Abstract

Quasi-symmetric functions show up in an approach to solve the Kadomtsev-Petviashvili (KP) hierarchy. This moreover features a new nonassociative product of quasi-symmetric functions that satisfies simple relations with the ordinary product and the outer coproduct. In particular, supplied with this new product and the outer coproduct, the algebra of quasi-symmetric functions becomes an infinitesimal bialgebra. Using these results we derive a sequence of identities in the algebra of quasi-symmetric functions that are in formal correspondence with the equations of the KP hierarchy.

1 Introduction

Quasi-symmetric functions [1–8] in a set of commuting variables extend the ring of symmetric functions [9] and show up in various branches of mathematics, most notably in combinatorics. The theory of symmetric functions has many applications in mathematics and physics. In particular, Schur polynomials play an important role in the \( \tau \)-function formulation of the famous Kadomtsev-Petviashvili (KP) hierarchy (see e.g. [10]) of completely integrable nonlinear partial differential equations. In this work we demonstrate the appearance of quasi-symmetric functions for the potential form of the KP hierarchy, and more generally for its noncommutative version (see e.g. [11]), where the dependent variable has values in a noncommutative associative algebra (typically an algebra of matrices of functions). Moreover, we show that the noncommutative KP hierarchy has an algebraic counterpart in the algebra of quasi-symmetric functions. These results involve a nonassociative product of quasisymmetric functions that satisfies nice relations with the ordinary product and the outer coproduct. Here we meet a weak form of nonassociativity characterized as follows [12, 13].

Definition 1.1. Let \( \mathbb{A} \) be a nonassociative ring (or algebra over a commutative ring) with product \( \bullet \). \( \mathbb{A} \) is called weakly nonassociative if

\[ (a, b \bullet c, d) = 0 \quad \forall a, b, c, d \in \mathbb{A}, \]

where \( (a, b, c) = (a \bullet b) \bullet c - a \bullet (b \bullet c) \) is the associator.

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Section 2 demonstrates the appearance of quasi-symmetric functions, and the new product $\bullet$ of the latter, in an approach to solve the KP hierarchy. Section 3 recalls some basic facts on quasi-symmetric functions needed in the sequel. In section 4 we study the new product in more detail. In particular, we find that the algebra $QSym$ of quasi-symmetric functions is generated by the identity element via $\bullet$. Section 5 establishes relations between the new product and the outercoproduct [3], which we denote by $\Delta$. In particular, $(QSym, \bullet, \Delta)$ turns out to be an infinitesimal bialgebra [16, 17]. Section 6 reveals a simple relation between the product $\bullet$ and the antipode of the Hopf algebra of quasi-symmetric functions. Section 7 derives a sequence of “KP identities” in $QSym$.

The analysis of the KP equation (and more generally the KP hierarchy) in section 2 actually exhibits a generalization of quasi-symmetric functions to quasi-supersymmetric functions, and section 8 briefly treats this extension. The algebra of quasi-supersymmetric functions extends the algebra of supersymmetric functions [9,18–21], and an appearance of the latter in the context of the KP hierarchy has been noted in [22,23]. Section 9 contains some concluding remarks.

### 2 From the KP equation to quasi-symmetric functions

The noncommutative KP equation is the partial differential equation

$$4\phi_{t_1t_3} - 3\phi_{t_2t_2} - \phi_{t_1t_1t_1t_1} = 6(\phi_{t_1}\phi_{t_2})_{t_1} - 6[\phi_{t_1}, \phi_{t_2}],$$

where $\phi$ has values in a (typically non-commutative) associative algebra $A$, supplied with a structure to define differentiability with respect to independent variables $t_n$, $n = 1, 2, 3$. $\phi_{t_n}$ denotes the partial derivative of $\phi$ with respect to $t_n$. Inserting the (formal) power series ansatz [11, 24–26]

$$\phi = \sum_{n \geq 1} \epsilon^n \phi^{(n)},$$

and reading off coefficients of powers of the parameter $\epsilon$, we obtain the equations

$$4\phi^{(n)}_{t_1t_3} - 3\phi^{(n)}_{t_2t_2} - \phi^{(n)}_{t_1t_1t_1t_1} = 6 \sum_{r=1}^{n-1} \left[ (\phi^{(r)}_{t_1} \phi^{(n-r)}_{t_1})_{t_1} - \phi^{(r)}_{t_1} \phi^{(n-r)}_{t_2} + \phi^{(r)}_{t_2} \phi^{(n-r)}_{t_1} \phi^{(n-r)}_{t_1} \right].$$

Setting

$$\phi^{(n)} = \sum_{i_1, \ldots, i_n = 1}^N \frac{\phi_{i_1} \cdots \phi_{i_n}}{(y_{i_1} - x_{i_2}) \cdots (y_{i_k} - x_{i_k})},$$

with $N \in \mathbb{N}$ (the “soliton number”), constants $x_i, y_j$, and

$$\phi_k = c_k \epsilon^{\xi(t, y_k)} e^{-\xi(t, y_k)}, \quad \xi(t, x) = \sum_{n \geq 1} t_n x^n,$$

then (2) reduces to

$$4 p_1 p_3 - 3 p_2^2 - p_1^4 = -6 (p_1 \bullet p_1) + 6 (p_1 \bullet p_2 - p_2 \bullet p_1),$$

where

$$p_r = \sum_{k=1}^N (x_k^r - y_k^r) \quad r = 1, 2, \ldots$$
and
\[ p_r \cdot p_s = \sum_{1 \leq i < j \leq k \leq N} (x_i^r - y_i^r) x_j (x_k^s - y_k^s) - \sum_{1 \leq i < j < k \leq N} (x_i^r - y_i^r) y_j (x_k^s - y_k^s). \] (4)
By closer inspection, (3) turns out to be an identity for arbitrary \( N \) (see also section 7). It mirrors in an obvious way the structure of the KP equation (2) with the correspondence expressed by a linear map \( \sigma \) such that
\[ \sigma(p_n) = -\phi_{tn}, \quad \sigma(p_n a) = \partial_{tn}(\sigma(a)) = \sigma(a)_{tn}, \quad \sigma(a \cdot b) = \sigma(a) \sigma(b). \] (5)
Setting \( y_k = 0, k = 1, \ldots, N \), (3) becomes an identity involving quasi-symmetric polynomials in the “variables” \( x_k, k = 1, \ldots, N \). Since this is an identity for any \( N \in \mathbb{N} \), it is helpful to consider quasi-symmetric functions in an infinite set of variables, \( x_1, x_2, \ldots \). The products in the nonlinear terms of the KP equation then correspond to a new product of quasi-symmetric functions, e.g.
\[ \left( \sum_i x_i^r \right) \cdot \left( \sum_j x_j^s \right) = \sum_{i < j \leq k} x_i^r x_j x_k^s. \] (6)
The algebraic structure that emerges in this way, more generally from the KP hierarchy and certain extensions, has been elaborated in [26], but the relation with quasi-symmetric functions remained unrecognized. It appeared explicitly in a different, though related, approach in [12].

(5) defines the map \( \sigma \) on symmetric functions without a constant term, and their \( \cdot \)-products. Since the product \( \cdot \) leads out of the algebra of symmetric functions into the bigger algebra of quasi-symmetric functions, it is natural to ask for an extension of \( \sigma \) to the whole algebra \( \text{QSym} \). Such an extension indeed has been achieved in [26], and it necessitated the introduction of Moyal-type products in the target space (see also [27]).

Switching the second set of variables \( y_1, y_2, \ldots \) on, we are led to a generalization of quasi-symmetric functions to quasi-supersymmetric functions, see section 8. The focus of the present work is, however, on the use of the above new product in the theory of quasi-symmetric functions, and in particular on relations with familiar structures on \( \text{QSym} \).

3 Quasisymmetric functions

Let \( X \) be a countably infinite totally ordered set of commuting variables and \( \mathbb{Z}[[X]] \) the corresponding ring of formal power series over \( \mathbb{Z} \), which is unital with identity element 1. We denote the ordering relation by \( \leq \) and use \( < \) for the strict order. An element \( a \) of \( \mathbb{Z}[[X]] \) is a quasi-symmetric function if it is of bounded degree and if for \( x_1 < \cdots < x_k \) and \( y_1 < \cdots < y_k \) in \( X \), and for any choice of positive integers \( n_1, \ldots, n_k \), the monomials \( x_1^{n_1} \cdots x_k^{n_k} \) and \( y_1^{n_1} \cdots y_k^{n_k} \) have the same coefficient in \( a \) [1–3, 6]. (Our use of \( x_i \) and \( y_j \) differs from that of section 2) \( \text{QSym} \) is a unital subring of \( \mathbb{Z}[[X]] \). In the following, we consider \( \text{QSym} \) as an algebra over \( \mathbb{Q} \). A basis of \( \text{QSym} \) is given by 1 and
\[ M_C = \sum_{x_1 < \cdots < x_k} x_1^{n_1} \cdots x_k^{n_k}, \] (7)
where \( C \) denotes the composition \( (n_1, \ldots, n_k) \), a sequence of positive integers. The sum is over all \( x_1, \ldots, x_k \in X \), subject to the ordering condition indicated under the summation symbol. The weight of compositions supplies \( \text{QSym} \) with a natural grading. For the above composition \( C \), the weight is
$|C| = n_1 + \cdots + n_k$ and its length is $\ell(C) = k$. We set $M_{\emptyset} = 1$, where $\emptyset$ denotes the empty composition. Since

$$M_{(n)} = \sum_x x^n \quad n = 1, 2, \ldots ,$$

we have

$$M_{(m)} M_{(n)} = \sum_{x_1 < x_2} x_1^n x_2^m + \sum_{x_1 < x_2} x_1^n x_2^m = M_{(m,n)} + M_{(m+n)} + M_{(n,m)},$$

which generalizes to

$$M_{(m)} M_{(n_1,\ldots,n_k)} = M_{(m,n_1,\ldots,n_k)} + M_{(n_1,m,n_2,\ldots,n_k)} + M_{(n_1,m,n_2,\ldots,n_k)} + \cdots + M_{(n_1,n_2,\ldots,n_k,m)} . \quad (8)$$

$\text{QSym}$ has a natural Hopf algebra structure $[1, 3, 4]$ with the coassociative (outer $[3]$) coproduct defined by

$$\Delta(M_{C}) = \sum_{AB = C} M_{A} \otimes M_{B} , \quad (9)$$

where the sum is over all compositions $A, B$ that concatenate (with their concatenation denoted by $AB$) to $C$. The summation includes the empty composition. In particular,

$$\Delta(1) = 1 \otimes 1 ,$$
$$\Delta(M_{(n)}) = 1 \otimes M_{(n)} + M_{(n)} \otimes 1 ,$$
$$\Delta(M_{(n_1,n_2)}) = 1 \otimes M_{(n_1,n_2)} + M_{(n_1)} \otimes M_{(n_2)} + M_{(n_1,n_2)} \otimes 1 .$$

We see that $M_{(n)}$ is a primitive element of the Hopf algebra. The counit is determined by $\varepsilon(M_{C}) = \delta_{C,\emptyset}$. We will later meet the antipode $S$. $\text{QSym}$ admits another bialgebra structure $[1, 3]$ (with the “inner coproduct” [3]), but this will not be considered in this work.

An alternative basis of $\text{QSym}$ is given by 1 and

$$\tilde{M}_{(n_1,\ldots,n_r)} = \sum_{1 \leq x_2 \leq \cdots \leq x_r} x_1^{n_1} \cdots x_r^{n_r} ,$$

For a composition $C = (n_1, \ldots , n_r)$ of $n$, hence $|C| = n$, we define

$$F_{C} = \sum x_1 \cdots x_n ,$$

where the summation is over all $x_1, \ldots , x_n \in X$, subject to the conditions

$$x_1 \leq \cdots \leq x_1 < x_1 + 1 \leq \cdots \leq x_n + n_2 < \cdots < x_1 + \cdots + n_r + 1 \leq \cdots \leq x_n .$$

For the empty composition we set $F_{\emptyset} = 1$. Then $\{F_{C} \mid C \text{ composition} \}$ constitutes the fundamental basis of $\text{QSym}$ [1, 6]. Its elements are called quasi-ribbons in [5]. In particular, we have $F_{(n)} = \tilde{M}_{(n^n)}$ and $F_{(1^n)} = M_{(1^n)}$. Expressed in terms of the basis $\{M_{C}\}$, we have $F_{C} = \sum_{D \succeq C} M_{D}$, where the sum is over all compositions $D$ with weight equal to $|C|$ and which are finer than $C$ (including $D = C$). For example, $F_{(3,1)} = M_{(3,1)} + M_{(2,1,1)} + M_{(1,1,1,1)}$. With this notation, we also have $\tilde{M}_{C} = \sum_{C \succeq D} M_{D}$.
4 New products

We introduce a sequence of noncommutative and nonassociative products in \( \text{QSym} \) (\( \mathbb{Q} \)-linear maps \( \text{QSym} \otimes \text{QSym} \to \text{QSym} \)) via

\[
M_A \bullet_k M_B = \sum_{x_1 < \ldots < x_r < z \leq y_1 < \ldots < y_s} x_1^{n_1} \cdots x_r^{n_r} z^k y_1^{m_1} \cdots y_s^{m_s} \quad k = 1, 2, \ldots , \tag{10}
\]

where \( A = (n_1, \ldots, n_r) \) and \( B = (m_1, \ldots, m_s) \), and

\[
1 \bullet_k M_A = \sum_{z \leq x_1 < \ldots < x_r} z^k x_1^{n_1} \cdots x_r^{n_r} \quad M_A \bullet_k 1 = \sum_{x_1 < \ldots < x_r < z} x_1^{n_1} \cdots x_r^{n_r} z^k \quad 1 \bullet_k 1 = \sum_x x^k . \tag{11}
\]

For \( k = 1 \), we recover (6), i.e. the product that appeared in the above sketched approach to solve the KP equation (and more generally the KP hierarchy). With respect to the natural grading of \( \text{QSym} \), the product \( \bullet \) determines \( \mathbb{Q} \)-bilinear maps \( \text{QSym}^m \otimes \text{QSym}^n \to \text{QSym}^{m+n+k} \). If \( G_k \) denotes the semigroup \( (\mathbb{N} \cup \{0\}, +_k) \), where \( i +_k j = i + j + k \), then \( (\text{QSym}, \bullet_k) \) is a \( G_k \)-graded algebra. As a consequence of the above definitions, we have

\[
M_A \bullet_n 1 = M_{A(n)} ,
\]

and thus

\[
M_{(n_1, \ldots, n_k)} = M_{(n_1, \ldots, n_{k-1})} \bullet_n 1 = (\cdots ((1 \bullet_1 1) \bullet_2 1) \bullet_3 \cdots) \bullet_n 1 . \tag{12}
\]

Furthermore,

\[
M_A \bullet_n M_{(m)} = M_{A(n,m)} = M_{A(n+m)} \tag{13}
\]

where \( A, B \) may be empty. We will also use the notation

\[
a \bullet_k b = m_k(a \otimes b) \quad \forall a, b \in \text{QSym} .
\]

Restricted to the non-unital ring \( \text{QSym}' \), obtained from \( \text{QSym} \) without the constant elements, i.e. \( \text{QSym}' = \text{QSym}/\mathbb{Q}1 \), the new products are associative and also combined associative. But in general they are not associative. Indeed, we have the following property, which also shows that the products \( \bullet_k, k > 1 \), can all be expressed in terms of the first product \( \bullet = \bullet_1 \).

Lemma 4.1. For all \( a, b \in \text{QSym} \) and \( k, l = 1, 2, \ldots \) we have

\[
a \bullet_k (1 \bullet_l b) - (a \bullet_k 1) \bullet_l b = a \bullet_{k+l} b . \tag{14}
\]

Proof. By linearity it is sufficient to consider (14) for the elements of the basis \( \{ M_C \mid C \text{ composition} \} \). But for the latter, (14) is immediately verified by use of the definitions (7), (10) and (11).

\[\square\]

Proposition 4.2. \( \text{QSym} \) is generated by 1 and the nonassociative product \( \bullet \).

Proof. Since 1 and the elements \( M_{(n_1, \ldots, n_k)} \) constitute a basis of \( \text{QSym} \), the assertion follows from (12) and lemma 4.1.

\[\square\]

The following is also easily verified.
Proposition 4.3. QSym, supplied with any of the products \( \bullet_k, k \in \mathbb{N} \), is a weakly nonassociative algebra. More generally, we have

\[
(a \bullet_k (b \bullet_m c)) \bullet_n d = a \bullet_k ((b \bullet_m c) \bullet_n d)
\]

for all \( a, b, c, d \in \text{QSym} \) and all \( k, m, n = 1, 2, \ldots \)

The alternative basis elements \( \bar{M}_C \) of \( \text{QSym} \) are recursively determined by

\[
\bar{M}_{(n)C} = 1 \bullet_n \bar{M}_C.
\]

Furthermore, for any two compositions \( A, B \), we have

\[
\bar{M}_{A(m)} \bullet_n \bar{M}_B = \bar{M}_{A(m,n)B} - \bar{M}_{A(m+n)B}.
\]

In case of the fundamental basis elements, it is quite easily verified that

\[
F_A \bullet F_{(m)B} = F_{A(m+1)B}
\]

for any two compositions \( A, B \), and \( m = 1, 2, \ldots \). Of particular importance are compositions of the form \((m+1, 1^n)\), \(m, n = 0, 1, \ldots \), where \((1^n) = (1, \ldots , 1)\) (with \(n\) entries), for which we find

\[
F_{(m+1, 1^n)} = L_{1}^{m}R_{1}^{n}(1 \bullet 1),
\]

where \( L_{a}b = a \bullet b \) and \( R_{a}b = b \bullet a \). Note that on the right hand side of (19) the left and right multiplications commute as a consequence of the weak nonassociativity property (1) of the product \( \bullet \). Since any non-empty composition \( C \) that is not of the form \((m+1, 1^n)\) can be written as \( C = (m_{1-1}, 1^{n_{1}}, m_{2}+2, 1^{n_{2}}, \ldots , m_{r}+2, 1^{n_{r}})\) with \(m_1, n_1, \ldots , m_r, n_r \in \mathbb{N} \cup \{0\}\), as a consequence of (18) we have

\[
F_C = F_{(m+1, 1^{n+2})} \bullet F_{(m_{2}+2, 1^{n_{2}})} \bullet \cdots \bullet F_{(m_{r}+2, 1^{n_{r}})},
\]

where the right hand side has an associative structure due to (1). Hence, \( F_C \) factorizes into a \( \bullet \)-product of elementary basis elements (19).

Remark 4.4. Instead of setting the \( y \)'s to zero in (4), we may set the \( x \)'s to zero. With a change of sign, this leads to the alternative weakly nonassociative products in \( \text{QSym} \) determined by \( 1 \bullet_k 1 = \sum x^k \).

\[
M_{A} \bullet_k M_{B} = \sum_{x_1 < \cdots < x_r \leq z \leq y_1 < \cdots < y_s} x_1^{n_1} \cdots x_r^{n_r} z^{k} y_1^{m_1} \cdots y_s^{m_s},
\]

\[
1 \bullet_k M_{A} = \sum_{z < x_1 < \cdots < x_r} z^{k} x_1^{n_1} \cdots x_r^{n_r}, \quad M_{A} \bullet_k 1 = \sum_{x_1 < \cdots < x_r \leq z} x_1^{n_1} \cdots x_r^{n_r} z^{k},
\]

for which all results in this work have a counterpart. In particular, we obtain \( F_{(1^{m}, n+1)} = L_{1}^{m}R_{1}^{n}(1 \bullet 1) \) and any \( F_C \) with \( C \) not of the form \((1^{m}, n + 1)\) factorizes into a \( \bullet \)-product of such elementary elements.

5 Relations between the old and the new products, and the coproduct

The following result allows us to express the ordinary product of quasi-symmetric functions recursively in terms of the product \( \bullet \).


Proposition 5.1. For any composition $C$, any $a \in \text{QSym}$, and $k = 1, 2, \ldots$,
\[ M_{C(k)} a = \text{m}_k \left( (M_C \otimes 1) \Delta(a) \right). \]  

Proof. With $D = (m_1, \ldots, m_s)$ and $E = (n_1, \ldots, n_r)$, 
\[ M_D M_E = \left( \sum_{y_1 \cdots < y_s} y_1^{m_1} \cdots y_s^{m_s} \right) \left( \sum_{x_1 \cdots < x_r} x_1^{n_1} \cdots x_r^{n_r} \right) \]  
can be written as 
\[ \sum_{0 \leq i_1 \cdots \leq i_s \leq r} \sum x_1^{n_{i_1}} \cdots x_r^{n_{i_s}} y_1^{m_1} \cdots y_s^{m_s} = \sum x_1^{n_{i_1}} \cdots x_r^{n_{i_s}} y_1^{m_1} \cdots y_s^{m_s} \]  
where any expression of the form $x_k^{n_k} \cdots x_l^{n_l}$ should be replaced by 1 if $k > l$. The inner summation is over all $x_1, \ldots, x_r, y_1, \ldots, y_s \in X$, subject to the condition 
\[ x_1 < \cdots < x_i < y_1 \leq x_{i+1} < \cdots < y_{r-1} \leq x_{i+1} < \cdots < x_i < y_s \leq x_{i+1} < \cdots < x_r. \]  
For fixed $i_s$ is 
\[ \sum_{0 \leq i_1 \cdots \leq i_{s-1} \leq i_s} \sum x_1^{n_{i_1}} \cdots x_r^{n_{i_s}} y_1^{m_1} \cdots y_s^{m_s} \]  
equal to $(M_{D'} M_A) \bullet_{m_s} M_B$, where $A = (n_1, \ldots, n_{i_s}), B = (n_{i_s+1}, \ldots, n_r)$ and $D' = (m_1, \ldots, m_{s-1})$. Summing over $0 \leq i_s \leq r$, we thus obtain 
\[ M_D M_E = \sum_{AB = E} (M_{D'} M_A) \bullet_{m_s} M_B. \]

In particular, (2) can be expressed as 
\[ M_{(m)} M_{(n_1, \ldots, n_k)} = \text{m}_m \circ \Delta(M_{(n_1, \ldots, n_k)}) \]  
\[ = \text{m}_m M_{(n_1, \ldots, n_k)} + \text{m}_{(n_1)} \bullet_{m} M_{(n_2, \ldots, n_k)} + \cdots + M_{(n_1, \ldots, n_k)} \bullet_{m} 1. \]

Next we define a bimodule structure on $\text{QSym} \otimes \text{QSym}$ via 
\[ (a \otimes b) \bullet_k c = a \otimes (b \bullet_k c), \quad c \bullet_k (a \otimes b) = (c \bullet_k a) \otimes b, \]  
for all $a, b, c \in \text{QSym}$. As a consequence, the usual bimodule property holds, i.e. 
\[ a \bullet_k (m \bullet_l b) = (a \bullet_k m) \bullet_l b \quad \forall a, b \in \text{QSym}, \quad \forall m \in \text{QSym} \otimes \text{QSym}. \]

But we have the following restricted form of the usual left and right module properties (see also [12]), 
\[ a \bullet_k (b \bullet_l m) = (a \bullet_k b) \bullet_l m, \quad (m \bullet_k b) \bullet_l a = m \bullet_k (b \bullet_l a) \]  
\forall b \in \text{QSym}',

whereas 
\[ a \bullet_k (1 \bullet_l m) = (a \bullet_k 1) \bullet_l m + a \bullet_{k+l} m, \quad (m \bullet_k 1) \bullet_l a = m \bullet_k (1 \bullet_l a) - m \bullet_{k+l} a. \]

The following proposition turns $(\text{QSym}, \bullet, \Delta)$ into an infinitesimal bialgebra [16, 17, 28–33][8]. We should stress, however, that here $(\text{QSym}, \bullet)$ is not associative, so that we are not quite in the framework of the latter references. Furthermore, we note that $(\text{QSym}, \bullet)$ is not unital. In fact, any infinitesimal bialgebra possessing a unit and a counit is trivial [17].

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[8] This has to be distinguished from a “unital infinitesimal bialgebra” as defined in [34] (see also [35, 36]).
Proposition 5.2. The coproduct of QSym acts as a derivation on the products $\bullet_n$, i.e.

$$\Delta(a \bullet_n b) = \Delta(a) \bullet_n b + a \bullet_n \Delta(b) \quad n = 1, 2, \ldots,$$  \hfill (22)

for all $a, b \in \text{QSym}$.

Proof. We have

$$\Delta(M_A) \bullet_n 1 + M_A \bullet_n \Delta(1) = \sum_{CD=A} M_C \otimes (M_D \bullet_n 1) + (M_A \bullet_n 1) \otimes 1$$

$$= \sum_{CD=A} M_C \otimes M_D(n) + M_A(n) \otimes 1$$

$$= \sum_{CD=A(n)} M_C \otimes M_D = \Delta(M_A(n)) = \Delta(M_A \bullet_n 1),$$

and also

$$\Delta(M_A) \bullet_n M_{(m)B} + M_A \bullet_n \Delta(M_{(m)B})$$

$$= \sum_{CD=A} M_C \otimes (M_D \bullet_n M_{(m)B}) + \sum_{CD=(m)B} (M_A \bullet_n M_C) \otimes M_D$$

$$= \sum_{CD=A} M_C \otimes M_D(n,m)B + \sum_{CD=A} M_C \otimes M_D(n+m)B$$

$$+ M_A(n) \otimes M_{(m)B} + \sum_{CD=B} M_A(n,m)C \otimes M_D + \sum_{CD=B} M_A(n+m)C \otimes M_D$$

$$= \sum_{CD=A(n,m)B} M_C \otimes M_D + \sum_{CD=A(n+m)B} M_C \otimes M_D = \Delta(M_A \bullet_n M_{(m)B}).$$

Using (16) and (22), one easily proves by induction that

$$\Delta(\tilde{M}_C) = \sum_{AB=C} \tilde{M}_A \otimes \tilde{M}_B.$$  \hfill (23)

Proposition 5.3. The distributivity rule

$$c(a \bullet_m b) = m_m(\Delta(c)(a \otimes b))$$  \hfill (24)

holds for all $a, b, c \in \text{QSym}$ and $m = 1, 2, \ldots$.

Proof. Clearly, the assertion is true for $c = M_\emptyset = 1$. An application of (21) and (22) yields

$$M_{C(n)}(a \bullet_m b) = m_m((M_C \otimes 1) \Delta(a \bullet_m b))$$

$$= m_m((M_C \otimes 1)(\Delta(a) \bullet_m b + a \bullet_m \Delta(b)))$$

$$= m_m((M_C \otimes 1)(\sum_{(a)} a_{[1]} \otimes a_{[2]} \bullet_m b + \sum_{(b)} a \bullet_m b_{[1]} \otimes b_{[2]}))$$

$$= m_m(\sum_{(a)} M_C a_{[1]} \otimes a_{[2]} \bullet_m b + \sum_{(b)} M_C(a \bullet_m b_{[1]}) \otimes b_{[2]}))$$

$$= \sum_{(a)} (M_C a_{[1]} \bullet_n (a_{[2]} \bullet_m b) + \sum_{(b)} (M_C(a \bullet_m b_{[1]})) \bullet_n b_{[2]}).$$
Here we used the Sweedler notation $\Delta(a) = \sum_{(a)} a_{[1]} \otimes a_{[2]}$. We proceed by induction and assume that the assertion holds for $c = M_C$ with $\ell(C) = r$. Hence

$$M_{C(n)}(a \bullet_m b) = \sum_{(a)} (M_C a_{[1]}) \bullet_n (a_{[2]} \bullet_m b) + \sum_{(b), AB = C} \left( \sum_{AB = C} (M_A) \bullet_m (M_B b_{[1]}) \right) \bullet_n b_{[2]} ,$$

by use of (9). The first term on the right hand side of the last equation can then be rewritten as follows,

$$\sum_{(a)} (M_C a_{[1]}) \bullet_n (a_{[2]} \bullet_m b) = \sum_{(a)} (M_C a_{[1]}) \bullet_n a_{[2]} \bullet_m b + (M_C a) \bullet_n (1 \bullet_m b) ,$$

where the primed summation omits the term involving the summand $a \otimes 1$ of $\Delta(a)$, which (by recalling (13)) is the only term with a nonassociative structure (now the last term on the right hand side). Similarly, the second term on the right hand side of the previous equation can be written as

$$\sum_{(b), AB = C} ((M_A) \bullet_m (M_B b_{[1]})) \bullet_n b_{[2]} = \sum_{(b), AB = C} \left( [1] \bullet (M_A) \bullet_m (M_B b_{[1]}) \right) \bullet_n b_{[2]} + ((M_C a) \bullet_m 1) \bullet_n b ,$$

where the primed summation omits the only summand (corresponding to the summand $1 \otimes b$ of $\Delta(b)$ and $B = \emptyset$) that has a nonassociative structure. Using

$$(M_C a) \bullet_n (1 \bullet_m b) + ((M_C a) \bullet_m 1) \bullet_n b = ((M_C a) \bullet_n 1) \bullet_m b + (M_C a) \bullet_m (1 \bullet_n b) ,$$

which is an immediate consequence of (14), we find that

$$M_{C(n)}(a \bullet_m b) = \sum_{(a)} ((M_C a_{[1]}) \bullet_n a_{[2]} ) \bullet_m b + \sum_{(b), AB = C} (M_A) \bullet_m ((M_B b_{[1]}) \bullet_n b_{[2]} ) .$$

With the help of (21), this becomes

$$M_{C(n)}(a \bullet_m b) = (M_C(n) a) \bullet_m b + \sum_{AB = C} (M_A) \bullet_m (M_B(n) b) = \sum_{AB = C(n)} (M_A) \bullet_m (M_B) ,$$

which completes the induction step.

\[\square\]

### 6 Relation between the antipode and the new products

Let us define a linear map $S : \text{QSym} \to \text{QSym}$ by

$$S(M_C) = (-1)^{\ell(C)} \tilde{M}_C ,$$

(25)

where $\tilde{C}$ is the reverse of the composition $C$, i.e. $C$ written in reversed order. In particular, $S(1) = 1$.

**Proposition 6.1.**

$$S(a \bullet_n b) = -S(b) \bullet_n S(a) \quad \forall a, b \in \text{QSym}, \quad n = 1, 2, \ldots .$$

(26)

**Proof.**

$$S(M_C \bullet_n 1) = S(M_{C(n)}) = (-1)^{\ell(C)} \tilde{M}_{(n)C} = (-1)^{\ell(C)} 1 \bullet_n \tilde{M}_C = -S(1) \bullet_n S(M_C) .$$

Furthermore, we have

$$S(M_A \bullet_n M_{(m)B}) = S(M_{A(n,m)B} + M_{A(n+m)B}) = (-1)^{\ell(A)+\ell(B)} (\tilde{M}_{B(m,n)A} - \tilde{M}_{B(n+m)A} ) = (-1)^{\ell(A)+\ell(B)} \tilde{M}_{B(m)} \bullet_n \tilde{M}_A = -S(M_{(m)B}) \bullet_n S(M_A) ,$$

where we used (13) and (17).\[\square\]
Next we prove that $S$ is the antipode of the Hopf algebra of quasi-symmetric functions.

**Proposition 6.2.** If $\mu$ denotes the usual product of quasi-symmetric functions, and $\varepsilon$ the counit, then

$$\mu \left( \text{id} \otimes S \right) \Delta = 1 = \mu \left( S \otimes \text{id} \right) \Delta \tag{27}$$

**Proof.** Since $\mu \left( \text{id} \otimes S \right) \Delta(1) = 1 = \mu \left( S \otimes \text{id} \right) \Delta(1)$, it is sufficient to verify that both sides of (27) applied to $M_C$ vanish if $C$ is not empty. Hence we have to show that

$$\sum_{DE=C} M_D S(M_E) = 0 \quad \text{and} \quad \sum_{DE=C} S(M_D) M_E = 0,$$

for all $C$ different from the empty composition. We concentrate on the first relation and use induction on the length $\ell(C)$ of the composition $C$. For a composition of length 1, we obtain

$$\sum_{DE=(n)} M_D S(M_E) = 1 S(M(n)) + M(n) 1 = S(M(n)) + M(n) = S(1 \bullet_n 1) + M(n) = -S(1) \bullet_n 1 + M(n) = -M(n) + M(n) = 0.$$ 

Here we used (26). Let us now assume that the assertion holds for all $C$ with $\ell(C) \leq r$. Then we have

$$\sum_{DE=C(n)} M_D S(M_E) - M_C(n) = \sum_{DE=C} M_D S(M_E \bullet_n 1) = -\sum_{DE=C} M_D (1 \bullet_n S(M_E))$$

$$= -\sum_{ABE=C} M_A \bullet_n (M_B S(M_E)) = -\sum_{AD=C} M_A \bullet_n \left( \sum_{BE=D} M_B S(M_E) \right).$$

Besides (26), we applied (24). By induction hypothesis, $\sum_{BE=D} M_B S(M_E)$ vanishes, except for the case where $D$ is the empty composition, where this is equal to 1. Hence

$$\sum_{DE=C(n)} M_D S(M_E) = M_C(n) - M_C \bullet_n 1 = 0.$$

The second relation in (27) can be proved in the same way. \hfill $\Box$

Formula (25) for the antipode appeared in [3, 4]. As a side-result, we obtained a new proof of this expression for the antipode. Since $\text{QSym}$ is commutative with respect to the original product, the antipode satisfies $S^2 = \text{id}$ (see e.g. [37], p.15). We should mention that $S$ is not an antipode of an infinitesimal Hopf algebra as defined in [17].

(26) shows that (up to a sign) the antipode exchanges left and right multiplication with the product $\bullet$. This implies

$$S(F_{(m+1,1^n)}) = (-1)^{m+1+n} F_{(n+1,1^m)}$$

for $m, n = 0, 1, \ldots$. Using (20) and (26), we quickly recover the following result (see [3, 4]),

$$S(F_C) = (-1)^{|C|} F_{\omega(C)}.$$

The simple calculation determines the map $\omega$ of compositions as follows. If $C = (m+1,1^n)$, then $\omega(C) = (n+1,1^m)$. If a composition $C$ is not of this form, then it can be written uniquely as $C = (m_1 + 1, 1^{n_1}, m_2 + 2, 1^{n_2}, \ldots, m_r + r, 1^{n_r})$ with $m_1, n_1, \ldots, m_r, n_r \in \mathbb{N} \cup \{0\}$ and $r > 1$, and we set $\omega(C) = (n_r + 1, 1^{m_r}, n_{r-1} + 2, 1^{m_{r-1}}, \ldots, n_1 + 2, 1^{m_1})$. Note that $\omega^2 = \text{id}$.\hfill 10
7 KP identities

We recall that the symmetric functions form a subalgebra $\text{Sym}$ of $\text{QSym}$, and a basis is given by products of the (homogeneous) complete symmetric functions

$$h_n = \tilde{M}_{(1^n)} \quad \text{where} \quad (1^n) = (1, \ldots, 1),$$

including $h_0 = 1$. In the following, we write

$$p_n = M_{(n)} \quad n = 1, 2, \ldots,$$

which is the $n$-th power sum. By use of Newton’s identities

$$n h_n = \sum_{k=1}^{n} p_k h_{n-k} \quad n = 1, 2, \ldots, \quad (28)$$

$h_n, n > 0$, can be recursively expressed in terms of $p_k, k = 1, \ldots, n$.

Remark 7.1. Introducing the formal sums

$$h(\zeta) = \sum_{n \geq 0} \zeta^n h_n, \quad p(\zeta) = \sum_{n \geq 1} \zeta^{n-1} p_n,$$

with an indeterminate $\zeta$, $\text{[28]}$ leads to

$$\frac{d}{d\zeta} h(\zeta) = p(\zeta) h(\zeta),$$

which integrates to

$$h(\zeta) = \exp \left( \sum_{n \geq 1} \frac{\zeta^n}{n} p_n \right).$$

It follows that $h_n$ can be expressed as

$$h_n = s_n(p_1, p_2/2, p_3/3, \ldots)$$

in terms of the elementary Schur polynomial $s_n$.

As a preparation for the main result of this section, we recall the divided power structure of the coproduct of $h_n$,

$$\Delta(h_n) = \sum_{k=0}^{n} h_k \otimes h_{n-k} \quad n = 1, 2, \ldots, \quad (29)$$

which is a special case of $\text{[23]}$.

Proposition 7.2. For $m, n = 1, 2, \ldots$, we have the following identities,

$$h_m h_{n+1} - h_m h_n = \sum_{k=1}^{m} h_k \cdot (h_{m-k} h_n) - \sum_{k=1}^{n} h_k \cdot (h_{n-k} h_m). \quad (30)$$
Proof. Using (24) and (29), we find

\[
h_m h_{n+1} = h_m (1 \bullet h_n) = m(\Delta(h_m) (1 \otimes h_n)) = \sum_{k=0}^{m} m((h_k \otimes h_{m-k})(1 \otimes h_n)) = \sum_{k=0}^{m} h_k \bullet (h_{m-k} h_n) .
\]

This implies (30). \(\square\)

(30) is a sequence of identities for symmetric functions, but in the space of quasi-symmetric functions, the product \(\bullet\) leads outside the subspace of symmetric functions. For \(m = 1, n = 2\), we obtain

\[
h_1 h_3 - h_2 h_2 = h_1 \bullet h_2 - h_1 \bullet h_1^2 - h_2 \bullet h_1 .
\]

Expressed in terms of \(p_n\) via (28), this takes the form

\[
4 p_1 p_3 - 3 p_2^2 - p_1^4 = -6 p_1 (p_1 \bullet p_1) + 6 (p_1 \bullet p_2 - p_2 \bullet p_1) ,
\]

where we used (24) to write \(p_1^2 \bullet p_1 + p_1 \bullet p_2^2 = p_1 (p_1 \bullet p_1)\). We observe that the identity (31) is the KP identity (3) (for vanishing \(y_1, y_2, \ldots, N = \infty\)). We already explained in section 2 how the KP equation can be reconstructed from this identity. For a partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)\), let

\[
p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_r} .
\]

Then \(\sigma(p_\lambda) = -\phi_{t_{\lambda_1} t_{\lambda_2} \cdots t_{\lambda_r}}\). Since the symmetric functions (32) form a basis of \(\text{Sym}\) (over \(\mathbb{Q}\)) [9], it follows that to any symmetric function \(f \in \text{Sym} / \mathbb{Q} 1\) there corresponds an expression \(\sigma(f) = -F \phi\) with a differential operator \(F(\partial_{t_1}, \partial_{t_2}, \ldots)\), having coefficients in \(\mathbb{Q}\) and no term of 0-th order. Furthermore, \(\sigma(f_1 \bullet f_2 \bullet \cdots \bullet f_r) = (-F_1 \phi) \cdots (-F_r \phi)\) for \(f_i \in \text{Sym} / \mathbb{Q} 1, i = 1, \ldots, r\). Applying \(\sigma\) to (30), one recovers the whole (noncommutative) KP hierarchy. A formulation of the KP hierarchy that corresponds to (30) in this way, can be found e.g. in [38, 39]. Of course, one can apply the procedure in section 2 of solving the KP equation more generally to any member of the KP hierarchy. The fact that (30) is a sequence of identities would then prove that the method indeed generates solutions of the whole KP hierarchy.

We conjecture that any identity in QSym, that is built from symmetric functions in \(\text{Sym} / \mathbb{Q} 1\) and only the product \(\bullet\), corresponds to a partial differential equation that is satisfied as a consequence of the KP hierarchy.

There is another way to describe the correspondence between the identities (30) and the equations of the KP hierarchy. According to (24), the primitive element \(p_n = M_{(n)}\) acts on a product \(a \bullet_k b\) as a derivation. Hence

\[
\delta_n (a) = m_n \circ \Delta(a) = p_n a \quad \forall a \in \text{QSym}
\]

defines a sequence of commuting derivations on QSym with respect to (any of) the products \(\bullet_k\), i.e.

\[
\delta_n (a \bullet_k b) = \delta_n (a) \bullet_k b + a \bullet_k \delta_n (b), \quad \delta_m \delta_n = \delta_n \delta_m .
\]

This makes contact with the framework developed in [12,13] for weakly nonassociative algebras. Expressing (31) in terms of these derivations, we obtain

\[
4 \delta_1 \delta_3 (1) - 3 \delta_2^2 (1) - \delta_1^4 (1) = -6 \delta_1 (\delta_1 (1) \bullet \delta_1 (1)) + 6 (\delta_1 (1) \bullet \delta_2 (1) - \delta_2 (1) \bullet \delta_1 (1)) ,
\]

which becomes the KP equation (2) via \(1 \mapsto -\phi, \delta_n \mapsto \partial_{t_n}\), and with \(\bullet\) replaced by the product in the associative algebra where \(\phi\) takes its values. This correspondence extends to the whole KP hierarchy [12,13].
8 Quasi-supersymmetric functions

(4) shows that the product corresponding to the nonlinearities of the KP equation involves two sets of parameters, the \( x \)'s and the \( y \)'s. Using the usual product, the expressions \( p_r \) defined in (4) with \( N = \infty \) generate supersymmetric functions [19–23] (called “bisymmetric functions” in [18]), see also [9] (example 23 of section I.3). Hence we should expect to encounter more generally “quasi-supersymmetric functions”. Such a generalization of supersymmetric functions has not yet appeared in the literature, according to our knowledge.

Let \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) be two countably infinite sequences of commuting variables, and let \( \mathbb{Q}[[x, y]] \) be the algebra of formal power series in the latter with coefficients in \( \mathbb{Q} \). For any monomial of bounded degree, \( a = z_{i_1}^{n_1} \cdots z_{i_r}^{n_r} \) where \( z_i \) is either \( x_i \) or \( y_i \), let \( M(a) \) denote the minimal respectively maximal element of \( \{i_1, \ldots, i_r\} \). We extend the previously defined products by setting

\[
1 \bullet_n 1 = \sum_i (x_i^n - y_i^n),
\]
\[
1 \bullet_n a = \sum_{i \leq M(a)} x_i^n a - \sum_{i < M(a)} y_i^n a,
\]
\[
a \bullet_n 1 = \sum_{M(a) < i} a x_i^n - \sum_{M(a) \leq i} a y_i^n,
\]
\[
a \bullet_n b = \sum_{M(a) < i \leq m(b)} a x_i^n b - \sum_{M(a) \leq i < m(b)} a y_i^n b,
\]

for monomials \( a, b \) of bounded degree. Here a sum contributes zero if there is no index \( i \) satisfying the conditions underneath the respective summation symbol. These definitions extend to the whole of \( \mathbb{Q}[[x, y]] \) by linearity. Again, this defines weakly nonassociative products that satisfy (14) and (15). Also in this case all products can be expressed in terms of the first. Based on further developments of the theory of weakly nonassociative algebras, we will show in a separate work that the weakly nonassociative subalgebra of \( \mathbb{Q}[[x, y]] \), generated by \( 1 \) via \( \bullet \), is closed under the usual multiplication, and that it contains the supersymmetric functions. A substitution \( x_i = t \) and \( y_i = t \) for the same \( i \) results in expressions independent of \( t \) (a central property of supersymmetric functions [20, 21]). Moreover, the space of quasi-supersymmetric functions is spanned by \( M_\emptyset = 1 \) and the elements defined recursively by

\[
M_{C(n)} = M_C \bullet_n 1,
\]

for any composition \( C \). By use of these results, one derives a sequence of KP identities in the algebra of quasi-supersymmetric functions, which are in one-to-one correspondence with identities derived in section 7 and of which (3) is the simplest (non-trivial).

9 Final remarks

We supplied the algebra of quasi-symmetric functions with a weakly nonassociative product and studied its relations with the ordinary product and the coproduct. The new product \( \bullet \) turned out to be a useful tool in the theory of quasi-symmetric functions, despite of the fact that it introduces a weak form of nonassociativity. It should be of interest to study more generally weakly nonassociative algebras that admit an infinitesimal coproduct, and in this way to extend the results in [12, 13]. Such a generalization will be elaborated in a separate work, including an exploration of the algebra of quasi-supersymmetric functions.

The Hopf algebra of quasi-symmetric functions is the graded dual of the Hopf algebra \( \text{NSym} \) of non-commutative symmetric functions (see e.g. [3, 5, 40]). An exploration of the dual of the product \( \bullet \) would then be a further interesting route to pursue.
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