SCALAR CURVATURE AND INTRINSIC FLAT CONVERGENCE

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ABSTRACT. Herein we present open problems and survey examples and theorems concerning sequences of Riemannian manifolds with uniform lower bounds on scalar curvature and their limit spaces. Examples of Gromov and of Ilmanen which naturally ought to have certain limit spaces do not converge with respect to smooth or Gromov-Hausdorff convergence. Thus we focus here on the notion of Intrinsic Flat convergence, developed jointly with Wenger. This notion has been applied successfully to study sequences that arise in General Relativity. Gromov has suggested it should be applied in other settings as well. We first review intrinsic flat convergence, its properties, and its compactness theorems, before presenting the applications and the open problems.

1. Introduction

Gromov proved that sequences of Riemannian manifolds with nonnegative sectional curvature have subsequences which converge in the Gromov-Hausdorff sense to Alexandrov spaces with nonnegative Alexandrov curvature [Gro81]. Burago-Gromov-Perelman proved that such spaces are rectifiable [BGP92]. Building upon Gromov’s Compactness Theorem, Cheeger-Colding proved that sequences of Riemannian manifolds with nonnegative Ricci curvature have subsequences which converge in the metric measure sense to metric measure spaces with generalized nonnegative Ricci curvature which are also rectifiable [CC97] [Gro81].

Sequences of manifolds with nonnegative scalar curvature need not have subsequences which converge in the Gromov-Hausdorff or metric measure sense. Gromov has suggested that perhaps under the right conditions a subsequence will converge in the intrinsic flat sense to a metric space with generalized nonnegative scalar curvature [Gro14b]. This is an open question: the notion of generalized scalar curvature has not yet been defined.

Intrinsic flat convergence was first defined by the author and Wenger in [SW11]. The limits obtained under this convergence are countably $H^m$ rectifiable metric spaces called integral current spaces. We review the definitions of these notions within this chapter along with various continuity and compactness theorems by the author, Perales, Portegies, Matsu, Munn [PS14][Per15b][Per15a][MP15][Mun14]. We also review applications of intrinsic flat convergence to study sequences of manifolds with nonnegative scalar curvature that arise in General Relativity by the author, Huang, Jauregui, Lee, LeFloch, and Stavrov [Jau16] [LS14] [LS15].

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We present many examples and state a number of open problems concerning limits of manifolds with nonnegative scalar curvature.

Recall that a Riemannian manifold, $M^n$, is endowed with a metric tensor, $g : T M \times T M \rightarrow \mathbb{R}$. One can then define lengths of curves and distances,

$$L(C) = \int_0^1 g(C', C')^{1/2} \, dt, \quad d(p, q) = \inf \{ L(C) : C(0) = p, \, C(1) = q \}.$$  

If $M$ is compact the distances are achieved as the lengths of curves called geodesics. Given any $p \in M$ and any vector $V \in T_p(M)$ there is a geodesic,

$$\gamma(t) = \exp_p(tV)$$

such that $\gamma(0) = p$ and $\gamma'(0) = V$.

Scalar curvature can also be computed using volumes of balls:

$$\text{Scalar}_p = \lim_{r \to 0} 6(m + 2) \left( \frac{\omega_m t^m - \text{Vol}(B(p, r))}{\omega_m r^{m+2}} \right).$$

In particular,

$$\text{Scalar}_p > 0 \iff \exists r > 0 \ s.t. \ \forall r < r_p \ \text{Vol}(B(p, r)) < \omega_m r^m.$$  

This control on volume is too local to apply to prove any global results. All properties of manifolds with lower bounds on their scalar curvature are built using the fact that curvature is defined using tensors as in (3). While it may be tempting to define generalized positive scalar curvature on a limit space using (6) it is unlikely to lead to any consequences because the limit spaces are not smooth and have no tensors. We need a stronger definition which implies (6) and other properties.

Schoen and Yau applied the three dimensional version of (3) to study minimal surfaces in manifolds with positive scalar curvature. They proved that a strictly stable closed minimal surface in a manifold with $\text{Scalar} \geq 0$ is diffeomorphic to a sphere in [SY79b]. In [SY79c], they applied minimal surface techniques to prove the Positive Mass Theorem: if $M^3$ is an asymptotically flat Riemannian manifold with nonnegative scalar curvature then $m_{\text{ADM}}(M^3) \geq 0$. They also proved the following Positive Mass Rigidity Theorem:

$$\text{Scalar} \geq 0 \quad \text{and} \quad m_{\text{ADM}}(M^3) = 0 \quad \implies M^3 \text{ is isometric to } \mathbb{R}^3.$$  

Here $\mathbb{R}^3$ is Euclidean space and the ADM mass is the limit of the Hawking masses of asymptotically expanding spheres $m_{\text{ADM}}(M) = \lim_{r \to \infty} m_H(\Sigma_r)$ where

$$m_H(\Sigma) = \sqrt{\frac{\text{Area}(\Sigma)}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma} H^2 \, d\sigma \right).$$
Geroch proved that if \( N_t : S^2 \rightarrow M^3 \) evolves by inverse mean curvature flow and \( M^3 \) has Scalar \( \geq 0 \) then then the Hawking mass, \( m_H(N_t) \), is nondecreasing. Huisken-Ilmanen introduced weak inverse mean curvature flow, proving it also satisfies Geroch monotonicity and \( \lim_{t \rightarrow \infty} m_H(N_t) = m_{ADM}(M) \). They applied this to prove the Penrose Inequality:

\[
m_{ADM}(M^3) \geq m_H(\partial M^3) = \sqrt{\frac{\text{Area}(\Sigma)}{16\pi}}
\]

when \( M^3 \) is asymptotically flat with a connected *outermost minimizing boundary* (e.g. \( \partial M \) is a minimal surface and there are no other closed minimal surfaces in \( M \)). Bray extended their result to have boundaries with more than one connected component in [Bra01]. In addition, there is the Penrose Rigidity Theorem:

\[
m_{ADM}(M^3) = m_H(\partial M^3) \implies M^3 \text{ is isometric to } M_{Sch,m}^3
\]

where \( M_{Sch,m}^3 \) is the Riemannian Schwarzschild space with mass \( m = m_{ADM}(M^3) \).

In addition to Hawking mass, there are other quasilocal masses defined on manifolds with Scalar \( \geq 0 \) including the Brown-York mass (which has nice properties proven by Shi-Tam in [ST02]) and the Bartnik mass [Bar89]. It is not a simple task to define and apply these quasilocal masses on limit spaces because they all involve the mean curvatures of surfaces. Perhaps more promising is Huisken’s new isoperimetric quasilocal mass of a region \( \Omega \subset M^3 \),

\[
m_{ISO}(\Omega) = \frac{2}{\text{Area}(\partial \Omega)} \left( \text{Vol}(\Omega) - \frac{\text{Area}(\partial \Omega)^{3/2}}{6\sqrt{\pi}} \right),
\]

and \( m_{ISO}(M) = \lim sup_{r \rightarrow \infty} m_{ISO}(\Omega_r) \) introduced in [Hui09]. Miao has proven that \( m_{ISO}(M) = m_{ADM}(M) \) using volume estimates of Fan-Shi-Tam in [FST09]. See also work of Jauregui, Lee, Carlotto, Chodosh, and Eichmair [JL16] [CCE16].

Gromov-Lawson applied the Lichnerowicz formula to prove many things (cf. [LM89]) including the Scalar Torus Rigidity Theorem [GL80a] in all dimensions:

\[(12) \text{ Scalar } \geq 2 \text{ and } \text{MinA}(M^3) = 4\pi \implies M^3 \text{ is isometric to } S^2 \times \mathbb{R} \]

Witten applied it to prove the Positive Mass Theorem for Spin manifolds [Wit81].

Hamilton’s Ricci flow leads to a precise control on the scalar curvature as well as the areas of minimal surfaces in the evolving manifolds (cf. [Ham95]). It was applied to prove the following rigidity theorems about minimal surfaces. Let

\[
\text{MinA}(M^3) = \inf \{ \text{Area}(\Sigma^2) : \Sigma^2 \text{ is a closed min surf in } M^3 \} \text{ and }
\]

\[
\text{MinA}_1(M^3) = \inf \{ \text{Area}(\Sigma^2) : \Sigma^2 \text{ is a min noncontr sphere in } M^3 \}.
\]

Note that these invariants are infinite if there are no qualifying minimal surfaces in \( M^3 \). Bray, Brendle, and Neves proved the Cover Splitting Rigidity Theorem:

\[(15) \text{ Scalar } \geq 2 \text{ and } \text{MinA}_1(M^3) = 4\pi \implies \tilde{M}^3 \text{ is isometric to } S^2 \times \mathbb{R} \]

where \( \tilde{M}^3 \) is the universal cover of \( M^3 \) [BBN10]. Bray, Brendle, Eichmair and Neves proved the \( \mathbb{RP}^3 \) Rigidity Theorem in [BBEN10]: If \( M^3 \) is diffeomorphic to \( \mathbb{RP}^3 \) then

\[(16) \text{ Scalar } \geq 6 \text{ and } \text{MinA}(M^3) = 2\pi \implies M^3 \text{ is isometric to } \mathbb{RP}^3. \]
They have in fact proven theorems which imply these two more simply stated theorems as corollaries.

Recall that a rigidity theorem has a statement in the following form:

\[(17) \quad M \text{ satisfies an hypothesis } \implies M \text{ isometric to } M_0.\]

The corresponding almost rigidity theorem (if it exists) would then be:

\[(18) \quad M \text{ almost satisfies an hypothesis } \implies M \text{ is close to } M_0.\]

The almost rigidity theorem can also be stated as follows:

\[(19) \quad M_j \text{ closer and closer to satisfying an hypothesis } \implies M_j \to M_0.\]

Within we describe conjectured almost rigidity theorems for each of the rigidity theorems described above. All of those conjectures remain open although some have been proven under additional hypothesis.

First one needs to define closeness for pairs of Riemannian manifolds and convergence of sequences of Riemannian manifolds. A pair of compact Riemannian manifolds, \(M_1\) and \(M_2\), may be mapped into a common metric space, \(Z\), via distance preserving maps \(\varphi_i: M_i \to Z\) which satisfy

\[(20) \quad d_Z(\varphi_j(x), \varphi_j(y)) = d_{M_j}(x, y) \quad x, y \in M_j.\]

Once they lie in a common metric space, \(Z\), then one may use the Hausdorff distance or the flat distance to measure the distance between the images with respect to the extrinsic space, \(Z\). We review these extrinsic distances which depend on both \(Z\) and the location of the \(M_j\) within \(Z\) in Section 3.

However, an intrinsic notion of distance between \(M_1\) and \(M_2\) can only depend on intrinsic data about these spaces and not on how they may be embedded into some extrinsic \(Z\). Thus Gromov defined his “intrinsic Hausdorff distance” in \(\text{[Gro81]}\), now known as the Gromov-Hausdorff distance, by taking the infimum over all distance preserving maps into arbitrary compact metric spaces, \(Z\), of the Hausdorff distance, \(d_{GH}\), between the images:

\[(21) \quad d_{GH}(M_1, M_2) = \inf_{Z, \varphi_i} \left\{ d_Z(\varphi_1(M_1), \varphi_2(M_2)) \middle| \varphi_i: M_i \to Z \right\}.\]

Many almost rigidity theorems have been proven for manifolds with nonnegative Ricci curvature using the Gromov-Hausdorff distance (cf. \(\text{[Col97]}\), \(\text{[CC96]}\) and \(\text{[Sor04]}\)).

For manifolds with nonnegative scalar curvature, one does not obtain Gromov-Hausdorff closeness in the almost rigidity theorems. Counterexamples will be described in Section 2. Gromov has suggested that intrinsic flat convergence may be more well suited towards proving an Almost Rigidity for the Torus Rigidity Theorem in \(\text{[Gro14a]}\). Indeed some progress has been made by the author, Lee, LeFloch, Huang, and Stavrov proving special cases of Almost Rigidity for the Positive Mass Theorem in \(\text{[LS14]}\), \(\text{[LS15]}\), \(\text{[HLS16]}\), \(\text{[SS16b]}\).

The intrinsic flat distance between compact oriented Riemannian manifolds was defined by the author with Wenger in \(\text{[SW11]}\) with an infimum over all distance
preserving maps into arbitrary complete metric spaces, $Z$, of the flat distance, $d^F_Z$, between the images:

$$d^F(M_1, M_2) = \inf_{Z, \varphi_i} \left\{ d^F_Z(\varphi_1\#[M_1], \varphi_2\#[M_2]) : \varphi_i : M_i \to Z \right\}. \tag{22}$$

Intuitively this distance is measuring the filling volume between the two spaces. One may also consider the intrinsic volume flat distance:

$$d_{\text{Vol}}^F(M_1, M_2) = d^F(M_1, M_2) + |\text{Vol}(M_1) - \text{Vol}(M_2)| \tag{23}$$

Full details about the intrinsic flat distance and limits obtained under intrinsic flat convergence are provided in Section 4 after a review of Ambrosio-Kirchheim theory in Section 3.

There are a few methods that can be applied to prove almost rigidity theorems. To apply the explicit control method one provides enough controls on the $M$ in (18) so that one can explicitly construct an embedding of $M$ and of $M_0$ into a common metric space and explicitly estimate the distance between them. This technique was applied to prove GH almost rigidity theorems by Colding in [Col97] and by Cheeger-Colding in [CC96]. It was also applied to prove the $F$ almost rigidity of the Positive Mass Theorem under additional hypothesis in joint work with Lee [LS14] and in joint work with Stavrov [SS16b]. Lakzian and the author have proven a theorem which provides such a construction and estimate if one can show $M$ and $M_0$ are close on large regions in [LS13]. See Section 5.

A second technique used to prove almost rigidity theorems is the compactness and weak rigidity method. One first provides enough controls on $M_j$ in (19) so that a subsequence converges to a limit space $M_\infty$. Then one proves the limit space satisfies the hypothesis in some weak sense. Finally one proves the rigidity theorem in that weak setting. This technique was applied by the author to prove a GH almost rigidity theorem in [Sor04] using Gromov’s Compactness Theorem.

Wenger’s Compactness Theorem [Wen11] states that

$$\text{Diam}(M_j) \leq D, \text{Vol}(M_j) \leq V, \text{Vol}(\partial M_j) \leq A \implies M_j \xrightarrow{F} M_\infty. \tag{25}$$

Huang, Lee and the author prove Almost Rigidity of the Positive Mass Theorem for graph manifolds using Wenger’s Compactness Theorem combined with an Arzela-Ascoli Theorem and a number of other theorems concerning intrinsic flat convergence in [HLS16]. We will review $F$ compactness and Arzela-Ascoli theorems in Section 6.

We begin with Section 2 surveying examples of sequences of manifolds with nonnegative scalar curvature. These examples reveal that one cannot simply use intrinsic flat convergence to handle all the problems that arise when trying to prove almost rigidity theorems involving nonnegative scalar curvature. There is a phenomenon called bubbling. One may also have tiny tunnels and construct sequences of manifolds through a process called sewing developed by the author with Basilio
in [BST16], which lead to limit spaces that do not even satisfy (5). These examples with bubbling and sewing have $\text{MinA}(M_j) \to 0$.

We next present the general theory of Intrinsic Flat convergence and Integral Current Spaces and survey the key theorems proven in this area. We begin with Section 3 by reviewing work of Federer-Flemming and Ambrosio-Kirchheim on integral currents in Euclidean space and metric spaces. In Section 4 we rigorously define Intrinsic Flat Convergence and Integral Current Spaces and survey known compactness theorems and proposed compactness theorems. In Section 5 we present various methods that may be used to estimate the intrinsic flat distance between two spaces and describe how these estimates have been used to prove almost rigidity theorems using the explicit control method. In Section 6 we present theorems about intrinsic flat convergence including theorems about disappearing and converging points, converging balls, semicontinuity theorems, Arzela-Ascoli Theorems and Intrinsic Flat Volume Convergence and mention how these results have been applied to prove almost rigidity theorems using the compactness and weak rigidity method.

We close with Section 7 which includes statements of conjectures, surveys of partial solutions to the conjectures and recommended related problems. We discuss the Almost Rigidity of the Positive Mass Theorem, the Bartnik Conjecture, the Almost Rigidity of the Scalar Torus Theorem, the Almost Rigidity of Rigidity Theorems proven using Ricci Flow, Gromov’s Prism Conjecture and the Regularity of Limit Spaces. Throughout one hopes to devise a generalized notion of nonnegative scalar curvature on limit spaces. Conjectures and problems are interspersed throughout the paper. If a reader is interested in studying any of these questions, please contact the author. More details can be provided and the author can coordinate the research of those working on these problems.

The author must apologize up front that there is no possible way to mention all the fundamentally important papers that have been written concerning manifolds with scalar curvature bounds. The results mentioned here have been selected because the author has read them and developed some idea as to how they may applied to study the intrinsic flat convergence of manifolds with nonnegative scalar curvature.

2. Examples with Positive Scalar Curvature

In this section we survey examples of sequences of three dimensional Riemannian manifolds, $M_j$, with lower scalar curvature bounds and describe their intrinsic flat limits. Many of these examples were found by mathematicians interested in applications to General Relativity. Manifolds with positive scalar curvature can be viewed as time symmetric spacelike slices of spacetime satisfying the positive energy condition. Such manifolds are curved by matter and can have gravity wells and/or black holes with horizons that are minimal surfaces.

We do not provide the explicit details or the proofs for these examples but instead provide references. We also propose new examples as open problems that
could be written up and published by an interested reader. We present these examples before presenting intrinsic flat convergence because they provide some intuitive understanding of intrinsic flat convergence and what may occur when one has a sequence of manifolds with nonnegative scalar curvature.

2.1. Examples with Wells. Arbitrarily thin arbitrarily deep wells can be constructed with positive scalar curvature. By the Positive Mass Theorem, one cannot attach such wells smoothly to Euclidean space. However they may be glued to spheres of constant positive sectional curvature. In fact, the Ilmanen Example, which initially inspired the definition of intrinsic flat convergence, consists of a sequence of spheres with increasingly many increasingly thin wells as in Figure 1. This sequence converges in the intrinsic flat sense to a standard sphere because a sphere with many thin holes and a standard sphere can be mapped into a common metric space, and the flat distance between them, which is intuitively a filling volume between them, will be very small.

Example 2.1. Lakzian has explicitly constructed sequences of spheres with one increasingly thin well in [Lak16]. He has also explicitly constructed the Ilmanen Example of sequences of spheres with increasingly many increasingly thin wells in the same paper. He proves both sequences converge to the standard sphere. He proves the sequence with one increasingly thin well converges in the GH sense to a sphere with a line segment attached and the Ilmanen example has no GH limit.

Example 2.2. In joint work with Lee, the author has constructed examples of asymptotically flat rotationally symmetric manifolds of positive scalar curvature with arbitrarily thin and arbitrarily deep wells and $m_{\text{ADM}}(M_j) \to 0$ [LS14]. They have no smooth limits and the the pointed GH limits of such examples are Euclidean spaces with line segments of arbitrary length attached. Lee and the author have also constructed sequences which are not rotationally symmetric that have $m_{\text{ADM}}(M_j) \to 0$ and increasingly many increasingly thin wells [LS12]. Such sequences have no GH converging subsequences because they have increasingly many disjoint balls [Gro81]. Thus almost rigidity of the positive mass theorem cannot be proven with GH or smooth convergence, only $F$ convergence.

Example 2.3. It is possible to construct $M_j$ with $S\text{calar} \geq -1/j$ that are diffeomorphic to tori and contain balls of radius $1/2$ that are isometric to balls in rescaled standard spheres. This will appear in work of the author with Basilio [BS16]. One may then attach an increasingly thin well of arbitrary depth to such $M_j$ that have positive scalar curvature. These examples would $F$ converge to a standard flat torus and would GH converge to a standard flat torus with a line segment attached. One may also attach increasingly many increasingly thin wells of arbitrary depth.
to the $M_j$ and still $F$ converge to a standard flat torus but there will be no GH limit of such a sequence. Thus one must use intrinsic flat convergence to prove almost rigidity for the Scalar Torus Theorem.

2.2. Tunnels and Bubbling. Gromov-Lawson and Schoen-Yau constructed tunnels diffeomorphic to $S^2 \times [0,1]$ with positive scalar curvature which attach smoothly on either end to the standard spheres [GL80b] [SY79a]. These tunnels may be arbitrarily thin and long or thin and short. At the center of the tunnel, there is a closed minimal surface diffeomorphic to a sphere. Sometimes these tunnels are called necks.

Example 2.4. Using these tunnels one may construct sequences of $M_j$ which consist of a pair of standard spheres joined by increasingly thin tunnels of length $L_j$. If $L_j \to 0$, then the GH and $F$ limit can be shown to be a pair of standard spheres joined at a point as in Figure 2. This effect is called bubbling. Note that in this example, $\text{MinA}(M_j) \to 0$.

Example 2.5. If $L_j \to L_\infty > 0$, then the GH limit is a pair of standard spheres joined by a line segment of length $L$ and the $F$ limit is just the pair of spheres without the line segment with the restricted distance from the GH limit. Examples similar to these are described by Wenger and the author in [SW11]. Notice that the $F$ limit is not geodesic.

Example 2.6. In fact one may add increasingly many bubbles with increasingly short and thin tunnels, and the sequence will have no GH limit and no $F$ limit. This does not contradict Wenger’s Compactness Theorem as in (25) because the volume is diverging to infinity even though the diameter is bounded and there is no boundary. Examples similar to these with many bubbles of various sizes and tunnels of various lengths converging to rectifiable limit spaces appear in [SW11].

Example 2.7. One may also have bubbling in asymptotically flat sequences of $M_j$ with $m_{\text{ADM}}(M_j) \to 0$ obtaining limits which are Euclidean planes with spheres attached. This can be done by attaching bubbles instead of wells to the sequences in Example 2.2. Such an example demonstrates that any almost rigidity theorem for the Positive Mass Theorem must somehow avoid bubbling. In joint work with Lee, we require that the manifolds have outward minimizing boundaries just as in the Penrose Inequality [LS14]. This is effectively cutting off the bubbles. One could alternately eliminate bubbling by requiring the sequence to have a uniform lower bound on the area of the smallest closed minimal surface, $\text{MinA}(M_j) \geq A_0 > 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{bubble.png}
\caption{Bubbling. Image owned by the author.}
\end{figure}
Example 2.8. One may add a bubble to each $M_j$ of Example 2.3 with $\text{Scalar}_j \geq -1/j$ that are diffeomorphic to tori and contain balls of constant sectional curvature isometric to balls in rescaled standard spheres. Such sequences would $\mathcal{F}$ converge to standard flat tori with a sphere of arbitrary radius attached at a point. This will appear in work of the author with Basilio [BS16]. One could eliminate such examples by requiring the sequence have a uniform lower bound on the area of the smallest closed minimal surface, $\text{MinA}(M_j) \geq A_0 > 0$. One cannot require that there are no closed minimal surfaces here since the manifolds are diffeomorphic to tori.

2.3. Cancellation and Doubling. The next two examples are described by the author and Wenger in [SW10] [SW11]. Intrinsic flat limit spaces may be the 0 or a rectifiable space with integer weight.

Example 2.9. There are sequences of manifolds $M_j^3$ with positive scalar curvature which have a $\mathcal{F}$ limit which is the 0 space, while converging in the GH sense to a standard three sphere. This cancelling sequence can be constructed with positive scalar curvature by taking a pair of standard three spheres and connecting them by increasingly dense increasingly small tunnels. These sequences converge to the 0 space because their filling volumes converge to 0. In fact they are the totally geodesic boundaries of four dimensional manifolds whose volume converges to 0.

Example 2.10. If in the previous example all the tunnels are cut and glued back together with reversed orientation, then the GH limit is still a standard three sphere and the $\mathcal{F}$ limit is a sphere with weight two everywhere.

2.4. Sewing Manifolds. In upcoming joint doctoral work of Basilio with the author, the notion of sewing Riemannian manifolds is introduced [BS16]. One starts with a three dimensional manifold, $M$, that contains a curve, $C: [0,1] \rightarrow M$, such that a tubular neighborhood around the curve has constant positive sectional curvature. One then creates a sequence of manifolds sewn along this curve. That is short thin tunnels are attached along the curve pulling the points on the curve closer together. The GH and $\mathcal{F}$ limit of such a sequence is then the original manifold with a pulled thread along $C$. That is, all the points in the image of $C$ have been identified. One can also sew entire regions with constant positive sectional curvature to obtain sequences converging to the original manifold with the entire region identified as a single point. If the original manifold has positive scalar curvature, then so does the sequence. In addition one may consider sequences of $M_j$ and sew along curves or in regions of those $M_j$. Using this construction, Basilio and the author construct the following examples.

Example 2.11. If one takes the $M_j$ of Example 2.3 one may sew along curves lying in the balls of radius $1/2$ that have constant sectional curvature to obtain a sequence of manifolds, $M'_j$ with $\text{Scalar}_j \geq -1/j$ that are no longer tori but converge to a limit which is the standard flat torus with a contractible circle pulled to a point. Or a contractible sphere pulled to a point. Or a ball of radius $1/2$ pulled to a point. These examples demonstrate that limits of manifolds with $\text{Scalar}_j \geq -1/j$
may fail to have generalized nonnegative scalar curvature in the sense that limit in \(5\) fails to be nonnegative. These limits can be biLipschitz to tori and still not be isometric to a flat tori. Like all examples created with this sewing construction, \(\text{MinA}(M_j) \to 0\).

3. Integral Currents on a Metric Space

Before we can rigorously define intrinsic flat convergence and describe the limit spaces obtained under intrinsic flat convergence, we need to review Ambrosio and Kirchheim’s notion of currents and convergence of currents on a complete metric space \([AK00]\). Note that like Federer-Fleming’s earlier work on the flat and weak convergence of submanifolds viewed as currents in Euclidean space, the flat and weak convergence of Ambrosio-Kirchheim’s currents are extrinsic notions of convergence, depending very much on the way in which the submanifold or current lies within an extrinsic space.

3.1. Federer-Fleming currents on Euclidean Space. In \([FF60]\) Federer and Fleming first introduced the notion of a current on Euclidean space as a generalization of the notion of an oriented submanifold, \(\varphi : M \to \mathbb{E}^N\), which views \(M\) as a linear functional, \(T = \left[ M \right]\), on differential forms:

\[
T(\omega) = \left[ M \right]\omega = \int_M \varphi^* \omega.
\]

In particular

\[
T(f \,d\pi_1 \wedge \cdots \wedge \,d\pi_m) = \int_M (f \circ \varphi_i) \,d(\pi_1 \circ \varphi_i) \wedge \cdots \wedge \,d(\pi_m \circ \varphi_m).
\]

Observe that this is perfectly well defined when \(\varphi : M \to \mathbb{E}^N\) is only Lipschitz. This linear functional captures the notion of boundary,

\[
\partial T(\omega) = \int_{\partial M} \omega = \int_M d\omega = T(d\omega).
\]

So that

\[
\partial T(f \,d\pi_1 \wedge \cdots \wedge \,d\pi_{m-1}) = T(1 \,d\pi_1 \wedge \cdots \wedge \,d\pi_{m-1}).
\]

Federer and Fleming then studied sequences of submanifolds by considering the weak limits of their corresponding linear functionals. They applied this to study the Plateau Problem: searching for the submanifold of smallest area with a given boundary. They proved sequences of submanifolds approaching the smallest area converge in the weak sense to a limit which they called an integral current.
3.2. **Ambrosio-Kirchheim Integer Rectifiable Currents.** In [AK00], Ambrosio and Kirchheim defined currents on Euclidean space to integral currents on any complete metric space, $Z$. In Federer-Fleming, currents were defined as linear functionals on differential forms [FF60]. Since there are no differential forms on a metric space, Ambrosio and Kirchheim’s currents are multilinear functionals which act on DiGiorgi’s $m+1$ tuples [DeG95]. A tuple $(f, \pi_1, \ldots, \pi_m)$ is in $\mathcal{D}^m(Z)$ iff $f: Z \to \mathbb{R}$ is a bounded Lipschitz function and $\pi_i: Z \to \mathbb{R}$ are Lipschitz. These tuples have no antisymmetry properties.

In [AK00] Ambrosio-Kirchheim began their work by defining currents. As we do not need the notion of a current in this paper, we jump directly to their notion of an integer rectifiable current applying their Theorems 9.1 and 9.5 as an explanation rather than using their definition.

A linear functional $T: \mathcal{D}^m(Z) \to \mathbb{R}$ is an $m$ dimensional integer rectifiable current, denoted $T \in I^m(Z)$ if and only if it can be parametrized as follows

$$T(f, \pi_1, \ldots, \pi_m) = \sum_{i=1}^{\infty} \left( \theta_i \int_{A_i} (f \circ \varphi_i) d(\pi_1 \circ \varphi_i) \land \cdots \land d(\pi_m \circ \varphi_i) \right)$$

where $\theta_i \in \mathbb{Z}$ and $\varphi_i: A_i \to \varphi_i(A_i) \subset Z$ are biLipschitz maps defined on precompact Borel measurable sets, $A_i \subset \mathbb{R}^m$, with pairwise disjoint images such that

$$\sum_{i=1}^{\infty} |\theta_i| \mathcal{H}_m(\varphi_i(A_i)) < \infty \quad \text{where } \mathcal{H}_m \text{ is the Hausdorff measure.}$$

A 0 dimensional integer rectifiable current can be parametrized by a finite collection of distinct weighted points

$$T(f) = \sum_{i=1}^{N} \theta_i f(p_i) \text{ where } \theta_i \in \mathbb{Z} \text{ and } p_i \in Z.$$

Observe that we then have the following antisymmetry property,

$$T(f, \pi_1, \ldots, \pi_m) = \text{sgn}(\sigma) T(f, \pi_{\sigma(1)}, \ldots, \pi_{\sigma(m)})$$

for any permutation $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$. In addition, $T(f, \pi_1, \ldots, \pi_m) = 0$ if $f$ is the zero function or one of the $\pi_i$ is constant. So while the tuples do not have the properties of differential forms, the action of the integer rectifiable currents on the tuples has these properties.

Ambrosio-Kirchheim’s mass measure $\|T\|$ of a current $T$, is the smallest Borel measure, $\mu$, such that

$$|T(f, \pi)| \leq \int_X |f| d\mu \quad \forall (f, \pi) \text{ where } \text{Lip}(\pi_i) \leq 1.$$

In Theorem 9.5 of [AK00], the mass measure is explicitly computed. For the purposes of this paper we need only the following consequence of their theorem:

$$m^{-m/2} H_T(A) \leq \|T\|(A) \leq \frac{2^m}{\omega_m} H_T(A)$$
where

\[ H_T(A) = \sum_{i=1}^{\infty} |\theta_i| \mathcal{H}_m(\varphi_i (A_i) \cap A). \]

Furthermore the mass measure of a 0 dimensional integer rectifiable current satisfies

\[ \|T\|(A) = \sum_{\rho_i \in A} |\theta_i|. \]

The Ambrosio-Kirchheim mass of \( T \) is defined

\[ M(T) = \|T\| (Z). \]

By the definition of the Ambrosio-Kirchheim mass we have

\[ T(f, \pi_1, ..., \pi_m) \leq \sup |f| \prod_{i=1}^{m} \text{Lip}(\pi_i) M(T). \]

The restriction of a current \( T \) by a \( k+1 \) tuple \( \omega = (g, \tau_1, ..., \tau_k) \in \mathcal{D}^k(Z) \) with \( k < m \) is defined by

\[ (T \sqcap \omega)(f, \pi_1, ..., \pi_m) := T(f \cdot g, \tau_1, ..., \tau_k, \pi_1, ..., \pi_m). \]

Given a Borel set, \( A \),

\[ T \sqcap A(f, \pi_1, ..., \pi_m) := T(1_A \cdot f, \pi_1, ..., \pi_m) \]

where \( 1_A \) is the indicator function of the set. Observe that \( T \sqcap \omega \) is an integer rectifiable current of dimension \( m - k \) and that

\[ M(T \sqcap \omega) = \|T\|(A). \]

Given a Lipschitz map, \( \varphi : Z \rightarrow Z' \), the push forward of a current \( T \) on \( Z \) to a current \( \varphi_\#T \) on \( Z' \) is given by

\[ \varphi_\#T(f, \pi_1, ..., \pi_m) := T(f \circ \varphi, \varphi_1 \circ \varphi, ..., \varphi_m \circ \varphi) \]

which is clearly still an integer rectifiable current. Observe that

\[ (\varphi_\#T) \sqcap (f, \pi_1, ..., \pi_k) = \varphi_\#(T \sqcap (f \circ \varphi, \varphi_1 \circ \varphi, ..., \varphi_k \circ \varphi)) \]

and

\[ (\varphi_\#T) \sqcap A = (\varphi_\#T) \sqcap (1_A) = \varphi_\#(T \sqcap (1_A \circ \varphi)) = \varphi_\#(T \sqcap \varphi^{-1}(A)). \]

In (2.4) of [AK00], Ambrosio-Kirchheim show that

\[ \|\varphi_\#T\| \leq \text{Lip}(\varphi)^m |\varphi_\#||T||, \]

so that when \( \varphi \) is an isometric embedding

\[ \|\varphi_\#T\| = \varphi_\#||T|| \quad \text{and} \quad M(T) = M(\varphi_\#T). \]

In [AK00][Theorem 4.6] Ambrosio-Kirchheim define the (canonical) set of a current, \( T \), as the collection of points in \( Z \) with positive lower density:

\[ \text{set}(T) = \{ p \in Z : \Theta_{\ast m} (||T||, p) > 0 \}, \]
where the definition of lower density is

\[
\Theta_{sm}(\mu, p) = \lim inf_{r \to 0} \frac{\mu(B_p(r))}{\omega_m r^m}.
\]

When \( T \) is an integer rectifiable current then \( \text{set}(T) \) is countably \( \mathcal{H}^m \) rectifiable, which means there exists a collection of biLipschitz maps, \( \varphi_i : A'_i \to \text{set}(T) \subset Z \), defined on Borel sets \( A'_i \subset \mathbb{R}^m \) such that

\[
\mathcal{H}_m \left( \text{set}(T) \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i) \right) = 0.
\]

These \( \varphi_i \) can be taken from the parametrization of \( T \) with \( A'_i \subset A_i \subset \bar{A}_i \).

3.3. Ambrosio-Kirchheim Integral Currents. The boundary of \( T \) is defined

\[
\partial T(f, \pi_1, ..., \pi_{m-1}) := T(1, f, \pi_1, ..., \pi_{m-1}).
\]

Note that \( \varphi_B(\partial T) = \partial(\varphi_B T) \) and it can easily be shown that \( \partial \partial T = 0 \). The boundary of an integer rectifiable current is not necessarily an integer rectifiable current.

An integer rectifiable current \( T \in \mathcal{I}_m(Z) \) is an integral current, denoted \( T \in \mathcal{I}_m(Z) \), if \( \partial T \) is an integer rectifiable current. This includes the zero current

\[
0(f, \pi_1, ..., \pi_m) := 0 \text{ with } \partial 0(f, \pi_1, ..., \pi_{m-1}) = 0(1, f, \pi_1, ..., \pi_{m-1}) = 0.
\]

Note that Ambrosio-Kirchheim define an integral current as an integer rectifiable current whose boundary has finite mass and the more easily applied statement we have here is their Theorem 8.6 in [AK00].

Given an oriented Riemannian manifold with boundary, \( M^m \), such that \( \text{Vol}_m(M) < \infty \) and \( \text{Vol}_{m-1}(\partial M) < \infty \), and given a Lipschitz map \( \varphi : M \to Z \), we can define an integral current \( \varphi_B[M] \in \mathcal{I}_m(Z) \) as follows

\[
\varphi_B[M](f, \pi_1, ..., \pi_m) = \int_M (f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi).
\]

Note that \( \partial \varphi_B[M] = \varphi_B[\partial M] \) where \( \partial M \) is the boundary of \( M \) and

\[
\mathcal{M}(\varphi_B[M]) = \text{Vol}_m(\varphi(M)).
\]

If \( \text{Vol}_m(M) < \infty \) and \( \text{Vol}_{m-1}(\partial M) = \infty \), then \([M]\) is only integer rectifiable and not integral.

Whenever, \( T \) is an integer current, \( \partial \partial T = 0 \), and

\[
\partial : \mathcal{I}_m(Z) \to \mathcal{I}_{m-1}(Z).
\]

In addition, if \( \varphi : Z_1 \to Z_2 \) is Lipschitz, then by \( \overset{43}{\text{(43)}} \)

\[
\varphi_B : \mathcal{I}_m(Z_1) \to \mathcal{I}_m(Z_2).
\]

The restriction of an integral current defined in \( \overset{40}{\text{(40)}} \) need not be an integral current. However, the Ambrosio-Kirchheim Slicing Theorem implies that

\[
T \subseteq B(p, r) \text{ is an integral current for almost every } r > 0
\]

where \( B(p, r) = \{ x : d(x, p) < r \} \).
3.4. **Convergence of Currents in a Metric Space.** In Definition 3.6 of [AK00], Ambrosio and Kirchheim state that a sequence of integral currents $T_j \in \mathcal{I}_m(Z)$ lying in a complete metric space, $Z$, is said to converge weakly to a current $T$, denoted $T_j \to T$, iff the pointwise limits satisfy
\[
\lim_{j \to \infty} T_j(f, \pi_1, ..., \pi_m) = T(f, \pi_1, ..., \pi_m)
\]
for all bounded Lipschitz $f : Z \to \mathbb{R}$ and Lipschitz $\pi_i : Z \to \mathbb{R}$. Ambrosio-Kirchheim next observe that if $T_j$ converges weakly to $T$, then the boundaries converge
\[
\partial T_j \to \partial T,
\]
and the mass is lower semicontinuous
\[
\liminf_{j \to \infty} M(T_j) \geq M(T).
\]
Thus the weak limit of a sequence of integer rectifiable currents with a uniform upper bound on mass is an integer rectifiable current:
\[
T_j \in \mathcal{I}_m(Z), \quad M(T_j) \leq V_0 \quad \text{and} \quad T_j \to T \quad \implies \quad T \in \mathcal{I}_m(Z).
\]
Similarly for integral currents we have
\[
T_j \in \mathcal{I}_m(Z), \quad M(T_j) \leq V_0, \quad M(\partial T_j) \leq A_0 \quad \text{and} \quad T_j \to T \quad \implies \quad T \in \mathcal{I}_m(Z).
\]
For any open set, $A \subset Z$, if $T_j \to T$ then
\[
\liminf_{j \to \infty} \|T_j\|(A) \geq \|T\|(A).
\]
However $T_j \subset A$ need not converge weakly to $T \subset A$ (cf. Example 2.21 of [Sor14]).

Ambrosio-Kirchheim prove the following compactness theorem:

**Theorem 3.1.** [AK00] Given any complete metric space $Z$, a compact set $K \subset Z$ and $A_0, V_0 > 0$. Given any sequence of integral currents $T_j \in \mathcal{I}_m(Z)$ satisfying
\[
M(T_j) \leq V_0, \quad M(\partial T_j) \leq A_0 \quad \text{and} \quad \text{set}(T_j) \subset K,
\]
there exists a subsequence, $T_{j_i}$, which converges weakly to $T \in \mathcal{I}_m(Z)$.

It is possible that the limit obtained in this theorem is the 0 integral current. Observe that whenever the sequence of currents is **collapsing**, we have
\[
M(T_j) \to 0,
\]
then by (39) we have
\[
\|T_j(f, \pi_1, ..., \pi_m)\| \leq \sup \|f\| \prod_{i=1}^m \text{Lip}(\pi_i) M(T_j) \to 0
\]
and so $T_j$ converges weakly to 0.

It is also possible for $T_j$ to converge weakly to 0 without collapsing. This can occur due to **cancellation**, when the $T_j$ fold over on themselves as in Example 3.2.

We include this example in detail because it inspires the notion of flat convergence and will be referred to repeatedly in this paper.
Example 3.2. Let $T_j = \varphi_{j\#}[M] \subset I_2(\mathbb{R}^3)$ where
\begin{equation}
\varphi_j(s, t) = (s, t/j, |t|b_j/j) \text{ where } b_j = \sqrt{j^2 - 1}
\end{equation}
on $M = \{(s, t) : s \in [-1, 1], t \in [-1, 1]\}$. Since $M(\varphi_{j\#}[M]) = \text{Vol}(M)$ does not converge to $0$, this sequence is not collapsing. Observe that $T_j = A_j + \partial B_j$ where
\begin{equation}
B_j = \{((x, y, z) : |x| \leq 1, |y| \leq 1/j, z \in [\|y\|b_j, b_j/j])\} \in I_3(\mathbb{R}^3) \text{ and }
\end{equation}
\begin{equation}
A_j = A_j^- + A_j^+ + A_j^0 \in I_2(\mathbb{R}^3) \text{ where }
\end{equation}
\begin{align}
A_j^- &= -\{((-1, y, z) : y \in [-1/j, 1/j], z \in [\|y\|b_j, b_j/j])\} \\
A_j^+ &= \{(+1, y, z) : y \in [-1/j, 1/j], z \in [\|y\|b_j, b_j/j])\} \\
A_j^0 &= \{((x, y, b_j/j) : x \in [-1, 1], y \in [-1/j, 1/j])\}.
\end{align}
\begin{equation}
\text{Since }
\end{equation}
\begin{equation}
M(B_j) \leq (4/j) \text{ and } M(A_j) \leq (2/j) + (2/j) + 4/j,
\end{equation}
we have $B_j \to 0$ and $A_j \to 0$. By (59) we have $\partial B_j \to \partial 0 = 0$, and thus
\begin{equation}
T_j = A_j + \partial B_j \to 0.
\end{equation}
Sometimes part of a sequence disappears under weak convergence and part remains. This happens in the following example:

Example 3.3. Let $T_j = \varphi_{j\#}[D^2] \subset I_2(\mathbb{R}^3)$ with $D^2 = \{(x, y) : x^2 + y^2 \leq 1\}$ and
\begin{equation}
\varphi_{j\#}(x, y) = (x, y, f_j(\sqrt{x^2 + y^2})),
\end{equation}
where $f_j : [0, 1] \to [0, 1]$ is a smooth cutoff function such that $f_j(r) = 1$ near $r = 0$ and $f_j(r) = 0$ for $r \geq 1/j$. Then $\partial T_j = \varphi_{j\#}[S^1]$ is constant and so the sequence does not disappear. In fact $T_j$ converges weakly to $T_\infty = \varphi_{\infty\#}[D^2]$ where
\begin{equation}
\varphi_{\infty\#}(x, y) = (x, y, 0)
\end{equation}
since $T_j - T_\infty = \partial B_j$ where
\begin{equation}
B_j = \{(x, y, z) : x^2 + y^2 \leq 1, 0 \leq z \leq f_j(\sqrt{x^2 + y^2})\}.
\end{equation}
\begin{equation}
\text{Since }
\end{equation}
\begin{equation}
M(B_j) \leq \pi(1/j)^2 \to 0
\end{equation}
we have $B_j \to 0$ and thus $T_j - T_\infty \to 0$ and $T_j \to T_\infty$. 

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3.5. The Flat distance vs the Hausdorff distance. In [Wen07], Wenger defines the flat distance between two integral currents, $T_1, T_2 \in I_m(Z)$, lying in a common complete metric space, $Z$, to be

$$d^F_Z(T_1, T_2) = \inf \{ M(A) + M(B) : A + \partial B = T_1 - T_2 \}$$

(79)

where the infimum is taken over all $A \in I_m(Z)$ and $B \in I_{m+1}(Z)$ such that $A + \partial B = T_1 - T_2$. This is the same definition given by Federer and Fleming in [FF60] building on work of Whitney [Whi57] for the flat distance in Euclidean space, where it is a norm, $|T_1 - T_2|$. The lack of scaling in (80) is a result of setting the flat distance to be a norm on Euclidean space. A scalable version of the flat distance might be defined for $Z$ with a finite diameter $\text{Diam}(Z) = D$ as follows (cf. [LS15]):

$$d^2_{DF}(T_1, T_2) = \inf \{ DM(A) + M(B) : A + \partial B = T_1 - T_2 \}$$

(80)

Observe that if two oriented hypersurfaces share a boundary, then the flat distance between them is bounded above by the volume between them. In Example 3.3, we have

$$d^3_F(T_j, T_\infty) \to 0$$

(81)

by taking $B_j$ as in (77) so that $T_j - T_\infty = \partial B_j$. Then $d^2_F(T_j, 0) \leq M(B_j)$ which converges to 0 as $j \to \infty$ by (78). In Example 3.2, we produce a sequence of integral currents $T_j$ in Euclidean space such that

$$d^3_F(T_j, 0) \to 0$$

(82)

by taking $B_j$ as in (68) and $A_j$ as in (69) so that $T_j - 0 = \partial B_j + A_j$. Then $d^2_F(T_j, 0) \leq M(A_j) + M(B_j)$ which converges to 0 as $j \to \infty$ by (73).

In [Wen07], Wenger proves that when

$$\text{M}(T_j) \leq V_0 \text{ and } M(\partial T) \leq A_0$$

then weak and flat convergence are equivalent:

$$T_j \to T \text{ if and only if } d^2_F(T_j, T) \to 0.$$  

(84)

One should contrast the flat distance between submanifolds viewed as integral currents with the Hausdorff distance between submanifolds viewed as subsets, $X_i = \varphi_i(M_i)$. Note that the Hausdorff distance is defined to be

$$d^H_f(X_1, X_2) = \inf \{ r > 0 : X_1 \subset T_r(X_2), X_2 \subset T_r(X_1) \}$$

(85)

where $T_r(X) = \{ z : \exists x \in X \text{ s.t. } d(x, z) < r \} \subset Z$. There is no notion of a disappearing Hausdorff limit. The Hausdorff limit of a collapsing sequence of sets like $[0, 1/j] \times [0, 1] \subset \mathbb{R}^2$ is easily seen to be $[0] \times [0, 1] \subset \mathbb{R}^2$, which is simply a lower dimensional set. The Hausdorff limit of the sequence of cancelling submanifolds, $\varphi_j(M_j)$ in Example 3.2, is easily seen to be the set $[-1, 1] \times [0] \times [0, 1]$. No points in a Hausdorff limit can disappear. In Example 3.3, the Hausdorff limit of $\varphi_j(D^2)$ is a disk with a line segment attached: $(D^2 \times \{0\}) \cup ([0, 0] \times [0, 1]) \subset \mathbb{R}^3$. 


One reason Federer and Fleming introduced integral currents and flat convergence was to solve the Plateau problem of finding a minimal surface with a given boundary, \( \Gamma \). Suppose for example that

\[
\Gamma = \{(\cos(t), \sin(t), 0) : t \in \mathbb{S}^1 \} \subset \mathbb{E}^3.
\]

One must find the surface \( \varphi(D^2) \) such that \( \partial(\varphi(D^2)) = \Gamma \) of smallest area. One may try to find this minimal surface by taking a sequence of such surfaces, \( \varphi_j(D^2) \), with area decreasing to the infimum of these areas, and look for a limit. In Example 3.3 we have such a sequence with a thinner and thinner spine so that the Hausdorff limit is a disk with a line segment attached, not a minimal surface. Even worse, one may have a sequence of \( \varphi_j(D^2) = Y_j \) with increasingly many increasingly dense spines as in Figure 3 so that the Hausdorff limit is

\[
Y = \{(x, y, z) : x^2 + y^2 \leq 1, \ z \in [0, 1]\}.
\]

This Hausdorff limit has no notion of boundary and is no longer two dimensional. There is no smooth or even \( C_0 \) limit of such \( \varphi_j \).

On the other hand the flat limit of a sequence, \( \varphi_j(D^2) \), with area decreasing to the infimum of these areas does exist and is the standard disk

\[
\varphi_\infty : \{(x, y) : x^2 + y^2 \leq 1\} \to \mathbb{E}^3 \text{ with } \varphi_\infty(x, y) = (x, y, 0).
\]

This was proven in the case with one spline in Example 3.3. This can be seen in Figure 3 because the volume between the \( \varphi_j(M_j) \) and \( \varphi_\infty(M_\infty) \) converges to 0.

![Figure 3. A troublesome Hausdorff limit. Image owned by the author.](image)

Even on a compact metric space, \( Z \), flat convergence is well suited to the Plateau problem, where one is given \( \Gamma \in I_{m-1}(Z) \) and asked to find an integral current \( T \in I_m(Z) \), such that \( \partial T = \Gamma \) and

\[
M(T) = M_0 = \inf(M(T) : \partial T = \Gamma).
\]

One takes \( T_j \in I_m(Z) \) such that \( \partial T_j = \Gamma \), and \( M(T_j) \to M_0 \). By Ambrosio Kirchheim’s Compactness Theorem [Theorem 3.1 above] and (84), a subsequence converges in the weak and flat sense to some \( T_\infty \in I_m(Z) \). Since \( \partial T_\infty = \Gamma \) by (59) and \( M(T_\infty) = M_0 \) by (60), we have a desired solution to the Plateau problem.
4. Integral Current Spaces and Intrinsic Flat Convergence

In this section we provide the rigorous definition for the intrinsic flat convergence of a sequence of oriented Riemannian manifolds or, more generally, a sequence of integral current spaces \([SW11]\). It is crucial to remember that the manifolds in the sequence are not submanifolds of any common Euclidean space. This is an intrinsic notion about the intrinsic geometry of the Riemannian manifolds.

As described in the introduction, the intrinsic flat distance is defined much like the Gromov-Hausdorff distance, by taking an infimum over all distance preserving maps, \(\varphi_j : M_j^m \to Z\) into any common complete metric space, \(Z\):

\[
d_F(M_1, M_2) = \inf_{Z, \varphi_j} \left\{ d_Z(\varphi_1#(M_1), \varphi_2#(M_2)) \right\}
\]

where this is now rigorously defined using (53) and (80). In fact we can define the intrinsic flat distance between \(M_1\) and an abstract \(0\) space as well:

\[
d_F(M_1, 0) = \inf_{Z, \varphi_j} \left\{ d_Z(\varphi_1#(M_1), 0) \right\}.
\]

Keep in mind that by the Kuratowski Embedding Theorem any pair of separable metric spaces can be isometrically embedded into a Banach space, \(Z\), so these infima are always finite (cf. \([SW11]\)). In \([SW11]\), the author and Wenger prove that for \(M_j\) compact, we have \(d_F(M_1, M_2) = 0\) if and only if there is an orientation preserving isometry between them.

It is essential to remember that the \(\varphi_j\) are distance preserving maps or isometric embeddings in the sense of Gromov as in (20). They are not Riemannian isometric embeddings which only preserve lengths of curves. For example, the Riemannian isometry from the standard circle, \(S^1\), to the boundary of the closed Euclidean disk, \(D^2\), is not a distance preserving map. The Riemannian isometry from the standard circle, \(S^1\), to the boundary of the hemisphere, \(S^2_+\), is a distance preserving map. In Example 3.2 we have a single flat square, \(M = [-1, 1] \times [-1, 1]\), with a sequence of \(\varphi_j : M \to \mathbb{R}^2\) which preserve lengths of curves, and yet the flat limit of the images is 0 due to cancellation. If the intrinsic flat distance were defined using such maps, then \(d_F(M_1, 0) = 0\), and similarly the intrinsic flat distance between any pair of oriented manifolds would be 0.

In this section we introduce a larger class of spaces, integral current spaces, which are metric spaces with an additional structure. These spaces include oriented Riemannian manifolds with boundary and their intrinsic flat limits. We then define the intrinsic flat distance between this larger class of spaces and review fundamental theorems about intrinsic flat convergence.

4.1. Integral Current Spaces. Unlike the Gromov-Hausdorff distance, the intrinsic flat distance cannot be defined between an arbitrary pair of metric spaces, \(M_j = (X_j, d_j)\). One needs an additional structure which guarantees that the isometric embeddings of the \(M_j\) into \(Z\) may be viewed as integral currents. Thus the author and Wenger introduced the following notion in \([SW11]\):
Definition 4.1. An $m$ dimensional metric space $M = (X, d, T)$ is called an integral current space if it has an integral current structure $T \in \mathcal{I}_m(\hat{X})$ where $\hat{X}$ is the metric completion of $X$ and $\text{set}(T) = X$. Also included in the $m$ dimensional integral current spaces is the $0$ space, denoted $0 = (0, 0, 0)$. We say two such spaces are equal, $M_1 = M_2$, if there is a current preserving isometry, $F : M_1 \rightarrow M_2$:

$$d_2(F(x), F(t)) = d_1(x, y)$$
and $F_\#T_1 = T_2$.

The mass of the integral current space is, $\text{M}(M) = \text{M}(T)$. The diameter is $\text{Diam}(M) = \sup \{d(x, y) : x, y \in X\}$ if $M \neq 0$ and $\text{Diam}(0) = 0$.

Any $m$ dimensional integral current space $(X, d, T)$ is countably $\mathcal{H}^m$ rectifiable in the sense that there exists biLipschitz charts $\varphi_i : A_i \rightarrow X$ where $A_i \subset \mathbb{R}^m$ are Borel and

$$\mathcal{H}^m \left(X \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i)\right) = 0.$$ 

In fact these charts can be viewed as oriented with weights $\theta_i \in \mathbb{N}$ as in (30). Any 0 dimensional integral current space, $(X, d, T)$ is a finite collection of points, $X = \{p_1, ..., p_N\}$, with a metric $d$ and with weights $\theta_i \in \mathbb{Z}$ so that $T(f) = \sum_{i=1}^{N} \theta_i f(p_i)$.

A compact oriented Riemannian manifold with boundary, $(M^m, g)$, is an integral current space, where $X = M^m$, where $d = d_g$ is the standard metric on $M$,

$$d_g(p, q) = \inf \{L_g(C) : C(0) = p, C(1) = q\}$$

where

$$L_g(C) = \int_0^1 g(C'(t), C'(t))^{1/2} dt,$$
and where $T = [M]$ is integration over $M$,

$$T(f, \pi_1, ..., \pi_m) = \int_M f \, d\pi_1 \wedge \cdots \wedge d\pi_m.$$ 

In this setting $\text{M}(M) = \text{Vol}(M)$. Note that if $(M_1, g_1)$ and $(M_2, g_2)$ are diffeomorphic then they have the same integral current structure up to a sign. In fact they need only be biLipschitz equivalent. They do not have the same mass unless there is a volume preserving diffeomorphism between them. They are not viewed as the same integral current space unless there is an orientation preserving isometry between them.

If $M$ is a precompact oriented Riemannian manifold with boundary, $(M^m, g)$, then we can define an integral current space $(X, d, T)$, by taking the metric completion $\hat{X} = \hat{M}$, defining $d_g$ as the continuous extension to $\hat{X}$ of (94), and defining $T \in \mathcal{I}_m(\hat{X})$ exactly as in (96). Then $X = \text{set}(T) \subset \hat{X}$. This set is called the settled completion of $M$ and is denoted, $M'$. In particular

$$M' = \{x \in \hat{M} : \liminf_{r \rightarrow 0} \text{Vol}_m(B(x, r) \cap M) / r^m > 0\}.$$ 

So if $M$ is a manifold with a singular point removed, that point is always included in the metric completion $M$ but it is not included in the settled completion if it is a cusp singularity.
The boundary of an integral current space, \((X, d, T)\), is the integral current space:

\[
\partial (X, d_X, T) := (\text{set}(\partial T), d_X, \partial T)
\]

where the distance on the boundary is \(d_X\) which is restricted from the distance on the metric completion \(\bar{X}\). If \(\partial T = 0\) then one says \((X, d, T)\) is an integral current without boundary. The 0 space has no boundary.

Note that the boundary of \((M, d_g, [M])\) when \(M\) is an oriented Riemannian manifold with boundary is \((\partial M, d_g, \partial [M])\) endowed with the restricted distance, \(d_g\), defined on \(M\) as in \((94)\). It is only a geodesic space if \(\partial M\) is totally geodesic in \(M\). For example, when \(M = D^2 \subset \mathbb{R}^2\), then \(\partial M = (S^1, d_{S^1}, [S^1])\) is not a geodesic space because there are no curves whose length is equal to the distance between the points,

\[
d_{S^1}(p, q) = |p - q| < d_{S^1}(p, q) = \cos^{-1}(1 - |p - q|^2/2).
\]

The boundary of the upper hemisphere, \(M = (S^2_+, d_{S^2_+}, [S^2_+])\), is a geodesic integral current space, \(\partial M = (S^1, d_{S^1}, [S^1])\).

In [Sor14] the author proves that a ball in an integral current space, \(M = (X, d, T)\), with the current restricted from the current structure of \(M\) is an integral current space itself,

\[
S(p, r) := (\text{set}(T \cup B(p, r)), d, T \cup B(p, r))
\]

for almost every \(r > 0\). Furthermore,

\[
B(p, r) \subset \text{set}(S(p, r)) \subset \bar{B}(p, r) \subset X.
\]

Note that the outside of the ball, \((M \setminus B(p, r), d, T - S(p, r))\), and the sphere,

\[
\partial S(p, r) := (\text{set}(\partial T \cup B(p, r)), d, \partial(T \cup B(p, r))),
\]

are integral current spaces for the same values of \(r > 0\). If \(\partial M = 0\) then

\[
\text{set}(\partial(T \cup B(p, r))) \subset \{x : d(x, p) = r\}.
\]

In [PS14], Portegies and Sormani investigate the notion of the filling volume

\[
\text{FillVol}(\partial M) = \inf\{M(N) : \partial N = \partial M\}
\]

where the infimum is over all integral current spaces, \(N\), such that there is a current preserving isometry from \(\partial N\) to \(\partial M\). This notion of filling volume does not quite agree with Gromov’s notion of Filling Volume in [Gro83] because in our notion there is a larger collection of candidates, \(N\), for filling the manifold, because we require a current preserving isometry on the boundary, and because we use the Ambrosio-Kirchheim mass. With our notion, we immediately have

\[
M(M) \geq \text{FillVol}(\partial M).
\]

Applying the filling volume to balls, we have for almost every \(r > 0\) that

\[
\|T\|_1(B(p, r)) = M(T \cup B(p, r)) = M(S(p, r)) \geq \text{FillVol}(\partial S(p, r)).
\]

In particular, if a point \(x \in \bar{X}\) satisfies

\[
\lim_{r \to 0} \inf \text{FillVol}(\partial S(p, r))/r^m > 0
\]
then \( x \in \text{set}(T) = X \). This idea was first applied jointly with Wenger in [SW10] before the notion of integral current space was precisely defined in [SW11]. Further exploration of filling volume and a new notion called the sliced filling volume appeared in [PS14] with Portegies.

4.2. Intrinsic Flat Convergence. The intrinsic flat distance between integral current spaces was first defined by the author and Wenger in [SW11]:

**Definition 4.2.** For \( M_1 = (X_1, d_1, T_1) \) and \( M_2 = (X_2, d_2, T_2) \) let the intrinsic flat distance be defined:

\[
d^F(M_1, M_2) := \inf d^Z(\varphi_1\#T_1, \varphi_2\#T_2),
\]

where the infimum is taken over all complete metric spaces \((Z, d)\) and distance preserving maps \( \varphi_j : (\bar{X}_j, d_j) \to (Z, d) \).

When \( M_j \) are precompact integral current spaces we prove the infimum in this definition is obtained [SW11][Thm 3.23] and consequently \( d^F \) is a distance [SW11][Thm 3.27] on the class of precompact integral current spaces up to current preserving isometries as in (92). In particular, it is a distance on the class of oriented compact manifolds with boundary of a given dimension.

We say

\[
M_j \xrightarrow{\mathcal{F}} M_\infty \text{ iff } d^F(M_j, M_\infty) \to 0.
\]

By the definition \( M_j \xrightarrow{\mathcal{F}} M_\infty \) if and only if there exists distance preserving maps to complete metric spaces, \( \varphi_j : M_j \to Z_j \) and \( \varphi'_j : M_\infty \to Z_j \), and integral currents, \( B_j \in \mathcal{I}_{m+1}(Z_j) \) and \( A_j \in \mathcal{I}_m(Z_j) \), such that

\[
\varphi_j\#T_j - \varphi'_j\#T_\infty = \partial B_j + A_j
\]

and

\[
d^F(M_j, M_\infty) \leq d^Z(\varphi_j\#T_j, \varphi'_j\#T_\infty) \leq M(B_j) + M(A_j) \to 0.
\]

We could then replace \( Z_j \) with \( Z'_j \) that are closures of countably \( \mathcal{H}^{m+1} \) rectifiable spaces by taking

\[
Z'_j = \text{Cl}(\text{set}(B_j) \cup \text{set}(A_j)) \subset Z_j.
\]

So in fact the \( Z \) in the infimum of the Definition 4.2 may be chosen in this class.

Note that if \( M_j \xrightarrow{\mathcal{F}} M_\infty \) then using the same distance preserving maps we have

\[
\varphi_j\#\partial T_j - \varphi'_j\#\partial T_\infty = \partial A_j
\]

and

\[
d^F(\partial M_j, \partial M_\infty) \leq d^Z(\varphi_j\#\partial T_j, \varphi'_j\#\partial T_\infty) \leq M(A_j) \to 0.
\]

So \( \partial M_j \xrightarrow{\mathcal{F}} \partial M_\infty \).

The following theorem in [SW11] is an immediate consequence of Gromov’s Embedding Theorem and Ambrosio-Kirchheim’s Compactness Theorems:
Theorem 4.3. Given a sequence of precompact $m$ dimensional integral current spaces $M_j = (X_j, d_j, T_j)$ such that

\[(\tilde{X}_j, d_j) \overset{GH}{\longrightarrow} (Y, d_Y), \quad M(M_j) \leq V_0 \quad \text{and} \quad M(\partial M_j) \leq A_0\]

then a subsequence converges in the intrinsic flat sense

\[(X_{j_i}, d_{j_i}, T_{j_i}) \overset{\mathcal{F}}{\longrightarrow} (X, d_X, T)\]

where either $(X, d_X, T)$ is the 0 integral current space or $(X, d_X, T)$ is an $m$ dimensional integral current space such that $X \subset Y$ with the restricted metric $d_X = d_Y$.

Immediately one notes that if $Y$ has Hausdorff dimension less than $m$, then $(X, d_X, T) = 0$. In Section 4.3 we survey theorems in which it is proven under additional hypothesis that the intrinsic flat and GH limits agree. There are many examples with nonnegative scalar curvature where these limits do not agree presented in [SW11]. In fact one may not even have a GH converging subsequence for a sequence with an intrinsic flat limit (see Section 2.1).

Gromov’s Embedding Theorem which is applied to prove Theorem 4.3, states that if $(X_j, d_j) \overset{GH}{\longrightarrow} (X_\infty, d_\infty)$ then there is a compact metric space $Z$ and a collection of isometric embeddings $\varphi_j : X_j \to Z$ such that

\[d_Z^H(\varphi_j(X_j), \varphi_\infty(X_\infty)) \to 0.\]

Note that without his embedding theorem one needs different $Z_j$ for each term in the sequence and then one would not be able to apply the Ambrosio-Kirchheim Compactness Theorem (cf. Theorem 3.1) to complete the proof of Theorem 4.3.

In [SW11] Thms 4.2-4.3, the author and Wenger prove similar embedding theorems for sequences which converge in the intrinsic flat sense: if

\[M_j = (X_j, d_j, T_j) \overset{\mathcal{F}}{\longrightarrow} M_0 = (X_\infty, d_\infty, T_\infty),\]

then there is a common separable complete metric space, $Z$, and distance preserving maps $\varphi_j : X_j \to Z$ such that

\[d_F^Z(\varphi_j\#T_j, \varphi_\infty\#T_\infty) \to 0.\]

In the case where $M_0 = 0$ then we have (119) as well with $\varphi_\infty\#T_\infty = 0$ and can find $z \in Z$ and $x_j \in X_j$ such that $\varphi_j(x_j) = z$. In fact this $Z$ can be chosen to be the closure of a countably $T^{m+1}$ rectifiable metric space and is glued together from the $Z_j'$ in (112).

These embedding theorems do not require uniform bounds on the masses or volumes of the $M_j$ and $\partial M_j$. Combining them with Ambrosio-Kirchheim’s lower semicontinuity of mass (60) we see that

\[M_j \overset{\mathcal{F}}{\longrightarrow} M_\infty \implies \liminf_{i \to \infty} M(M_i) \geq M(M_\infty).\]

In [Sor14] the author proves lower semicontinuity of the diameter as well:

\[M_j \overset{\mathcal{F}}{\longrightarrow} M_\infty \implies \liminf_{i \to \infty} \operatorname{Diam}(M_i) \geq \operatorname{Diam}(M_\infty),\]
In [PST14], the author and Portegies prove that

\[ \partial M_j \xrightarrow{\mathcal{F}} \partial M_\infty \implies \text{FillVol}(\partial M_j) \to \text{FillVol}(\partial M_\infty). \]

This idea was first observed in joint work of the author with Wenger [SW10]. Since

\[ M(M) \geq \text{FillVol}(\partial M) \]

one can use filling volumes to provide a lower bound on the mass of the limit

\[ M(M_\infty) \geq \text{FillVol}(\partial M_\infty) = \lim_{j \to \infty} \text{FillVol}(\partial M_j). \]

Portegies and the author also introduce the notion of a sliced filling volume in [PST14] and prove that it is continuous with respect to intrinsic flat convergence and also provides a lower bound for mass.

### 4.3. Compactness Theorems for Intrinsic Flat Convergence

The first compactness theorem for intrinsic flat convergence is stated by the author with Wenger in [SW11]. It is a combination of Gromov’s Compactness and Embedding Theorems with Ambrosio-Kirchheim Compactness, to say that if \( M_j = (X_j, d_j, T_j) \) satisfy the hypothesis of Gromov’s Compactness Theorem and of Ambrosio-Kirchheim’s Compactness theorem, then a subsequence converges in the GH sense and the \( \mathcal{F} \) sense where the \( \mathcal{F} \) limit is a subset of the GH limit (cf. Theorem 4.3). There are a number of theorems which apply Gromov’s Compactness theorem combined with this theorem and then prove the GH and \( \mathcal{F} \) limits agree under additional hypothesis including noncollapsing, \( M(M_j) \geq V_0 > 0 \). We call these \( \mathcal{F} \) =GH compactness theorems.

The author and Wenger prove in [SW11] a \( \mathcal{F} \) =GH compactness theorem for sequences of manifolds without boundary that either have uniform linear contractibility functions or are noncollapsing with \( \text{Ric} \geq 0 \). Perales has extended this to allow boundaries with various conditions on the boundary in [Per15a]. Matveev-Portegies have extended the result without boundary to uniform negative lower bounds on Ricci curvature in [MP15]. The author, Huang and Lee have proven a \( \mathcal{F} \) =GH compactness theorem for sequences of integral current spaces, \((X, d_j, T)\), with varying bounded distance functions \( d_j \) in the Appendix to [HLS16]. Li and Perales have proven a \( \mathcal{F} \) =GH compactness theorem for noncollapsing integral current spaces \((X_j, d_j, T_j)\) with nonnegative Alexandrov curvature (including manifolds with nonnegative sectional curvature) in [LP15]. It is unknown whether integral current spaces satisfying various generalized notions of Ricci curvature have \( \mathcal{F} \) =GH compactness theorems.

In the setting with Scalar \( \geq 0 \), we do not in general have GH limits and so we need compactness theorems with weaker hypothesis that do not imply GH convergence of subsequences. Wenger’s Compactness Theorem was proven in [Wen11] and stated in the following form in [SW11]:

**Theorem 4.4. Wenger Compactness**

If \( M_j \) are integral current spaces of dimension \( m \) satisfying the following

\[ M(M_j) \leq V_0 \quad M(\partial M_j) \leq A_0 \quad \text{Diam}(M_j) \leq D_0 \]
then there exists a subsequence $M_{jk} \xrightarrow{\mathcal{F}} M_\infty$ where $M_\infty$ is an integral current spaces of dimension $m$ possibly 0.

Perales has applied this theorem in [Per15b] to prove two compactness theorems. One assumes the given sequence of oriented manifolds satisfies

\begin{equation}
\text{Ric}_j \geq 0 \quad \text{Vol}(\partial M_j) \leq A_0 \quad H_{\partial M_j} \geq H_0 \quad \text{Diam}(M_j) \leq D_0
\end{equation}

and the other assumes the given sequence satisfies

\begin{equation}
\text{Ric}_j \geq 0 \quad \text{Vol}(\partial M_j) \leq A_0 \quad H_{\partial M_j} \geq H_0 > 0 \quad \text{Diam}(\partial M_j) \leq D_0.
\end{equation}

Here $H$ is the mean curvature with respect to the outward pointing normal. Note that in (127) the only condition on the interior of the manifold is $\text{Ric} \geq 0$.

Observe that in both of these theorems, we could renormalize the manifolds to have $\text{Vol}(\partial M_j) = A_0$. When $H_0 \geq 0$, these sequences have Hawking mass as in (8) uniformly bounded above:

\begin{equation}
m_H(\partial M_j) \leq m_0 = \sqrt{\frac{A_0}{16\pi}} \left(1 - \frac{1}{16\pi} A_0 H_0^2\right).
\end{equation}

This leads naturally to the following conjecture which could be a step towards proving almost rigidity of the Positive Mass Theorem or Bartnik’s Conjecture [Bar89]:

**Conjecture 4.5. Hawking Mass Compactness**

**Given** a sequence of three dimensional oriented manifolds $M^3_j$ satisfying

\begin{equation}
\text{Vol}(M_j) \leq V_0 \quad \text{Vol}(\partial M_j) = A_0 \quad \text{Diam}(M_j) \leq D_0.
\end{equation}

\begin{equation}
\text{Scalar}_j \geq 0 \quad H_{\partial M_j} \geq 0 \quad m_H(\partial M_j) \leq m_0
\end{equation}

and either no closed interior minimal surfaces or $\text{MinA}(M_j) \geq A_1 > 0$, then a subsequence converges in the intrinsic flat sense

\begin{equation}
M_{jk} \xrightarrow{\mathcal{F}} M_\infty \quad \text{and} \quad \text{M}(M_{jk}) \to \text{M}(M_\infty)
\end{equation}

and $M_\infty$ satisfies (130) in some generalized sense (cf. Section 7.5). One might replace Hawking mass with another quasilocal mass in this conjecture.

LeFloch and the author have proven this Hawking Mass Compactness Conjecture in the rotationally symmetric setting assuming that there are no closed interior minimal surfaces in [LS15]. This is shown by proving $H^1_{\text{loc}}$ convergence of a subsequence of the manifolds with a well chosen gauge and then proving the $H^1_{\text{loc}}$ limit is a $\mathcal{F}$ limit using Theorem 5.2. In general it is unknown whether $H^1_{\text{loc}}$ convergence implies $\mathcal{F}$ convergence, but here there is also monotonicity of the Hawking mass to help. Since the limit space is a rotationally symmetric manifold with a metric tensor $g \in H^1_{\text{loc}}$, it is possible to define generalized notions of nonnegative scalar curvature using (3) in a weak sense and also to define Hawking mass and show (130) hold on the limit spaces as well.

Gromov has conjectured vaguely that intrinsic flat convergence may preserve some notion of $\text{Scalar} \geq 0$ in [Gro14b] and [Gro14a]. Considering the examples
and the above conjecture, we propose the following Scalar Compactness Theorem which requires a uniform lower bound on the area of a closed minimal surface:

**Conjecture 4.6. Scalar Compactness**

*Given a sequence of oriented manifolds $M^3_j$ with $\partial M^3_j = \emptyset$ satisfying*

\[
\begin{align*}
\text{Vol}(M_j) &\leq V_0, \\
\text{Diam}(M_j) &\leq D_0, \\
\text{Scalar}_j &\geq 0, \\
\text{MinA}(M_j) &\geq A_1 > 0
\end{align*}
\]

*then a subsequence converges in the intrinsic flat sense*

\[
M_{j_{k}} \overset{\mathcal{F}}{\rightarrow} M_{\infty} \quad \text{and} \quad M(M_{j_{k}}) \rightarrow M(M_{\infty})
\]

*and $M_{\infty}$ has $\text{Scalar}_{\infty} \geq 0$ in some generalized sense (cf. Section 7.5).*

A proof of this Scalar Compactness Theorem in the rotationally symmetric case might imitate the proof of the Hawking Compactness Theorem of the author with LeFloch, however the work in [LS15] very strongly uses that there is a boundary to choose a gauge. Nevertheless a very similar proof should work and would make a nice problem for a doctoral student. Quite a different technique would be needed to handle other settings. In the graph case as in [HLS16] there is no $H^{1}_{loc}$ convergence.

See Section 7.5 for more about how generalized $\text{Scalar}_{\infty} \geq 0$ might be defined.

### 5. Theorems which imply Intrinsic Flat Convergence

In this section we present theorems which have been applied to prove sequences of spaces converge in the intrinsic flat sense. In the previous section we have already presented compactness theorems which imply intrinsic flat convergence of subsequences. Here we present theorems where geometric constraints and relationships between a pair of spaces are used to bound the intrinsic flat distance between them.

Before we begin, note that in Section 5 of [SW11], the author and Wenger proved that if the Gromov Lipschitz distance between two Riemannian manifolds is small, then the intrinsic flat distance is small. In particular if the manifolds are close in the $C_{0}$ sense then they are close in the intrinsic flat sense. This theorem may now be viewed as a special case of Theorem 5.2 included below. More general statements about pairs of integral current spaces with such bounds also appear in [SW11].

#### 5.1. Using Riemannian Embeddings to estimate $D_{\mathcal{F}}$.

Recall that in the definition of intrinsic flat convergence, one must find distance preserving maps of the pair of manifolds into a common complete metric space, $Z$, before estimating the flat distance between the images. If, however, one only has Riemannian isometric embeddings of the manifolds into a common Riemannian manifold, then one may apply the following theorem proven by Lee and the author in [LS14] to estimate the intrinsic flat distance between the spaces.

**Theorem 5.1.** If $\phi_{i} : M^{m}_{i} \rightarrow N^{m+1}$ are Riemannian isometric embeddings with embedding constants $C_{M_{i}}$ where

\[
C_{M} := \sup_{p,q \in M} |d_{M}(p, q) - d_{N}(\phi(p), \phi(q))|,
\]

...
and if they are disjoint and lie in the boundary of a region \( W \subset N \) then
\[
\begin{align*}
135 & \quad d_F(M_1, M_2) \leq S_{M_1}(\text{Vol}_n(M_1) + \text{Vol}_{n-1}(\partial M_1)) \\
136 & \quad + S_{M_2}(\text{Vol}_n(M_2) + \text{Vol}_{n-1}(\partial M_2)) \\
137 & \quad + \text{Vol}_{n+1}(W) + \text{Vol}_n(V)
\end{align*}
\]
where \( V = \partial W \setminus (\varphi_1(M_1) \cup \varphi_2(M_2)) \) where \( S_{M_i} = \sqrt{C_{M_i}(\text{Diam}(M_i) + C_{M_i})} \).

This theorem is proven in [LS14] by explicitly constructing a geodesic metric space,
\[
138 \quad Z = W_0 \cup W_1 \cup W_2 \subset N \times [0, S_M],
\]
where \( W_0 = \{(x, 0) : x \in N \} \) and
\[
139 \quad W_i = \{(x, s) : x \in \varphi_i(M_i), s \in [0, S_{M_i}] \}
\]
and proving \( \psi_i(x) = (\varphi_i(x), S_{M_i}) \) are distance preserving maps into \( Z \). Then taking \( B = [W_0] + [W_1] - [W_2] \) and \( A = V_0 + V_1 - V_2 \) where \( V_0 = \{((x, 0) : x \in V) \} \) and
\[
140 \quad V_i = \{((x, s) : x \in \varphi_i(\partial M_i), s \in [0, S_{M_i}] \}.
\]
with the appropriate orientations, we get \( \psi_1#M_1 - \psi_2#M_2 = A + \partial B \). The estimate then follows because \( d_F(M_1, M_2) \leq M(A) + M(B) \).

5.2. Smooth Convergence Away from Singular Sets. If one has a pair of Riemannian manifolds containing subregions that are close in the \( C_0 \) sense, then one can estimate the intrinsic flat distance between these manifolds using estimates on the volumes of the regions where they are different and additional information as shown by Lakzian and the author in [LS13]:

**Theorem 5.2.** Suppose \( M_1 = (M, g_1) \) and \( M_2 = (M, g_2) \) are oriented precompact Riemannian manifolds with diffeomorphic subregions \( U_i \subset M_i \) and diffeomorphisms \( \psi_i : U \rightarrow U_i \) such that the following hold
\[
141 \quad (1 + \epsilon)^2 \psi_2^*g_2(V, V) < \psi_1^*g_1(V, V) < (1 + \epsilon)^2 \psi_2^*g_2(V, V) \quad \forall V \in TU,
\]
\[
142 \quad D_{U_i} = \sup\{\text{Diam}_{M_i}(W) : W \text{ is a connected component of } U_i\},
\]
\[
143 \quad \lambda = \sup \{d_{M_1}(\psi_1(x), \psi_1(y)) - d_{M_2}(\psi_2(x), \psi_2(y))\}.
\]
Then the Gromov-Hausdorff distance between the metric completions is bounded,
\[
144 \quad d_{GH}(\bar{M}_1, \bar{M}_2) \leq a + 2h + \max \{d_H^U(U_1, M_1), d_H^U(U_2, M_2)\}
\]
and the intrinsic flat distance between the settled completions is bounded,
\[
145 \quad d_F(M_1', M_2') \leq \text{Vol}_n(M_1 \setminus U_1) + \text{Vol}_n(M_2 \setminus U_2) \\
146 \quad + (2h + a) \left( \text{Vol}_{n-1}(\partial U_1) + \text{Vol}_{n-1}(\partial U_2) \right) \\
147 \quad + (2h + a) \left( \text{Vol}_n(U_1) + \text{Vol}_n(U_2) \right),
\]
where \( a = a(\epsilon, D_{U_1}, D_{U_2}) \) converges to 0 as \( \epsilon \rightarrow 0 \) for fixed values of \( D_{U_i} \), and where \( h = h(\epsilon, \lambda, D_{U_1}, D_{U_2}) \) converges to 0 as both \( \epsilon \rightarrow 0 \) and \( \lambda \rightarrow 0 \) for fixed \( D_{U_i} \).
Explicit formulas for \( a(\epsilon, D_{U_1}, D_{U_2}) \) and \( h(\epsilon, \lambda, D_{U_1}, D_{U_2}) \) are given in [LS13].
Theorem 5.2 is proven in [LS13] by explicitly constructing a common metric space
\begin{equation}
Z = M_1 \sqcup_{U_1} (U \times [0, 2\tilde{h} + a]) \sqcup_{U_2} M_2
\end{equation}
where $U \times [0, 2\tilde{h} + a]$ is a product manifold with a precisely given metric $g'$, and $M_1$ is glued to it along $U_1 \subset M_1$ which is isometric to $U \times \{0\}$ and $M_2$ is glued to it along $U_2 \subset M_2$ which is isometric to $U \times \{2\tilde{h} + a\}$ as in Figure 4. The metric $g'$ is chosen so that $\varphi_1 : M_1 \to Z$ are distance preserving. In particular $g' = dt^2 + g_1$ on $U \times [0, \tilde{h}]$ where $\tilde{h}$ is chosen as in the theorem statement to guarantee that there are no short paths between points in $M_1$ that run through $U \times [0, 2\tilde{h} + a]$. Similarly, $g' = dt^2 + g_2$ on $U \times [a + \tilde{h}, a + 2\tilde{h}]$. On the middle, $U \times [\tilde{h}, a + \tilde{h}]$, there is a hemispheric warping between the metrics $g_1$ and $g_2$. Once an explicit $Z$ has been found, then an explicit $A \in \mathbb{I}_m(Z)$ is found where $\mathcal{M}(A)$ is the sum of the terms in (145)-(146) and an explicit $B \in \mathbb{I}_{m+1}(Z)$ is found where $\mathcal{M}(B)$ is the term in (147). The details are easy to follow in [LS13].

Theorem 5.2 has been applied in work of the author and Lakzian to prove the rotationally symmetric Hawking Mass Compactness Theorem [LS15]. It has been applied by Lakzian to study Ricci flow through neck pinch singularities in [Lak15a]. It has been applied by Lakzian in [Lak16] to prove $\mathcal{F}$ convergence of sequences of manifolds which converge smoothly away from singular sets. In particular, Lakzian proves that if $M_j = (M, g_j)$ be a sequence of compact oriented Riemannian manifolds with a set $S \subset M$ such that $\mathcal{H}^n-1(S) = 0$, and a connected precompact exhaustion, $W_k$, of $M \setminus S$ satisfying
\begin{equation}
\tilde{W}_k \subset W_{k+1} \cap \bigcup_{k=1}^\infty W_k = M \setminus S
\end{equation}
with $g_j$ converge smoothly to $g_\infty$ on each $W_k$,
\begin{equation}
\text{Diam}_{M_j}(W_k) \leq D_0 \quad \forall i \geq j,
\end{equation}
\begin{equation}
\text{Vol}_{g_j}(\partial W_k) \leq A_0,
\end{equation}
and
\begin{equation}
\text{Vol}_{g_j}(M \setminus W_k) \leq V_k \frac{\text{lim}}{k \to \infty} V_k = 0.
\end{equation}
Then
\[ \lim_{j \to \infty} d_F(M_j', N') = 0. \]
where \( N' \) is the settled completion of \( N = (M \setminus S, g_\infty) \) as in (97). If one also assumes \( \text{Ric}_j \geq H \), then
\[ \lim_{j \to \infty} d_{GH}(M_j, \bar{N}) = 0. \]

Other theorems about smooth convergence away from singular sets are proven in \[\text{Lak16}\] as well. KAHLER

### 5.3. Pairs of Integral Current Spaces

The following theorem concerns pairs of integral current spaces which share the same space, \( X \), and the same current structure, \( T \), but have different distance functions. This happens for example when one has a pair of oriented Riemannian manifolds with an orientation preserving diffeomorphism between them but can also be applied to pairs of integral current spaces with a biLipschitz current preserving map between them. The Gromov-Hausdorff part of this theorem was proven by Gromov in \[\text{Gro81}\] and the intrinsic flat part was proven by the author with Huang and Lee in the Appendix to \[\text{HLS16}\].

**Theorem 5.3.** Fix a precompact \( m \)-dimensional integral current space \((X, d_0, T)\) with \( \partial T = 0 \) and fix \( \lambda > 0 \). Suppose that \( d_1 \) is another distance on \( X \) such that
\[ \epsilon = \sup \{|d_1(p, q) - d_0(p, q)| : p, q \in X\}. \]
Then we have the following:
\[ d_{GH}((X, d_1), (X, d_0)) \leq 2\epsilon \]
\[ d_F((X, d_1, T), (X, d_0, T)) \leq 2^{(m+1)/2} \lambda^{m+1}(2\epsilon) M(T). \]

Note that the hypothesis of this theorem arises when an integral current space, \((X, d, T)\), has an almost distance preserving map into another metric space, \( F : X \to Y \), and one defines a new distance \( d_1 \) on \( X \) as \( d_1(p, q) = d_Y(F(p), F(q)) \). It was applied by the author with Huang and Lee in \[\text{HLS16}\] as one of the final steps towards proving the almost rigidity for the Positive Mass Theorem in the graph setting.

**Problem 5.4.** It seems natural that one could apply Theorem 5.3 to prove a more general theorem about pairs of integral current spaces, \((X_i, d_i, T_i)\), which contain a pair of regions \( U_i \subset X_i \) with a current preserving isometry between the regions. If one simply uses the restricted distances then there is no need for an embedding constant or a \( \lambda \) as in the prior two sections. However, one needs to extend the Theorem 5.3 to consider the setting with boundary.
6. Theorems about Intrinsic Flat Limits

In this section we assume we have a sequence, \( M_j \xrightarrow{\text{GH}} M_\infty \) or \( M_j \xrightarrow{\mathcal{F}} M_\infty \) and present theorems about limits of points in these spaces, limits of functions on these spaces, continuity and semicontinuity of various quantities on these spaces. Recall that we have already mentioned a number of such results in Subsection 4.2 and we will be referring to those results here as well. We close this section with a discussion of the setting where one has intrinsic flat convergence with volume continuity.

6.1. Limits of Points and Points with no Limits. When studying sequences of converging Riemannian manifolds, \( M_j \), one often wishes to understand what happens to points, \( x_j \in M_j \): where do they converge and when do they disappear? One is also interested in understanding limit points, \( x_\infty \in M_\infty \), by considering \( x_j \in M_j \) that converge to these points. This can easily be understood when the \( M_j \) are a sequence of converging submanifolds lying in a given space, and it is clear that under flat convergence of submanifolds some sequences of points will disappear in the limit. Defining converging sequences of points when \( M_j \) are distinct Riemannian manifolds is much more difficult.

Under Gromov-Hausdorff convergence (158) (\( X_j, d_j \xrightarrow{\text{GH}} (X_\infty, d_\infty) \)), one can apply Gromov’s Embedding Theorem (117) to define what it means to say \( x_j \to x_\infty \) for \( x_j \in X_j \) as follows: there exists a compact metric space, \( Z \), and distance preserving maps, \( \varphi_j : X_j \to Z \), such that
\[(159) \quad d_Z(\varphi_j(x_j), \varphi_\infty(x_\infty)) \to 0 \quad \text{and} \quad d_Z(\varphi_j(x_j), \varphi_\infty(x_\infty)) \to 0.
\]
Note that \( x_\infty \) is not unique: if \( F : X_\infty \to X_\infty \) is an isometry then \( F(x_\infty) \) is also a limit of \( x_j \). This is a natural consequence of the fact that the GH distance is between isometry classes of metric spaces. Gromov shows that
\[(160) \quad \forall x_\infty \in X_\infty \quad \exists x_j \in X_j \text{ such that } x_j \to x_\infty.
\]
In fact, there exist functions, \( H_j : X_\infty \to X_j \), and distance preserving maps, \( \varphi_j : X_j \to Z \), satisfying (159) such that if \( x_j = H_j(x_\infty) \) then \( x_j \to x_\infty \) and
\[(161) \quad d_H(H_j(x), H_j(y)) \to d_\infty(x, y).
\]
Furthermore, for any \( r > 0 \), we have converging closed balls
\[(162) \quad x_j \to x_\infty \implies (\bar{B}(x_j, r), d_j) \xrightarrow{\text{GH}} (\bar{B}(x_\infty, r), d_\infty).
\]
By the compactness of \( Z \), there is a Bolzano-Weierstass Theorem:
\[(163) \quad x_j \in X_j \implies \exists j_k \text{ s.t. } x_{j_k} \to x_\infty \in X_\infty.
\]
Combining (161) with (163) we have diameter continuity:
\[(164) \quad \lim_{j \to \infty} \text{Diam}(X_j) = \text{Diam}(X_\infty).
\]
See for example [BBI01] and [Sor14] for more details.
Now suppose we have intrinsic flat convergence,

\begin{equation}
M_j = (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty).
\end{equation}

In \cite{Sor14}, the author applied the Intrinsic Flat Embedding Theorem as in (119) to say a sequence \( x_j \in X_j \) is Cauchy if there exists a complete metric space, \( Z \), a point, \( z_\infty \in Z \), and distance preserving maps, \( \varphi_j : X_j \to Z \), such that

\begin{equation}
\delta_F^Z(\varphi_j#T_j, \varphi_\infty#T_\infty) \to 0
\end{equation}

and \( \varphi_j(x_j) \to z_\infty \). One says the sequence has no limit in \( \bar{X}_\infty \) if

\begin{equation}
z_\infty \notin \varphi_\infty(\bar{X}_\infty).
\end{equation}

One says the points converge \( x_j \to x_\infty \) in \( X_\infty \) (or respectively in \( \bar{X}_\infty \)) if there exists \( x_\infty \in X_\infty \) (or respectively in \( \bar{X}_\infty \)) such that

\begin{equation}
z_\infty \in \varphi_\infty(X_\infty) = \text{set}(\varphi_\infty#T_\infty) \quad \text{(or respectively } z_\infty \in \varphi_\infty(\bar{X}_\infty) \text{)}.\end{equation}

Note that \( x_\infty \) is not unique: if \( F : X_\infty \to X_\infty \) is a current preserving isometry then \( F(x_\infty) \) is also a limit of \( x_j \). This is a natural consequence of the fact that the \( \mathcal{F} \) distance is between current preserving isometry classes of integral current spaces.

Below we will provide additional conditions (176) and (177) which demonstrate that Cauchy sequences of points which disappear with respect to one sequence of \( \varphi_j \) satisfying (166) cannot be Cauchy sequences of points that converge with respect to another sequence of \( \varphi_j \) satisfying (166) and visa versa.

In \cite{Sor14} the author proves that if (165) then

\begin{equation}
\forall x_\infty \in \bar{X}_\infty \ \exists x_j \in X_j \text{ such that } x_j \to x_\infty.
\end{equation}

In fact there exist convergence functions, \( H_j : X_\infty \to X_j \), and distance preserving maps, \( \varphi_j : X_j \to Z \), satisfying (166) such that \( H_j(x_\infty) \to x_j \) as in (159) with

\begin{equation}
d_j(H_j(x), H_j(y)) \to d_\infty(x, y) \text{ and } H_j(\partial M_\infty) \subset \partial M_j.
\end{equation}

In fact the author proves in \cite{Sor14} [Theorem 5.1] the following:

\begin{equation}
\text{if } M_j \xrightarrow{\mathcal{F}} M_\infty \text{ where } M_\infty \text{ is nonzero and precompact, then there exists } N_j \subset M_j \text{ such that } N_j \xrightarrow{\text{GH}} \bar{M}_\infty.
\end{equation}

Additional properties of the \( N_j \) are described there. In general intrinsic flat limits need not be precompact and need not have finite diameter \cite{SW11}.

Since \( Z \) is only complete, we do not have a simple Bolzano-Weierstrass Theorem. In fact points may disappear under flat convergence even in a compact \( Z \). So we do not have continuity of diameter. However we have semicontinuity of diameter

\begin{equation}
\lim \inf_{j \to \infty} \text{Diam}(M_j) \geq \text{Diam}(M_\infty).
\end{equation}

and depth

\begin{equation}
\lim \inf_{j \to \infty} \text{Depth}(M_j) \geq \text{Depth}(M_\infty).
\end{equation}
where
\[ \text{Depth}(M) = \sup \{ d_X(x, y) : x \in X, y \in \text{set}(\partial T) \} \in [0, \infty]. \]

Recall that for almost every \( r > 0 \) one may view a ball in an integral current spaces as an integral current space itself \( S(x, r) = \left( \text{set}(T \cap B(x, r)), d, T \cap B(x, r) \right) \).

and examine the convergence of balls when \( M_j \rightarrow M_\infty \). In \[\text{Sor14}\] the author proves that if \( x_j \rightarrow x_\infty \) as in (169), then there is a subsequence \( j_k \) such that
\[ S(x_{j_k}, r) \overset{\mathcal{F}}{\rightarrow} S(x_\infty, r) \neq 0 \quad \text{for almost every } r > 0. \]

If a Cauchy sequence of points, \( x_j \in M_j \), has no limit in \( \bar{X}_\infty \) as in (167) then
\[ \exists \delta > 0 \text{ s.t. } S(x_{j_k}, r) \overset{\mathcal{F}}{\rightarrow} 0 \quad \text{for almost every } r \in (0, \delta). \]

Since (176) and (177) are intrinsic notions which do not depend on a choice of distance preserving maps, one concludes that a Cauchy sequence of points which has no limit in \( \bar{X}_\infty \) with respect to one sequence of \( \varphi_j \) satisfying (166) cannot be a Cauchy sequence of points that converges with respect to another sequence of \( \varphi_j \) satisfying (166) and visa versa. Far more subtle is determining when exactly a sequence of points converges to a point in \( \bar{X}_\infty \setminus X_\infty \).

Let us further consider converging sequences of points, \( x_j \rightarrow x_\infty \). As a consequence of mass semicontinuity as in (60), we have for almost every \( r > 0 \)

\[ \liminf_{k \rightarrow \infty} M(S(x_{j_k}, r)) \geq M(S(x_\infty, r)). \]

As a consequence of (59), we have convergence of spheres for almost every \( r > 0 \)

\[ \partial S(x_{j_k}, r) \overset{\mathcal{F}}{\rightarrow} \partial S(x_\infty, r) \]

Combining sphere convergence as in (179) with filling volume continuity as in (122), Portegies and the author have proven that for almost every \( r > 0 \)

\[ \text{FillVol}(\partial S(x_{j_k}, r)) \overset{\mathcal{F}}{\rightarrow} \text{FillVol}(\partial S(x_\infty, r)) \]

as well as continuity of another notion called the sliced filling volume in \[\text{PS14}\]. In Theorem 4.27 of that paper, this continuity is applied to determine when a sequence converges in \( X_\infty = \text{set}(T_\infty) \). The filling volume case of this theorem observes that
\[ M(S(x_\infty, r)) \geq \text{FillVol}(\partial S(x_\infty, r)) = \lim_{k \rightarrow \infty} \text{FillVol}(\partial S(x_{j_k}, r)), \]

which implies that if there is a uniform lower bound \( C > 0 \) such that
\[ \text{FillVol}(\partial S(x_j, r)) \geq Cr^m \quad \forall j \in \mathbb{N} \]

then by (48), \( x_\infty \in \text{set}(T_\infty) = X_\infty \). An idea similar to this one was applied earlier in an extrinsic way by the author and Wenger in \[\text{SW10}\] to show that when a sequence of manifolds has nonnegative Ricci curvature or has a linear contractibility function, then the GH and \( \mathcal{F} \) limits agree. This method has been applied to prove intrinsic flat and GH limits agree under a variety of different conditions by the author with Portegies, by Munn, by Perales, by Perales-Li, and by Matveev-Portegies \[\text{PS14}{\text{Mun14}}\text{Per15a}{\text{LP15}}\text{MP15}].
With only lower scalar curvature bounds on the sequence one does not in general have GH and intrinsic flat limits that agree (see examples below) FILLIN and in fact one may not have any GH limit even for a subsequence. Nevertheless this method might be applied to determine which points in the sequence of manifolds are disappearing and which remain.

In fact the author and Portegies prove a Bolzano-Weierstrass Theorem for sequences of points with bounds on their filling or sliced filling volume [PS14] [Theorem 4.30]. One consequence of this theorem is that if a sequence satisfies (181) then a subsequence is Cauchy (using an argument involving the fact that the limit space has finite mass) and then by the above method, the sequence converges to a point in $X_{\infty}$. There is also a Bolzano-Weierstrass Theorem for sequences of points which does not require a lower bound on the filling volumes of spheres, but instead requires

$$\exists 0 > 0 \ s.t. \ \lim_{j \to \infty} \inf d_F(S(x_j, r), 0) \geq h(r) > 0 \ \text{for a.e. } r \in (0, r_0)$$

(183)

to obtain a subsequence which is Cauchy and converges in $\tilde{X}_{\infty}$ [Sor14] [Theorem 7.1]. Note that the theorems and proofs in [Sor14] are very easy to read. The more technically difficult results in [PS14] involving filling volumes and sliced filling volumes are better in that they provide more precise controls under weaker hypothesis.

6.2. Limits of Functions. Recall that when using the compactness and weak rigidity method to prove an almost rigidity theorem as in (19), one proves that $M_j \to M_0$, where $M_0$ is a specific given rigid space by first using a compactness theorem to show a subsequence, $M_{j_k} \to M_\infty$ and then proving $M_\infty = M_0$. One way to prove that $M_\infty = M_0$ is to construct an isometry between these spaces as a limit of functions from $M_{j_k}$ to $M_0$. Theorems which produce limits of subsequences of functions are called Arzela-Ascoli Theorems.

First let us describe what we mean by a limit function. Suppose $F_j : X_j \to Y_j$ are functions, and $X_j \to X_\infty$ and $Y_j \to Y_\infty$ in the GH or $F$ sense then we say $F_\infty : X_\infty \to Y_\infty$ is their limit, denoted $F_j \to F_\infty$, if

$$F_j(x_j) \to F_\infty(x_\infty) \ \text{whenever } x_j \to x_\infty$$

(184)

More precisely, $F_j \to F_\infty$ if there exists convergence functions, $H_j : X_\infty \to X_j$ and $H'_j : Y_\infty \to Y_j$, as in (170) such that

$$F_j \circ H_j(x) = H'_j \circ F_\infty(x) \ \forall x \in X_\infty.$$ 

(185)

The Gromov-Hausdorff Arzela-Ascoli Theorem states that if one has compact metric spaces, $X_j \to X_\infty$, $Y_j \to Y_\infty$, and if $F_j : X_j \to Y_j$ are equicontinuous

$$\forall \epsilon > 0 \ \exists \delta_\epsilon > 0 \ \text{such that } d_{X_j}(x, x') < \delta_\epsilon \ \Rightarrow \ d_{Y_j}(F_j(x), F_j(x')) \leq \epsilon.$$ 

(186)

then $F_j \to F_\infty$ where $F_\infty : X_\infty \to Y_\infty$ satisfies (186). This theorem is a direct consequence of Gromov’s Embedding Theorem combined with the standard proof of the classical Arzela-Ascoli Theorem (cf. [Sor14]). Furthermore, if $F_j$ are surjective then the limit $F_\infty$ is surjective. If the $F_j$ are isometries on balls of radius
r_0 > 0$, then the limit is an isometry on balls of radius $r_0 > 0$. This was applied by the author in joint work with Wei to prove the GH limits of manifolds with Ricci $\geq 0$ have universal covering spaces [SW01] and that the covering spectrum is continuous with respect to Gromov-Hausdorff convergence in [SW04].

It is not absolutely necessary that the sequence of functions be equicontinuous. Gromov proves the same result for $\epsilon_j$ almost isometries $F_j : X_j \to Y_j$.

\begin{equation}
|d_Y(F_j(p), F_j(q)) - d_X(p, q)| < \epsilon_j \text{ and } Y_j \subset T_{\epsilon_j}(X_j)
\end{equation}

with $\epsilon_j \to 0$, producing a limit $F_\infty : X_\infty \to Y_\infty$ which is an isometry [Gro81]. See also the Burago-Burago-Ivanov text [BBI01]. Almost isometries are applied to prove the GH almost rigidity theorems of Colding in [Col97] and of Cheeger-Colding in [CC96] by constructing almost isometries from the $M_j$ to the rigid space, $M_0$. In [Sor04], a GH Arzela-Ascoli Theorem which only requires almost equicontinuity is proven by the author in order to prove another GH almost rigidity theorem.

One cannot hope for a $\mathcal{F}$ Arzela-Ascoli Theorem which is as powerful as the GH Arzela-Ascoli Theorem. In Example 2.5 one has no limit for the geodesics $\gamma_j : [0, 1] \to M_j$ running through the increasingly thin tunnels. Nevertheless two useful Arzela-Ascoli Theorems with additional hypotheses were proven in [Sor14].

Suppose $F_j : M_j \to M_j'$ are (surjective) current preserving isometries on balls of radius $r_0 > 0$ and $M_j \xrightarrow{\text{GH}} M_\infty \neq 0$ and $M_j' \xrightarrow{\text{GH}} M_\infty' \neq 0$. Then a subsequence of $F_j$ converges to $F_\infty : M_\infty \to M_\infty$ which is also a (surjective) current preserving isometry on balls of radius $r_0 > 0$ [Sor14]. This theorem has been applied in joint work of the author with Sinaei to study the intrinsic flat convergence of covering spaces and the covering spectrum [SS16a].

Suppose $F_j : M_j \to Y_j$ are equicontinuous maps as in (186) where $M_j$ are integral current spaces and $Y_j$ are compact metric spaces such that $M_j \xrightarrow{\text{GH}} M_\infty \neq 0$ and $Y_j \xrightarrow{\text{GH}} Y_\infty$. Then $F_j$ converge to $F_\infty : M_\infty \to Y_\infty$ which also satisfies (186). Furthermore, if $F_j$ are surjective then the limit $F_\infty$ is surjective [Sor14]. Keep in mind that this includes equicontinuous functions, $F_j : M_j \to [a, b]$, and embeddings in compact regions in Euclidean space, $F_j : M_j \to \mathbb{B}^N$. This theorem has been applied jointly with Huang and Lee to prove Almost Rigidity of the Positive Mass Theorem in the graph setting [HLS16].

Conjectured related Arzela-Ascoli theorems are suggested in [Sor14] and the proofs there are not difficult to read. These theorems are particularly useful when proving almost rigidity theorems to try to construct isometries from the limit $M_\infty$ of a subsequence $M_j$ to the desired rigid space, $M_0$.

**Problem 6.1.** Suppose $F_j : M_j \to Y$ has $\text{Lip}(F_j) \leq 1$ with $Y$ compact (including Riemannian embeddings and graphs), and $M_j$ satisfy the hypothesis of Wenger’s Compactness Theorem as in (125). Then we know a subsequence $M_j \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$. We can see $F_{j\delta}[M_j]$ satisfies the hypothesis of the Ambrosio-Kirchheim Compactness as in (64), so we know a subsequence $F_{j\delta}[M_j]$ converges in the flat sense to some $S_\infty \in \mathbf{L}(Y)$. If $M_\infty \neq 0$ we know the Arzela-Ascoli
Theorem above that a subsequence, also denoted $F_j$, converges to $F_\infty : X_\infty \to Y$. We conjecture that

$$M_\infty = (X_\infty, d_\infty, T_\infty) \text{ with } F_{\infty#} T_\infty = S_\infty.$$  

This conjecture captures the key steps used in joint work of the author with Huang and Lee to prove the Almost Rigidity of the Positive Mass Theorem in the graph setting in [HLS16]. In that proof we have $M^n_j$ which satisfy our almost rigidity conditions, and show they satisfy the hypothesis of Wenger’s Compactness Theorem and then study their images as graphs in $Y = \mathbb{R}^{m+1}$ and obtain (188). Some of the techniques there may be useful towards proving this conjecture in general.

6.3. Intrinsic Flat with Volume Convergence. Recall that intrinsic flat convergence does not imply volume convergence. One does have semicontinuity, $M_j \overset{\mathcal{F}}{\to} M_\infty$ implies $\liminf_{j \to \infty} M(M_j) \geq M(M_\infty)$, but even when $M_j$ and $M_\infty$ are Riemannian manifolds the volumes need not converge (as seen in the examples with cancellation). Note that all the examples presented above with wells, bubbling and sewing do have volume convergence. We introduce the following:

**Definition 6.2.** The intrinsic flat volume distance between two integral current spaces, $M_j = (X_j, d_j, T_j)$ is

$$d_{\mathcal{VF}}(M_1, M_2) = d_{\mathcal{F}}(M_1, M_2) + |M(M_1) - M(M_2)|$$

So $M_j \overset{\mathcal{VF}}{\to} M_\infty$ if $M(M_j) \to M(M_\infty)$ and $M_j \overset{\mathcal{F}}{\to} M_\infty$. Note that points may still disappear as in the Ilmanen Example. However there is no cancellation.

Portegies studied the properties of $M_j = (X_j, d_j, T_j)$ such that $M_j \overset{\mathcal{VF}}{\to} M_\infty$. Applying his Lemma 2.7 in [Por15], we see that the metric measure spaces, $(X_j, d_j, \|T_j\|)$ converge in a measured sense. That is, there are distance preserving maps, $\varphi_j : X_j \to Z$ such that

$$\varphi_j \|T_j\| \to \varphi_j \|T_\infty\| \text{ weakly as measures in } Z.$$  

Here $Z$ is only complete and one need not have GH convergence (as in Ilmanen’s Example). Portegies then applied this to prove that the Laplace eigenvalues of converging sequences of manifolds are upper semicontinuous,

$$\limsup_{j \to \infty} \lambda_j(M_j) \leq \lambda_j(M_\infty).$$

He presents examples showing this is false without $M(M_j) \to M(M_\infty)$.

Note that the Hawking Mass Compactness Conjecture and Scalar Compactness Conjecture both propose that a subsequence $M_{j_k} \overset{\mathcal{VF}}{\to} M_\infty$ as is shown in the compactness theorem proven jointly with LeFloch [LS15]. Also note that in the work of the author with Lee, Huang, and Stavrov proving various special cases towards almost rigidity of the Positive Mass Theorem, it is proven that $M_j \overset{\mathcal{VF}}{\to} M_0$ [LS14] [HLS16] [SS16b]. Matveev and Portegies prove $M_{j_k} \overset{\mathcal{VF}}{\to} M_\infty$ when Ricci curvature is uniformly bounded below [MP15][Theorem 4.1].
Keep in mind that $M_{j} \xrightarrow{VF} M_\infty$ alone will not suffice to achieve generalized Scalar $> 0$ properties on $M_\infty$ as seen in the problematic examples involving sewing of a single curve to a point will have volume convergence and yet the limit in $\{5\}$ will fail to be nonnegative for a ball about that limit point. Note however that if one assumes $\{5\}$ holds at every point $p$ in every $M_j$ with a uniform lower bound on $r_p \geq r_0 > 0$, then $\{5\}$ can be shown to hold with inequalities on $M_\infty$.

7. Results and Conjectures about Limits of Manifolds with Nonnegative Scalar Curvature

In this final section of the paper we state our almost rigidity conjectures precisely and survey known results towards proving these conjectures. We suggest special cases which might be proven more easily. Note that completely proving any almost rigidity theorem is significantly more difficult that proving the corresponding rigidity theorem. One must either reprove the rigidity theorem in a quantitative way obtaining $\{18\}$ using the explicit control method as described in the introduction. Or one must prove the Rigidity Theorem on a generalized class of spaces obtaining $\{19\}$ using the compactness and weak rigidity method. Recall that we have already proposed two compactness conjectures in Section 4.3. In the final two subsections of this paper we discuss generalized notions of nonnegative scalar curvature as proposed by Gromov and regularity theory.

7.1. Almost Rigidity of the Positive Mass Theorem and the Bartnik Conjecture. Consider the class, $\mathcal{M}$, of asymptotically flat three dimensional Riemannian manifolds with nonnegative scalar curvature and no interior closed minimal surfaces and either no boundary or the boundary is an outermost minimizing surface. This is the physically natural class of spaces used to prove the Penrose Inequality as discussed in the introduction. By the Positive Mass Theorem, if $M \in \mathcal{M}$ has $m_{\text{ADM}}(M) = 0$ then $M$ is isometric to Euclidean space. In $\text{[LS14]}$, Lee and the author proposed that almost rigidity for the Positive Mass Theorem should be provable using intrinsic flat convergence and demonstrated that it is false for GH convergence.

Conjecture 7.1. Fix $D > 0$, $r_0 > 0$. Let $M^3_j \in \mathcal{M}$ and let $\Sigma_j \subset M^3_j$ be special surfaces with $\text{Area}(\Sigma_j) = 4\pi r_0^2$ and $\Sigma_\infty = \partial B(0, r_0) \subset \mathbb{R}^3$. We conjecture that

\begin{equation}
\lim_{j \to \infty} m_{\text{ADM}}(M_j) = 0 \implies d_{\text{VolF}} \left( T_D(\Sigma_j), T_D(\Sigma_\infty) \right) \to 0.
\end{equation}

where $T_D(\Sigma)$ is the tubular neighborhood of radius $D$ around $\Sigma$ or alternatively

\begin{equation}
\lim_{j \to \infty} m_{\text{ADM}}(M_j) = 0 \implies d_{\text{VolF}} \left( \Omega_j, \Omega_\infty \right) \to 0.
\end{equation}

where $\Omega_j$ is the interior of $\Sigma_j$ with $\text{Depth}(\Omega_j) \leq D$.

Note that Lee and the author were being deliberately vague as to what a special surface, $\Sigma_j$, should be in this conjecture. A number of more precisely stated special cases of this conjecture were provided in the final section of $\text{[LS14]}$ along with brief ideas as to how one might approach the proof of the conjecture in those cases. Most of these special cases are still open.
Lee and the author proved that (192) holds when $M_j^3$ have metric tensors of the form $g_\j = dr^2 + f_\j(r)^2 g_S^2$ and $\Sigma_\j = r^{-1}(r_0)$. The proof uses Geroch monotonicity and Theorem 5.1 to obtain explicit controls on $T_D(\Sigma_\j)$ [LS14]. LeFloch and the author examined these explicit controls in more detail in [LS15]. The author and Stavrov proved this conjecture when $M_j^3$ are Brill-Lindquist geometrostatic manifolds and $\Sigma_\j$ are large spheres in [SS16]. That proof is also completed using explicit controls: bounds on the metric tensor are found on carefully selected regions within the manifolds followed by an application of Theorem 5.2.

Huang, Lee and the author proved (193) when $\Omega_\j$ are graph manifolds, which have Riemannian embeddings, $\Psi_\j : M_j^3 \to \mathbb{B}^4$ as graphs. We assumed controls on the $\Sigma_\j$ and other technical properties on the $M_j^3$ [HLS16]. Using the properties of graph manifolds with $\text{Scalar} \geq 0$ and $m_{\text{ADM}}(M_j) \to 0$ we first proved that $\text{Vol}(\Omega_\j) \to \text{Vol}(B(0, r_0))$. We applied Wenger’s Compactness Theorem and an Arzela Ascoli Theorem to prove a subsequence of the $\Omega_j \to \Omega_\infty$ and $\Psi_j$ converge to $\Psi_\infty : \Omega_\infty \to \mathbb{B}^4$ with $\text{Lip}(\Psi_\infty) \leq 1$. By the lower semicontinuity of mass we showed

$$
\text{M}(\Psi_\infty(\Omega_\infty)) \leq \text{M}(\Omega_\infty) \leq \text{Vol}(B(0, r_0)).
$$

(194)

We used the controls on $\Sigma_j$ to prove they Lipschitz converge to $\partial \Omega_\infty$ and that $\Psi_\infty : \partial \Omega_\infty \to \partial B(0, r_0) \times \{0\}$ is biLipschitz. Thus $\partial \Psi_\infty(\Omega_\infty) = [\partial B(0, r_0) \times \{0\}]$. Combining this with (194) implies $P_{\text{iso}(\Omega_\infty)}$ solves the Plateau Problem. So

$$
\Psi_\text{iso}(\Omega_\infty) = [B(0, r_0) \times \{0\}] \text{ and } \text{M}(\Psi_\text{iso}(\Omega_\infty)) = \text{Vol}(B(0, r_0))(4/3).
$$

(195)

This implies equality in (194) so $\Psi_\infty$ must be an isometry: $\Omega_\infty = B(0, r_0)$ [HLS16].

**Remark 7.2.** One might consider applying the Huisken isoperimetric mass as in [11] to prove Conjecture 7.1 with $\Sigma_j$ chosen to be uniformly asymptotically spherical in $M_j$ with $m_{\text{ADM}}(M_j) \to 0$ so that

$$
m_{\text{IS}O}(\Omega_j) = \frac{2}{\text{Area}(\partial \Omega_j)} \left( \text{Vol}(\Omega_j) - \frac{\text{Area}(\partial \Omega_j)^{3/2}}{6 \sqrt{\pi}} \right) \to 0.
$$

(196)

Observe that (196) immediately implies

$$
\text{Vol}(\Omega_j) \to A_0^{3/2} / (6 \sqrt{\pi}) = (4/3) \pi r_0^3 = \text{Vol}(B(0, r_0))
$$

(197)

So by Wenger’s Compactness Theorem a subsequence converges $\Omega_j \to \Omega_\infty$. By lower semicontinuity of mass we have $\text{M}(\Omega_\infty) \leq \text{Vol}(B(0, r_0))$.

Suppose we also assume that there exist Riemannian isometric embeddings $\Psi_j : \Omega_j \to \mathbb{B}^N$ with the property that when $\Psi_j$ restricted to $\partial \Omega_j$ is uniformly biLipschitz to $\partial B(0, r_0) \times \{0, ..., 0\}$. Then exactly as in the above description of the proof in [HLS16] we have $\Omega_\infty$ isometric to $B(0, r_0)$. Here the only place we used Scalar $\geq 0$ was when we replaced $m_{\text{ADM}}(M_j) \to 0$ by (196) considering that Miao’s proof that $m_{\text{IS}O}(\Omega)$ is close to $m_{\text{ADM}}(M)$ for large round $\partial \Omega$ in [FST09] uses Scalar $\geq 0$. Note that $m_{\text{IS}O}(\Omega)$ for $M^3 \in M$ does not imply $\Omega = B(0, r_0)$; one needs to impose some asymptotic roundness on the $\partial \Omega$ as well as Scalar $\geq 0$ even to obtain rigidity.
Problem 7.3. Shi and Tam proved in [ST02] that the Brown-York Mass, $m_{BY}(\partial \Omega)$, is nonnegative if \( \text{Scalar} \geq 0 \) on $\Omega$ and $\partial \Omega$ has positive Gauss curvature, and
\[
m_{BY}(\partial \Omega) = 0 \implies \Omega \subset \mathbb{E}^3.
\]
This mass (which agrees with the Liu-Yau mass in this setting) is defined using a Riemannian isometric embedding $\Psi : \partial \Omega \to \mathbb{E}^3$, the mean curvature, $H$, of $\partial \Omega \subset M^3$ and the mean curvature, $H_0$, of $\Psi(\partial \Omega) \subset \mathbb{E}^3$ as follows:
\[
m_{BY}(\partial \Omega) = \frac{1}{8\pi} \int_{\partial \Omega} H_0 - H \, d\sigma
\]
(198)
(199)

The Arzela-Ascoli Theorems proven above might be helpful towards proving Conjecture 4.5 for the Brown-York Mass including semicontinuity of the Brown-York mass under $\mathcal{F}$-convergence and almost rigidity of the Shi-Tam Rigidity Theorem. This might also be applied to prove the almost rigidity of the Positive Mass Theorem or the Bartnik Conjecture. One of the biggest difficulties here is that mean curvature must be defined in a generalized way and controlled under intrinsic flat convergence.

Problem 7.4. Huisken and Ilmanen defined a weak notion of mean curvature in their proof of the Penrose Inequality (which can also be applied to prove the Positive Mass Theorem) [HI01]. Perhaps one might try to use their method to prove Almost Rigidity of the Positive Mass Theorem. One might consider a limit space $M_\infty$ and attempt to define Huisken-Ilmanen’s weak inverse mean curvature flow and prove Geroch monotonicity on $M_\infty$. What regularity is needed on $M_\infty$? What notion of nonnegative scalar curvature is required?

Recall that in [Ger73], Geroch proved that if $N_t : \mathbb{S}^2 \to M^3$ evolves by smooth inverse mean curvature flow,
\[
dt x = \frac{\nu}{H} \quad \text{where} \quad x = N_t(p), \quad \nu \text{ is the normal to } N_t \text{ at } x,
\]
\[
and H \text{ is the mean curvature of } N_t \text{ at } x,
\]
and $M^3$ has Scalar $\geq 0$ then then the Hawking mass, $m_H(N_t)$, is nondecreasing.
\[
t_2 > t_1 \implies m_H(N_{t_2}) \geq m_H(N_{t_1}).
\]
(200) (201) (202)

In [HI01], Huisken-Ilmanen introduced weak inverse mean curvature flow,
\[
N_t = \partial \{ x : \ u(x) < t \} \quad \text{where} \quad \nabla_M \left( \frac{\nabla u}{|\nabla u|} \right) = |\nabla u|,
\]
(203)
proving it also satisfies Geroch monotonicity as in (202), and with the right boundary conditions $\lim_{t \to \infty} m_H(N_t) = m_{ADM}(M)$. They defined a weak mean curvature for the level sets of $u$ to be $H = \nabla u$ almost everywhere. One may naturally ask what regularity was needed on the limit space to prove these results.

Alternatively one might apply Huisken-Ilmanen’s method on the sequence $M_j$ rather than on $M_\infty$. One might consider sequences of $u = u_j$ satisfying (203) on $M_j$ and consider limits $u_j \to u_\infty$, if it is difficult to define (203) on $M_\infty$ itself. On each $M_j$ one can define (203) and then one has $m_H(N_j)$ nearly constant. This was a key step in the proof of the Almost Rigidity of the Positive Mass Theorem
in the rotationally symmetric case \cite{LS14}. It would be interesting to investigate this even in the setting with smooth inverse mean curvature flow. In the setting where one only has weak inverse mean curvature flow, then the $N_i$ may skip over entire regions. In private communication with the author, Huisken has suggested that it might be possible to bound the volume of the skipped regions using his isoperimetric masses. These same techniques might also be applied to prove the Almost Rigidity of the Penrose Inequality.

**Problem 7.5.** Recall that the Penrose Inequality for $M^3 \in M$:

\begin{equation}
\label{eq:PenroseInequality}
m_{\text{ADM}}(M^3) \geq m_{\text{H}}(\partial M^3) = \sqrt{\frac{\text{Area}(\Sigma)}{16\pi}}
\end{equation}

and Penrose Rigidity:

\begin{equation}
\label{eq:PenroseRigidity}
m_{\text{ADM}}(M^3) = m_{\text{H}}(\partial M^3) \implies M^3 \text{ is isometric to } M_{S_{ch,m}}
\end{equation}

where $M_{S_{ch,m}}$ is the Riemannian Schwarzschild space with mass $m = m_{\text{ADM}}(M^3)$:

\begin{equation}
\label{eq:Schwarzschild}
M_{S_{ch,m}} = (\mathbb{R}^3 \setminus \{0\}, (1 + \frac{m}{2r})^2 \delta).
\end{equation}

Almost rigidity for the Penrose Inequality was conjectured jointly with Lee in \cite{LS12} where it was proven in the rotationally symmetric setting. This has not yet been explored for graph manifolds nor for Brill-Lindquist Geometrostatic manifolds. These are perhaps easy enough to assign as a first project to a doctoral student as the techniques in \cite{HLS16} and \cite{SS16b} should directly apply. Proving it in general would involve all the same difficulties as proving the almost rigidity for the Positive Mass Theorem and more.

### 7.2. The Bartnik Conjecture

Bartnik’s quasilocal mass of a region $\Omega_0 \subset M^3$ where $M_0^3 \in M$ and $\partial M_0^3 \subset \Omega_0$ was defined in \cite{Bar86} as an infimum of the ADM masses of extensions, $M$, of $\Omega_0$:

\begin{equation}
\label{eq:BartnikMass}
m_B(\Omega_0) = \inf \{m_{\text{ADM}}(M) | M \in \mathcal{P}M(\Omega_0)\}
\end{equation}

where $M \in \mathcal{P}M \subset M$ if it contains an isometric image of $\Omega$: $\Omega \subset M$. Bartnik conjectured that this infimum is achieved by what he called the minimal mass extension and that this minimal mass extension is scalar flat and static. Significant research in this area has been completed by Corvino in \cite{Cor00} in which the properties of a minimal mass extension are proven assuming it exists. Miao has searched for static extensions using perturbative methods in \cite{Mia03}. To prove the minimal mass extension exists in general one may consider the following approach:

**Conjecture 7.6 (Bartnik Conjecture).** There exists a sequence

\begin{equation}
\label{eq:BartnikConjecture}\tag{208}
M_j \in \mathcal{P}M(\Omega_0) \text{ such that } m_{\text{ADM}}(M_j) \to m_B(\Omega_0).
\end{equation}

with a limit $M_j \to M_\infty$ such that $m_{\text{ADM}}(M_\infty) = m_B(\Omega_0) = \lim_{j \to \infty} m_{\text{ADM}}(M_j)$, where $M_\infty$ is asymptotically flat, smooth, scalar flat and static.

By the definition of Bartnik Mass we know there is a sequence satisfying (208). We propose that one may be able to prove an intrinsic flat compactness theorem
for \( \Omega_{R,j} \subset M_j \) where \( \Omega_{R,j} = T_R(\partial \Omega) \) or perhaps \( \Omega_{R,j} \supset \Omega_0 \) with well chosen \( \partial \Omega_{R,j} = \Sigma_{R,j} \subset M_j \). Note that by Wenger’s Compactness Theorem we only need
\[
\text{(209)} \quad \text{Vol}(\Omega_{R,j}) \leq V_R \quad \text{and} \quad \text{Vol}(\partial \Omega_{R,j}) \leq A_R
\]
to obtain a subsequence \( \Omega_{R,j} \rightarrow \Omega_{R,\infty} \) since \( \text{Diam}(\Omega_{R,j}) \leq 2R + \text{Diam}(\Omega_0) \). Then we could glue together an \( M_\infty \) from these limit regions \( \Omega_{R,\infty} \) and \( \Omega_0 \subset M_\infty \). Our conjectured Hawking mass compactness theorem would imply that \( M_\infty \) has generalized Scalar \( \geq 0 \) and a well controlled Hawking mass.

However one must be warned that an \( M_\infty \) obtained in this manner need not be asymptotically flat. In [LS12], Lee and the author proved that a sequence \( M_j \) approaching equality in the Penrose Inequality may develop a longer and longer neck. The recent examples of Mantoulidis and Schoen which satisfy \( \text{Vol}(\Omega_{R,j}) = V_0 \) and \( \text{Vol}(\partial \Omega_{R,j}) \leq A_0 > 0 \).

Conjecture 7.7. Let \( M^3_j \) be diffeomorphic to \( \mathbb{T}^3 \) with \( \text{Scalar}(M^3_j) \geq -1/j \) and
\[
\text{(210)} \quad \text{Vol}(M^3_j) = V_0 \quad \text{Diam}(M^3_j) \leq D_0 \quad \text{MinA}(M^3_j) \geq A_0 > 0.
\]
Then a subsequence \( M_{j_k} \rightharpoonup M_0 \) where \( M_0 \) is a flat torus. Possibly \( M_{j_k} \rightharpoonup M_\infty \).

Note that by Wenger’s Compactness Theorem, we know \( M_{j_k} \rightharpoonup M_\infty \), and by semicontinuity we have \( \text{Diam}(M_\infty) \leq D_0 \) and \( \text{M}(M_\infty) \leq V_0 \). We do not know if \( M_\infty \) is the \( 0 \) integral current space. So even proving that much would be interesting.
Some points will disappear as seen in Example 2.3, so one must find a sequence of points which doesn’t disappear.

Even assuming that $M_\infty \neq 0$ one needs to prove that it has some sort of generalized Scalar $\geq 0$ which is strong enough to prove torus rigidity. Gromov suggests a few such notions which should be strong enough in [Gro14a] and prove $C^0$ limits of $M_j$ as above satisfy these conditions. Examining his proofs and considering filling volumes and sliced filling as defined in joint work of the author with Portegies [PS14] might be helpful. Note that Bamler has proven Scalar $\geq 0$ persists under $C^0$ convergence using Ricci flow in [Bam16].

Perhaps a more approachable problem would be to consider first $M_j$ which are graphs over the standard $\mathbb{T}^3$ and try to prove $M_j \xrightarrow{F} \mathbb{T}^3$ using methods similar to those used in joint work of the author with Huang and Lee [HLS16]. Another possibility is to consider $M_j$ with metric tensors $g_j = dx^2 + dy^2 + f_j(x,y)dz^2$ or $g_j = a_j(z)^2 dx^2 + b_j(z)^2 dy^2 + dz^2$ and first try to prove the warping functions $a_j, b_j$ and $f_j$ converge in the $H^1_{loc}$ sense to a metric with generalized Scalar $\geq 0$ as in joint work of the author with LeFloch [LS15]. It is possible that spinors and the Lichnerowetz formula might be formulated weakly for an $H^1_{loc}$ metric tensor. More likely one can find a partial differential inequality on the warping functions which implies the rigidity theorem and can be shown to persist weakly under $H^1_{loc}$ convergence and the intrinsic flat limit.

7.4. Almost Rigidity Theorems and Ricci Flow. Bray, Brendle and Neves proved the Cover Splitting Rigidity Theorem as in (15) and Bray, Brendle, Eichmair and Neves have proven the $\mathbb{R}P^3$ Rigidity Theorem as in (16) using Ricci flow [BBN10] [BBEN10]. Here we propose the following:

**Conjecture 7.8** (Almost Rigidity of the $\mathbb{R}P^3$ Rigidity Theorem).

Given a sequence of $M_j$ that are diffeomorphic to $\mathbb{R}P^3$ with

\[(211) \quad \text{Scalar}_j \geq \frac{6j}{j-1}, \quad \text{MinA}(M^3) = 2\pi, \quad \text{Vol}(M^3_j) \leq V_0, \quad \text{Diam}(M^3_j) \leq D_0 \]

then $M_j \xrightarrow{F} M_0 = \mathbb{R}P^3$ or possibly even $M_j \xrightarrow{WF} M_0 = \mathbb{R}P^3$.

One may construct examples with wells demonstrating that $M_j \xrightarrow{GH} \mathbb{R}P^3$ may fail.

**Conjecture 7.9** (Almost Rigidity of Cover Splitting Theorem).

Given a sequence of $M_j$ that contain noncontractible $S^2$ and have

\[(212) \quad \text{Scalar}_j \geq \frac{2j}{j-1}, \quad \text{MinA}(M^3) = 4\pi, \quad \text{Vol}(M^3_j) \leq V_0, \quad \text{Diam}(M^3_j) \leq D_0 \]

then a subsequence $M_{j_k} \xrightarrow{F} M_0$ where $\tilde{M}_0$ is isometric to $S^2 \times \mathbb{R}$.

**Problem 7.10.** Can one construct $M_j$ with $\text{Scalar} \geq 2j/(j - 1)$ that are diffeomorphic to $S^2 \times S^1$ and contain balls of radius $1/4$ that are isometric to balls in rescaled standard spheres? If so then one can attach wells and have no GH limit.
One approach to proving these conjectures would be to use volume renormalized Ricci flow, \( M_{j,t} \), of \( M_j \) and show \( M_{j,t} \) flows as \( t \to \infty \) to an \( M_{j,\infty} \) which is isometric to \( M_0 \). One may simplify things by assuming a smooth Ricci flow exists for all time, or try to prove this, or attempt to deal with Ricci flow through singularities. If

\[
d_{Vol}(M_{j,t}, M_{j,s}) < \epsilon\left(\frac{1}{t} - \frac{1}{s}\right) \quad \text{where} \quad \lim_{\delta \to 0} \epsilon(\delta) = 0
\]

where \( \epsilon \) is independent of \( j \), then we have the conjecture. Continuity of Ricci flow with respect to the intrinsic flat distance has been studied by Lakzian in [Lak15b] but only to analyze the Ricci flow through a neck pinch singularity. The author believes one might be able to obtain (213) constructing an explicit

\[
Z = \{(x, r) | x \in M_{j(1/r)}, \ r \in [(1/t), (1/s)]\} \quad \text{with} \quad g = dt^2 + f(r)^2 g_{j(1/r)}.
\]

Again one might consider special cases where \( M_j \) are known to be warped products.

7.5. Gromov’s Prisms and Generalized Nonnegative Scalar Curvature. Gromov suggested in [Gro14b] that if a sequence of \( M_j \xrightarrow{F} M_\infty \) has Scalar \( \geq 0 \) then \( M_\infty \) might have Scalar \( \geq 0 \) in some generalized sense. In [Gro14a], Gromov proved the Gauss Bonnet Prism Inequality for prisms in manifolds with Scalar \( \geq 0 \):

\[
3 \sum_{i=1}^{3} \alpha_i \geq \pi \quad \text{and} \quad \sum_{i=1}^{3} \alpha_i = \pi \quad \implies \ P \text{ is a prism in } \mathbb{R}^3
\]

where \( \alpha_i \) are bounds on the dihedral angles between the sides of the prism which are minimal surfaces. He suggests that the Prism Inequality could be used to define generalized Scalar \( \geq 0 \) and then be applied to prove the torus rigidity theorem on such limit spaces.

Before we can define such a generalized notion of nonnegative scalar curvature on integral current spaces, we would need to prove more regularity on the \( F \) limit spaces. Ordinarily there is no notion of angle on an integral current space, nor is an integral current space a geodesic space. In fact it might even be the zero space. It should be noted that Ilmanen, the author and Wenger conjectured in [SW11][Conjecture 4.18] that “a converging sequence of three dimensional Riemannian manifolds with positive scalar curvature, a uniform lower bound on volume, and no interior closed minimal surfaces converges without cancellation to a nonzero integral current space.” Combining Gromov’s ideas with this one we introduce the following conjecture:

**Conjecture 7.11.** Suppose \( M^3_j \xrightarrow{F} M_\infty \) or \( M^3_j \xrightarrow{VF} M_\infty \) where Scalar \( \geq 0 \) and

\[
\text{MinA}(M_j) \geq A_0 > 0 \quad \text{Vol}(M_j) \in [V_0, V_1] \subset (0, \infty) \quad \text{Diam}(M_j) \leq D_0.
\]

Then we have the following:

(a) \( M_\infty \) is a nonzero integral current space. In fact there is no cancellation without collapse: If \( p_j \in M_j \) has no limit in \( M_\infty \) then \( \exists \delta > 0 \) s.t. \( \text{Vol}(B(p_j, \delta)) \to 0. \)
(b) $M_\infty$ is geodesic: If $p, q \in M_\infty$ there exists $p_j \to p$ and $q_j \to q$ with midpoints $x_j$ that converge to $x_\infty$ which is a midpoint between $p$ and $q$ in $M_\infty$.

(c) There is a notion of angle between geodesics emanating from a point.

(d) There is a notion of dihedral angle between two surfaces at $p \in \Sigma \cap \Sigma' \subset M_\infty$.

(e) Gromov’s Gauss-Bonnet Prism Inequality as in (215) holds on $M_\infty$.

(f) $\forall p \in M_\infty \exists r_p > 0$ s.t. $\forall r < r_p \ V_p(r) = M(B(p,r))/(4\pi r^3/3) \leq 1$.

Note that (a)-(d) are regularity properties for our limit spaces. As seen in Examples above, they do not hold on limits of manifolds with $\text{Scalar}_j \geq 0$ unless the $\text{MinA}(M_j) \geq A_0 > 0$. The notion of sliced filling volume developed in joint work with Portegies in [PS14] might be helpful towards proving (a) and (b). It is quite possible that (c) is false but that one can still prove (d) using the limit process to define the dihedral angle. In the special case where there exists Riemannian embeddings $\Psi_j : M_j \to \mathbb{E}^N$, one can use an Arzela-Ascoli Theorem to obtain $\Psi_\infty : M_j \to \mathbb{E}^N$ and then use $\mathbb{E}^N$ to define angles as needed in (c) or (d) and examine the semicontinuity of such angles. Note that any Riemannian manifold satisfies (a)-(d) so they do not capture a generalized notion of nonnegative scalar curvature.

Now (e)-(f) are properties which capture $\text{Scalar} \geq 0$. Property (e) proposed by Gromov needs a notion of angle so one needs to prove (d) first or use embeddings into some large $\mathbb{E}^N$ to even define what this means. Then one need only prove semicontinuity of the angles. The power of property (e) is described in [Gro14a] including ideas towards the possibility that (e) implies (f). Recall that (f) on a Riemannian manifold is equivalent to $\text{Scalar} \geq 0$ but was not powerful enough to prove any global properties of such spaces. Nor is (f) continuous with respect to $\text{VolF}$ convergence unless one assumes a uniform lower bound $r_p \geq r_0 > 0$ for all $p \in M_j$ for all $j \in \mathbb{N}$. Nevertheless any natural notion of $\text{Scalar} \geq 0$ on an integral current space ought to imply (f). Note that Gromov also proves hyperbolic and spherical prism inequalities on spaces with lower bounds on scalar curvature which are negative or positive respectively and that such inequalities might be used to generalize these lower bounds on scalar curvature and one might prove they imply the appropriate limit as in (5) on $M_\infty$.

**Remark 7.12.** In Remark 7.3 we proposed that the Brown York mass might be semicontinuous with respect to $\text{F}$ convergence. One might devise a way to define the Brown-York mass, $m_{BY}(\partial P)$, where $P$ is a prism. These definitions are likely only to involve integrals of dihedral angles. It is quite possible that it would be easier to study the limits of Brown-York masses of prisms than to even define dihedral angles on the limit spaces. If one can do this, then might try to replace (e) with

(1.4) **Shi-Tam Nonnegativity of the Brown-York Mass of Prisms:** $m_{BY}(\partial P) \geq 0$.

Perhaps the consequences Gromov devises using (e) might be concluded from (1.4).
7.6. **Almost Rigidity of the Positive Mass Theorem and Regularity of \( \mathcal{F} \) Limits.** It should be noted that in the famous work of Cheeger-Colding on the properties of metric measure limits of Riemannian manifolds with lower bounds on Ricci curvature, a key step in proving regularity and the existence of Euclidean tangent spaces at regular points, was the proof of their Almost Splitting Theorem [CC97]. In their work a point \( p \) in a limit space \( M_\infty \) has a Euclidean tangent space if the sequence of rescaled balls

\[
(B(p, r_j), d_j/r_j) \text{ where } r_j \to 0
\]

converges in the GH sense to a ball \( B(0, 1) \) in Euclidean space. The Toponogov Splitting Theorem was similarly used to prove regularity results for Alexandrov Spaces by Burago-Gromov-Perelman [BGP92]. However there is no splitting theorem for manifolds with nonnegative scalar curvature.

**Remark 7.13.** Let us consider \( M_j \xrightarrow{\mathcal{F}} M_\infty \) where the \( M_j \) satisfy conditions like those in the proposed Scalar Compactness Conjecture,

\[
\text{Scalar}_j \geq 0 \text{ and } \text{MinA}(M_j) \geq A_0 > 0
\]

and \( p \in M_\infty \). If one has proven a local version of the almost rigidity of the Positive Mass Theorem then one can determine settings in which

\[
(B(p, r_j), d_j/r_j, [B(p, r_j)]) \xrightarrow{\mathcal{F}} (B(0, 1), d_{\mathbb{E}^3}, [B(0, 1)]).
\]

This may possibly then be applied in the place of a splitting theorem to prove some sort of regularity on the limit space, \( M_\infty \).

Note that without the assumptions in (218), an intrinsic flat limit may have no regular points in the sense described in (219). In the original paper defining intrinsic flat convergence by the author with Wenger [SW11], an example of a sequence of \( M_j \) is given which converges in the intrinsic flat sense to taxicab space,

\[
M_j^2 = ([0, 1] \times [0, 1], d_{\text{taxi}}, [0, 1] \times [0, 1])
\]

where

\[
d_{\text{taxi}}((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.
\]

Such a space has no notion of angles and no prism properties. It is possible to imagine how sewing methods could be used to construct a sequence of \( M_j^3 \) with \( \text{Scalar}_j \geq -1/j \) that converges to \( M^3_{\text{taxi}} \). However, the \( \text{MinA}(M_j) \geq A_0 > 0 \) condition fails on such examples. The taxicab limit example in [SW11] satisfies \( \text{MinA}(M_j) \geq A_0 > 0 \) but contains points for which the scalar curvature decreases to negative infinity.

Given \( M_\infty \) as in Remark 7.13 one may even be able to prove that almost every point \( p \in M_\infty \) satisfies (219). Thus it may well be worthwhile to examine to what extent one may use this kind of regularity to define dihedral angles, mean curvatures, quasilocal masses and a generalized notion of nonnegative scalar curvature on \( M_\infty \).
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References

[AK00] Luigi Ambrosio and Bernd Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.

[Bam16] Richard Bamler. A ricci flow proof of a result by gromov on lower bounds for scalar curvature. *Mathematical Research Letters*, 23(2):325–337, 2016.

[Bar86] Robert Bartnik. The mass of an asymptotically flat manifold. *Comm. Pure Appl. Math.*, 39(5):661–693, 1986.

[Bar89] Robert Bartnik. New definition of quasilocal mass. *Phys. Rev. Lett.*, 62(20):2346–2348, 1989.

[BBEN10] H. Bray, S. Brendle, M. Eichmair, and A. Neves. Area-minimizing projective planes in 3-manifolds. *Comm. Pure Appl. Math.*, 63(9):1237–1247, 2010.

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. A course in metric geometry, volume 33 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001.

[BBN10] Hubert Bray, Simon Brendle, and Andre Neves. Rigidity of area-minimizing two-spheres in three-manifolds. *Comm. Anal. Geom.*, 18(4):821–830, 2010.

[BGP92] Yu. Burago, M. Gromov, and G. Perelman. A. D. Aleksandrov spaces with curvatures bounded below. *Uspekhi Mat. Nauk*, 47(2(284)):3–51, 222, 1992.

[Bra01] Hubert L. Bray. Proof of the Riemannian Penrose inequality using the positive mass theorem. *J. Differential Geom.*, 59(2):177–267, 2001.

[BS16] Jorge Basilio and Christina Sormani. Sequences of three-dimensional manifolds with positive scalar curvature. *preprint to appear*, 2016.

[CC96] Jeff Cheeger and Tobias H. Colding. Lower bounds on Ricci curvature and the almost rigidity of warped products. *Ann. of Math. (2)*, 144(1):189–237, 1996.

[CC97] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. *1. J. Differential Geom.*, 46(3):406–480, 1997.

[CCE16] A Carlotto, O Chodosh, and M Eichmair. Effective versions of the positive mass theorem. *Inventiones mathematicae*, 2016.

[Col97] Tobias H. Colding. Ricci curvature and volume convergence. *Ann. of Math. (2)*, 145(3):477–501, 1997.

[Cor00] Justin Corvino. Scalar curvature deformation and a gluing construction for the Einstein constraint equations. *Comm. Math. Phys.*, 214(1):137–189, 2000.

[DeG95] E. DeGiorgi. Problema di plateau generale e funzionali geodetici. *Atti Sem. Mat. Fis. Univ. Modena*, 43:285–292, 1995.

[FF60] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.

[FST09] Xu-Qian Fan, Yuguang Shi, and Luen-Fai Tam. Large-sphere and small-sphere limits of the Brown-York mass. *Comm. Anal. Geom.*, 17(1):37–72, 2009.

[Ger73] R. Geroch. Energy extraction. *Ann. New York Acad. Sci.*, 224:108D117, 1973.
[GL80a] Mikhael Gromov and H. Blaine Lawson, Jr. Spin and scalar curvature in the presence of a fundamental group. I. Ann. of Math. (2), 111(2):209–230, 1980.

[GL80b] Mikhael Gromov and H. Blaine Lawson, Jr. Spin and scalar curvature in the presence of a fundamental group. I. Ann. of Math. (2), 111(2):209–230, 1980.

[Gro81] Mikhael Gromov. Structures Métriques pour les Variétés Riemanniennes, volume 1 of Textes Mathématiques [Mathematical Texts]. CEDIC, Paris, 1981. Edited by J. Lafontaine and P. Pansu.

[Gro83] Mikhael Gromov. Filling Riemannian manifolds. J. Differential Geom., 18(1):1–147, 1983.

[Gro14a] Misha Gromov. Dirac and Plateau billiards in domains with corners. Cent. Eur. J. Math., 12(8):1109–1156, 2014.

[Gro14b] Misha Gromov. Plateau-Stein manifolds. Cent. Eur. J. Math., 12(7):923–951, 2014.

[Ham95] Richard S. Hamilton. The formation of singularities in the Ricci flow. In Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), pages 7–136. Int. Press, Cambridge, MA, 1995.

[H101] Gerhard Huisken and Tom Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. J. Differential Geom., 59(3):353–437, 2001.

[HLS16] Lan-Hsuan Huang, Dan Lee, and Christina Sormani. Stability of the positive mass theorem for graphical hypersurfaces of euclidean space. to appear Journal fur die Riene und Angewandte Mathematik, 2016.

[Hui09] G Huisken. An isoperimetric concept for the mass in general relativity. Presentation at IAS, 2009.

[Jau16] J Jauregui. On the lower semicontinuity of the adm mass. arXiv:1411.3699, 2016.

[JL16] J Jauregui and Dan A. Lee. Lower semicontinuity of mass under $C^0$ convergence and huijskens’s isoperimetric mass. arXiv:1602.00732, 2016.

[Lak15a] Sajjad Lakzian. Intrinsic flat continuity of Ricci flow through neckpinch singularities. Geom. Dedicata, 179:69–89, 2015.

[Lak15b] Sajjad Lakzian. Intrinsic flat continuity of Ricci flow through neckpinch singularities. Geom. Dedicata, 179:69–89, 2015.

[Lak16] Sajjad Lakzian. On diameter controls and smooth convergence away from singularities. Differential Geom. Appl., 47:99–129, 2016.

[LM89] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. Spin geometry, volume 38 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1989.

[LP15] Nan Li and Raquel Perales. On the sormani-wenger intrinsic flat convergence of alexandrov spaces. arXiv:1411.6854, 2015.

[LS12] Dan A. Lee and Christina Sormani. Near-equality of the Penrose inequality for rotationally symmetric Riemannian manifolds. Ann. Henri Poincaré, 13(7):1537–1556, 2012.

[LS13] Sajjad Lakzian and Christina Sormani. Smooth convergence away from singular sets. Comm. Anal. Geom., 21(1):39–104, 2013.

[LS14] Dan A. Lee and Christina Sormani. Stability of the positive mass theorem for rotationally symmetric riemannian manifolds. Journal fur die Riene und Angewandte Mathematik (Crelle’s Journal), 686, 2014.

[LS15] Philippe G. LeFloch and Christina Sormani. The nonlinear stability of rotationally symmetric spaces with low regularity. J. Funct. Anal., 268(7):2005–2065, 2015.

[Mia03] Pengzi Miao. On existence of static metric extensions in general relativity. Comm. Math. Phys., 241(1):27–46, 2003.

[MP15] Rostislav Matveev and Jacobus Portegies. Intrinsic flat and Gromov-Hausdorff convergence of manifolds with Ricci curvature bounded below. arXiv:1510.07547, 2015.

[MS15] Christos Mantoulidis and Richard Schoen. On the bartnik mass of apparent horizons. Classical and Quantum Gravity, 32(20):205002, 2015.

[Mun14] Michael Munn. Intrinsic flat convergence with bounded ricci curvature. arXiv:1405.3313, 2014.
[Per15a] Raquel Perales. Convergence of manifolds and metric spaces with boundary. arXiv:math/1505.01792, 2015.

[Per15b] Raquel Perales. Volumes and limits of manifolds with ricci curvature and mean curvature bounds. arXiv:math/1404.0560, 2015.

[Por15] Jacobus W. Portegies. Semicontinuity of eigenvalues under intrinsic flat convergence. Calc. Var. Partial Differential Equations, 54(2):1725–1766, 2015.

[PS14] Jacobus Portegies and Christina Sormani. Properties of the intrinsic flat distance. arXiv:1210.3895, 2014.

[Sor04] Christina Sormani. Friedmann cosmology and almost isotropy. Geom. Funct. Anal., 14(4):853–912, 2004.

[Sor14] Christina Sormani. Intrinsic flat Arzela-Ascoli theorems. preprint on arXiv: 1402.6066, 2014.

[SS16a] Zahra Sinaei and Christina Sormani. Limits of covering spaces under intrinsic flat convergence. to appear in Geometriae Dedicata, 2016.

[SS16b] C Sormani and I Stavrov. Geometrostatic manifolds of small adm mass. to appear, 2016.

[ST02] Yuguang Shi and Luen-Fai Tam. Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature. J. Differential Geom., 62(1):79–125, 2002.

[SW01] Christina Sormani and Guofang Wei. Hausdorff convergence and universal covers. Trans. Amer. Math. Soc., 353(9):3585–3602 (electronic), 2001.

[SW04] Christina Sormani and Guofang Wei. The covering spectrum of a compact length space. J. Differential Geom., 67(1):35–77, 2004.

[SW10] Christina Sormani and Stefan Wenger. Weak convergence and cancellation, appendix by Raanan Schul and Stefan Wenger. Calculus of Variations and Partial Differential Equations, 38(1-2), 2010.

[SW11] Christina Sormani and Stefan Wenger. Intrinsic flat convergence of manifolds and other integral current spaces. Journal of Differential Geometry, 87, 2011.

[SY79a] R. Schoen and S. T. Yau. On the structure of manifolds with positive scalar curvature. Manuscripta Math., 28(1-3):159–183, 1979.

[SY79b] R. Schoen and Shing Tung Yau. Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. Ann. of Math. (2), 110(1):127–142, 1979.

[SY79c] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. Comm. Math. Phys., 65(1):45–76, 1979.

[Wen07] Stefan Wenger. Flat convergence for integral currents in metric spaces. Calc. Var. Partial Differential Equations, 28(2):139–160, 2007.

[Wen11] Stefan Wenger. Compactness for manifolds and integral currents with bounded diameter and volume. Calculus of Variations and Partial Differential Equations, 40(3-4):423–448, 2011.

[Whi57] Hassler Whitney. Geometric integration theory. Princeton University Press, Princeton, N. J., 1957.

[Wit81] Edward Witten. A new proof of the positive energy theorem. Comm. Math. Phys., 80(3):381–402, 1981.

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