On Stable Pareto Laws in a Hierarchical Model of Economy

A. M. Chebotarev

Quantum Statistics Department, Moscow State University,
Moscow 119899, Russia

Abstract

This study considers a model of the income distribution of agents whose pairwise interaction is asymmetric and price-invariant. Asymmetric transactions are typical for chain-trading groups who arrange their business such that commodities move from senior to junior partners and money moves in the opposite direction. The price-invariance of transactions means that the probability of a pairwise interaction is a function of the ratio of incomes, which is independent of the price scale or absolute income level. These two features characterize the hierarchical model. The income distribution in this class of models is a well-defined double-Pareto function, which possesses Pareto tails for the upper and lower incomes. For gross and net upper incomes, the model predicts definite values of the Pareto exponents, \( a_{\text{gross}} \) and \( a_{\text{net}} \), which are stable with respect to quantitative variation of the pair-interaction. The Pareto exponents are also stable with respect to the choice of a demand function within two classes of status-dependent behavior of agents: linear demand (\( a_{\text{gross}} = 1 \), \( a_{\text{net}} = 2 \)) and unlimited slowly varying demand (\( a_{\text{gross}} = a_{\text{net}} = 1 \)). For the sigmoidal demand that describes limited returns, \( a_{\text{gross}} = a_{\text{net}} = 1 + \alpha \), with some \( \alpha > 0 \) satisfying a transcendental equation. The low-income distribution may be singular or vanishing in the neighborhood of the minimal income; in any case, it is \( L_1 \)-integrable and its Pareto exponent is given explicitly.

The theory used in the present study is based on a simple balance equation and new results from multiplicative Markov chains and exponential moments of random geometric progressions.

1 Introduction

1.1 A brief historical survey

The orthodox interpretation of statistical distributions in economics assumes a certain stochastic process whose invariant probability measure characterizes the given distribution. Champernowne [1] made a remarkable attempt to model the Paretoian distribution, and obtained several discrete versions of the distribution as stationary probability

*This work is supported by DFG 436 RUS 113/779/0-1.
measures for a class of Markov chains on half-line. Starting from an empirical analysis of UK tax records for the period 1951–52, Champernowne reconstructed a (12 × 15)-approximation of the stochastic matrix of the Markov chain, and noted a definite “tendency for the lowest income to shift upwards” [1]. In a sense, his basic assumptions (asymmetry and homogeneity of the transition probability) are inherited by the model described in the present study. To ensure the probabilistic interpretation and to obtain a closed equation for the stationary distribution, Champernowne introduced a boundary condition for the occupation number of the lowest income range. This fare for mathematical correctness was not reasonably explained in economic terms, but a similar model of multiplicative stochastic process with “reset” events (i.e., with stochastic renewal of the process at the boundary) [2] was successfully applied in [3] for a Monte Carlo simulation of variations in income distribution within Japan. In the present paper, only the continuity assumption is imposed at the boundary (at infinity and at the point corresponding to the minimal income) that defines a stationary density distribution.

To fix notation, recall that the cumulative probability distribution $\Pr(x \geq s)$ of a random variable $x$ has the Pareto tail with exponent $a$ if $\Pr(x \geq s) = O(s^{-a})$, $a > 0$, for large $s$.

A number of empirical facts in economics concerning the Pareto distribution can be qualitatively explained using generalizations of the Yule stochastic process [4], which was proposed in 1937 as a simple dynamical model of biological taxa. In view of a certain similarity between the present model and Yule’s model, it is worth explaining the basics of Yule’s model in a macroeconomic context. Suppose that the starting capital equals 1 and a nonrandom consumer demand provides the business community with fixed $1 + m$ units of income per unit of time. Each step of the time-evolution consists of two events: (1) a new agent obtains the starting capital and joins the business community; (2) the remaining $m$ units of income are randomly distributed among the existing agents according to their status-demand, for example, with partial probabilities proportional to the existing distribution of capital (linear demand). The construction of a stationary solution in a biological context is discussed in [5]. The cumulative stationary distribution has the power tail with the following exponent:

$$a = 1 + \alpha, \quad \alpha = \frac{1}{m}.$$ 

If the business society only rarely recruits new agents, $\alpha$ is small and $a \approx 1^2$.

In addition to linear demand ($\ell$-demand), the class of slowly varying functions is considered. From an empirical viewpoint, in the linear case the size of a business is far from market limitations or the constraints of resources. The slowly varying demand ($u$-demand) carries features of utility and sigmoidal functions that describe the diminishing returns of investments and the effects of market or production constraints.

---

1In fact, the conventional terminology is not completely correct: Zipf constructed his plots with the probability axis $y$, while Pareto used $x$ for the same purpose [5].

2Let $n$ be the number of time units per year. Then, $nm \approx n(1 + m) = \sigma$ is the size of the market and $\nu = n$ is the intensity of the creation of vacancies. If $a \geq 1$, it is considered that $\alpha = a - 1$ is the horizon of the power tail, as the heaviest tail with $a = 1$ normally overshadows the remaining components of a sampling. In these terms, Yule’s law states: “Horizon × market size ∼ intensity of vacancies’ creation”.

---
Empirical studies of the top income distribution show that the Pareto exponent \( a \) of the cumulative distribution normally belongs in the interval \([1, 3]\). There is evidence for both stability \([6]–[10]\) and instability \([11, 12]\) for this indicator over time. In the latter two reviews, the authors discuss the relationship between tax legislation and the distribution of personal income. The reviews present variation of the Pareto exponent in the range \([1.3, 3.5]\), based on tax records from Great Britain and the Netherlands for the years 1907–1999, and records from the US, Canada, and France over shorter periods. The detailed records enable us to monitor details of income distribution for the top 0.05% to 10% of earners. The Pareto index \( a = 1 \) for the net income of firms and corporations has been observed by many authors (see \([9, 17, 18]\)), including the author of the present paper \([19, 20]\). These tails are the least informative in the sense that they overshadow the possibly important details of less heavy tails that contribute to the general picture.

A solvable model for the Pareto income distribution was recently proposed by Reed \([13]\). In fact, Reed’s approach is based on the Poincare ergodic conjecture, which states that the time-average coincides with the average over the ensemble. The ensemble considered by both Reed and Gabaix \([14]\) consists of noninteracting agents whose income is described by geometric Brownian motion and whose life-time is an exponentially distributed random variable. By averaging the individual income distribution over the random life-time, the double-Pareto distribution \([13]\) is derived\(^3\). According to the Poincare conjecture, the obtained distribution describes the income distribution over the ensemble of ideal (i.e., noninteracting) agents.

The present paper considers an ensemble of asymmetrically interacting agents. The balance equation that describes the stationary distribution of income is derived using arguments that are similar to those used by Bouchaud \([15]\) for symmetric pair-interactions. The present equation enables the calculation of the pair of Pareto exponents \( a \) and \( d > -1 \)

\(^3\)A correspondent informed me that in 1949, E. Fermi described a similar mechanism in an astrophysical context, but I have been unable to confirm this.
for the top and bottom income tails. The value of the Pareto exponent $a_{\text{net}} = 1$, derived below for hierarchical models with $u$-demand, is normal for the top revenues of firms and corporations in Japan (see [3, 6]) and is in agreement with estimates for Europe, Britain, and the USA (see [10, 12, 13, 16]). Indirect reconstruction of the gross income distribution for the top 5% of householders, based on sale statistics for the national car market and direct estimates of the capital distribution among the 200 largest banks in Russia, yields the same value [19, 20]. A possible qualitative explanation for this rather general phenomena is the existence of a dominating corporate subculture of profit-making communities. The aim of this paper is to consider the extreme case of a corporate community: the hierarchical structure. A sketch of this approach was published in [21].

1.2 Assumptions and results

In the business world, the hierarchical environment maintains and permanently reproduces the specific circulation of money and commodities; corporations, network-marketing and chain-trading structures, shadow economy groups, etc., aim to organize supply and demand such that commodities and money move in opposite directions. Within the hierarchy, an agent consumes the commodities distributed by senior partners, while at the same time, he distributes them to junior agents [22].

Clearly, the above picture is not strictly true for agents with low income who are receiving money (e.g., low wages or pensions, welfare subsidies or grants) from the state. To simplify mathematical considerations, taxation and the redistribution policy of the state is ignored, and a toy model is considered that describes the stable distribution of unilateral money flow in a hierarchical structure under the assumption that the intensity $R(s, x)$ of transactions between the pair of agents is an asymmetric and price-invariant function of incomes $s$ and $x$ of these agents; that is, $R(s, x) = R(s/x) \geq 0$ if $s \geq x$, and $R(s/x) = 0$ if $s < x$. These assumptions characterize this class of hierarchical models. We also assume that income is (mathematically) unlimited and that the minimal income $s_0$ is positive, i.e., $s \in [s_0, \infty)$, $s_0 > 0$.

The money flow controlled by an agent with net income $s$ consists of the welfare payment $P(s_0, s)$ (cash benefits, social grants, etc.) and the contribution $B(n, s)$ from subordinate agents. Here, $n = n(s) \in L_1(s_0, \infty)$ is the density of agents with net income $s$, and $B(n, s)$ is a function of $s$ and a functional of $n$. The hierarchy assumption means that $B(n, s)$ depends on the values of $n(x)$ restricted to $x \in [s_0, s]$, and $B(n, s) \to 0$ as $s \to s_0$. The welfare $P(s_0, s)$ decreases and vanishes for all sufficiently large $s$. The conjectured price-invariance of the model implies that $P(s_0, s) = P(s/s_0)$.

To maintain the social status, an agent with net income $s$ pays out $H = H(n, s)$ to senior partners. We assume that the hierarchy is free of competition; in other words, we conjecture that (at least in the neighborhood of the stable distribution) variations in the number of senior agents does not affect consumer demand and price levels. This assumption implies that the individual consumption costs $H = H(s)$ are independent of $n$. Thus, the net income is equal to

$$s = P(s/s_0) + B(n, s) - H(s),$$

and the sum $g(s) = P(s/s_0) + B(n, s) = s + H(s)$ is the gross income. Suppose that the
consumption costs $H(s)$ decrease as $s \downarrow s_0$. Hence, the gross income $g(s) = s + H(s)$ is an increasing function of $s$ and the equation

$$s_0 : s_0 + H(s_0) = P(1)$$

has a unique solution. Furthermore, (2) is regarded as an analytical definition of the minimal income $s_0$, while (1) is considered as an equation with respect to $n(s)$.

In Section 2, $B$ and $H$ are represented in terms of the pair-interaction intensity $R(x)$ and the demand function $\sigma(s)$. The main statement of this section is a conditional proposition: if the solution $n(s)$ of equation (1) can be represented in the form

$$n(s) = x^b (1 - x)^d \rho(x) / \sigma(x^{-1}), \quad x = s_0 / s \in (0, 1]$$

where $\rho$ is a continuous, strictly positive, and uniformly bounded function on $[0, 1]$, then

$$d = -1 + \left( \frac{R(1)}{C(1)} \right) > -1, \quad C(1) = 1 - P'(1) + 2R_0 > 0, \quad R_0 \overset{\text{def}}{=} \int_1^\infty R(x) \, dx. \quad (4)$$

In addition, it is demonstrated that $a_{\text{gross}} = 1$, $a_{\text{net}} = 2$ for the linear demand, $a_{\text{gross}} = a_{\text{net}} = 1$ for any slowly varying demand, and $a_{\text{gross}} = a_{\text{net}} = 1 + \alpha$ for the sigmoidal demand, where $\alpha$ is a unique solution of the transcendental equation

$$\alpha : \frac{1}{R_0} \int_1^\infty s^{\alpha} R(s) \, ds = 1 + \frac{1}{R_0 S_0}, \quad S_0 = \lim_{s \to \infty} \sigma(s).$$

Section 3 considers the results of numerical simulation of $\rho$ that illustrate various opportunities of the model. The existence of the income distribution $n(s)$ in the form (3) is proved in Section 4. $\rho$ is first represented in terms of the pair of multiplicative Markov chains, $\{\xi_n\}$ and $\{\eta_n\}$, that describe the function $\rho(s)$ either as $s \to \infty$ or $s \to s_0$. In Section 5, possible economical issues are discussed. Section 6 consists of a mathematical appendix that contains four basic lemmas on multiplicative Markov chains. It is anticipated that mathematical facts considered in the last section will be useful for the readers who use multiplicative Markov processes in econophysics and mathematical finance.

## 2 Balance equation and continuity conditions

### 2.1 Balance equation

Suppose that the money flow from agents with income $x$ to $n(s)$ agents with higher income $s$ is proportional to the density $n(x)$, consumer demand $\sigma(x)$, and the pair-potential $R(s/x)$ of interactions. As this income is shared among $n(s)$ senior agents, the personal inflow functional $B(n, s)$ can be written as follows:

$$B(n, s) = \frac{1}{n(s)} \int_{s_0}^s \sigma(x) R(s/x) n(x) \, dx.$$  

The given rational agent behaves in exactly the same way toward the senior partners as his junior partners: the cost of his demand to agents with higher income $x$ is proportional to the intensity of interactions and to his demand function $\sigma(s)$:

$$H(s) = \sigma(s) \int_s^\infty R(x/s) \, dx = s \sigma(s) R_0, \quad R_0 \overset{\text{def}}{=} \int_1^\infty R(y) \, dy.$$
Taking into account the welfare income $P$, the balance equation (1) is rewritten as follows:

$$ s = P(s/s_0) + \frac{1}{n(s)}\int_{s_0}^{s} \sigma(x) R(s/x) n(x) \, dx - \sigma(s) \int_{s}^{\infty} R(x/s) \, dx. $$

This relation is considered as an equation for the stationary distribution $n(s)$ of the net income $s$. The equation $g = s \left(1 + \sigma(s) R_0\right)$ is an obvious statement on the relation between the gross and net income, $g$ and $s$, which readily follows from (5).

For further use of the model, it is convenient to rewrite (5) as a linear homogeneous integral equation:

$$ \int_{s_0}^{s} \sigma(x) R(s/x) n(x) \, dx = \left(s - P(s) + s \sigma(s) R_0\right) n(s). $$

The choice of a smooth monotone demand $\sigma(s) \leq s$ is restricted to the following three classes:

$$(\ell): \sigma(s) = s \text{ for all } s > 1, \quad (s): \sigma(s) = s \text{ for } s = O(1); \sigma(s) \to S_0 \text{ as } s \to \infty,$$

$$(u): \sigma(s) = s \text{ for } s = O(1); \frac{\sigma(at)}{\sigma(s)} \to 1, \sigma(s) \to \infty \text{ as } s \to \infty.$$ 

The slowly varying functions $\sigma(s) = 1 + \ln s, s > 1$ (utility function), $\sigma(x) = S_0 s/(S_0 + s - 1)$, and $S_0 > 1$ (sigmoidal function) are examples of $u$- and $s$-demands. These functions describe various kinds of diminishing returns.

Because the contribution from junior partners, $B(n, s)$, vanishes as $s \downarrow s_0$, to ensure consistency in equations (5) and (6), equation (2) is rewritten as follows:

$$ s_0 - P(1) + s_0 \sigma(s_0) R_0 = 0 $$

and consider the equation to be the definition of $s_0$. Equations (6) and (7) represent the analytical setup of the model. The aim is to describe the asymptotic behavior of $n(s)$ as $s \to \infty$ and $s \to s_0$ and to prove the uniqueness of the solution. For simplicity of notation, the price scale is chosen such that $s_0 = 1$.

### 2.2 Consequences of the continuity conjecture

To study the problem on a compact set, the variable $s \in [1, \infty)$ is changed to $x = 1/s \in (0, 1]$ in (6) and (7). According to (7), the integral in the left-hand side of (6) vanishes at the point $s = x = 1$. Hence, the right-hand side of (6) can be rewritten as the following product:

$$ s - P(s) + s \sigma(s) R_0 = \sigma(s) (s - 1) C(s^{-1}) = x^{-1} \sigma(x^{-1})(1 - x) C(x), $$

where $\sigma(x^{-1}) = x^{-1}$ for $x = O(1)$.

As $s \to \infty$ and $x = 1/s \to 0$, the leading terms in the left- and right-hand sides of (8) are $x^{-1} \sigma(x^{-1}) C(0)$ and $x^{-1}(1 + \sigma(x^{-1}) R_0)$, respectively. Therefore, for $C(0) = R_0 + \sigma(\infty)^{-1}$, we have

$$ C(0) = \begin{cases} R_0 & \text{for } \ell- \text{ and } u-\text{demands}, \\ R_0 + S_0^{-1} & \text{for } s-\text{demand}, \end{cases} $$

6
because $\sigma(\infty) = \infty$ for $\ell$ and $u$-demands, and $\sigma(\infty) = S_0$ for $s$-demand. As $x \to 1$, we obtain

$$C(1) = 1 + 2R_0 - P'(1) \geq 1 + 2R_0 > 0, \quad (10)$$

where $P'(1) \leq 0$ because the welfare $P(s)$ is normally a decreasing (or finite) function. Therefore, the left-hand side of (8) is an increasing function that vanishes only at $s = 1$. For this reason, it is supposed that $c_\ast > 0$ exists such that

$$c(x) \overset{\text{def}}{=} C(x)/C(0) \geq c_\ast \ \forall x \in (0, 1].$$

Suppose that $P(x^{-1})$ vanishes for sufficiently small $x$, say for $x \in (0, x_s]$, and $\sigma(s) = s$ for $s \in [1, 1/x_s]$. It is clear that for $x \in (0, x_s]$, the assumption $\sigma(s) \leq s$ implies

$$\frac{(1 - x)C(x)}{R_0} = 1 + \frac{1}{\sigma(x^{-1})R_0} \geq \begin{cases} 1 + x/R_0 & \text{for } \ell- \text{ and } u\text{-demands}, \\
1 + 1/S_0R_0 & \text{for } s\text{-demand}.
\end{cases} \quad (11)$$

Let us search for a positive solution of the balance equation (6) in the factorized form

$$n(1/x) = x^b (1 - x)^d \rho(x)/\sigma(x^{-1}), \quad x \in (0, 1], \quad b > 1, \ d > -1. \quad (12)$$

As the normalization of $n$ is not essential, $\rho(1) = 1$ is used. Let us derive an equation for $\rho$ and calculate $a \text{ priori}$ values of $b$ and $d$. Set

$$r(s) \overset{\text{def}}{=} R(s)/R_0, \quad m_0 \overset{\text{def}}{=} \int_1^\infty s^a r(s) \, ds = 1, \quad m_0 = 1 \quad (13)$$

and suppose that the probability density $r(x)$ is a bounded continuous function, $r(1) > 0$.

**Theorem 1** The relation $d = R(1)/C(1) - 1$ is necessary for the continuity of a strictly positive bounded function $\rho(x)$ at the point $x = 1$ for all continuous demand functions such that $\sigma(1) = 1$. Continuity at the point $x = 0$ implies $b_\ell = 3, b_u = 2,$ and $b_s > 2$ for $s$-demand; more precisely, $b_s$ satisfies the transcendental equation

$$b_s : \int_1^\infty s^{b_s - 2} r(s) \, ds = 1 + \frac{1}{S_0 R_0}. \quad (14)$$

**Proof.** According to equations (8) and (12), rewrite equation (6) in the variables $x = 1/s$, $y = 1/z$:

$$n(s) \left( s - P(s) + s \sigma(s) R_0 \right) = R_0 x^{b-1} (1 - x)^{d+1} c(x) \rho(x)$$

$$= \int_1^s \sigma(y) R(s/y) n(y) \, dy = \int_1^1 \sigma(z^{-1}) z^{-2} R(sz) n(z^{-1}) \, dz$$

$$= \int_x^1 z^{b-2} (1 - z)^d R(z/x) \rho(z) \, dz.$$
This integral equation for $\rho(x)$ can be represented in two equivalent forms which are independent of $\sigma$:

$$
\rho(x) = \frac{1}{x(1-x)C(x)} \int_{x}^{1} \left( \frac{z}{x} \right)^{b-2} \left( \frac{1-z}{1-x} \right)^{d} R(z/x) \rho(z) \, dz 
$$

(15)

$$
= \frac{1}{(1-x)C(x)} \int_{1/x}^{1} y^{b-2} \left( \frac{1-xy}{1-x} \right)^{d} R(y) \rho(xy) \, dy.
$$

(16)

If $d > -1$ and $x > 0$ is fixed, the measure $\mu(x, dz) = (d+1)(1-z)^d/(1-x)^{d+1} \, dz$ is a probability measure on $(x, 1]$. If $\rho(x)$ is a bounded measurable function that is continuous at the point $x = 1$, so is the product

$$
p(x, z) = (z/x)^{b-2} R(z/x) \rho(z), \quad z \in (x, 1)
$$

and $\lim p(x, z) = \lim p(x, z) = R(1) \rho(1)$ as $x \to 1$, $z \in (x, 1]$. Denote with $\mathbb{E}_x$ the mathematical expectation with respect to the probability measure $\mu(x, dz)$. Hence,

$$
\lim \mathbb{E}_x p(x, \cdot) = \lim \mathbb{E}_x p(x, \cdot) = R(1) \rho(1).
$$

Multiplying (15) by $d + 1$, we obtain the condition that is necessary for the continuity of $\rho$:

$$(d+1) \rho(1) = \lim_{x \to 1} \mathbb{E}_x p(x, \cdot)/(x C(x)) = R(1) \rho(1)/C(1).$$

(17)

If $\rho(1) \neq 0$, this equation yields (4): $d = R(1)/C(1) - 1 = r(1)/c(1) - 1 > -1$.

As $x \to 0$, the left-hand side of (16) clearly converges to $\rho(0)$; the Lebesgue theorem on dominated convergence implies that the right-hand side of equation (16) converges to

$$
\frac{\rho(0)}{C(0)} m_{b-2} = \rho(0), \quad m_{\alpha} \triangleq \int_{1}^{\infty} s^{\alpha} r(s) \, ds.
$$

As $C(0)/R_0 = 1$ for $\ell$- and $u$-demands, and $C(0)/R_0 = 1 + 1/S_0 R_0$ for $s$-demand, the necessary continuity condition at $x = 0$ is the equation with respect to $b$

$$
m_{b-2} = \begin{cases} 
1 & \text{for } \ell \text{- and } u \text{-demands}, \\
1 + 1/R_0 S_0 & \text{for } s \text{-demand}
\end{cases}
$$

(18)

provided $\rho(0) \neq 0$ (see (9) and (13)). The moment $m_{\alpha}$ is an increasing function of $a$ because $r$ is a regular probability density on $[1, \infty)$; hence, equations (14) or (18), in each case, have a unique solution. \hfill \Box

Remark 1. As $\sigma((xy)^{-1})/\sigma(x^{-1}) \to 1$ as $x \to 0$, this result remains valid if $\rho(x) = g > 0$. This will be the case for $u$-demand. It is also clear that the Pareto exponent for $n(s)O(\sigma(x^{-1})^{-g})$ is the same as that for $n(s)$.

Remark 2. If $\delta \triangleq 1/S_0 R_0$ is a small constant and the measure $r(s) \, ds$ has at least one moment, the equation $m_{\alpha} = 1 + \delta$ can be approximately solved:

$$
\alpha = \frac{\delta}{\int_{1}^{\infty} r(s) \ln s \, ds} + O(\delta^2), \quad \delta = 1/S_0 R_0,
$$
i.e., $a = 1 + \alpha$, similar to the Yule case\textsuperscript{5}.

Thus, it is demonstrated that the continuity and strict positivity of $n(s)$ implies the expansion:

$$u(x^{-1}) = x^b(1 - x)^d \rho(x)/\sigma(x^{-1}), \quad b - 1 = a_{\text{net}} = \begin{cases} 1 & \text{for } \ell- \text{ and } u\text{-demands}, \\ 1 + \alpha & \text{for } s\text{-demand}. \end{cases}$$

In addition, for all kinds of demands, equation (15) can finally be rewritten as follows:

$$\rho(x) = \frac{1}{x(1 - x)C(x)} \int_x^1 \left( \frac{z}{x} \right)^{\alpha} \left( \frac{1 - z}{1 - x} \right)^d R(z/x) \rho(z) dz,$$

(19)

where $\alpha = 0$ for $\ell$- and $u$-demands, and $C$ is the only function that is dependent on $\sigma$ (see equation (8)).

\subsection*{2.3 Pareto exponents for gross income}

The relationship $g = s + \sigma(s) R_0$ between gross and net income allows one to calculate the Pareto exponents for the gross income distribution. First consider the $\ell$-demand. Denote the unique positive root of the equation $g = s + s^2 R_0$ with $s(g)$, where $g$ is gross income. The density distribution of $g$ then equals $N_\ell(g)$:

$$N_\ell(g) = n_\ell(s(g)) s'(g), \quad s(g) = 1/\sqrt{1 + 4R_0 g} = O(g^{1/2}),$$

where $n_\ell(s) = O(s^{-3})$, $n_\ell(s(g)) = O(g^{-3/2})$, and $s'(g) = O(g^{-1/2})$. Finally, we obtain $N_\ell(g) = O(g^{-2})$, i.e., the Pareto exponent of the gross income $d_{\text{gross}} = a_{\text{net}} - 1 = 1$.

Now consider the $u$-demand. Because $\sigma(s)$ is a slowly varying and increasing function, there exists a finite constant $\sigma_\varepsilon$ such that $\sigma(s) \leq \sigma_\varepsilon s^\varepsilon$ for any $\varepsilon > 0$. Now denote the unique positive root of the equation $g = s (1 + \sigma(s) R_0)$ with $s(g)$, and let $N_u(g)$ be the probability density of $g$. The monotonicity of $\sigma(s)$ implies the following inequalities:

$$g \geq s(g) = \frac{g}{1 + \sigma(s(g)) R_0} \geq \frac{g}{1 + \sigma(g) R_0} \geq \frac{g}{1 + \sigma_\varepsilon g^\varepsilon R_0}. \quad (20)$$

By definition of the cumulative probability, we have

$$\int_y^\infty N_u(g) dg = \int_y^\infty n_u(s(g)) s'(g) dg = \int_{s(g)}^\infty n_u(g) dg.$$

As $n_u(g) = O(g^{-2})$, the above inequality (20) yields the two-sided estimates for the cumulative probability $P_u\{g \geq y\}$:

$$O(y^{-1}) = \int_y^\infty n_u(g) dg \leq \int_{s(y)}^\infty n_u(g) dg = \int_y^\infty N_u(g) dg = Z^{-1} P_u\{g \geq y\} \leq \int_{y/(1 + \sigma_\varepsilon g^\varepsilon R_0)}^\infty n_u(g) dg = y^{-1} O(1 + \sigma_\varepsilon g^\varepsilon R_0),$$

The constant $R_0$ characterizes an intensity of pairwise interactions. In this context, $1/R_0$ is the average time between interactions. On the other hand, $S_0$ can be interpreted as the volume of a “market”. The difference $a - 1$ between the Pareto exponent $a$ ($a > 1$) and the exponent of the “heaviest” tail is termed the horizon of the distribution. In these terms, the Pareto exponent that corresponds to the sigmoidal demand satisfies the following law: “Horizon $\times$ market volume $\sim$ expectation-of-transaction time.”
where \( Z \) is the normalizing constant and \( \varepsilon \) is arbitrarily small. Hence, passing to the limit as \( y \to \infty, \varepsilon \to 0 \), we have

\[
a_{\text{gross}} = -\lim \ln \left( \mathbb{P}\{g \geq y\} \right)/\ln y = -\lim \ln \left( \mathbb{P}\{g \geq y\} \right)/\ln y = 1 = a_{\text{net}}.
\]

For \( s \)-demand, gross and net income have the same order: \( g(s) = s(1 + \sigma(s) R_0) = s + O(s) \), as \( \sigma(s) \) is bounded by \( S_0 \). Hence,

\[
a_{\text{net}} = a_{\text{gross}} = 1 + \alpha, \quad \alpha : 1 + \frac{1}{\Pi_0 R_0} = \int_1^{\infty} s^\alpha r(s) \, ds.
\]

The above estimates of \( b \) and \( d \) define meaningful asymptotical properties of \( n(s) \) provided that \( \rho(x) \) is bounded from above and below. The proof of this fact is a nontrivial mathematical problem that will be solved via a probabilistic representation of the solution of equations (6) and (7); however, first the results of an empirical study of this problem are considered.

### 3 Numerical simulation of \( \rho \)

With respect to the analytical proof of the boundedness and the strict positivity of \( \rho(x) \) etc., a computer simulation is an easier approach that convinced us that equation (19) is consistent. Although the operator \( A : C[0, 1] \to C[0, 1] \)

\[
(A\rho)(x) = \frac{1}{x(1-x)c(x)} \int_x^1 \left( \frac{z}{x} \right)^\alpha \left( \frac{1-z}{1-x} \right)^{r(1)/c(1)-1} r\left( \frac{z}{x} \right) \rho(z) \, dz
\]

is not a contraction in the uniform topology\(^6\), the iteration procedure \( \rho_n = A\rho_{n-1} \) converges to a smooth and strictly positive solution \( \rho(x) \) for all tested probability distributions \( r(s) \), initial approximations \( \rho_0 \), and values of \( d \in (-1, \infty) \). Figures 2-4 show the numerical solutions \( \rho(x) \) and \( n(s) \) of equation (19) for the pair-interaction \( R(s) = \lambda e^{-\lambda(s-1)} \) and “earning” \( P(s) = 2/(1 + (s - 1)^3) \); black, red and blue curves correspond to \( d = 0 \), \( d > 0 \), and \( d < 0 \), respectively.

It is worthwhile noting that the solution for equation (19) can be calculated on any interval \( (y, 1] \subset (0, 1) \). This means that problem (6) remains well-posed on any finite interval \([s_{\text{min}}, s_{\text{max}}]\) and \( n(s_{\text{max}}) = O(s_{\text{max}}^{-b}) \).

It was observed that the Fourier or Chebyshev polynomial approximation of the solution \( \rho(x) \) does not provide a fast convergence. By differentiating equation (19), one can derive a pair of homogeneous boundary conditions of the form \( \rho'(x) = \gamma_x \rho(x), \, x = 0, 1 \). Note that the standard orthogonal polynomials do not fulfill the boundary conditions of this type; this is a possible explanation of the inefficiency of the polynomial approximation. Nevertheless, the finite dimensional approximation of operator (22) in the Fourier basis shows that the spectrum \( \{\lambda_k\} \) of \( I - A_n \) is distributed more or less uniformly over a smooth “semicircle” curve \( \Re \lambda_k \in [-1, 0], \, \Im \lambda_k \in [0, 1/2] \) (see Fig. 5).

\(^6\)The estimate given in Remark 3 implies that \( ||A||_C \leq F. \Pi_1 \Pi_2 < \infty \), and the numerical spectral estimates show that \( ||A||_C \geq 1 \).
Figure 2: For $\ell$-demand, we set $\sigma(s) = s$ and $n(s) = n_0 s^{-3}(1 - s)^d \rho(1/s)$; more precisely, $\lambda = 3$ for $d = 0$ (black curve), $\lambda = 4$ for $d = 1/3$ (red curve), $\lambda = 2$ for $d = -1/2$ (blue curve).

Figure 3: In order to illustrate $\rho$ and $n$ for $\sigma$-demand, we set $\sigma(s) = 20/(1 + 19 s^{-2})$. Solving equation (14), we find $b = 2.22741$ for $d = 0$ and $\lambda = 3.9$, $b = 2.25672$ for $d = 1/3$ and $\lambda = 4$, $b = 2.13304$ for $d = -19/39$ and $\lambda = 2$, so that $n(s) = n_0 s^{-b}(1 - s)^d \rho(1/s)/\sigma(s)$.

Figure 4: For $u$-demand, we have $\sigma(s) = 1 + \log s$ and $n(s) = n_0 s^{-2}(1 - s)^d \rho(1/s)/\sigma(s)$. Simple computation implies $\lambda = 3$ for $d = 0$, $\lambda = 4$ for $d = 0.3$, and $\lambda = 2$ for $d = -1/3$. 

11
Figure 5: The spectrums of the operator $I - A_n$ in $\ell$-cases for $d = 0$ and $d = -0.5$, in the complex plain for $n = 100$. There is no reason to expect the existence of a spectral gap; it is most likely that the limit spectrum is continuous.

The spectrum has no spectral gap near the minimal eigenvalue $\lambda_0 \approx 0$, however, in a finite dimensional approximation at least, the minimal eigenvalue is simple. Thus, the spectral approximation provides no helpful hints as to the proof of the uniqueness theorem based on the existence of a spectral gap. In contrast, Fig. 4 shows that all eigenvalues $\lambda$, such that $|1 - \lambda| > 1$, are complex; the corresponding eigenfunctions must also be complex. This observation illustrates and extends the statement which will be proved in Theorem 3 below: a positive bounded solution of equation (6) is unique, while all other solutions are complex. This statement is a kind of Perron-Frobenius theorem.

Now we must pay attention to the interesting mathematical background to the problem (6)-(7). The consideration below is self-consistent and does not refer the reader to mathematical literature.

4 Markov chains and a priori bounds for $\rho(x)$

4.1 Transition probabilities near 0 and 1

Consider the Markov chain $\{x_n\}$ on $(0, 1]$ with transition probability $P(x, A)$ from $x$ to $A \subset [x, 1]$ given by a pair of measures describing the process in the neighborhoods of the points 0 and 1:

$$P(x, A) = \begin{cases} \frac{1}{xZ_\alpha(x)} \int_A (y/x)^\alpha r(y/x) \, dy, & \text{if } x \in (0, x_*], \\ -\int_A d \left(\frac{1-y}{1-x}\right)^{d+1}, & \text{if } x \in (x_*, 1], \end{cases}$$

(23)

where $\alpha = 0$ for $\ell$- and $u$-demands; for $s$-demand, $\alpha$ satisfies equation (21); and $Z_\alpha$ is a local normalizing factor:

$$Z_\alpha(x) \overset{\text{def}}{=} \int_{[1, 1/x]} s^\alpha r(s) \, ds \leq 1.$$  

(24)

Note that $Z_0(x) \leq 1$ and $Z_\alpha(x) \leq 1 + 1/S_0 R_0$ (see (11)), but $Z_\alpha(x)/(1 - x)c(x) \leq 1$ in any case on $(0, x_*]$. 

12
The associated function $F(x, y)$ on the set $x, y : 1 > y \geq x$

$$F(x, y) = \begin{cases} \frac{Z_s(x)}{(1-y)e(x)} \left(\frac{1-y}{1-x}\right)^d, & \text{if } x \in [0, x_*], \\ \frac{1}{x} \left(\frac{r(y/x)}{r(1)}\right)^\alpha, & \text{if } x \in (x_*, 1], \end{cases}$$

is defined such that equation (19) is equivalent to the integral equation

$$\rho(x) = \mathbb{E}_x F(x, x_1) \rho(x_1), \quad x \in (0, 1]. \tag{25}$$

As the mathematical expectation $\mathbb{E}_x$, $x \in (0, 1]$, is the well-defined integral operator, any measurable bounded solution $\rho(x)$ is absolutely continuous; moreover, equation (25) can be rewritten as

$$\rho(x) = \mathbb{E}_x \prod_{k=1}^n F(x_{k-1}, x_k) \rho(x_n).$$

According to Lemma 1 from the Appendix, the Markov chain exits any interval $(x, y) \subset (0, 1]$ with probability one. Hence, if the solution $\rho$ is bounded, it is continuous and $\rho(x_n) \to \rho(1) = 1$ as $n \to \infty$ with probability one; therefore,

$$\rho(x) = \mathbb{E}_x \prod_{n=1}^\infty F(x_{n-1}, x_n), \quad x_0 = x, \tag{26}$$

where $\{x_n\}_1^\infty$ is the monotone discrete-time Markov chain to be defined by (23).

Consider a priori bounds for mathematical expectation (26). For simplicity, it is supposed that the probability density $r(s)$ is uniformly bounded. Then $x \leq y$ on the domain of $F(x, y)$. It is clear that $F(x, y) \to 1$, if either $y \to 0$ or $x \to 1$. Set

$$F_* = \max_{0 \leq x \leq x_* \leq y \leq 1} F(x, y)$$

and consider the integer-valued exit time $n_* = n(x, x_*) \geq 0$ from the segment $(0, x_*]$ for the Markov chain $\{x_n\}_{n=0}^{x_*}$ starting at $x_0 = x$: $x_{n_*-1} \leq x_* < x_{n_*}$. We have

$$\prod_{n=1}^\infty F(x_{n-1}, x_n) = F(x_{n_*-1}, x_{n_*}) \prod_{n=1}^{n_*-1} F(x_{n-1}, x_n) \prod_{m=1}^\infty F(y_{m-1}, y_m)$$

$$\leq F_* \prod_{n=1}^{n_*-1} F(x_{n-1}, x_n) \prod_{m=1}^\infty F(y_{m-1}, y_m),$$

where $\{y_m\}$ is the Markov chain starting at the point $y = y_0 = x_{n_*} \in [x_*, 1]$. Thus,

$$\mathbb{E}_x \prod_{n=1}^\infty F(x_{n-1}, x_n) \leq F_* \mathbb{E}_x \prod_{n=1}^{n_*-1} F(x_{n-1}, x_n) \sup_{y \geq x_*} \mathbb{E}_y \prod_{m=1}^\infty F(y_{m-1}, y_m).$$

Therefore, it suffices to prove the uniform boundedness of $\Pi_1(x) = \mathbb{E}_x \prod_n F(x_{n-1}, x_n)$ for $x_k \in (0, x_*]$ and $\Pi_2(x) = \mathbb{E}_x \prod_m F(y_{n-1}, y_n)$ for $y_m \in [x_*, 1]$. 

13
4.2 Upper bound and the uniqueness theorem

Consider the uniform upper bound for $\Pi_1$. According to (11) and (24),

$$Z_\alpha(x)/(1 - x)c(x) \leq 1 \quad \forall x \leq x_*.$$

The Markov chain $\{x_n\}$ increases monotonically and exits any interval $(x, y) \subset (0, x_*)$ with probability one (see Appendix, Lemma 1). Thus, we have

$$\Pi_1(x) = \mathbb{E}_x \prod_{n=1}^{n-1} F(x_{n-1}, x_n) \leq \left( \frac{1 - x_{n-1}}{1 - x} \right)^d \leq \begin{cases} 1, & \text{if } d \geq 0, \\ (1 - x_*)^d, & \text{if } d < 0 \end{cases} < \infty.$$

Consider the upper bound for $\Pi_2$. Note that $F(1, 1) = 1$, and suppose that

$$F^{*} \overset{\text{def}}{=} \max_{x \geq x_*, y \in [x, 1]} \frac{1}{1 - x} \left| 1 - F(x, y) \right| = \max_{x \geq x_*, y \in [x, 1]} \frac{1}{1 - x} \left| 1 - \frac{c(1)}{x} \frac{r(y/x)}{r(1)} \left( \frac{y}{x} \right)^\alpha \right| < \infty. \quad (27)$$

For example, if $R(s)$ is a monotone function and $\alpha = 0$, then $r(y/x) \leq r(1)$, and

$$F^{*} \leq \frac{1}{x_* c_*} \left( c(1) + \max_{x \in [x_*, 1]} \frac{|c(1) - c(x)|}{1 - x} \right), \quad (28)$$

where $c_*$ is $\min c(x)$. In any event, $F(y_{n-1}, y_n) \leq 1 + (1 - y_{n-1}) F^*$, and the estimate

$$\Pi_2(y) = \mathbb{E}_y \prod_{m=1}^{\infty} F(y_{m-1}, y_m) \leq \mathbb{E}_y \exp \{ F^* \sum_{m=1}^{\infty} (1 - y_m) \}, \quad y_0 = y$$

can be rewritten in terms of the independent variables $\xi_n \overset{\text{def}}{=} (1 - y_n)/(1 - y_{n-1}) \in (0, 1]$ as the sum of a stochastic geometrical progression:

$$\sum_{m=0}^{\infty} (1 - y_m) = (1 - y) \left( 1 + \sum_{m=1}^{\infty} \prod_{k=1}^{m} \xi_n \right), \quad \xi_m \in (0, 1].$$

**Theorem 2** Let $F^*$ be defined by (27). Then, $\rho(x)$ is a bounded continuous function if

$$d + 1 < F^* \frac{2 - e^{F^*}}{e^{F^*} - 1 - F^*}, \quad F^* < \ln 2. \quad (29)$$

**Proof.** As proved above, $\sup_{x \in [0, x_*]} \Pi_1(x) < \infty$ and $F_* < \infty$ under rather general assumptions. To prove that $\sup_{y \in (x_*, 1)} \Pi_2(y) < \infty$, it suffices to ensure the finiteness of the exponential moment

$$\Pi_2(y) = \mathbb{E}_y e^{(1-y)F^* (1+G)} < \infty, \quad G = \sum_{m=1}^{\infty} \prod_{k=1}^{m} \eta_n. \quad (30)$$
According to (23), the factors $\eta_n \in [0, 1]$ of the random geometrical progression (30) are independent random variables with the following probability distribution:

$$ P(\eta_m \in B) = \gamma \int_B \eta^d \, d\eta, \quad \gamma = d + 1 > 0. $$

(31)

Note that $\mu_n = \gamma/(\gamma + n)$ for random variable (31), and that according to (36) from Lemma 2 (see Appendix), the exponential moment of the geometric progression (30) is finite if

$$ \sup_{k \geq 1} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{1}{1 - \mu_{n+k}} = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \left(1 + \frac{\gamma}{n+1}\right) \leq e^\lambda - 1 + \frac{1}{\lambda^\gamma} \left(e^\lambda - 1 - \lambda\right) < 1 $$

for $\lambda = F^*$. This inequality can be resolved with respect to $\gamma$: $\gamma < \lambda (2 - e^\lambda)/(e^\lambda - 1 - \lambda)$. Integral equation (25) implies the absolute continuity of a bounded solution $\rho$. \[\square\]

**Remark 3** Condition (29) is the most restrictive assumption. For example, as $R(s)$ is a monotone function in cases (i–iii) considered in Section 3, estimate (28) can be used to evaluate $F_* = 20$. The convergence still holds despite the fact that the sufficient condition $F_* \leq \ln 2 \approx 0.693$ (see (29)) is crudely violated, i.e., the sufficient condition is rather far from an unknown necessary criteria. It appears that assumption (29) can be relaxed after a more detailed study of exponential moments.

A remarkable fact to arise from this analysis is that a solution, bounded from above, is unique, independently of the lower bound.

**Theorem 3** The class of strictly positive bounded continuous solutions of equation (15) consists of the unique function.

**Proof.** Suppose that there exists two nonnegative bounded continuous solutions, $\rho_1(x)$ and $\rho_2(x)$, that satisfy the condition $\rho(1) = 1$. The continuity assumption implies that $d_1 = d_2$ and $b_1 = b_2$ for $n_1(s)$ and $n_2(s)$, respectively.

Consider the bounded and uniformly continuous function $\Delta(x) = |\rho_1(x) - \rho_2(x)|$ on $[0, 1]$. It is clear that

$$ \max_{x \in [x, y]} P(x, y) = - \min_{x \in [x, y]} \int_x^y d \left(\frac{1-y}{1-x}\right)^{d+1} = 1 - \left(\frac{1-y}{1-x}\right)^{d+1} < 1 $$

majorizes the probability for the Markov chain to remain in the segment $[x, y]$, $y < 1$, $x \in [x, y]$ after one step, and hence the chain exits any segment $[x, y]$ with a probability of one (see also Lemma 1 in Appendix). In contrast, $\mathbb{E}_x \prod_{n=1}^{\infty} F(x_{n-1}, x_n) \leq \Pi_2(x) < \infty$, where $\Pi_2(x)$ is uniformly bounded in $[x_*, 1)$ (see Theorem 2). Equation (25) implies that

$$ \Delta(x) \leq \mathbb{E}_x \prod_{n=1}^{n_y} F(x_{n-1}, x_n) \max_{y \in [x_{n_y}, 1]} \Delta(y) \leq \Pi_2(x) \max_{\xi \in [y, 1]} \Delta(\xi), $$

where $n_y$ is the exit time from $[x_*, y]$: $x_{n_y-1} < y \leq x_{n_y}$. As $y$ can be arbitrarily chosen to be close to 1 and

$$ \lim_{y \to 1 \xi \in [y, 1]} \max_{\xi \in [y, 1]} \Delta(\xi) = 0, $$

(because of the uniform continuity) the previous estimate implies that $\Delta(x) = 0$ for all $x \geq x_*$. The uniform boundedness of $\Pi_1(y)$, $y \in (0, x_*]$ enables us to apply similar arguments to prove that $\Delta(y) = 0$, $y \in (0, x_*]$. \[\square\]
4.3 Lower bounds

This subsection provides a proof for the existence of a strictly positive lower bound for \( \rho(x) \) in the \( \ell \)- and \( s \)- cases. For \( u \)-demand, in the particular case \( \sigma(s) = 1 + \ln s \), a weaker estimate of \( \rho(x) = O((\ln x^{-1})^{-g}) \), \( g > 0 \) will be established in the neighborhood of the origin (see Lemma 4 in Appendix). This estimate does not change the value of the Pareto exponent, and because of Remark 1, the necessary continuity condition \( b_u = 2 \) also remains valid.

As \( \rho(x) \) is continuous and \( \rho(1) = 1 \), it follows that \( \rho \) is strictly positive on a certain segment \([x_*, 1]\), i.e., there exists \( \rho_* \) such that

\[
\min_{x \in [x_*, 1]} \rho(x) \geq \rho_* > 0.
\]

For simplicity, suppose that the point \( x_* \) is the same as that in equations (11) and (23). Let the \( s \)-demand be given by a smooth increasing bounded function \( \sigma(s) \) and that the tail of the probability distribution \( r(s) \) is not too heavy, i.e., there exist \( \mu \) such that

\[
\frac{Z_\alpha(x)}{(1 - x) c(x)} = \frac{1}{(1 - x) c(x)} \int_1^{1/x} s^\alpha r(s) ds \geq \frac{1}{1 + \mu x} \quad \forall x \in (0, x_*]. \tag{32}
\]

For \( \ell \)- and \( s \)-demands, this estimate holds for \( r(s) \) with any tail dominated by \( O(s^{-\alpha-2}) \). Under the same assumption on \( r(s) \), for \( u \)-demand we have

\[
\frac{Z_\alpha(x)}{(1 - x) c(x)} \geq \frac{1}{1 + \mu x} \left(1 + \frac{1}{R_0 \sigma(x^{-1})}\right)^{-1}. \tag{33}
\]

In the following theorem, the lower bounds are established for \( \rho \) under assumptions (32)–(33) for any demand, but in \( u \)-case, as a typical demand function \( \sigma(s) = 1 + \ln s \) will be considered. The proof remains valid for a wider class of slowly varying functions that inherit the property of the logarithm \( \sigma(\prod s_k) \leq C \sum \sigma(s_k) \), with a constant \( C \) that is independent of the number of factors \( \{s_k\} \).

**Theorem 4** For \( \ell \)- and \( s \)-demands, \( \rho(x) \) is strictly positive, while for \( u \)-demand, \( \rho(x) \) is bounded below by \( 1/f(x^{-1}) \), where \( f(s) \) a slowly varying function. More precisely, \( f(s) = (\ln s)^g \) and

\[
\rho(x) \geq \rho_* e^{-C (\ln x^{-1})^{-g}}
\]

with positive constants \( C \) and \( g \) that are specified in the proof.

**Proof.** The proof does not differ for \( \ell \)- and \( s \)-cases. Lemma 1 implies the a.s. existence of a finite exit time \( n_* = n_*(x, x_*) \) from the segment \((x, x_*)\) for the Markov chain (23) \( \{x_n\} \) that starts at the point \( x_0 = x : x_{n_*-1} \leq x_* < x_{n_*} \). Suppose that \( d \leq 0 \). Hence, we have

\[
\rho(x) = \mathbb{E}_x \prod_{n=1}^{n_*} F(x_{n-1}, x_n) = \mathbb{E}_x \mathbb{E}_{x_{n_*}} \prod_{n=1} F(x_{n-1}, x_n)
\]

\[
= \mathbb{E}_x \prod_{n=1}^{n_*} F(x_{n-1}, x_n) \mathbb{E}_{x_{n_*}} F(x_{n_*-1}, x_{n_*}) \rho(x_{n_*}) \geq \rho_* \mathbb{E}_x \prod_{n=0}^{n_*-1} \frac{1}{1 + \mu x_n}.
\]
Taking into account the inequalities $\mathbb{E} e^\xi \geq e^{E \xi}$ and $\ln(1+x) \leq x$, we obtain the estimate

$$\rho(x) \geq \rho_* \exp\left\{-\mu \mathbb{E}_x \sum_{n=0}^{x_*} x_n \right\}, \quad x \in (0, x_*],$$

(34)

where the points $x_n$ can be expressed as the product of factors $\xi_k = x_{k+1}/x_k \in (1, \infty)$, whose distribution is given by the first equation (23):

$$x_n = x \prod_{k=1}^{n} \frac{x_k}{x_{k-1}} = x_{n-1} \prod_{k=n-1}^{n-1} \left(\frac{x_k}{x_{k-1}}\right)^{-1} \leq x_* \prod_{k=n-1}^{n-1} \xi_k^{-1},$$

$$\mathbb{P}_x\{\xi > 1/x\} = \frac{1}{Z_\alpha(x)} \int_{1/x}^{1} \xi^\alpha r(\xi) d\xi.$$

Note that the function $\mathbb{E}_x \xi^{-1}$ increases in $x \in (0, x_*]$, i.e., $\mathbb{E}_x \xi^{-1} \leq \mathbb{E}_x, \xi^{-1} = \epsilon_* < 1$, and the Markov property implies the estimate

$$\mathbb{E}_x x_n = \mathbb{E}_x x_{n-1} \prod_{k=n-1}^{n-1} \mathbb{E}_x \xi_k^{-1} \leq x_* \sum_{n} \epsilon_n^\alpha \leq (1 - \epsilon_*), \quad x_* < 1.$$

Hence, the series $\sum_n x_n$ converges uniformly in $x$; this fact implies the strict positivity of $\rho$: $\rho(x) \geq \rho_* \exp\{-\mu/(1 - \epsilon_*)\} > 0$, where $\mu$ is defined by (32).

Consider the case $d > 0$. By the Markov property, we have

$$\mathbb{E}_x \left(\frac{1-x_{n+1}}{1-x}\right)^d \prod_{n=0}^{n-1} \frac{1}{1+\mu x_n} = \mathbb{E}_x \prod_{n=0}^{n-1} \frac{1}{1+\mu x_n} \mathbb{E}_x x_{n-1} \left(\frac{1-x_{n+1}}{1-x}\right)^d \geq \inf_{x_m \lesssim x_*} \mathbb{E}_x (1-x_{m+1})^d \prod_{n=0}^{n-1} \frac{1}{1+\mu x_n},$$

(35)

where the expectation of the last product is finite (see the proof for $d \leq 0$) and the first infimum is strictly positive:

$$\inf_{x_m \lesssim x_*} \mathbb{E}_x (1-x_{m+1})^d = \inf_{x \lesssim x_*} \frac{1}{Z_\alpha(x)} \int_{1/x}^{1} s^\alpha r(s) (1 - sx)^d \, ds \geq \inf_{x \lesssim x_*} \frac{1}{Z_\alpha(x/2)} \frac{1}{2^d} \int_{1}^{1/2x} s^\alpha r(s) \, ds = \frac{Z_\alpha(2x_*)}{2^d Z_\alpha(x_*)} > 0$$

because the integral normalized by $Z_\alpha$ in the last line of the equation is a decreasing function of $x$.

Consider the $u$-case with $\sigma(s) = 1 + \ln s$. For simplicity, suppose that $d \leq 0$; in the case $d > 0$, the proof is similar to that in the previous paragraph. Under the assumption $r(s) \leq O(s^{-2})$, there exists $\mu > 0$ such that

$$\frac{Z_\alpha(x)}{(1-x)c(x)} \geq \frac{1}{1+\mu x} \frac{1}{1 + 1/R_0 \ln x^{-1}}$$
(see (33)). By Lemma 4, the exit time can be restricted by \( N(x, x) \) with a probability greater than or equal to \((2Y^*(x)/Y_* - 1)^{-1}\), where \( Y^*(x) = \ln(x_*/x) \) and \( Y_* = \mathbb{E}_x \ln \xi \).

Then, similarly to (34), we get

\[
\rho(x) \geq \rho_* \mathbb{E}_x \exp \left\{ - \sum_{n=0}^{N(x, x_*)} \left( \mu x_n + \frac{1}{R_0 \ln x_n} \right) \right\} \\
\geq \frac{\rho_*}{2Y^*(x)/Y_* - 1} \mathbb{E}_x \exp \left\{ - \sum_{n=0}^{N(x, x_*)} \left( \mu x_n + \frac{1}{R_0 \ln x_n} \right) \right\} \\
\geq \frac{\rho_*}{2Y^*(x)/Y_* - 1} \exp \left\{ - \sum_{n=0}^{N(x, x_*)} \left( \mu \mathbb{E}_x x_n + \mathbb{E}_x \frac{1}{R_0 \ln x_n} \right) \right\},
\]

where the first sum is uniformly bounded in \( x \) (see the above proof for the \( \ell^- \) and \( s^- \)cases).

As \( \ln(1/x) \) is a concave function on \( x \in (0, e^{-2}] \), it is assumed that \( x_* \leq e^{-2} \) and apply the Jensen inequality to estimate the last sum:

\[
\mathbb{E}_x \frac{1}{\ln x_n^{-1}} \leq \left( \frac{\ln \mathbb{E}_x}{x_n \mathbb{E}_x \prod_{k=n}^{N(x, x_*)} \xi_k} \right)^{-1}, \quad \xi_k = x_k/x_{k-1}, \ N = N(x, x_*).
\]

Note that the function

\[
\mathbb{E}_x \xi^{-1} = \frac{1}{\mathbb{E}_x} \int_{1/x}^{1} \xi^{-1+\alpha} r(s) \, ds
\]

increases in \( x \). Hence, \( \mathbb{E}_x \xi^{-1} \leq \mathbb{E}_x \xi^{-1} \), and therefore

\[
\ln \frac{1}{x_n \mathbb{E}_x \prod_{k=n}^{N(x, x_*)} \xi_k} \leq \ln \frac{1}{x_n} + \ln \frac{1}{\prod_{k=n}^{N(x, x_*)} \mathbb{E}_x \xi^{-1}} \leq (N(x, x_*) - n + 1) \lambda_*
\]

where \( \lambda_* = \ln(\mathbb{E}_x \xi^{-1})^{-1} > 0 \). This estimate enables completion of the proof:

\[
\sum_{n=0}^{N(x, x_*)} \frac{\mathbb{E}_x}{R_0 \ln x_n^{-1}} \leq \frac{1}{\lambda_*} \frac{N(x, x_*)}{R_0} \sum_{n=1}^{N(x, x_*)} \frac{1}{n} \leq 1 + \ln \frac{N(x, x_*)}{\lambda_* R_0},
\]

where \( N(x, x_*) \leq \frac{2}{Y_*} \ln \frac{1}{x} \), i.e., the assertion of the theorem \( \rho(x) \geq \rho_* e^{-C (\ln x^{-1})^{-g}} \) holds with

\[
g = \frac{1}{\lambda_* R_0}, \quad C = \frac{\mu}{1 - e_*} + \frac{1 + \ln \frac{2}{Y_*}}{\lambda_* R_0}.
\]

It is clear that the factor \((\ln x^{-1})^{-g}\), which is a slowly varying function, does not disturb the Pareto exponent. \( \square \)

## 5 Conclusions

The aim of this paper is to develop a qualitative model that implies stable income distributions with credible Pareto tails. To this end, the extreme case of economical cooperation has been considered and the basic features of chain-trading structures with rigid corporative hierarchy have been axiomatized. An analysis of Section 2 shows that the qualitative definition of demand functions determines the values of Pareto exponents for gross and
net income. In general, the choice of the demand function is predeter-
ned by the economic environment and the scale of the business. Hence,
the qualitative behavior of the demand function can be motivated by
estimates of market size and other macroeconomic constraints. The
observations that are valid in terms of our model may provide some hints
for the interpretation of facts established in empirical studies. For
example, according to a number of reports (see [8]–[9], [17]–[18]), the
graph of the net income distribution drops down from the Pareto exponent $a = 1$
to a greater value $1 + \alpha$. This phenomena can be explained as
a transition from a long-range u-demand to s-limitation of the market
reached by the largest corporations.

The existence of the peak of the density distribution in the neigh-
borhood of the minimal income ($d < 0$) depends on the amplitude
$R_0 = \int R(s) \, ds$; the accumulation of the poverty cannot be controlled by
the only amplitude $P(1)$ of the social welfare. To remove the peak of poverty (to make
$d \geq 0$) or at least reduce it, according to equation (4) it is better to decrease the gradient
of the social aid ($P'(1) \rightarrow 0$), increase the amplitude of pair-interactions $R(1)$ (i.e.,
to increase the economic activity of agents), and to decrease $R_0/R(1) = \int R(x) / R(1) \, dx$
to decrease the range of spending). These expectable issues mean that the mathematical
caricature (6)–(7) is potentially consistent. The quotation “If I give out my money to the
poor in the morning, all money will be repaid by tonight” (apparently made by George
Soros) aphoristically expresses the concept of the hierarchical model and motivates the
basic equations of this paper.

6 Mathematical Appendix: Four lemmas on multi-

Lemma 1 (On the a.s. finiteness of the exit time) The Markov chain $\{x_n\}$ with transition
probabilities (23) starting at the point $x_0 = x$ exits any interval $(x, y) \subset (0, 1)$ in a finite
number of steps a.s.

Proof. (a) If $x \in (0, x_*]$, then by definition (23),

$$
\mathbb{P}\{x_{n+1} \geq (1 + \varepsilon)x_n \mid x_n\} = \frac{1}{x_n Z_\alpha(x_n)} \int_{(1+\varepsilon)x_n}^1 s^\alpha r(y/x) \, dy = \frac{1}{Z_\alpha(x_n)} \int_{1+\varepsilon}^{1/x_n} s^\alpha r(s) \, ds,
$$

where the right-hand side is a decreasing function of $x_n$, as the derivative in $x_n$ of
the right-hand side of the equation is nonpositive. If $\int_{1/x_*}^{1/x_*} r(s) \, ds = 0$, the Markov chain exits
$(x, x_*]$ in the first step, with a probability of one. Otherwise, set $\xi_{n+1} = x_{n+1}/x_n \geq 1$. As
$\int_{1/x_*}^{1/x_*} s^\alpha r(s) \, ds > 0$ and $r \in L_1$, there exists $\varepsilon > 0$ such that

$$
\mathbb{P}\{\xi_{n+1} \geq 1 + \varepsilon\} \geq \frac{1}{Z_\alpha(x_*)} \int_{1+\varepsilon}^{1/x_*} s^\alpha r(s) \, ds \geq \delta_\varepsilon > 0, \quad \varepsilon < x_*^{-1} - 1
$$

independently of the previous events. The probability $P_\varepsilon(n, N)$ of the event “$n$ random
variables $\xi_k$ among $N$ take values larger than $1 + \varepsilon$” can be estimated from the binomial
distribution:
\[
P_\varepsilon(n, N) = \mathbb{P}\left(\prod_{1}^{N} \xi_k \geq (1 + \varepsilon)^n \right) \geq \sum_{k=0}^{N} C_N^k \delta_\varepsilon^k (1 - \delta_\varepsilon)^{N-k} \\
= 1 - \sum_{k=0}^{n} C_N^k \delta_\varepsilon^k (1 - \delta_\varepsilon)^{N-k} = 1 - C_N^n \delta_\varepsilon^n (1 - \delta_\varepsilon)^{N-n} \frac{\delta_\varepsilon (N + 1 - n)}{(N + 1)\delta_\varepsilon - n} \\
= 1 - O\left(\exp\{n \ln N - N \ln(1 - \delta_\varepsilon)^{-1}\}\right) \to 1
\]
as \(N \to \infty\) for any finite \(n\) and \(x < x_*\) ([24], Ch.6, §3). Hence, the Markov chain exits the interval \((x, x_*)\) with probability one.

(b) If \(x \in (x_*, y)\), then by (23), for \(\varepsilon < 2/(1 + y)\) we have
\[
\mathbb{P}\{x_{n+1} \geq (1 + \varepsilon)x_n|x_n\} = - \int_{(1+\varepsilon)x_n}^{1} d \left( \frac{1 - y}{1 - x} \right)^{\gamma} = \left( 1 + \varepsilon - \frac{\varepsilon}{1 - x_n} \right)^{\gamma}
\]
which is clearly a decreasing function of \(x_n\). Hence,
\[
\mathbb{P}\{x_{n+1} \geq (1 + \varepsilon)x_n|x_n\} \geq \mathbb{P}\{x \geq (1 + \varepsilon)y|y\} = \left( 1 + \varepsilon - \frac{\varepsilon}{1 - y} \right)^{\gamma} = \delta_\varepsilon > 0
\]
and \(\mathbb{P}\{x_1 \geq (1 + \varepsilon)|y\} \to 1\) as \(N \to \infty\).

**Lemma 2** (On the finiteness of exponential moments) Let \(\{\xi_n\}\) be i.i.r.v. with moments \(\mu_n = \mathbb{E}\xi^n < 1\) and let \(G\) be the random geometric progression
\[
G = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \xi_k = \xi_1 (1 + \xi_2 (1 + \xi_3 (1 + \ldots))).
\]
Then \(\mathbb{E} e^{\lambda G} < \infty\) for all \(\lambda \in \mathbb{C}\) such that
\[
\lambda : \sigma(\lambda) \equiv \sup_{k \geq 1} \sum_{n=1}^{\infty} \frac{|\lambda|^n}{n!} \frac{1}{1 - \mu_{n+k}} < 1. \quad (36)
\]

**Proof.** It suffices to prove the statement in the case \(\lambda \geq 0\). Let us regard sum (30) as a product of two random variables, \(\xi\) and \(1 + \tilde{G}\),
\[
G = \xi_1 (1 + \xi_2 (1 + \ldots)) = \xi_1 (1 + \tilde{G}),
\]
where \(\xi_1\) and \(\tilde{G}\) are independent, \(G\) and \(\tilde{G}\) are identically distributed. Denote by \(M_n\) the moments of the random variable \(G\): \(M_n \equiv \int G^n P(dG)\). Therefore, the expansion
\[
\mathbb{E} e^{\lambda G} = \sum_{n=0}^{\infty} \frac{\lambda^n M_n}{n!} = \int_{\mathbb{R}} \mathbb{E} e^{\lambda \xi (1+\tilde{G})} d\xi = \sum_{n=0}^{\infty} \frac{\lambda^{n} \mu_{n} \mathbb{E} (1 + G)^n}{n!}
\]
implies a solvable recurrent system of equations for the moments \(M_n\):
\[
M_n = \frac{1}{1 - \mu_n} \sum_{k=0}^{n-1} C_n^k M_k, \quad M_0 = 1, \quad (37)
\]
where \( C_n^k = \frac{n!}{k!(n-k)!} \). Using the recurrent definition (37) of \( M_n \), we obtain

\[
S \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{M_n \lambda^n}{n!} = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{1}{1-\mu_n} \left( 1 + \sum_{k=1}^{n-1} M_k C_n^k \right) \\
= \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{1}{1-\mu_n} + \sum_{k=1}^{\infty} \frac{\lambda^k M_k}{k!} \sum_{n=k+1}^{\infty} \frac{\lambda^{n-k}}{(n-k)!} \frac{1}{1-\mu_n} \\
= \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{1}{1-\mu_n} + \sum_{k=1}^{\infty} \frac{\lambda^k M_k}{k!} \sum_{m=1}^{\infty} \frac{\lambda^{m}}{m!} \frac{1}{1-\mu_n+1} \\
\leq L + S \sup_{k \geq 1} \sum_{m=1}^{\infty} \frac{\lambda^m}{m!} \frac{1}{1-\mu_{k+1}} = L + \sigma(\lambda) S, \quad L \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} (1-\mu_n). 
\]

Thus, \( S \leq L (1 - \sigma(\lambda))^{-1} < \infty \), provided \( \sigma(\lambda) < 1 \). \( \square \)

**Lemma 3** (Anti-Chebyshev inequality) Let \( \{Y_n\} \) be a sequence of positive finite random variables such that \( Y_n \leq Y^* < \infty \) and \( \inf_n Y_n \leq \ldots \leq Y_m = Y_s > 0 \). Then,

\[
\mathbb{P}\left\{ \sum_{k=1}^{n} Y_k > \frac{1}{2} n Y_s \right\} \geq \frac{1}{2Y^*/Y_s - 1}.
\]

**Proof.** For \( X = \sum_{k=1}^{n} Y_k \), we have \( n Y^* \geq X^* \geq \mathbb{E} X \geq n Y_s \). Hence, for any \( \varepsilon \in (0,1) \), we have \( \mathbb{P}\{ X > \varepsilon n Y_s \} \geq \mathbb{P}\{ X > \varepsilon \mathbb{E} X \} \). In addition,

\[
\mathbb{E} X \leq X^* \mathbb{P}\{ X > \varepsilon \mathbb{E} X \} + \varepsilon \mathbb{E} X (1 - \mathbb{P}\{ X > \varepsilon \mathbb{E} X \}).
\]

Therefore, by setting \( \varepsilon = 1/2 \), we obtain

\[
\mathbb{P}\{ X > \varepsilon n Y_s \} \geq \mathbb{P}\{ X > \varepsilon \mathbb{E} X \} \geq \frac{1 - \varepsilon}{X^*/\mathbb{E} X - \varepsilon} \geq \frac{1 - \varepsilon}{Y^*/Y_s - \varepsilon} \bigg|_{\varepsilon=1/2} = \frac{1}{2Y^*/Y_s - 1}. \quad \square
\]

**Lemma 4** (On the distribution of the exit time) The Markov chain \( \{x_n\} \) with transition probabilities (23) exits the interval \((x, x_s)\) in \( N(x, x_s) \) steps with the probability

\[
\mathbb{P}\{ n_s(x) \geq N(x, x_s) \} \geq \frac{1}{2Y^*(x)/Y_s - 1} \overset{\text{def}}{=} O(\ln x^{-1})^{-1},
\]

where \( Y^*(x) = \ln(x_s/x) \leq \ln x^{-1} \), and

\[
N(x, x_s) = \left[ \frac{2}{Y_s} \ln \frac{x_s}{x} \right] + 1 = O(\ln x^{-1}), \quad Y_s = \frac{1}{Z_\alpha(x_s)} \int_1^{1/x_s} \ln s s^a r(s) \, ds > 0.
\]

**Proof.** Note that the function

\[
\mathbb{E}_{x_n} \ln \frac{x_{n+1}}{x_n} = \frac{1}{Z_\alpha(x_n)} \int_1^{1/x_n} \ln \xi \xi^a r(\xi) \, d\xi \overset{\text{def}}{=} \mathbb{E}_{x_n} \ln \xi
\]

is a decreasing function of the initial point \( x_n \in (0, x_s] \); hence, \( \mathbb{E}_{x_n} \ln \frac{x_{n+1}}{x_n} \geq \mathbb{E}_{x_s} \ln \xi \), and for the random variables \( Y_n = \ln \frac{x_{n+1}}{x_n} \), we have

\[
Y_s = \min \mathbb{E}_x \{Y_{n+1}|Y_1, \ldots, Y_n\} = \mathbb{E}_{x_s} \ln \xi.
\]
Hence, for $x_n = x \prod_{k=1}^{n} x_k / x_{k-1} = x \exp\{\sum_{k=0}^{n-1} \ln Y_k\}$, Lemma 3 implies
\[
\mathbb{P}\{x_n > x_\ast\} = \mathbb{P}\left\{\sum_{k=0}^{n-1} \ln Y_k > \ln x_\ast / x\right\} \\
\geq \mathbb{P}\left\{\sum_{k=0}^{n-1} \ln Y_k > n_\ast Y_\ast / 2\right\}_{n_\ast Y_\ast > 2 \ln (x_\ast / x)} = \frac{1}{2Y_\ast(x) / Y_\ast - 1},
\]
i.e., exit time $N(x, x_\ast) = \left[\frac{2}{Y_\ast} \ln \frac{x_\ast}{x}\right] + 1$ has a probability of not less than $(2Y_\ast(x) / Y_\ast - 1)^{-1}$.

\[\square\]

References

[1] D. G. Champernowne, Econ. J., 23 (1953) 318–351.

[2] S. C. Manrubia, D. H. Zanette, Phys. Rev. E, 59 (1999) 4945–4948.

[3] M. Nieri, W. Suoma, Working paper of Dept. of Economics of Utah State Univ., October 6 (2004).

[4] G. U. Yule, Philos. Trans. R. Soc. London B, 213 (1925) 21–87.

[5] M. E. J. Newman, Contemporary Physics, 46 (2005) 323–351.

[6] H. Aoyama, W. Souma, Y. Fujiwara, Physica A, 324 (2003) 352-358.

[7] M. Y. Sweezy, Economic Statistics, 21, no. 4 (1939) 178–184.

[8] H. Aoyama, W. Souma, Y. Nagahara, M. P. Okazaki, H. Takayasu, M. Takayasu, Fractals, 8, no. 3 (2000) 293–300.

[9] Y. Fujiwara, C. Di Guilmi, H. Aoyama, M. Galetti, W. Souma, arXiv:cond-mat/0310061, 2003.

[10] J. H. Ausubel, Technology in Society, 26 (2004) 343–360.

[11] A. B. Atkinson, W. Salverda, Journal of European Economic Association, 3, no. 4 (2005) 883–913.

[12] A. B. Atkinson, Income Tax and Top Incomes over the Twentieth Century, Plenary Lecture given at the XXVIII Meeting on Economic Analysis, Seville, 11–12 December 2003.

[13] W. J. Reed, Physica A, 319 (2003) 469–486.

[14] X. Gabaix, Quarterly Journal of Economics, CXIV (1999) 739–767.

[15] J.-Ph. Bouchaud, Power laws in economics and finance: some ideas from physics, Quantitative Finance, 1 (2001) 105–112.
[16] C. Kleiber, S. Kotz, Statistical Size Distributions in Economics and Actuarial Sciences, John Wiley & Sons, New York, 2003.

[17] T. Mizuno, M. Katori, H. Takayasu, M. Takayasu, In: Empirical Science of Financial Fluctuations - The Advent of Econophysics, Springer (2001) 321–330.

[18] T. Kaizoji, Y. Ikeda, H. Iyetomi, arXiv:physics/0512124, 2005.

[19] A. M. Chebotarev, Ukrainian Journal “Economist”, no. 7 (2003) 6–11.

[20] V. N. Baturin, S. G. Lebedev, V. P. Maslov, B. I. Sadovnikov, A. M. Chebotarev, Economics of Contemporary Russia, no. 4, (31) (2005) 57–62.

[21] A. M. Chebotarev, Ukrainian Journal “Economist”, no. 9 (2004) 54–57.

[22] L. Boggio, Riv. Mat. Sci. Econom. Social. 22 (1999), no. 1-2, 13–30.

[23] J. L. Gastwirth, Review of Economics and Statistics, 54, no. 3, (1972) 306–316.

[24] W. Feller, An Introduction to Probability Theory and Its Applications, Vol. 1, John Wiley & Sons, New York, 1966.