Two types of electric field enhancements by infinitely many circular conductors arranged closely in two parallel line

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Abstract
In this paper, we consider very high concentration of electric field in between infinitely many circular perfect conductors arranged closely in two rows. In stiff fiber-reinforced composite, shear stress concentrations occur in between neighboring fibers, and the electric field means shear stress in this paper. Due to material failure of composites, there have been intensive studies so far to estimate the field in between only a finite number of inclusions. Indeed, fiber-reinforced composites contain a large number of stiff fibers, and the concentration can be strongly enhanced by some combinations of inclusions. Thus, we establish some asymptotes and optimal blow-up rates for the field in narrow regions in between infinitely many conductors in two rows to describe the effects combined horizontally and vertically by a large number of inclusions. Especially, the one of blow-up rates is substantially different from the existing result in the case of finite inclusions.

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1 Introduction
In stiff fiber-reinforced composites, high shear stress concentrations occur in between closely spaced neighboring fibers [8]. In the anti-plane shear model, the out-of-plane displacement \( u \) satisfies a conductivity equation whose inclusions in the plane are the cross-sections of fibers, and the gradient \( \nabla u \) implies the shear stress tensor. The problem to estimate \( \nabla u \) in between inclusions was raised by Babuška in the study of material failure of composites [4]. Many studies on the gradient estimate have been successfully carried out due to such practical significance [18, 17, 16, 7]. The genetic blow-up rate of \( |\nabla u| \) is \( \frac{1}{\sqrt{\tilde{\epsilon}}} \) for small \( \tilde{\epsilon} > 0 \) when \( \tilde{\epsilon} \) is the distance between two neighboring inclusions [3, 2, 22, 23], and moreover, asymptotic behavior of \( \nabla u \) was also established [1, 10, 11]. The two dimensional problem has been generalized in various ways including high dimensions [5, 6, 11, 12, 13, 14, 15, 16, 19, 21, 24]. Especially, it has been shown in [20] that
the concentration of $\nabla u$ can be strongly enhanced by a small inclusion between inclusions. This means that some combinations of inclusions can have strong influence on the concentration. So far, such studies have considered the cases when only a finite number of inclusions exist.

This paper is mainly concerned with the concentration of $\nabla u$ enhanced by a combination of infinitely many inclusions, because composites contain a large number of stiff fibers. Thus, we consider infinitely many circular inclusions arranged closely in two rows to describe the horizontal and vertical effects of infinitely many inclusions. According to our results, one of effects is very strong enough to provide the blow-up rate substantially different from the existing rate in the case of finite number of inclusions.

We set up infinitely many circular perfect conductors arranged closely in two rows. For any integer number $n$, we choose a pair of unit open disks $D_{Rn}$ and $D_{Ln}$ spaced $\epsilon$ apart in the horizontal direction, and moreover the distances between $D_{Rn}$ and $D_{R_{n+1}}$, and between $D_{Ln}$ and $D_{L_{n+1}}$ both are $\delta$ in the vertical direction. The open disks $D_{Rn}$ and $D_{Ln}$ are defined as

$$D_{Rn} = \left\{(x, y) \mid (x - (1 + \frac{\epsilon}{2}))^2 + (y - n(2 + \delta))^2 < 1 \right\}$$

and

$$D_{Ln} = \left\{(x, y) \mid (x - (-1 - \frac{\epsilon}{2}))^2 + (y - n(2 + \delta))^2 < 1 \right\}.$$

Then, the domain $\mathbb{R}^2 \setminus \bigcup_{n=\infty}^{\infty} (D_{Rn} \cup D_{Ln})$ has a periodic structure with period $2 + \delta$ in the $y$ direction. In this paper, we suppose that $\epsilon$ and $\delta$ are sufficiently small and positive.

Dealing with the governing equation, let the symbol $H$ denote a harmonic function defined in $\mathbb{R}^2$ whose gradient is a periodic function with period $2 + \delta$ in the $y$ direction satisfying

$$\nabla H(x, y) = \nabla H(x, y + 2 + \delta) \text{ for any } (x, y) \in \mathbb{R}^2. \tag{1.1}$$

For example, a linear function

$$H(x, y) = ax + by$$

can be a harmonic function with a periodic gradient described above. For a such harmonic function $H$, we estimate the gradient $\nabla u$ of a solution $u$ to the equation

$$\Delta u = 0 \text{ in } \mathbb{R}^2 \setminus \bigcup_{n=\infty}^{\infty} (D_{Ln} \cup D_{Rn}) \tag{1.2}$$

with the conditions

$$\begin{cases}
  u = c_L & \text{on } \partial D_{L0}, \\
  u = c_R & \text{on } \partial D_{R0}, \\
  \int_{\partial D_{L0}} \partial_p u ds = \int_{\partial D_{R0}} \partial_p u ds = 0, \\
  \int_{\Omega \setminus \bigcup D_{L0} \cup D_{R0}} |\nabla(u - H)|^2 dx dy < \infty, \\
  \nabla u(x, y) = \nabla u(x, y + 2 + \delta) & \text{for } (x, y) \in \mathbb{R}^2 \setminus \bigcup_{n=\infty}^{\infty} (D_{Ln} \cup D_{Rn}).
\end{cases} \tag{1.3}$$
Here, the constants $c_L$ and $c_R$ depend on $H$, $\epsilon$ and $\delta$, and the normal unit vector $\nu$ points toward the inside of $D_{L0}$ or $D_{R0}$. The domain $\Omega$ denotes a horizontal area as

$$\Omega = \mathbb{R} \times \left( -1 - \frac{\delta}{2}, 1 + \frac{\delta}{2} \right),$$  \hspace{1cm} (1.4)$$

and the domain $\Omega_m$ is defined as

$$\Omega_m = \{ (x, y) \mid (x, y) \in \Omega \text{ and } |x| < m \}$$  \hspace{1cm} (1.5)$$

for $m \geq 3$ containing $D_{L0}$ and $D_{R0}$.

By definition, the gradient $\nabla u$ is a periodic function with period $2+\delta$ in the $y$ direction, and the solution $u$ has a constant Dirichlet boundary data on each of boundaries $\partial D_{Rn}$ and $\partial D_{Ln}$ for $n = 0, \pm 1, \pm 2, \cdots$. The existence of the solution $u$ for a harmonic function $H$ can be shown by considering $u(x, y) + u(x, -y)$ and $u(x, y) - u(x, -y)$ in $\Omega \setminus (D_{L0} \cup D_{R0})$.

It is worth noting that if $u_\alpha$ and $u_\beta$ are the solutions for the same harmonic function $H$, then there is a constant $c$ such that $u_\alpha = u_\beta + c$ and $\nabla u_\alpha = \nabla u_\beta$ in $\mathbb{R}^2 \setminus \bigcup_{n=-\infty}^{n=\infty} (D_{Ln} \cup D_{Rn})$.

In this paper, we establish some asymptotes and optimal blow-up rates for $\nabla u$ in two kinds of narrow regions in between $D_{Ln}$ and $D_{Rn}$, and in between $D_{Rn}$ and $D_{Rn+1}$. Theorem 1.1 provides asymptotes with a coefficient and an upper bound of the coefficient, and moreover Theorem 1.3 presents a specific asymptote with a lower bound for the coefficient in the case of a linear function $H(x, y) = ax + by$ to get the optimality of the gradient estimates.

**Theorem 1.1** For any harmonic function $H$ with (1.1), let $u$ be a solution to (1.2) with the condition (1.3). Let $N_v$ be a narrow vertical region in between $D_{L0}$ and $D_{R0}$, and let $N_h$ be a narrow horizontal region in between $D_{R0}$ and $D_{R1}$, defined as

$$N_v = \left\{ (x, y) \mid |x| < 1 + \frac{\epsilon}{2} - \sqrt{1 - y^2} \text{ and } |y| < \frac{\sqrt{3}}{2} \right\}$$

and

$$N_h = \left\{ (x, y) \mid y - 1 - \frac{\delta}{2} < 1 + \frac{\delta}{2} - \sqrt{1 - \left(x - 1 - \frac{\epsilon}{2}\right)^2} \text{ and } |x - 1 - \frac{\epsilon}{2}| < \frac{\sqrt{3}}{2} \right\}.$$

Then, there exist constants $\mu$ and $\lambda$ such that

$$\nabla u(x, y) = \lambda \left( 1 \left( 0, 1 \right) + \frac{1}{\sqrt{\delta}} S(x, y)(1, 0) \right) + R_1(x, y) \quad \text{for any } (x, y) \in N_h,$$

$$\nabla u(x, y) = \mu \frac{\sqrt{\epsilon}}{\epsilon + y^2} (1, 0) + R_2(x, y) \quad \text{for any } (x, y) \in N_v.$$  \hspace{1cm} (1.6)

(1.7)
Here, the constants $\lambda$ and $\mu$ satisfy
\[
\lambda = H(0,1) - H(0,-1)
\]
and
\[
|\mu| \leq C\|H\|_{L^\infty(\Omega_4)},
\]
the function $S$ is defined as
\[
S(x, y) = -2\sqrt{\delta} \frac{(x - 1 - \frac{\epsilon}{2})(y - 1 - \frac{\delta}{2})}{\left((x - 1 - \frac{\epsilon}{2})^2 + \delta\right)^2}
\]
with
\[
\|S(x, y)\|_{L^\infty(N_h)} \leq 2 \text{ and } S\left(1 + \frac{\epsilon}{2}, 1 + \frac{\delta}{2}\right) = 0,
\]
and the remainder terms $R_1$ and $R_2$ are bounded as
\[
\|R_1\|_{L^\infty(N_h)} + \|R_2\|_{L^\infty(N_v)} \leq C\|H\|_{L^\infty(\Omega_4)}
\]
for a constant $C$ regardless of $\epsilon$ and $\delta$.

It is worth noting that the constant $\mu$ depends on $\delta$ and $\epsilon$ in this paper, even though it is bounded regardless of $\delta$ and $\epsilon$. The proof of Theorem 1.1 is given in Section 3, based on the results in Section 2.

**Remark 1.2** We consider the behavior of $\nabla u$ in the domain
\[
\left(-1 - \frac{\epsilon}{2} + \frac{\sqrt{3}}{2}, 1 + \frac{\epsilon}{2} - \frac{\sqrt{3}}{2}\right) \times \left(\frac{\sqrt{3}}{2}, 2 + \delta - \frac{\sqrt{3}}{2}\right)
\]
which doesn’t belong to $N_v$ and $N_h$. Theorem 1.1 provides the boundedness of $|\nabla u|$ on its rectangular boundary regardless of $\epsilon$ and $\delta$. By the maximum principle, $|\nabla u|$ is bounded in the domain regardless of $\epsilon$ and $\delta$. Combined with Theorem 1.1 again, an asymptote for $\nabla u$ in
\[
\left(\left(-\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}\right) \times \mathbb{R}\right) \setminus \bigcup_{n=-\infty}^{\infty} (D_{Ln} \cup D_{Rn})
\]
is also obtained, since the gradient $\nabla u$ is periodic with period $2 + \delta$ in the $y$ direction.

**Theorem 1.3** Let $N_v$, $N_h$ and $S$ be as given in Theorem 1.1. Assume that $H$ is a linear function given as
\[
H(x, y) = ax + by
\]
for any $(x, y) \in \mathbb{R}^2$ and $u$ is a solution to (1.2) with the condition (1.3) for $H$. Then,
\[
\nabla u(x, y) = 2b \left(\frac{1}{\delta + (x - 1 - \frac{\epsilon}{2})^2}(0,1) + \frac{1}{\sqrt{\delta}}S(x, y)(1,0)\right) + R_1(x, y) \text{ for any } (x, y) \in N_h,
\]
(1.8)
and there is a constant $\mu_0$ such that
\[ \nabla u(x, y) = a\mu_0 \frac{\sqrt{\epsilon}}{\epsilon + y^2} (1, 0) + R_2(x, y) \text{ for any } (x, y) \in N_0. \] (1.9)

Here, the constant $\mu_0$ satisfies
\[ \frac{1}{C} < \mu_0 < C \] (1.10)
and the remainder terms $R_1$ and $R_2$ are bounded as
\[ \|R_1\|_{L^\infty(N_h)} + \|R_2\|_{L^\infty(N_h)} \leq C(|a| + |b|) \]
for a constant $C$ regardless of $\epsilon$ and $\delta$.

The proof of Theorem 1.3 is presented in Section 4.

From now on, the symbols $C$ and $C_n$ denote the constants regardless of small $\epsilon > 0$ and $\delta > 0$ for $n = 1, 2, \cdots$.

**Remark 1.4** Theorems 1.1 and 1.3 provide the generic blow-up rates of $\nabla u$ in between $D_{Ln}$ and $D_{Rn}$, and in between $D_{Ln}$ and $D_{Ln+1}$ which are
\[ \frac{1}{\sqrt{\epsilon}} \text{ and } \frac{1}{\delta}, \]
respectively.

**Corollary 1.5** For a harmonic function $H$ with a periodic gradient as above, let $w$ be a solution to the equation
\[ \Delta w = 0 \text{ in } \mathbb{R}^2 \setminus \bigcup_{n=-\infty}^{\infty} D_{Rn} \]
with the conditions
\[
\begin{cases}
  w = c & \text{on } \partial D_{R0}, \\
  \int_{\partial D_{R0}} \partial_w w ds = 0, \\
  \int_{\Omega \setminus D_{R0}} |\nabla (w - H)|^2 dx dy < \infty, \\
  \nabla w(x, y) = \nabla w(x, y + 2 + \delta) & \text{for any } (x, y) \in \mathbb{R}^2 \setminus \bigcup_{n=-\infty}^{\infty} D_{Rn},
\end{cases}
\]
where $c$ is a constant depending on $H$, $\epsilon$. Then, there exist a constant $\lambda$ such that
\[ \nabla w(x, y) = \lambda \left( \frac{1}{\delta + (x - 1 - \frac{\epsilon}{2})^2} (0, 1) + \frac{1}{\sqrt{\delta}} S(x, y)(1, 0) \right) + R(x, y) \text{ for any } (x, y) \in N_h, \]
where the constant $\lambda = H(0, 1) - H(0, -1)$ and the remainder term $R$ is bounded as
\[ \|R\|_{L^\infty(N_h)} \leq C\|H\|_{L^\infty(\Omega_4)} \]
for a constant $C$ regardless of $\delta > 0$. 

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This paper is organized as follows: Section 2 provides the potential differences of $u$ between $\partial D_{R_n}$ and $\partial D_{L_n}$, and between $\partial D_{R_0}$ and $\partial D_{R_{n+1}}$. In Section 3, two asymptotes (1.6) and (1.7) for $\nabla u$ in Theorem 1.1 result from the potential differences. In Section 4, we establish more descriptive asymptotes of $\nabla u$ for $H(x, y) = ax + by$ to prove Theorem 1.3 and to get the optimality of the blow-up rates in Remark 1.4. The proof of Corollary 1.5 is left as an exercise for the reader, since it is much the same as the proof of (1.6) and Proposition 2.1.

2 Estimates for potential differences

The potential differences play important roles in establishing the asymptotes and estimates for the gradient $\nabla u$. Once the potential difference is estimated, the methods in [11, 10] are modified to obtain the asymptotes. This section thus provides the estimates for potential differences $u|_{D_{R_{n+1}}} - u|_{D_{R_n}}$, $u|_{D_{L_{n+1}}} - u|_{D_{L_n}}$, and $u|_{D_{R_n}} - u|_{D_{L_n}}$.

**Proposition 2.1** Let $u$ be a solution to (1.2) with the condition (1.3) for any harmonic function $H$ with a periodic gradient satisfying (1.1). Then, the potential differences between $\partial D_{L_n}$ and $\partial D_{L_{n+1}}$, and between $\partial D_{R_n}$ and $\partial D_{R_{n+1}}$ are obtained as

$$u|_{D_{R_{n+1}}} - u|_{D_{R_n}} = u|_{D_{L_{n+1}}} - u|_{D_{L_n}} = H \left( 0, 1 + \frac{\delta}{2} \right) - H \left( 0, -1 - \frac{\delta}{2} \right)$$

for any $n = 0, \pm 1, \pm 2, \cdots$.

**Proof.** We begin by proving that

$$(u - H)(x, y) - (u - H)(x, y - 2 - \delta) = 0$$

(2.1)

for any $(x, y) \in \mathbb{R}^2 \setminus \bigcup_{n=\infty}^{\infty} (D_{L_n} \cup D_{R_n})$. The gradient $\nabla (u - H)$ is a periodic function with period $2 + \delta$ in the $y$ direction as given in (1.1) and (1.3). Then, $\nabla (u - H)(x, y) - \nabla (u - H)(x, y - 2 - \delta) = (0, 0)$ for any $(x, y) \in \mathbb{R}^2 \setminus \bigcup_{n=\infty}^{\infty} (D_{L_n} \cup D_{R_n})$. There exists a constant $d > 0$ such that

$$(u - H)(x, y) - (u - H)(x, y - 2 - \delta) = d$$

for any $(x, y) \in \mathbb{R}^2 \setminus \bigcup_{n=\infty}^{\infty} (D_{L_n} \cup D_{R_n})$. By the Jensen’s inequality, every $x \in [3, \infty)$ has the upper bound for $|d|^2$ as

$$|d|^2 \leq \left( \int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} |\partial_y (u - H)(x, y)| dy \right)^2 \leq (2 + \delta) \int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} |\nabla (u - H)(x, y)|^2 \, dy.$$  

Since $\int_{\Omega \setminus (D_{L_n} \cup D_{R_n})} |\nabla (u - H)|^2 \, dx \, dy < \infty$, it follows from the Fubini’s theorem that

$$d = 0$$

implying (2.1).

The periodic property (1.1) implies that

$$H(x, y) - H(x, y - 2 - \delta) = H \left( 0, 1 + \frac{\delta}{2} \right) - H \left( 0, -1 - \frac{\delta}{2} \right)$$

(2.2)
for any \((x, y) \in \mathbb{R}^2\), since \(\nabla (H(x, y) - H(x, y - 2 - \delta)) = (0, 0)\). The equality (2.1) yields this proposition as follows:

\[
H \left(0, 1 + \frac{\delta}{2}\right) - H \left(0, -1 - \frac{\delta}{2}\right) = H(x, y) - H(x, y - 2 - \delta) \\
= H(x, y) - H(x, y - 2 - \delta) + ((u - H)(x, y) - (u - H)(x, y - 2 - \delta)) \\
= u(x, y) - u(x, y - 2 - \delta)
\]

for any \((x, y) \in \mathbb{R}^2 \setminus \bigcup_{n=-\infty}^{\infty} (D_{Ln} \cup D_{Rn})\). This implies that

\[
H \left(0, 1 + \frac{\delta}{2}\right) - H \left(0, -1 - \frac{\delta}{2}\right) = u|_{D_{Ln+1}} - u|_{D_{Ln}} = u|_{D_{Rn+1}} - u|_{D_{Rn}}.
\]

\[
\square
\]

**Proposition 2.2** Let \(u\) be the solution to (1.2) satisfying (1.3) for \(H(x, y) = x\) for any \((x, y) \in \mathbb{R}^2\). Then, the potential difference between \(\partial D_{Ln}\) and \(\partial D_{Rn}\) is estimated as

\[
\frac{1}{C}\sqrt{\epsilon} < u|_{D_{Rn}} - u|_{D_{Ln}} < C\sqrt{\epsilon},
\]

and there are no potential differences between \(\partial D_{Ln}\) and \(\partial D_{Ln+1}\), and between \(\partial D_{Rn}\) and \(\partial D_{Rn+1}\), i.e.,

\[
|u|_{D_{Ln+1}} - u|_{D_{Ln}} = u|_{D_{Rn+1}} - u|_{R_{Ln}} = 0
\]

for any \(n = 0, \pm 1, \pm 2, \cdots\).

The proof of the proposition is presented in Subsection 2.1.

The potential difference of \(u\) between \(\partial D_{L0}\) and \(\partial D_{R0}\) can be expressed as an integral containing \(\partial \nu \phi\) in Proposition 2.4 motivated by the method in [22, 23]. Following lemma is used to modify the idea for the proposition.

**Lemma 2.3** Let \(h\) be a harmonic function as

\[
\begin{cases}
\triangle h = 0 & \text{in } (4, \infty) \times \left(1 - \frac{\delta}{2}, 1 + \frac{\delta}{2}\right), \\
\partial_y h = 0 & \text{on } (4, \infty) \times \left\{1 - \frac{\delta}{2}, 1 + \frac{\delta}{2}\right\}, \\
\int_{(4,\infty)\times\left(1-\frac{\delta}{2},1+\frac{\delta}{2}\right)} |\nabla h|^2 dx dy < \infty.
\end{cases}
\]

Then,

\[
\sup_{y \in \left(1 - \frac{\delta}{2}, 1 + \frac{\delta}{2}\right)} |h(x, y)| = O(1) \quad \text{and} \quad \sup_{y \in \left(1 - \frac{\delta}{2}, 1 + \frac{\delta}{2}\right)} |\nabla h(x, y)| = O\left(\exp\left(-\frac{\pi}{2 + \delta} x\right)\right)
\]

as \(x \to \infty\).
Proof. The function $h$ can be express as

$$h(x, y) = \sum_{n=0}^{\infty} a_n \cos \left( \frac{n\pi}{2 + \delta} \left( y + 1 + \frac{\delta}{2} \right) \right) \exp \left( -\frac{n\pi}{2 + \delta} x \right).$$

The estimates can be obtained immediately. □

**Proposition 2.4** There exists the harmonic function $\phi$ defined in $\Omega \setminus (D_{L0} \cup D_{R0})$ with the following conditions as

$$\begin{cases}
\partial_\nu \phi = 0 & \text{on } \mathbb{R} \times \left\{ y \mid y = -1 - \frac{\delta}{2} \text{ or } y = 1 + \frac{\delta}{2} \right\}, \\
\phi = c_0 & \text{on } \partial D_{R0}, \\
\phi = -c_0 & \text{on } \partial D_{L0}, \\
\int_{\partial D_{R0}} \partial_\nu \phi \, ds = -\int_{\partial D_{L0}} \partial_\nu \phi \, ds = 1, \\
\int_{\Omega \setminus (D_{L0} \cup D_{R0})} |\nabla \phi|^2 \, dxdy < \infty,
\end{cases} \tag{2.3}$$

where $c_0$ is a proper constant depending on $\epsilon$ and $\delta$. If $u$ is a solution to (1.2) satisfying the condition (1.3) for any harmonic function $H$ with (1.1), then

$$u \big|_{\partial D_{R0}} - u \big|_{\partial D_{L0}} = \int_{\partial D_{L0} \cup \partial D_{R0}} H \partial_\nu \phi \, ds. \tag{2.4}$$

Proof. First, we prove the existence of $\phi$. By the Lax-Milgram theorem, there exists the harmonic function $\phi_0$ defined in $\Omega \setminus (D_{L0} \cup D_{R0})$ with conditions:

$$\begin{cases}
\partial_\nu \phi_0 = 0 & \text{on } \mathbb{R} \times \left\{ y \mid y = -1 - \frac{\delta}{2} \text{ and } y = 1 + \frac{\delta}{2} \right\}, \\
\phi_0 = 1 & \text{on } \partial D_{R0}, \\
\phi_0 = -1 & \text{on } \partial D_{L0}, \\
\int_{\Omega \setminus (D_{L0} \cup D_{R0})} |\nabla \phi_0|^2 \, dxdy < \infty.
\end{cases}$$

We can construct a bijective conformal mapping $\Phi : B_1(0, 0) \to \Omega$ such that

$$\begin{cases}
\Delta \phi_0(\Phi) = 0 & \text{in } B_1(0, 0) \setminus \Phi^{-1}(D_{L0} \cup D_{R0}), \\
\phi_0(\Phi) = 1 & \text{on } \partial \Phi^{-1}(D_{R0}), \\
\phi_0(\Phi) = -1 & \text{on } \partial \Phi^{-1}(D_{L0}), \\
\phi_0(\Phi) \text{ belongs to } H^1(B_1(0, 0) \setminus \Phi^{-1}(D_{L0} \cup D_{R0})),
\end{cases}$$

since

$$\int_{B_1(0, 0) \setminus \Phi^{-1}(D_{L0} \cup D_{R0})} |\nabla (\phi_0(\Phi))|^2 \, d\xi d\eta = \int_{\Omega \setminus (D_{L0} \cup D_{R0})} |\nabla \phi_0|^2 \, dxdy < \infty.$$
By the maximum principle, \( \phi_0 \) has the maximal value 1 on \( \partial D_{R0} \) and also has the minimal value \(-1\) on \( \partial D_{L0} \), since \( \phi_0(\Phi) \) is a harmonic function defined in a bounded domain. By the Hopf's lemma,

\[
\int_{\partial D_{R0}} \partial_\nu \phi_0 ds = - \int_{\partial D_{L0}} \partial_\nu \phi_0 ds > 0,
\]

since the normal vector \( \nu \) points toward the inside of \( D_{L0} \) or \( D_{R0} \). Then,

\[
\phi = \frac{1}{\int_{\partial D_{R0}} \partial_\nu \phi_0 ds} \phi_0,
\]

which means the existence of \( \phi \). In addition, it can be easily shown that

\[
\phi(x, y) = \phi(x, -y)
\]

(2.5)

for \((x, y) \in \Omega \setminus \overline{D_{L0} \cup D_{R0}}\).

Second, we prove the equality (2.4). From definition, \( u \) is constant on each of boundaries \( \partial D_{R0} \) and \( \partial D_{L0} \), and

\[
\int_{\partial D_{R0}} \partial_\nu \phi ds = - \int_{\partial D_{L0}} \partial_\nu \phi ds = 1.
\]

Thus,

\[
\left| u \right|_{\partial D_{R0}} - \left| u \right|_{\partial D_{L0}} = \int_{\partial D_{L0} \cup \partial D_{R0}} u \partial_\nu \phi ds
\]

\[
= \int_{\partial D_{L0} \cup \partial D_{R0}} H \partial_\nu \phi ds + \int_{\partial D_{L0} \cup \partial D_{R0}} (u - H) \partial_\nu \phi ds.
\]

We shall use the divergence theorem to prove that

\[
\int_{\partial D_{L0} \cup \partial D_{R0}} (u - H) \partial_\nu \phi = 0.
\]

This immediately results in the desirable equality (2.4). To use the divergence theorem, we define \( \tilde{u} \) and \( \tilde{H} \) as even functions with respect to \( y \) as \( \tilde{u}(x, y) = \frac{1}{2} (u(x, y) + u(x, -y)) \) and \( \tilde{H}(x, y) = \frac{1}{2} (H(x, y) + H(x, -y)) \). By the periodic property of \( \nabla u \), \( \tilde{u} - \tilde{H} \) has zero Neumann data on two horizontal boundaries \( \partial \Omega \) so that

\[
\partial_y \left( \tilde{u} - \tilde{H} \right) \left( x, 1 + \frac{\epsilon}{2} \right) = \partial_y \left( \tilde{u} - \tilde{H} \right) \left( x, -1 - \frac{\epsilon}{2} \right) = 0
\]

for any \( x \in \mathbb{R} \). From definition of \( u \),

\[
\int_{\Omega \setminus \overline{D_{L0} \cup D_{R0}}} \left| \nabla \left( \tilde{u} - \tilde{H} \right) \right|^2 dxdy < \infty.
\]

By Lemma 2.3

\[
\tilde{u}(x, y) - \tilde{H}(x, y) = O(1) \text{ and } \nabla \left( \tilde{u}(x, y) - \tilde{H}(x, y) \right) = O \left( \exp -|x| \right)
\]

as \( |x| \to \infty \), and \( \phi \) and \( \nabla \phi \) also show the same behaviors as \( |x| \to \infty \). Thus, we can use
the divergence theorem so that by (2.5),
\[
\int_{\partial D_L \cup \partial D_R} (u - H) \partial_{\nu} \phi ds = \int_{\partial D_L \cup \partial D_R} (\tilde{u} - \tilde{H}) \partial_{\nu} \phi ds = \int_{\partial (\Omega \setminus D_L \cup D_R)} \phi \partial_{\nu}(\tilde{u} - \tilde{H}) ds = \phi|_{\partial D_L} \int_{\partial D_L} \partial_{\nu}(\tilde{u} - \tilde{H}) ds + \phi|_{\partial D_R} \int_{\partial D_R} \partial_{\nu}(\tilde{u} - \tilde{H}) ds = 0.
\]
Thus, we have done it. \(\square\)

2.1 Proof of Proposition 2.2

In this subsection, we suppose that \(H(x, y) = x\) for any \((x, y) \in \mathbb{R}^2\). The function \(u\) is the solution to (1.2) satisfying (1.3) for \(H\).

The integral equation (2.4) is mainly used to estimate the potential difference of \(u\) between \(\partial D_L\) and \(\partial D_R\). In (2.12), the function \(\phi\) is construct by a series of \(\phi_n\) whose property has been well known, and which is also given explicitly as in (2.10).

First, some maximum principles related to \(\phi\) are considered in Lemmas 2.5 and 2.7 before constructing \(\phi\). Let \(\Omega_R\) be the right-hand side of \(\Omega\) as
\[
\Omega_R = (0, \infty) \times \left( -1 - \frac{\delta}{2}, 1 + \frac{\delta}{2} \right).
\]

Lemma 2.5 There exists a harmonic function \(\phi_R\) defined on \(\Omega_R \setminus \overline{D_R}\) with the conditions
\[
\begin{align*}
\partial_{\nu} \phi_R &= 0 \quad \text{on } (0, \infty) \times \left\{ y \mid y = -\frac{\delta}{2} \text{ or } y = 1 + \frac{\delta}{2} \right\}, \\
\phi_R &= 0 \quad \text{on } \{0\} \times \left( -1 - \frac{\delta}{2}, 1 + \frac{\delta}{2} \right), \\
\phi_R &= 1 \quad \text{on } \partial D_R, \\
\int_{\Omega_R \setminus D_R} |\nabla \phi_R|^2 dxdy &< \infty.
\end{align*}
\]

The function \(\phi_R\) has the extreme values only on the boundary so that
\[
0 < \phi_R < 1 \text{ in } \Omega_R \setminus \overline{D_R}.
\]

Remark 2.6 It is obvious that
\[
\phi_R = \frac{1}{c_0} \phi
\]
in \(\Omega_R \setminus \overline{D_R}\), where the constant \(c_0\) is given in Proposition 2.4.
By the Lax-Milgram theorem, there exists the unique harmonic function \( \phi_R \) defined on \( \Omega_R \setminus \overline{D_{R_0}} \) with the boundary condition (2.7). As mentioned in the remark above, the existence of \( \phi_R \) results immediately from \( \phi \) given in Proposition 2.4 due to \( \phi_R = \frac{1}{c_0} \phi \).

We use a conformal map to prove that \( 0 < \phi_R < 1 \) in \( \Omega_R \setminus \overline{D_{R_0}} \).

There exists a bijective conformal mapping \( \Phi_R : B_1^+(0, 0) \to \Omega_R \) such that
\[
\begin{align*}
\Delta \phi_R(\Phi_R) &= 0 \quad \text{in } B_1^+(0, 0) \setminus \overline{\Phi_R^{-1}(D_{R_0})}, \\
\phi_R(\Phi_R) &= 1 \quad \text{on } \partial \Phi_R^{-1}(D_{R_0}), \\
\phi_R(\Phi_R) &= 0 \quad \text{on } \xi^2 + \eta^2 = 1 \text{ and } \xi > 0, \\
\partial_\nu (\phi_R(\Phi_R)) (0, \eta) &= 0 \quad \text{on } |\eta| < 1, \\
\phi_R(\Phi_R) &\text{ belongs to } H^1(B_1(0, 0) \setminus \overline{\Phi_R^{-1}(D_{R_0})}).
\end{align*}
\]

By the maximal principle, \( 0 < \phi_R(\Phi_R) < 1 \) in the bounded domain \( B_1^+(0, 0) \setminus \overline{\Phi_R^{-1}(D_{R_0})} \).

This implies that
\[
0 < \phi_R < 1 \quad \text{in } \Omega_R \setminus \overline{D_{R_0}}.
\]

The following lemma is derived easily by an argument, analogous to Lemma 2.5 where the same function \( \Phi_R : B_1^+(0, 0) \to \Omega_R \) is used.

Lemma 2.7 Let \( \rho \) be a harmonic function defined in \( \Omega_R \setminus \overline{D_{R_0}} \) with the boundary conditions:
\[
\begin{align*}
\partial_\nu \rho &= 0 \quad \text{on } (0, \infty) \times \{ y \mid y = -1 - \frac{\delta}{2} \text{ or } y = 1 + \frac{\delta}{2} \}, \\
\rho &= 0 \quad \text{on } \{ x \mid x = 0 \} \times (-1 - \frac{\delta}{2}, 1 + \frac{\delta}{2}), \\
\rho &> 0 \quad \text{on } \partial D_{R_0},
\end{align*}
\]
and also satisfying
\[
\int_{\Omega_R \setminus \overline{D_{R_0}}} |\nabla \rho|^2 \, dx dy < \infty.
\]

Then,
\[
0 < \rho \quad \text{in } \Omega_R \setminus \overline{D_{R_0}}.
\]

Second, we use a series of functions \( \phi_n \) to express \( \phi \), given in Proposition 2.4. Here,
the harmonic function $\varphi_n$ satisfies

$$
\begin{cases}
\Delta \varphi_n = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D_{L_n} \cup D_{R_n}} \\
\varphi_n = \text{a constant} & \text{on } \partial D_{L_n}, \\
\varphi_n = -\varphi_n \big|_{\partial D_{L_n}} & \text{on } \partial D_{R_n}, \\
\int_{\partial D_{R_n}} \partial_\nu \varphi_n \, ds = -\int_{\partial D_{L_n}} \partial_\nu \varphi_n \, ds = 1, \\
\varphi_n(x) = O \left( \frac{1}{|x|} \right) & \text{as } x \to \infty
\end{cases}
$$

(2.9)

for any integer $n$. Then, the function $\varphi_n$ is expressed explicitly as

$$
\varphi_n = \frac{1}{2\pi} \left( \log |x - (p, n(2 + \delta))| - \log |x - (p, n(2 + \delta))| \right)
$$

(2.10)

where

$$
p = \sqrt{\epsilon} + O(\epsilon)
$$

for small $\epsilon > 0$. Refer to [11, 19] for details. We define the sum $\tilde{\varphi}$ of the series of $\varphi_n$ in the manner as

$$
\tilde{\varphi} = \lim_{N \to \infty} \sum_{n=-N}^{N} \varphi_n.
$$

The function $\tilde{\varphi}$ is well defined in $\Omega \setminus \overline{D_{L_0} \cup D_{R_0}}$ by the help of a neutralization reaction between $\varphi_n$ and $\varphi_{-n}$, and satisfies

$$
\begin{cases}
\Delta \tilde{\varphi} = 0 & \text{in } \Omega \setminus \overline{D_{L_0} \cup D_{R_0}}, \\
\partial_\nu \tilde{\varphi} = 0 & \text{on } \partial \Omega = \mathbb{R} \times \left\{ y \mid y = -1 - \frac{\delta}{2} \text{ or } y = 1 + \frac{\delta}{2} \right\}, \\
\int_{\partial D_{R_0}} \partial_\nu \tilde{\varphi} \, ds = -\int_{\partial D_{L_0}} \partial_\nu \tilde{\varphi} \, ds = 1, \\
\int_{\Omega \setminus \overline{D_{L_0} \cup D_{R_0}}} |\nabla \tilde{\varphi}|^2 \, dxdy < \infty.
\end{cases}
$$

(2.11)

Here, $\tilde{\varphi}$ is not constant on each of $\partial D_{L_0}$ and $\partial D_{R_0}$, and $\Omega$ is as given at (1.4). There exists a harmonic function $\tilde{v}$ defined in $\Omega \setminus \overline{D_{L_0} \cup D_{R_0}}$ with conditions:

$$
\begin{cases}
\partial_\nu \tilde{v} = 0 & \text{on } \partial \Omega = \mathbb{R} \times \left\{ y \mid y = -1 - \frac{\delta}{2} \text{ or } y = 1 + \frac{\delta}{2} \right\}, \\
\tilde{\varphi} + \tilde{v} = \text{a constant } \tilde{c} & \text{on } \partial D_{R_0}, \\
\tilde{\varphi} + \tilde{v} = -\tilde{c} & \text{on } \partial D_{L_0}, \\
\int_{\partial D_{R_0}} \partial_\nu \tilde{v} \, ds = \int_{\partial D_{L_0}} \partial_\nu \tilde{v} \, ds = 0, \\
\int_{\Omega \setminus \overline{D_{L_0} \cup D_{R_0}}} |\nabla \tilde{v}|^2 \, dxdy < \infty
\end{cases}
$$

(2.11)
for a proper constant $\tilde{c}$. The existence of $\tilde{v}$ is derived in the same way as $u$ to \cite{1,2} for a given $H$ in the introduction. In another way, the existence of $\tilde{v}$ is also derived from the existence of $\phi$ shown in Proposition \cite{2,4} since $\tilde{\phi} + \tilde{v}$ satisfies all conditions of $\phi$ so that

$$\phi = \tilde{\phi} + \tilde{v}.$$  \hfill (2.12)

The function $\phi$ can be decomposed into three functions as

$$\phi = \alpha \phi + (\tilde{\phi} - \alpha \phi) + \tilde{v},$$

where the positive constant $\alpha$ is defined as

$$\alpha = \frac{\tilde{\phi}(\frac{\epsilon}{2},0) - \tilde{\phi}(-\frac{\epsilon}{2},0)}{\phi|_{\partial D_{R_0}} - \phi|_{\partial D_{L_0}}}.$$

**Lemma 2.8** There is a constant $C$ such that

$$0 < 1 - \alpha \leq C \sqrt{\epsilon}$$

for small $\epsilon > 0$.

**Proof.** First, we show that

$$0 < (\tilde{\phi} - \alpha \phi + \tilde{v})|_{\partial D_{R_0}} - (\tilde{\phi} - \alpha \phi + \tilde{v})|_{\partial D_{L_0}}$$

$$= \int_{\partial D_{L_0} \cup \partial D_{R_0}} (\tilde{\phi} - \alpha \phi) \partial_n \phi \, ds$$

$$\leq C_1 \sqrt{\epsilon} \int_{\partial D_{L_0} \cup \partial D_{R_0}} x \partial_n \phi \, ds.$$  \hfill (2.13)

for a positive constant $C_1$. In the same way as Proposition \cite{2,4} the integration by parts yields

$$\int_{\partial D_{L_0} \cup \partial D_{R_0}} \tilde{v} \partial_n \phi ds = \int_{\partial D_{L_0} \cup \partial D_{R_0}} \phi \partial_n \tilde{v} ds$$

$$= \left( \phi \right|_{\partial D_{L_0}} \int_{\partial D_{L_0}} \partial_n \tilde{v} ds + \left( \phi \right|_{\partial D_{R_0}} \int_{\partial D_{R_0}} \partial_n \tilde{v} ds = 0.$$

We thus have the equality

$$\left( \tilde{\phi} - \alpha \phi + \tilde{v} \right)|_{\partial D_{R_0}} - \left( \tilde{\phi} - \alpha \phi + \tilde{v} \right)|_{\partial D_{L_0}} = \int_{\partial D_{L_0} \cup \partial D_{R_0}} (\tilde{\phi} - \alpha \phi + \tilde{v}) \partial_n \phi \, ds$$

$$= \int_{\partial D_{L_0} \cup \partial D_{R_0}} (\tilde{\phi} - \alpha \phi) \partial_n \phi \, ds.$$
The integral can be decomposed into two terms as
\[
\phi \quad \text{and} \quad \phi
\]
By Lemma 2.3, the boundedness of \( \epsilon \) and \( \tilde{\epsilon} \) implies (2.13).
Second, the positivity of \( 1 - \alpha \) is derived simply from (2.13). We note that
\[
(\tilde{\phi} - \alpha \phi + \bar{v}) \bigg|_{\partial D_{R0}} = - (\tilde{\phi} - \alpha \phi + \bar{v}) \bigg|_{\partial D_{L0}} = (1 - \alpha) \phi \bigg|_{\partial D_{R0}}
\]
and \( \phi \bigg|_{\partial D_{R0}} > 0 \). It follows immediately from (2.13) that
\[
1 - \alpha > 0.
\]
Third, we consider \( \int_{\partial D_{L0} \cup \partial D_{R0}} x \partial_\nu \phi \, ds \) to estimate \( 1 - \alpha \). The inequality (2.13) implies
\[
\int_{\partial D_{L0} \cup \partial D_{R0}} x \partial_\nu \phi \, ds > 0.
\]
The integral can be decomposed into two terms as
\[
\int_{\partial D_{L0} \cup \partial D_{R0}} x \partial_\nu \phi \, ds = \alpha \int_{\partial D_{L0} \cup \partial D_{R0}} x \partial_\nu \phi \, ds + \int_{\partial D_{L0} \cup \partial D_{R0}} x \partial_\nu \left( \tilde{\phi} - \alpha \phi + \bar{v} \right) \, ds.
\]
By Lemma 2.3, the boundedness of \( \int_{\Omega \cup D_{L0} \cup D_{R0}} \nabla \left( \tilde{\phi} - \alpha \phi + \bar{v} \right)^2 \, dxdy \) implies the existence of a constant \( c_1 \) such that \( \left( \tilde{\phi} - \alpha \phi + \bar{v} \right) (x, y) \) converges to a constant \( c_1 \), or \(-c_1 \), as \( x \) approaches \( \infty \) or \(-\infty \), respectively, and also show that \( \partial_x \left( \tilde{\phi} - \alpha \phi + \bar{v} \right) (x, y) \) shrinks exponentially fast to 0, as \( |x| \) approaches \( \infty \). The integration by parts yields
\[
\int_{\partial D_{L0} \cup \partial D_{R0}} x \partial_\nu \left( \tilde{\phi} - \alpha \phi + \bar{v} \right) \, ds
\]
\[
= \lim_{t \to \infty} \int_{\partial D_{L0} \cup \partial D_{R0} \cup \{x|x=\pm t\} \times (-1-\delta/2, 1+\delta/2)} \partial_x \left( \tilde{\phi} - \alpha \phi + \bar{v} \right) \, ds
\]
\[
= \lim_{t \to \infty} \int_{\{t\} \times (-1-\delta/2, 1+\delta/2)} \tilde{\phi} - \alpha \phi + \bar{v} \, ds - \int_{\{-t\} \times (-1-\delta/2, 1+\delta/2)} \tilde{\phi} - \alpha \phi + \bar{v} \, ds,
\]
since \( \tilde{\phi} - \alpha \phi + \bar{v} \) is constant on \( \partial D_{L0} \) and on \( \partial D_{R0} \), respectively. Note that \( \tilde{\phi} - \alpha \phi + \bar{v} = (1 - \alpha) \phi \), \( (1 - \alpha) > 0 \), \( \phi(x, y) = -\phi(-x, y) \), and \( \phi = \frac{1}{c_0} \phi_R \) for \( (x, y) \in \Omega_R \setminus \overline{D_{R0}} \), where
φ_R and c_0 > 0 are defined in Lemma 2.3 and Proposition 2.4 respectively. The maximum principle in Lemmas 2.5 and (2.13) yield
\[
\left| \int_{\partial D_{L_0} \cup \partial D_{R_0}} x \partial_\nu \left( \tilde{\phi} - \alpha \phi + \tilde{v} \right) \, ds \right|
\]
\[
= 2 \lim_{l \to \infty} \left| \int_{\{l\} \times (-1-\delta/2, 1+\delta/2)} \tilde{\phi} - \alpha \phi + \tilde{v} \, ds \right|
\]
\[
\leq (2 + \delta) \left( \int_{\partial D_{R_0}} \left( \tilde{\phi} - \alpha \phi + \tilde{v} \right) - \left( \tilde{\phi} - \alpha \phi + \tilde{v} \right) \right) \, ds
\]
\[
\leq 3 \int_{\partial D_{L_0} \cup \partial D_{R_0}} \left( \tilde{\phi} - \alpha \phi \right) \partial_\nu \phi \, ds
\]
\[
\leq 3C_2 \sqrt{\epsilon} \int x \partial_\nu \phi \, ds,
\]
since \( \delta < 1 \). Applying this bound to the decomposition (2.14),
\[
0 < (1 - \alpha) \int_{\partial D_{L_0} \cup \partial D_{R_0}} x \partial_\nu \phi \, ds \leq 3C_2 \sqrt{\epsilon} \int_{\partial D_{L_0} \cup \partial D_{R_0}} x \partial_\nu \phi \, ds.
\]
Since \( \int_{\partial D_{L_0} \cup \partial D_{R_0}} x \partial_\nu \phi \, ds > 0 \), we are done.

Now, we take the last step to prove Proposition 2.2. Calculating \( \tilde{\phi}(x, y) \) directly from (2.10), there exists a positive constant \( C_* \) regardless of \( \epsilon \) and \( \delta \) such that
\[
\frac{1}{C_*} \sqrt{\epsilon} \leq \tilde{\phi}(x, y) \leq C_* \sqrt{\epsilon}
\]
for any \((x, y) \in \partial D_{R_0}\) containing \((\frac{\delta}{2}, 0)\). The definition and symmetric property imply \( \alpha = \frac{\tilde{\phi}(\frac{\delta}{2}, 0)}{\phi_{\partial D_{R_0}}} \), and Lemma 2.3 yields \( \frac{1}{2} \leq \alpha \leq 2 \). Thus, there is a constant \( C_{**} > 0 \) regardless of \( \epsilon \) and \( \delta \) such that
\[
\frac{1}{C_{**}} \tilde{\phi} \leq \phi \leq C_{**} \tilde{\phi} \quad \text{on} \ \partial D_{R_0}.
\]
By Lemma 2.7, this inequality on the bounday can be extended into \( \Omega_R \setminus \overline{D_{R_0}} \) so that
\[
\frac{1}{C_{**}} \tilde{\phi} \leq \phi \leq C_{**} \tilde{\phi} \quad \text{in} \ \Omega_R \setminus \overline{D_{R_0}}.
\]
By the divergence theorem,
\[
\int_{\partial D_{R_0}} x \partial_\nu \phi \, ds = \lim_{l \to \infty} \int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} \partial_y x\phi(l, y) \, dy = \lim_{l \to \infty} \int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} \phi(l, y) \, dy
\]
whose value is intermediate between
\[
\frac{1}{C_{**}} \lim_{l \to \infty} \int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} \tilde{\phi}(l, y) \, dy \quad \text{and} \quad C_{**} \lim_{l \to \infty} \int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} \tilde{\phi}(l, y) \, dy.
\]
By Lemma 2.7, 
\[ \frac{1}{C_{*}^{**}} (2 + \delta) \inf_{(x,y) \in \partial D_{R_0}} \tilde{\phi}(x,y) \leq \int_{\partial D_{R_0}} x \partial_{x} \phi ds \leq C_{*}^{**} (2 + \delta) \sup_{(x,y) \in \partial D_{R_0}} \tilde{\phi}(x,y). \]

Thus, 
\[ (2 + \delta) \frac{1}{C_{*} C_{*}^{**} \sqrt{\epsilon}} \leq \int_{\partial D_{R_0}} x \partial_{x} \phi ds \leq (2 + \delta) C_{*} C_{*}^{**} \sqrt{\epsilon}. \]

By the symmetric property, 
\[ \int_{\partial D_{L_0}} x \partial_{x} \phi ds = \int_{\partial D_{R_0}} x \partial_{x} \phi ds. \]
Therefore, the equality (2.4) implies the desirable result in Proposition 2.2. \( \square \)

3 Proof of Theorem 1.1

We begin by defining the domains as 
\[ \tilde{\Omega} = \mathbb{R} \times (-1 - \frac{1}{2} \delta, 3 + \frac{3}{2} \delta), \quad \tilde{\Omega}_4 = (-4, 4) \times (-1 - \frac{1}{2} \delta, 3 + \frac{3}{2} \delta), \quad \tilde{\Omega}_{4L} = (0, 4) \times (-1 - \frac{1}{2} \delta, 3 + \frac{3}{2} \delta) \]
which are used in this proofs. We assume that \( H \) is a harmonic function with a periodic gradient as (1.1) and \( u \) is a solution to (1.2) with the condition (1.3).

Lemma 3.1 There exists a constant \( C \) regardless of \( \delta \) and \( \epsilon \) such that 
\[ |u|_{\partial D_{R_0}} - |u|_{\partial D_{L_0}}| = |u|_{\partial D_{R_1}} - u|_{\partial D_{R_1}}| \leq C \| H \|_{L^\infty(\Omega_4)} \sqrt{\epsilon}. \]

Proof. Let \( \tilde{u} \) and \( r \) be defined by 
\[ \tilde{u}(x,y) = \frac{1}{2} (u(x,y) + u(x,-y)) \quad \text{and} \quad r(x,y) = \frac{1}{2} (u(x,y) - u(x,-y)) \]
and let 
\[ \tilde{H}(x,y) = \frac{1}{2} (H(x,y) + H(x,-y)). \]

Since \( u = \tilde{u} + r \) and \( r|_{\partial D_{L_0}} = r|_{\partial D_{R_0}} = 0 \), Proposition 2.4 implies 
\[ u|_{\partial D_{R_0}} - u|_{\partial D_{L_0}} = \tilde{u}|_{\partial D_{R_0}} - \tilde{u}|_{\partial D_{L_0}} = \int_{\partial D_{L_0} \cup \partial D_{R_0}} \tilde{H} \partial_{x} \phi ds = \int_{\partial D_{L_0} \cup \partial D_{R_0}} \left( \tilde{H}(x,y) - \tilde{H}(0,0) \right) \partial_{x} \phi ds. \]

It follows from \( \partial_y \tilde{H}(0,0) = 0 \) that 
\[ |\tilde{H}(x,y) - \tilde{H}(0,0)| \leq C \left( \| \nabla H \|_{L^\infty(\Omega_4)} + \| D^2 H \|_{L^\infty(\Omega_4)} \right) |x| \]
for any \((x, y) \in \partial D_{L0} \cup \partial D_{R0}\). Thus,

\[
|u|_{\partial D_{R0}} - |u|_{\partial D_{L0}} \leq C \left( \|\nabla H\|_{L^\infty(\Omega_3)} + \|D^2 H\|_{L^\infty(\Omega_3)} \right) \int_{\partial D_{L0} \cup \partial D_{R0}} x \partial \nu \varphi ds.
\]

Here, the standard gradient estimate for harmonic functions implies that

\[
\|\nabla H\|_{L^\infty(\Omega_3)} + \|D^2 H\|_{L^\infty(\Omega_3)} \leq C \|H\|_{L^\infty((-4,4) \times (-3,3))}
\]

for some \(C > 0\), since the domain \(\Omega_3\) has nonzero distance from \(\partial ((-4,4) \times (-3,3))\), and the periodic property of \(\nabla H\) or (2.2) imply

\[
\|H\|_{L^\infty((-4,4) \times (-3,3))} \leq 2 \|H\|_{L^\infty(\Omega_4)}.
\]

Thus, we are done. \(\square\)

The following lemma provides some maximal principles more general than Lemma 2.7.

**Lemma 3.2** Let \(\Omega_R\) be as defined in the proof of Proposition 2.2. Assume that \(\rho_{00}, \rho_{10}, \rho_{01}\) and \(\rho_{11}\) are harmonic functions defined on \(\Omega_R \setminus \overline{D_{R0}}\) with the boundary conditions:

\[
\begin{align*}
&\partial^i \nu \rho_{ij} = 0 \quad \text{on } (0, \infty) \times \left\{ y \bigg| y = -1 - \frac{\delta}{2} \text{ or } y = 1 + \frac{\delta}{2} \right\}, \\
&\partial^j \nu \rho_{ij} = 0 \quad \text{on } \{0\} \times (-1 - \frac{\delta}{2}, 1 + \frac{\delta}{2}),
\end{align*}
\]

and also satisfying

\[
\int_{\Omega_R \setminus \overline{D_{R0}}} |\nabla \rho_{ij}|^2 dxdy < \infty
\]

for any \(i, j = 0, 1\), where \(\partial^i u = \partial^i \nu u\) and \(\partial^j u = \partial^j \nu u\). Then,

\[
\|\rho_{ij}\|_{L^\infty(\Omega_R \setminus \overline{D_{R0}})} \leq \|\rho_{ij}\|_{L^\infty(\partial D_{R0})}.
\]

This lemma can also be derived easily by the same function \(\Phi_R : B^+_1(0,0) \to \Omega_R\) used in Lemmas 2.5 and 2.7, the maximum principle and the Hopf Lemma. The proof of this lemma is left as an exercise for the reader.

**Lemma 3.3** Let \(a_\ast\) be the constant defined as

\[
a_\ast = u \left( -\frac{\epsilon}{2}, 0 \right) - H \left( -\frac{\epsilon}{2}, 0 \right) = u|_{\partial D_{L0}} - H \left( -\frac{\epsilon}{2}, 0 \right).
\]

Then,

\[
\|u - H - a_\ast\|_{L^\infty(\overline{\Omega_4})} \leq C \|H\|_{L^\infty(\Omega_4)}
\]

and

\[
\|u - u|_{\partial D_{L0}}\|_{L^\infty(\overline{\Omega_4})} \leq C \|H\|_{L^\infty(\Omega_4)}.
\]
Proof. We define the notations \((v)_e\) and \((v)_o\) as follows:

\[
(v)_e(x, y) = \frac{1}{2} (v(x, y) + v(x, -y))
\]

and

\[
(v)_o(x, y) = \frac{1}{2} (v(x, y) - v(x, -y))
\]

for a function \(v\) defined in \(\overline{\Omega} \setminus (D_{L0} \cup D_{R0})\). Then,

\[
u - H = (u - H)_e + (u - H)_o. 
\]

First, we estimate \(\|(u - H)_e - a_s\|_{L^\infty(\Omega \setminus (D_{L0} \cup D_{R0}))}\). For any \((x, y) \in \partial D_{L0}\),

\[
(u - H)_e(x, y) - a_s
\]

\[
= (u - H)_e(x, y) - u|_{\partial D_{L0}} + H \left( -\frac{\epsilon}{2}, 0 \right)
\]

\[
= (u)_e(x, y) - u|_{\partial D_{L0}} - (H)_e(x, y) + (H)_e \left( -\frac{\epsilon}{2}, 0 \right)
\]

\[
= - (H)_e(x, y) + (H)_e \left( -\frac{\epsilon}{2}, 0 \right),
\]

and for any \((x, y) \in \partial D_{R0}\),

\[
(u - H)_e(x, y) - a_s
\]

\[
= (u - H)_e(x, y) - u|_{\partial D_{R0}} - (H)_e(x, y) + (H)_e \left( -\frac{\epsilon}{2}, 0 \right)
\]

Thus, the equality (2.4) implies that

\[
\|(u - H)_e - a_s\|_{L^\infty(\partial D_{L0} \cup \partial D_{R0})} \leq 4\|H\|_{L^\infty(\partial D_{L0} \cup \partial D_{R0})}. 
\]  \(\text{(3.1)}\)

Note that \((u - H)_e - a_s\) is a harmonic function in \(\Omega \setminus D_{L0} \cup D_{R0}\) with

\[
\begin{aligned}
\partial_y (u - H)_e - a_s &= 0 &\quad \text{on } (-\infty, \infty) \times \left\{ y \mid y = -1 - \frac{\delta}{2} \text{ or } y = 1 + \frac{\delta}{2} \right\}, \\
\int_{\Omega \setminus (D_{L0} \cup D_{R0})} |\nabla ((u - H)_e - a_s)|^2 \; dx \, dy &< \infty
\end{aligned}
\]

due to a periodic property of \(\nabla u\) and \(\nabla H\). Applying \((u - H)_e - a_s\) to \(\rho_{10} + \rho_{11}\) in Lemma 3.2 the bound (3.1) implies

\[
\|(u - H)_e - a_s\|_{L^\infty(\Omega \setminus (D_{L0} \cup D_{R0}))} \leq 4\|H\|_{L^\infty(\partial D_{L0} \cup \partial D_{R0})}. 
\]

Second, we estimate \(\|(u - H)_o\|_{L^\infty(\Omega \setminus (D_{L0} \cup D_{R0}))}\). On \(\partial D_{L0} \cup \partial D_{R0}\),

\[
(u - H)_o = -(H)_o,
\]

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since \(u\) is constant on \(\partial D_{L_0}\) and \(\partial D_{R_0}\), respectively. Meanwhile, the equality (2.1) in the proof of Proposition 2.1 means that the harmonic function \((u-H)_o\) satisfies
\[(u-H)_o(x,y) = 0\] on \((-\infty, \infty) \times \left\{ y \mid y = -\frac{\delta}{2}\right\}.
\]
Since \((u-H)_o = -(H)_o\) on \(\partial D_{L_0} \cup \partial D_{R_0}\) and \(\int_{\Omega(H_r)} (\nabla((u-H)_o - a^*_o))^2 \, dx \, dy < \infty\), the results on \(\rho_{01}\) and \(\rho_{01}\) in Lemma 3.2 yield
\[
\|u - H - a^*_o\|_{L^\infty(\Omega^c, D_{L_0} \cup D_{R_0})} \leq 5 \|H\|_{L^\infty(\partial D_{L_0} \cup \partial D_{R_0})}.
\]
Combining the first and second cases,
\[
\|u - H - a^*_o\|_{L^\infty(\Omega^c, D_{L_0} \cup D_{R_0})} \leq 5 \|H\|_{L^\infty(\partial D_{L_0} \cup \partial D_{R_0})}.
\]
The first inequality in this lemma can be derived by (2.1) as follows:
\[
\|u - H - a^*_o\|_{L^\infty(\Omega^c, D_{L_0} \cup D_{R_0} \cup D_{R_1})} = \|u - H - a^*_o\|_{L^\infty(\Omega^c, D_{L_0} \cup D_{R_0})} \leq 5 \|H\|_{L^\infty(\partial D_{L_0} \cup \partial D_{R_0})}.
\]
The second inequality also follows immediately so that
\[
\|u - u|_{\partial D_{L_0}}\|_{L^\infty(\Omega^c, D_{L_0} \cup D_{L_1} \cup D_{R_0} \cup D_{R_1})} \leq \|u - H - a^*_o\|_{L^\infty(\Omega^c, D_{L_0} \cup D_{L_1} \cup D_{R_0} \cup D_{R_1})} + \|H - H\left(-\frac{\epsilon}{2}, 0\right)\|_{L^\infty(\Omega^c, D_{L_0} \cup D_{L_1} \cup D_{R_0} \cup D_{R_1})} \leq 5 \|H\|_{L^\infty(\partial D_{L_0} \cup \partial D_{R_0})} + 2 \|H\|_{L^\infty(\Omega^c)} \leq 14 \|H\|_{L^\infty(\Omega^c)},
\]
due to (2.2). We are done. \(\square\)

3.1 The proof of (1.6)

The potential difference \(u|_{D_{R_1}} - u|_{D_{R_0}}\) was evaluated exactly in Proposition 2.1. The value has very different nature from the cases of finite number of inclusions, and also results in much stronger concentration than finite cases. In this proof, we establish an asymptote of \(\nabla u\) from the potential difference. Indeed, a nice method to get an asymptote was already introduced by Kang, Lim, Yun in the case of two circular inclusions in [11], and Bao, Li, Yin in [5] showed the boundedness of the gradient in the case of no potential difference. In this proof, we modify these methods to apply to our problem, and obtain an asymptote describing the stronger concentration. Hence, the potential difference evaluated in Proposition 2.1 plays the most important role in the result.

To establish the asymptote, we consider the decomposition of \(\nabla u\) into two terms as
\[
\nabla u = \alpha_h \nabla \phi_h + \nabla u_h,
\]
where $\alpha_h$, $\phi_h$ and $u_h$ are defined below. The function $\phi_h$ has a high concentration in between $D_{R0}$ and $D_{R1}$, and is also easy to handle. In this proof, we estimate the coefficient $\alpha_h$ and show that $\nabla u_h$ is bounded regardless of $\epsilon$ and $\delta$. Thus, we can establish the desirable asymptote (1.6).

We define $\alpha_h$, $\phi_h$ and $u_h$ and set the decomposition up. Let $\phi_h(x, y)$ be the unique solution to

\[
\begin{align*}
\Delta \phi_h &= 0 & \text{in } \mathbb{R}^2 \setminus \bar{D}_{R0} \cup \bar{D}_{R1}, \\
\phi_h &= \text{a constant} & \text{on } \partial D_{R1}, \\
\phi_h &= -\phi_h \big|_{\partial D_{R1}} & \text{on } \partial D_{R0}, \\
\int_{\partial D_{R1}} \partial_\nu \phi_h ds &= -\int_{\partial D_{R0}} \partial_\nu \phi_h ds = \frac{2\pi}{\sqrt{\delta}}, \\
\phi_h(x) &= O \left( \frac{1}{|x|} \right) & \text{as } |x| \to \infty.
\end{align*}
\]

in the same as (2.9) and (2.10). The solution can be expressed as

\[
\phi_h(x, y) = \frac{1}{\sqrt{\delta}} \left( \log |x, y| - \left( 1 + \frac{\epsilon}{2}, 1 + \frac{\delta}{2} - p_h \right) \right) - \log |x, y| - \left( 1 + \frac{\epsilon}{2}, 1 + \frac{\delta}{2} + p_h \right) \right)
\]

where

\[
p_h = \sqrt{\delta} + O(\delta)
\]

for small $\delta$. Let $\alpha_h$ be the constant as

\[
\alpha_h = \frac{u|_{\partial D_{R1}} - u|_{\partial D_{R0}}}{\phi_h|_{\partial D_{R1}} - \phi_h|_{\partial D_{R0}}},
\]

and we define a harmonic function $u_h$ as $u_h = u - \alpha_h \phi_h - (u - \alpha_h \phi_h)|_{\partial D_{R0}}$. The solution $u$ is decomposed into $\alpha_h \phi_h + (u - \alpha_h \phi_h)|_{\partial D_{R0}}$ and $u_h$ as follows:

\[
u = (\alpha_h \phi_h + (u - \alpha_h \phi_h)|_{\partial D_{R0}}) + u_h.
\]

Hence,

\[
\nabla u = \alpha_h \nabla \phi_h + \nabla u_h.
\]

From the definition of $\alpha_h$, two functions $\alpha_h \phi_h$ and $u$ have the same potential difference between $\partial D_{R1}$ and $\partial D_{R0}$, and $u_h$ has no difference between the boundaries so that $\alpha_h \phi_h|_{\partial D_{R1}} - \alpha_h \phi_h|_{\partial D_{R0}} = u|_{\partial D_{R1}} - u|_{\partial D_{R0}}$ and $u_h|_{\partial D_{R1}} - u_h|_{\partial D_{R0}} = 0$. Indeed, this means that $\nabla u$ is dominated by $\alpha_h \nabla \phi_h$. By direct calculation and the definition of $N_h$, there is a constant $C$ regardless of $\epsilon$ and $\delta$ such that

\[
\left| \nabla \phi_h - 2 \left( -2 \frac{\left( x - 1 - \frac{\epsilon}{2} \right) \left( y - 1 - \frac{\delta}{2} \right)}{\left( \left( x - 1 - \frac{\epsilon}{2} \right)^2 + \frac{1}{\delta} + \left( x - 1 - \frac{\epsilon}{2} \right)^2 \right)} \right) \right| \leq C
\]
and
\[
2\sqrt{\frac{\left(x - 1 - \frac{\delta}{2}\right)\left(y - 1 - \frac{\delta}{2}\right)}{(x - 1 - \frac{\delta}{2})^2 + \delta^2}} = |S(x, y)|
\]
\[
\leq \left|\frac{y - 1 - \frac{\delta}{2}}{(x - 1 - \frac{\delta}{2})^2 + \delta}\right| \leq 2 \left|\frac{(x - 1 - \frac{\delta}{2})^2}{(x - 1 - \frac{\delta}{2})^2 + \delta}\right| \leq 2
\]
in \(N_h\). By Proposition 2.1, direct calculation of \(\phi_h\) implies
\[
\left|\alpha_h - \frac{1}{2}(H(0, 1) - H(0, -1))\right| \leq C\delta\|\nabla H\|_{L^\infty(\Omega_4)}.
\]
By (2.2), a standard gradient estimate for harmonic functions yields \(\|\nabla H\|_{L^\infty(\Omega_3)} \leq C\|H\|_{L^\infty(\Omega_4)} \leq 3C\|H\|_{L^\infty(\Omega_4)}\) in the same way as the proof of Lemma 3.1. Hence,
\[
\left|\alpha_h - \frac{1}{2}(H(0, 1) - H(0, -1))\right| \leq C\delta\|H\|_{L^\infty(\Omega_4)}. \tag{3.3}
\]

The remainder of the proof is dedicated only to prove the boundedness of \(\nabla u_h\) such that
\[
\|\nabla u_h\|_{L^\infty(N_h)} \leq C\|H\|_{L^\infty(\Omega_4)}
\]
for some constant \(C\). Then, we obtain the desirable (1.6). Some properties of \(u_h\) are considered before proving the boundedness. From the definition of \(\alpha_h\),
\[
u_h|_{\partial D_{R_1}} - u_h|_{\partial D_{R_0}} = 0,
\]
\[
\alpha_h\phi_h|_{\partial D_{R_1}} - \alpha_h\phi_h|_{\partial D_{R_0}} = u|_{\partial D_{R_1}} - u|_{\partial D_{R_0}}
\]
and \(|\alpha_h| \leq 2\|H\|_{L^\infty(\Omega_4)}\) by (3.3). Lemma 3.3 implies that
\[
\|u_h\|_{L^\infty(\Omega_4)} \leq C\|H\|_{L^\infty(\Omega_4)}
\]
(3.4). From definition,
\[
u_h = 0 \text{ on } \partial D_{R_0} \cup \partial D_{R_1}. \tag{3.5}
\]

Dealing with the boundedness of \(\nabla u_h\), we decompose \(u_h\) into two functions \(u_+\) and \(u_-\) as
\[
u_h = u_+ + u_-,
\]
where \(u_+\) and \(u_-\) are the harmonic functions given as
\[
\begin{align*}
\Delta u_+ &= \Delta u_- = 0 & \text{in } \tilde{\Omega}_{4R} \setminus D_{R_0} \cup D_{R_1}, \\
u_+|_{\partial D_{R_0} \cup \partial D_{R_1}} &= u_-|_{\partial D_{R_0} \cup \partial D_{R_1}} = 0, \\
u_+(x, y) &= \max\{u_h(x, y), 0\} \geq 0 & \text{for any } (x, y) \in \partial\Omega_{4R}, \\
u_-(x, y) &= \min\{u_h(x, y), 0\} \leq 0 & \text{for any } (x, y) \in \partial\Omega_{4R}.
\end{align*}
\]
Then, 

$$u_+ \geq 0 \text{ and } u_- \leq 0 \text{ in } \tilde{\Omega}_{4R} \setminus (D_{R0} \cup D_{R1}).$$

In order to derive the boundedness of $\nabla u_+$ in $N_h$, we estimate $\nabla u_+$ on the boundary $\partial N_h$ which consists of four curves $\partial D_{R1} \cap \partial N_h$, $\partial D_{R0} \cap \partial N_h$, $\{1 + \frac{\sqrt{2}}{2} + \frac{\epsilon}{2}\} \times [\frac{1}{2}, \frac{3}{2} + \delta]$ and $\{1 - \frac{\sqrt{2}}{2} + \frac{\epsilon}{2}\} \times [\frac{1}{2}, \frac{3}{2} + \delta]$. We use $u_{+0}$ and $u_{+1}$ defined as

$$
\begin{aligned}
\triangle u_{+0} &= 0 \quad \text{in } \tilde{\Omega}_{4R} \setminus \overline{D_{R0}}, \\
u_0 &= u_+ \quad \text{on } \partial \tilde{\Omega}_{4R}, \\
u_0 &= 0 \quad \text{on } \partial D_{R0}.
\end{aligned}
$$

and

$$
\begin{aligned}
\triangle u_{+1} &= 0 \quad \text{in } \tilde{\Omega}_{4R} \setminus \overline{D_{R1}}, \\
u_1 &= u_+ \quad \text{on } \partial \tilde{\Omega}_{4R}, \\
u_1 &= 0 \quad \text{on } \partial D_{R1}.
\end{aligned}
$$

It follows from definitions and (3.3) that

$$
\|u_{+0}\|_{L^\infty(\tilde{\Omega}_{4R} \setminus \overline{D_{R0}})} + \|u_{+1}\|_{L^\infty(\tilde{\Omega}_{4R} \setminus \overline{D_{R1}})} 
\leq \frac{2}{\epsilon} \left| u_h \right|_{L^\infty(\tilde{\Omega}_{4R} \setminus (D_{R0} \cup D_{R1}))} \leq C \|H\|_{L^\infty(\Omega_4)}. \tag{3.6}
$$

Since $u_{+0} - u_+ = 0$ on $\partial \tilde{\Omega}_{4R} \cup \partial D_{R0}$ and $u_{+0} - u_+ \geq 0$ on $\partial D_{R1}$,

$$0 \leq u_+ \leq u_{+0} \quad \text{in } \tilde{\Omega}_{4R} \setminus \overline{D_{R0} \cup D_{R1}}. \tag{3.7}
$$

Since $u_{+0} - u_+ = u_+ = 0$ on $\partial D_{R0}$, the functions $u_{+0} - u_+$ and $u_+$ attain the minimal value 0 on $\partial D_{R0}$. The Hopf’s lemma thus implies that $0 \geq \partial_x u_+ \geq \partial_y u_{+0}$. Thus,

$$0 \leq |\nabla u_+| \leq |\partial_x u_{+0}| \quad \text{on } \partial D_{R0}. \tag{3.8}
$$

and similarly

$$0 \leq |\nabla u_+| \leq |\partial_y u_{+1}| \quad \text{on } \partial D_{R1}. \tag{3.9}
$$

Since $u_{+0} = 0$ on $\partial D_{R0}$ and $u_{+1} = 0$ on $\partial D_{R1}$, the Kelvin transform can extend the functions $u_{+0}$ and $u_{+1}$ into harmonic functions $\tilde{u}_{+0}$ and $\tilde{u}_{+1}$, defined open sets containing $\partial D_{R0}$ and $\partial D_{R1}$, respectively. For any $(x_0, y_0) \in \partial D_{R0} \cap \partial N_h$, the extended function $\tilde{u}_{+0}$ is defined in $B_{\frac{1}{2}}(x_0, y_0)$. A gradient estimate for harmonic functions and (3.10) yield

$$
|\nabla u_{+0}(x_0, y_0)| = |\nabla \tilde{u}_{+0}(x_0, y_0)| \leq C_1 \left( \frac{1}{\delta} \right)^{-1} \sup_{B_{\frac{1}{2}}(x_0, y_0)} |\tilde{u}_{+0}(x, y)|
\leq C_2 \|u_{+0}\|_{L^\infty(\tilde{\Omega}_{4R} \setminus \overline{D_{R0}})} \leq C_3 \|H\|_{L^\infty(\Omega_4)}.
$$

Thus, (3.8) implies

$$\|\nabla u_+\|_{L^\infty(\partial D_{R0} \cup \partial N_h)} \leq C \|H\|_{L^\infty(\Omega_4)}, \tag{3.10}
$$

and in the same way, (3.9) and (3.10) yield

$$\|\nabla u_+\|_{L^\infty(\partial D_{R1} \cap \partial N_h)} \leq C \|H\|_{L^\infty(\Omega_4)}. \tag{3.11}
$$

Hence, we have the upper bounds for $|\nabla u_+|$ on each boundaries $\partial D_{R1} \cap \partial N_h$ and $\partial D_{R0} \cap \partial N_h$ as above. Meanwhile, we estimate $|\nabla u_+|$ on two vertical line segments $\left\{1 + \frac{\sqrt{2}}{2} + \frac{\epsilon}{2}\right\} \times
\[ \frac{1}{2}, \frac{3}{2} + \delta \] and \( \left\{ 1 - \frac{\sqrt{3}}{2} + \delta \right\} \times \left[ \frac{1}{2}, \frac{3}{2} + \delta \right] \) which are the remainder boundaries of \( \partial N_h \). Since \( u_+ = 0 \) on \( \partial D_{R_0} \cup \partial D_{R_1} \), the Kelvin transform extends \( u_+ \) to a harmonic function \( \tilde{u}_+ \) defined in an open set containing \( \partial D_{R_0} \cup D_{R_1} \) as well as \( \tilde{\Omega}_{4R} \setminus (D_{R_0} \cup D_{R_1}) \). For any point \((x_0, y_0)\) on the vertical line segments above, the extended harmonic function \( \tilde{u}_+ \) is defined in the open disk \( B_{\frac{1}{2}}(x_0, y_0) \). A gradient estimate for harmonic functions and \( \tilde{h} \) thus yield

\[
|\nabla u_+(x_0, y_0)| = |\nabla \tilde{u}_+(x_0, y_0)| \leq C_1 \left( \frac{1}{8} \right)^{-1} \sup_{B_{\frac{1}{2}}(x_0, y_0)} |\tilde{u}_+(x, y)| \\
\leq C_2 \|u_+\|_{L^\infty(\tilde{\Omega}_{4R} \setminus (D_{R_0} \cup D_{R_1}))} \leq C_2 \|u_+\|_{L^\infty(\tilde{\Omega}_{4R} \setminus (D_{R_0} \cup D_{R_1}))}. \tag{3.12}
\]

By the definitions of \( u_{+0} \) and \( u_{+1} \), and by (3.4), \( \|u_{+0}\|_{L^\infty(\tilde{\Omega}_{4R} \setminus D_{R_0})} + \|u_{+1}\|_{L^\infty(\tilde{\Omega}_{4R} \setminus D_{R_1})} \leq C_3 \|H\|_{L^\infty(\Omega_4)}. \) Hence, (3.10), (3.11) and (3.12) result in a gradient estimate on the boundary \( \partial N_h \) as

\[
\|\nabla u_+\|_{L^\infty(\partial N_h)} \leq C_4 \|H\|_{L^\infty(\Omega_4)}. \]

By the maximal principle,

\[
\|\nabla u_+\|_{L^\infty(\Omega_4)} \leq C_4 \|H\|_{L^\infty(\Omega_4)}. \]

In the same way, we also get

\[
\|\nabla u_-\|_{L^\infty(\Omega_4)} \leq C_5 \|H\|_{L^\infty(\Omega_4)}. \]

Therefore, we obtain

\[
\|\nabla u_h\|_{L^\infty(\Omega_4)} \leq C_6 \|H\|_{L^\infty(\Omega_4)}. \]

We are done. \( \square \)

### 3.2 The proof of (1.7)

An estimate for the potential difference \( u|_{\partial D_{R_0}} - u|_{\partial D_{L_0}} \) was obtained in Lemma 3.1. We repeat the same method as the proof of (1.6) to establish the asymptote from the potential difference. Thus, this proof also begins at the decomposition as

\[
u = \beta_v \nabla \phi_v + \nabla u_v.\]

Here, \( \phi_v \) is the unique solution to

\[
\begin{aligned}
\Delta \phi_v &= 0 & \text{in } \mathbb{R}^2 \setminus D_{L_0} \cup D_{R_0} \\
\phi_v &= \text{a constant} & \text{on } \partial D_{R_0}, \\
\phi_v &= -\phi_v \big|_{\partial D_{R_0}} & \text{on } \partial D_{L_0}, \\
\int_{\partial D_{R_0}} \partial_v \phi_v ds &= -\int_{\partial D_{L_0}} \partial_v \phi_h ds = 2\pi, \\
\phi_v(x) &= O \left( \frac{1}{|x|} \right) & \text{as } |x| \to \infty.
\end{aligned} \tag{3.13}
\]
Then,

\[ \phi_v(x, y) = \log |(x, y) + (p_v, 0)| - \log |(x, y) - (p_v, 0)| \]

where

\[ p_v = \sqrt{\epsilon} + O(\epsilon). \]

Let \( \beta_v \) be the constant as

\[ \beta_v = \frac{u_{\partial D_{R_0}} - u_{\partial D_{L_0}}}{\phi_v_{\partial D_{R_0}} - \phi_v_{\partial D_{L_0}}}, \]

and we define a harmonic function \( u_v \) as

\[ u_v = u - \beta_v \phi_v - (u - \beta_v \phi_v)_{\partial D_{R_0}}. \]

The solution \( u \) is decomposed into

\[ \beta_v \phi_v + (u - \beta_v \phi_v)_{\partial D_{R_0}} \]

and \( u_v \) as

\[ u = (\beta_v \phi_v + (u - \beta_v \phi_v)_{\partial D_{R_0}}) + u_v. \]

Hence, we obtain the desirable decomposition

\[ u = \beta_v \nabla \phi_v + \nabla u_v. \]

By direct calculation, there is a constant \( C \) regardless of \( \epsilon \) and \( \delta_v \) such that

\[ \left| \nabla \phi_v - \frac{2\sqrt{\epsilon}}{\epsilon + y^2} (1, 0) \right| \leq C, \]

and Lemma 3.1 implies \( |\beta| \leq 3 \| H \|_{L^\infty(\Omega_4)}. \) If we prove the boundedness of \( \nabla u_v \) such that

\[ \| \nabla u_v \|_{L^\infty(\Omega_4)} \leq C \| H \|_{L^\infty(\Omega_4)} \]

for some constant \( C \), then we can obtain the main result (1.7).

The remainder of the proof is to prove the boundedness of \( \nabla u_v \). From the definition of \( \beta_v \), we have \( \beta_v \phi_v_{\partial D_{R_0}} - \beta_v \phi_v_{\partial D_{L_0}} = u_{\partial D_{R_0}} - u_{\partial D_{L_0}}, \ u_v_{\partial D_{R_0}} - u_v_{\partial D_{L_0}} = 0 \) and \( |\beta_v| \leq 3 \| H \|_{L^\infty(\Omega_4)} \) by Lemma 3.1. Similarly to (3.14), Lemma 3.3 implies that

\[ \| u_v \|_{L^\infty(\Omega_4 \setminus (D_{R_0} \cup D_{L_0}))} \leq C \| H \|_{L^\infty(\Omega_4)} \].

(3.14)

From definition,

\[ u_v = 0 \text{ on } \partial D_{L_0} \cup \partial D_{R_0}. \]

(3.15)

They are the conditions analogous to (3.4) and (3.5). In the same way as the proof of (1.6), we have

\[ \| \nabla u_v \|_{L^\infty(\Omega_4)} \leq C \| H \|_{L^\infty(\Omega_4)}. \]

We are done. \( \square \)
4 Proofs of Theorems 1.3

The proof is mainly concerned with the second equality (1.9) in $N_v$, since the first equality (1.8) in $N_h$ follows immediately from Theorem 1.1. Owing to the linearity of problem, we consider two cases when $H(x, y) = x$, and when $H(x, y) = y$, separately. Let $u_a$ and $u_b$ be the solutions for $H(x, y) = x$ and $H(x, y) = y$, respectively.

In the first case when $H(x, y) = x$ for $(x, y) \in \mathbb{R}^2$, Theorem 1.1 presents a constant $\mu_0$ satisfying

$$\nabla u_a(x, y) = \mu_0 \frac{\sqrt{\epsilon}}{\epsilon + y^2} (1, 0) + R_{a2}(x, y)$$

(4.1)

for $(x, y) \in N_v$, while $\|R_{a2}(x, y)\|_{L^\infty(N_v)}$ is bounded regardless of small $\epsilon > 0$ and $\delta > 0$. Proposition 2.2 provides a positive constant $C_1$ regardless of $\epsilon$ and $\delta$ such that

$$\frac{1}{C_1} \sqrt{\epsilon} \leq u_a|_{\partial D_{R_0}} - u_a|_{\partial D_L} \leq C_1 \sqrt{\epsilon}.$$ 

By the mean value theorem, there exists a point $(x_a, 0) \in N_v$ such that $-\frac{1}{2}\epsilon < x_a < \frac{1}{2}\epsilon$ and

$$\frac{1}{C_1} \frac{1}{\sqrt{\epsilon}} \leq \partial_x u_a(x_a, 0) \leq C_1 \frac{1}{\sqrt{\epsilon}}.$$ 

By (4.1), the coefficient $\mu_0$ is bounded below as

$$\mu_0 \geq \frac{1}{C_1} - \sqrt{\epsilon} \|R_{a2}(x, y)\|_{L^\infty(N_v)} \geq \frac{1}{2} \frac{1}{C_1}$$

for small $\epsilon > 0$ due to the boundedness of $\|R_{a2}(x, y)\|_{L^\infty(N_v)}$. Theorem 1.1 provides an upper bound for $\mu_a$ so to obtain a constant $C_2 > 0$ satisfying

$$\frac{1}{C_2} \leq \mu_0 \leq C_2$$

regardless of $\epsilon$ and $\delta$. Hence, we have the estimate (1.9) for $\nabla u_a$ in $N_v$ with (1.10).

In the second case when $H(x, y) = y$ for $(x, y) \in \mathbb{R}^2$, it follows from Theorem 1.1 that

$$\nabla u_b(x, y) = \mu_b \frac{\sqrt{\epsilon}}{\epsilon + y^2} (1, 0) + R_{b2}(x, y)$$

(4.2)

and for a proper constant constant $\mu_b$, and $\|R_{b2}\|_{L^\infty(N_v)}$ is bounded regardless of $\epsilon$ and $\delta$. By Proposition 2.4 for $H = y$,

$$\frac{1}{\sqrt{\epsilon}} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \partial_x u_b(x, 0)dx = \frac{1}{\sqrt{\epsilon}} (u_b|_{\partial D_{R_0}} - u_b|_{\partial D_L}) = 0.$$ 

Applying (4.2) to here,

$$|\mu_b| \leq C_3 \sqrt{\epsilon}.$$ 

Applying the inequality to (4.2), there exists a constant $C_4$ regardless of $\epsilon$ and $\delta$ such that

$$\|\nabla u_b\|_{L^\infty(N_v)} \leq C_4.$$
Therefore, in the case when $H(x, y) = ax + by$ in $\mathbb{R}^2$, we have the desirable asymptote as

\[ \nabla u = a\nabla u_a + b\nabla u_b = a\mu_0 \frac{\sqrt{\epsilon}}{\epsilon + y^2} (1, 0) + R_2 \]

in $N_0$ and the remainder term $R_2$ is bounded, since

\[ R_2 = aR_{a2} + b\nabla u_b + bR_{b2}. \]

\[ \square \]

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