Reductions of the \((4 + 1)\)-dimensional Fokas equation and their solutions

Yulei Cao · Jingsong He∗ · Yi Cheng · Dumitru Mihalache

Received: date / Accepted: date

Abstract An integrable extension of the Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) equations is investigated in this paper. We will refer to this integrable extension as the \((4 + 1)\)-dimensional Fokas equation. The determinant expressions of soliton, breather, rational, and semi-rational solutions of the \((4 + 1)\)-dimensional Fokas equation are constructed based on the Hirota’s bilinear method and the KP hierarchy reduction method. The complex dynamics of these new exact solutions are shown in both three-dimensional plots and two-dimensional contour plots. Interestingly, the patterns of obtained high-order lumps are similar to those of rogue waves in the \((1 + 1)\)-dimensions by choosing different values of the free parameters of the model. Furthermore, three kinds of new semi-rational solutions are presented and the classification of lump fission and fusion processes is also discussed. Additionally, we give a new way to obtain rational and semi-rational solutions of \((3 + 1)\)-dimensional KP equation by reducing the solutions of the \((4 + 1)\)-dimensional Fokas equation. All these results show that the \((4 + 1)\)-dimensional Fokas equation is a meaningful multidimensional extension of the KP and DS equations. The obtained results might be useful in diverse fields such as hydrodynamics, nonlinear optics and photonics, ion-acoustic waves in plasmas, matter waves in Bose-Einstein condensates, and sound waves in ferromagnetic media.

∗Corresponding author, Jingsong He: hejingsong@szu.edu.cn; jshe@ustc.edu.cn

Yulei Cao, Yi Cheng
School of Mathematical Sciences, USTC, Hefei, Anhui 230026, P. R. China

Jingsong He
Institute for Advanced Study, Shenzhen University, Shenzhen, Guangdong 518060, P. R. China

Dumitru Mihalache
Horia Hulubei National Institute of Physics and Nuclear Engineering, Magurele, RO 077125, Romania
Keywords 4D Fokas equation · KP hierarchy reduction method · Rational solution · Semi-rational solution · 3D KP equation

1 Introduction

The research field of solitary waves is in fact an interdisciplinary research area that has been deeply studied both theoretically and experimentally. Solitary waves in hydrodynamics originated from the accidental discovery of Russell in 1834 [1]. Nevertheless, he failed to give a rigorous proof of the existence of such special type of waves. More than 60 years later, Korteweg and de Vries [2] made a comprehensive analysis of these solitary waves and established a mathematical model of shallow water waves that adequately describes their complex dynamics. About 70 years later, in 1965, Zabusky and Kruskal [3] found numerically that such solitary waves have the property of elastic scattering, and called them “solitons”. This pioneering work of Zabusky and Kruskal was a milestone in the area of solitons and all that. Since then, the study of solitons has begun to flourish in various fields such as nonlinear optics and optical fibers [4, 5], condensed matter, fluid mechanics, and plasma physics. The current research mainly focuses on (1 + 1)-dimensional (1D) and (2 + 1)-dimensional (2D) systems. However the physical space in reality is (3 + 1)-dimensional (3D) and one of the most important open problems in soliton theory is to construct integrable nonlinear partial differential equations (NPDEs) in higher than two spatial dimensions. Therefore, the research value of high-dimensional nonlinear systems is enormous. In order to seek new high-dimensional integrable NPDEs, many researchers have made great efforts during the past decades [6–26]. But, there are still many meaningful open problems to be addressed. With the increasing number of variables, solving high-dimensional NPDEs will be very difficult. Therefore, it is a challenging work to obtain exact solutions of high-dimensional nonlinear systems. Furthermore, a natural problem is whether the exact solutions of high-dimensional NPDEs can be reduced to the exact solutions of low-dimensional NPDEs?

Inspired by the above problems, we consider the (4 + 1)-dimensional (4D) Fokas equation [27]:

\[ u_{x_1 t} - \frac{1}{4} u_{x_1} u_{x_1 x_1 x_2} + \frac{1}{4} u_{x_1 x_2} u_{x_2 x_2} + \frac{3}{2} (u^2)_{x_1 x_2} - \frac{3}{2} u y_{y_1 y_2} = 0. \]  

(1)

This equation was introduced by Fokas in 2006 [27], being an integrable extension of the Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) equations. Because of the important physical applications of KP and DS equations, the 4D Fokas equation may be used to describe surface and internal waves in rivers with different physical situations. Solitons [28, 29], quasi-periodic solutions [30], lumps [31, 32] and lump-soliton solutions [33] for the 4D Fokas equation have been investigated. However, these studies are far from being complete. To the best of authors’ knowledge, high-order rational and semi-rational solutions for the 4D Fokas equation have never been reported. In this
paper, we mainly focus on the new exact solutions of the 4D Fokas equation, and how to reduce the exact solutions of the 4D Fokas equation to the exact solutions of low-dimensional NPDEs.

The structure of this paper is as follows. In section 2 the determinant expressions of soliton and breather solutions are constructed by using the KP hierarchy reduction method. In sections 3 and 4 high-order rational and semi-rational solutions are generated for the 4D Fokas equation and the complex dynamic behavior of the corresponding solutions are shown by both three-dimensional plots and two-dimensional contour plots. Then in sections 5 a new way for obtaining rational and semi-rational solutions of 3D KP equation is presented. Finally, in section 6 we discuss and summarize our results.

2 Soliton and breather solutions in the determinant form

In this section, we introduce the determinant expression of soliton and breather solutions for the 4D Fokas equation. Through the following transformation:

\[ x = k_1 x_1 + k_2 x_2, \]

the 4D Fokas equation (1) becomes the following 3D equation

\[ u_{xt} + \frac{1}{4} k_2 (k_2^2 - k_1^2) u_{xxxx} + \frac{3k_2}{2} (u^2)_{xx} - \frac{3}{2k_1} u_{y_1 y_2} = 0. \] (2)

Additionally, if we further make the transformation

\[ y = k_3 y_1 + k_4 y_2, \]

then the 4D Fokas equation becomes

\[ u_{xt} + \frac{1}{4} k_2 (k_2^2 - k_1^2) u_{xxxx} + \frac{3k_2}{2} (u^2)_{xx} - \frac{3k_3 k_4}{2k_1} u_{yy} = 0. \] (3)

Now we make the variable transformation

\[ u = (k_2^2 - k_1^2) (\ln f)_{xx}. \] (4)

Then the 4D Fokas equation (1) is transformed into the following bilinear form:

\[ [D_x^4 + \frac{4}{k_2} (k_2^2 - k_1^2) D_x D_t - \frac{6k_3 k_4}{k_1 k_2} (k_2^2 - k_1^2) D_y^2] f \cdot f = 0, \] (5)

where \( D \) is Hirota’s bilinear differential operator \[34\]. Applying the change of independent variables

\[ z_1 = x, \quad z_2 = \sqrt{\frac{k_1 k_2 (k_2^2 - k_1^2)}{2k_3 k_4}} iy, \quad z_3 = -2k_2 (k_2^2 - k_1^2) t, \] (6)
the bilinear form (5) can be transformed into the following bilinear equation of the KP hierarchy [8]:

$$[D^4_{z_1} - 4D_{z_1}D_{z_2} + 3D^2_{z_2}]f \cdot f = 0$$  

(7)

According to Sato theory [35,36], we construct the Gram determinant solutions of the 4D Fokas equation.

**Theorem 1.** The 4D Fokas equation (1) admits the following soliton and breather solutions:

$$u = (k_2^2 - k_1^2)(\ln f)_{xx}, \quad x = k_1 x_1 + k_2 x_2,$$

(8)

with

$$f = \det_{1 \leq i,j \leq N}(m^{(n)}_{i,j}), \quad \text{where} \quad m^{(n)}_{i,j} = \delta_{ij} + \frac{1}{p_i + p_j} \left( -\frac{p_i}{p_j} \right)^n e^{\xi_i + \xi'_j},$$  

(9)

$$\xi_j = k_1 p_i x_1 + k_2 p_i x_2 + \sqrt{\frac{k_1 k_2 (k_2^2 - k_1^2)}{-2k_3 k_4}} p_i^2 y_1$$

$$+ \sqrt{\frac{k_1 k_2 (k_2^2 - k_1^2)}{-2k_3 k_4}} p_i^2 y_2 - k_2 (k_2^2 - k_1^2) p_i^3 t + \xi_0.$$  

(10)

Here $\delta_{ij} = 0, 1$, $p_i$ and $\xi_{i0}$ are arbitrary complex constants, $i, j$, and $N$ are arbitrary positive integers, and the asterisk denotes the complex conjugation. We must emphasize that $\frac{k_1 k_2 (k_2^2 - k_1^2)}{2k_3 k_4} > 0$ must hold. In order to prove Theorem 1, we first introduce the following Lemma.

**Lemma 1.** The bilinear equation of KP hierarchy (7) has solutions

$$\tau_n = \det_{1 \leq i,j \leq N}(m^{(n)}_{i,j}),$$  

(11)

with the matrix element $m^{(n)}_{i,j}$ satisfying the following differential and difference relations

$$\partial_{z_1} m^{(n)}_{i,j} = \varphi^{(n)}_i \psi^{(n)}_j,$$

$$m^{(n+1)}_{i,j} = m^{(n)}_{i,j} + \varphi^{(n)}_i \psi^{(n+1)}_j,$$

$$\partial_{z_2} m^{(n)}_{i,j} = \varphi^{(n+1)}_i \psi^{(n)}_j + \varphi^{(n)}_i \psi^{(n-1)}_j,$$

$$\partial_{z_3} m^{(n)}_{i,j} = \varphi^{(n+2)}_i \psi^{(n)}_j + \varphi^{(n+1)}_i \psi^{(n-1)}_j + \varphi^{(n)}_i \psi^{(n-2)}_j,$$

$$\partial_{z_4} \varphi = \varphi^{(n+k)}_i, \quad \partial_{z_4} \psi_i = -\psi^{(n-k)}_i, (k = 1, 2, 3).$$  

(12)

Here $m^{(n)}_{i,j} \varphi^{(n)}_i$, and $\psi^{(n)}_j$ are functions of the variables $z_1, z_2$, and $z_3$. 
Proof of Lemma 1. Reusing the differential of determinant and the expansion formula of bordered determinant \[35,36\], the derivatives of the \( \tau \) functions can be expressed by the following bordered determinants:

\[
\begin{align*}
\partial_{z_1}\tau_n &= \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n)} & 0 \end{vmatrix}, \\
\partial_{z_2}\tau_n &= \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ \psi_j^{(n)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n-1)} & 0 \end{vmatrix}, \\
\partial_{z_3}\tau_n &= \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+2)} \\ \psi_j^{(n)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ \psi_j^{(n-1)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n-2)} & 0 \end{vmatrix}, \\
\partial_{z_4}\tau_n &= \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+3)} \\ \psi_j^{(n)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ \psi_j^{(n-2)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n-3)} & 0 \end{vmatrix}, \\
\partial_{z_4}^2\tau_n &= \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+3)} \\ \psi_j^{(n)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ \psi_j^{(n-2)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+2)} \\ \psi_j^{(n-1)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n-3)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n-4)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n-5)} & 0 \end{vmatrix}.
\end{align*}
\]

As a result:

\((\partial_{z_1}^3 - 4\partial_{z_2} \partial_{z_3} + 3\partial_{z_2}^2)\tau_n \times \tau_n = 0,\)

\(4\partial_{z_2} \partial_{z_3} \tau_n - (\partial_{z_2}^2 \tau_n)^2 + 6(\partial_{z_1} \tau_n)^2 - 6(\partial_{z_2}^2 \tau_n)^2 = 0.\)

This completes the proof of Lemma 1. Then, we will prove Theorem 1 with Lemma 1.

Proof of Theorem 1. In order to construct soliton and breather solutions for the bilinear equation \[7\], we choose functions \(m_{i,j}^{(n)}, \varphi_i^{(n)},\) and \(\psi_j^{(n)}\) as follows

\[
\begin{align*}
m_{i,j}^{(n)} &= \delta_{ij} + \frac{1}{p_i + q_j} \varphi_i^{(n)} \psi_j^{(n)}, \\
\varphi_i^{(n)} &= p_i^n e^{\xi_i}, \\
\psi_j^{(n)} &= (-q_j)^n e^{\eta_j},
\end{align*}
\]

where

\[
\begin{align*}
\xi_i &= p_i z_1 + \frac{p_i^2}{2} z_2 + \frac{p_i^3}{3} z_3 + \xi_0, \\
\eta_j &= q_j z_1 - \frac{q_j^2}{2} z_2 + \frac{q_j^3}{3} z_3 + \eta_0,
\end{align*}
\]

and \(p_i, q_j, \xi_0,\) and \(\eta_0\) are arbitrary complex constants. Through the following restrictions:

\[
z_1 = k_1 x_1 + k_2 x_2, \quad z_2 = \sqrt{\frac{k_1 k_2 (k_3^2 - k_4^2)}{2 k_3 k_4}} (k_3 y_1 + k_4 y_2),
\]

...
and then setting \( f = \tau_0, \delta_{ij} = 0, 1 \), the solutions of bilinear equation \([9]\) can be transformed into the solutions of the 4D Fokas equation. This completes the proof of Theorem 1. Without losing generality, we take \( k_1 = 1, k_2 = 2, k_3 = 1, \) and \( k_4 = 1 \) in this section.

### 2.1 N-soliton solutions

Equation \([1]\) admits \( N \)-soliton solutions, assuming \( \delta_{ij} = 1 \) when \( i = j \), and \( \delta_{ij} = 0 \) when \( i \neq j \) in \([8]\). The one-soliton solution \( u_{1s} \) is generated by taking \( N = 1 \) and \( p_i = p_{1R} - ip_{1I} \):

\[
u_{1s} = \frac{24p^3_{1R}e^{2p_{1I}}[x_1 + 2x_2 - 2\sqrt{3}p_{1I}(y_1 + y_2) - 6(p^2_{1R} - 3p^2_{1I})t]}{(2p_{1I} + e^{2p_{1I}}[x_1 + 2x_2 - 2\sqrt{3}p_{1I}(y_1 + y_2) - 6(p^2_{1R} - 3p^2_{1I})t])^2}.
\]

From the above expressions, it is not difficult to calculate that the maximum amplitude of the one-soliton solution is \( 3\sqrt{3}p_{1R} \); when \( x_1, y_1 \to \pm \infty \) solution \( u_{1s} \) approaches to the constant background plane 0 in the \((x_1, y_1)\)-plane [see Fig.1(a)]. The velocity and center of the soliton are \( 6p_{1R}, 18p_{1I} \) and \( x_1 + 2x_2 - 2\sqrt{3}p_{1I}(y_1 + y_2) - 6(p^2_{1R} - 3p^2_{1I})t \), respectively. By taking the parameters \( N = 2, p_1 = 1 - \frac{i}{2} \) and \( p_2 = 1 + \frac{i}{2} \) in equation \([8]\) we obtain the expression of the two-soliton solution \( u_{2s} \) [see Fig.1(b)]:

\[
u_{2s} = \frac{120e^{4\xi_{11}} + 120e^{4\xi_{12}} + 960e^{14} + 2400e^{14} + 2400e^{15}}{(e^{\xi_{11}} + 10e^{\xi_{12}} + 10e^{\xi_{13}} + 20e^{\xi_{14}})^2},
\]

\[
\zeta_{11} = 4x_1 + 8x_2 + 2\sqrt{3}y_1 + 2\sqrt{3}y_2, \quad \zeta_{14} = 2\sqrt{3}y_1 + 2\sqrt{3}y_2 + 6t,
\]

\[
\zeta_{12} = 2x_1 + 4x_2 + 4\sqrt{3}y_1 + 4\sqrt{3}y_2 + 3t, \quad \zeta_{13} = 2x_1 + 4x_2 + 3t,
\]

\[
\zeta_{11} = \zeta_{11} + \zeta_{12}, \quad \zeta_{12} = \zeta_{11} + \zeta_{13}, \quad \zeta_{13} = \zeta_{12} + \zeta_{13},
\]

\[
\zeta_{14} = \zeta_{12} + \zeta_{14}, \quad \zeta_{15} = \zeta_{13} + \zeta_{14}.
\]

Additionally, taking the parameters \( N = 3, p_1 = 1 - \frac{i}{2}, p_2 = 1 + \frac{i}{2} \), and \( p_3 = 1 \) in equation \([8]\), the three-soliton solution is obtained. We also give the expression of the three-soliton solution in the \((x_1, y_1)\)-plane [see Fig.1(c)] in which \( f \) is expressed as

\[
f = 1 + \frac{1}{2}e^{2x_1 + 4x_2 + 2\sqrt{3}y_1 + 2\sqrt{3}y_2 - 3t} + \frac{1}{68}e^{4x_1 + 8x_2 + 2\sqrt{3}y_1 + 2\sqrt{3}y_2 - 15t}
\]

\[
+ \frac{1}{68}e^{4x_1 + 8x_2 - 2\sqrt{3}y_1 - 2\sqrt{3}y_2 - 15t} + \frac{1}{2}e^{2x_1 + 4x_2 - 2\sqrt{3}y_1 - 2\sqrt{3}y_2 - 3t}
\]

\[
+ \frac{1}{11560}e^{6x_1 + 12x_2 - 18t} + \frac{1}{2}e^{2x_1 + 4x_2 - 12t} + \frac{1}{20}e^{4x_1 + 8x_2 - 6t}.
\]
Fig. 1 Dynamic behavior of solutions of the 4D Fokas equation defined in equation (8). (a): one-soliton solution with parameters $N = 1, \delta_{11} = 1, p_1 = 1 - \frac{i}{2}, x_2 = 0, y_2 = 0$, and $t = 0$; (b): two-soliton solution with parameters $N = 2, \delta_{11} = 1, \delta_{12} = 0, \delta_{21} = 0, \delta_{22} = 1, p_1 = 1 - \frac{i}{2}, p_2 = 1 + \frac{i}{2}, x_2 = 0, y_2 = 0$, and $t = 0$; (c): three-soliton solution with parameters $N = 3, \delta_{jj} = 1, \delta_{ij} = 0 (i, j = 1, 2, 3), p_1 = 1 - \frac{i}{2}, p_2 = 1 + \frac{i}{2}, p_3 = 1, x_2 = 0, y_2 = 0$, and $t = 0$.

Fig. 2 A hybrid of a V-type soliton and one breather of the 4D Fokas equation with parameters $N = 2, p_1 = \frac{1}{2} + i, p_2 = \frac{1}{2} - i, \delta_{12} = 1 (i, j = 1, 2), x_2 = 0, y_2 = 0$, and $t = 0$ in equation (8). Panel (b) is the contour plot of panel (a).

2.2 A hybrid of a V-type soliton and breathers

In addition to the soliton solutions, equation (1) admits a hybrid of V-type soliton and breather solutions, assuming $N \geq 2$ and $\delta_{ij} = 1$ and some parameters $p_i$ are complex in equation (8). We first consider the case of $N = 2$ and $\delta_{ij} = 1$. The following parameters are further taken in equation (8):

$$p_1 = \frac{1}{2} + i, \quad p_2 = \frac{1}{2} - i, \quad \delta_{12} = 1, \quad \delta_{21} = 1,$$

and the mixed solution consisting of a V-type soliton and one breather solution is derived, see Fig. 2. For this mixed solution the expression of $f$ is as follows:

$$f = \cosh(\gamma_1 - 2\sqrt{3}(y_1 + y_2)) + \sinh(\gamma_1 - 2\sqrt{3}(y_1 + y_2)) + \cosh(\gamma_1 + 2\sqrt{3}(y_1 + y_2))$$

$$+ \sinh(\gamma_1 - 2\sqrt{3}(y_1 + y_2)) + \frac{4}{5}[\cosh(2\gamma_1) + \sinh(2\gamma_1)] - \frac{4}{5}[\cosh(\gamma_1)]$$

$$+ \sinh(\gamma_1)] \sin(\gamma_2) - \frac{2}{5}[\cosh(\gamma_1) + \sinh(\gamma_1)] \cos(\gamma_2),$$

(16)
where
\[ \gamma_1 = x_1 + 2x_2 + \frac{33}{2}t, \quad \gamma_2 = 2x_1 + 4x_2 + 3t. \] 

Furthermore, for larger \( N \), we can derive the mixed solution consisting of a V-type soliton and more breathers. For example, when we take the parameters \( N = 3, p_1 = \frac{1}{2} + i, p_2 = \frac{1}{2} - i, p_3 = \frac{3}{2}, \delta_{ij} = 1 (i, j = 1, 2, 3) \) in equation (8), the mixed solution consisting of a V-type soliton and two breathers is presented in Fig. 3.

![Fig. 3](image_url) A hybrid of a V-type soliton and two breathers of the 4D Fokas equation with parameters \( N = 3, p_1 = \frac{1}{2} + i, p_2 = \frac{1}{2} - i, p_3 = \frac{3}{2}, \delta_{ij} = 1 (i, j = 1, 2, 3), x_2 = 0, y_2 = 0, \) and \( t = 0 \) in (8). Panel (b) is the contour plot of panel (a).

### 3 Rational solutions in the determinant form

The rational solutions of low-dimensional integrable systems have been extensively investigated. However, there are few studies of rational solutions in high-dimensional systems. Inspired by the works of Ohta and Yang [36–38], the rational solutions of the 4D Fokas equation are constructed by introducing the following Lemma.

**Lemma 2.** The bilinear equation of KP hierarchy (7) has solutions
\[ \tau'_n = \det_{1 \leq i, j \leq N} (M_{2i-1,2j-1}^{(n)}), \] 
with the matrix element \( M_{i,j}^{(n)} \) satisfying the following differential and difference relations
\[
\begin{align*}
\partial_{z_1} M_{i,j}^{(n)} &= \Phi_i^{(n)} \psi_j^{(n)}, \\
M_{i,j}^{(n+1)} &= M_{i,j}^{(n)} + \Phi_i^{(n)} \psi_j^{(n+1)}, \\
\partial_{z_2} M_{i,j}^{(n)} &= \Phi_i^{(n+1)} \psi_j^{(n)} + \Phi_i^{(n)} \psi_j^{(n-1)}, \\
\partial_{z_3} M_{i,j}^{(n)} &= \Phi_i^{(n+2)} \psi_j^{(n)} + \Phi_i^{(n+1)} \psi_j^{(n-1)} + \Phi_i^{(n)} \psi_j^{(n-2)}, \\
\partial_{z_4} \Phi &= \Phi_i^{(n+k)}, \\
\partial_{z_4} \psi &= -\psi_i^{(n-k)}, \quad (k = 1, 2, 3).
\end{align*}
\]
Here \( M^{(n)}_{i,j}, \phi_j^{(n)}, \) and \( \psi_j^{(n)} \) are functions of the variables \( z_1, z_2, \) and \( z_3. \) The above relations were proven in [30], hence we omit here the proof. The functions \( \phi_j^{(n)}, \psi_j^{(n)}, \) and \( M^{(n)}_{i,j} \) are defined by
\[
\phi_j^{(n)} = A_i p^n e^{\xi}, \quad \psi_j^{(n)} = B_j (-q)^n e^{\eta}, \\
M^{(n)}_{i,j} = \int \phi_j^{(n)} \psi_j^{(n)} dz_1 = A_i B_j \frac{1}{p+q} (-\frac{p}{q})^n e^{\xi+\eta},
\]
where
\[
A_i = \sum_{k=0}^{i} \frac{c_k}{(i-k)!} (p \partial_p + \xi + j)^{i-k}, \quad B_j = \sum_{l=0}^{j} \frac{d_l}{(j-l)!} (q \partial_q + \eta - j)^{j-l},
\]
\[
\xi = p z_1 + p^2 z_2 + p^3 z_3 + \xi_0, \quad \eta = q z_1 - q^2 z_2 + q^3 z_3 + \eta_0.
\]
For simplicity, we can rewrite the functions \( M^{(n)}_{i,j} \) as
\[
M^{(n)}_{i,j} = \sum_{k=0}^{i} \frac{c_k}{(i-k)!} (p \partial_p + \xi' + n)^{i-k} \times \sum_{l=0}^{j} \frac{d_l}{(j-l)!} (q \partial_q + \eta' - n)^{j-l} \frac{1}{p+q},
\]
where
\[
\xi' = p z_1 + 2p^2 z_2 + 3p^3 z_3, \quad \eta' = q z_1 - 2q^2 z_2 + 3q^3 z_3,
\]
and \( p, q, c_k, \) and \( d_l \) are arbitrary complex constants. Further, taking the parameter constraints
\[
p = q = 1, \quad c_k = d^*_k,
\]
setting \( \tau_0 = f, \) \( z_1 = k_1 x_1 + k_2 x_2, \) \( z_2 = \sqrt{k_1 k_2 (k_1^2 - k_2^2)} (k_3 y_1 + k_4 y_2), \) and \( z_3 = -2k_3 (k_2^2 - k_1^2) f, \) the rational solutions of the 4D Fokas equation can be generated from equation (7). Based on the above results, the rational solutions of the 4D Fokas equation are presented in the following theorem.

**Theorem 2.** The \( (4+1) \)-dimensional Fokas equation (1) has rational solutions
\[
u = (k_2^2 - k_1^2) (\ln f), \quad x = k_1 x_1 + k_2 x_2,
\]
where
\[
f = \det_{1 \leq i,j \leq N} (M^{(n)}_{2i-1,2j-1}) \Big|_{n=0}.
\]

The matrix elements in \( f \) are defined by
\[
M^{(n)}_{i,j} = \sum_{k=0}^{i} \frac{c_k}{(i-k)!} (p^* \partial_{p^*} + \xi^* + n)^{i-k} \times \sum_{l=0}^{j} \frac{d^*_l}{(j-l)!} (p^* \partial_{p^*} + \eta^* - n)^{j-l} \frac{1}{p^* + p^*} \Big|_{p^* = 1},
\]
\[ \xi' = k_1 x_1 + k_2 x_2 + \sqrt{\frac{2k_1 k_3 (k_2^2 - k_4^2)}{-k_4}} y_1 + \sqrt{\frac{2k_1 k_2 (k_2^2 - k_4^2)}{-k_3}} y_2 - 3k_2(k_2^2 - k_4^2)t, \]

the asterisk denotes the complex conjugation, \( i, j, k, \) and \( l \) are arbitrary positive integers, and \( c_k \) and \( c_l \) are arbitrary complex constants. We take \( k_1 = 1, k_2 = \frac{6}{7}, k_3 = 1, \) and \( k_4 = 1 \) in this section.

3.1 Fundamental rational solution

According to Theorem 2, taking the parameters \( N = 1, c_0 = 1, \) and \( c_1 = 0 \) in equation (24), we first derive the fundamental rational solution of the 4D Fokas equation:

\[ u = 2(k_1^2 - k_2^2)k_3k_4 \frac{2k_1 k_2 (k_2^2 - k_4^2)(k_3 y_1 + k_2 y_2)^2 + k_3 k_4 l_{\text{lump}} - \frac{1}{2} k_3 k_4}{(18k_1 k_2 (k_2^2 - k_4^2)(k_3 y_1 + k_2 y_2)^2 + 9k_3 k_4 l_{\text{lump}} + \frac{9}{2} k_3 k_4)^2}, \]

where

\[ l_{\text{lump}} = k_1 x_1 + k_2 x_2 + 3(k_2^2 k_2 - k_3^2) + \frac{1}{2}. \]  

(27)

As can be seen from the above expressions, in order to ensure that the fundamental rational solution is non-singular, \( k_1 k_2 k_3 k_4(k_2^2 - k_4^2) > 0 \) must be held. The fundamental rational solution is a lump and has the following extreme points in the \((x_1, y_1)\)-plane, see Fig. 4:

\[ A_1 = (x_{11}, y_{11}) = \left( -\frac{k_2}{k_1} x_1 + 3\left(\frac{k_2}{k_1} - k_1 k_2\right)t + \frac{1}{2k_1}, y_1 = -\frac{k_4}{k_3} y_2 \right), \]

\[ A_2 = (x_{12}, y_{12}) = \left( -\frac{k_2}{k_1} x_1 + 3\left(\frac{k_2}{k_1} - k_1 k_2\right)t + 1 + \frac{3}{2k_1}, y_1 = -\frac{k_4}{k_3} y_2 \right), \]

\[ A_3 = (x_{13}, y_{13}) = \left( -\frac{k_2}{k_1} x_1 + 3\left(\frac{k_2}{k_1} - k_1 k_2\right)t + 1 - \frac{3}{2k_1}, y_1 = -\frac{k_4}{k_3} y_2 \right). \]

After simple calculations, we get a maximum value \( H_{\text{Max}} = H(x_{11}, y_{11}) = 8(k_2^2 - k_4^2) \) and two minimum values \( H_{\text{Min}} = H(x_{12}, y_{12}) = H(x_{13}, y_{13}) = k_2^2 - k_4^2 \) of the lump solution. The lump trajectory is \( k_1 x_1 + k_2 x_2 + 3(k_2^2 k_2 - k_3^2) + 1 = 0. \)

3.2 High-order rational solutions

In this section, we consider the high-order rational solutions of the 4D Fokas equation. \( N \)-order lump solutions are derived in the \((x_1, y_1)\)-plane from Theorem 2 for any given \( N \). For example, taking \( N = 2, k_1 = 1, k_2 = \frac{6}{7}, k_3 = 1, \)
Fig. 4 First-order rational solution for the 4D Fokas equation in the \((x_1, y_1)\)-plane with parameters \(N = 1, c_0 = 1, c_1 = 0, k_1 = 1, k_2 = \frac{6}{5}, k_3 = 1, k_4 = 1, x_2 = 0, y_2 = 0,\) and \(t = 0\) in equation (24). Panel (b) is the contour plot of panel (a).

Fig. 5 Second-order rational solutions of the 4D Fokas equation with parameters \(N = 2, c_1 = 0, c_2 = 0, k_1 = 1, k_2 = \frac{6}{5}, k_3 = 1, k_4 = 1, x_2 = 0, y_2 = 0,\) and \(t = 0\) in equation (24). (a): a fundamental pattern with \(c_0 = 1\) and \(c_3 = -\frac{1}{12};\) (b): a triangular pattern with \(c_0 = 1\) and \(c_3 = -30.\) Panels (c) and (d) are the contour plots of panels (a) and (b), respectively.

and \(k_4 = 1,\) the second-order lump solutions \(u_{2\text{ord}}\) are obtained

\[
u_{2\text{ord}} = (k_2^2 - k_1^2) \left( \ln \begin{vmatrix} M_{11}^{(0)} & M_{13}^{(0)} \\ M_{31}^{(0)} & M_{33}^{(0)} \end{vmatrix} \right)_{xx},
\]

(29)

where \(M_{i\ j}^{(n)}\) are defined in Theorem 2. As shown in Fig. 5, the second-order lumps have two types of patterns, which are controlled by four free parameters. Similarly, the third-order lump solutions are derived by taking \(N = 3, k_1 = 1, k_2 = \frac{6}{5}, k_3 = 1,\) and \(k_4 = 1.\) The third-order lumps have three types of patterns, which are controlled by six parameters, see Fig. 6. For larger values
of \( N \), as more free parameters will be generated, the patterns of the lumps will be more abundant and their dynamic behavior will be more complicated. We note that the pattern dynamics of high-order lumps is similar to rogue waves dynamics in \((1 + 1)\)-dimensional systems.

![Fig. 6](image)

**Fig. 6** Third-order rational solutions of the 4D Fokas equation with parameters \( N = 3, c_1 = 0, c_2 = 0, c_4 = 0, k_1 = 1, k_2 = \frac{5}{2}, k_3 = 1, k_4 = 1, x_2 = 0, y_2 = 0, \) and \( t = 0 \) in equation (24). (a): a fundamental pattern with \( c_0 = 1, c_3 = -\frac{1}{12} \) and \( c_5 = -\frac{1}{45} \); (b): a triangular pattern with \( c_0 = 1, c_3 = -\frac{25}{3} \) and \( c_5 = 0 \); (c): a ring pattern with \( c_0 = 1, c_3 = 0 \) and \( c_5 = 20 \). Panels (d), (e), and (f) are the contour plots of panels (a), (b), and (c), respectively.

### 4 Semi-rational solutions in the determinant form

In this section, we present a Theorem for constructing the semi-rational solutions of the 4D Fokas equation. In order to obtain the semi-rational solutions of the 4D Fokas equation, we will first introduce the following differential operators

\[
\Xi_i = \sum_{k=0}^{n_i} a_{ik}(p_i \partial_{p_i})^{n_i-k}, \quad \Omega_j = \sum_{l=0}^{n_j} a_{jl}(p_j \partial_{p_j})^{n_j-l}.
\]  

(30)
We choose the following functions
\[
\begin{align*}
\varphi^{(n)}_i &= \Xi_i p_i^n e^{\xi_i}, \\
\psi^{(n)}_j &= \Theta_j (-q_j)^{-n} e^{\eta_j}, \\
K^{(n)}_{i,j} &= \Xi_i \Theta_j \left[ \delta_{ij} + \left( -\frac{p_i}{p_j} \right)^n e^{\xi_i} e^{\xi_j} \right].
\end{align*}
\] (31)

The functions \(\varphi^{(n)}_i\) and \(\psi^{(n)}_j\) also satisfy the equation (11). For simplicity, we rewrite the matrix element \(K^{(n)}_{i,j}\) as
\[
K^{(n)}_{i,j} = \left( -\frac{p_i}{p_j} \right)^{\xi_i + \xi_j^*} \sum_{k=0}^{n_i} a_{ik} (p_i \partial p_i + \xi_i^* + n_i)^{n_i-k}
\times \sum_{l=0}^{n_j} a_{jl}^* (p_j^* \partial p_j^* + \xi_j^* - n_j)^{n_j-l} \frac{1}{p_i + p_j} + \delta_{ij} a_{in_i} a_{jn_j}^*.
\] (32)

where
\[
\begin{align*}
\xi_i &= p_i z_1 + p_i^2 z_2 + p_i^3 z_3 + \xi_{io}, \\
\xi_i^* &= p_i z_1 + 2p_i^2 z_2 + 3p_i^3 z_3.
\end{align*}
\] (33)

Here \(p_i\) and \(a_{ik}\) are arbitrary complex constants, \(\delta_{ij} = 0, 1\), and \(n_i\) are arbitrary positive integers. Furthermore, taking \(\tau_0 = f, z_1 = k_1 x_1 + k_2 x_2, z_2 = \sqrt{k_1 k_2 (k_2^2 - k_1^2)} i(k_3 y_1 + k_4 y_2), \) and \(z_3 = -2k_2 (k_2^2 - k_1^2) t\), then, the semi-rational solutions of the 4D Fokas equation would be derived. Thus, semi-rational solutions of 4D Fokas equation can be determined by the following Theorem.

**Theorem 3.** The \((4 + 1)\)-dimensional Fokas equation (1) has semi-rational solutions
\[
u = \left( k_2^2 - k_1^2 \right) \ln f, \quad x = k_1 x_1 + k_2 x_2,
\] (34)

where
\[
f = \det_{1 \leq i,j \leq N} \left( K^{(n)}_{i,j} \right) |_{n=0},
\] (35)

and the matrix elements in \(f\) are defined by
\[
K^{(n)}_{i,j} = \left( -\frac{p_i}{p_j} \right)^{\xi_i + \xi_j^*} \sum_{k=0}^{n_i} c_{ik} (p_i \partial p_i + \xi_i^* + n_i)^{n_i-k}
\times \sum_{l=0}^{n_j} c_{jl}^* (p_j^* \partial p_j^* + \xi_j^* - n_j)^{n_j-l} \frac{1}{p_i + p_j} + \delta_{ij} a_{in_i} a_{jn_j}^*.
\] (36)
By choosing different parameters $k$, calculate that the velocities of lump and soliton are $V_{\text{lump}}$ and $V_{\text{soliton}}$ respectively. From the above results, we can easily find that the semi-rational solutions will become rational solutions when $\delta_{ij} = 0$. We note that the patterns of rational solutions are similar to those corresponding to rational solutions of the Davey-Stewartson equation, reported in Refs. 39-41.

4.1 Lumps on one-soliton background

The semi-rational solution $u_{ls}$ consisting of a lump and a soliton is derived by taking $N = 1, n_1 = 1, a_{01} = 0, a_{11} = 1, p_1 = 1, \delta_{i1} = 1, and \delta_{ij} = 0 (i \neq j)$ in equation (34). The expression of $u_{ls}$ is as follows

$$u_{ls} = 2(k_1^2 - k_2^2)k_3k_4 e^{\xi},$$

where

$$\xi = 2k_1x_1 + 2k_2x_2 + 2(k_1^2k_2 - k_2^2)t, \quad l_{\text{temp}} = k_1x_1 + k_2x_2 + 3(k_3^2k_2 - k_2^2)t.$$

By choosing different parameters $k_1, k_2, k_3,$ and $k_4$, we derive a lump fusing into or fissioning from a dark soliton or from a bright soliton, see Fig. 7. The classification of four different types of interaction between lumps and one-soliton solutions is given in Table 1. From the above results, we can easily calculate that the velocities of lump and soliton are $V_{\text{lump}} = 3\frac{k_2^2k_2 - k_2^2}{k_3^2}$ and $V_{\text{soliton}} = \frac{k_2^2k_2 - k_2^2}{k_4^2}$, respectively. The velocity of lump is always greater than that of soliton, see Fig. 7.

The semi-rational solutions consisting of more lumps and a soliton are generated for $N = 1$ and $n_i \geq 2$ in equation (34). For example, taking $N = 1, n_i = 2, a_{01} = 0, a_{11} = 4, a_{12} = 1, p_1 = 1, \delta_{11} = 1, and \delta_{12} = 0$ in equation (34), we can derive the semi-rational solutions composed of two lumps and a soliton. There are also four distinct types of such semi-rational solutions. To
have an idea of the dynamics of such semi-rational solutions, we show here only the process of fission of two lumps from a bright soliton, see Fig. [8].

### 4.2 Lumps on multi-solitons background

For $N \geq 2$, $n_1 = 1$, $\delta_{11} = 1$, $\delta_{12} = 0$, $\delta_{21} = 0$, and $\delta_{22} = 1$ in equation (34), the semi-rational solutions consisting of more lumps and more solitons are derived.

For example, taking $N = 1$, $n_1 = 2$, $k_1 = 1$, $k_2 = 2$, $k_3 = 1$, $a_{10} = 1$, $a_{20} = 1$, $a_{11} = 1$, $a_{21} = 1$, $p_1 = \frac{1}{5}$, $p_2 = \frac{1}{3}$, $\delta_{11} = 1$, $\delta_{12} = 0$, $\delta_{21} = 0$, and $\delta_{22} = 1$ in equation (34), we obtain the interaction of local wave structures described by two lumps and two solitons. The exact expression of the corresponding
Table 1 Classification of four different types of interactions between lumps and one-soliton solution.

| Parameter condition (i) | Parameter condition (ii) | Results                          |
|-------------------------|--------------------------|----------------------------------|
| $k_3 k_4 > 0$           | $k_2 < k_1 < 0$          | a lump is annihilated            |
| $k_3 k_4 < 0$           | $k_2 > k_1 > 0$          | a lump is created                |
| $k_3 k_4 > 0$           | $k_1 > k_2 > 0$          | a lump is annihilated by a bright soliton |
| $k_3 k_4 < 0$           | $k_1 > k_2 < 0$          | a lump is created by a dark soliton |

solution $u_{2ls}$ is as follows

$$u_{2ls} = (k_2^2 - k_1^2) \left( \ln \begin{vmatrix} K_{11}^{(0)} & K_{12}^{(0)} \\ K_{21}^{(0)} & K_{22}^{(0)} \end{vmatrix} \right)_{xx}, \quad (38)$$

where

$$K_{11}^{(0)} = 1 + \left( \frac{3}{4} (y_1 + y_2)^2 + \frac{1}{4} (x_1 + 2x_2 - \frac{9}{2} t + 1)^2 + \frac{1}{4} \right) e^{x_1 + 2x_2 - \frac{9}{2} t},$$

$$K_{12}^{(0)} = \left( \frac{2(y_1 + y_2)^2}{5} + \frac{1}{5} (x_1 + 2x_2 - \frac{13}{4} t + \frac{13}{10})^2 - \frac{5}{16} (t + \frac{2}{5} t^2 + \frac{36}{125} + A) \right) e^{y_1},$$

$$K_{21}^{(0)} = \left( \frac{2(y_1 + y_2)^2}{5} + \frac{1}{5} (x_1 + 2x_2 - \frac{13}{4} t + \frac{13}{10})^2 - \frac{5}{16} (t + \frac{2}{5} t^2 + \frac{36}{125} - A) \right) e^{y_2},$$

Fig. 8 Time evolution of the process of fission of two lumps from a bright soliton of the 4D Fokas equation with parameters $N = 1$, $n_1 = 2$, $a_{01} = 0$, $a_{11} = 4$, $a_{12} = 1$, $p_1 = 1$, $\delta_{ii} = 1$, $\delta_{12} = 0$, $k_1 = 1$, $k_2 = 1.2$, $k_3 = 1$, $k_4 = 1$, $x_2 = 0$, and $y_2 = 0$ in equation (34).
Reductions of the (4 + 1)-dimensional Fokas equation and their solutions

\[
K_{22}^{(0)} = 1 + \left(\frac{2}{9}(y_1 + y_2)^2 + 6(x_1 + 2x_2 - 2t + \frac{3}{2})^2 + \frac{3}{8}\right) e^{\frac{4}{3}x_1 + \frac{4}{3}x_2 - \frac{4}{3}t},
\]

\[
A = \sqrt{-3}(y_1 + y_2)(\frac{1}{3}x_1 + \frac{2}{3}x_2 + t + \frac{9}{15}),
\]

\[
\varrho_1 = \frac{5}{6}x_1 + \frac{5}{3}x_2 - \frac{35}{36}t + \frac{5\sqrt{-3}}{36}(y_1 + y_2),
\]

\[
\varrho_2 = \frac{5}{6}x_1 + \frac{5}{3}x_2 - \frac{35}{36}t - \frac{5\sqrt{-3}}{36}(y_1 + y_2).
\]

The five panels in Fig. 9 describe the process of creation of two lumps from the background of two solitons. As shown in Fig. 9, with time evolution more peaks are created during the interaction between lumps and solitons around \(t = 2.5\), then two lumps and two solitons are completely separated around \(t = 10\).

4.3 A hybrid of two lumps, a breather, and a soliton

The third type of semi-rational solution consisting of two lumps, a breather, and a V-type soliton is derived for \(N = 2, n_i = 2\) and \(\delta_{ij} = 1\) in equation (34). The corresponding semi-rational solution is shown in Fig. 10.
the process of their interaction is elastic, the amplitudes and shapes of soliton, breather, and lumps did not change after the interaction. This type of semi-rational solution has never been reported elsewhere, to the best of our knowledge.

Fig. 10 Time evolution of semi-rational solution consisting of two lumps, a breather, and a V-type soliton of the 4D Fokas equation with parameters \( N = 2, n_i = 2, k_1 = 1, k_2 = 2, k_3 = 1, k_4 = 1, a_{10} = 1, a_{20} = 1, a_{11} = 1, a_{21} = 1, p_1 = 1 + \frac{1}{2}, p_2 = 1 - \frac{1}{2}, \delta_{11} = 1, \delta_{12} = 1, \delta_{21} = 1, \delta_{22} = 1, x_2 = 0, \) and \( y_2 = 0 \) in equation (34).

5 Rational and semi-rational solutions related to the 3D KP equation

The 3D KP equation can be read as follows [42]

\[
(W_\zeta + 6WW_\nu + W_{\nu\nu})_\nu - \alpha (W_{\nu\nu} + W_{\nu\nu}) = 0.
\]  
(39)

It describes the dynamic behavior of nonlinear waves and solitons in plasma and fluids [43, 44]. The rational and semi-rational solutions of the 3D KP equation can be expressed in Theorems 4 and 5 as follows.

**Theorem 4.** The 3D KP equation (39) has rational solutions

\[
W = 2(\ln f)_{\nu\nu},
\]  
(40)

where

\[
f = \det_{1 \leq i, j \leq N} (H^{(n)}_{2i-1, 2j-1}) \big|_{n=0},
\]  
(41)

and the matrix elements in \( f \) are defined by

\[
H^{(n)}_{i,j} = \sum_{k=0}^{i} \frac{d_k}{(i-k)!} (p^\nu \partial_p + E' + n)^{i-k} \times \sum_{l=0}^{j} \frac{d_l^*}{(j-l)!} (p^* \partial_{p^*} + E'^* - n)^{j-l} \frac{1}{p + p^*} \big|_{p=1},
\]  
(42)
Theorem 5. The 3D KP equation \([39]\) has semi-rational solutions
\[ W = 2(\ln f)_{uv}, \]  
where
\[ f = \det_{1 \leq i,j \leq N} (H^{(n)}_{i,j}) \bigg|_{n=0}, \]  
and the matrix elements in \(f\) are defined by
\[ H_{i,j}^{(n)} = (-\frac{p_{i}}{p_{j}}) e^{E_{i}+E_{j}'} \sum_{k=0}^{n_{j}} d_{ik}(p_{i}\partial p_{i} + E_{i}' + n)^{n_{i}-k} \]
\times \sum_{l=0}^{n_{j}} d_{jl}^{*}(p_{j}^{*}\partial p_{j}^{*} + E_{j}'^{*} - n)^{n_{j}-l} \frac{1}{p_{i} + p_{j}^{*}} + \delta_{ij} b_{in_{i}} b_{jn_{j}}^{*}, \]
\[ E_{i} = p_{i}v + ip_{i}^{2} \sqrt{\frac{3}{2\alpha}} \varphi + ip_{i}^{2} \sqrt{\frac{3}{2\alpha}} z - 4p_{i}^{2} \varsigma + E_{0}, \]
\[ E_{i}' = v + \sqrt{\frac{6}{\alpha}} i\varphi + \sqrt{\frac{6}{\alpha}} iz - 12\varsigma. \]

Here the asterisk denotes the complex conjugation and \(d_{k}, d_{ik}, d_{jl}, \) and \(d_{l}\) are arbitrary complex constants.

The proofs of Theorems 4 and 5 are similar to those of Theorems 2 and 3. It is not difficult to see that the rational and semi-rational solutions of the 4D Fokas equation can degenerate to the rational and semi-rational solutions of the 3D KP equation. The corresponding transformation is as follows
\[ W_{3DKP}(v, \varphi, Z, \varsigma) = \frac{2}{k_{2}^{2} - k_{1}^{2}} u_{4DFokas}(x_{1}, x_{2}, y_{1}, y_{2}, t), \]  
where
\[ k_{1}x_{1} + k_{2}x_{2} = v, \quad k_{1}y_{1} + k_{4}y_{2} = \sqrt{\frac{2k_{1}k_{4}}{k_{1}k_{2}(k_{2}^{2} - k_{1}^{2})}} (\varphi + Z), \quad t = \frac{4}{k_{2}(k_{2}^{2} - k_{1}^{2})} \varsigma. \]

6 Conclusions

In this paper, the determinant expression of \(N\)-solitons is constructed for the 4D Fokas equation by using the KP hierarchy reduction method. New types of mixed solutions composed of breathers and V-type solitons are obtained by choosing the appropriate parameters in Theorem 1 (see Fig. 2 and Fig. 3). High-order rational solutions of the 4D Fokas equation are also derived by means of Theorem 2, as well as we give the condition \(k_{1}k_{2}k_{3}k_{4}(k_{2}^{2} - k_{1}^{2}) > 0\) to ensure that the rational solutions are smooth. We show that the fundamental rational solution is a lump in the \((x_{1}, y_{1})\)-plane, which is a traveling wave.
localized in space and time, see Fig. 4. High-order rational solutions display
the interaction between several lumps in the \((x_1, y_1)\)-plane, and exhibit similar
dynamical patterns to those of rogue waves in the \((1+1)\)-dimensions by altering
the free parameters \(c_k\) in Theorem 2 (see Fig. 5 and Fig. 6).

Furthermore, three kinds of new semi-rational solutions of the 4D Fokas
equation are generated by introducing differential operators \(\Xi_i\) and \(\mathfrak{H}_j\). For \(N = 1, n_i \geq 1, \delta_{ii} = 1, \) and \(\delta_{ij} = 0 (i \neq j)\) in Theorem 3, the semi-rational
solutions composed of lumps and one-soliton solutions are derived. There are
four distinct dynamical patterns of these semi-rational solutions, which are
obtained by changing the values of parameters \(k_1, k_2, k_3, \) and \(k_4\) (see Fig. 7
and Fig. 8). The specific classification of these patterns is shown in Table 1.
For \(N \geq 1, n_i = 1, \delta_{ii} = 1 \) and \(\delta_{ij} = 0 (i \neq j)\) in Theorem 3, the semi-rational
solutions consisting of more lumps and more solitons are also generated (see
Fig. 9). Also a new kind of semi-rational solution composed of two lumps, a
breather, and a V-type soliton is derived, for \(N = 2, n_i = 2, \) and \(\delta_{ij} = 1.\) We
point out that the interaction between the mentioned entities of such semi-
rational solution is elastic. This kind of semi-rational solution that is illustrated
in Fig. 10 has never been reported elsewhere, to the best of our knowledge.

Additionally, using our rational and semi-rational solutions of the 4D Fokas
equation, we derived the rational and semi-rational solutions of the 3D KP
equation. These results indicate that the 4D Fokas equation is a valuable multi-
dimensional extension of the KP and DS equations. In addition, this paper
provides an idea for seeking the exact solutions of high-dimensional soliton
equations, and also provide a reference for how to reduce the exact solutions
of high-dimensional systems to the exact solutions of low-dimensional ones.
These results are useful to the study of the dynamics of nonlinear waves in
diverse physical settings in hydrodynamics, nonlinear optics and photonics,
plasmas, quantum gases (Bose-Einstein condensates), and solid state physics.

**Funding** This work is supported by the NSF of China under Grant No.
11671219 and No. 11871446.

**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

**References**

1. Russell, J.S.: Report of the committee on waves, Report of the 7th Metting of the British
   Association for the Advacement of Science. Liverpool, 417–496 (1838)
2. Korteweg, D.J., Vries, G. de: On the change of form of long waves advancing in a rect-
   angular canal, and a new type of long stationary waves. Philos. Mag. Ser. 5, 39, 422–443
   (1895)
3. Zabusky, N.J., Kruskal, M.D.: Interaction of solitons in a collisionless plasma and the recurrence of initial states. Phys. Rev. Lett. 15, 240–243 (1965)
4. Liu, W. J., Zhang, Y. J., Luan, Z. T., Zhou, Q., Mirzazadeh, M., Ekici, M., Biswas, A.: Dromion-like soliton interactions for nonlinear Schrödinger equation with variable coefficients in inhomogeneous optical fibers. Nonl. Dyn. 96, 729–736 (2019)
5. Liu, S. Z., Zhou, Q., Biswas, A., Liu, W. J.: Phase-shift controlling of three solitons in dispersion-decreasing fibers. Nonl. Dyn. 98, 395–401 (2019)
6. Ablowitz, M., Clarkson, P.: Soliton, nonlinear evolution equations and inverse scattering. Cambridge University Press, Cambridge, UK (1991)
7. Davey, A., Stewartson, K.: On Three-Dimensional Packets of Surface Waves. Proc. R. Soc. Lond. A 328, 101–110 (1974)
8. Jimbo, M., Miwa, T.: Solitons and infinite-dimensional Lie algebras. Publ. Res. Inst. Math. Sci. 19, 943–1001 (1983)
9. Lou, S.Y., Lin, J., Yu, J.: (3 + 1)-dimensional models with an infinitely dimensional Virasoro type symmetry algebra. Phys. Lett. A 201, 47–52 (1995)
10. Lou, S.Y.: Dromion-like structures in a (3+1)-dimensional KdV-type equation. J. Phys. A: Math. Gen. 29, 5989–6001 (1996)
11. Yu, S.J., Toda, K., Sasa, N., Fukuyama, T.: N soliton solutions to the Bogoyavlenskii-Schiff equation and a quest for the soliton solution in (3 + 1) dimensions. J. Phys. A: Math. Gen. 31, 3337–3347 (1998)
12. Geng, X.G.: Algebraic-geometrical solutions of some multidimensional nonlinear evolution equations. J. Phys. A: Math. Gen. 36, 2289–2303 (2003)
13. Fokas, A.S.: The D-bar method, inversion of certain integrals and integrability in 4 + 2 and 3 + 1 dimensions. J. Phys. A: Math. Theor. 41, 344006 (2008)
14. Fokas, A.S.: Soliton multimodal equations and integrable evolutions preserving Laplace’s equation. Phys. Lett. A 372, 1277–1279 (2008)
15. Fokas, A.S.: Nonlinear Fourier transforms, integrability and nonlocality in multidimensions. Nonlinearity 20, 2095–2113 (2007)
16. Fokas, A.S.: Nonlinear Fourier transforms and integrability in multidimensions. Contemp. Math. 458, 71–80 (2008)
17. Dimakos, M., Fokas, A.S.: Davey-Stewartson type equations in 4 + 2 and 3 + 1 possessing soliton solutions. J. Math. Phys. 54, 081504 (2013)
18. Fokas, A.S., van der Weele, M.C.: Complexification and integrability in multidimensions. J. Math. Phys. 59, 091413 (2018)
19. Wazwaz, A.M., El-Tantawy, S.A.: Solving the (3+1)-dimensional KP-Boussinesq and BKP-Boussinesq equations by the simplified Hirota’s method. Nonl. Dyn. 88, 3017–3021 (2017)
20. Liu, J.G., He, Y.: Abundant lump and lump-kink solutions for the new (3+1)-dimensional generalised Kadomtsev-Petviashvili equation. Nonl. Dyn. 92, 1103–1108 (2018)
21. Sergyeyev, A.: Integrable (3+1)-dimensional systems with rational Lax pairs. Nonl. Dyn. 91, 1677–1680 (2018)
22. Xu, G.Q., Wazwaz, A.M.: Characteristics of integrability, bidirectional solitons and localized solutions for a (3+1)-dimensional generalized breaking soliton equation. Nonl. Dyn. 96, 1989–2000 (2019)
23. Ding, C.C., Gao, Y.T., Deng, G.F.: Breather and hybrid solutions for a generalized (3+1)-dimensional B-type Kadomtsev-Petviashvili equation for the water waves. Nonl. Dyn. 97, 2023–2040 (2019)
24. Chen, S., Zhou, Y., Baronio, F., Mihalache, D.: Special types of elastic resonant soliton solutions of the Kadomtsev-Petviashvili II equation. Rom. Rep. Phys. 70, 102 (2018)
25. Kaur, L., Wazwaz, A.M.: Bright-dark lump wave solutions for a new form of the (3+1)-dimensional BKP-Boussinesq equation. Rom. Rep. Phys. 71, 102 (2019)
26. Malomed, B.A., Mihalache, D.: Nonlinear waves in optical and matter-wave media: A topical survey of recent theoretical and experimental results. Rom. J. Phys. 64, 106 (2019)
27. Fokas, A.S.: Integrable Nonlinear Evolution Partial Differential Equations in 4 + 2 and 3 + 1 Dimensions, Phys. Rev. Lett. 96, 190201 (2006)
28. Yang, Z.Z., Yan, Z.Y.: Symmetry Groups and Exact Solutions of New (4 + 1)-Dimensional Fokas Equation. Commun. Theor. Phys. 51, 876–880 (2009)
29. Lee, J., Saktihivel, R., Wazzan, L.: Exact Traveling wave solutions of a high-dimensional evolution equation. Mod. Phys. Lett. B 24, 1011–1021 (2010)
30. Wang, X.B., Tian, S.F., Feng, L.L., Zhang, T.T.: On quasi-periodic waves and rogue waves to the (4 + 1)-dimensional nonlinear Fokas equation. J. Math. Phys. 59, 073505 (2018)
31. Cheng, L., Zhang, Y.: Lump-type solutions for the (4 + 1)-dimensional Fokas equation via symbolic computations. Mod. Phys. Lett. B 31, 1750224 (2017)
32. Tan, W., Dai, Z. D., Xie, J. L., Qiu, D. Q.: Parameter limit method and its application in the (4 + 1)-dimensional Fokas equation. Comp. Math. Appl. 75, 4214–4220 (2018)
33. Sun, H.Q., Chen, A.H.: Interactional solutions of a lump and a solitary wave for two higher-dimensional equations. Nonl. Dyn. 94, 1753–1762 (2018)
34. Hirota, R.: The direct method in soliton theory. (Cambridge University Press, Cambridge, 2004)
35. Ohta, Y., Wang, D.S., Yang, J.K.: General N-Dark-Dark Solitons in the Coupled Nonlinear Schrödinger Equations. Stud. Appl. Math. 127, 345–371 (2011)
36. Ohta, Y., Yang, J.K.: General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation, Proc. R. Soc. A 468, 1716 (2012)
37. Ohta, Y., Yang, J.K.: Dynamics of rogue waves in the Davey-Stewartson II equation. J. Phys. A: Math. Theor. 46, 105202 (2013)
38. Ohta, Y., Yang, J.K.: Rogue waves in the Davey-Stewartson I equation, Phys. Rev. E 86, 036604 (2012)
39. Rao, J.G., Cheng, Y., He, J.S.: Rational and Semirational Solutions of the Nonlocal Davey-Stewartson Equations. Stud. Appl. Math. 139, 568–598 (2017)
40. Rao, J.G., Zhang, Y.S., Fokas, A.S., He, J.S.: Rogue waves of the nonlocal Davey-Stewartson I equation. Nonlinearity 31, 4099–4107 (2018)
41. Qian, C., Rao, J.G., Miliaache, D., He, J.S.: Rational and semi-rational solutions of the y-nonlocal Davey-Stewartson I equation. Computers and Math. with Appl. 75, 3317–3330 (2018)
42. Senatorski, A., Infeld, E.: Simulations of Two-Dimensional Kadomtsev-Petviashvili Soliton Dynamics in Three-Dimensional Space. Phys. Rev. Lett. 77, 14 (1996)
43. Infeld, E., Rowlands, G.: Three-dimensional stability of Korteweg-de Vries waves and solitons II. Acta Phys. Polon. A 56, 329–332 (1979)
44. Kuznietsov, E.A., Mush, S.L.: Effect of sound wave collapse on the structure of collisionless shock waves in a magnetized plasma. Zh. Eksp. Teor. Fiz. 91, 1605 (1986).