Gravitational allocation on the sphere

Nina Holden*, Yuval Peresb,1, and Alex Zhai*

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139; bMicrosoft Research, Redmond, WA 98052; and cDepartment of Mathematics, Stanford University, Stanford, CA 94305

This contribution is part of the special series of Inaugural Articles by members of the National Academy of Sciences elected in 2016.

Contributed by Yuval Peres, August 3, 2018 (sent for review December 11, 2017; reviewed by Michel Ledoux and Mikhail Sodin)

Given a collection $\mathcal{L}$ of $n$ points on a sphere $S^2$ of surface area $n$, a fair allocation is a partition of the sphere into $n$ parts each of area 1, and each is associated with a distinct point of $\mathcal{L}$. We show that, if the $n$ points are chosen uniformly at random and if the partition is defined by a certain “gravitational” potential, then the expected distance between a point on the sphere and the associated point of $\mathcal{L}$ is $O(\sqrt{\log n})$. We use our result to define a matching between two collections of $n$ independent and uniform points on the sphere and prove that the expected distance between a pair of matched points is $O(\sqrt{\log n})$, which is optimal by a result of Ajtai, Komlós, and Tusnády.

Formal Definitions and Main Result

Let $S^n_0 \subset \mathbb{R}^3$ denote the sphere centered at the origin with surface area $n$, so that we work in the scaling where each cell has unit area. Let $\lambda_n$ denote the surface area measure on $S^n_0$, so that $\lambda_n(S^n_0) = n$.

For any set $\mathcal{L} \subset S^n_0$, we call a matching $\psi : S^n_0 \rightarrow \mathcal{L}$ a fair allocation of $\lambda_n$ to $\mathcal{L}$ if it satisfies the following:

$$\lambda_n(\psi^{-1}(x)) = 0, \quad \lambda_n(\psi^{-1}(z)) = 1, \quad \forall z \in \mathcal{L}. \quad [1]$$

For $z \in \mathcal{L}$, we call $\psi^{-1}(z)$ the cell allocated to $z$.

Let us now describe gravitational allocation in particular. First, we define a potential function $U : S^n_0 \rightarrow \mathbb{R}$ given by

$$U(x) = \sum_{x \in \mathcal{L}} \log |x - z|, \quad [2]$$

where $| \cdot |$ denotes Euclidean distance in $\mathbb{R}^3$. For each location $x \in S^n_0$, let $F(x)$ denote the negative gradient of $U$ with respect to $x$.

Significance

Given a set $\mathcal{L}$ of $n$ points on the sphere, an allocation is a way to divide the sphere into $n$ cells of equal area, each associated with a point of $\mathcal{L}$. Given two sets of $n$ points $\mathcal{A}$ and $\mathcal{B}$ on the sphere, a matching is a bijective map from $\mathcal{A}$ to $\mathcal{B}$. Allocation and matching rules that minimize the distance between matched points are related to optimal transport and discretization of continuous measures. We define a matching and allocation rule by considering the gravitational field associated with the point configurations and show that they are optimal in expectation up to multiplication by a constant when our points are chosen independently and uniformly at random.

Author contributions: N.H., Y.P., and A.Z. performed research and wrote the paper.

Reviewers: M.L., University of Toulouse; and M.S., Tel Aviv University.

The authors declare no conflict of interest.

This open access article is distributed under Creative Commons Attribution-NonCommercial-NoDerivatives License 4.0 (CC BY-NC-ND).

See Profile on page 9646.

1 To whom correspondence should be addressed. Email: peres@microsoft.com.

Published online September 7, 2018.

*We note that many results are stated for points in a square or a 2D torus instead of the sphere. As $n \rightarrow \infty$, all of these settings are essentially equivalent. For the sake of consistency, in this article, we will state everything in terms of the sphere.
More generally, for any $p > 0$, there is a constant $C_p > 0$ depending only on $p$ such that

$$ E|\psi(x) - x|^p \leq C_p (\log n)^{p/2}. \quad [7] $$

### Why Is Gravitational Allocation a Fair Allocation?

The reader may find it somewhat surprising that the basins of attraction in gravitational allocation always have equal areas, even if a point in $L$ is crowded by many other points in $L$ (Fig. 3). As seen in Fig. 3, the surrounded point will still attract certain faraway points, so that its basin of attraction still has total area 1.

We give two explanations for this phenomenon. Both explanations rely on the fact that our potential $U$ satisfies the Poisson equation

$$ \Delta_S U(x) = -2\pi + 2\pi \sum_{z \in L} \delta_z, $$

where $\Delta_S$ denotes the spherical Laplacian (i.e., the Laplace–Beltrami operator on $S^n$).

The first explanation is based on the divergence theorem. Consider any $z \in L$ and its cell $B(z)$. Since $B(z)$ is a basin of attraction, $F$ must be parallel to $B(z)$ along its boundary. We can then apply the divergence theorem$^1$ to obtain

$$ 0 = -\int_{\partial B(z)} F \cdot n \, ds = \int_{B(z)} \nabla \cdot F \, d\lambda_n $$

$$ = \int_{B(z)} \Delta_S U \, d\lambda_n = 2\pi - 2\pi \lambda_n(B(z)). $$

It follows that $\lambda_n(B(z)) = 1$ as desired.

The second explanation is slightly longer, but it also provides a more detailed understanding of the flow under $F$. Imagine the surface area measure $\lambda_n$ as representing the density of grains of sand uniformly distributed on the sphere. The sand is flowing along $F$, so that a grain of sand at $x$ will be moved to location $Y_x(t)$ after time $t$.

In a small time $\epsilon$, the net change in the density of sand at a point $x$ will be approximately

$$ -\epsilon \nabla \cdot F(x) = \epsilon \Delta_S U(x) = -2\pi \epsilon + 2\pi \epsilon \sum_{z \in L} \delta_z(x). $$

Thus, the density is decreasing everywhere at a uniform rate, except at points of $L$, where sand is accumulating (at the same rate for each point). Integrating this over time, the density of sand at a time $t$ will be given by

$$ \lambda_{n,t} := e^{-2\pi t} \lambda_n + (1 - e^{-2\pi t}) \sum_{z \in L} \delta_z. $$

We find that $\lim_{t \to \infty} \lambda_{n,t} = \sum_{z \in L} \delta_z$, so that the amount of sand at each point in $L$ tends to one. Consequently, the area of each basin of attraction must have been one.

### Proof Outline of the Main Theorem

The proof of Theorem 1 is based on estimating the magnitude of the gradient force $F$. In the previous section, we saw that, after time $t$, all but a $e^{-2\pi t}$ proportion of the sphere will have reached one of the points in $L$, and therefore, the average time that it

---

$^1$Except for a set of measure zero.

$^2$Assuming various smoothness properties, which we do not justify here.
takes for a point to flow into a potential well is \( \int_0^\infty e^{-2\pi t} \, dt = 1/2\pi \). We can also estimate the average distance traveled in a similar way:

\[
\int_{S^2_\lambda} \int_0^\infty |F(Y_z(t))| \, dt \, d\lambda_n(x) = \int_0^\infty \int_{S^2_\lambda \cap \mathcal{L}} |F(x)| \, d\lambda_n(x,t) \, dt \\
= \int_0^\infty e^{-2\pi t} \int_{S^2_\lambda} |F(x)| \, d\lambda_n(x) \, dt \\
= \frac{1}{2\pi} \int_{S^2_\lambda} |F(x)| \, d\lambda_n(x). \tag{8}
\]

It remains to estimate the average magnitude of \( F(x) \), which is given by the following lemma.

**Lemma 2.** Fix any \( x \in S^2_\lambda \). Then, \( E|F(x)| = O(1/\log n) \), where the expectation is taken over the randomness of \( \mathcal{L} \).

Taking expectations in Eq. 8 and then integrating Lemma 2 over all \( x \in S^2_\lambda \) proves Theorem 1 in the case \( p = 1 \). Larger values of \( p \) can be handled in the same spirit, but it requires more involved estimates for \( F \) that we do not reproduce here (ref. 17 has details).

**Proof:** Let \( U_z(x) = \log |x - z| \) and \( F_z(x) = \nabla U_z(x) \), so that \( U(x) = \sum_{z \in \mathcal{L}} U_z(x) \) and \( F(x) = \sum_{z \in \mathcal{L}} F_z(x) \). Thus, \( F_z(x) \) represents the contribution to \( F(x) \) coming from the point \( z \in \mathcal{L} \).

To estimate \( F(x) \), it is convenient to decompose into the contributions of nearby and faraway points in \( \mathcal{L} \). For our purposes, “near” means points within the spherical cap of radius 1 around \( x \), which we denote by \( B(x,1) \). Then, we may write

\[
F(x) = \sum_{z \in \mathcal{L} \cap B(x,1)} F_z(x) + \sum_{z \in \mathcal{L} \setminus B(x,1)} F_z(x). \tag{9}
\]

When \( |z - x| = r \), an explicit computation shows that \( F_z(x) \) is of order \( 1/r \). It is also not hard to calculate that the expected number of points in \( \mathcal{L} \) with distance from \( x \) that is between \( r \) and \( r + dr \) is of order \( r \cdot dr \). By the triangle inequality, we can estimate \( F_{\text{near}}(x) \) as

\[
E|F_{\text{near}}(x)| \leq \sum_{z \in \mathcal{L} \cap B(x,1)} |F_z(x)| = \int_{\mathcal{L} \setminus B(x,1)} |F_y(x)| \, dy \\
= O\left( \int_1^{\sqrt{n}} \frac{1}{r} \cdot (r \, dr) \right) = O(1). \tag{10}
\]

To estimate the far term, the triangle inequality is too weak, because we expect much cancellation between the \( F_z(x) \). In fact, by symmetry, we have \( E|F_{\text{far}}(x)| = 0 \). Thus, we instead estimate the second moment

\[
E|F_{\text{far}}(x)|^2 = E \sum_{z \in \mathcal{L} \setminus B(x,1)} |F_z(x)|^2 = \int_{\mathcal{L} \setminus B(x,1)} |F_y(x)|^2 \, dy \\
= \mathcal{O} \left( \int_{1}^{\sqrt{n}} \frac{1}{r^2} \cdot (r \, dr) \right) = O(\log n). \tag{11}
\]

Combining Eqs. 9–11 yields

\[
E|F(x)| \leq E|F_{\text{near}}(x)| + \sqrt{E|F_{\text{far}}(x)|^2} = O(\sqrt{\log n}),
\]

which is the bound claimed in Lemma 2.

**A Heuristic Picture**

Lemma 2 also provides a good heuristic proof of Eq. 7. We know by Lemma 2 that, for a typical point \( x \), we have \( F(x) = \mathcal{O}(1/\log n) \), and moreover, our above analysis suggests that the value of \( F(x) \) is dominated by contributions from faraway points. Thus, we expect that direction and speed of travel for \( x \) under the flow induced by \( F \) will remain relatively constant.

However, \( x \) will not travel forever in this way; suppose that it passes within \( \mathcal{O}(1/\sqrt{\log n}) \) distance of a point \( z \in \mathcal{L} \). Then, the contribution \( F_z(x) \) from \( z \) to the overall “force” \( F \) will be of order \( \log n \), which may overpower the contribution from all other points, causing \( x \) to fall into the potential well at \( z \).

Consider a strip of width \( 1/\sqrt{\log n} \) around the path of \( x \) (Fig. 4). If there is a point \( z \in \mathcal{L} \) in this strip, then it is likely to “swallow” \( x \) (i.e., \( x \) will be allocated to \( z \)). The probability that any given region contains no points of \( \mathcal{L} \) decays exponentially in its area, which suggests the heuristic

\[
P \left( x \text{ travels distance at least } r \sqrt{\log n} \right) \\
\approx P \left( \text{no points of } \mathcal{L} \text{ in a strip of area roughly } \pi r \sqrt{\log n} \cdot (1/\sqrt{\log n}) = r \right) \\
\approx e^{-r},
\]

giving Eq. 7, because \( |\psi(x) - x| \) is bounded above by the distance traveled by \( x \).

**From Allocations to Matchings**

We now turn to the connection between fair allocations and optimal matchings. Suppose that \( A = \{ a_1, \ldots, a_n \} \) and \( B = \{ b_1, \ldots, b_n \} \) are two sets of \( n \) points in \( S^2_\lambda \). A matching from \( A \) to \( B \) is a bijective function \( \varphi : A \to B \). Recall that the matching

Fig. 3. The center point is surrounded by seven other nearby points (Left). Nevertheless, it turns out that its basin of attraction (light blue; Right) can slip past its neighbors in certain places.
problem is to find the matching that minimizes the total distance between matched points.

When the points of $A$ and $B$ are drawn uniformly at random, the asymptotic behavior of the minimal matching distance was identified by Ajtai, Komlós, and Tusnády (3), who proved the following theorem.

**Theorem 3** (Ajtai–Komlós–Tusnády). Suppose that $A$ and $B$ each consist of $n$ points drawn uniformly and independently at random from $[0, 1]$. Let

$$d_{\text{match}}(A, B) = \min_{\varphi: A \rightarrow B} \frac{1}{n} \sum_{a \in A} |\varphi(a) - a|.$$ 

Then, there are constants $C_1$, $C_2 > 0$ for which

$$\lim_{n \to \infty} \mathbb{P}\left( C_1 \sqrt{\log n} \leq d_{\text{match}}(A, B) \leq C_2 \sqrt{\log n} \right) = 1. \quad [12]$$

It turns out that the average displacement of a fair allocation gives an upper bound on the matching distance, as the next proposition shows.

**Proposition 4**. Let $A, B \subseteq S^2_n$ be two sets of $n$ points, and let $\psi_A$ and $\psi_B$ be fair allocations of $\lambda_n$ to $A$ and $B$, respectively. Then, there exists a matching $\varphi: A \rightarrow B$ such that

$$\sum_{a \in A} |a - \varphi(a)| \leq \int_{S^2_n} |x - \psi_A(x)| d\lambda_n(x) + \int_{S^2_n} |x - \psi_B(x)| d\lambda_n(x). \quad [13]$$

**Remark 5**: Consider the case where $A$ and $B$ are drawn uniformly at random, and suppose that we use gravitational allocation for $\psi_A$ and $\psi_B$ in Proposition 4. Then, the $p = 1$ case of Theorem 1 implies that the right-hand side of Eq. 13 has expectation of order $n \sqrt{\log n}$. Comparing with Theorem 3, this implies that the asymptotic rate of $\sqrt{\log n}$ in Theorem 1 is the best possible up to a constant factor. By Eq. 8, we also get that $\mathbb{E}[F(x)]$ is at least of order $\sqrt{\log n}$ for any fixed $x \in S^2_n$.

The triangle inequality for the linear Wasserstein distance justifies why we can pass from an allocation to a matching, but we choose to describe the connection explicitly. Let $A_i = \psi^{-1}_A(a_i)$ denote the cell allocated to $a_i$, and similarly, let $B_i = \psi^{-1}_B(b_i)$. Consider the $n \times n$ matrix $M = (M_{ij})_{i,j=1}^n$ given by

$$M_{ij} = \lambda_n(A_i \cap B_j).$$

We see that $M$ is a doubly stochastic matrix:

$$\sum_{j=1}^n M_{ij} = \sum_{j=1}^n \lambda_n(A_i \cap B_j) = \lambda_n(A_i) = 1,$n

$$\sum_{i=1}^n M_{ij} = \sum_{i=1}^n \lambda_n(A_i \cap B_j) = \lambda_n(B_j) = 1.$$
in gravitational allocation will be \( O\left(\sqrt{n/m \cdot \log m}\right) \), where the factor \( \sqrt{n/m} \) comes from rescaling \( S^2_m \) to \( S^2_n \). Thus,

\[
\sum_{k=1}^{n} E|a_k - \varphi(a_k)| \leq \sum_{m=2}^{n} O\left(\sqrt{n/m \cdot \log m}\right)
\]

\[
\leq O(\sqrt{n \log n}) \sum_{m=2}^{n} \frac{1}{\sqrt{m}} = O(n \sqrt{\log n}),
\]

which shows that, even in the online setting, one has similar asymptotics as in Theorem 3.

We remark that our online matching algorithm can be implemented efficiently using the well-known “fast multipole method” introduced by Rokhlin (19) and Greengard and Rokhlin (20). This entails precomputing estimates of the gravitational potential from clusters of points in \( \mathcal{B} \), and these computations can be reused as new points of \( \mathcal{A} \) are introduced.

**Gravitational Allocation for Other Point Processes**

So far, we have focused on the setting where our \( n \) points on \( S^2 \) are taken independently at random. However, one may also analyze other random point processes where the points are not independent, which allows them to be distributed more evenly over the sphere.

One example is given by the roots of a certain Gaussian random polynomial. Specifically, we look at the polynomial

\[
p(z) = \sum_{k=0}^{n} \zeta_k \frac{\sqrt{n(n-1) \cdots (n-k+1)}}{\sqrt{k!}} z^k,
\]

where \( \zeta_1, \ldots, \zeta_n \) are independent standard complex Gaussians. The roots \( \lambda_1, \ldots, \lambda_n \) of \( p \) are then \( n \) random points in the complex plane, which we can bring to the sphere via stereographic projection. More explicitly, let \( x_0 = (0, 0, 1) \). The function

\[
P : z \mapsto \sqrt{\frac{n}{4\pi}} x_0 + 2(z - x_0) \frac{1}{|z - x_0|^2}
\]

maps the horizontal plane in \( \mathbb{R}^3 \) to \( S^2 \). Then, viewing the \( \lambda_k \) as lying in the horizontal plane,

\[
\mathcal{L} = \{P(\lambda_k)\}_{k=1}^{n}
\]

is a rotationally equivariant random set of \( n \) points on \( S^2 \). [The rotational equivariance comes from the particular choice of coefficients for \( p \) (ref. 21, chapter 2.3).]

Heuristically, the points of \( \mathcal{L} \) are distributed more evenly than independent uniformly random points, because roots of random polynomials tend to “repel” each other (Fig. 6). This can be quantified as follows. Let \( \psi : S^2_n \to \mathcal{L} \) be the gravitational allocation. Then, we claim that

\[
\frac{1}{n} \mathbb{E} \int_{S^2_n} |x - \psi(x)| d\lambda_n(x) = O(1).
\]

To prove this, by Eq. 8 and rotational symmetry, it suffices to show that \( E|F(x)| = O(1) \) for any point \( x \in S^2_n \). It is convenient to pick \( x = (0, 0, -\sqrt{n/4\pi}) \). Then, in the notation of the Proof of Lemma 2, we may calculate that

\[
F_{\lambda_k}(x) = \sqrt{\frac{n}{m}} \frac{\bar{\lambda}_k}{|\zeta|},
\]

where we interpret the complex number on the right-hand side as a 2D vector. Thus, we have

\[
F(x) = \sqrt{\frac{n}{m}} \sum_{k=1}^{n} \frac{\bar{\lambda}_k}{\zeta} = \sqrt{\frac{n}{m}} \frac{\bar{\lambda}_k}{\zeta},
\]

which gives a simple expression for \( F \) in terms of two independent complex Gaussians. Taking expectations of the magnitude, we obtain

\[
E|F(x)| = \sqrt{\frac{n}{m}} E\left|\frac{\bar{\lambda}_k}{\zeta}\right| = \frac{\pi}{2},
\]

which establishes Eq. 16.

**Open Problems**

We conclude by describing two other matching algorithms for which we do not know a precise analysis.

First, one may consider a dynamic electrostatic version of gravitational allocation. Suppose that the points in \( \mathcal{A} (\mathcal{B}) \) are positive (negative) and that points of different (similar) kinds attract (repulse) each other. After some time, it seems that each point in \( \mathcal{A} \) will collide with a point in \( \mathcal{B} \), forming a matching. What will be the average distance between the original positions of matched pairs?

Second, in the online matching problem, instead of matching each new point \( a_k \) to a point in \( \mathcal{B} \) according to gravitational allocation, suppose that we simply match \( a_k \) to the closest point in \( \mathcal{B} \) that has not been matched already. Alternatively, we can reveal \( \mathcal{A} \) and \( \mathcal{B} \) simultaneously and iteratively match closest pairs of points. In other words, we choose \( i, j \in \{1, \ldots, n\} \) such that \( |a_i - b_j| \) is minimized, define \( \varphi(a_i) = b_j \), and we repeat with the sets \( \mathcal{A} \setminus \{a_i\} \) and \( \mathcal{B} \setminus \{b_j\} \). What will be the average matching distance in these settings? In the latter setting, ref. 16, theorem 6 suggests an upper bound for the matching distance of \( \int_0^{\infty} e^{-0.496 \cdots} dr = \Theta(n^{0.252 \cdots}) \). Can this be improved?
ACKNOWLEDGMENTS. We thank Manjunath Krishnapur for useful discussions as well as for sharing his code for producing simulations. Most of this work was carried out while N.H. and A.Z. were visiting Microsoft Research; we thank Microsoft Research for the hospitality.

1. Huesmann M, Sturm KT (2013) Optimal transport from Lebesgue to Poisson. Ann Probab 41:2426–2478.
2. Dereich S, Scheutzow M, Schottstedt R (2013) Constructive quantization: Approximation by empirical measures. Ann Inst Henri Poincaré Probab Stat 49:1183–1203.
3. Ajtai M, Komlós J, Tusnády G (1984) On optimal matchings. Combinatorica 4: 259–264.
4. Leighton T, Shor P (1989) Tight bounds for minimax grid matching with applications to the average case analysis of algorithms. Combinatorica 9:161–187.
5. Talagrand M (1994) Matching theorems and empirical discrepancy computations using majorizing measures. J Am Math Soc 7:455–537.
6. Dacorogna B, Moser J (1990) On a partial differential equation involving the Jacobian determinant. Ann de l'Institut Henri Poincaré Analyse Non Linéaire 7:1–26.
7. Nazarov F, Sodin M, Volberg A (2007) Transportation to random zeroes by the gradient flow. Geom Funct Anal 17:887–935.
8. Chatterjee S, Peled R, Peres Y, Romik D (2010) Gravitational allocation to Poisson points. Ann Math 172:617–671.
9. Chatterjee S, Peled R, Peres Y, Romik D (2010) Phase transitions in gravitational allocation. Geom Funct Anal 20:870–917.
10. Holroyd AE, Liggett TM (2001) How to find an extra head: Optimal random shifts of Bernoulli and Poisson random fields. Ann Probab 29:1405–1425.
11. Hoffman C, Holroyd AE, Peres Y (2006) A stable marriage of Poisson and Lebesgue. Ann Probab 34:1241–1272.
12. Ambrosio L, Stra F, Trevisan D (2018) A PDE approach to a 2-dimensional matching problem. Probab Theory Relat Fields. Available at https://link.springer.com/article/10.1007%2Fs00440-018-0837-x#citeas. Accessed September 4, 2018.
13. Caracciolo S, Lucibello C, Parisi G, Sicuro G (2014) Scaling hypothesis for the Euclidean bipartite matching problem. Phys Rev E 90:012118.
14. Holroyd AE, Peres Y (2005) Extra heads and invariant allocations. Ann Probab 33: 31–52.
15. Hoffman C, Holroyd AE, Peres Y (2009) Tail bounds for the stable marriage of Poisson and Lebesgue. Canad J Math 61:1279–1299.
16. Holroyd AE, Pemantle R, Peres Y, Schramm O (2009) Poisson matching. Ann Inst Henri Poincaré Probab Stat 45:266–287.
17. Holden N, Peres Y, Zhai A (2017) Gravitational allocation for uniform points on the sphere. arXiv:170408238. Preprint, posted April 26, 2017.
18. van Lint JH, Wilson RM (2001) A Course in Combinatorics (Cambridge Univ Press, Cambridge, UK), 2nd Ed.
19. Rokhlin V (1985) Rapid solution of integral equations of classical potential theory. J Comput Phys 60:187–207.
20. Greengard L, Rokhlin V (1987) A fast algorithm for particle simulations. J Comput Phys 73:325–348.
21. Hough JB, Krishnapur M, Peres Y, Virág B (2009) Zeros of Gaussian Analytic Functions and Determinantal Point Processes. University Lecture Series (American Mathematical Society, Providence, RI), Vol 51.