Closed Strings in the 2D Lorentzian Black Hole

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ABSTRACT: We revisit the spectrum of closed strings in the Lorentzian signature 2D black hole in string theory. Using the description of the black hole as a gauged WZNW model, we argue that the spectrum of the closed strings contain states from the spectrally flowed versions of the principal continuous and also the principal discrete series of $SL_2(\mathbb{R})$. We identify the string configurations that correspond to these states. Using vector-axial duality, we also find new localized states that are essentially stringy in origin.

KEYWORDS: 2D black hole, long strings, winding strings, spectral flow, CFT, coset CFT

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1 Introduction

Recent years have seen great progress in understanding entropy and grey-body factors of black holes using string theory. In one sense, the problem of accounting for the microscopic states responsible for the entropy has been solved. Not only the leading order Bekenstein-Hawking entropy, but also an infinite series of subleading corrections have been computed.
both in the microscopic description and in the gravitational description and shown to match term by term (for a review, see [1]).

However, it is perhaps fair to say that a direct relationship between the microstates that contribute to the entropy and the horizon, which is the classical and geometrical manifestation of this entropy, has not been demonstrated. This is, at least in part, due to the fact that the counting of the degrees of freedom is performed at $G_N = 0$ when the horizon vanishes.

The two dimensional black hole is particularly relevant in this context. Not only is it an exact solution of string theory, but it also admits a tractable CFT description in the form of a gauged WZNW model. Further, it has nonzero temperature and entropy [5, 6, 7, 8]. Because of this one could hope to tie up the relation between thermodynamics and the horizon more explicitly. Another closely related reason for interest is that the corresponding Euclidean black hole also admits an exact CFT description. This fact should help in understanding the relationship between the Euclidean and Lorentzian black holes.

Using the gauged sigma model description, we can study the spectrum of the coset theory that corresponds to the Lorentzian black hole (in this work, we concentrate on the Lorentzian sigma model). This was taken up in the paper of Dijkgraaf et al [9], and then a more detailed analysis appeared in the paper of Distler and Nelson [10]. In the latter, the authors studied the cohomology of the coset model in order to determine all candidate physical states of the coset theory (from the hermitean representations of the Kac-Moody algebra known at that time).

In spite of these investigations, the Minkowski black hole has not yet been completely understood. In particular, we do not know how to arrive at a modular invariant partition function (if that notion is still relevant. This is because of complications coming from the non-compact coset CFT since the WZNW model is based on the group $SL_2(\mathbb{R})$. In contrast, for the Euclidean black hole the partition function is known [11].

After the work of Maldacena and Ooguri [12], it was understood that string theory on $AdS_3$ (i.e., $SL_2(\mathbb{R})$) requires additional representations to form a modular invariant partition function. These new representations maybe generated from the usual ones by a transformation termed as “spectral flow”. Since the black hole is obtained as a coset of the $SL_2(\mathbb{R})$ CFT, a logical question is to ask what happens to these new representations? Do they give rise to new states of the black hole background?

Recall that the Euclidean black hole geometry looks like $R \times S^1$ asymptotically. Therefore, we can have strings winding on this circle. This winding number will however not be conserved (since a winding string can ”slip” off the tip). A short calculation, presented in the appendix, shows that these winding strings appear as the projection of the spectrally flowed representations of $SL_2(\mathbb{R})$ to the coset theory that describes the Euclidean black hole [12]. This suggests that one should consider the spectrally flowed representations in the Lorentzian case as well.

The Lorentzian black hole however, has no periodic direction and hence no winding strings. Therefore, it appears that upon “Wick rotation”, an entire tower of states disappears from the string spectrum!
Another motivation for searching for such states is the following. There are D-branes in the Euclidean black hole geometry which couple to the winding strings [13]. In [14], it was shown that there are corresponding D-branes in the Lorentzian black hole. In order to write the boundary states for these branes, we can expect that we require the Lorentzian counterparts of winding strings. Since the D-branes are not perturbative objects, this provides a “nonperturbative” justification for searching for the analogues of the winding strings.

From the work of [15], it is known that the winding strings of the Euclidean theory correspond to the non-singlet sector of the matrix model which is the field theory dual (in the sense of holography) to the black hole. In [16], Maldacena studied another set of nonsinglet modes in the form of folded “long strings” (which are dual to qq-states) in the asymptotically flat region of the Lorentzian black hole. Therefore, we can ask how those strings lift to the full black hole geometry (if at all).

However, to obtain the Lorentzian black hole, we need to gauge a hyperbolic direction in $SL_2(\mathbb{R})$. Therefore we will need to understand the action of spectral flow along this hyperbolic direction (in contrast to [12]). This problem has already been studied by Keski-Vakkuri and Hemming [17] in the context of BTZ black holes. The BTZ black hole is an orbifold of $SL_2(\mathbb{R})$, where the orbifolding action is along a hyperbolic direction of $SL_2(\mathbb{R})$ (i.e., we orbifold by a boost). Thus, the string spectrum of the black hole will contain states from the twisted sectors of the orbifold action. It was shown by them that the twisted sector states may be understood as a projection of spectrally flowed strings of the $SL_2(\mathbb{R})$ theory, where the spectral flow action is now along the hyperbolic direction. See also [19, 20] for closely related explorations.

Yet another perspective on the string theory of the black hole is based on holography. In the spirit of the AdS/CFT correspondence (and further extensions along the lines of Vasiliev theory and the O(N) model), we may regard the string theory of the black hole as a dual description of the high temperature (deconfined) phase of a matrix model that lives on the boundary. In this case the states of the string theory correspond to states of the matrix model, while operators of the matrix model should be dual to non-normalizable modes of the bulk fields.

In our work, we will investigate the spectrum of the Lorentzian black hole with careful attention to spectral flow. Rather than starting with the spectral flow operation itself, we first consider geodesics as representing point-like closed strings, and investigate if they satisfy the physical state conditions of the string theory. By this procedure, we are naturally led to spectrally flowed strings. Thus from the viewpoint of the 2D-black hole, we need to start with these “spectrally flowed” representations of the $SL_2(\mathbb{R})$ CFT. We then verify the earlier results about the spectrum of the string theory with some interesting qualifications. We show that the ‘tachyon’ occurs in a one parameter family, analogous to the tachyon in the Euclidean black hole (in that case, the parameter is the winding number). Thus we conclude that this new parameter (which arises as a spectral flow parameter in both cases) is the Lorentzian equivalent of the winding number. Secondly, we propose that the massless particle corresponds to a spectrally flowed version of the coset primary. Further, using the vector-axial self duality of the black hole sigma model, we find additional, essentially stringy
states by dualising the geodesics.

This manuscript is organized as follows. In Section 2, we briefly discuss the black hole and its thermodynamic properties to be self-contained. We will then recall its construction as a gauged sigma model and discuss co-ordinate charts on \( SL_2(\mathbb{R}) \) which project to various regions of the black hole geometry in Section 3.1. A careful reconsideration of the various geodesics of the black hole geometry in Section 4 gives us a handle on identifying the various classical string excitations. Section 7 uses the vector-axial duality of this sigma model to produce new, essentially string states which are ‘T-dual’ to the point-like states of the previous section. In section 8, we consider the physical state conditions, both classical and quantum to be satisfied by the modes constructed above. We conclude in Section 9 with a summary and some suggestive directions for future research. Two appendices present conventions employed in this paper and a short illustration of the manner in which spectrally flowed representations descend as winding strings of the Cigar geometry.

Part of the results in this paper has been published earlier, albeit in condensed form [21]. While this manuscript was being readied, the paper [22] appeared which also suggests an interior structure to these black holes.

2 The Lorentzian black hole

The 2-D black hole is described by the following line element and dilaton profile

\[
ds^2 = -(1 - \frac{M}{r})dt^2 + \frac{kdr^2}{4r^2(1 - \frac{M}{r})}, \quad e^\Phi = \sqrt{\frac{M}{r}} \tag{2.1}
\]

It follows that, at the horizon \( r = M \), the dilaton value is \( \Phi = 0 \) and that the string coupling blows up at the singularity \( r = 0 \). The dimensionful parameter \( k \) is proportional to the level of the \( SL_2(\mathbb{R}) \) coset theory described below. One can pass to global Kruskal co-ordinates by the change of variable \( uv = -\frac{r - M}{M} \) and \( \exp(t) = -\frac{u}{v} \). The metric then becomes

\[
ds^2 = -\frac{k}{1 - uv} \frac{du dv}{1 - uv} \tag{2.2}
\]

with the curvature singularity located at \( uv = 1 \) and the horizon at \( uv = 0 \).

This black hole is a solution of the equations of motion that follow from the action

\[
S = \int \sqrt{g} e^{-2\Phi} (R + 4(\nabla \Phi)^2 + \Lambda), \tag{2.3}
\]

where \( \Lambda = -8 \) is a negative cosmological constant. Far away from the black hole, the physics is controlled by the cosmological constant - hence this black hole solution is analogous to a black hole inside AdS-space [4].

The thermodynamics of this black hole has been studied in various works [5]-[8] using various techniques. The black hole has a finite temperature which can be easily determined by the examining the periodicity of Euclidean time. The ADM mass is read off from the metric and then one can appeal to the first law of thermodynamics \( E = TS \) to obtain an entropy. These are

\[
E = M, \quad T = \frac{1}{2\pi} \sqrt{\frac{M}{k}}, \quad S = 2\pi \sqrt{Mk} \tag{2.4}
\]
Note that $M$ is completely determined by the dilaton value at the horizon $e^{\Phi(r=M)} = 1$.

3 The Lorentzian black hole as a gauged sigma model

The Lorentzian black hole is obtained by gauging a non-compact axial $U(1)$ symmetry of the $SL(2, R)$ WZW model [3]. We shall briefly outline the procedure here, for details refer to the appendix.

The symmetry that is being gauged corresponds to a hyperbolic subgroup of $SL_2(R)$ which acts on $g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix} \in SL_2(R)$, as $\delta g = \epsilon(\sigma_3 g + g \sigma_3)$, i.e.,

\begin{align}
\delta a &= 2\epsilon a, \quad \delta u = 0, \\
\delta b &= -2\epsilon b, \quad \delta v = 0.
\end{align}

To obtain a target space interpretation, we have to gauge fix and integrate out the gauge fields. In the region $(1 - uv) > 0$, we have $ab > 0$, and hence a natural gauge fixing condition is $a = b$. Upon integrating out the gauge fields (which appear quadratically), we obtain the black hole sigma model

\[ L = -\frac{k}{4\pi} \int d^2x \sqrt{h} \frac{h^{ij} \partial_i u \partial_j v}{(1 - uv)} \]

In the region $(1 - uv) < 0$ however, a good gauge fixing condition is $a = -b$ (because $ab < 0$). When $uv = 1$, we have either $a = 0$ or $b = 0$ or both, hence we cannot gauge transform a generic field configuration to the gauge slice (for either gauge choice). Although the gauge fixing condition is singular, the sigma model is itself non-singular everywhere (it has been argued that the locus $uv = 1$ which maps to the singularity is, by itself, governed by a topological string theory [24]). Requiring conformal invariance generates a dilaton

\[ \Phi = \Phi_0 - \frac{1}{2} \ln(1 - uv), \]

where the parameter $\Phi_0$ is related to the mass $M$ of the black hole 2.1.

The target space geometry of the sigma model so obtained is shown in Fig: 1.

In the figure, the diagonal lines $uv = 0$ form the horizon, while $uv = 1$ is the singularity (the Ricci scalar diverges as $R \sim (1 - uv)^{-2}$). Regions I and II are asymptotically flat regions and in regions V and VI time flows “sideways”. Straight lines passing through the origin are constant time slices with time increasing from top to bottom in region II, and from bottom to top in region I. Thus the black hole singularity is in the fourth quadrant (in the figure the diagonal lines are the $u, v$ co-ordinate axes!).

To obtain the physical state conditions and for quantization, it is convenient to follow the BRST procedure [9]. As a result of this, the black hole sigma model Lagrangian becomes

\[ S_{BAH} = S_{WZNW}(\rho, t_L - \phi_L, t_R - \phi_R) - \frac{k}{2\pi} \int \partial_+ X \partial_- X + S_{ghosts}, \]
where $X = \phi_L - \phi_R$, and $A_\pm = \partial_x \phi_{L,R}$ are the gauge fields of the gauged WZW model. The original gauge symmetry manifests itself as invariance under a simultaneous shift of $t_{R,L}$ and $\phi_{R,L}$. We note that $\rho, t_{R,L}$ are the fields of the original WZNW model (before gauging) and hence are directly related to the target space variables of the black hole $(t = \frac{1}{2}(t_R - t_L))$. As a result of this procedure, we are left with a BRST constraint,

$$k\partial_\pm X = J^{(2)}_\pm,$$

where $J'$ in this equation is the conserved current of the WZNW Lagrangian $S'$. We also have a (classical) Virasoro constraint (since this is a string theory)

$$T^{tot}_{++} = T^{WZW'}_{++} - k(\partial_+ X)^2 = 0$$

(3.6)

It is a remarkable fact that, instead of gauging the axial action as above, if we gauge the vectorial action as in $\delta g = \epsilon (\sigma_3 g - g \sigma_3)$, we nevertheless obtain the same target space geometry. In this case, the diagonal entries $a, b$ are invariant under the gauging and form coordinates of the black hole spacetime.

### 3.1 Co-ordinate systems for $SL_2(\mathbb{R})$

It is useful to understand the correspondence between the various regions of the black hole geometry and coordinate charts of $SL_2(\mathbb{R})$. First, we recall that every matrix $g \in SL_2(\mathbb{R})$ with all entries nonzero can be written as a product [25]

$$g = d_1 (-e)^{t_1} s^{t_2} p d_2,$$

where $d_{1,2} = \text{diag}(e^{\theta_{1,2}}, e^{-\theta_{1,2}})$ and $\theta_{1,2} \in (-\infty, \infty)$, $e$ is the identity matrix, $s = i\sigma_2$ is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $p$ is one of two matrices

$$p_1 = \begin{pmatrix} \cosh \rho & -\sinh \rho \\ -\sinh \rho & \cosh \rho \end{pmatrix}, \quad \rho \in [-\infty, \infty], \quad \text{or} \quad p_2 = \begin{pmatrix} \cos \rho & \sin \rho \\ -\sin \rho & \cos \rho \end{pmatrix}, \quad \rho \in [-\frac{\pi}{4}, \frac{\pi}{4}],$$

It is a remarkable fact that, instead of gauging the axial action as above, if we gauge the vectorial action as in $\delta g = \epsilon (\sigma_3 g - g \sigma_3)$, we nevertheless obtain the same target space geometry. In this case, the diagonal entries $a, b$ are invariant under the gauging and form coordinates of the black hole spacetime.
and $\epsilon_{1,2} \in \{0,1\}$. Instead of using 4 trigonometric charts, we will extend the range $-\frac{\pi}{2} \leq \rho \leq \frac{\pi}{2}$ and drop the action of $i\sigma_2$. In a similar manner, matrices in $SL_2(\mathbb{R})$ with at least one zero entry can be written as a product

$$g = d (e^\rho, e^{-\rho})$$

where $d = \text{diag}(e^{\phi}, e^{-\phi})$.

The axial gauge symmetry that leads to the Lorentzian black hole acts as $\theta_{1,2} \rightarrow \theta_{1,2} + \epsilon$, and the time co-ordinate $t$ of the black hole geometry is related to the $\theta_i$ as $t = (\theta_1 - \theta_2)$. It is then easy to see how the various co-ordinate charts project down (upon gauging) to cover different regions of the black hole geometry.

For instance, setting $p = p_1 \epsilon_1 = \epsilon_2 = 0$ gives us $SL_2(\mathbb{R})$ matrices of the form

$$g = d_1 \begin{pmatrix} \cosh \rho & \sinh \rho \\ \sinh \rho & \cosh \rho \end{pmatrix} d_2.$$  

Gauging the axial $U(1)$ symmetry sets $d_2 = d_1^{-1} = e^{-\sigma_3 t/2}$ and projects to the $(u,v)$ co-ordinates. The matrices above are then seen to cover the $uw < 0$ regions of the black hole geometry with $u = -e^t \sinh \rho$ and $v = e^{-t} \sinh \rho$. We shall refer to the time coordinate $t$ as ‘Schwarzschild’ time and $\rho \leq 0$ will correspond to region I,II respectively. Note that $p = p_1$, $\epsilon_1 = 1$, $\epsilon_2 = 0$ also covers the same region of the coset (for the full range of $\rho \in \mathbb{R}$).

Thus we obtain the following covering diagram Fig. 2. The singularity $(ab = 0)$ is the dark (black) line in the figure, while the horizon is the diagonal (blue) line $uv = 0$. The matrices in $SL_2(\mathbb{R})$ with zero entries cover the horizon lines and the singularity and are not indicated in the figure. The region between the horizon and the singularity is covered by the four charts with $p_2$ type of matrices.

There are two $\mathbb{Z}_2$ operations which are important for our purposes, multiplication by $-I$ denoted as $R$ and $g \rightarrow i\sigma_2 g i\sigma_2$ denoted as $C$. $C$ reflects $u+v \rightarrow -(u+v)$ while leaving $u-v$ unchanged - and $R$ is a simple reflection $(u,v) \rightarrow (-u,-v)$.  

Figure 2. Covering diagram for the black hole geometry
In regions I and II, the $C$ operation is nothing but time reversal (of the ‘Schwarzschild’ time) and acts within each region. The $R$ operation maps region I to region II and vice versa. In region V, the chart is given by $u = e^t \cosh \rho \quad v = e^{-t} \cosh \rho$ and region VI is obtained by the action of $R$. However, in regions V and VI, the $C$ transformation interchanges the regions besides reversing the ‘Schwarzschild’ time. In contrast to regions I and II, the full range of $\rho$ covers each region twice separately.

Regions III and IV are also covered twice by $u = e^t \sin \rho, \quad v = e^{-t} \sin \rho$. $R$ and $C$ interchange regions III and IV, but $C$ reverses $t$ as well. The product $CR$, therefore, results in changing the sign of $\rho$ alone (which is the time direction in this region).

Thus, time reversal is implemented by $C$ in regions I and II, by $CR$ in regions III and IV, and by $R$ in regions V and VI. Also, to obtain a single cover of the black hole, we can restrict to $\rho \geq 0$, and require $C$ and $R$ to be implemented faithfully on the states. It must be noted that $C$ and $R$ are $SL_2(\mathbb{R})$ transformations, while time reversal is a property of the black hole geometry.

Thus for our choice of coordinate charts, straight lines passing through the origin represent constant time surfaces in regions I, II and V, VI.

4 Geodesics and their lifts

The perturbative spectrum of the Lorentzian black hole has been discussed in the literature (see [9, 10]). The states in the spectrum are a single massless scalar field, the “tachyon” in the principal continuous series of $SL_2(\mathbb{R})$ and possibly, some massive states in the principal discrete series. In the semiclassical limit, the particles that correspond to these states will move on null/timelike geodesics. Hence we may expect that the geodesics of the geometry solve the equations of motion of the sigma model of the black hole (the equations of the black hole sigma model reduce to geodesic equations for point-like configurations - the dilaton also plays a role only for stringy configurations). Thus, one way to study the spectrum would be to quantize the geodesics. It is of course possible that some of the states of the string theory do not have a classical limit i.e., are essentially “stringy”.

However, rather than study the solutions to the sigma model defined by the black hole metric, we will make use of gauged sigma model description (see [19] for a similar approach). One advantage of this description is that $\alpha'$ corrections can all be taken into account (and for this black hole the $\frac{1}{4}$ corrections are quite large [3]). Because the black hole geometry is obtained by a gauging procedure, the geodesics can be lifted to classical solutions of the ungauged theory, viz. the $SL_2(\mathbb{R})$ sigma model - upto gauge ambiguities.

We shall discuss the various geodesics of the black hole geometry (timelike geodesics and geodesic deviation for this black hole have been discussed in [23]). Uplifting these as solutions of the $SL_2(\mathbb{R})$ WZW model, we will find that the various geodesics all lift to “spectrally flowed” geodesic solutions of $SL_2(\mathbb{R})$. However, in this case, the various representations mix in a manner which is markedly different from that of the $AdS_3$ case [12].

The geodesic equations (of the leading order geometry in $l_s$) are
\[ \dot{\rho}^2 - \dot{t}^2 \tanh^2 \rho = \epsilon \quad \dot{t} \tanh^2 \rho = E \quad (4.1) \]
\[ \epsilon = 0, \mp \] accordingly as we are considering null, time-like or space-like geodesics. This form of the equations are relevant to the asymptotically flat regions I and II of the preceding section. We shall discuss the other regions of the black hole geometry in order to determine multiplicities of the current algebra states in the string theory spectrum. For simplicity, we will focus on one example in each case now (we will have occasion to examine the action of symmetries on the geodesics later).

We will proceed as follows - we first determine the geodesic solutions, and then construct the quantities 

\[ u = -e^{-t(\tau)} \sinh \rho(\tau) \quad \text{and} \quad v = e^{t(\tau)} \sinh \rho(\tau) \] (this is appropriate for Region I of the geometry).

We can then form an \( SL_2(\mathbb{R}) \) matrix as 

\[ \begin{pmatrix} a & u \\ -v & b \end{pmatrix} \] where \( a, b = \sqrt{1 - uv} \). The non-uniqueness of this procedure lies in the fact that only the product \( ab = 1 - uv \) is determined. This ambiguity is of course a gauge artefact - imposing the gauge fixing condition 3 will result in a unique lift.

We shall choose \( a, b \) in such a manner that it is easy to factorize the resultant \( SL_2(\mathbb{R}) \) matrix into a product \( g = g_+(\sigma^+) g_-(\sigma^-) \), and thus manifestly a solution of the \( SL_2(\mathbb{R}) \) CFT. It may happen that there are solutions of \( SL_2(\mathbb{R}) \) theory which are gauge inequivalent, but which nevertheless give rise to the same solution of the gauged model. In that case, we must treat the various solutions separately.

4.1 Null geodesics

The null geodesics of this geometry are the same as in flat two dimensional space (because the 2-D black hole is conformally flat). For instance, one family of null geodesic solutions are

\[ u = -1, \quad v = e^{2E(\tau - \tau_0)} - 1, \] (4.2)

where \( \tau \) is an affine parameter. For now, we set the integration constant \( \tau_0 = 0 \), but the general cases will be discussed in Section 6. Some null geodesics are shown as blue straight lines in Fig: 3.
This solution can be written as an $SL_2(\mathbb{R})$ matrix
\[
g = \begin{pmatrix}
e^{E(\tau-\sigma)} & -1 \\
-e^{2E\tau} & e^{E(\tau+\sigma)}\end{pmatrix},
\]
(4.3)
where, we have used the (gauge) ambiguity of the lifting procedure to introduce sigma dependence in the $(1, 1)$ and the $(2, 2)$ entries. The above matrix is not periodic in $SL_2(\mathbb{R})$ - however, it does represent a closed string in the coset theory because it is periodic up to a constant gauge transformation (or more trivially, the coset solution has no $\sigma$-dependence).

This matrix $g$ can be factorized into a product of $SL_2(\mathbb{R})$ matrices
\[
g = e^{-\frac{E+}{2}\sigma_3} g_+(\sigma^+) g_-(\sigma^-) e^{-\frac{E-}{2}\sigma_3},
\]
(4.4)
where
\[g_+(\sigma^+) = \begin{pmatrix}e^{\frac{\beta}{2}\sigma^+} & 0 \\
-e^{\frac{\beta}{2}\sigma^+} & e^{\frac{\beta}{2}\sigma^+}\end{pmatrix}, \quad g_-(\sigma^-) = \begin{pmatrix}e^{\frac{\beta}{2}\sigma^-} & -e^{\frac{\beta}{2}\sigma^-} \\
e^{\frac{\beta}{2}\sigma^-} & 0\end{pmatrix}.
\]

For the null geodesics under consideration, the parameter $\beta$ should be set equal to $E$.

Since $g$ can be written as a product of a purely right-moving and a purely left moving matrix, it solves the classical equations of motion of the $SL_2(\mathbb{R})$ WZNW model as well\(^1\).

The factorization above into four factors might seem somewhat arbitrary since the product of the first two is still purely right-moving and the product of the last two is purely left-moving. However, note that the $SL_2(\mathbb{R})$ matrix $\tilde{g} = g_+ g_-$ is also a solution of the WZNW model and furthermore
\[
\tilde{g} = \begin{pmatrix}1 & 0 \\
-1 & 1\end{pmatrix} \begin{pmatrix}e^{\beta \tau} & 0 \\
0 & e^{-\beta \tau}\end{pmatrix} \begin{pmatrix}1 & -1 \\
1 & 0\end{pmatrix}
\]
(4.5)
is a function of $\tau$ alone (whereas the product of all four is not).

Therefore the (null) geodesic is obtained in a two step procedure. First consider the $SL_2(\mathbb{R})$ solution $\tilde{g}$ above. This represents a point-like particle trajectory in $SL_2(\mathbb{R})$. Since the ‘proper time' $(\frac{d\tau}{d\sigma})^2 > 0$ (in $SL_2(\mathbb{R})$) (our sign conventions are presented in the appendix), $\tilde{g}$ represents a space-like geodesic in $SL_2(\mathbb{R})$. We then construct a new matrix
\[
g = \exp\left(\frac{w}{2} \sigma_3 \sigma_+\right) \tilde{g} \exp\left(-\frac{w}{2} \sigma_3 \sigma_-\right),
\]
(4.6)
where $\sigma_\pm = \tau \pm \sigma$. This operation leads to a new solution of the $SL_2(\mathbb{R})$ conformal field theory. This translation parallels the spectral flow operation in [12] - the difference being that we perform the flow along the noncompact direction of $SL_2(\mathbb{R})$. However, this is the same as the spectral flow relevant for the BTZ black hole [17].

Since we are gauging the axial symmetry, the sigma dependence coming from this spectral flow operation may be gauged away. The off diagonal entries of the resultant $SL_2(\mathbb{R})$ matrix are invariant under the gauging (3.1) that leads to the black hole. Hence,

\(^1\)A general solution to the equations of motion of a WZNW model is a product of a matrix whose entries are purely right moving with another matrix whose entries are purely left-moving.
they can be projected to the black hole geometry and interpreted as string configurations on the black hole. In this manner, we obtain the full solution after “spectral flow”. To obtain, the particular null geodesic of the previous section, we must interpret \( w \) above as \(-E\).

Note that since \( \tilde{g} \) is a pointlike trajectory in \( SL_2(\mathbb{R}) \), it might naturally be associated with the primaries of the \( SL_2(\mathbb{R}) \) conformal theory. This is because the vertex operators that correspond to point-like solutions can be regarded as eigenfunctions of the Laplacian operator on \( SL_2(\mathbb{R}) \). The coset states are then obtained by a projection condition.

The first thing to note is that the point-like \( SL_2(\mathbb{R}) \) solution above cannot be transformed by constant matrices into the null geodesic of the black hole. As a consequence, the state corresponding to the null geodesic of the black hole ("tachyon") is not directly the primary of the coset theory (associated with the spacelike geodesic in \( SL_2(\mathbb{R}) \)).

The second thing to note is that spectral flow in the hyperbolic direction seems to produce a string in \( SL_2(\mathbb{R}) \) that extends all along x-direction (parallel to the boundary). Thus it would correspond to a large excitation in the \( AdS_3 \) string theory. However, the current algebra charges are all finite for this string.

The third thing to note is that the pointlike geodesic in \( SL_2(\mathbb{R}) \) was written as the product \( g_+g_- \) where the \( g_\pm \) were not periodic in the \( \sigma \)-coordinate. This is not a serious shortcoming since the pointlike trajectory of \( SL_2(\mathbb{R}) \) maybe written using other forms for \( g_\pm \) - this will not alter the analysis, the key point being the coset solution is being obtained after spectral flow and using the spacelike geodesic in \( SL_2(\mathbb{R}) \).

For this solution, we can determine the left and right moving charges \( \tilde{J}_\pm^{(2)} \) of the current algebra, and also the world sheet stress tensors \( \tilde{T}_\pm \) (for conventions, see Appendix A). And similar quantities can be calculated for the matrix \( g \) (denoted without a tilde). These are as follows.

\[
\begin{align*}
\tilde{J}_\pm^{(2)} &= -\frac{k\beta}{2} ; \quad \tilde{T}_\pm = \frac{k}{4} \beta^2, \\
J_\pm^{(2)} &= \frac{k}{2} (w - \beta) ; \quad T_\pm = \frac{k}{4} (w - \beta)^2,
\end{align*}
\]

which obey the equations

\[
\begin{align*}
J_\pm^{(2)} &= \tilde{J}_\pm^{(2)} + kw, \\
T_\pm &= \tilde{T}_\pm + wJ_\pm^{(2)} + kw^2.
\end{align*}
\]  

with \( w \) being the spectral flow parameter. In contrast with the spectral flow operation in [12], since this operation is along the \( J^{(2)} \) direction, we have different signs in the above equations.

We can also determine the \( J^{(0)} \) quantum numbers for the above solution (which is related to the energy in global AdS coordinates - the \( J^{(0)} \) direction is the compact elliptic direction in \( SL_2(\mathbb{R}) \))

\[
\tilde{J}_\pm^{(0)} = \pm \frac{\beta}{2} \quad J_\pm^{(0)} = \pm \frac{\beta}{2} e^{w\sigma_\pm}.
\]

It is worth noting that before spectral flow, the matrix \( \tilde{g} \) is an ‘eigenstate’ of \( J^{(0)} \), while after spectral flow this is no longer the case.
4.2 The timelike geodesics

From the geodesic equations, it immediately follows that we have three families of massive geodesics depending on $E^2 - m^2$. They also fall into distinct representations of $SL_2(\mathbb{R})$ when we uplift them to classical solutions of the un-gauged theory.

**Case 1: Geodesics with $E^2 > m^2$**

These geodesic trajectories reach $J^\pm$ at late (early) times. They may thus be thought of as either particles falling into the black hole or particles that scatter out to asymptotic infinity. An example is

$$u = -\frac{e^{-E\tau}}{\sinh \phi} \sinh(\beta \tau + \phi), \quad v = \frac{e^{E\tau}}{\sinh \phi} \sinh(\beta \tau - \phi),$$

where $\beta = \sqrt{E^2 - m^2}$, $\tanh^2 \phi = \frac{m^2}{E^2}$. These geodesics, shown as a pair of black dashed lines in Fig: 4, satisfy $u(-\tau) = v(\tau)$.

This solution too can be lifted up to $SL_2(\mathbb{R})$ by choosing matrices $\tilde{g}_\pm$

$$\sqrt{\sinh 3\phi \sinh \phi} \quad \tilde{g}_+(\sigma^+) = \left( \begin{array}{cc} \sin(\frac{1}{2} \beta \sigma^+ - \phi) & -\sinh(\frac{1}{2} \beta \sigma^+ + 2\phi) \\ -\sinh(\frac{1}{2} \beta \sigma^+ - 2\phi) & \sinh(\frac{1}{2} \beta \sigma^+ + \phi) \end{array} \right),$$

and

$$\sqrt{\sinh 3\phi \sinh \phi} \quad \tilde{g}_-(\sigma^-) = \left( \begin{array}{cc} \sinh(-\frac{1}{2} \beta \sigma^- + 2\phi) & \sin(\frac{1}{2} \beta \sigma^- - \phi) \\ -\sin(\frac{1}{2} \beta \sigma^- + \phi) & \sinh(-\frac{1}{2} \beta \sigma^- - 2\phi) \end{array} \right).$$

The full solution is obtained after spectral flow on $\tilde{g}$ as before 4.6. The product of the two matrices gives

$$\tilde{g} = \frac{1}{\sinh \phi} \left( \begin{array}{cc} \sin \beta \tau & -\sin(\beta \tau + \phi) \\ -\sinh(\beta \tau - \phi) & \sinh \beta \tau \end{array} \right).$$
It is easily shown that \( \tilde{g} \) can be rewritten as
\[
\tilde{g} = U \begin{pmatrix} e^{\beta \tau} & 0 \\ 0 & e^{-\beta \tau} \end{pmatrix} V,
\]
where \( U, V \) maybe chosen to be
\[
U = \frac{1}{\sqrt{2 \sinh \phi}} \begin{pmatrix} e^{\frac{\phi}{2}} & -e^{-\frac{\phi}{2}} \\ -e^{-\frac{\phi}{2}} & e^{\frac{\phi}{2}} \end{pmatrix}
\]
and \( V = \frac{1}{\sqrt{2 \sinh \phi}} \begin{pmatrix} e^{-\frac{\phi}{2}} & -e^{\frac{\phi}{2}} \\ e^{\frac{\phi}{2}} & -e^{-\frac{\phi}{2}} \end{pmatrix} \).

Note that both the null and massive geodesics above are obtained from the spacelike geodesics of \( SL_2(\mathbb{R}) \) (before spectral flow). Therefore, one can transform the corresponding \( SL_2(\mathbb{R}) \) matrices into each other by a constant transformation in \( SL_2(\mathbb{R}) \).

For this solution, we can calculate the Kac-Moody charges:
\[
j^{(2)}_{\pm} = \frac{k \beta \coth \phi}{2} ; \quad \hat{T}^{\pm} = \frac{k}{4} \beta^2,
\]
\[
J^{(2)}_{\pm} = \frac{k}{2} \left( w - \beta \cot \phi \right) ; \quad T^{\pm} = \frac{k}{4} \left( \beta^2 + w^2 - 2w\beta \coth \phi \right).
\]

It may be observed that \( \hat{T} > 0 \) and \( J^{(2)}_{\pm} > -k\beta \) which together may be interpreted as defining the principal continuous representation of \( SL_2(\mathbb{R}) \) (in our sign conventions) — because \( \hat{T} > 0 \) only for this representation and \( J^{(2)} \) is bounded on one side.

Using the above solution, we find
\[
cosh^2 \rho = \frac{\sinh^2 \beta \tau}{\sinh^2 \phi} \sim \frac{e^{2|\beta|\tau}}{\sinh^2 \phi}.
\]
From the expressions above, we can see that \( \beta \), which is related to the quadratic Casimir \( T \) of \( SL_2(\mathbb{R}) \), determines the momentum along the \( \rho \) direction. Also,
\[
\exp(2t) = \exp(2w\tau) \sqrt{\frac{\sinh(\beta \tau - \phi)}{\sinh(\beta \tau + \phi)}} \sim \exp(2w\tau - 2\phi),
\]
which shows, rather surprisingly, that the spectral flow parameter \( w \) determines the energy of the state in the bulk. Note that if we had interpreted the parameter \( \beta \) as the energy of the state, and if we use the unitarity bound on the \( SL_2(\mathbb{R}) \) current algebra representations (of the continuous series), we would have concluded that the energy is bounded above. This is now no longer an issue since \( w \) is allowed to be any real number. Henceforth, we will use \( E \) to label the spectral flow parameter instead of \( w \). A related observation has already appeared in the literature [34].

A last point for consideration is that the solution, Eq. 4.10, for the full range of the affine parameter \( \tau \), actually represents a pair (see Fig: 4) of geodesics, one each in regions I and II. If the one in region I (solid black curve in the fourth quadrant) is interpreted as emanating from the singularity and going to \( J^+ \) (depending on the sign of \( E \)), then the other (solid black curve in the second quadrant) in region II falls into the singularity (and vice versa). Both members of a pair intersect at the singularity - which represents ‘the end of time’. There are, in fact, no time-like geodesics near the singularity on the other side as we will see later.
Under $C$ operation, which is time reversal, the solid curve in Fig: 4 with parameters $(E, \beta, J^2)$ is mapped to the dashed curve with parameters $(-E, \beta, -J^2)$. The same thing happens under the $R$ operation.

In the $SL_2(\mathbb{R})$ quantum theory, we therefore interpret the state $|E, \beta, J^2\rangle$ as representing this pair of particles (and not a single particle in isolation in region I).

Since both timelike and null geodesics are obtained from the same parent $SL_2(\mathbb{R})$ states by ‘spectral flow’, an outgoing null geodesic in region I must also be similarly ‘paired’ with another ingoing null geodesic with the same quantum numbers.

**Case 2: Geodesics with $E^2 < m^2$**

When the energy is lower than the mass, the particle geodesics become qualitatively different from the previous case. These solutions can be interpreted as analytical continuations in $\beta$, and therefore $\phi$, of the timelike geodesics of the previous section.

$$u = -\frac{e^{-Er}}{\sin \phi} \sin(\beta \tau + \phi), \quad v = \frac{e^{Er}}{\sin \phi} \sin(\beta \tau - \phi),$$  \hspace{1cm} (4.14)

where $\beta^2 + E^2 = m^2$, and $\tan^2 \phi = \frac{\beta^2}{E^2}$. Since, $1 > uv > -\cot^2 \phi$, these geodesics never reach asymptotic infinity and thus may be interpreted as bound states.

The $SL_2(\mathbb{R})$ matrices in this case can also be obtained by analytic continuation

$$\sqrt{\sin 3\phi \sin \phi} \tilde{g}_+(\sigma^+) = \begin{pmatrix} \sin\left(\frac{1}{2}\beta\sigma^+ - \phi\right) & -\sin\left(\frac{1}{2}\beta\sigma^+ + 2\phi\right) \\ -\sin\left(\frac{1}{2}\beta\sigma^+ - 2\phi\right) & \sin\left(\frac{1}{2}\beta\sigma^+ + \phi\right) \end{pmatrix},$$

and

$$\sqrt{\sin 3\phi \sin \phi} \tilde{g}_-(\sigma^-) = \begin{pmatrix} \sin\left(-\frac{1}{2}\beta\sigma^- + 2\phi\right) & \sin\left(\frac{1}{2}\beta\sigma^- - \phi\right) \\ -\sin\left(\frac{1}{2}\beta\sigma^- + \phi\right) & \sin\left(\frac{1}{2}\beta\sigma^- + 2\phi\right) \end{pmatrix},$$

with $g = \exp(-\frac{E}{2} \sigma_3 \sigma_+) \tilde{g} \exp(\frac{E}{2} \sigma_3 \sigma_-)$
Again, the matrix $\tilde{g} = \tilde{g}_+ \tilde{g}_-$ can be written as a product $U \begin{pmatrix} \cos \beta \tau & \sin \beta \tau \\ -\sin \beta \tau & \cos \beta \tau \end{pmatrix} V$, with the following choices for $U, V \in SL(2, \mathbb{R})$

$$
U = \frac{1}{\sqrt{\sin \phi}} \begin{pmatrix} \sin \phi/2 - \cos \phi/2 \\ \sin \phi/2 \cos \phi/2 \end{pmatrix}, \quad V = \frac{1}{\sqrt{\sin \phi}} \begin{pmatrix} \cos \phi/2 - \cos \phi/2 \\ \sin \phi/2 \sin \phi/2 \end{pmatrix}.
$$

Hence, these are obtained by spectrally flowing time-like geodesics of $SL_2(\mathbb{R})$ as can be verified by computing the ‘proper time’ $(\frac{d\tau}{\ddot{\tau}})^2 < 0$ in $SL_2(\mathbb{R})$.

For these solutions, we can calculate the Kac-Moody charges:

$$
J^{(2)}_{\pm} = -\frac{k\beta \cot \phi}{2}; \quad \tilde{T}_{\pm} = -\frac{k}{4} \beta^2,
$$

$$
J^{(2)}_{\pm} = \frac{k}{2} (E - \beta \cot \phi); \quad T_{\pm} = \frac{k}{4} (-\beta^2 + E^2 - 2E\beta \cot \phi).
$$

In this case, $\tilde{T}$ is bounded above, $\tilde{T} < 0$, which means that these geodesics fall into the discrete series representations of $SL_2(\mathbb{R})$ (since for these representations, the quadratic Casimir is bounded above).

Thus, we see that as we vary the energy of the massive string in the black hole geometry, we pass between time-like and space-like geodesics in $SL_2(\mathbb{R})$. However, all these map onto time-like geodesics of the black hole geometry. In [12], it was observed that as one increases the energy of strings in $AdS_3$, the parent configurations passed from time-like geodesics to space-like geodesics (i.e., short strings to long strings). The curious thing is that the energy in global $AdS_3$ is related to the compact direction in $SL_2(\mathbb{R})$ while the energy of the black hole geometry is related to a hyperbolic direction in $SL_2(\mathbb{R})$. Yet, a similar thing happens in this case.

In [12], it was argued that the spacelike geodesics of $AdS_3$ give rise to the (spectrally flowed) principal continuous representations of the $SL_2(\mathbb{R})$ Kac-Moody algebra. These long strings were scattered out to infinity. In our case also, we observe that the strings obtained from the spacelike geodesics of $SL_2(\mathbb{R})$ can reach the future/past infinities - and are thus visible in the asymptotic region of the black hole. The discrete representations on the other hand gave rise to states which are localized in the interior of $AdS_3$. This is true of the corresponding geodesics of the black hole geometry as well.

We can also construct geodesics with $E^2 = m^2$

$$
u = -e^{-\tau}(\tau + 1)$$

$$
v = e^\tau(\tau - 1)
$$

In this case, $\cosh^2 \rho = \tau^2$, and

$$
g = \begin{pmatrix} e^{\frac{1}{2} \sigma^+} & 0 \\ 0 & e^{\frac{1}{2} \sigma^-} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \sigma^+ & 1 \\ -\frac{1}{2} \sigma^+ + 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ \frac{1}{2} \sigma^- - \frac{1}{2} \sigma^+ & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2} \sigma^-} & 0 \\ 0 & e^{-\frac{1}{2} \sigma^-} \end{pmatrix}
$$

These solutions above are unusual in the sense that the solution do not seem to depend on any parameters at all.
4.3 Spacelike geodesics

In this case, the equations of motion have the following solution,

\[ u = \frac{e^{-E\tau}}{\cosh \phi} \sinh(\beta \tau + \phi), \quad v = \frac{e^{E\tau}}{\cosh \phi} \sinh(\beta \tau - \phi), \tag{4.18} \]

which bear a remarkable resemblance to the time-like geodesics the difference now being, \( \tanh^2 \phi = \frac{E^2}{\beta^2} \). These satisfy \( uv < \tanh^2 \phi < 1 \), and hence do not reach the singularity, but extend across both regions I and II of the black hole geometry, similar to the timelike geodesics (see the magenta curve in Fig 4).

For this solution, we can calculate the Kac-Moody charges:

\[ J_\pm^{(2)} = -\frac{k \beta \tanh \phi}{2}; \quad \tilde{T}_\pm = \frac{k}{4} \beta^2, \tag{4.19} \]

\[ J_\pm^{(2)} = \frac{k}{2} (E - \beta \tanh \phi) = 0; \quad T_\pm = \frac{k}{4} (\beta^2 + E^2 - 2E\beta \tanh \phi) \]

In this case, we again get \( \tilde{T}_\pm > 0 \), as expected from the continuous series. But since \(|J_\pm^{(2)}| < |\beta|\), these form a double sided representation of \( SL_2(\mathbb{R}) \).

It is also clear from the above that the spacelike geodesics of the black hole geometry are obtained from the spacelike geodesics of \( SL_2(\mathbb{R}) \) in the same manner as the null and the massive geodesics.

5 Regions V and VI: ‘Behind’ the singularity

In view of the observation that the physical geodesics constructed in the previous sections do not extend beyond \( uv = 1 \) into the region “behind” the singularity, it is of interest to examine the nature of the geodesics in this region.

Timelike geodesics

The ones with energy \( E^2 > m^2 \) are given by

\[ u = \frac{1}{\sinh \phi} e^{-E\tau} \cosh(\beta \tau - \phi), \quad v = \frac{1}{\sinh \phi} e^{E\tau} \cosh(\beta \tau + \phi) \tag{5.1} \]

where \( \tanh^2 \phi = \frac{E^2}{\beta^2} \). Using the above, we see that these reach infinity - i.e., represent scattering states as before. But, more interestingly, we get \( uv = \frac{\sinh^2 \beta \sigma^+ + \cosh^2 \phi}{\sinh^2 \phi} > 1 \), i.e, the geodesics never reach the singularity at \( uv = 1 \). This supports the idea that the geodesics in this region are independent of those in the regions “in front” of the singularity (as suggested by the spacelike geodesics of the previous section).

Again, we can find matrices

\[ \tilde{g}_V^Y(\sigma^+) = \begin{pmatrix} \cosh(\beta \sigma^+ / 2 - 2\phi) & \cosh(\beta \sigma^+ / 2 + \phi) \\ \cosh(\beta \sigma^+ / 2 - \phi) & \cosh(\beta \sigma^+ / 2 + 2\phi) \end{pmatrix}, \tag{5.2} \]

\[ \tilde{g}_V^Y(\sigma^-) = \begin{pmatrix} \sinh(\beta \sigma^- / 2 - \phi) & -\sinh(\beta \sigma^- / 2 - 2\phi) \\ -\sinh(\beta \sigma^- / 2 + 2\phi) & \cosh(\beta \sigma^- / 2 + \phi) \end{pmatrix} \]
which give rise to the geodesic above after spectral flow. The various quantum numbers for these timelike geodesics turn out to be

\[ \tilde{T}^+_+ = \tilde{T}^-_+ = \frac{k}{4} \beta^2; \quad \tilde{J}^+_+ = \frac{k}{2} \beta \coth \phi, \]

\[ T^+_+ = T^-_- = \frac{k}{4} (\beta^2 + E^2 + 2 \beta E \coth \phi); \quad J^+_+ = J^-_- = \frac{k}{2} (E + \beta \coth \phi), \]

implying that these also belong to the principal continuous series. And as before, these are obtained by transforming spacelike geodesics in $SL_2(\mathbb{R})$ with

\[ U = \begin{pmatrix} e^{-\phi} & -e^{\phi} \\ e^{-\phi} & -e^{\phi} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} -e^{\phi} & e^{-\phi} \\ e^{\phi} & -e^{-\phi} \end{pmatrix}. \]

**Null geodesics**

The null geodesics in the region $uv \geq 1$ are

\[ u = -k; \quad v = -(ke^{+2E\tau} + \frac{1}{k}) \]

\[ v = -k; \quad u = -(\frac{1}{k} + ke^{-2E\tau}) \]

and are distinguished from those in region I by the relative plus sign between the $e^{2E\tau}$ and the $\frac{1}{k}$ terms. Consider a particular null geodesic

\[ v = -(1 + e^{2\beta\tau}), \quad u = -1, \]

where the constants are chosen such that $\tau \to -\infty$ corresponds to the singularity $uv = 1$. Observe that the derivatives $\frac{d(uv)}{d\tau}$ and $\frac{d(u/v)}{d\tau}$ both vanish as we tend to the singularity. On the other side of the singularity, the null geodesic in region II ($uv < 1$) with the same quantum numbers given by

\[ v = (-1 + e^{-2\beta\tau}), \quad u = -1, \]

also has these derivatives vanishing as we approach the singularity (in the future). Thus, these two geodesics cannot be argued to be the continuation of each other through the singularity.

**Bound states**

It is easy to show that there are no solutions with $E^2 < m^2$ in this region of the geometry. The localised geodesics belonging to the discrete series live only in the regions in “front” of the singularity.

6 Building up representations

To summarise the results of the previous sections, observations of the timelike geodesics in regions V and VI of the black hole geometry suggest that these are independent states from
the corresponding geodesics in regions I and II. The spacelike geodesics of these regions also suggest a similar conclusion.

The timelike geodesics of region I continue across to region II - but are discontinuous at the singularity (see the dashed curves in Fig. 4). However, we observed that like the spacelike and timelike geodesics with \( E^2 > m^2 \), the null geodesics are also obtained from spacelike geodesics in \( SL_2(\mathbb{R}) \). Thus, we conclude that null geodesics of region I must also be paired with a corresponding null geodesic of region II. Also, as argued in the previous section, this ‘pair’ is independent of the null geodesic in regions V and VI. A similar ‘pairing’ must occur for the spacelike geodesics as well.

Thus, in the quantum theory we expect two copies of states with parameters \(|E, \beta, J^2\rangle\) - one representing the states in regions I and II, and the other representing the pair in regions V and VI. On the other hand, the absence of the timelike geodesics with \( E^2 < m^2 \) in regions V and VI implies that a single copy of states for these geodesics suffices. These observations are borne out by the representation theory of \( SL_2(\mathbb{R}) \) [28] as well.

By using the states of the quantum theory, we can construct wavepackets which “follow” the geodesic (coherent states). In this construction, the classical initial conditions will appear as parameters of the wavefunctions. Symmetry transformations acting on the classical solutions can be interpreted as changing the initial conditions. This action must therefore lift to an action on the wavefunctions as well. Hence, if there is some set of initial conditions which is closed under the action of the (symmetry) group - then this set will form a representation of the (symmetry) group upon quantization. We can therefore try to understand how the representation is filled out by examining the classical geodesics.

We first note that the trace of a matrix in \( SL_2(\mathbb{R}) \) is a conjugacy class invariant. Therefore, regions V and VI which correspond to \( uv > 0 \) form the conjugacy classes \( |Tr(g)| = |a + b| < 2 \). Hence, under the action of the vectorial symmetry (which is conjugation), the trajectory now viewed as an \( SL_2(\mathbb{R}) \) matrix cannot be moved into the other regions of the black hole geometry. If we regard the vectorial action as changing the initial conditions of the geodesic, then it follows that all points on the geodesics in region V and VI form a closed set of initial conditions (of the vectorial action).

Writing the classical solutions as \( SL_2(\mathbb{R}) \) matrices, we can easily see that only the following \( SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \) transformations commute with the spectral flow,\[ g \rightarrow e^{\sigma_3 \lambda} ge^{-\sigma_3 \lambda} , \]
\[ g \rightarrow -g , \]
\[ g \rightarrow i\sigma_2 g i\sigma_2 , \]
and compositions of these transformations. These, when applied to a geodesic produce another classical geodesic of the black hole. The first shifts \((u, v) \rightarrow (u\lambda^2, v)\) - which preserves the hyperbola \( uv = \text{constant} \). The second changes the sign of \( u, v \). Both produce new classical solutions with different initial conditions while keeping the quantum numbers \( E, \beta, J^{(2)} \) fixed. The third transformation interchanges \((u, v) \rightarrow (-v, -u)\) and results in flipping the signs of \( E, \beta \). Thus, it maps the outgoing (from the singularity) solution in
region I \((u > 0, v < 0)\) to different infalling trajectory in region II \((u < 0, v > 0)\) and vice versa.

The most general solution of the geodesic equations will have four integration constants. Two of those can be taken to be \(E\) and \(\beta\). The third parameter is the constant \(\lambda\) in the first of the transformations above - this shifts the origin of Schwarzschild time. The last constant is the parameter \(\tau_0\) which shifts the affine parameter \(\tau\) of the geodesics. Thus, once we have considered the action of the above transformations, we have accounted for all geodesics.

It should be noted that both \(\beta > 0\) and \(\beta < 0\) give rise to the same spacelike or timelike \((E^2 > m^2)\) geodesics (changing sign of \(\beta\) also changes the sign of \(\phi\)) provided we consider the entire trajectory. For null geodesics, this means we consider the pair together. In particular, the signs of both \(\frac{dt}{d\tau}\) and \(\frac{d\rho}{d\tau}\) are not affected by \(\beta \rightarrow -\beta\). In view of this, it is necessary to retain only \(\beta > 0\) states in the spectrum of the theory (or \(\beta < 0\), of course) to account for all the time like geodesics of the black hole geometry.

We can put together the above observations thus; for each set of quantum numbers, we have two states representing the pairs in regions I and II and V and VI respectively,

\[
C_E^\beta = \{ \begin{pmatrix} +E, \beta, -\lambda \\ -E, \beta, +\lambda \end{pmatrix}, \quad E, \lambda \in \mathbb{R}, \quad \beta > 0 \}. \tag{6.2}
\]

In this case, the \(J^{(2)}\) operator will act as \(\lambda \sigma_3\) where \(\lambda\) is the \(J^{(2)}\) eigenvalue. This is the doubled structure of the continuous series representations of \(so(2,1)\) argued for in [28]. In that work, it was argued that for the global time \(SL_2(\mathbb{R})\) translations to be properly represented, the generator \(J^a\) should be represented \(2 \times 2\) matrices along with being differential operators.

The above representation of the state is consistent with the observation that if one part of the paired timelike or null geodesics are infalling in region I (V), then the other is outgoing in region II (VI). We also note that the sign of \(E\) is correlated with that of \(J^{(2)}\) via spectral flow.

Under time reversal the ket \(|E, \beta, J^{(2)}\rangle\), is mapped to \(-E, \beta, -J^{(2)}\rangle\) and thus the above doubled state is mapped to itself (upto a phase)\(^2\). Time reversal transformation generated by \(i\sigma_2\) is equivalent to translation by \(\pi\) along the (compact) global time direction in \(SL_2(\mathbb{R})\) (\(AdS_3\)) which is generated by \(J^{(0)}\) (this is easily seen using global \(AdS_3\) charts as given in, say, [12]). Thus, \(J^{(0)}\) operator of \(SL_2(\mathbb{R})\) acts as \(J^{(0)} \otimes I\). This structure of the \(SL_2(\mathbb{R})\) (or current algebra) operators is also consistent with the assignment in [28].

All in all, we conclude that geodesics of the extended geometry fall into three representations of \(SL_2(\mathbb{R})\): the discrete series \(D_E^\beta\) and the continuous series \(C_E^\beta\) with \(\beta > 0\). In each of these representations, the \(J^{(2)}\) eigenvalue occurs twice and is understood as labelling the states in regions I,II and V,VI respectively. The \(D_E^\beta\) on the other hand does not have such a doubling, and represents the localised geodesics, which are not present in regions V and VI.

\(^2\)Time reversal should be represented as an anti-unitary operator, we consider only the unitary part of such an operator.
7 Vector-Axial duality and new states

It is well known that gauging either the axial action (which is what we have been considering until now) or the vectorial action of the diagonal $R$-subgroup in $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, one obtains the same (extended) target space. This is the analog of the spacetime operation $R \rightarrow \frac{1}{R}$ in the usual T-duality of string theory.

Under the vectorial gauging action, the $a, b$ entries of the $SL_2(\mathbb{R})$ matrix are invariant while $(u, v)$ transform to $(\Lambda^2 u, \Lambda^{-2} v)$ respectively. Thus, in this case, the black hole target space is described using the $a, b$ coordinates of $SL_2(\mathbb{R})$. Since $ab = 1 - uv$, the asymptotically flat region in “front” of the horizon $uv < 0$ is mapped to the region “behind” the singularity $ab = 1 - uv > 1$. Hence, we can interpret the entries $a(\tau, \sigma)$ and $b(\tau, \sigma)$ from all the $SL_2(\mathbb{R})$ matrices \[
\begin{pmatrix}
a & u \\
-v & b
\end{pmatrix}
\] representing the worldsheet solutions of the previous sections, as describing new sigma model solutions in the dual region of the black hole spacetime (in the $u, v$ coordinate chart). From the original point of view (i.e., axial gauging), this is equivalent to a right (or left) multiplication of the $SL_2(\mathbb{R})$ matrix worldsheet by \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\] and hence gives another solution of the sigma model equations. Clearly, this operation results in $J^{(2)}_\pm \rightarrow \pm J^{(2)}_\pm$.

We may alternately, not only perform the target space transformation above, but supplement it with a worldsheet $\tau \rightarrow \sigma$ interchange. This operation is analogous to flipping the sign of the right moving momentum in the usual circle T-duality considerations. If this T-duality is a symmetry of the full string theory, then this will map solutions to solutions (in the textbook example of T-duality on a circle, this sends the state $|n, w\rangle \rightarrow |w, n\rangle$) in our case, this will produce a new worldsheet solution.

For instance, in the matrix given by Eq. (4.3), representing a null geodesic in region I, right multiplication by $i\sigma_2$ gives \[
u = \exp(E(\tau - \sigma)), \quad v = \exp(E(\tau + \sigma)),\]
from which, after interchanging $\tau \rightarrow \sigma$, we get \[
u = \cosh^2 \rho = e^{2E\sigma}, \quad t = -E\tau,
\] which we could expect to be a solution in region V (provided the Virasoro conditions are met, of course). On the other hand, we start with the matrix representing a null geodesic in region V, we obtain a string worldsheet that covers all of regions I and II \[
u = \sinh^2 \rho = e^{-E\sigma} \quad t = -E\tau.
\] Motivated by the AdS/CFT correspondence, we propose that such worldsheets extending to the boundary should be interpreted as operators of the boundary theory.

Upon dualising, the timelike geodesics of region V (the black dashed curve in Fig:6) give worldsheets \[
u = -\frac{\cosh^2 \beta \sigma}{\sinh^2 \phi},
\]
which extend out to the boundary in the asymptotically flat regions I & II, but do not reach the horizon. These worldsheets, for a few instants of time, are shown as black segments in the right quadrant in Fig:6. These “folded strings” are similar to the long strings considered in [16] in that the tip scatters out from the region near the horizon. These worldsheets are naturally better interpreted as (single trace) operators of the boundary theory.

The implication of this duality is that for a given set of quantum numbers $E, \beta, J^{(2)}$ there are two solutions in each asymptotically flat region. We have the worldsheet representing the geodesic in this region, or the solution obtained by dualising the corresponding geodesic in region V which we can choose to have the opposite sign for $\beta$. Note that the solution and its ‘dual’ version do not exist simultaneously in the same asymptotically flat regions (I,II) or (V, VI).

Using the earlier assignment of quantum numbers for the geodesics, we might write these as $| + E, -\beta, \lambda \rangle$ with $\beta > 0$. This assignment of quantum numbers is natural in terms of the Seiberg-bound states and Seiberg anti-bound operators of the matrix model (which are indeed constructed by $j \rightarrow 1 - j$ in the quantum theory - $j = \frac{1}{2} - i\beta$).

7.1 Horizon Strings

We reserve the most interesting case for the last - the geodesics with $E^2 < m^2$ (belonging to the discrete series). In this case, the dual solution is

$$uv = \frac{\sin^2 \beta \sigma}{\sin^2 \phi}$$

which is always inside the horizon but extends across the singularity ($0 \leq uv \leq \csc^2 \phi$).

These worldsheets do not extend to infinity and since we might expect that $\beta$ is an integer, are doubly folded over at $\sigma = 0, 2\pi$ and thus meson-like. These string worldsheets (for a few instants of time) are shown as thin (blue) lines in Fig:7 that extend a little across the singularity (which is the red wavy hyperbola).

Thus, we might choose to regard the black hole as a condensate of “mesons” (to use AdS/CFT terminology) consistent with a holographic interpretation of the finite tempera-
ture state of the dual theory. Since these folded mesonic strings are visible only in regions V and VI, this interpretation is valid only for the boundary observer in this region(s).

Or else, the black hole can be considered as a bound state, with the localised geodesics being the internal degrees of freedom visible only to the external observer in regions I and II (in Fig: 7, these are the dashed curves in the middle).

These are complementary to each other in the sense of vector-axial duality and also possibly complementary to each other in the sense of black hole complementarity. However, only in the region between the horizon and the boundary, we have both descriptions as being simultaneously applicable.

It should be emphasized that these are complementary (in $\alpha'$) descriptions of the Hilbert space of states interior to the black hole and assumes that the vector-axial duality is a symmetry of the string theory.

8 Analysis of the spectrum

The conditions for a classical solution to be a physical string configuration is

$$T^{tot}_{++} = T^{WZW}_{++} - \frac{(J^{(2)})^2}{k} + T^{I}_{++} = 0, \quad J^{(2)}_+ = J^{(2)}_. \quad (8.1)$$

The first condition is, of course, the Virasoro condition for the total stress tensor where we have included an “internal” CFT with stress tensor $T^I$. A similar condition applies for the right moving $T_{--}$ as well - in what follows, we shall focus exclusively on the left moving currents. The full spectrum of the string theory is obtained by putting together the left and right moving states so that they satisfy the level matching condition given above.

We observer that, for any solution $g$, if $\tilde{T} - \frac{1}{k}(\tilde{J}^{(2)})^2 = 0$, then the solutions obtained by spectral flow, Eq. 4.9, along the $J^{(2)}$ direction satisfy

$$T - \frac{1}{k}(J^{(2)})^2 = \tilde{T} + E\tilde{J}^{(2)} + \frac{k}{4}E^2 - \frac{1}{k}(\tilde{J}^{(2)} + \frac{k}{2}E)^2 = 0$$

and thus give rise to a family of solutions labelled by $E$. 

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**Figure 7.** The near horizon worldsheets
The second condition on the currents $J_{\pm}^{(2)}$ in Eq. 8.1 comes from the gauging and is interpretable as a level matching condition. The spectral flow acts the same way on $J_{\pm}^{(2)}$ and hence, if this level matching condition is satisfied before spectral flow, it will be automatic afterwards.

8.1 Null Geodesics

We first point out that the matrix $\tilde{g}$ of Eq. 4.5 that represents a spacelike geodesic in $AdS_3$ itself satisfies the physical state conditions of the coset model. However, it does not represent a fixed energy trajectory of the black hole. That is to say, $\tanh^2 \rho \frac{dt}{\tau}$ is not a c-number. Yet, this configuration is an eigenstate of $J_{(2)}^{(2)}$ (that is $J_{(2)}^{(2)}$ is a c-number). This is an additional reason why $J_{(2)}^{(2)}$ must not be identified with the spacetime energy of the particle.

From the expressions given in Eq. 4.8, we can see that the Virasoro constraints of the gauged theory are satisfied for the matrix $g$ for any $E, \beta$. Thus, these are physical solutions of the black hole sigma model. These equations, not unexpectedly, resemble the dispersion relation for massless particles since it is known that there are massless particles in the spectrum of this theory. The important difference is that we have massless particles for every pair $(\beta, E)$.

Observing that $T_{++}$ of the WZW is bounded below, i.e., $T_{++} > 0$ (before “spectral flow”) allows to infer that these particles must be in the principal continuous representation of $SL_2(\mathbb{R})$ (for principal continuous series the quadratic Casimir of $SL_2(\mathbb{R})$ is bounded below by $\frac{1}{4} + \beta^2$).

In the case the string theory includes an ‘internal’ CFT - the massless states above continue to be physical, provided the internal CFT has states with $T_{++}^I = h = 0$ (since the condition is $T_{++} + T_{++}^I = 0$).

8.2 Timelike geodesics

For $E^2 > m^2$, the physical state condition is

$$\tilde{T}_{tot} = \tilde{T}^{SL_2} - \frac{1}{k} (J^2)^2 + h = -\frac{k\beta^2}{4} \text{cosech}^2 \phi + h = 0. \quad (8.2)$$

Thus, these massive geodesics can be physical in the classical string theory provided the internal CFT contributes a positive weight $h$. Similarly, for the geodesics with $E^2 < m^2$, we see that the total stress tensor is

$$\tilde{T}_{tot}^{++} = -\frac{k\beta^2 \text{sec}^2 \phi}{4} + h = 0,$$

which can be satisfied if $h > 0$. The situation corresponding to $\beta = 0$ is the case when we start with a constant matrix in $SL_2(\mathbb{R})$ and then perform a spectral flow on it

$$g = \begin{pmatrix} a e^{E\sigma} & u e^{-E\tau} \\ -v e^{E\tau} & e^{E(-\sigma)} \end{pmatrix}. \quad (8.3)$$
These solutions of the sigma model equations of motion correspond to $uv = \text{const}$. The existence of such solutions is analogous to similar solutions in [12]. As for the spacelike geodesics, these are never physical because

$$\tilde{T}^{\text{tot}} = \tilde{T}^{SL_2} - \frac{1}{k} (J^2)^2 + h = \frac{k\beta^2}{4} - \frac{k}{4} \beta^2 \tanh^2 \phi + h = \frac{k\beta^2}{4} \coth^2 \phi + h > 0,$$

as long as the “internal” CFT has $h > 0$.

### 8.3 New string solutions

The new string solutions are constructed from $SL_2(\mathbb{R})$ solutions, by right multiplying by $i\sigma_2$ and then interchanging $\tau, \sigma$. Under each operation the right moving currents $J^a$ pick up a negative sign and hence remain unchanged in the end. Thus, the analysis of the previous section can be carried over in toto and the new solutions are valid classical solutions if the old ones are.

### 8.4 Quantum analysis

The space of states of a coset conformal field theory is normally constructed by starting with representations of the parent CFT and writing the states in a basis adapted to the action of the quotienting subgroup. In this adapted basis, it is easy to impose the gauging conditions, and one can identify the states of the coset sigma model.

In the quantum theory, the physical state condition for an axially gauged coset model can be written [26, 27]

$$L_0 - L_0^{(SL_2)} = 1, \quad J_m^{(2)} = \bar{J}_m^{(2)} = 0, \forall m > 0, \quad J_0^{(2)} + \bar{J}_0^{(2)} = 0,$$

where, for $SL_2(\mathbb{R})$

$$\tilde{L}_0 = \frac{-j(j - 1)}{k - 2} + N, \quad \tilde{J}_0^{(2)} = \lambda$$

with $N$ being the Kac-Moody level. The third conditions arises from the gauging procedure. The $J_0^{(2)}$ quantum number $\lambda$ is unrelated to the Casimir (although in the Discrete series representations there are restrictions on the values of $J_0^{(2)}$ depending on $j$).

For the continuous series representations of $SL_2(\mathbb{R})$, we have $j = \frac{1}{2} + i\beta$, so that for the null and timelike geodesics with $E^2 > m^2$, we get

$$\frac{j^2 + \beta^2}{k - 2} + N + h - \frac{\lambda^2}{k} = 1$$

as the on-shell condition. If we assume that $N = 0$, that there is no “internal” CFT and set $k = \frac{9}{4}$, this reduces to the dispersion relation of the usual ‘tachyon’ of the critical ($c=26$) 2D black hole $\beta^2 = \frac{4}{9} \lambda^2$. However, we have a subtlety [28]. The states of the nonexceptional continuous series of representations, when written in terms of the eigenstates of $J_0^{(2)}$ are 'doubled'. That is to say, a single irrep of $SL_2(\mathbb{R})$ requires that each ‘momentum state’ $J_0^{(2)}|\lambda\rangle = \lambda|\lambda\rangle$ appear twice - so that a basis for this Hilbert space takes the form of a column vector

$$C_\beta = \text{Span}\{ \begin{pmatrix} |\beta, +\lambda\rangle \\ |\beta, -\lambda\rangle \end{pmatrix} | \lambda \in \mathbb{R}, \beta > 0 \}$$
On these vectors, the Hermitean $J^{(2)}_0$ operator acts as $J^{(2)}_0 = \lambda \otimes \sigma_3$. The other generators are also now written as $2 \times 2$ matrices - so that the $SL_2(\mathbb{R})$ commutation relations are satisfied.

This fits nicely with the conclusions drawn in section 6 from studying the geodesics which led to the suggestion that timelike and null geodesics in regions I and II are paired as a single state. The second copy then refers to regions V and VI. The difference in sign can be related to the observation that if a timelike (or null) geodesic is infalling in region I, then its partner in region II is outgoing. We have also remarked that we shall choose $\beta > 0$ for region I and $\beta < 0$ for region V (and similarly for regions II and VI). This differs slightly from the suggestion made in [28] that the doubling of $J^{(2)}$ could be related to the option of choosing either sign for $b$.

For the discrete series, we know that $\frac{1}{2} < j < \frac{k+1}{2}$ and hence the mass shell condition at level zero is

$$-j(j-1) + N + h - \frac{\lambda^2}{k} = 1.$$ 

These states exist for special values of the momentum [10] as determined by this condition, but the energy of these states is given by the spectral flow parameter and can take any value. This is possible because in the hyperbolic basis the eigenvalues of $J^{(2)}$ can also be pure imaginary in a hermitean representation.

Given the above matrix form of the zero modes of the KM generators $J^{(a)}_0$, we need to determine the form of the remaining modes $J^{a}_m$ of the currents so that the commutation relations

$$[J^2_m, J^2_n] = \frac{k}{2} n \delta_{m+n,0}, \quad [J^{(2)}_n, J^{(2)}_m] = \pm i J^{\pm}_n, \quad [J^+_{m}, J^{-}_{n}] = -2i J^{(2)}_{m+n} - kn \delta_{m+n,0}, \quad (8.8)$$

of the current algebra are preserved in the continuous series representation $C_\beta$. The simplest assumption is that the $J^a_m$ are tensored with the same $2 \times 2$ matrix as the corresponding $J^a_0$. This assumption is consistent with the spectral flow operation and is clearly also consistent with the physical state conditions Eq. 8.5 of the coset sigma model [27].

Bringing in spectral flow into the discussion does not alter the above conclusions about the mass shell conditions. This is because, as remarked above, if a state satisfies the mass shell condition before spectral flow, it will satisfy it after the spectral flow as well.

9 Summary and Discussion

To summarise the results presented, we have seen that the study of the action of symmetries on the space of geodesics has proven to be highly profitable. We have been able to cast several features of the representation theory of $SL_2(\mathbb{R})$ into properties exhibited by the geodesics. The doubling of the representations of the $J^{(2)}, J^+$ algebra to form a single irreducible representation of $SL_2(\mathbb{R})$ (for the Continuous series) was motivated in [28] as a requirement that the generator of global time translations be properly represented. Here, we find that it is better motivated as the requirement that time reversal be representable.

This doubling of the $J^{(2)}, J^+$ representation is absent for the Discrete series (these are absent in regions V, VI). These are localized in the near horizon region, and are captured by
the square integrable wavefunctions making up the Discrete series. The geodesic analysis clearly leaves open the physical interpretation of the eigenvalue of $J^{(2)}$ maybe along the lines of the phase shift studied in [29].

A primary issue is to understand whether the operation of “spectral flow” along the hyperbolic direction makes sense in the CFT. While we have argued for it classically, there does not appear to be any problem at a formal level in generalizing to the quantum theory. In fact, to obtain all possible values of $E$, we must allow all values for the spectral flow parameter $E$. A separate question is whether these flowed states are necessary. However, we can expect that the gravitational backreaction of any finite energy particle in 1+1-d will lead to large deformations of the asymptotic region. Thus, to preserve asymptotics we might impose the condition that $E = 0$.

We have also exhibited several new features of the states of this sigma model. The states arising from the discrete series play the role of the winding strings of the Euclidean ‘Cigar’ theory. However, these are only visible to one (pair) of the asymptotic regions. The winding strings of the T-dual ‘Trumpet’ geometry are mapped to the localized worldsheets visible only to regions V and VI (which form the asymptotics of the Trumpet geometry). This complementary view of the ‘interior’ is clearly a stringy effect and is worth exploring further. Similarly, the states of the continuous series have two descriptions - either as scattering states or as worldsheets ending at the boundary (and thus operators) and involving a $j \rightarrow 1 - j$ flip suggesting that this is a state-operator duality. The folded worldsheets ending at the boundary could be related to the additional non-singlet degrees of freedom used by [31] to obtain the KKK model. This suggests comparing correlation functions obtained from this matrix model and that of the strings described using ideas of holography.

These observations are pertinent if we wish to regard the 2D black hole as the high temperature phase of a boundary gauge theory (the matrix model). The geodesic analysis suggests that we should regard the boundary as two points - the asymptotics of region I and II (or dually regions V and VI). It has been suggested [4] that region II of the black hole geometry is mapped to the second asymptotic of the potential in the matrix model. This also suggests that we might search for a Hawking-Page type phase transition to a low temperature (linear dilaton) phase.

A related question is the issue of conserved charges carried by the various strings. In [32], the $W_\infty$ charge carried by the 2D black hole was computed. If the horizon strings are indeed the degrees of freedom, then they should also carry some part of the same $W_\infty$ charges.

It may be facile to expect that the special features noted above will extend in a simple manner to other dimensions. However, at least the existence of strings ending on the horizon, and their dual relation to localized geodesics seems generalizable [33].

One interesting question is to construct the characters of these representations and hence forming the partition function. One could further compare with the partition function of the Euclidean black hole [11] and trace the winding modes to the Lorentzian geometry. In this regard, a significant contribution to answering this question already appears in the work of [34]. In this work, the authors construct the partition function for Lorentzian $AdS_3$ by building upon characters of the gauged $SL_2(\mathbb{R})$ theory. Using this, they have
also constructed partition functions for various marginal deformations - in particular, the Lorentzian black hole (see eqns 5.15 and 5.17). It will be of much interest to read off the spectrum of states from the partition function. We should compare the structure of the spectrum with the work of [35]. In this work, the (worldsheet) elliptic genus of the Euclidean black hole which included states from the discrete series was argued to satisfy a curious identity. This identity should be related to the observation that while the timelike geodesics with $E^2 < m^2$ are absent in the region V,VI of the black hole, one can construct “T-dual” string configurations which satisfy the physical state conditions. The computation of various correlation functions - either from the point of view of the regions I and II or from the T-dual regions V and VI is another question. As in AdS holography, the geodesics and evaluation of the action can be used to obtain a saddle point approximation to correlation functions.

One can use similar techniques to study the third gauging of $SL_2(\mathbb{R})$ (i.e., gauging the lightcone (parabolic) direction) which gives rise to Liouville theory. We can again ask whether the spectrally flowed representations of $SL_2(\mathbb{R})$ survive in the coset theory. In this context, Balog et. al., [36] have found topological sectors from a study of the Virasoro co-adjoint orbits in the case of the Liouville theory.

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10 Appendices

10.1 Conventions

In this appendix, we present the details of the gauged sigma model to make our conventions clear. The action for the $SL_2(\mathbb{R})$ CFT is given by

$$S_{WZNW} = \frac{k}{8\pi} \int d^2 \sigma \sqrt{-h} \text{Tr} \left( \partial_a g \partial^a g^{-1} \right) + k \Gamma$$

(10.1)

$$\Gamma = \frac{1}{12\pi} \int_B e^{abc} \text{Tr} \left( \partial_a g g^{-1} \partial_b g g^{-1} \partial_c g g^{-1} \right)$$

(10.3)

and the trace is calculated in the two dimensional representation of $SL(2,\mathbb{R})$ and $\alpha' = 1$.

Parametrizing $g = e^{L_\sigma^3} e^{\rho \sigma_1} e^{t R \sigma_3}$, the kinetic term gives

$$L = -\frac{k}{4\pi} \int d^2 \sigma \sqrt{-h} \left( \partial_\rho \partial^\rho \rho + \partial_\alpha t_L \partial^\alpha t_L + \partial_\alpha t_R \partial^\alpha t_R + 2 \partial_\alpha t_R \partial^\alpha t_L \cosh 2 \rho \right),$$

(10.2)

and the WZ term gives

$$\Gamma = -\frac{k}{2\pi} \int d^2 \sigma \cosh 2 \rho e^{\alpha \beta} \partial_\alpha t_L \partial_\beta t_R.$$

(10.3)
From the kinetic term above, one can read off the metric of the WZW model which turns out to be
\[ ds^2 = k(d\rho^2 + \cosh^2 \rho d\phi^2 - \sinh^2 \rho dt^2) \quad t_{R,L} = \frac{\phi \pm t}{2} \] (10.4)
By contrast the metric in global \( SL_2(\mathbb{R}) \) (or \( AdS_3 \)) co-ordinates is \( ds^2 = k(d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho d\phi^2) \).

We chose the normalization of the of the \( SL_2(\mathbb{R}) \) generators such that \( Tr(\tau^a \tau^b) = \frac{1}{2} \eta^{ab} \) and where \( \eta_{ab} = \text{diag}(-1,1,1) \). More specifically, \( \tau^0 = \frac{i\sigma^2}{2}, \tau^1 = \frac{\sigma^1}{2}, \tau^2 = \frac{\sigma^3}{2} \). Note that \( \tau^0 \) is not Hermitean. The conserved currents of the WZNW model are defined as \( J^a_{\pm} = \frac{1}{k} \eta_{ab} J^b_{\pm} \) and the components are defined by \( J^a_{\pm} = Tr(\tau^a J_{\pm}) \).

The stress tensor (defined as \( T_{ab} = \frac{4\pi}{\sqrt{-g}} \delta S}{\delta g^{ab}} \) turns out to be \( T_{++} = \frac{1}{k} \eta_{ab} J^a_{+} J^b_{+} \) in terms of the current.

### 10.2 Gauging

Adding the two terms in the action (\( \epsilon_{01} = 1 \)), and rewriting in \( \sigma_{\pm} \)-variables on the world-sheet, we get
\[ L = \frac{k}{2\pi} (\partial_+ \rho \partial_- \rho + \partial_- t_L \partial_+ t_L + \partial_+ t_R \partial_- t_R + 2 \partial_+ t_R \partial_- t_L \cosh 2 \rho) \] (10.6)
where \( \partial_\pm = \frac{1}{2} (\partial_+ \pm \partial_-) \). The black hole sigma model is obtained by gauging a simultaneous translation of \( t_{L,R} \). This is accomplished by adding the extra terms involving gauge fields \( A_{\pm} \)
\[ L_{\text{gauge}} = \frac{k}{\pi} (A_+ (\partial_- t_R + \cosh 2 \rho \partial_- t_L) + A_- (\cosh 2 \rho \partial_+ t_R + \partial_+ t_L) - A_- A_+ (1 + \cosh 2 \rho)) \]
\[ = \frac{k}{\pi} \left( \frac{1}{k} (A_+ J^2_- - A_- J^2_+) - A_- A_+ (1 + \cosh 2 \rho) \right) \]
and prescribing the following gauge transformation properties for \( A_{\pm} \)
\[ \delta t_{L,R} = \Lambda \quad \delta A_{\pm} = \partial_{\pm} \Lambda \]

Solving the e.o.m for the gauge fields, we get
\[ A_+ = \frac{(\partial_+ t_L + \cosh 2 \rho \partial_+ t_R)}{(1 + \cosh 2 \rho)}, \quad A_- = \frac{(\partial_- t_R + \cosh 2 \rho \partial_- t_L)}{(1 + \cosh 2 \rho)} \] (10.7)
Substituting into the action, and gauge fixing by setting \( t_R = -t_L = t \), it is simple to see that we get the black hole sigma model as mentioned. We can instead reparametrise the gauge fields as \( A_+ = \partial_+ \phi_R \) and \( A_- = \partial_- \phi_L \) in terms of two noncompact scalars. The reason these are noncompact has to do with the noncompactness of the symmetry being gauged. Upon shifting \( t_{R,L} \) by \( \phi_{R,L} \) respectively, the Lagrangian becomes
\[ S = S_{\text{WZNW}}(t_L + \phi_L, \rho, t_R + \phi_R) - \frac{k}{4\pi} \int d^2\sigma \sqrt{-h} \partial_\alpha X \partial^\alpha X \] (10.8)
where $X = \frac{1}{2}(\phi_L - \phi_R)$. A quick way to see this is to start with the WZW model and then shift $t_{R,L}$.

Thus the classical stress tensor of this model is

$$T_{++} = \frac{1}{k} \eta_{ab} J_a^+ J_b^+ - k(\partial_+ X)^2$$  \hspace{1cm} (10.9)

and a similar expression for the $--$ components. Upon quantizing the $1/k$ term becomes $1/(k-2)$ coming from the standard WZNW renormalization. If we incorporate this first, and then undo the field redefinitions, we obtain the sigma model metric to all orders in $\alpha' [30]$.

Although the above Lagrangian is the sum of two non-interacting theories, there is a constraint relating the two. The constraint expressed in words implies the vanishing of the total current of the $H$-subgroup that is being gauged [26]. This constraint is easily obtained. The equations of motion of the gauge fields (which are really Lagrange multipliers enforcing the constraints) can be written as

$$J^{(2)}_+ = kA_+ (1 + \cosh 2\rho) \quad J^{(2)}_- = -kA_+ (1 + \cosh 2\rho)$$  \hspace{1cm} (10.10)

Shifting $t_{L,R}$ as before and parametrising $A_{\pm} = \partial_{\pm} \phi_{L,R}$, the above equations become

$$J^{(2)}_{\pm} = k \partial_{\pm} X$$  \hspace{1cm} (10.11)

which are the BRST constraints.

The level matching condition for the noncompact scalar $X$ forces the conclusion that the left and right $J^{(2)}$ quantum numbers ought to be equal.

### 10.3 Generators

We will choose the generators of $SL_2(\mathbb{R})$ as in [12]. However, we will label them so that $T^0$ is associated with the time direction of the $SL_2(\mathbb{R})$ geometry as

$$T^0 = \frac{i}{2} \sigma^2 \quad T^2 = \frac{1}{2} \sigma^3 \quad T^1 = \frac{1}{2} \sigma^1$$  \hspace{1cm} (10.12)

With this labelling, the generator $T^0$ is anti-hermitean. However, the black hole time is related to the $T^2$ direction and hence the energy will still be real. The commutation relations are

$$[T^2, T^1] = -T^0, \quad [T^2, T^0] = -T^1, \quad [T^1, T^0] = T^2.$$  \hspace{1cm} (10.13)

In order to map these commutation relations to those in [28] (note that the generators in this latter work are hermitean), we make the following identification

$$T^2 = J_1, \quad T^1 = J_2, \quad T^0 = iJ_3$$  \hspace{1cm} (10.14)

We will define the $SL_2(\mathbb{R})$ currents as

$$J^a_+ = -k Tr(T^a \partial_+ g g^{-1}) \quad J^a_- = k Tr(T^a g^{-1} \partial_- g)$$  \hspace{1cm} (10.15)

Given these conventions, the direction being gauged is $J^{(2)}_+ + J^{(2)}_-$ and the black hole time direction is $J^{(2)}_+ - J^{(2)}_-$. The raising and lowering operators are $T^\pm = T^1 \pm T^0$, respectively. But the commutation relations are a bit unusual and given by $[T^2, T^\pm] = \pm iT^\pm$. 

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10.4 Euclidean Black hole

In this section, we briefly illustrate how the winding strings of the cigar geometry can be obtained by operation of spectral flow. In this case, we gauge the axial symmetry $g \rightarrow e^{i\sigma_2 L} g e^{i\sigma_2 L}$ of the global $SL_2(\mathbb{R})$ coordinates $g = e^{i\sigma_2 (t+\theta)} e^{\sigma_1 \rho} e^{i\sigma_2 (t-\theta)}$. The coset geometry is obtained from gauge-fixed $SL_2(\mathbb{R})$ matrices of the form

$$g = \cosh \rho + \sinh \rho \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$ 

In this case, we perform the same “spectral flow operation” as in $SL_2(\mathbb{R})$ [12]

$$g \rightarrow e^{i\omega L \sigma_2 \sigma^+} g e^{-i\omega \sigma_2 \sigma^-}.$$ 

The axial gauge symmetry can be used to eliminate part of the spectral flow - but not the whole of it. The gauge invariant content of the spectral flow is

$$g \rightarrow e^{i(m\tau+n\sigma)\sigma_2} g e^{-i(m\tau+n\sigma)\sigma_2}$$

under which $\theta \rightarrow \theta - m\tau + n\sigma$. It is clear that the winding strings come from this ‘spectrally flowed’ sector.

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