Variability of polar flows

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Abstract. It is anticipated that the physical mechanism of the variability of polar currents may be attributed to temporal and spatial variations of the cyclonic mean flow due to circularly propagating nonlinear Rossby waves. A low order analytical model for the effect of large scale and low frequency variability of a polar current is presented. Rossby wave packet is shown to be an intrinsic mode of motion trapped between the major flows, which constitute the mean current. The Rossby wave pattern is predicted to rotate with a specific angular velocity, which depends on both the magnitude and width of the time averaged flow. Spatial structure of the rotating pattern, including its zonal wave number, is defined by the specific form of the stream function-vorticity relation. Theoretically predicted patterns capture the main features of the Antarctic Circumpolar Wave (ACW). The model is capable to predict the observed sequence of warm and cold patches in the ACW along with its zonal number.

1. Introduction
As changes in climatic temperatures drastically affect polar areas, the characteristics of the polar flows draw significant research interest. Available observations suggest that polar currents, for example, the Antarctic Circumpolar Current (ACC), contain a number of narrow temperature and density fronts, each separating water masses with distinct physical properties [1]. Their mean paths are now well defined [2], where it was noted that the major currents in the ACC are the Subantarctic Front (SAF) and the Polar Front (PF). The polar currents vary both temporally and spatially, which includes both the short-timescale eddies with relatively short spatial extent and lower-frequency patterns of large spatial extent. Modeling of the short-timescale variability is beyond the scope of the present research. Here the effects related to lower-frequency variability are addressed. An example of such low-frequency variability is the Antarctic Circumpolar Wave (ACW) [3], [4]. A mechanism leading to variability of the ACC due to the Rossby wave trains trapped between the SAF and PF is proposed. The patterns in the ACW were initially identified as zonal wave number 2 (ACW2) with the period of circulation of about 8-9 years leading to 4-4.5 years as the apparent period at any location [3], [4]. The aim of the present study is to show that the Rossby wave train between the SAF and PF is an important mode of motion in the Southern Ocean, and to present a simplified mathematical model for this mode. A model for this fundamental mode, including the prediction that the zonal number for the patterns is likely to be 2 or 3. A sequence of warmer and colder patches could be trapped inside such patterns.

2. Model
Let us consider the 2D flow of a fluid, which is rotated with an angular velocity $c$ around a geographical pole in a circular channel of the width $L$. The exact expression for the Coriolis
force is expanded into Taylor series about a pole with the help of the \( \gamma \) plane approximation. This approach differs from the widely used \( \beta \) plane approximation where expansions are around some mid-latitude location. The gamma plane approximation is thus specially designed for examination of effects in polar areas [5]. The stream function is introduced in the usual way

\[
u = \psi_\xi, \quad \xi = -\psi_\phi,
\]

where \( u, v \) are the velocities in the eastward and the northward directions; \( \xi \) is the dimensionless radial distance to the geographical pole scaled with the channel width \( L \); and \( \phi \) is the longitude. An important concept of the model is to consider a circular channel formed by the SAF and PF that act as impermeable boundaries that create a waveguide for the perturbations of the mean flow. The dimensionless equation for the stream function \( \psi \) in this case is as follows (e.g. [6], [7]),

\[
\nabla^2 \psi - \frac{\varepsilon^2 \xi^2}{R_0} = F(\psi - c\xi^2), \quad \varepsilon = L/R_E << 1.
\]

An arbitrary function \( F \) should be determined using the relation between the potential vorticity and the stream function at some cross sections of the flow. We consider the length scale of the flow to be much smaller that the radius of the Earth \( R_E \). \( Ro = U/2\Omega L \) is the Rossby number, \( \Omega \) stands for the rotation rate of the Earth, \( U \) is the typical circumferential velocity. The hydrostatic and rigid lid approximations are applied. Validity of these approximations for the problem was discussed in [8].

The simplest approach is to consider a linear form for \( F \) [9]. In this research, the potential vorticity-stream function relation \( F \) has a weakly nonlinear functional form thus extending the classical approach [9].

\[
F = -\lambda(\psi - c\xi^2) + \sigma^2 f(\psi - c\xi^2), \quad \sigma << 1.
\]

Function \( f \) determines the nonlinearity in the vorticity-stream function relation and \( \lambda \) is a constant. The introduced weakly nonlinear relation was experimentally confirmed in [6]. The equation (1) reduces to a linear equation for the stream function when \( \sigma = 0 \) in (2), even if the waves amplitude is not small. Consider focusing on the solutions of (1), whose radial variations appear at much shorter scales compared to the angular one

\[
\psi = \psi(\xi, \Phi), \quad \Phi = \delta \phi, \delta << 1.
\]

Asymptotic solutions of (1, 2) are constructed using power series in the small parameter \( \delta^2 \). Solutions of (1, 2) are sought in a circular channel. The inner and outer rigid boundaries of which are located at \( \xi = R_1 \) and \( \xi = R_2 \), respectively. So the boundary conditions are

\[
\psi_\Phi(R_1, \Phi) = 0, \quad \psi_\Phi(R_2, \Phi) = 0, \quad \psi(\xi, \Phi) = \psi(\xi, \Phi + 2\pi\delta),
\]

As shown in [8], the zeroth order solution of (1, 2) satisfying the boundary conditions (4) yields

\[
\psi^{(0)} = \left(\frac{\xi^2}{2} - \frac{2}{\lambda}\frac{\varepsilon^2}{\lambda R_0} + c\right) A(\Phi) W(\xi),
\]

where the function \( A \) in (5) has to be determined from the solvability condition applied to the first approximation. The constant \( \lambda \) and function \( W(\xi) \) can be determined for the prescribed values \( R_1 \) and \( R_2 \) as shown in [8]. The first term in (5) has physical meaning of an ambient mean current, which is supposed to be the ACC in the model. The second term represents a disturbance superimposed on the mean current, which corresponds to the ACW-like pattern.
The formula (5) indicates that the mean current can be approximated as the current with a constant angular velocity. Non-dimensional expression for that velocity $c_m$ and its dimensional expression, $c_{m,dim}$, are as follows,

$$c_m = \frac{\epsilon^2}{Ro \lambda} + c, \quad c_{m,dim} = \frac{L^2}{R^2} \frac{2\Omega}{\lambda} + c_{dim}.$$  

(6)

Taking value for $L$ from observations, then calculating non-dimensional distances $R_1$ and $R_2$ from the pole to the SAF and PF leads to the determination of $\lambda$ according to the technique presented in [8], the equation (6) determines the dimensional angular velocity for the rotating Rossby wave pattern if the mean current is determined using the field data. The spatial structure of the Rossby wave pattern can be resolved using the equation (1) along with the boundary conditions (4). The boundary conditions have to be applied to the first order stream function. The solvability condition (Fredholm alternative) for the first order stream function determines the spatial pattern of the zeroth order solution.

All terms in the first order equation have to be of the same order, so the scaling $\delta = \sigma$ should be set. Re-writing (1) in a self-adjoint form and applying the Fredholm alternative yields the equation for the amplitude function $A(\Phi)$,

$$A_{\Phi \Phi} \int_{R_1}^{R_2} \frac{W(\xi)^2}{\xi} d\xi = \int_{R_1}^{R_2} \xi W(\xi) f(A(\Phi)W(\xi) - \frac{c^2}{2}) d\xi$$  

(7)

The equations (5) and (7) define the spatial pattern for the stream function in the leading order. Laboratory experiments [6], have shown that the vorticity - stream function relation is weakly nonlinear. Many authors (e.g. [9]) make use a quadratic polynomial for this relation.

$$f(B) = s_1 B + s_2 B^2.$$  

(8)

This choice of $f$ reduces (7) to the Korteveg-de Vries type equation for the amplitude of nonlinear Rossby wave,

$$A_{\Phi \Phi} = I_0 + A^2 I_1 + A^2 I_2,$$  

(9)

where $I_0, I_1,$ and $I_2$ are constants. These constants are defined by $s_1, s_2, c,$ and $c_m$. Here the patterns with zero circulation are addressed, which means that

$$2 \pi \delta \int_0^{2\pi} A(\Phi) d\Phi = 0$$  

(10)

The straightforward analysis of (9) along with (10) leads to the expression for the Rossby wave amplitude as function of longitude. Analytical results for the case with zero mean flow were presented in [8], these results are extended here for more realistic case with ambient mean current. Equation (9) has solutions in the form of cnoidal waves with the wavelength $2\pi N$ (for the original angular coordinate $\phi$, $N = 1, 2, ..., \text{where} \ N$ is the zonal number. Specific choice of $N$ corresponds to a $2N$-pole, that is a vortex pattern with $N = 1$ is a dipole, with $N = 2$ is a quadrupole, while $N = 3$ is a hexapole.

The problem (4, 9, 10) has a unique solution once $s_1, s_2,$ and the magnitude of the ambient shear are prescribed [8]. Thus, the wave amplitude and spatial structure are uniquely defined by the specific form of the vorticity- stream function relation and the magnitude and the width of the ambient current. For small wave amplitudes, the resulted patterns are only slightly
asymmetric. For parameter sets that allow greater wave amplitudes, the wave tends to be more asymmetric, for example, in a dipolar solution, one vortex has larger angular extent compared to another vortex.

3. Application of the model to natural conditions
The study presented here aimed at explanation of the fundamentals for low frequency and large scale variability of the ACW, which is rotated around the Antarctic. Distances between PF and SAF are much smaller than the distance from either SAF or PF to the South pole, thus the SAF and PF paths can be approximated as two circumferences. As SAF and PF separate water masses with distinct physical properties, they could be conceptualized as rigid boundaries, which establish a circular channel. According to the observations, it is rational to set the internal circumference, at which the radial velocity is zero to be located at 3400 km from the pole.

The angular velocity of rotation of the vortical pattern is independent of the zonal number as predicted by (6). This deduction is supported by the observations [10], where variability was detected with time scale of about 5 years for the zonal mode 2, and 3.3 years, for the zonal mode 3, respectively. It results in the rotation period for the whole pattern to be 10 years for the zonal mode 1, and the same is correct for the modes 2 and 3. As can be seen from (6), the period of rotation of the vortical structure is independent of the specific coefficients $\sigma$, $s_1$, $s_2$ in (2). Nevertheless, it depends on the width and the magnitude of the ambient mean flow. The period of rotation of the vortical system as a function of the distance between the SAF and PF for various magnitudes of the mean current, is shown in Figure 1. It is seen that the Rossby waves pattern propagates against the current if the waveguide is sufficiently wide. In this case, the vortical patterns take 3-4 years to complete a full rotation around the Antarctic, and this period insignificantly depends on the magnitude of the mean current. As the distance between the SAF and PF fronts becomes shorter, the period of rotation increases, and reaches infinity, so there is no rotation is predicted at some width of the waveguide. When the distance between the SAF and PF further decreases, the sense of rotation changes to the opposite, and the pattern rotates in the same direction as the mean current does. When it happens, specific width is strongly affected by the magnitude of the mean current. The velocity of propagation of anomalies of the sea surface temperature observed in the ACC, is about 8 cm/s [10]. As the mean circulation due to the Rossby waves themselves is assumed to be zero here, therefore it is sensible to identify this velocity with the average velocity of ambient current. Magnitude of the mean current of 8 cm/s at the inner boundary, according to (6), leads to the 8.6 years for the period of rotation of the Rossby wave pattern, when the width of the waveguide is taken to be $L = 360$ km. This value is close to the observed values for the variability due to the ACC reported in [3] and [11]. According to their findings a period of $8 - 9$ years during 1985-1994 was observed. As the waveguide is quite narrow, there is no much difference between the mean current magnitudes at the inner and the outer boundaries of the waveguide according to (5). The choice of the magnitude of the ambient current and the length scale accepted in this study, leads to the flow rate of 121 Sv if the average depth of the oceans is taken to be 4 km. This value is close to 130 Sv, the flow rate in the ACC, reported by many authors. It can be shown [8] that solutions to the problem can exist only if $s_2 < 0$. Here, the case $s_2 = -1$ and set $\delta = \sigma = 0.3$ is examined in detail. These parameters affect the magnitude of the velocity field in the wave for given $s_1$. This choice of parameters leads to a reasonable agreement with available observations. Once all these parameters are set, analytical determination of the range of $s_1$ for which stationary rotating vortex patterns can exist is straightforward. Basically, this model we aggregates many physical effects into a couple of coefficients in the vorticity- stream function relation. Again, we recall that the The leading order stream function patterns can be found from (5), (6), (9) and (10). These patterns are shown in Figure 2 for the case of quadrupole ($N = 2$) and $s_1 = 0.912$. The pattern looks quite symmetric and the the predicted cnoidal
Figure 1. Period of rotation of the ACW in years versus the mean distance (in km) between the SAF and PF. Various magnitudes of the mean current are plotted. 12 cm/s - dash line, 8 cm/s - solid line, 2 cm/s - dots.

Figure 2. Streamline pattern for the case of quadrupole \((N = 2)\) with \(s_1 = 0.912\). Rossby wave packet is close to a sinusoidal one. However, linear theories cannot predict the wave amplitude and therefore do not predict the intensity of the velocity field in it. For the mentioned set of parameters, the theory predicts that the circumferential velocity due to the waves has the maximum of about 15 cm/s, while the magnitude of the the radial velocity is approximately 1 cm/s. The model also predicts the existence of the closed streamlines in the flow as seen in Figure 2. In nature it is associated with patches of the trapped fluid. Two regions of closed streamlines are attached to the inner boundary (PF), which are supposed to be cold bodies of water. These cold water bodies are expected to be caught from the regions affiliated with the Antarctica itself. Another two regions of closed streamlines are attached to the outer boundary (SAF), which separates the ACC and warmer waters from the North. These two patches are expected to be filled with warm waters caught from the outside the ACC. The ACW also has two trapped bodies of cold waters and two bodies of warm waters, and their sequence warm-cold-warm-cold is the same as predicted by the presented model. Circumferential velocity in the quadrupole under the conditions of Figure 2 is shown in Figure 3.
Figure 3. Circumferential velocity for the flow shown in Figure 2.

4. Conclusions
The variability of the polar flow due to rotation of nonlinear Rossby wave packets with zero circulation around the pole has been modeled. The rate of rotation does not influence by of the zonal number but it depends on the width of the waveguide and the strength of the mean current. Once the vorticity-streamfunction relation is set, the model predicts a unique nonlinear vortex structure for each zonal mode number. Modes 1, 2 and 3 can exist and described analytically, however they have realizable amplitudes in different range of parameters in the stream function-vorticity relation. It is predicted that the Rossby wave patterns include recirculation zones in which patches of trapped fluid can be transported. The theory is applied to the Antarctic Circumferential Wave propagating on the Antarctic Circumferrential Current. A reasonable agreement with observed features is shown despite the highly idealized approach.

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