THE $S^1$-EQUIVARIANT YAMABE INVARIANT OF 3-MANIFOLDS

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Abstract. We show that the $S^1$-equivariant Yamabe invariant of the 3-sphere, endowed with the Hopf action, is equal to the (non-equivariant) Yamabe invariant of the 3-sphere. More generally, we establish a topological upper bound for the $S^1$-equivariant Yamabe invariant of any closed oriented 3-manifold endowed with an $S^1$-action. Furthermore, we prove a convergence result for the equivariant Yamabe constants of an accumulating sequence of subgroups of a compact Lie group acting on a closed manifold.

1. Overview over the classical Yamabe invariant

The Yamabe constant $\mu(M, [g])$ of an $n$-dimensional conformal compact manifold $(M, [g])$ is the infimum of the restriction to the conformal class $[g]$ of the Einstein–Hilbert functional defined on the set of all Riemannian metrics as

$$h \mapsto \frac{\int_M \text{Scal}_h \, dv_h}{\text{vol}(M, h)^{\frac{n-2}{2}}}.$$ 

Aubin [9] proved that the Yamabe constant of $(M, [g])$ is bounded above by the Yamabe constant of the sphere, i.e. $\mu(M, [g]) \leq \mu(S^n, [g_{st}])$. The Yamabe invariant $\sigma(M)$ of a compact manifold $M$ is defined as

$$\sigma(M) := \sup_{[g] \in C(M)} \mu(M, [g]),$$

where $C(M)$ is the set of all conformal classes on $M$. It follows that $\sigma(M) \leq \sigma(S^n) = \mu(S^n, [g_{st}])$. In particular, the Yamabe invariant of any compact manifold is finite. The Yamabe invariant $\sigma(M)$ is positive if and only if a metric of positive scalar curvature exists on $M$.

In dimension 2, the Yamabe invariant is a multiple of the Euler characteristic. For $n \geq 3$ it is in general a difficult problem to compute the Yamabe invariant, and only in few cases it can be calculated explicitly. Aubin [9] proved for the $n$-dimensional sphere $\sigma(S^n) = \mu(S^n, [g_{st}]) = n(n-1)(\text{vol}(S^n, g_{st}))^{2/n}$. Kobayashi [18] and Schoen [29] proved that $\sigma(S^{n-1} \times S^1) = \sigma(S^n)$. For many closed manifolds $M$, one can show $\sigma(M) = 0$ as the existence of metrics with positive scalar curvature is obstructed, whereas conformal classes $[g_i]$ with $\mu(M, [g_i]) \to 0$ can be written down explicitly. For example the $n$-torus $T^n$ does not carry a metric of positive scalar curvature which can be shown with enlargeability type index obstructions by Gromov and Lawson or with the hypersurface obstruction by Schoen and Yau. For the standard metric $g_0$ we have $\mu(T^n, [g_0]) = 0$, so $\sigma(T^n) = 0$. Similarly we know $\sigma(M) = 0$ for all nilmanifolds, and quotients thereof.

In order to determine non-zero values for $\sigma$, many modern techniques were used: Ricci-flow, Atiyah-Singer index theorem, Seiberg-Witten theory, and the Bray-Huisken inverse mean curvature proof of the Penrose inequality. In dimension 3,
values for the Yamabe invariant of irreducible manifolds were already conjectured and partially studied in [3, 7].

For example, on a hyperbolic 3-manifold $H^3/\Gamma$ the supremum in the definition of the Yamabe invariant $\sigma(M)$ is attained in the conformal class of the hyperbolic metric $g_{hyp}$, and the infimum in the definition of $\mu(H^3/\Gamma, [g_{hyp}])$ is attained in $g_{hyp}$. More generally, it follows from Perelman’s work on the Ricci flow that for 3-manifolds with $\sigma(M) \leq 0$, the value of $\sigma(M)$ is determined by the volume of the hyperbolic pieces in the Thurston decomposition. We learned this from [17] Prop. 93.10 on page 2832, but ideas for this application go back to [8]. In the case $\sigma(M) > 0$, $n = 3$, $M$ is the connected sum of copies of quotients $S^2 \times S^1$ and of quotients of $S^3$. For connected sums of copies of $S^2 \times S^1$ we have $\sigma(M) = \sigma(S^3)$ but the precise value cannot be determined in most cases. Using inverse mean curvature flow, the Yamabe invariants of $\mathbb{R}P^3$ and some related spaces were determined in [13] and [1], e.g. $\sigma(\mathbb{R}P^3) = 2^{-2/3}\sigma(S^3)$. This is indeed a special case of Schoen’s conjecture explained below.

Also in higher dimensions the case of positive Yamabe invariant is notoriously difficult. In dimension $n \geq 5$ one does not know any $n$-dimensional manifold $M$ for which one can prove $0 < \sigma(M) < \sigma(S^n)$. In dimensions $n \leq 4$ there are some examples for which exact calculations can be carried out, even in the positive case. The values for $\mathbb{C}P^2$ and some related spaces were calculated by LeBrun [20] using Seiberg-Witten theory. The calculation then was simplified considerably by Gursky and LeBrun [14]. This proof no longer uses Seiberg-Witten theory, but only the index theorem by Atiyah and Singer. See also [13, 19, 21] for related results.

Recently, surgery techniques known from the work of Gromov and Lawson could be refined to obtain explicit positive lower bounds for the Yamabe invariant. Such bounds are easily obtained for special manifolds, e.g. for manifolds with Einstein metrics or connected sum of such manifolds. Namely, a theorem by Obata [24] states that the Einstein–Hilbert functional of an Einstein metric $g$ equals $\mu(M, [g])$, thus providing a lower bound for $\sigma(M)$. For instance, if $M$ is $S^n$, $T^n$, $\mathbb{R}P^n$ or $\mathbb{C}P^n$, the canonical Einstein metrics provide lower bounds for $\sigma(M)$. However, obtaining a lower bound for $\sigma(M)$ is difficult in general if $M$ carries a metric of positive scalar curvature but no Einstein metric. Using surgery theory, Petean and Yun have proven that $\sigma(M) \geq 0$ for all simply-connected manifolds of dimension at least 5, see [26, 27]. Stronger results can be obtained with the surgery formula developed in [2]. For example, it now can be shown, see [3] and [3, 5], that simply-connected manifolds of dimension 5 resp. 6 satisfy $\sigma(M) \geq 45.1$ resp. $\sigma(M) \geq 49.9$.

In order to find more manifolds with $0 < \sigma(M) < \sigma(S^n)$, it would be helpful to prove the following conjecture by Schoen [29]: it states that if $\Gamma$ is a finite group acting freely on $S^n$, then $\sigma(S^n/\Gamma) = \sigma(S^n)/(|\Gamma|)^{2/n}$. In particular, it would imply with [2] that for any odd $n \geq 5$ and sufficiently large $k := |\Gamma|$, every manifold $M$ representing the bordism class $[S^n/\Gamma] \in \Omega_n^{pin}(BT)$ with maps inducing isomorphisms $\pi_1(M) \cong \Gamma \cong \pi_1(S^n/\Gamma)$ has $\sigma(M) = \sigma(S^n)/(|\Gamma|)^{2/n}$ an many more similar conclusions. Unfortunately, besides the trivial cases $\Gamma = \{\text{id}\}$ or $n = 2$, this conjecture has only been proven in the particular case, when $n = 3$ and $|\Gamma| = 2$, which is the determination of $\sigma(\mathbb{R}P^3)$ by Bray and Neves in [13] mentioned above.

2. Overview over the $G$-equivariant Yamabe invariant

In this paper, we study the $G$-equivariant setting by taking the supremum and the infimum only among $G$-invariant metrics and conformal classes where $G$ is a compact Lie group acting on $M$, see Section 3.1 for details. The associated invariants are called the $G$-equivariant Yamabe constant or simply the $G$-Yamabe constant $\mu(M, [g]^G)$, and similarly the $G$-(equivariant) Yamabe invariant $\sigma^G(M)$. To
our knowledge the first reference for the $G$-equivariant Yamabe constant $\mu(M, [g]^G)$ is Bérard Bergery [11]. In particular, he formulated a $G$-equivariant version of the Yamabe conjecture, which was the main subject of an article by Hebey and Vaugon [16] and by the second author [22, 23]. In general neither $\sigma^G(M) \leq \sigma(M)$ nor $\sigma^G(M) \geq \sigma(M)$, see Example 3.

One motivation for the present article is to shed new light on Schoen’s conjecture which is equivalent to saying $\sigma^S(S^n) = \sigma(S^n)$. A proof of Schoen’s conjecture (or even partial results) would be very helpful, as it would provide interesting conclusions about the Yamabe invariant of non-simply connected manifolds. For example, if we were able to obtain an upper bound on $\sigma^I(S^n)$ which is uniform in $\Gamma$, then the Yamabe invariant would define interesting subgroups of the spin bordism and oriented bordism groups, see [2].

The simplest case of Schoen’s conjecture is when $\mathbb{Z}_k \subset S^1 \subset \mathbb{C}$ acts by complex multiplication on $S^3 \subset \mathbb{C}^2$, the so-called Hopf action. As it seems currently out of reach to show $\sigma^{\mathbb{Z}_k}(S^n) = \sigma(S^n)$ for $k > 2$, we study the limit $k \to \infty$ instead, and this leads to the following two questions:

1. Is $\sigma^{S^1}(S^3) = \sigma(S^3)$ true for the Hopf action?

2. Assume that a sequence $(H_i)$ of subgroups of $G$ “converges” to $G$. Can we conclude that $\sigma^{H_i}(M)$ converges to $\sigma^G(M)$?

The answer to the first question is answered affirmatively by our main theorem. More generally, we give an upper bound for the $S^1$-Yamabe invariant of any 3-dimensional closed oriented manifold $M$, endowed with an $S^1$-action. This upper bound depends only on the following topological invariants: the first Chern class of the associated line bundle and the Euler–Poincaré characteristic of the quotient space (see Theorem 9 for the precise statement).

Our strategy is to use the quotient space $M/S^1$. We distinguish the following three cases, since the isotropy group of any point is either $\{\text{id}\}$, $\mathbb{Z}_k$ or $S^1$. If the $S^1$-action has at least one fixed point, a result of Hebey and Vaugon [16] implies that $\sigma^{S^1}(M) \leq \sigma(S^3)$. If the $S^1$-action is free, then $M/S^1$ is a smooth surface. In order to find an upper bound in this case, we mainly use O’Neill’s formula relating the curvatures of the total space and the base space of a Riemannian submersion and the Gauss–Bonnet theorem. In the last case, when the $S^1$-action is neither free nor has fixed points (i.e. there exists at least one point with non-trivial finite isotropy group), the quotient space $M/S^1$ is a closed 2-dimensional orbifold. We proceed as in the free action case, since the Gauss–Bonnet theorem still holds on orbifolds (see [28]). In the two latter cases, we find a topological upper bound of $\sigma^{S^1}(M)$, which depends only on the Euler–Poincaré characteristic of $M/S^1$ and the first Chern number of the associated line bundle over $M/S^1$.

The last part of the article partially answers the second question. More precisely the statement of Corollary 13 is

$$\liminf_{i \to \infty} \sigma^{H_i}(M) \geq \sigma^G(M).$$

Unfortunately, the corresponding $\leq$-inequality which would allow the interesting application to Schoen’s conjecture still fails due to lack of curvature control.

3. Preliminaries, definitions and some known results

3.1. Definition of the $G$-equivariant Yamabe invariant. In this section we assume that a compact Lie group $G$ acts on the compact manifold $M$. All actions are supposed to be smooth.
We recall that the Einstein-Hilbert functional of \( M \) is given by
\[
J(\tilde{g}) := \frac{\int_M \text{Scal}_{\tilde{g}} \, d\tilde{g}}{\text{vol}(M, \tilde{g})^{\frac{2}{n}}}. \tag{1}
\]

We denote by \([\tilde{g}]^G\) the set of \( G \)-invariant metrics in the conformal class of \( \tilde{g} \) and by \( C^G(M) \) the set of all conformal classes containing at least one \( G \)-invariant metric.

**Definition 1** (G-Yamabe invariant). We define the \( G \)-equivariant Yamabe constant (or shorter: the \( G \)-Yamabe constant) by
\[
\mu(M, [\tilde{g}]^G) = \inf_{g' \in [\tilde{g}]^G} J(g') \tag{2}
\]
and the \( G \)-equivariant Yamabe invariant of \( M \) (or shorter: the \( G \)-Yamabe invariant) by
\[
\sigma^G(M) = \sup_{[\tilde{g}]^G \in C^G(M)} \mu(M, [\tilde{g}]^G) \in (-\infty, \infty].
\]

**Remark 2.** It follows for the solution of the equivariant Yamabe problem \cite{16} that \( \mu(M, [\tilde{g}]^G) > 0 \) if and only if \([\tilde{g}]\) contains a \( G \)-invariant metric of positive scalar curvature. It thus follows that \( \sigma^G(M) > 0 \) holds if and only if \( M \) carries a \( G \)-invariant metric of positive scalar curvature.

The following examples show that both \( \sigma^G(M) > \sigma(M) \) and \( \sigma^G(M) < \sigma(M) \) may arise.

**Example 3.** \( \sigma^G(M) \leq \sigma(M) \) nor \( \sigma^G(M) \geq \sigma(M) \). For example if \( S^1 \) acts on the \( S^1 \) factor of \( N \times S^1 \), \( \text{dim} \, N = n - 1 \), and if \( N \) is a compact manifold carrying a metric of positive scalar curvature, then \( \sigma^{S^1}(N \times S^1) = \infty \), whereas \( \sigma(N \times S^1) \leq \sigma(S^n) < \infty \). On the other hand, if \( M \) is a simply-connected circle bundle over a K3-surface then \( \sigma(M) > 0 \) but \( \sigma^{S^1}(M) = 0 \). Here \( \sigma(M) > 0 \) follows classically from work by Gromov and Lawson and the fact that every compact simply connected spin 5-manifolds is a spin boundary. For \( \sigma^{S^1}(M) \leq 0 \) we refer to \cite{32} Theorem 6.2. The inequality \( \sigma^{S^1}(M) \geq 0 \) follows from \( 3 \) by taking an \( S^1 \)-invariant metric \( g_1 \) on \( M \), we rescale the fibers by a factor \( \ell \) \( > 0 \) and obtain \( g_{\ell} \) and then \( \lim_{\ell \to 0} \mu(M, [g_{\ell}]^{S^1}) = 0 \).

The situation changes in the non-positive case. In the case \( \sigma^G(M) \leq 0 \) we have \( \mu(M, [\tilde{g}]^G) = \mu(M, [g]) \) for any \( G \)-invariant conformal class \([g]\), as the maximum principle implies that minimizers are unique up to a constant. Thus \( \sigma(M) \geq \sigma^G(M) \) in this case.

### 3.2. Some known results.

In \cite{16}, Hebey and Vaugon gave the following upper bound for the \( G \)-Yamabe constant:

**Proposition 4** (Hebey–Vaugon). Let \( M \) be an \( n \)-dimensional compact connected oriented manifold endowed with an action of a compact Lie group \( G \), admitting at least one orbit of finite cardinality. Then the following inequality holds:

\[
\sigma^G(M) \leq \sigma(S^n) \left( \inf_{p \in M} \text{card}(G \cdot p) \right)^{\frac{1}{n}}.
\]

Other results in the literature can be rephrased as follows.

**Proposition 5** (Bérard Bergery, \cite{11}). If \( G \) is a compact Lie group whose connected component of the identity is non-Abelian and which acts effectively on a closed manifold \( M \) with cohomogeneity 2. Then \( \sigma^G(M) > 0 \).
Proposition 6 (Béard Bergery [11] n = 3, Wiemeler [31] all n). Let an abelian Lie group $G$ act effectively on a closed connected manifold $M$ with a fix point component of codimension 2. Then $\sigma^G(M) > 0$.

More recent progress about the question whether $\sigma^G(M) > 0$ can be found in [15] and [32].

3.3. Scalar curvature of $S^1$-bundles. Let $M^n$ be a compact oriented and connected manifold, which is an $S^1$-bundle over $N$, let $\pi : M \to N$ be the projection, let $\tilde{g}$ be an $S^1$-invariant metric on $M$ and $g$ its projection under $\pi$ on $N$. Let $K$ denote the tangent vector field induced by the $S^1$-action and let $\ell$ be its length (with respect to $\tilde{g}$) and $e_0 := \frac{K}{\ell}$. We define the $(2,1)$-tensor fields $A$ and $T$ on $M$ as in [12, 9.C.], i.e. for all vector fields $U, V$ on $M$:

$A_UV = \nabla_{\mathcal{H}U}V + \mathcal{V}_{\mathcal{H}U}V,$

$T_UV = \nabla_{\mathcal{V}U}V + \mathcal{V}_{\mathcal{V}U}V,$

where $\mathcal{H}$ and $\mathcal{V}$ denote the horizontal, resp. vertical part of a vector field. The tensor $A$ measures the non-integrability of the horizontal distribution, whereas $T$ is essentially the second fundamental form of the $S^1$-orbits. Cf. [12, 9.37], the following formula relating the scalar curvatures of $(M, \tilde{g})$ and $(N, g)$ holds:

$\tilde{\text{Scal}} = \text{Scal} - \frac{1}{4} |A|^2 - |T|^2 - |T_{e_0}e_0|^2 - 2\hat{\delta}(T_{e_0}e_0),$ 

where $\hat{\delta}$ is the codifferential in the horizontal direction. For any vector fields $X, Y$ on $N$ with horizontal lifts $\tilde{X}, \tilde{Y}$, the vertical part of $[\tilde{X}, \tilde{Y}]$ equals $\Omega(X, Y)K := 2A_{\tilde{X}}\tilde{Y}$.

We compute:

$|A|^2 = \frac{\ell^2}{4} |\Omega|^2,$

$T_K \tilde{X} = \nabla_K \tilde{X} = \nabla_{\tilde{X}}K = \frac{\partial_{\tilde{X}} \ell}{\ell} K,$

$T_{e_0}e_0 = -\frac{\text{grad} \ell}{\ell} = -\text{grad} \log \ell,$

which yield

$\text{Scal}_{\tilde{g}} = \text{Scal}_g - \frac{\ell^2}{4} |\Omega|^2 - 2 \frac{|d\ell|^2}{\ell^2} + 2\Delta_g(\log \ell) = \text{Scal}_g - \frac{\ell^2}{4} |\Omega|^2 + 2 \frac{\Delta_g \ell}{\ell}, \quad (3)$

where $\Delta_g$ is the Laplacian of the base $(N, g)$.

3.4. An analytical ingredient. We recall that the following classical result still holds on orbifolds:

Lemma 7. Let $(\Sigma, g)$ be a closed 2-dimensional orbifold. Let $f \in C^k(\Sigma)$ be a function with $\int_{\Sigma} f dv_{g} = 0$. Then there exists a solution $u \in C^{k+2}(\Sigma)$ of the equation $\Delta_g u = f$, which is unique up to an additive constant.

The proof of Lemma 7 is analogously to the classical case.

4. The $S^1$-Yamabe invariant

In this section we always have $G = S^1$, and we use the notation $N = M/S^1$ similar to Section 3.3. Here $N$ may have singular points, i.e. orbifold points or boundary points.
4.1. Yamabe functional on $S^1$-bundles. If the action of $S^1$ is free, then by (3), we obtain from (1):

$$J(\tilde{g}) = \frac{2\pi \int_{\Sigma} (\Scal_{\tilde{g}} - \frac{2}{n-2} \omega^2 \omega^2) dv_{\tilde{g}}}{\int_{\Sigma} 2\pi \ell dv_{\tilde{g}}},$$

(4)
since the length of any fibre is $2\pi \ell$. The Yamabe functional of $(M, \tilde{g})$ is the restriction of the Einstein-Hilbert functional to the conformal class of $\tilde{g}$. It can be equivalently written as follows:

$$J(u \mapsto \tilde{g}) = \frac{\int_{\Sigma} 2\pi \ell \left(\frac{1}{n-2} |du_{\tilde{g}}|^2 + \Scal_{\tilde{g}} u^2\right) dv_{\tilde{g}}}{\int_{\Sigma} 2\pi \ell u dv_{\tilde{g}}},$$

(5)

where $\Scal_{\tilde{g}}$ is given by (3).

4.2. Classification of 3-manifolds with $\sigma^{S^3}(M) > 0$. It is completely understood, under which condition there is an $S^3$-invariant metric of positive scalar curvature, in other words, when $\sigma^{S^3}(M) > 0$.

**Theorem 8** ([11, Theorem 12.1]). Let $M$ be a compact connected 3-dimensional manifold with a smooth $S^1$-action on $M$.

a) If the action has a fixed point, then $\sigma^{S^3}(M) > 0$.

b) If the action has no fixed point, then $\sigma^{S^3}(M) > 0$ if and only if $M$ is a finite quotient of $S^3$ or of $S^2 \times S^1$.

Note that every finite quotient of $S^3$ by a freely acting subgroup of $SO(4)$ admits a non-trivial $S^1$-action [25, Sec. 6, Theorem 5].

4.3. Oriented 3-manifolds. From now on, we assume that $M$ is a 3-dimensional compact oriented connected manifold endowed with an $S^1$-action. If this $S^1$-action has at least one fixed point, Proposition 4 implies that the Yamabe invariant of $S^3$ is an upper bound for the $S^1$-Yamabe invariant: $\sigma^{S^3}(M) \leq \sigma(S^3)$.

We want to determine an upper bound for the $S^1$-Yamabe invariant in the complementary case, i.e. we consider $S^1$-actions without fixed points. This implies that $M$ is an $S^1$-principal (orbi)bundle over $\Sigma := M/S^1$, which is a 2-dimensional orbifold (a smooth surface, if the action is free). As usually, we use the correspondence between $S^1$-principal bundles and complex line bundles defined by $\Sigma \mapsto L := \Sigma \times_{S^1} \mathbb{C}$.

We write $c_1(L, \Sigma) := \langle c_1(L), [\Sigma] \rangle \in \mathbb{Q}$, where $c_1(L) \in H^2(\Sigma, \mathbb{Q})$ is the first rational Chern class of $L$ in the orbifold sense. Let $\chi(\Sigma) = c_1(T\Sigma, \Sigma)$ be the (orbi) Euler-Poincaré characteristic of $\Sigma$.

We are now ready to state our main result:

**Theorem 9.** Let $M$ be a 3-dimensional compact connected oriented manifold endowed with an $S^1$-action without fixed points. With the above notation, the following assertions hold:

i) If $\chi(\Sigma) > 0$ and $c_1(L, \Sigma) \neq 0$, then

$$0 < \sigma^{S^1}(M) \leq \sigma(S^3) \left(\frac{\chi(\Sigma)}{2|c_1(L, \Sigma)|}\right)^\frac{1}{2}.$$

ii) If $\chi(\Sigma) > 0$ and $c_1(L, \Sigma) = 0$, then $\sigma^{S^1}(M) = \infty$.

iii) If $\chi(\Sigma) \leq 0$, then $\sigma^{S^1}(M) = 0$.

In particular, $\sigma^{S^1}(M)$ is positive if and only if $\chi(\Sigma)$ is positive. This coincides with the characterization in [11], as explained in Section 4.2.
Proof of Theorem 9. Let \([\tilde{g}]^{S^1}\) be the class of \(S^1\)-invariant metrics conformal to \(\tilde{g}\) on \(M\). Without loss of generality, we assume that the length of the vector field \(K\) generating the \(S^1\)-action \(\ell := |K|_\tilde{g}\) is constant (otherwise we take a different representant of the class \([g]^{S^1}\)). Let \(g\) be the projection of the metric \(\tilde{g}\) on \(\Sigma\), so that \((M, \tilde{g}) \rightarrow (\Sigma, g)\) is a Riemannian submersion. Since \(\ell\) is constant, the O’Neill formula (3) yields that \(\text{Scal}_g = \text{Scal}_{\tilde{g}} - \frac{1}{2}\Omega_\tilde{g}^2\). Using the Gauß–Bonnet theorem, we compute the Yamabe functional as follows:

\[
J(\tilde{g}) = \frac{2\pi \int_\Sigma (\text{Scal}_g - \frac{1}{2}\Omega_\tilde{g}) \ell \, dv_{\tilde{g}}}{(2\pi)^\frac{\ell}{4}(\int_\Sigma \ell \, dv_{\tilde{g}})^\frac{\ell}{4}}
\]

\[
= (2\pi)^\frac{\ell}{4} \frac{\ell (\int_\Sigma \text{Scal}_g \, dv_g) - \frac{1}{2}\ell (\int_\Sigma |\Omega_\tilde{g}|^2 \, dv_{\tilde{g}})}{\ell^2 (\int_\Sigma \, dv_g)^\frac{\ell}{4}}
\]

\[
= \left( \frac{\pi^2}{16\text{vol}(\Sigma, g)} \right)^\frac{1}{4} (16\pi \chi(\Sigma) \ell^\frac{\ell}{4} - ||\Omega||_2^2 \ell^\frac{\ell}{4}).
\]

If we have \(\chi(\Sigma) \geq 0\) and \(||\Omega||_2 > 0\), then the maximal value of this expression as a function in \(\ell\) is attained for \(\ell = \sqrt{4\pi \chi(\Sigma)} ||\Omega||_2^{-1}\) and its maximal value equals

\[
3 \cdot 2^\frac{\ell}{4} \pi^2 (\text{vol}(\Sigma, g))^{-\frac{\ell}{4}} ||\Omega||_2^{-\frac{\ell}{4}}.
\]

We now consider cases i) to iii) in the theorem.

i) Note that in this case \(c_1(L, \Sigma) \neq 0\) implies \(||\Omega||_2 > 0\). By the Cauchy–Schwarz inequality, it further follows that

\[
J(\tilde{g}) \leq 3 \cdot 2^\frac{\ell}{4} \pi^2 \chi(\Sigma) ||\Omega||_1^{-\frac{\ell}{4}}.
\]

On the other hand, we claim that \(||\Omega||_1 \geq 2\sqrt{2\pi}|c_1(L, \Sigma)|\), since

\[
\frac{1}{\sqrt{2}} \int_\Sigma |\Omega|_g \, dv_g \geq \left| \int_\Sigma \Omega \right| = 2\pi|c_1(L, \Sigma)|,
\]

where the volume form \(dv_g\) has length \(\sqrt{2}\), by convention. Using \(\sigma(S^3) = 3 \cdot 2^{5/3} \pi^{4/3}\) it follows that \(J(\tilde{g}) \leq \sigma(S^3) \chi(\Sigma) ||\Omega||_2^{-\frac{\ell}{4}} ||\Omega||_1^{-\frac{\ell}{4}}\), for all \(S^1\)-invariant metrics \(\tilde{g}\) on \(M\) with \(\ell = |K|_\tilde{g}\) constant. This yields

\[
\mu(M, [\tilde{g}]^{S^1}) \leq \sigma(S^3) \chi(\Sigma) ||\Omega||_2^{-\frac{\ell}{4}} ||\Omega||_1^{-\frac{\ell}{4}},
\]

for all \(S^1\)-invariant conformal classes \([\tilde{g}]^{S^1} \in \text{Conf}^{S^1}(M)\).

Now, we show that \(\sigma^{S^1}(M)\) is positive. The function \(f := \frac{2\pi}{\text{vol}(\Sigma, g)} \chi(\Sigma) - \frac{1}{2}\text{Scal}_\tilde{g}\) has zero average over \(\Sigma\). By Lemma 7, there exists a solution \(u\) of the equation \(\Delta_g u = f\). Therefore the scalar curvature of \(g_u := e^{2u}g\) is given by

\[
\text{Scal}_{g_u} = 2e^{-2u}(\Delta_g u + \frac{1}{2}\text{Scal}_g) = \frac{4\pi}{\text{vol}(\Sigma, g)} \chi(\Sigma) e^{-2u}.
\]

Hence, the scalar curvature of \(g_u\) is positive. Using the identity (3) and choosing the length of the \(S^1\)-fibre constant and sufficiently small, we construct an \(S^1\)-invariant metric \(\tilde{g}_u\) (which is not necessarily conformal to \(\tilde{g}\)) with positive scalar curvature. Therefore, the Yamabe constant \(\mu(M, [\tilde{g}_u]^{S^1})\) is positive, so \(\sigma^{S^1}(M) > 0\).

ii) If \(c_1(L, \Sigma) = 0\), then there exists an \(S^1\)-equivariant finite covering \(S^1 \times \tilde{\Sigma}\) of \(M\) of degree \(d\), where \(\tilde{\Sigma}\) is a smooth compact surface finitely covering \(\Sigma\) (for more details, see e.g. [30 Lemma 3.7]). Since \(\chi(\Sigma) > 0\), we see that \(\tilde{\Sigma}\) is diffeomorphic to \(S^2\). As in the previous case, we know that a metric of positive Gauß curvature exists on \(\Sigma\). The product metric \(\tilde{g}_u\) of its lift to \(\tilde{\Sigma}\) with a rescaled standard metric on \(S^1\) of length \(2\pi\ell\) is invariant under the deck transformation.
group of $S^1 \times \Sigma \to M$. As this deck transformation group commutes with the $S^1$-action, $\hat{g}$ descends to an $S^1$-invariant metric $g_\ell$ on $M$. From (5), we get $\mu(S^1 \times \Sigma, [\hat{g}_\ell]^{S^1}) = \mu(S^1 \times \Sigma, [g]^{S^1})^{\ell/3}$. Obviously we have $\mu(S^1 \times \Sigma, [\hat{g}_\ell]^{S^1}) \leq d^{3/2} \mu(M, [g]^{S^1})$. Then $\mu(M, [g]^{S^1})$ converges to $\infty$ for $\ell \to \infty$, which implies the statement.

iii) Assume that the Euler-Poincaré characteristic of $\Sigma$ is nonpositive. By (6), we have

$$\mu(M, [\hat{g}]^{S^1}) \leq J(\hat{\ell}^{-2} \hat{g}) \leq 2(2\pi)^{\frac{m}{2}} \chi(\Sigma) \text{vol}(\Sigma, \hat{g}_\Sigma)^{-\frac{1}{2}} \leq 0,$$

for any $S^1$-invariant Riemannian metric $\hat{g}$ on $M$, where $\hat{\ell} := |K|_{\hat{g}}$. This yields $\sigma^{S^1}(M) \leq 0$. Moreover, if we fix a Riemannian metric $\hat{g}_\Sigma$ on $\Sigma$, we define $(\hat{g}_j)$ to be a sequence of metrics on $M$ with constant functions $\hat{\ell}_j := |K|_{\hat{g}_j} \leq 1$ converging to 0 and $\pi^* \hat{g}_\Sigma = \hat{g}_j$. From (5) and using the Hölder inequality, we obtain

$$\mu(M, [\hat{g}_j]^{S^1}) \geq -(2\pi \hat{\ell}_j)^{\frac{m}{2}} (\|\text{Scal}_{\hat{g}_j}\|_2 + \frac{1}{4^{\frac{m}{2}}} \|\Omega\|^2_2).$$

Hence, when $j$ goes to $+\infty$, it follows that $\sigma^{S^1}(M) \geq 0$. We conclude that $\sigma^{S^1}(M) = 0$.

4.4. The case of $S^3$. We now consider the special case of $S^1$-actions on $S^3 \subset \mathbb{C}^2$.

For $m_1, m_2 \in \mathbb{N}$ assumed to be relatively prime as long as $m_1 m_2 \neq 0$, we define

$$\phi_{m_1, m_2} : S^1 \to \text{Diff}(S^3), \quad \phi_{m_1, m_2}(x)(z_1, z_2) := (x^{m_1}z_1, x^{m_2}z_2). \quad (9)$$

With this notation, the Hopf action of $S^1$ on $S^3$ corresponds to $\phi_{1,1}$. These are the only possible smooth $S^1$-actions on $S^3$ up to diffeomorphisms (see e.g. [25]). Note that such an action has fixed points if and only if $m_1 m_2 = 0$.

**Theorem 10.** For the Hopf action of $S^1$ on $S^3$ it holds:

$$\sigma^{S^1}(S^3) = \sigma(S^3).$$

Moreover, the $S^1$-equivariant Yamabe invariant of any $S^1$-action $\phi_{m_1, m_2}$ on $S^3$ satisfies the following:

i) If $m_1 m_2 = 0$, then $\sigma^{S^1}(S^3) = \sigma(S^3) = 6 \cdot 2^{\frac{m}{2}} \cdot \pi^{\frac{m}{2}}$.

ii) If $m_1 m_2 \neq 0$, then

$$\sigma(S^3) \leq \sigma^{S^1}(S^3) \leq \sigma(S^3) \left(\frac{m_1 + m_2}{2 \sqrt{m_1 m_2}}\right)^{\frac{1}{4}}.$$

**Proof.** Let us first remark that, since the standard metric $g_{st}$ on $S^3$ is $S^1$-invariant for any $S^1$-action $\phi_{m_1, m_2}$, it follows that $\mu(S^3, [g_{st}]^{S^1}) \geq \mu(S^3, [g_{st}]) = \sigma(S^3)$. Hence, we obtain the inequality: $\sigma^{S^1}(S^3) \geq \sigma(S^3)$.

i) If $m_1 m_2 = 0$, then the $S^1$-action has fixed points and by Proposition 4 we also obtain the reverse inequality: $\sigma^{S^1}(S^3) \leq \sigma(S^3)$.

ii) If $m_1 m_2 \neq 0$, then the quotient orbifold is the so-called 1-dimensional weighted projective space denoted by $\mathbb{P}^1(m_1, m_2)$. In order to use the upper bound provided by Theorem 9, we need to compute $\chi(\mathbb{P}^1(m_1, m_2))$ and $\ell_1(L, \mathbb{P}^1(m_1, m_2))$. Using the Seifert invariants of $S^1$-bundles (see e.g. [25], [50]), one obtains:

$$\chi(\mathbb{P}^1(m_1, m_2)) = \frac{1}{m_1} + \frac{1}{m_2} \quad \text{and} \quad |\ell_1(L, \mathbb{P}^1(m_1, m_2))| = \frac{1}{m_1 m_2}. \quad \text{Alternatively,}$$

we give in the Appendix an explicit geometric computation of this topological invariants. Substituting these values in Theorem 9, i), we obtain the desired inequality.

The first statement of the theorem follows from ii) for $m_1 = m_2 = 1$. □
5. Convergence result

Definition 11. Let $G$ be a Lie group, and let $(H_i)_{i \in \mathbb{N}}$ be a sequence of subgroups. We say that $h \in G$ is an accumulation point for $(H_i)_{i \in \mathbb{N}}$ if there is a sequence $(h_i)_{i \in \mathbb{N}}$ with $h_i \in H_i$ and $h_i \to h$. The set of accumulation points is a closed subgroup of $G$. We say that $(H_i)_{i \in \mathbb{N}}$ is accumulating, if every element of $G$ is an accumulation point.

Proposition 12. Assume that a compact Lie group $G$ acts on a closed manifold $M$. Let $(H_i)_{i \in \mathbb{N}}$ be an accumulating sequence of subgroups of $G$. Then for any $G$-equivariant conformal class $[g]$ we get
\[
\lim_{i \to \infty} \mu(M, [g]^{H_i}) = \mu(M, [g]^G).
\]

Proof. We distinguish the following two cases:

- If the (non equivariant) Yamabe constant satisfies $\mu(M, [g]) \leq 0$, then there is, up to a multiplicative constant, a unique metric $u_{\infty}^\frac{-2}{n} g$ of constant scalar curvature and $u_{\infty}$ is $G$-invariant. This implies $\mu(M, [g]) = \mu(M, [g]^G) = \mu(M, [g]^{H_i})$.
- Now we assume that the Yamabe constant satisfies $\mu(M, [g]) > 0$. Set $\mu_i := \mu(M, [g]^{H_i}), \mu := \mu(M, [g]^G)$. Obviously $\mu_1 \leq \mu$. After passing to a subsequence we can assume that $\mu_i$ converges to a number $\mu \leq \mu$ and it remains to show that $\bar{\mu} < \mu$ leads to a contradiction. For an orbit $O$ we will use the convention that $\#O$ takes values in $\mathbb{N} \cup \{\infty\}$, i.e. we do not distinguish between infinite cardinalities. We claim that $\lim_{i \to \infty} \#(H_i \cdot p) = \#(G \cdot p)$, for any $p \in M$. The inequality $\#(H_i \cdot p) \leq \#(G \cdot p)$ is obvious as $H_i \subset G$.

To prove the claim in the case $\#(G \cdot p) < \infty$, we choose pairwise disjoint neighborhoods of all the $G$-orbit points of $p$ and for $i$ sufficiently large, we find in each such neighborhood an element of the $H_i$-orbit of $p$, showing that $\#(H_i \cdot p) \geq \#(G \cdot p)$. If $\#(G \cdot p) = \infty$, then we apply the previous argument to a finite subset of the $G$-orbit of $p$ and then let its cardinality converge to $\infty$. This shows that $\lim_{i \to \infty} \#(H_i \cdot p) = \infty$.

Without loss of generality, we assume that $\mu_i \leq \bar{\mu} := (\mu + \bar{\mu})/2 < \mu$. Let $k$ be the cardinality of the smallest $G$-orbit, again sloppily written as $\infty$ in the case that $k$ is infinite. Then by Proposition 4, we have $\mu \leq \sigma(S^n)k^{2/n}$. Hence, by the above claim, we obtain the following inequality $\mu_i \leq \bar{\mu} < \sigma(S^n)\min_{p \in M} \#(H_i \cdot p)^{2/n}$, for $i \geq i_0$, where $i_0$ is sufficiently large. By Hebey and Vaugon [16], it follows that, for $i \geq i_0$, there exists a sequence $\left(\frac{u_i}{\mu_i}^\frac{-2}{n} g\right)_{i \in \mathbb{N}}$ of $H_i$-invariant metrics, which minimizes the functional $J$ among all $H_i$-invariant metrics in $[g]$. Furthermore $u_i$ is a positive smooth $H_i$-invariant solution of the Yamabe equation, and we may assume $\|u_i\|_{L^\infty} = 1$. The sequence $(u_i)_{i \in \mathbb{N}}$ is uniformly bounded in $H^1(M)$. Hence there exists a nonnegative function $u_{\infty} \in H^1(M)$, such that $(u_i)_{i \in \mathbb{N}}$ converges strongly in $L^q(M)$, for $1 \leq q < \frac{2n}{n-2}$, and weakly in $H^1(M)$ to $u_{\infty}$. We now claim, that $u_i$ is bounded in the $L^\infty$-norm. Suppose that it is not bounded. Then we find a sequence of $x_i \in M$ such that $u_i(x_i) \to \infty$, and after taking a subsequence $x_i$ converges to a point $\bar{x}$. For any point $g\bar{x}$ in its orbit, there is a sequence of $h_i \in H_i$ with $h_i x_i \to g\bar{x}, u_i(h_i x_i) \to \infty$. If the orbit $G \cdot \bar{x}$ contains at least $k$ points, then we can do classical blowup-analysis in $k$ points, which would yield $\bar{\mu} \geq \sigma(S^n)k^{2/n}$ (see for example [10], Chapter 6.5.). This implies $\bar{\mu} \geq \sigma(S^n)k^{2/n}$ which contradicts $\bar{\mu} < \sigma(S^n)k^{2/n}$. We obtain the claim, i.e. the boundedness of $u_i$ in $L^\infty$. By a standard bootstrap argument this yields the boundedness of $u_i$ in $C^{2,\alpha}$ for $0 < \alpha < 1$, and thus $u_i$ converges to $u_{\infty}$ in $C^2$. It follows that $u_{\infty}$ is a smooth, positive $G$-invariant solution of the Yamabe equation, with $\|u_{\infty}\|_{L^\infty} = 1$ and $J(u_{\infty}^\frac{-2}{n} g) = \bar{\mu}$. Thus $\mu \leq \bar{\mu}$. 


Corollary 13. Assume that a compact Lie group $G$ acts on a closed manifold $M$. Let $(H_i)_{i \in \mathbb{N}}$ be an accumulating sequence of subgroups of $G$. Then

$$\liminf_{i \to \infty} \sigma^{H_i}(M) \geq \sigma^G(M).$$

□

APPENDIX A.

A.1. Computation of $c_1(L, \mathbb{C}P^1(m_1, m_2))$. We consider the action of $S^1$ on $S^3 \subset \mathbb{C}^2$ given by

$$\phi_{m_1, m_2} : e^{i\theta} \mapsto ((z_1, z_2) \mapsto (e^{im_1\theta}z_1, e^{im_2\theta}z_2)),$$

where $m_1$ and $m_2$ are two positive relatively prime integers. Let $\pi : S^3 \to S^3/S^1$ denote the projection, where the quotient $S^3/S^1 =: \mathbb{C}P^1(m_1, m_2)$ is the one dimensional weighted projective space. We consider the round metric of $S^3$ induced by the standard metric on $\mathbb{R}^4 \simeq \mathbb{C}^2$: $(z_1, z_2), (w_1, w_2)) = \text{Re}(z_1\bar{w}_1 + z_2\bar{w}_2)$. The vector field induced by the $S^1$-action is given by:

$$K_p = i(m_1z_1, m_2z_2) \in T_pS^3 = p^\perp, \text{ where } p = (z_1, z_2) \in S^3.$$

The vector field $K$ vanishes nowhere, since $|K_p|^2 = m_1^2|z_1|^2 + m_2^2|z_2|^2 > 0$, for all $p \in S^3$. For $p \in S^3 \setminus \{0\} \times S^1 \cup S^1 \times \{0\}$, the orthogonal complement of $K_p$ in $T_pS^3$ (w.r.t. the round metric) is spanned by the horizontal vector fields

$$\tilde{X}_1(p) := i(m_2|z_2|^2z_1, -m_1|z_1|^2z_2), \quad \tilde{X}_2(p) := (|z_2|^2z_1, -|z_1|^2z_2),$$

which are also $S^1$-invariant. Hence they project to the vector fields $X_1$, resp. $X_2$ on $\mathbb{C}P^1(m_1, m_2)$.

We define the connection 1-form $\omega$ on $S^3$ whose kernel is given by the orthogonal complement of $K$ and normed such that $\omega(K) = 1$, $\omega := \frac{(K, \cdot)}{|K|^2}$. The 2-form $\Omega := d\omega$ is $S^1$-invariant and thus projects onto a 2-form on $\mathbb{C}P^1(m_1, m_2)$, which we denote by the same symbol. It follows that

$$\Omega_{\pi(p)}(X_1, X_2) = -\omega_p([\tilde{X}_1, \tilde{X}_2]) = \frac{2m_1m_2|z_1|^2|z_2|^2}{m_1^2|z_1|^2 + m_2^2|z_2|^2},$$

since we have $d\tilde{X}_1(\tilde{X}_2) - d\tilde{X}_2(\tilde{X}_1) = -2i|z_1|^2|z_2|^2(m_2z_1, m_1z_2)$.

We now introduce the following complex coordinates on $\mathbb{C}P^1(m_1, m_2) \setminus \{[0 : 1]\}$.

$$\varphi : \mathbb{C}P^1(m_1, m_2) \setminus \{[0 : 1]\} \to \mathbb{C}, \quad [z_1 : z_2] \mapsto z := \frac{m_1}{m_2}z_1.$$

It follows that for any $p \in S^3 \setminus \{0\} \times S^1$, the tangent linear map of the projection is given by

$$\pi_* (p) = \begin{pmatrix} m_2 & z_1 \frac{m_1}{m_2} \\ z_1 \frac{m_2}{m_1} & m_1 \frac{z_1}{m_2} \end{pmatrix}^{-1}$$

and the vector fields $X_1$ and $X_2$ are

$$X_1(z) = -(m_2^2|z_2|^2 + m_1^2|z_1|^2)iz, \quad X_2(z) = -(m_2|z_2|^2 + m_1|z_1|^2)z.$$
These together imply the following:
\[
\Omega_z = \frac{-m_1 m_2 |z_1|^2 |z_2|^2}{(m_2^2 |z_2|^2 + m_1^2 |z_1|^2)(2m_2 |z_2|^2 + m_1 |z_1|^2)|z|^2} i \, dz \wedge d\bar{z},
\]
\[
e_1(L, \mathbb{C}P^1(m_1, m_2)) = \frac{1}{2\pi} \int_{\mathbb{C}P^1(m_1, m_2)} \Omega = -\int_0^{\infty} \frac{2m_1 m_2 r(1-r)}{\rho} \frac{d\rho}{m_1 m_2}
\]
where \( r = |z_1|^2, \rho = |z| \) and \( \rho = \frac{(1-r)^{m_2}}{r^{m_1}} \).

A.2. Computation of \( \chi(\mathbb{C}P^1(m_1, m_2)) \). The quotients metric \( g \) induced on \( \mathbb{C}P^1(m_1, m_2) \) by the standard metric of \( S^3 \) is uniquely determined by setting that the following two vector fields of the tangent space of \( \mathbb{C}P^1(m_1, m_2) \) at \( z \in \mathbb{C} \setminus \{0\} \) build an orthonormal base:
\[
e_1(z) := \frac{X_1(z)}{|X_1(z)|} = \lambda_1(z)iz, \quad e_2(z) := \frac{X_2(z)}{|X_2(z)|} = \lambda_2(z)z,
\]
where \( \lambda_j(z) := \tilde{\lambda}_j \circ \gamma^{-1}(|z|^2), \tilde{\lambda}_j(t) := -\sqrt{m_2^2 m_1^2 + \rho^2} \), \( \tilde{\lambda}_2(t) := \frac{(m_1-m_2)t + m_2}{\sqrt{(1-t)}} \)
and \( \gamma \) is the diffeomorphism \( \gamma(r) := \frac{(1-r)^{m_1}}{r^{m_2}} \), for \( r \in (0, 1) \) and \( |z|^2 = \frac{(1-|z_1|^2)^2}{|z_2|^2} \).
We consider \( \Theta \) to be the Levi-Civita connection 1-form. We have
\[
\Theta(v) := g(\nabla_v e_2, e_1) = g([e_1, e_2], v).
\]
We first compute the Lie bracket:
\[
[e_1, e_2] = \lambda_1 \lambda_2 i[z, z] + \lambda_1 d\lambda_2 (iz) z - \lambda_2 d\lambda_1 (iz) iz = -\frac{d\lambda_1(e_2)}{\lambda_1} e_1,
\]
since \( [i z, z] = 0 \) and \( d\lambda_j = 2(\tilde{\lambda}_j \circ \gamma^{-1})'(\cdot |z|^2)z, \) for \( j = 1, 2 \), which implies \( d\lambda_2(iz) = 0 \). Secondly, we compute the Gaussian curvature of \( \mathbb{C}P^1(m_1, m_2) \):
\[
\kappa = d\theta(e_1, e_2) = -d(g([e_1, e_2], e_1))(e_2) - \Theta([e_1, e_2]) = d\left( \frac{d\lambda_1(e_2)}{\lambda_1} \right) - \frac{d\lambda_1(e_2)}{\lambda_1} \right)^2.
\]
Hence \( d\theta = \kappa e_1^* \wedge e_2^* = -\frac{\kappa}{\lambda_1 \lambda_2} dx \wedge dy \). By the orbifold Gauss–Bonnet theorem, it follows that
\[
\chi(\mathbb{C}P^1(m_1, m_2)) = 2\pi \int_{\mathbb{C}P^1(m_1, m_2)} \kappa = -\frac{1}{2} \int_0^{\infty} \frac{\kappa}{|z_1|^2 |z_2|^2} \, d|z|^2 = \frac{1}{2} \int_0^{1} \frac{\kappa(r) \gamma'(r)}{\lambda_1(r) \lambda_2(r) \gamma(r)} \, dr,
\]
since the functions \( \lambda_j \) are radial and thus \( \kappa \) is also radial. Substituting \( \kappa \) in the last integral, we get
\[
\chi(\mathbb{C}P^1(m_1, m_2)) = 2\pi \int_0^{1} \left( \frac{\tilde{\lambda}_2 \tilde{\lambda}_1 \gamma'}{\lambda_1 \gamma'} - \frac{\tilde{\lambda}_2 \tilde{\lambda}_1 \gamma^2}{\lambda_1 \gamma} \right) \frac{\gamma'}{\lambda_1 \lambda_2} \, dr
\]
\[
= 2 \int_0^{1} \frac{\tilde{\lambda}_2 \tilde{\lambda}_1 \gamma'}{\lambda_1} - \frac{\tilde{\lambda}_2 \tilde{\lambda}_1 \gamma^2}{\lambda_1} \, dr = 2 \left[ \frac{\tilde{\lambda}_2 \tilde{\lambda}_1 \gamma^2}{\lambda_1 \gamma'} \right]_0^1 = \frac{1}{m_1} + \frac{1}{m_2}.
\]
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