A $K$-THEORETIC REFINEMENT OF TOPOLOGICAL REALIZATION OF UNSTABLE ALGEBRAS

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Abstract. In this paper we propose and partially carry out a program to use $K$-theory to refine the topological realization problem of unstable algebras over the Steenrod algebra. In particular, we establish a suitable form of algebraic models for $K$-theory of spaces, called $\psi^p$-algebras, which give rise to unstable algebras by taking associated graded algebras mod $p$. The aforementioned problem is then split into (i) the algebraic problem of realizing unstable algebras as mod $p$ associated graded of $\psi^p$-algebras and (ii) the topological problem of realizing $\psi^p$-algebras as $K$-theory of spaces. Regarding the algebraic problem, a theorem shows that every connected and even unstable algebra can be realized. We tackle the topological problem by obtaining a $K$-theoretic analogue of a theorem of Kuhn and Schwartz on the so-called Realization Conjecture.

1. Introduction

Let $A$ be the Steenrod algebra associated to a prime $p$; the mod $p$ cohomology of a topological space is then an $A$-algebra. In fact, it is an unstable $A$-algebra. This means that if $p$ is an odd prime, then $P^i x = x^p$ if $2i = |x|$ and $P^i x = 0$ if $2i > |x|$, and there are similar conditions for the prime 2. The problem of which unstable $A$-algebras (or $A$-modules) can be realized as the cohomology of a space has a long history and is one of the central problems in algebraic topology. For instance, about four decades ago, Steenrod asked which polynomial algebras over $\mathbb{Z}/p$ can be realized by the cohomology of a space, and the famous work of Adams and Wilkerson on their embedding theorem is one of the many papers on this $A$-algebra realization problem.

More recently, the topological realization problem of $A$-modules has been studied by Kuhn and Schwartz. In Kuhn made an interesting conjecture, the Realization Conjecture, which says that if a finitely generated $A$-module is topologically realizable, then it must be finite as a set. Thus, Kuhn’s conjecture predicts that a finitely generated $A$-module that is not finitely generated as a $\mathbb{Z}/p$-module cannot arise as the cohomology of a space. This conjecture is partially verified by Kuhn and is proved in its entirety by Schwartz. Meanwhile, Blanc approached the question of topologically realizing $A$-algebras from an obstruction theory point of view. He established an obstruction theory which one can use, in principle, to decide whether any $A$-algebra of finite type is topologically realizable.

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The purpose of this paper is to propose and partially carry out a program in which $K$-theory is used to refine the topological realization problem of unstable $\mathcal{A}$-algebras. In order to describe our program, we first need to recall a result of Atiyah relating the unstable structures of $K$-theory and mod $p$ cohomology of torsionfree spaces.

Let $X$ be a torsionfree space; that is, a space that has no torsion in integral cohomology. Then the even dimensional part of its integral cohomology ring $H^*(X; \mathbb{Z})$ can be identified with the associated graded ring of its $K$-theory:

$$\text{Gr}^* K(X) \cong H_{\text{even}}^*(X; \mathbb{Z}) \quad (1.1)$$

Here $\text{Gr}^*(-)$ denotes the associated graded ring and the filtration on $K(X)$ arises from a skeletal filtration on $X$. That is, letting $X_n$ denote the $n$th skeleton of $X$, the $i$th filtration ideal of $K(X)$ is the kernel $I^i(X) = \ker(K(X) \to K(X_{i-1}))$ of the restriction map. Throughout this paper we will use $\otimes$ to denote tensor product over $\mathbb{Z}$ and $\psi^n$ to denote Adams operations in $K$-theory.

Atiyah [2, 5.6 and 6.5] showed that, in fact, operations in $K$-theory determine those in mod $p$ cohomology.

**Theorem 1.1** (Atiyah). Let $p$ be a prime and let $X$ be a torsionfree space. If $\alpha \in K(X)$ lies in filtration $2q$, then there exist elements $\alpha_i \in K(X)$ ($i = 0, 1, \ldots, q$) in filtration $2q + 2i(p - 1)$ such that

$$\psi^p(\alpha) = \sum_{i=0}^{q} p^{q-i} \alpha_i \quad (1.2)$$

where $\alpha_q = \alpha^p$ if $q > 0$. This yields well-defined functions

$$\mathbb{P}^i : (\text{Gr}^* K(X)) \otimes \mathbb{Z}/p \to (\text{Gr}^{*+2i(p-1)} K(X)) \otimes \mathbb{Z}/p,$$

sending $\bar{\alpha}$ (the image of $\alpha$ in the mod $p$ associated graded ring) to $\bar{\alpha}_i$. With the identification of eq. (1.1) mod $p$, these functions $\mathbb{P}^i$ are precisely the Steenrod operations (with $\mathbb{P}^i = Sq^{2i}$ when $p = 2$) in mod $p$ cohomology.

This theorem of Atiyah suggests that there should be some suitable algebraic objects modeling $K$-theory of spaces, whose associated graded algebras mod $p$ are unstable $\mathcal{A}$-algebras with Steenrod operations induced by some “Adams operation” $\psi^p$. As we will see shortly, there is indeed an algebraic model for $K$-theory with the desired property, and we call it $\psi^p$-algebra.

Our program to use $K$-theory to refine the topological realization problem of unstable $\mathcal{A}$-algebras can now be summarized by the following diagram.

$$\begin{array}{ccc}
\psi^p(\text{-}) & \to & \text{Gr}^* K(-) \otimes \mathbb{Z}/p \\
\downarrow & & \\
H^*(-) & \to & \text{Gr}^* (-) \otimes \mathbb{Z}/p \\
\end{array}$$

(1.3)
In other words, the topological realization problem of unstable $A$-algebras, which corresponds to the bottom arrow, splits into two separate problems corresponding to the other two arrows:

(i) The *algebraic* problem of realizing unstable $A$-algebras as associated graded algebras mod $p$ of $\psi^p$-algebras.

(ii) The *topological* problem of realizing $\psi^p$-algebras as $K$-theory of spaces.

In this paper we concentrate mostly on the algebraic realization problem, but we will also present a result about the $K$-theoretic topological realization problem.

A discussion of some of the main results of this paper follows. We begin with a result which establishes a certain kind of algebraic model for $K$-theory of spaces. In Atiyah’s Theorem 1.1, the Adams operation $\psi^p$ is the only $K$-theory operation used when relating $K$-theory and mod $p$ cohomology of torsion-free spaces. This suggests that we should model $K$-theory by algebras having a self-map $\psi^p$ that behaves like the Adams operation on $K$-theory and that splits into a sum of the form (1.2) when applied to elements in the algebra. More precisely, we define a *pre-$\psi^p$-algebra* to be a commutative, filtered ring $R$ with a distinguished endomorphism $\psi^p : R \to R$ such that the following two conditions hold.

(i) The $2n$th filtration of $R$ coincides with the $(2n-1)$st filtration for all $n$.

(ii) Let $r \in R$ be in filtration $2q$ for some $q$. If $q > 0$, then there exist elements $r_i \in R$ ($0 \leq i \leq q$) in filtration $2q + 2i(p-1)$ such that

$$\psi^p(r) = p^q r_0 + p^{q-1} r_1 + \cdots + p r_{q-1} + r_q$$

where $r_q = r^p$. If $q = 0$, then there exists an element $r' \in R$ in filtration 0 such that

$$\psi^p(r) = pr' + r^p.$$ 

Moreover, the elements $r_i$ are required to be well-defined in the associated graded algebra mod $p$ of $R$.

The $K$-theory of a space together with its Adams operation $\psi^p$ is clearly a pre-$\psi^p$-algebra, at least if the space has integral cohomology that is torsion-free and is concentrated in even dimensions. Each of equation (1.4) and (1.5) is called an *Atiyah formula* for $r$. Notice that the elements $r_i$ in an Atiyah formula for $r$ are not unique.

Our first main result is then the following theorem, which says that the associated graded algebra mod $p$ of a pre-$\psi^p$-algebra is “close” to being an unstable $A$-algebra.

**Theorem 1.2.** Let $p$ be a fixed prime and let $R$ be a pre-$\psi^p$-algebra. Then there exist operations

$$P^i : \text{Gr}^* R \otimes \mathbb{Z}/p \to \text{Gr}^{*+2i(p-1)} R \otimes \mathbb{Z}/p \quad (i \geq 0)$$

(1.6)
Moreover, these operations have the following four properties.

1. Each $P^i$ is additive.
2. For any $q \geq 0$, $P^q$ is the $p$th power map from $\text{Gr}^{2q} R \otimes \mathbb{Z}/p$ to $\text{Gr}^{2pq} R \otimes \mathbb{Z}/p$.
3. If $|\tau| = 2q$ then $P^i\tau = 0$ for every $i > q$.
4. For any two elements $\tau$ and $\sigma$ in $\text{Gr}^* R \otimes \mathbb{Z}/p$, $P^i(\tau\sigma) = \sum_{i+k=i} (P^i\tau)(P^k\sigma)$.

Thus, the associated graded algebra mod $p$ of a pre-$\psi^p$-algebra has operations $P^i$ ($i \geq 0$) that are additive and satisfies the unstable conditions and the Cartan formula. However, missing from the above list are the Adem relations, however, are satisfied for trivial reasons.

**Example 1.3.** For any prime $p$, there exists a pre-$\psi^p$-algebra structure on the ring $\mathbb{Z}[\varepsilon]$ ($\varepsilon^2 = 0$) of dual numbers, where $\varepsilon$ lies in filtration precisely 4, in which the function $P^0$ as in (1.6) is not equal to the identity function. The Adem relations, however, are satisfied for trivial reasons.

**Example 1.4.** For any prime $p > 2$, there exists a pre-$\psi^p$-algebra structure on the polynomial ring $\mathbb{Z}/(p)[x]$ over the $p$-local integers, where $x$ lies in filtration precisely $2(p-1)$, in which the Adem relations are not satisfied. It is the case that $P^0 = \text{Id}$.

In view of these examples, we define a $\psi^p$-algebra to be a pre-$\psi^p$-algebra $R$ for which the operations $P^i$, $i > 0$, (as in Theorem 1.2) on $\text{Gr}^* R \otimes \mathbb{Z}/p$ satisfies the Adem relations and the property that $P^0 = \text{Id}$. We remark that it is possible to describe the Adem relations in terms of elements appearing in Atiyah formula (see Remark 3.3). A $\psi^p$-algebra map $f : R \rightarrow S$ between two $\psi^p$-algebras is a filtered algebra map $f : R \rightarrow S$ (that is, an algebra map that preserves the respective filtrations) which is compatible with the endomorphisms $\psi^p$ of $R$ and $S$, in the sense that $\psi^p f = f \psi^p$. It follows from Atiyah’s Theorem 1.1 that the $K$-theory $K(X)$ with its Adams operation $\psi^p$ is a $\psi^p$-algebra, provided $X$ is a space whose integral cohomology is concentrated in even dimensions and is torsionfree.

The following result is now an immediate consequence of Theorem 1.2.

**Corollary 1.5.** For any prime $p$ there is a functor

$$\text{Gr}^*(-) \otimes \mathbb{Z}/p : (\psi^p \text{-algebras}) \rightarrow (\text{unstable A-algebras})$$
which associates to each $\psi^p$-algebra $R$ the unstable $A$-algebra $\text{Gr}^* R \otimes \mathbb{Z}/p$ with Steenrod operations $P^i \ (i \geq 0)$ as in Theorem 1.2.

Henceforth whenever $R$ is a $\psi^p$-algebra, the symbol $\text{Gr}^* R \otimes \mathbb{Z}/p$ will mean the unstable $A$-algebra with Steenrod operations $P^i \ (i \geq 0)$ as in Theorem 1.2.

Having established that $\psi^p$-algebras are appropriate algebraic models for $K$-theory for our purpose, we now turn to the algebraic realization problem concerning $\psi^p$-algebras and unstable $A$-algebras. We would like to answer the following question:

Given an unstable $A$-algebra is there always a “lift” of it back to a $\psi^p$-algebra?

Before we can answer this question, we first need to make precise what we mean by a “lift” of an unstable $A$-algebra.

For certain technical reasons, we need to work with “connected” objects. An unstable $A$-algebra $H^*$ is said to be connected if $H^0 = \mathbb{Z}/p$ and $H^< 0 = 0$. It is called an even unstable $A$-algebra if it is trivial in odd dimensions. Let $H^*$ be an even, connected unstable $A$-algebra. A $\psi^p$-algebra $R$ is said to be connected if the degree 0 part $\text{Gr}^0 R \otimes \mathbb{Z}/p$ of its associated graded algebra $\text{mod } p$ is isomorphic to $\mathbb{Z}/p$. Then a lift of $H^*$ is defined to be a connected $\psi^p$-algebra $R$ for which there exists an isomorphism

$$H^* \cong \text{Gr}^* R \otimes \mathbb{Z}/p$$

of unstable $A$-algebras.

The following result shows that every even, connected unstable $A$-algebra has a lift that is canonical up to a choice of $A$-algebra generators and relations. This result, in particular, answers the question stated above, at least if the unstable $A$-algebra is even and connected.

**Theorem 1.6.** Let $p$ be a fixed prime and let $H^*$ be an even, connected unstable $A$-algebra. Then for each choice of a set of $A$-algebra generators and relations for $H^*$, there exist a corresponding canonical construction of a lift $B = B_{H^*}$ of $H^*$ and a canonical unstable $A$-algebra isomorphism

$$\rho_{H^*} : \text{Gr}^* B_{H^*} \otimes \mathbb{Z}/p \xrightarrow{\cong} H^*.$$  

Moreover, suppose that a fixed choice of $A$-algebra generators and relations for $H^*$ is made and that $B_{H^*}$ and $\rho_{H^*}$ are, respectively, the corresponding lift and isomorphism. Suppose that

$$\phi : H^* \xrightarrow{\cong} K^*$$

is an isomorphism of even, connected unstable $A$-algebras and that the lift $B_{K^*}$ is constructed with the $A$-algebra generators and relations for $K^*$ corresponding via $\phi$ to those of $H^*$ already chosen. Then there exists a canonical isomorphism

$$\varphi : B_{H^*} \xrightarrow{\cong} B_{K^*} \quad (1.8)$$
of connected $\psi^p$-algebras such that the following diagram of unstable $A$-algebras and isomorphisms commutes:

$$
\begin{array}{ccc}
\text{Gr}^* B_{H^*} \otimes \mathbb{Z}/p & \xrightarrow{\varphi^*} & \text{Gr}^* B_{K^*} \otimes \mathbb{Z}/p \\
\rho_{H^*} & & \rho_{K^*} \\
H^* & \phi & K^*
\end{array}
$$

This finishes the presentation of our main results concerning the algebraic realization problem.

As for the $K$-theoretic topological realization problem, we have a $K$-theoretic analogue of a theorem of Kuhn [5] and Schwartz [7] about the so-called Realization Conjecture that was mentioned earlier in the Introduction (see Proposition 5.3). Our $K$-theoretic result is about a simpler type of algebraic objects, $\psi^p$-modules, which corresponds to $A$-modules. Since we do not want to encumber this Introduction with too many definitions, we will not discuss this result in this section.

1.1. Organization of the paper. This paper is organized as follows.

In §2 we set the stage by discussing some preliminary materials on unstable $A$-algebras, filtered algebras, and pre-$\psi^p$-algebras. Sections 3 contains the proofs of Theorems 1.2, Examples 1.3 and 1.4, and Corollary 1.5. The proof of Theorem 1.6 is given in §4.

The final section, §5, contains our $K$-theoretic analogue of Kuhn’s Realization Conjecture.

2. Unstable $A$-algebra, filtered algebra, and pre-$\psi^p$-algebra

2.1. Unstable $A$-algebra. Here we briefly recall the definition of an unstable algebra over the Steenrod algebra. The reader can consult the books [1], [8, Ch. 1] for more information on this subject. The field of $p$ elements is denoted by $\mathbb{Z}/p$.

2.1.1. The Steenrod algebra. Let $p$ be a fixed prime and denote by $A$ the mod $p$ Steenrod algebra. We do not decorate $A$ with a subscript $p$, as some authors do, since we usually work with a fixed prime $p$ in mind.

The Steenrod algebra $A$ is the free, graded, associative $\mathbb{Z}/p$-algebra generated by the Bockstein $\beta$ in degree 1 and the Steenrod operations $P^i$ in degree $2i(p-1)$ (resp. $Sq^i$ when $p = 2$) $(i \geq 0)$. They are subject to the conditions:

(i) $P^0 = \text{Id}$ (resp. $Sq^0 = \text{Id}$ when $p = 2$).
(ii) $\beta^2 = 0$.
(iii) The Adem relations: If $p > 2$, then for any $i, j > 0$

$$P^i P^j = \sum_{t=0}^{[i/p]} (-1)^{i+t} \binom{(p-1)(j-t)-1}{i-pt} P^{i+j-t} P^t \quad \text{if } i < pj,$$
and
\[
P^i \beta P^j = \sum_{t=0}^{\lfloor i/p \rfloor} (-1)^{i+t} \binom{(p-1)(j-t)}{i-pt} \beta P^{i+j-t} P^t
\]
\[
- \sum_{t=0}^{\lfloor (i-1)/p \rfloor} (-1)^{i+t-1} \binom{(p-1)(j-t)-1}{i-pt-1} \beta P^{i+j-t} P^t
\]
if \(i \leq pj\).

There are similar Adem relations for the prime 2. (Here \(\lfloor \frac{m}{n} \rfloor\) denotes the integer part of \(\frac{m}{n}\), that is, the largest integer not exceeding it.)

2.1.2. \(A\)-modules and \(A\)-algebras. A module over \(A\) is assumed to be graded by the integers. An \(A\)-algebra is a \(\mathbb{Z}/p\)-algebra \(M\) with an \(A\)-module structure such that the Steenrod operations satisfy the Cartan formula. That is, for any integer \(l \geq 0\) and elements \(m, m' \in M\), the equality
\[
P^l (mm') = \sum_{i+j=l} P^i(m)P^j(m')
\]
holds; there is a similar Cartan formula when \(p = 2\).

An unstable \(A\)-algebra is an \(A\)-algebra \(M\) which satisfies the unstable conditions:

(i) \(P^i(m) = mp\) if \(2i = |m|\), the degree of \(m\), and
(ii) \(\beta^\varepsilon P^i(m) = 0\) if \(2i + \varepsilon > |m|\) (\(\varepsilon = 0, 1\)).

There are analogous conditions when \(p = 2\). An even unstable \(A\)-algebra is an unstable \(A\)-algebra \(H^*\) that is concentrated in even dimensions, that is, \(H^{odd} = 0\). An unstable \(A\)-algebra is connected if its degree 0 part is \(\mathbb{Z}/p\) and \(H^{<0} = 0\).

Example 2.1. The mod \(p\) cohomology \(H^*(X; \mathbb{Z}/p)\) of a topological space \(X\) is an unstable \(A\)-algebra. If \(X\) is a spectrum, then its mod \(p\) cohomology is, in general, not an unstable \(A\)-algebra but merely an \(A\)-module.

2.2. Filtered objects.

2.2.1. Filtered group. A filtered abelian group is an abelian group \(M\) together with a deceasing filtration of subgroups
\[
M = M^0 \supset M^1 \supset \cdots
\]
A map of filtered abelian groups is required to preserve the filtrations; that is, the image of the \(n\)th filtration in the domain is a subset of the \(n\)th filtration in the target.

2.2.2. Filtered algebra. A filtered algebra is a commutative ring \(R\) with unit together with a multiplicative, decreasing filtration
\[
R = I^0 \supset I^1 \supset I^2 \supset \cdots
\]
of ideals, called filtration ideals. We usually speak of a filtered algebra \(R\) and leave the filtration implicit. A map of filtered algebras is required to preserve the filtrations.
A filtration ideal $I$ of a filtered algebra $R$ is said to be closed under dividing by $p$ if, for any element $r$ in $R$, $pr$ lies in $I$ implies that $r$ lies in $I$. This notion will be useful when we prove the examples in the Introduction and Theorem 1.6.

If $R$ is a filtered algebra, then the associated graded algebra is defined to be the (non-negatively) graded algebra

$$\text{Gr}^* R \equiv \bigoplus_{n=0}^{\infty} I^n/I^{n+1}.$$ 

A map of filtered algebras induces naturally a map between the associated graded algebras. If $r$ is an element in $R$, then the corresponding element in the associated graded algebra (or its tensor product with $\mathbb{Z}/p$) is denoted by $r$, and we say that $r$ is a lift of $\overline{r}$.

**Example 2.2.** One example of a filtered algebra is the complex $K$-theory of a topological space $X$ having the homotopy type of a CW complex. The filtration ideals of $K(X)$ are the kernels

$$I^n \equiv \ker(K(X) \xrightarrow{i^n} K(X_{n-1})),$$

where $i: X_{n-1} \hookrightarrow X$ is the inclusion of the $n$th skeleton.

2.3. **Pre-$\psi^p$-algebra.** The $K$-theory $K(X)$ of a space $X$ is not just a filtered algebra; it also has $K$-theory operations. Among these are the Adams operations $\psi^n$, $n \geq 1$, satisfying the conditions:

(i) $\psi^1 = \text{Id}$ and $\psi^n \psi^m = \psi^m \psi^n = \psi^{mn}$ for all $m$ and $n$, and

(ii) if $p$ is a prime, $\psi^p r \equiv r^p \pmod{pK(X)}$ for all elements $r$.

As discussed in §1, Atiyah’s Theorem 1.1 says that for a space whose integral cohomology is torsionfree and is concentrated in even dimensions, the Adams operation $\psi^p$ on its $K$-theory determines the Steenrod operations $P^i$ ($i \geq 0$) on its mod $p$ cohomology. Since we would like to model $K$-theory of spaces by algebraic objects which corresponds to unstable $A$-algebras, as $K$-theory corresponds to mod $p$ cohomology, Atiyah’s result suggests that we consider the following.

We already stated the definition of a pre-$\psi^p$-algebra in §1, but we will repeat it here.

**Definition 2.3.** Let $p$ be an arbitrary but fixed prime. Define a pre-$\psi^p$-algebra to be a filtered algebra $R$ that comes equipped with a distinguished endomorphism $\psi^p: R \to R$ such that the following two conditions hold:

(i) The $2n$th filtration of $R$ coincides with the $(2n - 1)$st filtration for all $n$.

(ii) Let $r \in R$ be in filtration $2q$ for some $q$. If $q > 0$, then there exist elements $r_i \in R$ ($0 \leq i \leq q$) in filtration $2q + 2i(p - 1)$ such that

$$\psi^p(r) = p^q r_0 + p^{q-1} r_1 + \cdots + p r_{q-1} + r_q \quad (2.1)$$
where \( r_q = r^p \). If \( q = 0 \), then there exists an element \( r' \in R \) in filtration 0 such that

\[
\psi^p(r) = pr' + r^p. \tag{2.2}
\]

Moreover, the elements \( r_i \) in (2.1) are required to be well-defined in the associated graded algebra mod \( p \) of \( R \).

Equation (2.1) (or (2.2), if \( q = 0 \)) is called an Atiyah formula for \( r \). Note that the elements \( r_i \) in an Atiyah formula for \( r \) are not unique.

**Remark 2.4.** We now pause to make a few remarks about this definition.

(i) An element \( r \in R \) can be considered to lie in different filtrations, since if \( r \) lies in filtration \( 2q \geq 2 \) then it also lies in filtration \( 2(q-1) \). Thus, condition (ii) above has to be interpreted to mean that for every element \( r \in R \), regardless of what filtration (say, \( 2q \)) it is considered to be in, \( \psi^p(r) \) can be decomposed into the form (2.1) or (2.2) with the \( r_i \) well-defined in the associated graded algebra mod \( p \).

(ii) If \( r \) lies in filtration \( 2q \) with \( q > 0 \), then an Atiyah formula for \( r \) also yields an Atiyah formula when \( r \) is considered to be in filtration \( 2(q-1) \), since we can rewrite (2.1) as

\[
\psi^p(r) = p^{q-1}(pr_0) + \cdots + p^2(pr_{q-3}) + p(pr_{q-2} + r_{q-1}) + r^p.
\]

(iii) If we think of an element \( r \in R \) to lie in filtration, say, \( 2q \), then \( \psi^p(r) \) is considered to lie in the same filtration. In particular, if \( q > 0 \) then

\[
\psi^p(r) = 0
\]

in \( \text{Gr}^{2q} R \otimes \mathbb{Z}/p \).

(iv) The requirement that the \( r_i \) be well-defined in the associated graded algebra mod \( p \) means that if \( \psi^p(r) \) admits another decomposition of the form (2.1), say,

\[
\psi^p(r) = \sum_{i=0}^{q} p^{q-i} r_i',
\]

then

\[
\overline{r_i} = \overline{r_i'}
\]

in \( \text{Gr}^{2q+2i(p-1)} R \otimes \mathbb{Z}/p \) for all \( i \).

The \( K \)-theory of a space together with its Adams operation \( \psi^p \) is clearly a pre-\( \psi^p \)-algebra, at least if the space has integral cohomology that is torsionfree and is concentrated in even dimensions.

If \( R \) and \( S \) are pre-\( \psi^p \)-algebras, then a pre-\( \psi^p \)-algebra map

\[
f : R \to S
\]

is a map of the underlying filtered algebras that is compatible with \( \psi^p \), in the sense that

\[
f \psi^p = \psi^p f.
\]
An ideal $I \subset R$ is said to be a $\psi^p$-ideal if it is closed under $\psi^p$. If $I \subset R$ is a $\psi^p$-ideal and that $R/I$ inherits from $R$ the structure of a pre-$\psi^p$-algebra, then the natural map

$$R \rightarrow R/I$$

is a pre-$\psi^p$-algebra map.

Now we make a few preliminary observations concerning Atiyah formula; they will be used below to show that certain filtered algebras with a distinguished endomorphism $\psi^p$ admit pre-$\psi^p$-algebra structures. The first preliminary result says that if two elements “admit Atiyah formulas,” then so does their sum.

**Lemma 2.5.** Let $p$ be a prime and let $R = (R, \{I^n\})$ be a filtered algebra with a distinguished endomorphism $\psi^p$ and in which the $2n$th filtration coincides with the $(2n - 1)$st filtration for all $n$. Suppose that $r$ and $s$ are elements in $R$ with $r \in I^q \setminus I^{q+2}$ and $s \in I^v \setminus I^{v+2}$ for some integers $q < v$. Then:

(i) If $q = 0$ and if $\psi^p(r)$ and $\psi^p(s)$ admit decompositions of the forms (2.2) and (2.1) (where the $s_i$ are not-necessarily well-defined) then $\psi^p(r+s)$ also admits a decomposition of the form (2.2).

(ii) If $q > 0$ and if both $\psi^p(r)$ and $\psi^p(s)$ admit decompositions of the form (2.1) (where the $r_i$ and $s_i$ are not-necessarily well-defined), then so does $\psi^p(r+s)$.

**Proof.** The proofs for the two statements are similar, with the first one being easier, so we will only present the argument for the second statement.

Let us write $t = r + s$ and note that $t$ lies in $I^q \setminus I^{q+2}$. By assumption we can write

$$\psi^p(r) = \sum_{i=0}^{q} p^{q-i} r_i$$

and

$$\psi^p(s) = \sum_{i=0}^{v} p^{v-i} s_i.$$

Define the following elements

$$t_i = \begin{cases} 
    r_0 + s' & \text{if } i = 0 \\
    r_i + s_i & \text{if } 1 \leq i \leq q - 2 \\
    r_{q-1} + s_{q-1} - c & \text{if } i = q - 1 \\
    (r+s)^q & \text{if } i = q.
\end{cases}$$

Here $s'$ and $c$ are given by

$$s' = \sum_{i=0}^{v-q} p^{v-q-i} s_i$$

and

$$c = \sum_{i=1}^{p-1} \frac{1}{p^i} \binom{p}{i} p^{p-c} s'.$$
Note that $c$ satisfies the equation

$$(r + s)^p - pc = r^p + s^p.$$ 

Then we have that

$$\psi^p(t) = \sum_{i=0}^{q} p^{q-i}r_i + \sum_{i=0}^{v} p^{v-i}s_i$$

$$= p^q(r_0 + s') + \sum_{i=1}^{q-2} p^{q-i}(r_i + s_i)$$

$$+ p(r_{q-1} + s_{q-1} - c) + (r + s)^p$$

$$= \sum_{i=0}^{q} p^{q-i}t_i.$$

The Lemma now follows. \(\square\)

The following lemma gives a sufficient condition in order that a filtered algebra with a distinguished endomorphism $\psi^p$ be a pre-$\psi^p$-algebra. Recall that a filtration ideal $I$ in a filtered algebra $R$ is closed under dividing by $p$ if, for any element $r$ in $R$, $pr \in I$ implies that $r \in I$.

**Lemma 2.6.** Let $p$ be a prime and let $R$ be a filtered algebra with a distinguished endomorphism $\psi^p$ in which every filtration ideal is closed under dividing by $p$. Assume that for some element $r \in R$ in filtration $2q$ with $q > 0$, $\psi^p(r)$ admits two decompositions of the form (2.1):

$$\psi^p(r) = \sum_{i=0}^{q} p^{q-i}r_i = \sum_{i=0}^{q} p^{q-i}r'_i. \quad (2.3)$$

Then

$$\overline{r_i} = \overline{r'_i}$$

in $\text{Gr}^{2q+2i(p-1)} R \otimes \mathbb{Z}/p$ for $0 \leq i \leq q$.

**Proof.** Let $I^n$ denote the $n$th filtration ideal of $R$. It follows from (2.3) that we have, modulo filtration $2q + 2(p-1)$, the equality

$$p^q(r'_0 - r_0) = 0.$$ 

Since $I^{2q+2(p-1)}$ is closed under dividing by $p$, the above equality implies that

$$r'_0 - r_0 = s_0$$

for some element $s_0 \in I^{2q+2(p-1)}$. In particular, the Lemma is true in the case $i = 0$.

We will now show by induction that for $1 \leq k \leq q - 1$, there exist elements $s_k$ in filtration $2q + 2(k+1)(p-1)$ such that

$$r'_k - r_k = s_k - ps_{k-1}, \quad (2.4)$$

in which $s_0$ is as in the previous paragraph. For the case $k = 1$, (2.3) implies that, modulo filtration $2q + 2(2)(p-1)$, the equality

$$p^q r_0 + p^{q-1}r_1 = p^q r'_0 + p^{q-1}r'_1$$
holds, and so by rearranging terms we see that
\[ 0 = p^{q-1}(ps_0 + r'_1 - r_1). \]
Since \( I^{2q + 2(2)(p-1)} \) is closed under dividing by \( p \), it follows that
\[ ps_0 + r'_1 - r_1 = s_1 \]
for some element \( s_1 \in I^{2q + 2(2)(p-1)}. \)

Suppose that (2.4) has been proved for all \( k, 1 \leq k \leq n \), for some \( n < q - 1 \). Then, working modulo filtration \( 2q + 2(n + 2)(p-1) \), (2.3) implies that
\[
0 = \sum_{i=0}^{n+1} p^q i (r'_i - r_i) = \sum_{i=0}^{n} p^q i (s_i - ps_{i-1}) + p^q i (r'_{n+1} - r_{n+1}) = p^q s_n + p^q i (r'_{n+1} - r_{n+1}) = p^q - n (ps_n + r'_{n+1} - r_{n+1}).
\]
Thus, since \( I^{2q + 2(n + 2)(p-1)} \) is closed under dividing by \( p \), the case \( k = n + 1 \) follows. Therefore, (2.4) holds for all \( k, 1 \leq k \leq q - 1. \)

In particular, since the case \( i = q \) is trivially true (because \( r_q = r^p = r'_q \)), we see that
\[ r_i = s_i \]
for \( 0 \leq i \leq q \), thereby proving the Lemma. \( \square \)

3. Proof of Theorem 1.2, Examples 1.3 and 1.4, and Corollary 1.5

We begin with the proof of Theorem 1.2.

3.1. Proof of Theorem 1.2. First we prove that the operations \( P^i \), as defined in eq. (1.7), are well-defined. We will use the notation in Theorem 1.2, so \( \overline{\tau} \) is an element in \( Gr^{2q} R \otimes \mathbb{Z}/p \). The cases \( q = 0 \) and \( q > 0 \) need to be considered separately. However, the arguments are very similar, so we will only present the argument for the more complicated case \( q > 0 \).

For this case, it suffices to prove the following statement:

Let \( s \) be another lift of \( \overline{\tau} \) back to \( R \), so that
\[ s = r + ph + f \]
for some elements \( h \) and \( f \) in filtrations, respectively, \( 2q \) and \( 2q + 2n \) for some \( n \geq 1 \). Then there exists an Atiyah formula
\[ \psi^p(s) = p^q s_0 + p^q s_1 + \cdots + ps_{q-1} + s_q \]
for \( s \) (in filtration \( 2q \)) such that
\[ \overline{\tau}_i = \overline{s}_i \]
in \( Gr^{2q + 2i(p-1)} R \otimes \mathbb{Z}/p \) for \( i = 0, \ldots, q. \)
To begin proving this statement, let us first write down Atiyah formulas for the elements $h$ and $f$,

\[
\psi^p(h) = p^q h_0 + p^{q-1} h_1 + \cdots + ph_{q-1} + h^p
\]

\[
\psi^p(f) = p^{q+n} f_0 + p^{q+n-1} f_1 + \cdots + pf_{q+n-1} + f^p,
\]

with $h_i$ in filtration $2(q + i(p-1))$ and $f_i$ in filtration $2(q + n + i(p-1))$. Define an element $\gamma$ in $R$ by the equation

\[
r^p + ph^p + f^p = s^p + pr^p = (r + ph + f)^p + pr^p,
\]

and define elements $s_i$ as follows:

\[
s_i = \begin{cases} 
  r_i + ph_i + p^nf_i & \text{if } 0 \leq i \leq q - 2 \\
  r_{q-1} + ph_{q-1} + \gamma + \sum_{j=q-1}^{q+n-1} p^{q+n-j-1}f_j & \text{if } i = q - 1 \\
  s^p & \text{if } i = q.
\end{cases}
\]

Now we calculate

\[
\psi^p(s) = \psi^p(r) + p\psi^p(h) + \psi^p(f)
\]

\[
= \left( \sum_{i=0}^{q-1} p^{q-i} r_i \right) + p \left( \sum_{i=0}^{q-1} p^{q-i} h_i \right)
\]

\[
+ \left( \sum_{i=0}^{q+n-1} p^{q+n-i} f_i \right) + r^p + ph^p + f^p
\]

\[
= \left( \sum_{i=0}^{q-1} p^{q-i} r_i \right) + p \left( \sum_{i=0}^{q-1} p^{q-i} h_i \right) + \left( \sum_{i=0}^{q+n-1} p^{q+n-i} f_i \right) + s^p + pr^p
\]

\[
= \sum_{i=0}^{q-2} p^{q-i} (r_i + ph_i + p^nf_i)
\]

\[
+ p \left( r_{q-1} + ph_{q-1} + \gamma + \sum_{j=q-1}^{q+n-1} p^{q+n-j-1}f_j \right) + s^p
\]

\[
= \sum_{i=0}^{q} p^{q-i} s_i.
\]

It is not hard to see that the elements $s_i$ satisfy the required properties. For instance,

\[
\overline{s}_{q-1} = \overline{s}_{q-1}
\]

because $\gamma$ lies in filtration $2pq$, $f_j$ (for $j \geq q - 1$) lies in filtration $2(q + n + (q - 1)(p - 1))$, and $ph_{q-1}$ is $p$-divisible.

Thus, the functions $P^i$ as defined in eq. (1.7) are well-defined.

Now we consider the four properties of the $P^i$ listed in Theorem 1.2. The first three of them are immediate consequences of the definition (eq. (1.7)) of the functions $P^i$.

As for the last statement, let $\overline{s}$ and $\overline{r}$ be elements of $\text{Gr}^* R \otimes \mathbb{Z}/p$ in degrees $2m$ and $2n$, respectively. Without loss of generality we may assume that $m \leq n$. We will denote by $r$ and $s$ arbitrary but fixed lifts of $\overline{s}$ and $\overline{r}$,
respectively, to $R$ in filtrations precisely $2m$ and $2n$. Note that $rs$ is a lift of $rs$.

The case when both $m$ and $n$ are equal to 0 is immediate. So let us now consider the case when $m = 0$ and $n > 0$. We write down Atiyah formulas for $r$ and $s$:

$$
\psi^p(r) = r^p + pr'
$$
$$
\psi^p(s) = p^n s_0 + \cdots + ps_{n-1} + s^p.
$$
Here $r'$ is some element in $R$ in filtration 0. Using the fact that $\psi^p$ is multiplicative, we have that

$$
\psi^p(rs) = (r^p + pr')\sum_{i=0}^n p^{n-i}s_i \quad (s_n \equiv s^p)
$$

$$
= \left(\sum_{i=0}^{n-1} p^{n-i}s_i(r^p + pr')\right) + s^p(r^p + pr')
$$

Since

$$
P_i^p = \begin{cases} 
\mathfrak{p}^p & \text{if } i = 0 \\
0 & \text{if } i > 0,
\end{cases}
$$

the last statement of the theorem when $m = 0$ and $n > 0$ follows.

Finally, we consider the case when both $m$ and $n$ are positive. The Atiyah formula for $s$ is as above, but that for $r$ looks like

$$
\psi^p(r) = p^m r_0 + \cdots + pr_{m-1} + r^p.
$$

Therefore, we have that

$$
\psi^p(rs) = \sum_{i=0}^{m+n} p^{m+n-i}c_i,
$$
where

$$
c_i = \sum_{l+k=i} r_l s_k.
$$
The case when $m, n > 0$ follows. This finishes the proof of statement (iv) in the theorem.

The proof of Theorem 1.2 is complete.

**Remark 3.1.** The proof of statement (iv) in Theorem 1.2 above also shows that if two elements “admit Atiyah formulas,” then so does their product. This is the multiplicative analogue of Lemma 2.5.

3.2. Proof of Examples 1.3 and 1.4. Now we prove the two examples in the Introduction.

**Proof of Example 1.3.** Fix a prime $p$ and let $R$ be the filtered ring $\mathbb{Z}[\varepsilon]$ ($\varepsilon^2 = 0$) of dual numbers with the $\varepsilon$-adic filtration, where $\varepsilon$ lies in filtration precisely 4. Let $k$ be any integer and define the endomorphism $\psi^p_k$ on $R$ by

$$
\psi^p_k(\varepsilon) = p^2 k \varepsilon.
$$

(3.1)
Using Lemmas 2.5 and 2.6 it is readily checked that \( R \) is a pre-\( \psi_p \)-algebra (Definition 2.3) with \( \varepsilon_0 = k \varepsilon \). Thus, in \( \text{Gr}^4 R \otimes \mathbb{Z}/p \) we have
\[
P^0 \varpi = k \varpi,
\]
which is equal to \( \varpi \) if and only if
\[
k \equiv 1 \pmod{p}.
\]
In other words, this congruence condition is equivalent to the condition that \( P^0 = \text{Id} \). In particular, if \( k \not\equiv 1 \pmod{p} \), then \( P^0 \neq \text{Id} \).

It is worth pointing out that the Adem relation is satisfied in \( \text{Gr}^* R \otimes \mathbb{Z}/p \), since only \( P^0 \) can be non-zero. \( \square \)

Proof of Example 1.4. Fix a prime \( p > 2 \) and let \( \mathbb{Z}(p) \) denote the ring of integers localized at \( p \). Let \( R \) be the filtered polynomial ring \( \mathbb{Z}(p)[x] \) with the \( x \)-adic filtration, where \( x \) lies in filtration precisely \( 2(p - 1) \). Define the endomorphism \( \psi_p \) on \( R \) by the equation
\[
\psi_p(x) = -p^{p-2}x^2 + \sum_{i=1}^{p} p^{p-i}x^i
\]
\[
= p^{p-1}x + p^{p-3}x^3 + \cdots + px^{p-1} + x^p.
\]
It follows from Lemmas 2.5 and 2.6 and Remark 3.1 that \( R \) is a pre-\( \psi_p \)-algebra.

Now the operation \( P^i \) (\( 0 \leq i \leq p-1 \)) takes \( \varpi \in \text{Gr}^{2(p-1)} R \otimes \mathbb{Z}/p \) to
\[
P^i \varpi = \begin{cases} 
\varpi^{i+1} & \text{if } i \neq 1 \\
0 & \text{if } i = 1.
\end{cases}
\]
In particular, we have that
\[
P^1 P^1 \varpi = P^1 0 = 0,
\]
which is not equal to
\[
2P^2 \varpi = 2\varpi^3,
\]
since \( p > 2 \). In other words, \( P^1 P^1 \neq 2P^2 \).

In summary, \( R \) is a pre-\( \psi_p \)-algebra for which the operations \( P^i \) on \( \text{Gr}^* R \otimes \mathbb{Z}/p \) do not satisfy the Adem relation \( P^1 P^1 = 2P^2 \).

This finishes the proof of Example 1.4. \( \square \)

3.3. Proof of Corollary 1.5. In view of Examples 1.3 and 1.4, to make the associated graded algebra mod \( p \) of a pre-\( \psi_p \)-algebra into an unstable \( A \)-algebra, we should add the assumptions that the operations \( P^i \) satisfy \( P^0 = \text{Id} \) and the Adem relations. Thus, we make the following definition.

Definition 3.2. A \( \psi_p \)-algebra is a pre-\( \psi_p \)-algebra \( R \) for which the operations \( P^i \) on the associated graded algebra mod \( p \) (as in Theorem 1.2) satisfy the Adem relations and the property \( P^0 = \text{Id} \). A \( \psi_p \)-algebra map is a map of the underlying pre-\( \psi_p \)-algebras. A \( \psi_p \)-algebra \( R \) is connected if there is an isomorphism
\[
\text{Gr}^0 R \otimes \mathbb{Z}/p \cong \mathbb{Z}/p.
\]
Remark 3.3. We pause to make a few comments about this definition.

(i) It follows from Theorem 1.2 that the associated graded algebra mod $p$, \( \text{Gr}^* R \otimes \mathbb{Z}/p \), of a $\psi^p$-algebra $R$ is an unstable $A$-algebra in which the Steenrod operations are given by (1.7).

(ii) Atiyah’s Theorem 1.1 implies that the $K$-theory $K(X)$ with its Adams operation $\psi^p$ is a $\psi^p$-algebra, provided that $X$ is a space whose integral cohomology is concentrated in even dimensions and is torsionfree.

(iii) Examples 1.3 and 1.4 show that there are pre-$\psi^p$-algebras that are not $\psi^p$-algebras.

(iv) The condition that the operations $P^i$ satisfy the Adem relations can be rephrased in terms of elements appearing in Atiyah formulas, as follows: For each element $r \in R$ in filtration, say, $2q$ with $q > 0$, there exist elements $r_i$ ($0 \leq i \leq q$) and $r_{i,j}$ ($0 \leq j \leq q + i(p-1)$) in $R$ in filtrations, respectively, $2q + 2i(p-1)$ and $2q + 2(i+j)(p-1)$ such that

\[
\psi^p(r) = \sum_{i=0}^{q} p^{q-i} r_i
\]

\[
\psi^p(r_i) = \sum_{j=0}^{q+(p-1)} p^{q+i(p-1)-j} r_{i,j}
\]  

(3.2)

These elements are required to satisfy the following condition. Whenever $i,j > 0$ and $i < pj$, the following equality holds in $\text{Gr}^{2q+(i+j)(p-1)} R \otimes \mathbb{Z}/p$:

\[
\tau_{j,i} = \sum_{t=0}^{\lfloor i/p \rfloor} (-1)^t \cdot \left( (p-1)(j-t) - 1 \right) \cdot \tau_{t,i+j-t} \quad \text{if} \quad p > 2
\]

\[
\tau_{j,i} = \sum_{t=0}^{\lfloor i/2 \rfloor} \left( 2j - 2t - 1 \right) \cdot \tau_{2i-4t, i+j-t} \quad \text{if} \quad p = 2.
\]  

(3.3)

(v) If $f: R \to S$ is a pre-$\psi^p$-algebra map, then the induced map on the associated graded algebras mod $p$,

\[ f_*: \text{Gr}^* R \otimes \mathbb{Z}/p \to \text{Gr}^* S \otimes \mathbb{Z}/p, \]

commutes with the operations $P^i$. In particular, this applies when $f$ is a map of $\psi^p$-algebras.

Proof of Corollary 1.5. This is an immediate consequence of Theorem 1.2, Definition 3.2, and Remark 3.3 (v). \qed

4. Proof of Theorem 1.6

Before giving the proof, we first state the definition of a lift of an even and connected unstable $A$-algebra.
**Definition 4.1.** Let $p$ be a prime and let $H^*$ be an even and connected unstable $A$-algebra. Define a lift of $H^*$ to be a connected $\psi^p$-algebra (see Definition 3.2) $R$ for which there exists an isomorphism

$$H^* \cong \text{Gr}^* R \otimes \mathbb{Z}/p$$

of unstable $A$-algebras.

We are now ready for the proof of Theorem 1.6.

4.1. **Construction of $B_{H^*}$.**

4.1.1. **Choosing generators and relations for $H^*$.** We begin the proof with the construction of the lift $B_{H^*}$ of $H^*$.

Let us choose a set of $A$-algebra generators $\{x_\theta\}_{\theta \in \Theta}$ and relations $\{f_\delta\}_{\delta \in \Delta}$ (all in positive, even dimensions) for the connected, even unstable $A$-algebra $H^*$. Denote the degree $|x_\theta|$ of $x_\theta$ by $2d_\theta$ for some positive integer $d_\theta$. Thus, every element in $H^*$ can be written as a polynomial, with coefficients in $\mathbb{Z}/p$, in elements of the form

$$P^1 x_\theta = P^{i_n} \cdots P^{i_1} x_\theta.$$  \hfill (4.1)

Here $n \geq 0$, $\theta \in \Theta$, and, for $0 \leq k \leq n - 1$, $i_{k+1}$ is in the range

$$0 \leq i_{k+1} \leq d_\theta + (p - 1) \sum_{j=1}^{k} i_j.$$  \hfill (4.2)

The relations among the $x_\theta$ and the Adem relations (when applied to these elements) are contained in the set $\{f_\delta\}$. Denoting the ideal in $H^*$ generated by the $f_\delta$ by $\langle f_\delta : \delta \in \Delta \rangle$, we have the identification

$$H^* = \mathbb{Z}/p \left[ \{P^{i_n} \cdots P^{i_1} x_\theta : \theta \in \Theta, n \geq 0\} \right] / \langle f_\delta : \delta \in \Delta \rangle$$  \hfill (4.3)

in which the $i_{k+1}$ are in the range (4.2).

4.1.2. **Construction of $B_{H^*}$.** The desired lift $B_{H^*}$ is going to be a quotient of the filtered polynomial algebra

$$\Pi \overset{\text{def}}{=} \mathbb{Z} \left[ \{X_{(\theta, i_1, \ldots, i_n)}\} \right].$$

Here $n \geq 0$, $\theta \in \Theta$, and the $i_k$ are in the range (4.2). The indeterminate $X_{(\theta, i_1, \ldots, i_n)}$ is in filtration exactly

$$2d_\theta + 2(p - 1) \sum_{j=1}^{n} i_j,$$

which is equal to the degree of the element in (4.1). Define an endomorphism $\psi^p$ on $\Pi$ as follows: Given a multi-index $(\theta, i) = (\theta, i_1, \ldots, i_n)$, let

$$\sigma = d_\theta + (p - 1) \sum_{j=1}^{n} i_j$$

for $\Pi$. Then define $\psi^p : \Pi \to \Pi$ by

$$\psi^p (X_{(\theta, i_1, \ldots, i_n)}) = X_{(\theta, i_1, \ldots, i_n)}^p.$$
and define
\[ \psi^p X(\theta,i) = p^\sigma X(\theta,i) + \left( \sum_{i=1}^{p-1} p^{\sigma-i} X(\theta,i,i) \right) + X^p(\theta,i,i). \] (4.4)

As in Examples 1.3 and 1.4, using Lemmas 2.5 and 2.6 and Remark 3.1, it is readily checked that Π together with the endomorphism \( \psi^p \) is a pre-\( \psi^p \)-algebra.

Now for each \( \delta \in \Delta \) let \( f_\delta \) also denote the element in Π obtained canonically from the \( f_\delta \) in \( H^* \) by replacing each nonzero coefficient \( a \) (mod \( p \)) by the unique positive integer \( a \) not exceeding \( p - 1 \) and the element in (4.1) by \( X_{(\theta,i_1,\ldots,i_n)} \). The \( k \)-fold composite, for any non-negative integer \( k \), of \( \psi^p \) is denoted \( \psi^{pk} \), where \( \psi^0 \) is defined to be the identity map. Let \( J \subset \Pi \) be the ideal generated by the set of elements
\[ \{ \psi^{pk} f_\delta : k \geq 0, \delta \in \Delta \}. \]

Then the proposed lift \( B_{H^*} \) is defined to be the quotient
\[ B = B_{H^*} = \Pi / J. \] (4.5)

We now make a few observations about \( B \).

(i) An argument similar to the proof of Lemma 2.6 shows that \( B \) inherits from \( \Pi \) the structure of a pre-\( \psi^p \)-algebra.

(ii) For any \( \delta \in \Delta \) and \( k \geq 1 \), the image \( \psi^{pk} f_\delta \) of the element \( \psi^{pk} f_\delta \) in \( \text{Gr}^* B \otimes \mathbb{Z}/p \) is 0. This is because
\[ \psi^{pk} f_\delta = p\alpha + \alpha' \]
for some elements \( \alpha, \alpha' \in B \) with \( \alpha \) in some positive filtration and \( \alpha' \) in a filtration strictly greater than that of \( \alpha \).

(iii) The unstable \( A \)-algebra generators and relations for \( H^* \) were the only choices we made in the process of defining the pre-\( \psi^p \)-algebra \( B \). Thus, the construction of the proposed lift \( B \) is canonical up to a choice of generators and relations for \( H^* \).

4.1.3. \( B_{H^*} \) is a lift of \( H^* \). Now we show that the pre-\( \psi^p \)-algebra \( B = B_{H^*} \) constructed above is in fact a lift of \( H^* \).

First of all, the associated graded algebra mod \( p \) of \( \Pi \) is the polynomial algebra
\[ \text{Gr}^* \Pi \otimes \mathbb{Z}/p = \mathbb{Z}/p \left[ \{ X_{(\theta,i_1,\ldots,i_n)} \} \right] \]
in which \( n \geq 0 \), \( \theta \in \Theta \), and the \( i_k \) are in the range (4.2). The generator \( X_{(\theta,i_1,\ldots,i_n)} \) has degree \( 2d_\theta + 2(p - 1) \sum_{j=1}^{n} i_j \). (We omit the bars above these elements for typographical reasons.) In view of observation (ii) above, it follows that the associated graded algebra mod \( p \) of \( B \) is the quotient
\[ \text{Gr}^* B \otimes \mathbb{Z}/p = \mathbb{Z}/p \left[ \{ X_{(\theta,i_1,\ldots,i_n)} \} \right] / \langle f_\delta : \delta \in \Delta \rangle. \] (4.6)
Its operations $P^i$ are given on the generators by the equation

$$P^i X(\theta, i_1, \ldots, i_n) = \begin{cases} X(\theta, j_1, \ldots, j_n) & \text{if } i = 0 \\ X(\theta, j_1, \ldots, j_n, i) & \text{if } 1 \leq i \leq d_\theta - 1 + (p - 1) \sum_{j=1}^n i_j \\ X^p(\theta, i_1, \ldots, i_n) & \text{if } i = d_\theta + (p - 1) \sum_{j=1}^n i_j \\ 0 & \text{otherwise.} \end{cases}$$

Combining this description of $\text{Gr}^* B \otimes \mathbb{Z}/p$ with the identification (4.3) of $H^*$, we see that the map

$$\rho H^*: \text{Gr}^* B \otimes \mathbb{Z}/p \rightarrow H^*$$

defined on the generators by

$$\rho H^* X(\theta, i_1, \ldots, i_n) = P^{i_n} \cdots P^{i_1} x_\theta$$

is a graded $\mathbb{Z}/p$-algebra isomorphism that commutes with the operations $P^i$. Now since the Adem relations and the identity $P^0 = \text{Id}$ hold in $H^*$, it follows that these two properties also hold in $\text{Gr}^* B \otimes \mathbb{Z}/p$. In particular, $B$ is, in fact, a $\psi^p$-algebra and is a lift of $H^*$ via the canonical map $\rho H^*$.

Furthermore, the second assertion in Theorem 1.6 is a consequence of the constructions of $B_H^*$ and $\rho H^*$ (see, in particular, (4.4) – (4.7)).

The proof of Theorem 1.6 is complete.

5. Kuhn’s Realization Conjecture

Our last result is a $K$-theoretic analogue of a conjecture of N. Kuhn. This is our first attempt to tackle the problem of realizing $\psi^p$-algebras as $K$-theory of spaces. Here we consider the simpler notion of a $\psi^p$-module.

In Kuhn made an interesting conjecture, the Realization Conjecture, about the size of the mod $p$ cohomology of topological spaces: The mod $p$ cohomology of a space should be either finite as a set or infinitely generated as a module over the mod $p$ Steenrod algebra. To rephrase it, the conjecture predicts that a finitely generated $A$-module that is not finitely generated as a $\mathbb{Z}/p$-module cannot be realized as the mod $p$ cohomology of a space. Kuhn verified this conjecture in the case when the Bockstein is zero in sufficiently high degrees [Theorem 0.1]. Using reduction steps in Kuhn’s paper [7], the Realization Conjecture was proved recently by L. Schwartz [5].

One naturally wonders if there are analogous results concerning the size of spaces in other cohomology theories. Using Atiyah’s Theorem 1.1 and Kuhn’s original result, we establish such an analogue for $K$-theory. To generalize the result of Kuhn and Schwartz, we first introduce a $K$-theoretic notion which corresponds to a module over the Steenrod algebra.

**Definition 5.1.** Let $p$ be a prime. We define a $\psi^p$-module to be an ordered pair $((M, \{I_n\}), \psi^p)$ (or simply $(M, \psi^p)$ or even just $M$) consisting of a filtered abelian group $(M, \{I_n\})$ (see §2.2.1) and a distinguished endomorphism $\psi^p$, which is required to satisfy the following condition: If $\alpha \in M$ lies
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in filtration, say, \(2q\), then there exist elements \(\alpha_i (i = 0, \ldots, q)\) in filtration \(2q + 2i(p - 1)\) such that

\[
\psi^p(\alpha) = \sum_{i=0}^{q} p^{q-i} \alpha_i. \tag{5.1}
\]

An expression as in (5.1) is referred to as an *Atiyah formula* for \(\alpha\). For example, the \(K\)-theory \(K(X)\) with its Adams operation \(\psi^p\) is a \(\psi^p\)-module, provided that \(X\) is a space whose integral cohomology is concentrated in even dimensions and is torsionfree.

Now we can ask what a \(K\)-theoretic analogue of a *finitely generated* \(A\)-module is. The \(A\)-linear multiples of an element in an \(A\)-module are the finite sums of iterated Steenrod operations acting on that element. Since Atiyah’s result above tells us that the Steenrod operations on \(H^*(X; \mathbb{Z}/p)\) come from Atiyah formula decomposition (eq. (1.2)) of \(\psi^p\), a \(K\)-theoretic analogue of \(A\)-linear multiples should involve iterated applications of \(\psi^p\) on Atiyah formula. We arrive at the following \(K\)-theoretic finiteness condition, which corresponds to \(H^*(X; \mathbb{Z}/p)\) being a finitely generated \(A\)-module.

**Definition 5.2.** Let \((M, \psi^p)\) be a \(\psi^p\)-module and let \(m_1, \ldots, m_n\) be elements in \(M\). We say that \(M\) is \(\psi^p\)-finitely generated by \(m_1, \ldots, m_n\) if the following condition is true: There exist Atiyah formulas

\[
\begin{align*}
\psi^p m_* &= \sum_{j_1=0}^{q_*} p^{q_*-j_1} m_{(\ast,j_1)} \quad (1 \leq \ast \leq n) \\
\psi^p m_{(1,j_1)} &= \sum_{j_2=0}^{q_1+j_1(p-1)} p^{q_1+j_1(p-1)-j_2} m_{(1,j_1,j_2)} \quad (0 \leq j_1 \leq q_1) \\
\vdots \\
\psi^p m_{(n,j_1)} &= \sum_{j_2=0}^{q_n+j_1(p-1)} p^{q_n+j_1(p-1)-j_2} m_{(n,j_1,j_2)} \quad (0 \leq j_1 \leq q_n)
\end{align*}
\]

(5.2)

etc., etc. such that \(M\) is generated as an abelian group by the elements \(m_{(i,j_1,\ldots,j_r)}\) \((1 \leq i \leq n, r \geq 0)\). The \(\psi^p\)-module \(M\) is said to be \(\psi^p\)-finitely generated if there exists a finite set of elements \(m_1, \ldots, m_n\) in \(M\) with the above property.

Having a \(K\)-theoretic analogue of a finitely generated \(A\)-module, we are now ready for the promised \(K\)-theoretic analogue of Kuhn’s Realization Conjecture, or Kuhn-Schwartz Theorem. As for the original Kuhn-Schwartz Theorem, we think of the following result as an assertion about non-realization (by “nice” spaces) of \(\psi^p\)-modules that are \(\psi^p\)-finitely generated but whose underlying abelian groups are not finitely generated.
Proposition 5.3. Let $X$ be a torsionfree space of finite type whose integral cohomology is concentrated in even dimensions. If there exists a prime $p$ for which the $\psi^p$-module $(K(X), \psi^p)$ is $\psi^p$-finitely generated, then the underlying abelian group of $K(X)$ must be finitely generated.

Proof. We begin with three reductions.

Reduction step 1. To show that $K(X)$ is a finitely generated abelian group, it suffices to show that its associated graded group

$$\text{Gr}^* K(X) = H^*(X; \mathbb{Z})$$

is such. To see this, first note that $K(X)$ with the topology induced by the filtration ideals

$$I^n = \ker(K(X) \to K(X_{n-1})),$$

where $X_{n-1}$ denotes the $n$th skeleton of $X$, is Hausdorff; that is, the intersection $\cap_n I^n$ is 0. Indeed, an element $\alpha$ in $\cap_n I^n$ is represented by a map

$$\alpha: X \to BU$$

whose restriction to each skeleton $X_{n-1}$ is nullhomotopic; that is, $\alpha$ is a phantom map from $X$ to $BU$. But since $H^n(X; \mathbb{Q})$ and $\pi_{n+1} BU \otimes \mathbb{Q}$ cannot be simultaneously nonzero for any integer $n$, there can be no non-nullhomotopic phantom maps from $X$ to $BU$ (see [6] for more information about phantom maps). Therefore, $\alpha$ must be 0 and so

$$\bigcap_n I^n = 0,$$

that is, $K(X)$ is Hausdorff.

Now if $\text{Gr}^* K(X) = H^*(X; \mathbb{Z})$ is a finitely generated abelian group, then there exists an integer $N > 0$ such that

$$H^*(X; \mathbb{Z}) = H^{< N}(X; \mathbb{Z}) \quad \text{and} \quad H^{\geq N}(X; \mathbb{Z}) = 0.$$

It follows that $K(X)$ admits the finite filtration

$$K(X) = I^0 \supset I^1 \supset \cdots \supset I^{N-1} \supset I^N = \cdots = \cap_j I^j = 0. \quad (5.3)$$

Moreover, in this filtration of $K(X)$ both $I^{N-1} = H^{N-1}(X; \mathbb{Z})$ and each successive quotient are finitely generated abelian groups. Thus an easy (reverse) induction argument implies that $K(X)$ itself is a finitely generated abelian group.

Reduction step 2. To show that the associated graded $\text{Gr}^* K(X) = H^*(X; \mathbb{Z})$ is a finitely generated abelian group, it suffices to show that the mod $p$ associated graded group

$$\text{Gr}^* K(X) \otimes \mathbb{Z}/p = H^*(X; \mathbb{Z}/p)$$

is a finite dimensional $\mathbb{Z}/p$-vector space. This is because of the torsionfree hypothesis on $H^*(X; \mathbb{Z})$.

Reduction step 3. By Kuhn’s theorem discussed above, to show that $H^*(X; \mathbb{Z}/p)$ is a finite dimensional $\mathbb{Z}/p$-vector space it suffices to show that it is a finitely generated $A$-module.
Now suppose that $K(X)$ is $\psi^p$-finitely generated by the elements $m_1, \ldots, m_n$. The image of an element in the mod $p$ associated graded is in general given by the same name with a bar above it. We claim that the elements $\overline{m}_1, \ldots, \overline{m}_n$ generate $H^*(X; \mathbb{Z}/p)$ as an $A$-module. What this means is that every element in $H^*(X; \mathbb{Z}/p)$ can be written as a finite sum of elements of the form

$$P^{j_r} \cdots P^{j_1} \overline{m}_i \quad (P^j = \text{Sq}^j \text{ if } p = 2) \quad (5.4)$$

which is equal to $\overline{m}_{(i,j_1,\ldots,j_r)}$. This is, of course, implied by the hypothesis that $K(X)$ is $\psi^p$-finitely generated by $m_1, \ldots, m_n$.

This finishes the proof of the proposition. \hfill \Box

Remark 5.4. As in the case of modules over the Steenrod algebra, purely algebraic counterexamples are easily constructed. For example, let $p$ be an arbitrary prime, and consider the abelian group

$$A = \bigoplus_{n=0}^{\infty} \mathbb{Z} \langle x^{p^n} \rangle$$

with $x^{p^n}$ in filtration $2p^n$ and the endomorphism

$$\psi^p(x^{p^n}) = x^{p^{n+1}}.$$ 

Using a slight modification of Lemma 2.3, it is easy to check that $A$ is a $\psi^p$-module and is $\psi^p$-finitely generated by $\{x\}$, and yet it is not finitely generated as an abelian group. Thus, Proposition 5.3 tells us that many algebraically allowed $\psi^p$-modules cannot be realized as the $K$-theory of spaces.

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