LONG-RUN RISK SENSITIVE IMPULSE CONTROL

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Abstract. In this paper we consider long-run risk sensitive average cost impulse control applied to a continuous-time Feller-Markov process. Using the probabilistic approach, we show how to get a solution to a suitable continuous-time Bellman equation and link it with the impulse control problem. The optimal strategy for the underlying problem is constructed as a limit of dyadic impulse strategies by exploiting regularity properties of the linked risk sensitive optimal stopping value functions.

Key words. Impulse control, Bellman equation, risk sensitive control, multiplicative Poisson equation, risk sensitive criterion

AMS subject classifications. 93E20, 49N25, 93C10, 60J25

1. Introduction. In this paper we consider long-run risk sensitive impulse control problem

\[
\limsup_{T \to \infty} \frac{1}{T} \ln \mathbb{E}(x,V) \left[ \exp \left( \int_0^T f(X_s) ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq T\}} c(X_{\tau_i}, \xi_i) \right) \right],
\]

where the Feller-Markov process \(X = (X_t)\) starting in \(x\) is controlled by impulse strategies \(V\) for a reward function \(f\) and a shift-cost function \(c\). The impulse strategies \(V = (\tau_i, \xi_i)_{i \in \mathbb{N}}\) are described by the sequences \((\tau_i)\) and \((\xi_i)\) indicating impulse times and after-impulse states of the process, respectively, and \(X_{\tau_i}\) denotes the state of \(X\) right before the impulse. Hence, up to time \(\tau_1\) the process \(X\) evolves according to its usual dynamics, then it is shifted to \(\xi_1\) and starts its evolution again. Moreover, we assume that \(f\) and \(c\) are continuous and bounded, while shifts take values in a compact set. Expectation \(\mathbb{E}(x,V)\) is defined on a probability space corresponding to the controlled process; see Section 2 for details. We refer to \([13, 16, 19]\) where similar framework has been studied.

The main aim of this paper is to find optimal control that minimise (1.1). The optimal strategy is constructed as a limit of dyadic impulse strategies by exploiting results from \([16]\) and \([6]\), i.e. by combining discrete time existence results that are based on the span-contraction approach with regularity properties of risk sensitive optimal stopping value functions. More explicitly, using change of measure technique based on the Multiplicative Poisson Equation (MPE) solution, we establish a link between impulse control Bellman equation and the optimal stopping problem considered in \([6]\). As expected, the strategy is characterised by relation between optimal values of the dyadic problems and the type of semigroup associated with the underlying Markov process. To get the characterisation result, we show that the finite time horizon equivalent of (1.1) could be rewritten as a limit of problems in which dyadic strategies with finite number of impulses are considered; this could be of independent interest to the reader.

The risk sensitive control could be considered as a non-linear extension of the classical risk-neutral case; see \([23, 13]\) and references therein. In particular, its application to finance have been extensively studied in the literature; see e.g. \([3, 9, 15]\).

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While impulse control is among the most popular forms of control, its coverage in the risk-sensitive case is limited. In particular, the application of standard methods from the risk-neutral case often leads to difficult problems involving quasi variational inequalities and differential equations techniques; see [2, 10, 4, 1] and references therein. In this paper we follow the probabilistic approach and extend the results from [18], where the risk-neutral optimal stopping problem is linked with impulse control; see also [14, 11, 12, 5, 7].

This paper is organised as follows: In Section 2 we establish the framework and the linked notation that will be used throughout the paper. Section 3 discusses the dyadic case; this section incorporates the results from [16]. In Section 4 we consider finite time horizon version of the problem. Next, in Section 5 we solve problem (1.1); Theorem 5.1 and Theorem 5.3 might be seen as the main contribution of this paper.

2. Preliminaries. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a continuous time filtered probability space. We assume that $\mathbb{F} := (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{T} := \mathbb{R}_+$, $\mathcal{F}_0$ trivial, $\mathcal{F} = \bigcup_{t \in \mathbb{T}} \mathcal{F}_t$, and the usual conditions are satisfied. We use $X := (X_t)$ to denote a continuous-time standard Markov process taking values in a locally compact separable metric space $E$ endowed with a metric $\rho$ and Borel $\sigma$-field $\mathcal{E}$; see Definition 4 in [20, Section 1.4]. Throughout the paper, $C(E)$ denotes the set of continuous and bounded functions on $E$, while $C_0(E) \subset C(E)$ denotes functions that are additionally vanishing at infinity. We assume that $X$ satisfies the $C_0$-Feller property, i.e. the transition semigroup of $X$ transforms $C_0(E)$ into itself; we refer to [6, Proposition 2.1] for a discussion of properties implied by the Feller property.

As in (1.1), with slight abuse of notation, we also use $X$ to denote a controlled process. For any $V = (\tau_i, \xi_i)_{i \in \mathbb{N}}$, we assume that the sequence of stopping times $(\tau_i)$ is increasing while the shifts $(\xi_i)$ are adapted to the filtration and take values in a compact set $U \subseteq E$, i.e. that $\xi_i \in \mathcal{F}_\tau_i$-measurable and $\xi_i \in U$, for any $i \in \mathbb{N}$. For brevity, we use $\mathcal{V}$ to denote the set all (admissible) control strategies; see [16] for details. For any $x \in E$ and $V \in \mathcal{V}$ we consider controlled process probability space that is constructed following the logic from [17]. In a nutshell, we consider a countable product of canonical spaces of càdlàg functions with values in $E$, and with the inductively defined filtration. For $t \in [\tau_i-1, \tau_i)$ we set $X_t = x_{i-1}^t$, where $x_{i-1}^t$ corresponds to $(i - 1)$th coordinate of the canonical process; see [21, Section 2] and [13] for more details. For clarity, we use $\mathbb{P}_{(x,V)}$ and $\mathbb{E}_{(x,V)}$ to denote the probability measure and expectation operator corresponding to particular choice of $x \in E$ and $V \in \mathcal{V}$. For brevity, we also use $\mathbb{E}_x$ and $\mathbb{P}_x$ in reference to the uncontrolled process dynamics.

Our aim is to maximise the long-run version of the risk sensitive criterion with negative (risk averse) parameter $\gamma < 0$; see [16] for details. After risk aversion parameter standardisation, our objective is to minimise the functional

$$\begin{equation}
J(x, V) := \limsup_{T \to \infty} \frac{1}{T} \ln \mathbb{E}_{(x,V)} \left[ \exp \left( \int_0^T f(X_s)ds + \sum_{i=1}^\infty 1_{\{\tau_i \leq T\}} c(X_{\tau_i^-}, \xi_i) \right) \right],
\end{equation}$$

where $x \in E$ is the starting point, $f : E \to \mathbb{R}$ and $c : \mathbb{E} \times U \to \mathbb{R}$ are the (normalised) reward and shift-cost functions, $X_{\tau_i^-}$ is the state of the process right before the $i$th impulse, and $V \in \mathcal{V}$.

Let $\mathbb{T}$ denote the family of almost surely finite stopping times taking values in $\mathbb{T}$, and let $\mathbb{T}_m \subset \mathbb{T}$ denote the dyadic stopping times defined on the time-grid \{0, $\delta_m, 2\delta_m, \ldots$\}, where $\delta_m := (1/2)^m$ and $m \in \mathbb{N}$. By $\mathcal{V}_m \subset \mathcal{V}$ we denote the family of impulse control strategies with stopping times restricted to $\mathbb{T}_m$. 


In order to solve (2.1), we show the existence of a function \( w \in C(E) \) and a constant \( \lambda \in \mathbb{R} \) that satisfy Bellman equation

\[
e^{w(x)} = \inf_{\tau} \mathbb{E}_x \left[ e^{\int_0^\tau (f(X_s) - \lambda) ds + Mw(X_\tau)} \right],
\]

where the operator \( M : C(E) \to C(E) \) is defined as

\[
Mw(x) := \inf_{\xi \in U} (c(x, \xi) + w(\xi)).
\]

Of course, one needs additional assumptions imposed on the process \( X \) and related functions in order to have a proper non-degenerate solution to (2.2) that could be linked to the original problem (2.1). Our approach to solve (2.2) is based on the dyadic approximations for the linked (discrete) stopping time problem; see [16].

Based on [16] and [19], we introduce the following set of assumptions:

(A.1) (Reward function constraints.) The function \( f : E \to \mathbb{R} \) is continuous and bounded.

(A.2) (Cost function constraints.) The function \( c : E \times U \to \mathbb{R} \) is continuous and bounded. Moreover, we assume that the cost function is bounded away from zero by \( c_0 > 0 \) and satisfies triangle inequality, i.e.

\[
c_0 \leq c(x, y) \leq c(x, z) + c(z, y), \quad x, y, z \in U.
\]

Also, \( c \) satisfies uniform limit at infinity condition, i.e.

\[
\lim_{\|x\|, \|y\| \to \infty} \sup_{\xi \in U} |c(x, \xi) - c(y, \xi)| = 0.
\]

(A.3) (Process ergodicity.) For any \( t > 0 \) there exists a constant \( a_t > 0 \) and a probability measure \( \nu_t \), such that \( \nu_t(U) > 0 \) and

\[
\inf_{x \in E} \mathbb{P}_x(X_t \in A) \geq a_t \nu_t(A), \quad A \in \mathcal{E}.
\]

(A.4) (Existence of MPE solution.) There exists \( v \in C(E) \) satisfying

\[
e^{v(x)} = \mathbb{E}_x \left[ e^{\int_0^t (f(X_s) - r(f)) ds + v(X_t)} \right], \quad \forall t \geq 0,
\]

where \( r(f) := \inf_{t>0} \frac{1}{t} \ln \left( \sup_{h \in C(E)} \frac{\|P^f_t h\|}{\|h\|} \right) \) is the type of the semigroup \( P^f_t \) given by

\[
P^f_t h(x) := \mathbb{E}_x \left[ e^{\int_0^t f(X_s) ds} h(X_t) \right], \quad h \in C(E).
\]

Assumption (A.1) is a classic reward function constraint and directly reflects assumption (A.1) in [16].

Assumption (A.2) imposes multiple restrictions on the cost function and partly reflects assumption (A.2) in [16]. For completeness, let us provide exemplary class of functions which satisfy all conditions from (A.2). Let \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) be bounded, continuous, non-decreasing and subadditive, i.e. \( h(x + y) \leq h(x) + h(y), \) \( x, y \geq 0 \). Then, the function given by

\[
c(x, \xi) := h(\rho(x, \xi)) + c_0,
\]
where $\rho$ is the underlying $E$-space metric, satisfies assumption (A.2). For example, we can set $h_1(x) = x \wedge K$, $h_2(x) = \frac{x}{x+1}$, or $h_3(x) := \frac{1}{1+x}$.

Assumption (A.3) corresponds to assumption (A.4) in [16] and refers to ergodic properties of the underlying uncontrolled process $X$; note that in our setting assumption (A.3) from [16] is trivially satisfied, as we act in the bounded framework.

Assumption (A.4) requires the existence of a solution to the Multiplicative Poisson Equation (MPE) which is tightly linked to the problem (2.2). Exemplary sufficient condition for the existence of $v$ include existence of density $p_t$ with respect to some probability $\nu_t$, for $t > 0$, such that

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x, y)\nu_t(dy), \quad A \in \mathcal{E},$$

and $0 < a_t \leq p_t(x, y) \leq A_t$, for $0 < a_t \leq A_t < \infty$ and $x, y \in E$; we refer to [22, Corollary 2] or [19, Lemma 3.2] for details. In particular, this condition is satisfied for regular reflected diffusions, see [8, Remark 2.1] and references therein.

Before we proceed, let us provide further comment about the implications of Assumption (A.4). In fact, it allows us to apply change of measure technique that substantially simplifies problem (2.2) by reducing it to the optimal stopping setting considered in [6]. Assuming (A.4) holds, for any $x \in E$ we can define the measure $\mathbb{Q}_x$ via a Radon-Nikodym derivative

$$d\mathbb{Q}_x|_t := Y_t(x)d\mathbb{P}_x|_t, \quad t \geq 0,$$

where

$$Y_t(x) := e^{-\nu(x)}e^{\int_0^t(f(X_s) - \lambda X_s)ds + v(X_t)}, \quad t \geq 0;$$

note that it can be easily shown that $Y_t(x)$ is a martingale with $\mathbb{E}_x[Y_t(x)] = 1$. For brevity, by analogy to $\mathbb{E}_x$, we use $\mathbb{E}_x^\mathbb{Q}$ to denote the expectation with respect to $\mathbb{Q}_x$. By introducing measure $\mathbb{Q}_x$ we are able to reduce the problems of the type of (2.2) to the situation, where $(f(\cdot) - \lambda)$ is replaced by a constant. This is based on identity

$$\mathbb{E}_x \left[ e^{\int_0^t(f(X_s) - \lambda X_s)ds + G(X_T)} \right] = e^{v(x)} \mathbb{E}_x^\mathbb{Q} \left[ e^{r(f(\cdot) - \lambda \tau + G(X_T) - v(X_T))} \right],$$

that holds for any $x \in E, \lambda \in \mathbb{R}, G \in C(E)$, and $\tau \in \mathcal{T}$.

3. Dyadic impulse control. In this section we study the dyadic version of the optimal impulse control problem introduced in (2.1). Recall that for any $m \in \mathbb{N}$ and time-step $\delta_m = \frac{1}{2^m}$, the corresponding family of dyadic stopping times taking values in \{0, $\delta_m, 2\delta_m, \ldots\} is denoted by $\mathcal{T}_m$. For any $m \in \mathbb{N}$, the dyadic version of the Bellman equation (2.2) is given by

$$e^{w_m(x)} = \inf_{\tau \in \mathcal{T}_m} \mathbb{E}_x \left[ e^{\int_0^{\tau_m}(f(X_s) - \lambda_m)ds + Mw_m(x_{\tau_m})} \right],$$

where $w_m \in C(E), \lambda_m \in \mathbb{R},$ and $M$ is defined in (2.3). In fact, due to the dyadic nature of the problem, one could consider the associated one-step equation given by

$$e^{w_m(x)} = \min \left( \mathbb{E}_x \left[ e^{\int_0^m(f(X_s) - \lambda_m)ds + w_m(X_{\delta_m})} \right], e^{Mw_m(x)} \right).$$

We now show that, under suitable assumptions, solution to (3.2) exists and solves the dyadic version of the optimal control problem, i.e.

$$\inf_{V \in \mathcal{V}_m} J(x, V),$$
where $V_m$ is the set of admissible control strategies defined on the time-grid spanned by $\delta_m$; note we will link the solution of (3.2) to (3.1) in Proposition 3.2.

**Theorem 3.1.** Under Assumptions (A.1)–(A.3), for any $m \in \mathbb{N}$ there exists a unique (up to an additive constant) function $w_m \in C(E)$ and a constant $\lambda_m \in \mathbb{R}$ satisfying (3.2). Moreover, $\lambda_m$ is the optimal value of the problem (3.3).

*Proof.* The proof relies on the results presented in [16]; it should be noted that while in [16, Equation 2.5] the maximisation problem is considered, it directly corresponds to standardised problem (3.3) due to negative sign of the risk aversion parameter in the entropic utility measure. Also, note that in our setting all assumptions listed in [16, Section 2] are satisfied. Please recall that [16, Assumption A.3] is naturally satisfied in the bounded framework, while other assumptions have a direct link to assumptions stated in this paper; see Section 2 for details.

First, from [16, Proposition 3.4] we get that there exists a unique (up to an additive constant) function $w_m \in C(E)$ and a constant $\lambda_m \in \mathbb{R}$ satisfying equation

\[
e^{w_m(x)} = \min \left( E_x \left[ e^{\int_0^t (f(X_s) - \lambda_m)ds + w_m(X_{s\delta})} \right] , \right.
\]

\[\left. \inf_{\xi \in U} e^{c(x,\xi)} E_x \left[ e^{\int_0^t (f(X_s) - \lambda_m)ds + w_m(X_{s\delta})} \right] \right).\]

Second, let us show that

\[
\inf_{\xi \in U} e^{c(x,\xi)} E_x \left[ e^{\int_0^t (f(X_s) - \lambda_m)ds + w_m(X_{s\delta})} \right] = e^{Mw_m(x)}.
\]

For brevity, we use notation

\[Z(x) := E_x \left[ e^{\int_0^t (f(X_s) - \lambda_m)ds + w_m(X_{s\delta})} \right], \quad x \in E.\]

From (3.4), we get $Z(x) \geq e^{w_m(x)}$ for any $x \in E$. Therefore, for any $x \in E$, we get

\[
\inf_{\xi \in U} e^{c(x,\xi)} Z(\xi) \geq \inf_{\xi \in U} e^{c(x,\xi)} e^{w_m(\xi)}.
\]

On the other hand, due to (2.4) and (3.4), we get

\[
\inf_{\xi \in U} e^{c(x,\xi)} Z(\xi) = \inf_{\xi \in U} e^{c(x,\xi)} Z(\xi) \wedge \inf_{\xi \in U} e^{c(x,\xi)} Z(\xi)
\]

\[\leq \inf_{\xi \in U} \left( e^{c(x,\xi)} Z(\xi) \wedge \inf_{\xi \in U} e^{c(x,\xi)} e^{c(\xi,\xi)} Z(\xi) \right)
\]

\[= \inf_{\xi \in U} e^{c(x,\xi)} \left( Z(\xi) \wedge \inf_{\xi \in U} e^{c(\xi,\xi)} Z(\xi) \right)
\]

\[= \inf_{\xi \in U} e^{c(x,\xi)} e^{w_m(\xi)},\]

which proves (3.5). Combining (3.4) and (3.5) we know that $\lambda_m$ and $w_m$ are solutions to (3.2).

Finally, using [16, Proposition 4.3] we can link the constant $\lambda_m$ from (3.2) with the optimal value of the dyadic control problem (3.3), which concludes the proof. □

Next, we link the solution of (3.2) with the type of the semigroup $r(f)$; see (A.4) for details.
Proposition 3.2. For any $m \in \mathbb{N}$ it follows that $\lambda_m \leq r(f)$. Moreover,
1. if $\lambda_m = r(f)$, then the no impulse strategy is optimal for $\mathbb{V}_m$;
2. if $\lambda_m < r(f)$, then $w_m$ defined in (3.2) satisfies (3.1).

Proof. Let us fix $m \in \mathbb{N}$. First, we show that for any $x \in E$, we get $\lambda_m \leq r(f)$. Using [22, Proposition 1], the type of the semigroup may be rewritten as

$$r(f) = \lim_{T \to \infty} \frac{1}{T} \ln \left( \sup_{h \in C(E)} \frac{\|P_T^f h\|}{\|h\|} \right).$$

Thus, for any $x \in E$ and no impulse strategy $V_0 \in \mathbb{V}$, we get

$$J(x, V_0) = \limsup_{T \to \infty} \frac{1}{T} \ln |P_T^f 1|(x) \leq \limsup_{T \to \infty} \frac{1}{T} \ln \sup_{x \in E} |P_T^f 1|(x) \leq r(f).$$

Using Theorem 3.1, we know that for $\lambda_m$ we get

$$\lambda_m = \inf_{V \in \mathbb{V}_m} J(x, V) \leq J(x, V_0) \leq r(f),$$

which concludes the proof of $\lambda_m \leq r(f)$.

Now, if $\lambda_m = r(f)$, then from (3.7) we get $r(f) = J(x, V_0)$, which implies that the no impulse strategy is optimal.

Next, let us assume that $\lambda_m < r(f)$. To show that $w_m$ defined in (3.2) satisfies (3.1) we use the change of measure technique based on the Multiplicative Poisson Equation. In this way we can replace the term $(f(\cdot) - \lambda_m)$ in (3.2) by some positive constant and use the results from [6, Section 3]. Recalling $v$ from (2.6), $w_m$ from (3.2), and $\mathbb{Q}_x$ from (2.8), we set

$$u_m(x) := w_m(x) - v(x) + \|w_m\| + \|v\| + \|Mw_m\|,$n, m

$$G_m(x) := Mw_m(x) - v(x) + \|w_m\| + \|v\| + \|Mw_m\|,$n, m

$$S^m h(x) := \mathbb{E}_x^{\mathbb{Q}} \left[ e^{(r(f) - \lambda_m)\delta_m} h(X_{\delta_m}) \right] \wedge e^{G_m(x)} , \ h \in C(E).$$n, m

Using (2.9) and (3.2) we get $S^m e^{u_m} = e^{u_m}$. Also, noting that $G_m \in C(E), G_m(\cdot) \geq 0$, $r(f) - \lambda_m > 0$, and using [6, Proposition 3.3], we get

$$e^{u_m(x)} = \inf_{\tau \in \mathbb{F}_m} \mathbb{E}_x^{\mathbb{Q}} \left[ e^{\tau(r(f) - \lambda_m) + G_m(X_{\tau})} \right].$$n, m

Recalling (2.9) and (3.8) we conclude that $w_m$ satisfies (3.1).

4. Finite horizon impulse control. Before we present how to extend dyadic results presented in Section 3 to generic continuous time setting, we need to show some auxiliary results for the finite time horizon version of (2.1). Namely, in this section we consider

$$J_T(x, V) := \ln \mathbb{E}_{(x, V)} \left[ \exp \left( \int_0^T f(X_s) ds + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq T\}} c(X_{\tau_i}, \xi_i) \right) \right],$$n, m

where $T \geq 0$, $x \in E$, $V \in \mathbb{V}$, and the remaining notation is aligned with (2.1); note that for simplicity we dropped the normalising constant $1/T$ in (4.1). For any $n, m \in \mathbb{N},$ let $\mathbb{V}_n \subset \mathbb{V}$ and $\mathbb{V}_m \subset \mathbb{V}_m$ denote the families of impulse control strategies and $\delta_m$-dyadic impulse control strategies that have at most $n$ impulses, respectively.
The main goal of this section is to show that the optimal value of (4.1) could be approximated using strategies from $V^*_{m,n}$; see Proposition 4.2. This will be later used in the proof of Theorem 5.3.

Let us fix $T \in \mathbb{N}$ and define the family $(w^n)_{n \in \mathbb{N}}$ of functions $w^n: [0,T] \times E \to \mathbb{R}$, which are given recursively by

$$w^0(t,x) := \ln E_x \left( \exp \left( \int_0^{T-t} f(X_s)ds \right) \right),$$

(4.2) $$w^n(t,x) := \inf_{\tau \leq T-t} \ln E_x \left[ \exp \left( \int_0^{\tau} f(X_s)ds + \tilde{M}w^{n-1}(\tau,X_{\tau}) \right) \right],$$

for $\tilde{M}: C([0,T] \times E) \to C([0,T] \times E)$ being the operator defined as

$$\tilde{M}h(t,x) = \min_{\xi \in U} (c(x,\xi) + h(t,\xi), h(t,x)).$$

Similarly, for any $m \in \mathbb{N}$, we define the family $(w^m_n)_{n \in \mathbb{N}}$ as

$$w^0_m(t,x) := \ln E_x \left( \exp \left( \int_0^{T-t} f(X_s)ds \right) \right),$$

(4.3) $$w^n_m(t,x) := \inf_{\tau \leq T-t} \ln E_x \left[ \exp \left( \int_0^{\tau} f(X_s)ds + \tilde{M}w^{n-1}_m(\tau,X_{\tau}) \right) \right], \quad n \in \mathbb{N}.$$

Before we link $w^n$ and $w^m_n$ to (4.1), let us state an auxiliary result that may be seen as an extension of Proposition 4.2 and Proposition 4.3 from [6]. To ease the notation we set

$$v^n(t,h,x) := \inf_{\tau \leq T-t} \ln E_x \left[ \exp \left( \int_0^{\tau} f(X_s)ds + \tilde{M}w^{n-1}(\tau+h,X_{\tau}) \right) \right],$$

(4.4) where $n \in \mathbb{N}_+$, $t \in [0,T]$, $h \in [0,t]$, and $x \in E$.

**Lemma 4.1.** For any $n \in \mathbb{N}$, the map $(t,x) \mapsto w^n(t,x)$ is jointly continuous. Furthermore, if $n \geq 1$, then the optimal stopping time for $w^n(t,x)$ is given by

$$\tau^n_t := \inf \left\{ s \geq 0 : v^n(t+s,s,X_s) = \tilde{M}w^{n-1}(s,X_s) \right\},$$

(4.5)

**Proof.** For transparency, we split the proof into two steps: (1) proof of joint continuity of $(t,x) \mapsto w^0(t,x)$; (2) proof of joint continuity of $(t,x) \mapsto w^n(t,x)$ and optimality of $\tau^n_t$, for $n \geq 1$.

**Step 1.** We prove that $(t,x) \mapsto w^0(t,x)$ is jointly continuous. Noting that

$$|\ln z - \ln y| \leq \frac{1}{\min(z,y)} |z - y| \quad \text{and} \quad |e^z - e^y| \leq e^{\max(z,y)} |z - y|,$$

for $z, y > 0$ and $z, y \in \mathbb{R}$, respectively, and using boundedness of $f$, we get

$$|w^0(t,y) - w^0(s,y)| \leq e^{|t-s|\|E_0\|} \left| e^{\int_0^{T-t} f(X_u)du} - e^{\int_0^{T-s} f(X_u)du} \right| \leq L|t-s|,$$
for $t, s \in [0, T]$, $y \in E$, and $L := e^{2T\|/\|f\|}$. By Part 4 of [6, Proposition 2.1] we get that the map $x \mapsto w^0(t, x)$ is continuous for any fixed $t \in [0, T]$. Thus, for any sequence $((tk, x_k))_{k \in \mathbb{N}}$ converging to $(t, x)$, we get

$$|w^0(tk, x_k) - w^0(t, x)| \leq |w^0(tk, x_k) - w^0(t, x_k)| + |w^0(t, x_k) - w^0(t, x)| \to 0,$$

as $k \to \infty$.

**Step 2.** We show that $(t, x) \mapsto w^n(t, x)$ is jointly continuous and $\tau^n_t$ is the optimal stopping time for $w^n(t, x)$, for $n \geq 1$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space, where $\Omega := [0, \infty) \times \Omega$, the $\sigma$-field is given by $\mathcal{F} := \mathcal{B}(0, \infty) \otimes \mathcal{F}$, and the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, \infty)}$, is defined as $\mathcal{F}_t := \mathcal{B}(0, \infty) \otimes \mathcal{F}_t$. Also, let $\mathcal{E} := [0, \infty) \times E$ and $\mathcal{E} := \mathcal{B}(0, \infty) \otimes \mathcal{E}$. Let us define the space-time process $\tilde{X} = (\tilde{X}_t)_{t \in [0, \infty)}$ by

$$X_t(s, \omega) := (t + s, X_t(\omega)), \quad t \geq 0, (s, \omega) \in [0, \infty) \times \Omega.$$  

Using standard argument one can show that $\tilde{X}$ is a standard Markov process on the filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ that takes values in $(\tilde{\mathcal{E}}, \tilde{\mathcal{E}})$ and satisfies $C_0(\tilde{\mathcal{E}})$-Feller property; the measure $\tilde{\mathbb{P}}$ is given via the kernel transition functions corresponding to the family $(\tilde{\mathbb{P}}_{(h,x)})_{(h,x)}$, where $\tilde{\mathbb{P}}_{(h,x)} := \delta_h \otimes \mathbb{P}_x$, $\delta_h$ denotes the Dirac measure, $h \in [0, \infty)$, and $x \in E$. Note that $\tilde{\mathbb{P}}_{(h,x)}$ might be linked to the distribution of $\tilde{X}$ starting from $(h,x)$. We refer to [20, Section 1.4.6] for a discussion of a similar homogenisation procedure.

For any $n \in \mathbb{N}$ let

$$\tilde{v}^n(t, h, x) := \inf_{\tau \leq T - t} \tilde{E}_{(h,x)} \left[ \exp \left( \int_0^T \tilde{g}(\tilde{X}_s)ds + \tilde{G}^n(\tilde{X}_s) \right) \right],$$

where $t \in [0, T], h \in [0, t], x \in E, \tilde{g}(h,x) := f(x)$ and $\tilde{G}^n(h, x) := \tilde{M}w^n-1(h, x)$.

To show joint continuity of $(t, x) \mapsto w^n(t, x)$ we use induction argument. For $n = 0$, the continuity of $(t, x) \mapsto w^0(t, x)$ was shown in Step 1. For $n \geq 1$ let us assume that that $(t, x) \mapsto w^{n-1}(t, x)$ is jointly continuous. Then, we get $\tilde{g} \in C(\tilde{\mathcal{E}})$ and $\tilde{G}^n \in C(\tilde{\mathcal{E}})$. Using Proposition 4.3, Remark 4.4, and Remark 4.5 from [6], we get that $(t, h, x) \mapsto \tilde{v}^n(t, h, x)$ is jointly continuous and the optimal stopping time for $\tilde{v}^n(t, h, x)$ is given by

$$\tau^n_t := \inf \left\{ s \geq 0 : \tilde{v}^n(t + s, \tilde{X}_s) = \exp(\tilde{G}^n(\tilde{X}_s)) \right\}.$$

Also, we get $w^n(t, h, x) = \ln \tilde{v}^n(t, h, x)$ and $\tilde{v}^n(t, 0, x) = w^n(t, x)$, which follows from definition of $\tilde{F}_{(h,x)}$. Thus, from joint continuity of $(t, h, x) \mapsto \tilde{v}^n(t, h, x)$, we get joint continuity of $(t, x) \mapsto w^n(t, x)$. Also, recalling (4.5), we get $\tilde{\tau}^n(0, \omega) = \tau^n_t(\omega)$ for any $\omega \in \Omega$. Noting that $\tilde{\tau}^n(0, \cdot)$ is the optimal stopping time for $w^n(t, x)$, we conclude the proof.

We are ready to state the main result of this section.

**Proposition 4.2.** For any $x \in E$ and $T \in \mathbb{N}$ it follows that

$$\lim_{n \to \infty} \lim_{m \to \infty} \inf_{V \in \mathcal{V}_m^n} J_T(x, V) = \inf_{V \in \mathcal{V}} J_T(x, V).$$
Proof. For transparency, we split the proof into four steps: (1) proof of the equalities \( w^n(0, x) = \inf_{V \in \mathcal{V}^n} J_T(x, V) \) and \( w^n_m(0, x) = \inf_{V \in \mathcal{V}^n_m} J_T(x, V) \) for any \( n, m, T \in \mathbb{N} \) and \( x \in E \); (2) proof of \( w^n_m(0, x) \to w^n(0, x) \) as \( m \to \infty \) for any \( n \in \mathbb{N} \) and \( x \in E \); (3) proof of \( \lim_{n \to \infty} \inf_{V \in \mathcal{V}^n} J_T(x, V) = \inf_{V \in \mathcal{V}^n} J_T(x, V) \) for any \( T \in \mathbb{N} \) and \( x \in E \); (4) proof of \( (4.7) \).

**Step 1.** Let \( n, m, T \in \mathbb{N} \) and \( x \in E \) be fixed. We show that
\[
w^n(0, x) = \inf_{V \in \mathcal{V}^n} J_T(x, V) \quad \text{and} \quad w^n_m(0, x) = \inf_{V \in \mathcal{V}^n_m} J_T(x, V).
\]
For brevity, we present the proof only for \( w^n \); the proof for \( w^n_m \) is analogous. Clearly, for \( n = 0 \), the claim is straightforward so we can assume that \( n \geq 1 \).

First, we show that \( w^n(0, x) = J_T(x, \hat{V}) \), where \( \hat{V} \in \mathcal{V}^n \) is some fixed strategy. To ease the notation, we set \( \tilde{c}(x, \xi) := 1_{\{x \neq \xi\}} c(x, \xi) \), \( x \in E \), \( \xi \in U \). Then, recalling that \( c(x, \xi) \geq c_0 > 0 \), we get \( \tilde{M} h(t, x) = \inf_{\xi \in U} (\tilde{c}(x, \xi) + h(t, \xi)) \). Recalling (4.4), for any \( i = 1, 2, \ldots, n \), let us define recursively\(^1\)
\[
\bar{\tau}_i := \inf \left\{ t \geq \bar{\tau}_{i-1} : v^{n-i+1}(t, t, X_t) = \tilde{M} w^{n-i}(t, X_t) \right\},
\]
where \( \bar{\tau}_0 = 0 \); note that \( \bar{\tau}_n \leq T \). Next, we set \( \hat{V} := (\bar{\tau}_i, \hat{\xi}_i)_{i=1}^n \in \mathcal{V}^n \), where
\[
\hat{\tau}_i := \inf \left\{ \hat{\tau}_j \geq \hat{\tau}_{j-1} : \inf_{\xi \in U} \left( \tilde{c}(X_{\hat{\tau}_j^{-}}, \xi) + w^{n-j}(\hat{\tau}_j, \xi) \right) < w^{n-j}(\bar{\tau}_j, X_{\bar{\tau}_j}) \right\},
\]
\[
\hat{\xi}_i := \arg \min_{\xi \in U} \left( \tilde{c}(X_{\bar{\tau}_i^{-}} \land T, \xi) + w^{n-i}(\bar{\tau}_i \land T, \xi) \right),
\]
is defined recursively for \( \hat{\tau}_0 = 0 \); we used convention \( \inf\{\emptyset\} = +\infty \). Note that \( (\hat{\tau}_i) \) is a modification of \( (\bar{\tau}_i) \) which takes into account only situation where the process is shifted. Recalling Lemma 4.1 and using recursive arguments combined with strong Markov property, we get
\[
w^n(0, x) = \ln \mathbb{E}_{(x, \hat{V})} \left[ \exp \left( \int_0^{\bar{\tau}_1} f(X_s) ds + \tilde{c}(X_{\bar{\tau}_1^{-}}, \hat{\xi}_1) + w^{n-1}(\bar{\tau}_1, \hat{\xi}_1) \right) \right]
\]
\[
= \ln \mathbb{E}_{(x, \hat{V})} \left[ \exp \left( \int_0^T f(X_s) ds + \sum_{i=1}^{n} \tilde{c}(X_{\bar{\tau}_i^{-}}, \hat{\xi}_i) \right) \right]
\]
\[
= \ln \mathbb{E}_{(x, \hat{V})} \left[ \exp \left( \int_0^T f(X_s) ds + \sum_{i=1}^{n} 1(\hat{\tau}_i \leq T) \tilde{c}(X_{\hat{\tau}_i^{-}}, \hat{\xi}_i) \right) \right],
\]
which concludes the proof of \( w^n(0, x) = J_T(x, \hat{V}) \).

Second, using similar argument one can show that \( w^n(0, x) \leq \inf_{V \in \mathcal{V}^n} J_T(x, V) \); see [19, Proposition 2.3] for details. Thus, \( w^n(0, x) = \inf_{V \in \mathcal{V}^n} J_T(x, V) \), which concludes the proof of this step.

**Step 2.** We show that \( w^n_m(0, x) \to w^n(0, x) \) as \( m \to \infty \) for any \( n \in \mathbb{N} \) and \( x \in E \). In fact, due to recursive nature of \( w^n \), we prove that \( w^n_m(t, x) \to w^n(t, x) \) as \( m \to \infty \) uniformly in \([0, T] \times \Gamma\), where \( \Gamma \subset E \) is a compact set.

\(^{1}\)For simplicity, we assume that there are unique minimisers \( \hat{\xi}_i \) and later in \( \tilde{\xi}_i \).
Let us fix compact set $\Gamma \subset E$ and proceed by induction. The claim for $n = 0$ is straightforward as $w^0_m = w^0$ for any $m \in \mathbb{N}$. Let $n \in \mathbb{N}$ and assume that the assertion holds for $w^n$; we show it for $w^{n+1}$. The proof is based on the argument used in [6, Lemma 4.1].

For any $m \in \mathbb{N}$, $t \in [0, T]$ and $x \in E$, we set
\[
h^{n+1}(t, x) := \exp \left( w^{n+1}(t, x) \right),
\]
Noting that $\min \left( h^{n+1}(t, x), h^{n+1}_m(t, x) \right) \geq \exp \left( -T(n+2) \| f \| \right)$ for any $m \in \mathbb{N}$, $t \in [0, T]$ and $x \in E$, and using inequality $|\ln y - \ln z| \leq \frac{1}{(\min(y, z)^2(y - z)}$, for $y, z > 0$, we get
\[
0 \leq w^{n+1}_m(t, x) - w^{n+1}(t, x) \leq e^{T(n+2) \| f \|} (h^{n+1}_m(t, x) - h^{n+1}(t, x)).
\] (4.8)

Thus, it is enough to show that $h^{n+1}_m(t, x) \to h^{n+1}(t, x)$ uniformly on $[0, T] \times \Gamma$. Before we do that, we need to introduce some auxiliary notation and results.

Consider any $t \in [0, T]$ and $x \in \Gamma$. Let $\varepsilon > 0$ and $\tau^{(\varepsilon, t, x)}$ be an $\varepsilon$-optimal stopping time for $h^{n+1}(t, x)$. For any $m \in \mathbb{N}$, we set
\[
\tau^{(\varepsilon, t, x)} := \pi_{[T - \varepsilon, \varepsilon]^2} \left( \sum_{j = 1}^{(T - \varepsilon)^2} \left\lfloor \frac{1}{\varepsilon} \right\rfloor \frac{j}{2m} \right)
\]
In the following, for brevity, we write $\tau$ and $\tau_m$ instead of $\tau^{(\varepsilon, t, x)}$ and $\tau_m^{(\varepsilon, t, x)}$, and use notation
\[
Z^n(\tau) := \exp \left( \int_0^T f(X_s) ds + \tilde{M}w^n(\tau, X_\tau) \right),
\]
\[
A^n_m(s, y) := |w^n_m(s, y) - w^n(s, y)|,
\]
\[
B^n_m(s, u, y, z) := \left| \tilde{M}w^n_m(s, y) - \tilde{M}w^n(u, z) \right|,
\]
\[
C^n(s, u, y, z) := \sup_{\xi \in U} |c(y, \xi) - c(z, \xi)| + \sup_{\xi \in U} |w^n(s, \xi) - w^n(s, \xi)| + |w^n(s, y) - w^n(u, z)|,
\]
for $m \in \mathbb{N}$, $s, u, \in [0, T]$, and $y, z \in E$. Note that, by Lemma 4.1, the function $C^n$ is jointly continuous and bounded. Moreover, by induction assumption, $A^n_m(s, y) \to 0$ as $m \to \infty$ uniformly in $[0, T] \times \Gamma$ where $\Gamma \subset E$ is compact. Also, for any $m \in \mathbb{N}$, $s, u, \in [0, T]$, and $y, z \in E$, we get
\[
B^n(s, u, y, z) \leq \sup_{\xi \in U} |c(y, \xi) - c(z, \xi)| + \sup_{\xi \in U} |w^n_m(s, \xi) - w^n_m(u, \xi)| + |w^n_m(s, y) - w^n(u, z)|
\]
\[
\leq C^n(s, u, y, z) + \sup_{v \in [0, T]} \sup_{\xi \in U} |w^n_m(v, \xi) - w^n(v, \xi)| + A^n_m(s, y).
\] (4.9)

Next, recalling that $X$ is $C_0$-Feller and using Part 2 of [6, Proposition 2.1] we can find $R > 0$ such that
\[
\sup_{x \in \Gamma} \sup_{s \in [0, T]} \rho(X_s, x) > R \leq \varepsilon.
\] (4.10)

Let $B := \{x \in E: \rho(x, \Gamma) \leq R + 1\}$. Using induction assumption and compactness of $U$, we may find $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$ we get
\[
\sup_{s \in [0, T]} \sup_{y \in B \cup U} A^n_m(s, y) \leq \varepsilon.
\] (4.11)
Also, noting that $C^n$ is uniformly continuous on $[0, T] \times B \times B$ and $C^n(s, s, y, y) = 0$ for any $s \in [0, T]$ and $y \in E$, we may find $r_1 > 0$ and $m_1 \in \mathbb{N}$ such that for any $m \geq m_1$ we get

\begin{equation}
(4.12) \quad \sup_{s, u \in [0, T]} \sup_{\{s - u \leq 2^{-m}\}} C^n(s, u, y, z) \leq \varepsilon.
\end{equation}

Let $r := \min(r_1, \frac{1}{2})$. Using Part 3 of [6, Proposition 2.1], we may find $m_2 \in \mathbb{N}$ such that for any $m \geq m_2$ we get

\begin{equation}
(4.13) \quad \sup_{x \in B} \mathbb{P}_x \left[ \sup_{s \in [0, 2^{-m}]} \rho(X_s, x) \geq r \right] \leq \varepsilon \quad \text{and} \quad \|f\| 2^{-m} \leq \varepsilon.
\end{equation}

Finally, it is useful to note that $\|w_h\| \leq T\|f\|$ and $\|w_h^k\| \leq T\|f\|$, for $k, m \in \mathbb{N}$; these follow from Step 1 and the fact that the cost of no impulse strategy is bounded from above by $T\|f\|$.

Using \((4.9)-(4.11)\), for any $t \in [0, T]$, $x \in \Gamma$ and $m \geq \max(m_0, m_1, m_2)$, we get

\begin{equation}
(4.14) \quad 0 \leq h^{n+1}_m(t, x) - h^{n+1}_m(t, x)
\end{equation}

\begin{equation}
\leq \mathbb{E}_x \left[ \exp \left( \int_0^{\tau_m} f(X_s)ds + \tilde{M}w^n_m(\tau_m, X_{\tau_m}) \right) \right] - \mathbb{E}_x \left[ Z^n(\tau) \right] + \varepsilon.
\end{equation}

Noting that $Z^n(\tau) \leq e^{\|e\|^{2+T}\|f\|}$ and $\|B^n_m\| \leq 2\|e\| + 4T\|f\|$ for any $m \in \mathbb{N}$, and using \((4.9)-(4.11)\), for any $t \in [0, T]$, $x \in \Gamma$ and $m \geq \max(m_0, m_1, m_2)$, we get

\begin{equation}
(4.15) \quad \mathbb{E}_x \left[ Z^n(\tau)e^{B^n_m(\tau_m, \tau_m, X_{\tau_m})} \right] \leq e^{2}\mathbb{E}_x \left[ 1_{\{X_{\tau} \leq R\}} Z^n(\tau)e^{A^n_m(\tau_m, X_{\tau_m})} + A^n_m(\tau_m, X_{\tau_m}) \right] + \varepsilon K_1.
\end{equation}

where $K_1 := e^{2\|e\|^{2+T}\|f\|}$. Let $D := \{ \sup_{s \in [0, \delta_m]} \rho(X_s, x) \leq r \}$, where $\delta_m = 2^{-m}$.

Using \((4.11)-(4.13)\), on the set $\{X_{\tau} \leq R\}$, we get

\begin{equation}
\mathbb{E}_x \left[ 1_D \exp \left( \sup_{v \in [0, \delta_m]} \sup_{|x - u| \leq \delta_m} C^n(s, u, X_v, X_0) + \sup_{v \in [0, \delta_m]} \sup_{s \in [0, T]} A^n_m(s, X_v) \right) \right] \leq e^{2r},
\end{equation}

\begin{equation}
\mathbb{E}_x \left[ 1_{D'} \exp \left( \sup_{v \in [0, \delta_m]} \sup_{|x - u| \leq \delta_m} C^n(s, u, X_v, X_0) + \sup_{v \in [0, \delta_m]} \sup_{s \in [0, T]} A^n_m(s, X_v) \right) \right] \leq e^{2\varepsilon K_2},
\end{equation}

where $K_2 := e^{2\|e\|^{2+T}\|f\|}$. Thus, using strong Markov property, for any $t \in [0, T]$ and $x \in \Gamma$, we get

\begin{equation}
(4.16) \quad \mathbb{E}_x \left[ 1_{\{X_{\tau} \leq R\}} Z^n(\tau)e^{A^n_m(\tau_m, X_{\tau_m})} \right] \leq e^{2r}\mathbb{E}_x \left[ Z^n(\tau) \right] + \varepsilon K_3,
\end{equation}

where $K_3 := K_2e^{\|e\|^{2+T}\|f\|}$. Combining \((4.15)\) and \((4.16)\) with \((4.14)\) and recalling that $\|f\| 2^{-m} \leq \varepsilon$, for any $t \in [0, T]$ and $x \in \Gamma$, we finally get

\begin{equation}
0 \leq h^{n+1}_m(t, x) - h^{n+1}_m(t, x) \leq (e^{4\varepsilon} - 1)\mathbb{E}_x \left[ Z^n(\tau) \right] + e^{2r}K_3 + \varepsilon K_3 + \varepsilon + \varepsilon K_1 + \varepsilon.
\end{equation}
Noting that the upper bound is independent of \( t \) and uniform on \( \Gamma \), and recalling that \( \varepsilon > 0 \) was arbitrary, we get \( h_{m+1}^n(t, x) \to h^{n+1}(t, x) \) as \( m \to \infty \) uniformly on \([0, T] \times \Gamma\). Thus, recalling (4.8), we conclude the proof of this step.

**Step 3.** We show that \( \lim_{n \to \infty} \inf_{V \in \mathcal{V}} J_T(x, V) = \inf_{V \in \mathcal{V}} J_T(x, V) \) for any fixed \( T \in \mathbb{N} \) and \( x \in E \). In fact, using monotonicity and continuity of the exponent function, it is enough to show

\[
(4.17) \quad \lim_{n \to \infty} \inf_{V \in \mathcal{V}} \exp(J_T(x, V)) = \inf_{V \in \mathcal{V}} \exp(J_T(x, V)),
\]

Let \( \varepsilon > 0 \) and \( V^\varepsilon = (\tau_i, \xi_i)_{i=1}^\infty \in \mathcal{V} \) be an \( \varepsilon \)-optimal strategy for \( \inf_{V \in \mathcal{V}} e^{J_T(x, V)} \). For any \( n \in \mathbb{N} \), let \( V_n^\varepsilon \in \mathcal{V}^n \) denote the restriction of \( V^\varepsilon \) to the first \( n \) impulses. Then, for any \( n \in \mathbb{N} \), we get

\[
(4.18) \quad 0 \leq \inf_{V \in \mathcal{V}^n} e^{J_T(x, V)} - \inf_{V \in \mathcal{V}} e^{J_T(x, V)} \leq e^{J_T(x, V_n^\varepsilon)} - e^{J_T(x, V^\varepsilon)} + \varepsilon.
\]

For clarity, we use \( X \) and \( Y \) to denote the processes controlled by the strategies \( V^\varepsilon \) and \( V_n^\varepsilon \), respectively. Furthermore, we set \( N_T := \sum_{i=1}^\infty 1_{\{\tau_i \leq T\}} \) and

\[
A_n(x) := \mathbb{E}_{(x, V_n^\varepsilon)} \left[ 1_{\{N_T > n\}} \exp \left( \int_0^T f(Y_s) ds + \sum_{i=1}^n 1_{\{\tau_i \leq T\}} c(Y_{\tau_i}, \xi_i) \right) \right] 
- \mathbb{E}_{(x, V^\varepsilon)} \left[ 1_{\{N_T > n\}} \exp \left( \int_0^T f(X_s) ds + \sum_{i=1}^\infty 1_{\{\tau_i \leq T\}} c(X_{\tau_i}, \xi_i) \right) \right],
\]

for any \( n \in \mathbb{N} \) and \( x \in E \). Noting that, for any \( n \in \mathbb{N} \), we get

\[
\mathbb{E}_{(x, V_n^\varepsilon)} \left[ 1_{\{N_T \leq n\}} \exp \left( \int_0^T f(Y_s) ds + \sum_{i=1}^n 1_{\{\tau_i \leq T\}} c(Y_{\tau_i}, \xi_i) \right) \right] = \mathbb{E}_{(x, V^\varepsilon)} \left[ 1_{\{N_T \leq n\}} \exp \left( \int_0^T f(X_s) ds + \sum_{i=1}^\infty 1_{\{\tau_i \leq T\}} c(X_{\tau_i}, \xi_i) \right) \right],
\]

and recalling (4.18), we get

\[
0 \leq \inf_{V \in \mathcal{V}^n} e^{J_T(x, V)} - \inf_{V \in \mathcal{V}} e^{J_T(x, V)} \leq |A_n(x)| + \varepsilon.
\]

Next, using boundedness of \( f \), non-negativity of \( c \) and the fact that \( V_n^\varepsilon \) is the restriction of \( V^\varepsilon \), we get

\[
|A_n(x)| \leq \mathbb{E}_{(x, V^\varepsilon)} \left[ 1_{\{N_T > n\}} \exp \left( T\|f\| + \sum_{i=1}^\infty 1_{\{\tau_i \leq T\}} c(X_{\tau_i}, \xi_i) \right) \right].
\]

Consequently, noting that \( \mathbb{E}_{(x, V^\varepsilon)} \left[ \sum_{i=1}^\infty 1_{\{\tau_i \leq T\}} c(X_{\tau_i}, \xi_i) \right] < \infty \) and \( 1_{\{N_T > n\}} \to 0 \) as \( n \to \infty \), and using bounded convergence theorem, we get \( |A_n(x)| \to 0 \), as \( n \to \infty \), which concludes the proof of this step.

**Step 4.** We are finally ready to show (4.7). Combining Step 1 and Step 2 we get

\[
\lim_{m \to \infty} \inf_{V \in \mathcal{V}_m} J_T(x, V) = \lim_{m \to \infty} w_m^n(0, x) = w^n(0, x) = \inf_{V \in \mathcal{V}} J_T(x, V).
\]

Next, letting \( n \to \infty \) and recalling Step 3, we conclude the proof.
5. Continuous time impulse control. In this section we study the impulse control problem introduced in (2.1) and linked Bellman equation (2.2). Our approach is based on the approximation of the Bellman equation by its dyadic version discussed in Section 3. We end this section by showing the connection between a solution to (2.2) and the optimal control strategy for (2.1).

Let $(\lambda_m)$, $m \in \mathbb{N}$, be a sequence of solutions to the dyadic Bellman equations (3.1); from Theorem 3.1 we know that this sequence exists. Since $(\lambda_m)$ is decreasing and $\lambda_m \geq -\|f\|$ we can define the finite limit

$$\lambda := \lim_{m \to \infty} \lambda_m.$$ (5.1)

Using Proposition 3.2, we get $\lambda \leq r(f)$. In Theorem 5.1, we show that if $\lambda < r(f)$, then there exists a solution to (2.2).

**Theorem 5.1.** Assume (A.1)-(A.4) and let $\lambda$ be given by (5.1). If $\lambda < r(f)$, then there exists a function $w \in C(E)$, such that $w$ and $\lambda$ are solutions to (2.2).

**Proof.** For transparency, we split the proof into two steps: (1) proof of the fact that $(Mw_m) \to \phi$ uniformly to some $\phi \in C(E)$, for predefined subsequence $m_n$; (2) proof of the identity $\phi = Mw$ for a suitable $w$. In the end we comment how to combine those steps to get (2.2) and conclude the proof.

**Step 1.** Let $(w_m)$ be a sequence of functions $w_m \in C(E)$ given by

$$w_m(x) := \tilde{w}_m(x) - \inf_{\xi \in U} \tilde{w}(\xi),$$ (5.2)

where $\tilde{w}_m$ is a solution to the Bellman equation (3.2); note that $\tilde{w}_m$ exists due to Theorem (3.1), and $(w_m)$ is also a solution to (3.2). Now, we show that using Arzelà-Ascoli Theorem one can choose uniformly convergent subsequence of $(Mw_m)$, where $M$ is given in (2.3).

First, we show that $(Mw_m)$ is uniformly bounded on $E$. For any $m \in \mathbb{N}$ and $\xi \in U$, we get $w_m(\xi) \geq 0$. Consequently, for $x \in E$, we get $Mw_m(x) \geq 0$, i.e. we found the uniform lower bound for $(Mw_m)$. On the other hand, recalling (3.2) and definition (5.2), for any $x \in E$ and $m \in \mathbb{N}$, we get $Mw_m(x) \leq 2\|c\|$ as

$$w_m(x) \leq M\tilde{w}_m(x) - \inf_{\xi \in U} \tilde{w}_m(\xi) \leq (\|c\| + \inf_{\xi \in U} \tilde{w}_m(\xi)) - \inf_{\xi \in U} \tilde{w}_m(\xi) = \|c\|.$$

Second, as equicontinuity of $(Mw_m)$ follows directly from the inequality

$$|Mw_m(x) - Mw_m(y)| \leq \sup_{\xi \in U} |c(x, \xi) - c(y, \xi)|, \quad x, y \in E, m \in \mathbb{N}$$ (5.3)

and continuity of $c$, we can use Arzelà-Ascoli Theorem. For any $N \in \mathbb{N}$ and a compact set

$$B(N) := \{x \in E : \|x\| \leq N\},$$

we can find a subsequence of $(Mw_m)$, say $(Mw_{m_n})$, and $\phi_N \in C(E)$ such that

$$Mw_{m_n} \to \phi_N, \quad n \to \infty,$$ (5.4)

uniformly on $B(0, N)$.

Third, using diagonal argument we show that the limit may be chosen independently of $N$. Indeed, using recursive procedure, and taking consecutive subsequences.
\( \{m_n^N\} \subseteq \{m_{n-1}^N\} \), we can find a sequence \((\phi_N)\), such that \(\phi_N(x) = \phi_{N-1}(x)\) for \(x \in B(N - 1)\). Then, for any \(x \in E\), we set \(N_x = \inf \{N \in \mathbb{N} : x \in B(N)\}\) and define \(\phi \in C(E)\) by

\[
\phi(x) := \phi_{N_x}(x).
\]

Also, for the diagonal sequence \((m_n)\) given by \(m_n := m_n^N\), we get

\[
M w_{m_n} \rightarrow \phi, \quad n \rightarrow \infty,
\]

uniformly on \(B(0, N)\), for any \(N \in \mathbb{N}\). Using (5.3), we get that \(\phi\) satisfies

\[
|\phi(x) - \phi(y)| \leq \sup_{\xi \in U} |c(x, \xi) - c(y, \xi)|, \quad x, y \in E.
\]

Finally, we can show that the convergence \(M w_{m_n} \rightarrow \phi\) is (globally) uniform. Let \(\varepsilon > 0\). From (2.5) we know that there exists \(N_{\varepsilon} \in \mathbb{N}\) such that

\[
\sup_{\xi \in U} |c(x, \xi) - c(y, \xi)| \leq \varepsilon, \quad x, y \not\in B(N_{\varepsilon}).
\]

Since \(M w_{m_n} \rightarrow \phi\) uniformly on \(B(N_{\varepsilon} + 1)\), it is sufficient to show that

\[
\sup_{x \not\in B(N_{\varepsilon})} |M w_{m_n}(x) - \phi(x)| \rightarrow 0, \quad n \rightarrow \infty.
\]

Consider \(x \not\in B(N_{\varepsilon})\) and \(y \in B(N_{\varepsilon} + 1) \setminus B(N_{\varepsilon})\). Recalling (5.3) and (5.6) we get

\[
|M w_{m_n}(x) - \phi(x)| \leq |M w_{m_n}(x) - M w_{m_n}(y)| + |M w_{m_n}(y) - \phi(y)| + |\phi(x) - \phi(y)|
\]

\[
\leq |M w_{m_n}(y) - \phi(y)| + 2 \sup_{\xi \in U} |c(x, \xi) - c(y, \xi)|.
\]

Since \(y \in B(N_{\varepsilon} + 1)\), starting from some \(n_0 \in \mathbb{N}\), we get \(|M w_{m_n}(y) - \phi(y)| \leq \frac{\varepsilon}{4}\) for \(n \geq n_0\). As the choice of \(\varepsilon\) was arbitrary, this inequality, together with (5.7) and (5.9), concludes the proof of (5.8).

**Step 2.** For brevity, we drop the subscript \(n\) from the diagonal sequence \((m_n)\) given in Step 1, i.e. we assume that \(M w_m \rightarrow \phi\) uniformly. We show that

\[
\phi(x) = M w(x), \quad x \in E,
\]

where \(w: E \rightarrow \mathbb{R}\) is given by

\[
e^{w(x)} := \inf_{\tau \in T} \mathbb{E}_x \left[ e^{\int_0^\tau (f(X_s) - \lambda) ds + \phi(X_s)} \right].
\]

The idea is to use [6, Theorem 5.1] in order to show that \(M w_m \rightarrow M w\) uniformly. Then, (5.10) will follow from the uniqueness of the limit.

As in the proof of Proposition 3.2 we use the change of measure technique to transform the problem (5.11) into the setting, where the assumptions of [6, Theorem 5.1] are satisfied. For any \(m \in \mathbb{N}\) and \(x \in E\) let us define

\[
G_m(x) := M w_m(x) - v(x) + \|M w_m\| + \|v\|, \quad G(x) := \phi(x) - v(x) + \|\phi\| + \|v\|,
\]

\[
d_m := r(f) - \lambda_m, \quad d := r(f) - \lambda,
\]

\[
\hat{w}_m(x) := w_m(x) - v(x) + \|M w_m\| + \|v\|, \quad \hat{w}(x) := w(x) - v(x) + \|\phi\| + \|v\|.
\]
Observe that $G_m \to G$ uniformly, $d_m \uparrow d$, and $d > 0$. Moreover, using (2.9), Proposition 3.2, and (5.11), we get

$$e^{\bar{w}_m(x)} = \inf_{\tau \in T_m} E^x_\tau e^{\tau d_m + G_m(X_\tau)} \quad \text{and} \quad e^{\bar{w}(x)} = \inf_{\tau \in T} E^x_\tau e^{\tau d + G(X_\tau)}.$$ 

Hence, using [6, Theorem 5.1], we get $\bar{w}_m \to \bar{w}$ uniformly on compact sets and consequently $w_m \to w$ uniformly on compact sets. Moreover, from [6, Theorem 4.7], we get $\bar{w} \in C(E)$ and consequently $w \in C(E)$. Finally, recalling (2.3), we get

$$\sup_{x \in E} |Mw_m(x) - Mw(x)| \leq \sup_{\xi \in U} |w_m(\xi) - w(\xi)| \to 0, \quad m \to \infty.$$ 

This implies uniform convergence of $Mw_m \to Mw$. From this we immediately get $\phi(x) = Mw(x)$, $x \in E$. Recalling (5.11), we conclude the proof of (2.2).

**Remark 5.2.** The optimal stopping time for (2.2) in the case that $\lambda < r(f)$ is given by $\hat{\tau} = \inf\{t \geq 0 : Mw(X_t) = Mw(X_{\hat{\tau}})\}$. Moreover, one could easily show that $v(t) = e^{\lambda t}(g(x(t) - \lambda t)ds + w(X_t))$, $t \geq 0$, is a submartingale while $(v(t \wedge \tau))$ is a martingale; cf. [6, Remark 4.9]. Those conclusions follow from [6, Theorem 4.7] by applying reasoning as in the proof of Theorem 5.1.

Finally, we are ready to link the constant $\lambda$ given by (5.1) with the optimal strategy and the optimal value of the problem (2.1); see Theorem 5.3. In case $\lambda < r(f)$, Theorem 5.1 guarantees the existence of a solution to Bellman equation (2.2) that might be used to construct the optimal impulse control strategy. For the degenerate case $\lambda = r(f)$, we show that the no impulse strategy is the optimal choice; note that in this case we additionally assumed that $E = U$, which allowed us to use the finite horizon results from Section 4.

Before we state the next result, let us introduce some auxiliary notation. Let us fix $w \in C(E)$ that is a solution to (2.2) satisfying $w \geq 0$; from Theorem 5.1 we know that such $w$ exists if $\lambda < r(f)$. Let $\hat{V} := (\hat{\tau}_i, \hat{\xi}_i)_{i=1}^\infty$ be a strategy given recursively by

\begin{equation}
\begin{cases}
\hat{\tau}_i := \inf\{t \geq \hat{\tau}_{i-1} : w(X_t) = Mw(X_t)\}, \\
\hat{\xi}_i := \arg\min_{\xi \in U} \left[c(X_{\hat{\tau}_{i-1}}, \xi) + w(\xi)\right],
\end{cases}
\end{equation}

for $i = 1, 2, \ldots$, where $\hat{\tau}_0 := 0$. Note that in (5.12) a slight abuse of notation was used since, for each recursive step, $X_t$ refers to a process for which $ith$ impulse is not yet applied. More formally, (5.12) could be rewritten as $\hat{\tau}_i = \hat{\sigma} \circ \theta_{\hat{\tau}_{i-1}} + \hat{\tau}_{i-1}$, where $\hat{\sigma} := \inf\{t \geq 0 : w(X_t) = Mw(X_t)\}$ and $\theta$ is the Markov shift operator; see e.g. [20, Section 1.4.3] for details.

**Theorem 5.3.** Let $\lambda$ be given by (5.1). Then,

1. If $\lambda < r(f)$, then $\lambda = \inf_{V \in \mathcal{V}} J(x, V)$ for any $x \in E$, and the strategy defined in (5.12) via Bellman equation (2.2) is optimal.

2. If $\lambda = r(f)$ and $E = U$, then $\lambda = \inf_{V \in \mathcal{V}} J(x, V)$ for any $x \in E$, and the no impulse strategy is optimal.

**Proof.** For transparency, we split the proof into two parts: (1) when $\lambda < r(f)$ and (2) when $\lambda = r(f)$ and $E = U$.

**Proof of 1.** We adapt the arguments from [19, Proposition 2.3] to the continuous time case. Given strategy $V \in \mathcal{V}$, for any $i \in \mathbb{N}$, we use the notation

$$X^i_t := 1_{(t < \tau_{i+1})}X_t + 1_{(t \geq \tau_{i+1})}X_{\tau_{i+1}}.$$
where $X_{i+1}^-$ denotes the state of the process $(X_t)$ right before the $(i + 1)$th impulse.

First, we show that $\lambda = J(x, \hat{V})$ for any $x \in E$. By Remark 5.2 we know that

$$e^{\int_0^T (f(X_s) - \lambda)ds + w(X_{i+1}^0)}, \quad T \geq 0,$$

is a martingale under $\mathbb{P}_{(x, \hat{V})}$. As $w(X_{i+1}^0) = Mw(X_{i+1}^0) = c(X_{i+1}^0, \hat{\xi}_i) + w(\hat{\xi}_i)$, we get

$$e^{w(x)} = \mathbb{E}_{(x, \hat{V})} \left[ e^{\int_0^T (f(X_s) - \lambda)ds + w(X_{i+1}^0)} \right] = \mathbb{E}_{(x, \hat{V})} \left[ e^{\int_0^T (f(X_s) - \lambda)ds + 1_{\{\tau_i \leq T\}}(X_{i+1}^0, X_{i+1}^0) + 1_{\{\tau_i > T\}}w(X_{i+1}^0)} \right].$$

Acting recursively we get

$$(5.13)\quad e^{w(x)} = \mathbb{E}_{(x, \hat{V})} \left[ e^{\int_0^T (f(X_s) - \lambda)ds + \sum_{i=1}^n 1_{\{\tau_i \leq T\}}c(X_{i+1}^{i-1}, \hat{\xi}_i) + w(X_{i+1}^n(T))} \right],$$

where

$$X_{\tau_n}^n(T) := \begin{cases} X_{\tau_n}^n, & \tau_n \leq T, \\ X_T^n, & \tau_n \leq T < \tau_{k+1}. \end{cases}$$

Using Fatou Lemma and boundedness of $f$ and $w$, we get

$$\infty > e^{w(x)} \geq \mathbb{E}_{(x, \hat{V})} \left[ \liminf_{n \to \infty} e^{\int_0^T (f(X_s) - \lambda)ds + \sum_{i=1}^n 1_{\{\tau_i \leq T\}}c(X_{i+1}^{i-1}, \hat{\xi}_i) + w(X_{i+1}^n(T))} \right] \geq \mathbb{E}_{(x, \hat{V})} \left[ e^{T\|f\|-\|\lambda\| + \sum_{i=1}^n 1_{\{\tau_i \leq T\}}c(X_{i+1}^{i-1}, \hat{\xi}_i) - \|w\|} \right].$$

This implies

$$\mathbb{E}_{(x, \hat{V})} \left[ e^{\sum_{i=1}^n 1_{\{\tau_i \leq T\}}c(X_{i+1}^{i-1}, \hat{\xi}_i)} \right] < \infty,$$

for any $T \geq 0$. Thus, recalling that by (A.2) the cost function $c$ is bounded away from zero, we conclude that $\hat{\tau}_n \uparrow \infty$. Consequently, we get $X_{\tau_n}^n(T) \to \hat{X}_T$, where $\hat{X}_T(\omega) := X_T^\infty(\omega)$ on $\{\hat{\tau}_k(\omega) \leq T < \hat{\tau}_{k+1}(\omega)\}$. Finally, by bounded convergence theorem applied to (5.13), we get

$$(5.14)\quad e^{w(x)} = \mathbb{E}_{(x, \hat{V})} \left[ e^{\int_0^T (f(X_s) - \lambda)ds + \sum_{i=1}^\infty 1_{\{\tau_i \leq T\}}c(X_{i+1}^{i-1}, \hat{\xi}_i) + w(X_{\tau_n}^\infty(T))} \right].$$

Taking logarithm of both sides, dividing by $T$, and recalling that $0 \leq w(\cdot) \leq \|w\|$, we get

$$\lambda = \limsup_{T \to \infty} \frac{1}{T} \ln \mathbb{E}_{(x, \hat{V})} \left[ e^{\int_0^T (f(X_s) - \lambda)ds + \sum_{i=1}^\infty 1_{\{\tau_i \leq T\}}c(X_{i+1}^{i-1}, \hat{\xi}_i)} \right],$$

which concludes the proof of equality $\lambda = J(x, \hat{V})$, for any $x \in E$.

Second, we show that for any $x \in E$ and strategy $V = (\tau_i, \xi_i) \in \mathcal{V}$ we get $\lambda \leq J(x, V)$. Clearly, we can restrict our attention to the strategies for which

$$(5.15)\quad \mathbb{E}_{(x, V)} \left[ e^{\sum_{i=1}^\infty 1_{\{\tau_i \leq T\}}c(X_{i+1}^{i-1}, \xi_i)} \right] < \infty, \quad T \geq 0.$$

By Remark 5.2 we know that $e^{\int_0^T (f(X_s) - \lambda)ds + w(X_T)}$ is a submartingale. Hence,

$$e^{w(x)} \leq \mathbb{E}_{(x, V)} \left[ e^{\int_0^T (f(X_s) - \lambda)ds + w(X_{\tau_n}^\infty(T))} \right].$$
Using the fact that \( w(X_{n}^{x}) - M w(X_{n}^{x}) \leq c(X_{n}^{x}, \xi_n) + w(\xi_n) \), we get
\[
e^{w(x)} \leq E_{(x, V)} \left[ e^{\int_{0}^{T} f(x_s) ds + \sum_{\tau_i \leq T} 1_{\tau_i \leq T} c(X_{\tau_i}^{x}, X_{\tau_i}^{x}) + w(X_{n}^{x}(T))} \right].
\]
Recalling (5.15) and letting \( n \to \infty \), we get
\[
e^{w(x)} \leq E_{(x, V)} \left[ e^{\int_{0}^{T} f(x_s) ds + \sum_{\tau_i} 1_{\tau_i \leq T} c(X_{\tau_i}^{x}, X_{\tau_i}^{x}) + \|w\|} \right].
\]
As before, we get \( \lambda \leq J(x, V) \), which concludes this part of the proof.

**Proof of 2.** Using Theorem 3.1 and Proposition 3.2 we get that the cost of the no impulse strategy equals \( r(f) \). Thus, it is sufficient to show that for any \( x \in E \) we get \( \inf_{V \in \mathcal{V}} J(x, V) \geq r(f) \). On the contrary, suppose that \( \inf_{V \in \mathcal{V}} J(x_0, V) < r(f) \) for some \( x_0 \in E \). Then, for some \( \varepsilon > 0 \), we get
\[
\limsup_{T \to \infty} \frac{1}{T} \inf_{V \in \mathcal{V}} J_T(x_0, V) \leq \inf_{V \in \mathcal{V}} J(x_0, V) < r(f) - \varepsilon,
\]
where \( J_T \) is given by (4.1). Next, we can find \( T_0 \in \mathbb{N} \) big enough to get
\[
\inf_{V \in \mathcal{V}} \frac{1}{T_0} J_{T_0}(x_0, V) \leq \limsup_{T \to \infty} \frac{1}{T} \inf_{V \in \mathcal{V}} J_T(x_0, V) + \varepsilon\frac{4}{4} \quad \text{and} \quad \frac{\|c\|}{T_0} \leq \varepsilon\frac{4}{4}.
\]
Using Proposition 4.2 we can find \( n \in \mathbb{N}, m \in \mathbb{N} \), and a strategy \( \tilde{V} \in \mathcal{V}_m^n \), such that
\[
\frac{1}{T_0} J_{T_0}(x_0, \tilde{V}) \leq \inf_{V \in \mathcal{V}} \frac{1}{T_0} J_{T_0}(x_0, \tilde{V}) + \varepsilon\frac{4}{4}.
\]
Define the strategy \( \tilde{V} \) in the following way: for the period \( [kT_0, (k + 1)T_0] \), \( k \in \mathbb{N} \), we follow the strategy \( \tilde{V} \) and at \( (k + 1)T_0 \) we shift the process to \( x_0 \). It should be noted that \( \tilde{V} \in \mathcal{V}_m \), as \( E = U \) and \( T_0 \in \mathbb{N} \). Then, we get
\[
J(x_0, \tilde{V}) = \limsup_{k \to \infty} \frac{1}{kT_0} \ln \left( E_{(x_0, \tilde{V})} \left[ e^{\int_{0}^{T_0} f(x_s) ds + \sum_{\tau_i \leq T_0} 1_{\tau_i \leq T_0} c(X_{\tau_i}^{x}, \xi_{\tau_i}) + c(X_{T_0}^{x_0})} \right] \right) \]
\[
\leq \frac{1}{T_0} J_{T_0}(x_0, \tilde{V}) + \frac{\|c\|}{T_0}.
\]
Combining (5.16)–(5.18) with (5.19) we get \( J(x_0, \tilde{V}) < r(f) - \varepsilon \). This leads to contradiction due to Proposition 3.2, i.e. the property \( \inf_{V \in \mathcal{V}_m} J(x_0, V) = r(f) \).  

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