DEDEKIND ZETA FUNCTIONS OF CERTAIN REAL QUADRATIC FIELDS

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Abstract. Using analytic and modular transformation methods, we represent the value of the product of two Dedekind zeta functions of certain real quadratic number fields at $-3$ by Dedekind sums of high rank in this paper.

1. Introduction and Results

The values of Dedekind zeta function of a number field $K$ at rational integers are closely related with the algebraic character of the number field $K$ itself. To represent these values as clearly as possible is one of the important tasks of algebraic number theory. In history many mathematicians had some work on this project. Hasse (see ref.[1]) expressed Dedekind zeta function of a number field as product of Riemann zeta function and usual Dirichlet L-functions. Siegel (see ref.[2]) got some properties of explicit values of Dedekind zeta functions of quadratic number fields at negative integers, and a particular interesting case is at $-1$, using modular transformation method. Zagier (see ref.[3]) also obtained another expression of the values of Dedekind zeta functions of real quadratic fields at negative integers using Kronecker limit formula. Shintani (see ref.[4, 5]) using astonishing linear programming method expressed Dedekind zeta functions as a sum of Dirichlet series of some real cones.

In reference [6], we represented the value of the product of two Dedekind zeta functions of certain real quadratic number fields at $-1$ by Dedekind sums of high rank. Using the reciprocity law of Dedekind sums (see ref.[7]) and software of Mathematica 4.0, we got

**Theorem 1.** If the class number of the real quadratic number field $\mathbb{Q}(\sqrt{q})$ is $1$, with prime $q = 4n^2 + 1$. Then

$$\zeta_{\mathbb{Q}(\sqrt{q})}(-1) = \frac{1}{45} (26n^3 - 41n \pm 9),$$

if $n \equiv \pm 2 \pmod{5}$.

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The main result of this paper is the following Theorem 2.

**Theorem 2.** Let real quadratic fields \( K_1 = \mathbb{Q}(\sqrt{p}) \), \( K_2 = \mathbb{Q}(\sqrt{q}) \), where \( p \equiv q \equiv 1 \pmod{4} \) be different primes. Let the class number of \( K_2 \) be 1, and write \( c = \frac{1 - \sqrt{p}}{2} \) and \( K_3 = \mathbb{Q}(\sqrt{p}) \), then we have

\[
14400 \sqrt{p} \delta_{K_2} K_3 (-3) \delta_{K_3} (-3) = \frac{3q^2 p^6}{35840} \left( \frac{64q^3 U^3}{3T} - 16q^2 \frac{U}{T} \right) (p - \frac{1}{p}) B_8 + \frac{1}{8!} \sum_{m=0}^{4} \left( \begin{array}{c} 8 \\ m \end{array} \right) S_{m, 8-m} \left( \frac{T - U}{2}, \frac{U}{p} \right) \cdot (9qU^2(m - 1)(m - 7) - 540 + 45T^2) + \frac{1}{T^3-m} \left( \begin{array}{c} \frac{8}{2f} \\ m \end{array} \right) 2^5 - 2f(-U^2 q)^f 
\]

\[
\frac{90}{8!} \sum_{m=0}^{4} \left( \begin{array}{c} 8 \\ m \end{array} \right) S_{m, 8-m} \left( \frac{T - U}{2}, \frac{U}{p} \right) \cdot (4 - m) \sum_{f=1}^{\frac{8}{2f} - 1} (2f - 1) 2^6 - 2f(-U^2 q)^f 
\]

\[
+ \frac{T}{576} \left( \frac{T - U}{2}, \frac{U}{p} \right) - \sum_{(U, p) = 1} \chi(l^2 + l + c) S_{4, 4} \left( \frac{T - U}{2} - lU, pU \right) (-81qU^2 - 540 + 45T^2) 
\]

\[
+ \frac{p^6}{8!} \sum_{(U, p) = 1} \chi(l^2 + l + c) \sum_{m=0}^{4} \left( \begin{array}{c} 8 \\ m \end{array} \right) S_{m, 8-m} \left( \frac{T - U}{2}, lU, pU \right) + S_{m, 8-m} \left( \frac{T + U}{2}, lU, pU \right) \cdot \left( \frac{T - U}{2} \right) \left( \frac{T + U}{2} \right) \left( \frac{T - U}{2} - lU, pU \right) \left( \frac{T + U}{2} - lU, pU \right) + S_{m, 8-m} \left( \frac{T + U}{2}, lU, pU \right) \frac{1}{T^3-m} \left( \begin{array}{c} \frac{8}{2f} \\ m \end{array} \right) 2^6 - 2f(-U^2 q)^f 
\]

where \( S_{k, l}(u, m) = \sum_{v \pmod{m}} B_k(\frac{v}{m}) B_l(\frac{u + v}{m}) \) be Dedekind sum; and \( B_n(x) \) be the usual Bernoulli polynomial, with \([x]\) and \( \{x\} \) denote the integral part and fractional part of \( x \) respectively; \( \delta_{K_2} = \frac{\log x}{\log \epsilon} \), with \( \epsilon \) and \( \epsilon_+ \) denote the fundamental and totally positive fundamental unit of \( K_2 \) respectively; \( \chi \) be the Kronecker symbol mod \( p \); \( \epsilon_j^2 = -\frac{T + U}{U} \epsilon \) with positive integer \( j \) such that \( p \mid U \). From the define equation of Dedekind sums one can see that \( S_{k, l}(u_1, m) = S_{k, l}(u_2, m) \), if \( u_1 \equiv u_2 \pmod{m} \); \( S_{k, l}(u, m) = S_{k, l}(\overline{u}, m) \), if \( u \overline{m} \equiv 1 \pmod{m} \).
Of course Theorem 2 is a effective computing formulae in the case of the the conditions in Theorem 1.

2. Main Lemma

Let \( K = \mathbb{Q}(\sqrt{\Delta}) \) be a real quadratic number field with basic discriminant \( \Delta \), and let \( A = [a, \frac{b+\sqrt{\Delta}}{2}, \frac{b-\sqrt{\Delta}}{2}] \) be an integral ideal of \( K \). Set \( A^* = \sqrt{\Delta}A \), \( B = [a, -b+\sqrt{\Delta}, -b-\sqrt{\Delta}] \), and \( c = a^2 - b^2 - 4ac \). Let \( F_A(Z) = aZ^2 - bZ + c \) and \( F_B(Z) = aZ^2 + bZ + c \). Denote \( \omega = \frac{b+\sqrt{\Delta}}{2}, \omega' = \frac{b-\sqrt{\Delta}}{2}, \omega = -\frac{b+\sqrt{\Delta}}{2}, \omega' = -\frac{b-\sqrt{\Delta}}{2} \), let \( \epsilon^+ \) be a totally positive fundamental unit of the number field \( K \), For positive rational integer \( j \), set

\[
\epsilon_j^+ = \frac{T_j + U_j\sqrt{\Delta}}{2}, \quad \rho_j = \frac{\epsilon_j^+ + i\epsilon_j^-}{\epsilon_j^+ - i\epsilon_j^-},
\]

\[
Z_{A,j} = \frac{b}{2a} + \frac{\sqrt{\Delta}}{2a} \rho_j, \quad Z_{B,j} = -\frac{b}{2a} + \frac{\sqrt{\Delta}}{2a} \rho_j,
\]

\[
\overline{Z}_{A,j} = \frac{b}{2a} + \frac{\sqrt{\Delta}}{2a} \rho_j, \quad \overline{Z}_{B,j} = -\frac{b}{2a} + \frac{\sqrt{\Delta}}{2a} \rho_j.
\]

Let \( \Gamma \) denote the upper half circle with center \( \frac{b}{2a} \) and radius \( \frac{\sqrt{\Delta}}{2a} \), and \( \Gamma_{A,j} \) denote the arc of \( \Gamma \) located between \( Z_{A,j} \) and \( \overline{Z}_{A,j} \).

We write \( Z = X + iY \), where \( X \) and \( Y \) denote real and imaginary part of \( Z \) respectively. Let \( \chi \) be a real primitive Dirichlet character of mod \( k \). Define

\[
\mathcal{L}(s, \chi, A) = \sum_{\lambda > 0 \atop \lambda \in A/\epsilon^+} \frac{\chi(N((\lambda)))/N(A)}{(N((\lambda)))/N(A)^s}, \quad \text{Re}(s) > 1,
\]

(1)

where \( N \) denote the norm map of \( K/\mathbb{Q} \). Obviously, such defined \( \mathcal{L}(s, \chi, A) \) is a ideal class function of \( A \).

We got the following Lemma 1 in ref. [8]:

**Lemma 1.** With notations above, and let \( s \) be complex variable with \( \text{Re}(s) > 1 \), then we have

\[
j(\mathcal{L}(s, \chi, A) + \chi(-1)\mathcal{L}(s, \chi, A^*)) = -\frac{\Gamma(s)\Delta^{\frac{s+1}{2}}2\Gamma(\frac{s}{2})^2}{24\pi^s} \int_{\Gamma_{A,j}} \frac{E(s, Z, \chi, A)}{F_A(Z)} \, dZ,
\]

(2)

where the Eisenstein series

\[
E(s, Z, \chi, A) = \sum_{\substack{(m, n) \neq (0, 0) \atop m, n \neq -\infty}} \frac{\chi(am^2 + bmn + cn^2)Y^s}{|m + nZ|^{2s}}.
\]
We got the Fourier expansion of $E(s, Z, A, \chi)$ in ref. [7], i.e.

$$E(s, Z, \chi, A) = 2Y^s\chi(a)\zeta(2s) \prod_{p\mid k}(1 - p^{-2s}) + \frac{2\sqrt\pi sGamma(s - \frac{1}{2})Y^{1-s}}{sGamma(s)}$$

$$\times \sum_{n=1}^{+\infty} n^{1-2s} \sum_{m \mod k} \chi(am^2 + bmn + cn^2) + \frac{8\pi^s k^{-s-\frac{1}{2}} Y^{\frac{1}{2}}}{Gamma(s)} \sum_{u=1}^{+\infty} u^{-\frac{1}{2}} K_{s-\frac{1}{2}}(\frac{2\pi u Y}{k})$$

$$\times \sum_{1 \leq n \leq u} n^{1-2s} \sum_{m \mod k} \chi(am^2 + bmn + cn^2) \cos \frac{2\pi u (\frac{m}{k} + X)}{k}.$$

(3)

where $K_s(z)$ be Bessel function, i.e.

$$K_s(z) = \frac{1}{2} \int_0^{+\infty} \exp^{-\frac{1}{2}z(t+\frac{1}{2})} t^{s-1} dt, \quad z > 0.$$  

It is easy to know that the Eisenstein series $E(s, Z, A, \chi)$ have the analytic continuations to the whole complex plane by (3). It is well known that $\mathcal{L}(f, \chi, A)$ and $\Gamma(s)$ could also have the analytic continuations to the whole complex plane.

Taking the limit of both sides of (2) when $s \rightarrow -3$, substituting (3) into (2), and then write the R.H.S of (2) as three summands, i.e. $I_1 + I_2 + I_3$, and let’s compute each summand individually.

Firstly, by the well-known functional equation of $\zeta(s)$ and $\lim_{s \rightarrow -3} \Gamma(\frac{s}{2})^2 = \frac{16\pi}{9}$ we get

$$\lim_{s \rightarrow -3} I_1 = -\frac{135\chi(a)\Delta^2\zeta(7)}{(2\pi)^7} (1 - p^6) \int_{\Gamma_{A,j}} \frac{Y^{-3}}{F_A(Z)} dZ.$$  

(4)

It is not difficult to get $Y^s \frac{dF_A(Z)}{dZ} = -iF_A(Z)$ for $Z \in \Gamma_{A,j}$. Hence substituting it in (4) we get

$$\lim_{s \rightarrow -3} I_1 = i \frac{135\chi(a)\Delta^2\zeta(7)}{(2\pi)^7} \prod_{p \mid \ell}(p^\ell - 1)(2aF_A - (Z + \Delta 3F_A^3(Z))) \int_{\Gamma_{A,j}} \frac{\bar{Z}_{A,j}}{Z_{A,j}}$$  

(5)

Secondly, let’s calculate $\lim_{s \rightarrow -3} I_2$. It is easy to get $\sum_{n=1}^{+\infty} n^{1-2s} \sum_{m \mod k} \chi(am^2 + bmn + cn^2) = 1, m,n \leq k \chi(\psi^2 + \psi + \chi^2)\zeta(2s-1, n/k)k^{1-2s}$, where $\zeta(\psi, \chi)$ be the Hurwitz zeta function. We know that $\zeta(-7, n/k) = -\frac{1}{8} B_9(n/k)$ and $\Gamma(-7/2) = 16\sqrt{\pi}/105$, so through a not difficult computation, we have

$$\lim_{s \rightarrow -3} I_2 = \frac{3\Delta^2 k^6}{35840a^3} \sum_{1 \leq m, n \leq k} \chi(am^2 + bmn + cn^2) B_{8}(\frac{n}{k})(\frac{64\Delta^3 U_2}{3T_2} - 16\Delta^3 T_2)$$  

(6)

Finally, we deal with $\lim_{s \rightarrow -3} I_3$. In ref.[9], we get $K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2}} \exp(-z) \sum_{l=0}^{\infty} \frac{(n+1)^l}{l!(n-l)!} (2z)^{-l}$. So applying the similar integral techniques as in ref.[6] though, through a
long but not tough calculation we have:

\[
\lim_{s \to -3} I_3 = -\frac{9\Delta^2 k^3}{16\pi^4} \sum_{u=1}^{+\infty} u^{-4} \sum_{1 \leq n \leq u} n^7 \sum_{m \pmod{k}} \chi(am^2 + bmn + cn^2)
\]

\[
\left( \left( \frac{k}{2\pi i u F_A(Z)} + \frac{5k^2 F'_A(Z)}{4\pi^2 u^2 F'_A(Z)} + \frac{30ak^2 i}{8\pi^4 u^3 F'_A(Z)} + \frac{5\Delta k^3 i}{8\pi^4 u^3 F'_A(Z)} \right)
\]

\[
\cdot \exp \left( \frac{2\pi i u}{k} \left( \frac{m}{n} + Z \right) \right) \left[ Z_{A,j} \right]^{\tilde{Z}_{A,j}}
\]

\[
\left( \left( \frac{k}{2\pi i u F_B(Z)} + \frac{5k^2 F'_B(Z)}{4\pi^2 u^2 F'_B(Z)} + \frac{30ak^2 i}{8\pi^4 u^3 F'_B(Z)} + \frac{5\Delta k^3 i}{8\pi^4 u^3 F'_B(Z)} \right)
\]

\[
\cdot \exp \left( \frac{2\pi i u}{k} \left( \frac{-m}{n} + Z \right) \right) \left[ Z_{B,j} \right]^{\tilde{Z}_{B,j}}
\]

(7)

By (2), (3), (5), (6), and (7) we have:

**Main Lemma.** Notations as explained above,

\[
\lim_{s \to -3} j(L(f, \chi, A) + \chi(-\infty)L(f, \chi, A^*))
\]

\[
= \frac{135}{(2\pi)^7} \sum_{p \mid k} (p^6 - 1)(2aF_A^{-2}(Z) + \Delta A^{-3}(Z)) \left[ Z_{A,j} \right]^{\tilde{Z}_{A,j}}
\]

\[
+ \frac{3\Delta^2 k^6}{35840\pi^4} \sum_{1 \leq m, n \leq k} \chi(am^2 + bmn + cn^2) B_8 \left( \frac{n}{k} \right) \left( \frac{64\Delta^3 U_{2j}^3}{3T_{2j}} - 16\Delta^2 U_{2j} \right)
\]

\[
- \frac{9\Delta^2 k^3}{16\pi^4} \sum_{u=1}^{+\infty} u^{-4} \sum_{1 \leq n \leq u} n^7 \sum_{m \pmod{k}} \chi(am^2 + bmn + cn^2)
\]

\[
\left( \left( \frac{k}{2\pi i u F_A(Z)} + \frac{5k^2 F'_A(Z)}{4\pi^2 u^2 F'_A(Z)} + \frac{30ak^2 i}{8\pi^4 u^3 F'_A(Z)} + \frac{5\Delta k^3 i}{8\pi^4 u^3 F'_A(Z)} \right)
\]

\[
\cdot \exp \left( \frac{2\pi i u}{k} \left( \frac{m}{n} + Z \right) \right) \left[ Z_{A,j} \right]^{\tilde{Z}_{A,j}}
\]

\[
\left( \left( \frac{k}{2\pi i u F_B(Z)} + \frac{5k^2 F'_B(Z)}{4\pi^2 u^2 F'_B(Z)} + \frac{30ak^2 i}{8\pi^4 u^3 F'_B(Z)} + \frac{5\Delta k^3 i}{8\pi^4 u^3 F'_B(Z)} \right)
\]

\[
\cdot \exp \left( \frac{2\pi i u}{k} \left( \frac{-m}{n} + Z \right) \right) \left[ Z_{B,j} \right]^{\tilde{Z}_{B,j}}
\]

(8)

3. Proof of Theorem 2

\( L(s, \chi, A) \) and \( \chi \) as in the last section. We define \( L(s, \chi) = \sum_A L(s, \chi, A) \), where \( A \) runs through representative set of the narrow ideal class group of the real quadratic
number field $K = \mathbb{Q}(\sqrt{\Delta})$ with basic discriminant $\Delta$. Obviously, $\mathcal{L}(s, \chi)$ is analytic for $\text{Re}(s) > 1$, we also proved

**Theorem 3.** Symbols as explained above, we have

$$L(s, \chi)L(s, \chi\Delta) = \mathcal{L}(s, \chi),$$

(9)

where $\chi_\Delta$ is the Kronecker symbol $(\frac{\Delta}{\cdot})$, and $L(s, \chi)$ together with $L(s, \chi\Delta)$ is the usual Dirichlet $L$-function.

To prove Theorem 2, we need to apply Theorem 3 to the Main Lemma.

In the Main Lemma, we may assume $k$ to be a prime $p$ with $p \equiv 1 (mod 4)$ and $g.c.d(p, \Delta) = 1$, and take an integral ideal $A$ described in the beginning of the second section satisfying $a = b = 1, c = \frac{1-\Delta}{4}$. If the class number of the real quadratic number field $K$ is 1, and $\chi$ be a real primitive Dirichlet character of module $p$ then using the well-known fact that $\zeta(s)L(s, \chi\Delta) = \zeta_{\mathbb{Q}(\sqrt{\Delta})}(s)$ and $\zeta(s)L(s, \chi^\Delta) = \zeta_{\mathbb{Q}(\sqrt{\Delta})}(s) \zeta(-3) = 1/120$, we can easily get as $s \to -3$

$$\text{L.H.S of (2)} = 14400j\delta_K\zeta_K(-3)\zeta_{\mathbb{Q}(\sqrt{\Delta})}(-3)$$

(10)

To calculate the R.H.S of (2), we set

$$Q_m(z) = \frac{1}{2^2}Q_m(z) + \sum_{n=1}^{+\infty} \frac{\sigma_{2m+1}(n)}{n^{2m+1}} \exp(2\pi inz), \text{Im}(z) > 0,$$

(11)

Using standard summation techniques in the R.H.S of (2), we have

$$\lim_{s \to -3} I_1 + I_3 = \frac{9\Delta^2p^3}{(2\pi i)^3} \left( \frac{1}{F_A(Z)} \right) \left[ p^{-1}Q_\delta(pZ) + p^2 \sum_{m(\text{mod } p)} \chi(m^2 + m + c)Q_{\delta} \left( \frac{m + Z}{p} \right) \right] \tilde{Z}_{A,j}$$

$$- \frac{5F_A(Z)}{F_A(Z)} \left[ p^{-2}Q_\delta(pZ) + p^2 \sum_{m(\text{mod } p)} \chi(m^2 + m + c)Q_{\delta} \left( \frac{m + Z}{p} \right) \right] \tilde{Z}_{A,j}$$

$$+ \frac{30}{F_A(Z)} \left[ p^{-3}Q_\delta(pZ) + p^3 \sum_{m(\text{mod } p)} \chi(m^2 + m + c)Q_{\delta} \left( \frac{m + Z}{p} \right) \right] \tilde{Z}_{A,j}$$

$$+ \frac{5\Delta}{F_A(Z)} \left[ p^{-3}Q_\delta(pZ) + p^3 \sum_{m(\text{mod } p)} \chi(m^2 + m + c)Q_{\delta} \left( \frac{m + Z}{p} \right) \right] \tilde{Z}_{A,j}$$
Lemma 1. \( Q_k(Z) \) as in (10), for \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) be an element in modular group, with \( C > 0 \), we have

\[
(QZ + D)^{2k}Q_k(M < Z >) = Q_k(Z) + \frac{(-1)^k+1(2\pi)^{2k+1}}{2(2k+1)!} \sum_{m=0}^{2k+2} \binom{2k+2}{m}(QZ + D)^{m-1}S_{m,2k+2-m}(D,C), \quad Im(Z) > 0. \tag{13}
\]

For \( M \) as in the condition of Lemma 1, Fixing \( k = 3 \) and taking derivations of \( Z \) on both sides of (13) we get

\[
Q'_3(M < Z >) = \frac{27Q'_3(Z)(CZ + D)^2}{(CZ + D)^2} - \frac{6CQ'_3(Z)(CZ + D)^3}{(CZ + D)^4} - \frac{(2\pi i)^7}{2 \cdot 8!} \sum_{m=0}^{8} (-1)^m \binom{8}{m}(m - 7)(CZ + D)^{m-6}S_{m,8-m}(D,C), \tag{14}
\]

and

\[
Q''_3(M < Z >) = \frac{Q''_3(Z)(CZ + D)^2}{(CZ + D)^2} - \frac{10CQ'_3(Z)(CZ + D)^3}{(CZ + D)^4} + \frac{30C^2Q'_3(Z)(CZ + D)^4}{(CZ + D)^4} + \frac{(2\pi i)^7}{2 \cdot 8!} \sum_{m=0}^{8} (-1)^m \binom{8}{m}(m - 7)(CZ + D)^{m-6}S_{m,8-m}(D,C). \tag{15}
\]

It is easy to prove the following two lemmas:

Lemma 2. Let \( F_A(Z) \), \( \omega \), and \( \omega' \) as explained in the beginning of Section 2. Matrix \( M \) as in Lemma 1 above, in addition that \( \omega \) and \( \omega' \) are fixed by \( M \), then

\[
(CZ + D)^2F_A(M < Z >) = F_A(Z), \quad Im(Z) > 0. \tag{16}
\]
Lemma 3. Let $F_B(Z)$, $\varpi$, and $\varpi'$ as explained in the beginning of Section 2. Matrix $M$ as in Lemma 1 above, in addition that $\varpi$ and $\varpi'$ are fixed by $M$, then

$$(CZ + D)^2F_B(M < Z >) = F_B(Z), Im(Z) > 0. \quad (17)$$

Finally, we have already found the modular matrix translate the variables in (12); let $\epsilon_+ = \frac{T_1 + U_1 + Z}{2}$ denote the fundamental and totally positive fundamental unit of $K$, set $\epsilon^l = \frac{T_1 + U_1 + Z}{2}$ and matrix $M = (\frac{aU_1}{2T_2 - 4U_1})$. For any integer $l$, set matrix $M_l = M^l = (\frac{bU_1}{2T_2 - 4U_1})$.

Let’s choose an positive integer $j$ such that $p|U_j$, then all the following matrixes are in the modular group:

$M_{2j}, M_{2j}^{(p)} = \left(\frac{T_{2j} + bU_{2j}}{2}, \frac{-cpU_{2j}}{2aU_{2j}}\right), M_{2j}^{(p,m)} = \left(\frac{T_{2j} + bU_{2j}}{2}, \frac{maU_{2j}}{2aU_{2j}}\right)$

$\overline{M}_{2j} = \left(\frac{T_{2j} - bU_{2j}}{2}, \frac{-cU_{2j}}{2aU_{2j}}\right), \overline{M}_{2j}^{(p)} = \left(\frac{T_{2j} - bU_{2j}}{2}, \frac{-cpU_{2j}}{2aU_{2j}}\right)$.

And it is not difficult to check that these matrixes transfer $Z_{A,j}, pZ_{A,j}, \frac{m + Z_{A,j}}{p}, Z_{B,j}, pZ_{B,j}, \frac{m + Z_{B,j}}{p}$, respectively. And it is also easy to check that these matrixes satisfy the conditions in Lemma 1, and Lemma 2 or Lemma 3. So by a long and tedious calculation, using corresponding modular transformation (13)-(17) in (12), and we feel at ease at the end, for the irrational part disappears. Comparing (10) and (12), and the R.H.S of (2) equal $I_1 + I_2 + I_3$, further more let the fundamental discriminant $\Delta$ be a prime $q \equiv 1 \pmod{4}$ we get the proof of Theorem 2.

At the end of this paper, we would add a remark, though the methods are a little similar to Siegel’s [2], our start point is different from his, and the results obviously are different from his.

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