Stability of Self-similar Solutions to Geometric Flows

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Abstract
We show that self-similar solutions for the mean curvature flow, surface diffusion and Willmore flow of entire graphs are stable upon perturbations of initial data with small Lipschitz norm. Roughly speaking, the perturbed solutions are asymptotically self-similar as time tends to infinity. Our results are built upon the global analytic solutions constructed by Koch and Lamm [25], the compactness arguments adapted by Asai and Giga [2], and the spatial equi-decay properties on certain weighted function spaces. The proof for all of the above flows are achieved in a unified framework by utilizing the estimates of the linearized operator.

1 Introduction

We analyze in this paper the long-time asymptotics of various geometric flows, in particular the stability of self-similar solutions. From the point of view of calculus of variations, many geometric flows can be seen as the negative gradient flows of some geometric functionals with respect to certain underlying metric. Heuristically, the gradient descent nature of the flows evolves general initial data toward a critical point of the corresponding functional. These evolutions are often modeled by nonlinear parabolic partial differential equations. The long time asymptotics of the solution is one of the key questions to be investigated. For instance, in the celebrated work [30] of Leon Simon, the asymptotics of a large class of such geometric evolution equations are studied by infinite dimensional version of the Lojasiewicz inequalities combined with the Liapunov–Schmidt reduction. It is also worth pointing out that in [13] Eells and Sampson used the long-time limit of heat flows to construct harmonic mappings between Riemannian manifolds under certain curvature assumptions.

The geometric flows studied in this paper is of curvature driven type which arises from energy minimization of the surface area functional. This naturally leads to evolutions involving mean curvature which is the first variation of the surface area. These motions appear often in the modeling of materials science such as phase transitions and grain growth [1, 28]. It is also used in describing the bending of membranes in red blood cells [20, 29]. The underlying equations are related to mean curvature flows (MCF), surface diffusion (SD) and Willmore flows (WF) which are the three equations analyzed in this paper.
One mathematical point to note is that the equations to be analyzed include fourth order flows which are much harder to handle than their second order counterparts due to the lack of maximum or comparison principle. On the other hand, these flows enjoy certain invariant property leading to the existence of self-similar solutions. The main goal of the current paper is to analyze the stability of these solutions. More precisely, under fairly general initial conditions, we will show that the solutions to these equations converge to some self-similar form. In order to take advantage of a general unified approach, we restrict ourselves to entire graph solutions relying very much on linearized analysis.

One can also interpret this phenomena of self-similarity using the renormalization group method as in [4]. The key idea is that after rescaling or zooming out in the spatial variable, suppose the initial data converges to a scale invariant function which is determined by the behavior of the data at infinity, then the solution will converge to a scale invariant solution, or so-called self-similar solution. In other words, the long-time asymptotics are determined by the rescaling limit of the initial data. Hence we expect that if the initial data is perturbed without changing the scaling limit, then the corresponding solution will more and more looked like the self-similar solution corresponding to the unperturbed scale invariant initial data. There is also a huge literature where such a phenomena is proved for semilinear heat equations - see for example [6, 19, 23, 27], just to name a few. Another technique extensively used in the case of MCF is the monotonicity formula. It has been used in this case to characterize the form of self-similar solutions and the convergence to them [22, 11]. This is also the pre-cursor to the more recent entropy method to characterize self-similar shrunkers [10].

In this paper, we will investigate the stability of self-similar solutions corresponding to MCF, SD and WF. Note that global-in-time existence of classical solutions to these geometric flows with general initial data does not hold due to the possibility of finite time blow-ups. On the other hand, in the case of graph setting, it is possible to have long time solutions. For MCF, this is comprehensively analyzed in [11, 12]. In a very interesting paper [25], Koch and Lamm has constructed a unique global-in-time solution to these geometric flows under small Lipschitz norm assumption on the initial data. This is in contrast to those existence results of classical solutions making use of maximal regularity property of elliptic operators where the initial data are required to be $C^{1,\alpha}$ or $C^{2,\alpha}$ (depending on the order of the equation) – see [16, 17, 31] for examples of such results. The main technique of [25], originated from Koch–Tataru [26] for incompressible Navier-Stokes equations, is a fixed point argument on some scale invariant function spaces. Even though it can only handle the case of graphs, all the above geometric flows in general dimensions can be tackled in a unified framework. In addition, the approach does not rely on maximum principle which only works for second order scalar PDEs. Thus, it is applicable for PDE systems and higher order equations.

Another relevant work is Asai-Giga [2] which establishes a stability result for self-similar solutions to a one dimensional surface diffusion with bounded initial data. It uses a compactness argument in some Hölder spaces. An earlier work [3] proves a similar result but it seems the technique is only applicable to the one dimensional curve case. From an application point of view, these two works touch upon the celebrated model called thermal grooving first described by Mullins [28]. Combining the techniques of [25] and [2], we are able to show a local-in-space stability result (Theorem 2.1) and also a global-in-space result (Theorem 2.3). The latter is achieved in the setting of some weighted function spaces. Qualitatively, we
have extended the result of [2, 3] to higher dimensions with unbounded initial data.

This paper is organized as follows. In Section 2, we introduce the geometric flows, the definition of self-similar solutions, and the statement of our main results. Then we outline the strategy of proof. Section 3 is devoted to the proof of Theorem 2.1 which asserts the local-in-space convergence of the perturbed solution. Next in Section 4, we prove our global-in-space convergence result (Theorem 2.3) under a spatial decaying assumption on the initial perturbation. We make a remark in Section 5 on the generalization to polyharmonic flows. The proofs of technical lemmas like Lemma 4.4 and Lemma 4.7 are put in the Appendix.

Before getting into the technical details, we introduce one notation to be used throughout this paper. We write for any two positive quantities that \( A \lesssim B \) if there is a universal constant \( C \) such that \( A \leq CB \). The value of the constant is not relevant in the argument and can change from one line to the other.

## 2 Geometric flows

Let \( \Sigma \) be a closed hypersurface in \( \mathbb{R}^{n+1} \). The area functional of \( \Sigma \) is given by

\[
A(\Sigma) = \int_\Sigma 1d\mu_g, \tag{2.1}
\]

where \( g \) is the induced metric from the immersion and \( d\mu_g \) is the corresponding area element. The aim of this paper is to investigate the \( (L^2\text{-} \text{ and } H^{-1}) \) negative gradient flows of (2.1). More precisely, we consider a time dependent hypersurface \( \Sigma_t \) given by immersions \( f: \Sigma \times \mathbb{R}_+ \to \mathbb{R}^{n+1} \) which evolves according to

**Mean curvature flow (MCF)**

\[
\partial_t f = \mathcal{H} := -\nabla_{L^2} A, \quad \tag{2.2}
\]

**Surface diffusion (SD)**

\[
\partial_t f = -\Delta_g \mathcal{H} = -\nabla_{H^{-1}} A, \quad \tag{2.3}
\]

where \( \mathcal{H} \) represents the mean curvature vector and \( \Delta_g \) is the Laplace–Beltrami operator with respect to the induced metric \( g \). Note that MCF and SD can be recast as the negative gradient flows to \( A \) with respect to the \( L^2 \) and \( H^{-1} \)-metric. See [34, 5] for more details about the derivation.

We will also consider the following Willmore functional for two dimensional surfaces \( n = 2 \) in \( \mathbb{R}^3 \):

\[
W(f) = \frac{1}{4} \int_\Sigma |\mathcal{H}|^2 d\mu_g. \tag{2.4}
\]

The negative \( L^2 \)-gradient flow of (2.4) is then given as follows:

**Willmore flow (WF)**

\[
f_t = -\Delta_g \mathcal{H} - \frac{1}{2} \mathcal{H}^3 + 2 \mathcal{H} \mathcal{K} =: -\nabla_{L^2} W, \tag{2.5}
\]
where $K$ is the Gauss curvature of $\Sigma$. We refer the reader to [24] for detail of the derivation.

As mentioned earlier, in this paper, we consider the case that $\Sigma_t$ given by an entire graph, i.e. there exists a function $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ such that $\Sigma_t = \{(x, u(x,t)) | x \in \mathbb{R}^n, t \in \mathbb{R}_+\}$. For concreteness, we write down the graph equations for (2.2), (2.3) and (2.5):

**MCF**

$$\frac{\partial u}{\partial t} = \sqrt{1 + |\nabla u|^2} \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right),$$  

(2.6)

**SD**

$$\frac{\partial u}{\partial t} = -\text{div} \left[ \sqrt{1 + |\nabla u|^2} \left( I - \frac{\nabla u \otimes \nabla u}{1 + |\nabla u|^2} \right) \nabla \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \right],$$  

(2.7)

**WF**

$$\frac{\partial u}{\partial t} = -w \text{div} \left[ \frac{1}{w} \left( I - \frac{\nabla u \otimes \nabla u}{w^2} \right) \nabla (wH) - \frac{1}{2} \mathcal{H}^2 \nabla u \right].$$  

(2.8)

In the above, we have used the following notations and representation

$$w = \sqrt{1 + |\nabla u|^2} \quad \text{and} \quad \mathcal{H} = \text{div} \left( \frac{\nabla u}{w} \right).$$

To simplify the above equations, we borrow the contraction operator $\star$ from [25] for all possible contractions between derivatives of $u$, for example, we use $\nabla^2 u \star \nabla u \star \nabla u$ to indicate any expression of the form $\nabla_{ij} u \nabla_k u \nabla_l u$ with $1 \leq i, j, k, l \leq n$. They are all treated equally in terms of analysis. Moreover, we use $P_k(\nabla u)$ to denote some $k$-th power contraction of $\nabla u$, i.e.,

$$P_k(\nabla u) = \nabla u \star \cdots \star \nabla u = \Pi_{j=1}^k \nabla_{ij} u, \quad \text{for some} \ 1 \leq i_j \leq n.$$

As derived in [25], we can rewrite the equation (2.2), (2.3) and (2.5) using the above convention as follows:

**MCF**

$$\partial_t u - \Delta u = w^{-2} \nabla^2 u \star P_2(\nabla u),$$  

(2.9)

**SD**

$$\partial_t u + \Delta^2 u = \nabla_{ij} f_{ij}^1 [u] + \nabla_{ij} f_{ij}^2 [u],$$  

(2.10)

**WF**

$$\partial_t u + \Delta^2 u = f_0 [u] + \nabla_{ij} f_{ij}^1 [u] + \nabla_{ij} f_{ij}^2 [u],$$  

(2.11)

where

$$f_0 [u] = \nabla^2 u \star \nabla^2 u \star \nabla^2 u \star \sum_{k=1}^{4} w^{-2k} P_{2k-2}(\nabla u),$$  

(2.12)

$$f_1 [u] = \nabla^2 u \star \nabla^2 u \star \sum_{k=1}^{4} w^{-2k} P_{2k-1}(\nabla u),$$  

(2.13)

$$f_2 [u] = \nabla^2 u \star \sum_{k=1}^{2} w^{-2k} P_{2k}(\nabla u).$$  

(2.14)
Under the assumption that $|\nabla u| \lesssim 1$, the following crude bounds for the nonlinear terms play crucial roles in our analysis:

$$|f_0[u]| \lesssim |\nabla^2 u|^3, \quad |f_1[u]| \lesssim |\nabla^2 u|^2, \quad \text{and} \quad |f_2[u]| \lesssim |\nabla^2 u|.$$  \hfill (2.15)

Abstractly, we can write (2.9), (2.10), and (2.11) in the following form

$$\begin{aligned}
\left\{ \begin{array}{l}
\partial_t u + Au = N[u], \quad (x,t) \in \mathbb{R}^n \times (0, \infty), \\
u(x,0) = u_0(x), \quad x \in \mathbb{R}^n,
\end{array} \right.
\end{aligned} \quad (2.16)$$

where $A = -\Delta$ or $\Delta^2$, and $N[u]$ is the nonlinear term in the right hand sides of (2.9), (2.10), or (2.11). We say $u(x,t)$ is a mild solution to (2.16) if it satisfies the following integral equation

$$u(x,t) = e^{-At}u_0(x) + \int_0^t e^{-(t-s)A}N[u](x,s)ds, \quad (x,t) \in \mathbb{R}^n \times (0, \infty).$$  \hfill (2.17)

where $e^{-At}$ is the semigroup generated by $-A$. If the Lipschitz norm of $u_0$ is small, the global well-posedness of mild solution to (2.16) is obtained by Koch–Lamm [25].

One of the most important features of these equations is their scale invariant property. More precisely, for any positive constant $\lambda$, if we define $\Sigma_\lambda := \lambda^{-1}\Sigma$, then

$$\mathcal{H}_{\Sigma_\lambda} = \lambda \mathcal{H}_\Sigma, \quad \mathcal{K}_{\Sigma_\lambda} = \lambda^2 \mathcal{K}_\Sigma, \quad \text{and} \quad \Delta_{\Sigma_\lambda} = \lambda^2 \Delta_\Sigma.$$  

In terms of equation, let $u$ be a mild solution to (2.16). If we similarly define $u_\lambda(x,t) := \lambda^{-1}u(\lambda x, \lambda^\alpha t)$, where $\alpha = 2$ if $A = -\Delta$ and $\alpha = 4$ if $A = \Delta^2$. Then $u_\lambda$ solves the same PDE but with rescaled initial data, i.e.,

$$\begin{aligned}
\left\{ \begin{array}{l}
\partial_t u_\lambda + Au_\lambda = \lambda N[u_\lambda], \quad (x,t) \in \mathbb{R}^n \times (0, \infty), \\
u_\lambda(x,0) = \lambda^{-1}u_0(\lambda x), \quad x \in \mathbb{R}^n.
\end{array} \right.
\end{aligned} \quad (2.18)$$

Note that with $y = \lambda x$, then $\nabla_y u_\lambda = \nabla_y u$, $\nabla_y^2 u_\lambda = \lambda \nabla_y^2 u$ and so forth. The powers of $\nabla_y^2 u$ in the nonlinear terms $f_i$ are such that $f_i(u_\lambda) = \lambda^{3-i}f_i(u)$ for $i = 0,1,2$. Hence $\nabla^i f_i(u_\lambda) = \lambda^{3-i}\nabla^i f_i(u)$. They indeed give the corresponding scale invariance with $\alpha = 4$ for SD and WF. For MCF, we only have the term $f_2(u) \sim \nabla^2 u$, corresponding to $\alpha = 2$.

The above naturally leads to the notion of self-similar solutions $v$ which satisfy $v_\lambda(x,t) = v(x,t)$. Setting $t = 0$, then the initial data necessarily has the property that $v(x,0) = \lambda^{-1}v(\lambda x,0)$. Conversely, let $v$ be the solution of (2.16) with self-similar initial data $v_0(x) = |x|\psi \left( \frac{x}{|x|} \right)$ for some function $\psi : S^{n-1} \rightarrow \mathbb{R}$ so that $v_0$ is indeed self-similar, $v_0(x) = \lambda^{-1}v_0(\lambda x)$. Since $v_\lambda$ solves the same equation and initial data, by the uniqueness of solution, it holds that $v_\lambda(x,t) = v(x,t)$. Upon introducing $\Psi(y) = v(y,1)$, we then have

$$v(x,t) = v_{t^{-\frac{1}{\alpha}}}(x,t) = t^{\frac{1}{\alpha}}v(xt^{-\frac{1}{\alpha}},1) =: t^{\frac{1}{\alpha}}\Psi(xt^{-\frac{1}{\alpha}}).$$  \hfill (2.19)

The function $\Psi$ is called a self-similar profile and it satisfies the following equation:

$$A\Psi(y) + \frac{1}{\alpha}\Psi(y) - \frac{1}{\alpha}y \cdot \nabla \Psi(y) = N(\Psi(y)).$$

The main objective of this paper is to study the stability of self-similar solutions under bounded (and small) perturbation of self-similar initial data. Our main results are given as follows:
Theorem 2.1. There exists an $\varepsilon > 0$ such that if $u(x,t)$ is a global mild solution to (2.16) with perturbed self-similar initial data of $u_0(x) = v_0(x) + p(x)$ such that $\|p\|_{L^\infty(\mathbb{R}^n)} < \infty$ and $\|\nabla v_0\|_{L^\infty(\mathbb{R}^n)}$, $\|\nabla p\|_{L^\infty(\mathbb{R}^n)} < \varepsilon$, then for any compact subset $K$ of $\mathbb{R}^n$, it holds that
\[
\lim_{t \to \infty} \left\| t^{-\frac{1}{n}} u(t^{\frac{1}{n}} x,t) - \Psi(x) \right\|_{C^k(K)} = 0, \quad \forall k \in \mathbb{N}^+. \tag{2.20}
\]

The next example demonstrates the validity of Theorem 2.1.

Example 2.2. Consider a shifting perturbation on initial self-similar data $v_0(x)$ by $a \in \mathbb{R}^n$, i.e., $u_0(x) = v_0(x - a)$. In this case, $p(x) = u_0(x) - v_0(x) = v_0(x - a) - v_0(x)$, which satisfies the condition of Theorem 2.1. In fact,
\[
\begin{align*}
\|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} + \|\nabla p\|_{L^\infty(\mathbb{R}^n)} &\leq \|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} + \|\nabla (v_0(x - a) - v_0(x))\|_{L^\infty(\mathbb{R}^n)} \\
&\leq 3\|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} < 3\varepsilon,
\end{align*}
\]
and
\[
\|p(x)\|_{L^\infty(\mathbb{R}^n)} = \|v_0(x - a) - v_0(x)\|_{L^\infty(\mathbb{R}^n)} < \|\nabla v_0\|_{L^\infty(\mathbb{R}^n)}|a| < \infty.
\]
From the uniqueness of the mild solution to (2.16), we have
\[
u(x,t) = v(x - a,t) = t^\frac{1}{2} \Psi \left( (x - a)\alpha^{-\frac{1}{2}} \right),
\]
then we can show
\[
\lim_{t \to \infty} \left\| t^{-\frac{1}{n}} u(t^{\frac{1}{n}} x,t) - \Psi(x) \right\|_{C^k(K)} = \lim_{t \to \infty} \left\| \Psi \left( x - at^{-\frac{1}{2}} \right) - \Psi(x) \right\|_{C^k(K)} = 0, \quad \forall k \in \mathbb{N}^+,
\]
since $\Psi$ is smooth.

We also have the following result on the global convergence under perturbation with spatial decay.

Theorem 2.3 (Global stability with spatial decay). There exists an $\varepsilon > 0$ such that if $u(x,t)$ is a global mild solution to (2.16) with perturbed self-similar initial data of $u_0(x) = v_0(x) + p(x)$ such that $\|p\|_{L^\infty(\mathbb{R}^n)} < \infty$ and $\|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} + \|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} + |1 + |x|^2)\nabla p\|_{L^\infty(\mathbb{R}^n)} < \varepsilon$, then we have
\[
\lim_{t \to \infty} \left\| t^{-\frac{1}{n}} u(t^{\frac{1}{n}} x,t) - \Psi(x) \right\|_{C^1(\mathbb{R}^n)} = 0. \tag{2.21}
\]

Remark 2.4. It seems possible to also prove higher order global-in-space convergence results. The main technical step is to generalize Lemma 4.4 and Lemma 4.7 to higher order estimates. The work [25] uses analytic Banach fixed point theorem to obtain higher order regularity. For the reason of conciseness and space, we omit this step in this paper.
The following result (global well-posedness for initial data with small Lipschitz norm) for (2.16) and the technique to prove it provide a starting point for our investigation. (The definition of the function space $X_\infty$ will be given in Section 4.)

**Theorem 2.5** (Koch–Lamm [25], Theorem 3.1 & 5.1). There exists $\varepsilon > 0, C > 0$ such that for every $u_0$ with $\|\nabla u_0\|_\infty < \varepsilon$ there exists an analytic solution $u \in X_\infty$ of (2.16) with $u(\cdot,0) = u_0$ which satisfies $\|u\|_{X_\infty} \leq C\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}$. The solution is unique in the ball $B^n_{C\varepsilon}(0) := \{u \in X_\infty\|u\|_{X_\infty} \leq C\varepsilon\}$. Moreover, there exist $R > 0, c > 0$ such that for every $k \in \mathbb{N}_0$ and multi-index $\gamma \in \mathbb{N}_0^n$, we have the estimate

$$\sup_{x \in \mathbb{R}^n} \sup_{t > 0} \left| \frac{t^\frac{\gamma}{2}}{\gamma!} \frac{\partial^\gamma}{\partial t^\gamma/2} \nabla u(x,t) \right| \leq c\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)} R^{\|\gamma\| + k}(\|\gamma\| + k)!.$$  \hspace{1cm} (2.22)

Furthermore, $u$ depends analytically on $u_0$.

Note that even though the estimate resembles those coming from linear parabolic equations and is consistent with the scale invariant property, it is highly nontrivial to establish for nonlinear equations. The fact that the estimates are expressed in terms of the Lipschitz norm of the initial data is particularly useful as self-similar initial data is necessarily only Lipschitz. Furthermore, note the following gradient bound for the solution ($\gamma = 0, k = 0$):

$$\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}$$  \hspace{1cm} (2.23)

implies that the smallness of the Lipschitz norm is preserved in time. This fact is crucial if we want to work in the graph setting because for surface diffusion, it has been shown by [15] that in general the graph property might not be preserved.

For the rest of this section, we outline the strategy of the proof of Theorem 2.1. Such an approach is also described in [18, Chapter 1] by M.-H. Giga, Y. Giga and J. Saal. First, note that upon setting $\lambda = \frac{t^\frac{1}{2}}{\sqrt{t}}$, then $u_\lambda(x,1) = u_\frac{t^\frac{1}{2}}{\sqrt{t}}(x,1) = \frac{t^\frac{1}{2}}{\sqrt{t}} u(\frac{t^\frac{1}{2}}{\sqrt{t}},t)$. Hence (2.20) is equivalent to

$$\lim_{\lambda \to \infty} \|u_\lambda(x,1) - v_\lambda(x,1)\|_{C^k(K)} = \lim_{\lambda \to \infty} \|u_\lambda(x,1) - v(x,1)\|_{C^k(K)} = 0, \quad \forall k \in \mathbb{N}^+.$$  \hspace{1cm} (2.24)

Thus all we need is to estimate at time $t = 1$ the difference between the two solutions $u_\lambda$ and $v_\lambda \equiv v$. Now let $\Phi_\lambda := u_\lambda - v$. Then it satisfies

$$\Phi_\lambda(x,t) = e^{-At}p_\lambda(x) + \int_0^t e^{-(t-s)A}(N[v + \Phi_\lambda] - N[v])(x,s)ds,$$  \hspace{1cm} (2.25)

where we have used the fact that the difference between the initial data is given by $u_\lambda(x,0) - v(x,0) = \frac{1}{t^\frac{1}{2}} p(\lambda x) := p_\lambda(x)$.

Next, the following estimate from Theorem 2.5 is applicable to both $u_\lambda$ and $v$:

$$|\nabla \partial^k_t \nabla u(x,t)| \leq C t^{-\left\lfloor \frac{\|\gamma\| + k}{2} \right\rfloor} \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}.$$  \hspace{1cm} (2.26)

Putting (2.25) and (2.26) together, we can apply Arzela–Ascoli compactness theorem to show that there is a subsequence $\{\Phi_\lambda_k\}, \lambda_k \to \infty$ and $\Phi_\infty \in C^\infty(\mathbb{R}^n \times (0,1])$ such that the following statements hold.
1. (Convergence) For any compact subset $K$ of $\mathbb{R}^n$,
\[
\lim_{\lambda_k \to \infty} \|\Phi_{\lambda_k}(x, 1) - \Phi_\infty(x, 1)\|_{C^k(K)} = 0, \quad \forall k \in \mathbb{N}.
\] (2.27)

2. (Regularity) For any $t \in (0, 1]$,
\[
\|\nabla^\gamma \partial_t^k \nabla \Phi_\infty(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\left(\frac{\gamma}{\alpha} + k\right)} \left(\|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} + \|\nabla p\|_{L^\infty(\mathbb{R}^n)}\right).
\] (2.28)

3. (Integral equation) $\Phi_\infty(x, t)$ solves the following integral equation:
\[
\Phi_\infty(x, t) = \int_0^t e^{-(t-s)A} \left( N[v + \Phi_\infty] - N[v] \right)(x, s) ds, \quad (x, t) \in \mathbb{R}^n \times (0, 1].
\] (2.29)

As the last step, we conclude the proof of (2.24) by showing that every solution $\Phi_\infty$ of (2.29) satisfying the property $\|\nabla \Phi_\infty\|_{L^\infty(\mathbb{R}^n)} \ll 1$ and the regularity estimate (2.28) must be equal to 0.

We would like to emphasize that the above strategy is very simple and robust. See again [18] for a general exposition of this strategy. Despite the fact that the results are restricted to the graph setting, it is applicable to all the geometric evolutions under consideration here. Another advantage is that maximum or comparison principle is not used in the current approach. See for example the results for MCF [32, 9, 7] that do rely on such a principle.

As a last remark before presenting the proof, note that WF has one more term, $f_0[u]$, than SD. Thus in the current work, we will only consider MCF and WF for simplicity.

### 3 Stability Result - Local Version

In this section, we will prove Theorem 2.1. As outlined above, we will first establish uniform estimates and compactness of $\Phi_\lambda$. In all of the following result, we are working in the regime of small Lipschitz norm. More precisely, there exist an $\epsilon \ll 1$ such that $\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} \ll 1$.

#### 3.1 MCF

In this case, we have $A = -\Delta$, $\alpha = 2$. Thus equation (2.25) for $\Phi_\lambda$ becomes
\[
\Phi_\lambda(x, t) = e^{t\Delta} p_\lambda(x) + \int_0^t e^{(t-s)\Delta} \left( N[u_\lambda] - N[u] \right)(x, s) ds.
\] (3.1)

The nonlinear term $N[u]$ can be estimated as:
\[
|N[u]| = (1 + |\nabla u|^2)^{-1} |\nabla u \ast \nabla u \ast \nabla^2 u| \lesssim |\nabla u|^2 |\nabla^2 u| \lesssim |\nabla^2 u|.
\] (3.2)

We also recall the heat kernel and its associated semigroup:
\[
h(x, t) := \frac{1}{(4\pi t)^\frac{n}{2}} \exp \left( -\frac{|x|^2}{4t} \right), \quad \text{and} \quad e^{\Delta t} f(x) := \int_{\mathbb{R}^n} h(x - y, t) f(y) dy.
\] (3.3)
3.1.1 Uniform estimates and Compactness for $\Phi_\lambda$

We first note several useful facts. By the $L^1$-bound of the heat kernel, we get
\[
\sup_{\lambda > 1} \sup_{t \geq 0} \| e^{\Delta t} p_\lambda(x) \|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\lambda > 1} \| p_\lambda(x) \|_{L^\infty(\mathbb{R}^n)} < \infty. \tag{3.4}
\]
Furthermore, the Lipschitz norm is invariant under the rescaling:
\[
\| \nabla p_\lambda \|_{L^\infty(\mathbb{R}^n)} = \| \nabla p \|_{L^\infty(\mathbb{R}^n)}. \tag{3.5}
\]
From the regularity estimate \([2.26]\), we have
\[
\sup_{\lambda > 1} \| \partial_t^k \nabla^\gamma \nabla u_\lambda(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} \leq t^{-\frac{|\gamma|}{2} - k} \sup_{\lambda > 1} \| \nabla (v_0 + p_\lambda) \|_{L^\infty(\mathbb{R}^n)} \tag{3.6}
\]
and similarly for $v_\lambda = v$,
\[
\sup_{\lambda > 1} \| \partial_t^k \nabla^\gamma \nabla v_\lambda(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} \leq t^{-\frac{|\gamma|}{2} - k} \| \nabla v_0 \|_{L^\infty(\mathbb{R}^n)}. \tag{3.7}
\]
Now we estimate
\[
\sup_{\lambda > 1} \| \Phi_\lambda(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\lambda > 1} \| e^{\Delta t} p_\lambda(\cdot) \|_{L^\infty(\mathbb{R}^n)} + \sup_{\lambda > 1} \left\| \int_0^t e^{-(t-s)\Delta} N[u_\lambda] - N[v](\cdot, s) \|_{L^\infty(\mathbb{R}^n)} ds \right\|,
\]
\[
\leq \sup_{\lambda > 1} \| e^{\Delta t} p_\lambda(\cdot) \|_{L^\infty(\mathbb{R}^n)} + \sup_{\lambda > 1} \left\| \int_0^t \int_{\mathbb{R}^n} |h(\cdot - y, t - s)| (|N[u_\lambda]| + |N[v]|)(y, s) dy ds \right\|_{L^\infty(\mathbb{R}^n)} \tag{3.8}
\]
\[
\leq \sup_{\lambda > 1} \| p_\lambda \|_{L^\infty(\mathbb{R}^n)} + (\| \nabla v_0 \|_{L^\infty(\mathbb{R}^n)} + \| \nabla p \|_{L^\infty(\mathbb{R}^n)}) \left\| \int_0^t \int_{\mathbb{R}^n} h(\cdot - y, t - s) s^{-\frac{1}{2}} dy ds \right\|_{L^\infty(\mathbb{R}^n)} \tag{3.9}
\]
\[
\leq \sup_{\lambda > 1} \| p_\lambda \|_{L^\infty(\mathbb{R}^n)} + (\| \nabla v_0 \|_{L^\infty(\mathbb{R}^n)} + \| \nabla p \|_{L^\infty(\mathbb{R}^n)}) \leq \infty.
\]
In the above, we used the estimate
\[
\| N[u_\lambda(\cdot, s)] \|_{L^\infty(\mathbb{R}^n)} \leq \| \nabla u_\lambda(\cdot, s) \|_{L^\infty(\mathbb{R}^n)} \| \nabla^2 u_\lambda(\cdot, s) \|_{L^\infty(\mathbb{R}^n)} \lesssim s^{-\frac{1}{2}} \tag{3.10}
\]
By the higher order regularity estimate \([3.6]\) and \([3.7]\), we have for any $k \in \mathbb{N}, \gamma \in \mathbb{N}^{n},$
\[
\sup_{\lambda > 1} \| \nabla^\gamma \partial_t^k \nabla \Phi_\lambda(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} \lesssim \sup_{\lambda > 1} \| \nabla^\gamma \partial_t^k \nabla u_\lambda(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} + \| \nabla^\gamma \partial_t^k \nabla v(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} \tag{3.11}
\]
\[
\lesssim (\| \nabla v_0 \|_{L^\infty(\mathbb{R}^n)} + \| \nabla p \|_{L^\infty(\mathbb{R}^n)}) t^{-\frac{|\gamma|}{2} - k}.
\]
With the above uniform estimates for $\Phi_\lambda$, we can apply the Arzela–Ascoli theorem to extract a subsequence $\{\Phi_{\lambda_k}\}$ and $\Phi_\infty(x, t) \in C^\infty(\mathbb{R}^n \times (0, 1])$ such that for any $\delta > 0$, compact subset $K \subset \mathbb{R}^n$, and $k \in \mathbb{N}$, we have
\[
\lim_{\lambda_k \to \infty} \sup_{\delta \leq t \leq 1} \| \Phi_{\lambda_k} - \Phi_\infty \|_{C^k(K)} = 0. \tag{3.12}
\]
Then \([2.27]\) and \([2.28]\) follow.
3.1.2 Equation for $\Phi_\infty$

Here we verify (2.29) by passing the limit $\lambda_k \to \infty$ in (3.1). First note that

$$\lim_{\lambda \to \infty} \sup_{t \geq 0} \|e^{\Delta t} p_\lambda(\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \lim_{\lambda \to \infty} \|p_\lambda\|_{L^\infty(\mathbb{R}^n)} = \lim_{\lambda \to \infty} \frac{1}{\lambda} \|p\|_{L^\infty(\mathbb{R}^n)} = 0. \quad (3.11)$$

Second, from (3.10), we know that for any $\delta > 0$ and any compact subset $K \subset \mathbb{R}^n$,

$$\lim_{\lambda \to \infty} \sup_{\delta \leq t \leq 1} \|N[v + \Phi_\lambda] - N[v + \Phi_\infty]\|_{C^k(K)} = 0. \quad (3.12)$$

Now note that

$$\left| \int_0^t e^{(t-s)N} (N[v + \Phi_\lambda] - N[v + \Phi_\infty])(x, s) ds \right| \leq \int_0^t \int_{\mathbb{R}^n} h(t-s, x-y) \left( |N[v + \Phi_\lambda]| + |N[v + \Phi_\infty]| \right)(y, s) dy ds.$$

By the formula of the heat kernel (3.3) and the estimate for the nonlinear term (3.9), the integrand can be estimated as:

$$h(t-s, x-y) \left( |N[v + \Phi_\lambda]| + |N[v + \Phi_\infty]| \right)(y, s) \lesssim (t-s)^{-\frac{n}{2}} \exp \left( -\frac{|x-y|^2}{4(t-s)} \right) s^{-\frac{1}{2}}$$

which is integrable:

$$\int_0^t \int_{\mathbb{R}^n} (t-s)^{-\frac{n}{2}} \exp \left( -\frac{|x-y|^2}{4(t-s)} \right) s^{-\frac{1}{2}} dy ds \lesssim t^\frac{n}{4}.$$

Hence (2.29) follows by the Lebesgue Dominated Convergence Theorem.

3.2 WF

In this case, we have $A = \Delta^2, \alpha = 4,$ and

$$N[u] = f_0[u] + \nabla_i f_1^i[u] + \nabla^2_{ij} f_2^{ij}[u].$$

First we introduce the heat kernel of biharmonic operator $b(x, t)$:

$$b(x, t) = t^{-\frac{n}{4}} g \left( \frac{x}{t^{\frac{1}{4}}} \right), \quad \text{where} \quad g(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi \cdot k - |k|^4} dk, \quad \xi \in \mathbb{R}^n.$$

Furthermore, it satisfies the following decaying estimates (see [14, Chapter 9, Theorem 7], [25]) which play a very important role in this paper:

$$|b(x, t)| \lesssim t^{-\frac{n}{4}} \exp \left( -C|\frac{|x|^2}{t^{\frac{1}{4}}} \right), \quad (3.13)$$
\[ |\nabla^k b(x, t)| \lesssim t^{-n+k} \exp \left(-C_k \frac{|x|^\frac{4}{n}}{t^4}\right), \quad \forall k \geq 1, \quad (3.14) \]

The integral equation for mild solutions \( u(x, t) \) to (2.11) now reads

\[
u(x, t) = \int_{\mathbb{R}^n} b(x - y, t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} b(x - y, t - s) f_0[u](y, s) dy ds
- \int_0^t \int_{\mathbb{R}^n} \nabla_i b(x - y, t - s) f_1^i[u](y, s) dy ds
+ \int_0^t \int_{\mathbb{R}^n} \nabla^2_{ij} b(x - y, t - s) f_2^{ij}[u](y, s) dy ds. \quad (3.15)\]

Given the uniform bound for \( \|\nabla u\|_{L^\infty(\mathbb{R}^n)} \lesssim 1 \), we note here the estimates for the nonlinear structures:

\[ |f_0[u]| \lesssim |\nabla^2 u|^3 \lesssim t^{-\frac{3}{4}}, \quad |f_1[u]| \lesssim |\nabla u|^2 \lesssim t^{-\frac{2}{3}}, \quad |f_2[u]| \lesssim |\nabla u| \lesssim t^{-\frac{1}{2}}. \quad (3.16)\]

Note also that in order to take advantage of the kernel decay, we perform integration by parts to eliminate the derivatives on \( f_1 \) and \( f_2 \). With this, we use the following \( L^1 \)-bound for \( b \),

\[ \|\nabla^k b(\cdot, t)\|_{L^1(\mathbb{R}^n)} \lesssim t^{-\frac{k}{4}} \quad \text{for } k = 0, 1, 2. \quad (3.17) \]

3.2.1 Uniform estimates and convergence for \( \Phi_\lambda \)

Using the estimates for \( b \), we first establish \( L^\infty \) bound for \( \Phi_\lambda \). For \( e^{-\Delta^2_t} p_\lambda \), we have

\[
sup_{\lambda > 1} \sup_{t \geq 0} \left\| e^{-\Delta^2_t} p_\lambda \right\|_{L^\infty} = \sup_{\lambda > 1} \left\| \int_{\mathbb{R}^n} b(\cdot, t) p_\lambda(y) dy \right\|_{L^\infty}
\leq \sup_{\lambda > 1} \sup_{t \geq 0} \| p_\lambda \|_{L^\infty} \| b(\cdot, t) \|_{L^1}
\lesssim \sup_{\lambda > 1} \| p_\lambda \|_{L^\infty} < \infty. \quad (3.18) \]

From the regularity estimates (2.20) we have

\[
sup_{\lambda > 1} \left\| \partial^k_t \nabla^\gamma \nabla u_\lambda(\cdot, t) \right\|_{L^\infty} \lesssim t^{-\frac{|\gamma|}{4} - k} \sup_{\lambda > 1} \| \nabla (v_0 + p_\lambda) \|_{L^\infty}
\lesssim t^{-\frac{|\gamma|}{4} - k} \left( \| \nabla v_0 \|_{L^\infty(\mathbb{R}^n)} + \sup_{\lambda > 1} \| \nabla p_\lambda \|_{L^\infty(\mathbb{R}^n)} \right)
\lesssim t^{-\frac{|\gamma|}{4} - k} (\| v_0 \|_{L^\infty(\mathbb{R}^n)} + \| \nabla p \|_{L^\infty(\mathbb{R}^n)}) \quad (3.19) \]

and similarly for \( v_\lambda = v \),

\[
sup_{\lambda > 1} \left\| \partial^k_t \nabla^\gamma \nabla v_\lambda(\cdot, t) \right\|_{L^\infty} \lesssim t^{-\frac{|\gamma|}{4} - k} \| v_0 \|_{L^\infty(\mathbb{R}^n)} \quad (3.20) \]

For the \( L^\infty \)-estimate for \( \Phi_\lambda \), we combine (3.13), (3.14), (5.19) and (2.15) to give

\[
sup_{\lambda > 1} \left\| \Phi_\lambda(\cdot, t) \right\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\lambda > 1} \left\| e^{-\Delta^2_t} p_\lambda(\cdot) \right\|_{L^\infty(\mathbb{R}^n)}
\]
and regularity estimates (3.17), (3.19) and (3.20), we have

\[
\lambda > \| \Phi \| \approx \lambda \leq \sup_t |\nabla b(\cdot - y, t - s)(f_0[u_\lambda] - f_0[v])|(y, s)dyds \quad (3.21)
\]

Now we make use of the structure for the nonlinear terms (3.16) together with the kernel and regularity estimates (3.17), (3.19) and (3.20), we have

\[
\left\| \int_0^t \int_{\mathbb{R}^n} |b(\cdot - y, t - s)|(f_0[u_\lambda] + f_0[v])(y, s)dyds \right\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{\frac{n}{4}}
\]

\[
\left\| \int_0^t \int_{\mathbb{R}^n} |\nabla_i b(\cdot - y, t - s)|(f_1[u_\lambda] + f_1[v])(y, s)dyds \right\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{\frac{n}{4}}
\]

\[
\left\| \int_0^t \int_{\mathbb{R}^n} |\nabla_{ij} b(\cdot - y, t - s)|(f_2[u_\lambda] + f_2[v])(y, s)dyds \right\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{\frac{n}{4}}.
\]

Hence we have

\[
\sup_{\lambda > 1} \| \Phi_\lambda(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} \lesssim \sup_{\lambda > 1} \| p_\lambda \|_{L^\infty} + t^{\frac{n}{4}} \left( \| \nabla v_0 \|_{L^\infty(\mathbb{R}^n)} + \| \nabla p \|_{L^\infty(\mathbb{R}^n)} \right) < \infty.
\]
For higher order regularity estimates, by (3.19), we have
\[
\sup_{\lambda > 1} \| \nabla^2 \partial_t^k \nabla \Phi_{\lambda}(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} \leq C t^{-\frac{|k|}{2} - k} (\| \nabla v_0 \|_{L^\infty(\mathbb{R}^n)} + \| \nabla p \|_{L^\infty(\mathbb{R}^n)}).
\] (3.24)

As in the MCF case, we apply the Arzela–Ascoli theorem to extract a subsequence \( \{ \Phi_{\lambda_k} \} \) and \( \Phi_\infty(x, t) \in C^\infty(\mathbb{R}^n \times (0, 1]) \) such that for any \( \delta > 0 \) and any compact subset \( K \) of \( \mathbb{R}^n \),
\[
\lim_{\lambda_k \to \infty} \sup_{\delta \leq t \leq 1} \| \Phi_{\lambda_k}(\cdot, t) - \Phi_\infty(\cdot, t) \|_{C^k(K)} = 0, \quad \forall k \in \mathbb{N}^+,
\] (3.25)
and \( \Phi_\infty \) satisfies the regularity estimate (2.28).

### 3.2.2 Equation for \( \Phi_\infty \)

Here we check that \( \Phi_\infty \) satisfies (2.29). The strategy is similar to the MCF case.

Recall that \( \Phi_\lambda \) satisfies the following identity:
\[
\Phi_{\lambda}(x, t) = e^{-\Delta t} p_{\lambda}(x) + \int_0^t \int_{\mathbb{R}^n} b(x - y, t - s)(f_0[\Phi_{\lambda} + v] - f_0[v])(y, s)dyds
\]
\[
- \int_0^t \int_{\mathbb{R}^n} \nabla b(x - y, t - s)(f_1^1[\Phi_{\lambda} + v] - f_1^1[v])(y, s)dyds
\]
\[
+ \int_0^t \int_{\mathbb{R}^n} \nabla^2 b(x - y, t - s)(f_2^{ij}[\Phi_{\lambda} + v] - f_2^{ij}[v])dyds.
\] (3.26)

First, by the \( L^1 \)-bounded of \( b(\cdot, t) \), similar to (3.11), we have
\[
\lim_{\lambda \to \infty} \| e^{-\Delta t} p_{\lambda}(\cdot) \|_{L^\infty(\mathbb{R}^n)} \leq \lim_{\lambda \to \infty} \| p_{\lambda} \|_{L^\infty(\mathbb{R}^n)} \leq \lim_{\lambda \to \infty} \frac{1}{\lambda} \| p \|_{L^\infty(\mathbb{R}^n)} = 0.
\] (3.27)

Second, similar to the previous computations, in particular, the derivations of (3.21), (3.22), (3.23), the integrals of the nonlinear terms are all bounded by integrands that are integrable with bounds independent of \( \lambda \). Hence, (2.29) follows from the Lebesgue Dominated Convergence Theorem. We emphasize here again the crucial use of the estimates (3.16) for the nonlinear terms and the \( L^1 \)-bounds (3.17) for the derivatives of the bi-harmonic heat kernel.

### 3.3 Proof of \( \Phi_\infty = 0 \)

In this section, we will show that the integral equation (2.29) only admits the zero solution among the class of functions with small Lipschitz norm. This follows from a fixed point type argument.

Motivated by the translation and scaling invariance of the equation, the following functions space was introduced in [25]. Let \( T > 0 \).

1. For MCF with \( \alpha = 2 \),
\[
X_T := \left\{ f(x, t) : \mathbb{R}^n \times (0, T) \to \mathbb{R} \left| \| f \|_{X_T} := \sup_{0 < t < T} \| \nabla f(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \sup_{0 < R^2 < T} R^{\frac{n}{2} + \frac{2}{1}} \| \nabla^2 f \|_{L^{n+4}(B_R(x) \times (R^2, R^2))} < \infty \right. \right\}
\] (3.28)
2. For WF with $\alpha = 4$,

$$X_T := \left\{ f(x, t) : \mathbb{R}^n \times (0, T) \to \mathbb{R} \left| \| f \|_{X_T} = \sup_{0 < t < T} \| \nabla f(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n, 0 < R < T} R^{\frac{2}{n+4}} \| \nabla^2 f \|_{L^{n+6}(B_R(x) \times (R^4, R^4))} \right. \right\} \tag{3.29}$$

Note that the above norms are scale invariant:

$$\| f_\lambda \|_{X_T} = \| f \|_{X_{\lambda^\alpha T}} \quad \text{and} \quad \| f_\lambda \|_{X_\infty} = \| f \|_{X_\infty}.$$ 

We then have the following estimate.

**Lemma 3.1** (Koch–Lamm [25] Lemma 3.10 and 5.2). For any $0 < T \leq \infty$ and $0 < \delta < 1$ there exists $C(\delta) > 0$ s.t. for every $g_1, g_2 \in B^\alpha_\delta(0) := \{ g \in X_T \| g \|_{X_T} \leq \delta \}$, we have

$$\left\| \int_0^T e^{-(T-s)A} N[g_1](x, s)ds - \int_0^T e^{-(T-s)A} N[g_2](x, s)ds \right\|_{X_T} \leq C(\delta)(\| g_1 \|_{X_T} + \| g_2 \|_{X_T}) \| g_1 - g_2 \|_{X_T}. \tag{3.30}$$

The above is established through the following linearized estimate:

$$\left\| \int_0^T e^{-(T-s)A} g ds \right\|_{X_T} \leq \| g \|_{Y_T}$$

for some appropriate spatial-temporal function space $Y_T$ [25, Lemma 3.11, 5.3]. We will in fact present the proof of the above result in the setting of weighted function spaces, $X_T^\beta$ and $Y_T^\beta$—see Lemmas 4.4 and 4.7.

We apply the above lemma with $T = 1$, $g_1 = \Phi_\infty + v$ and $g_2 = v$. Suppose we can show that $\| g_1 \|_{X_T}, \| g_2 \|_{X_T} \ll 1$, then we would have

$$\| \Phi_\infty \|_{X_T} = \left\| \int_0^T e^{-(T-s)A}(N[\Phi_\infty + v] - N[v])(x, s)ds \right\|_{X_T} \ll \| \Phi_\infty \|_{X_T},$$

which implies $\| \Phi_\infty \|_{X_T} = 0$. Hence $\nabla \Phi_\infty \equiv 0$ leading to $N[\Phi_\infty + v] = N[v]$ as $N(\cdot)$ only involves the derivatives of $\Phi_\infty$. From (2.29), we conclude that $\Phi_\infty \equiv 0$.

Hence we are led to compute the $X_T$-norm of $g_1$ and $g_2$ under the regularity estimates given by (2.22) and (2.28).

For MCF, we have,

$$\| \Phi_\infty \|_{X_T} + \| v \|_{X_T} \leq (\| \nabla v_0 \|_{L^\infty(\mathbb{R}^n)} + \| \nabla p \|_{L^\infty(\mathbb{R}^n)}) \left( 1 + \sup_{0 < R < T} R^{\frac{2}{n+4}} \left( \int_{B_R(x) \times (R^2, R^2)} (t^{-\frac{n}{2}}) dt dy \right) \right)$$

$$\leq (\| \nabla v_0 \|_{L^\infty(\mathbb{R}^n)} + \| \nabla p \|_{L^\infty(\mathbb{R}^n)}) \left( 1 + \sup_{0 < R < T} R^{\frac{2}{n+4}} \left( \int_{R^2} t^{-\frac{n+4}{2}} dt \right) \right).$$
\[ \lesssim (\|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} + \|\nabla p\|_{L^\infty(\mathbb{R}^n)}) \left( 1 + \sup_{0 < R^2 < T} R^{\frac{2}{n+6}} \left( R^n R^{-n-2} \right)^{\frac{1}{n+6}} \right) \]

\[ \lesssim (\|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} + \|\nabla p\|_{L^\infty(\mathbb{R}^n)}). \]

For WF, we have,

\[ \|\Phi_\infty\|_{X_T} + \|v\|_{X_T} \]

\[ \lesssim (\|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} + \|\nabla p\|_{L^\infty(\mathbb{R}^n)}) \left( 1 + \sup_{0 < R^4 < T} R^{\frac{2}{n+6}} \left( R^n \int_{B_R(x) \times (R^4/2, R^4)} R^n (t - \frac{n+6}{4}) \frac{1}{n+6} dt \right) \right) \]

\[ \lesssim (\|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} + \|\nabla p\|_{L^\infty(\mathbb{R}^n)}) \left( 1 + \sup_{0 < R^4 < T} R^{\frac{2}{n+6}} \left( R^n R^{-n-2} \right)^{\frac{1}{n+6}} \right) \]

\[ \lesssim (\|\nabla v_0\|_{L^\infty(\mathbb{R}^n)} + \|\nabla p\|_{L^\infty(\mathbb{R}^n)}). \]

The above show that in order to obtain the desired result, we just need to take the Lipschitz norms of \( v_0 \) and \( p \) to be sufficiently small which is indeed assumed to be the case under the current setting.

4 Equi-decay and Global Uniform Convergence

Here we will tackle Theorem 2.3. In essence, if the gradient of initial perturbation is assumed to have some spatial decay, then we can obtain a global-in-space convergence result. The idea is to establish the equi-decay property of \( \{\Phi_\lambda\}_{\lambda > 1} \) via a contraction property of the nonlinear operators in some weighted spaces. For convenience, we recall here the weighted Lipschitz seminorm used in Theorem 2.3:

\[ [p]_\beta := \|(1 + |x|^\beta) \nabla p(x)\|_{L^\infty(\mathbb{R}^n)}. \]  

(4.1)

4.1 MCF

For the mean curvature flow case, we introduce the following function space which is the spatially weighted version of \( X_T \):

**Definition 4.1.** For every \( 0 < T < \infty \), we define the function space \( X_T^\beta \) by

\[ X_T^\beta = \left\{ u \mid \|u\|_{X_T^\beta} := \sup_{0 < t < T} \sup_{x \in \mathbb{R}^n} (1 + |x|^\beta)|\nabla u(t, x)| \right. \]

\[ + \left. \sup_{x \in \mathbb{R}^n} \sup_{0 < R^2 < T} (1 + |x|^\beta) R^{\frac{2}{n+4}} \| \nabla^2 u \|_{L^{n+4}(Q_R(x))} < \infty \right\}, \]

(4.2)

where

\[ Q_R(x) := B_R(x) \times (R^2/2, R^2). \]
Then we have the following linear estimate.

**Lemma 4.2.** For $k \geq 0$ and $0 < t < T$,

$$\left\| t^\frac{k}{2} \nabla^k e^{t \Delta p(x)} \right\|_{X^\beta_T} \lesssim [p]_\beta.$$  \hfill (4.3)

For the analysis of the nonlinear part, we introduce the weighted function spaces $Y^\beta_T$ as follows.

**Definition 4.3.** For every $0 < T \leq \infty$, we define the function space $Y^\beta_T$ by

$$Y^\beta_T = \left\{ g \left| \left\| g \right\|_{\beta Y^\beta_T} = \sup_{x \in \mathbb{R}^n} \sup_{0 < R < T} (1 + |x|^\beta) R^{\frac{n}{2}} \| g \|_{L^n(Q_R(x))} < \infty \right. \right\}.$$  

Now we define

$$Sg(x, t) := \int_0^t \int_{\mathbb{R}^n} h(x - y, t - s)g(y, s)dyds..$$  \hfill (4.4)

The following is the key technical estimate concerning $S$.

**Lemma 4.4.** For $0 < t < T < \infty$,

$$\sup_{0 < t < T} \| (1 + |x|^\beta) Sg(x, t) \|_{L^\infty(\mathbb{R}^n)} + \| Sg \|_{X^\beta_T} \lesssim \| g \|_{Y^\beta_T}.$$  

With the above, then we have the following result for the nonlinear functional.

**Lemma 4.5.** For every $0 < T < \infty$,

$$\| N[u] - N[v] \|_{Y^\beta_T} \lesssim \left(\| u \|_{X^\beta_T}^2 + \| v \|_{X^\beta_T}^2 \right) \| u - v \|_{X^\beta_T}.$$  \hfill (4.5)

In particular, there exist $\varepsilon > 0$ and $q < 1$ such that for all $[v_0] + [p]_\beta < \varepsilon$,

$$\left\| \int_0^t e^{(t-s)\Delta} (N[u] - N[v])(x, s)ds \right\|_{X^\beta_T} \leq q \| u - v \|_{X^\beta_T}.$$  \hfill (4.6)

We will give the proofs of Lemmas 4.2 and 4.3 here but that for Lemma 4.4 in the Appendix due to its length and technical nature.

**Proof of Lemma 4.2.** It suffices to show that there exists a $C > 0$ depending only on $T$, $n$, $\beta$ and $k$ such that if $[p]_\beta \leq 1$, then $\left\| e^{t \Delta p(x)} \right\|_{X^\beta_T} \leq C$. From the definition of $\| \cdot \|_{X^\beta_T}$, we need to estimate two terms.

First, consider

$$\left| t^\frac{k}{2} \nabla^k e^{t \Delta p(x)} \right|$$

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while for II, when $|y - x| \leq \frac{\sqrt{t}}{2\sqrt{t}} |x|$ implies that $|y| \geq \frac{|x|}{2}$ for $0 < t < T$. Hence,

$$
\frac{1}{1 + |y|^\beta} \leq \frac{1}{1 + |x|^\beta} = \frac{2^\beta}{2^\beta + |x|^\beta} \leq \frac{2^\beta}{1 + |x|^\beta}
$$

so that

$$
I \lesssim \frac{1}{1 + |x|^\beta} \int_{\{y: |y-x| \leq \frac{\sqrt{t}}{2\sqrt{t}} |x|\}} \left( \frac{1}{(4\pi t)^{\frac{n}{2}}} \right) |\mathcal{P}_k \left( \frac{x-y}{\sqrt{t}} \right) e^{-\frac{|x-y|^2}{4t}} dy
$$

$$
\lesssim \frac{1}{1 + |x|^\beta} \int_{\mathbb{R}^n} |\mathcal{P}_k(z)| e^{-|z|^2} dz \lesssim \frac{1}{1 + |x|^\beta},
$$

while for II, when $|y - x| \geq \frac{\sqrt{t}}{2\sqrt{t}} |x|$, we have

$$
e^{-\frac{|x-y|^2}{4t}} = e^{-\frac{|x-y|^2}{4t}} e^{-\frac{|x-y|^2}{4t}} \leq e^{-\frac{|x|^2}{4t}} e^{-\frac{|z|^2}{4t}},
$$

so that

$$
II \leq e^{-\frac{|x|^2}{4t}} \int_{\{y: |y-x| \geq \frac{\sqrt{t}}{2\sqrt{t}} |x|\}} \left( \frac{1}{(4\pi t)^{\frac{n}{2}}} \right) e^{-\frac{|x-y|^2}{4t}} |\mathcal{P}_k \left( \frac{x-y}{2\sqrt{t}} \right) | dy
$$

$$\lesssim e^{-\frac{|x|^2}{4t}} \int_{\mathbb{R}^n} |\mathcal{P}_k(z)| e^{-\frac{|z|^2}{2t}} dz \lesssim e^{-\frac{|x|^2}{2t}} \lesssim \frac{1}{1 + |x|^\beta}.
$$

Combining I and II, we have

$$
\left| \int_{\mathbb{R}^n} t^{\frac{k}{2}} \nabla^k \nabla e^{t\Delta} p(x) \right| \lesssim \frac{1}{1 + |x|^\beta}. \quad (4.7)
$$

Second, we estimate

$$
\sup_{x \in \mathbb{R}^n} \sup_{0 < R < T} (1 + |x|^\beta) R^{\frac{n}{2} + 1} \left\| t^{\frac{k}{2}} \nabla^k \nabla^2 e^{t\Delta} p(x) \right\|_{L^{n+4}(Q_R(x))}. \quad (4.8)
$$

Note that

$$
\left\| t^{\frac{k}{2}} \nabla^k \nabla^2 e^{t\Delta} p(x) \right\|_{L^{n+4}(Q_R(x))}^{n+4}
$$
Proof of Lemma 4.5. Recall the form (2.9) for the nonlinear term

Then we have

\[ \lVert t^{\frac{k}{2}} \nabla^k e^{-\frac{|y-z|^2}{4t}} p(z) \rVert_{2} \leq C, \]

which leads to that \( (4.8) \lesssim 1. \)

The above two parts combined give \( \lVert t^{\frac{k}{2}} \nabla^k e^{+\Delta} p(x) \rVert_{X^k} \leq C. \)

Proof of Lemma 4.5 Recall the form (2.9) for the nonlinear term \( N(u) \). First note that

\[ \left| (1 + |\nabla u|^2)^{-1} - (1 + |\nabla v|^2)^{-1} \right| \leq \frac{\left| (\nabla u + |\nabla v|) \nabla (u - v) \right|}{(1 + |\nabla u|^2)(1 + |\nabla v|^2)}. \]

Then we have

\[ |N[u] - N[v]| \]

\[ \leq \left| (1 + |\nabla u|^2)^{-1} \nabla u \ast \nabla u \ast \nabla^2 u - (1 + |\nabla v|^2)^{-1} \nabla v \ast \nabla v \ast \nabla^2 v \right| \]

\[ \lesssim (\nabla u + |\nabla v|)(\nabla^2 u + |\nabla^2 v|)|\nabla (u - v)| + (\nabla u + |\nabla v|)^2 |\nabla^2 (u - v)|. \]

Then estimate (4.5) follows from

\[ \|N[u] - N[v]\|_{X^k} \]

\[ = \sup_{x \in \mathbb{R}^n} \sup_{0 < R < T} (1 + |x|^\beta) R^{n+4} \|N[u] - N[v]\|_{L^{n+4}(Q_R(x))} \]

\[ \lesssim \sup_{0 < t < T} \left( \|\nabla u\|_{L^\infty(\mathbb{R}^n)} + \|\nabla v\|_{L^\infty(\mathbb{R}^n)} \right) \times \]

\[ \sup_{0 < t < T} \sup_{x \in \mathbb{R}^n} \left( \|\nabla^2 u\|_{L^{n+4}(Q_R(x))} + \|\nabla^2 v\|_{L^{n+4}(Q_R(x))} \right) \times \]

\[ \sup_{0 < t < T} \sup_{x \in \mathbb{R}^n} (1 + |x|^\beta)|\nabla(u - v)| \]

\[ + \sup_{0 < t < T} \left( \|\nabla u\|_{L^\infty(\mathbb{R}^n)} + \|\nabla v\|_{L^\infty(\mathbb{R}^n)} \right)^2 \sup_{x \in \mathbb{R}^n} (1 + |x|^\beta) \sup_{0 < R < T} R^{n+4} \|\nabla^2 (u - v)\|_{L^{n+4}(Q_R(x))}. \]
\( \lesssim (\|u\|_{X_T} + \|v\|_{X_T})^2 \|u - v\|_{X_T^\beta}. \)

For (4.6), using Lemma 4.4, we have that
\[
\|S(N[u] - N[v])\|_{X_T^\beta} \lesssim \|N[u] - N[v]\|_{X_T^\beta}
\lesssim (\|u\|_{X_T}^2 + \|v\|_{X_T}^2) \|u - v\|_{X_T^\beta}
\lesssim (\|p\|^2 + \|v_0\|^2) \|u - v\|_{X_T^\beta} \lesssim \varepsilon^2 \|u - v\|_{X_T^\beta}.
\]

Note that we have used unweighted version Theorem 2.5 to estimate the \( \| \cdot \|_{X_T} \) norms by the initial data. Hence, (4.7) holds if we take \( \varepsilon \) to be sufficiently small.

### 4.2 WF case

The strategy here is similar to the MCF case. We again introduce the following weighted function space:

\[ X_T^{\beta} = \left\{ u \left| \|u\|_{X_T^{\beta}} := \sup_{0 < t < T} \sup_{x \in \mathbb{R}^n} (1 + |x|^\beta) |\nabla u(x, t)| + \sup_{x \in \mathbb{R}^n} (1 + |x|^\beta) R^{n+6} \| \nabla^2 u \|_{L^\infty(Q_R(x))} < \infty \right. \right\}, \]

where \( Q_R(x) := B_R(x) \times (R^4/2, R^4) \).

**Lemma 4.6.** For \( k \geq 0 \),
\[
\left\| t^k \nabla e^{-t \Delta^2} p(x) \right\|_{X_T^{\beta}} \lesssim [p]_{\beta}. \tag{4.9}
\]

Anticipating the forms of the nonlinear terms in (2.11), we introduce the following weighted function spaces \( Y_{0,T}^{\beta}, Y_{1,T}^{\beta} \) and \( Y_{2,T}^{\beta} \), where

\[
\|g_0\|_{Y_{0,T}^{\beta}} = \sup_{x \in \mathbb{R}^n} \sup_{0 < R^4 \times T} \left( 1 + |x|^\beta \right) R^{n+6} \|g_0\|_{L^\infty(Q_R(x))},
\]
\[
\|g_1\|_{Y_{1,T}^{\beta}} = \sup_{x \in \mathbb{R}^n} \sup_{0 < R^4 \times T} \left( 1 + |x|^\beta \right) R^{n+6} \|g_1\|_{L^\infty(Q_R(x))},
\]
\[
\|g_2\|_{Y_{2,T}^{\beta}} = \sup_{x \in \mathbb{R}^n} \sup_{0 < R^4 \times T} \left( 1 + |x|^\beta \right) R^{n+6} \|g_2\|_{L^\infty(Q_R(x))}. 
\]

Now consider the following operator:
\[
S g(x, t) := \int_0^t e^{-(t-s)\Delta^2} g ds = \int_0^t \int_{\mathbb{R}^n} b(x - y, t - s) g(y, s) dy ds. \tag{4.10}
\]

The key estimate is the following lemma:

**Lemma 4.7.** For every \( 0 < t < T < \infty \),
\[
\sum_{l=0}^2 \left( \sup_{0 < t < T} \| (1 + |x|^\beta) \nabla^l S g_l (\cdot, t) \|_{L^\infty(\mathbb{R}^n)} + \| \nabla^l S g_l \|_{X_T^{\beta}} \right) \lesssim \sum_{l=0}^2 \| g_l \|_{Y_{l,T}^{\beta}}. \tag{4.11}
\]
Lemma 4.8. For every $0 < T < \infty$,\[
\sum_{l=0}^{2} \left\| (f_l[u] - f_l[v]) \right\|_{Y_{\alpha, T}} \lesssim \left( \left\| u \right\|_{X_T} + \left\| v \right\|_{X_T} \right) \left\| u - v \right\|_{X_{\alpha, T}^T} \tag{4.12}
\]
(Recall the forms (2.12), (2.14) for the $f_l$'s.) In particular, there exist $\varepsilon > 0$ and $q < 1$ such that for all $[v_0] + [p] < \varepsilon$,
\[
\sum_{l=0}^{2} \int_0^t e^{-(t-s)\Delta^2} \left( \nabla^l f_l(u) - \nabla^l f_l(v) \right) ds \lesssim q \left\| u - v \right\|_{X_{\alpha, T}^T}. \tag{4.13}
\]

Proof of Lemma 4.6. It suffices to show that there exists a $C > 0$ depending only on $T, n, \beta$ and $k$ such that if $[p]_{\beta} \leq 1$, then $\left\| e^{-t\Delta^2} p(x) \right\|_{X_{\alpha, T}^T} \leq C$. Again, we need to estimate two terms.

First, by the estimate (3.14) for the biharmonic kernel $b$, for any $k \in \mathbb{N}_+$, there exists $c_k > 0$ such that
\[
\left| t^k \nabla^k \nabla e^{-t\Delta^2} p(x) \right| = \int_{\mathbb{R}^n} t^k \nabla_x \nabla b(t, x - y)p(y)dy \leq \int_{\mathbb{R}^n} t^k \nabla_y b(x - y, t) \left| \nabla_y p(y) \right| dy \lesssim \left( \int_{\left\{ y : |y - x| \leq \frac{1}{2} t^{\frac{3}{4}} \right\}} + \int_{\left\{ y : |y - x| \geq \frac{1}{2} t^{\frac{3}{4}} \right\}} \right) t^{-\frac{n}{4}} e^{-c_k (x-y)t^{\frac{3}{4}}} \frac{1}{1 + |y|^\beta} dy
\]
\[=: I + II,
\]
where similar to the MCF case, we have
\[
I \lesssim \frac{1}{1 + |x|^\beta} \int_{\left\{ y : |y - x| \leq \frac{1}{2} t^{\frac{3}{4}} \right\}} t^{-\frac{n}{4}} e^{-c_k (x-y)t^{\frac{3}{4}}} \frac{1}{1 + |y|^\beta} dy \lesssim \frac{1}{1 + |x|^\beta},
\]
\[
II \lesssim e^{-C|x|^\frac{3}{4}} \int_{\left\{ y : |y - x| \geq \frac{1}{2} t^{\frac{3}{4}} \right\}} t^{-\frac{n}{4}} e^{-c_k (x-y)t^{\frac{3}{4}}} \frac{1}{1 + |y|^\beta} dy \lesssim \frac{1}{1 + |x|^\beta}
\]
so that
\[
\sup_{0 < t < T} \sup_{x \in \mathbb{R}^n} \left( 1 + |x|^\beta \right) \left| t^k \nabla^k \nabla e^{-t\Delta^2} p(x) \right| \lesssim 1. \tag{4.14}
\]
Second, we compute
\[
\left\| t^k \nabla^k \nabla^2 e^{-t\Delta^2} p(x) \right\|_{L^{n+6}(Q_{\alpha}(x))} = \int_{\mathbb{R}^n} \int_{B_{R}(x)} \left| t^k \nabla^k \nabla b(y - z, t) \nabla p(z) dz \right|^{n+6} dy dt
\]
\[
\lesssim \int_{\mathbb{R}^n} \int_{B_{R}(x)} \left| t^{-\frac{n}{4}} t^{-\frac{n}{4}} e^{-c_k (y-z)t^{\frac{3}{4}}} \frac{1}{1 + |z|^\beta} dz \right|^{n+6} dy dt
\]
\[
\lesssim \int_{\mathbb{R}^n} \int_{B_{R}(x)} \left| \frac{t^{-\frac{1}{4}}}{1 + |y|^\beta} dz \right|^{n+6} dy dt
\]
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\[
\int_{R^4/2} \int_{B_R(x)} \frac{1}{(1 + |y|^2)^{n+6}} dy 
\]

which implies that

\[
\sup_{x \in R^n} \sup_{0 < R < T} (1 + |x|^2) R^{n+6} \left\| t^{\frac{n}{2}} \nabla^k \nabla^2 e^{-t \Delta^2} P \right\|_{L^{n+6}(Q_R(x))} \leq 1. 
\]

(4.15)

Combining (4.14) and (4.15) then gives Lemma 4.6.

Proof of Lemma 4.8

It is similar to that of Lemma 4.3. We will just highlight some key computations, though mostly at the symbolic level.

Recall the form of \( f_0 \): \( f_0(u) = (\nabla^2 u)^3 P(\nabla u) \) where \( P \) is some polynomial. Then

\[
f_0(u) - f_0(v) = ((\nabla^2 u)^3 - (\nabla v)^3) P(\nabla u) + (\nabla^2 v)^3 (P(\nabla u) - P(\nabla v)) 
\]

\[
\approx P(\nabla u) ((\nabla^2 u)^2 + (\nabla v)^2) (\nabla^2 (u - v)) + (\nabla^2 v)^3 P'(\nabla u) (\nabla (u - v)) 
\]

so that

\[
\|f_0(u) - f_0(v)\|_{L^{n+6}(Q_R(x))} 
\]

\[
\lesssim \|P(\nabla u)\|_{L^\infty(R^n)} \left( \|(\nabla^2 u)^2 + (\nabla v)^2\|_{L^{n+6}(Q_R(x))} \right) \|\nabla^2 (u - v)\|_{L^{n+6}(Q_R(x))} 
\]

\[
+ \|\nabla^2 v\|_{L^n(\nabla^2 (u - v))} \|P'(\nabla v)\|_{L^\infty(R^n)} \|\nabla (u - v)\|_{L^\infty(R^n)} 
\]

and hence

\[
\|f_0(u) - f_0(v)\|_{Y_0^\beta_{1,T}} \lesssim \left( \|u\|^2_{X_T} + \|v\|^2_{X_T} \right) \|u - v\|_{X_T^\beta}. 
\]

Similarly, for \( f_1(u) = (\nabla^2 u)^2 P(\nabla u) \) and \( f_2(u) = (\nabla^2 u) P(\nabla u) \), we have

\[
\|f_1(u) - f_1(v)\|_{L^{n+6}(Q_R(x))} 
\]

\[
\lesssim \|P(\nabla u)\|_{L^\infty(R^n)} \left( \|(\nabla^2 u)^2\|_{L^{n+6}(Q_R(x))} + \|\nabla^2 v\|_{L^{n+6}(Q_R(x))} \right) \|\nabla^2 (u - v)\|_{L^{n+6}(Q_R(x))} 
\]

\[
+ \|\nabla^2 u\|_{L^n(\nabla^2 (u - v))} \|P'(\nabla v)\|_{L^\infty(R^n)} \|\nabla (u - v)\|_{L^\infty(R^n)} 
\]

and

\[
\|f_2(u) - f_2(v)\|_{L^{n+6}(Q_R(x))} \lesssim \|P(\nabla u)\|_{L^\infty(R^n)} \|\nabla^2 (u - v)\|_{L^{n+6}(Q_R(x))} 
\]

\[
+ \|\nabla^2 u\|_{L^n(\nabla^2 (u - v))} \|P'(\nabla v)\|_{L^\infty(R^n)} \|\nabla (u - v)\|_{L^\infty(R^n)} 
\]

so that

\[
\|f_1(u) - f_1(v)\|_{Y_1^\beta_{1,T}} + \|f_2(u) - f_2(v)\|_{Y_2^\beta_{1,T}} \lesssim \left( \|u\|^2_{X_T} + \|v\|^2_{X_T} \right) \|u - v\|_{X_T^\beta} 
\]

and hence completing the proof of (4.12).

Again, we postpone the proof of Lemma 4.7 to the Appendix due to its technicality.
4.3 Conclusion of the Proof of Theorem 2.3

For simplicity, we just write down the steps for WF as it involves more terms. Recall the equation for $\Phi_\lambda$:

$$\Phi_\lambda = e^{-\Delta^2 t} p_\lambda + \sum_{l=0}^{2} (\mathcal{N}_l[v + \Phi_\lambda] - \mathcal{N}_l[v])$$  \hspace{1cm} (4.16)

where

$$\mathcal{N}_l(g) = \int_0^t e^{-\Delta^2 (t-s)} \nabla f_l(g) \, ds.$$  

First, taking the $X_T^{\beta}$ norm of both sides of the equation, by Lemma 4.7 and (4.13) of Lemma 4.8 we get

$$\|\Phi_\lambda\|_{X_T^{\beta}} \leq \|e^{-\Delta^2 t} p_\lambda\|_{X_T^{\beta}} + \sum_{l=0}^{2} \|\mathcal{N}_l[\Phi_\lambda + v] - \mathcal{N}_l[v]\|_{X_T^{\beta}}$$

$$\leq \|e^{-\Delta^2 t} p_\lambda\|_{X_T^{\beta}} + \sum_{l=0}^{2} \|f_l(\Phi_\lambda + v) - f_l[v]\|_{Y^{\beta}_{l,T}}$$

$$\leq \|e^{-\Delta^2 t} p_\lambda\|_{X_T^{\beta}} + q \|\Phi_\lambda\|_{X_T^{\beta}}.$$  

Hence, upon choosing $[v_0], [p] < \varepsilon$ small enough, we will have $q < 1$ which implies a uniform bound for $\Phi_\lambda$ in $X_T^{\beta}$. More precisely,

$$\|\Phi_\lambda\|_{X_T^{\beta}} \lesssim \|e^{-\Delta^2 t} p_\lambda\|_{X_T^{\beta}} \lesssim [p_\lambda]^{\beta}. \hspace{1cm} (4.17)$$

Second, from Lemmas 4.7 and 4.8 again, we have that

$$\sum_{l=0}^{2} \sup_{0 \leq t < T} \|(1 + |x|^{\beta})(\mathcal{N}_l[\Phi_\lambda + v] - \mathcal{N}_l[v])\|_{L^\infty(\mathbb{R}^n)} \lesssim \sum_{l=0}^{2} \|f_l(\Phi_\lambda + v) - f_l[v]\|_{Y^{\beta}_{l,T}} \lesssim \|\Phi_\lambda\|_{X_T^{\beta}} \lesssim [p_\lambda]^{\beta}.$$  

When $\lambda > 1$, we have $[p_\lambda]^{\beta} \leq [p]^{\beta}$. Hence

$$\sup_{\lambda > 1} \sum_{l=0}^{2} \|(1 + |x|^{\beta})(\mathcal{N}_l[\Phi_\lambda + v] - \mathcal{N}_l[v])(x, T)\|_{L^\infty(\mathbb{R}^n)} \lesssim [p]^{\beta}. \hspace{1cm} (4.18)$$

With the above, we can prove the global $C^1$-convergence. Upon setting $T = 1$ in (4.18), we have the $\left\{\Phi_\lambda(\cdot, 1) - e^{-\Delta^2} p_\lambda(\cdot) = \sum_{l=0}^{2} \mathcal{N}_l(\Phi_\lambda + v) - \mathcal{N}_l(v)\right\}_{\lambda > 1}$ satisfies the equi-decay property, i.e.

$$\lim_{R \to \infty} \sup_{\lambda > 1} \sup_{|x| < R} \left|\Phi_\lambda(x, 1) - e^{-\Delta^2} p_\lambda(x, 1)\right| = 0.$$  

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From (3.24) (with $\gamma = k = 0$) and (3.18) (with the latter applied to $\nabla p_\lambda$) we have
\[
\left\| \nabla \left( \Phi_\lambda (\cdot, 1) - e^{-\Delta^2} p_\lambda (\cdot, 1) \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \left\| \nabla \Phi_\lambda (\cdot, 1) \right\|_{L^\infty(\mathbb{R}^n)} + \left\| e^{-\Delta^2} \nabla p_\lambda (\cdot, 1) \right\|_{L^\infty(\mathbb{R}^n)} < \infty.
\]

Finally, recall (3.27). Hence by Arzela-Ascoli Theorem, we can conclude that $\Phi_{\lambda_j} \to \Phi_\infty$ in $C^0(\mathbb{R}^n)$ for a subsequence $\lambda_j \to \infty$. The proof of $\Phi_\infty \equiv 0$ is the same as in Section 3.3 for the spatially un-weighted case.

For the convergence of $\nabla \Phi_\lambda$, by 4.17, we have that $\nabla \Phi_\lambda$ is equi-decay, i.e,
\[
\lim_{R \to \infty} \sup_{\lambda > 0} \sup_{|x| < R} \left| \nabla \Phi_\lambda (x, 1) \right| = 0.
\]

From (3.24) (with $\gamma = 1, k = 0$), we further have,
\[
\sup_{\lambda > 0} \left\| \nabla^2 \Phi_\lambda (\cdot, 1) \right\|_{L^\infty(\mathbb{R}^n)} < \infty.
\]

Hence, we deduce that $\nabla \Phi_{\lambda_j} \to \nabla \Phi_\infty \equiv 0$ uniformly in $\mathbb{R}^n$.

The overall $C^1$-convergence of $u_\lambda = \Phi_\lambda + v$ to $v$ is thus established.

5 Generalization to Polyharmonic Flows

As a future perspective and direction, we use this section to illustrate the robustness of the current approach and outline an abstract framework for the stability of self-similar solutions to possible higher order polyharmonic flows. Suppose the polyharmonic flow, in the graphical setting, takes the following form
\[
\begin{align*}
\partial_t u + Au &= N[u], & \text{on } \mathbb{R}^n \times (0, \infty), \\
u(x, 0) &= u_0(x), & \text{in } \mathbb{R}^n,
\end{align*}
\]
where $A = (-\Delta)^m$, $m \geq 2$, and $N[u]$ is the nonlinear term – see [21] for an example of the form of $N$. Furthermore, we assume that (5.1) is invariant under the rescaling
\[
u(x, t) = u_{\frac{\lambda}{\lambda^m}} (x, \frac{t}{\lambda^2}) = t^{\frac{1}{2m}} v(x t^{-\frac{1}{2m}}, 1) =: t^{\frac{1}{2m}} \Psi(x t^{-\frac{1}{2m}}).
\]

One could follow Koch–Lamm’s method to find a unique analytic solution to (5.1) with initial data of small Lipschitz norm in the following scale invariant function space:
\[
X_T := \left\{ f(x, t) : \mathbb{R}^n \times (0, T) \to \mathbb{R} \left\| f \right\|_{X_T} := \sum_{k=0}^{m-2} \sup_{0 < t < T} \frac{t^{\frac{k}{2m}}}{t^{\frac{k}{2m}}} \left\| \nabla^k \nabla f (x, t) \right\|_{L^\infty(\mathbb{R}^n)} \right\}
\]
Moreover, by putting the difference \( u - v \) will also be used: for all \((z,s)\) where the last is from the theory of singular integral [33]. The following pointwise estimate

\[
\text{Proof of Lemma 4.4.}
\]

which follows from the scaling property of the heat kernel.

We anticipate that a similar procedure as in this paper can show the stability of the self-similar solution \( v \) under bounded (and small) perturbation, more specifically, for \( u_0 = v_0(x) + p(x) \) with \( \|p\|_{L^\infty(\mathbb{R}^n)} < \infty \) and \( \|\nabla p\|_{L^\infty(\mathbb{R}^n)} < \varepsilon \), it holds that

\[
\lim_{t \to \infty} \left\| t^{-\frac{1}{2m}} u(t \frac{1}{2m} x, t) - \Psi(x) \right\|_{C^k_{\text{loc}}(\mathbb{R}^n)} = 0, \quad \forall k \in \mathbb{N}^+. \tag{5.4}
\]

Moreover, by putting the difference \( u - v \) in the following weighted space:

\[
X_T^\beta := \left\{ f(x,t) : \mathbb{R}^n \times (0,T) \to \mathbb{R} \left| f \right|_{X_T^\beta} := \sum_{k=0}^{m-2} \sup_{0 < t < T} t^{\frac{k}{2m}} \| (1 + |x|^\beta) \nabla^k f(x,t) \|_{L^\infty(\mathbb{R}^n)} \right\}
\]

we gain the equi-decay property which leads to the global convergence

\[
\lim_{t \to \infty} \left\| t^{-\frac{1}{2m}} u(t \frac{1}{2m} x, t) - \Psi(x) \right\|_{C^k(\mathbb{R}^n)} = 0 \tag{5.6}
\]

provided the perturbation is small in the weighted space, i.e., \( \| (1 + |x|^\beta) \nabla p \|_{L^\infty(\mathbb{R}^n)} < \varepsilon \).

\section{A Proof of Lemma 4.4}

Before the proof, we first recall some \( L^p \)-estimates concerning the heat kernel (3.3) \( h(x,t) \): for \( 0 < t < \infty \),

\[
\|h\|_{L^p(\mathbb{R}^n \times [0,t])} \lesssim t^{\frac{(n+2)-pn}{2p}}, \quad \text{for } 1 \leq p < \frac{n+2}{n}, \tag{A.1}
\]

\[
\|\nabla h\|_{L^p(\mathbb{R}^n \times [0,t])} \lesssim t^{\frac{(n+2)pn+(n+1)p}{2p}}, \quad \text{for } 1 \leq p < \frac{n+2}{n+1}, \tag{A.2}
\]

\[
\left\| \int_0^t \int_{\mathbb{R}^n} \nabla^2 h(z-y,t-s)g(y,s)\,dy\,ds \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)} \lesssim \|g\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)}, \quad \text{for } 1 \leq p < \infty, \tag{A.3}
\]

where the last is from the theory of singular integral [33]. The following pointwise estimate will also be used: for all \((z,s) \in \mathbb{R}^n \times (0,t) \setminus B_{\sqrt{t}}(0) \times (0,\frac{s}{2})\), it holds that

\[
|h(z,s)| + \sqrt{t} |\nabla h(z,s)| + t |\nabla^2 h(z,s)| \leq Ct^{-\frac{n}{2}} \exp \left(-c \frac{|z|}{\sqrt{t}} \right) \tag{A.4}
\]

which follows from the scaling property of the heat kernel.

\textit{Proof of Lemma 4.4.} It suffices to show that if \( \|g\|_{Y_T^\beta} \leq 1 \), then

\[
\sup_{0 < t < T} \|(1 + |x|^\beta)Sg(x,t)\|_{L^\infty(\mathbb{R}^n)} + \|Sg\|_{X_T^\beta} \lesssim 1.
\]
For this purpose, we need to estimate \(|Sg(x,t)|\), \(|\nabla Sg(x,t)|\), and \(|\nabla^2 Sg|_{L^{n+4}(Q_R(x))}\). We recall the notation \(Q_R(x) = B_R(x) \times (\frac{R^2}{2}, R^2)\) and further let \(Q'_R(x) := B_R(x) \times (0, \frac{R^2}{2})\). Without loss of generality, we fix \(T = 1\).

**Estimate for \(Sg\).** We decompose

\[
|Sg(x,t)| = \left| \int_0^t \int_{\mathbb{R}^n} h(x-y, t-s)g(y,s)dyds \right|
\leq \int_{Q_{\sqrt{T}}(x)} + \int_{\mathbb{R}^n \times (0,T) \setminus Q_{\sqrt{T}}(x)} \left| h(x-y, t-s)g(y,s) \right|dyds
:= I_1 + I_2.
\]

For \(I_1\), by Hölder inequality and the heat kernel estimate (A.1) with \(p = \frac{n+4}{n+3} < \frac{n+2}{n}\), we have,

\[
I_1 \leq \|h\|_{L^{\frac{n+4}{n+3}}(Q'_{\sqrt{T}}(0))}\|g\|_{L^{n+4}(Q_{\sqrt{T}}(x))} \leq \|h\|_{L^{\frac{n+4}{n+3}}(\mathbb{R}^n \times (0,T/2))}\|g\|_{L^{n+4}(Q_{\sqrt{T}}(x))}
\lesssim t^{\frac{n+4}{n+3}}\|g\|_{L^{n+4}(Q_{\sqrt{T}}(x))} = t^{\frac{1}{2}}t^{\frac{1}{n+4}}\|g\|_{L^{n+4}(Q_{\sqrt{T}}(x))}
\lesssim \frac{1}{1 + |x|^3}.
\]

For \(I_2\), we estimate it as follows:

\[
I_2 = \int_{\mathbb{R}^n \times (0,T) \setminus Q_{\sqrt{T}}(x)} \left| h(x-y, t-s)g(y,s) \right|dyds
\lesssim \sum_{m=0}^{\infty} \sum_{z \in 2^{-\frac{m}{2}}Q_{\sqrt{T}}(x)} \int_{2^{-m-1}}^{2^{-m}} \int_{B_{2^{-\frac{m}{2}}}} t^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4t}} |g(y,s)|dyds
= \sum_{m=0}^{\infty} \sum_{z \in 2^{-\frac{m}{2}}Q_{\sqrt{T}}(x)} \int_{Q_{2^{-\frac{m}{2}}}} t^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4t}} |g(y,s)|dyds
\lesssim \left( \sum_{m=0}^{\infty} \sum_{z \in 2^{-\frac{m}{2}}Q_{\sqrt{T}}(x)} + \sum_{m=0}^{\infty} \sum_{z \in 2^{-\frac{m}{2}}Q_{\sqrt{T}}(x)} \right) \int_{Q_{2^{-\frac{m}{2}}}} t^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4t}} |g(y,s)| dyds
:= I_{21} + I_{22}.
\]

To estimate \(I_{21}\), we compute,

\[
I_{21} \lesssim \sum_{m=0}^{\infty} \sum_{z \in 2^{-\frac{m}{2}}Q_{\sqrt{T}}(x)} e^{-\frac{|x-z|^2}{4t}} \int_{Q_{2^{-\frac{m}{2}}}} t^{-\frac{n}{2}} |g(y,s)|dyds
\lesssim \sum_{m=0}^{\infty} \left( \sup_{z \in 2^{-\frac{m}{2}}Q_{\sqrt{T}}(x)} \int_{Q_{2^{-\frac{m}{2}}}} t^{-\frac{n}{2}} |g(y,s)|dyds \right) \left( \sum_{z \in 2^{-\frac{m}{2}}Q_{\sqrt{T}}(x)} e^{-\frac{|x-z|^2}{4t}} \right),
\]

\[25\]
where we have used the estimate \( | \sum_z a(z) b(z) | \leq \sup_z |a(z)| \sum_z |b(z)| \). Note that

\[
\sum_{z \in 2^{-m} \sqrt{T} \mathbb{Z}^n} e^{-\frac{|z|^2}{4T}} \leq \sum_{z \in 2^{-m} \sqrt{T} \mathbb{Z}^n} e^{-c |z|^2} = \sum_{z \in \mathbb{Z}^n} e^{-c |z|^2} 2^{\frac{m}{2}} \int_{\mathbb{R}^n} e^{-c |z|^2} 2^{\frac{m}{2}} \, d^n z \approx 2^{\frac{m}{2}}
\]

while

\[
\sup_{z \in 2^{-m} \sqrt{T} \mathbb{Z}^n} \int_{Q_{2^{-m}} \sqrt{T} \mathbb{Z}^n} t^{-\frac{3}{2}} |g(y, s)| \, dy \, ds \leq t^{-\frac{3}{2}} \sup_{z \in 2^{-m} \sqrt{T} \mathbb{Z}^n} \| 1 \|_{L^{\infty}} (Q_{2^{-m}} \sqrt{T} \mathbb{Z}^n) \| g \|_{L^{n+4}(Q_{2^{-m}} \sqrt{T} \mathbb{Z}^n)}
\]

\[
\leq t^{-\frac{3}{2}} 2^{\frac{m(2(n+2)(n+3))}{2(n+4)}} \sup_{z \in 2^{-m} \sqrt{T} \mathbb{Z}^n} \left( 2^{\frac{m}{2}} \sqrt{T} \right)^{\frac{1}{2(n+4)}} \| g \|_{L^{n+4}(Q_{2^{-m}} \sqrt{T} \mathbb{Z}^n)}
\]

\[
\leq t^{\frac{1}{2}} 2^{\frac{m(2(n+2)(n+3))}{2(n+4)}} \sup_{z \in 2^{-m} \sqrt{T} \mathbb{Z}^n} \frac{1}{1 + |z|^\beta} \lesssim \frac{t^{\frac{1}{2}} 2^{\frac{m(2(n+2)(n+3))}{2(n+4)}}}{1 + |x|^\beta}.
\]

Hence

\[
I_{21} \leq \frac{t^{\frac{1}{2}}}{1 + |x|^\beta} 2^{\frac{m(2(n+2)(n+3))}{2(n+4)}} 2^{\frac{m}{2}} = \frac{t^{\frac{1}{2}}}{1 + |x|^\beta} \sum_{m=0}^{\infty} 2^{\frac{m}{2}} \lesssim \frac{t^{\frac{1}{2}}}{1 + |x|^\beta}.\quad (A.6)
\]

For \( I_{22} \), we estimate it as

\[
I_{22} \lesssim \sum_{m=0}^{\infty} \sum_{z \in 2^{-m} \sqrt{T} \mathbb{Z}^n} e^{-\frac{3}{2} |z|^2} \int_{Q_{2^{-m}} \sqrt{T} \mathbb{Z}^n} t^{-\frac{3}{2}} e^{-\frac{c}{2} \sqrt{T} \sqrt{x-y}} |g(y, s)| \, dy \, ds
\]

\[
\lesssim e^{-\frac{3}{2} |z|^2} \sum_{m=0}^{\infty} \sum_{z \in 2^{-m} \sqrt{T} \mathbb{Z}^n} \int_{Q_{2^{-m}} \sqrt{T} \mathbb{Z}^n} t^{-\frac{3}{2}} e^{-\frac{c}{2} \sqrt{T} \sqrt{x-y}} |g(y, s)| \, dy \, ds
\]

\[
\lesssim e^{-\frac{3}{2} |z|^2} \sum_{m=0}^{\infty} \left( \sup_{z \in 2^{-m} \sqrt{T} \mathbb{Z}^n} \int_{Q_{2^{-m}} \sqrt{T} \mathbb{Z}^n} t^{-\frac{3}{2}} |g(y, s)| \, dy \, ds \right) \left( \sum_{z \in 2^{-m} \sqrt{T} \mathbb{Z}^n} e^{-\frac{3}{2} |z|^2} \right).
\]

Then similar to the computation for \( I_{21} \), we arrive at

\[
I_{22} \lesssim e^{-\frac{3}{2} |z|^2} t^{\frac{1}{2}} \sum_{m=0}^{\infty} \left( \sup_{z \in 2^{-m} \sqrt{T} \mathbb{Z}^n} \frac{1}{1 + |z|^\beta} \right) 2^{\frac{m}{2}} \lesssim e^{-\frac{3}{2} |z|^2} t^{\frac{1}{2}} \lesssim \frac{t^{\frac{1}{2}}}{1 + |x|^\beta}.\quad (A.7)
\]
Combining (A.5), (A.6), and (A.7), we obtain
\[
\sup_{0 < t < T} \| (1 + |x|^\beta) Sg(x, t) \|_{L^\infty(\mathbb{R}^n)} \lesssim t^{\frac{1}{\beta}} \lesssim 1.
\] (A.8)

We re-state the estimate \( I_2 \) here for future usage:
\[
I_2 = \int_{\mathbb{R}^n \times (0, t)} |h(x - y, t - s) g(y, s)| dyds \lesssim \int_{\mathbb{R}^n \times (0, t)} t^{-\frac{n}{2}} e^{-\frac{|x-y|}{\sqrt{t}}} |g(y, s)| dyds \lesssim \frac{t^{\frac{1}{\beta}}}{1 + |x|^\beta}.
\] (A.9)

**Estimate for \( \nabla Sg \).**
\[
|\nabla Sg(x, t)| = \left| \int_0^t \int_{\mathbb{R}^n} \nabla h(x - y, t - s) g(y, s) dyds \right| \leq \int_{Q_r(x)} + \int_{\mathbb{R}^n \times (0, t) \setminus Q_r(x)} |\nabla h(x - y, t - s) g(y, s)| dyds := J_1 + J_2.
\]

For \( J_1 \), by Hölder inequality, using the heat kernel estimate (A.2) with \( p = \frac{n+4}{n+3} < \frac{n+2}{n+1} \), we can derive
\[
J_1 \lesssim \| \nabla h \|_{L^{n+4}(Q_r(x))} \| g \|_{L^{n+4}(Q_r(x))} \leq \| \nabla h \|_{L^{n+4}(\mathbb{R}^n \times (0, t/2))} \| g \|_{L^{n+4}(Q_r(x))} \lesssim \frac{1}{1 + |x|^\beta}.
\] (A.10)

For \( J_2 \), we can exactly follow the derivation of (A.9). The only change is the appearance of \( t^{-\frac{n}{2}} \) due to the point-wise estimate of \( \nabla h \) in (A.4).
\[
J_2 = \int_{\mathbb{R}^n \times (0, t)} |\nabla h(x - y, t - s) g(y, s)| dyds \lesssim \int_{\mathbb{R}^n \times (0, t) \setminus Q_r(x)} t^{-\frac{n}{2}} e^{-\frac{|x-y|}{\sqrt{t}}} |g(y, s)| dyds \lesssim \frac{1}{1 + |x|^\beta}.
\] (A.11)

Combining (A.10) and (A.11), we have
\[
\sup_{0 < t < 1} \sup_{x \in \mathbb{R}^n} (1 + |x|^\beta) |\nabla Sg(x, t)| \lesssim 1.
\] (A.12)

**Estimate for \( \nabla^2 Sg \).** For this, we need to show
\[
\sup_{0 < R^2 < 1} R^{\frac{2}{n+4}} \| \nabla^2 Sg \|_{L^{n+4}(Q_R(x))} \leq \frac{1}{1 + |x|^\beta}.
\] (A.13)

For this purpose, we compute
\[
R^{\frac{2}{n+4}} \| \nabla^2 Sg(z, t) \|_{L^{n+4}(Q_R(x))}
\] 27
\[ = R_{n+1} \left\| \int_0^t \int_{\mathbb{R}^n} \nabla^2 h(z - y, t - s)g(y, s)dyds \right\|_{L^{n+4}(Q_R(x))} \]
\[ = R_{n+1} \left\| \int_{\mathbb{R}^n \times (0,t) \setminus B_{2R}(x) \times (R^2/4, R^2)} + \int_{B_{2R}(x) \times (R^2/4, R^2)} \nabla^2 h(z - y, t)g(y, s)dyds \right\|_{L^{n+4}(Q_R(x))} \]
\[ = R_{n+1} \left\| \int_{\mathbb{R}^n \times (0,t) \setminus B_{2R}(x) \times (R^2/4, R^2)} \nabla^2 h(z - y, t - s)g(y, s)dyds \right\|_{L^{n+4}(Q_R(x))} \]
\[ + R_{n+1} \left\| \int_{B_{2R}(x) \times (R^2/4, R^2)} \nabla^2 h(z - y, t - s)g(y, s)dyds \right\|_{L^{n+4}(Q_R(x))} := K_1 + K_2. \]

For \( K_1 \), we have
\[ K_1 = R_{n+1} \left\| \int_{\mathbb{R}^n \times (0,t) \setminus B_{2R}(x) \times (R^2/4, R^2)} \nabla^2 h(z - y, t - s)g(y, s)dyds \right\|_{L^{n+4}(Q_R(x))} \]
\[ \lesssim R_{n+1} \left\| \frac{t^{-\frac{3}{2}}}{1 + |z|^\beta} \right\|_{L^{n+4}(Q_R(x))} \lesssim \frac{1}{1 + |x|^\beta}, \tag{A.14} \]

where we have used again the estimate \( [A.9] \) for \( I_2 \) but with \( h \) replaced by \( \nabla^2 h \). The \( t^{-\frac{3}{2}} \) factor is due to the pointwise estimate for \( \nabla^2 h \) from \( [A.4] \). Note also \( \frac{R^2}{2} < t < R^2 \).

For \( K_2 \),
\[ K_2 := R_{n+1} \left\| \int_{B_{2R}(x) \times (R^2/4, R^2)} \nabla^2 h(z - y, t - s)g(y, s)dyds \right\|_{L^{n+4}(Q_R(x))} \]
\[ \lesssim R_{n+1} \left\| \chi_{B_{2R}(x) \times (R^2/4, R^2)} g(z, t) \right\|_{L^{n+4}(\mathbb{R}^n \times \mathbb{R}^+)} \]
\[ \lesssim R_{n+1} \left\| g \right\|_{L^{n+4}(B_{2R}(x) \times (R^2/4, R^2))} \lesssim \frac{1}{1 + |x|^\beta}. \tag{A.15} \]

where the second inequality is due to \( [A.3] \).

Hence \( [A.13] \) holds upon combining \( [A.14] \) and \( [A.15] \). \( \square \)

## B Proof of Lemma 4.7

The strategy here is very similar to Lemma 4.4. The main difference is the usage of the estimates of the biharmonic kernel \( b \) and also the fact that we need to deal with \( g_t \) for \( l = 0, 1, 2 \). For \( 0 \leq k \leq 3 \) and \( t > 0 \), we have from \( [A.14] \) that,
\[ \left\| \nabla^k b \right\|_{L^p(\mathbb{R}^n \times (0,t))} \leq Ct^{(n+4)-p(n+k)} \frac{1}{t^p}, \quad 1 \leq p < \frac{n + 4}{n + k}, \tag{B.1} \]
while for \( k = 4 \), the following comes from the theory of singular integrals \( [33] \):
\[ \left\| \int_0^t \int_{\mathbb{R}^n} \nabla^4 b(z - y, t - s)g(y, s)dyds \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)} \lesssim \left\| g \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)} \tag{B.2} \]
Furthermore, from the scaling property of the kernel, the following pointwise estimates hold,
\[
\sum_{k=0}^{4} \left| (\sqrt[4]{t} \nabla)^k b(z, s) \right| \lesssim t^{-\frac{4}{p}} \exp \left( -c \frac{|z|}{\sqrt[4]{t}} \right), \quad \forall (y, s) \in \mathbb{R}^n \times (0, t) \setminus Q'_T(0). \tag{B.3}
\]
where we recall the notation, \( Q_R(x) = B_R(x) \times \left( \frac{R^4}{2}, R^4 \right) \) and \( Q'_R(x) = B_R(x) \times \left( 0, \frac{R^4}{2} \right) \).

**Proof of Lemma 4.7** The proof follows a similar paradigm as in the previous section. It suffices to show that there exists a \( C > 0 \) such that if \( \sum_{t=0}^{2} \| g_t \|_{Y^\beta_{1/T}} \leq 1 \), then
\[
\sum_{t=0}^{2} \sup_{0 < t < T} \left\| (1 + |x|^\beta) \nabla^l S g_t(x, t) \right\|_{L^\infty(\mathbb{R}^n)} + \left\| \nabla^l S g_t \right\|_{X^\beta_T} \leq C.
\]
Without loss of generality, we fix \( T = 1 \). Note also \( Q_{\frac{1}{T}}(x) := B_{\frac{1}{T}}(x) \times (t/2, t) \) and \( Q'_{\frac{1}{T}}(x) := B_{\frac{1}{T}}(x) \times (0, t/2) \). Now we estimate the relevant quantities.

**Estimate for** \( S g_l \) (0 \( \leq l < 2 \)). We compute,
\[
\left| \nabla^l S g_l(x, t) \right| \leq \left| \int_0^t \int_{\mathbb{R}^n} \nabla^l b(x - y, t - s) g_l(y, s) dy ds \right|
\leq \left( \int_{Q'_{\frac{1}{T}}(0)} + \int_{\mathbb{R}^n \times (0, t) \setminus Q_{\frac{1}{T}}(x)} \right) \left| \nabla^l b(x - y, t - s) g_l(y, s) \right| dy ds
= I_1 + I_2.
\]

For \( I_1 \), by the Hölder inequality, using the kernel estimate \([\text{B.1}] \) with \( p = \frac{n+6}{n+3+l} < \frac{n+4}{n+l} \), we arrive at
\[
I_1 \leq \left\| \nabla^l b \right\|_{L^\frac{n+6}{n+3+l}(Q'_{\frac{1}{T}}(0))} \left\| g_l \right\|_{L^\frac{n+6}{n+3+l}(Q_{\frac{1}{T}}(x))} \leq \left\| \nabla^l b \right\|_{L^\frac{n+6}{n+3+l}(\mathbb{R}^n \times (0, t))} \left\| g_l \right\|_{L^\frac{n+6}{n+3+l}(Q_{\frac{1}{T}}(x))}
\lesssim t^{-\frac{n+4}{4p} - \frac{(n+l+2)}{n+6}} t^{-\frac{1}{4p} - \frac{(n+l+2)}{n+6}} \left\| g_l \right\|_{L^\frac{n+6}{n+3+l}(Q_{\frac{1}{T}}(x))} \lesssim t^\frac{1}{4} \frac{1}{1 + |x|^\beta}.
\tag{B.4}
\]

For \( I_2 \), we make use of \([\text{B.3}] \) and compute
\[
I_2 \lesssim \int_{\mathbb{R}^n \times (0, t) \setminus Q_{\frac{1}{T}}(x)} \left| \nabla^l b(x - y, t - s) g_l(y, s) \right| dy ds
\lesssim \sum_{m=0}^{\infty} \sum_{z \in 2^{-m} \frac{1}{T} \mathbb{Z}^n} \int_{Q_{2^{-m} \frac{1}{T}}(x)} t^{-\frac{n+1}{4}} e^{-\frac{|z-y|}{\sqrt{T}}} \left| g_l(y, s) \right| dy ds
\lesssim \left( \sum_{m=0}^{\infty} \sum_{z \in 2^{-m} \frac{1}{T} \mathbb{Z}^n} \int_{Q_{2^{-m} \frac{1}{T}}(x)} t^{-\frac{n+1}{4}} e^{-\frac{|z-y|}{\sqrt{T}}} \left| g_l(y, s) \right| dy ds \right) \int_{Q_{2^{-m} \frac{1}{T}}(x)} t^{-\frac{n+1}{4}} e^{-\frac{|z-y|}{\sqrt{T}}} \left| g_l(y, s) \right| dy ds.
\]
Again, similar to the previous section, we have

$$I_{21} = \sum_{m=0}^{\infty} \sum_{z \in 2^{-m+\frac{4}{3}} \frac{Q_2}{\sqrt{\nu}} |z-x| \leq \frac{1}{2}} \int_{Q_2 - \frac{m}{6} \frac{Q_2}{\sqrt{\nu}}} t^{-\frac{n+l}{4}} e^{-\frac{c\sqrt{|z-x|}}{\sqrt{\nu}}} |g_l(y, s)| dy ds$$



$$\leq t^{-\frac{n+l}{4}} \sum_{m=0}^{\infty} \left( \sum_{z \in 2^{-m+\frac{4}{3}} \frac{Q_2}{\sqrt{\nu}} |z-x| \leq \frac{1}{2}} e^{-\frac{c\sqrt{|z-x|}}{\sqrt{\nu}}} \right) \left( \sup_{z \in 2^{-m+\frac{4}{3}} \frac{Q_2}{\sqrt{\nu}} |z-x| \leq \frac{1}{2}} \int_{Q_2 - \frac{m}{6} \frac{Q_2}{\sqrt{\nu}}} |g_l(y, s)| dy ds \right)$$



$$\leq t^{-\frac{n+l}{4}} \sum_{m=0}^{\infty} 2^{\frac{m}{4}} 2^{-m+\frac{4}{3}} \left( \sup_{z \in 2^{-m+\frac{4}{3}} \frac{Q_2}{\sqrt{\nu}} |z-x| > \frac{1}{2}} \|1\|_{L^{\frac{n+l+6}{n}}(Q_2 - \frac{m}{6} \frac{Q_2}{\sqrt{\nu}})} \|g_l\|_{L^{\frac{n+l+6}{n}}(Q_2 - \frac{m}{6} \frac{Q_2}{\sqrt{\nu}})} \right)$$



$$\leq t^{-\frac{1}{4}} \sum_{m=0}^{\infty} 2^{-\frac{m(1+l)}{4}} \frac{1}{1 + |x|^\beta} \lesssim \frac{t^{\frac{1}{4}}}{1 + |x|^\beta}$$  \hspace{2cm} (B.5)

while for $I_{22}$,

$$I_{22} = \sum_{m=0}^{\infty} \sum_{z \in 2^{-m+\frac{4}{3}} \frac{Q_2}{\sqrt{\nu}} |z-x| > \frac{1}{2}} \int_{Q_2 - \frac{m}{6} \frac{Q_2}{\sqrt{\nu}}} t^{-\frac{n+l}{4}} e^{-\frac{c\sqrt{|z-x|}}{\sqrt{\nu}}} |g_l(y, s)| dy ds$$



$$\leq t^{-\frac{n+l}{4}} e^{-\frac{c\sqrt{|z-x|}}{\sqrt{\nu}}} \sum_{m=0}^{\infty} \left( \sum_{z \in 2^{-m+\frac{4}{3}} \frac{Q_2}{\sqrt{\nu}} |z-x| > \frac{1}{2}} e^{-\frac{c\sqrt{|z-x|}}{\sqrt{\nu}}} \right) \left( \sup_{z \in 2^{-m+\frac{4}{3}} \frac{Q_2}{\sqrt{\nu}} |z-x| > \frac{1}{2}} \int_{Q_2 - \frac{m}{6} \frac{Q_2}{\sqrt{\nu}}} |g_l(y, s)| dy ds \right)$$



$$\leq t^{-\frac{n+l}{4}} e^{-\frac{c\sqrt{|z-x|}}{\sqrt{\nu}}} \sum_{m=0}^{\infty} 2^{\frac{m}{4}} 2^{-m+\frac{4}{3}} \left( \sup_{z \in 2^{-m+\frac{4}{3}} \frac{Q_2}{\sqrt{\nu}} |z-x| > \frac{1}{2}} \|1\|_{L^{\frac{n+l+6}{n}}(Q_2 - \frac{m}{6} \frac{Q_2}{\sqrt{\nu}})} \|g_l\|_{L^{\frac{n+l+6}{n}}(Q_2 - \frac{m}{6} \frac{Q_2}{\sqrt{\nu}})} \right)$$



$$\leq t^{\frac{1}{4}} e^{-\frac{c\sqrt{|z-x|}}{\sqrt{\nu}}} \sup_{z \in 2^{-m+\frac{4}{3}} \frac{Q_2}{\sqrt{\nu}} |z-x| > \frac{1}{2}} \frac{1}{1 + |x|^\beta} \lesssim \frac{t^{\frac{1}{4}}}{1 + |x|^\beta}.$$  \hspace{2cm} (B.6)

Combining (B.4), (B.5) and (B.6) leads to

$$\sup_{0 < t < 1} \sup_{x \in \mathbb{R}^n} (1 + |x|^\beta) \sum_{l=0}^{2} |\nabla^l S g_l(x, t)| \lesssim \frac{t^{\frac{1}{4}}}{1 + |x|^\beta} \sum_{l=0}^{2} \|g_l\|_{Y^\beta} \lesssim \sum_{l=0}^{2} \|g_l\|_{Y^\beta}.$$  \hspace{2cm} Estimate for $\nabla S g_l$ (0 ≤ l < 2). The same computation leads to

$$\sup_{0 < t < 1} \sup_{x \in \mathbb{R}^n} (1 + |x|^\beta) \sum_{l=0}^{2} |\nabla^l \nabla S g_l(x, t)| \lesssim \sum_{l=0}^{2} \|g_l\|_{Y^\beta}.$$

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This is essentially the same as going from (A.8) to (A.12). Hence we just outline the key computation.

\[
||\nabla^l S g_l(x,t)|| = \left| \int_0^t \int_{\mathbb{R}^n} \nabla^l b(x-y,t-s) g_l(y,s) dy ds \right|
\leq \left( \int_{Q_{s/2}(0)} + \int_{Q_{s/2}(0) \setminus Q_{s/2}(x)} \right) ||\nabla^l b(x-y,t-s) g_l(y,s)|| dy ds
:= J_1 + J_2.
\]

For \( J_1 \), by the Hölder inequality, using the kernel estimate (B.1) with \( p = \frac{n+6}{n+3+l} < \frac{n+4}{n+l+1} \), \( l = 0, 1, 2 \) so that we can derive

\[
J_1 \leq ||l^{l+1}b||_{L^{\frac{n+6}{n+3+l}}(Q_{s/2}(0))} ||g_l||_{L^{\frac{n+6}{n+4}}(Q_{s/2}(x))} \leq ||l^{l+1}b||_{L^{\frac{n+6}{n+3+l}}(R^n \times (0,t))} ||g_l||_{L^{\frac{n+6}{n+4}}(Q_{s/2}(x))}
\leq \frac{1}{1 + |x|^\beta}.
\]

For \( J_2 \), the computation is similar. The extra factor \( t^{-\frac{1}{4}} \) coming from \( \nabla^{l+1} b \) is absorbed by the \( t^{\frac{1}{4}} \) in (B.4), (B.5), and (B.6).

**Estimate for \( \nabla^2 S g_l \) \( (0 \leq l < 2) \).** For this, we need to show

\[
\sup_{0 < R^2 < 1} R^{\frac{n+6}{n+6}} ||\nabla^{2+l} S g_l(z,t)||_{L^{n+6}(Q_R(x))} \lesssim \frac{1}{1 + |x|^\beta}, \quad l = 0, 1, 2. \tag{B.7}
\]

We first compute,

\[
||\nabla^{2+l} S g_l||_{L^{n+6}(Q_R(x))}
= \left| \int_{R^n \times (0,t) \setminus B_{2R}(x) \times (R^2/4,R^2)} + \int_{B_{2R}(x) \times (R^2/4,R^2)} \nabla^{2+l} b(z-y,t-s) g_l(y,s) dy ds \right|_{L^{n+6}(Q_R(x))}
\leq \left| \int_{R^n \times (0,t) \setminus B_{2R}(x) \times (R^2/4,R^2)} \nabla^{2+l} b(z-y,t-s) g_l(y,s) dy ds \right|_{L^{n+6}(Q_R(x))}
+ \left| \int_{B_{2R}(x) \times (R^2/4,R^2)} \nabla^{2+l} b(z-y,t-s) g_l(y,s) dy ds \right|_{L^{n+6}(Q_R(x))}
:= K_1 + K_2.
\]

For \( K_1 \), using the same arguments as the \( K_1 \) in the previous section, we get the pointwise bound

\[
\int_{R^n \times (0,t) \setminus B_{2R}(x) \times (R^2/4,R^2)} |\nabla^{2+l} b(z-y,t-s) g(y,s)| dy ds \lesssim \frac{t^{-\frac{1}{2}}}{1 + |z|^\beta}
\]

so that

\[
R^{\frac{n}{n+6}} \left| \frac{t^{-\frac{1}{2}}}{1 + |z|^\beta} \right|_{L^{n+6}(Q_R(x))} \lesssim R^{\frac{n}{n+6}} \frac{t^{-\frac{1}{2}}}{1 + |x|^\beta} (R^n R^4)_{\frac{1}{n+6}} \approx \frac{1}{1 + |x|^\beta}. \tag{B.8}
\]
where we have used the fact that \( \frac{R^4}{t} < t < R^4 \).

For \( K_2 \), we can focus on the \( L^{n+6} \)-estimate for \( \nabla^{2+} Sg_l \) with \( g_l \) supported in \( Q_{2R}(x) \). First we recall the Young inequality:

\[
\| f \ast g \|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)} \leq C \| f \|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)} \| g \|_{L^q(\mathbb{R}^n \times \mathbb{R}^+)},
\]

where \( 0 \leq p, q, m \leq \infty \), and \( p^{-1} + q^{-1} = 1 + m^{-1} \). Applying the inequality with \( m = n + 6 \), \( p = \frac{n+6}{n+4} \), \( q = \frac{n+6}{n+3} \), we get

\[
\| \nabla^2 Sg_0 \|_{L^{n+6}(\mathbb{R}^n \times (0,1))} \leq \| g_0 \|_{L^{\frac{n+6}{n-4}}(\mathbb{R}^n \times \mathbb{R}^+)} = \| g_0 \|_{L^{\frac{n+6}{n+3}}(B_{2R}(x) \times (R^4/4, R^4))},
\]

and

\[
\| \nabla^3 Sg_1 \|_{L^{n+6}(\mathbb{R}^n \times (0,1))} \leq \| g_1 \|_{L^{\frac{n+6}{n-2}}(\mathbb{R}^n \times \mathbb{R}^+)} = \| g_1 \|_{L^{\frac{n+6}{n+2}}(B_{2R}(x) \times (R^4/4, R^4))}.
\]

Hence

\[
R^{\frac{2}{n+6}} \| \nabla^2 Sg_0 \|_{L^{n+6}(\mathbb{R}^n \times (0,1))}, \quad R^{\frac{2}{n+6}} \| \nabla^2 Sg_1 \|_{L^{n+6}(\mathbb{R}^n \times (0,1))} \lesssim \frac{1}{1 + |x|^\beta}.
\] (B.9)

For the \( L^{n+6} \) norm of \( \nabla^4 Sg_2 \), by the singular integral estimate (B.2) with \( p = n + 6 \), we have that

\[
R^{\frac{2}{n+6}} \left\| \int_{B_{2R}(x) \times (R^4/4, R^4)} \nabla^4 b(z-y, t-s) g_2(y, s) dyds \right\|_{L^{n+6}(Q_R(x))} \lesssim \frac{1}{1 + |x|^\beta}.
\] (B.10)

Combining (B.8), (B.9), and (B.11) gives (B.7), completing the proof. \( \square \)

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