Quantum structure of T-dualized models with symmetry breaking

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Abstract

We study the principal $\sigma$-models defined on any group manifold $G_L \times G_R/G_D$ with breaking of $G_R$, and their T-dual transforms. For arbitrary breaking we can express the torsion and Ricci tensor of the dual model in terms of the frame geometry of the initial principal model. Using these results we give necessary and sufficient conditions for the dual model to be torsionless and prove that the one-loop renormalizability of a given principal model is inherited by its dual partner, who shares the same $\beta$ functions. These results are shown to hold also if the principal model is endowed with torsion. As an application we compute the $\beta$ functions for the full Bianchi family and show that for some choices of the breaking parameters the dilaton anomaly is absent: for these choices the dual torsion vanishes. For the dualized Bianchi V model (which is torsionless for any breaking), we take advantage of its simpler structure, to study its two-loops renormalizability.
1 Introduction

The subject of classical versus quantum equivalence of T-dualized $\sigma$-models has been strongly studied in recent years, and extensive reviews covering abelian, non-abelian dualities and their applications to string theory and statistical physics are available [2], [3], [19]. More recent developments on the geometrical aspects of duality can be found in [1]. After the proof that T-duality is indeed a canonical transformation [24], [25] relating two classically equivalent theories, the most interesting problem is to study this equivalence at the quantum level. This was done mostly for dualizations of Lie groups, with emphasis put on $SU(2)$. For this model the one-loop equivalence was established in [16], [18].

The way towards the general case was cleaned up with the derivation of the classical structure of the non-abelian dual for any group [18], [3], [2], [21]. However the analysis of Bianchi V in [20] revealed that for some renormalizable dual theories the divergences could not be absorbed by a re-definition of the dilaton field! It was further realized that this phenomenon occurs for non semi-simple Lie groups with traceful structure constants ($f_{\alpha \beta} = 0$), and that it can be interpreted as a mixed gravitational-gauge anomaly [3].

A further decisive progress was made by Tyurin [26], who generalized the one-loop equivalence to an arbitrary Lie group and derived the general structure of the dilaton anomaly. However, as pointed out in [7], his analysis considers only models with explicit invariance under the left group action (whose existence is crucial for the dualization process) leaving aside the right action and the possible symmetry breaking schemes for it. The one-loop equivalence problem in this more general setting has been examined recently [7], [22] for the group manifold $SU(2)_L \times SU(2)_R/SU(2)_D$, where $SU(2)_R$ is broken down to a $U(1)$. The renormalizability and dilatonic properties do survive despite the lowering of the right isometries. It is the purpose of the present article to analyze the geometry of the dualized model for a large class of models built on $G_L \times G_R/G_D$, with arbitrary breaking of $G_R$. While in [26] supersymmetry considerations à la Busher [8], [9] were convenient to derive the dualized geometry, we will show that a direct computation in local coordinates is fairly efficient to extract the Ricci tensor in the presence of symmetry breaking.

The content of this article is the following: after setting the notations, in section 2 we study the geometry of the group manifold $(G_L \times G_R)/G_D$. This is most conveniently done using frames and, despite symmetry breaking, one obtains a manageable form for the Ricci tensor. In section 3 the dualized theory is examined and its torsion and Ricci tensor are computed, exhibiting their dependence with respect to the geometrical quantities of the principal model. The possibility of torsionless dualized models is discussed. In section 4 we use the previous results to show that the one-loop renormalizability of the principal model is inherited by its T-dual. In section 5 we generalize the previous analyses to deal with a principal model endowed with torsion. In section 6 we examine the models in the Bianchi class, compute their beta functions, and for the non semi-simple algebras discuss the dilaton anomaly. For some breaking choices this anomaly may vanish and in these cases the dual models are torsionless. Since any dualized Bianchi V model is torsionless, we study in section 7, for the simplest breaking, its two-loops renormalizability.
2 Geometry of the broken principal models

Since we have in view perturbative applications, our considerations will be of a local nature. Let us consider a Lie algebra \( \mathcal{G} = \{X_i, \ i=1, \ldots, \nu\} \) with structure constants

\[
[X_i, X_j] = f_{ij}^s X_s.
\]

Denoting by \( z^i \) the local coordinates in a neighbourhood of the origin, we exponentiate to the group by \( g = \exp(z \cdot T) \), and define

\[
g^{-1} \partial_\mu g = J^i_\mu X_i.
\]

For further use we introduce the adjoint representation by

\[
(T_i)_j^k \equiv (\text{ad} X_i)_j^k = -f_{ij}^k,
\]

which allows to write the Jacobi identity

\[
[T_i, T_j] = f_{ij}^s T_s, \quad i,j,s = 1, \ldots \nu = \text{dim}(\mathcal{G}).
\]

Then the action of the corresponding principal model can be written

\[
S = \frac{1}{2} \int d^2 x B_{ij} \eta^{\mu\nu} J^i_\mu J^j_\nu,
\]

where the matrix \( B \) is symmetric and invertible. For field theoretic applications one should add the restriction that \( B \) is positive definite [6], while this does not seem to be necessary for stringy applications. This restriction implies, in the semi-simple case that its simple components have to be compact. Our analysis will not make use of this positivity hypothesis.

Taking the curl of the first relation in (1) gives the Bianchi identity

\[
M^i_{\mu\nu}(J) \equiv \partial_\mu J^i_\nu - \partial_\nu J^i_\mu + f_{st}^i J^s_\mu J^t_\nu = 0 \quad \iff \quad \epsilon^{\mu\nu} M^i_{\mu\nu}(J) = 0.
\]

2.1 Isometries

Let us proceed to a discussion of the isometries of the action (4). The groups \( G_L \times G_R \) and \( G_D \) act on \( g \) according to

\[
g \rightarrow g' = G_L g G_R^{-1}, \quad g \rightarrow g' = G_D g G_D^{-1}.
\]

As a consequence

\[
g^{-1} \partial_\mu g \rightarrow G_R g^{-1} \partial_\mu g G_R^{-1},
\]

and specializing to infinitesimal transformations one gets

\[
G_R \approx \mathbb{I} + \epsilon^i R_i, \quad \Rightarrow \quad \delta J^i_\mu = f_{ij}^k \epsilon^i R_i J^j_\mu.
\]

It follows that the action (4) is invariant under \( G_L \), while the matrix \( B_{ij} \) will generally break \( G_R \) down to some subgroup \( H \) (possibly trivial). Denoting by \( \{T_s, \ s = 1, \ldots, h\} \) the generators of its Lie algebra \( \mathcal{H} \), these should satisfy

\[
(T_s)_i^k B_{kj} + (T_s)_j^k B_{ik} = 0, \quad \forall \ T_s \in \mathcal{H}.
\]
Let us emphasis that the metric $B$ can be freely chosen (as far as it is symmetric and invertible!), but, if $\mathcal{G}$ is simple, the most symmetric choice is given by the bi-invariant metric

$$B_{ij} = \frac{1}{\rho} g_{ij}, \quad g_{ij} = \text{Tr} (T_i T_j) = \tilde{\rho} \text{Tr} (t_i t_j),$$

(9)

where $g_{ij}$ is the Killing metric and the $t_i$ the defining representation of the simple algebra under consideration. In the simple compact case we have

$$\tilde{\rho} = (n-2)/2n \quad \text{and} \quad 2(n+1)$$

and with the standard normalization of the generators $\text{Tr} (t_i t_j) = -\delta_{ij}$, we see that the choice $\rho = -2\tilde{\rho}$ gives $B_{ij} = \delta_{ij}$. In the simple non-compact case the same choice of $\rho$ gives $B_{ij} = \eta_{ij}$, which is diagonal, with $\eta_{ii} = +1$ for a compact generator $t_i$ and $\eta_{ii} = -1$ for a non-compact one.

The bi-invariant metric has for isometry group the full $G_L \times G_R$ because $(T_i)_k^k g_{kl} = -f_{silt}$ is fully skew-symmetric and therefore (8) is true for all the generators of $G_R$.

For a semi-simple $\mathcal{G}$ the situation is not very different, since it can be split into a direct sum of simple algebras

$$\mathcal{G} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_k, \quad [\mathcal{S}_i, \mathcal{S}_j] = 0 \quad i \neq j.$$

### 2.2 Geometry of frames

In order to have a better insight into the geometry of the principal models with action (4), it is convenient to use a vielbein formalism, through the identification

$$B_{ij} \eta^{\mu\nu} J^i_\mu J^j_\nu \quad \leftrightarrow \quad B_{ij} e^i e^j,$$

and now the Bianchi identities appear as the Maurer-Cartan equations

$$de^i + \frac{1}{2} f_{st}^i e^s \wedge e^t = 0. \quad (10)$$

We follow the notations of [12] and define the spin-connection $\omega^j_i$ by

$$de^i + \omega^j_s \wedge e^s = 0, \quad \omega^j_i = \omega^j_i e^s.$$

The frame indices are lowered or raised using the metric $B_{ij}$ and its inverse $B^{ij} = B^{-1}_{ij}$. A straightforward computation gives

$$2\omega_{ij,k} = f_{ij,k} + f_{ik,j} - f_{jk,i}, \quad f_{ij,k} = f_{ij}^s B_{sk}. \quad (11)$$

For further use let us point out two consequences

$$\omega^j_{i,k} - \omega^j_{k,i} = -f_{jk}^i, \quad \omega_i^s = -f_{is}^s. \quad (12)$$

The curvature and the Ricci tensor are defined by

$$R^i_j = d\omega^i_j + \omega^i_s \wedge \omega^s_j = \frac{1}{3} R^i_{j, st} e^s \wedge e^t, \quad r_i^c_j = R^s_{i, sj}. $$
It follows that
\[ R^i_{j,st} = -\omega^i_{j,a} f_{st}^a - \omega^i_{a,t} \omega^a_{j,s} + \omega^i_{a,s} \omega^a_{j,t}. \] (13)

In the Ricci tensor the first two terms are gathered using (12) and give
\[ \text{ric}_{ij} = -\omega^s_{i,t} \omega^t_{j,s} + \omega^t_{s,t} \omega^s_{i,j}. \] (14)

The $i \leftrightarrow j$ symmetry of the first term is obvious while for the second it follows from
\[ \omega^t_{s,t} (\omega^s_{i,j} - \omega^s_{j,i}) = f_{st}^t f_{ij}^s = 0, \] (15)
where the last equality is obtained by taking the trace of the Jacobi identity (3).

One can give the following explicit form of the Ricci tensor
\[ \text{ric}_{ij} = \frac{1}{2} B_{st} (A^s B^{-1} A^t)_{ij} - \frac{1}{4} B_{is} \text{Tr} (B^{-1} A^s B^{-1} A^t) B_{tj} \]
\[ -\frac{1}{2} \text{Tr} (T_i T_j) + \frac{1}{2} \text{Tr} (T_s) \left( f_{s,i,j}^s + f_{s,j,i}^s \right), \]
\[ f_{s,i,j}^s = (B^{-1})_{st} f_{i,j}^s, \] (16)
which exhibits that it is an homogeneous function of degree 0 in the breaking matrix $B$.

The scalar curvature $R = (B^{-1})_{ij} \text{ric}_{ij}$ is a constant, as it should for homogeneous spaces.

A drastic simplification takes place for the bi-invariant metric (9), for which we have
\[ \text{ric}_{ij} = -\frac{\rho}{4} B_{ij}. \] (17)

The metric is therefore Einstein, and such a simple structure will have a counterpart in the dualized theory.

### 2.3 Dualization

For the reader's convenience we present a quick derivation [18], [2] of the dualized model. The essence of the dualization process is to switch from the coordinates on the group, which parametrize $g$, to new coordinates $\psi_i$ defined as the Lagrange multipliers of the Bianchi identities. Concretely this transformation is carried out starting from the action
\[ S = \frac{1}{4} \int d^2 x \left\{ B_{ij} \epsilon^{\mu\nu} J_i^j J^j_{\mu} - \epsilon^{\mu\nu} \psi_i M^{\mu\nu}_{ij}(J) \right\}. \]

Using light-cone coordinates, with the following conventions
\[ x_\pm = \frac{x^0 \pm x^1}{\sqrt{2}}, \quad \epsilon_{01} = 1, \quad \epsilon^{\mu\nu} \epsilon_{\sigma\tau} = \delta^\mu_\nu, \quad J_\pm = \frac{J_0 \pm J_1}{\sqrt{2}}, \]
one has
\[ S = \frac{1}{2} \int d^2 x \left\{ (B + A \cdot \psi)_{ij} J^j_+ J^j_- - \psi_i (\partial_+ J^j_- + \partial_- J^j_+) \right\} \] (18)
with
\[ (A^s)_{ij} = (T_i)_{j}^s, \quad (A \cdot \psi)_{ij} = (A^s)_{ij} \psi_s. \] (19)

The field equations obtained from the variations with respect to the currents $J_\pm^i$ give
\[ J^i_- = (B + A \cdot \psi)^{is} \partial_- \psi_s, \] \[ J^i_+ = -\partial_+ \psi_s (B + A \cdot \psi)^{si}, \] \[ (B + A \cdot \psi)^{is} (B + A \cdot \psi)_{sj} = \delta^i_s. \]
Using minkowskian coordinates on the worldsheet one has
\[ J^\mu = B^{ij} \epsilon^{\mu\nu} \left( \partial_\nu \psi_j - (A \cdot \psi)_{jk} J_k^\nu \right), \quad B^{is} B_{sk} = \delta^i_k. \]
Using this relation, the action \([18]\) can be written, up to total derivatives
\[ S = \frac{1}{2} \int d^2 x \partial_+ \psi_i J^i_+ = \frac{1}{2} \int d^2 x \partial_+ \psi_i (B + A \cdot \psi)^{ij} \partial_- \psi_j. \quad (20) \]
Comparing this action with the one given in relation (4.16) of \([26]\) we see that in this reference only the unbroken case \(B_{ij} = \delta_{ij}\) has been considered.

Let us emphasize the following points:

1. Before dualization, all the field dependence on the coordinates chosen to parametrize \(G\) must be hidden in expressions involving solely the currents \(J^i_\mu\). If this is not the case the dualization process is not possible.

2. The dualized action is completely defined by the breaking matrix \(B\) and the field matrix \(A \cdot \psi \in sl(\nu)\). There are as many coordinates as generators in \(G\).

3. In the process of dualization the isometries corresponding to \(G_L\) (which leave the \(J^i_\mu\) invariant) are lost. This has for consequence that starting from an homogeneous metric, we are led to a non-homogeneous one.

### 3 Geometry of the dualized theory

In \([23]\) we come back to standard notations and change the coordinates \(\psi_i\) to \(\psi^i\). Let us write the dual action
\[ S = \frac{1}{2} \int d^2 x G_{ij} \partial_+ \psi^i \partial_- \psi^j, \quad G_{ij} = (B + A \cdot \psi)^{ij}_-. \quad (21) \]
For further use we define the matrices
\[ G^\pm = (B \pm A \cdot \psi)^{-1}, \quad G \equiv G^+, \quad \Gamma^\pm = B \pm A \cdot \psi, \quad (A \cdot \psi)_{ij} = -f_{ij}^s \psi^s. \]
Writing the dual action \([23]\) in minkowskian coordinates
\[ S = \frac{1}{2} \int d^2 x \left\{ g_{ij} \eta^{\mu\nu} \partial_\mu \psi^i \partial_\nu \psi^j + h_{ij} \epsilon^{\mu\nu} \partial_\mu \psi^i \partial_\nu \psi^j \right\}, \quad (22) \]
gives for metric and torsion potential
\[ g_{ij} = \frac{1}{2} (G_{ij} + G_{ji}), \quad h_{ij} = \frac{1}{2} (G_{ij} - G_{ji}), \quad G_{ij} = g_{ij} + h_{ij}. \]
Using matrix notations we have
\[ g = G^+ B G^- = G^- B G^+, \quad h = -g (A \cdot \psi) B^{-1}, \quad (23) \]
and for the inverse metric:
\[ g^{-1} = \Gamma^+ B^{-1} \Gamma^- = \Gamma^- B^{-1} \Gamma^+. \quad (24) \]
The determinant of the metric is
\[ \det g = \frac{\det B}{(\det \Gamma^\pm)^2} = \det B \cdot (\det G^\pm)^2. \]
### 3.1 Connection

We work with the standard conventions

\[
\Gamma^i_{jk} = \gamma^i_{jk} + T^i_{jk}, \quad T^i_{jk} = g^{is}T^s_{jk}, \quad T^i_{ijk} = \frac{1}{2}(\partial_i h_{jk} + \partial_k h_{ij} + \partial_j h_{ki}), \quad \partial_i \equiv \frac{\partial}{\partial \psi^i},
\]

or using differential forms

\[
H = \frac{1}{2!} h_{ij} d\psi^i \wedge d\psi^j, \quad T = \frac{1}{3!} T^s_{ijk} d\psi^i \wedge d\psi^j \wedge d\psi^k = \frac{1}{2} dH.
\]

The torsion potential is not uniquely defined since the following gauge transformation leaves invariant the torsion:

\[
H \rightarrow H + dA, \quad A = A_i d\psi^i, \quad \iff \quad h_{ij} \rightarrow h_{ij} + \partial_i [A_j].
\]

The connection is given by

\[
\Gamma^i_{jk} = \frac{1}{2} (g^{-1})_{is} (\partial_j G_{ks} + \partial_k G_{sj} - \partial_s G_{kj}).
\]

Using the relation

\[
\partial_i G_{jk} = f^i_{st} G_{js} G_{tk} = -(G A^i G)_{jk},
\]

one gets

\[
\Gamma^i_{jk} = -\frac{1}{2} f^i_{st} (\Gamma^+ B^{-1})_{is} G_{kt} - \frac{1}{2} f^k_{st} (B^{-1} \Gamma^+)_{ti} G_{sj} + \frac{1}{2} (g^{-1})_{iu} f^u_{st} G_{sj} G_{kt}.
\]

The next step is to simplify the last term in (29). To this end we combine Jacobi identity and the definition (11) to prove the identity

\[
f^s_{ij} \Gamma^{(\pm)}_{sk} - f^s_{kj} \Gamma^{(\pm)}_{si} = 2\omega_{ik,j} - f^u_{ik} \Gamma^{(\pm)}_{uj}.
\]

Starting from relation (24) for the inverse metric we can write

\[
(g^{-1})_{iu} f^u_{st} = (\Gamma^+ B^{-1})_{iv} f^u_{st} \Gamma^+_{uv},
\]

and use (30) to interchange the indices \( s \leftrightarrow v \). Several simplifications occur then in relation (29) and one is left with the simple result

\[
\Gamma^i_{jk} = (f^k_{is} \omega^t_{s,u} G^+_t G_{ku}) G_{sj}.
\]

The same procedure, using the second writing of \( g^{-1} \) in relation (24), gives another interesting form

\[
\Gamma^i_{jk} = (-f^s_{ij} + \omega^t_{s,u} G^-_{tu} G_{uj}) G_{ks}.
\]
3.2 Torsion

To get a useful form for the torsion we use relation (31) to compute

$$2T_{jk}^i \Gamma^+_{jr} \Gamma^+_{ks} = f_{ir}^k \Gamma^+_{ks} - \omega^\gamma_{r,\beta} \Gamma^+_{i\gamma}(\Gamma^- G)_{s\beta} - (r \leftrightarrow s).$$

The identity (30) and the easy relation $\Gamma^- G = 2BG - I$, transform the previous relation into

$$T_{jk}^i \Gamma^-_{jr} \Gamma^-_{ks} = -\omega^\gamma_{r,\beta} \Gamma^+_{i\gamma}(BG)_{s\beta} - (r \leftrightarrow s) - \omega_{rs,i}.$$

It is natural to multiply both sides by $(BG)^{-1}(BG)^{-1}$. Observing that $g^{-1} = (BG)^{-1}\Gamma^-$, we get

$$T^{ijk} = (\Gamma^+ B^{-1})_{ks} \omega^\gamma_{s,j} \Gamma^+_{i\gamma} - (j \leftrightarrow k) + (\Gamma^+ B^{-1})_{js} (\Gamma^+ B^{-1})_{kl} \omega_{rs,i}.$$

This result shows that this tensor is much simpler than $T_{ijk}$ since it is a polynomial in the fields $\psi$. The coefficient of the linear term vanishes from Jacobi’s identity and we are left with

$$T^{ijk} = \frac{1}{2} f_{ij,k} - (A \cdot \psi)_{i\alpha} (A \cdot \psi)_{j\beta} \omega^\alpha_{s,i} + \cdots,$$

where the dots indicate circular permutations of the indices $i, j, k$. We expand the spin connection according to (11) and use the identity (30) to end up with

$$2T^{ijk} = f_{ij,k} - (A \cdot \psi B^{-1})_{ut} (A \cdot \psi B^{-1})_{ju} f_{tu,k} - f_{ij,s} (A \cdot \psi B^{-1} A \cdot \psi)_{sk} + \cdots \tag{33}$$

Now we can discuss a possibility not yet considered in the literature: the vanishing of the torsion in the dual model. The terms which are independent of $\psi$ require $f_{[ij,k]} = 0$, a first condition which mixes the structure constants and the breaking matrix. Using this relation and the Jacobi identity one can check that the last two terms in (33) are equal. We conclude that the torsion vanishes iff

$$f_{[ij,k]} = 0, \quad \text{and} \quad f_{\alpha s}^{(u} (B^{-1})^{st} f_{t[i}^{v]} f_{ij]}^\alpha = 0, \quad \forall (u, v) [ijk]. \tag{34}$$

Clearly for a simple algebra, the first constraint never holds, but for solvable algebras both conditions may be satisfied, as will be seen in section 5 for the Bianchi family.

Let us conclude with an example of Lie algebra, for which the torsion vanishes for any choice of the breaking matrix. Let its generators be $\{X_i, i = 1, \cdots, \nu\}$ and take

$$[X_1, X_i] = X_i, \quad i = 2, \cdots, \nu, \quad [X_i, X_j] = 0, \quad i \neq j \neq 1.$$  

3.3 Ricci tensor

The covariant derivatives are defined by

$$D_i v^j = \partial_i v^j + \Gamma^j_{is} v^s = \nabla_i v^j + T^j_{is} v^s, \quad D_i v_j = \partial_i v_j - \Gamma^s_{ij} v_s = \nabla_i v_j - T^s_{ij} v_s \tag{35}$$

and the Riemann curvature by

$$[D_k, D_l] v^i = R^i_{s,kl} v^s - 2T^s_{kl} D_s v^i.$$

Its explicit form is given by

$$R^i_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{ks} \Gamma^s_{lj} - \Gamma^i_{ls} \Gamma^s_{kj}.$$
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The Ricci tensor follows from

\[ R_{ij} = \mathcal{R}^s_{i,sj} = \partial_s \Gamma^s_{ji} - \partial_j \Gamma^s_{si} + \Gamma^s_{st} \Gamma^t_{ji} - \Gamma^s_{jt} \Gamma^t_{si}. \]  (36)

Using

\[ \Gamma^s_{st} = \gamma^s_{st} = \partial_t (\ln \sqrt{\det g}), \]

we get for it a useful form

\[ R_{ij} = \partial_s \Gamma^s_{ji} - \partial_j \Gamma^s_{si} - D_j D_i (\ln \sqrt{\det g}). \]  (37)

In order to compute the first two terms in this relation, we use (31) for the first two connections and (32) for the third one. Apart from trivial cancellations one has to use the identity

\[ \omega^s_{i,u} \Gamma^+_{as} + \omega^s_{a,t} \Gamma^-_{as} = f^s_{al} \Gamma^-_{su} - f^s_{ua} \Gamma^+_{st} \]  (38)

in order to obtain further strong cancellations of terms, with the final simple result

\[ \partial_s \Gamma^s_{ji} - \Gamma^s_{jt} \Gamma^t_{si} = -G_{is} \text{ric} \Gamma_{ij} + 2 f^s_{st} \omega^t_{u,v} G_{iu} G_{vj}. \]  (39)

Using (32) and (33), one can check that the last term can be written

\[ 2 f^s_{st} \omega^t_{u,v} G_{iu} G_{vj} = D_j V_i, \quad V_i = -2 \text{ric} f^s_{st}. \]

Therefore we end up with

\[ R_{ij} = -G_{is} \text{ric} \Gamma_{ij} + D_j V_i, \quad V_i = V_i - \partial_i \ln (\sqrt{\det g}). \]  (40)

This relation, which displays the relation between the frame geometry of the principal model and the geometry of its dual, will play an essential role in the next section.

Let us conclude with some remarks:

1. This result is different, although related to the ones by Tyurin [24] and Alvarez [1], who expressed the frame geometry of the dual model in terms of the frame geometry of the principal model. The first reference uses supersymmetry while the second uses purely frames. Our approach, using mainly local coordinates computations is valid for any breaking matrix \( B \), while the previous authors have considered only the case \( B = I \). Note also that, in view of the complexity of the dualized vielbein it’s a long way from the vielbein components of the Ricci to our relation (40).

2. If we consider a simple algebra \( G \), equipped with its bi-invariant metric \( I \). Relation (17) shows that the corresponding principal model is Einstein and we will prove that the dual metric is quasi-Einstein. To this aim we insert relation (17) into (40), use \( f^s_{si} = 0 \) to get

\[ R_{ij} = -\frac{\rho}{4} (BG)_i^j + D_j v_i. \]

Using relation (32) one can check that

\[ D_j \lambda_i = \frac{1}{2} G_{ij} + \frac{1}{2} (BG)_i^j, \quad \lambda_i = (B^{-1})_{is} \psi^s \]  (41)

from which we deduce

\[ R_{ij} = -\frac{\rho}{4} G_{ij} + D_j V_i, \quad V_i = \partial_i \left( -\ln (\sqrt{\det g}) + \frac{\rho}{4} (B^{-1})_{st} \psi^s \psi^t \right), \]  (42)

which establishes the desired result.
3. One further important point, with respect to string theory, is the dilatonic property of the dualized geometry, i.e. whether the vector $V_i$ is a gradient or not. For the semi-simple groups the dilatonic property does hold since we have $f^s_{st} = 0$.

The failure of this property was first discovered for the dualized Bianchi V metric [20] (see also [13]). In [21], [26] it was shown to appear when the isometries are not semi-simple and have traceful structure constants $f^s_{st} \neq 0$, and its interpretation as an anomaly was worked out in [3].

4 One loop divergences of the dualized models

We are now in position to discuss the quantum properties of the dualized models at the one loop level.

Let us first consider the broken principal models with classical action (4). Its one loop counterterm, first computed by Friedan [17], is

$$
\int d^2x \, ric_{ij} \eta^\mu\nu J_i^\mu J_j^\nu, \quad d = 2 - \epsilon,
$$

where the Ricci components are computed in the vielbein basis.

Renormalizability in the strict field theoretic sense requires that these divergences have to be absorbed by (field independent) deformations of the coupling constants $\hat{\rho}_s$ hidden in the matrix $B$ and possibly a non-linear field renormalization. The renormalizability of the classical theory is ensured by

$$
ric_{ij} = \hat{\chi}_s(\rho) \frac{\partial}{\partial \hat{\rho}_s} B_{ij}. 
$$

The one loop renormalizability is clear for two extreme choices of metrics:

1. The bi-invariant metric, for which relation (44) shows that the principal model is Einstein.

2. The maximally broken metric, for which the matrix $B$ contains $\nu(\nu + 1)/2$ independent coupling constants $\hat{\rho}_s$. Since the Ricci is also a symmetric matrix, it can always be absorbed by a deformation of the coupling constants.

For partial breakings of the group $G_R$, relation (44) may fail to hold and is indeed a constraint which mixes conditions involving the breaking matrix $B$ and the algebra through its structure constants.

In order to compare to the renormalization properties of the dualized theory, let us recall that the most general conditions giving one loop renormalizability are

$$
\begin{align*}
Ric_{(ij)} &= \hat{\chi}_s \frac{\partial}{\partial \hat{\rho}_s} g_{ij} + D_i u_{ij}, \\
Ric_{[ij]} &= \hat{\chi}_s \frac{\partial}{\partial \hat{\rho}_s} h_{ij} + u_s T^s_{ij} + \partial_i U_{ij},
\end{align*}
$$

where the $\hat{\rho}_s$ are the coupling constants in the principal model we started from, appearing now in a non trivial way in the dualized model. The only constraint on the functions $\hat{\chi}_s$ is that they should be field independent.
These relations can be gathered into the single one
\[ \text{Ric}_{ij} = \hat{\chi}_s \frac{\partial}{\partial \hat{\rho}_s} G_{ij} + D_j u_i + \partial_i (u + U)_{jj}, \] (46)

We are now in position to prove that the one-loop renormalizability of the principal model implies the one-loop renormalizability of its dual. For the reader’s convenience we recall relation (40)
\[ \text{Ric}_{ij} = -G_{is} \text{ric}_{st} G_{tj} + D_j v_i, \quad v_i = -2 G_{it} f^s_{st} - \partial_i \ln(\sqrt{\det g}), \]
in which we insert (44) to get
\[ \text{Ric}_{ij} = \hat{\chi}_l G_{is} \frac{\partial}{\partial \hat{\rho}_l} B_{st} G_{tj} + D_j v_i. \] (48)

The first term is reduced using the identity
\[ \frac{\partial}{\partial \hat{\rho}_l} G_{ij}(B, \psi) = -G_{is}(B, \psi) \left( \frac{\partial}{\partial \hat{\rho}_l} B_{st} \right) G_{tj}(B, \psi), \] (47)
to the final form
\[ \text{Ric}_{ij} = \hat{\chi}_s \frac{\partial}{\partial \hat{\rho}_s} G_{ij} + D_j v_i. \] (48)

Comparing with relation (46) we conclude to the one-loop renormalizability of the dual model. Furthermore the vectors \( u_i \) and \( U_i \), defined in relation (45), which could be independent, are in fact related up to a gauge transformation by
\[ U_i = -u_i + \partial_i \tau. \]

Our next task is to prove that the \( \beta \) functions are the same, so we need a precise definition of the coupling constants. To do this let us switch from the couplings \( \{\hat{\rho}_i, i = 1, \ldots, c\} \) to new couplings \( (\lambda, \rho_i) \) defined by
\[ \hat{\rho}_1 = \frac{1}{\lambda}, \quad \hat{\rho}_{i+1} = \frac{\rho_i}{\lambda}, \quad i = 1, \ldots, c - 1. \] (49)

We scale similarly the breaking matrix
\[ B_{ij}(\hat{\rho}) = \frac{1}{\lambda} S_{ij}(\rho), \]
where, for simplicity, the matrix \( S \) can be taken linear in the couplings \( \rho_s \). Then relation (44) becomes
\[ \begin{cases} \text{ric}_{ij}(B) = \text{ric}_{ij}(S) = \left( \chi_\lambda + \sum_s \chi_s \frac{\partial}{\partial \rho_s} \right) S_{ij}(\rho), \\ \chi_\lambda = \hat{\chi}_1, \quad \chi_i = \hat{\chi}_i - \rho_i \hat{\chi}_1, \quad i = 1, \ldots, c - 1. \end{cases} \] (50)

The full one loop action is therefore
\[ \frac{1}{\lambda} \int d^2 x \left[ \left( 1 + \frac{\lambda \chi_\lambda}{2 \pi \epsilon} \right) S_{ij}(\rho) + \frac{\lambda}{2 \pi \epsilon} \sum_s \chi_s \frac{\partial}{\partial \rho_s} S_{ij}(\rho) \right] J^i \mu J^j_\nu, \quad \epsilon = 2 - d, \] (51)
from which we see that the divergences can be absorbed through coupling constant renormalizations:
\[ \lambda_0 = \mu' \lambda Z_\lambda, \quad Z_\lambda = 1 - \frac{\lambda \chi_\lambda}{2 \pi \epsilon}, \quad \rho^{(0)}_i = \rho_i Z_i, \quad \rho_i Z_i = 1 + \frac{\lambda \chi_i}{2 \pi \epsilon}. \]
It follows that the corresponding beta functions are
\[ \beta_\lambda = \mu \frac{\partial \lambda}{\partial \mu} = \lambda^2 \frac{\partial}{\partial \lambda} Z_\lambda^{(1)} = -\frac{\lambda^2}{2\pi} \chi_\lambda, \quad \beta_i = \mu \frac{\partial \rho_i}{\partial \mu} = \lambda \frac{\partial}{\partial \lambda} (\rho_i Z_i^{(1)}) = \frac{\lambda}{2\pi} \chi_i. \] (52)

For a principal model built with the bi-invariant metric given by (9) one has just the single coupling \( \lambda \), and
\[ \beta_\lambda = \frac{\lambda^2}{2\pi} \frac{\rho}{4}. \] (53)

In order to compute the divergences of the dualized theory in terms of the coupling constants defined in (49) we start from the dual classical action
\[ G_{ij}(B, \tilde{\psi}) \partial_+ \tilde{\psi}^i \partial_- \tilde{\psi}^j, \]
which we transform according to
\[ G(B, \tilde{\psi}) = \lambda G(S, \psi), \quad \psi^i = \lambda \tilde{\psi}^i, \quad \rightarrow \quad \frac{1}{\lambda} G_{ij}(S, \psi) \partial_+ \psi^i \partial_- \psi^j. \]
The one-loop counterterms follow from the ricci. We start from relation (40) written
\[ \text{Ric}_{ij} = \frac{\lambda}{2} \left[ -G_{is}(S, \psi) \text{ric}_{st} G_{tj}(S, \psi) + D_j v_i \right]. \]
Using (50) we write the first term
\[ -\chi_\lambda G_{is}(S, \psi) S_{st} G_{tj}(S, \psi) - \sum_u \chi_u G_{is}(S, \psi) \frac{\partial S_{st}}{\partial \rho_u} G_{tj}. \]
While the second term is reduced using the identity (17), the first term requires more work. One has first to define the vectors
\[ w_i = g_{is} \psi^s, \quad W_i = \psi^s G_{si}, \] (54)
then check the relation
\[ D_j w_i + \partial_i W_j = G_{ij} + \frac{1}{2} \psi^s (\partial_j G_{is} + \partial_i G_{sj}) - \Gamma^t_{ji} g_{ts} \psi^s, \]
which upon use of (27) becomes
\[ D_j w_i + \partial_i W_j = G_{ij} + \frac{1}{2} \psi^s \partial_s G_{ij}. \]
Eventually relation (28) gives
\[ D_j w_i + \partial_i W_j = \frac{1}{2} G_{ij} + \frac{1}{2} (GBG)_{ij}. \] (55)
Scaling appropriately this identity, we have
\[ \left[ \chi_\lambda G_{ij}(S, \psi) + \sum_u \chi_u \frac{\partial}{\partial \rho_u} G_{ij}(S, \psi) + D_j (v_i - 2\chi_\lambda w_i) - 2\chi_\lambda \partial_i W_j \right] \partial_+ \psi^i \partial_- \psi^j. \]
We end up with the renormalized dual theory

\[
\left\{ \frac{1}{\lambda} \int d^2x \left[ \left( 1 + \frac{\lambda \chi}{2\pi \varepsilon} + \frac{\lambda}{2\pi \varepsilon} \sum_s \chi_s \frac{\partial}{\partial \rho_s} \right) G_{ij} + \frac{\lambda}{2\pi \varepsilon} \left( D_j \mathcal{V}_i - 2\chi \partial_\rho [W_j] \right) \right] \partial_+ \psi^i \partial_- \psi^j, \right. 
\]

\[
\mathcal{V}_i = v_i - 2\chi \lambda w_i. 
\]

Comparing this relation with (51) we conclude that, up to the non-linear field re-definition described by the vector \( \mathcal{V}_i \) and the gauge transformation described by \( W_i \), the coupling constants renormalization are exactly the same as in the principal model we started from. We have thus established, at the one-loop level, that the principal \( \sigma \)–model renormalizability implies the renormalizability, in the strict field theoretic sense, of its dual and proved that their \( \beta \) functions do coincide.

**Remarks :**

1. What is really new with respect to [26] is that, even working with renormalizability in the strict field theoretic sense, the (possibly strong) breaking of the right isometries \( G_R \) does not jeopardize the one-loop renormalizability, and even in this extreme situation the \( \beta \) functions of the principal model and its dual remain the same. This was not obvious since the symmetry breaking is a “hard” breaking, by couplings of power counting dimension two.

2. As already observed in section 2.3, the isometries of \( G_L \) are lost in the dualization process. Hence for the maximal breaking of \( G_R \), no trace seems to remain of the original isometries in the dualized theory. These dual theories constitute a nice example of non-homogeneous metrics with torsion, with no isometries to account for their one-loop renormalizability. Our computation, which puts forward an experimental fact (the one-loop renormalizability) needs some basic theoretical explanation since we know that renormalizability is never accidental but the result of some underlying deeper symmetry.

3. As first observed in [7] for the dualized \( SU(2) \) model with symmetry breaking, there appears in the final form of the divergences (56) a gauge transformation \( W_i \). Indeed in this case we have the identities

\[
\psi^s G_{si} = G_{is} \psi^s = (B^{-1})_{is} \psi^s \quad \Rightarrow \quad w_i = W_i = \partial_i \left( \frac{1}{2}(B^{-1})_{st} \psi^s \psi^t \right) = \lambda_i, 
\]

which implies \( \partial_\rho [W_j] = 0 \). Then the general identity (53) reduces, for this particular case, to relation (41).

4. The situation at the two-loop level is still unclear since despite negative results in several models [22], [7] a more promising and new approach to the problem [23] seems to yield a positive answer.

5. It is well known that the unbroken principal models are integrable (for a review see [27]). On the contrary the broken ones are not believed to be generically integrable, a notable exception being \( SU(2) \), whose integrability was shown in [10] for the most general breaking. If this belief is confirmed, our results show that the one-loop quantum equivalence survives to symmetry breaking and therefore the root of this equivalence cannot be integrability.
5 Extension to principal models with torsion

The previous results can be generalized to cover principal models with torsion, with action

$$S = \frac{1}{2} \int d^2x \left( B_{ij} \eta^{\mu\nu} + C_{ij} \epsilon^{\mu\nu} \right) J^i_\mu J^j_\nu, \quad C_{ij} = -C_{ji},$$

where the matrix $C$ has constant components. Taking into account the vielbein interpretation of the currents, we define the torsion $t_{ijk}$ as usual by

$$t_{ijk} = \frac{1}{3!} t_{ijk} e^i \wedge e^j \wedge e^k = \frac{1}{2} dC, \quad C = \frac{1}{2} C_{ij} e^i \wedge e^j,$$

which gives

$$t_{ijk} = -\frac{1}{2} \left( f_{ij}^s C_{sk} + f_{jk}^s C_{si} + f_{ki}^s C_{sj} \right).$$

One should first observe that the parallelizing torsion (52) is not of this kind, and second that we have to exclude the case where

$$C_{ij} = f_{si}^s \gamma^s.$$

Indeed, if this relation holds the Bianchi identity (5) gives

$$C_{ij} \epsilon^{\mu\nu} J^i_\mu J^j_\nu = -2 \gamma^s \epsilon^{\mu\nu} \partial_\mu J^s_\nu, \quad \text{which is a total divergence.}$$

Correspondingly the torsion vanishes as a consequence of the Jacobi identity.

Even if (57) is valid for a semi-simple algebra $G$, it is not valid for any algebra. To see this let us suppose that the center of $G$, is non-trivial, i.e. there is some generator $X_\alpha$ which commutes with all the other generators. It follows that $f_{si}^s \gamma^s \equiv 0$ for all values of $i$, while $C_{ai}$ can be non-vanishing.

Let us describe briefly how our analysis can be generalized.

The spin connection $\Omega^i_j$ now verifies

$$de^i + \Omega^i_j e^j = B^i_j t_j, \quad t_i = t_{ist} e^s \wedge e^t.$$
which is strikingly similar to (40).

Let us denote by $\rho_s^B$ the couplings present in the matrix $B$, by $\rho_s^C$ the couplings present in the matrix $C$, and $\rho_s$ the couplings present in both matrices. The renormalizability of the principal model with torsion is ensured by

\[
\left\{
\begin{array}{l}
ric_{(ij)} = \left( \chi_s \frac{\partial}{\partial \rho_s^B} + \eta_s \frac{\partial}{\partial \rho_s} \right) B_{ij}, \\
ric_{[ij]} = \left( \eta_s \frac{\partial}{\partial \rho_s} + \xi_s \frac{\partial}{\partial \rho_s^C} \right) C_{ij}.
\end{array}
\right.
\]

Inserting relation (59) into (58) one ends up with

\[
Ric_{ij} = \left( \chi_s \frac{\partial}{\partial \rho_s^B} + \eta_s \frac{\partial}{\partial \rho_s} + \xi_s \frac{\partial}{\partial \rho_s^C} \right) G_{ij} + D_j v_i.
\]

It follows, by the same arguments as in section 5, that the dual model is also renormalizable and has the same $\beta$ functions as the initial principal model with torsion.

6 Dualized Bianchi metrics

Particular dualized models in the Bianchi family have been studied with emphasis either put on the renormalizability properties of the dualized models with symmetry breaking [4], [22] or on the dilaton anomaly [20], [13]. The aim of this section is to give some detailed analysis of both aspects for the full family.

All the Lie algebras with 3 generators were classified by Bianchi (1897). In a modern presentation [14], [15] these algebras are described in terms of the parameter $a$ and the vector $\vec{n} = (n_1, n_2, n_3)$ according to

\[
[X_1, X_2] = aX_2 + n_3X_3, \quad [X_2, X_3] = n_1X_1, \quad [X_3, X_1] = n_2X_2 - aX_3, \quad f_{st}^b = -2a\delta_{1t}.
\]

The Jacobi identity requires $a \cdot n_1 = 0$.

The algebras of interest appear in the following table

| Class A : $a = 0$ | Class B : $n_1 = 0, a > 0$ |
|-------------------|--------------------------|
| type | $n_1$ | $n_2$ | $n_3$ | type | $a$ | $n_2$ | $n_3$ |
|-------|-------|-------|-------|-------|-----|-------|-------|
| I     | 0     | 0     | 0     | V     | 1   | 0     | 0     |
| II    | 1     | 0     | 0     | IV    | 1   | 0     | 1     |
| VI$_0$| 0     | 1     | -1    | III   | 1   | 1     | -1    |
| VII$_0$| 0    | 1     | 1     | VI$_a$| $a \neq 1$ | 1 | -1 |
| VIII  | $su(1,1)$ | 1  | 1     | -1   | VII$_a$| 1   | 1     |
| IX    | $su(2)$    | 1  | 1     |      |       |     |       |

The adjoint representation is given by

\[
T_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & -a & -n_3 \\
0 & n_2 & -a
\end{pmatrix}, \quad T_2 = \begin{pmatrix}
a & 0 & n_3 \\
0 & 0 & 0 \\
-n_1 & 0 & 0
\end{pmatrix}, \quad T_3 = \begin{pmatrix}
0 & -n_2 & a \\
n_1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
The Killing metric \( g_{ij} = \text{Tr} (T_i T_j) \) is diagonal with
\[
g_{11} = 2(a^2 - n_3 n_1), \quad g_{22} = -2n_3 n_1, \quad g_{33} = -2n_1 n_2.
\]
It follows that B VIII and B IX are semi-simple (in fact, simple). Among the remaining non semi-simple algebras, only those in class B have traceful structure constants.

To simplify matters, still keeping the main peculiarities of symmetry breaking, we take the diagonal metric \( B_{ij} = r_i \delta_{ij} \). The dual metric tensor is then
\[
G = \frac{1}{\Delta_+} \begin{pmatrix}
  r_2 r_3 + x^2 & r_3 z - xy & -r_2 y - zx \\
  -r_3 z - xy & r_3 r_1 + y^2 & -r_1 x + yz \\
  r_2 y - zx & r_1 x + yz & r_1 r_2 + z^2
\end{pmatrix}, \quad \begin{cases} x = n_1 \psi^1, \\
y = n_2 \psi^2 - a \psi^3, \\
z = a \psi^2 + n_3 \psi^3.
\end{cases}
\]
with
\[
\Delta_+ = r_1 r_2 r_3 + r_1 x^2 \pm (r_2 y^2 + r_3 z^2).
\]

From (24) we get the torsion
\[
\begin{align*}
T_{ijk} &= t \epsilon_{ijk}, \\
t &= \frac{N}{\Delta_+}, \\
N &= \nu \Delta_+ + 2r_2 r_3 (n_3 y^2 + n_2 z^2 - n_1 r_1^2), \\
\nu &= r_1 n_1 + r_2 n_2 + r_3 n_3.
\end{align*}
\]
This result shows that for Bianchi V the dualized metric is torsion free!

Relation (16) gives for the non-vanishing vielbein components of the initial Ricci tensor
\[
\begin{align*}
\text{ric}_{11} &= -2a^2 + \frac{n_1^2 r_1^2 - (n_2 r_2 - n_3 r_3)^2}{2 r_2 r_3}, \\
\text{ric}_{22} &= -2a^2 \frac{r_2}{r_1} + \frac{n_2^2 r_2^2 - (n_3 r_3 - n_1 r_1)^2}{2 r_3 r_1}, \\
\text{ric}_{33} &= -2a^2 \frac{r_3}{r_1} + \frac{n_3^2 r_3^2 - (n_1 r_1 - n_2 r_2)^2}{2 r_1 r_2}, \\
\text{ric}_{23} &= \text{ric}_{32} = a \frac{(n_2 r_2 - n_3 r_3)}{r_1}.
\end{align*}
\]

### 6.1 Class A dual models and their \( \beta \) functions

We see at a glance from the Ricci that the class A principal models, with 3 independent diagonal couplings are renormalizable at one-loop. From the the previous section this ensures the renormalizability of the dualized model, with the same \( \beta \) functions.

For Bianchi I and II we define
\[
r_1 = \frac{1}{\lambda}, \quad r_2 = \frac{g}{\lambda}, \quad r_3 = \frac{g'}{\lambda}.
\]
Then one gets for the \( \beta \) functions
\[
\begin{align*}
\text{Bianchi I:} \quad & \beta_\lambda = \beta_g = \beta_{g'} = 0, \\
\text{Bianchi II:} \quad & \beta_\lambda = -\frac{\lambda^2}{4 \pi g g'}, \quad \beta_g = -\frac{\lambda}{8 \pi g'}, \quad \beta_{g'} = -\frac{\lambda}{8 \pi g}.
\end{align*}
\]
The result for Bianchi I is obvious, since its metric is flat.
For Bianchi IX (with $\sigma = +1$) and Bianchi VIII (with $\sigma = -1$), we parametrize the couplings according to

$$
\begin{align*}
    r_1 &= \frac{\sigma}{\lambda}, \\
    r_2 &= \frac{\sigma(1 + g)}{\lambda}, \\
    r_3 &= \frac{1 + g'}{\lambda}.
\end{align*}
$$

With three independent couplings, the $SU(2)_R$ isometries are fully broken. If one takes $g = g'$ the corresponding model has a residual $U(1)_R$ isometry and has been studied in [7], where the quantum equivalence was proved at the one-loop order.

Using (64) and (52) it is a simple matter to compute

$$
\begin{align*}
    \beta_\lambda &= \frac{\lambda^2}{4\pi} \frac{(1 + g - g')(1 - g + g')}{(1 + g)(1 + g')}, \\
    \beta_g &= \frac{\lambda}{2\pi} \frac{g(1 + g - g')}{(1 + g')}, \\
    \beta_{g'} &= \frac{\lambda}{2\pi} \frac{g'(1 - g + g')}{(1 + g)}.
\end{align*}
$$

(65)

For $g = 0$ and $\sigma = 1$ these results agree with [7].

For the remaining models we parametrize the couplings according to

$$
\begin{align*}
    r_1 &= \frac{1}{\lambda}, \\
    r_2 &= \frac{1 + g}{\lambda}, \\
    r_3 &= \frac{1 + g'}{\lambda}
\end{align*}
$$

where $\sigma = +1$ (resp. $\sigma = -1$) correspond to Bianchi VII$_0$ (resp. Bianchi VI$_0$). We get for the $\beta$ functions

$$
\begin{align*}
    \beta_\lambda &= \frac{\lambda^2}{4\pi} \frac{(g - g')^2}{(1 + g)(1 + g')}, \\
    \beta_g &= \frac{\lambda}{2\pi} \frac{(1 + g)(g - g')}{(1 + g')}, \\
    \beta_{g'} &= -\frac{\lambda}{2\pi} \frac{(1 + g')(g - g')}{(1 + g)}.
\end{align*}
$$

(66)

Let us observe that for $g' = g$ the metric of the principal model is flat, which explains the vanishing of all the $\beta$ functions.

### 6.2 Class B dual models and their $\beta$ functions

Let us begin with Bianchi V, which has diagonal ricci, and is therefore renormalizable with three independent couplings

$$
\begin{align*}
    r_1 &= \frac{1}{\lambda}, \\
    r_2 &= \frac{1 + g}{\lambda}, \\
    r_3 &= \frac{1 + g'}{\lambda}.
\end{align*}
$$

One gets

$$
\begin{align*}
    \beta_\lambda &= \frac{\lambda^2}{\pi}, \\
    \beta_g &= \beta_{g'} = 0.
\end{align*}
$$

(67)

For the remaining models in this class the ricci is not diagonal, therefore we conclude to the non-renormalizability of the remaining models with three independent couplings.

However, if we restrict ourselves to two couplings, tuned in such a way to have $ric_{23} = 0$, most of the class B models become renormalizable:

$$
\begin{align*}
    \text{Bianchi III} & \quad r_1 = \frac{1}{\lambda}, \quad r_2 = \frac{1 + g}{\lambda}, \quad r_3 = \frac{-1 + g}{\lambda}, \\
    \text{Bianchi VI$_{a}(\sigma = -1)$, VII$_{a}(\sigma = +1)$} & \quad r_1 = \frac{1}{\lambda}, \quad r_2 = \frac{1 + g}{\lambda}, \quad r_3 = \sigma\frac{1 + g}{\lambda}.
\end{align*}
$$

1In the $g = g' = 0$ limit we recover the bi-invariant metrics.
Their beta functions are
\[ \beta_\lambda = \frac{\lambda^2}{\pi} a^2, \quad \beta_g = 0. \]  \hfill (68)

For Bianchi IV no choice of diagonal breaking matrix leads to renormalizability.

### 6.3 Class B dual models and dilaton anomaly

Let us first get a convenient characterization of the absence of the dilaton anomaly. Using relation (28) one has the equivalence

\[ V_i = -2G_{it} f^s_{st} = \partial_i \Phi \iff \partial_i V_j - \partial_j V_i = 0 \iff G_{su} f^v_{vu} (f^t_{st} G_{jt} - f^j_{st} G_{it}) = 0. \]

Upon multiplication by \( \Gamma^a_{ai} \Gamma^b_{bj} \) and use of (15), (38) one gets

\[ G_{su} f^v_{vu} (f^t_{sb} \Gamma^a_{at} - f^t_{ta} \Gamma^b_{bt}) = 0, \to \omega_{ab,s} G_{st} f^u_{ut} = 0. \]

It follows that the equivalence becomes

\[ V_i = -2G_{it} f^s_{st} = \partial_i \Phi \iff \omega_{ab,s} V_s = 0, \forall a, b. \]  \hfill (69)

Despite the convenient form of the final relation (69), it is fairly difficult to discuss in general. Let us simply observe that the matrices \( \omega_a \), with matrix elements defined by \( (\omega_a)^{bs} = \omega_{ab,s} \) are singular. So the analysis of (69) depends strongly on the size of the kernel of the \( \omega_a \) and therefore of the algebra and of the breaking matrix considered.

To discuss this point for the class B of the Bianchi family, we will consider the most general breaking matrix \( B \) and we denote its off-diagonal terms by

\[ B_{12} = s_3, \quad B_{23} = s_1, \quad B_{31} = s_2, \quad \det B = r_1 r_2 r_3 - r_1 s_1^2 - r_2 s_2^2 - r_3 s_3^2 + 2 s_1 s_2 s_3 \neq 0. \]

Let us notice that this last condition forbids the simultaneous vanishing of \( s_1, r_2 \) and \( r_3 \).

The matrices \( \omega_a \) are given generally by

\[ (\omega_l)_{jk} = \omega_{ij,k} = -\frac{\nu}{2} \epsilon_{ijk} + \sum_n n_s B_{sk} \epsilon_{sij} + a_i B_{jk} - a_j B_{ik}, \quad a_i = a \delta_{i1}, \quad \nu = \sum_n n_s B_{ss}. \]

For class B we have \( \nu = n_2 r_2 + n_3 r_3 \). Taking into account the relations

\[ G_{11} = \frac{(r_2 r_3 - s_1^2)}{\det \Gamma}, \quad G_{21} = -\frac{(r_3 s_3 - s_1 s_2 + s_1 y + r_3 z)}{\det \Gamma}, \quad G_{31} = \frac{(s_3 s_1 - r_2 s_2 + r_2 y + s_1 z)}{\det \Gamma}, \]

\[ \det \Gamma = \det B + r_2 y^2 + r_3 z^2 + 2 s_1 y z, \]

it is a purely algebraic matter, using (69), to prove that the dilaton anomaly is absent iff

\[ \nu = n_2 r_2 + n_3 r_3 = 0 \quad \text{and} \quad \mu = s_1^2 - r_2 r_3 = 0. \]  \hfill (70)

These constraints show that Bianchi VII\( _a \) is always anomalous, but also that an appropriate choice of the couplings can get rid of the anomaly in the other models!

One can summarize the constraints (70) for the class B models and their possibly non-vanishing ricci component:
The geometry of dualized principal chiral models

| model        | constraint \((\epsilon = \pm 1)\) | \(ric_{11}\)           | \(\det B \neq 0\) |
|--------------|------------------------------------|------------------------|-------------------|
| Bianchi III  | \(r_3 = r_2, \ s_1 = \epsilon r_2\) | \(2(\epsilon - 1)\)  | \(r_2(s_2 - \epsilon s_3) \neq 0\) |
| Bianchi IV   | \(r_3 = 0, s_1 = 0\)               | 0                      | \(r_2 \cdot s_2 \neq 0\) |
| Bianchi V    | \(s_1 = \epsilon \sqrt{r_2r_3}\) | 0                      | \(\sqrt{|r_2|s_2 - \epsilon \sqrt{|r_3|s_3} \neq 0\) |
| Bianchi VIa  \((a \neq 1)\) | \(r_3 = r_2, s_1 = \epsilon r_2\) | \(2(a\epsilon - 1)\) | \(r_2(s_2 - \epsilon s_3) \neq 0\) |
| Bianchi VIIa | impossible                          |                        |                   |

It follows that the models B IV, B V and B III with \(\epsilon = +1\) are flat.

We want to show that the restrictions (70) are equivalent to the vanishing of the torsion. To see this we use the constraints (34), which give, when specialized to class B:

\[
3 f_{[ij,k]} = \nu, \quad 3 (A \cdot \psi B^{-1} A \cdot \psi)_{s[k} f_{i]s} = \mu \frac{(a^2 + n_2n_3)(n_2(\psi^2)^2 + n_3(\psi^3)^2)}{\det B}.
\]

In this case it is interesting to compare the vectors \(V_i = -2G_{st} f_{st}^a\) and \(g_i = D_i \ln(\sqrt{\det g})\). One can check that the difference \(V_i - 2g_i\) is then covariantly constant, giving for final geometry

\[
Ric_{ij} = -G_{is} ric_{st} G_{tj} + D_j D_i \ln(\sqrt{\det g}).
\]

7 Dualized Bianchi V model at two loops

As observed in the previous sections, dualized models may be torsionless: it is therefore important to ascertain which models lead to this phenomenon. To this end we use the constraints (34). Algebraic computations lead to the following conclusions:

1. For the class A models, no choice of the non-singular matrix \(B\) leads to vanishing torsion.

2. For the class B models, except Bianchi V, the necessary and sufficient conditions for vanishing torsion are given by the relations (70).

3. Among all the class B models only Bianchi V has a vanishing torsion for an arbitrary breaking matrix \(B\). In this case the torsion potential is an exact 2-form with

\[
H = \frac{1}{2} dA, \quad \nu_2 = \frac{s_3s_1 - r_2s_2}{s_1^2 - r_2r_3}, \quad \nu_3 = \frac{s_1s_2 - r_3s_3}{s_1^2 - r_2r_3},
\]

\[
A = \gamma \left( d\psi^1 - \nu_3 d\psi^2 - \nu_2 d\psi^3 \right), \quad \gamma = \ln(\sqrt{\det g}).
\]

The case where \(s_1^2 = r_2r_3 \neq 0\) is special, with

\[
A = \gamma d\psi^1 + \frac{1}{r_2} \left( s_3 \ln |x^2 - \alpha^2| - \ln \left| \frac{x + \alpha}{x - \alpha} \right| \cdot \psi^2 \right) d\psi^2, \quad x = r_2\psi^3 - s_1\psi^2, \quad \alpha = s_1s_3 - r_2s_2.
\]

It follows that the dual model, at least perturbatively, can be analyzed as if it had no WZW coupling! This situation is fairly original: the principal Bianchi V model, which is homogeneous and torsionless, is mapped by T-duality to an inhomogeneous but still torsionless \(\sigma\)-model. It is therefore attractive to check the two-loop equivalence of the models using the firmly established
counterterms given by Friedan \cite{17}. Let us consider the simplest Bianchi V dual model, with $B_{ij} = r \delta_{ij}$. Its dualized metric, taken from (62), reads:

$$g = \frac{r}{\Delta}(d\psi^1)^2 + \frac{1}{\Delta} \left[ r^2(d\psi^2)^2 + r^2(d\psi^3)^2 + (\psi^3 d\psi^2 - \psi^2 d\psi^3)^2 \right], \quad \Delta = (\psi^2)^2 + (\psi^3)^2 + r^2. \quad (71)$$

Following \cite{13} we take for new coordinates

$$\psi^1 = z, \quad \psi^2 + i\psi^3 = \rho e^{i\phi}, \quad \Rightarrow \quad g = \frac{r}{\rho^2 + r^2}(dz^2 + d\rho^2) + \frac{\rho^2}{r} (d\phi)^2, \quad (72)$$

which bring the metric to a simple diagonal form, with the obvious vielbein

$$g = \sum_{a=1}^{3} e_a^2, \quad e_1 = \frac{\sqrt{r}}{\sqrt{\rho^2 + r^2}} dz, \quad e_2 = \frac{\sqrt{r}}{\sqrt{\rho^2 + r^2}} d\rho, \quad e_3 = \frac{\rho}{\sqrt{r}} d\phi. \quad (73)$$

One can prove that this metric has two isometries, described by the vector fields $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \phi}$.

The geometrical quantities of interest are

$$\begin{cases}
\omega_{23} = -\frac{1}{\sqrt{r}} \frac{\sqrt{\rho^2 + r^2}}{\rho} e_3, & \omega_{12} = -\frac{1}{\sqrt{r}} \frac{\rho}{\sqrt{\rho^2 + r^2}} e_1, \\
R_{23} = \frac{1}{r} e_2 \wedge e_3, & R_{31} = \frac{1}{r} e_3 \wedge e_1, & R_{12} = -\frac{\sigma}{r} e_1 \wedge e_2, & \sigma = \frac{\rho^2 - r^2}{\rho^2 + r^2}, \\
Ric_{11} = \frac{1 - \sigma}{r}, & Ric_{22} = \frac{1 + \sigma}{r}, & Ric_{33} = 0, \\
R = Ric_{ss} = -2 \frac{\sigma}{r}.
\end{cases} \quad (74)$$

The one-loop renormalizability relations

$$Ric_{ij} = \chi^{(1)} \frac{\partial}{\partial r} g_{ij} + \nabla_i (v_j),$$

become, using vielbein components

$$\begin{cases}
Ric_{ab} = \chi^{(1)} \left( (e^{-1})_b^j \frac{\partial}{\partial r} e_{aj} + (e^{-1})_a^j \frac{\partial}{\partial r} e_{bj} \right) + D(a v_b), \\
D_a v_b = \hat{\partial}_a v_b + \omega_{bs,a} v_s, \quad \hat{\partial}_a = (e^{-1})_a^j \partial_j.
\end{cases} \quad (75)$$

Relation (75) works with

$$\chi^{(1)} = -2, \quad v \equiv v_a e_a = -\frac{2}{\sqrt{r}} \frac{\rho}{\sqrt{\rho^2 + r^2}} d\rho. \quad (76)$$

Let us remark that while $\chi^{(1)}$ is uniquely defined, the vector $v_i$ is not unique and we took its simplest form. As it should, the renormalization of the coupling constant $r$ is the same as in the principal model as can be seen from relation (64).
The geometry of dualized principal chiral models

The two-loops counterterms, first computed by Friedan [17], are

$$\frac{1}{16\pi^2\epsilon} \int d^2x \ R_{is, tu} R_{js, tu} \eta^{\mu\nu} J_i^j J_j^\nu,$$

where the $R_{is, tu}$ are the vielbein components of the Riemann tensor.

For three dimensional geometries, this counterterm is most easily obtained from the identity

$$(RR)_{ab} \equiv \frac{1}{2} R_{as, tu} R_{bs, tu} = R Ric_{ab} - (Ric^2)_{ab} + \left(\text{Tr} \ (Ric^2) - \frac{R^2}{2}\right) \delta_{ab},$$

which gives

$$(RR)_{11} = (RR)_{22} = \frac{1}{r^2}(1 + \sigma^2), \quad (RR)_{33} = \frac{2}{r^2}.\quad (77)$$

In order to prove renormalizability we have to solve for $\chi^{(2)}$ and $\rho$ such that

$$(RR)_{ab} = \chi^{(2)} \left( (e^{-1})^{ij}_b \partial_{\rho} e_{aj} + (e^{-1})^{ij}_a \partial_{\rho} e_{bj} \right) + D_{(a}w_{b)}.\quad (78)$$

Explicitly, these equations give the differential system

$$\begin{cases}
\frac{1}{r^2} (1 + \sigma^2) - \chi^{(2)} \frac{\sigma}{r} = \hat{\partial}_1 w_1 + \omega_{12,1} w_2, & 0 = \hat{\partial}_1 w_2 + \hat{\partial}_2 w_1 - \omega_{12,1} w_1, \\
\frac{1}{r^2} (1 + \sigma^2) - \chi^{(2)} \frac{\sigma}{r} = \hat{\partial}_2 w_2, & 0 = \hat{\partial}_3 w_1 + \hat{\partial}_1 w_3, \\
\frac{2}{r^2} + \chi^{(2)} \frac{1}{r} = \hat{\partial}_3 w_3 - \omega_{23,3} w_2, & 0 = \hat{\partial}_3 w_2 + \hat{\partial}_2 w_3 + \omega_{23,3} w_3. \quad (79)
\end{cases}$$

Integrating some relations with respect to the variable $\rho$ we obtain

$$\begin{cases}
w_1 = -w_{1}^{(0)}(\rho) \partial_{z} W_2(z, \phi) + \frac{W_1(z, \phi)}{\sqrt{\rho^2 + r^2}}, & \frac{d}{d\rho} \left( \sqrt{\rho^2 + r^2} w_{1}^{(0)}(\rho) \right) = \sqrt{\rho^2 + r^2}, \\
w_2 = w_{2}^{(0)}(\rho) + W_2(z, \phi), & \frac{d}{d\rho} w_{2}^{(0)}(\rho) = \frac{\sqrt{r}}{\sqrt{\rho^2 + r^2}} \left( \frac{1 + \sigma^2}{r^2} - \chi^{(2)} \frac{\sigma}{r} \right), \\
w_3 = -\sqrt{\rho^2 + r^2} \partial_{\phi} W_2(z, \phi) + \rho W_3(z, \phi). \quad (80)
\end{cases}$$

Inserting these relations into the last left relation of (79) one has

$$\frac{2}{r^2} + \chi^{(2)} \frac{1}{r} - \sqrt{r} \partial_{\phi} W_3 = M, \quad W_2 - \partial_{\phi}^2 W_2 = N, \quad w_{2}^{(0)}(\rho) + N = \sqrt{r} \frac{\rho}{\sqrt{\rho^2 + r^2}} M. \quad (81)$$

where $M$ and $N$ are coordinate independent. Differentiating this last relation with respect to $\rho$ yields a constraint which does not hold, irrespectively of the values taken for $M$ and $\chi^{(2)}$.

The failure of relations (78) means that the two-loops quantum extension chosen for the dual model does not lift the classical equivalence to the quantum level.

In fact we should consider $g_{ij}$ the whole family of metrics

$$g_{ij} \rightarrow g_{ij} + \gamma_{ij},$$

We thank G. Bonneau for suggesting to us this idea.
where $\gamma_{ij}$ is a one-loop deformation of the classical metric $g_{ij}$ which describes different possible quantum extensions of the same classical dual model. For this modified theory we get an extra contribution at the two-loops level which is

$$\frac{1}{4\pi\epsilon} \int d^2x \left[ Ric_{ij}(g + \gamma) - Ric_{ij}(g) \right] \eta^{\mu\nu} J^i_{\mu} J^j_{\nu}.$$ 

Let us examine whether the two-loops renormalizability can be implemented or not. As is well known, one has

$$Ric_{ij}(g + \gamma) - Ric_{ij}(g) = -\frac{1}{2} \Delta_L \gamma_{ij} + \nabla(\alpha_j), \quad \alpha_i = \nabla^s \gamma_{si} - \frac{1}{2} \nabla^i \gamma^s_s,$$

where $\Delta_L$ is Lichnerowicz’s laplacian

$$\Delta_L \gamma_{ij} = \nabla^s \nabla_s \gamma_{ij} + 2R_{is,jt} \gamma_{st} - Ric_{is} \gamma_{sj} - Ric_{js} \gamma_{si}.$$ 

The connection $\nabla$, the Riemann tensor and the raising or lowering of indices are related to the unperturbed metric $g$.

Up to a scaling of $\gamma$, the two-loops renormalizability constraints become

$$(RR)_{ij} = \Delta_L \gamma_{ij} + \chi^{(2)} \frac{\partial}{\partial r} g_{ij} + \nabla(iw_j).$$  \hspace{1cm} (82)

We will exhibit a solution of these equations for any choice of $\chi^{(2)}$. For this we consider the vielbein components of the deformation

$$\gamma = \frac{1}{r} \left( \gamma_1 e_1^2 + \gamma_2 e_2^2 + \gamma_3 e_3^2 \right),$$

and we use the notations

$$\chi^{(2)} = \frac{2}{r}(1 - \chi), \quad x = \frac{\rho^2}{\rho^2 + r^2} \in [0, 1[, \quad \Phi(x) = \frac{3 + \chi}{2(1 - x)^2} \int_0^x \frac{\ln(1 - u)}{u} du.$$

One should notice that for the principal Bianchi V model at two-loops we have $\chi = 0$.

Let us define

$$\left\{ \begin{array}{l}
\gamma_1(x) = -\frac{1 + x}{1 - x} \gamma_2(0) - \frac{4(4 + 3\chi)x - (5\chi - 1)x^2}{8(1 - x)^2} - \frac{3 + \chi - x}{2(1 - x)} \ln(1 - x) + \Phi(x), \\
\gamma_2(x) = \gamma_2(0) + \frac{4(11 + 4\chi)x + (11 + 5\chi)x^2 - 4x^3}{8(1 - x)^2} \\
\gamma_3(x) = -\gamma_1(x).
\end{array} \right.$$ 

The vector vielbein components are

$$w_1 = w_3 = 0, \quad r^{3/2} \sqrt{x} w_2 = 4(1 - \chi)x - 2x^2 - (2 + \chi) \ln(1 - x) - 2x(1 - x) \frac{d}{dx} \gamma_2(x).$$

The reader can check that the deformation and the vector given above are indeed solution of (82) for any value of $\chi$. 

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Two main points need to be checked. The first one is the analyticity of the $\gamma_i(x)$ in a neighbourhood of $x = 0$. This follows from the analyticity of $\Phi(x)$ and is explicit on the other terms. The second point is that we are using polar coordinates; in order to secure an analytic dependence with respect to the cartesian coordinates $\psi^1, \psi^2 \approx 0$ we have imposed $\gamma_3(0) = \gamma_2(0)$. The free parameters in this solution are $\gamma_2(0)$ and $\chi$.

As a side remark, let us observe that the deformation obtained above, cannot be written in the form

$$\gamma_{ij} = A \frac{\partial}{\partial r} g_{ij} + D_i \mathcal{W}_j,$$

which means that it cannot be interpreted as a finite renormalization of the initial metric $g_{ij}$.

So we can conclude that it is always possible to have a quantum extension of the dualized Bianchi V which does preserve the two-loops renormalizability. Unfortunately nothing, in this process, enforces $\chi$ to have the same value as in the principal model we started from. This shows that further constraints are needed to define uniquely the two-loops quantum dual theory.

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