Light propagation through optical media using metric contact geometry

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In this work, we show that the orthogonality between rays and fronts of light propagation in a medium is expressed in terms of a suitable metric contact structure of the optical medium without boundaries. Moreover, we show that considering interfaces (modeled as boundaries) orthogonality is no longer fulfilled, leading to optical aberrations and in some cases total internal reflection. We present some illustrative examples of this latter point.

I. INTRODUCTION

Geometric optics has acquired a renewed interest in theoretical and applied physics. The inception of geometric techniques and methods to control the path of light inside a medium provided us with the foundations of a new material science. Arising in the context of gravitational lensing, the geometric studies regarding Fermat’s and Huygens’ principles brought a new perspective in the understanding and modelling of wave phenomena in non-trivial media.

Soon after the advent of general relativity, it was observed that the gravitational field bends the path followed by a light ray. This was confirmed by Eddington after noticing that the apparent position of the stars are shifted from their expected position in the sky when observed during a solar eclipse. This bending phenomenon is analogous to the deviation of light rays while traversing a medium whose refractive index changes from one place to the other. In this sense, it was shown that a class of optical media can be modeled by means of a metric tensor encoding its electromagnetic properties. Furthermore, this analogy has evolved into the very active field of transformation optics, where the techniques and tools of differential geometry have been insightful in gravitation physics – have found its way into the more applied area of material science. In addition, this has also been used in modelling analogue gravitational spacetimes such as black holes and cosmological solutions, or to approximate curved spaces by N-dimensional simplices in order to describe light propagation on them.

Fermat’s principle poses a well known problem in the calculus of variations. In the non-relativistic setting – where the notion of time is absolute and universal – it states that light travels between two given spatial locations following a path minimising a time functional. Such a perspective is clearly untenable in the context of General Relativity, where it is commonly replaced by the assumption that the path taken by light in traveling from one location to another corresponds to a null geodesic. However, albeit it remains a variational problem, its precise formulation is far more elaborate (see Theorem 7.3.1 in).

Similarly (cf. Theorem 7.1.2 in), Huygens’ principle is centered in the idea of light emissions being simultaneous for different observers at each moment in time. In this way, light emission is described by well localised wavefronts, as they travel through three dimensional space. In the relativistic setting, the electromagnetic field at a precise event in spacetime depends on initial conditions originated its past null cone. This has led to explore different wave phenomena where Huygens’ principle is not fulfilled, most of which are due to dissipation processes where the wave distribution has not converging tails.

In recent years, contact geometry has become a unifying framework for various physical theories, eg. it provides a solid foundation for thermodynamics, non-conservative classical mechanics, electromagnetic fields among other applications. In particular, it has been used to exhibit an explicit correspondence between Fermat’s and Huygens’ principles. In this sense, the aim of this work is to explore the use of contact transformations in the description of light propagation in optical media.

Here we assume that an optical medium can be represented by a Riemannian manifold (, ) where is considered to be the space occupied by a material embedded in spacetime while is its corresponding optical metric. Moreover, the geodesic flow in its unitary tangent bundle can be represented by a contact transformation acting on its space of contact elements. This fact allows us to describe the wavefront evolution in an optical medium solely in terms of the contact transformation and to reconstruct the geodesic flow from the Reeb vector field. This provides us with a way of constructing wavefronts in optical media without directly solving the wave equation. This approach is particularly useful to explore the relationship between rays and fronts as they move across interfaces. While this is simply expressed as Snell’s Law in homogeneous materials, we will show that this construction helps in visualising phenomena such as aberration and total internal reflection in a broader class of geometries. Such techniques may prove to be useful from astrophysics, as light propagates through the intergalactic/galactic media; to the geometric optics of interfaces.

The manuscript is structured as follows. In section II we
recall the construction of a material manifold as the quotient space given by the orbits of the static lab observer. Then, in section III we establish the relation between the unitary tangent bundle and the space of contact elements. In sections IV - VII we provide three specific examples for wavefront evolution in an optical medium by means of a contact transformation. The first two are optical media with Euclidian geometry in two and three dimensions. The results agree with Fermat’s principle and recreate Snell’s law of refraction. The third example, is a two dimensional optical medium endowed with an hyperbolic geometry.

II. THE MATERIAL MANIFOLD

To model an optical medium will use a Riemannian manifold \( (\mathcal{B}, g) \) submersed in a bi-metric Lorentzian spacetime \( (M, g_0, \bar{g}) \), where \( g_0 \) is the background vacuum spacetime metric while \( \bar{g} \) represents the metric encoding the material properties of an optical medium. In this setting, the four velocity of the static observer in the lab frame, \( u \in TM \) satisfies

\[
g_0(u, u) = -1 \quad \text{and} \quad \bar{g}(u, u) = -\frac{1}{n^2},
\]

where \( n > 1 \) is the refractive index of the medium. In this way, the speed of light in the medium is clearly less than that in the vacuum background spacetime. We use the vacuum metric \( g_0 \) to raise an lower indices by means of the musical isomorphisms

\[
g_0^b : TM \rightarrow T^*M \quad \text{and} \quad g_0^* : T^*M \rightarrow TM.
\]

Every non-spacelike vector field \( w \in TM \) can be decomposed in terms of the optical metric \( \bar{g} \) as

\[
w = \text{Hor}(w) + \text{Ver}(w)
\]

where \( \text{Hor}(w) \) and \( \text{Ver}(w) \) represent the transverse and parallel components of \( w \) with respect to the lab frame \( u \), respectively. Thus, every null vector field \( v \) such that

\[
\text{Ver}(v) = \Omega u
\]

for some non-vanishing function \( \Omega \) on \( M \), satisfies

\[
\bar{g}(v, v) = \bar{g}(\text{Hor}(v), \text{Hor}(v)) + \bar{g}(\text{Ver}(v), \text{Ver}(v))
\]

\[
= \bar{g}(\text{Hor}(v), \text{Hor}(v)) + \bar{g}(\Omega u, \Omega u)
\]

\[
= \bar{g}(\text{Hor}(v), \text{Hor}(v)) - \frac{\Omega^2}{n^2} = 0.
\]

so that

\[
\bar{g}(\text{Hor}(v), \text{Hor}(v)) = \frac{\Omega^2}{n^2}.
\]

Now, let us consider the quotient manifold defined by the orbits of the lab frame four-velocity, namely

\[
\mathcal{B} = M/G_u,
\]

where \( G_u \) represents the translation group associated to the flow of the vector field \( u \). One can equip this manifold with a Riemannian metric \( g \) such that for every null vector field \( v \) satisfying \( (4) \),

\[
g(\Pi_1 v, \Pi_1 v) = \frac{\Omega^2}{n^2},
\]

where \( \Pi : M \rightarrow \mathcal{B} \) is a local trivialization. The pair \( (\mathcal{B}, g) \) is called a material manifold Fig. I (cf. definition of body manifold in\(^{26}\) and\(^{22}\)).

The optical metric can be decomposed in terms of the material manifold Riemannian metric and the observer’s four velocity as

\[
\bar{g} = g \circ (\Pi_1 \circ \Pi_1) - \frac{1}{n^2} u^\flat \otimes u^\flat,
\]

where \( u^\flat \equiv g_0^b(u) \) [cf equation \( (2) \)]. In the rest of the manuscript, we will consider the normalization factor \( \Omega = 1 \). Thus, expression \( (9) \) corresponds to the well known Gordon’s optical metric\(^{11}\).

In the following section, we will consider the unitary tangent bundle \( ST \mathcal{B} \) together with \( PT^* \mathcal{B} \) representing the cotangent bundle of \( \mathcal{B} \) with the zero-section \( 0_{\mathcal{B}} \) removed\(^{22}\). This allows us to represent the dual notions of light rays and wave fronts on the optical medium \( (\mathcal{B}, g) \)\(^{28,29}\). The former one represents light rays as the projected null geodesic flow of \( \bar{g} \) while the elements of \( PT^* \mathcal{B} \) are the tangent planes to the wavefront\(^{30}\). Moreover, there is a one to one correspondence between the elements of these two manifolds relating a geodesic flow in the unitary tangent space to a contact transformation in the space of contact elements\(^{25}\). Finally, since contact transformations are symmetries of a contact distribution, its associated contact elements remain invariant when projected to the base manifold \( \mathcal{B} \).

III. MATHEMATICAL STRUCTURES FOR RAYS AND FRONTS.

In this section, we explicitly exhibit the dual nature of light rays and wave fronts from the contact geometric perspective. Let us consider the tangent bundle \( T \mathcal{B} \) of an \( m \)-dimensional
material manifold \( (\mathcal{B}, g) \). Its associated unitary tangent bundle is defined as

\[
ST \mathcal{B} \equiv \{ v \in T \mathcal{B} \mid g(v, v) = 1 \},
\]

where \( g \) is the bundle metric of \( g_{\mathcal{B}} \), (see Exercise 2 of Chapter 3, Section 4 in \[13\]). It follows from the unitary constraint that

\[
\dim(ST \mathcal{B}) = 2m - 1.
\]

for \( m \) the dimension of \( \mathcal{B} \). Indeed, the defining property in \[10\] corresponds to the zero level set of a function on \( T \mathcal{B} \), namely, \( g(v, v) - 1 = 0 \).

Similarly, the unitary co-tangent bundle \( ST^* \mathcal{B} \) is defined as

\[
ST^* \mathcal{B} = \{ p \in T^* \mathcal{B} \mid g^{-1}(p, p) = 1 \}.
\]

Note that the restriction of \( ST^* \mathcal{B} \) to \( PT^* \mathcal{B} \) corresponds to a ray-optical structure – as defined by Perlick (see Definition 5.1.1 and Proposition 5.1.1 in \[13\]) – that is,

\[
\mathcal{N} = \{ p \in PT^* \mathcal{B} \mid g^{-1}(p, p) = 1 \}.
\]

The co-tangent bundle \( T^* \mathcal{B} \) carries a natural symplectic structure, that is, a non-degenerate, closed 2-form \( \omega \). Consider a vector field \( L \in T(T^* \mathcal{B}) \) such that

\[
\mathcal{L}_L \omega = \omega,
\]

where \( \mathcal{L} \) denotes the Lie derivative and \( L \) is a Liouville vector field. One can define a 1-form \( \lambda \in T^*(T^* \mathcal{B}) \) generating the symplectic structure in terms of the Liouville vector field as

\[
\lambda = i_L \omega.
\]

It follows from Cartan’s identity and definition \[15\] that

\[
\mathcal{L}_L \omega = i_L d\omega + di_L \omega = -d i_L \omega = -d \lambda = \omega.
\]

In this way, one can define a contact 1-form for \( ST^* \mathcal{B} \), namely,

\[
\eta = i^* \lambda.
\]

Here \( i^* : T^* T^* \mathcal{B} \rightarrow T^* \mathcal{N} \) is the pullback induced by the inclusion map \( i : \mathcal{N} \rightarrow T^* \mathcal{B} \). Let \( D = \ker(\eta) \) be the contact distribution generated by \( \eta \), then \( (\mathcal{N}, D) \) is a contact manifold, usually referred as the space of contact elements or the contact bundle of \( \mathcal{B} \). The restricted bundle projection

\[
\pi|_\mathcal{N} : \mathcal{N} \subset T^* \mathcal{B} \longrightarrow \mathcal{B}
\]

maps the hyperplanes of the contact structure \( D \) to the contact elements of \( \mathcal{B} \). Note that the vector fields tangent to affinely parametrized geodesics on \( (\mathcal{B}, g) \) are sections of \( ST \mathcal{B} \). In this sense, we can establish a relation between \( ST \mathcal{B} \) and \( \mathcal{N} \) by demanding that

\[
g^{-1} (\Pi_n \circ i_s) (\xi) = \frac{1}{n} (\pi_s(v))
\]

where \( v \in ST \mathcal{B}, n \) is a non-vanishing real number and \( \xi \) is the Reeb vector field defined by

\[
i_\xi \eta = 1 \quad \text{and} \quad i_\xi d\eta = 0.
\]

Thus, the required condition \[19\] expresses that the geodesics of \( (\mathcal{B}, g) \) are transverse to \( \mathcal{N} \) and that \( \xi \) is the spatial part of a null vector field in \( TM \) [cf. equation \[3\]]. Indeed,

\[
g(\xi, \xi) = g(\xi_1, \xi_1) = \frac{1}{n^2}
\]

for each \( x \in \mathcal{B} \).

This, however, does not imply that the light rays should be metric orthogonal to \( \mathcal{N} \). The orthogonal condition implies the existence of an almost contact structure

\[
\phi : T \mathcal{N} \longrightarrow T \mathcal{N},
\]

where

\[
\phi^2 = -1 + \eta \otimes \xi \quad \text{and} \quad \eta \circ \phi = 0,
\]

such that \( (\mathcal{N}, \eta, g|_\mathcal{N}) \) is a contact metric manifold. That is, the restricted bundle metric \( g|_\mathcal{N} = \eta \otimes \eta + d\eta \circ (\phi \otimes 1) \) can be written as

\[
g|_\mathcal{N} = \eta \otimes \eta + d\eta \circ (\phi \otimes 1).
\]

Indeed,

\[
g|_\mathcal{N} (\xi, \xi) = \eta(\xi) \eta(\xi) + d\eta \circ (\phi \xi, \xi) = 0,
\]

where the last equality follows from the transversality condition \[20\] and the definition \( \phi \), equation \[23\] where \( \phi \mathcal{D} = \mathcal{D} \).

In the next section we will see that the presence of boundaries (interfaces between different media) breaks the orthogonality condition.

Now, let us recall the definition of a wavefront centered at the point \( b \in \mathcal{B} \) as the surface

\[
F_b(t) = \{ b_t \in \mathcal{B} | \gamma(0) = b, \gamma(t) = b_t \}
\]

where \( \gamma \) is a geodesic on \( (\mathcal{B}, g) \) (see \[25\]). Observe that for a point \( b_t \in F_b(t) \) the contact element \( \pi(b_t) \) is tangent to \( F_b(t) \). The bundle projector \( \pi \) projects Legendrian submanifolds of \( \mathcal{N} \) to wavefronts in \( \mathcal{B} \).

Consider a geodesic vector flow on \( ST \mathcal{B} \) given by \( \gamma(t) \in \mathcal{B} \) and \( \gamma(t) \in ST\gamma(t) \mathcal{B} \). There is a unique contact element such that

\[
g|_\mathcal{N} (\mathcal{D} b_t, \xi)|_b = 0.
\]

The Reeb flow induces a strict contact transformation

\[
\mathcal{L}_\xi \lambda = 0.
\]

Thus, by the duality of the Reeb’s and the geodesic flow (see Theorem 1.5.2 in \[32\]), there exists a 1-parameter family of contact transformations \( \phi_t : \mathcal{N} \rightarrow \mathcal{N} \), such that for every \( p \in \mathcal{N} \) where \( p = \gamma(t) \),

\[
\phi_t(p) = g^\lambda (\gamma(t))
\]
The unitary condition (10) leads to
\[ p_y = \sqrt{n^2 - p_x^2}. \]  

(33)

Consider the transformation \( \phi_{ab} : T\mathbb{R}^2 \rightarrow ST\mathbb{R}^2 \) defined by
\[ [x = x, y = y, p_x = -\sin \theta, p_y = \cos \theta] \]  

(34)
to change into polar coordinates \( \{x, y, \theta\} \). In this coordinates, the Liouville 1-form (15) becomes
\[ \lambda_{ab} = -\sin \theta \, dx + \sqrt{\cos^2 \theta + n^2 - 1} \, dy, \]  

(35)
and the Reeb vector field associated to \( \lambda_{ab} \) is
\[ \xi_{ab} = \frac{1}{n^2} \left( -\sin \theta \frac{\partial}{\partial x} + \sqrt{\cos^2 \theta + n^2 - 1} \frac{\partial}{\partial y} \right). \]  

(36)

A 1-parameter family of strict contact transformations induced by the Reeb’s flow is given by
\[ \phi_t = \begin{bmatrix} x = x - \frac{t \sin \theta}{n^2}, y = y + \frac{t \cos \theta}{n^2}, \theta = \theta \end{bmatrix}. \]  

(37)

The flow of the Reeb vector field, as proved by Geiges, results dual to the geodesic flow. Observe that (37) is indeed a strict contact transformation, as \( \phi_t^* (\lambda) = \lambda \). In terms of (37), the contact distribution transforms into
\[ \chi = \text{span} \left\{ \frac{1}{n^2} \sqrt{\cos^2 \theta + n^2 - 1} \frac{\partial}{\partial x} - \sin \theta \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta} \right\}. \]

(38)
The bundle projection of the contact distribution \( \pi(\chi) \) are the contact elements of \( \mathbb{R}^2 \). In this case, this corresponds to the first vector of (38). For a point \( q \in \mathbb{R}^2 \) and a fixed \( \theta_0 \) there is a unique geodesic passing through \( q \) and the direction of its tangent vector is precisely \( \theta_0 \). Thus, the contact element is the positive normal line generated by the vector \( \pi(\chi) \).

The geodesic flow is now written in terms of the 1-parameter family of strict contact transformations (37). Observe that for vacuum (\( \varepsilon_0 \mu_0 = 1 \)) the contact transformation will induce a flow given by a family of straight lines with slope \( m = -(\tan \theta_0)^{-1} \), recalling that \( \theta_0 \) is the angle of the contact element positively orthogonal to the geodesic. As the geodesic curve is the trajectory of a light beam, when it passes to a different medium (\( \varepsilon \mu \neq 1 \)), the light will be refracted with \( n = \sqrt{\varepsilon \mu} \) as expected by Snell’s law. Furthermore, we can reconstruct the wavefronts form the contact elements without solving directly the wave equation. In Fig. 4 we can observe how the wavefronts are deformed and slowed down when passing through a different medium. In Fig. 4 the source of light is in a medium with a larger refractive index. Note that when the light passes through the interface, total internal reflection can be observed in the light rays.
V. OPTICAL MEDIUM IN EUCLIDEAN $\mathbb{R}^3$

In the same manner as before, let us consider an optical medium in $\mathbb{R}^3$ with body metric

$$g = n^2 \sum_{i=1}^{3} dx^i \otimes dx^i.$$  \hfill (39)

Again, we propose the coordinate transformation to the unitary tangent bundle $\phi_{ub}: T\mathbb{R}^3 \rightarrow ST\mathbb{R}^3$

$$[x = x, \quad y = y, \quad z = z, \quad p_x = \sin \theta \cos \phi, \quad p_y = \sin \theta \sin \phi, \quad p_z = \cos \theta]$$ \hfill (40)

to get spherical coordinates $\{x, y, z, \theta, \phi\}$. The Liouville 1-form transforms to

$$\lambda_{ub} = \cos \phi \sin \theta \, dx + \sin \phi \sin \theta \, dy + \sqrt{\cos^2 \theta + n^2 - 1} \, dz,$$ \hfill (41)

for which the Reeb vector field is

$$\xi_{ub} = \frac{1}{n^2} \left( \cos \phi \sin \theta \frac{\partial}{\partial x} + \sin \phi \sin \theta \frac{\partial}{\partial y} \right.$$

$$\left. + \sqrt{\cos^2 \theta + n^2 - 1} \frac{\partial}{\partial z} \right)$$ \hfill (42)

while the contact distribution is expressed as

$$\chi = \text{span} \left\{ -\sin \theta \sin \phi \frac{\partial}{\partial x} + \cos \phi \sin \theta \frac{\partial}{\partial y}, \right.$$ \hfill (43)

$$\left. -\sqrt{\cos^2 \theta + n^2 - 1} \frac{\partial}{\partial x} + \cos \phi \sin \theta \frac{\partial}{\partial y} + \cos \phi \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial}{\partial \phi} \right\}.$$ \hfill 

We find the 1-parameter family of contact transformations induced by the Reeb vector field \hfill (42)

$$\phi = \left[ x = x - t \cos \phi \sin \theta \frac{n^2}{\sqrt{\cos^2 \theta + n^2 - 1}}, \quad y = y + \frac{t \sin \phi \sin \theta}{n^2}, \right.$$ \hfill (44)

$$\left. z = z + t \sqrt{\cos^2 \theta + n^2 - 1} \frac{n^2}{\cos \theta}, \quad \theta = \theta, \quad \phi = \phi \right].$$ \hfill 

As expected, the projection of the contact distribution to $\mathbb{R}^3$ is a set of 2-dimensional planes, which are the contact elements of $\mathbb{R}^3$. Each plane is tangent to a wavefront surface sphere.

VI. OPTICAL MEDIUM WITH HYPERBOLIC GEOMETRY ON $\mathbb{R}^2$ HALF PLANE

We consider now a 2-dimensional optical medium endowed with a hyperbolic geometry represented by the upper half plane model $\mathbb{H}^2 = \{ z = x + iy, \quad y > 0 \}$ with the usual metric

$$\tilde{g} = \frac{1}{y^2} \left( \sum_{i=1}^{2} dx^i \otimes dx^i - \frac{1}{n^2} dr \otimes dr \right).$$ \hfill (45)
As before, we will work with the body metric given by

$$g = \left( \frac{n}{y} \right)^2 \sum_{i=1}^{2} dx^i \otimes dx^i.$$  

The unitary tangent bundle of the hyperbolic space is naturally identified with $PSL(2, \mathbb{R})$, which is different from $ST^n \mathbb{R}$ used in the previous examples. Nevertheless, our construction is sufficiently general to work in either space, as $PSL(2, \mathbb{R})$ can also be endowed with coordinates $\{x, y, \phi\}$, for $(x, y)$ cartesian coordinates and $\phi$ an angular coordinate. In terms of these coordinates, the Liouville 1-form is

$$\lambda_{\text{st b}} = n y \left( -\cos \phi \; dx + \sin \phi \; dy \right).$$

The associated Reeb vector field is

$$\xi_{\text{st b}} = \frac{1}{n} \left( -y \cos \phi \frac{\partial}{\partial x} + y \sin \phi \frac{\partial}{\partial y} - \cos \phi \frac{\partial}{\partial \phi} \right),$$

and the contact distribution

$$\chi = \text{span} \left\{ \frac{y}{n} \left( \cos \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial y} \right), \frac{\partial}{\partial \phi} \right\}.$$  

Once again, we find the 1-parameter family of contact transformations in $PT^* \mathcal{B}$ induced by the Reeb’s flow with the natural identification with $ST^* \mathbb{H}^2$.

$$\phi = \left[ x = \frac{(x \sin \phi + y \cos \phi - x)e^{2\pi} - x \sin \phi - y \cos \phi - x}{\cos \phi - 1} \right],$$

$$y = -\frac{2y e^{2\pi}}{(\sin \phi - 1)e^{2\pi} - \sin \phi - 1},$$

$$\phi = \arctan \left( -\frac{(-\sin \phi + 1)e^{2\pi} - \sin \phi - 1}{2e^{2\pi} \cos \phi} \right).$$

The mapping (50) is indeed a strict contact transformation as $\phi^* (\lambda) = \lambda$. As expected, the projection of the Reeb’s flow which induced (50) are semicircles of radius

$$r = \frac{y_0}{\cos \phi_0},$$

and center

$$\left( \frac{x_0 \cos \phi_0 - y_0 \sin \phi_0}{\cos \phi_0}, 0 \right),$$

which are precisely the geodesic curves in $\mathbb{H}^2$.

For interfaces between media with different refractive index, we solve (48) using numerical methods. The solution observed in Fig. 5 and 6 represents the refraction of the light rays mapped to the Poincaré disc, the first one represents a horizontal layer of a different refractive index as interface, while the second represents a vertical interface.

Finally, we use the same numerical methods to obtain the waves and fronts of a light source in the medium with larger refractive index. The result, shown in Fig. 7 presents no total internal reflection.

In the gravitational context, the anti-deSitter model has been closely related with the hyperbolic geometry. It is known that AdS admits closed timelike curves and achronal surfaces can be observed. In Fig. 7 we can observe some of this geometric properties in light rays which are closer to the infinity circle boundary of the disc. Where one light ray can intersect more than once the same wavefront.
The Reeb flow allows one to reconstruct the trajectories and wavefronts of light while propagating through an optical medium. Using this result, we explored two and three dimensional Euclidean optical media and a two dimensional hyperbolic medium. In the case of a 2-dimensional Euclidean medium, it was possible to recreate Snell’s law of refraction together with the total internal reflection phenomenon. In the case of the hyperbolic medium, we obtained the refraction patterns from vertical and horizontal interfaces mapped to the Poincaré disc. No total internal reflection was observed when the light source was placed in a medium with a larger refractive index. Nevertheless, we saw that light rays can intersect the same wavefront more than once. This is a surprising effect which deserves further exploration.

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