Stokes Matrices for the Quantum Cohomologies of Grassmannians

Kazushi Ueda

Abstract

We prove the conjectural relation between the Stokes matrix for the quantum cohomology of $X$ and an exceptional collection generating $D^b\text{coh}(X)$ when $X$ is the Grassmannian $\text{Gr}(r, n)$. The proof is based on the relation between the quantum cohomology of the Grassmannian and that of the projective space.

1 Introduction

Gromov-Witten invariants of homogeneous spaces contain enumerative information such as the number of nodal rational curves of a given degree passing through a given set of points in general position. The theory of Frobenius manifold allows a systematic treatment of these invariants. A Frobenius manifold is a complex manifold whose tangent bundle has a holomorphic bilinear form and an associative commutative product with certain compatibility conditions. From these compatibility conditions, it follows that there is a function on the Frobenius manifold, called the potential, whose third derivatives give the structure constants of the product.

Given a symplectic manifold $X$, one can endow a Frobenius structure on its total cohomology group $H^*(X; \mathbb{C})$. In this case, the holomorphic bilinear form is given by the Poincaré pairing and the potential is the generating function of the genus-zero Gromov-Witten invariants. The product structure in this case is called the quantum cohomology ring. It is a deformation of the cohomology ring parametrized by $H^*(X; \mathbb{C})$ itself.

Given a Frobenius manifold, one can construct the following isomonodromic family of ordinary differential equations on $\mathbb{P}^1$:

$$\frac{\partial \Phi}{\partial \hbar} = \left( \frac{1}{\hbar} U + \frac{1}{\hbar^2} V \right) \Phi,$$

(1)
\[ \hbar \frac{\partial \Phi}{\partial t_\alpha} = \frac{\partial}{\partial t_\alpha} \circ \Phi, \quad \alpha = 0, \ldots, N - 1. \quad (2) \]

Here, \( \Phi \) is the unknown function on \( \mathbb{P}^1 \) times the Frobenius manifold taking value in the tangent bundle of the Frobenius manifold, \( \hbar \) is the coordinate on \( \mathbb{P}^1 \), \( N \) is the dimension of the Frobenius manifold and \( \{t_\alpha\}_{\alpha=0}^{N-1} \) is the flat coordinate of the Frobenius manifold. The circle denotes the product on the tangent bundle and \( U, V \) are certain operators acting on sections of the tangent bundles. See Dubrovin [3] for details. Note that \( z \) loc. cit. is \( 1/\hbar \) in this paper. (1) is an ordinary differential equation on \( \mathbb{P}^1 \) with a regular singularity at infinity and an irregular singularity at the origin, and (2) gives its isomonodromic deformation. If a point on the Frobenius manifold is semisimple, i.e., if there are no nilpotent elements in the product structure on the tangent space at this point, one can define the monodromy data of (1) at this point, consisting of the monodromy matrix at infinity, the Stokes matrix at the origin and the connection matrix between infinity and the origin. These data do not depend on the choice of a semisimple point because of the isomonodromicity.

The following conjecture, originally due to Kontsevich, developed by Zaslov [12], and formulated into the following form by Dubrovin [4], reveals a striking connection between the Gromov-Witten invariants and the derived category of coherent sheaves:

**Conjecture 1.1.** The quantum cohomology of a smooth projective variety \( X \) is semisimple if and only if the bounded derived category \( D^b_{\text{coh}}(X) \) of coherent sheaves on \( X \) is generated as a triangulated category by an exceptional collection \((\mathcal{E}_i)_{i=1}^N\). In such a case, the Stokes matrix \( S \) for the quantum cohomology of \( X \) is given by

\[ S_{ij} = \sum_k (-1)^k \dim \text{Ext}^k(\mathcal{E}_i, \mathcal{E}_j). \quad (3) \]

An exceptional collection appearing above is the following:

**Definition 1.2.** 1. An object \( \mathcal{E} \) in a triangulated category is exceptional if

\[ \text{Ext}^i(\mathcal{E}, \mathcal{E}) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases} \]

2. An ordered set of objects \((\mathcal{E}_i)_{i=1}^N\) in a triangulated category is an exceptional collection if each \( \mathcal{E}_i \) is exceptional and \( \text{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0 \) for any \( i > j \) and any \( k \).
To our knowledge, Conjecture 1.1 was previously known to hold only for projective spaces \([5], [7]\). The main result in this paper is:

**Theorem 1.3.** Conjecture 1.1 holds for the Grassmannian \(Gr(r, n)\) of \(r\)-dimensional subspaces in \(\mathbb{C}^n\).

The proof consists of explicit computations on both sides of (3). The computation on the left hand side relies on the following two results: The first is a conjecture of Hori and Vafa [8], proved by Bertram, Ciocan-Fontanine and Kim [2], describing the solution of (1) for the Grassmannian \(Gr(r, n)\) in terms of that of the product of projective spaces \((\mathbb{P}^{n-1})^r\). The second is the Stokes matrix for the quantum cohomology of projective spaces obtained by Dubrovin [3] for the projective plane and by Guzzetti [7] in any dimensions. By combining these two results, we can compute the Stokes matrix for the quantum cohomology of the Grassmannian.

On the right hand side, we have an exceptional collection generating \(D^b\text{coh}(Gr(r, n))\) by Kapranov [9]. It consists of equivariant vector bundles on \(Gr(r, n)\) and Ext-groups between them can be computed by the Borel-Weil theory.

Both of the above computations can be carried out for any \(r\) and \(n\), and Conjecture 1.1 reduces to the combinatorial identity in Corollary 4.3.

**Acknowledgements:** We thank A. N. Kirillov for providing the proof of the identity in Corollary 4.3 and for allowing us to include it in this paper. We also thank H. Iritani, T. Kawai, Y. Konishi, T. Maeno, K. Saito and A. Takahashi for valuable discussions and comments. The author is supported by JSPS Fellowships for Young Scientists No.15-5561.

## 2 Stokes matrix from the Hori-Vafa conjecture

Let us begin with the discussion of the Stokes matrix. Fix a semisimple point on a Frobenius manifold. The differential equation (1) has a regular singularity at infinity and an irregular singularity at the origin, and the Stokes matrix is the monodromy data for the irregular singularity at the origin, defined as follows: First, fix a formal fundamental solution \(\Phi_{\text{formal}}\) of the form

\[
\Phi_{\text{formal}}(h) = \Psi R(h) \exp[U/h]
\]

where

\[
U = \text{diag}(u_1, \ldots, u_N),
\]
\{u_i\}_{i=1}^N is the canonical coordinate, \(\Psi\) is the coordinate transformation matrix from the flat coordinate to the normalized canonical coordinate and \(R(h) = (1 + R_1 h + R_2 h^2 + \cdots)\) is a formal series satisfying

\[ R'(h)R(-h) = 1. \]

Here, \(\dagger\) denotes the transpose of a matrix. By \([5]\), Lemma 4.3., such \(R(h)\) exists uniquely. Here we have taken the local trivialization of the tangent bundle given by the normalized canonical coordinate and regarded \(\Phi\) as an \(n \times n\) matrix-valued function.

**Definition 2.1.** For \(0 \leq \phi < \pi\), a straight line \(l = \{h \in \mathbb{C}^\times \mid \arg(h) = \phi, \phi - \pi\}\) passing through the origin is called admissible if the line through \(u_k\) and \(u_k'\) is not orthogonal to \(l\) for any \(k \neq k'\).

Fix such a line, and choose a small enough number \(\epsilon > 0\) so that any line passing through the origin with angle between \(\phi - \epsilon\) and \(\phi + \epsilon\) is admissible. Define

\[
D_{\text{right}} = \{h \in \mathbb{C}^\times \mid \phi - \pi - \epsilon < \arg(h) < \phi + \epsilon\}, \\
D_{\text{left}} = \{h \in \mathbb{C}^\times \mid \phi - \epsilon < \arg(h) < \phi + \pi + \epsilon\}, \\
D_- = \{h \in \mathbb{C}^\times \mid \phi - \pi - \epsilon < \arg(h) < \phi - \pi + \epsilon\}. 
\]
Since the singularity at the origin is irregular, the formal solution $\Phi_{\text{formal}}(h)$ does not converge. Nevertheless, by [4], Theorem 4.2., there exist unique solutions $\Phi_{\text{right}}(h)$ and $\Phi_{\text{left}}(h)$, defined on the angular domains $D_{\text{right}}$ and $D_{\text{left}}$ respectively, which asymptote to the same formal solution:

$$
\Phi_{\text{right/left}} \sim \Phi_{\text{formal}} \text{ as } h \to 0 \text{ in } D_{\text{right/left}}.
$$

Since these two solutions satisfy the same linear differential equation on $D_-$, there exists a matrix $S$ independent of $h$ such that

$$
\Phi_{\text{right}}(h) = \Phi_{\text{left}}(h)S, \quad h \in D_-
$$

This matrix $S$ is called the Stokes matrix. Although locally on the Frobenius manifold this Stokes matrix does not depend on the choice of a semisimple point by [5], Isomonodromicity Theorem (second part), it undergoes a discrete change as we vary the point on the Frobenius manifold so that it crosses the point where the line $l$ we have fixed at the beginning is not admissible any more. This change in the Stokes matrix is described by an action of the braid group $B_N$ (the number of strands is the dimension of the Frobenius manifold).

In the case of the projective space $\mathbb{P}^{n-1}$, semisimplicity of the quantum cohomology is well-known. The solution to (1), (2) has an integral representation by Givental:

**Theorem 2.2 (Givental [6]).** Let

$$
W(x_1, \ldots, x_{n-1}) = x_1 + \cdots + x_{n-1} + \frac{e^t}{x_1 \cdots x_{n-1}}
$$

be a function on $(\mathbb{C}^*)^{n-1}$ depending on a parameter $t \in \mathbb{C}$ and choose a basis $\{\Gamma_i\}_{i=1}^n$ of the space of flat sections of the relative homology bundle (the flat bundle on the $h$-plane whose fiber over $h \in \mathbb{C}^*$ is $H_{n-1}((\mathbb{C}^*)^{n-1}, \mathfrak{R}(W/h) = -\infty)$). Let $\{p^\alpha\}_{\alpha=0}^{n-1}$ be the basis of $H^*(\mathbb{P}^{n-1}; \mathbb{Z})$ such that $p^\alpha \in H^{2\alpha}(\mathbb{P}^{n-1}; \mathbb{Z})$. For $k = 1, \ldots, n$, define a cohomology-valued function $I_k$ by

$$
I_k = \sum_{\alpha=0}^{n-1} p^\alpha \int_{\Gamma_k} (h\frac{d}{dt})^\alpha \exp[W(x_1, \ldots, x_{n-1})/h] \frac{dx_1 \cdots dx_{n-1}}{x_1 \cdots x_{n-1}}. \quad (6)
$$

Then $(I_k)_{k=1}^n$ gives a fundamental solution to (1), (2) where $t$ is the coordinate of $H^2(\mathbb{P}^{n-1}; \mathbb{C})$ and all the other flat coordinates are set to zero.

Note that since the relative homology bundle has a monodromy, $\Gamma_k$’s (and hence $I_k$’s) cannot be defined globally. The above integral representation is related to the Stokes matrix in the following way: Fix $\phi$ and $\epsilon$
such that any line passing through the origin with angle between $\phi - \epsilon$ and $\phi + \epsilon$ is admissible. There are $n$ critical points and their critical values are the canonical coordinate $\{u_i\}_{i=1}^n$. Order these critical points so that $\Re[\exp(-\sqrt{-1}\phi)u_i] > \Re[\exp(-\sqrt{-1}\phi)u_j]$ if $i < j$. Take the Lefschetz thimble (the descending Morse cycle for a suitable choice of a Riemannian metric on $(\mathbb{C}^\times)^{n-1}$) for $\Re(W/\hbar)$ at $\hbar = \exp[\sqrt{-1}(\phi - \pi/2)]$ starting from the $i$-th critical point of $W$ and extend it to a flat section of the relative homology bundle on $D_{\text{right}}$. Let us call this section $\Gamma_{i,\text{right}}$ and let $I_{i,\text{right}}$ be the integral as in (4) with $\Gamma_{i,\text{right}}$ as the integration cycle. Now form the row vector $(I_{i,\text{right}})_{i=1}^n$ and think of it as an $n \times n$ matrix by regarding an element in the cohomology group as a column vector by the normalized canonical coordinate. Then we can see that $(I_{i,\text{right}})_{i=1}^n$ asymptotes on $D_{\text{right}}$ to the formal solution of the form (4) as $\hbar \to 0$ by the saddle-point method. In the same way, starting from the Lefschetz thimble at $\hbar = \exp[\sqrt{-1}(\phi + \pi/2)]$, we obtain a solution $(I_{i,\text{left}})_{i=1}^n$ defined on $D_{\text{left}}$ which asymptotes to the same formal solution as $(I_{i,\text{right}})_{i=1}^n$. Since the integrand is single-valued, the monodromy of $I_{i}$'s comes solely from the monodromy of the integration cycles and the Stokes matrix is given by

$$\Gamma_{i,\text{right}} = \sum_{j=1}^n \Gamma_{j,\text{left}} S_{ji}. $$

The Stokes matrix for the quantum cohomology of $\mathbb{P}^{n-1}$ has been computed by Dubrovin [5] for $n \leq 3$ and by Guzzetti [7] for general $n$. See also [11].

**Theorem 2.3 (Dubrovin, Guzzetti).** The Stokes matrix $S$ for the quantum cohomology of the projective space $\mathbb{P}^{n-1}$ is given by

$$S_{ij} = \binom{n-1 + j - i}{j - i}$$

up to the braid group action. Here, $\binom{n}{r}$ is the binomial coefficient.

Since $(\mathcal{O}_{\mathbb{P}^{n-1}}(i))_{i=0}^{n-1}$ is an exceptional collection generating $D^b\text{coh}\mathbb{P}^{n-1}$ by Beilinson [11] and

$$\binom{n-1 + j - i}{j - i} = \sum_k (-1)^k \dim \text{Ext}^k(\mathcal{O}_{\mathbb{P}^{n-1}}(i), \mathcal{O}_{\mathbb{P}^{n-1}}(j)),$$

the Conjecture [11] holds for projective spaces.
Now let us move on to the Grassmannian case. Let Gr($r,n$) be the Grassmannian of $r$-dimensional subspaces in $\mathbb{C}^n$. The semisimplicity of the quantum cohomology in this case is also known. The following theorem is proved by Bertram, Ciocan-Fontanine and Kim (see the proof of Theorem 3.3 in [2]):

**Theorem 2.4 (Bertram–Ciocan-Fontanine–Kim).** For a choice of a basis \( \{ \phi_\alpha \}_{\alpha=0}^{N-1} \) of \( H^*(\text{Gr}(r,n); \mathbb{C}) \) where \( N = \binom{n}{r} = \dim H^*(\text{Gr}(r,n); \mathbb{C}) \), there exists a set \( \{ \varphi_\alpha(x_{11}, \ldots, x_{r,n-1}; t, h) \}_{\alpha=0}^{N-1} \) of functions of \( (x_{11}, \ldots, x_{r,n-1}) \in (\mathbb{C}^*)^{r(n-1)} \), \( t \in \mathbb{C} \), and \( h \in \mathbb{C}^* \) such that

\[
\left( \sum_{\alpha=0}^{N-1} \phi_\alpha \int_{\Gamma_{k_1} \times \cdots \times \Gamma_{k_r}} e^{W/h} \varphi_\alpha(x_{11}, \ldots, x_{r,n-1}; t, h) \prod_{j=1}^r \frac{dx_{j1} \cdots dx_{j,n-1}}{x_{j1} \cdots x_{j,n-1}} \right)_{1 \leq k_1 < k_2 < \cdots < k_r \leq n}
\]

forms a fundamental solution to (1), (2) where \( t \) is the coordinate of \( H^2(\text{Gr}(r,n); \mathbb{C}) \) and all the other flat coordinates are set to zero. Here,

\[
W(x_{11}, \ldots, x_{r,n-1}) = \sum_{j=1}^r \left( x_{j1} + \cdots + x_{j,n-1} + \frac{e^t}{x_{j1} \cdots x_{j,n-1}} \right)
\]

and \( \{ \Gamma_{k_i} \}_{i=1}^n \) is the basis of the flat sections of the relative homology bundle as in Theorem 2.2.

By construction, \( \varphi_\alpha(x_{11}, \ldots, x_{r,n-1}; t, h) \) is anti-symmetric with respect to the exchange of \( (x_{i1}, \ldots, x_{i,n-1}) \) and \( (x_{j1}, \ldots, x_{j,n-1}) \) for any \( 1 \leq i < j \leq r \). Therefore, if we define \( H^*(\text{Gr}(r,n); \mathbb{C}) \)-valued functions \( I_K(t, h) \) for \( K = (k_1, \ldots, k_r) \), \( 1 \leq k_i \leq n \), \( i = 1, \ldots, r \) by

\[
I_K = \sum_{\alpha=0}^{N-1} \phi_\alpha \int_{\Gamma_{k_1} \times \cdots \times \Gamma_{k_r}} e^{W/h} \varphi_\alpha(x_{11}, \ldots, x_{r,n-1}; t, h) \prod_{j=1}^r \frac{dx_{j1} \cdots dx_{j,n-1}}{x_{j1} \cdots x_{j,n-1}},
\]

then \( I_K \) is totally anti-symmetric in \( k_1, \ldots, k_r \). Hence it follows that if we put

\[
\Gamma_K = \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn} \sigma \Gamma_{k_{\sigma(1)}} \times \cdots \times \Gamma_{k_{\sigma(r)}}
\]

where \( S_r \) is the symmetric group of degree \( r \) and \( \text{sgn} \sigma \) is the signature of \( \sigma \), then we have

\[
I_K = \sum_{\alpha=0}^{N-1} \phi_\alpha \int_{\Gamma_K} e^{W/h} \varphi_\alpha(x_{11}, \ldots, x_{r,n-1}; t, h) \prod_{j=1}^r \frac{dx_{j1} \cdots dx_{j,n-1}}{x_{j1} \cdots x_{j,n-1}}.
\]
We can use the above result to compute the Stokes matrix for the quantum cohomology of $Gr(r, n)$ from that of $\mathbb{P}^{n-1}$ as follows: By Theorem 2.3, there exists a choice $\{\Gamma_{i,\text{right}}\}_{i=1}^{n}$ and $\{\Gamma_{i,\text{left}}\}_{i=1}^{n}$ of bases of flat sections of the relative homology bundle on $D_{\text{right}}$ and $D_{\text{left}}$ respectively such that

$$\Gamma_{i,\text{right}} = \sum_{j=1}^{n} \Gamma_{j,\text{left}} S_{ji}$$

on $D_{-}$ for $S_{ij} = \binom{n - 1 + j - i}{j - i}$. Then the monodromy for $\Gamma_{K}$ is given by

$$\Gamma_{K,\text{right}} = \sum_{1 \leq l_1 < l_2 < \cdots < l_r \leq n} I_{L,\text{left}} S_{L,K}.$$ 

where $K = (k_1, \ldots, k_r)$, $L = (l_1, \ldots, l_r)$, and

$$S_{L,K} = \det(S_{l_j,k_i})_{1 \leq i,j \leq r} = \det\left(\binom{n + l_i - k_j - 1}{l_i - k_j}\right)_{1 \leq i,j \leq r}. \quad (7)$$

3 Derived category of coherent sheaves

In this section, we use the presentation

$$Gr(r, n) = GL_n(\mathbb{C})/P$$

of the Grassmannian as a homogeneous space, where

$$P = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid A \in GL_r(\mathbb{C}), B \in M_{r,n-r}(\mathbb{C}), D \in GL_{n-r}(\mathbb{C}) \right\}$$

is a parabolic subgroup of $GL_n(\mathbb{C})$. A representation of $GL_r(\mathbb{C})$ gives a representation of $P$ through the projection $P \ni \begin{pmatrix} A \\ 0 \end{pmatrix} \mapsto A \in GL_r(\mathbb{C})$, hence a $GL_n(\mathbb{C})$-equivariant bundles on $Gr(r, n)$ associated to the principal $P$-bundle $GL_n(\mathbb{C}) \to Gr(r, n)$. Let $E_\rho$ denote the equivariant bundle on $Gr(r, n)$ corresponding to a representation $\rho$ of $GL_r(\mathbb{C})$ in this way.

Let

$$\Lambda = \{ (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \mid n - r \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0 \}$$

be a set of weights of $GL_r(\mathbb{C})$. Given a weight $\lambda$, let $\rho_\lambda$ denote the irreducible representation of $GL_r(\mathbb{C})$ with highest weight $\lambda$. We abbreviate $E_{\rho_\lambda}$ as $E_\lambda$. 


Theorem 3.1 (Kapranov [9]). \( \{ \mathcal{E}_\lambda \}_{\lambda \in \Lambda} \) is an exceptional collection generating \( D^b_{\text{coh}}(\text{Gr}(r,n)) \).

Kapranov also proved that \( \text{Ext}^k(\mathcal{E}_\lambda, \mathcal{E}_\mu) = 0 \) for any \( \lambda, \mu \in \Lambda \) and any \( k \neq 0 \). \( \text{Hom}(\mathcal{E}_\lambda, \mathcal{E}_\mu) \) is calculated as follows: Decompose the tensor product \( \rho^\vee_\lambda \otimes \rho_\mu \) of the dual representation of \( \rho_\lambda \) and \( \rho_\mu \) into the direct sum of irreducible representations

\[
\rho^\vee_\lambda \otimes \rho_\mu = \bigoplus_\nu \rho^\vee_\nu \otimes \tilde{N}^\nu_{\lambda\mu}.
\]

Here, \( \tilde{N}^\nu_{\lambda\mu} \) is the multiplicity of \( \rho_\nu \) in \( \rho^\vee_\lambda \otimes \rho_\mu \) and \( \nu \) runs over all weights of \( \text{GL}_r(\mathbb{C}) \). Define

\[
N^\nu_{\lambda\mu} = \begin{cases} 
\tilde{N}^\nu_{\lambda\mu} & \text{if } \nu_r \geq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

For a weight \( \lambda \in \mathbb{Z}^r \) of \( \text{GL}_r(\mathbb{C}) \), let \( R_\lambda \) be the irreducible representation of \( \text{GL}_n(\mathbb{C}) \) with highest weight \( (\lambda_1, \ldots, \lambda_r, 0, \ldots, 0) \in \mathbb{Z}^n \). Then

\[
\text{Hom}(\mathcal{E}_\lambda, \mathcal{E}_\mu) = H^0(\mathcal{E}_\lambda^\vee \otimes \mathcal{E}_\mu) \\
= H^0(\mathcal{E}_\rho^\vee_\lambda \otimes \rho_\mu) \\
= \bigoplus_\nu H^0(\mathcal{E}_\nu^\vee) \oplus \tilde{N}^\nu_{\lambda\mu} \\
= \bigoplus_\nu R^\oplus_{\nu} N^\nu_{\lambda\mu},
\]

where the last equality follows from the Borel-Weil theory.

4 A combinatorial identity

The content of this section is due to A. N. Kirillov. Fix two integers \( r, n \) such that \( r < n \). Let \( A = \{(k_1, \ldots, k_r) \in \mathbb{Z}^r \mid 1 \leq k_1 < \cdots < k_r \leq n\} \). \( A \) and \( \Lambda \) defined in the previous section are bijective by the correspondence

\[
\Lambda \ni (\lambda_i)_{i=1}^r \mapsto (\lambda_{r-i+1} + i)_{i=1}^r \in A.
\]

For \( n \) variables \( x = (x_1, \ldots, x_n) \), let \( s_\lambda(x) = \det(h_{\lambda_i-i+j}(x))_{1 \leq i, j \leq n} \) be the Shur function, where \( h_i(x) \) is the complete symmetric function (the sum of all monomials of degree \( i \)). For generalities on symmetric functions, see, e.g., [10]. Define integers \( c^\lambda_{\mu\nu} \)'s by

\[
s_\mu(x)s_\nu(x) = \sum_\lambda c^\lambda_{\mu\nu}s_\lambda(x)
\]
and the skew Shur function $s_{\lambda/\mu}(x)$ by

$$s_{\lambda/\mu}(x) = \sum_{\nu} c_{\mu\nu}^\lambda s_{\nu}(x).$$

Then

$$s_{\lambda/\mu}(x) = \det(h_{\lambda_1-\mu_j-i+j}(x))_{1 \leq i,j \leq n}.$$  

**Lemma 4.1.** Let $\mu$, $\nu$ and $\lambda$ be partitions such that $\mu_1 \leq \nu_r$. Define $\mu^c = (\mu_1 - \mu_r, \mu_1 - \mu_{r-1}, \ldots, \mu_1 - \mu_2, 0)$ and $\tilde{\nu} = (\nu_1 - \mu_1, \nu_2 - \mu_1, \ldots, \nu_r - \mu_1)$. Then

$$c_{\nu^c \lambda} = c_{\mu \tilde{\nu}}^\lambda.$$  

**Proof.**

$$c_{\mu \tilde{\nu}}^\lambda = \dim \text{Hom}_{\text{GL}_r(\mathbb{C})}(\rho_\lambda \otimes \rho_{\mu^c}; \rho_{\nu})$$

$$= \dim \text{Hom}_{\text{GL}_r(\mathbb{C})}(\rho_\lambda \otimes \rho_{\nu}^c \otimes \rho_{\nu})$$

$$= \dim \text{Hom}_{\text{GL}_r(\mathbb{C})}(\rho_\lambda \otimes (\rho_{\nu}^c \otimes \text{det}^{\otimes \mu_1}) \otimes \rho_{\nu})$$

$$= \dim \text{Hom}_{\text{GL}_r(\mathbb{C})}(\rho_\lambda \otimes \rho_{\nu}^c \otimes \rho_{\nu}^c)$$

$$= \dim \text{Hom}_{\text{GL}_r(\mathbb{C})}(\rho_\lambda \otimes \rho_{\nu}^c \otimes \rho_{\nu})$$

where $\rho_0$ is the trivial representation and det is the determinant representation (the irreducible representation with highest weight $(1, \ldots, 1) \in \mathbb{Z}^r$).

**Theorem 4.2.** $s_{\lambda/\mu}(x) = \sum_{\nu} N_{\lambda\mu}^\nu s_{\nu}(x).$

**Proof.**

$$\sum_{\nu} N_{\lambda\mu}^\nu s_{\nu}(x) = \sum_{\nu} c_{\mu \tilde{\nu}}^\lambda s_{\nu}(x)$$

$$= \sum_{\nu} c_{\nu^c \lambda} s_{\nu}(x)$$

$$= s_{\lambda/\mu}(x).$$

By substituting $x_1 = \cdots = x_n = 1$ in Theorem 4.2 and using $h_r(1, \ldots, 1) = \binom{n+r-1}{r}$, we obtain the following:

**Corollary 4.3.** For $\lambda, \mu \in \Lambda$, let $k = (\lambda_{r-i+1} + i)^{r}_{i=1}$, $l = (\mu_{r-i+1} + i)^{r}_{i=1}$. Then

$$\text{det}\left(\begin{array}{c} n + l_i - k_j - 1 \\ l_i - k_j \end{array}\right)_{1 \leq i,j \leq r} = \sum_{\nu} N_{\lambda\mu}^\nu \dim R_{\nu}.$$
The left hand side is the component of the Stokes matrix from (7) and the right hand side is the Euler number in the derived category of coherent sheaves form (8). This proves Conjecture 1.1 in the case of Grassmannians.

References

[1] A. A. Beĭlinson. Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra. Funktsional. Anal. i Prilozhen., 12(3):68–69, 1978.

[2] A. Bertram, I. Ciocan-Fontanine, and B. Kim. Two proofs of a conjecture of Hori and Vafa. math.AG/0304403.

[3] Boris Dubrovin. Geometry of 2D topological field theories. In Integrable systems and quantum groups (Montecatini Terme, 1993), volume 1620 of Lecture Notes in Math., pages 120–348. Springer, Berlin, 1996.

[4] Boris Dubrovin. Geometry and analytic theory of Frobenius manifolds. In Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), number Extra Vol. II, pages 315–326 (electronic), 1998.

[5] Boris Dubrovin. Painlevé transcendents in two-dimensional topological field theory. In The Painlevé property, CRM Ser. Math. Phys., pages 287–412. Springer, New York, 1999.

[6] Alexander B. Givental. Equivariant Gromov-Witten invariants. Internat. Math. Res. Notices, (13):613–663, 1996.

[7] Davide Guzzetti. Stokes matrices and monodromy of the quantum cohomology of projective spaces. Comm. Math. Phys., 207(2):341–383, 1999.

[8] K. Hori and C. Vafa. Mirror symmetry. hep-th/0002222.

[9] M. M. Kapranov. On the derived categories of coherent sheaves on some homogeneous spaces. Invent. Math., 92(3):479–508, 1988.

[10] I. G. Macdonald. Symmetric functions and Hall polynomials. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[11] Susumu Tanabé. Invariant of the hypergeometric group associated to the quantum cohomology of the projective space. Bull. Sci. Math., 128(10):811–827, 2004.
[12] Eric Zaslow. Solitons and helices: the search for a math-physics bridge. *Comm. Math. Phys.*, 175(2):337–375, 1996.