The giant in random graphs is almost local

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Abstract: Local convergence techniques have become a key methodology to study sparse random graphs. However, convergence of many random graph properties does not directly follow from local convergence. A notable, and important, such random graph property is the size and uniqueness of the giant component. We provide a simple criterion that guarantees that local convergence of a random graph implies the convergence of the proportion of vertices in the maximal connected component. We further show that, when this condition holds, the local properties of the giant are also described by the local limit. We give several examples where this method gives rise to a novel law of large numbers for the giant, based on results proved in the literature. Aside from these examples, we apply our novel method to the configuration model as a proof of concept, reproving a well-established result. As a side result of this proof, we identify the small-world nature of the configuration model.

MSC2020 subject classifications: 60K37, 05C81, 82C27, 37A25.

Keywords and phrases: Random graphs, Local convergence, Giant component, Configuration model.

1. INTRODUCTION

Local convergence techniques, as introduced by Aldous and Steele [3] and Benjamini and Schramm [11], have become the main methodology to study random graphs in sparse settings where the average degree remains bounded. Local convergence roughly means that the proportion of vertices whose neighborhoods have a certain shape converges to some limiting value, which is to be considered as a measure on rooted graphs. We refer to [43, Chapter 2] for details.

The giant component problem has received enormous attention ever since the very first and seminal results by Erdős and Rényi on the Erdős-Rényi random graph [34], see also [17, 53] for detailed further results for this model. The simplest form of this question is whether there exists a linear-size connected component or not, in many cases a sharp transition occurs depending on a certain graph parameter. See [6, 27, 33, 37, 38, 42, 43, 55], as well as the references therein, for details.

This paper combines these two threads by investigating the size of the giant component when the random graph converges locally. Consider a sequence of random graphs \( (G_n)_{n \geq 1} \), where we will simplify the notation by assuming that \( G_n = (V(G_n), E(G_n)) \) is such that \( |V(G_n)| = n \). For \( v \in V(G_n) \), we let \( \mathcal{C}(v) \) denote its connected component. We let

\[
|\mathcal{C}_{\text{max}}| = \max_{v \in V(G_n)} |\mathcal{C}(v)|
\]  

(1.1)

denote the maximal component size. When the random graph converges locally to some limit, one would expect that also \( |\mathcal{C}_{\text{max}}|/n \overset{p}{\to} \zeta \), where \( \zeta \) is the survival probability of the local limit. However, while the number of connected components is well behaved in the local topology, the proportion of vertices in the giant is not. Indeed, since local convergence is all about proportions of vertices whose finite (but arbitrarily large) neighborhoods converge to a limit, there is an enormous gap between surviving locally and being in the giant. In this paper, we identify the extra condition that is necessary and sufficient for the above natural implication to hold.

2. ASYMPTOTICS AND PROPERTIES OF THE GIANT

In this section, we investigate the behavior of the giant component \( |\mathcal{C}_{\text{max}}| \) for random graphs that converge locally. In Section 2.1, we introduce the notion of local convergence in probability.
In Section 2.2, we study the asymptotics of \(|G_{\text{max}}|\) for random graphs that converge locally in probability under necessary and sufficient ‘giant-is-almost-local’ condition, and in Section 2.3, we investigate local properties of the giant. In Section 2.4, we provide examples for the law of large numbers of the giant, obtained through bounds on the second largest component proved elsewhere. In Section 2.5, we give an alternative for the ‘giant-is-almost-local’ condition that is often more convenient to establish. We start by introducing some useful notation used throughout this paper.

**Notation.** We abbreviate left- and right-hand side by lhs and rhs, respectively. A sequence of random variables \((X_n)_{n \geq 1}\) converges in probability to a random variable \(X\), denoted as \(X_n \xrightarrow{p} X\), if, for all \(\varepsilon > 0\), \(\Pr(|X_n - X| > \varepsilon) \to 0\). We write \(X_n = o(1)\) when \(X_n \xrightarrow{p} 0\). A sequence of events \((\mathcal{E}_n)_{n \geq 1}\) is said to hold with high probability (whp) if \(\lim_{n \to \infty} \Pr(\mathcal{E}_n) = 1\).

### 2.1. Local convergence in probability

Local weak convergence was introduced independently by Aldous and Steele in [3] and Benjamini and Schramm in [11]. The purpose of Aldous and Steele in [3] was to describe the local structure of the so-called ‘stochastic mean-field model of distance’, meaning the complete graph with i.i.d. exponential edge weights. This local description allowed Aldous to prove the celebrated \(\zeta(2)\) of the so-called ‘stochastic mean-field model of distance’, meaning the complete graph with i.i.d. exponential edge weights. Benjamini and Schramm in [11] instead used local weak convergence to show that limits of planar graphs are with probability one recurrent. Since its conception, local convergence has proved a key ingredient in random graph theory. In this section, we provide some basics of local convergence. For more detailed discussions, we refer the reader to [21] or [43, Chapter 2]. Let us start with some definitions.

A rooted graph is a pair \((G, o)\), where \(G = (V(G), E(G))\) is a graph with vertex set \(V(G)\) and edge set \(E(G)\), and \(o \in V(G)\) is a vertex. Further, a rooted or non-rooted graph is called locally finite when each of its vertices has finite degree (though not necessarily uniformly bounded). Two (finite or infinite) graphs \(G_1 = (V(G_1), E(G_1))\) and \(G_2 = (V(G_2), E(G_2))\) are called isomorphic, which we write as \(G_1 \cong G_2\), when there exists a bijection \(\phi: V(G_1) \to V(G_2)\) such that \(\{u, v\} \in E(G_1)\) precisely when \(\{\phi(u), \phi(v)\} \in E(G_2)\). Similarly, two rooted (finite or infinite) graphs \((G_1, o_1)\) and \((G_2, o_2)\), with \(G_i = (V(G_i), E(G_i))\) for \(i \in \{1, 2\}\), are called isomorphic, abbreviated as \((G_1, o_1) \cong (G_2, o_2)\), when there exists a bijection \(\phi: V(G_1) \to V(G_2)\) such that \(\phi(o_1) = o_2\) and \(\{u, v\} \in E(G_1)\) precisely when \(\{\phi(u), \phi(v)\} \in E(G_2)\). These notions can easily be adapted to multigraphs (which we will need below), for which \(G = (V(G), (x_{i,j})_{i,j \in V(G)})\), where \(x_{i,j}\) denotes the number of edges between \(i\) and \(j\), and \(x_{i,i}\) the number of self-loops at \(i\). There, instead, the isomorphism \(\phi: V(G) \to V(G')\) is required to satisfy that \(x_{i,j} = x'_{\phi(i), \phi(j)}\), where \((x'_{i,j})_{i,j \in V(G')}\) are the edge multiplicities of \(G\) and \(G'\) respectively.

We let \(\mathcal{G}_r\) be the space of (possibly infinite) connected rooted graphs, where we consider two rooted graphs to be equal when they are isomorphic. Thus, we consider \(\mathcal{G}_r\) as the set of equivalence classes of rooted graphs modulo isomorphisms. The space \(\mathcal{G}_r\) of rooted graphs is a nice metric space, with an explicit metric, see [43, Chapter 2 and Appendix A] for details.

For a graph \(G\) and \(u, v \in V(G)\), we let \(d_G(u, v)\) denote the graph distance between \(u\) and \(v\), where, by convention, \(d_G(u, v) = \infty\) when \(u\) and \(v\) are not connected in \(G\). For a rooted graph \((G, o)\), we let \(B_r^{(G)}(o)\) denote the (rooted) subgraph of \((G, o)\) of all vertices at graph distance at most \(r\) away from \(o\). Formally, this means that \(B_r^{(G)}(o) = (V(B_r^{(G)}(o)), E(B_r^{(G)}(o)), o)\), where

\[
V(B_r^{(G)}(o)) = \{u : d_G(o, u) \leq r\},
\]

\[
E(B_r^{(G)}(o)) = \{(u, v) \in E(G) : d_G(o, u), d_G(o, v) \leq r\}.
\]

We say that the graph sequence \((G_n)_{n \geq 1}\) converges locally in probability to a limit \((G, o) \sim \mu\), when, for every \(r \geq 0\) and \(H^* \in \mathcal{G}_r\),

\[
\frac{1}{|V(G_n)|} \sum_{v \in V(G_n)} \mathbb{1}_{\{B_r^{(G_n)}(v) \simeq H^*\}} \xrightarrow{p} \mu(B_r^{(G)}(o) \simeq H^*).
\]

This means that the subgraph proportions in the random graph \(G_n\) are close, in probability, to those given by \(\mu\). Let \(o_n \in V(G_n)\) be chosen uniformly at random (uar) in \(V(G_n)\). Then, (2.2) is
that

There are related notions of local convergence, such as local weak convergence, where (2.2) is replaced by the convergence of expectations, and local almost sure convergence, where the convergence holds almost surely. For our purposes, local convergence in probability is the most convenient, for example since it implies that the neighborhoods of two uniformly chosen vertices are asymptotically independent (see e.g., [43, Corollary 2.18]), which is central in our proof.

2.2. ASYMPTOTICS OF THE GIANT

Given a random graph sequence \(G_n\) that converges locally in probability to \((G, o) \sim \mu\), one would expect that \(|\mathcal{E}_{\text{max}}|/n \xrightarrow{p} \zeta := \mu(|\mathcal{E}(o)| = \infty)\). However, the proportion of vertices in the largest connected component \(|\mathcal{E}_{\text{max}}|/n\) is not continuous in the local convergence topology, as it is a global object. In fact, also \(|\mathcal{E}(o_n)|/n\) does not necessarily converge in distribution when \(G_n\) converges locally in probability to \((G, o) \sim \mu\). However, local convergence still tells us a useful story about the existence of a giant, as well as its size. Indeed, Corollary 2.1 shows that the upper bound as well:

\(\mathbb{P}(|\mathcal{E}_{\text{max}}| \leq n(\zeta + \varepsilon)) \to 1.\) (2.4)

In particular, Corollary 2.1 implies that \(|\mathcal{E}_{\text{max}}|/n \xrightarrow{p} 0\) when \(\zeta = 0\), so that there can only be a giant when the local limit has a positive survival probability.

\textbf{Proof.} Define

\[ Z_{\geq k} = \sum_{v \in V(G_n)} \mathbb{1}_{\{|\mathcal{E}(v)| \geq k\}}. \] (2.5)

Assume that \(G_n\) converges locally in probability to \((G, o) \sim \mu\) as defined in (2.2). Since \(|\mathcal{E}(v)| \geq k\} = \{|B^{(G)}_k(v)| \geq k\}, \text{ with } \zeta_{\geq k} = \mu(|\mathcal{E}(o)| \geq k),

\[ \frac{Z_{\geq k}}{n} \xrightarrow{p} \zeta_{\geq k}. \] (2.6)

For every \(k \geq 1,\)

\[ \{|\mathcal{E}_{\text{max}}| \geq k\} = \{|Z_{\geq k}| \geq k\}, \] (2.7)

and, on the event that \(Z_{\geq k} \geq 1\), also \(|\mathcal{E}_{\text{max}}| \leq Z_{\geq k}\). Note that \(\zeta = \lim_{k \to \infty} \zeta_{\geq k} = \mu(|\mathcal{E}(o)| = \infty)\).

We take \(k\) so large that \(\zeta \geq \zeta_{\geq k} \cdot \varepsilon/2\). Then, for every \(k \geq 1, \varepsilon > 0, \text{ and all } n\) large enough such that \(k \leq n(\zeta + \varepsilon),\)

\[ \mathbb{P}(|\mathcal{E}_{\text{max}}| \geq n(\zeta + \varepsilon)) \leq \mathbb{P}(Z_{\geq k} \geq n(\zeta + \varepsilon)) \leq \mathbb{P}(Z_{\geq k} \geq n(\zeta_{\geq k} + \varepsilon/2)) = o(1). \] (2.8)

We conclude that while local convergence cannot determine the size of the largest connected component, it \textit{does} prove an upper bound on \(|\mathcal{E}_{\text{max}}|\). There are many results that extend this to \(|\mathcal{E}_{\text{max}}|/n \xrightarrow{p} \zeta\) (see Section 4 for pointers to the literature), but this is no longer a consequence of local convergence alone. Therefore, in general, more involved arguments must be used. We next prove that one, relatively simple, condition suffices. In its statement, and for \(x, y \in V(G_n),\) we write \(x \leftrightarrow y\) when \(x \notin \mathcal{E}(y)\):
Theorem 2.2 (The giant is almost local). Let $G_n = (V(G_n), E(G_n))$ denote a random graph of size $|V(G_n)| = n$. Assume that $G_n$ converges locally in probability to $(G, o) \sim \mu$. Assume further that

$$\lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n^2} E \left[ \# \{ (x, y) \in V(G_n) \times V(G_n): |\mathcal{C}(x)|, |\mathcal{C}(y)| \geq k, x \leftrightarrow y \} \right] = 0. \quad (2.9)$$

Then, with $\mathcal{C}(2)$ denoting the second largest connected component (with ties broken arbitrarily),

$$\frac{|\mathcal{C}_{\text{max}}|}{n} \xrightarrow{p} \zeta = \mu(|\mathcal{C}(o)| = \infty), \quad \frac{|\mathcal{C}_{(2)}|}{n} \xrightarrow{p} 0. \quad (2.10)$$

Theorem 2.2 shows that a relatively mild condition as in (2.9) suffices for the giant to have the expected limit. In fact, we will see that it is necessary and sufficient (see Remark 2.3 below).

In the proof below, for $k \geq 1$, it will be convenient to write $X_{n,k} = o_{k,p}(1)$ when

$$\lim_{k \to \infty} \limsup_{n \to \infty} \mathbb{P}(|X_{n,k}| > \varepsilon) = 0. \quad (2.11)$$

Proof of Theorem 2.2. By Corollary 2.1, it suffices to prove Theorem 2.2 for $\zeta > 0$, which we assume from now on.

Let the vector $(|\mathcal{C}_{(i)}|)_{i \geq 1}$ denote the component sizes ordered in size, from large to small with ties broken arbitrarily, so that $\mathcal{C}_{(i)} = \mathcal{C}_{\text{max}}$. Since $|\mathcal{C}_{\text{max}}| \geq |\mathcal{C}_{(i)}|$ for all $i \geq 1$, we can bound

$$\frac{|\mathcal{C}_{\text{max}}|}{n} \geq \frac{1}{n^2} \sum_{i \geq 1} |\mathcal{C}_{(i)}|^2 \mathbb{1}_{\{|\mathcal{C}_{(i)}| \geq k\}} \geq \frac{1}{n^2} \sum_{i \geq 1} |\mathcal{C}_{(i)}| \mathbb{1}_{\{|\mathcal{C}_{(i)}| \geq k\}} \mathbb{1}_{\{|\mathcal{C}_{(i)}| \geq k\}}. \quad (2.12)$$

We investigate both numerator and denominator of this expression. For the denominator, by (2.6),

$$\frac{1}{n} \sum_{i \geq 1} |\mathcal{C}_{(i)}| \mathbb{1}_{\{|\mathcal{C}_{(i)}| \geq k\}} = \frac{1}{n} \sum_{v \in V(G_n)} \mathbb{1}_{\{|\mathcal{C}(v)| \geq k\}} = \frac{1}{n} Z_{\geq k} \xrightarrow{p} \zeta_{\geq k}, \quad (2.13)$$

where we recall that $\zeta_{\geq k} = \mu(|\mathcal{C}(o)| \geq k)$. Thus, since $\zeta_{\geq k} \to \zeta$ as $k \to \infty$,

$$\frac{1}{n} \sum_{i \geq 1} |\mathcal{C}_{(i)}| \mathbb{1}_{\{|\mathcal{C}_{(i)}| \geq k\}} = \zeta + o_{k,p}(1). \quad (2.14)$$

For the numerator, we let

$$X_{n,k} = \frac{1}{n^2} \sum_{i \geq 1} |\mathcal{C}_{(i)}|^2 \mathbb{1}_{\{|\mathcal{C}_{(i)}| \geq k\}} - \left( \frac{1}{n} \sum_{i \geq 1} |\mathcal{C}_{(i)}| \mathbb{1}_{\{|\mathcal{C}_{(i)}| \geq k\}} \right)^2, \quad (2.15)$$

so that the numerator in (2.12) equals

$$\left( Z_{\geq k}/n \right)^2 + X_{n,k} = \zeta^2 + X_{n,k} + o_{k,p}(1). \quad (2.16)$$

It remains to investigate $X_{n,k}$, which we rewrite as

$$X_{n,k} = \frac{1}{n^2} \sum_{i,j \geq 1, i \neq j} |\mathcal{C}_{(i)}| |\mathcal{C}_{(j)}| \mathbb{1}_{\{|\mathcal{C}_{(i)}|, |\mathcal{C}_{(j)}| \geq k\}}$$

$$= \frac{1}{n^2} \# \{ (x, y) \in V(G_n) \times V(G_n): |\mathcal{C}(x)|, |\mathcal{C}(y)| \geq k, x \leftrightarrow y \}. \quad (2.17)$$

By the Markov inequality,

$$\lim_{k \to \infty} \limsup_{n \to \infty} \mathbb{P}(X_{n,k} \geq \varepsilon) \leq \lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{\varepsilon n^2} E[X_{n,k}] = 0,$$
by our main assumption in (2.9). As a result, $X_{n,k} = o_{\kappa}(1)$. We conclude that, by (2.12),

$$\frac{|\mathcal{E}_{\max}|}{n} \geq \frac{\zeta^2 + o_{\kappa}(1)}{\zeta + o_{\kappa}(1)} = \zeta + o_{\kappa}(1).$$  

(2.18)

Since $k$ is arbitrary, this proves that $|\mathcal{E}_{\max}| \geq n\zeta(1 + o_{\kappa}(1))$. Further, by Corollary 2.1, also $|\mathcal{E}_{\max}| \leq n\zeta(1 + o_{\kappa}(1))$. Therefore, $|\mathcal{E}_{\max}|/n \xrightarrow{p} \zeta$.

In turn, since $|\mathcal{E}_{\max}|/n \xrightarrow{p} \zeta$ and by (2.13),

$$\frac{1}{n} |\mathcal{E}_{\max}^{(2)}| \mathbb{1}_{\{|\mathcal{E}_{\max}^{(2)}| \geq k\}} \leq \frac{1}{n} \sum_{i \geq 2} |\mathcal{E}_{\max}^{(i)}| \mathbb{1}_{\{|\mathcal{E}_{\max}^{(i)}| \geq k\}} = \frac{1}{n} \sum_{i \geq 1} |\mathcal{E}_{\max}^{(i)}| \mathbb{1}_{\{|\mathcal{E}_{\max}^{(i)}| \geq k\}} - \frac{|\mathcal{E}_{\max}|}{n} = o_{\kappa}(1),$$

(2.19)

which, again since $k$ is arbitrary, implies that $|\mathcal{E}_{\max}^{(2)}|/n \xrightarrow{p} 0$. \qed

**Remark 2.3** (Proof of necessity of (2.9)). Condition (2.9) is also necessary for $|\mathcal{E}_{\max}|/n \xrightarrow{p} \zeta$ to hold. Indeed, (2.17) implies that when (2.9) fails, there exists a $\kappa > 0$ such that

$$\limsup_{k \to \infty} \sup_{n \to \infty} \mathbb{E}[X_{n,k}] = \kappa > 0.$$

(2.20)

Then, there exists a subsequence $(n_i)_{i \geq 1}$ for which $\lim_{l \to \infty} \mathbb{E}[X_{n_i,k}] = \kappa$, and, by (2.13) and (2.15),

$$\lim_{l \to \infty} \frac{1}{n_l^2} \mathbb{E}[|\mathcal{E}_{\max}^{(2)}|^2 \mathbb{1}_{\{|\mathcal{E}_{\max}^{(2)}| \geq k\}}] \leq \lim_{l \to \infty} \frac{1}{n_l^2} \mathbb{E}\left[\sum_i |\mathcal{E}_{\max}^{(i)}|^2 \mathbb{1}_{\{|\mathcal{E}_{\max}^{(i)}| \geq k\}}\right]$$

$$= \lim_{l \to \infty} \frac{1}{n_l^2} \mathbb{E}[Z_{\geq k}^2 - X_{n_i,k}] = \zeta^2 - \kappa.$$

(2.21)

Obviously, the same holds for $\mathbb{E}[|\mathcal{E}_{\max}^{(i)}|^2]/n_l^2$. Letting $k \to \infty$ and using that $\zeta_{\geq k} \to \zeta$, we obtain

$$\lim_{l \to \infty} \frac{1}{n_l^2} \mathbb{E}[|\mathcal{E}_{\max}^{(2)}|^2] \leq \zeta^2 - \kappa.$$

(2.22)

By (2.22), we conclude that $|\mathcal{E}_{\max}|/n \xrightarrow{p} \zeta$ cannot hold, as by bounded convergence, this would imply that also $\mathbb{E}[|\mathcal{E}_{\max}^{(2)}|^2]/n^2 \to \zeta^2$. \qed

### 2.3. Local properties of the giant

We next extend Theorem 2.2 by investigating the structure of the giant. For this, we first let $v_k(\mathcal{E}_{\max})$ denote the number of vertices with degree $k$ in the giant component, and $|E(\mathcal{E}_{\max})|$ the number of edges in the giant component. Further, for a graph $G$ and $v \in V(G)$, we write $d_v^{(G)}$ for the degree of $v$:

**Theorem 2.4** (Properties of the giant). Under the assumptions of Theorem 2.2, when $\zeta = \mu(|\mathcal{E}(o)| = \infty) > 0$,

$$\frac{v_k(\mathcal{E}_{\max})}{n} \xrightarrow{p} \mu(|\mathcal{E}(o)| = \infty, d_v^{(G)} = k).$$

(2.23)

Further, assume that $D_n = d_v^{(G_n)}$ is uniformly integrable. Then,

$$\frac{|E(\mathcal{E}_{\max})|}{n} \xrightarrow{p} \frac{1}{2} \mathbb{E}_\mu\left[d_v^{(G)} \mathbb{1}_{\{|\mathcal{E}(o)| = \infty\}}\right].$$

(2.24)

**Proof.** We now define, for $k \geq 1$ and $A \subseteq \mathbb{N}$ and with $d_v^{(G_n)}$ the degree of $v$ in $G_n$,

$$Z_{A,\geq k} = \sum_{v \in [n]} \mathbb{1}_{\{|\mathcal{E}(v)| \geq k, d_v^{(G_n)} \in A\}}.$$

(2.25)

Assume that $G_n$ converges locally in probability to $(G,o) \sim \mu$. Then, as in (2.13),

$$\frac{Z_{A,\geq k}}{n} \xrightarrow{p} \mu(|\mathcal{E}(o)| \geq k, d_v^{(G)} \in A).$$

(2.26)
Since $|\mathcal{E}_{\max}| \geq k$ whp by Theorem 2.2, we thus obtain, for every $A \subseteq \mathbb{N}$, on the high-probability event $\{|\mathcal{E}_{\max}| \geq k\}$,
\[
\frac{1}{n} \sum_{a \in A} \nu_a(\mathcal{E}_{\max}) \leq \frac{Z_{A,k}}{n} \overset{p}{\longrightarrow} \mu(|\mathcal{E}(o)| \geq k, d_{o}^{(G)} \in A).
\] (2.27)

Applying this to $A = \{\ell\}$, we obtain that, for all $\varepsilon > 0$,
\[
\lim_{n \to \infty} \mathbb{P}\left(\frac{1}{n} \left[|\mathcal{E}_{\max}| - \nu_{\ell}(\mathcal{E}_{\max})\right] \leq \mu(|\mathcal{E}(o)| \geq k, d_{o}^{(G)} = \ell) + \varepsilon/2\right) = 1.
\] (2.28)

We argue by contradiction. Suppose that, for some $\ell$,
\[
\limsup_{n \to \infty} \mathbb{P}\left(\frac{\nu_{\ell}(\mathcal{E}_{\max})}{n} \leq \mu(|\mathcal{E}(o)| = \infty, d_{o}^{(G)} = \ell) - \varepsilon\right) = \kappa > 0.
\] (2.29)

Then, along the subsequence $(n_{t})_{t \geq 1}$ that attains the limsup in (2.29), with asymptotic probability $\kappa > 0$, and using (2.28) and (2.29),
\[
\frac{|\mathcal{E}_{\max}|}{n} \to \frac{1}{n} \left[|\mathcal{E}_{\max}| - \nu_{\ell}(\mathcal{E}_{\max})\right] + \frac{\nu_{\ell}(\mathcal{E}_{\max})}{n} \leq \mu(|\mathcal{E}(o)| = \infty) - \varepsilon/2,
\] (2.30)

which contradicts Theorem 2.2. We conclude that (2.29) cannot hold, so that (2.23) follows.

For (2.24), we note that
\[
|\mathbb{E}(\mathcal{E}_{\max})| = \frac{1}{2} \sum_{\ell \geq 1} \ell \nu_{\ell}(\mathcal{E}_{\max}).
\] (2.31)

We divide by $n$ and split the sum over $\ell$ in $\ell \in [K] = \{1, \ldots, K\}$ and $\ell > K$ as
\[
\frac{|\mathbb{E}(\mathcal{E}_{\max})|}{n} = \frac{1}{2n} \sum_{\ell \in [K]} \ell \nu_{\ell}(\mathcal{E}_{\max}) + \frac{1}{2n} \sum_{\ell > K} \ell \nu_{\ell}(\mathcal{E}_{\max}).
\] (2.32)

For the first term in (2.32), by (2.23),
\[
\frac{1}{2n} \sum_{\ell \in [K]} \ell \nu_{\ell}(\mathcal{E}_{\max}) \overset{p}{\longrightarrow} \frac{1}{2} \sum_{\ell \in [K]} \ell \mu(|\mathcal{E}(o)| = \infty, d_{o}^{(G)} = \ell)
\] (2.33)

\[= \frac{1}{2} \mathbb{E}_{\mu}\left[d_{o}^{(G)} \mathbb{1}_{\{|\mathcal{E}(o)| = \infty, d_{o}^{(G)} \in [K]\}}\right].
\]

For the second term in (2.32), we bound, with $n_{\ell}$ the number of vertices in $G_{n}$ of degree $\ell$,
\[
\frac{1}{2n} \sum_{\ell > K} \ell \nu_{\ell}(\mathcal{E}_{\max}) \leq \frac{1}{2} \sum_{\ell > K} \frac{n_{\ell}}{n} = \frac{1}{2} \mathbb{E}\left[d_{o}^{(G_{n})} \mathbb{1}_{\{d_{o}^{(G_{n})} > K\}} \mid G_{n}\right] - K_{n}.
\] (2.34)

By uniform integrability of $(d_{o}^{(G_{n})})_{n \geq 1}$,
\[
\lim_{K \to \infty} \limsup_{n \to \infty} \mathbb{E}\left[d_{o}^{(G_{n})} \mathbb{1}_{\{d_{o}^{(G_{n})} > K\}} \mid G_{n}\right] = 0.
\] (2.35)

As a result, by the Markov inequality and for every $\varepsilon > 0$, there exists a $K = K(\varepsilon) < \infty$ such that
\[
\mathbb{P}\left(\frac{\mathbb{E}\left[d_{o}^{(G_{n})} \mathbb{1}_{\{d_{o}^{(G_{n})} > K\}} \mid G_{n}\right]}{K} > \varepsilon\right) \to 0.
\] (2.36)

This completes the proof of (2.24). \qed

It is not hard to extend the above analysis to the local convergence in probability of the giant, as well as its complement, as formulated in the following theorem:
2.4 Examples through bounds on second largest component

**Theorem 2.5** (Local limit of the giant). Under the assumptions of Theorem 2.2, when $\zeta = \mu(|E(o)| = \infty) > 0$,

$$
\frac{1}{n} \sum_{v \in G_{\max}} \mathbb{1}_{\{|B_{v}|(v) \geq H_*\}} \xrightarrow{p} \mu(|E(o)| = \infty, \ B_{v}(o) \simeq H_*), \tag{2.37}
$$

and

$$
\frac{1}{n} \sum_{v \in G_{\max}} \mathbb{1}_{\{|B_{v}|(v) \geq H_*\}} \xrightarrow{p} \mu(|E(o)| < \infty, \ B_{v}(o) \simeq H_*). \tag{2.38}
$$

**Proof.** The convergence in (2.38) follows from that in (2.37) combined with the fact that, by assumption,

$$
\frac{1}{n} \sum_{v \in G_{\max}} \mathbb{1}_{\{|B_{v}|(v) \geq H_*\}} \xrightarrow{p} \mu(B_{v}(o) \simeq H_*). \tag{2.39}
$$

The convergence in (2.37) can be proved as for Theorem 2.4, now using that

$$
\frac{1}{n} Z_{x_{\max}, k} \equiv \frac{1}{n} \sum_{v \in V(G_{n})} \mathbb{1}_{\{|E(v)| \geq k, B_{v}(o) \in \mathcal{E}_{n}\}} \xrightarrow{p} \mu(|E(v)| \geq k, B_{v}(o) \in \mathcal{E}_{n}), \tag{2.40}
$$

and, since $|G_{\max}|/n \xrightarrow{p} \zeta > 0$ by Theorem 2.2, on the high-probability event $\{|E_{\max}| \geq k\}$,

$$
\frac{1}{n} \sum_{v \in G_{\max}} \mathbb{1}_{\{|B_{v}|(v) \in \mathcal{E}_{n}\}} \leq \frac{1}{n} Z_{x_{\max}, k}. \tag{2.41}
$$

We leave the details to the reader. \qed

2.4 Examples through bounds on second largest component

In this section, we give examples of how we can apply our results to obtain a law of large numbers for the giant, given that a bound on the second largest component has been proved. These examples will follow from the following corollary:

**Corollary 2.6** (Law of large numbers giant given bound second largest component). Let $(G_{n})_{n \geq 1}$ be a sequence of graphs having size $|V(G_{n})| = n$. Assume that $G_{n}$ converges locally in probability to $(G, o) \sim \mu$. Write $\zeta = \mu(|E(o)| = \infty)$ for the survival probability of the limiting graph $(G, o)$. Assume further that

$$
|E_{(2)}|/n \xrightarrow{p} 0. \tag{2.42}
$$

Then, as $n \to \infty$,

$$
|E_{\max}|/n \xrightarrow{p} \zeta. \tag{2.43}
$$

Further, Theorem 2.5 and (2.23) in Theorem 2.4 also hold, while (2.24) in Theorem 2.4 holds when $D_{n} = d_{\max}(o)$ is uniformly integrable.

**Proof.** Recall $X_{n,k}$ from (2.15), and recall also (2.17), so that

$$
X_{n,k} = \frac{1}{n} \sum_{i \neq j} \mathbb{1}_{\{|E_{i}|, |E_{j}|, |E_{i,j}| \geq k\}} \leq \frac{2|E_{(o)}|}{n} \frac{1}{n} \sum_{j \geq 1} \mathbb{1}_{\{|E_{i}|, |E_{i,j}| \geq k\}} \leq \frac{2|E_{(o)}|}{n} \xrightarrow{p} 0. \tag{2.44}
$$

Therefore, by bounded convergence, also $E[X_{n,k}] = o(1)$, so that (2.9) holds. The conclusion on $|E_{\max}|$ follows from Theorem 2.2, while the other conclusions follow from the proofs of Theorems 2.4 and 2.5. \qed
Remark 2.7 (Examples). There are many examples for which linear size lower bounds on the giant have been proved, jointly with a sublinear upper bound on the second component. The latter then shows that the giant is unique, and implies that $\zeta > 0$. Further assuming local convergence in probability, Corollary 2.6 immediately implies the law of large numbers of the giant, which is obviously very interesting. For example, for the geometric inhomogeneous random graph studied in [26], under appropriate conditions, the local limit was identified in [46], and the upper bound on the second largest component proved in [26]. For the hyperbolic random graph, again the local limit was identified in [46], the lower bound on the giant proved in [15], and the bound on the second largest component in [60]. Such results highlight the power of the ‘giant is almost local’ method.

2.5. The ‘giant is almost local’ condition revisited

The ‘giant is almost local’ condition \((2.9)\) is sometimes not so convenient to verify directly, and we now give an alternative form that is often easier to work with:

Lemma 2.8 (Condition \((2.9)\) revisited). Consider \((G_n)_{n \geq 1}\) under the conditions of Theorem 2.2. Assume further that there exists \(r = r_k \to \infty\) such that

\[
\mu(|\mathcal{C}(o)| \geq k, |\partial B_r^{(G)}(o)| < r_k) \to 0, \quad \mu(|\mathcal{C}(o)| < k, |\partial B_r^{(G)}(o)| \geq r_k) \to 0. \tag{2.45}
\]

Then, the ‘giant is almost local’ condition in \((2.9)\) holds when

\[
\lim_{r \to \infty} \limsup_{n \to \infty} \frac{1}{n^2} \mathbb{E} \left[ \# \{ (x, y) \in V(G_n) \times V(G_n) : |\partial B_r^{(G_n)}(x)|, |\partial B_r^{(G_n)}(y)| \geq r, x \leftrightarrow y \} \right] = 0. \tag{2.46}
\]

Proof. Denote

\[
P_k = \# \{ (x, y) \in V(G_n) \times V(G_n) : |\mathcal{C}(x)|, |\mathcal{C}(y)| \geq k, x \leftrightarrow y \},
\]

\[
P_r^{(2)} = \# \{ (x, y) \in V(G_n) \times V(G_n) : |\partial B_r^{(G_n)}(x)|, |\partial B_r^{(G_n)}(y)| \geq r, x \leftrightarrow y \}. \tag{2.47}
\]

Then,

\[
|P_k - P_r^{(2)}| \leq 2n [Z_{<r \geq k} + Z_{>r < k}], \tag{2.49}
\]

where

\[
Z_{<r \geq k} = \sum_{v \in V(G_n)} 1_{\{ |\partial B_r^{(G_n)}(v)| < r, |\mathcal{C}(v)| \geq k \}}, \quad Z_{>r < k} = \sum_{v \in V(G_n)} 1_{\{ |\partial B_r^{(G_n)}(v)| \geq r, |\mathcal{C}(v)| < k \}}. \tag{2.50}
\]

Therefore, by local convergence in probability,

\[
\frac{1}{n^2} |P_k - P_r^{(2)}| \leq \frac{2}{n} \left[ Z_{<r \geq k} + Z_{>r < k} \right] \to 2\mu(|\mathcal{C}(o)| \geq k, |\partial B_r^{(G)}(o)| < r) + 2\mu(|\mathcal{C}(o)| < k, |\partial B_r^{(G)}(o)| \geq r). \tag{2.51}
\]

Take \(r = r_k\) as in \((2.45)\), so that the rhs of \((2.51)\) vanishes, and, by Dominated Convergence, also

\[
\lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n^2} \mathbb{E} \left[ |P_k - P_r^{(2)}| \right] = 0. \tag{2.52}
\]

We arrive at

\[
\lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n^2} \mathbb{E} \left[ P_k \right] \leq \lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n^2} \mathbb{E} \left[ P_r^{(2)} \right] = 0, \tag{2.53}
\]

by \((2.46)\) and since \(r_k \to \infty\) when \(k \to \infty\). \qed
Remark 2.9 (Alternative ‘giant is almost local’ condition). The assumption in (2.46) is sometimes more convenient than (2.9), as it requires that most pairs of vertices with many vertices at distance $r$ are connected to one another. In many random graphs, there is some weak dependence between $B^o(x)$ and the graph outside of it. This is more complicated when dealing with $|\mathcal{E}(x)| \geq k$.

The assumption in (2.45) on the local limit is often easily verified. For example, for the unimodular branching processes with bounded offspring, to which we will apply it below, we can take $r_k = k$ and use that, on the event of survival, $\nu^{-1}\partial B_k(o) \overset{a.s.}\to M$, where $(G, o)$ denotes the unimodular branching process with bounded offspring, $M \geq 0$ on the event of survival by [42, Theorem 3.9], and $\nu$ the expected offspring of this branching process. Therefore, $\mu(|\mathcal{E}(o)| \geq k, |\partial B_k(o)| < k) \to 0$ as $k \to \infty$. Further, $\mu(|\mathcal{E}(o)| < k), |\partial B_k(o)| \geq k) = 0$ trivially. However, there are examples where (2.45) fails, so that (2.9) and (2.46) may not be equivalent then. ▷

3 Application to the configuration model

In this section, we apply our results to the configuration model as introduced by Bollobás [16] in the context of random regular graphs. The giant in the configuration model has a long history. It was first investigated by Molloy and Reed [63, 64] in a setting where the degrees are general. The problem was revisited by Janson and Luczak [54] and Bollobás and Riordan [19], amongst others. This section is organised as follows. We start by introducing the model and stating our results in Section 3.1. We then prove the main result in Section 3.2, and state a consequence on typical graph distances, that is proved along the way, in Section 3.3. Some proofs of technical ingredients are deferred to Appendix A.

3.1. Model definition and results

The configuration model has the nice property that, when conditioned on being simple, it yields a uniform random graph with the prescribed degree distribution. We refer to [42, Chapter 7] for an extensive introduction.

Model definition and assumptions. Fix an integer $n$ that will denote the number of vertices in the random graph. Consider a sequence of degrees $d = (d_i)_{i \in [n]}$. Without loss of generality, we assume throughout this paper that $d_j \geq 1$ for all $j \in [n]$, since when $d_j = 0$, vertex $j$ is isolated and can be removed from the graph. We assume that the total degree $\ell_n = \sum_{j \in [n]} d_j$ is even.

To construct the multigraph where vertex $j$ has degree $d_j$ for all $j \in [n]$, we have $n$ separate vertices and incident to vertex $j$, we have $d_j$ half-edges. We number the half-edges in an arbitrary order from 1 to $\ell_n$, and start by randomly connecting the first half-edge with one of the $\ell_n - 1$ remaining half-edges. Once paired, two half-edges form a single edge of the multigraph, and the half-edges are removed from the list of half-edges that need to be paired. We continue the procedure of randomly choosing and pairing the half-edges until all half-edges are connected, and call the resulting graph the configuration model with degree sequence $d$, abbreviated as $CM_n(d)$. A careful reader may worry about the order in which the half-edges are being paired. In fact, this ordering turns out to be irrelevant since the random pairing of half-edges is completely exchangeable. It can even be done in a random fashion, which will be useful when investigating neighborhoods in the configuration model. See e.g., [42, Definition 7.5 and Lemma 7.6] for more details.

We denote the degree of a uniformly chosen vertex $o$ in $[n]$ by $D_n = d_o$. The random variable $D_n$ has distribution function $F_n$ given by

$$F_n(x) = \frac{1}{n} \sum_{j \in [n]} 1\{d_j \leq x\}, \quad (3.1)$$

which is the empirical distribution of the degrees. Equivalently, $\mathbb{P}(D_n = k) = n_k/n$, where $n_k$ denotes the number of vertices of degree $k$. We assume that the vertex degrees satisfy the following regularity conditions:
Condition 3.1 (Regularity conditions for vertex degrees).
(a) Weak convergence of vertex weight. There exists a distribution function $F$ such that, as $n \to \infty$,
\[ D_n \xrightarrow{d} D, \]
where $D_n$ and $D$ have distribution functions $F_n$ and $F$, respectively. Further, we assume that $F(0) = 0$, i.e., $P(D \geq 1) = 1$.
(b) Convergence of average vertex degrees. As $n \to \infty$,
\[ E[D_n] \to E[D] < \infty, \]
where $D_n$ and $D$ have distribution functions $F_n$ and $F$ from part (a), respectively.

Note that Conditions 3.1(a)-(b) are equivalent to Condition 3.1(a) and uniform integrability of $(D_n)_{n \geq 1}$. While Conditions 3.1(a)-(b) may appear to be quite strong, in fact, they are quite close to assuming uniform integrability of $(D_n)_{n \geq 1}$. Indeed, when $(D_n)_{n \geq 1}$ is uniformly integrable, then there is a subsequence along which $D_n$ converges in distribution as in Condition 3.1(a), and along this subsequence also Condition 3.1(b) holds. We can then apply our results along this subsequence.

The giant in the configuration model. We now come to our result on the connected components in the configuration model $CM_n$, where the degrees $d = (d_i)_{i \in [n]}$ satisfy Conditions 3.1(a)-(b). For a graph $G$, we write $v_k(G)$ for the number of vertices of degree $k$ in $G$, and $|E(G)|$ for the number of edges. The main result concerning the size and structure of the largest connected components of $CM_n(d)$ is the following:

Theorem 3.2 (Phase transition in $CM_n(d)$). Suppose that Conditions 3.1(a)-(b) hold and consider the random graph $CM_n(d)$, letting $n \to \infty$. Assume that $p_2 = P(D = 2) < 1$.

(a) If $\nu = E[D(D - 1)]/E[D] > 1$, then there exist $\xi \in [0, 1), \zeta \in (0, 1]$ such that
\[ \frac{|C_{\max}|}{n} \xrightarrow{P} \zeta, \]
\[ \frac{v_k(C_{\max})}{n} \xrightarrow{P} p_k(1 - \xi^k) \quad \text{for every } k \geq 0, \]
\[ \frac{|E(C_{\max})|}{n} \xrightarrow{P} \frac{1}{2}E[D](1 - \xi^2). \]

while $|C_2|/n \xrightarrow{P} 0$ and $|E(C_2)|/n \xrightarrow{P} 0$.

(b) If $\nu = E[D(D - 1)]/E[D] \leq 1$, then $|C_{\max}|/n \xrightarrow{P} 0$ and $|E(C_{\max})|/n \xrightarrow{P} 0$.

We prove Theorem 3.2 in Section 3.2 below. We now remark upon the result and on the conditions arising in it.

Local structure of $C_{\max}$. Theorem 3.2 is the main result for $CM_n(d)$ proved in [54]. The result in [19] only concerns $|C_{\max}|/n$. Related results are in [63, 64, 67]. Many of these proofs use an exploration of the connected components in the graph, in [63, 64] in discrete time and in [54] in continuous time. Further, [19] relies on concentration inequalities combined with a sprinkling argument. For $CM_n(d)$, [19, Theorem 25] identifies the local structure of the giant that follows from Theorem 2.5. [19, Theorem 25] also applies to uniform simple graphs with given degrees.

Local convergence of configuration models. The fact that the configuration model converges locally in probability is well-established, and is the starting point for the proof of Theorem 3.2 using Theorems 2.2 and 2.4. We refer to Appendix A.1 for more details on local convergence of $CM_n(d)$. Dembo and Montanari [28] crucially rely on it in order to identify the limiting pressure for the Ising model on the configuration model. Many alternative proofs exist. We use some of the ingredients in [43, Proof of Theorem 4.1], since these are helpful in the proof of Theorem 3.2 as well. We also refer to the lecture notes by Bordenave [21] for a nice exposition of local convergence proofs for the configuration model. Further, Bordenave and Caputo [22] prove that the neighborhoods in the configuration model satisfy a large deviation principle at speed $n$ in the context where the degrees of the configuration model are bounded.
Let us now describe the local limit of $\text{CM}_n(d)$ subject to Conditions 3.1(a)-(b). The root has offspring distribution $(p_k)_{k \geq 1}$ where $p_k = \mathbb{P}(D = k)$ and $D$ is from Condition 3.1(a), while all other individuals in the tree have offspring distribution $(p^*_k)_{k \geq 0}$ given by

$$p^*_k = \frac{k + 1}{\mathbb{E}[D]} \mathbb{P}(D = k + 1). \quad (3.4)$$

The distribution $(p^*_k)_{k \geq 0}$ has the interpretation of the forward degree of a uniform edge. The above branching process is a so-called unimodular branching process (recall [2]) with root offspring distribution $(p_k)_{k \geq 1}$.

**Reformulation in terms of branching processes.** We next interpret the results in Theorem 3.2 in terms of our unimodular branching processes. In terms of this, $\xi$ is the extinction probability of a branching process with offspring distribution $(p^*_k)_{k \geq 0}$, and $\zeta$ as the survival probability of the unimodular branching process with root offspring distribution $(p_k)_{k \geq 1}$. Thus, $\zeta$ satisfies

$$\zeta = \sum_{k \geq 1} p_k (1 - \xi^k), \quad (3.5)$$

with $\xi$ the smallest solution to

$$\xi = \sum_{k \geq 0} p^*_k \xi^k. \quad (3.6)$$

Clearly, $\xi = 1$ precisely when $\nu \leq 1$, where

$$\nu = \sum_{k \geq 0} k p^*_k = \frac{1}{\mathbb{E}[D]} \sum_{k \geq 0} k(k + 1)p_{k+1} = \mathbb{E}[D(D - 1)]/\mathbb{E}[D]. \quad (3.7)$$

by (3.4). This explains the condition on $\nu$ in Theorem 3.2(a). Further, to understand the asymptotics of $v_k(\mathcal{C}_{\max})$, we note that there are $n_k \approx np_k$ vertices with degree $k$. Each of the $k$ direct neighbors of a vertex of degree $k$ survives with probability close to $1 - \xi$, so that the probability that at least one of them survives is close to $1 - \xi^k$. When one of the neighbors of the vertex of degree $k$ survives, the vertex itself is part of the giant component, which explains why $v_k(\mathcal{C}_{\max})/n \xrightarrow{p} p_k(1 - \xi^k)$. Finally, an edge consists of two half-edges, and an edge is part of the giant component precisely when one of the vertices incident to it is, which occurs with asymptotic probability $1 - \xi^2$. There are in total $\ell_n/2 = n\mathbb{E}[D_n]/2 \approx n\mathbb{E}[D]/2$ edges, which explains why $|E(\mathcal{C}_{\max})|/n \xrightarrow{p} \frac{1}{2}\mathbb{E}[D](1 - \xi^2)$. Therefore, all results in Theorem 3.2 have a simple explanation in terms of the branching process approximation of the connected component for $\text{CM}_n(d)$ of a uniform vertex in $[n]$.

**The condition $\mathbb{P}(D = 2) = p_2 < 1$.** The case $p_2 = 1$, for which $\nu = 1$, is quite exceptional, and quite different limits for $|\mathcal{C}_{\max}|/n$ can occur, see [43, Section 4.2] and [35] for more details.

### 3.2. The ‘giant component is almost local’ proof

In this section, we prove Theorem 3.2 using the ‘giant is almost local’ results in Theorems 2.2 and 2.4. We start by setting the stage.

**Setting the stage for the proof of Theorem 3.2.** Theorem 3.2(b) follows directly from Corollary 2.1 combined with the local convergence in probability discussed below Theorem 3.2 and the fact that, for $\nu \leq 1$, the unimodular branching process with root offspring distribution $(p_k)_{k \geq 0}$ given by $p_k = \mathbb{P}(D = k)$ dies out.

Theorem 3.2(a) follows from Theorem 2.4, together with the facts that, for the unimodular branching process with root offspring distribution $(p_k)_{k \geq 0}$ given by $p_k = \mathbb{P}(D = k)$,

$$\mu(|\mathcal{C}(o)| = \infty, d_o = \ell) = p_k(1 - \xi^k), \quad (3.8)$$
and
\[ E_\mu \left[ d_x^\star \mathbb{1}_{\{|\mathcal{V}(o)|=\infty\}} \right] = E[D] (1 - \xi^2). \] (3.9)

Thus, it suffices to check the assumptions to Theorem 2.4. The uniform integrability of \( D_n \) follows from Conditions 3.1(a)-(b). For the conditions in Theorem 2.2, the local convergence in probability is discussed below Theorem 3.2, so we are left to proving the crucial hypothesis in (2.9), which the remainder of the proof will do.

We start by proving the result for configuration models with \textit{bounded degrees}, i.e., for now we assume that \( \max_{v \in [n]} d_v \leq b \). To start with our proof of (2.9), applied to \( \text{CM}_n(d) \), we first use the alternative formulation from Lemma 2.8, and note that (2.45) holds for the limiting unimodular branching process (recall Remark 2.9). Then, with \( o_1, o_2 \in [n] \) chosen independently and urn,
\[
\frac{1}{n^2} \mathbb{E} \left[ \# \{(x, y) \in V(G_n) : |\partial B_r^{(G_n)}(x)|, |\partial B_r^{(G_n)}(y)| \geq r, x \leftrightarrow y \} \right]
= \mathbb{P}(|\partial B_r^{(G_n)}(o_1)|, |\partial B_r^{(G_n)}(o_2)| \geq r, o_1 \leftrightarrow o_2). \] (3.10)

Thus, our main aim is to show that
\[
\lim_{r \to \infty} \limsup_{n \to \infty} \mathbb{P}(|\partial B_r^{(G_n)}(o_1)|, |\partial B_r^{(G_n)}(o_2)| \geq r, o_1 \leftrightarrow o_2) = 0. \] (3.11)

\textbf{Coupling to \( n \)-dependent branching process.} We next relate the neighborhood in a random graph to a certain \( n \)-dependent unimodular branching process where the root has offspring distribution \( D_n \). Such a coupling has previously appeared in [14]. Since the branching process is unimodal, all other individuals have offspring distribution \( \min D_n = 1 \), where
\[
\mathbb{P}(D_0^* = k) = \frac{k}{\mathbb{E}[D_n]} \mathbb{P}(D_n = k), \quad k \in \mathbb{N}, \] (3.12)
is the size-biased distribution of \( D_n \). Denote this branching process by \( (\text{BP}_n(t))_{t \in \mathbb{N}_0} \). Here, \( \text{BP}_n(t) \) denotes the branching process when precisely \( t \) vertices have been explored, and we explore it in the breadth-first order. Clearly, by Conditions 3.1(a)-(b), \( D_n \xrightarrow{d} D \) and \( D_n^* \xrightarrow{d} D^* \), which implies that \( \text{BP}_n(t) \xrightarrow{d} \text{BP}(t) \) for every \( t \) finite, where \( \text{BP}(t) \) is the restriction of the unimodal branching process \( \text{BP} \) with root offspring distribution \( (p_k)_{k \geq 1} \) for which \( p_k = \mathbb{P}(D = k) \) to its first \( t \) individuals.

Below, we extend the coupling of the graph exploration in \( \text{CM}_n(d) \) and \( (\text{BP}_n(t))_{t \in \mathbb{N}_0} \) significantly. For this, we let \( (G_n(t))_{t \in \mathbb{N}_0} \) denote the graph exploration process from a uniformly chosen vertex \( o \in [n] \). Here \( G_n(t) \) is the graph exploration after pairing \( t \) half-edges, in the breadth-first manner. In particular, from \( (G_n(t))_{t \in \mathbb{N}_0} \) we can retrieve \( (B_r^{(G_n)}(o))_{r \in \mathbb{N}_0} \). The following lemma shows that we can couple the graph exploration to the unimodal branching process in such a way that \( (G_n(t))_{0 \leq t \leq m_n} \) is equal to \( (\text{BP}_n(t))_{0 \leq t \leq m_n} \) whenever \( m_n \to \infty \) sufficiently slowly. In the statement, we write \( (\tilde{G}_n(t), \tilde{\text{BP}}_n(t))_{t \in \mathbb{N}_0} \) for the coupling of \( (G_n(t))_{0 \leq t \leq m_n} \) and \( (\text{BP}_n(t))_{0 \leq t \leq m_n} \):

\textbf{Lemma 3.3} (Coupling graph exploration and branching process). \textit{Subject to Conditions 3.1(a)-(b), there exists a coupling \( (\tilde{G}_n(t), \tilde{\text{BP}}_n(t))_{t \in \mathbb{N}_0} \) of \( (G_n(t))_{0 \leq t \leq m_n} \) and \( (\text{BP}_n(t))_{0 \leq t \leq m_n} \) such that}
\[
\mathbb{P}\left( (\tilde{G}_n(t))_{0 \leq t \leq m_n} \neq (\tilde{\text{BP}}_n(t))_{0 \leq t \leq m_n} \right) = o(1), \] (3.13)
\textit{when \( m_n \to \infty \) sufficiently slowly.}

Lemma 3.3 also implies that the proportion of vertices whose \( r \) neighborhood is isomorphic to a specific tree converges to the probability that the unimodal branching process with root offspring distribution \( (p_k)_{k \geq 1} \) is isomorphic to this tree, which is a crucial ingredient in local convergence.

\textbf{Remark 3.4} (Extensions). Here we discuss some extensions of Lemma 3.3. First, the proof shows that any \( m_n = o(\sqrt{n/d_{\max}}) \) is allowed. Here \( d_{\max} = \max_{i \in [n]} d_i \) is the maximal vertex degree in the graph. Secondly, Lemma 3.3 can easily be extended to explorations from \textit{two} independent sources
The following lemma investigates the asymptotics of $(b_k^{(2)}(o_1))_{k \leq \ell_n}$, where we can still take $m_n = o(\sqrt{n/d_{\text{max}}})$, and the two branching processes to which we couple the exploration from two sources, denoted by $(\hat{B}_n^{(1)}(t))_{0 \leq t \leq m_n}$ and $(\hat{B}_n^{(2)}(t))_{0 \leq t \leq m_n}$, are independent.

See Appendix A.3 for the proof of Lemma 3.3 and Remark 3.4.

Take an arbitrary $m_n = o(\sqrt{n})$, then Remark 3.4 shows that whp we can perfectly couple $(G_n^{(2)}(o_1))_{k \leq \ell_n}$ to a unimodular branching processes $(\hat{B}_n^{(1)})_{k \leq \ell_n}$ with root offspring distribution $(\mathbb{P}(D_n = k))_{k \geq 1}$. Here $|B_k^{(1)}|$ is the size of the $k$th generation in our $n$-dependent branching process, while

$$k_n = \inf \{k : |B_k^{(1)}| \geq m_n\}. \quad (3.14)$$

Since all degree are bounded, $|B_k^{(1)}| \leq b m_n = \Theta(m_n)$. Below, we will need that we can even do this up to $k_n + 1$, since $|B_{k_n+1}^{(1)}| \leq (b + (b-1)^2) m_n = \Theta(m_n)$, and then the same bounds hold for $(G_n^{(2)}(o_1))_{k \leq \ell_n}$ on the event of perfect coupling. Let $C_n(1)$ denote the event that this perfect coupling happens, so that

$$C_n(1) = \{(|B_k^{(1)}(o_1)|_{k \leq \ell_n+1}) = (|B_k^{(1)}|_{k \leq \ell_n+1})\}, \quad \text{and} \quad \mathbb{P}(C_n(1)) = 1 - o(1). \quad (3.15)$$

We extend the above coupling to also deal with vertex $o_2$, for which we explore a little further. For this, we start by defining the necessary notation. Let $(\hat{B}_k^{(2)})_{k \geq 0}$ be an $n$-dependent unimodular branching process independent of $(\hat{B}_k^{(1)})_{k \geq 0}$. For $m_n \geq m_n$, let

$$k_n = \inf \{k : |B_k^{(2)}| \geq \bar{m}_n\},$$

and, again since all degrees are bounded, $|B_k^{(2)}| \leq b m_n = \Theta(m_n)$. Further, for $\delta > 0$, we let

$$C_n(2) = \{(|B_k^{(2)}(o_2)|) - |B_k^{(2)}| \leq (\bar{m}_n/\ell_n)^{1+\delta} \forall k \in [k_n + 1]\}. \quad (3.17)$$

With $\bar{m}_n = o(\sqrt{n})$, we will later pick $\bar{m}_n$ such that $m_n \bar{m}_n \gg n$ to reach our conclusion. The following lemma shows that also $C_n(2)$ occurs whp:

**Lemma 3.5** (Coupling beyond Lemma 3.3). Consider $CM_n(d)$ and let $\bar{m}_n^2/\ell_n \rightarrow \infty$. Then, for every $\delta > 0$,

$$\mathbb{P}(C_n(2)) = 1 - o(1). \quad (3.18)$$

For the proof, we refer to Appendix A.3. We now define the successful coupling event $C_n$ to be

$$C_n = C_n(1) \cap C_n(2), \quad \text{so that} \quad \mathbb{P}(C_n) = 1 - o(1). \quad (3.19)$$

**Branching process neighborhood growth.** The previous step relates the graph exploration process to two independent $n$-dependent unimodular branching process $(B_k^{(1)}, B_k^{(2)})_{k \geq 1}$. In this step, we investigate the growth of these branching processes. Fix $r$, and denote $\hat{b}_k^{(i)} = |B_k^{(i)}|$, which we assume to be at least $r$ (and which is true on the event $C_n \cap \{\partial B_k^{(1)}(o_1)|, |\partial B_k^{(2)}(o_2)| \geq r\}$).

Let $\nu_n = \frac{1}{k_n} \sum_{v \in [n]} d_v (d_v - 1)$ denote the expected forward degree of a uniform half-edge in $CM_n(d)$, which also equals the expected offspring of the branching processes $(B_k^{(i)})_{k \geq 0}$. Define

$$D_n = \{b_k^{(i)} \leq B_k^{(i)}_{r+1} \leq \hat{b}_k^{(i)} \forall i \in [2], k \geq 0\},$$

where $(\hat{b}_k^{(i)})_{k \geq 0}$ and $(\bar{b}_k^{(i)})_{k \geq 0}$ satisfy the recursions $\hat{b}_0^{(i)} = \bar{b}_0^{(i)} = \bar{b}_0^{(i)}$, while, for some $\alpha \in (\frac{1}{2}, 1)$,

$$\hat{b}_{k+1}^{(i)} = \nu_n \hat{b}_k^{(i)} - (\hat{b}_k^{(i)})^\alpha, \quad \bar{b}_{k+1}^{(i)} = \nu_n \bar{b}_k^{(i)} + (\bar{b}_k^{(i)})^\alpha. \quad (3.21)$$

The following lemma investigates the asymptotics of $(\hat{b}_k^{(i)})_{k \geq 1}$ and $(\bar{b}_k^{(i)})_{k \geq 1}$:

**Lemma 3.6** (Asymptotics of $\hat{b}_k^{(i)}$ and $\bar{b}_k^{(i)}$). Assume that $\lim_{n \rightarrow \infty} \nu_n \nu = \nu > 1$, and assume that $\hat{b}_0^{(i)} = B_0^{(i)} \geq r$. Then, there exists an $A = A_r > 1$ with $A_r < \infty$ for $r$ large enough, such that, for all $k \geq 0$,

$$\hat{b}_k^{(i)} \leq A \hat{b}_0^{(i)} \nu_n, \quad \bar{b}_k^{(i)} \geq b_0^{(i)} \nu_n/A. \quad (3.22)$$
Proof. First, obviously, $\bar{b}^{(i)}_k \geq \bar{b}^{(i)}_0 \nu_n^{k}$. Thus, since $\alpha < 1$ and also using that $\nu_n > 1$,

$$\bar{b}^{(i)}_{k+1} = \nu_n \bar{b}^{(i)}_k + (\bar{b}^{(i)}_k) \alpha = \nu_n \bar{b}^{(i)}_k (1 + (\bar{b}^{(i)}_k) \alpha^{-1}) \leq \nu_n \bar{b}^{(i)}_k (1 + r^{-(1-\alpha)} \nu_n^{-(1-\alpha)k}), \quad (3.23)$$

By iteration, this implies the upper bound with $A$ replaced by $\bar{A}$, given by

$$\bar{A} = \prod_{k \geq 0} (1 + r^{-(1-\alpha)} \nu_n^{-(1-\alpha)k}) < \infty. \quad (3.24)$$

Further, note that $\lim_{n \to \infty} \limsup_{n \to \infty} \bar{A} = 1$. For the lower bound, we use that $\bar{b}^{(i)}_k \leq \bar{A} \nu_n^{k}$ to obtain

$$\bar{b}^{(i)}_{k+1} \geq \nu_n \bar{b}^{(i)}_k - \bar{A} \nu_n^{k}. \quad (3.25)$$

We now use induction to show that

$$\bar{b}^{(i)}_k \geq a_k \nu_n^{k}, \quad (3.26)$$

where $a_0 = 1$ and

$$a_{k+1} = a_k - \bar{A} r^{-(1-\alpha)} \nu_n^{(1-\alpha)k-1}. \quad (3.27)$$

The initialization follows, since $\bar{b}^{(i)}_0 = \bar{b}^{(i)}_0$ and $a_0 = 1$. To advance the induction hypothesis, we substitute the induction hypothesis to obtain that

$$\bar{b}^{(i)}_{k+1} \geq a_k \nu_n^{k} r^{1-\alpha} \nu_n^{(1-\alpha)k} \quad (3.28)$$

by (3.27). Finally, $a_k$ is decreasing, and thus $a_k \downarrow a \equiv 1/\bar{A}$, where

$$\bar{A} = 1 - \sum_{k \geq 0} \bar{A} r^{-(1-\alpha)} \nu_n^{-(1-\alpha)k} < \infty$$

for $r$ large enough, so that the claim follows with $A = \bar{A}$, $\alpha = \max\{\bar{A}, \bar{A}_r\}$. Note further that

$$\lim_{n \to \infty} \limsup_{n \to \infty} A_r = 1.$$ 

The following lemma shows that $D_n = \{\bar{b}^{(i)}_k \leq |BP^{(i)}_{r+k}| \leq \bar{b}^{(i)}_k \forall i \in [2], \ k \geq 0\}$ occurs whp when first $n \to \infty$ followed by $r \to \infty$:

**Lemma 3.7** ($D_n$ occurs whp). Assume that $\bar{b}^{(i)}_0 = |BP^{(i)}_r| \geq r$. Then

$$\lim_{r \to \infty} \liminf_{n \to \infty} P(D_n) = 1. \quad (3.29)$$

**Proof.** We will show that $\lim_{r \to \infty} \limsup_{n \to \infty} P(D_n) = 0$. The inequalities $\bar{b}^{(i)}_k \leq |BP^{(i)}_r| \leq \bar{b}^{(i)}_k$ hold by definition. We thus write

$$P(D_n) \leq \sum_{k=1}^\infty P(D_{n,k} \cap D_{n,k-1}), \quad (3.30)$$

where

$$D_{n,k} = \{\bar{b}^{(i)}_k \leq |BP^{(i)}_{r+k}| \leq \bar{b}^{(i)}_k \forall i \in [2]\}. \quad (3.31)$$

Note that, when $|BP^{(i)}_{r+k}| > \bar{b}^{(i)}_k$ and $|BP^{(i)}_{r+k-1}| \leq \bar{b}^{(i)}_k$,

$$|BP^{(i)}_{r+k}| - \nu_n |BP^{(i)}_{r+k-1}| \geq \bar{b}^{(i)}_k - \nu_n \bar{b}^{(i)}_{k-1} = (\bar{b}^{(i)}_{k-1})^\alpha, \quad (3.32)$$

while, when $|BP^{(i)}_{r+k}| < \bar{b}^{(i)}_k$ and $|BP^{(i)}_{r+k-1}| \geq \bar{b}^{(i)}_k$,

$$|BP^{(i)}_{r+k}| - \nu_n |BP^{(i)}_{r+k-1}| < \bar{b}^{(i)}_k - \nu_n \bar{b}^{(i)}_{k-1} = (\bar{b}^{(i)}_{k-1})^\alpha, \quad (3.33)$$
Thus,
\[ D_{r,k}^c \cap D_{n,k-1} \subseteq \{ |BP_{r+k}^{(1)}| - \nu_n|BP_{r+k-1}^{(1)}| \geq (\bar{b}_{k-1}^{(1)})^\alpha \} \cup \{ |BP_{r+k}^{(2)}| - \nu_n|BP_{r+k-1}^{(2)}| \geq (\bar{b}_{k-1}^{(2)})^\alpha \}. \]  
(3.34)

By the Chebychev inequality, conditionally on \( D_{n,k-1} \),
\[ \mathbb{P}\left( |BP_{r+k}^{(1)}| - \nu_n|BP_{r+k-1}^{(1)}| \geq (\bar{b}_{k-1}^{(1)})^\alpha \mid D_{n,k-1} \right) \leq \frac{\text{Var}(BP_{r+k}^{(1)} \mid |BP_{r+k-1}^{(1)}|)}{(\bar{b}_{k-1}^{(1)})^{2\alpha}} \leq \frac{\sigma_n^2 BP_{r+k}^{(1)}}{(\bar{b}_{k-1}^{(1)})^{2\alpha}} \leq \sigma_n^2 (\bar{b}_{k-1}^{(1)})^{1-2\alpha}, \]  
(3.35)

where \( \sigma_n^2 \) is the variance of the offspring distribution given by
\[ \sigma_n^2 = \frac{1}{\ell_n} \sum_{v \in [n]} d_v (d_v - 1)^2 - \nu_n^2, \]  
(3.36)

which is uniformly bounded. Thus, by the union bound for \( i \in \{1, 2\} \),
\[ \mathbb{P}(D_{r,k}^c \cap D_{n,k-1}) \leq 2\sigma_n^2 (\bar{b}_{k-1}^{(i)})^{1-2\alpha}, \]  
(3.37)

and we conclude that
\[ \mathbb{P}(D_{r,n}^c) \leq 2\sigma_n^2 \sum_{k=1}^{\infty} (\bar{b}_{k-1}^{(1)})^{1-2\alpha} + (\bar{b}_{k-1}^{(2)})^{1-2\alpha}. \]  
(3.38)

The claim now follows from Lemma 3.6 and the fact that \( \sigma_n^2 \leq b(b-1)^2 \) remains uniformly bounded. \( \square \)

**Completion of the proof for bounded degrees.** Recall (3.11). Also recall the definition of \( \mathcal{C}_n \) in (3.19), (3.15) and (3.17), and that of \( \mathcal{D}_n \) in (3.20). Let \( \mathcal{G}_n = \mathcal{C}_n \cap \mathcal{D}_n \) be the good event. By (3.19) and Lemma 3.7,
\[ \lim_{r \to \infty} \limsup_{n \to \infty} \mathbb{P}(\partial B_r^{(G_n)}(o_1), |\partial B_r^{(G_n)}(o_2) \geq r, o_1 \leftrightarrow o_2; \mathcal{G}_n^c) = 0, \]  
(3.39)

so that it suffices to investigate \( \mathbb{P}(\partial B_r^{(G_n)}(o_1), |\partial B_r^{(G_n)}(o_2) \geq r, o_1 \leftrightarrow o_2; \mathcal{G}_n) \). On \( \mathcal{G}_n \) (recall (3.17)),
\[ |\partial B_r^{(G_n)}(o_2)| - |BP_{k_n^{(2)}}^{(2)}| \geq -(\bar{m}_n^2 / \ell_n^{1+\delta}). \]  
(3.40)

Further, on \( \mathcal{G}_n \) (recall (3.15)),
\[ |\partial B_r^{(G_n)}(o_1)| - |BP_{k_n^{(1)}}^{(1)}| = 0. \]  
(3.41)

On the event \( \{o_1 \leftrightarrow o_2\} \), we must have that \( \partial B_r^{(G_n)}(o_1) \cap \partial B_r^{(G_n)}(o_2) = \emptyset \). By Lemma 3.6, on \( \mathcal{G}_n \) and when \( \bar{m}_n^2 / \ell_n \to \infty \) sufficiently slowly, \( |\partial B_r^{(G_n)}(o_1)| = \Theta(m_n) \) and \( |\partial B_r^{(G_n)}(o_2)| = \Theta(m_n) \). The same bounds hold for the number of half-edges \( Z_{k_n^{(1)}}^{(1)} \) and \( Z_{k_n^{(2)}}^{(2)} \) incident to \( \partial B_r^{(G_n)}(o_1) \) and \( \partial B_r^{(G_n)}(o_2) \), respectively, since \( Z_{k_n^{(1)}}^{(1)} \geq |\partial B_r^{(G_n)}(o_1)| \) and \( Z_{k_n^{(2)}}^{(2)} \geq |\partial B_r^{(G_n)}(o_2)| \), so that, on \( \mathcal{G}_n \), also \( Z_{k_n^{(1)}}^{(1)} = \Theta(m_n) \) and \( Z_{k_n^{(2)}}^{(2)} = \Theta(m_n) \).

Conditionally on having paired some half-edges incident to \( \partial B_r^{(G_n)}(o_1) \), conditionally on none of them being paired to half-edges incident to \( \partial B_r^{(G_n)}(o_2) \), each further such half-edge has probability at least \( 1 - Z_{k_n^{(2)}}^{(2)} / \ell_n \) to be paired to a half-edge incident to \( \partial B_r^{(G_n)}(o_2) \), thus creating a path between \( o_1 \) and \( o_2 \). The latter conditional probability is independent of the pairing of the earlier half-edges. Thus, the probability that \( \partial B_r^{(G_n)}(o_1) \) is not directly connected to \( \partial B_r^{(G_n)}(o_2) \) is at most
\[ \left(1 - \frac{Z_{k_n^{(2)}}^{(2)}}{\ell_n}\right)^{Z_{k_n^{(1)}}^{(1))}/2}, \]  
(3.42)
since at least $Z_n^{(1)}/2$ pairings need to be performed. This probability vanishes when $m_n \gg n$. As a result, as $n \to \infty$,
\[
\mathbb{P}(|\partial B^{(G_n)}_r(o_1)|, |\partial B^{(G_n)}_r(o_2)| \geq r, o_1 \leftrightarrow o_2; G_n) = o(1),
\]
as required. This completes the proof of (2.9) for $CM_n(d)$ with bounded degrees, and even shows that $\text{dist}_{CM_n(d)}(o_1, o_2) \leq 2r + k_n + k_n + 1$ whp on the event that $|\partial B^{(G_n)}_r(o_1)|, |\partial B^{(G_n)}_r(o_2)| \geq r$. \hfill \Box

**Extension of the proof to unbounded degrees.** To extend the proof to configuration models whose degrees satisfy Conditions 3.1(a)-(b), we apply a degree-truncation technique that allows us to go from such a configuration model to one whose degrees are uniformly bounded. This result makes the proof of the ‘giant is almost local’ condition in (2.46) simpler, and is interesting in its own right:

**Theorem 3.8 (Degree truncation for configuration models).** Consider $CM_n(d)$ with general degrees. Fix $b \geq 1$. There exists a related configuration model $CM_n(d')$ that is coupled to $CM_n(d)$ and satisfies that

(a) the degrees in $CM_n(d')$ are truncated versions of those in $CM_n(d)$, i.e., $d'_v = (d_v \wedge b)$ for $v \in [n]$; and $d'_v = 1$ for $v \in [n'] \setminus [n]$;

(b) the total degree in $CM_n(d')$ is the same as that in $CM_n(d)$, i.e., $\sum_{v \in [n']} d'_v = \sum_{v \in [n]} d_v$;

(c) for all $u, v \in [n]$, if $u$ and $v$ are connected in $CM_n(d')$, then so are $u$ and $v$ in $CM_n(d)$.

The proof of Theorem 3.8 is in Section A.2. We will use Theorem 3.8 with $b = b(\varepsilon)$ so large that
\[
\mathbb{P}(D'_n > b) = \frac{1}{\ell_n} \sum_{v \in [n]} d_v \mathbb{1}_{\{d_v > b\}} \leq \frac{\varepsilon n}{\ell_n}.
\]
This implies that $CM_n(d')$ has at most $(1 + \varepsilon)n$ vertices, and the (at most $\varepsilon n$) vertices in $[n'] \setminus [n]$ all have degree 1, while the vertices in $[n]$ have degree $d'_v \leq b$. As a result, it suffices to prove Theorem 3.2(b) for $CM_n(d')$ instead, and, in the remainder of this proof, we assume that $d'^*_n \leq b$ is uniformly bounded. We thus apply Theorem 2.4 to $CM_n(d')$. We denote the parameters of $CM_n(d')$ by $(p'_k)_{k \geq 1}, \xi'$ and $\zeta'$, respectively, and note that, for $\varepsilon$ as in (3.44), when $\varepsilon \searrow 0$,
\[
p_k' \to p_k, \quad \xi' \to \xi, \quad \zeta' \to \zeta,
\]
Further, with $\mathcal{C}'_{\text{max}}$ the largest connected component in $CM_n(d')$, by (3.44),
\[
|\mathcal{C}'_{\text{max}}| \leq |\mathcal{C}_{\text{max}}| + \varepsilon n,
\]
so that
\[
\mathbb{P}(|\mathcal{C}_{\text{max}}| \geq n(\zeta' - 2\varepsilon)) \to 1,
\]
since $|\mathcal{C}'_{\text{max}}|/n \xrightarrow{\mathbb{P}} \zeta'$. This proves the required lower bound on $|\mathcal{C}_{\text{max}}|$, while the upper bound follows from Corollary 2.1. \hfill \Box

### 3.3. Small-world nature of the configuration model

In this section, we use the above proof to study the **typical distances** in $CM_n(d)$. Such distances have attracted considerable attention, see e.g., [30, 31, 44, 45, 47]. Here, we partially reprove a result from [44]:

**Theorem 3.9 (Typical distances in $CM_n(d)$ for finite-variance degrees).** Consider the configuration model $CM_n(d)$ subject to Conditions 3.1(a)-(b) and where $\mathbb{E}[D^2_n] \to \mathbb{E}[D^2] < \infty$ with $\nu \in (1, \infty)$. Then, conditionally on $\text{dist}_{CM_n(d)}(o_1, o_2) < \infty$,
\[
\text{dist}_{CM_n(d)}(o_1, o_2)/\log n \xrightarrow{\mathbb{P}} 1/\log \nu.
\]
Proof. Recall that we condition on \( \text{dist}_{\text{CM}_{\ell n}(d)}(o_1, o_2) < \infty \). Since \( |\mathcal{G}_{(z)}|/n \xrightarrow{z} 0 \), the majority of pairs of vertices \( u, v \) satisfying \( \text{dist}_{\text{CM}_{\ell n}(d)}(o_1, o_2) < \infty \) satisfy that \( u, v \in \mathcal{G}_{\text{max}} \). Thus, from now on, we will assume that \( o_1, o_2 \in \mathcal{G}_{\text{max}} \), and, particularly, their component sizes are large.

We again start by proving the result for bounded degrees. The proof in (3.39) shows that for pairs of vertices with large connected components, whp as first \( n \to \infty \) followed by \( r \to \infty \),

\[
\text{dist}_{\text{CM}_{\ell n}(d)}(o_1, o_2) \leq k_n + \bar{k}_n + 1 = \frac{\log n}{\log \nu_n} (1 + o_r(1)).
\]

Indeed, Lemma 3.6 implies that \( k_n = (1 + o_r(1)) \log m_n / \log \nu_n \) and \( \bar{k}_n = (1 + o_r(1)) \log \bar{m}_n / \log \nu_n \) on the good event \( \mathcal{G}_n = \mathcal{C}_n \cap \mathcal{D}_n \), so that

\[
k_n + \bar{k}_n + 1 = (1 + o_r(1)) \log (m_n \bar{m}_n) / \log \nu_n = (1 + o_r(1)) \log n / \log \nu_n
\]

when \( m_n \bar{m}_n / n \to \infty \) slowly enough. Further, \( \nu_n \to \nu \), so that, for every \( \epsilon > 0 \),

\[
\mathbb{P}(\text{dist}_{\text{CM}_{\ell n}(d)}(o_1, o_2) / \log n \leq (1 + \epsilon) / \log \nu \mid \text{dist}_{\text{CM}_{\ell n}(d)}(o_1, o_2) < \infty) = 1 - o(1).
\]

This proves the upper bound on \( \text{dist}_{\text{CM}_{\ell n}(d)}(o_1, o_2) \) for uniformly bounded degrees.

We next extend this argument to degrees under Conditions 3.1(a)-(b) and when \( \text{E}[D^2_n] \to \infty \). We again rely on the degree-truncation argument in Theorem 3.8, and let \( \text{CM}_{\ell n}(d') \) be the graph obtained in Theorem 3.8. Recall from (3.44) that \( \text{CM}_{\ell n}(d') \) has at most \((1 + \epsilon) n \) vertices, and the (at most \( \epsilon n \)) vertices in \([n'] \setminus [n] \) all have degree 1, while the vertices in \([n] \) have degree \( d'_v \leq b \).

Let \( \mathcal{G}'_{\text{max}} \) be the giant in \( \text{CM}_{\ell n}(d') \), and \( \mathcal{G}_{\text{max}} \) that in \( \text{CM}_{\ell n}(d) \). By Theorem 3.8, pairs \( u, v \) of vertices in \([n] \) that are connected in \( \mathcal{G}_{\text{max}} \) are also connected in \( \mathcal{G}_{\text{max}} \). Thus, all vertices in \( \mathcal{G}'_{\text{max}} \) are also connected in \( \text{CM}_{\ell n}(d) \), and therefore are also part of \( \mathcal{G}_{\text{max}} \). As a result, differences in \( \mathcal{G}'_{\text{max}} \) and \( \mathcal{G}_{\text{max}} \) only arise through the addition (at most \( \epsilon n \)) degree-one vertices in \([n'] \setminus [n] \).

We conclude that the giants \( \mathcal{G}'_{\text{max}} \) in \( \text{CM}_{\ell n}(d') \) and \( \mathcal{G}_{\text{max}} \) in \( \text{CM}_{\ell n}(d) \) satisfy that \( |\mathcal{G}'_{\text{max}} \Delta \mathcal{G}_{\text{max}}| \leq \epsilon n \) whp for \( b \) sufficiently large, where \( \Delta \) denotes the symmetric difference between sets. Thus, whp, a pair of vertices in \( \mathcal{G}_{\text{max}} \) is also in \( \mathcal{G}'_{\text{max}} \).

Finally, let \( \nu'_n = \sum_{v \in [n']} d'_v (d'_v - 1) / \sum_{v \in [n']} d_v \) denote the expected forward degree of a half-edge for \( \text{CM}_{\ell n}(d') \). When Conditions 3.1(a)-(b) hold and \( \text{E}[D^2_n] \to \infty \), also \( \nu'_n \to \nu' = \nu' (\epsilon) \), where \( \nu' (\epsilon) \to \nu \) when \( \epsilon \searrow 0 \). We conclude that the upper bound on \( \text{dist}_{\text{CM}_{\ell n}(d)}(o_1, o_2) \) for unbounded degrees under Conditions 3.1(a)-(b) follows from that for uniformly bounded degrees.

The lower bound on \( \text{dist}_{\text{CM}_{\ell n}(d)}(o_1, o_2) \) applies more generally, and follows from the fact that

\[
\mathbb{P}(\text{dist}_{\text{CM}_{\ell n}(d)}(o_1, o_2) \leq k) = \mathbb{E}[B^{|\mathcal{G}_{\text{max}}|}(o_1)/n],
\]

together with the fact that, for \( k \geq 1 \),

\[
\mathbb{E}[|\partial B^{|\mathcal{G}_{\text{max}}|}(o_1)|] \leq \frac{\ell_n}{n} n^{-k-1}
\]

by [52, Lemma 5.1]. Thus, with \( k_n = \lfloor (1 - \epsilon) \log n / \log \nu_n \rfloor \),

\[
\mathbb{P}(\text{dist}_{\text{CM}_{\ell n}(d)}(o_1, o_2) \leq (1 - \epsilon) \log n / \log \nu_n) \leq \frac{1}{n} + \sum_{k=1}^{k_n} \frac{\ell_n}{n} n^{-k-1} \leq \frac{1}{n} + \frac{\ell_n}{n} \frac{\nu_n - 1}{\nu_n - 1} = o(1),
\]

as required.

4. Discussion and open problems

In this section, we discuss our proof and results, and state some open problems.
Minimal conditions for the proof. Inspection of the proof of Corollary 2.1 and Theorem 2.2 shows that \(Z_{2k}/n \to \zeta_k\), with \(\zeta_k \leq \zeta\), combined with (2.9), suffices to obtain Theorem 2.2. Theorem 2.4 requires a little more (see (2.26)), while only Theorem 2.5 requires the full local convergence in probability. The use of \(Z_{2k}\) to study connected components (particularly close to criticality) has a long history, and is implicitly present in [25], while being formally introduced in the high-dimensional percolation context in [23], see [24, 49, 50] for applications to percolation on the \(n\)-cube. [41] used it to study the critical behavior for rank-1 inhomogeneous random graphs, and [42] for the phase transition on the Erdős-Rényi random graph.

Related models. We use the configuration model as a proof of concept for the method in this paper, and gave several further examples using Corollary 2.6 in Remark 2.7. This shows that our approach is quite versatile, and here we give some further examples in the literature where the method is used. In [43, Section 2.5], the method is performed for the Erdős-Rényi random graph. That proof can easily be adapted to finite-type inhomogeneous random graphs as studied in the seminal paper by Bollobás, Janson and Riordan [18], as performed in [43, Chapters 3 and 6]. In turn, such an approach also yields the type distribution of the giant, as identified in greater generality in [56, Lemma 4.10]. Further, and in line with Remark 2.7, our results were recently used as a by-product of the work [58] (see also [59]) on component tails in spatial inhomogeneous random graphs. This work, combined with the local convergence in [46], shows that the survival probability of the local limit indeed is the limit of the proportion of vertices in the giant. Here, the local limit is not a tree. This shows that our result has the potential to be widely applicable. Further, the identification of the local structure of the giant that follows from Theorem 2.5 now follows immediately, as it was done in [19, Theorem 25] for \(CM_n(d)\), and applies for example also to spatial inhomogeneous random graphs as studied in [58].

Local convergence. There are many models for which local convergence has been established, and thus our results might be applied. For inhomogeneous random graphs, local convergence was not established explicitly in the seminal paper by Bollobás, Janson and Riordan [18], but the methodology is very related and does allow for a simple proof of it (see [43, Chapter 3] for a formal proof, and [28] for a proof for Erdős-Rényi random graph). For the configuration model, and the related uniform random graph with prescribed degrees, it was proved explicitly by Dembo and Montanari [28], see also [21, 29] as well as [43, Chapter 4] for related proofs. Berger, Borgs, Chayes and Saberi [12] establish local convergence for preferential attachment models (see also [43, Chapter 5] for some extensions). A related preferential attachment model with conditionally independent edges is treated by Dereich and Mörters [32], who again do not formally prove local convergence, but state a highly related result. Random intersection graphs are treated in [61], see also [48]. In most of these models, the size of the giant is established, though mostly in ways that are quite different from the current proof (see [43] for an extensive overview). It would be of interest to see whether the current proof simplifies some of these analyses.

Limiting properties that do or do not follow from local convergence. As stated in the introduction, local convergence is a versatile tool. The number of connected components, the clustering coefficient, local neighborhoods, and edge-neighborhoods are all local quantities (see [43, Chapter 2] for many examples of local quantities). In some cases, some extra local conditions (typically on the degree distribution) need to be made in order to be able to use such results. Some less obvious properties also converge when the graph converges locally. An important and early example is the Ising model partition function [28]. Also the PageRank distribution is local [40]. A property that is almost local is the density of the densest subgraph in a random graph, as shown by Anantharam and Salez [7]. Lyons [62] shows that the exponential growth rate of the number of spanning trees of a finite connected graph can be computed through the local limit. See also [68] for weighted spanning subgraphs, [39] for maximum-weight independent sets, and [13] for the limiting spectral distribution of the graph adjacency matrix.

Our approach shows that while the proportion of vertices in the giant is not a local property, it is ‘almost local’ in the sense that one additional necessary and sufficient condition does identify the limiting proportion of vertices in the giant in terms of the local limit. It would be interesting
to investigate whether the idea of ‘almost local’ properties can be extended to other properties as well. For example, while Theorem 3.9 suggests that also logarithmic distances are an almost local property, the statement in Theorem 3.9 follows from the proof of Theorem 3.2, rather than from a general ‘almost local’ result as in Theorem 2.2. It would be interesting to explore this further, to see whether also related typical distance results might be proved in a similar way (see [30, 31, 44, 45, 47] for such related results).

Percolation on finite graphs. Another area where our results may be useful is percolation on finite graphs. There, it is often not even clear how to precisely define the critical value or critical behavior. We refer to Janson and Warnke [57] and Nachmias and Peres [66] for extensive discussions on the topic. Percolation on random graphs has also attracted substantial attention, see for example Janson [51] or Fountoulakis [36] for the derivation of the limiting percolation threshold for the configuration model. This is related to the locality of the percolation threshold as investigated by Benjamini, Nachmias and Peres [10] in the context of $d$-regular graphs that have large girth, and inspired by a question by Oded Schramm in 2008. Indeed, Schramm conjectured that the critical percolation threshold on a converging sequence of infinite graphs $G_n$ should converge to that of the graph limit. In a transitive setting, local convergence can be used also on infinite graphs (as every vertex is basically the same, thus skipping the necessity of drawing a root uar). See also [5, 9, 65] for related work on sharp threshold for the existence of a giant in the context of expanders.

Recently, Alimohammadi, Borgs and Saberi [4] brought the discussion significantly forward, by showing that the percolation critical value is indeed local for expanders. Further, interestingly, they applied their results to the Barabási-Albert preferential attachment model [8], in the version of Bollobás et al. in [20], to show that $p_c = 0$ and identify the giant for all $p > 0$. The proof relies on a sprinkling argument. It would be interesting to study the relation between our central assumption in (2.9) and $(G_n)_{n \geq 1}$ being a sequence of expander graphs. Further, it would be very interesting to see whether local convergence has consequences for the critical behavior of percolation on locally convergent graph sequences.

Appendix A: Further ingredients for the configuration model

In this section, we provide proofs of technical ingredients used in the proof of Theorem 3.2. This section is organised as follows. In Section A.1, we give some more details on local convergence in probability for the configuration model. In Section A.2, we prove Theorem 3.8 that shows that it suffices to prove Theorem 3.2 for uniformly bounded degrees. In Section A.3, we investigate the coupling of the graph exploration process for $CM_n(d)$ and the $n$-dependent unimodular branching process.

A.1. Local convergence of the configuration model

As mentioned already below Theorem 3.2, there are several ways in which local convergence in probability, as formalised in (2.2), can be proved. Such proofs always consist of two steps. In the first, we investigate the expected number of vertices whose $r$-neighborhood is isomorphic to a specific tree, in the second, we prove that this number is highly concentrated. Convergence of the mean follows from the coupling result in Lemma 3.3 below (see the discussion right below it). Concentration can be proved using martingale techniques (by pairing edges one by one and using Azuma-Hoefding), see [28] or [21]. Alternatively, one can use a second moment method and results similar to Lemma 3.3. See [43, Section 4.1] for such an approach.

A.2. A useful degree-truncation argument for the configuration model

In this section, we prove Theorem 3.8. The proof relies on an ‘explosion’ or ‘fragmentation’ of the vertices $[n]$ in $CM_n(d)$ inspired by [51]. Label the half-edges from 1 to $\ell_n$. We go through the vertices $v \in [n]$ one by one. When $d_v \leq b$, we do nothing. When $d_v > b$, then we let $d'_v = b$, and we keep the $b$ half-edges with the lowest labels. The remaining $d_v - b$ half-edges are exploded from vertex $v$, in that they are incident to vertices of degree 1 in $CM_n(d')$, and are given vertex
labels above \( n \). We give the exploded half-edges the remaining labels of the half-edges incident to \( v \). Thus, the half-edges receive labels both in \( \text{CM}_n(d) \) as well as in \( \text{CM}_n(d') \), and the labels of the half-edges incident to \( v \in [n] \) in \( \text{CM}_n(d') \) are a subset of those in \( \text{CM}_n(d) \). In total, we thus create an extra \( n^+ = \sum_{v \in [n]} (d_v - b)_+ \) ‘exploded’ vertices of degree 1, and \( n' = n + n^+ \).

We then pair the half-edges randomly, in the same way in \( \text{CM}_n(d) \) as in \( \text{CM}_n(d') \). This means that when the half-edge with label \( x \) is paired to the half-edge with label \( y \) in \( \text{CM}_n(d') \), the half-edge with label \( x \) is also paired to the half-edge with label \( y \) in \( \text{CM}_n(d) \), for all \( x, y \in [n] \).

We now check parts (a)-(c). Obviously parts (a) and (b) follow. For part (c), we note that all created vertices have degree 1. Further, for vertices \( u, v \in [n] \), if there exists a path in \( \text{CM}_n(d') \) connecting them, then the intermediate vertices have degree at least 2, so that they cannot correspond to exploded vertices. Thus, the same path of paired half-edges also exists in \( \text{CM}_n(d) \), so that \( u \) and \( v \) are connected in \( \text{CM}_n(d) \) as well.

\[ \square \]

A.3. Coupling of Configuration Neighborhoods and Branching Processes

In this section, we prove Lemma 3.3 and Remark 3.4, as well as Lemma 3.5.

**Proof of Lemma 3.3 and Remark 3.4.** Fix \( m \). We next explain how to jointly construct \((\hat{\mathcal{G}}_n(t), \hat{\mathcal{BP}}_n(t))\) for \( t \in [m] \), given that we have already constructed \((\hat{\mathcal{G}}_n(t), \hat{\mathcal{BP}}_n(t))\) for \( t \in [m-1] \). Further, for each half-edge, we record its status as real or ghost, where the real half-edges correspond to the ones present in both \( \mathcal{G}_n(m - 1) \) and \( \hat{\mathcal{BP}}_n(m - 1) \), while the ghost half-edges are only present in \( \hat{\mathcal{G}}_n(m - 1) \) or \( \hat{\mathcal{BP}}_n(m - 1) \).

To obtain \( \hat{\mathcal{G}}_n(m) \), we take the first unpaired half-edge \( x_m \) in \( \hat{\mathcal{G}}_n(m - 1) \). When this half-edge has the ghost status, then we draw a uniform unpaired half-edge \( y'_m \) and pair \( x_m \) to \( y'_m \) to obtain \( \hat{\mathcal{G}}_n(m) \), and we give all sibling half-edges of \( y'_m \) the ghost status (where we recall that the sibling half-edges of a half-edge \( y \) are those half-edges unequal to \( y \)).

When the half-edge has the real status, it needs to be paired both in \( \hat{\mathcal{G}}_n(m) \) and \( \hat{\mathcal{BP}}_n(m) \). To obtain \( \hat{\mathcal{G}}_n(m) \), this half-edge needs to be paired to a uniform ‘free’ half-edge, i.e., one that has not been paired so far. For \( \hat{\mathcal{BP}}_n(m) \), this restriction does not hold. We now show how these two choices can be conveniently coupled.

For \( \hat{\mathcal{BP}}_n(m) \), we draw a uniform half-edge \( y_m \) from the collection of all half-edges, independently of the past. Let \( U_m \) denote the vertex to which \( y_m \) is incident. We then let the \( m \)th individual in \((\hat{\mathcal{BP}}_n(t))_{t \in [m-1]} \) have precisely \( d_{U_m} - 1 \) children. Note that \( d_{U_m} - 1 \) has the same distribution as \( D_n^* - 1 \) and, by construction, the collection \( (d_{U_i} - 1)_{i \geq 1} \) is iid. This constructs \( \hat{\mathcal{BP}}_n(m) \), except for the statuses of the sibling half-edges incident to \( U_1 \), which we describe below.

For \( \hat{\mathcal{G}}_n(m) \), when \( y_m \) is still free, i.e., has not yet been paired in \((\hat{\mathcal{G}}_n(t))_{t \in [m-1]} \), then we let \( x_m \) be paired to \( y_m \) in \( \hat{\mathcal{G}}_n(m) \), and we have thus also constructed \((\hat{\mathcal{G}}_n(t), \hat{\mathcal{BP}}_n(t))_{t \in [m]} \). We give all the other half-edges of \( U_m \) the status real when \( U_m \) has not yet appeared in \( \mathcal{G}_n(m - 1) \), and otherwise we give them the ghost status. The latter case implies that a cycle appears in \((\hat{\mathcal{G}}_n(t))_{t \in [m]} \). By construction, such a cycle does not occur in \((\hat{\mathcal{BP}}_n(t))_{t \in [m]} \), where re-used vertices are simply repeated several times.

A difference in the coupling arises when \( y_m \) has already been paired in \((\hat{\mathcal{G}}_n(t))_{t \in [m-1]} \), in which case we give all the sibling half-edges of \( U_1 \) the ghost status. For \( \hat{\mathcal{G}}_n(m) \), we draw a uniform unpaired half-edge \( y'_m \) and pair \( x_m \) to \( y'_m \) instead to obtain \( \hat{\mathcal{G}}_n(m) \), and we give all the sibling half-edges of \( y'_m \) the ghost status. Clearly, this might give rise to a difference between \( \hat{\mathcal{G}}_n(m) \) and \( \hat{\mathcal{BP}}_n(m) \).

We continue the above exploration algorithm until it terminates at some time \( T_n \). Since each step pairs exactly one half-edge, we have that \( T_n = |E(C(0))| \), so that \( T_n \leq \ell_n / 2 \) steps. The final result is then \((\hat{\mathcal{G}}_n(t), \hat{\mathcal{BP}}_n(t))_{t \in [\ell_n]} \). At this moment, however, the branching process tree \((\hat{\mathcal{BP}}_n(t))_{t \geq 1} \) has not been fully explored, since all the tree vertices corresponding to ghost half-edges in \((\hat{\mathcal{BP}}_n(t))_{t \geq 1} \) have not been explored. We complete the tree exploration \((\hat{\mathcal{BP}}_n(t))_{t \geq 1} \) by iid drawing children of all the ghost tree vertices until the full tree is obtained.
A.3 Coupling of configuration neighborhoods and branching processes

We emphasise that the law of \((\hat{B}P_n(t))_{t \geq 1}\) obtained above is not the same as that of \((BP_n(t))_{t \geq 1}\), since the order in which half-edges are paired is chosen so that \((\hat{G}_n(t))_{t \in [T_n]}\) has the same law as the graph exploration process \((G_n(t))_{t \in [T_n]}\). However, with \(\sigma_n\) the first moment that a ghost half-edge is paired, we have that \((\hat{B}P_n(t))_{t \in [\sigma_n]}\) does have the same law as \((BP_n(t))_{t \geq 1}\).

There are two sources of differences between \((\hat{G}_n(t))_{t \geq 0}\) and \((BP_n(t))_{t \geq 0}\):

**Half-edge re-use:** In the above coupling, a half-edge re-use occurs when \(y_m\) had already been paired and is being re-used in the branching process. As a result, for \((\hat{G}_n(t))_{t \in [\sigma_n]}\), we need to redraw \(y_m\) to obtain \(y_m^\prime\) that is instead used in \((\hat{G}_n(t))_{t \leq [\sigma_n]}\);

**Vertex re-use:** A vertex re-use occurs when \(U_m = U_m\) for some \(m < m\). In the above coupling, this means that \(y_m\) is a half-edge that has not yet been paired in \((\hat{G}_n(t))_{t \in [\sigma_n-1]}\), but it is incident to a half-edge that has already been paired in \((\hat{G}_n(t))_{t \in [\sigma_n-1]}\). In particular, the vertex \(U_m\) to which it is incident has already appeared in \((\hat{G}_n(t))_{t \in [\sigma_n-1]}\), and it is being re-used in the branching process. In this case, a copy of \(U_m\) appears in \((\hat{B}P_n(t))_{t \in [\sigma_n]}\), while a cycle appears in \((\hat{G}_n(t))_{t \in [\sigma_n]}\).

We continue by providing a bound on both contributions:

**Half-edge re-use.** Up to time \(m - 1\), exactly \(2m - 1\) half-edges are forbidden to be used by \((\hat{G}_n(t))_{t \leq m}\). The probability that the half-edge \(y_m\) equals one of these these \(2m - 1\) previously chosen half-edges is at most

\[
\frac{2m - 1}{\ell_n}. \tag{A.1}
\]

Hence the expected number of half-edge re-uses before time \(m_n\) is at most

\[
\sum_{m=1}^{m_n} \frac{2m - 1}{\ell_n} = \frac{m_n^2}{\ell_n} = o(1), \tag{A.2}
\]

when \(m_n = o(\sqrt{n})\).

**Vertex re-use.** The probability that vertex \(i\) is chosen in the \(m\)th draw is equal to \(d_i/\ell_n\). The probability that vertex \(i\) is drawn twice before time \(m_n\) is at most

\[
\frac{m_n(m_n - 1)}{2\ell_n} \left(\frac{d_i^2}{\ell_n^2}\right). \tag{A.3}
\]

By the union bound, the probability that a vertex re-use has occurred before time \(m_n\) is at most

\[
\frac{m_n(m_n - 1)}{2\ell_n} \sum_{i \in [n]} \frac{d_i^2}{\ell_n} \leq m_n^2 \frac{d_{\max}}{\ell_n} = o(1), \tag{A.4}
\]

by Conditions 3.1(a)-(b) and when \(m_n = o(\sqrt{n}/d_{\max})\). This completes the coupling part of Lemma 3.3, including the bound on \(m_n\) as formulated in Remark 3.4. It is straightforward to check that the exploration can be performed from the two sources \((o_1, o_2)\) independently, thus establishing the requested coupling to two independent branching processes claimed in Remark 3.4.

Finally, we adapt the above argument to prove Lemma 3.5:

**Proof of Lemma 3.5.** Define \(a_n = (\tilde{m}_n^2/\ell_n)^{1+\delta}\) where \(\delta > 0\). Recall from (3.17) that

\[
C_n(2) \equiv C_n(2, 1) \cap C_n(2, 2) = \{ (|\partial B_k^{G_n}(o_2)|)_{k \leq \xi_n} = (|BP_k^{(2)}|)_{k \leq \xi_n} \} \cap \{ (|\partial B_k^{G_n}(o_2)| - |BP_k^{(2)}|) \leq (\tilde{m}_n^2/\ell_n)^{1+\delta} \forall k \in [\xi_n + 1] \}, \tag{A.5}
\]

where \(C_n(2, 1)\) occurs whp. Also recall the definition of \(D_n\) right above Lemma 3.7.
We apply a first-moment method and bound
\[ P(C_n(2)^c \cap C_n(2, 1) \cap D_n) \leq \frac{1}{a_n} \mathbb{E} \left[ \mathbb{I}_{C_n(2, 1)} \left( |B_{k_n}^{(G_n)}(\phi_2)| - |\overline{B}_{k_n}^{(G_n)}| \right) \right] \]
\[ = \frac{1}{a_n} \mathbb{E} \left[ \mathbb{I}_{C_n(2, 1) \cap D_n} \left( |B_{k_n}^{(G_n)}(\phi_2)| - |\overline{B}_{k_n}^{(G_n)}| \right) + \left( |B_{k_n}^{(G_n)}(\phi_2)| - |\overline{B}_{k_n}^{(G_n)}| \right)_{+} \right] \]
\[ \leq \frac{1}{a_n} \mathbb{E} \left[ \mathbb{I}_{C_n(2, 1) \cap D_n} \left( |\bar{G}_n(m_n)| - |\overline{B}_{n}^{(m_n)}| \right)_{+} + \mathbb{I}_{C_n(2, 1) \cap D_n} \left( |\bar{G}_n(m_n) - |\overline{B}_{n}^{(m_n)}| \right)_{-} \right], \]
where the notation is as in Lemma 3.3 and \( |\bar{G}_n(m_n)| \) and \( |\overline{B}_{n}^{(m_n)}| \) denote the number of half-edges and individuals found up to the \( m_n \)th step of the exploration starting from \( \phi_2 \), and \( m_n = (b+1)\bar{m}_n \), while \( x_+ = \max\{0, x\} \) and \( x_- = \max\{0, -x\} \).

To bound these expectations, we adapt the proof of Lemma 3.3 to our setting. We start with the first term in (A.6), for which we use the exploration up to size \( \bar{m}_n \) used in the proof of Lemma 3.3. We note that the only way that \( |\bar{G}_n(t+1)| - |\bar{G}_n(t)| \) can be larger than \( |\overline{B}_{n}(t+1)| - |\overline{B}_{n}(t)| \) is when a half-edge re-use occurs. This gives rise a primary ghost, which then possibly gives rise to some further secondary ghosts, i.e., ghost vertices that are found by pairing a ghost half-edge. Since all degrees are bounded by \( b \), on \( C_n(2, 1) \) where the first \( k_n \) generations are perfectly coupled, the total number of secondary ghosts, together with the single primary ghost, is at most
\[ c_n = \sum_{k=0}^{\bar{k}_n - k_n} (b-1)^k = \frac{(b-1)^{\bar{k}_n - k_n} + 1}{b - 2}, \]
which tends to infinity. On \( D_n \), we can let \( \bar{k}_n - k_n \to \infty \) arbitrarily slowly. Therefore, on \( C_n(2, 1) \cap D_n \),
\[ |\bar{G}_n(\bar{m}_n)| - |\overline{B}_{n}(\bar{m}_n)| \leq c_n \#\{\text{half-edge re-uses up to time } \bar{m}_n\}. \]

Thus, by (A.2),
\[ \mathbb{E} \left[ \mathbb{I}_{C_n(2, 1) \cap D_n} \left( |\bar{G}_n(\bar{m}_n)| - |\overline{B}_{n}(\bar{m}_n)| \right)_{+} \right] \leq c_n \bar{m}_n^2. \]

We continue with the second term in (A.6), which is similar. We note that \( |\bar{G}_n(t+1)| - |\bar{G}_n(t)| \) can be smaller than \( |\overline{B}_{n}(t+1)| - |\overline{B}_{n}(t)| \) when a half-edge re-use occurs, or when a vertex re-use occurs. Thus, again using that the total number of secondary ghosts, together with the single primary ghost, is at most \( c_n \), on \( C_n(2, 1) \cap D_n \),
\[ |\overline{B}_{n}^{(m_n)}| - |\bar{G}_n(\bar{m}_n)| \leq c_n \#\{\text{half-edge and vertex re-uses up to time } \bar{m}_n\}. \]

As a result, by (A.2) and (A.4),
\[ \mathbb{E} \left[ \mathbb{I}_{C_n(2, 1) \cap D_n} \left( |\bar{G}_n(\bar{m}_n)| - |\overline{B}_{n}^{(m_n)}| \right)_{+} \right] \leq c_n \bar{m}_n^2 \ell_n. \]

We conclude that
\[ P(C_n(2)^c) \leq \frac{1}{a_n} 2c_n \bar{m}_n^2 \ell_n = O\left( \left( \frac{\ell_n}{\bar{m}_n^2} \right)^{3/2} \right) = o(1), \]
when taking \( c_n \to \infty \) such that \( c_n = o((\bar{m}_n^2/\ell_n)^{3/2}) \).

\[ \Box \]

Acknowledgements. The work in this paper was supported by the Netherlands Organisation for Scientific Research (NWO) through Gravitation-grant NETWORKS-024.002.003. I thank Joel Spencer for proposing the ‘easiest solution’ to the giant component problem for the Erdős-Rényi random graph, which inspired this paper. I thank Shankar Bhamidi for useful and inspiring discussions, Souvik Dhara for literature references, and Christian Borgs for suggestions on local convergence and encouragement.
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