GROUP-LIKE ELEMENTS IN QUANTUM GROUPS
AND FEIGIN’S CONJECTURE

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Abstract

Let $A$ be an arbitrary symmetrizable Cartan matrix of rank $r$, and $n = n_+$ be the standard maximal nilpotent subalgebra in the Kac-Moody algebra associated with $A$ (thus, $n$ is generated by $E_1, \ldots, E_r$ subject to the Serre relations). Let $U_q(n)$ be the completion (with respect to the natural grading) of the quantized enveloping algebra of $n$. For a sequence $i = (i_1, \ldots, i_m)$ with $1 \leq i_k \leq r$, let $P_i$ be a skew polynomial algebra generated by $t_1, \ldots, t_m$ subject to the relations $t_l t_k = q^{C_{i_k,i_l}} t_k t_l$ $(1 \leq k < l \leq m)$ where $C = (C_{ij}) = (d_i a_{ij})$ is the symmetric matrix corresponding to $A$. We construct a group-like element $e_i \in P_i \otimes U_q(n)$. This element gives rise to the evaluation homomorphism $\psi_i : C_q[N] \to P_i$ given by $\psi_i(x) = x(e_i)$, where $C_q[N] = U_q(n)^0$ is the restricted dual of $U_q(n)$. Under a well-known isomorphism of algebras $C_q[N]$ and $U_q(n)$, the map $\psi_i$ identifies with Feigin’s homomorphism $\Phi(i) : U_q(n) \to P_i$. We prove that the image of $\psi_i$ generates the skew-field of fractions $\mathcal{F}(P_i)$ if and only if $i$ is a reduced expression of some element $w$ in the Weyl group $W$; furthermore, in the latter case, $\text{Ker} \ \psi_i$ depends only on $w$ (so we denote $I_w := \text{Ker} \ \psi_i$). This result generalizes the results in [5], [6] to the case of Kac-Moody algebras. We also construct an element $R_w \in (C_q[N]/I_w) \otimes U_q(n)$ which specializes to $e_i$ under the embedding $C_q[N]/I_w \hookrightarrow P_i$. The elements $R_w$ are closely related to the quazi-$R$-matrix studied by G. Lusztig in [8]. If $i, i'$ are reduced expressions of the same element $w \in W$, we have a natural isomorphism $R_{i'}^i : \mathcal{F}(P_i) \to \mathcal{F}(P_{i'})$ such that $(R_{i'}^i \otimes \text{id})(e_i) = e_{i'}$. This leads to identities between quantum exponentials. The maps $R_{i'}^i$ are $q$-deformations of Lusztig’s transition maps [8]. The existence of the maps $R_{i'}^i$ leads to a surprising combinatorial corollary about skew-symmetric matrices associated with reduced expressions (cf. [12]).
0. Introduction and main results

It is well-known that a quantum group is not a group. One of the goals of this chapter is to introduce group-like elements for quantum deformations of certain nilpotent algebraic groups. In this section, we sketch our main results; more details will be given in subsequent sections.

Consider a maximal unipotent subgroup $N$ in a complex simple algebraic group $G$. The group-like elements will be obtained as quantum deformation of certain morphisms $\pi_i : \mathbb{C}^m \to N$ defined as follows.

Let $E_1, E_2, \ldots, E_r$ be standard generators of $n$, the Lie algebra of $N$. For any sequence $i = (i_1, \ldots, i_m)$ of indices (possibly with repetitions), one defines a map $\mathbb{C}^m \to N$ by the formula

$$\pi_i(t_1, \ldots, t_m) = \exp(t_1 E_{i_1}) \exp(t_2 E_{i_2}) \cdots \exp(t_m E_{i_m})$$

where $\exp : n \to N$ is the exponential map. Note that $\pi_i$ is a regular (algebraic) map.

It is well-known that $\pi_i$ is a birational isomorphism $\mathbb{C}^m \cong N$ if $i = (i_1, \ldots, i_m)$ is a reduced expression of $w_0$, the longest element in the Weyl group $W$ of $G$. Furthermore, if $i$ is a reduced expression of $w \in W$, then the closure in $N$ of the image of $\pi_i$ depends only on $w$.

To construct a $q$-deformation of $\pi_i$ we interpret the evaluation homomorphism $\pi_i^* : \mathbb{C}[N] \to \mathbb{C}[t_1, \ldots, t_m]$ as follows. First, we think of the product in (0.1) as an element

$$\tilde{\pi}_i \in \mathbb{C}[t_1, \ldots, t_m] \otimes \hat{U}(n),$$

where $\hat{U}(n)$ is the completion of the universal enveloping algebra of $n$ with respect to the natural grading.

Second, $\mathbb{C}[N]$ can be identified with $U(n)^0$, the restricted dual Hopf algebra. This gives rise to a natural pairing $\mathbb{C}[N] \times \hat{U}(n) \to \mathbb{C}$. Extending scalars from $\mathbb{C}$ to $P = \mathbb{C}[t_1, \ldots, t_m]$ we see that each $f \in \mathbb{C}[N]$ becomes a linear form on $P \otimes \hat{U}(n)$ with values in $P$. Then we have

$$\pi_i^*(f) = f(\tilde{\pi}_i).$$

We construct the deformation of $\tilde{\pi}_i$ in a more general situation when $n$ is the standard maximal nilpotent Lie subalgebra in a Kac-Moody algebra $g$. Let us briefly introduce necessary definitions and notation.

Let $A = (a_{ij})$ be a symmetrizable Cartan matrix of size $r \times r$. Denote by $C = (C_{ij}) = (d_i a_{ij})$ the corresponding symmetric matrix. Let $\mathcal{U}$ be the associative algebra over $\mathbb{C}(q)$ generated by $E_1, \ldots, E_r$ subject to the quantum Serre relations (this is the quantized universal enveloping algebra $U_q(n)$ of the nilpotent part of the Kac-Moody algebra corresponding to $A$). The algebra $\mathcal{U}$ is graded by $\mathbb{Z}_+^r$ via $\deg(E_i) = \alpha_i$, the
standard basis vector in $\mathbb{Z}_r^+$. Denote by $\hat{U}$ the completion with respect to the grading. Following [8], Chapter 2, we consider $U$ with the structure of a braided bialgebra with the braided coproduct $\Delta : U \to U \otimes U$. Namely, $\Delta(E_i) = E_i \otimes 1 + 1 \otimes E_i$, and $\Delta$ is a homomorphism of $\mathbb{Z}_r^+$-graded algebras, where the algebra structure on the tensor square of $U$ differs from the standard one by a twist (see Section 2 below for more details). It follows that $\hat{U}$ is a complete bialgebra with the coproduct $\hat{\Delta} : \hat{U} \to \hat{U} \hat{\otimes} \hat{U}$. The quantum group $A$ is the restricted dual algebra of $U$ (if $A$ is of finite type then $A$ can be identified with the $q$-deformed ring of polynomial functions $\mathbb{C}_q[N]$). The natural evaluation pairing $A \times \hat{U} \to \mathbb{C}(q)$ will be denoted by $(x, E) \mapsto x(E)$.

Let $i = (i_1, \ldots, i_m)$ be a sequence of integers with $1 \leq i_k \leq r$. Denote by $P_i$ the $\mathbb{C}(q)$-algebra generated by $t_1, \ldots, t_m$ subject to the relations

$$t_l t_k = q^{C_{i_k+i_l}} t_k t_l, \quad 1 \leq k < l \leq m. \quad (0.3)$$

Let $\hat{U}_i = P_i \otimes \hat{U}$ be the space of all series of the form

$$\sum_{\gamma \in \mathbb{Z}_r^+} t_\gamma \otimes E_\gamma$$

where $t_\gamma \in P_i$ and $E_\gamma \in U$ is a homogeneous element of degree $\gamma$. There is a standard algebra structure on $\hat{U}_i$. We identify $P_i \otimes 1$ with $P_i$ and $1 \otimes \hat{U}$ with $\hat{U}$ so that $tE = Et = t \otimes E$ in $\hat{U}_i$. Note also that $\hat{U}_i$ is a $P_i$-bimodule in the standard way. The coproduct $\hat{\Delta}$ on $\hat{U}$ extends naturally to the $P_i$-bilinear map

$$\hat{\Delta}_i : \hat{U}_i \to \hat{U}_i \hat{\otimes} \hat{U}_i$$

by the formula: $\hat{\Delta}_i(\sum_\gamma t_\gamma E_\gamma) = \sum_\gamma t_\gamma \Delta(E_\gamma)$. We call $e \in \hat{U}_i$ a group-like element if $\hat{\Delta}_i(e) = e \otimes e$.

Finally, we define the $q$-exponential

$$\exp_q(u) = \sum_{n \geq 0} \frac{u^n}{[n]_q \cdot [n]_q !} \quad (0.4)$$

where $[n]_q! = [1]_q [2]_q \cdots [n]_q$, $[l]_q = 1 + q + \cdots q^{l-1}$.

Now we can state our first main result.

**Theorem 0.1.** For any sequence $i = (i_1, \ldots, i_m)$ and any $c_1, \ldots, c_m \in \mathbb{C}(q)$, the product

$$\exp_{q_{i_1}}(c_1 t_1 E_{i_1}) \exp_{q_{i_2}}(c_2 t_2 E_{i_2}) \cdots \exp_{q_{i_m}}(c_m t_m E_{i_m})$$
is a group-like element in \( \hat{U}_i \), where \( q_i = q^{C_{ii}} \) for \( i = 1, \ldots, r \).

We prove Theorem 0.1 in Section 1 for more general braided bialgebras.

We denote
\[
e_i = \exp_{q_{i1}}(t_1 E_{i1}) \exp_{q_{i2}}(t_2 E_{i2}) \cdots \exp_{q_{im}}(t_m E_{im}) .
\]

(0.5)

As in the commutative case, we extend the evaluation pairing \( A \times \hat{U} \to \mathbb{C}(q) \) to the \( P_i \)-linear pairing \( A \times \hat{U}_i \to P_i \). As an analogue of (0.2) we define the map \( \psi_i : A \to P_i \) by
\[
\psi_i(x) := x(e_i) .
\]

(0.6)

**Corollary 0.2.** The map \( \psi_i \) is an algebra homomorphism.

**Proof.** The definition of the pairing \( (x, E) \to x(E) \) implies that \( (xy)(u) = (x \otimes y)(\hat{\Delta}(u)) \) for all \( x, y \in A \) and \( u \in \hat{U}_i \), where \( (x \otimes y)(u_1 \otimes u_2) := x(u_1)y(u_2) \) for any \( u_1, u_2 \in \hat{U}_i \). Thus, we have
\[
\psi_i(xy) = (xy)(e_i) = (x \otimes y)(\hat{\Delta}_i(e_i)) = (x \otimes y)(e_i \otimes e_i) = x(e_i)y(e_i) = \psi_i(x)\psi_i(y) .
\]

Corollary 0.2 is proved. \(<\)

Expanding (0.5), we obtain the following formula for \( \psi_i \):
\[
\psi_i(x) = \sum_{a_1, a_2, \ldots, a_m \geq 0} x(E_{i1}^{[a_1]} E_{i2}^{[a_2]} \cdots E_{im}^{[a_m]}) t_1^{a_1} t_2^{a_2} \cdots t_m^{a_m} \]  
(0.7)

where \( E_{i}^{[n]} = \frac{1}{[n]_{q_i}} E_i^n \). Note that the sum in (0.7) is always finite.

Under a well-known isomorphism \( A \cong \mathcal{U} \) the homomorphism \( \psi_i \) becomes Feigin’s homomorphism \( \Phi(i) : \mathcal{U} \to P_i \) ([5] and Section 2 below).

Using the homomorphism \( \psi_i \), we can express the group-like element \( e_i \) in terms of the universal element \( R \in A \otimes \hat{U} \) defined as follows. Under the canonical isomorphism between \( A \otimes \hat{U} \) and the space of linear maps \( \mathcal{U} \to \hat{U} \), the element \( R \) corresponds to the inclusion \( \mathcal{U} \hookrightarrow \hat{U} \).

**Proposition 0.3.** For any sequence \( i \) as above, we have
\[
(\psi_i \otimes \text{id})(R) = e_i .
\]

(0.8)

The element \( R \) is uniquely determined by the equations (0.8) for all \( i \).

**Proof.** Let \( B \) be a homogeneous basis in \( \mathcal{U} \) (that is, \( B \) is compatible with the \( \mathbb{Z}_+ \)-grading), and \( \{ b^* \} \) be the dual basis in \( A \) (so that \( b^*(b') = \delta_{b,b'} \)). By definition,
\[
R = \sum_{b \in B} b^* \otimes b .
\]

(0.9)
We have

\[(\psi_1 \otimes \text{id})(\mathcal{R}) = \sum_{b \in B} \psi_1(b^*) \otimes b = \sum_{b \in B} b^*(e_1) \otimes b = e_1\]

by definition (0.6) of \(\psi_1\) and by the formula \(\sum_{b \in B} b^*(u) \otimes b = u\) for any \(u \in \hat{U}_i\).

It remains to check the uniqueness. Assume that there is another \(\mathcal{R}'\) satisfying (0.8) for all \(i\). The equations (0.8) imply that \((\mathcal{R} - \mathcal{R}') \in \text{Ker} \ \psi_1 \otimes \hat{U}\). By (0.7), an element \(x \in \mathcal{A}\) is in the kernel of each \(\psi_i\) if and only if \(x\) vanishes at all monomials in \(E_1, \ldots, E_r\). Hence, \(\mathcal{R}' = \mathcal{R}\), and we are done. \(\triangle\)

The element \(\mathcal{R}\) was studied in [8], Chapter 4 in a slightly different setting; it is a version of the universal \(R\)-matrix for the “braided” quantum double of \(\hat{U}\).

Let \(W\) be the Weyl group generated by simple reflections \(s_1, \ldots, s_r : \mathbb{Z}^r \to \mathbb{Z}^r\) defined by \(s_i(\alpha_j) = \alpha_i - \alpha_{ij}\alpha_i\) for all \(i, j\). We say that \(i = (i_1, \ldots, i_m)\) is a reduced expression of \(w \in W\) if \(w = s_{i_1} \cdots s_{i_m}\) and this factorization of \(w\) is the shortest possible. We denote by \(R(w)\) the set of all reduced expressions of \(w\). We also reserve notation \(w_0\) for the longest element in \(W\) if \(W\) is finite.

For a sequence \(i = (i_1, \ldots, i_m)\) let \(\mathcal{U}(i)\) be the subspace in \(\mathcal{U}\) spanned by all monomials \(E_{i_1}^{a_1} E_{i_2}^{a_2} \cdots E_{i_m}^{a_m}\). It is well-known ([7], Section 4.4, or [9]) that if \(i \in R(w)\) then \(\mathcal{U}(i)\) depends only on \(w\). So we denote \(\mathcal{U}(w) := \mathcal{U}(i)\) for all \(i \in R(w)\). It is also well-known that \(\mathcal{U}(w_0) = \mathcal{U}\) if \(W\) is finite. Further, if \(i\) is not reduced then there is a subsequence \(i'\) of \(i\) such that \(i'\) is a reduced expression and \(\mathcal{U}(i) = \mathcal{U}(i')\). For the convenience of the reader we will prove these assertions in Section 2.

Now we can give a complete description of \(\text{Ker} \ \psi_i\) using (0.7) and the above discussion.

**Lemma 0.4.** The kernel of \(\psi_i\) is the orthogonal complement of \(\mathcal{U}(i)\):

\[\text{Ker} \ \psi_i = \{x \in \mathcal{A} : x(u) = 0 \text{ for all } u \in \mathcal{U}(i)\} .\]

Furthermore,

(i) If \(i\) is not reduced then there is a reduced subsequence \(i'\) of \(i\) such that \(\text{Ker} \ \psi_i = \text{Ker} \ \psi_{i'}\).

(ii) For every \(w \in W\) and \(i, i' \in R(w)\) we have \(\text{Ker} \ \psi_i = \text{Ker} \ \psi_{i'}\).

(iii) If \(W\) is finite then \(\text{Ker} \ \psi_i = \{0\}\) for any \(i \in R(w_0)\).

Denote \(I_w := \text{Ker} \ \psi_i\) for \(i \in R(w)\). Our next main result is the following.

**Theorem 0.5.** For every \(w \in W\) and \(i \in R(w)\), the image of \(\psi_i\) generates the (skew) field of fractions of \(P_i\). Hence, \(\psi_i\) induces an isomorphism of the fields of fractions

\[\overline{\psi_i} : \mathcal{F}(\mathcal{A}/I_w) \overset{\sim}{\longrightarrow} \mathcal{F}(P_i) .\]

In particular, if \(W\) is finite and \(w = w_0\) then \(\overline{\psi}_i\) is an isomorphism between \(\mathcal{F}(\mathcal{A})\) and \(\mathcal{F}(P_i)\).
Here the symbol $\mathcal{F}$ stands for the skew-field of fractions. We review the necessary definitions and results in the appendix below.

The last statement in Theorem 0.5 coincides with Feigin’s conjecture ([5], [6]) (stated for $\mathcal{U}$ rather than for $A$). The conjecture was proved in [5] for the type $A_r$ and $w = w_0$ by a direct computation involving some specific reduced word $i \in R(w_0)$. The conjecture was further generalized by A. Joseph ([6]) to any $w \in W$ in the assumption that $W$ is finite, and proved by geometric arguments.

We give an algebraic proof of Theorem 0.5 in Section 2 without the assumption that $W$ is finite. The following proposition plays the crucial role in the proof.

**Proposition 0.6.** For every reduced expression $i$ of some $w \in W$, there is an element $x = x(i) \in A$ such that $\psi_i(x) = t_1^{a_1}t_2^{a_2} \cdots t_m^{a_m}$ with $a_1 > 0$.

We prove Proposition 0.6 in Section 3.

**Corollary 0.7.** The skew-field $\mathcal{F}(\psi_i(A))$ coincides with $\mathcal{F}(P_i)$ if and only if $i$ is a reduced expression.

**Proof.** We need to prove the “only if” part (the “if” part is the assertion of Theorem 0.5). Assume that $i = (i_1, \ldots, i_m)$ is not reduced. Then, by Lemma 0.4(i) and Theorem 0.5, there is a reduced subsequence $i' = (i_{k_1}, \ldots, i_{k_l})$ of $i$ such that $\text{Im } \psi_{i'} \cong \text{Im } \psi_i$. It follows that $\mathcal{F}(\text{Im } \psi_i) \cong \mathcal{F}(\text{Im } \psi_{i'}) \cong \mathcal{F}(P_{i'})$. But $\mathcal{F}(P_{i'}) \not\cong \mathcal{F}(P_i)$ since these skew-fields have different Gel'fand-Kirillov dimensions (see [12], Proposition 2.18). Corollary 0.7 is proved.

Let us illustrate the above results in the case when $A$ is of type $A_{n-1}$. In this case, one can show that $A$ is generated by the elements $x_{ij}$ with $1 \leq i < j \leq n$ subject to the following relations (cf. [2], [4]):

\[ x_{ij} x_{kl} = x_{kl} x_{ij}, \quad x_{il} x_{jk} = x_{jk} x_{il} \quad (1 \leq i < j < k < l \leq n), \]

\[ x_{ik} = \frac{q x_{ij} x_{jk} - x_{jk} x_{ij}}{q - q^{-1}}, \quad x_{ij} x_{ik} = q x_{ik} x_{ij}, \quad x_{jk} x_{ik} = q^{-1} x_{ik} x_{jk} \quad (1 \leq i < j < k \leq n). \]

The elements $x_{ij}$ are $q$-deformations of the matrix entries considered as polynomial functions on the group $N$ of the unipotent upper-triangular matrices. So we arrange the $x_{ij}$ into the matrix $X = I + \sum_{i < j} x_{ij} E_{ij}$, where $I$ is the identity matrix, and the $E_{ij}$ are the matrix units. Let $\psi_i(X)$ be the $n \times n$-matrix (over $P_i$) obtained from $X$ by applying $\psi_i$ to each matrix entry. The following proposition will be proved in Section 2.

**Proposition 0.8.** For any sequence $i = (i_1, \ldots, i_m)$, the matrix $\psi_i(X)$ admits the following matrix factorization

\[ \psi_i(X) = (I + t_1 E_{i_1,i_1+1})(I + t_2 E_{i_2,i_2+1}) \cdots (I + t_m E_{i_m,i_m+1}). \]
If \( i \in R(w_0) \) then, applying the inverse isomorphism \((\psi_1)^{-1} : F(P_1) \to F(A)\) to the factorization (0.12), we obtain the factorization of the matrix \( X \) over \( F(A) \):

\[
X = (I + \tilde{t}_1 E_{i_1,i_1+1})(I + \tilde{t}_2 E_{i_2,i_2+1}) \cdots (I + \tilde{t}_m E_{i_m,i_m+1}) ,
\]

where \( \tilde{t}_k = (\psi_1)^{-1}(t_k) \). This factorization is a \( q \)-deformation of the one studied in [1]; it can be shown that such a factorization is unique. The explicit formulas for \( \tilde{t}_k \) in terms of the matrix entries \( x_{ij} \) will be given in a separate publication. The above factorizations of the matrix \( X \) (and, more generally, \( R \) for quantum groups of finite type) were studied in [11].

Let us return to the general situation and discuss some corollaries of Theorem 0.5. For every \( w \in W \) and \( i, i' \in R(w) \) there is an isomorphism of skew fields

\[
R^\prime_i : F(P_1) \cong F(P_\prime)
\]

defined by \( R^\prime_i = \psi_\prime \circ (\psi_1)^{-1} \). We extend it to the isomorphism \( R^\prime_i \otimes id : F(P_1) \otimes U \to F(P_\prime) \otimes U \). In the following proposition, every element \( e_i \) given by (0.5) is regarded as an element of \( F(P_1) \otimes U \).

**Proposition 0.9.** For every \( i, i' \in R(w) \), we have \((R^\prime_i \otimes id)(e_i) = e_i\).

**Proof.** Let \( p_w : A \to A/I_w \) be the canonical projection. Denote \( \mathcal{R}_w = (p_w \otimes id)(\mathcal{R}) \). Note that, similarly to \( \mathcal{R} \), the element \( \mathcal{R}_w \in A/I_w \otimes U \) corresponds to the inclusion \( U(w) \hookrightarrow U \).

Proposition 0.3 implies that

\[
(\psi_1 \otimes id)(\mathcal{R}_w) = e_i \tag{0.13}
\]

for every \( i \in R(w) \). We are done since \( R^\prime_i \otimes id = (\psi_\prime \otimes id) \circ (\psi_1 \otimes id)^{-1} \).

Proposition 0.9 implies some identities between quantum exponentials. For two reduced expressions \( i = (i_1, \ldots, i_m) \) and \( i' = (i'_1, \ldots, i'_m) \) of an element \( w \in W \), let \( t_1, \ldots, t_m \) (resp. \( t'_1, \ldots, t'_m \)) be the standard generators of \( P_1 \) (resp. \( P_\prime \)).

**Corollary 0.10.** The following identity holds in the algebra \( F(P_1) \otimes U \):

\[
\exp_{q_{i_1}}(t_1 E_{i_1}) \cdots \exp_{q_{i_m}}(t_m E_{i_m}) = \exp_{q_{i'_1}}(p_1 E_{i'_1}) \cdots \exp_{q_{i'_m}}(p_m E_{i'_m}) ,
\]

where \( p_k = R^\prime_i(t'_k) \) for \( k = 1, \ldots, m \). Identity (0.14) remains true under the rescaling \( E_i \mapsto c_i E_i \) for any \( c_i \in \mathbb{C}(q) \), \( i = 1, \ldots, r \).

**Example 1.** Let \( A \) be the Cartan matrix of type \( A_2 \). Then the Weyl group \( W \) is the symmetric group \( S_3 \), and \( w_0 = s_1 s_2 s_1 = s_2 s_1 s_2 \). Take \( i = (121), i' = (212), \) and denote by \( t_1, t_2, t_3 \) and \( t'_1, t'_2, t'_3 \) the generators of \( P_{(121)} \) and \( P_{(212)} \) respectively. Then \( R_{(212)}(t'_k) = p_k \) for \( k = 1, 2, 3 \), where

\[
p_1 = t_2 t_3 (t_1 + t_3)^{-1}, \quad p_2 = t_1 + t_3, \quad p_3 = (t_1 + t_3)^{-1} t_1 t_2 .
\]
This is a consequence of the matrix equation
\[(I + p_1 E_{23})(I + p_2 E_{12})(I + p_3 E_{23}) = (I + t_1 E_{12})(I + t_2 E_{23})(I + t_3 E_{12}),\]
which follows from the factorization (0.12).

The identity (0.14) takes the form
\[\exp_q^2(c_1 t_1 E_1)\exp_q^2(c_2 t_2 E_2)\exp_q^2(c_1 t_3 E_1) = \exp_q^2(c_2 p_1 E_2)\exp_q^2(c_1 p_2 E_1)\exp_q^2(c_2 p_3 E_2)\]
for any \(c_1, c_2 \in \mathbb{C}(q)\). Expanding both sides of (0.15) and comparing the components of degree \(2\alpha_1 + \alpha_2\), we obtain the quantum Serre relation
\[E_1^2 E_2 - (q + q^{-1})E_1 E_2 E_1 - E_2 E_1^2 = 0.\]

We also note that setting \(c_1 = 1\) and \(c_2 = 0\) in (0.15) yields the familiar rule
\[\exp_q^2(t_1 E_1)\exp_q^2(t_3 E_1) = \exp_q^2((t_1 + t_3)E_1).\]

The identity (0.15) appeared in [11], Section 10.4; it was proved there by a straightforward computation.

We conclude the introduction by a surprising combinatorial consequence of the above results. To a sequence \(i = (i_1, \ldots, i_m)\) we associate a skew-symmetric \(m \times m\)-matrix \(S(i)\) by the formula:
\[S(i) = \sum_{1 \leq k < l \leq m} C_{i_k, i_l} (E_{kl} - E_{lk}).\]

We say that two \(m \times m\) matrices \(S\) and \(S'\) are equivalent if there is a matrix \(T \in SL_m(\mathbb{Z})\) such that \(S' = TST^t\) (where \(T^t\) is the transpose of \(T\)).

**Proposition 0.11.** For every \(w\) and \(i, i' \in R(w)\), the matrices \(S(i)\) and \(S(i')\) are equivalent.

This follows from the fact that the skew-fields \(\mathcal{F}(P_l)\) and \(\mathcal{F}(P_{l'})\) are isomorphic, in view of a general result by A. Panov [12] (see also Section 2 below). Proposition 0.11 essentially says that there exists an isomorphism \(\mathcal{F}(P_{l'}) \to \mathcal{F}(P_l)\) which takes every generator \(t_{k}'\) to a monomial in \(t_1, \ldots, t_m\). (Note that the isomorphism \(R[l]_i\) considered above, in general does not have this property).

The material is organized as follows. In Section 1 we introduce braided bialgebras and prove some results about them, including the generalization of Theorem 0.1. The quantum group \(\mathcal{A}\) associated with a symmetric Cartan matrix is studied in Section 2, which contains the proofs of Lemma 0.4, Theorem 0.5 (modulo Proposition 0.6), and
Propositions 0.8 and 0.11. Section 3 is devoted to the proof of Proposition 0.6; our proof is based on the properties of extremal vectors in simple $U_q(g)$-modules. In Appendix we review necessary definitions and results about non-commutative fields of fractions.

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1. Results on braided bialgebras

Let $k$ be a field and $U$ be a $\mathbb{Z}^r_+$-graded $k$-algebra: $U = \bigoplus U(\gamma)$, the sum over $\gamma \in \mathbb{Z}^r_+$. We assume that $U(0) = k$ and every $U(\gamma)$ is finite-dimensional. Let $q = (q_{ij})$, $1 \leq i, j \leq r$ be a $r \times r$-matrix with all $q_{ij} \in k, q_{ij} \neq 0$. Following G. Lusztig ([8]) we associate with $q$ an algebra structure on the vector space $U \otimes U$. For any two homogeneous elements $b \in U(m_1, \ldots, m_r)$ and $c \in U(n_1, \ldots, n_r)$ we set

$$Q(b, c) = Q((m_1, \ldots, m_r), (n_1, \ldots, n_r)) = \prod_{i,j=1}^{r} q_{ij}^{m_i n_j}. \quad (1.1)$$

We define the $q$-braided multiplication in $U \otimes q U$ by

$$(a \otimes b)(c \otimes d) := Q(b, c)(ac \otimes bd) \quad (1.2)$$

for any homogeneous elements $b, c$ of $U$ and any $a, d \in U$.

It is easy to see that (1.2) makes $U \otimes q U$ into a $\mathbb{Z}^r_+$-graded associative algebra (with the standard grading $(U \otimes_k U)(\gamma) = \bigoplus_{\gamma'} U(\gamma') \otimes U(\gamma - \gamma')$). This algebra will be denoted by $U \otimes_q U$ and called the $q$-braided tensor square of $U$.

We call $U$ a $q$-braided bialgebra if

(i) there is a homomorphism of $\mathbb{Z}^r_+$-graded algebras $\Delta : U \rightarrow U \otimes_q U$ satisfying the coassociativity constrain (we call $\Delta$ the coproduct);

(ii) There is a counit homomorphism of algebras $\varepsilon : U \rightarrow U(0) = k$ satisfying

$$(\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}, \quad \varepsilon(1) = 1. \quad (1.3)$$

This definition implies that for every $u \in U$,

$$\Delta(u) = u \otimes 1 + 1 \otimes u + \sum_{n} u_n \otimes u'_n \quad (1.4)$$
where all \( u_n, u'_n \) are homogeneous elements of nonzero degrees. In particular, \( \Delta(u) = u \otimes 1 + 1 \otimes u \) for every \( x \in \mathcal{U}(\alpha_1) \bigoplus \cdots \bigoplus \mathcal{U}(\alpha_r) \) where \( \alpha_1, \ldots, \alpha_r \) is the standard basis in \( Z'_+ \). Another consequence of this definition is that \( \varepsilon(x) = 0 \) for any \( x \in \mathcal{U}(\gamma), \gamma \neq 0 \).

Note that the algebra \( \mathcal{U} \) from the introduction, associated to a symmetrizable Cartan matrix \( A \), is a \( \mathbf{q} \)-braided algebra, where \( q_{ij} = q^{C_{ij}} \). Another example is the free algebra generated by \( E_1, \ldots, E_r \), where \( \mathbf{q} \) is arbitrary.

Let \( \hat{\mathcal{U}} \) be the completion of \( \mathcal{U} \) with respect to the grading, that is, the space of all formal series \( \hat{u} = \sum \gamma \in Z'_+ u_\gamma \), where \( u_\gamma \in \mathcal{U}(\gamma) \). Clearly, \( \hat{\mathcal{U}} \) is an algebra. The coproduct in \( \mathcal{U} \) extends to \( \hat{\Delta} : \hat{\mathcal{U}} \to \mathcal{U} \otimes \mathcal{U} \) so \( \hat{\mathcal{U}} \) becomes a complete bialgebra.

Now we fix a positive integer \( m \) and consider a sequence \( i = (i_1, i_2, \ldots, i_m) \) of integers with \( 1 \leq i_k \leq r \). Let \( \mathbf{q} = (q_{ij}) \) be the matrix used in the definition of \( \mathcal{U} \). Consider a \( k \)-algebra \( P_i = P_i, \mathbf{q} \) generated by \( t_1, \ldots, t_m \) subject to the following relations:

\[
t_l t_k = q_{i_k, i_l} t_k t_l
\]

for all \( 1 \leq k < l \leq m \).

Define \( \hat{\mathcal{U}}_i = P_i \bigotimes_k \hat{\mathcal{U}} \), the space of formal series of the form \( \sum_{\gamma} t_{\gamma} \otimes u_\gamma \), where \( t_\gamma \in P_i \) and \( u_\gamma \in \mathcal{U}(\gamma) \). We consider \( \hat{\mathcal{U}}_i \) with the standard algebra structure (so we can write \( t u = u t = t \otimes u \)).

Consider the completed tensor square \( \mathcal{V}_i = \hat{\mathcal{U}}_i \bigotimes_k \hat{\mathcal{U}}_i \) where the left factor is regarded as a right \( P_i \)-module and the right factor as a left \( P_i \)-module. Note that \( \mathcal{V}_i \) is a \( P_i \)-bimodule. In \( \mathcal{V}_i \), we can write \( t(u \otimes v) = (tu) \otimes v = u \otimes (tv) = (u \otimes v)t \) for any \( u, v \in \mathcal{U}, t \in P_i \). Under the standard identification \( \mathcal{V}_i \cong P_i \bigotimes_k \mathcal{U} \bigotimes_k \mathcal{U} \) this bimodule \( \mathcal{V}_i \) becomes an algebra.

There is a natural morphism of \( P_i \)-bimodules

\[
\hat{\Delta}_i : \hat{\mathcal{U}}_i \to \mathcal{V}_i
\]

which is the \( P_i \)-linear extension of the coproduct \( \hat{\Delta} \) on \( \hat{\mathcal{U}} \). Clearly, \( \hat{\Delta}_i \) is an algebra homomorphism.

Let \( \mathbf{E} = (E_1, \ldots, E_m) \) be the family of elements \( E_k \in \mathcal{U}(\alpha_i) \). We define an element \( e_i = e_i, \mathbf{E} \in \hat{\mathcal{U}}_i \) as follows:

\[
e_i, \mathbf{E} = \exp_{q_1}(t_1 E_1) \exp_{q_2}(t_2 E_2) \cdots \exp_{q_m}(t_m E_m)
\]

where \( q_k = q_{i_k, i_k} \) for \( k = 1, \ldots, m \), and \( \exp_{q_k} \) stands for the quantum exponential defined by (0.4).

The following result extends Theorem 0.1 to arbitrary \( \mathbf{q} \)-braided algebras.

**Theorem 1.1.** For any sequence \( i \) and any family \( \mathbf{E} = (E_k) \) as above the element \( e_i = e_i, \mathbf{E} \) is a group-like element in \( \hat{\mathcal{U}}_i \), i.e., \( \hat{\Delta}_i(e_i) = e_i \otimes e_i \).

**Proof.** We need the following.
Lemma 1.2.
(a) Each factor $e_k = \exp_{q_k}(t_k E_k)$ of $e_i$ is a group-like element in $\hat{U}_q$.
(b) $(1 \otimes e_k)(e_l \otimes 1) = (e_l \otimes 1)(1 \otimes e_k)$ for any $1 \leq k < l \leq m$.

Proof. (a) Denote $E = t_k E_k$. Since $\Delta(E_k) = E_k \otimes 1 + 1 \otimes E_k$, for each $k$ we have

$$
\hat{\Delta}_1(E) = t_k (E_k \otimes 1 + 1 \otimes E_k) = E \otimes 1 + 1 \otimes E.
$$

Denote $x = E \otimes 1, y = 1 \otimes E$. Let us show that $yx = q xy$ where $q := q_k = q_{i_k,i_k}$.

Indeed,

$$
yx = (1 \otimes E)(E \otimes 1) = (1 \otimes t_k E_k)(t_k E_k \otimes 1) = t_k^2 (1 \otimes E_k)(E_k \otimes 1)
$$

$$
= Q(E_k,E_k)t_k^2 (E_k \otimes E_k) = q_{i_k,i_k} (t_k E_k \otimes t_k E_k) = q xy.
$$

Further, we obtain

$$
\hat{\Delta}_1(\exp_q(E)) = \exp_q(\hat{\Delta}_1(E)) = \exp_q(x + y)
$$

and

$$
\exp_q(E) \otimes \exp_q(E) = (\exp_q(E) \otimes 1)(1 \otimes \exp_q(E))
$$

$$
= (\exp_q(E \otimes 1))(\exp_q(1 \otimes E)) = \exp_q(x)\exp_q(y).
$$

Then the well-known rule for the quantum exponentials.

$$
\exp_q(x + y) = \exp_q(x)\exp_q(y)
$$

(provided that $yx = q xy$) implies that $\hat{\Delta}_1(\exp_q(E)) = \exp_q(E) \otimes \exp_q(E)$. Part (a) is proved.

(b) Denote $E = t_k E_k$ and $E' = t_l E_l$. By definition of $U \otimes_q U$,

$$
(1 \otimes E)(E' \otimes 1) = t_k t_l (1 \otimes E_k)(E_l \otimes 1) = q_{i_k,i_l} t_k t_l (E_l \otimes E_k).
$$

The commutation relations (1.5) imply that

$$
(1 \otimes E)(E' \otimes 1) = t_k t_l (1 \otimes E_k)(E_l \otimes 1) = t_l t_k (E_l \otimes E_k) = E' \otimes E = (E' \otimes 1)(1 \otimes E).
$$

It follows that $(1 \otimes f(E))(g(E') \otimes 1) = f(E') \otimes g(E) = (f(E') \otimes 1)(1 \otimes g(E))$ for any polynomials $f$ and $g$ in one variable. Passing to the completion, we see that $f$ and $g$ can also be power series in the above formula. Taking $f(E) := e_k = \exp_{q_k}(E)$ and $g(E') := e_l = \exp_{q_l}(E')$ completes the proof of part (b). Lemma 1.2 is proved. \(\diamondsuit\)

We are ready to complete the proof of Theorem 1.1 now. Recall that we use the shorthand $e_k = \exp_{q_k}(t_k E_k)$ so $e_1 = e_1 e_2 \cdots e_m$. 

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Using Lemma 1.2 and the fact that \((a \otimes 1)(1 \otimes b) = a \otimes b\) for any \(a, b \in \hat{\mathcal{U}}_i\), we obtain
\[
\hat{\Delta}(e_i) = \hat{\Delta}(e_1 e_2 \cdots e_m) = \hat{\Delta}(e_1) \hat{\Delta}(e_2) \cdots \hat{\Delta}(e_m) = (e_1 \otimes e_1)(e_2 \otimes e_2) \cdots (e_m \otimes e_m)
\]
\[
= (e_1 \otimes 1)(1 \otimes e_1)(1 \otimes e_2) \cdots (e_m \otimes 1)(1 \otimes e_m).
\]
Using the commutativity property in Lemma 1.2(b), we obtain
\[
\hat{\Delta}(e_i) = ((e_1 \otimes 1)(e_2 \otimes 1) \cdots (e_m \otimes 1))((1 \otimes e_1)(1 \otimes e_2) \cdots (1 \otimes e_m))
\]
Finally, using the identities \((u \otimes 1)(v \otimes 1) = uv \otimes 1, (1 \otimes u)(1 \otimes v) = 1 \otimes uv\) for any \(u, v \in \hat{\mathcal{U}}\), we obtain \(\hat{\Delta}(e_i) = (e_1 \otimes 1)(1 \otimes e_1) = e_i \otimes e_i\). Theorem 1.1 is proved. \(\checkmark\)

Now we define the restricted dual algebra \(A = \mathcal{U}^0\) of \(\mathcal{U}\). As a vector space, \(A\) is the set of all \(k\)-linear forms \(x : \mathcal{U} \rightarrow k\) such that \(x\) vanishes on \(\mathcal{U}(\gamma)\) for all but finitely many \(\gamma \in \mathcal{Z}_+^r\). In other words, \(A \cong \bigoplus_{\gamma} A(\gamma)\) where \(A(\gamma) = \text{Hom}_k(\mathcal{U}(\gamma), k)\).

We define the multiplication \(A \otimes A \rightarrow A\) by the formula \((xy)(u) = (x \otimes y)(\Delta(u))\) where \((x \otimes y)(u_1 \otimes u_2) = x(u_1)y(u_2)\). Thus, \(A\) becomes a \(\mathcal{Z}_+^r\)-graded algebra (with the unit \(k \rightarrow A\) dual to the counit \(\varepsilon : \mathcal{U} \rightarrow k\)).

Denote by \((x, u) \mapsto x(u)\) the natural non-degenerate evaluation pairing \(A \times \mathcal{U} \rightarrow k\). Furthermore, we define the pairing \(A \times \hat{\mathcal{U}}_i \rightarrow P_i\) by the formula \(x(\sum t_\gamma u_\gamma) = \sum x(u_\gamma)t_\gamma\). (The sum is finite by the definition of \(A\).)

For every family \(E\) as above define a map \(\psi_i = \psi_{i,E} : A \rightarrow P_i\) by the formula \(\psi_i(x) := x(e_i)\). Expanding \(e_i\) into a power series we obtain
\[
\psi_i(x) = \sum_{a_1, \ldots, a_m \in \mathcal{Z}_+^r} x(E_1^{[a_1]} E_2^{[a_2]} \cdots E_m^{[a_m]})t_1^{a_1}t_2^{a_2} \cdots t_m^{a_m} \quad (1.7)
\]
where \(E_k^{[n]} = \frac{E_k^n}{[n]_{q_k}}\). Note that the sum in (1.7) is always finite because \(x\) vanishes on all but finitely many monomials \(E_1^{a_1} \cdots E_m^{a_m}\). Define a \(\mathcal{Z}_+^r\)-grading on \(P_i\) by \(\text{deg}(t_k) = \alpha_{i_k}\) and denote by \(P_i(\gamma)\) the graded component of degree \(\gamma\) in \(P_i\).

**Corollary 1.3.** For any sequence \(i = (i_1, \ldots, i_m)\) and a family \(E\) of elements \(E_k \in \mathcal{U}(\alpha_{i_k})\) \((k = 1, \ldots, m)\), the map \(\psi_{i,E} : A \rightarrow P_i\) defined by (1.7) is a homomorphism of \(\mathcal{Z}_+^r\)-graded algebras.

The proof of Corollary 1.3 repeats that of Corollary 0.2. \(\checkmark\)

**Remark 1.** One can prove (see e.g. [10]) that \(A\) is a \(q^t\)-braided bialgebra (where \(q^t\) is the transpose of \(q\)). Moreover, starting with an arbitrary \(q^t\)-braided algebra \(A\), one recovers \(\mathcal{U}\) as the restricted dual of \(A\). So the result of Corollary 1.2 holds for any \(q^t\)-braided bialgebra \(A\).
Remark 2. Let $A_1 = \bigoplus_{i=1}^r A(\alpha_i)$. Corollary 1.2 implies that any morphism $A_1 \to \bigoplus_{i=1}^r P(\alpha_i)$ of $\mathbb{Z}_+^r$-graded vector spaces extends to an algebra homomorphism. If $A$ is generated by $A_1$, then this extension is unique. Thus, in the latter case all the homomorphisms $A \to P_1$ of $\mathbb{Z}_+^r$-graded algebras are parametrized by the space $\bigoplus_{i=1}^r (U(\alpha_i) \otimes P_1(\alpha_i))$.

We define the universal element $R \in A \otimes \hat{U}$ as follows. The tensor product $A \otimes \hat{U}$ is canonically identified with the space of all linear maps $U \to \hat{U}$. Then $R$ is the element in $A \otimes \hat{U}$ corresponding to the inclusion $U \hookrightarrow \hat{U}$.

**Proposition 1.4.** The element $R$ satisfies $(\psi_{1,E} \otimes \text{id})(R) = e_{1,E}$ for any $i$ and $E$ as above.

The proof of Proposition 1.4 coincides with that of Proposition 0.3. $\triangleleft$

Clearly, the correspondence

$$c_1t_1 + \cdots + c_mt_m \mapsto \exp_{q_1}(c_1 t_1 E_1) \cdots \exp_{q_m}(c_m t_m E_m)$$

is a map from the $m$-dimensional “quantum” affine space $(P_1)_1 = \bigoplus_{i=1}^m k \cdot t_i$ to the set of group-like elements in $\hat{U}_i$. This map can be regarded as a deformation of the morphism (0.1).

Now let us turn to the fields of fractions. For an algebra $B$ without zero divisors, $\mathcal{F}(B)$ is a vector space of right fractions (see Appendix). Its elements can be written as $ab^{-1}$, where $a, b \in B$ and $b \neq 0$. As shown in the Appendix, for any sequence $i$ and any subalgebra $B \subset P_1$, the space $\mathcal{F}(B)$ is a skew-field. Note that $\psi_1$ induces an embedding of skew fields $\overline{\psi}_1 : \mathcal{F}(A/\text{Ker } \psi_1) \hookrightarrow \mathcal{F}(P_1)$.

Now consider two elements $e_{i,E}$ and $e_{i',E'}$ corresponding via (1.6) to two sequences of indices $i = (i_1, \ldots, i_m)$ and $i' = (i_1', \ldots, i'_m)$ and two families of elements $E = (E_1, \ldots, E_m)$ and $E' = (E'_1, \ldots, E'_m)$. Let $t_1, \ldots, t_m$ (resp. $t'_1, \ldots, t'_m$) be the standard generators of $P_1$ (resp. $P_{i'}$).

**Proposition 1.5.** Assume that $\text{Ker } \psi_{1,E} = \text{Ker } \psi_{i',E'}$, and $\overline{\psi}_{i',E'}$ is an isomorphism of skew-fields $\mathcal{F}(A/\text{Ker } \psi_{i',E'})$ and $\mathcal{F}(P_{i'})$. Then the map $R := \overline{\psi}_{1,E} \circ (\overline{\psi}_{i',E'})^{-1}$ is an embedding $\mathcal{F}(P_{i'}) \hookrightarrow \mathcal{F}(P_1)$, and the following identity holds in $\mathcal{F}(P_1) \otimes \hat{U}$:

$$\exp_{q_1}(t_1 E_1) \cdots \exp_{q_m}(t_m E_m) = \exp_{q'_1}(p_1 E'_1) \cdots \exp_{q'_m}(p_n E'_n),$$

where $q_k = q_{i_k,i_k}$ for $k = 1, \ldots, m$, and $q'_l = q'_{i'_l,i'_l}$, $p_l = R(t'_l)$ for $l = 1, \ldots, n$.

**Proof.** We omit subscripts $E$ and $E'$ in the formulas below. Let $p_i : A \to A/\text{Ker } \psi_i$ be the canonical projection. Denote $R_i = (p_i \otimes \text{id})(R)$. Then Proposition 1.4 implies that

$$\overline{\psi}_i \otimes \text{id})(R_i) = e_i$$

(1.9)
for every $i \in R(w)$. We are done since $R \otimes \text{id} = (\psi_i \otimes \text{id}) \circ (\psi_i' \otimes \text{id})^{-1}$. Proposition 1.5 is proved. ◽

2. Feigin’s conjecture and other results for quantum groups

Throughout this section we will work over the field $k = k(q)$ where $k$ is a field of characteristic 0 (say, $k = \mathbb{C}$ as in the introduction), and $q$ is a variable (or a purely transcendental element over $k$). Let $A = (a_{ij})$ be a symmetrizable Cartan matrix of size $r \times r$, and $C = (C_{ij})$ be the corresponding symmetric matrix with integer entries. In this section we consider a matrix $q$ of the form $q = (q_{ij}) = (q^{C_{ij}})$. We denote $q_i := q_{ii} = q^{C_{ii}}$ for all $i$.

Similarly to [8], Chapter 1, we define the quantized enveloping algebra $U$ and the quantum group $A$ associated with $A$ as follows. First, let $U$ be the free algebra over $k(q)$ generated by $E_1, \ldots, E_r$. We make $U$ into a $q$-braided bialgebra (see Section 1). Second, the restricted dual algebra $A$ of $U$ is defined as in Section 1. Define a homomorphism $f : U \rightarrow A$ by $f(E_i) = x_i$ where $x_i$ is the only element in $A(\alpha_i)$ such that $x_i(E_i) = 1$. Finally, define $\hat{A} := \overline{U}/\text{Ker } f$ and $A := \text{Im } f$, and keep the above notation for the generators. In particular, $U \cong A$ via $E_i \mapsto x_i$. It is well-known that the right kernel of the evaluation pairing $A \otimes \hat{U} \rightarrow k(q)$ coincides with $\text{Ker } f$. Hence the induced pairing $A \otimes U \rightarrow k(q)$ is non-degenerate, so we identify $A$ with the restricted dual algebra to the $q$-braided bialgebra $U$ (and denote the evaluation pairing (2.1) by $(x, u) \mapsto x(u)$). Note that the generators $E_1, \ldots, E_r$ of $U$ (as well as the generators $x_1, \ldots, x_r$ of $A$) are subject to the quantum Serre relations ([8], Section 1.4.3, or Section 3 below).

The algebra $U$ is $\mathbb{Z}_r^+$-graded via $\text{deg } E_i = \alpha_i$. The pairing $A \times U \rightarrow k(q)$ extends to the $P_1$-linear pairing $A \times \hat{U} \rightarrow k(q)$ (we denote it by $(x, u) \mapsto x(u)$), where $\hat{U} := P_1 \otimes \hat{U}$ and $P_1$ is a $k(q)$-algebra generated by $t_1, \ldots, t_m$ subject to the relations $t_l t_k = q^{C_{ik} - C_{il}} t_k t_l$ for $1 \leq k < l \leq m$.

For the convenience of the reader, we summarize the results from Section 1 for the quantum groups $A$ and $U$ in the following theorem.

**Theorem 2.1.** Let $i = (i_1, \ldots, i_m)$ be any sequence. Then

(a) the element

$$e_i = \exp_{q_{i_1}}(t_1 E_{i_1}) \cdots \exp_{q_{i_m}}(t_m E_{i_m})$$

is a group-like element in $\hat{U}_i$;

(b) the element $e_i$ gives rise to an algebra homomorphism $\psi_i : A \rightarrow P_1$ defined by $\psi_i(x) := x(e_i)$;

(c) there is a unique element $\mathcal{R} \in A \otimes \hat{U}$ satisfying $(\psi_i \otimes \text{id})(\mathcal{R}) = e_i$ for all $i$;
(d) the homomorphism $\psi_1$ satisfies

$$\psi_1(x) = \sum_{a_1, \ldots, a_m \geq 0} x(E_{i_1}^{[a_1]}E_{i_2}^{[a_2]} \cdots E_{i_m}^{[a_m]})l_1^{a_1}l_2^{a_2} \cdots l_m^{a_m},$$

where $E_i^{[n]} = \frac{1}{[n]_q!}E_i^n$, and $[n]_q!$ is defined in (0.4). In particular, for $i = 1, \ldots, r$ we have $\psi_1(x_i) = \sum_{k:i_k=i} t_k$, and this determines $\psi_1$ uniquely;

(e) Ker $\psi_1 = \{x \in A : x(E_{i_1}^{a_1}E_{i_2}^{a_2} \cdots E_{i_m}^{a_m}) = 0$ for all $a_1, \ldots, a_m \in \mathbb{Z}_+\}$. \[1992 \text{ (see e.g. [5] and [6]).} \]

Remark. After the identification $A \cong U$ as above, $\psi_1$ coincides with Feigin’s homomorphism $\Phi(i) : U \to P_1$. B. Feigin introduced this homomorphism in his talk at RIMS in 1992 (see e.g. [5] and [6]).

Let $W$ be the Weyl group associated with the Cartan matrix $A$. By definition, $W$ is generated by simple reflections $s_1, \ldots, s_r : \mathbb{Z}^r \to \mathbb{Z}^r$ where $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. We call a sequence $\mathbf{i} = (i_1, \ldots, i_m)$ of indices a reduced expression of $w \in W$ if $w = s_{i_1}s_{i_2} \cdots s_{i_m}$, and the above expression of $w$ is the shortest (we call $\mathbf{i}$ simply a reduced expression if $w$ is not specified). We set $l(w) := m$ and call $l(w)$ the length of $w$. Denote by $R(w)$ the set of all reduced expressions of $w$. It is well-known that $W$ is a Coxeter group, so the defining relations between $s_1, \ldots, s_r$ are of the form $(s_i s_j)^l = 1$ where $l \in \{2, 3, 4, 6\}$. (More precisely, $l = a_{ij}^2 a_{ji} + 2$ if $a_{ij}a_{ji} < 3$ and $l = 6$ if $a_{ij}a_{ji} = 3$.) It follows that every two reduced expressions of an element $w \in W$ are connected by a chain of moves

$$(i_1, (i, j, i, \ldots), i_2) \mapsto (i_1, (j, i, j \ldots), i_2)$$

where each fragment in parentheses has the length $l$. If the Weyl group $W$ is finite then there is a unique element of the maximal length in $W$ which we denote by $w_0$.

Let us study the kernel of $\psi_1$. According to Theorem 2.1(e), Ker $\psi_1$ is the orthogonal complement of the subspace $U(\mathbf{i}) \subset U$ spanned by all monomials $E_{i_1}^{a_1}E_{i_2}^{a_2} \cdots E_{i_m}^{a_m}$.

Lemma 2.2.

(a) For every sequence $\mathbf{i}$ there is a reduced expression $\mathbf{i}'$ such that $U(\mathbf{i}) = U(\mathbf{i}')$. Moreover, $\mathbf{i}'$ can always be chosen as a subsequence of $\mathbf{i}$.

(b) For any $w \in W$ and $\mathbf{i}, \mathbf{i}' \in R(w)$, we have $U(\mathbf{i}) = U(\mathbf{i}')$.

(c) $U(\mathbf{i})$ contains the subalgebra in $U$ generated by all $E_i$ such that $l(ws_i) = l(w) - 1$.

Therefore, $U(\mathbf{i}) = U$ for every $\mathbf{i} \in R(w_0)$ when $W$ is finite.

Proof. The collection of the subspaces $\{U(\mathbf{i})\}$ is a multiplicative semigroup with respect to the product of vector subspaces in $U$. By definition,

$$U(i_1, i_2, \ldots, i_m) = U(i_1) \cdot U(i_2) \cdots U(i_m)$$

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where \( \mathcal{U}(i) \) is a subalgebra in \( \mathcal{U} \) generated by \( E_i, i = 1, \ldots, r \).

We have \( \mathcal{U}(i)\mathcal{U}(i) = \mathcal{U}(i) \), and for every pair \( (i, j) \) with \( a_{ij}a_{ji} < 4 \) the following relation holds:

\[
\mathcal{U}(i) \cdot \mathcal{U}(j) \cdot \mathcal{U}(i) \cdots = \mathcal{U}(j) \cdot \mathcal{U}(i) \cdot \mathcal{U}(j) \cdots
\]

where each product contains \( l \) factors. The identity (2.3) can be proved by the standard arguments for the algebras \( \mathcal{U} \) whose Cartan matrices are of types \( A_1 \times A_1, A_2, B_2 \) or \( G_2 \).

It follows that every \( \mathcal{U}(i) \) equals to \( \mathcal{U}(i') \) for some reduced subsequence \( i' \) of \( i \), which proves (a). Part (b) also follows because the braid relation (2.3) can be used to move from any reduced expression of \( w \in W \) to any other one.

(c) Let \( J = J_w \) be the the set of all \( i \) satisfying \( l(ws_i) = l(w) - 1 \). For each \( i \in J \), there exists \( i \in R(w) \) such that \( i \) ends with \( i \). Using (b) we see that \( \mathcal{U}(i)E_i \subset \mathcal{U}(i) \) for any \( i \in R(w), i \in J \). This completes the proof of Lemma 2.2. \( \triangleleft \)

We define \( I_w := \text{Ker} \psi_1 \) for any \( i \in R(w) \). Since \( A/I_w \) is isomorphic to \( \psi_1(A) \), it follows that \( \mathcal{F}(A/I_w) \) is a skew field (see Appendix).

**Theorem 2.3.** For every \( w \in W \) and \( i \in R(w) \) the map \( \psi_1 \) induces an isomorphism of skew fields

\[
\overline{\psi}_1 : \mathcal{F}(A/I_w) \cong \mathcal{F}(P_1) .
\]

Taking \( w = w_0 \) we obtain the following

**Corollary 2.4.** (Feigin’s conjecture). For any \( i \in R(w_0) \), the homomorphism \( \psi_1 : A \rightarrow P_1 \) is an embedding, and it induces an isomorphism of skew-fields

\[
\overline{\psi}_1 : \mathcal{F}(A) \cong \mathcal{F}(P_1) .
\]

**Proof of Theorem 2.3.** It is enough to prove that for any \( i \in R(w) \) the image of \( \psi_1 \) generates \( \mathcal{F}(P_1) \), that is, \( t_1, \ldots, t_m \) belong to \( \mathcal{F}(\text{Im} \psi_1) \). We will deduce this statement from Proposition 0.6. Then we need the following.

**Proposition 2.5.** For each element \( x \in A \) satisfying

\[
\psi_1(x) = t_1^{a_1}t_2^{a_2} \cdots t_m^{a_m}
\]

with \( a_1 > 0 \), there is an element \( y \in A \) such that \( \psi_1(y) = ct_1^{a_1-1}t_2^{a_2} \cdots t_m^{a_m} \) where \( c \in k(q), c \neq 0 \).

**Proof of Proposition 2.5.** For \( i = 1, \ldots, r \), let \( E_i^* : A \rightarrow A \) be the adjoint operator of the left multiplication operator \( E \mapsto E_iE \) in \( \mathcal{U} \). Thus, the element \( E_i^*(x) \) is determined by the equations \( (E_i^*(x))(E) = x(E_iE) \) for every \( E \in \mathcal{U} \).

We will show that \( y \) can be chosen as \( y = E_i^*(x) \).
Indeed, (2.5) means that the right hand side of the expansion (2.2) for \( \psi_1(x) \) reduces to one summand or, equivalently,
\[
x(\underbrace{E_{i_1}^{[b_1]} \cdots E_{i_m}^{[b_m]}}) = 0
\]
unless \((b_1, \ldots, b_m) = (a_1, \ldots, a_m)\).

By (2.2), we have
\[
\psi_1(y) = \psi_1(E^*_{i_1}(x)) = \sum_{b_1, \ldots, b_m \in \mathbb{Z}_+} (E^*_{i_1}(x)) (E_{i_1}^{[b_1]} E_{i_2}^{[b_2]} \cdots E_{i_m}^{[b_m]}) t_1^{b_1} t_2^{b_2} \cdots t_m^{b_m}
\]
\[
= \sum_{b_1, \ldots, b_m \in \mathbb{Z}_+} x(E_{i_1}^{[b_1]} E_{i_2}^{[b_2]} \cdots E_{i_m}^{[b_m]}) t_1^{b_1} t_2^{b_2} \cdots t_m^{b_m}.
\]

In view of (2.6),
\[
\psi_1(y) = x(E_{i_1}^{[a_1-1]} E_{i_2}^{[a_2]} \cdots E_{i_m}^{[a_m]}) t_1^{a_1-1} t_2^{a_2} \cdots t_m^{a_m} = ct_1^{a_1-1} t_2^{a_2} \cdots t_m^{a_m}
\]
with \( c \neq 0 \) as desired. \(<\)

Taking \( x \) and \( y \) as in Proposition 2.5, we see that
\[
t_1 = c\psi_1(x)(\psi_1(y))^{-1} \in \mathcal{F}(\text{Im } \psi_1).
\]

To complete the proof of Theorem 2.3, we proceed by induction on \( m \). If \( m = 1 \) then \( t_1 \in \text{Im } \psi_1 = P_1 \). So let \( m \geq 2 \), denote \( i' = (i_2, \ldots, i_m) \) and assume that Theorem 2.3 holds for \( i' \), that is,
\[
t_2, t_3, \ldots, t_m \in \mathcal{F}(\text{Im } \psi_{i'}).
\]

Note that \( P_{i'} \) is naturally embedded into \( P_1 \) as a subalgebra generated by \( t_2, \ldots, t_m \).

In view of the formula for \( \psi_1(x_i) \) in Theorem 2.1(d),
\[
\psi_{i'}(x_i) = \psi_1(x_i) \ (i \neq i_1), \ \psi_{i'}(x_{i_1}) = \psi_1(x_{i_1}) - t_1.
\]

Using (2.7), we see that
\[
\psi_{i'}(x_i) \in \mathcal{F}(\text{Im } \psi_1), \ (i = 1, \ldots, r)
\]
hence
\[
\mathcal{F}(\text{Im } \psi_{i'}) \subset \mathcal{F}(\text{Im } \psi_1).
\]

Combining this with the inductive assumption (2.8), we conclude that \( t_2, \ldots, t_m \in \mathcal{F}(\text{Im } \psi_1) \). Since \( t_1 \) also belongs to \( \mathcal{F}(\text{Im } \psi_1) \), Theorem 2.3 is proved. \(<\)

**Proof of Proposition 0.8.** Let \( \mathcal{B} \) be the algebra of the upper triangular \( n \times n \)-matrices over \( \mathbb{C}(q) \) (with the unity \( I \), the identity matrix). Let \( \rho : \mathcal{U} \to \mathcal{B} \) be a representation of \( \mathcal{U} \) given by \( \rho(E_i) = E_{i,i+1} \), where \( E_{ij} \) is the matrix unit. The representation \( \rho \) extends naturally to \( \text{id} \otimes \rho : \mathcal{A} \otimes \mathcal{U} \to \mathcal{A} \otimes \mathcal{B} \). We identify the latter algebra with \( \mathcal{B}(\mathcal{A}) \), the algebra of upper triangular matrices over \( \mathcal{A} \).
Lemma 2.6. We have $X = (\text{id} \otimes \rho)(R)$, where $R$ is the universal element in $A \otimes \hat{U}$.

Proof. Note that $B$ is a $\mathbb{Z}_{+}^{n-1}$-graded algebra via $\deg(E_{ii}) = 0$, $\deg(E_{ij}) = \alpha_{ij} = \alpha_{i} + \cdots \alpha_{j-1}$ ($1 \leq i < j \leq n$), and $\rho$ preserves the $\mathbb{Z}_{+}^{n-1}$-grading ($n = r + 1$). Therefore, the formula (0.9) for $\rho$ implies

$$(\text{id} \otimes \rho)(R) = I + \sum_{i < j} \sum_{b \in B_{ij}} b^* \rho(b).$$

where $B_{ij}$ is a basis in $U(\alpha_{ij})$ and $\{b^*\}$ is the dual basis in $A(\alpha_{ij})$. We choose $B_{ij}$ to consist of the products (in any order) of the generators $E_{i}, E_{i+1}, \ldots, E_{j-1}$. It is easy to see that $\rho(b) = E_{ij}$ for the element $b = b_{ij} = E_{i}E_{i+1} \cdots E_{j-1}$ in $B_{ij}$, and $\rho(b) = 0$ if $b \in B_{ij}, b \neq b_{ij}$. Denote $x_{ij} = (b_{ij})^*$. To identify these $x_{ij}$ with those in Section 0 we have to verify the relations (0.11). As an algebra, $A$ is generated by the $x_{i} := x_{i,i+1}$ ($i = 1, \ldots, r$) subject to the quantum Serre relations. The relations (0.11) can be verified similarly to those between the $t_{ij}$ in [2], Section 3 (they also follow from the relations in [4]). Thus, $(\text{id} \otimes \rho)(R) = I + \sum_{i < j} x_{ij}E_{ij} = X$. Lemma 2.6 is proved. <

To complete the proof of Proposition 0.8, note that, for every $i$ we have $\psi_{i}(X) = (\psi_{i} \otimes \text{id}) \circ (\text{id} \otimes \rho)(R) = (\text{id} \otimes \rho) \circ (\psi_{i} \otimes \text{id})(R) = (\text{id} \otimes \rho)(e_{i})$, by (0.8). The formula (0.12) follows since $(\text{id} \otimes \rho)(\exp_{q_{ik}}(t_{k}E_{ik})) = I + t_{k}E_{ik,i,k+1}$ for all $k$. Proposition 0.8 is proved. <

We have the following obvious corollary of Theorem 2.3.

Corollary 2.7. For any $w \in W$ and $i, i' \in R(w)$ there is an isomorphism of skew-fields

$$R_{i}' : \mathcal{F}(P_{i}) \cong \mathcal{F}(P_{i'})$$

defined by $R_{i}'' := \overline{\psi}_{i'} \circ (\overline{\psi}_{i})^{-1}$.

The “transition maps” $R_{i}'$ lead to identities between quantum exponentials given by Corollary 0.10. To compute each $p_{k}$ in (0.14), it is enough (in principle) to do this for the following pairs:

$$i = (i, j, i, \ldots), \quad i' = (j, i, j, \ldots),$$

of the length $l$ each, where $l$ is the order of $s_{i}s_{j}$ in $W$. Recall that $l = 2, 3, 4$, or 6 for any Weyl group. In the following proposition, we compute $R_{i}'$ for these $i, i'$ with $l = 2, 3$, or 4 (when $l = 6$ the explicit expressions for $p_{k}$ are more complicated, so we do not present them here).

Proposition 2.8. Let $t_{1}, \ldots, t_{l}$ (resp. $t_{1}', \ldots, t_{l}'$) be standard generators of $P_{(iji\ldots)}$ (resp. $P_{(ji\ldots)}$). We denote $p_{k} = R_{(ji\ldots)}^{(iji\ldots)}(t_{k}')$ ($k = 1, \ldots, l$).
(a) If \( l = 2 \) then \((p_1, p_2) = (t_2, t_1)\).
(b) If \( l = 3 \) then \((p_1, p_2, p_3) = (t_2 t_3 (t_1 + t_3)^{-1}, t_1 + t_3, (t_1 + t_3)^{-1} t_1 t_2)\).
(c) If \( l = 4 \) and \( a_{ij} = -2, a_{ji} = -1 \) then \( p_1, p_2, p_3, p_4 \) are determined by the following equations:
\[
\begin{align*}
p_{2} p_{3} &= t_{1} t_{2} + t_{1} t_{4} + t_{3} t_{4}, & p_{2} p_{3} p_{4} &= t_{1} t_{2} t_{3}, \\
p_{2}^{2} p_{3} &= t_{1}^{2} t_{2} + (t_{1} + t_{3})^{2} t_{4}, & p_{1} p_{2}^{2} p_{3} &= t_{2} t_{3}^{2} t_{4}.
\end{align*}
\]
Proof. (a) By definition, \( \psi_{(ij)} : (x_i, x_j) \mapsto (t_1, t_2) \) and \( \psi_{(ji)} : (x_i, x_j) \mapsto (t_2', t_1') \). Thus, \( p_1 = t_2 \) and \( p_2 = t_1 \) as claimed.

Part (b) is proved in Section 0 (see Example 1).

(c) Let \( \mathcal{U}_{ij} \) be the subalgebra of \( \mathcal{U} \) generated by \( E_i \) and \( E_j \). Define \( \mathcal{B}_{ij} \) as the quotient algebra of \( \mathcal{U}_{ij} \) modulo the relations \( E_i^3 = E_j^2 = 0 \) (we keep the same notation for generators). It is easy to see ([8], or Section 3 below) that the following are all the defining relations in \( \mathcal{B}_{ij} \):
\[
E_i^3 = E_j^2 = E_j E_i E_j = 0, \quad E_i^2 E_j E_i = E_i E_j E_i^2.
\]
Using these relations, it is easy to prove that the homogeneous components \( \mathcal{B}_{ij}(\alpha_i + \alpha_j), \mathcal{B}_{ij}(2\alpha_i + \alpha_j), \) and \( \mathcal{B}_{ij}(2\alpha_i + 2\alpha_j) \) of \( \mathcal{B}_{ij} \) have the following bases: \( \{E_i E_j, E_j E_i E_i\} \) for \( \mathcal{B}_{ij}(\alpha_i + \alpha_j) \), \( \{E_i^2 E_j, E_j E_j E_i, E_j E_i^2\} \) for \( \mathcal{B}_{ij}(2\alpha_i + \alpha_j) \), and \( \{E_j E_i^2 E_j\} \) for \( \mathcal{B}_{ij}(2\alpha_i + 2\alpha_j) \).

Applying the projection \( \rho : \mathcal{U}_{ij} \rightarrow \mathcal{B}_{ij} \) to both sides of (0.14), we obtain the following equation in \( \mathcal{F}(P_{ij}) \otimes \mathcal{B}_{ij} \):
\[
(1 + p_1 E_j)(1 + p_2 E_i + \frac{p_2^2 E_i^2}{1 + q^2})(1 + p_3 E_j)(1 + p_4 E_i + \frac{p_4^2 E_i^2}{1 + q^2})
\]
\[
= (1 + t_1 E_i + \frac{t_1^2 E_i^2}{1 + q^2})(1 + t_2 E_j)(1 + t_3 E_i + \frac{t_3^2 E_i^2}{1 + q^2})(1 + t_4 E_j).
\]
(2.11)
The desired expressions for \( p_2 p_3, p_2 p_3 p_4, p_2^2 p_3, \) and \( p_1 p_2^2 p_3 \) can be obtained from (2.11) by comparing the coefficients of \( E_j E_j, E_i E_j E_i, E_i^2 E_j, \) and \( E_j E_i E_j \) respectively on both sides of (2.11). Proposition 2.9 is proved. \( \triangleright \)

Remark. Taking in the identities of Proposition 2.8 the homogeneous components of degrees \( \alpha_i + (1 - a_{ij}) \alpha_j \) and \( \alpha_j + (1 - a_{ji}) \alpha_i \) yields quantum Serre relations between \( E_i \) and \( E_j \).

We conclude this section by a proof of Proposition 0.11. For a skew-symmetric \( m \times m \)-matrix \( S = (S_{kl}) \) with integer entries let \( P_S \) be a \( k(q) \)-algebra generated by \( t_1, \ldots, t_m \) subject to the relations
\[
t_i t_k = q^{S_{ki}} t_k t_i.
\]
(2.12)
Note that \( P_i = P_{S(i)} \) where the matrix \( S(i) \) is defined in (0.16). Recall that two \( m \times m \)-matrices \( S \) and \( S' \) are called equivalent if there is a matrix \( T = T_{kl} \in SL_m(\mathbf{Z}) \) such that
$S' = TST^t$. It is easy to see that $\mathcal{F}(P_S) \cong \mathcal{F}(P_{S'})$ if $S$ and $S'$ are equivalent: one can choose such an isomorphism $\mathcal{F}(P_{S'}) \sim \mathcal{F}(P_S)$ by sending each generator $t'_k$ of $P_{S'}$ to the monomial $t_1^{T_{k,1}}t_2^{T_{k,2}}\cdots t_m^{T_{k,m}}$ in the generators of $P_S$. The converse statement was proved by A. Panov.

**Proposition 2.9 ([11], Theorem 2.19).** Let $S, S'$ be skew-symmetric $m \times m$ matrices with the integer entries. Then $\mathcal{F}(P_S) \cong \mathcal{F}(P_{S'})$ if and only if $S$ and $S'$ are equivalent.

Thus, Proposition 2.9 means that the existence of any isomorphism $R : \mathcal{F}(P_{S'}) \rightarrow \mathcal{F}(P_S)$ implies that of a monomial isomorphism $M : \mathcal{F}(P_{S'}) \rightarrow \mathcal{F}(P_S)$, that is, $M$ takes each generator $t'_k$ of $P_{S'}$ to a monomial in generators $t_1, \ldots, t_m$ of $P_S$. Taking $S = S(i), S' = S(i')$, and $R = R_i^1 : \mathcal{F}(P_V) \rightarrow \mathcal{F}(P_i)$ with $i, i' \in R(w)$ for some $w \in W$, we obtain, in particular, the statement of Proposition 0.11. Note that $R_i^1$ is not monomial in general.

One can prove that there exists a local monomial isomorphism $M = M_{i,i'} : \mathcal{F}(P_V) \rightarrow \mathcal{F}(P_i)$. Namely, for $i, i'$ of the form

$$i = (i_1, \ldots, i_{a-l}; i, j, i, \ldots; i_{a+1}, \ldots, i_m)$$

$$i' = (i_1, \ldots, i_{a-l}; j, i, j, \ldots; i_{a+1}, \ldots, i_m),$$

$M(t'_k) = t_k$ if $k \leq a - l$ or $k > a$, and each $M(t'_k)$ for $k = a - l + 1, \ldots, a$ depends only on $t_{a-l+1}, \ldots, t_a$. We will present such $M$ elsewhere.

By Proposition 2.9, the equivalence class of $S(i)$ for $i \in R(w)$ depends only on $w$. If we choose some representative $S(w)$ of this class then, by Theorem 2.3, there is an isomorphism

$$\mathcal{F}(A/I_w) \sim \mathcal{F}(P_{S(w)}).$$

(2.16)

The well-known normal form for skew-symmetric matrices shows that $S(w) = (S_{kl})$ can be chosen uniquely subject to the following requirements:

(i) $S_{kl} = 0$ unless $k + l \neq m + 1$;

(ii) there is a sequence $c_1, c_2, \ldots$ of nonnegative integers such that $S_{k,m+1-k} = c_1c_2\cdots c_k$

for $1 \leq k \leq \frac{w}{2}$.

It would be interesting to compute the invariants $c_1, \ldots, c_m$ in terms of $w$, and to find a direct way to describe the isomorphism (2.16).

3. Extremal vectors in $A$ and proof of Proposition 0.6

We retain terminology and notation of Section 2. Recall that $\alpha_1, \ldots, \alpha_r$ is the standard basis in $\mathbb{Z}_+^r$. We define a bilinear form in $\mathbb{Z}^r$ by the formula $(\alpha_i, \alpha_j) = C_{ij}$ for all $i, j$. 

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Let us fix \( \lambda = (l_1, \ldots, l_r) \in \mathbb{Z}_+^r \). For \( i = 1, \ldots, r \) define linear operators \( F_i = F_{i, \lambda} : \mathcal{A} \to \mathcal{A} \) by the formula:

\[
F_i \cdot x = \frac{v_i^{l_i} q^{-\gamma(\alpha_i)} xx_i - v_i^{-l_i} x_i x}{v_i - v_i^{-1}}
\]

for \( x \in \mathcal{A}(\gamma) \) (where \( v_i = q^{rac{C_i}{2}} \)).

We identify \( \lambda \) with a linear form on the coroot lattice \( \mathbb{Z} \alpha_1^\vee \oplus \cdots \oplus \mathbb{Z} \alpha_r^\vee \) defined by \( \lambda(\alpha_i^\vee) := l_i \) (recall that \( \alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)} \)). For each reduced \( \mathbf{i} = (i_1, \ldots, i_m) \) we define a sequence of integers \( a_1, \ldots, a_m \) by the formula

\[
a_1 = \lambda(\alpha_{i_1}^\vee), \ a_2 = \lambda(s_{i_1}(\alpha_{i_2}^\vee)), \ldots, a_m = \lambda(s_{i_1} s_{i_2} \cdots s_{i_{m-1}}(\alpha_{i_m}^\vee)).
\]

It is well-known that \( a_k \in \mathbb{Z}_+ \) for all \( k \).

Define the element \( v(\mathbf{i}) = v(\mathbf{i})^\lambda \in \mathcal{A} \) by:

\[
v(\mathbf{i}) = F_{i_m}^{a_m} F_{i_{m-1}}^{a_{m-1}} \cdots F_{i_1}^{a_1} 1.
\]

The following result refines Proposition 0.6.

**Theorem 3.1.** In the above notation, we have \( \psi_1(v(\mathbf{i})) = ct_1^{a_1} t_2^{a_2} \cdots t_m^{a_m} \) where \( c \in k(q), c \neq 0 \).

Proposition 0.6 follows by taking any \( \lambda \) with \( \lambda(\alpha_{i_1}^\vee) = l_{i_1} > 0 \) (and \( x := c^{-1} v(\mathbf{i}) \)).

**Proof of Theorem 3.1.** Let us reformulate our statement in terms of modules over the quantized enveloping algebra \( \mathbf{U} = U_q(\mathfrak{g}) \). The \( k(q) \)-algebra \( \mathbf{U} \) is generated by \( F_1, \ldots, F_r, E_1, \ldots, E_r \) and the invertible pairwise commuting elements \( K_1, \ldots, K_r \) subject to the following relations (see [8]):

\[
K_i E_j K_i^{-1} = q^{C_{ij}} E_j, \ K_i F_j K_i^{-1} = q^{-C_{ij}} E_j K_i, \ E_i F_j - F_j E_i = \delta_{ij} \frac{K - K^{-1}}{v_i - v_i^{-1}}; \quad (3.4)
\]

\[
\sum_{p+p' = 1 - a_{ij}} (-1)^p v_i^{pp'} E_i^{[p]} E_j E_i^{[p']} = 0, \quad \sum_{p+p' = 1 - a_{ij}} (-1)^p v_i^{pp'} F_i^{[p]} F_j F_i^{[p']} = 0. \quad (3.5)
\]

The relations (3.5) are quantum Serre relations; they hold for all \( i \neq j \) where \( E_i^{[n]} \) means the same as in (2.2), and \( v_i = q^{rac{C_i}{2}} \).

The between \( E_1, \ldots, E_r \) (the same relations between \( F_1, \ldots, F_r \)):

We identify the subalgebra generated by \( E_1, \ldots, E_r \) with \( \mathcal{U} \). Note that \( \mathbf{U} \) is a \( \mathbb{Z}' \)-graded algebra via \( \deg(K_i) = \deg(K_i^{-1}) = 0, \ \deg(E_i) = \alpha_i, \ \deg(F_i) = -\alpha_i \) for \( i = 1, \ldots, r \). Note also, that each triple \((E_i, F_i, K_i)\) generates the subalgebra in \( \mathbf{U} \) isomorphic to \( U_{v_i}(sl_2) \).
We will denote by the same symbol \( E_i \) the operator \( E_i : A \rightarrow A \) adjoint of the operator of the right multiplication \( E \rightarrow EE_i \) in \( \mathcal{U} \); for every \( x \in A \) the element \( E_i \cdot x \in A \) is defined by the equations \((E_i \cdot x)(E) = x(EE_i)\) for all \( E \in \mathcal{U} \). We also define the operator \( K_i : A \rightarrow A \) (depending on \( \lambda \)) by

\[
K_i \cdot x = K_{i, \lambda} \cdot x := q^{\lambda(\alpha_i) - (\gamma, \alpha_i)}x
\]

for all \( x \in A(\gamma) \), \( i = 1, \ldots, r \).

The following result is well-known, (for type \( A_r \) it can be found e.g., in [3]).

**Proposition 3.2.** For every \( \lambda \) as above, the operators \( F_i \) defined in (3.1) together with the \( K_i \) and \( E_i \), give rise to an action \( \mathcal{U} \times A \rightarrow A \).

Denote by \( V_\lambda \) the cyclic \( \mathcal{U} \)-submodule in \( A \) generated by the unit \( 1 \in A \). It is well-known (cf. [3], [8]) that \( V_\lambda \) is an integrable simple \( \mathcal{U} \)-module. Note also that the vector \( v = 1 \in V_\lambda \) is a highest weight vector of weight \( \lambda \) since \( E_i \cdot v = 0 \) and \( K_i(v) = q^{\lambda(\alpha_i)}v \) for all \( i \).

It is also known (see [8]), that the element \( v(i) \) given by (3.3) depends only on \( w \). Such elements are called extremal vectors in \( V_\lambda \). Denote \( i_k := (i_1, \ldots, i_k) \) for \( k = 1, \ldots, m \). It is also well-known that for all \( k \) we have

\[
F_{i_k} \cdot v(i_k) = 0, \quad E_{i_k}^a \cdot v(i_k) = c_{k;a}F_{i_{k-1}}^{a_{k-1}}v(i_{k-1}), \quad E_{i_k} \cdot v(i_{k-1}) = 0
\]

for some \( c_{k;a} \in k(q) \setminus \{0\} \), and \( a = 0, 1, \ldots, a_k \) (with the agreement \( v(i_0) := v \)).

In view of (2.2), Theorem 3.1 is equivalent to the following.

**Proposition 3.3.** There is a unique sequence \( b = (b_1, \ldots, b_m) \) such that

\[
E_{i_1}^{b_1} \cdots E_{i_m}^{b_m} \cdot v(i) = cv
\]

for some nonzero scalar \( c \in k(q) \), namely, \( (b_1, \ldots, b_m) = (a_1, \ldots, a_m) \), where \( a_1, \ldots, a_m \) are given by (3.2).

**Proof of Proposition 3.3.** We proceed by induction on \( m \).

Assume that our statement is true for every reduced expression of length < \( m \), in particular, for \( i_{m-1} \).

**Step 1.** Let us prove that the equality (3.8) implies that \( b_k = a_k \) for all \( k \) with \( i_k \neq i_m \).

We will use the following identity in \( \mathcal{U} \), which is a straightforward consequence of the relations (3.4):

\[
E_{i_1}^{b_1} E_{i_2}^{b_2} \cdots E_{i_m}^{b_m} F_{i_m}^{a_m} = \sum_{b'} F_{i_m}^{|b'| - |b| + a_m} E_{i_1}^{b_1'} E_{i_2}^{b_2'} \cdots E_{i_m}^{b_m'} p_{b'}
\]

(3.9)
where the sum is over all $b' = (b'_1, \ldots, b'_{m'}) \in \mathbb{Z}^m_+$ such that $b'_k = b_k$ if $i_k \neq i_m$, and $b'_k \leq b_k$ if $i_k = i_m$; each $p_{b'}$ is a Laurent polynomial of $K_{i_m}$, and $|b| = b_1 + \cdots + b_m$.

Using (3.9) and the fact that $v(i) = F^{a_m}_{i_m} \cdot v(i_{m-1})$, we rewrite (3.8) as follows:

$$cv = E^{b_1}_{i_1} \cdots E^{b_m}_{i_m} F^{a_m}_{i_m} \cdot v(i_{m-1}) = \sum F^{b'_1 - |b| + a_m}_{i_1} E^{b'_2}_{i_1} \cdots E^{b'_{m-1}}_{i_{m-1}} p_{b'} \cdot v(i_{m-1})$$

(3.10)

where the sum is over all $(b'_1, \ldots, b'_{m'})$ such that $b'_k = b_k$ whenever $i_k \neq i_m$.

It follows that, for some $b'$, we have

$$F^{b'_1 - |b| + a_m}_{i_1} E^{b'_2}_{i_1} \cdots E^{b'_{m-1}}_{i_{m-1}} p_{b'} \cdot v(i_{m-1}) = c' v$$

with $c' \in \mathbb{k}(q), c' \neq 0$. By (3.7), we have $|b'| - |b| + a_m = 0$ and $b_{m'} = 0$; also,

$$E^{b'_1}_{i_1} \cdots E^{b'_{m-1}}_{i_{m-1}} \cdot v(i_{m-1}) = c'' v$$

with $c'' \neq 0$. Remembering the inductive assumption, we see that $b'_k = a_k$ for all $k \leq m - 1$. Thus, $b_k = b'_k = a_k$ for all $k \leq m - 1$ such that $i_k \neq i_m$. This completes Step 1.

**Step 2.** Let us prove that $b_m = a_m$. If $i_k \neq i_m$ for $k = 1, \ldots, m - 1$ then the equality $b_m = a_m$ follows by comparing degrees. So we can assume that $i_k = i_m$ for some $k < m$.

Let $k < m$ be the maximal index such that $i_k = i_m$. Clearly, $k \leq m - 2$ since $i$ is reduced. By Step 1, we have $b_{k+1} = a_{k+1}, b_{k+2} = a_{k+2}, \ldots, b_{m-1} = a_{m-1}$. Combining this observation with (3.7), we can rewrite the left hand side of (3.10) as follows.

$$cv = dE^{b_1}_{i_1} \cdots E^{b_{m-1}}_{i_{m-1}} F^{a_{m-1}}_{i_{m-1}} \cdot v(i_{m-1}) = dE^{b_1}_{i_1} \cdots E^{b_{k+1}}_{i_{k+1}} E^{a_{k+1}}_{i_{k+1}} F^{a_{m-1}}_{i_{m-1}} \cdot v(i_{m-1})$$

for some $d \in \mathbb{k}(q), d \neq 0$.

Then, by the commutativity property $E_{i_l} F_{i_m} = F_{i_m} E_{i_l}$ for $k < l < m$, the previous expression is equal to

$$c v = d' E^{b_1}_{i_1} \cdots E^{b_{k}}_{i_{k}} F^{a_{m-1}}_{i_{m-1}} E^{a_{k+1}}_{i_{k+1}} F^{a_{m-1}}_{i_{m-1}} \cdot v(i_{m-1}) = d' E^{b_1}_{i_1} \cdots E^{b_k}_{i_k} F^{a_{m-1}}_{i_{m-1}} \cdot v(i_{m-1})$$

(3.11)

(we again used the property (3.7) of the extremal vectors). Since $i_k = i_m$, it follows that $F_{i_m} \cdot v(i_k) = 0$. Hence, the right hand side of (3.11) is zero unless $a_m - b_m = 0$. This completes Step 2.

**Step 3.** Now we are able to complete the proof. Since $b_m = a_m$, (3.7) and (3.11) imply that

$$c_m E^{b_1}_{i_1} \cdots E^{b_{m-1}}_{i_{m-1}} \cdot v(i_{m-1}) = cv$$

with some nonzero constants $c, c_m$. We conclude that $(b_1, \ldots, b_{m-1}) = (a_1, \ldots, a_{m-1})$ by the inductive assumption. Combining this with Step 2, we see that $(b_1, \ldots, b_m) = (a_1, \ldots, a_m)$. 

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Appendix. Skew-fields of fractions and skew polynomials

Let $A$ be an associative ring with unit without zero-divisors. As in [7], A.2, we say that $A$ satisfies the right Ore condition if $aA \cap bA \neq \{0\}$ for any non-zero $a, b \in A$. The set of right fractions $F(A)$ is defined as the set of all pairs $(a, b)$ with $a, b \in A, b \neq 0$ modulo the following equivalence relation: $(a, b) \sim (c, d)$ if there are $f, g \in A \setminus \{0\}$ such that $af = cg$ and $bf = dg$. The equivalence class of $(a, b)$ in $F(A)$ is denoted by $ab^{-1}$. The ring $A$ is naturally embedded into $F(A)$ via $a \mapsto (a,1)$. It is well known that if $A$ satisfies the right Ore condition then the addition and multiplication in $A$ extend to $F(A)$ so that $F(A)$ becomes a skew-field.

Now we suppose that $A$ is an algebra over a field $k$ with an increasing filtration $(k = A_0 \subset A_1 \subset \cdots)$, where each $A_k$ is a finite dimensional $k$-vector space, $A_k A_l \subset A_{k+l}$, and $A = \bigcup A_k$. We say that $A$ has polynomial growth if for all $n \geq 0$ we have $\dim A_n \leq p(n)$, where $p(x)$ is a polynomial. For the convenience of the reader, we will present a proof of the following well known lemma (see, e.g., [5]).

**Lemma A1.** Any algebra of polynomial growth without zero-divisors satisfies the right Ore condition.

**Proof.** Assume, on the contrary, that $aA \cap bA = \{0\}$ for some non-zero $a, b \in A$. Denote $I_n = I \cap A_n$ for any subspace $I \subset A$. Choose some $k$ such that $a, b \in A_k$. Then $(aA)_{n+k} \supset aA_n$ and $(bA)_{n+k} \supset bA_n$, which implies

$$\dim(aA)_{n+k} \geq \dim A_n, \quad \dim(bA)_{n+k} \geq \dim A_n .$$

On the other hand, since $aA \cap bA = \{0\}$, it follows that

$$\dim A_{n+k} \geq \dim(aA)_{n+k} + \dim(bA)_{n+k} \geq 2 \dim A_n$$

for all $n$. Iterating this inequality, we see that $\dim A_{mk} \geq 2^m$ for $m \geq 0$. This contradicts the condition that $A$ has polynomial growth. Lemma A1 is proved. \(\diamond\)

Lemma A1 implies that any subalgebra of an algebra $A$ of polynomial growth without zero-divisors also satisfies the right Ore condition.

In particular, consider the $k$-algebra $P$ of skew polynomials generated by $t_1, \ldots, t_m$ subject to the relations $t_l t_k = q_{lk} t_l t_k$ for $1 \leq k < l \leq m$, where the $q_{kl}$ are some non-zero elements of $k$. It is easy to see that $P$ has no zero-divisors and has polynomial growth with respect to the filtration ($k = P_0 \subset P_1 \subset \cdots$), where $P_n$ is the linear span of all monomials in $t_1, \ldots, t_m$ of degree $\leq n$. We see that every subalgebra $\mathcal{B}$ of $P$ satisfies the right Ore condition. Therefore, $F(\mathcal{B})$ is a skew subfield of $F(P)$. 

Proposition 3.3 and Theorem 3.1 are proved. \(\diamond\)
References

[1] A. Berenstein, S. Fomin, A. Zelevinsky, Parametrizations of canonical bases and totally positive matrices. To appear in Adv. in Math.

[2] A. Berenstein, A. Zelevinsky, String bases for quantum groups of type $A_r$, Advances in Soviet Math., 16, Part 1 (1993), 51–89.

[3] A. Berenstein, A. Zelevinsky, Canonical bases for the quantum group of type $A_r$ and piecewise-linear combinatorics. To appear in Duke Math. J..

[4] L. D. Faddeev, N. Yu. Reshetikhin, L. A. Takhtadzhyan, Quantization of Lie groups and Lie algebras. (Russian) Algebra i Analiz, 1 (1989), no. 1, 178–206.

[5] K. Iohara, F. Malikov, Rings of skew polynomials and Gel’fand-Kirillov conjecture for quantum groups, Commun. Math. Phys. 164 (1994), 217–238.

[6] A. Joseph, Sur une conjecture de Feigin. C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), no. 12, 1441–1444.

[7] A. Joseph, Quantum groups and their primitive ideals, Springer-Verlag, Berlin, 1995.

[8] G. Lusztig, Introduction to quantum groups, Birkhauser, Boston, 1993.

[9] G. Lusztig, Problems on canonical bases. Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991), 169–176, Proc. Sympos. Pure Math., 56, Part 2, Amer. Math. Soc., Providence, RI, 1994.

[10] S. Majid, Algebras and Hopf algebras in braided categories. Advances in Hopf algebras (Chicago, IL, 1992), 55–105, Lecture Notes in Pure and Appl. Math., 158, Dekker, New York, 1994.

[11] A. Morozov, L. Vinet, Free-field representation of group element for simple quantum group. Preprint ITEP-M3/94, CRM-2202, hep-th 9409093.

[12] A. Panov, Skew fields of twisted rational functions and the skew field of rational functions on $GL_q(n, K)$. St. Petersburg Math. J., 7 (1996), no. 1.