TWO-SIDED BOUNDS FOR COST FUNCTIONALS OF TIME-PERIODIC PARABOLIC OPTIMAL CONTROL PROBLEMS

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ABSTRACT. In this paper, a new technique is applied on deriving computable, guaranteed lower bounds of functional type (minorants) for two different cost functionals subject to a parabolic time-periodic boundary value problem. Together with previous results on upper bounds (majorsants) for one of the cost functionals, both minorants and majorsants lead to two-sided estimates of functional type for the optimal control problem. Moreover, a different cost functional is discussed with the same PDE-constraints. Both upper and lower bounds are derived. The time-periodic optimal control problems are discretized by the multiharmonic finite element method leading to large systems of linear equations having a saddle point structure. The derivation of preconditioners for the minimal residual method for the new optimal control problem is discussed in more detail. Finally, several numerical experiments for both optimal control problems are presented confirming the theoretical results obtained.

1. Introduction

The optimal control of partial differential equations is the subject matter of lots of different works (see, e.g., [39, 16, 48, 6] and the references therein) due to their importance in research and application. Optimal control problems with time-periodic parabolic state equations often arise in different practical applications such as in electromagnetics, chemistry, biology, or heat transfer. We refer as examples to the recent works [1] and [17] considering problems in electromagnetics and biochemistry, respectively. The multiharmonic finite element method (MhFEM) – also called harmonic-balanced finite element method – is a natural approach for the solution of parabolic time-periodic problems, where the state functions are expanded into truncated Fourier series in time. The space-dependent Fourier coefficients are then approximated by the finite element method (FEM). The MhFEM was successfully used for the simulation of electromagnetic devices described by nonlinear eddy current problems with harmonic excitations, see, e.g., [50, 3, 4, 5, 11]. Later, the MhFEM has been applied to linear time-periodic parabolic boundary value and optimal control problems [10, 21, 22, 51] and to linear time-periodic eddy current problems and the corresponding optimal control problems [21, 22]. Recent works on robust preconditioners for time-periodic parabolic and eddy-current optimal control problems are discussed in [32] and [2], respectively.

A posteriori estimates of functional type provide a useful machinery to derive computable and guaranteed quantities for the desired unknown solution. These estimation techniques were earlier introduced by S. Repin, see, e.g., the papers on parabolic problems [12, 14]. Recent works on new estimates for parabolic problems and parabolic optimal control problems can be found in [35] and [46], respectively. A posteriori estimates of functional type for elliptic optimal control problems can be found in [14, 13, 43, 34]. First functional type estimates for inverse problems can be found in [44, 10]. Moreover, recent results on guaranteed computable estimates for convection-dominated diffusion problems are presented in [36].

In [30], majorsants (fully computable upper bounds) for the optimality system and the cost functional of a time-periodic parabolic optimal control were presented. In this work, we want to extend the analysis by deriving the corresponding minorants (fully computable lower bounds) for this cost functional using the technique presented in [49] applying also earlier results derived by Mikhlin [37] (see also [43] for a more recent discussion). We mention here that Mali [33] presents a different approach for the derivation of a lower bound for a class of elliptic optimal control problems.

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problems. Moreover, we extend the analysis and consider a second cost functional with respect to the same parabolic time-periodic boundary value problem. The results on computable lower bounds together with the upper bounds lead to two-sided estimates which can be used to derive majorants for the discretization error in state and control. These majorants and minorants provide a new formulation of the optimization problems since they can, in principle, be used as objects of direct minimization on their difference. In this work, robust preconditioners for the preconditioned minimal residual (MinRes) method (see [10]) are discussed for the second minimization problem, which are new for this case.

The paper is organized as follows: In Section II we discuss both time-periodic parabolic optimal control problems including some preliminary results. The majorants and new minorants for minimization problem I are presented in Section III followed by the results for minimization problem II in Section IV. The multiharmonic finite element discretization of the optimization problems is considered in Section V. Section VI presents robust preconditioners for applying the preconditioned MinRes method on the problems discretized by the MhFEM. Finally, Section VII discusses detailed majorants and new minorants for minimization problem I are presented in Section III followed by the results for minimization problem II.

2. Two Cost Functionals with Respect to Time-Periodic Parabolic PDE-Constraints

Let \( Q_T := \Omega \times (0, T) \) denote the space-time cylinder and \( \Sigma_T := \Gamma \times (0, T) \) its lateral surface, where \( (0, T) \) is a given time interval. The spatial domain \( \Omega \subset \mathbb{R}^d \), \( d = \{1, 2, 3\} \), is assumed to be a bounded Lipschitz domain with boundary \( \Gamma := \partial \Omega \). The minimization problems are both subject to the following parabolic time-periodic boundary value problem:

\[
\begin{aligned}
\sigma(x) \partial_t y(x, t) - \text{div} (\nu(x) \nabla y(x, t)) &= u(x, t) \quad (x, t) \in Q_T, \\
y(x, t) &= 0 \quad (x, t) \in \Sigma_T, \\
y(x, 0) &= y(x, T) \quad x \in \Omega,
\end{aligned}
\]

(1)

where \( y \) and \( u \) denote the state and control, respectively. The coefficients \( \sigma(\cdot) \) and \( \nu(\cdot) \) are uniformly bounded and satisfy the conditions

\[
0 < \underline{\sigma} \leq \sigma(x) \leq \overline{\sigma}, \quad \text{and} \quad 0 < \underline{\nu} \leq \nu(x) \leq \overline{\nu}, \quad x \in \Omega,
\]

(2)

where \( \underline{\sigma}, \overline{\sigma}, \underline{\nu}, \overline{\nu} \) are positive constants. In the following, we present a proper functional space setting for time-periodic problems. The notation is similar to that was used in [27, 28]. We define the Hilbert spaces

\[
\begin{align*}
H^{1,0}(Q_T) &= \{ v \in L^2(Q_T) : \nabla v \in [L^2(Q_T)]^d \}, \\
H^{0,1}(Q_T) &= \{ v \in L^2(Q_T) : \partial_t v \in L^2(Q_T) \}, \\
H^{1,1}(Q_T) &= \{ v \in L^2(Q_T) : \nabla v \in [L^2(Q_T)]^d, \partial_t v \in L^2(Q_T) \},
\end{align*}
\]

which are endowed with the norms

\[
\begin{align*}
\| v \|_{1,0} &= \left( \int_{Q_T} \left( v(x, t)^2 + |\nabla v(x, t)|^2 \right) \, dx \, dt \right)^{1/2}, \\
\| v \|_{0,1} &= \left( \int_{Q_T} \left( v(x, t)^2 + |\partial_t v(x, t)|^2 \right) \, dx \, dt \right)^{1/2}, \\
\| v \|_{1,1} &= \left( \int_{Q_T} \left( v(x, t)^2 + |\nabla v(x, t)|^2 + |\partial_t v(x, t)|^2 \right) \, dx \, dt \right)^{1/2},
\end{align*}
\]

respectively. Here, \( \nabla = \nabla_x \) is the spatial gradient and \( \partial_t \) denotes the weak derivative with respect to time. Also we define subspaces of the above introduced spaces by putting subindex zero if the functions satisfy the homogeneous Dirichlet condition on \( \Sigma_T \) and subindex \( per \) if they satisfy the time-periodical condition \( v(x, 0) = v(x, T) \). All inner products and norms in \( L^2 \) related to the whole space-time domain \( Q_T \) are denoted by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \), respectively. If they are associated with the spatial domain \( \Omega \), then we write \( \langle \cdot, \cdot \rangle_\Omega \) and \( \| \cdot \|_\Omega \). The symbols \( \langle \cdot, \cdot \rangle_{1,\Omega} \) and \( \| \cdot \|_{1,\Omega} \) denote the standard inner products and norms of the space \( H^1(\Omega) \).
The functions used in our analysis are presented in terms of Fourier series, e.g., the Fourier representation of the function \( v(x, t) \) is given by the series
\[
(3) \quad v(x, t) = v_0(x) + \sum_{k=1}^{\infty} (v_k(x) \cos(k\omega t) + v_k^*(x) \sin(k\omega t)),
\]
where
\[
v_0(x) = \frac{1}{T} \int_0^T v(x, t) \, dt,
\]
\[
v_k(x) = \frac{2}{T} \int_0^T v(x, t) \cos(k\omega t) \, dt \quad \text{and} \quad v_k^*(x) = \frac{2}{T} \int_0^T v(x, t) \sin(k\omega t) \, dt
\]
are the Fourier coefficients, \( T \) denotes the period, and \( \omega = 2\pi/T \) is the frequency. The choice of this type of representation is natural because the problem has time-periodical conditions. We also introduce the spaces
\[
H_{\text{per}}^{0,\frac{1}{2}}(Q_T) = \{ v \in L^2(Q_T) : \| \partial_t^{1/2} v \| < \infty \},
\]
\[
H_{\text{per}}^{1,\frac{1}{2}}(Q_T) = \{ v \in H^{1,0}(Q_T) : \| \partial_t^{1/2} v \| < \infty \},
\]
\[
H_{0,\text{per}}^{1,\frac{1}{2}}(Q_T) = \{ v \in H_{\text{per}}^{1,\frac{1}{2}}(Q_T) : v = 0 \text{ on } \Sigma_T \},
\]
where \( \| \partial_t^{1/2} v \| \) is defined in the Fourier space by the relation
\[
(4) \quad \| \partial_t^{1/2} v \| := |v|^2_{\frac{1}{2}} := \frac{T}{2} \sum_{k=1}^{\infty} k\omega \| v_k \|_{L_2}^2.
\]
Here, \( v_k = (v_k^0, v_k^1)^T \) for all \( k \in \mathbb{N} \), see also [31]. These spaces can be considered as Hilbert spaces if we introduce the following (equivalent) products:
\[
\langle \partial_t^{1/2} y, \partial_t^{1/2} v \rangle := \frac{T}{2} \sum_{k=1}^{\infty} k\omega (v_k^0, v_k^1)_\omega, \quad \langle \sigma \partial_t^{1/2} y, \partial_t^{1/2} v \rangle := \frac{T}{2} \sum_{k=1}^{\infty} k\omega (\sigma v_k^0, v_k^1)_\omega.
\]
The above introduced spaces allow us to operate with “symmetrized” formulations of the time-periodic problems. The seminorm and norm in \( H_{\text{per}}^{1,\frac{1}{2}}(Q_T) \) are defined by the relations
\[
|v|^2_{1,\frac{1}{2}} := \| v \|^2 + \| \partial_t^{1/2} v \|^2 = T \| \nabla v_0 \|^2_{L_2} + \frac{T}{2} \sum_{k=1}^{\infty} (k\omega \| v_k \|_{L_2}^2 + \| \nabla v_k \|_{L_2}^2)
\quad \text{and}
\]
\[
\| v \|^2_{1,\frac{1}{2}} := \| v \|^2 + |v|^2_{1,\frac{1}{2}} = T \left( \| v_0 \|^2_{L_2} + \| \nabla v_0 \|^2_{L_2} + \frac{T}{2} \sum_{k=1}^{\infty} ((1 + k\omega) \| v_k \|_{L_2}^2 + \| \nabla v_k \|_{L_2}^2) \right),
\]
respectively. Using Fourier type series, it is easy to define the function “orthogonal” to \( v \):
\[
v^\perp(x, t) := \sum_{k=1}^{\infty} (-v_k^0(x) \sin(k\omega t) + v_k^1(x) \cos(k\omega t)) = \sum_{k=1}^{\infty} \left(v_k^0(x), -v_k^1(x)\right)_{\omega} \left(\cos(k\omega t) \quad \sin(k\omega t)\right).
\]
Obviously, \( \| u_k \|_\Omega = \| u_k \|_\Omega \) and we find that
\[
\| \partial_t^{1/2} v^\perp \|^2 = \frac{T}{2} \sum_{k=1}^{\infty} k\omega \| v_k^1 \|^2_{L_2} = \frac{T}{2} \sum_{k=1}^{\infty} k\omega \| v_k \|^2_{L_2} = \| \partial_t^{1/2} v \|^2 \quad \forall v \in H_{\text{per}}^{0,\frac{1}{2}}(Q_T).
\]
The identities
\[
(5) \quad \langle \sigma \partial_t^{1/2} y, \partial_t^{1/2} v \rangle = \langle \sigma \partial_t y, v^\perp \rangle \quad \text{and} \quad \langle \sigma \partial_t^{1/2} y, \partial_t^{1/2} v^\perp \rangle = \langle \sigma \partial_t y, v \rangle
\]
are valid for all \( y \in H_{\text{per}}^{1,0}(Q_T) \) and \( v \in H_{\text{per}}^{0,\frac{1}{2}}(Q_T) \), see also [29]. Also, we define the product
\[
(6) \quad \langle \xi, \partial_t^{1/2} v \rangle := \frac{T}{2} \sum_{k=1}^{\infty} (k\omega)^{1/2} (\xi_k, v_k)_\omega.
\]
We recall the orthogonality relations

\begin{align}
\langle \sigma \partial_y y, y \rangle &= 0 & \langle \sigma y^\perp, y \rangle &= 0 & \forall y \in H_{per}^{0,1}(Q_T), \\
\langle \sigma \partial_t^{1/2} y, \partial_t^{1/2} y^\perp \rangle &= 0 & \langle \nu \nabla y, \nabla y^\perp \rangle &= 0 & \forall y \in H_{per}^{1,1}(Q_T),
\end{align}

and the identity (in the sense of Fourier series)

\begin{align}
\int_{Q_T} \xi \partial_t^{1/2} v^\perp \, dx \, dt &= - \int_{Q_T} \partial_t^{1/2} \xi^\perp v \, dx \, dt & \forall \xi, v \in H_{per}^{0} (Q_T),
\end{align}

see [31]. We note that for functions presented in terms of Fourier series the standard Friedrichs inequality holds in $Q_T$:

\begin{align}
\| \nabla u \|^2 &= T \| \nabla u_0 \|^2 + \frac{T}{2} \sum_{k=1}^{\infty} \| \nabla u_k \|^2 \geq \frac{1}{c_F^2} \left( T \| u_0 \|^2 + \frac{T}{2} \sum_{k=1}^{\infty} \| u_k \|^2 \right) = \frac{1}{c_F^2} \| u \|^2.
\end{align}

In the following, the parameter $\lambda > 0$ denotes the regularization or cost parameter.

### 2.1. Minimization problem I.

In the first case, we want to minimize the following cost functional with respect to the unknown state $y$ and control $u$:

\begin{align}
\mathcal{J}(y, u) := \frac{1}{2} \| y - y_d \|^2 + \frac{\lambda}{2} \| u \|^2
\end{align}

subject to the time-periodic boundary value problem (1), where $y_d \in L^2(Q_T)$ is the desired given state. The cost functional $\mathcal{J}$ defined in (10) can be written as

\begin{align}
\mathcal{J}(y, u) &= T \mathcal{J}_0(y_0, u_0) + \frac{T}{2} \sum_{k=1}^{\infty} \mathcal{J}_k(y_k, u_k),
\end{align}

where $\mathcal{J}_0(y_0, u_0) = \frac{1}{2} \| y_0 - y_d0 \|^2 + \frac{\lambda}{2} \| u_0 \|^2$ and $\mathcal{J}_k(y_k, u_k) = \frac{1}{2} \| y_k - y_{dk} \|^2 + \frac{\lambda}{2} \| u_k \|^2$. In [30], the corresponding optimality system is derived, which is given in weak formulation as follows: Given $y_d \in L^2(Q_T)$, find $y, p \in H_{per}^{1,1}(Q_T)$ such that

\begin{align}
\begin{cases}
\int_{Q_T} \left( y z - \nu(x) \nabla p \cdot \nabla z + \sigma(x) \partial_t^{1/2} p \partial_t^{1/2} z^\perp \right) \, dx \, dt = \int_{Q_T} y_d \, z \, dx \, dt, \\
\int_{Q_T} \left( \nu(x) \nabla y \cdot \nabla q + \sigma(x) \partial_t^{1/2} y \partial_t^{1/2} q^\perp + \lambda^{-1} p \cdot q \right) \, dx \, dt = 0,
\end{cases}
\end{align}

for all test functions $z, q \in H_{per}^{1,1}(Q_T)$. Using the Fourier series ansatz [3] in (12) and exploiting the orthogonality of $\cos(k\omega t)$ and $\sin(k\omega t)$, we arrive at the following problem: Find $y_k, p_k \in \mathcal{V} := V \times V = (H_0^1(\Omega))^2$ such that

\begin{align}
\begin{cases}
\int_{\Omega} \left( y_k \cdot z_k - \nu(x) \nabla p_k \cdot \nabla z_k + k \omega \sigma(x) p_k \cdot z_k^\perp \right) \, dx = \int_{\Omega} y_{dk} \cdot z_k \, dx, \\
\int_{\Omega} \left( \nu(x) \nabla y_k \cdot \nabla q_k + k \omega \sigma(x) y_k \cdot q_k^\perp + \lambda^{-1} p_k \cdot q_k \right) \, dx = 0,
\end{cases}
\end{align}

for all test functions $z_k, q_k \in \mathcal{V}$. The system (13) must be solved for every mode $k \in \mathbb{N}$. For $k = 0$, we obtain a reduced problem: Find $y_0^0, p_0^0 \in V$ such that

\begin{align}
\begin{cases}
\int_{\Omega} \left( y_0^0 \cdot z_0^0 - \nu(x) \nabla p_0^0 \cdot \nabla z_0^0 \right) \, dx = \int_{\Omega} y_{d0}^0 \cdot z_0^0 \, dx, \\
\int_{\Omega} \left( \nu(x) \nabla y_0^0 \cdot \nabla q_0^0 + \lambda^{-1} p_0^0 \cdot q_0^0 \right) \, dx = 0,
\end{cases}
\end{align}

for all test functions $z_0^0, q_0^0 \in \mathcal{V}$. The problems (13) and (14) have unique solutions, see [31]. In [30], the a posteriori error analysis for the optimality system as well as majorants for the cost functional $\mathcal{J}$ are presented. This work extends and deepens these results by deriving minorants for this cost functional as well as for a different one, which is introduced in the next subsection.
2.2. Minimization problem II. In the second case, we want to minimize the following cost functional with respect to the unknown state \( y \) and control \( u \):

\[ \tilde{J}(y, u) := \frac{1}{2} \| \nabla y - g_d \|^2 + \frac{\lambda}{2} \| u \|^2 \]  

subject to the time-periodic boundary value problem \( (1) \), where \( g_d \in [L^2(Q_T)]^d \) is the given desired gradient. The optimality conditions are given in weak form as follows: Given \( \eta, \tau \in H^{1,2}_{0,\text{per}}(Q_T) \), the Lagrange functional

\[ \tilde{L}(y, u, p) = \tilde{J}(y, u) - \int_{Q_T} (\sigma(x) \partial_t y - \text{div} (\nu(x) \nabla y) - u) p \, dx \, dt. \]

Only the equation \( \partial_y \tilde{L}(y, u, p) = 0 \) is in the optimality conditions for problem II. The optimality conditions are given in weak form as follows: Given \( g_d \in [L^2(Q_T)]^d \), find \( y, p \in H^{1,2}_{0,\text{per}}(Q_T) \) such that

\[ \begin{aligned} 
\int_{Q_T} \left( \nabla y \cdot \nabla z - \nu(x) \nabla p \cdot \nabla z + \sigma(x) \partial_t^{1/2} p \partial_t^{1/2} z^\perp \right) \, dx \, dt &= \int_{Q_T} g_d \cdot \nabla z \, dx \, dt, \\
\int_{Q_T} \left( \nu(x) \nabla y \cdot \nabla q + \sigma(x) \partial_t^{1/2} y \partial_t^{1/2} q^\perp + \lambda^{-1} p q \right) \, dx \, dt &= 0, 
\end{aligned} \]  

for all test functions \( z, q \in H^{1,2}_{0,\text{per}}(Q_T) \). The optimality systems corresponding to every mode \( k \) are analogously derived as for minimization problem I (similar to (13) and (14)). In Section 3, we will derive the two-sided bounds for minimization problem II.

3. Two-sided Bounds for Minimization Problem I

3.1. Majorant for cost functional \((10)\). First, the results of \((10)\) on upper bounds for minimization problem I are presented, since they are needed later to derive the two-sided estimate. Let \( y = y(v) \) be the corresponding state to an arbitrary control \( v \). The following upper bound can be proved:

\[ J(y(v), v) \leq J^{\oplus}(\alpha, \beta; \eta, \tau, v) \quad \forall v \in L^2(Q_T), \]

for arbitrary \( \alpha, \beta > 0, \eta \in H^{1,1}_{0,\text{per}}(Q_T) \) and \( \tau \in H(\text{div}, Q_T) := \{ \tau \in [L^2(Q_T)]^d : \text{div}_x \tau(\cdot, t) \in L^2(\Omega) \text{ for a.e. } t \in (0, T) \} \),

where, for any \( \tau \in H(\text{div}, Q_T) \), the identity

\[ \int_\Omega \text{div} \tau w \, dx = - \int_\Omega \tau \cdot \nabla w \, dx \quad \forall w \in H^{1,1}_0(\Omega) \]

is valid. The guaranteed and fully computable majorant is given by

\[ \begin{aligned} J^{\oplus}(\alpha, \beta; \eta, \tau, v) := & \frac{1 + \alpha}{2} \| \eta - g_d \|^2 + \frac{(1 + \alpha)(1 + \beta)C_F^2}{2 \alpha \mu_1^2} \| R_2(\eta, \tau) \|^2 \\
&+ \frac{(1 + \alpha)(1 + \beta)C_F^4}{2 \alpha \beta \mu_1^2} \| R_1(\eta, \tau, v) \|^2 + \frac{\lambda}{2} \| v \|^2, \end{aligned} \]

where \( \mu_1 = \sqrt{\frac{1}{2} \min \{ \alpha, \beta \} } \) and \( C_F > 0 \) is the constant coming from the Friedrichs inequality. The parameters \( \alpha, \beta > 0 \) have been introduced in order to obtain a quadratic functional by applying Young’s inequality. The arbitrary functions \( \eta \in H^{1,1}_{0,\text{per}}(Q_T) \) and \( v \in L^2(Q_T) \) can be taken as the approximate solutions of the optimal control problem \((10)\) subject to \((1)\) and \( \tau \in H(\text{div}, Q_T) \) represents the image of the exact flux \( \nu \nabla \eta \). For the derivation of \((13)\), the following estimate for the approximation error has been used:

\[ |y(v) - \eta|_{1/2} \leq \frac{1}{\mu_1^2} (C_F \| R_1(\eta, \tau, v) \| + \| R_2(\eta, \tau) \|), \]
where

\[(20) \quad R_1(\eta, \tau, v) := \sigma \partial_t \eta - \text{div} \tau - v \quad \text{and} \quad R_2(\eta, \tau) := \tau - v \nabla \eta.\]

The derivation of estimate (19) can be found in [29]. The function \( J^\oplus(\alpha, \beta; \eta, \tau, v) \) is a sharp upper bound on \( J(y(v), v) \) for arbitrary but fixed \( v \) as well as on the optimal value \( J(y(u), u) \), i.e.,

\[(21) \quad \inf_{\eta \in H^{1,1}_{0,\text{per}}(Q_T), \tau \in H(\text{div}; Q_T)} J^\oplus(\alpha, \beta; \eta, \tau, v) = J(y(u), u), \]

since the infimum is attained for the optimal control \( u \), its corresponding state \( y(u) \) and its exact flux \( \nu \nabla y(u) \), and for \( \alpha \) going to zero. Therefore, we have the estimate

\[(22) \quad J(y(u), u) \leq J^\oplus(\alpha, \beta; \eta, \tau, v) \quad \forall \eta \in H^{1,1}_{0,\text{per}}(Q_T), \tau \in H(\text{div}; Q_T), v \in L^2(Q_T), \alpha, \beta > 0.\]

3.2. Minorant for cost functional (10). In this work, we complement the guaranteed upper bounds for the discretization error in state and control of minimizing cost functional \( J \) defined in (10) subject to (1). This is done by obtaining fully computable lower bounds for \( \eta \) as well as on the optimal value \( J(y(u), u) \). For any \( \eta \in H^{1,1}_{0,\text{per}}(Q_T) \), let \( y = y(u) \) be the optimal state corresponding to the optimal control function \( u \in L^2(Q_T) \), which is connected with the adjoint state \( p = p(u) \) by the identity \( u = -\lambda^{-1}p(u) \). Then \( y = y(u) \) is the solution of the variational formulation

\[(23) \quad \int_{Q_T} \left( \nu \nabla y \cdot \nabla q + \sigma \partial_t^{1/2} y \partial_t^{1/2} q^\perp + \lambda^{-1} p q \right) dx \, dt = 0 \quad \forall q \in H^{1,1}_{0,\text{per}}(Q_T) \]

of the boundary value problem (11) (see also (24)). For any \( \eta \in H^{1,1}_{0,\text{per}}(Q_T) \), we have that

\[J(y(u), u) = \frac{1}{2} \|y - \eta\|^2 + \int_{Q_T} (y - \eta) (\eta - y_d) \, dx \, dt + \frac{1}{2} \|\eta - y_d\|^2 + \lambda \|u\|^2.\]

Since \( \frac{1}{2} \|y - \eta\|^2 \geq 0 \) and using the identity \( u = -\lambda^{-1}p(u) \), we can estimate \( J \) from below by

\[(24) \quad J(y(u), u) = J(y(u), p(u)) \geq \frac{1}{2} \|\eta - y_d\|^2 + \frac{1}{2\lambda} \|p\|^2 + \int_{Q_T} (y - \eta) (\eta - y_d) \, dx \, dt.\]

For \( \eta \in H^{1,1}_{0,\text{per}}(Q_T) \), let \( p_\eta, \tilde{p}_\eta \in H^{1,1}_{0,\text{per}}(Q_T) \) be the solutions to the equations

\[(25) \quad \int_{Q_T} \left( \nu \nabla p_\eta \cdot \nabla z - \sigma \partial_t^{1/2} p_\eta \partial_t^{1/2} z^\perp \right) dx \, dt = \int_{Q_T} (\eta - y_d) z \, dx \, dt,\]

\[(26) \quad \int_{Q_T} \left( \nu \nabla \eta \cdot \nabla q + \sigma \partial_t^{1/2} \eta \partial_t^{1/2} q^\perp + \lambda^{-1} \tilde{p}_\eta q \right) dx \, dt = 0,\]

for all test functions \( z, q \in H^{1,1}_{0,\text{per}}(Q_T) \), respectively.

Remark 1. Note that we assumed that \( \eta \in H^{1,1}_{0,\text{per}}(Q_T) \) according to the derivation of the majorant, but so far the assumption \( \eta \in H^{1,1}_{0,\text{per}}(Q_T) \) would be enough.

Adding and subtracting \( p_\eta \) in (24) together with \( \frac{1}{2\lambda} \|p - p_\eta\|^2 \geq 0 \) yields the estimate

\[J(y(u), u) = J(y(u), p(u)) \geq \frac{1}{2} \|\eta - y_d\|^2 + \frac{1}{2\lambda} \|p_\eta\|^2 + \int_{Q_T} (y - \eta) (\eta - y_d) \, dx \, dt + \int_{Q_T} \lambda^{-1} (p - p_\eta) p_\eta \, dx \, dt.\]
By using (25) and identity (8), it follows that

\[
\mathcal{J}(y(u), u) \geq \frac{1}{2} \| \eta - y_d \|^2 + \frac{1}{2\lambda} \| p \eta \|^2 + \int_{Q_T} \lambda^{-1} (p - p_\eta) p_\eta \, dx \, dt
\]

\[
+ \int_{Q_T} \left( \nu \nabla p_\eta \cdot \nabla (y - \eta) - \sigma \partial_t^{1/2} p_\eta \partial_t^{1/2} (y - \eta) \right) \, dx \, dt
\]

\[
= \frac{1}{2} \| \eta - y_d \|^2 + \frac{1}{2\lambda} \| p \eta \|^2 + \int_{Q_T} \lambda^{-1} (p - p_\eta) p_\eta \, dx \, dt
\]

\[
+ \int_{Q_T} \left( \nu \nabla y \cdot \nabla \eta + \sigma \partial_t^{1/2} y \partial_t^{1/2} p_\eta + \lambda^{-1} p p_\eta \right) \, dx \, dt
\]

\[
= \frac{1}{2} \| \eta - y_d \|^2 + \frac{1}{2\lambda} \| p \eta \|^2
\]

\[
+ \int_{Q_T} \left( \nu \nabla \eta \cdot \nabla p_\eta + \sigma \partial_t^{1/2} \eta \partial_t^{1/2} p_\eta + \lambda^{-1} p p_\eta \right) \, dx \, dt.
\]

Using the equations (28) and (29) leads to the estimate

\[
\mathcal{J}(y(u), u) \geq \frac{1}{2} \| \eta - y_d \|^2 + \frac{1}{2\lambda} \| \eta \|^2 + \int_{Q_T} \lambda^{-1} (p_\eta - p_\eta) p_\eta \, dx \, dt.
\]

We introduce now the arbitrary function \( \zeta \in H_{0, \text{per}}^{1,1}(Q_T) \). Note that at the moment \( \zeta \in H_{0, \text{per}}^{1,1}(Q_T) \) would be enough, but the higher regularity in time will be needed in another step. This goes along with the higher regularity assumption on \( \eta \) (see Remark 1). Since \( \frac{1}{2\lambda} \| \eta - \zeta \|^2 \geq 0 \), we have that

\[
\mathcal{J}(y(u), u) \geq \frac{1}{2} \| \eta - y_d \|^2 + \frac{1}{2\lambda} \| \zeta \|^2 + \int_{Q_T} \lambda^{-1} (p_\eta - \zeta) + \lambda^{-1} \zeta^2 \, dx \, dt.
\]

Now we add and subtract \( \lambda^{-1} \bar{p}_\eta \zeta \) in the last integral as well as use equation (29) again. Moreover, we exploit the fact that we have assumed that \( \eta \in H_{0, \text{per}}^{1,1}(Q_T) \), hence, we can apply the identities (3). Altogether these steps yield the estimate

\[
\mathcal{J}(y(u), u) \geq \frac{1}{2} \| \eta - y_d \|^2 + \frac{1}{2\lambda} \| \zeta \|^2 - \int_{Q_T} \left( \nu \nabla \eta \cdot \nabla \zeta + \sigma \partial_t \eta \zeta + \lambda^{-1} \zeta^2 \right) \, dx \, dt
\]

\[
+ \int_{Q_T} \lambda^{-1} \zeta (p_\eta - \bar{p}_\eta) \, dx \, dt.
\]

In the following, we need to estimate the last integral of this expression in order to formulate a computable lower bound for the cost functional. For that let us first prove a computable upper bound for the error in the adjoint state, which is presented in the following theorem. Note that here we will need the higher regularity assumption (in time) on \( \zeta \).

**Theorem 1.** Let \( y_d \in L^2(Q_T) \) be given and let \( p_\eta \in H_{0, \text{per}}^{1,1}(Q_T) \) meet equation (29) with \( \eta \in H_{0, \text{per}}^{1,1}(Q_T) \) satisfying assumptions (3). For any \( \zeta \in H_{0, \text{per}}^{1,1}(Q_T) \), we have that

\[
\| \nabla (p_\eta - \zeta) \| \leq \frac{1}{\mu_1} \left( C_F \| \mathcal{R}_3(\zeta, \rho, \eta) \| + \| \mathcal{R}_4(\zeta, \rho) \| \right),
\]

where \( \mu_1 = \frac{1}{\lambda} \min \{ \lambda, 2 \} \), \( \mathcal{R}_3(\zeta, \rho, \eta) = \eta - y_d + \text{div } \rho + \sigma \partial_t \zeta \) and \( \mathcal{R}_4(\zeta, \rho) = \rho - \nu \nabla \zeta \) with \( \rho \in H(\text{div}, Q_T) \) and \( C_F > 0 \) is the constant coming from the Friedrichs inequality.
Proof. Let us consider the adjoint equation (26). Adding and subtracting \( \zeta \in H_{0,per}^{1,1}(Q_T) \) in the left-hand side of the equation leads to

\[
\begin{array}{c}
\left\{ \int_{Q_T} (\nu \nabla (p_\eta - \zeta) \cdot \nabla z - \sigma \partial_t \frac{1}{2} (p_\eta - \zeta) \partial_t \frac{1}{2} z^+) dx dt = \int_{Q_T} (\eta - y_\rho) z dx dt \\
- \int_{Q_T} \nu \nabla \zeta \cdot \nabla z dx dt + \int_{Q_T} \sigma \partial_t \frac{1}{2} \zeta \partial_t \frac{1}{2} z^- dx dt.
\end{array}
\]

(31)

Next we introduce the auxiliary variable \( \rho \in H(div, Q_T) \). Together with using that \( \zeta \in H_{0,per}^{1,1}(Q_T) \) as well as applying the Cauchy-Schwarz and Friedrichs inequalities, the following estimate for the right-hand side of (31) can be obtained:

\[
\begin{align*}
\sup_{0 \neq z \in H_{0,per}^{1,1}(Q_T)} & \int_{Q_T} (\eta - y_\rho + \nu \nabla \zeta) \cdot \nabla z dx dt + \int_{Q_T} (\rho - \nu \nabla \zeta) \cdot \nabla z dx dt \\
\leq & \sup_{0 \neq z \in H_{0,per}^{1,1}(Q_T)} \| \eta - y_\rho + \nu \nabla \zeta \| \| z \| + \| \rho - \nu \nabla \zeta \| \| \nabla z \|
\leq & \sup_{0 \neq z \in H_{0,per}^{1,1}(Q_T)} C_F \| \eta - y_\rho + \nu \nabla \zeta \| \| \nabla z \| \\
\leq & C_F \| \eta - y_\rho + \nu \nabla \zeta \|.
\end{align*}
\]

Using (23) and the orthogonality relations (4), we can prove the following estimate from below for the left-hand side of (31):

\[
\begin{align*}
\sup_{0 \neq z \in H_{0,per}^{1,1}(Q_T)} & \int_{Q_T} (\nu \nabla (p_\eta - \zeta) \cdot \nabla z - \sigma \partial_t \frac{1}{2} (p_\eta - \zeta) \partial_t \frac{1}{2} z^+) dx dt \\
\geq & \mu_1 \| p_\eta - \zeta \|_{0, \frac{1}{2}}.
\end{align*}
\]

Combining now both estimates together with \( \| \nabla (p_\eta - \zeta) \| \leq \| p_\eta - \zeta \|_{0, \frac{1}{2}} \) we finally derive estimate (30).

Now we have all the tools in order to estimate the last term of (26) as follows

\[
\begin{array}{c}
\int_{Q_T} \lambda^{-1} (\zeta - p_\eta) (p_\eta - \zeta + \zeta - \tilde{p}_\eta) dx dt \\
= \int_{Q_T} (\lambda^{-1} (\zeta - p_\eta)(p_\eta - \zeta) + \lambda^{-1} (\zeta - p_\eta)(\zeta - \tilde{p}_\eta)) dx dt \\
= -\lambda^{-1} \| \zeta - p_\eta \|^2 + \int_{Q_T} (\lambda^{-1} (\zeta - p_\eta)(\zeta - \tilde{p}_\eta)) dx dt \\
= -\lambda^{-1} \| \zeta - p_\eta \|^2 + \int_{Q_T} \left( \nu \nabla \eta \cdot \nabla (\zeta - p_\eta) + \sigma \partial_t \frac{1}{2} \eta \partial_t \frac{1}{2} (\zeta - p_\eta) \right) dx dt \\
+ \int_{Q_T} \lambda^{-1} \zeta (\zeta - p_\eta) dx dt \\
= -\lambda^{-1} \| \zeta - p_\eta \|^2 + \int_{Q_T} (\sigma \partial_t \eta - \text{div} \tau + \lambda^{-1} \zeta) (\zeta - p_\eta) dx dt \\
+ \int_{Q_T} (\nu \nabla \eta \cdot \nabla (\zeta - p_\eta)) dx dt \\
= -\lambda^{-1} C_F^2 \| \nabla (\zeta - p_\eta) \|^2 - (C_F \| R_1(\eta, \tau, -\lambda^{-1} \zeta) \| + \| R_2(\eta, \tau) \|) \| \nabla (\zeta - p_\eta) \| \\
\geq & -\frac{C_F^2}{\mu_1 \lambda} (C_F \| R_3(\zeta, \rho, \eta) \| + \| R_4(\zeta, \rho) \|)^2 \\
& - \frac{1}{\mu_1} (C_F \| R_1(\eta, \tau, -\lambda^{-1} \zeta) \| + \| R_2(\eta, \tau) \|) (C_F \| R_3(\zeta, \rho, \eta) \| + \| R_4(\zeta, \rho) \|),
\end{array}
\]

(32)
where \( \tau, \rho \in H(\text{div}, Q_T) \) and we have used equation (26), relations (4), Cauchy-Schwarz’ and Friedrichs’ inequalities, estimate (31) and that \( \eta \in H_{0, \text{per}}^{1, 1}(Q_T) \).

Finally, we obtain the following estimate from below:

\[
J(y(u), u) \geq J^{\ominus}(\eta, \zeta, \tau, \rho) \quad \forall \eta, \zeta \in H_{0, \text{per}}^{1, 1}(Q_T), \tau, \rho \in H(\text{div}, Q_T),
\]

where the minorant is given by

\[
\begin{align*}
J^{\ominus}(\eta, \zeta, \tau, \rho) &= \frac{1}{2}||\eta - y_d||^2 + \frac{1}{2\lambda}\|\zeta\|_2^2 - \int_{Q_T} (\nu \nabla \eta \cdot \nabla \zeta + \sigma \partial_t \eta \zeta + \lambda^{-1}\zeta^2) \, dx \, dt \\
&\quad - \frac{C_F}{\mu(2\lambda)} (C_F \| R_3(\zeta, \rho) \| + \| R_4(\zeta, \rho) \|)^2 \\
&\quad - \frac{1}{\mu} (C_F \| R_1(\eta, \tau, -\lambda^{-1}\zeta) \| + \| R_2(\eta, \tau) \| ) (C_F \| R_3(\zeta, \rho) \| + \| R_4(\zeta, \rho) \|).
\end{align*}
\]

Note that the minorant is fully computable.

**Theorem 2.** The exact upper bound of the minorant \( J^{\ominus} \) defined in (37) coincides with the optimal value of the cost functional (10) subject to (1), or, equivalently, of the optimality system (12), i.e.,

\[
\sup_{\eta, \zeta \in H_{0, \text{per}}^{1, 1}(Q_T), \tau, \rho \in H(\text{div}, Q_T)} J^{\ominus}(\eta, \zeta, \tau, \rho) = J(y(u), u).
\]

**Proof.** For the exact solution \( u, \eta = y(u), \zeta = p(u), \tau = \nu \nabla y(u) \) and \( \rho = \nu \nabla p(u) \), the estimate is sharp, i.e.,

\[
J^{\ominus}(y(u), p(u), \nu \nabla y(u)), \nu \nabla p(u)) = \frac{1}{2}||y - y_d||^2 + \frac{1}{2\lambda}\|p\|_2^2 = \frac{1}{2}||y - y_d||^2 + \frac{\lambda}{2}\|u\|^2 = J(y(u), u).
\]

\[\square\]

**Remark 2.** Note that similar to the convergence results shown in (12), sequences of \( (\eta, \zeta, \tau, \rho) \) converge in finite-dimensional subspaces that are limit dense in \( H_{0, \text{per}}^{1, 1}(Q_T) \) and \( H(\text{div}, Q_T) \) to the exact solution \( (y(u), p(u), \nu \nabla y(u), \nu \nabla p(u)) \) of the continuous problem. Moreover, corresponding sequences of the minorant \( J^{\ominus} \) converge to the exact value of the cost functional \( J \).

### 3.3. A posteriori error estimates for control and state.

In this section, we present guaranteed upper bounds for the discretization errors of the control and the state measured in the following norm:

\[
|||u - v|||^2 := \frac{1}{2}||y(u) - y(v)||^2 + \frac{\lambda}{2}\|u - v\|^2,
\]

making use of the ideas based on the work by Mikhlin [37] but generalized for the class of optimal control problems, see also [43]. Other results in the context of functional a posteriori estimates for optimal control problems can be found, e.g., in [12, 13, 33, 39]. The next theorem was proved for the elliptic case (together with control constraints) in [49]. The norm \(|| \cdot |||\) defined in (36) can be represented in terms of the state and the adjoint state (instead of the control), since there are no control constraints imposed. Hence, \( u = -\lambda^{-1}p(u), v = -\lambda^{-1}p(v) \), and

\[
|||u - v|||^2 = \frac{1}{2}||y(u) - y(v)||^2 + \frac{1}{2\lambda}\|p(u) - p(v)\|^2.
\]

**Theorem 3.** For any control function \( v \in L^2(Q_T) \), we have the identity

\[
|||u - v|||^2 = J(y(v), v) - J(y(u), u).
\]
Proof. We compute the difference
\[
\mathcal{J}(y(v), v) - \mathcal{J}(y(u), u) = \frac{1}{2} ||y(v) - y_d||^2 - \frac{1}{2} ||y(u) - y_d||^2 + \frac{\lambda}{2} ||v||^2 - \frac{\lambda}{2} ||u||^2
\]
\[
= \frac{1}{2} \int_{Q_T} (y(v) + y(u) - 2y_d)(y(v) - y(u)) \, dx \, dt + \frac{\lambda}{2} \int_{Q_T} (v + u)(v - u) \, dx \, dt
\]
\[
= \frac{1}{2} \int_{Q_T} (y(v) - y(u)) + 2y_d)(y(v) - y(u)) \, dx \, dt + \frac{\lambda}{2} \int_{Q_T} (v - u + 2u)v - u) \, dx \, dt
\]
\[
= \frac{1}{2} ||y(u) - y(v)||^2 + \int_{Q_T} (y(u) - y_d)(y(v) - y(u)) \, dx \, dt
\]
\[
+ \frac{\lambda}{2} ||u - v||^2 + \lambda^{-1} \int_{Q_T} p(u)(p(v) - p(u)) \, dx \, dt,
\]
where we also used the identities \( u = -\lambda^{-1}p(u) \) and \( v = -\lambda^{-1}p(v) \). Since the adjoint states \( p(u), p(v) \in H^{1,1}_{0,per}(Q_T) \) fulfill (12) for the corresponding states \( y(u), y(v) \in H^{1,1}_{0,per}(Q_T) \), we obtain
\[
\mathcal{J}(y(v), v) - \mathcal{J}(y(u), u)
\]
\[
= \frac{1}{2} ||y(u) - y(v)||^2 + \int_{Q_T} (\nu \nabla p(u)(\nabla y(v) - \nabla y(u)) - \sigma \partial^{1/2}_i p(u) \partial^{1/2}_i (y(v) - y(u)) \, dx \, dt
\]
\[
+ \frac{\lambda}{2} ||u - v||^2 + \lambda^{-1} \int_{Q_T} p(u)(p(v) - p(u)) \, dx \, dt
\]
\[
= \frac{1}{2} ||y(u) - y(v)||^2 + \int_{Q_T} (\nu \nabla y(v) \cdot \nabla p(u) + \sigma \partial^{1/2}_i y(v) \partial^{1/2}_i p(u)) \, dx \, dt
\]
\[
+ \frac{\lambda}{2} ||u - v||^2 + \lambda^{-1} \int_{Q_T} p(u)(p(v) - p(u)) \, dx \, dt,
\]
which finally leads to the equality relation □

Using the result of Theorem 3, we can derive the majorant for the discretization errors of control and state measured in the norm (38).

Theorem 4. We obtain the following error majorant for any control function \( v \in L^2(Q_T) \):

\[
||u - v||^2 \leq \mathcal{M}^\square(\alpha, \beta; \eta, \zeta, \tau, \rho, v) := \mathcal{J}^\square(\alpha, \beta; \eta, \zeta, \tau, \rho) - \mathcal{J}^\square(\eta, \zeta, \tau, \rho)
\]

with

\[
\mathcal{M}^\square(\alpha, \beta; \eta, \zeta, \tau, \rho, v) = \frac{\alpha}{2} ||\eta - y_d||^2 + \frac{(1 + \alpha)(1 + \beta)C_F^2}{2\alpha \mu_1^2} ||R_2(\eta, \tau)||^2
\]
\[
+ \frac{(1 + \alpha)(1 + \beta)C_F^2}{2\alpha \beta \mu_1^2} ||R_1(\eta, \tau, v)||^2 + \frac{\lambda}{2} ||v||^2 - \frac{1}{2\lambda} ||\zeta||^2
\]
\[
+ \int_{Q_T} (\nu \nabla \eta \cdot \nabla \zeta + \sigma \partial_\eta \zeta + \lambda^{-1} \zeta^2) \, dx \, dt + \frac{C_F^2}{\mu_1^2 \lambda} (C_F ||R_3(\zeta, \rho, \eta)|| + ||R_4(\zeta, \rho)||)^2
\]
\[
+ \frac{1}{\mu_1} (C_F ||R_1(\eta, \tau, -\lambda^{-1} \zeta)|| + ||R_2(\eta, \tau)||) (C_F ||R_3(\zeta, \rho, \eta)|| + ||R_4(\zeta, \rho)||)
\]

for arbitrary \( \eta, \zeta \in H^{1,1}_{0,per}(Q_T) \), \( \tau, \rho \in H(\text{div}, Q_T) \) and \( \alpha, \beta > 0 \), and where \( \mu_1 = \frac{1}{\sqrt{2}} \min\{\nu, \zeta\} \).

Proof. Applying (38) together with (22) and (33) finally leads to the estimate (39).

□

Proposition 1. The majorant \( \mathcal{M}^\square(\alpha, \beta; \eta, \zeta, \tau, \rho, v) \) defined in (39) attains the exact lower bound on the exact solution of the optimal control problem (17) w.r.t. (7), or, equivalently, of the optimality system (12), i.e.,

\[
\inf_{\eta, \zeta \in H^{1,1}_{0,per}(Q_T), \tau, \rho \in H(\text{div}, Q_T), \ v \in L^2(Q_T), \alpha, \beta > 0} \mathcal{M}^\square(\alpha, \beta; \eta, \zeta, \tau, \rho, v) = 0.
\]
The infimum is attained for \( v = u, \eta = y(u), \zeta = p(u) = -\lambda u, \tau = \nu \nabla y(u) \) and \( \rho = \nu \nabla p(u) \).

**Proof.** We have that
\[
\mathcal{M}(\alpha, \beta; y(u), p(u), \nu \nabla y(u), \nu \nabla p(u), u) = \frac{\alpha}{2} ||y(u) - y_d||^2,
\]
which is zero if we let \( \alpha \) go to zero. \( \square \)

Although the majorant \( \mathcal{M} \) is a guaranteed, computable and sharp upper estimate for the discretization error in the combined norm, it only decreases with order \( h \) when discretizing the mesh. However, the combined norm \( || . || \) is an \( L^2 \)-norm, and, hence, decreases with order \( h^2 \). So the majorant \( \mathcal{M} \) is an overestimation for the combined norm. Following the idea from [49], we introduce another norm which is a weighted \( H^1 \)-norm for the state and decreases with the same order as the majorant. More precisely, we derive an estimate for the discretization error measured in the following norm:

\[
||u - v||^2_1 := \frac{1}{2} ||y(u) - y(v)||^2 + \frac{2\lambda h^2}{C_F^2} ||y(u) - y(v)||^2_1,
\]

**Theorem 5.** For any control function \( v \in L^2(Q_T) \), we have the estimate

\[
||u - v||^2_1 \leq \mathcal{J}(y(v), v) - \mathcal{J}(y(u), u) + \frac{3\lambda}{2C_F^2}(C_F||R_1(\eta, \tau, v)|| + ||R_2(\eta, \tau)||)^2,
\]

where \( R_1(\eta, \tau, v) := \sigma \partial_t \eta - \text{div} \tau - v \) and \( R_2(\eta, \tau) := \tau - \nu \nabla \eta \).

**Proof.** Let \( \delta > 0 \) be an arbitrary but fixed parameter. Adding and subtracting \( \eta \) as well as applying triangle inequality for \( \frac{\mu_1^2}{C_F^2}|y(u) - y(v)|^2_1 \), we derive the following estimate:

\[
\frac{\mu_1^2}{C_F^2}|y(u) - y(v)|^2_1 \leq \frac{\mu_1^2}{C_F^2} \left( |y(u) - \eta|^2_1 + |y(v) - \eta|^2_1 \right).
\]

Using [20], adding and subtracting \( v \) as well as applying twice triangle inequality, we arrive at the estimate

\[
\frac{\mu_1^2}{C_F^2}|y(u) - y(v)|^2_1 \leq \frac{1}{2C_F^2} \left( \left( ||\tau - \nu \nabla \eta|| + C_F||\sigma \partial_t \eta - \text{div} \tau - v|| + C_F||u - v|| \right)^2 \right.
\]

\[
+ \left. \left( ||\tau - \nu \nabla \eta|| + C_F||\sigma \partial_t \eta - \text{div} \tau - v|| \right)^2 \right)
\]

\[
\leq \frac{3}{4C_F^2} (||\tau - \nu \nabla \eta|| + C_F||\sigma \partial_t \eta - \text{div} \tau - v||)^2 + \frac{1}{4\delta} ||u - v||^2.
\]

By using (58) and the previous estimate, we derive the inequality

\[
||u - v||^2 + \frac{\mu_1^2}{C_F^2}|y(u) - y(v)|^2_1 \leq \frac{1}{4\delta} ||u - v||^2
\]

\[
= \frac{1}{2} ||y(u) - y(v)||^2 + \frac{\mu_1^2}{C_F^2}|y(u) - y(v)|^2_1 + \left( \frac{\lambda}{2} - \frac{1}{4\delta} \right) ||u - v||^2
\]

\[
\leq \mathcal{J}(y(v), v) - \mathcal{J}(y(u), u) + \frac{3\lambda}{2C_F^2}(C_F||R_1(\eta, \tau, v)|| + ||R_2(\eta, \tau)||)^2.
\]

Finally, choosing \( \delta = 1/(2\lambda) \) yields the estimate (41).

\( \square \)

This theorem directly leads to the following two results presented in Propositions [2] and [3].

**Proposition 2.** The following error majorant for any control function \( v \in L^2(Q_T) \) is obtained:

\[
||u - v||^2_1 \leq \mathcal{M}(\alpha, \beta; \eta, \zeta, \tau, \rho, v) := \mathcal{J}(\alpha, \beta; \eta, \tau, v) - \mathcal{J}(\alpha, \beta; \eta, \zeta, \tau, \rho)
\]

\[
+ \frac{3\lambda}{2C_F^2}(C_F||R_1(\eta, \tau, v)|| + ||R_2(\eta, \tau)||)^2
\]
with
\[
\mathcal{M}^\oplus_1(\alpha, \beta; \eta, \zeta, \tau, \rho, v) = \frac{\alpha}{2} \|\eta - y_d\|^2 + \frac{(1 + \alpha)(1 + \beta)C_2^2}{2\alpha\beta\mu^2} \|\mathcal{R}_2(\eta, \tau)\|^2 \\
+ \frac{(1 + \alpha)(1 + \beta)C_2^2}{2\alpha\beta\mu^2} \|\mathcal{R}_1(\eta, \tau, v)\|^2 + \frac{\lambda}{2} \|v\|^2 - \frac{1}{2\lambda} \|\zeta\|^2 \\
+ \int_{Q_T} (\nu \nabla \eta \cdot \nabla \zeta + \sigma \partial_0 \eta \zeta + \lambda^{-1} \zeta^2) \, dx \, dt + \frac{C_2^2}{\mu^2} (C_F \|\mathcal{R}_3(\zeta, \rho, \eta)\| + \|\mathcal{R}_4(\zeta, \rho)\|)^2 \\
+ \frac{1}{\mu} (C_F \|\mathcal{R}_1(\eta, \tau, -\lambda^{-1} \zeta)\| + \|\mathcal{R}_2(\eta, \tau)\|) (C_F \|\mathcal{R}_3(\zeta, \rho, \eta)\| + \|\mathcal{R}_4(\zeta, \rho)\|) \\
+ \frac{3\lambda}{2C_2^2} (C_F \|\mathcal{R}_1(\eta, \tau, v)\| + \|\mathcal{R}_2(\eta, \tau)\|)^2
\]
for arbitrary \(\eta, \zeta \in H_{0, \text{per}}(Q_T), \tau, \rho \in H(\text{div}, Q_T)\) and \(\alpha, \beta > 0\), and where \(\mu = \sqrt{2} \min\{\mu_1, \mu_2\}\).

**Proposition 3.** The majorant \(\mathcal{M}^\oplus_1(\alpha, \beta; \eta, \zeta, \tau, \rho, v)\) defined in (47) attains the exact lower bound on the exact solution of the optimal control problem (19) w.r.t. (3), or, equivalently, of the optimality system (22), i.e.,
\[
\inf_{\eta, \zeta \in H_{0, \text{per}}(Q_T), \tau, \rho \in H(\text{div}, Q_T), \alpha, \beta > 0} \mathcal{M}^\oplus_1(\alpha, \beta; \eta, \zeta, \tau, \rho, v) = 0.
\]
The infimum is attained for \(v = u, \eta = y(u), \zeta = p(u) = -\lambda u, \tau = \nu \nabla y(u)\) and \(\rho = \nu \nabla p(u)\).

4. Two-sided Bounds for Minimization Problem II

In this section, we analogously derive the majorants and minorants for the second cost functional. However, we will skip most of the details, since the derivation is very similar as in Section 3.

4.1. Majorant for cost functional (20). Adding and subtracting \(\nabla \eta\) in the cost functional \(\mathcal{J}(y(v), v)\), applying the triangle inequality as well as using the estimate
\[
\|\nabla y(v) - \nabla \eta\|^2 \leq |y(v) - \eta|^2 = \|\nabla y(v) - \nabla \eta\|^2 + \|\partial_1^{1/2} y(v) - \partial_1^{1/2} \eta\|^2,
\]
we conclude that
\[
\mathcal{J}(y(v), v) \leq \frac{1}{2} \left(\|\nabla \eta - g_d\| + |y(v) - \eta|, \frac{1}{\mu}\right)^2 + \frac{\lambda}{2} \|v\|^2.
\]
Together with (19) this leads to the estimate
\[
\mathcal{J}(y(v), v) \leq \frac{1}{2} \left(\|\nabla \eta - g_d\| + \frac{1}{\mu} \|\mathcal{R}_2(\eta, \tau)\| + \frac{C_F}{\mu} \|\mathcal{R}_1(\eta, \tau, v)\|\right)^2 + \frac{\lambda}{2} \|v\|^2,
\]
where again \(\mu = \sqrt{2} \min\{\mu_1, \mu_2\}\) as well as \(\mathcal{R}_1(\eta, \tau, v)\) and \(\mathcal{R}_2(\eta, \tau)\) are defined as in (20). Finally, introducing parameters \(\alpha, \beta > 0\) and applying Young’s inequality, we can reformulate the estimate such that the right-hand side is given by a quadratic functional as follows
\[
\mathcal{J}(y(v), v) \leq \mathcal{J}^\oplus(\alpha, \beta; \eta, \tau, v) \quad \forall v \in L^2(Q_T),
\]
where
\[
\mathcal{J}^\oplus(\alpha, \beta; \eta, \tau, v) := \frac{1 + \alpha}{2} \|\nabla \eta - g_d\|^2 + \frac{(1 + \alpha)(1 + \beta)C_2^2}{2\alpha\beta\mu^2} \|\mathcal{R}_2(\eta, \tau)\|^2 \\
+ \frac{(1 + \alpha)(1 + \beta)C_2^2}{2\alpha\beta\mu^2} \|\mathcal{R}_1(\eta, \tau, v)\|^2 + \frac{\lambda}{2} \|v\|^2.
\]
The majorant (33) provides a guaranteed upper bound of the cost functional, which can be computed for any approximate control and state functions. Moreover, minimization of this functional
with respect to \( \eta, \tau, v \) and \( \alpha, \beta > 0 \) yields the exact value of the cost functional. Analogously to (21), we can show that
\[
\inf_{\eta \in H^{1,1}_{0,\text{per}}(Q_T), \tau \in H(\text{div}, Q_T)} \tilde{J}^\oplus(\alpha, \beta; \eta, \tau, v) = \tilde{J}(y(u), u),
\]
and that
\[
\tilde{J}(y(u), u) \leq \tilde{J}^\oplus(\alpha, \beta; \eta, \tau, v) \quad \forall \eta \in H^{1,1}_{0,\text{per}}(Q_T), \tau \in H(\text{div}, Q_T), v \in L^2(Q_T), \alpha, \beta > 0.
\]

### 4.2. Minorant for cost functional

Let us derive now the minorant. For any \( \eta \in H^{1,1}_{0,\text{per}}(Q_T) \), we have that
\[
\tilde{J}(y(u), v) = \frac{1}{2} \| \nabla y - \nabla \eta \|^2 + \int_{Q_T} (\nabla y - \nabla \eta) \cdot (\nabla \eta - g_d) \, dx \, dt + \frac{1}{2} \| \nabla \eta - g_d \|^2 + \frac{\lambda}{2} \| v \|^2
\]
for all \( v \in L^2(Q_T) \). The first norm is again greater or equal to zero, together with the identity \( v = -\lambda^{-1} p(v) \), we can estimate \( \tilde{J} \) from below by
\[
\tilde{J}(y(v), v) = \tilde{J}(y(v), p(v)) \geq \frac{1}{2} \| \nabla \eta - g_d \|^2 + \frac{1}{2\lambda} \| p \|^2 + \int_{Q_T} (\nabla y - \nabla \eta) \cdot (\nabla \eta - g_d) \, dx \, dt.
\]

Note that Remark 1 applies here as well. For \( \eta \in H^{1,1}_{0,\text{per}}(Q_T) \), let \( p_\eta, \tilde{p}_\eta \in H^{1,1}_{0,\text{per}}(Q_T) \) be the solutions to the equations
\[
\int_{Q_T} \left( \nu \sqrt{p_\eta} \cdot \nabla \sqrt{z} - \sigma \sqrt{p_\eta} \cdot z \right) \, dx \, dt = \int_{Q_T} (\nabla \eta - g_d) \cdot \nabla z \, dx \, dt,
\]
\[
\int_{Q_T} \left( \nu \nabla \eta \cdot \nabla q + \sigma \sqrt{p_\eta} \cdot z \right) \, dx \, dt = 0,
\]
for all test functions \( z, q \in H^{1,1}_{0,\text{per}}(Q_T) \). The minorant for the second minimization functional is derived in the same way as for the first problem (see Subsection 3.2). Adding and subtracting \( p_\eta \) together with \( \frac{1}{2\lambda} \| p - p_\eta \|^2 \geq 0 \) yields the estimate
\[
\tilde{J}(y(v), u) \geq \frac{1}{2} \| \nabla \eta - g_d \|^2 + \frac{1}{2\lambda} \| p_\eta \|^2 + \int_{Q_T} \lambda^{-1} (p_\eta - p_\eta) p_\eta \, dx \, dt.
\]

By using equation (41), identity (8) (analogously to (27)) and then using equations (25) and (47) leads to the estimate
\[
\tilde{J}(y(u), u) \geq \frac{1}{2} \| \nabla \eta - g_d \|^2 + \frac{1}{2\lambda} \| p_\eta \|^2 + \int_{Q_T} \lambda^{-1} (\tilde{p}_\eta - p_\eta) p_\eta \, dx \, dt.
\]

Now we introduce again the arbitrary function \( \zeta \in H^{1,1}_{0,\text{per}}(Q_T) \) and perform the estimation following (29), applying Theorem 1 and using (32). This leads to the following estimate from below:
\[
\tilde{J}(y(u), u) \geq \tilde{J}^\ominus(\eta, \zeta, \tau, \rho) \quad \forall \eta, \zeta \in H^{1,1}_{0,\text{per}}(Q_T), \tau, \rho \in H(\text{div}, Q_T),
\]
where the fully computable minorant is given by
\[
\tilde{J}^\ominus(\eta, \zeta, \tau, \rho) = \frac{1}{2} \| \nabla \eta - g_d \|^2 + \frac{1}{2\lambda} \| \zeta \|^2 - \int_{Q_T} (\nu \nabla \eta \cdot \nabla \zeta + \sigma \partial_\eta \zeta + \lambda^{-1} \zeta^2) \, dx \, dt
\]
\[
- \frac{C_F}{\mu^2 \lambda} \left( C_F \| R_3(\zeta, \rho, \eta) \| + \| R_4(\zeta, \rho) \| \right)^2
\]
\[
- \frac{1}{\mu} \left( C_F \| R_1(\eta, \tau, -\lambda^{-1} \zeta) \| + \| R_2(\eta, \tau) \| \right) (C_F \| R_3(\zeta, \rho, \eta) \| + \| R_4(\zeta, \rho) \|).
\]

The following theorem is analogous to Theorem 2 and so is its proof.
Theorem 6. The exact upper bound of the minorant \( \tilde{J} ^\circ \) defined in (50) coincides with the optimal value of the cost functional \( J \) subject to (4), or, equivalently, of the optimality system (10), i.e.,

\[
\sup_{\eta, \zeta \in H^1_0, \per, Q_T, \tau, \rho \in H(\text{div}, Q_T)} \tilde{J} ^\circ (\eta, \zeta, \tau, \rho) = \tilde{J} (y(u), u).
\]

Note that Remark 2 can be applied for minimization problem II as well.

4.3. A posteriori error estimates for control and state. We present guaranteed upper bounds for the discretization errors of the control and the state similar as in Subsection 3.3, but measured in the norm

\[
|||u - v|||^2 := \frac{1}{2} \| \nabla y(u) - \nabla y(v) \|^2 + \frac{\lambda}{2} \| u - v \|^2
\]

(52)

All the statements and estimates are derived completely analogously to the minimization problem I. We obtain the equation

\[
|||u - v|||^2 = \tilde{J} (y(v), v) - \tilde{J} (y(u), u)
\]

(53)

for any control function \( v \in L^2(Q_T) \). Using the equation (53), we obtain the following error majorant:

\[
|||u - v|||^2 \leq \tilde{M} ^\beta (\alpha, \beta; \eta, \zeta, \tau, \rho, v) := \tilde{J} ^\beta (\alpha, \beta; \eta, \tau, v) - \tilde{J} ^\circ (\eta, \zeta, \tau, \rho)
\]

with

\[
\tilde{M} ^\beta (\alpha, \beta; \eta, \zeta, \tau, \rho, v) = \frac{\alpha}{2} \| \nabla \eta - g_d \|^2 + \frac{(1 + \alpha)(1 + \beta)}{2 \alpha \mu \lambda^2} \| R_2 (\eta, \tau) \|^2 + \frac{\lambda}{2} \| v \|^2 - \frac{1}{2 \lambda} \| \xi \|^2
\]

(54)

\[
+ \int_{Q_T} (\nu \nabla \eta \cdot \nabla \zeta + \sigma \partial_t \eta \zeta + \lambda^{-1} \xi^2) \, dx \, dt + \frac{C_F^2}{\mu \lambda^2} (C_F \| R_3 (\zeta, \rho, \eta) \| + \| R_4 (\zeta, \rho) \|)^2
\]

\[
+ \frac{1}{\mu} (C_F \| R_1 (\eta, \tau, -\lambda^{-1} \xi) \| + \| R_2 (\eta, \tau) \|) (C_F \| R_3 (\zeta, \rho, \eta) \| + \| R_4 (\zeta, \rho) \|)
\]

for arbitrary \( \eta, \zeta \in H^1_0, \per, Q_T \), \( \tau, \rho \in H(\text{div}, Q_T) \), and \( \alpha, \beta > 0 \), and where \( \mu = \sqrt{\frac{\mu_1}{\alpha \beta}} \). This majorant attains the exact lower bound on the exact solution of the optimal control problem (15) w.r.t. (I), or, equivalently, of the optimality system (10), i.e.,

\[
\inf_{\eta, \zeta \in H^1_0, \per, Q_T, \tau, \rho \in H(\text{div}, Q_T), v \in L^2(Q_T), \alpha, \beta > 0} \tilde{M} ^\beta (\alpha, \beta; \eta, \zeta, \tau, \rho, v) = 0.
\]

The infimum is attained for \( v = u, \eta = y(u), \zeta = p(u) = -\lambda u, \tau = \nu \nabla y(u) \) and \( \rho = \nu \nabla p(u) \).

Analogously, defining

\[
|||u - v|||^2 := \frac{1}{2} \| \nabla y(u) - \nabla y(v) \|^2 + \frac{2 \lambda \mu}{C_F^2} \| y(u) - y(v) \|^2
\]

(55)

we derive the estimate

\[
|||u - v|||^2 \leq \tilde{M} ^\beta_1 (\alpha, \beta; \eta, \zeta, \tau, \rho, v) := \tilde{J} ^\beta (\alpha, \beta; \eta, \tau, v) - \tilde{J} ^\circ (\eta, \zeta, \tau, \rho)
\]

\[
+ \frac{3 \lambda}{2 C_F^2} (C_F \| R_1 (\eta, \tau, v) \| + \| R_2 (\eta, \tau) \|)^2.
\]

(56)
where now
\[ \mathcal{M}_i^o(\alpha, \beta; \eta, \zeta, \tau, \rho, \nu) = \frac{\alpha}{2} \| \nabla \eta - g_d \|^2 + \frac{(1 + \alpha)(1 + \beta)C^2_f}{2\alpha \mu'} \| R_2(\eta, \tau) \|^2 \]
\[ + \frac{1}{2\lambda} \| \tau \|^2 + \frac{1}{2\lambda} \| \eta \|^2 \]
\[ + \int_{Q_T} (\nu \nabla \eta \cdot \nabla \zeta + \sigma \partial_t \eta \zeta + \lambda^{-1} \eta^2) \, dx \, dt + \frac{C^2_f}{\mu^2 \lambda} (C_f \| R_3(\zeta, \rho, \eta) \| + \| R_4(\zeta, \rho) \|)^2 \]
\[ + \frac{1}{\mu^2} (C_f \| R_1(\eta, \tau, -\lambda^{-1} \zeta) \| + \| R_2(\eta, \tau) \|) (C_f \| R_3(\zeta, \rho, \eta) \| + \| R_4(\zeta, \rho) \|)^2 \]
\[ + \frac{3\lambda}{2C^2_f} (C_f \| R_1(\eta, \tau, v) \| + \| R_2(\eta, \tau) \|)^2 . \]

All other results similar to Propositions \[ \Box \] and \[ \Box \] follow completely.

5. Multiharmonic Finite Element Discretization

Since the desired state \( y_d \) and desired gradient \( g_d \) belong to \( L^2(Q_T) \) and \( [L^2(Q_T)]^d \), respectively, they can be represented as Fourier series. Henceforth, we assume that the approximations \( \eta \) and \( \zeta \) of the exact state \( y \) and the adjoint state \( p \), respectively, are also represented in terms of truncated Fourier series as well as the vector-valued functions \( \tau \) and \( \rho \). Examples for them are
\[ \eta(x, t) = \eta_0(x) + \sum_{k=1}^{N} \eta_k(x) \cos(k\omega t) + \eta_k(x) \sin(k\omega t) , \]
\[ \tau(x, t) = \tau_0(x) + \sum_{k=1}^{N} \tau_k(x) \cos(k\omega t) + \tau_k(x) \sin(k\omega t) , \]
where all the Fourier coefficients belong to the space \( L^2(\Omega) \). We also have
\[ \partial_t \eta(x, t) = \sum_{k=1}^{N} (k\omega \eta_k(x) \cos(k\omega t) - k\omega \eta_k(x) \sin(k\omega t)) , \]
\[ \nabla \eta(x, t) = \nabla \eta_0(x) + \sum_{k=1}^{N} (\nabla \eta_k(x) \cos(k\omega t) + \nabla \eta_k(x) \sin(k\omega t)) , \]
\[ \text{div} \tau(x, t) = \text{div} \tau_0(x) + \sum_{k=1}^{N} (\text{div} \tau_k(x) \cos(k\omega t) + \text{div} \tau_k(x) \sin(k\omega t)) . \]

For the numerical treatment, all Fourier series are truncated at a finite index \( N \) and the unknown Fourier coefficients \( y_k = (y_{k,x}, y_{k,t})^T, p_k = (p_{k,x}, p_{k,t})^T \in \mathbb{V} \) are approximated by finite element (FE) functions \( y_{kh} = (y_{k,xh}, y_{k,th})^T, p_{kh} = (p_{k,xh}, p_{k,th})^T \in \mathbb{V}_h = V_h \times V_h \subset \mathbb{V} \), where \( V_h = \text{span} \{ \varphi_1, \ldots, \varphi_n \} \) with \( \{ \varphi_i(x) : i = 1, 2, \ldots, n_h \} \) is a conforming FE space. We denote by \( h \) the usual discretization parameter such that \( n = n_h = \text{dim} V_h = O(h^{-d}) \), and we use continuous, piecewise linear finite elements on a regular triangulation \( \mathcal{T}_h \) to construct \( V_h \) and its basis (see, e.g., \[ \Box \] \[ \Box \] \[ \Box \] \[ \Box \]

5.1. Minimization problem I. The MhFE discretization leads to the following discrete saddle point system corresponding to problem \[ \Box \] for every single mode \( k = 1, 2, \ldots, N \):
\[ \begin{pmatrix} M_h & 0 & -K_{h,\nu} & k\omega M_{h,\sigma} & k\omega M_{h,\nu} & K_{h,\nu} \\ 0 & M_h & -k\omega M_{h,\nu} & -K_{h,\sigma} & -K_{h,\nu} & 0 \\ -K_{h,\nu} & -K_{h,\sigma} & -k\omega M_{h,\sigma} & -k\omega M_{h,\nu} & 0 & -K_{h,\nu} \\ k\omega M_{h,\sigma} & -K_{h,\nu} & -k\omega M_{h,\nu} & -k\omega M_{h,\sigma} & 0 & -K_{h,\nu} \\ k\omega M_{h,\nu} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{kh}^e \\ y_{kh}^o \\ M_{h}y_{kh}^c \\ M_{h}y_{kh}^p \\ 0 \end{pmatrix} = \begin{pmatrix} M_h y_{kh}^e \\ M_h y_{kh}^o \\ M_{h}y_{kh}^c \\ M_{h}y_{kh}^p \\ 0 \end{pmatrix} , \]
which has to be solved with respect to the nodal parameter vectors
\[ y_{k,i}^e = (y_{k,i}^e)_{i=1,\ldots,n}, y_{k,i}^o = (y_{k,i}^o)_{i=1,\ldots,n}, p_{k,i}^c = (p_{k,i}^c)_{i=1,\ldots,n}, p_{k,i}^p = (p_{k,i}^p)_{i=1,\ldots,n} \in \mathbb{R}^n \]
of the FE approximations $y_{kh}^c(x) = \sum_{i=1}^n y_{ki}^c \varphi_i(x)$ and $y_{kh}^s(x) = \sum_{i=1}^n y_{ki}^s \varphi_i(x)$. Similarly, $p_{kh}^c(x) = \sum_{i=1}^n p_{ki}^c \varphi_i(x)$ and $p_{kh}^s(x) = \sum_{i=1}^n p_{ki}^s \varphi_i(x)$. The matrices $M_h$, $M_{h,\sigma}$, and $K_{h,\nu}$ denote the mass matrix, the weighted mass matrix and the stiffness matrix, respectively. Their entries are defined by the integrals

$$M_{ij} = \int_\Omega \varphi_i \varphi_j \, dx, \quad M_{ij}^{\sigma} = \int_\Omega \sigma \varphi_i \varphi_j \, dx, \quad K_{ij}^{\nu} = \int_\Omega \nu \nabla \varphi_i \cdot \nabla \varphi_j \, dx$$

and the right-hand side vectors have the form

$$M_{h}^{y_{jk}} = \left[ \int_\Omega y_{jk}' \varphi_j \, dx \right]_{j=1, \ldots, n} \quad \text{and} \quad M_{h}^{y_{jk}} = \left[ \int_\Omega y_{jk}' \varphi_j \, dx \right]_{j=1, \ldots, n}.$$  

For $k = 0$, the problem (54) generates a reduced system of linear equations, i.e.,

$$\begin{pmatrix} M_h & -K_{h}\nu & -\lambda^{-1} M_h \\ -K_{h}\nu & -K_{h,\nu} & -\lambda^{-1} M_h \\ -\lambda^{-1} M_h & -\lambda^{-1} M_h & 0 \end{pmatrix} \begin{pmatrix} y_{jk}^c \\ y_{jk}^s \\ \tau_{jk} \end{pmatrix} = \begin{pmatrix} M_h y_{jk}^c \\ 0 \end{pmatrix}. \tag{59}$$

Fast and robust solvers for the systems (55) and (59) can be found in [20, 24, 31] which we use in order to obtain the MhFE approximations

$$\begin{align*}
y_{Nh}(x, t) &= y_{0h}(x) + \sum_{k=1}^N (y_{kh}^c(x) \cos(k\omega t) + y_{kh}^s(x) \sin(k\omega t)), \\
p_{Nh}(x, t) &= p_{0h}(x) + \sum_{k=1}^N (p_{kh}^c(x) \cos(k\omega t) + p_{kh}^s(x) \sin(k\omega t)).
\end{align*} \tag{60}$$

Both, majorant (18) and minorant (31) of the cost functional $J$ can be computed by choosing the MhFE approximations $y_{Nh}$, $p_{Nh}$ and $u_{Nh} = -\lambda^{-1} p_{Nh}$ as $\eta$, $\zeta$ and $v$, respectively. The arbitrary functions $\tau$ and $\rho$ can also be represented in form of multiharmonic functions $\tau_{Nh}$ and $\rho_{Nh}$. Hence, majorant (18) and minorant (31) have a multiharmonic structure too. Since all the terms corresponding to every single mode $k$ are again decoupled, we arrive at majorants $J_k^{\alpha}$ and minorants $J_k^{\beta}$. Moreover, for the majorant, we can, of course, introduce positive parameters $\alpha_k$ and $\beta_k$ for every single mode $k$. Finally, the majorant (18) can be written as

$$J^{\alpha}(\alpha_{N+1}, \beta_N; y_{Nh}, p_{Nh}, \tau_{Nh}) = T J_0^{\alpha}(\alpha_0, \beta_0; y_{0h}, p_{0h}, \tau_{0h})$$

$$+ \frac{T}{2} \sum_{k=1}^N \left\{ J_k^{\alpha}(\alpha_k, \beta_k; y_{kh}, p_{kh}, \tau_{kh}) + \frac{1 + \alpha_{N+1}}{2} \mathcal{E}_N \right\} \tag{61}$$

where $\alpha_{N+1} = (\alpha_0, \ldots, \alpha_{N+1})^T$, $\beta_N = (\beta_0, \ldots, \beta_N)^T$, and

$$\begin{align*}
J_0^{\alpha}(\alpha_0, \beta_0; y_{0h}, p_{0h}, \tau_{0h}) &= \frac{1 + \alpha_0}{2} \| y_{0h} \|_\Omega^2 - \| y_{0h}^c \|_\Omega^2 + \frac{1}{2\lambda} \| p_{0h} \|_\Omega^2 \\
&+ \left( 1 + \alpha_0 \right) \left( 1 + \beta_0 \right) C_{\mathcal{E}}^2 \| \mathcal{R}_{20}^c \|_\Omega^2 + \left( 1 + \alpha_0 \right) \left( 1 + \beta_0 \right) C_{\mathcal{E}}^4 \| \mathcal{R}_{10}^c \|_\Omega^2, \\
J_k^{\alpha}(\alpha_k, \beta_k; y_{kh}, p_{kh}, \tau_{kh}) &= \frac{1 + \alpha_k}{2} \| y_{kh} \|_\Omega^2 - \| y_{kh}^c \|_\Omega^2 + \frac{1}{2\lambda} \| p_{kh} \|_\Omega^2 \\
&+ \left( 1 + \alpha_k \right) \left( 1 + \beta_k \right) C_{\mathcal{E}}^2 \| \mathcal{R}_{2k}^c \|_\Omega^2 + \left( 1 + \alpha_k \right) \left( 1 + \beta_k \right) C_{\mathcal{E}}^4 \| \mathcal{R}_{1k}^c \|_\Omega^2 \tag{62, 63}\end{align*}$$

The terms $\mathcal{R}_{i0}^c = \text{div} \tau_{0h}^c - \lambda^{-1} p_{0h}^c$, $\mathcal{R}_{20}^c = \tau_{0h}^c - \nu \nabla y_{0h}^c$,

$$\mathcal{R}_{1k} = k \omega \sigma y_{kh} + \text{div} \tau_{kh} - \lambda^{-1} p_{kh}$$

$$= (\mathcal{R}_{1k}^c, \mathcal{R}_{1k}^s)^T = (-k \omega \sigma y_{kh}^c + \text{div} \tau_{kh}^c - \lambda^{-1} p_{kh}^c, k \omega \sigma y_{kh}^s + \text{div} \tau_{kh}^s - \lambda^{-1} p_{kh}^s)^T$$

and

$$\mathcal{R}_{2k} = \tau_{kh} - \nu \nabla y_{kh} = (\mathcal{R}_{2k}^c, \mathcal{R}_{2k}^s)^T = (\tau_{kh}^c - \nu \nabla y_{kh}^c, \tau_{kh}^s - \nu \nabla y_{kh}^s)^T.$$
The remainder term of truncation

\begin{align}
\mathcal{E}_N := \frac{T}{2} \sum_{k=N+1}^{\infty} \| y_d_k \|^2_\Omega = \frac{T}{2} \sum_{k=N+1}^{\infty} \left( \| y_d_k \|^2_\Omega + \| y_d_k \|^2_\Omega \right)
\end{align}

can always be computed with any desired accuracy, since \( y_d \) is known. The minorant \( \mathcal{J}^{\ominus} \) can be written as

\begin{align}
\mathcal{J}^{\ominus}(y_{Nh}, p_{Nh}, \tau_{Nh}, \rho_{Nh}) = T \mathcal{J}^{\ominus}_0(y_{0h}, p_{0h}, \tau^{\ominus}_{0h}, \rho^{\ominus}_{0h}) + \frac{T}{2} \sum_{k=1}^{N} \mathcal{J}^{\ominus}_k(y_{kh}, p_{kh}, \tau_{kh}, \rho_{kh}) + \frac{\mathcal{E}_N}{2},
\end{align}

where

\begin{align}
\mathcal{J}^{\ominus}_0(y_{0h}, p_{0h}, \tau^{\ominus}_{0h}, \rho^{\ominus}_{0h}) &= \frac{1}{2} \| y_{0h} - y_{0h} \|^2_\Omega + \frac{1}{2\lambda} \| p_{0h} \|^2_\Omega \\

- \int_{\Omega} \left( \nu \nabla y_{0h} \cdot \nabla p_{0h} + \lambda^{-1} (p_{0h})^2 \right) \, dx - \frac{C^2_F}{\mu^2_1} (C_F \| R_{30} \|_{\Omega} + \| R_{40} \|_{\Omega})^2 \\
- \frac{1}{\mu_1} (C_F \| R_{10} \|_{\Omega} + \| R_{20} \|_{\Omega}) (C_F \| R_{30} \|_{\Omega} + \| R_{40} \|_{\Omega})
\end{align}

and

\begin{align}
\mathcal{J}^{\ominus}_k(y_{kh}, p_{kh}, \tau_{kh}, \rho_{kh}) &= \frac{1}{2} \| y_{kh} - y_{dk} \|^2_\Omega + \frac{1}{2\lambda} \| p_{kh} \|^2_\Omega \\

- \int_{\Omega} \left( \nu \nabla y_{kh} \cdot \nabla p_{kh} - k \omega \sigma y_{kh} \cdot p_{kh} + \lambda^{-1} (p_{kh})^2 \right) \, dx - \frac{C^2_F}{\mu^2_1} (C_F \| R_{3k} \|_{\Omega} + \| R_{4k} \|_{\Omega})^2 \\
- \frac{1}{\mu_1} (C_F \| R_{1k} \|_{\Omega} + \| R_{2k} \|_{\Omega}) (C_F \| R_{3k} \|_{\Omega} + \| R_{4k} \|_{\Omega})
\end{align}

The terms \( R_{30} = \text{div} \rho^{\ominus}_{0h}, R_{40} = \rho^{\ominus}_{0h} - \nu \nabla p^{\ominus}_{0h}, \)

\begin{align}
R_{3k} &= k \omega \sigma p_{kh}^+ + \text{div} \rho_{kh} + y_{kh} - y_{dk} \\
= (R_{3k}^+, R_{3k}^+)^T &= (-k \omega \sigma p_{kh}^+ + \text{div} \rho_{kh}^+ + y_{kh} - y_{dk}, k \omega \sigma p_{kh}^+ + \text{div} \rho_{kh}^+ + y_{kh} - y_{dk})^T
\end{align}

and

\begin{align}
R_{4k} &= \rho_{kh} - \nu \nabla p_{kh} = (R_{4k}^+, R_{4k}^+)^T = (\rho_{kh}^+ - \nu \nabla p_{kh}^+, \rho_{kh}^+ - \nu \nabla p_{kh}^+)^T.
\end{align}

The fluxes \( \tau_{0h}, \rho_{0h} \) and \( \tau_{kh}, \rho_{kh} \) for all \( k = 1, \ldots, N \), have to be reconstructed, which we denote by

\begin{align}
\tau_{kh} = R_{h}^{\text{flx}}(\nu \nabla y_{kh}) \quad \text{and} \quad \rho_{kh} = R_{h}^{\text{flx}}(\nu \nabla p_{kh}).
\end{align}

This can be done by various techniques. In [29] [30], Raviart-Thomas elements of the lowest order have been used (see also [11] [12] [13]), in order to regularize the fluxes by a post-processing operator, which maps the \( L^2 \)-functions into \( H(\text{div}, \Omega) \). Collecting all the fluxes corresponding to the modes together yields the reconstructed flux

\begin{align}
\tau_{Nh} = R_{h}^{\text{flx}}(\nu \nabla y_{Nh}) \quad \text{and} \quad \rho_{Nh} = R_{h}^{\text{flx}}(\nu \nabla p_{Nh}).
\end{align}

We also perform a simple minimization of the majorant \( \mathcal{J}^{\ominus} \) with respect to the positive parameters leading to the optimized \( \alpha_{N+1} \) and \( \beta_{N} \). Finally, the majorant \( \mathcal{J}^{\ominus} \) and the minorant \( \mathcal{J}^{\ominus} \) provide guaranteed upper and lower bounds for the cost functional. Altogether, the infimum of the majorant and the supremum of the minorant coincide with the optimal value of the cost functional, see [30].
5.2. Minimization problem II. For the second problem, we only summarize the main results and changes. The MhFE discretization leads to the following discrete saddle point system corresponding to problem [13] for every single mode $k = 1, 2, \ldots, N$:

\begin{equation}
\begin{pmatrix}
K_h & 0 & -K_{h,\nu} & \omega M_{h,\sigma} \\
0 & K_h & -\omega M_{h,\sigma} & -K_{h,\nu} \\
-K_{h,\nu} & -\omega M_{h,\sigma} & -\lambda^{-1}M_h & 0 \\
\omega M_{h,\sigma} & -K_{h,\nu} & 0 & -\lambda^{-1}M_h \\
\end{pmatrix}
\begin{pmatrix}
y^c_k \\
y^c_0 \\
0 \\
0 \\
\end{pmatrix}
= \begin{pmatrix}
K_h g^c_{d^k h} \\
K_h g^c_{d^0 h} \\
0 \\
0 \\
\end{pmatrix},
\end{equation}

and for $k = 0$:

\begin{equation}
\begin{pmatrix}
K_h & -K_{h,\nu} \\
-K_{h,\nu} & -\lambda^{-1}M_h \\
\end{pmatrix}
\begin{pmatrix}
y^c_0 \\
0 \\
\end{pmatrix}
= \begin{pmatrix}
K_h g^c_{d^0 h} \\
0 \\
\end{pmatrix}.
\end{equation}

Now, the right-hand sides vectors have the form

\[ K_h g^c_{d^k h} = \left[ \int_{\Omega} g_{d^k h} \cdot \nabla \varphi_j \, dx \right]_{j=1,\ldots,n} \quad \text{and} \quad K_h g^c_{d^0 h} = \left[ \int_{\Omega} g_{d^0 h} \cdot \nabla \varphi_j \, dx \right]_{j=1,\ldots,n}. \]

We summarize the discrete majorant [13] and minorant [50] of the cost functional $\tilde{J}$ computed by choosing the MhFE approximations for all the (arbitrary) functions. The majorant [13] can be written as

\begin{equation}
\tilde{J}^\oplus(\alpha_{N+1}, \beta_N; y_{Nh}, p_{Nh}, \tau_{Nh}) = T \tilde{J}_0^\oplus(\alpha_0, \beta_0; y_{0h}, p_{0h}, \tau_{0h})
\end{equation}
\begin{equation}
+ \frac{T}{2} \sum_{k=1}^{N} \tilde{J}_k^\oplus(\alpha_k, \beta_k; y_{kh}, p_{kh}, \tau_{kh}) + \frac{1 + \alpha_{N+1}}{2} \tilde{\varepsilon}_N,
\end{equation}

where $\alpha_{N+1} = (\alpha_0, \ldots, \alpha_{N+1})^T$, $\beta_N = (\beta_0, \ldots, \beta_N)^T$, and

\begin{equation}
\tilde{J}_k^\oplus(\alpha_0, \beta_0; y_{0h}, p_{0h}, \tau_{0h}) = \frac{1 + \alpha_0}{2} \| \nabla y_{0h} - g_{d_0} \|^2_{\Omega} + \frac{1}{2\lambda} \| p_{0h} \|^2_{\Omega}
\end{equation}
\begin{equation}
+ \frac{(1 + \alpha_0)(1 + \beta_0)C_F^2}{2\alpha_0\mu_1^2} \| \mathcal{R}_{20} \|^2_{\Omega}^2 + \frac{(1 + \alpha_0)(1 + \beta_0)C_F^4}{2\alpha_0\beta_0\mu_1^2} \| \mathcal{R}_{40} \|^2_{\Omega},
\end{equation}

\begin{equation}
\tilde{J}_k^\oplus(\alpha_k, \beta_k; y_{kh}, p_{kh}, \tau_{kh}) = \frac{1 + \alpha_k}{2} \| \nabla y_{kh} - g_{d_k} \|^2_{\Omega} + \frac{1}{2\lambda} \| p_{kh} \|^2_{\Omega}
\end{equation}
\begin{equation}
+ \frac{(1 + \alpha_k)(1 + \beta_k)C_F^2}{2\alpha_k\mu_1^2} \| \mathcal{R}_{2k} \|^2_{\Omega}^2 + \frac{(1 + \alpha_k)(1 + \beta_k)C_F^4}{2\alpha_k\beta_k\mu_1^2} \| \mathcal{R}_{4k} \|^2_{\Omega}.
\end{equation}

The remainder term of truncation is now given by

\begin{equation}
\tilde{\varepsilon}_N := \frac{T}{2} \sum_{k=N+1}^{\infty} \| g_{d_k} \|^2_{\Omega} = \frac{T}{2} \sum_{k=N+1}^{\infty} \left( \| g_{d_k} \|^2_{\Omega} + \| g_{d_k} \|^2_{\Omega} \right).
\end{equation}

The minorant [50] can be written as

\begin{equation}
\tilde{J}^\ominus(y_{Nh}, p_{Nh}, \tau_{Nh}) = T \tilde{J}_0^\ominus(y_{0h}, p_{0h}, \tau_{0h}) + \frac{T}{2} \sum_{k=1}^{N} \tilde{J}_k^\ominus(y_{kh}, p_{kh}, \tau_{kh}) + \frac{\tilde{\varepsilon}_N}{2},
\end{equation}

where

\begin{equation}
\tilde{J}_0^\ominus(y_{0h}, p_{0h}, \tau_{0h}) = \frac{1}{2} \| \nabla y_{0h} - g_{d_0} \|^2_{\Omega} + \frac{1}{2\lambda} \| p_{0h} \|^2_{\Omega}
\end{equation}
\begin{equation}
- \int_{\Omega} (\nu \nabla y_{0h} \cdot \nabla p_{0h} + \lambda^{-1}(p_{0h})^2) \, dx - \frac{C_F^2}{\mu_1^2\lambda} \left( C_F \| \mathcal{R}_{20} \|_{\Omega} + \| \mathcal{R}_{40} \|_{\Omega} \right)^2
\end{equation}
\begin{equation}
- \frac{1}{\mu_1} \left( C_F \| \mathcal{R}_{40} \|_{\Omega} + \| \mathcal{R}_{60} \|_{\Omega} \right) \left( C_F \| \mathcal{R}_{50} \|_{\Omega} + \| \mathcal{R}_{60} \|_{\Omega} \right).
\end{equation}
and
\[
\begin{cases}
\tilde{J}_k^\sigma(y_{kh}, p_{kh}, \tau_{kh}, \nu_{kh}) = \frac{1}{2} \|\nabla y_{kh} - g_{dk}\|_2^2 + \frac{1}{2\lambda} \|p_{kh}\|_2^2 \\
- \int_\Omega (\nu \nabla y_{kh} \cdot \nabla p_{kh} - k\omega \sigma y_{kh}^1 \cdot p_{kh} + \lambda^{-1} p_{kh}^2) \, dx - \frac{C_F}{\mu_1} (C_F \|R_{3h}\|_\Omega + \|R_{4h}\|_\Omega)^2 \\
- \frac{1}{\mu_1} (C_F \|R_{1h}\|_\Omega + \|R_{2h}\|_\Omega) (C_F \|R_{3h}\|_\Omega + \|R_{4h}\|_\Omega).
\end{cases}
\]

(76)

\section{Robust Preconditioners for the Minimal Residual Method}

In order to solve the saddle point systems \cite{20}, \cite{61}, \cite{62} and \cite{63} we can use the preconditioned minimal residual (MinRes) method, see \cite{40}. A convergence result for the preconditioned MinRes method can be found in \cite{15} stating that the convergence rate of the preconditioned MinRes method depends on the condition number of the preconditioned system. The derivation of preconditioners for problems \cite{63} for \(k = 1, \ldots, N\) and \cite{63} for \(k = 0\) have already been presented and discussed in \cite{60, 61} given by
\[
P_k = \text{diag}(D_k, D_k, \lambda^{-1} D_k, \lambda^{-1} D_k)
\]
respectively, where \(D_k = \sqrt{\lambda} K_{h,v} + k\omega \sqrt{\lambda} M_{h,\sigma} + M_{h}\) and \(D_0 = M_{h} + \sqrt{\lambda} K_{h,v}\). In \cite{20}, preconditioners are derived following the operator (matrix) interpolation technique presented in \cite{51}.

In this section, we present robust preconditioners for the problem matrices in \cite{63} for \(k = 1, \ldots, N\) and in \cite{63} for \(k = 0\) in order to solve minimization problem II. Here, we assume that \(\sigma\) and \(\nu\) are constant, which we choose also in the numerical results in Section 7. Hence, \(M_{h,\sigma} = \sigma M_{h}\) and \(K_{h,v} = \nu K_{h}\). For an inexact realization of these block-diagonal preconditioners the robust algebraic multilevel iteration (AMLI) method presented in \cite{23} is used. The AMLI preconditioned MinRes solver is robust and of optimal complexity which is proved in \cite{24}. This can be also observed in the numerical results’ section. We consider the saddle point system \cite{63}.

The derivation for \cite{63} is completely analogous. Defining the following block matrices
\[
A := \begin{pmatrix} K_h & 0 \\
0 & K_h \end{pmatrix}, \quad B := \begin{pmatrix} -\nu K_h & -k\omega \sigma M_h \\
\kappa \omega \sigma M_h & -\nu K_h \end{pmatrix}, \quad C := \begin{pmatrix} \lambda^{-1} M_h & 0 \\
0 & \lambda^{-1} M_h \end{pmatrix}
\]
and vectors
\[
\mathbf{f} := \begin{pmatrix} K_h \mathbf{y}_k^c \\
K_h \mathbf{y}_k^d \end{pmatrix}, \quad \mathbf{u} := \begin{pmatrix} \mathbf{y}_k^c \\
\mathbf{y}_k^d \end{pmatrix}, \quad \mathbf{p} := \begin{pmatrix} \mathbf{p}_k^c \\
\mathbf{p}_k^d \end{pmatrix},
\]
leads to the following structure
\[
\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \mathbf{u} \\
\mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\
0 \end{pmatrix},
\]
with the symmetric and positive definite matrices \(A\) and \(C\). We define the negative Schur complements
\[
S = C + BA^{-1}B^T \quad \text{and} \quad R = A + B^TC^{-1}B
\]
yielding the preconditioners
\[
P^0 = \begin{pmatrix} A & 0 \\
0 & S \end{pmatrix} \quad \text{or} \quad P^1 = \begin{pmatrix} R & 0 \\
0 & C \end{pmatrix}.
\]

The negative Schur complements are given by
\[
S = \begin{pmatrix} \nu K_h + \lambda^{-1} M_h + k^2 \omega^2 \sigma^2 M_h K_h^{-1} M_h & 0 \\
0 & \nu K_h + \lambda^{-1} M_h + k^2 \omega^2 \sigma^2 M_h K_h^{-1} M_h \end{pmatrix}
\]
and
\[
R = \begin{pmatrix} K_h + k^2 \omega^2 \sigma^2 \lambda M_h + \nu^2 \lambda K_h M_h^{-1} K_h & 0 \\
0 & K_h + k^2 \omega^2 \sigma^2 \lambda M_h + \nu^2 \lambda K_h M_h^{-1} K_h \end{pmatrix}.
\]
Both Schur complement preconditioners (79) can be chosen leading to fast and robust convergence rates, see [26] and [38]. Altogether, the following Schur complement preconditioners for minimization problem II are derived:

\[ \tilde{P}_k^0 = \text{diag}(K_h, K_h, \tilde{D}_k^0, \tilde{D}_k^0) \quad \text{and} \quad \tilde{P}_h^0 = \text{diag}(K_h, \nu K_h + \lambda^{-1} M_h) \]

where \( \tilde{D}_k^0 = \nu K_h + \lambda^{-1} M_h + k^2 \omega^2 \sigma^2 M_h K_h^{-1} M_h \), and

\[ \tilde{P}_k^1 = \text{diag}(\tilde{D}_k^1, \tilde{D}_k^1, \lambda^{-1} M_h) \quad \text{and} \quad \tilde{P}_h^1 = \text{diag}(\lambda^{-1} M_h, K_h + \nu^2 \lambda K_h M_h^{-1} K_h), \]

where \( \tilde{D}_k^1 = K_h + k^2 \omega^2 \sigma^2 \lambda M_h + \nu^2 \lambda K_h M_h^{-1} K_h \), for the saddle point systems [38] and [63], respectively. In the numerical results of this work, we choose the preconditioners [30]. Finally, for every mode \( k = 0, 1, 2, \ldots, N \), we have determined a preconditioner such that the corresponding system can be solved by the preconditioned MinRes method with a robust convergence rate.

7. Numerical Results

In this section, we present numerical results for computing the minorants and majorants of the two optimal control problems with cost functionals (10) and (15) and for different cases of given data. We consider the following cases for problem I:

1. the desired state is periodic and analytic in time, but not time-harmonic,
2. the desired state is analytic in time, but not time-periodic, and
3. the desired state is a non-smooth function in space and time.

These numerical results enrich the numerical tests presented in [30]. The convergence and other properties of numerical approximations generated by the MhFEM have already been studied in [20] and [31] for the same three cases. However, the numerical experiments on the minorant are new.

The numerical results for problem II are all new. We perform numerical experiments for the same three cases as for problem I but now applied on the desired gradient:

4. the desired gradient is periodic and analytic in time, but not time-harmonic,
5. the desired gradient is analytic in time, but not time-periodic, and
6. the desired gradient is a non-smooth function in space and time.

The computational domain is chosen to be \( \Omega = (0, 1) \times (0, 1) \) with the Friedrichs constant \( C_F = 1/(\sqrt{2} \pi) \) using a uniform simplicial mesh and standard continuous, piecewise linear finite elements. The material coefficients are chosen \( \sigma = \nu = 1 \). In the first two examples (for both minimization problems), we choose the cost parameter \( \lambda = 0.1 \), and in the third and sixth one, we choose \( \lambda = 0.01 \). As mentioned in the section before, we choose the MhFE approximations (77) for \( \eta, \zeta \) and \( \tau, \rho \), and we reconstruct the fluxes by the standard lowest-order Raviart-Thomas (RT0-) extension of normal fluxes

\[ RT^0(T_h) := \{ \tau \in (L^2(T))^2 : \forall T \in T_h \quad \exists a, b, c \in \mathbb{R} \quad \forall x \in T, \tau(x) = (a, b)^T + c x \quad \text{and} \quad [\tau|_E \cdot n_E = 0 \quad \forall \text{interior edges } E \}, \]

where \( [\tau|_E \) denotes the jump of \( \tau \) across the edge \( E \) shared by two neighboring elements on a triangulation \( T_h \). Altogether, it follows an averaged flux from \( H(\text{div}, \Omega) \), i.e.,

\[ \tau_h^k = G_{RT}(\nabla \eta_h^k), \quad \tau_h^k = G_{RT}(\nabla \eta_h^k), \quad G_{RT} : L^2(\Omega) \rightarrow H(\text{div}, \Omega), \]

see also [26] for further details. The numerical experiments were computed on grids of mesh sizes 16\times16 to 256\times256 as well as 512\times512 to obtain finer grid solutions for Examples 3 and 6. The algorithms were implemented in C++. The preconditioned MinRes iteration was stopped after 8 iteration steps in all computations using the AMLI preconditioner with 4 inner iterations. The numerical experiments for Examples 1, 2, 4 and 5 were performed on a laptop with Intel(R) Core(TM) i5-6267U CPU @ 2.90GHz processor and 16 GB 2133 MHz LPDDR3 memory. Since the laptop does not provide enough memory for computing the finer grid solutions in addition, the numerical experiments for Examples 3 and 6 were performed on a CPU server with a Tumbleweed distribution having 64 cores and 1 terabyte memory. Note that the presented CPU times \( t_{\text{sec}} \) in seconds include the computational times for computing the majorants and minorants, which are
very small in comparison to the computational times of the solver. The computational times of Examples 3 and 6 exclude the computation of the solution on the finer grid (512×512).

7.1. Numerical results for minimization problem I. In Example 1, we set the desired state

\[ y_d(x, t) = \frac{e^t \sin(t)}{10} \left( (12 + 4\pi^4) \sin^2(t) - 6 \cos^2(t) - 6 \sin(t) \cos(t) \right) \sin(x_1 \pi) \sin(x_2 \pi), \]

where \( T = 2\pi/\omega \) and \( \omega = 1 \). The Fourier coefficients of \( y_d \) can be computed analytically, and, then, they are approximated by the FEM. Next the systems (58) and (59) are solved for all \( k \in \{0, \ldots, 8\} \). After a proper reconstruction of the fluxes (by RT\(_0\)-extension), finally, the majorants and minorants are computed. The exact state is given by

\[ y(x, t) = e^t \sin(t)^3 \sin(x_1 \pi) \sin(x_2 \pi). \]

In Table 1, we present the CPU times \( t_{\text{cpu}} \), the majorants \( J^0 \) and minorants \( J^0 \) as defined in (63) and (67) as well as the corresponding efficiency indices \( I_{\text{eff}}^0 = J^0 / J_0 \), \( I_{\text{eff}}^0 = J^0 / J_0 \) and \( I_{\text{eff}}^0 = J^0 / J^0 \) obtained on grids of different mesh sizes. Table 2 presents the CPU times \( t_{\text{cpu}} \), the majorants \( J^1 \) and minorants \( J^1 \) as defined in (63) and (67) for \( k = 1 \) and, finally, the corresponding efficiency indices \( I_{\text{eff}}^1 = J^1 / J_1 \), \( I_{\text{eff}}^1 = J^1 / J_1 \) and \( I_{\text{eff}}^1 = J^1 / J^1 \) obtained on grids of different mesh sizes. Moreover, we present the efficiency indices for \( M_t^0 \) given for the modes by

\[ I_{\text{eff}}^{M_t,0} = \sqrt{\frac{M_t^0(\alpha_0, \beta_0; y_{0h}, p_{0h}, \tau_{0h}, \rho_{0h})}{\| y_0 - y_{0h} \|^2_{1,0}}} \quad \text{and} \quad I_{\text{eff}}^{M_{t,1}} = \sqrt{\frac{M_t^0(\alpha_1, \beta_1; y_{kh}, p_{kh}, \tau_{kh}, \rho_{kh})}{\| y_k - y_{kh} \|^2_{1,k}}} \]

The error norms for the modes are given by

\[ \| y_0 - y_{0h} \|^2_{1,0} = \frac{1}{2} \| y_0 - y_{0h} \|^2_{1,0} + \frac{2\lambda_{\mu_1}^2}{C_F^2} \| \nabla y_0 - \nabla y_{0h} \|^2_{1,0} \]

and

\[ \| y_k - y_{kh} \|^2_{1,k} = \left( 1 + \frac{2\lambda_{\mu_1}^2}{C_F^2} \right) \| y_k - y_{kh} \|^2_{1,k} + \frac{2\lambda_{\mu_1}^2}{C_F^2} \| \nabla y_k - \nabla y_{kh} \|^2_{1,k} \]

leading the representation

\[ \| u - v \|^2_1 = T \| y_0 - y_{0h} \|^2_{1,0} + \frac{T}{2} \sum_{k=1}^{N} \| y_k - y_{kh} \|^2_{1,k} + \mathcal{F}_N \]

with the remainder term

\[ \mathcal{F}_N := \frac{T}{2} \sum_{k=N+1}^{\infty} \| y_k \|^2_{1,k}. \]

For the numerical experiments, we can estimate the efficiency index for \( M_t^0 \) from above by estimating \( \mathcal{F}_N \) from below ignoring the remainder term \( \mathcal{F}_N \) leading to the overall efficiency index for \( M_t^0 \)

\[ I_{\text{eff}}^{M_t} = \sqrt{\frac{M_t^0(\alpha, \beta; \eta, \zeta; \tau, \rho, v)}{T \| y_0 - y_{0h} \|^2_{1,0} + \frac{T}{2} \sum_{k=1}^{N} \| y_k - y_{kh} \|^2_{1,k}}} \]
The corresponding majorants are given by
\[
\mathcal{M}_{i,o}^{(0)}(\alpha_0, \beta_0; y_{i,0}, \rho_{i,0}, \tau_{i,0}, \rho_{i,0}) = \mathcal{J}_{i,o}^{(0)}(\alpha_0, \beta_0; y_{i,0}, \rho_{i,0}, \tau_{i,0}, \rho_{i,0}) - \mathcal{J}_{i,o}^{(0)}(y_{i,0}, \rho_{i,0}, \tau_{i,0}, \rho_{i,0}) \\
+ \frac{3\lambda}{2C_F} (C_F \| R_{1,0} \|_{\Omega} + \| R_{2,0} \|_{\Omega})^2 \\
= \frac{\alpha_0}{2} \| y_{i,0} - y_{a,0} \|^2 + \frac{(1 + \alpha_0)(1 + \beta_0)C_F}{2\alpha_0\beta_0\mu_1^2} \| R_{2,0} \|^2_{\Omega} + \frac{(1 + \alpha_0)(1 + \beta_0)C_F}{2\alpha_0\beta_0\mu_1^2} \| R_{1,0} \|^2_{\Omega} \\
+ \int_{\Omega} (\nu \nabla y_{i,0} \cdot \nabla p_{i,0} + \lambda^{-1}(p_{i,0})^2) \, dx + \frac{C_F^2}{\mu_1^2\lambda} (C_F \| R_{3,0} \|_{\Omega} + \| R_{4,0} \|_{\Omega})^2 \\
+ \frac{1}{\mu_1} (C_F \| R_{1,0} \|_{\Omega} + \| R_{2,0} \|_{\Omega})(C_F \| R_{3,0} \|_{\Omega} + \| R_{4,0} \|_{\Omega}) + \frac{3\lambda}{2C_F} (C_F \| R_{1,0} \|_{\Omega} + \| R_{2,0} \|_{\Omega})^2
\]
and
\[
\mathcal{M}_{i,k}^{(0)}(\alpha_k, \beta_k; y_{k,0}, \rho_{k,0}, \tau_{k,0}, \rho_{k,0}) = \mathcal{J}_{i,k}^{(0)}(\alpha_k, \beta_k; y_{k,0}, \rho_{k,0}, \tau_{k,0}, \rho_{k,0}) - \mathcal{J}_{i,k}^{(0)}(y_{k,0}, \rho_{k,0}, \tau_{k,0}, \rho_{k,0}) \\
+ \frac{3\lambda}{2C_F} (C_F \| R_{1,k} \|_{\Omega} + \| R_{2,k} \|_{\Omega})^2 \\
= \frac{\alpha_k}{2} \| y_{k,0} - y_{a,k} \|^2 + \frac{(1 + \alpha_k)(1 + \beta_k)C_F}{2\alpha_k\beta_k\mu_1^2} \| R_{2,k} \|^2_{\Omega} + \frac{(1 + \alpha_k)(1 + \beta_k)C_F}{2\alpha_k\beta_k\mu_1^2} \| R_{1,k} \|^2_{\Omega} \\
+ \int_{\Omega} (\nu \nabla y_{k,0} \cdot \nabla p_{k,0} - k \omega \sigma y_{k,0} \cdot p_{k,0} + \lambda^{-1}(p_{k,0})^2) \, dx + \frac{C_F^2}{\mu_1^2\lambda} (C_F \| R_{3,k} \|_{\Omega} + \| R_{4,k} \|_{\Omega})^2 \\
+ \frac{1}{\mu_1} (C_F \| R_{1,k} \|_{\Omega} + \| R_{2,k} \|_{\Omega})(C_F \| R_{3,k} \|_{\Omega} + \| R_{4,k} \|_{\Omega}) + \frac{3\lambda}{2C_F} (C_F \| R_{1,k} \|_{\Omega} + \| R_{2,k} \|_{\Omega})^2.
\]
The results obtained for larger $k$ are illustrated in Table \ref{table:results}. This table compares the results for the

| grid | $\ell$-sec | $J_{ii}^{(0)}$ | $J_{eff}^{(0)}$ | $J_{eff}^{(0)}$ | $J_{eff}^{(0)}$ | $J_{eff}^{(0)}$ | $J_{eff}^{(0)}$ |
|------|-----------|---------------|----------------|----------------|----------------|----------------|----------------|
| 16 × 16 | 0.02 | 1.13e+05 | 0.90 | 1.26e+05 | 1.01 | 1.12 | 1.53 |
| 32 × 32 | 0.07 | 1.14e+05 | 0.90 | 1.27e+05 | 1.00 | 1.11 | 1.47 |
| 64 × 64 | 0.24 | 1.14e+05 | 0.90 | 1.27e+05 | 1.00 | 1.11 | 1.44 |
| 128 × 128 | 1.16 | 1.14e+05 | 0.90 | 1.27e+05 | 1.00 | 1.11 | 1.43 |
| 256 × 256 | 4.51 | 1.14e+05 | 0.90 | 1.27e+05 | 1.00 | 1.11 | 1.42 |

Table 1. Example 1. The majorant $J_{ii}^{(0)}$ and minorant $J_{ii}^{(0)}$, and their efficiency indices.

modes $k \in \{0, \ldots, 8\}$ computed on the $256 \times 256$-mesh and presents the overall functional error estimates. For that, the remainder term $\mathcal{E}_N$ is precomputed exactly. For $N = 3$, $N = 4$, $N = 6$ and $N = 8$, they are $\mathcal{E}_3 = 63694.86$, $\mathcal{E}_4 = 9316.23$, $\mathcal{E}_6 = 640.25$ and $\mathcal{E}_8 = 106.06$, respectively. It can be observed in Table \ref{table:results} that the values of the overall efficiency indices $J_{eff}^{(0)}$, $J_{eff}^{(0)}$ and $I_{eff}^{(0)}$ are all about the same, which is a demonstration for the robustness of the method with respect to the modes. However, the efficiency indices for the combined error norm $I_{eff}^{(0)}$ indicate that the modes $k = 1$ and $k = 4$ are the most significant to represent the solution by its multiharmonic approximation. Comparing the last four lines of Table \ref{table:results} shows that the value for representing the cost functional of the exact solution is already sufficiently accurate for a truncation index $N = 3$. One of the reasons for this is that the remainder term $\mathcal{E}_N$ can be precomputed exactly.

In Example 2, we choose the time-analytic, but not time-periodic desired state function

\[
y_d(x, t) = \frac{e^t}{10} (-2\cos(t) + (10 + 4\pi^4)\sin(t)) \sin(x_1\pi) \sin(x_2\pi),
\]
where $T = 2\pi/\omega$ with $\omega = 1$. We compute the MhFE approximation of the desired state and solve the systems \ref{eq:example2} and \ref{eq:example2} for all $k \in \{0, \ldots, 10\}$. The exact state is given by

\[
y(x, t) = e^t \sin(t) \sin(x_1\pi) \sin(x_2\pi).
\]
The results related to computational expenditures and efficiency indices are quite similar to those for Example 1. Therefore, we present only numerical results in the form similar to Table 3 now presented in Table 4. We compare the overall values of majorants and minors for different truncation indices $N = 6$, $N = 8$ and $N = 10$. The remainder terms for $N = 6$, $N = 8$ and $N = 10$ are $E_6 = 44094.84$, $E_8 = 19869.30$ and $E_{10} = 10597.20$, respectively. The overall efficiency indices for the cost functional are all close to one. This demonstrates the accuracy of the majorants and minors for the cost functional. Moreover, one can see from the last three lines that the truncation index $N = 6$ suffices already to provide an accurate enough approximate solution.

In Example 3, we set

\begin{equation}
\label{eq:example3}
y_d(x, t) = \chi(\frac{x}{\pi})(t) \chi(\frac{1}{4})z(x),
\end{equation}

where $\chi$ denotes the characteristic function in space and time. Let $T = 1$, then $\omega = 2\pi$. Again the coefficients of the Fourier expansion associated with $y_d$ can be found analytically. They are

\begin{equation}
\label{eq:example3coeff}
y^{s}_{dk}(x) = \left(\frac{\sin(k \pi x) + \sin(\frac{m \pi x}{4})}{k \pi}\right) \chi(\frac{x}{4}),
\end{equation}

and $y^{s}_{dk}(x) = 0$ for all $k \in \mathbb{N}$. For $k = 0$, $y^{s}_{d0}(x) = \chi(\frac{x}{4})z(x)/2$. Since the exact solution cannot be computed analytically, we compute its MhFE approximation on a finer mesh ($512 \times 512$-mesh). Since the modes $y^{s}_{dk}(x) = 0$ for all even $k \in \mathbb{N}$, it suffices to show the results for odd modes as well as for $k = 0$. Table 5 presents the results for a truncation index $N = 29$. The results regarding the efficiency indices, especially regarding the combined norm, are similar for higher modes. We added also the results for the modes $k = 41$ and $k = 81$ to the table as examples. The majorants of the combined norm stay approximately in the same range for $k \geq 1$. The majorants and minors for the cost functional are close to 1, which demonstrates their efficiency also in this numerical example, where the given data has jumps in space and time.
7.2. Numerical results for minimization problem II. We compute the numerical results for the three same cases as in problem I but now applied on the desired gradient. In Example 4, we set the desired gradient to be

\[ g_d(x,t) = e^{t} \sin(t)(-3 \cos(t)\cos(t) + \sin(t)) + (10\pi^2 + 1 + 2\pi^4)\sin(t)^2 \]

\[ \left( \frac{\cos(x_1\pi)\sin(x_2\pi)}{\sin(x_1\pi)\cos(x_2\pi)} \right) \]
In this case, the exact state is given by \( \tilde{\mathcal{M}}_1 \). Moreover, we present the efficiency indices for \( \tilde{\mathcal{M}}_1 \) given for the modes by

\[
I_{\text{eff}}^{\tilde{\mathcal{M}}_1,0} = \sqrt{\frac{\mathcal{M}_1(\alpha, \beta; y_0^0, \bar{p}_0^0, \tau_0^0, \rho_0^0)}{|||y_0^0 - \bar{y}_0^0|||_{1,0}^2}} \quad \text{and} \quad I_{\text{eff}}^{\tilde{\mathcal{M}}_1,k} = \sqrt{\frac{\mathcal{M}_1(\alpha, \beta; y_{kh}, \bar{p}_{kh}, \tau_{kh}, \rho_{kh})}{|||y_k - y_{kh}|||_{1,k}^2}}
\]

The error norms for the modes are given by

\[
|||y_0^0 - \bar{y}_0^0|||_{1,0}^2 = \frac{1}{2} \left( 1 + \frac{2\lambda_{1/2}^0}{C_p^2} \right) |||\nabla y_0^0 - \nabla \bar{y}_0^0|||_{1}^2 \quad \text{and}
\]

\[
|||y_k - y_{kh}|||_{1,k}^2 = \frac{2\kappa \omega \lambda_{1/2}^k}{C_p^2} |||y_k - y_{kh}|||_{1}^2 + \frac{1}{2} \left( 1 + \frac{2\lambda_{1/2}^0}{C_p^2} \right) |||\nabla y_k - \nabla y_{kh}|||_{1}^2 \quad \text{leading the representation}
\]

\[
(91) \quad |||u - v|||_{1}^2 = T |||y_0^0 - \bar{y}_0^0|||_{1,0}^2 + \frac{T}{2} \sum_{k=1}^{N} |||y_k - y_{kh}|||_{1,k}^2 + \tilde{\mathcal{F}}_N
\]

with the remainder term

\[
(92) \quad \tilde{\mathcal{F}}_N := \frac{T}{2} \sum_{k=1}^{\infty} |||y_k|||_{1,k}^2.
\]

For the numerical experiments, the efficiency index for \( \tilde{\mathcal{M}}_1 \) above by estimating \( 91 \) from below ignoring the remainder term \( 92 \) leading to the overall efficiency index for \( \tilde{\mathcal{M}}_1 \)

\[
I_{\text{eff}}^{\tilde{\mathcal{M}}_1} = \sqrt{\frac{\tilde{\mathcal{M}}_1(\alpha, \beta, \eta, \zeta, \tau, \rho, \nu)}{T |||y_0^0 - \bar{y}_0^0|||_{1,0}^2 + \frac{T}{2} \sum_{k=1}^{N} |||y_k - y_{kh}|||_{1,k}^2}}.
\]

The corresponding majorants are given by

\[
\tilde{\mathcal{M}}_1(\alpha, \beta; y_0^0, \bar{p}_0^0, \tau_0^0, \rho_0^0) = \tilde{\mathcal{F}}_0(\alpha, \beta; y_0^0, \bar{p}_0^0, \tau_0^0, \rho_0^0) - \tilde{\mathcal{F}}_0(\alpha, \beta; y_0^0, \bar{p}_0^0, \tau_0^0, \rho_0^0)
\]

\[
\quad + \frac{3\lambda}{2C_p^2} \left( C_F \||R_{10}^0||_{1} + \||R_{20}^0||_{1} \right)^2
\]

\[
= \frac{\alpha}{2} \sqrt{|\nabla y_0^0 - g_{d0}^0|||_{1}^2} + \frac{(1 + \alpha \kappa)(1 + \beta)}{2\alpha \omega \mu_1^2} |||\nabla p_{0^0}|||_{1}^2 + \frac{(1 + \alpha \kappa)(1 + \beta \lambda)}{2\alpha \omega \mu_1^2} |||\nabla \rho_{0^0}|||_{1}^2
\]

\[
+ \int_{\Omega} \left( \nu \nabla y_0^0 \cdot \nabla p_{0^0} + \lambda^{-1} (p_{0^0}^2) \right) dx + \frac{C_p^2}{\mu_1^2} \left( C_F \||R_{10}^0||_{1} + \||R_{20}^0||_{1} \right)^2
\]

\[
+ \frac{1}{\mu_1^2} \left( C_F \||R_{10}^0||_{1} + \||R_{20}^0||_{1} \right) \left( C_F \||R_{20}^0||_{1} + \||R_{40}^0||_{1} \right) + \frac{3\lambda}{2C_p^2} \left( C_F \||R_{10}^0||_{1} + \||R_{20}^0||_{1} \right)^2
\]

and

\[
\tilde{\mathcal{M}}_1(\alpha, \beta; y_{kh}, \bar{p}_{kh}, \tau_{kh}, \rho_{kh}) = \tilde{\mathcal{F}}_k(\alpha, \beta; y_{kh}, \bar{p}_{kh}, \tau_{kh}, \rho_{kh}) - \tilde{\mathcal{F}}_k(\alpha, \beta; y_{kh}, \bar{p}_{kh}, \tau_{kh}, \rho_{kh})
\]

\[
\quad + \frac{3\lambda}{2C_p^2} \left( C_F \||R_{1k}||_{1} + \||R_{2k}||_{1} \right)^2
\]

\[
= \frac{\alpha}{2} \sqrt{|\nabla y_k - g_{dk}|||_{1}^2} + \frac{(1 + \alpha \kappa)(1 + \beta)}{2\alpha \omega \mu_1^2} \left( C_F \||R_{2k}||_{1} \right)^2 + \frac{(1 + \alpha \kappa)(1 + \beta \lambda)}{2\alpha \omega \mu_1^2} \left( C_F \||R_{4k}||_{1} \right)^2
\]

\[
+ \int_{\Omega} \left( \nu \nabla y_k \cdot \nabla p_{kh} - k \omega \sigma y_{kh} p_{kh} + \lambda^{-1} p_{kh}^2 \right) dx + \frac{C_p^2}{\mu_1^2} \left( C_F \||R_{3k}||_{1} + \||R_{4k}||_{1} \right)^2
\]

\[
+ \frac{1}{\mu_1^2} \left( C_F \||R_{1k}||_{1} + \||R_{2k}||_{1} \right) \left( C_F \||R_{2k}||_{1} + \||R_{4k}||_{1} \right) + \frac{3\lambda}{2C_p^2} \left( C_F \||R_{1k}||_{1} + \||R_{2k}||_{1} \right)^2.
\]
We present the numerical results for the modes $k = 0$ and $k = 1$ for different mesh sizes in Tables 6 and 7. The efficiency indices for the majorants and minorants are very close to 1.00. Also the efficiency indices for $M_{1,0}$ show a good accuracy.

| grid     | $\rho^{\text{sec}}$ | $\tilde{J}_{0}^{\ominus}$ | $I_{\text{eff}}^{\tilde{J}_{0}^{\ominus}}$ | $\tilde{J}_{1}^{\ominus}$ | $I_{\text{eff}}^{\tilde{J}_{1}^{\ominus}}$ | $I_{\text{eff}}^{\tilde{J}_{0}^{\ominus}}$ | $I_{\text{eff}}^{\tilde{J}_{1}^{\ominus}}$ |
|----------|----------------------|-----------------------------|------------------------------------------|-----------------------------|------------------------------------------|------------------------------------------|------------------------------------------|
| $16 \times 16$ | 0.02                 | 8.92e+03                    | 0.99                                     | 9.85e+03                    | 1.09                                     | 1.10                                     | 2.04                                     |
| $32 \times 32$ | 0.06                 | 9.24e+03                    | 0.99                                     | 9.95e+03                    | 1.07                                     | 1.08                                     | 1.97                                     |
| $64 \times 64$ | 0.24                 | 9.32e+03                    | 0.99                                     | 9.97e+03                    | 1.05                                     | 1.07                                     | 1.94                                     |
| $128 \times 128$ | 1.04                | 9.34e+03                    | 0.98                                     | 9.98e+03                    | 1.05                                     | 1.07                                     | 1.93                                     |
| $256 \times 256$ | 4.39                | 9.35e+03                    | 0.98                                     | 9.98e+03                    | 1.05                                     | 1.07                                     | 1.92                                     |

Table 6. Example 4. The majorant $\tilde{J}_{0}^{\ominus}$ and minorant $\tilde{J}_{0}^{\ominus}$, and their efficiency indices.

| grid     | $\rho^{\text{sec}}$ | $\tilde{J}_{1}^{\ominus}$ | $I_{\text{eff}}^{\tilde{J}_{1}^{\ominus}}$ | $\tilde{J}_{0}^{\ominus}$ | $I_{\text{eff}}^{\tilde{J}_{0}^{\ominus}}$ | $I_{\text{eff}}^{\tilde{J}_{1}^{\ominus}}$ | $I_{\text{eff}}^{M_{1,1}}$ |
|----------|----------------------|-----------------------------|------------------------------------------|-----------------------------|------------------------------------------|------------------------------------------|------------------------------------------|
| $16 \times 16$ | 0.02                 | 3.22e+04                    | 0.95                                     | 3.51e+04                    | 1.03                                     | 1.09                                     | 1.52                                     |
| $32 \times 32$ | 0.06                 | 3.39e+04                    | 0.96                                     | 3.58e+04                    | 1.02                                     | 1.05                                     | 1.33                                     |
| $64 \times 64$ | 0.26                 | 3.45e+04                    | 0.97                                     | 3.60e+04                    | 1.01                                     | 1.04                                     | 1.25                                     |
| $128 \times 128$ | 1.08                | 3.48e+04                    | 0.97                                     | 3.62e+04                    | 1.01                                     | 1.04                                     | 1.21                                     |
| $256 \times 256$ | 4.31                | 3.51e+04                    | 0.98                                     | 3.65e+04                    | 1.01                                     | 1.04                                     | 1.19                                     |

Table 7. Example 4. The majorant $\tilde{J}_{1}^{\ominus}$ and minorant $\tilde{J}_{1}^{\ominus}$, and their efficiency indices.

Table 8 compares the results for the modes $k \in \{0, \ldots, 8\}$ computed on the 256 $\times$ 256-mesh presenting the overall functional error estimates, where the remainder term $\mathcal{E}_{N}$ is precomputed exactly. The values of the efficiency indices vary for different modes $k$. For example, the results for $M_{1,4}$ indicate that the mode $k = 4$ is essential to represent the solution accurately. The values for $I_{\text{eff}}^{\tilde{J}_{0}^{\ominus}}$ and $I_{\text{eff}}^{\tilde{J}_{1}^{\ominus}}$ indicate that the minorants require a different refinement for a more accurate representation. However, in these cases the majorants give a good representation for the minimization functional. Finally, comparing the last two lines of Table 8 again shows that the overall value for representing the cost functional of the exact solution is already sufficiently accurate for a truncation index $N = 6$.

| grid     | $\rho^{\text{sec}}$ | $\tilde{J}_{0}^{\ominus}$ | $I_{\text{eff}}^{\tilde{J}_{0}^{\ominus}}$ | $\tilde{J}_{1}^{\ominus}$ | $I_{\text{eff}}^{\tilde{J}_{1}^{\ominus}}$ | $I_{\text{eff}}^{\tilde{J}_{0}^{\ominus}}$ | $I_{\text{eff}}^{\tilde{J}_{1}^{\ominus}}$ | $I_{\text{eff}}^{M_{1,1}}$ |
|----------|----------------------|-----------------------------|------------------------------------------|-----------------------------|------------------------------------------|------------------------------------------|------------------------------------------|------------------------------------------|
| $k = 0$   | 4.39                 | 9.35e+03                    | 0.98                                     | 9.98e+03                    | 1.05                                     | 1.04                                     | 1.97                                     |
| $k = 1$   | 4.31                 | 3.51e+04                    | 0.98                                     | 3.65e+04                    | 1.01                                     | 1.04                                     | 1.19                                     |
| $k = 2$   | 4.42                 | 9.23e+03                    | 0.63                                     | 1.57e+04                    | 1.06                                     | 1.70                                     | 1.65                                     |
| $k = 3$   | 4.43                 | 2.85e+03                    | 0.58                                     | 5.06e+03                    | 1.03                                     | 1.78                                     | 1.08                                     |
| $k = 4$   | 4.44                 | 5.89e+02                    | 0.98                                     | 6.47e+02                    | 1.07                                     | 1.10                                     | 5.91                                     |
| $k = 5$   | 4.36                 | 5.85e+01                    | 0.75                                     | 7.78e+01                    | 1.00                                     | 1.33                                     | 2.20                                     |
| $k = 6$   | 4.36                 | 1.74e+01                    | 0.97                                     | 2.82e+01                    | 1.58                                     | 1.62                                     | 2.32                                     |
| $k = 7$   | 4.41                 | 1.40e+00                    | 0.35                                     | 3.99e+00                    | 1.00                                     | 2.85                                     | 1.61                                     |
| $k = 8$   | 4.33                 | 2.99e-01                    | 0.23                                     | 1.37e+00                    | 1.05                                     | 4.59                                     | 1.97                                     |
| overall with $N = 6$ | –                   | 2.09e+05                    | 0.89                                     | 2.45e+05                    | 1.04                                     | 1.17                                     | 2.46                                     |
| overall with $N = 8$ | –                   | 2.09e+05                    | 0.89                                     | 2.45e+05                    | 1.04                                     | 1.17                                     | 2.46                                     |

Table 8. Example 4. The overall majorant $\tilde{J}_{0}^{\ominus}$ and minorant $\tilde{J}_{1}^{\ominus}$, their parts, and their efficiency indices computed on a 256 $\times$ 256-mesh.

In Example 5, we choose the desired gradient

\[ g_{d}(x, t) = -e^{t} \sin(t)(0.1 \cos(t) - \pi^{2}(1 + 2\pi^{2}0.1)) \left( \frac{\cos(x_{1}\pi) \sin(x_{2}\pi)}{\sin(x_{1}\pi) \cos(x_{2}\pi)} \right) \]
leading to the time-analytic, but not time-periodic exact state \( \mathbf{g} \). Again we compute the MhFE approximation of the desired gradient and solve the systems \( \mathbf{E} \) and \( \mathbf{M} \) for all \( k \in \{0, \ldots, 10\} \). Table 4 presents the numerical results for different modes on a \( 256 \times 256 \)-mesh. The remainder terms for \( N = 6 \), \( N = 8 \) and \( N = 10 \) are \( E_6 = 4796.54 \), \( E_8 = 2158.78 \) and \( E_{10} = 1149.65 \), respectively. The efficiency indices for the overall majorant and minorant show that a truncation index of \( N = 6 \) already gives a sufficiently accurate approximation for the overall cost functional. Note that the efficiency index for \( M_{1,2} \) indicates that the mode \( k = 2 \) is essential for the multiharmonic approximation giving an accurate representation of the solution.

| grid | \( t^\text{sec} \) | \( \tilde{J}^\oplus \) | \( I^\text{eff}_\text{eff} \) | \( \tilde{J}^\ominus \) | \( I^\text{eff}_\text{eff} \) | \( \tilde{J}^\ominus \) | \( I^\text{eff}_\text{eff} \) | \( I^M_\text{eff} \) |
|------|-----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( k = 0 \) | 4.31 | 2.63e+04 | 1.00 | 2.79e+04 | 1.06 | 1.06 | 1.36 |
| \( k = 1 \) | 4.30 | 8.49e+04 | 1.00 | 8.60e+04 | 1.02 | 1.01 | 1.00 |
| \( k = 2 \) | 4.39 | 2.08e+04 | 0.98 | 2.21e+04 | 1.04 | 1.06 | 2.83 |
| \( k = 3 \) | 4.32 | 4.85e+03 | 0.97 | 5.25e+03 | 1.05 | 1.08 | 1.77 |
| \( k = 4 \) | 4.35 | 1.58e+03 | 0.96 | 1.75e+03 | 1.06 | 1.11 | 1.74 |
| \( k = 5 \) | 4.37 | 6.18e+02 | 0.90 | 7.10e+02 | 1.04 | 1.15 | 1.69 |
| \( k = 6 \) | 4.36 | 2.93e+02 | 0.87 | 3.53e+02 | 1.05 | 1.20 | 1.63 |
| \( k = 7 \) | 4.41 | 1.48e+02 | 0.81 | 1.90e+02 | 1.04 | 1.28 | 1.40 |
| \( k = 8 \) | 4.37 | 8.95e+01 | 0.82 | 1.20e+02 | 1.10 | 1.34 | 1.08 |
| \( k = 9 \) | 4.42 | 5.71e+01 | 0.83 | 8.16e+01 | 1.18 | 1.43 | 1.30 |
| \( k = 10 \) | 4.34 | 3.27e+01 | 0.71 | 5.23e+01 | 1.14 | 1.60 | 1.19 |
| overall with \( N = 6 \) | | 5.23e+05 | 1.00 | 5.43e+05 | 1.04 | 1.04 | 2.00 |
| overall with \( N = 8 \) | | 5.22e+05 | 1.00 | 5.42e+05 | 1.04 | 1.04 | 2.00 |
| overall with \( N = 10 \) | | 5.22e+05 | 1.00 | 5.42e+05 | 1.04 | 1.04 | 2.00 |

Table 9. Example 5. The overall majorant \( \tilde{J}^\oplus \) and minorant \( \tilde{J}^\ominus \), their parts, and their efficiency indices computed on a \( 256 \times 256 \)-mesh.

In Example 6, we set

\[
g_d(x, t) = \begin{pmatrix} \chi_{\lfloor \frac{t + 1}{2} \rfloor} \chi_{\lfloor \frac{x + 1}{4} \rfloor} \\ \chi_{\lfloor \frac{t + 1}{2} \rfloor} \chi_{\lfloor \frac{x + 1}{4} \rfloor} \end{pmatrix},
\]

where \( \chi \) denotes the characteristic function in space and time. Here, \( T = 1 \), then \( \omega = 2\pi \). Also the coefficients of the Fourier expansion associated with \( g_d \) can be found analytically. They are as in Example 3 given by \( \mathbf{m} \) for each direction of the gradient \( \mathbf{m} \). Again the exact solution cannot be computed analytically and hence its MhFE approximations are computed on a finer mesh (512 \times 512-mesh). Table 10 presents the results for modes up to truncation index \( N = 29 \) as well as for \( k = 41 \) and \( k = 81 \) analogously to Example 3. The results reflected by the efficiency indices show the good representation by using the minorants and majorants, especially, considering the efficiency indices in the last two columns of Table 10. This again demonstrates the efficiency of the minorants and majorants for data having jumps in space and time but now for minimization problem II.

8. Conclusions and Outlook

In \cite{58}, the authors derived functional-type a posteriori error estimates for MhFE approximations to linear parabolic time-periodic optimal control problems. Upper bounds for the state, the adjoint state, the control and for the cost functional are presented. In this work, the a posteriori error analysis has been extended by deriving also lower bounds, called minorants, for the cost functional leading to an upper estimate for the error norm of the state and control or equivalently in state and adjoint state.

The multiharmonic approximations for linear problems lead to a decoupling of computations corresponding to different modes. Hence, we can in principle use different meshes for different modes. Moreover, we can build up the meshes independently by applying an adaptive finite element
method for approximating the respective Fourier coefficients. We could clearly see the need and opportunity for implementing the adaptive finite element method in the numerical experiments of this work. In order to assure the quality of approximations constructed in this way, we need fully computable a posteriori estimates, which provide guaranteed bounds of global errors and reliable indicators of errors associated with the modes. By prescribing certain bounds, we then can finally filter out the Fourier coefficients, which are important for the numerical solution of the respective problem. This systematic approach describes an adaptive multiharmonic finite element method (AMhFEM) that will provide complete adaptivity in space and time. The detailed description and analysis of such an AMhFEM exceeds the range of this paper, and the results of this paper as described before as well as of [30] will strongly serve as the basis for the AMhFEM.

The functional a posteriori estimates of [30] and this work can also be derived for distributed time-harmonic eddy current optimal control problems as studied in [21, 22] as well as in the recent work [2].

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