A New Fate of a Warped 5D FLRW Model with a U(1) Scalar Gauge Field

Reinoud Jan Slagter\textsuperscript{a} and Supriya Pan\textsuperscript{b}

\textsuperscript{a}Asfyon and Department of Physics, University of Amsterdam, The Netherlands
\textsuperscript{b}Department of Mathematics, Jadavpur University, Kolkata-700032, India

If we live on the weak brane with zero effective cosmological constant in a warped 5D bulk space-time, gravitational waves and brane fluctuations can be generated by a part of the 5D Weyl tensor and carries information of the gravitational field outside the brane. We consider on a cylindrical symmetric warped FLRW background a U(1) self-gravitating scalar-gauge field without bulk matter. It turns out that “branons” can be formed dynamically, due to the modified energy-momentum tensor components of the cosmic string. As a result, we find that the late-time behavior could be significant deviate from the standard evolution of the universe. The effect is triggered by the time-dependent warpfactor of the form $\sqrt{c_1 e^{\tau t} + c_2 e^{-\tau t}}$ (with $\tau, c_i$ constants) and the modified brane equations comparable with a dark energy effect. This is a brane-world mechanism, not present is standard 4D FLRW, where the large disturbances are rapidly damped as the expansion proceed. Because gravity can propagate in the bulk, the cosmic string can build up a huge angle deficit (or mass per unit length) by the warpfactor. Disturbances in the spatial components of the stress-energy tensor cause cylindrical symmetric waves, amplified due to the presence of the bulk space and warpfactor. They could survive the natural damping due to the expansion of the universe. This long range effect could also explain the recently found spooky alignment of quasars in vast structures in the cosmic web.

\textsuperscript{a} info@asfyon.com
\textsuperscript{b} span@research.jdvu.ac.in & pansupriya051088@gmail.com
I. INTRODUCTION

It is conjectured that the expansion of our universe is accelerating. Recent observations provide strong evidence of this acceleration. The explanation of this remarkable phenomenon is rather difficult: one needs a dark energy field with an effectively negative pressure, $p < -\frac{1}{3}\rho$. From severe independent observational data, one finds $\Omega_\Lambda \approx 0.7$ and $\Omega_M \approx 0.3$, where $\Omega_M$ and $\Omega_\Lambda$ stand for the energy densities of matter and dark energy respectively with respect to the critical ones $[1-5]$. We should live now in the cosmological constant dominated era (and approximately flat, $\Omega_0 = \Omega_{M0} + \Omega_{\Lambda0} \approx 1$), while at earlier times there was a radiation and matter dominated era.

There are some fundamental question. First, will the acceleration last forever. Secondly, why there is the huge discrepancy between $\rho_\Lambda \approx 10^{-123}$ in Planck units and vacuum energy density $\langle \rho_V \rangle \approx 10^{-3}$ which is $10^{120}$ times greater than the value we need to accelerate the expansion of the universe. Thirdly, the value must be incredibly fine-tuned, $\Omega_\Lambda \sim \Omega_M$. So it would be a logical step to try to explain the late-time acceleration without the need for dark energy $[6]$. It is also conjectured that one needs an inflaton field in the very early stage of our universe, to solve the flatness and horizon problem in the standard model of cosmology. This is the inflationary cold dark matter model with cosmological constant (LCDM). It could also predict the existence of fluctuations we observe in the CMB shortly before the end of inflation. The inflaton-field could be the well-known scalar-Higgs field with the mexican-hat potential. This model has lived up to his reputation. It was successful in the explanation of superconductivity, i.e., the Ginsburg-Landau theory, in the standard model of particle physics, in the general relativistic solution of the self-gravitating Nielsen-Olesen vortex (cosmic string) $[7,9]$ and could play a fundamental role in warped spacetimes $[10-12]$. Cosmic strings are U(1) scalar-gauge vortex solutions in general relativity (GR)$[9,13,14]$. It is conjectured that in any field theory which admits cosmic strings, a network of strings inevitable forms at some point during the early universe. However it is doubtful if they persist to the present time. Evidence of these objects would give us information at very high energies in the early stages of the universe. From recent observations by COBE, WMAP and Planck satellites, it was concluded that cosmic strings cannot provide satisfactory explanation for the magnitude of the initial density perturbations from which galaxies and clusters grew. The interest in cosmic strings faded away, mainly because of the inconsistencies with the power spectrum of CMB.

Cosmological cosmic strings can also be investigated on a FLRW background. However, the string-cosmology spacetime essentially looks like a scaled version of a string in a vacuum spacetime and the corrections in the field equations due to cylindrical gravitational radiation are rapidly damped and are negligible in any physical regime $[15]$. The reason of this result comes from the notion that the string radius $r_{CS}$
is much smaller than the Hubble radius $R_H$ in the post-inflationary era. The ratio $r_{CS}/R_H = \partial_t C/C$, where $C$ is the factored out overall scale factor in the metric, is then of the order $10^{-57}$. The resulting field equations look like the generalized Nielsen-Olesen vortex system. Only in the pre-inflationary epoch, radiative corrections could be very large. While the inflaton field plays a crucial role in the early stage of our universe, it could play a comparable role at much later times, if we modify gravity by considering warped brane-world models. It could be possible that there exists a correlation between the accelerating universe and large extra dimensions in brane-world models.

When it was realized that cosmic strings could be produced within the framework of superstring theory inspired cosmological models, a revival of cosmic strings occurred. These so-called cosmic superstrings can play the role of cosmic strings in the framework of string theory or M-theory, i.e., brane-world models. Supersymmetric GUT’s can even demand the existence of cosmic strings. Physicists speculate that extra spatial dimensions could exist in addition to our ordinary 4-dimensional spacetime. The idea that spacetime could have more than four dimensions was first proposed by Kaluza and Klein (KK) in the early 20th century [16,17]. These theories can be used to explain several of the shortcomings of the Standard Model, i.e., the unknown origin of dark energy and dark matter and the weakness of gravity (hierarchy problem). In these models, the weakness of gravity might be fundamental. One might naively imagine that these extra dimensions must be very small, i.e., curled up and never observable. Super-massive strings with $G\mu > 1$, could be produced when the universe underwent phase transitions at energies much higher than the GUT scale. Recently there is growing interest in the so-called brane-world models, first proposed by Arkani-Hamed, Dimopoulos and Dvali (ADD) [18,19] and I. Antoniadis et al. [20], which was extended by Randall and Sundrum (RS) [21]. In these models, the extra dimension can be very large compared to the ones predicted in string theory, i.e., of order of millimeters. The difference with the standard superstring model is that the compactification rely on the curvature of the bulk. The huge discrepancy between the electro-weak scale, $M_{EW} = 10^3 GeV$ and the gravitational mass scale $M_{Pl} = 10^{19} GeV$ will be suppressed by the volume of the extra dimension $y$, or the curvature in that region. This effect can also be achieved in the RS models by a warpfactor. In the RS-2 model, there are two branes, the visible and the gravity brane at $y = 0$. The branes have equal and opposite tensions $\pm \Lambda_4$. At low energy, a negative bulk cosmological constant will prevent gravity to leak into the extra dimension, $\Lambda_5 = -6/R_0^2 = -6/\mu^2$, with $\mu$ the corresponding energy scale and $R_0$ the compactification radius. The $\Lambda_5$ squeeze the gravitational field closer to the brane at $y = 0$. In the RS–1 model, one pushes the negative tension brane $y \to \infty$. If one fine-tunes the $\Lambda_4 = 3M_{Pl}^2/4\pi R_0^2$, then this ensures a zero effective cosmological constant on the brane. The infinite extra dimension makes, however, a finite contribution to the 5D volume due to the warpfactor. Because of the finite separation of the branes in the RS-2 model, one obtains so-called effective 4D modes (KK-modes) of the perturbative 5D
graviton on the 4D brane. These KK-modes will be massive from the brane viewpoint. In the RS-1 model, the discrete spectrum disappears and will form a continuous spectrum [22–24].

It will be clear that compact objects, such as black holes and cosmic strings, could have tremendous mass in the bulk, while their warped manifestations in the brane can be consistent with observations. So, brane-world models could overcome the observational bounds one encounters in cosmic string models. $G\mu$ could be warped down to GUT scale, even if its value was at the Planck scale. Although static solutions of the U(1) gauge string on a warped spacetime show significant deviation from the classical solution in 4D [25], one is interested in the dynamical evolution of the effective brane equations. Wavelike disturbances triggered by the huge mass of the cosmic string in the bulk, could have observational effects in the brane. One conjectures that these disturbances could act as an effective dark energy field. The question is if the pulse-like cylindrical waves have the desired asymptotic behavior at null infinity. The recently found large-scale alignment of quasar polarizations [26] could be explained by this warped brane-effect.

Here, we will investigate the late-time evolution of a warped 5D FLRW model when a U(1) scalar gauge field is present. In section II, we outline the model and in section III, we present some numerical solutions of the model. Finally, we have summarized our results in section IV.

II. THE FIELD EQUATIONS

It is problematic to embed U(1) cosmic strings in a Friedmann–Lemaitre–Robertson–Walker (FLRW) model, even when rewritten in cylindrical form, because the string spacetime is invariant under boosts in the $z$ direction, whereas the FLRW spacetime is not. However, for a radiating cylindrically symmetric space-time, one can analyze the behavior of gravitational waves generated by the local strings on the expansion of the universe if we consider the zero-thickness limit of non-singular spacetimes containing a cylindrical distribution of stress-energy embedded on a cosmological background.

The importance of cylindrical symmetric gravitational waves was first noticed by Beck [27] and re-discovered by Einstein and Rosen (ER) [28]. A substantial part of the present knowledge on gravitational waves originated from their wavelike solutions of the fully non-linear gravitational field equations. However, these original ER cylindrical wave solutions of the Einstein’s vacuum field equations inhabit a universe that is conical rather than flat at infinite cylindrical radius [29]. To overcome this problem, one could impose artificially non-conicality constraints [30, 31] in order to obtain a Kasner-like spacetime. In a warped 5-dimensional brane-world model with a U(1) gauge cosmic string, these constraints are superfluous because the effective brane spacetime is non-conical [10]. Much insight into wavelike solutions on a cylindrically spacetime can also be obtained from well known stationary axially symmetric solutions, like the Weyl,
Papapetrou, Lewis-van Stockum, and Kerr solution, through a complex transformation $z \rightarrow it, t \rightarrow iz$ \[32\]. The cylindrical symmetric metric is (with two killing vectors $\partial / \partial z, \partial / \partial \phi$, both spacelike and hypersurface orthogonal)

$$
 ds^2 = e^{2(\gamma(t,r) - \psi(t,r))}(-dt^2 + dr^2) + e^{2\psi(t,r)}dz^2 + r^2e^{-2\psi(t,r)}d\phi^2. 
$$

(1)

Without any matter source, one then obtains a wave equation for $\psi$, which is decoupled from $\gamma$. Once $\psi$ is solved, $\gamma$ can be solved by quadratures and the behavior of $\psi$ and $\gamma$ at null infinity is well known, i.e., the Einstein-Rosen pulse waves. One can construct incoming and outgoing radiation solutions \[33, 34\] in terms of the retarded and advanced coordinates $u = t - r$ and $v = t + r$. For a non-vacuum situation, the equations will not decouple.

It was found by Gregory \[15\] that a U(1) cosmic string can be embedded into a flat FLRW spacetime along the polar axis, when the Hubble radius is much larger than the string-core. The approximated spacetime is

$$
 ds^2 = a(t)^2[-dt^2 + dr^2 + K(r)^2dz^2 + (1 - 4\pi G\mu)^2S(r)^2d\phi^2],
$$

(2)

with $\mu$ the mass per unit length of the string, or angle deficit. One can match this spacetime on the well-known FLRW spacetime by a suitable transformation \[35\] and one should like to impose matching conditions across the boundary $r_{CS}$ of the interior and exterior spacetime (although for radiating strings, this will be difficult). For late-time solutions, it is not necessary to consider the matching conditions between the interior string solutions and the exterior FLRW spacetime. It turns out that when the width of the strings $r_{CS}$ is smaller than the Hubble radius, the disturbances are neglectable \[15\].

On a warped spacetime, where the warpfactor has a significant effect on the cosmic string solution \[10, 11\], disturbances can be significant. In the bulk, the mass of the cosmic string will be huge, inducing a back reaction on the brane equations.

### A. The Bulk Equations

Following Shiromizu et al. \[36\], we have 5-dimensional Einstein equations with a bulk cosmological constant $\Lambda_5$ and brane energy-momentum as source

$$
(5)G_{\mu\nu} = -\Lambda_5(5)g_{\mu\nu} + \kappa_5^2\delta(y)(-\Lambda_4(4)g_{\mu\nu} + (4)T_{\mu\nu}),
$$

(3)

with $\kappa_5 = 8\pi(5)G = 8\pi/(5)M_{pl}^3$, $\Lambda_4$ the brane tension, $x^\mu = (t, x^i, y)$, $(4)g_{\mu\nu} = (5)g_{\mu\nu} - n_\mu n_\nu$, and $n^\mu$ the unit normal to the brane. The $(5)M_{pl}$ is the fundamental 5D Planck mass, which is much smaller than the
effective Planck mass on the brane, \( \sim 10^{19} \) GeV. We consider here the matter field \((^{4})T_{\mu \nu}\) confined to the brane, i.e., the U(1) scalar-gauge field, written in the form \[ \Phi = \eta X(t, r)e^{i\varphi}, \quad A_\mu = \frac{1}{e} \left[ P(t, r) - 1 \right] \nabla_\mu \varphi, \] with \( \eta \), the vacuum expectation value of the scalar field. Let us consider the cylindrically symmetric warped spacetime
\[ ds^2 = \mathcal{W}(t, r, y)^2 \left[ e^{2(\gamma(t, r) - \psi(t, r))} (-dt^2 + dr^2) + e^{2\psi(t, r)} dz^2 + r^2 e^{-2\psi(t, r)} d\varphi^2 \right] + dy^2, \] with \( \mathcal{W} \) be the warpfactor and \( y \) be the bulk space coordinate.

From the 5D equations we obtain from the \((t, y)\) component of the Einstein equations (a dot means \( \frac{\partial}{\partial t} \))
\[ \partial_y \mathcal{W} = \frac{\mathcal{W} \partial_r \mathcal{W}}{\mathcal{W}}, \] with general solution \( \mathcal{W}(t, r, y) = W_1(t, r)W_2(y) \). We can then separate the \( W_2 \) equations. They become
\[ \partial_y W_2 = -\frac{\left( \partial_r W_2 \right)^2}{W_2} - \frac{1}{3} \Lambda_5 W_2 - \frac{c_1}{W_2}, \quad \left( \partial_r W_2 \right)^2 = -\frac{1}{6} \Lambda_5 W_2^2 + c_2, \] with a simplified solution (for the moment, \( c_1, c_2 = 0 \))
\[ W_2(y) = e^{\sqrt{-\Lambda_5}(y - y_0)}. \] We see that only a negative bulk cosmological constant make sense. For \( c_1, c_2 \neq 0 \) one obtains a solution with positive bulk cosmological constant. The equation for \( W_1(t, r) \) becomes
\[ \ddot{W}_1 = W_1'' + \frac{1}{W_1} (W_1'^2 - W_1^2) + \frac{2}{r} W_1', \]

**FIG. 1.** Typical solutions of the warpfactor \( W_1 \). Left the two different possibilities for the time-dependent part: a minimum or an inflection point. Right a 3D plot for some values of \( \tau \) and \( d_\tau \) [see Eq. (10)].
with a prime $\frac{d}{dr}$. A typical solution is

$$W_1(t, r) = \frac{1}{\sqrt{\tau r}} \sqrt{ \left( d_1 e^{(\sqrt{2\tau}r)} - d_2 e^{-(\sqrt{2\tau}r)} \right) \left( d_3 e^{(\sqrt{2\tau}r)} - d_4 e^{-(\sqrt{2\tau}r)} \right)},$$

(10)

with $\tau, d_i$ some constants. Figure 1 represents the typical plots of $W_1(t, r)$ for different sets of the constants $\tau$ and $d_i$. In general, $W_1$ can possess a saddle-point or extremal values.

We shall see in Section 2.2 that these solutions for the warpfactor are consistent with the supplementary equations following from the 5D Einstein and the Bianchi equations. It turns out that the equations for $W_1(t, r)$ can not be isolated from the effective 4D brane equations, indicating that $W_1$ is a really a warpfactor-effect.

**B. The Effective Brane Equations**

The field equations induced on the brane can be derived using the Gauss–Codazzi equations together with the Israel–Darmois junction conditions at the brane and the $Z_2$ symmetry [37]. The modified Einstein equations become

$$(4)G_{\mu\nu} = -\Lambda_{\text{eff}}(4)g_{\mu\nu} + \kappa_4^2(4)T_{\mu\nu} + \kappa_5^4 J_{\mu\nu} - \epsilon_{\mu\nu},$$

(11)

where $\Lambda_{\text{eff}} = \frac{1}{2}(\Lambda_5 + \kappa_4^2 \Lambda_4)$ and $\Lambda_4$ is the vacuum energy in the brane (brane tension). $\Lambda_{\text{eff}} = 0$ for the RS fine-tuning. The first correction term $J_{\mu\nu}$ is the quadratic term in the energy-momentum tensor arising from the extrinsic curvature terms in the projected Einstein tensor

$$J_{\mu\nu} = \frac{1}{12}(4)T^{\alpha\gamma}(4)T_{\alpha\beta} - \frac{1}{4}(4)T_{\mu\alpha}(4)T^{\alpha\beta}_{\nu} + \frac{1}{24}(4)g_{\mu\nu} \left[ 3(4)T_{\alpha\beta}(4)T^{\alpha\beta} - (4)T^2 \right].$$

(12)

The second correction term $\epsilon_{\mu\nu}$ is given by

$$\epsilon_{\mu\nu} = (5)C_{\alpha\beta\gamma\delta}n^{\gamma}(4)T_{\alpha\beta}(4)g_{\mu\nu}^{\alpha\beta},$$

(13)

and is a part of the 5D Weyl tensor and carries information of the gravitational field outside the brane and is constrained by the motion of the matter on the brane, i.e., the Codazzi equation

$$(4)\nabla_\mu K^\mu_{\nu} - (4)\nabla_{\nu} K = (5)R_{\mu\rho}(4)g_{\nu}^{\mu} n^\rho.$$  

(14)

Further, we have for the extrinsic curvature from the junction conditions

$$K_{\mu\nu} = -\frac{1}{2}\kappa_5^2 \left( (4)T_{\mu\nu} + \frac{1}{3}(\lambda_4 - (4)T)(4)g_{\mu\nu} \right),$$

(15)
and the 4D contracted Bianchi equations

\[(4)\nabla^v \varepsilon_{\mu v} = \kappa_5^4 (4) \nabla^v \varepsilon_{\mu v}.\]  

(16)

From the 5D Einstein en Bianchi equations one obtains supplementary equations

\[\mathcal{L}_n K_{\mu \nu} = K_{\mu \alpha} K_{\nu}^{\alpha} - \varepsilon_{\mu \nu} - \frac{1}{6} \Lambda_5 (4) g_{\mu \nu},\]  

(17)

\[\mathcal{L}_n \varepsilon_{\mu \nu} = (4) \nabla^\alpha \mathcal{B}_{\alpha (\mu \nu)} + \frac{1}{6} \Lambda_5 (K_{\mu \nu} - (4) g_{\mu \nu} K) + K^{\alpha \beta} R_{\mu \alpha \nu \beta} + 3K^{\alpha}_{(\mu \varepsilon \nu)\alpha} - K^\varepsilon_{\mu \nu} + (K_{\mu \alpha} K_{\nu \beta} - K_{\alpha \beta} K_{\mu \nu}) K_{\alpha \beta},\]  

(18)

\[\mathcal{L}_n \mathcal{B}_{\mu \nu \alpha} = -2(4) \nabla_{[\mu \varepsilon \nu] \alpha} + K^{\beta}_{\alpha} \mathcal{B}_{\mu \nu \beta} - 2 \mathcal{B}_{\alpha [\mu} K^{\beta}_{\nu]},\]  

(19)

with \(\mathcal{B}\) the "magnetic" part of the bulk Weyl tensor. From these supplementary equations one can also separate the equations in Eq. (7) and in Eq. (9) for \(W_1(t, r)\) and \(W_2(y)\) which proves that the supplementary equations are consistent with the bulk equations. From the 4D scalar-gauge field equations we obtain

\[\ddot{\psi} - \dot{P} = -\frac{P'}{r} + 2(P' \psi' - \dot{\psi}) - e^2 W_1^2 e^{2\gamma - 2\psi} P X^2,\]  

(20)

\[\ddot{X} - \dot{X}' = \frac{X'}{r} + 2\left(\frac{W_1 W'}{W_1} - \frac{W_1 X}{W_1}\right) - \frac{e^{2\gamma} XP^2}{r^2} - \frac{1}{2} W_1^2 e^{2\gamma - 2\psi} \beta X (X^2 - \eta^2),\]  

(21)

which are consistent with those obtained by Gregory [15]. From the Einstein equations we obtain

\[\dot{\gamma} = \frac{1}{W_1} \left[\frac{W'}{r} - \gamma W' - \frac{1}{2r} \gamma' W_1 - \frac{1}{2W_1} (W_1^2 + 3W_1^2) + \frac{1}{2W_1} (\psi' \psi^2 + \psi^2) \right] + W_1 \psi + \dot{W_1} \psi + \frac{3}{8} \kappa_5^2 \left(\frac{e^{2\psi} (\dot{\beta}^2 + P^2)}{e^{2\psi} W_1^2} + W_1 (X^2 + X^2)\right) + \frac{\kappa_5^4}{128} \left(3\beta (X^2 - \eta^2)^2 + 8 e^{2\psi} X^2 P^2 r^2 W_1^2 + 4 \frac{e^{2\psi} - 2\gamma (X^2 - X^2)}{W_1^2} \right) \left(\frac{e^{2\psi} (\dot{\beta}^2 + P^2)}{e^{2\psi} W_1^2} + W_1 (X^2 + X^2)\right),\]  

(22)

\[\ddot{\psi} - \dot{\psi}' = \frac{1}{r} \psi' - \frac{W_1 W'}{W_1} + 2 \frac{W_1 (\dot{W_1} \psi' - \dot{\psi}_1)}{W_1} + \frac{3}{4r^2} \left(\frac{e^{2\psi} (\dot{\beta}^2 + P^2)}{e^{2\psi} W_1^2} - X^2 P^2 e^{2\gamma}\right) + \frac{\kappa_5^4}{64 r^2} \left(3\beta (X^2 - \eta^2)^2 + 4 \frac{e^{2\psi} X^2 P^2 r^2 W_1^2}{W_1^2} - 8 \frac{e^{2\psi} - 2\gamma (X^2 - X^2)}{W_1^2} \right) \left(\frac{e^{2\psi} (\dot{\beta}^2 + P^2)}{e^{2\psi} W_1^2} - X^2 P^2 e^{2\gamma}\right).\]  

(23)

In the 4D case [15], one could safely omit the terms \(\dot{W_1}/W_1\), because \(r_s/R_H = \dot{W_1}/W_1 << 1\). In our case, this is not allowed. By the warpfactor, one obtains a huge angle deficit, or equivalently, a tremendous mass per unit length \(G\mu \sim 1\), while the warped manifestation in the brane will be warped down to GUT scale, consistent with observations. The time dependent part of the warpfactor \((W_1/W_1)\) causes disturbances of the order much larger than the expected values in the 4D case.
III. NUMERICAL SOLUTION

We solved numerically the set of PDE’s and plotted in figure 2 a typical solution for the choice of $W_1 = 0.1 \sqrt{\frac{e^{r-t}}{r}}$. We used value-boundary conditions at $r = r_0$ and Neumann boundary conditions at $r = r_{\text{end}}$. For the initial values we choose: $\psi(0, r) = \frac{1}{2} e^{-2r} \sin 3r$, $\gamma(0, r) = 0$, $P(0, r) = e^{-0.5r}$ and $X(0, r) = 1 - e^{-0.5r}$. It is evident that the emission of retarded wave energy of the scalar-gauge field, triggered by $W_1$, changes

FIG. 2. Solution of the metric components, the scalar-gauge fields, the energy momentum tensor component $T_{rr}$, the C-energy and the flux of C-energy.
the evolution of the induced brane metric. This is a non-local effect from the bulk. The relevant components of the energy-momentum tensor \( {T^{(4)}}_{\mu\nu} \) are \( T_{rr} \) and \( T_{\phi\phi} \):

\[
{({T^{(4)}}_{rr})} = -\frac{1}{8} B W_1^2 e^{2\gamma - 2\psi} (X^2 - \eta^2)^2 + \frac{1}{2} (X^2_r + X^2_t) + \frac{1}{2} \frac{e^{2\psi}}{e^2 W_1^2} (P_r^2 + P_t^2) + \frac{1}{2} \frac{X^2 P^2 e^{2\gamma}}{r^2},
\]

\[
{({T^{(4)}}_{\phi\phi})} = -\frac{1}{8} B W_1^2 e^{-2\gamma} r^2 (X^2 - \eta^2)^2 + \frac{1}{2} e^{-2\gamma} (X^2_t - X^2_r) - \frac{1}{2} \frac{e^{2\psi - 2\gamma}}{e^2 W_1^2} (P_t^2 - P_r^2) + \frac{1}{2} X^2 P^2.
\]

We observe that there are two terms which will become dominant for different epochs of cosmic time, i.e., the potential term \( (X^2 - \eta^2)^2 \), acting as a negative tension and the gauge field term \( (P_t^2 \pm P_r^2) \). They both contain the \( W_1^2 \)-term in the numerator and denominator respectively. The same arguments hold for the quadratic term \( \mathcal{J}_{\mu\nu} \). In figure 2 we also plotted the Brown–York quasi-local C-energy per unit Killing length \( z \) [38], \( 1 - e^{-2\gamma} \) and the advanced and retarded C-energy flow along the two null-directions, \( (\gamma_t + \gamma_r) \) and \( (\gamma_t - \gamma_r) \). Further, we plotted the warped equivalents of the C-energy and fluxes, i.e., \( W_1^2 (1 - e^{-2\gamma}) \).

The difference is manifest. Because \( \gamma \) doesn’t decouple from the equation for \( \psi \), we have two degrees of freedom for the gravitational waves (in the Einstein-Rosen case there is one degree of freedom), i.e., \( \psi \) and \( W_1 \). It turns out that the wavelike behavior depends critically on the ratio of the scalar to gauge masses, \( \alpha \equiv \sqrt{\frac{m_\psi}{m_\gamma}} \). We took \( \alpha > 1 \). Otherwise we are dealing with a global string in stead of a gauge string [11].

We observe a regular behavior at future null infinity. We also observe from the behavior of \( e^{-2\psi} = \frac{\text{gsec}}{\text{pl}} \) in figure 2 that our spactime is asymptotically non-conical: it approaches a constant value \( > 1 \). It was found by Gowdy [31] in the vacuum situation, that the peak of the pulse wave of \( \psi \) occurs at \( r = t \) (as is the case in our situation) and has asymptotically a \( \frac{\ln(r)}{r} \) behavior. These pulses are not localized and fill the \( r = t \) event horizon of the universe. This anomalous behavior can be scaled away, in order to obtain the expected \( \frac{1}{r} \). In our case we can fine-tune the parameters of the warpfactor \( W_1 (t, r) \), i.e., \( \tau \) and \( d_i \) in Eq. (10). They are determined by the parameters \( \alpha, \beta \) and \( \eta \). Further, we see an evolution of a single pulse wave into a kind of “notched” wave, with the minimum moving downwards. This was also found by Gowdy [31].

**IV. CONCLUSION**

We find a significant new late-time behavior of the warped FLRW cosmological model, when a self-gravitating U(1) scalar-gauge field (cosmic string) is present on the brane and the effective cosmological constant on the brane is zero (RS fine-tuning). Compared with the 4D counterpart model, the cylindrically symmetric disturbances have a significant impact on the late-time expansion rate of the universe. This is caused by the fact that one has no bound on the mass per unit length of the cosmic string, as is the case in the 4D model of Gregory [15]. We find an exact solution for the warpfactor, \( \psi' = W_1 (t, r) W_2 (y) \). The
time-dependent part, which plays the role of a “scale” factor, is a monotone increasing function with an inflection point or has a minimum. We don’t need the conventional dark energy generated by an effective brane cosmological constant.

The wave propagation in the warped model will survive the expansion, as can be seen from the warped C-energy and retarded flux in the numerical solution of figure 2. These disturbances could act as an effective dark energy field. The amplitude of the cylindrical pulse wave can fall off asymptotically at null infinity as $\frac{1}{r}$ by suitable choices of the parameters of the model.

---

[1] Spergel, D. N. et al.: First-Year WMAP Observations: determination of cosmological parameters. Astrophys. J. Suppl. 148, 175 (2003) [arXiv: astro-ph/0302209]
[2] Peiris, H. V. et al.: First-Year WMAP observations: Implications for inflation. Astrophys. J. Suppl. 148, 213 (2003) [arXiv: astro-ph/0302225]
[3] Spergel, D. N. et al.: Three-Year WMAP observations: Implications for cosmology. Astrophys. J. Suppl. 170, 377 (2007) [arXiv: astro-ph/0603449]
[4] Perlmutter, S.: et al.: Measurements of $\Omega$ and $\Lambda$ from 42 High-Redshift Supernovae Astrophys. J. 517, 565 (1999) [arXiv: astro-ph/9812133]
[5] Riess, A. G. et al.: New Hubble Space Telescope Discoveries of Type Ia Supernovae at $z \geq 1$: narrowing constraints on the early behavior of dark energy. Astrophys. J. 659, 98 (2007) [arXiv: astro-ph/0611572]
[6] Maartens, R.: Dark Energy and Dark Gravity J. Phys. Conf. Ser. 68, 012046 (2007), Dark energy from braneworld gravity. Lect. Notes. Phys. 720, 323 (2007) [arXiv: astro-ph/0602415v1]
[7] Felsager, B.: Geometry, Particles and Fields. Odense University Press, Odense (1981)
[8] Nielsen, H. B., Olesen, P.: Vortex-line models for dual strings. Nucl. Phys. B 61, 45 (1973)
[9] Garfinkle, D.: General relativistic strings. Phys. Rev. D 32, 1323 (1985)
[10] Slagter, R. J., Masselink, D.: Warped self-gravitating U(1) gauge cosmic strings in 5D. Int. J. Mod. Phys. D 21, 1250060 (2012) [arXiv:gr-qc/1202.4487]
[11] Slagter, R. J.: Time evolution of a warped cosmic string. Int. J. Mod. Phys. D 10, 1237 (2014)
[12] Slagter, R. J.: Dark energy generated by warped cosmic strings. arXiv: gr-qc/14077505. (2014)
[13] Laguna-Castillo, P., E. A. Matzner, E. A.: Coupled field solutions for U(1) gauge cosmic strings. Phys. Rev. D 36, 3663 (1987)
[14] Laguna-Castillo, P., Garfinkle, D.: Spacetime of supermassive U(1) gauge cosmic strings. Phys. Rev. D 40, 1011 (1989)
[15] Gregory, R.: Cosmological cosmic strings. Phys. Rev. 39, 2108 (1989)
[16] Kaluza, T.: Zum unitäts problem in der physik. Sitzungsber. K. Preuss. Akad. Wiss. Berlin, 966 (1921)
[17] Klein, O.: Quantentheorie und fünfdimensionale relativitätstheorie. Zeitsch. Phys. 37, 895 (1926)
[18] Arkani-Hamed, N, Dimopoulos, S., Dvali, G.: The hierarchy problem and new dimensions at a millimeter. Phys.
[19] Arkani-Hamed, N., Dimopoulos, S., Dvali, G.: Phenomenology, astrophysics and cosmology of theories with submillimeter dimensions and TeV scale quantum gravity. Phys. Rev. D 59, 086004 (1999) (arXiv: hep-ph/9807344)

[20] Antoniadis, I., Arkani-Hamed, N., Dimopoulos, S., Dvali, G.: New dimensions at a millimeter to a Fermi and superstrings at a TeV. Phys. Lett. B 436, 257 (1998) [arXiv: hep-ph/9804398]

[21] Randall, L., Sundrum, R.: Large mass hierarchy from a small extra dimension. Phys. Rev. Lett. 83, 3370 (1999) [arXiv: hep-ph/9905221]; An alternative to compactification. Phys. Rev. Lett. 83, 4690 (1999) [arXiv: hep-th/9906064]

[22] Maartens, R.: Brane-world cosmological perturbations: a covariant approach. Prog. Theor. Phys. Suppl. 148, 213 (2003) [arXiv: gr-qc/0312059]

[23] Maartens, R., Koyama, K.: Brane-world gravity. Liv. Rev. Relt 13, 5 (2010) [arXiv: gr-qc/10043962v2]

[24] Shiromizu, T., Maeda, K., Sasaki, M.: The Einstein equations on the 3-brane world. Phys. Rev. D 62, 024012 (2000)

[25] Slagter, R. J.: Cosmic strings in brane world models. In: Dabrowski, M. P., et al. (ed.) Proceedings of the conference Multiverse and Fundamental Cosmology, pp.39-42. American Inst. Phys., NewYork. (2012)

[26] Hutsemekers, D., Braibant, L., Pelgrims, V., Sluse, D.: Alignment of quasar polarizations with large-scale structures. Astron. Astrophys. 572, A 18 (2014) [arXiv: astro-ph/1409.6098]

[27] Beck, G.: Zur theorie binarer gravitationsfelder. Z. Physik 33, 713 (1925)

[28] Einstein, A., Rosen, N. JK.: On gravitational waves. J. Franklin Inst. 223, 43 (1937)

[29] Weber, J., Wheeler, J. A.: Reality of the cylindrical gravitational waves of Einstein and Rosen. Rev. Mod. Phys. 29, 509 (1957)

[30] Gowdy, R. A., Edmonds, B. D.: Cylindrical gravitational waves in expanding universes: models for waves from compact sources. Phys. Rev. D75, 084011 (2007)

[31] Gowdy, R. A.: Cylindrical gravitational waves in expanding universes: explicit pulse solutions. arXiv: gr-qc/07100987v1 (2007)

[32] H. Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C., Herlt, E.: In: Landshoff, P. V., et al.(ed.) Exact Solutions of Einstein’s Field Equations. Cambridge University Press, Cambridge, (2009)

[33] Stachel, J. J.: Cylindrical gravitational waves. J. Math. Phys. 7, 1321 (1968)

[34] Ashtekar, A., Bicak, J., Schmidt, B. G.: Behavior of Einstein-Rosen waves at null infinity. Phys. Rev. D 55, 687 (1997) [arXiv: gr-qc/9608041v1]

[35] Anderson, M. R.: In: Foster, B., et al. (ed.) The Mathematical Theory of Cosmic Strings. Institute of Physics Publishing, Bristol, (2003)

[36] Shiromizu, T., Maeda, K., Sasaki, M.;: The Einstein equations on the 3-brane world. Phys. Rev. D 62, 024012 (2000) [arXiv: gr-qc/9910076]

[37] Sasaki, M., Shiromizu, T., Maeda, K.;: Gravity, stability and energy conservation on the Randall-Sundrum brane world. Phys. Rev. D 62, 024008 (2000) [arXiv: hep-th/9912233]
[38] Goncalves, M. C. V.: Unpolarized radiative cylindrical space-times: Trapped surfaces and quasilocal energy. Class. Quant. Grav. 20, 37-49 (2003) [arXiv: gr-qc/0212125]