SECANT PLANES OF A GENERAL CURVE VIA DEGENERATIONS

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Abstract. We study linear series on a general curve of genus \( g \), whose images are exceptional with respect to their secant planes. Each such exceptional secant plane is algebraically encoded by an included linear series, whose number of base points computes the incidence degree of the corresponding secant plane. With enumerative applications in mind, we construct a moduli scheme of inclusions of limit linear series with base points over families of curves of compact type, which we then use to compute combinatorial formulas for the number of secant-exceptional linear series when the spaces of linear series and of inclusions are finite.

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1. Introduction

Determining when an abstract curve $C$ comes equipped with a nondegenerate map of degree $m$ to $\mathbb{P}^s$ is central to curve theory. The Brill–Noether theorem of Griffiths and Harris gives a complete solution to this problem when $C$ is general in $\mathcal{M}_g$, but there are also infinitely many variations on this basic problem. One of these involves requiring that the image of $C$ intersect a linear subspace of prescribed dimension in the ambient $\mathbb{P}^s$, with prescribed incidence degree. The linear subspaces are then \textit{secant planes} of the image of $C$, and are associated with inclusions of linear series of the form

\begin{equation}
\gamma_{m-d+r}^s + p_1 + \cdots + p_d \hookrightarrow \gamma_m^s
\end{equation}

in which $d$ is the incidence degree and $(d - r - 1)$ is the dimension of the secant plane. Here the points $p_1, \ldots, p_d$ in $C$ are base points of the included series.

In [Cot11, Cot12, Far08], the first author and G. Farkas studied these questions using Eisenbud and Harris’ theory of \textit{limit linear series} for nodal curves of compact type. Farkas’ contribution is to the dimension-theoretic understanding of (spaces of) secant planes of linear series, while the first author’s results primarily deal with the issue of counting these.

In the intervening ten years, a couple of interesting developments have occurred. B. Osserman has developed an alternative theory of limit linear series, closely related to that of Eisenbud and Harris (and indeed the definitions are set-theoretically equivalent for curves of compact type). Osserman’s theory has somewhat better functorial properties than that of Eisenbud and Harris; it is also more cumbersome to use. On the other hand, it has become increasingly clear that \textit{chains of elliptic curves} behave well (with respect to linear series) as surrogates for a general curve. Our aim in this paper is put both of these developments to good use, and initiate a program for counting secant planes of linear series on a general curve, by first counting inclusions of limit linear series as in (1.1) and then lifting the result to a general curve via a \textit{smoothing} theorem.

To prove our smoothing theorem, we appeal to the theory of limit linear series developed by Osserman in [Oss17]. Although our focus is on limit linear series over curves of compact type, Osserman’s (more general) theory provides a natural template for constructing proper moduli spaces, and for proving smoothing theorems. We in fact give two distinct constructions of a moduli space of inclusions of (limit) linear series; the first space is proper, while the second (which agrees set-theoretically with the first along an open locus) has a structure more amenable to local dimension estimates. The second construction makes use of \textit{linked chains of flags}, which we introduce as (flagged) generalizations of the linked chains described by Murray and Osserman. Associated with linked chains of flags are \textit{linked determinantal loci}, of which inclusions of limit linear series (1.1) are archetypal examples.

When it comes to \textit{counting} (inclusions of) limit linear series, Eisenbud–Harris’ theory of limit linear series is more combinatorially advantageous. On the other hand, it is known that the two theories (schematically) agree when we restrict ourselves to \textit{refined} limit linear series, a fact that we will leverage to count $d$-secant $(d - r - 1)$-planes to (the image of) a fixed linear series $\gamma_m^s$ of well-behaved combinatorial type on a general curve when the spaces of linear series $\gamma_m^s$ (resp, inclusions (1.1) associated with a fixed $\gamma_m^s$) are both zero-dimensional. Our method produces enumerative
formulas whose structure is qualitatively different from the usual formulas obtained via classical intersection theory described in [ACGH85] and [Cot11, Cot12]. Crucially, the formulas that arise from inclusions of limit linear series on elliptic chains are manifestly positive.

1.1. Roadmap. The plan for the remainder of this paper is as follows. In Section 2, we review Osserman’s theory of limit linear series; for the reader’s convenience, we include a proof of the fact that Osserman and Eisenbud–Harris limit linear series on a fixed curve $X$ are equivalent when $X$ is of compact type; see Proposition 2.17. Section 3 develops the theory of linked chains of flags. A key subsidiary notion is that of $\underline{r}$-strictness, in which $\underline{r}$ is a rank vector. Theorem 3.6 establishes that a linked determinantal locus has the expected dimension along its $\underline{r}$-strict locus.

In Section 4, we first provide a construction of the moduli scheme of inclusion of linear series on a non-singular curve in Proposition 4.3. After giving our definition of inclusion of limit linear series in Definition 4.5, we describe two constructions of a projective moduli scheme of inclusion of limit linear series in subsections 4.3.1 and 4.3.2. The upshot is that the induced reduced structure of two moduli schemes agree. Using the second construction, we then manage to prove a smoothing theorem for inclusion of limit linear series in the case $\rho = \mu = 0$ (Theorem 4.12), provided that the ambient series has certain nice properties. This is the theoretical foundation for enumerating inclusion of linear series via degeneration.

In Section 5, we give a general algorithm for counting inclusions (1.1) of limit linear series in cases where the two fundamental invariants $\rho$ (which computes the dimension of the space of $g^m_s$) and $\mu$ (which computes the dimension of the space of inclusions (1.1) associated with a fixed $g^m_s$) are both equal to zero. Our results, like the classical intersection-theoretic results are most explicit when $r = 1$; see Theorem 5.19. We introduce the notion of shift poset, which precisely dictates how base points force the included $g^{s-d+r}_m$, viewed as a point of the Grassmannian $\text{Gr}(s-d+r+1, s+1)$, to move. It seems reasonable to speculate that the coefficients in our formulas are related in some as-yet-unknown way to Schubert calculus.

Finally, in Section 6, we present an example due to Melody Chan, which shows that the moduli space of included limit linear series we constructed in Section 4 may have components of unexpectedly large dimension. A deeper understanding of this phenomenon as it relates to smoothability is crucial to extending the enumerative analysis carried out in this paper to situations in which the total dimension $\rho + \mu$ of the space of linear series with secant planes is zero, but $\rho$ is strictly positive.

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2. Basic definitions

Throughout this section, $X_0$ will denote a proper, reduced and connected nodal curve, whose irreducible components are all smooth. In particular, we do not assume $X_0$ to be of compact type a priori. Notationally, $d$ will always denote a positive integer, while $\Gamma$ will always denote the dual graph of $X_0$. For the sake of convenience, we assume $\Gamma$ is directed. Directedness arises whenever
we need to specify a reference component for $X_0$. For example, when $X_0$ is of compact type, it is natural to specify that $\Gamma$ be a rooted tree, with all edges pointing away from the root. The irreducible components of $X_0$ are always denoted by $(C_v)_{v \in V(\Gamma)}$, unless stated otherwise.

The main premise of Osserman’s theory of limit linear series is that one should consider all possible multidegrees of line bundles on a nodal curve, and not just concentrated multidegrees considered in the classical theory of Eisenbud and Harris \(^1\). While this extra freedom comes at the expense of additional complexity, it will turn out to be crucial in studying inclusions of limit linear series with base points.

We now review the basic ingredients of Osserman’s theory.

**Definition 2.1.** The multidegree of a line bundle $\mathcal{L}$ over $X_0$, $\text{md}(\mathcal{L})$, is the integer-valued function $V(\Gamma) \to \mathbb{Z} : v \mapsto \deg(\mathcal{L}|_{C_v})$. A multidegree is concentrated whenever $f$ takes positive value at exactly one $v$. The total degree of $\mathcal{L}$ is then $\sum_{v \in V(\Gamma)} \text{md}(\mathcal{L})(v)$.

Distinct line bundles of the same total degree are not independent in general. To systematize how line bundles of the same total degree relate to one another, we start by fixing a choice of pseudo-divisors $\{(\mathcal{O}_v, C_v, s_v)\}_{v \in V(\Gamma)}$ over $X_0$. Each of these is a triple satisfying the following conditions:

1. $\mathcal{O}_v|_{C_v} \cong \mathcal{O}_{C_v}(-(C_v \cap C_v^c))$;
2. $\bigotimes_{v \in V(\Gamma)} \mathcal{O}_v \cong \mathcal{O}_{X_0}$;
3. $s_v \in \Gamma(X_0, \mathcal{O}_v)$ is a section vanishing precisely along $C_v$.

Following [Oss17], we call $\{(\mathcal{O}_v, C_v, s_v)\}_{v \in V(\Gamma)}$ an enriched structure on $X_0$.

**Notation 2.2.** Given $\{\mathcal{O}_v\}_{v \in V(\Gamma)}$, let $f_v : \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_v$ denote the natural morphism sending $s$ to $s \otimes s_v$. More generally, for any sequence $\overrightarrow{v} := (v_1, ..., v_n) \in V(\Gamma)^\times n$, let $f_{\overrightarrow{v}} : \mathcal{L} \to \mathcal{L} \otimes \bigotimes_{j=1}^n \mathcal{O}_{v_j}$ denote the morphism defined by $s \mapsto s \otimes s_{v_1} \otimes ... \otimes s_{v_n}$.

**Notation 2.3.** Two multidegrees $\text{md}_1$ and $\text{md}_2$ are similar, as denoted by $\text{md}_1 \sim \text{md}_2$, whenever $\text{md}_1 - \text{md}_2 = \sum_{j=1}^n \text{md}(\mathcal{O}_{v_j})$, for some choice of $v_1, ..., v_n \in V(\Gamma)$.

**Definition 2.4.** Similarly, two line bundles $\mathcal{L}_1, \mathcal{L}_2$ are similar whenever $\mathcal{L}_2 \cong \mathcal{L}_1 \otimes (\bigotimes_{j=1}^n \mathcal{O}_{v_j})$ for some $v_1, ..., v_n \in V(\Gamma)$.

**Remark 2.5.** Given $\mathcal{L}$ and a multidegree $\text{md}_0$ similar to $\text{md}(\mathcal{L})$, there is up to isomorphism precisely one line bundle $\mathcal{L}'$ for which $\mathcal{L}' \sim \mathcal{L}$ and $\text{md}(\mathcal{L}') = \text{md}_0$. If we further require that $\mathcal{L}' = \mathcal{L} \otimes (\bigotimes_{j=1}^n \mathcal{O}_{v_j})$ such that $n$ is minimal, then the twist $\mathcal{L}'$ is unique; see [Oss17, Prop. 2.12].

**Notation 2.6.** Let $\text{md}_1, \text{md}_2$ be two similar multi-degrees. Let $\mathcal{O}_{\text{md}_2 - \text{md}_1}$ denote the tensor product $\bigotimes_{i=1}^n \mathcal{O}_i$ of line bundles $\mathcal{O}_i \in \{\mathcal{O}_v\}_{v \in V(\Gamma)}$ for which

$$\mathcal{L} \otimes \mathcal{O}_{\text{md}_2 - \text{md}_1}$$

is the unique line bundle of multi-degree $\text{md}_2$ similar to $\mathcal{L}$ whenever $\mathcal{L}$ is of multi-degree $\text{md}_1$, as per Remark 2.5.

\(^1\)For example, if $X_0$ is compact type, a line bundle on $X_0$ with concentrated mutidegree restricts to a degree zero line bundle over all but one component.
**Notation 2.7.** Given a line bundle \( \mathcal{L} \) over \( X_0 \) and for any multidegree \( \text{md}_0 \) similar to \( \text{md}(\mathcal{L}) \), let \( \mathcal{L}(\text{md}_0) \) denote the line bundle of multidegree \( \text{md}_0 \) and similar to \( \mathcal{L} \).

**Remark 2.8.** Remark 2.5 has the following useful consequence. Given two similar line bundles \( \mathcal{L} \) and \( \mathcal{L}' \), let \( n \) be the smallest integer for which there exist \( v_1, ..., v_n \in V(\Gamma)^\times \) and

\[
\mathcal{L}' \cong \mathcal{L} \otimes (\bigotimes_{j=1}^n \mathcal{O}_{\mathcal{V}_{v_j}}).
\]

There is then a well-defined morphism

\[
\mathcal{L} \to \mathcal{L}'
\]

that maps a section \( s \) to \( s \otimes s_{v_1} \otimes ... \otimes s_{v_n} \).

**Definition 2.9.** Given similar line bundles \( \mathcal{L} \) and \( \mathcal{L}' \), we shall refer to the morphism defined in 2.8 as the **natural** morphism from \( \mathcal{L} \) to \( \mathcal{L}' \).

**Lemma 2.10.** Similarity of line bundles is an equivalence relation. Further, whenever \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are similar to one another, the natural morphisms \( f: \mathcal{L}_1 \to \mathcal{L}_2 \) and \( g: \mathcal{L}_2 \to \mathcal{L}_1 \) satisfy

\[
f \circ g = g \circ f = 0.
\]

**Proof.** The proof of Lemma 2.10 is given in [Oss17], but for the convenience of the reader, we recall the basic idea here. Namely, if \( \mathcal{L}_2 \cong \mathcal{L}_1 \otimes \mathcal{O}_{v} \), then

\[
\mathcal{L}_1 \cong \mathcal{L}_2 \otimes (\bigotimes_{v' \neq v} \mathcal{O}_{v'})
\]

since \( \bigotimes_{v \in V(\Gamma)} \mathcal{O}_v \cong \mathcal{O}_{X_0} \). Expanding upon this idea gives the first assertion.

The second assertion follows from the fact that \( f \) and \( g \) are defined by multiplying by the sections \( s_v \) given by the chosen pseudo-divisors \( \{\mathcal{O}_v, s_v\} \), and \( s_v \) vanishes on \( C_v \). \( \square \)

**Definition 2.11.** A **line bundle class** of degree \( d \) over \( X_0 \) is an equivalence class of degree \( d \) line bundles; we let \( [\mathcal{L}] \) denote the class of \( \mathcal{L} \). A morphism from \( \mathcal{L}_1 \) to \( \mathcal{L}_2 \) of multi-degree \( w \) consists of a family of morphisms

\[
\{ \mathcal{L}_{w_1} \to \mathcal{L}_{w_2} \}_{w_1, w_2}
\]

such that \( w_2 - w_1 = w \) for all \( w_1, w_2 \) and \( \mathcal{L}_{w_i} \) varies over all line bundles similar to \( \mathcal{L}_k \), for \( k = 1, 2 \).

Clearly any given morphism of line bundles induces a morphism between the corresponding classes, for a specific pair of multidegrees.

**Definition 2.12.** A **limit** \( g_r \) over \( X_0 \) consists of the following data \( (\mathcal{L}, (V_v)_{v \in V(\Gamma)}) \).

1. \( \mathcal{L} \) is a line bundle over \( X_0 \) of total degree \( d \);
2. For each \( v \in V(\Gamma) \), there is some fixed choice of concentrated multidegree \( \text{md}_v \sim \text{md}(\mathcal{L}) \) such that \( V_v \subset \Gamma(C_v, \mathcal{L}(\text{md}_v)|_{C_v}) \) is an \( (r + 1) \)-dimensional subspace of sections;
3. For all \( \text{md}_0 \sim \text{md}(\mathcal{L}) \), the kernel of the linear map

\[
\Gamma(X_0, \mathcal{L}(\text{md}_0)) \to \bigoplus_{v \in V(\Gamma)} \Gamma(C_v, \mathcal{L}(\text{md}_v)|_{C_v})/V_v
\]

has dimension at least \( r + 1 \).
2.1. Equivalent formulation of the theory for curves of compact type. In this section, we will show that the definition (2.12) of limit linear series given in the preceding section is in fact equivalent to the definition of Eisenbud–Harris whenever the underlying curve $X_0$ is of compact type. This will be useful for the enumerative applications to come later.

**Notation 2.13.** Suppose $X_0$ is a curve of compact type and $v_0$ is the vertex in $\Gamma = \Gamma(X_0)$ corresponding to the base component. For all $v \neq v_0$, let $P_v^*$ denote the unique node on the $v$-th component that corresponds to the unique incoming edge at $v \in \Gamma$.

**Lemma 2.14.** When $X_0$ is of compact type, any two multidegrees $m_1, m_2$ of the same total degree $d$ are similar.

Proof. Any given multidegree of total degree on $X_0$ may be encoded as a divisor on the graph $\Gamma$, in the sense of [BN07]. The desired result then follows from the fact that similarity of multidegrees corresponds to linear equivalence of divisors on $\Gamma$. But $\Gamma$ is a tree, so any two divisors of $\Gamma$ are linearly equivalent.

Alternatively, we may argue directly as follows (according to the dictionary of the preceding paragraph, this amounts to constructing an explicit linear equivalence of graphical divisors). Recall that $\Gamma$ is in fact a directed tree, with all edges pointing away from its root, $v_0$. It then suffices to show that an arbitrary multidegree $m'$ of total degree $d$ is similar to the concentrated multidegree $m_0$, for which $m_0(v_0) = d$ and $m_0(v) = 0$ for $v \neq v_0$.

To do so, we make use of the enriched structure of $X_0$. Note that for $v \in V(\Gamma)$, $m_0(\mathcal{O}_v)$ takes value $-\text{val}(v)$ at $v$, 1 at any $v'$ adjacent to $v$, and 0 elsewhere. In particular, if $v$ is a leaf of $\Gamma$, it takes value $-1$ at $v$ and 1 at the unique vertex $v'$ adjacent to $v$.

On the other hand, for every vertex $v$, there is a unique directed path from $v_0$ to $v$ in $\Gamma$. Let $d(v)$ be the number of edges on this path. The desired result follows by downward induction on $d(v)$. Namely, suppose $v_1, \ldots, v_n$ are the vertices realizing the maximal $d(v)$ among all $v \in V(\Gamma)$, and let $v'_j$ be the vertex adjacent to $v_j$. It is easy to see that for some choice of $m_j, n_j \geq 0$,

\[
m_0' + \sum_j (n_j m_0(\mathcal{O}_{v_j}) + m_j m(\mathcal{O}_{v'_j}))\]

has value zero at all $v_j$. In general, suppose $m_0'$ has value zero at all $v$ such that $d(v) > N$. There then exists some choice of $n_v \geq 0$ such that $m_0' + \sum_{d(v) \geq N-1} n_v \cdot m_0(\mathcal{O}_v)$ has value zero at all $v$ with $d(v) \geq N$.

**Remark 2.15.** Lemma 2.14 is clearly false for arbitrary nodal curves, as graphs with cycles have nontrivial Jacobians in general.

Consequently, when $X_0$ is of compact type, we say a multidegree $m_0$ is concentrated at $v$ only when $m_0(v) = d$ and $m_0(v') = 0$ for $v' \neq v$.

We now recall the Eisenbud-Harris definition of limit linear series for curves of compact type.

**Definition 2.16.** Assume $X_0$ is compact type. A limit $g^d_{\text{lim}}$ over $X_0$ consists of the data $(\mathcal{L}_v, V_v)_{v \in V(\Gamma)}$, where each $(\mathcal{L}_v, V_v)$ is a $g^d_{\text{lim}}$ over the smooth curve $C_v$, satisfying the following compatibility condition for vanishing sequences at points of intersection $C_v \cap C_{v'} = P$ of smooth components.
Namely, when writing the vanishing sequence $\text{Van}_P(V_v)$ (resp. $\text{Van}_P(V_{v'})$) as an increasing (resp. decreasing) sequence, we have

$$\text{Van}_P(V_v) + \text{Van}_P(V_{v'}) \geq (d, d, \ldots, d).$$

An important point is that for curves of compact type, the definitions 2.16 and 2.12 agree with one another.

**Proposition 2.17.** Let $X_0$ be a nodal curve of compact type. Then 2.16 and 2.12 are equivalent.

**Proof.** This is a special case of Theorem 5.9 in [Oss17], where Osserman considered curves of pseudo-compact type, a class of curves strictly containing curves of compact type. □

2.2. **Subbundles of push-forwards of vector bundles.** In order to facilitate the discussion of (limit) linear series in families, Osserman introduced the following notion of subbundles in [Oss14]:

**Definition 2.18.** Let $\pi : X \rightarrow B$ be a proper morphism that is locally of finite presentation, and let $E$ be a quasicoherent sheaf on $X$, locally finitely presented and flat over $B$. A subsheaf $V$ is a subbundle of $\pi_*E$ if $V$ is locally free of finite rank, and for any $S \rightarrow B$, the natural pullback map $V_S \rightarrow \pi_*E_S$ is injective.

**Remark 2.19.** It is straightforward to see that the map $V_S \rightarrow \pi_*E_S$ realizes $V_S$ as a subbundle of $\pi_*E_S$.

3. **Linked chains of flags**

In this section we introduce linked chains of flags, which generalize the linked chains described in [MO16]. We give a dimension estimate for the linked determinantal locus associated to a linked chain of flags, which we will apply later to calculate the dimension of the moduli space of inclusion of (limit) linear series that we construct in Section 4; see also the proof of Theorem 4.12.

The basic set-up is as follows. Let $S$ be a scheme, and fix a choice of positive integers $d$ and $n$. Suppose that $E_1, \ldots, E_n$ are vector bundles of rank $d$ on $S$ and that we are given morphisms

$$f_i : E_i \rightarrow E_{i+1}, \quad f^i : E_{i+1} \rightarrow E_i$$

for each $i = 1, \ldots, n-1$. For each index $i = 1, \ldots, n$, suppose moreover that

$$\mathcal{E}_i^m \hookrightarrow \mathcal{E}_i^{m-1} \hookrightarrow \cdots \hookrightarrow \mathcal{E}_i^1 = \mathcal{E}_i$$

is a flag of subbundles of $\mathcal{E}_i$, in which $\text{rank}(\mathcal{E}_i^j) = d_j$ and $d_1 = d$.

**Definition 3.1.** Given $s \in H^0(S, \mathcal{O}_S)$, we say that $(\{\mathcal{E}_i^j\}_{1 \leq j \leq m}, f_*, f^*)$ is an $s$-linked chain of flags of index $(d_1, \ldots, d_m)$ whenever

$$f_i(\mathcal{E}_i^j) \subset \mathcal{E}_i^{i+1} \text{ and } f^i(\mathcal{E}_i^{i+1}) \subset \mathcal{E}_i^j$$

for every $1 \leq i \leq n-1$ and $1 \leq j \leq m$, and the tuple $(\mathcal{E}_i^j, f_*, f^*)$ is an $s$-linked chain in the sense of [MO16].
Now suppose that for the two extremal values \( i = 1, n \) we have (a compatible system of) morphisms
\[
\begin{array}{cccc}
\mathcal{E}_i^m & \rightarrow & \mathcal{E}_i^{m-1} & \rightarrow & \cdots & \rightarrow & \mathcal{E}_i^1 \\
\downarrow g_i^m & & \downarrow g_i^{m-1} & & \ddots & & \downarrow g_i^1 \\
\mathcal{G}_i^m & \rightarrow & \mathcal{G}_i^{m-1} & \rightarrow & \cdots & \rightarrow & \mathcal{G}_i^1
\end{array}
\]
in which \( \mathcal{G}_i^j \) is a vector bundle of rank \( d_j - r_i^j \) for every \( 1 \leq j \leq m \).

**Definition 3.2.** Let \( \mathcal{G}_i^j \) and \( g_i^j \) be as above, and let \( r_1, ..., r_m \) be a (non-strictly) decreasing sequence of nonnegative integers. The **linked determinantal locus** associated to the \( s \)-linked chain of flags \( \{ (\mathcal{E}_i^j)_{1 \leq j \leq m}, f_\bullet, f_\bullet^* \} \) is the closed subscheme of \( S \) along which the rank of the induced morphisms
\[
\pi_i^j: \mathcal{E}_i^j \rightarrow \mathcal{G}_i^j \oplus \mathcal{G}_n^j
\]
is at most \( d_j - r_j \) for all \( 1 \leq j \leq m \) and \( 1 \leq i \leq n \). In other words, it is the intersection of the \( r_j \)-th vanishing loci of the \( \pi_i^j \) as \( i \) and \( j \) are allowed to vary.

By \( r_j \)-th vanishing locus we mean the subscheme of \( S \) cut out by all \( (d_j - r_j + 1) \)-minors of \( \pi_i^j \); see [Oss14, Definition B.1.1] for a formal definition.

**Definition 3.3.** Given a tuple \( \underline{r} = (r_m, ..., r_1) \) of nonnegative integers \( r_i \) as in Definition 3.2, we say that the \( s \)-linked chain of flags \( \{ (\mathcal{E}_i^j)_{1 \leq j \leq m}, f_\bullet, f_\bullet^* \} \) is **\( \underline{r} \)-strict** at \( x \in S \) with respect to the maps \( \pi_i^j \) of Definition 3.2, for all \( 1 \leq j \leq m \) and \( 1 \leq i \leq n \), provided that
(a) \( (K_i^j)_x \cap (\mathcal{E}_i^j)_x = (K_i^j)_x \) for all \( j \) and \( l \geq j \); and
(b) \[
\sum_{i=1}^{n} \text{dim}_{k(x)}(\tilde{K}_i^j)_x \leq (n-1)r_j \quad \text{for every } 2 \leq j \leq m
\]
where
\[
(K_i^j)_x = \ker(f_i|_{(\mathcal{E}_i^j)_x}) \oplus \Im(f_{i-1}|_{(\mathcal{E}_{i-1}^j)_x}) \quad \text{and} \quad (\tilde{K}_i^j)_x = \ker(\pi_i^j|_x) \cap (K_i^j)_x.
\]

Note that if \( s \) is nonzero at \( x \), then item (a) is satisfied automatically, and the strictness condition is equivalent to requiring that \( \text{dim}_{k(x)}(\pi_i^j)_x \leq r_j \) for all \( 2 \leq j \leq m \).

**Remark 3.4.** For every \( l < i \), set \( f_{i,l} := f_{i-1} \circ f_{i-2} \circ \cdots \circ f_l \) and \( f^{l,i} := f_l \circ f^{l+1} \circ \cdots \circ f^{i-1} \).
According to Lemma 2.10 of [MO16], there is a distinguished set of vectors \( S_i^1 \subset (\mathcal{E}_i^j)_x \) for which
\[
\text{span}(S_i^1) \cap (K_i^j)_x = \emptyset
\]
and the images \( \{ f_{i,l}S_i^j \}_{1 \leq i} \cup \{ f^{l,i}S_i^j \}_{1 \leq i} \) in \( (\mathcal{E}_i^j)_x \) and \( S_i^j \) are linearly independent and generate \( (\mathcal{E}_i^j)_x \). Item (b) in the \( \underline{r} \)-strict condition of Definition 3.3 is used to ensure that, when \( x \) is \( \underline{r} \)-strict and contained in the linked determinantal locus in Definition 3.2, we may similarly find, for every \( 1 \leq j \leq m - 1 \), sets of vectors \( \{ S_i^j \subset (\mathcal{E}_i^j)_x \}_{1 \leq i \leq n} \) that are disjoint with \( (K_i^j)_x \), and whose images in \( (\mathcal{E}_i^j)_x \) are linearly independent and span a subspace of \( \ker(\pi_i^j|_x) \) of dimension at least \( r_j \). This actually follows from the fact that
\[
\sum_{i=1}^{n} \text{dim}_{k(x)}(\pi_i^j|_x) \geq nr_j
\]
as we are free to choose $\tilde{S}_i^j$ such that $\ker(\pi_i^j|_x) = \text{span}\tilde{S}_i^j \oplus (\tilde{K}_i^j)_x$. Note that the fact that $\text{span}(\tilde{S}_i^j) \cap (K_i^j)_x = \emptyset$ ensures that the images of $\tilde{S}_i^j$ in $(E_i^j)_x$ are linearly independent. On the other hand, since

$$\dim_{k(x)}(\tilde{K}_i^j)_x \geq \sum_{l \neq i} |\tilde{S}_l^j|$$

we have that

$$\sum_{i=1}^n \dim_{k(x)}(\tilde{K}_i^j)_x \geq (n-1) \sum_{i=1}^n |\tilde{S}_i^j| \geq (n-1)r_j.$$ 

Item (b) of Definition 3.3 then implies that in fact $\dim_{k(x)}(\pi_i^j|_{x}) = \sum_{i=1}^n |\tilde{S}_i^j| = r_j$ for every $2 \leq j \leq m$. This along with item (a) guarantees that passing to the quotient $\{E_i^j/\ker\pi_i^j\}_{1 \leq j \leq l}$ induces a linked chain of flags over $\mathfrak{r}$-strict points, for every $2 \leq l \leq m$; see the proof of Theorem 3.6.

**Non-example 3.5.** To illustrate Definition 3.3, we construct an example of a non-$\mathfrak{r}$-strict point, as follows. Let $m = 2$, $r_2 = 3$, and $n = 4$, and let $X = X_1 \cup X_2$ be the union of two rational curves glued along a point $P$. Choose parametrizations $x_i$ of $X_i$, $i = 1, 2$ such that $P$ corresponds to a point on each $X_i$, and prescribe line bundles $\mathcal{L}_i$ on $X$ according to their restrictions $\mathcal{L}_i|_{X_i} = \mathcal{O}_{X_i}((4-i)P)$ and $\mathcal{L}_1|_{X_2} = \mathcal{O}_{X_2}((i-1)P)$, for $1 \leq i \leq 4$. The global sections of $\mathcal{L}_i$ are then given by:

$$H^0(X, \mathcal{L}_1) = \text{span}\{(1, 0), (x_1, 0), (x_1^2, 0), (x_2^3, 1)\};$$

$$H^0(X, \mathcal{L}_2) = \text{span}\{(1, 0), (x_2, 0), (x_1^2, x_2), (0, 1)\};$$

$$H^0(X, \mathcal{L}_3) = \text{span}\{(1, 0), (x_1, x_2^3), (0, 1), (0, x_2)\};$$

$$H^0(X, \mathcal{L}_4) = \text{span}\{(1, x_2^3), (0, 1), (0, x_2), (0, x_2^3)\}.$$

Now let $E_i^2$ be the pushforward of $\mathcal{L}_i$ to a point $x$. The (non-trivial) twisting maps between fibers of $E_i^2$ are then given by:

$$(1, 0) \leftarrow (1, 0) \leftarrow (1, x_2^3);$$

$$(x_1, 0) \leftarrow (x_1, 0) \leftarrow (x_1, x_2^2) \rightarrow (0, x_2^2);$$

$$(x_2^3, 0) \leftarrow (x_2^3, x_2) \rightarrow (0, x_2) \rightarrow (0, x_2);$$

and

$$(x_2^3, 1) \rightarrow (0, 1) \rightarrow (0, 1) \rightarrow (0, 1).$$

These maps define a 0-linked chain on $x$. Now let

$$\mathcal{G}_i^2 = E_i^2/\text{span}\{(1, 0), (x_1, 0), (x_2^3, 1)\}$$

and

$$\mathcal{G}_i^2 = E_i^2/\text{span}\{(1, x_2^3), (0, x_2), (0, 1)\}.$$ 

It is easy to check that $\dim(\mathcal{K}_i^2)_x = 2$ for $i = 1, 4$ and $\dim(\mathcal{K}_i^2)_x = 3$ for $i = 2, 3$. Hence $x$ is not $\mathfrak{r}$-strict. Even though $\dim \ker \pi_i^2 = 3$ for every $1 \leq i \leq 4$, the quotients $\{E_i^2/\ker \pi_i^2\}_{1 \leq i \leq 4}$ do not comprise a linked chain.

**Theorem 3.6.** Suppose $S$ is Noetherian. Let $\{\{E_i^j\}_{1 \leq j \leq m}, f_*, f^*\}$ be a linked chain of flags, and let $\Delta \subset S$ denote the rank-$\mathfrak{r}$ linked determinantal locus with respect to the induced morphisms $\pi_i^j$. Assume that $r_j + d_j - r_1^j - r_2^j \geq 0$ for every $j$. Then near any $\mathfrak{r}$-strict point $x$ in $S$, $\Delta$ has codimension at most

$$\sum_{j=1}^m (r_j - r_{j+1})(r_j + d_j - r_1^j - r_2^j)$$

where by convention we set $r_{m+1} = 0$. 
Proof. We may assume $x$ lies in $\Delta$. Let $S' \subset S$ be the subscheme along which the morphism $\pi^m_i$ has rank at most $d_m - r_m$ for every $1 \leq i \leq n$. According to [MO16, Theorem 1.3], every irreducible component of $S'$ has codimension at most $r_m(r_m + d_m - r_1^m - r_2^m)$.

We begin by showing that there are distinguished linearly independent vectors

$$x_{i,1}, \ldots, x_{i,k}, y_{i,1}, \ldots, y_{i,l} \subset (\ell_i^m)_x$$

for which

1. $\sum_{i=1}^n k_i + l_i = d_m$ and $\sum_{i=1}^n l_i = r_m$;
2. (a) the set of vectors

$$\{f_{a,i}(X^m_i \cup Y_a), f^{i,b}(X^m_i \cup Y_b)|a < b\} \cup X^m_i \cup Y_i$$

generates $(\ell_i^m)_x$, where $X^m_i := \{x_{i,1}, \ldots, x_{i,k}\}$ and $Y_i := \{y_{i,1}, \ldots, y_{i,l}\}$; and

(2b) the set

$$\{f_{a,i}(Y_a), f^{i,b}(Y_b)|a < b\} \cup Y_i$$

generates $\ker(\pi^m_i|_x)$.

Indeed, the first equality of condition (1) and (2a) are simultaneously satisfied whenever

$$\text{span}(X^m_i) \oplus \text{span}(Y_i) \oplus (K^m_i)_x = (\ell_i^m)_x,$$

in which case we set $l_i = \dim \ker(\pi^m_i|_x) - \dim (K^m_i)_x$ and choose any $l_i$ independent vectors $y_{i,\lambda}, \lambda \in \ker(\pi^m_i|_x)\setminus K^m_i$, $1 \leq \lambda \leq l_i$. The set of vectors in (2b) then spans a subspace of $\ker(\pi^m_i|_x)$ of dimension $\sum_{i=1}^n l_i$, which is at least $r_m$ by $\xi$-strictness. According to Remark 3.4, we have $\dim \ker(\pi^m_i|_x) = r_m = \sum_{i=1}^n l_i$, from which the generation statement in (2b) follows.

For each $1 \leq j \leq m - 1$ we now extend $X^m_i$ inductively to $X_j^i$, say $X_j^i = X_j^{m+1} = Z_j^i$, such that $\text{span}(X_j^i) \oplus \text{span}(Y_i) \oplus (K_j^i)_x = (\ell_j^i)_x$. This is possible because $x$ is $\xi$-strict. Hence (2a) remains true when we replace $m$ by $j$. Further, we have $\sum_{i=1}^n |X_j^i| = d_j - r_m$. Let $d_j' := d_j - r_m$, $1 \leq j \leq m - 1$.

Let $U$ be an open neighborhood $U$ of $x$ in $S'$ over which (the restriction of) each $\ell_j^i$ becomes free, the kernel of $\pi^m_i$ at each fiber has dimension exactly $r_m$, and each $X_j^i$ lifts to a section in $\ell_j^i(U)$. Fix such a lift and let $X_j$ denote the set of lifts of all $X_j^i$. Let $(\ell_j^i)'$ be the subsheaf of $\ell_j^i$ over $U$ generated by the image of $X_j$. We claim that, after shrinking $U$, there is an induced linked-chain-of-flags structure on $\{(\ell_j^i)\}$ which serves as the “quotient” of $\{\ell_j^i\}$ by the kernel of $\pi^m_i$.

To this end, we first restrict $U$ to the complement of the closed subscheme along which the maps

$$\mathcal{O}_U^{d_j'} \to \ell_j^i$$

induced by the images of $X_j'$ in $\ell_j^i(U)$ fail to have full rank. By the same argument as that used in [Oss11, Lemma 2.5], $(\ell_j^i)'$ is a free module over $U$ of rank $d_j'$ and there is an inclusion of fibers $(\ell_j^i)'_y \hookrightarrow (\ell_j^i)_y$ for every $y \in U$. It is straightforward to check that $\{(\ell_j^i)'\}_{1 \leq j \leq m - 1, \ell', f, f'}$ is a linked chain of flags.

Next shrink $U$ so that the map $\pi^m_i : (\ell_j^m)' \to \mathcal{G}_1^m \oplus \mathcal{G}_n^m$ has full rank on $U$. Further, assume the sub-bundle of $\ell_j^i$ generated by the images of $X_j \setminus X_m$ has trivial intersection with $\ell_j^m$, i.e. that
the induced map from the sub-bundle of \( E_i^t/E^m \) has full rank. Then for every \( 1 \leq j \leq m-1 \), the map \( \pi_j^t \) has kernel of dimension at least \( r_j \) on \( U \) if and only if \( \pi_j^t \mid (E_i^t) \) has kernel of dimension at least \( r'_j = r_j - r_m \).

We now check that the linked chain of flags \( (\{(E_i^t)\}_1 \leq j \leq m-1, f_*, f^* \) is \( r \)-strict at \( x \). By construction, it is easy to see that \( (K_l^t)_x \cap (E_i^t)_x = (K_l^t)'_x \) for every \( l \geq j \). On the other hand, we have \( (K_l^t)'_x + \hat{Y}_i \subset (\hat{K}_t^i)_x \), where \( \hat{Y}_i \) is the span of \( \{f_{a,i}(Y_a), f_i^b(Y_b) | a < i, b > i \} \). Hence

\[
\sum_{i=1}^n \dim(\hat{K}_t^i)_x \leq \sum_{i=1}^n (\hat{K}_t^i)_x - (n-1)r_m \leq (n-1)r'_j
\]

for \( 1 \leq j \leq m-1 \) because \( x \) is \( r \)-strict with respect to \( (\{(E_i^t)\}_1 \leq j \leq m, f_*, f^*) \).

By induction, the determinantal locus of \( (\{(E_i^t)\}_1 \leq j \leq m-1, f_*, f^*, \pi) \) has codimension at most

\[
\sum_{j=1}^{m-1} (r'_j + d^*_j - (r'_j + r^*_j)) = \sum_{j=1}^{m-1} (r_j - r_{j+1})(r_j + d_j - r_1^* - r_2^*)
\]

in \( S' \). It therefore has codimension at most

\[
\sum_{j=1}^{m-1} (r_j - r_{j+1})(r_j + d_j - r_1^* - r_2^*) + r_m(r_m + d_m - r_1^* - r_2^*) = \sum_{j=1}^{m} (r_j - r_{j+1})(r_j + d_j - r_1^* - r_2^*)
\]

in \( S \).

\[\Box\]

4. A MODULI SCHEME FOR INCLUSIONS OF (LIMIT) LINEAR SERIES

4.1. Moduli of inclusions of linear series on a smooth curve. In this section, we briefly review the moduli problem for inclusion of linear series over a smooth curve. The main point is to exhibit the moduli scheme as a determinantal locus. Throughout, \( C \) will denote an irreducible, smooth, projective curve over some algebraically closed field \( K \).

Hereafter, we fix positive integers \( d_1 > d_2 \) and \( r_1 > r_2 \) and we denote the \((d_1 - d_2)\)-th symmetric product of \( C \) by \( C_{d_1 - d_2} \). We let \( G_{d_1}^{r_1}(C) \) denote the moduli scheme of linear series \( g_{d_1}^{r_1} \) over \( C \). It is well-known that on a general curve, \( G_{d_1}^{r_1}(C) \) is smooth of dimension

\[
\rho(g, r_1, d_1) := g - (r_1 + 1)(r_1 + g - d_1).
\]

We also let

\[
\mu(d_1 - d_2, r_2, r_1) := (d_1 - d_2) - (d_1 - d_2 - r_1 + r_2)(r_2 + 1)
\]

denote the expected dimension of the space of \((d_1 - d_2)\)-secant \((r_1 - r_2 - 1)\)-planes to the image of a fixed \( g_{d_1}^{r_1} \).

We now rigorously define the functor of points associated to our moduli problem.

**Definition 4.1.** Let \( C \) be a smooth, irreducible projective curve. For every \( K \)-scheme \( S \), let \( G_{d_1, d_2, C}(S) \) denote the set of equivalence classes of objects of the form \((Y, (L, V_2, V_1))\), where \( Y \) is an effective divisor of relative degree \( d_1 - d_2 \) on \( C \times S \), \((L, V)\) is an \( S \)-family of linear series \( g_{d_1}^{r_1} \), and \((L(-D), V_2)\) is an \( S \)-family of series \( g_{d_2}^{r_2} \). Here \((Y, (L, V_2, V_1))\) and \((Y, (L', V_2', V_1'))\) are equivalent
whenever there exists a line bundle \( F \) on \( S \) and a line bundle isomorphism \( \phi : L \to L' \otimes p^* F \) for which \( p_* \phi(V_i) = V_i' \), for \( i = 1, 2 \).

We shall define a determinantal scheme \( G_{d_1,d_2}^{r_1,r_2}(C) \) that represents \( G_{d_1,d_2}^{r_1,r_2}(C) : (\text{Sch}/K) \to (\text{Sets}) \). To do so, we begin by reserving the following notation for future reference.

- Let \( \mathcal{L} \) denote a Poincaré line bundle over \( C \times \text{Pic}^1_d(C) \) and let \( \tilde{\mathcal{L}} \) denote its pullback to \( C \times C_{d_1-d_2} \times \text{Pic}^{d_1}(C) \).
- Fix a reduced effective divisor \( Z \) of \( C \) of sufficiently large degree and \( Z' \) and \( Z'' \) denote the respective pullbacks \( Z' = Z \times \text{Pic}^{d_1}(C), \ Z'' = Z \times C_{d_1-d_2} \times \text{Pic}^{d_1}(C) \).
- Let \( E \subset C \times C_{d_1-d_2} \times \text{Pic}^{d_1}(C) \) denote the pull-back of the universal divisor of degree \( d_1 - d_2 \).
- Let \( \text{Flag}_{r_1,r_2}^*(\pi_+ \mathcal{L}(Z')) \) denote the flag scheme over \( \text{Pic}^{d_1}(C) \) of two-term flags of subbundles of \( \pi_+ \mathcal{L}(Z') \) and let \( \mathcal{V}_* = \mathcal{V}_1 \supset \mathcal{V}_2 \) denote the universal flag over it.
- Set \( X := (C_{d_1-d_2} \times \text{Pic}^{d_1}(C)) \times_{\text{Pic}^{d_1}(C)} \text{Flag}_{r_1,r_2}^*(\pi_+ \mathcal{L}(Z')) \), and let \( \mathcal{V}_* \) denote the pullback of \( \mathcal{V}_* \) to \( X \).

These objects fit into the following commutative diagram:

\[
\begin{array}{ccc}
C \times C_{d_1-d_2} \times \text{Pic}^{d_1}(C) & \xrightarrow{\tilde{\pi}} & C_{d_1-d_2} \times \text{Pic}^{d_1}(C) \\
\downarrow{\pi} & & \downarrow{\pi}
\end{array} \quad \begin{array}{ccc}
X & \xleftarrow{q_1} & C_{d_1-d_2} \times \text{Pic}^{d_1}(C) \\
\downarrow{\pi} & & \downarrow{\pi}
\end{array} \quad \begin{array}{ccc}
\text{Flag}_{r_1,r_2}^*(\pi_+ \mathcal{L}(Z')) & \xrightarrow{\mathcal{V}_*} & \text{Pic}^{d_1}(C) \\
\downarrow{s} & & \downarrow{s}
\end{array}
\]

**Definition 4.2.** With notation as above, let \( G_{d_1,d_2}^{r_1,r_2}(C) \) denote the subscheme of \( X \) defined the intersection of the following two determinantal conditions:

1. \( \tilde{\mathcal{V}}_1 \to t^* \pi_+ \mathcal{L}(Z') \mid_{Z'} \) is zero, where \( t = u \circ q_1 = s \circ q_2 \); and
2. \( \tilde{\mathcal{V}}_2 \to q_1^* \pi_+ \mathcal{L}(Z'')/q_1^* \pi_+ \mathcal{L}(Z'' - E) \) is zero.

We briefly justify the existence of the morphism involved in the second condition, as follows. The natural morphism \( \tilde{\mathcal{V}}_2 \to \mathcal{V}_1 \to s^* \pi_+ \mathcal{L}(Z') \) pulls back to a morphism \( \tilde{\mathcal{V}}_2 \to t^* \pi_+ \mathcal{L}(Z') \). Pulling back the natural map \( u^* \pi_+ \mathcal{L}(Z') \to \tilde{\pi}_+ p^* \mathcal{L}(Z') = \tilde{\pi}_+ \tilde{\mathcal{L}}(Z'') \) along \( q_1 \), we get a map \( t^* \pi_+ \mathcal{L}(Z') \to q_1^* \pi_+ \mathcal{L}(Z'') \). Composing this map with the previous one, we get a map \( \tilde{\mathcal{V}}_2 \to q_1^* \pi_+ \mathcal{L}(Z'') \), which induces the map in item (2).

**Proposition 4.3.** The functor \( G_{d_1,d_2}^{r_1,r_2}(C) : (\text{Sch}/K) \to (\text{Sets}) \) is represented by \( G_{d_1,d_2}^{r_1,r_2}(C) \).

**Proof.** We need to check that each \( S \)-valued point of our functor defines an \( S \)-family \((Y,(L,V_2,V_1))\) of inclusion of linear series as specified in Definition 4.1. To this end, fix a morphism \( f : S \to \)
$G_{d_1,d_2}^{r_1,r_2}(C)$. For the sake of clarity, we update our previous commutative diagram to

$$
\begin{array}{ccc}
C \times S & \xrightarrow{\pi_S} & S \\
\downarrow & & \downarrow q_1 \\
C \times C_{d_1-d_2} \times \text{Pic}^{d_1}(C) & \xrightarrow{\tilde{\pi}} & \text{Flag}_{r_1,r_2}(\pi_*\mathcal{L}(Z')) \\
\downarrow & & \downarrow s \\
C \times \text{Pic}^{d_1}(C) & \xrightarrow{\pi} & \text{Pic}^{d_1}(C)
\end{array}
$$

Notice that the two rectangles on the left are Cartesian.

By appealing to the universal properties of $C_{d_1-d_2}$, $\text{Pic}^{d_1}(C)$ and $\text{Flag}_{r_1,r_2}(\pi_*\mathcal{L}(Z'))$, respectively, and composing $f$ with the relevant morphisms, we get a tuple of the form $(Y,L,V_2,V_1)$, where $Y$ is a relative divisor of degree $d_1 - d_2$, $L$ is a line bundle of relative degree $d_1$ over $C \times S$ and $V_2 \subset V_1 \subset (\pi_*\mathcal{L}(Z'))_S$. In particular, we have $V_1 = (\mathcal{V}_1)_S$ (where the restriction to $S$ is via pullback along $q_1 \circ f$), $L = (\mathcal{L})_{C \times S}$ (via pullback along the left vertical arrow), and $Y = (\mathcal{Z})_{C \times S}$ (via pullback along $C \times S \to C \times C_{d_1-d_2} \times \text{Pic}^{d_1}(C) \to C \times C_{d_1-d_2}$).

We then need to check that $(L,V_1)$ and $(L(-Y),V_2)$ are $S$-families of $\mathfrak{g}_{d_1}^{r_1}$ and $\mathfrak{g}_{d_2}^{r_2}$, respectively. Condition (1) in Definition 4.2 implies that the natural morphism $V_1 \to \pi_*\mathcal{L}$ is zero; since $V_1 = (\mathcal{V}_1)_S$ is a subbundle of $\pi_*\mathcal{L}(Z')_{C \times S}$ to begin with (see Remark 2.19), it follows that $V_1$ is a sub-bundle of $\pi_*\mathcal{L}_{C \times S}$ of rank $r_1 + 1$ in the sense of Definition 2.18, i.e. $(L,V_1)$ is an $S$-family of $\mathfrak{g}_{d_1}^{r_1}$. It follows that $V_2$ is a subbundle of $\pi_*\mathcal{L}_{C \times S}$, and condition (2) in Definition 4.2 implies that $V_2 \to \pi_*\mathcal{L}_{C \times S}$ factors through $\pi_*\mathcal{L}_{C \times S}(-Y)$ and hence $(L(-Y),V_2)$ is an $S$-family of $\mathfrak{g}_{d_2}^{r_2}$. □

4.2. Inclusions of limit linear series. Motivated by the theory on smooth curves, we now give a functorial definition of the main object of study of this paper. Note that in general it is not natural to assume that our objects consist of limit $\mathfrak{g}_{d_2}^{r_2}$ embedded in limit $\mathfrak{g}_{d_1}^{r_1}$, where $d_2 < d_1$. A priori, we also want to take into consideration objects of the form $(\mathcal{L} \otimes \mathcal{I}_Z, \{V_v\}_v)$ embedded inside $(\mathcal{L}, \{V_v\}_v)$, where $\mathcal{I}_Z$ is the ideal sheaf of a non-Cartier effective divisor of a nodal curve.

We make the following assumption on the (families of) curves we consider in the remainder of the paper:

**Situation 4.4.** Hereafter, $X/B$ denotes a regular smoothing family; in particular, $X$ is projective over $B$ ([Lic68, §2]). We assume that the special fiber $X_0$, whose dual graph we denote by $\Gamma$, is of compact type and is equipped with a fixed enriched structure $(\mathcal{O}_v)_{v \in V(\Gamma)}$. We denote by $Z_v$ the component of $X_0$ corresponding to $v \in V(\Gamma)$, and by $Z_v^c$ the closure of $X_0 \setminus Z_v$. Let $Z_v^\circ := Z_v \setminus Z_v^c$.

**Definition 4.5.** Given a quadruple of nonnegative integers $(r_1, r_2, d_1, d_2)$ with $r_1 > r_2$ and $d_1 > d_2$, an inclusion of limit linear series of type $(r_1, r_2, d_1, d_2)$ over $X_0$ consists of the data

$$(\mathcal{L}, Z, \{V_v\} \cup \{V_{v,2} \subset V_{v,1}\}_{v \in V(\Gamma)})$$

in which

1. $\mathcal{L}$ is a line bundle of total degree $d_1$ over $X_0$;
(2) for all \( v \in V(\Gamma) \), \( V_{v,2} \subset V_{v,1} \subset \Gamma(X_0, \mathcal{L}(\mathcal{M}_v)) \) are subspaces of sections for which \((\mathcal{L}, (V_{v,1})_{v \in V(\Gamma)})\) is a limit \( g^*_d \) in the sense of Definition 2.12 and \((\mathcal{L}, (V_{v,2})_{v \in V(\Gamma)})\) is a limit \( g^*_{d_1} \); and

(3) \( Z \) is a finite sub-scheme of length \( d_1 - d_2 \) such that for every \( v \in V(\Gamma) \) the evaluation map

\[ e_{v,Z} : V_{v,2} \rightarrow \Gamma(X_0, \mathcal{L}(\mathcal{M}_v) \otimes \mathcal{O}_Z) \]

is zero.

### 4.3. Moduli of inclusions of limit linear series

Continuing with the notation in Situation 4.4, in this subsection we construct the moduli space over \( B \) of inclusion of (limit) linear series on \( X \) with base points following an idea of [Oss17, §3]. To this end, let \( d_1 > d_2 > 0, r_1 > r_2 > 0, \) \( d = d_1 - d_2 \) such that \( d_1 - d_2 > r_1 - r_2 \). We have the following theorem, whose proof is carried out in §4.3.1.

**Theorem 4.6.** Let \( X \) and \( B \) be as above. There exists a proper moduli space \( \mathcal{G} \) over \( B \). Its generic fiber \( \mathcal{G}_\eta \) is isomorphic to \( G_{d_1,d_2}^{r_1,r_2}(X_\eta) \) and parameterizes all inclusions of linear series of type \((r_1,r_2,d_1,d_2)\) over \( X_\eta \), while its special fiber contains an open subset parameterizing all inclusions of limit linear series of type \((r_1,r_2,d_1,d_2)\) over \( X_0 \).

Let \( w_0 \) be a multidegree on \( \Gamma \) with total degree \( d_1 \), and let \( M_{w_0} \) denote the set of multidegrees on \( \Gamma \) that can be obtained from \( w_0 \) by twisting in a finite sequence of vertices. Note that there is a directed graph \( G(w_0) \) whose vertices are identified with \( M_{w_0} \) and edges are induced by twisting; see [Oss17, Notation 2.11].

Now let \( w_v \in M_{w_0} \) be the naive multidegree concentrated on \( Z_v \), i.e. that which has degree \( d_1 \) on \( Z_v \) and is zero elsewhere. Let \( \mathcal{M}_{w_0} = \mathcal{M}_{w_0,v} \) denote the union over all adjacent pairs \( (v,v') \subset V(\Gamma) \) of vertices \( w \in V(G(w_0)) \) that lie between \( w_v \) and \( w_{v'} \) (so the divisor indexed by \( w \) is effective and has degree zero on \( V(\Gamma) \setminus \{v,v'\} \)). See [Oss17, Definition 5.5] for a construction of \( \mathcal{M}_{w_0,v} \) for more general concentrated degrees \( (w_v) \). We will use the terminology limit \( g^*_{w_0} \) to mean a limit series of rank \( r \) and multidegree \( w_0 \) on \( X_0 \).

Finally, let \( \text{Hilb}_d(X/B) \) denote the “relative” Hilbert scheme parameterizing closed finite degree \( d \) subschemes of the fibers of \( X \) over \( B \). Let \( \text{Hilb}^*_d(X/B) \subset \text{Hilb}_d(X/B) \) be the open subscheme parameterizing subschemes of \( X \) supported away from the nodes of \( X_0 \). Given \( w \in M_{w_0} \), let \( \text{Pic}^w(X/B) \) denote the moduli schemes of line bundles of degree \( d_1 \) that have multidegree \( w \) on \( X_0 \).

### 4.3.1. The schematic structure of the moduli space

The moduli space of limit linear series constructed in [Oss17] is isomorphic as a scheme to the Eisenbud–Harris limit linear series moduli space for compact-type curves over the locus of refined series. As we build on Osserman’s construction, we will freely use the notions of “subbundles of pushforwards” and “generalized determinantal loci” introduced in [Oss14, Appendix B].

To begin, let \( G_{w_0,v}^{r_1}(X/B) \) (resp., \( G_{w_0,v}^{r_2}(X_0) \)) denote the moduli space of (limit) linear series on \( X \) with multidegree \( w_0 \) on \( X_0 \) (resp., the moduli space of limit \( g_{w_0}^{r_1} \)'s on \( X_0 \)) with respect to the fixed enriched structure \( (\mathcal{O}_v)_v \) as in Situation 4.4. These spaces are introduced in [Oss17, Definition 3.7, Notation 3.13] as closed subschemes of the \( \text{Pic}^w(X/B) \)-fiber product of \( G_{w_0,v}^{r_1}(X) \) for all \( v \).
Here $G^v_w(X)$ is the moduli scheme of pairs $(L, V)$, where $L$ is in $\text{Pic}^w(X/B)$ and $V$ is an $(r + 1)$-dimensional space of global sections of $L$. So let $\mathcal{P} = \text{Hilb}_d(X/B) \times_B G^v_{w_0}(X)$. We have the following diagram:

$$
\begin{array}{ccc}
\mathcal{P} \times_B X & \longrightarrow & G^v_{w_0}(X) \times_B X \\
\downarrow & & \downarrow \\
\mathcal{P} \times_B X & \longrightarrow & \mathcal{P} \\
\downarrow & & \downarrow \\
\text{Hilb}_d(X/B) \times_B X & \longrightarrow & \text{Hilb}_d(X/B).
\end{array}
$$

(4.3)

For notational simplicity, we abusively omit writing pullbacks whenever doing so is unambiguous. Our construction of course depends on various naturally-arising bundles lying over the schemes in the basic diagram (4.3). Accordingly, let $L$ denote the universal (Poincaré) line bundle over $G^v_{w_0}(X/B) \times_B X$, and let $\mathcal{I}^v_1 \subset \tilde{p}_* \mathcal{L}_w$, denote the universal subbundle, where $\mathcal{L}_w$, is the universal bundle with multidegree $w_v$ on $X_0$ obtained from $L$ by twisting with line bundles in the enriched structure.

Let $\psi_v : G_v := Gr(r_2, q^* \mathcal{I}_v^1) \to \mathcal{P}$ denote the relative Grassmann bundle of $r_2$-dimensional subspaces in the fibers of $q^* \mathcal{I}_v^1$, with universal subbundle $\mathcal{I}_v^2 \subset \psi_v^* q^* \mathcal{I}_v^1$. Let $G^1$ be the fiber product of all $G_v$ over $\mathcal{P}$ with projection map $\tilde{\psi}_v : G^1 \to G_v$, and let $\psi := \psi_v \circ \tilde{\psi}_v$.

It is easy to check that $G^1_0$ parameterizes generic tuples $(L(-E) \to L_v(W^2_v)_{v \in V(\Gamma)}, W_1)$, in which $E \in \text{Hilb}_d(X_0)$, $(L, W_1) \in G^d_1(X_0)$ and $W^2_0 \in Gr(r_2, W_1)$. Similarly $G^1_0$ parameterizes special tuples $(L' \oslash I_{E'}, \to L'_v(V^2_v, V^1_v)_{v \in V(\Gamma)})$, in which $(L'_v, (V^1_v)_v) \in G^{r_1}_{w_0}(X_0)$ is a limit $G^{r_1}_{w_0}$ on $X_0$, $I_{E'}$ is the ideal sheaf of $E' \subset \text{Hilb}_d(X_0)$ and $V^2_v \subset Gr(r_2, V^1_v)$.

Let $G^2 \subset G^1$ denote the intersection of the (generalized) $(r_2 + 1)$-st vanishing loci of the maps

$$
\tilde{p}_* \psi \circ \tilde{q}_v \mathcal{L}_w \rightarrow \bigoplus_{v \in V(\Gamma)} \tilde{p}_* \psi \circ \tilde{q}_v^* \mathcal{I}_v \mathcal{L}_w / \psi_v^* \mathcal{I}_v^2
$$

(4.4)

over all $w \in M_{w_0}$. Then $G^2_0$ corresponds to generic tuples with $W_0^2$ independent of $v$, and $G^2_0$ parameterizes special tuples satisfying $(L'_v,(V^2_v)_v) \in G^{r_2}_{w_0}(X_0)$.

Let $\mathcal{I}$ be the universal ideal sheaf over $\text{Hilb}_d(X/B) \times_B X$. Then $\psi^* q^* \mathcal{I}$, as a coherent sheaf on $G^1 \times_B X$, is flat over $G^1$. Note that $\tilde{\psi}_v^* \mathcal{I}_v^2$ is a subbundle of $\tilde{p}_* \psi_v^* \tilde{q}_v^* \mathcal{L}_w$, and we have an exact sequence

$$
0 \rightarrow \psi^* \tilde{q}_v^* \mathcal{L}_w \otimes \psi^* q^* \mathcal{I} \rightarrow \psi^* \tilde{q}_v^* \mathcal{L}_w \rightarrow \mathcal{L}_w \mid_{\mathcal{I}} := \psi^* \tilde{q}_v^* \mathcal{L}_w / (\psi^* \tilde{q}_v^* \mathcal{L}_w \otimes \psi^* q^* \mathcal{I}) \rightarrow 0
$$

on $G^1 \times_B X$ that is preserved under the base change $T \to G^1$. Furthermore, the pushforward $\tilde{p}_* \mathcal{L}_w \mid_{\mathcal{I}}$ is a vector bundle of rank $d$ over (the preimage in $G^1$ of) $\text{Hilb}_d(X/B)$ by [FK94, §0.5].

Our moduli space over $B$ will be the locus $G$ in $G^2$ along which the composition

$$
\tilde{\psi}_v^* \mathcal{I}_v^2 \rightarrow \tilde{p}_* \psi_v^* \tilde{q}_v^* \mathcal{L}_w \rightarrow \tilde{p}_* \mathcal{L}_w \mid_{\mathcal{I}}
$$

(4.5)

vanishes identically.
By construction, a $T$-valued point of $\mathcal{G}^1$ is a tuple $(L_T, E_T, (V^2_v)_v, (V^1_v)_v)$ where

- $L_T$ is a line bundle over $T \times_B X$ with multidegree $w_0$ on $T \times_B X_0$;
- $E_T$ is a closed subscheme of $T \times_B X$, flat and with Hilbert polynomial $d$ over $T$;
- $V^1_v$ is a rank-$(r_1 + 1)$ subbundle of $\tilde{p}_*(L_{T,w})$ for which the map
  \[
  \tilde{p}_*L_{T,w} \to \bigoplus_{v \in V(\Gamma)} \tilde{p}_*L_{T,w_v}/V^1_v
  \]
  has $(r_1 + 1)$-st vanishing locus equal to $T$ for all $w \in M_{w_0}$; and
- $V^2_v$ is a rank-$(r_2 + 1)$ subbundle of $V^1_v$.

Here $L_{T,w}$ (and similarly for $L_{T,w_v}$) is defined in the same manner as $\mathcal{L}_{w_v}$. Such a tuple is a $T$-valued point of $\mathcal{G}$ whenever the map

\[
\tilde{p}_*L_{T,w} \to \bigoplus_{v \in V(\Gamma)} \tilde{p}_*L_{T,w_v}/V^2_v
\]

has $(r_2 + 1)$-st vanishing locus equal to $T$ for all $w \in M_{w_0}$, and

\[
V^2_v \to \tilde{p}_*(L_{T,w_v}/L_{T,w_v} \otimes I_E))
\]

vanishes identically on $T$ for all $v \in V(\Gamma)$, where $I_E$ is the ideal sheaf of $E$.

Remark 4.7. For technical convenience, our moduli functor keeps track of the base-locus of the sub-series as well as the sub-series itself. One can certainly recover the moduli problem of (limit) linear series admitting sub-series with prescribed amount of base points by applying the obvious forgetful functor.

Finally, let $\mathcal{G}^\circ$ denote the preimage of $\text{Hilb}_d^\circ(X/B)$ in $\mathcal{G}$. Then $\mathcal{G}^\circ_0 \subset \mathcal{G}_0$ is the open subset described in Theorem 4.6.

4.3.2. An alternative construction. In this subsection we give another construction of a moduli space over $B$ of inclusion of (limit) linear series, which at least set-theoretically agrees with $\mathcal{G}^\circ$. We construct the alternative space as an intersection of (linked) determinantal loci of vector bundles over a regular ambient space. The alternative construction makes estimating the dimension of $\mathcal{G}^\circ$ more transparent and is also helpful in proving the smoothing of certain well-behaved inclusions of limit linear series; see §4.4.

To begin, let $\mathcal{L}_w$ denote the universal bundle over $\text{Pic}^w(X/B) \times_B X$. Fix a choice of effective divisor $D = \sum_{v \in V(\Gamma)} D_v$ on $X$ of relative degree $d = \sum \deg D_v$ in which $D_v$ is a union of sections of $X/B$ passing through $Z^\circ_v$ and $d_v = \deg D_v$ is sufficiently large relative to $d$ and $g_v = g(Z_v)$. In fact, we will see that it suffices to choose $D_v$ with $d_v - d > 2g_v - 2$.

Now set

\[
\tilde{\mathcal{P}} := \text{Hilb}_d^\circ(X/B) \times_B \prod_{v \in V(\Gamma)} \text{Pic}^w(X/B)
\]
where the product of Picard varieties is over $\text{Pic}^{w_0}(X/B)$. The following commutative diagram will be useful in tracking (the various projections involved in) our construction.

$$
\begin{array}{ccc}
\hat{P} \times_B X & \longrightarrow & \text{Pic}^{w_0}(X/B) \times_B X \\
\downarrow & & \downarrow \pi_v \\
\hat{P} & \longrightarrow & \text{Pic}^{w_0}(X/B)
\end{array}
$$

Let $L_v := q_v^*(\mathcal{L}_{w_v} \otimes \pi_v^*\mathcal{O}_X(D))$ and $E_v := p_{v*}L_v$. Similarly, let $L'_v := q'_v(\mathcal{L}_{w_v} \otimes \pi_v^*\mathcal{O}_X(D)|_{D_v})$ and $E'_v := p_{v*}L'_v$ over $\hat{P}$. By virtue of our choice of $d_v$, $E_v$ (resp., $E'_v$) is a rank-$(d_1 + d - g + 1)$ (resp., rank-$(d_1)$) vector bundle.

Let $F_v := \text{Flag}(r_2 + 1, r_1 + 1; \mathcal{L}_v)$ denote the relative flag variety over $\hat{P}$, and let $\mathcal{T}^0$ denote the universal ideal sheaf on $\text{Hilb}^0(X/B) \times_B X$. Let $\tilde{E}_v := p_{v*}(L_v \otimes \tilde{q}^*\mathcal{T}^0)$ and $\tilde{E}'_v := p_{v*}(L'_v \otimes \tilde{q}'^*\mathcal{T}^0)$; note that $p = p_v$ on $\hat{P} \times_B X$ for each $v$. We have $\tilde{E}_v \hookrightarrow E_v$ and it follows from our choice of $d_v$ and [FK94, §0.5] that $\tilde{E}_v$ (resp., $\tilde{E}'_v$) is a vector bundle of rank $d_2 + d - g + 1$ (resp., $d_1$).

**The total space.** The generic fiber of the $\hat{P}$-fiber product

$$G^1_D := \coprod_{v \in V(\Gamma)} F_v$$

over $B$ parameterizes generic $D$-tuples $(L(-E) \hookrightarrow \mathcal{L}, (W^2_v, W^1_v)_{v \in V(\Gamma)})$ in which $E \in \text{Hilb}_d(X_\eta)$, $L \in \text{Pic}^{d_1}(X_\eta)$ and

$$(W^2_v, W^1_v) \in \text{Flag}(r_2 + 2, r_1 + 1; H^0(X_\eta, L(D_\eta))).$$

Similarly the special fiber of $G^1_D$ parameterizes special $D$-tuples $(L'(-E') \hookrightarrow L', (V^2_v, V^1_v)_{v \in V(\Gamma)})$, in which $L' \in \text{Pic}^{w_0}(X_0)$, $E' \in \text{Hilb}^0(X_0)$ and

$$(V^2_v, V^1_v) \in \text{Flag}(r_2 + 1, r_1 + 1; H^0(X_0, L'_{w_v}(D_0))).$$

Here $L'_{w_v}$ is the line bundle with multidegree $w_v$ obtained from $L'$ by twisting.

**Remark 4.8.** We will realize the alternative moduli space as a closed subscheme of $G^1_D$, rather than $G^1$. We do so for two reasons. First, it enables us to describe the moduli space as a determinantal locus associated with several linked chains of flags (see the construction below), which in turn enables us to leverage the dimension estimate of Section 3 to obtain a lower bound for its dimension. On the other hand, to prove the universal openness of the moduli space in Theorem 4.12 we need an ambient space that is smooth over $B$; see also [Oss15, Corollary 5.1].

Now consider the projection

$$G^1_D \xrightarrow{\phi_v} F_v \xrightarrow{\phi_{v*}} \hat{P}.$$  

Let $\phi := \phi_v \circ \tilde{\phi}_v$, and let $\mathcal{V}^2_v \hookrightarrow \mathcal{V}^1_v \hookrightarrow \phi^*_v\mathcal{E}_v$ denote the universal flag over $F_v$ whose pullback to $G^1_D$ we denote by $\mathcal{V}^2_v \hookrightarrow \mathcal{V}^1_v \hookrightarrow \phi^*\mathcal{E}_v$.

**Conditions imposed by** $W^1_v \subset H^0(X_\eta, L(D_\eta - (D_v)_\eta))$ and $V^1_v \subset H^0(X_0, L_{w_v}(D_0 - (D_v)_\eta))$. The locus $G^2_D \subset G^1_D$ of points over which the map $\mathcal{V}^1_v \rightarrow \phi^*\mathcal{E}_v$ vanishes identically for each $v \in V(\Gamma)$
has generic fiber parameterizing generic $D$-tuples such that $W_v^1 \subset H^0(X_\eta, L(D_\eta - (D_v)_\eta))$ and special fiber parameterizing special $D$-tuples such that $V_v^1 \subset H^0(X_0, L_{w_v}(D_0 - (D_v)_0))$ for each $v \in V(\Gamma)$.

**Conditions imposed by $W_v^2 \subset H^0(X_\eta, L(-E))$ and $V_v^2 \subset H^0(X_0, L_{w_v}'(-E'))$.** On $\hat{\mathcal{P}} \times_B X$, we have an exact sequence

$$0 \to \hat{q}^*T^\circ \to \mathcal{O}(D) \to \mathcal{O}(D)/\hat{q}^*T^\circ \to 0$$

where $\mathcal{O}(D) = q_v^*\pi_v^*\mathcal{O}_X(D)$. Tensoring with $q_v^*\mathcal{L}_{w_v}$, we get

$$0 \to q_v^*\mathcal{L}_{w_v} \otimes \hat{q}^*T^\circ \to \mathcal{L}_v = q_v^*\mathcal{L}_{w_v} \otimes \mathcal{O}(D) \to q_v^*\mathcal{L}_{w_v} \otimes (\mathcal{O}(D)/\hat{q}^*T^\circ) \to 0.$$

Note that $\mathcal{O}(D)/\hat{q}^*T^\circ$ is flat over $\hat{\mathcal{P}}$, as

$$0 \to \mathcal{O}(\mathcal{P} \times_B X)/\hat{q}^*T^\circ \to \mathcal{O}(D)/\hat{q}^*T^\circ \to \mathcal{O}(D)/\mathcal{O}(\mathcal{P} \times_B X) \to 0$$

is exact. It follows that $p_v^*(\mathcal{L}_{w_v} \otimes (\mathcal{O}(D)/\hat{q}^*T^\circ))$ is a vector bundle on $\hat{\mathcal{P}}$ of rank $\tilde{d} + d$. Let $\mathcal{G}_D^3 \subset \mathcal{G}_D^2$ denote the locus over which the composition

$$\mathcal{V}_v^2 \to \mathcal{E}_v = \phi^*p_v^*(q_v^*\mathcal{L}_{w_v} \otimes \mathcal{O}(D)) \to \phi^*p_v^*(q_v^*\mathcal{L}_{w_v} \otimes (\mathcal{O}(D)/\hat{q}^*T^\circ))$$

vanishes identically.

Similarly, by applying [FK94, §0.5] we see that $p_v^*(\mathcal{L}_v/\mathcal{L}_v \otimes \hat{q}^*T^\circ)$ is an vector bundle over $\hat{\mathcal{P}}$ of rank $d$ whose fiber at a $B_0$-point (resp., $B_0$-point) is $H^0(X_\eta, L(D_\eta))/H^0(X_\eta, L(D_\eta - E))$ (resp., $H^0(X_0, L_{w_v}'(D_0))/H^0(X_0, L_{w_v}'(D_0 - E'))$) for some $L$ and $E$ (resp., $L'$ and $E'$). Let $\mathcal{G}_D^3 \subset \mathcal{G}_D^2$ denote the locus along which the map $\mathcal{V}_v^2 \to \phi^*p_v^*(\mathcal{L}_v/\mathcal{L}_v \otimes \hat{q}^*T^\circ)$ vanishes identically. Then $\mathcal{G}_D^3$ parameterizes generic $D$-tuples such that $W_v^2 \subset H^0(X_\eta, L(D_\eta - E))$ and special $D$-tuples such that $V_v^2 \subset H^0(X_0, L_{w_v}'(D_0 - E'))$. The space $\mathcal{G}_D^3$ will rejoin our discussion at the end of this subsection.

**Conditions imposed by the linked chain of flags.** By construction, $\mathcal{V}_v^2$ is a subbundle of $\phi^*\mathcal{E}_v = p_v^*\phi^*(\mathcal{L}_v \otimes \hat{q}^*T^\circ)$ of rank $r_2 + 1$ over $\mathcal{G}_D^3$. Hence $\phi^*\mathcal{E}_v/\mathcal{V}_v^2$ is a vector bundle of rank $d_2 + \tilde{d} - g - r_2$ over $\mathcal{G}_D^3$. Note also that for each $w \in M_{w_0}$ we have a commutative diagram

$$\begin{array}{ccc}
\hat{\mathcal{P}} \times_B X & \longrightarrow & \text{Pic}^w(X/B) \times_B X \\
\downarrow & & \downarrow \\
\hat{\mathcal{P}} & \xrightarrow{q_v} & \text{Pic}^{w_v}(X/B)
\end{array}$$

in which $q_v^w$ is defined by twisting. Let $\mathcal{G}^3_D \subset \mathcal{G}_D^3$ denote the locus along which the morphism

$$(4.6) \quad \phi^*p_v^* q_v^w (\mathcal{L}_{w_v} \otimes \pi_v^* \mathcal{O}_X(D)) \to \phi^*\mathcal{E}_v/\mathcal{V}_v^1 \oplus \phi^*\mathcal{E}_v'/\mathcal{V}_v^1'$$

induced by twisting with the enriched structure (see also [Oss17, Remark 2.22]) has rank at most $d_1 + \tilde{d} - g - r_1$ for every $w \in M_{w_0}$ between $w_v$ and $w_v'$, where $(v, v')$ varies over all pairs of adjacent vertices in $\Gamma$. This determinantal condition was used by Osserman to construct the moduli space of limit linear series [Oss17, §3]. Note also that the domain vector bundle in (4.6) is independent of $v$, as $q_v^w$ factors through

$$\text{Pic}^w(X/B) \to \text{Pic}^{w_w}(X/B) \to \text{Pic}^w(X/B).$$
Finally, let $\mathcal{G}_D \subset \mathcal{G}_D^1$ denote the locus along which the morphism
\begin{equation}
\phi^*p_{v'}^*(q_v^*\mathcal{T}_v \otimes \mathcal{O}_X(D)) \otimes q^*\mathcal{T} \to \phi^*\mathcal{E}_v^2/V_v^2 \oplus \phi^*\tilde{\mathcal{E}}_v^2/V_v^2
\end{equation}
has rank at most $d_2 + d - g - r_2$ for any $w \in \overline{M}_{u_0}$ between $w_v$ and $w_{v'}$, where $v$ and $v'$ are as above. Arguing as in [Oss17, Theorem 6.1], we see that the generic tuples (resp., special tuples) in $\mathcal{G}_D^1$ are such that $W^1_v \subset H^0(X, L_v)$ and $W_v^1$ is independent of $v$ (resp., $V^1_v \subset H^0(X, L'_w)$ and $(L', V^1_{v'})$ is a limit linear series on $X_0$ of multidegree $w_0$). In addition, the generic tuples (resp., special tuples) in $\mathcal{G}_D$ are such that $W_v^2 \subset H^0(X, L(-E))$ and $W_v^2$ is independent of $v$ (resp. (i) $V_v^2 \subset H^0(X, L'_w, (-E'))$ and (ii) $(L'(-E'), (V_v^2)_{v'})$ is a limit linear series on $X_0$, where, given (i), (ii) is equivalent to that $(L', (V_v^2)_{v'})$ is a limit linear series on $X_0$). Summing up, we have proved the following proposition:

**Proposition 4.9.** The constructions of Subsections 4.3.1 and 4.3.2 agree set-theoretically, i.e. we have $|\mathcal{G}| = |\mathcal{G}_D|$.

**Remark 4.10.** It is unclear whether $\mathcal{G}$ and $\mathcal{G}_D$ are isomorphic as schemes. The main subtlety is that the determinantal condition in (4.4) for $\mathcal{G}$ is imposed for all $w \in M_{u_0}$, whereas in (4.7) it is only imposed for $w \in \overline{M}_{u_0}$. This is already significant at the level of the moduli space of limit linear series; see, for example, the proof of [LO18, Proposition 3.2.7].

**Remark 4.11.** Let $\tilde{\mathcal{G}}_D$ denote the subscheme of $\tilde{\mathcal{G}}_D^1$ cut out by the same determinantal condition as $\mathcal{G}_D$ in $\tilde{\mathcal{G}}_D^1$ (here note that $\mathcal{V}_0^2$ is also a subbundle of $\phi^*(\tilde{\mathcal{E}}_v)$ over $\tilde{\mathcal{G}}_D^1$). Let
\[
\text{Hilb}_d^\Box(X/B) \subset \text{Hilb}_d^\Box(X/B)
\]
denote the open subscheme parameterizing subschemes of $X$ that avoid both $D_0$ and the nodes of $X_0$. Then the preimage of $\text{Hilb}_d^\Box(X/B)$ in $\mathcal{G}_D$ agrees set-theoretically with its preimage in $\tilde{\mathcal{G}}_D$, both of which parameterize inclusions of linear series (resp., limit linear series) with base points disjoint with $D_0$ (resp., $D_0$). Indeed, for a generic $D$-tuple and $E \in \text{Hilb}_d^\Box(X/B)$, if $W_v^2 \subset H^0(X, L(D_\eta - E_\eta))$ and $W_v^1 \subset H^0(X, L)$, then $W_v^2 \subset H^0(X, L(-E_\eta))$, and similarly for the special $D$-tuples.

4.4. A smoothing theorem. In this subsection we assume that $X_0$ is a general chain of $g$ smooth elliptic curves. We also assume that $\rho(g, r_1, d_1) = \mu(d_1 - d_2, r_2, r_1) = 0$.

**Theorem 4.12.** The moduli scheme $\mathcal{G}$ is universally open and flat and reduced at every point $x \in \mathcal{G}_0$ corresponding to an inclusion of limit linear series on $X_0$ for which the ambient series is of word type $\left\lfloor \frac{12 \cdots (r_1 + 1)}{g/(r_1+1) \text{ times}} \right\rfloor$.

In particular, every such $x$ lifts to a unique inclusion of linear series on $X_\eta$.

**Proof.** The results of Section 5 imply that $x$ is isolated in $\mathcal{G}_0$, and that both the ambient and included limit linear series of $x$ are refined. By [Oss15, Proposition 3.7] it therefore suffices to show that (1) the map $\mathcal{G} \to B$ has universal relative dimension at least 0 at $x$; and (2) $\mathcal{G}_0$ is reduced at $x$. 
Proposition 4.9 together with [Oss15, Corollary 5.1] reduces item (1) to showing that $G_D$ has dimension at least 1 at $x$. Accordingly, choose $D$ so that $|D_0|$ avoids the set of base points of $x$. We then have

$$\dim G_D^1 = 1 + g + d + |V(\Gamma)|((r_2 + 1)(r_1 - r_2) + (r_1 + 1)(d_1 + d - g - r_1)).$$

It follows that each irreducible component of $G_D^2$ has dimension at least

$$\dim G_D^1 - \sum_v d_v(r_1 + 1) = \dim G_D^1 - \tilde{d}(r_1 + 1),$$

since it has codimension at most $\sum_v d_v(r_1 + 1) = \tilde{d}(r_1 + 1)$. Similarly $G_D^3$ has dimension at least

$$\dim G_D^1 - \tilde{d}(r_1 + 1) - d|V(\Gamma)|(r_2 + 1)$$

near $x$. On the other hand, since the included limit linear series corresponding to $x$ is refined, the linked chain of flags that cuts out $G_D$ inside $G_D^2$ is $(r_2 + 1, r_1 + 1)$-strict at $x$, and the codimension of $G_D$ inside $G_D^3$ is at most

$$|E(\Gamma)|((r_2 + 1)(d_2 + d - r_2 - g) + (r_1 - r_2)(d_1 + d - r_1 - g))$$

near $x$ by Theorem 3.6. A straightforward calculation now shows that $G_D$ has dimension at least $\rho_{g,r_1,d_1} + \mu + 1$ near $x$, as does $G_D$. So item (1) is proved.

We now prove item (2). To this end, let $(L_x, E_x, (V_{x,v}^2, V_{x,v}^1)_{v \in V(\Gamma)})$ be the special tuple represented by $x$. For each $v \in V(\Gamma)$ let $e_v$ denote the degree of $E_x$ in $Z_v$. According to Theorem 5.3 of Section 5, we have $e_v \in \{0,1\}$ for every $v$. Let $V(\Gamma)$ denote the set of vertices $\tilde{v}$ for which $e_{\tilde{v}} = 1$. We know that

$$x_1 := (L_x, (V_{x,v}^1)_{v}) \in G_{w_0}^r(X/B)_0 = G_{w_0}^r(X_0)$$

is a refined limit $G_{w_0}^r$ on $X_0$. But [Oss14, Proposition 4.2.6, Proposition 4.2.9] establishes that the moduli space $G_{w_0}^r(X_0)$ is (schematically) isomorphic to the Eisenbud–Harris moduli space of limit linear series over refined locus. In particular, $G_{w_0}^r(X/B)_0$ is reduced at $x_1$.

By construction, $G_D^2$ is locally isomorphic near $x$ to the closed subscheme of

$$\text{Hilb}^2_d(X_0) \times \prod_{v \in V(\Gamma)} Gr(r_2, V_{x,v}^1)$$

cut out by the (refined) ramification condition on $\prod_{v \in V(\Gamma)} Gr(r_2, V_{x,v}^1)$ induced by $(L_x, (V_{x,v}^2)_{v}) \in G_{w_0}^r(X_0)$, where $V_{x,v}^1$ denotes the restriction of $V_{x,v}$ to $Z_v$. Let $G_2^2$ be a neighborhood of $x$ in $G_0^2$ that witnesses this (local) isomorphism. Then by construction for each $\tilde{v} \in V(\Gamma)$ and each point $y = (E_{\tilde{v}}, (V_{\tilde{v},v}^2)_{v})$ in $G_2^2$, we have that $V_{\tilde{v},v}^2 \subset V_{x,v}^1$ contains a fixed vector $f_{\tilde{v},v} \in V_{x,v}^1$, determined by $x$.

On the other hand, any sufficiently small neighborhood $U_x$ of $x$ in $G_0^1$ is locally isomorphic to

$$\prod_{v \in V(\Gamma)} \text{Hilb}^1_{\tilde{e}}(Z_{\tilde{v}}) \times \prod_{v \in V(\Gamma)} Gr(r_2, V_{x,v}^1).$$

Here $\text{Hilb}^1_{\tilde{e}}(Z_{\tilde{v}}) = Z_{\tilde{v}}^e$. Thus we see that the vanishing locus of (4.5) on $U_x$ is contained in the vanishing locus of

$$\varphi_v^{-} y_{x,v}^2 \to \tilde{\delta}_v \delta_v^a L_x^v = H^0(Z_v, L_x^v) \otimes \mathcal{O}_{U_x} \to \tilde{\delta}_v^a (\delta_v^a L_x^v / (\delta_v^a L_x^v \otimes \delta_v^a X_v^e)).$$

(4.8)
Here $\mathcal{F}_{x,v}^1 \subset \mathcal{V}_{x,v}^1 \otimes O_{Gr(r_2, V_{x,v}^1)}$ is the universal subbundle, $L_v^r$ is the restriction of $(L_x)_{u_0}$ to $Z_v$, and $\mathcal{L}_v$ is the universal ideal sheaf on $\text{Hilb}_c^\circ(Z_v) \times Z_v$. Note that the determinantal locus is compatible with pullback, and locally it contains the condition imposed by vanishing of the image $\Delta_{\tilde{\delta}_v}$.

Hence $\mathcal{F}_{x,v}^1 \subset \mathcal{V}_{x,v}^1 \otimes O_{Gr(r_2, V_{x,v}^1)}$ is contained in $\mathcal{L}_v^r$.

Restricting (4.8) to $G^0_v$, we see that for every $\tilde{v} \in V^1(\Gamma)$, the determinantal ideal always contains the condition imposed by vanishing of the image $\tilde{f}_{x,\tilde{v}} \in H^0(G^0_v, \tilde{\delta}_v)^{\circ}((\tilde{\delta}_v^* L_v^r \otimes \sigma_v^* \mathcal{I}_v^c))$ of $\tilde{\delta}_v \sigma_v^* f_{x,v}$. Let $i_{\tilde{v}} : \text{Hilb}_c^\circ(Z_{\tilde{v}}) \hookrightarrow Z_{\tilde{v}}$ denote the natural inclusion, which induces a diagonal map $\Delta_{\tilde{v}} : \text{Hilb}_c^\circ(Z_{\tilde{v}}) \to \text{Hilb}_c^\circ(Z_{\tilde{v}}) \times Z_{\tilde{v}}$. We then have

$$\begin{array}{c}
\text{Hilb}_c^\circ(Z_{\tilde{v}}) \xrightarrow{i_{\tilde{v}}} Z_{\tilde{v}} \\
\sigma_v \downarrow \quad \delta_v \downarrow \\
U_x \xrightarrow{\Delta_{\tilde{v}}} U_x \times Z_{\tilde{v}} \xrightarrow{\tilde{\delta}_v} U_x \\
\text{Hilb}_c^\circ(Z_{\tilde{v}}) \xrightarrow{\Delta_{\tilde{v}}} \text{Hilb}_c^\circ(Z_{\tilde{v}}) \times Z_{\tilde{v}} \xrightarrow{\tilde{\delta}_v} \text{Hilb}_c^\circ(Z_{\tilde{v}}).
\end{array}$$

Since $\tilde{\delta}_v \circ \Delta_{\tilde{v}}$ is the identity map on $\text{Hilb}_c^\circ(Z_{\tilde{v}})$, and $\mathcal{L}_v^r$ is the ideal sheaf of $\Delta_{\tilde{v}}(\text{Hilb}_c^\circ(Z_{\tilde{v}}))$, it follows that

$$\tilde{\delta}_v^* (\delta_v^* L_v^r / (\sigma_v^* \mathcal{I}_v^c)) = \Delta_{\tilde{v}}^* \sigma_v^* L_v^r$$

and $\tilde{f}_{x,\tilde{v}} = \sigma_v^* i_{\tilde{v}}^* f_{x,v}$. Since $f_{x,v}$ cuts out a single point $P_{x,\tilde{v}}$ in $Z_{\tilde{v}}^0$ by Theorem 5.3, we see that, locally at $x$, $G^0_v$ is contained in

$$\tilde{G}_x = \prod_{\tilde{v} \in V^1(\Gamma)} \left( P_{x,\tilde{v}} \times Gr(r_2, \tilde{V}_{x,\tilde{v}}^1(-P_{x,\tilde{v}})) \right) \times \prod_{v \in V(\Gamma) \setminus V^1(\Gamma)} \text{Gr}(r_2, \tilde{V}_{x,v}^1).$$

Now imposing the ramification condition that defines $G^0_v$ inside $\tilde{G}_x$, we get a product over $\tilde{v} \in V^1(\Gamma)$ (resp., $v \in V(\Gamma) \setminus V^1(\Gamma)$) of intersections of pairs of Schubert varieties in $Gr(r_2, \tilde{V}_{x,\tilde{v}}^1(-P_{x,\tilde{v}}))$ (resp., $Gr(r_2, \tilde{V}_{x,v}^1)$), that is zero-dimensional. According to [KL04, Theorem 4.3.1], it is reduced; and locally it contains $G^0_v$. Hence $G^0_v$ is reduced at $x$. \hfill \Box

5. Combinatorics of inclusions of limit linear series

In this section we will describe (and partially implement) a general algorithm for computing the number of inclusions

$$g_{m-d-r}^s + p_1 + \cdots + p_d \hookrightarrow g_m^s$$

whenever $\rho = \mu = 0$ and the ambient $g_m^s$ is of combinatorial word type $\underbrace{12 \cdots s + 1}_u$ times, where $u \geq 1$.

Here the fact that $\mu = 0$ implies that $1 \leq s \leq r + 1$. The extremal cases $r = 1$ and $r = s - 1$ are distinguished, and accordingly we will pay particular attention to these.
**Definition 5.1.** A *good* inclusion of limit linear series on a curve $X$ of compact type is an inclusion (5.1) for which

1. No base point $p_i$ lies along a point of attachment of $X$; and
2. At most one base point $p_i$ of the included $(s-d+r)$-dimensional series lies along any given component.

**Situation 5.2.** In the remainder of this section, we will always assume $X$ is a general chain of elliptic curves and $\rho = \mu = 0$.

A crucial point is that whenever the ambient $g^*_{m}$ is of combinatorial word type $(12\cdots s+1)$, all inclusions (5.1) that satisfy item (1) in Definition 5.1 automatically satisfy item (2).

**Claim 5.4.** Fix a choice of positive integers $s < m$ and let $i_0$ and $M$ be positive integers for which $M \geq 2$ and $2 \leq i_0 \leq s$. Let $L(i_0, s, m)$ denote the unique increasing nearly-consecutive sequence of nonnegative integers of length $s+1$ beginning in $0$ and ending in $s$ with distinguished index $i_0$, and let $L^* = L^*(i_0, s, m)$ denote the $m^*$-complementary sequence defined by $L^*_i = m - 1 - L_i$ for all $i \neq i_0$, and $L^*_{i_0} = m - L_{i_0}$. Further choose an increasing subsequence $L^*$ of $L$ with length $s^* \leq s$. Then every decreasing subsequence $Q$ of $L^*$ for which

$$L^* + Q \leq (m - M - 1, \ldots, m - M - 1, m - M)$$

is obtained from the $m^*$-complement $(L^*)^c$ by shifting in at least $M s^* + 1$ places.

Note that the assumption that $L$ begin in 0 merely represents a convenient choice of normalization, and corresponds to removing the base points concentrated in $p$ of the ambient $g^*_{m}$. 
Proof of Claim 5.4. We treat first the case $M \geq s^* + 3$, which geometrically corresponds to the situation in which the number $M$ of base points is large relative to the dimension $s^* = s - d + r$ of the included series. The point is that in this regime, we have $(M - 1)s^* + M - 2 \geq Ms^* + 1$, so it suffices to show that $(M - 1)s^* + M - 2$ shifts are forced.

To do so, assume that the distinguished index $i_0$ does not belong to $L^*$. Then $L^*_i + (L^*)^+_i = m - 1$ for all $i$. It is clear that in passing from $(L^*)^+_i$ to $Q_i$, at least $M - 2$ shifts are required, irrespective of how large $M$ is relative to $i_0 - L^*_i$. But in fact at most one $(L^*)^+_i$ may shift by exactly $M - 2$ places. The analysis when $i_0$ belongs to $L^*$ is analogous (and easier); we conclude immediately.

Now say $2 \leq M \leq s^* + 2$. Given an arbitrary increasing length-$(s^* + 1)$ subsequence $L^*$ of $L$, let $S_1 = S_1(L^*)$ denote the collection of elements of $L^*$ lying either to the right of $L^*_{i_0}$, or at least $M$ places to the left of $L^*_{i_0}$, and let $S_2$ denote the complement of $S_1$ in $L^*$. Let $a^* = a^*(L^*)$ denote the cardinality of $S_1$ Note that every element of $S_1$ (resp., $S_2$) except for possibly one contributes $M$ forced shifts; the exceptional element, if it exists, contributes $M - 1$ (resp., $M - 2$) shifts. Because there is at most one exceptional element, it follows that $L^*$ is associated with at least $\nu(a^*) := (M - 1)(s^* + 1 - a^*) + Ma^* - 1$ forced shifts, and we may conclude provided $a^* \geq s^* + 3 - M$.

Indeed, $\nu$ is an increasing function of $a^*$, and by rewriting we see that $\nu(s^* + 3 - M) = Ms^* + 1$.

It remains to handle those cases in which $a^* < s^* + 3 - M$. Note that $\#S_2 \leq M$ by construction, so $a^* \geq s + 1 - M$ is automatic. Accordingly there are two basic situations, depending upon whether $a^* = s^* + 1 - M$ or $a^* = s^* + 2 - M$. In the first situation, the $M$ elements $L_{i_0 - j + 1}$, $j = 1, \ldots, M$ belong to $L^*$. More precisely,

$$S_2 = \{L_{i_0 - M + 1}, \ldots, L_{i_0}\} = \{i_0 - M, \ldots, i_0 - 2, i_0\}.$$ 

It is easy to check that $S_1$ induces at least $(s^* + 1 - M)M - \epsilon$ shifts, while $S_2$ minimally induces $M(M - 1) + 1 + \epsilon$ shifts, where the value of $\epsilon$ is 1 (resp., 0) when distinguished index of $L^*$ belongs to $S_1$ (resp., $S_2$).

The case in which $a^* = s^* + 2 - M$ is related to the preceding case by exchanging an element of $S_2$ for an element of $S_1$. It is easy, if tedious, to see that in so doing the number of induced shifts does not decrease.

Lemma 5.3 follows immediately from Claim 5.4. □

Lemma 5.5. The included $(s - d + r)$-dimensional series in any good inclusion (5.1) is refined.

Proof. Because our ambient $g_m^s$ is refined by construction, refinedness of the included $g_m^{s-d+r}$ amounts to the statement that the sum of the vanishing sequences of the included $(s - d + r)$-dimensional aspect is always maximal possible, i.e. either $(m - 2, \ldots, m - 2, m - 1)$ or $(m - 1, \ldots, m - 1, m)$ depending upon whether or not a (simple) base point lies along a given component. It also clearly suffices to prove maximality of vanishing on components containing base points $p_i$. Accordingly we may argue by induction on the index $1 \leq i \leq d$ of the given base point. Since the assertion is clear when $i = 1$, it suffices to show that maximality-of-vanishing is preserved in the presence of a base point. Much as in the proof of Lemma 5.3, this in turn reduces to a purely combinatorial statement about nearly-consecutive sequences of integers that we leave to the reader. □
Lemma 5.6. The position of a base point \( p_i \) of (the included \( g_m^{s-d+r} \) of) any good inclusion (5.1) along an elliptic component \( E_j \) is uniquely prescribed.

Proof. The maximality-of-vanishing property established in the proof of Lemma 5.5 implies, in particular, that there is always a pair of aligned vanishing orders \((a_p(k), a_q(k))\) of the included series along each component containing a base point \( p_i \). Letting \( \mathcal{O}(\alpha p + \beta q) \) denote the line bundle underlying the aspect of our limit linear series along \( E_j \), it then follows that the degree 0 line bundle \( \mathcal{O}(\alpha p + \beta q - a_p(k)p - a_q(k)q - p_i) \) has a nonzero global section \( \sigma_j \). This, in turn, may only happen if \( p_i \) is linearly equivalent to \((\alpha - a_p(k))p + (\beta - a_q(k))q\).

Remark 5.7. The global section \( \sigma_j \) of the aspect line bundle on \( E_j \) singled out in the proof of Lemma 5.6 is a key ingredient in the proof of our smoothing theorem 4.12 for inclusions of limit linear series in which the ambient series is of type \((12 \cdots s + 1)\).

Lemma 5.8. Assume that \( X \) is a general chain of \( g \) elliptic curves. The only inclusions of limit linear series (5.1) for which the ambient \( g^u_m \) is of combinatorial type \((12 \cdots s + 1)\) are good.

Proof. We will show that no inclusions (5.1) exist for which the ambient \( g^u_m \) is of combinatorial type \((12 \cdots s + 1)\) and some base point \( p_i \) of the included series \( g^{s-d+r}_m \) is supported at a node of \( X \). For this it suffices to show that no inclusions (5.1) exist along a curve \( \bar{X} \) obtained from \( X \) via blow-ups in the nodes for which the ambient \( g^s \) is of combinatorial type \((12 \cdots s + 1)\) and some \( p_i \) belongs to the interior of a rational component of \( \bar{X} \). Just as in the proofs of Lemma 5.3, it suffices to show that the placement of \( M \) putative base points (with multiplicities) along a rational component imposes at least \( M(s - d + r) + 1 \) shifts among indices of the included \( g^{s-d+r}_m \). And this, in turn, may be distilled to a claim about nearly-consecutive sequences, as follows.

Claim 5.9. Fix a choice of positive integers \( s < m \) and let \( i_0 \) and \( M \) be positive integers for which \( M \geq 2 \) and \( 2 \leq i_0 \leq s \). Let \( L(i_0, s, m) \) denote the unique increasing nearly-consecutive sequence of nonnegative integers of length \( s + 1 \) beginning in 0 and ending in \( s \) with distinguished index \( i_0 \), and let \( L^\circ = L^\circ(i_0, s, m) \) denote the \( m^\circ \)-complementary sequence defined by \( L^\circ_i = m - L_i \) for all \( i \). Further choose an increasing subsequence \( L^* \) of \( L \) with length \( s^* \leq s \). Then every decreasing subsequence \( Q \) of \( L^\circ \) for which

\[
L^* + Q \leq (m - M, \ldots, m - M)
\]

is obtained from the \( m^\circ \)-complement \((L^*)^\circ \) of \( L^* \) by shifting in at least \( Ms^* + 1 \) places.

Proof of Claim 5.9. The proof follows the same lines as the proof of Claim 5.4. It is clear, first of all, that every element of \( L^* \) contributes at least \( M - 1 \) shifts, so the issue is controlling the number of elements that contribute exactly \( M - 1 \) of these. But in fact there can be at most \( M - 1 \) of these, corresponding to elements of \( L \) that lie strictly less than \( M \) places to the left of \( L_{i_0} \). Accordingly, we see that at least \( (M - 1)(M - 1) + M(s^* + 1 - (M - 1)) = Ms^* + 1 \) shifts are forced.

Lemma 5.8 follows immediately from Claim 5.9.
5.1. The case r = 1. Assume that X is a general chain of g elliptic curves, that the ambient $g_\infty^n$ is of combinatorial type $(12\cdots s + 1)$, and that $r = 1$. Our basic inclusion (5.1) then becomes

\begin{equation}
(5.2) \quad g_\infty^{t(u+1)-(t+1)} + p_1 + \cdots + p_{t+1} \mapsto g_\infty^{2t(u+1)}
\end{equation}

and the genus of our elliptic chain X is $g = (2t + 1)u$. We characterize the set of good inclusions on X as follows.

**Theorem 5.10.** The set of inclusions (5.2) on X is indexed by the elements of the set

$S = \{(j_1, \ldots, j_{t+1}) : 1 \leq j_1 < \cdots < j_{t+1} \leq (2t+1)u \text{ and } j_k \neq 2k-1 \pmod{2t+1} \text{ for all } k = 1, 2, \ldots, t+1\}$.

**Proof.** The result follows easily from the assertion that when $r = 1$, the index of the unique aligned section of the included aspect $g_\infty^{t-d+1}$ along the $j_k$th component (along which the $k$th base point $p_k$ lies, by definition) is $(2k-1) \pmod{2t+1}$. To see this, note that when $r = 1$, the index of the aligned section, viewed as a section of the ambient series, necessarily increases by precisely 2 along each component containing a base point of the included series. But because the key assertion is clear when $k = 1$, by induction it also holds in the case of arbitrary $k$. □

**Remark 5.11.** We expect more generally that when $r = 1$, $\rho = \mu = 0$, and our ambient series is of type w, the set of good inclusions is indexed by

$S(w) = \{(j_1, \ldots, j_{t+1}) : 1 \leq j_1 < \cdots < j_{t+1} \leq (2t+1)u \text{ and } w_{j_k} \neq 2k-1, j = 1, 2, \ldots, t+1\}$.

A useful conceptualization of S is as follows. Let $G = G(S)$ denote a $d \times g$ (i.e. $(t+1) \times (2t+1)u$) grid of (vertices labeled by) positive integers between 1 and $s + 1$, whose $j$th row corresponds to the (placement of) the $j$th base point, and in which each row the (same) word $w = (12\cdots s + 1)\, \overset{u \text{ times}}{\smile}$ is written. The cardinality of S is then the number of positive traversals of G starting from an arbitrary vertex in the first row and ending on an arbitrary vertex in the $(t + 1)$th row, where positive means that each individual movement in a traversal is down (by a single unit) and to the right (by a nonzero number of units).

**Example 5.12.** Say $s = d = 2$. The corresponding grid G is then given by

```
* 2 3 * 2 3 ··· * 2 3
1 2 * 1 2 * ··· 1 2 *
```

in which asterisks denote prohibited positions. Each traversal may be thought of as a length-2 word, namely either 21, 22, 31, or 32. It is clear, furthermore, that the number of traversals is independent of the length-2 word chosen. So it suffices to compute the number $N(2, u - 1)$ of positive traversals associated with a fixed choice of length-2 word. The quantity $N(2, u - 1)$, in turn, computes the number of top-to-bottom paths in a directed graph $\Gamma(2, u - 1)$, which is a special case of a more general construction that we present next. It follows easily that the number of positive traversals of G is $4\binom{u}{2}$, which is the predicted value.

**Construction 5.13.** Given positive integers $d$ and $e$, we construct a graph $\Gamma(d, e)$ as follows.

(i) The vertices of $\Gamma(d, e)$ are the vertices of a $d \times e$ grid with integer-valued coordinates $(i, j)$, where $1 \leq i \leq d$ and $1 \leq j \leq e$. Each vertex is labeled by its row, i.e., by its first coordinate.
(ii) Edges in $\Gamma(d, e)$ link only vertices in distinct and adjacent rows, and an edge links $(i, j)$ with $(i + 1, k)$ if and only if $k \geq j$.

(iii) We orient $\Gamma(d, e)$ according to the convention that smaller labels point towards larger labels, i.e. each edge is oriented “from top to bottom”.

For example, $\Gamma(2, 4)$ is the following graph (the orientations on edges are omitted):

```
  1 1 1 1
e 2 2 2 2
```

Remark 5.14. Note that $s^d$ is the number of possible $d$-tuples $(\alpha_1, \ldots, \alpha_d)$ that label possible words corresponding to traversals of our original grid. The particularity of the $s = d = 2$ case is that the number of possible positive traversals is independent of the choice of $(\alpha_1, \alpha_2) \in \{(2, 1), (2, 2), (3, 1), (3, 2)\}$, but this is not true in general. On the other hand, we always have

$$N(d, e) = \binom{d + e - 1}{d}.$$  

Indeed, (5.3) follows after observing that the path counts $N(d, e)$ satisfy the binomial-type recursion $N(d, e) = N(d - 1, e) + N(d, e - 1)$ whenever $d, e \geq 2$, and that $N(1, e) = e$ and $N(d, 1) = 1$.

The essential question, then, is understanding how $d$-tuples are stratified according to their associated graphs $\Gamma(d, e)$, i.e. which values $e$ are possible. To do so, we introduce an auxiliary graph $\Gamma^s(d)$. Here $\Gamma^s(d)$ has integer coordinate-valued vertices $(i, j)$ where $1 \leq i \leq d$ and $i \leq j \leq s+i-1$. Vertices with coordinates $(i, j)$ are labeled by $1 + (i + j - 1) \mod s$. We now consider the set of all (not-necessarily positive) top-to-bottom traversals of $\Gamma^s(d)$, starting from a vertex in the first (top) row and ending in a vertex in the bottom ($d$th) row. Those that involve precisely $k$ backwards edges are indexed by $d$-tuples with associated graphs $\Gamma(d, u - 1 - k)$. In general, we have $0 \leq k \leq d - 2$.

Putting all of this together, we conclude the following.

**Theorem 5.15.** When $\rho = \mu = 0$ and $r = 1$, the number of inclusions (5.2) for a fixed ambient $g^2_{2t(u+1)}$ of combinatorial word type $(12 \ldots s + 1)$ on $X$ is given by

$$N_1(t, u) = \sum_{k=0}^{d-2} \binom{d + u - 2 - k}{d}$$

where $d = t + 1$, $s = 2t$, and

$$N_k^s(d) := \#\{\text{traversals of } \Gamma^s(d) \text{ with precisely } k \text{ backwards edges}\}.$$  

Remark 5.16. Since traversals of $\Gamma^s(d)$ are in bijection with elements of the set of $d$-tuples

$$W^s(d) := \{(x_1, \ldots, x_d) : j \leq \sum_{i=1}^{j} x_i \leq s + j - 1 \text{ for all } j = 1, \ldots, d\}$$
it follows that
\[ N^s_k(d) = \# \{ \text{elements of } W^s(d) \text{ with precisely } k \text{ negative entries} \}. \]

In general, it is unclear if closed formulas for the coefficients \( N^s_k(d) \) exist. However, they are relatively straightforward to compute, as we will see shortly.

**Example 5.17.** Say \( s = 4 \) and \( d = 3 \). In this case, the associated diagram \( \Gamma^4(3) \) is as follows:

\[
\begin{array}{cccc}
2 & 3 & 4 & 5 \\
4 & 5 & 1 & 2 \\
1 & 2 & 3 & 4
\end{array}
\]

Traversals in the diagram with backwards edges linking vertices in the first and second rows correspond to the words \( 44m \), \( 54m \), and \( 55m \), where \( 1 \leq m \leq 4 \); the traversal corresponding to \( 441 \) is drawn. There are 12 such traversals; by symmetry, it follows there are 24 traversals with backwards edges in the diagram. Since there are \( 4^3 = 64 \) traversals in total, there are 40 traversals without backwards edges. It follows that the total number of inclusions (5.2) in this case is \( 40 \binom{4}{3} + 24 \binom{4}{3} \).

5.2. **Diagrammatic traversals with prescribed numbers of backwards edges.** By working a bit harder, we can write explicit formulas for the number \( N^{2d-2}_j(d) \) of traversals of a \( d \times (2d - 2) \) grid \( \Gamma^{2d-2}(d) \) as above with precisely \( k \) backwards edges.

We now codify each traversal as a binary string on \( (d-1) \) bits, in which instances of 1 correspond to backwards steps. These form a poset \( P \) with \( (d-1) \) levels, in which level \( k \) consists of those vertices whose binary strings involve precisely \( k \) instances of 1; there are \( (d-1) \) levels because the minimal (resp., maximal) possible number of backwards edges in a traversal is clearly 0 (resp., \( (d-2) \)).

The number of traversals with exactly the maximal number \( k = d - 2 \) of backwards edges is easy to compute. Indeed, each such traversal is classified by its associated binary string, which in turn describes which rows in the grid are associated with backwards edges. It is not hard to see in this case that the set of rows involving backwards edges is partitioned into at most two connected components, and that each component involving \( j \) rows contributes a factor of \( N(j, 2d - 2j) \). It follows that

\[
N^{2d-2}_{d-2}(d) = \sum_{j=1}^{d-1} N(j, 2d - 2j) \cdot N(d - j, 2d - 2(d - j)) = \sum_{j=1}^{d-1} \binom{2d - j - 1}{j} \binom{d + j - 1}{d - j}.
\]

Each of the products in (5.4) represents the contribution of a binary string in \( P \). More generally, each vertex in \( P \) indexed by a string \( s \) has a natural multiplicity given by the number of traversals of a \( d \times (2d - 2) \) grid involving at least those backwards edges that are specified by \( s \); we may then use a process of inclusion-exclusion to determine the exact number of traversals involving (only) those backwards edges that are specified by \( s \).

Moreover, the multiplicity of a given string \( s \) at level \( k \) in \( P \) depends only on the underlying (unordered) partition \( \lambda \) of \( k \) to which it is associated; we denote this number by \( m_d(\lambda) \). Explicitly, letting \( \ell \) denote the length of \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) (i.e. the total number of nonzero parts, which are
allowed to be nondistinct) we have

$$m_d(\lambda) = \prod_{i=1}^{\ell} N(\lambda_i + 1, 2d - 2\lambda_i) \cdot N(1, 2d - 2)^{d-\ell-|\lambda|}$$

where by convention $N(1, 2d - 2)^{d-\ell-||\lambda||} = 0$ whenever the exponent $d - \ell - |\lambda|$ is negative.

In order to compute $N_{2d-2}^{\ell+j}(d)$, another crucial combinatorial ingredient is the number $c_d(\lambda)$ of binary strings on $(d-1)$ bits associated to a given partition $\lambda$. Before deriving the general formula for $c_d(\lambda)$, we consider a typical situation in which all features of the general case are already present. Namely, say $\lambda = (1^j)$. We then want to count strings of type

$$*10*10\cdots*10*1*$$

in which there are $d - 2j$ asterisks that may either be empty or occupied by zeros. Accordingly, we see that $c_d((1^j))$ is equal to the coefficient of $x^{d-2j}$ in $(1 + x + \cdots + x^{d-2j})^{d-2j}$, from which it follows easily that

$$c_d((1^j)) = [x^{d-2j}][(1-x^{-1})^{d-2j}] = [x^{d-2j}](1-x)^{2j-d} = \left(\begin{array}{c}2j-d \\ d-2j \end{array}\right).$$

In the more general situation, write $\lambda = (\lambda_1^e_1, \ldots, \lambda_m^e_m)$ where $\sum_{i=1}^{m} e_i = \ell$. We then have, accounting for reorderings of the (a priori unordered) partition $\lambda$:

$$c_d(\lambda) = \left(\begin{array}{c}\ell \\ e_1, \ldots, e_m \end{array}\right) [x^{d-|\lambda|-\ell}](1-x)^{\ell+|\lambda|-d} = \left(\begin{array}{c}\ell \\ e_1, \ldots, e_m \end{array}\right) \cdot \left(\begin{array}{c}\ell + |\lambda| - d \\ d - |\lambda| - \ell \end{array}\right).$$

Now set

\begin{equation}
N_{2d-2}^{\ell+j}(d) := \sum_{\lambda:|\lambda|=j} \left(\begin{array}{c}\ell \\ e_1, \ldots, e_m \end{array}\right) \cdot \left(\begin{array}{c}\ell + |\lambda| - d \\ d - |\lambda| - \ell \end{array}\right) \cdot \prod_{i=1}^{\ell} N(\lambda_i + 1, 2d - 2\lambda_i) \cdot N(1, 2d - 2)^{d-\ell-|\lambda|}
\end{equation}

\begin{equation}
= \sum_{\lambda:|\lambda|=j} \left(\begin{array}{c}\ell \\ e_1, \ldots, e_m \end{array}\right) \cdot \left(\begin{array}{c}\ell + |\lambda| - d \\ d - |\lambda| - \ell \end{array}\right) \cdot (2d-2)^{d-\ell-|\lambda|} \cdot \prod_{i=1}^{\ell} \left(\begin{array}{c}2d - \lambda_i + 1 \\ \lambda_i + 1 \end{array}\right).
\end{equation}

It follows from the discussion above that the number of traversals of $\Gamma^{2d-2}(d)$ involving exactly $j$ backwards edges is given by

\begin{equation}
N_{2d-2}^{\ell+j}(d) = \sum_{j \leq k \leq d-2} (-1)^{k-j} \gamma_j^k(d) \cdot N_{2d-2}^{\ell+j}(d)
\end{equation}

where the coefficients $\gamma_j^k(d)$ are positive integers specified by inclusion-exclusion carried out over $\mathcal{P}$. Explicitly, we have $\gamma_j^j(d) = 1$, and whenever $k > j$, the value of $\gamma_j^k(d)$ is prescribed by the requirement that

\begin{equation}
\sum_{j \leq k \leq d-2} (-1)^{k-j} \binom{k}{j} \gamma_j^k(d) = 0.
\end{equation}

5.3. Comparison with Macdonald’s formula. An inclusion (5.2) of linear series on a smooth curve $C$ codifies a $d$-secant $(d-2)$-plane to the image of $C$ in $\mathbb{P}^{2d-2}$. The (virtual) number $N_d = N_d(g, m)$ of these is computable via a classical formula of Macdonald’s.
Proposition 5.18 ([ACGH85], Proposition VIII.4.2). The virtual number of included $g_{m-d+r}^{s-d+r}$ inside a fixed $g_m$ on a general genus-$g$ curve is

$$N(g, s, m, d, r) = \frac{(-1)^{\binom{d}{2}}}{r!} \prod_{i=1}^{r} \left[ (1-t_i)^{g+s-m} \left( 1 + \sum_{i} t_i \right)^g \Delta(t)^2 \right]_{(t_1 t_2 \ldots t_r)^{s-d+2r}}$$

(5.8)

where $\Delta(t) = \prod_{i>j} (t_i - t_j)$ and $[F(t_1, ..., t_n)]_{i,e}$ denotes the coefficient of $t^d$ in the multi-variable polynomial $F(t_1, ..., t_n)$.

Note that in the statement of Proposition VIII.4.2 in loc. cit. there is a misprint in the second version of the formula, and the $s+d+2r$ appearing in the subscript should be replaced by $s-d+2r$. When $r = 1$, the first version of Macdonald’s formula specializes to the statement that

$$N_d(g, m) = \sum_{i=0}^{d} (-1)^i \binom{g+2d-2-m}{i} \binom{g}{d-i}.$$

(5.9)

Of course, when $\rho = \mu = 0$, the parameters $d$, $g$, and $m$ specialize to $d = t + 1$, $g = (2t+1)u$, and $m = 2t(u+1)$. Applying our smoothing theorem 4.12 for good inclusions, together with Lemma 5.8 (“all inclusions are good”), we immediately deduce the following result.

Theorem 5.19. Assume that $r = 1$ and $\rho = \mu = 0$, so that $d = t + 1$, $g = (2t+1)u$, and $m = 2t(u+1)$. The Macdonald numbers $N_d(g, m)$ of (5.9) agree with the numbers $N(t,u)$ of Theorem 5.15, i.e. we have

$$\sum_{k=0}^{d-2} N_{k}^{2d-2}(d) \binom{d+u-2-k}{d} = \sum_{i=0}^{d} (-1)^i \binom{u}{i} \binom{(2d-1)u}{d-i}.$$  

(5.10)

where the coefficients $N_{k}^{2d-2}(d)$ are determined by equations 5.5 and 5.6.

Remark 5.20. From Theorem 5.19 it follows that Macdonald’s formula, together with Theorem 5.10, determines the coefficients $N_{2d-2}^{2d-2}(d)$ uniquely. To see why this is interesting, note that in [Cot11] we computed the exponential generating function for Macdonald’s secant plane numbers via intersection theory on the $d$th Cartesian product of a smooth curve. In that approach, the class formula is modeled on the combinatorics of the complete graph on $d$ vertices. So Theorem 5.19 gives a bridge between the combinatorics of complete graphs and that of the graphs $\Gamma(d,g)$.

5.4. Calculation of secant plane numbers $N_1(d)$ for small values of $d$. Here we compute the coefficients $N_{k}^{2d-2}(d)$ for all $2 \leq d \leq 6$, $k = 0, \ldots, d - 2$ by explicitly applying the formulas of the preceding subsection.

- $d = 2$. We have $N_0^2(2) = 4$, so (5.10) reduces to the statement that

$$4 \binom{u}{2} = 2u^2 - 2u.$$  

- $d = 3$. We have $N_0^4(3) = 24$ and $N_0^6(3) = 4^3 - N_0^4(3) = 40$, so (5.10) reduces to

$$40 \binom{u+1}{3} + 24 \binom{u}{3} = \frac{32}{3} u^3 - 12u^2 + \frac{4}{3} u.$$  

- $d = 4$. We have $N_0^8(4) = 120$ and $N_0^{10}(4) = 4^4 - N_0^8(4) = 640$, so (5.10) reduces to

$$640 \binom{u+2}{4} + 120 \binom{u}{4} = \frac{160}{3} u^4 - 40u^3 + 4u^2.$$  

- $d = 5$. We have $N_0^{12}(5) = 5040$ and $N_0^{14}(5) = 4^5 - N_0^{12}(5) = 24192$, so (5.10) reduces to

$$24192 \binom{u+3}{5} + 5040 \binom{u}{5} = \frac{5040}{5} u^5 - 200u^4 + 4u^3.$$  

- $d = 6$. We have $N_0^{18}(6) = 40320$ and $N_0^{20}(6) = 4^6 - N_0^{18}(6) = 699840$, so (5.10) reduces to

$$699840 \binom{u+4}{6} + 40320 \binom{u}{6} = \frac{40320}{6} u^6 - 280u^5$$
• $d = 4$. Applying (5.4), we find $N_2^4(4) = 148$. We next compute $N_2^6(4)$ by applying the inclusion-exclusion relation (5.6), which yields

$$N_1^6(4) = N_1^{1,+}(4) - 2 \cdot N_2^{6,+}(4) = 3 \cdot N(2, 4) \cdot 6^2 - 2 \cdot (2 \cdot N(3, 2) \cdot 6 + N(2, 4)^2) = 784.$$ 

It follows that $N_2^6(4) = 64^2 - 932 = 364$ and (5.10) reduces to the statement that

$$364\left(\frac{u + 2}{4}\right) + 784\left(\frac{u + 1}{4}\right) + 148\left(\frac{u}{4}\right) = 54u^4 - 72u^3 + 20u^2 - 2u.$$

• $d = 5$. We will check by explicit calculation that

$$\sum_{j=0}^{3} N^5_j(8) \left(\frac{u + 3 - j}{5}\right) = \frac{4096}{15}u^5 - \frac{1280}{3}u^4 + \frac{556}{3}u^3 - \frac{100}{3}u^2 + \frac{8}{5}u.$$ 

To do so, we first apply (5.4), obtaining $N_1^5(8) = 920$. Next, applying (5.6), we compute

$$N_2^5(8) = N_2^{1,+}(8) - 3 \cdot N_3^{5,+}(8) = 3 \cdot N(3, 4) \cdot 8^2 + 3 \cdot N(2, 6) \cdot 8 - 3 \cdot N_3^{3}(8) = 11664$$

and

$$N_3^5(8) = N_3^{1,+}(8) - 2 \cdot N_4^{5,+}(8) + 3 \cdot N_5^{5,+}(8) = 4 \cdot N(2, 6) \cdot 8^3 - 2(3 \cdot N(3, 4) \cdot 8^2 + 3 \cdot N(2, 6)^2 \cdot 8) + 3 \cdot N_3^{3}(8) = 16920;$$

it follows that $N_3^5(8) = 8^5 - (N_3^1(8) + N_3^2(8) + N_3^3(8)) = 3264$.

• $d = 6$. We will explicitly check that

$$\sum_{i=0}^{3} N^6_i(10) \left(\frac{u + 4 - i}{6}\right) = \frac{12500}{9}u^6 - 2500u^5 + \frac{13100}{9}u^4 - 386u^3 + \frac{392}{9}u^2 - 2u.$$ 

Applying (5.4) yields $N_1^6(10) = 5776$. Next via (5.6) we compute

$$N_2^6(10) = 3 \cdot N_3^{1,+}(10) - 4 \cdot N_3^{6,+}(10) = 155012;$$

$$N_3^6(10) = 4 \cdot N_4^{1,+}(10) - 3 \cdot N_4^{6,+}(10) + 6 \cdot N_4^{6,+}(10);$$

and

$$N_4^6(10) = 5 \cdot N_4^{1,+}(10) - 2 \cdot N_4^{6,+}(10) + 3 \cdot N_5^{6,+}(10) - 2 \cdot N_4^{6,+}(10) = 308044.$$ 

It follows that $N_6^6(10) = 10^6 - (N_1^6(10) + N_2^6(10) + N_3^6(10) + N_4^6(10)) = 29260$.

5.5. The case $r = s - 1$. Here the basic inclusion (5.1) specializes to

$$(5.11) \quad g_{(r+1)(u+1)-2r} = p_1 + \cdots + p_{2r} \mapsto g_{(r+1)(u+1)}^{r+1}.$$ 

The combinatorics in this regime is more complicated. Its distinguishing feature is that the shifting of the distinguished indices (associated to the included limit linear pencil) induced by the presence of base points is not deterministic as in the $r = 1$ case; rather, shifting exhibits branching, in a way that is explicitly predicted by the Plücker poset of $\text{Gr}(s - d + r + 1, s + 1) = \text{Gr}(2, r + 2)$.

More precisely, the set of Plücker coordinates of $\text{Gr}(2, r + 2)$ is indexed by pairs of numbers from the index set $[r + 2] = \{1, \ldots, r + 2\}$. On the other hand, each component of our elliptic chain that contains a base point of the included linear series is associated to a pair of pairs $\pi_1, \pi_2 \in [r + 2]^2$ which share a common edge in the Plücker poset. In particular, $\pi_1$ and $\pi_2$ have an element in common in $[r + 2]$. The important point is now the following.

Lemma 5.21. Assume the ambient series $g_{(r+1)(u+1)}^{r+1}$ is of word type $(12 \cdots s + 1)^u$, and that

$(\pi_1, \pi_2)$ is an edge of the Plücker poset for $\text{Gr}(2, r + 2)$, with $\nu = \pi_1 \cap \pi_2$. Every index $j = j(\pi_1, \pi_2)$ of a component of a general elliptic chain along which an inclusion (5.11) has a base point whose local evolution (i.e., shifting) is described by $(\pi_1, \pi_2)$ satisfies $j \not\equiv \nu \pmod{s + 1}$.
Proof. The index $\nu$, by definition, describes the unique aligned section of the aspect of the included limit linear pencil along the $j$th component. On the other hand, because the ambient series $g_{(r+1)(u+1)}^{r+1}$ is of word type $(12\cdots s+1)$, $j \mod (s+1)$ is the distinguished index of the ambient aspect $g_m^*$ associated with the unique section that vanishes to total order $m$ in the points of attachment of the $j$th component. It follows immediately that $j \not\equiv \nu \mod (s+1)$. □

It is not hard to see, moreover, that $j$ being congruent to $\nu$ modulo $s+1$ is the only local numerical obstruction to the placement of a base point along the $j$th component of our elliptic chain. Accordingly, counting good inclusions in the $r = s - 1$ case reduces to resolving a graphical enumeration problem analogous to the one described in the $r = 1$ case for each top-to-bottom traversal of the Plücker poset $\mathcal{P}$, and then summing over all traversals of the Plücker poset in order to obtain the total number of inclusions (5.11).

Explicitly, each traversal of $\mathcal{P}$ is indexed by a prohibition sequence $\Lambda$ comprised of the $d$ forbidden base point indices in Lemma 5.21. To each fixed choice of $\Lambda$ we associate a $d \times g$ grid $G(\Lambda)$ whose $j$th row is of the form $(12\cdots s+1)$, but in which no entry $\Lambda(j)$ may be traversed. From the prohibition grid $G(\Lambda)$ we next extract a diagram $\Gamma^*(d; \Lambda)$ whose traversals with prescribed numbers $k$ of backwards edges (nearly) stratify the space of $d$-tuples on the alphabet $[s]$ according to their associated $u$-binomial contributions. The total number of positive traversals of $G(\Lambda)$ is a sum of (multiples of) $u$-binomials $(u+d-2-k)$, $k = 0, \ldots, d - 2$, plus a finite leftover term that we label $R_\Lambda$. Summing over all top-to-bottom traversals $\Gamma$ of $\mathcal{P}$ yields the following result.

**Theorem 5.22.** When $\rho = \mu = 0$ and $r = s - 1$, the number of inclusions (5.11) for a fixed ambient $g_{(r+1)(u+1)}^{r+1}$ of combinatorial word type $(12\cdots s+1)$ on $X$ is given by

$$N_{s-1}(t,u) = \sum_{k=0}^{d-2} N_k^*(d) \binom{d+u-2-k}{d} + R$$

in which $d = 2r$, $s = r + 1$,

$$N_k^*(d) := \# \{\text{traversals of } \Gamma^*(d; \Lambda) \text{ with precisely } k \text{ backwards edges} \}$$

and $R := \sum_{\Lambda} R_\Lambda$. Here $\Lambda$ varies over all top-to-bottom traversals of the Plücker poset $\mathcal{P}$ of $Gr(2,s+1)$.

5.5.1. **Example.** Say $d = 6$ and $s = 4$. In this case we are counting inclusions

$$g_{4u-2}^1 + p_1 + \cdots + p_6 \hookrightarrow g_{4u+4}^4$$

along a chain of genus $g = 5u$. Here the relevant Grassmannian is $Gr(2,5)$, whose Plücker poset is drawn below.
There are five top-to-bottom traversals of the Plücker poset $\mathcal{P}$ of $\text{Gr}(2, 5)$, corresponding to five prohibition sequences $\Lambda$. A typical traversal of $\mathcal{P}$ is $\Lambda = (1, 3, 2, 4, 3, 5)$. The corresponding finite diagram $\Gamma^4(6; (1, 3, 2, 4, 3, 5))$ is given by

$$
\begin{array}{cccccc}
2 & 3 & 4 & 5 & \ast \\
4 & 5 & 1 & 2 & \ast \\
5 & 1 & \ast & 3 & 4 \\
1 & 2 & 3 & \ast & 5 \\
2 & \ast & 4 & 5 & 1 \\
3 & 4 & \ast & 1 & 2
\end{array}
$$

and the coefficient $N^*_s(d; (1, 3, 2, 4, 3, 5))$ of $\binom{d+u-2-k}{j}$ contributed by prohibition sequence $(1, 3, 2, 4, 3, 5)$ is equal to the number of top-to-bottom traversals of $\Gamma^4(6)$ with (exactly) $0 \leq k \leq d-2$ backwards arrows, plus $4$, which is the number of nonnegative traversals of the following leftover diagram corresponding to the $u = 2$ case:

$$
\begin{array}{cccc}
2 & 3 \\
4 & \\
5 & \\
1 & \\
2 & \\
3 & 4
\end{array}
$$

5.6. Comparison with Macdonald’s formula. When $r = s - 1$, the second version of Macdonald’s formula 5.8 establishes that the virtual number of $2r$-secant $(r - 2)$-planes to a smooth curve in $\mathbb{P}^{r-1}$ is given by

$$(5.12) \quad \frac{(-1)^{\ell(z)}}{r!} \left[ (1 + t_1)(1 + t_2) \right]^{m-g-s} (1 + t_1 + t_2)^g (t_1 - t_2)^2 \left[ t_1^{r+1} t_2^{r+1} \right].$$

When $\rho = \mu = 0$, so that $g = u(s + 1)$ and $m = (r + 1)(u + 1)$, Theorem 4.12 and Lemma 5.8 imply that the expression in (5.12) agrees precisely with our good inclusion number $N_{s-1}(t, u)$.

**Theorem 5.23.** Assume $\rho = \mu = 0$, and let $N_{s-1}(t, u)$ denote the number of inclusions (5.11) on a general elliptic chain of genus $g = u(s + 1)$ for which the ambient series is of combinatorial type
Secant planes of a general curve via degenerations

We have

\[ N_{s-1}(t, u) = \frac{(-1)^t}{r!} \left[ ((1 + t_1)(1 + t_2))^{-u}(1 + t_1 + t_2)^{u(r+2)}(t_1 - t_2)^2 \right]_{t_1^r} t_2^{r+1} \]

where \( N_{s-1}(t, u) \) is as defined in the statement of Theorem 5.22.

5.7. The general case. Counting included series inside a fixed ambient series \((12 \cdots s + 1)\) times when \(\rho = \mu = 0\) and \(1 \leq r \leq s - 1\) is arbitrary follows the basic template established in the discussion of the \(r = s - 1\) case; the caveat is that one must replace the Plücker poset by the appropriate subposet determined by the collection of possible shift sequences induced by the placement of base points along the (interiors of) components of a general elliptic chain curve \(X\). In this way we obtain an obvious generalization of Theorems 5.19 and 5.23.

Remark 5.24. The monodromy action on \(G_m^*(C)\) induced by a degeneration of a general genus-\(g\) curve \(C\) to a flag curve is well-known to be transitive whenever \(\rho(g, s, m) = 0\) by a celebrated result of Eisenbud and Harris; it follows that the number of \(d\)-secant \((d - r - 1)\)-planes to (the image of) each \(g_m^*\) is constant whenever \(\mu(d, r) = 0\). In particular, the total number of \(d\)-secant \((d - r - 1)\)-planes to series \(g_m^*\) on a general curve is simply \(\eta\) times the Macdonald number \(N(g, s, m, d, r)\), where \(\eta = \eta(g, s, m)\) is the generalized Catalan number

\[ \eta = g! \cdot \prod_{i=0}^{s} \frac{i!}{(g - m + s + i)!}. \]

Our results thus show that when \(\rho = \mu = 0\) all secant planes on a general curve are “detected” by inclusions of limit linear series. A next logical line of inquiry would be to extend our counting scheme to compute counts of secant planes along a general curve in situations for which the sum \(\rho + \mu\) is zero, but \(\rho\) itself is positive.

6. A DIMENSION-THEORETIC MODULI PATHOLOGY

What follows is an example, due to Melody Chan, that shows that the moduli space of included limit linear series constructed in Section 4 may have unexpectedly large dimension. This already occurs near the points whose ambient limit linear series has word type distinct from that used in our smoothing theorem 4.12.

Example 6.1. Let \((g, d_1, d_2, r_1, r_2) = (12, 10, 7, 2, 1)\). Here \(\rho(g, r_1, d_1) = 0\) and \(\mu = -1\), so we expect the corresponding moduli space of inclusions of limit linear series to be empty. We will show that this is not the case in general. Indeed, let \(E\) be a chain of \(13\) curves \(Z_i\) in which all but the middle component are elliptic, and the remaining component \(Z_7\) is rational. There is a one-dimensional family of inclusions of limit linear series with three base points supported on the rational component. In this family, the ambient limit linear series is constant and its aspects are specified by the following vanishing sequences at the nodes:
The included linear series on each elliptic component $Z_j$, $j \neq 7$, is specified unambiguously by the underlined vanishing orders, however there is a one-dimensional ambiguity along $Z_7$. To see this, let $(x, y)$ be projective coordinates on $Z_7$, and let $P = 0 = Z_6 \cap Z_7$ and $Q = \infty = Z_7 \cap Z_8$ be the two distinguished points of attachment on $Z_7$. Letting $V_1^1 := \text{span}\{x^8y^2, x^5y^5, x^3y^8\}$ and $V_1^2 := \text{span}\{x^8y^2 + \lambda x^5y^5, x^5y^5 + \lambda x^3y^8\}$, we obtain an inclusion of linear series $V_1^2 \hookrightarrow V_1^1$ for every $\lambda \neq 0$. So associated with every $\lambda \neq 0$ there is an inclusion of limit linear series with three base points $\{ (\omega : 1) | \omega^3 = -\lambda \}$ on $Z_7$.

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