S$ lies in the unit sphere. Moreover,

$$\mathcal{O}(Sf_1 - Sf_2, A) = \mathcal{O}(f_1^* - f_2^*, I),$$

where the interval $I$ is the image of $J$ under the map $r \mapsto \omega_n r^n$. The length of this interval is bounded by $|I| \leq n \omega_n R^{n-1} |J|$. \qed

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REFERENCES

[1] Aubin, T. Problèmes isopérimétriques et espaces de Sobolev. J. Differential Geom. 11 (1976), no. 4, 573-598.
[2] Baernstein, A., II. Symmetrization in analysis. With David Drasin and Richard S. Laugesen. With a foreword by Walter Hayman. New Mathematical Monographs, 36. Cambridge University Press, Cambridge (2019)
[3] Bennett, C.; DeVore, R. A.; Sharpley, R. Weak-$L^\infty$ and BMO. Ann. of Math. (2) 113 (1981), no. 3, 601-611.
[4] Bennett, C.; Sharpley, R. Interpolation of operators. Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988. xiv+469 pp.
[5] Bourgain, J.; Mirek, M.; Stein, E. M.; Wróbel, B. On dimension-free variational inequalities for averaging operators in $\mathbb{R}^d$. Geom. Funct. Anal. 28 (2018), no. 1, 58-99.
[6] Bourgain, J.; Mirek, M.; Stein, E. M.; Wróbel, B. Dimension-free estimates for discrete Hardy-Littlewood averaging operators over the cubes in $\mathbb{Z}^d$. *Amer. J. Math.* 141 (2019), no. 4, 857-905.

[7] Lieb, E. H. Sharp Constants in the Hardy-Littlewood-Sobolev and Related Inequalities. *Ann. Math.* 118 (1983), no. 2, 349-374.

[8] Burchard, A.; Dafni, G.; Gibara, R. A note on decreasing rearrangement and mean oscillation on measure spaces. *Proc. Amer. Math. Soc.* to appear. DOI: 10.1090/proc/15505

[9] Carlen, E. A.; Loss, M. Extremals of functionals with competing symmetries. Journal of Functional Analysis 88, no. 2 (1990): 437-456.

[10] Coifman, R. R.; Rochberg, R. Another characterization of BMO. *Proc. Amer. Math. Soc.* 79 (1980), no. 2, 249-254.

[11] Cwikel, M.; Sagher, Y.; Shvartsman, P. A new look at the John-Nirenberg and John-Strömberg theorems for BMO. *J. Funct. Anal.* 263 (2012), no. 1, 129-166.

[12] Dafni, G.; Gibara, R. BMO on shapes and sharp constants. *Advances in harmonic analysis and partial differential equations*, 1-33, Contemp. Math., 748, *Amer. Math. Soc.*, Providence, RI, 2020.

[13] Dafni, G.; Hytönen, T.; Korte, R.; Yue, H. The space $JN_p$: nontriviality and duality. *J. Funct. Anal.*, 275 (2018), 577-603.

[14] Fefferman, C. Characterizations of bounded mean oscillation. *Bull. Amer. Math. Soc.* 77 (1971), 587-588.

[15] Fefferman, C.; Stein, E. M. $H^p$ spaces of several variables. *Acta Math.* 129 (1972), no. 3-4, 137-193.

[16] Folland, G. B. Real analysis. Modern techniques and their applications. Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. *John Wiley & Sons, Inc.*, New York, 1999. xvi+386 pp.

[17] John, F.; Nirenberg, L. On functions of bounded mean oscillation. *Comm. Pure Appl. Math.* 14 (1961), 415-426.

[18] Klemes, I. A mean oscillation inequality. *Proc. Amer. Math. Soc.* 93 (1985), no. 3, 497-500.

[19] Korenovskii, A. A. The connection between mean oscillations and exact exponents of summability of functions. (Russian) *Mat. Sb.* 181 (1990), no. 12, 1721-1727; translation in *Math. USSR-Sb.* 71 (1992), no. 2, 561-567.

[20] Korenovskii, A. A. The Riesz “rising sun” lemma for several variables, and the John-Nirenberg inequality (in Russian). *Mat. Zametki* 77 (2005), no. 1, 53-66; translation in *Math. Notes* 77 (2005), no. 1-2, 48-60.

[21] Korenovskii, A. Mean oscillations and equimeasurable rearrangements of functions. Lecture Notes of the Unione Matematica Italiana, 4. *Springer, Berlin: UMI, Bologna*, 2007. viii+188 pp.

[22] Korenovskyy, A. A.; Lerner, A. K.; Stokolos, A. M. On a multidimensional form of F. Riesz’s “rising sun” lemma. *Proc. Amer. Math. Soc.* 133 (2005), no. 5, 1437-1440.

[23] Stein, E. M. Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 *Princeton University Press*, Princeton, N.J. 1970 xiv+290 pp.

[24] Stein, E. M.; Weiss, G. Introduction to Fourier analysis on Euclidean spaces. Princeton Mathematical Series, No. 32. *Princeton University Press*, Princeton, N.J., 1971. x+297 pp.

[25] Stolyarov, D. M.; Vasyunin, V. I.; Zaititskiy, P. B. Monotonic rearrangements of functions with small mean oscillation. *Studia Math.* 231 (2015), no. 3, 257-267.

[26] Talenti, G. Best constant in Sobolev inequality. Annali di Matematica pura ed Applicata 110 (1976) No.1, 353-372.

[27] Wik, I. On John and Nirenberg’s theorem. *Ark. Mat.* 28 (1990), no. 1, 193-200.
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