The Petrov type of the five-dimensional Myers-Perry metric

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**Abstract:** We point out that the Myers-Perry metric in five dimensions is algebraically special. It has Petrov type $22$, which is the Petrov type of the five-dimensional Schwarzschild metric.

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1. Introduction

In this article, we calculate the Petrov type of the five-dimensional Myers-Perry metric [1]. The five-dimensional Myers-Perry (MP) metric is the generalization to five dimensions of the four-dimensional Kerr-metric. The Kerr-metric, which describes the gravitational field of a rotating star, and its static limit, the Schwarzschild metric, have both Petrov type $D$. It is remarkable to see that, although the Kerr-metric has fewer symmetries than the Schwarzschild metric, it does still have the same Petrov type.

We show that the same holds in five dimensions. The five-dimensional MP metric, which describes a rotating black hole in five dimensions, and its static limit, the five-dimensional Schwarzschild metric, have both Petrov type $22$. Again, we see that although the five-dimensional MP metric has fewer isometries than the five-dimensional Schwarzschild metric, it does have the same Petrov type. The remainder of the article is organized as follows. In Section 2, we give a review of the five-dimensional Petrov classification. The Petrov type of the MP metric is given in Section 3. We conclude in Section 4.

2. Review of the five-dimensional Petrov classification

We only give a brief review of this classification, a longer discussion can be found in ref. [2]. We need to introduce two objects, the Weyl spinor and the Weyl polynomial. The Weyl spinor $\Psi_{abcd}$ is the spinorial translation of the Weyl tensor $C_{ijkl}$

$$\Psi_{abcd} = (\gamma_{ij})_{ab}(\gamma_{kl})_{cd}C^{ijkl}.$$ 

Here, $\gamma_{ij} = \frac{1}{2}[\gamma_i, \gamma_j]$, where $\gamma_i$ are the $\gamma$-matrices in five dimensions. In this article, we use the following representation $\gamma_1 = i\sigma_1 \otimes 1$, $\gamma_2 = \sigma_2 \otimes 1$, $\gamma_3 = \sigma_3 \otimes \sigma_1$, $\gamma_4 = \sigma_3 \otimes \sigma_2$ and
\[ \gamma_5 = \sigma_3 \otimes \sigma_3. \] The Weyl spinor is symmetric in all its indices. The Weyl polynomial \( \Psi \) is a homogeneous polynomial of degree four in four variables:

\[ \Psi = \Psi_{abcd} x^a x^b x^c x^d. \]

The Petrov type of a given Weyl tensor is the number and multiplicity of the irreducible factors of its corresponding Weyl polynomial \( \Psi \). In this way, we obtain 12 different Petrov types, which are depicted in figure 1. We use the following notation. The number denotes the degree of the irreducible factors and underbars denote the multiplicities. For example, a Weyl polynomial which can be factorized into two different factors, each having degree 2, has Petrov type 22. If the two factors of degree 2 are the same, the Petrov type is 22.

3. The Myers-Perry metric has Petrov type 22

The five-dimensional rotating black hole is described by the Myers-Perry metric [1]

\[
\begin{align*}
  ds^2 &= -dt^2 + \frac{2m}{\rho^2} \left[ dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi \right]^2 \\
  &\quad + \frac{\rho^2}{R^2} dr^2 + \rho^2 d\theta^2 + \Sigma_a^2 \sin^2 \theta d\phi^2 + \Sigma_b^2 \cos^2 \theta d\psi^2,
\end{align*}
\]

where \( \rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta \), \( \Sigma_a^2 = r^2 + a^2 \), \( \Sigma_b^2 = r^2 + b^2 \) and

\[
R^2 = \left[ \Sigma_a^2 \Sigma_b^2 - 2mr^2 \right] / r^2.
\]
We choose the following tetrad

\[ e_1 = \frac{1}{r^2 R \rho} \left[ \Sigma^2 \Sigma^2 \partial_t + a \Sigma^2 \partial_{\phi} + b \Sigma^2 \partial_{\psi} \right] \]

\[ e_2 = \frac{1}{r \rho \sin \theta} \left[ \Sigma_b \left( \partial_{\phi} + a \sin^2 \theta \partial_t \right) + b \Sigma_a V \right] \]

\[ e_3 = \frac{1}{r \rho \cos \theta} \left[ \Sigma_a \left( \partial_{\psi} + b \cos^2 \theta \partial_t \right) - a \Sigma_b V \right] \]

\[ e_4 = \frac{R}{\rho} \partial_r \]

\[ e_5 = \frac{1}{\rho} \partial_{\theta}. \]

Here, we have used the vectorfield

\[ V = \frac{1}{\Sigma_a \Sigma_b + r \rho} \left( a \sin^2 \theta \partial_{\psi} - b \cos^2 \theta \partial_{\phi} \right). \]

A straightforward calculation gives the Weyl polynomial

\[ \Psi = -\frac{48mr^2}{\rho^6} \left( x^2 - y^2 + z^2 - t^2 - 2f(xy + zt) - 2ig(xz + yt) \right)^2, \]

where

\[ f = \frac{b \sin \theta \left( a^2 \Sigma_b \rho + (b^2 - a^2) r \Sigma_a \cos \theta \right)}{r \Sigma^2 \Sigma_b - r^2 \rho^2}, \]

\[ g = \frac{a \cos \theta \left( b^2 \Sigma_a \rho + (a^2 - b^2) r \Sigma_b \sin \theta \right)}{r \Sigma^2 \Sigma_b - r^2 \rho^2}. \]

This polynomial is the square of a polynomial of degree 2. Therefore, the MP metric has Petrov type \( 22 \).

4. Conclusions and topics for further research

In this article, we showed that the five-dimensional MP metric has the same Petrov type as its static limit, namely Petrov type \( 22 \). Some open problems are the following.

- Recently, Emparan and Reall found a black rotating ring [3], see ref. [4] for easier coordinates. It would be nice to know its Petrov type.

- In four dimensions, adding electric charge to the rotating star does not change its Petrov type; the Kerr-Newmann metric has Petrov type \( D \). In five dimensions, the story is more complicated. The metric of an electrically charged rotating black hole is only known in five dimensions when there is a specific Chern-Simons term in the action. The particular form of this Chern-Simons term is dictated by supersymmetry. The charged rotating black hole in this theory is described by the BMPV metric [5], which was found
by using duality. It would be good to calculate its Petrov type. The charged rotating black hole is not known when this Chern-Simons term has an arbitrary (or even zero) coefficient. It remains to be seen if it can be found within the class of algebraically special metrics.

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