WELL-POSEDNESS AND ILL-POSEDNESS FOR THE FOURTH ORDER CUBIC NONLINEAR SCHRÖDINGER EQUATION IN NEGATIVE SOBOLEV SPACES

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ABSTRACT. We consider the Cauchy problem for the fourth order cubic nonlinear Schrödinger equation (4NLS). The main goal of this paper is to prove low regularity well-posedness and mild ill-posedness for (4NLS). We prove three results. First, we show that (4NLS) is locally well-posed in $H^s(\mathbb{R})$, $s \geq -\frac{1}{2}$ using the Fourier restriction norm method. Second, we show that (4NLS) is globally well-posed in $H^s(\mathbb{R})$, $s \geq -\frac{1}{2}$. To prove this, we use the $I$-method with the correction term strategy presented in Colliander-Keel-Staffilani-Takaoka-Tao [7]. Finally, we prove that (4NLS) is mildly ill-posed in the sense that the flow map fails to be locally uniformly continuous in $H^s(\mathbb{R})$, $s < -\frac{1}{2}$. Therefore, these results show that $s = -\frac{1}{2}$ is the sharp regularity threshold for which the well-posedness problem can be dealt with an iteration argument.

1. Introduction

1.1. Fourth order cubic nonlinear Schrödinger equation. In this paper, we consider the following one-dimensional fourth order cubic nonlinear Schrödinger equation (4NLS):

\[
\begin{cases}
    i\partial_t u = \partial^4_x u \pm |u|^2 u, & (t, x) \in \mathbb{R} \times \mathbb{R} \\
    u(x, 0) = u_0(x) \in H^s(\mathbb{R}),
\end{cases}
\]

where $u$ is a complex-valued function. The equation (4NLS) is also known as the biharmonic NLS and was studied in the context of stability of solitons in magnetic materials (for more physical background, see [10, 11]). The (4NLS) has been extensively studied in recent years. For instance, see [1, 9, 20, 19, 17, 16, 18, 15]. In the following, we make no distinction between the defocusing or focusing nature of (4NLS) and hence we assume that it is defocusing, that is, with the + sign in (4NLS).

It is well known that (4NLS) enjoys the scaling symmetry. More precisely, if $u(t, x)$ is a solution to (4NLS) with an initial condition $u_0$, then

\[
u_\lambda(t, x) := \lambda^2 u(\lambda^4 t, \lambda x), \quad \lambda > 0
\]

is also a solution to (4NLS) with the $\lambda$-scaled initial condition $u_{0,\lambda}(x) := \lambda^2 u_0(\lambda x)$. Associated with this scaling symmetry, there is the so-called scaling critical regularity $s_c := -\frac{3}{2}$ such that the homogeneous $H^{s_c}$-norm is invariant under the scaling symmetry (1.1). In general, we have

\[
\|u_{0,\lambda}\|_{H^{s_c}(\mathbb{R})} = \lambda^{s_c + \frac{3}{2}} \|u_0\|_{H^{s_c}(\mathbb{R})}.
\]
As in the case of the classical NLS, (4NLS) is a Hamiltonian PDE with the following Hamiltonian
\begin{equation}
H(u(t)) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x^2 u(t)|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}} |u(t)|^4 \, dx.
\end{equation}

We also define the mass
\begin{equation}
M(u(t)) = \int_{\mathbb{R}} |u(t)|^2 \, dx.
\end{equation}

Under the flow of (4NLS), the Hamiltonian (1.3) and the mass (1.4) are conserved.

The main goal of this paper is to solve low regularity well-posedness problem for (4NLS). A small modification of [23] with the mass conservation provides global well-posedness of (4NLS) in $L^2(\mathbb{R})$. Therefore, it is natural to ask whether the well-posedness also holds in negative Sobolev spaces between scaling critical space $H^{-\frac{1}{2}}(\mathbb{R})$ and $L^2(\mathbb{R})$. We want to investigate which regularity is optimal for the local and global well-posedness for (4NLS). In this paper, we prove that (4NLS) is locally and globally well-posed in $H^s(\mathbb{R}), s \geq -\frac{1}{2}$. Also, we show that (4NLS) is mildly ill-posed in the sense that the solution map fails to be locally uniformly continuous on $H^s(\mathbb{R}), s < -\frac{1}{2}$. This means that $s = -\frac{1}{2}$ is the sharp regularity threshold for which the well-posedness can be handled by an iteration argument.

Remark 1.1. The one-dimensional cubic NLS is given by
\begin{equation}
\begin{cases}
  i \partial_t u = \partial_x^2 u \pm |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R} \\
  u(x,0) = u_0(x) \in H^s(\mathbb{R}).
\end{cases}
\end{equation}

In [23], Tsutsumi proved that (NLS) is globally well-posed in $L^2(\mathbb{R})$. We note that the equation (NLS) admits the Galilean invariance: if $u$ is a solution of (NLS) with initial data $u_0$, then
\begin{equation}
u_v(t,x) = e^{ixv}e^{-it|v|^2}u(t,x-2vt)
\end{equation}
is also a solution to the same equation (NLS) with initial data $e^{ixv}u_0(x)$. As a consequence of the Galilean invariance, the flow map cannot be locally uniformly continuous in $H^s, s < 0$ (i.e. mild ill-posedness). We refer to [14, 3] for more details. In view of the failure of local uniform continuity, one can observe that in negative Sobolev spaces it is impossible to prove well-posedness of (NLS) via a contraction argument. As for (4NLS), thanks to the lack of the Galilean invariance, there is a hope to prove local well-posedness of (4NLS) by a contraction argument in negative Sobolev spaces.

Remark 1.2. The fourth order cubic nonlinear Schrödinger equation on $\mathbb{T}$ is given by
\begin{equation}
\begin{cases}
  i \partial_t u = \partial_x^4 u \pm |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{T} \\
  u(x,0) = u_0(x) \in H^s(\mathbb{T}).
\end{cases}
\end{equation}

In [17] Appendix A], Oh and Tzvetkov proved that (1.6) is globally well-posed in $L^2(\mathbb{T})$. However, below $L^2(\mathbb{T})$ Oh and Wang [18] proved a nonexistence result for (1.6). Despite this ill-posedness, by considering the renormalized cubic 4NLS
to the renormalized cubic 4NLS (1.7) for any initial data $u$ conservation laws. More precisely, by using short-time Fourier restriction norm method, Oh and Wang [18] showed the existence of a global solution in $H^s(\mathbb{T})$ for any initial data $u_0 \in H^s(\mathbb{T})$, $s \in (-\frac{9}{20}, 0)$. Moreover, by exploiting an infinite iteration of normal form reductions, they showed that the renormalized cubic 4NLS (1.7) is globally well-posed in $H^s(\mathbb{T})$ for $s \in (-\frac{9}{20}, 0)$ with enhanced uniqueness. Later, by an adaptation of Takaoka and Tsutsumi’s argument [21], Kwak [15] proved that the renormalized cubic 4NLS (1.7) is locally well-posed in $H^s(\mathbb{T})$ for $-\frac{1}{4} \leq s < 0$. This result extends local well-posedness of (1.7) to the endpoint regularity $s = -\frac{1}{4}$.

1.2. Local well-posedness. In this subsection, we present our first main result which is local-wellposedness in $H^s(\mathbb{R})$, $s \geq -\frac{1}{2}$. Before we state Theorem 1.3 we briefly look into the local well-posedness of (4NLS) in $L^2(\mathbb{R})$. In Proposition 2.5 the Strichartz estimates associated with the (linear) biharmonic Schrödinger equation are given by
\[
(1.8) \quad \left\| D^\frac{\alpha}{2} (1 - \frac{\alpha}{2}) e^{it\partial_x^4} u_0 \right\|_{L^q_t L^r_x (\mathbb{R} \times \mathbb{R})} \lesssim q, \quad \|u_0\|_{L^2_x (\mathbb{R})},
\]
for $0 \leq \alpha \leq 1$, $r \geq 2$, $q \geq \frac{8}{(1+4\alpha)}$ and $\frac{1}{q} + \frac{1+\alpha}{r} = \frac{1+2\alpha}{2}$. In contrast to the case of linear Schrödinger equation, one can see that derivative gains occur in the estimates (1.8), thanks to the stronger dispersive effect in the high frequency mode. See Section 2 for more details. Using the Strichartz estimates (1.8), one can easily show the local well-posedness of (4NLS) for the regular initial data $u_0 \in H^s(\mathbb{R})$, $s \geq 0$ as in [23]. Moreover, global well-posedness of (4NLS) in $H^s(\mathbb{R})$, $s \geq 0$ follows from the conservation laws.

The first main result of this paper is the following local well-posedness result in $H^s(\mathbb{R})$, $s \geq \frac{1}{2}$.

**Theorem 1.3.** Let $s \geq \frac{1}{2}$. Then, (4NLS) is locally well-posed in $H^s(\mathbb{R})$. More precisely, for any $u_0 \in H^s(\mathbb{R})$, there exists $T = T (\|u_0\|_{H^s(\mathbb{R})}) > 0$ and a solution $u \in C([0, T]; H^s(\mathbb{R}))$ to (4NLS). This solution is unique in $X^{s, \frac{1}{2}+}$-space depending on the choice of a time cutoff function. Moreover, the flow map from data to solutions is locally Lipschitz continuous.

**Remark 1.4.** In our formulation, the uniqueness depends on the choice of the cutoff function in the Duhamel formulation (3.12). In particular, given two solutions to the Duhamel formulation (3.12) with different cutoff $\eta$ and $\tilde{\eta}$, both belonging to $X^{s, \frac{1}{2}+}$, they agree on $[0, \delta]$ where $\delta > 0$ denotes the shorter one of the local existence times of these two solutions.

To prove Theorem 1.3 we use the contraction mapping argument. A natural choice of the iteration space is the $X^{s, b}$-space. The $X^{s, b}$-space were simultaneously introduced by Klainerman and Machedon in the context of wave equations and Bourgain [2, 3] in the context of Schrödinger equations and the KdV equation. In the contraction mapping argument, the main part is to obtain a suitable trilinear
estimate (Proposition 3.5). In Proposition 3.5, we prove that the trilinear estimate holds for $s \geq -\frac{1}{2}$. We point out that below $L^2(R)$, nonlinear interactions generate a loss of derivatives in the trilinear estimate and hence the Strichartz estimates (1.8) are not enough to get around this loss of derivatives. To deal with this loss, we strongly use the nature of $X^{s,b}$-space which captures dispersive smoothing effects.

Observing the non-resonant case in the nonlinear interactions, one can detect a dispersive smoothing effect, which is crucial to overcome the loss of derivatives. By exploiting this dispersive smoothing effects, one can prove the trilinear estimate in the negative regularity regime (Proposition 3.5). However, in Remark 3.6, we present a counterexample for the trilinear estimate when $s < -\frac{1}{2}$ (we point out that the failure of the trilinear estimate is due to the resonant interaction of high-high-high to high). Therefore, $s = -1/2$ is the optimal regularity for the trilinear estimate to hold. More details are presented in Remark 3.6.

1.3. Global well-posedness. In this subsection, we present our second main result which is global-wellposedness in $H^s(R), s \geq -\frac{1}{2}$.

**Theorem 1.5.** Let $s \geq -\frac{1}{2}$. The (4NLS) is globally well-posed in $H^s(R)$.

We observe that $L^2$ solutions of (4NLS) satisfy the mass conservation law (1.4). This conservation law allows us to extend the local solution to the global one for $L^2$ data. However, it is non-trivial to obtain global well-posedness below $L^2$, due to the absence of a conservation law. We make use of the $I$-method to obtain global well-posedness in $H^s, s < 0$. The $I$-method was introduced by Colliander-Keel-Staffilani-Takaoka-Tao [5, 6, 7]. We briefly describe their approach. We introduce a radial $C^\infty$, monotone multiplier $m$, taking values in $[0, 1]$, and

$$m(\xi) := \begin{cases} 1, & |\xi| < N \\ \left(\frac{\xi}{N}\right)^s, & |\xi| > 2N, \end{cases}$$

(1.9)

Here, $N$ is a large parameter to be determined later. We define an operator $I$ by

$$\hat{I}u(\xi) := m(\xi) \hat{u}(\xi).$$

Note that we have the estimate

$$\|u\|_{H^s} \lesssim \|Iu\|_{L^2} \lesssim N^{-s} \|u\|_{H^s}.$$ 

Thus, the operator $I$ acts as the identity for low frequencies and as a smoothing operator of order $|s|$ on high frequencies. That is, it maps $H^s$ solutions to $L^2$. Observe that $Iu \to u$ as $N \to \infty$. Therefore, it is intuitively plausible that when $N$ is large enough, $\|Iu(t)\|_{L^2}$ almost follows mass conservation law i.e. by regularizing a solution $u$ to $Iu$, $Iu$ approximately satisfies mass conservation law. According to this idea, we will prove Lemma 4.3 (the almost conservation law). Indeed, Lemma 4.3 shows that there is a tiny increment in $\|Iu(t)\|_{L^2}$ as $t$ evolves from 0 to $\delta$ ($\delta \ll 1$) if $N$ is very large.

The basic structure of our argument to prove Lemma 4.3 (the almost conservation law) follows the argument introduced in [7]. As in [7], we carry out our energy estimate on a modified energy (4.5). This introduction of a modified energy is essential to exhibit a hidden dispersive smoothing effect.
1.4. Mild ill-posedness below $H^{-\frac{1}{4}}(\mathbb{R})$. In this subsection, we discuss our third main result which is the mild ill-posedness. We show that \([\text{NLS}]\) is mildly ill-posed in the sense that the data-to-solution map fails to be locally uniformly continuous on $H^s(\mathbb{R})$, $s < -\frac{1}{2}$. This implies that $s = -\frac{1}{2}$ is the optimal regularity threshold for which the well-posedness can be dealt with an iteration argument. Our method is inspired by Christ-Colliander-Tao [4].

**Theorem 1.6.** Let $-\frac{15}{13} < s < -\frac{1}{2}$. Then the solution map of the fourth order NLS equation \([\text{4NLS}]\) fails to be locally uniformly continuous in $H^s(\mathbb{R})$. More precisely, there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$, $T > 0$ and $0 < \varepsilon < \varepsilon_0$, there are two solutions $u, v$ to \([\text{NLS}]\) such that

\[
\begin{align*}
(1.10) & & \|u(0)\|_{H^s} + \|v(0)\|_{H^s} \lesssim \varepsilon, \\
(1.11) & & \|u(0) - v(0)\|_{H^s} \lesssim \delta, \\
(1.12) & & \sup_{0 \leq t \leq T} \|u(t) - v(t)\|_{H^s} \gtrsim \varepsilon.
\end{align*}
\]

We present a heuristic idea of the proof of Theorem 1.6. The key idea is to construct an approximate solution to \([\text{4NLS}]\) by using the solution of \([\text{NLS}]\) and use this approximate solution to transfer the mild ill-posedness result (Theorem 1.6) of \([\text{NLS}]\) to \([\text{4NLS}]\).

We point out that time-localized solutions to \([\text{NLS}]\) have spacetime Fourier transform near the parabola $\tau = \xi^2$, but time-localized solutions to \([\text{4NLS}]\) have spacetime Fourier transform near the quartic $\tau = \xi^4$. Choose an $N \gg 1$. Let $u(t, x)$ be a linear solution to $(i\partial_t - \partial_x^4) u = 0$ with $u(0) = u_0$. We use the following change of variables

\[\xi := N + \frac{\xi'}{\sqrt{6N}}.\]

Then, $\tau = \xi^4$ leads to $\tau = N^4 + \frac{4}{\sqrt{6}} N^2 \xi' + \tau'$ where

\[\tau' = (\xi')^2 + \frac{2}{3\sqrt{6}N^2} \xi^3 + \frac{1}{36N^4} \xi'^4.\]

By using these change of variables, we have

\[
(1.13) \quad u(t, x) = \int_{\mathbb{R} \times \mathbb{R}} e^{it\tau} e^{ix\xi} \hat{u}_0(\xi) \, d\xi d\tau
= \int_{\mathbb{R} \times \mathbb{R}} e^{it(N^4 + \frac{4}{\sqrt{6}} N^2 \xi' + \tau')} e^{ix(N + \xi')} \hat{u}_0(\xi') \, d\xi' d\tau'
= e^{iN^4 + i\xi N} \int_{\mathbb{R} \times \mathbb{R}} e^{i\tau' + i\xi'} \left( \sqrt{N^2 + \frac{4}{\sqrt{6}} N^2 t} \right) \hat{u}_0(N + \xi') \, d\tau' d\xi'.
\]

Thus, for $|\xi'| \ll N$, this change of variables converts $\tau = \xi^4$ to an approximate $\tau' \approx \xi'^2$. Therefore, we obtain an approximate solution

\[u(t, x) \approx e^{iN^4 + i\xi N} v \left( t, \frac{x}{\sqrt{6N}} + \frac{4}{\sqrt{6}} N^2 t \right).\]

to the fourth order linear Schrödinger equation where $v(t, x)$ solves the linear Schrödinger equation. Indeed, if $v$ solves \([\text{NLS}]\), then the function

\[
(1.14) \quad u(t, x) := e^{iN^4 + i\xi N} v \left( t, \frac{x}{\sqrt{6N}} + \frac{4}{\sqrt{6}} N^2 t \right)
\]
is an approximate solution to (4NLS). We present the details in Section 5.

**Organization of paper.** The rest of the paper is organized as follows: In Section 2 we collect the estimates that capture the linear dispersive effects. In Sections 3 and 4 we prove that (4NLS) is locally and globally well-posed in $H^s(\mathbb{R}), s \geq -1/2$ respectively. In Section 5 we show that (4NLS) is mildly ill-posed in the sense that the solution map fails to be locally uniformly continuous in $H^s(\mathbb{R}), s < -1/2$.

**Notation.** We use $A \lesssim B$ if $A \leq CB$ for some $C > 0$ and $A = O(B)$ if $A \lesssim B$. We use $X \sim Y$ when $X \lesssim Y$ and $Y \lesssim X$. Moreover, we use $A \ll B$ if $A \leq \frac{1}{C}B$, where $C$ is a sufficiently large constant. We also write $A^\pm$ to mean $A^{\pm \varepsilon}$ for any $\varepsilon > 0$.

Given $p \geq 1$, we let $p'$ be the Hölder conjugate of $p$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. We denote $L^p = L^p(\mathbb{R}^d)$ be the usual Lebesgue space. We also define the Lebesgue space $L^q(I, L^r)$ be the space of measurable functions from an interval $I \subset \mathbb{R}$ to $L^r$ whose $L^q(I, L^r)$ norm is finite, where

$$
\|u\|_{L^q(I, L^r)} = \left( \int_I \|u(t)\|_{L^r}^q \right)^{\frac{1}{q}}.
$$

We may write $L^q(I)^r(I \times \mathbb{R})$ instead of $L^q(I, L^r)$.

We denote the space time Fourier transform of $u(t, x)$ by $\hat{u}(t, \xi)$ or $\mathcal{F}u$

$$
\hat{u}(t, \xi) = \mathcal{F}u(t, \xi) = \int e^{-it\tau - i\xi \cdot \tau} d\tau dx.
$$

On the other hand, the space Fourier transform of $u(t, x)$ is denoted by

$$
\hat{u}(t, \xi) = \mathcal{F}_x u(t, \xi) = \int e^{-i\xi \cdot x} u(t, x) dx.
$$

The fractional differential operators are defined by

$$
\hat{D}^\alpha u(\xi) = |\xi|^\alpha \hat{u}(\xi),
$$

$$
\hat{(D)}^\alpha u(\xi) = \langle \xi \rangle^\alpha \hat{u}(\xi), \quad \alpha \in \mathbb{R},
$$

and the biharmonic Schrödinger semigroup is defined by

$$
e^{-it\partial_x^4} g = \mathcal{F}_x^{-1}(e^{-it|\xi|^4} \mathcal{F}_x g)
$$

for any tempered distribution $g$.

Lastly, for each dyadic number $N \in 2\mathbb{Z}$, the Littlewood-Paley projection $P_N, P_{\leq N}, P_{> N}$ are smoothed out projections to the regions $|\xi| \sim N, |\xi| \leq 2N, |\xi| > N$ respectively. More precisely, let $\varphi(\xi)$ be a bump function supported on the set $\{|\xi| \leq 2\}$ which equals 1 on the unit ball $\{|\xi| \leq 1\}$. For any dyadic number $N = 2^k, k \in \mathbb{Z}$, we define the following Littlewood-Paley projections:

$$
\hat{P}_{\leq N} u(\xi) = \varphi(\xi/N) \hat{u}(\xi),
$$

$$
\hat{P}_{> N} u(\xi) = (1 - \varphi(\xi/N)) \hat{u}(\xi),
$$

$$
\hat{P}_N u(\xi) = (\varphi(\xi/N) - \varphi(2\xi/N)) \hat{u}(\xi).
$$

They commute with derivative operators $D^\alpha, \langle D \rangle^\alpha$ and the semigroup $e^{it\partial_x^4}$. We also use the notation $u_N = P_N u$ if there is no confusion. Furthermore, they obey
the following easily verified Bernstein inequalities for $1 \leq p \leq q \leq \infty$:

\begin{align}
\|P_N f\|_{L^q_x(\mathbb{R})} \lesssim_{q,p} N^{\frac{1}{q} - \frac{1}{p}} \|P_N f\|_{L^p_x}
\end{align}

\begin{align}
\|P_N f\|_{L^q_x(\mathbb{R})} \lesssim_{q,p} N^{\frac{1}{q} - \frac{1}{p}} \|P_N f\|_{L^p_x}
\end{align}

We also use $a+$ (and $a-$) to denote $a + \eta$ (and $a - \eta$, respectively) for arbitrarily small $\eta \ll 1$.

Acknowledgements. The author would like to appreciate his advisor Soonsik Kwon for helpful discussion and encouragement. The author is also grateful to Chulkwang Kwak for pointing out an unclear portion in the proof of Lemma 4.3 and helpful discussion. The author is also grateful to Justin forloan for helpful discussion related to writing introduction part. The author is also grateful to the anonymous referee for their helpful comments that have improved the presentation of this paper. The author is partially supported by NRF-2018R1D1A1A09083345 (Korea).

2. Linear estimates

In this section, we collect the estimates that capture the linear dispersive effects. As in NLS or KdV, the proofs are fairly standard, but there are no places written for our purpose. Therefore, the proofs are self-contained. We follow the argument in [8].

2.1. Dispersive estimate. In this subsection, we prove the following dispersive estimates.

Lemma 2.1. For any $\alpha \in [0, 1]$ and $t \in \mathbb{R}$, we have

\begin{align}
\left\|D^\alpha e^{it\partial_x^4} u_0\right\|_{L^\infty_x} \lesssim C_{\alpha} |t|^{-\frac{\alpha}{2}} \|u_0\|_{L^1_x}.
\end{align}

Remark 2.2. We observe that obtaining the $L^1 \rightarrow L^\infty$ estimate is more involved since there is no explicit formula for linear biharmonic Schrödinger operator. It is important to notice that the dispersion is stronger for high frequencies and weaker for low frequencies compared to the usual linear Schrödinger evolution.

Before we prove Lemma (2.1), we first prove the following oscillatory integral estimate.

Lemma 2.3. Let $\phi$ be a smooth cutoff function supported in $\{\frac{1}{2} \leq |\xi| \leq 2\}$. Then for each $x, t \in \mathbb{R}$, we have

\begin{align}
\left| \int_{\mathbb{R}} e^{it\xi^4 + i\xi x} \phi(\xi) \, d\xi \right| \lesssim \langle t \rangle^{-\frac{1}{2}},
\end{align}

and if $|x| \gg |t|$ or $|x| \ll |t|$, then

\begin{align}
\left| \int_{\mathbb{R}} e^{it\xi^4 + i\xi x} \phi(\xi) \, d\xi \right| \lesssim \frac{1}{\max(|x|^2, |t|^2)}.
\end{align}

Proof. Observe that the second derivative of the phase function on $\{\frac{1}{2} \leq |\xi| \leq 2\}$ is given by

\begin{align}
\left| \partial_x^2 \left( \xi^4 + \frac{\xi x}{t} \right) \right| = 12|\xi|^2 \gtrsim 1.
\end{align}
Hence, by the Vander Corput lemma, we have
\[ \left| \int \mathbb{R} e^{it\xi^4 + ix\xi} \phi(\xi) \, d\xi \right| \lesssim (t)^{-\frac{1}{2}}. \]
For $|x| \gg |t|$ or $|x| \ll |t|$, the first derivative of the phase function is given by
(2.4) \[ \left| \partial_\xi \left( \xi^4 t + \xi x \right) \right| = |4\xi^3 t + x| \gtrsim |t| \max(|x|, |t|). \]

By two integration by parts and the bound (2.4), we have
\[ \left| \int \mathbb{R} e^{it\xi^4 + ix\xi} \phi(\xi) \, d\xi \right| = \left| \int \mathbb{R} \partial_\xi \left( \frac{\phi(\xi)}{4it\xi^3 + ix} \right) e^{it\xi^4 + ix\xi} \, d\xi \right| \]
\[ \lesssim \sup_{\frac{1}{2} \leq |\xi| \leq 2} \left| \partial_\xi \left( \frac{1}{4it\xi^3 + ix} \partial_\xi \left( \frac{\phi(\xi)}{4it\xi^3 + ix} \right) \right) \right| \]
\[ \lesssim \left( \frac{1}{(4\xi^3 t + x)^2} + \frac{|t|}{4|\xi^3 t + x|^3} + \frac{t^2}{(4\xi^3 t + x)^4} \right) \]
\[ \lesssim \max(|x|^2, |t|^2). \]

Hence, we obtain the desired result. \( \square \)

**Proof of Lemma 2.1** Fix $\alpha \in [0, 1]$. We define the kernel $K_t (x)$ of the operator $D^\alpha e^{it\partial_x^4}$ as follows:
\[ K_t (x) = \mathcal{F}^{-1} \left( |\xi|^\alpha e^{it|\xi|^4} \right) (x). \]

The functions $K_t (x)$ are defined a priori as distributions. Later, we will show that they are in fact functions. For any tempered distribution $T \in \mathcal{S}^*$ and invertible linear transformation $L$, we have $\mathcal{F} (T \circ L) = |\det L|^{-1} (\mathcal{F} T) \circ (L^*)^{-1}$. Therefore we obtain
\[ K_t (x) = t^{-\frac{\alpha+1}{4}} K_1 \left( \frac{x}{t^\frac{1}{4}} \right). \]

Note that
\[ D^\alpha e^{it\partial_x^4} u_0 = \mathcal{F}^{-1} \left( |\xi|^\alpha e^{it|\xi|^4} \right) \ast u_0 = K_t \ast u_0. \]

Hence, by Young’s inequality, it is enough to show that $K_1 (x)$ is a bounded function. For any Schwartz function $f \in \mathcal{S} (\mathbb{R})$, we have $P_{\leq 1} f + \sum_{k=1}^n P_k f \to f$ in $\mathcal{S} (\mathbb{R})$. Here $P_k f$ is the standard Littlewood-Paley projectors to the regions $\{2^{k-1} \leq |\xi| \leq 2^{k+1} \}$. 
Therefore, we have
\[
\langle K_1, f \rangle = \langle F^{-1} \left( |\xi|^\alpha e^{i\xi^4} \right), f \rangle
\]
\[
= \langle F^{-1} \left( |\xi|^\alpha e^{i\xi^4} \right), P_{\leq 1} f \rangle + \lim_{n \to \infty} \sum_{k=1}^{n} \langle F^{-1} \left( |\xi|^\alpha e^{i\xi^4} \right), P_k f \rangle
\]
\[
= \langle F^{-1} \left( |\xi|^\alpha e^{i\xi^4} \varphi(\xi) \right), f \rangle + \lim_{n \to \infty} \sum_{k=1}^{n} \langle F^{-1} \left( |\xi|^\alpha e^{i\xi^4} \psi \left( \frac{\xi}{2^k} \right) \right), f \rangle. \]

Here \( \varphi \) is a bump function supported on \( \{ |\xi| \leq 2 \} \) which equals 1 on \( \{ |\xi| \leq 1 \} \) and
\( \psi \) is the function given by
\[
\psi(\xi) := \varphi(\xi) - \varphi(2\xi). \]
Therefore, we can identify \( K_1(x) \) as the distributional limits:
\[
K_1(x) = F^{-1} \left( |\xi|^\alpha e^{i\xi^4} \varphi(\xi) \right)(x) + \lim_{n \to \infty} \sum_{k=1}^{n} F^{-1} \left( |\xi|^\alpha e^{i\xi^4} \psi \left( \frac{\xi}{2^k} \right) \right)(x). \]

Obviously, the first summand is a bounded function. The second term is
\[
\sum_{k=1}^{\infty} \int_{\mathbb{R}} e^{i\xi^4 + i\xi x} |\xi|^\alpha \psi \left( \frac{\xi}{2^k} \right) d\xi = \sum_{k=1}^{\infty} 2^{k(1+\alpha)} \int_{\mathbb{R}} e^{i2^{4k} \xi^4 + i2^k \xi} |\xi|^\alpha \psi(\xi) d\xi.
\]
By using the oscillatory integral estimate (2.2) and (2.3), we have
\[
\sum_{k=1}^{\infty} 2^{k(1+\alpha)} \int_{\mathbb{R}} e^{i2^{4k} \xi^4 + i2^k \xi} |\xi|^\alpha \psi(\xi) d\xi \lesssim \sum_{k=1}^{\infty} 2^{2k} \cdot 2^{-8k} + \sum_{2^k \ll |x|} 2^{2k} \cdot 2^{-2k} |x|^{-2} + \sum_{2^k \sim |x|} 2^{2k} \cdot 2^{-2k}
\]
\[
\lesssim \sum_{k=1}^{\infty} 2^{-6k} + \sum_{k \geq 1} 1 \lesssim 1.
\]
This completes the proof of Lemma 2.4. \( \square \)

**Lemma 2.4** (Dispersive decay estimate). For any \( r \geq 2 \) and \( \alpha \in [0, 1] \), we have
\[
(2.5) \quad \left\| D^{\alpha(1 - \frac{1}{r})} e^{i\alpha \tau z} u_0 \right\|_{L^r_x(X)} \lesssim \frac{1}{|z|^{\frac{\alpha}{1 + \alpha}}} \left\| u_0 \right\|_{L^r_x(X)}.
\]

**Proof.** Note that
\[
\left\| D^{\alpha} e^{i\alpha \tau z} u_0 \right\|_{L^\infty_x} \lesssim \frac{1}{|z|^{\frac{\alpha}{1 + \alpha}}} \left\| u_0 \right\|_{L^1_x},
\]
\[
\left\| e^{i\alpha \tau z} u_0 \right\|_{L^\infty_x} \lesssim \left\| u_0 \right\|_{L^2_x}.
\]
Consider the analytic family of operators
\[
T_z = |z|^{\frac{\alpha}{1 + \alpha}} D^{\alpha} e^{i\alpha \tau z},
\]
where \( z = x + iy, x \in \mathbb{R}, y \in [0, 1] \).
Therefore, complex interpolation between the lines \( \text{Re}(z) = 0 \) and \( \text{Re}(z) = 1 \) yields the theorem. \( \square \)
2.2. Strichartz estimates. In this subsection, we present the following Strichartz estimates.

**Lemma 2.5** (Strichartz estimates). For any \( \alpha \in [0, 1] \), we call a triplet \((q, r, \alpha)\) admissible exponents if \( r \geq 2 \), \( q \geq \frac{s}{1+\alpha} \) and \( \frac{4}{q} + \frac{4 \alpha}{r} = \frac{4}{1+\alpha} \). Then, for any admissible exponents \((q, r, \alpha)\) and \((\tilde{q}, \tilde{r}, \tilde{\alpha})\), we have

\[
the estimates \quad (2.6) \quad \left\| D^{\frac{\alpha}{2}} e^{it\partial_x^\alpha} u_0 \right\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R})} \lesssim q, r \left\| u_0 \right\|_{L_t^2(\mathbb{R})},
\]

\[
the estimates \quad (2.7) \quad \left\| \int_{\mathbb{R}} D^{\frac{\alpha}{2}} e^{-it\partial_x^\alpha} F(t') \ dt' \right\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}(\mathbb{R} \times \mathbb{R})} \lesssim \tilde{q}, \tilde{r} \left\| F \right\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R} \times \mathbb{R})},
\]

\[
the estimates \quad (2.8) \quad \left\| \int_{\mathbb{R}} D^{\alpha}(1 - \frac{\alpha}{2}) e^{i(t-t')\partial_x^\alpha} F(t') \ dt' \right\|_{L_t^q L_x^{r'}(\mathbb{R} \times \mathbb{R})} \lesssim q, r, \tilde{r} \left\| F \right\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R} \times \mathbb{R})}.\]

**Proof.** From the standard \( TT^* \) argument, it suffices to show the last estimate. Note that

\[
T := D^{\frac{\alpha}{2}} e^{it\partial_x^\alpha},
T^* := \int_{\mathbb{R}} D^{\frac{\alpha}{2}} e^{it\partial_x^\alpha} dt.
\]

Then, by the Minkowski and Hardy-Littlewood sobolev inequality, we have the following \( TT^* \) estimate

\[
\left\| \int_{\mathbb{R}} D^{\alpha}(1 - \frac{\alpha}{2}) e^{i(t-t')\partial_x^\alpha} F(t') \ dt' \right\|_{L_t^q L_x^{r'}(\mathbb{R} \times \mathbb{R})} \lesssim \left\| \int_{\mathbb{R}} \frac{1}{|t-t'|^{\frac{4 \alpha}{1+\alpha}}} \left\| F(t') \right\|_{L_x^{\tilde{r}'}} \ dt' \right\|_{L_t^q} \lesssim \left\| F \right\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R} \times \mathbb{R})}.
\]

This completes the proof of Lemma 2.5. \( \square \)

**Remark 2.6.** In particular, we mainly use the following estimates:

\[
\left\| D^{\frac{\alpha}{2}} e^{it\partial_x^\alpha} u_0 \right\|_{L_t^q L_x^{r'}(\mathbb{R} \times \mathbb{R})} \lesssim \left\| u_0 \right\|_{L_t^2(\mathbb{R})}, (2.9)
\]

\[
\left\| e^{it\partial_x^\alpha} u_0 \right\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R} \times \mathbb{R})} \lesssim \left\| u_0 \right\|_{L_t^2(\mathbb{R})}, (2.10)
\]

\[
\left\| e^{it\partial_x^\alpha} u_0 \right\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R} \times \mathbb{R})} \lesssim \left\| u_0 \right\|_{L_t^2(\mathbb{R})}. (2.11)
\]

2.3. Bilinear Strichartz estimate. In a low-high interaction, we need the following bilinear Strichartz estimates.

**Lemma 2.7** (Bilinear Strichartz estimate). Let \( N_1, N_2 \) be dyadic numbers with \( N_1 \leq \frac{N_2}{8} \). Then, we have

\[
\left\| e^{it\partial_x^\alpha} \phi_{N_1} e^{it\partial_x^\alpha} \phi_{N_2} \right\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R})} \lesssim N_2^{-\frac{2}{3}} \left\| \phi_{N_1} \right\|_{L_t^2 L_x^2(\mathbb{R})} \left\| \phi_{N_2} \right\|_{L_t^2 L_x^2(\mathbb{R})}. (2.12)
\]
This completes the proof of Lemma 2.7. □

3. Local well-posedness of the Fourth Order NLS

In this section, we prove local well-posedness of (4NLS). We use the contraction principle on the \( X^{s,b} \)-space.

3.1. Duhamel formulation and \( X^{s,b} \)-space. In this subsection, we introduce the following Duhamel formulation and \( X^{s,b} \)-space.

By expressing (4NLS) in the Duhamel formulation, we have

\[
(3.1) \quad u(t) = e^{it\partial_x^4} u_0 \mp \int_0^t e^{i(t-t') \partial_x^4} F(u)(t') \, dt',
\]
where \(F(u) = |u|^2u = u\overline{u} u\). Let \(\eta\) be a smooth cutoff function supported on \([-2, 2]\), \(\eta = 1\) on \([-1, 1]\) and \(\eta_b(t) = \eta(t/\delta)\). If \(u\) satisfies

\[
(3.2) \quad u(t) = \eta(t) e^{it\partial_x^2} u_0 + \eta(t) \int_0^t e^{i(t-t')\partial_x^2} \eta_b(t') F(\eta u)(t') \, dt'
\]

for some \(\delta \ll 1\), then it also satisfies (3.1) on \([-\delta, \delta]\). Hence, we consider (3.2) in the following.

Next, let us recall some standard notations and facts. We denote the \(X^{s, b}_{r=\xi} (\mathbb{R} \times \mathbb{R})\) by \(X^{s, b}\). The \(X^{s, b}\) space is defined to be the closure of the Schwartz functions \(S(\mathbb{R} \times \mathbb{R})\) under the norm

\[
\|u\|_{X^{s, b}_{r=\xi}} := \|\langle \xi \rangle^{s} (\tau + \xi^4) \vec{u}(\tau, \xi)\|_{L^2_{\tau \xi}(\mathbb{R} \times \mathbb{R})},
\]

Next, we state the standard facts related to \(X^{s, b}\) space.

**Lemma 3.1.** Let \(b > \frac{1}{2}\) and \(s \in \mathbb{R}\). Then for any \(u \in X^{s, b}_{r=\xi} (\mathbb{R} \times \mathbb{R})\), we have

\[
(3.3) \quad \|u\|_{C_t H^s_x(\mathbb{R} \times \mathbb{R})} \lesssim \|u\|_{X^{s, b}_{r=\xi}}.
\]

Furthermore, \(X^{s, b}\) space enjoy the same Strichartz estimate that free solutions do.

For the proof of this lemma, see [22].

**Lemma 3.2** (Strichartz estimates and bilinear Strichartz estimates). Let \((q, r, \alpha)\) be an admissible exponent. Then, for \(b > \frac{1}{2}\) and \(N_1 \ll N_2\), we have

\[
(3.4) \quad \|D^b (1 - \frac{1}{2}) u\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R})} \lesssim \|u\|_{X^{0, b}(\mathbb{R} \times \mathbb{R})},
\]

\[
(3.5) \quad \|P_{N_1} u_1 P_{N_2} u_2\|_{L^2_t L^2_x(\mathbb{R} \times \mathbb{R})} \lesssim N_2^{-\frac{3}{2}} \|P_{N_1} u_1\|_{X^{0, b}(\mathbb{R} \times \mathbb{R})} \|P_{N_2} u_2\|_{X^{0, b}(\mathbb{R} \times \mathbb{R})}.
\]

In particular, we have

\[
(3.6) \quad \left\|D^b u\right\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R})} \lesssim \|u\|_{X^{0, 0}(\mathbb{R} \times \mathbb{R})},
\]

\[
(3.7) \quad \|u\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R})} \lesssim \|u\|_{X^{0, b}(\mathbb{R} \times \mathbb{R})},
\]

\[
(3.8) \quad \|u\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R})} \lesssim \|u\|_{X^{0, 0}(\mathbb{R} \times \mathbb{R})}.
\]

**Proof.** It follows directly from the transference principle. For the proof, see [22].

The following is a \(X^{s, b}\) energy estimate of time cut-off solutions.

**Lemma 3.3.** Let \(b > \frac{1}{2}\), \(s \in \mathbb{R}\) and let \(u\) solve the inhomogeneous fourth order NLS equation \(i\partial_t u - \partial_x^4 u = F\). Then we have

\[
(3.9) \quad \|\eta(t) u\|_{X^{s, b}} \lesssim \|u_0\|_{H^s} + \|F\|_{X^{0, b-1}}.
\]

For the proof of this lemma, see for instance [22]. To gain a small time factor in a nonlinear term, we need the following lemma.

**Lemma 3.4.** For any \(-\frac{1}{2} < b' < b < \frac{1}{2}\) and \(s \in \mathbb{R}\), we have

\[
(3.10) \quad \|\eta(t/\delta) u\|_{X^{s, b'}} \lesssim \delta^{b-b'} \|u\|_{X^{s, b}}.
\]

For the proof of this lemma, see [22].
3.2. **Trilinear estimate.** To prove Theorem 1.3 the last thing we need is the following trilinear estimate.

**Proposition 3.5.** Let $-\frac{1}{2} \leq s, \frac{1}{2} < b \leq \frac{7}{8}$. Then, for each time localized function $u_j$, we have

$$(3.11) \quad \|u_1 \overline{u_2} u_3\|_{X^{s,b-1}_{\tau=-\xi^4}} \leq \|u_1\|_{X^{s,b}_{\tau=-\xi^4}} \|u_2\|_{X^{s,b}_{\tau=-\xi^4}} \|u_3\|_{X^{s,b}_{\tau=-\xi^4}}.$$ 

Combining (3.9), (3.10) and (3.11), one can prove that the operator

$$(3.12) \quad \Phi(u)(t) := \eta(t) e^{it\partial_x^4} u_0 \mp \eta(t) \int_0^t e^{i(t-t')\partial_x^4} \eta_4(t') F(\eta u)(t') \, dt'$$

is a contraction mapping on a ball of $X^{s,b}$ space

$$B := \{ u \in X^{s,b} : \|u\|_{X^{s,b}} \leq 2R \}$$

for $R > 0$ and $\|u_0\|_{H^s} \leq R$. Since our proof is via the contraction principle, we also obtain that the solution map is locally Lipschitz continuous. Therefore, it suffices to prove the trilinear estimate (3.11). Before we give the proof of the trilinear estimate (3.11), let us remark an example that the trilinear estimate fails in the $X^{s,b}$ space for $s < -1/2$.

**Remark 3.6.** We present the example that for $s < -1/2$, the trilinear estimate fails

$$\|u_1 \overline{u_2} u_3\|_{X^{s,b-1}_{\tau=-\xi^4}} \leq \|u_1\|_{X^{s,b}_{\tau=-\xi^4}} \|u_2\|_{X^{s,b}_{\tau=-\xi^4}} \|u_3\|_{X^{s,b}_{\tau=-\xi^4}}.$$ 

In particular, the nonlinear interaction high $\times$ high $\times$ high $\rightarrow$ high is a sources that makes a trouble. We follow the argument presented in Kenig-Ponce-Vega [13]. For a fixed large $N$, set $A = \{(\tau, \xi) \in \mathbb{R}^2 : |\tau - \xi^4| \leq 1, N \leq \xi \leq N + \frac{1}{N}\}$ and $B = -A = \{(\tau, \xi) \in \mathbb{R}^2 : (-\tau, -\xi) \in A\}$. We define functions $u, v$ and $w$ by Fourier transform

$$\hat{u}(\tau, \xi) = 1_A(\tau, \xi),$$

$$\hat{v}(\tau, \xi) = 1_B(\tau, \xi),$$

and

$$\hat{w}(\tau, \xi) = 1_A(\tau, \xi).$$

Note that $\|\hat{u}\|_{L^2_{\tau,\xi}} \approx N^{-\frac{1}{2}}, \|\hat{v}\|_{L^2_{\tau,\xi}} \approx N^{-\frac{1}{2}}$ and $\|\hat{w}\|_{L^2_{\tau,\xi}} \approx N^{-\frac{1}{2}}$. Moreover by the definiton of the convolution we have

$$\hat{u} \ast \hat{v}(\tau, \xi) \approx \frac{1}{N} 1_R,$$

where $R = \{(\tau, \xi) : |\tau - 4N^3\xi| \lesssim 1, |\xi| \leq \frac{1}{N}\}$ is a low frequency region. Hence the high $\times$ high interaction gives us a low frequency localized function $\hat{u} \ast \hat{v}$. We also have

$$(\hat{u} \ast \hat{v}) \ast \hat{w} \approx \frac{1}{N^2} 1_A.$$
Note that on the sets $A, B, R, (\tau - \xi^4) \approx 1$ and hence we have

\[
\begin{align*}
\| \langle \xi \rangle^s (\tau - \xi^4)^{b-1} (\hat{u} \ast \hat{v} \ast \hat{w}) (\tau, \xi) \|_{L^2_{\tau, \xi}} & \approx N^s N^{-2} N^{-\frac{b}{2}}, \\
\| \langle \xi \rangle^s (\tau - \xi^4)^{b} \hat{w}(\tau, \xi) \|_{L^2_{\tau, \xi}} & \approx N^s N^{-\frac{b}{2}}, \\
\| \langle \xi \rangle^s (\tau - \xi^4)^{b} \hat{v}(\tau, \xi) \|_{L^2_{\tau, \xi}} & \approx N^s N^{-\frac{b}{2}}, \\
\| \langle \xi \rangle^s (\tau - \xi^4)^{b} \hat{u}(\tau, \xi) \|_{L^2_{\tau, \xi}} & \approx N^s N^{-\frac{b}{2}}.
\end{align*}
\]

Hence, in order for the trilinear estimate to hold, we need to have

\[
(3.13) \quad N^s N^{-2} N^{-\frac{b}{2}} \lesssim \left( N^{-\frac{b}{2}} N^s \right)^3.
\]

Since $N$ is very large, (3.13) is possible only if $s \geq -1/2$. That is, there is a counter example of the trilinear estimate when the regularity $s$ is less than $-\frac{1}{2}$.

Now, we give the proof of Proposition 3.5. Before proceeding further, we simplify some of the notations. Let us suppress the smooth time cut-off function $\eta$ from $\eta u_j$ and simply denote them by $u_j$.

**Proof of Proposition 3.5.** We may assume $-1/2 \leq s < 0$. By duality and Plancherel theorem,

\[
\|u_1 \overline{u_2} u_3\|_{X^{s,b-1}_{\tau=-\xi^4}} = \sup_{\|v\|_{X^{-s,1-b}_{\tau=\xi^4}}} \left| \int_{\mathbb{R}^2} u_1 \overline{u_2} u_3 \overline{v} \, dx \, dt \right| = \sup_{\|v\|_{X^{-s,1-b}_{\tau=\xi^4}}} \left| \int_{\tau_1+\cdots+\xi_4=0} \overline{u_1}(\tau_1, \xi_1) \overline{u_2}(\tau_2, \xi_2) \overline{u_3}(\tau_3, \xi_3) \overline{v}(\tau_4, \xi_4) \right|
\]

Since $u_1, u_2$ and $u_3$ are functions localized at time $|t| \lesssim 1$, we may assume that $v$ is also the function localized at time $|t| \lesssim 1$. Now set

\[
\begin{align*}
f_1 (\tau_1, \xi_1) &= |\overline{u_1}(\tau_1, \xi_1)| \langle \xi_1 \rangle^s \langle \tau_1 + \xi_1^4 \rangle^{\frac{1}{2}}, \\
f_2 (\tau_2, \xi_2) &= |\overline{u_2}(\tau_2, \xi_2)| \langle \xi_2 \rangle^s \langle \tau_2 - \xi_2^4 \rangle^{\frac{1}{2}}, \\
f_3 (\tau_3, \xi_3) &= |\overline{u_3}(\tau_3, \xi_3)| \langle \xi_3 \rangle^s \langle \tau_3 + \xi_3^4 \rangle^{\frac{1}{2}},
\end{align*}
\]

and

\[
f_4 (\tau_4, \xi_4) = |\overline{v}(\tau_4, \xi_4)| \langle \xi_4 \rangle^{-s} \langle \tau_4 - \xi_4^4 \rangle^{1-b}.
\]

Hence, we need to show that for nonnegative $L^2$ functions $f_j$, we have

\[
\int_{\Gamma_4} \langle \xi_4 \rangle^s \langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \langle \tau_1 + \xi_1^4 \rangle^{\frac{1}{2}} \langle \tau_2 - \xi_2^4 \rangle^{\frac{1}{2}} \langle \tau_3 + \xi_3^4 \rangle^{\frac{1}{2}} \langle \tau_4 - \xi_4^4 \rangle^{1-b} \prod_{j=1}^4 f_j (\tau_j, \xi_j)
\]

\[
\leq \prod_{j=1}^4 \|f_j\|_{L^2_{\tau, \xi}},
\]

where $\Gamma_4 = \{\xi_1 + \cdots + \xi_4 = 0, \tau_1 + \cdots + \tau_4 = 0\}$ is the hyperplane. Note that the variables $\xi_1$ and $\xi_3$ have symmetry. We define $|\xi_{\text{max}}|, |\xi_{\text{sub}}|, |\xi_{\text{thr}}|, |\xi_{\text{min}}|$ to be the
Therefore, by using Hölder's inequality, we have

\[ m(\xi_1, \ldots, \xi_4) = \frac{(\xi_4)^{\alpha}}{(\xi_1)^{\alpha} (\xi_2)^{\alpha} (\xi_3)^{\alpha}}. \]

In order to obtain the trilinear estimate (3.11), we need to split the domain of integration (3.13) in several cases:

**Case 1.** \( \Omega_1 = \{ |\xi_1|, \ldots, |\xi_4| \leq 1 \}. \)

**Case 2.** \( \Omega_2 = \{ |\xi_{\max}| \geq 1 \text{ and } |\xi_{\max}| \sim |\xi_{\sub}| \gg |\xi_{\thd}||\xi_{\min}| \} . \)

**Case 3.** \( \Omega_3 = \{ |\xi_{\max}| \gg 1 \text{ and } |\xi_{\max}| \sim |\xi_{\sub}| \sim |\xi_{\thd}| \gg |\xi_{\min}| \} . \)

**Case 4.** \( \Omega_4 = \{ |\xi_{\max}| \gg 1 \text{ and } |\xi_{\max}| \sim |\xi_{\sub}| \sim |\xi_{\thd}| \sim |\xi_{\min}| \} . \)

**Case 1.** \( \Omega_1 = \{ |\xi_1|, \ldots, |\xi_4| \leq 1 \}. \) Here, \( P_{\leq 1} u_i \) is defined by \( \hat{P}_{\leq 1} u_i (\xi) = 1_{|\xi| \leq 1} \hat{u}_i (\xi). \) As we mentioned above, we may assume \( v \) is localized at time \(|t| \leq 1 \). Therefore, by using Hölder’s inequality, we have

\[
\left\| \int_{\mathbb{R} \times \mathbb{R}} P_{\leq 1} u_1 P_{\leq 1} \overline{P} P_{\leq 1} u_3 P_{\leq 1} \overline{P} \, dx \right\|
\leq \|v\|_{L_x^2 L_t^2} \|P_{\leq 1} u_1 P_{\leq 1} \overline{P} P_{\leq 1} u_3\|_{L_t^4 L_x^2}
\leq \|v\|_{L_x^2 L_t^2} \|P_{\leq 1} u_1\|_{L_t^4 L_x^2} \|P_{\leq 1} \overline{u}_2\|_{L_t^4 L_x^2} \|P_{\leq 1} u_3\|_{L_t^4 L_x^2}
\leq \|v\|_{X_r^{s,-1,4}} \|P_{\leq 1} u_1\|_{L_t^4 L_x^2} \|P_{\leq 1} \overline{u}_2\|_{L_t^4 L_x^2} \|P_{\leq 1} u_3\|_{L_t^4 L_x^2}.
\]

Note that from the Strichartz estimates (3.7), (3.8) and using the low frequency range \(|\xi| \lesssim 1 \), we have

\[
\|P_{\leq 1} u_1\|_{L_t^4 L_x^2} \lesssim \|P_{\leq 1} u_1\|_{X_r^{s,-1,4}} \lesssim \|u_1\|_{X_r^{s,4}} ,
\|P_{\leq 1} \overline{u}_2\|_{L_t^4 L_x^2} \lesssim \|P_{\leq 1} \overline{u}_2\|_{X_r^{s,4}} \lesssim \|u_2\|_{X_r^{s,4}} ,
\|P_{\leq 1} u_3\|_{L_t^4 L_x^2} \lesssim \|P_{\leq 1} u_3\|_{X_r^{s,4}} \lesssim \|u_3\|_{X_r^{s,4}} .
\]

In this case, the trilinear estimate holds for all negative \( s < 0 \).

**Case 2.** \( \Omega_2 = \{ |\xi_{\max}| \geq 1 \text{ and } |\xi_{\max}| \sim |\xi_{\sub}| \gg |\xi_{\thd}||\xi_{\min}| \} . \) We split the set \( \Omega_2 \) into the following subsets:

**Subcase 2.a** \( \Omega_{2,a} = \{ |\xi_{\max}| \geq 1 \text{ and } |\xi_2| \sim |\xi_4| \gg |\xi_1|, |\xi_3| \} . \)

**Subcase 2.b** \( \Omega_{2,b} = \{ |\xi_{\max}| \geq 1 \text{ and } |\xi_2| \sim |\xi_3| \gg |\xi_1|, |\xi_4| \} . \)

**Subcase 2.c** \( \Omega_{2,c} = \{ |\xi_{\max}| \geq 1 \text{ and } |\xi_3| \sim |\xi_1| \gg |\xi_2|, |\xi_4| \} . \)

**Subcase 2.d** \( \Omega_{2,d} = \{ |\xi_{\max}| \geq 1 \text{ and } |\xi_3| \sim |\xi_2| \gg |\xi_1|, |\xi_4| \} . \)

**Subcase 2.a** \( \Omega_{2,a} = \{ |\xi_{\max}| \geq 1 \text{ and } |\xi_2| \sim |\xi_4| \gg |\xi_1|, |\xi_3| \} . \) Note that on \( \Omega_{2,a} \), we have

\[ m(\xi_1, \xi_2, \xi_3, \xi_4) \lesssim (\xi_1)^{\frac{1}{2}} (\xi_3)^{\frac{1}{2}}. \]

On \( \{ |\xi_1|, |\xi_3| \geq 1 \} \), there is no difference between \(|\xi_i|^{\frac{1}{2}}\) and \((\xi_i)^{\frac{1}{2}}\), \( i = 1, 3 \). This means that on the high frequency region there is no difference between homogeneous
derivative and inhomogeneous derivative. We consider the region

\[ \Omega_{2,a,high,high} = \Omega_{2,a} \cap \{ |\xi_1|, |\xi_3| \gg 1 \} , \]
\[ \Omega_{2,a,high,low} = \Omega_{2,a} \cap \{ |\xi_1| \gg 1, |\xi_3| \lesssim 1 \} , \]
\[ \Omega_{2,a,low,high} = \Omega_{2,a} \cap \{ |\xi_1| \lesssim 1, |\xi_3| \gg 1 \} , \]
\[ \Omega_{2,a,low,low} = \Omega_{2,a} \cap \{ |\xi_1|, |\xi_3| \lesssim 1 \} . \]

Then, on \( \Omega_{2,a,high,high} \), from the Strichartz estimates \( \text{(3.6)} \) and \( \text{(3.8)} \), we have

\[
\int_{\mathbb{R}^4 \cap \Omega_{2,a,high,high}} \left( \frac{\langle \xi_1 \rangle^{a} \langle \tau_4 - \xi_4 \rangle^{b-1}}{\langle \xi_1^{4} \rangle^{2} \langle \xi_2^{4} \rangle^{2} \langle \xi_3^{4} \rangle^{2}} \langle \tau_1 + \xi_1^{4} \rangle^{2+} \langle \tau_2 - \xi_2^{4} \rangle^{2+} \langle \tau_3 + \xi_3^{4} \rangle^{2+} \right) \prod_{j=1}^{4} f_j (\tau_j, \xi_j) \]
\[
\lesssim \left\| D^\frac{1}{2} \mathcal{F}^{-1} \left( \frac{f_1}{\langle \tau_1 + \xi_1^{4} \rangle^{2+}} \right) \right\|_{L^4_x L^\infty_t} \left\| \mathcal{F}^{-1} \left( \frac{f_2}{\langle \tau_2 - \xi_2^{4} \rangle^{2+}} \right) \right\|_{L^4_x L^\infty_t} \times \left\| D^\frac{1}{2} \mathcal{F}^{-1} \left( \frac{f_3}{\langle \tau_3 + \xi_3^{4} \rangle^{2+}} \right) \right\|_{L^4_x L^\infty_t} \left\| f_4 \right\|_{L^1_t L^2_x} \]
\[
\lesssim \| u_1 \|_{X^{\frac{1}{2}+1}} \| u_2 \|_{X^{\frac{1}{2}+1}} \| u_3 \|_{X^{\frac{1}{2}+1}} .
\]

On \( \Omega_{2,a,high,low} = \Omega_{2,a} \cap \{ |\xi_1| \gg 1, |\xi_3| \lesssim 1 \} \), we consider the integral

\[
\text{(3.15)} \int_{\mathbb{R}^4} P_{\gg 1} u_1 P_{\gg 1} \overline{u_2} P_{\lesssim 1} u_3 P_{> 1} v \, dx dt ,
\]
where \( \widehat{P_{\gg 1} u_1} = 1_{|\xi| \gg 1} \hat{u}_1 \) and \( \widehat{P_{\lesssim 1} u_3} = 1_{|\xi| \lesssim 1} \hat{u}_3 \). Therefore, by using Hölder’s inequality, we have

\[ \text{(3.15)} \lesssim \| u \|_{L^2_t L^2_x} \| P_{\gg 1} u_1 P_{\gg 1} \overline{u_2} P_{\lesssim 1} u_3 \|_{L^1_t L^2_x} \lesssim \| u \|_{X^{\frac{1}{2}+1}} \| P_{\gg 1} u_1 P_{\gg 1} \overline{u_2} P_{\lesssim 1} u_3 \|_{L^2_t L^2_x} .
\]

Note that

\[
| \mathcal{F} ( P_{\gg 1} u_1 P_{\gg 1} \overline{u_2} P_{\lesssim 1} u_3 ) (\xi) | \]
\[
\lesssim \left| \int_{\xi_1 + \xi_2 + \xi_3 = \xi} \widehat{P_{\gg 1} u_1 (\xi_1) P_{\gg 1} \overline{u_2 (\xi_2)} P_{\lesssim 1} u_3 (\xi_3)} \right| \]
\[
\lesssim \left( \langle \xi \rangle^{\frac{1}{2}} 1_{|\xi| \gg 1} (\xi)^{+} |\hat{u}_1| \right) \ast \left( \langle \xi \rangle^{\frac{1}{2}} 1_{|\xi| \gg 1} (\xi)^{+} |\hat{u}_2| \right) \ast \left( 1_{|\xi| \lesssim 1} |\hat{u}_3| \right) (\xi) .
\]

Therefore, from Strichartz estimates \( \text{(3.6)}, \text{(3.8)} \) and using the low frequency range \( |\xi| \lesssim 1 \), we have

\[
\| P_{\gg 1} u_1 P_{\gg 1} \overline{u_2} P_{\lesssim 1} u_3 \|_{L^2_t L^2_x} \lesssim \| D^\frac{1}{2} P_{\gg 1} |D|^s \mathcal{F}^{-1} |\hat{u}_1|_{L^1_t L^\infty_x} \| D^\frac{1}{2} P_{\gg 1} |D|^s \mathcal{F}^{-1} |\overline{u_2}|_{L^1_t L^\infty_x} \| P_{\lesssim 1} \mathcal{F}^{-1} |\hat{u}_3|_{L^2_t L^\infty_x} \|
\[
\lesssim \| u_1 \|_{X^{\frac{1}{2}+1}} \| u_2 \|_{X^{\frac{1}{2}+1}} \| u_3 \|_{X^{\frac{1}{2}+1}} .
\]

For the case \( \Omega_{2,a,low,high} \), it is essentially the same as \( \Omega_{2,a,high,low} \) by symmetry.

On \( \Omega_{2,a,low,low} = \Omega_{2,a} \cap \{ |\xi_1|, |\xi_3| \lesssim 1 \} \), we consider the integral

\[
\text{(3.16)} \int_{\mathbb{R}^4} P_{\lesssim 1} u_1 P_{\gg 1} \overline{u_2} P_{\lesssim 1} u_3 P_{> 1} v \, dx dt .
\]
Recall that \( v \) is time localized function. Hence, by using Hölder’s inequality, we have

\[
\begin{align*}
\|P_{\xi_1} u_1 P_{\xi_2} u_2 P_{\xi_3} u_3\|_{L^1_t L^2_x} &\lesssim \|u\|_{L^\frac{5}{3}_t L^\infty_x} \|P_{\xi_1} u_1 P_{\xi_2} u_2 P_{\xi_3} u_3\|_{L^1_t L^2_x} \\
&\lesssim \|u\|_{L^2_t L^2_x} \|P_{\xi_1} u_1 P_{\xi_2} u_2 P_{\xi_3} u_3\|_{L^1_t L^2_x} \\
&\lesssim \|u\|_{X^{-1,-\frac{1}{2}}} \|P_{\xi_1} u_1 P_{\xi_2} u_2 P_{\xi_3} u_3\|_{L^1_t L^{\infty}_x}.
\end{align*}
\]

Note that

\[
|\mathcal{F}(P_{\xi_1} u_1 P_{\xi_2} u_2 P_{\xi_3} u_3)(\xi)| \lesssim \int_{\xi_1 + \xi_2 + \xi_3 = \xi} |\mathcal{F}(P_{\xi_1} u_1)(\xi_1)\mathcal{F}(P_{\xi_2} u_2)(\xi_2)\mathcal{F}(P_{\xi_3} u_3)(\xi_3)|
\lesssim (1_{|\xi| \leq 1} |\mathcal{F}(u)|) * (|\xi^{\frac{1}{2}}| 1_{|\xi| \geq 1} (|\xi|^s |\mathcal{F}(u)|)) * (1_{|\xi| \leq 1} |\mathcal{F}(u)|)(\xi).
\]

Therefore, from Strichartz estimates (3.6), (3.8) and using the low frequency range \(|\xi| \lesssim 1\), we have

\[
\begin{align*}
&&
\|P_{\xi_1} u_1 P_{\xi_2} u_2 P_{\xi_3} u_3\|_{L^1_t L^2_x} \\
&\lesssim \|P_{\xi_1} u_1 \mathcal{F}^{-1} |\mathcal{F}(u)| L^\infty_x L^\infty_y \| D^\frac{1}{2} P_{\xi_1} (D^s \mathcal{F}^{-1} |\mathcal{F}(u)|) \| L^\infty_t L^\infty_x \|P_{\xi_3} \mathcal{F}^{-1} |\mathcal{F}(u)|\|_{L^\infty_t L^\infty_x} \\
&\lesssim \|u_1\|_{X^{-\frac{1}{2},-\frac{1}{2}}} + \|u_2\|_{X^{-\frac{1}{2},-\frac{1}{2}}} + \|u_3\|_{X^{-\frac{1}{2},-\frac{1}{2}}}.\nonumber
\end{align*}
\]

**Subcase 2.2** \( \Omega_{2,b} = \{ |\xi_{\text{max}}| \gg 1 \text{ and } |\xi_2| \sim |\xi_3| \gg |\xi_1|, |\xi_4| \} \text{ or } \{ |\xi_{\text{max}}| \gg 1 \text{ and } |\xi_2| \sim |\xi_3| \gg |\xi_1|, |\xi_4| \} \). By symmetry of \( \xi_1 \) and \( \xi_3 \), we may assume \( \Omega_{2,b} = \{ |\xi_{\text{max}}| \gg 1 \text{ and } |\xi_2| \sim |\xi_3| \gg |\xi_1|, |\xi_4| \} \). We split the set \( \Omega_{2,b} \) into the following subsets:

\[
\begin{align*}
\Omega_{2,b,i} &= \{ |\xi_{\text{max}}| \gg 1 \text{ and } |\xi_2| \sim |\xi_3| \gg |\xi_1| \gg |\xi_1| \} \\
\Omega_{2,b,ii} &= \{ |\xi_{\text{max}}| \gg 1 \text{ and } |\xi_2| \sim |\xi_3| \gg |\xi_1| \gg |\xi_4| \}.
\end{align*}
\]

On \( \Omega_{2,b,i} \) the multiplier is estimated by

\[
m(\xi_1, \ldots, \xi_4) = \frac{\langle \xi_4 \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s} = \left( \frac{\langle \xi_1 \rangle}{\langle \xi_4 \rangle} \right)^{-s} \left( \frac{\langle \xi_2 \rangle}{\langle \xi_3 \rangle} \right)^{-s} \lesssim (\xi_2)^{\frac{s}{2}} (\xi_3)^{\frac{s}{2}} \sim |\xi_2|^{\frac{s}{2}} |\xi_3|^{\frac{s}{2}}.
\]

Then, on \( \Omega_{2,b,i} \), from Strichartz estimates (3.6) and (3.8), we have

\[
\begin{align*}
&\int_{\Gamma_{j \in \Omega_{2,b,i}}} \frac{\langle \xi_4 \rangle^s \langle \tau_4 - \xi_4 \rangle^{b-1}}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s (\tau_1 + \xi_1^{\frac{1}{2}+}) (\tau_2 - \xi_2^{\frac{1}{2}+}) (\tau_3 + \xi_3^{\frac{1}{2}+})} \prod_{j=1}^4 f_j(\tau_j, \xi_j) \\
&\lesssim \left\| \mathcal{F}^{-1} \left( \frac{f_1}{(\tau_1 + \xi_1^{\frac{1}{2}+})} \right) \right\|_{L^\infty_t L^2_x} \left\| \mathcal{F}^{-1} \left( \frac{f_2}{(\tau_2 - \xi_2^{\frac{1}{2}+})} \right) \right\|_{L^\infty_t L^2_x} \times \left\| \mathcal{F}^{-1} \left( \frac{f_3}{(\tau_3 + \xi_3^{\frac{1}{2}+})} \right) \right\|_{L^\infty_t L^2_x} \left\| f_4 \right\|_{L^2_t L^\infty_x} \\
&\lesssim \|u_1\|_{X^{-\frac{1}{2},-\frac{1}{2}}} + \|u_2\|_{X^{-\frac{1}{2},-\frac{1}{2}}} + \|u_3\|_{X^{-\frac{1}{2},-\frac{1}{2}}}.
\end{align*}
\]

\[
\]
On $\Omega_{2.b.ii}$, we need to observe that the interaction is nonresonant. More precisely, either the output or at least one of the inputs should have large modulation. Note that on the hyperplane $\{\xi_1 + \cdots + \xi_4 = 0, \tau_1 + \cdots + \tau_4 = 0\}$, we have

$$
(3.17) \quad |\tau_1 + \xi_1^4| + |\tau_2 - \xi_2^4| + |\tau_3 + \xi_3^4| + |\tau_4 - \xi_4^4| \\
\geq \left| (\xi_1 + \xi_2)(\xi_2 + \xi_3)(\xi_3^4 + \xi_3^2 + \xi_3^2 + (\xi_1 + \xi_2 + \xi_3)^2 + 2(\xi_1 + \xi_3)^2) \right|.
$$

For the proof of the factorization, see \[17\]. Therefore on $\Omega_{2.b.ii}$ we have

$$
\max\left( |\tau_1 + \xi_1^4|, |\tau_2 - \xi_2^4|, |\tau_3 + \xi_3^4|, |\tau_4 - \xi_4^4| \right) \geq |\xi_1 + \xi_2| |\xi_2^2 + \xi_3| |\xi_{\text{max}}^3|
$$

First, we consider the case $|\tau_4 - \xi_4^4| = \max\left( |\tau_1 + \xi_1^4|, |\tau_2 - \xi_2^4|, |\tau_3 + \xi_3^4|, |\tau_4 - \xi_4^4| \right)$. For $\frac{1}{2} < b \leq \frac{7}{8}$, the multiplier is estimated by

$$
\frac{m(\xi_1, \ldots, \xi_4)}{(\tau_4 - \xi_4^4)^{1-b}} \lesssim \frac{\langle \xi_4 \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \langle \xi_1 \rangle^{4(1-b)}} \\
\lesssim \langle \xi_1 \rangle^{\frac{b}{2}} \langle \xi_2 \rangle^{\frac{b}{2}} \langle \xi_3 \rangle^{\frac{b}{2}} \frac{1}{\langle \xi_1 \rangle^{4(1-b)}} \\
\lesssim \langle \xi_2 \rangle^{\frac{b}{2}} \langle \xi_3 \rangle^{\frac{b}{2}} \langle \xi_1 \rangle^{\frac{b}{2}} \lesssim |\xi_2|^{\frac{b}{2}} |\xi_3|^{\frac{b}{2}}.
$$

Hence, by using Strichartz estimates (3.6) and (3.8), one can proceed as in case $\Omega_{2.b.i}$. If $|\tau_1 + \xi_1^4| = \max\left( |\tau_1 + \xi_1^4|, |\tau_2 - \xi_2^4|, |\tau_1 + \xi_1^4|, |\tau_4 - \xi_4^4| \right)$, then we have

$$
\langle \tau_4 - \xi_4^4 \rangle^{1-b} \langle \tau_1 + \xi_1^4 \rangle^{\frac{b}{2}} \gtrsim \langle \tau_4 - \xi_4^4 \rangle^{\frac{b}{2}} \langle \tau_1 + \xi_1^4 \rangle^{1-b}.
$$

Therefore, by using (3.18), we consider the integral

$$
(3.19) \quad \int_{\Gamma_4 \cap \Omega_{2.b.ii}} \langle \xi_4 \rangle^s \langle \tau_1 + \xi_1^4 \rangle^{b-1} \langle \tau_2 - \xi_2^4 \rangle^{\frac{b}{2}} \langle \tau_3 + \xi_3^4 \rangle^{\frac{b}{2}} \langle \tau_4 - \xi_4^4 \rangle^{\frac{b}{2}} \prod_{j=1}^{4} f_j(\tau_j, \xi_j).
$$

For $\frac{1}{2} < b \leq \frac{7}{8}$, the multiplier is estimated by

$$
\frac{m(\xi_1, \ldots, \xi_4)}{(\tau_1 + \xi_1^4)^{1-b}} \lesssim \frac{\langle \xi_4 \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \langle \xi_1 \rangle^{4(1-b)}} \\
\lesssim \langle \xi_1 \rangle^{\frac{b}{2}} \langle \xi_2 \rangle^{\frac{b}{2}} \langle \xi_3 \rangle^{\frac{b}{2}} \frac{1}{\langle \xi_1 \rangle^{4(1-b)}} \\
\lesssim \langle \xi_2 \rangle^{\frac{b}{2}} \langle \xi_3 \rangle^{\frac{b}{2}} \langle \xi_1 \rangle^{\frac{b}{2}} \lesssim |\xi_2|^{\frac{b}{2}} |\xi_3|^{\frac{b}{2}}.
$$
Therefore, from Strichartz estimates (3.6) and (3.8), we have

\[ 0 \leq f_1 \parallel \mathcal{F}^{-1} \left( \frac{f_2}{\langle \tau_2 - \xi_2^2 \rangle^{\frac{1}{2}}} \right) \parallel_{L^1_tL^\infty_x} \times \parallel \mathcal{F}^{-1} \left( \frac{f_3}{\langle \tau_3 + \xi_3^2 \rangle^{\frac{1}{2}}} \right) \parallel_{L^1_tL^\infty_x} \parallel \mathcal{F}^{-1} \left( \frac{f_4}{\langle \tau_4 + \xi_4^2 \rangle^{\frac{1}{2}}} \right) \parallel_{L^\infty_tL^2_x} \leq \parallel u_1 \parallel_{X^{\frac{1}{2}+}} \parallel u_2 \parallel_{X^{\frac{1}{2}+}} \parallel u_3 \parallel_{X^{\frac{1}{2}+}}. \]

If \( |\tau_3 + \xi_3^2| = \max \{ |\tau_1 + \xi_1^2|, |\tau_2 - \xi_2^2|, |\tau_1 + \xi_1^2|, |\tau_4 - \xi_4^2| \} \), then we have

\[ \langle \tau_4 - \xi_4^2 \rangle^{1-b} \langle \tau_3 + \xi_3^2 \rangle^{\frac{1}{2}+} \gtrsim \langle \tau_4 - \xi_4^2 \rangle^{\frac{1}{2}+} \langle \tau_3 + \xi_3^2 \rangle^{1-b}. \]

Therefore, we consider the integral

\[ \langle \xi_4 \rangle^s \langle \tau_3 + \xi_3^2 \rangle^{b-1} \prod_{j=1}^4 \langle \xi_j \rangle^s \langle \xi_j \rangle \]

On \( \Omega_{2,b,ii} = \{ |\xi_{\text{max}}| \gg 1 \text{ and } |\xi_2| \sim |\xi_3| \gg |\xi_1| \gg |\xi_4| \} \), we may assume \( |\xi_1| \) is in the high frequency region \( |\xi_1| \gg 1 \). If \( |\xi_1| \) is in the low frequency region, we can proceed as in the case \( \Omega_{2,a,high,low} \).

For \( \frac{1}{2} b \leq \frac{2}{5} \), the multiplier is estimated by

\[ \frac{m(\xi_1, \ldots, \xi_4)}{\langle \tau_3 + \xi_3^2 \rangle^{\frac{1}{2}}} \lesssim \frac{\langle \xi_4 \rangle^s}{\langle \xi_1 \rangle^s} \frac{\langle \xi_2 \rangle^s}{\langle \xi_3 \rangle^s} \frac{\langle \xi_1 \rangle^{1-b} \langle \xi_3 \rangle^{3(1-b)}}{\langle \xi_1 \rangle^{1-b} \langle \xi_3 \rangle^{3(1-b)}} \lesssim \langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}}. \]

Therefore, from the Strichartz estimates (3.8) and (3.9), we have

\[ \lesssim \parallel D^\frac{1}{2} \mathcal{F}^{-1} \left( \frac{f_2}{\langle \tau_2 - \xi_2^2 \rangle^{\frac{1}{2}}} \right) \parallel_{L^1_tL^\infty_x} \parallel D^\frac{1}{2} \mathcal{F}^{-1} \left( \frac{f_3}{\langle \tau_3 + \xi_3^2 \rangle^{\frac{1}{2}}} \right) \parallel_{L^1_tL^\infty_x} \parallel D^\frac{1}{2} \mathcal{F}^{-1} \left( \frac{f_4}{\langle \tau_4 + \xi_4^2 \rangle^{\frac{1}{2}}} \right) \parallel_{L^\infty_tL^2_x} \leq \parallel u_1 \parallel_{X^{\frac{1}{2}+}} \parallel u_2 \parallel_{X^{\frac{1}{2}+}} \parallel u_3 \parallel_{X^{\frac{1}{2}+}}. \]

For the case \( |\tau_2 - \xi_2^2| = \max \{ |\tau_1 + \xi_1^2|, |\tau_2 - \xi_2^2|, |\tau_1 + \xi_1^2|, |\tau_4 - \xi_4^2| \} \), we can proceed as in the case \( |\tau_1 + \xi_1^2| = \max \{ |\tau_1 + \xi_1^2|, |\tau_2 - \xi_2^2|, |\tau_1 + \xi_1^2|, |\tau_4 - \xi_4^2| \} \).

**Subcase 2.c \( \Omega_{2,c} = \{ |\xi_{\text{max}}| \gg 1 \text{ and } |\xi_3| \sim |\xi_4| \gg |\xi_1|, |\xi_2| \} \) or \( \{ |\xi_{\text{max}}| \gg 1 \text{ and } |\xi_1| \sim |\xi_4| \gg |\xi_3|, |\xi_2| \} \).** By symmetry we may assume \( \Omega_{2,c} = \{ |\xi_{\text{max}}| \gg 1 \text{ and } |\xi_1| \sim |\xi_4| \gg |\xi_3|, |\xi_2| \} \). Note that

\[ m(\xi_1, \xi_2, \xi_3, \xi_4) \lesssim \langle \xi_1 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}}. \]

This subcase can be handled as in Subcase 2.a.
Subcase 2.d $\Omega_{2,d} = \{ |\xi_{\max} | \gg 1 \text{ and } |\xi_3 | \sim |\xi_1 | \gg |\xi_2 |, |\xi_4 | \}$. As in the case Subcase 2.b, we decompose the set $\Omega_{2,d}$ into the set $\{ |\xi_3 | \sim |\xi_1 | \gg |\xi_4 | \gg |\xi_2 | \}$ and $\{ |\xi_3 | \sim |\xi_1 | \gg |\xi_2 | \gg |\xi_4 | \}$. The remaining part can be proceed as in Subcase 2.b.

**Case 3.** $\Omega_3 = \{ |\xi_{\max} | \gg 1 \text{ and } |\xi_{\max} | \sim |\xi_{\text{sub}} | \sim |\xi_{\text{thd}} | \gg |\xi_{\text{min}} | \}$. Note that $|\xi_{\max} | \sim |\xi_{\text{sub}} | \sim |\xi_{\text{thd}} | \gg |\xi_{\text{min}} |$ and $\xi_1 + \cdots + \xi_4 = 0$ imply that

\[
\begin{align*}
|\xi_1 | = |\xi_{\text{min}} | \Rightarrow |\xi_1 + \xi_2 |,& \xi_3 | \xi_4 |^2 \sim |\xi_2 |,|\xi_4 |^2 \gtrsim |\xi_{\max} |^4, \\
|\xi_2 | = |\xi_{\text{min}} | \Rightarrow |\xi_1 + \xi_2 |,& \xi_3 | \xi_4 |^2 \sim |\xi_1 |,|\xi_3 |,|\xi_4 |^2 \gtrsim |\xi_{\max} |^4, \\
|\xi_3 | = |\xi_{\text{min}} | \Rightarrow |\xi_1 + \xi_2 |,& \xi_3 | \xi_4 |^2 \sim |\xi_1 |,|\xi_2 |,|\xi_4 |^2 \gtrsim |\xi_{\max} |^4, \\
|\xi_4 | = |\xi_{\text{min}} | \Rightarrow |\xi_1 + \xi_2 |,& \xi_3 | \xi_4 |^2 \sim |\xi_1 |,|\xi_2 |,|\xi_3 | \gtrsim |\xi_{\max} |^4.
\end{align*}
\]

Therefore, we have

\[
\max \left( |\tau_1 + \xi_1 |, \ldots, |\tau_4 - \xi_4 | \right) \gtrsim |\xi_{\max} |^4.
\]

We may assume $|\tau_4 - \xi_4 | = \max \left( |\tau_1 + \xi_1 |, \ldots, |\tau_4 - \xi_4 | \right)$ and hence $\langle \tau_4 - \xi_4 | \gtrsim |\xi_{\max} |^4$. The remaining case can be handled as in Subcase 2.b. Hence, we have

\[
\begin{align*}
\int_{\Gamma_{\xi} \cap \Omega_3} & \left( \frac{\langle \xi_4 |^4 \langle \xi_3 |^4 \langle \xi_2 |^4 \langle \xi_1 |^4 \langle \tau_1 + \xi_1 \rangle \langle \tau_2 - \xi_2 \rangle \langle \tau_3 + \xi_3 \rangle \langle \tau_4 - \xi_4 \rangle }{\prod_{j=1}^{4} f_j (\tau_j, \xi_j)} \right) d\tau_j d\xi_j \\
\lesssim & \int_{\Gamma_{\xi} \cap \Omega_3} \left( \frac{\langle \xi_{\max} |^4 \langle \xi_{\text{sub}} \rangle^2 \langle \xi_{\text{thd}} \rangle^2 \langle \tau_4 - \xi_4 |^4 \langle \tau_1 + \xi_1 \rangle \langle \tau_2 - \xi_2 \rangle \langle \tau_3 + \xi_3 \rangle }{\prod_{j=1}^{4} f_j (\tau_j, \xi_j)} \right) d\tau_j d\xi_j \\
\lesssim & \int_{\Gamma_{\xi} \cap \Omega_3} \left( \frac{\langle \xi_{\max} |^4 \langle \xi_{\text{sub}} \rangle^2 \langle \xi_{\text{thd}} \rangle^2 }{\langle \tau_1 + \xi_1 \rangle \langle \tau_2 - \xi_2 \rangle \langle \tau_3 + \xi_3 \rangle } \right) d\tau_j d\xi_j.
\end{align*}
\]

Therefore, from the Strichartz estimates (3.4) and (3.5), we obtain the desired result.

**Case 4.** $\Omega_4 = \{ |\xi_{\max} | \gg 1 \text{ and } |\xi_{\max} | \sim |\xi_{\text{sub}} | \sim |\xi_{\text{thd}} | \sim |\xi_{\text{min}} | \}$. Note that the multiplier is estimated by

\[
m (\xi_1, \xi_2, \xi_3, \xi_4) \lesssim \langle \xi_1 |^\frac{1}{2} \langle \xi_2 |^\frac{1}{2} \lesssim \langle \xi_1 |^\frac{1}{2} \langle \xi_2 |^\frac{1}{2}.
\]

Therefore, by using Strichartz estimates (3.6) and (3.5), we obtain the desired result. \(\blacksquare\)

4. **Global well-posedness on \(H^{-\frac{1}{2}}(\mathbb{R})\) and correction term strategy**

We introduce an even, smooth, and monotone multiplier \(m\) taking values in \([0, 1]\), and

\[
m (\xi) := \begin{cases}
1, & |\xi| < N \\
\left( \frac{|\xi|}{N} \right)^s, & |\xi| > 2N.
\end{cases}
\] (4.1)

Here, \(N\) is a large parameter to be determined later. We define an operator \(I\) by

\[
\tilde{I} u (\xi) := m (\xi) \hat{u} (\xi).
\]

Note that we have the estimate

\[
\| u \|_{H^{s}} \lesssim \| I u \|_{L^2} \lesssim N^{-s} \| u \|_{H^{s}}.
\]
Since \( \xi \)

\[ \text{By differentiating this, we have} \]

Note that from the Plancherel’s theorem, we obtain

\[ \text{(4.4)} \]

We define a multilinear form

\[ \Lambda_{n} (M_{n}; u_{1}, \ldots, u_{n}) := \int_{\xi_{1} + \cdots + \xi_{n} = 0} M_{n} (\xi_{1}, \ldots, \xi_{n}) \prod_{j=1}^{n} \hat{u}_{j} (\xi_{j}). \]

Usually we apply \( \Lambda_{n} \) on \( n \) same functions which are all \( u \). We will use the following notation:

\[ \Lambda \ (M_{n}) := \Lambda \ (M_{n}; u, \bar{u}, u, \bar{u}, \ldots, u, \bar{u}) \]

\[ = \int_{\xi_{1} + \cdots + \xi_{n} = 0} M_{n} (\xi_{1}, \ldots, \xi_{n}) \hat{u} (\xi_{1}) \hat{u} (\xi_{2}) \hat{u} (\xi_{3}) \hat{u} (\xi_{4}) \cdots \hat{u} (\xi_{n-1}) \hat{u} (\xi_{n}). \]

Note that \( \Lambda_{n} (M_{n}) \) is invariant under permutations of the even \( \xi_{j} \) indices, or of the odd \( \xi_{j} \) indices. We shall often write \( \xi_{ij} \) for \( \xi_{i} + \xi_{j}, \xi_{ijk} \) for \( \xi_{i} + \xi_{j} + \xi_{k} \). We also write \( \xi_{i-j} \) for \( \xi_{i} - \xi_{j}, \xi_{ij-klm} \) for \( \xi_{ij} - \xi_{klm} \).

Define the modified energy \( E_{I}^{2} (t) \)

\[ \text{(4.4)} \]

Note that from the Plancherel’s theorem, we obtain

\[ E_{I}^{2} (t) = \|Iu(t)\|_{L_{2}^{2}}^{2} \]

\[ = \int_{\xi_{1} + \xi_{2} = 0} m (\xi_{1}) m (\xi_{2}) \hat{u} (\xi_{1}) \hat{u} (\xi_{2}) \]

\[ = \Lambda_{2} (m (\xi_{1}) m (\xi_{2}); u). \]

By differentiating this, we have

\[ \frac{d}{dt} E_{I}^{2} (t) = \int_{\xi_{1} + \xi_{2} = 0} i (\xi_{1}^{4} - \xi_{2}^{4}) m (\xi_{1}) m (\xi_{2}) \hat{u} (\xi_{1}) \hat{u} (\xi_{2}) \]

\[ + \int_{\xi_{1} + \xi_{2} + \xi_{3} + \xi_{4} = 0} i (m^{2} (\xi_{1}) - m^{2} (\xi_{4})) \hat{u} (\xi_{1}) \hat{u} (\xi_{2}) \hat{u} (\xi_{3}) \hat{u} (\xi_{4}). \]

Since \( \xi_{1} + \xi_{2} = 0 \), the first term vanishes. By symmetrizing the second term, we have

\[ \frac{d}{dt} E_{I}^{2} (t) = \frac{1}{2} \int_{\xi_{1} + \xi_{2} + \xi_{3} + \xi_{4} = 0} i (m^{2} (\xi_{1}) - m^{2} (\xi_{2}) + m^{2} (\xi_{3}) - m^{2} (\xi_{4})) \hat{u} (\xi_{1}) \hat{u} (\xi_{2}) \hat{u} (\xi_{3}) \hat{u} (\xi_{4}) \]

\[ = \frac{i}{2} \Lambda_{4} (m^{2} (\xi_{1}) - m^{2} (\xi_{2}) + m^{2} (\xi_{3}) - m^{2} (\xi_{4})); u). \]

We set

\[ M_{4} (\xi_{1}, \ldots, \xi_{4}) = m^{2} (\xi_{1}) - m^{2} (\xi_{2}) + m^{2} (\xi_{3}) - m^{2} (\xi_{4}). \]
Hence, we obtain
\[
\frac{d}{dt} E_2(t) = i \Lambda_4(M_4) .
\]

Define a new modified energy
\[
E_4^i(t) = E_2^i(t) + \Lambda_4(\sigma_4) ,
\]
where the function \( \sigma_4 \) is symmetric under the even \( \xi_j \) indices, or of the odd \( \xi_j \) indices. The \( \sigma_4 \) will be determined later. Observe that
\[
\frac{d}{dt} \Lambda_4(\sigma_4) = \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} \sigma_4(\xi_1, \ldots, \xi_4) i (\xi_1^4 - \xi_2^4 + \xi_3^4 - \xi_4^4) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4)
\]
\[
+ \int_{\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6=0} 2i (\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5 + \xi_6) - \sigma_4(\xi_1, \xi_2, \xi_3 + \xi_4 + \xi_5, \xi_6)) \times \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4) \hat{u}(\xi_5) \hat{u}(\xi_6) .
\]

By differentiating the new modified energy \( E_4^i(t) \), we obtain
\[
\frac{d}{dt} E_4^i(t) = \Lambda_4(M_4) + \Lambda_4(\sigma_4 i (\xi_1^4 - \xi_2^4 + \xi_3^4 - \xi_4^4)) + 4 \Re(\Lambda_6(\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5 + \xi_6))).
\]

Define \( \alpha_n, \sigma_4 \) by
\[
\alpha_n := i (\xi_1^4 - \xi_2^4 + \cdots + (-1)^{n+1}\xi_n^4)
\]
and
\[
\sigma_4 := - \frac{M_4}{i \alpha_4}
\]
such that the two first terms in (4.6) are canceled. We define
\[
M_6(\xi_1, \ldots, \xi_6) = i \sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5 + \xi_6).
\]

Then, we have
\[
\frac{d}{dt} E_4^i(t) = 4 \Re(\Lambda_6(M_6)).
\]

We will prove two main properties. One is to show that modified energy \( E_4^i(t) \) is almost conserved. The other is to prove that it is close to \( E_2^i(t) \). To control the increment of \( E_4^i(t) \), it suffices to estimate its derivative
\[
\frac{d}{dt} E_4^i(t) = 4 \Re(\Lambda_6(M_6)).
\]

Hence, in the following subsections, we consider the multiplier \( M_6 \) and multilinear form \( \Lambda_6(M_6) \).
4.1. **Multiplier estimates.** In this subsection, we present multiplier estimates. If $m$ is of the form (4.1), then $m^2$ satisfies

\[
\begin{align*}
&\text{(4.9)} \\
&\begin{cases}
m^2(\xi) \sim m^2(\xi'), \quad |\xi| \sim |\xi'|, \\
(m^2)'(\xi) = O \left( \frac{m^2(\xi)}{|\xi|^{4}} \right), \\
(m^2)''(\xi) = O \left( \frac{m^2(\xi)}{|\xi|^{4}} \right).
\end{cases}
\end{align*}
\]

We will use two mean value formulas: if $|\eta|, |\lambda| \ll |\xi|$, then

\[
|a(\xi + \eta) - a(\xi)| \lesssim |\eta| \sup_{|\xi'| \sim |\xi|} |a'(\xi')|,
\]

and

\[
|a(\xi + \eta + \lambda) - a(\xi + \eta) - a(\xi + \lambda) + a(\xi)| \lesssim |\eta||\lambda| \sup_{|\xi'| \sim |\xi|} |a''(\xi')|.
\]

**Lemma 4.1.** Let $m$ be the multiplier of (4.1). On the area $|\xi_i| \sim N_i$, we have

\[
|\sigma_4(\xi_1, \ldots, \xi_4)| = \frac{M_4}{\alpha_4} \lesssim \frac{m^2(\min(N_i))}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}.
\]

**Proof.** In the following, we use the notation $|\xi_j + \xi_k| \sim N_{jk}$, where $N_{jk}$ are dyadic numbers. If $|\xi_{\text{max}}| \ll N$, then $m^2(\xi_1) - m^2(\xi_2) + m^2(\xi_3) - m^2(\xi_4) = 0$. Hence, we may assume that $|\xi_{\text{max}}| \sim |\xi_{\text{sub}}| \gtrsim N$. Recall that

\[
\frac{M_4}{\alpha_4} = \frac{m^2(\xi_1) - m^2(\xi_2) + m^2(\xi_3) - m^2(\xi_4)}{i(\xi_1 - \xi_2 + \xi_3 - \xi_4)}.
\]

Note that the resonance function on the hyperplane $\{\xi_1 + \cdots + \xi_4 = 0\}$ is given by

\[
\xi_1^4 - \xi_2^4 + \xi_3^4 - \xi_4^4 = (\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_3)(\xi_1 + \xi_2 + \xi_3 + 2(\xi_1 + \xi_3)^2).
\]

Therefore, on the hyperplane $\{\xi_1 + \cdots + \xi_4 = 0\}$ we consider

\[
\frac{m^2(\xi_1) - m^2(\xi_2) + m^2(\xi_3) - m^2(\xi_4)}{(\xi_1 + \xi_2)(\xi_1 + \xi_4)(\xi_1 + \xi_4)(\xi_1 + \xi_2 + \xi_3 + 2(\xi_1 + \xi_3)^2)}.
\]

We may assume that $N_{\text{max}} = N_1$. By symmetry, we can also assume $|\xi_{12}| \leq |\xi_{14}|$.

We consider the two cases.

**Case 1.** $|\xi_{14}| \ll N_1$.
**Case 2.** $N_1 \lesssim |\xi_{14}|$.

**Case 1.** $|\xi_{14}| \ll N_1$. In this case it suffices to show that

\[
|m^2(\xi_1) - m^2(\xi_2) + m^2(\xi_3) - m^2(\xi_4)| \lesssim |\xi_{12}||\xi_{14}| \frac{m^2(N_1)}{N_1^2}.
\]

By using the double mean value theorem (4.11), we have

\[
|m^2(\xi_1) - m^2(\xi_2) + m^2(\xi_3) - m^2(\xi_4)|
\]

\[
= |m^2(\xi_1 - \xi_{12} - \xi_{14}) - m^2(\xi_1 - \xi_{12}) - m^2(\xi_1 - \xi_{14}) + m^2(\xi_1)|
\]

\[
\lesssim |\xi_{12}||\xi_{14}| \frac{m^2(N_1)}{N_1^2}.
\]
Then, by the mean value theorem (4.10), we have
\[ N \| \frac{m^2(\xi_1) - m^2(\xi_2)}{\xi_1 + \xi_2} \| \quad \text{and} \quad \| \frac{m^2(\xi_3) - m^2(\xi_4)}{\xi_3 + \xi_4} \|. \]
If \( N \| \xi_1 \| \| N \| \xi_2 \| \), then we obtain the result directly. Hence, we may assume \( |\xi_1| \ll N \). Then, by the mean value theorem (4.10), we have
\[ |m^2(\xi_1) - m^2(\xi_2)| = |m^2(\xi_1) - m^2(\xi_1 - \xi_1)| \ll |\xi_1| \| \frac{m^2(N)}{N}. \]

By symmetry, we now assume \( N \| \xi_1 \| \| N \| \xi_2 \| \). As we did in the previous step, if \( N \| \xi_1 \| = |\xi_3| \), then we obtain the desired result. If \( |\xi_3| \ll N \), then by the mean value theorem (4.10) we have
\[ |m^2(\xi_3) - m^2(\xi_4)| = |m^2(\xi_3) - m^2(\xi_3 - \xi_3)| \ll |\xi_3| \| \frac{m^2(N)}{N}. \]

Note that if \( N \| \xi_1 \| \| N \| \xi_2 \| \), then we have \( m^2(\xi_3) - m^2(\xi_4) = 1 - 1 = 0 \). Therefore, it is enough to consider the case \( N \| \xi_1 \| \| N \| \xi_2 \| \) and hence we have \( N \| N \| N \) and hence we have \( N \| N \| N \).

By applying the above lemma, the estimate of \( M_6 \) follows directly from the estimate \( M_4 \).

**Proposition 4.2.** If \( m \) is of the form (4.1), then
\[ |M_6(\xi_1, ..., \xi_6)| \ll \frac{m^2(\min(N, N_{ijkl}))}{(N + N_1)(N + N_2)(N + N_3)(N + N_{456})}, \]
where \( N, N_{ijkl} \) are dyadic numbers and \( N \| \xi \| \| N \| \xi_{ijkl} \| \| \xi_{k + \xi + \xi} \| \).

**4.2. Almost conservation law.** In this subsection, we present the following almost conservation law.

**Lemma 4.3** (Almost conservation law.). Let \( 0 < \delta \leq 1 \). Then, for \( m \) (given by (4.7)) with \( s = -\frac{1}{2} \), we have
\[ |E^4_I(\delta) - E^4_J(0)| \ll \int_0^\delta \|4\Re_6(M_6)\| \, dt \ll N^{-3} \|I(\eta u)\|_x^{\epsilon, \delta} \|_x^{\epsilon, \delta} \]
where \( \eta \) is a smooth time cutoff function which satisfies \( \eta = 1 \) on \([0, \delta]\).

Before proving Lemma 4.3, we simplify the notation. Let us suppress the smooth time cut-off function \( \eta \) from \( \eta u \) and simply denote them by \( u \).

**Proof.** We may assume the functions \( \hat{u}_j \) are nonnegative. We project \( \hat{u}_j \) onto a dyadic piece \( |\xi| \sim N_j \) where \( N_j = 2^{k_j} \) is a dyadic number for \( k_j \in \{0, 1, \ldots, \} \). Note that we are not decomposing the frequencies \( |\xi| \leq 1 \) here. By symmetry, it suffices to show that
\[
\left| \int_0^\delta \Lambda_6 \left( \frac{m^2 \left( \min(N_i, N_{ijkl}) \right)}{(N + N_1)(N + N_2)(N + N_3)(N + N_{456})m(N_1)\cdots m(N_6)} \right) \, dt \right| < N^{-3} \|u\|_{X^{0,\frac{1}{+}}}^6
\]

By symmetry, we may assume \(N_1 \geq N_2 \geq N_3\) and \(N_4 \geq N_5 \geq N_6\). Observe that \(M_4\) vanishes if \(|\xi_i| \ll N\) for \(i = 1, 2, 3, 4\). Therefore, we assume at least one and hence two of the \(N_i \gtrsim N\). Note that

\[
\left| \int_0^\delta \Lambda_6 \left( \frac{m^2 \left( \min(N_i, N_{ijkl}) \right)}{(N + N_1)(N + N_2)(N + N_3)(N + N_{456})m(N_1)\cdots m(N_6)} \right) \, dt \right| < \left| \int_0^\delta \Lambda_6 \left( \prod_{j=1}^3 \frac{1}{(N + N_j)m(N_j)m(N_4)m(N_5)m(N_6)} \right) \, dt \right|.
\]

Note that with \(s = -\frac{1}{2}, \frac{1}{(N+N_j)m(N_j)} \lesssim N^{-\frac{1}{2}}N_i^{-\frac{1}{2}}\). We decompose the cases as follows:

**Case 1.** \(N \gg N_4 \geq N_5 \geq N_6\).
**Case 2.** \(N_4 \gtrsim N \gg N_5 \geq N_6\).
**Case 3.** \(N_4 \geq N_5 \gtrsim N \gg N_6\).
**Case 4.** \(N_4 \geq N_5 \geq N_6 \gtrsim N\).

**Case 1.** \(N \gg N_4 \geq N_5 \geq N_6\). Since the two largest frequencies should be comparable and one of \(N_1, N_2, N_3\) and \(N_{456}\) must be \(N_i \gtrsim N\), we have \(N_1 \sim N_2 \gtrsim N \gg N_4 \geq N_5 \geq N_6\). In this case, we have \(m(N_4) = m(N_5) = m(N_6) = 1\). Hence, we consider the integral

\[
\sum_{N_1, \ldots, N_6 \geq 1} N^{-\frac{3}{2}} \left| \int_0^\delta \Lambda_6 \left( N_1^{-\frac{1}{2}}N_2^{-\frac{1}{2}}N_3^{-\frac{1}{2}}N_4^{-\frac{1}{2}}N_5^{-\frac{1}{2}}N_6^{-\frac{1}{2}} \right) \, dt \right|.
\]

Note that \(\frac{N_1}{N_1^2 N_2^2 N_3^2 N_4^2 N_5^2 N_6^2} \gtrsim 1\). By multiplying this term, it suffices to bound

\[
\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left| \int_0^\delta \Lambda_6 \left( N_1^{-\frac{1}{2}}N_2^{-\frac{1}{2}}N_3^{-\frac{1}{2}}N_4^{-\frac{1}{2}}N_5^{-\frac{1}{2}}N_6^{-\frac{1}{2}} \right) \, dt \right| \lesssim N^{-3} \|u\|_{X^{0\frac{1}{+}}}^6.
\]
By the bilinear Strichartz estimate (3.5) and Bernstein (1.17), we have

\[
\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left| \int_0^\delta \mathcal{A}_6 \left( N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_1 N_2 N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) \, dt \right| \lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left( N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_1 N_2 N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right)
\]
\[
\times \| u_{N_1} u_{N_4} \|_{L^2_{t,x}} \| u_{N_2} u_{N_5} \|_{L^\infty_{t,x}} \| u_{N_3} \|_{L^\infty_{t,x}} \| u_{N_6} \|_{L^\infty_{t,x}}
\]

\[
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \| u \|_{X^{0, \frac{1}{2}+}}^6
\]

\[
\lesssim N^{-3} \| u \|_{X^{0, \frac{1}{2}+}}^6.
\]

**Case 2.** \( N_1 \gtrsim N \gg N_5 \geq N_6. \) In this case, we have \( m(N_5) = m(N_6) = 1. \) Since the two largest frequencies must be comparable, we obtain \( N_1 \gtrsim N_4 \) and hence \( N_1 \gtrsim N \gg N_5 \geq N_6. \) We consider the integral

\[
\sum_{N_1, \ldots, N_6 \geq 1} N^{-2} \left| \int_0^\delta \mathcal{A}_6 \left( N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) \, dt \right|.
\]

Note that \( \frac{N_1}{N_2 N_3} \frac{N_1}{N_4 N_5} \gtrsim 1. \) Then, by multiplying this term, it suffices to bound

\[
\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left| \int_0^\delta \mathcal{A}_6 \left( N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_1 N_2 N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) \, dt \right| \lesssim N^{-3} \| u \|_{X^{0, \frac{1}{2}+}}^6.
\]

From the bilinear Strichartz estimate (3.5) and Bernstein (1.17), we have

\[
\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left| \int_0^\delta \mathcal{A}_6 \left( N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_1 N_2 N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) \, dt \right| \lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left( N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_1 N_2 N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right)
\]
\[
\times \| u_{N_1} u_{N_4} \|_{L^2_{t,x}} \| u_{N_2} u_{N_5} \|_{L^\infty_{t,x}} \| u_{N_3} \|_{L^\infty_{t,x}} \| u_{N_6} \|_{L^\infty_{t,x}}
\]

\[
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left( N_1^{-\frac{1}{2}} N_1 N_2 N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) N_1^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_6^{-\frac{1}{2}}
\]
\[
\times \| u_{N_1} \|_{X^{0, \frac{1}{2}+}} \| u_{N_2} \|_{X^{0, \frac{1}{2}+}} \| u_{N_4} \|_{X^{0, \frac{1}{2}+}} \| u_{N_5} \|_{X^{0, \frac{1}{2}+}} \| u_{N_6} \|_{X^{0, \frac{1}{2}+}} \| u_{N_2} \|_{L^\infty_{t,x}} \| u_{N_4} \|_{L^\infty_{t,x}} \| u_{N_6} \|_{L^\infty_{t,x}}
\]

\[
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} N_1^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \| u \|_{X^{0, \frac{1}{2}+}}^6
\]

\[
\lesssim N^{-3} \| u \|_{X^{0, \frac{1}{2}+}}^6.
\]
Case 3. $N_4 \geq N_5 \gtrsim N \gg N_6$. In this case we have $m(N_6) = 1$. We consider an integral

$$\sum_{N_1, \ldots, N_6 \geq 1} N^{-\frac{3}{2}} \left| \int_{0}^{\delta} \mathcal{A}_6 \left( N_1^{-\frac{3}{2}} N_2^{-\frac{3}{2}} N_3^{-\frac{3}{2}} N_4^{-\frac{3}{2}} N_5^{-\frac{3}{2}} \right) dt \right|.$$ 

Note that $\frac{N_5}{N_2^\frac{3}{2} N_6^\frac{3}{2}} \gtrsim 1$. Therefore, by multiplying this term, it suffices to bound

$$\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left| \int_{0}^{\delta} \mathcal{A}_6 \left( N_1^{-\frac{1}{2}} N_2^{-\frac{3}{2}} N_3^{-\frac{3}{2}} N_4^{-\frac{3}{2}} N_5^{-\frac{3}{2}} N_6^{-\frac{3}{2}} \right) dt \right| \lesssim N^{-3} \| u \|_{X^{0, \frac{1}{3} +}}^{6}.$$ 

Subcase 3.a. $N_1 \gg N_4$. Since the two largest frequencies must be comparable, we have $N_1 \sim N_2 \gg N_4 \geq N_5 \gg N_6$. By using the bilinear Strichartz \cite{3.5} and Bernstein’s inequality \cite{1.17}, we have

$$\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left| \int_{0}^{\delta} \mathcal{A}_6 \left( N_1^{-\frac{1}{2}} N_2^{-\frac{3}{2}} N_3^{-\frac{3}{2}} N_4^{-\frac{3}{2}} N_5^{-\frac{3}{2}} N_6^{-\frac{3}{2}} \right) dt \right| \lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left( N_1^{-\frac{1}{2}} N_2^{-\frac{3}{2}} N_3^{-\frac{3}{2}} N_4^{-\frac{3}{2}} N_5^{-\frac{3}{2}} N_6^{-\frac{3}{2}} \right)$$

$$\times \| u_{N_1} u_{N_4} \|_{L_{t,x}^{6}} \| u_{N_2} u_{N_5} \|_{L_{t,x}^{6}} \| u_{N_3} \|_{L_{t,x}^{6}} \| u_{N_6} \|_{L_{t,x}^{6}}$$

$$\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left( N_1^{-\frac{1}{2}} N_2^{-\frac{3}{2}} N_3^{-\frac{3}{2}} N_4^{-\frac{3}{2}} N_5^{-\frac{3}{2}} N_6^{-\frac{3}{2}} \right)$$

$$\times \| u_{N_1} \|_{X^{0, \frac{1}{3} +}} \| u_{N_4} \|_{X^{0, \frac{1}{3} +}} \| u_{N_2} \|_{X^{0, \frac{1}{3} +}} \| u_{N_5} \|_{X^{0, \frac{1}{3} +}} \| u_{N_6} \|_{L_{t,x}^{6}}$$

$$\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \| u \|_{X^{0, \frac{1}{3} +}}^{6}.$$ 

Subcase 3.b. $N_4 \gg N_1$. Since the two largest frequencies must be comparable, we have $N_1 \sim N_5 \gg N_1 \geq N_2 \geq N_3$. By using the bilinear Strichartz \cite{3.5} and Bernstein’s inequality \cite{1.17}, we have

$$\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left| \int_{0}^{\delta} \mathcal{A}_6 \left( N_1^{-\frac{1}{2}} N_2^{-\frac{3}{2}} N_3^{-\frac{3}{2}} N_4^{-\frac{3}{2}} N_5^{-\frac{3}{2}} N_6^{-\frac{3}{2}} \right) dt \right| \lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left( N_1^{-\frac{1}{2}} N_2^{-\frac{3}{2}} N_3^{-\frac{3}{2}} N_4^{-\frac{3}{2}} N_5^{-\frac{3}{2}} N_6^{-\frac{3}{2}} \right)$$

$$\times \| u_{N_1} u_{N_4} \|_{L_{t,x}^{6}} \| u_{N_2} u_{N_5} \|_{L_{t,x}^{6}} \| u_{N_3} \|_{L_{t,x}^{6}} \| u_{N_6} \|_{L_{t,x}^{6}}$$

$$\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left( N_1^{-\frac{1}{2}} N_2^{-\frac{3}{2}} N_3^{-\frac{3}{2}} N_4^{-\frac{3}{2}} N_5^{-\frac{3}{2}} N_6^{-\frac{3}{2}} \right)$$

$$\times \| u_{N_1} \|_{X^{0, \frac{1}{3} +}} \| u_{N_4} \|_{X^{0, \frac{1}{3} +}} \| u_{N_2} \|_{X^{0, \frac{1}{3} +}} \| u_{N_5} \|_{X^{0, \frac{1}{3} +}} \| u_{N_6} \|_{L_{t,x}^{6}}$$

$$\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \| u \|_{X^{0, \frac{1}{3} +}}^{6}.$$
Subcase 3.c. $N_4 \sim N_1$. Note that $N_1 \sim N_4 \geq N_5 \gg N_6$. From the bilinear Strichartz estimate (3.5), Strichartz estimate (3.4) and Bernstein inequality (1.17), we have

$$
\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \int_0^\delta \Lambda_6 \left(N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) dt \\
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left(N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) \\
\times \|u_{N_1} u_{N_6}\|_{L^2_{t,x}} \|u_{N_2}\|_{L^\infty_{t} L^2_x} \|u_{N_3}\|_{L^\infty_{t} L^2_x} \|u_{N_4}\|_{L^\infty_{t} L^2_x} \|u_{N_6}\|_{L^\infty_{t} L^2_x} \\
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left(N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) N_1^{-\frac{1}{2}} \|u\|_{\dot{X}^{0,-\frac{1}{2}}.}}$

Case 4. $N_4 \geq N_5 \geq N_6 \gg N$.

Subcase 4.a. $N_1 \gg N_4$. Since the two largest frequencies must be comparable, we have $N_1 \sim N_2 \gg N_4 \geq N_5 \geq N_6$. In this case, we consider the integral

$$
\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \int_0^\delta \Lambda_6 \left(N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) dt.
$$

By the bilinear Strichartz estimate (3.3) and Bernstein inequality (1.17), we have

$$
\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \int_0^\delta \Lambda_6 \left(N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) dt \\
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left(N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) \\
\times \|u_{N_1} u_{N_4}\|_{L^2_{t,x}} \|u_{N_2} u_{N_6}\|_{L^2_{t,x}} \|u_{N_3}\|_{L^\infty_{t} L^2_x} \|u_{N_4}\|_{L^\infty_{t} L^2_x} \|u_{N_6}\|_{L^\infty_{t} L^2_x} \\
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left(N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \\
\times \|u_{N_1}\|_{\dot{X}^{0,-\frac{1}{2}}} \|u_{N_4}\|_{\dot{X}^{0,-\frac{1}{2}}} \|u_{N_2}\|_{\dot{X}^{0,-\frac{1}{2}}} \|u_{N_6}\|_{\dot{X}^{0,-\frac{1}{2}}} \|u_{N_3}\|_{L^\infty_{t} L^2_x} \|u_{N_4}\|_{L^\infty_{t} L^2_x} \|u_{N_6}\|_{L^\infty_{t} L^2_x} \\
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} N_1^{-2} N_2^{-2} N_3^{-2} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \|u\|_{\dot{X}^{0,-\frac{1}{2}}} \\
\lesssim N^{-3} \|u\|_{\dot{X}^{0,-\frac{1}{2}}}.
$$

Subcase 4.b $N_4 \gg N_1$. Since the two largest frequencies must be comparable, we have $N_4 \sim N_5 \gg N_1 \geq N_2 \geq N_3$. In this case, it suffices to consider the integral

$$
\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \int_0^\delta \Lambda_6 \left(N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \right) dt.
$$
By the bilinear Strichartz estimate (4.3) and Bernstein’s inequality (1.17), we have

\[
\sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \int_0^\delta \Lambda_6 \left( N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^\frac{\lambda}{2} N_5^{\frac{\lambda}{2}} N_6^{\frac{\lambda}{2}} \right) dt \\
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left( N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^\frac{\lambda}{2} N_5^{\frac{\lambda}{2}} N_6^{\frac{\lambda}{2}} \right) \\
\times \| u_{N_1} u_{N_4} \|_{L_t^\infty L_x^6} \| u_{N_2} u_{N_5} \|_{L_t^\infty L_x^2} \| u_{N_3} \|_{L_t^\infty L_x^\infty} \\
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} \left( N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^\frac{\lambda}{2} N_5^{\frac{\lambda}{2}} N_6^{\frac{\lambda}{2}} \right) N_4^{-\frac{3}{2}} N_5^{-\frac{1}{2}} N_6^{\frac{\lambda}{2}} \\
\times \| u_{N_1} \|_{X_0^\lambda \dot{B}^0_6} \| u_{N_4} \|_{X_0^\lambda \dot{B}^0_6} \| u_{N_2} \|_{X_0^\lambda \dot{B}^0_6} \| u_{N_5} \|_{X_0^\lambda \dot{B}^0_6} \| u_{N_3} \|_{L_t^\infty L_x^2} \| u_{N_6} \|_{L_t^\infty L_x^\infty} \\
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} N^{-3} N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^\frac{\lambda}{2} N_5^{\frac{\lambda}{2}} N_6^{\frac{\lambda}{2}} \| u \|_{X_0^\lambda \dot{B}^0_6}^6 \\
\lesssim N^{-3} \| u \|_{X_0^\lambda \dot{B}^0_6}^6.
\]

**Subcase 4.c** \( N_4 \sim N_1 \). We consider the integral

\[
N^{-3} \int_0^\delta \Lambda_6 \left( N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^\frac{\lambda}{2} N_5^{\frac{\lambda}{2}} N_6^{\frac{\lambda}{2}} \right) dt.
\]

From the Strichartz estimate (4.3), we have

\[
N^{-3} \int_0^\delta \Lambda_6 \left( N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^\frac{\lambda}{2} N_5^{\frac{\lambda}{2}} N_6^{\frac{\lambda}{2}} \right) dt \\
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} \left( N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^\frac{\lambda}{2} N_5^{\frac{\lambda}{2}} N_6^{\frac{\lambda}{2}} \right) \left( N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^\frac{\lambda}{2} N_5^{\frac{\lambda}{2}} N_6^{\frac{\lambda}{2}} \right) \\
\times \| u_{N_1} \|_{L_t^\infty L_x^\infty} \| u_{N_2} \|_{L_t^\infty L_x^2} \| u_{N_3} \|_{L_t^\infty L_x^\infty} \| u_{N_4} \|_{L_t^\infty L_x^\infty} \| u_{N_5} \|_{L_t^\infty L_x^\infty} \| u_{N_6} \|_{L_t^\infty L_x^\infty} \\
\lesssim \sum_{N_1, \ldots, N_6 \geq 1} \left( N_1^{-\frac{3}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^\frac{\lambda}{2} N_5^{\frac{\lambda}{2}} N_6^{\frac{\lambda}{2}} \right) \| u \|_{X_0^\lambda \dot{B}^0_6}^6 \\
\lesssim N^{-3} \| u \|_{X_0^\lambda \dot{B}^0_6}^6.
\]

This completes the proof of Lemma 4.3.

\[\square\]

4.3. **Correction term estimate.** The following lemma shows that \( E_1^I(t) \) is very close to \( E_2^I(t) \).

**Lemma 4.4.** Let \( I \) be defined with the multiplier \( m \) of the form (4.1) and \( s = -\frac{1}{2} \). Then,

\[
| E_1^I(t) - E_2^I(t) | \lesssim \| I u(t) \|_{L_x^2}^4.
\]

**Proof.** Note that \( E_1^I(t) = E_2^I(t) + \Lambda_4 (\sigma_4) \) and

\[
| \sigma_4 (\xi_1, \ldots, \xi_4) | \lesssim \frac{m^2 (\min (N_i))}{(N + N_1) (N + N_2) (N + N_3) (N + N_4)}
\]
with $|\xi_i| \sim N_i$. We restrict our attention to the contribution arising from $|\xi_i| \sim N_i$. It suffices to bound

$$|\Lambda_4(\sigma_4)| \lesssim \prod_{j=1}^{4} \|Iu_j(t)\|_{L^2_x}.$$ 

We may assume the functions $\hat{u}_j$ are nonnegative. We project $\hat{u}_j$ to a dyadic piece $|\xi_j| \sim N_j$ where $N_j = 2^k_j$ is a dyadic number for $k_j \in \{0, 1, \ldots\}$. Note that we are not decomposing the frequencies $|\xi| \leq 1$ here. We may assume $N_1 \geq N_2 \geq N_3 \geq N_4$ with $N_1 \sim N_2 \gtrsim N$. Therefore, for $s = -\frac{3}{2}$, it suffices to bound

\[(4.12) \quad \sum_{N_1, \ldots, N_4 \geq 1} \Lambda_4 \left( \frac{1}{(N + N_1) (N + N_2) (N + N_3) (N + N_4)} \frac{1}{m(N_1) m(N_2) m(N_3) m(N_4)} \right) \leq \prod_{j=1}^{4} \|u_j\|_{L^2_x} \]

The multiplier appearing in (4.12) is

$$\frac{N^{4s}}{N_1^{\frac{3}{2} + s} N_2^{\frac{3}{2} + s} N_3^{\frac{3}{2} + s} N_4^{\frac{3}{2} + s}} \lesssim N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}}.$$ 

Then, by Bernstein inequality (1.17), we have

$$|\Lambda_4(\sigma_4)| \lesssim \sum_{N_1 \geq N_2 \geq N_3 \geq N_4 \geq 1} N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} \|u_{N_1}\|_{L^2_x} \|u_{N_2}\|_{L^2_x} \|u_{N_3}\|_{L^2_x} \|u_{N_4}\|_{L^2_x} \leq \sum_{N_1 \geq N_2 \geq N_3 \geq N_4 \geq 1} N_1^{-\frac{1}{2}} N_2^{-\frac{1}{2}} \|u_1\|_{L^2_x} \|u_2\|_{L^2_x} \|u_3\|_{L^2_x} \|u_4\|_{L^2_x} \lesssim \prod_{j=1}^{4} \|u_j\|_{L^2_x}.$$ 

This completes the proof of Lemma 4.4. \qed

4.4. Well-posedness of $I$-system. Before we construct a global solution, we show the following modified local well-posedness.

**Proposition 4.5.** Consider the initial value problem

\[
\begin{aligned}
&i\partial_t Iu + I\partial_x^3 u = I(|u|^2 u) \\
&Iu(0, x) = Ig(x) \in L^2(\mathbb{R}).
\end{aligned}
\]

Then, the initial value problem is locally well-posed (in $L^2(\mathbb{R})$) on an interval $[-\delta, \delta]$ with $\delta \sim \|Ig\|_{L^2}^{\alpha}$ for some $\alpha > 0$ and satisfies the bound

$$\|I(\eta u)|\|_{X^{\alpha, \frac{1}{2}+, \frac{1}{2}+}} \lesssim \|Ig\|_{L^2},$$

where $\eta$ is a smooth time cutoff function which satisfies $\eta = 1$ on $[-\delta, \delta]$.

**Proof.** Let us suppress the smooth time cut-off function $\eta$ from $\eta u$ and simply denote them by $u$. It suffices to show the following trilinear estimate

$$\|I(|u|^2 u)\|_{X^{\alpha, \frac{1}{2}, \frac{1}{2}}+} \lesssim \|Iu\|_{X^{\alpha, \frac{1}{2}+}}.$$
Note that

\[ \|I(\|u\|^2 u)\|_{X^{0,-\frac{1}{2}}} = \sup_{\|v\|_{X^{0,1-b}} = 1} \left| \int_{\mathbb{R} \times \mathbb{R}} I(u\bar{u}v) \, dx \, dt \right| \]

\[ = \sup_{\|v\|_{X^{0,1-b}} = 1} \left| \int_{\mathbb{T}_{1}}^{\mathbb{T}_{4}} m(\xi_1)\hat{u}_1(\tau_1, \xi_1)\hat{u}_2(\tau_2, \xi_2)\hat{u}_3(\tau_3, \xi_3)\hat{v}(\tau_4, \xi_4) \right| . \]

Now set

\[ f_1(\tau_1, \xi_1) = |\hat{u}_1(\tau_1, \xi_1)| \langle \tau_1 + \xi_1^4 \rangle^{\frac{1}{2}} m(\xi_1), \]
\[ f_2(\tau_2, \xi_2) = |\hat{u}_2(\tau_2, \xi_2)| \langle \tau_2 - \xi_2^4 \rangle^{\frac{1}{2}} m(\xi_2), \]
\[ f_3(\tau_3, \xi_3) = |\hat{u}_3(\tau_3, \xi_3)| \langle \tau_3 + \xi_3^4 \rangle^{\frac{1}{2}} m(\xi_3), \]
\[ f_4(\tau_4, \xi_4) = |\hat{v}(\tau_4, \xi_4)| \langle \tau_4 - \xi_4^4 \rangle^{\frac{1}{2}} . \]

Therefore, it suffices to show that for nonnegative $L^2$ functions $f_j$

\[ \int_{\xi_1+\cdots+\xi_4=0, \tau_1+\cdots+\tau_4=0} \frac{m(\xi_4) \langle \tau_4 - \xi_4^4 \rangle^{-\frac{1}{2}}}{m(\xi_1) m(\xi_2) m(\xi_3) \langle \tau_1 + \xi_1^4 \rangle^{\frac{1}{2}} \langle \tau_2 + \xi_2^4 \rangle^{\frac{1}{2}} \langle \tau_3 + \xi_3^4 \rangle^{\frac{1}{2}}} \prod_{j=1}^{4} f_j(\tau_j, \xi_j) \]
\[ \lesssim \prod_{j=1}^{4} \|f_j\|_{L^2_{\tau,\xi}} . \]

Note that $m(\xi) \langle \xi \rangle^{\frac{1}{2}}$ is increasing in $\xi$ and $m(\xi) \langle \xi \rangle^{\frac{1}{2}} \ge 1$ for all $\xi \in \mathbb{R}$. By symmetry, we may assume that $|\xi_1| \ge |\xi_2| \ge |\xi_3|$. Also, $\xi_1 + \cdots + \xi_4 = 0$ implies $|\xi_{\max}| \sim |\xi_{\sub}|$ and $|\xi_1| \ge |\xi_4|$. Note that

\[ \frac{m(\xi_4)}{m(\xi_1) m(\xi_2) m(\xi_3) \langle \tau_1 + \xi_1^4 \rangle^{\frac{1}{2}} \langle \tau_2 + \xi_2^4 \rangle^{\frac{1}{2}} \langle \tau_3 + \xi_3^4 \rangle^{\frac{1}{2}}} \lesssim \frac{1}{m(\xi_2) m(\xi_3) \langle \xi_3 \rangle^{\frac{1}{2}}} \lesssim 1. \]

Therefore, it is enough to show that

\[ \int_{\xi_1+\cdots+\xi_4=0, \tau_1+\cdots+\tau_4=0} \langle \xi_1 \rangle^{\frac{1}{4}} \langle \xi_2 \rangle^{\frac{1}{4}} \langle \xi_3 \rangle^{\frac{1}{4}} \langle \tau_4 - \xi_4^4 \rangle^{-\frac{1}{2}} \prod_{j=1}^{4} f_j(\tau_j, \xi_j) \]
\[ \lesssim \prod_{j=1}^{4} \|f_j\|_{L^2_{\tau,\xi}} . \]

Hence, the trilinear estimate follows directly by just repeating the proof of Proposition 3.5.

\[ \square \]

4.5. Proof of global well-posedness for (INLS). In this subsection, we prove global well-posedness of (INLS). For any given $u_0 \in H^s$, $s \ge -\frac{1}{4}$ and $T > 0$, our goal is to construct a solution on $[0, T]$. Suppose that $u$ is a solution to (INLS) with initial data $u_0$. Then, for any $\lambda > 0$, $u_\lambda(x, t) = \lambda^{-2} u(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ is also a solution to (INLS) with initial data $u_{0, \lambda} = \lambda^{-2} u_0(\frac{\cdot}{\lambda})$. Recall that

\[ \|u\|_{H^s} \lesssim \|Iu\|_{L^2} \lesssim N^{-s} \|u\|_{H^s} . \]
Straightforward calculation shows
\[ \|Iu_{0, \lambda}\|_{L^2} \lesssim \lambda^{-\frac{2}{3} - s} N^{-s} \|u_0\|_{H^{\frac{1}{2}}} . \]
The parameter \( N \) will be chosen later but we take \( \lambda \) now such that
\[ \lambda^{-\frac{2}{3} - s} N^{-s} \|u_0\|_{H^{\frac{1}{2}}} = \varepsilon_0 < 1 \quad \Rightarrow \quad \lambda \sim N^{-\frac{2}{3+s}} . \]
For simplicity of notations, we still denote \( u_\lambda \) by \( u, u_{0, \lambda} \) by \( u_0 \) and assume \( \|Iu_0\|_{L^2} \leq \varepsilon_0 \). The goal is to construct solutions on \([0, \lambda^4 T]\). We already have the local solution on \([0, 1]\) and need to extend the solution on \([0, \lambda^4 T]\). It suffices to control the modified energy \( E_1^2(t) = \|Iu\|_{L^2}^2 \).

First, we control \( E_1^2(t) \) for \( t \in [0, 1] \). By a standard bootstrapping argument, we may assume \( E_1^2(t) < 4\varepsilon_0^2 \) for \( t \in [0, 1] \). By Lemma 4.3, we know that modified energies \( E_1^2(t) \) and \( E_1^2(t) \) are very close, i.e.
\[ E_1^2(0) = E_1^2(0) + O(\varepsilon_0^4) , \]
and
\[ E_1^2(t) = E_1^2(t) + O(\varepsilon_0^4) . \]
for \( t \in [0, 1] \).

Also, it follows from Lemma 4.3 that for all \( t \in [0, 1] \), we have
\[ E_1^2(t) \leq E_1^2(0) + C\varepsilon_0^6 N^{-3} . \]
Then, our rescaled solution satisfies
\[ \|u(1)\|_{H^{\frac{1}{2}}} \leq \|Iu(1)\|_{L^2} = E_1^2(1) + O(\varepsilon_0^4) \leq E_1^2(0) + C\varepsilon_0^6 N^{-3} + O(\varepsilon_0^4) \leq \varepsilon_0^2 + C\varepsilon_0^6 N^{-3} + O(\varepsilon_0^4) \leq 4\varepsilon_0^2 . \]

Now it suffices to do an iteration. We now consider the initial value problem for \( u(1) \) with initial data \( u(1) \). By the above bound \( \|Iu(1)\|_{L^2} < 4\varepsilon_0^2 \), the local solution extend on \([0, 2]\). By iterating this procedure \( M \) steps, we obtain
\[ E_1^2(t) \leq E_1^2(0) + MC\varepsilon_0^6 N^{-3} . \]
for all \( t \in [0, M + 1] \). As long as \( MN^{-3} \sim 1 \), we have the bound
\[ \|u(M)\|_{H^{\frac{1}{2}}} \leq \|Iu(M)\|_{L^2} = E_1^2(t) + O(\varepsilon_0^4) \leq \varepsilon_0^2 + O(\varepsilon_0^4) + CM\varepsilon_0^6 N^{-3} < 4\varepsilon_0^2 , \]
and the lifetime span of the local result remains uniformly of size 1. We choose \( M \sim N^3 \). This process extends the local solution to the time interval \([0, N^3]\). Take \( N \) such that
\[ N^3 \sim \lambda^4 T \sim N^{-\frac{3}{3+s}} T \quad \text{or} \quad N^{3+\frac{s}{3+s}} \sim T , \]
which is certainly done for \( s \geq -\frac{1}{2} \). Therefore, the solution \( u \) is extended on \([0, \lambda^4 T]\) for arbitrary fixed time \( T \). This completes the proof of global well-posedness for \( \text{4NLS} \) in \( H^s(\mathbb{R}), s \geq -\frac{1}{2} \).

In the end of this section, we prove some properties of the global solution. By rescaling of our global solution, we get
\[ \sup_{[0, T]} \|u(t)\|_{H^{\frac{1}{2}}} \sim \lambda^{\frac{3}{2} + s} \sup_{t \in [0, \lambda^4 T]} \|u_\lambda(t)\|_{H^{\frac{1}{2}}} \leq \lambda^{\frac{3}{2} + s} \sup_{t \in [0, \lambda^4 T]} \|Iu_\lambda(t)\|_{L^2} , \]
and

$$\|Iu_{0,\lambda}\|_{L^2} \lesssim N^{-s}\|u_{0,\lambda}\|_{H^s} \sim N^{-s}\lambda^{-\frac{2}{3}-s}\|u_0\|_{H^s}.$$ 

From the above local well-posedness iteration argument, we have

$$\sup_{t \in [0, \lambda^4T]} \|Iu_{\lambda}(t)\|_{L^2} \lesssim \|Iu_{0,\lambda}\|_{L^2},$$

and hence

$$\sup_{t \in [0, T]} \|u(t)\|_{H^s} \lesssim N^{-s}\|u_0\|_{H^s}.$$ 

Recall that the parameter $\lambda$ was chosen such that $\|Iu_{0,\lambda}\|_{L^2} \sim \varepsilon_0$. Thus we get $\lambda \sim N^{-\frac{3}{2}}$. Also the parameter $N$ is chosen such that $N^3 \sim \lambda^4T$ or $N^\frac{14}{13+s} \sim T$. This shows that the selection of $N$ is polynomial in $T$. So this gives a polynomial growth bound on $\|u(t)\|_{H^s}$

$$\|u(t)\|_{H^\frac{1}{2}} \lesssim t^\frac{1}{2}\|u_0\|_{H^{-\frac{1}{2}}}.$$ 

5. Mild ill-posedness

In this section we give the proof of Theorem 1.6. We follow the argument introduced by Christ-Colliander-Tao [4]. In [4], Christ-Colliander-Tao established two solutions to (NLS) breaking the uniform continuity of the flow map for $s < 0$. The method presented in Christ-Colliander-Tao [4] can be applied to both focusing and defocusing cases. They used the Galilean and scale invariances to construct the counter example. For the mKdV case, instead of using Galilean invariance, they used the approximate solution to the mKdV equation by exploiting the solution of (NLS). Likewise, we approximate (4NLS) by (NLS) to prove mild ill-posedness.

5.1. Mild ill-posedness result for the cubic NLS. First, we state the ill-posedness result for (NLS) in [4].

**Theorem 5.1 ([4]).** The solution map of the cubic NLS in $H^s$ for $s < 0$ fails to be uniformly continuous. More precisely, there exists $\varepsilon_0 > 0$ such that for any $\delta > 0, T > 0$ and $\varepsilon < \varepsilon_0$, there are two solutions $u_1, u_2$ to (5.5) such that

(5.1) $\|u_1(0)\|_{H^s}, \|u_2(0)\|_{H^s} \lesssim \varepsilon$, 

(5.2) $\|u_1(0) - u_2(0)\|_{H^s} \lesssim \delta$, 

(5.3) $\sup_{0 \leq t < T} \|u_1(t) - u_2(t)\|_{H^s} \gtrsim \varepsilon$.

This implies the solution map is not uniformly continuous from the ball $\{u_0 \in H^s_x : \|u_0\|_{H^s_x} \lesssim \varepsilon\}$ to $C^\infty([0, T]; H^s_x)$.

**Remark 5.2.** In the case of focusing cubic NLS, there is another way to show mild ill-posedness. Kenig-Ponce-Vega [14] demonstrated mild ill-posedness of the focusing cubic NLS in the sense that the solution map fails to be locally uniformly continuous for $s < 0$. They used the Galilean invariance on the soliton solutions to obtain mild ill-posedness result. Recall that if $u(t, x)$ is a solution of the cubic NLS with initial data $u_0$, then $u_N(t, x) = e^{iN^2x^2 - iN^2u(t, x - 2N)}$ is another solution with initial data $u_N(0, x) = e^{iN^2x^2u_0(x)}$. Observe that for $s < 0$, we have

(5.4) $\lim_{N \to \infty} \|u_N(0, x)\|_{H^s(\mathbb{R})} = 0.$
By applying the Galilean symmetry to the soliton solution \( u \), we consider the solutions \( u_{N_1}, u_{N_2} \). By using (5.4), we can make \( \| u_{N_1}(0, \cdot) - u_{N_2}(0, \cdot) \|_{H^s} \) sufficiently small by choosing \( N_1 \) and \( N_2 \) large enough. Notice that \( u_{N_1}, u_{N_2} \) move with different speeds. Therefore, the difference \( \| u_{N_1}(t) - u_{N_2}(t) \|_{H^s} \) is bounded below by some fixed constant.

5.2. **Approximate solution.** In the following, we only consider defocusing (4NLS). The same argument can be applied in the focusing case. First, we find the approximate solution to defocusing (4NLS) by using the solution of the cubic NLS

\[
i \partial_t u = \partial_x^2 u - |u|^2 u.
\]

Assume that \( u(s, y) \) solves (5.5). We use the following change of variable

\[
(s, y) := \left( t, \frac{x + 4N^3 t}{\sqrt{6N}} \right).
\]

Then, we define the approximate solution

\[
U_{ap}(t, x) := e^{iN^4 t} e^{iN^4 x} u(s, y),
\]

where \( N \gg 1 \) will be chosen later.

We want to show that \( U_{ap} \) is an approximate solution to the defocusing (4NLS). A straightforward calculation shows that

\[
i \partial_t U_{ap}(t, x) = i e^{iN^4 x} (iN^4) e^{iN^4 x} u \left( t, \frac{x}{\sqrt{6N}} + \frac{4N^2 t}{\sqrt{6}} \right) + i e^{iN^4 x} e^{iN^4 t} \partial_x u \left( t, \frac{x}{\sqrt{6N}} + \frac{4N^2 t}{\sqrt{6}} \right)
\]

\[
+ i e^{iN^4 x} e^{iN^4 t} \partial_y u \left( t, \frac{x}{\sqrt{6N}} + \frac{4N^2 t}{\sqrt{6}} \right) 4N^2 \left( \frac{1}{\sqrt{6N}} \right)^2
\]

\[
\partial_x^4 U_{ap}(t, x) = (iN)^4 e^{iN^4 x} e^{iN^4 t} u(s, y) + e^{iN^4 x} e^{iN^4 t} \partial_y u(s, y) \left( \frac{1}{\sqrt{6N}} \right)^4
\]

\[
+ \frac{4}{1} (iN)^2 e^{iN^4 x} e^{iN^4 t} \partial_y^2 u(s, y) \left( \frac{1}{\sqrt{6N}} \right)^2
\]

\[
+ \frac{4}{3} (iN)^3 e^{iN^4 x} e^{iN^4 t} \partial_y u(s, y) \frac{1}{\sqrt{6N}}
\]

and hence we have

\[
(i \partial_t - \partial_x^4) U_{ap}(t, x) = e^{iN^4 x} e^{iN^4 t} (i \partial_x u(s, y) - \partial_y^2 u(s, y))
\]

\[
+ \frac{1}{36} N^{-4} e^{iN^4 x} e^{iN^4 t} \partial_y^4 u(s, y) + \frac{4i}{6^2} N^{-2} e^{iN^4 x} e^{iN^4 t} \partial_y^3 u(s, y).
\]

Moreover, we note that

\[
|U_{ap}(t, x)|^2 U_{ap}(t, x) = |u(s, y)|^2 e^{iN^4 x} e^{iN^4 t} u(s, y).
\]
Since \( u \) is a solution of (5.5), we obtain
\[
(i\partial_t - \partial_x^4) U_{ap}(t,x) - |U_{ap}(t,x)|^2 U_{ap}(t,x) = e^{iNx} e^{iN^4t} \left( i\partial_s u(s,y) - \partial_y^2 u(s,y) + |u(s,y)|^2 u(s,y) \right)
\]
\[
+ \frac{1}{36} N^{-4} e^{iNx} e^{iN^4t} \partial_y^4 u(s,y) + \frac{4i}{6^2} N^{-2} e^{iNx} e^{iN^4t} \partial_y^3 u(s,y)
\]
\[
= \frac{1}{36} N^{-4} e^{iNx} e^{iN^4t} \partial_y^4 u(s,y) + \frac{4i}{6^2} N^{-2} e^{iNx} e^{iN^4t} \partial_y^3 u(s,y)
\]
\[
= E,
\]
where the error term \( E \) is a linear combination of the expressions
\[
E_1 := N^{-4} e^{iNx} e^{iN^4t} \partial_y^4 u(s,y), \\
E_2 := N^{-2} e^{iNx} e^{iN^4t} \partial_y^3 u(s,y).
\]

5.3. Error estimate. In this subsection, we prove the following error estimates for \( E_1 \) and \( E_2 \).

Lemma 5.3. For each \( j = 1, 2 \), let \( e_j \) be the solution to the initial value problem
\[
(\partial_t - \partial_x^4) e_j = E_j, \quad e_j(0) = 0.
\]
Let \( \eta(t) \) be a smooth time cut-off function taking value 1 near the origin and compactly supported. Then, we have
\[
\|\eta(t) e_j\|_{X^{-\frac{1}{2},b}} \lesssim \varepsilon N^{-2}.
\]

For the proof of the above lemma, we need the following lemma.

Lemma 5.4 ([4]). Let \( \sigma \in \mathbb{R}^+ \) and \( u \in H^\sigma(\mathbb{R}) \). For any \( M > 1, \tau \in \mathbb{R}^+, x_0 \in \mathbb{R} \) and \( A > 0 \) let
\[
v(x) = A e^{iMx} u((x - x_0) / \tau).
\]
(i) Suppose \( s \geq 0 \). Then, there exists a constant \( C_1 < \infty \), depending only on \( s \), such that whenever \( M \tau \geq 1 \),
\[
\|v\|_{H^s} \leq C_1 |A|^{\frac{1}{2}} M^s \|u\|_{H^s}
\]
for all \( u, A, x_0 \).

(ii) Suppose that \( s < 0 \) and that \( \sigma \geq |s| \). Then there exists a constant \( C_1 < \infty \) depending only on \( s \) and on \( \sigma \), such that whenever \( 1 \leq \tau \cdot M^{1+|s|/\sigma} \),
\[
\|v\|_{H^s} \leq C_1 |A|^{\frac{1}{2}} M^s \|u\|_{H^s}
\]
for all \( u, A, x_0 \).

(iii) There exists \( c_1 > 0 \) such that for each \( u \), there exists \( C_u < \infty \) such that
\[
\|v\|_{H^s} \geq c_1 |A|^{\frac{1}{2}} M^s \|u\|_{L^2}
\]
whenever \( \tau \cdot M \geq C_u \).
Proof. Observe that
\[ A^{-2}\tau^{-1}M^{-2s}\|u\|_{H^s}^2 = c\tau^{-1}M^{-2s}\int_{\mathbb{R}} \left(1 + |\xi|^2\right)^s \tau^2|\hat{u}(\tau (\xi - M))|^2 \, d\xi \]
\[ = c\int_{\mathbb{R}} \left(\frac{\tau^2 + |M\tau + \eta|^2}{\tau^2 M^2}\right)^s |\hat{u}(\eta)|^2 \, d\eta \]
\[ \lesssim \int_{|\eta|\leq \tau M/2} |\hat{u}(\eta)|^2 + \int_{\tau M/2 \leq |\eta| \leq 2\tau M} M^{-2s}|\hat{u}(\eta)|^2 \]
\[ + \int_{|\eta| \geq 2\tau M} \frac{|\eta|^{2s}}{(\tau M)^{2s}} |\hat{u}(\eta)|^2 \]
\[ = I + II + III. \]

Term I is \( \lesssim \|u\|_{L^2}^2 \). If \( s \geq 0 \), then \( M^{-2s} \leq 1 \), so \( II \lesssim \|u\|_{H^s}^2 \) and \( III \lesssim \|u\|_{H^s}^2 \), because \( \tau M \geq 1 \).

If \( s < 0 \), then \( III \lesssim \|u\|_{L^2}^2 \), since \( |\eta|/\tau M \gtrsim 1 \). Moreover, \( II \lesssim M^{-2s} (\tau M)^{-2\sigma} \|u\|_{H^s}^2 \), which is \( \lesssim \|u\|_{H^s}^2 \) under the further hypothesis \( 1 \leq \tau \cdot M^{1+(s/\sigma)} \).

To obtain \((III)\), it suffices to consider term I: for any \( u \), \( \int_{|\eta| \leq \tau M/2} |\hat{u}(\eta)|^2 \) approaches \( c\|u\|_{L^2}^2 \) as \( \tau M \to \infty \).

Now we are ready to prove the error estimates.

Proof. For \( \frac{1}{2} < b < 1 \), from Lemma 5.3 and \( e_j(0) = 0 \), we have
\[ \|\eta(t)e_j\|_{X^{-b+1}} \lesssim \|\eta(t)E_j\|_{X^{-b+1}} \]
\[ = \|\langle \xi + \frac{1}{2}\rangle^{b-1} \langle \xi \rangle^{-\frac{1}{2}} \eta(t)E_j\|_{L^2_{\xi}} \]
\[ \lesssim \|\langle \xi \rangle^{-\frac{1}{2}} \eta(t)E_j\|_{L^2_{\xi}} \]
\[ = \|\eta(t)\langle \xi \rangle^{-\frac{1}{2}} F_x E_j(t, \xi)\|_{L^2_{\xi}} \]
\[ \lesssim \|\langle \xi \rangle^{-\frac{1}{2}} F_x E_j(t, \xi)\|_{L^2_{\xi}[0,1] \times \mathbb{R}}. \]

Hence, it suffices to show that
\[ \sup_{0 \leq t \leq 1} \|E_j(t)\|_{H^{\frac{1}{2}}_x} \lesssim \varepsilon N^{-2}. \]

The above error estimate then follows by Lemma 5.4, the fact that \( H^s \) is a Banach algebra for \( s > \frac{1}{2} \), and (5.7). More precisely, we apply Lemma 5.4 with \( M = N, \tau = N \) and \( A = N^{-4} \) or \( N^{-2} \).

\[ \|E_1(t)\|_{H^{\frac{1}{2}}_x} \lesssim N^{-4} N^{\frac{1}{2}} N^{-\frac{1}{2}} \|u(t)\|_{H^s}, \]
\[ \|E_2(t)\|_{H^{\frac{1}{2}}_x} \lesssim N^{-2} N^{\frac{1}{2}} N^{-\frac{1}{2}} \|u(t)\|_{H^s} \]
for all \( t \in \mathbb{R} \).

5.4. Perturbation lemma. In this subsection, we prove the following perturbation lemma.

Lemma 5.5. Let \( u \) be a smooth solution to (4NLS) and \( v \) be a Schwartz solution to the approximate fourth order NLS equation
\[ i\partial_t v - \partial_x^4 v - |v|^2 u = E \]
for some error function $E$. Let $v$ be the solution to the inhomogeneous problem
\[ i\partial_t v - \partial_x^4 v = E, \quad v(0) = 0. \]

Suppose that
\[ u(0) \|_{H^{\frac{1}{2}}} \| v(0) \|_{H^{\frac{1}{2}}} \lesssim \varepsilon, \quad \text{and} \quad \| \eta(t) e \|_{X^{\frac{1}{2}, \varepsilon}} \lesssim \varepsilon. \]

Then we have
\[ \| \eta(t) (u - v) \|_{X^{\frac{1}{2}, \varepsilon}} \lesssim \| u(0) - v(0) \|_{H^{\frac{1}{2}}} + \| \eta(t) e \|_{X^{\frac{1}{2}, \varepsilon}}. \]

Then we have
\[ \sup_{0 \leq t \leq 1} \| u(t) - v(t) \|_{H^{\frac{1}{2}}} \lesssim \| u(0) - v(0) \|_{H^{\frac{1}{2}}} + \| \eta(t) e \|_{X^{\frac{1}{2}, \varepsilon}}. \]

In particular, we have
\[ \sup_{0 \leq t \leq 1} \| u - v \|_{H^{\frac{1}{2}}} \lesssim \| u(0) - v(0) \|_{H^{\frac{1}{2}}} + \| \eta(t) e \|_{X^{\frac{1}{2}, \varepsilon}}. \]

Proof. The proof is a standard perturbation argument. See [1, Lemma 5.1]. We give only a sketch. We write the Duhamel formula for $v$ with a time-cut off function $\eta(t)$
\[ \eta(t)v(t) = \eta(t) e^{it\partial_x^4} u(0) - \eta(t) e + \eta(t) \int_0^t e^{i(t-t')\partial_x^4} |v|^2 v(t') \ dt'. \]

We use (5.5), (5.11) and a continuity argument assuming that $\varepsilon$ is very small. Then, we obtain
\[ \| \eta(t) \|_{X^{\frac{1}{2}, \varepsilon}} \lesssim \varepsilon. \]

We repeat the same argument on the difference $w = u - v$ to get the desired result. \qed

Let $U$ be the global Schwartz solution to the fourth order NLS equation (4NLS) with initial datum $U(0, \cdot) = U_{ap}(0, \cdot)$. By applying the above two Lemma 5.3 and 5.5 we obtain
\[ \sup_{k \leq t \leq k+1} \| U(t) - U_{ap}(t) \|_{H^{\frac{1}{2}}} \lesssim \| U(k) - U_{ap}(k) \|_{H^{\frac{1}{2}}} + \varepsilon N^{-2} \]
\[ \lesssim \sup_{k-1 \leq t \leq k} \| U(t) - U_{ap}(t) \|_{H^{\frac{1}{2}}} + \varepsilon N^{-2}. \]

Therefore, by applying the induction on $k$ for $k \lesssim \log N$ and using $U(0) = U_{ap}(0)$, we conclude that for any $\eta > 0$
\[ \sup_{0 \leq t \leq \log N} \| U(t) - U_{ap}(t) \|_{H^{\frac{1}{2}}} \lesssim \varepsilon N^{-2+\eta}, \]
uniformly for all $N \gg 1$.

5.5. Proof of Theorem 1.6. In this subsection, we prove Theorem 1.6. We follow the argument in [4]. Before we start proving the theorem, let us recall the following. In [1, (3.16), (3.19), (3.20)], Christ-Colliander-Tao constructed the global solution $u^{(aw)}$ to (NLS) for all $a \in [1/2, 1]$, where $w(x) = \varepsilon \exp(-x^2)$ for some parameter $0 < \varepsilon \ll 1$, such that
\[ \sup_{0 \leq t < \infty} \| u^{(aw)}(t) \|_{X^{\frac{1}{2}}} \lesssim \varepsilon, \]
\[ \| u^{(aw)}(0) - u^{(a'w)}(0) \|_{X^{\frac{1}{2}}} \lesssim \varepsilon |a - a'|, \]
\[ \limsup_{t \to +\infty} \| u^{(aw)}(t) - u^{(a'w)}(t) \|_{L_2^2} \geq 1. \]
for sufficiently large integer $K \geq 5$. The details of solutions $u^{(aw)}$ are presented in [4, Section 3]. The Galilean invariances and scale symmetry can be used to transform such solutions $u^{(aw)}$ into solutions $u_1, u_2$ in Theorem 5.1. They also used these solutions $u^{(aw)}$ to establish approximate solution of the mKdV equation to show mild ill-posedness of the mKdV equation. In the proof of Theorem 1.6, we also use these solutions $u^{(aw)}$.

**Proof of Theorem 1.6.** Let $0 < \delta \ll \varepsilon < 1$ and $T > 0$ be given. Note that we have two global solutions $u_1, u_2$ of defocusing NLS (5.5) satisfying (5.7), (5.8) and (5.9).

Define $U_{ap,1}$ and $U_{ap,2}$ by

$$
U_{ap,1}(t, x) := e^{iN^4 t} e^{iN x} u_1(s, y),
$$

$$
U_{ap,2}(t, x) := e^{iN^4 t} e^{iN x} u_2(s, y)
$$

and let $U_1, U_2$ be global Schwartz solutions with initial data $U_{ap,1}(0, \cdot), U_{ap,2}(0, \cdot)$, respectively. Let $\lambda \gg 1$ be a large parameter to be chosen later. Let $U_j^{\lambda}$ denote the function

$$
U_j^{\lambda}(t, x) := \lambda^2 U_j(\lambda^4 t, \lambda x).
$$

Since $U_j$ is a global solution to fourth order NLS, so is $U_j^{\lambda}$. Similarly, we define

$$
U_{ap,j}^{\lambda}(t, x) := \lambda^2 U_{ap,j}(\lambda^4 t, \lambda x)
$$

for $j = 1, 2$. Note that

$$
U_j^{\lambda}(0, x) = U_{ap,j}^{\lambda}(0, x) = \lambda^2 U_{ap,j}(0, \lambda x) = \lambda^2 e^{iN^4 x} u_j(0, \frac{\lambda x}{\sqrt{N}}).
$$

From Lemma 5.4 we have

$$
\|U_{ap,j}^{\lambda}(0)\|_{H^s} \lesssim \lambda^{s + \frac{3}{4}} \|U_j(\cdot, 0)\|_{H^s}
$$

provided that its hypothesis $1 \leq \tau M^{1+(s/K)}$ is satisfied with $A = \lambda^2, M = N\lambda$, and $\tau = N/\lambda$. We define $\lambda$ by

$$
\lambda := N^{-\frac{3}{4+2s}}.
$$

The condition $1 \leq \tau M^{1+(s/K)}$ then becomes $1 \leq N^{-\frac{3}{4+2s}} \cdot \left(N^{\frac{3}{2+3s}}\right)^{s/K}$. Hence, for any $s > -3/2$ this condition is satisfied with sufficiently large $N, K$.

Then, from (5.7), we have $\|U_j^{\lambda}(0)\|_{H^s} \lesssim \varepsilon$ for $j = 1, 2$. Similarly by using (5.8) instead of (5.7), we have $\|U_1^{\lambda}(0) - U_2^{\lambda}(0)\|_{H^s} \lesssim \delta$.

Now we show $\sup_{0 \leq t \leq T} \|U_j(t) - U_{ap,j}(t)\|_{H^s} \lesssim \varepsilon N^{-2+\eta}$. Recall that we have

$$
\sup_{0 \leq t \leq \log N} \|U_j(t) - U_{ap,j}(t)\|_{H^s} \lesssim \varepsilon N^{-2+\eta}
$$

A scaling calculation shows

$$
\|U_j^{\lambda}(t) - U_{ap,j}^{\lambda}(t)\|_{H^s} \lesssim \lambda^{\max(0, s) + \frac{3}{4}} \|U_j(\lambda^4 t) - U_{ap,j}(\lambda^4 t)\|_{H^s} \lesssim \lambda^{\max(0, s) + \frac{3}{4}} \|U_j(\lambda^4 t) - U_{ap,j}(\lambda^4 t)\|_{H^s} \lesssim \lambda^{\max(0, s) + \frac{3}{4}} \varepsilon N^{-2+\eta}
$$
for $0 < \lambda^4 t \lesssim \log N$. By using (5.10) and $s > -\frac{15}{14}$, we have
\begin{equation}
\|U_\lambda^\pm(t) - U_{ap,\pm}^\pm(t)\|_{H^s_x} \lesssim \varepsilon \tag{5.12}
\end{equation}
for sufficiently small $\eta > 0$ and large $N \gg 1$. From (5.9), we can find $t_0$ such that
\begin{equation}
\|u_1(t_0) - u_2(t_0)\|_{L^2_x} \gtrsim \varepsilon. \tag{5.13}
\end{equation}
Fix this $t_0$. We choose $N$ so large that $t_0 \ll \log N$. Using Lemma 5.4 and (5.13) as before, we have
\begin{equation}
\|U_\lambda^{ap,1}(t_0/\lambda^4) - U_\lambda^{ap,2}(t_0/\lambda^4)\|_{H^s_x} \gtrsim \lambda^{s + \frac{3}{2}} N^{s + \frac{1}{2}} \|u_1(t_0) - u_2(t_0)\|_{L^2_x} \gtrsim \varepsilon. \tag{5.14}
\end{equation}
Hence, from (5.12), (5.14), and triangle inequality, we have
\begin{equation}
\|U_\lambda^1(t_0/\lambda^4) - U_\lambda^2(t_0/\lambda^4)\|_{H^s_x} \gtrsim \varepsilon.
\end{equation}
By choosing $N$ (and hence $\lambda$) large enough that $t_0/\lambda^4 < T$, we obtain the desired result.
\[\square\]

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