Optimal Accelerated Degradation Testing Based on Bivariate Gamma Process with Dependent Components

Helmi Shat*, Norbert Gaffke*

*Institute for Mathematical Stochastics, Otto-von-Guericke University Magdeburg, PF 4120, 39016 Magdeburg, Germany

Abstract

Accelerated degradation testing (ADT) is one of the major approaches in reliability engineering which allows accurate estimation of reliability characteristics of highly reliable systems within a relatively short time. The testing data are extrapolated through a physically reasonable statistical model to obtain estimates of lifetime quantiles at normal use conditions. The Gamma process is a natural model for degradation, which exhibits a monotone and strictly increasing degradation path. In this work, optimal experimental designs are derived for ADT with two response components. We consider the situations of independent as well as dependent marginal responses where the observational times are assumed to be fixed and known. The marginal degradation paths are assumed to follow a Gamma process where a copula function is utilized to express the dependence between both components. For the case of independent response components the optimal design minimizes the asymptotic variance of an estimated quantile of the failure time distribution at the normal use conditions. For the case of dependent response components the $D$-criterion is adopted to derive $D$-optimal designs. Further, $D$- and $c$-optimal designs are developed when the copula-based models are reduced to bivariate binary outcomes.

Keywords: Accelerated degradation testing, Gamma process, Frank copula, Gaussian copula, $D$- and $c$-optimal designs, multiplicative algorithm.

1. Introduction

In recent years, the increasing demand for highly reliable products has motivated a noticeable growth of research interest in the area of degradation testing. For systems with high reliability, it is time consuming to do reliability assessment based on traditional degradation tests. Hence, ADT ensures an efficient reliability and life time assessment within relatively short testing times by statistically extrapolating the obtained actual degradation data. In fact, the majority of research on ADT has considered the case of one performance characteristic or the case of multiple but independent failure modes. For example, (Chen-Mao Liao and Sheng-Tsaing Tseng [2006]) used a stochastic diffusion process to model a typical step stress ADT problem with a single failure mode under the constraint that the total experimental cost does not exceed a predetermined budget. The optimal settings of the design variables were obtained by minimizing the asymptotic variance of an estimated quantile of the product’s lifetime distribution. (Huang and Askin [2003]) presented a reliability analysis of electronic devices with independent competing failure modes involving performance aging degradation. The authors used Weibull distributions to describe the time-to-failure of a catastrophic failure mode and that of a degradation failure mode. (Bai and Chun [1991]) introduced optimal simple step stress for products with competing causes of failure, where the life times for the different failure causes were assumed to be independent and exponentially distributed. The authors presented optimal plans which minimize the sum over all failure causes of asymptotic variances of the estimated log mean lives at design stress. Modern products usually have complex structure with multiple failure mechanisms as well as multiple degradation measures. Thus, it is realistic to assume some kind of dependence among different failure components. In the past decade, copula-based modelling has become an efficient tool in many areas of applied statistics, see (AghaKouchak et al. [2010]) and (Embrechts et al. [2001]). For instance, (Perrone and Müller [2016]) has provided an equivalence theorem for binary bivariate copula models that allows applications of efficient design algorithms and quick checks of whether a design is optimal or at least efficient. With an application in cancer clinical
trials, (Yin and Yuan, 2009) proposed a Bayesian adaptive design for dose finding that is based on a Clayton copula model to account for the synergistic effect of two or more drugs in combination. Considering modern complex system, Levy stochastic processes, i.e. Gamma process, Wiener process (Lim and Yum, 2011) and (Xiao and Ye, 2016), and Inverse Gaussian process (Peng et al., 2014 and Ye et al., 2014), were used to model the degradation path. For instance, Tsai et al. (2012) and (Amini et al., 2016) discuss the problem of optimal design for degradation testing based on a Gamma degradation process with random effects. Tsai et al. (2012) considered several decision variables such as the sample size, inspection frequency, and measurement numbers in order to find the c-optimal decision variables. Duan and Wang (2019) addressed the optimal design problems for constant stress ADT based on Gamma processes with fixed effects and random effects. For D-, V- and A-optimality criteria, the authors proved that optimal constant stress ADT plans with multiple stress only use the minimum and the maximum stress levels.

The Archimedean, Clayton, Frank and Gumbel copulas are intensively used to describe the dependence among different failure components when the marginal degradation paths correspond to Levy stochastic processes, see (Mireh et al., 2019). For example, (Zhou et al., 2010) and (Guo and Li, 2017) followed a similar approach through considering a system with multiple failure components where the marginal degradation paths are governed by Gamma processes. They utilized the Frank copula to describe the dependence of failure components. Furthermore, the authors used the Bayesian MCMC method in order to efficiently evaluate the maximum likelihood estimator. In addition, (Adegbola and Yuan, 2019) proposed a multivariate Gamma process to model dependent deterioration phenomena that collectively define the service life of infrastructure assets. Lin et al. (2014) developed a reliability model for systems with s-dependent degradation processes using several Archimedean copulas. The marginal degradation processes were assumed to be inverse Gaussian with a time scale transformation. Furthermore, the authors incorporated a random drift to account for a possible heterogeneity in population, with an application to fit the crack length growth problem. Considering a Wiener process, Pan et al. (2015) and (Pan et al., 2011) presented a bivariate Gamma process that the dependence of the performance characteristics were described by a Frank copula. In addition, the authors used MCMC to jointly estimate the parameters of the two performance characteristics as well as the parameter of the Frank copula. In order to provide a more flexible dependence structure between competing failure modes, (Wang and Pham, 2011) introduced time-varying copulas to develop an s-dependent competing risk model for systems subject to multiple degradation processes and random shocks. Moreover, Tang et al. (2013) investigated the effect of various copulas for modeling dependence structures between variables on reliability under incomplete information. The authors formulated a reliability problem and a direct integration method for calculating the probability of failure. Mercier et al. (2012) discussed the intervention scheduling of a railway track, based on the observation of two dependent randomly increasing degradation components. The authors used trivariate reduction for constructing a bivariate Gamma process that describes the dependency between the two components. Further, they utilized an EM-algorithm to compute the maximum likelihood estimators of the model parameters. In regards to ALT, Howe et al. (2017) utilized the Frank copula to model the general dependence structure between the conceptual lifetimes of system with multiple competing risks. With an application to finance, Semeraro (2008) proposed a generalised bivariate variance Gamma process by subordinating a multivariate Wiener process with independent components by a multivariate Gamma subordinator. With an application to toxicity trials, Denman et al. (2011) derived locally D-optimal designs for dependent bivariate binary data, where several Archimedean, i.e. Clayton, Frank and Gumbel, copulas were utilized to describe the dependence among the marginal regression models. Further, Mireh et al. (2019) proposed a simulation-based reliability analysis for systems with dependent Gamma degradation processes and Weibull distributed hard failure times. The authors used the Frank copula to represent the dependence between failure modes. Pan et al. (2016) introduced a copula based bias correction approach to address model uncertainty in a defined product design. In addition, the results were illustrated by a modified vehicle side impact response case study.

The rest of the present paper is organized as follows. In Section 2 we obtain an optimal experimental design for a bivariate Gamma model with independent marginal components. In Section 3 we develop D-optimal designs for bivariate Gamma models with dependent responses based on the Frank copula function or the Gaussian copula function, respectively. Section 4 introduces D- and c-optimal designs for ADT with dependent failure modes when the copula-based model is reduced to bivariate binary outcomes. The numerical computations were made by using the R programming language (R Core Team, 2020).
2. Bivariate Gamma process with independent components

2.1. Model construction

The Gamma process is a natural stochastic model for degradation processes in which degradation occurs gradually over time in a sequence of independent increments. In this section, we assume that the testing unit has two failure modes where the marginal degradation paths are given by Gamma processes in terms of a standardized continuous time variable \( t \geq 0 \), and the two marginal Gamma processes are independent. It is further assumed that each of the marginal (standardized) stress levels \( x_l \), \( l = 1, 2 \), is a scalar in the standardized interval \([0, 1] \). The joint stress variable \( x = (x_1, x_2) \) can be chosen by the experimenter from the experimental region \( \mathcal{X} = [0, 1]^2 \). Below we clarify the approximation of the Gamma model with a generalized linear model approach. For (locally) optimal design, the information matrix as a function of \( x \) (at given values of the model parameters) is of basic interest and will be derived in Subsection 2.2. Locally c-optimal designs will be presented in Subsection 2.3, where the particular c-criterion expresses the asymptotic variance of an estimated quantile of the failure time distribution.

A Gamma process \( Z^{(l)}_t \), \( t \geq 0 \), considering the response component \( l = 1, 2 \) is a stochastic process with independent and Gamma distributed increments. A degradation increment \( Z^{(l)}_t - Z^{(l)}_s \), \( 0 \leq s < t \), is Gamma distributed with shape parameter \( \gamma_l \cdot (t - s) \) and scale parameter \( \nu_l \). The scale parameter \( \nu_l \) is a known positive constant, while the shape rate \( \gamma_l \) is a positive function of the stress variable \( x_l \) and some further model parameters (see below). The two marginal processes \( Z^{(1)}_t \), \( t \geq 0 \), and \( Z^{(2)}_t \), \( t \geq 0 \), are assumed to be independent. The bivariate degradation process \( Z_t = (Z^{(1)}_t, Z^{(2)}_t) \) is observed at \( k \) subsequent time points \( t_j \), \( j = 1, \ldots, k \), \( 0 < t_1 < \ldots < t_k \), which are prescribed in advance. Equivalently, the bivariate increments \( Y_{j} = (Y^{(1)}_j, Y^{(2)}_j) \), \( j = 1, \ldots, k \), are observed, where \( Y^{(l)}_j = Z^{(l)}_{t_{j+1}} - Z^{(l)}_{t_j} \), \( l = 1, 2 \), and \( t_0 = 0 \). By the above assumptions, the bivariate increments are independent, and the components \( Y^{(1)}_j \) and \( Y^{(2)}_j \) are independent for each \( j \). The density of a marginal increment \( Y^{(l)}_j \) is given by

\[
f^{(l)}_{jl}(y_{jl}) = \frac{y_{jl}^{\gamma_l\Delta_j-1} e^{-y_{jl}/\nu_l}}{\gamma_l(\nu_l)\Delta_j^{\gamma_l\Delta_j}} , \quad y_{jl} \in (0, \infty),
\]  

(2.1)

where \( \Delta_j = t_j - t_{j-1} \) and \( \Gamma(u) = \int_0^\infty x^{u-1} e^{-x} \, dx \), \( u > 0 \), is the complete Gamma function, see Qi and Chen (2004). In accordance with the work of Shat and Schwabe (2019) for the univariate Gamma process, for the marginal shape rate \( \gamma_l = \gamma_l(x_l) \) as a function of the stress variable we consider the particular case

\[
\gamma_l(x_l) = \exp(\beta_{l1} + \beta_{l2}x_l) \tag{2.2}
\]

where the intercept and slope parameters \( \beta_{l1} \) and \( \beta_{l2} \) are to be estimated. Hence, the mean of a marginal increment is given by

\[
\mu_{jl}(x_l) = E(Y_{jl}) = \gamma_l(x_l)\Delta_j\nu_l = \exp(\beta_{l1} + \beta_{l2}x_l)\Delta_j\nu_l. \tag{2.3}
\]

Thus the mean is linked to the linear predictor \( \beta_{l1} + \beta_{l2}x_l \) by the (non-linear) log link, and the present model is related to a generalized linear model with Gamma distributed response variables.

When an accelerated degradation test is run under a stress setting \( x = (x_1, x_2) \), measurements of the bivariate degradation process at the prescribed time points \( t_j \), \( j = 1, \ldots, k \), are made. So the increments \( Y_{j} = (y_{j1}, y_{j2}) \), \( j = 1, \ldots, k \), \( l = 1, 2 \), of the bivariate degradation path are obtained, which follow the model of independent bivariate random variables \( Y_{jl} = (Y^{(1)}_j, Y^{(2)}_j) \), \( j = 1, \ldots, k \), with Gamma distributed components \( Y^{(l)}_{jl} \) according to (2.1) and (2.2). Thus, under the stress setting \( x \) and given the incremental data \( y = (y_1, \ldots, y_k) \), the log-likelihood of the parameter vector \( \beta = (\beta^T, \beta^T_2) \) \( l = 1, 2 \), is given by

\[
\ell(\beta; x, y) = \sum_{j=1}^{k} \sum_{l=1}^{2} \exp(\ell_{jl}(y_{jl})) = \sum_{j=1}^{k} \sum_{l=1}^{2} \left( \gamma_l(x_l)\Delta_j - 1 \right) \exp(y_{jl}) - \frac{y_{jl}}{\nu_l} - \exp(\Gamma(\gamma_l(x_l)\Delta_j)) - \gamma_l(x_l)\Delta_j\exp(\nu_l),
\]

where \( \gamma_l(x_l) = \exp(\beta_{l1} + \beta_{l2}x_l) \), \( l = 1, 2 \).
Usually, an accelerated degradation test is conducted at \( n \) distinct testing units \( i = 1, \ldots, n \) at stress settings \( x_1, \ldots, x_n \), respectively. Note that the stress settings \( x_i, i = 1, \ldots, n \) may not all be distinct. Under the assumption of independence of the testing units, the joint log-likelihood equals the sum of the log-likelihoods over the units,

\[
\ell(\beta; x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} \ell(\beta; x_i, y_i).
\]

The collection \( x_1, \ldots, x_n \) constitutes the experimental design of the test. Since the ordering of the design points \( x_i \) (along with the response vector \( y_j \)) is of no importance, a design is usually described by the set of of distinct points \( x'_1, \ldots, x'_m \) among the collection \( x_1, \ldots, x_n \) and the corresponding frequencies \( n_1, \ldots, n_m \) of their occurrence among \( x_1, \ldots, x_n \). In optimal design theory, when the sample size \( n \) is kept fixed, it has become standard to use the relative frequencies \( \omega_j = n_j / n \), \( j = 1, \ldots, m \), and defining an exact design for sample size \( n \) by

\[
\xi_n = \left( \begin{array}{cccc} x'_1 & \cdots & x'_m \\ \omega_1 & \cdots & \omega_m \end{array} \right),
\]

where \( m \in \mathbb{N} \), \( x'_1, \ldots, x'_m \in \mathcal{X} \), and \( \omega_1, \ldots, \omega_m \) are positive integer multiples of \( 1/n \) with \( \sum_{j=1}^{m} \omega_j = 1 \). Note that the positive integer \( m \), called the support size of \( \xi_n \), may vary with the design. As a mathematical relaxation one dispenses the discrete character of the weights, allowing any positive weights \( \omega_j > 0, j = 1, \ldots, m \), with \( \sum_{j=1}^{m} \omega_j = 1 \). Then the r.h.s. of (2.5) defines an approximate design, for short: a design \( \xi \). The weight \( w_i \) given by \( \xi \) to the support point \( x'_i \) will also be denoted by \( \xi(x'_i) \). In what follows, we will employ the approximate design theory for deriving optimal designs, (see e. g. [Silvey 1980]).

2.2. Information matrix

By the log-likelihood \( \ell(\beta; x, y) \) from (2.4) the elemental Fisher information matrix of \( x \) at \( \beta \) is given by either of following two representations,

\[
M(x, \beta) = E \left( \left[ \frac{\partial \ell(\beta; x, Y)}{\partial \beta} \right] \left[ \frac{\partial \ell(\beta; x, Y)}{\partial \beta} \right]^T \right) = -E \left( \frac{\partial^2 \ell(\beta; x, Y)}{\partial \beta \partial \beta^T} \right).
\]

Using the latter representation, direct calculations and observing (2.2) yield a block structure because of independence of the components as

\[
M(x, \beta) = \left( \begin{array}{cc} M_1(x_1, \beta_1) & 0 \\ 0 & M_2(x_2, \beta_2) \end{array} \right),
\]

where \( x = (x_1, x_2) \), \( \beta_l = (\beta_{1l}, \beta_{2l})^T \), \( l = 1, 2 \), and

\[
M_l(x_i, \beta_l) = \lambda_l(x_i, \beta_l) \left( \begin{array}{c} 1 \\ x_i \\ x_i^2 \end{array} \right), \quad l = 1, 2,
\]

with \( \lambda_l(x_i, \beta_l) = \gamma^2_i(x_i) \sum_{j=1}^{k} \Delta_j \psi_1(\gamma_i(x_i) \Delta_j) \),

where \( \psi_1 \) denotes the Tri-Gamma function, i.e., \( \psi_1(z) = d^2 \exp \Gamma(z)/dz^2 \), \( z > 0 \). As usual in the approximate design theory, for any (approximate) design

\[
\xi = \left( \begin{array}{cccc} x_1 & \cdots & x_m \\ \omega_1 & \cdots & \omega_m \end{array} \right),
\]

where \( m \in \mathbb{N} \), \( x_i \in \mathcal{X} \), \( \omega_i > 0 \), \( 1 \leq i \leq m \), and \( \sum_{i=1}^{m} \omega_i = 1 \), the information matrix of \( \xi \) at a parameter point \( \beta \) is given by

\[
M(\xi, \beta) = \sum_{i=1}^{m} \omega_i M(x_i, \beta).
\]

By the block-diagonal structure of the elemental information matrices (2.7), the information matrix of \( \xi \) is again block-diagonal where the blocks are given by the information matrices of the marginal designs w.r.t. the marginal models,

\[
M(\xi, \beta) = \left( \begin{array}{cc} M_1(\xi_1, \beta_1) & 0 \\ 0 & M_2(\xi_2, \beta_2) \end{array} \right),
\]

where \( M_l(\xi_l, \beta_l) = \sum_{i=1}^{m} \omega_i M_l(x_{il}, \beta_l), \quad l = 1, 2. \)
Recall that $X = [0, 1]^2$, hence $x_i = (x_{i1}, x_{i2})$ with $x_{i1}, x_{i2} \in [0, 1]$, $i = 1, \ldots, m$. The designs $\xi_1$ and $\xi_2$ on $[0, 1]$ are the marginal designs of $\xi$, which are defined as the projections on the corresponding components (in a measure theoretic sense).

2.3. Optimality criterion based on failure time distribution

In ADT one considers some characteristics of the failure time distribution due to degradation under normal use condition $x_m = (x_{m1}, x_{m2})$. Note that typically the normal use conditions $x_{m1}$ and $x_{m2}$ are outside the normalized interval $[0, 1]$ of the possible stress values $x_1$ and $x_2$ in ADT. Usually, one has $x_{m1} < 0, l = 1, 2$. It is assumed that the marginal Gamma process $Z_{u,t}^{(l)}$ describing the degradation under normal use condition $x_{m1}$ has the rate $\gamma_l(x_{m1}) = \exp(\beta_{1l} + \beta_{2l}x_{m1})$ according to (2.2) and scale $\nu_l$. A soft failure due to degradation is defined by exceedance of the marginal degradation paths over some failure thresholds. The marginal failure time $T_l$ under normal use condition $x_{m1}$ is expressed as the first time $t$ the degradation path $Z_{u,t}^{(l)}$ reaches or exceeds a given positive value $z_{10}$, i.e.

$$T_l = \inf\{t \geq 0; Z_{u,t}^{(l)} \geq z_{10}\}. \quad (2.11)$$

Its distribution function is given by $F_{T_l}(t) = \mathbb{P}(T_l \leq t)$, $t \geq 0$. By (2.11), $T_l \leq t$ if and only if $Z_{u,t}^{(l)} \geq z_{10}$, hence

$$F_{T_l}(t) = \mathbb{P}(Z_{u,t}^{(l)} \geq z_{10}) = \frac{1}{\Gamma(\gamma_l(x_{m1}))} \int_{z_{10}}^{\infty} (z_l/\nu_l)^{\gamma_l(x_{m1})-1} e^{-z_l/\nu_l} d\nu_l \quad (2.12)$$

where $Q(s, z) = \Gamma(s, z)/\Gamma(s)$ is the regularized Gamma function and $\Gamma(s, z) = \int_{z}^{\infty} x^{s-1} e^{-x} dx$ is the upper incomplete Gamma function, see (Wang et al., 2015).

As opposed to (Shat and Schwabe, 2019) we assume a parallel system, that is, the system fails as soon as both marginal components have failed. Denote by $T$ the joint failure time, $T = \max\{T_1, T_2\}$. By independence of the components its distribution function is given by

$$F_T(t) = \mathbb{P}(Z_{u,t}^{(1)} \geq z_{10}, Z_{u,t}^{(2)} \geq z_{20}) = Q(\gamma_1(x_{m1})t, z_{10}/\nu_1) \cdot Q(\gamma_2(x_{m2})t, z_{20}/\nu_2). \quad (2.13)$$

For a given $\alpha \in (0, 1)$ let $t_\alpha$ be the $\alpha$-quantile of the failure time distribution of $T$, that is $F_T(t_\alpha) = \alpha$. This quantile represents the time up to which on the average, under normal use conditions, $\alpha \cdot 100$ percent of the testing units will fail and $(1 - \alpha) \cdot 100$ percent of the units will persist. It is worth noting that the distribution function $F_T$ as well as its quantile $t_\alpha$ depend on the parameter vector $\beta$, though not expressed by our notations. The performance of the maximum likelihood estimator $\hat{t}_\alpha$ is measured by its asymptotic variance $a\operatorname{Var}(\hat{t}_\alpha)$, and design optimization will be conducted with respect to minimizing $a\operatorname{Var}(\hat{t}_\alpha)$. This $c$-criterion is commonly used in planning degradation tests when experimenters are interested in accurately estimating reliability properties of a system over its life cycle. However, it should be noted that the asymptotic variance will depend on $\beta$, and thus, as a common feature of non-linear models, one is concerned with local design optimality at some given parameter point $\beta$. Under a design $\xi$ the asymptotic variance of $\hat{t}_\alpha$ is given by

$$a\operatorname{Var}(\hat{t}_\alpha) = c(\beta)^T \mathbf{M}(\xi, \beta)^{-1} c(\beta), \quad (2.14)$$

where $c(\beta) = \frac{\partial t_\alpha}{\partial \beta}$. \quad (2.15)

A criterion given by the r.h.s. of (2.14) is called a (local) $c$-criterion. Efficient algorithms have been developed to compute a $c$-optimal design, see the numerical example in Subsection 2.4 below. However, a more explicit formula of the coefficient vector $c(\beta)$ of the criterion has to be provided. Due to the implicit definition of $t_\alpha$ as the unique solution of $F_T(t_\alpha) = \alpha$, the following identity is ensured by the implicit function theorem, see (Krantz and Parks, 2012)

$$\frac{\partial t_\alpha}{\partial \beta} = \frac{\partial F_T(t_\alpha)}{\partial \beta} / f_T(t_\alpha), \quad \text{where} \quad f_T(t_\alpha) = \frac{\partial F_T(t)}{\partial t} \bigg|_{t=t_\alpha} > 0. \quad (2.16)$$
From (2.13) and (2.2) one obtains, denoting $Q_1(s,z) = \frac{\partial Q(s,z)}{\partial s}$,
\[
\frac{\partial F_T(t_{\omega})}{\partial \beta_{11}} = Q_1(\gamma_1(x_{1u})t_{\omega}, z_{10}/\nu_1) Q(\gamma_2(x_{2u})t_{\omega}, z_{20}/\nu_2) \gamma_1(x_{1u})t_{\omega},
\]
\[
\frac{\partial F_T(t_{\omega})}{\partial \beta_{21}} = x_{1u} \frac{\partial F_T(t_{\omega})}{\partial \beta_{11}},
\]
\[
\frac{\partial F_T(t_{\omega})}{\partial \beta_{12}} = Q(\gamma_1(x_{1u})t_{\omega}, z_{10}/\nu_1) Q_1(\gamma_2(x_{2u})t_{\omega}, z_{20}/\nu_2) \gamma_2(x_{2u})t_{\omega},
\]
\[
\frac{\partial F_T(t_{\omega})}{\partial \beta_{22}} = x_{2u} \frac{\partial F_T(t_{\omega})}{\partial \beta_{21}}.
\]
Hence, the coefficient vector from (2.15) reads as
\[
c(\beta) = (f_T(t_{\omega}))^{-1}(c_1(\beta) (1, x_{1u}) c_2(\beta) (1, x_{2u}))^T, \quad \text{where} \quad (2.17)
\]
c(\beta) = \frac{\partial F_T(t_{\omega})}{\partial \beta_{11}}, \quad \text{if} \quad l = 1, 2.
\]
Together with the block-diagonal structure (2.9) of the information matrices, the c-criterion from (2.14) becomes
\[
c(\beta)^T M(\xi, \beta)^{-1} c(\beta) = (f_T(t_{\omega}))^{-2} \sum_{l=1}^{2} c_l^2 (1, x_{ul}) M_l(\xi_l, \beta_l)^{-1} (1, x_{ul})^T. \quad (2.18)
\]
It follows that a design $\xi^*$ is c-optimal w.r.t. the coefficient vector $c(\beta)$, that is, $\xi^*$ minimizes (2.18) over all designs $\xi$ on $\mathcal{X} = [0, 1]^2$, if and only if its marginal designs $\xi^*_l$, $l = 1, 2$, are c-optimal w.r.t. the coefficient vectors $c_l = (1, x_{ul})^T$, $l = 1, 2$, respectively, that is $\xi^*_l$ minimizes $(1, x_{ul}) M_l(\xi_l, \beta_l)^{-1} (1, x_{ul})^T$ over all designs $\xi_l$ on $[0, 1]$, $l = 1, 2$. In particular, c-optimality w.r.t. the coefficient vector $c(\beta)$ does not depend on $\alpha$. It should be noted that, under the assumption of independent components, the result can be readily extended to $r > 2$ components and to any s-out-of-r system, see Shat (2021) for further details in this regard. Under the premise that the locally optimal designs $\xi^*_l$ are supported on the endpoints of the design region $[0, 1]$, i.e., they are of the form $\xi^*_l = \xi_0^*_l$, where $\xi_0$ denotes a design with weight $w_{ul} = w_{l1}$ on $x_{ul} = 0$ and weight $w_{ul} = 1 - w_{l1}$ on $x_{ul} = 1$. Shat and Schwabe (2019) stated that the marginal optimal weight $w^*_l$ can be determined analytically by Elfving’s theorem Elfving (1952),
\[
w^*_l = \frac{(1 + |x_{ul}|) \sqrt{\lambda_l(1, \beta_l)}}{(1 + |x_{ul}|) \sqrt{\lambda_l(1, \beta_l)} + |x_{ul}| \sqrt{\lambda_l(0, \beta_l)}}. \quad (2.19)
\]
2.4. Numerical example
The distribution function $F_T(t)$ from (2.13) is plotted for illustration in Figure 2.4 under the nominal values given in Table 1, the normal use conditions $x_{ul} = -0.60$ and $x_{ul} = -0.50$, and the failure thresholds $z_{10} = 4.6$ and $z_{20} = 6.25$. The median $t_{0.5} = 2.11$ is indicated by a dashed vertical line. Also, the distribution functions $F_T(t)$ from (2.12) are shown in the figure. We assume that units are observed according to a time plan with $k = 4$ time points, and $t_1 = 0.02$, $t_2 = 0.04$, $t_3 = 0.06$, $t_4 = 0.1$. For computing optimal marginal designs $\xi^*_l$ minimizing $(1, x_{ul}) M_l(\xi_l, \beta_l)^{-1} (1, x_{ul})^T$, $l = 1, 2$, with nominal values of parameters and constants from Table 1, the multiplicative algorithm (Torsney and Martín-Martín 2009) was applied. The marginal design interval $[0, 1]$ was replaced by an equidistant grid with increment equal to 0.05. The obtained optimal marginal designs $\xi^*_l$ and $\xi^*_s$ are as follows,
\[
\xi^*_l = \left( \begin{array}{c} 0.79 \\ 0.09 \\ 0.21 \\ 0.09 \end{array} \right) \quad \text{and} \quad \xi^*_s = \left( \begin{array}{c} 0.72 \\ 0.07 \\ 0.19 \\ 0.02 \end{array} \right).
\]
So the locally c-optimal designs at $\beta$ are given by those designs $\xi^*$ on $\mathcal{X} = [0, 1]^2$ (actually on the product grid of the employed marginal grids) whose marginal designs are equal to $\xi^*_l$ and $\xi^*_s$ from (2.20). One of them is the product design
\[
\xi^* = \xi^*_l \otimes \xi^*_s = \left( \begin{array}{c} (0, 0) \\ 0.72 \\ (0, 1) \\ 0.07 \\ (1, 0) \\ 0.19 \\ (1, 1) \\ 0.02 \end{array} \right).
\]
Note that the locally c-optimal design is not unique: the set of all designs with marginal designs given by (2.20) consists of all designs $\xi^*$ supported by the points $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$ with weights
\[
\xi^*(0, 0) = \omega, \quad \xi^*(0, 1) = 0.79 - \omega, \quad \xi^*(1, 0) = 0.91 - \omega, \quad \xi^*(1, 1) = \omega - 0.70, \quad \text{where} \quad 0.70 \leq \omega \leq 0.79.
\]
Table 1: Nominal values of the Gamma model with independent marginal components

| $\beta_{11}$ | $\beta_{12}$ | $\nu_1$ | $\beta_{21}$ | $\beta_{22}$ | $\nu_2$ |
|--------------|--------------|----------|--------------|--------------|----------|
| 1.80         | 1.60         | 1.24     | 2.80         | 3.13         | 1.17     |

Figure 1: Failure time distribution $F_T(t)$ at the bivariate Gamma model for Example 2.4, dashed line: $F_{T1}(t)$, dotted line: $F_{T2}(t)$

Figure 2: Optimal weights in dependence on $x_{u1}$ for Example 2.4

Figure 3: Optimal weights in dependence on $x_{u2}$ for Example 2.4

For $0.70 < \omega < 0.79$ the four weights of $\xi^*$ are positive and $\xi^*$ is actually a four-point design. The particular value $\omega = 0.72$ yields the above product design. The boundary values $\omega = 0.70$ and $\omega = 0.79$ yield three-point designs supported by $(0, 0)$, $(0, 1)$, $(1, 0)$ and by $(0, 0)$, $(1, 0)$, $(1, 1)$, respectively.

When the value of normal use conditions $x_{ul}$, $l = 1, 2$ are altered within some in intervals of the negative half-line, while keeping all other parameters fixed to their nominal values in Table 1 the optimal marginal designs $\xi^*_l$, $l = 1, 2$, computed by the algorithm are again supported by the boundary values 0 and 1. The optimal weight $\omega_1 = \xi^*_1(0)$ as a function of $x_{u1}$ is plotted in Figure 2 and the optimal weight $\omega_2 = \xi^*_2(0)$ as a function of $x_{u2}$ is plotted in Figure 3.

Finally, we examine the influence of varying normal use conditions on the efficiencies of some particular marginal designs $\xi_l$, $l = 1, 2$. The efficiency of a marginal design $\xi_l$ at a normal use
condition \( x_{u1} \), where all other parameters are kept fixed according to Table 1, is defined by

\[
\text{eff}(\xi; x_{u1}) = \frac{(1, x_{u1}) \mathbf{M}_l(\xi^{(x_{u1})}, \beta_l)^{-1}(1, x_{u1})^T}{(1, x_{u1}) \mathbf{M}_l(\xi, \beta_l)^{-1}(1, x_{u1})^T},
\]

where \( \xi^{(x_{u1})} \) denotes a locally optimal design at \( \beta_l \), that is, \( \xi^{(x_{u1})} \) minimizes \((1, x_{u1}) \mathbf{M}_l(\xi, \beta_l)^{-1}(1, x_{u1})^T\) over all marginal designs \( \xi \) on \([0, 1]\), and the current marginal efficiencies may serve as lower bounds for the combined efficiency \( \text{eff}(\xi; x_{u1}) \) of the combined design \( \xi \). In Figure 4 and Figure 5, we plot, respectively, the efficiencies of the locally optimal designs \( \xi^*_1 \) and \( \xi^*_2 \) from (2.20) (solid line), the efficiencies of the design \( \xi_2 \) (dashed line) which assigns equal weights 1/2 to the points 0 and 1, and the design \( \xi_3 \) (dashed line) which assigns equal weights 1/3 to the marginal stress levels 0, 0.5 and 1. Note that the latter designs \( \xi_2 \) and \( \xi_3 \) may serve as standard designs. The nominal values for \( x_{u1} \) and \( x_{u2} \) from Table 1 are indicated in the figures by vertical dotted lines. The efficiencies of the optimal designs \( \xi^*_1 \) and \( \xi^*_2 \) from (2.20) seem to perform quite well over the ranges of \( x_{u1} \) and \( x_{u2} \), respectively. The design \( \xi_2 \) is preferable for small values of \( x_{u1} \) while the design \( \xi_3 \) performs worse throughout for reasonable values of both \( x_{u1} \) and \( x_{u2} \).

3. Bivariate Gamma model with dependent components

Again, let the system under study have two failure modes corresponding to two degradation components, but independence of the components will no longer be assumed. How to model the case of dependent degradation components? One would like to have as a model the following. Each marginal degradation component should follow a Gamma process \( Z_i^{(l)}, l = 1, 2 \), as explained in subsection 2.1. The joint degradation path of both failure modes \( Z_i = (Z_i^{(1)}, Z_i^{(2)}) \) should be a process with independent increments, and the distribution function \( F^{(h)}(y_1, y_2) \) of an increment \( Z_{i+h} - Z_i, \ t \geq 0, h > 0 \), should be given by a fixed copula \( C(r, s) \), \( 0 \leq r, s \leq 1 \), describing the dependence structure between the marginal processes,

\[
F^{(h)}(y_1, y_2) = C(F_1^{(h)}(y_1), F_2^{(h)}(y_2)), \quad y_1, y_2 > 0, \tag{3.1}
\]

where \( F_i^{(h)}(y_i), l = 1, 2 \), denotes the distribution function of the increment \( Z_i^{(l)} \) of the marginal Gamma process. Note that (3.1) implies that the bivariate process has stationary increments. The reason for using a copula is its ability to provide a flexible and convenient method for combining marginal distributions in a multivariate distribution, see Pan et al. [2011], see also Sklar [1959] for Sklar’s Theorem. Two particular copulas are the Frank copula and the Gaussian copula, employed in recent work on degradation modeling, see the corresponding definitions in Subsection 3.3 below. However, a copula \( C(r, s) \) such that a bivariate process as described exists, is unknown, unless the independence copula \( C(r, s) = rs \) which retrieves the case of independent components. Note that, by the assumption of independent increments of the bivariate process, the family of bivariate
distributions $Q_{h}$, $h > 0$, given by (3.1) must form a convolution semi-group which, however, is unknown and even not known to exist (unless, of course, in case of the independence copula). As a way out, we do no longer consider processes (marginal or bivariate processes), but restrict to a simple model considering degradations and their increments only at $k$ fixed time points.

3.1. A simple bivariate copula model

Let $k \geq 1$ time points be given, $0 < t_{1} < \cdots < t_{k}$. Denote $\Delta_{j} = t_{j} - t_{j-1}$, $j = 1, \ldots, k$, where $t_{0} = 0$. Consider the degradation $Z_{0}^{(l)}$ at time $t_{j}$ of the $l$th component and the increments $Y_{jl} = Z_{0}^{(l)} - Z_{l-1}^{(l)}$, $j = 1, \ldots, k$, $l = 1, 2$, where $Z_{0}^{(l)} = 0$. For each $l = 1, 2$, the increments $Y_{1l}, \ldots, Y_{kl}$ are independent and Gamma distributed with parameters as in Section 2. In particular, the shape parameter of the Gamma distribution of $Y_{jl}$ is given by $\gamma(x_{1}) \Delta_{j}$, where $x = (x_{1}, x_{2})$ is a normalized bivariate stress variable chosen from the experimental region $[0, 1]^{2}$, and

$$\gamma(x_{1}) = \exp(\beta_{11} + \beta_{12}x_{1}).$$

The bivariate increments $Y_{j} = (Y_{j1}, Y_{j2})$ of the bivariate degradations $Z = (Z_{j1}, Z_{j2}), j = 1, \ldots, k$, are assumed to be independent and follow a distribution according to (3.1), that is, the distribution function of $Y_{j}$ is given by

$$F_{j}(y_{1}, y_{2}) = C(F_{j1}(y_{1}), F_{j2}(y_{2})), \quad y_{1}, y_{2} > 0,$$

where $C$ is a given copula and $F_{jl}$ denotes the distribution function of the Gamma distribution with shape parameter $\gamma(x_{1}) \Delta_{j}$ and scale $\nu_{l}$. The copula is assumed to be smooth (sufficiently often continuously differentiable), and thus it has a density

$$c(r, s) = \frac{\partial^{2}C(r, s)}{\partial r \partial s}, \quad 0 < r, s < 1.$$  

Hence it follows that the bivariate increment $Y_{j}$ has a density

$$f_{j}(y) = c(F_{j1}(y_{1}), F_{j2}(y_{2})) f_{j1}(y_{1}) f_{j2}(y_{2}), \quad y = (y_{1}, y_{2}) \in (0, \infty)^{2},$$

where $f_{jl}$ denotes the Gamma density with shape $\gamma(x_{1}) \Delta_{j}$ and scale $\nu_{l}$.

By (3.4) and by independence of the increments, the log-likelihood for the parameter vector $\beta = (\beta_{11}, \beta_{12}, \beta_{13}, \beta_{22})^{T}$ given the values $y_{1}, \ldots, y_{k}$ of the increments $Y_{1}, \ldots, Y_{k}$ and under the stress condition $x = (x_{1}, x_{2}) \in [0, 1]^{2}$, reads as

$$\ell(\beta; y_{1}, \ldots, y_{k}, x) = \sum_{j=1}^{k} \left[ \exp \left( c(F_{j1}(y_{1}), F_{j2}(y_{2})) \right) \right] + \sum_{l=1}^{2} \exp \left( f_{jl}(y_{jl}) \right).$$

The following definitions present two particular copulas (in two dimensions) to be considered in further applications: the Frank copula and the Gaussian copula.

**Definition 3.1.** The Frank copula, which is a very common Archimedean copula for bivariate data, is utilized to describe the dependence relation between marginal failure modes. The bivariate Frank copula is defined as

$$C(r, s) = -\frac{1}{\kappa} \exp \left( 1 + \frac{(e^{-\kappa r} - 1)(e^{-\kappa s} - 1)}{e^{-\kappa} - 1} \right)$$

where $\kappa \in (-\infty, \infty) \setminus \{0\}$ is a fixed copula dependence parameter. The density from (3.3) becomes

$$c(r, s) = \frac{\kappa(1 - e^{-\kappa r})(1 - e^{-\kappa s})}{(1 - e^{-\kappa} - (1 - e^{-\kappa r})(1 - e^{-\kappa s}))^{2}}.$$  

**Definition 3.2.** The Gaussian copula employs a correlation parameter $\rho$ defining a positive definite correlation matrix

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \quad -1 < \rho < 1.$$  

Denote by $\Phi$ the standard normal distribution function, and denote by $F_{0, \Sigma}$ the distribution function of the bivariate normal distribution with expectation $0$ and covariance matrix $\Sigma$, that is,

$$F_{0, \Sigma}(a, b) = (2\pi)^{-1}(\det(\Sigma))^{-1/2} \int_{-\infty}^{a} \int_{-\infty}^{b} \exp \left( -\frac{1}{2} z^{T} \Sigma^{-1} z \right) dz.$$  

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Then, the Gaussian copula reads as

\[ C(r, s) = F_0 \Sigma (\Phi^{-1}(r), \Phi^{-1}(s)), \quad r, s \in (0, 1). \]  

(3.9)

Its density according to (3.3) is given by

\[ c(r, s) = \frac{(2\pi)^{-1} \det(\Sigma)^{-1/2} \exp \left( -\frac{1}{2} (\Phi^{-1}(r), \Phi^{-1}(s)) \Sigma^{-1} (\Phi^{-1}(r), \Phi^{-1}(s))^T \right)}{\phi(\Phi^{-1}(r)) \phi(\Phi^{-1}(s))}, \]  

(3.10)

where \( \phi \) denotes the standard normal density. The normal copula space provides a flexible and convenient method for combining marginal distributions in a multivariate distribution, see Pan et al. [2011]. Using the Gaussian copula in our bivariate Gamma model, the resulting density (3.4) of a bivariate increment was employed in Adegbola and Yuan [2019].

### 3.2. Information matrix

From the log-likelihood (3.5) we calculate the elemental Fisher information matrix of \( x \) at \( \beta \),

\[ \mathbf{M}(x, \beta) = -\mathbb{E} \left( \frac{\partial^2 \ell(\beta; Y_1, \ldots, Y_k, x)}{\partial \beta \partial \beta^T} \right). \]  

(3.11)

The symbol \( \mathbf{M}_{\text{ind}}(x, \beta) \) will be used for the elemental information matrix from the model with independent components studied in Section 2. In fact, on the r.h.s. of (3.5), the second term (double sum over \( j = 1, \ldots, k \) and \( l = 1, 2 \) yields, after (twice) partial differentiation, taking the expectation and putting a minus sign in front, the information matrix from (2.7) since the expectation of that term depend only on the marginal distributions of \( Y_j \), \( j = 1, \ldots, k \), which are the same Gamma distributions as in Section 2. It remains to calculate the matrix

\[ \mathbb{E} \left( \frac{\partial^2 \exp(\mathcal{C}(F_{11}(Y_{11}), F_{12}(Y_{12})))}{\partial \beta \partial \beta^T} \right). \]  

(3.12)

Here \( F_{jl} \) denotes the distribution function of the Gamma distribution with shape \( \gamma_l(x_l) \) \( \Delta_j \) and scale \( \nu_l \), \( F_{jl} \) denotes its density, and \( \gamma_l(x_l) = \exp(\beta_{1l} + \beta_{2l}x_l) \). Formulas for (3.12) are derived in Appendix A which involve two-dimensional integrals. From this, the information matrix (3.11) reads as

\[ \mathbf{M}(x, \beta) = \mathbf{H}(x, \beta) + \mathbf{M}_{\text{ind}}(x, \beta), \]  

(3.13)

where

\[ \mathbf{H}(x, \beta) = \begin{bmatrix} \mathbf{H}_1(x, \beta) & \mathbf{H}_{12}(x, \beta) \\ \mathbf{H}_{12}(x, \beta) & \mathbf{H}_2(x, \beta) \end{bmatrix}, \]

\[ \mathbf{H}_l(x, \beta) = \varphi_l(x, \beta) (1, x_l)^T (1, x_l), \quad l = 1, 2, \quad \mathbf{H}_{12}(x, \beta) = \varphi_{12}(x, \beta) (1, x_1)^T (1, x_2), \]

\[ \varphi_l(x, \beta) = \gamma_l(x_l) \sum_{j=1}^k \int_0^\infty \int_0^\infty \frac{c_l(F_{j1}(y_1), F_{j2}(y_2))}{c(F_{j1}(y_1), F_{j2}(y_2))} \left( \frac{\partial F_{j1}(y_1)}{\partial y_1} \right)^2 f_{j1}(y_1) f_{j2}(y_2) \mathrm{d}y_1 \mathrm{d}y_2, \quad l = 1, 2, \]

and

\[ \varphi_{12}(x, \beta) = \gamma_1(x_1) \gamma_2(x_2) \sum_{j=1}^k \int_0^\infty \int_0^\infty \left[ \frac{c_1(F_{j1}(y_1), F_{j2}(y_2))}{c(F_{j1}(y_1), F_{j2}(y_2))} \frac{\partial F_{j1}(y_1)}{\partial y_1} \frac{\partial F_{j2}(y_2)}{\partial y_2} f_{j1}(y_1) f_{j2}(y_2) \right. \]  

\[ \left. - c((F_{j1}(y_1), F_{j2}(y_2)) \frac{\partial f_{j1}(y_1)}{\partial y_1} \frac{\partial f_{j2}(y_2)}{\partial y_2} \right] \mathrm{d}y_1 \mathrm{d}y_2. \]

such that

\[ c_1(r, s) \] and \( c_2(r, s) \) denote the first order partial derivatives of the copula density \( c(r, s) \), that is,

\[ c_1(r, s) = \frac{\partial c(r, s)}{\partial r} \] and \( c_2(r, s) = \frac{\partial c(r, s)}{\partial s}, \quad 0 < r, s < 1. \]

Formulas for the partial derivatives \( \frac{\partial F_{jl}(y_l)}{\partial y_l} \) and \( \frac{\partial f_{jl}(y_l)}{\partial y_l} \) are given in Appendix A. Note that in case of equidistant time points \( t_1, \ldots, t_k \), that is, \( \Delta_j = \Delta \) for \( j = 1, \ldots, k \), the distribution functions and densities \( F_{jl} \) and \( f_{jl} \), respectively, are independent of \( j \), and the above formulas simplify in that case.
Table 2: Nominal values of the bivariate Gamma model with Copula function

| \( \beta_{11} \) | \( \beta_{12} \) | \( \nu_1 \) | \( \beta_{21} \) | \( \beta_{22} \) | \( \kappa \) | \( \rho \) |
|---|---|---|---|---|---|---|
| 0.30 | 0.90 | 1.17 | 0.80 | 0.10 | 1.15 | -0.40 | -0.10 |

As usual, if \( \xi \) is an (approximate) design on \([0, 1]^2\) with support points \( x_1, \ldots, x_m \) and corresponding weights \( w_i, i = 1, \ldots, m \), the information matrix of \( \xi \) at a parameter point \( \beta \) is given by

\[
M(\xi, \beta) = \sum_{i=1}^{m} w_i M(x_i, \beta).
\]

(3.14)

In contrast to the settings of independent response components in Section 2, the D-optimality criterion will be applied, instead of the \( c \)-criterion, for the current settings of Copula-based bivariate degradation models. The main reason behind that is the difficulty to accurately define the continuous failure time variable \( T \), and, hence, the quantile \( t_\alpha \), under the assumptions of dependent marginal failure modes based on Copula functions. Accordingly, we are adopting the D-criterion for the numerical calculations in Example 1 and Example 2.

3.3. Local D-optimality

For a given parameter point \( \beta \), a design \( \xi^* \) is called locally D-optimal at \( \beta \) if \( \xi^* \) maximizes \( \det(M(\xi, \beta)) \) over all designs \( \xi \). For numerical computation of a locally D-optimal design we used the multiplicative algorithm, where the design region \([0, 1]^2\) is discretized by a grid with 0.05 increments in both dimensions. The elemental information matrices from (3.13) were computed by numerical integration in two dimensions. We employed the Frank copula and the Gaussian copula from based on 3.1 and 3.2 respectively.

Example 1. Let \( C(r, s) \) be the Frank copula from (3.6). Its density \( c(r, s) \) is given by (3.7). By straightforward calculations, one obtains the first order partial derivatives \( c_1(r, s) = \hat{c}(r, s)/\hat{c}_r \) and \( c_2(r, s) = \hat{c}(r, s)/\hat{c}_s \),

\[
c_1(r, s) = \frac{\kappa^2 (1 - e^{-\kappa}) e^{-\kappa (r + s)} \left[(1 + e^{-\kappa}) (1 - e^{-\kappa^2 r}) (1 - e^{-\kappa^2 s}) - (1 - e^{-\kappa})\right]}{1 - e^{-\kappa} - (1 - e^{-\kappa^2}) \left[(1 - e^{-\kappa^2}) (1 - e^{-\kappa})\right]^2}, \quad c_2(r, s) = c_1(s, r).
\]

(3.15)

Choosing \( k = 4 \) equidistant time points \( t_1 = 0.05, t_2 = 0.10, t_3 = 0.15, t_4 = 0.20 \), and the nominal values of the parameter vector \( \beta \) in Table 3, numerical computations with the multiplicative algorithm were done for local D-optimal design. The obtained locally D-optimal design is a uniformly weighted 6-point design,

\[
\xi_d^* = \begin{pmatrix}
(0, 0) & (0, 1) & (0.5, 0) & (0.5, 1) & (1, 0) & (1, 1) \\
0.166 & 0.166 & 0.166 & 0.166 & 0.166 & 0.166
\end{pmatrix}
\]

(3.16)

Example 2. Let \( C(r, s) \) be the Gaussian copula from (3.9) with parameter value \( \rho = -0.1 \). Its density is given by (3.10), and the first order partial derivatives of the latter are given by

\[
c_1(r, s) = \frac{\rho}{1 - \rho^2} c(r, s) \frac{\Phi^{-1}(s) - \rho \Phi^{-1}(r)}{\phi(\Phi^{-1}(r))}, \quad c_2(r, s) = c_1(s, r).
\]

As is the preceding example, we choose \( k = 4 \) equidistant time points \( t_1 = 0.05, t_2 = 0.10, t_3 = 0.15, t_4 = 0.20 \), and the nominal values of the parameter vector \( \beta \) from Table 3. The locally D-optimal design obtained with the multiplicative algorithm has the same six support points as that for Example 1 with non-uniform weights, as

\[
\xi_d^* = \begin{pmatrix}
(0, 0) & (0.1) & (0.5, 0) & (0.5, 1) & (1, 0) & (1, 1) \\
0.20 & 0.20 & 0.16 & 0.16 & 0.18 & 0.09
\end{pmatrix}
\]

(3.17)

Due to the difficulty of accurately deriving the information matrix (3.13) for the Copula-based models 3.1 and 3.2 with multiple observations, we consider in Section 4 a simplified approach with binary outcomes which facilitates the derivations of the corresponding information matrix and, hence, considerably reduce the calculations time.
4. Copula-based gamma model with binary outcomes

4.1. Model formulation

In this section, we consider the model from Section 3 but now the measurements of bivariate degradations \( \mathbf{Z}_j = (Z_j^{(1)}, Z_j^{(2)}) \), \( j = 1, ..., k \), are reduced to the information on whether or not the marginal degradation paths have reached or exceeded given thresholds \( z_{10} > 0 \) and \( z_{20} > 0 \), respectively, at each time \( t_j \), \( j = 1, ..., k \). This information is equivalently reflected by two discrete variables \( U \) and \( V \) with values in \( \{1, ..., k, k+1\} \), where \( U \) (resp. \( V \)) gives the first time label \( j \) such that the marginal degradation \( Z_j^{(1)} \) (resp. \( Z_j^{(2)} \)) has reached or exceeded the threshold \( z_{01} \) (resp. \( z_{02} \)), and the value \( k+1 \) expresses that failure did not occur until time \( t_k \). That is, we define

\[
U = \min \left\{ j \in \{1, ..., k\} : Z_j^{(1)} \geq z_{10} \right\},
\]

\[
V = \min \left\{ j \in \{1, ..., k\} : Z_j^{(2)} \geq z_{20} \right\},
\]

where the minimum of the empty set is defined to be \( k+1 \). The joint distribution of \( U, V \) is given by the probabilities \( Q_{u,v} = \Pr(U = u, V = v) \), \( u, v \in \{1, ..., k, k+1\} \). Below we will see that their calculation involves multi-dimensional integrals over polyhedral regions which are difficult to handle theoretically as well as numerically. A slight simplification of the integration regions is gained by considering the probabilities

\[
Q_{u,v} = \Pr(U \leq u, V \leq v) \quad \text{for} \quad 1 \leq u, v \leq k+1.
\]

Note that \( Q_{k+1,v} = \Pr(V \leq v) \) and \( Q_{u,k+1} = \Pr(U \leq u) \), and especially \( Q_{k+1,k+1} = 1 \). The probabilities \( P_{u,v} \) are obtained from the \( Q_{u,v} \) by

\[
P_{u,v} = Q_{u,v} - Q_{u,v-1} - Q_{u-1,v} + Q_{u-1,v-1} \quad \text{for} \quad 1 \leq u, v \leq k+1
\]

(4.1)

where \( Q_{0,0} = Q_{u,v} = 0 \) for \( 1 \leq u, v \leq k+1 \). By the two equivalences, for any \( u, v \in \{1, ..., k\} \),

\[
U \leq u \iff Z^{(1)}_u \geq z_{10}, \quad V \leq v \iff Z^{(2)}_v \geq z_{20},
\]

and writing the degradations as sums of increments, \( Z^{(1)}_u = \sum_{j=1}^u Y_{j1} \) and \( Z^{(2)}_v = \sum_{j=1}^v Y_{j2} \), we get for all \( u, v \in \{1, ..., k\} \),

\[
Q_{u,v} = \int_{A_{u,v}} \prod_{j=1}^k f_j(Y_j) \, dy_1 \cdots dy_k,
\]

(4.2)

where

\[
A_{u,v} = \left\{ (y_1, ..., y_k) \in (0, \infty)^{2k} : \sum_{j=1}^u y_{j1} \geq z_{10}, \sum_{j=1}^v y_{j2} \geq z_{20} \right\},
\]

and \( f_j \) denotes the density of the bivariate increment \( Y_j = (Y_{j1}, Y_{j2}) \) from [3,4]. For \( u = k+1 \) or \( v = k+1 \), a calculation of \( Q_{k+1,v} \) or \( Q_{u,k+1} \) involves only the marginal degradations, which are Gamma distributed,

\[
Q_{k+1,v} = \Pr(Z^{(2)}_v \geq z_{20}) = \frac{\Gamma(\gamma_2(x_2) t_v, y_{2/v_2})}{\Gamma(\gamma_2(x_2) t_v)}, \quad 1 \leq v \leq k,
\]

\[
Q_{u,k+1} = \frac{\Gamma(\gamma_1(x_1) t_u, y_{1/v_1})}{\Gamma(\gamma_1(x_1) t_u)}, \quad 1 \leq u \leq k.
\]

4.2. Information matrix

The log likelihood of the bivariate discrete variable \((U, V)\) is given by

\[
\ell(\beta; u, v, x) = \exp P_{u,v}(x, \beta)
\]

(4.3)

where now we observe the dependence of the probabilities \( P_{u,v} \) \( 1 \leq u, v \leq k+1 \), on the design variable \( x = (x_1, x_2) \) and the parameter vector \( \beta = (\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})^T \). The elemental information matrix of \( x \) at a parameter point \( \beta \) is given by

\[
M(x, \beta) = \mathbb{E} \left[ \left( \frac{\partial^2 \ell(\beta; u, v, x)}{\partial \beta \partial \beta} \right) \left( \frac{\partial \ell(\beta; u, v, x)}{\partial \beta} \right)^T \right].
\]

(4.4)
We can decompose $\ell$, as a function of $\beta$, according to

$$
\beta \longrightarrow \gamma = (\gamma_1, \gamma_2)^T \longrightarrow P = (P_{1u}, \ldots, P_{uv}, \ldots, P_{k+1,k+1})^T \longrightarrow \ell,
$$

where the $P_{uv}, 1 \leq u, v \leq k + 1$, have been arranged in lexicographic order, say, to form the vector $P$. By the chain rule a factorization of the gradient $\partial \ell(\beta; u, v, x)/\partial \beta$ results,

$$
\frac{\partial \ell(\beta; u, v, x)}{\partial \beta} = ABC,
$$

with matrices $A, B$ and a column vector $C$,

$$
A = A(x, \beta) = \frac{\partial \gamma}{\partial \beta} = \begin{bmatrix} \gamma_1 & 0 \\ x_1\gamma_1 & 0 \\ 0 & \gamma_2 \\ 0 & x_2\gamma_2 \end{bmatrix},
$$

$$
B = B(x, \beta) = \frac{\partial P}{\partial \gamma} = \begin{bmatrix} \frac{\partial P_{uv}}{\partial \gamma_1}(1 \leq u, v \leq k + 1) \\ \frac{\partial P_{uv}}{\partial \gamma_2}(1 \leq u, v \leq k + 1) \end{bmatrix},
$$

$$
C = C(x, \beta, u, v) = \frac{\partial \ell}{\partial P_{(1 \leq u, v \leq k + 1)}} = \left( \frac{1}{P_{uv}} \right)^T.
$$

Note that the two rows of $B$ and the column vector $C$ have components indexed by the pairs $(u, v)$ arranged in lexicographic order. It follows that

$$
M(x, \beta) = AB E(CC^T) B^T A^T,
$$

and

$$
E(CC^T) = \text{diag}\left( \frac{1}{P_{uv}}(1 \leq u, v \leq k + 1) \right).
$$

Again, for a design $\xi$ with support points $x_i$ and weights $w_i, i = 1, \ldots, m$, the information matrix of $\xi$ at $\beta$ is given by

$$
M(\xi, \beta) = \sum_{i=1}^{m} w_i M(x_i, \beta).
$$

In order to obtain explicit formulas for the entries of $B(x, \beta)$, that is, the partial derivatives $\partial P_{uv}/\partial \gamma_l$, we consider the corresponding partial derivatives of the probabilities $Q_{uv}$ from (4.2). One gets

$$
\frac{\partial Q_{uv}}{\partial \gamma_l} = \int_{A_{uv}} \frac{\partial}{\partial \gamma_l} \prod_{j=1}^{k} f_j(Y_j) \, dy_1 \ldots dy_k
$$

$$
= \int_{A_{uv}} \sum_{i=1}^{k} \left[ \prod_{j \neq i} f_j(Y_j) \right] \frac{\partial f_i(Y_i)}{\partial \gamma_l} \, dy_1 \ldots dy_k,
$$

(4.6)

and by (3.4)

$$
\frac{\partial f_j(Y_j)}{\partial \gamma_1} = c_1 (F_{j}(y_{j1}), F_{j2}(y_{j2})) \frac{\partial F_{j1}(y_{j1})}{\partial \gamma_1} f_{j1}(y_{j1}) + c(F_{j1}(y_{j1}), F_{j2}(y_{j2})) \frac{\partial f_{j1}(y_{j1})}{\partial \gamma_1} f_{j2}(y_{j2}),
$$

$$
\frac{\partial f_j(Y_j)}{\partial \gamma_2} = c_2 (F_{j}(y_{j1}), F_{j2}(y_{j2})) \frac{\partial F_{j2}(y_{j2})}{\partial \gamma_2} f_{j2}(y_{j2}) + c(F_{j1}(y_{j1}), F_{j2}(y_{j2})) \frac{\partial f_{j2}(y_{j2})}{\partial \gamma_2} f_{j1}(y_{j1}).
$$

(4.7)

However, due to the $2k$-dimensional integration in (4.6) the calculation of information matrices is not tractable when $k > 1$. Therefore, we consider now the simple case $k = 1$ of a single measurement. Then, we have one bivariate increment $Y = (Y_1, Y_2)$, and the distribution function of $Y$ is given by

$$
C(F_1(y_1); F_2(y_2)), \quad y_1, y_2 \in (0, \infty).
$$

The probabilities $P_{uv}, u, v \in \{1, 2\}$, can be expressed by the latter joint distribution function and the marginal distribution functions $F_1$ and $F_2$,

$$
P_{2,2} = P(Y_1 < z_{10}, Y_2 < z_{20}) = C(F_1(z_{10}), F_2(z_{20})),
$$

$$
P_{1,2} = P(Y_1 \geq z_{10}, Y_2 < z_{20}) = F_2(z_{20}) - C(F_1(z_{10}), F_2(z_{20})),
$$

$$
P_{2,1} = P(Y_1 < z_{10}, Y_2 \geq z_{20}) = F_1(z_{10}) - C(F_1(z_{10}), F_2(z_{20})),
$$

$$
P_{1,1} = P(Y_1 \geq z_{10}, Y_2 \geq z_{20}) = 1 - F_1(z_{10}) - F_2(z_{20}) + C(F_1(z_{10}), F_2(z_{20})).
$$

(4.8)
The partial derivatives \( \partial P_{u,v}/\partial \alpha_l \) are easily obtained from the partial derivatives \( \partial F_{l}(z_{10})/\partial \alpha_l \) and the partial derivatives of the copula, \( C_1(r,s) = \partial C(r,s)/\partial r \) and \( C_2(r,s) = \partial C(r,s)/\partial s \), since by the chain rule
\[
\frac{\partial}{\partial \gamma_l} C(F_1(z_{10}), F_2(z_{20})) = C_1(F_1(z_{10}), F_2(z_{20})) \frac{\partial F_1(z_{10})}{\partial \gamma_l}, \quad l = 1, 2. \tag{4.9}
\]
In particular, when \( C \) is the Frank copula with parameter \( \kappa \), then by straightforward calculation,
\[
C_1(r,s) = \frac{e^{-\kappa r} (e^{-\kappa s} - 1)}{e^{-\kappa s} - 1 + (e^{-\kappa r} - 1)(e^{-\kappa s} - 1)} \quad \text{and} \quad C_2(r,s) = C_1(s,r). \tag{4.10}
\]
When \( C \) is the Gaussian copula with correlation parameter \( \rho \), then one obtains (see Appendix A)
\[
C_1(r,s) = \Phi\left(\frac{\Phi^{-1}(s) - \rho \Phi^{-1}(r)}{\sqrt{1 - \rho^2}}\right) \quad \text{and} \quad C_2(r,s) = C_1(s,r). \tag{4.11}
\]

4.3. Local D- or c-optimal designs when \( k = 1 \)

For our simple binary model \( (k = 1) \) employing the Frank copula or the Gaussian copula, locally D- or c-optimal designs are presented in the example below. A locally D-optimal design \( \xi_D^* \) at a given parameter point \( \beta \) maximizes \( \det(M(\xi_D, \beta)) \) over all designs \( \xi \). A locally c-optimal design \( \xi_c^* \) at \( \beta \) minimizes \( c^T M(\xi, \beta)^{-1} c \) over all designs \( \xi \), where \( c \) is a given nonzero column vector of dimension four. Here the coefficient vector \( c \) is chosen such that the c-criterion represents the asymptotic variance of the maximum likelihood estimator \( \hat{P}_{11} \) of the joint failure probability \( P_{11} = P_{11}(x_u, \beta) \) at normal use conditions \( x_u = (x_{u1}, x_{u2}) \). That is,
\[
c = \frac{\partial P_{11}(x_u, \beta)}{\partial \beta} = (c_1.(1, x_{u1}), c_2.(1, x_{u2}))^T,
\]
where
\[
c_l = \gamma_l(x_{u_l}) \Delta_1 \frac{\partial P_{11}}{\partial \gamma_l}, \quad l = 1, 2.
\]

The partial derivatives \( \partial P_{11}/\partial \gamma_l \) can be evaluated using formulas \( (4.8), (4.9), \) and \( (A.8) \).

**Example 3.** For obtaining numerically optimal designs, the multiplicative algorithm with an equidistant grid of 0.05 marginal increments over the standardized design region \( X = [0,1]^2 \) is employed. The single point time plan is chosen as \( t_1 = \Delta_1 = 0.3 \). The resulting optimal designs are derived in regards to the nominal values of parameters are given in Table 2, the normal use conditions \( x_{u1} = -0.40 \) and \( x_{u2} = -0.60 \), and the failure thresholds \( z_{10} = 2.56 \) and \( z_{20} = 2.37 \).

The D-optimal designs computed by the algorithm are the following four-point designs, which nearly coincide for the two copula,

- **Frank copula:** \( \xi_D^* = \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ 0.24 & 0.24 & 0.26 & 0.26 \end{pmatrix} \);  
- **Gaussian copula:** \( \xi_D^* = \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ 0.22 & 0.23 & 0.27 & 0.28 \end{pmatrix} \).

The c-optimal designs from the algorithm are again four-point designs, which nearly coincide on the location of support points and the optimal weights of extremal points with some differences in the optimal weights of the two middle points,

- **Frank copula:** \( \xi_c^* = \begin{pmatrix} (0,0) & (0,1) & (0.5,1) & (1,1) \\ 0.09 & 0.18 & 0.46 & 0.27 \end{pmatrix} \);  
- **Gaussian copula:** \( \xi_c^* = \begin{pmatrix} (0,0) & (0,1) & (0.5,1) & (1,1) \\ 0.11 & 0.22 & 0.39 & 0.28 \end{pmatrix} \).

To evaluate the behaviour of the resulting optimal designs we consider the variations of the optimal weights when the underlying nominal values are misspecified. For brevity we consider the c-optimal design \( \xi_c^* \) on the basis of the Gaussian copula function under deviations of the normal use condition \( x_{u1} \), and the correlation parameter \( \rho \). The four optimal weights \( w_1^*, \omega_2^*, \omega_3^* \) and \( \omega_4^* \) are plotted in Figure 7 in dependence on \( x_{u1} \) where all parameters are held fixed to their nominal values and in Figure 8 in dependence on \( \rho \) where all parameters are held fixed to their nominal values. Figure 4 shows that the optimal weights of the middle point, i.e. \( w_2^* \) and \( w_3^* \), considerably vary under changes
of $x_u$ where the optimal weights of the external point, i.e. $w^*_2$ and $w^*_3$, are nearly constant throughout. Figure 8 indicates that the resulting optimal design is more robust against misspecification of the correlation parameter $\rho$. The nominal value for $x_u$ and $\rho$ at $\xi^*_c$ are indicated by vertical dotted lines in the corresponding figure. Define by

$$\text{eff}(\xi) = \frac{e^T M(\xi^*_c, \beta)^{-1} c}{e^T M(\xi, \beta)^{-1} c},$$

the efficiency of a design $\xi$ in terms of $\xi^*_c$ where $e^T M(\xi^*_c, \beta)^{-1} c$ indicates the asymptotic variance for estimating $P_{11}$ under the optimal design $\xi^*_c$. Figure 7 and Figure 9 show, respectively, the efficiencies in dependence on $x_u$ and $\rho$ together with the efficiency of the $\xi_2$ which assigns equal weights $1/4$ to the same support points of $\xi^*_c$, and the design $\xi_3$ which assigns equal weights $1/4$ to the vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Again, the nominal values for $x_u$ for $x_u$ and $\rho$ at $\xi^*_c$ are indicated by vertical dotted lines in the corresponding figure. In total, Figure 7 and Figure 9 indicate that the optimal design $\xi^*_c$ performs quite well over the range of $x_u$ and $\rho$ when compared to $\xi_2$ and $\xi_3$, which indicate that the optimal design is robust against changes of the normal use conditions as well as the nominal values. The existing results of the sensitivity analysis of $\xi^*_c$ on the basis of the Frank copula nearly coincides with the obtained results in regards to the Gaussian copula, and, hence, the latter results have been removed to avoid redundancy.

5. Concluding remarks

Reliability engineers are demanded to provide a sophisticated assessment of the reliability related properties during the design stage of highly reliable systems. Accelerated degradation testing (ADT) is a common approach to handle this issue. Accelerated degradation tests have the advantage to give
an estimation of lifetime and reliability characteristics of the system under study in a relatively short testing time. In this work, we introduced optimal experimental designs for accelerated degradation tests with two response components and repeated measures with or without dependence between marginal components. The marginal degradation paths are expressed using Gamma process models. In the current models for ADT, we assume that stress remains constant within each unit during the whole test but may vary between units. Further, the same time plan for measurements is used for all units in the test.

In the case of independent components, it is desirable to estimate certain quantiles of the joint failure time distribution as a characteristic of the reliability of the product. Hence, the purpose of optimal experimental design is to find the best settings for the stress variable to obtain most accurate estimates of the quantiles.

On the other hand, the Frank copula as well as the Gaussian copula are separately adopted to represent the dependence relation in bivariate Gamma models when dependence is assumed between response components. The $D$-criterion is considered for locally optimal designs in both cases. The resulting optimal designs coincide in terms of the optimal support points but differ in their weights allocated to the points.

We developed further $D$- and $c$-optimal designs when the two Copula-based models are reduced to binary responses. A sensitivity analysis showed that the resulting locally optimal designs are quite efficient against deviations from the assumed nominal values.

Throughout, Gamma process models were considered as marginal degradation models. As a topic for future research, the results should be extended to other marginal failure models, e.g. Wiener process, inverse Gaussian process or non-linear mixed-effects degradation models.
Appendix A. Derivation of the Information Matrix in Subsection 3.2

We derive formulas (3.11) by developing the double integral in (3.12). Since the index $j$ will be fixed in our derivations, we simply write $F_l, f_l, Y_l$ instead of $F_{lj}, f_{lj}, Y_{lj}$, respectively, $l = 1, 2$. Recall the partitioning of $\beta$ as $\beta = (\beta_1^T, \beta_2^T)^T$, where $\beta_l = (\beta_{l1}, \beta_{l2})^T$, $l = 1, 2$. By $c_1(r,s), c_2(r,s), c_{11}(r,s), c_{22}(r,s)$, and $c_{12}(r,s)$ we denote the partial derivatives of $c(r,s)$,

$$
c_1(r,s) = \frac{\partial c(r,s)}{\partial r}, \quad c_2(r,s) = \frac{\partial c(r,s)}{\partial s},
$$

$$
c_{11}(r,s) = \frac{\partial^2 c(r,s)}{\partial r^2}, \quad c_{22}(r,s) = \frac{\partial^2 c(r,s)}{\partial s^2}, \quad c_{12}(r,s) = \frac{\partial^2 c(r,s)}{\partial r \partial s}.
$$

By straightforward calculation,

$$
\frac{\partial \ln c(F_1(Y_1), F_2(Y_2))}{\partial \beta_l} = \frac{c_1(F(Y_1), F_2(Y_2))}{c(F_1(Y_1), F_2(Y_2))} \frac{\partial F_1(Y_1)}{\partial \beta_l}, \quad l = 1, 2;
$$

$$
\frac{\partial^2 \ln c(F_1(Y_1), F_2(Y_2))}{\partial \beta_1 \partial \beta_2} = \begin{bmatrix}
    \frac{c_{11}(F_1(Y_1), F_2(Y_2))}{c(F_1(Y_1), F_2(Y_2))} & \frac{c_2(F_1(Y_1), F_2(Y_2))}{c(F_1(Y_1), F_2(Y_2))} \\
    \frac{c_2(F_1(Y_1), F_2(Y_2))}{c(F_1(Y_1), F_2(Y_2))} & \frac{c_{22}(F_1(Y_1), F_2(Y_2))}{c(F_1(Y_1), F_2(Y_2))}
\end{bmatrix} \frac{\partial^2 F_1(Y_1)}{\partial \beta_1 \partial \beta_1} \left( \frac{\partial F_1(Y_1)}{\partial \beta_2} \right)^T + \frac{c_1(F_1(Y_1), F_2(Y_2))}{c(F_1(Y_1), F_2(Y_2))} \frac{\partial^2 F_1(Y_1)}{\partial \beta_1 \partial \beta_2} \frac{\partial^2 F_1(Y_1)}{\partial \beta_2 \partial \beta_2}, \quad l = 1, 2;
$$

$$
\frac{\partial^2 \ln c(F_1(Y_1), F_2(Y_2))}{\partial \beta_1 \partial \beta_2} = \frac{c_2(F_1(Y_1), F_2(Y_2))}{c(F_1(Y_1), F_2(Y_2))} \frac{\partial F_1(Y_1)}{\partial \beta_1} \left( \frac{\partial F_1(Y_1)}{\partial \beta_2} \right)^T.
$$

We show that

$$
E \left( \frac{c_{11}(F_1(Y_1), F_2(Y_2))}{c(F_1(Y_1), F_2(Y_2))} \frac{\partial F_1(Y_1)}{\partial \beta_1} \left( \frac{\partial F_1(Y_1)}{\partial \beta_1} \right)^T \right) = 0, \quad l = 1, 2.
$$

Using the joint density of $Y = (Y_1, Y_2)$ from (3.4), the expectation on the l.h.s. of (A.4) rewrites, when $l = 1$, as

$$
\int_0^\infty \int_0^\infty c_{11}(F_1(y_1), F_2(y_2)) \frac{\partial F_1(y_1)}{\partial \beta_1} \left( \frac{\partial F_1(y_1)}{\partial \beta_1} \right)^T f_1(y_1) f_2(y_2) dy_2 \, dy_1 = \int_0^\infty \frac{\partial F_1(y_1)}{\partial \beta_1} \left( \frac{\partial F_1(y_1)}{\partial \beta_1} \right)^T f_1(y_1) \left( \int_0^\infty c_{11}(F_1(y_1), F_2(y_2)) f_2(y_2) \, dy_2 \right) \, dy_1.
$$

For any fixed $y_1$, the inner integral becomes, by substituting $s = F_2(y_2)$ and interchanging integral and derivatives,

$$
\int_0^1 c_{11}(F_1(y_1), s) \, ds = \frac{\partial^2}{\partial r^2} \int_0^1 c(r,s) \, ds \bigg|_{r=F_1(y_1)} = 0,
$$

where the last equation follows from $\int_0^1 c(r,s) \, ds = 1$ for all $0 < r < 1$. Hence (A.4) follows for $l = 1$, and the case $l = 2$ can be proved analogously. Next we show that

$$
E \left( \frac{c_{11}(F_1(Y_1), F_2(Y_2))}{c(F_1(Y_1), F_2(Y_2))} \frac{\partial^2 F_1(Y_1)}{\partial \beta_1 \partial \beta_2} \right) = 0, \quad l = 1, 2.
$$

Again using the density from (3.4) and restricting to $l = 1$ (the case $l = 2$ is analogous), the expectation on the l.h.s. of (A.5) rewrites as

$$
\int_0^\infty \int_0^\infty c_{11}(F_1(Y_1), F_2(Y_2)) \frac{\partial^2 F_1(y_1)}{\partial \beta_1 \partial \beta_2} f_1(y_1) f_2(y_2) \, dy_2 \, dy_1 = \int_0^\infty \frac{\partial^2 F_1(y_1)}{\partial \beta_1 \partial \beta_2} f_1(y_1) \left( \int_0^\infty c_{11}(F_1(Y_1), F_2(Y_2)) f_2(y_2) \, dy_2 \right) \, dy_1,
$$

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and for any fixed $y_1$ the inner integral is equal to
\[
\int_0^1 c_1(F_1(y_1), s) \, ds = \left. \frac{\partial}{\partial r} \int_0^1 c(r, s) \, ds \right|_{r=F_1(y_1)} = 0.
\]
From \((A.4)\), \((A.5)\), and \((A.1)\) it follows that
\[
E \left( \frac{\partial^2 \ln c(F_1(Y_1), F_2(Y_2))}{\partial \beta_1 \partial \beta_1} \right) = E \left( \frac{\partial^2 F_1(Y_1)}{\partial \beta_1} \left( \frac{\partial^2 F_1(Y_1)}{\partial \beta_1} \right)^T \right)
\]
\[
- \int_0^\infty \int_0^\infty \frac{\partial^2 F_1(Y_1)}{\partial \beta_1} \left( \frac{\partial^2 F_1(Y_1)}{\partial \beta_1} \right)^T f_1(y_1) f_2(y_2) \, dy_1 dy_2, \quad l = 1, 2. \quad (A.6)
\]
Next we show that
\[
E \left( \frac{c_{12}(F_1(Y_1), F_2(Y_2))}{c(F_1(Y_1), F_2(Y_2))} \frac{\partial F_1(Y_1)}{\partial \beta_1} \left( \frac{\partial F_2(Y_2)}{\partial \beta_2} \right)^T \right) = \int_0^\infty \int_0^\infty \frac{c(F_1(y_1), F_2(y_2))}{c(F_1(Y_1), F_2(Y_2))} \frac{\partial f_1(y_1)}{\partial \beta_1} \left( \frac{\partial f_2(y_2)}{\partial \beta_2} \right)^T f_1(y_1) f_2(y_2) \, dy_1 dy_2. \quad (A.7)
\]
The expectation on the l.h.s. of \((A.7)\) equals
\[
\int_0^\infty \int_0^\infty c_{12}(F_1(y_1), F_2(y_2)) \frac{\partial F_1(y_1)}{\partial \beta_1} \left( \frac{\partial F_2(y_2)}{\partial \beta_2} \right)^T f_1(y_1) f_2(y_2) \, dy_1 dy_2.
\]
Writing
\[
c_{12}(F_1(y_1), F_2(y_2)) \frac{\partial F_1(y_1)}{\partial \beta_1} f_1(y_1) = \frac{\partial}{\partial \beta_1} \left[ c_2(F_1(y_1), F_2(y_2)) f_1(y_1) \right] - c_2(F_1(y_1), F_2(y_2)) \frac{\partial f_1(y_1)}{\partial \beta_1},
\]
the last double integral rewrites as
\[
\int_0^\infty \left\{ \int_0^\infty \left[ \frac{\partial}{\partial \beta_1} \left[ c_2(F_1(y_1), F_2(y_2)) f_1(y_1) \right] - c_2(F_1(y_1), F_2(y_2)) \frac{\partial f_1(y_1)}{\partial \beta_1} \right] \, dy_1 \left( \frac{\partial F_2(y_2)}{\partial \beta_2} \right)^T f_2(y_2) \, dy_2.
\]
Now for any fixed $y_2$,
\[
\int_0^\infty \frac{\partial}{\partial \beta_1} \left[ c_2(F_1(y_1), F_2(y_2)) f_1(y_1) \right] \, dy_1 = \frac{\partial}{\partial \beta_1} \int_0^1 c_2(r, F_2(y_2)) \, dr = 0,
\]
since the last integral does not depend on $\beta_1$. We have obtained that the expectation on the l.h.s. of \((A.7)\) is equal to
\[
- \int_0^\infty \int_0^\infty c_2(F_1(y_1), F_2(y_2)) \frac{\partial f_1(y_1)}{\partial \beta_1} \left( \frac{\partial f_2(y_2)}{\partial \beta_2} \right)^T f_2(y_2) \, dy_1 dy_2
\]
\[
= - \int_0^\infty \frac{\partial f_1(y_1)}{\partial \beta_1} \left\{ \int_0^\infty c_2(F_1(y_1), F_2(y_2)) \left( \frac{\partial f_2(y_2)}{\partial \beta_2} \right)^T f_2(y_2) \, dy_2 \right\} \, dy_1.
\]
Writing
\[
c_2(F_1(y_1), F_2(y_2)) \left( \frac{\partial f_2(y_2)}{\partial \beta_2} \right)^T f_2(y_2) = \left( \frac{\partial}{\partial \beta_2} \left[ c(F_1(y_1), F_2(y_2)) f_2(y_2) \right] \right)^T - c(F_1(y_1), F_2(y_2)) \left( \frac{\partial f_2(y_2)}{\partial \beta_2} \right)^T,
\]
and observing that for any fixed $y_1$
\[
\int_0^\infty \frac{\partial}{\partial \beta_2} \left[ c(F_1(y_1), F_2(y_2)) f_2(y_2) \right] \, dy_2 = \frac{\partial}{\partial \beta_2} \int_0^\infty c(F_1(y_1), s) \, ds = 0,
\]
we get \((A.7)\). From \((A.3)\) and \((A.7)\) we get
\[
E \left( \frac{\partial^2 \ln c(F_1(Y_1), F_2(Y_2))}{\partial \beta_1 \partial \beta_2} \right) = \int_0^\infty \int_0^\infty \left[ c(F_1(y_1), F_2(y_2)) \frac{\partial f_1(y_1)}{\partial \beta_1} \frac{\partial f_2(y_2)}{\partial \beta_2} - \frac{c_1(F_1(y_1), F_2(y_2)) c_2(F_1(y_1), F_2(y_2))}{c(F_1(y_1), F_2(y_2))} \frac{\partial F_1(y_1)}{\partial \beta_1} \frac{\partial F_2(y_2)}{\partial \beta_2} f_1(y_1) f_2(y_2) \right] \, dy_1 dy_2. \quad (A.8)
\]
Observing that

\[ \frac{\partial F_i(y_l)}{\partial \beta_{l_1}} = \frac{\partial F_i(y_l)}{\partial \beta_{l_1}}, \quad \frac{\partial F_i(y_l)}{\partial \beta_{l_1}} = \frac{\partial F_i(y_l)}{\partial \beta_{l_1}}, \quad (A.9) \]

and

\[ \frac{\partial \gamma_l}{\partial \beta_{l_1}} = \gamma_l(x_1)(1, x_1)^T, \quad l = 1, 2, \quad (A.10) \]

formulas (3.11) in Subsection 3.2 follow from (A.6) and (A.8).

The derivatives \( \frac{\partial F_i(y_l)}{\partial \gamma_l} \) and \( \frac{\partial F_i(y_l)}{\partial \gamma_l} \) are given by

\[ \frac{\partial}{\partial \gamma_l} f_i(y_l) = \frac{\delta_l \left( \ln(y_l) - \ln(\nu_l) - \psi(\kappa_i) \right)}{\Gamma(\kappa_i)\nu_l^{\kappa_i}}, \quad (A.11) \]

\[ \frac{\partial}{\partial \gamma_l} f_i(y_l) = \frac{\Gamma(\kappa_i, y_l/\nu_l)}{\Gamma(\kappa_i)} = \Delta \left( -\Gamma(\kappa_i)(y_l/\nu_l)^{\kappa_i} \hat{F}_2(\kappa_i, \kappa_i; \kappa_i + 1, \kappa_i + 1; -y_l/\nu_l) \right), \quad (A.13) \]

\[ -\psi(\kappa_i) \frac{\Gamma(\kappa_i, y_l/\nu_l)}{\Gamma(\kappa_i)} - \exp(y_l/\nu_l) \frac{\Gamma(\kappa_i, y_l/\nu_l, 0)}{\Gamma(\kappa_i)}, \quad (A.14) \]

such that \( \psi(\kappa) = \frac{\Gamma'(\kappa)}{\Gamma(\kappa)} \) indicates the digamma function, \( \Gamma(s, z, 0) = \Gamma(s, z) - \Gamma(s) \), \( \Gamma(\kappa_i, y_l/\nu_l) \) refers to the lower incomplete Gamma function, and \( \hat{F}_2 \) denotes the regularized hypergeometric function which is extended from the generalized hypergeometric function \( 2\hat{F}_2(\kappa, \kappa; \kappa + 1, \kappa + 1; -y/\nu) \) and given by

\[ 2\hat{F}_2(\kappa, \kappa; \kappa + 1, \kappa + 1; -y/\nu) = \frac{1 + \sum_{k=1}^{\infty} \left( \frac{\nu}{\kappa + k} \right)^k \frac{(-y/\nu)^k}{k!}}{\Gamma(\kappa + 1)^2}. \]

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