Equational theories of profinite structures

Michał Skrzypczak

University of Warsaw

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http://www.mimuw.edu.pl/~mskrzypczak/docs/
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Equational theories
What properties of languages can be expressed by (some) equations?
Definition

A framework is a pair \( \langle \Phi, W \rangle \) such that:

- \( \Phi \) is a countable set of recognisers \( \varphi \in \Phi \),
- \( W \) is a countable set of objects \( w \in W \),
- a recogniser \( \varphi \in \Phi \) is a function \( \varphi : W \to K_\varphi \) to a finite set \( K_\varphi \).
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Running example

Let \( \mathcal{W} = A^* \) be a set of all finite words and let \( \Phi \) be the set of all homomorphisms into finite monoids: for every finite monoid \( M \) and any homomorphism \( \varphi : A^* \to M \) let \( \varphi \in \Phi \).
A set $L \subseteq \mathbb{W}$ is \textit{recognisable} if there exists a recogniser $\varphi \in \Phi$ and a set $V \subseteq K_\varphi$ such that

$$L = \varphi^{-1}(V).$$
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A set $L \subseteq W$ is *recognisable* if there exists a recogniser $\varphi \in \Phi$ and a set $V \subseteq K_\varphi$ such that

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**Assumptions**

Additionally we assume:

a) Each object $w \in W$ is totally described by some recogniser (that is $\{w\}$ is recognisable).

b) Recognisable sets are closed under intersections.
Examples

- Let $\mathcal{W}$ be the set of all finite models of a fixed relational signature $\Sigma$.
- Let $\Phi$ be the set of all first order formulas over $\Sigma$.
- A formula $\varphi$ is a function $\varphi : \mathcal{W} \to \{\bot, \top\}$. 

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- Let $\mathcal{W}$ be the set of all finite labelled trees over a finite alphabet $A$.
- Let $\Phi$ be the set of all morphisms into finite tree algebras.

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Definition

Let

\[ X = \prod_{\varphi \in \Phi} K_{\varphi}. \]

\( X \) is a compact topological space. Let

\[ w \in W \mapsto \mu(w) = (\varphi_1(w), \varphi_2(w), \varphi_3(w), \ldots) \]

Since \( \mu \) is 1-1 we can identify \( w \) with \( \mu(w) \) and write \( W \subseteq X \).

Let

\[ \hat{W} = cl(W) \subseteq X. \]
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A virtual object $w' \in \hat{\mathbb{W}} \setminus \mathbb{W}$ is just a list of its properties $(v_1, v_2, \ldots)$ that are finitely realisable by real objects.
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- There is an *virtual* graph $w_\infty \in \widehat{\mathbb{W}}$ such that $w_n \rightarrow w_\infty$. 

Observe that $w_\infty$ is not so virtual — it can be seen as infinite empty graph. This is not a coincidence — Compactness Theorem.
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Fact

A set \( L \subseteq \hat{W} \) is recognisable iff it is closed and open.
Definition

For $u, v \in \hat{W}$ we say that a recognisable language $L \subseteq \hat{W}$ satisfies equation $u \rightarrow v$ iff.

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Lemma

If $I \subseteq \mathcal{L}$ and $K = \bigcup I$ is recognisable then $K \in \mathcal{L}$.
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**Sketch of the proof ($\iff$)**

Take any lattice $\mathcal{L}$ and let $\mathcal{E}$ contain all equations satisfied by $\mathcal{L}$. Take any language $L$ satisfying all $\mathcal{E}$ and show that $L \in \mathcal{L}$. Use above Lemma to approximate $L$ from inside and from outside. If it fails, then there is an equation $u \rightarrow v$ not satisfied by $L$ — a contradiction.
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Mai Gehrke, Serge Grigorieff, and Jean-Éric Pin. A topological approach to recognition. In Automata, Languages and Programming, volume 6199 of LNCS, pages 151–162. Springer Berlin / Heidelberg, 2010.