Gradedness of the set of rook placements in $A_{n-1}$

Mikhail V. Ignatev

Abstract. A rook placement is a subset of a root system consisting of positive roots with pairwise non-positive inner products. To each rook placement in a root system one can assign the coadjoint orbit of the Borel subgroup of a reductive algebraic group with this root system. Degenerations of such orbits induce a natural partial order on the set of rook placements. We study combinatorial structure of the set of rook placements in $A_{n-1}$ with respect to a slightly different order and prove that this poset is graded.

1 Introduction

Denote by $G = \text{GL}_n(\mathbb{C})$ the group of all invertible $n \times n$ matrices over the complex numbers. Let $B$ be the Borel subgroup of $G$ consisting of all invertible upper-triangular matrices, $U$ be the unipotent radical of $B$ (it consists of all upper-triangular matrices with 1’s on the diagonal), and $T$ be the subgroup of all invertible diagonal matrices (it is the maximal torus of $G$ contained in $B$). Next, let $\mathfrak{b}$ and $\mathfrak{n}$ be the Lie algebras of $B$ and $U$ respectively.

Let $\Phi$ be the root system of $G$ with respect to $T$, $\Phi^+$ be the set of positive roots with respect to $B$, $\Delta$ be the set of simple roots, and $W$ be the Weyl group of $\Phi$ (for basic facts on algebraic groups and root systems, see [3], [4] and [5]). The root system $\Phi$ is of type $A_{n-1}$; as usual, we identify the set of positive roots with the subset of the Euclidean space $\mathbb{R}^n$ of the form

$$A^+_{n-1} = \{\epsilon_i - \epsilon_j, \ 1 \leq i < j \leq n\}.$$ 

Under this identification, $\Delta$ consists of the roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq n-1$ ($\{\epsilon_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$).

2020 MSC: 06A07, 17B22, 17B08.

Key words: Root system, rook placement, Borel subgroup, coadjoint orbit, graded poset

During the work on the final version of the paper I was partially supported by FAPESP grant no. 19/00563-1. A part of this work was done during my stay at the Oberwolfach Research Institute for Mathematics in February–March 2018 via the program “Research in pairs” together with Alexey Petukhov. I thank Alexey for fruitful discussions. I also thank the Oberwolfach Research Institute for Mathematics for hospitality and financial support.

Affiliation:
Samara National Research University, ul. Ak. Pavlova, 1, 443011, Samara, Russia
E-mail: mihail.ignatev@gmail.com
Definition 1. A rook placement is a subset $D \subseteq \Phi^+$ such that $(\alpha, \beta) \leq 0$ for all distinct $\alpha, \beta \in D$. (Here $(\cdot, \cdot)$ denotes the standard inner product on $\mathbb{R}^n$.)

Example 1. Let $n = 6$. Below we draw the rook placement $D = \{\epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_6, \epsilon_3 - \epsilon_5\}$. If a root $\epsilon_i - \epsilon_j$ is contained in $D$, then we put the symbol $\otimes$ in the $(j, i)$th entry of the $n \times n$ chessboard. If we interpret these symbols as rooks, then it follows from the definition that the rooks do not hit each other.

We denote the set of all rook placement in $A_{n-1}$ by $\mathcal{R}(n)$. Further, let $\mathcal{I}(n)$ be the set of all orthogonal rook placements. Below we describe two closely related partial orders on these sets.

The Lie algebra $\mathfrak{n}$ has the basis $\{e_{\alpha}, \alpha \in \Phi^+\}$ consisting of the root vectors: for $\alpha = \epsilon_i - \epsilon_j$, $e_{\alpha}$ is nothing but the elementary matrix $e_{i,j}$. Denote by $\{e_{\alpha}^*, \alpha \in \Phi^+\}$ the dual basis of the dual space $\mathfrak{n}^*$. Given a rook placement $D$, put

$$f_D = \sum_{\beta \in D} e_{\beta}^* \in \mathfrak{n}^*.$$ 

The group $B$ acts on its Lie algebra $\mathfrak{b}$ by the adjoint action, and $\mathfrak{n}$ is an invariant subspace. Hence one has the dual action of the groups $B$ and $U$ on the space $\mathfrak{n}^*$; we call this action coadjoint. We say that the $B$-orbit $\Omega_D \subset \mathfrak{n}^*$ of the linear form $f_D$ is associated with the rook placement $D$.

Such orbits play an important role in the A.A. Kirillov’s orbit method [14], [15] describing representations of $B$ and $U$. For $D \in \mathcal{I}(n)$, such orbits were studied by A.N. Panov in [18] and by me in [6]. One can define analogues of such orbits for other root systems, see [7], [8], [9] for the case of $\mathcal{I}(n)$. For arbitrary rook placements in $\mathcal{R}(n)$, such orbits were considered in [10]; see also [1], [2], where C. Andre and A. Neto used rook placements to construct so-called supercharacter theory for the group $U$. Note that in [16], [17], A. Melnikov studied the adjoint $B$-orbits of elements of the form $\sum_{\beta \in D} e_{\beta}, D \in \mathcal{I}(n)$.

Given a subset $A \subseteq \mathfrak{n}^*$, we will denote by $\overline{A}$ its closure with respect to the Zarisski topology. There exists a natural partial order on the set $\mathcal{R}(n)$ induced by the degenerations of associated orbits: we will write $D_1 \leq_B D_2$ if $\Omega_{D_1} \subseteq \overline{\Omega_{D_2}}$. We need to introduce one more partial order on the set of rook placements. Namely, given an arbitrary $D \in \mathcal{R}(n)$, denote by $R_D$ the $n \times n$ matrix defined by

$$(R_D)_{i,j} = \begin{cases} 
\#\{\epsilon_a - \epsilon_b \in D \mid a \leq j, b \geq i\}, & \text{if } i > j, \\
0, & \text{otherwise.}
\end{cases}$$
Put $D_1 \leq D_2$ if $(R_{D_1})_{i,j} \leq (R_{D_2})_{i,j}$ for all $i, j$.

**Example 2.** Let $n = 4$, $D_1 = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_4\}$, $D_2 = \{\epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_4\}$. Then

$$D_1 = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\otimes &  &  & \\
3 &  &  & \\
\otimes &  &  & \\
4 &  &  & \\
1 & 2 & 3 & 4
\end{array}, \quad R_{D_1} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix},$$

$$D_2 = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
& & \otimes & \\
&  &  & \\
&  & \otimes & \\
&  &  & \\
1 & 2 & 3 & 4
\end{array}, \quad R_{D_2} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix}.$$

We conclude that $D_1 \leq D_2$. On the other hand, it is easy to check that $D_1 \not\leq_B D_2$, see [10, Remark 1.6 (iii)], so these two partial orders on $\mathcal{R}(n)$ do not coincide.

Nevertheless, it turns out that these orders are closely related to each other. Precisely, given rook placements $D_1, D_2 \in \mathcal{R}(n)$, it follows from $D_1 \leq_B D_2$ that $D_1 \leq D_2$ [10, Theorem 1.5]. Furthermore, if $D_1, D_2 \in \mathcal{I}(n)$ then the conditions $D_1 \leq_B D_2$ and $D_1 \leq D_2$ are equivalent [6, Theorem 1.7]. Besides, given a rook placement

$$D = \{\epsilon_{i_1} - \epsilon_{j_1}, \ldots, \epsilon_{i_l} - \epsilon_{j_l}\},$$

we denote by $w_D \in S_n$ the permutation, which is equal to the product of transpositions

$$w_D = (i_1, j_1) \ldots (i_l, j_l).$$

Now, both of the conditions above (for orthogonal rook placements $D_1, D_2$) are equivalent to the condition that $w_{D_1}$ is less or equal to $w_{D_2}$ with respect to the Bruhat order [6, Theorem 1.1]. Similar facts are true for orthogonal rook placements in the root system $C_n$, see [7]. Note that these results are in some sense “dual” to A. Melnikov’s results.

In the paper [12], F. Incitti studied the order on $\mathcal{I}(n)$ induced by the Bruhat order on the elements $w_D$, $D \in \mathcal{I}(n)$, from purely combinatorial point of view (see also [11] for other classical root systems). In particular, given an orthogonal rook placement $D$, he explicitly described the set of its immediate predecessors (it consists of $D' \in \mathcal{I}(n)$ such that there exists an edge from $D'$ to $D$ in the Hasse diagram of this poset). The set of immediate predecessors for the partial order $\leq$ on $\mathcal{I}(n)$ and $\mathcal{R}(n)$ was described by me in [6, Lemmas 3.6, 3.7, 3.8] and by A.S. Vasyukhin and me in [10, Theorem 3.3] respectively. (In the case of $\mathcal{I}(n)$, the set of immediate predecessors for $\leq$ coincides with the set described by F. Incitti, which implies that those two partial orders coincide.)
Furthermore, F. Incitti proved that the poset $I(n)$ is graded and calculated its Möbius function. Recall that a finite poset $X$ is called graded if it has the greatest and the lowest elements and all maximal chains in $X$ have the same length. Gradedness is equivalent to the existence of a rank function. By definition, it is a (unique) function $\rho$ on $X$, which value on the lowest element is zero, such that if $x$ is an immediate predecessor of $y$ then $\rho(y) = \rho(x) + 1$. In [12, Theorem 5.2], F. Incitti constructed the rank function on $I(n)$. As the main result of this paper, we prove the gradedness of the poset $R(n)$.

The main tool used in the proof is so-called Kerov placements (see [13]). To each rook placement $D \in R(n)$ one can assign a certain orthogonal rook placement $K(D) \in I(2n - 2)$. We prove that if rook placements $D_1$ is an immediate predecessor of $D_2$ in $R(n)$ then $K(D_1)$ is an immediate predecessor of $K(D_2)$ in $I(2n - 2)$ (and vice versa), see Theorem 3. As a corollary, we construct a rank function on $R(n)$ and prove the gradedness of this poset, see Corollary 1.

The structure of the paper is as follows. In the next section we describe the set of immediate predecessors of a given rook placement for $I(n)$ and $R(n)$. In the third section we introduce the Kerov map

$$K : R(n) \to I(2n - 2)$$

and show that it preserves the property “to be an immediate predecessor”. This allows us to use F. Incitti’s results to construct a rank function on $R(n)$, which implies the gradedness of this poset.

## 2 Immediate predecessors

To prove that the set $R(n)$ is graded with respect to the partial order introduced above, we need to describe the set of immediate predecessors of a given rook placement in $R(n)$ and $I(n)$. Such a description for $R(n)$ was provided in [10], while for $I(n)$ it was presented in F. Incitti’s work [12]. Recall that a rook placement $D \in R(n)$ is called an immediate predecessor of a rook placement $T \in R(n)$ if $D < T$ and there are no $S \in R(n)$ such that $D < S < T$. (As usual, $D < T$ means that $D \leq T$ and $D \neq T$.) In other words, there exists an oriented edge from $D$ to $T$ in the Hasse diagram of the poset $R(n)$. The definition of immediate predecessors for $I(n)$ is literally the same.

We denote the set of all immediate predecessors in $R(n)$ (respectively, in $I(n)$) of a rook placement $D \in R(n)$ (respectively, of an orthogonal rook placement $D \in I(n)$) by $L_R(D)$ (respectively, by $L_I(D)$). This set consists of rook placements of several types, which we will describe now. First, we will consider the set $L_R(D)$ in details.

It is convenient to introduce the following notation. We will write simply $(i, j)$ instead of $\epsilon_j - \epsilon_i$, $i > j$. Besides, for each $k$ from 1 to $n$, we put

$$R_k = \{(k, s) \in \Phi^+ \mid 1 \leq s < k\}, \quad C_k = \{(r, k) \in \Phi^+ \mid k < r \leq n\}.$$

**Definition 2.** The sets $R_k$, $C_k$ are called the $k$th row and the $k$th column of $\Phi^+$ respectively. We will write $\text{row}(\alpha) = k$ and $\text{col}(\alpha) = k$ if $\alpha \in R_k$ and $\alpha \in C_k$ respectively. Note that, for $D \in R(n)$, one has

$$|D \cap R_k| \leq 1 \text{ and } |D \cap C_k| \leq 1 \text{ for all } 1 \leq k \leq n.$$
Furthermore, if $D \in \mathcal{I}(n)$ then

$$|D \cap (\mathcal{R}_k \cup \mathcal{C}_k)| \leq 1 \text{ for all } 1 \leq k \leq n.$$  

There exists a natural partial order on the set of positive roots: given $\alpha, \beta \in \Phi^+$, by definition, $\alpha \leq \beta$ if $\beta - \alpha$ is a (probably, empty) sum of positive roots. In the other words,

$$(a, b) \leq (c, d) \text{ if } c \geq a \text{ and } d \leq b.$$  

Given a rook placement $D \in \mathcal{R}(n)$, denote by $\tilde{M}(D)$ the set of minimal roots from $D$ (with respect to $\leq$). Now, we set

$$M_{\mathcal{R}}(D) = \{(i, j) \in \tilde{M}(D) \mid D \cap \mathcal{R}_k \neq \emptyset \text{ and } D \cap \mathcal{C}_k \neq \emptyset \text{ for all } j < k < i\},$$

$$N_{\mathcal{R}}^{-}(D) = \{D_{(i,j)}^-, (i, j) \in M_{\mathcal{R}}(D)\},$$

where $D_{(i,j)}^- = D \setminus \{(i, j)\}$.

Next, fix a root $(i, j) \in D$. Denote

$$m = \min\{k \mid j < k < i \text{ and } D \cap \mathcal{C}_k = \emptyset\}.$$  

Suppose that such a number $m$ exists. Assume that $D \cap \mathcal{R}_k \neq \emptyset$ for all $k$ from $j + 1$ to $m$. Assume, in addition, that there are no $(p, q) \in D$ such that $(i, j) > (p, q)$ and $(i, m) \neq (p, q)$. The set of all roots $(i, j) \in D$ satisfying these conditions is denoted by $A_{\mathcal{R}}^{-}(D)$; given $(i, j) \in A_{\mathcal{R}}^{-}(D)$, we put

$$D_{(i,j)}^{+, \mathcal{R}} = (D \setminus \{(i, j)\}) \cup \{(i, m)\}.$$  

Similarly, suppose that there exists a number

$$m' = \max\{k \mid j < k < i \text{ and } D \cap \mathcal{R}_k = \emptyset\}.$$  

Assume also that $D \cap \mathcal{C}_k \neq \emptyset$ for $m' + 1 \leq k \leq i - 1$ and that there are no $(p, q) \in D$ such that $(i, j) > (p, q)$ and $(m', j) \neq (p, q)$. Denote the set of all such $(i, j)$s by $A_{\mathcal{R}}^{+}$; given $(i, j) \in A_{\mathcal{R}}^{+}$, we put

$$D_{(i,j)}^{+, \mathcal{R}} = (D \setminus \{(i, j)\}) \cup \{(m', j)\}.$$  

Now, let $B_{(i,j)}^{\mathcal{R}}(D)$ be the set of roots $(\alpha, \beta) \in D$ such that $(\alpha, \beta) > (i, j)$ and there are no $(p, q) \in D$ satisfying $(i, j) < (p, q) < (\alpha, \beta)$. For each $(\alpha, \beta) \in B_{(i,j)}^{\mathcal{R}}(D)$ we set

$$D_{(i,j)}^{(\alpha, \beta), \mathcal{R}} = (D \setminus \{(i, j), (\alpha, \beta)\}) \cup \{(i, \beta), (\alpha, j)\}.$$  

By definition, let

$$N_{\mathcal{R}}^{0}(D) = \left\{D_{(i,j)}^{+, \mathcal{R}}, (i, j) \in A_{\mathcal{R}}^{+}\right\} \cup \left\{D_{(i,j)}^{-, \mathcal{R}}, (i, j) \in A_{\mathcal{R}}^{-}\right\} \cup \bigcup_{(i,j) \in D} \left\{D_{(i,j)}^{(\alpha, \beta), \mathcal{R}}, (\alpha, \beta) \in B_{(i,j)}^{\mathcal{R}}(D)\right\}.$$
Example 3. Let \( n = 8 \) and \( D = \{(3,1), (6,2), (7,3), (5,4), (8,5)\} \). Clearly, \( M_{\mathcal{R}}(D) = \{(5,4)\}, (8,5) \in A_{(5,4)}^\mathcal{R}, (3,1) \in A_{(5,4)}^\mathcal{R} \) and \( 6,2 \) \( \in B_{(5,4)}^\mathcal{R}(D) \). On the picture below we draw the rook placements \( D, D_{(5,4)}^{(6,2), \mathcal{R}}, D_{(3,1)}^{\uparrow, \mathcal{R}} \) and \( D_{(8,5)}^{\rightarrow, \mathcal{R}} \).

Next, fix a root \((i,j) \in D\), and consider a pair \((\alpha,\beta) \in \mathbb{Z} \times \mathbb{Z}\). Suppose that \( i > \beta \geq \alpha > j \), \( D \cap \mathcal{R}_\alpha = D \cap C_\beta = \emptyset \), \( D \cap \mathcal{R}_k \neq \emptyset \), \( D \cap C_k \neq \emptyset \) for all \( \alpha < k < \beta \), and the conditions \((p,q) \in D, (i,j) > (p,q), (\alpha, j) \not> (p,q) \) imply \((i, \beta) > (p,q)\). Moreover, assume that if \( \alpha \neq \beta \) then \( D \cap \mathcal{R}_\beta \neq \emptyset \) and \( D \cap C_\alpha \neq \emptyset \). Denote the set of all such pairs \((\alpha, \beta)\) by \( C_{(i,j)}^\mathcal{R}(D) \). For an arbitrary pair \((\alpha, \beta) \in C_{(i,j)}^\mathcal{R}(D)\), we put

\[
D_{(i,j)}^{\alpha,\beta, \mathcal{R}} = (D \setminus \{(i,j)\}) \cup \{(i,\beta), (\alpha, j)\}.
\]

By definition, let

\[
N_{\mathcal{R}}^+(D) = \bigcup_{(i,j) \in D} \left\{ D_{(i,j)}^{\alpha,\beta, \mathcal{R}}, (\alpha, \beta) \in C_{(i,j)}^\mathcal{R}(D) \right\}.
\]
Example 4. Let \( n = 6 \) and \( D = \{(4, 1), (6, 2), (5, 4)\} \), then \((3, 3) \in C_{(6, 2)}^R(D)\). On the picture below we draw the rook placements \( D \) and \( D_{(6, 2)}^{3,3, R} \).

Finally, we set
\[
N_R(D) = N_R^-(D) \cup N_R^0(D) \cup N_R^+(D).
\]

The set of immediate predecessors of a given rook placement from \( R(n) \) is described as follows.

**Theorem 1 ([10, Theorem 3.3]).** Let \( D \in R(n) \). Then \( L_R(D) = N(D) \).

Now we turn to the description of immediate predecessors for \( I(n) \). Given an orthogonal rook placement \( D \in R(n) \), put
\[
M_I(D) = \{(i, j) \in \tilde{M}(D) \mid D \cap (R_k \cup C_k) \neq \emptyset \text{ for all } j < k < i\},
\]
\[
N_I^-(D) = \{D_{(i,j)}^-, (i,j) \in M_I(D)\},
\]
where \( D_{(i,j)}^- = D \setminus \{(i,j)\} \), as above.

Let \( D \in I(n) \), \((i,j) \in D \). Denote
\[
m = \min\{k \mid j < k < i \text{ and } D \cap C_k = D \cap R_k = \emptyset\}.
\]

Suppose that such a number \( m \) exists. Assume that there are no \((p,q) \in D\) such that \((i,j) > (p,q)\) and \((i,m) \not> (p,q)\). The set of all \((i,j) \in D\) satisfying these conditions is denoted by \( A_{(i,j)}^L(D)\); given \((i,j) \in A_{(i,j)}^L(D)\), we set
\[
D_{(i,j)}^{i,j, L} = (D \setminus \{(i,j)\}) \cup \{(i,m)\}.
\]

Similarly, suppose that there exists
\[
m' = \max\{k \mid j < k < i \text{ and } D \cap R_k = D \cap C_k = \emptyset\},
\]
and there are no \((p,q) \in D\) such that \((i,j) > (p,q)\) and \((m',j) \not> (p,q)\). The set of all such \((i,j)\)'s is denoted by \( A_{(i,j)}^L(D)\); given \((i,j) \in A_{(i,j)}^L(D)\), we set
\[
D_{(i,j)}^{i,j, L} = (D \setminus \{(i,j)\}) \cup \{(m',j)\}.
\]
Next, let \( B^I_{(i,j)}(D) \) be the set of roots \((\alpha, \beta) \in D\) such that \( j < \beta < i < \alpha\),
\[
D \cap (R_r \cup C_r) \neq \emptyset
\]
for all \( \beta < r < i \) and there are no \((p, q) \in D\) for which \( j < q < \beta < p < i \) or \( \beta < q < i < p < \alpha\) (in other words, for which \((i, j) > (p, q)\) and \((\beta, j) \not\succ (p, q)\), or \((\alpha, \beta) > (p, q)\) and \((\alpha, i) \not\succ (p, q)\)). To each \((\alpha, \beta) \in B^I_{(i,j)}(D)\) we assign the set
\[
D^{(\alpha, \beta), I}_{(i,j)} = (D \setminus \{(i, j), (\alpha, \beta)\}) \cup \{(\beta, j), (\alpha, i)\}.
\]

Now, let
\[
N^0_I(D) = \left\{ D^{(\alpha, \beta), I}_{(i,j)}, (i, j) \in A^I_{(i,j)} \right\} \cup \left\{ D^{(\alpha, \beta), I}_{(i,j)}, (i, j) \in A^+_{(i,j)} \right\}.
\]

\[
\bigcup_{(i,j) \in D} \left\{ D^{(\alpha, \beta), R}_{(i,j)}, (\alpha, \beta) \in B^{R}_{(i,j)}(D) \right\} \cup \bigcup_{(i,j) \in D} \left\{ D^{(\alpha, \beta), I}_{(i,j)}, (\alpha, \beta) \in B^{I}_{(i,j)}(D) \right\}.
\]

**Example 5.** If \( n = 8, D = \{(5, 1), (6, 2), (8, 4)\}\), then \((8, 4) \in B^{I}_{6, 2}(D)\), hence

\[
D = \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 &  &  &  &  &  &  & \\
3 &  &  &  &  &  &  & \\
4 &  &  &  &  &  &  & \\
5 &  &  &  &  &  &  & \\
6 &  &  &  &  &  &  & \\
7 &  &  &  &  &  &  & \\
8 &  &  &  &  &  &  & \\
\end{array}, \quad D^{(8, 4), I}_{(6, 2)} = \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 &  &  &  &  &  &  & \\
3 &  &  &  &  &  &  & \\
4 &  &  &  &  &  &  & \\
5 &  &  &  &  &  &  & \\
6 &  &  &  &  &  &  & \\
7 &  &  &  &  &  &  & \\
8 &  &  &  &  &  &  & \\
\end{array}.
\]

Besides, denote by \( C^I_{i,j}(D) \) the set of pairs \((\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}\) such that \( i > \beta > \alpha > j\),
\[
D \cap (R_\alpha \cup C_\alpha) = D \cap (R_\beta \cup C_\beta) = \emptyset,
\]
\[
D \cap (R_k \cup C_k) \neq \emptyset \text{ for all } \beta > k > \alpha, \text{ and if } (p, q) \in D, \ (i, j) > (p, q), (\alpha, j) \not\succ (p, q) \text{ then } (i, \beta) > (p, q). \text{ For each pair } (i,j) \in C^I_{i,j}(D) \text{, we put}
\]
\[
D^{\alpha, \beta, I}_{(i,j)} = (D \setminus \{(i, j)\}) \cup \{(i, \beta), (\alpha, j)\}.
\]
Example 6. Let $n = 8$, $D = \{(4, 1), (8, 2), (7, 6)\}$, then $(3, 5) \in C^{\mathcal{I}}_{(8, 2)}(D)$, so

Finally, we denote

$$N^+_{\mathcal{I}}(D) = \bigcup_{(i, j) \in D} \left\{ D_{(i, j)}^{\alpha, \beta, \mathcal{I}} : (\alpha, \beta) \in C^{\mathcal{I}}_{(i, j)}(D) \right\},$$

$$N_{\mathcal{I}}(D) = N^-_{\mathcal{R}}(D) \cup N^0_{\mathcal{R}}(D) \cup N^+_{\mathcal{I}}(D).$$

Immediate predecessors in $\mathcal{I}(n)$ are described by the following F. Incitti’s theorem (see also [6, Subsection 2.4]).

Theorem 2 ([12, Theorem 5.1]). Let $D \in \mathcal{I}(n)$. Then $L_{\mathcal{I}}(D) = N_{\mathcal{I}}(D)$.

3 Kerov map and the main result

In this section, we introduce our main technical tool, Kerov orthogonal rook placements, and, using them, prove that $\mathcal{R}(n)$ is graded.

Definition 3. Let $n \geq 3$, and $D$ be a rook placement from $\mathcal{R}(n)$. A Kerov rook placement corresponding to $D$ is, by definition, the orthogonal rook placement $K(D) \in \mathcal{I}(2n - 2)$ constructed by the following rule: if

$$D = \{(i_1, j_1), \ldots, (i_s, j_s)\},$$

then

$$K(D) = (2i_1 - 2, 2j_1 - 1) \cdots (2i_s - 2, 2j_s - 1).$$

(Kerov rook placements were introduced in the paper [13]). We call the map $K: \mathcal{R}(n) \to \mathcal{I}(2n - 2)$ given by the rule $D \mapsto K(D)$ the Kerov map.

Example 7. If $n = 8$ and $D = \{(3, 1), (6, 2), (7, 3), (5, 4), (8, 6)\} \in \mathcal{R}(8)$, then

$$K(D) = (4, 1) \cdot (10, 3) \cdot (12, 5) \cdot (8, 7) \cdot (14, 11)$$

$$= \left( \begin{array}{ccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
4 & 2 & 10 & 1 & 12 & 6 & 8 & 7 & 9 & 3 & 14 & 5 & 13 & 11
\end{array} \right) \in \mathcal{I}(14).$$

The following proposition is evident.
Proposition 1. Let \( D, T \in \mathcal{R}(n) \). Then the conditions \( T \leq D \) and \( K(T) \leq K(D) \) are equivalent.

The following theorem plays the crucial role in the proof of the main result.

Theorem 3. Let \( D, T \in \mathcal{R}(n) \) be rook placements. Then the conditions \( T \in L_T(D) \) and \( K(T) \in L_T(K(D)) \) are equivalent.

Proof. Clearly, \( K(T) \in L_T(D) \) implies \( T \in L_T(D) \). Indeed, since there are no orthogonal involutions from \( \mathcal{T}(2n-2) \) between \( K(T) \) and \( K(D) \), we conclude that, in particular, there are no Kerov involutions between them. It remains to prove that the converse is also true.

Assume that \( T \in L_T(D) \). By Theorem 1, this is equivalent to

\[
T \in N_T(D) = N^-(D) \cup N^0_T(D) \cup N^+_{T}(D).
\]

We will consider these variants case-by-case.

First, suppose that \( T \in N^-_T(D) \). This means that \( T = D^-_{(i,j)} \) for a certain root \((i,j) \in M(D)\). Automatically, \( K(T) = K(D) \setminus \{(2i - 2, 2j - 1)\} \). It follows immediately from \((i,j) \in \bar{M}(D)\) that \((2i - 2, 2j - 1) \in \bar{M}(K(D))\). Since \((i,j) \in M(D)\), we see that \(D \cap \mathcal{R}_k \) and \(D \cap \mathcal{C}_k \) are nonempty if \( i < k < j \). This shows that \( K(D \cap \mathcal{R}_{2k-2}) \) and \( K(D) \cap \mathcal{C}_{2k-1} \) are nonempty for all such \( k \). Thus,

\[(2i - 2, 2j - 1) \in M(K(D))\]

i.e., \( K(T) \in N^-_T(K(D)) \). By Theorem 2, \( K(T) \in L_T(K(D)) \).

Next, assume that \( T \in N^0_T(D) \). If \( T = D^{(\alpha, \beta), \mathcal{R}}_{(i,j)} \) for some \((i,j) \in D\), \((\alpha, \beta) \in \mathcal{B}^{\mathcal{R}}_{(i,j)}(D)\), then it is easy to see that

\[(2\alpha - 2, 2\beta - 1) \in \mathcal{B}^{\mathcal{R}}_{(2i-2,2j-1)}(K(D))\]

and

\[K(T) = K(D)^{(2\alpha - 2, 2\beta - 1), \mathcal{R}}_{(2i-2,2j-1)} \in N^0_T(K(D))\]

hence

\[K(T) \in N^0_T(D) \subset L_T(K(D))\].

Now consider the case when \( T = D^{\rightarrow, \mathcal{R}}_{(i,j)} \) for some \((i,j) \in A^\mathcal{R}_i\). (The case \( T = D^{\leftarrow, \mathcal{R}}_{(i,j)} \), \((i,j) \in A^\mathcal{R}_i\) can be considered similarly.) Let \( T = (D \setminus \{(i,j)\}) \cup \{(i,m)\} \), then

\[K(T) = (K(D) \setminus \{(2i - 2, 2j - 1)\}) \cup \{(2i - 2, 2m - 1)\}\].

Since there are no root in \( D \) which is less than \((i,j)\) but not less than \((i,m)\), we have a similar condition for \( K(D) \). Since \( D \cap \mathcal{C}_k \neq \emptyset \) for \( P \). Heymans: Pfaffians and skew-symmetric matrices \( j < k < m \), one has \( K(D) \cap \mathcal{C}_{2k-1} \neq \emptyset \) for such \( k \). On the other hand, \( D \cap \mathcal{R}_k \) is nonempty for \( j < k \leq m \), so \( K(D) \cap \mathcal{R}_{2k-2} \) is also nonempty for such \( k \). Thus, \( K(D) \cap (\mathcal{R}_k \cup \mathcal{C}_k) \neq \emptyset \) for \( 2j - 1 < k < 2m - 1 \), which means that \( (2i - 2, 2j - 1) \in A^\mathcal{R}_j \) and \( K(T) = K(D)^{\rightarrow, \mathcal{R}}_{(2i-2,2j-1)} \). Hence, by Theorem 2, \( K(T) \in L_T(K(D)) \), as required.
Finally, suppose that $T \in N^+_R(D)$, i.e., $T = D^{\alpha, \beta}_R$ for certain $(i, j) \in D$ and $(\alpha, \beta) \in C^R_{(i, j)}(D)$. Since $i > \beta \geq \alpha > j$, we have

$$2i - 2 > 2\beta - 1 > 2\alpha - 2 > 2j - 1.$$ 

It follows from $D \cap R_\alpha = D \cap C_\beta = \emptyset$ that

$$K(D) \cap R_{2\alpha - 2} = K(D) \cap C_{2\beta - 1} = \emptyset.$$ 

Since $K(D)$ is a Kerov rook placement, the condition

$$K(D) \cap C_{2\alpha - 2} = K(D) \cap R_{2\beta - 1} = \emptyset$$

is satisfied automatically. If $\alpha = \beta$ then there is nothing to prove. If $\beta > \alpha$ then $D \cap R_k \neq \emptyset$ and $D \cap C_k \neq \emptyset$ for all $k$ from $\alpha + 1$ to $\beta - 1$, hence $K(D) \cap R_{2k - 2} \neq \emptyset$ and $K(D) \cap C_{2k - 1} \neq \emptyset$ for all such $k$. Furthermore, $D \cap R_\beta$ and $D \cap C_\alpha$ are nonempty, which implies that $K(D) \cap R_{2\beta - 2}$ and $D \cap C_{2\alpha - 1}$ are also nonempty. Thus, we obtain $K(D) \cap (R_k \cap C_k) \neq \emptyset$ for all $k$ from $2\alpha - 1$ to $2\beta - 2$, as required. We conclude that $(2\alpha - 2, 2\beta - 1) \in C^R_{(2i - 2, 2j - 1)}(D)$ and $K(T) = K(D)^{2\alpha - 2, 2\beta - 1, I}_R$. Theorem 2 guarantees that $K(T) \in L_I(K(D))$. The proof is complete.

Corollary 1. For each $n \geq 2$ the poset $R(n)$ is graded with the rank function

$$\rho(D) = \frac{l(w_{K(D)}) + |D|}{2},$$

where $l(w)$ is the length of a permutation $w$ in the corresponding symmetric group.

Proof. As we mentioned in the introduction, F. Incitti showed that the set $I(2n-2)$ of orthogonal rook placements is graded. Precisely [11, Theorem 5.3.2], the rank function on this poset has the form

$$\rho(D) = \frac{l(w_D) + |D|}{2}.$$ 

Applying Theorem 3, we see that the restriction of this rank function to $K(R(n))$ in fact provided the rank function of the required form on $R(n)$. This concludes the proof.

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Received: 11 October 2019
Accepted for publication: 7 November 2019
Communicated by: Ivan Kaygorodov