O(n, n) invariance and Wald entropy formula
in the Heterotic Superstring effective action
at first order in $\alpha'$

Tomás Ortín$^a$

Instituto de Física Teórica UAM/CSIC
C/ Nicolás Cabrera, 13–15, C.U. Cantoblanco, E-28049 Madrid, Spain

Abstract

We perform the toroidal compactification of the full Bergshoeff-de Roo version of the Heterotic Superstring effective action to first order in $\alpha'$. The dimensionally-reduced action is given in a manifestly-O(n, n)-invariant form which we use to derive a manifestly-O(n, n)-invariant Wald entropy formula which we then use to compute the entropy of $\alpha'$-corrected, 4-dimensional, 4-charge, static, extremal, supersymmetric black holes.

$^a$Email: tomas.ortin[at]csic.es
Introduction

In a recent paper [1] we have performed the dimensional reduction of the Heterotic Superstring effective action in a circle to first-order in \( \alpha' \) with two goals in mind:

1. To study T duality in the dimensionally-reduced theory and the effect that the first-order \( \alpha' \) corrections have in it. In particular, we wanted to recover the first-order in \( \alpha' \) corrections to the Buscher T duality rules \([2,3]\) found in Ref. [4],\(^1\) and show explicitly that the whole action is invariant under them to that order.

2. To derive a T duality-invariant formula for the Wald entropy using the Iyer-Wald prescription developed in Refs. [7–9].\(^2\)

\(^1\)See also [5,6].
\(^2\)A discussion of the caveats in the direct use of this prescription in the Heterotic Superstring effective action can be found in the Introduction of Ref. [1]. On the other hand, it is clear that the entropy formula derived in that reference by using this prescription gives results which coincide with those obtained by microstate counting \([10]\) and also satisfy the fundamental thermodynamic relation \( \frac{\partial S}{\partial M} = \frac{1}{T} \) in black hole with finite temperature \([11]\).
This formula, though, can only be applied to black holes which can be obtained from a solution of the 10-dimensional theory by non-trivial and several trivial dimensional reductions over circles. This severely limits its applicability to 5-dimensional black holes and certain 4-dimensional ones.

It is natural to try to extend those results to non-trivial toroidal compactifications, testing the $O(n,n)$ invariance of the dimensionally-reduced action to first order in $\alpha'$ and obtaining a manifestly $O(n,n)$-invariant Wald entropy formula that can be applied to more general black-hole solutions such as, for instance, the heterotic version of the 4-dimensional, 4-charge, static, extremal black holes whose microscopic entropy was first computed in Refs. [13, 14].

Earlier work on the effect of $\alpha'$ corrections on the T duality invariance of the Heterotic Superstring effective action more or less complete in different forms and schemes [15], including Double Field Theory, can be found in Refs. [16–22] some of which we will comment upon in the main body of this paper. Here we will use the Bergshoeff-de Roo action Ref. [23] obtained by supersymmetric completion of the Lorentz Chern-Simons terms in the Kalb-Ramond field strength [24].

This paper is organized as follows: in Section 1 we introduce the 10-dimensional Heterotic Superstring effective action in the Bergshoeff-de Roo formulation. In Section 2, as a warm-up exercise, we review the toroidal dimensional reduction of the zeroth-order action, rewriting it in a manifestly $O(n,n)$-invariant form. In Section 3 we add Yang-Mills fields, and rewrite the dimensionally-reduced action in an apparently manifestly $O(n,n+n_V)$-invariant form, reproducing, in the Abelian case (when that invariance is real), the results of Maharana and Schwarz [12]. In Section 4 we consider the full $O(\alpha')$ action, which amounts to the addition of the torsionful spin-connection terms. The full action can be regarded, formally, as that of the previous section with more gauge fields and a gauge group which is the direct product of the Yang-Mills gauge group and the 10-dimensional Lorentz group $SO(1,9)$ [24] and, in a first stage (Section 4.1.1), we can simply use the results of the previous section. This cannot be the final result, though, because, as different from the Yang-Mills group, the 10-dimensional Lorentz group is broken into the $(10-n)$-dimensional one and $O(n)$. Thus, in a second stage (Section 4.1.2), we perform this decomposition leaving the dimensionally-reduced action in a manifestly gauge-, $(10-n)$-dimensional Lorentz-, diffeomorphism- and $O(n,n)$-invariant form. Then, in Section 5 we derive from that action a Wald entropy formula that we test on 4-dimensional 4-charge black holes. We present our conclusions in Section 6. Appendix A contains relevant formulae concerning the $O(n,n)/(O(n) \times O(n))$ coset space that we use in the manifestly-$O(n,n)$-invariant action.

3If the $n_V$ 10-dimensional gauge fields are Abelian, the theory is expected to have a larger duality group: $O(n,n+n_V)$ [12], but this group is obviously broken when they are non-Abelian, since they cannot be rotated into the Abelian Kaluza-Klein and winding vector fields. Here we will focus mostly on the $O(n,n)$ duality group which is expected to always be present.
1 The Heterotic Superstring effective action to $O(\alpha')$

Let us first introduce the Heterotic Superstring effective action to $O(\alpha')$, where $\alpha'$ is the Regge slope parameter, in the formulation of Ref. [23] but using the conventions of Ref. [25].

The torsionful spin connection and Kalb-Ramond field strength, which are two fundamental ingredients of the action, can be constructed recursively order by order in $\alpha'$. At zeroth-order, the field strength of the Kalb-Ramond 2-form $B_{\mu\nu}$ is defined as

$$H^{(0)}_{\mu\nu\rho} \equiv 3 \partial_{[\mu} B_{\nu\rho]} , \quad (1.1)$$

and it is added as torsion to the (torsionless, metric-compatible) Levi-Civita spin connection 1-form $\omega_{\mu}{}^{a}{}_{b}$ as

$$\Omega^{(0)}_{(\pm)}{}_{\mu}{}^{a}{}_{b} = \omega_{\mu}{}^{a}{}_{b} \pm \frac{1}{2} H^{(0)}_{\mu}{}^{a}{}_{b} , \quad (1.2)$$

to construct the zeroth-order torsionful spin connections.

The corresponding zeroth-order Lorentz curvature 2-forms and Chern-Simons 3-forms are defined as

$$R^{(0)}_{(\pm)}{}_{\mu\nu}{}^{a}{}_{b} = 2 \partial_{[\mu} \Omega^{(0)}_{(\pm)}{}_{|v]}{}^{a}{}_{b} - 2 \Omega^{(0)}_{(\pm)}{}_{[\mu}{}^{a}{}_{c} \Omega^{(0)}_{(\pm)}{}_{|v]}{}^{b}{}_{c} , \quad (1.3)$$

$$\omega^{L(0)}_{(\pm)} = 3 R^{(0)}_{(\pm)}{}_{[\mu|v]}{}^{a}{}_{b} \Omega^{(0)}_{(\pm)}{}_{|v]}{}^{b}{}_{a} + 2 \Omega^{(0)}_{(\pm)}{}_{[\mu}{}^{a}{}_{b} \Omega^{(0)}_{(\pm)}{}_{|v]}{}^{b}{}_{c} \Omega^{(0)}_{(\pm)}{}_{|v]}{}^{c}{}_{a} . \quad (1.4)$$

At first order in $\alpha'$ we also have to take into account the Yang-Mills fields. The gauge field is denoted by $A^{A}{}_{\mu}$, where $A, B, C, \ldots$ are the adjoint gauge indices of some group that we will not specify. The corresponding gauge field strength and the Chern-Simons 3-forms are defined by

$$F^{A}{}_{\mu\nu} = 2 \partial_{[\mu} A^{A}{}_{|v]} + f^{ABC}{}_{[\mu} A^{B}{}_{|v]} A^{C}{}_{|v]} , \quad (1.5)$$

$$\omega^{YM} = 3 F_{A}{}_{[\mu|v]} A^{A}{}_{|v]} - f_{ABC} A^{A}{}_{[\mu} A^{B}{}_{|v]} A^{C}{}_{|v]} , \quad (1.6)$$

where we have lowered the adjoint group indices using the Killing metric of $K_{AB}$: $f_{ABC} \equiv f_{AB}{}^{D} K_{D}{}^{BC}$ and of the gauge fields $F_{A}{}_{\mu\nu} \equiv K_{A}{}_{B} F^{B}{}_{\mu\nu}$.

Then, the first-order Kalb-Ramond field strength is given by

$$H^{(1)}_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]} + \frac{\alpha'}{4} \left( \omega^{YM}{}_{\mu\nu\rho} + \omega^{L(0)}_{(\pm)}{}_{|v]}{}^{a}{}_{b} \right) , \quad (1.7)$$

$^4$The Regge slope parameter is related to the string length $\ell_{S}$ by $\alpha' = \ell_{S}^{2}$.

$^5$The relation between the normalizations of the fields in Ref. [23] and here can be found in Ref. [26].
and now it is the torsion of the first-order torsionful spin connection
\[ \Omega^{(1)}_{(\pm)\mu} b = \omega^{a}_{\mu} b \pm \frac{1}{2} H^{(1)}_{\mu} a b, \tag{1.8} \]
whose curvature \( R^{(1)}_{(\pm)\mu\nu} a b \) and Chern-Simons form \( \omega^{L(1)} \) are now used to define the second-order Kalb-Ramond field strength \( H^{(2)} \) and so on.

Only \( \Omega^{(0)}_{(\pm)\mu} R^{(0)}_{(\pm)\mu\nu} b \omega^{L(0)}_{(\pm)\nu\rho} \) and \( H^{(1)}_{\mu\nu\rho} \) (plus the Yang-Mills fields) occur in the \( \mathcal{O}(\alpha') \) action and, in terms of these objects plus the dilaton field \( \phi \) and the Ricci scalar \( R \) of the metric \( g_{\mu\nu} \), the first-order in \( \alpha' \) Heterotic Superstring effective action in the string frame takes the form
\[ S = \frac{g_{s}^{2}}{16\pi G_{N}^{(10)}} \int d^{10}x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^{2} + \frac{1}{12} H^{2} - \frac{\alpha'}{8} \left[ F_{A} \cdot F^{A} + R_{(-)} a_{b} \cdot R_{(-)} b a \right] \right\}, \tag{1.9} \]

here \( g_{s} \) is the Heterotic Superstring coupling constant, which is given by the vacuum expectation value of \( e^{\phi} \), the dot indicates the contraction of the indices of 2-forms: \( F_{A} \cdot F^{A} \equiv F_{A\mu\nu} F^{A\mu\nu} \) and the 10-dimensional Newton constant \( G_{N}^{(10)} \) is related to the string length and coupling constants by
\[ G_{N}^{(10)} = 8\pi G_{s}^{2} \ell_{s}^{8}. \tag{1.10} \]

## 2 Dimensional reduction on \( T^{n} \) at zeroth order in \( \alpha' \)

As a warm-up exercise (and also because of the recursive definition of the action), we review the well-known dimensional reduction of the action at zeroth order in \( \alpha' \) using the Scherk-Schwarz formalism [27]. We add hats to all the 10-dimensional objects (fields, indices, coordinates) and split the 10-dimensional world and tangent-space indices as \((\hat{\mu}) = (\mu, m)\) and \((\hat{a}) = (a, i)\), with with \( \mu, \nu, \ldots \) and \( a, b, \ldots = 0, 1, \ldots, 9 - n \) and \( m, n, \ldots \) and \( i, j, \ldots = 1, \ldots, n \).

The Zehnbein and inverse-Zehnbein components \( \hat{e}_{\mu} \hat{a} \) and \( \hat{e}_{\hat{a}} \hat{\mu} \) can be put in an upper-triangular form by a local Lorentz transformation and, then, they can be decomposed in terms of the \( 10 - n \)-dimensional Vielbein and inverse Vielbein components \( e_{\mu}^{a}, e_{a}^{\mu} \), Kaluza-Klein (KK) vectors \( A_{\mu}^{m} \) and internal (\( T^{n} \)) metric Vielbeins and inverse Vielbein \( e_{m}^{i}, e_{i}^{m} \)
\[ \left( \hat{e}_{\mu} \hat{a} \right) = \left( \begin{array}{c} e_{\mu}^{a} \ A_{\mu}^{m} e_{m}^{i} \\ 0 \ e_{i}^{m} \end{array} \right), \quad \left( \hat{e}_{\hat{a}} \hat{\mu} \right) = \left( \begin{array}{c} e_{a}^{\mu} \ -A_{a}^{m} \ e_{m}^{i} \\ 0 \ e_{i}^{m} \end{array} \right). \tag{2.1} \]
where $A^m_a = e^m_a A^a_{\mu}$. We will always assume that all the $(10 - n)$-dimensional fields with Lorentz indices are $(10 - n)$-dimensional world tensors contracted with the $(10 - n)$-dimensional Vielbeins. For instance, the field strengths of the KK vector fields $F^m_{ab}$ are

$$
F^m_{ab} = e^m_a e^v_b F^m_{\mu v}, \quad F^m_{\mu v} = 2\partial_{[\mu} A^m_{\nu]}.
$$

We denote the internal metric by

$$
G_{mn} \equiv e^i_m e^n_j \delta_{ij}.
$$

The relation between the components of the 10-dimensional metric and $(10 - n)$-dimensional KK fields is

$$
\hat{g}_{\mu\nu} = g_{\mu\nu} - G_{mn} A_{\mu}^m A_{\nu}^n, \quad \hat{g}_{\mu m} = -G_{mn} A_{\mu}^n, \quad \hat{g}_{mn} = -G_{mn},
$$

The components of the 10-dimensional spin connection $\hat{\omega}_{abc}$ decompose into those of the $(10 - n)$-dimensional one $\omega_{abc}$, the KK vector field strengths $e_{im} F^m_{ab}$ and the pullback of the $O(n)$ connection 1-form $A^i_j$ defined in Eq. (A.10), as follows:

$$
\hat{\omega}_{abc} = \omega_{abc}, \quad \hat{\omega}_{abi} = -\frac{1}{2} e_{im} F^m_{ab}, \quad \hat{\omega}_{ibc} = -\hat{\omega}_{bci}, \quad \hat{\omega}_{aij} = A^i_j a,
$$

$$
\hat{\omega}_{ij} = -\frac{1}{2} e^m_i e^n_j \partial_b G_{mn},
$$

where we have used

$$
e_{(i}^m \partial_{a} e_{|m|j]} = -\frac{1}{2} e^m_i e^n_j \partial_a G_{mn}.
$$

Then, using the Palatini identity, it is not difficult to see that the first two terms in the action Eq. (1.9) take the following $(10 - n)$-dimensional form (up to a total derivative):

$$
\int d^{10}x \sqrt{|\hat{g}|} e^{-2\hat{\phi}} \left\{ \hat{R} - 4(\partial\hat{\phi})^2 \right\}
$$

$$
= \int d^nz \int d^{10-n}x \sqrt{|\hat{g}|} e^{-2\phi} \left\{ R - 4(\partial \phi)^2 - \frac{1}{4} \partial_a G_{mn} \partial^a G^{mn} - \frac{1}{4} G_{mn} F^m \cdot F^n \right\},
$$
where the \((10 - n)\)-dimensional dilaton field is related to the 10-dimensional one by

\[
\phi \equiv \hat{\phi} - \frac{1}{2} \log |G|, \quad |G| \equiv \det(G_{mn}). \quad (2.8)
\]

At zeroth order in \(\alpha'\), the last term that we have to reduce is the kinetic term of the Kalb-Ramond 2-form \(\sim (\hat{H}^{(0)})^2\). Following Scherk and Schwarz, we consider the Lorentz components of the 3-form field strength, because they are automatically combinations of gauge-invariant objects. These are given in terms of the world-indices components by

\[
\hat{H}_{ijk} = e_i^me_j^n e_k^p \hat{H}_{mnp},
\]

\[
\hat{H}_{aij} = e_i^me_j^n e_a^\mu \left[ \hat{H}^{(0)}_{\mu mn} - A^p_\mu \hat{H}_{p mn} \right],
\]

\[
\hat{H}_{abi} = e_i^m e_a^\mu e_b^\nu \left[ \hat{H}^{(0)}_{\mu vm} - 2A^n_{[\nu} \hat{H}^{(0)}_{\mu] nm} + A^n_{[\mu} A^p_{\nu]} \hat{H}_{pmn} \right],
\]

\[
\hat{H}_{abc} = e_a^\mu e_b^\nu e_c^\rho \left[ \hat{H}^{(0)}_{\mu \nu \rho} - 3A^m_{[\mu} \hat{H}^{(0)}_{\nu \rho] m} + 3A^m_{[\mu} A^n_{\nu]} \hat{H}^{(0)}_{\rho] mn} - A^m_{[\mu} A^n_{\nu} A^p_{\rho]} \hat{H}_{mn} \right] ,
\]

in general. At zeroth order in \(\alpha'\), \(\hat{H}^{(0)}_{mn} = 0\) and the above expressions are simplified to

\[
\hat{H}^{(0)}_{ijk} = 0 ,
\]

\[
\hat{H}^{(0)}_{aij} = e_i^m e_j^n e_a^\mu \hat{H}^{(0)}_{\mu mn} = e_i^m e_j^n \partial_a B^{(0)}_{mn} ,
\]

\[
\hat{H}^{(0)}_{abi} = e_i^m e_a^\mu e_b^\nu \left[ \hat{H}^{(0)}_{\mu vm} - 2A^n_{[\nu} \hat{H}^{(0)}_{\mu] nm} \right] = e_i^m \left[ G^{(0)}_{m ab} - B_{mn} F^n_{ab} \right] ,
\]

\[
\hat{H}^{(0)}_{abc} = e_a^\mu e_b^\nu e_c^\rho \left[ \hat{H}^{(0)}_{\mu \nu \rho} - 3A^m_{[\mu} \hat{H}^{(0)}_{\nu \rho] m} + 3A^m_{[\mu} A^n_{\nu]} \hat{H}^{(0)}_{\rho] mn} \right] = H^{(0)}_{abc} ,
\]

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where we have defined the potentials

\[ B_{mn} \equiv B^{(0)}_{mn}, \]  
\[ B^{(0)}_{m \mu} \equiv \hat{B}_{\mu m} + \hat{B}_{mn} A^n_\mu, \]  
\[ B^{(0)}_{\mu \nu} \equiv \hat{B}_{\mu \nu} + A^m_{[\mu} \hat{B}_{\nu]} m, \]

and the field strengths

\[ G^{(0)}_{m \mu \nu} \equiv 2 \partial_{[\mu} B^{(0)}_{m \nu]} \equiv 2 \partial_{[\mu} B^{(0)}_{m \nu]} \]  
\[ H^{(0)}_{\mu \nu \rho} \equiv 3 \partial_{[\mu} B^{(0)}_{\nu \rho]} - \frac{3}{2} A^m_{[\mu} G^{(0)}_{m \nu \rho]} - \frac{3}{2} B^{(0)}_{m [\mu F^m_{\nu \rho]}}. \]

Then, the reduction of the Kalb-Ramond kinetic term gives

\[ H^{(0)}_{\mu \nu \rho} = H^{(0)}_{\mu \nu \rho} - 3 G^{mn} (G^{(0)}_{m} - B^{(0)}_{mp} F^p) \cdot (G^{(0)}_{n} - B^{(0)}_{nq} F^q) \]

\[ + 3 G^{mn} G^{PQ} \partial_{[n} B^{(0)}_{m] P} \partial^Q B^{(0)}_{n q}, \]

and, after integrating over the length of the compact coordinates \( z_m \) \((2\pi \ell_s \text{ by convention})\) it can be checked that the whole \( \mathcal{O}(1) \) action can be written in the compact form

\[ S^{(0)} = \frac{g_s^{(10-n)} 2^{16\pi G_N^{(10-n)}}}{16 \pi^2} \int d^{10-n} x \sqrt{|g|} \ e^{-2\phi} \left\{ R - 4 (\partial \phi)^2 - \frac{1}{8} \text{Tr} \left( \partial_a M^{(0)} - 1 \partial^a M^{(0)} \right) \right\} \]

\[ - \frac{1}{4} \mathcal{F}^{(0)}_{MN} M^{(0) -1} \mathcal{F}^{(0)} + \frac{1}{12} H^{(0)}_{\mu \nu \rho} \]

\[ = \frac{g_s^{2} V_n / (2\pi \ell_s)^{n} g_s^{(10-n)2}}{16 \pi^2} \int d^n z = \frac{g_s^{2} (2\pi \ell_s)^{n}}{16 \pi^2} \frac{g_s^{(10-n)2}}{16 \pi^2} \]

The 10-dimensional string coupling constant \( g_s \) and Newton constant \( G_N^{(10)} \) and the \((10-n)\)-dimensional ones \( g_s^{(10-n)} \) and \( G_N^{(d)} \) are related by

\[ g_s^2 = V_n / (2\pi \ell_s)^{n} g_s^{(10-n)2}, \]  
\[ G_N^{(10)} = G_N^{(10-n)} V_n, \]

where \( V_n \) is the volume of the \( n \)-dimensional compact space. Then,

\[ \frac{g_s^{2} V_n / (2\pi \ell_s)^{n} g_s^{(10-n)2}}{16 \pi^2} \int d^n z = \frac{g_s^{2} (2\pi \ell_s)^{n}}{16 \pi^2} \frac{g_s^{(10-n)2}}{16 \pi^2}. \]
where $M^{(0)}$ is the $O(n,n)$ matrix defined in Eq. (A.4) of Appendix A with $B_{mn}$ replaced by $B^{(0)}_{mn}$ and where we have defined the $O(n,n)$ vectors of 1-forms and 2-form field strengths

$$A^{(0)} \equiv \begin{pmatrix} A^m \\ B^{(0)}_{mn} \end{pmatrix}, \quad \mathcal{F}^{(0)} \equiv \begin{pmatrix} F^m \\ G^{(0)}_{mn} \end{pmatrix}. \quad (2.17)$$

It is easy to show that $M^{(0)}$ is, indeed, an $O(n,n)$ matrix

$$M^{(0)} \Omega^{(0)} M^{(0) T} = 1, \quad (2.18)$$

and rewrite the Kalb-Ramond field strength in the manifestly $O(n,n)$-invariant form

$$H^{(0)}_{\mu \nu \rho} = 3 \partial_{[\mu} B^{(0)}_{\nu \rho]} - \frac{3}{2} A^{(0) T} [\mu \Omega^{(0)}_{\nu \rho}] \cdot \quad (2.19)$$

Actually, as it is well-known, the zeroth-order action $S^{(0)}$ given in Eq. (2.16) is manifestly invariant under $O(n,n)$ transformations which are understood as T-duality transformations from the 10-dimensional point of view.

### 3 Dimensional reduction on $T^n$ with Yang-Mills fields and Heterotic Supergravity

In this section we are only going to take into account the addition of the Yang-Mills fields which occur at first order in $\alpha'$, ignoring for the moment the terms that involve the torsionful spin connection. This truncation, which constitutes an intermediate step towards our final goal, is interesting by itself because it corresponds to the bosonic sector of a theory with exact local supersymmetry: $\mathcal{N} = 1, d = 10$ supergravity coupled to non-Abelian vector supermultiplets, also known as Heterotic Supergravity. The action of this theory is

$$S^{(h)} = \frac{g_s^2}{16 \pi G_N^{(10)}} \int d^{10}x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial \phi)^2 + \frac{1}{12} H^{(h)2} - \frac{\alpha'}{8} F^A \cdot F^A \right\}, \quad (3.1)$$

where

$$H^{(h)\mu \nu \rho} = 3 \partial_{[\mu} B_{\nu \rho]} + \frac{\alpha'}{4} \omega^{YM}_{\mu \nu \rho}, \quad (3.2)$$

and $F^A$ and $\omega^{YM}$ are defined in Eqs. (1.5) and (1.6), respectively.

Notice that the $O(\alpha'^2)$ terms of this action have to be kept in order to have exact local supersymmetry.

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\textsuperscript{7}That is, the action is exactly invariant, not just up to terms of higher order in $\alpha'$. 

9
The toroidal dimensional reduction of this theory in the case in which the gauge group is Abelian was carried out along the same lines we are going to follow here in Ref. [28, 12]. In the second of these references the O\((n, n + n_V)\) global symmetry of the resulting action was related to the T-duality transformations of the Heterotic Superstring. In the non-Abelian case, the gauge fields coming form the 10-dimensional gauge fields cannot be rotated into Kaluza-Klein and winding vector fields coming from the 10-dimensional metric and Kalb-Ramond fields. As a result, O\((n, n + n_V)\) is broken to O\((n, n)\), or O\((n, n + n_A)\) where \(n_A\) is the number of Abelian gauge fields.

The reduction of the Einstein-Hilbert term and of the scalar kinetic term are not modified by the inclusion of \(\alpha'\) corrections. The definitions of \((10 - n)\)-dimensional metric \(g_{\mu\nu}\), dilaton \(\phi\), KK vectors \(A^m_\mu\) and scalars \(G_{mn}\) in terms of the 10-dimensional fields are not modified by them either and they are still given by Eqs. (2.4) and (2.8).

Because of the additional Yang-Mills Chern-Simons term in the Kalb-Ramond field strength, we do expect modifications in the definitions of the definitions of the \((10 - n)\)-fields that originate in the Kalb-Ramond 2-form, namely the \((10 - n)\)-dimensional Kalb-Ramond 2-form \(B^{(h)}_{\mu\nu}\), the winding vectors \(B^{(h)}_{m\mu}\), with respect to their zeroth-order counterparts defined in Eqs. (2.11).

### 3.1 Reduction of the Yang-Mills fields

It is convenient to start by studying the dimensional reduction of the Yang-Mills fields. The Lorentz-indices decomposition of the gauge field is

\[
\hat{A}^A_i = e_i^m \hat{A}^A_m \equiv \varphi^A_i, \quad (3.3a)
\]

\[
\hat{A}^A_a = \hat{e}^\mu_a \hat{A}^A_\mu = e_a^\mu \left( \hat{A}^A_\mu - \hat{A}^A_m A^m_\mu \right) \equiv e_a^\mu \hat{A}^A_\mu, \quad (3.3b)
\]

which leads to the definition of the \((10 - n)\)-dimensional adjoint scalars \(\varphi^A_i\) and gauge vectors

\[
\varphi^A_i \equiv e_i^m \hat{A}^A_m, \quad (3.4a)
\]

\[
\hat{A}^A_\mu \equiv \hat{A}^A_\mu - \hat{A}^A_m A^m_\mu. \quad (3.4b)
\]

The components of 10-dimensional gauge field strength can be decomposed in terms of these fields as follows:

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8We use the superscript \((h)\) to indicate that these are the fields that arise in the reduction of Heterotic Supergravity and that possible contributions from the torsionful spin connection have not been taken into account.
\[
\hat{F}_{ij}^A = f_{ABC} \varphi^B_i \varphi^C_j, \quad (3.5a)
\]

\[
\hat{F}_{ai}^A = D_a \varphi^A_i + \frac{1}{2} \varphi_{i}^A e^j_m \varphi^B_j \partial_a G^{mn}, \quad (3.5b)
\]

\[
\hat{F}_{ab}^A = F_{ab}^A + \varphi_{i}^A e^j_m F_{mn}^i, \quad (3.5c)
\]

where \(F_{\mu\nu}^A\) is the standard Yang-Mills gauge field strength for the \((10-n)\)-dimensional gauge fields \(A_{\mu}^A\) and \(D\) is the Yang-Mills and \(O(n)\)-covariant derivative

\[
D_a \varphi^A_i = \partial_a \varphi^A_i + f_{ABC}^A \varphi^A_i \varphi^B_j + A_i^j \varphi^A_j, \quad (3.6)
\]

where the \(SO(n)\) composite connection is given in Eq. (A.10).

\[
\hat{F}_A \cdot \hat{F}^A = F_A \cdot F^A + 2 \varphi_i e^j_m F_{mn}^i \cdot F^j - \frac{1}{2} \varphi_i e^j_m e^k_n \partial^A \partial^B \partial^C - 2 D_a \varphi^A_i D_a \varphi^A_i \quad (3.7)
\]

Our next goal is the reduction of the Kalb-Ramond 3-form field strength. It is convenient to start with the reduction of the Yang-Mills Chern-Simons term

\[
\hat{\omega}^Y_{ijk} = 2 f_{ABC} \varphi^A_i \varphi^B_j \varphi^C_k, \quad (3.8a)
\]

\[
\hat{\omega}^Y_{aij} = 2 D_a \varphi^A_i [\varphi A_j], \quad (3.8b)
\]

\[
\hat{\omega}^Y_{abi} = \left( 2 F_{ab}^A + \varphi_i e^j_m F_{mn}^j \right) \partial_a A_i - 2 \varphi_{i}^A e^j_m \partial_a A_i, \quad (3.8c)
\]

\[
\hat{\omega}^Y_{abc} = \omega^Y_{abc} + 3 f_{ABC} \varphi^A_i e^j_m A_{[a}^A M_{bc]} \quad (3.8d)
\]

### 3.2 Reduction of the Kalb-Ramond field

Combining the results in the reduction of \(\hat{H}^{(0)}\) with the reduction of the Yang-Mills fields, we find
\[ \hat{H}^{(h)}_{ijk} = \frac{\alpha'}{2} f_{ABC} \phi^A_i \phi^B_j \phi^C_k, \]  
\[ (3.9a) \]

\[ \hat{H}^{(h)}_{aij} = \epsilon_i^m e_j^n \partial_a B_{mn} + \frac{\alpha'}{2} \partial_a \phi^A [i | \phi_A | j], \]  
\[ (3.9b) \]

\[ \hat{H}^{(h)}_{abi} = \epsilon_i^m e_{[a}^\mu e_{b]}^\nu \left\{ 2 \partial_\mu \left[ \hat{B}_{\nu m} + B_{mn} A^n_\nu - \frac{\alpha'}{4} A^A_m \hat{A}_A \right] \right. \]
\[ - \left[ B_{mn} - \frac{\alpha'}{4} \hat{A}_m^A \hat{A}_A n \right] F^n_{\mu \nu} + \frac{\alpha'}{2} \hat{A}_m A^A \right\}, \]
\[ (3.9c) \]

\[ \hat{H}^{(h)}_{abc} = \epsilon_{[a}^\mu e_{b]}^\nu e_{c]}^\rho \left\{ 3 \partial_\mu \left[ \hat{B}_{\nu p} + A^m_\nu \hat{B}_{pm} + \frac{\alpha'}{4} \hat{A}_m A^{m}_\nu \right] \right. \]
\[ - 3 A^m_\mu \partial_\nu \left[ \hat{B}_{\rho m} + B_{mn} A^n_\rho - \frac{\alpha'}{4} A^A_\rho \hat{A}_A \right] \]
\[ - 3 \left[ \hat{B}_{\mu m} + B_{mn} A^n_\mu - \frac{\alpha'}{4} A^A_\mu \hat{A}_A m \right] \partial_\nu A^m_\rho + \frac{\alpha'}{4} \omega_{\mu \nu} \right\}. \]
\[ (3.9d) \]

This result suggests the following definitions of \((10 - n)\)-dimensional fields:

\[ B^{(h)}_{mn} \equiv \hat{B}_{mn} - \frac{\alpha'}{4} \hat{A}_m A^A_n, \]  
\[ (3.10a) \]

\[ B^{(h)}_{m \mu} \equiv \hat{B}_{m \mu} + B_{mn} A^n_\mu - \frac{\alpha'}{4} A^A_\mu \hat{A}_A m \]
\[ = \hat{B}_{m \mu} + \left( \hat{B}_{mn} - \frac{\alpha'}{4} \hat{A}_m A^A_n \right) \hat{g}^{n \mu} \hat{g}^{\rho \mu} - \frac{\alpha'}{4} \hat{A}_A m \hat{A}_A^A_\mu, \]  
\[ (3.10b) \]

\[ B^{(h)}_{\mu \nu} \equiv \hat{B}_{\mu \nu} + A^m_{[\mu} \hat{B}_{\nu] m} + \frac{\alpha'}{4} \hat{A}_m A^m_{[\mu} A^A_{\nu]} \]
\[ = \hat{B}_{\mu \nu} + \hat{g}^{mn} \hat{g}_{m [\mu} \hat{B}_{\nu] n} + \frac{\alpha'}{4} \hat{A}_m \hat{g}^{mn} \hat{g}_{n [\mu} A^A_{\nu]}, \]  
\[ (3.10c) \]
and \((10 - n)\)-dimensional field strengths

\[
G^{(h)}_{m \mu \nu} \equiv 2 \partial_{[\mu} B^{(h)}_{m] \nu}, \quad (3.11a)
\]

\[
H^{(h)}_{\mu \nu \rho} \equiv 3 \partial_{[\mu} B^{(h)}_{\nu \rho]} - \frac{3}{2} A^m_{[\mu} G^{(h)}_{m] \nu \rho} - \frac{3}{2} B^{(h)}_{m [\mu} F^n_{\nu \rho]} + \frac{\alpha'}{4} Y^M_{\mu \nu \rho}, \quad (3.11b)
\]

This allows us to rewrite the components of the Kalb-Ramond field strength in the form

\[
\hat{H}^{(h)}_{ijk} = \frac{\alpha'}{2} f_{ABC} \varphi^A i \varphi^B j \varphi^C k, \quad (3.12a)
\]

\[
\hat{H}^{(h)}_{aij} = e_i^m e_j^n \partial_a B^{(h)}_{mn} + \frac{\alpha'}{2} D_a \varphi^A_i \varphi_A j, \quad (3.12b)
\]

\[
\hat{H}^{(h)}_{abi} = e_i^m \left( G^{(h)}_{m ab} - B^{(h)}_{mn} F^n_{ab} \right) + \frac{\alpha'}{2} F^A_{ab} \varphi_A i, \quad (3.12c)
\]

\[
\hat{H}^{(h)}_{abc} = H^{(h)}_{abc}, \quad (3.12d)
\]

so that the reduction of the kinetic term is

\[
\hat{H}^{(h)} = \frac{e_i^m e_j^n \partial_a B^{(h)}_{mn} - \alpha'}{2} F^A_{ab} \varphi_A i + \frac{\alpha'}{4} f_{ABC} f_{A'B'C'} \varphi^A_i \varphi^{A'}_j \varphi^{B'} k \varphi^{C'} k. \quad (3.13)
\]

Collecting all the terms (that is: Eqs. (2.7), (3.7) and (3.13)), we get
\begin{align}
S^{(h)} &= \frac{g^{(10-n)}_s}{16\pi G_N^{(10-n)}} \int d^{10-n}x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^2 \right. \\
&\quad - \frac{1}{4} \partial_a G_{mn} \partial^a G^{mn} + \frac{\alpha'}{16} \phi^A_i \phi_A j e^i_m e^J_n G_{pq} \partial^a G^{mp} \partial_a G^{nq} + \frac{1}{4} G^{mn} G^{pq} \partial^a B_{m p}^{(h)} \partial_a B_{n q}^{(h)} \\
&\quad + \frac{\alpha'}{4} \partial^a \phi^A_i \partial_a \phi_A i + \frac{\alpha'^2}{16} \phi^A_i \phi^B_j \partial^a \phi_A i \partial_a \phi_B j \\
&\quad + \frac{\alpha'}{4} \partial^a \phi_A i \phi^A j e^i_m e^J_n \partial_a B_{mn}^{(h)} + \frac{\alpha'}{4} \partial^a \phi^A_i \phi_A j e^i_m \partial_a G^{mn} \\
&\quad - \frac{1}{4} \left( G_{mn} - B_{m p}^{(h)} G^{pq} B_{n q}^{(h)} \right) F^m \cdot F^n \\
&\quad - \frac{1}{4} G^{mn} G^{(h)}_{m p} G^{(h)}_{n q} + \frac{1}{2} G^{mp} B_{p n}^{(h)} G^{(h)}_{m q} \cdot F^n \\
&\quad - \frac{\alpha'}{8} \left( K_{AB} + \frac{\alpha'}{2} \phi_A i \phi_B j \right) F^A \cdot F^B - \frac{\alpha'}{4} \phi_A i e^i_m G^{mn} G^{(h)}_{n q} \cdot F^A \\
&\quad - \frac{\alpha'}{4} \phi_A i e^i_m \left( \delta^m_n - G^{mp} B_{p n}^{(h)} \right) F^n \cdot F^A + \frac{1}{12} H^{(h)} \cdot H^{(h)} - V(\phi) \right\}. 
\end{align}

(3.14)

where \( K_{AB} \) is the Killing metric of the gauge group and we have defined the scalar potential

\[ V(\phi) \equiv \frac{\alpha'}{8} f_{ABC} f^A_D f^B_E f^C_F \phi^D_i \phi^E_j \phi^F_k + \frac{\alpha'^2}{48} f_{ABC} f_{A'B'C'} \phi^A_i \phi^B_j \phi^C_k \phi^C'_{k'}. \]  

(3.15)

Defining the scalar matrices

\[ G \equiv (G_{mn}), \quad B^{(h)} \equiv (B^{(h)}_{mn}), \quad \phi \equiv (\phi^A_i \epsilon^i_m), \quad K \equiv \frac{\alpha'}{2} (K_{AB}), \]

(3.16)

and the \( O(n, n + n_V) \) vector of 2-form field strengths

\[ \mathcal{F}^{(h)}_{\mu \nu} \equiv \left( \begin{array}{c} F_{\mu \nu}^m \\ G_{m \mu \nu}^{(h)} \\ F_{A \mu \nu}^{(h)} \end{array} \right), \]

(3.17)
we can rewrite their kinetic terms in the form
\[-\frac{1}{4}F^{(h)^T}M^{(h)} \cdot F^{(h)} ,\]
where $M^{(h)}$ is the symmetric matrix
\[
M^{(h)^{-1}} = \left( \begin{array}{ccc}
G + B^{(h)^T}G^{-1}B^{(h)} + \varphi^T K \varphi & -B^{(h)^T}G^{-1} (\mathbb{1}_{n \times n} - B^{(h)^T}G^{-1}) \varphi^T K \\
-G^{-1}B^{(h)} & G^{-1} & G^{-1} \varphi^T K \\
K \varphi (\mathbb{1}_{n \times n} - G^{-1}B^{(h)}) & K \varphi G^{-1} & K + K \varphi G^{-1} \varphi^T K 
\end{array} \right).
\]
This is an $O(n, n+n_V)$ matrix (for $n_V$ gauge fields) because it satisfies
\[M^{(h)} \Omega^{(h)} M^{(h)} \Omega^{(h)} = 1, \quad \text{with} \quad \Omega^{(h)} \equiv \left( \begin{array}{ccc}
0 & \mathbb{1}_{n \times n} & 0 \\
\mathbb{1}_{n \times n} & 0 & 0 \\
0 & 0 & -K 
\end{array} \right),\]
in a basis in which the Killing metric is not simply the identity.

The kinetic terms of the scalar fields can be written in the form
\[-\frac{1}{8} \mathcal{D}^a M^{(h)} \mathcal{D}_a M^{(h)^{-1}} ,\]
where the covariant derivative only acts on the gauge group.\(^9\)

The total action takes a form very similar to the zeroth-order one Eq. (2.16):
\[
S^{(h)} = \frac{g^2 (10-n)^2}{16 \pi G_N^{(10-n)}} \int d^{10-n}x \sqrt{|g|} e^{-2\varphi} \left\{ R - 4(\partial \varphi)^2 - \frac{1}{8} \text{Tr} \left( \mathcal{D}^a M^{(h)} \mathcal{D}_a M^{(h)^{-1}} \right) \right\} \]
\[-\frac{1}{4} F^{(h)^T} M^{(h)^{-1}} \cdot F^{(h)} + \frac{1}{16} H^{(h)^2} - V^{(h)}(\varphi) \right\} .
\]

At first sight, this action is formally $O(n, n+n_V)$-invariant except for the scalar potential. However, we cannot transform Abelian into non-Abelian fields and vice-versa and the $O(n, n+n_V)$ invariance is broken in the Chern-Simons term of $H^{(h)}$, in the kinetic term of the vector fields and also in the kinetic terms of the scalars and therefore, generically, the invariance is broken to just $O(n, n)$. If the gauge group is Abelian, the scalar potential disappears, the covariant derivatives of the scalars can be rewritten entirely in terms of partial derivatives and $H^{(h)}$ takes the manifestly $O(n, n+n_V)$-invariant form
\[H^{(h)^{\mu
u}} = 3\partial_{[\mu} B^{(h)^{\nu]} + \frac{3}{2} A^{(h)^{[\mu}} \Omega^{(h)} F^{(h)^{\nu]}} .\]

\(^9\)There are no free $SO(n)$ indices in $M^{(h)}$.
4 Complete dimensional reduction on $T^n$ to $O(\alpha')$

4.1 Torsionful spin connection

The reduction of the terms involving the torsionful spin connection $\hat{\Omega}^{(0)}_{(-) \hat{a} \hat{b}}$ can be carried out in two steps: first we just treat it as just another Yang-Mills field but with the particular gauge group $SO(1,9)$. Then, we decompose the gauge group indices into $SO(1,9 - n) \times SO(n)$ indices. As a matter of fact, we can just take the results of the previous section and assume that the gauge group has been extended to include $SO(1,9)$.

Let us carry out the first step.

4.1.1 First step

As in the general Yang-Mills case, the reduction of the torsionful spin connection gives two fields

$$\varphi_{i}^{\hat{a} \hat{b}} \equiv e_{i}^{m} \hat{\Omega}^{(0)}_{(-) m} \hat{a} \hat{b}, \quad (4.1a)$$

$$A^{\hat{a} \hat{b} \mu} \equiv \hat{\Omega}^{(0)}_{(-) \mu} \hat{a} \hat{b} - \hat{\Omega}^{(0)}_{(-) m} \hat{a} \hat{b} A^{m \mu}, \quad (4.1b)$$

and the different components of its curvature give

$$\hat{R}^{(0)}_{(-) \hat{a} \hat{b} i j} = -2 \varphi_{i}^{\hat{c} \hat{b}} [i \varphi_{j}^{\hat{c} \hat{b}}], \quad (4.2a)$$

$$\hat{R}^{(0)}_{(-) \hat{a} \hat{b} c i} = \overline{\nabla}_{c} \varphi_{i}^{\hat{a} \hat{b}} + \frac{1}{2} \varphi_{i e}^{\hat{a} \hat{b}} g^{e i m} \partial_{m} e_{c m}, \quad (4.2b)$$

$$\hat{R}^{(0)}_{(-) \hat{a} \hat{b} c d} = F_{c d}^{\hat{a} \hat{b}} + \varphi_{i}^{\hat{a} \hat{b}} g^{i m} F_{c d}^{m}. \quad (4.2c)$$

It is worth stressing that $F_{c d}^{\hat{a} \hat{b}}$ is the $(10 - n)$-dimensional curvature of the $(10 - n)$-dimensional $SO(1,9)$ gauge field $A^{\hat{a} \hat{b} \mu}$

$$F_{\mu \nu}^{\hat{a} \hat{b}} \equiv 2 \partial_{[\mu} A^{\hat{a} \hat{b} \nu]} - 2 A^{\hat{c} \hat{d}} [\mu A^{\hat{c} \hat{d} \nu}], \quad (4.3)$$

and $\overline{\nabla}_{c}$ is a $SO(1,9) \times SO(n)$ covariant derivative with the same connection plus the composite $SO(n)$ connection in Eq. (A.10):

$$\overline{\nabla}_{c} \varphi_{i}^{\hat{a} \hat{b}} = \partial_{c} \varphi_{i}^{\hat{a} \hat{b}} - 2 A_{[i}^{\hat{d} [\hat{a} |d| \partial_{c} \varphi_{j]^{\hat{b}]} + A_{ij}^{\hat{a} \hat{b}} \varphi_{i}^{\hat{b}}. \quad (4.4)$$
At this stage we can use the results of the previous section (the Heterotic Supergravity case) to write the \((10 - n)\)-dimensional action that one gets after completing the first step, because it has exactly the same form the same form as that of the Heterotic Supergravity case Eq. (3.22) if we define a new gauge index \(X\) that includes the \(10\)-dimensional adjoint gauge group index \(A\) and the adjoint \(10\)-dimensional Lorentz index \([\hat{a} \hat{b}]\): \(X = A, [\hat{a} \hat{b}]\). We can write, directly and formally

\[
S^{(1)} = \frac{G_s^{(10-n)}}{16\pi G_N^{(10-n)}} \int d^{10-n} x \sqrt{|\mathcal{g}|} e^{-2\phi} \left\{ R - 4(\partial \phi)^2 - \frac{1}{8} \mathrm{Tr} \left( \bar{\mathcal{D}}^a \mathcal{M}^{(1)} \mathcal{D}_a \mathcal{M}^{(1)} \right)^{-1} \right\}
\]

where the covariant derivative \(\bar{\mathcal{D}}\) only acts on \(X\) indices, \(\mathcal{F}^{(1)}\) is the vector of \((10 - n)\)-dimensional 2-form field strengths

\[
\mathcal{F}^{(1)} = \begin{pmatrix} F^m \\ G^{(1) m} \\ F^X \end{pmatrix}, \quad \text{with} \quad (F^X) \equiv \begin{pmatrix} F^A \\ F^{\hat{a} \hat{b}} \end{pmatrix},
\]

where \(F^m\) has been defined in Eq. (2.2), \(F^A\) is the field strength of \((10 - n)\)-dimensional Yang-Mills field, \(F^{\hat{a} \hat{b}}\) is the field strengths of the \((10 - n)\)-dimensional \(\text{SO}(1, 9)\) gauge field defined in Eq. (4.3) and

\[
G^{(1) m \mu \nu} = 2\partial_{[\mu} B^{(1) m | \nu]}.
\]

\(\mathcal{M}^{(1)}\) is the matrix

\[
\tilde{\mathcal{M}}^{(1)} = \begin{pmatrix} G + B^{(1)T} G^{-1} B^{(1)} + \phi^T K \phi & -B^{(1)T} G^{-1} & \left( 1_{n \times n} - B^{(1)T} G^{-1} \right) \phi^T K \\ -G^{-1} B^{(1)} & G^{-1} & G^{-1} \phi^T K \\ K \phi \left( 1_{n \times n} - G^{-1} B^{(1)} \right) & K \phi G^{-1} & K + K \phi G^{-1} \phi^T K \end{pmatrix},
\]

where

\(10\)With the connection \(A^A\) on YM indices and with the connection \(A^{\hat{a} \hat{b}}\) on \(\text{SO}(1, 9)\) indices.
\begin{equation}
G \equiv (G_{mn}), \quad B^{(1)} \equiv (B^{(1)}_{mn}), \quad \varphi \equiv (\varphi^X i_e^j m) = (\varphi^{A} i_e^j m, \varphi^{h} i_e^j m),
\end{equation}

\begin{equation}
K \equiv \frac{\alpha'}{2} (K_{XY}) = \frac{\alpha'}{2} \left( \begin{array}{cc} K_{AB} & 0 \\ 0 & -\hat{g}^{\hat{a}\hat{b}} \hat{c} \hat{d} \end{array} \right).
\end{equation}

Finally, the \((10-n)\)-dimensional Kalb-Ramond field strength \(\tilde{H}^{(1)}\) and the “scalar potential”\(^{11}\) \(V^{(1)}\) are given by\(^ {12}\)

\begin{equation}
\tilde{H}^{(1)}_{\mu\nu} \equiv 3\partial_{[\mu} B^{(1)}_{\nu]} - \frac{3}{2} A^m_{[\mu} C^{(1)} m_{\nu]} - \frac{3}{2} B^{(1)} m_{[\mu} F^m_{\nu]} \\
+ \frac{\alpha'}{4} (\omega_{YM}^{\mu\nu\rho} + \tilde{\omega}_{(-)}^{L(0)})_{\mu\nu\rho},
\end{equation}

\begin{equation}
V^{(1)}(\varphi) \equiv \frac{\alpha'}{8} \left( 2\varphi_{(\hat{a}\hat{b})} \varphi_{(\hat{c}\hat{d})} \varphi_{(\hat{e})_{(\hat{f})}} \varphi_{(\hat{g})_{(\hat{h})}} - 2\varphi_{(\hat{a}\hat{b})} \varphi_{(\hat{c})_{(\hat{d})}} \varphi_{(\hat{e})_{(\hat{f})}} \varphi_{(\hat{g})_{(\hat{h})}} \\
+ f_{ABC} f^{A DE} \varphi_{(\hat{a})_{(\hat{b})}} \varphi_{(\hat{c})_{(\hat{d})}} \varphi_{(\hat{e})_{(\hat{f})}} \varphi_{(\hat{g})_{(\hat{h})}} \right).
\end{equation}

In the above expressions we have used the \((10-n)\)-dimensional fields \(B^{(1)}_{mn}, B^{(1)}_{m\mu}\) and \(B^{(1)}_{\mu\nu}\). They are defined in terms of the 10-dimensional ones by

\begin{equation}
B^{(1)}_{mn} \equiv \hat{B}_{mn} - \frac{\alpha'}{4} \left( \hat{A}^A_{n} \hat{A}_{A} + \hat{\Omega}^{(0)}_{(-)} m \hat{a}_{(-)} n \hat{b}_{(-)} \right),
\end{equation}

\begin{equation}
B^{(1)}_{m\mu} \equiv \hat{B}_{m\mu} + \left[ \hat{B}_{mn} - \frac{\alpha'}{4} \left( \hat{A}^A_{n} \hat{A}_{A} + \hat{\Omega}^{(0)}_{(-)} m \hat{a}_{(-)} n \hat{b}_{(-)} \right) \right] \hat{g}^{np} \hat{g}_{p\mu},
\end{equation}

\begin{equation}
B^{(1)}_{\mu\nu} \equiv \hat{B}_{\mu\nu} + \hat{g}^{mn} \hat{g}_{m|\nu} \hat{B}_{n]\mu} - \frac{\alpha'}{4} \left( \hat{A}^A_{m} \hat{A}_{A |\mu} + \hat{\Omega}^{(0)}_{(-) m \hat{a}_{(-)} m \hat{b}_{(-)} |\mu} \hat{b}_{(-)} \right) \hat{g}^{mn} \hat{g}_{\nu|n}.
\end{equation}

The action Eq. (4.5) contains implicitly \(O(\alpha'^2)\) terms such as \(H^{(1)2}\), as the original action Eq. (1.9), but it is convenient to keep them in order to have more compact and

\(^{11}\)The variables \(\varphi^{A}_{i}\) become true \((10-n)\)-dimensional scalars with \(O(n)\) and adjoint gauge indices, but the variables \(\varphi^{h}_{i}\) become both \((10-n)\)-dimensional tensors and scalars, see Eqs. (4.13).

\(^{12}\)We have neglected the \(O(\alpha'^2)\) terms in \(V^{(1)}\).
gauge-invariant expressions. Eliminating all the $O(\alpha'^2)$ terms we would get an action which is gauge invariant only to $O(\alpha')$, but this gauge invariance and possible duality invariance (which we are going to discuss next) would not be manifest.

Although the action Eq. (4.5) is apparently manifestly $O(n,n)$-invariant, this is not so clear because the statement assumes that all the terms not directly affected by the linear $O(n,n)$ transformations remain invariant. However, some of those terms, such as $F_{\hat{a} \hat{b}} \cdot F_{\hat{a} \hat{b}}$, for instance, depend on the internal Vielbein $e^i_m$ and/or the KK vectors $A^m_\mu$ which are not invariant. We have to move to the second phase and expand the terms that depend on the fields with $SO(1,9)$ indices in terms of $(10 - n)$-dimensional fields.

4.1.2 Second step

The fields with $SO(1,9)$ indices $\varphi^{\hat{a} \hat{b}}_i$ and $A^{\hat{a} \hat{b}}_\mu$ are further reduced as follows:

$$
\varphi^{\hat{a} \hat{b}}_i \rightarrow \begin{cases}
\varphi^{ab}_i = -\frac{1}{2} K_{i ab}^{(0)}, \\
\varphi^{ai}_j = -P_{ij}^{(0)} a, \\
\varphi^{ij}_k = 0,
\end{cases} \quad (4.13)
$$

and

$$
A^{\hat{a} \hat{b}}_\mu \rightarrow \begin{cases}
A^{ab}_\mu = \Omega^{ab}_\mu, \\
A^{ai}_\mu = -\frac{1}{2} K_{i a}^{(0)}, \\
A^{ij}_\mu = A^{ij}_\mu,
\end{cases} \quad (4.14)
$$

where the 2-form $K_{i \mu}^{(0)}$, the $O(n)$ connection 1-form $A_{i \mu}^{(0)}$, and the Vielbein $P_{ij}^{(0)}$ we have been defined in Eq. (A.20) with $B_{mn}$ replaced by $B_{mn}^{(0)}$.

Taking into account these expressions, the components of the $(10 - n)$-dimensional $SO(1,9)$ field strength $F^{\hat{a} \hat{b}}_{cd}$ that occurs in the reduction of the curvature of the torsion-
ful spin connection Eq. (4.2) are decomposed as follows:

\[ F^{ab}_{\mu\nu} = R^{(0)}_{(-)\mu\nu} + \frac{1}{2} K^{(0)}_{(-)\mu} \epsilon^{(0)i}_a K^{(0)i}_v b, \]  
\[ F^{ai}_{\mu\nu} = \mathcal{D}^{(0)}_{\mu} [K^{(0)i}_v]^{a}, \]  
\[ F^{ij}_{\mu\nu} = F^{(0)ij}_{\mu\nu} + \frac{1}{2} K^{(0)i}_{\mu} a K^{(0)j}_v a, \]  

where \( \mathcal{D}_{(-)} \) is the \( \text{SO}(1,9-n) \times \text{O}(n) \) covariant derivative with the “(0)(−)” connections, that is

\[ \mathcal{D}^{(0)}_{\mu} [K^{(0)i}_v]^{a} = \partial^{[\mu} K^{(0)i}_{\nu]} a + A^{(0)i}_{\mu} (K^{(0)}_{\nu})^{a} - \Omega^{(0)}_{\mu} [\nu] a K^{(0)i}_v b. \]  

Now we use all these decompositions into \( \tilde{\omega}^{L(0)}_{(-)abc} \) the \((10-n)\)-dimensional Lorentz Chern-Simons 3-form of the \( \text{SO}(1,9) \) connection, obtaining

\[ \tilde{\omega}^{L(0)}_{(-)\mu\nu\rho} = \omega^{L(0)}_{(-)\mu\nu\rho} - \omega^{O(n)}_{(-)\mu\nu\rho} + 3 \mathcal{D}^{(0)}_{\mu} [K^{(0)i}_v] a K^{(0)i}_v a, \]  
\[ \omega^{O(n)}_{(-)\mu\nu\rho} \equiv 3 F^{(0)ij}_{\mu\nu} [A^{(0)i}]_{\rho} j + 2 A^{(0)i}_{\mu} A^{(0)jk}_{\nu} A^{(0)ki}_{\rho}. \]  

According to the discussion in Appendix A the 3-form \( \mathcal{D}^{(0)}_{\mu} [K^{(0)\mathcal{K}_{\nu}^{(-)}}] a \) is \( \text{O}(n,n) \)-invariant. Since it is also gauge and Lorentz-invariant it is natural to eliminate it from the definition of the \((10-n)\)-dimensional Kalb-Ramond field strength \( H^{(1)} \):

\[ \tilde{H}^{(1)}_{\mu\nu\rho} = H^{(1)}_{\mu\nu\rho} + \frac{3\alpha'}{4} \mathcal{D}^{(0)}_{\mu} [K^{(0)i}_v] a K^{(0)i}_v a, \]  
\[ H^{(1)}_{\mu\nu\rho} \equiv 3 \partial_{[\mu} B^{(1)\nu\rho]} - \frac{3}{2} A^{(m)\mu}_{\nu} G^{(1)}_{m|\rho} - \frac{3}{2} B^{(1)\mu\nu m} F_{m|\rho} \]  
\[ + \frac{\alpha'}{4} \left( \omega^{YM}_{\mu\nu\rho} + \omega^{L(0)}_{(-)\mu\nu\rho} - \omega^{O(n)}_{(-)\mu\nu\rho} \right). \]  

Observe that the Chern-Simons 3-form of the composite \( \text{O}(n)(-) \) connection \( A^{(0)ij}_{(-)} \) is not \( \text{O}(n,n) \)-invariant, because it is not invariant under the compensating local \( \text{O}(n)(-) \) transformation (it is invariant up to a total derivative, as any Chern-Simons 3-form).13

\[ ^{13} \text{The presence of this term has been observed in Ref. [21]} \]
It can be compensated with a standard Nicolai-Townsend transformation of the \((10 - n)\)-dimensional 2-form \(B^{(1)}\),\(^{14}\) though. Thus, as different from the \(n = 1\) case, this field is now not \(T\)-duality invariant to first order in \(\alpha'\). Observe that these transformations do not affect \(B^{(0)}\), which is the field that occurs in \(\Omega^{(0)}_{(n)} a_{b}^{\mu} \) and \(A^{(0)}_{ij}\).

Next, let us consider the decomposition of the "scalar potential" \(V^{(1)}(\varphi)\). Using Eqs. (4.13), we get

\[
V^{(1)}(\varphi) = \frac{\alpha'}{8} \left\{ \frac{1}{8} K^{(0) i}_{(+) ab} K^{(0) j}_{(+) cd} K^{(0) j bd}_{(+)} + \frac{1}{2} K^{(0) i}_{(+) ab} K^{(0) k}_{(+) cd} K^{(0) j bd}_{(+)} + \frac{1}{8} K^{(0) i}_{(+) ab} K^{(0) j}_{(+) cd} K^{(0) j bd}_{(+)},
\]

(4.19)

where we have used the identity Eq. (A.15). All terms are manifestly \(O(n, n)\)-invariant.

Next, we have to decompose the kinetic term for the vector field strengths into its \(O(n, n)\) invariant form and the rest:

\[
\tilde{\mathcal{F}}^{(1)} T \tilde{M}^{(1)} = \mathcal{F}^{(1)} T M^{(1)} + \frac{\alpha'}{2} \left\{ F_{A} \cdot F^{A} + R_{(-)}^{(0)} [a b] \cdot R_{(-)}^{(0)} [b a] + R_{(-)}^{(0)} [a b c d] \cdot (K^{(0) i ac}_{(-)} K^{(0) j bd}_{(-)} + K^{(0) i ab}_{(-)} K^{(0) c d}) + \frac{1}{4} K^{(0) i}_{(-) ab} K^{(0) j}_{(-) cd} K^{(0) j kd}_{(-)} + \frac{1}{8} \left( K^{(0) i}_{(-)} \cdot K^{(0) j}_{(-)} \right)^{2} - \frac{1}{8} \left( K^{(0) i}_{(-)} \cdot K^{(0) j}_{(-)} \right) \right\},
\]

(4.20)

where the covariant derivative \(\nabla^{(0)}_{(-)}\) contains the \(O(n)_{(-)}\) connection \(A^{(0)}_{(-) ij}\) and the torsionful Lorentz connection \(\Omega^{(0)}_{(-) ab}^{\mu}\), and where, now, the matrix \(M^{(1)} = 1\) is given by the upper-left corner \(2 \times 2\) part of \(\tilde{M}^{(1)}\) in Eq. (4.8), namely

\(^{14}\)The Nicolai-Townsend transformations were first found in Ref. [29] in a different context. They were shown to be necessary in the coupling of vector multiplets to \(\mathcal{N} = 1, d = 10\) supergravity in [30].
\[ M^{(1)-1} \equiv \begin{pmatrix} G + B^{(1)^T}G^{-1}B^{(1)} + \varphi^T K \varphi & -B^{(1)^T}G^{-1} \\ -G^{-1}B^{(1)} & G^{-1} \end{pmatrix} \], \quad (4.21)

and

\[ \mathcal{F}^{(1)} \equiv \begin{pmatrix} F_m \\ G^{(1)}_m \end{pmatrix}. \quad (4.22) \]

Finally, let us consider the scalar’s kinetic term. Expanding this term, reducing and keeping only terms of first order in \( \alpha’ \) we arrive at

\[
\text{Tr} \left( \bar{D}^a M^{(1)} \bar{D}_a M^{(1)-1} \right) = \text{Tr} \left( \partial^a M^{(1)} \partial_2 M^{(1)-1} \right) - \alpha' \mathcal{D}^{(0)}_c \left( U^{(+)}_i \varphi A_i \right)^T \Omega^{(1)} \mathcal{D}^{(0)}_c \left( U^{(+)}_i \varphi^A A_i \right) \\
- \frac{\alpha'}{4} \mathcal{D}^{(0)}_c \left( U^{(+)}_i K^{(0)}_{(+)} i_a a b \right)^T \Omega^{(1)} \mathcal{D}^{(0)}_c \left( U^{(+)}_i K^{(0)}_{(+)} b a \right) \\
- 2\alpha' \mathcal{D}^{(0)}_c \left( U^{(+)}_i P^{(0)}_{(+)} a i j \right)^T \Omega^{(1)} \mathcal{D}^{(0)}_c \left( U^{(+)}_i P^{(0)}_{(+)} a k j \right). \quad (4.23)
\]

Finally, combining all the partial results we get the final, manifestly-\( \mathcal{O}(n, n) \)-invariant form of the \((10 - n)\)-dimensional action
\[ S(1) = \frac{g_s^{(10-n)2}}{16\pi G_N^{(10-n)}} \int d^{10-n}x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial \phi)^2 - \frac{1}{8} \text{Tr} \left( \partial^a M(1) \partial_a M(1)^{-1} \right) \right\} \]

\[-\frac{1}{4} F^{(1)T} M^{(1)} \cdot F^{(1)} + \frac{1}{12} H^{(1)2} - \frac{k'}{8} \left[ F_A \cdot F^A + R^{(0)}_{a b} \cdot R^{(0) b a} \right] \]

\[+ R^{(0)}_{a b c d} K^{(0)}_{i a c b} K^{(0)}_{i d} + R^{(0)}_{a b} K^{(0)}_{i a} K^{(0) i a} \]

\[-\frac{1}{8} K^{(0)}_{i} i j a b K^{(0)}_{i} j b K^{(0)}_{i} j b - \frac{1}{8} K^{(0)}_{i} i j a b K^{(0)}_{i} j b K^{(0)}_{i} j b \]

\[+ \frac{1}{8} K^{(0)}_{i} i j a b K^{(0)}_{i} j b K^{(0)}_{i} j b + \frac{1}{8} K^{(0)}_{i} i j a b K^{(0)}_{i} j b K^{(0)}_{i} j b \]

\[+ 2 \mathcal{D}^{(0)}_{\mu a} K^{(0)}_{i} + 4 P^{(0)}_{a i j} K^{(0)}_{i} j a \mu K^{(0)}_{i} j a \mu \]

\[+ 2 \varphi_{A i} F^A_i + 2 \varphi_{A i} F^A_i + 2 \varphi_{A i} F^A_i + 2 \varphi_{A i} F^A_i \]

\[= \mathcal{D}^{(0)}_{\mu a} K^{(0)}_{i} + 4 P^{(0)}_{a i j} K^{(0)}_{i} j a \mu K^{(0)}_{i} j a \mu \]

\[+ \mathcal{D}^{(0)}_{\mu a} K^{(0)}_{i} + 4 P^{(0)}_{a i j} K^{(0)}_{i} j a \mu K^{(0)}_{i} j a \mu \]

This dimensionally-reduced action has two main immediate uses: the derivation of equations of motion and the derivation of an entropy formula. The equations of motion are very complicated and involve, in principle, derivatives higher than 2. According to the lemma proven in Ref. [23], the terms of higher-order in derivatives (possibly

23
in combination with some terms of lower order) are, actually, proportional to $\alpha'$ and combinations of the zeroth-order equations of motion and can, in be disregarded in practice. However, in order to find all the terms which can be ignored one has to use the lower-dimensional version of the lemma, which requires the identification of all the fields which originate, purely, on the 10-dimensional torsionful spin connection. Therefore, it is far easier to deal with the 10-dimensional equations of motion and perform the dimensional reduction of the solution using the rules derived in this paper. The entropy formula, though, can be readily obtained and used in $(10 - n)$ dimensions, as we are going to show in the next section.

5 Entropy formula

If we change the index 10 by $D$, the action Eq. (1.9) is identical to the action one would obtain by trivial dimensional reduction of the 10-dimensional theory on a $m \equiv (10 - D)$-dimensional torus. Then, with the same change, Eq. (4.24) is the action obtained by a fully non-trivial dimensional reduction on $T^n$, from $D$ to $d = D - n = 10 - n - m$ dimensions. The solutions of the equations of motion of this $d$-dimensional theory are solutions which, uplifted to $d + n = D$ dimensions with the rules derived in this paper, are, then, solutions of the equations of motion of the original action Eq. (1.9) with 10 replaced by $D$ and which can be trivially uplifted again to 10 dimensions.

An important example of solutions of this type are the heterotic version of the 4-dimensional, 4-charge extremal black holes whose microscopic entropy was originally computed in Refs. [13, 14] (see also Ref. [31])\(^\text{15}\) and whose first-order $\alpha'$ corrections were recently computed in Refs. [37, 38].\(^\text{16}\)

If these 4-dimensional solutions are uplifted to 6 dimensions ($n = 2$), they are solutions of our original action Eq. (1.9) with $D = 6$ ($m = 4$). Thus, they are solutions of the equations of motion of the action Eq. (4.24) with 10 replaced by 6 and $n = 2$ and we can use that action to compute their Wald entropy using the Iyer-Wald prescription [8, 9].

The direct application of this prescription to Eq. (4.24) with 10 replaced by a general dimension $D$ yields the following string-frame entropy formula for $d = (D - n)$-dimensions:

\[^{15}\text{These 4-dimensional solutions can be embedded either in the Heterotic or in type II theories. They were first found in Ref. [32] and further discussed in Ref. [33, 34]. They were rediscovered in Ref. [35] in the context of the so-called STU model [36] in a form that made it easier to identify harmonic functions and charges to 10-dimensional string theory extended objects.}\]

\[^{16}\text{In these references the first-order in $\alpha'$ corrections to the complete geometry were computed. Earlier work in which only the corrections to the near-horizon geometry were computed and then, used to compute the corrections to the entropy can be found in Refs. [39–44]. Some of the drawbacks of these methods, such as the problem of identification of asymptotic charges and the possible incompleteness of the higher-order terms considered have been discussed in Ref. [38, 46].}\]
dimensional black holes:

\[ S = -2\pi \int_\Sigma d^{d-2}x \sqrt{|h|} \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd}, \]  

\[ \frac{\partial L}{\partial R_{abcd}} = \frac{e^{-2(\phi - \phi_\infty)}}{16\pi G_N^{(d)}} \left\{ g^{ab,cd} - \frac{\alpha'}{8} \left[ H^{(0)}_{abg} \left( \omega^c_{\ell} - H^{(0)}_{\ell c} \right) \right] ight. 
\left. - 2R^{(0)}_{abcd} + K^{(-)} i [a|e K^{(-)} i |b] d + K^{(+) j a b} K^{(+) j c d} \right\}, \]  

where \(|h|\) is the absolute value of the determinant of the metric induced over the event horizon, \(g^{ab,cd} = \frac{1}{2} \left( g^{ac} g^{bd} - g^{ad} g^{bc} \right)\), \(\epsilon^{ab}\) is the event horizon’s binormal normalized so that \(\epsilon_{ab} \epsilon^{ab} = -2\) and \(R_{abcd}\) is the Riemann tensor.

This manifestly \(O(n, n)\)-invariant entropy formula reduces, for \(n = 1\) to the formula found in Ref. [1] and which has been used to compute the Wald entropy of the \(\alpha'\)-corrected Reissner-Nordström black hole of Ref. [11] and of the \(\alpha'\)-corrected heterotic version of the Strominger-Vafa black hole of Ref. [45].

5.1 The Wald entropy of the \(\alpha'\)-corrected 4d 4-charge black holes

The \(\alpha'\)-corrected 4d 4-charge black-hole solutions correspond to the following 10-dimensional solutions of the action Eq. (1.9):

\[ ds^2 = \frac{2}{Z_+} du \left[ dv - \frac{1}{2} Z_+ du \right] - Z_0 d\sigma^2 - dy^i dy^j, \quad i, j = 1, 2, 3, 4, \]  

\[ H = dZ_+^{-1} \wedge du \wedge dv + \star_4 dZ_0, \]  

\[ e^{-2\phi} = e^{-2\phi_\infty} \frac{Z_-}{Z_0}, \]

where \(d\sigma^2\) is the Gibbons-Hawking metric

\[ d\sigma^2 = H^{-1}(d\eta + \chi)^2 + H dx^i dx^j, \quad x, y, z = 1, 2, 3, \quad dH = \star_3 d\chi, \]

where \(\star_3\) denotes the Hodge dual in \(\mathbb{E}^3\). This last equation implies that \(H\) is harmonic in \(\mathbb{E}^3\). An appropriate choice of harmonic function \(H\) for single, spherically-symmetric, asymptotically flat black holes is
\[ H = 1 + \frac{\tilde{q}}{r}, \tag{5.4} \]

and, then, the Gibbons-Hawking metric is that of a Kaluza-Klein monopole, which, in spherical coordinates, takes the local form

\[ d\sigma^2 = H^{-1}(d\eta + \tilde{q}\cos\theta d\phi)^2 + H \left( dr^2 + r^2 d\Omega_2^2 \right), \tag{5.5} \]

where

\[ d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2, \tag{5.6} \]

and where \( \eta \) parametrizes a circle of radius \( R \). The Kaluza-Klein charge \( \tilde{q} \) has to be quantized according to

\[ \tilde{q} = \frac{WR}{2}, \quad W = 1, 2, \ldots \tag{5.7} \]

in order to avoid Dirac-Misner strings.

For this choice of \( H \) and to describe single, spherically-symmetric, asymptotically-flat black holes, the functions \( Z_+, Z_-, Z_0, H \) must take the explicit form

\[ Z_+ = 1 + \frac{\tilde{q}_+}{r} + \frac{\alpha' \tilde{q}_+ + \tilde{r}^2 + r(\tilde{q}_0 + \tilde{q}_+ + \tilde{q}) + \tilde{q}_0 \tilde{q}_0 + \tilde{q}_0 \tilde{q}_- + \tilde{q}_0 \tilde{q}}{2\tilde{q}_0 (r - \tilde{q})(r + \tilde{q}_0)(r + \tilde{q}_-)} + O(\alpha'^2), \tag{5.8a} \]

\[ Z_- = 1 + \frac{\tilde{q}_-}{r} + O(\alpha'^2), \tag{5.8b} \]

\[ Z_0 = 1 + \frac{\tilde{q}_0}{r} - \alpha' \left[ \frac{(r + \tilde{q})(r + 2\tilde{q}_0) + \tilde{q}_0^2}{4\tilde{q}(r + \tilde{q})(r + \tilde{q}_0)^2} + \frac{(r + \tilde{q})(r + 2\tilde{q}) + \tilde{q}^2}{4\tilde{q}(r + \tilde{q})^3} \right] + O(\alpha'^2). \tag{5.8c} \]

As we discussed at the beginning of this section, we can compactify trivially this solution in the \( T^4 \) parametrized by the coordinates \( y^1, \ldots, y^4 \). Relabeling \( u = k_\infty z, \eta = \ell_\infty w, \) with \( \ell_\infty \equiv R/\ell_s \) and \( v = t \), the 6-dimensional solution can be conveniently compactified.

---

\( ^{17} \)We normalize the periods of the compact coordinates that parametrize the internal circles, \( z \) and \( w \), to \( 2\pi\ell_s \). The information about the size of these circles is carried by the internal metric and the corresponding 4-dimensional moduli, which are dimensionless. Thus \( G_{zz \infty} = k_\infty^2 = (Rz/\ell_s)^2 \) and \( G_{ww \infty} = \ell_\infty^2 = (R/\ell_s)^2 \).
written in the following form:

\[
ds^2 = \frac{1}{Z_+ Z_-} dt^2 - Z_0 \mathcal{H} \left( dr^2 + r^2 d\Omega_2^2 \right) - k_\infty^2 \frac{Z_+}{Z_-} \left( dz - \frac{1}{k_\infty} dt \right)^2 - \ell^2 \frac{Z_0}{\mathcal{H}} \left( dw + \tilde{q} / \ell_\infty \cos \theta d\phi \right)^2 ,
\]

\(\hat{\mathcal{H}} = d \left( - k_\infty Z_0 dt \right) \wedge dz + \ell_\infty^2 Z_0' d \cos \theta \wedge d\phi \wedge dw ,\)

\(e^{-2(\phi - \phi_\infty)} = \frac{Z_-}{Z_0},\)

where a prime indicates a derivative with respect to the radial coordinate \(r\). In this form we can immediately identify the KK scalars and vector fields

\[
G = \begin{pmatrix} k_\infty^2 Z_+ / Z_- & 0 \\ 0 & \ell^2 \mathcal{H} \end{pmatrix} , \quad \left( A^M_{\mu} dx^\mu \right) = \begin{pmatrix} -dt / (k_\infty Z_+) \\ \tilde{q} / \ell_\infty \cos \theta d\phi \end{pmatrix} ,
\]

and the 4-dimensional string-frame metric

\[
ds^2 = \frac{1}{Z_+ Z_-} dt^2 - Z_0 \mathcal{H} \left( dr^2 + r^2 d\Omega_2^2 \right).
\]

The 4-dimensional dilaton field is

\(e^{-2(\phi - \phi_\infty)} = \sqrt{\frac{Z_+ Z_-}{Z_0 \mathcal{H}}},\)

where

\(e^{-2\phi_\infty} = e^{-2\phi_\infty} k_\infty \ell_\infty,\)

and the (modified) Einstein-frame metric is given by

\[
ds^2_E = e^{2(\phi - \phi_\infty)} ds^2 = e^{-2U} dt^2 - e^{2U} d\mathbf{x}^2 ,
\]

\(e^{2U} = (Z_+ Z_- Z_0 \mathcal{H})^{1/2} .\)

Let us now consider the Kalb-Ramond field strength and its decomposition. First, we choose the following Vielbein basis
\[ e^0 = \frac{1}{\sqrt{Z_+ Z_-}} dt, \quad e^1 = \sqrt{Z_0 H} dr, \]
\[ e^2 = \sqrt{Z_0 H} r d\theta, \quad e^3 = \sqrt{Z_0 H} r \sin \theta d\phi, \]
\[ e^4 = k\infty \sqrt{Z_+ / Z_-} \left( dz - \frac{1}{k\infty \sqrt{Z_+}} dt \right), \quad e^5 = \ell\infty \sqrt{Z_0 / H} \left( dw + \bar{q} / \ell\infty \cos \theta d\phi \right), \]
in terms of which, the non-vanishing components of \( \hat{H} \) are
\[ \hat{H}_{104} = \frac{1}{\sqrt{Z_0 H}} (\log Z_-)' , \quad \hat{H}_{235} = \frac{1}{\sqrt{Z_0 H}} (\log Z_0)' . \]

This implies that the 4-dimensional Kalb-Ramond field strength vanishes identically and there are two non-vanishing 4-dimensional winding vector field strengths plus, perhaps, scalars. Computing the 4-dimensional Kalb-Ramond 2-form \( B^{(1)}_{\mu\nu} \), the winding vectors \( B^{(1)}_{m\mu} \) and the scalars \( B^{(1)}_{mn} \) is very complicated because of the \( \alpha' \) corrections in their definitions. Fortunately for us, only their zeroth-order values contribute to the entropy formula. At this order
\[ \hat{H}^{(0)} = d \left( -\frac{k\infty}{Z_-} dt \right) \land dz + (-\ell\infty \bar{q}_0 \cos \theta d\phi) \land dw , \]
from which we read the only non-vanishing fields descending from the Kalb-Ramond field:
\[ \left( B^{(0)}_{m\mu}d\chi^n \right) = \begin{pmatrix} -k\infty / Z_- dt \\ -\ell\infty \bar{q}_0 \cos \theta d\phi \end{pmatrix} . \]

This allows us to compute the nonvanishing components of the 2-forms \( K^{(0)}_{i} \):
\[ K^{(0)\,401}_{(\pm)} = \frac{1}{\sqrt{Z_0 H}} \{ (\log Z_+)' \pm (\log Z_-)' \} , \]
\[ K^{(0)\,523}_{(\pm)} = \frac{1}{\ell^2 (Z_0 H)^{3/2}} (\bar{q}Z_0 \mp \bar{q}_0 H) . \]

Taking into account the vanishing of the 4-dimensional Kalb-Ramond field strength, the entropy formula takes the simple form
\[
S = -\frac{1}{8G_N^{(4)}} \int d^2x \sqrt{|h|} e^{-2(\phi - \phi_\infty)} \varepsilon_{ab} \varepsilon_{cd} \left\{ S_{ab,cd}^{\phi} \right\} 
- \frac{\alpha'}{8} \left\{ -2R(0)^{abcd} + K(-) i ac K(-) i bd + K(+) j ab K(+) j cd \right\} 
= \frac{1}{4G_N^{(4)}} \int d^2x \sqrt{|h|} e^{-2(\phi - \phi_\infty)} \left\{ 1 + \frac{\alpha'}{4} \left[ -2R(0)^{abcd} + (K(-) 401)^2 + (K(+) 401)^2 \right] \right\} 
= \frac{1}{4G_N^{(4)}} \left\{ A_H + 2\pi \zeta \lim_{r \to 0} e^{2U_r} \left[ -\sqrt{Z_+ Z_- Z_0 H} \left[ \frac{1}{\sqrt{Z_0 H}} \left( \frac{1}{\sqrt{Z_+ Z_-}} \right) \right] \right] + \frac{1}{Z_0 H} \left[ \left( \frac{Z'_+}{Z_+} \right)^2 + \left( \frac{Z'_-}{Z_-} \right)^2 \right] \right\} 
= \frac{A_H}{4G_N^{(4)}} \left\{ 1 + \frac{\alpha'}{2\tilde{q}_0 \tilde{q}} \right\} \tag{5.20}
\]
where we have assumed that all the charges \( \tilde{q}_i \) are different from zero\(^{18} \) and where

\[
A_H = 4\pi \sqrt{\tilde{q}_+ \tilde{q} - \tilde{q}_0 \tilde{q}} , \tag{5.21}
\]

is the area of the horizon, which does not receive any corrections to first order in \( \alpha' \).

The charges \( \tilde{q}_i \) are related to the numbers of solitonic 5-branes \( N \), units of momentum \( n \), winding number \( w \) and KK charge \( W \) by \([38]^{19} \)

\[
\tilde{q}_0 = \frac{\alpha'}{2R} N , \quad \tilde{q}_+ = \frac{\alpha'}{2RR^2} n , \quad \tilde{q}_- = \frac{\alpha'}{2R^2} w , \quad q = \frac{WR}{2} , \tag{5.22}
\]

and, using the value of the 10-dimensional Newton constant in Eq. (1.10) and its relation with the 4-dimensional one Eq. (2.14b) with \( V_6 = (2\pi)^6 \alpha'^2 RR^2 \) we get

\[
S = 2\pi \sqrt{NnWw} \left\{ 1 + \frac{2}{NW} \right\} . \tag{5.23}
\]

\(^{18}\)We have assumed the positivity of all those charges because, as a general rule, if these parameters have negative values, there are naked singularities. However, the \( \alpha' \) corrections can sometimes eliminate these singularities, as shown in Ref. [47].

\(^{19}\)Here we are using the notation of Ref. [46].
Finally, using the relation between the numbers of objects and the asymptotic charges found in Ref. [46]

\begin{align}
Q_+ &= n \left( 1 + \frac{2}{NW} \right), \\
Q_- &= w, \\
Q_0 &= N \left( 1 - \frac{2}{NW} \right), \\
Q &= W,
\end{align}

we find that

\[ S = 2\pi \sqrt{Q_+Q_- (Q_0Q + 4)}. \]

This is the entropy obtained by microstate counting in Ref. [48].

6 Conclusions

In this paper we have shown that the complete Heterotic Superstring effective action compactified on $T^n$ is $O(n,n)$ invariant to first-order in $\alpha'$.\textsuperscript{20} The $(10 - n)$-dimensional action is not really suitable for the derivation of the $(10 - n)$-dimensional equations of motion because it is not easy to apply the Bergshoeff-de Roo lemma [23] to it. Nevertheless, one can always work in 10 dimensions, where it is easy to apply it, reducing afterwards the solutions to $(10 - n)$ using the formulae obtained in this paper.\textsuperscript{21} Furthermore, the action can be used to obtain a Wald entropy formula which we have used to compute the entropy of the most basic stringy 4-dimensional, 4-charge black holes. In order to apply it to more general black holes, with more vector and scalar fields or with angular momentum one first needs to find their $\alpha'$ corrections, which can be a non-trivial task.

Our results leave, however, some important questions unanswered:

- If the vector fields are Abelian, does one recover $O(n,n + n\gamma)$ invariance in the presence of all the $\alpha'$ corrections (that is: adding the torsionful spin connection terms)? At first sight the answer should be yes: the situation is not qualitatively

\textsuperscript{20}A similar result using the effective action of Ref. [15] and without Yang-Mills fields has been recently published in Ref. [22]. Furthermore, it has been argued in Ref. [49] that $O(n,n)$ is present at all orders in $\alpha'$.

\textsuperscript{21}one can replace 10 by $D$ in this discussion, as explained at the beginning of Section 5.
different from having a number of gauge fields $n_A$ of which are Abelian and $n_N$ of which are non-Abelian, where one should have $O(n, n + n_A)$ invariance. Nevertheless, it would be convenient to rewrite this case in a manifestly $O(n, n + n_V)$-invariant form.

- In the 4-dimensional case, is S duality preserved (as expected) [50]? How is it realized? Work in these directions is already in progress [51].

- Why is the Iyer-Wald prescription so successful in this setting? After all, the entropy formula we have obtained is manifestly not invariant under local Lorentz transformations. A more rigorous derivation, performed in Ref. [52] shows the presence of an additional term not captured by the Iyer-Wald prescription that may restore the (expected) local Lorentz invariance of the Wald entropy formula. This term may not contribute in many relevant cases. However, the lack of information about the explicit form of this term does not allow us to give a final answer and we feel that determining explicitly this term is completely necessary to guarantee the validity of the macroscopic calculations of extremal black-hole entropies which are later compared with the microscopic ones. Work in this direction is also well under way [53].

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A  $O(n, n)/(O(n) \times O(n))$ coset space

Since this coset space occurs repeatedly in the main body of this paper, we describe it here in some detail. First, we define the $n \times n$ matrices $E \equiv (e^i_m)$, $G \equiv (G_{mn})$ and $B \equiv (B_{mn})$. Evidently $E^{-1} = (e^m_i)$ and $E^T E = G$. With them we construct the $2n \times 2n$
Vielbein $U$ and its inverse $U^{-1}$

$$U = -\frac{1}{\sqrt{2}} \begin{pmatrix} E^{-1} & E^{-1} \\ E^T + BE^{-1} & -E^T + BE^{-1} \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} e^m_i & -e^m_i \\ -e_m^i + B_{mn}e^n_i & -e_m^i - B_{mn}e^n_i \end{pmatrix},$$

(A.1a)

$$U^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -E + E^{-1T}B & -E^{-1T} \\ -E - E^{-1T}B & E^{-1T} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^i_m - e^m_i B_{nm} & e^i_m \\ e_m^i - e^m_i B_{nm} & e^m_i \end{pmatrix}$$

(A.1b)

where we have used the metric $-\delta_{ij}$ to raise and lower SO$(n)$ indices $i,j$, consistently with our mostly minus convention for the 10-dimensional spacetime metric.

In terms of the non-diagonal $O(n,n)$ metric $\Omega = \Omega^{-1} = \begin{pmatrix} 0 & I_{n\times n} \\ I_{n\times n} & 0 \end{pmatrix}$ we can define $O(n,n)$ transformations $\Lambda$ as those satisfying

$$\Lambda^T \Omega \Lambda = \Omega.$$  \quad (A.2)

Under a $O(n,n)$ transformation $\Lambda$ acting from the left, the Vielbein $U$ transforms as

$$\Lambda U = U' R, \quad R = \begin{pmatrix} R(+) & 0 \\ 0 & R(-) \end{pmatrix} \in O(n)(+) \times O(n)(-) \quad (A.3)$$

Thus, using $U$, we can construct a symmetric matrix $M$

$$M \equiv UU^T = \begin{pmatrix} G^{-1} & -G^{-1}B \\ -B^TG^{-1} & G + B^TG^{-1}B \end{pmatrix}$$

(A.4)

which transforms under $O(n,n)$ as

$$M' = \Lambda M \Lambda^T.$$  \quad (A.5)

The inverse of $M$ is given by

$$M^{-1} = U^{-1T}U^{-1} = \begin{pmatrix} G + B^TG^{-1}B & -B^TG^{-1} \\ -G^{-1}B^T & G^{-1} \end{pmatrix},$$

(A.6)

and it is not difficult to check that

$$M^{-1} = \Omega M \Omega,$$  \quad (A.7)

which implies that $M$ is a $O(n,n)$ matrix itself.

The left-invariant Maurer-Cartan 1-form is
\[-\mathcal{U}^{-1}d\mathcal{U} = \begin{pmatrix} A_{(+)i}^j & P_{(+)}^{ij} \\ P_{(-)}^{ij} & A_{(-)}^i_j \end{pmatrix}, \tag{A.8}\]

where the O(n)$_{(+)}$ connection $A_{(+)i}^j$, the O(n)$_{(-)}$ connection $A_{(-)}^i_j$ and the Vielbein $P_{(+)}^{ij} = P_{(-)}^{ij}$ ($n^2$ degrees of freedom) are given by

\[
A_{(\pm)}^{ij} \equiv A^{ij} \mp \frac{1}{2}e^{im}e^{in}dB_{mn}, \tag{A.9a}\]

\[
P_{(\pm)}^{ij} \equiv \frac{1}{2}e^{im}e^{in}d(G_{mn} \mp B_{mn}), \tag{A.9b}\]

where, in its turn

\[
A^{ij} \equiv -e^{i|m}de_m^n d^n. \tag{A.10}\]

Observe that the Vielbein transforms under both O(n) groups:

\[
P'_{(\pm)} = R_{\pm}P_{(\pm)}R_{\mp}. \tag{A.11}\]

The Maurer-Cartan equations, obtained by taking the exterior derivative of the left-invariant Maurer-Cartan 1-forms lead to the following identities:

\[
F_{(\pm)}^{ij} = P_{(\pm)}^{ik} \wedge P_{(-)}^{k}^j, \tag{A.12a}\]

\[
F_{(-)}^{ij} = P_{(-)}^{ik} \wedge P_{(\pm)}^{k}^j, \tag{A.12b}\]

\[
\mathcal{D}_{(\mp \pm)}P_{(\pm)}^{ij} = 0, \tag{A.12c}\]

where

\[
F_{(\pm)}^{ij} = dA_{(\pm)}^{ij} + A_{(\pm)}^{ik} \wedge A_{(\pm)}^{kj}, \tag{A.13}\]

are the curvatures of the two O(n) connections and where the covariant derivative $\mathcal{D}_{(\mp \pm)}$ uses the (\mp) connection on the first O(n) index and the (\pm) connection on the second of $P_{(\pm)}^{ij}$, that is:

\[
\mathcal{D}_{(\mp \pm)}P_{(\pm)}^{ij} = dP_{(\pm)}^{ij} + A_{(\pm)}^{ik} \wedge P_{(\pm)}^{kj} + P_{(\pm)}^{ik} \wedge A_{(\pm)}^{kj}. \tag{A.14}\]

From the Maurer-Cartan Eqs. (A.12a) and (A.12b) we find

\footnote{We have made use of Eq. (2.6).}
\[ F_{(\pm)}^{ij} \cdot F_{(\pm)}^{ij} = 2 P_{(\pm)}^{a} P_{(\pm)}^{b} P_{(\pm)}^{0} + 2 P_{(\pm)}^{0} P_{(\pm)}^{a} P_{(\pm)}^{0} - 2 P_{(\pm)}^{0} P_{(\pm)}^{a} P_{(\pm)}^{b} P_{(\pm)}^{b} \cdot k, \quad (A.15) \]

and the relation \( P_{(\pm)}^{ij} = P_{(\mp)}^{ji} \) implies that

\[ F_{(+)}^{ij} \cdot F_{(+)}^{ij} = F_{(-)}^{ij} \cdot F_{(-)}^{ij}. \quad (A.16) \]

The scalar kinetic term is given by

\[
P_{(+)}^{ij} P_{(-)}^{ij} = \frac{1}{4} G^{mn} G^{pq} \partial_{\mu} (G_{mp} + B_{mp}) \partial_{\mu} (G_{nq} - B_{nq})
\]

\[ = -\frac{1}{4} \partial_{\mu} G_{mn} \partial_{\mu} G_{mn} - \frac{1}{4} G^{mn} G^{pq} \partial_{\mu} B_{mp} \partial_{\mu} B_{nq}, \quad (A.17) \]

or by the equivalent expression

\[ -\frac{1}{2} \text{Tr} \left( \partial_{\mu} M \partial^{\mu} M^{-1} \right). \quad (A.18) \]

Using the \( O(n, n) \) vector

\[ \mathcal{F} \equiv \left( \begin{array}{c} F_{m} \\ G_{m} \end{array} \right), \quad (A.19) \]

one can construct the following combinations that occur naturally in some of the expressions in the main paper:

\[
\mathcal{U}^{-1} \mathcal{F} = -\frac{1}{\sqrt{2}} \left( \begin{array}{c} e^{i}_{m} \left( F_{m} + G^{mn} G^{0}_{n} - G^{mn} B_{np} F_{p} \right) \\ e^{i}_{m} \left( F_{m} - G^{mn} G^{0}_{n} + G^{mn} B_{np} F_{p} \right) \end{array} \right) = -\frac{1}{\sqrt{2}} \left( \begin{array}{c} K_{(+)}^{i} \\ K_{(-)}^{i} \end{array} \right). \quad (A.20) \]

Observe that \( K_{(\pm)}^{i} \) only transform under \( O(n)_{\pm} \) rotations, respectively. We can construct \( O(n, n) \) invariants by building \( O(n)_{\pm} \) invariants. For instance:

\[ K_{(\pm)}^{i} K_{(\pm)}^{i} = \mathcal{F}^{T} M^{-1} \mathcal{F} \pm 2 \mathcal{F}^{T} \Omega \mathcal{F}, \quad (A.21) \]

which is clearly consistent with

\[ (\mathcal{U}^{-1} \mathcal{F})^{T} \mathcal{U}^{-1} \mathcal{F} = \mathcal{F}^{T} M^{-1} \mathcal{F}. \quad (A.22) \]

Other examples of \( O(n, n) \) invariants built in the same way that arise in the main text are

\[ \mathcal{D}_{(-)}^{\mu} K_{(-)}^{i} v_{\rho \sigma} K_{(-)}^{i} L_{\lambda}. \quad \text{and} \quad P_{(-)}^{a} J_{(-)}^{i} \mathcal{D}_{(-)}^{\mu} K_{(-)}^{i} L_{\lambda}. \quad (A.23) \]
On the other hand, if we have a $O(n)_\pm$ vector $\xi^i$, we can construct a $O(n,n)$ vector using the Vielbein

$$U_i^\pm \xi^i = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^m_i \xi^i \\ \pm e^m_{i\sigma} - B_{mn} e^n_i \xi^i \end{pmatrix}, \quad (A.24)$$

and, then, the $O(n,n)$ invariants

$$\left(U_i^\pm \xi^i\right)^T \Omega U_i^\pm \xi^i = \pm \xi^i \xi_i, \quad (A.25a)$$

$$\mathcal{D} \left(U_i^\pm \xi^i\right)^T \Omega \mathcal{D} U_j^\pm \xi^j = -\mathcal{D} \left(e^m_{i\sigma} \xi^i\right) \mathcal{D} \left(\pm e^m_{i\sigma} - B_{mn} e^n_i \xi^i\right), \quad (A.25b)$$

the first of which is trivial. The second occurs in the main text with $\xi_i = \varphi^A_i$ transforming under $O(n)_\pm$. With this assignment,

$$\varphi^A_i K_i^{(\pm)} , \quad (A.26)$$

is also $O(n,n)$-invariant.

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