THREE AMALGAMS WITH REMARKABLE NORMAL
SUBGROUP STRUCTURES

DIEGO RATTAGGI

Abstract. We construct three groups \( \Lambda_1, \Lambda_2, \Lambda_3 \), which can all be decomposed as amalgamated products \( F_9 * F_{81} F_9 \) and have very few normal subgroups of finite or infinite index. Concretely, \( \Lambda_1 \) is a simple group, \( \Lambda_2 \) is not simple but has no non-trivial normal subgroup of infinite index, and \( \Lambda_3 \) is not simple but has no proper subgroup of finite index.

1. Introduction

Motivated by expected analogies between cocompact lattices in products of automorphism groups of regular trees and cocompact lattices in higher rank semisimple Lie groups, Burger and Mozes discovered in their study of groups acting on products of trees the first examples of finitely presented torsion-free simple groups \[ \text{\cite{5, 7}} \]. These groups are moreover amalgamated products of finitely generated non-abelian free groups, thus answering Neumann’s question \[ \text{\cite{10}} \] on the existence of simple amalgams of free groups. One crucial step in the construction of Burger-Mozes is a deep theorem, which states that certain cocompact lattices in the product of automorphism groups of locally finite trees \( \text{Aut}(T_1) \times \text{Aut}(T_2) \) cannot have non-trivial normal subgroups of infinite index. Applying this theorem to a cocompact lattice which contains as a subgroup a non-residually finite group constructed by Wise in \[ \text{\cite{14}} \], we give an example of a finitely presented torsion-free simple group \( \Lambda_1 \) of the form \( F_9 * F_{81} F_9 \), where \( F_k \) denotes the free group of rank \( k \). See \[ \text{\cite{12}} \] for a list of 32 other finitely presented torsion-free simple groups emerging from the same method. Note that the simple groups of Burger-Mozes are also explicitly given in principle, but not very manageable in practice, because of their extremely long finite presentations. In addition to the simple group \( \Lambda_1 \), we construct two other groups \( \Lambda_2 \) and \( \Lambda_3 \), also having amalgam decompositions \( F_9 * F_{81} F_9 \). They are not simple, but \( \Lambda_2 \) is virtually simple and \( \Lambda_3 \) has no non-trivial finite quotients. An amalgam \( F_3 * F_{13} F_3 \) without proper subgroups of finite index has already been constructed by Bhatacherjee in \[ \text{\cite{3}} \], using different techniques. Our search for groups with the desired properties was made possible by several computer programs written in GAP \[ \text{\cite{8}} \]. See \[ \text{\cite{11} Appendix B} \] for the program code used to construct the examples. We refer to \[ \text{\cite{6, 7, 11} and 14} \] for detailed background on automorphism groups of trees, lattices in products of trees, and square complexes.

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2. Definition of the groups $\Gamma_i$ and $\Lambda_i$

Let always $i \in \{1, 2, 3\}$. Our groups $\Lambda_i$ will be normal subgroups of index 4 of groups $\Gamma_i$ defined by their finite presentations

$$\Gamma_i = \langle a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 \mid r_1, \ldots, r_{25} \rangle,$$

where the relators $r_1, \ldots, r_{25}$ (depending on $i$) are given in Table 1. Capital letters in this table indicate inverses, for example $r_1 = a_1 b_1 A_2 B_2 = a_1 b_1 a_2^{-1} b_2^{-1}$.

| $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ |
|---------------------------------|------------|------------|------------|
| $r_1$                           | $a_1 b_1 A_2 B_2$ | $a_1 b_4 a_2 B_5$ | $a_1 b_4 A_4 b_5$ |
| $r_2$                           | $a_1 b_2 A_1 B_3$ | $a_1 b_5 A_3 b_4$ | $a_1 b_5 A_2 B_5$ |
| $r_3$                           | $a_1 b_3 A_2 B_4$ | $a_1 b_5 a_2 B_4$ | $a_1 B_5 A_1 B_4$ |
| $r_4$                           | $a_1 B_3 A_2 B_2$ | $a_1 B_4 a_2 b_5$ | $a_1 B_4 A_2 b_5$ |
| $r_5$                           | $a_1 B_1 A_3 b_5$ | $a_2 b_4 A_3 b_5$ | $a_2 B_4 A_3 b_5$ |
| $r_6$                           | $a_2 b_2 A_3 B_1$ | $a_2 b_5 a_2 b_3$ | $a_2 B_5 A_2 b_3$ |
| $r_7$                           | $a_3 b_2 A_3 b_2$ | $a_3 B_3 a_4 b_4$ | $a_3 B_3 A_2 b_4$ |
| $r_8$                           | $a_3 b_2 A_3 b_5$ | $a_3 B_3 A_5 b_5$ | $a_3 B_3 A_5 b_5$ |
| $r_9$                           | $a_4 b_4 A_2 B_1$ | $a_4 B_5 A_4 b_5$ | $a_4 B_5 A_2 b_5$ |
| $r_{10}$                        | $a_5 B_3 A_4 b_2$ | $a_5 b_1 A_3 b_5$ | $a_5 b_1 A_3 b_5$ |
| $r_{11}$                        | $a_5 b_2 A_4 b_3$ | $a_5 b_3 A_4 b_4$ | $a_5 b_3 A_3 b_4$ |
| $r_{12}$                        | $a_5 b_2 A_4 b_1$ | $a_5 b_4 A_3 b_5$ | $a_5 b_4 A_3 b_5$ |

Table 1. The 25 relators of $\Gamma_1$, $\Gamma_2$, $\Gamma_3$

Observe that the twelve relators $r_1, \ldots, r_{12}$ are the same for each group $\Gamma_i$. The reason for this will become clear in the proof of Theorem 1 in Section 3.

To describe the geometric nature of $\Gamma_i$, we recall the following general construction which associates to a finite presentation of a group $G$ its standard 2-complex $X$ with fundamental group $\pi_1(X)$. By definition, the one-skeleton of $X$ has a single vertex $x$ and an oriented loop for each generator of the given presentation of $G$. Furthermore, for each relator $r$, a 2-cell with boundary labelled by $r$ is glued into this one-skeleton to get $X$. Then $G = \pi_1(X, x)$. By construction of the 25 relators of $\Gamma_i$, its associated standard 2-complex $X_i$ is a finite square complex (all relators have length four, hence all 2-cells are squares) having the additional property that its universal cover $\tilde{X}_i$ is the affine building $T_{10} \times T_{10}$, the product of two 10-regular trees. Equivalently, this property requires that to each pair $(a, b) \in A \times B$, there is a uniquely determined pair $(\tilde{a}, \tilde{b}) \in A \times B$ such that $ab = b\tilde{a}$ in $\Gamma_i$, where $A := \{a_1, \ldots, a_5\}^{\pm 1}$ and $B := \{b_1, \ldots, b_5\}^{\pm 1}$. This can be easily verified for our three given examples. In the terminology of [7], $X_i$ is a finite 1-vertex VH-T-square complex, and in the terminology of [14] (12), $\Gamma_i = \pi_1(X_i)$ is a $(10, 10)$-group. The group of automorphisms $\text{Aut}(T_{10})$, equipped with the usual topology of simple convergence, is a locally compact group. Taking the product topology, $\Gamma_i$ can be seen as a discrete subgroup of $\text{Aut}(T_{10}) \times \text{Aut}(T_{10})$ with compact quotient, in other words as a cocompact lattice. A crucial role in deducing interesting results on the normal subgroup structure of $\Gamma_i$ play the so-called local groups of $\Gamma_i$. The idea to define them is the following: take the projection of $\Gamma_i$ to one factor of
 induces a finite permutation group $\Gamma_i < S_{10}$ on the 10 neighbouring vertices of $x_h$ in $T_{10}$ (or more generally, for $k \in \mathbb{N}$, subgroups $P_h^{(k)}(\Gamma_i)$ of the symmetric group $S_{10,9k-1}$, taking the induced action on the $k$-sphere in $T_{10}$ around $x_h$). The same procedure can be done with the second projection $\pi_2$ to get local groups $P^{(k)}(\Gamma_i) < S_{10,9k-1}$. It is important to note that these local groups (more precisely, their generators in $S_{10,9k-1}$) can be directly computed, given the relators $r_1, \ldots, r_{25}$ of Table II see [3] Chapter 1 or [14] Section 1.4 for details. Here, we get for $k = 1$ the groups

\[
P^{(1)}_{h}(\Gamma_1) = \langle (7,8)(9,10), (1,2)(3,4), (1,2)(3,4)(7,8)(9,10), \\
(1,8,4,5)(2,7,3,10), (1,9,4,8)(3,10,6,7) \rangle = A_{10},
\]

\[
P^{(1)}_{h}(\Gamma_2) = \langle (7,8)(9,10), (1,2)(3,4), (1,2)(3,4)(7,8)(9,10), \\
(1,8,4,9)(2,10,7,3), (1,9,8,6,4)(2,7,5,3,10) \rangle = A_{10},
\]

\[
P^{(1)}_{h}(\Gamma_3) = \langle (5,6)(7,8)(9,10), (1,2)(3,4), (1,2)(3,4)(7,8)(9,10), \\
(1,4,8,9,2,3,7,10)(5,6), (1,9,2,10)(3,5,7)(4,6,8) \rangle,
\]

\[
P^{(1)}_{v}(\Gamma_1) = \langle (1,2)(4,6,7,5)(8,10,9), (1,2,3)(4,5,7,6)(9,10), \\
(1,2)(4,5,7,6)(8,10,9), (1,2,3)(4,6,7,5)(9,10), (1,3,10,8)(2,4,6,9,7,5) \rangle = A_{10},
\]

\[
P^{(1)}_{v}(\Gamma_2) = \langle (1,2)(4,6,7,5)(8,10,9), (1,2,3)(4,6,5,7)(9,10), \\
(1,2)(4,6,5,7)(8,10,9), (1,2,3)(4,6,5,7)(9,10), (1,2,4,3,10,9,7,8)(5,6) \rangle = A_{10},
\]

\[
P^{(1)}_{v}(\Gamma_3) = \langle (1,2)(4,7,5,6)(8,10,9), (1,2,3)(4,7,5,6)(9,10), \\
(1,2)(4,5,6,7)(8,10,9), (1,2,3)(4,5,6,7)(9,10), (1,7)(2,8)(3,9)(4,10)(5,6) \rangle = S_{10}.
\]

The transitivity of the permutation groups given above will be important in the proof of Theorem II. Recall that a group $G < S_{10}$ is transitive if for any pair $m, n \in \{1, \ldots, 10\}$ there exists a $g \in G$ such that $g(m) = n$. Moreover, $G$ is called 2-transitive if for any $m_1, m_2, n_1, n_2 \in \{1, \ldots, 10\}$ with $m_1 \neq m_2$ and $n_1 \neq n_2$ there is an element $g \in G$ such that $g(m_1) = n_1$ and $g(m_2) = n_2$. Note that the group $P^{(1)}_{h}(\Gamma_3)$ is a transitive (but not 2-transitive) subgroup of $S_{10}$ of order 3840, whereas the alternating group $A_{10}$ and the symmetric group $S_{10}$ are obviously 2-transitive.

We define now $\Lambda_i$ to be the kernel of the surjective homomorphism

\[
\Gamma_i \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}
\]

\[
a_1, \ldots, a_5 \mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z})
\]

\[
b_1, \ldots, b_5 \mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}),
\]

where $\Gamma_i$ is given by its finite presentation described above. Each group $\Lambda_i$ can be decomposed in two ways as amalgamated products $F_9 *_{F_{s1}} F_9$, such that $F_{s1}$ has index 10 in both factors $F_9$. More precisely, this means that for any $i \in \{1, 2, 3\}$ there exist injective homomorphisms $j_1, j_3 : F_{s1} \to F_9 \cong \langle s_1, \ldots, s_9 \rangle$ and $j_2, j_4 : F_{s1} \to F_9 \cong \langle t_1, \ldots, t_9 \rangle$ such that

\[
[F_9 : j_1(F_{s1})] = [F_9 : j_2(F_{s1})] = [F_9 : j_3(F_{s1})] = [F_9 : j_4(F_{s1})] = 10.
\]
and

\[ \Lambda_i \cong \langle s_1, \ldots, s_9, t_1, \ldots, t_9 \mid j_1(u_1) = j_2(u_1), \ldots, j_9(u_81) = j_4(u_81) \rangle \]

where \( \{u_1, \ldots, u_{81}\} \) are the free generators of \( F_{81} \). This is a direct consequence of a result of Wise (see [14, Theorem I.1.18]), describing each of the two decompositions of certain square complex groups \( \Gamma \) as a fundamental group of a finite graph of finitely generated free groups (in the language of the Bass-Serre theory). If the local groups of \( \Gamma \) are “sufficiently transitive” (which always happens in our examples), the two finite graphs corresponding to \( \Lambda_i \) in Wise’s construction each consist of two vertices and a single edge. Therefore we get amalgams of finitely generated free groups. It is well-known that amalgams of free groups are always torsion-free, since every element of finite order in an amalgam is conjugate to an element of finite order in one of the two factors (see for example [9, Theorem IV.2.7]). Note that following Wise’s proof of [14, Theorem I.1.18], it is not difficult (but quite laborious by hand) to give explicit descriptions of the injective homomorphisms \( F_{81} \to F_9 \) in the amalgam decompositions of \( \Lambda_i \).

3. Results and Proofs

In the following theorem, we discuss the normal subgroups of \( \Lambda_i \).

**Theorem 1.** Let \( \Lambda_1, \Lambda_2, \Lambda_3 \) be the groups defined in Section 2. Then

1. \( \Lambda_1 \) is simple.
2. Every non-trivial normal subgroup of \( \Lambda_2 \) has finite index, but \( \Lambda_2 \) is not simple.
3. \( \Lambda_3 \) has no proper subgroups of finite index, but is not simple.

**Proof.** Let \( W \) be the group with finite presentation

\[ \langle a_1, a_2, a_3, a_4, b_1, b_2, b_3 \mid r_1, \ldots, r_{12} \rangle, \]

where the relators \( r_1, \ldots, r_{12} \) are again taken from Table 1. Wise showed in [14, Main Theorem 5.5], that the non-trivial element \( w := a_2a_1^{-1}a_3a_4^{-1} \in W \) is contained in each finite index subgroup of \( W \). In particular, \( W \) is non-residually finite. Moreover, \( W < \text{Aut}(T_6) \times \text{Aut}(T_6) \) is the fundamental group of a 1-vertex VH-T square complex which embeds into the square complex \( X_i \) associated to \( \Gamma_i \) \((i = 1, 2, 3)\), inducing an injection on the level of fundamental groups, i.e., \( W < \Gamma_i = \pi_1(X_i) \) (the fact that we get an injection can be deduced from the non-positive curvature of the product of trees \( T_{10} \times T_{10} \), see [4, Proposition II.4.14(1)]). Hence we have

\[ 1 \neq w \in \bigcap_{i} N < \bigcup_{i} N \triangleleft \Gamma_i, \]

where “f.i.” stands for “finite index”. In particular, \( \Gamma_i \) (and hence its finite index subgroup \( \Lambda_i \)) is non-residually finite. Observe that \( w \in \Lambda_i \triangleleft \Gamma_i \). One important point in the construction of \( \Gamma_i \) is to guarantee that the normal closure of \( w \) in \( \Gamma_i \), denoted by \( \langle \langle w \rangle \rangle_{\Gamma_i} \), has finite index in \( \Lambda_i \). (Note that however \( [W : \langle \langle w \rangle \rangle_{W}] = \infty \).) This already implies that \( \langle \langle w \rangle \rangle_{\Gamma_i} \) has no proper subgroups of finite index. Indeed,
assume that $M < \langle \langle w \rangle \rangle_{\Gamma_i}$ is a subgroup of finite index. Then

$$\bigcap_{N^f_i \leq \Gamma_i} N < M < \langle \langle w \rangle \rangle_{\Gamma_i} < A_i < \Gamma_i.$$ 

Using

$$\langle \langle w \rangle \rangle_{\Gamma_i} < \bigcap_{N^f_i \leq \Gamma_i} N = \bigcap_{N^f_i \leq \Gamma_i} N,$$

we get

$$M = \langle \langle w \rangle \rangle_{\Gamma_i} = \bigcap_{N^f_i \leq \Gamma_i} N.$$ 

We proceed now separately for the three groups $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$.

1. We have $\langle \langle w \rangle \rangle_{\Gamma_1} = \Lambda_1$. This can be checked by hand, or more easily, using a computer algebra system like GAP, which shows that adding the relator $w$ to the presentation of $\Gamma_1$ gives the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of order 4. It remains to prove that $\Lambda_1$ has no non-trivial normal subgroups of infinite index. But this follows directly from the normal subgroup theorem of Burger-Mozes [17, Theorem 4.1, Corollary 5.4] applied to the “irreducible” cocompact lattice $\Gamma_1 < \text{Aut}(T_{10}) \times \text{Aut}(T_{10})$ with local groups $P_h^{(1)}(\Gamma_1) \cong P_v^{(1)}(\Gamma_1) = A_{10}$, and applied to its finite index subgroup $\Lambda_1 < \Gamma_1$.

2. For the second group, we compute $[\Lambda_2 : \langle \langle w \rangle \rangle_{\Gamma_2}] = 2$, thus $\Lambda_2$ is not simple. By exactly the same argument as in part (1), every non-trivial normal subgroup of $\Gamma_2$ (and of $\Lambda_2$, respectively) has finite index. Observe that $\langle \langle w \rangle \rangle_{\Gamma_2}$ is a simple group with amalgam decomposition $F_{17} \ast_{F_{161}} F_{17}$. In particular, $\Gamma_2$ and $\Lambda_2$ are virtually simple groups.

3. As in part (1), $\langle \langle w \rangle \rangle_{\Gamma_3} = \Lambda_3$ proves that $\Lambda_3$ has no proper subgroup of finite index. However, in contrast to what happens in part (1) and (2), the local group $P_h^{(1)}(\Gamma_3)$ is transitive, but not 2-transitive. Therefore, the normal subgroup theorem of Burger-Mozes cannot be applied here. Indeed, $\Lambda_3$ is not simple, since $1 \neq \langle \langle a_5^4 \rangle \rangle_{\Lambda_3} \neq \Lambda_3$. This comes from the fact that $a_5^4$ acts trivially on the second factor of $T_{10} \times T_{10}$. In other words, $a_5^4 \in \ker(\text{pr}_2) < \Gamma_3$.

To see this, let

$$A' := \{(a_1a_2^{-1})^2, (a_2^{-1}a_1)^2, (a_3a_4^{-1})^2, (a_4^{-1}a_3)^2, a_5^{±1}\}$$

and check that for all $a' \in A'$ and $b \in B = \{b_1, \ldots, b_9\}^{±1}$, we have $b^{-1}a'b \in A'$. This in fact implies that $A' \subset \ker(\text{pr}_2)$. Note that no element of $\Gamma_3$ acts trivially on the first factor of $T_{10} \times T_{10}$ by [6, Proposition 3.1.2, 1] and [6, Proposition 3.3.2]). As a consequence, $\Lambda_3$ has two decompositions $F_9 \ast_{F_{31}} F_9$, where one amalgam is effective and the other one is not effective.

We conclude by giving two remarks:

**Remark 2.** Recall that a group $G$ is called *SQ-universal* if every countable group can be embedded in a quotient of $G$. It is mentioned in [11] Chapter 9.15] that Ilya Rips can prove any amalgamated product $A \ast_C B$ to be SQ-universal, provided that $B \neq C$ and the number of double cosets $|C \setminus A/C|$ is at least 3 (if $C$ is seen as usual as a subgroup of $A$ and $B$ via the two injections $j_1 : C \to A$ and $j_2 : C \to B$.
in the amalgam), but there is no published proof as far as we know. If Rips’ statement is true, we could apply it to exactly one decomposition $F_9 \star_{F_{81}} F_9$ of $\Lambda_3$ (to the effective one), where $|F_{81}\backslash F_9/F_{81}| = 3$. Note however that in the second decomposition of $\Lambda_3$ (where the corresponding local group $P^{(1)}_v(\Gamma_3)$ is $S_{10}$) and in both decompositions of $\Lambda_1$ and $\Lambda_2$, we always have $|F_{81}\backslash F_9/F_{81}| = 2$, since their local actions on $T_{10}$ are 2-transitive.

**Remark 3.** By construction, the three groups $\Lambda_1$, $\Lambda_2$, $\Lambda_3$ are non-residually finite. As a contrast, if one takes a double $F_9 \star_{F_{81}} F_9$ (i.e. an amalgam where the two injections $j_1, j_2 : F_{81} \to F_9$ are identical), such that $F_{81}$ has finite index in both factors $F_9$ (consequently index 10 = $(81 - 1)/(9 - 1)$), then one directly gets a surjective homomorphism $F_9 \star_{F_{81}} F_9 \to F_9$ (the obvious folding map), and moreover $F_9 \star_{F_{81}} F_9$ contains by [2] Theorem 1.4 a subgroup of finite index which is a direct product of two non-abelian free groups of finite rank. In particular, such a double $F_9 \star_{F_{81}} F_9$ is SQ-universal and residually finite. The residual finiteness also follows from [13].

**References**

[1] Bass, Hyman; Lubotzky, Alexander, *Tree lattices*, with appendices by Bass, L. Carbone, Lubotzky, G. Rosenberg and J. Tits. Progress in Mathematics, 176. Birkhäuser Boston, Inc., Boston, MA, 2001.

[2] Benakli, Nadia; Dasbach, Oliver T.; Glasner, Yair; Mangum, Brian, *A note on doubles of groups*, J. Pure Appl. Algebra 156(2001), no. 2-3, 147–151.

[3] Bhattacharjee, Meenaxi, *Constructing finitely presented infinite nearly simple groups*, Comm. Algebra 22(1994), no. 11, 4561–4589.

[4] Bridson, Martin R.; Haefliger, André, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999.

[5] Burger, Marc; Mozes, Shahar, *Finitely presented simple groups and products of trees*, C. R. Acad. Sci. Paris Sér. I Math. 324(1997), no. 7, 747–752.

[6] Burger, Marc; Mozes, Shahar, *Groups acting on trees: from local to global structure*, Inst. Hautes Études Sci. Publ. Math. No. 92 (2001), 113–150.

[7] Burger, Marc; Mozes, Shahar, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math. No. 92 (2001), 151–194.

[8] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2005. [http://www.gap-system.org](http://www.gap-system.org)

[9] Lyndon, Roger C.; Schupp, Paul E., *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89. Springer-Verlag, Berlin-New York, 1977.

[10] Neumann, Peter M., *The SQ-universality of some finitely presented groups*, Collection of articles dedicated to the memory of Hanna Neumann, I., J. Austral. Math. Soc. 16(1973), 1–6.

[11] Rattaggi, Diego, *Computations in groups acting on a product of trees: normal subgroup structures and quaternion lattices*, Ph.D. thesis, ETH Zürich, 2004.

[12] Rattaggi, Diego, *A finitely presented torsion-free simple group*, Preprint, 2004, available at [arXiv:math.GR/0411546](http://arxiv.org/abs/math.GR/0411546)

[13] Stebe, Peter F., *On free products of isomorphic free groups with a single finitely generated amalgamated subgroup*, J. Algebra 11(1969), 359–362.

[14] Wise, Daniel T., *Non-positively curved squared complexes, aperiodic tilings, and non-residually finite groups*, Ph.D. thesis, Princeton University, 1996.

Université de Genève, Section de mathématiques, 2–4 rue du Lièvre, CP 64, CH–1211 Genève 4, Switzerland

E-mail address: rattaggi@math.unige.ch