Third Order Tree Contributions in the Causal Approach

D. R. Grigore,\textsuperscript{1}

Department of Theoretical Physics,
Institute for Physics and Nuclear Engineering “Horia Hulubei”
Bucharest-Măgurele, P. O. Box MG 6, ROMÂNIA

Abstract

We consider the general framework of perturbative quantum field theory for the pure Yang-Mills model developed in \cite{9} and prove that the tree contributions do not give anomalies. We will provide a more general form of this gauge invariance property.

\textsuperscript{1}e-mail: grigore@theory.nipne.ro
1 Introduction

The most natural way to arrive at the Bogoliubov axioms of perturbative quantum field theory (pQFT) is by analogy with non-relativistic quantum mechanics [8], [10], [4], [5]: in this way one arrives naturally at Bogoliubov axioms [2], [7], [14], [15]. We prefer the formulation from [5] and as presented in [9]; for every set of monomials $A_1(x_1), \ldots, A_n(x_n)$ in some jet variables (associated to some classical field theory) one associates the operator-valued distributions $T^{A_1, \ldots, A_n}(x_1, \ldots, x_n)$ called chronological products; it will be convenient to use another notation: $T(A_1(x_1), \ldots, A_n(x_n))$.

The Bogoliubov axioms express essentially some properties of the scattering matrix understood as a formal perturbation series with the “coefficients” the chronological products:

- (skew)symmetry properties in the entries $A_1(x_1), \ldots, A_n(x_n)$;
- Poincaré invariance;
- causality;
- unitarity;
- the “initial condition” which says that $T(A(x))$ is a Wick monomial.

So we need some basic notions on free fields and Wick monomials. One can supplement these axioms by requiring

- power counting;
- Wick expansion property.

It is a highly non-trivial problem to find solutions for the Bogoliubov axioms, even in the simplest case of a real scalar field. The simplest way is, in our opinion the procedure of Epstein and Glaser; it is a recursive construction for the basic objects $T(A_1(x_1), \ldots, A_n(x_n))$ and reduces the induction procedure to a distribution splitting of some distributions with causal support. In an equivalent way, one can reduce the induction procedure to the process of extension of distributions [12].

In the next Section we give some introductory facts concerning Wick products, Wick submonomials, Wick theorem and pure Yang-Mills fields. The original results are given in Section 3 where we investigate chronological products of the form $T(A_1^{(2)}(x_1), A_2^{(2)}(x_2), A_3^{(1)}(x_3))$. The tree contribution in the third order of the perturbation theory is a sum of such expressions. So third order gauge invariance in the tree sector must be

$$sT_{\text{tree}}(A_1(x_1), A_2(x_2), A_3(x_3)) = sT(A_1^{(2)}(x_1), A_2^{(2)}(x_2), A_3^{(1)}(x_3))$$

$$+ sT(A_1^{(1)}(x_1), A_2^{(2)}(x_2), A_3^{(2)}(x_3)) + sT(A_1^{(2)}(x_1), A_2^{(1)}(x_2), A_3^{(1)}(x_3))$$ (1.1)

(with appropriate Grassmann signs). We will prove that a stronger result is true, namely the every one of the third terms above is null.
2 Perturbative Quantum Field Theory

There are two main ingredients in the construction of a perturbative quantum field theory (pQFT): the construction of the Wick monomials and the Bogoliubov axioms. For a pQFT of Yang-Mills theories one needs one more ingredient, namely the introduction of ghost fields and gauge charge.

2.1 Wick Products

We consider a classical field theory on the Minkowski space $M \simeq \mathbb{R}^4$ (with variables $x^\mu, \mu = 0, \ldots, 3$ and the metric $\eta$ with $\text{diag}(\eta) = (1, -1, -1, -1)$) described by the Grassmann manifold $\Xi_0$ with variables $\xi_a, a \in A$ (here $A$ is some index set) and the associated jet extension $J^r(M, \Xi_0)$, $r \geq 1$ with variables $x^\mu, \xi_{a;1\ldots r}, n = 0, \ldots, r$; we denote generically by $\xi_p, p \in P$ the variables corresponding to classical fields and their formal derivatives and by $\Xi_r$ the linear space generated by them. The variables from $\Xi_r$ generate the algebra $\text{Alg}(\Xi_r)$ of polynomials.

To illustrate this, let us consider a real scalar field in Minkowski space $M$. The first jet-bundle extension is $J^1(M, \mathbb{R}) \simeq M \times \mathbb{R} \times \mathbb{R}^4$ with coordinates $(x^\mu, \phi, \phi^\mu)$, $\mu = 0, \ldots, 3$.

If $\varphi : M \to \mathbb{R}$ is a smooth function we can associate a new smooth function $j^1\varphi : M \to J^1(M, \mathbb{R})$ according to $j^1\varphi(x) = (x^\mu, \varphi(x), \partial_\mu \varphi(x))$.

For higher order jet-bundle extensions we have to add new real variables $\phi_{\{\mu_1, \ldots, \mu_r\}}$ considered completely symmetric in the indexes. For more complicated fields, one needs to add supplementary indexes to the field i.e. $\phi \to \phi_a$ and similarly for the derivatives. The index $a$ carries some finite dimensional representation of $SL(2, \mathbb{C})$ (Poincaré invariance) and, maybe a representation of other symmetry groups. In classical field theory the jet-bundle extensions $j^r\varphi(x)$ do verify Euler-Lagrange equations. To write them we need the formal derivatives defined by

$$d_\nu \phi_{\{\mu_1, \ldots, \mu_r\}} \equiv \phi_{\{\nu, \mu_1, \ldots, \mu_r\}}. \quad (2.1)$$

We suppose that in the algebra $\text{Alg}(\Xi_r)$ generated by the variables $\xi_p$ there is a natural conjugation $A \to A^\dagger$. If $A$ is some monomial in these variables, there is a canonical way to associate to $A$ a Wick monomial: we associate to every classical field $\xi_a, a \in A$ a quantum free field denoted by $\xi_a^\text{quant}(x), a \in A$ and determined by the 2-point function

$$< \Omega, \xi_a^\text{quant}(x), \xi_b^\text{quant}(y)\Omega > = -i \ D_{ab}^+(x, y) \times 1. \quad (2.2)$$

Here

$$D_{ab}(x) = D_{ab}^+(x) + D_{ab}^-(x) \quad (2.3)$$

is the causal Pauli-Jordan distribution associated to the two fields; it is (up to some numerical factors) a polynomial in the derivatives applied to the Pauli-Jordan distribution. We understand by $D_{ab}^{\pm}(x)$ the positive and negative parts of $D_{ab}(x)$. From (2.2) we have

$$[\xi_a(x), \xi_b(y)] = -i \ D_{ab}(x - y) \times 1 \quad (2.4)$$

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where by $[\cdot, \cdot]$ we mean the graded commutator.

The $n$-point functions for $n \geq 3$ are obtained assuming that the truncated Wightman functions are null: see [3], relations (8.74) and (8.75) and proposition 8.8 from there. The definition of these truncated Wightman functions involves the Fermi parities $|\xi_p|$ of the fields $\xi_p, p \in P$.

Afterwards we define

$$\xi_{a;\mu_1,\ldots,\mu_n}^\text{quant}(x) \equiv \partial_{\mu_1} \cdots \partial_{\mu_n} \xi_a^\text{quant}(x), a \in \mathcal{A}$$

which amounts to

$$[\xi_{a;\mu_1,\ldots,\mu_n}(x), \xi_{b;\nu_1,\ldots,\nu_n}(y)] = (-1)^n i \partial_{\mu_1} \cdots \partial_{\mu_n} \partial_{\nu_1} \cdots \partial_{\nu_n} D_{ab}(x - y) \times 1. \tag{2.5}$$

More sophisticated ways to define the free fields involve the GNS construction.

The free quantum fields are generating a Fock space $\mathcal{F}$ in the sense of the Borchers algebra: formally it is generated by states of the form $\xi_{a_1}^\text{quant}(x_1) \cdots \xi_{a_n}^\text{quant}(x_n)\Omega$ where $\Omega$ the vacuum state. The scalar product in this Fock space is constructed using the $n$-point distributions and we denote by $\mathcal{F}_0 \subset \mathcal{F}$ the algebraic Fock space.

One can prove that the quantum fields are free, i.e. they verify some free field equation; in particular every field must verify Klein Gordon equation for some mass $m$

$$(\square + m^2) \xi_a^\text{quant}(x) = 0 \tag{2.6}$$

and it follows that in momentum space they must have the support on the hyperboloid of mass $m$. This means that they can be split in two parts $\xi_a^\text{quant(\pm)}$ with support on the upper (resp. lower) hyperboloid of mass $m$. We convene that $\xi_a^\text{quant(+)}$ resp. $\xi_a^\text{quant(-)}$ correspond to the creation (resp. annihilation) part of the quantum field. The expressions $\xi_p^\text{quant(+)}$ resp. $\xi_p^\text{quant(-)}$ for a generic $\xi_p, p \in P$ are obtained in a natural way, applying partial derivatives. For a general discussion of this method of constructing free fields, see ref. [3] - especially prop. 8.8. We will follow essentially the presentation from [9]. The Wick monomials are leaving invariant the algebraic Fock space.
2.2 Yang-Mills Fields

First, we can generalize the preceding formalism to the case when some of the scalar fields are odd Grassmann variables. One simply insert everywhere the Fermi sign. The next generalization is to arbitrary vector and spinorial fields. If we consider for instance the Yang-Mills interaction Lagrangian corresponding to pure QCD then the jet variables \( \xi_a, a \in \Xi \) are \((v_\mu^a, u_a, \tilde{u}_a), a = 1, \ldots, r \) where \( v_\mu^a \) are Grassmann even and \( u_a, \tilde{u}_a \) are Grassmann odd variables.

The interaction Lagrangian is determined by gauge invariance. Namely we define the *gauge charge* operator by

\[
d_Q v_\mu^a = i \, d\mu u_a, \quad d_Q u_a = 0, \quad d_Q \tilde{u}_a = -i \, d\mu v_\mu^a, \ a = 1, \ldots, r
\]

where \( d\mu \) is the formal derivative. The gauge charge operator squares to zero:

\[
d_Q^2 \simeq 0
\]

where by \( \simeq \) we mean, modulo the equation of motion. Now we can define the interaction Lagrangian by the relative cohomology relation:

\[
d_Q T(x) \simeq \text{total divergence}.
\]

If we eliminate the corresponding coboundaries, then a tri-linear Lorentz covariant expression is uniquely given by

\[
T = f_{abc} \left( \frac{1}{2} v_{a\mu} v_{b\nu} F_{c}^{\mu\nu} + u_a v_\mu^b d_\mu \tilde{u}_c \right)
\]

where

\[
F_{a}^{\mu\nu} \equiv d^\mu v_\nu^a - d^\nu v_\mu^a, \ \forall a = 1, \ldots, r
\]

and \( f_{abc} \) are real and completely anti-symmetric. (This is the tri-linear part of the usual QCD interaction Lagrangian from classical field theory.)

Then we define the associated Fock space by the non-zero 2-point distributions are

\[
< \Omega, v_\mu^a(x_1)v_\nu^b(x_2)\Omega >= i \, \eta^{\mu\nu} \, \delta_{ab} \, D_0^{(+)}(x_1 - x_2), \\
< \Omega, u_a(x_1)\tilde{u}_b(x_2)\Omega >= -i \, \delta_{ab} \, D_0^{(+)}(x_1 - x_2), \\
< \Omega, \tilde{u}_a(x_1)u_b(x_2)\Omega >= i \, \delta_{ab} \, D_0^{(+)}(x_1 - x_2).
\]

and construct the associated Wick monomials. Then the expression (2.10) gives a Wick polynomial \( T^{\text{quant}} \) formally the same, but: (a) the jet variables must be replaced by the associated quantum fields; (b) the formal derivative \( d^\mu \) goes in the true derivative in the coordinate space; (c) Wick ordering should be done to obtain well-defined operators. We also have an associated *gauge charge* operator in the Fock space given by

\[
[Q, v_\mu^a] = i \, \partial^\mu u_a, \quad \{Q, u_a\} = 0, \quad \{Q, \tilde{u}_a\} = -i \, \partial_\mu v_\mu^a \quad Q\Omega = 0.
\]
Then it can be proved that $Q^2 = 0$ and

$$[Q, T^{\text{quant}}(x)] = \text{total divergence}$$  \hfill (2.14)

where the equations of motion are automatically used because the quantum fields are on-shell. From now on we abandon the super-script \textit{quant} because it will be obvious from the context if we refer to the classical expression (2.10) or to its quantum counterpart.

In (2.12) we are using the Pauli-Jordan distribution

$$D_m(x) = D_m^{(+)}(x) + D_m^{(-)}(x)$$  \hfill (2.15)

where

$$D_m^{(\pm)}(x) = \pm \frac{i}{(2\pi)^3} \int dp e^{-ip \cdot x} \theta(\pm p_0) \delta(p^2 - m^2)$$  \hfill (2.16)

and

$$D^{(-)}(x) = -D^{(+)}(-x).$$  \hfill (2.17)

We conclude our presentation with a generalization of (2.14). In fact, it can be proved that (2.14) implies the existence of Wick polynomials $T^\mu$ and $T^{\mu\nu}$ such that we have:

$$[Q, T^I] = i\partial^I T^I$$  \hfill (2.18)

for any multi-index $I$ with the convention $T^\emptyset \equiv T$. Explicitly:

$$T^\mu = f_{abc} u_a v_{b\nu} F^{\nu\mu}_c - \frac{1}{2} u_a u_b d^\mu \tilde{u}_c$$  \hfill (2.19)

and

$$T^{\mu\nu} = \frac{1}{2} f_{abc} u_a u_b F^{\mu\nu}_c.$$  \hfill (2.20)

Finally we give the relation expressing gauge invariance in order $n$ of the perturbation theory. We define the operator $\delta$ on chronological products by:

$$\delta T(T^{I_1}(x_1), \ldots, T^{I_n}(x_n)) \equiv \sum_{m=1}^n (-1)^{s_m} \partial^\mu_m T(T^{I_1}(x_1), \ldots, T^{I_{m\mu}}(x_m), \ldots, T^{I_n}(x_n))$$  \hfill (2.21)

with

$$s_m \equiv \sum_{p=1}^{m-1} |I_p|,$$  \hfill (2.22)

then we define the operator

$$s \equiv dQ - i\delta.$$  \hfill (2.23)

Gauge invariance in an arbitrary order is then expressed by

$$sT(T^{I_1}(x_1), \ldots, T^{I_n}(x_n)) = 0.$$  \hfill (2.24)
2.3 A More Precise Version of Wick Theorem

For simplicity, we assume in this Section that the variables $\xi_a$ (see Subsection 2.1) are commutative. We also use the summation convention over the dummy indices. In [9] we have proved the following result.

**Theorem 2.1** Let us consider that

$$A_1 = \frac{1}{3!} \ f_{pqr} \xi_p \xi_q \xi_r, \quad f_{pqr} = \text{completely symmetric} \quad (2.25)$$

and $A_2, \ldots, A_n$ are arbitrary. We define

$$T(A_1^{(1)}(x_1), \ldots, A_n(x_n)) \equiv T_1(A_1(x_1), \ldots, A_n(x_n)) = \frac{1}{2} f_{pqr} : \xi_p(x_1) T_0(\xi_q(x_1) \xi_r(x_1), A_2(x_2), \ldots, A_n(x_n)) : \quad (2.26)$$

$$T(A_1^{(2)}(x_1), \ldots, A_n(x_n)) \equiv T_2(A_1(x_1), \ldots, A_n(x_n)) = \frac{1}{2} f_{pqr} : \xi_p(x_1) \xi_q(x_1) T(\xi_r(x_1), A_2(x_2), \ldots, A_n(x_n)) : \quad (2.27)$$

$$T(A_1^{(3)}(x_1), \ldots, A_n(x_n)) \equiv T_3(A_1(x_1), \ldots, A_n(x_n)) = \frac{1}{3!} f_{pqr} : \xi_p(x_1) \xi_q(x_1) \xi_r(x_1) T(A_2(x_2), \ldots, A_n(x_n)) : \quad (2.28)$$

and

$$T_0 \equiv T - T_1 - T_2 - T_3. \quad (2.29)$$

Then $T_0$ is of Wick type only in $A_2, \ldots, A_n$.

We can iterate the arguments above in the entries $A_2, \ldots, A_n$ and obtain the following version of Wick theorem:

$$T(A_1(x_1), \ldots, A_n(x_n)) = \sum T(A_1^{(k_1)}(x_1), \ldots, A_n^{(k_n)}(x_n)) \quad (2.30)$$

where the sum runs over $k_1, \ldots, k_n = 0, \ldots, 3$ for $A_1, \ldots, A_n$ tri-linear. This formula can be written in a more transparent way if we use Hopf algebra notions - see [9].
2.4 Wick submonomials in the pure Yang-Mills case

We notice that in (2.13) and in (2.18) we have a pattern of the type:

\[ d_Q A = \text{total divergence}. \]  

(2.31)

This pattern remains true for Wick submonomials. We consider the expressions (2.10), (2.19) and (2.20) from the pure Yang-Mills case and define:

\[ B_{\alpha\mu} \equiv \tilde{u}_{\alpha,\mu} \cdot T = -f_{abc} u_b v_{c\mu} \]
\[ C_{\alpha\mu} \equiv v_{\alpha\mu} \cdot T = f_{abc} (v_{b\nu} F_{c\nu\mu} - u_b \tilde{u}_{c,\mu}) \]
\[ D_{\alpha} \equiv u_{\alpha} \cdot T = f_{abc} v_{b\mu} \tilde{u}_{c,\mu} \]
\[ E_{\alpha\mu\nu} \equiv v_{\alpha\mu,\nu} \cdot T = f_{abc} v_{b\mu} v_{c\nu} \]
\[ C'_{\alpha\nu\mu} \equiv v_{\alpha\nu} \cdot T_{\mu} = -f_{abc} u_b F_{c\nu\mu}. \]

(2.32)

We also have

\[ u_{\alpha} \cdot T = -C_{\alpha\mu} \]
\[ v_{\alpha\rho,\sigma} \cdot T_{\mu} = \eta_{\mu\sigma} B_{\alpha\rho} - \eta_{\mu\rho} B_{\alpha\sigma} \]
\[ u_{\alpha} \cdot T_{\mu\nu} = -C_{\alpha\mu\nu} \]  

(2.33)

If we define

\[ B_{\alpha} \equiv \frac{1}{2} f_{abc} u_b u_c \]  

(2.34)

we also have

\[ \tilde{u}_{\alpha,\nu} \cdot T_{\mu} = \eta_{\mu\nu} B_{\alpha} \]
\[ v_{\alpha\rho,\sigma} \cdot T_{\mu\nu} = (\eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\nu\sigma} \eta_{\mu\rho}) B_{\alpha}. \]

(2.35)

Then we try to extend the structure (2.31) to the Wick submonomials defined above. We have:

\[ d_Q B_{\alpha}^\mu = i d^\mu B_{\alpha} \]
\[ d_Q C_{\alpha}^{\mu\nu} = i d_\nu C_{\alpha}^{\mu} \]
\[ d_Q D_{\alpha} = -i d^\mu C_{\alpha}^{\mu} \]
\[ d_Q E_{\alpha}^{\mu\nu} = i (d^\nu B_{\alpha}^\mu - d^\mu B_{\alpha}^\nu + C_{\alpha}^{\mu\nu}) \]
\[ d_Q B_{\alpha} = 0 \]
\[ d_Q C_{\alpha}^{\mu\nu} = 0. \]  

(2.36)

So we see that the pattern (2.31) is broken only for \( E_{\alpha}^{\mu\nu} \). We fix this in the following way. We have the formal derivative

\[ \delta A \equiv d_\mu A^\mu \]  

(2.37)

used in the definition of gauge invariance (2.21) + (2.23); we also define the derivative \( \delta' \) by

\[ \delta' E_{\alpha}^{\mu\nu} = C_{\alpha}^{\mu\nu}. \] 

(2.38)
and 0 for the other Wick submonomials (2.32) and (2.34). Finally

\[ s \equiv d_Q - i\delta, \quad s' \equiv s - i\delta' = d_Q - i(\delta + \delta'). \]  

(2.39)

Then we have the structure

\[ s'A = 0 \]  

(2.40)

for all expressions \( A = T^I, B_{a\mu}, C_{a\mu}, \) etc. and also for the basic jet variables \( v_{a\mu}, u_a, \bar{u}_a. \)
2.5 Second Order Gauge Invariance. Tree Contributions

We first have:

\[ D(v^\mu_a(x_1), v^\nu_b(x_2)) \equiv [v^\mu_a(x_1), v^\nu_b(x_2)] = i \eta^{\mu\nu} \delta_{ab} D_0(x_1 - x_2), \]
\[ D(u_a(x_1), \bar{u}_b(x_2)) \equiv [u_a(x_1), \bar{u}_b(x_2)] = -i \delta_{ab} D_0(x_1 - x_2), \]
\[ D(\bar{u}_a(x_1), u_b(x_2)) \equiv [\bar{u}_a(x_1), u_b(x_2)] = i \delta_{ab} D_0(x_1 - x_2). \]  

(2.41)

where in the left hand side we have the graded commutator. The causal splitting \( D = D^{\text{adv}} - D^{\text{ret}} \) is unique because the degree of singularity of \( D_0 \) is \( \omega = -2 \) and we obtain

\[ T(u_a(x_1)^{(0)}, v_b^\nu(x_2)^{(0)}) = i \eta^{\mu\nu} \delta_{ab} D^{F}_0(x_1 - x_2), \]
\[ T(u_a(x_1)^{(0)}, \bar{u}_b(x_2)^{(0)}) = -i \delta_{ab} D^{F}_0(x_1 - x_2), \]
\[ T(\bar{u}_a(x_1)^{(0)}, u_b(x_2)^{(0)}) = i \delta_{ab} D^{F}_0(x_1 - x_2). \]  

(2.42)

From the previous relations we also have uniquely:

\[ T(\xi_{a,\mu}^0(x_1), \xi_{b,\nu}^0(x_2)) = \partial^1_D D^F(\xi_{a}(x_1), \xi_{b}(x_2)) \]
\[ T(\xi_{a}^0(x_1), \xi_{b,\nu}^0(x_2)) = \partial^2_D D^F(\xi_{a}(x_1), \xi_{b}(x_2)). \]

(2.43)

However the causal splitting of \( T(\xi_{a,\mu}(x_1)^{(0)}, \xi_{b,\nu}(x_2)^{(0)}) \) is not unique because the distribution has the degree of singularity \( \omega = 0 \). This was noticed for the first time in [1] and [6]. A possible choice is the \textit{canonical} splitting, following from (2.5):

\[ T(\xi_{a,\mu}^0(x_1), \xi_{b,\nu}^0(x_2)) = i \partial^1_D \partial^2_D T(\xi_{a}^0(x_1), \xi_{b}^0(x_2)). \]

(2.44)

From these formulas we can determine now if gauge invariance is true; in fact, we have anomalies, as it is well known, but they can be eliminated according to:

**Theorem 2.2** We have

\[ sT(T^{I_1}(x_1), T^{I_2}(x_2)) = s'T(T^{I_1}(x_1)^{(2)}, T^{I_2}(x_2)^{(2)}) \]  

(2.45)

and the anomalies given in the previous theorem can be eliminated if and only if the constants \( f_{abc} \) verify the Jacobi identity

\[ f_{cab} f_{ecd} + f_{eab} f_{cde} + f_{eca} f_{ebd} = 0 \]  

(2.46)

using the finite renormalizations:

\[ T(A_1(x_1), A_2(x_2)) \rightarrow T^{\text{ren}}(A_1(x_1), A_2(x_2)) = T(A_1(x_1), A_2(x_2)) + \delta(x_1 - x_2) N(A_1, A_2)(x_2) \]  

(2.47)
where

\[
N(T, T) \equiv i \frac{1}{2} E_{a}^{\mu \nu} E_{a}^{\mu \nu}
\]

\[
N(T^{\mu}, T) = N(T, T^{\mu}) \equiv -i B_{a}^{\mu} E_{a}^{\mu \nu}
\]

\[
N(T^{\mu}, T^{\nu}) \equiv i B_{a}^{\mu} B_{a}^{\nu}
\]

\[
N(T^{\mu \nu}, T) = N(T, T^{\mu \nu}) \equiv -i B_{a} E_{a}^{\mu \nu}
\]

\[
N(T^{\mu \nu}, T^{\rho}) = N(T^{\rho}, T^{\mu \nu}) = 0
\]

\[
N(T^{\mu \nu}, T^{\rho \sigma}) = 0
\]

The previous finite renormalizations can be obtained performing the finite renormalization

\[
N(v_{a \mu, \nu}, v_{b \rho, \sigma}) = i \frac{1}{2} \eta_{\mu \rho} \eta_{\nu \sigma} \delta_{ab}.
\]
3 Tree Contributions in the Third Order

If we apply \( (2.30) \) in the third order of the perturbation theory we obtain the following tree contributions

\[
T(T^{(2)}(x_1), T^{(2)}(x_2), T^{(1)}(x_3)) =
\]
\[
T(v^{(0)}_{a_1\mu_1}(x_1), v^{(0)}_{a_2\mu_2}(x_2), C^{\mu_3(0)}_{a_3}(x_3)) C^{\mu_1}_{a_1}(x_1) C^{\mu_2}_{a_2}(x_2) v_{a_3\mu_3}(x_3) 
\]
\[
+ \frac{1}{2} T(v^{(0)}_{a_1\mu_1}(x_1), v^{(0)}_{a_2\mu_2}(x_2), E^{\rho\sigma(0)}_{a_3}(x_3)) E^{\rho\sigma \sigma \lambda}_{a_1}(x_1) E^{\rho\sigma \sigma \lambda}_{a_2}(x_2) F_{a_3\sigma\rho}(x_3) 
\]
\[
+ \frac{1}{4} T(F^{(0)}_{a_1\sigma_1\rho_1}(x_1), F^{(0)}_{a_2\sigma_2\rho_2}(x_2), C^{\mu_3(0)}_{a_3}(x_3)) E^{\rho_1\sigma_1 \sigma_1 \lambda}_{a_1}(x_1) E^{\rho_2\sigma_2 \sigma_2 \lambda}_{a_2}(x_2) v_{a_3\mu_3}(x_3) 
\]
\[
+ \frac{1}{8} T(F^{(0)}_{a_1\lambda \sigma_1\rho_1}(x_1), F^{(0)}_{a_2\sigma_2\rho_2}(x_2), E^{\rho_3\sigma_1(0)}_{a_3}(x_3)) E^{\rho_1\sigma_1 \sigma_1 \lambda}_{a_1}(x_1) E^{\rho_2\sigma_2 \sigma_2 \lambda}_{a_2}(x_2) F_{a_3\sigma_3\rho_3}(x_3) 
\]
\[
+ \frac{1}{2} T(v^{(0)}_{a_1\nu}(x_1), F^{(0)}_{a_2\sigma\rho}(x_2), C^{\mu(0)}_{a_3}(x_3)) C^{\nu \lambda}_{a_1}(x_1) E^{\nu \rho \sigma \sigma \lambda}_{a_2}(x_2) v_{a_3\mu \rho \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
+ \frac{1}{4} T(v^{(0)}_{a_1\lambda}(x_1), F^{(0)}_{a_2\rho \sigma}(x_2), E^{\mu(0)}_{a_3}(x_3)) C^{\lambda \lambda}_{a_1}(x_1) E^{\rho \sigma \nu \mu \lambda}_{a_2}(x_2) F_{a_3\nu \mu \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
+ T(v^{(0)}_{a_1\nu}(x_1), u^{(0)}_{a_2 \nu}(x_2), D^{(0)}_{a_3}(x_3)) C^{\rho \nu \sigma \rho \lambda}_{a_1}(x_1) D^{\rho \nu \sigma \rho \lambda}_{a_2}(x_2) u_{a_3 \rho \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
+ \partial_{\mu}^{2} T(v^{(0)}_{a_1\nu}(x_1), \tilde{u}^{(0)}_{a_2 \nu}(x_2), B^{\mu(0)}_{a_3}(x_3)) C^{\rho \nu \sigma \rho \lambda}_{a_1}(x_1) B^{\rho \nu \sigma \rho \lambda}_{a_2}(x_2) \partial_{\mu} \tilde{u}_{a_3 \rho \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
+ \frac{1}{2} T(F^{(0)}_{a_1\rho \sigma}(x_1), u^{(0)}_{a_2 \nu}(x_2), D^{(0)}_{a_3}(x_3)) E^{\rho \sigma \nu \mu \lambda}_{a_1}(x_1) D^{\rho \sigma \nu \mu \lambda}_{a_2}(x_2) u_{a_3 \mu \nu \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
+ \frac{1}{2} \partial_{\nu}^{2} T(F^{(0)}_{a_1\rho \sigma}(x_1), \tilde{u}^{(0)}_{a_2 \nu}(x_2), B^{\mu(0)}_{a_3}(x_3)) E^{\rho \sigma \nu \mu \lambda}_{a_1}(x_1) B^{\rho \sigma \nu \mu \lambda}_{a_2}(x_2) \partial_{\nu} \tilde{u}_{a_3 \mu \nu \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
+ \frac{1}{4} T(v^{(0)}_{a_1\lambda}(x_1), F^{(0)}_{a_2\rho \sigma}(x_2), E^{\mu(0)}_{a_3}(x_3)) C^{\lambda \lambda}_{a_1}(x_1) E^{\rho \sigma \nu \mu \lambda}_{a_2}(x_2) F_{a_3\nu \mu \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
+ \frac{1}{2} \partial_{\mu}^{2} T(v^{(0)}_{a_1\nu}(x_1), \tilde{u}^{(0)}_{a_2 \nu}(x_2), B^{\mu(0)}_{a_3}(x_3)) C^{\rho \nu \sigma \rho \lambda}_{a_1}(x_1) B^{\rho \nu \sigma \rho \lambda}_{a_2}(x_2) \partial_{\mu} \tilde{u}_{a_3 \rho \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
+ \frac{1}{2} \partial_{\nu}^{2} T(v^{(0)}_{a_1\nu}(x_1), \tilde{u}^{(0)}_{a_2 \nu}(x_2), B^{\mu(0)}_{a_3}(x_3)) D^{\rho \nu \sigma \rho \lambda}_{a_1}(x_1) B^{\rho \nu \sigma \rho \lambda}_{a_2}(x_2) v_{a_3 \mu \nu \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]

\[
T(T^{(2)}(x_1), T^{(2)}(x_2), T^{\mu(1)}(x_3)) =
\]
\[
T(v^{(0)}_{a_1\mu_1}(x_1), v^{(0)}_{a_2\mu_2}(x_2), C^{\mu_3(0)}_{a_3}(x_3)) C^{\mu_1}_{a_1}(x_1) C^{\mu_2}_{a_2}(x_2) u_{a_3 \mu_3}(x_3) 
\]
\[
+ \frac{1}{4} T(F^{(0)}_{a_1\sigma_1\rho_1}(x_1), F^{(0)}_{a_2\sigma_2\rho_2}(x_2), C^{\mu_3(0)}_{a_3}(x_3)) E^{\rho_1\sigma_1 \sigma_1 \lambda}_{a_1}(x_1) E^{\rho_2\sigma_2 \sigma_2 \lambda}_{a_2}(x_2) u_{a_3 \mu_3}(x_3) 
\]
\[
+ \partial_{\mu}^{2} T(u^{(0)}_{a_1 \nu}(x_1), u^{(0)}_{a_2 \nu}(x_2), B^{\nu \mu \sigma \rho \lambda}_{a_3}(x_3)) C^{\rho \lambda \sigma \nu \rho \lambda}_{a_1}(x_1) B^{\rho \lambda \sigma \nu \rho \lambda}_{a_2}(x_2) v_{a_3 \rho \lambda \nu \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
+ \frac{1}{2} T(u^{(0)}_{a_1 \nu}(x_1), \tilde{u}^{(0)}_{a_2 \nu}(x_2), B^{\nu(0)}_{a_3}(x_3)) C^{\rho \nu \sigma \rho \lambda}_{a_1}(x_1) B^{\rho \nu \sigma \rho \lambda}_{a_2}(x_2) F_{a_3\rho \nu \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
+ \frac{1}{2} \partial_{\nu}^{2} T(u^{(0)}_{a_1 \nu}(x_1), \tilde{u}^{(0)}_{a_2 \nu}(x_2), B^{\mu(0)}_{a_3}(x_3)) E^{\rho \sigma \nu \mu \lambda}_{a_1}(x_1) B^{\rho \sigma \nu \mu \lambda}_{a_2}(x_2) v_{a_3 \mu \nu \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
+ \frac{1}{2} \partial_{\nu}^{2} T(u^{(0)}_{a_1 \nu}(x_1), \tilde{u}^{(0)}_{a_2 \nu}(x_2), C^{\mu(0)}_{a_3}(x_3)) D^{\rho \nu \sigma \rho \lambda}_{a_1}(x_1) B^{\rho \nu \sigma \rho \lambda}_{a_2}(x_2) u_{a_3 \mu \nu \lambda}(x_3) + (x_1 \leftrightarrow x_2) 
\]
\[
(3.1)
\]
\[ T(T^{(1)}(x_1), T^{(2)}(x_2), T^{\mu(2)}(x_3)) = \]
\[ T(C^{\rho(0)}_{\alpha_1}(x_1), v^{\alpha_2}_a(x_2), v^{\alpha_3}_a(x_3)) \, v_{a_1,\rho}(x_1) \, C^{\alpha_\sigma}_{a_2}(x_2) \, C^{\nu\mu}_{a_3}(x_3) \]
\[ -T(C^{\rho(0)}_{\alpha_1}(x_1), v^{(0)}_{\sigma}, F^{\mu(0)}_{\alpha_3}(x_3)) \, v_{a_1,\rho}(x_1) \, C^{\alpha_\sigma}_{a_2}(x_2) \, B_{a_3 \nu}(x_3) \]
\[ + \frac{1}{2} \, T(C^{\Lambda(0)}_{\alpha_1}(x_1), F^{\sigma}_{a_2 \rho}(x_2), v^{(0)}_{\alpha_3}(x_3)) \, v_{a_1,\lambda}(x_1) \, E^{\rho}_{a_2}(x_2) \, C^{\nu\mu}_{a_3}(x_3) \]
\[ - \frac{1}{2} \, T(C^{\rho(0)}_{\alpha_1}(x_1), F^{\delta}_{a_2 \rho}(x_2), F^{\mu\rho(0)}_{\alpha_3}(x_3)) \, v_{a_1,\lambda}(x_1) \, E^{\rho}_{a_2}(x_2) \, B_{a_3 \nu}(x_3) \]
\[ + \partial_3^\mu T(C^{\rho(0)}_{\alpha_1}(x_1), u^{\alpha_2}_a(x_2), \bar{u}^{\alpha_3}_a(x_3)) \, v_{a_1,\rho}(x_1) \, D_{a_2}(x_2) \, B_{a_3 \nu}(x_3) \]
\[ - \partial_3^\sigma T(C^{\rho(0)}_{\alpha_1}(x_1), \bar{u}^{\alpha_2}_a(x_2), u^{\alpha_3}_a(x_3)) \, v_{a_1,\rho}(x_1) \, B_{a_2}^{\sigma}(x_2) \, C^{\nu\mu}_{a_3}(x_3) \]
\[ + \frac{1}{2} \, T(E^{\rho(0)}_{a_1}(x_1), \nu^{(0)}_{\alpha_2}(x_2), v^{\nu(0)}_{\alpha_3}(x_3)) \, F_{a_1,\sigma\rho}(x_1) \, C^{\lambda}_{a_2}(x_2) \, C^{\nu\mu}_{a_3}(x_3) \]
\[ - \frac{1}{2} \, T(E^{\rho(0)}_{a_1}(x_1), \nu^{(0)}_{\alpha_2}(x_2), F^{\nu\mu(0)}_{a_3}(x_3)) \, F_{a_1,\sigma\rho}(x_1) \, C^{\lambda}_{a_2}(x_2) \, B_{a_3 \nu}(x_3) \]
\[ + \frac{1}{4} \, T(E^{\rho(0)}_{a_1}(x_1), F^{(0)}_{a_2 \sigma \rho_2}(x_2), v^{(0)}_{\alpha_3}(x_3)) \, F_{a_1,\sigma_1 \rho_1}(x_1) \, E^{\rho_2 \sigma_2}(x_2) \, C^{\nu\mu}(x_3) \]
\[ - \frac{1}{4} \, T(E^{\rho(0)}_{a_1}(x_1), F^{(0)}_{a_2 \sigma \rho_2}(x_2), F^{\nu\mu(0)}_{a_3}(x_3)) \, F_{a_1,\sigma_1 \rho_1}(x_1) \, E^{\rho_2 \sigma_2}(x_2) \, B_{a_3 \nu}(x_3) \]
\[ + T(D^{(0)}_{a_1}(x_1), \nu^{(0)}_{\alpha_2}(x_2), u^{(0)}_{\alpha_3}(x_3)) \, u_{a_1,\lambda}(x_1) \, C^{\alpha}_{a_2}(x_2) \, C^{\nu\mu}_{a_3}(x_3) \]
\[ + \frac{1}{2} \, T(D^{(0)}_{a_1}(x_1), F^{(0)}_{a_2 \sigma \rho}(x_2), u^{(0)}_{\alpha_3}(x_3)) \, u_{a_1,\rho}(x_1) \, E^{\rho}_{a_2}(x_2) \, C^{\nu\mu}_{a_3}(x_3) \]
\[ + T(D^{(0)}_{a_1}(x_1), F^{(0)}_{a_2 \sigma \rho}(x_2), u^{(0)}_{\alpha_3}(x_3)) \, u_{a_1,\lambda}(x_1) \, C^{\alpha}_{a_2}(x_2) \, B_{a_3 \nu}(x_3) \]
\[ -T(D^{(0)}_{a_1}(x_1), \nu^{(0)}_{\alpha_2}(x_2), F^{\nu\mu(0)}_{a_3}(x_3)) \, u_{a_1,\lambda}(x_1) \, D_{a_2}(x_2) \, C^{\nu\mu}_{a_3}(x_3) \]
\[ - \partial_3^\lambda T(B^{(0)}_{a_1}(x_1), \nu^{(0)}_{a_2}(x_2), u^{(0)}_{a_3}(x_3)) \, \partial_\lambda \bar{u}_{a_1}(x_1) \, C^{\sigma}_{a_2}(x_2) \, B_{a_3}(x_3) \]
\[ - \frac{1}{2} \, \partial_3^\rho T(B^{(0)}_{a_1}(x_1), F^{(0)}_{a_2 \sigma \rho}(x_2), \bar{u}^{(0)}_{a_3}(x_3)) \, \partial_\lambda \bar{u}_{a_1}(x_1) \, E^{\rho}_{a_2}(x_2) \, B_{a_3}(x_3) \]
\[ + \partial_3^\rho T(B^{(0)}_{a_1}(x_1), \bar{u}^{(0)}_{a_2}(x_2), v^{(0)}_{a_3}(x_3)) \, \partial_\lambda \bar{u}_{a_1}(x_1) \, B^{\sigma}_{a_2}(x_2) \, C^{\nu\mu}_{a_3}(x_3) \]
\[ - \partial_3^\rho T(B^{(0)}_{a_1}(x_1), \bar{u}^{(0)}_{a_2}(x_2), F^{\nu\mu(0)}_{a_3}(x_3)) \, \partial_\lambda \bar{u}_{a_1}(x_1) \, B^{\sigma}_{a_2}(x_2) \, B_{a_3 \nu}(x_3) \] (3.3)
\[
T(T^{\mu_1(2)}(x_1), T^{\mu_2(2)}(x_2), T^{(1)}(x_3)) =
T(v^{(0)}_{a_1\nu}(x_1), v^{(0)}_{a_2\nu}(x_2), C^{(0)}_{a_3}(x_3)) C^{\mu_1\mu_2}_{a_1}(x_1) C^{\nu_2\mu_2}_{a_2}(x_2) u_{a_3\rho}(x_3)
\]

\[
+ \frac{1}{2} T(v^{(0)}_{a_1\nu}(x_1), v^{(0)}_{a_2\nu}(x_2), E^{\rho\sigma(0)}_{a_3}(x_3)) C^{\mu_1\mu_2}_{a_1}(x_1) C^{\nu_2\mu_2}_{a_2}(x_2) F_{\alpha_3\sigma\rho}(x_3)
\]

\[
+ T(F^{\nu_1\rho\mu_1(0)}_{a_1}(x_1), F^{\nu_2\mu_2(0)}_{a_2}(x_2), C^{(0)}_{a_3}(x_3)) B_{a_1\nu_1}(x_1) B_{a_2\nu_2}(x_2) u_{a_3\rho}(x_3)
\]

\[
+ \frac{1}{2} T(F^{\nu_1\rho\mu_1(0)}_{a_1}(x_1), F^{\nu_2\mu_2(0)}_{a_2}(x_2), E^{\rho\sigma(0)}_{a_3}(x_3)) B_{a_1\nu_1}(x_1) B_{a_2\nu_2}(x_2) F_{a_3\sigma\rho}(x_3)
\]

\[
T(T^{\mu_1(2)}(x_1), T^{\mu_2(2)}(x_2), T^{(2)}(x_3)) =
\]

\[
- \partial^3_{\rho} T(u^{(0)}_{a_1\nu}(x_1), C^{\nu_2\mu_2}_{a_2}(x_2), \bar{u}^{(0)}_{a_3}(x_3)) C^{\mu_1\nu_1}_{a_1}(x_1) u_{a_2\nu}(x_2) B^\rho_{a_3}(x_3)
\]

\[
+ T(v^{(0)}_{a_1\nu}(x_1), C^{\mu_2\nu_2}_{a_2}(x_2), v^{(0)}_{a_3\rho}(x_3)) C^{\nu_1\mu_1}_{a_1}(x_1) u_{a_2\nu}(x_2) C^\rho_{a_3}(x_3)
\]

\[
+ \frac{1}{2} T(v^{(0)}_{a_1\nu}(x_1), C^{\mu_2\nu_2}_{a_2}(x_2), F^{(0)}_{a_3\sigma\rho}(x_3)) C^{\mu_1\nu_1}_{a_1}(x_1) u_{a_2\nu}(x_2) E^{\sigma\rho}_{a_3}(x_3)
\]

\[
+ \partial^3_{\rho} T(v^{(0)}_{a_1\nu}(x_1), C^{\mu_2\nu_2}_{a_2}(x_2), u^{(0)}_{a_3}(x_3)) C^{\nu_1\mu_1}_{a_1}(x_1) v_{a_2\rho}(x_2) B^\rho_{a_3}(x_3)
\]

\[
+ \partial^3_{\rho} T(v^{(0)}_{a_1\nu}(x_1), B^{(0)}_{a_2\rho}(x_2), \bar{u}^{(0)}_{a_3}(x_3)) C^{\nu_1\mu_1}_{a_1}(x_1) F^{\nu_2\mu_2}_{a_2}(x_2) B^\rho_{a_3}(x_3)
\]

\[
- T(F^{\nu_1\mu_1(0)}_{a_1}(x_1), C^{\nu_2\mu_2(0)}_{a_2}(x_2), v^{(0)}_{a_3\rho}(x_3)) B_{a_1\nu_1}(x_1) u_{a_2\nu}(x_2) C^\rho_{a_3}(x_3)
\]

\[
- \frac{1}{2} T(F^{\nu_1\mu_1(0)}_{a_1}(x_1), C^{\mu_2\mu_2(0)}_{a_2}(x_2), F^{(0)}_{a_3\sigma\rho}(x_3)) B_{a_1\nu_1}(x_1) u_{a_2\nu}(x_2) E^{\sigma\rho}_{a_3}(x_3)
\]

\[
- \partial^3_{\rho} T(F^{\nu_1\mu_1(0)}_{a_1}(x_1), C^{\mu_2\mu_2(0)}_{a_2}(x_2), \bar{u}^{(0)}_{a_3}(x_3)) B_{a_1\nu_1}(x_1) v_{a_2\rho}(x_2) B^\rho_{a_3}(x_3)
\]

\[
- \partial^3_{\rho} T(F^{\nu_1\mu_1(0)}_{a_1}(x_1), B^{(0)}_{a_2\rho}(x_2), \bar{u}^{(0)}_{a_3}(x_3)) B_{a_1\nu_1}(x_1) F^{\nu_2\mu_2}_{a_2}(x_2) B^\rho_{a_3}(x_3)
\]

\[
+ \partial^3_{\rho} T(\bar{u}^{(0)}_{a_1}(x_1), C^{\nu_2\mu_2}_{a_2}(x_2), u^{(0)}_{a_3}(x_3)) B_{a_1}(x_1) u_{a_2\nu}(x_2) D^\rho_{a_3}(x_3)
\]

\[
- \partial^3_{\rho} T(\bar{u}^{(0)}_{a_1}(x_1), B^{(0)}_{a_2\rho}(x_2), v^{(0)}_{a_3\sigma}(x_3)) B_{a_1}(x_1) v_{a_2\rho}(x_2) C^\sigma_{a_3}(x_3)
\]

\[
- \frac{1}{2} \partial^3_{\mu_1} T(\bar{u}^{(0)}_{a_1}(x_1), C^{\nu_2\mu_2}_{a_2}(x_2), F^{(0)}_{a_3\sigma\rho}(x_3)) B_{a_1}(x_1) v_{a_2\nu}(x_2) E^{\sigma\rho}_{a_3}(x_3)
\]

\[
- \partial^5_{\mu_1} T(\bar{u}^{(0)}_{a_1}(x_1), B^{(0)}_{a_2\rho}(x_2), v^{(0)}_{a_3\sigma}(x_3)) B_{a_1}(x_1) F^{\mu_2\nu_2}_{a_2}(x_2) C^\sigma_{a_3}(x_3)
\]

\[
- \frac{1}{2} \partial^5_{\mu_1} T(\bar{u}^{(0)}_{a_1}(x_1), B^{(0)}_{a_2\rho}(x_2), F^{(0)}_{a_3\sigma\rho}(x_3)) B_{a_1}(x_1) F^{\nu_2\mu_2}_{a_2}(x_2) E^{\rho\sigma}_{a_3}(x_3)
\]

\[
- \partial^5_{\mu_1} \partial^3_{\rho} T(\bar{u}^{(0)}_{a_1}(x_1), B^{(0)}_{a_2\rho}(x_2), \bar{u}^{(0)}_{a_3}(x_3)) B_{a_1}(x_1) \partial^3_{\rho} \bar{u}_{a_2}(x_2) B^\rho_{a_3}(x_3)
\]
\[ T(T^{(2)}(x_1), T^{(2)}(x_2), T^\mu \nu(x_3)) = \]
\[-\partial_\rho \partial_\sigma^2 T(\bar{v}_{a_1}^{(0)}(x_1), \bar{u}_{a_2}^{(0)}(x_2), B_{a_3}^{(0)}(x_3)) B_{a_1}^{\rho} (x_1) B_{a_2}^{\sigma} (x_2) F_{a_3}^{\mu \nu}(x_3) \]
\[-\partial_\sigma^2 T(v_{a_1}^{(0)}(x_1), \bar{\bar{u}}_{a_2}^{(0)}(x_2), C_{a_3}^{\mu \nu}(x_3)) C_{a_1}^\rho (x_1) B_{a_2}^\sigma (x_2) u_{a_3} (x_3) + (x_1 \leftrightarrow x_2) \]
\[-\frac{1}{2} \partial_\lambda T(F_{a_1}^{\rho \sigma} (x_1), \bar{\bar{u}}_{a_2}^{(0)}(x_2), C_{a_3}^{\mu \nu}(x_3)) E_{a_1}^{\rho \sigma} (x_1) B_{a_2}^\lambda (x_2) u_{a_3} (x_3) + (x_1 \leftrightarrow x_2) \]  
(3.6) 

\[ T(T^{(1)}(x_1), T^{(2)}(x_2), T^{\mu \nu}(x_3)) = \]
\[ T(C_{a_1}^{\rho \sigma} (x_1), v_{a_2}^{(0)}(x_2), F_{a_3}^{\mu \nu}(x_3)) v_{a_1 \rho} (x_1) C_{a_2}^{\sigma} (x_2) B_{a_3} (x_3) \]
\[ + \frac{1}{2} T(C_{a_1}^{\lambda \rho}(x_1), F_{a_2}^{\rho \sigma}(x_2), F_{a_3}^{\mu \nu}(x_3)) v_{a_1 \lambda}(x_1) E_{a_2}^{\rho \sigma}(x_2) B_{a_3}(x_3) \]
\[ + \frac{1}{2} T(C_{a_1}^{\mu \nu}(x_1), \bar{u}_{a_2}^{(0)}(x_2), u_{a_3}^{(0)}(x_3)) v_{a_1 \rho}(x_1) B_{a_2}^\rho (x_2) C_{a_3}^{\mu \nu}(x_3) \]
\[ + \frac{1}{2} T(E_{a_1}^{\rho \sigma}(x_1), \bar{u}_{a_2}^{(0)}(x_2), F_{a_3}^{\mu \nu}(x_3)) F_{a_1 \rho \sigma}(x_1) C_{a_2}^{\lambda \rho}(x_2) B_{a_3}(x_3) \]
\[ + \frac{1}{4} T(e_{a_1}^{\rho \sigma \tau}(x_1), F_{a_2}^{\mu \nu}(x_2), F_{a_3}^{\mu \nu}(x_3)) F_{a_1 \rho \sigma \tau}(x_1) E_{a_2}^{\rho \sigma \tau}(x_2) B_{a_3}(x_3) \]
\[ - T(D_{a_1}^{\rho \sigma}(x_1), v_{a_2}^{(0)}(x_2), u_{a_3}^{(0)}(x_3)) u_{a_1}(x_1) C_{a_2}^{\rho \sigma}(x_2) C_{a_3}^{\mu \nu}(x_3) \]
\[ - \frac{1}{2} T(D_{a_1}^{\rho \sigma}(x_1), F_{a_2}^{\rho \sigma}(x_2), u_{a_3}^{(0)}(x_3)) u_{a_1}(x_1) E_{a_2}^{\rho \sigma}(x_2) C_{a_3}^{\mu \nu}(x_3) \]
\[ + T(D_{a_1}^{\rho \sigma}(x_1), u_{a_2}^{(0)}(x_2), F_{a_3}^{\mu \nu}(x_3)) u_{a_1}(x_1) D_{a_2}^{\rho \sigma}(x_2) B_{a_3}(x_3) \]
\[ + \frac{1}{2} T(B_{a_1}^{\rho \sigma}(x_1), \bar{u}_{a_2}^{(0)}(x_2), F_{a_3}^{\mu \nu}(x_3)) \partial_\rho \bar{u}_{a_1}(x_1) B_{a_2}^\rho (x_2) B_{a_3}(x_3) \]  
(3.7)
\[
T(T^{\mu_1 \nu_1}(x_1), T^{\mu_2 \nu_2}(x_2), T^{\mu_3 \nu_3}(x_3)) =
\]
\[
T(v_{a_1}^{(0)}(x_1), v_{a_2}^{(0)}(x_2), C^{(0)}_{a_3}(x_3)) C_{a_1}^{\mu_1}(x_1) C_{a_2}^{\mu_2}(x_2) u_{a_3}(x_3)
\]
\[
+ T(F^{\mu_1 \nu_1}(x_1), F^{\mu_2 \nu_2}(x_2), C^{(0)}_{a_3}(x_3)) B_{a_1} v_{a_1}(x_1) B_{a_2} v_{a_2}(x_2) u_{a_3}(x_3)
\]
\[
- \partial_1^\mu \partial_2^\nu (\tilde{T}_{a_1}^{(0)}(x_1), \tilde{T}_{a_2}^{(0)}(x_2), B_{a_3}(x_3)) B_{a_1}(x_1) B_{a_2}(x_2) \partial_3^\mu \tilde{u}_{a_3}(x_3)
\]
(3.8)

\[
T(T^{\mu \nu}(x_1), T^\rho(x_2), T^{(1)}(x_3)) =
\]
\[
- T(F_{a_1}^{\mu \nu}(x_1), u_{a_2}(x_2), D_{a_3}(x_3)) B_{a_1}(x_1) C_{a_2}^{\nu}(x_2) u_{a_3}(x_3)
\]
\[
+ \frac{1}{2} T(F_{a_1}^{\mu \rho}(x_1), v_{a_2}(x_2), C_{a_3}^{\rho}(x_3)) B_{a_1}(x_1) C_{a_2}^{\sigma}(x_2) F_{a_3}^{\sigma \rho}(x_3)
\]
\[
- \frac{1}{2} T(F_{a_1}^{\mu \rho}(x_1), F_{a_2}^{\rho \sigma}(x_2), E_{a_3}^{\beta}(x_3)) B_{a_1}(x_1) B_{a_2} C_{a_3}^{\rho \sigma \beta}(x_3)
\]
\[
+ \partial_1^\rho T(F_{a_1}^{\mu \nu}(x_1), \tilde{u}_{a_2}(x_2), B_{a_3}(x_3)) B_{a_1}(x_1) B_{a_2} \tilde{u}_{a_3}(x_3)
\]
\[
+ \frac{1}{2} T(u_{a_1}(x_1), v_{a_2}(x_2), D_{a_3}(x_3)) C_{a_1}^{\mu}(x_1) C_{a_2}^{\sigma}(x_2) u_{a_3}(x_3)
\]
\[
- T(u_{a_1}(x_1), F_{a_2}^{\rho \sigma}(x_2), D_{a_3}(x_3)) C_{a_1}^{\mu}(x_1) B_{a_2} C_{a_3}^{\mu \sigma}(x_3)
\]
\[
- \partial_2^\sigma T(u_{a_1}(x_1), \tilde{u}_{a_2}(x_2), C_{a_3}^{\mu}(x_3)) C_{a_1}^{\mu}(x_1) B_{a_2} u_{a_3}(x_3)
\]
(3.9)

\[
T(T^{\mu \nu}(x_1), T^\rho(x_2), T^{(2)}(x_3)) =
\]
\[
T(F_{a_1}^{\mu \rho}(x_1), C_{a_2}^{\rho}(x_2), v_{a_3}(x_3)) B_{a_1}(x_1) u_{a_2}(x_2) C_{a_3}^{\mu}(x_3)
\]
\[
+ \frac{1}{2} T(F_{a_1}^{\mu \rho}(x_1), C_{a_2}^{\rho}(x_2), F_{a_3}^{\rho \beta}(x_3)) B_{a_1}(x_1) u_{a_2}(x_2) F_{a_3}^{\rho \beta}(x_3)
\]
\[
+ \partial_3^\beta T(F_{a_1}^{\mu \rho}(x_1), C_{a_2}^{\rho \sigma}(x_2), \tilde{u}_{a_3}(x_3)) B_{a_1}(x_1) v_{a_2}(x_2) B_{a_3}(x_3)
\]
\[
+ \partial_3^\alpha T(F_{a_1}^{\mu \rho}(x_1), B_{a_2}^{\rho \sigma}(x_2), \tilde{u}_{a_3}(x_3)) B_{a_1}(x_1) F_{a_2}^{\rho \sigma}(x_2) B_{a_3}(x_3)
\]
\[
- \partial_3^\tau T(u_{a_1}(x_1), C_{a_2}^{\rho}(x_2), \tilde{u}_{a_3}(x_3)) C_{a_1}^{\mu}(x_1) u_{a_2}(x_2) B_{a_3}(x_3)
\]
(3.10)
\[
T(T^{\mu
u(1)}(x_1), T^{\nu(2)}(x_2), T^{(2)}(x_3)) = \partial_3^\lambda T(C^{\mu\nu(0)}_{a_1}(x_1), v^{(0)}_{a_2}(x_2), \tilde{u}^{(0)}_{a_3}(x_3)) u_{a_1}(x_1) C^{a_2}_a(x_2) B^{a_3}_a(x_3) + \partial_3^\lambda T(C^{\mu\nu(0)}_{a_1}(x_1), F^{\sigma(0)}_{a_2}(x_2), \tilde{u}^{(0)}_{a_3}(x_3)) u_{a_1}(x_1) B_{a_2\sigma}(x_2) B^{a_3}_a(x_3) + \partial_2^\rho T(C^{\mu\nu(0)}_{a_1}(x_1), \tilde{u}^{(0)}_{a_2}(x_2), v^{(0)}_{a_3}(x_3)) u_{a_1}(x_1) B_{a_2}(x_2) C^{a_3}_a(x_3) + \frac{1}{2} \partial_2^\rho T(C^{\mu\nu(0)}_{a_1}(x_1), \tilde{u}^{(0)}_{a_2}(x_2), F^{(0)}_{a_3\beta}(x_3)) u_{a_1}(x_1) B_{a_2}(x_2) F^{\alpha\beta}_{a_3}(x_3) - \partial_2^\rho \partial_3^\sigma T(B^{(0)}_{a_1}(x_1), \tilde{u}^{(0)}_{a_2}(x_2), \tilde{u}^{(0)}_{a_3}(x_3)) F^{\mu\nu}_{a_1}(x_1) B_{a_2}(x_2) B_{a_3\sigma}(x_3)
\]

(3.11)

All the coefficients from the 11 equations from above are in fact chronological products of the type \(T(\xi^{(0)}_1(x_1), \xi^{(0)}_2(x_2), (\eta_1 \eta_2)^{(0)}(x_3))\). In the simplest case when all variables are even we have

\[
T(\xi^{(0)}_1(x_1), \xi^{(0)}_2(x_2), (\eta_1 \eta_2)^{(0)}(x_3)) = D^F(\xi_1(x_1), \eta_1(x_3)) D^F(\xi_2(x_2), \eta_2(x_3)) + D^F(\xi_1(x_1), \eta_2(x_3)) D^F(\xi_2(x_2), \eta_1(x_3))
\]

(3.12)

and in the general case one has to introduce appropriate signs taking care of the Grassmann parities.

One can see from this formula that there will be two contributions: a canonical splitting contributions given by (2.44) i.e. we can pull out the derivatives from the chronological product, and a finite renormalization contribution due to (2.49) and from this relation

\[
N(F^{\mu\nu}_a, F^{\nu\sigma}_b) = i \left( \eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\nu\rho} \eta^{\mu\sigma} \right) \delta_{ab}.
\]

(3.13)

From these formulas we get

\[
T(F^{\mu\nu(0)}_{a_1}(x_1), v^{(0)}_{a_2}(x_2), C^{\sigma(0)}_{a_3}(x_3))^{\text{ren}} = \partial_1^\mu T(v^{(0)}_{a_1}(x_1), v^{(0)}_{a_2}(x_2), C^{\sigma(0)}_{a_3}(x_3)) - (\mu \leftrightarrow \nu) + f_{a_1a_2a_3} \eta^{\mu\rho} \eta^{\nu\sigma} \delta(x_1 - x_3) D^F_\rho(x_2 - x_3) - (\mu \leftrightarrow \nu)
\]

(3.14)

and

\[
T(F^{\mu\nu(0)}_{a_1}(x_1), F^{\rho\sigma(0)}_{a_2}(x_2), C^{\lambda(0)}_{a_3}(x_3))^{\text{ren}} = [\partial_1^\mu \partial_1^\rho T(v^{(0)}_{a_1}(x_1), v^{(0)}_{a_2}(x_2), C^{\lambda(0)}_{a_3}(x_3)) - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma) - f_{a_1a_2a_3} \left( \left( \eta^{\mu\rho} \eta^{\nu\lambda} \right) D^F_\rho(x_1 - x_3) \delta(x_2 - x_3) - (\mu \leftrightarrow \nu) \right) - (\rho \leftrightarrow \sigma)
\]

(3.15)

and

\[
T(\tilde{u}^{(0)}_{a_1}(x_1), F^{\mu\nu(0)}_{a_2}(x_2), C^{\rho\sigma(0)}_{a_3}(x_3))^{\text{ren}} = \partial_1^\rho T(\tilde{u}^{(0)}_{a_1}(x_1), v^{(0)}_{a_2}(x_2), C^{\rho\sigma(0)}_{a_3}(x_3)) - (\mu \leftrightarrow \nu) + f_{a_1a_2a_3} \eta^{\mu\rho} \eta^{\nu\sigma} D^F_\rho(x_1 - x_3) \delta(x_2 - x_3) - (\mu \leftrightarrow \nu)
\]

(3.16)

where one can see the two contributions. Using these three relations we can obtain the finite renormalization contributions to the relations (3.11) - (3.11).
Theorem 3.1  Following the finite renormalization

\[ D(v_{a\mu,\nu}(x), v_{b\rho,\sigma}(y))^f = \frac{i}{2} \eta_{\mu\rho} \eta_{\nu\sigma} \delta_{ab} \delta(x - y) \Rightarrow \]

\[ D(F_{a}^{\mu\nu}(x), F_{b}^{\rho\sigma}(y))^f = i (\eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\rho\sigma} \eta_{\mu\nu}) \delta_{ab} \delta(x - y) \]  

(3.17)

we have the following finite renormalizations of the expressions (3.1) - (3.11):

\[ T(T^{(2)}(x_1), T^{(2)}(x_2), T^{(1)}(x_3))^f = \]

\[ f_{a_1a_2a_3} \left\{ \delta(x_1 - x_3) \frac{D_F^a}{D_0^a} (x_2 - x_3) (E_{a_1}^{\mu\rho} v_{a_3\rho})(x_1) C_{a_2}^{\sigma} (x_2) \right\} 
\]

\[-\delta(x_1 - x_3) \frac{\partial_{\mu} D_F^a}{D_0^a} (x_2 - x_3) (E_{a_1}^{\rho\sigma} \nu_{a_3})(x_1) F_{a_2}^{\nu\mu}(x_2)] + (x_1 \leftrightarrow x_2) \]  

(3.18)

\[ T(T^{(2)}(x_1), T^{(2)}(x_2), T^{\mu(1)}(x_3))^f = \]

\[ f_{a_1a_2a_3} \left\{ -\delta(x_1 - x_3) \frac{D_F^a}{D_0^a} (x_2 - x_3) (E_{a_1}^{\mu\rho} u_{a_3})(x_1) C_{a_2\nu} (x_2) \right\} 
\]

\[-\delta(x_1 - x_3) \frac{\partial_{\nu} D_F^a}{D_0^a} (x_2 - x_3) [(E_{a_2}^{\rho\sigma} \nu_{a_3})(x_1) B_{a_1}^{\mu}(x_2) + (E_{a_1}^{\rho\sigma} u_{a_3})(x_1) E_{a_2}^{\nu\mu}(x_2)] \]  

(3.19)

\[ T(T^{(1)}(x_1), T^{(2)}(x_2), T^{\mu(2)}(x_3))^f = \]

\[ f_{a_1a_2a_3} \left\{ \delta(x_3 - x_1) \frac{D_F^a}{D_0^a} (x_2 - x_1) [(v_{a_1}^{\mu} B_{a_3\rho})(x_3) C_{a_2}^{\nu}(x_2) - (v_{a_1}^{\rho} B_{a_3\nu})(x_3) C_{a_2}^{\mu}(x_2)] \right\} 
\]

\[-\delta(x_2 - x_1) \frac{D_F^a}{D_0^a} (x_3 - x_1) (v_{a_1}^{\lambda} E_{a_2\nu})(x_2) C_{a_3}^{\mu}(x_3) \]  

(3.20)

\[ T(T^{\mu_1(2)}(x_1), T^{\mu_2(2)}(x_2), T^{(1)}(x_3))^f = \]

\[-f_{a_1a_2a_3} \left\{ \delta(x_1 - x_3) \frac{D_F^a}{D_0^a} (x_2 - x_3) [(B_{a_2}^{\rho\sigma} v_{a_3\rho})(x_1) C_{a_1}^{\mu\nu}(x_2) - (B_{a_2}^{\mu\nu} v_{a_3})(x_1) C_{a_1}^{\rho\sigma}(x_2)] \right\} 
\]

\[-\delta(x_1 - x_3) \frac{\partial_{\mu} D_F^a}{D_0^a} (x_2 - x_3) [(B_{a_1\rho}^{\nu} v_{a_3})(x_1) B_{a_2}^{\mu}(x_2) - (B_{a_1}^{\mu\nu} v_{a_3\rho})(x_1) B_{a_2}^{\nu\mu}(x_2)] \]  

(3.21)

\[ T(T^{\mu_1(2)}(x_1), T^{\mu_2(2)}(x_2), T^{(2)}(x_3))^f = \]

\[ f_{a_1a_2a_3} \left\{ \delta(x_3 - x_2) \frac{D_F^a}{D_0^a} (x_1 - x_2) C_{a_1}^{\mu\nu}(x_1) (u_{a_2} E_{a_3}^{\nu\mu})(x_3) \right\} 
\]

\[-\delta(x_1 - x_2) \frac{\partial_{\mu} D_F^a}{D_0^a} (x_3 - x_2) [(B_{a_2}^{\rho\sigma} u_{a_3})(x_1) C_{a_1}^{\mu\nu}(x_2) - (B_{a_2}^{\mu\nu} u_{a_3\rho})(x_1) C_{a_1}^{\rho\sigma}(x_2)] \]  

(3.22)
\[ T(T^{(2)}(x_1), T^{(2)}(x_2), T_{\mu\nu}^{(1)}(x_3))^f = \\
fa_{12} [\delta(x_1 - x_2) \partial_\mu D_0^F(x_2 - x_3) (E_{a_1}^{\mu\nu} u_{a_3})(x_1) B_{a_2}^\nu(x_2) + (x_1 \leftrightarrow x_2) ] (3.23) \]

\[ T(T^{(1)}(x_1), T^{(2)}(x_2), T_{\mu\nu}^{(2)}(x_3))^f = \\
fa_{12} [ -\delta(x_3 - x_1) \partial_\mu D_0^F(x_2 - x_1) (u_{a_1}^\nu B_{a_3})(x_3) C_{a_1}^\mu(x_2) \\
-\delta(x_3 - x_1) \partial_\nu D_0^F(x_2 - x_1) (u_{a_1}^\nu B_{a_3})(x_3) E_{a_2}^\mu(x_2) \\
-\delta(x_2 - x_1) \partial_\nu D_0^F(x_3 - x_1) (u_{a_1}^\nu E_{a_2}^\nu)(x_2) B_{a_3}(x_3)] - (\mu \leftrightarrow \nu) (3.24) \]

\[ T(T^{\mu_1(2)}(x_1), T^{\mu_2(2)}(x_2), T_{\mu_3}^{(1)}(x_3))^f = \\
-\fa_{12} \{ \delta(x_1 - x_3) D_0^F(x_2 - x_3) [\eta^{\mu_1\mu_3} (B_{a_2} u_{a_3})(x_1) C_{a_1}^{\mu_3}(x_2) \\
- (B_{a_2} u_{a_3})(x_1) C_{a_1}^{\mu_3}(x_2)] \\
+\delta(x_1 - x_3) \partial_{\mu_2} D_0^F(x_2 - x_3) (\eta^{\mu_1\mu_3} B_{a_1} u_{a_3} - B_{a_1} u_{a_3})(x_1) B_{a_2}(x_2) \\
+\eta^{\mu_1\mu_3} \delta(x_1 - x_3) \partial_\nu D_0^F(x_1 - x_2) (B_{a_1} u_{a_3})(x_1) B_{a_2}(x_2) \\
-\eta^{\mu_1\mu_3} \delta(x_1 - x_3) \partial_{\mu_2} D_0^F(x_1 - x_2) (B_{a_1} u_{a_3})(x_1) B_{a_2}(x_2) \\
-\eta^{\mu_1\mu_2} \delta(x_1 - x_3) \partial_\nu D_0^F(x_2 - x_3) (B_{a_1} u_{a_3})(x_1) B_{a_2}(x_2) \\
+\delta(x_1 - x_3) \partial_{\mu_2} D_0^F(x_2 - x_3) (B_{a_1} u_{a_3})(x_1) B_{a_2}(x_2) \} \\
-(x_1 \leftrightarrow x_2, \mu_1 \leftrightarrow \mu_2) (3.25) \]

\[ T(T^{\mu(2)}(x_1), T^{(2)}(x_2), T^{(1)}(x_3))^f = \\
\fa_{12} [ -\delta(x_1 - x_2) D_0^F(x_2 - x_3) (B_{a_1} u_{a_3})(x_1) C_{a_1}^{\mu}(x_2) \\
- \eta^{\mu} \delta(x_1 - x_3) \partial_\nu D_0^F(x_2 - x_3) (B_{a_1} u_{a_3})(x_1) B_{a_2}(x_2) \\
-\delta(x_1 - x_3) \partial_\nu D_0^F(x_2 - x_3) (B_{a_1} u_{a_3})(x_1) B_{a_2}(x_2) \\
+\delta(x_2 - x_3) \partial_\nu D_0^F(x_1 - x_3) (B_{a_1} u_{a_3})(x_1) B_{a_2}(x_2) \\
+\eta^{\mu} \delta(x_2 - x_3) \partial_\nu D_0^F(x_1 - x_3) (B_{a_1} u_{a_3})(x_1) C_{a_1}^{\mu}(x_2) \} \\
-(\mu \leftrightarrow \nu) (3.26) \]

\[ T(T^{\mu(2)}(x_1), T^{(1)}(x_2), T^{(2)}(x_3))^f = \\
\fa_{12} \{ -\delta(x_1 - x_2) D_0^F(x_3 - x_2) \eta^{\rho}(B_{a_1} u_{a_2})(x_1) C_{a_3}^{\rho}(x_3) \\
+\delta(x_1 - x_2) \partial_\lambda D_0^F(x_3 - x_2) \eta^{\rho}[(B_{a_1} u_{a_2})(x_1) E_{a_3}^{\mu}(x_3) + (B_{a_1} u_{a_2})(x_1) B_{a_3}^{\mu}(x_3)] \\
-\delta(x_3 - x_2) \partial_\nu D_0^F(x_1 - x_2) (u_{a_2} E_{a_3}^{\mu}(x_3)] \\
-(\mu \leftrightarrow \nu) (3.27) \]

\[ T(T^{\mu(1)}(x_1), T^{(2)}(x_2), T^{(2)}(x_3))^f = \\
\fa_{12} \{ \eta^{\rho} \delta(x_2 - x_1) \partial_\lambda D_0^F(x_3 - x_1) (u_{a_1} B_{a_2}^{\mu}(x_2) B_{a_3}^{\lambda}(x_3) - (\mu \leftrightarrow \nu)] \\
+\delta(x_3 - x_1) \partial_\nu D_0^F(x_2 - x_1) (u_{a_1} E_{a_3}^{\mu})(x_3) \} (3.28) \]
Now we can verify gauge invariance for the tree contributions in the third order. We first have

\[
sT(T^{(2)}(x_1), T^{(2)}(x_2), T^{(1)}(x_3))^f = \]

\[
[\delta(x_1 - x_3) \ D^F_0(x_2 - x_3) \ A_f(x_1, x_2) + \delta(x_1 - x_3) \ \partial_\mu D^F_0(x_2 - x_3) \ B^\mu_f(x_1, x_2) \\
+ \partial_\mu \delta(x_1 - x_3) \ D^F_0(x_2 - x_3) \ C^\mu_f(x_1, x_2) + \partial_\mu \delta(x_1 - x_3) \ \partial_\nu D^F_0(x_2 - x_3) \ D^\mu\nu_f(x_1, x_2) \\
+ \delta(x_1 - x_3) \ \partial_\mu \partial_\nu D^F_0(x_2 - x_3) \ E^{(\mu\nu)}_f(x_1, x_2)] + (x_1 \leftrightarrow x_2)
\]

\[
(3.29)
\]

where the expressions \(A, \ldots, F\) can be explicitly computed if we use the first three formulas from the preceding theorem. We only give explicitly

\[
F_f = -2 \ i \ f_{a_1 a_2 a_3} \ v_{a_1 \mu} E^{\mu\nu}_{a_2} B_{a_3 \nu}.
\]

(3.30)

Starting from (3.31) - (3.11) we can obtain a similar formula for the canonical contribution \(sT(T^{(2)}(x_1), T^{(2)}(x_2), T^{(1)}(x_3))^\text{can}\); we only notice that

\[
F^\text{can} = -F^f.
\]

(3.31)

In the end we have:

\[
sT(T^{(2)}(x_1), T^{(2)}(x_2), T^{(1)}(x_3)) = \]

\[
[\delta(x_1 - x_3) \ D^F_0(x_2 - x_3) \ A(x_1, x_2) + \delta(x_1 - x_3) \ \partial_\mu D^F_0(x_2 - x_3) \ B^\mu(x_1, x_2) \\
+ \delta(x_1 - x_3) \ \partial_\mu \partial_\nu D^F_0(x_2 - x_3) \ E^{(\mu\nu)}(x_1, x_2)] + (x_1 \leftrightarrow x_2)
\]

\[
(3.32)
\]

where

\[
A(x_1, x_2) = i \ P^\mu_a(x_1) \ C_{a\mu}(x_2)
\]

\[
B^\mu(x_1, x_2) = i \ P_{a\mu}(x_1) \ E^\mu_{a\nu}(x_2) + i \ Q^\mu_a(x_1) \ D_{a\nu}(x_2) \\
+ i \ Q^\nu_{a\mu}(x_1) \ C_{a\nu}(x_2) + i \ Q_a(x_1) \ B^\mu_{a\nu}(x_2)
\]

\[
E^{(\mu\nu)}_f(x_1, x_2) = i \ S_{\mu\nu} \ [Q^\rho_{a\mu}(x_1) \ E^\rho_{a\nu}(x_2)]
\]

(3.33)

and we have defined:

\[
P^\mu_a \equiv f_{abc} \ (C^\mu_b \ v_{c\nu} - C^\mu_b \ u_c + B_b \ \partial^\mu \bar{u}_c - \ B_b^\nu \ F^{\mu\nu}_c) \\
Q^\mu_a \equiv f_{abc} \ (B_b \ v^\mu_c + B_b^\mu \ u_c) \\
Q^\nu_{a\mu} \equiv f_{abc} \ (E^{\nu\mu}_b \ u_c + B_b^\nu \ v^\mu_c - B_b^\mu \ v^\nu_c) \\
Q_a \equiv f_{abc} \left( -C^\mu_b \ v_{c\rho} - \frac{1}{2} \ E_{b\rho\sigma} \ F^{\rho\sigma}_c + D_b \ u_c + B_b^\mu \ \partial^\mu \bar{u}_c \right).
\]

(3.34)

However, due to Jacobi identity one can prove that all previous expressions are null. So, in fact we have:

\[
sT(T^{(2)}(x_1), T^{(2)}(x_2), T^{(1)}(x_3)) = 0.
\]

(3.35)
From here we have

\[ sT_{\text{tree}}(T(x_1), T(x_2), T(x_3)) = sT(T^{(2)}(x_1), T^{(2)}(x_2), T^{(1)}(x_3)) + (x_1 \leftrightarrow x_3) + (x_2 \leftrightarrow x_3) = 0 \]  

(3.36)
i.e. there are no anomalies in the tree sector. We remark that (3.35) is stronger than (3.36). Also, we cannot prove (3.36) only from cohomology considerations because we cannot eliminate an anomaly of the form

\[ A(x_1, x_2, x_3) = \delta(x_1 - x_3) \delta(x_2 - x_3) F_f(x_3); \]  

(3.37)
indeed, if we use Jacobi identity we can prove that such an anomaly is a cocycle i.e. we have

\[ d_Q F_f = 0 \]  

(3.38)
but it is not a coboundary i.e. it cannot be written in the form \( A = d_Q B - i \partial_i B^\mu \) so it cannot be eliminated by a redefinition of the chronological products. One must perform an explicit computation and arrive at (3.31).

It is natural that (3.32) can be generalized in all other sectors: we have for instance

\[ sT(T^{(2)}(x_1), T^{(2)}(x_2), T^{\mu(1)}(x_3)) = \]

\[ \left[ \delta(x_1 - x_3) \partial_0 D_0^F(x_2 - x_3) A^\mu(x_1, x_2) + \delta(x_1 - x_3) \partial_\nu D_0^F(x_2 - x_3) B^{\mu\nu}(x_1, x_2) \right] + (x_1 \leftrightarrow x_2) \\
+ \delta(x_1 - x_3) \delta(x_2 - x_3) F^\mu(x_3) \]  

(3.39)
where

\[ A^\mu(x_1, x_2) = i R_a^{\mu
u}(x_1) C_{\alpha
u}(x_2) \]

\[ B^{\mu\nu}(x_1, x_2) = -i P_a^{\mu}(x_1) B_a^{\nu}(x_2) + i \left[ Q_a^{\nu}(x_1) C_a^{\mu}(x_2) - \eta^{\mu\nu} Q_a^{\rho}(x_1) C_{\alpha\rho}(x_2) \right] \\
+ i R_{a \rho}(x_1) E_a^{\rho\nu}(x_2) + i \eta^{\mu\nu} S_a(x_1) D_a(x_2) \]

\[ F^\mu = Q_{a
u} E_a^{\mu\nu}. \]  

(3.40)
Here we have used the previous notations plus

\[ R_a^{\mu\nu} \equiv -f_{abc} \left( C_b^{\mu\nu} u_c + B_b F_c^{\mu\nu} \right) \]

\[ S_a \equiv f_{abc} B_b u_c. \]  

(3.41)
These two expressions are also null due to Jacobi identity. We similarly have

\[ sT(T^{(1)}(x_1), T^{(2)}(x_2), T^{\nu(2)}(x_3)) = \]

\[ \delta(x_2 - x_1) D_0^F(x_3 - x_1) A^\mu(x_2, x_3) + \delta(x_3 - x_1) D_0^F(x_2 - x_3) A^\mu(x_2, x_3) + \delta(x_3 - x_1) \partial_\nu D_0^F(x_2 - x_3) B^{\mu\nu}(x_2, x_3) \]

\[ + \delta(x_2 - x_1) \partial_\nu D_0^F(\partial_0(x_3 - x_1) E^{\mu(\sigma)}(x_2, x_3) + \delta(x_3 - x_1) \partial_\nu D_0^F(x_2 - x_1) E^{\nu(\sigma)}(x_2, x_3) \]

\[ + \delta(x_2 - x_1) \delta(x_3 - x_1) F^\mu(x_1) \]  

(3.42)
where

\begin{align*}
A^\mu(x_2, x_3) &= i \, P_{\alpha\nu}(x_2) \, C^\nu_\alpha(x_3) \\
A^\mu_\nu(x_2, x_3) &= i \, R^\mu_\alpha(x_2) \, C^\nu_\alpha(x_3) \\
B^{\mu\nu}(x_2, x_3) &= i \, [ -P^\mu_\alpha(x_2) \, B_\nu(x_3) + \eta^{\mu\nu} \, P^\mu_\alpha(x_2) \, B_{\alpha\rho}(x_3) \\
&\quad - Q^\nu_\alpha(x_2) \, C^\rho\nu_\alpha(x_3) + \eta^{\mu\nu} \, Q_\alpha(x_2) \, B_\alpha(x_3) + Q^\rho_\alpha(x_2) \, C^\mu_\rho(x_3) ] \\
B^{\mu\nu}(x_2, x_3) &= i \, [ R^\mu_\alpha(x_3) \, E^\nu_\alpha(x_2) + Q^\rho_\alpha(x_3) \, C^\mu_\rho(x_2) \\
&\quad - \eta^{\mu\nu} \, Q_\alpha(x_2) \, C^\rho_\alpha(x_3) - P^\mu_\alpha(x_3) \, B^\nu_\alpha(x_2) - \eta^{\mu\nu} \, S_\alpha(x_3) \, D_\alpha(x_2) ] \\
E^{(\mu\nu)}(x_2, x_3) &= i \, S_{\rho\sigma} \{ Q^{\rho\sigma}_\alpha(x_2) \, B^\nu_\alpha(x_3) + \eta^{\mu\rho} \, Q^{\nu\lambda}_\alpha(x_2) \, B_{\alpha\lambda}(x_3) \\
&\quad E^{(\mu\nu)}(x_2, x_3) = i \, S_{\rho\sigma} \{ Q^{\rho\sigma}_\alpha(x_3) \, E^\sigma_\alpha(x_2) - \eta^{\mu\rho} \, Q_\alpha(x_3) \, E^\nu_\alpha(x_2) \\
F^\mu &= i \, f_{\alpha_1\alpha_2\alpha_3} \, [-v^\mu_\alpha \, B^\rho_\alpha + \eta^{\mu\rho} \, B^\rho_{\alpha 2}] \, B_{\alpha\rho} + v^\mu_\alpha \, E^\mu_\alpha \, D_{\alpha 3} \tag{3.43}
\end{align*}

This expression is null. Only the proof of $F^\mu = 0$ requires some computations; all other expressions have already appeared previously. So we have

\begin{align*}
sT_\text{tree}(T(x_1), T(x_2), T^\mu(x_3)) &= sT(T^{(2)}(x_1), T^{(2)}(x_2), T^{(1)}(x_3)) \\
&\quad + sT(T^{(1)}(x_1), T^{(2)}(x_2), T^{(2)}(x_3)) + (x_1 \leftrightarrow x_2) = 0. \tag{3.44}
\end{align*}

In the same way we have

\begin{align*}
sT(T^{(\mu_1)(2)}(x_1), T^{(\mu_2)(2)}(x_2), T^{(1)}(x_3)) &= [\delta(x_1 - x_3) \, D^F_\alpha(x_2 - x_3) \, A^{\mu_1\mu_2}(x_1, x_2) + \delta(x_1 - x_3) \, \partial_\sigma D^F_\alpha(x_2 - x_3) \, B^{\mu_1\mu_2\nu}(x_1, x_2) \\
&\quad + \delta(x_1 - x_3) \, \partial_\rho \partial_\sigma D^F_\alpha(x_2 - x_1) \, E^{\mu_1\mu_2(\rho\sigma)}(x_1, x_2) \\
&\quad - (x_1 \leftrightarrow x_2, \mu_1 \leftrightarrow \mu_2)] \\
&\quad + \delta(x_1 - x_3) \, \delta(x_2 - x_3) \, F^{\mu_1\mu_2}(x_3) \tag{3.45}
\end{align*}

which is null because

\begin{align*}
A^{\mu_1\mu_2}(x_1, x_2) &= i \, R^{\mu_1\nu}_\alpha(x_1) \, C^{\mu_2}_\alpha(x_2) \\
B^{\mu_1\mu_2\nu}(x_1, x_2) &= i \, [ -\eta^{\mu_1\nu} \, R^{\mu_1\rho}_\alpha(x_1) \, B_{\alpha\rho}(x_2) + R^{\mu_1\mu_2}_\alpha(x_1) \, B^\nu_\alpha(x_2) \\
&\quad - \eta^{\mu_1\nu} \, Q_\alpha(x_1) \, C^{\rho\nu}_\alpha(x_2) + Q^\rho_\alpha(x_1) \, C^{\mu_2}_\rho(x_2) \\
&\quad + \eta^{\mu_2\nu} \, P^{\mu_2}_\alpha(x_1) \, B_\alpha(x_2) + \eta^{\mu_1\nu} \, S_\alpha(x_1) \, C^{\mu_2}_\alpha(x_2) ] \\
E^{(\mu_1)(\rho\sigma)}(x_1, x_2) &= i \, S_{\rho\sigma} \{ \eta^{\mu_1\rho} \, Q^{\rho\sigma}_\alpha(x_1) \, B^\sigma_\alpha(x_2) - \eta^{\mu_1\rho} \, Q_\alpha(x_1) \, B^{\sigma\rho}_\alpha(x_2) \\
&\quad + \eta^{\mu_2\rho} \, Q^\rho_\alpha(x_1) \, B^{\mu_1}_\alpha(x_2) - \eta^{\mu_1\mu_2} \, Q^{\rho\sigma}_\alpha(x_1) \, B^\sigma_\alpha(x_2) ] \\
F^{\mu_1\mu_2} &= i \, f^{\mu_1}_\alpha \, v^{\mu_2}_\alpha - (\mu_1 \leftrightarrow \mu_2). \tag{3.46}
\end{align*}

are null. We have introduced

\[ T^\mu_\alpha \equiv f_{\alpha\beta\gamma} \, B^\mu_\beta B^\mu_\gamma. \tag{3.47} \]

Only $T^\mu_\alpha = 0$ must be checked.
We also have:

\[
sT(T^{\mu_1(2)}(x_1), T^{\mu_2(1)}(x_2), T^{(2)}(x_3)) =
\begin{align*}
[\delta(x_3 - x_2) & D_0^F(x_1 - x_2) A^{\mu_1 \mu_2}(x_1, x_3) + \delta(x_3 - x_2) \partial_\rho D_0^F(x_1 - x_2) B^{\mu_1 \mu_2 \rho}(x_1, x_3) \\
+\delta(x_3 - x_2) & \partial_\rho \partial_\sigma D_0^F(x_1 - x_2) E^{\mu_1 \mu_2 \rho \sigma}(x_1, x_3)
\end{align*}
\]

which is null because

\[
A^{\mu_1 \mu_2}(x_1, x_3) = i \left[ \frac{\eta^{\mu_1 \nu}}{\rho} B^\nu_a(x_1) \right] R^{\mu_2 \nu}_a(x_2) - \frac{\eta^{\mu_1 \nu}}{\rho} B^\nu_a(x_1) R^{\mu_2 \nu}_a(x_3)
\]

are null. As a consequence we have

\[
sT_{\text{tree}}(T^{\mu_1}(x_1), T^{\mu_2}(x_2), T(x_3)) = sT(T^{\mu_1(2)}(x_1), T^{\mu_2(2)}(x_2), T^{(1)}(x_3)) + sT(T^{\mu_1(2)}(x_1), T^{\mu_2(1)}(x_2), T^{(2)}(x_3)) - (x_1 \leftrightarrow x_2, \mu_1 \leftrightarrow \mu_2) = 0.
\]

Finally we have

\[
sT(T^{(1)}(x_1), T^{(2)}(x_2), T^{\mu \nu}(2)(x_3)) =
\begin{align*}
[\delta(x_1 - x_3) & \partial_\rho D_0^F(x_2 - x_3) B^{\mu \nu \rho}(x_1, x_2) \\
+\delta(x_1 - x_3) & \partial_\rho \partial_\sigma D_0^F(x_2 - x_3) E^{\mu \nu \rho \sigma}(x_1, x_2)] + (x_1 \leftrightarrow x_2)
\end{align*}
\]

which is null because

\[
B^{\mu \nu \rho}(x_1, x_2) = \left\{ [\eta^{\mu \rho} S_a(x_1) C^\rho_a(x_2) - (\mu \leftrightarrow \nu)] + R_a^{\mu \nu}(x_1) B^\rho_a(x_2) \right\}
\]

\[
E^{\mu \nu \rho \sigma}(x_1, x_2) = i S_{\rho \sigma}[(\eta^{\mu \rho} E^\sigma_a(x_2)) - (\mu \leftrightarrow \nu)]
\]

\[
F^{\mu \nu} = i S_a E^{\mu \nu}.
\]
We also have

\[ sT(T^{(1)}(x_1), T^{(2)}(x_2), T^{μν(2)}(x_3)) = δ(x_3 - x_1) \partial_μD_0^F(x_2 - x_1) B^{μνρ}(x_3, x_2) + δ(x_2 - x_1) \partial_μD_0^F(x_3 - x_1) B^{μνρ}(x_3, x_2) + δ(x_3 - x_1) \partial_μD_0^F(x_2 - x_1) E^{μν(ρσ)}(x_3, x_2) + δ(x_2 - x_1) \partial_μD_0^F(x_3 - x_1) E^{μν(ρσ)}(x_3, x_2) + δ(x_3 - x_1) δ(x_2 - x_1) F^{μν}(x_1) \] (3.53)

which is null because

\[ B^{μνρ}(x_3, x_2) = i \{[η^{μμ} S_α(x_2) C_α^μ(x_2) - (μ ↔ ν)] + R_α^μ(2) B_α^μ(2)\} \]

\[ B^{μνρ}(x_3, x_2) = i \{[η^{μμ} P_α^μ(x_2) B_α(x_3) - (μ ↔ ν)] - Q_α^μ(2) C_α^μ(x_3)\} \]

\[ E^{μν(ρσ)}(x_1, x_2) = i S_ρ[η^{μμ} S_α(x_3) E_α^{μν}(x_2) - (μ ↔ ν)] \]

\[ E^{μν(ρσ)}(x_1, x_2) = i S_ρ[η^{μμ} Q_α^{ρσ}(x_2) B_α(x_3) - (μ ↔ ν)] \]

\[ F^{μν} = i T_α^μ T_α^ν - (μ ↔ ν). \] (3.54)

It follows that

\[ sT_{tree}(T(x_1), T(x_2), T^{μν}(x_3)) = sT(T^{(2)}(x_1), T^{(2)}(x_2), T^{μν(1)}(x_3)) + sT(T^{(1)}(x_1), T^{(2)}(x_2), T^{μν(1)}(x_3)) + (x_1 ↔ x_2) = 0. \] (3.55)

4 Conclusions

We notice in all formulas of the type \[\boxed{332} + \boxed{333}\] a certain factorization property in two factors emerges; moreover one of the factors is null because of gauge invariance in the second order. Such a property is lost for the more general case of the standard model where one do get anomalies in the third order of the tree sector; the cancelation of this anomaly fixes the Higgs coupling \[\boxed{15}\]. This computation will be addressed in further papers.

It would be interesting to generalize these ideas in higher orders of perturbation theory.

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