Quantum Theory of Dilaton Gravity in 1+1 Dimensions

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ABSTRACT

We discuss the quantum theory of 1+1 dimensional dilaton gravity, which is an interesting model with analogous features to the spherically symmetric gravitational systems in 3+1 dimensions. The functional measures over the metrics and the dilaton field are explicitly evaluated and the diffeomorphism invariance is completely fixed in conformal gauge by using the technique developed in the two dimensional quantum gravity. We argue the relations to the ADM formalism. The physical state conditions reduce to the usual Wheeler-DeWitt equations when the dilaton $\varphi^2 (= e^{-2\phi})$ is large enough compared with $\kappa = (N - 51/2)/12$, where $N$ is the number of matter fields. This corresponds to the large mass limit in the black hole geometry. A singularity appears at $\varphi^2 = \kappa (> 0)$. The final stage of the black hole evaporation corresponds to the region $\varphi^2 \sim \kappa$, where the Liouville term becomes important, which just comes from the measure of the metrics. If $\kappa < 0$, the singularity disappears.
1. Introduction

The quantization of gravity is a long standing issue in theoretical physics and a sufficient solution has not been available for a long time. In recent years, however, we got the complete solutions in the two dimensional quantum gravity. The model was solved in the lattice gravity (or the matrix model) method\(^1\)\(^{,}2\) and the field theoretical (or Liouville gravity) method\(^3\)\(^{,}4\)\(^,\)\(^5\)\(^,\)\(^6\) in which the functional integration over metrics is carried out explicitly. Unfortunately the model is now solved only in the case that the dynamical degrees of freedom are very few.

A dynamical model of the gravitation in 1+1 dimensions has been proposed by Callan et al.\(^7\), which is called the dilaton gravity. The features of the model are very similar to the spherically symmetric gravitational system in 3+1 dimensions. An advantage of this model is very simple to manage. The several authors\(^8\)\(^,\)\(^9\) solved the model and estimated the back-reaction of the Hawking radiation in the semi-classical approximation.

The quantum gravity becomes very important at the final stage of the black hole evaporation. As a quantization method of the gravitation, there is the Arnowitt-Deser-Misner (ADM) formalism or Wheeler-DeWitt approach. Recently Tomimatsu\(^10\) derived the interesting results for the spherically symmetric black hole. He showed that by solving the Wheeler-Dewitt equations in the local mini-superspace approximation near the apparent horizon, the mass of the black hole \(M\) decreases with the law \(<\dot{M}> \propto -M^{-2}\) inferred by Hawking. The similar arguments can be done in the parallel way in the case of the 1+1 dimensional dilaton gravity.

However, there are some problems in the ADM formalism, the issues of measures and orderings. In fact, if we apply the ADM formalism to the two dimensional gravity without dilaton, the Hamiltonian and the momentum constraints become trivial. The non-trivial contributions exactly come from the functional measure over metrics.

In this paper we consider the quantization of the 1+1 dimensional dilaton gravity. Then we explicitly evaluate the contributions of measures. We first argue
the measure of the dilaton field in Sect.2. The dilaton field is coupled to the curvature, so we have to estimate it carefully. Following the procedure of David-Distler-Kawai (DDK)\cite{4}, we determine the measure of metrics in confomal gauge in Sect.3. From the gauge fixed action the physical state conditions are derived in Sect.4. Then we compare with the ADM formalism. Our physical state conditions reduce to the usual Wheeler-DeWitt equations in the large $\phi (= e^{-\phi})$ limit, which corresponds to the large mass limit of the black hole geometry.

### 2. Evaluation of the measure of dilaton field

The theory of 1+1 dimensional dilaton gravity is defined by the following action

$$I(g, \varphi, f) = I_D(g, \varphi) + I_M(g, f),$$

$$I_D(g, \varphi) = \frac{2}{\pi} \int d^2x \sqrt{-g} (g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + \xi R_g \varphi^2 + \lambda^2 \varphi^2), \quad (2.1)$$

$$I_M(g, f) = -\frac{1}{4\pi} \sum_{j=1}^{N} \int d^2x \sqrt{-g} g^{\alpha\beta} \partial_\alpha f_j \partial_\beta f_j,$$

where $\varphi = e^{-\phi}$ is the dilaton field and $f_j$’s are $N$ matter fields. $\lambda$ is the cosmological constant and $\xi = 1/4$. $R_g$ is the curvature of the metrics $g$. The classical equations of motion can be easily solved and one obtains, for instance, the black hole geometry

$$\varphi^2 = e^{-2\rho} = \frac{M}{\lambda} - \lambda^2 x^+ x^- , \quad f_j = 0, \quad (2.2)$$

where $g_{\alpha\beta} = e^{2\rho} \eta_{\alpha\beta}$, $\eta_{\alpha\beta} = (-1, 1)$ and $x^\pm = x^0 \pm x^1$. $M$ is the mass of the black hole.

The corresponding quantum theory is defined by

$$Z = \int \frac{Dg(Dg(\varphi)Dg(f))}{Vol(Diff.)} e^{iI(g, \varphi, f)}, \quad (2.3)$$

where $Vol(Diff.)$ is the gauge volume. The functional measures are defined from
the following norms
\[
\| \delta g \|^2_g = \int d^2 x \sqrt{-g} g^{\alpha \gamma} g^{\beta \delta} \delta g_{\alpha \beta} \delta g_{\gamma \delta},
\]
\[
\| \delta \varphi \|^2_g = \int d^2 x \sqrt{-g} \delta \varphi \delta \varphi,
\]
\[
\| \delta f_j \|^2_g = \int d^2 x \sqrt{-g} \delta f_j \delta f_j.
\]
(2.4)

We first consider the functional integration over \( \varphi \), which gives the following determinant
\[
\int D_g(\varphi) e^{iD(g, \varphi)} = L[\det gD]^{-1/2},
\]
(2.5)

where the operator \( D \) is defined by
\[
D = \Delta_g + \xi R_g + \lambda^2
\]
(2.6)

and \( L \) is a constant factor. The Laplacian is \( \Delta = -\nabla^\alpha \nabla_\alpha \). For a while we treat \( \xi \) as a free parameter. Now we decompose the metrics into the Weyl mode \( \sigma \) and the background metrics \( \hat{g} \) as \( g = g_\sigma \equiv e^{2\sigma} \hat{g} \). From now on we concentrate on the \( \sigma \)-dependence of \( \det g_\sigma D \). Noting the relation \( R_{g_\sigma} = e^{-2\sigma}(\hat{R} + 2\hat{\Delta} \sigma) \), one gets
\[
D = e^{-2\sigma} \hat{\Delta} + \xi e^{-2\sigma}(\hat{R} + 2\hat{\Delta} \sigma) + \lambda^2.
\]
(2.7)

The variation of \( D \) with respect to \( \sigma \) is given by
\[
\delta_\sigma D = -2\delta \sigma D + 2\xi e^{-2\sigma} \hat{\Delta} \delta \sigma + 2\lambda^2 \delta \sigma.
\]
(2.8)

Therefore the \( \sigma \)-dependent terms of \( \det g_\sigma D \) are evaluated as
\[
\delta_\sigma \log \det g_\sigma D = \int \frac{dt}{\varepsilon} \text{Tr}_{g_\sigma} \left( \delta_\sigma D e^{-itD} \right)
\]
\[
= -2\text{Tr}_{g_\sigma} (\delta \sigma e^{-i\varepsilon D}) + 2\xi \text{Tr}_{g_\sigma} (\Delta \delta \sigma D^{-1})
\]
\[
+ 2\lambda^2 \text{Tr}_{g_\sigma} (\delta \sigma D^{-1}),
\]
(2.9)

where \( \varepsilon \) is a infinitesimal parameter to regularize divergences. And we also evaluate the integral by giving a small imaginary cosmological constant as \( \lambda^2 - i\varepsilon' \). The
trace is

\[ Tr_g(F) = \int d^2x \sqrt{-g} < x | F | x >^g_g , \quad < x | x' >^g_g = \frac{1}{\sqrt{-g}} \delta^2(x - x') . \quad (2.10) \]

The first term of eq.(2.9) can be easily evaluated. The Kernel $< x | e^{-itD} | x >^g_g$ was already calculated (see for instance ref.[11]). The result is

\[ < x | e^{-itD} | x >^g_g = \frac{i}{4\pi} e^{-i\lambda^2 t} \frac{1}{it} (a_0(x) + a_1(x)it + a_2(x)(it)^2 + \cdots) , \]

\[ a_0(x) = 1 , \]
\[ a_1(x) = \left( \frac{1}{6} - \xi \right) R_g , \quad (2.11) \]
\[ a_2(x) = \left[ \frac{1}{2} \left( \frac{1}{6} - \xi \right)^2 + \frac{1}{360} \right] R_g^2 - \frac{1}{6} \left( \frac{1}{5} - \xi \right) \Delta_g R_g , \]
\[ \cdots . \]

From this we obtain

\[ -2Tr_{g_\sigma}(\delta \sigma e^{-i\varepsilon D}) = \frac{1}{2\pi} \left( -\frac{1}{\varepsilon} + i\lambda^2 \right) \int d^2x \sqrt{-g_\sigma} \delta \sigma \]
\[ - \frac{i}{2\pi} \left( \frac{1}{6} - \xi \right) \int d^2x \sqrt{-g_\sigma} R_{g_\sigma} \delta \sigma + O(\varepsilon) \quad (2.12) \]

The second and the third terms of (2.9) are not so easy, which are evaluated only in the curvature expansion (2.11).

However, we want to know only the Weyl mode dependence of the measure. So we also evaluate the following determinant

\[ \int D_{\hat{g}}(\varphi) e^{iL_D(g_\sigma, \varphi)} = L[\det_{\hat{g}} e^{2\sigma} D]^{-1/2} \equiv L[\det_{\hat{g}} \hat{D}]^{-1/2} , \quad (2.13) \]

and compare with the result (2.9), where $D_{\hat{g}}(\varphi)$ is defined by the norm (2.4) with $g = \hat{g}$. The variation of the operator $\hat{D} = \hat{\Delta} + \xi (\hat{R} + 2\hat{\Delta} \sigma) + \lambda^2 e^{2\sigma}$ with respect to
\( \sigma \) is given by
\[
\delta \sigma \hat{D} = 2 \xi \Delta \delta \sigma + 2\lambda^2 e^{2\sigma} \delta \sigma .
\tag{2.14}
\]
Thus the \( \sigma \)-dependent terms of \( \text{det} \hat{g} \hat{D} \) are
\[
\delta \sigma \log \text{det} \hat{g} \hat{D} = 2 \xi \text{Tr} \hat{g} (\Delta \delta \sigma \hat{D}^{-1}) + 2\lambda^2 \text{Tr} \hat{g} (e^{2\sigma} \delta \sigma \hat{D}^{-1})
\tag{2.15}
\]
where we use the relation for the propagators
\[
< x | \hat{D}^{-1} | x' > \hat{g} = < x | D^{-1} | x' > g_s .
\tag{2.16}
\]
The difference of the expressions (2.9) and (2.15) is what we want. Combining eqs.(2.9), (2.15) and (2.12), we get
\[
\delta_\sigma \log \text{det} g_\sigma D - \delta_\sigma \log \text{det} \hat{g} \hat{D}
= -2\text{Tr} g_s (\delta \sigma e^{-i\varepsilon D})
= \delta_\sigma \left[ \frac{1}{4\pi} \left( \frac{1}{\varepsilon} - i\lambda^2 \right) \int d^2x \sqrt{-\hat{g}} e^{2\sigma} - \frac{i}{12\pi} (1 - 6\xi) \int d^2x \sqrt{-\hat{g}} (\sigma \Delta \sigma + \hat{R}) \right] .
\tag{2.17}
\]
This means that the following relation is realized,
\[
\int D e^{2\sigma} \hat{g}(\varphi) e^{iI_D(e^{2\sigma} \hat{g}, \varphi)}
= \exp \left[ \frac{1}{8\pi} \left( \frac{1}{\varepsilon} - i\lambda^2 \right) \int d^2x \sqrt{-g} e^{2\sigma} + i \frac{c_\varphi}{12\pi} S_L(\sigma, \hat{g}) \right] \int D \hat{g}(\varphi) e^{iI_D(e^{2\sigma} \hat{g}, \varphi)} ,
\tag{2.18}
\]
where \( S_L \) is the Liouville action
\[
S_L(\sigma, \hat{g}) = \frac{1}{2} \int d^2x \sqrt{-\hat{g}} (\hat{g}^{\alpha\beta} \partial_\alpha \sigma \partial_\beta \sigma + \hat{R})
\tag{2.19}
\]
and \( c_\varphi \) is defined by
\[
c_\varphi = 1 - 6\xi .
\tag{2.20}
\]
At \( \sigma = 0 \), the both sides of the expression (2.18) should be equal. So the divergent term of the form \( \Lambda \int d^2x \sqrt{-g} e^{2\sigma} \) should be renormalized properly.
3. Conformal gauge fixing

The functional measure over metrics $g_{\alpha\beta}$ is defined by the norm (2.4). We decompose integrations over metrics $g_{\alpha\beta}$ into integrations over vector fields $v_\alpha$ which generate infinitesimal diffeomorphisms and an integration over the Weyl mode $\rho$, $g = e^{2\rho} \hat{g}$. The integrations over $v_\alpha$ cancel out the gauge volume. The Jacobian can be represented by a functional integral over ghosts $b$, $c$. Thus the partition function (2.3) becomes

$$Z = \int Dg_\rho (\rho) Dg_\rho (\varphi) Dg_\rho (f) Dg_\rho (c) e^{i I_D (g_\rho, \varphi) + i I_M (g_\rho, f) + i I_{gh} (g_\rho, b, c)},$$  \hfill (3.1)

where $g_\rho \equiv e^{\rho} \hat{g}$ and $I_{gh}$ is the well-known ghost action. The measure $Dg_\rho (\rho)$ is defined from the norm (2.4) by

$$\|\delta \rho\|^2_{g_\rho} = \int d^2 x \sqrt{-g_\rho (\delta \rho)^2} = \int d^2 x \sqrt{-\hat{g} e^{2\rho} (\delta \rho)^2}. \hfill (3.2)$$

This measure is very inconvenient because the norm is not invariant under the shift $\rho \rightarrow \rho + h$. According to DDK[4], we rewrite the measure $Dg_\rho (\rho)$ into the convenient one $D\hat{g} (\rho)$ which is defined by the norm

$$\|\delta \rho\|^2_{\hat{g}} = \int d^2 x \sqrt{-\hat{g} (\delta \rho)^2}. \hfill (3.3)$$

It is invariant under the shift of $\rho$. To rewrite the measure we need the Jacobian, which is assumed from the experience of the two dimensional quantum gravity as

$$Dg_\rho (\rho) = D\hat{g} (\rho) \exp \left[ i \frac{A}{12\pi} S_L (\rho, \hat{g}) \right]. \hfill (3.4)$$

The action $S_L (\rho, \hat{g})$ is defined by eq.(2.19) with $\sigma = \rho$. The parameter $A$ is determined by the consistency.
The transformation property of the measure $D_{g_\rho}(\varphi)$ under Weyl rescalings is calculated in the previous section. For the measures of the matters and the ghost fields the transformation property is well-known and given by

$$D_{g_\rho}(f)D_{g_\rho}(b)D_{g_\rho}(c) = \exp \left[ i \frac{N - 26}{12\pi} S^L(\rho, \hat{g}) \right] D_{\hat{g}}(f)D_{\hat{g}}(b)D_{\hat{g}}(c) . \quad (3.5)$$

Here note that the actions of the matters and the ghost fields are invariant under Weyl rescalings. Using these expressions we can rewrite the partition function as

$$\int D_{\hat{g}}(\rho)D_{\hat{g}}(\varphi)D_{\hat{g}}(f)D_{\hat{g}}(b)D_{\hat{g}}(c) \exp \left[ i \frac{B}{12\pi} S^L(\rho, \hat{g}) \right. \right.$$\[+ i I_D(e^{2\sigma} \hat{g}, \varphi) + i I_M(\hat{g}, f) + i I_{gh}(\hat{g}, b, c) \left. \right], \quad (3.6)

where

$$B = A + c_\varphi + N - 26 . \quad (3.7)$$

Note that, when we rewrite the partition function (2.3) into this form, the divergent terms appear, which always have the form proportional to $\int d^2 x \sqrt{-g}$. To renormalize the divergences we introduce a bare term $\mu_0 \int d^2 x \sqrt{-g}$ and cancel out these by adjusting the bare constant $\mu_0$. The renormalized constant is set to zero. In the following we do not consider this type of divergence.

To determine the parameter $A$ we use the fact that the original theory depends only on the metrics $g_\rho = e^{2\rho} \hat{g}$. This means that the theory should be invariant under the simultaneous shift

$$\rho \to \rho - \sigma , \quad \hat{g} \to e^{2\sigma} \hat{g} . \quad (3.8)$$

Applying the shift to the partition function, we obtain

$$\int D_{e^{2\sigma} \hat{g}}(\rho)D_{e^{2\sigma} \hat{g}}(\varphi)D_{e^{2\sigma} \hat{g}}(f)D_{e^{2\sigma} \hat{g}}(b)D_{e^{2\sigma} \hat{g}}(c) \exp \left[ i \frac{B}{12\pi} S^L(\rho - \sigma, e^{2\sigma} \hat{g}) \right.$$

\[+ i I_D(e^{2\sigma} \hat{g}, \varphi) + i I_M(\hat{g}, f) + i I_{gh}(\hat{g}, b, c) \left. \right], \quad (3.9)

where we use the fact that the measure $D_{e^{2\sigma} \hat{g}}(\rho)$ is invariant under the shift of $\rho$. 

8
This expression should return to the original form (3.6).

Let us argue changing the measures on the metrics $g_{\sigma} \equiv e^{2\sigma} \hat{g}$ into the measures on the background metrics $\hat{g}$. It is easily done for the measures of the matters and the ghost fields by using the relation (3.5) (by replacing $\rho$ with $\sigma$). The Weyl transformation property of the measure of $\varphi$ is given, from the previous calculation, by

$$
\int D \mathcal{e}^{2\sigma} \hat{g}(\varphi) e^{iD(e^{2\sigma} \hat{g}, \varphi)} = \exp \left[ \frac{i C_{\varphi}}{12\pi} S_{L}(\sigma, \hat{g}) \right] \int D \hat{g}(\varphi) e^{iD(e^{2\sigma} \hat{g}, \varphi)} . \tag{3.10}
$$

In Sect.2 we discussed the case of $\rho = \sigma$. So we must show that the expression (3.10) holds good in the case of $\rho \neq \sigma$. It is realized by seeing the $\rho$-dependence of the both sides of eq.(3.10). The left hand side gives the determinant $L[\det \mathcal{g} \hat{D}]^{-1/2}$, where $\hat{D} = e^{-2\sigma} e^{2\rho} D_{g_{\rho}}$ and $D_{g_{\rho}} = \Delta_{g_{\rho}} + \xi R_{g_{\rho}} + \lambda^2$. While the functional integration of the right hand side gives $L[\det \mathcal{g} \hat{D}_{\rho}]^{-1/2}$ where $\hat{D}_{\rho} = e^{2\rho} D_{g_{\rho}} = \hat{\Delta} + \xi (\hat{R} + 2\hat{\Delta}_{\rho}) + \lambda^2 e^{2\rho}$.

Then we obtain the following relation with respect to the $\rho$-dependence,

$$
\delta_{\rho} \log \det \mathcal{g} \hat{D} = \int d^2 x \sqrt{-g_{\sigma}} (2\xi e^{-2\sigma} \hat{\Delta}_{\rho} + 2\lambda^2 e^{2(\rho - \sigma)} \delta_{\rho}) < x \hat{D}^{-1} | x >_{g_{\sigma}}
$$

$$
= \int d^2 x \sqrt{-g} (2\xi \hat{\Delta}_{\rho} + 2\lambda^2 e^{2\rho} \delta_{\rho}) < x \hat{D}_{\rho}^{-1} | x >_{\hat{g}} \tag{3.11}
$$

$$
= \delta_{\rho} \log \det \mathcal{g} \hat{D}_{\rho} .
$$

This means that the difference of the functional integrations in eq.(3.10) depends only on $\sigma$, which is determined from the previous result at $\rho = \sigma$.

By using the expressions (3.5) and (3.10) and also the relation for the Liouville action

$$
S_{L}(\rho - \sigma, e^{2\sigma} \hat{g}) = S_{L}(\rho, \hat{g}) - S_{L}(\sigma, \hat{g}) , \tag{3.12}
$$

the partition function (3.9) reduces to the following form

$$
\exp \left[ -i \frac{A}{12\pi} S_{L}(\sigma, \hat{g}) \right] \int D \mathcal{e}^{2\sigma} \hat{g}(\rho) D \hat{g}(\varphi) D \hat{g}(f) D \hat{g}(b) D \hat{g}(c) \times \exp \left[ i \frac{B}{12\pi} S_{L}(\rho, \hat{g}) + i D(e^{2\rho} \hat{g}, \varphi) + i M(\hat{g}, f) + i I_{gh}(\hat{g}, b, c) \right] . \tag{3.13}
$$
Finally consider changing the measure $D\alpha^\sigma\hat{g}(\rho)$ into $D\hat{g}(\rho)$. As the measure is invariant under the shift of $\rho$, we can replace $\rho$ with $\rho' = \rho + h$, where $h = \left(1/2\right)\hat{\Delta}^{-1}\left[\hat{R} + (96\xi/B)\hat{\Delta}\varphi^2\right]$. Then the $\rho'$ integration becomes that of the single free boson $\dagger$. Since the shift $h$ is independent of $\rho$ and $\sigma$, the Weyl dependence of the $\rho$-measure corresponds to the case of central charge 1. Therefore, if we set

$$A = 1\,,$$

(3.14)

the Liouville action $S_L(\sigma, \hat{g})$ completely cancels out and the partition function reduces to the original form.

Setting $\xi = 1/4$, we finally get the conformal gauge fixed action of the 1+1 dimensional dilaton gravity

$$\hat{I} = \frac{1}{2\pi} \int d^2x \sqrt{-\hat{g}} \left[4\hat{g}^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + 4\hat{g}^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \rho + \hat{R} \varphi^2 + 4\lambda^2 \varphi^2 e^{2\rho} + N - 51/2 \left(\hat{g}^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho + \hat{R} \rho\right) - \frac{1}{2} \sum_{j=1}^{N} \hat{g}^{\alpha\beta} \partial_\alpha f_j \partial_\beta f_j \right] + I_{gh}(\hat{g}, b, c).$$

(3.15)

We showed that the theory is invariant under the simultaneous shift (3.8). Since the measure of $\rho$ defined by (3.3) is invariant under local shifts of $\rho$, the theory is invariant under conformal changes of the background metrics $\hat{g}$.

$\dagger$ Here we do not care the cosmological constant term. From the experience of the 2 dimensional quantum gravity, it probably does not contribute to the value of the central charge.
4. Physical state conditions

Now we carry out the canonical quantization of the gauge-fixed 1+1 dimensional dilaton gravity. Since the functional measures are defined on the background metrics \( \hat{g}_{\alpha\beta} \), we can set the canonical commutation relations in usual way. Here we choose the flat metric \( \eta_{\alpha\beta} \) as the background metric \( \hat{g}_{\alpha\beta} \). Then the canonically conjugate momentums for \( \varphi, \rho \) and \( f_j \) are given by

\[
\Pi_\varphi = -\frac{4}{\pi} \dot{\varphi} - \frac{2}{\pi} \varphi \dot{\rho} , \\
\Pi_\rho = -\frac{N - 51/2}{12\pi} \rho - \frac{2}{\pi} \varphi \dot{\varphi} , \\
\Pi_{f_j} = \frac{1}{2\pi} \dot{f}_j ,
\]

(4.1)

where the dot and the prime stand for the derivative with respect to the time and space coordinate respectively.

The physical state condition is defined from the independence of how to choose the background metrics \( \hat{g}_{\alpha\beta} \). So we set\(^\dagger\)

\[
\left\langle \frac{\delta \dot{\hat{I}}}{\delta g_{\alpha\beta}} \right\rangle = 0
\]

(4.2)

or

\[
\hat{T}_{00} \Psi = \hat{T}_{01} \Psi = 0 ,
\]

(4.3)

where the energy-momentum tensor \( \hat{T}_{\alpha\beta} \) is defined by \( \hat{T}_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta \dot{\hat{I}}}{\delta g_{\alpha\beta}} \). \( \Psi \) is a physical state. The condition for \( \hat{T}_{11} \) reduces to the one for \( \hat{T}_{00} \) by using the \( \rho \)-equation of motion. In the flat background metric, the physical state conditions

\(^\dagger\) In the case of the two dimensional gravity without dilaton, this corresponds to the heighest state conditions of the Virasoro algebra: \( L_n|\text{phys} >= L_n|\text{phys} >= 0 \) for \( n \geq 0 \), which means that for arbitrary non-singular function \( \epsilon \) and \( \epsilon' \), \( \int \epsilon \hat{T}_{00}|\text{phys} >= \int \epsilon' \hat{T}_{00}|\text{phys} >= 0 \).
(4.3) become

\[
\left[ \frac{\pi^2}{2\varphi^2 - \kappa} \left( \Pi^2_\rho - \varphi \Pi_\varphi \Pi_\rho + \frac{\kappa}{4} \Pi^2_\varphi \right) + 2 \left( \varphi \varphi'' - \varphi' \rho' - \lambda^2 \varphi^2 e^{2\rho} \right) - \frac{\kappa}{2\pi} \left( \rho'' - 2 \rho'' \right) + \sum_{j=1}^{N} \left( \pi \Pi^2_{f_j} + \frac{1}{4\pi} f^2_{j} \right) \right] \Psi = 0
\]

(4.4)

and

\[
(\varphi' \Pi_\varphi + \rho' \Pi_\rho - \Pi'_\rho + \sum_{j=1}^{N} \Pi_{f_j} f'_j) \Psi = 0,
\]

(4.5)

where \( \kappa \) is defined by

\[
\kappa = \frac{N - 51/2}{12}.
\]

(4.6)

If we rewrite the canonical momentums as the differential operators

\[
\Pi_\rho = \frac{\delta}{i\delta \rho}, \quad \Pi_\varphi = \frac{\delta}{i\delta \varphi}, \quad \Pi_{f_j} = \frac{\delta}{i\delta f_j},
\]

(4.7)

the eqs. (4.4) and (4.5) give the differential equations similar to the Wheeler-DeWitt equations*. The difference between the usual Wheeler-DeWitt equations and ours is just the Liouville term. If \( \kappa > 0 \), there is a singularity at finite \( \varphi^2 = \kappa \). Our physical state conditions reduce to the usual form of the Wheeler-DeWitt equations in the limit \( \varphi^2 \gg \kappa \). In the black hole geometry this means that the mass of the black hole \( M \) is large enough compared to \( \lambda \kappa \) or the thinking region is far from the singularity. So the usual Wheeler-DeWitt equations seem to be correct in the semi-classical region. The final stage of the black hole evaporation corresponds to the region \( \varphi^2 \sim \kappa \), where the Liouville term becomes important.

The region \( \kappa > \varphi^2 > 0 \) is called the Liouville region, in which the sign of the kinetic term of eq. (4.4) changes. The existence of the Liouville region is mysterious.

\footnote{‡ For the zero-mode part, we maybe need the further arguments.}

\footnote{* See for example ref.[12], in which the spherically symmetric gravitational system of 3+1 dimensions is discussed. Application to the 1+1 dimensional dilaton gravity is straightforward.}
\( \kappa = 0 \) is special. In this case the Liouville action disappears and the physical state conditions reduce to the usual form of Wheeler-DeWitt equations. If \( \kappa < 0 \), the situation drastically changes. In this region the singularity disappears.

**Note added:** After completing the calculation of the gauge fixing, we received the preprints [13], in which the value of \( c_\phi \) is different of ours. It seems that their calculation is essentially in the case of \( \xi = 0 \).

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