Gerber-Shiu Function in a Discrete-time Risk Model with Dividend Strategy

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Abstract

In this paper, a discrete-time risk model with dividend strategy and a general premium rate is considered. Under such a strategy, once the insurer’s surplus hits a constant dividend barrier \( b \), dividends are paid off to shareholders at \( \alpha \) instantly. Using the roots of a generalization of Lundberg’s fundamental equation and the general theory on difference equations, two difference equations for the Gerber-Shiu discounted penalty function are derived and solved. The analytic results obtained are utilized to derive the probability of ultimate ruin when the claim sizes is a mixture of two geometric distributions. Numerical examples are also given to illustrate the applicability of the results obtained.

Keywords: Compound binomial model; two-step premium; defective renewal equation; Gerber-Shiu discounted penalty function; dividend strategy.

1 Introduction

Risk theory has a long development time, Lundberg [1] and Gramer [2] established the connection of risk theory. The compound binomial model that was first proposed by Gerber [3] have received considerable attention. For
instance, Shiu [4], Willmot [5] and Dickson [6] have analyzed the compound binomial model. Markov chain is understood to be a stochastic process in discrete time possessing a certain conditional independence property. The state space may be finite, countably infinite or even more general. Cossette et al. [7] consider the so-called compound Markov binomial model which introduces dependency between claim occurrences. For an generalization of the classical risk model see Landriault [8]. Furthermore, in the discrete time risk model, the issue related to dividend is also widely considered.

Dividend strategies for insurance risk models were first proposed by DeFinetti [9] to reflect more realistically the surplus cash flows in an insurance portfolio. Because of the certainty of ruin for a risk model with a constant dividend barrier, the calculation of the Gerber-Shiu discounted penalty function is a major problem of interest in the context. Among the class of discrete-time risk models, Tan and Yang [10] derived a recursive algorithm to compute a particular class of Gerber-Shiu penalty functions in the framework of the compound binomial model with randomized dividend payments. Landriault [11] then generalized Tan and Yang’s model to consider the compound binomial model with a multi-threshold dividend structure and randomized dividend payments. In the discrete time risk model, He and Yang [12] considered that dividends are paid randomly to shareholders and policyholders in the framework of the compound binomial model. In the framework of a discrete semi-Markov risk model, a randomized dividend policy is studied by Yuen et al. [13]. Zhang and Liu [14] consider a discrete-time risk model with a mathematically tractable dependence structure between interclaim times and claim sizes in the presence of an impulsive dividend strategy.

The paper is structured as follows: a brief description of the discrete-time model and the introduction of the Gerber-Shiu discounted penalty function are considered in Section 2. In section 3, we obtain and solve a non-homogeneous difference equation satisfied by the Gerber-Shiu discounted penalty function $m(u,c,b)$. Closed-form solutions for $m(u)$ are obtained when the claim sizes is a mixture of two geometric distributions and corresponding numerical examples are also provided in Section 4.

2 The model

Throughout, denote by $N$ the set of natural numbers and $N^+ = N \cap \{0\}$. In the compound binomial model, the claim number process $\{N_k, k \in N\}$ is assumed to be a renewal process with independent and identically distributed (i.i.d.) interclaim times $\{W_j, j \in N^+\}$ having probability mass function (p.m.f.) $f_W(l) = q(l-1)p^{l-1}$ for $l \in N^+$. Equivalently, the probability of having a claim is $p(0 < p < 1)$ and the probability of no claim is $q = 1-p$. The individual claim amount r.v.’s(random variables) $\{X_j, j \in N^+\}$ form a sequence of strictly positive, integer-valued and i.i.d. r.v.’s. We suppose that the r.v.’s $\{X_j, j \in N^+\}$ are distributed as a generic r.v. $X$ with p.m.f. $f(x)$, probability generating function (p.g.f.) $\tilde{f}(x)$. Moreover, it is assumed that the r.v.’s $W_1,W_2,…$ and $X_1,X_2,…$ are mutually independent. Let $S_k = \sum_{i=1}^{N_k} X_i$ be the total amount of settled claims up the end of the kth time period with $S_0 = 0$.

Suppose that premiums are received at the beginning of each time period, and claims are paid out at the end of each time period. Denote $u \geq 0$ to be the initial surplus, $b > 0$ the constant barrier level, and $c_1 > 0$ the annual premium. Under such a strategy, let $\alpha(0 < \alpha \leq c_1)$ be the annual dividend rate, once the insurer’s surplus at time $k$ hits or exceeds a constant dividend barrier $b$, dividends are paid off to shareholders at $\alpha$ instantly. In this case, the net premium after dividend payments is $c_2 = c_1 - \alpha \geq 0$. The corresponding surplus of the insurer at the end of the kth time period is $U_s(k)$ for $k = 1,2,\cdots$ can be described as
\[ U_b(k) = \begin{cases} U_b(k-1) + c_1 - \eta_k X_k, & U_b(k-1) \leq b \\ U_b(k-1) + c_2 - \eta_k X_k, & U_b(k-1) > b. \end{cases} \]  

where \( U_b(0) = u \). \( \{\eta_k, k \in N\} \) is an independent and identically distributed Bernoulli sequence, we denote by \( \eta_k = 1 \) the event of having a claim at the time \( k \) and denote by \( \eta_k = 0 \) the event that no claim at the time \( k \). We assume that \( P(\eta_k = 1) = p \) and \( P(\eta_k = 0) = 1 - p = q \) and surplus process \( U_b(k) \) has a positive drift by letting \( c_2 > pE[X] \) (known as the positive security loading condition in ruin theory).

Define \( \tau_b = \min\{k : U_b(k) < 0\} \) to be the time of ultimate ruin. Let \( \nu \) be a constant annual discount rate for each period. When ruin occurs, \( U_b(\tau - 1) \) is the surplus one period prior to ruin and \( |U_b(\tau)| \) is the deficit at ruin. For \( \nu \in (0,1] \), the well-known Gerber-Shiu discounted penalty function is then defined as

\[ m(u; b) = E\left[ e^{\nu \tau_b} \omega(U_{\tau_b-1} \mid U_{\tau_b} = U_0 = u)^{1}\right], \]  

where \( \omega: N \times N^+ \rightarrow R \) is a penalty function and \( I_{\{Q\}} \) is the indicator function of an event \( Q \). Also, we consider some special cases of (2) with successively simplified the penalty functions. If \( \omega(n_1, n_2) = 1 \) for \((n_1, n_2) \in N \times N^+\), we get the generating function of the time to ruin, i.e.

\[ m_b(u) = E\left[ e^{\nu \tau_b} I_{\{\tau_b < \infty\}} U_0 = u \right]. \]

3. The Gerber-Shiu discounted penalty function

In this section, we derive two difference equations for the Gerber-Shiu discounted penalty function: one for the initial surplus below the barrier level \( b \) and the other for the initial surplus above the barrier level \( b \). Clearly, the Gerber-Shiu discounted penalty function \( m(u; b) \) behaves differently, depending on whether its initial surplus \( u \) is below or above the barrier level \( b \). Hence, we write

\[ m(u; b) = \begin{cases} m_1(u), & 0 \leq u < b \\ m_2(u), & u \geq b. \end{cases} \]

In order to identify the structural form of the solution for the Gerber-Shiu discounted penalty function, three cases will be considered separately.

3.1 For initial surpluses less than the barrier \( b \)

In the first scenario, the initial surplus below the barrier \( b \), for \( u = 0, 1, \ldots, b-c_1-1 \), we have

\[ m_1(u) = v q m_1(u + c_1) + v p \sum_{j=2}^{u+c_1} m_1(u+c_1-j) f(j) + v p \sum_{j=u+c_1+1}^{\infty} \omega(u+c_1; j-u-c_1) f(j) \]

\[ = v q m_1(u + c_1) + v p (m_1 * f)(u+c_1) + \gamma_1(u), \]  

where
\[
gamma_1(u) = vp \sum_{j=0}^{\infty} \omega(u + c_1; j - u - c_1) f(j).
\]

and \(m_1 \ast f\) holds for the convolution product of \(m_1\) and \(f\).

To state that (3) is a non-homogeneous difference equation of order \(c_1\), we re-express (3) according to the forward difference operator \(\Delta\) and its property (see Chapter 2 of Kelly & Peterson [15]),

\[
m(u + c) = \sum_{j=0}^{\infty} \binom{c}{j} \Delta^j m(u),
\]

(4)

substituting (4) into (3) shows

\[
m_1(u) = vq \sum_{j=0}^{c_1} \binom{c_1}{j} \Delta^j m_1(u) + vp \sum_{j=0}^{c_1} \binom{c_1}{j} \Delta^j (m_1 \ast f)(u) + \gamma_1(u),
\]

(5)

for \(u = 0,1,\ldots,b - c_1 - 1\). (5) can be simplified to

\[
\sum_{j=0}^{c_1} a_{1,j} \Delta^j m_1(u) = \sum_{j=0}^{c_1} b_{1,j} \Delta^j (m_1 \ast f)(u) + \gamma_1(u),
\]

(6)

where

\[
a_{1,j} = I_{(j=0)} - vq \binom{c_1}{j}, b_{1,j} = vp \binom{c_1}{j}.
\]

and \(A_1(z), B_1(z)\) are polynomials (in \(z\)) defined as

\[
A_1(z) = \sum_{j=0}^{c_1} a_{1,j} z^j, B_1(z) = \sum_{j=0}^{c_1} b_{1,j} z^j.
\]

becomes

\[
A_1(\Delta) m_1(u) = B_1(\Delta)(m_1 \ast f)(u) + \gamma_1(u), \quad u = 0,1,\ldots,b - c_1 - 1.
\]

(7)

We know from (7) that \(m_1(u)\) satisfies a non-homogeneous difference equation of order \(c_1\). From the general theory on difference equations, every solution to a \(c_1\)-th order difference equation can be expressed as a particular solution to this difference equation plus a linear combination of \(c_1\) linearly independent solutions to the associated homogeneous difference equation (cf. Elaydi [16], Theorem 2.30). Therefore, for \(u = 0,1,\ldots,b - 1\), the Gerber-Shiu discounted penalty function can be expressed as

\[
m_1(u) = \phi_1(u) + \sum_{j=0}^{c_1 - 1} \alpha_{1,j} \gamma_{1,j}(u), \quad u = 0,1,\ldots,b - 1.
\]

(8)
where \( \{ y_{i,j}(u) \}_{u=0}^{\infty} \) (\( j = 0,1,\ldots, c_1 - 1 \)) are \( c_1 \) fundamental solutions to the following homogeneous difference equation

\[
A_i(\Delta)y_i(u) = B_i(\Delta)(y_i \ast f)(u) \quad u \geq 0. \tag{9}
\]

\( \{ \phi_i(u) \}_{u=0}^{\infty} \) is a particular solution to

\[
A_i(\Delta)\phi_i(u) = B_i(\Delta)(\phi_i \ast f)(u) + y_i(u) \quad u \geq 0. \tag{10}
\]

Combining (3) and (9), we get

\[
y_i(u) = vqy_i(u + c_1) + vp(y_i \ast f)(u + c_1). \tag{11}
\]

Multiplying (11) by \( z^{u+c_1} \) and then summing over \( u \) from 0 to \( \infty \) lead to

\[
\sum_{u=0}^{\infty} z^{u+c_1} y_i(u) = vq \sum_{u=0}^{\infty} z^{u+c_1} y_i(u + c_1) + vp \sum_{u=0}^{\infty} z^{u+c_1} (y_i \ast f)(u + c_1), \tag{12}
\]

routine calculations lead to

\[
z^\alpha \tilde{y}_i(z) = vq \left[ \tilde{y}_i(z) - \sum_{u=0}^{\infty} z^{u} y_i(u) \right] + vp \left[ \tilde{y}_i(z) \tilde{f}(z) - \sum_{u=0}^{\infty} z^{u} (y_i \ast f)(u) \right].
\]

After some algebra, one could see that (12) can be written as

\[
\tilde{y}_i(z) = \frac{-vq \sum_{u=0}^{\infty} z^{u} y_i(u) + p \sum_{u=0}^{\infty} z^{u} (y_i \ast f)(u)}{z^\alpha - vq - vp \tilde{f}(z)}. \tag{13}
\]

By choosing \( y_{i,j}(u) = I_{j\ast u} \) for \( j, u \in \{0,1,\ldots, c_1 - 1\} \). According to (13), the generating function associated to the fundamental solution \( \{ y_{i,j}(u) \}_{u=0}^{\infty} \) is

\[
\tilde{y}_{i,j}(z) = \frac{-vq z^j + p \sum_{u=j+1}^{c_1} z^{u} f(u - j)}{z^\alpha - vq - vp \tilde{f}(z)} = \frac{-R_{i,j}(z)}{\tilde{h}_{i,j}(z) - \tilde{h}_{i,j}(z)} \quad u \geq 0, \tag{14}
\]

where

\[
\tilde{h}_{i,1}(z) = z^\alpha, \quad \tilde{h}_{i,2}(z) = vq + vp \tilde{f}(z), \quad R_{i,j}(z) = vq z^j + p \sum_{u=j+1}^{c_1} z^{u} f(u - j) \nonumber \]

\( \text{Lemma 3.1} \) : When \( v \in (0,1) \), the denominator in (14) has exactly \( c_1 \) zeros, say \( \{ z_i \}_{i=1}^{c_1} \) inside the unit circle \( C = \{ z : |z| = 1 \} \).
Lemma 3.2: When $v = 1$, the denominator in (14) has exactly $c_1 - 1$ zeros, say $\{z_{i_j}\}_{j=1}^{c_1-1}$ inside the unit circle $C = \{z : |z| = 1\}$ and another trivial root $z_{c_1} = 1$.

For the rest of the paper, we assume that all $\{z_{i_j}\}_{j=1}^{c_1}$ are distinct, since the analysis of the multiple roots of Lundberg’s generalized equation leads to tedious derivations.

Let $\pi_i(z) = \prod_{j=0}^{c_1} (z - z_j)$ and $\pi_i'(z_k) = \prod_{j 
eq k}^{c_1} (z_k - z_j)$, from Liu and Bao [17], we have

$$
\frac{\tilde{h}_{i,1}(z) - \tilde{h}_{i,2}(z)}{\pi_i(z)} = 1 - v p T_{z_1} \cdots T_{z_n} f(c_1),
$$

where $T_z$ is an operator (see Li [18]) defined as

$$
T_z y(c) = \sum_{u=0}^{\infty} z^u y(u + c) = \sum_{u=c}^{\infty} z^{u-c} y(u).
$$

(14) can be rewritten as

$$
\tilde{y}_{i,j}(z) = \frac{-R_{i,j}(z)}{\pi_i(z)} - \frac{\tilde{h}_{i,1}(z) - \tilde{h}_{i,2}(z)}{\pi_i(z)}. \tag{16}
$$

Regarding the numerator in (16), partial fractions yield the equivalent representation

$$
\frac{-R_{i,j}(z)}{\pi_i(z)} = \sum_{k=1}^{c_1} \frac{R_{i,j}(z_k)}{\pi_i'(z_k)} \frac{1}{z_k - z}. \tag{17}
$$

By inserting (15) and (17) into (16), we obtain

$$
\tilde{y}_{i,j}(z) = v p \tilde{y}_{i,j}(z) T_{z_1} \cdots T_{z_n} f(c_1) + \sum_{k=1}^{c_1} \frac{R_{i,j}(z_k)}{\pi_i'(z_k)} \frac{1}{z_k - z}. \tag{18}
$$

Theorem 3.1: For $j = 0, 1, \ldots, c_1 - 1$, $y_{i,j}(u)$ satisfies the following defective renewal equation

$$
y_{i,j}(u) = \zeta_i \sum_{n=0}^{u} y_{i,j}(u-n) \chi_i(n) + \zeta_i'(u), \tag{19}
$$

where

$$
\zeta_i = v p T_{z_1} \cdots T_{z_n} f(c_1), \quad \chi_i(n) = \frac{T_{z_2} \cdots T_{z_{c_1}} f(c_1 + n)}{T_{z_1} \cdots T_{z_{c_1}} f(c_1)}, \quad \zeta_i'(u) = \sum_{k=1}^{c_1} \frac{R_{i,j}(z_k)}{\pi_i'(z_k)} \left( \frac{1}{z_k} \right)^{u+1}
$$

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Now, we turn our attention to the calculation of the particular solutions $\{\phi_i(u)\}_{i=0}^{\infty}$, combining (3) and (10), $\{\phi_i(u)\}_{i=0}^{\infty}$ satisfies

$$
\phi_i(u) = vq\phi_i(u + c_i) + vp(\phi_i * f)(u + c_i) + \gamma_i(u).
$$

(20)

We use a solution procedure analogous to the fundamental solutions, we get

$$
\tilde{\phi}_i(z) = \frac{z^c_1\tilde{\gamma}_i(z) - v\left(\sum_{n=0}^{c_i-1} z^n \phi_i(u) + p \sum_{n=0}^{c_i-1} z^n (\phi_i * f)(u)\right)}{z^c_1 - vq - vp \tilde{f}(z)}
$$

(21)

where $Q_{i,j}(z) = v\left(\sum_{n=0}^{c_i-1} z^n \phi_i(u) + p \sum_{n=0}^{c_i-1} z^n (\phi_i * f)(u)\right)$ is a polynomial of degree $c_i - 1$ (or less) in $z$. It is known from (35) in Liu and Zhang [14] that

$$
\frac{z^c_1\tilde{\gamma}_i(0) - Q_{i,j}(z)}{\pi_1(z)} = T_{z_1} T_{z_2} \ldots T_{z_k} \tilde{\gamma}_i(0).
$$

(22)

By substituting (15) and (22) into (21), we get

$$
\tilde{\phi}_i(z) = \frac{T_{z_1} T_{z_2} \ldots T_{z_k} \tilde{\gamma}_i(0)}{1 - vpT_{z_1} T_{z_2} \ldots T_{z_k} T_{z_1} f(c_i)} = \frac{\tilde{\phi}_i(z)}{1 - vpT_{z_1} T_{z_2} \ldots T_{z_k} T_{z_1} f(c_i)}.
$$

(23)

**Theorem 3.2**: For $u \in N$, it holds that

$$
\phi_i(u) = \xi_1 \sum_{n=0}^{u} \phi_i(u - n) \chi_i(n) + \delta_i(u).
$$

(24)

where $\delta_i(u) = T_{c_i} \ldots T_{c_2} T_{c_i} \gamma_i(u)$.

In the second scenario, for $u = b - c_i, \ldots, b - 1$,

$$
m_i(u) = vq m_i(u + c_i) + vp(\phi_i * f)(u + c_i) + \gamma_i(u).
$$

(25)

**3.2 For initial surpluses equal to or more than the barrier $b$**

The last scenario, for $u \geq b$,

$$
m_2(u) = vq m_2(u + c_2) + vp(\phi_i * f)(u + c_2) + \gamma_2(u),
$$

(26)
where \( \gamma_2(u) = vp \sum_{j=u+c_2+1}^{\infty} \omega(u + c_2; j - u - c_2) f(j) \).

The structural form (8) for \( m_1(u) \) is expressed in terms of the \( \alpha_{1,j} \), and also depends on \( m_2(u) \) in (26). In order to drive the solutions of \( m(u) \), shifting the argument \( u \) in (26) by \( b \) units, for \( u \geq 0 \), (26) can be rewritten as

\[
m_2(u + b) = v q m_2(u + b + c_2) + vp \sum_{j=b}^{u+b-c_2} m_2(j)f(u + b + c_2 - j) + vp \sum_{j=0}^{b-1} m_1(j)f(u + b + c_2 - j) + \gamma_2(u + b),
\]

Let \( \xi_2(u) \equiv m_2(u + b) \), (26) becomes

\[
\xi_2(u) = v q \xi_2(u + c_2) + vp(\xi_2 * f)(u + c_2) + \eta(u),
\]

where \( \eta(u) = vp \sum_{j=0}^{b-1} m_1(j)f(u + b + c_2 - j) + \gamma_2(u + b) \).

We use a solution procedure analogous to that of Section 3.1, \( \xi_2(u) \) satisfies

\[
A_2(\Delta)\xi_2(u) = B_2(\Delta)(\xi_2 * f)(u) + \eta(u) \quad u \geq 0,
\]

where

\[
A_2(z) = \sum_{j=0}^{c_2} a_{2,j} z^j, B_2(z) = \sum_{j=0}^{c_2} b_{2,j} z^j, a_{2,j} = I_{(j=0)} - vq \binom{c_2}{j}, b_{2,j} = vp \binom{c_2}{j}.
\]

From the general theory on difference equations, can be expressed as

\[
m_2(u + b) \equiv \xi_2(u) = \phi_2(u) \quad u = 0,1, \ldots
\]

where \( \{\phi_2(u)\}_{u=0}^{\infty} \) satisfies

\[
A_2(\Delta)\phi_2(u) = B_2(\Delta)(\phi_2 * f)(u) + \eta(u) \quad u \geq 0.
\]

Some solution procedures are omitted, similar discussions can be find in Section 3.1. Generating function of the particular solution \( \phi_2(u) \) is

\[
\phi_2(z) = \frac{z^{c_2} \bar{\eta}(z) - v q \sum_{u=0}^{c_2-1} z^u \phi_2(u) + p \sum_{u=0}^{c_2-1} z^u (\phi_2 * f)(u)}{z^{c_2} - vq - vp \bar{f}(z)}
\]

\[
= \frac{z^{c_2} T z \eta(0) - R_{2,j}(z)}{h_{2,j}(z) - h_{2,j}(z)},
\]

\[

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Where
\[ \tilde{h}_{2,1}(z) = z^{c_1}, \quad \tilde{h}_{2,2}(z) = \nu q + v p \tilde{f}(z), \quad R_{2,j}(z) = \nu \left\{ q \sum_{n=0}^{\tilde{z}-1} z^n \phi_2(u) + p \sum_{n=0}^{\tilde{z}-1} z^n (\phi_2 \ast f)(u) \right\}. \]

**Theorem 3.3**: For \( u \in \mathbb{N} \), it holds that
\[ \phi_2(u) = \zeta_2 \sum_{n=0}^{u} \phi_2(u-n) \chi_2(n) + \vartheta_2(u), \tag{31} \]
where
\[ \zeta_2 = \nu p T_{1,2} \cdots T_{z_2, z_1} f(c_2), \quad \chi_2(n) = \frac{T_{z_2, z_2} \cdots T_{z_1, z_1} f(c_2 + n)}{T_{1, z_2} \cdots T_{z_1, z_1} f(c_2)} , \quad \vartheta_2(u) = T_{z_2, z_2} \cdots T_{z_1, z_1} \eta(u). \]

So for \( u \geq b \),
\[ m_2(u) \equiv \xi_2(u-b) = \phi_2(u-b). \tag{32} \]

**4 Numerical results**

It is well-known that \( f(x) = (1 - \rho) \rho^{x-1} \) is a geometric distribution. In this section, it is further assumed that \( f(x) \) is a mixture of two geometric distributions with \( f(x) = \theta (1 - \rho_1) \rho_1^{x-1} + (1 - \theta)(1 - \rho_2) \rho_2^{x-1} \).

Obviously, probability generating function is \( \tilde{f}(x) = \frac{z[(1 - \rho_1)(1 - \rho_2) + \beta(1-z)]}{(1 - \rho_1 z)(1 - \rho_2 z)} \) where
\[ \beta = \theta \rho_2 (1 - \rho_1) + (1 - \theta) \rho_1 (1 - \rho_2), \] and mean is \( \mu = \frac{\theta}{1 - \rho_1} + \frac{1 - \theta}{1 - \rho_2} \), we rewrite (16) as
\[ \tilde{y}_{1,i}(z) = -\frac{R_{1,i}(z)(1 - \rho_1 z)(1 - \rho_2 z)}{\Lambda_1(z)}. \tag{33} \]
where
\[ \Lambda_i(z) = z^i (1 - \rho_1 z)(1 - \rho_2 z) - \nu q(1 - \rho_1 z)(1 - \rho_2 z) - v p[z(1 - \rho_1)(1 - \rho_2) + \beta(1-z)](i = 1, 2) \]

Since \( \Lambda_i(z) \) is a polynomial of degree \( c_i + 1 \), with leading coefficient \( \rho_1 \rho_2 \), it can be expressed as
\[ \Lambda_i(z) = \rho_1 \rho_2 \pi_i(z) \prod_{j=1}^{h} (z - \xi_j), \]
where \( \xi_j \) are solutions of \( \Lambda_i(z) \) on the complex plane. It is notable that \( \xi_j \) have a module larger than 1, from performing partial fraction, we have
\[
\frac{\pi_i(z)(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_i(z)} = \pi_i(z)(1-\rho_1 z)(1-\rho_2 z) = 1 + \sum_{i=1}^{h} \frac{\omega_i}{\xi_i - z},
\]

where
\[
\omega_i = \frac{\prod_{k=1}^{b}(\rho_k^{-1} - \xi_i)}{\prod_{j=1, j\neq i}^{b}(\xi_k - \xi_i)}.
\]

For \(i = 1\), substituting (34) into (16) shows
\[
\tilde{y}_{1,j}(z) = \frac{(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_1(z)} \sum_{k=1}^{c_i} \frac{R_{i,j}(z_k)}{\pi_1(z_k)} \frac{\pi_i(z)}{z_k - z} = \sum_{k=1}^{c_i} \frac{R_{i,j}(z_k)}{\pi_1(z_k)} \left(1 + \sum_{i=1}^{h} \frac{\omega_i}{\xi_i - z}\right) \frac{1}{z_k - z}.
\]

Upon inversion, we obtain from (35) that
\[
y_{1,j}(u) = \sum_{k=1}^{c_i} \frac{R_{i,j}(z_k)}{\pi_1(z_k)} \left[z_k^{-(a+1)} + \sum_{i=0}^{h} \omega_i \sum_{i=0}^{h} \xi_i^{-(a+1-i)}z_k^{-(i+1)}\right]
\]
\[
= \sum_{k=1}^{c_i} \frac{R_{i,j}(z_k)}{\pi_1(z_k)} \left(1 - \sum_{i=1}^{h} \frac{\omega_i}{z_k - \xi_i}\right) \xi_i^{-(a+1)} + \sum_{k=1}^{c_i} \frac{R_{i,j}(z_k)}{\pi_1(z_k)} \sum_{i=1}^{h} \frac{\omega_i}{z_k - \xi_i} \xi_i^{-(a+1)}.
\]

Use the same method, we obtain from (30) and (34) that,
\[
\tilde{\phi}(z) = \frac{\pi_i(z)(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_j(z)} \tilde{g}(z) = \left(1 + \sum_{j=1}^{b} \frac{\omega_j}{\xi_j - z}\right) \tilde{g}(z),
\]
\[
\phi(u) = \tilde{g}(u) + \sum_{j=1}^{b} \omega_j \sum_{i=0}^{u} \xi_j^{-(u+1-i)} g_{i}(l).
\]

**Example:** Suppose \(c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, v = 0.95, \rho_1 = 0.3, \rho_2 = 0.6\). From (33)
\[
\Lambda_1(z) = z^2(1-\rho_1 z)(1-\rho_2 z) - vq(1-\rho_1 z)(1-\rho_2 z) - vp(z(1-\rho_1)(1-\rho_2) + \beta(1-z)),
\]
\[
\Lambda_2(z) = z(1-\rho_1 z)(1-\rho_2 z) - vq(1-\rho_1 z)(1-\rho_2 z) - vp(z(1-\rho_1)(1-\rho_2) + \beta(1-z)).
\]
And the relatively safety loading condition \(c_2 - p\mu > 0\) holds for all \(\theta \in (0,1)\). Hence, \(\theta\) is chosen to be \(0.1, 0.3, 0.5, 0.7, 0.9\), respectively. By solving Lundberg’s equation \(\Lambda_1(z) = 0\), we obtain the values of \(z_1\)’s and \(\xi_j\)’s, see Table 1. By solving Lundberg’s equation \(\Lambda_2(z) = 0\), we obtain the values of \(z_2\)’s and \(\xi_j\)’s, see Table 2.
Explicit expressions for $y_{1,j}(u)$ is determined by (36), so we obtain the values of $y_{1,j}(u)$ for $\theta = 0.5, c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, \rho_1 = 0.3, \rho_2 = 0.6, v = 0.95, b = 10$.

For instance, one has for $\theta = 0.5$,

$$y_6(u) = -0.42464 \times (-0.88633)^u + 0.47519 \times 0.96901^u - 0.05183 \times 1.56583^u + 0.00129 \times 3.40325^u$$

$$y_1(u) = 0.42071 \times (-0.88633)^u + 0.55572 \times 0.96901^u - 0.04023 \times 1.56583^u + 0.00056 \times 3.40325^u$$

Then solve a system of linear equations with $\alpha_{1,j}$, Table 3 lists the values of $\alpha_{1,j}$'s.

| $\theta$ | $\alpha_{1,0}$ | $\alpha_{1,1}$ |
|--------|---------------|--------------|
| 0.1    | 0.001602      | 0.001703     |
| 0.3    | 0.00103       | 0.000109     |
| 0.5    | 0.00059       | 0.00063      |
| 0.7    | 0.00027       | 0.00029      |
| 0.9    | 5.109949×10^{-3} | 6.318151×10^{-3} |

Explicit expressions for $\phi_2(u)$ is determined by (38) so we get the values of $\phi_2(u)$ for $\theta = 0.5, c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, \rho_1 = 0.3, \rho_2 = 0.6, v = 0.95, b = 10$, see Table 4.

| $u$ | $\phi_2(u)$ | $\phi_2(u)$ |
|-----|-------------|-------------|
| 10  | 7.22942×10^{-4} | 1.8993×10^{-4} |
| 11  | 5.02092×10^{-4} | 8.31054×10^{-5} |
| 12  | 3.49790×10^{-4} | 5.80441×10^{-5} |
| 13  | 2.44063×10^{-4} | 4.05411×10^{-5} |
| 14  | 1.70395×10^{-4} | 2.83162×10^{-5} |
Especially, when $\omega(N_1,N_2) = 1, b = 10$. Fig. 1 and Fig. 2 depict the generating function of the time to ruin $m_b(u)$ as functions of $u$. Observing Fig. 1 and Fig. 2, for each fixed $\theta$ it is easy that a larger $u$ corresponds to a smaller expected ruin time and $m_b(u)$ is a decreasing function of $\theta$ when $u$ is fixed.

![Fig. 1. Numerical results of $m_b(u)$ for $b = 10, u < b$](image1)

![Fig. 2. Numerical results of $m_b(u)$ for $b = 10, u \geq b$](image2)

5. Conclusion

In this paper, we consider the compound binomial model with general premium rate and a constant dividend barrier. Using the roots of a generalization of Lundberg’s fundamental equation and the general theory on difference equations, we derive an explicit expression for the Gerber-Shiu discounted penalty function up to the time of ruin. In particular, a numerical example is provided to show that the formulae are readily programmable in practice. From the numerical example given above, we can see that the barrier level has a negative effect on the total Gerber-Shiu discounted penalty function.

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Competing Interests

Authors have declared that no competing interests exist.

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