Group-Strategyproof mechanisms for facility location with Euclidean distance

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Abstract

We characterize the class of group-strategyproof mechanisms for single facility location game in Euclidean space. A mechanism is group-strategyproof, if no group of agents can misreport so that all its members are strictly better off.

We show that any deterministic, unanimous, group-strategyproof mechanism must be dictatorial, and that any randomized, unanimous, translation-invariant, group-strategyproof mechanism must be 2-dictatorial. Here a randomized mechanism is 2-dictatorial if the lottery output of the mechanism must be distributed on the line segment between two dictators’ inputs. A mechanism is translation-invariant if the output of the mechanism follows the same translation of the input.

Based on the characterizations, we obtain tight bounds of approximately optimal group-strategyproof mechanisms under both maximum and social cost objectives.

1 Introduction

In a single facility location game, every agent reports a location, and the designer chooses a facility location. In the model considered in this paper, the true locations of the agents are unconstrained and given in a Euclidean space. The cost of each agent is the distance (i.e. $L_2$-norm) between the facility location and her true location.

Such setting is partly motivated by the multi-issue voting problem [7] where agents aim to elect a candidate who aligns with their opinions on multiple political issues, each represents a single-peaked preference domain. As a result, each agent and candidates can be simply represented as a point in the Euclidean space and the degree of misalignment between a voter and a candidate is measured by the Euclidean distance between them.

Note that it is not necessary for an agent to report her true location. We aim to find truthful mechanisms, where on one can be better off by misreporting. By the revelation principle, every social choice function that can be implemented in dominant strategies can be implemented by a truthful and direct mechanism [6]. In other words, the studies on truthful mechanisms are without loss of generality. In the literature, such mechanisms are also known as strategyproof mechanisms.

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A closely related but more stringent condition is called group-strategyproofness, where no group of agents can jointly misreport their preferences so that all members are strictly better off [10].

Procaccia and Tennenholtz [10] study approximately optimal strategyproof mechanisms for one dimensional facility games under both social (the sum of expected individual costs) and maximum cost (the maximum of expected individual costs) objectives. They introduce the so-called median mechanism, which is group-strategyproof and minimizes social cost. They also propose an approximately optimal randomized strategyproof mechanism for the maximum cost objective, which $3/2$-approximates the optimum, and ensures group-strategyproofness.

The model and problem studied in this paper can be regarded as a generalization of Procaccia and Tennenholtz’s work to some high dimensional Euclidean space. Moulin [9] provide a complete characterization of strategy-proof social choice functions over one-dimensional single-peaked preferences, known as the median voter schemes. Border and Jordan [3] extend Moulin’s result to Euclidean space and show that it induces to median voter schemes in each dimension. However, to the best of our knowledge, none of the previous works studies randomized mechanisms in the Euclidean space. In fact, similar to the Border and Jordan’s work, some strategyproof mechanisms can indeed be generalized to the multi-dimensional domain and preserve the property of strategyproofness. Unfortunately, they are no longer group-strategyproof, which is the focus of this paper.

1.1 Our Results

We first study the design of deterministic mechanisms, and we show that a deterministic mechanism is unanimous and group-strategyproof if and only if it is dictatorial. A mechanism is unanimous if all agents report the same point, the mechanism must choose that point as well. In fact, a mechanism that is non-unanimous can be unbounded in terms of approximation of both social and maximum cost objectives.

For randomized mechanisms, we first introduce a condition called transnational invariance, which says that if we apply a translation to the inputs, the mechanism must output a location that is the result of the same translation to the original output. This condition is not only arguably natural for the facility location domain but is also essential in guaranteeing desirable approximation bound (see Proposition 4.1). Our main theorem (Theorem 2) in this part is that any unanimous, translation-invariant, and group-strategyproof mechanisms must be 2-dictatorial.

Based on the characterizations above, we study the approximation ratio of group-strategyproof mechanisms. The results are summarized in Table 1. All bounds are tight, except for the small gap of randomized mechanisms for the social cost objective.

To prove these results, we develop a series of technical tools to facilitate our characterization. First, when some results are obtained in some small-scale cases, we use induction to generalize them over all $n$ under group-strategyproofness. We successfully find some of the agents to form a temporary coalition, while the reduced mechanism is still unanimous and group-strategyproof (see the proof of Theorem 1 and Theorem 2 for more details). Second, we find the convexity of Euclidean distance very helpful to reduce the space of randomized distributions, as selecting the centroid in general makes everyone better off. Third, we use continuous expected distance (Lemma 3.4) in a smart way. Continuous expected distance is a strong local property of the outputs, but by moving in small steps, in many proofs we further prove other local properties and generalize them to the whole space. Note that continuous expected distance also holds for strategyproof mechanisms, which shows its potential for future works.
### Table 1: Summary of our results. The cells contain the lower and upper bounds of group-strategyproof mechanisms under different conditions, while minimizing either maximum cost (MC) or social cost (SC).

|       | Deterministic | Randomized |
|-------|---------------|------------|
| **MC** | n = 2         | [2, 2]     |
|       | n ≥ 3         | [2, 2]     |
| **SC** | n = 2         | [1, 1]     |
|       | n = 3         | [2, 2]     |
|       | n ≥ 4         | [n − 1, n − 1] | \([\frac{n}{2} - 1, \frac{n}{2}]\) |

### 1.2 Related Works

Our work can be regarded as a multi-dimensional extension of the work by Procaccia and Tennenholtz [10]. In their setting, agents are located on a line. The purpose is to find the best approximation ratio of strategyproof mechanisms under the social cost or maximum cost objective. They provide analysis of both deterministic and randomized mechanisms for a single facility, and show some bounds of their extensions, e.g., two facilities, and multiple locations per agent. In the later works, they further extend the location space to a network, and then obtain some bounds of some special cases (e.g., tree, circle) and general cases for maximum cost [1]. Many other works are concerned with the bounds of facility games on a line or a network, including many facilities and non-linear costs (see, e.g., [8, 4, 5]). Note that our setting is not a special case of a general network, which is discrete in location space. In Euclidean space, there are many more interesting properties that are not covered in the network case.

A closely related setting was introduced by Sui [12], in a more algorithmic and practical manner. In the unconstrained Euclidean space, Sui [12] show that GMMs (generalized median mechanisms) are strategyproof but are not group-strategyproof, and the incentive of group misreport is unbounded. Yet, the lower bound or possibility of group-strategyproofness is still unknown.

Beyond the field of approximate (randomized) mechanism design, there are at least two seminal works in earlier years. Schummer and Vohra [11] study strategyproof deterministic mechanisms on a general network and obtain that an onto mechanism must be dictatorial if the graph contains a cycle. Unfortunately, it cannot generalize to Euclidean space. Border and Jordan [3] provide complete characterizations of strategyproof deterministic mechanisms in Euclidean space, based on the work by Moulin [9]. However, for randomized mechanisms, this type of result is still lacking. Our analysis of randomized mechanisms complements this literature.

### 2 Settings

We consider the facility location game with \(n\) \((n \geq 2)\) agents \(N = \{1, \ldots, n\}\). All agents are located in a \(d\)-dimensional Euclidean space \(\mathbb{R}_d\). Let agent \(i\)'s location be \(x_i \in \mathbb{R}_d\) for each \(i \in N\). Let a vector of agents' location \(x = (x_1, \ldots, x_n)\) be a location profile.

A deterministic mechanism is a function \(f : \mathbb{R}_d^n \rightarrow \mathbb{R}_d\), mapping a location profile to a location of facility. Given a location of facility \(f(x) = y \in \mathbb{R}_d\), the cost of agent \(i\) is the Euclidean distance between \(x_i\) and \(y\), i.e., \(\|x_i - y\|\). We say \(f(x)\) is the output of \(x\).

A randomized mechanism is a function mapping a location profile \(x \in \mathbb{R}_d^n\) to a probability distribution over \(\mathbb{R}_d\). Given a probability distribution of facility \(f(x) = P\), the cost of agent \(i\) is
the expected distance between \(x_i\) and the facility, i.e.,

\[
\|x_i - P\| \triangleq \mathbb{E}_{y \sim P} \|x_i - y\|.
\]

In what follows, we formally define several constraints and/or properties of a mechanism, which will be thoroughly discussed in the paper.

**Definition 2.1** (Strategyproofness). A mechanism is strategyproof if and only if no agent can benefit from misreporting the location, that is, for all \(x \in \mathbb{R}^n_d\), for all \(i \in N\), and for all \(x_i' \in \mathbb{R}_d\),

\[
\|f(x) - x_i\| \leq \|f(x_i', x_{-i}) - x_i\|
\]

where \(x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\) is the location profile without \(x_i\).

**Definition 2.2** (Group-strategyproofness). A mechanism is group-strategyproof if and only if for all \(S \subseteq N\), there is no \(x'_S \in \mathbb{R}_d^{|S|}\) such that all agents in \(S\) can gain from misreporting, that is, for all \(x \in \mathbb{R}^n_d\), for all \(S \subseteq N\), for all \(x'_S \in \mathbb{R}_d^{|S|}\), there exists \(i \in S\) such that

\[
\|f(x) - x_i\| \leq \|f(x'_S, x_{-S}) - x_i\|
\]

where \(x_{-S}\) is the location profile without agents in \(S\).

**Definition 2.3** (Unanimity). A mechanism \(f\) is unanimous if and only if

\[
x_1 = \cdots = x_n = x \implies f(x) = x.
\]

That is, if all agents report the same point, the mechanism must choose that point as well.

**Definition 2.4** (Translational invariance). A mechanism \(f\) is translation-invariant if and only if

\[
\forall x \in \mathbb{R}^n_d, \forall a \in \mathbb{R}_d, f(x) + a = f(x + a),
\]

where \(x + a = (x_1 + a, \ldots, x_n + a)\). Namely, if we apply a translation to the inputs, the mechanism must output a location that is the result of the same translation to the original output.

**Definition 2.5** (Dictatorship). A mechanism \(f\) is dictatorial if and only if \(\exists i \in N, \forall x, f(x) = x_i\). We say agent \(i\) is the dictator.

**Definition 2.6** (2-Dictatorship). A mechanism \(f\) is 2-dictatorial if and only if \(\exists i, j \in N, \forall x, f(x)\) lies on the segment between \(x_i\) and \(x_j\). We say agents \(i\) and \(j\) are the 2-dictators.

In this paper, one of our goals is to design a group-strategyproof mechanism while minimizing one of the following common objectives — expected maximum cost or expected social cost, that is,

\[
mc(P, x) = \mathbb{E}_{y \sim P} \left[ \max_{i \in N} \|x_i - y\| \right],
\]

or

\[
sc(P, x) = \mathbb{E}_{y \sim P} \left[ \sum_{i \in N} \|x_i - y\| \right].
\]

Also, we slightly abuse the notation, such that for \(y \in \mathbb{R}_d\), we use \(mc(y, x)\) and \(sc(y, x)\) to denote the deterministic version of the objective functions.

We say a mechanism is an \(\alpha\)-approximation of the optimum with respect to an objective \(\text{obj}\) if for all \(x \in \mathbb{R}^n_d\),

\[
\text{obj}(f(x), x) \leq \alpha \min_{y \in \mathbb{R}^n_d} \text{obj}(y, x).
\]
3 Deterministic Mechanisms

In this section, we aim to characterize deterministic group-strategyproof mechanisms in Euclidean space.

First, we observe that the simple dictatorial mechanism is group-strategyproof and 2-approximation for maximum cost, but it reaches $n-1$ for social cost in the worst case (see Figure 1). Unfortunately, we will show that these are truly the tight bounds of deterministic mechanisms.

Figure 1: Worst case of the dictatorial mechanism for social cost. Agent 1 is the dictator.

We start with a lemma which quickly follows by the definition of unanimity. (See the proof in Appendix A.1.)

**Lemma 3.1.** All constant approximation mechanisms are unanimous.

In some papers, they use onto as one of the constraints instead of unanimous (e.g., [11]). In fact, these two constraints are equivalent when $f$ is deterministic and strategyproof.

**Proposition 3.2.** Suppose $f$ is deterministic and strategyproof, then $f$ is unanimous if and only if $f$ is onto.

We give a short proof of this proposition in Appendix A.2.

Using the same method, we obtain the following lemma, which is simple but useful, and also holds for randomized mechanisms. (See the proof in Appendix A.3.)

**Lemma 3.3** (Uncompromising property). Let $f$ be a strategyproof mechanism. Let $x$ be a profile such that the output is deterministic, and let $x = f(x)$. We claim that $f(x') = x$, if either $x_i' = x_i$ or $x_i = x$ holds for all $i \in N$.

Derived from strategyproofness, we then find a general property that characterizes the relation between one’s movement and the cost. In the following lemma, we will show that the distance of the output cannot have a sudden change when an agent is moving slowly. Since it also holds for randomized mechanisms, we will consider the expected distance. (See the proof in Appendix A.4.)

**Lemma 3.4** (Continuous expected distance). Let $f$ be a strategyproof mechanism. $\forall i \in N$, for any fixed $x_{-i} \in \mathbb{R}^{n-1}_d$, the distance between $x_i$ and the output of $(x_i, x_{-i})$

$$h(x_i) \triangleq \|f(x_i, x_{-i}) - x_i\|$$

is a continuous function. Moreover, $\forall x_i, x_i' \in \mathbb{R}_d$,

$$\|h(x_i) - h(x_i')\| \leq \|x_i - x_i'\|.$$

For convenience, we begin with the case when $n = 2$. The result of this base case will be used afterward.

**Lemma 3.5.** Suppose $f$ is deterministic, unanimous, and group-strategyproof. When $n = 2$, $\forall x$, $f(x)$ lies on $x_1x_2$. \[1\]

\[1\] For any $x \in \mathbb{R}_d$, $y \in \mathbb{R}_d$, we denote $xy$ as the segment between $x$ and $y$, that is, the set $\{\xi x + (1-\xi)y \mid \xi \in [0,1]\}$. 
Proof. Suppose for the sake of contradiction that there exists \( x \) such that \( f(x) \) does not lie on \( x_1x_2 \). Because of unanimity, we can assume \( x_1 \neq x_2 \), which leaves two cases:

- \( f(x) \) lies on the line of \( x_1 \) and \( x_2 \). Without loss of generality, assume \( x_1 \) lies on the segment between \( f(x) \) and \( x_2 \). Then, agent 2 would misreport \( x_1 \) instead of \( x_2 \) and violate strategyproofness.

- Otherwise. Let \( x \) be the projection point from \( f(x) \) to the line of \( x_1 \) and \( x_2 \). In this case, agents 1 and 2 would both misreport \( x \) and violate group-strategyproofness (see Figure 2).

![Figure 2: A case in the proof of Lemma 3.5.](image)

Lemma 3.5 states that the output always lies between the agents, from which we observe that the facility location is determined by the distance between the output and an agent. Combined with continuous expected distance, a corollary quickly follows.

**Corollary 3.6.** When \( n = 2 \), if \( f \) is deterministic, unanimous, and group-strategyproof, then \( f(x) \) is a continuous function.

In the following lemma, we give more insights into the domination of dictatorship — if it is dictatorial for some profile, then it is dictatorial over all profiles. (See the proof in Appendix A.5)

**Lemma 3.7.** Suppose \( f \) is deterministic, unanimous, and group-strategyproof. When \( n = 2 \), if there exist \( x_1 \neq x_2 \) such that \( f(x) = x_1 \), then agent 1 is the dictator in all profiles.

Then we show the dictatorship when \( n = 2 \), which is the special case of Theorem 1.

**Lemma 3.8.** When \( n = 2 \), if \( f \) is deterministic, unanimous, and group-strategyproof, then \( f \) is dictatorial.

**Proof.** Let \( x \) be any profile such that \( x_1 \neq x_2 \), and \( x = f(x) \). However, if \( x \neq x_1 \) and \( x \neq x_2 \), by uncompromising property, we obtain \( f(x, x_2) = x \). Therefore, by Lemma 3.7, agent 1 is the dictator in all profiles, which contradicts to \( x \neq x_1 \).

Therefore, either \( x = x_1 \), or \( x = x_2 \), so \( f \) is dictatorial (by Lemma 3.7). □

In the theorem below, we will generalize the result for any \( n \geq 2 \). Intuitively, we divide agents into two non-empty groups. If in each group, every agent has the same location, we may conclude that one of them is the group of dictators. Particularly, we first show that there exists a group of dictators containing \( n - 1 \) agents, and then reduce the \( n \)-agent game into an \( (n - 1) \)-agent problem by fixing the location of the non-dictator. By induction, we then prove that there must be a dictator in the group. In the last step, we prove that this dictator keeps to be the same agent regardless of the location of the non-dictator.
Theorem 1 (Universal dictatorship). If \( f \) is deterministic, unanimous, and group-strategyproof, then \( f \) is dictatorial.

Proof. We prove by induction on \( n \). By Lemma 3.8 it holds when \( n = 2 \).

When \( n \geq 3 \), assume it holds for \( n - 1 \) by induction.

First, we construct two new mechanisms \( g_1, g_2 \), where there are only two agents for each: \( \forall y = (y_1, y_2) \), let

\[
g_1(y_1, y_2) = f(y_1, y_2, \ldots, y_2) \]
\[
g_2(y_1, y_2) = f(y_1, \ldots, y_1, y_2).
\]

In short, we bind the two groups of agents \( \{2, \ldots, n\} \), \( \{1, \ldots, n-1\} \) respectively and then construct \( g_1, g_2 \).

It is clear that \( g_1, g_2 \) are deterministic, unanimous, and group-strategyproof, so \( g_1, g_2 \) are dictatorial (by Lemma 3.5). If agent 1 \((y_1)\) is the dictator of \( g_1 \), then agent 2 \((y_2)\) is not the dictator of \( g_2 \), because, by uncompromising property,

\[
f(y_1, y_2, \ldots, y_2) = y_1 \Rightarrow f(y_1, \ldots, y_1, y_2) = y_1.
\]

Thus, at least one of the following two conditions holds: agent 2 is the dictator of \( g_1 \), or agent 1 is the dictator of \( g_2 \). Without loss of generality, assume agent 2 is the dictator of \( g_1 \).

Intuitively, we can then reduce the game into \( n - 1 \) players. We construct a set of mechanisms, where there are exactly \( n - 1 \) agents for each: \( \forall x_1 \), \( \forall x_{-1} \), let

\[
f_{x_1}(x_{-1}) = f(x_1, x_{-1}).
\]

It is clear that each new mechanism \( f_{x_1} \) is deterministic and group-strategyproof, while its unanimity comes from the dictatorship of \( g_1 \). Therefore, \( \forall x_1 \), \( f_{x_1} \) is dictatorial (by induction assumption).

Then it suffices to show that all \( f_{x_1} \) have a common dictator. Suppose for contradiction that there exist \( x_1, x'_1 \) such that \( f_{x_1}, f_{x'_1} \) have different dictators. Without loss of generality, assume agents 2, 3 are the dictators of \( f_{x_1}, f_{x'_1} \), respectively. Consider the following profiles:

\[
x = (x_1, x'_1, x_1, x_{-\{2,3\}}),
\]
\[
x' = (x'_1, x'_1, x_1, x_{-\{2,3\}}),
\]

where \( x_{-\{2,3\}} \) can be arbitrary. In this case, agent 1 would misreport \( x'_1 \) instead of truthfully reporting \( x_1 \), as \( f(x) = x'_1 \) and \( f(x') = x_1 \), which leads to a contradiction.

Therefore, there exists \( i \in \{2, \ldots, n\} \) such that \( \forall x_1 \), \( f_{x_1} \) is dictatorial and agent \( i \) is the dictator. That is, \( f \) is dictatorial. \( \square \)

Theorem 1 provides complete characterizations of deterministic mechanisms. It also shows an impossibility result of anonymous deterministic mechanisms, as the following corollary states.

Corollary 3.9. No deterministic, anonymous, group-strategyproof mechanism has a constant approximation ratio.

4 Randomized Mechanisms

Compared to deterministic mechanisms, it seems that randomized mechanisms are more potential to achieve better approximations. In this section, we focus on the performance of randomized group-strategyproof mechanisms.

We first consider the following mechanism, which is a variant of the prototype proposed by Procaccia and Tennenholtz [10].
Mechanism 1. Given $x$, return $x_1$ with $1/4$ probability, $x_2$ with $1/4$ probability, and $(x_1 + x_2)/2$ with $1/2$ probability.

In one-dimensional setting, the prototype of Mechanism 1 where $x_1, x_2$ are replaced by the leftmost point and the rightmost point respectively, is group-strategyproof and $3/2$-approximation for maximum cost, which has been proved to be the best (see [10]).

In Euclidean space, when $n = 2$, Mechanism 1 is group-strategyproof and $3/2$-approximation as well (similar to the proof by Procaccia and Tennenholtz [10]). However, for all $n \geq 3$, it reduces to a trivial $2$-approximation. We know that it is hard to find the leftmost point and the rightmost point in Euclidean space, but we need far more details to prove that $2$-approximation is exactly the lower bound. For social cost, it ensures an $n/2$-approximation (see Appendix C.1 for the proof), but finally we will show that it is almost tight as well.

Under deterministic setting, translational invariance is not used since dictatorship actually implies translational invariance. However, after we remove the deterministic constraints, there exist unanimous and group-strategyproof mechanisms which are not translation-invariant.

Mechanism 2. Given $x$, let $r$ be the first coordinate of $x_1$, and $y$ be the point on $x_1 x_2$ such that $\|x_1 - y\| = \min\{|r - a|, \|x_1 - x_2\|\}$. Similarly, let $y'$ be the point on $x_1 x_3$ such that $\|x_1 - y'\| = \min\{|r - a|, \|x_1 - x_3\|\}$.

If $r \geq a$, return $x_1$ with $2/3$ probability, and $y$ with $1/3$ probability. Otherwise, return $x_1$ with $2/3$ probability, and $y'$ with $1/3$ probability.

![Figure 3: An illustration of Mechanism 2. These are some possible cases, according to the description.](image)

For any fixed constant $a$, Mechanism 2 is unanimous and group-strategyproof (see Appendix C.2 for the proof). However, it is not translation-invariant. For example, supposing $r \geq a$ and $r - a < \|x_1 - x_2\|$, when $r$ increases, $y$ gets more and more close to $x_2$. Intuitively, if a mechanism is not translation-invariant, it might be related to some constant (e.g., $a$ in Mechanism 2), which is not a desirable property.

More importantly, the following proposition shows that translational invariance cannot harm approximation bounds for any translation-invariant objective function $\text{obj}$. Here, $\text{obj}$ is translation-invariant if and only if $\forall y \in \mathbb{R}_d, \forall x \in \mathbb{R}^d_n, \forall a \in \mathbb{R}_d$,

$$\text{obj}(y, x) = \text{obj}(y + a, x + a).$$

Obviously, maximum cost and social cost are both translation-invariant.
Proposition 4.1. For any unanimous and group-strategyproof mechanism \( f \), there exists an unanimous, translation-invariant and group-strategyproof mechanism \( f' \) such that the approximation ratio of \( f' \) is no worse than \( f \) with respect to any translation-invariant objective function \( \text{obj} \).

See Appendix [B.1] for the proof.

For better illustration, we use the two notations below — centroid and radius of a distribution. Let the centroid of a distribution \( P \) be

\[
C(P) = \mathbb{E}_{y \sim P} y,
\]

and the radius of \( P \) be

\[
R(P) = \mathbb{E}_{y \sim P} \| y - C(P) \| = \| P - C(P) \|.
\]

Since \( f(x) \) in general is a distribution over the whole space, the following lemma reduces every group-strategyproof and unanimous mechanism to a distribution on a line segment, derived from the convexity of centroid.

Lemma 4.2. Suppose \( f \) is unanimous and group-strategyproof. \( \forall x \), if \( R(f(x)) = 0 \) (that is, the output is deterministic), then \( f(x) \in \text{conv}(x) \); otherwise, \( \exists i,j \in N \) such that the support of \( f(x) \) lies on the segment between \( x_i \) and \( x_j \).

For the deterministic case, if \( f(x) \notin \text{conv}(x) \), we prove that there exists a point \( y \) which is very close to \( f(x) \), while the distance from any \( x_i \) to \( y \) is shorter than the distance to \( f(x) \). Thus, agents may all misreport \( y \) and break group-strategyproofness (by unanimity). For the randomized case, we consider the situation where all agents misreport to \( C(f(x)) \). With high probability, all agents would get better off, because of the convexity of Euclidean distance. We further make detailed analysis for some extreme cases. For a full proof, see Appendix [B.2].

Corollary 4.3. When \( n = 2 \), if \( f \) is unanimous and group-strategyproof, then \( \forall x, f(x) \) lies on \( \overline{x_1x_2} \).

Combined with continuous expected distance, we get the extented version of Corollary 3.6.

Corollary 4.4. When \( n = 2 \), if \( f \) is unanimous and group-strategyproof, then \( g(x_1, x_2) = C(f(x_1, x_2)) \) is a continuous function.

Although the output space is already reduced to a line between some agents, the situation is still very complicated. The following lemma shows that under some conditions, the output will be relatively very close to the input of some agent. This is a pivot step of our final result.

Lemma 4.5. When \( n = 3 \), if \( f \) is unanimous and group-strategyproof, then \( \forall x \in \mathbb{R}_d, \forall \ell > 0, \exists i \in N, \forall \epsilon > 0, \) there exists \( x \) such that

\[
\|f(x) - x_j\| < \epsilon \|x_i - x_j\| < \ell
\]

holds for some \( j \neq i \) where \( \|x_j - x\| < \ell \).

To prove the lemma above, we construct a sequence of profiles such that each profile forms an isosceles triangle, while the output keeps on the base of the triangle. Moreover, we ensure that as the agents move, the legs of the triangle get longer, while the base gets shorter. Thus, we can

\[\text{conv}(x) \text{ stands for the convex hull of } \{x_1, \ldots, x_n\}, \text{ i.e., sets of all convex combinations of } x_1, \ldots, x_n.\]
easily find a profile in the sequence such that \( \| f(x) - x_j \| < \epsilon \| x_i - x_j \| \), where \( x_i \) is the apex of the triangle. To make the output remains on the base, a clever application of continuous expected distance is needed. For details, see Appendix B.3.

In what follows, we aim to the case where \( n = 3 \), \( f \) is unanimous and group-strategyproof. We first introduce some notations.

Let \( \chi : \mathbb{R}_d \times \mathbb{R} \rightarrow N \) be a function such that \( \chi(x, \ell) \) corresponds to the agent \( i \) chosen by Lemma 4.5 with respect to given \( x \) and \( \ell \).

Let \( \Phi : \mathbb{R}_d \rightarrow 2^N \) be a function mapping a location to a subset of \( N \), representing the convergence of \( \chi(x, \ell) \) as \( \ell \rightarrow 0 \). Formally, \( \forall x \in \mathbb{R}_d, \forall i \in N, i \in \Phi(x) \) if and only if either there are an infinite number of \( m \in \mathbb{N}^+ \) such that \( \chi(x, 1/m) = i \), or there exists \( x_i \neq x \) such that \( f(x_i, x_{-i}) = x \) where \( x_{-i} = (x, \ldots, x) \). \( \forall i \in N \), let \( S_i = \{ x_i \mid i \in \Phi(x) \} \). Clearly, \( \Phi(x) \neq \emptyset \), and \( S_1 \cup S_2 \cup S_3 = \mathbb{R}_d \).

Specially, if \( f \) is translation-invariant, then \( \chi(x, l) \) and \( \Phi(x) \) are invariant to \( x \), so \( S_i \) is either an empty set or exactly \( \mathbb{R}_d \).

Now we have an infinite number of profiles such that the output has nice properties. In the next lemma, from a set of convergent profiles, we prove there exist two agents such that the output is exactly their input as long as they share the same input. (See the proof in Appendix B.4)

**Lemma 4.6.** Consider the case when \( n = 3 \), \( f \) is unanimous, translation-invariant, and group-strategyproof. If \( S_i = \mathbb{R}_d \), then \( \forall x \) such that \( x_{-i} \) are identical, we have \( f(x) = x \), where \( x \) is the location of \( x_{-i} \).

By Lemma 4.6 we then complete the special case where \( n = 3 \).

**Lemma 4.7.** When \( n = 3 \), if \( f \) is unanimous, translation-invariant, and group-strategyproof, then \( f \) is 2-dictatorial.

**Proof.** Under translational invariance, we have already shown that \( S_i \) is either an empty set or exactly \( \mathbb{R}_d \), while \( S_1 \cup S_2 \cup S_3 = \mathbb{R}_d \). Assume \( S_i = \mathbb{R}_d \). Let \( j, k \) be the other two agents. We claim that agents \( j, k \) are the 2-dictators.

Assume for contradiction that there exists \( x \) such that \( f(x) \) does not lie on \( x_j x_k \). By the convexity in Euclidean space, there exists \( x \in \mathbb{R}_d \) such that \( \| x - x_j \| < \| f(x) - x_j \| \) and \( \| x - x_k \| < \| f(x) - x_k \| \) (similar to Lemma 4.2). Let \( x' \) be a profile where \( x'_i = x_i \) and \( x'_j = x'_k = x \). By Lemma 4.6 \( f(x') = x \), so agents \( j \) and \( k \) would both misreport \( x \), contradicting group-strategyproofness. \( \Box \)

Similar to Theorem 1 in the theory below, we will generalize the result for any \( n \geq 3 \). Using the same manner, we divide the agents into three non-empty groups, among which two will be the group of 2-dictators. Then the \( n \)-agent game can be reduced into an \( (n - 1) \)-agent problem by fixing the relative location of the non-dictator to another agent. By induction, we use the fact that there must be 2-dictators among the \( n - 1 \) agents. In accordance with group-strategyproofness, all reduced mechanisms (with \( n - 1 \) agents) are expected to have common 2-dictators, which finally concludes that the original mechanism is 2-dictatorial.

**Theorem 2** (Universal 2-dictatorship). If \( f \) is unanimous, translation-invariant, and group-strategyproof, then \( f \) is 2-dictatorial.

**Proof.** We prove by induction on \( n \).

When \( n = 2 \), by Corollary 4.3, the only two agents are the 2-dictators.

When \( n \geq 3 \), assume it holds for \( n - 1 \).
First, we divide all agents into three non-empty, nonintersecting groups \( N_1, N_2, N_3 \), where \( N_1 \cup N_2 \cup N_3 = N \). \( \forall y_1, y_2, y_3 \in \mathbb{R}_d \), denote \( x(y_1, y_2, y_3) \) as the profile where \( x_i = y_1 \) for all \( i \in N_1 \), \( x_j = y_2 \) for all \( j \in N_2 \), and \( x_k = y_3 \) for all \( k \in N_3 \).

Then we construct a mechanism \( g \), where there are only three agents: \( \forall y = (y_1, y_2, y_3) \), let

\[
g(y) = f(x(y_1, y_2, y_3)).
\]

In short, we bind the three groups of agents respectively and then construct \( g \).

It is clear that \( g \) is unanimous, translation-invariant and group-strategyproof, so \( g \) is 2-dictatorial (by Lemma 4.7). Without loss of generality, assume agents 2, 3 are the 2-dictators of \( g \), and \( x_1 \in N_1 \).

Similarly, we can reduce the game into \( n - 1 \) players. We construct a set of mechanisms, where there are exactly \( n - 1 \) agents for each: \( \forall a \in \mathbb{R}_d, \forall x_{-1} \), let

\[
f_a(x_{-1}) = f(x_2 + a, x_{-1}).
\]

It is clear that each \( f_a \) is translation-invariant and group-strategyproof, while its unanimity comes from the 2-dictatorship of \( g \) and uncompromising property. Therefore, \( \forall a \in \mathbb{R}_d, f_a \) is 2-dictatorial (by induction assumption).

We claim that all \( f_a \) have common 2-dictators. Assuming for contradiction, the cases are quite complicated, and they are detailed in Appendix B.5.

Therefore, there exist \( i, j \in \{2, \ldots, n\} \) such that \( \forall a, f_a \) is 2-dictatorial and agents \( i, j \) are the 2-dictators. That is, \( f \) is 2-dictatorial.

\[\square\]

Theorem 2 directly shows that no anonymous, translation-invariant, group-strategyproof mechanism has a constant approximation ratio, for all \( n \geq 3 \). Moreover, if there exists an anonymous, group-strategyproof mechanism which is not translation-invariant, then we can construct a translation-invariant one, as Proposition 4.1 did. Thus, this condition can be relaxed, as the following corollary states.

**Corollary 4.8.** No anonymous, group-strategyproof mechanism has a constant approximation ratio, for all \( n \geq 3 \).

We have already shown that our consideration on translation-invariant mechanisms are sufficient in the approximation view (see Proposition 4.1). By universal 2-dictatorship, we can move the 2-dictators at one side away from the others, and thus the approximation bounds are obtained.

**Corollary 4.9.** For all \( n \geq 3 \), no group-strategyproof mechanism can do better than 2-approximation for maximum cost.

**Corollary 4.10.** For all \( n \geq 3 \), no group-strategyproof mechanism can do better than \((n/2 - 1)\)-approximation for social cost.

## 5 Discussion

Our characterization of deterministic group-strategyproof mechanism is complete (Theorem 1). For randomized mechanisms, we have already discovered strong results (Theorem 2), but there is still a little gap between the lower bound \( n/2 - 1 \) and the upper bound \( n/2 \) (for social cost). We may have more works to do, e.g., one open problem is whether or not the distribution between the 2-dictators could be affected by the other agents.
In this paper, we do not consider the class of strategyproof mechanisms, which is much more larger than the class of group-strategyproof mechanisms. The deterministic case is already resolved by Border and Jordan [3], namely, generalized median voter schemes (see [2]). We suggest that selecting a median in each dimension is approximately efficient for social cost, but it also shows that 2-approximation is already the tight bound for maximum cost.

Yet there is no result for randomized mechanisms, even in an approximation view. We propose the following mechanism.

**Mechanism 3.** Given $\mathbf{x}$, output $(x_1 + \cdots + x_n)/n$ with $1/2$ probability, and each $x_i$ with $1/2n$ probability.

Mechanism 3 is strategyproof, and $(2 - 1/n)$-approximation for maximum cost (see Appendix C.3). It breaks the limit of 2, the tight bound of group-strategyproof mechanisms, and provides positive implications. However, when $n$ goes large, the approximation ratio becomes $2 - o(1)$, which is not a significant breakthrough. On the other hand, the lower bound is also unknown. In short, characterizing randomized strategyproof mechanisms would be a much more challenging but attractive task.
Appendix A  Deterministic Mechanisms

A.1 Proof of Lemma 3.1

Lemma 3.1. All constant approximation mechanisms are unanimous.

Proof. For all \( x \in \mathbb{R}^d \), consider the case when \( x_1 = \cdots = x_n = x \).

For both maximum cost and social cost,

\[
mc(x,x) = sc(x,x) = 0,
\]

so it must be

\[
mc(f(x),x) = sc(f(x),x) = 0.
\]

By definition of \( mc(\cdot) \) and \( sc(\cdot) \), \( f(x) = x \), i.e., \( f \) is unanimous.

A.2 Proof of Proposition 3.2

Proposition 3.2. Suppose \( f \) is deterministic and strategyproof, then \( f \) is unanimous if and only if \( f \) is onto.

Proof. If \( f \) is unanimous, \( \forall x, f(x) = x \) when \( x_1 = \cdots = x_n = x \), so \( f \) is onto.

If \( f \) is onto, then \( \forall x \), there exists a profile \( x \) such that \( f(x) = x \). For all \( i \in N \), when we move \( x_i \) to \( x \), because of strategyproofness, \( \|f(x,x_{-i}) - x\| \leq \|f(x) - x\| = 0 \), that is, \( f(x,x_{-i}) = x \). After we move all agents to \( x \) (one by one), the facility location still stays unchanged. Therefore, \( f \) is unanimous.

A.3 Proof of Lemma 3.3

Lemma 3.3. Let \( f \) be a strategyproof mechanism. Let \( x \) be a profile such that the output is deterministic, and let \( x = f(x) \). We claim that \( f(x') = x \), if either \( x'_i = x_i \) or \( x'_i = x \) holds for all \( i \in N \).

Proof. \( \forall i \in N \), we claim that \( f(x,x_{-i}) = x \), that is, if \( x_i \) moves to the output, the output will not change. If not, agent \( i \) would gain by misreporting \( x_i \) instead of \( x \). Formally,

\[
\|f(x,x_{-i}) - x\| \leq \|f(x) - x\| = 0.
\]

By applying the claim above multiple times, we can move multiple agents to \( x \) one by one, while the output stands still. Therefore, \( f(x') = x \).

A.4 Proof of Lemma 3.4

Lemma 3.4. Let \( f \) be a strategyproof mechanism. \( \forall i \in N \), for any fixed \( x_{-i} \in \mathbb{R}^{n-1} \), the distance between \( x_i \) and the output of \( (x_i,x_{-i}) \)

\[
h(x_i) \triangleq \|f(x_i,x_{-i}) - x_i\|
\]

is a continuous function. Moreover, \( \forall x_i,x'_i \in \mathbb{R}^d \),

\[
\|h(x_i) - h(x'_i)\| \leq \|x_i - x'_i\|.
\]
Proof. Assume for contradiction that \( \exists x_i, x'_i \in \mathbb{R}_d \) such that \( \| h(x_i) - h(x'_i) \| > \| x_i - x'_i \| \). Without loss of generality, assume \( h(x_i) - h(x'_i) > \| x_i - x'_i \| \). If agent \( i \) misreports \( x'_i \) instead of \( x_i \), then

\[
\| f(x'_i, x_{-i}) - x_i \| \leq \| f(x'_i, x_{-i}) - x'_i \| + \| x_i - x'_i \|
\]

\[
= h(x'_i) + \| x_i - x'_i \|
\]

\[
< h(x_i)
\]

\[
= \| f(x_i, x_{-i}) - x_i \|
\]

which disobeys strategyproofness. Therefore, \( \forall x_i, x'_i, \| h(x_i) - h(x'_i) \| \leq \| x_i - x'_i \| \).

A.5 Proof of Lemma \[3.7\]

In order to prove Lemma \[3.7\], we first introduce a weaker lemma, that is, by changing the location of the other agent, the dictator remains the same.

**Lemma A.1.** Suppose \( f \) is deterministic, unanimous, and group-strategyproof. When \( n = 2 \), if there exist \( x_1 \neq x_2 \) such that \( f(x) = x_1 \), then for all \( x'_2 \), \( f(x_1, x'_2) = x_1 \).

**Proof.** If \( x'_2 = x_1 \), by uncompromising property, \( f(x_1, x'_2) = x_1 \).

Otherwise, by Lemma \[3.5\] \( f(x_1, x'_2) \) lies on the segment between \( x_1 \) and \( x'_2 \). If \( \| x'_2 - x_2 \| < \| x_1 - x_2 \| \), \( \| f(x_1, x'_2) - x_2 \| \geq \| f(x_1, x_2) - x_2 \| \) if and only if \( f(x_1, x'_2) = x_1 \). By strategyproofness, we can obtain that \( f(x_1, x'_2) = x_1 \).

Now we only need to consider the case that \( \| x'_2 - x_2 \| < \| x_1 - x_2 \| \). In that case, it is actually easy to construct a sequence of points which starts with \( x_2 \), ends with \( x'_2 \), and for any two consecutive points \( y \) and \( z \), \( \| z - y \| < \| x_1 - y \| \). E.g., first move agent 2 from \( x_2 \) to the ray directed from \( x_1 \) to \( x'_2 \) while keeping the radius, then move straight to \( x'_2 \) (see Figure 4). Then we can inductively prove that for any point \( y \) in the sequence, \( f(x_1, y) = x_1 \). Finally, we obtain \( f(x_1, x'_2) = x_1 \).

---

**Figure 4:** Proof of Lemma \[A.1\] It shows a possible route that starts from \( x_2 \) and ends at \( x'_2 \), while keeping the dictator agent 1 staying at the center \( x_1 \).
Lemma 3.7. Suppose \( f \) is deterministic, unanimous, and group-strategyproof. When \( n = 2 \), if there exist \( x_1 \neq x_2 \) such that \( f(x) = x_1 \), then agent 1 is the dictator in all profiles.

Proof. \( \forall x' = (x'_1, x'_2) \), we prove \( f(x') = x'_1 \) by the following three steps.

1. Move agent 2 to \( \hat{x}_2 \), such that \( \hat{x}_2 \neq x'_1 \) and \( x'_1 \) is on \( \overline{x_1 x_2} \). By Lemma A.1, \( f(x_1, \hat{x}_2) = x_1 \).

2. Move agent 1 to \( x'_1 \).
   Suppose \( f(x'_1, \hat{x}_2) \neq x'_1 \). \( \forall \xi \in [0, 1] \), let
   \[
   h(\xi) = \| f(\xi x_1 + (1 - \xi)x'_1, \hat{x}_2) - \hat{x}_2 \|.
   \]
   By now we know that
   \[
   h(0) = \| f(x'_1, \hat{x}_2) - \hat{x}_2 \| < \| x'_1 - \hat{x}_2 \| \leq \| x_1 - \hat{x}_2 \| = h(1).
   \]
   Because \( h \) is a continuous function (by Lemma 3.4), there exists \( \xi_0 \in (0, 1) \) such that \( h(\xi_0) = \| x'_1 - \hat{x}_2 \| \). Let \( x = \xi_0 x_1 + (1 - \xi_0)x'_1 \). We have \( \| f(x, \hat{x}_2) - \hat{x}_2 \| = \| x'_1 - \hat{x}_2 \| \), and \( x \) is on \( \overline{x_1 x_2} \) (by Lemma 3.5), so \( f(x, \hat{x}_2) = x'_1 \). Consequently, agent 1 would misreport \( x \) instead of \( x'_1 \), which disobeys strategyproofness.

   Thus, \( f(x'_1, \hat{x}_2) = x'_1 \).

3. Move agent 2 from \( \hat{x}_2 \) to \( x'_2 \). By Lemma A.1, \( f(x') = x'_1 \).

\[\square\]

Appendix B Randomized Mechanisms

B.1 Proof of Proposition 4.1

Proposition 4.1. For any unanimous and group-strategyproof mechanism \( f \), there exists an unanimous, translation-invariant and group-strategyproof mechanism \( f' \) such that the approximation ratio of \( f' \) is no worse than \( f \) with respect to any translation-invariant objective function \( \text{obj} \).

Proof. Let \( f'(x) = f(x - x_1) + x_1 \).

Clearly, \( f' \) is unanimous and translation-invariant. For group-strategyproofness, assume for the sake of contradiction that there exist \( S \subseteq N \) and \( x'_S \in \mathbb{R}^{|S|}_d \), such that \( \forall i \in S \),

\[
\| f'(x) - x_i \| > \| f'(x'_S, x_{-S}) - x_i \|.
\]

That is, \( \forall i \in S \),

\[
\| f(x - x_1) - (x_i - x_1) \| > \| f(x' - x_1) - (x_i - x_1) \|,
\]

where \( x' = (x'_S, x_{-S}) \). In other words, in mechanism \( f \), a coalition in profile \( x - x_1 \) would misreport profile \( x' - x_1 \), which contradicts to the group-strategyproofness of \( f \).

Secondly, suppose \( f \) is an \( \alpha \)-approximation for objective \( \text{obj} \). Then for all \( x \in \mathbb{R}^n_d \),

\[
\text{obj}(f'(x), x) = \text{obj}(f(x - x_1) + x_1, x)
= \text{obj}(f(x - x_1), x - x_1)
\leq \alpha \min_{y \in \mathbb{R}^d} \text{obj}(y, x - x_1)
= \alpha \min_{y \in \mathbb{R}^d} \text{obj}(y + x_1, x)
= \alpha \min_{y \in \mathbb{R}^d} \text{obj}(y, x).
\]

Therefore, \( f' \) is also an \( \alpha \)-approximation for \( \text{obj} \). \[\square\]
B.2 Proof of Lemma 4.2

Lemma 4.2. Suppose \( f \) is unanimous and group-strategyproof. \( \forall x, \) if \( R(f(x)) = 0 \) (that is, the output is deterministic), then \( f(x) \in \text{conv}(x) \); otherwise, \( \exists i, j \in N \) such that the support of \( f(x) \) lies on the segment between \( x_i \) and \( x_j \).

Proof. First consider the case where \( f(x) \) is deterministic. By contradiction, we assume \( f(x) \) lies outside of \( \text{conv}(x) \), namely, there exists an unit vector \( e \), such that for any \( i \in N, \langle e, x_i - f(x) \rangle > 0 \). Let \( x' = f(x) + \epsilon e \) where

\[
\epsilon = \frac{1}{2} \max_{i \in N} \langle e, x_i - f(x) \rangle,
\]

then we still have \( \forall i \in N, \)

\[
\langle e, x_i - x' \rangle = \langle e, x_i - f(x) - \epsilon e \rangle = \langle e, x_i - f(x) \rangle - \epsilon > 0.
\]

Therefore, \( \|x_i - f(x)\| > \|x_i - x'\| \). According to the unanimity, if all agents collaborate to misreport \( x' \), the output must be \( x' \), which is a strictly better choice for all. It contradicts to group-strategyproofness, so \( f(x) \) must be in \( \text{conv}(x) \).

Since \( f \) is group-strategyproof, at least one agent in \( N \) cannot gain when every agent misreport the same location \( x \). That is, \( \forall x \in \mathbb{R}_d, \) there exists \( i \in N \) such that

\[
\|f(x) - x_i\| \leq \|x - x_i\|.
\]

(1)

Now consider the case where \( f(x) \) is strictly randomized and \( x = C(f(x)) \). Let \( N_1 \) be the set of agents that satisfy the inequality \( \|x_i - f(x)\| > \|x_i - x'\| \). Let \( N_2 = N \setminus N_1 \).

\( \forall i \in N_1, \) due to the convexity of distance in Euclidean space, \( \|f(x) - x_i\| \geq \|x - x_i\| \). Moreover, \( \|f(x) - x_i\| = \|x - x_i\| \) if and only if \( x_i \) and the support of \( f(x) \) lies on the same line, and the support of \( f(x) \) lies on the same side to \( x \) on the line. Formally, \( \|f(x) - x_i\| = \|x - x_i\| \) if and only if there exists a unit vector \( e \) such that \( \forall y \in \text{supp}(f(x)), \langle y - x_i, e \rangle = \|y - x_i\| \).

If all agents in \( N_1 \) share the same unit vector \( e \) defined above, then we can find \( \epsilon \) where \( \epsilon < \|f(x) - x_j\| - \|x - x_j\| \) for all \( j \in N_2, \) and \( \epsilon < \|x - x_i\| \) for all \( i \in N_1 \). Let \( x' = x - \epsilon \cdot e \). Then, \( \forall j \in N_2, \)

\[
\|x' - x_j\| \leq \|x' - x\| + \|x - x_j\| = \epsilon + \|x - x_j\| < \|f(x) - x_j\|,
\]

and \( \forall i \in N_1, \)

\[
\|x' - x_j\| = \|x - x_j\| - \epsilon = \|f(x) - x_i\| - \epsilon < \|f(x) - x_i\|.
\]

Thus, it also violates group-strategyproofness if all agents misreport \( x' \).

Otherwise, there exist agents \( i, j \in N_1 \) with two opposite unit vectors \( e \) and \( -e \), which means the support of \( f(x) \) lies on the different sides to \( x_i \) and \( x_j \) on the same line, that is, the support of \( f(x) \) lies on the segment between \( x_i \) and \( x_j \). \( \square \)

B.3 Proof of Lemma 4.5

Lemma 4.5. Suppose \( f \) is unanimous and group-strategyproof. Even for \( n = 3, \forall x \in \mathbb{R}_d, \forall \ell > 0, \exists i \in N, \forall \epsilon > 0, \) there exists \( x \) such that

\[
\|f(x) - x_j\| < \epsilon \|x_i - x_j\| < \ell
\]

holds for some \( j \neq i \) where \( \|x_j - x\| < \ell \).
Proof. For all \( x \) and \( \ell \), consider an equilateral triangle \( \Delta x_1x_2x_3 \) with edges of length \( \ell \), such that \( \|x_i - x\| \leq \ell \) for \( i \in N \). By Lemma 4.2, there are two cases: \( f(x_1, x_2, x_3) \) is deterministic and is in the triangle; or there exist \( i, j \), such that \( f(x_1, x_2, x_3) \) lies on \( x_i x_j \). For the first case, let \( y = f(x_1, x_2, x_3) \), without loss of generality, we assume \( x_3 \neq y \). By uncompromising property, \( f(y, y, x_3) = y \). That is, the lemma is satisfied for \( i = 3 \). For the second case, without loss of generality, we assume \( f(x_1, x_2, x_3) \) lies on \( x_1 x_2 \). Let \( x_3 \) be the “\( x_i \)” in the lemma. We will construct a sequence of location profiles such that

(a) \( \Delta x_1x_2x_3 \) remains to be an isosceles triangle;

(b) in each iteration, \( x_3 \) becomes further away from \( x_1 \) and \( x_2 \), or \( x_1 \) and \( x_2 \) get closer, i.e., either \( \|x_3 - x_1\| = \|x_3 - x_2\| \) increases, or \( \|x_1 - x_2\| \) decreases;

(c) \( \|x_1 - x\| \leq \ell, \|x_2 - x\| \leq \ell \);

(d) \( \|x_3 - x_1\| = \|x_3 - x_2\| < \ell / \epsilon \).

(e) the output \( f(x) \) keeps on \( x_1 x_2 \).

Particularly, in each round, we will find \( x_1', x_2', x_3' \), and inductively prove that \( \exists i \in N \), the location profile \( (x_i', x_{-i}) \) satisfies the properties above.

For all \( \delta_1, \delta_2, \delta_3 \), we can easily find \( x_1', x_2', x_3' \) such that \( \|x_1 - x_1'\| = \delta_1, \|x_2 - x_2'\| = \delta_2, \|x_3 - x_3'\| = \delta_3 \), and \( \forall i \in N \) the location profile \( (x_i', x_{-i}) \) satisfies the property (a)-(c). For property (d), if \( \|x_3 - x_1\| = \|x_3 - x_2\| > \ell / 2 \epsilon \), then we already have

\[
\min\{\|f(x) - x_1\|, \|f(x) - x_2\|\} \leq \ell / 2 = \epsilon \times \ell / 2 \epsilon < \epsilon \|x_2 - x_1\|.
\]

Therefore, we can assume \( \|x_3 - x_1\| \leq \ell / 2 \epsilon \), namely, if we set \( \delta_3 < \ell / 2 \epsilon \), property (d) is also satisfied. For property (e), by similar argument before, we first rule out the case where \( f(x_i', x_{-i}) \) is deterministic and strictly inside the triangle. In that case, we can move agents 1, 2 to the output. Since the output remains the same, the lemma is satisfied for any \( \epsilon \). In what follows, we only consider the case that \( f(x_i', x_{-i}) \) lies on some edge.

Let \( P_1 = f(x_1, x_2, x_3), P_2 = f(x_1, x_1', x_3), P_3 = f(x_1, x_2, x_3') \) and \( P_4 = f(x_1, x_2', x_3') \). If \( P_3 \) lies on \( x_1 x_2 \), we are done. If not, without loss of generality, we assume \( P_3 \) lies on \( x_2 x_3' \). Now consider \( P_2 \), if \( P_2 \) lies on \( x_1 x_2' \), we are done. Otherwise we will consider the following two cases separately:

1. \( P_2 \) lies on \( x_2' x_3 \) (see Figure 5(a)). Then we have

\[
\|x_3' - P_3\| \leq \|x_3 - P_2\| + \delta_3 \leq \|x_2' - P_2\| + \delta_3 \leq \|x_2' - x_3\| - \|x_2 - P_1\| + \delta_2 + \delta_3. \tag{continuous expected distance}
\]

Similarly,

\[
\|x_2' - P_4\| \leq \|x_2 - P_3\| + \delta_2 \leq \|x_2 - x_3'\| - \|x_3 - P_1\| + \delta_2 + \delta_3. \tag{continuous expected distance}
\]

Adding two inequalities together,

\[
\|x_2' - x_3\| = \|x_3' - P_4\| + \|x_2' - P_4\| \\
\leq \|x_2' - x_3\| - \|x_2 - P_1\| + \|x_2 - x_3\| - \|x_3 - P_1\| + 2\delta_2 + 2\delta_3 \\
\leq \|x_2' - x_3\| + \|x_2 - x_3\| - (\|x_2 - P_1\| + \|x_3 - P_1\|) + 3\delta_2 + 3\delta_3. \tag{2}
\]
Let $C_1 = C(P_1)$, $a = \|x_2 - x_3\|$, $b = \|x_2 - C_1\| = \|x_2 - P_1\|$, $c = \|x_2 - C_1\| \leq \|x_2 - P_1\|$. I.e.,

$\Delta x_2 x_3 C_1$ is a triangle with edges of length $a, b, c$. Since $\angle x_3 x_2 C_1 \geq \pi/3$,

$$a^2 + b^2 - c^2 = 2ab \cos \angle x_2 x_2 C_1 \leq \sqrt{3}ab.$$ 

Therefore $c \geq \sqrt{a^2 + b^2 - \sqrt{3}ab} \geq a - \sqrt{\frac{3}{2}}b$. As a result,

$$\|x_2 - P_1\| + \|x_3 - P_1\| - \|x_2 - x_3\| \geq b + c - a \geq \frac{2 - \sqrt{3}}{2}b.$$ 

Combining with (2), we have

$$3\delta_2 + 3\delta_3 \geq \frac{2 - \sqrt{3}}{2} \|x_2 - P_1\|.$$

2. $P_2$ lies on $x_1 x_3$ (see Figure 5(b)). Let $C_1 = C(P_1)$, $D$ be a point on $x_1 C_1$ such that $\|D - C_1\| = 2\delta_2$. Let $C_2 = C(P_2)$, $a = \|x_3 - C_2\| = \|x_3 - P_2\|$, $b = \|D - C_2\|$, $c = \|x_3 - D\| \leq \|x_3 - C_1\| + 2\delta_2 \leq \|x_3 - P_1\| + 2\delta_2$. I.e., $\Delta C_1 D x_3$ is an triangle with edges of length $a, b, c$.

By continuous expected distance,

$$\|x_2 - C_2\| \leq \|x_2 - P_2\| \leq \|x'_2 - P_2\| + \delta_2 \leq \|x_2 - P_1\| + 2\delta_2 = \|x_2 - C_1\| + 2\delta_2 = \|x_2 - D\|.$$ 

Therefore, $\angle C_2 D x_1 \geq \pi/2$, which indicates

$$b = \|C_2 - D\| \geq \|x_1 - D\| \tan \angle C_2 x_1 D \geq \sqrt{3}\|x_1 - D\| = \sqrt{3}(\|x_1 - P_1\| - 2\delta_2).$$ 

Also, $\angle x_3 C_2 D \geq \frac{\pi}{3}$, which shows that

$$c = \sqrt{a^2 + b^2 - 2ab \cos \angle x_3 C_2 D} \geq \sqrt{a^2 + b^2 + \sqrt{3}ab} \geq a + \sqrt{\frac{3}{2}}b.$$ 

![Figure 5: Proof of Lemma 4.5](image)
As a result,
\[ \| x_3 - P_1 \| \geq c - 2\delta_2 \geq a + \frac{\sqrt{3}}{2} b - 2\delta_2 \geq \| x_3 - P_2 \| + \frac{3}{2}\| x_1 - P_1 \| - 5\delta_2. \]

Then we have
\[ \| x_3 - P_4 \| \leq \| x_3 - P_2 \| + \delta_3 \] (continuous expected distance)
\[ \leq \| x_3 - P_1 \| - \frac{3}{2}\| x_1 - P_1 \| + 5\delta_2 + \delta_3. \]

On the other hand,
\[ \| x_2 - P_4 \| \leq \| x_2 - P_3 \| + \delta_2 \] (continuous expected distance)
\[ = \| x_2 - x_3' \| - \| x_3' - P_3 \| + \delta_2 \]
\[ \leq \| x_2 - x_3' \| - \| x_3 - P_1 \| + \delta_3 + \delta_2. \] (continuous expected distance)

Adding two inequalities above,
\[ \| x_2' - x_3' \| = \| x_2' - P_4 \| + \| x_3' - P_4 \| \]
\[ \leq \| x_2 - x_3' \| - \frac{3}{2}\| x_1 - P_1 \| + 6\delta_2 + 2\delta_3 \]
\[ \leq \| x_2' - x_3' \| - \frac{3}{2}\| x_1 - P_1 \| + 7\delta_2 + 2\delta_3. \]

That is,
\[ 7\delta_2 + 2\delta_3 \geq \frac{3}{2}\| x_1 - P_1 \|. \]

To sum up, as long as we set \( \delta_2 \) and \( \delta_3 \) small enough such that
\[ 3\delta_2 + 3\delta_3 < \frac{2}{\sqrt{2}}\| x_2 - P_1 \| \]
and
\[ 7\delta_2 + 2\delta_3 < \frac{3}{2}\| x_1 - P_1 \|, \]
then either \( P_2 \) lies on \( x_1 x_2' \), or \( P_3 \) lies on \( x_1 x_2 \), that is, property (e) is satisfied.

Moreover, let \( C_1 = C(P_1), r = \min\{\| x_2 - P_1 \|, \| x_1 - P_1 \|\}. \) Then
\[ \mathcal{R}(P_1) = \mathbb{E}_{y \sim P_1} \| y - C_1 \| \leq 2r. \]

If we move \( x_1, x_2 \) to \( C_1 \), i.e., consider \( x' = (C_1, C_1, x_3) \), then we have
\[ \| f(x') - C_1 \| \leq \| f(x) - C_1 \| = \| P_1 - C_1 \| = \mathcal{R}(P_1) \leq 2r. \]

If \( 2r \leq \epsilon\| x_3 - x_1 \| \), then the Lemma is proved. Therefore, we can assume \( r \leq \epsilon\| x_3 - x_1 \|/2 \leq \epsilon\ell/2. \)

In that case, we can set \( \delta_2 = \delta_3 = \epsilon\ell/100. \) As the sequence grows, either \( \| x_3 - x_1 \| > \ell/2\epsilon \), or \( \| x_2 - x_1 \| < \epsilon\ell. \)
B.4 Proof of Lemma 4.6

In order to prove Lemma 4.6, we introduce Lemma B.1 and Lemma B.2.

**Lemma B.1.** Suppose $f$ is unanimous and group-strategyproof. When $n = 2$, $\forall \delta \geq 0$, if there exist $x_1 \neq x_2$ such that $\|f(x) − x_1\| \leq \delta$, then for all $x'_2$ where $\|x'_2 − x_2\| < \|x_2 − x_1\|$, $\|f(x_1, x'_2) − x_1\| \leq \frac{\delta}{1 - \frac{x'_2 − x_2}{x_2 − x_1}}$.

**Proof.** Let $r = \|x_2 − x_1\|$, $d = \|x'_2 − x_2\|$. By Corollary 4.3, $f(x_1, x'_2)$ lies on $x_1x'_2$. $\forall x \in [0, r]$, let $h(x) = \|\frac{x}{r}x_1 + \left(1 - \frac{x}{r}\right)x'_2 - x_2\|$. Clearly, $h(0) = r$, $h(r) = d$. Due to the convexity of distance in Euclidean space, $\forall x \in [0, r]$, $h(x) \leq \left(1 - \frac{x}{r}\right)r + \frac{x}{r} \cdot d = r - \left(1 - \frac{d}{r}\right)x$.

Thus, by strategyproofness,

\[
\begin{align*}
r - \delta & \leq \|f(x_1, x'_2) − x_2\| \\
& \leq \|f(x_1, x'_2) − x_2\| \\
& = h(\|f(x_1, x'_2) − x_1\|) \\
& \leq r - \left(1 - \frac{d}{r}\right)\|f(x_1, x'_2) − x_1\|,
\end{align*}
\]

and we can solve that $\|f(x_1, x'_2) − x_1\| \leq \frac{\delta}{1 - \frac{d}{r}}$.

**Lemma B.2.** Suppose $f$ is unanimous and group-strategyproof. When $n = 2$, $\forall \delta \geq 0$, if there exist $x_1 \neq x_2$ such that $\|f(x) − x_1\| \leq \delta$, then for all $x'_2$ where $\ell \geq r$,

\[
\|f(x_1, x'_2) − x_1\| \leq \frac{100\ell}{r}\delta.
\]

Here, $r = \|x_2 − x_1\|$, $\ell = \|x'_2 − x_1\|$.

**Proof.** The proof consists of two steps.

1. If $r = \ell$, then $\|f(x_1, x'_2) − x_1\| \leq 50\delta$.

   We can move agent 2 from $x_2$ to $x'_2$ in several steps, while keeping the radius unchanged. For each step, we move a distance of at most $\frac{r}{4}$, then it will arrive in no more than 13 steps. By Lemma B.1,

   $$\|f(x_1, x'_2) − x_1\| \leq \left(\frac{4}{3}\right)^{13} \delta \leq 50\delta.$$
2. If $x_2$ lies on $\overline{x_1x_2}$, then $\|f(x_1, x'_2) - x_1\| \leq 2\ell\delta/r$.

From $x_2$ straight to $x'_2$, assume each time we move a distance of $\epsilon$ times the distance between $x_1$ and the current location of agent 2. As $\epsilon \to 0$, by Lemma B.1,

$$\|f(x_1, x'_2) - x_1\| \leq \delta(1 - \epsilon)^{-1-\log_{1+r}\frac{\ell}{\delta}} \to \frac{\ell}{r}\delta,$$

so there exists $\epsilon$ such that $\|f(x_1, x'_2) - x_1\| \leq 2\ell\delta/r$.

We can follow the two steps and then obtain the statement for all $x'_2$. It is similar to the route shown in Figure 4.

**Lemma 4.6.** Consider the case when $n = 3$, $f$ is unanimous, translation-invariant, and group-strategyproof. If $S_i = \mathbb{R}_d$, then $\forall x$ such that $x_{-i}$ are identical, we have $f(x) = x$, where $x$ is the location of $x_{-i}$.

**Proof.** If $x_i = x$, then it holds by unanimity.

Otherwise, suppose $f(x) \neq x$. Let $\delta = \min(\|f(x) - x\|, 1) > 0$. By definition, $\ell$ can be infinitely small while satisfying $\chi(x, \ell) = i$. Let $\epsilon = \delta/200$. Applying Lemma 4.5 with respect to any $\ell < \epsilon\delta/2$ where $\chi(x, \ell) = i$, there exists $\hat{x}$ such that

$$\|f(\hat{x}) - \hat{x}_j\| < \epsilon\|\hat{x}_i - \hat{x}_j\| < \frac{\epsilon\delta}{2}$$

holds for some $j \neq i$ where $\|\hat{x}_j - x\| < \epsilon\delta/2$.

Let $x' = \hat{x}_j$. Now we move the agent other than $i, j$ to $x'$ (coincides with agent $j$), and these conditions still hold (by strategyproofness). Let $x'$ be a profile where $x'_i = x_i$ and $x'_{-i}$ are identical at $x'$. Because $\|\hat{x}_i - \hat{x}_j\| < \delta/2 < \|x_i - x'\|$, we can apply Lemma B.2 by considering agent $i$ and the others as a two-player game, so $\|f(x') - x'\| \leq \delta/2$,

$$\|f(x') - x\| \leq \|f(x') - x'\| + \|x' - x\| < \delta.$$ 

It leads to a contradiction, as agent $i$ would misreport $x'$ instead of $x$.

**B.5 Common 2-Dictators in the Proof of Theorem 2**

**Proof.** Recall that $\forall a \in \mathbb{R}_d$, $\forall x_{-1}$,

$$f_a(x_{-1}) = f(x_2 + a, x_{-1}).$$

Now we prove that all $f_a$ have common 2-dictators. Assume for contradiction. Then, it must be included by the following six cases (without loss of generality), where there exist $a, a'$ such that:

1. $a \neq 0$, agents 2, 3 are the dictators of $f_a, f_{a'}$, respectively. Let

$$x = (a, 0, a, x_{-\{1,2,3\}}),$$
$$x' = (a', 0, a, x_{-\{1,2,3\}}),$$

where $x_{-\{1,2,3\}}$ is indifferent. Agent 1 would misreport $a'$ instead of $a$.

2. $a = 0$, agents 2, 3 are the dictators of $f_a, f_{a'}$, respectively. Let

$$x = (0, 0, -a', x_{-\{1,2,3\}}),$$
$$x' = (a', 0, -a', x_{-\{1,2,3\}}),$$

where $x_{-\{1,2,3\}}$ is indifferent. Agent 1 would misreport $a$ instead of $a'$. 

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3. Agents 3, 4 are the dictators of \( f_a, f_{a'} \), respectively. Let

\[
\begin{align*}
\mathbf{x} &= (a, 0, a', a, \mathbf{x}_{\{1,2,3,4\}}), \\
\mathbf{x}' &= (a', 0, a', a, \mathbf{x}_{\{1,2,3,4\}}),
\end{align*}
\]

where \( \mathbf{x}_{\{1,2,3,4\}} \) is indifferent. Agent 1 would misreport \( a' \) instead of \( a \).

4. \( f_a \) is not dictatorial but \( f_{a'} \) is dictatorial, and the dictator of \( f_{a'} \) is not one of the 2-dictators of \( f_a \). This case is similar to cases 1, 2, 3, by binding the 2-dictators together.

5. \( f_a, f_{a'} \) are not dictatorial, and \( f_a, f_{a'} \) do not share any 2-dictator. This case is similar to cases 1, 2, 3 as well, by binding each of the 2-dictators together.

6. \( a \neq 0, f_a, f_{a'} \) are not dictatorial, and \( f_a, f_{a'} \) share exactly one common 2-dictator. Let agents \( i, j \) be the 2-dictators of \( f_a \), and agents \( i, k \) be the 2-dictators of \( f_{a'} \).

We claim that there exists \( \mathbf{x}_{-1} \) such that \( R(f_a(\mathbf{x}_{-1})) \neq 0 \). If not, then \( f_a \) is deterministic, unanimous, and group-strategyproof, so \( f_a \) is dictatorial (by Theorem 1), contradicting our assumption.

Let \( \mathbf{x} = (x_2 + a, \mathbf{x}_{-1}) \). Denote \( \mathbf{x}_{-1}(x_i, x_j, x_k) \) as a profile where \( x_i, x_j, x_k \) are given and \( \mathbf{x}_{-\{1,i,j,k\}} = \mathbf{x}_{-\{1,i,j,k\}} \). Denote

\[
\begin{align*}
\mathbf{x}(x_i, x_j, x_k) &= (x_2 + a, \mathbf{x}_{-1}(x_i, x_j, x_k)), \\
\mathbf{x}'(x_i, x_j, x_k) &= (x_2 + a', \mathbf{x}_{-1}(x_i, x_j, x_k)).
\end{align*}
\]

If \( \mathbf{x}_{ij} \) does not (partly) coincide with \( \mathbf{x}_{ij}, \mathbf{x}_{ik}, \mathbf{x}_{jk} \), then we can find \( x \in \mathbb{R}^d \) such that \( x \) is strictly better than \( f(\mathbf{x}) \) for agents 1, \( i, k \) (similar to Lemma 4.2), so a coalition of agents 1, \( i, k \) would misreport \( \mathbf{x}'(x, \mathbf{x}_{ij}, x) \) instead of \( \mathbf{x} \), where \( f(\mathbf{x}'(x, \mathbf{x}_{ij}, x)) = x \), contradicting group-strategyproofness.

Otherwise, we can slightly move agent \( j \) to \( \hat{x}_j' \) (if \( j = 2 \), agent 1 follows.\), such that \( \|\hat{x}_j' - \hat{x}_j\| < \|f(\mathbf{x}) - \hat{x}_i\| \) and \( \hat{x}_j' \) lies out of the triangle \( \Delta \hat{x}_i \hat{x}_j \hat{x}_k \). Let \( P = f_a(\mathbf{x}(\hat{x}_i, \hat{x}_j', \hat{x}_k)) \). By Corollary 4.4, namely continuous centroid, in the situation that the two-player game formed by agents \( i, j \) is unanimous and group-strategyproof, we obtain that \( C(P) \neq \hat{x}_i \), i.e., \( C(P) \) lies out of the triangle \( \Delta \hat{x}_i \hat{x}_j \hat{x}_k \). Then we can find \( x \in \mathbb{R}^d \) such that \( x \) is strictly better than \( P \) for agents 1, \( i, k \) (similar to Lemma 4.2), so a coalition of agents 1, \( i, k \) would misreport \( \mathbf{x}'(x, \mathbf{x}_{ij}', x) \) instead of \( \mathbf{x}(\hat{x}_k, \mathbf{x}_{ij}', x) \), where \( f(\mathbf{x}'(x, \mathbf{x}_{ij}', x)) = x \), contradicting group-strategyproofness.

\[ \square \]

Appendix C  Proofs of Mechanisms

C.1 Approximation Bound of Mechanism 1

Mechanism 1. Given \( \mathbf{x} \), return \( x_1 \) with 1/4 probability, \( x_2 \) with 1/4 probability, and \( (x_1 + x_2)/2 \) with 1/2 probability.

Proposition C.1. Mechanism 1 is \( n/2 \)-approximation for social cost.

Proof. For any \( y_1 \neq y_2 \), consider the following profile

\( \mathbf{x}^* = (y_1, y_2, \ldots, y_2) \).
Because \( sc(f(x^*), x^*) = \frac{n}{2} \| y_1 - y_2 \| \) and the optimal is \( sc(y_1, x^*) = \| y_1 - y_2 \| \), the approximation ratio is at least \( n/2 \).

Now we prove it to be the upper bound. For any \( y \in \mathbb{R}_d \), assume \( y \) is the optimal location that minimizes the social cost. Let

\[
d_1 = \| x_1 - y \|, \quad d_2 = \| x_2 - y \|, \quad C = \sum_{i=3}^{n} \| x_i - y \|.
\]

Due to the convexity of Euclidean distance, \( \forall x \),

\[
sc(f(x), x) \leq \frac{sc(x_1, x) + sc(x_2, x)}{2},
\]

and

\[
\frac{sc(f(x), x)}{sc(y, x)} \leq \frac{\frac{n}{2} sc(x_1, x) + \frac{n}{2} sc(x_2, x)}{sc(y, x)} \leq \frac{\frac{n}{2} (d_1 + d_2 + C)}{d_1 + d_2 + C} \leq \frac{n}{2}.
\]

As this inequality holds for all \( y \in \mathbb{R}_d \), we conclude that Mechanism 1 is exactly \( n/2 \)-approximation for social cost.

\[\square\]

### C.2 Group-Strategyproofness of Mechanism 2

**Mechanism 2** Given \( x \), let \( r \) be the first coordinate of \( x_1 \), and \( y \) be the point on \( x_1x_2 \) such that \( \| x_1 - y \| = \min\{\| r - a \|, \| x_1 - x_2 \| \} \). Similarly, let \( y' \) be the point on \( x_1x_3 \) such that \( \| x_1 - y' \| = \min\{\| r - a \|, \| x_1 - x_3 \| \} \).

If \( r \geq a \), return \( x_1 \) with 2/3 probability, and \( y \) with 1/3 probability. Otherwise, return \( x_1 \) with 2/3 probability, and \( y' \) with 1/3 probability.

**Proposition C.2.** Mechanism 2 is unanimous and group-strategyproof.

**Proof.** We only prove the group-strategyproofness for all \( r \geq a \), as the other case is symmetric.

In this case, the output lies on \( x_1x_2 \), and determined by \( x_1 \) and \( x_2 \). Therefore, a group that contains neither agent 1 nor agent 2 cannot violate group-strategyproofness. On the other hand, a group that contains both of agents 1, 2 cannot violate group-strategyproofness either, because \( \| x_1 - f(x) \| + \| x_2 - f(x) \| = \| x_1 - x_2 \| \) has already reached the minimum.

Consider a group that contains agent 2 but not agent 1. If \( \| x_2 - x_1 \| < r - a \), then \( y = x_2 \), so agent 2 is truthful. Otherwise, let \( y' \) be the “\( y \)” after misreporting, where \( \| y' - x_1 \| \leq r - a \).

As a result, \( \| x_2 - y' \| \geq \| x_2 - x_1 \| - (r - a) = \| x_2 - y \| \). Therefore, this group cannot violate group-strategyproofness.

Consider a group that contains agent 1 but not agent 2. Let \( y' \) be the “\( y \)” after agent 1 misreports \( x_1' \). Let \( d = \| x_1 - x_1' \| \), and \( r' \) be the first coordinate of \( x_1' \). Then we have \( r' - a \geq r - a - d \) and
\[\|y' - x'_1\| \geq \|y - x_1\| - d.\] Therefore, \[\|y' - x_1\| \geq \|y - x_1\| - 2d,\] and
\[
\|f(x') - x_1\| = \frac{2}{3} \|x'_1 - x_1\| + \frac{1}{3} \|y' - x_1\|
\geq \frac{2}{3} d + \frac{1}{3} (\|y - x_1\| - 2d)
= \|y - x_1\|
= \|f(x) - x_1\|.
\]
Thus, this group cannot violate group-strategyproofness either.

Besides, the unanimity is clear. To sum up, Mechanism 2 is unanimous and group-strategyproof.

C.3 Strategyproofness and Approximation Bound of Mechanism 3

Mechanism 3. Given \(x\), output \((x_1 + \cdots + x_n)/n\) with 1/2 probability, and each \(x_i\) with 1/2\(n\) probability.

Proposition C.3. Mechanism 3 is strategyproof.

Proof. As all agents are symmetric, we will show that agent 1 cannot benefit from misreporting. \(\forall x, \forall x'_1 \in \mathbb{R}_d,\)
\[
\|f(x'_1, x_{-1}) - x_1\| = \frac{1}{2} \left\| \frac{x'_1 + \sum_{i=2}^{n} x_i}{n} - x_1 \right\| + \frac{1}{2n} \|x'_1 - x_1\| + \frac{1}{2n} \sum_{i=2}^{n} \|x_i - x_1\|
\geq \frac{1}{2} \left\| \frac{\sum_{i=1}^{n} x_i}{n} - x_1 \right\| + \frac{1}{2n} \sum_{i=2}^{n} \|x_i - x_1\|
= \|f(x) - x_1\|.
\]
Agent 1 cannot benefit from misreporting, and thus Mechanism 3 is strategyproof.

Proposition C.4. Mechanism 3 is \((2 - 1/n)\)-approximation for maximum cost.

Proof. For any \(y_1 \neq y_2\), consider the following profile
\[x^* = (y_1, y_2, \ldots, y_2)\].

In this case,
\[\text{mc}(f(x^*), x^*) = \frac{n-1}{2n} \|y_1 - y_2\| + \frac{1}{2} \|y_1 - y_2\| = \left(1 - \frac{1}{2n}\right) \|y_1 - y_2\|,\]
while the optimal is \(\text{mc}((y_1 + y_2)/2, x^*) = \|y_1 - y_2\|/2\), so the approximation ratio is at least \(2 - 1/n\).

Now we prove it to be the upper bound. Assume for contradiction that \(\exists x \in \mathbb{R}_d^n, \exists y \in \mathbb{R}_d\) such that
\[\frac{\text{mc}(f(x), x)}{\text{mc}(y, x)} > 2 - \frac{1}{n}\]
Let \(r = \text{mc}(y, x)\). Let \(\overline{x} = (x_1 + \cdots + x_n)/n\). Then,
\[(2 - \frac{1}{n}) r < \text{mc}(f(x), x) < \frac{1}{2} \text{mc}(\overline{x}, x) + 1,\]
so there exists $i \in N$ such that

$$\|x_i - \bar{x}\| = mc(x, \bar{x}) > \left(2 - \frac{2}{n}\right)r.$$ 

Also we have

$$\|x_i - \bar{x}\| = \left\|\frac{1}{n} \sum_{j \neq i} x_i - x_j\right\| \leq \frac{1}{n} \sum_{j \neq i} \|x_i - x_j\|,$$

so there exists $j \in N$ such that

$$\|x_i - x_j\| \geq \frac{n}{n - 1} \|x_i - \bar{x}\| > 2r.$$ 

On the other hand,

$$\|x_i - x_j\| \leq \|x_i - y\| + \|x_j - y\| \leq 2r,$$

which makes a contradiction. 

\[\square\]

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