Benchmark for a quasi-bound state of the $K^-pp$ system

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We present three-body nonrelativistic calculations within the framework of a potential model for the kaonic cluster $K^-pp$ using two completely different methods: the method of hyperspherical harmonics in the momentum representation and the method of Faddeev equations in configuration space. To perform a numerical benchmark, different $NN$ and antikaon-nucleon interactions are applied. The results of the calculations for the ground state energy for the $K^-pp$ system obtained by both methods are in reasonable agreement. Although the ground state energy is not sensitive to the $NN$ interaction, it shows very strong dependence on the $KN$ potential. We show that the dominant clustering of the $K^-pp$ system in the configuration $\Lambda(1405) + p$ allows us to calculate the binding energy to good accuracy within a simple cluster approach for the differential Faddeev equations. The theoretical discrepancies in the binding energy and width for the $K^-pp$ system related to the different $NN$ and $KN$ interactions are addressed.

I. INTRODUCTION

The theoretical and experimental studies of composite systems of mesons and baryons is a challenging issue in hadronic and nuclear physics. The study of the $KN$ interaction and the formation of the $\Lambda(1405)$ as a quasi-bound state of $\bar{K}N$ [1, 2] started in the 60s. Early suggestions for kaon condensate in dense matter [3, 4] motivated the search for bound states of kaons in nuclei, since the kaon–nucleus interaction could answer the question of whether kaon condensation takes place in the core of neutron stars. Kaonic systems have attracted much attention in the last decade. In Ref. 4 it was shown that kaonic nuclei could be produced by the $(K^-, p)$ and $(K^-, n)$ reactions. However, the renaissance in this field started after the prediction of deeply bound $K^-p$ states by calculations performed by Akaishi and Yamazaki in Ref. 8 and followed by several subsequent publications [9–13]. The experimental search was initiated at the KEK 12 GeV proton synchrotron in Japan by the finding of peaks in the nucleon spectra of $(K^-p)$ absorption in $^4$He [14, 15].

The $KN$ interaction has been found to be strongly attractive in the isospin $I = 0$ channel and, as a consequence of that, nucleons are attracted by the $\bar{K}$ meson to form dense nuclear states. In 1963, in Ref. 16 Nogami first examined three possible isospin configurations of the $KNN$ system and discussed the possible existence of the bound state using a rather crude calculation. The primary focus was on the lightest kaonic system $K^-pp$ because it has two $K^-p$ pairs with isospin $I = 0$ that can be considered as a building block of a three-body kaonic cluster. Regardless of the fact that two protons are not bound, there is an expectation that $KN$ interactions can be strong enough to bound the $K^-pp$ system. Interestingly enough, even two protons can be bound by a single $K$ meson and it is assumed that this would be the strongest binding cluster among the three-body systems.

On the experimental side, several experiments have been performed to search for the kaonic $K^-pp$ cluster using various reactions. The first measurement reported by the FINUDA collaboration was the measurement of the invariant mass distribution of the $p$ produced by $K^-\bar{p}$ absorption by $^{6}\text{Li}$, $^7\text{Li}$, and $^{12}\text{C}$ at the $e^+e^-$ rings of the DAFNE machine in Frascati [17]. The analysis of the invariant mass distribution of the $p$ produced by $K^-\bar{p}$ absorption gave the value $(2255\pm9)$ MeV/$c^2$ for the mass of the peak, corresponding to the binding energy $115^{+6}_{-5}(\text{stat})^{+2}_{-4}(\text{syst})$ MeV and the width $67^{+28}_{-18}(\text{stat})^{+15}_{-20}(\text{syst})$ MeV. However, the authors of Refs. 18, 19 claimed that the FINUDA data can be explained without postulating the existence of the $K^-pp$ bound state and suggested that the peaks of the proton spectra come from the $K^-\bar{p}$ absorption on a pair of nucleons, leaving the rest of the nucleons as spectators. Interestingly enough, even though such a mechanism cannot be completely excluded, there exists the further counter-arguments [20] against the interpretations of stopped-$K^-$ experimental data of the FINUDA group by [19]. The OBEIX experiment at LEAR-CERN [21] also suggested the observation of the $K^-pp$ state with invariant-mass spectroscopy of stopped $\bar{p}$ reactions on $^4\text{He}$. However, a less significant signal was reported in the stopped-$\bar{p}$ reaction. After the FINUDA collaboration observation [17], there have been reports of possible experimental evidence for the kaonic $K^-pp$ state in heavy ion collisions. Results were obtained by the FOPI group experiment [22] at GSI in the study of Ni+Ni and Al+Al collisions with the production of $\Lambda p$ and $\Lambda d$. The DISTO collaboration at the SATURRE
machine have reanalyzed their dataset of the experiment on the exclusive \( pp \rightarrow p \Lambda K^+ \) reaction at 2.85 GeV to search for a strongly-bound compact \( K^-pp \) state formed from the \( p + p \rightarrow K^+ + (K^-pp) \) reaction \cite{23,24}. The enormous large bump structure observed in the \( K^+ \) missing mass and the \( pA \) invariant-mass spectra at 2.85 GeV indicate a possible candidate for the formation of a compact \( K^-pp \) cluster with large binding energy. The binding energy and width were determined to be \( 103\pm5(\text{stat})\pm5(\text{syst}) \) MeV and \( 118\pm8(\text{stat})\pm10(\text{syst}) \) MeV, respectively. Therefore, two collaborations \cite{17} and \cite{23,24} claim that \( K^-pp \) is strongly bound with a binding energy of more than 100 MeV, although the decay width is rather different between these two. The first search results using the \( \gamma d \rightarrow K^+\pi^-X \) reaction at the range of photon energy 1.5–2.4 GeV were reported in Ref. \cite{23}. A statistically significant peak structure was not observed in the differential cross section of the \( K^+\pi^- \) photo-production in the region from 2.22 to 2.36 GeV/c\(^2\) in the inclusive missing mass spectrum. However, the upper limits of the differential cross section of the \( K^-pp \) bound state production were determined with the assumed widths of 20 MeV, 60 MeV and 100 MeV.

Ref. \cite{20} reported on preliminary results of the J-PARC E15 experiment aiming to search for the bound state of \( K^-pp \) via the in-flight \( ^3\text{He}(K^-, n) \) reaction at 1 GeV/c and the first data collection was performed with \( 5\times10^8 \) incident kaons on the \( ^3\text{He} \) target. The experiment investigates the \( K^-pp \) bound state exclusively both in the formation via missing-mass spectroscopy and its decay via invariant-mass spectroscopy using the emitted neutron and the expected decay, \( K^-pp \rightarrow \Lambda pp \rightarrow \pi^-pp \), respectively. Data analyses of the semi-inclusive and exclusive channels are in progress. At the same time, in Ref. \cite{26} have been reported results for the in-flight kaon-induced reaction on \( ^3\text{He}: K^- + ^3\text{He} \rightarrow K^-pp + n \). In particular, the semi-inclusive analysis of the neutron missing-mass spectrum, no significant peak was observed in the region corresponding to \( K^-pp \) binding energy larger than 80 MeV, where a bump structure was reported in the \( \Lambda p \) final state in different reactions by FINUDA and DISTO collaborations. Thus, the situation is still controversial and the existence of the \( K^-pp \) bound state has not been established yet. New experiments using different reactions could help to resolve this controversial situation.

Such experiments are being planned and performed by the HADES collaboration \cite{28}. It was recently proposed to search for kaonic nuclei at the future SuperB factory, which will be built in the Tor Vergata University Campus in Rome. The \( K^-pp \) cluster can be identified through its \( \Lambda p \) decay mode, searching for a narrow peak at a mass of about 2.25 GeV/c\(^2\) in the \( \Lambda p \) invariant mass spectrum and measuring the \( \Lambda - p \) angular correlations, which can give important hints on the nature of the event \cite{27}. One of the advantages of searching for light \( K^- \) nuclear clusters at the future SuperB factory is that the search could be extended from the nuclear medium to the vacuum, looking for its production in the strong decays of \( \Upsilon(1S) \) by taking advantage of the high luminosity of these machines. Also, all problems related to the influence of the medium as well as the final state interaction will hopefully be avoided.

On the theoretical side, there are many reports related to \( K^- \) nuclear systems. Today, refinements of the existing methods for studying few-body systems, on one hand, and developments of new methods, on the other hand, and advances in computational facilities enable very precise calculations for few-body \( K^- \) nuclear systems. The light kaonic \( KNN \) cluster represents a three-body system and has been treated in the framework of various theoretical approaches such as variational methods \cite{9,13,31,37}, the method of Faddeev equations \cite{33,34} and the method of hyperspherical harmonics (HH) \cite{51,52}.

In the framework of the Brueckner–Hartree–Fock theory, by defining the \( KN \) \( g \)-matrix in a nuclear medium as well as using antisymmetrized molecular dynamics for the spatial functions, the \( K^-pp \) cluster has been studied in Refs. \cite{3,10,37}. Variational calculations of the \( K^-pp \) system using several versions of energy-dependent effective \( KN \) interactions derived from chiral SU(3) dynamics were performed in Refs. \cite{33,34}. A different ansatz was used in Ref. \cite{32} for the trial wave function of strongly correlated nucleons in the variational calculations of the binding energy and width for the \( K^-pp \) cluster. Recently in Ref. \cite{37} the \( K^-pp \) cluster was investigated by combining a coupled-channel Complex Scaling Method with the Feshbach resonance method allowing one to effectively treat a coupled-channel problem as a single channel problem. The authors use an energy-dependent chiral-theory based potential and employ the correlated Gaussian function \cite{38}. They report that the \( K^-pp \) cluster is shallowly bound with a binding energy of around 20–35 MeV and the half decay width ranges from 20 to 65 MeV.

The strongly bound nature of the \( K^-pp \) system is shown in Refs. \cite{39,41,50} by using the coupled-channel Faddeev equations in the Alt-Grassberger-Sandhas form \cite{53} and solving for all two-body separable potentials to find the resonance energies and widths. Similar calculations have been done in Refs. \cite{41,42} and \cite{44}. The \( K^-pp \) cluster is studied in Refs. \cite{45,48} using the fixed center approximation to the Faddeev equations that relies upon shell two body amplitudes. Thus, the structure and the production mechanism of the \( K^-pp \) bound state have been investigated using various theoretical approaches. All the aforementioned approaches predict the existence of a bound state for the \( KNN \) system. While on the one hand, various theoretical approaches predict a strong bound state of the \( K^-pp \) system, on the other hand, various theories based on chiral dynamics predict a shallow bound state for the \( K^-pp \) cluster. The predicted values for the binding energy and the width are in considerable disagreement: 9–95 MeV and 20–110 MeV, respectively. The serious theoretical discrepancies concerning \( K^-pp \) binding and width essentially come from different starting ansatzes: phenomenological \( K \)-nucleon potential constructed using \( KN \) scattering data and the data of kaonic hydrogen atomic shift together with the ansatz that \( \Lambda(1405) \) resonance is a bound state of \( KN \).
system, a chiral ansatz for $\bar{K}N$ interaction that leads to a substantially shallow $\bar{K}N$ potential, and the calculation methods. Further theoretical investigations are apparently needed to resolve this controversial situation that perhaps stems from the ambiguity of the $\bar{K}N$ interaction, the importance of $NN$ interaction or the different procedures for three-body calculations.

We present three-body nonrelativistic calculations within the framework of a potential model for the kaonic cluster $K^-pp$ applying two completely different methods: the method of hyperspherical harmonics in momentum representation and the Faddeev equations in configuration space. Calculations for a binding energy and width of the kaonic three-body system are performed using three different potentials for the $NN$ interaction, as well as two different potentials for the description of the kaon-nucleon interaction. One is the energy independent phenomenological $\bar{K}N$ potential from Ref. [8], and the other is the energy dependent chiral $\bar{K}N$ interaction. Such approach allows one to understand the dependence of the bound state and width of the kaonic three-body system on the method of calculations, the importance of nucleon-nucleon interaction, and the key role of the kaon-nucleon interactions.

This paper is organized as follows. In Sec. II we describe the effective potentials of the $K^-pp$ interaction and present the theoretical framework to solve the three-body problem for the $K^-pp$ system. We apply the method of hyperspherical functions in the momentum representation for solving the three-body Schrödinger equation and the method of Faddeev equations in configuration space to find bound states for the kaonic $K^-pp$ system. The results of the three-body calculation for the binding energy and the width for the $K^-pp$ cluster and the analysis of the structure of this system are presented in Sec. 3. We also compare our results with those obtained within variational methods, the method of hyperspherical functions in the coordinate representation and the Faddeev equations in momentum representation. Finally, in the Sec. 4 we present the summary of this work and draw conclusions.

II. THEORETICAL FRAMEWORK

A. Interactions

The Hamiltonian of the three nonrelativistic particles for the $\bar{K}NN$ system reads

$$H = \hat{T} + V_{NN} + V_{\bar{K}N_1} + V_{\bar{K}N_2}, \quad (1)$$

where $\hat{T}$ is the operator of the kinetic energy, $V_{NN}$ is the nucleon-nucleon potential and $V_{\bar{K}N_1} + V_{\bar{K}N_2}$ is the sum of a pairwise effective antikaon interaction with the first and second nucleon, respectively. The effective interactions of the $\bar{K}N$, $KN$, $KK$ and $\bar{K}\bar{K}$ two-body subsystems are discussed in detail in Refs. [8, 13, 30, 34, 54–56]. Below, we use two effective $\bar{K}N$ interactions that were derived in different ways. The effective $\bar{K}N$ interactions can be constructed based on a phenomenological approach so as to reproduce the existing experimental data for the $\bar{K}N$ scattering length, the mass and width of the $\Lambda(1405)$ hyperon and the $1s$ level shift caused by the strong $\bar{K}N$ interaction in the kaonic hydrogen atom. The effective $\bar{K}N$ interaction can be derived within the chiral SU(3) effective field theory, that presents the low-energy realization of QCD with strange quarks and identifies the Tomozawa-Weinberg terms as the main contribution to the low-energy $\bar{K}N$ interaction [34]. The potential for the description of the $\bar{K}N$ interactions was derived in Refs. [8, 13] phenomenologically using $\bar{K}N$ scattering and kaonic hydrogen data and reproducing the $\Lambda(1405)$ resonance as a $K^-p$ bound state at 1405 MeV. We refer to this as the Akaishi-Yamazaki (AY) potential. The $\Lambda(1405)$ potential is energy independent. The other $\bar{K}N$ interaction given in Ref. [34] was derived based on the chiral unitary approach for the $s$–wave scattering amplitude with strangeness $S = -1$ and reproduces the total cross sections for the elastic and inelastic $K^-p$ scattering, threshold branching ratios, and the $\pi\Sigma$ mass spectrum associated with the $\Lambda(1405)$. We refer to this energy dependent potential for the parametrization [57] as the HW potential and in calculations with this potential, we are following a procedure [34, 56]. Both potentials are local and constructed in coordinate space.

The potentials for both above mentioned effective $\bar{K}N$ interactions can be written in the one-range Gaussian form as

$$V_{\bar{K}N}(r) = \sum_{I=0,1} U_I \exp \left[ -\left( \frac{r}{b} \right)^2 \right] P^I_{\bar{K}N}, \quad (2)$$

where $r$ is the distance between the antikaon and the nucleon, $b$ is the range parameter and $P^I_{\bar{K}N}$ is the isospin projection operator. The values of the potential depth $U^{I=0}$ and $U^{I=1}$ for each interaction are given in Refs. [8, 13, 34] and the range parameter is chosen to be $b = 0.66$ fm for the AY potential and $b = 0.47$ fm for the HW potential.
To describe a nucleon-nucleon interaction, we use three $NN$ potentials: the realistic Argonne V14 (AV14) $^{50}$, the semi-realistic Malfliet and Tjon MT-I-III (MT) $^{60}$ potential and the Tamagaki G3RS potential $^{61}$ that we refer to as the T potential. The T potential was used in Ref. $^{13}$ for variational calculations and that allows us to compare our results with the latter work. We chose the MT $s$-wave potential to compare how the binding energy depends on the shape of the $s$-wave $NN$ interaction. The MT $s$-wave potential has only two components in spin-isospin channels $(s, t)$=$(1,0)$, and $(s, t)$=$(0,1)$. The potential allows one to reproduce and obtain an acceptable description of experimental data for bound states and scattering in three and four nucleon systems $^{62, 64}$. The realistic AV14 potential perfectly fits both the $pp$ data, as well as the $np$ data, low-energy $nn$ scattering parameters and the deuteron properties. This potential was chosen in order to study the influence of the orbital partial wave components of the $NN$ interaction (by the non central $L^2$ operator) with $l > 0$ on the energy of the $K^{-}pp$ cluster. In our considerations, the other components (spin-orbital, tensor and other) of the realistic potential have not been taken into account due to their weak effect on the system. Moreover, in the model with one spin-isospin channel $(s, t)$=$(0,1)$, the other components of the potential do not make contributions to the nucleon-nucleon interaction. Thus, the use of all of these potentials allows the validity test against various $NN$ potentials.

B. Formalism of HH in momentum representation

Let’s introduce the trees of Jacobi coordinates in configuration and momentum spaces for a system of three particles with unequal masses $m_1, m_2,$ and $m_3$ having positions $R_1, R_2$ and $R_3$ and momenta $k_1, k_2,$ and $k_3$ as follows

$$x_i = \sqrt{\frac{m_j m_k}{m_j + m_k}}(r_j - r_k),$$

$$y_i = \sqrt{\frac{m_j (m_j + m_k)}{M}} \left( -r_i + \frac{m_j r_j + m_k r_k}{m_j + m_k} \right),$$

$$R = (m_1 r_1 + m_2 r_2 + m_3 r_3), \quad M = m_1 + m_2 = m_3, \quad i \neq j \neq k = 1, 2, 3. \quad (3)$$

The conjugate sets of the Jacobi momenta ($q_i, p_i, P$) for the partition $i$ are defined as:

$$q_i = \frac{1}{\sqrt{m_j m_k(m_j + m_k)}}(m_k j - m_j k),$$

$$p_i = -\frac{1}{\sqrt{m_i M(m_i + m_k)}}[-m_i (k_j + k_k) + (m_i + m_k)k_i],$$

$$P_i = \frac{1}{M}(k_1 + k_2 + k_3), \quad i \neq j \neq k = 1, 2, 3. \quad (4)$$

We introduce the hyperspherical coordinates as the hyperradii $\rho$ and $\kappa$, and two sets of five angles denoted by $\tilde{\Omega}_i = (\alpha_i, \tilde{x}_i, \tilde{y}_i)$ and $\Omega_i = (\beta_i, \tilde{q}_i, \tilde{p}_i)$ which define the direction of the vector $\rho$ and vector $\kappa$, in the six dimensional configuration and momentum spaces, correspondingly, so that

$$\rho = \sqrt{x_i^2 + y_i^2}, \quad x_i = \rho \cos \alpha_i, \quad y_i = \rho \sin \alpha_i,$$

$$dx_i dy_i = \rho^5 d\rho \sin^2 \alpha \cos^2 \alpha d\alpha d\tilde{x}_i d\tilde{y}_i \equiv \rho^5 d\rho d\tilde{\Omega}_i;$$

$$\kappa = \sqrt{p_i^2 + q_i^2}, \quad p_i = \kappa \sin \beta_i, \quad q_i = \kappa \cos \beta_i,$$

$$dp_i dq_i = \rho^5 d\rho \sin^2 \beta \cos^2 \beta d\beta d\tilde{p}_i d\tilde{q}_i \equiv \kappa^5 d\kappa d\Omega_i. \quad (6)$$

The hyperradii $\rho$ and $\kappa$ are invariant under three-dimensional rotations and independent of the partition $i$.

One can write the Schrödinger integral equation describing three bound particles in the momentum representation as

$$\Psi(p, q) = -\frac{1}{(2\pi)^6} \int G(p, q) < p' \cdot q' | V_{123} | pq > \Psi(p', q') dp' dq'. \quad (7)$$
where

\[ \langle p' q' | V_{123} | pq \rangle = \frac{1}{(2\pi)^6} \int V_{123} \exp \left[ i(q - q')x + i(p - p')y \right] dx dy, \]  

(8)
is the Fourier transformation of the \( V_{123} \), which is defined as the sum of the pair-wise nucleon-nucleon and effective antikaon-nucleon interactions. In Eq. (7), the Green function has the form

\[ G(p, q) = \frac{2M}{\hbar^2} \left( \frac{1}{p^2 + q^2 + \hbar^2} \right), \]

(9)

where \( \hbar^2 = \frac{2ME}{\hbar^2} \), and \( E \) is the bounding energy of the three-particle system.

For the next step, we follow the procedure given in Refs. [65, 66] and expand the wave function of three bound particles in terms of the antisymmetrized hyperspherical harmonics \( \Phi_{\mu_1}^{l_1}(\Omega_1, \sigma, \tau) \) in the momentum representation:

\[ \Psi(\kappa, \bar{\Omega}) = \sum_{\mu l q} u_{\mu l q}^{l_1L}(\kappa) \Phi_{\mu_1}^{l_1L}(\Omega_1, \sigma, \tau), \]

(10)

where \( \mu \) is the grand angular momentum, \( L \) is total orbital momentum and \( l_p \) and \( l_q \) are the angular momenta corresponding to the momenta \( p \) and \( q \). In Eq. (10) the antisymmetrized hyperspherical functions \( \Phi_{\mu_1}^{l_1L}(\Omega_1, \sigma, \tau) \) are written as a sum of products of spin and isospin functions and hyperspherical functions \( \Phi_{\mu}^{l m}(\Omega) \) which are hyperspherical harmonics, using the Raynal-Revai coefficients [68]. The HH \( \Phi_{\mu}^{l_1m_1m_2}(\Omega_1) \) are the eigenfunctions of the angular part of the six-dimensional Laplace operator in momentum space with eigenvalue \( \mu(\mu + 4) \), and are expressible in terms of spherical harmonics and Jacobi polynomials [64, 67]. By substituting Eq. (10) into the integral Schrödinger equation in the momentum representation (7) and taking into account (9) one obtains a system of coupled integral equations for the hyperradial functions \( u_{\mu l q}^{l_1L}(\kappa) \):

\begin{align*}
(\kappa^2 + \hbar^2)u_{\mu}^{l_1L}(\kappa) &= -\frac{2m}{\hbar^2} \sum_{\mu'} \sum_{l'_{\mu'} l'_{\mu} q} j^{(l'_{\mu'} l'_{\mu}, q)} \int dp J_{\mu + 2}(\kappa p) J_{\mu' + 2}(\kappa p') \\
&\quad \times \Phi_{\mu'}^{l'_{\mu} q}(\kappa_1) V_{\mu}(x_1) \Phi_{\mu}^{l_{\mu} q}(\kappa_1) u_{\mu'}^{l'_{\mu} L}(\kappa') d\kappa d\kappa',
\end{align*}

(11)

where the index \( A \) is related to the type of interaction \( A \in NN, KN \). In Eq. (11) \( j^{(l'_{\mu'} l'_{\mu}, q)} \) are the Raynal-Revai coefficients [68] for the unitary transformation of HH from one set of Jacobi coordinate to another, \( J_{\mu + 2}(\kappa p) \) are the spherical Bessel functions and

\[ \Phi_{\mu}^{l_{\mu} q}(\kappa_1) = N_{\mu}^{l_{\mu} q}(\cos \alpha_1)^{l_{\mu}} (\sin \alpha_1)^{l_{\mu}} P^{l_{\mu} + 1/2}_{n}(\cos 2\alpha_1), \]

(12)

where \( 2n = \mu - l_q - l_p, n=1, 2, 3, ... \) and \( P^{l_{\mu} q}_{n} \) is the Jacobi polynomial. By solving the coupled integral equations (11) one can find the hyperradial functions \( u_{\mu l q}^{l_1L}(\kappa) \) for a given \( L \).

C. Formalism of Faddeev equation in configuration space

The wave function of the three–body system can be obtained by solving the Schrödinger equation with the single channel Hamiltonian (11). Alternatively, in the Faddeev method the total wave function can be decomposed into three components: \( \Psi = \Phi_1 + \Phi_2 + \Phi_3 \). The Faddeev components \( \Phi_i \) correspond to the separation of particles into configurations \( i + (k\ell) \), \( i \neq k \neq \ell = 1, 2, 3 \). Each Faddeev component \( \Phi_i = \Phi_i(x_i, y_i) \) depends on its own set of the Jacobi coordinates (3). The components satisfy the Faddeev equations in the coordinate representation written in the form (71, 72):

\[ \left( H^0_0 + v_i(x_i) - E \right) \Phi_i(x_i, y_i) = -v_i(x_i) \left( \Phi_k(x_k, y_k) + \Phi_l(x_l, y_l) \right). \]

(13)
In this case the total wave function of the system is decomposed into the sum of the Faddeev components $U$ and $W$ corresponding to the $(pp)K^-$ and $(K^-p)p$ types of rearrangements: $\Psi = U + W - PW$, where $P$ is the permutation operator for two identical fermions. For a three-body system that includes two identical fermions the set of the Faddeev equations can be reduced to the system of two equations for the components $U$ and $W$:

$$
(H_0^u + V_{pp} - E)U = -V_{pp}(W - PW),
(H_0^w + V_{K^-p} - E)W = -V_{K^-p}(U - PW),
$$

where the potentials for $pp$ and $K^-p$ pairs are defined by $V_{pp}$ and $V_{K^-p}$, respectively. The partial wave analysis of the differential Faddeev equations (DFE) in the $LS$ basis is performed by the general scheme described in Refs. 72, 73, 75. The $LS$ basis allows us to restrict the model space to the states with total spin $S = 0$ (when the spin projections of protons are anti-parallel) and total isospin $I = \frac{3}{2}$. The possible isospin configurations with $I = 1$ are not taken into account in our calculations. According to the evaluations of different authors the total contribution of the $I=1$ configuration is about 5%.

We consider two cases for the solution of the system: the s-wave approach and the cluster approach. In the latter case we assume the dominant clustering of the $K^-pp$ system in the form $\Lambda(1405) + p$, that allows us to present the Faddeev components as the product of the eigenfunctions of the two-particle subsystems and functions of the relative motion of pair and the third particle.

1. s-wave approach

Let us consider the s-wave approach. Above, we mentioned that the nucleon-nucleon MT and T potentials and $K^-p$ interactions applied have only the s-wave components for the spin-isospin state $s=0$, $t=1$ in the pair $NN$ and $t=0,1$ in the pair $K^-p$. We consider the s-wave approach for the differential Faddeev equations. The s-wave DFE can be written as

$$
(H_0^s + v_{pp}^s(x) - E)U(x,y) = -v_{pp}^s(x) \int_{-1}^{1} du \frac{xy}{x_1y_1} AW(x_1', y_1'),
$$

$$
(H_0^w + V_{K^-p}^s(x) - E)W(x,y) = -\frac{1}{2}V_{K^-p}^s(x) \left[ \int_{-1}^{1} du \frac{xy}{x_2y_2} ATU(x_2', y_2') + \right.
$$

$$
+ \left. \int_{-1}^{1} du \frac{xy}{x_2y_2} DW(x_2', y_2') \right],
$$

where the numbers of particles in the system are labeled as 2 and 3 for protons and 1 for the kaon, $m_2 = m_3 = m$ is the nucleon mass, and $m_1$ is the mass of the kaon. $v_{pp}^s(x)$ is the singlet-triplet ($s=0$ and $t=1$) component of the $NN$ potential, $V_{K^-p} = \text{diag}\{v_{K^-p}^s, v_{K^-p}^t\}$ with $v_{K^-p}^s$ and $v_{K^-p}^t$ the singlet and triplet isospin components of $KN$
potential, respectively. In Eqs. (16) \(u=\cos(\vec{x}\vec{y})\), where \(\vec{x}\vec{y}\) is the angle between \(x\) and \(y\), and we define

\[
H_0 = -\frac{\hbar^2}{2M(1)} \frac{\partial^2}{\partial y^2} - \frac{\hbar^2}{2m(1)} \frac{\partial^2}{\partial x^2}, \quad H_0^w = -\frac{\hbar^2}{2M(2)} \frac{\partial^2}{\partial y^2} - \frac{\hbar^2}{2m(2)} \frac{\partial^2}{\partial x^2},
\]

\[
A = \left( -\frac{\sqrt{3}}{2}, \frac{1}{2} \right), \quad D = \left( \frac{1}{2\sqrt{2}}, \frac{1}{2} \right),
\]

\[
W(x, y) = (W^s(x, y), W^t(x, y))^T.
\]

In (16), the appropriate transformation of coordinates and reduced masses are given by

\[
x_1' = \left( \frac{m(1)m}{m}x + y^2 + 2 \frac{m(1)}{m}xyu \right)^{1/2}, \quad y_1' = \frac{m_1}{m_1 + m} \left( \frac{m}{M(1)}x + y^2 - 2 \frac{m}{M(1)}xyu \right)^{1/2},
\]

\[
x_2' = \left( \frac{m(2)}{m}x^2 + y^2 + 2 \frac{m(2)}{m}xyu \right)^{1/2}, \quad y_2' = \frac{1}{2} \left( \frac{m}{M(2)}x^2 + y^2 - 2 \frac{m}{M(2)}xyu \right)^{1/2},
\]

\[
x_2'' = \left( \frac{m(2)}{m_1}x^2 + y^2 - 2 \frac{m(2)}{m_1}xyu \right)^{1/2}, \quad y_2'' = \frac{m}{m_1 + m} \left( \frac{m_1}{M(2)}x^2 + y^2 + 2 \frac{m_1}{M(2)}xyu \right)^{1/2}.
\]

\[
M(1) = 2 \frac{m_1 m}{m_1 + 2m}, \quad M(2) = \frac{m(m_1 + m)}{m_1 + 2m},
\]

where \(m(1) = \frac{m_1}{2}\), \(m(2) = \frac{m_1 m}{m_1 + m}\) are reduced masses the \(pp\) pair and the pair and third particle \((K^-)\). Spin-isospin configurations of the subsystems of the \(K^-pp\) system, corresponding to (16) are graphically presented in Fig. 1.

![Fig. 1: Spin-isospin configurations of the subsystems of the \(K^-pp\) system. a) \(K^-(pp)\) with the singlet spin state of the pair \((pp)\). b) \((K^-p)\) + \(p\) with the singlet isospin state of the pair \((K^-p)\). c) \((K^-p)\) + \(p\) with the triplet isospin state of the pair \((K^-p)\).](image)

2. Averaged potential

The \(\bar{K}\) meson combines two nucleons into the bound state for two isospin configurations that are energetically favorable. The effective \(\bar{K}N\) interactions have strong attraction for the \(I = 0\) channel and weak attraction in the \(I = 1\) channel. In this section, we consider an approach for the case when one of the pair potentials has different components for spin-isospin states. Following, for instance Refs. [75] and [56], for Eqs. (11) and (16), one can consider an effective potential obtained by averaging the initial potential over isospin variables. Each component of the initial potential is substituted by the averaged potential. The isospin averaged potential \(V_{K^-p}(r)\) is defined by the following form:

\[
V_{K^-p}(r) = \frac{3}{4} V_{K^-p,(0)}(r) + \frac{1}{4} V_{K^-p,(1)}(r).
\]

This potential has moderate attraction in comparison with the component for \(I = 0\). Using this potential we reduce, for example, the set of equations (16) to two equations. The averaged potential replaces the \(V_{K^-p}\) so that

\[
V_{K^-p} = \text{diag}\{v_{K^-p,(0)}, v_{K^-p,(1)}\}
\]

and \(W(x, y)\) becomes \(W(x, y) = -\frac{\sqrt{2}}{2} W_1 - \frac{1}{2} W_2\). Note that this simplification changes the two-body threshold in Eq. (16), which is not related to the \(V_{K^-p}\) bound state as \(\Lambda(1405)\). The value of the two-body threshold for the effective potential is about -9.6 MeV instead of -30.2 MeV for the \(\Lambda Y\) potential \((I = 0)\).
3. Cluster approach

Within the formalism of the Faddeev equations, one may use dominant clustering of the $K^{-}pp$ system in the form $\Lambda(1405) + p$ to calculate the binding energy. In the framework of such an approach, the s-wave Faddeev components are decomposed as a product of the eigenfunctions of the Hamiltonians of two-particle subsystems and functions of the relative motion of the pair and the third particle.

Here, we present a brief description of the method based on the general form for the Faddeev equations \[[13]\]. The Faddeev components $U_{\alpha}$, $\alpha = 1,2,3$ are written in the following form \[[74]\]:

$$U_{\alpha}(x, y) = \phi_{\alpha}(x)f_{\alpha}(y). \quad (20)$$

In \[[20]\], the functions $\phi_{\alpha}$ are the solutions of the two-body Schrödinger equations for subsystems with minimal eigenvalue

$$\left(-\frac{\hbar^2}{2m_{(\alpha)}} \partial^2_x + v_{\alpha}(x)\right)\phi_{\alpha}(x) = \varepsilon_{\alpha}\phi_{\alpha}(x).$$

Substituting \[[20]\] into the Faddeev equations and projecting, one may obtain the set of integro-differential equations for the functions $f_{\alpha}(y)$ describing the relative motion of clusters:

$$\left(-\frac{\hbar^2}{2M(\alpha)} \partial^2_y + \varepsilon_{\alpha} - E\right)f_{\alpha}(y) = -\frac{1}{2} \langle \phi_{\alpha}(x) | v_{\alpha}(x) \int_{-1}^{1} du \left(\frac{x y}{x_{\beta} y_{\beta}} \phi_{\beta}(x_{\beta}) f_{\beta}(y_{\beta}) + \frac{x y}{x_{\gamma} y_{\gamma}} \phi_{\gamma}(x_{\gamma}) f_{\gamma,k}(y_{\gamma})\right)\rangle. \quad (21)$$

In these equations, $\langle \cdot | \cdot \rangle$ means the integration over the variables $x$, and the indexes $\alpha \neq \beta \neq \gamma = 1,2,3$. The functions $f_{\alpha}(y)$ satisfy the following boundary conditions $f_{\alpha}(y) \sim 0$, when $y \to \infty$. For a system including particles with spin and isospin, the number of equations in the set depends on the number of terms of the Faddeev components \[[20]\] in the spin-isospin basis. For the $K^{-}pp$ system considered within the framework of the s-wave approach, we have the set of three equations corresponding to \[[15]\]. The set of eigenvalues of pair subsystems $\epsilon_{i}$, $i = 1,2,3$ includes $\epsilon_{2}$ as the bound state energy of the $K^{-}p$ pair in singlet isospin state assumed to be the $\Lambda(1405)$ hyperon.

III. RESULTS AND DISCUSSION

Results of our calculations for the $K^{-}pp$ cluster are presented in this section. For the calculations of the binding energy and the width with the methods of HH and DFE we use for the $NN$ interaction MT, T and AV14 potentials, while for the $KN$ interaction we use the single-channel phenomenological $AY$ $KN$ potential with the range parameter $b = 0.66$ fm and the energy dependent effective HW potential with the range parameter $b = 0.47$ fm. Such an approach allowed us to examine how the $K^{-}pp$ cluster’s structure depends on different choices of the $KN$ interactions for the same $NN$ potential, as well as dependence from different choices of the $NN$ interaction for the same $KN$ interaction. This enable us to understand the sensitivity of the system to the input interactions. By solving the system of equations \[[11]\] one finds the binding energy of the $K^{-}pp$ cluster. The results of the calculations with the method of HH strongly depend on the number of terms in the expansion of the wave function \[[10]\]. The convergences of binding energy calculations for the ground state of the $K^{-}pp$ system as a function of the grand angular momentum $\mu$ are shown in Fig. 2 for different input potentials. Reasonable convergence is reached for $\mu_{max} = 10$ and we limit our considerations to this value. As is seen from Fig. 2 much faster convergence occurs for considered $NN$ potentials when the $AY$ potential is chosen for the $KN$ interaction, while for the HW $KN$ interaction the binding energy calculations converge more slowly. The accuracy reached is about 0.2 MeV and our expectation is that consideration of the higher values of the grand angular momentum $\mu$ would not dramatically change the binding energy of the $K^{-}pp$ cluster.

The results for the binding energy and the width of the $K^{-}pp$ system using the method of HH for different $KN$ and $NN$ interactions are presented in Table 1. $E_{K^{-}p}$ is the two-body energy when only the $K^{-}p$ pair are interacting, while the interaction between the two protons is neglected. The HH method allows one to obtain the wave function of the $K^{-}pp$ cluster. By solving the system of equations \[[11]\] one finds the binding energy as well as the corresponding hyperradial functions. The latter allows one to construct the wave function $\Psi$ for the $K^{-}pp$ system. Using the wave function, the width of the bound state can be evaluated in a perturbative way from the imaginary part of the $KN$ interaction as $\Gamma = -2 \langle \Psi | \text{Im} \{ V_{K^{-}p}(t_{12}) + V_{K^{-}p}(t_{13}) \} | \Psi \rangle$. Since $|\text{Im} V_{K^{-}p}(r)| \ll |\text{Re} V_{K^{-}p}(r)|$, this is a reasonable
FIG. 2: The convergence of the ground state energies $E$ of the $K^-pp$ system as a function of $\mu_{\text{max}}$ for different $NN$ potentials and $K^-N$ interactions.

TABLE I: The binding energy $B$ and width $\Gamma$ for the $K^-pp$ system calculated in the framework of the method of HH in the momentum representation for different interactions. $NN$ potentials: MT [60] and T [61]. $KN$ interactions: AY [8] and HW [30]. $E_{K^-p}$ is two-body energy of the $K^-p$.

|          | MT+AY | T+AY | MT+HW | T+HW |
|----------|-------|------|-------|------|
| $B$, MeV | 46.5  | 46.3 | 20.5  | 20.6 |
| $\Gamma$, MeV | 84.3  | 74.5 | 48.1  | 49.5 |
| $E_{K^-p}$, MeV | 29.9  | 10.9 |       |      |

approximation for the width. As it is stated in Gal’s review [77], as well as demonstrated in the recent calculations of the width for the $K^-pp$ system [78] using a coupled-channel complex scaling method with Feshbach projection, this is a good approximation. For an approximate evaluation of the width the imaginary part of the complex potential has often been treated perturbatively by many authors [11, 12, 32–35, 51, 52, 56]. Recently this approach was used in Ref. [80] for few-body calculations of $\eta$-nuclear quasibound states. We use the latter expression to find the width and the calculated values of the width for different $KN$ and $NN$ interactions are listed in Table I. The analysis of the results presented in Table I shows that, for the chiral HW $KN$ interaction, the binding energy is less than half that of the phenomenological AY interaction and the HW interaction also leads to a much smaller width. However, the binding energy and the width are not sensitive to the form of $NN$ potentials for the same type of $KN$ interaction. The binding energy turns out to be about 1 MeV, while the width ranges from $\sim$75 to $\sim$84 MeV for the AY potential and turns out to be within about 1.5 MeV for the HW $KN$ interaction.

We will now discuss the results when we employ the Faddeev technique. The results for the binding energy of the $K^-pp$ system for the orbital momentum configurations of $(pp)K^-$ and $(K^-p)p$ rearrangements are presented in Table II. The calculations are performed using the AV14 $NN$ potential for the AY and HW $KN$ interactions. As can be seen from Table II, the orbital contributions with $(l, \lambda)$ are equal $(2, 2)$ and $(4, 4)$ are small enough and the $s$-wave consideration is very reasonable. The numerical convergence for the binding energy calculations is fast with increasing model space. The visible contribution that provides the bound state of the system comes from the $p$-wave of the $pp$ pair. For the AV14 potential, we also calculated the binding energy using the averaged potential approach as well as the cluster approach. The results with the averaged potential [19] are 33.7 MeV and 12.5 MeV for the AY and HW $KN$ interactions, respectively. We used dominant clustering of the $K^-pp$ system in the configuration $\Lambda(1405) + p$ to calculate the binding energy within the cluster approach for the DFE. The Faddeev components are decomposed as a product of the eigenfunctions of the Hamiltonians for the two particle subsystems and the function of the relative motion of the pairs and the third particle. Our results for the functions associated with the relative motion are shown in Fig. 3 for the nucleon-nucleon MT potential and the AY and HW effective $KN$ interactions. One can evaluate the dominant contribution of the $(K^-p)^{\text{singlet}} + p$ configuration to the total wave function of the system, which is represented as a sum of the Faddeev components. The calculations for other $NN$ potentials have shown the same behavior and relative contributions of the functions associated with relative motion. In the cluster approach, the binding energy increases for both $K^-p$ interactions and are 49.5 MeV and 17.7 MeV, respectively.

Comparing the binding energies obtained by using the averaged potential and the cluster approaches with the $s$-wave DFE calculations, one can conclude that the averaged potential approach gives the better result for the relatively weaker HW potential. At the same time, the cluster approach is more accurate for the stronger AY potential. We also calculated the proton separation energy within the averaged potential, the cluster and the $s$-wave DFE approaches.
TABLE II: Orbital momentum configurations of \((pp)K^-\) and \((K^-p)p\) rearrangements and the ground state energy of the \(K^-pp\) system calculated with the HW and AY \(\bar{K}N\) potentials. The \(NN\) interaction is given by the AV14 potential. \(l\) is the orbital momentum of the pair of particles, and \(\lambda\) is the orbital momentum associated with the relative motion of the third particle with respect of the center of mass of the pair.

| \((pp)K^-\) | \((K^-p)p\) | HW | AY |
|------------|------------|----|----|
| \((l, \lambda)\) | (0,0) | -21.15 | -46.97 |
| \((l, \lambda)\) | (0,0) | -21.54 | -47.33 |
| \((l, \lambda)\) | (0,0) | -21.56 | -47.34 |

FIG. 3: The function of relative motion within the cluster approach for the \(K^-pp\) system: the dot-dashed curve shows calculation for the configuration \(K^- + (pp)\); the solid curve shows calculation for the configuration \((K^-p) + p\) with the singlet isospin state of the pair \((K^-p)\); the dashed curve shows calculation for the configuration \((K^-p) + p\) with the triplet isospin state of the pair \((K^-p)\). The configurations are related to the set of Eqs. (21). The calculations are performed with the nucleon-nucleon MT interaction for a) the AY effective \(\bar{K}N\) potential and b) the HW effective \(\bar{K}N\) potential.

The results are presented in Table III.

The summary of the results for the binding energy of the \(K^-pp\) system in the framework of the method of DFE for all considered \(NN\) and \(\bar{K}N\) interactions are presented in Table IV. The analysis of the calculations presented in Table IV shows that the AY potential as the \(\bar{K}N\) interaction input fall into the 46 - 47 MeV range for the binding energy of the \(K^-pp\) cluster, while the chiral HW \(\bar{K}N\) potential gives about 20.4 - 21.6 MeV for the binding energy. Now one can address the theoretical discrepancies in the binding energy for the \(K^-pp\) system related to the different \(NN\) and \(\bar{K}N\) interactions. The comparison of the results of calculations presented in Tables II and IV for the binding energy for the \(K^-pp\) system obtained by both methods are in reasonable agreement. The ground state energy is not sensitive to the \(NN\) interaction. However, it shows very strong dependence on the kaon-nucleon potential. The energy of the ground state, as well as the width calculated for the energy-independent \(\bar{K}N\) interaction are more than twice bigger than for the energy-dependent chiral \(\bar{K}N\) potential. Therefore, the highest binding energies are those that are obtained based on the phenomenological AY potential. Discrepancies obtained for the binding energy

TABLE III: The proton separation energy \(E_{(K^-p)\rightarrow p}\) calculated for the MT \(NN\) potential and the HW and AY \(\bar{K}N\) interactions. Energies are given in unit of MeV.

| \(\bar{K}N\) interaction | Averaged potential | Cluster DFE approach |
|--------------------------|-------------------|---------------------|
| AY                       | 24.1               | 19.2                |
| HW                       | 11.2               | 6.5                 |


TABLE IV: The binding energy $B$ for the $K^- pp$ system calculated in the framework of the DFE method for different $NN$ and $KN$ interactions. $NN$ potentials: AV14 [59], MT [60] and T [61]. $KN$ interactions: AY [8] and HW [30]. $a$ is the two-body scattering length and $\langle r^2 \rangle^{1/2}$ is the root-mean-squared distance in the $K^- p$ system. $E_{K^- p}$ is the two-body energy of $K^- p$.

| $KN$ | $a$, fm | $\langle r^2 \rangle^{1/2}$, fm | $E_{K^- p}$, MeV | $NN$ | $B$, MeV |
|------|--------|-----------------|-----------------|------|--------|
| AY   | 1.88   | 1.35            | 30.26           | AV14 | 47.3   |
|      |        |                 |                 | MT   | 46.0   |
|      |        |                 |                 | T    | 46.3   |
| HW   | 2.68   | 1.94            | 11.16           | AV14 | 21.6   |
|      |        |                 |                 | MT   | 20.4   |
|      |        |                 |                 | T    | 20.6   |

TABLE V: Summary of the theoretical studies for the $Kpp$ cluster.

| Method            | $B(K^- pp)$ | Width, $\Gamma$ | $KN$ References |
|-------------------|-------------|-----------------|-----------------|
| Variational       | 48          | 61              | AY [8, 9, 13]   |
| Methods of        | 20±3        | 40-70           | Chiral model    |
|                   | 40-80       | 40-85           | Separable       |
|                   | 20-35       | 20-65           | Chiral model    |
|                   | 124         | 12              | AY [36]         |
| Faddeev equations | ~32         | ~50-65          | Separable En. Dep. |
|                   | 45-95       | 45-80           | Separable En. Indep. |
|                   | 9-16        | 34-40           | Separable En. Dep. |
|                   | 30-40       | 50-80           | Separable En. Dep. |
|                   | ~52         |                 | Separable En. Indep. |
| Methods of HH     | ~16         | ~41             | Chiral model    |
|                   | 15-17       | 36-43           | Chiral model    |
|                   | 40-48       | 75-96           | AY [52]         |

using the same potentials but different methods, - the method of HH in momentum representation and the method of Faddeev equations in configuration space, - are mostly related to a problem of an equivalent representation of the potentials in momentum and configuration spaces.

Now let us compare our results with those obtained using variational methods, the method of Faddeev equations in the momentum representation and the method of hyperspherical functions in configuration space. The summary of the results for the binding energy and the width with different theoretical approaches and models, obtained for different kinds of $NN$ and $KN$ interactions, are presented in Table V. We start the comparison with variational calculations. In Ref. [13], the variational calculations were carried out by constructing a wave function for the three-body system with correlation functions for each of the constituent pairs on the basis of multiple scattering theory. The binding energy and the width for the $K^- pp$ cluster was calculated by employing the AY potential as the $KN$ interaction and the bare Tamagaki G3RS potential [61] as the $NN$ interaction. Thus, the authors of [13] used the same $s$-wave interactions, and this allows us to compare our results with the previous one. The binding energy found with the DFE and HH methods are in good agreement with the one obtained with the variational method. This is a good sign that the binding energy does not depend significantly on the method of calculation. The prediction in Ref. [35] for the binding energies of $I_{tot} = 1/2, I_{NN} = 1$ states given by the $s$-wave interactions and described by multiple scattering in the single $KN$ channel falls into the 40 - 80 MeV range with the parametrization for the Argonne AV18 potential from Ref. [58]. These results are consistent with our calculations using the AV14 potential [59] and the results from [13]. The differences within this range are mostly due to a different $KN$ input and possibly slightly due to the $NN$ input.

Different variational approaches used in Refs. [13] and [34] are of comparable quality in their high degree of consistency. Doté, Hyodo and Weise [34] employed several versions of energy-dependent effective $KN$ interactions
derived from chiral SU(3) dynamics together with the realistic Argonne 18 NN potential. They found that the antikaonic dibaryon $K^-pp$ is not deeply bound and obtained 20±3 MeV for the binding energy. Our calculations in the framework of the DFE and HH methods when we employ the effective energy-dependent chiral-theory based HW potential for $\bar{K}N$ interaction and different $NN$ interactions, as inputs, also predict a shallowly bound $K^-pp$ cluster. This is consistent with results from \cite{30, 33} and recent calculations \cite{37}.

The first calculations with Faddeev equations in Alt-Grassberger-Sandhas form \cite{93} for the three-body system with coupled $\bar{K}NN$ and $\pi\Sigma N$ channels were performed in Refs. \cite{39, 40} for the study of the $\bar{K}NN$ system with separable two-body potentials yield large bindings. In Refs. \cite{39, 40} authors obtained 55–70 MeV, and 95–110 MeV for the binding energy and the width, respectively. With a similar approach, Ikedda and Sato \cite{41} calculated $B \sim 80$ Mev and $\Gamma \sim 73$ MeV. Later, two of the authors of \cite{41} repeated their calculation in \cite{42, 43} using two models with the energy-independent and energy-dependent potentials for the $s$-wave $\bar{K}N$ interaction, and their calculations yield smaller values for the binding energy 44–58 MeV and width 34–40 MeV \cite{44}. Recently in Ref. \cite{50}, the Faddeev calculations for the $\bar{K}NN$ quasi-bound state with the two phenomenological and the energy-dependent chiral-motivated models of the $\bar{K}N$ interaction were carried out. The binding energy for the $K^-pp$ cluster obtained was 32 MeV with the chirally motivated models and 47 - 54 MeV with the phenomenological $\bar{K}N$ potentials. The width is about 50 - 65 MeV. Therefore, we can conclude that the Faddeev calculations for the energy-independent models for the $\bar{K}N$ interaction \cite{39, 40, 42, 44} predict a deeper binding energy than that of the energy-dependent description of the $\bar{K}N$ interaction \cite{44, 50}. Our calculations obtained with both methods confirm that the effective $\bar{K}N$ interaction derived from chiral SU(3) dynamics yields a shallowly bound $K^-pp$ cluster, while the phenomenological energy independent $\bar{A}Y$ potential predicts much deeper binding energy for all considered $NN$ interactions. A more detailed comparison of our results with the Faddeev calculations \cite{39, 42, 44, 50} is not easy because these calculations use separable potentials to describe the $\bar{K}N$ and $NN$ interactions.

The fixed center approximation (FCA) to the Faddeev equations \cite{43, 48} leads to the binding energy of the $K^-pp$ system with $S = 0$ 30-40 MeV and the width 50-80 MeV that includes the recent improvement of the method \cite{47, 48} by including the charge exchange mechanisms in the $K^-$ rescattering and absorption which have been ignored in previous works within this approximation. Due to the absorption of $K^-$ by two nucleons, the width of the bound $K^-pp$ cluster is increased by about 30 MeV \cite{48}. Although the FCA to the Faddeev equation makes a static picture of the two nucleon and does not consider the recoil of the spectator nucleon, this approach provides reasonable values for the binding energy and width, as seen from Table \ref{tab:binding_energy} and qualitatively corroborates findings done with other methods which are technically much more involved.

The study of $K^-pp$ with $S = 0$ was performed using the method of hyperspherical functions in configuration space \cite{51} and in the momentum representation \cite{52}. In Ref. \cite{51}, the binding energies and widths of the three-body $K^-pp$ cluster are calculated using the realistic AV4 \cite{79} $NN$ potential and a subthreshold energy-dependent chiral $\bar{K}N$ interaction derived with a chiral model \cite{30}. The results are in good agreement with previous $K^-pp$ calculations \cite{33, 34} with an energy-dependent chiral $\bar{K}N$ interaction as input. Calculations of the binding energy and the width with the method of HH in the momentum representation \cite{52} reproduced the results of Ref. \cite{51}, and we assume the slight difference arisen from the difference between using AV4 and AV18 potentials and perhaps is related to the conversion of the hyperspherical expansion. However, most importantly, our results and those of the previous HH calculations \cite{51, 52} are in good agreement and support the conclusion that the key role in binding the $K^-pp$ system is played by the $\bar{K}N$ interaction and the $\bar{K}N$ potential obtained based on chiral SU(3) dynamics leads to binding energies of relatively low values. Also, these calculations show that the binding energies are small and the widths are more than twice that of the binding energies. Therefore, all model calculations with the $\bar{K}N$ interaction derived based on the chiral unitary approach predict a shallow binding state with very large width for the $K^-pp$ cluster. This is a precautionary indication that it may be difficult to experimentally observe the $K^-pp$ cluster.

**Conclusion**

Within the framework of a potential model for the kaonic cluster $K^-pp$, we perform nonrelativistic three-body calculations using two methods: the method of hyperspherical harmonics in the momentum representation and the method of Faddeev equations in configuration space. We examine how the quasi-bound state of the $K^-pp$ cluster depends on different choices of the $\bar{K}N$ and $NN$ interactions. Our consideration includes the realistic AV14 \cite{50}, the semi-realistic Malfliet and Tjon MT-I-III \cite{60} and Tamagaki G3RS \cite{61} potentials as input for the $NN$ interaction and we employ the phenomenological $A\bar{Y}$ potential \cite{8, 13} and HW potential \cite{30} based on chiral SU(3) dynamics as input for the $\bar{K}N$ interaction. The results for the binding energy of the $K^-pp$ system obtained by the method of HH and the DFE method are in reasonable agreement. A discrepancy obtained is related to a problem of the equivalent representation of the potentials in momentum and configuration spaces. For all types of considered $NN$ interactions, the binding energy for the $K^-pp$ cluster obtained was 32 MeV with the chirally motivated models and 47 - 54 MeV with the phenomenological $\bar{K}N$ potentials. The width is about 50 - 65 MeV. Therefore, we can conclude that the Faddeev calculations for the energy-independent models for the $\bar{K}N$ interaction predict a deeper binding energy than that of the energy-dependent description of the $\bar{K}N$ interaction. Our calculations obtained with both methods confirm that the effective $\bar{K}N$ interaction derived from chiral SU(3) dynamics yields a shallowly bound $K^-pp$ cluster, while the phenomenological energy independent $\bar{A}Y$ potential predicts much deeper binding energy for all considered $NN$ interactions. A more detailed comparison of our results with the Faddeev calculations is not easy because these calculations use separable potentials to describe the $\bar{K}N$ and $NN$ interactions.
interactions, both methods predict deeply bound states for the $\Lambda Y K\bar{N}$ interaction and a relatively shallowly bound $K^-pp$ cluster for the effective $K\bar{N}$ interactions derived from chiral SU(3) dynamics. Moreover, the $K^-pp$ cluster is the most strongly quasi-bound three-body system. The results of our calculations within the DFE and HH methods show that the binding energy of the $K^-pp$ system depends entirely on the ansatz for the $K\bar{N}$ interaction and substantially changes when we use a phenomenologically constructed $K\bar{N}$ potential \cite{13} and a $K\bar{N}$ potential obtained within the framework of the chiral unitary approach \cite{30, 54}. Perhaps the ambiguity of the $K\bar{N}$ interaction stems from an accuracy of description of experimental data for the energy shift in the kaonic hydrogen atom and $K^-p$ scattering by the $K\bar{N}$ potentials. Related to the sensitivity of the binding energy to the details of the $NN$ potentials Ref. \cite{30} mentioned that as long as the $K^-pp$ system is only weakly bound, the dependence on different types of $NN$ interactions is weak. In fact, our study confirms this conclusion and, moreover, shows that the dependence on different types of $NN$ interactions is also weak if the $K^-pp$ system is strongly bound. Using the formalism of the Faddeev equations, we may separate channel configurations of the total wave function of the system. We have shown that the configuration $(K^-p) + p$ with the singlet isospin state of the pair $K^-p$ dominates in the system due to strong $K^-p$ interaction. We have seen that, on the background of the $K\bar{N}$ potential, different $NN$ interactions weakly influence to the bound state energy. It has to be noted that the so-called $s$-wave models, with the simplifications taken, provide reasonable description of the system. The analysis of data presented in Table \[\eref{V} shows that the width is always larger than the binding energy. Particularly for some calculations, the width is more than twice as much as the binding energy. Thus, we are facing a situation in which the states have a much larger width than binding energy, which makes the experimental observation challenging \cite{48} and it may be hard to identify the resonance. However, the continuation of the experimental search for the quasi-bound kaonic cluster still remains important.

Acknowledgements

This work is supported by the NSF (HRD-1345219) and NASA (NNX09AV07A). R.Ya.K is partially supported by MES RK, the grant 3106/GF4. The numerical calculations were performed at the High Performance Computing Center of the North Carolina State University and the Center for Theoretical Physics of New York City College of Technology, CUNY.

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