QUANTUM AND CLASSICAL MESSAGE IDENTIFICATION VIA QUANTUM CHANNELS

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(Dedicated to Alexander S. Holevo on his 60th birthday)

We discuss concepts of message identification in the sense of Ahlswede and Dueck via general quantum channels, extending investigations for classical channels, initial work for classical–quantum (cq) channels and “quantum fingerprinting”. We show that the identification capacity of a discrete memoryless quantum channel for classical information can be larger than that for transmission; this is in contrast to all previously considered models, where it turns out to equal the common randomness capacity (equals transmission capacity in our case): in particular, for a noiseless qubit, we show the identification capacity to be 2, while transmission and common randomness capacity are 1. Then we turn to a natural concept of identification of quantum messages (i.e. a notion of “fingerprint” for quantum states). This is much closer to quantum information transmission than its classical counterpart (for one thing, the code length grows only exponentially, compared to double exponentially for classical identification). Indeed, we show how the problem exhibits a nice connection to visible quantum coding. Astonishingly, for the noiseless qubit channel this capacity turns out to be 2: in other words, one can compress two qubits into one and this is optimal. In general however, we conjecture quantum identification capacity to be different from classical identification capacity.

1 Introduction

The theory of identification of messages via (noisy) channels was initiated by Ahlswede and Dueck and has resulted in interesting developments: it has a strong connection to the theory of common randomness, and it has triggered the theory of “approximation of output statistics”. Put briefly, whereas in Shannon’s famed theory of communication, the object is to transmit a message over a channel, described by a stochastic map $W: X \rightarrow Y$, by block coding of length $n$, for identification the receiver should only be able to answer questions “Is the message $i$ sent equal to one $j$ that I have in mind?”. It turns out that this achieves much larger codes than transmission: while the latter has an exponentially growing (in $n$) optimal message set, the former allows for double exponential growth. Rabin and Yao — see also Kushilevitz and Nisan, Example 3.6 and the Bibliographic Notes — have shown that the randomised communication complexity of the equality function of $n$–bit strings is $O(\log n)$ bits. The achievement of Ahlswede and Dueck was to determine the constant in the exponent: it is the Shannon capacity of the channel — but can be larger in models with feedback. However, in
this paper we will only consider discrete memoryless (classical and quantum) channels with no feedback or other helpers, and with an i.i.d. time structure.

Formally, consider a discrete memoryless quantum channel

\[ T : \mathcal{A}_1 \rightarrow \mathcal{A}_2, \]

modelled as a completely positive, trace preserving map between C*-algebras \(\mathcal{A}_1, \mathcal{A}_2\) which in this paper are always finite dimensional. It is well–known that a finite C*-algebra \(A\) is isomorphic to a direct sum of full matrix algebras, and we can think of it as the subalgebra of the operator algebra \(\mathcal{B}(\mathcal{H})\) on a finite dimensional Hilbert space \(\mathcal{H}\), of operators commuting with some selfadjoint operator \(A\):

\[ A = \{ X : XA = AX \} = \bigoplus_{i=1}^{r} \mathcal{B}(\mathcal{H}_i), \]

with the eigenspaces \(\mathcal{H}_i\) of \(A\). The states on \(A\) we identify with the semidefinite operators in \(A\) of trace 1, the set of which we denote \(\mathcal{S}(A)\). Following Holevo\[19\] we call \(T\) a cq–channel if \(\mathcal{A}_1\) is commutative (=classical), and a qc–channel if \(\mathcal{A}_2\) is commutative: in the first case the action of \(T\) is determined by its images \(W_x = T(x)\) on the (finitely many) minimal idempotents \(x\), in the second case this applies to the adjoint map \(T^* : \mathcal{A}_2 \rightarrow \mathcal{A}_1\), and the images \(M_y = T^*(y)\) of the minimal idempotents \(y\) form a positive operator valued measure (POVM). The channel is classical if both \(\mathcal{A}_1\) and \(\mathcal{A}_2\) are commutative.

We shall consider block coding for \(n\) copies of \(T\): an \((n, \lambda)\)-code for this channel is a collection \(\{ (\pi_i, D_i) : i = 1, \ldots, M \}\) of (without loss of generality: pure) states \(\pi_i\) on \(\mathcal{A}_1^\otimes n\), and positive operators \(D_i \in \mathcal{A}_2^\otimes n\) which sum to \(\mathbb{1}\) (i.e., a POVM), such that

\[ \forall i \quad \text{Tr}(T^\otimes n(\pi_i)D_i) \geq 1 - \lambda. \] (1)

The maximum such \(M\) will be denoted \(M(n, \lambda)\). If all the \(\pi_i\) are product states with respect to \(\mathcal{A}_1^\otimes n\), the code is called separable, and the corresponding maximal \(M\) is denoted \(M_1(n, \lambda)\). The knowledge about these quantities is summarised in the HSW–theorem (\(H(\rho) = -\text{Tr}\rho \log \rho\) is the von Neumann entropy):

**Theorem 1** (Holevo\[18\], Schumacher, Westmoreland\[32\], Ogawa, Nagaoka\[29\], Winter\[35\],\[36\]) For all \(0 < \lambda < 1\), one has the coding theorem and strong converse,

\[ \lim_{n \to \infty} \frac{1}{n} \log M_1(n, \lambda) = \chi(T), \]

with the Holevo capacity of the channel

\[ \chi(T) = \max_{(p_i, \pi_i)} \left\{ \left[ H\left( \sum_i p_i T(\pi_i) \right) - \sum_i p_i H(T(\pi_i)) \right] \right\}. \]
For general codings, the capacity and the weak converse are given by

\[
C(T) = \inf_{\lambda > 0} \liminf_{n \to \infty} \frac{1}{n} \log M(n, \lambda)
\]

\[
= \inf_{\lambda > 0} \limsup_{n \to \infty} \frac{1}{n} \log M(n, \lambda) = \lim_{n \to \infty} \frac{1}{n} \chi(T^\otimes n).
\]

(Here and elsewhere in the paper log and exp are to basis 2.)

It is widely conjectured that \(\chi\) is additive with respect to tensor products, which would imply \(C(T) = \chi(T)\). Furthermore, it is conjectured that the strong converse also holds for \(M(n, \lambda)\) (which is known for cq–channels since there \(M(n, \lambda) = M_1(n, \lambda)\)).

Following Lőber and generalising the classical definition an \((n, \lambda_1, \lambda_2)–ID\) code for \(T\) is a collection of pairs \(\{ (\rho_i, D_i) : i = 1, \ldots, N \}\) of states \(\rho_i\) on \(A_1^\otimes n\) and operators \(0 \leq D_i \leq 1\) in \(A_2^\otimes n\), such that

\[
\forall i \quad \text{Tr}(T^\otimes n(\rho_i)D_i) \geq 1 - \lambda_1, \\
\forall i \neq j \quad \text{Tr}(T^\otimes n(\rho_i)D_j) \leq \lambda_2.
\]

The bounds \(\lambda_1 \) and \(\lambda_2\) are called error probabilities of first and second kind, respectively, since the problem is a coding variant of hypothesis testing. For the identity channel \(\text{id}_A\) of a system \(A\) (Kuperberg calls this a hybrid quantum memory), we call an ID code also a code on the algebra \(A^\otimes n\), or more generally on an algebra \(\tilde{A}\) if the \(\rho_i\) and \(D_i\) are elements of \(\tilde{A}\).

The code is called simultaneous if the binary observables (POVMs) \((D_i, 1 - D_i)\) are all co–existent, in the sense of Ludwig. If there exists a POVM \((E_k)_{k=1,...,K}\) such that for all \(i\) a set \(D_i \subset \{1, \ldots, K\}\) can be found with

\[
D_i = \sum_{k \in D_i} E_k.
\]

Denote the maximum \(N\) such that an \((n, \lambda_1, \lambda_2)–ID\) code (a simultaneous \((n, \lambda_1, \lambda_2)–ID\) code) of length \(N\) exists by \(N(n, \lambda_1, \lambda_2)\) (\(N_{\text{sim}}(n, \lambda_1, \lambda_2)\)). Clearly, \(N_{\text{sim}}(n, \lambda_1, \lambda_2) \leq N(n, \lambda_1, \lambda_2)\).

**Theorem 2 (Lőber)*** For all \(\lambda_1, \lambda_2 > 0\),

\[
\liminf_{n \to \infty} \frac{1}{n} \log \log N_{\text{sim}}(n, \lambda_1, \lambda_2) \geq C(T),
\]

with the Holevo (transmission) capacity \(C(T)\) of the channel. If the channel \(T\) satisfies the strong converse and in addition the technical condition that on block length \(n\) its input states may be restricted to an alphabet of size \(2^{2^{o(n)}}\), then for \(0 < \lambda_1, \lambda_2, \lambda_1 + \lambda_2 < 1\),

\[
\lim_{n \to \infty} \frac{1}{n} \log \log N_{\text{sim}}(n, \lambda_1, \lambda_2) = C(T).
\]
The first part is proved by concatenating a transmission code for $T$ with the following construction:

**Proposition 3 (Ahlswede, Dueck)** Let $\mathcal{M}$ be a set of cardinality $M$, $\lambda > 0$ and $\epsilon$ such that $\lambda \log \left( \frac{1}{\epsilon} - 1 \right) > 2$. Then there exist $N \geq 2^{\epsilon M} / M$ subsets $\mathcal{M}_i \subset \mathcal{M}$ of cardinality $\lfloor \epsilon M \rfloor$, such that

$$\forall i \neq j \quad |\mathcal{M}_i \cap \mathcal{M}_j| \leq \lambda \lfloor \epsilon M \rfloor.$$ 

In other words, the pairs $\{(P_i, \mathcal{M}_i) : i = 1, \ldots, N\}$, with the uniform distribution $P_i$ on $\mathcal{M}_i$, form an ID code with error probability of first kind $0$, and of second kind $\lambda$.

The technical condition in theorem 2 is true for example for cq–channels ($2^{O(n)}$ input strings) but not for the ideal qubit channel $\text{id}_{C2}$: to approximate every input state on $n$ qubits requires $2^{2^{\text{const} \cdot n}}$ pure states. It is not known if the condition is necessary for the conclusion of the theorem but our construction following below in section 2 certainly violates it.

Regarding the necessity of the simultaneity condition, we have

**Theorem 4 (Ahlswede, Winter)** For a cq–channel $T$ and $0 < \lambda_1, \lambda_2$, $\lambda_1 + \lambda_2 < 1$,

$$\lim_{n \to \infty} \frac{1}{n} \log \log N(n, \lambda_1, \lambda_2) = C(T) = \chi(T).$$

The same holds for general channels and codes of separable states, as this restricts us effectively to a cq–channel.

This left open, however, the understanding of the precise role of the simultaneity condition in general. The present work will push this question further a bit: we determine the capacity to be 2 for the noiseless qubit channel (theorem 9), and give a formula for general hybrid quantum memories (corollary 13), based on a new construction to extend an identification code. We furthermore conjecture the capacity to be different for simultaneous identification: specifically, that it is 1 for the noiseless qubit.

While all this is concerned with identification of classical messages, the second part of the paper (section 3) will study concepts of quantum message identification. These can be related to visible codings and decodings for quantum channels. Roughly, quantum message identification aims at encoding the pure states $|\phi\rangle$ of a Hilbert space, such that a quantum mechanical test of “Is it $|\theta\rangle$?” may performed on the output: a binary measurement with distribution close to $\left( (|\theta\rangle |\phi\rangle|^2, 1 - |\theta\rangle |\phi\rangle|^2 \right)$. I.e., the fidelity test can be simulated, which has been argued to be the quantum analogue of the identity predicate. Our findings are in sharp contrast to the case of classical message identification: we do not find the doubly exponential growth of code length (here: dimension) with the block length, but only exponential growth, like in transmission. Still, we find a capacity different from the quantum transmission capacity: it is 2 for the noiseless qubit channel (theorem 19). We conclude with a discus-
sion of our results which puts them in their right context, and highlight open questions and conjectures.

2 Classical message identification

The results of Löber\cite{27} and of Ahlswede and the author\cite{5} seem to indicate that the (simultaneous and non–simultaneous) identification capacity of a memoryless quantum channel is equal to its transmission capacity.

We now show that for a general channel — in fact we may take a noiseless qubit — the (non–simultaneous) identification capacity can exceed the transmission capacity. We begin with a result about pure state ID codes:

**Proposition 5 (“Quantum fingerprinting”\cite{12})** On $B(C^d)$ there exists an ID code \{$(\psi_i, \psi_i) : i = 1, \ldots, N$\} of $N \geq 2^{[cd]/d}$ pure states $\psi_i = |\psi_i\rangle \langle \psi_i|$, with error of first kind 0 and of second kind $\lambda$, with $\lambda \log \left( \frac{1}{\epsilon} - 1 \right) > 4$.

**Proof.** We could simply take the construction of Buhrman et al.\cite{12} but we prefer to reduce it to proposition 3:

Take the probability distributions $P_i$ on $\{1, \ldots, d\}$ of an ID code \{$(P_i, M_j)$\} such as proposition 3 (with error probability of second kind $\lambda/2$) and define the states $|\psi_i\rangle = \sum_{k=1}^{d} \sqrt{P_i(k)} |k\rangle$. Note that defining $D_i = |\psi_i\rangle \langle \psi_i|$ will make the error probability of first kind equal to 0. For the error of second kind, note that it is (for $i \neq j$)

$$|\langle \psi_i | \psi_j \rangle|^2 = \left( \sum_k \sqrt{P_i(k)P_j(k)} \right)^2.$$ 

Using well–known relations between distinguishability measures of distributions (see e.g. Fuchs and van de Graaf\cite{15}) this is upper bounded by $\lambda$.  

Using mixed states, one can increase $N$ dramatically:

**Proposition 6** For every $0 < \lambda < 1$ and $\epsilon > 0$ such that $\lambda \log \left( \frac{1}{\epsilon} - 1 \right) > 8$, there exists on the quantum system $B(C^d)$ an ID code with

$$N \geq \frac{1}{K(\lambda)d^2} \exp \left( \epsilon K(\lambda)d^2 \right)$$

messages, with error probability of first kind equal to 0 and error probability of second kind bounded by $\lambda$. The constant $K(\lambda)$ may be chosen

$$K(\lambda) = \frac{(\lambda/100)^4}{4 \log(100/\lambda)}.$$ 

**Proof.** We concatenate quantum fingerprinting, proposition 3 with the main result of the next section, proposition 17 according to it one can encode the pure states on $\mathbb{C}^S$, $S = [K(\lambda)d^2]$, into $B(C^d)$ such that the measurements $(\pi, 1 - \pi)$ can be implemented with accuracy $\lambda/2$ in the individual output probabilities. We simply encode the states from a pure state ID code with
error of second kind $\lambda/2$. Looking at the proof of proposition 17 we see that we can assume that the fingerprinting states are part of the net of states (see the following lemma) on which the error of first kind does not increase — it stays 0.

Using the following lemma we can show that the cardinalities of the codes in propositions 5 and 6 are asymptotically optimal.

**Lemma 7 (Bennett et al. lemma 4)** For $\epsilon > 0$, there is a set $\mathcal{M}$ of pure states of pure states in $d$-dimensional Hilbert space with $|\mathcal{M}| \leq (5/\epsilon)^{2d}$, such that for all pure states $| \varphi \rangle$ there is $| \tilde{\varphi} \rangle \in \mathcal{M}$ with $\| | \varphi \rangle \langle \varphi | - | \tilde{\varphi} \rangle \langle \tilde{\varphi} | \|_1 \leq \epsilon$. (We call such a set an $\epsilon$-net.)

**Proposition 8** Let $\{(\rho_i, D_i) : i = 1, \ldots, N\}$ be an ID code on $B(C^d)$ with error probabilities $\lambda_1, \lambda_2$ of first and second kind, respectively, with $\lambda_1 + \lambda_2 < 1$. Then,

if all $\rho_i$ are pure, $\quad N \leq \left( \frac{5}{1 - \lambda_1 - \lambda_2} \right)^{2d}$,

for general $\rho_i, \quad N \leq \left( \frac{5}{1 - \lambda_1 - \lambda_2} \right)^{2d^2}$.

**Proof.** The key insight is that for $i \neq j$, $\frac{1}{2} \| \rho_i - \rho_j \|_1 \geq 1 - \lambda_1 - \lambda_2$, because we have an ID code.

In the first case of pure $\rho_i$, fix an $\epsilon$-net $\mathcal{M}$ in the pure states, with $\epsilon < 1 - \lambda_1 - \lambda_2$, according to lemma 7. This net decomposes the set of pure states into (Voronoi) cells of radius $\leq \epsilon$, so no two $\rho_i$ can be in the same cell. Hence $N \leq \left( \frac{2}{\epsilon} \right)^{2d}$, and as $\epsilon$ was arbitrary we obtain the first upper bound.

For the general case pick an $\epsilon$-net in $C^d \otimes C^d$, according to lemma 7. Since every state on $C^d$ has a purification on $C^d \otimes C^d$ and the trace distance is monotonic under partial traces, we obtain an $\epsilon$-net in $S(C^d)$ of cardinality $\left( \frac{2}{\epsilon} \right)^{2d^2}$. From here we argue as in the pure state case. \square

We can now determine the identification capacity of the noiseless qubit:

**Theorem 9** For the ideal qubit channel $id_{C^2}$ and $0 < \lambda_1, \lambda_2, \lambda_1 + \lambda_2 < 1$,

$$\lim_{n \to \infty} \frac{1}{n} \log \log N(n, \lambda_1, \lambda_2) = C_{ID}(id_{C^2}) = 2.$$ 

**Proof.** Immediate by putting together propositions 5 and 6 for $d = 2^n$. \square

This shows that for a general channel $T$, the (classical message) identification capacity is at least the double of its quantum transmission capacity $Q(T)$ (i.e., the capacity of the channel to transmit quantum information). Of course, by the general construction from proposition 3 it is also at least as large as its classical capacity $C(T)$. In fact, we can show a bit more, with the help of the following generalisation of the constructions of Ahlswede and Dueck.
Proposition 10 Let \( \{ (\rho_i, D_i) : i = 1, \ldots, N \} \) be an ID code on \( \mathcal{A} \) with error probabilities \( \lambda_1, \lambda_2 \) of first and second kind, respectively, and let \( \mathcal{C} \) be a classical system (commutative algebra) of dimension \( M \). Then, for \( \epsilon > 0 \), there exists an ID code \( \{(\sigma_f, \tilde{D}_f) : f = 1, \ldots, N'\} \) on \( \mathcal{A} \otimes \mathcal{C} \) with error probabilities \( \lambda_1, \lambda_2 + \epsilon \) of first and second kind, respectively, and \( N' \geq \left( \frac{1}{2}N' \right)^M \).

Proof. Denote the minimal idempotents of \( \mathcal{C} \) as \([k], k = 1, \ldots, M\). The new code elements and decoding operators will be constructed iteratively via (random) functions \( f : \{1, \ldots, M\} \to \{1, \ldots, N\} \):

\[
\sigma_f = \frac{1}{M} \sum_k \rho_{f(k)} \otimes [k], \quad \tilde{D}_f = \sum_k D_{f(k)} \otimes [k].
\]

Clearly, the error probability of first type is \( \lambda_1 \). Let a maximal code of this sort, with error probability of second kind \( \lambda_2 + \epsilon \), be already constructed, with functions \( f_1, \ldots, f_{N'} \) as above. Pick the new function \( f \) randomly, i.e. all values \( f(k) \) are uniform on \( \{1, \ldots, N\} \) and independent, and construct \( \sigma_f \) and \( \tilde{D}_f \) according to our prescription.

It is easy to check that the error probability of second kind is bounded as

\[
\text{Tr}(\sigma_f \tilde{D}_f) > \epsilon \leq \exp(-M D(\epsilon \| 1/N)) \leq \exp(-M (\epsilon \log N - 1)).
\]

Fixing \( j \) for the moment, we introduce the Bernoulli variables \( X_k \) with \( X_k = 1 \) if \( f(k) = f_j(k) \) and 0 otherwise. These are evidently independent and all have expectation \( 1/N \). By Sanov’s theorem \(^{13} \) (with the binary relative entropy \( D(p \| q) = p \log \frac{p}{q} + (1 - p) \log \frac{1 - p}{1 - q} \)),

\[
\Pr\left\{ \frac{1}{M} \sum_k X_k > \epsilon \right\} \leq \exp(-M D(\epsilon \| 1/N)) \leq \exp(-M (\epsilon \log N - 1)).
\]

With the union bound and using eq. \(^4\), we can bound the probability that one of the error probabilities of second kind exceeds \( \lambda_2 + \epsilon \), by

\[
N' \exp(-M (\epsilon \log N - 1));
\]

if this is less than 1 we can hence enlarge our code, contradicting the maximality assumption. This implies the lower bound on \( N' \).

\[\blacksquare\]

Remark 11 We can make a connection to the second construction of Ahlswede and Dueck \(^{14}\), based on common randomness, observing that the distribution on \( \{1, \ldots, M\} \) is always uniform; in other words, the code can be used with equal effect if the uniformly distributed \( k \in \{1, \ldots, M\} \) is known both to the sender and the receiver. Ahlswede and Dueck showed how to build an ID code of rate \( R \), using negligible communication, from rate \( R \) of common randomness; Proposition \(^{14}\) shows more generally how to increase the rate of any given ID code by \( R \).
This construction (see also how we use it in the proof of theorem 12 below) shows in generality that any additional classical capacity besides an ID code increases the identification capacity; hence, an optimal ID code for a channel does not allow for any remaining transmission rate.

In this connection we can ask the interesting question if the analogue to theorem 9 of the above-mentioned result by Ahlswede and Dueck holds with prior shared entanglement: do \( n \text{E} \) ebits and negligible communication yield an identification rate of \( 2 \text{E} \)?

**Theorem 12** If the channel \( T \) simultaneously transmits quantum information at rate \( Q \) and classical information at rate \( R \), then \( C_{ID}(T) \geq 2Q + R \).

The joint quantum–classical capacity region was recently determined by Devetak and Shor giving a lower bound of

\[
C_{ID}(T) \geq \lim_{n \to \infty} \frac{1}{n} \max_{\sigma^{XABn}} \left\{ 2I_c(A|BX) + I(X;B) \right\},
\]

where the maximisation is over all states \( \sigma^{XAB} = \sum_{x} p_x |x\rangle \langle x| \otimes \sigma_x^{AB} \), with \( \sigma_x^{AB} = (\text{id} \otimes T^{\otimes n}) \phi_x \) for some bipartite pure states \( \phi_x \). In the formula, \( I(X;B) = H(\sigma_B) - \sum_{x} p_x H(\sigma_{xB}^{P}) \) is the familiar Holevo quantity and \( I_c(A|BX) = H(\sigma^{X|B}) - H(\sigma^{XAB}) \) is the coherent information.

**Proof.** For sufficient block length \( n \), the channel can transmit a Hilbert space of dimension \( d = 2^{n(Q-\epsilon)} \) and simultaneously \( K = 2^{n(R-\epsilon)} \) messages (both with fidelity loss/error bounded by \( \epsilon \)).

Using an ID code for \( C_d \) according to proposition 6 and concatenating it with the construction of proposition 10 we obtain an \( (n,\lambda,\lambda) \)-ID code of \( N' \geq 2^{2nQ+R-\lambda} \) messages, with \( \lambda \to 0 \) as \( n \to \infty \) and \( \epsilon \to 0 \).

**Corollary 13** For a hybrid quantum memory \( \mathcal{A} = \bigoplus_{i=1}^{r} B(C_{d_i}) \),

\[
C_{ID}(\text{id}_{\mathcal{A}}) = \max_{(p_1, \ldots, p_r)} \left\{ 2 \sum_{i} p_i \log d_i + H(p) \right\} = \log \left( \sum_d d_i^2 \right).
\]

**Proof.** Denoting the maximum in the theorem by \( \Gamma \), even the strong converse holds: for \( \lambda_1 + \lambda_2 < 1 \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \log N(n, \lambda_1, \lambda_2) = \Gamma.
\]

First, the lower bound follows directly from theorem 12.

The strong converse is proved by building an \( \epsilon \)-net for \( S(A^{\otimes n}) \), and then arguing as in proposition 8 in detail, we first observe

\[
A^{\otimes n} \simeq \bigoplus_{P \text{ n-type}} B\left(C^{2^n \Sigma_i p_i \log d_i} \right) \otimes C_T(P),
\]

where the \( n \)-types are distributions on \( \{1, \ldots, r\} \) with probability values \( P_i \in \mathbb{N}/n \) and \( C_T(P) \) is a commutative algebra of dimension \( T(P) \leq 2^n H(P) \), the
number of sequences of type (i.e. empirical distribution) $P$. The states in this algebra are obtained from states on

$$B\left(\bigoplus_P C^{2^n \sum_i p_i \log d_i} \otimes C^{2^n \sum_i p_i \log d_i} \otimes C^T(P)\right)$$

by dephasing different $P$ (this is a conditional expectation), tracing out the first tensor factor in each product, and subjecting the third factor to a complete von Neumann measurement. All this can be described by a quantum channel $R : B(C^N) \rightarrow A \otimes n$, with $N \leq (n+1)^2 2^{n+1}$, since there are $\leq (n+1)^r$ many $n$--types. Now we can invoke lemma $H$ to construct an $\epsilon$--net for $S(C^N)$ which by virtue of $R$ gives an $\epsilon$--net for $S(A \otimes n)$.

**Remark 14** Now that we see that the identification capacity can be larger than the capacity for transmission, we may want to re–examine the significance of the condition of simultaneity: it effectively replaces the quantum channel by a classical one, with, on block length $n$, a doubly exponentially large net of input states, and classical output which is post processed for identification.

Nevertheless, the single measurement may have the effect of likening the simultaneous model to transmission, and in particular we conjecture that for the ideal qubit channel the capacity is only $1$: with $\lambda_1 + \lambda_2 < 1$,

$$C_{\text{sim-ID}}(\text{id}_{C^2}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \log N_{\text{sim}}(n, \lambda_1, \lambda_2) = 1.$$  

A justification for this may come from work on channel simulations$^{11}$ it can be shown that any measurement on $n$ qubits can, for all purposes of the output, be well approximated by a convex combination of measurements each of which has only $N = 2^{n+o(n)}$ many outcomes. The convex combination represents randomisation at the output and while in general an ID code cannot be derandomised at the output, the quantum situation may be sufficiently special to allow such a conclusion. If that works, we would be done, since the output probabilities of a measurement of $N$ outcomes permits a net of $2^O(N)$ points.

### 3 Quantum message identification

All the previous theory concerned identification of classical messages, though via quantum channels, and is thus analogous to classical information transmission via quantum channels.$^{21,12}$ But just as there is also a theory of coding quantum information for quantum channels (see the review by Bennett and Shor$^9$), we can discuss the possibilities of quantum message identification.

The following concepts appear rather natural: for the quantum channel $T : A_1 \rightarrow A_2$ we define an $(n, \lambda)$--quantum--ID code to be a pair $(\varepsilon, D)$ of maps, such that

$$\varepsilon : B(M) \rightarrow A_1^{\otimes n}$$

(5)
is a quantum channel and

\[ D : \mathcal{P}(\mathcal{M}) \rightarrow \mathcal{A}_2^\otimes n \]  

maps pure states \( \tau = |\theta\rangle\langle\theta| \) to operators \( 0 \leq D_\tau \leq \mathbb{1} \), with the properties that for all pure states \( \pi = |\phi\rangle\langle\phi| \), \( \tau = |\theta\rangle\langle\theta| \) on \( \mathcal{B}(\mathcal{M}) \):

\[
|\text{Tr}(\pi \tau) - \text{Tr}\left( \left(T^\otimes n \varepsilon\right)\pi \right) D_\tau| \leq \lambda/2.
\]  

This means: the encoding is such that for the pure states \( \pi \) and \( \tau \) a test (the binary POVM \( (D_\tau, \mathbb{1} - D_\tau) \)) may be performed on the output signal \( (T^\otimes n \varepsilon)\pi \) which is stochastically almost equivalent to the test \( (\tau, \mathbb{1} - \tau) \) on the original state \( \pi \). Why should we regard this as “quantum message identification”? The conception that we ought to make a (highly error–free) decision if the state \( \pi \) equals \( \tau \) we dismiss as unphysical: at the output we cannot expect higher distinguishability than at the input. For the input, however, the quantum mechanical version of the test “Is \( \pi \) equal to \( \tau \) or not?” has been argued to be just \( (\tau, \mathbb{1} - \tau) \). Also, restricted to a subset of mutually orthogonal (or almost orthogonal) states on \( \mathcal{M} \), i.e. for classical messages, the concept reduces to the classical message identification codes discussed in section 2.

Now define \( L(n, \lambda) \) to be the largest \( \text{dim} \mathcal{M} \) such that an \((n, \lambda)\)–quantum–ID code exists. For the identity channel on some system \( \tilde{A} \) we adopt the convention to speak of codes on \( \tilde{A} \), similar to the classical case; with en– and decoders \( \varepsilon : \mathcal{B}(\mathcal{M}) \rightarrow \tilde{A} \) and \( D : \mathcal{P}(\mathcal{M}) \rightarrow \tilde{A} \).

The above definition, eq. (7), should be compared with the usual definition of a quantum transmission code: there \( D \) in eq. (6) is (the restriction of) a completely positive, unit preserving map (the adjoint of the decoder channel \( \delta \))

\[
\delta^* : \mathcal{B}(\mathcal{M}) \rightarrow \mathcal{A}_2^\otimes n, \text{ such that } D_\tau = \delta^*(\tau).
\]  

Notice that in this case eq. (8) implies the familiar fidelity condition, that for all \( \pi \), \( \text{Tr}\left[ ((\delta T^\otimes n \varepsilon)\pi) \pi \right] \geq 1 - \lambda/2 \). On the other hand it is implied by

\[
\forall \pi \quad \left\| ((\delta T^\otimes n \varepsilon)\pi) - \pi \right\|_1 \leq \lambda.
\]  

In this sense our concept of quantum–ID code (eqs. (5), (6)) is a “visible decoder” version of quantum transmission: to be precise, we shall call it ID–visible. A decoder of the form eq. (8) we call blind. Noticing that a blind decoder allows to perform any measurement, not only the fidelity–test, on the input state (\( \delta^* \) translates POVMs into POVMs), we are motivated to also define (general) visible decoding as a map

\[
D : \text{POVM}(\mathcal{M}) \rightarrow \text{POVM}(\mathcal{A}_2^\otimes n)
\]

\[
\mathcal{M} = (M_y)_y \mapsto D(\mathcal{M}) = (M'_y)_y,
\]  

with the error criterion that for all \( \pi \in \mathcal{S}(\mathcal{M}) \),

\[
\sum_y \left| \text{Tr}(\pi M_y) - \text{Tr}\left[ ((T^\otimes n \varepsilon)\pi) M'_y \right] \right| \leq \lambda.
\]
Note that this is implied by eq. 9 and in turn implies eq. 7.

All this motivates us to also distinguish blind and visible encoders as well (see Barnum et al.7 and Bennett et al.8 for a discussion of these concepts in quantum source coding and remote state preparation): blind is our definition, eq. 6 above, visible is an encoder map

\[ E : \mathcal{P}(\mathcal{M}) \rightarrow \mathcal{S}(\mathcal{A}_1^{\otimes n}) \]  

from the pure states on \( \mathcal{M} \) into the state space of \( \mathcal{A}_1^{\otimes n} \).

The type of encoder/decoder we will indicate by a two–letter code, such as “bv” or “vV”: the first letter refers to the encoder (b: blind, v: visible), the second to the decoder (b: blind, V: (general) visible, v: ID–visible). Our original definition above is a bv–code, and this will be the default, unless otherwise stated.

We begin our investigation observing that all the \( L(n, \lambda) \)–functions exhibit only exponential growth:

**Proposition 15** For all \( 0 < \lambda < 1 \) and \( \epsilon, \mu > 0 \), such that \( \mu \log \left( \frac{1}{\epsilon} - 1 \right) > 4 \),

\[
1 + 2 \epsilon \log N(n, \lambda/2, \lambda/2 + \mu) \geq L_{vv}(n, \lambda) \geq L_{vV}(n, \lambda) \geq L_{vb}(n, \lambda) \\
\geq L_{bV}(n, \lambda) \geq L_{bb}(n, \lambda). 
\]

**Proof.** Only the first inequality is nontrivial. For it, simply concatenate a given \((n, \lambda)\)–quantum–ID code of maximal dimension \( L = L_{vv}(n, \lambda) \) with a pure state classical ID code with error probabilities of first and second kind \( 0 \) and \( \mu \), respectively: this gives an \((n, \lambda/2, \lambda/2 + \mu)\)–ID code of cardinality \( 2^{\epsilon L}/L \), and observing \( L_{vv}(n, \lambda) \leq N(n, \lambda/2, \lambda/2 + \mu) \) the claim follows. \( \Box \)

This means that the definition of the various capacities \( xy \in \{ vv, vV, vb, bv, bV, bb \} \) has to be

\[
Q_{xy}(T) := \inf_{\lambda > 0} \liminf_{n \to \infty} \frac{1}{n} \log L_{xy}(n, \lambda),
\]

like ordinary transmission capacity \( Q(T) \). \( Q_{ID}(T) \) will denote the default \( Q_{bv}(T) \). Note that we adopt the “pessimistic” notion of capacity: achievable rates have to be met for every sufficiently large block length \( n \). It is conceivable that the “optimistic” definition, where the rate has to be met only infinitely often (lim sup in the above formula) is larger, not to mention the question of the strong converse (allowing fixed positive \( \lambda \)).

**Remark 16** For every \( T \), \( Q_{bb}(T) = Q(T) \), the transmission capacity. This identity extends to \( Q_{vb} \) in some cases: \( Q_{vb}(\text{id}_{C_2}) = 1 \), for entanglement-breaking \( T \), \( Q_{vb}(T) = 0 = Q(T) \). Both statements are implied by a result in Bennett et al.8 Theorem 24 (Appendix C), which states that to transmit a state of \( n \) qubits visibly, using \( n - c \) qubits requires \( \Omega(2^n) \) classical bits: in the first case there is no classical channel available, so \( c = 0 \); in the second case the channel consists of a measurement followed by a state
preparation\cite{22} so it cannot be better than a classical channel of bounded capacity. This even gives the strong converse for these capacities ($\lambda < 1$).

For every $T$, $Q_{vv}(T) \leq C_{1D}(T)$: simply concatenate a $vv$–quantum–ID code with the construction of proposition\cite{23}.

For every $T$, $Q_{vV}(T) \leq C(T)$: for the states sent through the channels once could take an orthogonal basis of $\mathcal{M}$, and as the POVM the corresponding von Neumann measurement, which yields an $(n, \lambda)$–code for $T$. If $T$ is entanglement–breaking, then $Q_{bV}(T) = Q_{vV}(T) = 0$: as above the channel is not better than a classical channel of finite capacity, but for this we can invoke a result of Ambainis et al.,\cite{24} which implies that to classically simulate the statistics of arbitrary POVMs on arbitrary states of $n$ qubits requires transmission of $2^{\Omega(n)}$ bits.

The following result is the centre–piece of this paper:

**Proposition 17** For $0 < \lambda < 1$, there exists on $\mathcal{B}(\mathbb{C}^d)$ a quantum–ID code of error $\lambda$ and $\dim \mathcal{M} = S = \left\lfloor \frac{d^2 (\lambda/100)^4}{3 \log(100/\lambda)} \right\rfloor$.

The proof uses the following slight modification of a result in Bennett et al., Lemma 3 (Appendices A and B):

**Lemma 18** Let $\psi$ be a pure state, $P$ a projector of rank (at most) $r$ and let $U \in U(d)$ be a random variable, distributed according to the Haar measure. Then for $\epsilon > 0$,

$$\Pr \left\{ \Tr(U\psi U^* P) \geq (1 + \epsilon) \frac{r}{d} \right\} \leq \exp \left( -r \epsilon - \ln(1 + \epsilon) \ln 2 \right).$$

**Proof.** From Bennett et al.\cite{8} proof of Lemma 3, we take that the probability in question is bounded by $\exp \left( -r \frac{1}{\ln 2} \Lambda^*(1 + \epsilon) \right)$, with the rate function (see Dembo and Zeitouni\cite{13} for definitions) $\Lambda^*$ of the square of a real Gaussian distributed random variable of mean 0 and variance 1. This is calculated in Bennett et al.\cite{8} Lemma 23 (Appendix A), to

$$\Lambda^*(x) = \begin{cases} \frac{1}{2} (x - 1 - \ln x) & : x > 0, \\ \infty & : x \leq 0, \end{cases}$$

which gives the claim. \hfill \Box

**Proof of proposition\cite{23}** Pick an $\eta$–net, with $\eta = \lambda/8$, in $\mathcal{P}(\mathbb{C}^S)$ of cardinality $\left( \frac{5}{7} \right)^{2S}$, according to lemma\cite{7}. The encoder will be an isometry (with $a < d$ and $S < ad$) $V : \mathbb{C}^S \rightarrow \mathbb{C}^d \otimes \mathbb{C}^a$, followed by the partial trace over the second system $\mathbb{C}^a$: $\varepsilon(\pi) = \Tr_{\mathbb{C}^a}(V\pi V^*)$, while the decoder is simply given by $D_\tau = \text{supp} \varepsilon(\tilde{\tau})$, the support projector onto the image of $\tilde{\tau}$, the nearest state to $\tau$ in our $\eta$–net. We will fix $S$ and $a$ later.

We shall pick the isometry $V$ randomly (uniformly, i.e. according to the invariant measure), and show that with high probability it will yield a code of error $\eta$, at least for states from the $\eta$–net. Then, for arbitrary states $\pi, \tau$.
and their nearest neighbours $\tilde{\pi}, \tilde{\tau}$ in the net,

$$|\text{Tr}\pi\tau - \text{Tr}\varepsilon(\pi)D_\tau| \leq |\text{Tr}\pi(\tau - \tilde{\tau})| + |\text{Tr}(\pi - \tilde{\pi})\tilde{\tau}|$$

$$+ |\text{Tr}(\varepsilon(\pi) - \varepsilon(\tilde{\pi}))D_\tau| + |\text{Tr}\tilde{\pi}\tilde{\tau} - \text{Tr}\varepsilon(\tilde{\pi})D_\tau|$$

$$\leq 4\eta = \lambda/2,$$

using triangle inequality, $D_\tau = D_{\tilde{\tau}}$ and the nonincrease of the trace norm under the partial trace.

So, fix an ordered pair $(\tau = |\theta\rangle\langle\theta|, \pi = |\phi\rangle\langle\phi|)$ of states from the $\eta$-net, and write $|\phi\rangle = \sqrt{\alpha}|\theta\rangle + \sqrt{1 - \alpha}|\theta^\perp\rangle$, with $\alpha = |\langle\theta|\phi\rangle|^2$. Then we can write, with independent random unit vectors $|v\rangle, |w\rangle$,

$$V|\theta\rangle = |v\rangle,$$

$$V|\theta^\perp\rangle = \frac{|w\rangle - \langle v|w\rangle|v\rangle}{\||w\rangle - \langle v|w\rangle|v\rangle\|_2}.$$

(Note that the denominator vanishes with probability 0.) Now holding $|v\rangle$ constant for the moment, so that $D_\tau = \text{supp \text{Tr}}_\eta |v\rangle\langle v|$ is a constant, we have

$$\text{Tr}(\varepsilon(\pi)D_\tau) = \langle \phi|V^*(D_\tau \otimes \mathbb{1})V|\phi\rangle$$

$$= \left(\sqrt{\alpha} - \sqrt{1 - \alpha}\frac{|v\rangle\langle w|}{t}\langle v| + \sqrt{1 - \alpha}\frac{|w\rangle\langle v|}{t}\right)$$

$$D_\tau \otimes \mathbb{1} \left(\left(\sqrt{\alpha} - \sqrt{1 - \alpha}\frac{|v\rangle\langle w|}{t}\right)|v\rangle + \sqrt{1 - \alpha}\frac{|w\rangle\langle v|}{t}\right)$$

$$= \left|\sqrt{\alpha} - \sqrt{1 - \alpha}\frac{|v\rangle\langle w|}{t}\right|^2 + \frac{1 - \alpha}{t^2} \langle w|D_\tau \otimes \mathbb{1}|w\rangle$$

$$+ \left(\sqrt{\alpha} - \sqrt{1 - \alpha}\frac{|w\rangle\langle v|}{t}\right)\sqrt{1 - \alpha}\frac{|v\rangle\langle v|}{t}\langle v|w\rangle$$

$$+ \left(\sqrt{\alpha} - \sqrt{1 - \alpha}\frac{|v\rangle\langle w|}{t}\right)\sqrt{1 - \alpha}\frac{|w\rangle\langle w|}{t}\langle w|v\rangle$$

$$= \alpha + \frac{1 - \alpha}{t^2} \left[\langle w|D_\tau \otimes \mathbb{1}|w\rangle - \langle w|\langle v|v\rangle\right]\$$

where we have denoted $t := \||w\rangle - \langle v|w\rangle|v\rangle\|_2$ and used that $|v\rangle\langle v| \leq D_\tau \otimes \mathbb{1}$. Hence, if

$$\langle w|\langle v|v\rangle\rangle \leq \langle w|D_\tau \otimes \mathbb{1}|w\rangle \leq \epsilon := (\eta/2)^2,$$

we can conclude that $|\text{Tr}\varepsilon(\pi)D_\tau - \alpha| \leq \eta$. It remains to bound the probability of the event in eq. (13): according to lemma (13), putting $a = |\epsilon|/2$, we have

$$\text{Pr}\{\langle w|D_\tau \otimes \mathbb{1}|w\rangle > \epsilon\} \leq \exp\left(-a^2\frac{1 - \ln 2}{\ln 2}\right).$$

Note that $D_\tau$ can have rank $a$ at most; we will assume $a \geq 1$ from now, and will see at the end that the case where $a$ would be zero is trivial. Putting this
and eq. (13) together with the union bound, this gives us

\[
\Pr\left\{ \exists \pi, \tau \text{ from the } \eta \text{-net} \left| \text{Tr}_\varepsilon(\pi)D_\tau - \text{Tr}_\pi \tau \right| > \eta \right\} \leq \left( \frac{5}{\eta} \right)^{4S} \exp \left( -\frac{1 - \ln 2}{\ln 2} \left| \frac{cd}{2} \right|^2 \right),
\]

which is smaller than 1 by our choice of parameters: \( \eta = \lambda/8, \epsilon = (\eta/2)^2, S = \left[ d^2 \frac{(\lambda/100)^4}{4 \log(100/\lambda)} \right] \). That is, except when \( S = 0 \) (which is implied by \( a = 0 \)), in which case the proposition is trivial.

**Theorem 19** \( Q_{bv}(id^{C_2}) = Q_{vv}(id^{C_2}) = 2 \), and in fact the strong converse holds: for all \( 0 < \lambda < 1 \),

\[
\lim_{n \to \infty} \frac{1}{n} \log L_{bv}(n, \lambda) = \lim_{n \to \infty} \frac{1}{n} \log L_{vv}(n, \lambda) = 2.
\]

**Proof.** This follows immediately by invoking proposition 17 for the direct part on \( d = 2^n \) dimensions, and remark 16 for the converse. Looking at proposition 15 and the strong converse for \( N(n, \lambda_1, \lambda_2) \) (proposition 5) yields the strong converse. \( \square \)

**Corollary 20** For every channel \( T \),

\[
Q_{bv}(T) \geq 2Q_{bv}(T), \quad Q_{vv}(T) \geq 2Q_{vv}(T).
\]

**Proof.** We begin by observing that by linearity both blind and visible encoders can be understood as mappings \( S(M) \to S(A_1^{\otimes n}) \). The claim follows by understanding that one can concatenate a given bV– or vV–code with the construction of proposition 17 this translates a given pure state on \( M' \) into a mixed state on \( M \) which is then processed further by the encoder to a state on \( A_1^{\otimes n} \); it translates a fidelity–test on \( M' \) into a binary POVM on \( M \) which the visible decoder turns into a POVM on \( A_2^{\otimes n} \). \( \square \)

Let us summarise what we can say about two special channels, \( id^{C_2} \) (a noiseless qubit) and \( id_2^c \) (the completely dephasing channel, i.e. a noiseless classical bit):

\[
\begin{array}{c|c|c|c|c}
T = id^{C_2}: & x & y & b & V \\
C_{vD} = 2 & b & 1 & 1 & 2 \\
C_{ID} = 1 & v & 1 & 1 & 2 \\
\hline
T = id_2^c: & x & y & b & V \\
C_{vD} = 1 & b & 0 & 0 & 1 \\
C_{ID} = 1 & v & 0 & 0 & 1
\end{array}
\]

4 Conclusions

Consideration of identification problems via quantum channels has yielded surprising results: we proved that the unrestricted identification capacity can
be larger than the transmission capacity by explicitly computing it for hybrid quantum memories. These findings are in marked contrast to the classical situation where the identification capacity shows a strong connection to the capacity $\text{CR}(T)$ of the channel to create common randomness between sender and receiver.\cite{24,18} (It is possible to find classical channels with $C_{\text{ID}}(T) \neq \text{CR}(T)$ if one allows exponentially growing alphabet or memory.\cite{23}) Indeed, using the Holevo information bound\cite{13} it is not difficult to show that the common randomness capacity $\text{CR}(T)$ is at most $C(T)$; since transmission can always be used to set up correlation, the other inequality is trivial, so $\text{CR}(T) = C(T)$ (see also a forthcoming work by the author).

This increase (by a factor of 2 for the noiseless qubit channel) may be connected to dense coding, i.e., entanglement–assisted transmission.\cite{10} If so, we would learn that the technical condition in L"ober’s theorem\cite{2} on alphabet size is necessary; also, that something with the conjecture in remark \cite{14} is wrong — note that entanglement increases the capacity of typical qc–channels (e.g. from 0.585 to 1 for the trine measurement\cite{17}), and that ID coding for qc–channels is automatically simultaneous due to the built–in measurement. If not related to dense coding, there is another possible, geometric, explanation for the effect: it may ultimately have to do with the geometry of state space, where the dimension of the manifold of mixed states is roughly the square of the dimension of the pure state manifold.

Then we went further and considered identification of quantum states: also there we showed that the capacity can exceed the quantum transmission capacity, and the same factor 2 occurs for the noiseless qubit. Our construction may well be of independent interest, as it essentially consists of a “random noisy channel”

$$R_t^{(u)} : B(\mathbb{C}^s) \xrightarrow{V \cdot V^*} B(\mathbb{C}^t \otimes \mathbb{C}^u) \xrightarrow{\text{Tr}_{\mathbb{C}^u}} B(\mathbb{C}^t),$$

with an (isotropic) random isometry $V$ and $s \leq tu$. E.g. for $s = 1$ we recover the generation of random mixed states by partial trace from a random pure state (for a recent work see Życzkowski and Sommers\cite{39}) — one can rewrite our construction of a classical ID code (proposition \cite{6}) in these terms, with $t = O(u \log u)$; for $u = 1$ we are selecting a random embedding which is used in proofs of the quantum channel coding theorem.

We had to leave many open questions: is $C_{\text{sim–ID}} \neq C_{\text{ID}}$ and what is $C_{\text{ID}}(T)$ in general? Could it be equal to the lower bound given in theorem\cite{12}? What is $Q_{\text{ID}}(T)$ in general — in particular, can it be larger than $2Q(T)$? We conjecture that one needs quantum transmission to have $Q_{\text{ID}}(T) \neq 0$ and in particular that $Q_{\text{ID}}(\text{id}_A) = Q_{\text{ID}}(\text{id}_A) = \max \{2 \log d_i : 1 \leq i \leq r\}$ for the hybrid quantum memory $A = \bigoplus_{i=1}^r B(\mathbb{C}^{d_i})$. Note that this would also imply that (bounded) common randomness between the users does not increase the quantum identification capacities, unlike the classical case\cite{4} (see remark \cite{11}).
this is certainly true for $Q_{bb}$ (and likely for $Q_{vb}$, see Bennett et al.\textsuperscript{8} Theorem 24 (Appendix C) for combinations of noiseless quantum and classical channels) because we can phrase the error condition as an average pure state fidelity, and this can always be achieved without randomisation. Similar questions arise in connection with the other visible coding capacities that we defined; for example, it seems that visible encoding and bounded capacity classical side channels cannot increase our quantum capacities (unbounded classical side channels will certainly trivialise the visible encoder variants). How could one prove this in general? A possible strategy that would solve this and relates to the above question about $Q_{\text{ID}}(T) \geq 2Q(T)$ could be the construction of a simulation of $T$ for visible transmission, using $Q(T)$ qubits and some finite rate of classical bits. Entanglement binding channels however might pose a difficulty here (note that they are the reason that a simulation can only possibly exist for visibly given input states).

The role of the general visible decoding (“V”) is not yet very clear, but extremely challenging: we conjecture that this model is as hard as usual transmission but it seems one needs methods beyond those of Schumacher et al.\textsuperscript{31} to prove any bound. One such approach could be based on a further investigation of the state space geometric relations in ID coding: to begin with, classical identification is an almost-isometry (under statistical distance) from the vertices of an $O(2^d)$–probability simplex into a $d$–probability simplex; quantum fingerprinting maps similarly into the pure states on $\mathbb{C}^d$ (with trace distance). Our quantum–ID codes for the noiseless channel are almost–isometries (under trace distance) from the set of pure states on $\mathbb{C}^{O(d^2)}$ into the mixed states on $\mathbb{C}^d$; note that here even the dimensions of the two state spaces as manifolds are comparable! In fact, what is more important, they have comparable Vapnik–Chervonenkis dimension (also called covering dimension). For the V– and b–variants of decoders stricter metric constraints apply, extending to larger groups of states, not only pairwise distances, which accounts for the capacities being smaller in these models.

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