Approximative Wigner function for the Helium atom and dissipation

H. Dessano,†‡ R.G.G. Amorim,†‡ S. C. Ulhoa,† and A. E. Santana†§

†International Center for Condensed Matter Physics, Instituto de Física, Universidade de Brasília, 70910-900, Brasília, DF, Brazil
‡Faculdade Gama, Universidade de Brasília, 72444-240, Brasília, DF, Brazil.

The Schrödinger equation in phase space is used to calculate the Wigner function for the Helium atom in the approximation of a system of two oscillators. Dissipation effect is analysed and the non-classicality of the state is studied by the non-classicality indicator of the Wigner function, which is calculated as a function of the dissipation parameter.

PACS numbers: 03.65.Ca; 03.65.Db; 11.10.Nx

I. INTRODUCTION

Dissipative systems in quantum mechanics have raised interest for many years [1, 2]. A particular reason is that such systems are temporally irreversible; i.e. there is a preferred time direction in their evolution. Irreversible phenomena, such as dissipation, emerges out of interaction of a system with its neighborhoods, such that the energy flows irreversibly [3, 4]. Thus for instance a dissipative force as friction could find a microscopic explanation. In a broad perspective, many microscopic phenomena are described by irreversible models [5–9]. It is important to emphasize that the study of irreversible systems in such a framework can shed some light into the very structure of matter, in particular in experimental apparatus with atoms where the neighborhood effect is the cause of dissipation.

This is the case of some procedures with fermions following in parallel with the production and detection of Bose-Einstein condensates, using the expedients such as laser cooling, magnetic and magneto-optic traps [10, 11].

The fermion counterpart has been accomplished by considering a degenerate Fermi gas as well as condensates of rare isotopes [12–15]. In addition, there is a great deal of interest in studying entangled multipartite fermion states for quantum communication [16, 17]. A simple but intricate example of such a fermion system, at the level of electronic structure, is the Helium atom considered as a few-fermion system taken in a external field, which can in turn be considered as a dissipative effect.

For practical analysis, it is interesting to note that a combination of two damped quantum harmonic oscillators can describe an atom with two electrons. In this case, the Schrödinger equation has no known exact solution. The results obtained for such systems are based on approximative methods or variational formalisms. However, due to the resemblance between the gaussian wave function of the spherically symmetric harmonic oscillator and the 1s state of the hydrogen atom, some models are used to study solutions of the Schrödinger equation for Helium atom. It consists in changing the Coulomb interactions by the harmonic oscillator potential. In particular, in the work of Kestner [18, 19], the electron-nuclei interactions were replaced by a harmonic oscillator potential but the electron-electron interaction was Coulombic. Then it was shown that the energy values obtained were very close to experimental data. In the presented work we address this problem, in order to study the non-classicality of such states, considering in addition, dissipation effects. We proceed by using the quantum mechanics in phase space in order to analyse the Wigner function [20–24].

The analysis in phase space is important in order to track evidences of chaos as well as the statistical nature of quantum states. In this case, the Wigner formalism is physically appealing, in particular in experiments for the reconstruction of quantum states, in quantum tomography and for the direct measurement of the Wigner function [25–31].

However, there are difficulties with the direct use of the Wigner function. One is that, there is no gauge symmetry associated with the Wigner function, since it is a real function. This aspect is also a problem for considering superposition effects in phase space. Another difficulty is related with technical reasons: the Liouville-von Neumann equation in phase space has an intricate nature, with no practical perturbative solution. For instance, a perturbative theory for fermions is a non-trivial task and remains to be formulated consistently. This type of problem has lead to an intense search for the analysis of the Wigner formalism [32, 33], and some advancements have been made; it includes the study of representations of quantum equations directly in phase-space [34]. After some preliminary attempts [35, 36], exploring a representation of the Schrödinger equation in phase space [37, 38], a consistent formalism has been introduced [39], by using the notion of quasi-amplitude of probabilities, which is associated with the Wigner function by the Moyal (or star) product in phase space [40, 41]. This notion of symplectic structure and Weyl product have been explored to study unitary representations of Galilei group, leading to a symplectic (phase space) representation of Schrödinger
This approach provides an interesting procedure to deriving the Wigner function, by using consistently the gauge invariance [34]. This symplectic representation was applied in kinetic theory and extended to the relativistic context; giving rise to the Klein-Gordon and the Dirac equations in phase space [57] [61]. Here we use this symplectic quantum mechanics to analyse the behaviour of the Wigner function for the Helium atom, considering dissipation. In this case, we study the non-classicality of the state with the non-classicality (negativity) indicator of the Wigner function [62] as a function of the dissipation parameter.

The paper is organized in the following way. In Section II, we present an outline of the symplectic representation of quantum mechanics in phase space. In Section III, we solve the Schrödinger equation in phase space, for the Helium atom, using this symplectic quantum mechanics to analyse the non-classicality of the state with the non-classicality (negativity) indicator of the Wigner function [62].

II. OUTLINE ON SCHRODINGER EQUATION IN PHASE SPACE

In this section we present a brief outline of the construction of the Schrödinger equation in phase space, emphasizing the association of phase space amplitude of probability with the Wigner function. We consider initially a one-particle system described by the Hamiltonian $H = \hat{p}^2/2m$, where $m$ and $\hat{p}$ are the mass and the momentum, respectively, of the particle. The Wigner formalism for such a system is constructed from the Liouville-von Neumann equation [20] [23]

$$i\hbar \partial_t \rho(t) = [H, \rho],$$

where $\rho(t)$ is the density matrix. The Wigner function, $f_W(q, p)$, is defined by

$$f_W(q, p) = (2\pi\hbar)^{-3} \int d\zeta \exp\left(\frac{i\hbar}{\hbar} \zeta \cdot \left( q - \frac{\zeta}{2} \rho + \frac{\zeta}{2} \right) \right),$$

and satisfies the equation of motion

$$i\hbar \partial_t f_W(q, p, t) = \{H_W, f_W\}_M,$$

where $H_W$ is the Wigner Hamiltonian and $\{a, b\}_M = a \ast b - b \ast a$ is the Moyal bracket, such that the star-product $a \ast b$ is given by

$$a \ast b = a(q, p) e^{i\hbar \Lambda} b(q, p)$$

with $\Lambda = \frac{i\hbar}{\partial q} \frac{\partial}{\partial q} - \frac{i\hbar}{\partial p} \frac{\partial}{\partial p}$. The functions $a(q, p)$ are defined in a manifold $\Gamma$, using the basis $(q, p)$ with the physical content of the phase space. In this formalism an operator, say $A$, defined in the Hilbert space $\mathcal{H}$, is represented by the function

$$A(q, p) = \int d\zeta \exp\left(\frac{i\hbar}{\hbar} \zeta \cdot \left( q - \frac{\zeta}{2} \rho + \frac{\zeta}{2} \right) \right),$$

such that the product of two operators, $AB$, reads

$$(AB)(q, p) = A(q, p) e^{i\hbar A B(q, p)} = A(q, p) \ast B(q, p).$$

The average of the operator $A$ in a state $\psi \in \mathcal{H}$ is given by

$$\langle A \rangle = \langle \psi | A | \psi \rangle = \int dq dp A(q, p) f_W(q, p) = Tr \rho A.$$

Now we proceed in order to introduce the symplectic representation of quantum mechanics in phase space. First we introduce a Hilbert space associated to the phase space $\Gamma$, by considering the set of function $\psi(q, p)$ in $\Gamma$, such that

$$\int dq dp \langle \psi | q, p \rangle \psi(q, p) < \infty$$

is a bilinear real form. This Hilbert space is denoted by $\mathcal{H}(\Gamma)$. Unitary mappings, $U(\alpha)$, in $\mathcal{H}(\Gamma)$ are naturally introduced by using the star-product, i.e.

$$U(\alpha) = \exp(\alpha \hat{A}),$$

where

$$\hat{A} = A(q, p) \ast A(q, p) \exp \left[ \frac{i\hbar}{2} \left( \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q} \right) \right]$$

$= A(q + \frac{i\hbar}{2} \partial_q, p - \frac{i\hbar}{2} \partial_p).$

Let us consider some examples. For the basic functions $q$ and $p$ (3-dimensional Euclidian vectors), we have

$$\hat{q}_i = q_i \ast q_i = q_i + \frac{i\hbar}{2} \partial_{q_i}, \quad (3)$$

$$\hat{p}_i = p_i \ast p_i = p_i - \frac{i\hbar}{2} \partial_{q_i}. \quad (4)$$

These operators satisfy the Heisenberg relations $[\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}$. Then we introduce a Galilei boost by defining the boost generator $\hat{v}_i = m q_i \ast -t p_i = m q_i - t \hat{p}_i$, $i = 1, 2, 3$, such that

$$\exp \left( -i v \cdot \hat{k} / \hbar \right) \hat{q}_j \exp \left( i v \cdot \hat{k} / \hbar \right) = \hat{q}_j + v_j t,$n

$$\exp \left( -i v \cdot \hat{k} / \hbar \right) \hat{p}_j \exp \left( i v \cdot \hat{k} / \hbar \right) = \hat{p}_j + m v_j.$$

These results, with the commutation relations, show that $\hat{q}$ and $\hat{p}$ are physically the position and momentum operators, respectively.

We introduce the operators $\hat{Q}$ and $\hat{P}$, such that $[\hat{Q}, \hat{P}] = 0$, $\hat{Q}(q, p) = q(q, p)$ and $\hat{P}(q, p) = p(q, p)$, with

$$\langle q, p | q', p' \rangle = \delta(q - q') \delta(p - p'),$$

and $\int dq dp |q, p\rangle = 1$. From a physical point of view, we observe the transformation rules:

$$\exp(-i \hat{k} / \hbar) 2Q \exp(i v \hat{k} / \hbar) = 2Q + vt1,$$
and
\[ \exp(-iv\frac{\hat{k}}{\hbar})2\mathcal{P}\exp(iv\frac{\hat{k}}{\hbar}) = 2\mathcal{P} + mv\mathbf{1}. \]

Then \( Q \) and \( \mathcal{P} \) are transformed, under the Galilei boost, as position and momentum, respectively. Therefore, the manifold defined by the set of eigenvalues \( (q, p) \) has the content of a phase space. However, the operators \( \hat{Q} \) and \( \mathcal{P} \) are not observables, since they commute with each other.

Considering a homogeneous systems satisfying the Galilei symmetry, the commutations relation between \( \hat{k} \) and \( \hat{H} \) is \( [\hat{k}, \hat{H}] = i\hat{P}_j \). Explicitly, we have
\[ [mq_j + i\hbar \frac{\partial}{\partial p_j}, H(q, p)*] = ip_j + \frac{\hbar}{2} \frac{\partial}{\partial q_j}. \]

A solution, providing a general form to \( \hat{H} = H(q, p)* \), is
\[ \hat{H} = \frac{p^2}{2m} + V(q)* = \frac{p^2}{2m} - \frac{\hbar^2 \partial^2}{8m \partial q^2} - \frac{i\hbar p}{2m} \frac{\partial}{\partial q} + V(q*). \tag{5} \]

This is the Hamiltonian of a one-body system in an external field.

Consider the time evolution of a state \( \psi(q, p; t) \), that is given by \( \psi(q, p; t) = U(t, t_0)\psi(q, p; t_0) \), where \( U(t, t_0) = \exp(-i\hbar(t-t_0)\hat{H}) \). This result leads to a Schrödinger-like equation written in phase-space, i.e. \[ i\hbar \partial_t \psi(q, p; t) = \hat{H}\psi(q, p; t) \] \tag{6}

Now physical meaning of the state \( \psi(q, p; t) \) has to be identified. This is done, by associating \( \psi(q, p; t) \) with the Wigner function. From Eq. \((6)\), one can prove that \( g(q, p) = \psi(q, p; t)\star \psi^\dagger(q, p; t) \) satisfies Eq. \((2)\). \[ dqdp\psi(q, p; t)\star \psi^\dagger(q, p; t) = dqdp\psi(q, p; t)\psi^\dagger(q, p; t), \]

we have
\[ \langle A \rangle = \langle \psi | A | \psi \rangle = \int dqdp\psi(q, p; t)A(q, p; t) \]
\[ = \int dqdp\psi(q, p; t)\tilde{A}(q, p)\psi^\dagger(q, p; t) \]
where \( \tilde{A}(q, p) = A(q, p)* \) is an observable. Thus, the Wigner function can be calculated by using
\[ f_W(q, p) = \psi(q, p)\star \psi^\dagger(q, p). \tag{7} \]

It is to be noted also that the eigenvalue equation,
\[ H(q, p)\star \psi = E\psi, \tag{8} \]
results in \( H(q, p)\star f_W = E f_W \). Therefore, \( \psi(q, p) \) and \( f_W(q, p) \) satisfy the same differential equation. These results show that Eq. \((6)\) is a fundamental starting point for the description of quantum physics in phase space, fully compatible with the Wigner formalism.

### III. Helium-like system in phase space

Although Schrödinger equation can not be accurately solved, there are Helium-like systems that admit exact solutions. In this section we consider the Helium-like system, such that the Coulomb interaction is replaced by Hooke-like forces, including electron-electron interaction \[ 18 \, 19 \, 63. \]

Then we uses the Schrödinger equation in phase space to obtain the Wigner function for the Helium-like atom in this approximation. The classical counterpart of the Helium-like atom Hamiltonian is then written as \[ 63 \]
\[ H = \frac{p^2}{2m} + \frac{P_2}{2m} + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) - \frac{\xi}{4}(x_1 - x_2)^2, \tag{9} \]

where the the sub-indices 1 and 2 refer to the electrons and \( \xi \) is a small parameter. We restrict our analysis to the one-dimensional case. Using the variables,
\[ u = \frac{x_1 + x_2}{\sqrt{2}}, \]
\[ v = \frac{x_1 - x_2}{\sqrt{2}}, \]
\[ p_u = \frac{p_1 + p_2}{\sqrt{2}}, \]
\[ p_v = \frac{p_1 - p_2}{\sqrt{2}}, \]
Eq. \((9)\) is written as
\[ H = \frac{p_u^2}{2m} + m\omega^2u^2 + \frac{p_v^2}{2m} + \frac{1 - \xi}{2}m\omega^2v^2. \tag{10} \]

It is convenient to write \( H = H_u + H_v \), where
\[ H_u = \frac{p_u^2}{2m} + m\omega^2u^2, \]
and
\[ H_v = \frac{p_v^2}{2m} + \frac{1 - \xi}{2}m\omega^2v^2. \]

The time-independent Schrödinger equation in phase space is written as
\[ H \star \psi(u, v, p_u, p_v) = E\psi(u, v, p_u, p_v). \tag{11} \]

In order to solve this equation, we take
\[ \psi(u, v, p_u, p_v) = \varphi(u, p_u)\chi(v, p_v), \]
and
\[ E = E_u + E_v. \]
It is important to consider the relations,
\[ u^\star = u + \frac{i\hbar}{2} \frac{\partial}{\partial p_u}, \]
\[ p_{u^\star} = p_u - \frac{i\hbar}{2} \frac{\partial}{\partial u}, \]
\[ v^\star = v + \frac{i\hbar}{2} \frac{\partial}{\partial p_v}, \]
\[ p_{v^\star} = p_v - \frac{i\hbar}{2} \frac{\partial}{\partial v}, \]
which are obtained from the star product in Eq. \([11]\). In this sense, the resulting equations are solved by starting from the equation for the coordinates \(u\) and \(p_u\); i.e.
\[ \left( \frac{p_{u^\star}^2}{2m} + m\omega^2 u^2 \right) \varphi_n = E_u \varphi_n. \] \([12]\)
Writing
\[ H_u^\star = \frac{m\omega^2}{2} \left( u^\star + \frac{i}{m\omega} p_{u^\star} \right) \left( u^\star - \frac{i}{m\omega} p_{u^\star} \right) - \hbar \omega, \] \([13]\)
we then introduce the operators
\[ a_{u^\star} = \sqrt{\frac{m\omega}{2\hbar}} \left( u^\star + \frac{i}{m\omega} p_{u^\star} \right), \] \([14]\)
\[ a_{u^\star}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( u^\star - \frac{i}{m\omega} p_{u^\star} \right), \] \([15]\)

satisfying the relations,
\[ [a_{u^\star}, a_{u^\star}^\dagger] = 1, \]
\[ a_{u^\star} \varphi_n \propto \varphi_{n-1}, \]
\[ a_{u^\star}^\dagger \varphi_n \propto \varphi_{n+1}, \]
where \(n = 0, 1, 2, \ldots\) and \(a_{u^\star} \varphi_0 = 0, \varphi_{-1} = 0\).

The Hamiltonian given in Eq. \([13]\) is then written as
\[ H_u^\star = \hbar \omega (a_{u^\star} a_{u^\star}^\dagger + \frac{1}{2}). \] \([16]\)
In this way, we have
\[ \sqrt{\frac{m\omega}{2\hbar}} \left( u^\star + \frac{i}{m\omega} p_{u^\star} \right) \varphi_0 = 0. \] \([17]\)

Substituting \(u^\star = u + \frac{i\hbar}{2} \frac{\partial}{\partial p_u}\) and \(p_{u^\star} = p_u - \frac{i\hbar}{2} \frac{\partial}{\partial u}\) in Eq. \([17]\) we obtain
\[ \sqrt{\frac{m\omega}{2\hbar}} \left[ u + \frac{i\hbar}{2} \frac{\partial}{\partial p_u} + \frac{i}{m\omega} \left( p_u - \frac{i\hbar}{2} \frac{\partial}{\partial a} \right) \right] \varphi_0 = 0. \] \([18]\)

Separating the real and imaginary part of Eq. \([18]\), and considering \(\varphi_0(u, p_u) = \varphi_0^e(u) \varphi_0^i(p_u)\), we can show that real part satisfies the differential equation
\[ u \varphi_0^e + \frac{\hbar}{2m\omega} \frac{\partial \varphi_0^e}{\partial u} = 0, \] \([19]\)
with a solution given by
\[ \varphi_0^e(u) = \exp \left( -\frac{2m\omega}{\hbar} u^2 \right). \] \([20]\)

For the imaginary part, we have
\[ \frac{\hbar}{2} \frac{\partial \varphi_0^i}{\partial u} + \frac{p_u}{m\omega} \varphi_0^i = 0, \] \([21]\)
with the solution
\[ \varphi_0^i(p_u) = \exp \left( -\frac{2}{\hbar m\omega} p_u^2 \right). \] \([22]\)

Then, we get
\[ \varphi_0(u, p_u) \sim \exp \left( -\frac{2m\omega}{\hbar} u^2 - \frac{2}{\hbar m\omega} p_u^2 \right). \] \([23]\)

Similarly, the solution of the equation for \(\chi\) is obtained, i.e.
\[ \left( \frac{p_{v^\star}^2}{2m} + \frac{(1 - \xi)}{m\omega^2} u^2 \right) \chi_n = E_v \chi_n. \] \([24]\)

This leads to
\[ H_v^\star = \frac{(1 - \xi m\omega^2)}{2} \left( v^\star + \frac{i}{m\omega(1 - \xi)^{1/2} p_v^\star} \right) \times \left( v^\star - \frac{i}{m\omega(1 - \xi)^{1/2} p_v^\star} \right) - \frac{\hbar \omega}{2} (1 - \xi)^{1/2}. \] \([25]\)

Then, we define
\[ a_v^\star = \sqrt{\frac{m\omega}{2\hbar}} \left( v^\star + \frac{i}{m\omega(1 - \xi)^{1/2} p_v^\star} \right), \] \([26]\)
and
\[ a_v^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( v^\star - \frac{i}{m\omega(1 - \xi)^{1/2} p_v^\star} \right). \] \([27]\)

These operators satisfy the relations,
\[ [a_v^\star, a_v^\dagger] = 1, \]
\[ a_v^\star \chi_n \propto \chi_{n-1}, \]
\[ a_v^\dagger \chi_n \propto \chi_{n+1}, \]
where \(n = 0, 1, 2, \ldots\).

The operator given in Eq. \([25]\) has the form
\[ H_v^\star = \hbar \omega (a_v^\dagger a_v^\star - \frac{(1 - \xi)^{1/2}}{2}). \] \([28]\)

We can show that \(a_v^\star \chi_0 = 0\), such that
\[ \sqrt{\frac{m\omega(1 - \xi)^{1/2}}{2\hbar}} \left( v^\star + \frac{i}{m\omega(1 - \xi)^{1/2} p_v^\star} \right) \chi_0 = 0. \] \([29]\)

The real and imaginary part of Eq. \([29]\) leads, respectively, to the solutions
\[ \chi_0 = \exp \left( -\frac{2m\omega(1 - \xi)^{1/2}}{\hbar} u^2 \right). \] \([30]\)
and
\[ \chi_0 = \exp \left( -\frac{2}{\hbar \omega (1 - \xi) \sqrt{2} P_c^2} \right), \]  
(31)
such that
\[ \chi_0(v, p_v) \sim \exp \left( -\frac{2m \omega (1 - \xi)^{1/2} v^2}{\hbar} - \frac{2}{\hbar \omega (1 - \xi)^{1/2} P_c^2} \right). \]  
(32)

Therefore, the zero order solution of the Schrödinger equation is

\[ \psi_0(u, v, p_u, p_v) = \frac{2e}{\pi \hbar} \exp \left( -\frac{2m \omega [u^2 + (1 - \xi)^{1/2} v^2]}{\hbar} \right) \times \exp \left( -\frac{2}{m \omega \hbar} [p_u^2 + (1 - \xi)^{-1/2} p_v^2] \right), \]

where we have used the normalization condition
\[ \int dq dp \psi_n^\dagger(u, v, p_u, p_v) \psi_n(u, v, p_u, p_v) = 1. \]

To obtain higher order wave functions, we use the relation
\[ \psi_n(u, v, p_u, p_v) = (a_u^* \times a_v^*)^n \psi_0(u, v, p_u, p_v). \]  
(33)

The Wigner function is found from
\[ \mathcal{f}_W^{(n)}(u, v, p_u, p_v) = \psi_n(u, v, p_u, p_v) \star \psi_n^\dagger(u, v, p_u, p_v). \]

In particular for \( n = 0 \) we obtain
\[ \mathcal{f}_W^{(0)}(q_1, q_2, p_1, p_2) = \frac{(2e}{\pi \hbar} \exp \left( -\frac{2m \omega (x_1 + x_2)^2}{\hbar} \right) \times \exp \left( -\frac{2m \omega (1 - \xi)^{1/2} (x_1 - x_2)^2}{\hbar} \right) \times \exp \left( -\frac{2}{m \omega \hbar} (p_1 + p_2)^2 \right) \times \exp \left( -\frac{2}{m \omega \hbar} (1 - \xi)^{-1/2} (p_1 - p_2)^2 \right). \]

Hence, the energy of the fundamental state is given by
\[ E_0 = \hbar \omega (1 - \xi). \]

These results are interesting in a double sense. First, we have calculated analytically the Wigner function for Helium-like atom. Second, the Wigner function has many applications, among them one stands out quantum computing. So, such a procedure to study the Wigner function for Helium-like atom opens up new possibilities for analyzing entanglement. In this context, in experiments, the dissipation due to the effect of external fields, are a crucial factor. In the next section, in order to consider the Helium atom in a non-conservative external field, we add to the Hooke-like force, a linear dissipation.

### IV. Damped Quantum Oscillator

In order to analyze dissipation effect of neighborhood, here, we solve the Schrödinger equation in phase space for Hooke-like system with a damped interaction. This stands for the Helium atom in a dissipative field. We considers a one dimensional system, where Hamiltonian with a dissipative term is (see a different treatment for such a model in Refs [64, 65])

\[ \hat{H} = \frac{1}{2} \left( \hat{P}^2 + \hat{Q}^2 \right) - \frac{\lambda}{2} \left( \hat{Q} \hat{P} + \hat{P} \hat{Q} \right), \]  
(34)

where \( \lambda < 1 \).

Using the operators given in Eqs. (3) and (4) with \( \hbar = 1 \),

\[ \hat{Q} = q + \frac{i}{2} \partial_q, \]  
(35)

and

\[ \hat{P} = p - \frac{i}{2} \partial_p, \]  
(36)

Eq. (34) becomes

\[ \hat{H} = \frac{1}{2} \left( p^2 + q^2 - i p \partial_q + i q \partial_p - \frac{1}{4} \partial_q^2 - \frac{1}{4} \partial_p^2 \right) - \frac{\lambda}{2} \left( 2q p - i q \partial_q + i p \partial_p + \frac{1}{2} \partial_q \partial_p \right). \]

Applying this Hamiltonian in the eigenvalue equation \( \hat{H} \psi(q, p) = E \psi(q, p) \), we obtain

\[ (p^2 + q^2) \psi(q, p) - \frac{1}{4} \partial_q^2 \psi(q, p) - \frac{1}{4} \partial_p^2 \psi(q, p) - \frac{\lambda}{2} \partial_q \partial_p \psi(q, p) - 2 q p \psi(q, p) - 2 E \psi(q, p) = 0. \]

Introducing the new variable
\[ z = \frac{1}{2} (p^2 + q^2) - \lambda q p, \]  
(37)

we obtain

\[ \frac{1}{2} (\lambda^2 - 1) z \partial_z^2 \psi(z) + \frac{1}{2} (\lambda^2 - 1) \partial_z \psi(z) + 2 (z - E) \psi(z) = 0. \]  
(38)

Taking \( \alpha = \frac{1}{2} (1 - \lambda^2) \) and using the ansatz

\[ \psi(z) = e^{-\sqrt{\alpha/2} \omega(z)}, \]  
(39)

we have, after the changing of variables \( y = 2 \sqrt{\frac{2}{\alpha}} z \), the following expression

\[ y \partial_y^2 \omega(y) + (1 - y) \partial_y \omega(y) - \left[ \frac{1}{2} - \frac{E/2}{\sqrt{\alpha/2}} \right] \omega(y) = 0. \]  
(40)
The solution of Eq. (40) is given by the Kummer function (a confluent hypergeometric function, i.e.),

$$\omega(z) = F\left(\frac{1}{2} - \frac{E/2}{\sqrt{a/2}}; 1; 2 \sqrt{\frac{a}{2}} z\right).$$  \hspace{1cm} (41)

In this way, we have the solution

$$\psi(z) = e^{\frac{1}{2} \sqrt{a/2}} F\left(\frac{1}{2} - \frac{E/2}{\sqrt{a/2}}; 1; 2 \sqrt{\frac{a}{2}} z\right),$$  \hspace{1cm} (42)

where $z$ is given in Eq. (37). The confluent hypergeometric function condition is such that

$$\frac{1}{2} - \frac{E/2}{\sqrt{a/2}} = -n,$$

where $n \in \mathbb{Z}$. This relation gives

$$E_n = (1 - \lambda^2)^{1/2} \left[n + \frac{1}{2}\right].$$  \hspace{1cm} (43)

Note that if $\lambda = 0$ we obtain the result $E_n = (n + 1/2)$.

The Wigner function can be calculate by

$$f_W(q, p, t) = \psi \star \psi^\star.$$  \hspace{1cm} (44)

In this case we calculate the Wigner functions given in Eq. (45) using a MAPLE routine. The behavior of the stationary Wigner function for $\lambda = 0.1$ are shown in Figs. (1)-(4) and for $\lambda = 0.9$ are shown in Fig. (5)-(8).

A measure of non-classicality of quantum states is defined on the volume of the negative part of Wigner function, which may be interpreted as a signature of quantum interference. In this sense, the non-classicality (negativity) indicator is given by

$$\eta(\psi) = \int \int |W_\psi(q, p)| - W_\psi(q, p)| dq dp - 1.$$  \hspace{1cm} (45)

This indicator represents the doubled volume of the integrated part of the Wigner function. In sequence, we calculated numerically this indicator for damped oscillator. The results of this calculation are shown in Table 1 below. A surprising result is that the parameter $\eta(\psi)$ does not depend on $\lambda$.

| $n$  | $\eta(\psi)$          |
|------|-----------------------|
| 0    | 0                     |
| 1    | 0.426122634263795     |
| 2    | 0.7289892587057898    |
| 3    | 0.976673079293403     |
| 4    | 1.1913424288065964    |
| 5    | 1.3834384856692004    |
| 6    | 1.5588521972493026    |
| 7    | 1.7212938335545317    |
| 8    | 1.873265816082318     |
| 9    | 2.016572434609475     |

\[\text{FIG. 1. Wigner function, } n = 0, \lambda = 0.1\]

\[\text{FIG. 2. Wigner function, } n = 1, \lambda = 0.1\]

Table 1. The non-classicality indicator as a function of the order of the Wigner function, the parameter $n$.

In Fig. (9), the dependence of non-classicality indicator $\eta(\psi)$ and the order $n$ of Wigner function for damped oscillator is plotted. This allows us to conclude that the magnitude of the parameter $\lambda$, that represents the degree of damping, has no effect in the volume of the negative part of the Wigner function.

\[\text{V. CONCLUDING REMARKS}\]

In this work, the Wigner function for the Helium atom is calculated in the approximation of two-harmonic oscillators, considering also dissipation. Regarding the value of energy, this approximation has provided satisfactory results with the experiments [18, 19, 63]. Here we have considered the statistical nature of such quantum states, by analyzing the non-classicality through the Wigner function. We have proceeded by formulating the problem with the Schrödinger equation in phase space, such
that the state, called a quasi-amplitude of probability, is associated with the Wigner function by the Moyal product. In this context, we study a damped oscillator in phase space. Using Wigner functions, a non-classicality indicator is calculated as a function of the dissipation parameter. In this case, the non-classicality behavior is independent of the dissipation parameter.

ACKNOWLEDGEMENTS

This work was partially supported by CNPq of Brazil.

[1] R. W. Hasse, J. Math. Phys. 16, 2005 (1975).
[2] E. B. Davies, Quantum Theory of Open Systems (Academic Press, New York, 1976).
[3] H. Dekker, Phys. Rep. 80,1 (1981).
[4] K. H. Li, Phys. Rep. 134, 1 (1986).
[5] Y. N. Srivastava, G. Vitiello, Ann. Phys. (N.Y.) 238, 2001 (1995).
[6] A. Iorio, G. Vitiello, A. Widom, Ann. Phys. (N.Y.) 241, 496 (1995).
[7] M. Blasone, P. Jizba, G. Vitiello, Phys. Lett. A 287, 205 (2001).
[8] E. Kanai, Prog. Theor. Phys. 3, 440 (1948).
[9] H. Bateman, Phys. Rev. 38, 815 (1931).
[10] H.T.C. Stoof, M. Houbiers, C.A. Sackett and R.G. Hulet, Phys. Rev. Lett. 76, 10 (1996).
[11] C.J. Myatt, E.A. Burt, R.W. Ghrist, E.A. Cornell and C.E. Wieman, Phys. Rev. Lett. 78, 586 (1997).
[12] E. Hodby, S.T. Thompson, C.A. Regal, M. Greiner, A.C. Wilson, D.S. Jin, E.A. Cornell, C.E. Wieman, Phys. Rev. Lett. 94, 120402 (2005).
[13] K.E. Strecker, G.B. Partridge and R.G. Hulet, Phys. Rev. Lett. 91, 080406 (2003).
[14] J. Cubizolles, T. Bourdel, S.J.J.M.F. Kokkelmans, G.V. Shlyapnikov, and C. Salomon, Phys. Rev. Lett. 91, 240401 (2003).
[15] T. Bourdel, L. Khaykovich, J. Cubizolles, J. Zhang, F. Chevy, M. Teichmann, L. Tarruell, S.J.J.M.F. Kokkelmans, and C. Salomon, Phys. Rev. Lett. 93, 050401 (2004).
[16] A. Shimony, Measures of Entanglement, in The Dilemma of Einstein, Podolsky and Rosen – 60 Years Later, Edited by A. Mann and M. Revzen (IOP, Bristol, 1996).
[17] G. Alber, T. Beth, M. Horodecki, P. Horodecki, R. Horodecki, M. Rötteler, H. Weinfurter, R. Werner, and A. Zeilinger, Quantum Information (Springer-Verlag, Berlin, 2001).
[18] N.R. Kestner, J. Chem. Phys. 45, 213 (1966).
[19] N.R. Kestner, O Sinanoglu, Phys. Rev. 128, 2687 (1962).
[20] E.P. Wigner, Z. Phys. Chem. B 19, 749 (1932).
[21] M. Hillery, R. F. O ’Connell, M. O. Scully, E. P. Wigner, Phys. Rep. 106, 121 (1984).
[22] Y.S. Kim, M.E. Noz, Phase Space Picture and Quantum Mechanics - Group Theoretical Approach (W. Scientific, London, 1991).
FIG. 5. Wigner function, \( n = 0, \lambda = 0.9 \)

FIG. 6. Wigner function, \( n = 1, \lambda = 0.9 \)

FIG. 7. Wigner function, \( n = 5, \lambda = 0.9 \)

FIG. 8. Wigner function, \( n = 10, \lambda = 0.9 \)

FIG. 9. The non-classicality indicator versus quantum number for damped oscillator \( n \leq 50 \)

[23] T. Curtright, D. Fairlie, C. Zachos, Phys. Rev. D 58, 25002 (1998).
[24] D. Galetti and A.F.R. de Toledo Piza, Physica A 214, 207 (1995).
[25] D. T. Smithey, M. Beck, M. G. Raymer, A. Faridani, Phys. Rev. Lett. 70, 1244 (1993).
[26] D. Leibfried, D. M. Meekhof, B. E. King, C. Monroe, W. M. Itano, D. J. Wineland, Phys. Rev. Lett. 77, 4281 (1996).
[27] L. G. Lutterbach, L. Davidovich, Phys. Rev. Lett. 78, 2547 (1997).
[28] R. L. de Matos Filho, W. Vogel, Phys. Rev. A 58, R1661 (1998).
[29] A. Ibort, V. I. Man’ko, G. Marmo, A. Simoni, F. Ventriglia, Phys. Scripta 79, 065013 (2009) [arXiv:0904.4439 [quant-ph]].
[30] A. Ibort, V. I. Man’ko, G. Marmo, A. Simoni, F. Ventriglia, Phys. Lett. A 374, 2614 (2010) [arXiv:1004.0102 [quant-ph]].
[31] A. Ibort, A. Lopez-Yela, V. I. Man’ko, G. Marmo, A. Simoni, E. C. G. Sudarshan, F. Ventriglia, [arXiv:1202.3275 [math-ph]].
[32] V.V. Dodonov, Phys. Lett. A 364, 368 (2007).
[33] V. V. Dodonov, O.V. Man’ko, V. I. Man’ko, Phys. Rev. A 50, 813 (1994).
[34] V. V. Dodonov, O.V. Man'ko, V. I. Man'ko, Phys. Rev. A 49, 2993 (1994).
[35] L. S. F. Olavo, Phys. Rev. A, 61, 052109 (2000).
[36] L. S. F. Olavo, Found. Phys. 34, 891 (2004).
[37] F. C. Khanna, A. P. C. Malbouisson, J. M. C. Malbouisson, A. E. Santana, Thermal Quantum Field Theory: Algebraic Aspects and Applications (W. Scientific, Singapore, 2009).
[38] S. A. Smoliansky, A. V. Prozorkevich, G. Maino, S. G. Mashnic, Ann. Phys. (N.Y.) 277, 193 (1999).
[39] L.M. Abreu, A.E. Santana, A. Ribeiro Filho, Ann. Phys. (N.Y.) 297, 396 (2002).
[40] M.C.B. Fernandes, J.D.M. Vianna, Braz. J. Phys. 28, 2 (1999).
[41] M.C.B. Fernandes, A. E. Santana, J. D. M. Vianna, J. Phys. A: Math. Gen. 36, 3841 (2003).
[42] A.E. Santana, A. Matos Neto, J.D.M. Vianna, F.C. Khanna, Physica A 280, 405 (2001).
[43] M.C.B. Andrade, A.E. Santana, J.D.M. Vianna, J. Phys. A: Math. Gen. 33, 4015 (2000).
[44] M.A. Alonso, G.S. Pogosyan, K.B. Wolf, J. Math. Phys. 43, 5857 (2002).
[45] J. Dito, J. Math. Phys. 33, 791 (1992).
[46] G. Torres-Vega, J.H. Frederick, J. Chem. Phys. 93, 8862 (1990).
[47] G. Torres-Vega, J.H. Frederick, J. Chem. Phys. 98, 3103 (1993).
[48] O. F. Davi, L. T. Kelleyane, Mod. Phys. Lett. A 17, 1937 (2002) [hep-th/0202062].
[49] M.A. de Gosson, Bull. Sci. Math. 121, 301 (1997).
[50] M.A. de Gosson, Ann. Inst. H Poincaré 70, 547 (1999).
[51] M.A. de Gosson, J. Phys. A: Math. Gen. 37, 7297 (2004).
[52] M.A. de Gosson, Lett. Math. Phys. 72, 293 (2005).
[53] M.D. Oliveira, M.C.B. Fernandes, F.C. Khanna, A.E. Santana and J.D.M. Vianna, Ann. Phys. (N.Y.) 312, 492 (2004).
[54] R.G.G. Amorim, F.C. Khanna, A.P.C. Malbouisson, J.M.C. Malbouisson, A.E. Santana, Realization of the Noncommutative Seiberg-Witten Gauge Theory by Fields in Phase Space, arXiv:1402.1446 [hep-th].
[55] H. Weyl, Z. Phys. 46, 1 (1927).
[56] J.E. Moyal, Proc. Camb. Phil. Soc. 45, 99 (1949).
[57] R.G.G. Amorim, M.C.B. Fernandes, F.C. Khanna, A.E. Santana, J.D.M. Vianna, Phys. Lett. A 361, 464 (2007).
[58] R.G.G. Amorim, F.C. Khanna, A.E. Santana, J.D.M. Vianna, Physica A 388, 3771 (2009).
[59] R.G.G. Amorim, M.C.B. Fernandes, F.C. Khanna, A.E. Santana, J.D.M. Vianna, Int. J. Mod. Phys. A 28, 1350013 (2013).
[60] R.G.G. Amorim, S.C. Ulhoa, A.E. Santana, Braz. J. Phys. 43, 78 (2013).
[61] M.C.B. Fernandes, F.C. Khanna, M.G.R. Martins, A.E. Santana, J.D.M. Vianna, Physica A, 389, 3409 (2010).
[62] A. Kenfack, K. Zyczkowski, J. Opt. B: Quantum Semiclass. Opt. 6, 396 (2004).
[63] R. Crandall, R. Whitnell, R. Bettega, Am. J. Phys. 52, 439 (1984).
[64] R. Cordero-Soto, E. Suazo, S. K. Suslov, J. Phys. Math. 1, 1 (2009).
[65] A. Isar, A. Sandulescu, W. Scheid, Int. J. Mod. Phys. B 10, 2767 (1996).