Perfect State Transfer in $\mathcal{PT}$-symmetric Non-Hermitian Networks

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We systematically study the $\mathcal{PT}$ (parity-time reversal)-symmetric non-Hermitian version of a quantum network proposed in the work of Christandl et al. [Phys. Rev. Lett. 92, 187902 (2004)]. The exclusive nature of this model show that it is a nice paradigm to demonstrate the complex quantum mechanics theory for the relationship between the pseudo-Hermitian Hamiltonian and its Hermitian counterpart, as well as a candidate in experimental realization to simulate the $\mathcal{PT}$ symmetry breaking. We also show that this model allows conditional perfect state transfer within the unbroken $\mathcal{PT}$ symmetry region, but not arbitrary. This is due to the fact that the evolution operator at certain period is equivalent to the $\mathcal{PT}$ operator for the real-valued wavefunction in the elaborate $\mathcal{PT}$-symmetric Hilbert space.

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I. INTRODUCTION

The transmission of quantum state through a solid state data bus with minimal spatial and dynamical control is an experimental challenging and a theoretically necessary task for implementing a scalable quantum computation based on realistic silicon devices. S. Bose [1] first demonstrated the possibility that in a solid state based quantum computer, local interactions can apply entangling gates between distant qubits. In principle, perfect transfer of quantum state can be implemented by specifically engineering chain [2–5]. It is also showed that a quantum system possessing a commensurate structure of energy spectrum matched with the corresponding symmetry can ensure the perfect quantum state transfer [3, 4, 6].

The aim of this paper is to extend these findings to the non-Hermitian system. It is motivated by the interest in complex potentials in both theoretical and experimental aspects. Much effort has been devoted to establish a parity-time ($\mathcal{PT}$) symmetric quantum theory as a complex extension of the conventional quantum mechanics [7–14] since the seminal discovery by Bender [7]. It is found that the non-Hermitian Hamiltonian with simultaneous $\mathcal{PT}$ symmetry can have an entirely real quantum mechanical energy spectrum and has profound theoretical and methodological implications. Researches and findings relevant to the spectra of the $\mathcal{PT}$-symmetric systems were presented, such as exceptional points [15], spectral singularities for complex scattering potentials [16], as well as complex crystal and other specific models [17]. Furthermore, quantum dynamics in open systems and $\mathcal{PT}$-symmetric non-Hermitian systems were investigated [18–22]. At the same time the $\mathcal{PT}$ symmetry is also of great relevance to the technological applications based on the fact that the imaginary potential could be realized by complex index in optics [18, 22, 25]. In fact, such $\mathcal{PT}$ optical potentials can be realized through a judicious inclusion of index guiding and gain/loss regions, and the most interesting aspects associated with $\mathcal{PT}$-symmetric system were observed during the dynamic evolution process [26–29].

According to the complex quantum mechanics, a non-Hermitian Hamiltonian with entire real spectrum can be transformed into a Hermitian Hamiltonian via the well-established metric operator formalism [30]. Nevertheless, the example is rare in which explicit derivations can be performed to evaluate the equivalent Hermitian counterpart.

In this paper, we systematically study the $\mathcal{PT}$ (parity-time reversal)-symmetric non-Hermitian version of a quantum network proposed in the work of Christandl et al. [2, 3]. The main results are as follows: (i) The non-Hermitian version of a quantum network is exact solved and the critical behavior is analytically studied at the exceptional point. (ii) We provide an explicit mapping between the proposed pseudo-Hermitian Hamiltonian and its equivalent Hermitian Hamiltonian. The present model is a nice paradigm to demonstrate the relationship between the non-Hermitian Hamiltonian and its Hermitian counterpart, since all of Hamiltonians with different parameters, no matter Hermitian or non-Hermitian, are local and have the equally spaced spectra. (iii) Furthermore, we investigate the connection between the present model and the $\mathcal{PT}$-symmetric hypercube graph. (iv) We also show that such an extended model still allows conditional perfect state transfer within the unbroken $\mathcal{PT}$ symmetry region, but not arbitrary. This is due to the fact that the evolution operator at certain period is equivalent to the $\mathcal{PT}$ operator for the real-valued wavefunction in the elaborate $\mathcal{PT}$-symmetric Hilbert space.

This paper is organized as follows. In Section II we present the model and the solutions. In Section III we explore the basic properties of the model in its non-Hermitian version. In Section IV we investigate the metric and Hermitian counterpart for the extended model. In Section V we investigate the connection between the present model and the hypercube graph. Section VI is devoted to the dynamics of the model. Finally, we give a
II. THE MODEL

We start with the Hamiltonian for an N-site tight-binding chain with linear potentials,

\[ H = \frac{1}{2} \sum_{l=1}^{N-1} \sqrt{l(N-l)} (a_l^\dagger a_{l+1} + \text{H.c.}) + \sum_{l=1}^{N} \left[ \frac{1}{2} (N+1-l) \gamma a_l^\dagger a_l \right] \]

where \( a_l^\dagger \) (\( a_l \)) is the creation (annihilation) operator at site \( l \). For the sake of simplicity we take the units of coupling constant as 1. The conclusion of this paper is valid for both fermion and boson systems. In the case of \( \gamma = 0 \), it is reduced to the model in Ref. \[2, 3\], which has been shown \[3\] to guarantee that the Hamiltonian \( H (\gamma = 0) \) evolves states \(|\phi\rangle\) into \( \mathbb{P} |\phi\rangle \) at the time \( \pi \), no matter what these states are. Here parity operator \( \mathbb{P} \) is given by \( \mathbb{P} a_l^\dagger \mathbb{P}^{-1} = a_{N-l}^\dagger \). This due to the fact that the Hamiltonian \( H (\gamma = 0) \) is \( \mathbb{P} \)-symmetric and possesses an equally spaced spectrum \[2\]. In the case of nonzero \( \gamma \), the \( \mathbb{P} \) symmetry is broken. In the following, we will show that it still has an equal-spaced spectrum. Here the idea is to treat the Hamiltonian as an angular momentum in an external magnetic field.

Defining the operators

\[ J^+ = (J^-)^\dagger = \sum_{l=1}^{N-1} \sqrt{l(N-l)} a_l^\dagger a_{l+1}, \]
\[ J_x = \frac{J^+ + J^-}{2}, \quad J_y = \frac{J^+ - J^-}{2i}, \]
\[ J_z = \sum_{l=1}^{N} \left( \frac{N+1}{2} - l \right) a_l^\dagger a_l, \]

which satisfy the following angular momentum commutation relations

\[ [J^+, J^-] = 2J_z, \quad [J^+, J^\pm] = \pm J^\pm. \]

Then \( J_x, y, z \) acts as the angular momentum operator, and the Hamiltonian can be rewritten as

\[ H = J_x + \gamma J_z = \vec{J} \cdot \vec{B} \]

where

\[ \vec{B} = (B_x, B_y, B_z) = (1, 0, \gamma). \]

Obviously, it can be diagonalized as

\[ H = \sqrt{1 + \gamma^2} J_n' \]

with \( J_n' \) being a \( n \)-component of angular momentum operator, where \( n = \vec{B} / |\vec{B}| \) is the unit vector in the field direction. In this paper, we concentrate on the single-particle invariant subspace, which corresponds to the angular momentum system with \( J = (N-1)/2 \). Then the energy levels are still equally spaced. In this subspace, the eigenvector of \( H \) can be obtained from that of \( H (\gamma = 0) \) by the rotation operator, i.e.,

\[ |\psi_n\rangle = e^{-i[\beta(\gamma) - \pi/2]J_y} |\psi_n (\gamma = 0)\rangle \]

\[ (n = 1, 2, ... N). \]

Here \( \beta (\gamma) \) is an angle

\[ \beta (\gamma) = \arctan \left( \frac{1}{\gamma} \right) \]

and \( |\psi_n (\gamma = 0)\rangle \) is the eigenvector of \( H (\gamma = 0) \), i.e.,

\[ H (\gamma = 0) |\psi_n (\gamma = 0)\rangle = \frac{N+1}{2} - n |\psi_n (\gamma = 0)\rangle \]

which can be further expressed as

\[ |\psi_n (\gamma = 0)\rangle = \sum_l d_{n,l} \left( \frac{\pi}{2} \right) a_l^\dagger |0\rangle. \]

Then we have

\[ |\psi_n\rangle = \sum_l d_{n,l} (\beta) a_l^\dagger |0\rangle \]

where \( d_{n,l} (\beta) \) is Winger \( d \)-functions

\[ d_{n,l} (\beta) = d^{(N-1)/2} (\beta) \]

\[ = [(N-n)! (n-1)! (N-l)! (l-1)!]^{1/2} \times \sum_{\nu} \frac{(-1)^\nu \left( \cos \frac{\beta}{2} \right)^{N-1+l-n-2\nu} \left( -\sin \frac{\beta}{2} \right)^{n-\nu}}{(l-1-\nu)! (N-n-\nu)! (\nu+n-l)! \nu!}. \]

We define \( \psi_n (l) = d_{n,l} (\beta) \) to express the orthonormal relation as

\[ \sum_l \psi_m (l) \psi_n (l) = \delta_{mn}. \]

It is important to note that the above relation is still true for imaginary \( \gamma \) except the points \( \gamma = \pm i \).

III. NON-HERMITIAN \( \mathbb{P}T \)-SYMMETRIC HAMILTONIAN

Now we consider the Hamiltonian of Eq. (11) with imaginary linear potentials by taking \( \gamma \rightarrow i \gamma \), which can be written as

\[ \mathcal{H} = \frac{1}{2} \sum_{l=1}^{N-1} \sqrt{l(N-l)} (a_l^\dagger a_{l+1} + \text{H.c.}) \]

\[ + i \gamma \sum_{l=1}^{N} \left[ \frac{1}{2} (N+1-l) \right] a_l^\dagger a_l. \]
Mathematically, all the solutions of $H$ can be extended to that of $\mathcal{H}$ by simply taking $\gamma \to i\gamma$ except for the points $|\gamma| = 1$, since it does not induce any singularity in the rotation operator $e^{-i\beta(\gamma)t}$. This fact accords with complex quantum mechanics. Note that the non-Hermitian Hamiltonian $\mathcal{H}$ is $\mathcal{PT}$-symmetric, i.e., $[\mathcal{PT}, \mathcal{H}] = 0$. The antilinear time-reversal operator is defined as $T^iT^{-1} = -i$. The phase diagram of this system is determined by critical (exceptional) points $|\gamma| = 1$. This model exhibits two phases: an unbroken symmetry phase with a purely real energy spectrum

$$
\varepsilon_n = \sqrt{1 - \gamma^2} \left( \frac{N + 1}{2} - n \right), \quad n = 1, 2, ..., N \quad (15)
$$

when the potentials are in the region $|\gamma| < 1$ and a spontaneously broken symmetry phase with an imaginary spectrum when the potentials are in the region $|\gamma| > 1$.

In this paper, we only focus on the $\mathcal{PT}$-symmetric region. Although the spectrum is real, the corresponding eigenfunctions are no longer orthonormal with respect to the Dirac inner product. One can establish the complete biorthogonal set by the eigenfunctions of the Hamiltonian $\mathcal{H}^\dagger$. Denoting

$$
|\phi_n\rangle = \sum_l \phi_n (l) a_l^\dagger |0\rangle = \sum_l \psi_n^* (l) a_l^\dagger |0\rangle, \quad (16)
$$

we have

$$
\mathcal{H}^\dagger |\phi_n\rangle = \varepsilon_n |\phi_n\rangle. \quad (17)
$$

Then Eq. (13) leads to the orthonormal relation

$$
\langle \phi_m | \psi_n \rangle = \sum_l \phi_m^* (l) \psi_n (l) = \delta_{mn}. \quad (18)
$$

Furthermore, the complete set obeys the relation

$$
\mathcal{PT} |\psi_n\rangle = (-1)^n |\psi_n\rangle. \quad (19)
$$

In the case of $\gamma = 0$, the above relation is reduced to

$$
\mathcal{P} |\psi_n (\gamma = 0)\rangle = (-1)^n |\psi_n (\gamma = 0)\rangle. \quad (20)
$$

It is worth to note that Eqs. (19) and (20) have different implications: If one obtains a set of eigenfunctions satisfying Eq. (9), they will obey Eq. (20) spontaneously. However, Eq. (19) is not necessary for the eigenfunctions of the Hamiltonian $\mathcal{H}(\gamma)$. This is due to the fact that operator $\mathcal{P}$ is an antilinear operator. This issue will be elaborated in the next section when we investigate the application of Eq. (19) associated with the dynamic process.

Now we consider the case of $|\gamma| = 1$, which are singular points for $\psi_n (l)$. However, one can construct the eigenfunctions of the Hamiltonians $\mathcal{H}_\pm = \mathcal{H} (\gamma = \pm 1)$ as the form

$$
|\varphi_\pm\rangle = \sum_l \varphi_\pm (l) a_l^\dagger |0\rangle
$$

$$
= \sum_l \left( \sqrt{2} \right)^{1-N} \sqrt{C_{N-1}^{l-1}} (\pm i)^{n-l} a_l^\dagger |0\rangle. \quad (21)
$$

where

$$
\varphi_\pm (l) = \left( \sqrt{2} \right)^{1-N} \sqrt{C_{N-1}^{l-1}} (\pm i)^{n-l} \quad (22)
$$

Acting the Hamiltonians $\mathcal{H}_\pm$ on the states $|\varphi_\pm\rangle$, straightforward algebra shows that

$$
\mathcal{H}_\pm |\varphi_\pm\rangle = 0. \quad (23)
$$

Eigenstates $|\varphi_\pm\rangle$ are zero-norm states, i.e.,

$$
\sum_l \varphi_\pm^2 (l) = 0. \quad (24)
$$

We can see that $|\gamma| = 1$ is the boundary of two phases, possessing the characteristics of exceptional point: The spectrum exhibits square-root-type level repulsion in the vicinity of $|\gamma| = 1$. On the other hand, states $|\varphi_\pm\rangle$ should be the coalescence of the eigenfunctions $\{ |\psi_n\rangle \}$ as $|\gamma| \to 1$. To demonstrate this point, we introduce the dressed eigenfunctions $\{ \tilde{\psi}_n \}$ by multiplying factor $g_n (\gamma)$ to the original eigenfunctions $\{ |\psi_n\rangle \}$.

$$
\tilde{\psi}_n (l) = g_n (\gamma) |\psi_n (l)\rangle \quad (25)
$$

where

$$
g_n (\gamma) = \left( \sqrt{2} \cos \frac{\beta}{2} \right)^{1-N} \left( C_{N-1}^{l-1} \right)^{-\frac{1}{2}}. \quad (26)
$$

FIG. 1. (Color online) Plots of the dressed eigenfuctions as functions of $N = 10$ chain. The solid (empty) circle indicates the real (imaginary) part of the wavefunction for (a) $\gamma = 0.2$, (b) $\gamma = 0.7$, (c) $\gamma = 0.9$, and (d) $\gamma = 0.998$, respectively. It can be observed that as $\gamma$ tends to 1, all the wavefunctions approach to the same function Eq. (21) for $N = 10$, which is plotted in (d) (triangles). Note that the scales of the subfigures for various $\gamma$ are different.
We notice that
\[
\lim_{\pm \gamma \to 1} \psi_n (l) = \varphi_{\pm} (l),
\] (27)
i.e., all the dressed eigenstates coalesce with \(|\varphi_{\pm}\rangle\) at the critical point. For illustration, numerical simulation of finite size chains with various values of \(\gamma\) is performed. In Fig. 4 we plot the dressed eigenfunctions including real and imaginary parts for \(N = 10\) chain, respectively. It shows that all the eigenfunctions tend to \(|\varphi_{\pm}\rangle\) as \(\gamma\) approaches the critical point.

Finally, we would like to point out that the nature of the \(\mathcal{PT}\) symmetry breaking in such a model is exclusive. The entire spectrum becomes imaginary and the \(\mathcal{PT}\) symmetry of all the eigenstates is broken at the point \(|\gamma| = 1\) simultaneously. The degrees of \(\mathcal{PT}\) symmetry breaking is defined as the fraction of eigenvalues that become complex [31], the degree of our system is 1. This feature should lead to clear signatures in the dynamics of the wavepacket. Such a model is a good candidate to simulate the critical behavior in experiments.

IV. METRIC AND HERMITIAN COUNTERPART

Another theoretical interest in the non-Hermitian \(\mathcal{PT}\)-symmetric system is the physical meaning of the Hamiltonian \(\mathcal{H}\). When speaking of the physical significance of a non-Hermitian Hamiltonian, one of the ways is to seek its Hermitian counterparts [32–34] possessing the same real spectrum. According to the complex quantum mechanics, a non-Hermitian \(\mathcal{PT}\)-symmetric Hamiltonian can be transformed into a Hermitian Hamiltonian. This is achieved by introducing a metric, a bounded positive-definite Hermitian operator \(\eta\) which can be constructed via the eigenstates of \(\mathcal{H}^\dagger\) [30]. However, the obtained equivalent Hermitian Hamiltonian is usually quite complicated [31, 32]. It is tough to provide an explicit mapping of a pseudo-Hermitian Hamiltonian to its equivalent Hermitian Hamiltonian. Fortunately, the present model is a nice paradigm to demonstrate the relationship between the pseudo-Hermitian Hamiltonian and its Hermitian counterpart. In this section, we will illustrate the basic ideas via the above analytically solution.

Consider the single-particle case, the metric operator \(\eta\) can be constructed via the eigenstates of \(\mathcal{H}^\dagger\) as

\[
\eta = \sum_{n} |\phi_n\rangle \langle \phi_n| \quad \text{(28)}
\]

\[
= \sum_{n,l,l'} d_{n,l' \tau} (\beta) d_{n,l}^* (\beta) a_\tau^\dagger |0\rangle \langle 0| a_{l'}
= e^{-i(\beta^* - \beta) J_y},
\]

which guarantees the relation

\[
\eta \mathcal{H} \eta^{-1} = \mathcal{H}^\dagger. \quad \text{(29)}
\]

It is noticed that the matrix representation of \(\eta \mathcal{H}\) and \(\eta\) based on the orthonormal basis, say \(\{a_\tau^\dagger |0\rangle\}\) under the Dirac inner product, are Hermitian matrices. Operator \(\eta\) is called the metric operator since it can be used to define the biorthogonal inner product, under which the unitary evolution can be obtained. Furthermore, let

\[
\rho = \sqrt[\gamma]{\eta} = e^{-\frac{i}{2}(\beta^* - \beta) J_y}
\] (30)
be the unique positive-definite square root of \(\eta\). Then the Hermitian operator \(\rho\) acts as a similarity transformation to map the non-Hermitian Hamiltonian \(\mathcal{H}\) onto its equivalent Hermitian counterpart \(h\) by

\[
h = \bar{\rho} \mathcal{H} \rho^{-1} = \sqrt{(1 - \gamma^2)} J_y
= \sqrt{(1 - \gamma^2)} \mathcal{H}(\gamma = 0).
\] (31)

It can be regarded as the Hermitian counterpart of the non-Hermitian Hamiltonian \(\mathcal{H}(\gamma = 0)\). It is fortunate that both the non-Hermitian and the Hermitian Hamiltonian have simple structure: they possess the localized couplings and have equally spaced spectra. The physics of both the original non-Hermitian Hamiltonian and its equivalent Hermitian counterpart are clear: they can be regarded as either the lattice model with nearest-neighbor couplings (engineered chain with imaginary linear on-site potentials) or the angular momentum coupled to the external complex field. Additionally, the physics of \(\rho\) is also obviously, which presents the operation of two successive rotations, \(e^{-i(\beta^*/2) J_y}\) and \(e^{i(\beta/2) J_y}\), which denote the rotations about \(y\) axis with the angle \(\beta^*/2\) and \(-\beta/2\), respectively.

V. \(\mathcal{PT}\)-SYMMETRIC HYPERCUBE

It has been pointed that the one-dimensional system \(\mathcal{H}(\gamma = 0)\) can be achieved by the projection of the hypercube [3]. In this section, we will extend this approach to the non-Hermitian regime and investigate the solutions for all the possible projections of the hypercube.

We start the investigation with an ensemble of non-interacting spins in the complex magnetic field \(\vec{B}\) in Eq. 5. The Hamiltonian reads

\[
H_{\text{spin}} = \sum_{l=1}^{d} H_l \quad \text{(32)}
\]

\[
= \sum_{l=1}^{d} \vec{s}_l \cdot \vec{B},
\]

where \(\vec{s}_l\) is spin-\(\frac{1}{2}\) operator for the \(l\)th particle of the \(d\)-particle ensemble. In \(z\)-component spin basis, the matrix representation of \(H_{\text{spin}}\) has the form

\[
M_{\text{spin}} = \bigoplus_{l=1}^{d} M_l, \quad \text{(33)}
\]
Now we turn to the dynamics of the non-Hermitian system. We notice that the imaginary linear potentials break the $P$ symmetry but retains the $PT$ symmetry.

In the following, we will explain how to use such system to realize the perfect state transfer. The time evolution of the initial state $|\Phi(0)\rangle$ can be expressed as

$$|\Phi(t)\rangle = e^{-iHt} |\Phi(0)\rangle = \sum_n C_n e^{-iE_n t} |\psi_n\rangle$$

where $C_n = \langle \phi_n | \Phi(0) \rangle$ can be obtained from the Eq. (37). At time $t = \tau = \pi/\sqrt{1-\gamma^2}$, from Eqs. (15) and (19) we have

$$|\Phi(\tau)\rangle = \sum_n C_n (-1)^n |\psi_n\rangle$$

where $C_n = \langle \phi_n | \Phi(0) \rangle$ can be obtained from the Eq. (37). At time $t = \tau = \pi/\sqrt{1-\gamma^2}$, from Eqs. (15) and (19) we have

$$|\Phi(\tau)\rangle = \sum_n C_n PT |\psi_n\rangle$$

For the state with all real $C_n$, we have

$$|\Phi(\tau)\rangle = PT |\Phi(0)\rangle$$

i.e., at the instant $\tau$, the time evolution operator acts as $PT$ operator. For $\gamma = 0$ case, Eq. (37) reduces to $|\Phi(\tau)\rangle = P |\Phi(0)\rangle$ for arbitrary states $|\Phi(0)\rangle$. As we mentioned in the last section, Eq. (19) is not necessary for a given eigenstate. Nevertheless, Eq. (37) can provide a way to construct the state to satisfy the Eq. (19). In the following we will exemplify this point and its application.
FIG. 3. (Color online) Plots of the time evolution for real-valued quantum states in $N = 100$ chain. The initial state has the form of Eq. (37). The black (red) line indicates the real (imaginary) part of the wavefunction for (a) $\gamma = 0.05$, $l = 1$, (b) $\gamma = 0.05$, $l = 25$, (c) $\gamma = 0.1$, $l = 1$, (d) $\gamma = 0.05$, $l = 100$, respectively. It shows that the width of the initial state becomes wider as $l$ or $\gamma$ increases. We can see that the final state at the time $t = \tau$ is the $\mathcal{PT}$ function of the initial state. In (a, b, c) the amplitudes of the evolved state increase during the interval $[0, \tau/2]$, then decrease during $[\tau/2, \tau]$. In contrast, it decreases during $[0, \tau/2]$, then increases during $[\tau/2, \tau]$ in (d). Note that the scales of the subfigures are different.

With the same mechanism of the action of operator $\mathcal{P}$, operator $\mathcal{PT}$ also takes the role of perfect quantum state transfer. The flaw of this scheme is that it only applicable for some specific state. However, if there exists local state satisfying Eq. (37), it has a potential for future applications.

In the following, we will show that the local state can be constructed to perform perfect state transfer for small $|\gamma|$. Consider a local state at the end of the chain $a_1^\dagger |0\rangle$, which can be expanded with eigenstates

$$a_1^\dagger |0\rangle = \sum_m |\psi_m\rangle \langle \phi_m | a_1^\dagger |0\rangle$$

$$= \sum_m d_{m,1}^{N-1} (\beta) |\psi_m\rangle.$$ 

Obviously, such a state does not satisfy Eq. (37) and cannot be perfectly evolved to the state $a_N^\dagger |0\rangle$. Otherwise, one can construct a state satisfying Eq. (37) based
on the state $a_1 |0\rangle$ in the way

$$|\bar{\Omega}| = \frac{1}{\sqrt{2}} \sum_m \left\{ d_{m,1}^{[\gamma]} (\beta) + d_{m,1}^{[\gamma]} (\beta) \right\} |\psi_m\rangle, \quad (39)$$

where $\Omega_1 = 2 + (1 - \gamma^2)^{1-N}$ is the normalization factor. Note that the expansion coefficients of $|\bar{\Omega}|$ are all real, so that it can evolve to the state $PT |\bar{\Omega}|$. Now we will prove that such a state is local in the case of $|\gamma| \ll 1$. Actually, rewriting the state $|\bar{\Omega}|$ in the basis $\{ a_1 |0\rangle \}$ and neglecting the high-order terms of the Taylor expansion, we obtain

$$|\bar{\Omega}| = a_1 |0\rangle - \frac{\gamma}{2} \sqrt{(N-1)} a_2 |0\rangle, \quad (40)$$

which is a local state. At time $t = \tau = \pi/\sqrt{(1 - \gamma^2)}$, it evolves to state $\left( a_N^1 + i \gamma \sqrt{(N-1)/2} a_1^{1-1} \right) |0\rangle$.

In order to demonstrate and verify the above analysis, we perform the numerical simulation for a finite $N$-site system. The initial wave function has the form

$$|\bar{\Omega}| = \frac{1}{\sqrt{2}} \sum_m \left\{ d_{m,1}^{[\gamma]} (\beta) + d_{m,1}^{[\gamma]} (\beta) \right\} |\psi_m\rangle, \quad (41)$$

where $\Omega_l = 2 + 2 \sum_{m=1}^N d_{m,1}^{[\gamma]} (\beta) d_{m,1}^{[\gamma]} (\beta)$ is the normalization factor. The evolved wave function

$$\Phi(n, t) = \langle 0 | a_n e^{-i \mathcal{H} t} |\bar{\Omega}| \rangle \quad (42)$$

is computed via exact diagonalization method. We plot the real and imaginary parts of $\Phi(n, t)$ as function of time in Fig. 9 respectively. It shows that the evolution process is different from that in a Hermitian system: the Dirac inner product of the evolved state is not conservative. The time evolution during intervals $[0, \tau]$ and $[\tau, 2\tau]$ are completely different processes.

VII. SUMMARY AND DISCUSSION

In this paper, we have shown that adding the $PT$-symmetric potentials on the well-studied Hermitian quantum network [2,3] constructs a exactly solvable non-Hermitian model which allows conditional perfect state transfer within the unbroken $PT$ symmetry region, but not arbitrary. This model has applicability and relevant to the physical situations since there exist local states which can be transferred perfectly across long distance. In the theoretical aspect, this work provide a nice paradigm to demonstrate the relationship between a pseudo-Hermitian Hamiltonian and its Hermitian counterpart in the framework of the complex quantum mechanics. On the other hand, the simultaneity of the onset of the $PT$ symmetry breaking for the whole eigenstates should lead to remarkable phenomena in the dynamics of a wavepacket. This result suggests the evident observation of the $PT$ symmetry breaking in optical system with complex index.

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