Measurement of Tidal Deformability in the Gravitational Wave Parameter Estimation for Nonspinning Binary Neutron Star Mergers

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One of the main targets for the ground-based gravitational wave (GW) detectors such as Advanced LIGO (Laser interferometer gravitational wave observatory) and Virgo is coalescences of neutron star (NS) binaries. Even though NS’s macroscopic properties such as mass and radius have been obtained by electro-magnetic wave observations, its internal structure has been studied mainly by theoretical approaches. However, with the advent of Advanced LIGO and Virgo, tidal deformability of NS which depends on the internal structure of NS has been obtained recently by GW observations. Therefore, it is important to reduce the measurement error of tidal deformability as small as possible in the GW parameter estimation. In this study, we introduce a post-Newtonian (PN) gravitational waveform model in which the tidal deformability contribution appears from 5 PN order, and use the Fisher matrix (FM) method to calculate parameter measurement errors. Since the FM is computed semi-analytically using the wave function, the measurement errors can be obtained much faster than those of practical parameter estimations based on Monte-Carlo simulations. We investigate the measurement errors for mass and tidal deformability by applying the FM to the nonspinning TaylorF2 waveform model. We show that if the tidal deformability corrections are considered up to the 6 PN order, the measurement error for the dimensionless tidal deformability can be reduced to about \(75\%\) compared to that obtained by considering only the 5 PN order correction.

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I. INTRODUCTION

The presence of gravitational waves (GWs) was predicted by Einstein in 1916 [1] and 1918 [2]. The strain of GWs, however, is extremely small, hence numerous efforts to detect real GWs have failed for the past decades. Finally, the first GW signal, named as GW150914, was captured by the two advanced LIGO (Laser Interferometer Gravitational wave Observatory) detectors in September 2015 and several GW signals from binary black hole mergers have been detected by the network of Advanced LIGO and Virgo [3][7]. Eventually, a multi-messenger astronomy era has begun with the observation of a GW signal from a binary neutron star (NS) merger in 2017 [8][9]. Since KAGRA will be operating in the near future [10][11], the expected detection rate of binary NS mergers is \(3.2\sim9.2\) times per year with the three detector network [12].

The main target of Advanced LIGO and Virgo is compact binary coalescences (CBCs) in which compact binary means black hole-black hole, black hole-NS, or NS-NS binary. In data analysis for GWs of CBCs, a signal can be distinguished from noises by matching the detector data with theoretical waveforms. Once a GW signal is identified in the detection pipeline, a more detailed analysis is carried out in the parameter estimation pipeline to seek the physical parameters such as mass and spin of the system. The result of the parameter estimation analysis is given by probability density functions for the parameters considered, and this process typically requires a very long time and a large computing resources. On the other hand, if the incident GW signal is strong enough and buried in Gaussian noise, the measurement errors can be easily calculated by using the Fisher matrix (FM) method which is a semi-analytic approach used to estimate the parameter estimation result approximately.

Recently, the tidal deformability of NS has been measured from the GWs generated by the merger of a neutron star binary [8][13]. This parameter describes the response of a NS to the external tidal field generated by its companion star. In the current GW parameter estimations, the observational constraints on the NS radii are well estimated, however, the measurement errors for tidal deformability are not small enough to precisely distinguish various EOSs [8][13].

In this work, we introduce a post-Newtonian (PN) waveform model, which is expressed in the frequency domain and valid for the inspiral waveforms emitted from CBCs. This model incorporates 3.5 PN order point-particle contributions and 5 PN and 6 PN tidal corrections. We use the FM method and calculate the measurement errors of mass and tidal parameters for a nonspinning, equal-mass binary NS assuming the Advanced LIGO detector sensitivity. We investigate how much the 6 PN tidal correction can improve the accuracy in measuring the tidal deformability.
II. POST-NEWTONIAN GRAVITATIONAL WAVEFORMS

The physical parameters used to define waveforms from CBC systems are divided into two groups. One group includes the intrinsic parameters such as mass and spin, which are directly related to the dynamics of a binary and the shape of the waveform. The other includes the extrinsic parameters such as luminosity distance, sky position, and inclination of the orbital axis. Unlike the intrinsic parameters, the extrinsic parameters only affect the wave amplitude, hence they do not need to be considered in our analysis. Generally, the correlation values between intrinsic and extrinsic parameters are much less than those between the intrinsic parameters [14].

The dynamics of CBCs in the relativistic region can be calculated from numerical simulations which typically cost a large computing resources. However, in the region of slow orbital velocity ($v/c \ll 1$) and weak gravitational field $(GM/(rc^2) \ll 1$), the post-Newtonian (PN) approximation is good enough to describe the CBC dynamics and model the gravitational waveforms from the CBC inspirals, hence it requires much less computing resources.

The frequency-domain PN waveform model obtained by using stationary phase approximation is called TaylorF2. The TaylorF2 waveform can be written as

$$\tilde{h}(f) = Af^{-7/6}e^{i\psi(f)},$$

where $A$ is the wave amplitude consisting of the chirp mass ($M_c \equiv (m_1m_2)^{3/5}/M^{1/5}$ where $M$ is the total mass), and the intrinsic parameters. The phase $\psi(f)$ is expressed as the PN expansion:

$$\psi(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3}{128\eta v^6} \left( \Psi_{3.5\text{PN}} + \Psi_{\text{Tidal}} \right),$$

where $t_c$ and $\phi_c$ are the coalescence time and the coalescence phase; they can be chosen arbitrarily, and $\eta \equiv m_1m_2/M^2$ is the symmetric mass ratio. In the bracket, $\Psi_{3.5\text{PN}}$ is the point-particle contribution incorporated up to 3.5 PN order [15] and $\Psi_{\text{Tidal}}$ is the contribution by the quadrupolar tidal interaction, which is discussed in the next section.

III. TIDAL DEFORMABILITY

In a merging binary NS system, one NS can be deformed by the tidal field generated by the companion NS. In the linear order, the quadrupole moment $Q_{ij}$ induced by the external quadrupolar tidal field $\mathcal{E}_{ij}$ is given by [16–17]:

$$Q_{ij} = -\lambda \mathcal{E}_{ij},$$

which can be decomposed with the symmetric traceless tensor $Y_{ij}$ defined by spherical harmonics $Y_{lm}(\theta, \phi)$ [18]. The leading contribution $l = 2$ is only in our consideration, thus the constant $\lambda$ is the $l = 2$ tidal deformability. The quadrupole moment is related to density perturbation $\delta \rho$ as

$$Q_{ij} = \int d^3x \, \delta \rho(x) \, (x_i x_j - \delta_{ij} r^2/3),$$

and $\mathcal{E}_{ij}$ is expressed in terms of the external gravitational potential $\Phi_{\text{ext}}$ as

$$\mathcal{E}_{ij} = \frac{\partial^2 \Phi_{\text{ext}}}{\partial x^i \partial x^j}.$$

The tidal deformability is defined by [16–18]

$$\lambda = \frac{2}{3G} k_2 R^5,$$

where $k_2$ is Love number and $R$ is a radius of a NS. For a given NS mass, the tidal deformability values can vary according to the EOSs [19].

In a binary NS system, the tidal deformability contribution is given up to 6 PN order as [20]

$$\Psi_{\text{Tidal}}(f) = -\left[ \frac{39}{2} \eta^{\frac{1}{6}} + \left( \frac{3115}{64} - \frac{6595 \sqrt{1 - 4\eta} \delta \Lambda}{364} \right) \right]^{1/2},$$

where reduced tidal deformability $\tilde{\Lambda}$ and asymmetric tidal correction $\delta \tilde{\Lambda}$ are defined as [20]

$$\tilde{\Lambda} = 32 \frac{c^{10} \tilde{\Lambda}}{G^4 M^5} = \frac{16 c^{10}}{13 G^4} \left( \frac{m_1 + 12 m_2}{m_2} \right) \left( \frac{1}{m_1} + \frac{1}{m_2} \right) + \frac{8}{13} \left[ (1 + 7\eta - 31\eta^2)(\Lambda_1 + \Lambda_2) \right. - \left. \sqrt{1 - 4\eta}(1 + 9\eta - 11\eta^2)(\Lambda_1 - \Lambda_2) \right],$$

$$\delta \tilde{\Lambda} = \frac{1}{2} \left[ \sqrt{1 - 4\eta} \left( 1 - \frac{13272\eta}{1319} + \frac{8944\eta^2}{1319} \right) (\Lambda_1 + \Lambda_2) + \left( 1 - \frac{15910\eta}{1319} + \frac{32850\eta^2}{1319} + \frac{3380\eta^3}{1319} \right) (\Lambda_1 - \Lambda_2) \right],$$

where $m_1 \geq m_2$, and $\Lambda_i = c^{10} \lambda_i/(G^4 m_i^5)$ is the dimensionless tidal deformability of a single star. Since $\Lambda_1$ and $\Lambda_2$ are strongly correlated, $\tilde{\Lambda}$ and $\delta \tilde{\Lambda}$ are much more efficient in most analyses. Typically, $\delta \tilde{\Lambda}/\tilde{\Lambda}$ is about 0.01, hence the contribution of $\delta \tilde{\Lambda}$ is negligible [21]. In this work, we assume $m_1 = m_2$ and $\Lambda_1 = \Lambda_2$, then $\tilde{\Lambda} = \Lambda_1 = \Lambda_2$, therefore the $\delta \tilde{\Lambda}$ term vanishes in Eq. (7).

IV. MEASUREMENT ERROR

The main purpose of the GW detection pipeline is to decide whether the GW signal enters the data stream or not. Once a detection is made in the detection pipeline, the parameter estimation pipeline seeks the source properties based on Monte-Carlo simulations and this procedure typically requires a long computational time. On the other hand, the FM method has been used to predict the parameter estimation results approximately for high signal-to-noise ratio (SNR) signals [22–23] (for a general overview of the FM, refer to [24]).
The FM can be calculated semi-analytically by using the post-Newtonian waveform so that the measurement errors can be obtained much faster than performing the practical parameter estimation pipeline.

For a given theoretical waveform \( h(t) \) and detector data stream \( x(t) \) that can contain a real GW strain, the likelihood is determined by \[ L(x|\theta) = \exp[-(x-h(\theta)|x-h(\theta))/2], \] (10)
where \( \theta \) is a physical parameter set including a chirp mass and a symmetric mass ratio, etc. In the above, \( \langle x|h \rangle \) means the noise-weighted inner product given by
\[ \langle x|h \rangle = 2 \int_{f_{\text{min}}}^{f_{\text{max}}} \frac{\tilde{x}^*(f)\tilde{h}(f) + \tilde{x}(f)\tilde{h}^*(f)}{S_n(f)} df, \] (11)
where \( f_{\text{min}} \) and \( f_{\text{max}} \) are the minimum and the maximum cut-off frequencies, respectively, and \( S_n(f) \) is the noise power spectral density of the detector.

In the high SNR limit, the likelihood can be expressed as \[ L(\theta) \propto \exp[-\rho^2(1-\langle h_0|\tilde{h}(\theta)\rangle)] \] (12)
where \( \tilde{h} \) denotes the normalized waveform, \( \tilde{h} \equiv h/\sqrt{\langle h|h \rangle} \), \( \rho \) is the SNR, \( h_0 \) is the GW signal, and \( \Gamma_{ij} \) is the FM defined by
\[ \Gamma_{ij} = \left( \frac{\partial h}{\partial \theta_i} \frac{\partial h}{\partial \theta_j} \right)_{\theta_i=\theta_{\text{true}}}. \] (13)
The inverse of the FM corresponds to the covariance matrix of parameter errors. Thus, the measurement error \( \sigma_i \) and the correlation coefficient \( c_{ij} \) can be determined by
\[ \sigma_i = \sqrt{(\Gamma^{-1})_{ii}}, \quad c_{ij} = \frac{(\Gamma^{-1})_{ij}}{\sqrt{(\Gamma^{-1})_{ii}(\Gamma^{-1})_{jj}}}. \] (14)

**V. RESULT**

We adopt a nonspinning equal-mass binary NS with \( m_1 = m_2 = 1.4M_\odot \), and choose \( \Lambda_1 = \Lambda_2 = 300 \) following the result in [13]. We consider the Advanced LIGO noise curve given in [28] and assume \( f_{\text{min}} = 10 \) Hz. By using the TaylorF2 waveform, we calculate 5×5 FM whose components are \( \theta_i = \{M_e, \eta, \bar{t}, \bar{\psi}, \phi, \psi\} \) and present the measurement errors \( \sigma_{M_e}, \sigma_\eta, \) and \( \sigma_{\bar{\lambda}} \).

Figure 1 shows the cumulative phases as a function of \( f_{\text{max}} \) from 10 Hz to \( f_{\text{max}} \) due to the 5 PN tidal term (\( \psi_{5,6\text{PN}}^{\text{Tidal}} \)) and the sum of the 5 PN and the 6 PN tidal terms (\( \psi_{5,6,7\text{PN}}^{\text{Tidal}} \)), respectively. The fractional difference \( (\psi_{5,6\text{PN}}^{\text{Tidal}} - \psi_{5\text{PN}}^{\text{Tidal}})/\psi_{5\text{PN}}^{\text{Tidal}} \) is also given in Figure 1. We show the result only up to 450 Hz because our TaylorF2 model is fairly valid to 450 Hz [19]. Note that the fractional difference is about 20% at 450 Hz.

Next, we investigate how the 6 PN tidal term contributes to improving the accuracy in measuring the tidal parameter \( \bar{\lambda} \).

We take into account \( \psi_{5,6\text{PN}}^{\text{Tidal}} \) and \( \psi_{5,6,7\text{PN}}^{\text{Tidal}} \) in Eq. (2), respectively, and calculate the measurement errors for both cases. Figure 2 shows the ratio of the measurement errors between \( \psi_{5,6\text{PN}}^{\text{Tidal}} \) and \( \psi_{5,6,7\text{PN}}^{\text{Tidal}} \) waveform cases (see text). We adopt a nonspinning equal-mass binary NS with \( m_1 = m_2 = 1.4M_\odot \), and \( \Lambda_1 = \Lambda_2 = 300 \). We consider the Advanced LIGO noise curve and assume \( f_{\text{min}} = 10 \) Hz.

![Figure 1](image1.png)

**FIG. 1:** The cumulative phase due to the tidal terms \( \psi_{5\text{PN}}^{\text{Tidal}} \) and \( \psi_{5,6\text{PN}}^{\text{Tidal}} \) (upper panel), and their fractional difference \( (\psi_{5,6\text{PN}}^{\text{Tidal}} - \psi_{5\text{PN}}^{\text{Tidal}})/\psi_{5\text{PN}}^{\text{Tidal}} \) (lower panel). We assume \( f_{\text{min}} = 10 \).  

![Figure 2](image2.png)

**FIG. 2:** Ratio of the measurement errors between \( \psi_{5\text{PN}}^{\text{Tidal}} \) and \( \psi_{5,6\text{PN}}^{\text{Tidal}} \) waveform cases (see text). We adopt a nonspinning equal-mass binary NS with \( m_1 = m_2 = 1.4M_\odot \), and \( \Lambda_1 = \Lambda_2 = 300 \). We consider the Advanced LIGO noise curve and assume \( f_{\text{min}} = 10 \) Hz.
VI. SUMMARY AND DISCUSSION

We used the TaylorF2 waveform model that incorporates the 3.5 PN order point-particle contribution with the tidal corrections. By applying the FM method, we calculated the measurement errors of the mass and the tidal parameters for a nonspinning equal-mass binary NS. We found that the contribution of the 6 PN tidal correction to the cumulative phase from 10 to 450 Hz is about 20% of the contribution of the 5 PN tidal correction, and the measurement error of the tidal deformability \( \tilde{\Lambda} \) can be reduced to about 75% by including the 6 PN correction in the waveforms.

Tidal deformability is directly related with the NS EOS. Therefore, in order to restrict various theoretical EOS models, it is very important to reduce the measurement error of the tidal deformability. In this context, we briefly demonstrated the necessity of using higher-order tidal corrections in the GW parameter estimation. We assumed the maximum frequency cutoff to be 450 Hz in the overlap integration due to the validity of the waveform model. The changes in our result would be very small even if we consider much higher cutoff frequencies because the sensitivity curve of Advanced LIGO used in this work increases very rapidly around 450 Hz. However, future third-generation GW detectors such as Einstein telescope \([29, 31]\) are much more sensitive than Advanced LIGO, especially in the high-frequency region, therefore the impact of the higher-order tidal corrections on the accuracy in measuring the tidal deformability can be significantly increased by extending the cutoff frequency beyond 450 Hz.

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[1] V. A. Einstein, Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916, Nähерungsweise Integration der Feldgleichungen der Gravitation (1916).
[2] V. A. Einstein, Gesamtsitzung vom 14. Februar 1918. - Mitteilung vom 31. Januar, Über Gravitationswellen (1918).
[3] B. P. Abbott et al., (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 116, 061102 (2016).
[4] B. P. Abbott et al., (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 116, 241103 (2016).
[5] B. P. Abbott et al., (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 118, 221101 (2017).
[6] B. P. Abbott et al., (LIGO Scientific Collaboration and Virgo Collaboration), Astrophys. J. 851 L35 (2017).
[7] B. P. Abbott et al., (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 119, 141101 (2017).
[8] B. P. Abbott et al., (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 119, 161101 (2017).
[9] B. P. Abbott et al., Astrophys. J. Lett. 848, L12 (2017).
[10] K. Somiya et al., (KAGRA Collaboration), Classical Quantum Gravity 29, 124007 (2012).
[11] T. Akutsu et al., (KAGRA Collaboration), Proceedings for XV International Conference on Topics in Astroparticle and Underground Physics (arXiv:1710.04823).
[12] M. Dominik et al., Astrophys. J. 806 263 (2015).
[13] B. P. Abbott et al., (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 121, 161101 (2018).
[14] V. Raymond, ph.D. thesis, Northwestern University, (2012).
[15] A. Buonanno, B. R. Iyer, E. Ochsner, Y. Pan and B. S. Sathyaprakash, Phys. Rev. D 80, 084043 (2009).
[16] E. É. Flanagan and T. Hinderer, Phys. Rev. D 77, 021502 (2008).
[17] B. D. Lackey, ph.D. thesis, The University of Wisconsin-Milwaukee, (2012).
[18] T. Hinderer, Astrophys. J., 697 964 (2009).
[19] T. Hinderer, B. D. Lackey, R. N. Lang and J. S. Read, Phys. Rev. D 81, 123016 (2010).
[20] B. D. Lackey and L. Wade, Phys. Rev. D 91, 043002 (2015).
[21] M. Favata, Phys. Rev. Lett. 112, 101101 (2011).
[22] H.-S. Cho and C.-H. Lee, Classical Quantum Gravity 31, 235009 (2014).
[23] H.-S. Cho, J. Korean Phys. Soc. 66, 1637 (2015).
[24] M. Vallisneri, Phys. Rev. D 77, 042001 (2008).
[25] C. Cutler and É. É. Flanagan, Phys. Rev. D 49, 2658 (1994).
[26] L. S. Finn, Phys. Rev. D 46, 5236 (1992).
[27] H.-S. Cho, E. Ochsner, R. O’Shaughnessy, C. Kim and C.-H. Lee, Phys. Rev. D 87, 024004 (2013).
[28] P. Ajith and S. Bose, Phys. Rev. D 79, 084032 (2009).
[29] M. Punturo et al., Classical Quantum Gravity 27, 194002 (2010).
[30] S. Hild et al., Classical Quantum Gravity 28, 094013 (2011).
[31] B. Sathyaprakash et al., Classical Quantum Gravity 29, 124013 (2012).