MOMENT PROBLEMS AND THE CAUSAL SET APPROACH TO QUANTUM GRAVITY

AVNER ASH AND PATRICK MCDONALD

Abstract. We study a collection of discrete Markov chains related to the causal set approach to modeling discrete theories of quantum gravity. The transition probabilities of these chains satisfy a general covariance principle, a causality principle, and a renormalizability condition. The corresponding dynamics are completely determined by a sequence of nonnegative real coupling constants. Using techniques related to the classical moment problem, we give a complete description of any such sequence of coupling constants. We prove a representation theorem: every discrete theory of quantum gravity arising from causal set dynamics satisfying covariance, causality and renormalizability corresponds to a unique probability distribution function on the nonnegative real numbers, with the coupling constants defining the theory given by the moments of the distribution.

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1. Introduction

There are currently a number of approaches aimed at formulating a successful theory of quantum gravity undergoing development, the most familiar being String Theory. This note concerns an alternative to String Theory: the Causal Set approach to quantum gravity. In its current state of the development, the Causal Set approach provides a classical analog to a true quantum theory; work focussing on the development of a full quantum analog is currently underway (cf section 2 below and [1] for basic axioms of the Causal Set theory and [2] and [3] for physical discussions concerning the Causal Set approach). We study the Causal Set approach as a classical precursor to a theory of quantum gravity.

At first glance, the most natural way to combine quantum theory and general relativity would be to quantize the spacetime metric. As is well known, such a direct approach must contend with a number a significant obstructions, including the existence of unrenormalizable divergences. There is currently no clear consensus as to how these divergences are to be addressed. Many believe that the source of the problem (if not the solution) might lie in the basic assumptions involving the underlying structure of spacetime. More precisely, it has been suggested that treating spacetime as a discrete combinatorial object as opposed to a manifold could lead to insight towards removing the divergences in the quantum field theoretic approach, if not to a substitute for such an approach (cf [4], [5] and references therein).

Discrete approaches to gravity initially arose as an attempt to circumvent many of the difficulties arising in classical general relativity (eg, existence of singularities, the difficulty of solving Einstein’s field equations for general systems). Roughly speaking, the idea behind early discretization procedures involved replacing the space-time continuum with a triangulation, the construction being either a triangulation of 4-dimensional spacetime, or
later 3+1 in nature (triangulate a 3-dimensional hypersurface at a fixed time, triangulate a second hypersurface considered as a time evolution of the first hypersurface, and connect vertices between triangulated hypersurfaces). In such an approach the vertices are taken to be (discrete) events, the edges between vertices in different hypersurfaces spacelike or timelike curves, and the salient relation between two such events whether one can cause the other or not.

Over the last two decades, discretization procedures have been further developed and refined, and their applications in gravity greatly expanded (cf. [6] for a recent survey of discrete approaches to gravity, both classical and quantum). One particular line of development, pioneered by Sorkin and his co-workers [4], de-emphasized the role of the metric in favor of focussing on the causal structure of spacetime. This approach, the so-called Causal Set approach, is motivated in part by two observations. First, the causal structure of the spacetime continuum determines the topological, differentiable, and conformal Lorentzian metric structure of the spacetime continuum (cf. [4], [2], [5]). Second, the causal structure of the spacetime continuum and the corresponding discrete causal structure, are very simple mathematical objects: posets (partially ordered sets). Taking the primary relationship between two events to be causation, the Causal Set approach to gravity posits that the deep structure of spacetime should be modelled by the discrete causal structures which arise as natural abstractions of the posets occuring when the causal structure of the spacetime continuum is discretized (in the context of gravity, these posets are called causets). The Causal Set approach to gravity then seeks “natural” dynamics under which causets evolve. In [1], Rideout and Sorkin propose such dynamics (formulated probabilistically) for the (classical) evolution of causets.
Thus, the search for an appropriate dynamical framework for a quantum theory of gravity has recently led to interest in stochastic dynamical systems taking their values in certain locally finite partially ordered sets (causets). As discussed in [1], these systems can be realized as Markov chains whose transition probabilities are required to satisfy a discrete covariance principle and a discrete causality principle. We call such Markov chains “generic” if all of the transition probabilities which could be nonzero are positive (cf Definition 2.1). Given the appropriate mathematical formalism (cf [1], section 2 below), it is possible to classify all such generic chains: there is a 1-1 correspondence between generic chains satisfying covariance and causality and nonnegative sequences of real numbers, \( T = \{t_n\}_{n=0}^{\infty}, \) satisfying \( t_0 = 1 \) (the coupling constants \( t_n \) are given explicitly in terms of the Markov chain - cf [1] and section 2 below).

It is easy to see that an arbitrary sequence \( T \) is unlikely to have physical significance, and therefore we want to find additional natural conditions which restrict the collection of sequences under consideration to those sequences which are “physical.” Thus, in addition to covariance and causality one might expect, as first suggested in [7], that a discrete theory of quantum gravity should satisfy a cosmological renormalizability condition under cycles of expansion and contraction. Given the framework of [1], such a condition can be formulated as an additional constraint on the coupling constants defining the theory. To make this precise, we introduce the required notation.

We will denote by \( \mathcal{S} \) the collection of sequences of nonnegative real numbers. We will denote elements of \( \mathcal{S} \) by upper case roman letters, and, as above, we will use the corresponding lower case letter to denote specific elements of a given sequence. We will denote by \( \mathcal{S}_1 \) the subset of \( \mathcal{S} \) consisting of those sequences which begin with 1. We define a cosmological
renormalization operator $\mathcal{R} : \mathcal{S} \to \mathcal{S}$ by

\[(\mathcal{R}(T))_n = t_n + t_{n+1}.
\]

The operator $\mathcal{R}$ admits a stable manifold, $\text{Stab}(\mathcal{R}) \subset \mathcal{S}$, defined by

\[\text{Stab}(\mathcal{R}) = \left\{ T \in \mathcal{S} : T \in \bigcap_{k=0}^{\infty} \mathcal{R}^k(\mathcal{S}) \right\}.\]

We call elements of $\text{Stab}(\mathcal{R})$ stable sequences and we note that (cf. [7] and section 2 below) there is a 1-1 correspondence between generic chains satisfying causality, covariance and cosmological renormalizability under cycles of expansion and contraction and elements of $\mathcal{S} = \mathcal{S}_1 \cap \text{Stab}(\mathcal{R})$. Our main result, a representation theorem, gives a complete description of $\text{Stab}(\mathcal{R})$ in terms of measures on $\mathbb{R}^+ = [0, \infty)$:

**Theorem 1.1.** Let $T$ be a sequence of nonnegative real numbers. Then $T \in \text{Stab}(\mathcal{R})$ if and only if there is a nondecreasing function $\alpha : \mathbb{R}^+ \to \mathbb{R}$ such that

\[t_n = \int_0^{\infty} s^n d\alpha(s).
\]

For $T$ and $\alpha$ as in Theorem 1.1, we will say that $T$ is represented by $\alpha$.

Our theorem is motivated by an observation of [7]: transitive percolation, the theory which is determined by choosing $t \in \mathbb{R}^+$ and defining associated coupling constants by

\[t_n = t^n,
\]

defines a stable sequence (by convention $0^0 = 1$). Transitive percolation as given by (4) is represented by a probability measure on $\mathbb{R}^+$; a delta-mass of weight one concentrated at $t \in \mathbb{R}^+$ has moments which coincide with the sequence. This measure can in turn be represented by its probability distribution function, a translate of the Heaviside function. Our theorem can be seen as quantifying to what extent transitive percolation is representative of the
general behavior of stable sequences. Namely, any stable sequence is a “linear combination” of percolation sequences.

As is clear from the statement of Theorem 1.1, our result is closely related to the classical moment problem of Stieltjes type (cf section 3 below). As a consequence, Theorem 1.1 and its proof provide a means of applying the extensive collection of sophisticated mathematical tools developed in the context of the moment problem to questions related to quantum gravity. We provide a number of straightforward corollaries of our technique. These corollaries include an explicit representation of the transition probabilities associated to any generic Markov chain which defines a discrete theory satisfying covariance, causality and cosmological renormalizability, as well as a second representation theorem which associates to any such theory a natural self-adjoint nonnegative operator acting on a model Hilbert space (cf section 5 below).

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2. BACKGROUND AND DEFINITIONS FROM DISCRETE QUANTUM GRAVITY

In this section we present the mathematical formulation for a classical precursor of a discrete theory of quantum gravity. We follow the development of [1] and [7].

The fundamental object of study, a causet, is a locally finite partially ordered set. Throughout this note we will denote causets with upper case roman letters and, when needed, indicate the partial order relation using the symbol $\ll$. We assume throughout that $\ll$ is irreflexive.

An isomorphism of causets is a bijection which preserves the partial orders. Isomorphism defines an equivalence relation on causets. We will denote by $C_n$ the collection of equivalence classes of causets with $n$ elements indexed by $\{0, 1, \ldots, n - 1\}$, with partial order consistent
with indexing. Thus, up to equivalence,

\[ C_n = \{ C : C \text{ a causet }, C = \{ a_0, \ldots, a_{n-1} \}, a_k \ll a_l \Rightarrow k < l \}. \]  

We write

\[ C = \bigcup_{n \in \mathbb{N}} C_n \]  

and we note that \( C \) carries a natural partial order given as follows: \( C \prec D \) if and only if \( C \in C_n, D \in C_m \) with \( n < m \), and there exists an order preserving function \( f : C \rightarrow D \) such that \( f(C) \) is an initial segment of \( D \).

Informally, we can describe the dynamic evolution of causets as follows: Initially, the state of the system is given by the trivial causet consisting of a single point. At each increment of time, an element comes into existence as the “offspring” of elements already in existence. That is, at the beginning of the \( n \)th increment of time we have a causet \( C \in C_n \) which we evolve to a causet \( D \in C_{n+1} \) by adding an element to \( C \) together with relations between the new element and a subset of elements of \( C \) (those elements in the past of the new element, i.e., those which bear some causal relationship to the new element). The new relations are determined randomly; the probability that any given collection of relations is added is given by a collection of transition probabilities which define the theory. We can now proceed to formalize this description.

Given a causet \( C \) and an element \( x \in C \), we define the past of \( x \) by

\[ \text{past}_C(x) = \{ y \in C : y \ll x \}. \]  

We will regard \( \text{past}_C(x) \) as a poset with partial order given by the partial order of \( C \). A link in a partially ordered set is an irreducible relation (i.e., a relation that contains no other relation). A path in a partially ordered set is a sequence of elements of the set, each related to the next by a link.
Given $C \in \mathcal{C}_n$, we will define the family of $C$, denoted $F(C)$, as those elements $D \in \mathcal{C}_{n+1}$ such that $C \prec D$, where $\prec$ denotes the partial order of elements of $\mathcal{C}$:

(8) \hspace{1cm} F(C) = \{ D \in \mathcal{C}_{n+1} : C \prec D \}.

Given $C \in \mathcal{C}_n$ and $D \in F(C)$, the precursor set of the transition $C \rightarrow D$, denoted $\text{Prec}(C, D)$, is the past of the element $x \in D \setminus C$:

(9) \hspace{1cm} \text{Prec}(C, D) = \text{past}_D(x) \subset D.

Note that $\text{Prec}(C, D)$ is a poset with partial order given by its description as the past of an element $x \in D$. The collection of \emph{maximal elements associated to the transition} $C \rightarrow D$, is the collection of elements of $D$ with links to the element $x$:

(10) \hspace{1cm} \text{max}(C, D) = \{ y \in D : y \text{ linked to } x, \ x \in D \setminus C \}.

A special role in the theory will be played by those causets with no relations. We will denote the element of $\mathcal{C}_n$ with no relations by $A_n$:

(11) \hspace{1cm} A_n = ((a_0, a_1, \ldots, a_{n-1}), \emptyset).

We note that there is a natural path in $\mathcal{C}$ of length $n$ from $A_0$ to $A_n$.

We define a collection of Markov chains with state space $\mathcal{C}$ as follows:

\textbf{Definition 2.1.} We say that a Markov chain $M$ with state space $\mathcal{C}$ belongs to the collection $\mathcal{M}$ if the transition probabilities of $M$ satisfy:

(1) Given $C \in \mathcal{C}_n$, let $\text{Prob}(C \rightarrow D)$ denote the transition probability corresponding to an evolution from causet $C$ to causet $D$. Then $\text{Prob}(C \rightarrow D) = 0$ if $D \notin F(C)$ and $\sum_{D \in F(C)} \text{Prob}(C \rightarrow D) = 1$. 
(2) (General Covariance) Let $C \in \mathcal{C}_n$. Suppose $P_1$ and $P_2$ are two paths from $A_0$ to $C$ and write $P_i = \{l_{i1}, \ldots, l_{in}\}$ where the $l_{ij}$ are the links defining the path $P_i$. Then

\[ \prod_{k=1}^{n} \text{Prob}(l_{1k}) = \prod_{k=1}^{n} \text{Prob}(l_{2k}). \]

(3) (Causality) Suppose that $C \in \mathcal{C}_n$ and for $i = 1, 2$, suppose that $C_i \in F(C)$. Let $B \in \mathcal{C}_m$, $m \leq n$, be defined by

\[ B = \text{Prec}(C, C_1) \cup \text{Prec}(C, C_2) \]

with poset structure induced by that of $C$. Let $B_i \in \mathcal{C}_{m+1}$ be $B$ with an element added in the same manner as in the transitions $C \rightarrow C_i$. Then we require

\[ \frac{\text{Prob}(C \rightarrow C_1)}{\text{Prob}(C \rightarrow C_2)} = \frac{\text{Prob}(B \rightarrow B_1)}{\text{Prob}(B \rightarrow B_2)}. \]

It is a theorem of Rideout and Sorkin that any generic element of $\mathcal{M}$ is completely determined by a discrete collection of coupling constants given by transitions between causets with no relations. More precisely, let $M \in \mathcal{M}$, and suppose that $A_n$ is given as in (11). Associate to $M$ a sequence of positive coupling constants $\{q_n\}_{n=0}^{\infty}$ defined by

\[ \begin{align*}
q_0 & = 1 \\
q_n & = \text{Prob}(A_{n-1} \rightarrow A_n)
\end{align*} \]

where, as above, the expression appearing on the right hand side of (14) denotes the probability of transition from $A_{n-1}$ to $A_n$. In [1], Rideout and Sorkin prove that the sequence $\{q_n\}_{n=0}^{\infty}$ completely determines the theory associated to $M$. More precisely, given an element $C \in \mathcal{C}_n$, and $D \in F(C)$ (cf (8)), let max$(C, D)$ be the collection of maximal elements associated to the transition $C \rightarrow D$ (cf (10)) and let Prec$(C, D)$ be the precursor set of the transition $C \rightarrow D$ (cf (9)). Suppose the cardinality of Prec$(C, D)$ is $\rho$ and that the cardinality of max$(C, D)$ is $m$. Then the transition probability for the evolution $C \rightarrow D$ is given by
\[
\text{Prob}(C \rightarrow D) = q_n \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{q_{n-k}}
\]

which indicates that the Markov chain \( M \) is completely determined by the sequence of \( q_n \) defined as in (14).

Following [1], we define a sequence \( t_n \) by

\[
t_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{1}{q_k}.
\]

Then we can recover the coupling constants \( q_n \) from the sequence of \( t_n \):

\[
\frac{1}{q_n} = \sum_{k=0}^{n} \binom{n}{k} t_k.
\]

As in the introduction, let \( S_1 \) be defined by \( S_1 = \{ T \in S : t_0 = 1 \} \). There is a bijection between generic elements of \( \mathcal{M} \) and elements of \( S_1 \) given by associating to each element of \( S_1 \) the associated collection of coupling constants \( \{ q_n \}_{n=0}^{\infty} \) given by (17).

Amongst additional constraints that one might impose to restrict further the collection of chains that could serve as classical precursor for a discrete model of quantum gravity, there is a natural choice involving cosmological renormalizability under cycles of expansion and contraction. More precisely, given a causet \( C \), we call an element \( \gamma \in C \) a post, if every element of \( C \) is either in the past of \( \gamma \) or in the future of \( \gamma \) in \( C \) (denoted future\( C(\gamma) \)):

\[
C = \text{past}_C(\gamma) \cup \{ \gamma \} \cup \text{future}_C(\gamma).
\]

Physically, the occurrence of a post corresponds to a collapse of the universe to zero diameter, followed by re-expansion.

Given a causet \( C \) and a post \( \gamma \), there is a simple relationship between the coupling constants \( t_n \) governing the evolution of \( C \) and the coupling constants governing the evolution of the causet \( \{ \gamma \} \cup \text{future}_C(\gamma) \) (cf [1]): If \( p = |\text{past}_C(\gamma)| \), then the coupling constants for \( \{ \gamma \} \cup \text{future}_C(\gamma) \) are given by

\[
\frac{1}{q_p} = \sum_{k=0}^{p} \binom{p}{k} t_k.
\]
future\(_C(\gamma)\) are given by
\begin{equation}
\tilde{t}_n = \sum_{k=0}^{p} \binom{p}{k} t_{n+k}
\end{equation}
where \(n > 0\). This relationship is concisely described in terms of the cosmological renormalization operator \(\mathcal{R} : \mathcal{S} \to \mathcal{S}\) defined by
\begin{equation}
(\mathcal{R}(T))_n = t_n + t_{n+1}
\end{equation}
Using the renormalization operator we can write the right hand side of (19) as \((\mathcal{R}^p(T))_n\). We use this concise notation to define the collection of Markov chains which we intend to study.

**Definition 2.2.** We say that a Markov chain \(M\) with state space \(\mathcal{C}\) belongs to the collection \(\mathcal{M}\) if

1. \(M \in \mathcal{M}\) is generic, and
2. If \(M\) is represented by the sequence \(T \in \mathcal{S}\), then
\begin{equation}
T \in \bigcap_{n=0}^{\infty} \mathcal{R}^n(\mathcal{S}).
\end{equation}

As in the introduction, we call the right-hand-side of (21) the stable set of the renormalization operator and we write
\begin{equation}
\text{Stab}(\mathcal{R}) = \bigcap_{n=0}^{\infty} \mathcal{R}^n(\mathcal{S}).
\end{equation}
If we set \(\mathcal{S} = \mathcal{S}_1 \cap \text{Stab}(\mathcal{R})\), then it is clear from the definition that there is a bijection between elements of \(\mathcal{M}\) and elements of \(\mathcal{S}\). It is also clear that \(\text{Stab}(\mathcal{R})\) is a convex set.

As discussed in [1] and [7] and our introduction, there are a number of interesting special cases of processes which satisfy the conditions defining \(\mathcal{M}\). Of particular interest from our point of view are theories of transitive percolation defined by fixing \(t \in \mathbb{R}^+\) and setting
\begin{equation}
t_n = t^n.
\end{equation}
As mentioned in the introduction, the sequence defined by (23) can be represented by a probability measure on \( \mathbb{R} \): a delta-mass of weight 1 concentrated at \( t \in \mathbb{R}^+ \). This fact, together with the observed convexity of \( \text{Stab}(\mathcal{R}) \) suggests that we develop a representation of \( \text{Stab}(\mathcal{R}) \) in terms of the moments of probability measures on \( \mathbb{R}^+ \).

### 3. Moment Problems

In this section we develop material related to the classical moment problem which we will need in the sequel. References to this material include [8] and [9].

Let \( [a, b] \) be an interval in the real line, \( \alpha : [a, b] \to \mathbb{R} \) a function of bounded variation. Given \( t \in (a, b) \), we write

\[
\alpha(t^\pm) = \lim_{s \to t^\pm} \alpha(s).
\]

We say that \( \alpha \) is normalized if \( \alpha(a) = 0 \) and for all \( t \in (a, b) \),

\[
\alpha(t) = \frac{\alpha(t^-) + \alpha(t^+)}{2}.
\]

If \( f \) is continuous on \( [a, b] \) and \( \alpha \) is of bounded variation, we will denote the Stieltjes integral of \( f \) with respect to \( \alpha \) by \( \int_a^b f(s)d\alpha(s) \). Functions of bounded variation behave well with respect to Stieltjes integration: if \( \alpha \) is of bounded variation on \( [a, b] \), if \( f \) is continuous, and if \( c \in [a, b] \), then

\[
\beta(x) = \int_c^x f(s)d\alpha(s)
\]

defines a function of bounded variation. Moreover, if \( g \) is continuous, then

\[
\int_a^b g(s)d\beta(s) = \int_a^b g(s)f(s)d\alpha(s).
\]

Stieltjes integration behaves as expected under change of coordinates: if \( \alpha \) is of bounded variation on \( [a, b] \), if \( f \) is continuous on \( [a, b] \) and if \( \gamma \) is continuous and strictly increasing on
\[ \int_a^b f(s) \, d\alpha(s) = \int_c^d f(\gamma(s)) \, d\alpha(\gamma(s)) \]

where \( a = \gamma(c) \) and \( b = \gamma(d) \).

Stieltjes integration can be extended to improper integrals. For example if \( \alpha : \mathbb{R}^+ \to \mathbb{R} \) is of bounded variation, and \( f \) is continuous on \((0, \infty)\), we write

\[ \int_0^\infty f(s) \, d\alpha(s) = \lim_{R \to \infty, \epsilon \to 0} \int_\epsilon^R f(s) \, d\alpha(s) \]

when the limit exists and is finite. The formulas (26) and (27) are easily extended to improper integrals.

**Definition 3.1.** Let \( T \in \mathcal{S} \). We say that a nondecreasing function \( \alpha : [0, 1] \to \mathbb{R} \) is a solution of the Hausdorff Moment Problem for \( T \) if, for all \( n \),

\[ t_n = \int_0^1 s^n \, d\alpha(s). \tag{28} \]

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\[ t_n = \int_0^\infty s^n \, d\alpha(s). \tag{29} \]

The solution of the Stieltjes Moment Problem played a fundamental role in the development of modern analysis. We recall the material relevant to our purpose.

**Definition 3.2.** Let \( T \) be a sequence of real numbers. The difference operator, \( \Delta \), mapping sequences of real numbers to sequences of real numbers is defined by

\[ (\Delta(T))_n = t_{n+1} - t_n. \tag{30} \]

A sequence \( T \in \mathcal{S} \) is said to be completely monotonic if for all \( n \) and for all \( k \),

\[ (\Delta^k(T))_n \geq 0. \tag{31} \]
We can now state Hausdorff’s solution to the moment problem bearing his name:

**Theorem 3.1. (Hausdorff)** Suppose $T \in \mathcal{S}$. Then the Hausdorff Moment Problem for $T$ has a solution if and only if the sequence $T$ is completely monotonic. When $T$ is completely monotonic, the solution of the moment problem is unique.

The solution of the moment problem associated to Stieltjes is given in

**Theorem 3.2. (Stieltjes)** Suppose $T \in \mathcal{S}$. Then the Stieltjes Moment Problem for $T$ has a solution if and only if the Hankel determinants

\[
H_{0,n} = \begin{vmatrix}
    t_0 & t_1 & \ldots & t_n \\
    t_1 & t_2 & \ldots & t_{n+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    t_n & t_{n+1} & \ldots & t_{2n}
\end{vmatrix}
\]

(32)

\[
H_{1,n} = \begin{vmatrix}
    t_1 & t_2 & \ldots & t_{n+1} \\
    t_2 & t_3 & \ldots & t_{n+2} \\
    \vdots & \vdots & \ddots & \vdots \\
    t_{n+1} & t_{n+2} & \ldots & t_{2n+1}
\end{vmatrix}
\]

(33)

are nonnegative for all values of $n$.

4. Proof of the main result

We begin with a definition:

**Definition 4.1.** Let $X = (X_{i,j})$, $0 \leq i, j < \infty$ be a doubly infinite matrix with real entries. We say that $X$ is a tableau if

1. $X_{i,j} \geq 0$ for all $i, j$.

2. If $X_k = \{X_{k,j}\}_{j=0}^{\infty}$ is the sequence whose terms are given by the $k$th row of $X$ and $\mathcal{R}$ is the renormalization operator defined by (20), then $\mathcal{R}(X_k) = X_{k-1}$ for all $k$.

Given $n \in \mathbb{N}$, a partial $n$-tableau is a matrix of $n$ rows and an infinite number of columns which satisfies the two defining conditions of a tableau. If $\mathcal{P}_n$ is the collection of partial
n-tableau, if \( P \in \mathcal{P}_n \) and \( m \leq n \), the m-corner operator \( \mathcal{O}_m : \mathcal{P}_n \to \mathbb{R}^m \times \mathbb{R}^m \) is the map defined by truncation:

\[
\mathcal{O}_m(P) = (P_{i,j}), \quad 0 \leq i, j \leq m - 1.
\]

Tableaux are closely related to stable sequences: It is clear from Definition 2.2 and Definition 4.1 that if \( X \) is a tableau and \( X_0 = \{X_{0,n}\}^\infty_{n=0} \) is the first row of \( X \), then \( X_0 \in \text{Stab}(\mathcal{R}) \). Conversely,

**Lemma 4.1.** Suppose that \( T \in \text{Stab}(\mathcal{R}) \). Then there is a tableau whose first row is \( T \).

**Proof** Let \( T \in \text{Stab}(\mathcal{R}) \). For each \( n \in \mathbb{N} \) we can find an partial n-tableau with first row \( T \). We will create an infinite sequence, \( \{Y^\alpha\}^\infty_{\alpha=1} \). Each \( Y^\alpha \) is itself an infinite sequence of partial tableaux where the number of rows will tend to infinity as \( \alpha \to \infty \). Then we will use a diagonal trick to finish the proof.

Define a sequence of partial tableau, \( Y^1 = \{Y^1_n\}^\infty_{n=1} \), where for each \( n \), \( Y^1_n \) is a partial n-tableau with \( T \) as first row. Having chosen subsequences \( Y^{m-1} \subset Y^{m-2} \subset \cdots \subset Y^1 \), choose a subsequence \( Y^m \) of \( Y^{m-1} \) which satisfies

1. \( Y^m_n \) is a partial \( k_n \)-tableau with \( k_n \geq m \);

2. If \( \mathcal{O}_m \) is the m-corner operator defined in (34), then \( \mathcal{O}_m(Y^m_n) \) converges as \( n \to \infty \).

Consider the sequence of matrices \( Z_k = Y^k_k \). Then \( Z_k \) converges to a doubly infinite matrix with nonnegative entries and first row given by \( T \). That \( Z \) is a tableau follows from the continuity of the m-corner operator acting on \( Z_k \).

**Lemma 4.2.** Suppose that \( X \) is a tableau and let \( \{X_{k,n}\} = \{X_{k,n}\}^\infty_{k=0} \) be the sequence whose terms are given by the \( n \)th column of \( X \). Then \( \{X_{k,n}\} \) is a completely monotonic sequence.
Proof An explicit computation shows that the diagonal entries of $X$ are given by

$$X_{k,k} = \sum_{l=0}^{k} (-1)^l \binom{k}{l} X_{l,0}. \quad (35)$$

By assumption the terms of $X$ are all nonnegative. This proves that the first column of $X$ is completely monotonic. To finish the proof, note that tableau are stable under truncation of their first $n$ columns. Carrying out such a truncation, the argument above establishes that the $(n+1)$th column of $X$ (the first column of the truncated matrix) is completely monotonic. $\square$

Lemma 4.3. Suppose $\{y_i\}_{i=1}^{\infty}$ is a completely monotonic sequence. Let $\alpha : [0, 1] \to \mathbb{R}$ be the normalized nondecreasing function such that, for $1 \leq i,$

$$y_i = \int_0^1 s^{i-1}d\alpha(s). \quad (36)$$

Then there exists $y_0$ such that $\{y_i\}_{i=0}^{\infty}$ is completely monotonic if and only if

$$\int_0^1 s^{-1}d\alpha(s) \quad \text{converges.} \quad (37)$$

Moreover, if $\int_0^1 s^{-1}d\alpha(s) = L,$ then

$$L = \inf \{y_0 : \{y_i\}_{i=0}^{\infty} \text{ is completely monotonic} \}. \quad (38)$$

Proof Suppose that (37) holds. Define $\beta : [0, 1] \to \mathbb{R}$ by

$$\beta(t) = \int_0^t s^{-1}d\alpha(s).$$

Then $\beta$ is nondecreasing and for $i \geq 1,$

$$\int_0^1 s^id\beta(s) = \int_0^1 s^{i-1}d\alpha(s).$$

Setting $y_0 = \int_0^1 s^{-1}d\alpha(s),$ we see that there is a solution to the Hausdorff Moment Problem for the augmented sequence $\{y_i\}_{i=0}^{\infty}.$ By Hausdorff’s Theorem (cf Theorem 3.1), the augmented sequence is completely monotonic.
Conversely, suppose there is a $y_0 \in \mathbb{R}$ such that the augmented sequence $\{y_i\}_{i=0}^{\infty}$ is completely monotonic. Let $\beta : [0, 1] \to \mathbb{R}$ be the normalized nondecreasing solution of the Hausdorff Moment Problem for the augmented sequence. Then, for all $i \geq 1$,

$$(39) \quad \int_0^1 s^{i-1} \, d\alpha(s) = \int_0^1 s^{i-1} \, s \, d\beta(s).$$

Define continuous linear functionals, $L_\alpha$, $L_\beta$, on the space of continuous functions on $[0, 1]:$

$$L_\alpha(f) = \int_0^1 f(s) \, d\alpha(s)$$
$$L_\beta(f) = \int_0^1 f(s) \, s \, d\beta(s).$$

From (39) we conclude that $L_\alpha$ and $L_\beta$ agree on polynomials. By the Weierstrass theorem and continuity of the integral, we conclude that $L_\alpha = L_\beta$. Choose $f_n(s)$ the increasing sequence of nonnegative continuous functions equal to $\frac{1}{n}$ on $[\frac{1}{n}, 1]$ and equal to $n$ on $[0, \frac{1}{n}]$ so that

$$\lim_{n \to \infty} \int_0^1 f_n(s) \, s \, d\beta(s) = y_0.$$

Then,

$$(40) \quad \int_{\frac{1}{n}}^1 f_n(s) \, d\alpha(s) \leq \int_0^1 f_n(s) \, s \, d\beta(s).$$

Since the right hand side of (40) converges as $n \to \infty$, we conclude that (39) holds. Since the right hand side converges to $y_0$, we conclude that $L = \int_0^1 s^{-1} \, d\alpha(s)$ is a lower bound for any $y_0$ augmenting the original sequence. Since we have already established that when the integral converges, $y_0 = L$ gives a completely monotonic augmented sequence, we are done. □

Remark: With $\{y_i\}_{i=1}^{\infty}$ and $L$ as in Lemma 4.3, any $y_0 \geq L$ gives a completely monotonic augmented sequence.
Lemma 4.4. Let $X$ be a tableau. Then $X$ is determined by its first column. In fact, if $\{X_{n,0}\} = \{X_{n,0}\}_{n=0}^{\infty}$ is the first column of $X$ and $\alpha : [0,1] \to \mathbb{R}$ is the normalized nondecreasing function representing $\{X_{n,0}\}$:

$$X_{n,0} = \int_0^1 s^n d\alpha(s),$$

(41)

then

$$X_{0,p} = \int_0^1 s^{-p}(1-s)^p d\alpha(s).$$

(42)

**Proof** By Lemma 4.2 $\{X_{n,0}\} = \{X_{n,0}\}_{n=0}^{\infty}$ is a completely monotonic sequence and thus admits a representation by $\alpha$ as in (41). By definition of a tableau, $X_{n,0} + X_{n,1} = X_{n-1,0}$ for all $n \geq 1$, and thus for $n \geq 1$,

$$X_{n,1} = \int_0^1 s^{n-1}(1-s) d\alpha(s).$$

Since $\{X_{n,1}\}_{n=1}^{\infty}$ is represented as a moment sequence, by Hausdorff’s Theorem $\{X_{n,1}\}_{n=1}^{\infty}$ is completely monotonic. Since $\{X_{n,1}\}_{n=1}^{\infty}$ is part of a column of a tableau, $\{X_{n,1}\}_{n=1}^{\infty}$ extends to a completely monotonic sequence $\{X_{n,1}\}_{n=0}^{\infty}$. By Lemma 4.3, we conclude that

$$\int_0^1 s^{-1}(1-s) d\alpha(s)$$

converges and we set

$$L = \int_0^1 s^{-1}(1-s) d\alpha(s).$$

Let $\beta : [0,1] \to \mathbb{R}$ be a normalized nondecreasing function representing the completely monotone sequence $L, X_{1,1}, X_{2,1}, \ldots$. Let $\epsilon = X_{0,1} - L$ and let $h(t)$ be the Heaviside function:

$$h(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Define $\gamma : [0,1] \to \mathbb{R}$ by

$$\gamma(t) = \beta(t) + \epsilon h(t).$$

(43)
Then $\gamma$ is nondecreasing and for all $n \geq 0$,
\[ X_{n,1} = \int_0^1 s^n d\gamma(s). \]
Let $f_n(s)$ be as defined in the proof of Lemma 4.3. Consider the pair of columns $\{X_{n,1}\}$ and $\{X_{n,2}\}$. By the analysis given for the pair $\{X_{n,0}\}$ and $\{X_{n,1}\}$, we know that
\[ \int_0^1 s^{-1}(1-s)d\gamma(s) \]
converges and therefore $\int_0^1 f_n(s)(1-s)d\gamma(s)$ converges as $n \to \infty$. But $\int_0^1 f_n(s)(1-s)d\beta(s)$ is nonnegative and $\int_0^1 f_n(s)(1-s)dh(s)$ diverges as $n \to \infty$, from which we conclude that $\epsilon = 0$. Thus, $X_{0,1} = L$ and the column $\{X_{n,0}\}$ determines the column $\{X_{n,1}\}$. The lemma follows by induction. 

**Proof of Theorem 1.1** Suppose that $T \in S$ and suppose that $\alpha : \mathbb{R}^+ \to \mathbb{R}$ is a normalized nondecreasing function representing $T$:
\[ t_n = \int_0^\infty s^n d\alpha(s). \]
Fix $p \in \mathbb{N}$ and define $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ by
\[ \beta(s) = \int_0^s \frac{1}{(1+u)^p} d\alpha(u). \]
Then $\beta$ is nondecreasing on $\mathbb{R}^+$ and of bounded variation. Let $S$ be the sequence corresponding to the moments of $\beta$:
\[ s_n = \int_0^\infty u^n d\beta(u). \]
A direct computation using (20) gives $\mathcal{R}^p(S) = T$. This proves that every moment sequence is stable.

To establish the converse, suppose that $T$ is a stable sequence. By Lemma 4.1, there is a tableau, $X$, which has $T$ as its first row. By Lemma 4.4, $X$ is determined by its first column. By Lemma 4.2, the first column of $X$ is completely monotonic and thus,
by Hausdorff’s Theorem, there is a unique normalized nondecreasing \( \alpha : [0,1] \to \mathbb{R} \) which represents \( \{X_{n,0}\} \):

\[
X_{n,0} = \int_0^1 s^n d\alpha(s).
\]

Thus, \( T \) is determined by \( \alpha \). To complete the proof, we use \( \alpha \) to construct a measure on \( \mathbb{R}^+ \) with moments given by \( T \).

Write \( u = \frac{1-s}{s} \) and \( s = \frac{1}{1+u} \). The function \( \gamma : \mathbb{R}^+ \to \mathbb{R} \) defined as the composition \( \gamma(u) = -\alpha(s) \) is nondecreasing with total variation bounded by the variation of \( \alpha \). By Lemma 4.4,

\[
X_{0,n} = \int_0^\infty u^n d\gamma(u)
\]

which exhibits the first row of \( X \) as a moment sequence and completes our proof. \( \square \)

5. Applications

Theorem 1.1 provides for a representation of the transition probabilities for a Markov chain which provides a classical precursor for a discrete theory of quantum gravity satisfying causality, covariance and cosmological renormalizability:

**Corollary 5.1.** Suppose that \( M \in \mathbb{M} \) is a Markov chain satisfying Definition 2.2. Suppose that \( T \in \mathbb{S} \) is the sequence of coupling constants defining \( M \) and that \( \alpha : \mathbb{R}^+ \to \mathbb{R} \) is a nondecreasing function representing \( T \). Suppose that \( \{q_n\}_{n=0}^\infty \) are the transition probabilities defined in (14). Then

\[
\frac{1}{q_n} = \int_0^\infty (1+s)^n d\alpha(s).
\]

**Proof** This follows immediately from (17) and the binomial theorem. \( \square \)

Using Corollary 5.1 we obtain an attractive representation for general transition probabilities:
Corollary 5.2. Suppose that $C \in \mathcal{C}_n$ and that $D \in F(C)$. Suppose that the cardinality of $\text{Prec}(C,D)$ is $\rho$ and that the cardinality of $\text{max}(C,D)$ is $m$. Suppose that $M \in \mathcal{M}$ is a Markov chain satisfying Definition 2.2. Suppose that $T \in S$ is the sequence of coupling constants defining $M$ and that $\alpha : \mathbb{R}^+ \to \mathbb{R}$ is a nondecreasing function representing $T$. Then

$$\text{Prob}(C \to D) = \int_0^\infty \frac{s^m(1 + s)^{\rho - m}d\alpha(s)}{(1 + s)^\rho d\alpha(s)}.$$  

Proof. This follows immediately from Corollary 5.1, (15), and the binomial theorem applied to $s^m = ((1 + s) - 1)^m$. \hfill \Box

Our next result establishes that all positive sequences which grow fast enough are stable:

Corollary 5.3. Any monotonic sequence which grows sufficiently quickly defines an element of $\text{Stab}(\mathcal{R})$.

Proof. For a quickly growing sequence, the positivity condition on the Hankel determinants (32), (33), are trivially satisfied as the value of the determinant is controlled by the entry in the lower right hand corner. Thus, any monotonic sequence which grows sufficiently quickly is a moment sequence and Corollary 5.3 follows from Theorem 1.1. \hfill \Box

Corollary 5.2 and Corollary 5.3 provide a means of quantifying the evolution of causets under dynamics which provide for rapidly increasing coupling constants. We hope to return to this in a future paper.

Our final result uses Hankel determinants to associate to any stable sequence which is not a finite linear combination of percolation sequences, a model Hilbert space and a nonnegative self-adjoint operator. Our development follows that of Simon [10].

The Hankel determinants appearing in (32) and (33) are associated to quadratic forms which arise naturally in the analysis of the Stieltjes Moment Problem. More precisely, given
a sequence $T \in \text{Stab}(R)$, consider the sesquilinear forms $H^i_N : \mathbb{C}^N \to \mathbb{C}$ defined by

\begin{align}
H_0^i_N(\rho, \sigma) &= \sum_{0 \leq n, m \leq N-1} \bar{\rho}_n \sigma_m t_{n+m} \tag{47} \\
H_1^i_N(\rho, \sigma) &= \sum_{0 \leq n, m \leq N-1} \bar{\rho}_n \sigma_m t_{n+m+1}. \tag{48}
\end{align}

Let $H^i_N$ be the matrices associated to the forms $H^i_N$ via the relation

\begin{equation}
H^i_N(\rho, \sigma) = \langle \rho, H^i_N \sigma \rangle \tag{49}
\end{equation}

where the pairing is Euclidean. Then the Hankel determinants appearing in Theorem 3.2 are given by $\det(H^i_N) = H_{i,N}$ and the forms $H^i_N$ are strictly positive definite if and only if the corresponding Hankel determinants are positive (cf [10]). Following [10], we use this material to reformulate the Stieltjes result in the language of self-adjoint operators.

For the remainder of the paper we assume that the sequence $T$ is not a finite linear combination of percolation sequences, so that the Hankel determinants are all strictly positive definite.

Suppose that $\mathbb{C}[x]$, is the algebra of polynomials with complex coefficients. Define a positive definite inner product on $\mathbb{C}[x]$ by

\begin{equation}
\langle p, q \rangle = H_{0,N}(\rho, \sigma) \tag{50}
\end{equation}

where $p(x) = \sum_{n=0}^{N-1} \rho_n x^n$ and $q(x) = \sum_{n=0}^{N-1} \sigma_n x^n$. Using this inner product, we complete $\mathbb{C}[x]$ to a Hilbert space $\mathbb{H}_T$, where the subscript $T$ denotes the dependence on the moment sequence $T$.

Let $A$ be the operator with domain $D(A) = \mathbb{C}[x] \subset \mathbb{H}_T$ defined by

\begin{equation}
A(p)(x) = xp(x). \tag{51}
\end{equation}

Then $A$ is densely defined, symmetric and nonnegative. Thus, by the theory of von Neumann, $A$ admits self-adjoint extensions. Amongst the (possibly many) self-adjoint extensions of
A there is a distinguished extension, the \textit{minimal} nonnegative self-adjoint extension (the Friedrichs extension) of $A$ to an operator $A_F$ with domain contained in $\mathbb{H}_T$.

It is a theorem of Simon that the collection of such extensions of $A$ parameterizes solutions to the (nondegenerate) Stieltjes moment problem \cite{10}. We summarize these results in the following

\textbf{Theorem 5.1. (Simon)} Suppose that $T \in \mathcal{S}$ is a sequence which is not a finite linear combination of percolation sequences and whose corresponding Stieltjes problem admits a solution. Let $\mathbb{H}_T$ be the corresponding Hilbert space completion of the algebra of polynomials with inner product defined by (54), and let $A : D(A) \to \mathbb{C}[x]$ be the operator defined by (51). Then every solution to the Stieltjes problem for the sequence $T$ corresponds to a unique nonnegative self-adjoint extension of $A$ to an operator $\tilde{A} : \mathbb{H}_T \to \mathbb{H}_T$ with spectral measure $\mu_{\tilde{A}}$ satisfying

$$t_n = \int_0^\infty s^n d\mu_{\tilde{A}}(s).$$

With the conventions established above, we have the following corollary:

\textbf{Corollary 5.4.} To every sequence $T \in \mathcal{S}_1 \cap \text{Stab}(\mathcal{R})$, which is not a finite linear combination of percolation sequences, there corresponds a pair $(\mathbb{H}_T, A_F)$ where $\mathbb{H}_T$ is the Hilbert space completion of $\mathbb{C}[x]$ defined by inner products (57) and $A_F$ is the minimal nonnegative self-adjoint extension of the densely defined operator $A : \mathbb{C}[x] \to \mathbb{C}[x]$ defined in (54). Thus, there is a distinguished spectral measure whose moments are given by the sequence $T$.

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Boston College, Chestnut Hill, MA 02467

E-mail address: Avner.Ash@bc.edu

New College of Florida, Sarasota, FL 34243

E-mail address: ptm@virtu.sar.usf.edu