Department of Mathematics
Rutgers University
New Brunswick, NJ 08854, USA
email: akrol math.rutgers.edu
Variation on a theme of Selberg integrals

A. Kazarnovski-Krol

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Abstract

In this paper we calculate some Generalized Selberg integrals. The answer is expressed in terms of \( \Gamma \)-functions. Integrals of this type serve as normalization constants or directly via undoing 2-D integrals for determination of structural constants of operator algebra.

1 Notations

\( n, N \) two positive integers
\( \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \) simple roots of root system of type \( A_{n-1} \)
\( \alpha_1, \alpha_2, \ldots, \alpha_{N-1} \) simple roots of root system of type \( A_{N-1} \)
\( \Sigma_+ (N - 1) \) positive roots of root system of type \( A_{N-1} \)

\[
\rho = \frac{1}{\kappa} \sum_{\alpha \in \Sigma_+(N-1)} \alpha
\]

\[
q = e^{2\pi i \kappa}
\]

2 Introduction

The paper is a continuation of the series \[17], [18], [19], [20], [21] and contains an extension of the results of [19].

While in the previous papers we used conformal field theory for the needs of harmonic analysis (\( (n + 1)! \) dimensional space of solutions to the Heckman-Opdam hypergeometric system (wave functions in Calogero-Sutherland model) related to root system of type \( A_n \) is isomorphic to certain \( (n + 1)! \) dimensional space of conformal blocks in \( WA_n \) algebra, Harish-Chandra asymptotic solutions provide a basis in the space of conformal blocks, zonal spherical function is a particular conformal block), here the
accent is changed to applications to conformal field theory. The objective of conformal field theory is to construct Green’s function, i.e., monodromy invariant function which depends on $z = (z_1, z_2, \ldots, z_N)$ and conjugate variables. Green’s function is constructed out of conformal blocks:

$$G = \sum c_i |F_i(z)|^2$$

Conformal blocks are asymptotic solutions with the prescribed asymptotic behaviour at the singularity. Asymptotic solutions are normalized so that the leading asymptotic coefficient is equal to 1. On the other hand asymptotic solutions (conformal blocks) are provided by certain multidimensional integrals, and in order to calculate leading asymptotic coefficients value of Selberg-type integrals is needed. Also the Green’s function can be obtained by undoing 2D integrals cf. [3]. The coefficients $c_i$ should be properly normalized so that $\langle \phi, \phi \rangle = 1$. In this paper we calculate Selberg-type integrals related to the

$$V_\lambda \otimes V_{N\Lambda_1} \longrightarrow V_\mu.$$

Parameter $\lambda$ is assumed to be generic, $\Lambda_1$ is the first fundamental weight.

Note that in this case there is no multiplicities in the tensor product and the answer is expressed as a product of $\Gamma$-functions.

Here is organization of the paper. In theorem 2 we prove that certain type of integrals satisfy Heckman-Opdam hypergeometric system.

Then we collapse the arguments to the unity and using Opdam’s result obtain in theorem 3 the value of Selberg-type integrals. Here is organization of the paper. In section 3 we introduce the necessary combinatorics including two root systems. The first one is related to variables of integration, while the second one is related to the second factor of tensor product $N\Lambda_1$ (the row with $N$ boxes).

Then in section 4 we compare results with the usual Selberg integral, and finally in section 5 we emphasize one particular case, when after integration we actually obtain a monomial.

3

Fix two positive integers $n$ and $N$. We distinguish two root systems $A_{n-1}$ and $A_{N-1}$ with simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ and $\alpha_1, \alpha_2, \ldots, \alpha_{N-1}$, correspondingly. It is not important whether $n < N$ or $n \geq N$. We would like to realize this situation as follows. Let $m$ be the maximum of $N$ and $n$:

$$m = \text{Maximum}(n, N).$$
Consider $m$-dimensional Euclidean vector space $\mathbb{R}^m$ with $e_1, e_2, \ldots, e_m$ as orthonormal basis. Let $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{m-1} = e_{m-1} - e_m$.

The set of positive roots of $A_{N-1}$ shall be denoted by $\Sigma(N-1)$.

Let $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{m-1} = e_{m-1} - e_m$.

Let $z = (z_1, z_2, \ldots, z_N)$. We assume that $z_1, z_2, \ldots, z_N$ are real and

$$0 < z_1 < z_2 < \ldots < z_N$$

Let also $h_1 = \Lambda_1, h_2 = \Lambda_1 - \alpha_1, \ldots, h_{n+1} = \Lambda_1 - \alpha_1 - \ldots - \alpha_n$.

Let $V_{\lambda}$ denotes the irreducible highest weight module over quantum group $U_q(sl(n+1))$ with highest weight vector $\lambda$.

$$q = e^{\frac{2\pi i}{\kappa}}$$

Consider

$$V_{\lambda} \otimes V_{\Lambda_1} \xrightarrow{\otimes \rho_{\Lambda_1}} V_{\lambda+h_{i_1}} \xrightarrow{\otimes \rho_{\Lambda_1}} V_{\lambda+h_{i_1}+h_{i_2}} \ldots \xrightarrow{\otimes \rho_{\Lambda_1}} V_{\lambda+h_{i_1}+h_{i_2}+\ldots+h_{i_N}}$$  \hspace{1cm} (1)

Essentially we fix the ordered set of $N$ numbers (each of them not exceeding $n$).

$$I = \{i_1, i_2, \ldots, i_N\}.$$  \hspace{1cm} (2)

Let

$$s(k) = \#\left\{i_j \in I \mid i_j > k\right\},$$  \hspace{1cm} (3)

where by $\#$ we denote the cardinality of the corresponding set. Without loss of generality we assume

$$n = \text{Maximum}\left\{i_j - 1 \mid i_j \in I\right\}.$$  \hspace{1cm} (4)

Let

$$a(p) = \#\left\{i_j > 1 \mid j > p\right\}.$$  \hspace{1cm} (5)

Consider then the set of variables

$$\{t_{ij} \mid i = 1, \ldots, s(j), j = 1, \ldots, n\}$$
To each variable \( t_{ij} \) we assign a simple root \( \alpha_j \). To variables \( z_i \) we assign the first fundamental weight \( \Lambda_1 \), also we fix \( z_0 = 0 \) and assign to \( z_0 \) vector \( \lambda \), \( \lambda \in \text{span}\{\alpha_i | i = 1, \ldots, n - 1\} \).

Accordingly, we consider the integral

\[
\int \prod t_{ij}^{(\lambda - \alpha_j)} \prod (z_i - t_{i1})^{(\Lambda_1 - \alpha_1)} \prod (t_{i1} - t_{ij1})^{(\alpha_1, \alpha_{i1})} dt_{11} \ldots \frac{dt_{s(n), n}}{t_{s(n), n}} \tag{6}
\]

**Proposition 1** The leading asymptotic of the integral (4) over the cycle related to conformal block (3) is equal:

\[
z^\mu = z_1^{\mu_1} z_2^{\mu_2} \ldots z_N^{\mu_N} = \prod z_j^{\frac{(\lambda + h_{i1} + h_{i2} + \ldots + h_{ij-1} - \alpha_i) - (ij-1) + h_{i1} + h_{i2} + \ldots + h_{ij-1} - (j-1)\Lambda_1, \Lambda_1)}{\kappa} \tag{7}
\]

\[
\mu_j = \frac{(\lambda + h_{i1} + \ldots + h_{ij-1}, \sum_{i=1}^{ij-1} - \alpha_i) - (ij-1) + (h_{i1} + h_{i2} + \ldots + h_{ij-1} - (j-1)\Lambda_1, \Lambda_1)}{\kappa} \tag{8}
\]

The leading asymptotic coefficient is equal:

\[
\prod_{j=1}^N \prod_{p=1}^{ij-1} \exp(\pi i \frac{(\lambda + h_{i1} + h_{i2} + \ldots + h_{ij-1}, \sum_{i=p}^{ij-1} - \alpha_i) - (j-1)\Lambda_1, \Lambda_1)}{\kappa}) \times \frac{\Gamma(1 - \frac{1}{\kappa})(2\pi i)^{\frac{1}{\kappa}}}{\Gamma(\frac{(\lambda + h_{i1} + h_{i2} + \ldots + h_{ij-1}, \sum_{i=p}^{ij-1} - \alpha_i)}{\kappa} - \frac{p}{\kappa} + 1)\Gamma(\frac{(\lambda + h_{i1} + h_{i2} + \ldots + h_{ij-1}, \sum_{i=p}^{ij-1} - \alpha_i)}{\kappa} + \frac{p-1}{\kappa} + 1)} \tag{9}
\]

Moreover, for generic \( \lambda, \kappa \)

\[
\mu_i - \mu_j + \frac{1}{\kappa}(j - i) \notin \mathbb{Z} \tag{10}
\]

**Definition 1** Define \( \rho \) to be \( \frac{1}{\kappa} \) times half the sum of positive roots of root system of type \( A_{N-1} \):

\[
\rho = (\rho_1, \rho_2, \ldots, \rho_N) = \frac{1}{2\kappa}(N - 1, N - 3, \ldots, 3 - N, 1 - N) \tag{11}
\]
**Proof:** The calculation of the leading asymptotic coefficient inductively uses Dirichlet’s formula (and essentially corresponds to composition of vertex operators). Let

\[ \eta = \mu - \rho \]

\[ \eta_j = \mu_j - \rho_j \]

We want check that:

\[ \eta_s - \eta_j \notin \mathbb{Z} \]

for \( s \neq j \).

We consider the two cases: 1) \( i_s \neq i_j \) and 2) \( i_s = i_j \).

1). In the first case : \( i_s \neq i_j \) we get

\[ \eta_s - \eta_j = \frac{\lambda_i - \lambda_j}{\kappa} + \ldots \]

Thus (14) is satisfied for generic \( \lambda \).

2). In the second case let \( i_s = i_j = p + 1 \)

\[ \eta_j - \eta_s = \frac{1}{\kappa} \left( (h_{p+1} + h_{i_{s+1}} + h_{i_{s+2}} + \ldots + h_{i_{j-1}}, -(e_1 - e_{p+1})) \right. \]

\[ \left. + (h_{p+1} + h_{i_{s+1}} + \ldots + h_{i_{j-1}} - (j - s) \Lambda_1, \Lambda_1) + (j - s) \right) \neq 0 \]

As \( (h_{p+1}, -\alpha_1 - \ldots - \alpha_p) = 1 \). Moreover, (14) is positive for positive \( \kappa \) and thus the singularities of (14) can be avoided by assuming \( \kappa \) to be irrational. So from now on, we work in the above hypotheses on \( \kappa \) and \( \lambda \).

\[ \blacksquare \]

Let \( L \) be the following differential operator

\[ L = \sum_{i=1}^{N} (z_i \frac{\partial}{\partial z_i})^2 - k \sum_{i<j} \frac{z_j + z_i}{z_j - z_i} \left( z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right). \]

Explicitly, in the case of root system of type \( A_{N-1} \) the generators of the system of commuting operators were calculated in ref [27]. Namely, let

\[ D(\zeta, k) = \frac{1}{\prod_{i<j} (z_i - z_j)} \sum_{w \in S_N} \det(w) \prod_{j=1}^{N} z_j^{N-w(j)} \prod_{i=1}^{N} (\zeta + z_i \frac{\partial}{\partial z_i} + (w\delta, e_i)k) \]
Then

\[ D(\zeta, k) = \sum_{r=1}^{N} \zeta^r D_{N-r}^{(k)} \]  \hspace{1cm} (19)

Operators \( D_{N-r}^{(k)} \) commute with each other. Moreover, the system of hypergeometric differential equations can be written in the form:

\[ D(\zeta, k)\phi(\eta, k, z) = \prod_{i=1}^{N}(\zeta + \eta_i)\phi(\eta, k, z) \]  \hspace{1cm} (20)

Consider solutions of

\[ L\phi = ((\eta, \eta) - (\rho, \rho))\phi \]  \hspace{1cm} (21)

of the form

\[ \phi = \phi(\mu, k, z) = z^\mu(1 + \ldots) , \]  \hspace{1cm} (22)

where \( \mu = w\eta + \rho \).

Then for generic \( \eta \ ((\eta, \alpha) \notin \mathbb{Z}) \alpha \in \Sigma \) solutions \( \phi(w\eta + \rho, k, z) \) are uniquely defined, converge to analytic function in chamber \( 0 < |z_1| < |z_2| < \ldots < |z_N| \) linearly independent and provide a basis of linear space of solutions to the whole system of hypergeometric equations (20).

These solutions \( \phi(w\eta + \rho, k, z) \) will be referred to as Harish Chandra asymptotic solutions.

**Theorem 1 (Opdam)** The value of asymptotic solution \( \phi(\eta, k, z) \) at the unity is equal:

\[ \phi(\eta, k, 1) = \lim_{z \to 1} \phi(\eta, k, z) = \frac{\prod_{\alpha \in \Sigma_+ (N-1)} \frac{\Gamma((\eta, \alpha)+1)}{\Gamma((\eta, \alpha)-\frac{1}{2}+1)}}{\prod_{\alpha \in \Sigma_+ (N-1)} \frac{\Gamma((-\rho, \alpha)+1)}{\Gamma((-\rho, \alpha)-\frac{1}{2}+1)}} \]  \hspace{1cm} (23)

cf. \cite{25}, theorem 6.3.
Remark 1 Opdam’s result is actually in more general context, see also [17], [18], [20], [21].

Remark 2 The structure of the constant in theorem 1 is still very much similar to those of $c$-function of Harish-Chandra. It is not accidental, since it is obtained essentially using Harish Chandra decomposition for zonal spherical function and monodromy properties of zonal spherical function. Recall that $c$-function of Harish Chandra is needed for the Plancherel measure
\[ d\mu = \frac{d\lambda}{|c(\lambda, k)|^2}. \]

$c$-function was introduced by Harish Chandra, in the case of $SL(n, \mathbb{C})$ it was calculated by Gelfand and Naimark, in the case of $SL(n, \mathbb{R})$ by Bhanu Murti, and in general case by Gindikin and Karpelevich.

Theorem 2 The following integrals
\[
\int \prod_{i,j} t_{ij}^{\lambda_i - \alpha_j} \prod_{i,j} (z_i - t_{i_1})^{(\Lambda_1 - \alpha_1)_{i, j}} \prod_{i,j} (t_{i_1, j_1} - t_{i_2, j_2})^{(\alpha_{i_1, j_1}, \alpha_{i_2, j_2})_{i, j}} \frac{dt_{11}}{t_{11}} \cdots \frac{dt_{s(n), n}}{t_{s(n), n}} \tag{24}
\]

over appropriate cycles is a common eigenfunction of hypergeometric system of differential equations (20). Cycles should be chosen so that they are encoded as singular vectors of the tensor product of irreducible highest weight modules over quantum group $U_q(sl(n+1))$:

\[ V_\Lambda \otimes V_{\Lambda_1} \otimes V_{\Lambda_1} \otimes \cdots \otimes V_{\Lambda_1} \]

Proof: The theorem is proved with the help of integration by parts “upwards”. It is convenient to write the integral in the hierarchical form, and to organize variables of integration in the manner of Gelfand-Tsetlin patterns, s.t. the number of variables in the upper row is greater than or equal to the number of variables in the lower row. Since $(\alpha_i, \alpha_j) = 0$ if $|i - j| > 1$ and $(\Lambda_1, \alpha_i) = \delta_{i1}$, there is only interaction between the variables in the adjacent rows. It is quite instructive to think over what happens if the number of variables in the next row is the same as in the previous one. In this respect see example 1 below. In the same way we obtained an integral representation in [21]. $\square$
Remark 3. Integrals of the type considered in the theorem [3] are produced as conformal blocks of $W_n$ algebra, except that we do not have the factor $\prod_{i<j}(z_i-z_j)^{(\lambda_i, \lambda_j)}$ before the integral. One should notice, that this factor does not change the homological structure of the fiber, but affects Gauss-Manin connection.

Remark 4. Passage from "loops" to "tines" kills the kernel of the contravariant form and adds quantum Serre’s relations [29],[32].

Example 1. Triple integral for usual hypergeometric function. We consider the following situation:

$$V_\lambda \otimes V_\kappa \rightarrow V_{\lambda+\kappa} \otimes V_{\lambda+\kappa+\kappa}$$

$$\int \int \int \frac{(\lambda_1-a_1)}{\kappa} (z_1-t_1)^{-\frac{1}{\kappa}} (z_2-t_1)^{-\frac{1}{\kappa}} (z_2-t_2)^{-\frac{1}{\kappa}} (t_2-t_1)^{\frac{2}{\kappa}}$$

$$\times t_3^{-\frac{1}{\kappa}} (t_1-t_3)^{-\frac{1}{\kappa}} (t_2-t_3)^{-\frac{1}{\kappa}} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{dt_3}{t_3}$$

$$= \text{const} \frac{(\lambda_1-a_2)}{\kappa} \frac{2}{\kappa} \frac{(\lambda_1-a_1)}{\kappa} \frac{2}{\kappa} F\left(\frac{1}{\kappa}, \frac{(\lambda_1-a_2)}{\kappa}, \frac{(\lambda_1-a_2)}{\kappa}, \frac{1}{\kappa} + \frac{2}{\kappa} \right)$$

As a corollary we obtain the following identity involving 4-indices summation:

$$\frac{\Gamma(\frac{\lambda_1}{\kappa}) + 1}{\Gamma(\frac{\lambda_1}{\kappa} + \frac{1}{\kappa})} \frac{\Gamma(\frac{\lambda_2}{\kappa} - \frac{1}{\kappa} + 1)}{\Gamma(\frac{\lambda_2}{\kappa} - \frac{1}{\kappa})} \frac{\Gamma(\frac{(\lambda_1-a_2)}{\kappa}) + 1 - \frac{2}{\kappa}}{\Gamma(\frac{(\lambda_1-a_2)}{\kappa} + 1 - \frac{2}{\kappa})}$$

$$\times \sum_{m_1+m_2+m_3+m_4=M} \frac{\Gamma(\frac{(\lambda_1-a_2)}{\kappa} + m_1 + m_3 + m_4 - \frac{1}{\kappa})}{\Gamma(\frac{(\lambda_1-a_2)}{\kappa} + m_1 + m_3 + m_4 + 1 - \frac{1}{\kappa})}$$

$$\times \frac{\Gamma(\frac{(\lambda_1-a_2)}{\kappa} + m_4)}{\Gamma(\frac{(\lambda_1-a_2)}{\kappa} + m_4 + 1 - \frac{1}{\kappa})}$$

$$\times \frac{\Gamma\left(\frac{1}{\kappa} + m_1\right) \Gamma\left(\frac{1}{\kappa} + m_2\right) \Gamma\left(\frac{1}{\kappa} + m_3\right) \Gamma\left(\frac{1}{\kappa} + m_4\right)}{m_1! m_2! m_3! m_4!}$$

$$= \frac{\Gamma(\frac{(\lambda_1-a_2)}{\kappa} - \frac{1}{\kappa} + 1)}{\Gamma\left(\frac{1}{\kappa}\right)\Gamma\left(\frac{(\lambda_1-a_2)}{\kappa}\right)} \times \frac{\Gamma\left(\frac{1}{\kappa} + M\right) \Gamma\left(\frac{(\lambda_1-a_2)}{\kappa} + M\right)}{\Gamma\left(\frac{(\lambda_1-a_2)}{\kappa} - \frac{1}{\kappa} + 1 + M\right) M!}$$

(26)
Here is the main result of the paper.

**Theorem 3** The Generalized Selberg integral is equal:

\[
\int \prod t_{ij}^{(\lambda - \alpha_j)} \prod (1 - t_{ij})^{-\frac{1}{\kappa}} \prod (t_{i1j1} - t_{i2j2})^{(\alpha_j1 - \alpha_j2)} \frac{dt_{11}}{t_{11}} \cdots \frac{dt_{s(n),n}}{t_{s(n),n}} \\
= A \prod_{j=1}^{N} \prod_{p=1}^{i_j-1} \exp(\pi i(\frac{\lambda + h_{i1} + h_{i2} + \ldots + h_{i_j-1} + \sum_{i=i_j-p}^{i_j-1} - \alpha_i}{\kappa} - \frac{p - 1}{\kappa})) \times \Gamma(1 - \frac{1}{\kappa})(2\pi i) \\
\times \frac{\Gamma(\frac{\sum_{i=1}^{N} - \alpha_i}{\kappa} - \frac{2}{\kappa} + 1)\Gamma(\frac{\sum_{i=1}^{N} - \alpha_i}{\kappa} + \frac{p-1}{\kappa} + 1)}{\prod_{\alpha \in \Sigma_+(N-1)} \Gamma(\frac{\mu - \rho, \alpha + 1}{\kappa})} \times \prod_{\alpha \in \Sigma_+(N-1)} \Gamma(\frac{\mu - \rho, \alpha + 1}{\kappa}) (27)
\]

\[
A = e^{-\frac{\pi i}{\kappa} \sum_{p=1}^{N} a(p)} (28)
\]

Here \(a(p)\) is defined by (3), \(\mu\) by formula (8), and \(\rho\) by (11).

**Proof:** The theorem immediately follows from theorems 1, 2. The constant \(A\) takes into account the phase which is earned as \(z_i\) goes through some \(t_{ij}\) as all \(z_i\) collapse to the unity. \(|z_i| < |t_{ij}|\) changes to \(|t_{ij}| < |z_i|\).

**Example 2** Consider \(V_\lambda \otimes V_{\Lambda_1} \longrightarrow V_{\lambda+h_{i1}+h_{i2}+\ldots+h_{iN}}\). Accordingly we consider the integral

\[
\int t^{(\lambda - \alpha) - 1}(1 - t)^{-\frac{2}{\kappa}} dt = \frac{\Gamma(1 - \frac{3}{\kappa}) (2\pi i) e^{\pi i(\frac{\lambda - \alpha}{\kappa})}}{\Gamma(\frac{\lambda \alpha}{\kappa} + 1) \Gamma(\frac{\lambda - \alpha}{\kappa} + 1 - \frac{2}{\kappa})} (29)
\]

Here the contour of integration is chosen so that it starts and ends at 1 and encloses 0 anticlockwise. Now consider the following three cases.

a). \(V_\lambda \oplus V_{\Lambda_1} \longrightarrow V_{\lambda+h_{i1}+h_{i2}} \longrightarrow V_{\lambda+h_{i1}+h_{i2}+h_{i3}} \longrightarrow V_{\lambda+h_{i1}+2h_{i2}}\)

Then

\[
A = 1
\]
\[ \mu = \left( \frac{\lambda, -\alpha}{\kappa} - \frac{1}{\kappa'}, -\frac{1}{\kappa}, -\frac{1}{\kappa} \right) \]
\[ \rho = \left( \frac{1}{\kappa}, 0, -\frac{1}{\kappa} \right) \]
\[ \mu - \rho = \left( \frac{\lambda, -\alpha}{\kappa} - \frac{2}{\kappa'}, -\frac{1}{\kappa}, 0 \right) \]

So according to the theorem the Selberg integral in this case is equal:

\[
1 \times \frac{e^{\pi i \left( \frac{\lambda, -\alpha}{\kappa} \right)}}{\Gamma \left( \frac{\lambda, -\alpha}{\kappa} + 1 - \frac{1}{\kappa} \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} + 1 \right)} \times \frac{\Gamma \left( \frac{\lambda, -\alpha}{\kappa} - \frac{1}{\kappa'} + 1 \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} - \frac{1}{\kappa'} \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} + 1 \right)}{\Gamma \left( \frac{\lambda, -\alpha}{\kappa} + \frac{1}{\kappa'} + 1 \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} - \frac{1}{\kappa'} \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} + \frac{1}{\kappa'} \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} + 1 \right)}
\]

(30)

After simplifications we get (29).

b). Consider now the following choice of intermediate channels:

\[ V_\lambda \otimes V_{\Lambda_1} \rightarrow V_{\lambda + h_1} \otimes V_{\Lambda_1} \rightarrow V_{\lambda + h_2 + h_1} \rightarrow V_{\lambda + h_2 + 2h_1} \]

\[ A = e^{-\pi i} \]
\[ \mu = \left( 0, \frac{\lambda, -\alpha}{\kappa} - \frac{1}{\kappa}, -\frac{1}{\kappa} \right) \]
\[ \rho = \left( \frac{1}{\kappa}, 0, -\frac{1}{\kappa} \right) \]
\[ \mu - \rho = \left( -\frac{1}{\kappa'}, \frac{\lambda, -\alpha}{\kappa} - \frac{1}{\kappa'}, -\frac{1}{\kappa} \right) \]

So according to the theorem we get that the value of Selberg integral is equal:

\[
\frac{e^{\pi i} \Gamma(1 - \frac{1}{\kappa})}{\Gamma \left( \frac{\lambda, -\alpha}{\kappa} + 1 - \frac{1}{\kappa} \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} + 1 \right)} \times \frac{\Gamma \left( \frac{\lambda, -\alpha}{\kappa} - \frac{1}{\kappa'} + 1 \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} - \frac{1}{\kappa'} \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} + 1 \right)}{\Gamma \left( \frac{\lambda, -\alpha}{\kappa} + \frac{1}{\kappa'} + 1 \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} - \frac{1}{\kappa'} \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} + \frac{1}{\kappa'} \right) \Gamma \left( \frac{\lambda, -\alpha}{\kappa} + 1 \right)}
\]

(31)
Again in agreement with (29)

c).

\[ v_\lambda \otimes V_{\Lambda_1} \rightarrow V_{\lambda+h_1} \otimes V_{\Lambda_2} \rightarrow V_{\lambda+2h_1} \rightarrow V_{\lambda+2h_1+h_2} \]

\[ A = e^{-2\pi i \frac{\kappa}{\lambda}} \]

\[ \mu = (0, 0, \frac{(\lambda, -\alpha)}{\kappa} - \frac{3}{\kappa}) \]

\[ \mu - \rho = (-\frac{1}{\kappa}, 0, \frac{(\lambda, -\alpha)}{\kappa} - \frac{2}{\kappa}) \]

So according to the theorem the value of the Selberg integral is equal:

\[ e^{-2\pi i \frac{\kappa}{\lambda}} \times \frac{e^{\pi i \frac{(\lambda+2\lambda_1, -\alpha)}{\kappa}}}{\Gamma(1 - \frac{1}{\kappa})} \frac{(2\pi i)}{\Gamma(1 - \frac{1}{\kappa})} \]

\[ = e^{-2\pi i \frac{\kappa}{\lambda}} \times \frac{\Gamma(1 - \frac{1}{\kappa}) \Gamma(\frac{1}{\kappa} + 2 + \frac{1}{\kappa}) \Gamma(\frac{1}{\kappa} + 1)}{\Gamma(1 - \frac{1}{\kappa}) \Gamma(1 - \frac{1}{\kappa}) \Gamma(1 - \frac{1}{\kappa})} \]

(32)

Again after simplifications we get (29)

Remark 5 The Selberg type integrals are needed as normalization constants or directly undoing the 2D integrals for calculation of structural constants of operator algebra [3], [4].

4 Comparison with usual Selberg integral

In ref [26] Selberg considered the following multiple integral

\[ \int \ldots \int t_i^{a-1}(1-t_i)^{b-1} \prod_{i<j} |t_i - t_j|^{2c} dt_1 \ldots dt_m \]

\[ = \prod_{j=0}^{m-1} \frac{\Gamma(a + jc)\Gamma(b + jc)\Gamma((j+1)c)}{\Gamma(a + b + (n + j - 1)c)\Gamma(c)} \]

(33)

Here the integration is performed over domain:

\[ 0 \leq t_1 \leq t_2 \leq \ldots \leq t_m \leq 1 \]
Consider for example $m = 2$, 

$$V_\lambda \otimes V_{2\Lambda_1} \to V_{\lambda+2h_2}.$$ 

$$a = \frac{\langle \lambda, -\alpha \rangle}{\kappa}$$ 

$$b = 1 + \frac{(2\Lambda_1, -\alpha_1)}{\kappa} = 1 - \frac{2}{\kappa}$$ 

$$c = \frac{1}{2} \frac{(-\alpha_1, -\alpha_1)}{\kappa} = \frac{1}{\kappa}$$

Then the Selberg integral is equal:

$$\int \int (t_1 t_2) \frac{\langle \lambda, -\alpha \rangle}{\kappa} (1-t_1)^{-\frac{2}{\kappa}} (1-t_2)^{-\frac{2}{\kappa}} (t_1-t_2)^{\frac{2}{\kappa}} \frac{dt_1}{t_1} \frac{dt_2}{t_2} = \frac{\Gamma(-\frac{\langle \lambda, \alpha \rangle}{\kappa}) \Gamma(-\frac{\langle \lambda, \alpha \rangle}{\kappa} + \frac{1}{\kappa}) \Gamma(1-\frac{1}{\kappa}) \Gamma(\frac{1}{\kappa})}{\Gamma(-\frac{\langle \lambda, \alpha \rangle}{\kappa} + 1 - \frac{1}{\kappa}) \Gamma(-\frac{\langle \lambda, \alpha \rangle}{\kappa} + 1) \Gamma(\frac{2}{\kappa})} \Gamma(1-\frac{1}{\kappa}) \Gamma(\frac{1}{\kappa})$$

(34)

Here the integration is performed over $0 \leq t_1 \leq t_2 \leq 1$. Passing to another system of contours: contour for $t_1$ starts and ends at 1 and encloses $0$ anticolwise, contour for $t_2$ starts and adds at 1 encloses 0 and contour for $t_1$ anticolwise, we get the extra factor:

$$\left( e^{2\pi i \left(-\frac{\langle \lambda, \alpha \rangle}{\kappa}\right)} - 1 \right) \left( e^{2\pi i \left(-\frac{\langle \lambda, \alpha \rangle}{\kappa} + \frac{1}{\kappa}\right)} - e^{2\pi i \frac{1}{\kappa}} - e^{2\pi i \left(-\frac{\langle \lambda, \alpha \rangle}{\kappa} + \frac{1}{\kappa}\right)} - 1 \right)$$

$$= e^{\frac{2\pi i}{\kappa}} 2 \cos\left(\frac{\pi}{\kappa}\right) e^{\pi i \frac{\lambda_2 - \lambda_1 + 1}{\kappa}} (2i) \sin\left(\pi \left(\frac{\lambda_2 - \lambda_1}{\kappa} + 1\right)\right) e^{\pi i \frac{\lambda_2 - \lambda_1}{\kappa}} (2i) \sin\left(\pi \left(\frac{\lambda_2 - \lambda_1}{\kappa}\right)\right)$$

(35)

Multiplying formulas (34) by (35) and using

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

we get :

$$e^{2\pi i \left(-\frac{\langle \lambda, \alpha \rangle}{\kappa} + \frac{1}{\kappa}\right)} \Gamma(1-\frac{1}{\kappa}) (2\pi i)^2$$

$$\frac{\Gamma(\frac{\langle \lambda, \alpha \rangle}{\kappa} + 1 - \frac{1}{\kappa}) \Gamma(\frac{\langle \lambda, \alpha \rangle}{\kappa} + 1) \Gamma(\frac{\langle \lambda, \alpha \rangle}{\kappa} + 1 - \frac{1}{\kappa})}{\Gamma(\frac{\langle \lambda, \alpha \rangle}{\kappa} + 1 - \frac{1}{\kappa}) \Gamma(\frac{\langle \lambda, \alpha \rangle}{\kappa} + 1) \Gamma(\frac{\langle \lambda, \alpha \rangle}{\kappa} + 1 - \frac{1}{\kappa})} \Gamma(1-\frac{1}{\kappa}) \Gamma(\frac{1}{\kappa})$$

(36)

The value (36) coincides with the corresponding value provided by theorem 3. Phase factor $A$ takes into account the fact that first we had $0 < |t_1| < |z_1| < |t_2| < |z_2|$ and then $0 < |t_1| < |t_2| < z_1 = z_2 = 1$, i.e. $z_1$ goes through $t_2$ which results in the phase factor

$$A = e^{-\frac{\pi i}{\kappa}}$$

. Also, only the leading asymptotic coefficient is essential, while the value at the unity is equal to 1 in this example. This phenomena is explained in the next section.
5 Particular case

The following particular case deserves special attention. Let $i_1 = i_2 = i_3 = \ldots = i_N = n$.

$$V_\lambda \otimes V_{\lambda+2n} \otimes V_{\lambda+4n} \otimes \ldots \otimes V_{\lambda+2Nn}$$

Then

$$\int \prod t_{ij}^{(\lambda, -\alpha_j)} \prod (z_i - t_{i1})^{(\Lambda_1, -\alpha_1)} \prod (t_{i1,j1} - t_{i2,j2})^{(\alpha_1, \alpha_2)} dt_{i1} \ldots dt_{N,n-1} = \text{const}(z_1 z_2 \ldots z_N)^{\frac{(\lambda, a_1 - \alpha_1) + (n-1)}{\kappa}}$$ (37)

Again, the leading asymptotic coefficient is given by the formula (9).

And the Selberg type integral in this case is equal to the leading asymptotic coefficient (see also the previous section). Consider $N = n = 2$. Then

$$\int (t_1 t_2)^{\frac{(\lambda, -\alpha_1)}{\kappa} - 1} (z_1 - t_1)^{-\frac{1}{\kappa}} (z_2 - t_2)^{-\frac{1}{\kappa}} dt_1 dt_2 = \text{const}(z_1 z_2)^{\frac{\lambda_2 - \lambda_1 + 1}{\kappa}}$$ (38)

The constant is given by (9). As a corollary one obtains the following identity for $\Gamma$-functions for any integer $M (M > 0)$:

$$\sum_{m_1 + m_2 + m_3 = M > 0} \frac{\Gamma(\frac{\lambda, -\alpha}{\kappa} + m_1 + m_3)}{\Gamma(\frac{\lambda, -\alpha}{\kappa} + m_1 + m_3 + 1 - \frac{1}{\kappa})} \times \frac{\Gamma(\frac{\lambda, a}{\kappa} + m_2 + m_3)}{\Gamma(\frac{\lambda, a}{\kappa} + m_2 + m_3 + 1 - \frac{1}{\kappa})} \times \frac{\Gamma(\frac{1}{\kappa} + m_1) \Gamma(\frac{1}{\kappa} + m_2) \Gamma(-\frac{2}{\kappa} + m_3)}{m_1! m_2! m_3!} = 0$$ (39)

**Remark 6** Now one can collapse some adjacent variables $z_i$ and obtain an extension of formula (37).

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