ON PRODUCTS IN A REAL MOMENT-ANGLE MANIFOLD

LI CAI

Abstract. In this paper we give a necessary and sufficient condition for a (real) moment-angle complex to be a topological manifold. The cup and cap products in a real moment-angle manifold are studied: the Poincaré duality via cap products is equivalent to the Alexander duality of the defining complex $K$. Consequently, the cohomology ring (with coefficients integers) of a polyhedral product by pairs of disks and their bounding spheres is isomorphic to that of a differential graded algebra associated to $K$, and the dimensions of the disks.

1. Introduction

Let $m$ be a positive integer, and let $K$ be an abstract simplicial complex with vertex set $[m] = \{1, 2, \ldots, m\}$. Thus $\emptyset \in K$, and any subset of $\sigma \subset [m]$ is a simplex of $K$, if $\sigma$ is. We use the notation $|K|$ for the geometric realization of $K$. Denote by $(X, A)$ $m$ pairs of topological spaces $(X_i, A_i)$, $i = 1, 2, \ldots, m$, and let $\prod_{i=1}^{m} X_i$ be the $m$-fold Cartesian product of $X_i$. For $x = (x_i)_{i=1}^{m} \in \prod_{i=1}^{m} X_i$, define

$$\sigma_x = \{i \in [m] \mid x_i \in X_i \setminus A_i\},$$

then the corresponding polyhedral product (following [BBCG10a] and [BP14]) is given by

$$\left(\prod_{i=1}^{m} X_i, A_i\right)^K = \{x \in \prod_{i=1}^{m} X_i \mid \sigma_x \in K\}. \tag{1}$$

If all pairs $(X_i, A_i)$ are homeomorphic to a given one, $(X, A)$, then $\left(\prod_{i=1}^{m} X_i, A_i\right)^K$ shall be denoted by $(X, A)^K$.

$(D^1, S^0)^K$ is referred to as a real moment-angle complex (cf. [BP02, Section 6.6]). If it is a topological manifold, then we call it a real moment-angle manifold. Following [BP02], [GL13] and [BBCG12], in this paper we focus on two problems:

(P-1) the characterization of a real moment-angle manifold, and
(P-2) cup and cap products in its (co)homology.
(P-1) was answered by M. Davis, under the assumption that $K$ is a flag complex, i.e., any set of vertices of $K$ that are pairwise connected by edges spans a simplex of $K$ (cf. [Dav08, Theorem 10.6.1, p. 197]).\footnote{The assumption implies that $(D^1, S^0)^K$ is aspherical, whose fundamental group is isomorphic to the commutator subgroup of the associated right-angled Coxeter group; see [Dav08, pp. 11–12].} It follows that $(D^1, S^0)^K$ is a topological $n$–manifold if and only if $|K|$ is a generalized homology $(n−1)$–sphere (i.e., a homology $(n−1)$–manifold having the homology of an $(n−1)$–sphere), with $\pi_1(|K|) = 1$ when $n \neq 1, 2$.

In Section 2 we shall prove that Davis’s characterization theorem still holds, without assuming the flagness of $K$ (see Theorem 2.3). Together with the construction in [BBCG10b] given by Bahri, Bendersky, Cohen and Gitler (see Definition 2.6),\footnote{This construction was intensely used in [LdM89] and [GL13] in the language of simple polytopes.} it follows that a moment-angle complex $(D^2, S^1)^K$ is a topological $(n+m)$–manifold if and only if $|K|$ is a generalized homology $(n−1)$–sphere.

Now we turn to (P-2). Unless otherwise stated, we always suppose that all coefficients taken in (co)homology groups are integers.

The additive structure of $H^*((D^1, S^0)^K)$ is well-known.\footnote{See for example, [Dav83], [LdM89] from reflection groups, [BBCG10a] from homotopy theory, also the Goresky–MacPherson Formula [GM88] together with the fact that $(D^1, S^0)^K$ is the deformation retract of the coordinate subspace arrangement complement $(\mathbb{R}, \mathbb{R} \setminus \{0\})^K$ (see [BP02, Theorem 8.9]).} It turns out that, we have isomorphisms

$$H^p((D^1, S^0)^K) \cong \bigoplus_{\omega \subset [m]} H^{p−1}(K_{\omega})$$

in all dimensions $p \geq 0$, where $K_{\omega}$ is the full subcomplex with respect to $\omega$.

Let $J = (j_i)_{i=1}^m$ be an $m$-tuple of positive integers, and denote by $(D, S)^K$ the polyhedral product with respect to pairs $(D^{j_i}, S^{j_i−1})$ of $j_i$-disks and their bounding spheres, $i = 1, 2, \ldots, m$. With a general approach from homotopy theory, the information of the ring $H^*((D^1, S^0)^K)$ is given in [BBCG12], as well as its relation with other polyhedral products, including $(D, S)^K$.

Our approach here follows that of Baskakov, Buchstaber and Panov [BBP04] and Panov [Pan08] on $H^*((D^2, S^1)^K)$: we will show that $H^*((D^1, S^0)^K)$ is not commutative in any sense (see (38), (39)). Technically, we use the result of Whitney [Whi38] on the cup and cap products in a Cartesian product of simplicial complexes, and prove that it applies to the situation here (see Theorem 3.1). Consequently, we give product rules on the right-hand side of (2) which
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coincides with the cup products on the left (see (31) and (32)). Together with the method from [BBCG12], we will show that the ring $H^*((D,S)^K)$ can be understood uniformly (see Theorem 5.1, Remark 5.2).

In the language of the intersections of submanifolds, rules for the cup products were understood by Gitler and López de Medrano [GL13]. We shall follow their approach to make a comparison of the two rings $H^*((D^1,S^0)^K)$ and $H^*((D^2,S^1)^K)$ in Example 5.4 (compare [GPW01], [dL00]).

The paper is organized as follows. Section 2 is devoted to the characterization of a real moment-angle manifold; as a corollary, the characterization of a moment-angle manifold is given in Subsection 2.1. The cup and cap products in a real moment-angle complex, based on Whitney’s formulae (12) and (13), is given in Section 3; while the proof the main theorem, Theorem 3.1, is postponed to Section 6. Most of the explicit calculations are taken in Section 4, based on a special cellular (co)chain complex given in Subsection 3.2. As a by-product, we get an alternative proof of the Alexander duality (see Theorem 4.3). In Section 5, we describe explicitly the cup products in the polyhedral product $(D,S)^K$, via the multiplication in the corresponding differential graded algebra (see Theorem 5.1). Section 6 is devoted to the proof of Theorem 3.1.

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2. Davis’s characterization theorem

In this section, all manifolds are assumed to have no boundaries.

The dimension of a simplex $\sigma \in K$ is given by $\text{card}(\sigma) - 1$. Thus $\dim(\sigma) = -1$ if and only if $\sigma = \emptyset$. Let $K'$ be the derived complex of $K$, where a $k$-simplex is a chain $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ of simplices in $K$, each of non-negative dimension, such that $\sigma_i \subset \sigma_{i+1}$ is a proper face, $i = 0, 1, \ldots, k - 1$. Clearly $|K'|$ is the barycentric subdivision of $|K|$. Let $K'_+$ be the augmentation of $K'$: the dimension of the starting simplex in each chain of $K'_+$ can be negative. Denote by $|K'_+|$ the cone over $|K'|$, with the collapsed end point corresponding to $\emptyset$. (While $\emptyset \in K$ has no geometric meanings in $|K|$. ) In what follows, we treat $|K'|$ as a subcomplex of $|K'_+|$.

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4The cellular (co)chain complex here is used by Choi and Park [CP13b] to show that, any odd torsion can appear in the cohomology of a real toric manifold or a small cover.
Let $X_K$ be the intersection of $(D^1, S^0)^K$ with the first orthant of $\mathbb{R}^m$, namely the set $\{(x_i)_{i=1}^m \in \mathbb{R}^m \mid x_i \geq 0, i = 1, 2, \ldots, m\}$. By (1), $X_K$ can be decomposed as the union of cubes

$$\bigcup_{\sigma \in K} C_\sigma; \quad C_\sigma = \prod_{i=1}^m Y_i, \quad Y_i = \begin{cases} [0,1] & \text{if } i \in \sigma, \\ \{1\} & \text{otherwise.} \end{cases}$$

**Lemma 2.1** (cf. [BP02, Chapter 4, pp. 53–55]). Let $\varphi : K \to X_K$ be the map sending each simplex $\sigma$ to the point $(x_i)_{i=1}^m$, where $x_i = 0$ if $i \in \sigma$ and $x_i = 1$ if otherwise. Then the mapping $\varphi' : |K'_+| \to X_K$ which sends each $k$-simplex $|(\sigma_0, \sigma_1, \ldots, \sigma_k)|$ to the linear simplex spanned by $\varphi(\sigma_0), \varphi(\sigma_1), \ldots, \varphi(\sigma_k)$, yields a triangulation of $X_K$.

**Proof.** Suppose that $\sigma \in K$ is a $k$-simplex. Let $K'_{\leq \sigma} \subset K'_+$ be the subcomplex with maximal chains starting from $\emptyset$ and ending with $\sigma$, i.e., each is of the form

$$\sigma' = (\sigma_{-1}, \sigma_0, \ldots, \sigma_k)$$

with $\dim(\sigma_i) = i$, such that $\sigma_k = \sigma$. For $i = -1, 0, \ldots, k$, suppose $\varphi(\sigma_i) = (x_{i,j})_{j=1}^m$. By definition, $x_{i,j}$ has value of 0 or 1, $x_{i,j} \geq x_{i+1,j}$ and two adjacent points $\varphi(\sigma_i)$ and $\varphi(\sigma_{i+1})$ have Euclidean distance 1. We see that $\varphi'(|\sigma'|)$ is spanned by lattice points on a path connecting the two end-points on the diagonal of a cube. Then the standard way of triangulating a product of simplices implies that (cf. Eilenberg and Steenrod [ES52, p. 68]), the restriction $\varphi'_{|K'_{\leq \sigma}}$ triangulates $C_\sigma$. In this way $X_K$ is triangulated by $\varphi'$.

Let $\mathbb{R}^m$ be endowed with the $\mathbb{Z}_2^m$-action generated by $\{s_i\}_{i=1}^m$, with $s_i$ the reflection changing the sign of the $i$-th coordinate of each point. Denote by $X_i$ the subspace of $X_K$ fixed by $s_i$, $i = 1, 2, \ldots, m$. It can be checked that the inverse image of $X_i$ under $\varphi'$ in the lemma above is the star of $|(i)|$ in $|K'|$. Suppose that $Y_i = (\varphi')^{-1}(X_i)$. Clearly $|K'| = \bigcup_{i=1}^m Y_i$, and for $x \in |K'_+|$, denote by $I_x$ the set $\{i \in [m] \mid x \in Y_i\}$ (which can be empty). Let $U(\mathbb{Z}_2^m, |K'_+|)$ be the basic construction with respect to $\mathbb{Z}_2^m$, $|K'_+|$ and $\{Y_i\}_{i=1}^m$ (cf. [Dav08, Chapter 5]), which is a $\mathbb{Z}_2^m$-space given by

$$U(\mathbb{Z}_2^m, |K'_+|) = (\mathbb{Z}_2^m \times |K'_+|) / \sim,$$

where $(g, x) \sim (g', x')$ if and only if $x = x' \in |K'|$ and $g^{-1}g' \in (s_i)_{i \in I_x}$ ($g = g'$ if $I_x = \emptyset$), and the action follows $g' [g, x] = [g' g, x]$. It can be checked directly that the map

$$u_{\varphi'} : U(\mathbb{Z}_2^m, |K'_+|) \longrightarrow (D^1, S^0)^K$$

$$[g, x] \longmapsto g\varphi'(x),$$

is...
is a homeomorphism preserving the $\mathbb{Z}_2^m$-actions on both sides. Then it follows from Lemma 2.1 that $u_{\varphi'}$ triangulates $(D^1, S^0)^K$, such that $|K'|$ appears as the link of $\varphi'(\emptyset) = (1, 1, \ldots, 1)$.

**Definition 2.2.** A polyhedron $X$ is a homology $n$–manifold, if with any given triangulation, the link of each $p$-simplex, $0 \leq p \leq n$, has the homology of an $(n - 1 - p)$–sphere.\(^5\) A PL homeomorphism between two polyhedra is a simplicial homeomorphism, with suitable triangulations (and subdivisions) on both sides.

A polyhedral homology $n$–manifold $X$ is called a generalized homology $n$–sphere (resp. a PL $(n - 1)$–sphere), if it has the homology of an $n$–sphere (resp. is PL homeomorphic to the boundary of an $(n + 1)$-simplex). $X$ is a piecewise linear $n$–manifold, if with any given triangulation, the link of each vertex is a PL $(n - 1)$–sphere.

In what follows, let $(D^1, S^0)^K$ be equipped with the triangulation $u_{\varphi'}$ (see (5)). We see that the necessary condition for $(D^1, S^0)^K$ to be a homology $n$–manifold (resp. a PL $n$–manifold) is that $|K'|$ is a generalized homology $(n - 1)$–sphere (resp. a PL $(n - 1)$–sphere), where we can replace $|K'|$ by $|K|$ since they are PL homeomorphic to each other. In fact these conditions are also sufficient:

**Theorem 2.3.** The real moment-angle complex $(D^1, S^0)^K$ is a homology $n$–manifold (resp. a PL $n$–manifold), if and only if $|K|$ is a generalized homology $(n - 1)$–sphere (resp. a PL $(n - 1)$–sphere); suppose that $(D^1, S^0)^K$ is a homology $n$–manifold, then it is a topological manifold if and only if $|K|$ is simply connected when $n \geq 3$.

**Remark 2.4.** The argument of Panov and Ustinovsky [PU12, Theorem 2.2] can be used directly to proved that $(D^1, S^0)^K$ is homeomorphic to a smooth manifold, if $K$ is induced from a complete simplicial fan (see also Tambour [Tam12]), including the well-known case that $|K|$ bounds a convex polytope.

The famous Edwards–Freedman Theorem (see [Dav08, Theorem 10.4.10, p. 194] and references therein) asserts that, a polyhedral homology $n$–manifold $(n \geq 3)$ is a topological manifold, if and only if with any given triangulation, the link of each vertex is simply connected. Therefore, it suffices to check the link of each vertex.

We proceed with the proposition below, whose proof will be given after that of Theorem 2.3. Recall that for two disjoint polyhedra $X$ and $Y$ embedded in $\mathbb{R}^N$, their exterior join exists, if any two line segments, each joining two points

\(^5\)This is equivalent to the definition using local homology groups, see [Mun84, Exercise 64.1, p. 377].
Suppose that

\[ 2.4. \]

First if \( \sigma \in \text{the manifold is that of a vertex in the link of a codimension-1 face of} \]

Proof of Theorem 2.3. First if \( n \leq 2 \), then \( |K| \) can always be realized as the boundary of a convex polytope in \( \mathbb{R}^n \), thus the statement follows from Remark 2.4.

Suppose that \( n \geq 3 \). It is easy to see that the link of any \( k \)-simplex \( (k \geq 0) \) in a homology \( (n-1) \)-manifold (resp. a PL \( (n-1) \)-manifold) is a generalized homology \( (n-k-2) \)-sphere (resp. a PL \( (n-k-2) \)-sphere).\(^6\) Together with the fact that the exterior join of a generalized homology sphere (resp. a PL sphere) with \( S^0 \) will be a sphere of the same type,\(^7\) the statement follows from Proposition 2.5, and the Edwards–Freedman Theorem.

Proof of Proposition 2.5. Since \( \mathbb{Z}^m_2 \) acts on \( \mathcal{U}(\mathbb{Z}^m_2, |K'_+|) \) simplicially, it suffices to consider the links of each vertex in the image of \( \varphi' \). The link of \( \varphi'(|\emptyset|) \) is \( |K'| \). In what follows we consider other links.

Consider a vertex \( (\sigma) \in K'_+ \), where \( \sigma \in K \) is a \( k \)-simplex, \( k \geq 0 \). Let \( K'_{<\sigma} \) (resp. \( K'_{\geq \sigma} \)) be the subcomplex of \( K'_+ \) consisting of chains that ends with a proper face of \( \sigma \) (resp. begins with \( \sigma \)). By definition, \( \varphi'(|K'_{<\sigma}|) \ast \varphi'(|K'_{\geq \sigma}|) \) exists, which is \( \varphi'(|K'_{<\sigma} \ast K'_{\geq \sigma}|) \), where we see that \( K_{<\sigma} \ast K'_{\geq \sigma} \) is the subcomplex of chains containing \( \sigma \), i.e., it is the star of \( (\sigma) \) in \( K'_+ \). It follows that \( |(\sigma)| = \bigcap_{i \in \sigma} Y_i \), where \( \varphi'(Y_i) \) is fixed by \( s_i \) (see (4)), thus the star of \( |(\sigma)| \) in \( \mathcal{U}(\mathbb{Z}^m_2, |K'_+|) \) is given by

\[
(6) \quad \bigcup_{g \in (s_i)_{i \in \sigma}} g|K'_{<\sigma} \ast K'_{\geq \sigma}|.
\]

Since \( \varphi(|K'_{<\sigma}|) \) is invariant under the subgroup \( (s_i)_{i \in \sigma} \), the image of (6) under \( \varphi' \) is

\[
(7) \quad \bigcup_{g \in (s_i)_{i \in \sigma}} g\varphi'(|K'_{<\sigma}|) \ast \varphi'(|K'_{\geq \sigma}|) = \left( \bigcup_{g \in (s_i)_{i \in \sigma}} g\varphi'(|K'_{<\sigma}|) \right) \ast \varphi'(|K'_{\geq \sigma}|).
\]

\(^6\)For instance, by induction on dimension \( k \), with the observation that the link of a simplex \( \sigma \) in the manifold is that of a vertex in the link of a codimension-1 face of \( \sigma \).

\(^7\)The first case is easy; see [RS72, Proposition 2.23, pp. 23–24] for the PL case.
Let $\sigma' \in K'$ be any $k$-simplex (as a chain of length $k+1$) ending with $\sigma$. Then
\begin{equation}
\varphi'(|K'_{\leq \sigma}|) = \varphi'(|(\sigma)|) \ast \varphi'(|Lk(\sigma', K')|) = \varphi'(|(\sigma)|) \ast |Lk(\sigma', K')|.
\end{equation}
At last, let $K'_{\leq \sigma}$ be the subcomplex of $K'$ consisting of chains ending with a face of $\sigma$. It is straightforward to see that $\widetilde{C}_\sigma = \bigcup_{g \in (s_i)_{i \in \sigma}} g \varphi'(|K'_{\leq \sigma}|)$ is a $(k+1)$-cube centered at $\varphi'(|(\sigma)|)$, i.e., $C_\sigma$ in (3) with $[0,1]$ replaced by $[-1,1]$. It follows that $\bigcup_{g \in (s_i)_{i \in \sigma}} g \varphi'(|K'_{\leq \sigma}|)$ is the boundary of $\widetilde{C}_\sigma$, hence it is a $k$–sphere PL homeomorphic to the joins of $k + 1$ copies of $S^0$. Together with (8) and (7), Proposition 2.5 follows.

2.1. An application of Theorem 2.3. Suppose the vertex set of $K$ is $[m]$. A subset $\tau \subset [m]$ not contained in $K$ is called a missing face, if any proper subset of $\tau$ is a simplex of $K$. Clearly $K$ is determined by its missing faces.

Let $J = (j_k)_{k=1}^m$ be an $m$-tuple of positive integers, with $d(J) = \sum_{k=1}^m j_k$. For $i = 1, 2, \ldots, m$, let $B_i$ be the block of $j_i$ integers (we set $j_0 = 0$)
\[
\sum_{k=1}^{i-1} j_k + 1, \sum_{k=1}^{i-1} j_k + 2, \ldots, \sum_{k=1}^i j_k.
\]

Definition 2.6 (cf. [BBCG10b]). Suppose that $\tau = \{i_1, i_2, \ldots, i_l\}$ is a missing face of $K$. Denote by $\tau(J)$ the set $\{B_{i_1}, B_{i_2}, \ldots, B_{i_l}\}$. With respect to $K$ and $J$, we define an abstract simplicial complex $K(J)$ in such a way that, as $\tau$ runs through all missing faces of $K$, $\tau(J)$ gives all missing faces of $K(J)$. The vertex set of $K(J)$ is $[d(J)].$

In what follows, let $(\underline{D}, S)$ be the pairs $(D^{j_i}, S^{j_i-1})$, i.e., the unit $j_i$-disk with its boundary, $i = 1, 2, \ldots, m$.

Lemma 2.7 (cf. [BBCG10b]). The polyhedral product $(\underline{D}, S)^K$ is homeomorphic to the real moment-angle complex $(D^1, S^0)^{K(J)}$ (see (1) for definition).

Proof. Let $f = (f_1, f_2, \ldots, f_m)$ be the product of canonical homeomorphisms $f_i: P^i \to D^{j_i}$, $I = [-1,1]$. We write $f: I^{d(J)} \to \prod_{i=1}^m D^{j_i}$, which is clearly a homeomorphism. We claim that the restriction of $f$ to $(D^1, S^0)^{K(J)}$ induces a homeomorphism onto $(\underline{D}, S)^K$. By (1), $x = (x_i)_{i=1}^m \in (\underline{D}, S)^K$ if and only if $\sigma_x = \{i \in [m] \mid x_i \in D^{j_i} \setminus S^{j_i-1}\}$ does not contain any missing faces of $K$. This happens if and only if $\sigma_x(J) = \{B_i \mid f_i^{-1}(x_i) \in I^{j_i} \setminus \partial I^{j_i}\}$ does not contain any missing faces of $K(J)$. Therefore, the claim holds, from which the statement follows.  

\footnote{The definition here is slightly different from [BBCG10b, Definition 2.1]: we identify the vertices of $K(J)$ with $[d(J)]$ a priori and explicitly.}
Following [BBCG10b], \( K(J) \) can be understood in another way.

Recall that the simplicial join of two disjoint complexes \( K_1 \) and \( K_2 \) is the complex \( K_1 \ast K_2 = \{ \sigma_1 \cup \sigma_2 \mid \sigma_i \in K_i, i = 1,2 \} \). Suppose that \( v_i = \{ i \} \) is the \( i \)-th vertex of \( K_i \), \( i = 1,2,\ldots, m \). Let \( K(v_i) \) be the simplicial wedge of \( K \) on \( v_i \), which is a simplicial complex given by

\[
K(v_i) = \{ i \} \ast K[m]\{i\} \bigcup \{i+1\} \ast K[m]\{i\} \bigcup \{i, i+1\} \ast \text{Lk}(v_i, K),
\]

where \( K[m]\{i\} = \{ \sigma \in K \mid \sigma \subset [m] \setminus \{i\} \} \) is the full subcomplex containing the link \( \text{Lk}(v_i, K) \), such that for all \( j > i \), the label of the original \( j \)-th vertex is shifted to \( j+1 \) (labels \( \leq i \) are preserved). Thus the vertex set of \( K(v_i) \) is \( [m+1] \).

For \( i = 1,2,\ldots, m \), let \( J_i = (j_k)_{k=1}^m \) be the tuple with \( j_i = 2 \), and \( j_k = 1 \) for all \( k \neq i \). One can check that \( K(J_i) = K(v_i) \). In this way, \( K(J) \) can be obtained via consecutive simplicial wedge constructions. Moreover, it can be shown that \( |K(v_i)| \) is PL homeomorphic to the suspension \( S^0 \ast |K| \) (see for example, [CP13a, Proposition 2.2]). As a conclusion, we have the lemma below, the proof of which is omitted (see the proof of Theorem 2.3).

**Lemma 2.8.** \( |K(v_i)| \) is a generalized homology sphere (resp. a PL sphere) if and only if \( |K| \) is (see Definition 2.2).

**Proposition 2.9.** The polyhedral product \( (D^1, S^0)^{K(J)} \) is a homology manifold (resp. a PL manifold) of dimension \( n + d(J) - m \), if and only if \( |K| \) is a generalized homology sphere (resp. a PL sphere) of dimension \( n-1 \). Moreover, \( (D^1, S^0)^{K(J)} \) is a topological manifold if additionally \( d(J) > m \).

**Proof.** Since every simplicial wedge construction increases the dimension by one, the first statement follows directly from Theorem 2.3, and the lemma above. For the second one, from the Van Kampen theorem, we only need to consider the case when \( |K| \) is the 0–sphere \( (m = 2) \), with \( d(J) - m = 1 \). In this case \( |K(J)| \) bounds the 2-simplex, thus \( (D^1, S^0)^{K(J)} \) is the 2–sphere. \( \square \)

In particular, suppose that all pairs are taken as \( (D^2, S^1) \). Together with Lemma 2.7, we have (cf. [BP02, Problem 6.14]):

**Corollary 2.10.** The moment-angle complex \( (D^2, S^1)^K \) is a topological \( (n + m) \)-manifold if and only if \( |K| \) is a generalized homology \( (n-1) \)-sphere.

### 3. Cup and Cap Products

In this section we consider the cohomology of a real moment-angle complex \( (D^1, S^0)^K \), with products involved.
3.1. Whitney’s formulae. We say that an abstract simplicial complex $K$ is of finite type, if the number of simplices in each dimension is finite. Let $C_*(K)$ (resp. $C^*(K)$) denote the simplicial chain complex (resp. cochain complex) of $K$, where $C_*(K) = \bigoplus_{i=0}^\infty C_i(K)$ (resp. $C^*(K) = \bigoplus_{i=0}^\infty C^i(K)$). If $K$ is of finite type, then each $C_i(K)$ is finitely generated, whose dual basis generates $C^i(K) = \Hom(C_i(K), \mathbb{Z})$.

Suppose that $p = (p_i)_{i=1}^m$ and $q = (q_i)_{i=1}^m$ are two vectors of integers. The notation $(p, q)$ means the mod 2 integer from the shuffle

$$p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_m \rightarrow p_1, q_1, p_2, q_2, \ldots, p_m, q_m,$$

namely by interchanging adjacent integers to make the left sequence into the right one, each interchange yields a summand, being the product of the two integers involved, and $(p, q)$ is the sum of these summands. A straightforward calculation shows that

$$\sum_{i=1}^m q_i \sum_{j>i} p_j \mod 2. \tag{10}$$

Let $X = \prod_{i=1}^m [K_i]$ be a product of polyhedra, with each $K_i$ of finite type. Denote by $C_*(X) = \bigoplus_{p=0}^\infty C_p(X)$ the tensor product $\bigotimes_{i=1}^m C_*(K_i)$, which is a differential graded $\mathbb{Z}$-module with the boundary operator $\partial$ subject to

$$\partial(\otimes_{i=1}^m c_{p_i}) = (-1) \sum_{j<i} p_j c_{p_1} \otimes \ldots \otimes c_{p_{i-1}} \otimes \partial c_{p_i} \otimes c_{p_{i+1}} \otimes \ldots \otimes c_{p_m}, \tag{11}$$

where $c_{p_i} \in C_{p_i}(K_i)$. Let $(C^*(X), d)$ be the graded dual of $(C_*(X), \partial)$. By assumption, we have

$$C^p(X) = \bigoplus_{\sum_{i=1}^m p_i = p} \bigotimes_{i=1}^m C^{p_i}(K_i).$$

The notation $(C_*(X), C^*(X), \sim, \adm)$ means that $C_*(X)$ and $C^*(X)$ are endowed with products

$$\adm: C^*(X) \bigotimes C^*(X) \rightarrow C^*(X) \quad \text{and} \quad \sim: C^*(X) \bigotimes C_*(X) \rightarrow C_*(X)$$

respectively, such that

$$\adm(\otimes_{i=1}^m c_{p_i}) \sim (\otimes_{i=1}^m c_{q_i}) = (-1)^{(p,q)} \otimes_{i=1}^m c_{p_i} \sim c_{q_i} \tag{12}$$

with $c_{p_i} \in C^{p_i}(K_i)$, $c_{q_i} \in C^{q_i}(K_i)$, $p = (p_i)_{i=1}^m$ and $q = (q_i)_{i=1}^m$, and

$$\adm(\otimes_{i=1}^m c_{p_i}) \sim (\otimes_{i=1}^m c_{r_i}) = (-1)^{(r,p)} \otimes_{i=1}^m c_{p_i} \sim c_{r_i} \tag{13}$$

with $c_{r_i} \in C_{r_i}(K_i)$ and $r = (r_i)_{i=1}^m$, where each product $\adm$ and $\sim$ on the right-hand sides of (12) and (13) means the simplicial cup and cap products, respectively (see (22) and (23)).
We treat $X$ as a $CW$ complex, with each cell of the form $\prod_{i=1}^{m} |\sigma_i|$, $\sigma_i \in K_i$. Let $A$ be a subcomplex of $X$. Denote by $C_*(A)$ the restriction of $C_*(X)$ to $A$; clearly $C_*(A)$ is closed under $\partial$. Let $\nu_c: C_*(A) \to C_*(X)$ be the chain inclusion, and let $(C^*(A), d)$ be the graded dual of $(C_*(A), \partial)$. Since the dual $\nu^*_c$ is surjective, the quadruple $(C_*(A), C^*(A), \sim, \dashv)$ can be defined uniquely, such that diagrams

\[
\begin{array}{c}
C^*(X) \otimes C^*(X) \xrightarrow{\sim} C^*(X) \\
\downarrow \quad v^*_c \otimes v^*_c \quad \text{and} \\
C^*(A) \otimes C^*(A) \xrightarrow{\sim} C^*(A)
\end{array}
\quad \text{and} \quad
\begin{array}{c}
C^*(X) \otimes C^*(X) \xrightarrow{\sim} C^*(X) \\
\downarrow \quad v^*_c \otimes v^*_c \quad \text{and} \\
C^*(A) \otimes C^*(A) \xrightarrow{\sim} C^*(A)
\end{array}
\]

commute.

**Theorem 3.1.** Let $(H_*(A), H^*(A), \sim, \dashv)$ be the quadruple of the singular (co)homology of $A$, endowed with cup and cap products. Then we have an isomorphism

\[ (H_*(C_*(A), \partial), H^*(C^*(A), d), \sim, \dashv) \cong (H_*(A), H^*(A), \sim, \dashv) \]

on passage to (co)homology, preserving the products.

**Remark.** The quadruple $(H_*(C_*(X), \partial), H^*(C^*(X), d), \sim, \dashv)$ with formulae (12) and (13) were given in Whitney [Whi38, pp. 424–426].

Theorem 3.1 will be illustrated in what follows; the proof will given in Section 6.

3.2. **The (co)homology of a real moment-angle complex.** Henceforth, occasionally we will not distinguish a geometric cell (resp. geometric simplex) or the associated cellular chain (resp. simplicial chain).

Let $I = [-1, 1]$ be a simplicial complex with a single 1-simplex $u$ connecting the two endpoints $t = \{-1\}$ and $t = \{1\}$, respectively, and they are obviously oriented that $\partial u = t - t$. Let $I^m$ be the $CW$-complex, with each cell an $m$-fold product of simplices of the form above.

By (1), $(D^1, S^0)^K$ is a cellular subcomplex embedded in $I^m$, such that a cell $e = \prod_{i=1}^{m} e_i \subset I^m$ belongs to $(D^1, S^0)^K$ if and only if $\sigma_e = \{ i \in [m] \mid e_i = u \}$ is a simplex of $K$.

In the remainder of this section, let $(X, A)$ be the pair $(I^m, (D^1, S^0)^K)$.

Now we perform a change of basis for $C_*(A)$ and $C^*(A)$, respectively. For $(C_*(I), \partial)$ indicated above, denote by

\[ \varepsilon = \partial u = t - t. \]
then $C_*(I)$ is generated by $u$, $\varepsilon$ and $t$. After dualizing, for $(C^*(I), d)$, we use basis elements $u^*$, $t^*$ and

$$\delta^* = t^* + \ell^*, \tag{16}$$

where $u^*$, $\ell^*$ and $t^*$ are the dual simplices of $u$, $\ell$ and $t$, respectively. Immediately we have $dt^* = u^*$ and $du^* = 0 = d\delta^*$.

**Definition 3.2.** Given a basis element $c = \otimes_{i=1}^mc_i \in C_*(X)$, $c_i \in C_*(I)$, we define $\sigma_c = \{i \in [m] \mid c_i = u\}$ and $\tau_c = \{i \in [m] \mid c_i = \varepsilon\}$. Then we denote $c$ by the word

$$u_{\sigma_c}\varepsilon_{\tau_c}\ell_{[m]\setminus(\sigma_c\cup\tau_c)},$$

which we shall abbreviate as $u_{\sigma_c}\varepsilon_{\tau_c}$, omitting the obvious part $\ell_{[m]\setminus(\sigma_c\cup\tau_c)}$. Clearly each abbreviated word $u_{\sigma_c}\varepsilon_{\tau_c}$ with $\sigma \cap \tau = \emptyset$ corresponds uniquely to a chain of $C_*(X)$.

Analogously, we denote a dual basis element $c = \otimes c^i \in C^*(X)$, $c^i \in C^*(I)$, by the word

$$u^{\sigma_c}\varepsilon_{\tau_c}\ell^{[m]\setminus(\sigma_c\cup\tau_c)},$$

with $\sigma_c = \{i \in [m] \mid c^i = u^*\}$ and $\tau_c = \{i \in [m] \mid c^i = t^*\}$, and write it as $u^{\sigma_c}\varepsilon_{\tau_c}$.

By definition, we see that $C_*(A)$ and $C^*(A)$ are generated by $\{u_{\sigma}\varepsilon_{\tau} \mid \sigma \in K, \sigma \cap \tau = \emptyset\}$ and $\{u^{\sigma}\varepsilon_{\tau} \mid \sigma \in K, \sigma \cap \tau = \emptyset\}$ respectively, with the void word corresponding to $\ell_{[m]}$ or $\delta^{[m]}$.

For $i \in [m]$, we use the notation $(i, \sigma) = \text{card}\{j \in \sigma \mid j < i\}$. Now take $u_{\sigma}\varepsilon_{\tau} \in C_*(A)$ and $u^{\sigma}\varepsilon_{\tau} \in C^*(A)$ respectively, by (11) and its dual form, we have

$$\partial(u_{\sigma}\varepsilon_{\tau}) = \sum_{i \in \sigma}(-1)^{(i, \sigma)}u_{\sigma \setminus \{i\}}\varepsilon_{\tau \cup \{i\}}, \tag{17}$$

and

$$d(u^{\sigma}\varepsilon_{\tau}) = \sum_{\{\sigma \cup \{i\} \in K \mid i \in \tau\}}(-1)^{(i, \sigma)}u^{\sigma \cup \{i\}}\varepsilon_{\tau \setminus \{i\}}, \tag{18}$$

Recall that for each $\omega \subset [m]$, the full subcomplex $K_\omega$ is given by $\{\sigma \in K \mid \sigma \subset \omega\}$. Consider the augmented chain complex $(\bar{C}_*(K_\omega), \partial)$ and its dual $(\bar{C}^*(K_\omega), d)$. Each simplex $\sigma \in K_\omega$ can be treated as a subset of $[m]$ equipped with the natural ordering (thus $|\sigma|$ is positively oriented if the indices contained in $\sigma$ are written in the increasing order), then the simplicial boundary operator satisfies

$$\partial(\sigma) = \sum_{i \in \sigma}(-1)^{(i, \sigma)}\sigma \setminus \{i\}, \tag{19}$$
together with $\partial(\{i\}) = \emptyset$ for all $\{i\} \in K_\omega$, due to the augmentation. After dualizing, for a dual simplex $\sigma^* \in \tilde{C}^*(K_\omega)$, we have $d(\emptyset) = \sum_{i \in \omega} \{i\}$, and otherwise

$$d(\sigma^*) = \sum_{(\sigma \cup \{i\}) \in K_{i \in \omega \setminus \sigma}} (-1)^{(i, \sigma)} (\sigma \cup \{i\})^*.$$  

(20)

A comparison of (17) and (19) (resp. (18) and (20)) yields the following:

**Proposition 3.3.** The mapping

$$\mu: \bigoplus_{\omega \subset [m]} \tilde{C}_s(K_\omega) \longrightarrow C_s(A)$$

given by sending each $\sigma_\omega \in \tilde{C}_s(K_\omega)$ to the word $u_{\sigma \in \omega \setminus \sigma}$ yields a chain isomorphism that shifts the degrees up by one. Analogously, we have the degree-shifting cochain isomorphism

$$\eta: \bigoplus_{\omega \subset [m]} \tilde{C}^*(K_\omega) \xrightarrow{\cong} C^*(A)$$

sending each $\sigma^*_\omega \in \tilde{C}^*(K_\omega)$ to $u^{\sigma \in \omega \setminus \sigma}$. Combined with Theorem 3.1, we have isomorphisms

$$H_p(C^*(A), \partial) \cong \bigoplus_{\omega \subset [m]} \tilde{H}_{p-1}(K_\omega) \quad \text{and} \quad H^p(C^*(A), d) \cong \bigoplus_{\omega \subset [m]} \tilde{H}^{p-1}(K_\omega)$$

in each dimension $p \geq 0$.

Recall that, by definition, $\tilde{H}_{-1}(K_\omega)$ is non-trivial only when $\omega = \emptyset$: $\tilde{H}_{p-1}(K_{\emptyset})$ vanishes if $p > 0$, and is infinite cyclic if $p = 0$. Therefore, we see that $\tilde{H}_{-1}(K_{\emptyset})$ corresponds to $H_0(A)$; the representative of $\tilde{H}_{-1}(K_{\emptyset})$ is the “empty set” of $K_{\emptyset}$, which is sent by $\mu$ to the void word associated to $t_{[m]}$, i.e., the point in $I^m$ with constant coordinates $-1$.

4. Applications of Whitney’s formulae

In order to apply Whitney’s formulae (12) and (13) to the cup and cap products in a real moment-angle complex, by Theorem 3.1, it remains to understand the products in the simplicial complex $I = [-1, 1]$.

Recall that the *simplicial cup and cap products* in a simplicial complex $|K|$, i.e., $\cup: C^*(K) \otimes C^*(K) \rightarrow C^*(K)$ and $\cap: C^*(K) \otimes C_*(K) \rightarrow C_*(K)$, are given as follows. Choose a partial ordering on the vertex of $K$ which induces a
total ordering on each simplex, and let \([v_{i_0}, v_{i_1}, \ldots, v_{i_p}] \in C_*(K)\) denote such a \(p\)-simplex, with \(v_0 < v_1 < \ldots < v_p\) in the given ordering. Then we have
\[
(22) \quad (c^p \circ \partial^q)([v_{i_0}, \ldots, v_{i_p+q}]) = c^p([v_{i_0}, \ldots, v_{i_p}])c^q([v_{i_p}, \ldots, v_{i_p+q}])
\]
and
\[
(23) \quad c^p \circ [i_0, \ldots, i_r] = c^p([i_{r-p}, \ldots, i_r])[i_0, \ldots, i_{r-p}] \quad (c^p([i_{r-p}, \ldots, i_r]) \in \mathbb{Z}),
\]
for \(c^p \in C^p(K)\) and \(c^q \in C^q(K)\).

For instance, let \((C_*(I), \partial)\) and \((C^*(I), d)\) be the simplicial (co)chain complexes given in Subsection 3.2, endowed with bases \(\{u, \varepsilon, t\}\) and \(\{u^*, \delta^*, t^*\}\), respectively (see (15), (16)). It can be easily checked that the following formulae hold:
\[
t^* \cdot t^* = t^*, \quad t^* \cdot u^* = 0, \quad u^* \cdot t^* = u^*, \quad u^* \cdot u^* = 0,
\]
\[
\delta^* \cdot u^* = u^* \cdot \delta^* = u^*, \quad \delta^* \cdot t^* = t^* \cdot \delta^* = t^*, \quad \delta^* \cdot \delta^* = \delta^*;
\]
\[
\delta^* \cdot t = t, \quad u^* \cdot u = t, \quad u^* \cdot \varepsilon = 0, \quad u^* \cdot t = 0,
\]
\[
t^* \cdot \varepsilon = \varepsilon + t, \quad t^* \cdot u = u, \quad t^* \cdot t = 0, \quad \delta^* \cdot \varepsilon = \varepsilon, \quad \delta^* \cdot u = u.
\]

For \(\sigma \subset [m]\), let \(v(\sigma) = (v_i)_{i=1}^m \in \mathbb{Z}^m\) be the vector with \(v_i = 1\) if \(i \in \sigma\), and \(v_i = 0\) otherwise. By (12), (13) and the products presented above, choosing two cochains \(u^\sigma t^\tau\) and \(u^\sigma' t'^\tau'\) from \(C^*((D^1, S^0)^K)\) (see Definition 3.2), we have
\[
(24) \quad u^\sigma t^\tau \cdot u^\sigma' t'^\tau' = (-1)^{(v(\sigma), v(\sigma'))} u^\sigma \cup u^\sigma' t^\tau \cup (t'^\tau \setminus \sigma)
\]
if \(\sigma' \cap (\sigma \cup \tau) = \emptyset\) and \((\sigma \cup \sigma') \subset K\), otherwise the product vanishes. Additionally, choosing \(u_{\sigma'' \varepsilon \tau''} \in C_*((D^1, S^0)^K)\), then
\[
(25) \quad u^\sigma t^\tau \cdot u_{\sigma'' \varepsilon \tau''} = (-1)^{(v(\sigma'' \setminus \tau), v(\sigma)) \sum_{\gamma \subset (\tau \setminus \sigma'')} u_{\sigma'' \setminus \gamma \varepsilon \tau''} \gamma},
\]
provided that \(\sigma \subset \sigma''\) and \(\tau \subset (\sigma'' \cup \tau'')\); otherwise \(u^\sigma t^\tau \cdot u_{\sigma'' \varepsilon \tau''}\) vanishes. Suppose that \(\text{card}(\sigma) = p\) and \(\text{card}(\sigma'') = r\); it turns out that (see (50))
\[
(26) \quad \partial(u^\sigma t^\tau \cdot u_{\sigma'' \varepsilon \tau''}) = (-1)^{r-p} d(u^\sigma t^\tau) \cdot u_{\sigma'' \varepsilon \tau''} + u^\sigma t^\tau \cdot \partial(u_{\sigma'' \varepsilon \tau''}).
\]

It is convenient to consider \((C^*((D^1, S^0)^K), \cdot)\) as a differential graded algebra with \(2m\) generators, such that \(\delta^{[m]}\) is the unique identity.

**Example 4.1.** Let \(K\) be the pentagon with vertex set \([5]\), whose maximal simplices are
\[
\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}.
\]
Using mod 5 integers, it can be checked that full subcomplexes of \(K\) with non-vanishing reduced (co)homology are: \(K_{i,i+2}\) with \(i = 1, 2, 3\), \(K_{i,i+3}\) with
It can be checked directly that \( \Gamma = \{ \sum \alpha_i \in K \} \) is a simplicial complex with \( |\sigma| = 2 \), \( \sigma \in [7] \), such that \( \sum \alpha_i \in K \) contains three consecutive points of the form \( \{i, i+1, i+2\} \), \( i = 1, 2, \ldots, 7 \) mod 7 (see Figure 1). Therefore, \((D^1, S^0)^K\) is simply connected (see [Dav08, Chapter 1, p. 12]). Using (21), we can write down the orientation class of \((D^1, S^0)^K\) through \( H_3(K) \): after listing all 14 3-faces of \( |K| \) and assign to each of them a coefficient \( \pm 1 \) to make a cycle, we see that (see Proposition 3.3)

\[
\Gamma = \{ u_{1,2,4,5}^3 \epsilon_{3,6,7} - u_{1,2,4,6}^3 \epsilon_{3,5,7} + u_{1,2,3,4,6} \epsilon_{2,5,7} - u_{1,3,4,7}^3 \epsilon_{2,5,6} - u_{1,3,5,6}^2 \epsilon_{2,4,7} + u_{1,3,5,7}^2 \epsilon_{2,4,6} - u_{1,4,5,7} \epsilon_{2,3,6} + u_{2,3,5,6} \epsilon_{1,4,7} - u_{2,3,5,7} \epsilon_{1,4,6} + u_{2,3,6,7} \epsilon_{1,4,5} + u_{2,4,5,7} \epsilon_{1,3,6} - u_{2,4,6,7} \epsilon_{1,3,5} + u_{3,4,6,7}^2 \epsilon_{1,2,3} \}.
\]

\( i = 1, 2, K_{i+2,i+3} \) with \( i = 1, 2, \ldots, 5 \), together with \( K_0 \) and \( K_5 \). Therefore by (21), \( H^1((D^1, S^0)^K) \) is torsion-free with ten generators

\[
\alpha_1 = [u^1 t^3], \alpha_2 = [u^1 t^4], \alpha_3 = [u^2 t^4], \alpha_4 = [u^2 t^5], \alpha_5 = [u^3 t^5],
\]

and

\[
\beta_i = [u^i t^{i+2}(1 - t^{i+3})] = [u^i t^{i+2} - u^i t^{i+2,j} + j] \quad i = 1, 2, \ldots, 5 \mod 5;
\]

we see that \( H^2((D^1, S^0)^K) \) is generated by \( \gamma = [u^j t^{j+1} t^{j+2, j+3} + j] \), with \( j \) an arbitrary mod 5 integer. By Theorem 2.3, \((D^1, S^0)^K\) is an orientable surface of genus 5. Using (24) we can give a presentation of the cup products. Since \( H^1(K_o) \) is non-trivial only when \( \omega = \{1, 2, 3, 4, 5\} \), a direct calculation shows that for all mod 5 integers \( i, j \), \( \alpha_i \sim \alpha_j = \beta_i \sim \beta_j = 0 \), then

\[
\gamma = -\alpha_1 \sim \beta_2 = \alpha_2 \sim \beta_5 = -\alpha_3 \sim \beta_3 = \alpha_4 \sim \beta_1 = -\alpha_5 \sim \beta_4
\]

presents all non-trivial cup products.

**Example 4.2** (cf. [LdM89], [BLV13], [GL14]). Let \( \lambda = (\lambda_i)_{i=1}^7 \) be the 7-tuple with \( \lambda_i \in \mathbb{R}^2 \), consisting of the real and imaginary parts of the solutions of the equation \( z^7 = 1 \). Consider the variety \( Z(\Lambda) \subset \mathbb{R}^7 \) given by the intersections of quadrics

\[
\begin{cases}
\sum_{i=1}^7 \lambda_i x_i^2 = 0, \\
\sum_{i=1}^7 x_i^2 = 1.
\end{cases}
\]

It can be checked directly that \( Z(\Lambda) \) is a smooth 4–manifold.\(^9\)

Let \( K \) be a simplicial complex with vertex set \( [7] \), such that \( \sigma \subset [7] \) belongs to \( K \) if and only if the origin of \( \mathbb{R}^2 \) is in the relative interior of the convex hull \( \text{conv}(\lambda_i)_{i \in [7]} \setminus \sigma \). It can be shown that \( |K| \) bounds a convex polytope, and there is a piecewise differentiable homeomorphism \((D^1, S^0)^K \rightarrow Z(\Lambda)\).\(^10\)

Observe that any subset of \( [7] \) with cardinality 2 is a simplex of \( K \), and any subset of \( [8] \) not in \( K \) must contain three consecutive points of the form \( \{i, i+1, i+2\} \), \( i = \{1, 2, \ldots, 7\} \mod 7 \) (see Figure 1). Therefore, \((D^1, S^0)^K \) is simply connected (see [Dav08, Chapter 1, p. 12]). Using (21), we can write down the orientation class of \((D^1, S^0)^K \) through \( H_3(K) \): after listing all 14 3-faces of \( |K| \) and assign to each of them a coefficient \( \pm 1 \) to make a cycle, we see that (see Proposition 3.3)

\[
\Gamma = [u_{1,2,4,5}^3 \epsilon_{3,6,7} - u_{1,2,4,6}^3 \epsilon_{3,5,7} + u_{1,2,5,6} \epsilon_{3,4,7} + u_{1,3,4,5} \epsilon_{2,5,7} - u_{1,3,4,7} \epsilon_{2,5,6} - u_{1,3,5,6} \epsilon_{2,4,7} + u_{1,3,5,7} \epsilon_{2,4,6} - u_{1,4,5,7} \epsilon_{2,3,6} + u_{2,3,5,6} \epsilon_{1,4,7} - u_{2,3,5,7} \epsilon_{1,4,6} + u_{2,3,6,7} \epsilon_{1,4,5} + u_{2,4,5,7} \epsilon_{1,3,6} - u_{2,4,6,7} \epsilon_{1,3,5} + u_{3,4,6,7} \epsilon_{1,2,3}].
\]

\(^9\)See for example, [BM06, Lemma 0.3, p. 58].

\(^10\)See for example, [BM06, Lemma 0.12, p. 64] for the first statement, and [Cai14, Lemma 6.3] for the second.
By Alexander duality, we have

\[ H_{3-i} \cup K_\omega \cong H_i (K[7] \backslash \omega). \]

It follows that \( H^{*}(K_\omega) \) is non-trivial only when \( \text{card}(\omega) = 0, 3, 4, 7 \). In particular, \( H^1(K_\omega) \) is non-trivial if and only if \( \omega \) consists of three or four consecutive mod 7 integers (see Figure 2).

As a conclusion, \( H^2((D^1, S^0)^K) \) is torsion-free with 14 generators

\[ \alpha_i = [u^{i,i+1}t^{i+2}] \quad \text{and} \quad \beta_i = [u^{i+1,i+2}t^{i,i+3}], \]

\( i = 1, 2, \ldots, 7 \) mod 7, according to \( \text{card}(\omega) = 3 \) and \( \text{card}(\omega) = 4 \), respectively. It can be checked that \( \alpha_i \sim \alpha_j = 0 \), and \( \alpha_i \sim \beta_j \) is non-trivial if and only if \( j = i + 3 \), which generates \( H^4((D^1, S^0)^K) \). For instance, \( u^{1,2}t^3 \sim u^{5,6}t^{4,7} = u^{1,2,5,6}t^{4,7} \). Notice that \( \beta_i \sim \beta_{i+4} = \pm u^{i+1,i+2,i+5,i+6}t^{i+3,i+4,i+7} \), which is non-trivial. If we change each \( \beta_i \) into

\[ \beta_i' = [u^{i+1,i+2}t^i(1-t^{i+3})] = [u^{i+1,i+2}t^i - u^{i+1,i+2}t^{i,i+3}], \]

then a straightforward calculation shows that each \( \alpha_i \) has a unique pairing \( \beta_{i+3} \) to make a non-trivial cup product, and \( \beta_i' \sim \beta_j' \) vanish for all mod 7 integers \( i, j \). As a conclusion, we see that \( (D^1, S^0)^K \) and \( \sharp S^2 \times S^2 \) have isomorphic cohomology rings, hence the two 4–manifolds are homeomorphic, by the classification theorem of Freedman [Fre82] (an explicit diffeomorphism between them is given by Gutiérrez and López de Medrano [GL14]).
By Poincaré duality, the intersection of submanifolds can be understood through cup products. For instance, by (25), we have

\[
[u_{1,2} t^3 - \Gamma] = [u_{4,5} \varepsilon_{6,7} - u_{4,6} \varepsilon_{5,7} + u_{5,6} \varepsilon_{4,7} + u_{4,5} \varepsilon_{3,6,7} - u_{4,6} \varepsilon_{3,5,7} + u_{5,6} \varepsilon_{3,4,7}]
\]

because \( K_{3,4,5,6,7} \) is acyclic. Here the geometric meaning of the class on the right-hand side of (29) is as follows: consider the sphere

\[
S = \{(x_i)_{i=1}^7 \in (D^1, S^0)^K \mid x_1 = x_2 = x_3 = -1, x_7 = 1\}
\]

with suitable orientation, and let \( s_7: \mathbb{R}^7 \to \mathbb{R}^7 \) be the reflection changing the sign of the last coordinate, then \([u_{4,5} \varepsilon_{6,7} - u_{4,6} \varepsilon_{5,7} + u_{5,6} \varepsilon_{4,7}]\) corresponds to the class \([S] + s_7[S]\) (see Definition 3.2). Therefore, if we use the representative \([u_{1,2} t^3(1 - t^7)]\) in (28) instead of \([u_{1,2} t^3]\), we will get the class \(s_7[S]\), which is represented by a submanifold whose intersection with the sphere

\[
S' = \{(x_i)_{i=1}^7 \in (D^1, S^0)^K \mid x_4 = x_5 = x_6 = x_7 = -1\}
\]

is the point with constant coordinates \(-1\). In the same way, by expanding \([u^5 t^4(1 - t^7) - \Gamma]\) we see that it coincides with suitable orientation class \([S']\).

From the Poincaré duality we can read the plumbing of spheres \(s_7(S)\) and \(S'\), which can also be checked directly.

4.1. Alexander duality revisited. Let \(\widetilde{M}_s(K)\) and \(\widetilde{M}^s(K)\) be the differential graded \(\mathbb{Z}\)-modules

\[
\bigoplus_{\omega \subset [m]} \widetilde{C}_s(K_{\omega}) \quad \text{and} \quad \bigoplus_{\omega \subset [m]} \widetilde{C}^s(K_{\omega}),
\]
Suppose that unique (left) identity in the two products above.

\[
\text{Proof. Let } \sigma, \sigma' \text{ for each } m \in \mathbb{Z}. \text{Assume that Theorem 4.3.}
\]

Following theorem:

Example 4.2 coincides with the Alexander duality (27). Actually we have the projection, \( \omega \)

\[
\text{Using the language of the presentation class of } \tilde{\omega} \text{ given in Proposition 3.3, we can endow } \tilde{M}_s(K) \text{ and } \tilde{M}^s(K) \text{ with products } \sim' \text{ and } \sim'', \text{ such that diagrams}
\]

\[
(30) \quad C^*(A) \otimes C^*(A) \xrightarrow{\eta \otimes \eta} C^*(A) \quad C^*(A) \otimes C_s(A) \xrightarrow{\eta} C_s(A) \quad C^*(A) \otimes C_s(A) \xrightarrow{\eta \otimes \mu} C_s(A) \quad C^*(A) \otimes C_s(A) \xrightarrow{\eta \otimes \mu} C_s(A)
\]

\[
\tilde{M}^*(K) \otimes \tilde{M}^s(K) \xrightarrow{\sim'} \tilde{M}^s(K) \quad \tilde{M}^*(K) \otimes \tilde{M}_s(K) \xrightarrow{\sim''} \tilde{M}_s(K)
\]

commute, where \( A = (D^1, S^0)^K \).

For \( \sigma \in K \) contained in \( \omega \), let \( \sigma_\omega \in C_*(K_\omega) \) (resp. \( \sigma^*_\omega \in C^*(K_\omega) \)) denote the associated chain (resp. cochain) of the specified subcomplex \( K_\omega \). A straightforward calculation shows that (24) and (25) can be interpreted as

\[
(31) \quad \sigma^*_\omega \sim' \sigma''_\omega = \begin{cases} (-1)^{(v(\sigma),v(\sigma'))} (\sigma \cup \sigma')^*_\omega \cup \omega' & \text{if } \sigma' \cap \omega = \emptyset \text{ and } (\sigma \cup \sigma') \in K, \\ 0 & \text{otherwise,} \end{cases}
\]

and

\[
(32) \quad \sigma^*_\omega \sim'' \sigma''_\omega = \begin{cases} (-1)^{(v(\sigma'^\gamma(\sigma),v(\sigma)))} \sum_{\gamma \omega \setminus \sigma''} (\sigma'' \setminus \sigma)_{(\omega \setminus \sigma) \gamma} & \text{if } \sigma \subset \sigma'' \text{ and } \omega \subset \omega'', \\ 0 & \text{otherwise,} \end{cases}
\]

with \( \sigma, \sigma', \sigma'' \in K \) and \( \omega, \omega', \omega'' \subset [m] \). It is obvious that \( \emptyset^*_\emptyset \in C^*(K_\emptyset) \) is the unique (left) identity in the two products above.

Suppose that \( |K| \) has the orientation class \( \Gamma_K \). Let \( \pi_\omega: \tilde{M}_s(K) \to \tilde{C}_s(K_\omega) \) be the projection, \( \omega \subset [m] \), and let \( AD_{\omega}: \tilde{H}^*(K_\omega) \to \tilde{H}_s(K_{[m] \setminus \omega}) \) denote the map sending each element \([c^*_\omega] \) to \( \pi_{[m] \setminus \omega}([c^*_\omega \sim' \Gamma_K]) \), which is well-defined by (26).

Using the language of the \( \sim' \)-products, we see that the calculation (28) in Example 4.2 coincides with the Alexander duality (27). Actually we have the following theorem:

**Theorem 4.3.** Assume that \( |K| \) is a generalized homology \((n-1)\)-sphere with vertex set \([m]\). Then \( AD_{\omega} \) induces isomorphisms

\[
\tilde{H}^{p-1}(K_\omega) \cong \tilde{H}_{n-1-p}(K_{[m] \setminus \omega}),
\]

for each \( 0 \leq p \leq n-1 \).

**Proof.** Let \( \omega \subset [m] \) be fixed. By assumption, let \( \Gamma_K \in C_{n-1}(K) \) be the orientation class of \( |K|; \Gamma_K = [c_{n-1}] \), where \( c_{n-1} \in C_{n-1}(K) \) is a sum of all \((n-1)\)-simplices of \( |K| \), with coefficients \( \pm 1 \) for each of them. By Theorem
2.3. \((D^1, S^0)^K\) is a compact homology \(n\)-manifold, therefore by Poincaré duality (see [Mun84, Section 67, p. 394–397]), we have the commutative diagram

\[
\begin{array}{ccc}
H^p((D^1, S^0)^K) & \xrightarrow{\mu(T_K)} & H_{n-p}((D^1, S^0)^K) \\
\eta \uparrow \cong & & \mu \uparrow \cong \\
\tilde{H}^{p-1}(\tilde{M}^*(K)) & \xrightarrow{-\mu(T_K)} & \tilde{H}_{n-1-p}(\tilde{M}^*(K)) \\
\uparrow & & \downarrow \pi_{[m]o} \\
\tilde{H}^{p-1}(K_o) & \xrightarrow{AD_o} & \tilde{H}_{n-1-p}(K_{[m]\omega}).
\end{array}
\]

Now suppose \(\sigma' \in K\) is an \((n-1)\)-simplex, and \(\sigma_{\omega'}^* \in C^{p-1}(K_{\omega'})\) is the dual simplex of \(\sigma \in K_{\omega'}\) (the subscript specifies the subcomplex), such that \(\sigma \subset \sigma'\). It follows from (32) that

\[
(33) \quad c_{\omega'}^{p-1} \sim' c_{n-1} \in \bigoplus_{\gamma' \subset \omega'} \tilde{C}_{n-1-p}(K_{[m]\gamma'}).
\]

from which we see that for any \(c_{\omega'}^{p-1} \in \tilde{C}^{p-1}(K_{\omega'})\), we have

\[
(34) \quad c_{\omega'}^{p-1} \sim' c_{n-1} \in \bigoplus_{\gamma' \subset \omega'} \tilde{C}_{n-1-p}(K_{[m]\gamma'}).
\]

Now we prove that \(AD_o\) is surjective. Suppose \([c_{[m]\omega}] \in \tilde{H}_{n-1-p}(K_{[m]\omega})\) is an element with representative \(c_{[m]\omega} \in \tilde{C}_{n-1-p}(K_{[m]\omega})\). By Poincaré duality, there exists \(c_{\omega'}^{p-1} \in \tilde{H}^{p-1}(K_o)\) such that \(\pi_{[m]o}[c_{\omega'}^{p-1}] \sim' c_{n-1} = [c_{[m]\omega}]\). First by (34), denote \(c_{\omega'}^{p-1} = \sum_{\omega \subset \omega'} \pi_{\omega'}(c_{\omega'}^{p-1})\), then we have

\[
(35) \quad \pi_{[m]o}(c_{\omega'}^{p-1} \sim' c_{n-1}) = \pi_{[m]o}(c_{\omega'}^{p-1} \sim' c_{n-1}).
\]

For each \(\omega'\) containing \(\omega\) as a subset, let \(p_{\omega',\omega}: \tilde{C}^*(K_{\omega'}) \to \tilde{C}^*(K_o)\) be the map sending each \(\sigma_{\omega'}^*\) to \(\sigma_{\omega}^*\) if \(\sigma \subset \omega\) (i.e., changing the underlying subcomplex for each dual simplex), or to zero otherwise. Then we define \(c_{\omega}^{p-1} = \sum_{\omega \subset \omega'} p_{\omega',\omega} \pi_{\omega'}(c_{\omega'}^{p-1})\); clearly \(c_{\omega}^{p-1} \in \tilde{C}^{p-1}(K_o)\). Now by (33) we have

\[
(36) \quad \pi_{[m]o}(c_{\omega}^{p-1} \sim' c_{n-1}) = \pi_{[m]o}(c_{\omega}^{p-1} \sim' c_{n-1}).
\]

From (35) and (36) we see that \(AD_o\) is surjective.

It remains to prove the injectivity. Suppose a cocycle \(c_{\omega}^{p-1} \in \tilde{C}^{p-1}(K_o)\) has the property that \(\pi_{[m]o}(c_{\omega}^{p-1} \sim' c_{n-1})\) is a boundary in \(\tilde{C}_*(K_{[m]\omega})\). It suffices to show that \(c_{\omega}^{p-1}\) is a coboundary. We claim that associated to \(c_{\omega}^{p-1}\) there exists a cocycle \(c_{\omega}^{p-1} = \sum_{\omega' \subset \omega} c_{\omega'}^{p-1}, c_{\omega'}^{p-1} \in \tilde{C}^{p-1}(K_{\omega'})\), such that \(c_{\omega}^{p-1} - c_{\omega}^{p-1} \sim' c_{n-1}\)
is a boundary in $\widetilde{M}_u(K)$; that is to say, by (34), $\pi_{[m]\setminus\gamma}(d(c_{\omega}^{p-1} - c_{<\omega}^{p-1}) \prec c_{n-1})$ is a boundary in $C_*(K_{[m]\setminus\gamma})$, for all $\gamma \subset \omega$. Then by Poincaré duality, $c_{\omega}^{p-1} - c_{<\omega}^{p-1}$ is a coboundary, so is $\hat{c}_{\omega}^{p-1}$.

To prove the claim, note that for each $\gamma \subset \omega$ with $\text{card}(\gamma) = \text{card}(\omega) - 1$, by the surjectivity of $AD_\omega$, we can choose $c_\gamma^{p-1} \in \hat{C}^{p-1}(K,\gamma)$ so that $\pi_{[m]\setminus\gamma}(\hat{c}_{\omega}^{p-1} - c_\gamma^{p-1})$ is a coboundary. After we repeat this procedure for all $\gamma \subset \omega$, in the order that $\text{card}(\gamma)$ decreases, the claim follows. □

5. Cohomology of certain polyhedral products

Suppose $J = (j_i)_{i=1}^m$ is an $m$-tuple of positive integers. In this section we consider the relation between $H^*((D^1, S^0)^K)$ and $H^*((D^1, S^0)^{K(J)})$, where $K(J)$ is given in Definition 2.6.

Let $(\mathbb{Z}(\vec{v}^i, \vec{w}^j)_{i=1}^m, d)$ be the free algebra generated by $2m$ generators $\vec{v}^i$ with $\deg(\vec{v}^i) = j_i$ and $\vec{w}^j$ with $\deg(\vec{w}^j) = j_i - 1$, respectively, $i = 1, 2, \ldots, m$; for two homogeneous elements $x, y \in \mathbb{Z}(\vec{v}^i, \vec{w}^j)_{i=1}^m$, the differential $d$ satisfies

\begin{equation}
   d(xy) = (dx)y + (-1)^{\deg(x)}x(dy),
\end{equation}

with $d\vec{w}^i = \vec{v}^i$ and $d\vec{v}^i = 0$. Let $(R^*(J), d)$ be the quotient of $(\mathbb{Z}(\vec{v}^i, \vec{w}^j)_{i=1}^m, d)$, subject to relations

\begin{equation}
   x^i x^j = (-1)^{\deg(x^i) \deg(x^j)} x^j x^i, \quad (x^i)^2 = \begin{cases} x^i & \text{if } \deg(x^i) = 0, \\ 0 & \text{otherwise}, \end{cases}
\end{equation}

where $x^i$ (resp. $x^j$) is either $\vec{w}^i$ or $\vec{v}^i$ (resp. $\vec{w}^j$ or $\vec{v}^j$), for distinct $i, j = 1, 2, \ldots, m$, together with

\begin{equation}
   \vec{w}^i \vec{v}^i = 0, \quad \vec{w}^i \vec{w}^j = \begin{cases} \vec{v}^j & \text{if } \deg(\vec{w}^i) = 0, \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, 2, \ldots, m.
\end{equation}

For a simplicial complex $K$ with vertex set $[m]$, the corresponding Stanley-Reisner ideal $I_K(J) \subset R^*(J)$ is generated by all square-free monomials of the form $\vec{v}^\tau = \prod_{i \in \sigma} \vec{v}^i$, where $\tau$ is not a simplex of $K$.

Let $(R^*_K(J), d)$ be the quotient $R^*(J)/I_K(J)$. It is well-defined since $I_K(J)$ is closed under $d$. Clearly by relations (38) and (39), each monomial in $R^*_K(J)$ can be uniquely written in the square-free form $\vec{v}^\sigma \vec{u}^\tau = \prod_{i=1}^m x^i$, where $x^i = \vec{v}^i$ if $i \in \sigma$, $x^i = \vec{w}^i$ if $i \in \tau$ and $x^i = 1$ otherwise, $\sigma, \tau$ disjoint subset of $[m]$, $\sigma \in K$.

First let $J = (1)$ be constant with integers 1, and denote $R^*_K(1)$ simply by $R^*_K$. A comparison of $R^*_K$ with (24) shows that, if we identify $\vec{v}^i$ with $u^i$, $\vec{u}^i$ with $t^i$...
and the identity $1$ with the void word, then we have the cochain isomorphism $(R^*_K, \cdot) \cong (C^*((D^1, S^0)^K), -)$. Together with Theorem 3.1, we see that
\[
H^*(R^*_K(J), -) \cong H^*((D^1, S^0)^{(J)}, -).
\]

Notice that $R^*_K(J)$ has $2d(J)$ generators, $d(J) = \sum_{i=1}^{m} j_i$, which we shall still denote by $u^i$ of degree 1 and $t^i$ of degree 0, $i = 1, 2, \ldots, d(J)$. Actually, the algebra $H^*(R^*_K(J))$ can be refined as follows:

**Theorem 5.1.** For $k = 1, 2, \ldots, m$, if we set $\bar{k} = k + \sum_{r<k} (j_r - 1)$ and
\[
B_k = \{k + 1 + \sum_{r<k} (j_r - 1), k + 2 + \sum_{r<k} (j_r - 1), \ldots, k + j_k - 1 + \sum_{r<k} (j_r - 1)\},
\]
which is empty if $j_k = 1$, then the embedding $R^*_K(J) \to R^*_K(J)$ that sends $\bar{\sigma}$ and $\bar{\omega}$ to $u^\bar{k} u^\bar{B}_k$ and $t^\bar{k} u^\bar{B}_k$, respectively, induces an isomorphism
\[
H^*(R^*_K(J)) \cong H^*(R^*_K(J))
\]
of graded algebras. Moreover, by setting
\[
(\sigma, \omega) = \sum_{k \in \omega} (j_k - 1) \text{card}([i \in \sigma \mid i > k]) + \sum_{k \in \omega} (j_k - 1) \sum_{r \in \omega, r > k} (j_r - 1)
\]
for $\sigma \subset \omega \subset [m]$, the mapping (see Subsection 4.1 for definition)
\[
\eta_J: \widetilde{M}^*(K) \longrightarrow R^*_K(J)
\]
\[
\sigma_\omega \longmapsto (-1)^{(\sigma, \omega)} \bar{v}^\sigma \bar{u}^\omega \sigma
\]
yields a cochain map $(\sigma_\omega \in \widetilde{C}^*(K_\omega) = \text{a dual simplex})$ which induces an isomorphism $H^*(\widetilde{M}^*(K)) \cong H^*(R^*_K(J))$ of ungraded $\mathbb{Z}$-modules.

**Remark 5.2.** Together with Lemma 2.7, we see that the cohomology ring of the polyhedral product $(D, S)^K$ with $(D, S) = (D^j, S^{j-1})_{j=1}^m$ is isomorphic to $(H^*(R^*_K(J)), \cdot)$. When $j_i$ is even, $i = 1, 2, \ldots, m$, this can be deduced from [BBCG10b, Proposition 6.2]. In particular, when $J = \{2\}$ is constant with integers $2$, $R^*_K(2)$ is the well-known algebra given in [BBP04] and [Pan08] for the cohomology of the moment-angle complex $(D^2, S^1)^K$.

In order to prove Theorem 5.1, we consider the simplicial wedge $K(v_i)$ on the $i$-th vertex (see (9)). Denote by $\varpi_i: R^*_K \to R^*_K(v_i)$ the additive homomorphism such that for each monomial $u^\sigma t^\tau \in R^*_K$,
\[
\varpi_i(u^\sigma t^\tau) = \begin{cases} u^{\chi_i(\sigma)} t^{\chi_i(\tau)} \in R^*_K(v_i) & \text{if } i \notin \omega, \\ u^{\chi_i(\sigma)} t^{\chi_i(\tau)} u^i + 1 \in R^*_K(v_i) & \text{otherwise}, \end{cases}
\]
where $\chi_i$ is a label-shifting map such that each $\{i_k\}_{k=0}^l \subset [m]$ is sent to $\{i'_k\}_{k=0}^l \subset [m + 1]$, in which $i'_k = i_k$ if $i_k \leq i$ and $i'_k = i_k + 1$ otherwise (thus the label $i + 1$ is skipped in the image); $\chi_i(\emptyset) = \emptyset$. It can be checked easily that $\varpi_i$ is well-defined and preserves the differential $d$ on both sides.

In what follows, let $R^*_K|\omega$ be the subalgebra generated by $u^\sigma t^{\omega\setminus\sigma}$, $\sigma \subset \omega$. Clearly as a $\mathbb{Z}$-module, with $\omega$ running through subsets of $[m]$, $R^*_K$ is a direct sum with summands $R^*_K|\omega$.

**Lemma 5.3.** The mapping $\varpi_i$: $R^*_K \to R^*_K(v_i)$ induces an additive isomorphism on passage to cohomology. More precisely, $\varpi_i$ induces isomorphisms

\begin{equation}
H^p(R^*_K|\omega) \cong H^{p+1}(R^*_K(v_i)|\chi_i(\omega) \cup \{i+1\})
\end{equation}

if $i \in \omega$, and isomorphisms

\begin{equation}
H^p(R^*_K|\omega) \cong H^p(R^*_K(v_i)|\chi_i(\omega))
\end{equation}

if $i \not\in \omega$, for each $p \geq 0$.

**Proof.** First observe that if $i \not\in \omega$, then the full subcomplex $K(v_i)|\chi_i(\omega)$ and $K_\omega$ are simplicially isomorphic. Then (45) follows from Proposition 3.3. It remains to prove (44). Suppose $i \in \omega$, and consider the short exact sequence

\[ 0 \longrightarrow R^*_K|\omega \xrightarrow{\varpi_i} R^*_K|\chi_i(\omega) \cup \{i+1\} \longrightarrow Q^* \longrightarrow 0, \]

in which $Q^*$ is the graded quotient. Let $f$: $\tilde{C}^*(K(v_i)|\chi_i(\omega)) \to Q^*$ be the additive homomorphism sending each dual $p$-simplex $\sigma^*_\chi_i(\omega)$ to the element with representative $u^\sigma t^{\chi_i(\omega)\setminus\sigma}i^{i+1}$. We see that $f$ is a cochain isomorphism shifting the degrees up by one, since

\[ d(u^\sigma t^{i+1}) = d(u^\sigma t^i)i^{i+1} + (-1)^{p+1}u^\sigma t^i dt^{i+1}, \]

where the second summand vanishes in $Q^*$ (see (43)). It is easy to see that $K(v_i)|\chi_i(\omega)$ is the star of $v_i$ in $K(v_i)$, which is acyclic, so is $Q^*$. Then (44) follows. \qed

Now we specify a sequence of simplicial wedge constructions for $K(J)$: it can be checked that

\begin{equation}
K(J) = K(v_m)(v_{m+1}) \cdots (v_{m+jm-2})(v_{m-1})(v_m) \cdots (v_m+(jm-1)) \cdots \frac{(v_m-i_1+jm-2)(v_m-i_2)}{j_{m-i_1}} \frac{(v_m-i_2)}{j_{i_2-1}} \cdots \frac{(v_{m-i_1+jm-2})}{j_{m-i_1-1}} \cdots \frac{(v_{m-1+i_1+jm-2})}{j_{i_2-1}} \frac{(v_{m-1})}{j_{i_1-1}} \cdots \frac{(v_2+jj_2-2)}{j_{j_2-1}} \frac{(v_1)}{j_{j_1-1}} \cdots \frac{(v_1+jj_1-2)}{j_{j_1-1}}
\end{equation}
where the block marked by $j_i - 1$ is deleted if $j_i = 1$.

**Proof of Theorem 5.1.** With respect to (46), we define $m$ composite homomorphisms

$$\varpi_{j_k} = \varpi_{k+j_k-2} \varpi_{k+j_k-3} \cdots \varpi_{k+1} \varpi_k,$$

$k = 1, 2, \ldots, m$, and denote their composition by

$$\varpi_J = \varpi_{j_1} \varpi_{j_2} \cdots \varpi_{j_m}.$$

Suppose $\omega = \{i_1, i_2, \ldots, i_l\}$ is a subset of $[m]$, $i_1 < i_2 < \ldots < i_l$. Let $u^{\sigma t^{\omega \setminus \sigma}}$ ($\sigma \subset \omega$) be the monomial $\prod_{k=1}^l x^{i_k}$ with $x^{i_k} = u^{i_k}$ if $i_k \in \sigma$, and $x^{i_k} = t^{i_k}$ otherwise. By definition (43) and a straightforward calculation, we have

$$\varpi_J (u^{\sigma t^{\omega \setminus \sigma}}) = \varpi_{j_1} \varpi_{j_2} \cdots \varpi_{j_m} (x^{i_1} \cdots x^{i_l}) = x^{\tilde{i_1}} x^{\tilde{i_2}} \cdots x^{\tilde{i_l}} x^{\tilde{B}_1} x^{\tilde{B}_{i_1-1}} \cdots x^{\tilde{B}_{i_l}},$$

where $\tilde{k}$, $\tilde{B}_k$ are given in (40). Let $(\sigma, \omega)_J$ be the mod 2 integer such that

$$x^{\tilde{i_1}} x^{\tilde{i_2}} \cdots x^{\tilde{i_l}} x^{\tilde{B}_1} x^{\tilde{B}_{i_1-1}} \cdots x^{\tilde{B}_{i_l}} = (-1)^{(\sigma, \omega)_J} x^{\tilde{i_1}} x^{\tilde{i_2}} \cdots x^{\tilde{i_l}} x^{\tilde{B}_1} x^{\tilde{B}_{i_1}} \cdots x^{\tilde{B}_{i_l}}.$$

Clearly by (38), $(\sigma, \omega)_J$ coincides with (41). As a conclusion, by setting $\tilde{u}^k = u^k u^{\tilde{B}_k}$ and $\tilde{t}^k = t^k u^{\tilde{B}_k}$, $k = 1, 2, \ldots, m$, we see that $R^*_K(J)$ is embedded in $R^*_K$ as a subalgebra generated by the image of $\varpi_J$: $R^*_K \to R^*_K(J)$. Therefore, on passage to cohomology, the embedding above induces an isomorphism, by Lemma 5.3.

The second statement follows from the first one, and Proposition 3.3. Here we check it directly. Note that by (41), for $i \in \omega \setminus \sigma$, we have

$$(\sigma \cup \{i\}, \omega)_J - (\sigma, \omega)_J = \sum_{k \in \omega, k < i} (j_k - 1).$$
In particular, when $J$ is the constant tuple $(2)$, the sign $(\sigma, \omega)(J)$ was understood in Bosio and Meersseman [BM06, Theorem 10.1], using the intersection of submanifolds (see also [Pan08, Theorem 5.1]).

Example 5.4. Let $K$ be the complex dual to the truncated cube in Figure 3, such that each vertex is labeled by the number in the center of the associated 2-face. In [GL13, Theorem 3.1], it was proved that $H^*(R^*_{K})$ and $H^*(R^*_{K}(2))$ are not isomorphic as ungraded rings, even with $\mathbb{Z}_2$ coefficients. Following their approach, here we illustrate the difference of the two rings, by Theorem 5.1. First by [GL13], we have the homeomorphisms

$$(D^1, S^0)^K \cong (S^1 \times S^1 \times S^1) \not\cong (S^1 \times S^1) \not\cong (S^1 \times S^2),$$

and

$$(D^2, S^1)^K \cong \partial (M^9_{-1} \times D^2) \not\cong (S^3 \times S^7) \not\cong (S^4 \times S^6) \not\cong (S^5 \times S^5)$$

where $M^9_{-1}$ is the obtained from $S^3 \times S^3 \times S^3$ by subtracting an open ball.\footnote{\textit{(D^1, S^0)^K} and \textit{(D^2, S^1)^K} can be smoothed canonically to make the two homeomorphisms above into diffeomorphisms; see [GL13].}

It is easy to check that $\tilde{H}^*(K_{\omega})$ is non-trivial only when:

(I) $\omega = I_{1,2,3} \cup \{7\}$ where $I_{1,2,3}$ can be any non-empty subsets of $\{1,2,3\}$ (7 cases); $\omega = \{1,2,3,4,5,6\} \setminus I_{1,2,3}$ (7 cases);
Figure 3. The truncated cube

(II) \( \omega = \{1, 6, 7\} \setminus I_7 \), \( \omega = \{2, 4, 7\} \setminus I_7 \) and \( \omega = \{3, 5, 7\} \setminus I_7 \), where \( I_7 = \emptyset \) or \( I_7 = \{7\} \) (6 cases); \( \omega = \{1, 3, 5, 6, 7\} \setminus I_7 \), \( \omega = \{2, 3, 4, 5, 7\} \setminus I_7 \) and \( \omega = \{1, 2, 4, 6, 7\} \setminus I_7 \) (6 cases).

First we consider \( H^\ast(R^\ast K) \). With respect to (I), we have 7 pairs of representatives \( u_7 t_I \) of degree 1 and \( u_4 t_I \) of degree 2, respectively, with \( I \) running through non-empty subsets of \( \{1, 2, 3\} \), such that their product \( -u_4 t_I \) generates \( H^3(R^\ast K) \) \( \cong H^3(D^1, S^0)^K \). For (II), we choose representatives \( u_1 t_4 \) of degree 1, with their mutually two-fold products \( u_1 t_4 \) and \( u_3 t_5 \), respectively, together with the three-fold product \( u_1 t_4 t_5 t_6 \). Analogously, we choose representatives \( u_1 t_4 (1 - t_7) \), \( u_2 t_4 (1 - t_7) \) and \( u_3 t_5 (1 - t_7) \), together with their mutually two-fold and three-fold products. It can be checked that, in this way we have listed all non-trivial products in \( H^\ast(R^\ast K) \).

Next we turn to \( H^\ast(R^\ast K(2)) \). For (I), we can simply use the previous argument, replacing \( u^i t^i \) by \( \tilde{v}^i \) and \( \tilde{w}^i \), respectively. Notice that the degrees have changed accordingly. For (II), however, we have to modify the representatives for products, since the relations \( (\tilde{w}^i)^2 = \tilde{w}^i \) do not hold any longer. Therefore, any non-trivial three-fold product must come from a partition of \( \{7\} \) by three parts: two of them have cardinality 2 and the remainder has cardinality 3. For instance, if we have chosen \( \tilde{v}^1 \tilde{w}^6 \) and \( \tilde{v}^2 \tilde{w}^4 \) associated to \( \tilde{H}^0(K_{1,6}) \) and \( \tilde{H}^0(K_{2,4,7}) \) by \( \eta_2 \) (see (42)), respectively, then the remainder has to be \( \tilde{v}^3 \tilde{w}^5 \) associated to \( \tilde{H}^0(K_{3,5}) \).

6. Proof of Theorem 3.1

We will use the same notations as in the statement of the Theorem.
Let \( A^n \) be the \( n \)-th skeleton of \( A \) (we set \( A^{-1} = \emptyset \)), and let \((S_*(A), \partial_s)\) be the singular chain complex of \( A \) with differential \( \partial_s \).

**Proof of the additive isomorphism.** First we ignore those products. It suffices to show that, there is a chain map \( f_A: (C_*(A), \partial) \to (S_*(A), \partial_s) \) such that the image of \( C_n(A) \) generates \( H_n(A^n, A^{n-1}) \) in each dimension \( n \geq 0 \). We construct \( f_A \) by induction on dimension \( n \). This is clear when \( n = 0 \). Suppose \( n > 0 \) and \( f_A \) is well-defined for \( k < n \), with the desired property. Choosing a cell \( e_n \) of dimension \( n \), we see by hypothesis that the image of \( \partial(e_n) \) under \( f_A \) is carried by the boundary \( \text{Bd}(e_n) \), which is topologically an \((n-1)\)-sphere. Again by hypothesis, \( f_A(\partial(e_n)) \) generates the fundamental class of \( \text{Bd}(e_n) \). Therefore, by the long exact sequence of the pair \((e_n, \text{Bd}(e_n))\), we can choose from \( S_n(e_n) \) a representative of \( H_n(e_n, \text{Bd}(e_n)) \), say \( e_n \), such that \( \partial_s e_n = f_A(\partial(e_n)) \). Then we define \( f_A(e_n) = e_n \), and in this way we extend \( f_A \) over all \( n \)-cells. Then a standard argument of cellular homology shows that, \( f_A \) and its dual induce the isomorphisms \( H_*(C_*(A), \partial) \cong H_*(A) \) and \( H^*(C^*(A), d) \cong H^*(A) \), respectively. \( \square \)

On products, however, we have to work carefully at the (co)chain level. We shall proceed with singular (co)chains, and then compare it with simplicial ones.

Let \( \text{Top} \) be the category of topological spaces, and let \( \text{Top}^m \) be the category of \( m \)-fold Cartesian product spaces, with \( m \)-tuple of continuous maps as morphisms. Clearly \( \text{Top}^m \) is a subcategory of \( \text{Top} \). Denote by \( \text{C}(\text{Ab}) \) the category of chain complexes with chain maps as morphisms.

Let \( S_*: \text{Top} \to \text{C}(\text{Ab}) \) be the functor of singular chain complexes, assigning to a space \( X \) the singular chain complex \((S_*(X), \partial_s)\); let \( S_*^m: \text{Top}^m \to \text{C}(\text{Ab}) \) be the functor assigning to a product \( X = \prod_{i=1}^m X_i \) the tensor product \((\otimes_{i=1}^m S_*(X_i), \partial_{s'})\) (\( \partial_{s'} \) follows (11), with suitable chains replaced).

Recall that for an object \( X = \prod_{i=1}^m X_i \) from \( \text{Top}^m \), the associated Alexander-Whitney chain map \( T^m_X: (S_*(X), \partial_s) \to (S_*^m(X), \partial_{s'}) \) is defined by sending each singular \( p \)-simplex \( \sigma: \Delta^p \to X \) (the vertex set of \( \Delta^p \) is \( \{0, 1, \ldots, p\} \)) to the sum

\[
\sum_{0 = k_0 \leq k_1 \leq \ldots \leq k_m = p} \otimes_{i=1}^m \pi_i \sigma|_{[k_{i-1}, k_i]},
\]

where \( \pi_i \) is the projection onto the \( i \)-th component and \( \sigma|_{[k_{i-1}, k_i]} \) is the restriction of \( \sigma \) to the face spanned by \( \{k_{i-1}, k_{i-1} + 1, \ldots, k_i\} \). We see that \( T^m: S_*|_{\text{Top}^m} \to S_*^m \) is a natural transformation between the two functors. Moreover, by Eilenberg–Zilber Theorem [EZ53], \( T^m \) induces a natural chain-homotopy equivalence when the space is specified.
Let \((S^*(X), \delta_X)\) be the singular cochain complex of \(X\). For \(\mathbb{Z}\)-modules \(M_i\) with their duals \(M_i^* = \text{Hom}(M_i, \mathbb{Z})\), \(i = 1, 2, \ldots, m\), we denote by \(\theta_m : \bigotimes_{i=1}^m M_i^* \to \text{Hom}(\bigotimes_{i=1}^m M_i, \mathbb{Z})\) the \textit{evaluation map}, which is a homomorphism given by

\[
\theta_m(\otimes_{i=1}^m f_i)(\otimes_{i=1}^m c_i) = \prod_{i=1}^m f_i(c_i),
\]

where \(f_i \in M_i^*\) and \(c_i \in M_i\). Consider the sequence

\[
S_*(X) \xrightarrow{d_*} S_*(X \times X) \xrightarrow{\tau_{X \times X}} S_*^2(X),
\]

where \(d_*\) is induced by the diagonal map \(d : X \to X \times X\). We refer to the composition \(\tau_X = \tau_{X \times X} d_*\) as the \textit{Alexander-Whitney diagonal approximation}. By definition, we see that \(\tau : S_* \to S_*^2\) is a natural transformation between the two functors.

The cup product \(\sim : S^*(X) \otimes S^*(X) \to S^*(X)\) is defined as the composition \(\tau_X^\vee \theta_2\), where \(\tau_X^\vee\) is the dual of \(\tau_X\).

To formulate the cap product, we need another homomorphism

\[
h : M^* \otimes M \otimes M \xrightarrow{h} M
\]

\[
(m^*, m_1 \otimes m_2) \mapsto m^*(m_2)m_1,
\]

where \(M\) is a \(\mathbb{Z}\)-module with its dual \(M^*\). It can be checked by a straightforward calculation that for \(c^p \in S^p(X)\) and \(c'_r \in S^2_r(X \times X)\), we have

\[
\partial h(c^p, c'_r) = (-1)^{r-p} h(\delta_X c^p, c'_r) + h(c^p, \partial_X c'_r).
\]

The cap product \(\frown : S^*(X) \otimes S_*(X) \to S_*(X)\) is then given by

\[
c^p \frown c_r = h(c^p, \tau_X(c_r)).
\]

Now we consider functors from \(\text{Top}^m\) to \(\text{C}(\text{Ab})\). For a space \(X = \coprod_{i=1}^m X_i\), we have the diagram

\[
\begin{array}{cccc}
S_*(X) & \xrightarrow{\tau_X} & S_*(X) \otimes S_*(X) & \\
& & \downarrow{\tau_{X \otimes X}} & \\
& & T_X^m \otimes T_X^m & \\
& \downarrow{T_X^m} & & \\
S^m_*(X) & & S^m_*(X) \otimes S^m_*(X) & \\
& & \downarrow{T_{\text{Shuf}}} & \\
& & \otimes_{i=1}^m (S_*(X_i) \otimes S_*(X_i)) & \\
\end{array}
\]

in which the homomorphism \(T_{\text{Shuf}}\) is given by the shuffling:

\[
T_{\text{Shuf}}(\otimes_{i=1}^m (c_{p_i} \otimes c_{q_i})) = (-1)^{p \cdot q} (\otimes_{i=1}^m c_{p_i}) \otimes (\otimes_{i=1}^m c_{q_i}),
\]
where \( p = (p_i)_{i=1}^m, \ q = (q_i)_{i=1}^m \), \( c_{p_i} \in S_{p_i}(X_i) \) and \( c_{q_i} \in S_{q_i}(X_i), \ i = 1, 2, \ldots, m; (p,q) \) is given in (10). It can be checked directly that \( T_{Shuf} \) is a chain map. We see that chain maps
\[
\psi_X = (T^m_X \otimes T^m_X) \circ \tau_X \quad \text{and} \quad \psi'_X = T_{Shuf} \circ (\otimes_{i=1}^m \tau_{X_i}) \circ T^m_X
\]
can be treated as natural transformations \( \text{Top}^m \rightarrow \text{C}(\text{Ab}) \) which are specified to \( X \). Using Acyclic Model Theorem, with models as \( 2m \)-product of simplices (see [Spa66, Section 6, Chapter 5, p. 252] for the details of the case \( m = 2 \), and the proof for the general case is similar), it turns out that Diagram (51) commutes, up to natural chain homotopy. This means that there is a natural transformation \( D: \text{Top}^m \rightarrow \text{C}(\text{Ab}), \) such that
\[
(53) \quad \psi_X - \psi'_X = D_X \partial_s + \partial'_s D_X
\]
when \( X \) is specified.

**Lemma 6.1.** Let \( c^p = \otimes_{i=1}^m c^{p_i} \in \bigotimes_{i=1}^m S^{p_i}(X_i), c^q = \otimes_{i=1}^m c^{q_i} \in \bigotimes_{i=1}^m S^{q_i}(X_i) \) be two cochains of degrees \( |p| \) and \( |q| \), respectively. Then we have
\[
(54) \quad (T^m_X)^* \theta_m(c^p) - (T^m_X)^* \theta_m(c^q) - (-1)^{|p \cdot q|} (T^m_X)^* \theta_m(\otimes_{i=1}^m c^{p_i} \ominus c^{q_i}) = \delta_s B_1(c^p,c^q) + B_2(\delta_s c^p,c^q) + B_3(c^p,\delta_s c^q)
\]
where \( (T^m_X)^* \) is the dual of \( T^m_X \), \( \delta_s \) the coboundary operator of \( \bigotimes_{i=1}^m S^*(X_i) \), and \( B_i: \bigotimes_{i=1}^m S^*_i(X_i) \rightarrow S_s(X) \) are three bilinear forms associated to \( |p| \) and \( |q|, \ i = 1, 2, 3 \). Additionally, given \( c_r = \otimes_{i=1}^m c_{r_i}, \ c_{r_i} \in S_{r_i}(X_i), \) we have
\[
(55) \quad (T^m_X)^* \theta_m(c^p) - (T^m_X)^* \theta_m(c^q) - (-1)^{|r \cdot p \cdot q|} (T^m_X)^* \theta_m(\otimes_{i=1}^m c^{p_i} \ominus c_{r_i}) = \partial_s B'_1(c^p,c_r) + B'_2(\delta_s c^p,c_r) + B'_3(c^p,\partial_s c_r)
\]
with three bilinear forms \( B'_i: \bigotimes_{i=1}^m S^*(X_i) \bigotimes S^{r_m}_s(X) \rightarrow S^{r_m}_s(X) \) (with respect to \( |p|, |q| \), \( (T^m_X)^* \) the chain-homotopy inverse of \( T^m_X \).

**Proof.** This follows by a diagram chasing on Diagram (51). Let \( c^{p,q} \) denote the cochain \( c^p \otimes c^q \). By assumption, we have
\[
\psi^*_X (\theta_{2m}(c^{p,q})) = \tau^*_X \theta_2 (\{ (T^m_X)^* \theta_m(c^p) \otimes (T^m_X)^* \theta_m(c^q) \}) = (T^m_X)^* \theta_m(c^p) - (T^m_X)^* \theta_m(c^q),
\]
\[
(\psi'_X)^* (\theta_{2m}(c^{p,q})) = (T^m_X)^* \theta_m \left( -(-1)^{|p \cdot q|} \otimes_{i=1}^m \tau^*_X \theta_2 (c^{p_i} \otimes c^{q_i}) \right) = -(-1)^{|p \cdot q|} (T^m_X)^* \theta_m (\otimes_{i=1}^m c^{p_i} \ominus c^{q_i}).
\]
Then (54) follows from a comparison of the two equations above, together with (53). It remains to prove (55). It can be checked directly that

\[(T^m_X)^* \theta_m(c^p) \sim (T^m_X)^{-1}(c_r) = h ((T^m_X)^* \theta_m(c^p), \tau_X(T^m_X)^{-1}(c_r)) = (T^m_X)^{-1} h (h ((T^m_X)^* \theta_m(c^p), \tau_X(T^m_X)^{-1}(c_r)) + \partial_s B'_{11}(c^p, c_r) + B'_{12}(\delta_s c^p, c_r) + B'_{13}(c^p, \partial_s c_r),\]

where bilinear forms \(B'_{ij}\) comes from the chain-homotopy between \((T^m_X)^{-1} T^m_X\) and the identity of \(S_*(X)_i\), \(i = 1, 2, 3\), together with (50). On the other hand, again by (53), we see that

\[T^m_X h (h ((T^m_X)^* \theta_m(c^p), \tau_X(T^m_X)^{-1}(c_r)) = h (\theta_m(c^p), (T^m_X \otimes T^m_X) \tau_X(T^m_X)^{-1}(c_r)) = h (\theta_m(c^p), T_{\text{Shuf}}(\otimes_{i=1}^m \tau_{X_i}) (c_r)) + \partial_s B'_{21}(c^p, c_r) + B'_{22}(\delta_s c^p, c_r) + B'_{23}(c^p, \partial_s c_r).\]

By expanding \((\otimes_{i=1}^m \tau_{X_i}) (c_r)\) with (48) and then matching the degrees, we have

\[h (\theta_m(c^p), T_{\text{Shuf}}(\otimes_{i=1}^m \tau_{X_i}) (c_r)) = (-1)^{(r-p-p)} \otimes_{i=1}^m c^{p_i} \sim c_{r_i}.\]

Clearly (55) follows from the equations above. \(\square\)

Now let \((X, A)\) be the pair in the statement of Theorem 3.1. Additionally, suppose that each \(K_i\) is given a partial ordering on the vertex set, which induces a total ordering on each simplex. To complete the proof, it remains to compare the singular (co)chain complex with the simplicial one.

Recall that there is a chain-homotopy equivalence \(\iota_i : C_*(K_i) \to S_*(|K|)\) for each \(i = 1, 2, \ldots, m\), such that \(\iota_i\) sends each \(p\)-simplex \([v_0, v_1, \ldots, v_p]\) to the singular simplex linearly spanned by associated vertices, say \([v_0, v_1, \ldots, v_p]\), with \(v_0 < v_1 < \ldots < v_p\) in the given ordering; moreover, the chain-homotopy inverse \(\iota_i^{-1}\) satisfies \(\iota_i^{-1}([v_0, v_1, \ldots, v_p]) = [v_0, v_1, \ldots, v_p]\) (see [Mun84, Theorem 34.3, pp. 194–195]). It can be checked that diagrams (56)

\[S^{|K_1|} \otimes S^{|K_1|} \longrightarrow S^{|K_1|} \otimes S^{|K_1|} \longrightarrow S_*(|K_1|) \]

\[\downarrow \iota_1^{-1} \otimes \iota_1^{-1} \quad \downarrow \iota_1^{-1} \quad \text{and} \quad \downarrow \iota_1^{-1} \otimes \iota_1^{-1} \]

\[C^*(K_1) \otimes C^*(K) \longrightarrow C^*(K_1) \otimes C^*(K) \longrightarrow C_*(K_1) \]

commute, where the simplicial cup and cap products in the bottom rows are given by (22) and (23), respectively.
It follows that the map $\iota: (C_*(X), \partial) \to (S^m_*(X), \partial')$, $\iota = \otimes_{i=1}^m t_i$, is a chain-homotopy equivalence with its inverse $\iota^{-1} = \otimes_{i=1}^m t_i^{-1}$.

**Proof of Theorem 3.1.** Let $\nu_c: C_*(A) \to C_*(X)$ and $\nu_s: S_*(A) \to S_*(X)$ be the chain maps induced by the inclusion $A \to X$, respectively, and let $\nu^*_c$ and $\nu^*_s$ be their duals. We see that $\nu^*_c$ sends a dual basis $\otimes_{i=1}^m \sigma^p_{p_i}$ to the one of the same form, if $\prod_{i=1}^m |\sigma_{p_i}| \subset A$, and to zero otherwise (when using the dual basis of the form above, we have used the evaluation map $\theta_m$ implicitly). By definition, $\nu^*_c$ preserves the cup products given in (12).

Consider the composition $(T^m)^{-1} \nu_c: C_*(A) \to S_*(X)$, by the naturality of $(T^m)^{-1}$, it gives rise to a chain map $C_*(A) \to S_*(A)$ (since the image of $S_*(\prod_{i=1}^m |\sigma_{p_i}|)$ under $(T^m)^{-1}$ lies in $S_*(\prod_{i=1}^m |\sigma_{p_i}|)$). Thus we define

$$f_A = (T^m)^{-1} \nu_c;$$

it can be checked that $f_A$ satisfies the desired property in the proof of the additive isomorphism $H_*(C_*(A), \partial) \cong H_*(A)$. The argument above shows that the chain-homotopy inverse of $f_A$, say $f_A^{-1}$, is then given by $\iota^{-1} T^m \nu_s: S_*(A) \to C_*(A)$, whose dual

$$(f_A^{-1})^* = \nu^*_s(T^m_*)^* (\iota^{-1})^*: C^*(A) \to S^*(A)$$

induces the isomorphism $H^*(C^*(A), d) \cong H^*(A)$. Notice that $(f_A^{-1})^*$ can be defined over $C^*(X)$, which we shall denote by $(\tilde{f}_A^{-1})^*$: we have

$$(f_A^{-1})^* \nu^*_c = (\tilde{f}_A^{-1})^*.$$

Now we choose cochains $c^p = \otimes_{i=1}^m c^p_i \in C^{[p]}(X)$ and $c^q = \otimes_{i=1}^m c^q_i \in C^{[q]}(X)$, where $c^p_i \in C^{p_i}(|K_i|)$ and $c^q_i \in C^{q_i}(|K_i|)$. By (54) and Diagram (56), we have

$$(f_A^{-1})^* \nu^*_c(p) = (f_A^{-1})^* \nu^*_c(q) = (-1)^{[p][q]} (f_A^{-1})^* \nu^*_c \otimes \otimes_{i=1}^m c^p_i - c^q_i.$$

Now we compare cochains $c^p = \otimes_{i=1}^m c^p_i \in C^{[p]}(X)$ and $c^q = \otimes_{i=1}^m c^q_i \in C^{[q]}(X)$, where $c^p_i \in C^{p_i}(|K_i|)$ and $c^q_i \in C^{q_i}(|K_i|)$. By (54) and Diagram (56), we have

$$(f_A^{-1})^* \nu^*_c(p) = \nu^*_c(p) - (-1)^{[p][q]} (f_A^{-1})^* \nu^*_c \otimes \otimes_{i=1}^m c^p_i - c^q_i.$$

Now we compare cochains $c^p = \otimes_{i=1}^m c^p_i \in C^{[p]}(X)$ and $c^q = \otimes_{i=1}^m c^q_i \in C^{[q]}(X)$, where $c^p_i \in C^{p_i}(|K_i|)$ and $c^q_i \in C^{q_i}(|K_i|)$. By (54) and Diagram (56), we have

$$(f_A^{-1})^* \nu^*_c(p) = \nu^*_c(p) - (-1)^{[p][q]} (f_A^{-1})^* \nu^*_c \otimes \otimes_{i=1}^m c^p_i - c^q_i.$$

$$= \nu^*_s \delta_1 (t^{-1})^* (c^p) + \nu^*_s \delta_2 (t^{-1})^* (c^q) + \nu^*_s \delta_3 (t^{-1})^* (c^p) + \nu^*_s \delta_4 (t^{-1})^* (c^q)$$.
due the naturality of the chain-homotopy $D$ in (53), the summation in the last line above can be written as

\[(58) \quad (i^{-1})^*v^*_c(e^p), (i^{-1})^*v^*_c(e^q) + v^*_sB_2 ((i^{-1})^*v^*_c(d^p), (i^{-1})^*v^*_c(e^q)) + v^*_sB_3 ((i^{-1})^*v^*_c(e^p), (i^{-1})^*v^*_c(d^q)).\]

Therefore, we have proved part on cup products, since $v^*_c$ is surjective, and (58) will vanish on passage to cohomology (notice that $v^*_c(c^*) \in C^*(A)$ is closed means that $v^*_c(dc^*) = 0$, while $dc^*$ may not vanish in $C^*(X)$).

Now we prove the part on cap products. Suppose $c_r = \otimes_{i=1}^m c_{r_i} \in C([r](X), c_{r_i} \in C_{r_i}([K_i]))$, such that $c_r$ is, under $v_c$, the image of a chain of the same form in $C_*(A)$, which we shall denote as $v_c^*$. Then by definition (23), we see that under $v_c$, $\otimes_{i=1}^m c_{r_i}$ has a unique preimage $\otimes_{i=1}^m c_{r_i} \sim c_r \in C_*(A)$.

Analogously to the previous case, by (55) and the naturality, we have

\[
(f_A)^{-1} v^*_c(e^p) \sim f_A(c_r) - (-1)^{(r-p-p)} f_A \left( \otimes_{i=1}^m c_{r_i} \right) \\
= (f_A)^{-1} v^*_c(e^p) \sim (T^m_X)^{-1} v(c_r) - (-1)^{(r-p-p)} (T^m_X)^{-1} v (\otimes_{i=1}^m c_{r_i}) \\
= \partial_s B_1 ((i^{-1})^*v^*_c(e^p), \iota(c_r)) + B_2 (\delta_s^*(i^{-1})^*v^*_c(e^p), \iota(c_r)) + B_3 (\delta_s^*(i^{-1})^*v^*_c(e^p), \iota((\partial_\iota c_r))) \\
= \partial_s B_1 ((i^{-1})^*v^*_c(e^p), \iota(c_r)) + B_2 ((i^{-1})^*v^*_c(d^p), \iota(c_r)) + B_3 ((i^{-1})^*v^*_c(d^p), \iota((\partial_\iota c_r))),
\]

where the summation in the last line shall vanish on passage to (co)homology.

\[
\square
\]

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Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan

E-mail address: l-cai@imi.kyushu-u.ac.jp