A HEISENBERG DOUBLE ADDITION TO THE LOGARITHMIC KAZHDAN–LUSZTIG DUALITY

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ABSTRACT. For a Hopf algebra $B$, we endow the Heisenberg double $\mathcal{H}(B^*)$ with the structure of a module algebra over the Drinfeld double $\mathcal{D}(B)$. Based on this property, we propose that $\mathcal{H}(B^*)$ is to be the counterpart of the algebra of fields on the quantum-group side of the Kazhdan–Lusztig duality between logarithmic conformal field theories and quantum groups. As an example, we work out the case where $B$ is the Taft Hopf algebra related to the $\mathcal{U}_q\mathfrak{sl}(2)$ quantum group that is Kazhdan–Lusztig-dual to $(p, 1)$ logarithmic conformal models. The corresponding pair $(\mathcal{D}(B), \mathcal{H}(B^*))$ is “truncated” to $(\mathcal{U}_q\mathfrak{sl}(2), \mathcal{H}_q\mathfrak{sl}(2))$, where $\mathcal{U}_q\mathfrak{sl}(2)$ is a $\mathcal{U}_q\mathfrak{sl}(2)$-module algebra that turns out to have the form $\mathcal{U}_q\mathfrak{sl}(2) - \mathbb{C}_q[z, \partial] \otimes \mathbb{C}[\lambda]/(\lambda^{2p} - 1)$, where $\mathbb{C}_q[z, \partial]$ is the $\mathcal{U}_q\mathfrak{sl}(2)$-module algebra with the relations $z^p = 0$, $\partial^p = 0$, and $\partial z - q - q^{-1} + q^{-2}z\partial$.

1. INTRODUCTION

The “logarithmic” Kazhdan–Lusztig duality — a remarkable correspondence between logarithmic conformal field theories and quantum groups — is based on a Drinfeld double construction on the quantum group side [7]. The starting point is the Hopf algebra $B$ generated by the screening(s) in a logarithmic model and diagonal, “zero-mode-like” element(s) (see [8, 27, 28] for the two-screening case, which is relatively complicated by modern standards). The strategy is then to construct the Drinfeld double of this quantum group and to “slightly truncate” it, to produce the Kazhdan–Lusztig-dual quantum group. Various aspects of the “logarithmic” Kazhdan–Lusztig duality were developed in [29, 8, 27, 30, 31, 14, 15].

The resulting correspondence (ranging up to the coincidence) in the properties of the symmetry algebra of the logarithmic model and the dual quantum group is “circumstantial” in that it is seen to work nicely in particular cases, although no general argument for its existence has been developed or attempted. That the Drinfeld double of $B$ plays a crucial role in this correspondence was a serendipitous finding in [7]. Modulo the “slight truncation” mentioned above, the Drinfeld double is a counterpart of the symmetry alge-

1It has become impossible to list “all” papers on logarithmic conformal field theory. We note the pioneering works [1, 2, 3, 4], a prejudiced selection [5, 6, 7, 8, 9], a vertex-operator algebra trend in [10, 11, 12, 13, 14, 15, 16], and recent papers [17, 18, 19, 20, 21, 22, 23, 24, 25, 26], wherein further references can be found.
bra ("the" triplet \([3, 4, 32, 6, 10]\) or a higher one \([8]\)) of a given logarithmic conformal field model.

In this paper, we propose another algebraic object that may play a role in the logarithmic Kazhdan–Lusztig duality, being a counterpart of the algebra of fields in logarithmic models. We here mean the fields describing logarithmic models in manifestly quantum-group-invariant terms (i.e., "carrying quantum-group indices"), as a generalization of the symplectic fermions \([33]\). The necessary algebraic requirement is that the quantum group act "covariantly" on products of fields, which is expressed as the module algebra axiom $h \triangleright (\varphi \psi) = (h' \triangleright \varphi)(h'' \triangleright \psi)$, where we use the Sweedler notation $\Delta(h) = h' \otimes h''$ for the coproduct. We now describe a $\mathcal{D}(B)$-module algebra that is to play the role of fields on the algebraic side.

For a Hopf algebra $B$, the Drinfeld double $\mathcal{D}(B)$ is $B^* \otimes B$ as a vector space. The same vector space admits another characteristic algebraic structure, a (semisimple) associative algebra given by the smash product with respect to the (left) regular action of $B$ on $B^*$, or, in the established terminology traced back to \([34, 35, 36]\), a Heisenberg double (see, e.g., \([37, 38, 39]\)), specifically, the Heisenberg double $\mathcal{H}(B^*) = B^* \# B$ of $B^*$. The main observation in this paper is that for any Hopf algebra $B$ with bijective antipode, $\mathcal{H}(B^*)$ is a $\mathcal{D}(B)$-module algebra. This requires introducing a new $\mathcal{D}(B)$ action (which may be termed "heterotic" because it is obtained by combining, in a sense, a left and a right $\mathcal{D}(B)$ actions).

As is the case with the Drinfeld double $\mathcal{D}(B)$, the Heisenberg double $\mathcal{H}(B^*)$ turns out to be "slightly too big" for the correspondence with logarithmic models, but for the $2p^3$-dimensional quantum group $\overline{\mathfrak{u}}_q\mathfrak{sl}(2)$ at the $p$th root of unity dual to the $(p, 1)$ logarithmic conformal models, the corresponding $\mathcal{H}(B^*)$ nicely allows a "truncation" to a $2p^3$-dimensional $\overline{\mathfrak{u}}_q\mathfrak{sl}(2)$-module algebra.

We prove the general statement in Sec. 2 and detail the $\overline{\mathfrak{u}}_q\mathfrak{sl}(2)$ example in Sec. 3. The definition of the Drinfeld double is recalled in Appendix A. In Appendix B we collect some motivation coming from logarithmic conformal field theories.

2. $\mathcal{H}(B^*)$ as a $\mathcal{D}(B)$-Module Algebra

Let $B$ be a Hopf algebra. In this section, we make $\mathcal{H}(B^*)$ into a $\mathcal{D}(B)$-module algebra. For this, we combine two well-known $\mathcal{D}(B)$ actions, which can be taken from different sources, among which we prefer the beautiful paper \([40]\).

2.1. We use the "tickling" notation for the left and right regular actions: for a Hopf algebra $H$, its left and right regular actions on $H^*$ are respectively given by $h \rightarrow \beta = \beta(?h) = \langle \beta', h \rangle \beta'$ and $\beta \rightarrow h = \beta(h?)$, where $\beta \in H^*$ and $h \in H$. It follows that $H^*$ is an
$H$-bimodule under these actions (and $\langle , \rangle$ is the evaluation). We also have the left and right actions of $H^a$ on $H$, $\beta \mapsto a = \langle \beta , d'' \rangle d'$ and $a \mapsto \beta = \langle \beta , d \rangle d''$.

2.2. We recall that the Heisenberg double $\mathcal{H}(B^a)$ is the smash product $B^a \# B$ with respect to the left regular action of $B$ on $B^a$, which means that the composition in $\mathcal{H}(B^a)$ is given by

\[(\alpha \triangleleft a)(\beta \triangleleft b) = (\alpha(a \cdot \beta) \triangleleft a') \# a''b , \quad \alpha, \beta \in B^a, \quad a, b \in B.\]

We now describe the $\mathcal{D}(B)$ action on $\mathcal{H}(B^a)$ making it into a $\mathcal{D}(B)$-module algebra.

First, the $\mathcal{D}(B)$ action on $B^a$ — the first factor in $\mathcal{H}(B^a) = B^a \# B$ — is given by the restriction of the left regular action of $\mathcal{D}(B)$ on $\mathcal{D}(B)^a \cong B \otimes B^a$, which is

\[(\mu \otimes m) \rightarrow (a \otimes \alpha) = \langle \mu'' \cdot a \rangle \otimes \mu^m (m \rightarrow \alpha) S^{a^{-1}}(\mu').\]

Restricting this to $1 \otimes B^a$ gives

\[(\mu \otimes m) \rightarrow (a \otimes \alpha) = \mu''(m \rightarrow \alpha) S^{a^{-1}}(\mu'), \quad \mu \otimes m \in \mathcal{D}(B), \quad \alpha \in B^a,
\]

under which $B^a$ is an $R$-commutative $\mathcal{D}(B)$-module algebra [42] (also see [40]).

Second, the $\mathcal{D}(B)$ action on $B$ is obtained by restricting the right regular action of $\mathcal{D}(B)$ on $\mathcal{D}(B)^a \cong B \otimes B^a$ to $B \otimes e$ and using the antipode to convert it into a left action [44]. With the right regular action of $\mathcal{D}(B)$ on $\mathcal{D}(B)^a$ given by [41] [40]

\[(a \otimes \alpha) \leftarrow (\mu \otimes m) = S^{-1}(m''')(a \leftarrow \mu)m'' \otimes (\alpha \leftarrow m''),
\]

its restriction to $B$ is $a \leftarrow (\mu \otimes m) = S^{-1}(m''')(a \leftarrow \mu)m''$. Replacing $\mu \otimes m$ here with $S_2(\mu \otimes m) = (S(m'''') \rightarrow S^{a^{-1}}(\mu) \leftarrow S(m''')) \otimes S(m'')$, we readily calculate $a \leftarrow S_2(\mu \otimes m) = \langle S^{a^{-1}}(\mu), m''d'S(m''') \rangle S(m''') \# m''d''S(m''''),$ which defines the left action

\[(\mu \otimes m) \leftarrow a = (m''aS(m'''')) \rightarrow S^{a^{-1}}(\mu), \quad \mu \otimes m \in \mathcal{D}(B), \quad a \in B,
\]

under which $B$ is an $R$-commutative $\mathcal{D}(B)$-module algebra (also see [40]).

We now define a $\mathcal{D}(B)$ action on $\mathcal{H}(B^a)$, also denoted by $\triangleright$, simply by setting

\[(\mu \otimes m) \triangleright (\alpha \triangleleft a) = ((\mu \otimes m)' \rightarrow \alpha) \oplus ((\mu \otimes m)' \triangleright a),
\]

that is,

\[(\mu \otimes m) \triangleright (\alpha \triangleleft a) = \mu'''(m' \rightarrow \alpha) S^{a^{-1}}(\mu') \oplus (m''aS(m''')) \rightarrow S^{a^{-1}}(\mu'), \quad \mu \otimes m \in \mathcal{D}(B), \quad \alpha \triangleleft a \in \mathcal{H}(B^a),
\]

2 An algebra $A$ carrying an action of a quasitriangular Hopf algebra $H$ is called $R$-commutative, or quantum commutative [43] [40] if $ab = (R^{(2)}b)(R^{(1)}a)$ for all $a, b \in A$, where the dot denotes the action and $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ is the universal $R$-matrix.

3 The coproduct in (2.4) refers to $\mathcal{D}(B)$, and hence, in accordance with the Drinfeld double construction, $(\mu \otimes m)' \otimes (\mu \otimes m)'' = (\mu''' \otimes m') \otimes (\mu' \otimes m'')$, with the coproducts of $B^a$ and $B$ in the right-hand side.
and prove that $\mathcal{H}(B^e)$ is then a $\mathcal{D}(B)$-module algebra. Because each factor in $\mathcal{H}(B^e) = B^e \# B$ is already a $\mathcal{D}(B)$-module algebra, it suffices to show that
\[
((\mu \otimes m')_\varepsilon \triangleright (\varepsilon \# a))((\mu \otimes m)_\varepsilon \triangleright (\beta \# 1)) = (\mu \otimes m) \triangleright ((a' \# \beta) \# a'').
\]

We evaluate the left-hand side:
\[
((\mu'' \otimes m')_\varepsilon \triangleright (\varepsilon \# a))((\mu' \otimes m'')_\varepsilon \triangleright (\beta \# 1))
= ((\mu'' \otimes m')_\varepsilon \triangleright (\mu' \otimes m'' \triangleright \beta) \# m'')(a''')
(because $\Delta((\mu \otimes m) \triangleright a) = (m' \otimes S(m'' \triangleright S^{-1}(\mu)) \otimes m'' \triangleright S(m'''))
= (\mu'' \otimes m')_\varepsilon \triangleright (\beta' \# m'' \triangleright S^{-1}(\mu)) (m' \otimes m'' \triangleright \beta''', m'' \triangleright S(m'''))
(because $\Delta((\mu \otimes m) \triangleright a) = (a \otimes a) \triangleright (a' \# a'')(a'')$
= $\mu'' \otimes m' \triangleright (\beta' \# m'' \triangleright S^{-1}(\mu)) (m' \otimes m'' \triangleright \beta''', m'' \triangleright S(m'''))
= $\mu'' \otimes m' \triangleright (\beta' \# m'' \triangleright S^{-1}(\mu)) (m' \otimes m'' \triangleright \beta''', m'' \triangleright S(m'''))
$\mu'' \otimes m' \triangleright (\beta' \# m'' \triangleright S^{-1}(\mu)) (m' \otimes m'' \triangleright \beta''', m'' \triangleright S(m'''))
= \mu'' \otimes m' \triangleright (\beta' \# m'' \triangleright S^{-1}(\mu)) (m' \otimes m'' \triangleright (a' \# \beta) \# a'')
\]

which is the desired result.

2.3. Remark. As already noted, each of the subalgebras $B^e \otimes 1$ and $\varepsilon \otimes B$ in $\mathcal{H}(B^e)$ is known to be $R$-commutative with respect to the corresponding action (2.2) or (2.3) of $\mathcal{D}(B)$. But $\mathcal{H}(B^e)$ is not $R$-commutative with respect to the action in (2.4) in general: the $R$-commutativity axiom is satisfied for only “half” the cross-relations,
\[
(R \triangleright (\varepsilon \# b))(R \triangleright (\beta \# 1)) = (\alpha \# 1)(\varepsilon \# b) = \alpha \# b,
\]
but not for the other half: $(R \triangleright (\beta \# 1))(R \triangleright (\varepsilon \# a)) \neq (\varepsilon \# a)(\beta \# 1)$ in general.

3. The $(\overline{U}_q s\ell(2), \overline{F}_q s\ell(2))$ Pair

In this section, we consider the pair $(\mathcal{D}(B), \mathcal{H}(B^e))$ for the Taft Hopf algebra $B$ that underlies the Kazhdan–Lusztig correspondence with the $(p, 1)$ logarithmic conformal field theory models. By “truncation,” $\mathcal{D}(B)$ yields the $\overline{U}_q s\ell(2)$ quantum group that is Kazhdan–Lusztig-dual to the $(p, 1)$ logarithmic models. This quantum group first appeared in [45] and was rediscovered, together with its role in the Kazhdan–Lusztig correspondence, in [7]; its further properties were considered in [29, 46, 47, 48, 49] and, notably, recently
in [50] (also see [51] for a somewhat larger quantum group). We recall this in 3.1. We evaluate $H/D_4 \times B/D_5$ in 3.2, and in 3.3 “truncate” $(D(B), H(B^*))$ to a pair $(\mathcal{U}_q\mathfrak{sl}(2), \mathcal{U}_q\mathfrak{sl}(2))$, where $\mathcal{U}_q\mathfrak{sl}(2)$ is a $\mathcal{U}_q\mathfrak{sl}(2)$-module algebra. Its structure is detailed in 3.4.

3.1. $D(B)$ for the $4p^2$-dimensional Taft Hopf algebra $B$. For an integer $p \geq 2$, we set

$$q = e^{p^2}$$

and recall some of the results in [7].

3.1.1. The Taft Hopf algebra $B$. Let

$$B = \text{Span}(E^m k^n), \quad 0 \leq m \leq p - 1, \quad 0 \leq n \leq 4p - 1,$$

be the $4p^2$-dimensional Hopf algebra generated by $E$ and $k$ with the relations

$$kE = qEk, \quad E^p = 0, \quad k^{4p} = 1,$$

and with the comultiplication, counit, and antipode given by

$$\Delta(E) = 1 \otimes E + E \otimes k^2, \quad \Delta(k) = k \otimes k,$$

$$\epsilon(E) = 0, \quad \epsilon(k) = 1,$$

$$S(E) = -Ek^{-2}, \quad S(k) = k^{-1}.$$}

3.1.2. $B^a$ and $D(B)$. We next introduce elements $F, z \in B^a$ as

$$\langle F, E^m k^n \rangle = \delta_{m,1} \frac{q^{-n}}{q - q^{-1}}, \quad \langle z, E^m k^n \rangle = \delta_{m,0} q^{-n/2}.$$}

Then [7]

$$B^a = \text{Span}(F^a, z^b), \quad 0 \leq a \leq p - 1, \quad 0 \leq b \leq 4p - 1.$$}

Moreover, straightforward calculation shows [7] that the Drinfeld double $D(B)$ (see Appendix A) is the Hopf algebra generated by $E$, $F$, $k$, and $z$ with the relations given by

i) relations (3.2) in $B$,

ii) the relations

$$zF = qFz, \quad F^p = 0, \quad z^{4p} = 1$$

in $B^a$, and

iii) the cross-relations

$$kz = zk, \quad kFk^{-1} = q^{-1}F, \quad zEz^{-1} = q^{-1}E, \quad [E, F] = \frac{k^2 - z^2}{q - q^{-1}}.$$}

Here, in accordance with writing $D(B) = B^a \otimes B$, $E$ and $k$ are of course understood as $\epsilon \otimes E$ and $\epsilon \otimes k$, and $F$ and $z$ as $F \otimes 1$ and $z \otimes 1$. Then, for example, the last relation in (3.4) is to be rewritten as

$$[E, F] = \frac{k^2 - z^2}{q - q^{-1}}.$$
(ε ⊗ E)(F ⊗ 1) = F ⊗ E + \frac{1}{q - q^{-1}} ε ⊗ k^2 - \frac{1}{q - q^{-1}} x^2 ⊗ 1.

Dropping the ⊗ inside D(B) again, we have the Hopf-algebra structure (Δ_D, ε_D, S_D) given by (3.3) and

\begin{align*}
Δ_D(F) &= x^2 ⊗ F + F ⊗ 1, & Δ_D(ε) &= x ⊗ x, & ε_D(F) &= 0, & ε_D(ε) &= 1, \\
S_D(F) &= -x^{-2} F, & S_D(ε) &= x^{-1}
\end{align*}

(we reiterate that the coalgebra structure on D(B) is the direct product of those on B^cop and B).

3.2. The Heisenberg double \( H(B^a) \). For the above B, \( H(B^a) \) is spanned by

\begin{equation}
F^a x^b = E^c k^d, 
\end{equation}

where \( x^4 = 1, k^4 = 1, F^p = 0, \) and \( E^p = 0 \).

3.2.1. The composition law. To evaluate the product in \( H(B^a) \), defined in (2.1), we first write the left regular action of B on \( B^a \), \( b \mapsto β = β_0^a \langle β_1^a, b \rangle \):

\begin{equation}
E^m k^n \mapsto (F^a x^b) = \left[ \frac{m!}{a!} \right] q^{-(b+2a) \frac{n}{2} - m(a+b) + \frac{1}{2} m(m+1)} F^{a-m} x^b.
\end{equation}

It then follows that

\begin{equation}
(ε \neq E k^n)(F^a x^b \neq 1)
\end{equation}

\begin{align*}
&= \sum_{s \geq 0} q^{-\frac{1}{2}(s-1)} \left[ \frac{m!}{a!} \right] \left[ \frac{s!}{q - q^{-1}} \right] q^{-(b+2a) \frac{s}{2} + s(m-a-b)} F^{a-s} x^b \neq E^{m-s} k^{2s+n},
\end{align*}

(the sum is limited above by \( \min(m, a) \) due to the binomial coefficient vanishing). In particular,

\begin{equation}
(ε \neq E k^n)(F x^b \neq 1) = q^{-(b+2) \frac{n}{2}} F x^b \neq E k^n + \frac{1}{q - q^{-1}} q^{-(b+2) \frac{n}{2} - b} x^b \neq k^{n+2},
\end{equation}

and also \( (ε \neq k)(x \neq 1) = q^{-\frac{1}{2}} x \neq k, \) \( (ε \neq k)(F \neq 1) = q^{-1} F \neq k, \) and \( (ε \neq E)(x \neq 1) = x \neq E \). For the future reference, we write the general case, obtained from (3.7) immediately:

\begin{equation}
(F^r x^s \neq E^m k^n)(F^a x^b \neq E^c k^d)
\end{equation}

\begin{align*}
&= \sum_{a \geq 0} q^{-\frac{1}{2}(u-1)} \left[ \frac{m!}{u!} \right] \left[ \frac{u!}{q - q^{-1}} \right] q^{-\frac{1}{2} bn + cn + a(s-n) + u(2c-a-b+m-s)} \\
&\quad \times F^{a+r-u} x^{b+s} \neq E^{m+c-u} k^{n+d+2a}.
\end{align*}

(This is an associative product for generic q as well.)
3.2.2. The $\mathcal{D}(B)$ action. We next evaluate the $\mathcal{D}(B)$ action on $\mathcal{H}(B^\ast)$.

The $\mathcal{D}(B)$ action on $B^\ast$ in (2.2), rewritten in terms of the comultiplication and antipode of the double,

$$\langle \alpha'_{\mu_{D}^{}}, m \rangle | m \rangle \mu_{D}^{} \alpha_{D}^{\mu_{D}^{\prime}} (\mu_{D}^{\prime}), \quad \mu = F^{i} \kappa_{i}, \quad m = E^{m} k^{n},$$

factors into the action of $\varepsilon \otimes m$ in (3.6) times the action of $\mu \otimes 1$ given by

$$F^{i} \kappa_{i} \rightarrow (F^{\mu} \kappa^{\mu}) = q^{\frac{i}{2}(i+1-\mu)+a(i+j)}(1-q^{-1})^{j} \prod_{\ell=1}^{i} [\ell + a - 1 + \frac{b}{2}] F^{i+a} \kappa_{i}^{a}.$$

The $\mathcal{D}(B)$ action on $B$ in (2.3), $$(\mu \otimes m) \triangleright a = (m l a S(m'')) \rightarrow S_{\mu}^{\prime} (\mu),$$

with $\mu = F^{i} \kappa_{i}$ and $m = E^{m} k^{n}$, factors through the adjoint action of $\varepsilon \otimes m \in \varepsilon \otimes B$,

$$E^{m} k^{n} \triangleright (F^{\mu} \kappa^{\mu}) = q^{n^{2}+m(1-\mu+b)}(q-q^{-1})^{m} \prod_{\ell=1}^{m} [\ell - 1 + \frac{b}{2}] E^{a+m} k^{b-2m},$$

and the action of $\mu \otimes 1 \in B^{\ast} \otimes 1$, given by $\mu \triangleright a = \langle S_{\mu}^{\prime} (\mu), d' \rangle \triangleright d'$:

$$F^{i} \kappa_{i} \triangleright (F^{\mu} \kappa^{\mu}) = (-1)^{j} \prod_{i=1}^{[\mu]} \left[ q^{-1} \right]^{b} q^{-\frac{i}{2}(i+1)+a(j+a)} E^{a-i} k^{2i+b}.$$  

The action in (2.4) is therefore given by

$$E^{m} \triangleright (F^{\nu} \kappa^{\nu} \perp E^{c} k^{d}) = q^{m-\frac{3}{2}(m-1)} \sum_{s=0}^{m-1} (-1)^{s} q^{-\frac{2}{3}m^{2}+2m+2(2c-a-b)+\frac{1}{2}(m-s)} \left[ m \right] s \left[ s \right] \left[ s \right] ! \times \left( \prod_{\ell=1}^{m-s} [\ell - 1 + \frac{d}{2}] (q-q^{-1})^{m-2s} E^{a-s} k^{d-2m+2s},

\kappa \triangleright (F^{\nu} \kappa^{\nu} \perp E^{c} k^{d}) = q^{-\frac{2}{3}(m-1)} \left( F^{\nu} \kappa^{\nu} \perp E^{c} k^{d} \right),$$

$$\kappa \triangleright (F^{\nu} \kappa^{\nu} \perp E^{c} k^{d}) = q^{a+\frac{1}{3}} \left( F^{\nu} \kappa^{\nu} \perp E^{c} k^{d} \right),$$

$$F^{i} \triangleright (F^{\nu} \kappa^{\nu} \perp E^{c} k^{d}) = q^{\frac{1}{2}(i+1)} \left( -1 \right)^{s} q^{-s^{2}} \left[ s \right] \left[ s \right] \left[ s \right] ! q^{\frac{1}{2}(i-s)+a+i+as+sc} \times \left( \prod_{\ell=1}^{i-s} [\ell + a - 1 + \frac{b}{2}] (q-q^{-1})^{i-2s} F^{a+i-s} \kappa^{b} \perp E^{c-s} k^{d+2s},$$

3.3. From $\mathcal{D}(B)$ to $\mathcal{H}_{q} sl(2)$. The “truncation” whereby $\mathcal{D}(B)$ yields $\mathcal{H}_{q} sl(2)$ [7] consists of two steps: first, taking the quotient

$$(3.9) \quad \mathcal{D}(B) = \mathcal{D}(B)/(\kappa k - 1)$$

by the Hopf ideal generated by the central element $\kappa \otimes k - \varepsilon \otimes 1$ and, second, identifying $\mathcal{H}_{q} sl(2)$ as the subalgebra in $\mathcal{D}(B)$ spanned by $F^{\ell} E^{m} k^{2n}$ (tensor product omitted) with
from formulas for the antipode that $\tilde{\mathcal{U}}_q\mathfrak{sl}(2)$ is a Hopf algebra.}

In $\mathcal{H}(B^*)$, dually, we take a subalgebra and then a quotient, as follows.

First, dually to taking the quotient in (3.9), we identify the subspace $\mathcal{H}(B^*) \subset \mathcal{H}(B^*)$ on which $\mathfrak{z} \otimes k \in D(B)$ acts by unity. It follows from the above formulas for the $D(B)$ action that

$$\mathcal{H}(B^*) = \text{Span}(\Psi^{a,b,c}), \quad a, c = 0, \ldots, p - 1, \quad b \in \mathbb{Z}/(4p\mathbb{Z}),$$

$$\Psi^{a,b,c} = F^a \mathfrak{z}^b B^c k^{b-2c}.$$

Two nice properties immediately follow: from (3.8), $\mathcal{H}(B^*)$ is a subalgebra, and from (3.2) the $D(B)$ action restricts to $\mathcal{H}(B^*)$.

Second, dually to the restriction $\tilde{\mathcal{U}}_q\mathfrak{sl}(2) \subset \tilde{D}(B)$, we take a quotient of $\mathcal{H}(B^*)$. It follows from $k^2 \triangleright (F^a \mathfrak{z}^b B^c k^d) = q^{-2a-b+2c} F^a \mathfrak{z}^b B^c k^{d}$ that the eigenvalues of $(k^2)^b$ are not all different for $b \in \mathbb{Z}/(4p\mathbb{Z})$; we can impose the additional relation $\mathfrak{z}^{2p} \neq k^{2p} = 1$ in $\mathcal{H}(B^*)$ i.e., pass to the quotient by the relations

$$\Psi^{a,b+2p,c} = (-1)^b \Psi^{a,b,c}.$$

This defines the $2p^3$-dimensional algebra $\mathcal{H}_q\mathfrak{sl}(2)$, which is a $\mathcal{U}_q\mathfrak{sl}(2)$ module algebra.

### 3.4. The structure of $\mathcal{H}_q\mathfrak{sl}(2)$.

#### 3.4.1. Being a semisimple associative algebra, a Heisenberg double decomposes into matrix algebras. For our $\mathcal{H}(B^*)$, we choose the generators as $(\mathfrak{z}, \mathfrak{z}, \lambda, \partial)$, where $\mathfrak{z}$ is understood as $\mathfrak{z} \neq 1$ and we set

$$\mathfrak{z} = -(q-q^{-1})E \neq Ek^{-2},$$

$$\lambda = \mathfrak{z} \neq k,$$

$$\partial = (q-q^{-1})F \neq 1.$$

The relations in $\mathcal{H}(B^*)$ are then equivalent to

(3.10) \quad $\mathfrak{z}^{2p} = 1, \quad \lambda^{4p} = 1,$

(3.11) \quad $\mathfrak{z}^p = 0, \quad \partial^p = 0,$

(3.12) \quad $\mathfrak{z} \partial = (q-q^{-1}) 1 + q^{-2}z \partial,$

(3.13) \quad $\lambda \mathfrak{z} = \mathfrak{z} \lambda, \quad \lambda \partial = \partial \lambda.$

\footnote{It is actually a ribbon and (slightly stretching the definition) factorizable Hopf algebra [7, 29, 46] — the properties playing a crucial role in the Kazhdan–Lusztig correspondence.}

\footnote{The element $\Lambda = \mathfrak{z}^{2p} \neq k^{2p}$ is central in $\mathcal{H}(B^*)$, which suffices for our purposes, although it is not central in $\mathcal{H}(B^*)$, where $\Lambda F^a \mathfrak{z}^b B^c k^d = (-1)^b F^a \mathfrak{z}^b B^c k^d + 2p$ and $F^a \mathfrak{z}^b B^c k^d \Lambda = (-1)^d F^a \mathfrak{z}^b B^c k^d + 2p.$}
\[ \varepsilon z = q^{-1}z \varepsilon, \quad \varepsilon \lambda = q^{\frac{1}{2}} \lambda \varepsilon, \quad \varepsilon \partial = q \partial \varepsilon \]

(where the unity in (3.12) is of course \( \varepsilon \neq 1 \) in the detailed nomenclature used above).

Clearly, \( \lambda, z, \) and \( \partial \) generate a subalgebra, which is in fact \( \mathcal{H}(B^\sigma) \). Its quotient by \( \lambda^{2p} = 1 \) gives \( \overline{\mathcal{H}_qsl}(2) \). It follows that as an associative algebra,
\[
\overline{\mathcal{H}_qsl}(2) = \mathbb{C}_q[z, \partial] \otimes (\mathbb{C}[\lambda]/(\lambda^{2p} - 1)),
\]

where \( \mathbb{C}_q[z, \partial] \) is the \( p \)-dimensional algebra defined by relations (3.11) and (3.12). It is indeed isomorphic to the full matrix algebra \( \text{Mat}_p(\mathbb{C}) \) [49] (also see [52]); hence, \( \overline{\mathcal{H}_qsl}(2) \cong \text{Mat}_p(\mathbb{C}[\lambda]/(\lambda^{2p} - 1)) \).

The \( \overline{\mathcal{H}_qsl}(2) \) action on the new generators of \( \mathcal{H}(B^\sigma) \) is readily seen to be given by
\[
E \triangleright \varepsilon = 0, \quad k^2 \triangleright \varepsilon = q^{-1} \varepsilon, \quad F \triangleright \varepsilon = -q^{\frac{1}{q+1}} \partial \varepsilon, \quad E \triangleright \lambda^n = q^{-\frac{n}{2}}[\frac{n}{2}] \lambda^n z, \quad k^2 \triangleright \lambda^n = q^{-n} \lambda^n, \quad F \triangleright \lambda^n = -q^{\frac{n}{q+1}} \lambda^n \partial, \quad E \triangleright z^m = -q^m [m] z^{m+1}, \quad k^2 \triangleright z^m = q^m z^m, \quad F \triangleright z^m = [m] q^{1-m} z^{m-1},
\]
\[
E \triangleright \partial^n = q^{-n} [n] \partial^{n-1}, \quad k^2 \triangleright \partial^n = q^{-2n} \partial^n, \quad F \triangleright \partial^n = -q^{n} [n] \partial^{n+1}.
\]

As we have already noted (and as is very clearly seen now), the action restricts to \( \overline{\mathcal{H}(B^\sigma)} \) and then pushes forward to \( \overline{\mathcal{H}_qsl}(2) \). There, it restricts to the subalgebra \( \mathbb{C}_q[z, \partial] \), and the isomorphism
\[
\mathbb{C}_q[z, \partial] \cong \text{Mat}_p(\mathbb{C})
\]
is actually that of \( \overline{\mathcal{H}_qsl}(2) \)-module algebras [49].

\subsection*{3.4.2.}
Under the above action, \( \mathbb{C}_q[z, \partial] \) decomposes into indecomposable \( \overline{\mathcal{H}_qsl}(2) \) representations as [49]
\[
\mathbb{C}_q[z, \partial] = \mathcal{P}_1^+ \oplus \mathcal{P}_2^+ \oplus \cdots \oplus \mathcal{P}_v^+,
\]
where \( v = p - 1 \) if \( p \) is even and \( v = p \) if \( p \) is odd, and where \( \mathcal{P}_i^+ \) is the projective cover of the \( \overline{\mathcal{H}_qsl}(2) \) irreducible representation with weight \( q^{i-1} \) (in particular, \( \mathcal{P}_1^+ \) is the cover of the trivial representation; see [7] [29] for a detailed description). The \( 2p \)-dimensional projective module \( \mathcal{P}_1^+ \) in (3.15) has the remarkable structure
\[
\begin{align*}
\mathcal{P}_1^+ & \cong \mathcal{P}^{p-1} \cong \mathcal{P}^{p-2} \cong \cdots \cong \mathcal{P} \cong \partial^{p-1} \\
\end{align*}
\]

where the horizontal left–right arrows denote the action of \( E \) (to the left) and \( F \) (to the right) up to nonzero factors (and there are no maps inverse to the tilted arrows).
As regards all of $\mathfrak{H}_{q}\mathfrak{s}\ell(2)$, its decomposition into indecomposable $\mathfrak{H}_{q}\mathfrak{s}\ell(2)$ representations involves not just the “odd” projective modules as in (3.15) but actually all projective $\mathfrak{H}_{q}\mathfrak{s}\ell(2)$ modules with the multiplicity of each equal to the dimension of its irreducible quotient:

\[
(3.17) \quad \mathfrak{H}_{q}\mathfrak{s}\ell(2) = \bigoplus_{n=1}^{p} n\mathcal{P}^+_n \oplus \bigoplus_{n=1}^{p} n\mathcal{P}^-, \]

where $\mathcal{P}^+_r$ is the projective cover of the irreducible representation with weight $-q^{r-1}$. The multiplicities in (3.17) are identical to those in the regular representation decomposition. \[6\] The sum in (3.15) is nothing but the $\lambda$-independent subalgebra in $\mathfrak{H}_{q}\mathfrak{s}\ell(2)$.

Decomposition (3.17) follows by first noting the evident fact that the $\mathfrak{H}_{q}\mathfrak{s}\ell(2)$ action on $\mathfrak{H}_{q}\mathfrak{s}\ell(2)$ does not change the degree in $\lambda$, and then proceeding much as in [49]. For example, one of the two copies of $\mathcal{P}^+_2$ involved in (3.17) is given by

\[
(3.18) \quad t_+ \leftarrow t_- \quad \overset{E}{\leftarrow} \quad l_{p-2} \leftarrow \ldots \leftarrow l_1 \quad \overset{E}{\leftarrow} \quad r_1 \leftarrow \ldots \leftarrow r_{p-2} \quad \overset{F}{\leftarrow} \quad b_+ \leftarrow b_- \quad \overset{E}{\leftarrow}
\]

where

\[
t_+ = \frac{1}{q^i + 1} \sum_{i=1}^{p-2} \alpha_i C_i \lambda z^{i+1} \partial^i
\]

with

\[
\alpha_i = \sum_{j=1}^{i} \frac{q^j + q^{-j}}{[j - \frac{1}{2}]}, \quad C_i = q^i \prod_{n=1}^{i} \frac{[n - \frac{1}{2}]}{[n]},
\]

and

\[
l_1 = \frac{q^2}{q^2 + 1} \sum_{i=0}^{p-3} C_i \lambda z^{i+2} \partial^i, \quad b_+ = \sum_{i=0}^{p-2} C_i \lambda z^{i+1} \partial^i.
\]

This construction, being a linear-in-$\lambda$ analogue of (3.16), does not fully share its utmost simplicity, except possibly at one point: $l_{p-2}$ in (3.18) is proportional to $\lambda^{p-1}$; in the other copy of $\mathcal{P}^+_2$ in (3.17), linear in $\lambda^{-1}$, $r_{p-2}$ is proportional to $\lambda^{-1} \partial^{p-1}$.

We also note that the subspace of degree $p$ in $\lambda$ decomposes into the sum $\mathcal{P}^+_1 \oplus \mathcal{P}^+_3 \oplus \cdots \oplus \mathcal{P}^+_r$ of $\mathcal{P}^+_{2r+1}$ modules with multiplicities 1; because $\lambda^{2p} = 1$, there is the subalgebra

$\mathbb{C}_q[z, \partial] + \lambda^p \mathbb{C}_q[z, \partial] = \mathcal{P}^+_1 \oplus \mathcal{P}^+_3 \oplus \mathcal{P}^+_5 \oplus \cdots \oplus \mathcal{P}^+_p \oplus \mathcal{P}^-_r$.

Interestingly, the sum of projective modules with multiplicities in the right-hand side of (3.17) thus admits two different algebraic structures, one of which is actually a Hopf algebra and the other its module algebra.
of all “odd” projective modules in $\mathcal{F}_q sl(2)$.

### 3.4.3. We also recall from [49] that $C_q[z, \partial]$ extends to a differential $\mathcal{U}_q sl(2)$-module algebra $\Omega C_q[z, \partial]$ (a quantum de Rham complex of $C_q[z, \partial]$), which is the unital algebra with the generators $z$, $\partial$, $dz$, $d\partial$ and the relations (in addition to (3.11) and (3.12))

$$dzdz = 0, \quad d\partial d\partial = 0, \quad d\partial dz = -q^{-2}dzd\partial.$$  

$\Omega C_q$ is a differential $\mathcal{U}_q sl(2)$ module algebra, which is Kazhdan–Lusztig dual to the appropriate Drinfeld double, yielding the pair $\Omega C_q[z, \partial]$ and $\mathcal{F}_q$. (whence, in particular, $\Omega C_q$ and $\mathcal{F}_q$ are Kazhdan–Lusztig dual to the quantum group $\mathcal{F}_q$). The differential acting as

$$d(z) = dz, \quad d(\partial) = d\partial, \quad d(dz) = 0, \quad d(d\partial) = 0$$

(and $d(1) = 0$) commutes with the $\mathcal{U}_q sl(2)$ action if this is defined on $dz$ and $d\partial$ as

$$E \triangleright dz = -[2]z dz, \quad k^2 \triangleright dz = q^2 dz, \quad F \triangleright dz = 0, \quad E \triangleright d\partial = 0, \quad k^2 \triangleright d\partial = q^{-2}d\partial, \quad F \triangleright d\partial = -q^2[2]\partial d\partial,$$

and is then extended to all of $\Omega C_q[z, \partial]$ in accordance with the module algebra property.

In fact, the entire $\mathcal{F}_q sl(2)$ extends to a differential $\mathcal{U}_q sl(2)$-module algebra. Let $\Omega \mathcal{F}_q sl(2)$ be the algebra on $z$, $\partial$, $\lambda$, $dz$, $d\lambda$, and $d\partial$ with the relations given by those in $\Omega C_q[z, \partial]$ and $\mathcal{F}_q sl(2)$ and the following ones:

$$d(\lambda) = d\lambda, \quad d\lambda d\lambda = 0,$$

$$d\lambda \text{ commutes with } z \text{ and } \partial \text{ and anticommutes with } dz \text{ and } d\partial,$$

$$d\lambda \lambda = q^{-1}\lambda d\lambda \quad (\text{whence, in particular, } d(\lambda^2) = 0),$$

$$\lambda \text{ commutes with } dz \text{ and } d\partial.$$

Then $\Omega \mathcal{F}_q sl(2)$ endowed with the $\mathcal{U}_q sl(2)$ action

$$E \triangleright d\lambda = \frac{1}{q+1}(z d\lambda + \lambda dz), \quad k^2 \triangleright d\lambda = q^{-1}d\lambda, \quad F \triangleright d\lambda = -\frac{q}{q+1}(\partial d\lambda + \lambda d\partial)$$

is a differential $\mathcal{U}_q sl(2)$-module algebra.

### 4. Conclusion

We expect not only the Drinfeld double $\mathcal{D}(B)$ but also the pair $(\mathcal{D}(B), \mathcal{H}(B^*))$, with $\mathcal{H}(B^*)$ being a $\mathcal{D}(B)$-module algebra, to play a fundamental role on the quantum group side of the logarithmic Kazhdan–Lusztig duality. Based on the general recipe in Sec. [2], the contents of Sec. [3] must have a counterpart for the quantum group $\mathfrak{g}_{p, p'}$ that is Kazhdan–Lusztig-dual to the $(p, p')$ logarithmic conformal field models [27]; hopefully, a “truncation” of the appropriate Drinfeld double would also allow its dual version for the corresponding Heisenberg double, yielding the pair $(\mathfrak{g}_{p, p'}; \mathfrak{h}_{p, p'})$, where $\mathfrak{h}_{p, p'}$ is a $\mathfrak{g}_{p, p'}$-module algebra.
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Appendix A. Drinfeld Double

We recall that for a Hopf algebra $B$ with bijective antipode, its Drinfeld double $D(B)$ is $B^* \otimes B$ as a vector space, endowed with the structure of a quasitriangular Hopf algebra as follows. The coalgebra structure is that of $B^* \otimes B$, the algebra structure is

\begin{align}
(\mu \otimes m)(\nu \otimes n) &= \mu(m' \rightarrow \nu \leftarrow S^{-1}(m'')) \otimes m'n \\
\end{align}

for all $\mu, \nu \in B^*$ and $m, n \in B$, the antipode is given by

\begin{align}
S_D(\mu \otimes m) &= (\epsilon \otimes S(m))(S^*^{-1}(\mu) \otimes 1) = (S(m'') \rightarrow S^*^{-1}(\mu) \leftarrow m') \otimes S(n''),
\end{align}

and the universal $R$-matrix is

\begin{align}
R &= \sum_I (\epsilon \otimes e_I) \otimes (e^I \otimes 1),
\end{align}

where $\{e_I\}$ is a basis of $B$ and $\{e^I\}$ its dual basis in $B^*$.

Appendix B. LCFT Motivation

We proceed from the free-fermion description of the $(p, 1)$ logarithmic conformal field model. The starting point is the usual system of two free fermion fields $\xi(u)$ and
\( \eta(u) \) with the respective conformal weights 0 and 1, whose OPE is
\[
\xi(u) \eta(v) = \frac{1}{u-v}, \quad u, v \in \mathbb{C}.
\]
Virasoro generators with central charge \( c = -2 \) are the modes of the energy–momentum tensor
\[
T(u) = -\eta(u)d\xi(u) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},
\]
where \( d = \partial / \partial u \) and the normal-ordered product is understood in the right-hand side. It follows that two screenings — operators commuting with this Virasoro algebra — are given by
\[
E = \oint \eta = \eta_0
\]
(the “short” screening, which squares to zero for \( p = 2 \)) and
\[
f = \oint d\xi \xi
\]
(the “long” screening). The relevant complex of (Feigin–Fuchs) Virasoro modules is
\[
\text{(B.3)}
\]
where vertical arrows (directed towards submodules) indicate embedding of subquotients in Feigin–Fuchs modules. The picture continues to the left and to the right (and downward) indefinitely. The weight-2 fields \( d\eta \xi \) and \( d^2 \xi d\xi \) are the triplet algebra generators.

The algebra of fields is then extended by a field \( d^{-1} \eta(u) \) such that
\[
d^{-1} \eta(u) |
\]
\[
L_{-1} \eta(u)
\]
It is \( \delta(u) = d^{-1} \eta(u) \) and \( \xi(u) \) that are in fact the symplectic fermions \[^33\]. These weight-0 fields generate two standard first-order systems: our starting \((\eta(u), \xi(u))\) and \((\delta(u), d\xi(u))\) (cf. \[^29\]).

The fields \((\delta(u), \xi(u))\) allow constructing a logarithmic partner \( \Lambda(u) = \delta(u) \xi(u) \) of the identity operator; diagram \[(B.3)\] then extends such that the top level (split vertically
for visual clarity) becomes
\[(B.4)\]
\[
\begin{array}{c}
\delta(u) \xleftarrow{E} \xi(u) \xrightarrow{F} 1 \xleftarrow{E} \xi(u)
\end{array}
\]

There are also two characteristic diagrams of weight-1 fields. We recall that if the fermions are bosonized through a free bosonic field,
\[
\xi(u) = e^{\varphi(u)}, \quad \eta(u) = e^{-\varphi(u)}, \quad \eta(u)\xi(u) = -d\varphi(u),
\]
then the long-screening current (the “integrand” in \((B.2)\)) is \(e^{2\varphi}\) (which is a weight-1 field), and we have
\[(B.5)\]
\[
\begin{array}{c}
\delta(u) d\xi(u) \xrightarrow{F} e^{2\varphi(u)} \xleftarrow{E} \delta(u) d\xi(u)
\end{array}
\]

Similarly, there is an alternative bosonization through the scalar field introduced as \(d\phi(u) = \delta(u)d\xi(u)\). This gives the diagram
\[(B.6)\]
\[
\begin{array}{c}
\eta(u) \xrightarrow{E} \eta(u) \xleftarrow{F} e^{2\varphi(u)} \xrightarrow{E} \eta(u)
\end{array}
\]

(once again, \(\eta(u) = d\delta(u)\), which makes the two diagrams symmetric to each other).

The \((p = 2, 1)\) logarithmic model corresponds to \(q = \sqrt{-1}\) in \((3.1)\). Relations \((3.11)\) and \((3.12)\) in \(\mathbb{C}_q[z, \partial]\) are then indeed those mimicking free fermions:

\[
z^2 = 0, \quad \partial^2 = 0, \quad \partial z + z\partial = 2i.
\]

Based on the \(\mathfrak{U}_q\mathfrak{s}(2)\) symmetry, we conjecture that for general \(p, \mathbb{C}_q[z, \partial]\) similarly allows expressing the relations among the \((p, 1)\)-model fields “with an explicit quantum-group index.” Following [29], we call such fields parafermions (the term is somewhat overloaded by different meanings; its usage for fields transforming under a quantum group action goes back to \([53]\)).

On the quantum-group side, clearly, \((3.16)\) is the general-\(p\) counterpart of \((B.4)\) under the correspondence

\[
z^i \leftrightarrow \xi^{(i)}(u), \quad \partial^j \leftrightarrow \delta^{(j)}(u)
\]

for the \((p - 1)\)-component “parafermion” fields \(\delta^{(i)}(u)\) and \(\xi^{(i)}(u)\) generalizing the symplectic fermions \((\delta(u), \xi(u))\). The constituents of \((3.16)\) satisfy commutation relations generalizing the fermionic ones that occur for \(p = 2\): for general \(p\), we have

\[
\partial^j z^i = \sum_{\ell \geq 0} q^{-(2j-\ell)i+\ell j} \frac{\ell! j!}{\ell!}\frac{q - q^{-1}}{q - q^{-1}} \ell^j \ell^{-\ell}.
\]
Moreover, the counterparts of (B.5) and (B.6) for general $p$ are the diagrams that are easily established using (3.19), essentially by applying the differential to (3.16), with the resulting modules extended by the “cohomology corners” $z^{p-1}d_z$ and $\partial z^{p-1}d_\partial$:

\[
\sum_{i=1}^{p-1} \frac{1}{\partial_i} d(\partial_i) \quad \Longrightarrow \quad \partial^{p-1}d_\partial
\]

and

\[
\sum_{i=1}^{p-1} \frac{1}{\partial_i} d(z_i) \partial_i
\]

(as before, horizontal left–right arrows represent the action of $E$ and $F$ up to nonzero factors and tilted arrows have no inverse maps). The cohomology corners are Hopf-algebra counterparts of the screening currents in the two bosonizations, and the bottom elements, of the differentials $d\delta^{(1)}(u)$, $\ldots$, $d\delta^{(p-1)}(u)$ and $d\xi^{(p-1)}(u)$, $\ldots$, $d\xi^{(1)}(u)$.

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