Beta-Binomial stick-breaking non-parametric prior

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Abstract

A new class of nonparametric prior distributions, termed Beta-Binomial stick-breaking process, is proposed. By allowing the underlying length random variables to be dependent through a Beta marginals Markov chain, an appealing discrete random probability measure arises. The chain’s dependence parameter controls the ordering of the stick-breaking weights, and thus tunes the model’s label-switching ability. Also, by tuning this parameter, the resulting class contains the Dirichlet process and the Geometric process priors as particular cases, which is of interest for fast convergence of MCMC implementations.

Some properties of the model are discussed and a density estimation algorithm is proposed and tested with simulated datasets.

Keywords: Beta-Binomial Markov chain, Density estimation, Dirichlet process prior, Geometric process prior, Stick-breaking prior.
1 Introduction

Discrete random probability measures and their distributions play a key role in Bayesian nonparametric statistics. The availability of general classes of priors and their different representations are crucial for the study of theoretical properties, as well as for the proposal of simulation and estimation algorithms. This continuously encourages the search of competitive alternatives to the canonical model, Ferguson (1973) Dirichlet process. At the outset, one could consider a species sampling process (Pitman; 2006), over a measurable Polish space \((S, \mathcal{B}(S))\),

\[
\mu = \sum_{j \geq 1} w_j \delta_{\xi_j},
\]

where the atoms, \(\Xi = (\xi_j)_{j \geq 1}\), and the weights, \(W = (w_j)_{j \geq 1}\), are independent collections of random variables (r.v.’s), with \(\xi_j \overset{\text{iid}}{\sim} \mu_0\), a diffuse measure on \((S, \mathcal{B}(S))\), and \(\sum_{j \geq 1} w_j = 1\), almost surely (a.s.). To fully specify the law of \(\mu\), one could assume \(\mu_0\) and place a distribution over the infinite dimensional simplex \(\Delta_\infty = \{ (w_1, w_2, ...) : w_i \geq 0, \sum_{i \geq 1} w_i = 1 \}\).

An important aspect to note is that

\[
\sum_{j \geq 1} w_j \delta_{\xi_j} \overset{d}{=} \sum_{j \geq 1} w_{\rho(j)} \delta_{\xi_j}
\]

for every random permutation of \(\mathbb{N}, \rho\), independent of \(\Xi\). This means that once the atom’s distribution, \(\mu_0\), is fixed, there are infinitely many distributions over \(\Delta_\infty\) that lead to the exact same prior, hence the need to study orderings for the weights. In particular, one can consider the decreasing ordering of its elements, here denoted by \(W^\downarrow = (w_j^\downarrow)_{j \geq 1}\), with \(w_1^\downarrow > w_2^\downarrow > \cdots\) a.s., or the size-biased permutation, denoted by \(\tilde{W} = (\tilde{w}_j)_{j \geq 1}\), which satisfies \(\mathbb{P}[\tilde{w}_1 = w_j|W] = w_j\), and for \(n \geq 2\)

\[
\mathbb{P}[\tilde{w}_n = w_j|W, \tilde{w}_1, \ldots, \tilde{w}_{n-1}] = \frac{w_j}{1 - \sum_{i=1}^{n-1} \tilde{w}_i} \mathbb{1}_{\{w_j \notin \{w_1, \ldots, w_{n-1}\}\}}.
\]

Working with decreasing representations of the weights reduces the identifiability problem that arises from (2) in the sense that if \(\gamma_1, \gamma_2, \ldots\) is sampled i.i.d. from \(\mu\), conditionally given \(\mu\), then \(w_1^\downarrow\) corresponds to the atom that appears more frequently in the sequence, \(w_2^\downarrow\) corresponds to the second most frequent value, and so on (e.g., Mena and Walker; 2015). On the other hand, the size-biased permutation of the weights is of interest when the focus is in the clusters featured in the sample, i.e. if \(\gamma_j^s\) is the \(j\)th distinct value to appear in the sample, then the long-run proportion of elements in \(\{n : \gamma_n = \gamma_j^s\}\) coincides precisely with \(\tilde{w}_j\) (Pitman; 1996a).
Different techniques to place distributions on $\Delta_\infty$ are available (e.g. Ferguson; 1973; Blackwell and MacQueen; 1973; James et al.; 2009) and connections among such techniques are well known (e.g. Ishwaran and James; 2001; Ishwaran and Zarepour; 2002; Hjort et al.; 2010). Perhaps one of the most practical constructions is enjoyed by the so-called stick-breaking process (McCloskey; 1965; Sethuraman; 1994; Ishwaran and James; 2001) where the weights are decomposed as

$$w_1 = v_1, \quad w_j = v_j \prod_{i=1}^{j-1} (1 - v_i), \quad j \geq 2,$$

for some sequence taking values in $[0, 1]$, $V = (v_i)_{i \geq 1}$, hereinafter referred to as length variables (l.v.’s). The practical compromise inherent to (3) is relatively little, as most practical classes of priors have a stick-breaking representation, e.g. the Dirichlet process (Ferguson; 1973; Sethuraman; 1994), its two-parameter generalization (Pitman; 1992), the normalized inverse-Gaussian process (Favaro et al.; 2012) and the more general class of homogeneous normalized random measures with independent increments (Favaro et al.; 2016). In particular, the Dirichlet process is recovered when $v_i \overset{iid}{\sim} \text{Be}(1, \theta)$, for some $\theta > 0$, and, as shown by Pitman (1996b), the resulting weights coincide with the corresponding size-biased permutation of them, an ideal feature for clustering (Pitman; 1996a). A seemingly different stick-breaking prior is the Geometric process, introduced by Fuentes-García et al. (2010). For this case, the decreasing ordering of the weights takes the form

$$w_j = \lambda(1 - \lambda)^{j-1}, \quad j \geq 1,$$

for some $\lambda \sim \text{Be}(\alpha, \theta)$, with $\alpha, \theta > 0$. Here the random variables $(v_i)_{i \geq 1}$ are completely dependent, indeed identical, unlike for the Dirichlet process. As mentioned above, the ordering of the weights, or lack of it, is of high relevance when using Bayesian nonparametric priors for density estimation and/or clustering. The dependence on only one random variable makes the Geometric process an attractive choice from a numerical point of view, and also makes it quite simple to generalize to non-exchangeable settings (Fuentes-García et al.; 2009; Mena et al.; 2011; Hatjispyros et al.; 2018). Furthermore, as shown by Bissiri and Ongaro (2014), both the Dirichlet and the Geometric processes have full support.

We propose a new class of stick-breaking distributions over $\Delta_\infty$, featured by dependent l.v.’s driven by a strictly stationary Beta Markov chain, thus leading to a novel family of random probability measures, the Beta-Binomial stick-breaking (BBSB) priors. The Beta Markov chain in
question has a dependence parameter which modulates the ordering of the corresponding weights, allowing BBSB priors to enjoy a good trade-off between weights identifiability and mixing. For extreme values of the dependence parameter, we find that the Dirichlet process and the Geometric process priors are particular cases of our model. Furthermore, using an extension of the aforementioned result by Bissiri and Ongaro (2014), we will see that BBSB priors also have full support.

The remaining part of the article is organized as follows: In Section 2 we present the construction of the Markov chain with \( \text{Be}(\alpha, \theta) \) marginals. Inhere, we also analyze some special and limiting cases that will subsequently allow to recover the Dirichlet and Geometric processes. This Markov chain then assembles in Section 3 a sequence of l.v.'s, thus leading to Beta-Binomial stick-breaking priors. In Section 4 we derive a sampling scheme for density estimation and, in Section 5 we test it in simulated data. The proofs of the main results are deferred to the Appendix.

2 Beta-Binomial Markov chain

Following Pitt et al. (2002), given a joint density function \( \pi_{v,x}(v, x) \) with marginals \( \pi_v(v) \) and \( \pi_x(x) \), and whose conditional distributions are \( \pi_{v|x}(v|x) \) and \( \pi_{x|v}(x|v) \), it is possible to construct a couple of reversible Markov chains \((v_i)_{i \geq 1}\) and \((x_i)_{i \geq 1}\) with stationary distributions \( \pi_v \) and \( \pi_x \) respectively. The construction considers the law induced by \( v_1 \sim \pi_v \), and \( \{x_i | v_i\} \sim \pi_{x|v}(\cdot|v_i) \), \( \{v_{i+1} | x_i\} \sim \pi_{v|x}(\cdot|x_i) \), for \( i \geq 1 \). Arising from the Beta-Binomial conjugate model, we take

\[
\pi_{v,x}(v, x) = \text{Bin}(x | \kappa, v) \text{Be}(v | \alpha, \theta),
\]

for some \( \alpha, \theta > 0, \kappa \in \{0, 1, \ldots\} \), and where \( \text{Bin}(0, p) = \delta_0 \). Thus, the dependence induced by \( v_1 \sim \text{Be}(\alpha, \theta) \), and \( \{x_i | v_i\} \sim \text{Bin}(\kappa, v_i) \) and \( \{v_{i+1} | x_i\} \sim \text{Be}(\alpha + x_i, \theta + \kappa - x_i) \), for \( i \geq 1 \) generates Markov chains, \( V = (v_i)_{i \geq 1} \) and \( X = (x_i)_{i \geq 1} \), where the former has transition probabilities given by

\[
P[v_i \in A | v_{i-1}] = \int_A \sum_{x=0}^{\kappa} \text{Be}(s | x + \alpha, \theta + \kappa - x) \text{Bin}(x | \kappa, v_{i-1}) ds,
\]
and stationary distribution $\text{Be}(\alpha, \theta)$, and the latter

$$
\mathbb{P}[x_i = x|x_{i-1}] = \int_0^1 \text{Bin}(x|\kappa, p)\text{Be}(p|\alpha + x_{i-1}, \theta + \kappa - x_{i-1})dp
= \left(\frac{\kappa}{x}\right)\frac{(\alpha + x_{i-1})_\alpha(\theta + \kappa - x_{i-1})_\kappa}{(\alpha + \theta + \kappa)_\kappa},
$$

(5)

where $(y)_m = \prod_{j=0}^{m-1} (y + j)$, and its stationary distribution is

$$
\mathbb{P}[x_i = x] = \left(\frac{\kappa}{x}\right)\frac{(\alpha)_x(\theta)_{\kappa-x}}{(\alpha + \theta + \kappa)_{\kappa}}.
$$

(6)

To any Markov chains, $V$, $X$ and $(V, X) = (v_i, x_i)_{i \geq 1}$, we refer to them as $\text{Beta}$, $\text{Binomial}$ and $\text{Beta-Binomial}$ chains. See Nieto-Barajas and Walker (2002) and Mena and Walker (2009) for more on this kind of Markov chains. In what follows, we focus on the the Beta chain and some of its properties, specifically in how the parameter $\kappa$ affects the dependence of the chain. This will be relevant for our construction of the nonparametric prior in the following section.

**Proposition 2.1** Let $(V, X)$ be a Beta-Binomial chain with parameters $(\kappa, \alpha, \theta)$, then for the Beta chain, $V$, and for every $i \geq 1$, we have the following conditional moments

a) $\mathbb{E}[v_{i+1}|v_i] = \frac{\alpha + \kappa v_i}{\alpha + \theta + \kappa}$.

b) $\text{Var}(v_{i+1}|v_i) = \frac{(\alpha + \kappa v_i)(\theta + \kappa(1 - v_i)) + Nv_i(1 - v_i)(\alpha + \theta + \kappa)}{2(\alpha + \theta + \kappa + 1)}$.

c) $\text{Cov}(v_i, v_{i+1}) = \frac{\kappa \alpha \theta}{(\alpha + \theta)^2(\alpha + \theta + 1)}$.

d) $\rho_{v_i, v_{i+1}} = \frac{\text{Cov}(v_i, v_{i+1})}{\sqrt{\text{Var}(v_i)}\sqrt{\text{Var}(v_{i+1})}} = \frac{\kappa}{\alpha + \theta + \kappa}$.

Fixing the value of $\kappa$ and increasing either $\alpha$ or $\theta$, the correlation coefficient, $\rho_{v_i, v_{i+1}}$ goes to 0. Conversely, if we fix $\alpha$ and $\theta$, for large values of $\kappa$, $\rho_{v_i, v_{i+1}} \approx 1$. Also, if $\alpha$ and $\theta$ are very small with respect to $\kappa$

$$
\mathbb{E}[v_{i+1}|v_i] \approx v_i \quad \text{and} \quad \text{Var}(v_{i+1}|v_i) \approx \frac{2v_i(1 - v_i)}{\kappa + 1}.
$$

Hence, intuition tells us that the conditional distribution of $v_{i+1}$ given $v_i$, tends to $\delta_{v_i}$, as $\kappa$ grows, see Figure 1. The following result formalizes this intuition.
Proposition 2.2 Let $V^{(\kappa)} = \left(v_i^{(\kappa)}\right)_{i \geq 1}$ be a Beta-chain with parameters $(\kappa, \alpha, \theta)$.

(i) For $\kappa = 0$, $V^{(0)}$ is a sequence of i.i.d. random variables with distribution $\text{Be}(\alpha, \theta)$.

(ii) As $\kappa \to \infty$, $V^{(\kappa)}$ converges in distribution to $(\lambda, \lambda, \ldots)$, where $\lambda \sim \text{Be}(\alpha, \theta)$.

3 Beta-binomial stick-breaking prior

We call Beta-Binomial stick-breaking prior any species sampling process, $\mu$, with weights sequence as in (3) for some l.v.’s, $V$, driven by a Beta chain with transition density (4). As usual, the parameters of the l.v.’s are inherited to the prior, adding to the latter, the diffuse probability measure, $P_0$, as an additional parameter. The first property to check is that the corresponding weights add up to one.
Proposition 3.1 Let $W$ be as in equation (3), for some Beta chain, $V$. Then
\[ \sum_{j \geq 1} w_j^{a,s} = 1. \]
Moreover, notice that for every $0 < \delta < \varepsilon < 1$ and $n \geq 1$, any Beta-Binomial chain, $(V, X)$, with parameters $(\kappa, \alpha, \theta)$, satisfies
\[ P\left[ \bigcap_{i=1}^{n} (\delta < v_i < \varepsilon) \right] = E \left[ \prod_{i=1}^{n} P[\delta < v_i < \varepsilon | X] \right] \]
\[ = E \left[ P[\delta < v_1 < \varepsilon | x_1] \prod_{i=2}^{n} P[\delta < v_i < \varepsilon | x_{i-1}, x_i] \right] > 0, \]
insomuch as conditionally given $X$, the elements of $V$ are independent and Beta distributed. As shown by Bissiri and Ongaro (2014), the above observation shows that any Beta-Binomial prior has full support, and thus feasible for nonparametric inference. The following results, which follow from Proposition 2.2, motivate their study.

Theorem 3.2 Let $\mu^{(\kappa)}$ be a BBSB prior with parameters $(\kappa, \alpha, \theta, P_0)$ then

(i) For $\kappa = 0$ and $\alpha = 1$, $\mu^{(0)}$ is a Dirichlet process with parameters $(\theta, P_0)$.

(ii) For any $\alpha$ and $\theta$ fixed, as $\kappa \to \infty$, $\mu^{(\kappa)}$ converges in distribution to some Geometric process, $\mu$, with parameters $(\alpha, \theta, P_0)$.

In terms of the ordering of the corresponding weights, we have the following corollary.

Corollary 3.3 Let \( (w_j^{(\kappa)})_{j \geq 1} \) be as in equation (3), for some Beta chain, \( (v_i^{(\kappa)})_{i \geq 1} \), with parameters $(\kappa, \alpha, \theta)$. Then

(i) For $\alpha = 1$, $\kappa = 0$, and any choice of $\theta$, \( (w_j^{(\kappa)})_{j \geq 1} \) is size-biased ordered.

(ii) For any choices of $\alpha$ and $\theta$, and for every $j \geq 1$
\[ \lim_{\kappa \to \infty} P[\left. w_j^{(\kappa)} < w_j^{(\kappa)} \right] = 1. \]
Figure 2: Simulations of \((w_j)_{j=1}^{25}\) (A.2 and B.2) and their corresponding l.v.’s (A.1 and B.1 respectively) for distinct values of \(\kappa\). For the Beta chains in A.1, we fixed \(\alpha = 1\) and \(\theta = 1\), for the ones in B.1 we used the same value of \(\alpha\), whilst \(\theta = 10\). The chains in a single graph share the same initial r.v. for the sake of a simpler analysis.

If we fix \(\alpha = 1\), the choice \(\kappa = 0\) implies that \(W\) is size-biased ordered. In general for such sequences \(E[\tilde{w}_j] \geq E[\tilde{w}_{j+1}]\), but even though the weights are likely to be decreasing they are not in a almost sure form. On the other extreme, as \(\kappa \to \infty\) we found the decreasing ordering of the Geometric weights. Roughly speaking, by increasing the parameter \(\kappa\), we induce a stronger stochastic ordering to the weights. Figure 2 shows some simulations of \((w_j)_{j=1}^{25}\) (A.2 and B.2) and their corresponding l.v.’s (A.1 and B.1 respectively) that illustrate the aforementioned behaviour. Generally, a bigger value of \(\theta\), requires a larger value of \(\kappa\), to assure the weights are decreasing. The initial value \(v_1\) of the Beta chain strongly affects the behaviour of the complete sequence of weights, e.g. large initial values increase the rate at which the weights decrease, this is particularly evident for large values of \(\kappa\).
3.1 Distribution of the number of groups

When working with any species sampling process, $\mu$, such as a Dirichlet, BBSB or Geometric process, a r.v. of interest is the number of distinct values, $K_n$, that a sample $\{\gamma_1, ..., \gamma_n\}$ driven by $\mu$ exhibits. Although for some priors it is possible to compute or characterize the probabilistic behaviour $K_n$ (see for instance Pitman; 2006), in general this is not an easy task to do. Despite, whenever it is feasible to obtain samples from the weights sequence, $W$, as is the case of any BBSB prior, obtaining samples from $K_n$ can be easily achieved as follows: Sample $n$ independent $U(0,1)$ r.v.’s, $(u_k)_{k=1}^n$, and $(w_j)_{j=1}^\varphi$ where $\varphi$ is some constant satisfying $\sum_{j=1}^\varphi w_j > \max_k u_k$. For $k \in \{1, ..., n\}$ and $i \in \{1, ..., \varphi\}$, let $d_k = i$ if and only if $\sum_{j=1}^{i-1} w_j < u_k < \sum_{j=1}^i w_j$ (with the convention that the empty sum equals 1) then the number of distinct values $(d_1, ..., d_n)$ exhibits is precisely a sample from $K_n$.

![Frequency polygons](image)

Figure 3: Frequency polygons of samples of size 10000 from $K_{50}$ for distinct values of $\kappa$ and $\theta$ and fixing $\alpha = 1$. For the frequency polygons in A, B and C we fixed $\kappa$ to 0, 10 and 100 respectively, whilst the frequency polygons in D correspond to the Geometric prior, for each fixed value of $\kappa$, we vary $\theta$ in the set $\{0.5, 1, 3, 6\}$.

To understand how the parameters of a BBSB prior affect the distribution of $K_n$, we sampled
as aforementioned varying the values of $\kappa$ and $\theta$ and fixing $\alpha = 1$. As illustrated in Figure 3, for a fixed value of $\theta$, an increment on $\kappa$ contributes to the distribution of $K_n$ with a heavier right tail and thus a larger mean and variance, say less informative. If the value of $\theta$ is small, the effect of incrementing $\kappa$ is evident even for small values $\kappa$, on the contrary, if $\theta$ is bigger, it requires a larger value of $\kappa$ to visualize such flattening effect. For the Dirichlet process, $\kappa = 0$, it is well known that $E[K_n]$ increases when $\theta$ grows, this location behavior is also observed for other fixed values of $\kappa$.

4 Density estimation for Beta-Binomial mixtures

Given a BBSB prior, $\mu$, and a diffuse absolutely continuous density kernel $g(\cdot|s)$, with parameter space $S$, we can consider BBSB mixtures. Namely, we can model elements in $y(n) = \{y_1, ..., y_n\}$ as i.i.d. sampled from the random density

$$\phi(y) := \pi(y|W, \Xi) = \int_S g(y|s)\mu(ds) = \sum_{j\geq 1} w_j g(y|\xi_j).$$

(7)

For MCMC implementation purposes, and following Walker (2007), this random density can be augmented as

$$\pi(y, u|W, \Xi) = \sum_{j\geq 1} 1_{\{u < w_j\}} g(y|\xi_j),$$

(8)

where it can be easily deduced

$$\pi(u|W) = \sum_{j\geq 1} 1_{\{u < w_j\}}.$$  

(9)

As in the Dirichlet process case, given $u$, the number of components in the mixture is finite, with indexes being the elements of $A_u(W) = \{j : u < w_j\}$, that is

$$\pi(y|u, W, \Xi) = \frac{1}{|A_u(W)|} \sum_{j \in A_u(W)} g(y|\xi_j).$$

(10)

Using the membership variable $d$, i.e. $d = j$ iff $y$ is sampled from $g(\cdot|\xi_j)$, one can further consider the augmented joint density

$$\pi(y, u, d|W, \Xi) = 1_{\{u < w_d\}} g(y|\xi_d).$$

(11)
The complete data likelihood based on a sample of size \( n \) from (11) is easily seen to be
\[
\mathcal{L}_{\xi,w}((y_k, u_k, d_k)_{k=1}^n) = \prod_{k=1}^n 1_{\{u_k < w_{d_k}\}} g(y_k|\xi_{d_k}),
\]
and the full joint density of every variable involved is
\[
\pi((y_k, u_k, d_k)_{k=1}^n, (\xi_j)_{j \geq 1}, (v_i, x_i)_{i \geq 1}) = \left( \prod_{k=1}^n 1_{\{u_k < w_{d_k}\}} g(y_k|\xi_{d_k}) \right) \times \left( \text{Be}(v_1|\alpha, \theta) \prod_{i \geq 1} \text{Bin}(x_i|\kappa, v_i) \text{Be}(v_{i+1}|\alpha + x_i, \theta + \kappa - x_i) \right) \prod_{j \geq 1} P_0(\xi_j),
\]

(12)

recall \( w_{d_k} = v_{d_k} \prod_{i=1}^{d_k-1} (1 - v_i) \) with the convention that the empty product equals 1.

### 4.1 Full conditionals

The full conditional distributions, required for posterior inference via a Gibbs sampler implementation, are proportional to (12), and given as follows.

1. **Updating \( \Xi \):**

   \[
   \pi(\xi_j|\ldots) \propto P_0(\xi_j) \prod_{k \in D_j} g(y_k|\xi_j), \quad j \geq 1,
   \]
   where \( D_j = \{k \geq 1 : d_k = j\} \). If \( P_0 \) and \( g \) form a conjugate pair, the above is easy to sample from.

2. **Updating \((V, X)\) and \(U = (u_k)_{k=1}^n\) as a block:**

   \[
   \pi(U, (V, X)|\ldots) \propto \left( \prod_{k=1}^n w_{d_k}^{-1} 1_{\{u_k < w_{d_k}\}} w_{d_k} \right) \times \left( \text{Be}(v_1|\alpha, \theta) \prod_{i \geq 1} \text{Bin}(x_i|\kappa, v_i) \text{Be}(v_{i+1}|\alpha + x_i, \theta + \kappa - x_i) \right).
   \]

As \( w_{d_k} = v_{d_k} \prod_{i=1}^{d_k-1} (1 - v_i) \), with the convention \( \prod_{i=1}^{0} (\cdot) = 1 \), then

\[
\pi(U, (V, X)|\ldots) \propto \left[ \prod_{k=1}^n w_{d_k}^{-1} 1_{\{u_k < w_{d_k}\}} \right] \left[ v_1^{(\alpha + \sum_{k}(d_k=1)-1)} (1 - v_1)^{(\theta + \sum_{k}(d_k>1) -1)} \right] \times \left[ \prod_{i \geq 1} v_{i+1}^{(\alpha + x_i + \sum_{k}(d_k=i+1)-1)} (1 - v_{i+1})^{(\theta + \kappa - x_i + \sum_{k}(d_k>i+1) -1)} \text{Bin}(x_i|\kappa, v_i) \right].
\]
And can easily be seen that
\[
\pi(U, (V, X) | ...) = \left[ \prod_{k=1}^{n} U(u_k | 0, w_{d_k}) \right] \left[ \prod_{i \geq 1} \text{Be}(v_i | \alpha_i, \theta_i) \text{Bin}(x_i | \kappa, v_i) \right],
\]
where
\[
\alpha_1 = \alpha + \sum_{k=1}^{n} 1_{\{d_k = 1\}}, \quad \theta_1 = \theta + \sum_{k=1}^{n} 1_{\{d_k > 1\}},
\]
and for \(i \geq 2,\)
\[
\alpha_i = \alpha + x_{i-1} + \sum_{k=1}^{n} 1_{\{d_k = i\}}, \quad \theta_i = \theta + \kappa - x_{i-1} + \sum_{k=1}^{n} 1_{\{d_k > i\}}.
\]
Thus we update \((U, (V, X))\) as follows:

i) Sample \(v_1\) from a \(\text{Be}(\alpha_1, \theta_1)\) distribution and recursively for \(i \geq 1:\)

- Sample \(x_i\) from a \(\text{Bin}(\kappa, v_i)\) distribution.
- Sample \(v_{i+1}\) from a \(\text{Be}(\alpha_{i+1}, \theta_{i+1})\).

ii) Independently for \(k \in \{1, \ldots, n\},\) sample \(u_k\) from a \(U(0, w_{d_k})\) distribution.

3. Updating \(D = (d_k)_{k=1}^{n}:\)
\[
\pi(d_k = j | ...) \propto g(y_k | \xi_j) 1_{\{u_k < w_j\}}, \quad k \in \{1, \ldots, n\},
\]
which is a discrete distribution with finite support, hence easy to sample from.

**Remark 4.1 (For the updating of \(\Xi, V \) and \(X\))** As it is well-known for this algorithm, we do not need to sample \(v_j, x_j\) and \(\xi_j\) for every \(j \geq 1,\) it suffices to sample enough of them so that step 3 can take place. Explicitly, it suffices to sample \(\xi_j, v_j\) and \(x_j\) for \(j \leq \varphi,\) where \(\varphi\) is a constant such that \(\sum_{j=1}^{\varphi} w_j \geq \max_k (1 - u_k),\) then it is not possible that \(w_j > u_k\) for any \(k \leq n\) and \(j > \varphi.\)

4.2 Posterior distribution analysis

Given the samples, \(\left\{ \left( \xi_j^{(t)} \right)_j, \left( w_j^{(t)} \right)_j, \left( u_k^{(t)} \right)_k, \left( d_k^{(t)} \right)_k \right\}_{t=1}^{T},\) from \(\{\Xi, W, U, D|y^{(n)}\}\) obtained after \(T\) iterations of the Gibbs sampler, following (10) we estimate the density of the data by
\[
\mathbb{E}[\phi|y^{(n)}] \approx \frac{1}{T} \sum_{t=1}^{T} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{|A_k^{(t)}|} \sum_{j \in A_k^{(t)}} g \left( \cdot | \xi_j^{(t)} \right),
\]
(13)
where \( A_k^{(t)} = \{ j : u_k^{(t)} < w_j^{(t)} \} \). Furthermore, we can also estimate the posterior distribution of \( \{K_n|y^{(n)}\} \) through
\[
\mathbb{P}[K_n = m|y^{(n)}] \approx \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}_{\{K_n^{(t)} = m\}},
\]
(14)
where \( K_n^{(t)} \) is the number of distinct values \( \left( d_k^{(t)} \right)_k \) exhibits. As usual, when working with mixtures of densities, \( K_n \) is interpreted as the number of components of the mixture featuring the sample \( y^{(n)} \), that is the number of elements in \( \{g(\cdot|\xi_j)\}_{j \geq 1} \) such that \( y_k \) is sampled from \( g(\cdot|\xi_j) \), for some \( y_k \in y^{(n)} \). Thus, (14) favoring smaller values of \( m \), translates to the fact that fewer r.v.’s were needed to be sampled at each iteration of the Gibbs sampler. This way, the estimates (13) together with (14), give us information of how well a model performs for the given data set. Among the models for which (13) adjusts well to the data, those for which (14) favours smaller values of \( m \), might be preferred from a computational point of view. In the sense that just enough r.v.’s were needed to be sampled at each iteration of the Gibbs sampler without compromising the quality of (13).

5 Illustrations

In principle, every choice of \( \kappa \) leads to robust posterior MCMC estimates, after an appropriate burn-in period and enough valid iterations. However, depending on the sample, initial conditions, and current parameter values in the Gibbs sampler, the need to more/less ordered weights, thus different values of \( \kappa \), might be required. To test the performance of BBSB priors for density estimation, we designed a small experiment aimed to test the speed at which the model provides an acceptable estimation for distinct choices of the parameter \( \kappa \), thus no burn-in period was considered here.

We simulated two data sets the first one (database 1) having 13 modes equally spaced, and the second one (database 2) having 5 modes hard to recognize. We assume a Gaussian kernel with random location and scale parameters, i.e., for each \( j \geq 1 \), \( \xi_j = (m_j, p_j) \), and \( g(y|\xi_j) = N(y|m_j, p_j^{-1}) \).

To attain a conjugate pair for \( P_0 \) and \( g \), we assume \( P_0(\xi_j) = N(m_j|\vartheta, \tau p_j^{-1}) \text{Ga}(p_j|a, b) \).
5.1 Results for Database 1

In Figure 4 we observe that the Dirichlet process (A) struggles to recover the thirteen modes featured in the dataset, the three remaining models are able to capture the 13 well-separated modes. In terms of the speed at which the estimates recognize the modes, we observe that BBSB mixtures with larger values of $\kappa$ (C and D) perform better. As to $K_n$, consistently with the prior analysis, in Figure 5 we observe that for larger values of $\kappa$ the posterior mean and the posterior variance increase as $\kappa$ does. Comparing Figures 4 and 5 we note that the model with $\kappa = 10$ (B) mixes better the components of the mixture than the other ones in the sense that fewer r.v.'s need to be sampled at each step of the Gibbs sampler in order to provide a decent estimate of the density. Overall, the BBSB models with $\kappa = 10$ (B) and $\kappa = 100$ (C) appear to perform well for this data set, and the choice of one over the other depends on whether we prefer fewer mixing components or a faster convergence rate.

![Figure 4](image-url)

Figure 4: Evolution of the estimated densities for database 1, through the first 3000 iterations of the Gibbs sampler, for four distinct BBSB mixtures. The estimated densities in A, B, C and D correspond to BBSB mixtures with $\kappa$ fixed to 0, 10, 100 and 10000 respectively, in the four cases $\alpha = \theta = 1$. 

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Figure 5: Frequency polygon of the estimated posterior distribution of $K_n$ given database 1 for the four BBSB mixtures which share the parameters $\alpha = \theta = 1$, and differ on the parameter $\kappa$, same one that varies in the set $\{0, 10, 100, 10000\}$.

5.2 Results for Database 2

Figure 6: Evolution of the estimated densities for database 2, through the first 3000 iterations of the Gibbs sampler, for four distinct BBSB mixtures. The estimated densities in A, B, C and D correspond to BBSB mixtures with $\kappa$ fixed to 0, 10, 100 and 10000 respectively, in the four cases $\alpha = \theta = 1$.
Figure 7: Frequency polygon of the estimated posterior distribution of $K_n$ given database 2, for the four BBSB mixtures which share the parameters $\alpha = \theta = 1$, and differ on the parameter $\kappa$, same one that varies in the set \{0, 10, 100, 10000\}.

In Figure 6 we observe that although the Dirichlet model (A) seems to recognize the modes featured in the data set when the number of iterations is small, when more iterations are taken into account, this model only recovers two modes. The BBSB mixtures with parameter $\kappa > 0$ (A, B and C) seems to capture every mode at an excellent rate, the one with $\kappa = 10$ (B) excelling at this task. In the same figure, we also observe that after 3000 iterations, the BBSB mixture with $\kappa = 10000$ (D) appears to overestimate the second mode, whilst the mixture with $\kappa = 10$ slightly underestimates the fourth one. In Figure 7 we see that $P[K_n|\text{database 2}]$ exhibits an analogous behaviour that of $P[K_n|\text{database 1}]$ in terms of how the posterior mean and posterior variance are affected by varying $\kappa$. For this data set we also observe that the posterior distribution of $K_n$ for the cases $\kappa = 100$ and $\kappa = 10000$ are similar, differing in the right tail of the distribution. Overall, we can conclude that the three BBSB mixtures with $\kappa > 0$ perform well for this database, while the one with $\kappa = 10$ excels at the convergence rate and mixture of the components, the one with $\kappa = 100$ seems to provide the best estimation after 3000 iterations.
6 Discussion

Using Beta chains as the l.v.’s of some stick-breaking sequences, we were able to construct a new family of distributions over the infinite dimensional simplex, hence a new class of species sampling priors. The parameter, $\kappa$, that modulates the dependence among the elements of the Beta chain, also modulates the ordering of the corresponding weights. While the choice $\kappa = 0$ and $\alpha = 1$ recovers the size-biased permutation of the weights of Dirichlet processes, as $\kappa \to \infty$, we recover the decreasing ordered weights of Geometric processes, both classes of processes being models of interest. This approach to define priors also allows the construction of random measures that are hybrids between Dirichlet and Geometric processes. Furthermore, how similar is the BBSB prior to one model or the other is also modulated by the parameter $\kappa$. As to the prior distribution of $K_n$, generally speaking, we found that a larger value of $\kappa$ translates to a less informative prior. This in turn allows more flexible models in a density estimation context. In the sense that even if the parameters of the Gibbs sampler are not carefully chosen for a given data set, BBSB mixtures featuring a less informative prior distribution of $K_n$, seem to learn rapidly from the sample, thus provide decent density estimators after few iterations of the Gibbs sampler.

The present work gives rise to interesting questions, such as how to optimally choose $\kappa$ for a data set, given that the rest of the parameters are fixed. Or how to characterize the exchangeable partition structures corresponding to BBSB priors. From a theoretical point of view, it is also of interest to determine how large $\kappa$ needs to be in order to assure the weights are decreasing. Hopefully, the present paper motivates the study of stick-breaking sequences featuring dependent (or further, Markovian) l.v.’s, that might even lead to generalizations of BBSB priors.

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Appendix A.

Appendix A.1. Convergence of probability measures

To formally give the proof of the main results, we recall some topological details of measure spaces. For a Polish space \( S \), with Borel \( \sigma \)-algebra \( \mathcal{B}(S) \), we denote by \( \mathcal{P}(S) \) the space of all probability measures over \((S, \mathcal{B}(S))\). A well-known metric on \( \mathcal{P}(S) \) is the Lévy-Prokhorov metric given by

\[
d_L(P, P') = \inf\{\varepsilon > 0 : P(A) \leq P'(A^\varepsilon) + \varepsilon, \forall A \in \mathcal{B}(S)\},
\]

for any \( P, P' \in \mathcal{P}(S) \), and where \( A^\varepsilon = \{s \in S : d(s, A) < \varepsilon\} \), \( d(s, A) = \inf\{d(a, s) : a \in A\} \) and \( d \) is some complete metric on \( S \). For probability measures \( P, P_1, P_2, \ldots \) it is said that \( P_n \) converges weakly to \( P \), denoted by \( P_n \overset{w}{\rightarrow} P \), whenever \( \int_S f dP_n \rightarrow \int_S f dP \) for every continuous bounded function \( f : S \rightarrow [0, \infty) \). This condition is known to be equivalent to \( d_L(P_n, P) \rightarrow 0 \), and to \( \gamma_n \overset{d}{\rightarrow} \gamma \), whenever \( \gamma_n \sim P_n \) and \( \gamma \sim P \). \( \mathcal{P}(S) \), equipped with the topology of weak convergence, is Polish again. Its Borel \( \sigma \)-field, \( \mathcal{B}(\mathcal{P}(S)) \), can equivalently be defined as the \( \sigma \)-algebra generated by all the projection maps \( \{P \mapsto P(B) : B \in \mathcal{B}(S)\} \). In this sense the random probability measures (measurable mappings from a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) into \((\mathcal{P}(S), \mathcal{B}(S))\)), \( \mu, \mu_1, \mu_2, \ldots \), are said to converge weakly, a.s. whenever \( \mu_n(\omega) \overset{w}{\rightarrow} \mu(\omega) \) outside a \( \mathbb{P} \)-null set. Analogously, if \( \int_S f d\mu_n \overset{d}{\rightarrow} \int_S f d\mu \) for every continuous bounded function \( f : S \rightarrow [0, \infty) \), it is said that \( \mu_n \) converges weakly in distribution to \( \mu \), denoted by \( \mu_n \overset{dw}{\rightarrow} \mu \). Evidently, \( \mu_n \overset{w}{\rightarrow} \mu \) a.s. implies \( \mu_n \overset{dw}{\rightarrow} \mu \), which, in turns is a necessary and sufficient condition for \( \mu_n \overset{d}{\rightarrow} \mu \). For further details see for instance Parthasarathy (1967), Billingsley (1968) or Kallenberg (2017).

Appendix A.2. Proof of Proposition 2.1

a) Using elementary properties of conditional expectation and the fact that given \( x_i, v_{i+1} \) is conditionally independent of \( v_i \), be obtain

\[
\mathbb{E}[v_{i+1}|v_i] = \mathbb{E}[\mathbb{E}[v_{i+1}|x_i]|v_i] = \mathbb{E}\left[\frac{\alpha + x_i}{\alpha + \theta + \kappa} v_i\right] = \frac{\alpha + \kappa v_i}{\alpha + \theta + \kappa}.
\]

b) Notice that

\[
\text{Var}(v_{i+1}|v_i) = \mathbb{E}[(\text{Var}(v_{i+1}|x_i)|v_i] + \text{Var}(\mathbb{E}[v_{i+1}|x_i]|v_i),
\]

\( 18 \)
we first compute
\[
\text{Var}(\mathbb{E}[v_{i+1}\mid x_i]\mid v_i) = \text{Var}\left(\frac{\alpha + x_i}{\alpha + \theta + \kappa} \mid v_i\right) = \frac{v_i(1 - v_i)\kappa}{(\alpha + \theta + \kappa)^2},
\]
successfully, we note that
\[
\mathbb{E}((\alpha + x_i)(\theta + \kappa - x_i)\mid v_i) = \text{Cov}(\alpha + x_i, \theta + \kappa - x_i\mid v_i) + \mathbb{E}[\alpha + x_i\mid v_i]\mathbb{E}[\theta + \kappa - x_i\mid v_i]
\]
\[
= -\text{Var}(x_i\mid v_i) + (\alpha + \kappa v_i)(\theta + \kappa - \kappa v_i)
\]
\[
= -\kappa v_i(1 - v_i) + (\alpha + \kappa v_i)(\theta + \kappa(1 - v_i)),
\]
hence
\[
\mathbb{E}[\text{Var}(v_{i+1}\mid x_i)\mid v_i] = \mathbb{E}\left[\frac{(\alpha + x_i)(\theta + \kappa - x_i)}{(\alpha + \theta + \kappa)^2(\alpha + \theta + \kappa + 1)} \mid v_i\right] = \frac{-\kappa v_i(1 - v_i) + (\alpha + \kappa v_i)(\theta + \kappa(1 - v_i))}{(\alpha + \theta + \kappa)^2(\alpha + \theta + \kappa + 1)},
\]
and we can conclude the proof of b),
\[
\text{Var}(v_{i+1}\mid v_i) = \frac{-\kappa v_i(1 - v_i) + (\alpha + \kappa v_i)(\theta + \kappa(1 - v_i)) + v_i(1 - v_i)\kappa(\alpha + \theta + \kappa + 1)}{(\alpha + \theta + \kappa)^2(\alpha + \theta + \kappa + 1)}
\]
c) We first note that as a consequence of the joint reversibility of the Beta-Binomial chain, \( v_i \sim \text{Be}(\alpha + x_i, \theta + \kappa - x_i) \) conditionally given \( x_i \), thus
\[
\mathbb{E}[v_i v_{i+1}] = \mathbb{E}[\mathbb{E}[v_i v_{i+1}\mid x_i]] = \mathbb{E}[\mathbb{E}[v_i\mid x_i]\mathbb{E}[v_{i+1}\mid x_i]] = \mathbb{E}\left[\left(\frac{\alpha + x_i}{\alpha + \theta + \kappa}\right)^2\right],
\]
conditioning on \( v_i \), we obtain
\[
\mathbb{E}\left[\left(\frac{\alpha + x_i}{\alpha + \theta + \kappa}\right)^2\right] = \mathbb{E}\left[\mathbb{E}\left[\left(\frac{\alpha + x_i}{\alpha + \theta + \kappa}\right)^2\mid v_i\right]\right]
\]
\[
= \mathbb{E}\left[\frac{\alpha^2 + 2\alpha \mathbb{E}[x_i\mid v_i] + \mathbb{E}[x_i^2\mid v_i]}{(\alpha + \theta + \kappa)^2}\right]
\]
\[
= \frac{\alpha^2 + 2\alpha \kappa \mathbb{E}[v_i] + \kappa \mathbb{E}[v_i^2] + \kappa(\kappa - 1)\mathbb{E}[v_i^2]}{(\alpha + \theta + \kappa)^2}
\]
\[
= \left[\frac{\alpha^2 + \frac{\kappa(2\alpha^2 + \alpha)}{\alpha + \theta} + \frac{\kappa(\kappa - 1)\alpha(\alpha + 1)}{(\alpha + \theta)(\alpha + \theta + 1)}}{(\alpha + \theta + \kappa)^2}\right] (\alpha + \theta + \kappa)^{-2},
\]
hence
\[
\text{Cov}(v_i, v_{i+1}) = \mathbb{E}[v_i v_{i+1}] - \mathbb{E}[v_i] \mathbb{E}[v_{i+1}]
\]
\[
= (\alpha + \theta + \kappa)^{-2} \left[\frac{\alpha^2 + \frac{\kappa(2\alpha^2 + \alpha)}{\alpha + \theta} + \frac{\kappa(\kappa - 1)\alpha(\alpha + 1)}{(\alpha + \theta)(\alpha + \theta + 1)}}{(\alpha + \theta + \kappa)^2}\right] - \frac{\alpha^2}{(\alpha + \theta)^2}
\]
\[
= \frac{\kappa \alpha \theta}{(\alpha + \theta)^2(\alpha + \theta + 1)(\alpha + \theta + \kappa)}.
\]
d) The correlation simplifies as follows
\[
\rho_{v_i, v_{i+1}} = \frac{\text{Cov}(v_i, v_{i+1})}{\sqrt{\text{Var}(v_i)} \sqrt{\text{Var}(v_{i+1})}} = \frac{\kappa \alpha \theta (\alpha + \theta)^2 (\alpha + \theta + 1)}{\alpha \theta (\alpha + \theta)^2 (\alpha + \theta + 1)(\alpha + \theta + \kappa)} = \frac{\kappa}{\alpha + \theta + \kappa}.
\]

**Appendix A.3. Proof of Proposition 2.2**

To prove Proposition 2.2 we need some preliminary results.

**Lemma A.1** Let \((x_n)_{n \geq 1}\) be a sequence of random variables such that \(x_n \sim \text{Bin}(n, p_n)\) for every \(n \geq 1\) and where \(p_n \to p\) in \([0, 1]\). Then
\[
\frac{x_n}{n} \xrightarrow{L_2} p.
\]

**Proof:**

For \(n \geq 1,
\[
\mathbb{E} \left[ \left( \frac{x_n}{n} - p \right)^2 \right] = \frac{1}{n^2} \mathbb{E} \left[ x_n^2 \right] - \frac{2p}{n} \mathbb{E} [x_n] + p^2
\]
\[
= p_n (1 - p_n) + (p_n - p)^2. \tag{16}
\]

By taking limits as \(n \to \infty\) in (16) we obtain
\[
\lim_{n \to \infty} \mathbb{E} \left[ \left( \frac{x_n}{n} - p \right)^2 \right] = 0.
\]

**Lemma A.2** Let \(S\) and \(T\) be Polish spaces, \(\gamma, \gamma_1, \gamma_2, \ldots\) and \(\eta, \eta_1, \eta_2, \ldots\) be random elements taking values in \(S\) and \(T\) respectively and consider some regular versions, \(\pi(\cdot | \gamma)\) and \(\pi_n(\cdot | \gamma_n)\), of \(\mathbb{P}[\eta \in \cdot | \gamma]\) and \(\mathbb{P}[\eta_n \in \cdot | \gamma_n]\) respectively. If \(\gamma_n \xrightarrow{d} \gamma\) and for every \(s_n \to s\) in \(S\) we have that \(\pi_n(\cdot | s_n) \xrightarrow{w} \pi(\cdot | s)\), then \((\gamma_n, \eta_n) \xrightarrow{d} (\gamma, \eta)\).

**Lemma A.3** Let \(\gamma^n = (\gamma^n_1, \gamma^n_2, \ldots)\), \(\gamma = (\gamma_1, \gamma_2, \ldots)\) be random sequences taking values in a Polish space \(S\). Then \(\gamma^n \xrightarrow{d} \gamma\) if and only if
\[
(\gamma^n_1, \ldots, \gamma^n_i) \xrightarrow{d} (\gamma_1, \ldots, \gamma_i), \text{ for every } i \geq 1.
\]

Lemma A.3 and an analogue statement to that of Lemma A.2 are proven in Kallenberg (2002).

**Proof of Proposition 2.2:**

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(i) Insomuch as the corresponding spaces are Borel, we may construct on some probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})\) a Beta-Binomial chain \((\hat{V}, \hat{X})\) with parameters \((0, \alpha, \theta)\). Now, the elements of \(\hat{V}\) are conditionally independent given \(\hat{X}\), and given that \(\kappa = 0, \hat{X} \overset{a.s.}{=} (0, 0, \ldots)\), so we may think of \(\hat{X}\) as if it was deterministic, which implies that the elements of \(\hat{V}\) must be independent and \(\text{Be}(\alpha, \theta)\) distributed.

(ii): For every \(\kappa \geq 1\), let \(V^{(\kappa)} = (v^{(\kappa)}_i)_{i \geq 1}\) be a Beta chain with parameters \((\kappa, \alpha, \theta)\), and let \(\pi^{(\kappa)}_\cdot | v^{(\kappa)}_i\) be some regular version of \(P\left[v^{(\kappa)}_i + 1 \in \cdot | v^{(\kappa)}_i\right]\) (which clearly does not depend on \(i\)). Further let \(\lambda \sim \text{Be}(\alpha, \theta)\) and fix \(\pi(\cdot | \lambda) = \delta_\lambda\). The first thing we are interested in proving is that for every \(p_\kappa \to p\) in \([0, 1]\) we have that

\[
\pi^{(\kappa)}_\cdot | p_\kappa) \overset{w}{\to} \pi(\cdot | p).
\]

So, let \(p_\kappa \to p\) in \([0, 1]\), by Lemma A.1 and given that all the corresponding spaces are Borel, we may construct on a probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})\), with expectations \(\hat{E}[\cdot]\), some pairs of r.v.’s \((\hat{x}_\kappa, \hat{v}_\kappa)_{\kappa \geq 1}\) such that \(\hat{x}_\kappa \sim \text{Bin}(\kappa, p_\kappa)\), \(\{\hat{v}_\kappa | \hat{x}_\kappa\} \sim \text{Be}(\alpha + \hat{x}_\kappa, \theta + \kappa - \hat{x}_\kappa)\), and \(\hat{x}_\kappa / \kappa \overset{a.s.}{\to} p\). Note that marginally \(\hat{v}_\kappa \sim \pi(\cdot | p_\kappa)\) so to prove equation (17), it suffices to show \(\hat{v}_\kappa \overset{d}{\to} p\).

Conditionally given \(\hat{x}_\kappa\) the moment generator function of \(\hat{v}_\kappa\) is

\[
\hat{E}\left[e^{t\hat{v}_\kappa} | \hat{x}_\kappa\right] = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha + \hat{x}_\kappa + r}{\alpha + \theta + \kappa + r}\right) \frac{t^k}{k!}, \quad t \in \mathbb{R},
\]

by construction we have that \(\hat{x}_\kappa / \kappa \overset{a.s.}{\to} p\), which means that for every \(r \geq 0\),

\[
\frac{\alpha + \hat{x}_\kappa + r}{\alpha + \theta + \kappa + r} = \left(\frac{\alpha + r}{\kappa} + \frac{\hat{x}_\kappa}{\kappa}\right) \left(\frac{\alpha + \theta + r}{\kappa} + 1\right)^{-1} \overset{a.s.}{\to} p,
\]

as \(\kappa \to \infty\), hence by the tower property of conditional expectation, equations (18) and (19), and Lebesgue dominated convergence theorem (the corresponding functions are dominated by \(e^t\)) we obtain

\[
\lim_{\kappa \to \infty} \hat{E}\left[e^{t\hat{v}_\kappa} \right] = \lim_{\kappa \to \infty} \hat{E}\left[e^{t\hat{v}_\kappa} | \hat{x}_\kappa\right] = e^{tp},
\]
which proves altogether \( \hat{v}_\kappa \overset{d}{\to} p \) and equation (17).

Returning to the original Beta chains, we have that \( v^{(\kappa)}_1 \overset{d}{=} \lambda \) for every \( \kappa \geq 1 \), so trivially, \( v^{(\kappa)}_1 \overset{d}{\to} \lambda \), this together with equation (17) and the recursive application of Lemma A.2 allows us to obtain

\[
\left( v^{(\kappa)}_1, ..., v^{(\kappa)}_i \right) \overset{d}{\to} (\lambda, ..., \lambda), \quad i \geq 1,
\]

and by Lemma A.3 we can conclude \( V^{(\kappa)} = \left( v^{(\kappa)}_i \right)_{i \geq 1} \overset{d}{\to} (\lambda, \lambda, ...) \).

**Appendix A.4. Proof of Proposition 3.1**

For sequences that enjoy the decomposition (3) we may equivalently prove that

\[
\left( 1 - \sum_{i=1}^{j} w_i \right) = \prod_{i=1}^{j} (1 - v_i) \overset{a.s.}{\to} 0,
\]
as \( j \to \infty \) (see for instance Ghosal and van der Vaart; 2017). Further, these r.v.'s are non-negative and bounded by 1, thus it is enough to show that

\[
\lim_{j \to \infty} \mathbb{E} \left[ \prod_{i=1}^{j} (1 - v_i) \right] = 0. \quad (20)
\]

As the corresponding spaces are Borel, (after possibly enlarging the original probability space) it is possible to construct a Binomial chain \( X \) such that \((V, X)\) defines a Beta-Binomial chain.

Conditionally given \( X = \{x_i\}_{i \geq 1} \), the elements of \( V = \{v_i\}_{i \geq 1} \) are independent with, \( v_i|x_i \sim \text{Be}(\alpha + x_i, \theta + \kappa - x_i) \) and \( v_{i+1}|x_i, x_{i+1} \sim \text{Be}(\alpha + x_i + x_{i+1}, \theta + 2\kappa - x_i - x_{i+1}) \), for \( i \geq 1 \). Hence

\[
\mathbb{E} \left[ \prod_{i=1}^{j} (1 - v_i) \right] = \mathbb{E} \left[ \mathbb{E} \left[ \prod_{i=1}^{j} (1 - v_i) | X \right] \right]
\]

\[
= \mathbb{E} \left[ \mathbb{E}[(1 - v_1)|x_1] \prod_{i=2}^{j} \mathbb{E}[(1 - v_i)|x_{i-1}, x_i] \right]
\]

\[
= \mathbb{E} \left[ \frac{\theta + \kappa - x_1}{\alpha + \theta + \kappa} \prod_{i=2}^{j} \frac{\theta + 2\kappa - x_i - x_{i-1}}{\alpha + \theta + 2\kappa} \right].
\]

Recalling that \( 0 \leq x_i \leq \kappa \) a.s. we obtain

\[
\frac{\theta}{\alpha + \theta + \kappa} \left( \frac{\theta}{\alpha + \theta + 2\kappa} \right)^{j-1} \leq \mathbb{E} \left[ \prod_{i=1}^{j} (1 - v_i) \right] \leq \frac{\theta + \kappa}{\alpha + \theta} \left( \frac{\theta + 2\kappa}{\alpha + \theta + 2\kappa} \right)^{j-1},
\]

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for every \( j \geq 1 \). Finally by taking limits as \( j \to \infty \) in the last equation, (20) follows.

Appendix A.5. Proof of Theorem 3.2

To prove Theorem 3.2 we will first prove a couple of elementary results.

Lemma A.4 Let \( S \) be a Polish space and fix some distinct \( s_1, s_2, \ldots \in S \), let \( p = (p_1, p_2, \ldots) \) and \( q = (q_1, q_2, \ldots) \) be elements of \( \Delta_\infty \) and define \( P = \sum_{j \geq 1} p_j \delta_{s_j} \) and \( Q = \sum_{j \geq 1} q_j \delta_{s_j} \). Then for \( d_L \) as in equation (15)

\[
d_L(P, Q) \leq \sum_{j \geq 1} |p_j - q_j|.
\]

Proof:

Define \( \varepsilon(p, q) = \sum_{j \geq 1} |p_j - q_j| \), by definition of \( d_L \), it suffices to prove

\[
P(A) \leq Q \left( A^{\varepsilon(p, q)} \right) + \varepsilon(p, q), \quad \text{and} \quad Q(A) \leq P \left( A^{\varepsilon(p, q)} \right) + \varepsilon(p, q), \quad \forall A \in \mathcal{B}(S), \quad (21)
\]

So let \( A \in \mathcal{B}(S) \) and set \( M_A = \{ j \geq 1 : s_j \in A \} \), then

\[
P(A) = \sum_{j \in M_A} p_j \leq \sum_{j \in M_A} q_j + \sum_{j \in M_A} |p_j - q_j| \leq Q(A) + \varepsilon(p, q) \leq Q \left( A^{\varepsilon(p, q)} \right) + \varepsilon(p, q).
\]

Analogously, we have that \( Q(A) \leq P \left( A^{\varepsilon(p, q)} \right) + \varepsilon(p, q) \).

Lemma A.5 For fixed and distinct elements \( s_1, s_2, \ldots \in S \), the mapping,

\[
(w_1, w_2, \ldots) \mapsto \sum_{j \geq 1} w_j \delta_{s_j},
\]

from \( \Delta_\infty \) into \( \mathcal{P}(S) \) is continuous with respect to the weak topology.

Proof:
Let \(w^{(n)} = (w_1^{(n)}, w_2^{(n)}, \ldots)\) and \(w = (w_1, w_2, \ldots)\) be any elements of \(\Delta_\infty\) such that \(w_j^{(n)} \to w_j\), for every \(j \geq 1\). Define \(P^{(n)} = \sum_{j \geq 1} w_j^{(n)} \delta_{s_j}\) and \(P = \sum_{j \geq 1} w_j \delta_{s_j}\). By Lemma A.4

\[
d_L(P^{(n)}, P) \leq \sum_{j \geq 1} |w_j^{(n)} - w_j| \leq \sum_{j \geq 1} w_j^{(n)} + \sum_{j \geq 1} w_j = 2,
\]

and by the general Lebesgue dominated convergence theorem we obtain

\[
\lim_{n \to \infty} d_L(P^{(n)}, P) = \lim_{n \to \infty} \sum_{j \geq 1} |w_j^{(n)} - w_j| = \sum_{j \geq 1} \lim_{n \to \infty} |w_j^{(n)} - w_j| = 0,
\]

which means that the mapping \((w_1, w_2, \ldots) \mapsto \sum_{j \geq 1} w_j \delta_{s_j}\) is continuous.

**Remark A.6** Despite the choice of the metric, \(\rho\), in \(\Delta_\infty\), as long as \(\rho\) generates the Borel \(\sigma\)-algebra, \(\rho \left(w^{(n)}, w\right) \to 0\) implies \(|w_j^{(n)} - w_j| \to 0\), for every \(j \geq 1\). For this reason, in the above proof we did not discuss the details on the metric, of \(\Delta_\infty\), that is being used.

**Proof of Theorem 3.2:**

The proof of (i) follows directly from Proposition 2.2 (i). To prove (ii), note that by Proposition 2.2 (ii) and given that all the corresponding spaces are Borel, we may construct on a probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})\), Beta chains \(\hat{\mathbf{V}}^{(\kappa)} = (\hat{v}_i^{(\kappa)})_{i \geq 1}\) with parameters \((\kappa, \alpha, \theta)\) and a \(\hat{\Lambda} \sim \text{Be}(\alpha, \theta)\) such that \(\hat{v}_i^{(\kappa)} \overset{a.s.}{\to} \hat{\Lambda}\), for every \(i \geq 1\), further we may also define there, an independent sequence, \(\hat{\Xi} = (\hat{\xi}_j)_{j \geq 1}\), with \(\hat{\xi}_j \overset{\text{iid}}{\sim} P_0\). Now, for \(\kappa \geq 1\) define

\[
\hat{\mathbf{w}}_j^{(\kappa)} = \hat{v}_j^{(\kappa)} \prod_{i=1}^{j-1} \left(1 - \hat{v}_i^{(\kappa)}\right), \quad j \geq 1,
\]

and \(\hat{\boldsymbol{\mu}}^{(\kappa)} = \sum_{j \geq 1} \hat{w}_j^{(\kappa)} \delta_{\xi_j}\), with the convention that the empty product equals 1, also set \(\hat{\mu} = \sum_{j \geq 1} \lambda (1 - \lambda)^j \delta_{\xi_j}\), so that

\[
\hat{\boldsymbol{\mu}}^{(\kappa)} \overset{d}{=} \hat{\mu}^{(\kappa)}, \quad \kappa \geq 1 \quad \text{and} \quad \hat{\mu} \overset{d}{=} \mu.
\]

(22)

As the mapping, \((\hat{v}_1^{(\kappa)}, \ldots, \hat{v}_j^{(\kappa)}) \mapsto \hat{w}_j^{(\kappa)}\), is continuous we have that

\[
\hat{w}_j^{(\kappa)} \overset{a.s.}{\to} \lambda (1 - \lambda)^{j-1}, \quad j \geq 1,
\]

as to the sequence \(\hat{\Xi}\), the diffuseness of \(P_0\) implies that for \(i \neq j, \xi_i \neq \xi_j\) a.s., insomuch as we are dealing with a countable number of random variables, there exist some \(B \in \hat{\mathcal{F}}\) such that \(\hat{P}[B] = 1\)
and for every \( \omega \in B \)

\[
\hat{w}_j^{(\kappa)}(\omega) \to \lambda(\omega)(1 - \lambda(\omega))^{j-1}, \quad j \geq 1, \quad \text{and} \quad \xi_j(\omega) \neq \xi_i(\omega), \quad i \neq j
\]

By Lemma A.5

\[
\sum_{j \geq 1} \hat{w}_j^{(\kappa)}(\omega) \delta_{\xi_j(\omega)} \xrightarrow{w} \sum_{j \geq 1} \lambda(\omega)(1 - \lambda(\omega))^{j-1} \delta_{\xi_j(\omega)}, \quad \omega \in B
\]

that is, \( \hat{\mu}^{(\kappa)} \xrightarrow{w} \hat{\mu} \) a.s., implying \( \hat{\mu}^{(\kappa)} \xrightarrow{d} \hat{\mu} \). Finally, by equation (22), the result follows.

**Appendix A.6. Proof of Corollary 3.3**

The proof of (i) can be found on Pitman (1996a). To prove (ii) note that we may write

\[
w_1^{(k)} = v_1^{(k)}, \quad w_{j+1}^{(k)} = \frac{v_{j+1}^{(k)}(1 - v_j^{(k)})}{v_j^{(k)}} w_j^{(k)}, \quad j \geq 1,
\]

hence

\[
P[w_{j+1}^{(k)} < w_j^{(k)}] = P[v_{j+1}^{(k)}(1 - v_j^{(k)}) < v_j^{(k)}].
\]

By the second part of Proposition 2.2 and as the corresponding spaces are Borel, we may construct on some probability space, \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\), with expectations \(\hat{\mathbb{E}}[\cdot]\), Beta chains, \(\hat{v}_i^{(\kappa)}\), with parameters \((\kappa, \alpha, \theta)\), and a \(\hat{\lambda} \sim \text{Be}(\alpha, \theta)\) such that

\[
\left(\hat{v}_i^{(\kappa)}\right)_{i \geq 1} \to (\hat{\lambda}, \hat{\lambda}, ...) \quad \text{a.s.}
\]

This way, by Lebesgue dominated convergence theorem we have that

\[
\lim_{\kappa \to \infty} P[w_{j+1}^{(k)} < w_j^{(k)}] = \lim_{\kappa \to \infty} \mathbb{E}\left[1\left\{v_{j+1}^{(k)}(1 - v_j^{(k)}) < v_j^{(k)}\right\}\right] = \mathbb{E}\left[1\left\{\hat{v}_{j+1}^{(\kappa)}(1 - \hat{v}_j^{(\kappa)}) < \hat{v}_j^{(\kappa)}\right\}\right] = \mathbb{E}\left[1\left\{\hat{\lambda}(1 - \hat{\lambda}) < \hat{\lambda}\right\}\right] = 1.
\]
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