On tomography velocity uncertainty in relation with structural imaging

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ABSTRACT
Evaluating structural uncertainties associated with seismic imaging and target horizons can be of critical importance for decision-making related to oil and gas exploration and production. An important breakthrough for industrial applications has been made with the development of industrial approaches to velocity model building. We propose an extension of these approaches, sampling an equi-probable contour of the tomography posterior probability density function (pdf) rather than the full pdf, and using non-linear slope tomography (rather than standard tomographic migration velocity analysis as in previous publications). Our approach allows to assess the quality of uncertainty-related assumptions (linearity and Gaussian hypothesis within the Bayesian theory) and estimate volumetric migration positioning uncertainties (a generalization of horizon uncertainties), in addition to the advantages in terms of efficiency. We derive the theoretical concepts underlying this approach and unify our derivations with those of previous publications. As the method works in the full model space rather than in a preconditioned model space, we split the analysis into the resolved and unresolved tomography spaces. We argue that the resolved space uncertainties are to be used in further steps leading to decision-making and can be related to the output of methods that work in a preconditioned model space. The unresolved space uncertainties represent a qualitative byproduct specific to our method, strongly highlighting the most uncertain gross areas, thus useful for QCs. These concepts are demonstrated on a synthetic data. Complementarily, the industrial viability of the method is illustrated on two different 3D field datasets. The first one shows structural uncertainties on a merge of different seismic surveys in the North Sea. The second one shows the impact of structural uncertainties on gross-rock volume computation.

INTRODUCTION
Decision-making and risk mitigation are critical for oil and gas exploration and production (E&P). Relying only on maximum-likelihood (or single-valued) subsurface models can lead to drastic misinterpretations of the risk. Assessing uncertainties related to maximum-likelihood models is therefore necessary (Simpson et al., 2000), but is a challenging task. Indeed, such single-valued models are built by long and complex processes where various types of information are combined sequentially. Seismic migration is a central step within those processes, providing images of the general structure of the subsurface through a reflectivity model. The positioning uncertainties associated with the structures imaged in the reflectivity, or structural uncertainties, have been studied for decades.
The key step affecting migration structural uncertainties is velocity model building (VMB). According to Fournier et al. (2013), VMB is related to one of the biggest ambiguities impacting E&P. While we have seen over the last decade the development of full-wave VMB approaches (Virieux and Operto, 2009), ray-based reflection tomographic approaches remain an essential workhorse method (Woodward et al., 2008; Guillaume et al., 2013b; Lambaré et al., 2014). This is due to their inherent characteristics, i.e. efficient numerical implementations, “compressed” kinematic data (picks) and ability to digest prior information. These advantages are particularly appealing from the perspective of a structural uncertainties analysis in an industrial context. An important contribution in terms of theory, implementation and application has been delivered by the work of Osypov et al. (2008a,b, 2010, 2011, 2013). While there had been earlier investigations in the context of reflection tomography (Duffet and Sinoquet, 2006), to our knowledge Osypov et al. (2008b) were the first to implement a tool used in the industry to estimate structural uncertainties associated with ray-based tomography. Their approach is based on the tomography toolbox described by Woodward et al. (2008). The uncertainty analysis is performed around the maximum-likelihood tomography model within a Bayesian approach, assuming a linearized modeling and a Gaussian probability density function (pdf). A partial eigen-decomposition of the tomography data operator is performed in a “model preconditioned basis” (Osypov et al., 2008b). This allows the generation of perturbed tomography models related to a confidence level. Then map (or zero-offset kinematic) migrations of a target horizon within those models give a set of perturbed horizons. These are analyzed statistically to derive estimations of horizon error bars related to a confidence level. Osypov et al. (2010, 2011, 2013) give details on practical aspects (such as calibrating the regularizations) and discuss the impact of anisotropy emphasizing that it represents an important uncertainty component in the seismic images. They present applications relating to the assessment of oil reserves or well placement, and demonstrate the affordability and effectiveness of the corresponding horizon uncertainty analysis for industrial applications.

In continuation of these efforts, recent work on structural uncertainty analysis has been done by Messud et al. (2017a,b, 2018). The current paper concentrates on the theory underlying this work and the differences compared to previous works. The considerations are general and valid for any subsurface model including any form of anisotropy.

The first originality of our approach is, within the Bayesian formalism, to sample randomly a pdf’s equi-probable contour (related to a clear confidence level) rather than the full pdf (Osypov et al., 2013; Duffet and Sinoquet, 2002, 2006), providing efficient error bar estimates (Messud et al., 2017b,a, 2018). We define corresponding error bars for a given confidence level, accounting for the non-diagonal part of the covariance matrices (avoiding some underestimation).

Secondly, while the work of Osypov et al. (2008a, 2010, 2011, 2013) was based on the classical (linear) tomographic approach described by Woodward et al. (2008), our work is based on the non-linear slope tomography of Guillaume et al. (2008) and Lambaré et al. (2014). In the context of VMB, an advantage of non-linear tomography is the ability to compute all non-linear updates of the tomography model with only one picking step (Adler et al., 2008; Lambaré et al., 2014), whereas Woodward et al. (2008) requires a new picking step (thus a new migration) for each iteration (or linear update). Also, non-linear slope
Tomography has the advantage of belonging to the family of slope tomography, where the model is recovered from picks of locally coherent reflected events in the pre-stack unigrated domain (Lambard 2008). In brief, the approach can extract in a non-linear way the kinematic information contained in a dense set of local picks. Numerous versions have been proposed covering a large set of configurations, i.e. multi-layer tomography (Guillaume et al. 2013a), dip-constrained tomography (Guillaume et al. 2013a), high-definition tomography (Guillaume et al. 2011) and joint direct and reflected wave tomography (Alemmand et al. 2017). In the context of structural uncertainty analysis, one advantage of non-linear tomography together with pdf’s equi-probable contour sampling is to provide an efficient way to QC the assumptions made within the Bayesian formalism (linearity and Gaussian pdf hypothesis) (Messud et al. 2017a; Reinier et al. 2017). Another advantage is to make it possible to derive volumetric migration positioning error bars, a volumetric generalization of the horizon positioning error bars that provides migration uncertainties between horizons.

Thirdly, as we work in the full model space rather than in a preconditioned model space, we split the analysis into the resolved and unresolved tomography spaces. We argue that the resolved space uncertainties are to be used in further steps leading to decision-making and can be related to the output of methods that work in a preconditioned model space. The unresolved space uncertainties represent a qualitative byproduct specific to our method, giving an additional information reflects the priors and the illumination, strongly highlighting the most uncertain areas. It thus can be used for QC’s and may offer some possibility of exploring small-scale non-structural variations, that we cannot consider as fully improbable.

Note that these works on structural uncertainties can easily apply to full-waveform inversion (FWI)-derived models. Indeed, these models usually go through a last tomography pass (FWI “post-processing”) in order to obtain flatter common image gathers. The corresponding tomography uncertainty analysis can then naturally be performed to produce an estimate for FWI model kinematic-related uncertainties. This workflow is practical as long as rays can adequately describe the kinematics of FWI-derived models.

The main aim of this paper is to carefully details the theoretical developments associated to our method, providing a unifying framework to compare to other approaches (Osypov et al. 2008a, 2010, 2011, 2013; Duffet and Sinoquet 2006). Then, the concepts are demonstrated on a synthetic data. Finally, the industrial viability of the method is illustrated on two different 3D field datasets. The first one shows structural as well as volumetric migration uncertainties on a merge of different seismic surveys in the North Sea. The second one shows the impact of structural uncertainties on gross-rock volume (GRV) computation.

**TOMOGRAPHY AND BAYESIAN FORMALISM**

**Tomography inverse problem**

Non-linear slope tomography is a practical and efficient tool for velocity model building (Guillaume et al. 2013b). Its input (or observed) data $d_{obs}$ consists of a set of “invariants picks”, i.e. kinematic quantities that belong to the original (multi-dimensional) seismic data domain. Invariants picks are typically described by source and receiver positions, two-way travel-times and two time-slopes describing locally coherent reflection events in the original
seismic data domain (a time-slope in common offset gathers and a time-slope in common mid-point (CMP) gather), see Figure [1](left). Invariants picks can be “kinematically migrated” by a finite-offset ray-based method in any model \( m \) meeting ray theory assumptions \cite{Duffet and Sinoquet, 2006, Guillaume et al., 2008}, to be converted into locally coherent reflector events in the migrated domain, see Figure [1](right). Computed events are among others related to two depth-slopes (a depth-slope in common offset migrated gathers and a depth-slope in common image gather (CIG), the latter being the local residual move-out (RMO)). Conversely, locally coherent reflector events in the migrated domain can be picked, giving “migration picks”, and can be “kinematically demigrated” in \( m \) by a finite-offset ray-based method, to be converted into invariants picks \footnote{The invariant picks can be either directly obtained by picking in the original seismic data domain, or indirectly by a kinematic demigration of picks in the migrated domain. The latter is often preferable as the migrated domain has better signal to noise ratio and events separation.}.

Figure 1: Non-linear slope tomography concepts. Kinematic invariants represent a set of picks that belong to the original data domain. The invariants can be kinematically migrated in any sufficiently smooth model to deduce corresponding picks in the migrated domain. Conversely, picks in the migrated domain can be de-migrated in any sufficiently smooth model to deduce corresponding kinematic invariants.

The tomography model \( m \) consists of a set of parameters describing smooth velocity and anisotropy layers; following considerations are thus general and valid for any choice of contribution to \( m \) (velocity and/or any form of anisotropy). \( m \) is described by cardinal cubic Bspline (ccbs) basis \cite{Operto et al., 2003}. Ccbs have many advantages in terms of regularity for the ray tracing (continuous second derivative fields) and computational cost \cite{Boor, 1978, Virieux and Farra, 1991}; we will come back to them further. We denote by \( N_M \) the number of nodes that describe \( m \) (500,000 to 50 million in large scale applications).

Non-linear slope tomography finds the model \( m \) that reproduces the best the invariant picks, or equivalently that minimizes the local RMOs \cite{Chauris et al., 2002}. The concept of invariant picks allows the implementation of an iterative non-linear inversion scheme where each linearized update consist of (1) a non-linear modeling \( d(m) \) (kinematic migration) and a computation of tomography operator derivatives, and (2) a resolution of an inverse problem to deduce the update. After each linearized update, the data are modeled again non-linearly (by kinematic migration) before a new inversion starts again. The tomography model \( m \) can be refined by a decomposition into smooth layers (described by ccbs) separated by discontinuities called “horizons” (to represent major subsurface contrasts). Just like the migrated picks, the horizons describing the layer boundaries can be kinematically
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demigrated using zero-offset ray tracing to build zero-offset horizon invariant picks. For each tomography iteration, the horizons are repositioned in the updated model by “map migration” (zero-offset ray tracing) of the horizon invariant picks. The essence of non-linear slope tomography is thus the conversion between invariant picks and migrated picks, that allows non-linear model updates without having to repick at each iteration. Priors can easily be included and fine-tuned, possibly progressively reducing their level to let the data speak more as the number of tomography iterations increases.

This being said, we now formally describe the details of the inversion scheme. It outputs to the maximum-likelihood tomography model \( \mathbf{m}^* \), i.e. the model that fits the best the data under some prior constraints (Tarantola, 1986, 2005), defined by

\[
\mathbf{m}^* = \arg \min_{\mathbf{m}} \frac{1}{2} \left| \left| \mathbf{C}_D^{-1/2} (\mathbf{d}_{\text{obs}} - \mathbf{d} (\mathbf{m})) \right| \right|^2 + \frac{1}{2} \left| \left| \mathbf{C}_M^{-1/2} (\mathbf{m} - \mathbf{m}_{\text{prior}}) \right| \right|^2.
\]  

(1)

\( \mathbf{C}_D \) denotes the covariance matrix in the data space. It accounts for data (invariant picks) and modeling (kinematic) uncertainties. \( \mathbf{C}_M \) is the “prior” covariance matrix in the model space, associated with a prior model \( \mathbf{m}_{\text{prior}} \). It accounts for uncertainties on the prior model and helps regularization. Non-zero non-diagonal elements of a covariance matrix \( \mathbf{C}_i \) mean that corresponding nodes are correlated. The diagonal elements of a covariance matrix \( \mathbf{C}_i \) are called variances, and the square roots of the variances are called standard deviations. In the following, bold lowercase letter quantities represent vectors and all bold uppercase letter quantities represent matrices. Some details on our prior and data covariance matrices will be given further.

Equation 1 is solved by a non-linear local optimization method, updating iteratively the model by

\[
\mathbf{m}_{k+1} = \mathbf{m}_k + \delta \mathbf{m}_k,
\]

(2)

where \( k \in [0, n] \) denotes the iteration number and \( n \) the last iteration number. At the last tomography iteration, the obtained \( \mathbf{m}_n \) is considered to be the maximum-likelihood solution \( \mathbf{m}^* \). In practice, the updates \( \Delta \mathbf{m}_k \) in equation 2 are computed through a linearization of equation 1 at each iteration, solving

\[
\min_{\delta \mathbf{m}_k} \left| \left| \mathbf{A}_k \delta \mathbf{m}_k - \mathbf{b}_k \right| \right|^2,
\]

(3)

where \( \mathbf{A}_k \) is the Jacobian matrix and \( \mathbf{b}_k \) the “error vector”, containing information on the data and the prior (these terms are defined in the next paragraphs). \( \mathbf{A}_k \) is defined by

\[
\mathbf{A}_k = \begin{bmatrix} \mathbf{C}_D^{-1/2} \mathbf{G}_k \\ \mathbf{C}_R^{-1/2} \end{bmatrix},
\]

(4)

containing the tomography modeling Jacobian matrix at iteration \( k \)

\[
\mathbf{G}_k = \frac{\partial \mathbf{d}(\mathbf{m})}{\partial \mathbf{m}} \Big|_{\mathbf{m}=\mathbf{m}_k},
\]

(5)

This extension is called multi-layer tomography. Since the process preserves the travel-times, all the layers can be updated simultaneously, avoiding the downward propagation of errors of a conventional top-down approach, and a multi-scale technique avoids getting trapped into a local minimum (Guillaume et al., 2013b).
and $C_R^{1/2}$ gathers various contributions to the prior covariance in model space or constraints

$$C_R^{-1/2} = \begin{bmatrix}
\text{Damp} \\
\text{Other}_1 \\
\vdots \\
\text{Other}_{L-1}
\end{bmatrix}$$

$$C_M^{-1} = C_R^{-1/2} C_R^{-1/2} = \text{Damp}^\dagger \text{Damp} + \sum_{i=1}^{L-1} \text{Other}_i^\dagger \text{Other}_i,$$

where $\dagger$ denotes transpose. Each of the $L$ prior contributions ($\text{Damp}$ and $\text{Other}_i$) is represented by a square matrix of size $N_M \times N_M$, $C_R^{-1/2}$ being a matrix of size $(L \times N_M) \times N_M$. $\text{Damp}$ represents a “damping” diagonal matrix, scaled for each model contribution or subsurface parameter (velocity, various anisotropies, etc.). $\text{Other}_i$ represents other possible constraints, for instance:

- A 3D Laplacian to give smoothness to the model layers.
- A structural constraint to encourage the model to follow given geological structural dips (parameterized by a curvilinear 2D Laplacian along these dips directions).
- A coupling between pairs of model parameters contributions (velocity $V_p$, and different anisotropies like $\delta$ and $\epsilon$) to prevent outlier values from rock properties point of view. A coupling between $V_p$ and $\delta$ can be activated to preserve short spread focusing and a coupling between $\epsilon$ and $\delta$ can be activated to preserve $\eta$.
- Possibly a well-ties constraint, etc.

Some user-defined parameters enter into these constraints computation, that are tuned to produce the best tomography result. For instance, our damping is related to a user-defined “knob” tuned so as not to affect the relevant information in the tomography operator, i.e. to be representative of the tomography noise level.

We emphasize that, in our implementation, all prior contributions to $C_R^{1/2}$ are $N_M \times N_M$ square matrices. We do not use non-square matrices $P$ of size $p \times N_M$ with $p \ll N_M$ to parameterize constraints through $P^\dagger P$ ($\text{rank}(P^\dagger P) \leq p \ll N_M$). In other words, we do not use low-rank prior constraints that reduce the dimensionality of the problem, such as steering filters [Clapp et al. 1998]. As already mentioned, our models are described by ccbs grids that have many advantages in terms of regularity for the ray tracing and of computational cost [Boor 1978, Virieux and Farra 1991]. Dense ccbs grids can be used to represent detailed models up to the maximum resolution that can be obtained from the data. Also, even for large models parameterized by dense ccbs grids, efficient implementations of the tomography inversion can be achieved with current parallel computing capabilities, so that there is no real need for reducing the dimensionality of the problem. Performing the inversion on a dense ccbs grid can have the advantage of not drastically limiting a priori the space of investigated models. With this strategy, the space of investigated models is defined by the “level of prior” introduced by the users, selecting the most relevant combination of data and prior information through the user-defined parameters discussed above.
Denoting by $N_D$ the number of data, i.e. invariant picks, the data space covariance matrix $C_D^{-1/2}$ has size $N_D \times N_D$. In practice, is often taken diagonal, consisting of rescaled non-stationary quality weights on the picks (related among others to the semblances computed by the picking tool). The error vector has size $(N_D + L \times N_M)$ and is defined by

$$b_k = \begin{bmatrix} C_D^{-1/2}(d_{obs} - d(m_k)) \\ C_R^{-1/2}(m_k - m_{prior}) \end{bmatrix}. $$

$m_{prior}$ is often taken to be the model $m_k$ of the previous iteration, which helps convergence. This implies that, at the final tomography iteration, the prior tends to be equal to the maximum-likelihood model.

$A_k$ is a matrix of size $(N_D + L \times N_M) \times N_M$. The physical dimensions of the components of $A_k$ are the inverse of the physical dimensions of the model. Thus, in the multi-kinds of parameter case (i.e. velocity with anisotropy), the coefficients of $A_k$ do not have the same physical dimensions. A preconditioning can be used:

$$A_k' \rightarrow A_kD,$$

where $D$ is a square matrix that gives the same physical dimension and similar scaling to all coefficients of matrix $A_kD$ and can be used to re-weight $A_k$ to obtain better inversion results. $D$ is symmetric, $D^\dagger = D$, and should be invertible so that it does not increase the null space of the problem. A common preconditioning is to choose $D$ diagonal and put the inverse of the $L_2$ norm of each column of $A_k$ on the diagonal. We then solve, instead of equation (7)

$$\min_{\delta m_k'} ||A_k'\delta m_k' - b_k||^2.$$

and recover $\Delta m_k$ by

$$\delta m_k = D\delta m_k'.$$

SVD, EVD, LSQR and effective null space projector.

The solution of equation (9) is usually computed at each iteration using an approximation of $\delta m_k' = A_k'^{-g}b_k$, where $A_k'^{-g} = \lim_{\alpha \rightarrow 0^+}(A_{k'}^{-\alpha}A_k' + \alpha I_{N_M})^{-1}A_k'^{-\alpha}$ is the generalized (or Moore-Penrose pseudo-) inverse, where $A_k'^{-1/2}$ is called the Hessian matrix in the preconditioned domain.

An approximation of the generalized inverse can be computed for instance performing a partial singular value decomposition (SVD) of $A_k'$ or a partial eigenvalue decomposition (EVD) of the Hessian $A_k'^{-1/2}$. Suppose that we dispose of $p \leq N_M$ singular-values $\lambda_i$ and their corresponding left and right eigenvectors $u_i$ and $v_i$ (obtained for instance by $p$ iterations of SVD). We construct the $U_p = [u_1, etc., u_p]$ and $V_p = [v_1, etc., v_p]$ matrices, and the $p \times p$ matrix $A_k'^{-1/2}$ that contains the singular values on its diagonal. The partial SVD of $A_k'$ gives the best possible rank $p$-approximation of $A_k'$

$$A_k' \approx U_pA_p^{1/2}V_p^\dagger \Rightarrow A_k'^{-g} \approx V_pT_pA_p^{-1/2}U_p^\dagger.$$

$^3U_p$ has size $(N_D + L \times N_M) \times p$ and $V_p$ has size $N_M \times p$. Note that $U_p$ and $V_p$ are not unitary. They satisfy $V_p^\dagger V_p = U_p^\dagger U_p = I_p$. But $V_pV_p^\dagger \neq I_{N_M}$ for $p < N_M$, and $U_pU_p^\dagger \neq I_{N_D + L \times N_M}$. 

Similarly, the partial EVD of the Hessian is given by

$$A_k^\dagger A_k \approx V_p \Lambda_p V_p^\dagger \Rightarrow (A_k^\dagger A_k)^{-1} \approx A_k^{-g} \Lambda_k^{-g} \approx V_p T_p \Lambda_p^{-1} V_p^\dagger,$$

where

$$T_p = [\Lambda_p + \epsilon I_p]^{-1} \Lambda_p$$

is the Tikhonov regularization operator that stabilizes the inversion result in the event of very small components of $\Lambda_p$ (Zhang and McMechan, 1995; Zhang and Thurber, 2007). Note that this regularization is equivalent to including a “damping” level $\epsilon$ in a basis where all model components have the same units. In our case, we take $\epsilon = 0$ in $T_p$ as we already introduced a damping in our prior contributions, equation 6, that satisfies approximately $D^{-1} \text{Damp}^\dagger \text{Damp}D^{-1} \approx \epsilon I_{NM}$ in the preconditioned domain.

The $N_M - p$ eigenvectors not resolved by the iterative algorithms define the tomography “effective null space”, a cause of multiple equivalent effective solutions (depending among others on the initial model $m_0$) and thus of uncertainty. The effective null space projector is (Munoz and Rath, 2006)

$$\Pi_{null_p} = I_{NM} - V_p V_p^\dagger,$$

and the tomography “resolved” (or partial EVD spanned) space projector is (Munoz and Rath, 2006)

$$R_p = V_p V_p^\dagger.$$

In our implementation, the iterative Least-Squares Quadratic Relaxation (LSQR) algorithm is used as a solver for the tomography problem in equation 9, to approximate the effect of $A_k^{-g}$ (Paige and Saunders, 1982; Choi, 2006). It shares a close similarity with performing an iterative partial SVD or EVD, but is more efficient as the matrix $A_k'$ is sparse. It computes “Lanczos vectors”, related to the Lanczos tridiagonal matrix, that are not to be confused with eigenvectors (Zhang and Thurber, 2007). When we wish to recover eigenvectors, necessary for the following uncertainty analysis, we perform a diagonalization of the Lanczos matrix that gives Ritz vectors, a sufficient approximation of eigenvectors (Zhang and Thurber, 2007).

### Bayesian formalism for tomography model uncertainties

At the last tomography iteration ($k = n$), the obtained $m_n$ is considered to be the maximum-likelihood solution. The result is uncertain because the tomography input data, modeling, and constraints contain uncertainties. As a basis for uncertainty considerations, we will use the Bayesian formalism. It gives a clear definition of uncertainties in terms of physics plus a confidence level, or probability $P$ that the true model belongs to a region of the model space (Cowan, 1998). This is important for reservoir risk analysis.

We consider, within Bayesian theory, the “Gaussian posterior pdf in the model space” corresponding to equation 1. It is defined up to a proportionality constant by (Tarantola

$$R_p \Pi_{null_p} = \Pi_{null_p} R_p = I_{NM}.$$
Velocity and migration uncertainties

\[
\tilde{\rho}_M(m) \propto \exp \left[ -\frac{1}{2} \left( d(m) - d_{\text{obs}} \right)^\dagger C_D^{-1} \left( d(m) - d_{\text{obs}} \right) - \frac{1}{2} \left( m - m_{\text{prior}} \right)^\dagger C_M^{-1} \left( m - m_{\text{prior}} \right) \right]. \tag{16}
\]

Take \( m_{\text{prior}} = m_n \) in equation \( \text{(16)} \) (which implies that \( m_{\text{prior}} \) is chosen to be the previous iteration result, in agreement with most methods used to solve inverse problems, cf. section “Tomography inverse problem...”). Consider the first-order (linear) approximation

\[
d(m) \approx G_n(m - m_n) + d(m_n), \tag{17}
\]

that holds in some region around \( m_n \) (the weaker the non-linearity in \( d(m) \) the larger the region). Then, equation \( \text{(16)} \) can be rewritten as

\[
\tilde{\rho}_M(m) \propto \exp \left[ -\frac{1}{2} \left( m - m_n \right)^\dagger \tilde{C}_M^{-1} \left( m - m_n \right) \right], \tag{18}
\]

where \( \tilde{C}_M \) denotes the “posterior” covariance matrix in the model space, defined through

\[
\tilde{C}_M^{-1} = G_n^\dagger C_D^{-1} G_n + C_M^{-1} = H_n. \tag{19}
\]

Its inverse \( H_n \) is the posterior Hessian matrix. The maximum-likelihood model \( m_n \) does not represent the true model, but the most probable one according to the set of data and priors. Many other probabilistically pertinent models (or “admissible” model perturbations) exist. \( \tilde{\rho}_M(m) \) allows a characterization of these models in terms of confidence levels, giving information on the confidence region associated with a confidence level \( P \) (equal to the integral of \( \tilde{\rho}_M(m) \) over the confidence region). It thus represents a key to extract uncertainty information with a clear meaning. Of course, the quality of this interpretation depends on the approximations of the covariance matrices \( C_D \) and \( C_M \). In practical applications, this contributes to the fact that the following uncertainty evaluations will tend to remain somewhat qualitative. This will be further discussed in section “Global proportionality constraint...”.

Note that \( \tilde{C}_M \), defined through equation \( \text{(19)} \), should be computed using the same data and prior covariances than the ones of the (last) tomography pass. This is assumed in the following. In other terms, we suppose that all priors for the uncertainty analysis have been defined during the tomography pass (the ones that led to the “best” tomography result). It is of paramount importance that the model is coherent with the assumptions of the uncertainty analysis.

Using notations of section “Tomography inverse problem...”, the posterior inverse covariance matrix, equation \( \text{(19)} \) can be also computed through

\[
\tilde{C}_M^{-1} = A_n^\dagger A_n = D^{-1} + A_n^\dagger A_n' D^{-1}. \tag{20}
\]

The EVD of \( \tilde{C}_M \), also defined through the SVD of \( A_n' \), contains uncertainty information. Indeed, the principal axes of the Gaussian posterior pdf, equation \( \text{(18)} \) are given by the eigenvectors of \( \tilde{C}_M \) and the deformations of the pdf by the eigenvalues of \( \tilde{C}_M \). The “more extended” or poorly resolved directions correspond to smaller eigenvalues of \( \tilde{C}_M \).
Velocity and migration uncertainties

Migration structural uncertainties.

We are interested in computing migration structural uncertainties on target reflectors, i.e. migration uncertainties related to the kinematic part of the Kirchhoff operator. Tomography model uncertainties represent a main contributor to migration kinematic uncertainties. We can therefore first generate admissible tomography model perturbations, i.e. perturbed models from equations 18 and 20. Then, migrations using each perturbed model can be performed and analyzed. A study of the results would allow us to estimate migration kinematic uncertainty-related quantities.

Full Kirchhoff migrations may be considered, but it is difficult to separate structural uncertainties from other uncertainties in migrated images, such as those related to amplitudes. [Li et al. 2014] propose to use the Euclidean and Procrustes distances to somewhat perform such a separation. Here, we are interested mostly in the structural uncertainties related to target reflectors, a crucial component of migration uncertainties. We consider a kinematic approximation of the Kirchhoff operator, the kinematic or map migrations discussed above [Duffet and Sinoquet 2006; Guillaume et al. 2008]. h represents the result of the kinematic migration of an invariant pick. We have

\[ h = k(m), \]

where \( k(m) \) is the kinematic migration operator for an invariant pick, non linear with respect to the tomography velocity \( m \). We compute the maximum-likelihood position of a target reflector \( h_n \) related to the maximum-likelihood tomography model \( m_n \):

\[ h_n = k(m_n). \]

Let us consider a linearization of \( k(m) \) around \( m_n \):

\[ (h - h_n) \approx K^\dagger (m - m_n), \]

where \( K \) represents the linearized approximation (or Jacobian matrix) of the kinematic migration operator. The migration structural posterior covariance matrix related to \( h_n \) is then defined through

\[ \tilde{C}_K^{-1} = K^\dagger \tilde{C}_M^{-1} K, \]

\[ (h - h_n)^\dagger \tilde{C}_K^{-1} (h - h_n) \approx (m - m_n)^\dagger \tilde{C}_M^{-1} (m - m_n). \]

Using notation similar to equation 16, \( \tilde{C}_K \) defines the migration structural posterior pdf \( \tilde{\rho}_K(h) \) in a similar way to equation 18 and contains information on migration structural uncertainties (related to the tomography model uncertainties).

Map migrations (i.e. zero-offset kinematic migrations) of horizons or target reflectors invariant picks are most often used in practice [Duffet and Sinoquet 2006; Osypov et al. 2008b, 2013; Messud et al. 2017b,a]. Once a set of perturbed models \( \{m\} \) that follow equation 18 is generated (and possibly some spurious models removed), map migrations of horizons or target reflectors may be performed. This will give a set of perturbed horizons of target reflectors \( \{h\} \) that will be related to the migration structural posterior pdf according to equation 24. Some statistical analysis allows us to deduce structural uncertainty
Velocity and migration uncertainties

quantities related to a target reflector and a given confidence level $P$ (Duffet and Sinoquet, 2006; Osypov et al., 2008b, 2013).

Complementarily and specifically to non-linear slope tomography, kinematic migrations (using all offsets) of the invariant picks (using all data, not only target reflectors invariant picks) can be considered. This makes it possible to deduce positioning uncertainty quantities in the whole migrated volume, not only at target reflector positions. Details will be given in section “Specific to non-linear slope tomography...”.

Our analysis accounts for all the important sources of uncertainties related to the tomography model $\mathbf{m}$, that consists of a set of parameters describing velocity and anisotropy. Anisotropy, for example, has been pointed out as an important source of uncertainty by Osypov et al. (2008b, 2010, 2011), that is naturally part of the Bayesian formalism presented here. Of course, the obtained uncertainties should be interpreted in the light of the tomographic data that have been used to compute them (for instance, if faults are not inverted by the tomography, uncertainties on faults cannot be handled by the method). Also, the method is valid if:

- The tomography model $\mathbf{m}_n$ is a maximum likelihood model, i.e. that lies in the floor of the cost function valley. Then, perturbations of the model (that stay within the valley) are less probable. This is assumed in the following. Even if not fully rigorous, the analysis can also be performed for a model at a local minimum model of the cost function valley, but clearly not for a model on a “flank” of the valley.

- The linearization, equations 18 and 23, holds. The range of validity of the linearization in equation 23 should more or less be the same as that of the linearization in equation 17, which allowed us to define $\tilde{\mathbf{C}}_M$ in equation 18, that is coherent. This will be further studied below.

The main question now is: How to generate perturbed models from equation 18, together with a confidence level $P$? We first review previous work in a unifying framework, then highlight questions and finally present our contribution.

**SAMPLING A POSTERIOR PDF TO GENERATE A SET OF PERTURBED MODELS**

**Sampling the normal distribution**

Suppose we can find a matrix $\mathbf{E}$ that satisfies

$$\tilde{\mathbf{C}}_M^{-1} \approx \mathbf{E}^\dagger \mathbf{E} \quad \Rightarrow \quad \tilde{\mathbf{C}}_M \approx \mathbf{B} \mathbf{B}^\dagger \text{ where } \mathbf{B} = \mathbf{E}^{-g}. \quad (25)$$

Then, the posterior pdf, equation 18, can be rewritten

$$\tilde{\rho}_M(\mathbf{m}) \propto \exp \left[ -\frac{1}{2} \delta \mathbf{r}^\dagger \delta \mathbf{r} \right] \text{ where } \delta \mathbf{r} = \mathbf{E} \Delta \mathbf{m}. \quad (26)$$

$\Delta \mathbf{m}$ represents perturbations of the maximum-likelihood model

$$\Delta \mathbf{m} = \mathbf{m} - \mathbf{m}_n. \quad (27)$$
Equation 26 implies that the covariance matrix associated with the vector \( \delta r \) is the identity \( I \), i.e. that the \( \delta r \) coordinates are not correlated and all have a variance of 1. One method of generating tomography model perturbations \( \Delta m \) that follow the Gaussian distribution \( 18 \) is to draw \( \delta r \) from a normal distribution \( \mathcal{N}(0, I) \) and compute

\[
\Delta m = B \delta r,
\]  

(28)

where \( B \) is a matrix that contains the posterior covariance information and scaling. Once a set of perturbed models \( \{ \Delta m + m_n \} \) is generated (and possibly some spurious models discarded), migration structural or kinematic uncertainty quantities can be deduced using the method presented above.

We now discuss possible choices for \( B \) and corresponding dimensionality. The next three sections describe existing methods and their limitations, and underline questions still to be clarified. Then, the rest of the article presents our method.

**Cholesky decomposition and partial SVD-based methods**

Suppose we have computed \( \tilde{C}_M^{-1} = A_n^\dagger A_n \). [Duffet and Sinoquet, 2006] propose to perform a Cholesky decomposition of \( \tilde{C}_M^{-1} \). This implies finding a lower-triangular square matrix \( E \) of size \( N_M \times N_M \) that satisfies (exactly) equation 25 and computing (\( E \) being invertible as \( \tilde{C}_M^{-1} \) is)

\[
B = E^{-1},
\]

(29)

to generate model perturbations using equation 28. \( B \) is a matrix of size \( N_M \times N_M \) and \( \delta r \) is a vector of size \( N_M \) draw from \( \mathcal{N}(0, I_{N_M}) \). This scheme is costly and may lose accuracy if \( A_n \) is ill-conditioned. [Zhang and Thurber, 2007].

Another method is to use a partial SVD of \( A_n' \), equation 11. With equations 20 and 25, we deduce

\[
E = A_p^{1/2} V_p^\dagger D^{-1} \quad \Rightarrow \quad B = E^{-9} = D V_p T_p^{1/2} A_p^{-1/2},
\]

(30)

where Tikhonov regularization has been added for the generalized inverse computation. \( B \) is a matrix of size \( N_M \times p \) and \( \delta r \) a vector of size \( p \), denoted by \( \delta p \) to make the size explicit and draw from \( \mathcal{N}(0, I_p) \). This scheme reduces the space of model perturbations, usually to the degrees of freedom that can be resolved by tomography, which is numerically advantageous but implies an SVD-based low-rank approximation. (We did not find applications of this scheme in the literature.)

**Prior-based low-rank decomposition method**

This section details the formalism of [Osypov et al., 2013, 2011, 2008a] and the specific form obtained for \( B \). Let us consider prior contributions to the prior covariance matrix in equation 31, that are not represented by square matrices of size \( N_M \times N_M \) but by a non-square matrix \( P \) of size \( N_M \times p \) with \( p < N_M \)

\[
C_M \rightarrow PP^\dagger.
\]

(31)
As \( p \ll N_M \) is chosen in practice, low-rank prior constraints are considered, unlike in section “Tomography inverse problem...” and equation \( 6 \). The obtained \( C_M \) in equation \( 31 \) is not strictly invertible, i.e. not strictly speaking a covariance can be defined. \( P \) can for instance be a steering filter that contains information on the structures, to reduce the dimensionality of the problem (\( p \ll N_M \)) \cite{clapp1998}. \( P \) is called a preconditioner in \cite{osypov2013, osypov2011, osypov2008a}, but it has a very different role from the preconditioner \( D \) introduced in section “Tomography inverse problem...”. We thus here call it “model preconditioner”.

Gathering many constraints within this formalism is done in the model preconditioned basis (note that \( C_R^{1/2} \) is considered contrarily to equation \( 6 \) that considers \( C^{-1/2} \))

\[
C_R^{1/2} = \begin{bmatrix} PQ_1 & \cdots & PQ_L \end{bmatrix}
\]

\[
C_M = C_R^{1/2}C_R^{-1/2} = P(\sum_{i=1}^{L} Q_i^\dagger Q_i)P^\dagger,
\]

where \( Q_i \) is a \( p \times p \) matrix that can for instance contain the coupling between the various model contributions or subsurface parameters in the anisotropic case \cite{osypov2013, osypov2011, osypov2008a}. To lighten the notations, we do not consider such additional constraints in the following.

We have, using the generalized inverse

\[
\tilde{C}_M \approx [G_n^\dagger C_D^{-1}G_n + (PP^\dagger)^{-g}]^{-g} = P[(G_nP)^\dagger C_D^{-1}(G_nP) + I_p]^{-1}P^\dagger,\]

where \( (G_nP)^\dagger C_D^{-1}(G_nP) + I_p \) is a \( p \times p \) matrix that is invertible because it is positive definite (\( I_p \) kills the null eigenvalues). Now, let us perform the following partial EVD with \( q \leq p \) iterations (no additional preconditioning is needed as the matrices are dimensionless and the problem is well conditioned in the model preconditioned domain)

\[
(G_nP)^\dagger C_D^{-1}(G_nP) \approx V_q\Lambda_q V_q^\dagger.
\]

We deduce

\[
\tilde{C}_M \approx P[V_q\Lambda_q V_q^\dagger + I_p]^{-1}P^\dagger.
\]

The binomial inverse theorem recalled in Appendix A leads to

\[
\tilde{C}_M \approx P[I_p - V_q V_q^\dagger + V_q(\Lambda_q + I_q)^{-1}V_q^\dagger]P^\dagger.
\]

Using equation \( 25 \) \cite{osypov2013, osypov2011, osypov2008a} finally obtain

\[
B = P[I_p - V_q V_q^\dagger + V_q(\Lambda_q + I_q)^{-1/2}V_q^\dagger],
\]

allowing to generate perturbations in the model space by equation \( 28 \). As here \( \delta r \) is a vector of size \( p \ll N_M \), we denote it by \( \delta p \) to make the size explicit. \( \delta p \) must be drawn from \( \mathcal{N}(0, I_p) \). \( B \) is a matrix of size \( N_M \times p \) that can be split into

\[
B = B_q^{\text{res}} + B_q^{\text{an-res}}
\]

\[
B_q^{\text{res}} = PV_q(\Lambda_q + I_q)^{-1/2}V_q^\dagger
\]

\[
B_q^{\text{an-res}} = P[I_p - V_q V_q^\dagger]
\]

\[
\Delta m^{\text{modprec}} = B\delta p \quad \text{where} \quad \delta p \sim \mathcal{N}(0, I_p),
\]

where two terms appear:
• $B_{q}^{\text{res}}$ deals with the uncertainty information contained in the posterior covariance matrix and resolved by the partial EVD. Indeed, $V_{q}(A_{q} + I_{q})^{-1/2}V_{q}^{\dagger}$ represents the partial EVD of $C_{M}^{1/2}$ in the model preconditioned domain.

• $B_{q,p}^{\text{un-res}}$ contains a projector on an effective null space of dimension $p - q$, as $q$ EVD iterations have been used to approximate a $p \times p$ matrix. So, only the effective null space of the EVD in the model preconditioned domain (of size $p$) can be explored by this method. Even if this is a very limited part of the full null space of the tomography operator (because $q \leq p << N_{M}$), the advantage is that it allows to explore some part of the effective null space of the EVD while keeping geological structures and smoothness in the model perturbations (the perturbations are projected on $P$).

Questions

The methods described in the previous sections have advantages but some questions remain. In particular, the following points need to be clarified:

• A method to QC the validity of the Gaussian and linearized hypothesis. This is important to check if obtained uncertainties are pertinent. In the following, we propose an efficient sampling method that, together with non-linear slope tomography, will address this point.

• A link between the methods that work in the full model space and those that work in a preconditioned model space. We will define the resolved space uncertainty concept that allow to make the connection. We will also define the unresolved space uncertainties (considering the effective null space projector), that represent a byproduct specific to the following method, giving complementary qualitative information.

• A definition of error bars for the 68.3% (standard-deviation-like) confidence level, which accounts for the non-diagonal part of the posterior covariance matrices (as accounting only for the diagonal part would underestimate the errors). We will give such a definition.

• We will also discuss if the computed uncertainties be more than qualitative.

A NEW METHOD SAMPLING A GAUSSIAN EQUI-PROBABLE CONTOUR TO GENERATE A SET OF PERTURBED MODELS

Sampling a Gaussian equi-probable contour

Let us return to the tomography posterior Gaussian pdf, equation $18$, and propose a different sampling method that will reduce the sampled space and thus optimize the exploration. We consider an equi-probable contour

$$\Delta m^{\dagger}C_{M}^{-1}\Delta m = Q_{N_{M}}(P),$$ (39)

where $Q_{N_{M}}(P)$ is the quantile of order $P$ of the Chi-square distribution (Cowan 1998) (this distribution is among others used in confidence interval estimations related to Gaussian random variables).
Resolving (or sampling) equation \( \cdot \) for a given \( P \) value gives the set of maximum perturbations (or the boundary of the confidence region) associated with a confidence level \( P \); Figure 2 gives an illustration. The probability that the true model \( m \) lies within the \( N_M \)-dimensional hyper-ellipsoid of center \( m_n \) defined by equation \( \cdot \) is equal to \( P \). Restricting the sampled space to an equi-probable contour does not hamper the assessment of the uncertainties compared to the sampling of the full pdf because information on the full Gaussian is contained in one contour and the corresponding \( P \) value. Indeed, all hyper-ellipsoids defined by equation \( \cdot \) are related by a simple proportionality constant.

In the following, we consider a confidence level \( P = 68.3\% \). We call it the standard-deviation-like confidence level (Messud et al., 2017a) because it corresponds to a standard deviation interval when \( N_M = 1 \) or when model parameters are non-correlated and their single pdfs are considered independently, see Appendix B. For more generality, we define uncertainties through the tomography confidence region related to a probability \( P = 68.3\% \) and resolve equation \( \cdot \) to generate a set of admissible perturbations. In the spirit of the considerations of section “Sampling a normal distribution...”, the solution is

\[
\delta r^\dagger \delta r = Q_{N_M(68.3\%)} \quad \text{and} \quad \hat{C}_M \approx BB^\dagger \Rightarrow \Delta m = B\delta r, \tag{40}
\]

where \( \delta r \) is a vector of size \( N_M \) drawn from a uniform distribution and rescaled to have norm \( ||\delta r||_2 = \sqrt{Q_{N_M(68.3\%)}} \) (it must not be drawn from a Gaussian distribution here, as we sample only an equi-probable contour). Generating model perturbations from an equi-probable contour sampling has the advantage of reducing the sampled space to its most representative components, optimizing the computation. In our case, where we do not use low-rank prior constraints that reduce the dimensionality of the problem (\( B \) is a \( N_M \times N_M \) matrix), the method allows us to obtain stable uncertainty estimates with 200-500 random models, for large-scale applications. And especially, as detailed further, pdf’s equi-probable contour sampling together with non-linear tomography provide an efficient way to QC the assumptions made within the Bayesian formalism (linearity and Gaussian pdf hypothesis).

Let us now specify a problem concerning error bars. Why not simply define error bars by \( \pm \sqrt{\text{diag}(\tilde{C}_M)} \), where \( \text{diag}(\tilde{C}_M) \) denotes the vector containing the diagonal elements of \( \tilde{C}_M \)? This is sometimes used as a first tomography model uncertainty indicator (Osypov et al., 2013), but it has pathologies. When \( C_M \) is diagonal, a subset of the solutions of equation \( \cdot \) with \( P = 68.3\% \) is

\[
\Delta m = \pm \sqrt{\frac{Q_{N_M(68.3\%)}}{N_M}} \sqrt{\text{diag}(\tilde{C}_M)}, \tag{41}
\]

defining a confidence interval. So, firstly, using equation \( \cdot \) would be better than using \( \pm \sqrt{\text{diag}(\tilde{C}_M)} \) from a confidence level point of view (indeed, \( \pm \sqrt{\text{diag}(\tilde{C}_M)} \) is not associated with a constant confidence level \( P = 68.3\% \) as \( N_M \) varies). But, secondly, equation \( \cdot \) is still too restrictive, since:

- Even in the diagonal case, the full solution of equation \( \cdot \) with \( P = 68.3\% \) defines a larger (hyper-ellipsoidal) confidence region, encompassing the confidence interval defined by equation \( \cdot \). So, equation \( \cdot \) underestimates uncertainties.
- In the general case, \( \sqrt{\text{diag}(\tilde{C}_M)} \) is not sufficient to generate a set of admissible tomography models. Indeed, non-diagonal elements of \( \tilde{C}_M \) can have a strong contribution as
they describe correlations between model space nodes, which are crucial in tomography (because of the smoothness, structural conformity, etc. constraints on the model) [Duffet and Sinoquet, 2006]. Appendix B gives more formal details.

So, equation 40 must be resolved to represent the full solution of equation 39. It remains to define the B matrix and how to compute error bars from the equi-probable contour sampling.

Figure 2: Equi-probable contour of a posterior Gaussian pdf ($N_M = 2$) (red) and $\Delta m$ samples (triangles).

Separating the “resolved” space uncertainties

We consider an EVD of $\hat{C}^{-1}_M$ in the preconditioned domain (using notations of section “SVD, EVD...”)

$$D^T \hat{C}^{-1}_M D = \begin{bmatrix} V_p A_p V_p^\dagger + V_{NM-p} A_{NM-p} V_{NM-p}^\dagger & & \\ & & \end{bmatrix}$$

$$\approx \begin{bmatrix} V_p A_p V_p^\dagger + \epsilon I_{NM-p} & & \\ & & \end{bmatrix}$$

$$\quad \Lambda_{NM-p} \rightarrow \epsilon I_{NM-p}$$

$\epsilon$ being the noise level

The second line shares similarity with a damping and is related to the noise-contaminated effective null space of the tomography. Equation 42 is a partial EVD of $\hat{C}^{-1}_M$ in the preconditioned domain (all model components or parameters have the same units in this domain), stopped after $p$ iterations when the eigenvalues have reached a fixed prior level $\epsilon$ [Zhang and McMechan, 1995], and approximate the effect of the $N_M - p$ non-computed eigenvectors by $\epsilon I_{NM}$. The damping level $\epsilon$ is tuned by the user during the tomography pass, as discussed in section “Tomography inverse problem...” (remind also that the same priors than the ones of the (last) tomography pass must be used for the posterior covariance computation, as discussed in section “Bayesian formalism...”). Using the binomial inverse theorem, Appendix A, we obtain

$$\hat{C}_M \approx D \begin{bmatrix} V_p A_p V_p^\dagger + \epsilon I_{NM} \\ \epsilon I_{NM} \end{bmatrix}^{-1} D^\dagger$$

$$= D \begin{bmatrix} V_p (A_p + \epsilon I_p)^{-1} V_p^\dagger + \frac{1}{\epsilon} \Pi_{null} \end{bmatrix} D^\dagger,$$
where $\Pi_{null_p}$ is the effective null space (dimension $N_M - p$) projector, see section “SVD, EVD...”. Using equation 40 we can compute

$$B = D \left[ V_p \left( A_p + \epsilon I_p \right)^{-1/2} V_p^\dagger + \frac{1}{\sqrt{\epsilon}} \Pi_{null_p} \right],$$

which can be split into

$$B = B_{resolved}^p V_p^\dagger + B_{un-resolved}^{p,N_M}$$

$$B_{resolved}^p = D V_p \left( A_p + \epsilon I_p \right)^{-1/2}$$

$$B_{un-resolved}^{p,N_M} = \frac{1}{\sqrt{\epsilon}} D \Pi_{null_p}$$

$$\Delta m = B \delta r$$ where $\delta r$ is drawn from a uniform distribution and rescaled to have norm $\sqrt{Q_{NM}(68.3\%)}$.

The method allows us to separate the following two uncertainty contributions:

- $B_{resolved}^p$ deals with the effective uncertainty information contained in the posterior covariance matrix. It drives the contribution to $\Delta m$ of the eigenvectors with eigenvalues above the prior damping level $\epsilon$, which spans the so-called tomography “resolved” space (of dimension $p$). As those eigenvectors are greatly constrained by the tomography input data, so are the related resolved space perturbations that are defined by

$$\Delta m^{resolved} = B_{resolved}^p \delta p$$ where $\delta p = V_p^\dagger \delta r$ is a vector of size $p$.

- $B_{un-resolved}^{p,N_M}$ is related to the tomography full effective null space projector (note that an explicit orthonormalization of the eigenvectors like a Gram-Schmidt can be numerically important for an accurate computation of $\Pi_{null_p}$ (Zhang and Thurber 2007)). $B_{un-resolved}^{p,N_M}$ describes the contribution of eigenvectors with eigenvalues below $\epsilon$, which spans the tomography “unresolved” space (of dimension $N_M - p$). This space is mostly constrained by the priors and the illumination ($\Pi_{null_p}$ and $D$, equation 45). Unresolved space perturbations are defined by

$$\Delta m^{un-resolved} = B_{un-resolved}^{p,N_M} \delta r.$$  

- “Total” perturbations are defined by the sum of both contributions

$$\Delta m = \Delta m^{resolved} + \Delta m^{un-resolved}.$$  

Despite the formal similarities between the decomposition in equation 45 and the decomposition in equation 38 the content is somewhat different:

- In equation 38 tomography constraints are contained in the model preconditioner $P$ (a steering filter) through equation 31, whereas in the case of equation 45 the tomography constraints are contained in the eigenvectors and illumination information is contained in the preconditioner $D$. $B$ is a $N_M \times p$ matrix and $\delta p$ a $p$ size vector in equation 38 whereas $B$ is a $N_M \times N_M$ matrix and $\delta r$ a $N_M$ size vector in equation 45.
• The $B_{q,p}^{un-res}$ of equation 38 describes the effective null space (dimension $p - q$) of the EVD in the model preconditioned domain, not to be confused with the $B_{p,NM}^{un-resolved}$ of equation 45 that describes the tomography full effective null space (dimension $N_M - p$). Starting from equation 45 and considering that the corresponding $p$ is approximately the same as in equation 38, we can deduce a link between the two decompositions at the resolved space level:

$$
\Delta m_{resolved} = B_p^{resolved} \delta p \approx (B_q^{res} + B_{q,p}^{un-res}) \delta p = \Delta m_{modprec}^{resolved}
$$

This demonstrates that the resolved space perturbations $\Delta m_{resolved}$, computed with our scheme that works in the full model space, can be related to the perturbations $\Delta m_{modprec}$ computed with the previously described scheme that works in a model preconditioned domain. However, important to remark, $\delta p$ is constrained to be equal to $V_p^T \delta r$ in equation 49. The introduction of this projection constraint on $\delta p$ distinguishes our method, also from the one of section “Cholesky decomposition...”. An advantage of considering random perturbations $\delta r$ of dimension $N_M$ and projecting them on $V_p^T$ to compute the $p$-dimensional perturbations $\delta p$ is that the latter will already account for information present in the eigenvectors (structures, etc.), which will contribute to the resolved space results.

Figure 3 (b) gives an example of a resolved space perturbation $\Delta m_{resolved}$ for a field data. It tends to be organized, smooth (because tomography resolves the large wavelengths of the velocity model), and correlated to structures and to the tomography final model, as expected. It represents a basis of our uncertainty analysis method. Remind however that these perturbations represent “maximum” perturbations associated with a $P = 68.3\%$ confidence level (samples of the equi-probable contour that bounds the corresponding confidence region), hence the lateral variations visible in figure 3 (b). They should thus not be interpreted like samples of the corresponding normal distribution (more localized around the maximum likelihood for most of them).

• The introduction of $\Delta m_{un-resolved} = B_{p,NM}^{un-resolved} \delta r$ represents a byproduct specific to the method presented here, whose computation is little costly once the resolved space computed (i.e. the EVD performed).

Figure 3 (c) gives an example of a total space perturbation $\Delta m$ for a field data. It looks more random and of higher frequency than $\Delta m_{resolved}$, and has a quite larger magnitude. Indeed, $\Delta m$ is essentially driven by $\Delta m_{un-resolved}$ as the dimension $N_M - p$ of the unresolved space is quite larger than the dimension $p$ of the resolved space in most applications. This, together with the thresholding to $\epsilon$ in equation 42, makes the total space information fully qualitative. However, it is not without interest. It gives an additional information that reflects the priors and the illumination, strongly highlighting the most uncertain gross areas as illustrated further. It thus can be used for QCs and may also offer some possibility of exploring small-scale non-structural variations, that we cannot consider as fully improbable.

Finally, for completeness, we mention a last element regarding our scheme. Using notations of section “Prior-based low-rank ...” we have

$$
\delta r = R_p \delta r + \Pi_{null,p} \delta r.
$$

(50)
As \( \delta \mathbf{p} = \mathbf{V}_p^\dagger \delta \mathbf{r} \) in our scheme, we can deduce using equation (40)

\[
\delta \mathbf{r}^\dagger \delta \mathbf{r} = \delta \mathbf{p}^\dagger \delta \mathbf{p} + \delta \mathbf{r}^\dagger \Pi_{null} \delta \mathbf{r} = Q_{NM}(68.3\%). \tag{51}
\]

In the large \( NM \) case (like in tomography), where \( Q_{NM}(68.3\%) \) remains close to \( NM \), one may think the latter equation implies \( \delta \mathbf{p}^\dagger \delta \mathbf{p} = p \) and \( \delta \mathbf{r}^\dagger \Pi_{null} \delta \mathbf{r} = NM - p \), which would help to slightly simplify the sampling problem in equation (45). However, this is not the case as the amplitudes of the perturbations in resolved and unresolved subspaces are not fully independent from the total uncertainty point of view.

![Figure 3: North Sea field data. Maximum-likelihood velocity (Vp) model \( m_n \). (a) One resolved space perturbation \( \Delta \mathbf{m}_{resolved} \) and (b) corresponding total space perturbation \( \Delta \mathbf{m} \), displayed on sections and one horizon. Modified from Messud et al. (2018); see the article for more details.](image)

### Computing error bars

Uncertainty attributes can be computed statistically using the obtained set of perturbations, especially for the resolved space using the \( \{ \Delta \mathbf{m}_{resolved} \} \), and also possibly for the total space using the \( \{ \Delta \mathbf{m} \} \):

- **Tomography model 68.3% error bars on \( \mathbf{m}_n \) (velocity and anisotropy models):**
  Computed by considering the maximum possible variations of the model perturbations (\( i \) here represents the model grid coordinates):

  \[
  \sigma_i^{(\mathbf{m}_n)} = \max \{ \Delta m_i \}. \tag{52}
  \]

  The true model belongs to \( \mathbf{m}_n \pm \sigma^{(\mathbf{m}_n)} \) with a probability \( P \geq 68.3\% \) [Messud et al., 2017a]. [Reinier et al., 2017] give an illustration of such error bars for the velocity and the total space.

- **Maximum horizon perturbations within the 68.3% confidence region:**
  Map migrations can be performed using each model perturbation to obtain a set of equi-probable horizon perturbations \( \{ \Delta \mathbf{h} \} \) around the maximum-likelihood horizon position \( \mathbf{h}_n \), remind section “Migration structural uncertainties”. Note that those perturbations must not be interpreted as a migration structural posterior pdf sampling,
but as a sampling of the pdf’s 68.3% equi-probable contour, producing maximum possible perturbations within the 68.3% confidence region. The perturbations can be QCed, see Figure 4 for a field data example, and used to compute horizon error bars (next item), but they should not be used as such in reservoir workflows that need perturbed horizons sampled from the pdf as an input (as the latter samples are not similar, tending to be more concentrated around the maximum-likelihood). If needed, pdf sampling can easily be recovered from equi-probable perturbations (a Gaussian pdf can easily be reconstructed from one of its equi-probable contours, cf. section “Sampling a Gaussian equi-probable contour…”).

• Horizon position 68.3% error bars:
  Depth error bars can be defined as the maximum possible depth variation of the horizon perturbations (i here represents the horizon coordinates):
\[ \sigma_i^{(hn)} = \max\{\Delta h_i\}. \]

The horizon migrated points “move” vertically and laterally for each map migration; \( \sigma_i^{(hn)} \) considers the depth “envelope” of all migrated horizons and thus accounts for lateral displacements of migrated points. This gives 68.3% confidence level error-bars: the true horizon depth position belongs to \( h_n \pm \sigma^{(hn)} \) with a probability \( P \geq 68.3\% \) \cite{Messud2017a}. Lateral (x and y-directions) horizon error bars can be computed using the same principle, from differences of position between rays traced in \( m_n \) and rays traced in the perturbed model.

Note that the error bars \( (\sigma^{(mn)} \text{ and } \sigma^{(hn)}) \) should not be computed from standard-deviations of the perturbations but from a maximum, as we sample a pdf equi-probable contour. Also, our error bars definitions account for the non-diagonal elements of the tomography and migration structural posterior covariances, equations 19 and 24. Thus, they contain more uncertainty information than the diagonal elements of \( \tilde{C}_M^{1/2} \text{ and } \tilde{C}_H^{1/2} \) (i.e. the standard-deviations). They therefore can be considered as “generalized standard deviations”.

All previous error bars can be computed statistically the same way for the total space, using the \{\( \Delta m \)\} perturbations. However, as the latter are fully qualitative, it then seems difficult to keep a 68.3% confidence level interpretation. However, the hierarchy of the total space are not without interest as illustrated further.

The computational cost of the method resides in an EVD of the final tomography operator (equations 20 and 42) and map migrations of horizons in perturbed models (stable error bars are with 200-500 random models as the equi-probable contour sampling reduces the sampled space to its most representative components). It approximately equals the computational cost of one additional non-linear slope tomography iteration.
Specific to non-linear slope tomography: Migration volumetric positioning error bars and QC of the Gaussian hypothesis

The use of non-linear slope tomography, based invariants picks, provides unique advantages:

- Non-linear slope tomography provides an efficient way to assess the quality of the randomly generated model perturbations. Indeed, the cost functions related to the perturbations can be estimated automatically and non-linearly by kinematic migration of the invariant picks, for an affordable additional computational cost (this step involves computationally effective ray tracing only). Combined with the posterior pdf equi-probable contour sampling, it allows us to QC the validity of the Gaussian hypothesis done in section “Bayesian formalism...”. Indeed, these cost function values must equal the left side of equation 39 (up to an additive constant) in the linear approximation. In other terms, if the linear and Gaussian approximations are pertinent, computed perturbations should be equi-probable or equivalently iso-cost. This will be illustrated further.

- As discussed in section “Migration structural uncertainties”, full kinematic migrations (using all offsets) of the invariant picks (using all data, not only target reflectors invariant picks) can be performed on each model perturbation. We compute how much a migrated pick has moved compared to its position when kinematically migrated in the maximum likelihood model (both being related to the same invariant pick). Then, using a “maximum” principle as in section “Computing error bars...” allows us to deduce migration error bars at all migrated picks positions, not only at horizons positions. Interpolating corresponding error bars give error bars in the whole migrated

![Figure 4: North Sea field data. The tomography maximum-likelihood velocity model overlain with a subset of 20 (among several hundred) horizons perturbations, obtained by map migrations in perturbed velocity models (resolved space). Modified from Messud et al. (2017b); see the article for more details.](image)
volume, called migration volumetric 68.3% error bars in the following. This makes it possible to deduce positioning uncertainty quantities in the whole migrated volume, not only at target reflector positions. (We remind the error bars should be interpreted in the light of the data that have been used to compute them, here reflection events related to the migrated picks.) These error bars will be illustrated further.

Can computed uncertainties be more than qualitative?

We already mentioned that the results discussed in this paper depend on the approximations of the covariance matrices $C_D$ (quality of the invariant picks) and $C_M$ (various model space constraints) that enter into $\tilde{C}_M$ computation, leading to somewhat qualitative uncertainty evaluation in practical applications. Also, it is not possible to account for all possible sources of uncertainties, but only for some of them that we consider as the most important (for instance here, the tomography model is considered as a main source of uncertainty for the migrated domain). This reinforces the somewhat qualitative feature of the uncertainty evaluation.

Additionally, many aspects that only slightly affect the maximum-likelihood search (i.e. the minimum of the cost function) may affect the uncertainty computation (i.e. the curvatures of the cost function at the minimum). Often overlooked, those aspects tend to affect uncertainties up to a global proportionality constant:

- Bayesian uncertainty reasoning holds strictly if the diagonals of $C_D$ and $C_M$, that enter into $\tilde{C}_M$ computation, represent variances, i.e. are related to a confidence level of 68.3%. In practice, even good approximations of $C_D$ and $C_M$ tend to be defined up to a global proportionality constant, i.e. they are balanced together, so that they do not affect the maximum-likelihood. But the scaling of the global proportionality constant is not easy and itself uncertain.

- Data decimation will produce less “illumination” of each model node and therefore will tend to increase the uncertainties. Contrariwise, a larger model discretization step will produce more “illumination” of each model node and thus will tend to decrease the uncertainties. Those effects could theoretically be compensated by fine adaptation of the prior covariances, but this is not easy and basically requires knowledge of a large part of the inversion solution. Reasonable changes in data decimation and model discretization will tend to affect the uncertainties globally and linearly on average, i.e. up to a global proportionality constant.

The combination of these two effects will tend to affect uncertainties approximately only up to a global proportionality constant. This proportionality constant can be rescaled using posterior information external to the tomography, like wells, so that all well markers lie within the horizon error bars, see illustration in Figure 5. The resolved space error bars should of course be used for such a matching, as they contain the tomographic operator information; after rescaling they become less qualitative or more quantitative. However, they still will remain somewhat qualitative because of the various previously mentioned caveats. This is true for all methods discussed in this article.

The total space error bars always remain fully qualitative, among others as the null space exploration has been thresholded at the tomography noise level. They may be rescaled by
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the global constant found from the resolved space error bars (as is the case in next illustrations), but only their hierarchy is to be considered (not their values) as a complementary information.

Figure 5: Zoom-in around one well with Base Flett marker (cyan), migrated image section, Base Flett horizon (pink dotted line) and horizon depth error bars schematized by the pink lines. Modified from (Reinier et al., 2017).

ILLUSTRATION ON SYNTHETIC DATA

We consider a synthetic 2D case to illustrate the previously described method. Noise has been added to the exact invariant picks for more realism. We consider perturbations of the velocity model (Vp) (but, as previously mentioned, the formalism is general for any subsurface model including anisotropy). Figure 6 (a) shows the true velocity model \( \mathbf{m}^* \) with superimposed segments that represent locally coherent reflector events related to migrated picks, remind figure 1. Figure 6 (b) shows the best velocity model \( \mathbf{m}_n \) obtained after 20 iterations of non-linear slope tomography, having started with a constant velocity model. Inside the area delineated by the black/white line in figure 6, that schematizes the area illuminated by rays connected to the invariant picks, the tomography result is good but with some differences with the exact model. These are highlighted in figure 6 (c), representing \( |\mathbf{m}_n - \mathbf{m}^*| \) (where the absolute value is taken element-wise). Beyond the black/white line, the tomography is not expected to recover the model.
Velocity and migration uncertainties

Figure 6: Synthetic 2D velocity model. White small line segments represent locally coherent reflector events (related to migrated picks) and white/black line delineates the area illuminated by the tomography data. (a) True model $m^*$. (b) Best model $m_n$ obtained after 20 internal nonlinear iterations of slope tomography having started with constant velocity model ($m_0 = 2300 \text{ m/s}$). (c) Absolute difference $|m_n - m^*|$ (the absolute value is taken element-wise) between true and best inverted velocity models.

We now apply the previously described uncertainty analysis method, where $m_n$ is considered as the maximum likelihood model. Figure 7 shows two resolved space equi-probable velocity perturbations and figure 8 (a) the resolved space velocity error bars. Comparing these results magnitudes to figure 8 (c), we observe that the true velocity model $m^*$ lies within the range of perturbations of and error bars on $m_n$, which is satisfying.

Also, figure 8 (b) shows the (rescaled) standard-deviation estimated by the LSQR solver of tomography. It exhibits in this simple case relatively similar variations than the ones of the resolved space velocity error bars, figure 8 (a). This is satisfying as, even if our error bars contain more uncertainty information than only standard-deviations (related to diagonal elements of the posterior covariance), the standard-deviations tend to represent a first order contribution in many applications, especially in simple synthetic cases where non-local constraints like a structural ones were are not activated. However, note that the resolved space error bars contains a more subtle hierarchy, related to the correlations in $\tilde{C}_M$, depending on the contributions of the noise in the invariant picks and the form of the tomography modeling Jacobian matrix. We call this the uncertainty related to the “tomography discrimination power”.


Figure 9 shows the variations of mean velocity perturbations and of velocity error bars with the number of velocity perturbations, for the resolved space. The mean of the velocity perturbations tend towards zero when the number of realizations increases, as expected. The velocity error bars stabilize when the number of samples increases and we observe that 200 samples represents a good compromise. This is a typical value, also for large scale applications.

Figure 10 represents the non-linearly computed tomography cost function values (by kinematic migration of invariant picks) in 100 resolved space velocity perturbations. As discussed in section “Specific to non-linear slope tomography...”, if the linear and Gaussian approximations assumed in the uncertainty analysis are appropriate, perturbations should be iso-cost. We observe in figure 10 that the perturbations tend to be iso-cost, with only limited dispersion around the average of the cost function values of the perturbed models. This QC, specific to our method, allows to validate the linear and Gaussian hypothesis. Interestingly, we also see that no spurious perturbations related to too large variations of the cost function were generated; it is thus not necessary to have an additional step discarding spurious perturbations with the method discussed here (whereas it may be necessary with the method of section “Sampling a normal distribution...”).

Note that the dispersion range of the cost function values in the velocity perturbations depends on the shape of the cost function valley around the best model, decorrelated from the best model cost function value. In synthetic cases, where the problem is well constrained, the best model cost function is very low and the global minimum valley is sharp. This leads to more “uniqueness” of the solution and favors non-linearity, thus tends usually to produce a larger dispersion range for the cost function values in the perturbations.

We now interest ourselves to error bars in the migrated domain. In this synthetic case, it would not be pertinent to define an horizon and compute its perturbations by map migrations. This computation will rather be illustrated in next section on field data where target horizons make sense. So, we propose here to check the migration “volumetric” depth error bars, specific to our method, shown in figure 11 (a). These uncertainties make sense only inside the area illuminated by the tomography data, delineated by the black line. Outside this area, the migration “volumetric” depth error bars are just extrapolated until the edges, so that they are meaningless. These error bars allow to quantify how the perturbations of the best model, like in figure 7, as well as the “velocity stall” in the best model, figure 6 (b), affect the migrated space (here related to the migrated picks depths).

The migration “volumetric” depth error bars also allow to verify that the linear hypothesis assumed in the analysis for the migrations is pertinent, complementarily to figure 10 that allowed to verify that the linear hypothesis assumed in the velocity domain is pertinent. Figure 11 shows the effect of a proportionality constant equal to 1 (a) and 10 (b) applied in the velocity domain. The scaling factor in the velocity domain almost only translates into the same scaling factor in the migrated domain. This allow to QC that the linear hypothesis for the migrations is valid for our range of perturbations.

Finally, we discuss the total space by-product. Figure 12 compares the resolved and total spaces velocity error bars. The first thing we can note is that the total space error bars are here approximately three times larger than the resolved space error bars. This is because the dimension $N_M - p$ of the unresolved space is here approximately three times larger than the dimension $p$ of the resolved space; this makes the unresolved space perturbations dominate
the total space perturbations. Anyway, the total space error bars are fully qualitative so that only their hierarchy is to be considered, not their values.

Even if this synthetic case is simple and only allows to highlight subtle differences between total and resolved space error bars, we observe in figure 12 that the total space error bars delineate more precisely the area illuminated by the tomography data, i.e. the black line, than the resolved space error bars. The total space error bars may serve as a complementary QC, identifying areas where computing error bars is meaningless (especially the much less illuminated areas), this information being “attenuated” in the resolved space error bars as they concentrate more on the uncertainty related to the tomography discrimination power. Doing so, the latter give a more subtle hierarchy in the reasonably illuminated areas, highlighted by the dashed black arrows in figure 12. The resolved space error bars are more quantitative (even if still somewhat qualitative) and useful for further use in decision-makings.

Figure 7: Two resolved space equi-probable velocity perturbations (i.e. computed along the same posterior pdf contour) on $m_n$. Black line delineates the area illuminated by the tomography data.

Figure 8: Different types of velocity error bars on $m_n$. (a) Resolved space velocity error bars. (b) Velocity standard-deviation estimated by the LSQR solver of tomographic matrix (rescaled). Black line delineates the area illuminated by the tomography data.
Figure 9: Variations of mean velocity perturbations and of velocity error bars with the number of perturbations (resolved space). The velocity error bars are represented for (a) 20 perturbations, (b) 200 perturbations and (c) 900 perturbations. Similarly, the mean velocity perturbations are represented for (d) 20 perturbations, (e) 200 perturbations and (f) 900 perturbations.

Figure 10: Cost function values (in green) obtained by kinematic migration of invariant picks in the first 100 velocity perturbations (resolved space). Mean cost function value of all perturbations is 1.27. Cost function value for the tomography model is 0.48 (red dot) and 91.7 for initial model (constant velocity of 2300 m/s, blue dot).
Figure 11: Migration “volumetric” depth error bars (resolved space). Effect of a proportionality constant equal to 1 (a) and 10 (b) applied in the velocity domain. Black line delineates the area illuminated by the tomography data; outside this area, the error bars are just extrapolated until the edges, so that they are meaningless. The scaling factor in the velocity domain translates into the same scaling factor in the migrated domain, confirming that the linear hypothesis is valid for the range of perturbations.

Figure 12: Velocity error bars. (a) Resolved space and (b) total space. Black line delineates the area illuminated by the tomography data. Dashed black arrows highlight some difference in the hierarchy of resolved and total space error bars in the illuminated area.

ERROR BARS ILLUSTRATED ON 3D FIELD DATA

Industrial applications of the method presented in this paper have been published in Messud et al. (2017a,b, 2018), Reinier et al. (2017) and Coléou et al. (2019). In this section, we gather few examples illustrating error bars in the migrated domain. For further details, we invite the reader to refer to the aforementioned articles.

Let us illustrate our method on a first North Sea dataset merging four different overlapping narrow-azimuth towed-streamer surveys acquired over the years with different layout configurations. Figure 13 (a) shows a compounded fold map of the surveys labelled A to
D. The arrows indicate the different acquisition directions, and the overlapping parts with higher fold appear clearly. Figure 13 (b) displays the computed total space depth error bars for the top chalk horizon. One can observe a clear correlation between the illumination map and the total space depth error bars: the latter are smaller in overlap areas where the angle diversity (dip and azimuth) of raypaths is larger. On the other hand, lower-fold areas such as the rig zone inside survey C show relatively higher total space error bars correlated with the poorer angle diversity of raypaths and the reduced illumination. We can also observe much larger error bars on poorly illuminated survey edges. So, total space error bars highlight the combined effects of the acquisition fold and of the effective angle diversity that is in particular sensitive to structural complexities. These error bars are fully qualitative but give an information related to illumination issues that can be useful as a complementary QC. Indeed, they allow to better identify where computing error bars is meaningless, here mostly on the edges, this information being “attenuated” in the resolved space error bars as illustrated by the next example.

Figure 14 (a) shows total horizon depth error bars. Again, they highlight the acquisition illumination variations, allow to clearly identify where computing error bars is meaningless, again mostly on the edges. This information is strongly “attenuated” in the corresponding resolved space depth error bars, figure 14 (b). Within the area where computing resolved space error bars make sense, the resolved space error bars give a more detailed hierarchy, correlated to the uncertainty in the tomography discrimination power. Figure 14 (b) highlights the larger resolved space error bars in steeply dipping parts of the top Chalk horizon located below velocity features in the overburden. Comparing with figure 14 (c), we can observe the correlation between the velocity features in the overburden and the spatial distribution of the resolved space depth error bars at top chalk level. Also illustrated in the figure 4 in Messud et al. (2017b), resolved space error bars correlate very well with steeply dipping flanks or faults, and are stronger when shooting and dipping/fault plane directions are parallel, thus confirming that shooting along the dip direction is better for resolution than shooting strike. The resolved space error bars are more quantitative and useful for further use in decision-making and risk mitigation.

In these examples, horizon error bars were computed by map migration in model perturbations. However, as discussed in section “Specific to non-linear slope tomography...”, non-linear slope tomography allow computing migration volumetric error bars by full kinematic migration (using all offsets) of the invariant picks in model realizations. Figure 15 shows an example of migration volumetric depth error bars for the resolved space, extracted on vertical sections and along a horizon. These error bars exhibit layered and velocity-correlated variations having longer spatial wavelengths that the horizon error bars, as the full-offset range (not only zero-offset) is considered in the kinematic migrations. The advantage of the volumetric error bars is that they make it possible to track and understand the buildup of positioning uncertainties in the overburden and in-between horizons.

Figure 16 shows, for first 46 resolved space perturbations, the non-linearly computed tomography cost functions. Almost iso-cost, i.e. equi-probable, perturbations were generated. We observe limited variations around the average of the cost function values of the perturbed models, meaning that the Gaussian and linear hypothesis assumed in the analysis is appropriate. Again, we see that no spurious perturbations related to too large variation of the cost function were generated. It is not necessary to have an additional step discarding spurious perturbations with the method discussed here (whereas it may be necessary with
the method of section “Sampling a normal distribution...”).

Let us now illustrate the integration of structural uncertainties into a downstream gross-rock volume (GRV) calculation workflow on a second seismic dataset. In a conventional stochastic approach, structural uncertainties are known at well locations and inferred elsewhere using variogram models. The presented tomography-based method allows more control between wells and provides a realization-based way of assessing the positioning of reservoir boundaries. Figure 17 illustrates key milestones in the workflow which breaks down as follows:

- Estimating the tomography maximum-likelihood velocity model and computing target horizon error bars (resolved space) tuned to observed mis-ties at some well locations (Figure 17a),

- For the Top and Base reservoir horizons, describing the channel system of interest, calibrating horizon realizations to well markers (Figure 17b). By doing so, uncertainties between wells are reflected by the spatial variations of horizon depth error bars derived from the tomographic operator.

- Estimating spill point depending on closure assumption for various horizon realizations. Figure 17b shows three fault-driven types of closure: “four-way” closure made of dip or channel limits, “three-way” and “two-way” closures add one or two sealed bounding faults.

- Calculating GRV from Base and Top reservoir down to spill point closure level, with all these elements being affected by the error bars in the migrated domain and along well paths.

- Assessing, for each identified prospect, the closure probability map made from all plausible horizon realizations and defining the chance of finding closure above the measured spill point. Figure 17c shows the probability map for one prospect, demonstrating the presence of robust structural closures in the same channel system.

This example emphasizes the importance of structural uncertainties for providing error bars between well locations and allowing the generation of corresponding horizon realizations.
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Figure 13: Top Chalk horizon. (a) Illumination map with shooting direction of each survey indicated by the direction of the associated arrow. (b) Total space horizon depth error bars. Modified from Messud et al. (2017a); see the article for more details.

Figure 14: Horizon depth error bars. (a) Total space, (b) resolved space and (c) velocity (Vp), extracted above an horizon. Modified from Messud et al. (2018); see the article for more details.
Figure 15: Migration volumetric depth error bars (resolved space), displayed on vertical sections and on a horizon.

Figure 16: North Sea field data. (a) Tomography cost function values for the final model (i.e. maximum-likelihood model) and the first 46 generated perturbed models among 500. (b) Cost function values after the first tomography pass and the final tomography pass are represented with a larger nonlinear scale. Modified from Reinier et al. (2017).
(a) Spatial variations of horizon depth error bars provide an assessment of the potential area with higher uncertainty. (b) Display of realizations of some key horizons superimposed on estimated velocity parameters.

(c), (d), (e) Examples of reservoir contours for different spill point closure definitions and for different GRV scenarios (minimum case scenario on the left, average case in the center and maximum case on the right for the four-way closure type). For each scenario, the blue isoline represents the four-way closure, the red represents the three-way and the pink represents the two-way closure.
We proposed an extension of previous industrial works on the computation of migration structural uncertainties and provided a unifying framework. Firstly, we estimated error bars from the statistical analysis of perturbed models obtained from the sampling of an equiprobable contour of the posterior pdf (related to a clear confidence level), not of the full pdf like in previous works. Secondly, we developed the application in the context of non-linear slope tomography, based on the inversion of invariant picks. In addition to the advantages in terms of accuracy and efficiency of the VMB (compared to standard tomography) it provides the possibility to assess the quality of the linear and Gaussian assumptions. It also allows us to compute volumetric migration positioning error bars (using kinematic migration of all invariant picks and not only map migration of horizons invariant picks). Thirdly, we proposed to work in the full model space, not in a preconditioned model space (with smaller dimensionality). Splitting the analysis into the resolved and unresolved tomography spaces, we argued that the resolved space uncertainties are to be used in further steps leading to decision-making and can be related to the output of methods that work in a preconditioned model space. The unresolved space uncertainties represent a qualitative byproduct of our method that reflects priors and the illumination, strongly highlighting the most uncertain gross areas. The latter can be useful for QCs. These concepts were illustrated on a synthetic dataset. Complementarily, the industrial viability of the method was illustrated on two field 3D datasets where emphasis was placed on the importance of horizon error bars, among others for GRV computation and uncertainty evaluation between well locations.

The presented approach can be applied to various E&P topics. Also, the approach can easily apply to FWI-derived models. Indeed, state-of-the-art workflows involve interleaved FWI and tomography passes, ending with a tomography pass. The corresponding tomography uncertainty analysis can naturally be performed to produce an estimate for FWI model kinematic-related uncertainties.
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APPENDIX A

THE BINOMIAL INVERSE THEOREM AND THE WOODBURY MATRIX IDENTITY

Consider an invertible (thus square) matrix $A$ of size $N \times N$. Consider a matrix $V$ of size $N \times K_1$, a matrix $\Lambda$ of size $K_1 \times K_2$, and a matrix $W$ of size $K_2 \times N$. If the square matrix $I_{K_1} + AWA^{-1}$ of size $K_1 \times K_1$ is invertible, we have the following identity, called binomial inverse theorem

$$(A + VA)^{-1} = A^{-1} - A^{-1}V(I_{K_1} + AWA^{-1})^{-1}AWA^{-1}. \tag{A-1}$$

If the matrix $\Lambda$ is invertible (thus square $K_1 = K_2 = K$), the previous expression can be reduced to the Woodbury matrix identity

$$(A + VA)^{-1} = A^{-1} - A^{-1}V(A^{-1} + WA^{-1})^{-1}WA^{-1}. \tag{A-2}$$

Consider $A = I_N$, $\Lambda$ square but not necessarily invertible, and $W = V^\dagger$ that satisfies $V^\dagger V = I_K$. If the matrix $I_K + \Lambda$ is invertible, the binomial inverse theorem gives

$$(I_N + V\Lambda V^\dagger)^{-1} = I_N - V(I_K + \Lambda)^{-1}\Lambda V^\dagger. \tag{A-3}$$

Using the binomial theorem again under same conditions, we obtain $(I_K + \Lambda)^{-1}\Lambda = I_K - (I_K + \Lambda)^{-1}$, and finally the following identity

$$(I_N + V\Lambda V^\dagger)^{-1} = I_N - VV^\dagger + V(I_K + \Lambda)^{-1}V^\dagger. \tag{A-4}$$

APPENDIX B

STANDARD-DEVIATIONS AND ERROR BARS

The multi-dimensional Gaussian posterior pdf, equation 18, can be rewritten (we do not consider the normalization factor here to simplify the notations without loss of generality, so that the maximum of the pdf is always 1)

$$\tilde{\rho}_M(\Delta m) = \exp \left[ -\frac{1}{2} \Delta m^\dagger \tilde{C}_M^{-1} \Delta m \right]$$

$$= \prod_{i=1}^{N_M} \tilde{\rho}_{Mi}(\Delta m), \tag{B-1}$$

where the posterior pdf for one model space node $i$ is defined by

$$\tilde{\rho}_{Mi}(\Delta m) = \exp \left[ -\frac{1}{2} \tilde{C}_{Mi}^{-1} \Delta m_i^2 \right] \exp \left[ -\frac{1}{2} A_i(\Delta m) \Delta m_i \right]$$

$$A_i(\Delta m) = \sum_{j=1}^{N_M} \tilde{C}_{Mij}^{-1} \Delta m_j. \tag{B-2}$$

$A_i(\Delta m) \Delta m_i$ may be negative and can be neglected only when the correlations $\tilde{C}_{Mij}^{-1}$ are sufficiently small.
Consider the models related to a given value \( a \in [0, 1] \) of the un-normalized posterior pdf for one model space node, i.e. \( \forall i = 1..N_M : \tilde{\rho}_M(\Delta m) = a \). In other terms, \( a \) denotes a percentage of the maximum of each \( \tilde{\rho}_M \). Using equation B-1, this leads to

\[
\tilde{\rho}_M(\Delta m) = a^{N_M},
\]

\[
\Delta m^\dagger \tilde{C}_M^{-1} \Delta m = 2 \ln(1/a) \times N_M.
\]  

(B-3)

This equation defines the set of model perturbations \( \Delta m \) related to a chosen \( a \) value for the posterior pdfs of each model space node \( \tilde{\rho}_M(\Delta m) \). Equation B-3 is equivalent to

\[
\sum_{i=1}^{N_M} \tilde{C}^{-1}_{Mii} \Delta m_i^2 + \sum_{i=1}^{N_M} A_i(\Delta m) \Delta m_i = 2 \ln(1/a) \times N_M.
\]  

(B-4)

In the general case, one has to resolve the full equation B-4 to obtain the solutions \( \Delta m_i \). But if \( |\sum_{i=1}^{N_M} A_i(\Delta m) \Delta m_i| << |\sum_{i=1}^{N_M} \tilde{C}^{-1}_{Mii} \Delta m_i^2| \), the solutions become

\[
\Delta m_i = \pm \tilde{C}^{1/2}_{Mii} \sqrt{2 \ln(1/a)},
\]  

(B-5)

i.e. error bars are then related to the posterior standard deviations \( \tilde{C}^{1/2}_{Mii} \). Error bars become equal to \( \pm \tilde{C}^{1/2}_{Mii} \) if we select \( a = 0.6 \), that is related to a confidence probability \( P(a) = 68.3\% \) (computing the correctly normalized integral of \( \tilde{\rho}_M(\Delta m) \) inside the corresponding hyper-ellipsoid).
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