QUASI-PERIODIC SOLUTION OF QUASI-LINEAR
FIFTH-ORDER KDV EQUATION

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Abstract. We prove the existence of quasi-periodic small-amplitude solutions
for quasi-linear Hamiltonian perturbation of the fifth order KdV equation on
the torus in presence of a quasi-periodic forcing.

1. Introduction and main result. The existence of quasi-periodic solution of
Hamiltonian partial differential equations (HPDEs) has been studied for a long
time. The considered HPDEs are usually some linear or nonlinear integrable
equations with perturbations. According to the feature of them, the perturbations
can be classified into bounded and unbounded ones. The HPDEs with bounded
perturbations have been firstly studied by Kuksin,Wayne and Bourgain in [13],[25]
and [8]. In this direction there are too many references for us not to list here. In
the present paper, we focus on the HPDEs with unbounded perturbations.

If the perturbation is unbounded, the homological equation in the KAM iteration
reads as follows:

$$-i\omega \cdot \partial_x u + \lambda u + \mu(\phi) u = p(\phi), \quad \phi \in T^v,$$

(1)

here $\mu(\phi)$ has zero average, and $\mu(\phi) \approx \gamma$ is usually of large magnitude. The
equation of this type is called a small-denominator equation with large variable
coefficient. It is crucial to get appropriate estimation of such equation. Assuming
$\lambda \geq |\omega|^{1+\beta}$ for some $\beta > 0$, Kuksin gave a valid estimate of the solution in
[14], which is applied to the KdV equation and a whole hierarchy of so called higher
order KdV equations, see [12], [15] and [16]. Subsequently, Liu-Yuan [19] gave a new
estimate, including both $\beta > 0$ and $\beta = 0$, which extends the application of KAM
theory to 1-dimensional derivative NLS (DNLS) and Benjamin-Ono equations. See
[20],[28] and [31]. The case $\beta < 0$ is corresponding to quasi-linear or fully nonlinear
equations, for which there has not yet any clue to get a required estimation of the
solution for (1).

Recently, in a series of papers [1, 2, 3, 4, 6, 10, 11, 22], Baldi-Berti-Feola-Montalto
invented a sophisticated tool to deal with the case $\beta < 0$ for some quasi-linear or
fully nonlinear partial differential equations, such as KdV and water wave equation.
Take the fully nonlinear KdV equation

$$\partial_t u + u_{xxx} + f(\omega t, x, u, u_x, u_{xxx}) = 0, \quad x \in T := \mathbb{R}/2\pi\mathbb{Z},$$

(2)

2010 Mathematics Subject Classification. Primary: 37K55; Secondary: 37F50, 35Q53.
Key words and phrases. KAM theory, quasi-linear PDEs, fifth-order KdV equation, quasi-
periodic solutions.
Supported by NNSFC11421061.
as an example. By Nash-Moser iteration, the linearized homological equation can be seen as $L v = F$, where
\[ L = \omega \cdot \partial_\varphi + (1 + a_3(\varphi, x))\partial_x^3 + a_1(\varphi, x)\partial_x + a_0(\varphi, x), \quad \varphi \in \mathbb{T}^v. \] (3)

It is crucial to estimate the inverse of the linear operator $L$. Instead of directly reducing the linear operator $L$ to a diagonal operator, Baldi-Berti-Feola used some sophisticated way to reduce the linear operator $L$ to a diagonal operator plus a bounded perturbation by a set of regularization procedures. For example, the regularization procedures for the fully nonlinear KdV equation can be summarized as:

• To eliminate the space variable dependence of the coefficients of $\partial_x^3$ by a $\varphi$-dependent changes of variable. Then, to eliminate the time dependence of the coefficients of $\partial_x^3$ by a quasi-periodic time re-parametrization (See [2] for the detail).

The linear operator $L$ is thus reduced to
\[ L_1 = \omega \cdot \partial_\varphi + m_3\partial_x^3 + b_1(\varphi, x)\partial_x + b_0(\varphi, x), \] (4)
where $m_3 \in \mathbb{R}$ is a constant.

• The regularization procedure to dispose the coefficients of $\partial_x$ can be divided into two steps.

The first step is to use the space variable change $y = x + p(\varphi)$, by which the differential operator $\omega \cdot \partial_\varphi$ becomes
\[ \omega \cdot \partial_\varphi + \omega \cdot \partial_x p(\varphi) \partial_y. \] (5)
At this time, the linear operator $L_1 - \omega \cdot \partial_\varphi$ becomes
\[ m_3\partial_y^3 + b_1(\varphi, y - p(\varphi))\partial_y + b_0(\varphi, y - p(\varphi)). \] (6)

One sees that the coefficients of $\partial_y$ is $\omega \cdot \partial_x p(\varphi) + b_1(\varphi, y - p(\varphi))$. The term $\omega \cdot \partial_x p(\varphi)$ in (5) is used to amend the coefficients of $\partial_y$, which guarantees the spatial average of the coefficients of $\partial_y$ is a constant.

On the basis of the first step, one uses some pseudo-differential operator technique to reduce the linear operator $L_1$ to
\[ L_2 = \omega \cdot \partial_\varphi + m_3\partial_y^3 + m_1\partial_x + R, \] (7)
where $m_3, m_1 \in \mathbb{R}, R$ is a bounded remainder.

However, there are some difficulties to handle quasi-linear or fully nonlinear higher order KdV equations with the regularization method. Consider the fully nonlinear fifth order KdV equation
\[ \partial_t u + \partial_x^5 u + f(\omega t, x, u, \partial_x u, \partial_x^2 u, \partial_x^3 u, \partial_x^4 u) = 0, \quad x \in \mathbb{T}. \] (8)
The linearized homological equation still be seen as $L v = F$, with
\[ L = \omega \cdot \partial_\varphi + a_5\partial_x^5 + a_3\partial_x^3 + a_2\partial_x^2 + a_1\partial_x + a_0, \quad \varphi \in \mathbb{T}^v, \] (9)
where $a_i$ is the function of $(\varphi, x)$.

Applying the delicate variable change as above to $L$, then, the linear operator $L$ is reduced to
\[ L_1 = \omega \cdot \partial_\varphi + m_5\partial_x^5 + b_3\partial_x^3 + b_2\partial_x^2 + b_1\partial_x + b_0, \] (10)
where $m_5 \in \mathbb{R}$ is constant and $b_i$’s are function of $(\varphi, x)$.

When one tries to reduce the coefficient $b_3 = b_3(\varphi, x)$ of $\partial_x^3$ to constant, a new difficulty arises: the variable change $y = x + p(\varphi)$ to amend the coefficients of $\partial_x$ can not be applied to the coefficients of $\partial_x^3$. Therefore, a suitable regularization
procedure for the higher-order KdV equations seems to be more complicated and maybe need some new technical method.

In this paper we make attempt to deal with the problem by combing regularization method by Baldi-Berti-Feola and the unbounded reduction method by Kuksin [5, 16]. Consider the Hamiltonian quasi-linear fifth order KdV equation of the following form

$$
\partial_t u = X_H(u),
$$

(11)

with

$$
X_H(u) = \partial_x \nabla_{L^2} H(t, x, u, u_x, u_{xx}),
$$

(12)

and

$$
H(t, x, u, u_x, u_{xx}) = \int_T -\frac{1}{2} u^2_{xx} + 5uu_x^2 - \frac{5}{2} u^4 + g(\omega t, x, u, u_x, u_{xx}) dx.
$$

(13)

The perturbation $g(\omega t, x, u, u_x, u_{xx})$ is quasi-periodic about time variables and periodic about space variables, also a polynomial about $u, u_x, u_{xx}$. Here and in other places in this paper, $\int_{\Gamma_t}$ is short for the average $(\omega t, x, u, u_x, u_{xx})$, by a slight abuse of notation.

The primary fifth-order KdV equation without perturbation is

$$
u_t + \partial_x^5 u + 10u\partial_x^3 u + 20\partial_x u\partial_x^2 u + 30u^2 \partial_x u = 0,
$$

(14)

which is a special case of the general fifth-order KdV equation (fKdV) of the following

$$
u_t + \partial_x^5 u + \gamma u\partial_x^3 u + \beta \partial_x u\partial_x^2 u + \alpha u^2 \partial_x u = 0.
$$

(15)

The special case (14) is called the Lax case, which is characterized by $\beta = 2\gamma$ and $\alpha = \frac{3}{10}\gamma^2$. This general fifth-order KdV equation describes motions of long waves in shallow water under gravity. In a one-dimensional nonlinear lattice, it is an important mathematical model with wide application in quantum mechanics and nonlinear optics. Typical examples are widely used in various fields such as solid state physics, plasma physics, fluid physics, and quantum field theory. The relevant research can be found in [27, 26] and [18].

Since our purpose is to dispose quasi-linear equation, set

$$
g = u^3_{xx} + f(\omega t, x)u.
$$

The concrete form of the equation is

$$
u_t = -\partial_x^5 u - 10u\partial_x^3 u - 20\partial_x u\partial_x^2 u - 30u^2 \partial_x u + 6\partial_x^3 u\partial_x^2 u + 18\partial_x^3 u\partial_x^4 u + \partial_x f(\omega t, x),
$$

(16)

where $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, $\partial_x f(\omega t, x)$ is quasi-periodic in time with Diophantine frequency vector, namely

$$
\omega = \lambda \mathbb{Z}, \quad \lambda \in \Pi = \left\{ \frac{1}{2}, \frac{3}{2} \right\}, \quad |\omega \cdot \ell| \geq \frac{\alpha_0}{|\ell|^r}, \quad \forall \ell \in \mathbb{Z}^r \setminus \{0\},
$$

(17)

Clearly, if $\partial_x f(\omega t, x)$ is not identically zero, then $u = 0$ is not a solution of (16). Thus we look for non-trivial solutions $u(\varphi, x)$ of the fifth-order KdV equation in the analytical space $H_{s,p}(\mathbb{T}^{r+1})$

$$
\begin{cases}
    u(\omega t, x) = \sum_{(\ell, k) \in \mathbb{Z}^r \times \mathbb{Z}} u_{\ell, k} e^{i\omega t} e^{ikx}, \\
    \|u\|_{s,p}^2 := \sum_{(\ell, k) \in \mathbb{Z}^r \times \mathbb{Z}} |u_{\ell, k}|^2 e^{2(|\ell|+|k|)s} ([|\ell|] + [|k|])^{2p} < +\infty,
\end{cases}
$$

(18)
where
\[ |\ell| = |\ell_1| + \cdots + |\ell_v|, \quad [\ell] = \max(|\ell|, 1). \]  
(19)

The Banach space \( H_{s,p} \) can be extended to the analytic functions defined on \( T^b \) with any integer \( b > 0 \). If \( u(\varphi) = \sum_{k \in \mathbb{Z}^b} u_k e^{ik\varphi}, \varphi \in T^b, \) by the method in [23], we can denote
\[ (\|u\|_{s,p})^2 = \sum_{k \in \mathbb{Z}^b} |u_k|^2 e^{2(|k|s)[k]2p}. \]

For notation convenience, when \( p = 0, \) \( \|\cdot\|_{s,0} \) is simplified as \( \|\cdot\|_s \).

Our main result is

**Theorem 1.1.** Assume that \( \omega \) satisfies Diophantine Condition (17) and assume that there are constants \( q = q(v) > 0, \varepsilon_0 = \varepsilon_0(v) > 0 \) and \( s > 0 \) such that
\[ \|\partial_x f(\omega t, x)\|_{s,q} \leq \varepsilon \]
with \( \varepsilon \in (0, \varepsilon_0) \) and \( \varepsilon_0 = \varepsilon_0(v) > 0 \) being small enough. Then there exists a Cantor set \( \Pi_\varepsilon \subseteq \Pi \) of asymptotically full Lebesgue measure, i.e.,
\[ |\Pi_\varepsilon| \to 1 \] as \( \varepsilon \to 0, \)
such that for every \( \lambda \in \Pi_\varepsilon, \) the KdV equation (16) admits a solution \( u(t, x) \in C^\infty(\mathbb{R} \times T) \) which is quasi-periodic in time \( t \) with frequency \( \omega = \lambda \tilde{\omega}. \)

Although only the quasi-linear fifth-order KdV equation is investigated, the method in Theorem 1.1 can also be applied to other quasi-linear Hamiltonian higher order KdV equations, for example, the seventh order, even to some other quasi-linear or fully-nonlinear equations.

2. **Functional setting.** In this section, we introduce some notations, definitions and technical tools, which will be used in the following section.

The phase space of (16) is
\[ H^0_0(T) = \{ u(x) \in H_{0,1}(T, \mathbb{R}) : \int_T u(x) dx = 0 \} \]
endowed with non-degenerate symplectic form
\[ \Omega(u, v) = \int_T (\partial_x^{-1} u)v dx \quad \forall u, v \in H^0_0(T), \]
(21)
where \( \partial_x^{-1} u \) is the periodic primitive of \( u \) with zero average. The Hamiltonian vector field \( X_H(u) = \partial_x \nabla H(u) \) is the unique vector field satisfying the equality
\[ dH(u)[h] = (\nabla H(u), h)_{L^2(T)} = \Omega(X_H(u), h), \quad \forall u, h \in H^0_0(T), \]
(22)
where for all \( u, v \in L^2(T) \)
\[ (u, v)_{L^2(T)} = \int_T u(x)v(x) dx = \sum_{j \in \mathbb{Z}} u_j v_{-j}, \]
(23)
\[ u(x) = \sum_{j \in \mathbb{Z}} u_j e^{ijx}, \quad v(x) = \sum_{j \in \mathbb{Z}} v_j e^{ijx}. \]
(24)
Recall the Poisson bracket between two Hamiltonians \( F, G : H^0_0(T) \to \mathbb{R} \) are
\[ \{F(u), G(u)\} = \Omega(X_F, X_G) = \int_T \nabla F(u) \partial_x \nabla G(u) dx, \]
(25)
The function in this paper is quasi-periodic about time variables and periodic about space variable. It is also analytic for these variables in the domain of \( T^{s+1}_v, \)
where $\mathbb{T}_s^{v+1}$ be the complexified torus with $|\Im x_i| \leq s$. Therefore, the function will be of the form

$$u(\omega t, x) := \sum_{(\ell,k) \in \mathbb{Z}^s \times \mathbb{Z}} u_{\ell,k} e^{i\omega \ell \tau x} e^{i\ell k x} = \sum_{(\ell,k) \in \mathbb{Z}^s \times \mathbb{Z}} u_{\ell,k} e^{i\ell k x}. \quad (26)$$

Now, we define some important norms:

**Definition 2.1.** In contrast with Banach space $H_{s,p}(\mathbb{T}_s^{v+1})$ defined in (18), we define a new Banach space $H^s_{s,p}(\mathbb{T}^s \times \mathbb{T}^s)$ as

$$H^s_{s,p}(\mathbb{T}^s \times \mathbb{T}^s) := \left\{ u(\varphi, x) = \sum_{(\ell,k) \in \mathbb{Z}^s \times \mathbb{Z}} u_{\ell,k} e^{i\ell k x} : \left( \|u\|_{s,p}^2 \right)^2 = \sum_{(\ell,k) \in \mathbb{Z}^s \times \mathbb{Z}} |u_{\ell,k}|^2 e^{2(|\ell|+|k|)s[|k|]} \left| e^{2p \ell^2} \right| < +\infty \right\}, \quad (27)$$

where

$$|\ell| = |\ell_1| + \cdots + |\ell_v|, \quad |\ell| = \max(|\ell|, 1).$$

For analytic function $u$ defined on $\mathbb{T}_s^{v+1}$, the max norm plays important role in our paper

$$|u|_{s,p} = \max_{|k| \leq _p (\varphi, x) \in \mathbb{T}_s^{v+1}} |D^k u(\varphi, x)|.$$

As a notation, we denote $a < b$ as $a \leq Cb$, where $C$ is a constant depending on the form of equation, the number $v$ of frequencies, the diophantine exponent $\tau$ in the non-resonance condition. $a \approx b$ means $a \leq C b$ and $C a \geq b$.

When we consider a function $f : \Pi \to E$, $\lambda \to F(\lambda)$, where $(E, \|\cdot\|_E)$ is the Banach space and $\Pi$ is the subset of $\mathbb{R}$, we can define sup-norm and Lipschitz semi-norm below.

**Definition 2.2.**

$$\|f\|_{E,\Pi}^{{\sup}} = \sup_{\lambda \in \Pi} \|f\|_E, \quad \|f\|_{E,\Pi}^{Lip} = \sup_{\lambda_1, \lambda_2 \in \Pi, \lambda_1 \neq \lambda_2} \frac{\|f(\lambda_1) - f(\lambda_2)\|_E}{|\lambda_1 - \lambda_2|}.$$  

Then, the Lipschitz norm is

$$\|f\|_{E,\Pi}^{Lip} = \|f\|_{E,\Pi}^{Lip} = \|f\|_{E,\Pi}^{Lip} + \|f\|_{E,\Pi}^{Lip}.$$  

If $E$ is the space $H_{s,p}$, we denote $\|f\|_{E,\Pi}^{\Pi}$ by $\|f\|_{s,p}^{\Pi}$. When $E = H^s_{s,p}$, we denote $\|f\|_{E,\Pi}^{Lip}$ by $\|f\|_{s,p}^{Lip}$. Now, we show the relationship between Banach space $H_{s,p}$ and $H^s_{s,p}$.

**Lemma 2.3.**

$$\|u\|_{s,p}^2 \leq \|u\|_{s,2p} \leq 2^p \|u\|_{s,2p}^2. \quad (28)$$  

**Proof.** Notice that $2^p|k|p|\ell|^p \leq (|k| + |\ell|)^2p \leq 4^p|k|^{2p}|\ell|^{2p}$, we can get

$$\begin{align*}
\left( \|u\|_{s,2p}^2 \right)^2 &= \sum_{(\ell,k) \in \mathbb{Z}^s \times \mathbb{Z}} |u_{\ell,k}|^2 e^{2(|\ell|+|k|)s[|k|]} \left| e^{4p \ell^2} \right| \\
&\leq \frac{1}{4^p} \sum_{(\ell,k) \in \mathbb{Z}^s \times \mathbb{Z}} |u_{\ell,k}|^2 e^{2(|\ell|+|k|)s[|k|]} (|\ell| + |k|)^2p \\
&\leq (\|u\|_{s,2p}^2)^2.
\end{align*} \quad (29)$$

Then, the lemma is proved.

The algebra properties of Banach space $H_{s,p}$ and $H^2_{s,p}$ are also our concern.

**Lemma 2.4.** For all $p > \frac{n+1}{2}$, if $h_1, h_2 \in H_{s,p}(T^{n+1})$, then $h_1 h_2 \in H_{s,p}(T^{n+1})$.

Also, there are $c(p) > 0$, such that

$$||h_1 h_2||_{s,p} \leq c(p)||h_1||_{s,p}||h_2||_{s,p}. \quad (31)$$

If $h_1 = h_1(\lambda)$ and $h_2 = h_2(\lambda)$ depend in a Lipschitz way on the parameter $\lambda \in \Pi \subset \mathbb{R}$, then

$$||h_1 h_2||_{s,p,\text{Lip}} \leq c(p)||h_1||_{s,p,\text{Lip}}||h_2||_{s,p,\text{Lip}}. \quad (32)$$

**Proof.** $(31)$ is the same as Lemma 6.3. The proof of $(32)$ is standard.

**Lemma 2.5.** For all $p > \frac{n}{2}$, if $h_1, h_2 \in H^2_{s,p}(T^n \times T)$, then $h_1 h_2 \in H^2_{s,p}(T^n \times T)$.

Also, there are $c(p) > 0$, such that

$$||h_1 h_2||^2_{s,p} \leq c(p)||h_1||^2_{s,p}||h_2||^2_{s,p}. \quad (33)$$

If $h_1 = h_1(\lambda)$ and $h_2 = h_2(\lambda)$ depend in a Lipschitz way on the parameter $\lambda \in \Pi \subset \mathbb{R}$, then

$$||h_1 h_2||^2_{s,p,\text{Lip}} \leq c(p)||h_1||^2_{s,p,\text{Lip}}||h_2||^2_{s,p,\text{Lip}}. \quad (34)$$

**Proof.** If $h_i \in H^2_{s,p}(T^n \times T), i = 1, 2$, then $h_i = \sum_{k \in \mathbb{Z}} \hat{h}^i_k(\varphi) e^{ikx}$, with

$$||h_i||^2_{s,p} = \sum_{k \in \mathbb{Z}} e^{2|k|s}||\hat{h}^i_k(\varphi)||^2. \quad (35)$$

Let $\gamma_{j,k} = \frac{|j-k||k|}{|j|}$, By the Schwarz inequality, we have

$$\left| \sum_k x_k \right|^2 = \left| \sum_k \frac{\gamma_{j,k} x_k}{\gamma_{j,k}} \right|^2 \leq c_j^2 \sum_k \gamma_{j,k}^2 |x_k|^2, \quad c_j^2 = \sum_k \frac{1}{\gamma_{j,k}^2}, \quad (36)$$

where

$$c_j^2 = \sum_k \frac{1}{\gamma_{j,k}^2} \leq \sum_k \left( \frac{1}{|j-k|} + \frac{1}{|k|} \right)^2 |x_k|^2 \leq 4p \sum_k \frac{1}{|k|^{2p}} = c^2 < +\infty. \quad (37)$$
For the case $s = 0$, we have

\[
\left\| h_1 h_2 \right\|_{s,p}^2 = \sum_j [j]^{2p} \left( \sum_k \left\| \hat{h}_1^{j-k} \varphi \right\|_s \right)^2 \leq \sum_j [j]^{2p} \left( \sum_k \left\| \hat{h}_1^{j-k} \varphi \right\|_s \right)^2 \leq c^2 \cdot \varepsilon(p,v) \sum_j [j]^{2p} \sum_k \gamma_{j,k}^2 \left\| \hat{h}_1^{j-k} \varphi \right\|_s^2 \left\| \hat{h}_2^k \varphi \right\|_s^2 = c_1 \sum_j \sum_k [j-k]^{2p} \gamma_{j,k}^2 \left\| \hat{h}_1^{j-k} \varphi \right\|_s \left\| \hat{h}_2^k \varphi \right\|_s^2 = c_1 (\| h_1 \|_{s,p}^2) (\| h_2 \|_{s,p}^2).
\]

The case $s > 0$ is a simple variation. \[\square\]

2.1. Matrices with variable. Let $b \in \mathbb{N}$, and consider the exponential basis $e_i : i \in \mathbb{Z}^b$ of $L^2(\mathbb{T}^b)$, so that $L^2(\mathbb{T}^b)$ is the vector space $u = \sum u_i e_i$, $\sum |u_i|^2 < \infty$. Any linear operator $A : L^2(\mathbb{T}^b) \to L^2(\mathbb{T}^b)$ can be represented by the infinite dimensional matrix

\[
(A_i^s)_{i,i' \in \mathbb{Z}^b}, \quad A_i^s := (A e_{i'}, e_i)_{L^2(\mathbb{T})}. \quad Au = \sum_{i,i'} A_i^s u_i e_i.
\]

**Definition 2.6.** Consider an infinite dimensional matrix $A(\varphi)$ of time variables, where $A(\varphi)_{i_1}^{i_2} = (A e^{i_2 x}, e^{i_1 x})_{L^2(\mathbb{T})}$. Thus, we define a $(s,p)$-decay Banach space $B_{s,p}$ as

\[
B_{s,p} := \left\{ A : (|A|)_{s,p}^2 = \sum_{i \in \mathbb{Z}} e^{2|i|s} [i]^{2p} \left( \sup_{i_1 - i_2 = i} \left\| A(\varphi)_{i_1}^{i_2} \right\|_{s,p} \right) < +\infty \right\}. \tag{39}
\]

So, for parameter dependent matrices $A := A(\lambda), \lambda \in \Pi \subseteq \mathbb{R}$, we can also define Lipschitz norms as

\[
|A|_{s,p}^{Lip} = \sup_{\lambda_1 \neq \lambda_2} \frac{|A(\lambda_1) - A(\lambda_2)|_{s,p}}{|\lambda_1 - \lambda_2|}, \quad |A|_{s,p}^{Lip} = |A|_{s,p}^{sup} + |A|_{s,p}^{Lip}.
\]

Now, we show some properties of $(s,p)$-decay norm.

**Lemma 2.7.** (Multiplication operator) Let $p = \sum p_i(\varphi) e_i \in H_{s,2p}$, the multiplication operator $h \to ph$ is represented by the matrix with variables $T_i^j(\varphi) = p_{i-\varphi}(\varphi)$ and

\[
|T|_{s,p} \leq \|p\|_{s,2p}.
\]

Moreover, if $p = p(\lambda)$ is a Lipschitz family of functions,

\[
|T|_{s,p}^{Lip} \leq \|p\|_{s,2p}^{Lip}.
\]

**Proof.** According to Definition 2.6, we see

\[
|T|_{s,p}^2 = \sum_{k \in \mathbb{Z}} e^{2|k|s} [k]^{2p} \sup_{i-j=k} \left\| T_i^j(\varphi) \right\|_{s,p}^2
\]
Then, the lemma is proved.

\[\sum_{k \in \mathbb{Z}} \|p_k(\varphi)\|_{s,p}^2 e^{2|\ell| s} |k|^2 p\]
\[= \sum_{(\ell,k) \in \mathbb{Z}^2 \times \mathbb{Z}} |p_{\ell,k}|^2 e^{2|\ell| s} |\ell|^2 |k|^2 p\]
\[\leq \sum_{(\ell,k) \in \mathbb{Z}^2 \times \mathbb{Z}} |p_{\ell,k}|^2 e^{2(|\ell|+|k|) s} ([\ell] + |k|)^4 p\]
\[= \|p\|_{s,2p}^2\]

Then, the lemma is proved. \(\square\)

**Definition 2.8.** Given a \(A \in B_{s,p}, h = \sum_{k \in \mathbb{Z}} h_k(\varphi) e^{ikx} \in H_{s,p}\), we say that \(A\) is dominated by \(h\), and we write \(A \prec h\), if \(\|A(\varphi)_{i_1}^k\|_{s,p} \leq \|h(\varphi)_{i_1-i_2}\|_{s,p}\) for all \(i_1, i_2 \in \mathbb{Z}\).

It can be seen that
\[|A|_{s,p} = \min \{\|h\|_{s,p} : h \in H_{s,p}, A \prec h\} \quad \text{and} \quad \exists h \in H_{s,p}, |A|_{s,p} = \|h\|_{s,p}. \quad (41)\]

**Lemma 2.9.** For \(A_1, A_2 \in B_{s,p}\), we have
\[A_1 \prec h_1, A_2 \prec h_2 \Rightarrow |A_1A_2|_{s,p} \leq C(p)\|h_1\|_{s,p}^p\|h_2\|_{s,p}^p. \quad (42)\]

**Proof.** For all \(i_1, i_2 \in \mathbb{Z}, i_1 - i_2 = i\), we have
\[\|(A_1A_2(\varphi))_{i_1}^k\|_{s,p} = \|\sum_{k \in \mathbb{Z}} A_1(\varphi)_{i_1}^k A_2(\varphi)_{k}^i\|_{s,p} \leq \sum_{k \in \mathbb{Z}} \|A_1(\varphi)_{i_1}^k\|_{s,p} \|A_2(\varphi)_{k}^i\|_{s,p}\]
\[\leq \sum_{k \in \mathbb{Z}} \|(h_1(\varphi))_{i_1-k}\|_{s,p} \|h_2(\varphi)_{k-i_2}\|_{s,p}\]
\[= \sum_{k \in \mathbb{Z}} \|h_1(\varphi)_{k}\|_{s,p} \|h_2(\varphi)_{i-k}\|_{s,p}, \quad (43)\]
implies \(|A_1A_2|_{s,p} \leq C(p)\|h_1\|_{s,p}^p\|h_2\|_{s,p}^p\), following from the proof of Lemma 2.5. \(\square\)

**Lemma 2.10.** (Classical algebra property) For all \(p > \frac{2}{5}\), if \(A, B \in B_{s,p}\), then \(AB \in B_{s,p}\). Also, there are \(c(p) > 0\), such that
\[|AB|_{s,p} \leq c(p)|A|_{s,p}|B|_{s,p}. \quad (44)\]

If \(A = A(\lambda)\) and \(B = B(\lambda)\) depend in a Lipschitz way on the parameter \(\lambda \in \Pi \subset \mathbb{R}\), then
\[|AB|_{Lip}^{s,p} \leq c(p)|A|_{Lip}^{s,p}|B|_{Lip}^{s,p}. \quad (45)\]

**Proof.** We can immediately deduce (44) from Lemma 2.5 and Lemma 2.9. The proof of (45) is standard. \(\square\)

**Lemma 2.11.** For all \(p > \frac{2}{5}\), if \(A \in B_{s,p}, h \in H_{s,2p}\), then \(Ah \in H_{s,p}\). Also, there are \(c(p) > 0\), such that
\[\|Ah\|_{s,p} \leq c(p)|A|_{s,p}\|h\|_{s,2p}. \quad (46)\]

If \(A = A(\lambda)\) and \(h = h(\lambda)\) depend in a Lipschitz way on the parameter \(\lambda \in \Pi \subset \mathbb{R}\), then
\[\|Ah|_{Lip}^{s,p} \leq c(p)|A|_{Lip}^{s,p}\|h|_{Lip}^{s,2p}. \quad (47)\]
Proof. From Lemma 2.5 and Lemma 2.9, we can immediately get \( \|Ah\|_{s,p}^2 \leq c(p) |A|_{s,p}^2 \|h\|_{s,p}^2 \). To prove (46), observe that
\[
\|Ah\|_{s,p}^2 \leq c(p) \|Ah\|_{s,p}^2 \leq c_1(p) |A|_{s,p}^2 \|h\|_{s,p}^2 \leq c_2(p) |A|_{s,p}^2 \|h\|_{s,2p}.
\]
\( \square \)

Lemma 2.12. Let \( \Phi = e^\Psi \) with \( \Psi := \Psi(\lambda) \), depending in a Lipschitz way on the parameter \( \lambda \in \Pi \subset \mathbb{R} \), such that \( c(p)\|\Psi\|_{s,p}^{Lip} \leq \frac{1}{2} \). Then \( \Phi \) is invertible and, for all \( p > \frac{1}{2} \),
\[
|\Phi^{-1}_{s,p}|^{Lip} \leq 2, \quad |\Phi - I|_{s,p}^{Lip} \leq C|\Psi|_{s,p}^{Lip}, \quad |\Phi^{-1} - I|_{s,p}^{Lip} \leq C|\Psi|_{s,p}^{Lip}.
\]
If \( \Phi_i = e^{\Psi_i}, i = 1, 2, \) satisfy \( c(p)\|\Psi_i\|_{s,p}^{Lip} \leq \frac{1}{2} \), then
\[
|\Phi_2 - \Phi_1|_{s,p}^{Lip} \leq C|\Psi_2 - \Psi_1|_{s,p}^{Lip}, \quad |\Phi_2^{-1} - \Phi_1^{-1}|_{s,p}^{Lip} \leq C|\Psi_2 - \Psi_1|_{s,p}^{Lip}.
\]
Proof. (49) are from the power series of \( e^\Psi \) and Lemma 2.10. To prove (50), we see
\[
\Phi_2 - \Phi_1 = e^{\Psi_2} - e^{\Psi_1} = \sum_{n=1}^{\infty} \frac{1}{n!} [\Psi_2^n - \Psi_1^n]
\]
(51)
and
\[
\Phi_2^{-1} - \Phi_1^{-1} = \Phi_1^{-1}(\Phi_1 - \Phi_2)\Phi_2^{-1}.
\]
Then, use (49). \( \square \)

2.2. Linear time-dependent operator and Hamiltonian operators. In this section, we give some definitions and properties of the linear time-dependent Hamiltonian systems which will be used in the following section.

Definition 2.13. A time dependent linear vector field \( X(t) : H_0^1(\mathbb{T}) \to H_0^1(\mathbb{T}) \) is HAMILTONIAN if \( X(t) = \partial_x G(t) \) for some real linear operator \( G(t) \) which is self-adjoint with respect to the \( L^2 \) scalar product. The vector product is generated by the quadratic Hamiltonian
\[
H(t,h) = \frac{1}{2} (G(t)h, h)_{L^2(\mathbb{T})} = \frac{1}{2} \int_{\mathbb{T}} G(t)[h]h dx, \quad h \in H_0^1(\mathbb{T}).
\]
If \( G(t) = G(\omega t) \) is quasi-periodic in time, we say that the associate operator \( \omega \cdot \partial_x - \partial_x G(\varphi) \) is Hamiltonian.

Definition 2.14. A linear operator \( A : H_0^1(\mathbb{T}) \to H_0^1(\mathbb{T}) \) is SYMPLECTIC if
\[
\Omega(Au, Av) = \Omega(u, v), \quad \forall u, v \in H_0^1(\mathbb{T}).
\]
where the symplectic 2-form \( \Omega \) is defined in (21). Equivalently \( A^T \partial_x^{-1} A = \partial_x^{-1} \).

If \( A(\varphi), \forall \varphi \in \mathbb{T}^n \), is a family of symplectic maps we say that the operator \( A \) defined by \( Ah(\varphi,x) = A(\varphi)h(\varphi,x) \), acting on the functions \( h : \mathbb{T}^{n+1} \to \mathbb{R} \), is symplectic.

Under a time dependent family of symplectic transformations \( u = \Psi(t)v \) the linear Hamiltonian equation
\[
u_t = \partial_x G(t)u \quad \text{with Hamiltonian} \quad H(t,u) := \frac{1}{2} (G(t)u, u)_{L^2}
\]
(55)
transforms into the equation

\[ v_t = \partial_x E(t)v, \quad E(t) = \Psi(t)^T G(t) \Psi(t) - \Psi(t)^T \partial_x^{-1} \Psi_\epsilon(t) \]

with Hamiltonian

\[ K(t, v) = \frac{1}{2} (G(t) \Psi(t)v, \Psi(t)v)_{L^2} - \frac{1}{2} (\partial_x^{-1} \Psi_\epsilon(t)v, \Psi(t)v)_{L^2}. \]

Note that \( E(T) \) is self-adjoint with respect to the \( L^2 \) scalar product because \( \Psi^T \partial_x^{-1} \Phi_\epsilon + \Psi_\epsilon^T \partial_x^{-1} \Phi = 0 \). If the operators \( G(t), \Psi(t) \) are quasi-periodic in time. The Hamiltonian operator \( \omega \cdot \partial_x - \partial_x G(\varphi) \) transforms into the operator \( \omega \cdot \partial_x - \partial_x E(\varphi) \), which is still Hamiltonian, according to the Definition 2.14.

3. The regularization of the linearized operator. In this section, we perform a regularization procedure, which conjugates the linearized operator \( \mathcal{L}(u_n) \) defined in (61) to the operator \( \mathcal{L}(u_n) \) defined in (121), the coefficients of the highest order spatial derivative operator are constants. The method has been used in [1, 6, 2, 4, 10, 11]. Our existence proof is based on a modified Newton iteration. The main step concerns the invertibility of the linearized operator

\[ \mathcal{L}h = \mathcal{L}(\lambda, u, \varepsilon)h = \omega \cdot \partial_x h + a_5^* \partial_x^2 h + a_4^* \partial_x^1 h + a_3^* \partial_x^0 h + a_2^* \partial_x^2 h + a_1^* \partial_x h + a_0^* h, \]

obtained by linearizing (16) at any approximate (or exact) solution \( u \). The coefficients \( a_i^* (\varphi, x) = a_i^* (u)(\varphi, x) \) are periodic functions of \( (\varphi, x) \), depending on \( u \). Then, we have

\[ a_5^* = (1 - 6 \partial_x^2 u), \quad a_4^* = (-18 \partial_x^3 u), \quad a_3^* = (10u - 18 \partial_x^2 u), \]
\[ a_2^* = (20 \partial_x u - 6 \partial_x^2 u), \quad a_1^* = (20 \partial_x^2 u + 30 u^2), \quad a_0^* = (10 \partial_x^3 u + 60 u \partial_x u). \]

In the Hamiltonian case (11), the linearized operator (58) also has the form

\[ \mathcal{L}h = \omega \cdot \partial_x h + \partial_x \{ \partial_x^2 [(a_2(u)) \partial_x^2 h] + \partial_x [a_1(u) \partial_x h] + a_0(u) h \}, \]

where

\[ a_2(u) = 1 + a(u) = 1 - 6u_{xx}, \quad a_1(u) = 10u, \quad a_0(u) = 10u_{xx} + 30u^2. \]

The coefficients \( a_i \), together with their derivative \( \partial_u a_i[h] \) with respect to \( u \) in the direction \( h \), satisfy the following estimates:

**Lemma 3.1.** For all \( p > \frac{\nu + 1}{2} \), \( \| u \|_{s,p+2} \leq 1 \), we have, for \( i = 0, 1, 2 \),

\[ \| a_i(u) \|_{s,p} \leq C \| u \|_{s,p+2}, \]

\[ \| \partial_u a_i(u)[h] \|_{s,p} \leq C \| h \|_{s,p+2}. \]

Moreover, if \( \lambda \mapsto u(\lambda) \) is a Lipschitz family, and satisfying \( \| u \|_{s,p+2} \leq 1 \), then, we have

\[ \| a_i(u) \|_{s,p} \leq C \| u \|_{s,p+2}, \]

\[ \| \partial_u a_i(u)[h] \|_{s,p} \leq C \| h \|_{s,p+2}. \]

**Proof.** Notice

\[ \partial_u a_2(u)[h] = 6h_{xx}, \quad \partial_u a_1(u)[h] = 10h \]

and

\[ \partial_u a_0(u)[h] = 10h_{xx} + 60uh. \]

Then, these estimates are straightforward. \( \square \)
3.1. Change of space variable. We consider a \( \varphi \)-dependent family of space variable change of the form

\[
y = x + \beta(\varphi, x),
\]

where \( \beta \) is a (small) real analytic function, 2\( \pi \)-periodic in all its arguments. The change of variables (66) induces on the space of functions the linear operator

\[
(T h)(\varphi, x) = h(\varphi, x + \beta(\varphi, x)).
\]

The operator \( T \) is invertible, with inverse

\[
(T^{-1} h)(\varphi, y) = h(\varphi, y + \tilde{\beta}(\varphi, y)).
\]

where \( y \to y + \tilde{\beta}(\varphi, y) \) is the inverse of (66), namely

\[
x = y + \tilde{\beta}(\varphi, y) \iff y = x + \beta(\varphi, x).
\]

In the Hamiltonian case, in order to keep the Hamiltonian structure of linear operator, the operator \( T \) needs a slight change. The modified linear operator is

\[
(A h)(\varphi, x) = (1 + \beta_x(\varphi, x)) h(\varphi, x + \beta(\varphi, x)),
\]

\[
(A^{-1} h)(\varphi, y) = (1 + \tilde{\beta}_y(\varphi, y)) h(\varphi, y + \tilde{\beta}(\varphi, y)).
\]

Remark 1. By [2, remark 3.3], the modified change of variable and its inverse (69) are symplectic, for each \( \varphi \in \mathbb{T}^n \). Also, both \( A \) and \( A^{-1} \) are maps from \( H^1_0 \) to \( H^1_0 \), for each \( \varphi \in \mathbb{T}^n \).

Now, we calculate the conjugate \( A^{-1} LA \) of the linearized operator \( L \) in (58).

The conjugate \( A^{-1} aA \) of any multiplication operator \( a : h(\varphi, x) \mapsto a(\varphi, x) h(\varphi, x) \) is the multiplication operator \( (T^{-1} a) \) that maps \( v(\varphi, y) \mapsto (T^{-1} a)v(\varphi, y) \). The conjugate of differential operators are

\[
A^{-1} \omega \cdot \partial_x A = \omega \cdot \partial_x + \partial_y \{T^{-1} (\omega \cdot \partial_x \beta)\},
\]

\[
A^{-1}\{\partial_x a\} A = \partial_y \{T^{-1}[a(1 + \beta_x)]\},
\]

\[
A^{-1}\{\partial_x(\partial_x a)\} A = \partial_y \{\partial_y ([T^{-1}[a(1 + \beta_x)^3])\partial_y] + (T^{-1}[a_x \cdot \beta_{xx} + a \cdot \partial_x^2 \beta])\},
\]

\[
A^{-1}\{\partial_x(\partial_x^2 a)\} A = \partial_y \{\partial_y ([T^{-1}[a(1 + \beta_x)^5])\partial_y^2] + 
\]

\[
+ \partial_y ([T^{-1}[3a_x(1 + \beta_x)^2\beta_{xx} + 5a(1 + \beta_x)^2\partial_x^2 \beta])\partial_y] 
\]

\[
+ (T^{-1}[a_{xx} \cdot \partial_x^4 \beta + 2a_x \cdot \partial_x^3 \beta + a \cdot \partial_x^2 \beta])\},
\]

where all the coefficients \( \{T^{-1}\[\ldots\}\} \) are periodic functions of \( (\varphi, y) \).

Remark 2. We give out some calculation tricks which have been used above.

(1): \( A^{-1}\{\partial_x g\} = \partial_y \{T^{-1} g\} \), since

\[
\partial_y \{T^{-1} g\} = \partial_y g(y + \tilde{\beta}(\varphi, y), \varphi)
\]

\[
= (1 + \tilde{\beta}_y) \cdot \partial_x g(y + \tilde{\beta}(\varphi, y), \varphi) 
\]

\[
= A^{-1}\{\partial_x g\}.
\]

(2): \((1 + \tilde{\beta}_y(y, \varphi)\{T^{-1}(1 + \beta_x(x, \varphi))\} = 1 \).

Using (1)(2), the conjugate of differential operators \( \partial_x a \) and \( \omega \cdot \partial_x \) are obvious.
Remark 3. The calculation of the conjugate of differential operators $\partial_x \{\partial_x (a \partial_x)\}$ and $\partial_x \{\partial^2_x (a \partial^2_x)\}$ are slightly finicky. Take $\partial_x \{\partial_x (a \partial_x)\}$ as an example, we see
\[ A^{-1}\{\partial_x \{\partial_x (a \partial_x)\}Ah(y, \varphi)\} = \partial_y \{T^{-1}\{\partial_x (a \partial_x)\}Ah(y, \varphi)\} = \partial_y \{T^{-1}\{\partial_x (a \partial_x(1 + \beta_x)h(x + \beta_x, \varphi))\}\} = \partial_y \{T^{-1}\{\partial_x (a(1 + \beta_x)^2 \partial_y h + a \beta_x h)\}\} = \partial_y \{T^{-1}\{a(1 + \beta_x)^3 \partial^2_y h + [a_x(1 + \beta)^2 + 3a(1 + \beta_x)\beta_x] \partial_y h + (a_x \beta_x + a \partial^3_x (\beta) h)\}\} = \partial_y \{\partial_x \{(T^{-1}[a(1 + \beta_x)] \partial_y) + (T^{-1}[a \cdot \beta_x + a \cdot \partial^3_x (\beta)])\}\},
\]
because $T^{-1}[a_x(1 + \beta_x)^2 + 3a(1 + \beta_x)\beta_x] = \partial_y \{T^{-1}[a(1 + \beta_x)^3]\}$.

Now, we get
\[ \mathcal{L}_1 := A^{-1} \mathcal{L}A = \omega \cdot \partial_\varphi + \partial_\varphi \{\partial^2_y b_2(u) \partial^2_y\} + \partial_y [b_1(u) \partial_y b_0(u)] + b_0(u), \]
where
\[ b_2 = T^{-1}[a_2(1 + \beta_x)^5], \]
\[ b_1 = T^{-1}[a_2(1 + \beta_x)^3 + 3(1 + \beta_x)^2 \beta_x \partial_x a_2 + 5a_2(1 + \beta_x)^2 \partial^3_x \beta], \]
\[ b_0 = T^{-1}[\partial_x a_1 \beta_x + a_1 \partial^2_x \beta + \partial^2_y a_2 \partial^2_x \beta + 2\partial_y a_2 \partial^2_x \beta + a_2 \partial^2_x \beta \]
\[ + \omega \cdot \partial_\varphi \beta + a_0(1 + \beta_x)]. \]

For convenience, set $b_i = T^{-1}(u) b^*_i$, $i = 0, 1$. Now, we look for $\beta(\varphi, x)$ such that the coefficient $b_x(\varphi, y)$ do not depend on $y$, namely
\[ b_2(\varphi, y) = T^{-1}[(1 + a)(1 + \beta_x)^5] = 1 + b(\varphi). \]

Since $T$ only makes changes on the space variable, $T b = b$ for every function $b(\varphi)$ that is independent on $y$. Hence (74) is equivalent to
\[ (1 + a)(1 + \beta_x)^5 = 1 + b(\varphi), \]
namely
\[ \beta_x(\varphi, x) = p_0, \quad p_0(\varphi, x) = (1 + b(\varphi))^{\frac{1}{5}} (1 + a(\varphi, x))^{-\frac{1}{5}} - 1. \]

The equation (76) has a solution $\beta$, periodic in $x$, if and only if $\int_T p_0(\varphi, x) \, dx = 0$. This condition uniquely determines
\[ 1 + b(\varphi) = \left( \frac{\int_T (1 + a(\varphi, x))^{-\frac{1}{5}} \, dx}{dx} \right)^{-5}. \]

Then, we have a solution (with zero average) of (76)
\[ \beta(\varphi, x) = (\partial_x^{-1} p_0)(\varphi, x), \]
where $\partial_x^{-1}$ is defined by linearity as
\[ \partial_x^{-1} e^{ikx} = \frac{e^{ikx}}{ik}, \forall k \in \mathbb{Z} \setminus \{0\}, \quad \partial_x^{-1} 1 = 0. \]

In other words, $\partial_x^{-1}$ is the primitive of $h$ with zero average in $x$. Thus we obtain the operator $\mathcal{L}_1$ in (72), that $b_2(\varphi, x) = 1 + b(\varphi).$
Set $s_0$ an integer greater than $\frac{s+1}{2}$, $\|u\|_{s_p+2s_0+7}^{Lip} \ll \frac{1}{100}$. We have the following estimates:

3.1.1. Estimates of $b(\varphi)$. We prove $b(\varphi)$ satisfies the following estimates:

$$\|b\|_{s_p} \leq C\|a(\varphi, x)\|_{s_p} \leq C\|u(\varphi, x)\|_{s_p+2}, \quad (80)$$

$$\|b\|_{s_p}^{Lip} \leq C\|a(\varphi, x)\|_{s_p}^{Lip} \leq C\|u(\varphi, x)\|_{s_p+2}^{Lip}, \quad (81)$$

$$\|\partial_u b(u)[h]\|_{s_p}^{Lip} \leq C(\|h\|_{s_p+2}^{Lip})\|u\|_{s_p+2}^{Lip} + \|h\|_{s_p+2}^{Lip}. \quad (82)$$

Proof of (80) and (81): Write $b$ as

$$b = \psi(M[g(a) - g(0)]) - \psi(0),$$

where

$$\psi(t) = (1 + t)^{-5}, \quad Mh = \int_T h dx \quad g(t) = (1 + t)^{-\frac{\rho}{2}}.$$

If $u$ is small enough, we have

$$\|b(u)\|_{s_p} \leq C\|M[g(a) - g(0)]\|_{s_p} \leq C\|g(a) - g(0)\|_{s_p} \leq C\|a\|_{s_p}. \quad (83)$$

Since $u$ is small enough, $\psi(t)$ and $g(t)$ are well defined by its power series expansion, i.e. $g(t) = 1 - \frac{1}{5} t^2 + \frac{3}{25} t^4 + \cdots$. Hence we have

$$\|g(u) - 1\|_{s_p}^{Lip} = \|1 - \frac{1}{5} u^2 + \frac{3}{25} u^4 + \cdots\|_{s_p}^{Lip} \leq C\|u\|_{s_p}^{Lip}. \quad (84)$$

The first and last inequality of (83) can be proved in such way. The second inequality is a direct result of $\|Mg\|_{s_p} \leq C\|g\|_{s_p}$. 

Proof of (82): The derivative of $c$ with respect to $u$ in the direction $h$ is

$$\partial_u b(u)[h] = \psi'(M[g(a) - g(0)])M(g'(a)\partial_u a(u)[h]), \quad (85)$$

where

$$\psi' = -5(1 + t)^{-6}, \quad g' = -\frac{1}{5}(1 + t)^{-\frac{\rho}{2}}. \quad (86)$$

Using the same way as (83), by (62) and (63), we can get (82).

3.1.2. Estimates of $\beta(\varphi, x)$. Suppose $\zeta(t) = (1 + t)^{\frac{\rho}{2}}$. Considering $p_0$ defined in 76, we see

$$p_0 = g(a)\zeta(b) - 1. \quad (87)$$

Using the same way as (80), we can get

$$|\beta(\varphi, x)|_{s_p} \ll |\beta(\varphi, x)|_{s_p+s_0} \ll |p_0(\varphi, x)|_{s_p+s_0} \ll \|u\|_{s_p+s_0+2} \leq \frac{1}{100}. \quad (88)$$

and

$$|\beta(\varphi, x)|_{s_p}^{Lip} \ll \|u\|_{s_p+s_0+2}^{Lip} \leq \frac{1}{100}. \quad (89)$$

The derivative of $p_0$ with respect to $u$ in the direction $h$ is

$$\partial_u p_0[h] = g(a)(\zeta'(b)\partial_u b(u)[h] + (g'(a)\partial_u a(u)[h])\zeta(b)). \quad (90)$$

Using the same way as (83), the bounds (64), (65) and (82) imply

$$\|\partial_u \beta[h]\|_{s_p}^{Lip} \leq \|\partial_u p_0[h]\|_{s_p}^{Lip} \ll (1 + \|u\|_{s_p+2}^{Lip})\|h\|_{s_p+2}^{Lip}. \quad (91)$$

The inverse function $y \to y + \hat{\beta}(\varphi, y)$ is also under our consideration. By Lemma 6.8, one gets

$$|\hat{\beta}(\varphi, y)|_{s_p} \ll |\beta(\varphi, x)|_{s_p} \ll \|u\|_{s_p+s_0+2} \leq \frac{1}{100}. \quad (92)$$
and
\[ |\hat{\beta}(\varphi, y)|_{\text{Lip}}^{\text{Lip}} < |\beta(\varphi, x)|_{\text{Lip}}^{\text{Lip}} < \|u\|_{s, p+2}^{\text{Lip}} \leq \frac{1}{100}. \] (93)

Writing explicitly the dependence on \( u \), we have \( \hat{\beta}(\varphi, y; u) + \beta(\varphi, \hat{\beta}(\varphi, y; u); u) = 0 \). Differentiating this equality with respect to \( u \) in the direction \( h \) gives
\[ (\partial_u \hat{\beta})[h] = -T^{-1}(\partial_u \beta)[h]. \] (94)

Applying lemma 6.4 and 6.6 to cope with \( T^{-1} \), the bounds (89), (91) and (93) imply
\[ \| (\partial_u \hat{\beta})[h]\|_{\text{Lip}}^{\text{Lip}} \leq (1 + \|u\|_{s, p+2}^{\text{Lip}}) \|h\|_{s, p+2}^{\text{Lip}}. \] (95)

3.1.3. Estimates of \( T \) and \( T^{-1} \). Using Lemma 6.4,6.6,6.8, we can get the following estimates:
\[ \|T(u)g\|_{\text{Lip}}^{\text{Lip}} \leq \|g\|_{s, p+2}, \] (96)
\[ \|T(u)g\|_{\text{Lip}}^{\text{Lip}} \leq \|g\|_{s, p+2}^{\text{Lip}} + 1, \] (97)
\[ \|T(u)g\|_{s, p+2} \leq \|g\|_{s, p+2}, \] (98)
\[ \|T(u)g\|_{s, p+2} \leq \|g\|_{s, p+2}^{\text{Lip}}. \] (99)

Since \( T^{-1}(u)g = g(\varphi, y + \hat{\beta}(\varphi, y)) \), the derivative of \( T^{-1}(u)g \) in the direction \( h \) is the product \( \partial_u(T^{-1}(u)g) = (T^{-1}g_x)(\partial_u \hat{\beta})[h] \). The bounds (98), (99) and (95) imply
\[ \|\partial_u(T^{-1}(u)g)\|_{\text{Lip}}^{\text{Lip}} \leq \|g\|_{s, p+2}^{\text{Lip}} \|h\|_{s, p+2}^{\text{Lip}}(1 + \|u\|_{s, p+2}^{\text{Lip}}). \] (100)

3.1.4. Estimates of the coefficients \( b_i \). Consider the coefficients \( b_i^* \), \( b_i^0 \), which are given in (73). We have
\[ \|b_i^*\|_{s, p}^{\text{Lip}} \leq \|u\|_{s, p+4}^{\text{Lip}}, \] (101)
\[ \|b_i^0\|_{s, p}^{\text{Lip}} \leq \|u\|_{s, p+6}^{\text{Lip}}. \] (102)

Applying (98),(99) to (101), (102), we see, for \( i = 0, 1 \),
\[ \|b_i\|_{s, p}^{\text{Lip}} \leq \|u\|_{s, p+2}^{\text{Lip}}, \] (103)
and
\[ \|b_i\|_{s, p}^{\text{Lip}} \leq \|u\|_{s, p+2}^{\text{Lip}} + 7. \] (104)

Now, we estimate the derivative of \( b_1 \) with respect to \( u \). Write \( b_1 \) as \( T^{-1}(u)b_1^* \), where
\[ b_1^* = a_1(1 + \beta_x)^3 + 3(1 + \beta_x)^2 \partial_x^2 \beta \cdot \partial_x a_2 + 5a_2(1 + \beta_x)^2 \partial_x^2 \beta. \]
The bounds (64), (65) and (91) imply
\[ \|\partial_u b_1^*(u)[h]\|_{s, p}^{\text{Lip}} \leq \|h\|_{s, p+4}^{\text{Lip}}(1 + \|u\|_{s, p+4}^{\text{Lip}}). \] (105)

The derivative of \( b_1 \) in the direction \( h \) is
\[ \partial_u b_1(u)[h] = \partial_u(T(u)^{-1}b_1^*(u))[h] = (\partial_u T(u)^{-1})(b_1^*(u))[h] + T(u)^{-1}(\partial_u b_1^*(u))[h]. \] (106)

Then, (98), (99), (100), (105) and (109) imply
\[ \|\partial_u T(u)^{-1}(b_1^*(u))[h]\|_{s, p}^{\text{Lip}} \leq \|u\|_{s, p+2}^{\text{Lip}} \|h\|_{s, p+2}^{\text{Lip}}(1 + \|u\|_{s, p+2}^{\text{Lip}}). \] (107)
and
\[ \|T(u)^{-1}(\partial_u b_1^1(u))\|_{\text{Lip}} < \|h\|_{\text{Lip}} \|u\|_{s,p+2s_0+5}(1 + \|u\|_{s,p+2s_0+5}). \] (108)

Ultimately, (106), (107) and (108) imply that
\[ \|\partial_u b_1(u)[h]\|_{\text{Lip}} < \|h\|_{s,p+2s_0+5}(1 + \|u\|_{s,p+2s_0+6}). \] (109)

By the same way as \( b_1 \), we can get
\[ \|\partial_u b_0(u)[h]\|_{\text{Lip}} < \|h\|_{s,p+2s_0+7}(1 + \|u\|_{s,p+2s_0+8}). \] (110)

3.2. Time reparametrization. In this section, we will make constant the coefficient of the highest order spatial derivative operator of \( \mathcal{L}_1 \), by a quasi-periodic reparametrization of time. The change of variables has the form
\[ \varphi \mapsto \varphi + \omega \alpha(\varphi), \quad \varphi \in \mathbb{T}, \] (111)
where \( \alpha \) is a (small) real analytic function, 2\pi-periodic in all its arguments. The induced linear operator on the space of functions is
\[ (\mathcal{B} h)(\varphi, y) = h(\varphi + \omega \alpha(\varphi), y), \] (112)
whose inverse is
\[ (\mathcal{B}^{-1} h)(\theta, y) = h(\theta + \omega \hat{\alpha}(\theta), y). \] (113)
where \( \varphi = \theta + \omega \hat{\alpha}(\theta) \) is the inverse of \( \theta = \varphi + \omega \alpha(\varphi) \). Then, the time derivative operator becomes
\[ \mathcal{B}^{-1} \omega \cdot \partial_\varphi \mathcal{B} = \xi(\theta) \omega \cdot \partial_\theta, \quad \xi(\theta) = \mathcal{B}^{-1}(1 + \omega \cdot \partial_\varphi \alpha(\varphi)). \] (114)

The spatial derivative operator do not have any change in form. Thus, see
\[ \mathcal{B}^{-1} \mathcal{L}_1 \mathcal{B} = \xi(\theta) \omega \cdot \partial_\theta + \partial_y \{ \partial_y^2 [(\mathcal{B}^{-1} b_2(u)) \partial_y h] + \partial_y [(\mathcal{B}^{-1} b_1(u)) \partial_y h] + [\mathcal{B}^{-1} b_0(u)] h \}. \] (115)

We look for \( \alpha \) such that the coefficients of the highest order derivatives are proportional, namely
\[ [\mathcal{B}^{-1}(1 + b)](\theta) = m \xi(\theta) = m [\mathcal{B}^{-1}(1 + \omega \cdot \partial_\varphi \alpha)](\theta) \] (116)
for some constant \( m \in \mathbb{R} \). This is equivalent to require that
\[ 1 + b(\varphi) = m(1 + \omega \cdot \partial_\varphi \alpha(\varphi)). \] (117)
Integrating on \( \mathbb{T}^v \) determines the value of the constant \( m \),
\[ m = \int_{\mathbb{T}^v} (1 + b(\varphi)) d\varphi. \] (118)
We can find the unique solution of (117) with zero average
\[ a(\varphi) = \frac{1}{m}(\omega \cdot \partial_\varphi)^{-1}(1 + b - m)(\varphi), \] (119)
where \((\omega \cdot \partial_\varphi)^{-1}\) is defined by linearity
\[ (\omega \cdot \partial_\varphi)^{-1} e^{i\varphi} = \frac{e^{i\varphi}}{\omega \cdot \ell}, \ell \neq 0, \quad (\omega \cdot \partial_\varphi)^{-1} 1 = 0. \] (120)

With this choice of \( \alpha \), we have
\[ \mathcal{B}^{-1} \mathcal{L}_1 \mathcal{B} = \xi(\theta) \mathcal{L}, \quad \mathcal{L} = \omega \cdot \partial_\theta + m \partial_y^5 + \partial_y \{ \partial_y [(c_1(\theta, y) \partial_y)] + c_0(\theta, y)\}. \] (121)
where
\[ c_i = \frac{B^{-1}b_i}{\xi}, \quad i = 0, 1. \] (122)

Suppose \( \|u\|_{s,p+4s_0+n_0+9} \ll \frac{1}{100} \), we have these estimates below:

3.2.1. Estimates of \( m \). The coefficient \( m \), defined in (118), satisfy the following estimates:
\[
|m - 1| \leq C\|u\|_{s,p+2}, \quad |m - 1|^{Lip} \leq C\|u\|^{Lip}_{s,p+2},
\]
(123)
\[
|\partial_u m(u)[h]|^{Lip} \leq C\|h\|^{Lip}_{s,p+2}(1 + \|u\|^{Lip}_{s,p+2}).
\]
(124)
Using (80), (81), (123), and (124), we see
\[
|m - 1| = \int_{T^d} |b(\varphi)|d\varphi \leq \|b\|^{Lip}_{s,p} \leq C\|u\|^{Lip}_{s,p+2}
\]
(125)
Similarly we get the Lipschitz part of (123). The estimate (124) follows by (82), since
\[
|\partial_u m(u)[h]|^{Lip} \leq \int_{T^d} |\partial_u b(u)[h]|^{Lip}d\varphi \leq C\|\partial_u b(u)[h]\|^{Lip}_{s,p}.
\]
(126)

3.2.2. Estimates of \( \alpha \). The function \( \alpha(\varphi) \), defined in (119), satisfies
\[
|\alpha|_{s,p} \leq C\alpha_0^{-1}\|u\|_{s,p+\tau+2} \leq \frac{1}{100}.
\]
(127)
\[
|\alpha|^{Lip}_{s,p} \leq C\alpha_0^{-1}\|u\|^{Lip}_{s,p+\tau+2} \leq \frac{1}{100}.
\]
(128)
Remember that \( \omega = \lambda\bar{\omega} \), and \( |\bar{\omega} \cdot \ell| \geq \frac{\alpha_0}{|\ell|^2} \), \( \forall \ell \in \mathbb{Z}^n \setminus \{0\} \). By (80), (123) and (332), we see
\[
|\alpha|_{s,p} \ll \|u\|_{s,p+s_0} \leq C\alpha_0^{-1}\|b(\varphi) + (1 - m)\|_{s,p+s_0+\tau} \leq C\alpha_0^{-1}\|u\|_{s,p+s_0+\tau+2}.
\]
(129)
Providing (127). Then (128) holds similarly using (81) and \( (\omega \cdot \partial_\varphi)^{-1} = \lambda^{-1}(\omega \cdot \partial_\varphi)^{-1} \). Differentiating formula (119) with respect to \( u \) in the direction \( h \) gives
\[
\partial_u \alpha(u)[h] = (\lambda\bar{\omega} \cdot \partial_\varphi)^{-1}(\partial_u b(u)[h]m - (b(\varphi) + 1)\partial_u m(u)[h]).
\]
(130)
Then, (81), (123), and (124) imply that
\[
||\partial_u \alpha(u)[h]|^{Lip}_{s,p} \leq C(\|h\|^{Lip}_{s,p+\tau+2}\|u\|^{Lip}_{s,p+\tau+2} + \|h\|^{Lip}_{s,p+\tau+2})
\]
(131)
For the inverse change of variable (113), by Lemma 6.8, we have the following estimates:
\[
|\hat{\alpha}|^{s_0}_{\text{inv}, p} \ll |\alpha|_{s,p} \ll \|u\|_{s,p+s_0+\tau+2} \leq \frac{1}{100},
\]
(132)
\[
|\hat{\alpha}|^{Lip}_{\text{inv}, p} \ll |\alpha|^{Lip}_{s,p+1} \ll \|u\|^{Lip}_{s,p+s_0+\tau+3} \leq \frac{1}{100}.
\]
(133)
Writing explicitly the dependence on \( u \), we have \( \hat{\alpha}(\theta; u) + \alpha(\theta + \hat{\alpha}(\theta; u); u) = 0 \). Differentiating the equality with respect to \( u \) in \( h \) gives
\[
\partial_u \hat{\alpha}[h] = -B^{-1}(\partial_u \alpha(u)[h])(1 + \omega \cdot \partial_\varphi \alpha).
\]
(134)
Using Lemma 6.6 to cope with \( B^{-1} \), (127), (128) and (131) imply
\[
||\partial_u \hat{\alpha}[h]|^{Lip}_{\text{inv}, p} \ll \|u\|^{Lip}_{s,p+\tau+2s_0+4}\|h\|^{Lip}_{s,p+\tau+2s_0+3} + \|h\|^{Lip}_{s,p+2s_0+\tau+3}
\]
(135)
3.2.3. Estimates of $B$ and $B^{-1}$. Using Lemma 6.4, 6.6, 6.8, the transformations $B(u)$ and $B^{-1}(u)$, defined in (112), satisfy the following estimation:

\begin{align}
\|B(u)g\|_{\overline{\text{Lip}},p}^{100} & \ll \|g\|_{s,p+2s_0}, & (136) \\
\|B(u)g\|_{\text{Lip},p}^{100} & \ll \|g\|_{s,p+2s_0+1}, & (137) \\
\|B^{-1}(u)g\|_{\overline{\text{Lip}},p}^{100} & \ll \|g\|_{s,p+2s_0}, & (138) \\
\|B^{-1}(u)g\|_{\text{Lip},p}^{100} & \ll \|g\|_{s,p+2s_0+1}. & (139)
\end{align}

Differentiating $B^{-1}(u)g$ with respect to $u$ in the direction $h$ gives
\begin{equation}
\partial_u(B^{-1}(u)g[h]) = B^{-1}(u)(\omega \cdot \partial_x g) \cdot \partial_u \hat{\omega}[h] \tag{140}
\end{equation}

Then, the bounds (135) and (139) imply
\begin{equation}
\|\partial_u(B^{-1}(u)g)[h]\|_{\overline{\text{Lip}},p}^{100} \ll \|g\|_{s,p+2s_0+2}^{\overline{\text{Lip}},p} \|h\|_{s,p+2s_0+\tau+3}^{\text{Lip},p}(1+\|u\|_{s,p+2s_0+\tau+4}). \tag{141}
\end{equation}

3.2.4. Estimates of $\xi(\theta)$. The function $\xi$ is defined as $\xi(\theta) = B^{-1}(1+\omega \cdot \partial_x \alpha)$. Obviously, $\xi(\theta) - 1 = B^{-1}(\omega \cdot \partial_x \alpha)$. Then, the bounds (138) and (139) imply
\begin{equation}
\|\xi - 1\|_{\overline{\text{Lip}},p}^{100} \ll \|u\|_{s,p+2s_0+2}. \tag{142}
\end{equation}

Differentiating $\xi(\theta)$ with respect to $u$ in the direction $h$ gives
\begin{equation}
\partial_u \xi(u)[h] = \partial_u B(u)^{-1}(\omega \cdot \partial_x \alpha)[h] + B(u)^{-1}(\omega \cdot \partial_x (\partial_u \alpha[h])). \tag{144}
\end{equation}

By (135), (141) and (140), we get
\begin{equation}
\|\partial_u B(u)^{-1}(\omega \cdot \partial_x \alpha)[h]\|_{\overline{\text{Lip}},p}^{100} \ll \|u\|_{s,p+2s_0+4}^{\overline{\text{Lip}},p} \|h\|_{s,p+\tau+2s_0+3}^{\text{Lip},p}(1+\|u\|_{s,p+2s_0+\tau+4}). \tag{145}
\end{equation}

Using (139) and (131), we see
\begin{equation}
\|B(u)^{-1}(\omega \cdot \partial_x (\partial_u \alpha[h]))\|_{\overline{\text{Lip}},p}^{100} \ll \|h\|_{s,p+2s_0+\tau+3}^{\text{Lip},p}(1+\|u\|_{s,p+2s_0+\tau+4}). \tag{146}
\end{equation}

Finally, (145) and (146) imply
\begin{equation}
\|\partial_u \xi(u)[h]\|_{\overline{\text{Lip}},p}^{100} \ll \|h\|_{s,p+2s_0+\tau+3}^{\text{Lip},p}(1+\|u\|_{s,p+2s_0+\tau+4}). \tag{147}
\end{equation}

3.2.5. Estimates of the coefficients $c_i$. The coefficients $c_i$ defined in (122), for $i = 0, 1$, satisfy the following estimates:
\begin{equation}
\|c_i\|_{\overline{\text{Lip}},p}^{100} \ll \|u\|_{s,p+4s_0+\tau+6}. \tag{148}
\end{equation}
\begin{equation}
\|c_i\|_{\text{Lip},p}^{100} \ll \|u\|_{s,p+4s_0+\tau+8}. \tag{149}
\end{equation}

Differentiating $c_i$ with respect to $u$ in the direction $h$ gives
\begin{equation}
\partial_u c_i(u)[h] = \frac{1}{\xi} \partial_u [B^{-1}b_i] - \frac{1}{\xi^2} \partial_u \xi(u)[h](B^{-1}b_i). \tag{150}
\end{equation}

Now, we can obtain
\begin{equation}
\|\partial_u c_i(u)[h]\|_{\overline{\text{Lip}},p}^{100} \ll \|h\|_{s,p+4s_0+\tau+9}^{\overline{\text{Lip}},p}(1+\|u\|_{s,p+4s_0+\tau+9}). \tag{151}
\end{equation}

The definition of $c_i$ (122), (103), (138), (142) imply (148). Similarly, (104) (139), (143) imply (3.2.5). Finally, (151) follows from (3.2.5), (109), (110), (139), (147) and (131).
3.3. Estimates on $L$. Recall the procedure performed in the previous subsection, we have conjugated the operator $L$ to $L$, where

$$L = U_1^{-1}LU_2, \quad U_1 = AB\xi, \quad U_2 = AB. \quad (152)$$

In the following lemma, we would summarize the estimates for the linear operator $L$, $U_1$, and $U_2$, also define constants $p = 2s_0 + 5$, $\eta = 4s_0 + \tau + 9$, $k_1 = \frac{99}{101}$, $k_2 = \frac{10001}{10201}$.\(\quad (153)\)

Lemma 3.2. There exists $0 < \varepsilon \ll \frac{1}{100}$, such that for all $u(\lambda), h(\lambda)$ are Lipschitz-families, satisfying

$$\|u\|_{s,p+\eta}^{\text{Lip}} \leq \varepsilon. \quad (154)$$

(1) : Considering the transformation $U_i, i = 1, 2$, defined in (152), we have

$$\|U_ih\|_{k}\|s', p' < \|h\|_{k}, s', p' + \eta, \quad (155)$$

$$\|U_i^{-1}h\|_{k}, s', p' < \|h\|_{k}, s', p' + \eta. \quad (156)$$

(2) : The constant coefficient $m$, defined in (118), satisfies

$$|m - 1| \leq C\|u\|_{s,2}^{\text{Lip}}, \quad |m - 1|^{\text{Lip}} \leq C\|u\|_{s,2}^{\text{Lip}}, \quad (157)$$

$$|\partial u m(u)[h]|^{\text{Lip}} \leq C\|h\|_{s,2}^{\text{Lip}}(1 + \|u\|_{s,2}^{\text{Lip}}). \quad (158)$$

(3) : The variable coefficients $c_1$, defined in (122), satisfies

$$\|c_1\|_{k_1 s, p}^{\text{Lip}} \ll \|u\|_{s, p + \eta}, \quad (159)$$

$$\|c_1\|_{k_1 s, p}^{\text{Lip}} \ll \|u\|_{s, p + \eta}, \quad (160)$$

$$\|\partial u c_1(u)[h]\|_{k_1 s, p}^{\text{Lip}} \ll \|h\|_{s,\eta}^{\text{Lip}}(1 + \|u\|_{s, p + \eta}). \quad (161)$$

Proof. The detail of these estimates can be found in the previous subsection, we just give a summary here. \(\square\)

Remark 4. The $p'$ in (155) and (156) is any integer greater than $s_0$, $\hat{s}$ is any positive number smaller than $s$.

4. KAM step.

4.1. $\varepsilon_m$ approximate unbounded reducibility. In this section, we make a reduction to eliminate the unbounded perturbation of linear operator $L$ obtained in (121). The goal is to conjugate it to a diagonal operator $J$ plus a sufficient small remainder $R$. Before we apply the reducibility scheme, we will make some definitions, recall and revise some important lemmas.

Definition 4.1. From (16), set

$$F(u) = u_t + \partial^5_t u + 10u\partial^9_t u + 20\partial_x u \partial^9_x u + 30u^2 \partial_x u - 6\partial^2_x u \partial^9_x u - 18\partial^3_x u \partial^4_x u - \partial_x f(\omega t, x) \quad (162)$$

with $u$ quasi-periodic in time and periodic in space. If $\|F(u)\| \leq \varepsilon$ in a suitable Banach space, we say $u$ is an $\varepsilon$ approximate solution of the equation $F(u) = 0$.\(\quad (162)\)
**Definition 4.2.** Any linear operator \( A : H^1_0(T) \to H^1_0(T) \) can be represented by the infinite dimensional matrix \((A(\theta)_{i_1, i_2})_{i_1, i_2 \in \mathbb{Z}^2 \setminus \{(0,0)\}}\), where \( A(\theta)_{i_1} = (Ae^{2\pi i \theta x} e^{2\pi i x}) \). Now, we define a new \((s, p)\)-decay Banach space \( B^s_{s,p} \) as

\[
B^s_{s,p} := \left\{ A : (|A|_{s,p}^s)^2 = \sum_{i \in \mathbb{Z}} e^{2|s|x} i^s i^p \left( \sup_{i_1 - i_2 = i} \left\| A(\theta)_{i_1, i_2} \right\|_{s,p}^2 \right) < +\infty \right\}.
\]

**Definition 4.3.** We define some new \((s, p)\)-decay Banach space \( \tilde{B}_{s,p} \) and \( \hat{B}_{s,p} \) as

\[
\tilde{B}_{s,p} := \left\{ A : (|\tilde{A}|_{s,p}^s)^2 = \sum_{i \in \mathbb{Z}} e^{2|s|x} i^s i^p \left( \sup_{i_1 - i_2 = i} \left\| A(\theta)_{i_1, i_2} \right\|_{s,p}^2 \right) < +\infty \right\},
\]

and

\[
\hat{B}_{s,p} := \left\{ A : (|\hat{A}|_{s,p}^s)^2 = \sum_{i \in \mathbb{Z}} e^{2|s|x} i^s i^p \left( \sup_{i_1 - i_2 = i} \left\| A(\theta)_{i_1, i_2} \right\|_{s,p}^2 \right) < +\infty \right\}.
\]

The Banach space \( B^s_{s,p} \) can be denote as

\[
B^s_{s,p} := \left\{ A : |A|_{s,p}^s = \max\{|\tilde{A}|_{s,p}, |\hat{A}|_{s,p}, |A|_{s,p}\} < +\infty \right\}.
\]

**Lemma 4.4.** (Algebra property 1). For all \( p > \frac{v}{2} \), if \( A, B \in B^s_{s,p} \), then \( AB \in B^s_{s,p} \). Also, there are \( c(p) > 0 \), such that

\[
|AB|_{s,p}^s \leq c(p)|A|_{s,p}^s |B|_{s,p}^s.
\]

If \( A = A(\lambda) \) and \( B = B(\lambda) \) depend in a Lipschitz way on the parameter \( \lambda \in \Pi \subset \mathbb{R} \), then

\[
|AB|_{s,p}^{s,Lip} \leq c(p)|A|_{s,p}^{s,Lip} |B|_{s,p}^{s,Lip}.
\]

Proof. To prove (167), we will respectively prove the following three cases.

**Case 1.** If \( A, B \in \tilde{B}_{s,p} \), then \( AB \in \tilde{B}_{s,p} \) with \( |AB|_{s,p} \leq c(p)|A|_{s,p} |B|_{s,p} \). From Lemma 2.10, this case is simple.

**Case 2.** If \( A, B \in \hat{B}_{s,p} \), then \( AB \in \hat{B}_{s,p} \) with \( |\hat{AB}|_{s,p} \leq c(p)|\hat{A}|_{s,p} |\hat{B}|_{s,p} \). With regard to \( A, B \in \tilde{B}_{s,p} \), we can define \( Q_1, Q_2 \), where \( (Q_1)_i^j = \frac{A^i B^j}{i^2} \). Thus, \( A, B \) can be seen as \( \partial_{xx} Q_1, \partial_{xx} Q_2 \) and \( \partial_{xx} Q_2, \partial_{xx} Q_1, \partial_{xx} Q_1, \partial_{xx} Q_2 \), with \( |Q_1|_{s,p} = |\hat{A}|_{s,p} |Q_2|_{s,p} = |\hat{B}|_{s,p} \).

Then,

\[
|\hat{AB}|_{s,p} = |\partial_{xx} Q_1 Q_2 \partial_{xx} Q_2|_{s,p} = c(p)|\hat{A}|_{s,p} |\hat{B}|_{s,p}.
\]

**Case 3.** If \( A, B \in \hat{B}_{s,p} \), then \( AB \in \hat{B}_{s,p} \) with \( |\hat{AB}|_{s,p} \leq c(p)|\hat{A}|_{s,p} |\hat{B}|_{s,p} \). The proof of this case is almost the same as Case 2.

Now, (167) is proved. The proof of (168) is standard.

**Lemma 4.5.** (Algebra property 2) For all \( p > \frac{v}{2} \), if \( A, C \in B^s_{s,p} \), \( B \in C_{s,p} \), then \( ABC \in C_{s,p} \). Also, there are \( c(p) > 0 \), such that

\[
|ABC|_{s,p} \leq c(p)|A|_{s,p}^{|s|} |B|_{s,p}^{|s|} |C|_{s,p}^{|s|}.
\]
If $\mathcal{A} = \mathcal{A}(\lambda)$, $\mathcal{B} = \mathcal{B}(\lambda)$ and $\mathcal{C} = \mathcal{C}(\lambda)$ depend in a Lipschitz way on the parameter $\lambda \in \Pi \subset \mathbb{R}$, then

$$ |\mathcal{A}\mathcal{B}\mathcal{C}|_{s,p}^{\text{Lip}} \leq c(p) |\mathcal{A}|_{s,p}^{\text{g, Lip}} |\mathcal{B}|_{s,p}^{\text{g, Lip}} |\mathcal{C}|_{s,p}^{\text{g, Lip}}. \quad (171) $$

**Proof.** From Definition 2.6 and Definition 4.2. If $\mathcal{A}, \mathcal{C} \in \mathcal{B}_s^{g,p}$, they can be seen as $\partial_{xx} Q_1 \partial_{xx}^{-1}$ and $\partial_{x}^{-1} Q_2 \partial_{x}$, with $|Q_1|_{s,p} = |\mathcal{A}|_{s,p}^{g}, |Q_2|_{s,p} = |\mathcal{C}|_{s,p}^{g}$. If $\mathcal{B} \in \mathcal{B}_s^{g,p}$, it can be seen as $\partial_{xx} Q \partial_{x}$, with $|Q|_{s,p} = |\mathcal{B}|_{s,p}^{g}$. Thus,

$$ |\mathcal{A}\mathcal{B}\mathcal{C}|_{s,p}^{g} = |\partial_{xx} Q_1 Q_2 \partial_{x}|_{s,p}^{g} = |Q_1 Q_2|_{s,p}^{g} $$

$$ \leq c(p) |Q_1|_{s,p}^{g} |Q|_{s,p}^{g} |Q_2|_{s,p}^{g} = c(p) |\mathcal{A}|_{s,p}^{g} |\mathcal{B}|_{s,p}^{g} |\mathcal{C}|_{s,p}^{g}. \quad (172) $$

The proof of (171) is standard. \[\Box\]

**Lemma 4.6.** For all $p > \frac{\gamma}{2}$, if $\mathcal{A} \in \mathcal{B}_s^{g,p}$, $h \in \mathcal{H}_{s,2p}$, then $\mathcal{A}h \in \mathcal{H}_{s,p}$ with

$$ \|\mathcal{A}h\|_{s,p} \leq c(p) |\mathcal{A}|_{s,p}^{g} \|h\|_{s,2p}, \quad \|\mathcal{A}h\|_{s,p}^{\text{Lip}} \leq c(p) |\mathcal{A}|_{s,p}^{g} \|h\|_{s,2p}. \quad (173) $$

If $\mathcal{A} \in \mathcal{B}_s^{g,p}$, $h \in \mathcal{H}_{s,2p+1}$, then $\mathcal{A}h \in \mathcal{H}_{s,p-2}$ with

$$ \|\mathcal{A}h\|_{s,p-2} \leq c(p) |\mathcal{A}|_{s,p}^{g} \|h\|_{s,2p+1}, \quad \|\mathcal{A}h\|_{s,p-2}^{\text{Lip}} \leq c(p) |\mathcal{A}|_{s,p}^{g} \|h\|_{s,2p+1}^{\text{Lip}}. \quad (174) $$

**Proof.** The proof of (173) is standard. To prove (174), by Lemma 2.10 and Lemma 4.5, we see

$$ \|\mathcal{A}h\|_{s,p-2} = \|\partial_{xx} Q \partial_{x} h\|_{s,p-2} \leq \|Q \partial_{x} h\|_{s,p} $$

$$ \leq |Q|_{s,p} \|\partial_{x} h\|_{s,2p} \leq |\mathcal{A}|_{s,p}^{g} \|h\|_{s,2p+1} \quad (175) $$

\[\Box\]

Now, we recall the classical Kuksin’s lemma.

**Lemma 4.7 (Kuksin).** Consider the following first order partial differential equation

$$ -i \omega \cdot \partial_{y} u + du + \mu(\theta) u = p(\theta), \quad \theta \in \mathbb{T}^v, \quad (176) $$

for the unknown function $u$ defined on the torus $\mathbb{T}^v$, where $\omega = (\omega_1, \cdots, \omega_v) \in \mathbb{R}^v$ and $d \in \mathbb{R}$. We make the following assumption.

**Assumption A.** There are constants $\alpha, \gamma > 0$ and $\tau > v$ such that

$$ |\langle \ell, \omega \rangle| \geq \frac{\alpha}{|\ell|^\tau}, \quad (177) $$

$$ |\langle \ell, \omega \rangle + d| \geq \frac{\alpha \gamma}{|\ell|^\tau}, \quad (178) $$

for all $0 \neq \ell \in \mathbb{Z}^v$. Also, $|d| \geq \alpha \gamma$.

**Assumption B.** The function $\mu$ is analytic on some complex strip $D(s) = \{ \theta : |\Im \theta| < s \} \subset \mathbb{C}^v$ around $\mathbb{T}^v$ with mean value zero $\int_{\mathbb{T}} \mu(\theta) d\theta = 0$. Moreover,

$$ \ell_{s,\gamma} \mu_{s} \overset{\text{def}}{=} \sum_{\ell \in \mathbb{Z}^v} |\mu(\ell)| \ell^{s} e^{i \langle \ell, \gamma \rangle} \leq C \gamma. \quad (179) $$

for $\mu = \sum_{\ell \in \mathbb{Z}^v} \mu_{\ell} e^{i \ell \theta}$ with some $C > 0$.

**Assumption C.** $p(\theta)$ is analytic on the same complex strip $D(s)$, and $d \geq c_{\gamma}^{1+\beta}$, with some $\beta > 0, c > 0$. \[\Box\]}
Then the equation has a unique solution \( u(\theta) \) defined in a narrower domain \( D(s - \sigma) \), with \( 0 < \sigma < s \), which satisfies

\[
\| u(\theta) \|_{s-\sigma, s_0} \leq \frac{c e^{2(5/\sigma)^{1/\beta}}}{\alpha \gamma \sigma^{2(v + s_0 + 3)}} \| p(\theta) \|_{s, s_0},
\]

where \( c \) depend on \( v \) and \( \tau \).

**Proof.** The original proof can be found in [14] and [12]. Totally following the proof by Kuksin [14] and [12] and noting Lemma 6.4, (180) is verified. \( \square \)

The inverse of \( L(u_n) \) is our main concern. However, the inverse of \( L(0) \), a diagonal operator, is clearly. In order to make the structure of this paper much more simplicity, the initial approximate solution is \( u_1 \) other than \( u_0 \). Thus, we would estimate the inverse of \( L(0) \) and set the initial parameter .

**Lemma 4.8.** The linear equation \( L(0) v = F(0) \), for all \( \lambda \in \Gamma(u_0) \),

\[
\Gamma(u_0) := \{ \lambda : |\lambda \omega \cdot \ell + k^5| \geq \alpha_0 \frac{k^5}{|\ell|}, \quad \forall \ell \in \mathbb{Z}^v, k \in \mathbb{Z} \backslash \{0\} \}
\]

has a unique solution \( v \) with zero average, satisfying \( \| v \|_{s,p'} \leq \frac{1}{\alpha_0} \| F(0) \|_{s,p'}^{Lip} + 2 \tau + 1 \).

**Proof.** The equation \( L(0) v = F(0) \) is equivalent to

\[
\omega \cdot \partial_\phi v(\varphi, x) + \partial_5^5 v(\varphi, x) = F(\varphi, x),
\]

where

\[
v(\varphi, x) = \sum_{k \in \mathbb{Z} \backslash \{0\}} v_k(\varphi) e^{ikx}, \quad F(\varphi, x) = \sum_{k \in \mathbb{Z} \backslash \{0\}} F_k(\varphi) e^{ikx}.
\]

Thus, (182) can be transformed to

\[
\omega \cdot \partial_\phi v_k(\varphi) + (ik)^5 v_k(\varphi) = F_k(\varphi), \quad k \in \mathbb{Z} \backslash \{0\},
\]

whose solutions are

\[
v_k(\varphi) = \sum_{\ell \in \mathbb{Z}^v} v_{\ell,k} e^{ik\varphi} \quad \text{with coefficients}
\]

\[
v_{\ell,k} = \frac{F_{\ell,k}}{i\omega \cdot \ell + ik^5} \quad \forall \ell \in \mathbb{Z}^v, k \in \mathbb{Z} \backslash \{0\}.
\]

Using (181), we have

\[
\| v \|_{s,p'+\tau+1} \leq \frac{1}{\alpha_0} \| F(0) \|_{s,p'+2\tau+1}.
\]

Applying the operator \( \Delta v = v(\lambda_1) - v(\lambda_2) \) to (184), we have

\[
\omega \cdot \partial_\phi \Delta v_k(\varphi) + (ik)^5 \Delta v_k(\varphi) = \Delta F_k(\varphi) - \Delta \lambda \cdot \bar{\omega} \cdot \partial_\phi v_k(\varphi).
\]

Again using (181), we see

\[
\| \Delta v \|_{s,p'} \leq \frac{1}{\alpha_0} \| \Delta F(0) \|_{s,p'+\tau+1} + \Delta \lambda \cdot \frac{1}{\alpha_0} \| F(0) \|_{s,p'+\tau+1}.
\]

Combining (186) with (188), one gets

\[
\| v \|_{s,p'}^{Lip} \leq \frac{1}{\alpha_0^2} \| F(0) \|_{s,p'+2\tau+1}^{Lip}.
\]
Remark 5. Obviously, \( |F(0)|_{s,p+2r+1} = \|\partial_x f(\varphi, x)\|_{s,p+2r+1} \leq \varepsilon \). Set \( \varepsilon_1 = \frac{1}{3} \varepsilon \), \( v \) as \( v_1, v' \) as \( p + \eta \). Then, we have \( |v_1|^L_{s,p+\eta} \leq \varepsilon_1 \) and

\[
|F(u_1)|_{s,p+\eta-5} = |F(v_1)|_{s,p+\eta-5} \leq c(\|v_1\|^L_{s,p+\eta})^2 \leq \varepsilon_1^2,
\]

because

\[
F(v_1) = 10v_1\partial_x^4 v_1 + 20\partial_x v_1 \partial_x^2 v_1 + 30v_1^2 \partial_x v_1 - 6\partial_x^2 v_1 \partial_x^3 v_1 - 18\partial_x^3 v_1 \partial_x^4 v_1.
\]

KAM step. In this section, we will give the outline of the reducibility and show in detail one key step of the KAM iteration. The purpose is to define a transformation operator \( \Phi_m \) conjugating \( \Sigma_m \), a diagonal operator \( J_m \) plus a \( \varepsilon_m \) remainder \( R_m \), to \( \Sigma_{m+1} \), a diagonal operator \( J_{m+1} \) plus a \( \varepsilon_{m+1} \) remainder \( R_{m+1} \).

Now, we have already got the regularized linear operator \( \mathcal{L}(u_n) \) at the approximate solution \( u_n \), where

\[
\mathcal{L} = \omega \cdot \partial_\theta + m \partial_\theta^5 + \partial_y \{ \partial_y [c_1(\theta, y) \partial_y] + c_0(\theta, y) \}.
\]

The linear operator \( \mathcal{L} \) can be denote as

\[
\mathcal{L} = \mathcal{L}_1 = \omega \cdot \partial_\theta 1 + \mathcal{D} + \mathcal{R} = J + \mathcal{R},
\]

where

\[
\mathcal{D} = m \partial_\theta^5, \quad \mathcal{R} = \partial_y \{ \partial_y [c_1(\theta, y) \partial_y] + c_0(\theta, y) \}.
\]

According to Definition 4.2, we see \( |R|_{s,s,p} \leq \|c_1(\theta, y)\|_{s,p} + \|c_0(\theta, y)\|_{s,p} \).

Following from Lemma 3.2 and Lemma 6.7, the coefficients \( c_i(u_n) \) can be divided into \( n \) parts, where

\[
c_i(u_n) = c_i(u_1) + (c_i(u_2) - c_i(u_1)) + \cdots (c_i(u_n) - c_i(u_{n-1})).
\]

Set \( u_m - u_{m-1} = v_m \). From Lemma 4.9, the size of \( (c_i(u_m) - c_i(u_{m-1})) \) and its function space are controlled by \( v_m \). Thus, the linear operator \( \mathcal{L}(u_n) \) can be seen as

\[
\mathcal{L} = \omega \cdot \partial_\theta 1 + \mathcal{D} + \mathcal{K}^1 + \cdots + \mathcal{K}^n = \omega \cdot \partial_\theta 1 + \mathcal{D} + \mathcal{R}_1 + \mathcal{Q}_1
\]

where

\[
\mathcal{R}_1 = \mathcal{K}^1, \quad \mathcal{Q}_1 = \sum_{i=2}^{n} \mathcal{K}^i,
\]

\[
\mathcal{K}^i = \partial_y \{ \partial_y [((c_1(u_i) - c_1(u_{i-1})) \partial_y] + (c_0(u_i) - c_0(u_{i-1})) \}.
\]

Lemma 4.9. Assume \( \|u_m\|^L_{s_m,p+\eta} \ll \frac{1}{100} \), for all \( m \geq 1 \), we have

\[
|m(u_m) - m(u_{m-1})|^L_{s_m,\eta} \leq \|v_m\|^L_{s_m,\eta}.
\]

For \( i = 0, 1 \), we have

\[
\|c_i(u_m) - c_i(u_{m-1})|^L_{k_{1, p+\eta}} \leq \|v_m\|^L_{s_m, p+\eta},
\]

\[
|\mathcal{K}^m|^L_{k_{1, s_m, p+\eta}} \leq C \|v_m\|^L_{s_m, p+\eta}.
\]

The \( s_m \) will be defined later.

Proof. The estimates (199) and (200) are direct result of Lemma 6.7 and Lemma 3.2. Definition 4.2 implies (201).
The purpose of reducibility is to make the remainder of linear operator $\mathcal{L}_m$ much more small. If the remainder of $\mathcal{L}_m$ can be divided into $\mathcal{Q}_m$ and $\mathcal{R}_m$, that $\mathcal{R}_m$ lies in a more general function space and $\mathcal{Q}_m$ is small enough. The homological equation can only eliminate $\mathcal{R}_m$. Thus, the transformation operator $\Psi_m$ also lies in a much more general $(s, p)$-decay Banach space.

we would show in detail one step of reducibility. The transformation operator $\Psi_m = e^{\Phi_m}$ is acting on the operator $\mathcal{L}_m$:

$$\mathcal{L}_m = \omega \cdot \partial_\theta 1 + \mathcal{D}_m + \mathcal{R}_m + \mathcal{Q}_m, \quad (202)$$

where

$$\mathcal{D}_m = \text{diag}_{k \in \mathbb{Z}\setminus\{0\}}\{dk + \mu_k(\theta)\}, \quad dk \equiv k^5, \quad \int_{\mathbb{T}^d} \mu_k(\theta)d\theta = 0,$$

$$\mathcal{Q}_m = \prod_{i=m-1}^1 \Psi_i^{-1}\left[\sum_{i=m}^n \mathcal{K}_i\right] \prod_{i=1}^{m-1} \Psi_i, \quad (203)$$

Then, we can get

$$\Psi_m^{-1}\mathcal{L}_m \Psi_m = e^{-\Phi_m} [\omega \cdot \partial_\theta (e^{\Phi_m}) + \mathcal{D}_m e^{\Phi_m} + \mathcal{R}_m e^{\Phi_m} + \mathcal{Q}_m e^{\Phi_m}]$$

$$= \omega \cdot \partial_\theta + \mathcal{D}_m + \text{diag}[\mathcal{R}_m] + (\omega \cdot \partial_\theta \Psi_m + [\mathcal{D}_m, \Phi_m] + \mathcal{R}_m - \text{diag}[\mathcal{R}_m])$$

$$+ (e^{-\Phi_m} \mathcal{D}_m e^{\Phi_m} - \mathcal{D}_m - [\mathcal{D}_m, \Phi_m]) + (e^{-\Phi_m} \mathcal{R}_m e^{\Phi_m} - \mathcal{R}_m)$$

$$+ (e^{-\Phi_m} \omega \cdot \partial_\theta \Phi_m e^{\Phi_m} - \omega \cdot \partial_\theta \Phi_m)$$

$$+ \prod_{i=m}^1 \Psi_i^{-1}[\mathcal{K}_i] \prod_{i=1}^m \Psi_i + \prod_{i=m+1}^n \Psi_i^{-1}\left[\sum_{i=m+1}^n \mathcal{K}_i\right] \prod_{i=1}^m \Psi_i, \quad (204)$$

where $[\mathcal{D}_m, \Phi_m] = \mathcal{D}_m \Phi_m - \Phi_m \mathcal{D}_m$.

If we solve the homological equation

$$\omega \cdot \partial_\theta \Psi_m + [\mathcal{D}_m, \Phi_m] + \mathcal{R}_m = \text{diag}[\mathcal{R}_m], \quad (205)$$

$\mathcal{L}_{m+1}$ can be denote as

$$\mathcal{L}_{m+1} = \Psi^{-1}\mathcal{L}_m \Psi = \omega \cdot \partial_\theta 1 + \mathcal{D}_{m+1} + \mathcal{R}_{m+1} + \mathcal{Q}_{m+1}, \quad (206)$$

where

$$\mathcal{D}_{m+1} = \mathcal{D}_m + \text{diag}[\mathcal{R}_m], \quad (207)$$

$$\mathcal{R}_{m+1} = \mathcal{R}_m^+ + \prod_{i=m}^1 \Psi_i^{-1}[\mathcal{K}_{m+1}] \prod_{i=1}^m \Psi_i, \quad (208)$$

$$\mathcal{R}_m^+ = (e^{-\Phi_m} \mathcal{D}_m e^{\Phi_m} - \mathcal{D}_m - [\mathcal{D}_m, \Phi_m]) + (e^{-\Phi_m} \mathcal{R}_m e^{\Phi_m} - \mathcal{R}_m)$$

$$+ (e^{-\Phi_m} \omega \cdot \partial_\theta \Phi_m e^{\Phi_m} - \omega \cdot \partial_\theta \Phi_m), \quad (209)$$

$$\mathcal{Q}_{m+1} = \prod_{i=m}^1 \Psi_i^{-1}\left[\sum_{i=m+2}^n \mathcal{K}_i\right] \prod_{i=1}^m \Psi_i. \quad (210)$$

Before we give the iteration lemmas, we need the following iteration constants and domains.

**Iteration parameters.** Set $n \geq 1, m \geq 1$. Then, $(m, n)$ indicates the $m$th step KAM reduction for the linear operator $\mathcal{L}(u_n)$. 

\[ \varepsilon_1 = \frac{\varepsilon}{m^n} \]  
\[ \varepsilon_m = \left(\frac{1}{2}\right)^{m-1} \]  
which dominate the size of the perturbation \( R_m \) in KAM iteration, the modified function \( v_m \) and \( c_i(u_{m-1} + v_m) - c_i(u_{m-1}) \).

- \( s_n = \left(\frac{10}{77}\right)^{n-1} s_1, s_1 = s \), which dominate the width of \( u_n \).

- \( s'_n = \frac{99}{101} s_n, s_1 = s \), which dominate the width of the coefficients \( c_i(u_n) \) and \( R_n \).

- \( \sigma_m = \frac{1}{200} s_m \), which serve as a bridge from \( s_m \) to \( s_{m+1} \).

- \( \alpha_{mn} = \frac{99}{200} (1 + \frac{1}{2^m}) \), which dominate the measure of parameters removed in the \((m, n)\) step KAM iteration.

- \( C_{d,m} = \frac{1}{4} (1 - \frac{1}{2^m}) \), which \( \frac{1}{8} < C_{d,m} < \frac{1}{4} \).

- \( C_{\lambda,m} = (2 - \frac{1}{2^m}) \varepsilon \), which \( \varepsilon \leq C_{\lambda,m} \leq 2 \varepsilon \).

- \( C_{\mu,m} = c(2 - \frac{1}{2^m}) \varepsilon \), which \( \varepsilon \leq C_{\lambda,m} \leq 2 \varepsilon \).

Set these parameter
\[ 0 < \varepsilon \ll a_0 \ll \min\left\{\frac{1}{100}, s\right\}. \quad (211) \]

**Iteration lemmas.**  
**[H1]_n** Assume that \( q \geq p + \eta + 2 \tau + 1 \), \( \partial_x f \) satisfies the assumption of Theorem 1.1. Let \( \tau > v + 1 \), then, for all \( n \geq 1 \),

\( (P1)_n \) : There exist a function \( u_n : \lambda \subseteq \Lambda(u_n) \to u_n(\lambda) \), with \( \|u_n\|_{\Lambda(u_n) \to u_n(\lambda)} \leq 2 \varepsilon_1 \), where \( \Lambda(u_n) \) are cantor like subset of \( \Pi = \left[\frac{1}{4}, \frac{3}{4}\right] \).

The difference function \( v_n = u_n - u_{n-1} \), where, for convenience, \( v_0 = 0 \), satisfy
\[ \|v_n\|_{\Lambda(u_n) \to u_n(\lambda)} \leq \varepsilon_i. \quad (212) \]

\( (P2)_n \) : \( \|F(u_n)\|_{\Lambda(u_n) \to u_n(\lambda)} \leq \frac{\varepsilon}{n+1} \).

**[H2]_n** Let the regularized linear operator \( \mathcal{L} \) at the approximate solution \( u_n \) denote as
\[ \mathcal{L}_1 = \omega \cdot \partial_\theta 1 + \mathcal{D} + \mathcal{R}_1 + \mathcal{Q}_1. \quad (213) \]

where \( \mathcal{D} = m \partial^\theta_{\theta} \). \( \mathcal{R}_1, \mathcal{Q}_1 \) are defined in (197) and (198).

Then, for all \( m \geq 1 \):

\( (S1)_m \) : There exist an operator
\[ \mathcal{L}_m = \omega \cdot \partial_\theta 1 + \mathcal{D}_m + \mathcal{R}_m + \mathcal{Q}_m, \quad (214) \]

where
\[ \mathcal{D}_m = diag_k \{ k^5 + r_k^m k^3 \}, \quad (215) \]
\[ d_k^m(u_n) = m(u_n) k^5 + r_k^m k^3 \geq m(u_n) k^5 + r_k^{m-1} k^3 + [\mathcal{R}_{m-1}(k,k)], \quad (216) \]
is defined for all \( \lambda \in \Lambda_{m,n} \), where \( \Lambda_{m,n} = \Lambda(u_n) \) (is the domain of \( u_n \)), and, for \( m \leq n \),
\[ \Lambda_{m,n} := \left\{ \lambda \in \Lambda_{m-1,n} : |\ell \cdot \lambda \varpi + d_k^m(u_n) - d_j^m(u_n)| \geq \frac{\alpha_{mn}|i^5 - j^5|}{|\ell|^2} \right\}. \quad (217) \]

Also,
\[ \cdots < d_{-1}(\lambda) < 0 < d_{1}(\lambda) < \cdots, \quad |d_i^m(u_n) - d_j^m(u_n)| \geq (m(u_n) - C_{d,m})|i^5 - j^5|. \quad (218) \]
\( d^m_i(u_n) \) is Liphsitz – continuous in \( \lambda \), and fulfills the estimate:
\[
\sup_{\lambda_1, \lambda_2 \in \lambda} \frac{d^m(\lambda_1) - d^m(\lambda_2)}{\lambda_1 - \lambda_2} \leq m^{\text{Lip}}(u_n)\lambda^3 + C_{\lambda,m} \lambda^3.
\] (219)

\( \mu_k(\lambda, \theta) \) is real analytic in \( \theta \) and Liphsitz – continuous in \( \lambda \) of zero average. It also satisfies
\[
l_1 |\mu_k^m|_{s', \tau} \leq C_{\mu,m} \lambda^3, \quad \|\mu_k^m\|_{\text{Lip}} \leq C_{\lambda,m} \lambda^3.
\] (220)

\( Q_m \) is defined in (210), and \( Q_{n+1} = 0 \).

\[ (S2)_m : \text{The remainder } R_m \text{ is Liphsitz – continuous in } \lambda, \text{ and satisfies the estimate:} \]
\[
|R_m|_{s', \sigma, s_0} \leq C \varepsilon_m, \quad \forall m \leq n,
\] (221)
\[
|R_{n+1}|_{s', -2\sigma, s_0} \leq \frac{4}{\varepsilon_n}.
\] (222)

\( C \) is a constant depending on \( v \). Moreover, for \( m \geq 1 \), we have
\[
\Omega_{m+1} = \Psi^{-1}_m \Omega_m \Psi_m, \quad \Psi_m = e^{\Phi_m},
\] (223)
\[
|\Phi_m|_{s', -2\sigma, s_0} \leq \frac{5}{\varepsilon_m}.
\] (224)

**Remark 6.** We only make \( n \) step KAM reduction for the linear operator \( \Sigma(u_n) \), conjugating it to \( \Sigma_{n+1}(u_n) \).

**Remark 7.** In the Hamiltonian case \( \Phi_m \) is Hamiltonian, the transformation operator
\[
\Psi_m = e^{\Phi_m}
\]
is a symplectic map. The corresponding operator \( \Sigma_m, R_m \) are Hamiltonian, then, \( R_m(k, k) \) can be guaranteed to be pure imaginary.

**Corollary 1.** \( \forall \lambda \in \Lambda_{n,m}, \) the sequence
\[
\Omega_m = \prod_{i=1}^{m} \Psi_i = \Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_m
\] (225)
satisfies the following estimate
\[
|\Omega_m^{-1} - I|^{Lip}_{s', -2\sigma, s_0} + |\Omega_m - I|^{Lip}_{s', -2\sigma, s_0} \leq 1.
\] (226)

**Proof.** (167) and (224) imply (226). \( \square \)

Now, we would prove the iteration Lemma \([\mathbf{H2}]_{m}\).

**Proof.** For convenience, let \( \Sigma, R \) refer to \( \Sigma_{m}, R_{m}, \Sigma+, R+ \) refer to \( \Sigma_{m+1}, R_{m+1} \).

**Part1: Homological equation**
See \( D_{ii} = i(d_i + \mu_i(\varphi)) \). Multiply \(-i\) on both side of the following equation:
\[
\omega \cdot \partial \Phi + [D, \Phi] + R = \text{diag}[R].
\] (227)

Then, the homological equation is equivalent to
\( i = j: \)
\( \Phi_{ii} = 0, \)
\( (1) \)
\( i \neq j: \)
\[
- \omega \cdot \partial \Phi_{ij} + (d_i - d_j)\Phi_{ij} + (\mu_i(\theta) - \mu_j(\theta))\Phi_{ij} - iR_{ij} = 0.
\] (228)

See \( d_{ij} = d_i - d_j, \mu_{ij} = \mu_i(\theta) - \mu_j(\theta), \chi_{ij} = |i^3 - j^3| \). By (220), we have
\[
l_1 |\mu_{ij}|_{s', \tau} \leq C_{\mu,m} (i^3 + j^3).
\] (229)
Now, applying Kuksin’s lemma to (228), we get
\[
\|\Phi\|_{s_m'-\sigma_m,s_0} \leq \frac{c\varepsilon^{6(5/\sigma_m)^{1/3}}}{\alpha_m\chi_m\sigma_m^{2n+2r+s_0+3}} \|\mathcal{R}_{ij}\|_{s_m',s_0}.
\] (230)

Since
\[
\beta \geq \frac{1}{3}, \quad \sigma_m = \frac{1}{200} s_m = \frac{1}{200} \left( \frac{10}{11} \right)^m s,
\] (231)
we get
\[
e^{2(5/\sigma_m)^{1/3}} = c(s)^{\left( \frac{11}{10} \right)^3 (m-1)} < c(s)^{(4)\sigma} m^{-1},
\] (232)
c is a constant only depending on \(s\). Let \(c(s) = \frac{s}{m}\), we have
\[
\varepsilon^\frac{1}{m} \leq c(s)^{-1},
\] (233)
we have
\[
\|\Phi\|_{s_m'-\sigma_m,s_0} \leq \frac{\varepsilon^{-1/20}}{\alpha_m\sigma_m^{2n+2r+s_0+3}} \|\mathcal{R}_{ij}\|_{s_m',s_0}.
\] (234)

Then, consider the infinite matrices of elements
\[
\mathcal{R}_{ij} j^2, \quad \mathcal{R}_{ij} i, \quad \mathcal{R}_{ij} (i^2 + j^2) ^{1/2},
\] (235)
Combining (234), (235) with the Definition 4.2, we have the estimates of matrix \(\Phi\),
\[
\tilde{\Phi}_{s_m'-\sigma_m,s_0} \leq \frac{\varepsilon^{-1/20}}{\alpha_m\sigma_m^{2n+2r+s_0+3}} \|\mathcal{R}_{ij}\|_{s_m',s_0},
\] (236)
\[
\hat{\Phi}_{s_m'-\sigma_m,s_0} \leq \frac{\varepsilon^{-1/20}}{\alpha_m\sigma_m^{2n+2r+s_0+3}} \|\mathcal{R}_{ij}\|_{s_m',s_0},
\] (237)
\[
|\Phi|_{s_m'-\sigma_m,s_0} \leq \frac{\varepsilon^{-1/20}}{\alpha_m\sigma_m^{2n+2r+s_0+3}} \|\mathcal{R}_{ij}\|_{s_m',s_0},
\] (238)

Now, we need a bound on the Lipschitz semi-norm of \(\Phi\). Given a function \(\Phi\) of \(\omega = \lambda \tilde{\omega}\), set \(\Delta \Phi = \Phi(\lambda_1) - \Phi(\lambda_2)\). Then, applying the operator \(\Delta\) to the equation (228), we get
\[
-i(\lambda_1 \tilde{\omega} \cdot \partial_\theta (\Delta \Phi_{ij}) + d_{ij}(\lambda_1)(\Delta \Phi_{ij}) + \mu_{ij}(\theta, \lambda_1)(\Delta \Phi_{ij})) = i(\lambda_1 \tilde{\omega} - \lambda_2 \tilde{\omega}) \partial_\theta (\Phi_{ij}(\lambda_2)) - (\Delta d_{ij} + \Delta \mu_{ij}) \Phi(\lambda_2) - i\Delta \mathcal{R}_{ij},
\] (239)
where
\[
|\Delta d_{ij}| = |(d_i(\lambda_1) - d_j(\lambda_1)) - (d_i(\lambda_2) - d_j(\lambda_2))|
\leq |m(\lambda_1) - m(\lambda_2)||i^5 - j^5| + |r_i(\lambda_1) - r_i(\lambda_2)||i^3| + |r_j(\lambda_1) - r_j(\lambda_2)||j^3|
\leq c\varepsilon |i^5 - j^5||\Delta \lambda|.
\] (240)

Again applying Kuksin’s lemma, we have
\[
\|\Delta \Phi_{ij}\|_{s_m'-2\sigma_m,s_0} \leq \frac{\varepsilon^{-1/10}}{\alpha_m\sigma_m^{2n+2r+s_0+3}} (\|\Delta \lambda\|_{s_m',s_0} + \|\Delta \mathcal{R}_{ij}\|_{s_m',s_0}).
\] (241)

The bounds (236), (237), (238) and (241) imply that
\[
|\Phi|_{s_m'-2\sigma_m,s_0} \leq \varepsilon_0 \|\mathcal{R}_{ij}^{Lip}\|_{s_m',s_0},
\] (242)
Finally, we get
\[ |\Phi|_{s_m-2\sigma_m, s_0} \leq \varepsilon_m. \]  
(245)

(Part 2: New diagonal part)
We have already get the new linear operator
\[ \mathcal{L}_+ = \omega \cdot \partial_\theta 1 + \mathcal{D}_+ + \mathcal{R}_+ + \mathcal{Q}_+, \]  
(246)
where
\[ \mathcal{D}_+ := \text{diag}_{i \in \mathbb{N}} \{ d_i^+ + \mu_i^+(\varphi) \}, \]
(247)
\[ d_i^+ = d_i + \mathcal{R}_{ii}, \quad \mu_i^+(\varphi) = \mu_i(\varphi) + (R_{ii} - \mathcal{R}_{ii}), \quad \mathcal{R}_{ii} = \int_{\tau} R_{ii}(\theta) d\theta. \]  
(248)
By Lemma 6.2, we see
\[ l_1 |R_{ii} - \mathcal{R}_{ii}|_{s_m - \sigma_m, \tau} \leq \|R_{ii}\|_{s_m} \cdot (\sum_{k \in \mathbb{Z}^+ \setminus \{0\}} e^{-2k\sigma_m |k|^{2\tau}})^{1/2} \]
\[ \leq \frac{2\tau}{c} \sigma_m^{-\frac{1}{2} + \tau} (1 + e)^{\frac{1}{2}} \|R_{ii}\|_{s_m, s_0} \]
\[ \leq c_1 \frac{\varepsilon \tau}{s_m} \varepsilon m^{3}, \]  
(249)
and
\[ l_1 |\mu_i^+(\varphi)|_{s_m - \sigma_m, \tau} \leq l_1 |\mu_i(\varphi)|_{s_m, \tau} + l_1 |R_{ii} - \mathcal{R}_{ii}|_{s_m - \sigma_m, \tau} \]
\[ \leq c_1 (2 - \frac{1}{2m}) \varepsilon i^3 + c_1 \frac{\varepsilon}{2m+1} = C_{\mu, m} \varepsilon i^3. \]  
(250)

(Part 3: Estimates of New perturbed terms)
Consider the new perturbed terms \( \mathcal{R}_+ := \mathcal{H}_1 + \mathcal{H}_2, \) where
\[ \mathcal{H}_1 = (e^{-\Phi} \mathcal{R} e^\Phi - \mathcal{R}) + (e^{-\Phi} \mathcal{D} e^\Phi - \mathcal{D} - [\mathcal{D}, \Phi]) + (e^{-\Phi} \omega \cdot \partial_\theta e^\Phi - \omega \cdot \partial_\theta \Phi) \]
\[ = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3, \]  
(251)
\[ \mathcal{H}_2 = \prod_{i=m}^1 \Psi_i^{-1}(\mathcal{K}_{m+1}) \prod_{i=1}^m \Psi_i. \]  
(252)
Considering \( \mathcal{P}_1, \) by [5, Lemma 5.3], we get
\[ |\mathcal{P}_1|_{s_m - 2\sigma_m, s_0} \leq \|\mathcal{R}|_{s_m, s_0} \|_{s_m - 2\sigma_m, s_0} \]
\[ \leq \varepsilon_m. \]  
(253)
Considering \( \mathcal{P}_2, \) from the homological equation \([\mathcal{D}, \Phi] = -\omega \cdot \partial_\theta \Phi - (\mathcal{R} - \text{diag}[\mathcal{R}])\), we have
\[ |[\mathcal{D}, \Phi]|_{s_m - 3\sigma_m, s_0} \leq \frac{c(\nu)}{s_m} |\Phi|_{s_m - 3\sigma_m, s_0} + \|\mathcal{R}|_{s_m - 3\sigma_m, s_0} \]
\[ \leq \varepsilon_m. \]  
(254)
Moreover,
\[ \left| \left[ [\mathcal{D}, \Phi], \Phi \right]_{s_m^0-3\sigma_m,s_0}^{\text{Lip}} \right| \leq \left| \left[ [\mathcal{D}, \Phi], \Phi \right]_{s_m^0-3\sigma_m,s_0}^{\text{Lip}} \right|_{s_m^0-3\sigma_m,s_0}^{\text{Lip}} \leq \varepsilon_m. \] (255)

By the following formula
\[ e^{-\Phi \mathcal{D} \Phi} - \mathcal{D} - [\mathcal{D}, \Phi] = \int_0^1 \int_0^s e^{-s_1 \Phi} \left[ [\mathcal{D}, \Phi], \Phi \right] e^{s_1 \Phi} ds_1 ds, \] (256)
we get
\[ |P_2|_{s_m^0-3\sigma_m,s_0}^{\text{Lip}} \leq \frac{1}{3} \varepsilon_m. \] (257)

Considering \( P_3 \), by [5, Lemma 4.3], we have
\[ |P_3|_{s_m^0-3\sigma_m,s_0}^{\text{Lip}} \leq \frac{c}{\sigma_m} \left( (\Phi|_{s_m^0-2\sigma_m,s_0}^{\text{Lip}}) \right)^2 \] (258)
\[ \leq \frac{c}{\sigma_m} \varepsilon_m. \]

The bounds (253), (257) and (258) imply
\[ |H_1|_{s_m^0-3\sigma_m,s_0}^{\text{Lip}} \leq \frac{4}{3} \varepsilon_m. \] (259)

As we see in Corollary 1,
\[ \max \left\{ \left| \Omega_m^{-1} |_{s_m^0-2\sigma_m,s_0}^{\text{Lip}} \right|, \left| \Omega_m |_{s_m^0-2\sigma_m,s_0}^{\text{Lip}} \right| \right\} \leq 2, \]
then,
\[ |H_2|_{s_m^0-2\sigma_m,s_0}^{\text{Lip}} \leq \left| \Omega_m^{-1} |_{s_m^0-2\sigma_m,s_0}^{\text{Lip}} \right|, \left| \Omega_m |_{s_m^0-2\sigma_m,s_0}^{\text{Lip}} \right| \leq C \varepsilon_m. \] (260)

Finally,
\[ |R|_{s_m^0-3\sigma_m,s_0}^{\text{Lip}} \leq (C + 1) \varepsilon_m. \] (261)

\[ \Box \]

4.2. The \( \varepsilon_{n+1} \) approximate solution of linear equation \( F(u_n) + \mathcal{L}(u_n)v = 0 \). After \( n \)-step iteration, the linear operator \( \mathcal{L}(u_n) \) has been transformed to
\[ \mathcal{L}_{n+1} = \omega \cdot \partial_y 1 + \mathcal{D}_{n+1} + \mathcal{R}_{n+1}, \] (262)
where \( \mathcal{R}_{n+1} \) is relatively small linear operator with \( \| \mathcal{R}_{n+1} \|_{s_n^0-2\sigma_m}^{\text{Lip}} \leq \varepsilon_{n+1} \). Now, the main concern is the invertibility of \( \mathcal{J}_{n+1} = \omega \cdot \partial_y 1 + \mathcal{D}_{n+1} \). The same method can be found in [9, 29, 30].

**Lemma 4.10.** For all \( g \in H^0_{s_n^0-2\sigma_m,s_0} \) with zero space average and \( \lambda \in \Lambda_n, n \cap \Gamma(u_n), \)
\[ \Gamma(u_n) := \left\{ \lambda : |\lambda \omega \cdot \ell + d_k^{\varepsilon_{n+1}}(u_n)| \geq \alpha_{n+1} \frac{k^5}{|\ell|^5}, \forall \ell \in \mathbb{Z}^v, k \in \mathbb{Z} \setminus \{0\} \right\}, \] (263)
the equation \( \mathcal{J}_{n+1} v = g \) has a unique solution \( v \) with zero space average and satisfying
\[ \|v\|_{s_n^0-4\sigma_m,s_0}^{\text{Lip}} \leq \varepsilon_{n+1} \|g\|_{s_n^0-2\sigma_m,s_0}^{\text{Lip}}. \] (264)
Proof. Since \( \mathcal{J} \) is a diagonal linear operator and \( g \in H^1_0 \), the equation \( \mathcal{J}v = g \) can be transformed to
\[
-ik \cdot \partial_\theta v_i + d_i^{n+1} v_i + \mu_i^{n+1} (\theta) v_i(\theta) = -ig_i(\theta), \quad i \in \mathbb{Z}\setminus 0,
\]
where \( d_i^{n+1} \geq \frac{1}{2} i^5, \quad i^2 |\mu_i^{n+1}(\varphi)|_{\sigma_n-2\sigma_n, \tau} \leq 2 \varepsilon i^3, \quad \chi_i = i^5. \)

Applying Kuksin’s lemma to (265), we get
\[
\|v_i\|_{s'_n - 3\sigma_n, s_0} \leq \frac{ce^{2(5/\sigma_n)^{1/\beta}}}{\alpha_n \chi_i^{\sigma_n} + 2v + 1, j = 0} \|g_i\|_{s'_n - 2\sigma_n, s_0}.
\]

Since \( \beta \geq \frac{2}{3}, \quad \sigma_n = \frac{1}{200} n = \frac{1}{200} \left( \frac{10}{11} \right)^{n-1} s, \) we have
\[
e^{2(5/\sigma_n)^{1/\beta}} \leq c(s)^{\frac{1}{10}} \frac{1}{\sigma_n} < c(s)^{\frac{1}{10}},
\]
where \( c \) is a constant only depending on \( s \). Let \( \varepsilon \# \leq c(s)^{-1} \), we have
\[
\|v_i\|_{s'_n - 3\sigma_n, s_0} \leq \frac{ce^{2(5/\sigma_n)^{1/\beta}}}{\alpha_n \chi_i^{\sigma_n} + 2v + 1, j = 0} \|g_i\|_{s'_n - 2\sigma_n, s_0}.
\]

Applying the operator \( \Delta v = v(\lambda_1) - v(\lambda_2) \) to (265), one gets
\[
-1i\lambda_1 \nabla \cdot \partial_\theta \Delta v_i \quad + \quad d_i^{n+1}(\lambda_1) \Delta v_i + \mu_i^{n+1}(\theta)(\lambda_1) \Delta v_i(\theta)
\]
\[
= -i \Delta g_i(\varphi) - (\Delta d_i^{n+1} + \Delta \mu_i^{n+1})v_i(\lambda_2) + i \Delta \lambda \nabla \cdot \partial_\theta v_i(\lambda_2)
\]

(270)

Again applying Kuksin’s lemma, we have
\[
\|\Delta v_i\|_{s'_n - 4\sigma_n, s_0} \leq \frac{ce^{2(5/\sigma_n)^{1/\beta}}}{\alpha_n \chi_i^{\sigma_n} + 2v + 1, j = 0} \left( \|\Delta g_i\|_{s'_n - 2\sigma_n, s_0} + |\Delta \lambda| \|g_i\|_{s'_n - 2\sigma_n, s_0} \right)
\]

Finally, (269) and (271) imply
\[
\|v\|_{s^{Lip}} \leq \varepsilon_n^{1/10} \|g\|_{s^{Lip}}
\]

(272)

Now, we have conjugated the linearized operator \( \mathcal{L} \) to
\[
\mathcal{L}_{n+1} = \mathcal{J}_{n+1} + \mathcal{R}_{n+1} = W_1^{-1} \mathcal{L} W_2,
\]
where
\[
W_2 = AB \Omega_n, \quad W_1^{-1} = \Omega_n^{-1} \frac{1}{\xi(\theta)} B^{-1} A^{-1}.
\]

Also, we can see \( \mathcal{W}^ {\pm 1} \) are maps from \( H^1_0 \) to \( H^1_0 \).

Now, we can prove the first part \( (P1)_n \) of the iteration lemma \( [H1]_n \).

Lemma 4.11. For all \( \lambda \in \Lambda_n \cap \Gamma(u_n) \), the linear operator \( \mathcal{W}_1 \mathcal{J} \mathcal{W}_2^{-1} \) admits a right inverse of \( H^1_0 \). More precisely, for all Lipschitz family \( F(\lambda) \in H^1_0 \), the function
\[
v := (W_1 \mathcal{J} W_2^{-1})^{-1} F := W_2 \mathcal{J}^{-1} W_1^{-1} F
\]
is a solution of \( \mathcal{W}_1 \mathcal{J} \mathcal{W}_2^{-1} v = F. \) Moreover,
\[
\|v_{n+1}\|_{s_{n+1, p+\eta}} \leq \varepsilon_{n+1} \|F(u_n)\|_{s_{n, p+\eta-5}}.
\]

(273)
Proof. We have already get
\[ v_{n+1} = W_2 J^{-1} W_1^{-1} F(u_n). \]  
(275)

Applying Lemma 3.2 and Corollary 1, we see
\[ \| W_1^{-1} F(u_n) \|_{s_n - 2 \sigma_n, s_0}^{\Delta \text{Lip}} = \| W_1^{-1} F(u_n) \|_{s_n - 2 \sigma_n, s_0}^{\Delta \text{Lip}} \]
\[ \leq \| W_1^{-1} F(u_n) \|_{s_n - 2 \sigma_n, s_0}^{\Delta \text{Lip}} \]
\[ \leq \| W_1^{-1} F(u_n) \|_{s_n - 2 \sigma_n, s_0}^{\Delta \text{Lip}} \]
\[ \leq \| F(u_n) \|_{s_n - 2 \sigma_n, s_0}^{\Delta \text{Lip}} \]
(276)

Using Lemma 4.10, one gets
\[ \| J^{-1} W_1^{-1} F(u_n) \|_{s_n - 2 \sigma_n, s_0}^{\Delta \text{Lip}} \leq \| J^{-1} W_1^{-1} F(u_n) \|_{s_n - 2 \sigma_n, s_0}^{\Delta \text{Lip}} \]
\[ \leq \| F(u_n) \|_{s_n - 2 \sigma_n, s_0}^{\Delta \text{Lip}} \]
(277)

With reference to Corollary 1 and Lemma 6.4, we see
\[ \| W_1 J^{-1} W_1^{-1} F(u_n) \|_{s_n - 5 \sigma_n, p+2 \eta}^{\Delta \text{Lip}} \leq \| W_1 J^{-1} W_1^{-1} F(u_n) \|_{s_n - 5 \sigma_n, p+2 \eta}^{\Delta \text{Lip}} \]
\[ \leq \| W_1 J^{-1} W_1^{-1} F(u_n) \|_{s_n - 5 \sigma_n, p+2 \eta}^{\Delta \text{Lip}} \]
\[ \leq \| W_1 J^{-1} W_1^{-1} F(u_n) \|_{s_n - 5 \sigma_n, p+2 \eta}^{\Delta \text{Lip}} \]
\[ \leq \| F(u_n) \|_{s_n, p+\eta-5}^{\Delta \text{Lip}} \]
(278)

Using Lemma 3.2 again, we get
\[ \| W_2 J^{-1} W_1^{-1} F(u_n) \|_{k_1(s_n - 5 \sigma_n), p+\eta}^{\Delta \text{Lip}} \leq \| W_2 J^{-1} W_1^{-1} F(u_n) \|_{k_1(s_n - 5 \sigma_n), p+\eta}^{\Delta \text{Lip}} \]
\[ \leq \| W_2 J^{-1} W_1^{-1} F(u_n) \|_{k_1(s_n - 5 \sigma_n), p+\eta}^{\Delta \text{Lip}} \]
\[ \leq \| W_2 J^{-1} W_1^{-1} F(u_n) \|_{k_1(s_n - 5 \sigma_n), p+\eta}^{\Delta \text{Lip}} \]
\[ \leq \| F(u_n) \|_{k_1(s_n, p+\eta-5)}^{\Delta \text{Lip}} \]
(279)

Since \( k_1(s_n - 5 \sigma_n) > s_{n+1} \), we get
\[ \| v_{n+1} \|_{s_{n+1}, p+\eta}^{\Delta \text{Lip}} \leq \| v_{n+1} \|_{s_{n+1}, p+\eta}^{\Delta \text{Lip}} \]
\[ \leq \| v_{n+1} \|_{s_{n+1}, p+\eta}^{\Delta \text{Lip}} \]
(280)

Thus,
\[ \| v_{n+1} \|_{s_{n+1}, p+\eta}^{\Delta \text{Lip}} \leq \| v_{n+1} \|_{s_{n+1}, p+\eta}^{\Delta \text{Lip}} \]
(281)

4.3. The estimation of \( F(u_{n+1}) \) and \( v_{n+1} \).

Lemma 4.12. Assume \( u_{n+1} = u_n + v \), that \( v \) is the solution of \( W_1 J W_2^{-1} v = F(u_n) \). Then,
\[ F(u_{n+1}) = W_1 J W_2^{-1} F(u_n) + 10u \partial_x^2 v + 20 \partial_x v \partial_x^2 v + 30v^2 \partial_x v \]
\[ - 6 \partial_x^2 v \partial_x^2 v - 18 \partial_x^3 v \partial_x^2 v + 60 v \partial_x v + 30 v^2 \partial_x u_n. \]
(282)
Proof.

\[ F(u_{n+1}) = F(u_n) + L(u_n)v + 10v\partial_x^3v + 20\partial_xv\partial_x^2v + 30v^2\partial_xv - 60\partial_x^2v\partial_x^2v - 180\partial_xv\partial_x^2v + 60uv\partial_xv + 30v^2\partial_xu_n \]
\[ = F(u_n) + W_1JW_2^{-1}(v) + W_1RW_2^{-1}(v) + 10v\partial_x^3v + 20\partial_xv\partial_x^2v + 30v^2\partial_xv \quad (283) \]
\[ - 60\partial_x^2v\partial_x^2v - 180\partial_xv\partial_x^2v + 60uv\partial_xv + 30v^2\partial_xu_n \]
\[ = W_1RW_2^{-1}(v) + 10v\partial_x^3v + 20\partial_xv\partial_x^2v + 30v^2\partial_xv - 60\partial_x^2v\partial_x^2v - 180\partial_xv\partial_x^2v + 60uv\partial_xv + 30v^2\partial_xu_n \]
\]

Now, the whole necessary estimates have been prepared. We would prove the last piece \((P2)_n\) of iteration Lemma \([H1]_n\).

Proof. Considering the formula (283), an approximate estimate for \(W_1RW_2^{-1}(v_{n+1})\) is our main concern. Since \(W_2^{-1}v_{n+1} = J^{-1}W_1^{-1}F(u_n)\), by (277), we see
\[ \|W_2^{-1}v_{n+1}\|_{s_n^{-4\sigma_n,s_0}}^{Lip} = \|J^{-1}W_1^{-1}F(u_n)|_{s_n^{-4\sigma_n,s_0}}^{Lip} \ll \varepsilon_n + 1 \|F(u_n)|_{s_n,p+\eta}^{Lip}. \quad (284) \]
Then, by Lemma 6.4, we have
\[ \|W_2^{-1}v_{n+1}\|_{s_n^{-5\sigma_n,2s_0+1}}^{Lip} \ll \frac{1}{\sigma_n^{s_0+1}} \|W_2^{-1}v_{n+1}\|_{s_n^{-4\sigma_n,s_0}}^{Lip} \ll \frac{1}{\sigma_n^{s_0+1}} \|W_2^{-1}v_{n+1}\|_{s_n^{-5\sigma_n,s_0}}^{Lip}. \quad (285) \]
Using (274), we have
\[ \|RW_2^{-1}v_{n+1}\|_{s_n^{-5\sigma_n,s_0}}^{Lip} \ll \frac{1}{\sigma_n^{s_0+1}} \|W_2^{-1}v_{n+1}\|_{s_n^{-5\sigma_n,2s_0+1}}^{Lip} \ll \frac{2}{\sigma_n^{s_0+1}} \|F(u_n)|_{s_n,p+\eta}^{Lip}. \quad (286) \]
By Corollary 1 and Lemma 6.4, one gets
\[ \|\Omega_nRW_2^{-1}v_{n+1}|_{s_n^{-7\sigma_n,p+2\eta}}^{Lip} \ll \frac{1}{\sigma_n^{s_0+2\eta+5}} \|\Omega_nRW_2^{-1}v_{n+1}|_{s_n^{-6\sigma_n,s_0}}^{Lip} \ll \frac{1}{\sigma_n^{s_0+2\eta+5}} \|\Omega_nRW_2^{-1}v_{n+1}|_{s_n^{-6\sigma_n,s_0}}^{Lip} \ll \frac{1}{\sigma_n^{s_0+2\eta+5}} \|RW_2^{-1}v_{n+1}|_{s_n^{-6\sigma_n,s_0}}^{Lip} \ll \frac{1}{\sigma_n^{s_0+2\eta+5}} \|RW_2^{-1}v_{n+1}|_{s_n^{-6\sigma_n,s_0}}^{Lip} \ll \frac{1}{\sigma_n^{s_0+2\eta+5}} \|RW_2^{-1}v_{n+1}|_{s_n^{-5\sigma_n,s_0}}^{Lip} \ll \frac{1}{\sigma_n^{s_0+2\eta+5}} \|RW_2^{-1}v_{n+1}|_{s_n^{-5\sigma_n,s_0}}^{Lip} \ll \frac{1}{\sigma_n^{s_0+2\eta+5}} \|F(u_n)|_{s_n,p+\eta}^{Lip}. \quad (287) \]
By Lemma 3.2, we see
\[ \|W_1RW_2^{-1}v_{n+1}|_{k_1(s_n^{-7\sigma_n},p+\eta}} \ll \frac{1}{\sigma_n^{s_0+2\eta+8}} \|F(u_n)|_{s_n,p+\eta}^{Lip}. \quad (288) \]
Obviously, \( k_1(s'_n - 7\sigma_n) > s_{n+1} \). Finally, Combining (288) with the estimate of the rest part of \( F(u_{n+1}) \), we get

\[
\|F(u_{n+1})\|_{s_{n+1}, p+\eta-5} \leq \left( \left( \frac{\varepsilon_{n+1}}{s_n} \right) \|F(u_n)\|_{s_n, p+\eta-5} + (\|v_{n+1}\|_{s_{n+1}, p+\eta}^L)^2 \right)^{\frac{1}{2}} \leq \varepsilon_{n+2}^2.
\] (289)

\[\square\]

4.4. **Proof of main theorem.** Combining the iteration lemmas we have proved, we can give a precise proof of Theorem 1.1.

**Proof.** By iteration lemmas [H1] and [H2], we get a function \( u \), where \( u = \sum_{n \geq 1} v_n \). Since \( v_n \) is analytical on the complexified torus \( \mathbb{T}^{v+1} \), \( u(\omega t, x) \) is \( C^\infty(\mathbb{R} \times \mathbb{T}) \).

For any approximate solution \( u_n \), \( u_n = \sum_{i=1}^n v_i \), we already have \( \|F(u_n)\|_{s_n, p+\eta-5} \leq \varepsilon_{n+1} \). By Lemma 6.4, we see

\[
\|F(u_n)\|_{0, v} \leq \frac{c(v + 5 - p - \eta)}{s_n^{v+5-p-\eta}} \|F(u_n)\|_{s_n, p+\eta-5}.
\] (290)

\( \|F(u_n)\|_{0, v} \) is the Sobolev norm of \( F(u_n) \) on the torus \( \mathbb{T}^{v+1} \). For any finite number \( v \geq p + \eta - 5 \), there exists \( N \), if \( n \geq N \),

\[
\|F(u_n)\|_{0, v} \leq \varepsilon_{n+1}^2.
\]

Thus, \( \|F(u)\|_{0, v} = 0 \) \[\square\]

5. **Measure estimation.** For notational convenience, we extend the eigenvalues \( d_{ij}^m(u_n) \) defined for \( i \in \mathbb{Z} \setminus \{0\} \), to \( i \in \mathbb{Z} \), where \( d_{ij}^m(u_n) = 0, i = 0 \).

Set \( \Theta_{mn} = \bigcup_{i, j, \ell} R_{ij, \ell}^m(u_n), i, j \neq 0 \), where

\[
R_{ij, \ell}^m(u_n) = \{ \lambda \in \Pi : |\ell \cdot \lambda \varpi + d_{ij}^m(u_n) - d_{j}^m(u_n)| \leq \frac{\alpha_{mn}}{[\ell]^{\tau}}, i \neq j, m \leq n \}. \] (291)

Set \( \Gamma_n = \bigcup_{i, \ell} R_{i, 0, \ell}^m(u_n) = \bigcup_{i, \ell} R_{i, \ell}^m(u_n) \), where

\[
R_{i, \ell}^m(u_n) = \{ \lambda \in \Pi : |\ell \cdot \lambda \varpi + d_{i}^{m+1}(u_n)| \leq \frac{\alpha_{nn}}{[\ell]^{\tau}} \}. \] (292)

Although \( \Theta_{mn} \) and \( \Gamma_n \) seem different, the following two lemmas can apply them both.

**Lemma 5.1.** If \( R_{ij, \ell}^m(u_n) \neq \emptyset \), then \( |\hat{5} - \hat{j}| \leq 8|\varpi| \leq \ell |\leq \ell| \).

**Proof.** If \( R_{ij, \ell}^m(u_n) \neq \emptyset \), then there exists \( \lambda \in \Pi \), such that

\[
|\lambda \varpi + d_{ij}^m(u_n) - d_{j}^m(u_n)| \leq \frac{\alpha_{mn}}{[\ell]^{\tau}} |\hat{5} - \hat{j}|.
\]

Therefore,

\[
|d_{ij}^m - d_{j}^m| < \frac{\alpha_{mn}}{[\ell]^{\tau}} |\hat{5} - \hat{j}| + 2|\varpi| \leq \ell |\leq \ell|.
\] (293)

Moreover, by (218), for \( \varepsilon \) small enough,

\[
|d_{ij}^m - d_{j}^m| \geq \frac{1}{2} |\hat{5} - \hat{j}|.
\] (294)
Proof. Consider the function \( \phi(\lambda) : \Pi \rightarrow \mathbb{R} \) defined by
\[
\phi(\lambda) = \lambda \varpi \cdot \ell + d^n_j(u_n) - d^m_j(u_n),
\]
where
\[
|d^n_j(u_n) - d^m_j(u_n)|^{lip} \leq m_{lip}(u_n)|i^5 - j^5| + C_{\lambda, m}|j^3 - j^3| \leq C\varepsilon|i^5 - j^5|.
\]
Since \( \varepsilon \) is small enough, for any \( \lambda_1, \lambda_2 \in \Pi \), we see
\[
|\phi(\lambda_1) - \phi(\lambda_2)| \geq (\lambda_1 - \lambda_2)(|\varpi \cdot \ell| - |d^n_j(u_n) - d^m_j(u_n)|^{lip})
\]
\[
\geq \left( \frac{1}{8} - C\varepsilon \right)|i^5 - j^5||\lambda_1 - \lambda_2|
\]
\[
\geq \frac{|i^5 - j^5|}{9}|\lambda_1 - \lambda_2|.
\]
Then, one gets
\[
|R^n_{ij}(u_n)| \leq \frac{\alpha_{mn}|i^5 - j^5|}{|\ell|^r} \frac{9}{9|i^5 - j^5|} \leq \frac{9\alpha_{mn}}{|\ell|^r}.
\]
\]
Lemma 5.3. Let \( u_n(\lambda), u_{n-1}(\lambda) \) be Lipschitz families of analytic function, defined for \( \lambda \in \Pi \). Then, for \( v > 0, \forall \lambda \in \Lambda_{v, n} \cap \Lambda_{v, n-1} \),
\[
|R_v(u_n) - R_v(u_{n-1})|_{s_{\ell}, s_{\ell}} \leq C\varepsilon_v\varepsilon_n.
\]
Proof. Obviously, for \( v = 1 \), we have
\[
R_1(u_n) - R_1(u_{n-1}) = K_1 - K_1 = 0.
\]
By induction method, for \( v \leq m \), we have
\[
|\mu^v_j(u_n) - \mu^v_j(u_{n-1})| \leq (m(u_n) - m(u_{n-1}))(|i^5 - j^5| + |(r_i(u_n) - r_i(u_{n-1}))| + |(r_j(u_n) - r_j(u_{n-1}))|) + C\varepsilon_n|i^5 - j^5| + c\sum_{k \leq v-1} |R_k(u_n) - R_k(u_{n-1})|_{s_{k, s_{k}}}|i^3 + j^3|
\]
\[
\leq c\varepsilon_n|i^5 - j^5| + c\sum_{k \leq v-1} |R_k(u_n) - R_k(u_{n-1})|_{s_{k, s_{k}}}|i^3 + j^3|
\]
\[
\leq c_1\varepsilon_n|i^5 - j^5|
\]
and
\[
|\mu^v_j(u_n) - \mu^v_j(u_{n-1})| \leq c\sum_{k \leq v-1} |R_k(u_n) - R_k(u_{n-1})|_{s_{k, s_{k}}}|i^3 + j^3|
\]
\[
\leq c_2\varepsilon_n|i^5 - j^5|
\]
Now, we consider the situation \( v = m + 1 \). Set \( \Delta \Phi_{ij} = \Phi_{ij}(u_n) - \Phi_{ij}(u_{n-1}) \). Applying the operator \( \Delta \) to (228), we get
\[
-\omega \cdot \partial \theta(\Delta \Phi^m_{ij}) + d^m_{ij}(u_n)(\Delta \Phi^m_{ij}) + \mu_{ij}(u_n)(\Delta \Phi^m_{ij})
\]
\[
= -\Delta d_{ij} + \Delta \mu_{ij})\Phi^m_{ij}(u_{n-1}) - i\Delta R^m_{ij}.
\]
Applying Kuksin’s lemma to (304) again, we have
\[
\|\Delta \Phi_{ij}^m\|_{\sigma_m^2, s_m, s_0} < \frac{\varepsilon_m^{-1/10}}{d_m^{2\sigma_m^4+2\sigma_m^2+6}} \chi_{ij} \left( \varepsilon_n \|R_{ij}^m\|_{s_m', s_0} + \|\Delta R_{ij}^m\|_{s_m', s_0} \right),
\]
which indicates
\[
|\Delta \Phi_{ij}^m|_{s_m', s_0} \leq \varepsilon_n \frac{5}{m}.
\]
Recall the definition of \(\mathcal{R}_{m+1}\), we can get
\[
\Delta \mathcal{R}_{m+1} = \Delta \mathcal{P}_1 + \Delta \mathcal{P}_2 + \Delta \mathcal{P}_3 + \Delta \mathcal{H}_{m+1}.
\]
For convenience, we make some notations,
\[
|\Phi_{ij}^m|_{s_m^2, s_m, s_0} = \max\{|\Phi_{ij}(u_n)|_{s_m^2, s_m, s_0}, |\Phi_{ij}(u_n-1)|_{s_m^2, s_m, s_0}\}, \quad (308)
\]
\[
|\mathcal{R}_{m}|_{s_m^2, s_0} = \max\{|\mathcal{R}_{m}(u_n)|_{s_m^2, s_0}, |\mathcal{R}_{m}(u_n-1)|_{s_m^2, s_0}\}.
\]
By (50), we see
\[
|\Delta \Psi_{ij}^m|_{s_m^2, s_m, s_0} \leq C|\Delta \Phi_{ij}^m|_{s_m^2, s_m, s_0} \leq C\varepsilon_n \frac{5}{m},
\]
and
\[
|\Delta \Psi_{ij}^{-1}|_{s_m^2, s_m, s_0} \leq C|\Delta \Phi_{ij}^m|_{s_m^2, s_m, s_0} \leq C\varepsilon_n \frac{5}{m}.
\]
The estimate of \(\Delta \mathcal{R}_{m+1}\) can be divided into several parts.

**Part 1:** Firstly, we consider \(\Delta \mathcal{H}_{m+1}\). Since \(\Delta [\mathcal{K}_{m+1}] = 0\), one gets
\[
\Delta \mathcal{H}_{m+1} = \Delta \left( \prod_{i=m}^{m+1} \Psi_i \right) [\mathcal{K}_{m+1}] \prod_{i=1}^{m} \Psi_i + \prod_{i=m}^{m+1} \Psi_i^{-1} [\mathcal{K}_{m+1}] \Delta \left( \prod_{i=1}^{m} \Psi_i \right).
\]
Using (50), (310) and (311), we have
\[
|\Delta \mathcal{H}_{m+1}|_{s_m^2, s_0} \leq \left| [\Delta \Omega_m]_{[\mathcal{K}_{m+1}]} [\Omega_m] \right|_{s_m^2, s_0} \prod_{i=1}^{m} \Psi_i^{-1} |\mathcal{R}_{m}|_{s_m^2, s_0} \prod_{i=1}^{m} \Psi_i^{-1} |\mathcal{R}_{m}|_{s_m^2, s_0}
\]
\[
\leq C\varepsilon_n \varepsilon_n.
\]

**Part 2:** Considering \(\Delta \mathcal{P}_1\), we have
\[
\Delta \mathcal{P}_1 = (e^{-\Phi_m}(\Delta \mathcal{R}_m)e^{\Phi_m} - \Delta \mathcal{R}_m) + (\Delta e^{-\Phi_m}) \mathcal{R}_m e^{\Phi_m} + e^{-\Phi_m} \mathcal{R}_m (\Delta e^{\Phi_m}).
\]
Using [5, lemma 5.3] and (50), we have
\[
|\Delta \mathcal{P}_1|_{s_m^2, s_0} \leq C\varepsilon_m \varepsilon_n.
\]
and
\[
|\Delta \mathcal{P}_2 + \Delta \mathcal{P}_3 + \Delta \mathcal{H}_{m+1}|_{s_m^2, s_0} \leq C\varepsilon_m \varepsilon_n.
\]
Then, we have
\[
|\Delta \mathcal{P}_1|_{s_m^2, s_0} \leq C\varepsilon_m \varepsilon_n \leq \frac{1}{3} \frac{5}{m} \leq \frac{1}{3} \frac{5}{m} \leq \frac{1}{3} \frac{5}{m} \leq \frac{1}{3} \frac{5}{m} \leq \frac{1}{3} \frac{5}{m} \leq \frac{1}{3} \frac{5}{m} \leq \frac{1}{3} \frac{5}{m} .
\]

(315)
Part 3: Consider $\Delta P_2$, we have
\[
\Delta P_2 = \int_0^1 \int_0^s \Delta[e^{-s_1 \Phi_m}[[D_m, \Phi_m], \Phi_m]e^{s_1 \Phi_m}]ds_1 ds,
\] (316)
where
\[
\Delta[e^{-s_1 \Phi_m}[[D_m, \Phi_m], \Phi_m]e^{s_1 \Phi_m}] = (\Delta e^{-s_1 \Phi_m})[[D_m, \Phi_m], \Phi_m]e^{s_1 \Phi_m} \\
+ e^{-s_1 \Phi_m} [[D_m, \Phi_m], \Phi_m](\Delta e^{s_1 \Phi_m}) \\
+ e^{-s_1 \Phi_m} \Delta [[D_m, \Phi_m], \Phi_m]e^{s_1 \Phi_m}.
\] (317)
From the homological equation (228), we see $\Delta [D, \Phi] = -\omega \cdot \partial_z \Delta \Phi - (\Delta R - \Delta \text{diag}[R])$. Then, we get
\[
|\Delta [D_m, \Phi_m]|_{s_m - 3\sigma_m, s_0} \leq \frac{c(v)}{\sigma_m} \varepsilon_m \cdot \frac{\hat{m}}{\varepsilon_m} + c \varepsilon_n \cdot \varepsilon_m
\]
and
\[
|\Delta [D_m, \Phi_m], \Phi_m]|_{s_m - 3\sigma_m, s_0} \leq |\Delta [D_m, \Phi_m]|_{s_m - 3\sigma_m, s_0} |\Phi_m|_{s_m - 2\sigma_m, s_0}
\]
\[
+ |[D_m, \Phi_m]|_{s_m - 3\sigma_m, s_0} |\Delta \Phi_m|_{s_m - 2\sigma_m, s_0}
\]
\[
\leq \frac{\hat{m}}{\varepsilon_m} \varepsilon_n \cdot \varepsilon_m + \frac{\hat{m}}{\varepsilon_m} \varepsilon_n \cdot \varepsilon_m.
\] (318)
By (50), we see
\[
|(\Delta e^{-s_1 \Phi_m})[[D_m, \Phi_m], \Phi_m]e^{s_1 \Phi_m} + e^{-s_1 \Phi_m} [[D_m, \Phi_m], \Phi_m](\Delta e^{s_1 \Phi_m})|_{s_m - 2\sigma_m, s_0}
\]
\[
\leq \frac{2}{\varepsilon_m} \varepsilon_n \cdot \varepsilon_m.
\] (319)
The bounds (317), (318) and (319) imply
\[
|\Delta P_2|_{s_m - 3\sigma_m, s_0} \leq \frac{1}{3} \frac{\hat{m}}{\varepsilon_m} \varepsilon_n.
\] (320)
Part 4: For $\Delta P_3$, we see
\[
\Delta P_3 = (e^{-\Phi_m}(\omega \cdot \partial_z \Delta \Phi_m)e^{\Phi_m} - \omega \cdot \partial_z \Delta \Phi_m) + (\Delta(e^{-\Phi_m})\omega \cdot \partial_z \Phi_m e^{\Phi_m}) \\
+ (e^{-\Phi_m} \omega \cdot \partial_z \Phi_m \Delta(e^{\Phi_m})).
\]
Using [5, lemma5.3] and (50) again, one gets
\[
|e^{-\Phi_m}(\omega \cdot \partial_z \Delta \Phi_m)e^{\Phi_m} - \omega \cdot \partial_z \Delta \Phi_m|_{s_m - 3\sigma_m, s_0}
\]
\[
\leq |\Phi_m|_{s_m - 3\sigma_m, s_0} |\omega \cdot \partial_z \Delta \Phi_m|_{s_m - 3\sigma_m}
\]
\[
\leq \frac{c}{\sigma_m} |\Delta \Phi_m|_{s_m - 2\sigma_m, s_0} |\Phi_m|_{s_m - 2\sigma_m, s_0}
\]
\[
\leq \frac{\hat{m}}{\varepsilon_m} \varepsilon_n \cdot \varepsilon_m + \frac{\hat{m}}{\varepsilon_m} \varepsilon_n \cdot \varepsilon_m,
\] (321)
and
\[
|\Delta(e^{-\Phi_m})\omega \cdot \partial_z \Phi_m e^{\Phi_m} + e^{-\Phi_m} \omega \cdot \partial_z \Phi_m \Delta(e^{\Phi_m})|_{s_m - 3\sigma_m, s_0}
\]
\[
\leq C |\Delta \Phi_m|_{s_m - 3\sigma_m, s_0} |\omega \cdot \partial_z \Phi_m|_{s_m - 3\sigma_m, s_0}
\]
\[
\leq \frac{c}{\sigma_m} |\Delta \Phi_m|_{s_m - 2\sigma_m, s_0} |\Phi_m|_{s_m - 2\sigma_m, s_0}
\]
We see Proof.\footnote{\textit{i.e.}} Finally, \eqref{312}, \eqref{315}, \eqref{320} and \eqref{323} imply
\[ |\Delta R_{m+1}|_{s_{m+1},s_0}^\varepsilon \leq C\varepsilon_{m+1}\varepsilon_m. \] (324)
Then, the lemma is proved. \hfill \Box

**Lemma 5.4.** If $\varepsilon_0$ is small enough, for any $m \leq n - 1$ and $\ell$ satisfying
\[ |\ell|^r \leq \varepsilon_0^{-\frac{3}{2}} \leq \frac{1}{\varepsilon_n^2}, \]
we have
\[ R_{ij\ell}^m(u_n) \subseteq R_{ij\ell}^m(u_{n-1}). \]

**Proof.** By \eqref{302}, for any $i, j \in \mathbb{Z}$, we have
\[ |(d_i^m - d_j^m)(u_n) - (d_i^m - d_j^m)(u_{n-1})| \leq c|\ell^5 - j^5|\varepsilon_n. \] (325)
Then,
\[
|\lambda \varpi \cdot \ell + (d_i^m - d_j^m)(u_n)| \geq |\lambda \varpi \cdot \ell + (d_i^m - d_j^m)(u_{n-1})| \\
- |(d_i^m - d_j^m)(u_n) - (d_i^m - d_j^m)(u_{n-1})| \\
\geq \frac{\alpha_0}{2m} (1 + \frac{1}{2n-1-m}) \frac{|\ell^5 - j^5|}{|\ell|^r} - c|\ell^5 - j^5|\varepsilon_n \\
\geq \frac{\alpha_0}{2m} (1 + \frac{1}{2n-1-m}) \frac{|\ell^5 - j^5|}{|\ell|^r} - \frac{1}{2n} \frac{|\ell^5 - j^5|}{|\ell|^r} \tag{326} \\
= \alpha_{nm} \frac{|\ell^5 - j^5|}{|\ell|^r}.
\]

**Theorem 5.5.** The cantor like set $\Pi_\varepsilon \in \Pi$ is asymptotically full Lebesgue measure, \textit{i.e.}
\[ |\Pi \setminus \Pi_\varepsilon| \leq C\alpha_0. \]

**Proof.** We see
\[ \Pi \setminus \Pi_\varepsilon = (\bigcup_{n,m} \Gamma_n) \bigcup (\bigcup_{m,n} \Theta_{mn}), \]
where $m \leq n$. Obviously, we have $|\bigcup_{n} \Gamma_n| \leq C\alpha_0$. Consider the set $\bigcup \Theta_{mn}$ in a different view. Set
\[ \Lambda_m = \bigcup_{n \geq m} \Theta_{mn}, \]
where $\Lambda_m$ is set removed from the $m$-step reduction for all $\varepsilon(u_n), n \geq m$. By Lemma \ref{5.1}, $R_{ij\ell}^m(u_n) \neq \emptyset$ are confined in the ball $|i^4 + j^4| \leq 16|\varpi||\ell|$. Then, we have
\[
|\Theta_{mm}| = \left| \bigcup_{i,j,\ell} R_{ij\ell}^m(u_m) \right| \leq \sum_{\ell \in \mathbb{Z}^v} \sum_{i^4 + j^4 \leq 16|\varpi||\ell|} |R_{ij\ell}^m(u_m)| \\
\leq C \sum_{\ell \in \mathbb{Z}^v} \frac{\alpha_{mm}}{|\ell|^r} \leq C\alpha_{mm} = C \frac{\alpha_0}{2m-1}. \] (327)
\[ |\Theta_{m,n} \backslash \Theta_{m,n-1}| \leq C \sum_{|\ell| \geq \varepsilon_n} \frac{\alpha_{mn}}{|\ell|^2} \]
\[ \leq C \sum_{|\ell| \geq \varepsilon_n} \frac{\alpha_{mn}}{|\ell|^2} \]
\[ \leq C \frac{\alpha_0}{2m-1} \varepsilon_n^{0.5} (\varepsilon^{-0.5} - v) \]
\[ \leq C \frac{\alpha_0}{2m-1} \varepsilon_n^{0.5} . \]

Then, we can get \( |\Lambda_m| \leq C \frac{\alpha_0}{2m-1} . \) The lemma is proved. \( \Box \)

6. Technical lemmas. Suppose the function in this chapter real analytic on \( \mathbb{T}_n^\nu \), and \( s_0 \) an integer greater than \( \frac{n}{2} \).

**Definition 6.1.** Let \( p \) be an integer, the max norm of \( D^p u \) on \( \mathbb{T}_n^\nu \) is
\[ |D^p u|_\nu = \sum_{\alpha \in \mathbb{Z}^n, |\alpha| = p} |D^\alpha u|_\nu . \]

**Lemma 6.2 ([7]).** For \( \sigma > 0 \) and \( v > 0 \), the following inequalities holds:
\[ \sum_{k \in \mathbb{Z}^n} e^{-2|k|\sigma} \leq \frac{1}{\sigma^n} (1 + e)^n \] (329)
\[ \sum_{k \in \mathbb{Z}^n} e^{-2|k|\sigma} |k|^v \leq \left( \frac{\nu}{e} \right)^v \frac{1}{\sigma^{v+n}} (1 + e)^n \] (330)

**Proof.** The proof can be found in [7, p22]. \( \Box \)

**Lemma 6.3** (Appendix A. [17]). If \( s \geq 0 \) and \( p > \frac{n}{2} \), then \( \|uv(x)\|_{s,p} \leq c\|u(x)\|_{s,p} \|v(x)\|_{s,p} \) with a finite constant \( c \) depending on \( p \) and \( n \).

**Proof.** For \( n = 1 \), the detail of the proof can be found in [17, 24]. \( n > 1 \) is a simple variation. \( \Box \)

**Lemma 6.4.** For \( u \) be analytic on \( \mathbb{T}_n^\nu \), we have the following inequalities,
\[ \|u\|_{s-\sigma,p+v} \leq \frac{c(\nu)}{\sigma^\nu} \|u\|_{s,p} , \] (331)
and
\[ |u|_{s,p-s_0} \ll \|u\|_{s,p} \ll |u|_{s,p+s_0} . \] (332)

**Proof.** To prove \( (331) \), we see
\[ \|u\|_{s-\sigma,p+v}^2 = \sum_{k \in \mathbb{Z}^n} |u_k|^2 e^{2|k|(s-\sigma)} |k|^{2(p+v)} = \sum_{k \in \mathbb{Z}^n} |u_k|^2 e^{2|k|s} |k|^{2p} (e^{-|k|\sigma} |k|^{2v}). \]

Since \( e^{-|k|\sigma} |k|^v \leq e^{-\nu(e^{-|k|\sigma})} \), we get
\[ \|u\|_{s-\sigma,p+v} \leq \frac{c(\nu)}{\sigma^\nu} \|u\|_{s,p} \]
\[ c(\nu) = e^{-\nu e^\nu}. \]

Considering \( (332) \), since \( u \) is analytic on \( \mathbb{T}_n^\nu \), the Fourier coefficients \( u_k \) satisfy
\[ |u_k| \leq |u|_s e^{-|k|s} . \]
\(D^{\alpha}u\) is also an analytic function on \(\mathbb{T}^n\), the Fourier coefficients \((D^{\alpha}u)_k = u_k(i\mathbf{k})^\alpha\), \(k^\alpha = k_1^{\alpha_1} \cdots k_n^{\alpha_n}\), satisfy
\[
|(D^{\alpha}u)_k| = |u_k||i\mathbf{k})^\alpha| \leq |D^{\alpha}u|_s e^{-|\mathbf{k}|s}.
\]

If \(|\alpha| = p + s_0\), by Definition (6.1), we have
\[
|u_k||\mathbf{k})^{\alpha}| \leq |D^{p+s_0}u|_s e^{-|\mathbf{k}|s}.
\]

Now, we have
\[
||u||^2_{s,p} = \sum_{k \in \mathbb{Z}^n} |u_k|^2 e^{|k|s}|k|^{2p}
= |u_0|^2 + \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |u_k|^2 e^{|k|s}|k|^{2p}
\leq |u_0|^2 + \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |D^{p+s_0}u|_s^2 |k|^{-2s_0}
= |u_0|^2 + |D^{p+s_0}u|_s^2 \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{-2s_0}
\leq c|u|^2_{s,p,s_0}.
\]

To prove the left part of (332), we see
\[
|u|_{s,p,s_0} \leq c_1 \left\{ \sum_{k \in \mathbb{Z}^n} |u_k|e^{|k|s}|k|^{p-s_0} \right\}
\leq c_1 \left( \sum_{k \in \mathbb{Z}^n} |u_k|^2 e^{|2k|s}|k|^{2p} \right) \frac{1}{2} \left( \sum_{k \in \mathbb{Z}^n} |k|^{-2s_0} \right) \frac{1}{2}.
\]

\[
\text{Lemma 6.5 (21). If } f \text{ is analytic from the segment joining } z_1 \text{ and } z_0 \text{ defined on } \mathbb{C}^n \text{ to } \mathbb{C}^n. \text{ Then, there are point } w_1, w_2, \cdots w_{2n} \text{ on the segment such that}
\]
\[
f(z_1) - f(z_0) = (z_1 - z_0)(\lambda_1 Df(w_1) + \lambda_2 Df(w_2) + \cdots + \lambda_{2n} Df(w_{2n})),
\]
where \(\lambda_i \geq 0\) and \(\sum_{i=1}^{2n} \lambda_i = 1\).

\text{Proof. The detail of the proof can be found in [21].}

\text{Lemma 6.6. (Change of variable) Let } f \text{ be a real analytic function on } \mathbb{T}_s^n, \text{ with } |f|_{s,p} \leq \frac{1}{100}. \text{ Then, there are a constant } C \text{ depending on } n \text{ and } p, \text{ such that}

\text{(i). If } u \text{ is a real analytic function on } \mathbb{T}_s^n, \text{ } u(x + f(x)) \text{ is also a real analytic function on } \mathbb{T}_{\frac{n}{100}}^n \text{ and satisfies}
\]
\[
|u(x + f(x))|_{\frac{n}{100},p} \leq C|u(x)|_{s,p}.
\]

\text{(ii). Considering another analytic function } g \text{ on } \mathbb{T}_s^n \text{ with } |g|_{s,p} \leq \frac{1}{100}, \text{ we have}
\]
\[
|u(x + f(x)) - u(x + g(x))|_{\frac{n}{100},p} \leq C|u(x)|_{s,p+1} |f(x) - g(x)|_{s,p}.
\]

\text{(iii). Suppose } u = u_\lambda, f = f_\lambda \text{ depend in a Lipschitz way on the parameter } \lambda \in \Pi \subset \mathbb{R}, \text{ and } |f_\lambda|_{s,p} \leq \frac{1}{100}, \text{ for all } \lambda. \text{ Then, we have}
\]
\[
|u(x + f(x))|_{\frac{n}{100},p}^{Lip} \leq C|u(x)|_{s,p+1}^{Lip} (1 + |f(x)|_{s,p}^{Lip}.
\]
Finally, (339), (341) and (343) imply
\[ |f'(w_i)|_s \leq |f|_{s,p} \leq \frac{1}{100}, \]
that is equivalent to \( v(T_{100}) \subseteq \mathbb{T}_s \). Now, we would compare \( u(x) \) with \( u(x + f(x)) \).
Clearly, we can see
\[ |u \circ v(x)|_{100} \leq |u(x)|_s. \]
Differentiating \( u \circ v(x) \), one gets
\[ D(u \circ v)(x) = (Du)(v(x))[1 + Df(x)]. \]
By (338), we have
\[ |D(u \circ v)(x)|_{100} \leq |Du(x)|_s + |Du(x)|_s |Df(x)|_{100}. \]
(i) is proved for \( p = 0, 1 \). Considering the general situation, by the “chain rule”, the \( m \)th Fréchet derivative of the composition of functions \( u \circ v(x) \) is
\[ D^m(u \circ v)(x) = \sum_{k=1}^{m} \sum_{j_1 + \cdots + j_k = m} C_{k,j_1} \cdots C_{k,j_k} (D^k u)(v(x))[D^{j_1} v(x) \cdots D^{j_k} v(x)], \]
where \( j_1, \ldots, j_k \geq 1 \), and \( C_{k,j_1} \cdots C_{k,j_k} \) are constants depending on \( k, j_1, \ldots, j_k \).
For \( j_1 = 1 \), \( |Dv(x)|_s = |1 + Df(x)|_s < 2 \).
For \( 2 \leq j_i \leq m \), \( |D^{j_i} v(x)|_s = |f|_{s,m} \leq \frac{1}{100} \).
Collecting all the term in the sum, we have
\[ |D^m(u \circ v)(x)|_{100} \leq C \sum_{k=1}^{m} |D^k u(y)|_s. \]
Finally, (339), (341) and (343) imply
\[ |u(x + f(x))|_{100, p} \leq C|u|_{s,p}. \]
(ii) Set \( x + f(x) \) as \( v_1, x + g(x) \) as \( v_2, \) we see
\[ |(u \circ v_1 - u \circ v_2)(x)|_{100} \leq |Du(x)|_s|v_1(x) - v_2(x)|_{100} \]
\[ = |Du(x)|_s|f(x) - g(x)|_{100}. \]
Differentiating the left of inequality (345), we have
\[ D(u \circ v_1 - u \circ v_2)(x) \]
\[ = (Du) \circ v_1 + (Du) \circ v_1 \cdot Df(x) - (Du) \circ v_2 - (Du) \circ v_2 \cdot Dg(x) \]
\[ = (Du) \circ v_1 - (Du) \circ v_2 + ((Du) \circ v_1 - (Du) \circ v_2)Df(x) \]
\[ + (Du) \circ v_2(Df(x) - Dg(x)). \]
Then, we can get
\[ |D(u \circ v_1 - u \circ v_2)(x)|_{\text{101}} \]
\[ = |(Du) \circ v_1 - (Du) \circ v_2|_{\text{101}} + |(Du) \circ v_1 - (Du) \circ v_2|_{\text{101}} \left| Df(x) \right|_{\text{101}} \]
\[ + |(Du) \circ v_2|_{\text{101}} \left| Df(x) - Dg(x) \right|_{\text{101}} + |(Du) \circ v_2|_{\text{101}} \left| Df(x) \right|_{\text{101}} \]
\[ \leq |D^2 u(x)|_s |f(x) - g(x)|_{\text{101}} + |D^2 u(x)|_s |f(x) - g(x)|_{\text{101}} \left| Df(x) \right|_{\text{101}} \]
\[ + |D u(x)|_s |D(f(x) - g(x))|_{\text{101}} \]  \hspace{1cm} (347)

(ii) is proved for \( p = 0, 1 \). For the general form \( D^m(u \circ v_1 - u \circ v_2)(x) \), we have
\[ D^m(u \circ v_1 - u \circ v_2)(x) \]
\[ = \sum_{k=1}^m \sum_{j_1 + \cdots + j_k = m} C_{k,j} \left\{ (D^k u) \circ v_1[D^{j_1} v_1(x) \cdots D^{j_k} v_1(x)] \right. \]
\[ - (D^k u) \circ v_2[D^{j_1} v_2(x) \cdots D^{j_k} v_2(x)] \}
\[ = \sum_{k=1}^m \sum_{j_1 + \cdots + j_k = m} C_{k,j} \left\{ (D^k u) \circ (v_1 - v_2)[D^{j_1} v_1(x) \cdots D^{j_k} v_1(x)] \right. \]
\[ + (D^k u) \circ v_2[D^{j_1} (f - g)(x) \cdot D^{j_2} v_1(x) \cdots D^{j_k} v_1(x)] + \cdots \]
\[ \left. + (D^k u) \circ v_2[D^{j_1} (v_2(x) \cdots D^{j_k-1} v_2(x) \cdot D^{j_k} (f - g)(x)) \right\} \]  \hspace{1cm} (348)

From identity (348), we see
\[ |D^m(u \circ v_1 - u \circ v_2)(x)|_{\text{101}} \]
\[ \leq C |u|_{s,m+1} |f - g|_{\text{101},m} (1 + |f|_{\text{101},m} + |g|_{\text{101},m}) \]
\[ \leq C |u|_{s,m+1} |f - g|_{\text{101},m}, \]

because \( |f|_{s,m}, |g|_{s,m} \leq \frac{1}{100} \). (Some repeated details have been omitted, that are almost the same as (346) and (347).)

(iii) Let \( \lambda_1, \lambda_2 \in \Pi, u_1 = u_{\lambda_1}, u_2 = u_{\lambda_2}, x + f_{\lambda_1}(x) = v_1(x), x + f_{\lambda_2}(x) = v_2(x). \) Using (335) and (336), one gets
\[ |u_2 \circ v_2 - u_1 \circ v_1|_{\text{101},p} \]
\[ \leq |u_2 \circ v_2 - u_2 \circ v_1|_{\text{101},p} + |u_2 \circ v_1 - u_1 \circ v_1|_{\text{101},p} \]
\[ \leq C(|u_2|_{s,p+1} |v_2 - v_1|_{\text{101},p} + |u_2 - u_1|_{s,p} (1 + |v_1|_{\text{101},p})). \]  \hspace{1cm} (349)

Finally, (337) follows from (335) and (349). \( \square \)

**Lemma 6.7.** Let \( p > 0, \eta > 0, 0 < k < 1, u(\lambda), h(\lambda) \) be a Lipschitz family of function with \( \|h\|_{s,p+\eta}^{Lip} \leq 1 \). \( F \) be a \( C^1 \)-map satisfying
\[ \|F(u)|_{\text{ks,p}}^{Lip} \leq \|u|_{\text{s,p}}^{Lip}. \]  \hspace{1cm} (350)
\[ ||\partial_u F(u)[h]|_{\text{ks,p}}^{Lip} \leq ||h|_{\text{s,p+\eta}}^{Lip} (1 + \|u|_{\text{s,p+\eta}}^{Lip}). \]  \hspace{1cm} (351)

Then,
\[ \|F(u + h) - F(u)|_{\text{ks,p}}^{Lip} \leq ||h|_{\text{s,p+\eta}}^{Lip} (1 + \|u|_{\text{s,p+\eta}}^{Lip}). \]  \hspace{1cm} (352)
Proof. Since $F(u)$ be a $C^1$-map, we see
\[
F(u + h) - F(u) = \int_0^1 \partial_{(u+th)} F(u + th)[h] dt.
\]
Then, we have
\[
\|F(u + h) - F(u)\|_{\text{Lip}} \leq \int_0^1 \|\partial_{(u+th)} F(u + th)[h]\|_{\text{Lip}} dt
\]
\[
\leq \|h\|_{\text{Lip}} \|\partial_{(u+th)} F(u + th)[h]\|_{\text{Lip}} + \int_0^1 \|u + th\|_{\text{Lip}} dt
\]
\[
\leq \|h\|_{\text{Lip}} (1 + \|u\|_{\text{Lip}}) + \|h\|_{\text{Lip}} (1 + \|u\|_{\text{Lip}})
\]
because $\|h\|_{\text{Lip}} \leq 1$. 

Lemma 6.8. (The implicit function) Let $p$ be analytic on $\mathbb{T}_s^n$, with $|p|_{s,m} \leq \frac{1}{100}$. 
Set $f(x) = x + p(x)$. Then:

(i) $f$ is invertible, its inverse is $f^{-1}(y) = g(y) = y + q(y)$, where $q$ be real analytic on $\mathbb{T}_s^n$, and satisfying
\[
|q|_{s,m} \leq C|p|_{s,m},
\]
where the constant $C$ depends on $n$ and $m$.

(ii) Moreover, suppose that $p = p(\lambda)$ depends in a Lipschitz way by a parameter $\lambda \in \Pi \subset \mathbb{R}$, and $|p(\lambda)|_{s,m} \leq \frac{1}{100}$, for all $\lambda$. Then $q = q(\lambda)$ is also Lipschitz in $\lambda$, and
\[
|q|_{s,m+1} \leq C|p|_{s,m+1}.
\]
The constant $C$ depends on $n$ and $m$.

Proof. For symbolic simplicity, the following calculations only consider $n = 1$ in form. With regard to general situation, there is no difference.

(i) If we restrict $x$ to $\mathbb{R}$, by [1, Lemma B.4], $f(x)$ is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Considering $f(x)$ defined on $\mathbb{T}_s$, by Lemma 6.5, $f(x)$ is a one to one mapping from $\mathbb{T}_s$ to $f(\mathbb{T}_s)$.

Now, we would prove $\mathbb{T}_s^{99} \subseteq f(\mathbb{T}_s)$.

(1): Set $x = a + ib$, by (338), we can see
\[
\text{Im}(f(x)) = \text{Im}(f(x) - f(a)) = b + b \text{Re}(\lambda_1(Dp(w_1)) + \lambda_2(Dp(w_2))).
\]
Since $|Dp(w_1)|_s \leq |p|_{s,m} \leq \frac{1}{100}$, we have
\[
|\text{Im}(x + f(x))| \geq \frac{99}{100} s, \quad \text{for all } x \in \mathbb{T}_s.
\]

(2): Let $q(x) = g(y) - y$. Since $p(x)$ is periodic, $p(x + 2\pi m) = p(x)$. Then,
\[
f(x + 2\pi m) = x + 2\pi m + p(x + 2\pi m)
\]
\[
= x + 2\pi m + p(x)
\]
\[
= f(x) + 2\pi m
\]
for all $m \in \mathbb{Z}^n$. Note $g$ is the inverse of $f$, applying $g$ to this equality gives
\[
g \circ f(x + 2\pi m) = g(f(x) + 2\pi m),
\]
where
\[ g \circ f(x + 2\pi m) = x + 2\pi m = g(y) + 2\pi m. \]

On the other hand, \( g(f(x) + 2\pi m) = g(y + 2\pi m) \). This means that \( g(y) \) is periodic.

From (1), (2), \( g(y) \) is well defined on \( \mathbb{T}_{\frac{2\pi}{4\pi}} \). Now, we would compare \( p(x) \) with \( q(y) \).

By Neumann series, the matrix \( D f(x) = I + Dp(x) \) is invertible, where \( (Df(x))^{-1} = \sum_{n=0}^{\infty}(-Dp(x))^n \). Thus,
\[ Dq(f(x)) = \sum_{n=1}^{\infty}(-Dp(x))^n, \text{ for all } x \in \mathbb{T}_s. \] (355)

Since \( |Dp(x)|_s < |p|_{s,m} \leq \frac{1}{100} \), we see
\[ |Dq(f(x))| \leq \frac{100}{99}|Dp(x)| \leq \frac{1}{99}, \text{ for all } x \in \mathbb{T}_s. \] (356)

The identity \( f(g(y)) = y \) gives
\[ q(y) = -p(y + q(y)), \quad y \in \mathbb{T}_{\frac{2\pi}{4\pi}}. \] (357)

From (353) and (356), \( |Dq|_{\frac{2\pi}{4\pi}} \leq \frac{1}{99} \). Similarity with (338) and (338), we have
\[ |q|_{\frac{2\pi}{4\pi}} < |p|_s. \]

Clearly, \( |Dq|_{\frac{2\pi}{4\pi}} \leq C|Dp|_s \). (i) is proved for \( m = 0, 1 \).

Considering the general situation, suppose \( |q|_{\frac{2\pi}{4\pi},h} \leq C(h)|p|_{s,h} \), where \( h < m \).

Apply (342) to \( f \circ g \): since \( f(g(y)) = y, D^n(f \circ g) = 0 \) for all \( n \geq 2 \). Separate \( k = 1 \) from \( k \geq 2 \) in the sum (342) and solve for \( D^m \),
\[ D^m g = D^m q \text{ and } D^k f = D^k p, \text{ because } k \geq 2. \]

Since \( k \geq 2 \), it is \( 1 \leq j_i \leq m - 1 \) for all \( i = 1, \ldots, k \), because there are at least two \( j_1, j_2 \), each of them \( \geq 1 \), and \( \sum j_i = m \).

For \( k = m \), one has \( j_i = 1 \) for all \( i = 1, \ldots, m \), and the corresponding term in the sum is estimated
\[ |(D^m p) \circ g[Dg, \ldots, Dg]|_{\frac{2\pi}{4\pi}} \leq |D^m p|_s |Dg|_{\frac{2\pi}{4\pi}} \leq C|D^m p|_s, \] (359)

because \( |Dg|_{\frac{2\pi}{4\pi}} = |I + Dq|_{\frac{2\pi}{4\pi}} < 2 \).

For \( 2 \leq k \leq m - 1 \), at least one among \( j_1, \ldots, j_k \) is \( 2 \) (otherwise \( k = m \)). Let \( \ell \) be the number of indices \( j_i \) that are \( \geq 2 \), so that \( 1 \leq \ell \leq k \). It remains to estimate
\[ \sum_{k=2}^{m-1} \sum_{\ell=1}^{k} \sum_{\sigma_1+\cdots+\sigma_\ell=m-k+\ell} C_{k\ell\sigma}(D^k p)(g(y))[Dg(y)]^{k-\ell}[D^{\sigma_1} q(y) \cdots D^{\sigma_\ell} q(y)], \] (360)

where indices \( j_i \geq 2 \) have been renamed \( \sigma_1, \ldots, \sigma_\ell \), the number of indices \( j_i = 1 \) is \( k - \ell \), and \( D^{\sigma_1} g = D^{\sigma_1} q \) because \( \sigma_i \geq 2 \). Every factor \( Dg \) is estimated by \( |Dg|_{\frac{2\pi}{4\pi}} < 2 \). For the remaining factors
\[ |(D^k p) \circ g[D^{\sigma_1} q(y) \cdots D^{\sigma_\ell} q(y)]|_{\frac{2\pi}{4\pi}} \leq |D^k p|_s |[D^{\sigma_1} q(y) \cdots D^{\sigma_\ell} q(y)]|_{\frac{2\pi}{4\pi}} \leq C|D^k p|_s, \] (361)

because \( |D^{\sigma_i} q(y)|_{\frac{2\pi}{4\pi}} \leq |q|_{\frac{2\pi}{4\pi},m-1} \leq C|p|_{s,m-1} \leq C \).
Collecting all the terms in the sum, we can proved that
\[ |D^m q|_{\frac{2m}{m+1}} \leq C(m)|p|_{s,m}. \]

Finally, we have
\[ |q|_{\frac{2m}{m+1},m} \leq C(m)|p|_{s,m}. \] (362)

(ii) For the Lipschitz norm, we have
\[ q_\lambda(y) + p_\lambda(g_\lambda(y)) = 0, \quad \forall \lambda \in \Pi, y \in T_{\frac{2m}{m+1}}. \]

Let \( \lambda_1, \lambda_2 \in \Pi \), \( q_1 = q_{\lambda_1}, q_2 = q_{\lambda_2} \), and so on, then
\[ q_1 - q_2 = (p_1 \circ g_2 - p_1 \circ g_1) + (p_1 \circ g_2 - p_1 \circ g_1). \] (363)

Since \( |Dp_\lambda| < \frac{1}{100} \), for all \( \lambda \), \( g_\lambda(y) \) are well defined on \( T_{\frac{2m}{m+1}} \). From (338) and (345), we have
\[ |q_1 - q_2|_{\frac{2m}{m+1},m} \leq C|p_2 - p_1|_s + |Dp_1|_s|q_2 - q_1|_{\frac{2m}{m+1}}, \]
and \( (1 - |Dp_1|_s)|q_1 - q_2|_{\frac{2m}{m+1},m} \leq C|p_2 - p_1|_s \). Thus, we can get
\[ |q_1 - q_2|_{\frac{2m}{m+1},m} \leq C|p_2 - p_1|_s. \] (364)

Differentiating (363), by (345) and (347), we have
\[ |Dq_1 - Dq_2|_{\frac{2m}{m+1},m} \leq C|p_2 - p_1|_{s,1} + |D^2p_1|_s|q_2 - q_1|_{\frac{2m}{m+1}}|Dg_2|_{\frac{2m}{m+1},m} + |Dp_1|_s|Dq_1 - Dq_2|_{\frac{2m}{m+1},m}. \]
and \( (1 - |Dp_1|_s)|Dq_1 - Dq_2|_{\frac{2m}{m+1},m} \leq C(|p_2 - p_1|_{s,1} + |p_1|_{s,2}|p_2 - p_1|_s|Dg_2|_{\frac{2m}{m+1},m}). \) Thus, we can get
\[ |Dq_1 - Dq_2|_{\frac{2m}{m+1},m} \leq C|p_2 - p_1|_{s,1}(1 + |p_1|_{s,2}). \] (365)

(ii) is proved for \( m = 0, 1 \). Considering general situation, suppose \( |q_1 - q_2|_{\frac{2m}{m+1},m} \leq C(h)|p_2 - p_1|_{s,h}(1 + |p_1|_{s,h+1}), \) for all \( h < m \).

The estimates of \( D^m(q_1 - q_2) \) can be divided into the following two parts.

(A): Considering \( D^m(p_1 \circ g_2 - p_1 \circ g_1) \), we have
\[ D^m(p_1 \circ g_2 - p_1 \circ g_1) = \sum_{k=1}^{m} \sum_{j_1 + \cdots + j_k = m} C_{k,j} \left( (D^k p_1) \circ g_2[D^{j_1} g_2(y) \cdots D^{j_k} g_2(y)] 
- (D^k p_1) \circ g_1[D^{j_1} g_1(x) \cdots D^{j_k} g_1(y)] \right). \] (366)

For \( k = 1 \) one has \( j_1 = m \), and the corresponding term in the sum is estimated
\[ |(D^k p_1) \circ g_2 \cdot D^m q_2 - (D^k p_1) \circ g_1 \cdot D^m q_1|_{\frac{2m}{m+1}} \]
\[ \leq |(D^k p_1) \circ g_2 \cdot D^m q_2|_{\frac{2m}{m+1}} + |(D^k p_1) \circ g_2 \cdot D^m q_1|_{\frac{2m}{m+1}} \]
\[ + |D^k p_1|_s|D^m (q_2 - q_1)|_{\frac{2m}{m+1}} \]
\[ + |D^k p_1|_s|D^m q_2|_{\frac{2m}{m+1}} |D^m q_1|_{\frac{2m}{m+1}}. \] (367)
For $k \geq 2$, one has $j_i < m$. It remains to estimate
\[
\sum_{k=2}^{m} \sum_{j_1+\cdots+j_k=m} C_{kj}
\left\{
(D^k p_1) \circ g_2 - (D^k p_1) \circ g_1, [D^{j_1} g_2(y) \cdots D^{j_k} g_2(y)]
\right. \\
+ (D^k p_1) \circ g_1, [D^{j_1} (g_2 - g_1)(y) \cdot D^{j_2} g_2(y) \cdots D^{j_k} g_2(y)] + \cdots \\
\left. + (D^k p_1) \circ g_1, [D^{j_1} (g_1)(y) \cdots D^{j_{k-1}} g_1(y) \cdot D^{j_k} (g_2 - g_1)(y)].\right\}
\] (368)

Every factor $|D^q|_{\text{var}} < 2$, and $|(D^k p_1) \circ g_2 - (D^k p_1) \circ g_1|_{\text{var}} \leq |D^{k+1} p_1|_{s} \cdot |q_2(y) - q_1(y)|_{s}$. For the remaining factors,
\[
|\left((D^k p_1) \circ g_1\right) [D^i g_2 \cdots D^j (g_2 - g_1) \cdots D^{j_k} g_2]|_{\text{var}} \leq C|D^k p_1|_{s} |D^i (g_2 - q_1)(x)|_{\text{var}} \\
\leq C|D^k p_1|_{s} |q_2 - q_1|_{\text{var}}^{m-1}.
\] (369)

\textbf{(B):} Considering $D^m (p_2 \circ g_2 - p_1 \circ g_2)$, by (343), we have
\[
|D^m (p_2 \circ g_2 - p_1 \circ g_2)|_{\text{var}} \leq C|p_2 - p_1|_{s,m}.
\] (370)

Collecting all the terms above in the sum, we can see that
\[
|D^m q_1 - D^m q_2|_{\text{var}}^{m,m} \leq C|p_2 - p_1|_{s,m} + |p_1|_{s,m+1} |q_2 - q_1|_{\text{var}}^{m,m} \\
+ |Dp_1|_{s} |D^m q_1 - D^m q_2|_{\text{var}}^{m,m}.
\] (371)

Since $|Dp_1|_{s} \leq |p_1|_{s,m} \leq \frac{1}{100}$, we can see
\[
|D^m q_1 - D^m q_2|_{\text{var}}^{m,m} \leq C|p_2 - p_1|_{s,m} + |p_1|_{s,m+1} |p_2 - p_1|_{s,m-1},
\] (372)

and
\[
|q_1 - q_2|_{\text{var}}^{m,m} \leq C|p_2 - p_1|_{s,m} (1 + |p_1|_{s,m+1}).
\] (373)

Finally, the bounds (362) and (373) imply $|q|_{\text{var}}^{m,m} \leq C(m) |p|_{s,m+1}^{\text{Lip}}$. \hfill \Box

\textbf{Acknowledgment.} The first author would like thanks to W. Jian, Y. Shi for comments on earlier version of this manuscript. Special thanks is to H. Cong for many useful discussion.

\begin{thebibliography}{99}
\item 1. P. Baldi, Periodic solutions of fully nonlinear autonomous equations of Benjamri-Ono type, \textit{Annales De L'Institut Henri Poincare Non Linear Analysis}, 30 (2013), 33–77.
\item 2. P. Baldi, M. Berti and R. Montalto, KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation, \textit{Mathematissche Annalen}, 359 (2014), 471–536.
\item 3. P. Baldi, M. Berti and R. Montalto, KAM for autonomous quasi-linear perturbations of KdV, \textit{Annales De L'Institut Henri Poincare Non Linear Analysis}, 33 (2016), 1589–1638.
\item 4. P. Baldi, M. Berti and R. Montalto, KAM for autonomous quasi-linear perturbations of mKdV, \textit{Bollettino dell’Unione Matematica Italiana}, 9 (2016), 143–188.
\item 5. D. Bambusi and S. Graffi, Time quasi-periodic unbounded perturbations of schrödinger operators and KAM methods, \textit{Communications in Mathematical Physics}, 219 (2001), 465–480.
\item 6. M. Berti and R. Montalto, Quasi-periodic standing wave solutions of gravity-capillary water waves, preprint, \texttt{arXiv:1602.02411}.
\item 7. N. N. Bogoljubov, J. A. Mitropoliskii and A. M. Samoilenko, \textit{Methods of Accelerated Convergence in Nonlinear Mechanics}, Springer-verlag, New York, 1976.
\item 8. J. Bourgain, \textit{Green’s Function Estimates for Lattice Schrödinger Operators and Applications}, Annals of Mathematics Studies 158, Princeton University Press, 2005.
\item 9. H. Cong, L. Mi and X. Yuan, Positive quasi-periodic solutions to Lotka-Volterra system, \textit{Science China Mathematics}, 53 (2010), 1151–1160.
\end{thebibliography}
[10] R. Feola and M. Procesi, Quasi-periodic solutions for fully nonlinear forced reversible Schrödinger equations, *Journal of Differential Equations*, 259 (2015), 3389–3447.

[11] R. Feola, KAM for quasi-linear forced Hamiltonian NLS, preprint, arXiv:1602.01341.

[12] T. Kappeler and J. Pöschel, *KdV and KAM*, Springer-verlag, New York, 2003.

[13] S. Kuksin, *Nearly Integrable Infinite-Dimensional Hamiltonian Systems*, Springer-verlag, New York, 1993.

[14] S. Kuksin, On small-denominators equations with large variable coefficients, *Zeitschrift für angewandte Mathematik und Physik*, 48 (1997), 262–271.

[15] S. Kuksin, A KAM theorem for equations of the Korteweg-De Vries Type, *Reviews in Mathematical Physics*, 10 (1998), 1–64.

[16] S. Kuksin, *Analysis of Hamiltonian PDEs*, Oxford University Press, 2000.

[17] S. Kuksin and J. Pöschel, Invariant Cantor Manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation, *Annals of Mathematics*, 143 (1996), 149–179.

[18] P. D. Lax, Periodic solutions of the KdV equation, *Communications on Pure & Applied Mathematics*, 28 (1975), 141–188.

[19] J. Liu and X. Yuan, Spectrum for quantum duffing oscillator and small-divisor equation with large-variable coefficient, *Communications on Pure & Applied Mathematics*, 63 (2010), 1145–1172.

[20] J. Liu and X. Yuan, A KAM theorem for Hamiltonian partial differential equations with unbounded perturbations, *Communications in Mathematical Physics*, 307 (2011), 629–673.

[21] R. McLeod, Mean value theorems for vector valued functions, *Proceedings of the Edinburgh Mathematical Society*, 14 (1965), 197–209.

[22] R. Montalto, Quasi-periodic solutions of forced Kirchhoff equation, *Nonlinear Differential Equations and Applications*, 24 (2017), Art. 9, 71 pp.

[23] J. Pöschel, A KAM-theorem for some nonlinear partial differential equations, *Annali Della Scuola Normale Superiore di Pisa - Classe di Scienze*, 23 (1996), 119–148.

[24] J. Pöschel, Quasi-periodic solutions for a nonlinear wave equation, *Commentarii Mathematici Helvetici*, 71 (1996), 269–296.

[25] E. Wayne, Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory, *Communications in Mathematical Physics*, 127 (1990), 479–528.

[26] A. M. Wazwaz, Solitons and periodic solutions for the fifth-order KdV equation, *Applied Mathematics Letters*, 19 (2006), 1162–1167.

[27] A. M. Wazwaz, The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations, *Applied Mathematics & Computation*, 184 (2007), 1002–1014.

[28] X. Yuan and K. Zhang, A reduction theorem for time dependent Schrödinger operator with finite differentiable unbounded perturbation, *Journal of Mathematical Physics*, 54 (2013), 052701, 23 pp.

[29] E. Zehnder, Generalized implicit function theorems with applications to some small divisor problems, I, *Communications on Pure & Applied Mathematics*, 28 (1975), 91–140.

[30] E. Zehnder, Generalized implicit function theorems with applications to some small divisor problems, II, *Communications on Pure & Applied Mathematics*, 29 (1976), 49–111.

[31] J. Zhang, M. Gao and X. Yuan, KAM tori for reversible partial differential equations, *Nonlinearity*, 24 (2011), 1189–1228.

Received September 2017; revised February 2018.

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