Second order theory of \((j,0) \oplus (0,j)\) single high spins as Lorentz tensors

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ABSTRACT: We show that higher order differential equations and matrix spinor calculus are completely avoidable in the description of pure high spin-\(j\) Weinberg-Joos states, \((j,0) \oplus (0,j)\). The case is made on the example of \((\frac{3}{2},0) \oplus (0,\frac{3}{2})\), for the sake of concreteness and without loss of generality. Namely, we use as a vehicle for the aforementioned covariant single spin-\(\frac{3}{2}\) description the direct sum of \((\frac{3}{2},0) \oplus (0,\frac{3}{2})\) with the Dirac field, \(\psi \simeq (\frac{1}{2},0) \oplus (0,\frac{1}{2})\), on the one side, and \((\frac{1}{2},1) \oplus (1,\frac{1}{2})\), on the other, which amounts to the antisymmetric tensor of second rank with Dirac spinor components, \(\Psi_{[\mu\nu]} = B_{[\mu\nu]} \otimes \psi\). The \((\frac{3}{2},0) \oplus (0,\frac{3}{2})\) sector of interest is then tracked down in two steps. First we search for spin-\(\frac{3}{2}\) by means of a covariant spin projector constructed from the Casimir invariants of the Poincaré algebra, the squared four momentum, \(P^2\), and the squared Pauli-Lubanski vector, \(W^2\). This projector is second order in the momenta. Afterwords we identify the wanted irreducible representation space by means of a momentum independent (static) projector designed on the basis of the Casimir invariants of the Lorentz algebra. The latter projectors have the property to unambiguously identify any irreducible \(so(1,3)\) subspace of any Lorentz tensor and without rising the order of the differential equation. In this fashion, a Lagrangian that is second order in the momenta is furnished. The method proposed correctly reproduces the electromagnetic multipole moments earlier calculated for \((\frac{3}{2},0) \oplus (0,\frac{3}{2})\) in treating it in the standard way as eight dimensional spinor. We furthermore calculate Compton scattering off the pure spin-\(\frac{3}{2}\) under discussion, and show that the differential cross section satisfies unitarity in forward direction for a gyromagnetic ratio of \(g = \frac{3}{2}\). This result hints on the possible validity of Belinfante’s conjecture for pure spin-states, while the natural value of \(g = 2\) seems more likely to characterize the highest spins in the Rarita-Schwinger representation spaces. The scheme straightforwardly extends to any \((j,0) \oplus (0,j)\) Weinberg-Joos state and brings the advantage of avoiding rectangular matrix couplings between states of different spins, replacing them by simple Lorentz contractions.

KEYWORDS: Any spin, irreducible \(so(1,3)\) representation spaces, reducible Lorentz tensors, covariant spin projectors, static Lorentz projectors, second order Lagrangians
1 Methods for high-spin descriptions - introductory remarks

Particles of high-spins [1] continue being among the most enigmatic challenges in contemporary theoretical physics. The difficulties in their descriptions, both at the classical-, and the quantum-field theoretical levels, are well known and take their origin from the circumstance that such particles are most naturally described by differential equations of orders twice their respective spins [2],[3]. Higher-order theories are difficult to tackle and various strategies have been elaborated over the years to lower the order of the corresponding differential equations, the linear ones by Rarita-Schwinger...
[4] being the most popular so far. However, the latter framework is plagued by various inconsistencies, the acausal propagation within an electromagnetic environment [5], the violation of unitarity in Compton scattering in the ultraviolet in schemes with minimal gauge couplings [6], and the violation of Lorentz-symmetry upon quantization, being the most serious ones. In parallel, also second order spin-$\frac{1}{2}$ [7], [8], [9], [10] and spin-$\frac{3}{2}$ [11] fermion theories have been developed by different authors and shown to provide a reasonable compromise between the rigorous linear— and the natural higher-order descriptions in so far as they were able to circumvent the acausality problem simultaneously with the violation of unitarity in Compton scattering [12]. However, for spins higher than $\frac{3}{2}$ no second order theory has been developed so far. It is the goal of the present work to fill this gap. The interest in such a study is motivated by the observation that particles with spin-$j$ transforming according to distinct representation spaces of the Lorentz algebra describe particles of different physical properties. For example, because of the representation dependence of the boost operator, the electromagnetic quadrupole and octupole moments of fundamental particles with spin-$\frac{3}{2}$ transforming in the four-vector spinor come out different from those of particles transforming as $(\frac{3}{2},0) \oplus (0,\frac{3}{2})$ [13]. Same holds valid regarding spin-1 in the four-vector, $(\frac{1}{2},\frac{1}{2})$, versus the anti-symmetric tensor, $(1,0) \oplus (0,1)$.

In view of the expected production of new particles in the experiments run by the Large Hadron Collider it is important to have at ones disposal a reliable and comfortable to deal with universal calculation scheme of any high spin, be it bosonic, or fermionic. The present study is devoted to the elaboration of such a scheme. In the current section we present a concise review of the persisting techniques in high-spin description of frequent use in low and intermediate energy physics emphasizing on their differences and similarities. In due course we suggest our announced strategy for any high-spin second order formalism. It is based upon properly constructed Lorentz tensors for bosons, or Lorentz tensor-spinors for fermions, and the employment of momentum independent projectors on irreducible spaces (“irreps”) of the Lorentz algebra $so(1,3)$ in combination with second order mass-$m$ and spin-$j$ projectors constructed from the two invariants of the Poincaré algebra, the squared four momentum $P^2$ and the squared Pauli-Lubanski vector, $\mathcal{W}^2$.

**Totally symmetric tensor- and tensor-spinor representation spaces:**

Particles of high-spins, $j > \frac{1}{2}$, have been so far most frequently described in the literature in terms of multi-spin-parity $so(1,3)$ representation spaces given by totally symmetric Lorentz tensors of the type,

$$A_{\mu_1...\mu_j} \simeq \left(\frac{j}{2}, \frac{j}{2}\right),$$

for bosons, or Lorentz tensor-spinors,

$$\psi_{\mu_1...\mu_j-4} \simeq \left(\frac{j-\frac{1}{2}}{2}, \frac{j-\frac{1}{2}}{2}\right) \otimes \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right),$$

for fermions, respectively [4], [1]. They have been associated with the highest spins in the spaces under discussion, while their lower spin companions have been treated as redundant and had to be projected out in order to ensure the correct number of physical degrees of freedom for spin-$j$ description.

**Rarita-Schwinger’s wave equations:**
The wave equations for the latter case are obtained from the requirement that in any Lorentz index the field satisfies the Dirac equation, supplemented by certain auxiliary conditions according to

\[
(i \partial - m)\psi_{\mu_1...\mu_i...\mu_j} = 0, \\
\gamma^\mu\psi_{\mu_1...\mu_i...\mu_j} = 0, \\
\partial^\mu\psi_{\mu_1...\mu_i...\mu_j} = 0.
\]  

(1.3)

Specifically for spin-\(\frac{3}{2}\) one has to consider the four-vector–spinor, \(\psi_\mu\),

\[
\psi_\mu = A_\mu \otimes \psi \simeq \left( \frac{1}{2}, \frac{1}{2} \right) \otimes \left[ \left( \frac{1}{2}, 0 \right) \oplus \left( \frac{1}{2}, 0 \right) \right],
\]  

(1.4)

the direct product between the four vector, \(A_\mu\), and the Dirac spinor, \(\psi\), and solve the system of three linear differential equations

\[
(i \partial - m)\psi_\mu = 0, \\
\gamma^\mu\psi_\mu = 0, \\
\partial^\mu\psi_\mu = 0.
\]  

(1.5 – 1.7)

Multi-component representation spaces:

Alternatively, spin-\(j\) particles can be described in terms of the single spin valued non-tensorial representation spaces,

\[
\psi_B^{(j)} \simeq (j, 0) \oplus (0, j), \quad B \in [1, 2(2j + 1)].
\]  

(1.8)

Weinberg-Joos wave equations:

The wave functions of particles described in terms of the representation spaces in (1.8) satisfy higher order differential equations according to,

\[
\left( i^{2j} \left[ \gamma_{\mu_1\mu_2...\mu_{2j}} \right]_{AB} \partial^{\mu_1} \partial^{\mu_2} ... \partial^{\mu_{2j}} - m^{2j} \delta_{AB} \right) \psi_B^{(j)} (x) = 0,
\]  

(1.9)

where \(\psi_B^{(j)} (x)\) is the \(2(2j + 1)\)-component field \((j, 0) \oplus (0, j)\), \(\left[ \gamma_{\mu_1\mu_2...\mu_{2j}} \right]_{AB}\) are the elements of the generalized Dirac Hermitean matrices of dimensionality \([2(2j + 1)] \times [2(2j + 1)]\), which transform as Lorentz tensors of rank-\(2j\). The complete sets of such matrices have been extensively studied in the literature for the purpose of constructing all the possible field bilinears needed in the definitions of the generalized currents, both transitional and diagonal [14–15]. Though well elaborated, this so called Weinberg-Joos formalism has attracted comparatively less attention than the linear Rarita-Schwinger framework, mainly because of the difficult to handle higher order of the differential equations involved, on the one side, and the high dimensionality of the generalized Dirac matrices, on the other.

A possibility to put the Weinberg-Joos formalism on comparable footing with that by Rarita-Schwinger is to find a way to describe single-spin representation spaces by means of Lorentz-tensors. So far no such method for any spin, both integer and fractional, has been suggested in the literature, a circumstance that has presented over the years a serious obstacle in the employment of \((j, 0) \oplus (0, j)\)-fields in the description of physical processes. Instead, the Rarita-Schwinger spaces have been given the clear preference precisely for admitting comfortable couplings to the fundamental photon-proton system. However, particles of equal spins residing in different representation spaces can throughout
be characterized by distinct physical properties such as electromagnetic multipole moments, Compton scattering cross sections etc., this because of the non-trivial differences in the structures of the respective boost operators \([13],[16]\). To study such differences it is important to have at ones disposal a tool for the efficient description of single-spins in terms of Lorentz tensors. It is the goal of the present study to provide such a tool.

**Pairwise anti-symmetric tensor-spinor representation spaces:**

In \([17]\) a linear approach in analogy to the Rarita-Schwinger formalism has been elaborated for fractional high spins \(j\) as part of an anti-symmetric (in pairs of indexes) tensor of rank \((2j - 1)\),

\[
\psi_{[\mu_1\mu_2]...[\mu_i\nu_i]...[\mu_j,\nu_j,\ldots]} \simeq \left[(j - \frac{1}{2},0) \oplus (0,j - \frac{1}{2})\right] \oplus \left[(\frac{1}{2},0) \oplus (0,\frac{1}{2})\right]. \tag{1.10}
\]

**Niederle-Nikitin’s fermion wave equations:**

The corresponding wave equations are then

\[
(p - m)\psi_{[\mu_1\mu_2]...[\mu_i\nu_i]...[\mu_j,\nu_j,\ldots]} - \frac{1}{4j} \Sigma \mathcal{P} (\gamma^\mu_1 \gamma^\nu_1 - \gamma^\nu_1 \gamma^\mu_1) P_\lambda^\gamma \psi_{[\lambda\sigma]...[\mu_i\nu_i]...[\mu_j,\nu_j,\ldots]} = 0, \tag{1.11}
\]

where \(\mathcal{P}\) denotes permutations of \(\mu_i\) and \(\nu_i\), as well as of \([\mu_k\nu_k]\) with \([\mu_j\nu_j]\).

We here take a different path. Namely, we embed single spin- \(j\) Weinberg-Joos states, \((j,0) \oplus (0,j)\), into direct sums of properly selected irreducible \(so(1,3)\) representation spaces which are large enough as to allow to be equipped by Lorentz, and if needed, separate Dirac indexes. Then we identify the state of our interest in a two step procedure. First we search through the aforementioned direct sum for the spin of our interest using a covariant spin projector constructed from the Casimir invariants of the Poincaré algebra, the squared four momentum, \(P^2\), and the squared Pauli-Lubanski vector, \(W^2\). Afterwards we search for the irreducible representation space by means of a momentum independent (static) projector designed on the basis of the Casimir invariants of the Lorentz algebra. As long as the covariant spin projector is second order in the momenta, the emerging Lagrangian is of second order too. In this fashion, a second order formalism for any single-spin valued Weinberg-Joos state is furnished.

The paper is organized as follows. In the next section we present our suggested method. In section 3 we find wave equations and Lagrangians for all the spins residing in the antisymmetric tensor spinor and transforming according to one of the irreducible representation spaces, to be termed to in the following by \(irrep\). In section 4 we construct all the explicit degrees of freedom spanning the tensor-spinor space under consideration. In section 5 we gauge the spin-\(\frac{1}{2}\), and spin-\(\frac{3}{2}\) Lagrangians, find the electromagnetic currents of interest, and calculate the associated electromagnetic multipole moments. We show that the observables calculated in this fashion reproduce those earlier obtained in \([13]\) from considering pure spin-\(\frac{1}{2}\) state in the standard way as an eight-dimensional spinor. Also there we show that the pure spin-\(\frac{3}{2}\) sector of the antisymmetric tensor spinor describes a genuine Dirac particle, while the properties of the particles in \((\frac{3}{2},1) \oplus (1,\frac{3}{2})\) are same as those of the corresponding part of the Rarita-Schwinger four vector spinor. Section 6 is devoted to the evaluation of Compton scattering off \((\frac{3}{2},0) \oplus (0,\frac{3}{2})\). There we report on finite forward differential cross section in the ultraviolet for the gyromagnetic ratio taking the value of the inverse spin, i.e. for \(g = \frac{2}{3}\) and in accord with Belinfante’s conjecture. The paper closes with brief conclusions.
2 High-spins within the relativistic invariants (RInS) method: Covariant spin-irrep projectors

The method for high-spin description advocated in this work is based upon representation spaces which are different from those used in the three schemes highlighted in the introduction. While the representation spaces underlying the Weinberg-Joos formalism are of non-tensorial nature, those underlying the Rarita-Schwinger framework are totally symmetric tensors. Finally, the Niederle-Nikitin method is based on tensor-spinors that are antisymmetric in pairs of indexes, though symmetric with respect to the pairs. We here instead use Lorentz tensors of mixed symmetries. Then the irreducible tensor sector is tracked down by a projector operator designed on the basis of the momentum independent invariants of the Lorentz algebra, while the spin is identified by means of a projector constructed from the momentum dependent invariants of the Poincaré algebra. In the following, this method will be occasionally termed to as RInS-formalism.

Mixed-symmetric tensor (bosonic) and tensor-spinor (fermionic) representation spaces:

Our idea is to embed $(j, 0) \oplus (0, j)$ carrier spaces of the Lorentz algebra so(1,3) into finite direct sums of properly chosen auxiliary irreducible representation spaces with the aim to end up with a reducible representation space that is large enough as to allow to be equipped by Lorentz– (and if needed, separate Dirac) indexes, i.e.

$$\Psi_{\mu_1, \ldots, \mu_r} \simeq [(j, 0) \oplus (0, j)] \oplus \Sigma_{(k, l)} n_{(kl)} [(j_k, j_l) \oplus (j_l, j_k)].$$

(2.1)

Alternatively, spin-$j$ can be described in terms of two-spin valued representation spaces, $(j \mp \frac{1}{2}, \frac{1}{2}) \oplus (\frac{1}{2}, j \mp \frac{1}{2})$, in which case the carrier tensor-spinor can be designed as,

$$\Psi_{\mu_1, \ldots, \mu_r} \simeq \left( \begin{array}{c} s, \frac{1}{2} \end{array} \right) \oplus \left( \begin{array}{c} \frac{1}{2}, s \end{array} \right) \oplus \Sigma_{(k, l)} n_{(kl)} [(j_k, j_l) \oplus (j_l, j_k)], \quad j = s \pm \frac{1}{2},$$

(2.2)

where the generic coefficients $n_{(kl)}$ stand for the multiplicity of the attached states required to complete the Lorentz tensor (tensor-spinor) under construction. Contrary to the totally symmetric Rarita-Schwinger tensors in (1.4), the tensors in (2.1)-(2.2) can be antisymmetric in some of the indexes and symmetric in others.

For example, pure spin-$\frac{3}{2}$ can be embedded into the totally antisymmetric tensor of second rank with Dirac spinor components, $\Psi_{[\mu\nu]}$, a representation space that is reducible according to

$$\Psi_{[\mu\nu]} \simeq [1, 0] \otimes (0, 1) \oplus \left[ \left( \begin{array}{c} \frac{1}{2}, 0 \end{array} \right) \oplus \left( \begin{array}{c} \frac{1}{2}, 1 \end{array} \right) \right] \oplus \left[ \left( \begin{array}{c} 1, 0 \end{array} \right) \oplus \left( \begin{array}{c} 1, 1 \end{array} \right) \right] \oplus \left[ \left( \begin{array}{c} 3, 0 \end{array} \right) \oplus \left( \begin{array}{c} 3, 1 \end{array} \right) \right].$$

(2.3)

Spin-2 is part of the antisymmetric tensor-vector

$$\Phi_{[\mu\nu]\eta} \simeq B_{[\mu\nu]} \otimes A_\eta \simeq [1, 0] \otimes (0, 1) \otimes \left( \begin{array}{c} 1, 1 \end{array} \right) \oplus \left( \begin{array}{c} 1, \frac{3}{2} \end{array} \right) \oplus \left( \begin{array}{c} 1, -\frac{1}{2} \end{array} \right) \oplus \left( \begin{array}{c} 1, \frac{1}{2} \end{array} \right) \oplus \left( \begin{array}{c} 1, -\frac{3}{2} \end{array} \right) \oplus \left( \begin{array}{c} 1, -1 \end{array} \right) \oplus \left( \begin{array}{c} 3, 0 \end{array} \right) \oplus \left( \begin{array}{c} 3, 1 \end{array} \right) \oplus \left( \begin{array}{c} 3, 2 \end{array} \right) \oplus \left( \begin{array}{c} 3, 3 \end{array} \right) \oplus \left( \begin{array}{c} 3, -1 \end{array} \right) \oplus \left( \begin{array}{c} 3, -2 \end{array} \right) \oplus \left( \begin{array}{c} 3, -3 \end{array} \right).$$

(2.4)
Similarly, spin-$\frac{5}{2}$ can be embedded in the direct product of the antisymmetric tensor-vector from above with the Dirac spinor, giving the totally antisymmetric Lorentz tensor of second rank with four-vector-spinor components, $\Psi_{[\mu\nu]\eta}$, a representation space reducible according to

$$
\Phi_{[\mu\nu]} \otimes A_\eta \otimes \psi \simeq \Psi_{[\mu\nu]\eta} \simeq [(1, 0) \oplus (0, 1)] \otimes \left[ \left( \frac{1}{2}, \frac{1}{2} \right) \oplus \left( \frac{1}{2}, 0 \right) \oplus \left( 0, \frac{1}{2} \right) \right]
$$

$$
\rightarrow 2 \left[ \left( \frac{1}{2}, 0 \right) \oplus \left( 0, \frac{1}{2} \right) \right] \oplus 3 \left[ \left( \frac{1}{2}, \frac{1}{2} \right) \oplus \left( \frac{1}{2}, 1 \right) \right]
$$

$$
\oplus \left[ \left( 1, \frac{3}{2} \right) \oplus \left( \frac{3}{2}, 1 \right) \right] \oplus \left[ \left( 2, \frac{1}{2} \right) \oplus \left( \frac{1}{2}, 2 \right) \right],
$$

where the numbers in front of the irreps indicate their multiplicity in the reduction scheme.

**Fermion and boson wave equations within the RInS-formalism:**

In order to exclude the auxiliary irreducible sectors in the above large reducible representation spaces without rising the order of the wave equations, we employ static projectors constructed from the Casimir invariants of the Lorentz algebra which have the property to unambiguously identify anyone of the irreducible representation spaces, no matter whether single- or multiple-spin valued. To be specific,

- **Covariant momentum independent irrep projectors:**

  The Lorentz algebra has two Casimir operators, denoted by $F$ and $G$, and given by [18] as

  $$ [F]_{AB} = \frac{1}{4} [M^\mu]_A^C [M_\mu]_{CB}, $$

  $$ [G]_{AB} = \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} [M^\mu]_A^C [M^{\rho\sigma}]_{CB}, $$

  with $A, B, C$ standing for the generic indexes characterizing the representation space of interest. Their respective eigenvalue problems for generic irreducible representation spaces of the type $(j_1, j_2) \oplus (j_2, j_1)$, here denoted by $\psi^{(j_1, j_2)}$ are,

  $$ F \psi^{(j_1, j_2)} = \lambda_{(j_1, j_2)} \psi^{(j_1, j_2)}, $$

  $$ \lambda_{(j_1, j_2)} = \frac{1}{2} (j_1 (j_1 + 1) + j_2 (j_2 + 1)) = \frac{1}{2} (K(K + 2) + M^2) $$

  $$ G \psi^{(j_1, j_2)} = \eta_{(j_1, j_2)} \psi^{(j_1, j_2)}, $$

  $$ \eta_{(j_1, j_2)} = \frac{1}{2i} (j_1 (j_1 + 1) - j_2 (j_2 + 1)) = iM(K + 1) $$

  where

  $$ K = j_1 + j_2, \quad M = |j_1 - j_2|. $$

  The $F$ eigenvectors are of well defined parities, while those of $G$ are chiral states. In the following we choose to work with the $F$ invariant.

  On the basis of $F$ we design the following momentum independent Lorentz projector, $\mathcal{P}_F^{(j_1, j_2)}$,
\[ P_{F}^{(j_{1},j_{2})} \psi^{(j_{1},j_{2})} = \Pi_{kl} \otimes \left( \frac{F - \lambda^{(j_{k},j_{1})}}{\lambda^{(j_{1},j_{2})} - \lambda^{(j_{k},j_{1})}} \right) \psi^{(j_{k},j_{1})} = \psi^{(j_{1},j_{2})}, \quad (2.10) \]

where \( \lambda^{(j_{1},j_{2})} \) is the eigenvalue of the searched sector, while \( \lambda^{(j_{k},j_{1})} \) are the eigenvalues of the auxiliary sectors to be excluded. The mayor advantage of such projectors is that they are momentum independent and do not increase the order of the wave equations.

In what follows we shall consider only such reducible Lorentz tensors (or, tensors-spinors) which allow the spin-\( j \) of our interest to reside within an irreducible subspace of maximally two-spins, denoted by \( j \) and \( j' \), meaning that

\[ (j_{1},j_{2}) = \begin{cases} j_{2} = \frac{1}{2}, & \text{with } j = j_{1} + \frac{1}{2}, \quad j' = j_{1} - \frac{1}{2}, \\ j_{2} = 0, & \text{with } j = j_{1}. \end{cases} \quad (2.11) \]

**Covariant spin projectors, second order in the momenta:**

The dynamic into the irreducible sector from above carrying the spin of interest is then introduced by applying to it the appropriate covariant mass-\( m \) and spin-\( j \) projector, \( P_{W^{2}}^{(m,j)}(p) \), to be occasionally referred to as “Poincaré projector”, that expresses in terms of the Casimir invariants of the Poincaré algebra, the squared four momentum, \( P^{2} \), and the squared Pauli-Lubanski vector, \( W^{2}(p) \), as

\[ P_{W^{2}}^{(m,j)}(p) \psi^{(m,j)}(p) = \frac{P^{2}}{m^{2}} \left( \frac{W^{2}(p) - \epsilon_{j}}{\epsilon_{j} - \epsilon_{j'}} \right) \psi^{(m,j)}(p) = \psi^{(m,j)}(p). \quad (2.12) \]

Here, \( W^{\mu}(p) \) denotes the Pauli-Lubanski (pseudo)vector, defined as

\[ (W^{\mu})_{AB} (p) = \frac{1}{2} \epsilon_{\lambda\sigma\mu} (M^{\rho\sigma})_{ab} p_{\rho}, \quad (2.13) \]

where \( M^{\rho\sigma} \) are the generators of the Lorentz algebra in the representation space of interest, while \( A \) and \( B \) are the sets of indexes that characterize the dimensionality of that very representation space, \( \epsilon_{j} = -p^{2} j (j + 1) \) and \( \epsilon_{j'} = -p^{2} j' (j' + 1) \) are in their turn the eigenvalues corresponding to the spin-\( j \), or, spin- \( j' \) and mass-\( m \) eigenstates of the operators \( W^{2}(p) \), and \( P^{2} \), respectively. In case of \( j_{2} = 0 \), one sets \( \epsilon_{j'} = 0 \). In taking this path, one necessarily encounters Lagrangians that are second order in the momenta.

Second order fermion approaches have traditions in field theory \[7,8\], and are of growing popularity in QED as well as in QCD \[19,20,21\].

**Product spin-irrep projectors:**

Correspondingly, the master equation emerges from combining the covariant spin-irrep projectors as

\[ \left[ P_{W^{2}}^{(m,j)}(p) P_{F}^{(j_{0})} \right]_{\nu_{1}...\nu_{t}}^{\mu_{1}...\mu_{t}} \left[ \Psi^{(m,j)}(p) \right]_{\nu_{1}...\nu_{t}}^{\mu_{1}...\mu_{t}} = \left[ \Psi^{(m,j)}(p) \right]_{\nu_{1}...\nu_{t}}, \quad (2.14) \]
for pure spin-\(j\), or
\[
\left[ P^{(m,j)}_{\nu_1\nu_2} (p) \right] \left[ \Psi^{(m,j)} (s, \frac{1}{2}) \right]_{\mu_1...\mu_r} = \left[ \Psi^{(m,j)} (s, \frac{1}{2}) \right]_{\mu_1...\mu_r}, \quad j = s \pm \frac{1}{2}, \quad (2.15)
\]
for two-spin valued spaces \((2.2)\).

The present work focuses on the description of the pure spin-\(\frac{3}{2}\) Weinberg-Joos state, \((\frac{3}{2}, 0) \oplus (0, \frac{3}{2})\) as part of the antisymmetric tensor-spinor in \((2.3)\).

Though the representation spaces in \((1.10)\) underlying the scheme by Niederle and Nikitin are quite different from ours in \((2.1)\), occasionally a coincidence can occur, as it indeed happens for pure spin-\(\frac{1}{2}\) which in both approaches is described in terms of the anti-symmetric tensor spinor \((2.3)\). However, the assignment in \([17]\) of spin-\(\frac{1}{2}\) to the irreducible \((\frac{3}{2}, 0) \oplus (0, \frac{3}{2})\) sector of the anti-symmetric tensor spinor of second rank has not been made explicit, but might be hidden in the contraction by the Dirac matrices, possibly a remnant of a Lorentz projector. Also the wave equations in the two methods result different. Compared to \([17]\), our combined Lorentz-and Poincaré projector method has the advantage to apply both to bosons and fermions and to use in general tensors of lower ranks which is expected to significantly simplify calculations. To be specific, spin-\(\frac{1}{2}\) in \((2.5)\) within our framework can be described by a third rank tensor spinor, while the approach of \([17]\) relies upon a tensor spinor of fourth rank.

### 3 Covariant spin-irrep projectors within the antisymmetric tensor-spinor space

The anti-symmetric Lorentz tensor of second rank with spinor components, \(\Psi_{\mu \nu}\), is a two-spin valued reducible representation space of the \(so(1,3)\) algebra and has the three irreducible sectors \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\), \((\frac{1}{2}, 1) \oplus (1, \frac{1}{2})\), and \((\frac{3}{2}, 0) \oplus (0, \frac{3}{2})\) given in the above equation \((2.3)\). One spin-\(\frac{1}{2}\) state of say, of positive parity, resides in the Dirac sector, while the other, of negative parity, is part of \((\frac{1}{2}, 1) \oplus (1, \frac{1}{2})\). The latter space contains in addition a spin-\(\frac{3}{2}\) of same negative parity, opposite to the parity of the spin-\(\frac{1}{2}\) populating the remaining Weinberg-Joos sector, \((\frac{3}{2}, 0) \oplus (0, \frac{3}{2})\).

This section starts with the construction of the covariant spin-\(\frac{1}{2}\) and spin-\(\frac{3}{2}\) projector operators on the states with mass \(m\) within the anti-symmetric tensor-spinor space. As already announced in the introduction section, the pure-spin \(\frac{3}{2}\) component of this tensor is tracked down by first searching for its spin by means of a Poincaré covariant projector constructed along the lines of eq. \((2.12)\). Next, with the aid of eqs. \((2.10)\), we calculate the explicit form of the Lorentz projector that localizes the irreducible \((\frac{3}{2}, 0) \oplus (0, \frac{3}{2})\) subspace of interest. The wave equation is then obtained executing the prescription of \((2.14)\).

#### 3.1 The Pauli-Lubanski operator

We begin with constructing within the representation space of interest the Pauli-Lubanski vector, the key ingredient of the Poincaré covariant spin-\(\frac{3}{2}\) projector. The \(so(1,3)\) generators within the antisymmetric tensor-spinor (TS) are

\[
\begin{align*}
[M^{TS}_{\mu \nu}]_{AB} &= [M^T_{\mu \nu}]_{AB} 1^S_{ab} + 1^T_{AB} [M^S_{\mu \nu}]_{ab}, \quad A := [\alpha \beta], \quad B := [\gamma \delta], \\
M^S_{\mu \nu} &= \frac{1}{2} \sigma_{\mu \nu} = \frac{i}{4} [\gamma_\mu, \gamma_\nu],
\end{align*}
\]

\[(3.1)\]
where the capital letters \( A \) and \( B \) are the indexes within the \( B_{i\mu\nu} \sim (1,0) \odot (0,1) \) tensor \( (T) \) part, while \( a \) and \( b \) label the Dirac-spinor \( (S) \). In the following, the spinorial labels will be suppressed with the aim of simplifying notations. The \( [M_{\mu\nu}^T]_{AB} \) generators express in terms of the generators, \( [M_{\mu\nu}^{\frac{3}{2}, \frac{3}{2}}]_{\eta\tau} \) within the four-vector, \( (\frac{1}{2}, \frac{1}{2}) \), according to,

\[
[M_{\mu\nu}^T]_{[\alpha\beta][\gamma\delta]} = \frac{1}{2} \left( [M_{\mu\nu}^{\frac{3}{2}, \frac{3}{2}}]_{\alpha\gamma} g_{\beta\delta} + g_{\alpha\gamma} [M_{\mu\nu}^{\frac{3}{2}, \frac{3}{2}}]_{\beta\delta} - [M_{\mu\nu}^{\frac{3}{2}, \frac{3}{2}}]_{\gamma\delta} g_{\alpha\beta} - g_{\alpha\delta} [M_{\mu\nu}^{\frac{3}{2}, \frac{3}{2}}]_{\beta\gamma} \right),
\]

\[
= -2 \, \mathbf{1}_{\alpha\beta\gamma\delta} \epsilon_{\sigma} [M_{\mu\nu}^{\frac{3}{2}, \frac{3}{2}}]_{\alpha\sigma} \mathbf{1}_{\rho\sigma\gamma\delta},
\]

\[
[M_{\mu\nu}^{\frac{3}{2}, \frac{3}{2}}]_{\alpha\beta} = 2i \mathbf{1}_{\alpha\beta\mu\nu}.
\]

Now using eq. (2.13) for the Pauli-Lubanski vectors, the Poincaré projectors can be obtained along the line of (2.12), the task of the subsequent subsection.

### 3.2 The covariant spin-\( \frac{1}{2} \) and spin-\( \frac{3}{2} \) projectors from the Poincaré algebra invariants

The tensor-spinor representation space contains only two spin sectors, one corresponding to spin \( j = \frac{1}{2} \) and the other to \( j' = \frac{3}{2} \), then the corresponding projectors on mass-\( m \) and spin-\( j \) are found as [11]:

\[
P_{\mu\nu}^{(m, \frac{1}{2})}(p) \psi^{(m, \frac{1}{2})}(p) = \frac{p^2}{m^2} \left( \frac{W^2(p) - \epsilon_{\frac{1}{2}}}{\epsilon_{\frac{1}{2}} - \epsilon_{\frac{1}{2}}^2} \right) \psi^{(m, \frac{1}{2})}(p) = \psi^{(m, \frac{1}{2})}(p),
\]

\[
P_{\mu\nu}^{(m, \frac{3}{2})}(p) \psi^{(m, \frac{3}{2})}(p) = \frac{p^2}{m^2} \left( \frac{W^2(p) - \epsilon_{\frac{3}{2}}}{\epsilon_{\frac{3}{2}} - \epsilon_{\frac{3}{2}}^2} \right) \psi^{(m, \frac{3}{2})}(p) = \psi^{(m, \frac{3}{2})}(p).
\]

Here, \( \epsilon_j = -p^2 j (j + 1) \) is the \( W^2(p) \) eigenvalue for a generic state \( \psi^{(m,j)}(p) \), of mass-\( m \) and spin-\( j \). We calculate the following explicit expressions,

\[
\left[ P_{\mu\nu}^{(m, \frac{1}{2})} \right]_{\alpha\beta}[\gamma\delta] = \frac{2}{m^2} \epsilon_{\alpha\beta\gamma\delta} \prod_{\gamma\delta} + \chi_{\alpha\beta\gamma\delta} \prod_{\gamma\delta},
\]

\[
\left[ P_{\mu\nu}^{(m, \frac{3}{2})} \right]_{\alpha\beta}[\gamma\delta] = \frac{2}{m^2} \epsilon_{\alpha\beta\gamma\delta} \prod_{\gamma\delta} + \chi_{\alpha\beta\gamma\delta} \prod_{\gamma\delta},
\]

where

\[
\chi_{\alpha\beta\gamma\delta} = \frac{i}{2} \epsilon_{\alpha\beta\gamma\delta},
\]

is the chiral operator in the antisymmetric tensor representation [16], and the terms inside of the braces recover the momentum independent parts of the well known spin-\( \frac{1}{2} \) and spin-\( \frac{3}{2} \) projectors on the Rarita-Schwinger four vector-spinor \((VS) \) space [22]

\[
P_{\mu\nu}^{VS}(p, \frac{1}{2})_{\alpha\beta} = \frac{1}{p^2} \left( g_{\alpha\beta\mu\nu} - \frac{1}{3} \sigma_{\alpha\mu} \sigma_{\beta\nu} + \frac{1}{3} \sigma_{\alpha\nu} \sigma_{\beta\mu} \right) p^\mu p^\nu.
\]

\[
P_{\mu\nu}^{VS}(p, \frac{3}{2})_{\alpha\beta} = \frac{1}{p^2} \left( g_{\alpha\beta\mu\nu} - \frac{1}{3} \sigma_{\alpha\mu} \sigma_{\beta\nu} - g_{\alpha\mu} g_{\beta\nu} \right) p^\mu p^\nu.
\]
3.3 Covariant irrep-projectors from the Lorentz algebra invariants

The two Casimir invariants, $F$ and $G$ of the Lorentz algebra in (2.7) for the antisymmetric tensor-spinor are calculated using,

\[ [F]_{[\alpha\beta][\gamma\delta]} = \frac{1}{4} [M^\mu\nu]_{[\alpha\beta]} [\kappa\tau] [M_{\mu\nu}]_{[\kappa\tau][\gamma\delta]}, \]  
\[ (3.13) \]

\[ [G]_{[\alpha\beta][\gamma\delta]} = \frac{1}{4} \epsilon^{\mu\nu\sigma\rho} [M_{\mu\nu}]_{[\alpha\beta]} [\kappa\tau] [M_{\sigma\rho}]_{[\kappa\tau][\gamma\delta]}, \]  
\[ (3.14) \]

with the generators taken from (3.1). Their explicit forms are obtained as:

\[ [F]_{[\alpha\beta][\gamma\delta]} = -\frac{1}{8} (\sigma_{\alpha\beta}\sigma_{\gamma\delta} - \sigma_{\gamma\delta}\sigma_{\alpha\beta} - 22 \mathbf{1}_{\alpha\beta\gamma\delta}), \]  
\[ (3.15) \]

\[ [G]_{[\alpha\beta][\gamma\delta]} = \frac{i}{4} \chi_{\alpha\beta}^{\sigma\rho} (\sigma_{\sigma\rho}\sigma_{\gamma\delta} - \sigma_{\gamma\delta}\sigma_{\sigma\rho} - 16 \mathbf{1}_{\sigma\rho\gamma\delta}) - \frac{3}{2} i \gamma^{\beta} \mathbf{1}_{\alpha\beta\gamma\delta}. \]  
\[ (3.16) \]

The chiral operators $\gamma^5$ and $\chi$ change the parity of the states in the respective Dirac–and bi-vector sectors of representation space considered, so that the eigenstates of the $G$ invariant are parity-mixed chiral states. We here choose to work with states of well defined parities and construct the Lorentz projectors in terms of the $F$ invariant.

There are three Lorentz sectors of the type $(j_2, j_1) \oplus (j_1, j_2)$ in the antisymmetric tensor-spinor space, corresponding to $(j_2, j_1) = (\frac{3}{2}, 0)$, $(\frac{1}{2}, 1)$, and $(\frac{3}{2}, 0)$. The associated generic wave functions, $\psi(j_2, j_1)$, are characterized by their $\lambda(j_1, j_2)$ eigenvalues with respect to the $F$ invariant according to:

\[ F\psi(j_2, j_1) = \lambda(j_1, j_2)\psi(j_2, j_1) = \frac{1}{2} (K(K + 2) + M^2)\psi(j_1, j_2), \]  
\[ (3.17) \]

with

\[ K = j_1 + j_2, \quad M = |j_1 - j_2|. \]  
\[ (3.18) \]

All three eigenvalues are different and given by,

\[ \lambda(\frac{3}{2}, 0) = \frac{3}{4}, \quad \lambda(\frac{1}{2}, 1) = \frac{11}{4}, \quad \lambda(\frac{3}{2}, 0) = \frac{15}{4}. \]  
\[ (3.19) \]

Towards our goal, we define operators, $Q(j_1', j_2')$ and $Q(j_1'', j_2'')$ that suppress those $(j_1', j_2')$, and $(j_1'', j_2'')$ sectors within the tensor under investigation, which are different from the sector we are searching for,

\[ Q(j_1', j_2') = F - \lambda(j_1', j_2') \mathbf{1}. \]  
\[ (3.20) \]

In effect, the projector onto a selected irreducible Lorentz sector $(j_1, j_2)$ can be cast into the form,

\[ P_{F}^{(j_1, j_2)} = \frac{Q(j_1', j_2')Q(j_1'', j_2'')}{(\lambda(j_1, j_2) - \lambda(j_1', j_2'))(\lambda(j_1, j_2) - \lambda(j_1'', j_2'))}, \quad j_1', j_1'' \neq j_1, \quad j_2', j_2'' \neq j_2, \]  
\[ (3.21) \]

with $\lambda(j_1, j_2)$, $\lambda(j_1', j_2')$, $\lambda(j_1'', j_2'')$ from eq. (3.19). This way we find the following projectors forming a complete set:

\[ \left[ P_{F}^{(\frac{3}{2}, 0)} \right]_{\alpha\beta\gamma\delta} = \frac{1}{12} \sigma_{\alpha\beta}\sigma_{\gamma\delta}, \]  
\[ (3.22) \]

\[ \left[ P_{F}^{(\frac{1}{2}, 1)} \right]_{\alpha\beta\gamma\delta} = 1_{\alpha\beta\gamma\delta} - \frac{1}{8} (\sigma_{\alpha\beta}\gamma_{\delta} + \sigma_{\gamma\delta}\alpha_{\beta}), \]  
\[ (3.23) \]

\[ \left[ P_{F}^{(\frac{3}{2}, 0)} \right]_{\alpha\beta\gamma\delta} = \frac{1}{8} (\sigma_{\alpha\beta}\gamma_{\delta} + \sigma_{\gamma\delta}\alpha_{\beta}) - \frac{1}{12} \sigma_{\alpha\beta}\sigma_{\gamma\delta}. \]  
\[ (3.24) \]
3.4 Wave equations for particles belonging to the anti-symmetric tensor spinor irreps

The commutativity between the $F$ invariant of $so(1,3)$, on the one side, and the $P^2$ and $W^2(p)$ invariants of the Poincaré algebra, on the other, makes their diagonalizing in same generic basis (here denoted by $\psi_{(j_1,j_2)}^{(m,j)}(p)$) possible. In this manner, the spin- $j$ of interest is unambiguously assigned to the $(j_1,j_2) \oplus (j_2,j_1)$ representation space of interest, according to,

$$\Pi^{(j_1,j_2);j}(p) = P_F^{(j_1,j_2)} P_W^{(m,j)}(p), \quad \Pi^{(j_1,j_2);j}(p) \Psi^{(m,j)}_{(j_1,j_2)}(p) = \Psi^{(m,j)}_{(j_1,j_2)}(p), \quad j \in [j_1 - j_2, (j_1 + j_2)],$$

(3.25)

again, $\Psi^{(m,j)}_{(j_1,j_2)}(p)$ is a generic state of mass $m$, spin $j$ that transforms in the $(j_1,j_2)$ irrep. Within the antisymmetric tensor-spinor there are six different products of Lorentz and Poincaré projectors, denoted by $\Pi^{(j_1,j_2);j}(p)$, two of them vanishing :

$$\Pi^{(j_0,j_0);j}(p) = P_F^{(j_0,j_0)} P_W^{(m,j)}(p) = 0, \quad \Pi^{(j_0,j_0);j}(p) = P_F^{(j_0,j_0)} P_W^{(m,j)}(p) = 0.$$

(3.26)

(3.27)

The remaining four products can be summarized as follows:

$$\left[\Pi^{(j_1,j_2);j}(p)\right]^{[\alpha \beta]_{[\gamma \delta]}^{[\gamma \delta]}}(\Psi^{(m,j)}_{(j_1,j_2)}) = 0, \quad j = \frac{1}{2}, \frac{3}{2},$$

(3.28)

where

$$\left[\Pi^{(j_1,j_2);j}(p)\right]^{[\alpha \beta]_{[\gamma \delta]}^{[\gamma \delta]}} = \left[\Gamma^{(j_1,j_2);j}_{\mu \nu}\right]^{[\alpha \beta]_{[\gamma \delta]}^{[\gamma \delta]}} p^\mu p^\nu - m^2 1^{[\alpha \beta]_{[\gamma \delta]}}, \quad \left[\Gamma^{(j_1,j_2);j}_{\mu \nu}\right]^{[\alpha \beta]_{[\gamma \delta]}^{[\gamma \delta]}} p^\mu p^\nu = m^2 \left[\Pi^{(j_1,j_2);j}_{[\alpha \beta]_{[\gamma \delta]}}\right]^{[\sigma \rho]_{[\eta \zeta]}},$$

(3.29)

(3.30)

or equivalently

$$\left[\Gamma^{(j_1,j_2);j}_{\mu \nu}\right]^{[\alpha \beta]_{[\gamma \delta]}^{[\gamma \delta]}} p^\mu p^\nu = m^2 \left[\Pi^{(j_1,j_2);j}_{[\alpha \beta]_{[\gamma \delta]}}\right]^{[\sigma \rho]_{[\eta \zeta]}},$$

(3.31)

explicitly,

$$\left[\Gamma^{(j_0,j_0);j}_{\mu \nu}\right]^{[\alpha \beta]_{[\gamma \delta]}^{[\gamma \delta]}} = 4 \left[\Pi^{(1,1);j}_{[\alpha \beta]_{[\gamma \delta]}}\right]^{[\sigma \rho]_{[\eta \zeta]}},$$

(3.32)

$$\left[\Gamma^{(j_0,j_0);j}_{\mu \nu}\right]^{[\alpha \beta]_{[\gamma \delta]}^{[\gamma \delta]}} = 4 \left[\Pi^{(1,1);j}_{[\alpha \beta]_{[\gamma \delta]}}\right]^{[\sigma \rho]_{[\eta \zeta]}},$$

(3.33)

$$\left[\Gamma^{(j_0,j_0);j}_{\mu \nu}\right]^{[\alpha \beta]_{[\gamma \delta]}^{[\gamma \delta]}} = 4 \left[\Pi^{(1,1);j}_{[\alpha \beta]_{[\gamma \delta]}}\right]^{[\sigma \rho]_{[\eta \zeta]}},$$

(3.34)

$$\left[\Gamma^{(j_0,j_0);j}_{\mu \nu}\right]^{[\alpha \beta]_{[\gamma \delta]}^{[\gamma \delta]}} = 4 \left[\Pi^{(1,1);j}_{[\alpha \beta]_{[\gamma \delta]}}\right]^{[\sigma \rho]_{[\eta \zeta]}},$$

(3.35)

The propagators are found by inverting (3.30) as,

$$S^{(j_1,j_2);j}(p)^{[\alpha \beta]_{[\gamma \delta]}} = \left(\left[\Pi^{(j_1,j_2);j}_{[\alpha \beta]_{[\gamma \delta]}} - m^2 1^{[\alpha \beta]_{[\gamma \delta]}}\right]^{-1}\right),$$

(3.36)

$$\left[\Delta^{(j_1,j_2);j}_{[\alpha \beta]_{[\gamma \delta]}}\right]^{[\alpha \beta]_{[\gamma \delta]}} = \frac{1}{m^2} \left[\Gamma^{(j_1,j_2);j}_{[\alpha \beta]_{[\gamma \delta]}}\right]^{[\alpha \beta]_{[\gamma \delta]}},$$

(3.37)

$$\left[\Delta^{(j_1,j_2);j}_{[\alpha \beta]_{[\gamma \delta]}}\right]^{[\alpha \beta]_{[\gamma \delta]}} = \frac{1}{m^2} \left[\Gamma^{(j_1,j_2);j}_{[\alpha \beta]_{[\gamma \delta]}}\right]^{[\alpha \beta]_{[\gamma \delta]}},$$

(3.38)
4 The explicit irreducible degrees of freedom spanning the antisymmetric tensor spinor space

This section is devoted to the explicit construction of the 24 irreducible degrees of freedom spanning the antisymmetric tensor spinor space. In what follows we shall systematically omit the brackets indicating the antisymmetric indexes for the purpose of simplifying notations and hope that this will not create confusions.

4.1 Diagonalizing the covariant spin projector

4.1.1 Spin-1/2 states within the antisymmetric tensor spinor and Dirac spinors

The direct-product nature of the antisymmetric tensor spinor (2.3) allows to construct the spin states residing there making use of angular momentum addition theorems. There are two sorts of spin 1/2 states one can construct, the first kind is built up from positive parity spin-1 states in (1, 0) ⊕ (0, 1), here denoted by [η⁺(p, ℓ)]αβ, and Dirac’s u and v spinors of opposite parities, here denoted as u±(p, λ). The resulting spin-1/2 Clebsh-Gordan combination being

$$\begin{aligned}
\left[ U^{(+)}_\pm \left( p, \frac{1}{2}, \frac{1}{2} \right) \right]^{\alpha\beta} = & -\sqrt{\frac{1}{3}} [\eta⁺(p, 0)]^{\alpha\beta} u_\pm \left( p, \frac{1}{2} \right) + \sqrt{\frac{2}{3}} [\eta⁺(p, 1)]^{\alpha\beta} u_\pm \left( p, \frac{1}{2} \right), \\
\left[ U^{(+)}_\pm \left( p, \frac{1}{2}, -\frac{1}{2} \right) \right]^{\alpha\beta} = & \sqrt{\frac{1}{3}} [\eta⁺(p, 0)]^{\alpha\beta} u_\pm \left( p, -\frac{1}{2} \right) + \sqrt{\frac{2}{3}} [\eta⁺(p, -1)]^{\alpha\beta} u_\pm \left( p, -\frac{1}{2} \right),
\end{aligned}$$

(4.1)

and

$$\begin{aligned}
\left[ U^{(-)}_\pm \left( p, \frac{1}{2}, \frac{1}{2} \right) \right]^{\alpha\beta} = & -\sqrt{\frac{1}{3}} [\eta⁻(p, 0)]^{\alpha\beta} u_\mp \left( p, \frac{1}{2} \right) + \sqrt{\frac{2}{3}} [\eta⁻(p, 1)]^{\alpha\beta} u_\mp \left( p, -\frac{1}{2} \right), \\
\left[ U^{(-)}_\pm \left( p, \frac{1}{2}, -\frac{1}{2} \right) \right]^{\alpha\beta} = & \sqrt{\frac{1}{3}} [\eta⁻(p, 0)]^{\alpha\beta} u_\mp \left( p, -\frac{1}{2} \right) + \sqrt{\frac{2}{3}} [\eta⁻(p, -1)]^{\alpha\beta} u_\mp \left( p, -\frac{1}{2} \right).
\end{aligned}$$

(4.2)

There is a remarkable relationship between the basis states, $$A^{\mu}_\pm(p, \ell)$$, within the four-vector space (with the low case signs denoting the parity), and the associated Pauli-Lubanski vector, $$W^\mu(p)$$, on the one side, and the basis states within the anti-symmetric tensor spinor, on the other. Namely, these bases are intertwined as follows,

$$\begin{aligned}
[\eta⁺(p, \ell)]^{\alpha\beta} = & -\frac{1}{\sqrt{2m}} \left[ W_\mu(p) \right]^{\alpha\beta} A^{\mu}(p, \ell), \\
[\eta⁻(p, \ell)]^{\alpha\beta} = & \chi^{\alpha\beta\gamma\delta}[\eta⁺(p, \ell)]^{\gamma\delta},
\end{aligned}$$

(4.6)

where

$$\begin{aligned}
\left[ W(p) \right]^{\alpha\beta} = & \frac{1}{2} \epsilon^{\lambda\sigma\rho\mu}[M^{\alpha\beta\gamma\delta}]^{\gamma\delta} p_\mu.
\end{aligned}$$

(4.7)

In result, the eqs. (4.3) and (4.5) simplify to,

$$\begin{aligned}
\left[ U^{(+)}_\pm \left( p, \frac{1}{2} \right) \right]^{\alpha\beta} = & -\frac{1}{\sqrt{2m}} \left[ W_\mu(p) \right]^{\alpha\beta} \gamma^5 U^{\mu}_\pm \left( p, \frac{1}{2} \right).
\end{aligned}$$

(4.8)
Here, \( A^\alpha_\pm(p, \ell) \) and \( u^\pm(p, \lambda) \) have now been absorbed by the spin-\( \frac{3}{2} \) Rarita-Schwinger four-vector-spinor \( \mathcal{U}^\mu_\pm(p, \frac{1}{2}, \lambda) \). A further significant simplification is achieved by noticing that spin-\( \frac{3}{2} \) four-vector spinors, \( \mathcal{U}^\mu_\pm(p, \frac{1}{2}, \lambda) \), by themselves can be re-expressed in terms of the Dirac spinor and the Pauli-Lubanski vector, \( w^\alpha(p) \), in \( \left( \frac{1}{2}, 0 \right) \oplus \left( 0, \frac{1}{2} \right) \). Namely, the following relation holds valid,

\[
\mathcal{U}^\mu_\pm(p, \frac{1}{2}, \lambda) = \frac{2}{\sqrt{3}m}[\omega(p)]^\alpha \gamma^5 u^\pm(p, \lambda).
\]  \( \tag{4.10} \)

\[
[\omega(p)]^\alpha = -\frac{i}{2}\gamma^5 \sigma^{\alpha\nu} p_\nu.
\]  \( \tag{4.11} \)

In effect, the spin-\( \frac{1}{2} \) degrees of freedom within the antisymmetric tensor spinor equivalently rewrite as,

\[
\left[ U_\pm^{(+)} \left( p, \frac{1}{2}, \lambda \right) \right]^{\alpha\beta} = -\frac{2}{\sqrt{6}m^2}[\chi_\mu(p)]^{\alpha\beta} p^\nu \gamma^5 u^\pm(p, \lambda),
\]  \( \tag{4.12} \)

\[
\left[ U_\pm^{(-)} \left( p, \frac{1}{2}, \lambda \right) \right]^{\alpha\beta} = \frac{2}{\sqrt{6}m^2} \chi^{\alpha\beta} \gamma_5 \gamma_\mu(p) u^\pm(p, \lambda),
\]  \( \tag{4.13} \)

yielding the following compact expressions:

\[
\left[ U_\pm^{(+)} \left( p, \frac{1}{2}, \lambda \right) \right]^{\alpha\beta} = \frac{2}{\sqrt{6}m^2} \chi^{\alpha\beta} \gamma_\mu(p) u^\pm(p, \lambda),
\]  \( \tag{4.14} \)

\[
\left[ U_\pm^{(-)} \left( p, \frac{1}{2}, \lambda \right) \right]^{\alpha\beta} = \frac{2}{\sqrt{6}m^2} \chi^{\alpha\beta} \gamma_\mu(p) u^\pm(p, \lambda).
\]  \( \tag{4.15} \)

The conjugate states are now easy to define in terms of the Dirac conjugate spinors and read,

\[
\left[ \overline{U}_\pm^{(+)/(−)} \left( p, \frac{1}{2}, \lambda \right) \right]^{\alpha\beta} = \left[ \left[ \gamma^0 U_\pm^{(+)/(−)} \left( p, \frac{1}{2}, \lambda \right) \right]^{\alpha\beta} \right]^\dagger.
\]  \( \tag{4.16} \)

The explicit expressions are,

\[
\left[ \overline{U}_\pm^{(+)} \left( p, \frac{1}{2}, \lambda \right) \right]^{\alpha\beta} = \frac{2}{\sqrt{6}m^2} \chi_\mu(p, \lambda) \gamma^5 \gamma^\mu \gamma^\rho \chi^{\alpha\beta},
\]  \( \tag{4.17} \)

\[
\left[ \overline{U}_\pm^{(−)} \left( p, \frac{1}{2}, \lambda \right) \right]^{\alpha\beta} = \frac{2}{\sqrt{6}m^2} \chi_\mu(p, \lambda) \gamma^5 \gamma^\mu \gamma^\rho \chi^{\alpha\beta},
\]  \( \tag{4.18} \)

where we have used \( (\chi^{\alpha\beta\gamma})^* = -\chi^{\alpha\beta\gamma} \) and \( \gamma^5\gamma^\rho = -\gamma^\rho\gamma^5 \). The spin-\( \frac{1}{2} \) states are normalized as,

\[
\left[ \overline{U}_\pm^{(+)/(−)} \left( p, \frac{1}{2}, \lambda' \right) \right]^{\alpha\beta} \left[ U_\pm^{(+)/(−)} \left( p, \frac{1}{2}, \lambda \right) \right]^{\alpha\beta} = \pm \delta^\lambda_{\lambda'},
\]  \( \tag{4.19} \)

and orthogonal according to,

\[
\left[ \overline{U}_\pm^{(+)} \left( p, \frac{1}{2}, \lambda' \right) \right]^{\alpha\beta} \left[ U^{(+)}_\pm \left( p, \frac{1}{2}, \lambda \right) \right]^{\alpha\beta} = 0,
\]  \( \tag{4.20} \)

\[
\left[ \overline{U}_\pm^{(−)} \left( p, \frac{1}{2}, \lambda' \right) \right]^{\alpha\beta} \left[ U^{(−)}_\pm \left( p, \frac{1}{2}, \lambda \right) \right]^{\alpha\beta} = 0.
\]  \( \tag{4.21} \)
4.1.2 Spin-$\frac{3}{2}$ states within the antisymmetric tensor spinor and Rarita-Schwinger four-vector spinors

The spin-$\frac{3}{2}$ states arising from the coupling of the positive parity vectors spanning the $(1, 0) \oplus (0, 1)$ space with the Dirac $u$ and $v$ spinors (here denoted by $u_\pm$) are

$$U_{\pm}^{(+)}(p, \frac{3}{2}, \frac{3}{2})_{\alpha\beta} = [\eta_+(p, 1)]^{\alpha\beta} u_\pm(p, \frac{1}{2}),$$  (4.22)

$$U_{\pm}^{(+)}(p, \frac{3}{2}, \frac{1}{2})_{\alpha\beta} = \frac{1}{\sqrt{3}}[\eta_+(p, 1)]^{\alpha\beta} u_\pm(p, -\frac{1}{2}) + \sqrt{\frac{2}{3}}[\eta_+(p, 0)]^{\alpha\beta} u_\pm(p, \frac{1}{2}),$$  (4.23)

$$U_{\pm}^{(+)}(p, \frac{3}{2}, -\frac{1}{2})_{\alpha\beta} = \frac{1}{\sqrt{3}}[\eta_+(p, -1)]^{\alpha\beta} u_\pm(p, \frac{1}{2}) + \sqrt{\frac{2}{3}}[\eta_+(p, 0)]^{\alpha\beta} u_\pm(p, -\frac{1}{2}),$$  (4.24)

$$U_{\pm}^{(+)}(p, \frac{3}{2}, -\frac{3}{2})_{\alpha\beta} = [\eta_+(p, -1)]^{\alpha\beta} u_\pm(p, -\frac{1}{2}).$$  (4.25)

Again, making use of (4.6), allows for the simplifications,

$$U_{\pm}^{(+)}(p, \frac{3}{2}, \lambda)_{\alpha\beta} = -\frac{1}{\sqrt{2m}}W_\mu(\frac{3}{2}, \lambda)(p)_{\alpha\beta} u_\pm, \quad (4.26)$$

where $U^\mu_\pm(p, \frac{3}{2}, \lambda)$ are the standard Rarita-Schwinger spin-$\frac{3}{2}$ four-vector-spinors. Explicit use of the expression of $M_{\mu\nu}^{(\frac{3}{2})}$ in (3.5) to calculate $W_\mu(\frac{3}{2}, \lambda)(p)$ amounts again to a relationship between the spin-$\frac{3}{2}$ degrees within the anti-symmetric tensor spinor and those within the four-vector spinor,

$$U_{\pm}^{(+)}(p, \frac{3}{2}, \lambda)_{\alpha\beta} = -\frac{2}{\sqrt{2m}}\chi^{\alpha\beta \gamma\delta} U_{\pm}(p, \frac{3}{2}, \lambda)_{\gamma\delta},$$  (4.27)

$$U_{\pm}^{(-)}(p, \frac{3}{2}, \lambda)_{\alpha\beta} = \frac{2}{\sqrt{2m}}1^{\alpha\beta \mu\nu} U_{\pm}(p, \frac{3}{2}, \lambda)_{\gamma\delta}.$$  (4.28)

Finally, the couplings of the spin-1$^-$ vectors from $(1, 0) \oplus (0, 1)$ with the Dirac spinors emerge as

$$U_{\pm}^{(-)}(p, \frac{3}{2}, \lambda)_{\alpha\beta} = \chi^{\alpha\beta \gamma\delta} U_{\pm}^{(+)}(p, \frac{3}{2}, \lambda)_{\gamma\delta}.$$  (4.29)

The respective conjugate states are then found according to,

$$\overline{U}_{\pm}^{(+)}(p, \frac{3}{2}, \lambda)_{\alpha\beta} = -\frac{2}{\sqrt{2m}}\overline{U}_{\pm}(p, \frac{3}{2}, \lambda)_{\alpha\beta},$$  (4.30)

$$\overline{U}_{\pm}^{(-)}(p, \frac{3}{2}, \lambda)_{\alpha\beta} = -\frac{2}{\sqrt{2m}}1^{\alpha\beta \mu\nu} U_{\pm}(p, \frac{3}{2}, \lambda)_{\gamma\delta}.$$  (4.31)

The latter expressions allow for an easy calculation of the norms of the states under discussion as

$$\overline{U}_{\pm}^{ (+(-)}(p, \frac{3}{2}, \lambda')_{\alpha\beta} \overline{U}_{\pm}^{ (+(-)}(p, \frac{3}{2}, \lambda)_{\alpha\beta} = \pm 16 \delta_{\lambda},$$  (4.32)

and obey the following relationships,

$$\overline{U}_{\pm}^{(+)}(p, \frac{3}{2}, \lambda')_{\alpha\beta} \overline{U}_{\pm}^{(+)}(p, \frac{3}{2}, \lambda)_{\alpha\beta} = 0,$$  (4.33)

$$\overline{U}_{\pm}^{(-)}(p, \frac{3}{2}, \lambda')_{\alpha\beta} \overline{U}_{\pm}^{(-)}(p, \frac{3}{2}, \lambda)_{\alpha\beta} = 0.$$  (4.34)
4.2 Diagonalizing the covariant spin–irrep projectors

It is not difficult to verify that none of the sets of $U$ states diagonalizing a covariant spin projector is an eigenstate to the invariants of the Lorentz algebra from the above subsection 3.3. The spin-$\frac{1}{2}$ states that diagonalize the $F$ invariant, here denoted by $w_{\pm}^{(\frac{1}{2},0)}(p,\frac{1}{2},\lambda)$, with the lower case index, $\pm$, again indicating the parity, are found, modulo a constant, through the projections of the spin-$\frac{1}{2}$ states diagonalizing the Poincaré projector from the previous subsection on $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ as following,

$$
[w_{\pm}^{(\frac{1}{2},0)}(p,\frac{1}{2},\lambda)]^{\alpha\beta} = N \left[ P_F^{(\frac{1}{2},0)} \right]^{\alpha\beta\gamma\delta} \left[ U_{\pm}^{-}\left(\frac{1}{2},\lambda\right) \right]_{\gamma\delta}, \quad (4.35)
$$

$$
[w_{\pm}^{(\frac{1}{2},0)}(p,\frac{1}{2},\lambda)]^{\alpha\beta} = N \left[ P_F^{(\frac{1}{2},0)} \right]^{\alpha\beta\gamma\delta} \left[ U_{\pm}^{+}\left(\frac{1}{2},\lambda\right) \right]_{\gamma\delta}, \quad (4.36)
$$

where $N$ is a normalization factor and we have used

$$
[P_F^{(\frac{1}{2},0)}]^{\alpha\beta\rho\sigma} \chi_{\rho\sigma\gamma\delta} \gamma^5 = \left[ P_F^{(\frac{1}{2},0)} \right]^{\alpha\beta} \gamma_5 = \left[ P_F^{(\frac{1}{2},0)} \right]^{\alpha\beta\rho\sigma} \chi_{\rho\sigma\gamma\delta} \gamma^5, \quad (4.37)
$$

In a way similar, the irreducible states residing within $(\frac{1}{2},1) \oplus (1,\frac{1}{2})$ are defined according to,

$$
[w_{\pm}^{(\frac{1}{2},1)}(p,j,\lambda)]^{\alpha\beta} = -N \left[ P_F^{(\frac{1}{2},1)} \right]^{\alpha\beta\gamma\delta} \left[ U_{\pm}^{-}(p,j,\lambda) \right]_{\gamma\delta}, \quad (4.38)
$$

$$
[w_{\pm}^{(\frac{1}{2},1)}(p,j,\lambda)]^{\alpha\beta} = N \left[ P_F^{(\frac{1}{2},1)} \right]^{\alpha\beta\gamma\delta} \left[ U_{\pm}^{+}(p,j,\lambda) \right]_{\gamma\delta}, \quad (4.39)
$$

with $j = \frac{1}{2}, \frac{3}{2}$. Further use has been made of the following relationships,

$$
- \left[ P_F^{(\frac{1}{2},1)} \right]^{\alpha\beta\rho\sigma} \chi_{\rho\sigma\gamma\delta} \gamma^5 = \left[ P_F^{(\frac{1}{2},1)} \right]^{\alpha\beta} \gamma_5 = \left[ P_F^{(\frac{1}{2},1)} \right]^{\alpha\beta\rho\sigma} \chi_{\rho\sigma\gamma\delta} \gamma^5. \quad (4.40)
$$

Finally for the last sector $(\frac{3}{2},0) \oplus (0,\frac{3}{2})$ we have,

$$
[w_{\pm}^{(\frac{3}{2},0)}(p,\frac{3}{2},\lambda)]^{\alpha\beta} = N \left[ P_F^{(\frac{3}{2},0)} \right]^{\alpha\beta\gamma\delta} \left[ U_{\pm}^{-}\left(\frac{3}{2},\lambda\right) \right]_{\gamma\delta}, \quad (4.41)
$$

$$
[w_{\pm}^{(\frac{3}{2},0)}(p,\frac{3}{2},\lambda)]^{\alpha\beta} = N \left[ P_F^{(\frac{3}{2},0)} \right]^{\alpha\beta\gamma\delta} \left[ U_{\pm}^{+}\left(\frac{3}{2},\lambda\right) \right]_{\gamma\delta}. \quad (4.42)
$$

Here, use has been made of a relationship similar to that in (4.40) and given by,

$$
[P_F^{(\frac{3}{2},0)}]^{\alpha\beta\rho\sigma} \chi_{\rho\sigma\gamma\delta} \gamma^5 = \left[ P_F^{(\frac{3}{2},0)} \right]^{\alpha\beta} \gamma_5 = \left[ P_F^{(\frac{3}{2},0)} \right]^{\alpha\beta\rho\sigma} \chi_{\rho\sigma\gamma\delta} \gamma^5. \quad (4.43)
$$

In result, the states under discussion are related to the Rarita-Schwinger four vector-spinors, $\mathcal{U}_\pm^{\nu}(p,j,\lambda)$, as

$$
[u_{\pm}^{(j_1,j_2)}(p,j,\lambda)]^{\alpha\beta} = N [P_F^{(j_1,j_2)}]^{\alpha\beta\gamma\delta} \left[ U_{\pm}^{(j_1,j_2)}(p,j,\lambda) \right]_{\gamma\delta}, \quad (4.44)
$$

$$
[u_{\pm}^{(j_1,j_2)}(p,j,\lambda)]^{\alpha\beta} = \frac{2N}{\sqrt{2m}} [P_F^{(j_1,j_2)}]^{\alpha\beta\rho\sigma} \chi_{\rho\sigma\gamma\delta} \gamma^5 \mu_\nu p^\mu \mathcal{U}_\pm^{\nu}(p,j,\lambda). \quad (4.45)
$$
The constant factor $N$ has to be chosen in a way ensuring the normalization of these states to $\pm 1$ in dependence on their respective parity, positive versus negative. Such a normalization is guaranteed by $N = \sqrt{2}$. In effect, all the irreducible degrees of freedom within the antisymmetric tensor-spinor can be expressed in terms of four-vector-spinors according to:

$$[w_{\pm}^{(j_1,j_2)}(p, j, \lambda)]^{\alpha\beta} = [f^{(j_1,j_2)}(p)]^{\alpha\beta\mu} U_{\mu}^{\pm}(p, j, \lambda)$$

(4.46)

with the $[f^{(j_1,j_2)}(p)]^{\alpha\beta\mu}$ tensors being defined as

$$[f^{(j_1,j_2)}(p)]^{\alpha\beta\mu} = \frac{2}{m} [P_F^{(j_1,j_2)}]^{\alpha\beta\gamma\mu} p_\gamma,$$

(4.47)

and the projectors form eqs. (3.22)-(3.24). The explicit expressions for $[f^{(j_1,j_2)}(p)]^{\alpha\beta\mu}$ are,

$$[f^{(\pm,0)}(p)]^{\alpha\beta\mu} = \frac{1}{6m} (3\sigma_{\alpha\mu} \sigma_{\beta\nu}) p_\nu,$$

(4.48)

$$[f^{(\pm,1)}(p)]^{\alpha\beta\mu} = \frac{2}{m} (\sigma_{\alpha\mu} \sigma_{\gamma\nu} + \sigma_{\gamma\mu} \sigma_{\alpha\nu}) p_\nu,$$

(4.49)

$$[f^{(\pm,2)}(p)]^{\alpha\beta\mu} = \frac{1}{m} \left( \alpha\beta\nu \sigma_{\gamma\mu} p_\nu - \frac{1}{6m} (\sigma_{\alpha\mu} \sigma_{\gamma\nu} + \sigma_{\gamma\mu} \sigma_{\alpha\nu}) p_\nu \right).$$

(4.50)

The orthogonality of the Lorentz projectors implies orthogonality of the $f$-tensors according to,

$$[\mathcal{F}^{(j_1,j_2)}(p)]^{\alpha\beta\mu} [f^{(j_1,j_2)}(p)]^{\alpha\beta\mu} = 0,$$

(4.51)

for $(j_1', j_2') \neq (j_1, j_2)$, and $[\mathcal{F}^{(j_1,j_2)}(p)]^{\alpha\beta\mu} = \gamma^0 ([f^{(j_1,j_2)}(p)]^{\alpha\beta\mu} \gamma^0$. For $(j_1', j_2') = (j_1, j_2)$ the above tensors obey the relations

$$[f^{(j_1,j_2)}(p)]^{\alpha\beta\mu} [\mathcal{F}^{(j_1,j_2)}(p)]^{\gamma\delta\mu} p_\mu = \frac{p^2}{m^2} [P_F^{(j_1,j_2)}]^{\alpha\beta\gamma\delta},$$

(4.52)

and

$$[\mathcal{F}^{(\pm,0)}(p)]^{\alpha\beta\mu} [f^{(\pm,0)}(p)]^{\alpha\beta\nu} = \frac{1}{m^2} (\frac{1}{3} \sigma_{\alpha\mu} \sigma_{\beta\nu}) p^\mu p^\nu,$$

(4.53)

$$[\mathcal{F}^{(\pm,1)}(p)]^{\alpha\beta\mu} [f^{(\pm,1)}(p)]^{\alpha\beta\nu} = \frac{1}{m^2} (g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\mu} g_{\beta\nu}) p^\mu p^\nu,$$

(4.54)

$$[\mathcal{F}^{(\pm,2)}(p)]^{\alpha\beta\mu} [f^{(\pm,2)}(p)]^{\alpha\beta\nu} = \frac{1}{m^2} (g_{\alpha\beta} g_{\mu\nu} - \frac{1}{3} \sigma_{\alpha\mu} \sigma_{\beta\nu} - g_{\alpha\mu} g_{\beta\nu}) p^\mu p^\nu.$$

(4.55)

The proof of the normalization to $(\pm 1)$ of the Lorentz eigenstates follows upon recognizing here, the first product as the spin-$\frac{1}{2}$ projector over the vector-spinors, $\mathcal{U}_\pm^\mu (p, \frac{1}{2}, \lambda)$, the second product as the spin-$1^{-}$ projector over the four vectors $\lambda^\mu (p, \ell)$, and the third product as the spin-$\frac{3}{2}$ projector over the four vector-spinors, $\mathcal{U}_\pm^\mu (p, \frac{3}{2}, \lambda)$, thus ending up with

$$[\mathcal{W}^{(j_1,j_2)}(p, j, \lambda)]^{\alpha\beta} [w_{\pm}^{(j_1,j_2)}(p, j, \lambda)]^{\alpha\beta} = \mathcal{U}_\pm^{\alpha\beta}(p, j, \lambda) [\mathcal{F}^{(j_1,j_2)}(p)]^{\alpha\beta\mu} [f^{(j_1,j_2)}(p)]^{\alpha\beta\mu} \mathcal{U}_\pm^{\alpha\beta}(p, j, \lambda) = \mathcal{U}_\pm^{\alpha\beta}(p, j, \lambda) \mathcal{U}_\pm^{\alpha\beta}(p, j, \lambda) = \pm 1,$$

(4.56)

where $[\mathcal{W}^{(j_1,j_2)}(p, j, \lambda)]^{\alpha\beta} = \gamma^0 ([w_{\pm}^{(j_1,j_2)}(p, j, \lambda)]^{\alpha\beta} \gamma^0.$
Our pure spin-$\frac{3}{2}$ spinors, \( w(\frac{\pi}{2}, 0) (p, \frac{3}{2}, \lambda) \)\(^\alpha\beta \), following from the equation (4.46), satisfy the condition
\[
\gamma_\alpha \gamma_\beta \left[ w(\frac{\pi}{2}, 0) (p, \frac{3}{2}, \lambda) \right]^{\alpha\beta} = 0,
\]
but are not solutions to the Dirac equation due to the non-commutativity of the \( f \)-tensors in (4.50)–(4.55) with \( p \cdot \gamma \).

5 Electromagnetic properties of particles transforming according to the irreducible sectors of the antisymmetric tensor-spinor space

In order to transfer the formalism to position space, one introduces plane waves of the type, \( \psi_\pm^{(j_1, j_2)} (x) = \int \frac{d^4 p}{(2\pi)^3} \exp(i p \cdot x) \psi(p, j_1, j_2) \). In so doing, the momentum-space wave equations in (3.28)–(3.31) amount to,
\[
\left( \Gamma_{\mu\nu}^{(j_1, j_2)} \right)_{\alpha\beta\gamma\delta} \partial^\mu \partial^\nu \left[ \psi_\pm^{(j_1, j_2)} (x) \right]_{\alpha\beta} = 0,
\]
Minimal gauging is then standard and introduced by replacing ordinary by covariant derivatives,
\[
\partial^\mu \rightarrow D^\mu = \partial^\mu + ieA^\mu
\]
where \( e \) is the electric charge of the particle.

5.1 The particle’s Lagrangians

The free equations of motion (5.1) for positive parity states can be now derived from the following Lagrangians:
\[
\mathcal{L}^{(j_1, j_2)}_{\text{free}} = (\partial^\mu \left[ \psi^{(j_1, j_2)} (x) \right] A) \Gamma_{\mu\nu}^{(j_1, j_2)} A \partial^\nu \left[ \psi^{(j_1, j_2)} (x) \right] - m^2 \psi^{(j_1, j_2)} (x) A \psi^{(j_1, j_2)} (x) A,
\]
where we suppressed the arguments for the sake of simplifying notations. The gauged Lagrangians are then
\[
\mathcal{L}^{(j_1, j_2)} = \left( D^\mu \left[ \psi^{(j_1, j_2)} (x) \right] A \right) \Gamma_{\mu\nu}^{(j_1, j_2)} A \partial^\nu \left[ \psi^{(j_1, j_2)} (x) \right] - m^2 \psi^{(j_1, j_2)} (x) A \psi^{(j_1, j_2)} (x) A,
\]
whose decomposition into free and interacting parts is standard and reads,
\[
\mathcal{L}^{(j_1, j_2)} = \mathcal{L}^{(j_1, j_2)}_{\text{free}} + \mathcal{L}^{(j_1, j_2)}_{\text{int}},
\]
where
\[
\mathcal{L}^{(j_1, j_2)}_{\text{int}} = -j_{(j_1, j_2)} A^\mu + k_{(j_1, j_2)} A^\mu A^\nu.
\]

Back to momentum space, we find
\[
j_{(j_1, j_2)} (p, \lambda, p', \lambda') = e^2 \left[ \psi^{(j_1, j_2)} (p', j, \lambda') \right] A \left[ \psi^{(j_1, j_2)} (p, j, \lambda) \right] A B\left[ \psi^{(j_1, j_2)} (p, j, \lambda) \right] B,
\]
\[
k_{(j_1, j_2)} (p, \lambda, p', \lambda') = e^2 \left[ \psi^{(j_1, j_2)} (p', j, \lambda') \right] A \left[ \psi^{(j_1, j_2)} (p, j, \lambda) \right] A B\left[ \psi^{(j_1, j_2)} (p, j, \lambda) \right] B,
\]
the vertexes being given as
\[
\left[ \psi^{(j_1, j_2)} (p', p) \right]_{AB} = \left[ \Gamma_{\mu\nu}^{(j_1, j_2)} \right]_{AB} A B \nu + \left[ \Gamma_{\mu\nu}^{(j_1, j_2)} \right]_{AB} A B \nu,
\]
\[
\left[ \psi^{(j_1, j_2)} (p', p) \right]_{AB} = \frac{1}{2} \left( \left[ \Gamma_{\mu\nu}^{(j_1, j_2)} \right]_{AB} A B \nu + \left[ \Gamma_{\mu\nu}^{(j_1, j_2)} \right]_{AB} A B \nu \right).
\]
5.2 Electromagnetic multipole moments

5.2.1 Spin \( \frac{1}{2} \) in \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\)

The electromagnetic current, \(j_{\mu}^{(j_1, j_2); j}(p', \lambda', p, \lambda)\), of a particle with spin-\(j\) residing in the \((j_1, j_2) \oplus (j_2, j_1)\) irreducible sector of the antisymmetric tensor-spinor space and between states on their mass-shells for the case under consideration can be simplified with the aid of the \(f\)-tensors in (4.47) and cast exclusively in terms of conventional Dirac-spinors as

\[
j_{\mu}^{(\frac{1}{2}, 0); \frac{1}{2}}(p', \lambda', p, \lambda) = e\bar{u}(p', \lambda')(2m\gamma_{\mu})u(p, \lambda).
\]  

(5.12)

This current is identical to Dirac’s electromagnetic current and implies precisely same electromagnetic multipole moments as a genuine Dirac particle.

5.2.2 Spin \( \frac{3}{2} \) in \((\frac{1}{2}, 1) \oplus (1, \frac{1}{2})\)

In a way similar, the current of the lower spin-\(\frac{3}{2}\) companion to spin-\(\frac{1}{2}\) in \((\frac{1}{2}, 1) \oplus (1, \frac{1}{2})\) can be calculated and simplified to express in terms of Dirac-spinors only with the result,

\[
j_{\mu}^{(\frac{3}{2}, 1); \frac{3}{2}}(p', \lambda', p, \lambda) = \frac{1}{3}e\bar{u}(p', \lambda')(4(p' + p)^{\mu} - 2m\gamma_{\mu})u(p, \lambda).
\]  

(5.13)
As expected, also this bilinear is nothing else but the electromagnetic current of spin-$\frac{1}{2}$ in the non-Dirac sector of the four-vector spinor.

### 5.2.3 Spin $\frac{3}{2}$ in $(\frac{1}{2}, 1) \oplus (1, \frac{1}{2})$

For this case, the electromagnetic current is calculated to take the following form,

$$ j_\mu^{(\frac{3}{2}, 1) \oplus (1, \frac{1}{2})} (p', \lambda', p, \lambda) = e\bar{U}_+^\lambda (p', \frac{3}{2}, \lambda') (2m_{\alpha\beta\gamma\mu}) U_\beta^\lambda (p, \frac{3}{2}, \lambda), \quad (5.14) $$

where $U_\beta^\lambda (p, \frac{3}{2}, \lambda)$ are the conventional positive parity states in the Rarita-Schwinger four vector-spinor, meaning that the electromagnetic moments under discussion are identical to those of a Rarita-Schwinger particle (see Eq. (4.28) in [13]) and read,

\[ Q^0_E(\lambda)|_{RS} = e, \quad (5.15a) \]
\[ Q^1_M(\lambda)|_{RS} = \frac{2}{3} \left( \frac{e}{2m} \right) \langle S_z \rangle, \quad (5.15b) \]
\[ Q^2_E(\lambda)|_{RS} = \frac{1}{3} \left( \frac{e}{m^2} \right) \langle A \rangle, \quad (5.15c) \]
\[ Q^3_M(\lambda)|_{RS} = 2 \left( \frac{e}{2m^2} \right) \langle B \rangle. \quad (5.15d) \]

Here, $Q^2_E(\lambda)$ and $Q^3_M(\lambda)$ in turn denote the electric quadrupole, and magnetic octupole moments. Their explicit values correspond to a particular polarization are found from $\langle O \rangle \equiv \langle \lambda|O|\lambda \rangle$ by inserting the explicit form of the $A$ and $B$ operators (in obvious notations):

\[ A = 3S_z^2 - S^2, \quad (5.16) \]
\[ B = S_z \left( 15S_z^2 - \frac{41}{5} S^2 \right). \quad (5.17) \]

The gyromagnetic factor associated with (5.15) can only be identified via its dipole magnetic moment, $Q^1_M(\lambda)$, in (5.15b) as $g_{RS} = \frac{2}{3}$. Therefore, in combination with the result of the preceding subsubsection, our method correctly describes the Rarita-Schwinger four-vector spinor sector of the antisymmetric tensor-spinor space. However within the more general method of the Poincaré covariant projectors for spin-$\frac{3}{2}$ description within the four-vector spinor space [11], the gyromagnetic ratio, $g$, is identified at the level of the current and all the electromagnetic moments depend only on this parameter, because the currents within this method exhibit the general structure of two-term Gordon-decompositions. The general expressions reported within the latter method are essentially different from (5.15), specially in the relations between the highest moments with respect to the dipole magnetic moment, they can be found in [13] and read

\[ Q^0_E(\lambda)|_{VS} = e, \quad (5.18a) \]
\[ Q^1_M(\lambda)|_{VS} = g \left( \frac{e}{2m} \right) \langle S_z \rangle, \quad (5.18b) \]
\[ Q^2_E(\lambda)|_{VS} = (1 - g) \left( \frac{e}{m^2} \right) \frac{1}{3} \langle A \rangle, \quad (5.18c) \]
\[ Q^3_M(\lambda)|_{VS} = g \left( \frac{e}{2m^3} \right) \langle B \rangle. \quad (5.18d) \]

The $g$ value for the highest spin $\frac{3}{2}$ within the four-vector spinor space has been fixed in [11] to $g = 2$ from the requirement on causality of propagation within an electromagnetic environment.
5.2.4 Spin $\frac{3}{2}$ in $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$

Executing the same strategy as in the previous subsections, we calculate the electromagnetic multipole moments of a particle transforming in the single spin $\frac{3}{2}$ irreducible Weinberg-Joos sector of the antisymmetric tensor spinor of interest. In this case the current reads

\[
j_\mu^{(\frac{3}{2}, 0); \frac{3}{2}}(p', \lambda', p, \lambda) = e \left[ w^{(\frac{3}{2}, 0); \frac{3}{2}}(p', p) \right]_{AB} \left[ v^{(\frac{3}{2}, 0); \frac{3}{2}}(p, 3/2, \lambda) \right]_{AB} \tag{5.19}
\]

with the $v^{(\frac{3}{2}, 0); \frac{3}{2}}(p', p)$ vertex given in (5.9). The latter has been expressed in terms of the $[\Gamma^{(3/2, 0); \frac{3}{2}}_{\mu\nu}]_{AB}$ tensor in (3.35), and of the Lorentz-invariant irreps projector, $[P^{(3/2, 0)}_{AB}]$, in (3.24). This current can be further simplified in taking advantage of the equation (4.46), which relates tensor-spinors to vector-spinors. In so doing, the current in (5.19) re-expresses in terms of vector-spinors as:

\[
j_\mu^{(\frac{3}{2}, 0); \frac{3}{2}}(p', \lambda', p, \lambda) = \frac{38}{9} e U_+ \left( p', \frac{3}{2}, \lambda' \right) \left( (p' + p)_\mu g_{\alpha\beta} - mg_{\alpha\beta} \gamma_\mu \right)
+ \frac{20}{38m} (p_{\alpha} p'_\beta - p'_{\alpha} p_{\beta}) g_{\alpha\beta} \gamma_\mu U_+ \left( p, \frac{3}{2}, \lambda \right). \tag{5.20}
\]

The procedure to find the multipole moments from known currents is well established (see for example [23], [13] and references therein) and amounts to

\[
Q_0^E(\lambda) = e, \tag{5.21a}
\]
\[
Q_1^E(\lambda) = \frac{2}{3} \left( \frac{e}{2m} \right) \langle S_z \rangle, \tag{5.21b}
\]
\[
Q_2^E(\lambda) = -\frac{1}{3} \left( \frac{e}{m^2} \right) \langle A \rangle, \tag{5.21c}
\]
\[
Q_3^E(\lambda) = -2 \left( \frac{e}{2m^3} \right) \langle B \rangle. \tag{5.21d}
\]

The latter expressions fully coincide in form with those earlier reported in [13] and equivalent to the Weinberg-Joos formalism where the calculation have been carried out while treating the states under consideration as eight-component vectors. The difference is that here the gyromagnetic ratio is fixed to the inverse of the spin, $g = \frac{3}{2}$, and in accord with Belinfante's conjecture, while in [13], where only a covariant spin-projector has been used, $g$ had remained unspecified according to (see Table 1 in [13]),

\[
Q_0^E(\lambda)_{TS} = e, \tag{5.22a}
\]
\[
Q_1^E(\lambda)_{TS} = g \left( \frac{e}{2m} \right) \langle S_z \rangle, \tag{5.22b}
\]
\[
Q_2^E(\lambda)_{TS} = -(1 - g) \left( \frac{e}{m} \right) \langle A \rangle, \tag{5.22c}
\]
\[
Q_3^E(\lambda)_{TS} = -3g \left( \frac{e}{2m^3} \right) \langle B \rangle. \tag{5.22d}
\]

We conclude that the antisymmetric tensor-spinor is perfectly well suited for the adequate description of particles of spin $\frac{3}{2}$ transforming in the single-spin valued Weinberg-Joos representation space. Notice difference between the sets of observables in (5.22) and (5.18).

6 Compton scattering off spin-$\frac{3}{2}$ particles in $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$

The formalism developed in the present work provides a well defined and comfortably manageable technical tool for calculations of scattering cross sections in terms of matter fields as Lorentz tensors,
thus avoiding the cumbersome and computer time consuming matrix spinor calculus, which for the specific case under consideration would require the construction of $8 \times 8$ matrix invariants for diagonal processes, or, rectangular $4 \times 8$ bilinears for spin-$\frac{1}{2} \rightarrow \frac{3}{2}$ transitions. Above, we employed this tool to calculate the electromagnetic multipole moments of all the particles populating the irreducible sectors of the antisymmetric-tensor spinor space. However, the latter properties characterize the particles when they are at rest, while one also wants to know how they behave in dynamical processes such as collisions. For this purpose, we apply the method suggested in the study of processes involving particles in flight, as is the Compton scattering, the subject of this section.

The tree-level Compton scattering amplitude [24] contains contributions from three different channels (see Figs. 4, 5, 6). Here, $p$ and $p'$ to denote in turn the four-momenta of the incident and scattered single spin-$\frac{3}{2}$ target particles, while $q$ and $q'$ are the four-momenta of the incident and scattered photons, respectively. Then the amplitude has the following form

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3,$$  \hfill (6.1)
The contractions indicated in (6.4) are easier performed with the aid of the FeynCalc package giving as a result the following expression:

$$|\mathcal{M}|^2 = \frac{1}{162m^6(m^2-s)^2(m^2-u)^7} \sum_{k=1}^{7} m_k a_{2k},$$

(6.14)
where \( s, u \) are the standard Mandelstam variables and we are using the notations

\[
a_0 = 18s^2u^2(s + u)^3, \tag{6.15a}
\]
\[
a_2 = -9su(s + u)^2(7(s^2 + u^2) + 8su), \tag{6.15b}
\]
\[
a_4 = (s + u)(63(s^4 + u^4) + 348(s^3u + su^3) + 578s^2u^2), \tag{6.15c}
\]
\[
a_6 = -165(s^4 + u^4) - 588(s^3u + su^3) - 574s^2u^2, \tag{6.15d}
\]
\[
a_8 = 2(s + u)(5(s^2 + u^2)^2 - 142su), \tag{6.15e}
\]
\[
a_{10} = 2(105(s^2 + u^2)^2 - 158su), \tag{6.15f}
\]
\[
a_{12} = -165(s^4 + u^4), \tag{6.15g}
\]
\[
a_{14} = 912. \tag{6.15h}
\]

Now we can obtain the differential cross section in the laboratory frame from the standard formulas

\[
\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi m} \frac{\omega'}{\omega}\right)^2 |M|^2, \tag{6.16}
\]
\[
\omega' = \frac{m\omega}{m + (1 - \cos \theta)\omega}, \tag{6.17}
\]

where \( \omega \) and \( \omega' \) are the energies of the incident and scattered photons respectively, while \( \theta \) is the scattering angle in the laboratory frame. Furthermore, with

\[
s = m(m + 2\omega), \tag{6.18}
\]
\[
u = m(m - 2\omega'), \tag{6.19}
\]

and after some algebraic manipulations, the final result can be given the form of an expansion in powers of \( \eta = \omega/m \) according to,

\[
\frac{d\sigma(\eta, x)}{d\Omega} = \frac{r_0^2}{162(\eta(x - 1) - 1)^6} \sum_{k=0}^{6} \eta^k b_k, \tag{6.20}
\]

with \( r_0 = e^2/(4\pi m) = \alpha m, x = \cos \theta, \) and the expansion coefficients being,

\[
b_0 = -81(x^2 + 1), \tag{6.21}
\]
\[
b_1 = 243(x - 1)(x^2 + 1), \tag{6.22}
\]
\[
b_2 = -(x - 1)(243x^3 - 333x^2 + 338x - 468), \tag{6.23}
\]
\[
b_3 = (x - 1)^2(81x^3 - 261x^2 + 271x - 531), \tag{6.24}
\]
\[
b_4 = (x - 1)^3(90x^3 - 233x^2 + 440x - 459), \tag{6.25}
\]
\[
b_5 = 6(x - 1)^3(8x^2 - 20x + 39), \tag{6.26}
\]
\[
b_6 = 9(x - 1)^3(x^2 - 5x + 8). \tag{6.27}
\]

In the low energy limit, we recover as expected the Thompson differential cross section:

\[
\lim_{\eta \to 0} \frac{d\sigma(\eta, x)}{d\Omega} = \frac{1}{2^2} r_0^2 (x^2 + 1), \tag{6.28}
\]

while in forward direction, the differential cross section takes an energy independent value,

\[
\lim_{x \to 1} \frac{d\sigma(\eta, x)}{d\Omega} = r_0^2. \tag{6.29}
\]
and in accord with unitarity. In all other directions however, the differential cross section increases with energy. In the Figure 7 we present a plot of the quantity

\[
d\tilde{\sigma}(\eta, x) \equiv \frac{1}{r_0^2} \frac{d\sigma(\eta, x)}{d\Omega},
\]

as a function of the \( x = \cos \theta \) variable, at energies of \( \eta = 0 \) (solid curve), \( \eta = 1 \) (long dashed curve) and \( \eta = 2.5 \) (short dashed curve), here we see how the differential cross section approaches the classical limit at low energy (symmetric curve) and raises as the energy grows except in the forward direction.

\[d\tilde{\sigma}(\eta, x)\]

Figure 7. The differential cross section, \( d\tilde{\sigma}(\eta, x) \), as a function of \( x = \cos \theta \). The solid curve represents the classical limit at \( \eta = \omega/m = 0 \), the long dashed line corresponds to an energy comparable to the mass of the particle, \( \eta = 1 \), while the short dashed curve corresponds to \( \eta = 2.5 \). This cross section increases with energy, except in the forward direction, \( x = 1 \), where it approaches \( d\tilde{\sigma}(\eta, 1) = 1 \).

Integrating over the solid angle we find the total cross section as:

\[
\sigma(\eta) = \sum_{k=0}^{8} \frac{\eta^k c_k \sigma_T}{108 \eta^2 (2 \eta + 1)^4} + \sum_{\ell=0}^{4} \frac{\eta^\ell h_\ell \sigma_T \log(2 \eta + 1)}{216 \eta^3},
\]

being \( \sigma_T = (8/3)\pi r_0^2 \) the Thompson cross section and

\[
c_0 = 162, \quad c_1 = 1566, \quad (6.32a)
\]
\[
c_2 = 6217, \quad c_3 = 12796, \quad (6.32b)
\]
\[
c_4 = 14244, \quad c_5 = 8011, \quad (6.32c)
\]
\[
c_6 = 1794, \quad c_7 = 126, \quad (6.32d)
\]
\[
c_8 = 72, \quad h_0 = -162, \quad (6.32e)
\]
\[
h_1 = -432, \quad h_2 = -277, \quad (6.32f)
\]
\[
h_3 = -21, \quad h_4 = 27. \quad (6.32g)
\]

The total cross section (6.31) has the following limits,

\[
\lim_{\eta \to 0} \sigma(\eta) = \sigma_T, \quad (6.33)
\]
\[
\lim_{\eta \to \infty} \sigma(\eta) = \infty. \quad (6.34)
\]
This behavior is shown in the Figure 8, where we make a plot of

\[ \tilde{\sigma}(\eta) \equiv \frac{\sigma(\eta)}{\sigma_T}, \quad (6.35) \]

here we see the decreasing behavior of the cross section at low energies as well as its growing behavior at high energies.

![Figure 8](image)

Figure 8. The total cross section \( \tilde{\sigma}(\eta) \) as a function of \( \eta = \omega/m \). In the low energy limit the Thompson limit, \( \tilde{\sigma}(0) = 1 \), is recovered, otherwise the cross section grows with the energy increase.

7 Conclusions

In the present work we constructed the physical equivalent to the Weinberg-Joos theory for single spin-\( j \) particles by replacing the multi-component spinors by Lorentz tensors for bosons, or tensor-spinors for fermions and the higher order differential wave equations by such of second order. The theory is based on the relativistic invariants (RInS) of both the Lorentz- and Poincaré algebras, as well as on the fact that any irreducible sector of any Lorentz tensor, with or without Dirac spinor components, is equally good for the description of the elementary particle residing in it as a single irreducible representation. Stated differently, the individuality of the fundamental particles residing within given irreducible representation spaces of the Lorentz algebra are fully respected by any direct sum of them. Indeed, our approach, which we illustrated for the sake of concreteness and without loss of generality on the example of spin-\( \frac{3}{2} \) in \( [(1, 0) \oplus (0, 1)] \oplus [(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})] \), correctly reproduces the electromagnetic multipole moments of the particles in each one of the irreducible sectors of the anti-symmetric tensor spinor space, be them the Dirac, the Rarita-Schwinger, or the pure spin-\( \frac{3}{2} \) sectors, \( (\frac{1}{2}, 0) \oplus (0, \frac{3}{2}) \). We were able to show that unitarity is respected in Compton scattering off pure spin-\( \frac{3}{2} \) in forward direction within a minimal gauging scheme and without any need of invoking non-minimal couplings, and in parallel to the same behaviour of spin-\( \frac{3}{2} \) transforming within the four-vector spinor \[12\]. However, the gyromagnetic ratio for the case considered here has been found as the inverse of the spin, thereby matching Belinfante’s conjecture rather than the universal \( g = 2 \) value established for particles transforming as the highest spins in irreps of multiple spins and parities. This finding emphasizes once again the observation that fundamental particles residing in non-equivalent \( so(1,3) \) representation spaces can be equipped by distinct physical properties and are likely to participate in
different physical processes. The scheme elaborated here allows detecting such differences, is friendly
towards symbolic computational softwares such as Mathematica and FeynCalc, and significantly less
computer time consuming than the conventional matrix spinor calculus.
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