The $\frac{4}{3}$-variation of the derivative of the self-intersection Brownian local time and related processes

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Abstract

In this paper we compute the $\frac{4}{3}$-variation of the derivative of the self-intersection Brownian local time $\gamma_t = \int_0^t \int_0^u \delta'(B_u - B_s)dsdu, t \geq 0$, applying techniques from the theory of fractional martingales [3].

1 Introduction

Let $B = \{B_t, t \geq 0\}$ be a standard one-dimensional Brownian motion. In this paper we are interested in the process $\gamma = \{\gamma_t, t \geq 0\}$ formally given by

$$\gamma_t = \frac{d}{dy} \alpha_t(y)\big|_{y=0}, \quad \text{where} \quad \alpha_t(y) = \int_0^t \int_0^u \delta_y(B_u - B_s)dsdu.$$ It can be rigorously defined as the following limit in $L^2(\Omega)$

$$\gamma_t = \lim_{\epsilon \to 0} \int_0^t \int_0^u p_\epsilon'(B_u - B_s)dsdu,$$ (1.1)

where $p_\epsilon(x) = (2\pi \epsilon)^{-\frac{1}{2}} \exp(-x^2/(2\epsilon))$. This process has been studied by Rogers and Walsh in [5] and by Rosen in [6].

Let us recall the definition of the $\beta$-variation of a stochastic processes from [3].

Definition 1.1 Let $\beta \geq 1$ and let $X = \{X_t, t \geq 0\}$ be a continuous stochastic process. Denote

$$S_{\beta,n}^{[a,b]}(X) := \sum_{i=0}^{n-1} |X_{t_{i+1}^n} - X_{t_i^n}|^{\beta},$$ (1.2)

where $t_{i}^n = a + \frac{i}{n}(b-a)$ for $i = 0, \ldots, n$. If the limit of $S_{\beta,n}^{[a,b]}(X)$ exists in probability as $n$ tends to infinity, then we say that the $\beta$-variation of $X$ exists on the interval $[a, b]$ and the limit is denoted by $\langle X \rangle_{\beta,[a,b]}$. We say that the $\beta$-variations of $X$ on $[a, b]$ exists in $L^p$ if the limit of $S_{\beta,n}^{[a,b]}(X)$ exists in $L^p(\Omega)$, where $p \geq 1$. 

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For any $a < b < c$, if the $\beta$ variation of $X$ exist on the intervals $[a, b]$ and $[b, c]$, then it also exists on $[a, c]$ and

$$\langle X \rangle_{\beta, [a, c]} = \langle X \rangle_{\beta, [a, b]} + \langle X \rangle_{\beta, [b, c]}.$$ 

Denote by $\{L_t^x, t \geq 0, x \in \mathbb{R}\}$ a jointly continuous version of the Brownian local time. In the paper [5] Rogers and Walsh gave an explicit formula for the exact $\frac{4}{3}$-variation of the process $\gamma$, using Gebelein’s inequality for Gaussian random variables to bound the sums of powers of the increments of process $\gamma$. More precisely, they proved the following theorem.

**Theorem 1.2** The process $\gamma$ has a finite $\frac{4}{3}$-variation in $L^2$ on any interval $[0, T]$ given by

$$\langle \gamma \rangle_{\frac{4}{3}, [0, T]} = K \int_0^T (L_t^{B_r})^{\frac{2}{3}} dr,$$

where $K = E|B_1|^\frac{4}{3} E \left[ \int_\mathbb{R} (L_t^z)^2 dz \right]^\frac{2}{3}$.

The purpose of the present paper is to provide an alternative and simpler proof of Theorem 1.2 by using the methodology introduced by Hu, Nualart and Song in [3] to compute the $p$-variation of a fractional martingale. A basic ingredient in our approach is the stochastic integral representation of $\gamma_t$ obtained by Hu and Nualart in [2] through the Clark-Ocone formula:

$$\gamma_t = \int_0^t \left( \int_\mathbb{R} p_{t-r}(y) (L_r^y + B_r - L_r^B) dy \right) dB_r. \quad (1.3)$$

The main idea of the proof is as follows. By an approximation argument, and using the representation of the local time as a semimartingale in the space variable (see Perkins [4]), the problem is reduced to the computation of the $\frac{4}{3}$-variation of the process

$$X_t = \int_0^t \left( \int_\mathbb{R} p_{t-r}(y) W_y dy \right) dB_r, \quad (1.4)$$

where $W = \{W_y, y \in \mathbb{R}\}$ is a two-sided Brownian motion independent of $B$. Taking into account that $W$ is Hölder continuous of order almost $\frac{1}{2}$, the integral $\int_\mathbb{R} p_{t-r}(y) W_y dy$ behaves as $(t - r)^{\frac{3}{2}}$ as $r \uparrow t$. In this sense, the variation of the process $X$ is similar to the variation of the fractional Brownian motion with Hurst parameter $H = \frac{3}{4}$. Actually, we can compute easily the $\frac{4}{3}$-variation of the process $X$ applying the approach used for the fractional Brownian motion, based on the decomposition by Mandelbrot and Van Ness [1] and the ergodic theorem. Notice, however, that our proof shows only the existence of the $\frac{4}{3}$-variation in $L^1$, and we obtain a different expression for the constant $K$ in Theorem 1.2.

The paper is organized as follows. In the next section we derive the $\frac{4}{3}$-variation of the process $X$ given in (1.4) using ergodic theorem. Section 3 is devoted to the proof of Theorem 1.2, where the $\frac{4}{3}$-variation is considered in $L^1$. Finally, the appendix contains some technical lemmas. Along the paper we denote by $C$ a generic constant which may be different from line to line.

### 2 $\frac{4}{3}$-variation of a fractional-type process

Consider the stochastic process introduced in (1.4). This process can also be expressed as

$$X_t = \int_0^t E^\theta W_{\theta \sqrt{t-r}} dB_r,$$
where \( \theta \) is a \( N(0, 1) \) random variable, independent of \( B \), and \( E^\theta \) denotes the expectation with respect to \( \theta \). The following theorem is the main result of this section.

**Theorem 2.1** The process \( X = \{X_t, t \geq 0\} \) defined in (1.4) has a finite \( \frac{4}{3} \)-variation in \( L^1 \) given by

\[
(X)_{0, [a, b]} = K(b - a),
\]

where

\[
K = E((\theta|^{\frac{4}{3}}E) \left( \frac{1}{4} \int_0^\infty \int_0^\infty (x + y)^{-\frac{4}{3}}(B_{1+x} - B_x)(B_{1+y} - B_y) dx dy \right)^{\frac{2}{3}}. \tag{2.1}
\]

**Proof** The proof will be done in two steps. To simplify the presentation we assume that \([a, b] = [0, T]\).

**Step 1** Enlarging the probability space if necessary, we assume that \( B = \{B_t, t \in \mathbb{R}\} \) is a two-sided Brownian motion. Then we define

\[
Y_t = \int_{-\infty}^t E^\theta W_{\sqrt{t-r}} dB_r - \int_{-\infty}^0 E^\theta W_{\sqrt{t-r}} dB_r.
\]

This process is well defined because, using the fact that \( E(W_x W_y) = \frac{1}{2}(|x| + |y| - |x - y|) \), we can write

\[
E(Y_t^2) = E(W) \int_{\mathbb{R}} \left( E^\theta W_{\sqrt{(t-r)^+}} - E^\theta W_{\sqrt{(-r)^+}} \right)^2 dr
\]

\[
= \int_{\mathbb{R}} E^{\theta, \eta} E(W_{\sqrt{(t-r)^+}} - W_{\sqrt{(-r)^+}}) E(W_{\eta \sqrt{(t-r)^+}} - W_{\eta \sqrt{(-r)^+}}) dr
\]

\[
= \frac{\sqrt{2}}{2} E(|\theta|) \int_{\mathbb{R}} \left( \sqrt{2[(t-r)^+ + (-r)^+]} - \sqrt{(t-r)^+} - \sqrt{(-r)^+} \right) dr
\]

\[
= \frac{1}{\sqrt{\pi}} \left( \int_0^\infty \left( \sqrt{2t + 4r} - \sqrt{t + r} - \sqrt{r} \right) dr + \int_0^t \left( \sqrt{2(t-r)} + \sqrt{t-r} \right) dr \right) < \infty.
\]

We claim that the difference

\[
Y_t - X_t = \int_{-\infty}^0 \left( E^\theta W_{\sqrt{t-r}} - E^\theta W_{\sqrt{-r}} \right) dB_r \tag{2.2}
\]

has \( \frac{4}{3} \)-variation in \( L^1 \) equal to zero in any time interval \([0, T]\). In fact, if \( t_i = \frac{iT}{n} \), then from the Burkholder-Davis-Gundy inequality and the Jensen inequality, and using the notation (1.2), we have

\[
ES^{[0, T]}_{\frac{4}{3}, n} (Y - X) = \sum_{i=0}^{n-1} E \left| \int_{-\infty}^0 \left( E^\theta W_{\sqrt{t_i+1-r}} - E^\theta W_{\sqrt{t_i-r}} \right) dB_r \right|^{\frac{4}{3}}
\]

\[
\leq C \sum_{i=0}^{n-1} E \left( \int_{-\infty}^0 \left( E^\theta W_{\sqrt{t_i+1-r}} - E^\theta W_{\sqrt{t_i-r}} \right)^2 dr \right)^{\frac{2}{3}}
\]

\[
\leq C \sum_{i=0}^{n-1} \left( \int_{-\infty}^0 E \left( E^\theta W_{\sqrt{t_i+1-r}} - E^\theta W_{\sqrt{t_i-r}} \right)^2 dr \right)^{\frac{2}{3}}.
\]
By the same computations as above we obtain
\[
ES_t^{[0,T]}(Y - X) \leq C \sum_{i=0}^{n-1} \left( \int_0^\infty \left( \sqrt{2t_{i+1} + 2t_i + 4t} - \sqrt{t_{i+1} + t_i + 4t} \right) dr \right)^{\frac{2}{3}}
\]
\[
= C \sum_{i=0}^{n-1} \left( \int_0^{t_{i+1}} \int_0^{t_{i+1} - t_i} \left( x + y + t_i + T \right)^{\frac{3}{2}} dxdydr \right)^{\frac{2}{3}}
\]
\[
= C \sum_{i=0}^{n-1} \left( \int_0^{t_{i+1} - t_i} \left( x + t_i \right)^{\frac{3}{2}} dxdy \right)^{\frac{2}{3}}.
\]

For \( i \geq 1 \) we use the estimate \((x + y + t_i)^{-\frac{1}{2}} \leq t_i^{-\frac{1}{2}}\). In this way we can estimate the above sum for \( i \geq 1 \) by
\[
n^{-\frac{3}{4}} \sum_{i=1}^{n-1} \left( \frac{i}{n} \right)^{-\frac{1}{3}} = \frac{1}{n} \sum_{i=1}^{n-1} i^{-\frac{1}{3}},
\]
which clearly converges to zero as \( n \) tends to infinity.

**Step 2** From Step 1, it follows that to prove Theorem 2.1 it suffices to show
\[
(Y)^\frac{1}{3} \in [0,T] = KT. \tag{2.3}
\]

It is easy to verify that the process \( Y \) has stationary increments and is self-similar of order \( \frac{3}{4} \). As a consequence, the sequence \( \{Y_{t_{i+1}} - Y_{t_i}, i \geq 0\} \) has the same law as \( \{(\frac{t}{n})^{\frac{3}{4}} \xi_i, i \geq 0\} \), where
\[
\xi_i = \int_{-\infty}^{t_{i+1}} \int_{-\infty}^{t_{i+1} - t_i} E^\theta W_{\sqrt{t_{i+1} - r}} dB_r - \int_{-\infty}^{t_i} E^\theta W_{\sqrt{t_{i+1} - r}} dB_r.
\]

It suffices to show that \( \frac{1}{n} \sum_{i=0}^{n-1} |\xi_i|^{\frac{3}{4}} \) converges in \( L^1 \) to \( K \). By the ergodic theory, we know that,
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |\xi_i|^{\frac{3}{4}} = Z = E(|\xi_1|^{\frac{3}{4}} |T),
\]
in \( L^1 \), where \( T \) is the invariant \( \sigma \)-field. We claim that the random variable \( Z \) is a constant. To prove this we will show that both random variables \( E^W Z \) and \( E^B Z \) are constant, where \( E^W \) and \( E^B \) denote, respectively, the mathematical expectation with respect to the processes \( W \) and \( B \).

Let us first compute \( E^W Z \). Let \( C_0 = E[\theta^{\frac{1}{4}}] \). Then, we can write
\[
E^W |\xi_1|^{\frac{3}{4}} = C_0 \left( E^W \left( \int_{-\infty}^{t_{i+1}} \int_{-\infty}^{t_{i+1} - t_i} E^\theta W_{\sqrt{t_{i+1} - r}} dB_r - \int_{-\infty}^{t_i} E^\theta W_{\sqrt{t_{i+1} - r}} dB_r \right)^{\frac{2}{3}} \right).
\]

Let us first compute \( E^W Z \). Let \( C_0 = E[\theta^{\frac{1}{4}}] \). Then, we can write
\[
E^W \left( \int_{-\infty}^{t_{i+1}} \int_{-\infty}^{t_{i+1} - t_i} E^\theta W_{\sqrt{t_{i+1} - r}} dB_r - \int_{-\infty}^{t_i} E^\theta W_{\sqrt{t_{i+1} - r}} dB_r \right)^{\frac{2}{3}}.
\]
where the double integral \( \int \cdots dB_r dB_s \) with respect to \( B \) is a Stratonovich-type integral. Thus,

\[
E^W |\xi_i|^\frac{4}{3} = C_0 \left( \frac{1}{2} \int_{-\infty}^{i+1} \int_{-\infty}^{i+1} \left( -\sqrt{(i+1-s)+(i+1-r)} - \sqrt{(i-r)^+(i-s)^+} \\
+ \sqrt{(i+1-r) + (i-s)^+} + \sqrt{(i+1-s) + (i-r)^+} \right) dB_r dB_s \right)^\frac{2}{3}
\]

\[
= C_0 \left( \frac{1}{4} \int_{-\infty}^{i+1} \int_{-\infty}^{i+1} \int_{i+r}^{i+1-r} \int_{i+s}^{i+1-s} (x+y)^{-\frac{2}{3}} dy dx dB_r dB_s \right)^\frac{1}{3}.
\]

One can exchange the integration order of \( x, y \) and \( r, s \). The domain \(-\infty < r, s, < i+1, (i-r)^+ < x < i+1-r, (i-s)^+ < y < i+1-s\) can be written as \( 0 < x, y < \infty, i-x < r < i+1-x, i-y < s < i+1-y \). Thus, we have

\[
E^W |\xi_i|^\frac{4}{3} = C_0 \left( \frac{1}{4} \int_0^{\infty} \int_0^{\infty} (x+y)^{-\frac{2}{3}} (B_{i+1-x} - B_{i-x})(B_{i+1-y} - B_{i-y}) dy dx \right)^\frac{2}{3}
\]

\[
= C_0 \left( \frac{1}{4} \int_0^{\infty} \int_0^{\infty} \frac{1}{\Gamma(\frac{3}{2})} \int_0^{\infty} e^{-(x+y)z} z^{\frac{1}{2}} dz (B_{i+1-x} - B_{i-x})(B_{i+1-y} - B_{i-y}) dy dx \right)^\frac{2}{3}
\]

\[
= C_0 \left( \frac{1}{4\Gamma(\frac{3}{2})} \int_0^{\infty} \left( \int_0^{\infty} (B_{i+1-x} - B_{i-x}) e^{-xz} dx \right)^2 z^{\frac{1}{2}} dz \right)^\frac{2}{3}.
\]

For any fixed \( x \) and \( y \) in \( \mathbb{R} \), the correlation between the Gaussian random variables \( B_{i+1-x} - B_{i-x} \) and \( B_{i+1-y} - B_{i-y} \) is zero when \( i \) is sufficiently large. This implies that the sequence

\[
\int_0^{\infty} \left( \int_0^{\infty} (B_{i+1-x} - B_{i-x}) e^{-xz} dx \right)^2 z^{\frac{1}{2}} dz
\]

is stationary and ergodic. As a consequence, \( \frac{1}{n} \sum_{i=0}^{n-1} E^W |\xi_i|^\frac{4}{3} \) converges to the constant \( K \) given in (2.1).

Finally, we show that \( E^B Z \) is constant. We can write

\[
E^B |\xi_i|^\frac{4}{3} = C_0 \left( \int_{\mathbb{R}} \left( E^\theta W_{\sqrt{(i+1-r)^+}} - E^\theta W_{\sqrt{(i-r)^+}} \right)^2 dr \right)^\frac{2}{3}.
\]

For any fixed \( r \) and \( s \) in \( \mathbb{R} \), the covariance between the random variables \( \eta_0(s) \) and \( \eta_i(r) \), where

\[
\eta_i(r) = E^\theta W_{\sqrt{(i+1-r)^+}} - E^\theta W_{\sqrt{(i-r)^+}},
\]

is given by

\[
E^W(\eta_0(s)\eta_i(r)) = \frac{1}{2} E(|\theta|) \left( -\sqrt{(i+1-r)^+ + (1-s)^+} + \sqrt{(i+1-r)^+ + (-s)^+} \\
+ \sqrt{(i-r)^+ + (1-s)^+} - \sqrt{(i-r)^+ + (-s)^+} \right),
\]

and it converges to zero as \( i \) tends to infinity. Again, this implies that the sequence

\[
\int_{\mathbb{R}} \left( E^\theta W_{\sqrt{(i+1-r)^+}} - E^\theta W_{\sqrt{(i-r)^+}} \right)^2 dr
\]

is stationary and ergodic, and as a consequence, \( \frac{1}{n} \sum_{i=0}^{n-1} E^B |\xi_i|^\frac{4}{3} \) converges to a constant. ■
3 Proof of Theorem 1.2

In this section we proceed to the proof of Theorem 1.2, where the $\frac{1}{3}$-variation is in $L^1(\Omega)$, and the constant $K$ has the alternative expression given by (2.1).

Fix a partition $s_k = \frac{kT}{N}$, $k = 0, \ldots, N$. For any point $t$ we denote by $t(N)$ the maximum point of the partition on the left of $t$, namely, $t(N) = t_k$ if $s_k \leq t < s_{k+1}$. We approximate the process $\gamma_t$ defined in (1.3) by a sequence of processes obtained by freezing the time coordinate of $L^{y+B_y}_r - L^{B_y}_r$ at the point $r = r(N)$, that is,

$$\gamma_t^N = \int_0^t \int_\mathbb{R} p_{t-r}(y) \left( L^{y+B_y}_{r(N)} - L^{B_y}_{r(N)} \right) dy dB_r.$$

The proof will be divided into several steps.

Step 1 We claim that

$$\lim_{N \to \infty} \limsup_{n \to \infty} \sum_{k=0}^{N-1} ES_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} (\gamma - \gamma^N) = 0. \quad (3.1)$$

Consider a uniform partition of the interval $[kT/N, (k+1)T/N]$ denoted by $r_0 < r_1 < \cdots < r_n$, where $r_j = \frac{kT}{N} + \frac{j}{nN}$, $j = 0, 1, \ldots, n$. Then,

$$S_{n,N}^k := S_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} (\gamma - \gamma^N) = \sum_{j=0}^{n-1} |\Delta_j (\gamma - \gamma^N)|^{\frac{3}{4}}, \quad (3.2)$$

where $\Delta_j (\gamma - \gamma^N) = (\gamma - \gamma^N)_{r_{j+1}} - (\gamma - \gamma^N)_{r_j}$. Let $f_r^N(y) = L^{y+B_y}_r - L^{B_y}_r - L^{y+B_y}_{r(N)} + L^{B_y}_{r(N)}$. Then,

$$(\gamma - \gamma^N)_t = \int_0^t \int_\mathbb{R} p_{t-r}(y) f_r^N dy dB_r.$$

As a consequence,

$$S_{n,N}^k = \sum_{j=0}^{n-1} \left| \int_0^{r_{j+1}} \int_\mathbb{R} p_{r_{j+1}-r}(y) f_r^N dy dB_r - \int_0^{r_j} \int_\mathbb{R} p_{r_j-r}(y) f_r^N dy dB_r \right|^{\frac{3}{4}}$$

$$= \sum_{j=0}^{n-1} \left| \int_0^{r_{j+1}} \int_\mathbb{R} p_{r_{j+1}-r}(y) f_r^N dy dB_r + \int_0^{r_j} \int_\mathbb{R} [p_{r_{j+1}-r}(y) - p_{r_j-r}(y)] f_r^N dy dB_r \right|^{\frac{3}{4}}$$

$$\leq C \sum_{j=0}^{n-1} \left( \left| \int_0^{r_{j+1}} \int_\mathbb{R} p_{r_{j+1}-r}(y) f_r^N dy dB_r \right|^{\frac{4}{3}} + \left| \int_0^{r_j} \int_\mathbb{R} [p_{r_{j+1}-r}(y) - p_{r_j-r}(y)] f_r^N dy dB_r \right|^{\frac{4}{3}} \right)$$

$$= C \sum_{j=0}^{n-1} \left( |\Gamma_j^k|^{\frac{4}{3}} + |\Phi_j^k|^{\frac{4}{3}} \right), \quad (3.3)$$

where

$$\Gamma_j^k = \int_{r_j}^{r_{j+1}} \int_\mathbb{R} p_{r_{j+1}-r}(y) \left( L^{y+B_y}_{r(N)} - L^{B_y}_{r(N)} \right) dy dB_r$$

$$= \int_{r_j}^{r_{j+1}} E \left( L^{B_y}_{r_{j+1}} - L^{B_y}_{r(N)} - L^{B_y}_{r(N)} | \mathcal{F}_r \right) dB_r,$$

and

$$\Phi_j^k = \int_{r_j}^{r_{j+1}} \int_\mathbb{R} p_{r_{j+1}-r}(y) \left( L^{y+B_y}_{r(N)} - L^{B_y}_{r(N)} \right) dy dB_r.$$
and
\[ \Phi_j^k = \int_0^{r_j} \int_{\mathbb{R}} [p_{r_{j+1}}(y) - p_{r_j}(y)] \left( L_{r_j}^{B_{r_j}} - L_{r_j}^{B_{r_j+1}} + L_{r_j}^{B_{r_j+1}}|F_{r_j}| \right) dy dB_t. \]

Therefore,
\[ E S_{n,N}^k \leq C \left( \sum_{j=0}^{n-1} E(|\Gamma_j^k|^\frac{4}{3}) + \sum_{j=0}^{n-1} E(|\Phi_j^k|^\frac{4}{3}) \right). \]

Using the Burkholder inequality we obtain
\[ E(|\Gamma_j^k|^\frac{4}{3}) \leq C E \left( \int_{r_j}^{r_{j+1}} E(L_{r_j}^{B_{r_j+1}} - L_{r_j}^{B_{r_j}} - L_{r_{N_j}}^{B_{r_j+1}} + L_{r_{N_j}}^{B_{r_j}}|F_{r_j}|)^2 dr \right)^{\frac{2}{3}}, \]
and
\[ E(|\Phi_j^k|^\frac{4}{3}) \leq C E \left( \int_0^{r_j} E(L_{r_j}^{B_{r_j+1}} - L_{r_j}^{B_{r_j}} - L_{r_{N_j}}^{B_{r_j+1}} + L_{r_{N_j}}^{B_{r_j}}|F_{r_j}|)^2 dr \right)^{\frac{2}{3}}. \]

Let us first prove that
\[ \lim_{N \to \infty} \limsup_{n \to \infty} \sum_{k=0}^{N-1} \sum_{j=0}^{n-1} E(|\Gamma_j^k|^\frac{4}{3}) = 0. \]  

We shall use the notation \( L_{a,b}^x = L_b^x - L_a^x \). Then, we can write
\[ E(|\Gamma_j^k|^\frac{4}{3}) \leq C \left( E \int_{r_j}^{r_{j+1}} \left( L_{r_{N_j}}^{B_{r_j+1}} - L_{r_{N_j}}^{B_{r_j}} \right)^2 dr \right)^{\frac{2}{3}}. \]  

Consider the Brownian motion \( B_t - B_u \) where the parameter \( u \) goes backward from \( t \) to 0. Then, Tanaka’s formula applied to this Brownian motion says that for any \( s < t \)
\[ (B_t - B_s - x)_+ - (-x)_+ = -\int_s^t 1_{\{B_t - B_u > x\}} \tilde{d}B_u + \frac{1}{2} \int_s^t \delta_x(B_t - B_u) du, \]
where \( \tilde{d} \) denotes the backward Itô integral. Making the change of variable \( x = B_t - B_\tau, \tau > t \) yields
\[ (B_{r_j} - B_s)_+ - (B_s - B_t)_+ = -\int_s^t 1_{\{B_u < B_{r_j}\}} \tilde{d}B_u + \frac{1}{2} \int_s^t \delta_{B_\tau}(B_u) du. \]

Therefore, letting \( s = r(N), t = r \) and \( \tau = r_{j+1} \) in the above equality yields
\[ (B_{r_{j+1}} - B_{r(N)})_+ - (B_{r_{j+1}} - B_r)_+ = -\int_{r(N)}^r 1_{\{B_u < B_{r_{j+1}}\}} \tilde{d}B_u + \frac{1}{2} L_{r_{N_j}}^{B_{r_{j+1}}} \]
On the other hand, letting \( s = r(N) \) and \( t = \tau = r \) gives us
\[ (B_r - B_{r(N)})_+ = -\int_{r(N)}^r 1_{\{B_u < B_r\}} \tilde{d}B_u + \frac{1}{2} L_{r_{N_j}}^{B_r}. \]
This implies that
\[
\left| L_{[r(N),r]}^{B_{r_{j+1}}} - L_{[r(N),r]}^{B_r} \right| \leq 2 \left| (B_{r_{j+1}} - B_{r_{j+1}}(N)) - (B_r - B_{r_{j+1}}(N)) \right| + 2(B_{r_{j+1}} - B_r) + 2 \left| \int_{r(N)}^{r} \left( 1_{B_u < B_{r_{j+1}}} - 1_{B_u < B_r} \right) dB_u \right|
\]
\[
\leq 4 \left| B_{r_{j+1}} - B_r \right| + 2 \left| \int_{r(N)}^{r} \left( 1_{B_u < B_{r_{j+1}}} - 1_{B_u < B_r} \right) dB_u \right| .
\]
Therefore,
\[
E \left( L_{[r(N),r]}^{B_{r_{j+1}}} - L_{[r(N),r]}^{B_r} \right)^2 \leq 32(r_{j+1} - r) + 8 \int_{r(N)}^{r} E \left( 1_{B_u < B_{r_{j+1}}} - 1_{B_u < B_r} \right)^2 du. \tag{3.7}
\]
Notice that
\[
E \left( 1_{B_u < B_{r_{j+1}}} - 1_{B_u < B_r} \right)^2 = P(B_r < B_u < B_{r_{j+1}}) + P(B_r > B_u > B_{r_{j+1}}).
\]
Using the density of two-dimensional Gaussian random variables one can see that the probability
\[P(B_r \leq B_u \leq B_{r_{j+1}})\] is bounded by a constant times \(\sqrt{r_{j+1} - r} / \sqrt{r - u}\), which implies
\[
\int_{r(N)}^{r} E \left( 1_{B_u < B_{r_{j+1}}} - 1_{B_u < B_r} \right)^2 du \leq C \sqrt{r_{j+1} - r} N^{-\frac{1}{2}}. \tag{3.8}
\]
From (3.5), (3.7) and (3.8) we obtain
\[
E(\left| \Gamma_j^{k} \right|^2) \leq C \left( (r_{j+1} - r_j)^2 + (r_{j+1} - r_j)^3 N^{-\frac{1}{2}} \right)^{\frac{3}{2}}
\]
\[
\leq C \left( n^{-2} N^{-2} + n^{-\frac{3}{2}} N^{-2} \right)^{\frac{3}{2}}
\]
\[
\leq C \left( n^{-\frac{3}{2}} N^{-\frac{3}{2}} + n^{-1} N^{-\frac{3}{2}} \right),
\]
which implies (3.4).
To complete the proof of (3.1), we need to show that
\[
\lim_{N \to \infty} \limsup_{n \to \infty} \sum_{k=0}^{N-1} \sum_{j=0}^{n-1} E(\left| \Phi_j^{k} \right|^2) = 0. \tag{3.9}
\]
We continue to use the same notation as above. It is easy to obtain by using the Burkholder inequality
\[
E(\left| \Phi_j^{k} \right|^2) \leq \left( E \int_{0}^{r_j} \left( E(\left| L_{[r(N),r]}^{B_{r_{j+1}}} - L_{[r(N),r]}^{B_r} \right|^2) dr \right)^{\frac{3}{2}} \right).
\]
In order to deal with the above term, we use the backward Tanaka formula (3.6) again by taking \(\tau = r_{j+1}\) and \(r_j\). Subtracting the two obtained equations, we obtain
\[
L_{[r(N),r]}^{B_{r_{j+1}}} - L_{[r(N),r]}^{B_r} = C_j(r) + D_j(r), \tag{3.10}
\]
where
\[ C_j(r) = 2 (B_{r_{j+1}} - B_{r(N)})_+ - (B_{r_{j+1}} - B_r)_+ - (B_{r_j} - B_{r(N)})_+ + (B_{r_j} - B_r)_+ , \]
and
\[ D_j(r) = 2 \int_{r(N)}^{r} \left( \mathbf{1}_{\{B_u < B_{r_{j+1}}\}} - \mathbf{1}_{\{B_u < B_{r_j}\}} \right) \tilde{dB}_u. \]

Notice that
\[
E[(B_{r_{j+1}} - B_{r(N)})_+ - (B_{r_j} - B_{r(N)})_+ | \mathcal{F}_r] = E^\xi[(\sqrt{r_{j+1}} - r \xi + B_r - B_{r(N)})_+ - (\sqrt{r_j} - r \xi + B_r - B_{r(N)})_+],
\]
where \( \xi \) is \( N(0,1) \). Hence,
\[
|E[(B_{r_{j+1}} - B_{r(N)})_+ - (B_{r_j} - B_{r(N)})_+ | \mathcal{F}_r]| \leq C(\sqrt{r_{j+1}} - r - \sqrt{r_j} - r).
\]

Therefore, we obtain
\[
\int_0^{r_j} E(C_j(r) | \mathcal{F}_r)^2 dr \leq C \int_0^{r_j} (\sqrt{r_{j+1}} - r - \sqrt{r_j} - r)^2 dr \\
\leq C \int_0^{r_j} (r_{j+1} - r_j)^\frac{7}{4} (r_j - r)^{-\frac{3}{4}} dr \leq C(nN)^{-\frac{7}{4}}.
\]

As a consequence,
\[
\lim_{N \to \infty} \limsup_{n \to \infty} \sum_{k=0}^{N-1} \sum_{j=0}^{n-1} \left( E \int_0^{r_j} E(C_j(r) | \mathcal{F}_r)^2 dr \right)^{\frac{2}{7}} = 0. \tag{3.11}
\]

For the second term in the decomposition (3.10) we can write
\[
E \int_0^{r_j} E(D_j(r) | \mathcal{F}_r)^2 dr \leq \int_0^{r_j} \int_{r(N)}^{r} E \left( E \left( \mathbf{1}_{\{B_{r_j} < B_u < B_{r_{j+1}}\}} - \mathbf{1}_{\{B_{r_j} > B_u > B_{r_{j+1}}\}} | \mathcal{F}_r \right) \right)^2 dudr. \tag{3.12}
\]

From Lemma 4.1 it follows that
\[
E \left( E \left( \mathbf{1}_{\{B_{r_j} < B_u < B_{r_{j+1}}\}} - \mathbf{1}_{\{B_{r_j} > B_u > B_{r_{j+1}}\}} | \mathcal{F}_r \right) \right)^2 \\
\leq C(r - u)^{-\frac{3}{2}} \left( 2\sqrt{2(r_j - r) + \frac{T}{nN}} - \sqrt{2(r_j - r)} - \sqrt{2(r_j - r) + \frac{2T}{nN}} \right).
\]
Substituting this expression into (3.12) yields
\[
E\int_0^{r_j} E(D_j(r)|\mathcal{F}_r)^2 dr \\
\leq C \int_0^{r_j} \int_{r(N)}^r (r - u)^{-\frac{1}{2}} \times \left( 2\sqrt{2(r_j - r) + \frac{T}{nN} - \sqrt{2(r_j - r) - \sqrt{2(r_j - r) + 2\frac{T}{nN}}} \right) dudr \\
\leq CN^{-\frac{3}{2}} \int_0^{r_j} \left( 2\sqrt{2(r_j - r) + \frac{T}{nN} - \sqrt{2(r_j - r) - \sqrt{2(r_j - r) + 2\frac{T}{nN}}} \right) dr \\
\leq CN^{-\frac{3}{2}} \left( 2\left( \frac{k}{N} + \frac{j}{Nn} \right) + \frac{1}{Nn} \right)^{\frac{3}{2}} - \left( \frac{1}{Nn} \right)^{\frac{3}{2}} \\
- \left( 2\left( \frac{k}{N} + \frac{j}{Nn} \right) + \frac{1}{Nn} \right)^{\frac{3}{2}} + \left( \frac{1}{Nn} \right)^{\frac{3}{2}} \right) \\
\leq CN^{-2}n^{-\frac{3}{2}} \sup_{j, n} \left( 2(2(kn + j) + 1)^{\frac{3}{2}} - 2 - (2(kn + j))^{\frac{3}{2}} - (2(kn + j) + 2)^{\frac{3}{2}} + (2)^{\frac{3}{2}} \right)^{\frac{4}{3}} \\
= CN^{-2}n^{-\frac{3}{2}} \sup_{j} \left( 2(j + 1)^{\frac{3}{2}} - 2 - (2j)^{\frac{3}{2}} - (2j + 2)^{\frac{3}{2}} + (2)^{\frac{3}{2}} \right)^{\frac{4}{3}} \\
\leq CN^{-2}n^{-\frac{3}{2}}.
\]
Therefore,
\[
\sum_{k=0}^{N-1} \sum_{j=0}^{n-1} \left( E\int_0^{r_j} E(D_j(r)|\mathcal{F}_r)^2 dr \right)^{\frac{4}{3}} \leq CN^{-\frac{1}{3}},
\]
which implies
\[
\lim_{N \to \infty} \limsup_{n \to \infty} \sum_{k=0}^{N-1} \sum_{j=0}^{n-1} \left( E\int_0^{r_j} E(D_j(r)|\mathcal{F}_r)^2 dr \right)^{\frac{4}{3}} = 0. \tag{3.13}
\]
Then, (3.11) and (3.13) imply (3.9), which completes the proof of (3.1).

Step 2 Define
\[
\gamma_{t,N}^{N,1} = \int_{t(N)}^{t} \int_{\mathbb{R}} p_{t-r}(y) \left( L_{r(N)}^{y+B_r} - L_{r(N)}^{B_r} \right) dydB_r.
\]
We claim that, for each fixed $N$,
\[
\langle \gamma^N - \gamma^{N,1} \rangle_{[0,T]} = 0.
\]
It suffices to show that for each $k = 0, \ldots, N - 1$, the $\frac{4}{3}$ variation of $\gamma^N - \gamma^{N,1}$ over the interval $[kT/N, (k + 1)T/N]$ is zero. When $t \in [kT/N, (k + 1)T/N)$, $t(N) = kT/N$, and
\[
(\gamma^N - \gamma^{N,1})(t) = \int_0^{\frac{T}{N}} \int_{\mathbb{R}} p_{t-r}(y) \left( L_{r(N)}^{y+B_r} - L_{r(N)}^{B_r} \right) dydB_r.
\]
Applying the Burkholder inequality yields
\[ S_{n,N} := S^{|(k_T N N, (k_T+1)T)|}_{n/2} (\gamma^N - \gamma^{N,1}) = \sum_{j=0}^{n-1} |\Delta_j (\gamma^N - \gamma^{N,1})| \frac{2}{3}, \]
where
\[ \Delta_j (\gamma^N - \gamma^{N,1}) = \int_0^{k_T} \int_0^{r(N)} (p_{r_j+1-r}(y) - p_{r_j-r}(y)) \left( L_{r(N)}^{y+B_r} - L_{r(N)}^{B_r} \right) dy dB_r \]
\[ = \int_0^{k_T} \int_0^{r(N)} (p_{r_j+1-r}(B_r - B_s) - p_{r_j-r}(B_r - B_s)) ds dB_r. \]

Applying the Burkholder inequality yields
\[ E|\Delta_j (\gamma^N - \gamma^{N,1})| \frac{2}{3} \]
\[ \leq CE \left( \int_0^{k_T} \left( \int_0^{r(N)} (p_{r_j+1-r}(B_r - B_s) - p_{r_j-r}(B_r - B_s)) ds \right)^2 dr \right) \frac{2}{3} \]
\[ \leq C \left( \int_0^{k_T} E \left( \int_0^{r(N)} (p_{r_j+1-r}(B_r - B_s) - p_{r_j-r}(B_r - B_s)) ds \right)^2 dr \right)^{\frac{2}{3}}. \]

Then, for any \( u < s < r(N) < r \leq t(N) \leq r_j < r_{j+1} \) we can write, using Lemma 4.2
\[ E ((p_{r_j+1-r}(B_r - B_s) - p_{r_j-r}(B_r - B_s))(p_{r_j+1-r}(B_r - B_u) - p_{r_j-r}(B_r - B_u))) \]
\[ = ((r_j+1-s)(r_j+1-r+s-u) + (r_j+1-r)(r-s))^{-\frac{1}{2}} \]
\[ - ((r_j+1-s)(r_j-r+s-u) + (r_j+1-r)(r-s))^{-\frac{1}{2}} \]
\[ - ((r_j-s)(r_j+1-r+s-u) + (r_j-r)(r-s))^{-\frac{1}{2}} \]
\[ + ((r_j-s)(r_j-r+s-u) + (r_j-r)(r-s))^{-\frac{1}{2}} \]
\[ = -\frac{1}{2} \int_{r_j}^{r_{j+1}} ((r_j+1-s)(\theta - r + s - u) + (r_j+1-r)(r-s))^{-\frac{3}{2}} (r_j+1-s)d\theta \]
\[ + \frac{1}{2} \int_{r_j}^{r_{j+1}} ((r_j-s)(\theta - r + s - u) + (r_j-r)(r-s))^{-\frac{3}{2}} (r_j-s)d\theta. \]
Integrating in the variable \( u \) yields

\[
\int_0^s E\left( (p_{r_{j_1}+r}(B_r - B_s) - p_{r_j-r}(B_r - B_s))(p_{r_{j_1}+r}(B_r - B_u) - p_{r_j-r}(B_r - B_u)) \right) du
\]

\[
= - \int_{r_j}^{r_{j+1}} ((r_j+1-s)(\theta - r + s - u) + (r_{j+1} - r)(r - s))^\frac{1}{2} |u = 0| d\theta
\]

\[
+ \int_{r_j}^{r_{j+1}} ((r_j-s)(\theta - r + s - u) + (r_j - r)(r - s))^\frac{1}{2} |u = 0| d\theta
\]

\[
= - \frac{1}{2} \int_{r_j}^{r_{j+1}} \int_{r_j}^{r_{j+1}} ((\eta - s)(\theta - r + s) + (\eta - r)(r - s) - \frac{1}{2} \theta) d\eta d\theta
\]

\[
\leq C \int_{r_j}^{r_{j+1}} \int_{r_j}^{r_{j+1}} ((\eta - s)(\theta - r + s) + (\eta - r)(r - s) - \frac{3}{2} \eta)^2 d\eta d\theta
\]

\[
\leq C(r-s)^{-\frac{3}{2}} \left( \int_{r_j}^{r_{j+1}} (\eta - s)^{\frac{3}{2}} d\eta \right)^2
\]

\[
\leq C(r-s)^{-\frac{3}{2}} \left( (r_{j+1} - r)^{\frac{3}{4}} - (r_j - r)^{\frac{3}{4}} \right)^2
\]

\[
\leq C(r - r(N))^{-\frac{3}{4}} (r(N) - s)^{-\frac{3}{2}} (r_{j+1} - r_j)^{2-\frac{3}{2}\alpha} (r_j - r)^{-\frac{3}{2}(1-\alpha)},
\]

for any \( \alpha \in (0, 1) \). Choosing \( \alpha = \frac{1}{4} \) and integrating in the variables \( 0 < s < r(N) < r < t(N) \), we obtain

\[
E \left( \left( \int_0^{\frac{kT}{N}} \left( \int_0^{r(N)} (p_{r_{j_1}+r}(B_r - B_s) - p_{r_j-r}(B_r - B_s)) ds \right)^2 dr \right)^\frac{2}{3} \right) \leq C_{N}(r_{j+1} - r_j)^{-\frac{16}{15}}.
\]

As a consequence,

\[
E(S_{n,N}) \leq C_{N}n^{-\frac{16}{15}},
\]

which converges to zero as \( n \) tends to infinity.

**Step 3**

Let us compute the \( \frac{4}{3} \) variation of the process \( \gamma^{N,1} \) in the interval \( I_{k,N} := \left[ \frac{kT}{N}, \left( \frac{k+1}{N} \right)T \right] \). Set \( \tau_N = \frac{kT}{N} = t(N) \). By the results of [4], there exists a two-sided Brownian motion \( \{W_x, x \in \mathbb{R}\} \) independent of \( \{B_r, r \geq r_N, L^{B_N}_{\tau_N}\} \) such that for any \( x > y, x, y \in \mathbb{R} \),

\[
L^x_{\tau_N} - L^y_{\tau_N} = 2 \int_y^x \sqrt{L^z_{\tau_N} dW_z} + \int_y^x \alpha(z) dz.
\]

Using the fact that the random variables \( \{B_r, r \geq r_N, L^{B_N}_{\tau_N}\} \) are independent of \( W \) we can write for any \( r \geq \tau_N \),

\[
L^{B_{r+y}}_{\tau_N} - L^{B_r}_{\tau_N} = 2 \int_{B_r}^{B_{r+y}} \sqrt{L^z_{\tau_N} dW_z} + \int_{B_r}^{B_{r+y}} \alpha(z) dz.
\]
We decompose the process $\gamma_{N,1}$ as follows:

$$\gamma_{N,1} = \gamma_{N,2} + \gamma_{N,3} + \gamma_{N,4},$$

where

$$\gamma_{N,2} = \int_{\tau_N}^t E^\theta \left( \int_{B_r}^{B_r + \sqrt{t - \tau}} \alpha(z) dz \right) dB_r,$$

$$\gamma_{N,3} = \int_{\tau_N}^t E^\theta \left( \int_{B_r}^{B_r + \sqrt{t - \tau}} \left( \sqrt{L_z} - \sqrt{L_{B_r}^z} \right) dW_z \right) dB_r,$$

and

$$\gamma_{N,4} = \sqrt{L_{\tau_N r}} \int_{\tau_N}^t E^\theta (W(B_r + \theta \sqrt{t - r}) - W(B_r)) dB_r,$$

where here $\theta$ denotes a random variable with law $N(0,1)$, independent of $B$ and $W$. We claim that for any $k$,

$$\langle \gamma_{N,2} \rangle^{1/2}_{4/3} I_{k,N} = 0,$$  \hspace{1cm} (3.14)

and

$$\langle \gamma_{N,3} \rangle^{1/2}_{4/3} I_{k,N} = 0,$$  \hspace{1cm} (3.15)

**Proof of (3.14):** With the same notation as in Step 1, set

$$S_{n,N} := S_{4/3,n} \langle \gamma_{N,2} \rangle_{4/3} = \sum_{j=0}^{n-1} |\Delta_j(\gamma_{N,2})|^{4/3},$$

where $\Delta_j(\gamma_{N,2}) = \gamma_{r_{j+1},N} - \gamma_{r_j,N}$. Then

$$\sum_{j=0}^{n-1} E|\Delta_j(\gamma_{N,2})|^{4/3} = \sum_{j=0}^{n-1} E \left| \int_{\tau_N}^{r_{j+1}} E^\theta \left( \int_{B_r}^{B_r + \sqrt{t - \tau}} \alpha(y) dy \right) dB_r \right|^{4/3}$$

$$- \int_{\tau_N}^{r_j} E^\theta \left( \int_{B_r}^{B_r + \sqrt{t - \tau}} \alpha(y) dy \right) dB_r$$

$$= \sum_{j=0}^{n-1} E \left| \int_{\tau_N}^{r_j} E^\theta \left( \int_{B_r}^{B_r + \sqrt{t - \tau}} \alpha(y) dy \right) dB_r \right|^{4/3}$$

$$+ \int_{r_j}^{r_{j+1}} E^\theta \left( \int_{B_r}^{B_r + \sqrt{t - \tau}} \alpha(y) dy \right) dB_r$$

$$\leq C \sum_{j=0}^{n-1} \left\{ \left| \int_{\tau_N}^{r_j} E \left( E^\theta \int_{B_r}^{B_r + \sqrt{t - \tau}} \alpha(y) dy \right)^2 dr \right|^{2/3} \right\}$$

$$+ \left| \int_{r_j}^{r_{j+1}} E \left( E^\theta \int_{B_r}^{B_r + \sqrt{t - \tau}} \alpha(y) dy \right)^2 dr \right|^{4/3}$$

$$= A_n + B_n.$$
From [4], we have the following expression for the process $\alpha(y)$,

$$
\alpha(y) = I_{\{y \geq B_s\}} \left[ 2I_{\{y \leq 0\}} + 2I_{\{y \leq B_s\}} + I_{\{y \leq B_s\}} \right] \left( \frac{4I_{\{y \geq B_s\}}}{L(s, y) + 2y} - \frac{L(s, y) + 2y}{s - A(s, y)} \right)
$$

with

$$
\overline{B}_s = \sup\{B_u, u \leq s\}, \underline{B}_s = \inf\{B_u : u \leq s\}.
$$

Let $\gamma(y) = -I_{\{y \geq B_s\}}I_{\{y \leq \underline{B}_s\}}L(s, y)\frac{L(s, y) + 2y}{s - A(s, y)}$, and write $\alpha(y) = \beta(y) + \gamma(y)$. Then $\beta(y)$ is bounded, and from the result of section 3 (page 277 and 278) in [5], we can get that $E \int_\mathbb{R} |\gamma(y)|^p dy < \infty$ for all $p > 1$. As a consequence, by Lemma 4.3 we obtain

$$
\lim_{n \to \infty} \sum_{j=0}^{n-1} \left| \int_{\tau_N}^{r_j} E \left( E^\theta \int_{B_r + \sqrt{r_j + r\theta}}^{B_r + \sqrt{r_j + r\theta}} \beta(y) dy \right)^2 \right|^{\frac{2}{3}} \leq C \lim_{n \to \infty} \sum_{j=0}^{n-1} \left| \int_{\tau_N}^{r_j} \left( \sqrt{r_{j+1} + r} - \sqrt{r_j + r} \right)^2 dr \right|^{\frac{2}{3}} = 0.
$$

To handle the term containing $\gamma(y)$, we choose $p, q$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $p < \frac{4}{3}$. Then, again by Lemma 4.3

$$
\lim_{n \to \infty} \sum_{j=0}^{n-1} \left| \int_{\tau_N}^{r_j} E \left( E^\theta \int_{B_r + \sqrt{r_j + r\theta}}^{B_r + \sqrt{r_j + r\theta}} \gamma(y) dy \right)^2 \right|^{\frac{2}{3}} \leq \lim_{n \to \infty} \sum_{j=0}^{n-1} \left| \int_{\tau_N}^{r_j} E(|\theta|^{\frac{2}{3}}(\sqrt{r_{j+1} + r} - \sqrt{r_j + r})^2 E \left( \int_{\mathbb{R}} \gamma(y) dy \right)^\frac{2}{3} \right|^{\frac{2}{3}} = C \lim_{n \to \infty} \sum_{j=0}^{n-1} \left| \int_{\tau_N}^{r_j} \left( \sqrt{r_{j+1} + r} - \sqrt{r_j + r} \right)^2 dr \right|^{\frac{2}{3}} = 0.
$$

Hence we have $A_n$ goes to zero as $n$ goes to infinity. The convergence to zero of $B_n$ as $n$ tends to infinity follows from

$$
\lim_{n \to \infty} \sum_{j=0}^{n-1} \left| \int_{r_j}^{r_j+1} E \left( E^\theta \int_{B_r + \sqrt{r_{j+1} + r\theta}}^{B_r} \beta(y) dy \right)^2 \right|^{\frac{2}{3}} \leq C \lim_{n \to \infty} \sum_{j=0}^{n-1} \left| \int_{r_j}^{r_j+1} (r_j + 1 - r) dr \right|^{\frac{2}{3}} = C \lim_{n \to \infty} \sum_{j=0}^{n-1} \left( \frac{1}{n} \right)^{\frac{4}{3}} = 0,
$$

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and, choosing $p, q$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $p < 2$,

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} \left| \int_{r_j}^{r_{j+1}} E \left( E^\theta \int_{B_r+\sqrt{r_{j+1}-r}} \gamma(y) dy \right)^2 dr \right| \leq 0$$

and

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} \left| \int_{r_j}^{r_{j+1}} E(\frac{2}{p})(\sqrt{r_{j+1}-r})^\frac{2}{p} E \left[ \int_{R} |\gamma(y)|^q dy \right] \frac{2}{q} \frac{2}{3} dr \right| \leq C \lim_{n \to \infty} \sum_{j=0}^{n-1} \left( \frac{1}{n} \right)^{(\frac{1}{p}+1)\frac{2}{3}} = 0.$$

**Proof of (3.15):** With the same notation as in Step 1, set

$$S_{n,N} := S_{k,n}^N (\gamma_{N,3}) = \sum_{j=0}^{n-1} |\Delta_j(\gamma_{N,3})|^\frac{2}{3},$$

where $\Delta_j(\gamma_{N,3}) = \gamma_{r_{j+1}}^{N,3} - \gamma_{r_j}^{N,3}$. As in the proof of (3.14), applying the Burkholder inequality we obtain

$$\sum_{j=0}^{n-1} E|\Delta_j(\gamma_{N,3})|^\frac{2}{3} \leq C(C_n + D_n),$$

where

$$C_n = E \sum_{j=1}^{n-1} \left( \int_{r_j}^{r_{j+1}} \left( \int_{B_r+\sqrt{r_{j+1}-r}} \left( \sqrt{L_{r,N}^z} - \sqrt{L_{r,N}^{B_r}} \right) dW_z \right)^2 dr \right)^{2/3}$$

and

$$D_n = E \sum_{j=1}^{n-1} \left( \int_{r_j}^{r_{j+1}} \left( \int_{B_r+\sqrt{r_{j+1}-r}} \left( \sqrt{L_{r,N}^z} - \sqrt{L_{r,N}^{B_r}} \right) dW_z \right)^2 dr \right)^{2/3}.$$
where

$$\Phi(z) = \left( \sqrt{L_{T_N}^z} - \sqrt{L_{B^r_T}^z} \right) \int_\mathbb{R} (p_{r_{j+1} - r}(y) - p_{r_j - r}(y)) 1_{[B^r_r, B^r_r + y]}(z) dy.$$ 

As a consequence,

$$E\left( \int_\mathbb{R} \Phi(z) dW_z \right)^2 \leq E\left( G^2 \int_\mathbb{R} (p_{r_{j+1} - r}(y) - p_{r_j - r}(y)) (p_{r_{j+1} - r}(y') - p_{r_j - r}(y')) \times \int_{[B^r_r, B^r_r + y] \cap [B^r_r, B^r_r + y']} |z - B^r_r|^{\frac{1}{2} - \epsilon} dz dy dy' \right)^2 \leq C \left( \int_\mathbb{R} (p_{r_{j+1} - r}(y) - p_{r_j - r}(y)) |y|^{\frac{3}{2} - \frac{\epsilon}{2}} dy \right)^2 \leq C \left( \int_{r_j}^{r_{j+1}} (\theta - r) - \frac{3}{4} \frac{\epsilon}{4} d\theta \right)^2 \leq C(r_{j+1} - r_j)^{\frac{3}{4} + \epsilon} (r_j - r)^{-\frac{3}{4} - \frac{3\epsilon}{4}},$$

and we obtain

$$C_n \leq Cn^{-\frac{2}{3} \epsilon}.$$ 

This proves (3.15).

**Step 4**

Let us compute the $\frac{4}{3}$ variation of the process $\gamma^{N,4}$. By Theorem 2.1, the $\frac{4}{3}$ variation in $L^1$ of the process

$$Z_t = \int_0^t E^\theta(W_{B^r_t + \theta \sqrt{t-r}} - W_{B^r_r}) dB_r,$$

in an interval $[a, b]$ is $K(b - a)$. In fact, this process has the same distribution as

$$X_t = \int_0^t E^\theta(W_{\theta \sqrt{t-r}}) dB_r.$$

This follows from the fact that the processes

$$\{(B_t, W_{B^r_t + y} - W_{B^r_r}), t \geq 0, r \geq 0, y \in \mathbb{R}\}$$

and

$$\{(B_t, W_y), t \geq 0, r \geq 0, y \in \mathbb{R}\}$$

have the same law, as it can be easily seen by computing the characteristic function of the finite dimensional distributions of both processes. Therefore,

$$\langle \gamma^{N,4} \rangle_{\frac{4}{3}}[0,T] = K \sum_{k=0}^{N-1} \left( L_{B^r_{kT/N}}^{B^r_{kT/N}} \right)^{\frac{1}{2}} T \frac{T}{N}.$$
By Step 2 and Step 3, we have that \( \langle \gamma^N \rangle_{\frac{1}{2},[0,T]} = \langle \gamma^{N'} \rangle_{\frac{1}{2},[0,T]} \). Then the proof of Theorem 1.2 follows immediately from Step 1 and the fact that
\[
\lim_{N \to \infty} \langle \gamma^N \rangle_{\frac{1}{2},[0,T]} = K \int_0^T (L_r B_r)^2 dr.
\]

4 Appendix

Lemma 4.1 Let \( 0 \leq a < b < c < d \), and set \( x = b - a \), \( y = c - b \) and \( z = d - c \). Then,
\[
E \left[ E \left( 1_{\{B_c < B_a < B_d\}} - 1_{\{B_c > B_a > B_d\}} \right) \big| \mathcal{F}_b \right]^2 \leq Cx^{-\frac{1}{2}} \left( 2\sqrt{2y + z} - \sqrt{2y} - \sqrt{2y + 2z} \right).
\]

Proof Set
\[
B_a - B_b = \sqrt{x}X, \quad B_c - B_b = \sqrt{y}Y, \quad B_d - B_c = \sqrt{z}Z,
\]
where \( X, Y \) and \( Z \) are independent \( N(0,1) \) random variables. With this notation we can write
\[
E \left[ E \left( 1_{\{B_c < B_a < B_d\}} - 1_{\{B_c > B_a > B_d\}} \right) \big| \mathcal{F}_b \right]^2 = E \left[ P(\sqrt{y}Y < \sqrt{x}X < \sqrt{z}Z + \sqrt{y}Y | X) - P(\sqrt{y}Y > \sqrt{x}X > \sqrt{z}Z + \sqrt{y}Y | X) \right]^2 = \int_\mathbb{R} \left( \int_\mathbb{R} \phi(\eta) d\eta \int_{\sqrt{\frac{x}{y}} \theta - \sqrt{\frac{z}{y}} \eta}^{\sqrt{\frac{x}{y}} \theta} \phi(\xi) d\xi \right)^2 d\theta,
\]
where \( \phi(x) \) is the density of the law \( N(0,1) \). Set
\[
g(x,y,z,\theta) = \int_\mathbb{R} \phi(\eta) d\eta \int_{\sqrt{\frac{x}{y}} \theta - \sqrt{\frac{z}{y}} \eta}^{\sqrt{\frac{x}{y}} \theta} \phi(\xi) d\xi.
\]
Then,
\[
g(x,y,z,\theta) = \frac{1}{\sqrt{\gamma}} \int_0^{\sqrt{z}} \int_\mathbb{R} \phi(\eta) \phi(\sqrt{\frac{x}{y}} \theta - \frac{w}{\sqrt{\gamma}} \eta) \eta d\eta dw = \frac{1}{2\pi} \frac{1}{\sqrt{\gamma}} \int_0^{\sqrt{z}} \int_\mathbb{R} \exp \left( -\frac{1}{2} (\eta^2 + (\sqrt{\frac{x}{y}} \theta - \frac{z}{\sqrt{\gamma}} \eta)^2) \right) \eta d\eta dw = \frac{1}{2\pi} \int_0^{\sqrt{z}} \frac{w \sqrt{x} \theta}{(y + w^2)^{\frac{3}{2}}} \exp \left( -\frac{x \theta^2}{2(y + w^2)} \right) dw = \frac{1}{4\pi} \int_0^{\sqrt{z}} \frac{\sqrt{x} \theta}{(y + \xi)^{\frac{3}{2}}} \exp \left( -\frac{x \theta^2}{2(y + \xi)} \right) d\xi.
\]
Finally, integrating with respect to \( \theta \) yields

\[
\int_R g(x, y, z, \theta)^2 \phi(\theta) d\theta
\]

\[
= C x \int R \int_0^z \int_0^z \frac{\theta^2}{(y + \xi_1)^\frac{3}{2}(y + \xi_2)^\frac{3}{2}} \exp \left( -\frac{1}{2} \left( \frac{x\theta^2}{y + \xi_1} + \frac{x\theta^2}{y + \xi_2} \right) \right) d\xi_1 d\xi_2 \phi(\theta) d\theta
\]

\[
= C x \int_0^z \int_0^z \frac{1}{(y + \xi_1)^\frac{3}{2}(y + \xi_2)^\frac{3}{2}} \int_\mathbb{R} \theta^2 \exp \left( \frac{\theta^2}{2} \left( \frac{x}{y + \xi_1} + \frac{x}{y + \xi_2} + 1 \right) \right) d\theta d\xi_1 d\xi_2
\]

\[
= C x \int_0^z \int_0^z \frac{1}{(y + \xi_1)^\frac{3}{2}(y + \xi_2)^\frac{3}{2}} \left( \frac{x}{y + \xi_1} + \frac{x}{y + \xi_2} + 1 \right)^{-\frac{3}{2}} d\xi_1 d\xi_2
\]

\[
= C x \int_0^z \int_0^z \frac{1}{(y + \xi_1 + \xi_2)^\frac{3}{2}} (2y + \xi_1 + \xi_2)^{-\frac{3}{2}} d\xi_1 d\xi_2
\]

\[
\leq C x^{-\frac{1}{2}} \int_0^z \int_0^z \frac{1}{(y + \xi_1 + \xi_2)^\frac{3}{2}} (2y + \xi_1 + \xi_2)^{-\frac{3}{2}} d\xi_1 d\xi_2
\]

\[
= C x^{-\frac{1}{2}} \left[ 2\sqrt{2y + z} - \sqrt{2y - \sqrt{2y + 2z}} \right],
\]

which completes the proof of the lemma. 

**Lemma 4.2** Let \( \alpha, \beta > 0 \) and let \( X, Y \) be independent random variables with laws \( N(0, \sigma_1^2) \) and \( N(0, \sigma_2^2) \), respectively. Then 

\[
E[p_\alpha(X)p_\beta(X + Y)] = ((\alpha + \sigma_1^2)(\beta + \sigma_2^2) + \alpha\sigma_2^2)^{-\frac{1}{2}}.
\]

**Lemma 4.3** Suppose \( a < b \) and \( n \in \mathbb{N} \). Let \( r_j = a + \frac{j}{n}(b - a), j = 0, 1, \ldots, n \). Then, for any \( \beta > \frac{3}{2} \), we have

\[
\lim_{n \to \infty} \sum_{j=1}^n \left| \int_a^{r_j} (\sqrt{r_{j+1} - r} - \sqrt{r_j - r})^\beta dr \right|^\frac{1}{\beta} = 0.
\]

**Proof** It suffices to use the estimate

\[
\sqrt{r_{j+1} - r} - \sqrt{r_j - r} \leq C(r_{j+1} - r_j)^{\frac{1}{2} + \frac{\beta}{3n}}(r_j - r)^{-\frac{\beta}{3n}}.
\]

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