An Expansion of a Poset Hierarchy

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Abstract

This article extends a paper of Abraham and Bonnet which generalised the famous Hausdorff characterisation of the class of scattered linear orders. Abraham and Bonnet gave a poset hierarchy that characterised the class of scattered posets which do not have infinite antichains (abbreviated FAC for finite antichain condition). An antichain here is taken in the sense of incomparability. We define a larger poset hierarchy than that of Abraham and Bonnet, to include a broader class of “scattered” posets that we call $\kappa$-scattered. These posets cannot embed any order such that for every two subsets of size $< \kappa$, one being strictly less than the other, there is an element in between. If a linear order has this property and has size $\kappa$ we call this set $Q(\kappa)$. Such a set only exists when $\kappa^{<\kappa} = \kappa$. Partial orders with the property that for every $a < b$ the set $\{ x : a < x < b \}$ has size $\geq \kappa$ are called weakly $\kappa$-dense, and partial orders that do not have a weakly $\kappa$-dense subset are called strongly $\kappa$-scattered. We prove that our hierarchy includes all strongly $\kappa$-scattered FAC posets, and that the hierarchy is included in the class of all FAC $\kappa$-scattered posets. In addition, we prove that our hierarchy is in fact the closure of the class of all $\kappa$-well-founded linear orders under inversions, lexicographic sums and FAC weakenings. For $\kappa = \aleph_0$ our hierarchy agrees with the one from the Abraham-Bonnet theorem.  

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1 Introduction

Recall that a scattered order is one which does not embed the rationals. Hausdorff ([3], or see [2]) proved that the class of scattered linear orders is the least family of linear orderings which includes the ordinals and is closed under lexicographic sums and inversions. The paper [1] by Abraham and Bonnet proved that the class of scattered posets satisfying FAC (the finite antichain condition) is the least family of posets satisfying FAC which includes the well-founded posets and is closed under inversions, lexicographic sums and augmentations.

There are several routes for expansion on these results which centre around a generalisation of the concept of scattered to higher cardinalities. To this effect, one would consider a \( \kappa \)-scattered poset (or linear order) to be one which does not embed a \( \kappa \)-dense set. There are two definitions that one could give to a \( \kappa \)-dense set. The first was introduced by Hausdorff in 1908 as an \( \eta_\alpha \)-ordering for \( \kappa = \aleph_\alpha \). This is an order such that between any two subsets of size \( < \kappa \), one being strictly less than the other, there is an element in between. Orders with this property are here called strongly \( \kappa \)-dense. When an \( \eta_\alpha \)-ordering is linear and also has size \( \kappa \), we call it \( \mathbb{Q}(\kappa) \).

The other definition of \( \kappa \)-dense is a strictly weaker one in which between every two elements there is a subset of size \( \kappa \). We call this notion weakly \( \kappa \)-dense. Using either of the two definitions for \( \kappa \)-scattered, which we will call weakly \( \kappa \)-scattered (not embedding a strongly \( \kappa \)-dense set) and strongly \( \kappa \)-scattered (not embedding even a weakly \( \kappa \)-dense set), we can attempt to expand the characterisation results on linear orders or FAC posets. Note that the class of strongly \( \kappa \)-scattered is included in the class of weakly \( \kappa \)-scattered orders.

This paper builds on [1] and extends its results. As in [1], a class of posets

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is built in a hierarchical way so that for any regular $\kappa$ we have that $^\kappa \mathcal{H}$ is the least family of posets satisfying FAC which includes the $\kappa$-well founded posets and is closed under inversions, lexicographic sums and augmentations. We then close this class under FAC weakenings (the dual notion to augmentations, but retaining the FAC) to obtain the class $^\kappa \mathcal{H}^*$. We prove that class $^\kappa \mathcal{H}^*$ contains all strongly $\kappa$-scattered posets and is contained in the class of all weakly $\kappa$-scattered FAC posets. For $\kappa = \aleph_0$ where the two notions of scattered agree the two hierarchies agree, as follows by the Abraham-Bonnet theorem, and are both equal to the class of FAC scattered posets.

It is also shown that the class $^\kappa \mathcal{H}^*$ can be constructed in a simpler way, by starting with the $\kappa$-well founded linear orders and closing under inversions, lexicographic sums and FAC weakenings. It is proved that this is exactly the same class as the one constructed by posets.

A reader familiar with [1] may at this point wonder why it is that for $\kappa > \aleph_0$ we do not obtain the complete analogue of the Abraham-Bonnet theorem. There are two main difficulties, apart from the fact that the notions of weakly and strongly $\kappa$-scattered for $\kappa > \aleph_0$ are distinct, as opposed to what happens at $\kappa = \aleph_0$. The first one is that it is not necessarily the case that if all augmentations of a poset are weakly or strongly $\kappa$-scattered then the poset has the FAC, or at least we have not been able to prove this. The other difficulty is that we do not know how to prove that FAC posets which are not in the hierarchy defined above actually embed a strongly $\kappa$-dense set, although we can prove that they have an augmentation that embeds a weakly $\kappa$-dense subset.

It remains unknown whether every weakly $\kappa$-scattered poset is in the hierarchy $^\kappa \mathcal{H}^*$ or if $^\kappa \mathcal{H}^*$ and $^\kappa \mathcal{H}$ are in general equal. However, $^\kappa \mathcal{H}$ does contain examples of weakly $\kappa$-dense posets (as we will show in the final section), so it cannot be the case that $^\kappa \mathcal{H}$ only contains strongly $\kappa$-scattered posets.
2 On $\kappa$-scattered Posets

We start by explaining how Abraham and Bonnet’s theorem extends Hausdorff’s theorem. We first need several definitions. In this paper, we use ‘order’ to denote a ‘partial order’, and whenever we deal with linear orders we specify this.

**Definition 2.1.** (1) A (partial) order $P$ embeds an order $Q$ iff there is an order preserving one-to-one function from $Q$ to $P$.

(2) An order is said to be scattered iff it does not embed the rationals, $\mathbb{Q}$, with their usual ordering.

(3) If $(I, \leq_I)$ is a partial order and $\bar{P} = \langle (P_i, \leq_i) : i \in I \rangle$ is a sequence of partial orders, the lexicographic sum of $\bar{P}$ is the order whose universe is $\bigcup_{i \in I} P_i$, ordered by letting $p \leq q$ if and only if there exists $i \in I$ such that $p \leq_i q$ or there exists $i <_I j$ with $p \in P_i$ and $q \in P_j$.

(4) A poset $(P', \leq')$ is an augmentation of a poset $(P, \leq)$ if and only if $P = P'$, and for all $p, q \in P$, if $p \leq q$ then $p \leq' q$. We also say that $P$ is a weakening of $P'$.

(5) If $P$ is a subposet of $Q$ in which all relevant $Q$-relations are kept then $P$ is said to be a restriction of $Q$ to $P$ (written $Q \upharpoonright P$). In particular, for any $p_1, p_2 \in P$ we have $p_1 <_P p_2$ iff $p_1 <_Q p_2$.

(6) We say that a poset $P$ is $\kappa$-well founded if and only if $P$ does not have any decreasing sequences of size $\kappa$.

Note that the notion of $\kappa$-well founded was introduced by Zaguia as the extraction property for $\kappa$, see [1, §4.11.3].

The relevant theorem of Hausdorff in [5] (see also [2]) states that the scattered linear orders are exactly the closure of the class of all well orderings under inversions and lexicographic sums.

**Notation 2.2.** 1. Let $p \perp q$ denote that $p$ is incomparable with $q$.

2. We will say antichain when we mean an incomparable antichain, that is, a subset whose elements are pairwise incomparable.
3. Let FAC denote the finite antichain condition, so in an FAC poset all antichains are finite.

4. Similarly, let $\kappa$-AC denote the $\kappa$-antichain condition. That is, a poset has the $\kappa$-AC if and only if it does not have an antichain of size $\kappa$.

5. If $(P, \leq)$ is a poset and $S, T \subseteq P$, we write $S \leq T$ iff for all $s \in S$ and $t \in T$, we have $s \leq t$.

Clearly, linear orders are a special case of FAC posets. Abraham and Bonnet proved that the class of scattered FAC posets is the closure of the class of well founded FAC posets under augmentations, inverses and lexicographic sums. Some of the main tools they used were Hessenberg based operations on ordinals and the notion of the antichain rank $\rho$. We shall not need to reintroduce Hessenberg operation, and as for the antichain rank, basically it is a function that determines the length of the set of antichains in any given FAC poset. We include the definition given in [1] here; it will be needed in §3.

**Definition 2.3.** For any FAC poset $P$, let $(\mathcal{A}(P), \supset)$ be the poset of all non-empty antichains of $P$ under inverse inclusion. Since this is a well-founded poset, we can define the usual rank function on it which we will call the antichain rank of $P$ and denote by $\text{rk}_{\mathcal{A}}(P) \overset{\text{def}}{=} \text{rk}(\mathcal{A}(P))$.

Hausdorff’s theorem is in fact the restriction of the Abraham-Bonnet theorem to antichain rank 1. Let us now go on to define what we mean by weakly $\kappa$-scattered, by first defining the dual notion, strongly $\kappa$-dense.

**Definition 2.4.** (1) For a cardinal $\kappa \geq \aleph_0$ we say an order $(P, <^*)$ is strongly $\kappa$-dense iff

$$\forall S, T \in [P]^{<\kappa} \left[ S <^* T \implies (\exists x)S <^* x <^* T \right], \quad (*)^\kappa.$$  

(2) We denote by $Q(\kappa)$ a strongly $\kappa$-dense linear order of size $\kappa$ whenever this set exists and is unique up to isomorphism.

A linear order which satisfies $(*)^\kappa$ is also known as an $\eta_\alpha$-ordering for $\kappa = \aleph_\alpha$. Hausdorff proved in [3] that such an ordering exists for all regular cardinals.
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\[ \kappa \]. However, it can only be shown that these sets can have size \( \kappa \) when \( \kappa \) satisfies this stronger property, \( \kappa = \kappa^{< \kappa} \).

We know that the countable version of this set exists, namely the rationals satisfy this for \( \kappa = \aleph_0 \). It follows from Shelah’s work on the existence of saturated models for unstable theories (see [7], Theorem VIII 4.7) that \( Q(\kappa) \) exists iff \( \kappa^{< \kappa} = \kappa \). The specific instance of this result for a dense linear order with no endpoints is well-known.

Sierpinski showed that for general \( \kappa \) satisfying \( \kappa^{< \kappa} = \kappa \) the order \( Q(\kappa) \) may be constructed by induction (see [6] for details). The same proof also gives a more general construction of an order of size \( \lambda \) which satisfies \( (\star)^\kappa \) where \( \kappa \) is a regular cardinal and \( \kappa^{< \kappa} = \lambda \).

The obvious way to generalise the notion of scattered would be to say that an order is \( \kappa \)-scattered iff it does not embed the unique linear order \( Q(\kappa) \). However, since this set only exists given some strong cardinal arithmetic assumptions, Stevo Todorčević suggested that it is more natural to say that an order is \( \kappa \)-scattered iff it does not embed a strongly \( \kappa \)-dense set of any size. In this way, the notion makes sense whenever \( \kappa \) is a regular cardinal.

The next claim shows that whenever \( Q(\kappa) \) exists, the properties of being strongly \( \kappa \)-dense and embedding \( Q(\kappa) \) are equivalent.

**Claim 2.5.** Suppose \( Q(\kappa) \) exists and \( P \) is a strongly \( \kappa \)-dense order. Then there is \( Q \subseteq P \) such that \( Q \) is isomorphic to \( Q(\kappa) \).

**Proof of the Claim.** As shown above, \( Q(\kappa) \) exists whenever \( \kappa = \kappa^{< \kappa} \). Since \( P \) is strongly \( \kappa \)-dense, it follows that \( |P| \geq \kappa \). Let \( Q_0 \) be any subset of \( P \) of size \( \kappa \). In particular, \( |Q_0|^{< \kappa} = \kappa \).

Let \( \{A_\alpha, B_\alpha : \alpha < \kappa \} \) enumerate all \( A, B \in [Q_0]^{< \kappa} \) such that \( A < B \). For each \( \alpha \), there exists \( x_\alpha \in P \) such that \( A_\alpha < x_\alpha < B_\alpha \) by the fact that \( P \) is strongly \( \kappa \)-dense. Let \( Q_1 = Q_0 \cup \{x_\alpha : \alpha < \kappa \} \).

Repeat this process inductively, creating \( Q_\beta \) at each step \( \beta < \kappa \) taking the union at limit stages. Let \( Q = \bigcup_{\beta < \kappa} Q_\beta \).

We will show that the \( Q \) we have constructed is isomorphic to \( Q(\kappa) \). First at each successor stage \( \beta + 1 < \kappa \) we have added only \( \kappa \) many elements to
\( Q_\beta \) so \(|Q_{\beta+1}| = \kappa \). Note that no new elements are added at limit stages. Therefore, \( Q \) is the union of \( \kappa \) many sets of size \( \kappa \) so itself has size \( \kappa \).

For \( A, B \in Q \) such that \(|A|, |B| < \kappa \) and \( A < B \) there exists \( \beta < \kappa \) such that \( A, B \subseteq Q_\beta \) as \( \kappa \) is regular. By the construction, there exists \( x \in Q_{\beta+1} \) such that \( A < x < B \). Thus, \( Q \) is strongly \( \kappa \)-dense.

Thus for example, the notion of strongly \( \kappa \)-dense for \( \kappa = \aleph_0 \) agrees with the definition of dense as \( \mathbb{Q}(\aleph_0) = \mathbb{Q} \) and so an order which embeds a strongly \( \aleph_0 \)-dense set also embeds the rationals.

The following fact about strongly \( \kappa \)-dense sets will be useful to us.

**Lemma 2.6.** Any strongly \( \kappa \)-dense set necessarily has a \( \kappa \)-decreasing sequence and a \( \kappa \)-increasing sequence.

**Proof of the Lemma.** Let \( P \) be a strongly \( \kappa \)-dense set. We prove this for a \( \kappa \)-decreasing sequence, the proof for a \( \kappa \)-increasing sequence is similar. By induction on \( \alpha < \kappa \), we construct \( A = \{q_\alpha : \alpha < \kappa \} \) with \( q_\alpha >^* q_\beta \) if \( \alpha < \beta \). Let \( q_0 \) be any element of \( P \).

For any \( 0 < \alpha < \kappa \), assume \( q_\beta \) is defined for all \( \beta < \alpha \). Let \( T = \{q_\beta : \beta < \alpha \} \) and \( S = \emptyset \). Choose \( q_\alpha \) to be any \( q \) such that \( S <^* q <^* T \). \( \square \)

Now that we have proved the relevant properties of strongly \( \kappa \)-dense sets, we may turn to their opposite, the idea of weakly \( \kappa \)-scattered sets.

**Definition 2.7.** Suppose that \( \kappa \) is a regular cardinal. We say that a partial order is weakly \( \kappa \)-scattered if and only if it does not embed any strongly \( \kappa \)-dense set. We may omit the adjective ‘weakly’ when discussing this notion.

Hence for \( \kappa \) as in Definition 2.7 all orders of size \( < \kappa \), in particular finite orders, are \( \kappa \)-scattered. If \( \kappa > \aleph_0 \), then there are orders which are \( \kappa \)-scattered and not scattered, for example the rationals. Similarly, if \( \kappa_1 > \kappa_2 \) both satisfy the assumptions of Definition 2.7 then there are orders which are \( \kappa_1 \)-scattered without being \( \kappa_2 \)-scattered. In the other direction, every \( \kappa_2 \)-scattered order is \( \kappa_1 \)-scattered, as we can see that in these circumstances \( \mathbb{Q}(\kappa_2) \) embeds into \( \mathbb{Q}(\kappa_1) \) whenever these sets exist.
Our aim is to consider the Abraham-Bonnet theorem for \( \kappa \)-scattered partial orders which satisfy FAC for regular cardinals \( \kappa \) with \( \kappa \geq \aleph_0 \). We shall start by showing that strongly \( \kappa \)-dense sets have a property which might seem stronger than \((\ast)^\kappa\), but as the claim shows, is actually equivalent to it. The proof is similar to that of Lemma 2.6.

**Claim 2.8.** Suppose \( P \) is a poset satisfying \((\ast)^\kappa\). Then for all \( S, T \subseteq P \) with \(|S|, |T| < \kappa \) and \( S <_P T \) we have \(|\{q \in P : S <_P q <_P T\}| \geq \kappa \).

Moreover, if \( \kappa = \kappa^{<\kappa} \) and \( P \) is a linear order then \( P \) restricted to the suborder \( Q = \{q \in P : S <_P q <_P T\} \) is isomorphic to \( Q(\kappa) \).

**Proof of the Claim.** Let \( S, T \) satisfy the assumptions of the claim. By induction on \( \alpha < \kappa \), we construct \( \{q_\alpha : \alpha < \kappa\} \) such that \( \beta < \alpha \) implies \( q_\beta <_P q_\alpha \) and \( S <_P q_\alpha <_P T \). Let \( q_0 \) be any such \( q \), which exists by the assumption \((\ast)^\kappa\). Assume we are given \( \{q_\beta : \beta < \alpha\} \), by the induction hypothesis. Apply \((\ast)^\kappa\) to \( T^* = T \cup \{q_\beta : \beta < \alpha\} \) and \( S \), noticing that \( |T^*| < \kappa \) and \( S <_P T^* \) to obtain \( q_\alpha \).

If \( P \) satisfies \((\ast)^\kappa\), then the suborder \( Q \) as defined in the claim must also satisfy \((\ast)^\kappa\). Therefore if \( \kappa = \kappa^{<\kappa} \) then \( Q \) is a linear order of size \( \kappa \) which satisfies \((\ast)^\kappa\) and hence isomorphic to \( Q(\kappa) \).

In our main result we shall use a weaker notion of \( \kappa \)-density as well, so we define it here.

**Definition 2.9.** If a linear order \( L \) satisfies the property

\[
|L| \geq 2 \text{ and } (\forall s, t \in L)[s < t \implies |\{x : s < x < t\}| \geq \kappa]
\]

then we say that \( L \) is weakly \( \kappa \)-dense. We may omit the adjective ‘weakly’ when discussing this notion. (The first clause of the property is included to avoid trivialities.)

An order that does not embed a weakly \( \kappa \)-dense order is called strongly \( \kappa \)-scattered.

For \( \kappa > \aleph_0 \) it is easy to construct an example of a \( \kappa \)-dense linear order that is not strongly \( \kappa \)-dense, moreover there are \( \kappa \)-dense linear orders that are
\(\kappa\)-scattered and ones that do not even have a decreasing \(\kappa\)-sequence. See \[4\] Note that at \(\kappa = \aleph_0\), the two definitions of \(\kappa\)-density agree.

If an order is not \(\kappa\)-scattered for \(\kappa = \kappa^{<\kappa}\) then it embeds a copy of \(Q(\kappa)\) so clearly it has a suborder of size \(\kappa\) that is not \(\kappa\)-scattered. For future purposes we note that a similar statement is true about orders that are strongly \(\kappa\)-scattered for any cardinal \(\kappa\).

**Claim 2.10.** Suppose that \(P\) is an order that is not strongly \(\kappa\)-scattered for \(\kappa = \kappa^{<\kappa}\). Then \(P\) has a suborder of size \(\kappa\) that is not strongly \(\kappa\)-scattered.

**Proof of the Claim.** We shall define a suborder \(Q = \bigcup_{n<\omega} Q_n\) of \(P\) by defining \(Q_n\) by induction on \(n\). Let \(Q_0\) be any two-element linear suborder of \(P\), which exists by the definition of weak \(\kappa\)-density. Given \(Q_n\) of size \(\leq \kappa\) let us choose for any \(a < Q_n b\) a set \(S^n_{a,b} \subseteq P\) of size \(\kappa\) such that \(a < S^n_{a,b} < b\) and let \(Q_{n+1} = \bigcup_{a < Q_n b} S^n_{a,b} \cup Q_n\). It is easy to see that \(Q\) is as required. \(\Box\)

It is a well-known theorem of Bonnet and Pouzet and independently Galvin and McKenzie (see \[3\]), that every scattered partial order has a scattered linear extension. An important ingredient in the Abraham-Bonnet theorem is a lemma which says that an FAC partial order is scattered iff all its augmentations are scattered. In our situation, we shall not be able to get such a neat equivalence, but a chain of implications instead. To prove the mentioned equivalence, Abraham and Bonnet use a particular claim which relies heavily on the fact that the lexicographic sum along \(0 < 1\) (i.e. the union) of two scattered partial orders is still scattered. In our circumstances we need the following claim.

**Claim 2.11.** Assume that \(\kappa\) is a regular cardinal. Suppose that \((D, <_D)\) is a poset of size \(\kappa\) and \(D = \bigcup_{i<\kappa} D_i\) for some \(i^* < \kappa\), and each \(D_i\) is \(\kappa\)-scattered. Then \(D\) is \(\kappa\)-scattered.

**Proof of the Claim.** The proof is by induction on \(i^* < \kappa\). For \(i < i^*\) let \(R_i \overset{\text{def}}{=} \bigcup_{j<i} D_j\). By the induction hypothesis, each \(R_i\) is \(\kappa\)-scattered, and by definition, we have \(D = \bigcup_{i<i^*} R_i\), while \(j < i < i^*\) implies \(R_j \subseteq R_i\). Suppose for contradiction that \(D\) is not \(\kappa\)-scattered. As we may shrink \(D\), without loss of generality we will assume that \(D\) is strongly \(\kappa\)-dense. By
induction on $\zeta < \kappa$, we define $i(\zeta), S_\zeta, T_\zeta$ as in the following, if possible, and we stop at the first ordinal $\zeta^*$ for which such a choice is not possible.

$i(0)$. Let $i(0)$ be the first $i$ such that $R_i$ is of size $\geq \kappa$, which exists as $\kappa$ is regular. As $R_i(0)$ is $\kappa$-scattered, yet it has size $\geq \kappa$, we can find $S_0, T_0$ such that they are both subsets of $R_i(0)$ of size $< \kappa$, and $S_0 <_D T_0$, but for no $x \in R_i(0)$ do we have $S_0 <_D x <_D T_0$.

$i(\zeta + 1)$. Given $i(\zeta), S_\zeta, T_\zeta$ such that $S_\zeta$ and $T_\zeta$ are subsets of $R_i(\zeta)$ of size $< \kappa$ with $S_\zeta <_D T_\zeta$, yet for no $x \in R_i(\zeta)$ do we have $S_\zeta <_D x <_D T_\zeta$. As $D$ is strongly $\kappa$-dense we have that $B_\zeta \overset{\text{def}}{=} \{x \in D : S_\zeta <_D x <_D T_\zeta\}$ has size at least $\kappa$ by Claim 2.8. By the regularity of $\kappa$, there is $i < i^*$ such that $B_\zeta \cap R_i$ has size at least $\kappa$. As $B_\zeta \cap R_i = \emptyset$, we have that first such $i$ is greater than $i(\zeta)$. We let this $i$ be $i(\zeta + 1)$.

Now $B_\zeta \cap R_i(\zeta + 1)$ is $\kappa$-scattered, hence there are $S_{\zeta+1}$ and $T_{\zeta+1}$ exemplifying this. In other words, they are both subsets of $B_\zeta \cap R_i(\zeta + 1)$ of size $< \kappa$, and $S_{\zeta+1} <_D T_{\zeta+1}$, while for no $x \in B_\zeta \cap R_i(\zeta + 1)$ do we have $S_{\zeta+1} <_D x <_D T_{\zeta+1}$.

$i(\zeta)$ for $\zeta$ limit. Let $i^+ \overset{\text{def}}{=} \sup_{\zeta < \zeta} i(\zeta)$. Hence, either $i^+ = i^*$, in which case we stop the induction, or $i^+ < i^*$, in which case we let $i(\zeta) \overset{\text{def}}{=} i^+$ and $S_\zeta \overset{\text{def}}{=} \bigcup_{\zeta < \zeta} S_\xi$, and similarly for $T_\zeta$.

Notice that our induction must stop at some limit stage $\zeta^* < \kappa$ as $i^* < \kappa$.

Now let $S \overset{\text{def}}{=} \bigcup_{\zeta < \zeta^*} S_\zeta$, and similarly for $T$. By the construction, it follows that $S <_D T$. Hence, there is $x \in D$ with $S <_D x <_D T$. But then $x \in R_i(\zeta + 1)$ for some $\zeta$ (noticing that $D = \bigcup_{\zeta < \zeta^*} R_i(\zeta + 1)$ as the $R_i$’s are increasing). Therefore, $S_{\zeta+1} <_D x <_D T_{\zeta+1}$ and $x \in R_i(\zeta + 1)$, a contradiction. □

The analogue of the above claim is not true for strongly $\kappa$-scattered posets, even when only the union of $\aleph_0$ strongly $\kappa$-scattered posets is considered; this follows from the example in §4, see Claim 4.1. However a weaker claim is true.

Claim 2.12. The union of two strongly $\kappa$-scattered is strongly $\kappa$-scattered. Consequently, the union of any finite number of such posets is strongly $\kappa$-scattered.
Proof of the Claim. It clearly suffices to prove the first statement. So assume that \(D_1\) and \(D_2\) are strongly \(\kappa\)-scattered posets and assume that \(D = D_1 \cup D_2\) embeds a weakly \(\kappa\)-dense poset. By thinning out the sets, we can assume that \(D\) itself is weakly \(\kappa\)-dense. Then for all \(s, t \in D\) we have that the set \((s, t)_D = \{x \in D : s <_D x <_D t\}\) has size \(\geq \kappa\).

On the other hand, there exist \(s, t \in D_1\) such that \(|(s, t)_{D_1}| < \kappa\) as \(D_1\) is strongly \(\kappa\)-scattered. Consider \(T \overset{\text{def}}{=} (s, t)_{D_1} \upharpoonright D_2\). As \(T \subseteq D_2\), \(T\) is not weakly \(\kappa\)-dense, and hence there must be \(u <_{D_2} v \in T\) such that \(|(u, v)_{D_2}| < \kappa\). But then \((u, v)_D = (u, v)_{D_2} \cup (u, v)_{D_1} \upharpoonright D_1\) must have size \(< \kappa\), as \((u, v)_{D_1} \subseteq (s, t)_{D_1}\). This is a contradiction. \(\square\)

Now we can prove the following lemma which holds both for weakly and strongly \(\kappa\)-scattered posets. The version needed for the strongly \(\kappa\)-scattered case is to be read from within the square brackets.

Lemma 2.13. Assume that \(\kappa\) is a regular cardinal. For any poset \(P\), we have \((1) \implies (2) \implies (3)\):

1. \(P\) is [strongly] \(\kappa\)-scattered and satisfies FAC,
2. Every augmentation of \(P\) is [strongly] \(\kappa\)-scattered,
3. \(P\) is [strongly] \(\kappa\)-scattered and satisfies \(\lambda\)-AC where \(\lambda = \kappa^{<\kappa}\).

Proof of the Lemma. Let \(\kappa\) be as in the assumptions of the lemma.

(1) \(\implies\) (2) Assume the contrary; \(P\) is [strongly] \(\kappa\)-scattered and satisfies FAC, but \(P'\) is an augmentation of \(P\) which is not [strongly] \(\kappa\)-scattered. This implies that the size of \(P\) and hence \(P'\) is at least \(\kappa\).

For the next subclaim, let \((\bot, q)^S\) be the set of all elements of \(S\) that are incomparable with \(q\) in \(S\).

Subclaim 2.14. Suppose that \(S \subseteq P\) and \(A, B \subseteq S\) are such that \(A <_S B\), \(|A|, |B| < \kappa\) \([|A|, |B| = 1]\) and \((A, B)_S = \emptyset\) \([|(A, B)_S| < \kappa]\) while with \(S' = P' \upharpoonright S\) we have that \((A, B)_{S'}\) is not [strongly] \(\kappa\)-scattered \([|(A, B)_{S'}| = \kappa]\). Then there is \(q \in A \cup B\) such that \((A, B)_{S'} \upharpoonright (\bot, q)^S\) is not [strongly] \(\kappa\)-scattered.
Proof of the Subclaim. Let $A, B$ and $S$ be as in the assumptions. Let

$$D_A = \{c \in S : A <_{S'} c <_{S'} B \text{ and } a \perp_S c \text{ for some } a \in A\}$$
$$D_B = \{c \in S : A <_{S'} c <_{S'} B \text{ and } b \perp_S c \text{ for some } b \in B\},$$

so that $D_A \cup D_B = (A, B)_{S'}$. Since the union of two [strongly] $\kappa$-scattered posets is itself [strongly] $\kappa$-scattered by Claim 2.11 [Claim 2.12], then either $S' \upharpoonright D_A$ or $S' \upharpoonright D_B$ is not [strongly] $\kappa$-scattered. [Let $q$ be the unique element of $A$ or of $B$, depending on which of the two sets is not strongly $\kappa$-scattered, so finishing the proof in this case].

Now notice that

$$D_A = \bigcup_{a \in A} D_a,$$

where $D_a \overset{\text{def}}{=} \{c \in S : A <_{S'} c <_{S'} B \text{ and } a \perp_S c\}$, and similarly for $D_B$. Again by Claim 2.11 there is either $a \in A$ or $b \in B$ such that $D_a$ or $D_b$ is not $\kappa$-scattered. Let $q$ be either $a$ or $b$, whichever gives us the non-$\kappa$-scattered poset. Hence, $D_q = (A, B)_{S'} \upharpoonright (\perp q)^S$ is not $\kappa$-scattered. 

\[\square\]

Suppose that $S \subseteq P$ is such that $S' = P' \upharpoonright S$ is strongly [weakly] $\kappa$-dense. Such an $S$ exists because $P'$ embeds a strongly [weakly] $\kappa$-dense subset. Since $P \upharpoonright S$ is [strongly] $\kappa$-scattered and has size $\kappa$ there must be $A, B \subseteq S$ with $A <_P B$ and $|A|, |B| < \kappa$ with the property that the set $(A, B)_S$ is empty, as otherwise the condition $(\ast)$ would be satisfied by $S$ [there must be $A, B$ such that $|A|, |B| = 1$ and $|(A, B)_S| < \kappa$. By Claim 2.8 we must have $|\{c \in S' : A <_{S'} c <_{S'} B \text{ and } c \perp_S q\}| \geq \kappa$ [the analogue for strongly $\kappa$-scattered holds by the choice of $A, B$]. By induction on $n < \omega$ we shall choose $A_n, B_n, q_n, S_n$ so that

1. $A_0 = A, B_0 = B, q_0 = q, S_0 = S$,
2. $|A_n|, |B_n| < \kappa$ [\(A_n = \{a_n\}, B_n = \{b_n\}\)], $A_n, B_n \subseteq S_n$,
3. $S_n$ is [strongly] $\kappa$-scattered while $S'_n = P' \upharpoonright S_n$ is not,
4. $A_n <_{S'_n} B_n$ and $(A_n, B_n)_{S_n} = \emptyset$ [(a\(n, b_n)_{S_n} < \kappa \text{ and } |(a_n, b_n)_{S'_n}| \geq \kappa)],
5. $q_n \in A_n \cup B_n$ and $(A_n, B_n)_{S'_n} \upharpoonright \bigcap_{k \leq n} (\perp q_k)^{S_n}$ is not $\kappa$-scattered [(a\(n, b_n)_{S'} \upharpoonright (\perp q_n)^S$ is not strongly $\kappa$-scattered)].
6. \( A_{n+1} \cup B_{n+1} \subseteq (\perp q_n)^{S_n} \),
7. \( A_n < s'_n \ A_{n+1} < s'_n \ B_{n+1} < s'_n B_n \).

To start the induction we use the choices already made. Suppose that we are at the stage \( n + 1 \) of the induction. Since by the induction hypothesis \((A_n, B_n) s'_n \mid \bigcap_{\beta < n} (\perp q_{k})^{S_{n}}\) is not [strongly] \( \kappa \)-scattered, it includes an order \( T'_{n+1} \) which is strongly [weakly] \( \kappa \)-dense. Let \( T_{n+1} = P \mid T'_{n+1} \). Since \( T_{n+1} \) is [strongly] \( \kappa \)-scattered it does not satisfy \((\ast)^{\kappa}\) [it is not weakly \( \kappa \)-dense] and hence there are \( A_{n+1}, B_{n+1} \subseteq T_{n+1} \) such that

\[
|A_{n+1}|, |B_{n+1}| < \kappa \ [A_{n+1} = \{a_{n+1}\}, B_{n+1} = \{b_{n+1}\}] \text{ and } A_{n+1} <_{T_{n+1}} B_{n+1}
\]

but

\[
(A_{n+1}, B_{n+1})_{T_{n+1}} = \emptyset \ [|(a_{n+1}, b_{n+1})_{T_{n+1}}| < \kappa].
\]

Let \( S_{n+1} = S_n \cup T_{n+1} \) and hence \( S'_{n+1} \supseteq S'_n \cup T'_{n+1} = P' \mid S_{n+1} \).

We have \( A_{n+1}, B_{n+1} \subseteq S_{n+1} \) and \( A_n < s'_n \ A_{n+1} < s'_n \ B_{n+1} < s'_n B_n \). Also \( S_{n+1} \) as the union of two [strongly] \( \kappa \)-scattered orders is [strongly] \( \kappa \)-scattered by Claim 2.11 [Claim 2.12] while \( S'_{n+1} \) is not as it includes \( T'_{n+1} \) which is strongly [weakly] \( \kappa \)-dense. Note also that \( A_{n+1} \cup B_{n+1} \subseteq \bigcap_{k \leq n} (\perp q_k)^{S_n} \).

At any rate, Subclaim 2.14 applies to \( A_{n+1}, B_{n+1} \) and \( S_{n+1} \) in place of \( A, B \) and \( S \). Hence we can find \( q_{n+1} \in A_{n+1} \cup B_{n+1} \) such that

\[
(A_{n+1}, B_{n+1})_{S'_{n+1}} \mid (\perp q_{n+1})^{S_{n+1}} = (A_{n+1}, B_{n+1})_{S'_n} \mid \bigcap_{k \leq n} (\perp q_k)^{S_{n+1}}
\]

(as for \( k \leq n \) we have that \( (A_{n+1}, B_{n+1})_{S'_{n+1}} \subseteq (A_n, B_n)_{S'_n} \subseteq (\perp q_k)^{S_k} \subseteq (\perp q_k)^{S_{n+1}} \) is not [strongly] \( \kappa \)-scattered, hence satisfying all the requirements of the induction at this step.

Having finished the induction we obtain that if \( k < n \) then \( q_n \in A_n \cup B_n \subseteq (\perp q_{k})^{S_n} \subseteq (\perp q_{k})^{P} \), hence \( q_n \perp_{P} q_k \). Then the sequence \( \langle q_n : n < \omega \rangle \) forms an infinite antichain in \( P \), contradicting the fact that \( P \) is FAC.

(2) \( \implies \) (3) Suppose that every augmentation of \( P \) is [strongly] \( \kappa \)-scattered but \( P \) does not satisfy the \( \lambda \)-AC for \( \lambda = \kappa^{< \kappa} \). (\( P \) is automatically [strongly] \( \kappa \)-scattered, since trivially \( P \) is an augmentation of itself.) Take a subset
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$S \subseteq P$ such that $|S| = \lambda$ and $S$ is a $\lambda$-antichain. We can now embed any strongly [weakly] $\kappa$-dense set into $S$ so forming an augmentation of $P$ which is not [strongly] $\kappa$-scattered. □

**Remark 2.15.** For $\kappa = \aleph_0$ the three conditions in Lemma 2.13 are equivalent, as follows from the lemma. However for $\kappa > \aleph_0$ the disjoint sum of an ordinal $\kappa$ with an antichain of size $\aleph_0$ shows that (3) does not imply (1) even for posets of size $\kappa$ when $\kappa = \kappa^{<\kappa}$. The above proof does not seem to generalise to show that $(3) \implies (2)$ and we do not know if this is the case.

## 3 A Generalisation of the Classification

Here we will generalise the classification of [1] to $\kappa$-scattered FAC partial orders for regular $\kappa$. From now on we will fix such a cardinal $\kappa$. We remind the reader of the notion of the antichain rank of FAC posets, as introduced in Definition 2.3.

**Definition 3.1.** Fix some $\rho \geq 1$. By induction on $\alpha$, an ordinal, we define $\kappa H_\rho^\alpha$ as follows:

1. $\kappa H_0^\rho = \{1\}$.
2. $\kappa H_1^\rho$ is the class of all posets $P$ satisfying the FAC with $\text{rk}_A(P) \leq \rho$ such that either $P$ or its inverse, or both, are $\kappa$-well founded.
3. If $\alpha$ is a limit ordinal, then $\kappa H_\alpha^\rho = \bigcup_{\beta < \alpha} \kappa H_\beta^\rho$.
4. If $\alpha = \beta + 1$ for some $\beta > 0$, then $\kappa H_\alpha^\rho$ consists of all posets $P$ that are lexicographical sums of the the form $P = \sum_{i \in I} P_i$ where $P_i \in \kappa H_\beta^\rho$ and $I$ is in $\kappa H_1^\rho$.

In general, let for all ordinals $\alpha$ and $\rho$,

$$\kappa H^\rho = \bigcup_{\alpha \text{ an ordinal}} \kappa H_\alpha^\rho \quad \text{and} \quad \kappa H = \bigcup_{\rho \text{ an ordinal}} \kappa H^\rho.$$  

We let $\text{aug}(\kappa H^\rho)$ be the set of all augmentations of posets in $\kappa H^\rho$. 

Lemma 3.2. (1) The class $\kappa H^\rho$ is the least class that contains the $\kappa$-well founded FAC posets with antichain ranks $\leq \rho$ and is closed under lexicographical sums and inverses.

(2) Each $\kappa H^\rho_\alpha$ and $\kappa H^\rho$ is closed under restrictions and inverses.

(3) If $P \in \kappa H^\rho$ then $P$ is $\kappa$-scattered and satisfies the FAC.

(4) $\text{aug}(\kappa H^\rho)$ is closed under lexicographical sums, restrictions and augmentations. Every poset in $\text{aug}(\kappa H^\rho)$ is $\kappa$-scattered.

We remind the reader that $\kappa$-scattered is throughout used to refer to weakly $\kappa$-scattered orders.

Proof of the Lemma. (1) It is clear that $\kappa H^\rho$ contains all $\kappa$-well founded posets of antichain ranks $\leq \rho$ and their inverses, as $\kappa H^\rho_1$ does. The proof that $\kappa H^\rho$ is closed under lexicographical sums is the same as the one in [1] since we are holding $\kappa$ fixed. We will not include it here.

It remains to show that $\kappa H^\rho$ is closed under inverses. In fact we shall prove by induction on $\alpha$ that each $\kappa H^\rho_\alpha$ is closed under inverses. For $P \in \kappa H^\rho$ we shall use the notation $\alpha(P) = \min\{\alpha : P \in \kappa H^\rho_\alpha\}$. Let us commence the induction.

At $\alpha = 0$ the situation is trivial and at $\alpha = 1$, by definition $\kappa H^\rho_1$ contains all inverses of its members.

At $\alpha = \beta + 1$, if $\alpha(P) < \alpha$ then this case is covered by the induction hypothesis. So, assume that $\alpha(P) = \alpha$. Then, $P = \sum_{i \in I} P_i$ where $P_i \in \kappa H^\rho_\beta$ and $I \in \kappa H^\rho_1$. The inverse of $P$ is $P^* = \sum_{i \in I^*} P_i^*$ where $I^* \in \kappa H^\rho_1$ because $\kappa H^\rho_1$ is closed under inverses by definition, and $P_i^*$ is the inverse of $P_i$. We know that $P_i \in \kappa H^\rho_\beta$, thus $P_i^*$ is also in $\kappa H^\rho_\beta$ by the induction hypothesis and hence $P^*$ is in $\kappa H^\rho_\alpha$.

We know that $\alpha(P)$ is never a limit because $\alpha$ is a minimum. Therefore, for $\alpha$ a limit ordinal and any $P \in \kappa H^\rho_\alpha$, $\alpha(P)$ is strictly less than $\alpha$. Thus, this case is covered by the induction hypothesis.

Hence $\kappa H^\rho$ has the closure properties as required. Now we will show that it is the least such class. Suppose that $\mathcal{H}$ is another class with such properties. Again by induction on $\alpha$, we will show that $\mathcal{H}$ contains each $\kappa H^\rho_\alpha$. 
Thus, we will show $\mathcal{H} \supseteq \kappa \mathcal{H}^\rho$. The cases of $\alpha = 0$ and $\alpha = 1$ are trivial by definition. At $\alpha = \beta + 1$, all sets $P \in \kappa \mathcal{H}^\rho_\beta$ are of the form $P = \sum_{i \in I} P_i$ where each $P_i \in \kappa \mathcal{H}^\rho_\beta$. Since $\mathcal{H}$ contains all $\kappa \mathcal{H}^\rho_\beta$ by the induction hypothesis and is closed under lexicographical sums, all $P \in \kappa \mathcal{H}^\rho_\alpha$ must be in $\mathcal{H}$. Thus, $\kappa \mathcal{H}^\rho_\alpha \subseteq \mathcal{H}$. The case where $\alpha$ is a limit is similar since by definition, $\kappa \mathcal{H}^\rho_\alpha = \bigcup_{\beta < \alpha} \kappa \mathcal{H}^\rho_\beta$.

(2) We have already proved the closure under inverses in the proof of (1). By induction on $\alpha$, we will show that each $\kappa \mathcal{H}^\rho_\alpha$ is closed under restrictions. The case $\alpha = 0$ is trivial. For $\alpha = 1$, if $P \in \kappa \mathcal{H}^\rho_1$ then either $P$ or $P^*$ is $\kappa$-well founded. Suppose that $P$ is $\kappa$-well founded. Thus, if any restriction of $P$, call it $P^-$, had a $\kappa$-decreasing sequence, it would actually be in $P$, which is a contradiction. The same argument can be used for $P^*$, the inverse of any $\kappa$-well founded poset in $\kappa \mathcal{H}^\rho_1$.

At $\alpha = \beta + 1$, suppose we are given $P = \sum_{i \in I} P_i$ where each $P_i \in \kappa \mathcal{H}^\rho_\beta$ and $I \in \kappa \mathcal{H}^\rho_1$. By the induction hypothesis, all restrictions of $P_i$ are in $\kappa \mathcal{H}^\rho_\beta$. Any restriction, $P^-$, of $P$ can be expressed as a lexicographical sum of restrictions of the $P_i$s along a restriction of $I$. Thus $P^-$ is also in $\kappa \mathcal{H}^\rho_\alpha$. The limit case is obvious.

(3) Fix an ordinal $\rho$. By induction on $\alpha$, we will prove that any $P \in \kappa \mathcal{H}^\rho_\alpha$ is $\kappa$-scattered. The case $\alpha = 0$ is trivial. For $\alpha = 1$, notice that since any strongly $\kappa$-scattered order has a $\kappa$-decreasing sequence by Lemma 2.6, we have that no $\kappa$-well founded poset could embed such an order. Similarly, since by the same lemma strongly $\kappa$-dense orders have $\kappa$-increasing sequences, a poset whose inverse is $\kappa$-well founded also cannot embed such an order. The limit case of the induction is taken care of by the induction hypothesis.

For $\alpha = \beta + 1$, if $P \in \kappa \mathcal{H}^\rho_\alpha$ we can by the induction hypothesis let $P = \sum_{i \in I} P_i$ where each $P_i$ is $\kappa$-scattered and $I$ is $\kappa$-scattered. We will show that $P$ is $\kappa$-scattered. For the sake of contradiction, let $Q$ be a strongly $\kappa$-dense order and suppose $f : Q \to P$ is an order preserving embedding.

Case 1. For every $i \in I$, there is at most one $q \in Q$ such that $f(q) \in P_i$. Define $g : Q \to I$ by letting $g(q) = i$ iff $f(q) \in P_i$. This is well-defined by the assumptions of Case 1. We also have $q <^* r$ implies $f(q) <_P f(r)$ which implies $g(q) <_I g(r)$ by the definition of the lexicographic sum. Hence, $g$ is
an order preserving embedding, contradicting the fact that $I$ is $\kappa$-scattered.

**Case 2.** Not Case 1. There is an $i \in I$ and $q, r \in Q$ such that $q \neq r$ and $f(q), f(r) \in P_i$. Without loss of generality, take $q <^* r$. Because $f$ is an embedding, $f(q) <_{P_i} f(r)$. By the definition of the sum, we also have $f(x) \in P_i$ for all $x \in (q, r)$. However, $(q, r)$ is strongly $\kappa$-dense by Claim 2.8, so $P_i$ is not $\kappa$-scattered, which is a contradiction.

A similar proof shows that the second part of the claim in (3) is true.

(4) The second sentence has already been covered in Lemma 2.13 because (3) shows that every element of $^{\kappa}\mathcal{H}_\rho$ satisfies the statement (1) of that lemma. The first sentence of (4) is easily proven with each property requiring the same type of argument. For example, to prove that $\text{aug}(^{\kappa}\mathcal{H}_\rho)$ is closed under lexicographical sums, consider the following observations: suppose $P = \sum_{i \in J} P_i$ and each $P_i$ is an augmentation of $Q_i$, where $Q_i$ is in $^{\kappa}\mathcal{H}_\rho$ while $J$ is an augmentation of $I \in ^{\kappa}\mathcal{H}_\rho$. Then $P$ is an augmentation of $Q = \sum_{i \in J} Q_i$. As we have that $Q$ belongs to $^{\kappa}\mathcal{H}_\rho$ (by part (1)), we conclude that $P$ is in $\text{aug}(^{\kappa}\mathcal{H}_\rho)$. $\square$

The next theorem is virtually the same in claim and proof as Theorem 2.3 of [1]. The only modification is the larger classification $^{\kappa}\mathcal{H}_\rho$ that replaces the $\mathcal{H}_\rho$ in the paper. The proof for the larger classification is the same because we hold $\kappa$ fixed, as we have done with all other proofs of this nature. We will leave this theorem as a fact, rather than reiterating the proof.

Before we state the theorem, we need to draw attention to an unusual ordinal operation known as Hessenberg based exponentiation. This smoothly extends the Hessenberg product operation which in turn extends the natural sum operation. Since we do not need to know the exact value of the exponent for this paper, we refer the reader to [1] for a more precise definition. We denote the Hessenberg based exponentiation of $\alpha$ and $\beta$ by $\alpha^{H\beta}$.

**Theorem 3.3.** If $P \in ^{\kappa}\mathcal{H}_\alpha$ then $\text{rk}_A(P) \leq \rho^{H\alpha}$.

Hausdorff’s theorem [5] (or see [2]) and the Abraham-Bonnet generalisation in [1] are both characterisations of the class of linear and FAC posets, respectively, which do not embed the rationals. The latter class is exactly $^{8\alpha}\mathcal{H}$. 

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To prove something like that we would need to know that if \( P \notin \text{aug}(\kappa) \) is an FAC poset, then \( Q(\kappa) \) embeds into \( P \). Unfortunately we have not been able to prove such a claim for uncountable \( \kappa \), and the question if it is true even if we assume that \( \kappa \) has some large cardinal properties remains open. We shall instead prove a weaker claim, for which we shall fatten up our hierarchy a little.

**Definition 3.4.** Let \( {}^*\mathcal{H}^* \) denote the closure of \( \text{aug}(\mathcal{H}) \) under FAC weakenings, that is, the class obtained by taking all FAC orders \( P \) for which there is an order \( P' \) in \( {}^*\mathcal{H} \) such that \( P \) is a weakening of \( P' \).

We shall show that \( {}^*\mathcal{H}^* \) lies between the classes of strongly and weakly \( \kappa \)-scattered FAC partial orders. Let us first show the easy direction.

**Claim 3.5.** Every poset in \( {}^*\mathcal{H}^* \) is (weakly) \( \kappa \)-scattered and FAC.

**Proof of the Claim.** Let \( P \) be in \( {}^*\mathcal{H}^* \) and let \( P' \) in \( \text{aug}(\mathcal{H}) \) be such that \( P \) is an FAC weakening of \( P' \). Clearly \( P \) is FAC. If \( P \) were not to be weakly \( \kappa \)-scattered then some strongly \( \kappa \)-dense order would embed into \( P \) and hence into \( P' \), in contradiction with Lemma 3.2(3) and Lemma 3.2(4). \( \square \)

The heart of our main theorem lies in the following:

**Claim 3.6.** Every strongly \( \kappa \)-scattered FAC partial order belongs to \( {}^*\mathcal{H}^* \).

**Proof of the Claim.** Suppose for contradiction that \( P \) is a strongly \( \kappa \)-scattered FAC partial order which does not belong to \( {}^*\mathcal{H}^* \). Recalling that every poset has a linear augmentation, let \( Q \) be any linear augmentation of \( P \). By Lemma 2.13 \( Q \) is strongly \( \kappa \)-scattered, and by the definition of \( {}^*\mathcal{H}^* \) we have that \( Q \notin {}^*\mathcal{H} \) (and even \( Q \notin \text{aug}(\mathcal{H}) \)).

For \( a, b \in Q \) we define an equivalence relation \( a \equiv b \) iff the interval in \( Q \) between \( a \) and \( b \) is in \( {}^*\mathcal{H} \). It is easily seen that this indeed is an equivalence relation. For \( a \in Q \) let \( C_a = \{ b : a \equiv b \} \) be the equivalence class of \( a \).

**Subclaim 3.7.** Each \( C_a \) with the order induced from \( Q \) is in \( {}^*\mathcal{H} \).
Proof of the Subclaim. Given \( C_a \). By induction on \( \gamma \) an ordinal pick if possible \( a_\gamma \) and \( b_\gamma \) in \( C_a \) so that \( a_0 = b_0 = a, a_\gamma \) is \( Q \)-increasing with \( \gamma \) and \( b_\gamma \) is \( Q \)-decreasing with \( \gamma \). Since \( C_a \) is a set, there must be ordinals \( \alpha \), the first \( \gamma \) for which we cannot choose \( a_\gamma \) and \( \beta \), the first \( \gamma \) for which we cannot choose \( b_\gamma \). Then \( C_a \) is the lexicographic sum

\[
\sum_{i<\beta} [b_{i+1}, b_i] \oplus \{a_0\} \oplus \sum_{j<\alpha} (a_j, a_{j+1}].
\]

Note that each of the intervals mentioned above is \(^*H\), by the definition of \( C_a \) and the fact that \( \equiv \) is an equivalence relation. Since \(^*H\) is closed under lexicographic sums of the above kind, we obtain that \( C_a \in ^*H \).

Subclaim 3.8. If \( a, b \in Q \) are not \( \equiv \)-equivalent, and \( a <Q b \), then \( C_a <Q C_b \).

Proof of the Subclaim. Let \( c \in C_a \) and \( d \in C_b \). Clearly \( c \neq d \). Suppose for contradiction that \( d <Q c \) and distinguish two cases.

Case 1. \( c \leq_Q a \).

Then \( d <Q a \), so \( (a, b) \subseteq (d, b) \), which is a contradiction because the latter is a member of \(^*H\) while the former is not.

Case 2. \( c \geq_Q a \).

Then either \( d <Q a \), in which case we obtain a contradiction like in Case 1, or \( a \leq_Q d \). In the latter case we have that \( (d, c) \subseteq (a, c) \), contradicting the fact that \( (a, c) \in ^*H \) and \( (d, c) \notin ^*H \).

Let \( Q^* \) be a set of the representatives of the \( \equiv \)-equivalence classes ordered by the factor order (by Subclaim 3.8 this order agrees with the order in \( Q \)). Then \( Q \) is the lexicographic sum \( \sum_{a \in Q} C_a \) and since \( Q \notin ^*H \) we obtain by Subclaim 3.7 and the closure of \(^*H\) under lexicographic sums that \( Q^* \notin ^*H \).

In particular \( Q^* \) has size at least \( \kappa \).

Now note that by the choice of \( Q^* \) for every \( a <Q b \) in \( Q^* \) the interval \( (a, b)_Q \) is not in \(^*H\) (and that \( Q^* \) is a maximal such set). We claim that in fact \( (a, b)_{Q^*} \notin ^*H \) for such \( a, b \). Once we prove this we shall be done, because every poset of size \( < \kappa \) is easily seen to be in \(^*H\) and thus, \( Q^* \) is a weakly \( \kappa \)-dense subset of \( Q \).
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So suppose that \( a <_Q b \) are elements of \( Q^* \) but \( (a, b)_Q \) \( \in \mathcal{H} \). We then observe that \( (a, b)_Q \) is the lexicographic sum \( \Sigma_{c \in (a, b)_Q} C_c \), which would then have to be in \( \mathcal{H} \), a contradiction. \( \square \)

To finish our work we shall give a simpler description of the class \( \mathcal{H}^* \). We show that we do not need to start with \( \kappa \)-well-founded FAC posets in the formation of \( \mathcal{H}^1 \), we may start with \( \kappa \)-well founded linear orders and then the FAC posets get picked up when we form \( \mathcal{H}^* \).

**Claim 3.9.** Suppose that \( \kappa \) is a regular cardinal. Then \( \mathcal{H}^* \) is the closure of the class of all \( \kappa \)-well founded linear orders under inversions, lexicographic sums, FAC weakenings and augmentations.

**Proof of the Claim.** Let \( \mathcal{H} \) denote the closure of of the class of all \( \kappa \)-well founded linear orders under inversions, lexicographic sums, FAC weakenings and augmentations. Since \( \mathcal{H}^* \) is the closure of the class of \( \kappa \)-well founded FAC posets under these operations we have that \( \mathcal{H}^* \supseteq H \). On the other hand, if \( P \in \mathcal{H}^* \) then let \( Q \in \text{aug}(\mathcal{H}) \) be such that \( P \) is an FAC weakening of \( Q \) and let \( R \in \mathcal{H} \) be such that \( Q \) is an augmentation of \( R \). If \( R \in \mathcal{H} \) then \( Q \in \mathcal{H} \) by the closure of \( \mathcal{H} \) under augmentations and hence \( P \in \mathcal{H} \) by the closure of \( \mathcal{H} \) under FAC weakenings. Hence it suffices to show that \( \mathcal{H}^* \subseteq \mathcal{H} \). Let \( \rho \geq 1 \) be any ordinal, we shall show by induction on \( \alpha \) that \( \mathcal{H}^*_\alpha \subseteq \mathcal{H} \). We first need a subclaim.

**Subclaim 3.10.** Every augmentation of a \( \kappa \)-well founded FAC poset is \( \kappa \)-well founded.

**Proof of the Subclaim.** Let \( P \) be a \( \kappa \)-well founded FAC poset and \( Q \) an augmentation of \( P \). Suppose that \( \langle a_\alpha; \alpha < \kappa \rangle \) is a \( \leq_Q \)-decreasing sequence. For \( \alpha < \beta < \kappa \) define \( f(\alpha, \beta) = 1 \) if \( a_\alpha \) and \( a_\beta \) are comparable in \( P \) and let \( f(\alpha, \beta) = 0 \) otherwise. We now use the Dushnik-Miller theorem which says that either there is an infinite 0-homogeneous set or a 1-homogeneous set of type \( \kappa \). Since \( P \) is an FAC poset there cannot be an infinite 0-homogeneous set, but a 1-homogeneous set of type \( \kappa \) would contradict the fact that \( P \) is \( \kappa \)-well founded. This contradiction proves the subclaim. \( \square \)
We now proceed with the promised inductive proof. If \( \alpha = 0 \) the conclusion is clear. If \( P \in ^{\kappa}H_1 \) then \( P \) is FAC and either \( P \) or its inverse (or both) are \( \kappa \)-well-founded.

In the first case we can use the subclaim to find \( Q \) which is a \( \kappa \)-well-founded linear augmentation of \( P \). Hence \( Q \in H \) and as its FAC weakening, \( P \in H \). The other case is similar.

The case of \( \alpha \) a limit ordinal follows from the inductive hypothesis and the case \( \alpha = \beta + 1 \) for \( \beta > 0 \) follows by the closure of \( H \) under lexicographic sums.

Let us also observe the following:

**Observation 3.11.** Suppose that \( P \) is a linear order, \( Q \) is an FAC weakening of \( P \), and \( R \) is an augmentation of \( Q \). Then \( R \) is an FAC weakening of \( P \).

We conclude that the following theorem is true.

**Main Theorem 3.12.** Assume that \( \kappa \) is a regular cardinal. Let \( ^{\kappa}H^* \) denote the closure of the class of all \( \kappa \)-well founded linear orders under inversions, lexicographic sums and FAC weakenings. Equivalently,

1. \( ^{\kappa}H^* \) contains all strongly \( \kappa \)-scattered FAC posets.
2. \( ^{\kappa}H^* \) is contained in the class of all \( \kappa \)-scattered FAC posets.

If \( \kappa = \aleph_0 \) we obtain an equality between the notions of \( \kappa \)-dense and strongly \( \kappa \)-dense, so applying Theorem 3.12 to \( \kappa = \aleph_0 \) we obtain that \( H^* \) is the class of all scattered FAC posets. Since Abraham-Bonnet theorem already gives that this class of posets is described by \( H \) we have as a corollary

**Corollary 3.13.** \( ^{\aleph_0}H^* \) is exactly the Abraham-Bonnet class \( ^{\aleph_0}H \).

In general the two notions of density are not equivalent, as we illustrate in 4. Moreover, example 4.1 shows that for every uncountable \( \kappa \) with \( \kappa = \kappa^{<\kappa} \) there are members of \( ^{\kappa}H \) which are not strongly \( \kappa \)-scattered. We also do
not know for which uncountable $\kappa$ we obtain that $^\kappa \mathcal{H}^*$ is the same as $^\kappa \mathcal{H}$. Note that it is not to be expected that $^\kappa \mathcal{H}^\rho$ is closed under FAC weakenings as weakening a partial order generally adds larger antichains and hence increases the antichain rank.

When reduced to the class of linear orders the class $^\kappa \mathcal{H}^*$ can be replaced by a simpler class.

**Theorem 3.14.** Assume that $\kappa$ is a regular cardinal. Let $^\kappa \mathcal{L}^*$ denote the closure of the class of all $\kappa$-well founded linear orders under inversions and lexicographic sums. Then:

(1) $^\kappa \mathcal{L}^*$ contains all strongly $\kappa$-scattered linear orders.

(2) $^\kappa \mathcal{L}^*$ is contained in the class of all $\kappa$-scattered linear orders.

**Proof.** Linear orders are FAC posets with antichain rank $\leq 1$. By Lemma 3.2(1) the class $^\kappa \mathcal{H}^1$ is the least class that contains the $\kappa$-well founded linear orders and is closed under inversions and lexicographic sums, hence $^\kappa \mathcal{L}^* = ^\kappa \mathcal{H}^1$. Since every order in $^\kappa \mathcal{H}^1$ is linear we obtain $^\kappa \mathcal{H}^1 = \text{aug}(^\kappa \mathcal{H}^1)$, and hence Lemma 3.2(4) gives part (2) of the theorem.

To prove (1) we use the proof of Claim 3.6. We start with a strongly $\kappa$-scattered linear order $Q$ that does not belong to $^\kappa \mathcal{L}^* = ^\kappa \mathcal{H}^1$ and obtain a contradiction literally as in the proof of that claim. ✷

With $\kappa = \aleph_0$ Theorem 3.14 gives Hausdorff’s theorem.

The above theorems and remarks raise the following questions

**Question 3.15.** (1) For which uncountable $\kappa$ is $\text{aug}(^\kappa \mathcal{H}^\rho)$ exactly the class of all $\kappa$-scattered FAC posets with antichain rank $\leq \rho$?

(2) For which $\kappa$ is it true that any FAC poset all of whose subposets (or even just chains) of size $\kappa$ belong to $^\kappa \mathcal{H}$, is itself an element of $^\kappa \mathcal{H}$?

(3) For which $\kappa > \aleph_0$ is it true that every augmentation of a [strongly] $\kappa$-scattered $\kappa$-AC poset is [strongly] $\kappa$-scattered?

(4) For which $\kappa > \aleph_0$ is $^\kappa \mathcal{H}$ closed under FAC weakenings?
We comment that one may generalise Theorem 3.12 to the case of \( \lambda < \kappa \) where both \( \lambda \) and \( \kappa \) are equal to their weak powers, and consider the situation of posets of size \( \kappa \) that satisfy (strong) \( \lambda \)-density, obtaining the expected results.

4 Examples

For the sake of completeness we include some examples that illustrate the difference between weak \( \kappa \)-density and strong \( \kappa \)-density. We shall assume that \( \kappa \) is an uncountable regular cardinal.

An easy example of a linear order that is \( \kappa \)-dense but not strongly \( \kappa \)-dense is the lexicographic sum along \( \omega + \omega^* \) of any strongly \( \kappa \)-dense order. This order is clearly strongly \( \kappa \)-dense. We give an example of a \( \kappa \)-dense linear order which is weakly \( \kappa \)-scattered and moreover does not have a \( \kappa \)-decreasing sequence.

Let \( L_0 \) be the lexicographic sum along \( \omega^* \) of copies of \( \kappa \). By induction on \( n < \omega \) define \( L_n \) by letting \( L_{n+1} \) be the lexicographic sum along \( L_n \) of copies of \( L_0 \). We denote the order of \( L_n \) by \( \leq_n \). Let \( L = \bigcup_{n<\omega} L_n \) be ordered by letting \( p \leq q \iff p \leq_n q \) for the first \( n \) that contains both \( p \) and \( q \).

Claim 4.1. No \( L_n \) for \( n < \omega \) is \( \kappa \)-dense. \( L \) is \( \kappa \)-dense.

Proof. The first statements can easily be proven by induction. For the second one, let \( p < q \) and let \( n \) be the first such that \( p, q \in L_n \). By the definition of \( L_{n+1} \) there is a copy of \( L_0 \) in \( \{ x \in L_{n+1} : p < x < q \} \), so clearly the size of this set is \( \kappa \). \( \square \)

Claim 4.2. \( L \) does not have a decreasing sequence of size \( \kappa \).

Proof. Suppose it had such a decreasing sequence, call it \( S \). Then \( S = \bigcup_{n<\omega} S \cap (L_{n+1} \setminus L_n) \). By the regularity of \( \kappa > \aleph_0 \) there has to be \( n \) for which the size of \( S \cap (L_{n+1} \setminus L_n) \) is \( \kappa \), hence it suffices for us to show that no \( L_n \) can have a decreasing sequence of size \( \kappa \). This can be done by induction on \( n \). \( \square \)
Hence by Lemma 2.6 we have

**Corollary 4.3.** $L$ does not embed any strongly $\kappa$-dense order and $L$ is in $^\kappa H^\rho$ for any $\rho \geq 1$.

This shows that the boundary of $^\kappa H^\rho$ is somewhere in between weakly $\kappa$-scattered and strongly $\kappa$-scattered. We conjecture that $^\kappa H^\rho$ contains all strongly $\kappa$-scattered FAC posets.

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