Discrete $\Omega$-nets and Guichard nets via discrete Koenigs nets

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Abstract
We provide a convincing discretisation of Demoulin’s $\Omega$-surfaces along with their specialisations to Guichard and isothermic surfaces with no loss of integrable structure.

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1 | INTRODUCTION

1.1 | Background

Our topic begins with the work of Darboux, Bianchi, Guichard, and Demoulin in the early 20th century. This period saw rapid progress in surface geometry of a kind that we now recognise as a manifestation of the close relation of geometry to soliton theory. In particular, three surface classes, of increasing generality, were introduced: isothermic surfaces by Bour [9] in 1862, Guichard surfaces by Guichard [33] in 1900 and, finally, $\Omega$-surfaces by Demoulin [22] in 1911.

These classes have a common formulation in terms of duality. For this, let $x : \Sigma \to \mathbb{R}^3$ be an isometric immersion of a surface and recall that a Combesure transformation of $x$ is a second
immersion\(^\dagger\) \( \hat{x} : \Sigma \to \mathbb{R}^3 \) with parallel curvature directions to those of \( x \). Let \( \kappa_1, \kappa_2 \) be the principal curvatures of \( x \) and \( \hat{\kappa}_1, \hat{\kappa}_2 \) those of \( \hat{x} \). Now \( x \) is isothermic if and only if it has a Combescure transform \( \hat{x} \), the *Christoffel dual* [19], for which

\[
\frac{1}{\kappa_1 \hat{\kappa}_2} + \frac{1}{\kappa_2 \hat{\kappa}_1} = 0. \tag{1.1a}
\]

Meanwhile, Guichard’s original definition of his eponymous surfaces is that there should be a Combescure transform \( \hat{x} \), the *associate surface*, such that

\[
\frac{1}{\kappa_1 \hat{\kappa}_2} + \frac{1}{\kappa_2 \hat{\kappa}_1} = c \neq 0, \tag{1.1b}
\]

for some constant \( c \).

Finally, one of us [39, Theorem 5.1] observed that \( \Omega \)-surfaces may be similarly characterised in terms of two Combescure transforms \( \hat{x} \) and \( \hat{n} \), where the latter has principal curvatures \( \ell_1, \ell_2 \), for which

\[
\frac{1}{\kappa_1 \hat{\kappa}_2} + \frac{1}{\kappa_2 \hat{\kappa}_1} = \frac{1}{\ell_1} + \frac{1}{\ell_2}. \tag{1.1c}
\]

Observe that when \( \hat{n} = n \), the Gauss map of \( x \), \( (1.1c) \) reduces to (1.1b) so that Guichard surfaces are \( \Omega \) as Demoulin observed [23]. Again, when \( \hat{n} \) is constant, (1.1c) reduces to (1.1) and isothermic surfaces are seen to be \( \Omega \)-surfaces also, another result of Demoulin [22].

In each case, \( x \) and \( \hat{x} \) appear symmetrically so that \( \hat{x} \) is isothermic, Guichard or \( \Omega \) as \( x \) is.

These characterisations admit a reformulation that is very amenable to discretisation. The equations (1.1) are equivalent to:

\[
\begin{align*}
\text{d} x \wedge \text{d} \hat{x} &= 0 \tag{1.2a} \\
\text{d} x \wedge \text{d} \hat{x} + \text{d} n \wedge \text{d} n &= 0 \tag{1.2b} \\
\text{d} x \wedge \text{d} \hat{x} + \text{d} n \wedge \text{d} \hat{n} &= 0. \tag{1.2c}
\end{align*}
\]

Here, \( \wedge \) is exterior product of \( \mathbb{R}^3 \)-valued 1-forms using the wedge product of \( \mathbb{R}^3 \) to multiply coefficients so that (1.2) are equations on \( \wedge^2 \mathbb{R}^3 \)-valued 2-forms. As we shall see in Section 8.2, equations (1.2) show that \( \Omega \)-surfaces are \( O \)-surfaces in the sense of Schief–Konopelchenko [41].

On the down-side, these Euclidean formulations obscure the symmetry of the situation: both isothermic and Guichard surfaces are conformally invariant while \( \Omega \)-surfaces are Lie sphere invariant. Moreover, the fundamental role played by isothermic surfaces is not apparent. Demoulin’s original approach does not have these defects but requires a change of viewpoint to that of Lie sphere geometry. Here, the basic idea is to study a surface via the collection of 2-spheres tangent to that surface. The oriented 2-spheres in \( S^3 \) are parametrised by a 4-dimensional real quadric \( Q \) in such a way that two such are in oriented contact exactly when the corresponding points in \( Q \) lie on a (projective) line in \( Q \). Thus, each line in \( Q \) parametrises the 1-parameter family of oriented 2-spheres through a fixed point \( p \in S^3 \) and tangent to a fixed 2-plane in \( T_p S^3 \). Otherwise said, the contact elements of \( S^3 \) are parametrised by the 5-dimensional space \( Z \) of lines in \( Q \). One now studies surfaces in \( S^3 \) by replacing them with their Legendre lifts (their collection of

\(^\dagger\) The immersion requirement will be relaxed below.
contact elements), viewed as maps into $\mathcal{Z}$. See [18] for more details. With this in hand, Demoulin originally defined an $\Omega$-surface to be a surface in $S^3$ whose Legendre lift contains an isothermic surface in $Q$, otherwise said, the surface admits an enveloping sphere congruence that is isothermic $qua$ map into $Q$. This is a manifestly Lie sphere geometric characterisation of $\Omega$-surfaces. From here, one can show that a surface is $\Omega$ exactly when its Legendre lift is Lie applicable [38] which gives a second Lie sphere geometric characterisation. Additionally, Demoulin shows there is a second isothermic enveloping sphere congruence harmonically separated from the first by the curvature spheres so that $\Omega$-surfaces have Legendre lifts spanned by isothermic sphere congruences.

In this invariant formulation of isothermic and $\Omega$ surfaces, the duality of (1.2) manifests itself in the existence of closed Lie algebra-valued 1-forms whose existence characterises the surfaces in question [12, 20, 38]. These 1-forms are gauge potentials for pencils of flat connections that provide an efficient way into the integrable systems approach to these surfaces.

This integrability of all three surface classes is evidenced by their rich transformation theory as developed by Bianchi, Calapso, and Darboux [1, 2, 17, 21] for isothermic surfaces and Eisenhart [28–31] for Guichard and $\Omega$-surfaces. In fact, the transformations of Guichard and $\Omega$ surfaces are induced by the Darboux and Calapso transforms of their isothermic sphere congruences [13]. Indeed, even the duality of $\Omega$-surfaces is induced by the Christoffel duality of the sphere congruences.

Having reached an understanding of $\Omega$-surfaces, a basic question is how to identify isothermic and Guichard surfaces among them. In the Euclidean formulation, this is clear: one requires either that $\hat{n}$ be constant or that $\hat{n} = n$, respectively. To get an invariant picture that is compatible with the transformation theory, one can exploit the 1-parameter family of flat connections alluded to above. Requiring that these connections admit families of parallel sections depending affinely linearly on the parameter (linear conserved quantities) leads to such a characterisation of isothermic, Guichard, and indeed, subclasses of isothermic surfaces [13].

1.2 Manifesto

We have discussed the smooth theory of $\Omega$-surfaces and their sub-classes at such length because the aim of this paper is to provide a discrete theory that almost exactly replicates the smooth one. In particular, we shall define discrete Guichard and $\Omega$-nets (extending the well-known theory of isothermic nets along the way) as well as discrete applicable Legendre maps so that:

- Discrete isothermic, Guichard, and $\Omega$-nets are characterised by (1.2) and so have the same duality as the smooth case. For this, we will need to develop a discrete exterior calculus.
- $\Omega$-nets are discrete $O$-surfaces in the sense of Schief [40].
- Both isothermic nets and applicable Legendre maps are defined in terms of a closed Lie algebra valued 1-form.
- Applicable Legendre maps are generically spanned by isothermic sphere congruences.
- A net is $\Omega$ if and only if it is enveloped by an isothermic sphere congruence so that its Legendre lift is applicable.
- Isothermic and Guichard nets are characterised by the existence of linear conserved quantities just as in the smooth case [13] as are $L$-isothermic and $L$-Guichard nets.
- The transformation theory of isothermic nets induces a transformation theory of $\Omega$-nets, restricting to one of Guichard nets and the other subclasses.

† This is how Demoulin arrived at the dual $\Omega$-surface [24].
This match of smooth and discrete extends to fine detail. For example, the radii of the isothermic spheres congruences enveloping a Guichard net are reciprocal to those enveloping the associate net: a result of Demoulin [24] in the smooth case. Again, a classical formula of Eisenhart [29] relates distances between corresponding points of a Darboux pair of Guichard surfaces and their associates: this result holds equally for Guichard nets and, in fact, we extend it to $\Omega$-nets.

1.3 Roadmap

Let us briefly sketch the contents of the paper.

Section 2 is mostly preparatory in nature but has some novelty in that we develop a discrete exterior calculus that we use extensively in the sequel. Thus, we define discrete exterior forms, their exterior derivative and exterior product and show that these satisfy the anti-commutativity and Leibniz rules familiar from the smooth case. Versions of discrete exterior calculus are well known in the finite elements literature [25, 27, 35, 36] but the technology is used more rarely in discrete differential geometry.

Much of our paper is concerned with isothermic nets in the Lie quadric which is a 4-dimensional quadric with a signature (3, 1) conformal structure. The indefiniteness of the conformal structure means much of the well-known theory of isothermic nets in the definite case either needs adjustment (the family of flat connections) or is not available (cross-ratio factorising functions). These defects do not apply to the characterisation of Bobenko–Suris [7, 8] of isothermic nets as Koenigs nets (a purely projective notion) taking values in a quadric. We therefore start, in Section 3, by studying Koenigs nets in a projective space $\mathbb{P}(V)$. We find a new invariant characterisation of Koenigs nets in terms of a closed $\wedge^2 V$-valued 1-form $\eta$ and then define applicable line congruences similarly. The main result is that, in the presence of mild regularity assumptions, a line congruence is applicable exactly when it is spanned by a Koenigs–Moutard pair of Koenigs nets.

In Section 4, we apply this theory to isothermic nets in an arbitrary non-singular quadric. We find that the 1-form $\eta$ is exactly what is needed to extend the key points of the theory to the indefinite case. In particular, we use $\eta$ to construct the Christoffel transform and the family of flat connections and, from them, reach the transformation theory. A distinguishing characteristic of the indefinite case is the extension of Darboux transforms to include the case of infinite spectral parameter: it is precisely Darboux pairs of this kind that span applicable line congruences whose lines lie in the quadric.

Such line congruences are called applicable Legendre maps and are the topic of Section 5. We find that the transformation theory of the isothermic sphere congruences induces one on the applicable Legendre maps they span independently of choices. In particular, we find a duality of such line congruences. Once more, the closed 1-form $\eta$ plays an essential role.

The remainder of the paper applies the preceding theory to $\Omega$-nets (in Section 6) and Guichard nets (in Section 7.1). The duality of applicable Legendre maps induces the duality of $\Omega$-nets. We prove that our notion of $\Omega$-net coincides with that of [14] and that Guichard nets, defined by $n = \bar{n}$, are also characterised by polynomial conserved quantities. We also test the robustness of our discretisation by proving discrete analogs of the classical results of Demoulin and Eisenhart alluded to above.

Finally, we make contact in Section 8 with Schief’s theory of discrete $O$-surfaces for which our exterior calculus furnishes an efficient methodology. We show that our notion of Guichard net coincides with that of Schief and that $\Omega$-nets are $O$-surfaces.
2 | PRELIMINARIES

Our approach to discrete differential geometry is to follow the smooth situation as much as possible. We will therefore require both a discrete exterior calculus of forms and a discrete theory of bundles and connections. We begin by sketching both of these.

First some notation: our domain will be a subset $\Sigma$ of $\mathbb{Z}^N$. The latter is organised into vertices, edges, quadrilaterals (faces), and more generally, $k$-cubes for $0 \leq k \leq n$. We say that a $k$-cube belongs to $\Sigma$ if all of its $2^k$ vertices lie in $\Sigma$.

2.1 | Discrete exterior calculus

2.1.1 | Discrete $k$-forms

Definition 2.1 ($k$-form). A discrete $k$-form $\omega$ on $\Sigma$ is a real-valued function on the oriented $k$-cubes of $\Sigma$ with values that change sign when the orientation reverses: $\omega(-C) = -\omega(C)$.

We denote the vector space of $k$-forms on $\Sigma$ by $\Omega^k_{\Sigma}$.

Let us spell out what this means for the cases $k \leq 2$ that are of principal interest to us.

1. An orientation of a point is a sign so a 0-form $f$ on $\Sigma$ satisfies $f(-p) = -f(p)$, for $p \in \Sigma$, and so amounts to a function $f : \Sigma \rightarrow \mathbb{R}$.
2. View an oriented edge as an ordered pair $ji$ of vertices, read as the edge from $i$ to $j$, see Figure 1(a). Now a 1-form $\alpha$ on $\Sigma$ is a function on such pairs in $\Sigma$ with $\alpha_{ij} = -\alpha_{ji}$ (c.f. [7, Definition 2.23]).
3. View an oriented quadrilateral as a cyclic ordering $\ell k ji$ of its adjacent vertices, as in Figure 1(b), so that a 2-form $\omega$ on $\Sigma$ is a function of such cyclicly ordered quadrilaterals of $\Sigma$ with $\omega_{\ell k ji} = -\omega_{ijk \ell}$.

2.1.2 | Exterior derivative

Just as in the smooth case, we have an exterior derivative and exterior product which together satisfy Leibniz rule. For our purposes, we only need this structure on $\Omega^k_{\Sigma}$, for $k \leq 2$ so we restrict attention to this case below.

Definition 2.2 (Exterior derivative). The interior derivative $d : \Omega^k_{\Sigma} \rightarrow \Omega^{k+1}_{\Sigma}$, $k = 0, 1$ is defined by
1. \( df_{ji} = f_j - f_i \), for \( f \in \Omega^0_\Sigma \).

2. \( d\alpha_{\ell k ji} = \alpha_{i\ell} + \alpha_{\ell k} + \alpha_{k j} + \alpha_{ji} \), for \( \alpha \in \Omega^1_\Sigma \).

An easy computation gives:

**Proposition 2.3.** \( d \circ d = 0 \).

**Remark 2.4.** The exterior derivative can be extended to \( k \)-forms, \( k \geq 2 \), retaining \( d \circ d = 0 \), but we will have no need of such generality here.

### 2.1.3 Exterior algebra

**Definition 2.5** (Exterior product). The exterior product \( \wedge : \Omega^k_\Sigma \times \Omega^\ell_\Sigma \to \Omega^{k+\ell}_\Sigma \) is defined, for \( k + \ell \leq 2 \), as follows:

- For \( f, g \in \Omega^0_\Sigma \), \((f \wedge g)_i = f_i g_i \).
- For \( f \in \Omega^0_\Sigma \), \( \alpha \in \Omega^1_\Sigma \), \((f \wedge \alpha)_i j = (\alpha \wedge f)_{ij} = \frac{1}{2} (f_i + f_j) \alpha_{ij} \).
- For \( f \in \Omega^0_\Sigma \), \( \omega \in \Omega^2_\Sigma \), \((f \wedge \omega)_{\ell k ji} = (\omega \wedge f)_{\ell k ji} = \frac{1}{4} (f_i + f_j + f_k + f_\ell) \omega_{\ell k ji} \).
- For \( \alpha, \beta \in \Omega^1_\Sigma \),

\[
(\alpha \wedge \beta)_{\ell k ji} = \frac{1}{4} ((\alpha_{ji} + \alpha_{k\ell})(\beta_{\ell i} + \beta_{k j}) - (\alpha_{\ell i} + \alpha_{k j})(\beta_{ji} + \beta_{k\ell})).
\]

The key point of these definitions is that the exterior product retains the skew-symmetry and Leibniz rules of the smooth setting. Indeed, a routine computation gives:

**Proposition 2.6.** Let \( \alpha \in \Omega^k_\Sigma \) and \( \beta \in \Omega^\ell_\Sigma \). Then,

1. For \( k + \ell \leq 2 \), \( \alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha \);
2. For \( k + \ell \leq 1 \), \( d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \).

**Remark 2.7.** One can extend the exterior product to arbitrary \( k, \ell \) in such a way that Proposition 2.6 continues to hold. However, the product is not associative: we can have \( f \wedge (g \wedge \alpha) \neq (f \wedge g) \wedge \alpha \) even for functions \( f, g \) and a 1-form \( \alpha \). It seems possible that \( d \) and \( \wedge \) are the unary and binary operators for an \( A_\infty \)-algebra structure on \( \Omega_\Sigma \); see [27] for some evidence of this.

### 2.1.4 Vector-valued forms

In what follows, we shall often have recourse to forms taking values in a vector space \( V \) rather than simply \( \mathbb{R} \). We denote the space of \( k \)-forms on \( \Sigma \) with values in \( V \) by \( \Omega^k_\Sigma(V) \). Thus, \( \Omega^k_\Sigma(V) = \Omega^k_\Sigma \otimes_{\mathbb{R}} V \).

The exterior derivative \( d : \Omega^k(V) \to \Omega^{k+1}(V) \) is defined just as in Definition 2.2 but, for the exterior product, we need to be able to multiply values and that requires extra structure:

**Definition 2.8** (Exterior product of vector-valued forms). Let \( V, W, U \) be vector spaces and \( B : V \times W \to U \) a bilinear map. Then there is a bilinear map
\[ \Omega^k_\Sigma(V) \times \Omega^\ell_\Sigma(W) \to \Omega^{k+\ell}_\Sigma(U) \]
\[(\alpha, \beta) \mapsto B(\alpha \wedge \beta) \]

obtained by replacing all multiplications in the formulae of Definition 2.5 by applications of \(B\).

**Example 2.9.** For \(\alpha \in \Omega^1_\Sigma(V), \beta \in \Omega^1_\Sigma(W), B(\alpha \wedge \beta) \in \Omega^2_\Sigma(U)\) is given by

\[
B(\alpha \wedge \beta)_{\ell kj} = \frac{1}{4} \left( B(\alpha_{\ell j} + \alpha_{k\ell}, \beta_{\ell i} + \beta_{kj}) - B(\alpha_{\ell i} + \alpha_{kj}, \beta_{ji} + \beta_{kj}) \right).
\]

This product has Leibniz rule just as in Proposition 2.6(2) and, when \(V = W\), is graded-(anti)-commutative if \(B\) is (skew)-symmetric:

**Lemma 2.10.** Let \(B : V \times V \to U\) be bilinear and \(\alpha \in \Omega^k_\Sigma(V), \beta \in \Omega^\ell_\Sigma(V)\) then:

(a) if \(B\) is symmetric, \(\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha;\)

(b) if \(B\) is skew-symmetric, \(\alpha \wedge \beta = (-1)^{k\ell+1} \beta \wedge \alpha.\)

**Notation 2.11.** When \(B : V \times V \to \wedge^2 V\) is exterior product \(B(v, w) = v \wedge w\), we write \(\alpha \wedge \beta\) for \(B(\alpha \wedge \beta)\).

In view of Lemma 2.10(b), we have \(\alpha \wedge \beta = \beta \wedge \alpha,\) for \(\alpha, \beta \in \Omega^1_\Sigma(V)\).

Our exterior calculus of vector-valued forms gives a convenient approach to the mixed area\(^\dagger\) discussed by Bobenko–Suris [7, §4.5.2]. For this, let \(x : \Sigma \to V\) and contemplate an ordered quadrilateral \(\ell kj i\) of \(\Sigma\). The area \(A(x, x)_{\ell kj i} \in \wedge^2 V\) is given by

\[
A(x, x)_{\ell kj i} := \frac{1}{2} (x_i \wedge x_j + x_j \wedge x_k + x_k \wedge x_\ell + x_\ell \wedge x_i) = \frac{1}{2} (x_i - x_k) \wedge (x_j - x_\ell).
\]

However, we readily see that \(x_i \wedge x_j = (x \wedge dx)_{ji}\) so the first equation reads \(A(x, x) = \frac{1}{2} d(x \wedge dx)\) which is \(\frac{1}{2} dx \wedge dx\) by the Leibniz rule. The mixed area is obtained by polarising \(A\) on the space of maps \(y : \Sigma \to V\) edge-parallel to \(x\) (that is, \(dy_{ji} \parallel dx_{ji}\) on all edges \(ji\)), and we conclude:

**Lemma 2.12.** Let \(x, y : \Sigma \to V\) be edge-parallel. Then the mixed area of \(x\) and \(y\) is given by

\[
A(x, y) = \frac{1}{2} dx \wedge dy.
\]

### 2.2 Discrete gauge theory

The gauge theoretic approach to discrete differential geometry is well established, see, for example, [12, 15, 16]. Here, we simply recall the relevant definitions to set notation:

**Definition 2.13** (Bundles and connections). A **bundle** on \(\Sigma\) is a surjection of a set \(X\) onto \(\Sigma\) with fibres \(X_i, i \in \Sigma\).

\(^\dagger\) In fact, the mixed area in [7] is only defined for polygons lying in translates of a fixed 2-plane \(U\) and so takes values in a fixed line \(\wedge^2 U\) which is identified with a scalar via a choice of area form.
A \textit{connection} \( \Gamma \) on \( X \) is an assignment of a bijection \( \Gamma_{ji} : X_i \to X_j \) to each oriented edge \( ji \) of \( \Sigma \) such that \( \Gamma_{ij}\Gamma_{ji} = 1_{X_i} \) on all edges.

A \textit{gauge transformation} is a section \( T \) of \( \text{Aut}(X) \) the bundle of fibre automorphisms: thus, a map \( i \mapsto T_i \in \text{Aut}(X_i) \).

Gauge transformations act on connections by
\[
(T \cdot \Gamma)_{ji} = T_j\Gamma_{ji}T^{-1}_i.
\]

A section \( \sigma \) of \( X \) is \( \Gamma \)-parallel if, on each \( ji \), we have
\[
\sigma_j = \Gamma_{ji}\sigma_i.
\]

A connection \( \Gamma \) is \textit{flat} if, on each quadrilateral \( \ell kjl \), we have
\[
\Gamma_{k\ell}\Gamma_{ji} = \Gamma_{kj}\Gamma_{jl},
\]
a condition that is invariant under cyclic permutation of vertices.

The action of gauge transformations permutes flat connections.

\textit{Remark 2.14.} In our applications, our bundles will have more structure: the fibres will be projective lines or vector spaces with an inner product and our connections and gauge transformations will preserve that structure.

2.3 \hspace{1cm} \textbf{Simply connected subsets of} \( \mathbb{Z}^N \)

We want to integrate closed \( 1 \)-forms and flat connections on \( \Sigma \) which amounts to a topological requirement on \( \Sigma \). Here are the ingredients:

\textbf{Definition 2.15} (Paths and homotopy). A \textit{path in} \( \Sigma \) from \( p \) to \( q \) is a sequence of vertices \( (i_n \ldots i_0) \) in \( \Sigma \) such that \( i_0 = p \), \( i_n = q \) and each \( i_{j+1}i_j \) is an edge in \( \Sigma \).

\( \Sigma \) is \textit{connected} if there is a path between any two points of \( \Sigma \).

Two paths \( \gamma, \gamma' \) in \( \Sigma \) from \( p \) to \( q \) are \textit{based homotopic}, \( \gamma \simeq \gamma' \), if there is a sequence of paths \( \gamma_m \) in \( \Sigma \) from \( p \) to \( q \) with \( \gamma_0 = \gamma, \gamma_l = \gamma' \) and each \( \gamma_m \) differs from \( \gamma_{m+1} \) either:

(a) on a single edge by \( (iji) \) (Figure 2a), or
(b) on a single quadrilateral \( \ell kjl \) by \( (k\ell i) \) versus \( (kji) \) (Figure 2b).

\( \Sigma \) is \textit{simply connected} if it is connected and any two paths in \( \Sigma \) with the same end-points are based homotopic.
Example 2.16. Given a quadrilateral $\ell kji$, we have $(ji) \simeq (jk\ell i)$. Indeed, $(jk\ell i) \simeq (jkji)$ by (b) and then $(jkji) \simeq (ji)$ by (a).

A pleasant exercise provides sufficient examples for our needs:

**Proposition 2.17.**

(a) $\mathbb{Z}^N$ is simply connected.
(b) If $\Sigma \subset \mathbb{Z}^N$ is simply connected so is $\Sigma \times \{0, 1\} \subset \mathbb{Z}^{N+1}$.

We now have:

**Proposition 2.18.** Let $\Sigma$ be simply connected. Then,

(a) Closed 1-forms are exact: $\alpha \in \Omega^1_\Sigma(V)$ has $d\alpha = 0$ if and only if $\alpha = df$, for some $f \in \Omega^0_\Sigma(V)$, unique up to addition of a constant.

(b) Flat connections are trivialisable: if $\Gamma$ is a flat connection on a bundle $X \to \Sigma$ and $p \in \Sigma$ is fixed, there are isomorphisms $T_i : X_i \to X_p$, $i \in \Sigma$, such that $\Gamma_{ji} = T_j^{-1}T_i$, on all edges $ji$ of $\Sigma$.

Equivalently, there is a $\Gamma$-parallel section of $X$ through any point of $X_p$.

**Proof.** For (a), let $\alpha$ be a closed 1-form, choose $f(p)$ arbitrarily and define $f(q)$ by

$$f(q) = \sum_{j=0}^{n-1} \alpha_{j+1}^q_{ij},$$

for some path $i_n \ldots i_0$ from $p$ to $q$. We note that, $\alpha$ being a closed 1-form, we have

$$\alpha_{ij} + \alpha_{ji} = 0 \quad \alpha_{k\ell} + \alpha_{\ell i} = \alpha_{kj} + \alpha_{ji}$$

so that $f$ is well defined since $\Sigma$ is simply connected.

The proof of (b) is similar: define $T_i$ by

$$T_i = \Gamma_{i_0i_1} \ldots \Gamma_{i_{n-1}i_n},$$

for some path $i_0 \ldots i_n$ from $p$ to $i$. This is well defined as before, since $\Gamma$ is flat and $\Sigma$ is simply connected.

Finally, for $\sigma_p \in X_p$, $\sigma := T^{-1}_p \sigma_p$ is a $\Gamma$-parallel section of $X$ while, conversely, given sufficiently many parallel sections, define $T_i : \sigma_i \mapsto \sigma_p$ to construct $T$. ☐

3 | KOENIGS NETS AND APPLICABILITY

3.1 | Koenigs nets

Let $V$ be a vector space and let $s : \Sigma \to \mathbb{P}(V)$ be a map or, equivalently, a line subbundle $s \leq \Sigma \times V$, on which we impose, without further comment, the following regularity conditions:
**Assumption 3.1.** On each quadrilateral $\ell k ji$:

1. $s_i, s_j, s_k, s_\ell$ are pair-wise distinct;
2. $s_i, s_j, s_k, s_\ell$ are not collinear: $\dim\{s_i + s_j + s_k + s_\ell\} \geq 3$.

**Definition 3.2** (Koenigs net). A map $s : \Sigma \to \mathbb{P}(V)$ is Koenigs if there exists a $\wedge^2 V$-valued never-zero 1-form $\eta \in \Omega^1_\Sigma(\wedge^2 V)$ such that

1. $\eta_{ji} \in s_j \wedge s_i \leq \wedge^2 V$, for each edge $ij$;
2. $\eta$ is closed: $d\eta = 0$.

Remark that if $\eta$ satisfies these conditions, so does any non-zero constant scale of $\eta$.

We abuse notation and write $(s, \eta) : \Sigma \to \mathbb{P}(V)$ for the package of a Koenigs net with a particular choice of $\eta$.

Our first order of business is to reconcile this projectively invariant notion with the affine-geometric one of Bobenko–Suris [7, 8]. For the latter, we recall:

**Definition 3.3** (Koenigs dual). Let $F : \Sigma \to \mathbb{A}$ be a map to an affine space $\mathbb{A}$. A Koenigs dual to $F$ is a map $\tilde{F} : \Sigma \to \mathbb{A}$ which is

1. edge-parallel to $F$: $dF_{ij} \parallel d\tilde{F}_{ij}$ on each edge $ij$;
2. has parallel opposite diagonals to $F$: for each quadrilateral $\ell k ji$, $F_k - F_i \parallel \tilde{F}_\ell - \tilde{F}_j$ and $F_\ell - F_j \parallel \tilde{F}_k - \tilde{F}_i$, or, equivalently [7, Theorem 4.42], $A(F, \tilde{F}) = 0$.

For Bobenko–Suris, $s$ is Koenigs when its image in some affine chart has a Koenigs dual. This coincides with our notion thanks to the following:

**Proposition 3.4.** $s : \Sigma \to \mathbb{P}(V)$ is Koenigs if and only if $s$ has an affine lift $F \in \Gamma s$ which admits a Koenigs dual $\tilde{F}$.

In this case, we may take

$$\eta = d\tilde{F} \wedge F.$$  \hfill (3.1)

**Proof.** Let $\alpha \in V^*$ define a hyperplane $H = \mathbb{P}(\ker \alpha) \leq \mathbb{P}(V)$ which we may assume is disjoint from the image of $s$. Then $\alpha$ determines an affine lift $F \in \Gamma s$ with $\alpha(F) \equiv -1$ so that $F : \Sigma \to \mathbb{A} := \{v \in V \mid \alpha(v) = -1\}$.

Now suppose that $(s, \eta)$ is Koenigs and contemplate the interior product $i_\alpha \eta \in \Omega^1(\ker \alpha)$ which is closed, since $\eta$ is, and so of the form $d\tilde{F}$ for some $\tilde{F} : \Sigma \to \mathbb{A}$. We claim that $\tilde{F}$ is Koenigs dual to $F$.

For this, observe that on an edge $ij$, $\eta_{ji} = \lambda_{ji} F_j \wedge F_i$, $\lambda_{ij} \in \mathbb{R}$ so that

$$d\tilde{F}_{ji} = i_\alpha \eta_{ji} = \lambda_{ji}(F_j - F_i) = \lambda_{ji} dF_{ji}.$$  

Thus, we see first that $\tilde{F}$ is edge-parallel to $F$ and then that $\eta = d\tilde{F} \wedge F$. Taking the exterior derivative and using the Leibniz rule then yields

$$d\tilde{F} \wedge dF = 0,$$

or, equivalently by Lemma 2.12, $A(F, \tilde{F}) = 0$ which settles the claim.
For the converse, given an affine lift $F$ of $s$ with Koenigs dual $\tilde{F}$, define $\eta$ by (3.1). Then $\eta_{ji} \in s_i \wedge s_j$, since $d\tilde{F}_{ji}$ is parallel to $dF_{ji}$, while $\eta$ is closed since $A(F, \tilde{F}) = 0$. □

Our projectively invariant formulation of the Koenigs condition gives us a fast proof of a third characterisation, due to Bobenko–Suris, of Koenigs nets via Moutard lifts. For this, we need a simple application of Cartan’s Lemma:

**Lemma 3.5.** Let $a, b, c, d \in V$ and $r \in \mathbb{R}$ such that

$$a \wedge (c - rb) = d \wedge (c - b).$$

Then either $r = 1$ or $\dim \langle a, b, c, d \rangle \leq 2$, where, here and below, $\langle \cdot \rangle$ denotes linear span of vectors.

**Proof.** Suppose that $\dim \langle a, b, c, d \rangle \geq 3$. Then at least one of $a \wedge d$ and $b \wedge c$ is non-zero. If $a \wedge d \neq 0$, Cartan’s Lemma tells us that $c - b, c - rb \in U := \langle a, d \rangle$, and, unless $r = 1$, this gives $(1 - r)b \in U$ and thus $c \in U$: a contradiction.

If $a \wedge d = 0$ then $b \wedge c \neq 0$, and so, unless $r = 1$, $(c - rb) \wedge (c - b) \neq 0$. Now Cartan’s Lemma tells us that $a, d \in \langle c - rb, c - b \rangle = \langle b, c \rangle$ which is again a contradiction. □

With this in hand, we have:

**Theorem 3.6** [7, Theorem 2.32]. $s$ is Koenigs if and only if there exists $\mu \in \Gamma s^\infty$ satisfying the Moutard equation

$$d\mu \wedge d\mu = 0 \quad (3.2)$$

or, in more familiar terms,

$$(\mu_k - \mu_i) \wedge (\mu_\ell - \mu_j) = 0 \quad (3.3)$$

on each quadrilateral $ijkl$.

In this case, $\eta$ can be taken to be $d\mu \wedge \mu$ so that, on each edge $ij$,

$$\eta_{ji} = \mu_j \wedge \mu_i. \quad (3.4)$$

We call $\mu$ a Moutard lift of $s$.

**Proof.** First, assume that $(s, \eta)$ is Koenigs so that $\eta$ is a closed 1-form with $\eta_{ji} \in s_j \wedge s_i$ on each edge $ij$. Given $\mu_i \in s_i$, define $\mu_j \in s_j$ to ensure that (3.4) holds: $\eta_{ji} = \mu_j \wedge \mu_i$. To see that this is well defined, consider a quadrilateral $ijkl$: starting with $\mu_i$, we get $\mu_j \in s_j$ and $\mu_\ell \in s_\ell$ with

$$\eta_{ji} = \mu_j \wedge \mu_i, \quad \eta_{i\ell} = \mu_\ell \wedge \mu_i,$$

and then $\mu'_k, \mu''_k \in s_k$ with

$$\eta_{kj} = \mu'_k \wedge \mu_j, \quad \eta_k\ell = \mu''_k \wedge \mu_\ell.$$

The closedness of $\eta$ now reads

$$\mu_j \wedge (\mu_i - \mu'_k) = \mu_\ell \wedge (\mu_i - \mu''_k).$$
Write $\mu_k'' = r\mu_k'$ for some $r \in \mathbb{R}$ and apply Lemma 3.5 to see that $r = 1$ since $\dim(\mu_i, \mu_j, \mu_k', \mu_l') \geq 3$ by Assumption 3.1. Thus, $\mu_k' = \mu_k''$, and so $\mu$ is well defined and $\eta = d\mu \wedge \mu$. Now

$$0 = d\eta = -d\mu \wedge d\mu$$

so that $\mu$ is a Moutard lift.

Conversely, if $\mu \in \Gamma s^\times$ is a Moutard lift, set $\eta = d\mu \wedge \mu$ to conclude that $(s, \eta)$ is Koenigs. \qed

Remark 3.7. We see at once from (3.3) that Koenigs nets have planar quadrilaterals and so are $Q$-nets in the sense of Bobenko–Suris [7, Definition 2.1].

A central tenet of the Bobenko–Suris philosophy [7] is that transformations are the same as higher-dimensional nets. With this in mind, we introduce the following:

Notation 3.8. For maps $x^\pm : \Sigma \to X$, a set, we define $x^+ \cup x^- : \{0, 1\} \times \Sigma \to X$ by

$$(x^+ \cup x^-)|_{\{0\}\times\Sigma} = x^+,(x^+ \cup x^-)|_{\{1\}\times\Sigma} = x^-.$$ We visualise $\{0, 1\} \times \Sigma \subset \mathbb{Z}^{N+1}$ as two copies of $\Sigma$ stacked on top of each other and refer to edges $\{0, 1\} \times \{i\}$ and quadrilaterals $\{0, 1\} \times \{i, j\}$ as *vertical*.

We now have:

Definition 3.9 (Koenigs–Moutard transformation). Let $(s^\pm, \eta^\pm) : \Sigma \to \mathbb{P}(V)$ be two Koenigs nets and let $s = s^+ \cup s^- : \{0, 1\} \times \Sigma \to \mathbb{P}(V)$.

Assume that $s$ is regular so that $s^+_i, s^-_i$ are all distinct and $\dim s^+_i + s^-_i + s^-_j + s^-_j \geq 3$.

We say that $s^-$ is a *Koenigs–Moutard transformation* of $s^+$, or that $s^+, s^-$ are a *Koenigs–Moutard pair*, if $(s, \eta)$ is also Koenigs with $\eta$ satisfying

$$\eta|_{\{0\}\times\Sigma} = \eta^+, \eta|_{\{1\}\times\Sigma} = \eta^-.$$ For a more practical formulation, define $\tau \in \Gamma(s^+ \wedge s^-)$ by

$$\tau_i = \eta_{(1,i)(0,j)}$$

and use the closedness of $\eta$ on vertical quadrilaterals to conclude:

Proposition 3.10. Koenigs nets $(s^\pm, \eta^\pm)$ are a Koenigs–Moutard pair if and only if there is a section $\tau$ of $s^+ \wedge s^-$ such that

$$\eta^- = \eta^+ + d\tau.$$ (3.5)

Alternatively, Theorem 3.6 implies the following:

Corollary 3.11. $(s^\pm, \eta^\pm)$ are a Koenigs–Moutard pair if and only if there are Moutard lifts $\mu^\pm \in \Gamma s^\pm$ such that

$$\left(\mu^+_j - \mu^-_j\right) \wedge \left(\mu^+_i - \mu^-_i\right) = 0.$$ (3.6)
In this case, we may take \( \eta^\pm = d\mu^\pm \land \mu^\pm \) and then the section \( \tau \in \Gamma(s^+ \land s^-) \) of Proposition 3.10 is \( \mu^- \land \mu^+ \).

**Remark 3.12.** Corollary 3.11 says that two Koenigs nets \( s^\pm \) are a Koenigs–Moutard pair if and only if there are Moutard lifts \( \mu^\pm \in \Gamma s^\pm \) so that \( \mu^\pm \) are Moutard transformations [7, Definition 2.36] of each other. Therefore, [6, Theorem 2.7] implies that the Koenigs–Moutard transformation is three-dimensionally consistent, and hence multi-dimensionally consistent.

### 3.2 Applicable line congruences

Let \( G_2(V) \) be the Grassmannian of 2-planes in \( V \) or, equivalently, lines in \( \mathbb{P}(V) \).

**Definition 3.13** (Line congruence [26, Definition 2.1]). A map \( f : \Sigma \to G_2(V) \) is a line congruence if, on each edge \( ij \), \( f_i \cap f_j \neq \{0\} \).

For a Koenigs–Moutard pair \((s^+, s^-)\) of Koenigs nets, contemplate the map \( f = s^+ \oplus s^- : \Sigma \to G_2(V) \). In view of (3.6), we see that, on each edge \( ij \), \( \dim f_i + f_j \leq 3 \) so that \( f \) is a line congruence.

Line congruences of this kind are central to our programme and, in this section, we characterise them in terms of closed \( \land^2 V \)-valued 1-forms parallel to our definition of Koenigs nets.

So let \( f : \Sigma \to G_2(V) \) be a discrete line congruence in \( \mathbb{P}(V) \) and impose the following regularity conditions:

**Assumption 3.14.**

1. First order: on each edge \( ij \), \( f_{ij} := f_i + f_j \) is 3-dimensional so that \( s_{ij} := f_i \cap f_j \) has \( \dim s_{ij} = 1 \).
2. Second order: on each quadrilateral \( kji \), we have

\[
{s_{ij} \land s_{jk} \land s_{ki} \land s_{ei} \neq \{0\}.}
\]

It then follows that

\[
f_i = s_{ei} \oplus s_{ij}, \quad f_{ij} = s_{ei} \oplus s_{ij} \oplus s_{jk}
\]

and cyclic permutations of these.

**Remark 3.15.** In this case, as we will see in Lemma 3.21 below, a line subbundle \( s < f \) is regular in the sense of Assumption 3.1 so long as \( s_i \neq s_{ij} \) on any edge \( ij \).

**Definition 3.16** (Applicable line congruence). We say that \( f \) is applicable if there is a \( \land^2 V \)-valued 1-form \( \eta \in \Omega^1(\land^2 V) \) with (notation as in Assumption 3.14):

1. \( \eta \) is closed.
2. \( \eta_{ij} \in \land^2 f_{ij} = f_j \land f_i \).
3. (non-degeneracy) \( \eta_{ij} \land s_{ij} \neq \{0\} \), on each edge \( ij \).

**Example 3.17.** If \((s, \eta)\) is a Koenigs net with \( s < f \) then \( f \) is applicable via the same \( \eta \) since each \( s_i \land s_j < \land^2 f_{ij} \).
There is a gauge freedom in the choice of $\eta$: if $\eta$ satisfies the conditions of the definition, so does $\eta^\tau := \eta + d\tau$ for any section $\tau$ of $\wedge^2 f$. Indeed, $\eta^\tau$ is certainly closed; $\eta^\tau_{ji} = \eta_{ji} + \tau_j - \tau_i \in \wedge^2 f_{ij}$ since $\tau_j, \tau_i \in \wedge^2 f_{ij}$ while, for the non-degeneracy, note that $\tau_i \wedge s_{ij} = \tau_j \wedge s_{ij} = 0$. We denote by $[\eta]$ the equivalence class of all 1-forms that arise this way:

$$[\eta] = \{\eta + d\tau \mid \tau \in \Gamma \wedge^2 f\}.$$  

**Remark 3.18.** It would be interesting to know under what circumstances $f$ is applicable with respect to $\eta_1$ and $\eta_2$ with $[\eta_1] \neq [\eta_2]$. See, for example, Musso–Nicolodi [38] for the smooth case.

We are going to show in Theorem 3.23 that, up to gauge, all applicable line congruences $(f, [\eta])$ arise from a Koenigs net as in Example 3.17. This will require some preparation.

### 3.2.1 Flat connection of an applicable net

We begin by discussing the projective geometry of $\eta_{ji}$ on a single edge. Since $\dim f_{ij} = 3$, $\eta_{ji}$ is decomposable and so determines a 2-plane in $f_{ij}$ or, equivalently, a projective line in the projective plane $\mathbb{P}(f_{ij})$. We denote this 2-plane or projective line by $[\eta_{ji}]$, that is, $[a \wedge b] = \langle a, b \rangle$. The non-degeneracy condition tells us $s_{ij}$ does not lie on $[\eta_{ji}]$ so that $[\eta_{ji}]$ is distinct from both of the projective lines $\mathbb{P}(f_i)$ and $\mathbb{P}(f_j)$. We can therefore define $s_{ij}^i \in \mathbb{P}(f_i)$ and $s_{ij}^j \in \mathbb{P}(f_j)$ by

$$s_{ij}^i = [\eta_{ji}] \cap \mathbb{P}(f_i), \quad s_{ij}^j = [\eta_{ji}] \cap \mathbb{P}(f_j).$$

Then $\eta_{ji} \in s_{ij}^i \wedge s_{ij}^j$ and $s_{ij}^i \wedge s_{ij} \wedge s_{ij}^j \neq \{0\}$.

For $\tau_j \in \wedge^2 f_j, r \in \mathbb{R}$, not both zero, $[r\eta_{ji} + \tau_j]$ is a line in the pencil through $[\eta_{ji}]$ and $\mathbb{P}(f_j)$. Thus, replacing $\eta_{ji}$ with $r\eta_{ji} + \tau_j$ changes the intersection with $\mathbb{P}(f_i)$ and leaves that with $\mathbb{P}(f_j)$ untouched. In particular, define a map $g_{ij} : \mathbb{P}(\wedge^2 f_j \oplus \mathbb{R}) \to \mathbb{P}(f_i)$ by

$$g_{ij}([\tau_j, r]) = [r\eta_{ji} + \tau_j] \cap \mathbb{P}(f_i) \in \mathbb{P}(f_i).$$

The situation is illustrated in Figure 3.

From elementary projective geometry, we have:

**Lemma 3.19.** $g_{ij}$ is an isomorphism of projective lines with $g_{ij}([\tau_j, 0]) = s_{ij}$.

Now for the main construction of this section: write $\Sigma = \Sigma_b \sqcup \Sigma_w$ in such a way that each edge has one vertex in $\Sigma_b$ and one in $\Sigma_w$ and define two bundles of projective lines $X^b, X^w$ over $\Sigma$ as follows:

$$X^b|_{\Sigma_b} := \mathbb{P}(f)|_{\Sigma_b}, \quad X^b|_{\Sigma_w} := \mathbb{P}(\wedge^2 f \oplus \mathbb{R})|_{\Sigma_w}$$

$$X^w|_{\Sigma_b} := \mathbb{P}(\wedge^2 f \oplus \mathbb{R})|_{\Sigma_b}, \quad X^w|_{\Sigma_w} := \mathbb{P}(f)|_{\Sigma_w}.$$  

Now use the $g_{ij}$ and their inverses to define connections $\gamma^b, \gamma^w$ on the projective line bundles $X^b, X^w$ (c.f. Remark 2.14). In detail,
The key point is that these connections are flat:

**Proposition 3.20.** On any quadrilateral \(\ell kj i\), we have

\[
\gamma_{ij}^w \gamma_{jk}^w \gamma_{kl}^w = \text{id}_{X_i} = \gamma_{ij}^b \gamma_{jk}^b \gamma_{kl}^b.
\]

**Proof.** For \((X, \gamma)\) one of the bundles with connection under consideration, cyclically permute the indices if necessary to arrange that \(X_i = \mathbb{P}(f_i)\). Choose \(s_i \in \mathbb{P}(f_i)\) with \(s_i \neq s_{ij}, s_{\ell i}\). By continuity, it suffices to prove that

\[
(\gamma_{\ell i} \circ \gamma_j)(s_i) = (\gamma_{kj} \circ \gamma_{ji})(s_i).
\]

Call the left side \(s'_{\ell}\) and the right \(s''_{\ell}\). We want to prove that these coincide so suppose, for a contradiction, that they do not. Then, with \(\gamma_{\ell i}(s_i) = [\tau_{\ell i}, 1]\) and \(\gamma_{ji}(s_i) = [\tau_{ji}, 1]\), we have

\[
(\eta_{\ell i} + \tau_{\ell i}) = (\eta_{ji} + \tau_{ji}) \in s_i \wedge \mathbb{V}, \tag{3.7}
\]

while there are \(s_j = s^j_{kj} \in \mathbb{P}(f_j) \setminus s_{kj}\), \(s_\ell = s^\ell_{\ell i} \in \mathbb{P}(f_\ell) \setminus s_{k\ell}\) such that

\[
\eta_{\ell i} + \tau_{\ell i} \in s_\ell \wedge s'_{\ell}, \quad \eta_{kj} + \tau_j \in s_i \wedge s''_{kj}.
\]

Since \(\eta\) is closed we also have

\[
(\eta_{\ell i} + \tau_{\ell i}) - (\eta_{ji} + \tau_{ji}) = (\eta_{\ell k} + \tau_{\ell i}) - (\eta_{kj} + \tau_{j}). \tag{3.8}
\]

Now, thanks to (3.7), the left side of (3.8) lies in \(s_i \wedge \mathbb{V}\) and so is decomposable. The right side therefore satisfies the Plücker relation, and we deduce that \(s_\ell \wedge s'_{k} \wedge s_j \wedge s''_{k} = \{0\}\) so that \(s_\ell, s'_{k}, s_j, s''_{k}\)
span a 3-plane. However, this 3-plane contains \( f_k = s'_k \oplus s''_k = s_{jk} \oplus s_{\ell k} \), and so \( f_j = s_j \oplus s_{jk} \) and \( f_{\ell} = s_{\ell} \oplus s_{\ell k} \). Thus, our 3-plane contains all four intersections \( s_{ij} \), contradicting the second order regularity of \( f \).

\[ \square \]

### 3.2.2 Applicability via Koenigs–Moutard pair of Koenigs maps

Since the connections \( \gamma^b, \gamma^w \) are flat, we may choose a \( \gamma^b \)-parallel section \( x^b \) of \( X^b \) and a \( \gamma^w \)-parallel section \( x^w \) of \( X^w \) and, using these, define \( s < f \) by

\[
s|_{\Sigma_b} = x^b|_{\Sigma_b}, \quad s|_{\Sigma_w} = x^w|_{\Sigma_w}. \tag{3.9a}
\]

\( s \) is uniquely determined by its values at a pair of initial vertices, one black and one white. By choosing the initial values away from a countable set, we assume that \( s \) never coincides with an intersection \( s_{ij} \). We then define \( \tau \in \Gamma (\wedge^2 f) \) by

\[
[\tau, 1]|_{\Sigma_b} = x^w|_{\Sigma_b}, [\tau, 1]|_{\Sigma_w} = x^b|_{\Sigma_w}, \tag{3.9b}
\]

and our assumption on the intersections \( s_{ij} \) ensures that \( \tau \) is never \( \infty \) thanks to Lemma 3.19. With \( s, \tau \) so defined, we have, on each edge \( ij \), \( s_i = g_{ij}([\tau_j, 1]) \) so that

\[
\eta_{ji} + \tau_j \in s_i \wedge V. \tag{3.10}
\]

We are about to prove that \((s, \eta + d\tau)\) is a Koenigs net but first we show that it satisfies the regularity conditions of Assumption 3.1.

**Lemma 3.21.** Suppose that \( s < f \) with \( s_i \wedge s_j \neq 0 \), equivalently \( s_i \neq s_{ij} \), on each edge \( ij \). Then, for any quadrilateral \( \ell k ji \), \( s_i, s_j, s_k, s_{\ell} \) are pairwise distinct while \( \dim (s_i + s_j + s_k + s_{\ell}) \geq 3 \).

**Proof.** Since \( s_i \wedge s_j \neq \{0\} \), with \( U := s_i + s_j + s_k + s_{\ell} \), we have \( \dim U \geq 2 \). Suppose now that \( \dim U = 2 \). Then \( U = s_i \oplus s_j < f_{ij} \), and similarly \( U < f_{k\ell} \) so that \( U = f_{ij} \wedge f_{k\ell} = s_{ij} \wedge s_{k\ell} \). But the second order regularity gives \((s_{ij} \wedge s_{k\ell}) \wedge (s_{ij} \wedge s_{k\ell}) = \{0\} \), and so a contradiction.

Second order regularity also gives \( f_i \cap f_k = \{0\} = f_j \cap f_{\ell} \) so diagonal vertices are also pairwise distinct.

With this in hand, we have:

**Proposition 3.22.** Let \((f, \eta)\) be applicable and define \( s, \tau \) by (3.9). Then \((s, \eta + d\tau)\) is a Koenigs net.

**Proof.** The only thing to prove is that \((\eta + d\tau)_{ji} \in s_i \wedge s_j \) on each edge \( ij \). Clearly, \( \tau_i \in s_i \wedge V \) so that, by (3.10),

\[
(\eta + d\tau)_{ji} = (\eta_{ji} + \tau_j) - \tau_i \in s_i \wedge V.
\]

By the same argument, \((\eta + d\tau)_{ji} = -(\eta + d\tau)_{ij} \in s_j \wedge V \) so that \((\eta + d\tau)_{ji} \in (s_i \wedge V) \cap (s_j \wedge V) = s_i \wedge s_j \) as required.

\[ \square \]
We can say more: choose distinct initial conditions for two parallel sections each of $X^b, X^w$ to arrive at pointwise distinct maps $s^± < f$ and $τ^±$ with $η^± := η + dτ^±$ satisfying

$$η^±_{ij} ∈ s^±_j ∧ s^±_i$$
on all edges. Since $s^±$ are pointwise distinct, we have $f = s^+ ⊕ s^-$. Finally, set $τ = τ^- − τ^+$ so that

$$η^- = η^+ + dτ.$$Observe that $τ$ is never zero: indeed if $τ_j = 0$, and $ij$ is an edge, then $τ^±_j$ coincide so that $s^±_i = g_{ij}([τ^±_j, 1])$ coincide also. In view of Proposition 3.10, we have therefore arrived at the following characterisation of an applicable line congruence.

**Theorem 3.23.** Let $f : Σ → G_2(V)$. Then the following are equivalent:

1. $(f, η)$ is a regular applicable line congruence.
2. $f$ is spanned by a Koenigs–Moutard pair of Koenigs nets $s^± < f$ and regular.

In this case $[η] = [η^+] = [η^-]$.

**Proof.** The only thing left to prove is that the span of a Koenigs–Moutard pair is a line congruence. However, the Moutard equation (3.3) on vertical quadrilaterals assures us that $s^±_i$ and $s^±_j$ are coplanar so that $f_i$ and $f_j$ intersect. □

**Remark 3.24.** Of course, the Koenigs nets of Theorem 3.23 are far from unique: their values may be chosen freely on a pair of initial vertices, one black and one white, c.f. [14, Lemma 3.3].

### 3.2.3 Applicability via Koenigs dual lifts

Let $f : Σ → G_2(V)$ be a line congruence and suppose it is spanned by sections $σ^±$ which are Koenigs dual: thus $σ^± : Σ → V$ with

$$dσ^+_i ∧ dσ^-_i = 0,$$on each edge $ij$;

$$dσ^+ ∧ dσ^- = 0,$$or, equivalently, $A(σ^+, σ^-) = 0$. (3.11b)

We emphasise that here we do not require that $σ^±$ take values in an affine subspace of $V$ or even that they have planar quadrilaterals.

In this situation, set $s^± = ⟨σ^±⟩$ and then define $η^±, τ$ by

$$η^± = dσ^± ∧ σ^±$$

Then each $η^±_{ij} ∈ s^±_j ∧ s^±_i$ by (3.11a) while $dη^± = 0$ by (3.11b). Finally, $η^- = η^+ + dτ$ so that $s^±$ are a Koenigs–Moutard pair and $f$ is applicable.

If $μ^±$ are the corresponding Moutard lifts then Corollary 3.11 tells us that $τ = μ^- ∧ μ^+$ so that there is a function $r : Σ → ℜ^∞$ with

$$μ^± = r^±(σ^±).$$ (3.13)
With \( d\sigma^-_{ji} = \lambda_{ij} d\sigma^+_{ji} \), we have

\[
\begin{align*}
\mu^+_j \wedge \mu^+_i &= \eta^+_j = \frac{1}{2} d\sigma^-_{ji} \wedge (\sigma^+_i + \sigma^+_j) \\
\frac{1}{2} \lambda_{ij} d\sigma^+_{ji} \wedge (\sigma^+_i + \sigma^+_j) &= \lambda_{ij} \sigma^+_j \wedge \sigma^+_i
\end{align*}
\]

and so we arrive at the Christoffel formula:

\[
d\sigma^-_{ij} = r_i r_j d\sigma^+_{ij}.
\]

(3.14)

In terms of \( \mu^\pm \), this last reads

\[
\mu^-_j - \mu^+_i = \frac{r_i}{r_j} (\mu^-_i - \mu^+_j)
\]

(3.15)

which is a refinement of (3.6).

Conversely, if (3.15) holds, we can reverse this argument and starting from \( \mu^\pm \), obtain \( \sigma^\pm \) from (3.13) for which (3.12) holds so that \( \sigma^\pm \) are Koenigs dual.

In fact, (3.15) always holds and we have:

**Proposition 3.25.** Let \( f : \Sigma \to G_2(V) \) be a line congruence. Then \( f \) is spanned by Koenigs dual sections if and only if \( f \) is an applicable net.

**Proof.** Let \( f = s^+ \oplus s^- \) be applicable, and let \( \mu^\pm \in \Gamma s^\pm \) be Moutard lifts of \( f \) with (3.6), that is,

\[
\mu^-_j - \mu^+_i = b_{ij}(\mu^-_i - \mu^+_j)
\]

for some discrete function \( b_{ij} \) defined on oriented edges \( ij \) such that \( b_{ij} = 1/b_{ji} \). From the discussion above, we only need to show that there is some function \( r \) such that \( b_{ji} = r_i/r_j \), or, equivalently, that

\[
b_{ij} b_{\ell k} b_{kj} b_{ji} = 1
\]

on any quadrilateral \( \ell k ji \).

For this last, consider the following Moutard cube in the sense of [7]:

![Moutard cube diagram](image-url)
There is a Moutard equation (and so a $b$) relating the four vertices of each face of the cube. The content of equation (2.52) of [7, Theorem 2.34] is that suitable ratios of the $b$’s from any pair of opposite faces coincide. In our case, this yields

$$\frac{b_{k\ell}}{b_{ji}} = \frac{b_{kj}}{b_{\ell i}}$$

and so the desired conclusion.

Remark 3.26. Proposition 3.25 guarantees the existence of applicable congruences through any Koenigs net $s$: simply take the line spanned by an affine lift $F = \sigma^+$ of $s$ and the Koenigs dual affine lift $\tilde{F} = \sigma^-$ provided by Proposition 3.4.

We conclude this section with a summary of our discussion:

**Theorem 3.27.** Let $f$ be a regular line congruence. Then the following are equivalent:

- $f$ is applicable;
- $f$ is spanned by a Koenigs–Moutard pair of Koenigs nets;
- $f$ is spanned by Moutard sections satisfying (3.6);
- $f$ is spanned by Koenigs dual sections.

## 4 ISOTHERMIC NETS

We now restrict attention to nets and line congruences taking values in a non-singular quadric. The quadric reduces the ambient projective geometry to conformal geometry of some signature $(p, q)$. In our application to $\Omega$-nets, $(p, q) = (3, 1)$ so we will emphasise the case of indefinite signature.

So contemplate the pseudo-Euclidean space $\mathbb{R}^{p+1,q+1}$, a $(p + q + 2)$-dimensional space equipped with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ of signature $(p + 1, q + 1)$.

Let $L := \{ x \in \mathbb{R}^{p+1,q+1} \mid (x, x) = 0 \}$ be the light cone, and $Q = Q^{p,q} := \mathbb{P}(L) = \{ x \in \mathbb{P}(\mathbb{R}^{p+1,q+1}) \mid x \in L \setminus \{0\} \} \subset \mathbb{P}(\mathbb{R}^{p+1,q+1})$ be the projective light cone. Thus, $Q$ is a non-singular quadric. It carries an $O(p + 1, q + 1)$-invariant conformal structure of signature $(p, q)$.

We identify the Lie algebra $\mathfrak{o}(p + 1, q + 1)$ with the exterior algebra $\wedge^2 \mathbb{R}^{p+1,q+1}$ via

$$x \wedge y(z) = (x, z)y - (y, z)x$$

for $x, y, z \in \mathbb{R}^{p+1,q+1}$.

On maps $s : \Sigma \to Q \subset \mathbb{P}(\mathbb{R}^{p+1,q+1})$ we impose regularity assumptions extending those of Assumption 3.1:

**Assumption 4.1.** On each quadrilateral $\ell k j i$:

1. $s_i, s_j, s_k, s_\ell$ are pair-wise distinct;
2. $s_i, s_j, s_k, s_\ell$ are not collinear;
3. diagonals are non-isotropic. That is, the projective lines $s_i s_k$ and $s_j s_\ell$ do not lie in $Q$. 


4.1 Isothermic nets and the Moutard equation

**Definition 4.2** (Isothermic net). A map \( s : \Sigma \to Q \subset \mathbb{P}(\mathbb{R}^{p+1,q+1}) \) is *isothermic* if it is Koenigs as a map into \( \mathbb{P}(\mathbb{R}^{p+1,q+1}) \).

Thus, \( s \) is isothermic if and only if there is a closed, never-zero 1-form \( \eta \) with \( \eta_{ij} \in s_i \wedge s_j \leq \wedge^2 \mathbb{R}^{p+1,q+1} \), for each edge \( ij \).

According to Theorem 3.6, \( s \) is isothermic exactly when it admits a Moutard lift \( \mu \in \Gamma s \) and then \( \eta_{ji} = \mu_j \wedge \mu_i \). However, in our conformal setting, the Moutard equation (3.3) becomes much more rigid and reads:

\[
\mu_k - \mu_i = \frac{(\mu_i, \mu_\ell - \mu_j)}{(\mu_\ell, \mu_j)} (\mu_\ell - \mu_j).
\]

Indeed, (3.3) tells us that

\[
\mu_k = \mu_i + c(\mu_\ell - \mu_j),
\]

for some \( c \in \mathbb{R}^\times \). However, taking the inner product of (4.3) with itself and using the vanishing of \((\mu, \mu)\) repeatedly allows us to solve for \( c \) and arrive at (4.2).

Moreover, \( \mu \) gives rise to an edge-labeling \( m_{ij} := \frac{1}{(\mu_i, \mu_j)} \in \mathbb{R} \cup \{\infty\} \), that is \([7, \text{Theorem 4.5}]\),

\[
m_{ii} = m_{\ell\ell}, \quad m_i = m_j, \quad m_{i\ell} = m_{j\ell}.
\]

In addition, our regularity assumptions assure us that \( m_{ij} \neq m_{i\ell} \).

When \( m \) is finite (which is guaranteed by regularity of \( s \) if \( q = 0 \)) (4.2) tells us \( m \) is a cross ratio factorising function \([14, \text{Lemma 3.5}]\), that is, on any quadrilateral \( \ell k ji \), the vertices \( s_i, s_j, s_k, s_\ell \) lie on a nonsingular conic with cross ratio

\[
[s_i, s_j, s_k, s_\ell] = \frac{m_{jk}}{m_{ij}}.
\]

**Remark 4.3.** On an edge \( ij \) where \( m_{ij} \) is finite, \( m_{ij} \) is equivalent data to \( \eta_{ij} \) since \( s_i \wedge s_j \) is 1-dimensional and \( m_{ij} \) fixes the scale. In particular, if we scale \( \eta \), then \( m_{ij} \) scales reciprocally.

4.2 Koenigs–Moutard transformations of isothermic nets

Let \((s^\pm, \eta^\pm) : \Sigma \to Q\) be a Koenigs–Moutard pair of isothermic nets with Moutard lifts \( \mu^\pm \). Thus, there is an isothermic net \( s = s^+ \sqcup s^- : \{0, 1\} \times \Sigma \to Q \)

\[
s_{\{0\} \times \Sigma} = s^+, \quad s_{\{1\} \times \Sigma} = s^{-}
\]

and Moutard lift \( \mu = \mu^+ \sqcup \mu^- \). In particular, the edge-labeling property for \( \mu \) on vertical edges says that \((\mu_{(0,i)}, \mu_{(1,i)}) \) so that \( m := 1/(\mu^+, \mu^-) \) is constant. When \( m \) is finite, \( s^\pm, s_j^\pm \) lie on a non-singular conic while (4.5) on vertical faces reads

\[
[s_i^+, s_i^-, s_j^+, s_j^-] = \frac{m}{m_{ij}}
\]
so that we recognise that \( s^- \) is precisely a Darboux transform \([7, \text{Definition 4.7}]\) of \( s^+ \) with parameter \( m \). In view of this, we mildly extend the notion of Darboux transform to include the case \( m = \infty \):

**Definition 4.4** (Darboux transform, Darboux pair). Let \((s^+, \eta^+) : \Sigma \to Q \) be isothermic. A Koenigs–Moutard transform \((s^-, \eta^-) \) with \( 1/(\mu^+, \mu^-) \equiv m \in \mathbb{R} \cup \{ \infty \} \) is called a Darboux transform of \( s^+ \) with parameter \( m \) and \( s^+, s^- \) are called a Darboux pair with parameter \( m \).

An isotropic Darboux pair is a Darboux pair with parameter \( m = \infty \).

The nets \( s^\pm \) of an isotropic Darboux pair are orthogonal and so span an applicable congruence of lines lying in \( Q \) by Theorem 3.23. We shall have more to say about such congruences below in Section 5.

### 4.3 Duality for isothermic nets

#### 4.3.1 Circular nets in \( \mathbb{R}^{p,q} \)

Let \( o, q \in L \subseteq \mathbb{R}^{p+1,q+1} \) with \((o, q) = -1\). Set \( \mathbb{R}^{p,q} := \langle o, q \rangle \perp \) and let

\[
E = \{ v \in L \mid (v, q) = -1 \}.
\]

Then \( \phi : \mathbb{R}^{p,q} \to E \) given by

\[
\phi(x) := o + x + \frac{1}{2}(x, x)q
\]

is an isometry with inverse \( \psi := \pi|_E \) for \( \pi : \mathbb{R}^{p+1,q+1} \to \mathbb{R}^{p,q} \) orthoprojection. Meanwhile, the projection \( L \to \mathbb{P}(L) \) restricts to a conformal diffeomorphism \( E \cong \mathbb{P}(L) \setminus \mathbb{P}(L \cap q^\perp) \). Putting these together yields stereoprojection \( \mathbb{P}(L) \setminus \mathbb{P}(L \cap q^\perp) \cong \mathbb{R}^{p,q} \) with inverse \( x \mapsto \langle \phi(x) \rangle \).

We note that, for \( x_1, x_2 \in \mathbb{R}^{p,q} \) and \( y_i = \phi(x_i) \), we have

\[
(y_1, y_2) = -\frac{1}{2}(x_1 - x_2, x_1 - x_2).
\]

**Definition 4.5** (Euclidean lift). Let \( s : \Sigma \to Q \) with stereoprojection \( x : \Sigma \to \mathbb{R}^{p,q} \). We call \( y := \phi(x) = o + x + \frac{1}{2}(x, x)q \in \Gamma s \) the Euclidean lift of \( s \) (or \( x \)) with respect to \( q \).

It is the unique section \( y \) of \( s \) with \((y, q) \equiv -1\).

A circular net in \( \mathbb{R}^{p,q} \) is the stereoprojection of a \( Q \)-net in \( Q \setminus \mathbb{P}(L \cap q^\perp) \):

**Definition 4.6** (Circular net). \( x : \Sigma \to \mathbb{R}^{p,q} \) is called a circular net if \( x \) has non-collinear quadrilaterals and, on each such quadrilateral \( \ell k ji \), its inverse stereoprojection \( s : \Sigma \to Q \) has

\[
\dim s_{\ell k ji} = 3,
\]

where \( s_{\ell k ji} = s_i + s_j + s_k + s_{\ell} \).
Remarks 4.7.

1. It is easy to see that a circular net has (affine) planar quadrilaterals and so is a $Q$-net in $\mathbb{R}^{p,q}$.

2. A circle in $\mathbb{R}^{p,q}$ is the intersection of an affine 2-plane with a quadric cone of the form $\{ x \in \mathbb{R}^{p,q} | (x-c,x-c) = R \}$, for some $c \in \mathbb{R}^{p,q}$ and $R \in \mathbb{R}$. We note that $x \in \mathbb{R}^{p,q}$ lies on such a circle if and only if its Euclidean lift $\mathfrak{o} + x + \frac{1}{2}(c,c) - R \mathfrak{q}$ in $\mathbb{R}^{p+1,q+1}$.

In indefinite signature, circles need not be 1-dimensional: null affine 2-planes are circles.

One can show that $x$ is circular exactly when the vertices of each quadrilateral lie on a circle: an element of $s_{\ell kji}^{\perp} \setminus q^{\perp}$ is, up to scale, of the form $\mathfrak{o} + c + \frac{1}{2}((c,c) - R) \mathfrak{q}$.

For later use, we record:

**Lemma 4.8.** Let $x : \Sigma \to \mathbb{R}^{p,q}$ be a circular net with inverse stereoprojection $s$ and let $\ell kji$ be a quadrilateral. Set $U_{\ell kji} := \langle dx_{ij}, dx_{jk}, dx_{k\ell}, dx_{\ell i} \rangle$ and $W_{\ell kji} := s_{\ell kji} \cap q^{\perp}$.

Then $\dim U_{\ell kji} = \dim W_{\ell kji} = 2$ and orthoprojection $\pi : \mathbb{R}^{p+1,q+1} \to \mathbb{R}^{p,q}$ restricts to an isomorphism $W_{\ell kji} \cong U_{\ell kji}$.

In particular, $\wedge^2 \pi|_{\wedge^2 W_{\ell kji}} : \wedge^2 W_{\ell kji} \to \wedge^2 U_{\ell kji}, \wedge^2 \pi(a \wedge b) = \pi(a) \wedge \pi(b)$ is also an isomorphism.

**Proof.** First note that $s_{\ell kji}$ is not contained in $q^{\perp}$ since no $s_i$ lies in $q^{\perp}$. Thus, $\dim W_{\ell kji} = 2$. Moreover, $W_{\ell kji} = \langle dy_{ij}, dy_{jk}, dy_{k\ell}, dy_{\ell i} \rangle$ since $s_{\ell kji} = \langle y_i, y_j, y_k, y_\ell \rangle$, while $\pi(dy) = dx$ so that $\pi(W_{\ell kji}) = U_{\ell kji}$. Now the quadrilateral is non-collinear so that $\dim U_{\ell kji} = 2$ also, whence $\pi|_{W_{\ell kji}}$ is an isomorphism.

**Remark 4.9.** Note that

$$W_{\ell kji} \cap \ker \pi = s_{\ell kji} \cap q^{\perp} \cap \langle \mathfrak{o}, q \rangle = s_{\ell kji} \cap \langle q \rangle$$

so that a quadrilateral $\ell kji$ of $x$ is collinear exactly when $q \in s_{\ell kji}$. Thus, the stereoprojection of a $Q$-net $s$ with respect to $\mathfrak{o}, q$ is circular in our sense so long as $q \not\in s_{\ell kji}$ for any elementary quadrilateral $\ell kji$. This amounts to choosing $q$ off a set of measure zero in $Q$.

We conclude our present discussion of circular nets with a result that is “obvious” [7, page 156] in the definite case but less so (at least to us!) in the present setting:

**Proposition 4.10.** Let $x : \Sigma \to \mathbb{R}^{p,q}$ be a circular net and $\tilde{x} : \Sigma \to \mathbb{R}^{p,q}$ an edge-parallelnet. Then $\tilde{x}$ is also circular.

**Proof.** We work on a single quadrilateral $\ell kji$. By translation and scaling, we may assume without loss of generality that $x_i = \tilde{x}_i$ and $x_j = \tilde{x}_j$. Also without loss of generality, assume that $dx_{i\ell}, dx_{k\ell}$ span $U_{\ell kji}$ and write

$$dx_{ij} = \alpha dx_{i\ell} + \beta dx_{k\ell}$$

$$dx_{jk} = \gamma dx_{i\ell} + \delta dx_{k\ell},$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Then, for some $t \in \mathbb{R}$, $d\tilde{x}_{jk} = t dx_{jk} = t(\gamma dx_{i\ell} + \delta dx_{k\ell})$ from which we deduce, using $d\tilde{x}_{k\ell} \parallel dx_{k\ell}$ that
\[ \mathrm{d} \hat{x}_{i'} = (\alpha + \gamma t) \mathrm{d} x_{i'} \quad (4.8a) \]
\[ \mathrm{d} \hat{x}_{k'} = - (\beta + \delta t) \mathrm{d} x_{k'}. \quad (4.8b) \]

Now let \( y, \hat{y} \) be the Euclidean lifts of \( x, \hat{x} \) and contemplate
\[
p(t) := \hat{y}_i \wedge \hat{y}_j \wedge \hat{y}_k \wedge \hat{y}_\ell,
\]
a polynomial with values in \( \wedge^4 \mathbb{R}^{p+1,q+1} \) that vanishes exactly when \( \hat{x} \) is circular.

In view of (4.8), \( p(t) \) is cubic in \( t \) if \( \gamma \delta \neq 0 \) and quadratic otherwise. However, \( p(t) \) has roots at 0, when \( \hat{x}_k = \hat{x}_j \); at 1, when \( x = \hat{x} \); at \( -\alpha/\gamma \), when \( \hat{x}_i = \hat{x}_{\ell'} \), if \( \gamma \neq 0 \) and at \( -\beta/\delta \), when \( \hat{x}_k = \hat{x}_{\ell'} \), if \( \delta \neq 0 \). In any case, \( p(t) \) has four roots when it is cubic and three when quadratic and so vanishes identically. Thus, any edge-parallel \( \hat{x} \) is circular. \( \square \)

4.3.2 | The Christoffel dual

With these preparations in hand, we show that, just as in the definite case [7, Theorem 4.32], a net in \( Q \) is isothermic if and only if its stereoprojection is a circular Koenigs net. More precisely,

**Theorem 4.11.** Let \( (s, \eta) : \Sigma \to Q \setminus P(L \cap q^1) \) be a \( Q \)-net with (circular) stereoprojection \( x \). Then \( s \) is isothermic if and only if \( x \) has a Koenigs dual \( \hat{x} \): that is, \( \hat{x} \) is edge-parallel to \( x \) with
\[
\mathrm{d} x \wedge \mathrm{d} \hat{x} = 0. \quad (4.9)
\]

In this case, \( \hat{x} \) is also the stereoprojection of an isothermic net.

In view of this, we say that \( x : \Sigma \to \mathbb{R}^{p,q} \) is isothermic if it is a circular Koenigs net. We call \( \hat{x} \) a Christoffel dual of \( x \). It is determined up to translation and a constant scaling (and then \( \eta \) is scaled in the same way).

**Proof.** First, suppose that \( s \) is isothermic so that the Euclidean lift \( y \) of \( s \) takes values in the affine space \( A = \{ v \mid (q, v) = -1 \} \) and let \( \hat{y} \) be the Koenigs dual of \( y \) as in Proposition 3.4. Thus, \( \eta = \mathrm{d}\hat{y} \wedge y \). Set \( \hat{x} = \pi \hat{y} \). Then \( \hat{x} \) is edge-parallel to \( x \) since \( \hat{y} \) is edge-parallel to \( y \). Furthermore, we have \( \mathrm{d} y \wedge \mathrm{d}\hat{y} = 0 \) and taking \( \wedge^2 \pi \) of this yields (4.9).

For the converse, we must exploit the circularity of \( x \). Define a \( q^1 \)-valued 1-form \( \omega \) by
\[
\omega = \mathrm{d} \hat{x} + (x \wedge \mathrm{d} \hat{x}) q
\]
and observe that each \( \omega_{ji} \parallel \mathrm{d} y_{ji} = \mathrm{d} x_{ji} + (x \wedge \mathrm{d} x)_{ji} q \) since \( \mathrm{d} \hat{x}_{ji} \parallel \mathrm{d} x_{ji} \). We now define \( \eta := \omega \wedge y \) and deduce that each \( \eta_{ji} \parallel y_i \wedge y_j \) and so takes values in \( s_i \wedge s_j \). Moreover, on a quadrilateral \( \ell k j i \), both \( \omega \) and \( \mathrm{d} y \) take values in \( W_{\ell k j i} \) while
\[
\pi(\mathrm{d}\omega) = \mathrm{d}^2 \hat{x} = 0
\]
\[
\wedge^2 \pi(\omega \wedge \mathrm{d} y) = \mathrm{d} \hat{x} \wedge \mathrm{d} x = 0.
\]
Thus Lemma 4.8 tells us that \( \mathrm{d}\omega, \omega \wedge \mathrm{d} y \) and so \( \mathrm{d}\eta \) vanish. Thus \( s \) is isothermic.
Finally, thanks to Proposition 4.10, \( \hat{x} \) is also circular with Koenigs dual \( x \) and so is also the stereoprojection of an isothermic net.

Remark 4.12. Note that we may express \( d\hat{x} \) directly in terms of \( \eta \): the argument of Proposition 3.4 tells us that \( d\hat{y} = \eta q \) so that:

\[
\eta = (\eta q) \wedge y \tag{4.10a}
\]

\[
d\hat{x} = \pi(\eta q). \tag{4.10b}
\]

The scaling between \( dx \) and \( d\hat{x} \) can be expressed directly in terms of the edge labeling \( m_{ij} \):

**Proposition 4.13.** Let \( x : \Sigma \to \mathbb{R}^{p,q} \) be isothermic with edge-labeling \( m_{ij} \) and Christoffel dual \( \hat{x} \). Then, for each edge \( ij \),

\[
(d_{x_{ij}}, d_{\hat{x}_{ij}}) = -\frac{2}{m_{ij}}. \tag{4.11}
\]

**Proof.** Let \( s \) be the inverse stereoprojection of \( x \) and \( y, \mu \in \Gamma s \) the Euclidean and Moutard lifts. We have \( \mu = ry \) where \( r = -(\mu, q) \). Now (4.7) gives

\[
\frac{1}{m_{ij}} = (\mu_i, \mu_j) = r_ir_j(y_i, y_j) = -r_ir_j\frac{1}{2}(dx_{ij}, dx_{ij}), \tag{4.12}
\]

while, on the other hand,

\[
\eta_{ij}q = (\mu_i \wedge \mu_j)q = r_ir_j \, dy_{ij}
\]

so that (4.10b) gives

\[
d_{\hat{x}_{ij}} = r_ir_j \, dx_{ij}. \tag{4.13}
\]

Putting (4.12) and (4.13) together yields (4.11).

Since \( x \) is the Christoffel dual of \( \hat{x} \), we immediately learn:

**Corollary 4.14.** An isothermic net and its Christoffel dual have the same edge-labeling.

Proposition 4.13 can be interpreted in two interesting ways. When \( \Sigma = \mathbb{Z}^n \) and all \( m_{ij} \) are finite, we recover the well-known Christoffel formula [34, §5.7.7]:

\[
d_{\hat{x}_{ij}} = -\frac{2}{m_{ij}} \, dx_{ij}/(dx_{ij}, dx_{ij}).
\]

On the other hand, if \( \hat{x} \) is a Darboux transform of \( x \) on \( \mathbb{Z}^n \), we may apply Theorem 4.11 and Proposition 4.13 to \( x \sqcup \hat{x} \) on \( \Sigma = \{0, 1\} \times \mathbb{Z}^n \) to obtain a result due, in the classical smooth case, to Bianchi [1, p. 105] (see [34, §5.7.32] for the discrete definite case):

**Corollary 4.15.** Let \( x : \mathbb{Z}^n \to \mathbb{R}^{p,q} \) be isothermic and \( \hat{x} : \mathbb{Z}^n \to \mathbb{R}^{p,q} \) a Darboux transform with parameter \( m \in \mathbb{R}^\times \cup \{\infty\} \). Let \( \hat{x} \) be a Christoffel dual of \( x \).
Then there is a Christoffel dual $\hat{x}$ of $x$ which is simultaneously a Darboux transform of $x$ with parameter $m$. Moreover $\hat{x} - x$ and $\tilde{x} - \hat{x}$ are pointwise parallel and

$$(\hat{x} - x, \tilde{x} - \hat{x}) = -\frac{2}{m}. \quad (4.14)$$

Proof. Apply Theorem 4.11 to $x \sqcup \hat{x}$ and translate to get a Christoffel dual $(x \sqcup \hat{x})^\vee = x \sqcup \tilde{x}$ of $x \sqcup \hat{x}$ extending $\hat{x}$. Then $\tilde{x}$ is a Christoffel dual of $\hat{x}$ and also, thanks to Corollary 4.14, a Darboux transform of $\hat{x}$ with parameter $m$ (the edge label for vertical edges). Again, since $(x \sqcup \hat{x})^\vee$ is edge-parallel to $x \sqcup \hat{x}$ on vertical edges, we get that $(\hat{x} - x) \parallel (\tilde{x} - \hat{x})$ while Proposition 4.13 yields (4.14). $\square$

## 4.4 Families of flat connections

A defining characteristic of an isothermic net $s$ with finite cross-ratio factorising function $m$ is a 1-parameter family of flat connections $(\Gamma^s(t))_{t \in \mathbb{R}}$ on the trivial bundle $\Sigma \times \mathbb{R}^{p+1,q+1}$ that are defined as follows:

$$\Gamma^s(t)_{ji} := \Gamma_{s_i}^s(1 - t/m_{ij}), \quad (4.15)$$

where, for $\lambda \in \mathbb{R}^\times$,

$$\Gamma_{s_i}^s(\lambda) = \begin{cases} 
\lambda & \text{on } s_j \\
1 & \text{on } (s_i \oplus s_j)^\perp \\
1/\lambda & \text{on } s_i.
\end{cases}$$

Implicit in the discussion in [14, §3] is an extension of this to the case where $m_{ij} = \infty$ on one family of edges. We give an explicit self-contained argument here.

**Proposition 4.16.** Let $(s, \eta) : \Sigma \to Q$ be isothermic with edge-labeling $m$. Define connections $(\Gamma^s(t))_{t \in \mathbb{R}}$ on $\Sigma \times \mathbb{R}^{p+1,q+1}$ by (4.15) if $m_{ij}$ is finite and

$$\Gamma^s(t)_{ji} = \exp(t \eta_{ji}) \quad (4.16)$$

when $m_{ij} = \infty$.

Then each $\Gamma^s(t)$ is a flat connection: on each quadrilateral $\ell k ji$,

$$\Gamma^s(t)_{kj} \Gamma^s(t)_{ji} = \Gamma^s(t)_{k \ell} \Gamma^s(t)_{\ell i}. \quad (4.17)$$

Proof. We suppose that $m_{ij} = m_{k \ell} = \infty$. Assumption 4.1 assures us that $s_j, s_\ell$ are not orthogonal and that $m_{ij}$ is finite. We now follow the strategy for the case of finite $m$ in [12, Lemma 4.7] by proving that both sides of (4.17) equal $\Gamma_{s_j}^s(1 - t/m_{ij})$. With $L(t)$ denoting the left side, it is easy to see that $L(t)$ and $\Gamma_{s_j}^s(1 - t/m_{ij})$ agree on both $s_j$ and $s_j^\perp/s_j$ so that, since both lie in $O(p + 1, q + 1)$, it suffices to show that they agree on $s_\ell$, that is, with $\mu \in \Gamma s$, the Moutard lift,

$$\Gamma_{s_j}^s(1 - t/m_{jk}) \exp(t \eta_{ji}) \mu_\ell = (1 - t/m_{\ell i}) \mu_\ell,$$
or, equivalently,
\[
\exp(t\eta_{ji})\mu_{\varphi} = (1 - t/m_{\varphi i})\Gamma_{s,k}^s(1 - t/m_{jk})\mu_{\varphi}.
\]

However, a straightforward calculation using (4.1) and \(m_{\varphi i} = m_{jk}\) shows that this last amounts to (4.2). The equality for the right hand side is similar.

Remark 4.17. The connections \(\Gamma^s(t)\) and the 1-form \(\eta\) are equivalent data: given \((\Gamma^s(t))_{t\in\mathbb{R}}\) we recover \(\eta\) by
\[
\eta = \partial / \partial t \bigg|_{t=0} \Gamma^s(t).
\]

Conversely, given \(\eta\), we have
\[
\Gamma^s_{ji}(t) = \exp\left(-m_{ij} \log(1 - t/m_{ij})\eta_{ji}\right),
\]
where we use L'Hôpital's rule to interpret the right side when \(m_{ij} = \infty\). In particular, scaling \(\eta\) by a constant scales the parameter \(t\) also. In more detail, replacing \(\eta\) by \(\lambda \eta\), for \(\lambda \in \mathbb{R}^\times\) a constant, requires us to replace \(m_{ij}\) by \(m_{ij}/\lambda\) and \(\Gamma^s(t)\) by \(\Gamma^s(\lambda t)\).

These flat connections give an alternative perspective on the transformation theory of isothermic nets. In particular, we have a discrete analogue of Darboux’s linear system [21], see also [12, 14]:

Proposition 4.18. Let \(s : \Sigma \to Q\) be an isothermic net and \(m \in \mathbb{R}^\times\) not equal to any \(m_{ij}\). Let \(\hat{s} : \Sigma \to Q\) be pointwise non-orthogonal to \(s\).

Then \(\hat{s}\) is a Darboux transform of \(s\) with parameter \(m\) if and only if \(\hat{s}\) is \(\Gamma^s(m)\)-parallel:
\[
\hat{s}_j = \Gamma^s(m)_{ji}\hat{s}_i,
\]
for all edges \(i, j\).

In particular, \(\hat{s}\) is uniquely determined by its value at a single point of \(\Sigma\).

Proof. Let \(\mu \in \Gamma s\) be the Moutard lift with \(m_{ij} = 1/(\mu_i, \mu_j)\) and let \(\hat{\mu} \in \Gamma \hat{s}\) be the unique section with \((\mu, \hat{\mu}) = 1/m\). Then, for any edge \(ij\), including any with \(m_{ij} = \infty\), we have
\[
\Gamma^s(m)_{ji}\hat{\mu}_i = \hat{\mu}_i - \mu_j + \frac{(\mu_j, \hat{\mu}_i)}{(\mu_i, \hat{\mu}_i - \mu_j)}\mu_i.
\]

If \(\hat{s}\) is a Darboux transform with parameter \(m\), then \(\hat{\mu}\) is the Moutard lift of \(\hat{s}\) and the Moutard equation (4.2) on the vertical quadrilateral tells us that (4.18) is a multiple of \(\hat{\mu}_j\) and so takes values in \(\hat{s}_j\). Thus \(\hat{s}\) is \(\Gamma^s(m)\)-parallel.

Conversely, if \(\hat{s}\) is parallel, we have
\[
\hat{\mu}_i - \mu_j + \frac{(\mu_j, \hat{\mu}_i)}{(\mu_i, \hat{\mu}_i - \mu_j)}\mu_i = c\hat{\mu}_j,
\]
for some $c \in \mathbb{R}$. Taking the inner product with $\mu_j$ rapidly yields

$$c = \frac{(\mu_j, \hat{\mu}_i)}{(\mu_i, \hat{\mu}_i - \mu_j)}$$

so that $\mu, \hat{\mu}$ solve the Moutard equation on vertical quadrilaterals. That $s$ is a Darboux transform now follows at once from the multidimensional consistency of the Moutard equation. □

Again, the flat connections are responsible for the Calapso transformation, and we have the following extension of [15, §2] (see also [34, §5.7.16]) to include the case where $m_{ij} = \infty$:

**Proposition 4.19.** Let $(s, \eta) : \Sigma \to Q$ be isothermic with Moutard lift $\mu$, edge-labeling $m_{ij}$ and flat connections $\Gamma^s(t)$. Let $T(t) : \Sigma \to O(p + 1, q + 1)$ trivialise $\Gamma^s(t)$: $\Gamma^s(t)_{ji} = T(t)^{-1}T(t)_i$ on each edge $ij$.

Define $s(t) := T(t)s : \Sigma \to Q$. Then $s(t)$ is isothermic with Moutard lift $T(t)\mu$, edge-labeling $m(t)_{ij} = m_{ij} - t$ (interpreted as $\infty$ if $m_{ij} = \infty$) and flat connections given by

$$\Gamma^{s(t)}(u) = T(t) \cdot \Gamma^{s}(t + u). \quad (4.19)$$

We call $s(t)$ a Calapso transform of $s$. It is defined up to a constant element $g(t) \in O(p + 1, q + 1)$.

**Proof.** We start with the flat connections $T(t) \cdot \Gamma^{s}(t + u)$. A computation using $T(t)^{-1}T(t)_j = \Gamma^s(t)_{ij}$ reveals that

$$(T(t) \cdot \Gamma^{s}(t + u))_{ji} = \begin{cases} T(t)_j \Gamma^s_{ji} (1 - u/(m_{ij} - t))T(t)^{-1} & \text{if } m_{ij} \neq \infty \\ T(t)_j \exp(u \eta_{ji})T(t)^{-1} & \text{if } m_{ij} = \infty. \end{cases} \quad (4.20)$$

Define 1-forms $\eta(t)$ by

$$\eta(t) := \partial/\partial u|_{u=0} T(t) \cdot \Gamma^s(t + u)$$

and use (4.20) to get, in all cases,

$$\eta(t)_{ji} = \frac{1}{1 - t/m_{ij}} \text{Ad}_{T(t)_j} \eta_{ji} = (T(t)\mu)_j \wedge (T(t)\mu)_i,$$

where we have used

$$T(t)^{-1}T(t)_j\mu_i = \frac{1}{1 - t/m_{ij}} \mu_i. \quad (4.21)$$

Now $\eta(t)$ is closed since $T(t) \cdot \Gamma^{s}(t + u)_{ji}$ is flat for all $u$, so that $s(t)$ is isothermic with Moutard lift $\mu(t) = T(t)\mu$. Moreover, (4.21) rapidly yields $(\mu(t)_j, \mu(t)_j) = 1/(m_{ij} - t)$ which, together with (4.20), gives (4.19). □
5 | APPLICABLE LEGENDRE MAPS

For $p, q \geq 1$, let $\mathcal{Z} = \mathcal{Z}^{p,q}$ be the space of projective lines in $Q^{p,q}$ or, equivalently, the Grassmannian of null 2-planes in $\mathbb{R}^{p+1,q+1}$. Then $\mathcal{Z}$ is a contact manifold of dimension $2(p + q) - 3$.

**Definition 5.1** (Legendre map). A Legendre map is a discrete line congruence $f : \Sigma \to \mathcal{Z}$.

We study applicable Legendre maps and their transformations. The key observation is that, thanks to Theorem 3.23 and the discussion in Section 4.2, $f : \Sigma \to \mathcal{Z}$ is an applicable Legendre map if and only if it is spanned by an isotropic Darboux pair of isothermic nets.

5.1 | Duality for applicable Legendre maps

With notation as in Section 4.3, write $\mathbb{R}^{p+1,q+1} = \mathbb{R}^{p,q} \oplus \langle \mathfrak{o}, \mathfrak{q} \rangle$ and let $\mathcal{Z}_q$ denote the set of affine null lines in $\mathbb{R}^{p,q}$. Inverse stereoprojection identifies $\mathcal{Z}_q$ with the open subset of $\mathcal{Z}$ consisting of lines that do not lie in the quadric at infinity $\mathbb{F}(L \cap \mathfrak{q}^\perp)$ (which only contains lines if $p, q \geq 2$).

If $L : \Sigma \to \mathcal{Z}_q$ is the stereoprojection of $f$ then $f$ is Legendre exactly when $L_i, L_j$ are affine coplanar for each edge $ij$. We say that $L$ is applicable if $f$ is.

So let $(f, [\eta])$ be an applicable Legendre map with stereoprojection $L$. For any $\eta \in [\eta]$, $\eta \mathfrak{q}$ is closed and so there is $\hat{x}^\eta : \Sigma \to \mathbb{R}^{p,q}$, unique up to translation, with

$$d\hat{x}^\eta = \pi \eta \mathfrak{q}. \quad (5.1)$$

Moreover, for $\tau \in \Gamma \wedge^2 f$, we may take

$$\hat{x}^\eta + \tau = \hat{x}^\eta + \pi \tau \mathfrak{q}. \quad (5.2)$$

Let $x_1, x_2$ span $L$ with Euclidean lifts $y_1, y_2$ so that any $\tau \in \Gamma \wedge^2 f$ is of the form $\lambda y_2 \wedge y_1$, for some $\lambda : \Sigma \to \mathbb{R}$. Then (5.2) reads

$$\hat{x}^\eta + \tau = \hat{x}^\eta + \lambda(x_2 - x_1). \quad (5.3)$$

Thus, all $\hat{x}^\eta + \tau$ lie on the family $\hat{L}$ of affine null lines through $\hat{x}$ that are pointwise parallel to $L$. In particular, $\hat{L}$ contains a Christoffel dual of any isothermic net in $L$. Further, any section of $\hat{L}$ is of the form $\hat{x}^\eta + \tau$, for some $\tau \in \Gamma \wedge^2 f$.

We have:

**Theorem 5.2.** Let $(f, \eta) : \Sigma \to \mathcal{Z}$ be an applicable Legendre map with stereoprojection $L$, let $\hat{x}^\eta : \Sigma \to \mathbb{R}^{p,q}$ solve (5.1) and let $\hat{L} : \Sigma \to \mathcal{Z}_q$ consist of the lines through $\hat{x}^\eta$ parallel to $L$.

Then $\hat{L}$ is also the stereoprojection of an applicable Legendre map.

We call $\hat{L}$ the dual Legendre map to $L$ with respect to $\mathfrak{q}$. It is determined up to translation.

**Proof.** $f : \Sigma \to \mathcal{Z}$ is spanned by an isotropic Darboux pair $(s^\pm, \eta^\pm)$ of isothermic nets with $\eta^\pm \in [\eta]$. Then $\hat{x}^+ := \hat{x}^{\eta^+}$ is a Christoffel dual of $x^+$ and with $\hat{x}^-$ the simultaneous Christoffel dual
of \( x^- \) and Darboux transform of \( \hat{x}^+ \) provided by Corollary 4.15, we have that \( \hat{x}^- \) lies on the line through \( \hat{x}^+ \) in the direction \( x^+ - x^- \) which is \( \hat{L} \). Thus, \( \hat{L} \) is spanned by an isotropic Darboux pair of isothermic nets and so is the stereoprojection of an applicable Legendre map as required. □

5.2 Transformations of applicable Legendre maps

Let \( f : \Sigma \to \mathcal{Z} \) be an applicable Legendre map spanned by an isotropic Darboux pair \( s^\pm \). The key to Theorem 5.2 was to consider the isothermic net \( s = s^+ \cup s^- : \{0, 1\} \times \Sigma \to Q \) and then take the Christoffel dual of \( s \).

The same idea gives us Darboux and Calapso transformations of applicable Legendre maps:

5.2.1 Darboux transformations

**Proposition 5.3.** Let \( f = s^+ \oplus s^- : \Sigma \to \mathcal{Z} \) be an applicable Legendre map spanned by an isotropic Darboux pair of isothermic nets and set \( s = s^+ \sqcup s^- : \{0, 1\} \times \Sigma \to Q \).

Let \( m \in \mathbb{R}^\times \) be distinct from any \( m_{ij} \) of \( s \) and let \( \hat{s} = \hat{s}^+ \sqcup \hat{s}^- \) be a Darboux transform of \( s \) with parameter \( m \).

Then \( \hat{f} := \hat{s}^+ \oplus \hat{s}^- \) is an applicable Legendre map which we call a Darboux transform of \( f \) with parameter \( m \).

**Proof.** \( s \) and \( \hat{s} \) have the same edge labeling so that, in particular, the label on the vertical edges of \( \hat{s} \) is \( \infty \). It follows at once that \( \hat{s}^\pm \) are an isotropic Darboux pair and so span an applicable Legendre map. □

We give another characterisation of the Darboux transform \( \hat{f} \) which replicates the formulation of [13, §2.4.2] in the smooth case. It also shows that \( \hat{f} \) is independent of the choice of isothermic sphere congruences (see Remark 3.24):

**Proposition 5.4.** Let \( f : \Sigma \to \mathcal{Z} \) be an applicable Legendre map and \( s^+ < f \) an isothermic net. Let \( \hat{s}^+ \) be a Darboux transform of \( s^+ \) with parameter \( m \in \mathbb{R}^\times \) and set \( f := \hat{s}^+ \oplus \hat{s}^- \).

Then \( \hat{f} : \Sigma \to \mathcal{Z} \) is a Darboux transform of \( f \) with parameter \( m \) and all such arise this way.

**Proof.** Choose a Darboux transform \( s^- \) of \( s^+ \) so that \( f = s^+ \oplus s^- \) and let \( s = s^+ \sqcup s^- \). Fix some \( i_0 \in \Sigma \) and let \( \hat{s} \) be the Darboux transform of \( s \) with parameter \( m \) that coincides with \( s^+ \) at \( (0, i_0) \).

Then the uniqueness assertion in Proposition 4.18 tells us that \( \hat{s}_{|\{0\} \times \Sigma} = \hat{s}^+ \). Set \( \hat{s}^- = \hat{s}_{|\{1\} \times \Sigma} \). The Moutarde equation relating \( s^\pm \) and \( \hat{s}^\pm \) tells us that \( f_0 \) and \( \hat{s}^\pm \) are coplanar and so intersect, necessarily at \( f \sqcap (\hat{s}^\pm)^\perp \). It follows at once that \( \hat{f} \) and the Darboux transform \( \hat{s}^+ \oplus \hat{s}^- \) coincide. □

5.2.2 Calapso transformations

**Proposition 5.5.** Let \( f = s^+ \oplus s^- : \Sigma \to \mathcal{Z} \) be an applicable Legendre map spanned by an isotropic Darboux pair of isothermic nets and set \( s = s^+ \sqcup s^- : \{0, 1\} \times \Sigma \to Q \).

For \( t \in \mathbb{R} \), let \( s(t) = s^+(t) \sqcup s^-(t) \) be the Calapso transform of \( s \) and set \( f(t) := s^+(t) \oplus s^-(t) \).
Then \( f(t) : \Sigma \to \mathcal{L} \) is an applicable Legendre map that we call the Calapso transform of \( f \).

**Proof.** According to Proposition 4.19, the edge label on vertical edges of \( s(t) \) is \( \infty \) so that \( s^\pm(t) \) are again an isotropic Darboux pair and so their span \( f(t) \) is an applicable Legendre map. \( \square \)

Again, the construction is independent of the choice of isothermic nets in \( Q \). To see this and to make contact with the discussion in [14, §3], we observe that Proposition 4.16, applied to the vertical quadrilaterals of \( s \) yields:

**Lemma 5.6** (c.f. [14, Corollary 3.8]). Let \((s^\pm, \eta^\pm)\) be an isotropic Darboux pair of isothermic nets. Let \( \Gamma^\pm(t) \) be the corresponding families of flat connections and \( \tau \in \Gamma^+ \wedge \Gamma^- \) such that \( \eta^- = \eta^+ + d\tau \). Then, for all \( t \in \mathbb{R} \),

\[
(\exp t\tau) \cdot \Gamma^+(t) = \Gamma^-(t).
\]

With this in hand, we let \( T(t) = T^+(t) \cup T^-(t) : \{0, 1\} \times \Sigma \to O(p + 1, q + 1) \) trivialise \( \Gamma(t) \) and note that, by Lemma 5.6, we may take \( T^-(t) = T^+(t) \exp(-t\tau) \). Thus

\[
f(t) = T^+(t)s^+ \oplus T^-(t)s^- = T^+(t)(s^+ \oplus \exp -t\tau s^-) = T^+(t)(s^+ \oplus s^-) = T^+(t)f,
\]

since \( \exp -t\tau s^- = s^- \). We conclude that our Calapso transforms coincide with those of [14, Theorem 3.9]:

**Proposition 5.7.** Let \( f : \Sigma \to \mathcal{L} \) be an applicable Legendre map and \( s^+ < f \) an isothermic net with flat connections \( \Gamma^+(t) \) trivialised by \( T^+(t) : \Sigma \to O(p + 1, q + 1) \).

Then, the Calapso transform \( f(t) \) of \( f \) is given by

\[
f(t) = T^+(t)f.
\]

### 5.3 Edge-labeling of an applicable Legendre map

The edge labeling of an isothermic sphere congruence \( s < f \) is, in fact, an invariant of the applicable Legendre map \( f \) and can be computed from any \( \eta \in \Lambda f \).

**Proposition 5.8.** Let \((f, \eta) : \Sigma \to \mathcal{L} = \mathbb{Z}^{p,q} \) be applicable and, for each edge \( ij \), choose \( \sigma_i \in f_i, \sigma_j \in f_j \) such that \( \eta_{ji} = \sigma_j \wedge \sigma_i \) and set \( m^\eta_{ij} = 1/(\sigma_i, \sigma_j) \in \mathbb{R}^\times \cup \{\infty\} \) to get a well-defined function \( m^\eta \) on edges.

Then for any \( \tau \in \Gamma \wedge^2 f \),

\[
m^\eta + d\tau = m^\eta.
\]

**Proof.** On an edge \( ij \), there are \( \lambda_i, \lambda_j \in \mathbb{R} \) and \( \sigma_{ij} \in s_{ij} \) such that

\[
\tau_i = \lambda_i \sigma_i \wedge \sigma_{ij}, \quad \tau_j = \lambda_j \sigma_j \wedge \sigma_{ij}.
\]
Then a simple calculation gives

$$\eta_{ji} + d\tau_{ji} = (\sigma_j + \lambda_i\sigma_{ij}) \wedge (\sigma_i + \lambda_j\sigma_{ij}).$$

Now $s_{ij}$ is orthogonal to $f_i + f_j$ so that

$$(\sigma_i + \lambda_j\sigma_{ij}, \sigma_j + \lambda_i\sigma_{ij}) = (\sigma_i, \sigma_j),$$

whence the result. □

In particular, all isothermic sphere congruences $s < f$ share the same edge-labeling.

## 6 | Ω-nets

### 6.1 | Lie sphere geometry

For the rest of the paper, we restrict attention to the case $(p, q) = (3, 1)$ which is the setting for Lie sphere geometry. Here the Lie quadric $Q = Q^{3,1}$ parametrises oriented 2-spheres in $S^3 = Q^{3,0}$ and two points of $Q$ are orthogonal exactly when the corresponding 2-spheres are in oriented contact. It follows at once that $Z = Z^{3,1}$ parametrises oriented contact elements in $S^3$. All this requires the choice of a point sphere complex, that is $p \in \mathbb{R}^{4,2}$ with $(p, p) = -1$. Then $S^3 = \mathbb{P}(L \cap p^\perp)$ and the 2-sphere corresponding to $q \in Q$ is $\mathbb{P}(L \cap p^\perp \cap q^\perp)$.

We get a more practical take on these matters via stereoprojection onto $\mathbb{R}^{3,1}$: thus choose $o, q \in L \cap p^\perp$ with $(o, q) = -1$ and, as usual, set $\mathbb{R}^{3,1} = \langle o, q \rangle^\perp$. Then $p \in \mathbb{R}^{3,1}$ and we have a further orthogonal decomposition:

$$\mathbb{R}^{3,1} = \mathbb{R}^3 \oplus \langle p \rangle.$$

Now a point $z = c + rp \in \mathbb{R}^{3,1}$, with $c \in \mathbb{R}^3$, parametrises the oriented 2-sphere in $\mathbb{R}^3$ with centre $c$ and (signed) radius $r$. This is the intersection of $\mathbb{R}^3$ with the affine light-cone $\{v \in \mathbb{R}^{3,1} | (v - z, v - z) = 0\}$ centred at $z$. Again, a contact element $(x, n) \in \mathbb{R}^3 \times S^2$ of $\mathbb{R}^3$ corresponds to the affine null line through $x$ in the direction $n + p$. This is the Laguerre picture of Lie sphere geometry: see [18] for a more detailed discussion.

Now contemplate a Legendre map $f : \Sigma \rightarrow Z$ and choose $i, j$ so that:

1. $f_i^\perp \cap \langle p, q \rangle = \{0\}$, for all $i \in \Sigma$;
2. The curvature sphere $s_{ij} = f_i \cap f_j$ lies in neither $p^\perp$ nor $q^\perp$, for all edges $ij$.

In view of condition 1, we have sections $y, t$ of $f$ with

$$(y, q) = -1 \quad (y, p) = 0 \quad (t, q) = 0 \quad (t, p) = -1.$$  

Now condition 2 tells us that $y_i, t_i$ do not lie in any $s_{ij}$ so that $y, t$ both span $Q$-nets which are regular by Lemma 3.21.

† This is a countable number of open conditions.
Stereoprojection onto $\mathbb{R}^{3,1}$ now yields $x = \pi y : \Sigma \to \mathbb{R}^3$ and $n + p = \pi t$ with $n : \Sigma \to S^2$. On each edge $ij$, we have a principal curvature $\kappa_{ij} \in \mathbb{R}^\times$ with

$$\kappa_{ij}x_i + (n_i + p) = \kappa_{ij}x_j + (n_j + p), \quad (6.1)$$

this common value being the stereoprojection of $s_{ij}$. Otherwise said, $(x, n)$ comprise a principal (contact element) net (c.f. [7, Definition 3.24]). In particular, $x$ and $n$ are edge-parallel.

Conversely, given a principal net $(x, n)$, we take $y$ to be the Euclidean lift of $x$, set $t = n + p + (x, n)q$ and $f = \langle y, t \rangle$ to recover a Legendre map $f : \Sigma \to \mathcal{Z}$. For further discussion, see [7, §3.5] or [14, §2].

### 6.2 $\Omega$-nets

With these preparations in hand, we recall from [14, Definition 3.1]:

**Definition 6.1** ($\Omega$-net). A Legendre map $f : \Sigma \to \mathcal{Z}$ is an $\Omega$-net if it is spanned by Koenigs dual sections.

In view of Theorem 3.27, we have a number of alternative characterisations:

**Theorem 6.2.** Let $f : \Sigma \to \mathcal{Z}$ be a regular Legendre map. Then the following are equivalent:

- $f$ is an $\Omega$-net;
- $f$ is applicable;
- $f$ is spanned by an isotropic Darboux pair of isothermic sphere congruences $s^{\pm} < f$.

**Remark 6.3.** The characterisation via isothermic sphere congruences is a direct discrete analogue of Demoulin’s original formulation of $\Omega$-surfaces [22].

Here, we add another characterisation to the list which is also due to Demoulin in the smooth case [24]:

#### 6.2.1 $\Omega$-nets in $\mathbb{R}^3$ via associate nets

Let $f : \Sigma \to \mathcal{Z}$ be an $\Omega$-net with stereoprojection $L : \Sigma \to \mathcal{Z}_q$ and corresponding principal net $(x, n)$ so that $L$ is the affine line congruence through $x$ pointing along $n + p$.

Theorem 5.2 provides a dual $\Omega$-net $\hat{L}$ which cuts $\mathbb{R}^3$ in a net $\hat{x} : \Sigma \to \mathbb{R}^3$ so that $(\hat{x}, n)$ is also principal. We call $\hat{x}$ an associate net of $x$ and seek to characterise it in purely Euclidean terms.

We begin with a lemma:

**Lemma 6.4.** Let $(f, [\eta])$ be applicable with $f = \langle y, t \rangle$. Suppose that $(\eta q, p) = 0$. Then

$$\eta = \eta q \wedge y + \eta p \wedge t \quad (6.2)$$

with both $\eta q$ and $\eta p$ edge-wise parallel to $dy$. 
Proof. We work on an edge \( ij \). It is easy to see that a basis for \( f_{ij} \) is given by \( y_{ij} := \frac{1}{2}(y_i + y_j) \), \( t_{ij} := \frac{1}{2}(t_i + t_j) \), and \( dy_{ij} \) so that

\[
\eta_{ij} = \alpha_{ij} dy_{ij} \land y_{ij} + \beta_{ij} dy_{ij} \land t_{ij} + \gamma_{ij} y_{ij} \land t_{ij}.
\]

Now \( \gamma_{ij} = (\gamma q, p) = 0 \) and then

\[
\eta_{ij} q = \alpha_{ij} dy_{ij}, \quad \eta_{ij} p = \beta_{ij} dy_{ij},
\]

whence the result. \(\square\)

Now we have \( dx = \pi q \), for some \( \eta \in [\eta] \), so that \( (\eta q, p) = (dx, p) = 0 \) and Lemma 6.4 applies. Since \( \eta p \) and so \( \pi \eta p \) is closed, we have \( \hat{n} : \Sigma \to \mathbb{R}^{3,1} \) with \( d\hat{n} = \pi \eta p \). Moreover, \( (\hat{n}, p) = (\eta p, p) = 0 \) so that we may adjust the constant of integration to ensure that \( \tilde{n} : \Sigma \to \mathbb{R}^3 \). In view of Lemma 6.4, both \( \tilde{x} \) and \( \tilde{n} \) are edge-parallel to \( x \). Finally, the exterior derivative of (6.2) yields

\[
0 = -d\eta = \eta q \land dy + \eta p \land dt,
\]

the \( \land^2 \mathbb{R}^3 \)-component of which reads

\[
d\tilde{x} \land dx + d\tilde{n} \land dn = 0. \tag{6.3}
\]

Conversely, let \( f : \Sigma \to \mathcal{Z} \) be Legendre with principal net \((x, n)\) and suppose there are edge-parallel \( \tilde{x}, \tilde{n} \) satisfying (6.3). We define 1-forms

\[
\alpha := d\tilde{x} + (x \land d\tilde{x})q, \quad \beta := d\tilde{n} + (x \land d\tilde{n})q,
\]

both of which are edge-wise parallel to \( dy \). Thus, on any face \( \ell kj \), \( \alpha, \beta, dy, dt \) all take values in \( W_{\ell kj} \). Thus, we apply Lemma 4.8, first to conclude that \( \alpha, \beta \) are closed since \( d\tilde{x}, d\tilde{n} \) are and then, from (6.3), that

\[
\alpha \land dy + \beta \land dt = 0.
\]

Thus, setting \( \eta = \alpha \land y + \beta \land t \) gives a closed 1-form with \( \eta_{ij} \in \land^2 f_{ij} \). Finally, \( s_{ij} \) is spanned by \( \kappa_{ij} y_{ij} + t_{ij} \) so that \( \eta_{ji} \land s_{ij} \neq 0 \) if and only if

\[
\alpha_{ji} \land y_{ij} \land t_{ij} + \kappa_{ij} \beta_{ji} \land t_{ij} \land y_{ij} \neq 0,
\]

or, equivalently,

\[
d\tilde{x}_{ji} \neq \kappa_{ij} d\tilde{n}_{ji}. \tag{6.4}
\]

In this case, \((f, \eta)\) is an \( \Omega \)-net.

To summarise, we have the discrete analogue of the discussion in [39, Theorem 5.1] (c.f. [13, §2.5]):
Theorem 6.5. A principal net \((x, n) : \Sigma \to \mathbb{R}^3 \times S^2\) lifts to an \(\Omega\)-net if and only if there exist edge-parallel nets \(\hat{x}, \hat{n} : \Sigma \to \mathbb{R}^3\) satisfying (6.3), or, equivalently,

\[ A(\hat{x}, x) + A(\hat{n}, n) = 0, \]

together with (6.4).

In this case, \((\hat{x}, n)\) is also a principal net whose Legendre lift is an \(\Omega\)-net when it is regular. We call \(\hat{n}\) an associate Gauss map of \(x\).

Remark 6.6. A parallel net \(\tilde{x} := \hat{x} + cn\), for \(c \in \mathbb{R}\) constant, to an associate net is also an associate net: with \(\tilde{n} = \hat{n} - cx\) we still have

\[ A(x, \tilde{x}) + A(\tilde{n}, n) = 0. \]

This amounts to replacing \(\tilde{L}\) with \(L - c\gamma\).

We therefore obtain a one-parameter family of associate nets with associate Gauss maps.

We conclude our discussion of duality for \(\Omega\)-nets by proving a generalisation of Proposition 4.13 and so Bianchi’s formula Corollary 4.15 to the present setting. As we shall see below in Corollary 7.3, this will also generalise a discrete version of a formula of Eisenhart for Guichard surfaces.

From Proposition 5.8, we know that the edge-labeling of an isothermic sphere congruence \(s < f\) is an invariant of the applicable Legendre map \(f\). The geometry of this edge-labeling is given by the following generalisation of (4.14):

Theorem 6.7. Let \((x, n)\) be a principal net lifting to an \(\Omega\)-net with edge-labeling \(m_{ij}\), associate net \(\hat{x}\) and associate Gauss map \(\hat{n}\). Then, for each edge \(ij\),

\[ (dx_{ij}, d\hat{x}_{ij}) + (dn_{ij}, d\hat{n}_{ij}) = -\frac{2}{m_{ij}}. \]  

(6.5)

Proof. The \(\Omega\)-net \(f = (y, t)\) has an \(\eta\) with \(\pi \eta q = d\hat{x}, \pi \eta p = d\hat{n}\) and \((\eta q, \eta p) = 0\). On an edge \(ij\), we write \(\eta_{ji} = \sigma_j \wedge \sigma_i\) where

\[ \sigma_i = a_i y_i + b_i t_i \quad \sigma_j = a_j y_j + b_j t_j, \]

for constants \(a_i, a_j, b_i, b_j\). Then \(1/m_{ij} = (\sigma_i, \sigma_j)\) by Proposition 5.8.

Now \((\eta q, \eta p) = 0\) yields

\[ a_i b_j = a_j b_i \]  

(6.6)

and then contracting \(\eta_{ji}\) with \(q, p\) and projecting tells us that

\[ d\hat{x}_{ji} = a_i a_j \, dx_{ji} + a_i b_j \, dn_{ji} \]  

(6.7a)

\[ d\hat{n}_{ji} = a_j b_i \, dx_{ji} + b_i b_j \, dn_{ji}. \]  

(6.7b)
On the other hand, in view of (6.6),

$$(\sigma_i, \sigma_j) = a_i a_j (y_i, y_j) + a_i b_j ((y_i, t_j) + (y_j, t_i)) + b_i b_j (t_i, t_j)$$

while

$$(y_i, y_j) = -\frac{1}{2}(d x_{ij}, d x_{ij}),$$

$$(y_i, t_j) + (y_j, t_i) = -(d x_{ij}, d n_{ij}),$$

$$(t_i, t_j) = -\frac{1}{2}(d n_{ij}, d n_{ij}),$$

where the first identity is (4.7), the second is proved similarly as is the last using $(n, n) \equiv 1$. Thus, using (6.7), we get

$$(\sigma_i, \sigma_j) = -\frac{1}{2} (a_i a_j (d x_{ij}, d x_{ij}) + 2 a_i b_j (d x_{ij}, d n_{ij}) + b_i b_j (d n_{ij}, d n_{ij}))$$

$$= -\frac{1}{2} ((d x_{ij}, d \tilde{x}_{ij}) + (d n_{ij}, d \tilde{n}_{ij})),$$

whence the result. □

7 | GUICHARD NETS

In this section, we define Guichard nets via a direct analogue of Guichard’s original formulation [33]. We shall see in section 8.2 that our definition is equivalent to that of Schief via discrete $O$-surfaces [40], and to a formulation through special $\Omega$-nets of type 1 that is the discrete version of the discussion in [13], see Section 7.2.

7.1 | Guichard nets via associate surfaces

**Definition 7.1** (Guichard net). A principal net $(x, n) : \Sigma \to \mathbb{R}^3 \times S^2$ is **Guichard** if and only if there exists an edge-parallel net $\tilde{x} : \Sigma \to \mathbb{R}^3$ such that

$$d \tilde{x} \wedge dx + d n \wedge dn = 0,$$

(7.1)

or, equivalently,

$$A(\tilde{x}, x) + A(n, n) = 0.$$

(7.2)

In this case, $(\tilde{x}, n)$ is also Guichard. We call $\tilde{x}$ an **associate net of** $x$.

We see at once that Guichard nets are $\Omega$ (a result of Demoulin [23] in the smooth case) and are characterised among $\Omega$-nets by the requirement that the associate Gauss map $\tilde{n}$ may be taken to be $n$. 

In particular, Theorem 6.7 immediately yields:

**Theorem 7.2.** Let \((x, n)\) be a Guichard net with edge-labeling \(m_{ij}\) and associate net \(\dot{x}\). Then, on each edge \(ij\),

\[
(dx_{ij}, d\dot{x}_{ij}) + (dn_{ij}, d\dot{n}_{ij}) = -\frac{2}{m_{ij}}.
\]

(7.3)

This has a pretty reformulation due to Eisenhart [29, p. 210] in the smooth case. For this, recall that \(r_{ij} := 1/x_{ij}\) is the radius of the 2-sphere in oriented contact with both \((x_i, n_i)\) and \((x_j, n_j)\) so that

\[
dx_{ji} + r_{ij} dn_{ji} = 0
\]

and, similarly,

\[
d\dot{x}_{ji} + \dot{r}_{ij} dn_{ji} = 0.
\]

This, along with (7.3), yields:

**Corollary 7.3** (Eisenhart’s formula). Let \((x, n)\) be Guichard with associate net \(\dot{x}\) and edge-labeling \(m_{ij}\). On each edge \(ij\), the line segments from \(x_i\) to \(x_j\) and from \(\dot{x}_i\) to \(\dot{x}_j\) are parallel with directed lengths \(d_{ij}, \dot{d}_{ij}\). Then

\[
d_{ij}\dot{d}_{ij}\left(1 + \frac{1}{r_{ij}\dot{r}_{ij}}\right) = -\frac{2}{m_{ij}}.
\]

(7.4)

**Remark 7.4.** Eisenhart’s original formulation reads:

\[
d_{ij}\dot{d}_{ij} + \frac{d^2_{ij}}{r^2_{ij}} = -\frac{2}{m_{ij}},
\]

which is equivalent to our more symmetric version thanks to the identity \(d_{ij}/r_{ij} = \dot{d}_{ij}/\dot{r}_{ij}\).

**Example 7.5.** According to Bobenko–Pottmann–Wallner [5], a principal net \((x, n)\) in \(\mathbb{R}^3\) has mean curvature \(H\) and Gauss curvature \(K\) given by

\[
H = -A(x, n)/A(x, x), \quad K = A(n, n)/A(x, x).
\]

Thus, a principal net is linear Weingarten if there are constants \(\alpha, \beta, \gamma\), not all zero, with

\[
\alpha K + 2\beta H + \gamma = 0,
\]

or, equivalently,

\[
\alpha A(n, n) - 2\beta A(x, n) + \gamma A(x, x) = 0.
\]

Linear Weingarten nets are \(\Omega\) [14, Theorem 2.8] but more is true. If \(\alpha \neq 0\), then \((x, n)\) is Guichard with associate net \((\gamma x - 2\beta n)/\alpha\). In particular, when \(\beta = 0\), that is, \(K\) is constant, \(x\) is, up to scale, self-associate just as in the smooth case [29, p. 229].
On the other hand, if $\alpha = 0$, that is, $H$ is constant, $x$ is isothermic with Christoffel dual $\gamma x - 2\beta n$ which is $n + Hx$ up to scale. In particular, $(x, n)$ is minimal, $H = 0$, if and only if
\[ dx \wedge dn = 0, \]
(c.f. [4]).

Let $f = \langle y, t \rangle : \Sigma \to \mathcal{Z}$ be the Legendre lift of a Guichard net $(x, n)$. Thus $f$ is applicable with 1-form $\eta$ satisfying
\begin{align*}
\pi \eta q &= dx \\
\eta p &= dt. \tag{7.5a}
\end{align*}

Now let $s^+ < f$ with $(s^+, \eta^+)$ isothermic and $\eta^+ = \eta + d\tau^+$, for some $\tau^+ \in \Gamma \land^2 f$. Set $\sigma^- := t + \tau^+ p$. We have:
\[ d\sigma^- = \eta^+ p \]
\[ (\sigma^-, p) = -1. \]

Further let $\sigma^+ \in \Gamma s^+$ be the affine lift\(^1\) with $(\sigma^+, p) = -1$. The argument of Proposition 3.4 tells us that $\sigma^-$ is Koenigs dual to $\sigma^+$ so that $\eta^+ = d\sigma^- \land \sigma^+$. This proves one half of:

**Theorem 7.6.** An applicable Legendre map $f : \Sigma \to \mathcal{Z}$ lifts a Guichard net if and only if it contains\(^2\) Koenigs dual sections $\sigma^\pm$ with $(\sigma^\pm, p) \equiv -1$.

**Proof.** Let $f$ lift $(x, n)$. We need only prove the reverse implication. For this, write $\sigma^- = t + \tau^+ p$, for some $\tau^+ \in \Gamma \land^2 f$, and consider $\eta := \eta^- + d\tau^+$. Then
\[ \eta p = (d\sigma^- \land \sigma^+) p - d\tau^+ p = d\sigma^- - d\tau^+ p = dt. \]

In particular, $(\eta p, q) = 0$ and so we have $\tilde{x} : \Sigma \to \mathbb{R}^3$, edge-parallel to $x$, with $d\tilde{x} = \pi \eta q$. Thus, $\tilde{x}$ is associate to $x$ with associate Gauss map $n$. Otherwise said, $(x, n)$ is Guichard. \hfill \Box

This analysis yields more. Let $(x, n)$ be Guichard with Legendre lift $(f, \eta)$ satisfying (7.5). We have seen that $f$ contains isothermic sphere congruences $s^\pm$ with Koenigs dual lifts $\sigma^\pm$ and that there is $\tau^+ \in \Gamma \land^2 f$ with $\eta^+ = \eta + d\tau^+$ and $\sigma^- = t + \tau^+ p$. Now set $\tau^- := \tau^+ + \sigma^+ \land \sigma^-$. An easy computation shows that the situation is completely symmetric:
\begin{align*}
\eta^\pm &= \eta + d\tau^\pm \\
\sigma^\pm &= t + \tau^\pm p,
\end{align*}
where we have used (3.12) for the first identity. In particular, by (5.2), $x^\pm$ have Christoffel duals $\tilde{x}^\pm = \tilde{x} + \pi \tau^\pm q$. The corresponding 2-spheres in $\mathbb{R}^3$ have signed radii $r^\pm = -1/(\sigma^\pm, q)$ and

\(^1\) We cheerfully assume that $s^+$ is never orthogonal to $p$.

\(^2\) In general, $s^\pm$ will span $f$ but it is possible for $s^\pm$ to coincide: see [15, Lemma 3.11].
\( \hat{\tau}^\pm = - (\hat{x}^\pm, p) \). However,

\[
(\sigma^\pm, q) = (t + \tau^\pm p, q) = -(p, \tau^\pm q) = -(p, \hat{x} + \tau^\pm q) = -(p, \hat{x}^\pm).
\]

We therefore conclude, as does Demoulin [24] in the smooth case:

**Theorem 7.7.** Let \((x, n)\) be Guichard with associate net \(\hat{x}\). Then \((x, n)\) and \((\hat{x}, n)\) are enveloped by Christoffel dual isothermic sphere congruences \(x^\pm, \hat{x}^\pm\) with radii satisfying

\[
r^\pm \hat{r}^\mp = -1.
\]

### 7.2 Special \(\Omega\)-nets

There is another approach to Guichard nets and other reductions of \(\Omega\)-nets which arises by applying the theory of special isothermic nets, due to Bianchi [1] in the classical smooth case and developed in [15, 16]. The virtue of this viewpoint is that it plays well with the transformation theory.

Recall:

**Definition 7.8** (Special isothermic net, special \(\Omega\)-net). Let \(s : \Sigma \to \mathbb{Z}^{p+q}\) be isothermic with family of flat connections \(\Gamma^s(t)\). A polynomial conserved quantity of degree \(d \in \mathbb{N}\) is a family \(p(t) = \sum_{n=0}^d p(n) t^n\) of sections of the trivial bundle \(\Sigma \times \mathbb{R}^{p+1,q+1}\), with \(p^{(d)}\) not identically zero, such that \(p(t)\) is \(\Gamma^s(t)\)-parallel:

\[
\Gamma^s(t)_{ji}p_i(t) = p_j(t). \quad (7.6)
\]

If \(s\) admits such a polynomial conserved quantity, we say that \(s\) is a special isothermic net of type \(d\).

If \(f\) is an \(\Omega\)-net, we say that \(f\) is a special \(\Omega\)-net of type \(d\) if it is spanned by isothermic sphere congruences \(s^\pm\) for which \(s = s^+ \cup s^-\) is special isothermic of type \(d\).

The condition that \(f\) be special \(\Omega\) is independent of the choice of isothermic sphere congruence:

**Lemma 7.9.** \(f : \Sigma \to \mathbb{Z}\) is special \(\Omega\) of type \(d\) if and only if there is \(s^+ < f\) which has a degree \(d\) polynomial conserved quantity \(p^+(t)\) with \((p^+)^{(d)} \perp f\).

**Proof.** Let \(f = s^+ \oplus s^-\) be special \(\Omega\) so that \(s = s^+ \cup s^-\) has a degree \(d\) conserved quantity \(p(t) = p^+(t) \cup p^-(t)\). If \(\eta^- = \eta^+ + d\tau\) then \(p(t)\) being \(\Gamma^s(t)\)-parallel on vertical edges amounts to:

\[
\exp t\tau p^+(t) = p^-(t). \quad (7.7)
\]

Certainly, \(p^+(t)\) is a polynomial conserved quantity for \(s^+\). Moreover, the left side of (7.7) has a term of degree \(d + 1\) with coefficient \(\tau (p^+)^{(d)}\) which must vanish since \(\deg p^-(t) = d\). This is equivalent to the demand that \((p^+)^{(d)} \perp f\).

For the converse, given \(s^+, p^+(t)\), choose an \(m = \infty\) Darboux transform \(s^-\) so that \(f = s^+ \oplus s^-\). With \(\tau \in \Gamma s^+ \wedge s^-\) such that \(\eta^- = \eta^+ + d\tau\), define \(p^-(t)\) by (7.7) which is of degree \(d\) since \((p^+)^{(d)} \perp f\). Now set \(p(t) = p^+(t) \cup p^-(t)\) to get a degree \(d\) conserved quantity of \(s^+ \cup s^-\). \(\square\)

In this context, there is a useful reformulation when \(d = 1\):
Lemma 7.10. Let \((s^+, \eta^+): \Sigma \to Q\) be isothermic with flat connections \(\Gamma^+(t)\) and edge-labeling \(m_{ij} \neq \infty\). Then \(p^+(t) = c + t\xi\) is a polynomial conserved quantity if and only if
\[
\begin{align*}
\frac{dc}{dt} &= 0 \quad (7.8a) \\
\frac{d\xi}{dt} &= \eta^+ c \\
\xi &\perp s^+.
\end{align*}
\]

Proof. If \(p^+(t)\) is \(\Gamma^+(t)\)-parallel then evaluating (7.6) at \(t = 0\) yields (7.8a) while differentiating it at \(t = 0\) gives (7.8b). Finally, \(\Gamma^+(t)_{ji} p^+(t)_i\) has a quadratic term along the \(s^+_j\)-component of \(\xi_j\) and a simple pole along the \(s^+_i\)-component of \(p^+(m_{ij})_i\). These terms must both vanish, or, equivalently,
\[
\xi_j \perp s^+_j \quad p^+(m_{ij})_i \perp s^+_j. \tag{7.9}
\]

However, with \(\mu \in \Gamma s^+\) the Moutard lift, we use \(\eta^+_{ji} = \mu_j \wedge \mu_i\) and \(m_{ij} = 1/(\mu_i, \mu_j)\) along with (7.8b) to get
\[
(\xi_j, \mu_j) = \frac{1}{m_{ij}} (p^+(m_{ij})_i, \mu_j).
\]

Thus, given (7.8b), (7.8c) is equivalent to (7.9).

For the converse, if (7.8) holds then \(\Gamma^+(t)_{ji} p^+(t)_i - p^+(t)_j\) is a degree 1 polynomial in \(t\) with vanishing 1-jet at \(t = 0\) and so vanishes identically. □

Since the \(\Gamma(t)\) are metric connections, when \(p(t)\) is a degree \(d\) conserved quantity, \((p(t), p(t))\) is a constant coefficient polynomial in \(t\) of degree no more than \(2d\) which encodes much of the geometry of the situation. For example, let us take \(d = 1\) and suppose that \((p(t), p(t))\) is affine linear. We prove:

Theorem 7.11. Let \(f: \Sigma \to \mathcal{Z}\) be an \(\Omega\)-net. Then \(f\) lifts a Guichard net if and only if it is special \(\Omega\) with linear conserved quantity \(p(t)\) satisfying \((p(t), p(t)) = -1 - 2ct\), for \(c\) a non-zero constant.

Proof. If \(f\) lifts a Guichard net, let \(\sigma^\pm\) be the Koenigs dual sections provided by Theorem 7.6 and set \(s^+ = \langle \sigma^+ \rangle\). Then \(\eta^+ = d\sigma^- \wedge \sigma^+\) so that \(d\sigma^- = \eta^+ \mathfrak{p}\). Now \(p^+(t) := \mathfrak{p} + t\sigma^-\) is a polynomial conserved quantity for \((s^+, \eta^+)\) by Lemma 7.10 and \((p^+(t), p^+(t)) = -1 - 2t\). Since \(\sigma^- \perp f\), Lemma 7.9 tells us that \(f\) is special \(\Omega\) with a conserved quantity of the desired kind.

For the converse, let \(s = s^+ \sqcup s^-\) have linear conserved quantity \(p(t) = p^+(t) \sqcup p^-(t)\) with \((p(t), p(t)) = -1 - 2ct\). By scaling \(t\) (and so \(\eta, m_{ij}\), see Remark 4.17), if necessary, we may take \(c = 1\) and, without loss of generality, set the constant \(p^{(0)} = \mathfrak{p}\). Write \(p^+(t) = \mathfrak{p} + t\xi\) and note that \(\xi \in \Gamma f\) being a null section of \(f^\perp\). Moreover, let \(\sigma^+ \in \Gamma s^+\) with \((\sigma^+, \mathfrak{p}) = -1\). Since \((\xi, \mathfrak{p}) = -1\) also, (7.8b) tells us that \(\xi\) is Koenigs dual to \(\sigma^+.\) Thus \(f\) lifts a Guichard net by Theorem 7.6. □

7.2.1 Variations on the theme

We have just seen that Guichard surfaces are special \(\Omega\) nets of type 1 with affine linear \((p(t), p(t))\). Other specialisations of \(\Omega\)-nets are obtained by varying the possibilities for the linear function
\((p(t), p(t)) = a + bt\). We may argue as in Theorem 7.11 to conclude that such \(\Omega\)-nets contain Koenigs dual lifts \(\sigma^\pm\) with

\[(\sigma^+, p^{(0)}) = -1, \quad (\sigma^-, p^{(0)}) = b/2.\]

Moreover, we can scale \(p(t)\) by a constant to ensure that \(a^2 = 1\) or \(0\) and then a constant rescaling of \(t\) allows us to assume that \(b = -2\) or \(0\). This gives four possibilities:

1. \(a^2 = 1\) and \(b = -2\). For \(a = -1\), this is the Guichard case we have already studied while \(a = 1\) gives the entirely analogous theory of Guichard nets in \(\mathbb{R}^{2,1}\).
2. \(a^2 = 1\) and \(b = 0\). Here, we have

\[(\sigma^+, p) = -1, \quad (\sigma^-, p) = 0\]

where \(p = p^{(0)}\) with \((p, p) = a\).

Thus, \(\sigma^-\) spans an isothermic net with stereoprojection \(x\) in either \(\mathbb{R}^3\) or \(\mathbb{R}^{2,1}\). We therefore have a principal net \((x, n)\) with \(x\) isothermic or equivalently, \(\vec{n}\) is constant. Conversely, any such principal net arises this way.
3. \(a^2 = 0\) and \(b = -2\). Here, we may take \(p^{(0)} = q\) and then

\[(\sigma^\pm, q) = -1.\]

Equivalently, the corresponding principal net \((x, n)\) has \(\vec{x} = x\) so that

\[dx \wedge dx + d\vec{n} \wedge dn = 0.\]

We call such nets \(L\)-Guichard and remark that these nets are additionally characterised by having self-dual Legendre maps: \(L = \vec{L}\).
4. \(a^2 = b = 0\). Here, we have

\[(\sigma^+, q) = -1, \quad (\sigma^-, q) = 0\]

so that the tangent sphere congruence \(\langle t \rangle\) is isothermic. Equivalently, \(\vec{x}\) is constant so that

\[dn \wedge d\vec{n} = 0.\]

With Bobenko–Suris [6, §5], we call such nets \(L\)-isothermic \(^{†}\) and remark that they are characterised by the requirement that \((\vec{n}, n)\) is a minimal principal net, c.f. Example 7.5, or equivalently, that \(n\) is isothermic in \(S^2\) with Christoffel dual \(\vec{n}\), c.f. [6, Theorem 5.3]. See [28, 37] for the smooth theory of \(L\)-isothermic surfaces.

### 7.3 | Transformations

The transformation theory of special \(\Omega\)-nets \(f = s^+ \oplus s^-\) now proceeds by applying the results of [13, 15, 16] to the special isothermic net \(s = s^+ \cup s^-\).

\(^{†}\) For Bobenko–Suris, the \(L\)-isothermic net is the conical net in the space of affine 2-planes in \(\mathbb{R}^3\) given by the planes through \(x\) normal to \(n\).
7.3.1 Darboux transformations

According to Proposition 5.3, a Darboux transformation $\hat{f}$ of $f$ amounts to a Darboux transformation $\hat{s}$ of $s$ with parameter $m$. When $s$ has a polynomial quantity $p(t)$ of degree $d$ then $\hat{p}(t)$ given by

$$\hat{p}(t) := \Gamma s(1 - t/m)p(t)$$

is also a polynomial conserved quantity of degree $d$ so long as $\hat{s} \perp p(m)$ [16, Lemma 4]. Since $\hat{s}$ and $p(m)$ are both $\Gamma s(m)$-parallel, this last holds as soon as it holds at a single point. In this case, $p(t)$ and $\hat{p}(t)$ have the same constant term and the constant polynomials $(p(t), p(t))$ and $(\hat{p}(t), \hat{p}(t))$ coincide. Thus, so long as $p(m) \perp \hat{s}$, $\hat{f}$ lifts a Guichard, $L$-Guichard, isothermic or $L$-isothermic net if $f$ does. In this way, we obtain discrete versions of the Eisenhart transformations of Guichard surfaces [29] and the Bianchi–Darboux transformations of $L$-isothermic surfaces [3, 28, 37].

7.3.2 Calapso transformations

Again, Calapso transforms $f(t)$ of $f$ amount to Calapso transforms $s(t)$ of $s$ thanks to Proposition 5.5. If $s$ has polynomial conserved quantity $p(u)$ then it follows from Proposition 4.19 that $p(t)(u) := T(t)p(u + t)$ is a polynomial conserved quantity of the same degree [16, Theorem 3]. Here, however, the constant coefficient polynomials differ by a translation: $(p(t)(u), p(t)(u)) = (p(u + t), p(u + t))$. Thus, for example, we have:

**Proposition 7.12.** Let, $f$ lift a Guichard net in $\mathbb{R}^3$ with linear conserved quantity $p(t)$. Then $f(t)$ lifts a Guichard net in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$ or an $L$-Guichard net according to whether $(p(t), p(t))$ is negative, positive or zero, respectively.

8 O-SYSTEMS AND $\Omega$-NETS

The theory of $O$-surfaces developed by Konopelchenko–Schief [41] and discretised by Schief [49] offers a uniform approach to integrable surface geometry in $\mathbb{R}^3$ that fits well with our approach. We give a brief account of this theory in our framework and prove that $\Omega$-surfaces are indeed $O$-surfaces.

8.1 Combescure transformations

**Definition 8.1** (Combescure transformation). Let $\chi : \Sigma \to \mathbb{R}^{p,q}$. A Combescure transformation of $\chi$ is a map $\chi^* : \Sigma \to \mathbb{R}^{p,q}$ such that

1. $\chi^*$ is edge-parallel to $\chi$.
2. $(dx \wedge dx^*) = 0$, where $(,)$ is the inner product on $\mathbb{R}^{p,q}$.

In this case, we say that $\chi, \chi^*$ are a Combescure pair.
Generically, $x, x^*$ are a Combescure pair exactly when they are edge-parallel circular nets:

**Lemma 8.2.** Let $x, x^* : \Sigma \to \mathbb{R}^{p,q}$ be edge-parallel nets with non-collinear quadrilaterals. Define $\lambda_{ij} \in \mathbb{R}^x$ by $dx^*_{ij} = \lambda_{ij} dx_{ij}$ on each edge and let $\ell kji$ be a quadrilateral on which $\lambda_{ij}, \lambda_{jk}, \lambda_{k\ell}, \lambda_{\ell i}$ do not all coincide.

Then the following are equivalent:

1. $x$ is circular on $\ell kji$.
2. $x^*$ is circular on $\ell kji$.
3. $(dx \wedge dx^*)_{\ell kji} = 0$.

**Proof.** Begin by observing that $d^2x^*_{\ell kji} = 0$ reads

$$0 = \lambda_{ij} dx_{ij} + \lambda_{jk} dx_{jk} + \lambda_{k\ell} dx_{k\ell} + \lambda_{\ell i} dx_{\ell i} = \sum_i a_i x_i,$$

for $a_i = \lambda_{ij} - \lambda_{\ell i}$ and so on. Since $\sum_i a_i = 0$, we deduce that

$$\sum_i a_i(x_i + o) = 0 \quad (8.1)$$

with not all $a_i$ zero. Moreover, since our quadrilateral is non-collinear, the $a_i$ are uniquely determined up to scale by (8.1).

Now $x$ is circular if and only if the Euclidean lifts also satisfy $\sum_i a_i y_i = 0$ or, equivalently, we have $\sum_i a_i(x_i, x_i) = 0$ in addition.

However,

$$\sum_i a_i(x_i, x_i) = \lambda_{ij} d(x, x)_{ij} + \lambda_{jk} d(x, x)_{jk} + \lambda_{k\ell} d(x, x)_{k\ell} + \lambda_{\ell i} d(x, x)_{\ell i}$$

$$= 2(\lambda_{ij}(x \wedge dx)_{ij} + \lambda_{jk}(x \wedge dx)_{jk} + \lambda_{k\ell}(x \wedge dx)_{k\ell} + \lambda_{\ell i}(x \wedge dx)_{\ell i})$$

$$= 2 d(x \wedge dx^*)_{ijk\ell} = 2(dx \wedge dx^*)_{ijk\ell}.$$

This settles the equivalence of items 1 and 3 and the proposition follows by symmetry since $(dx^* \wedge dx) = -(dx \wedge dx^*)$. \qed

On the other hand, if $\lambda_{ij} = \lambda_{jk} = \lambda_{k\ell} = \lambda_{\ell i} = \lambda$, then

$$(dx \wedge dx^*)_{\ell kji} = \lambda(dx \wedge dx)_{\ell kji} = 0$$

automatically so that we have:

**Proposition 8.3.** Let $x, x^* : \Sigma \to \mathbb{R}^{p,q}$ be edge-parallel nets with non-collinear quadrilaterals. If $x$ is circular then $x, x^*$ are a Combescure pair.

Conversely, if $x, x^*$ are a Combescure pair and there is no quadrilateral on which $\lambda_{ij} = \lambda_{jk} = \lambda_{k\ell} = \lambda_{\ell i}$, then $x, x^*$ are both circular.
8.2 O-systems

Let $x^\alpha : \Sigma \to \mathbb{R}^{p,q}$, $\alpha = 1, \ldots, N$ be a family of mutually edge-parallel nets. Being edge-parallel is a linear condition so we may view the $x^\alpha$ as a single linear map $\Phi = \sum_{\alpha} x^\alpha \otimes w^\alpha : \Sigma \to \mathbb{R}^{p,q} \otimes W$ where $W$ is some $N$-dimensional vector space with basis $w_1, \ldots, w_N$. We recover the $x^\alpha$ from $\Phi$ by contracting $\Phi$ against the dual basis $w^1, \ldots, w^N$. That the $x^\alpha$ are edge-parallel is equivalent to the demand that each $d\Phi_{ji}$ be decomposable:

$$d\Phi_{ji} = X_{ji} \otimes Y_{ji},$$

for $X_{ji} \in \mathbb{R}^{p,q}$ and $Y_{ji} \in W$.

Now choose a basis $e_1, \ldots, e_{p+q}$ of $\mathbb{R}^{p,q}$ and write $\Phi = \sum_m e_m \otimes y^m$ to define $y^m : \Sigma \to W$ and observe that the $y^m$ are also edge-parallel: each $dy^m_{ji} \parallel Y_{ji}$. We call the $y^m$ the dual edge-parallel family.

With this in hand, we make the following:

**Definition 8.4 (O-system).** A family $x^\alpha : \Sigma \to \mathbb{R}^{p,q}$ of mutually Combescure nets is an **O-system** if the dual family $y^m$ are also mutually Combescure with respect to a non-degenerate inner product on $W$.

**Remark 8.5.** In view of Proposition 8.3, when $\Sigma = \mathbb{Z}^2$, an O-system generically comprises a family of O-surfaces in the sense of Schief [40, Definition 5.1].

We now give two characterisations of O-systems:

**Theorem 8.6.** Let $x^\alpha : \Sigma \to \mathbb{R}^{p,q}$ be a family of mutually Combescure nets with $\Phi = \sum_{\alpha} x^\alpha \otimes w^\alpha : \Sigma \to \mathbb{R}^{p,q} \otimes W$ as above.

Equip $W$ with a non-degenerate inner product $g$ and set $g_{\alpha \beta} := g(w^\alpha, w^\beta)$.

Then the following are equivalent:

1. $x^\alpha$ is an O-system with respect to $g$.
2. $\sum_{\alpha,\beta} g_{\alpha \beta} \; dx^\alpha \wedge dx^\beta = 0$.
3. $[d\Phi \wedge d\Phi] = 0$, where we identify $\mathbb{R}^{p,q} \otimes W$ with $\mathbb{R}^{p,q} \wedge W \leq \wedge^2(\mathbb{R}^{p,q} \oplus W) \cong \mathfrak{o}(\mathbb{R}^{p,q} \oplus W)$ via (4.1).

**Proof.** For $a \otimes v, b \otimes w \in \mathbb{R}^{p,q} \otimes W$ we have

$$[a \otimes v, b \otimes w] = (a, b)v \wedge g(w, w) + g(v, w)a \wedge b \in \wedge^2W \oplus \wedge^2\mathbb{R}^{p,q}.$$

Thus, since $\Phi = \sum_{\alpha} x^\alpha \otimes w^\alpha$,

$$[d\Phi \wedge d\Phi] = \sum_{\alpha,\beta} (dx^\alpha \wedge dx^\beta)w^\alpha \wedge w^\beta + g_{\alpha \beta} \; dx^\alpha \wedge dx^\beta.$$

\(^*\) Intrinsically, $W$ is the dual of the finite-dimensional subspace of $\text{Map}(\Sigma, \mathbb{R}^{p,q})$ spanned by the $x^\alpha$ and $\Phi$ is evaluation.
Since the $x^\alpha$ are mutually Combescure, the $\wedge^2 W$-component vanishes and we have established the equivalence of items 2 and 3. On the other hand, writing $\Phi = \sum_m e_m \otimes y^m$, the same argument tells us that the $\wedge^2 \mathbb{R}^{p,q}$-component of $[d\Phi \wedge d\Phi]$ can also be written as

$$
\sum_{m,n} g(dy^m \wedge dy^n)e_m \wedge e_n,
$$

which vanishes exactly when the $y^m$ are mutually Combescure. Thus items 1 and 3 are equivalent.

□

As a consequence, all the nets we have considered in this paper come from $O$-systems:

Examples 8.7.

1. Recall from Example 7.5 that a principal net $(x, n)$ is linear Weingarten if there are constants $\alpha, \beta, \gamma$, not all zero, such that

$$
\alpha dn \wedge dn - 2\beta dn \wedge dx + \gamma dx \wedge dx = 0.
$$

Theorem 8.6 now tells us that this happens exactly when $x, n$ comprise an $O$-system with

$$
(g_{\alpha\beta}) = \begin{pmatrix}
\gamma & -\beta \\
-\beta & \alpha
\end{pmatrix}.
$$

In particular, our notion of linear Weingarten net coincides with that of Schief [40, §5(b)(iv)].

2. (c.f. [40, §5(b)(ii)]) An isothermic net $x$ with Christoffel dual $\hat{x}$ comprise an $O$-system with $W = \mathbb{R}^{1,1}$ and

$$
(g_{\alpha\beta}) = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
$$

thanks to Theorem 4.11.

The same is true for $n, \hat{n}$ when $x$ is $L$-isothermic as in Section 7.2.1. We conclude that $x$ is $L$-isothermic if and only if it is a Combescure transform of a minimal net $(\hat{n}, n)$ as was observed in the smooth case by Schief–Szereszewski–Rogers [42, §6].

3. (c.f. [40, §5(b)(v)]) Let $(x, n)$ be a Guichard net with associate net $\hat{x}$. Then $x, \hat{x}, n$ comprise an $O$-system with $W = \mathbb{R}^{2,1}$ and

$$
(g_{\alpha\beta}) = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

thanks to Definition 7.1.

The same is true for $n, \hat{n}, x$ when $(x, n)$ is $L$-Guichard with associate Gauss map $\hat{n}$. 
4. Finally, let \((x, n)\) be the stereoprojection of an \(\Omega\)-net with associate net \(\hat{x}\) and associate Gauss map \(\hat{n}\). Then \(x, \hat{x}, n, \hat{n}\) comprise an \(O\)-system with \(W = \mathbb{R}^{2,2}\) and

\[
(g_{\alpha\beta}) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

by Theorem 6.5.

**Remark 8.8.** The equation \([d\Phi \land d\Phi] = 0\) is a straightforward discretisation of the curved flat system [32] in a version studied by Burstall [11, §3.1] (as \(\mathfrak{p}\)-flat maps) and Brück–Du–Park–Terng [10, Definition 6.6] (as \(n\)-tuples in \(\mathbb{R}^m\) of type \(O(n)\)). We may return to discrete curved flats elsewhere.

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**REFERENCES**

1. L. Bianchi, *Ricerche sulle superficie isoterme e sulla deformazione delle quadriche*, Ann. Mat. Pura Appl. 11 (1905), no. 1, 93–157.
2. L. Bianchi, *Complementi alle ricerche sulle superficie isoterme*, Ann. Mat. Pura Appl. 12 (1905), 19–54.
3. L. Bianchi, *Sulle superficie a rappresentazione isoterma delle linee di curvatura come inviluppi di rotolamento*, Rend. Acc. Naz. Lincei 24 (1915), 367–377.
4. A. I. Bobenko, U. Hertrich-Jeromin, and I. Lukyanenko, *Discrete constant mean curvature nets in space forms: Steiner’s formula and Christoffel duality*, Discrete Comput. Geom. 52 (2014), no. 4, 612–629. MR3279541
5. A. I. Bobenko, H. Pottmann, and J. Wallner, *A curvature theory for discrete surfaces based on mesh parallelity*, Math. Ann. 348 (2010), no. 1, 1–24. MR2657431
6. A. I. Bobenko and Y. B. Suris, *Isothermic surfaces in sphere geometries as Moutard nets*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 463 (2007), no. 2088, 3171–3193. MR2386657
7. A. I. Bobenko and Y. B. Suris, *Discrete differential geometry*, Graduate Studies in Mathematics, vol. 98, American Mathematical Society, Providence, RI, 2008. Integrable structure. MR2467378
8. A. I. Bobenko and Y. B. Suris, *Discrete Koenig nets and discrete isothermic surfaces*, Int. Math. Res. Not. IMRN 11 (2009), 1976–2012. MR2507107
9. E. Bour, *Théorie de la déformation des surfaces*, J. Éc. Impériale Polytech. 39 (1862), 1–148.
10. M. Brück, X. Du, J. Park, and C.-L. Terng, *The submanifold geometries associated to Grassmannian systems*, Mem. Amer. Math. Soc. 155 (2002), no. 735, viii+95. MR1875645
11. F. E. Burstall, *Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems, geometry, and topology (C.-L. Terng, ed.),* AMS/IP Stud. Adv. Math., vol. 36, Amer. Math. Soc., Providence, RI, 2006, pp. 1–82. MR2222512
12. F. E. Burstall, N. M. Donaldson, F. Pedit, and U. Pinkall, *Isothermic submanifolds of symmetric R-spaces*, J. Reine Angew. Math. 660 (2011), 191–243. MR2855825
13. F. E. Burstall, U. Hertrich-Jeromin, M. Pember, and W. Rossman, *Polynomial conserved quantities of Lie applicable surfaces*, Manuscripta Math. 158 (2019), no. 3-4, 505–546. MR3914961
14. F. E. Burstall, U. Hertrich-Jeromin, and W. Rossman, *Discrete linear Weingarten surfaces*, Nagoya Math. J. 231 (2018), 55–88. MR3845588
15. F. E. Burstall, U. Hertrich-Jeromin, W. Rossman, and S. Santos, *Discrete surfaces of constant mean curvature*, Developments in differential geometry of submanifolds (S.-P. Kobayashi, ed.), RIMS Kôkyûroku, vol. 1880, Res. Inst. Math. Sci. (RIMS), Kyoto, 2014, pp. 133–179.
16. F. E. Burstall, U. Hertrich-Jeromin, W. Rossman, and S. Santos, *Discrete special isothermic surfaces*, Geom. Dedicata 174 (2015), 1–11. MR3303037
17. P. Calapso, *Sulle superficie a linee di curvatura isoterme*, Rendiconti Circolo Matematico di Palermo 17 (1903), 275–286.
18. T. E. Cecil, *Lie sphere geometry: with applications to submanifolds*, Universitext, Springer, NY, p. xii+207, 2008. MR2361414
19. E. B. Christoffel, *Ueber einige allgemeine Eigenschaften der Minimumsflächen*, J. Reine Angew. Math. 67 (1867), 218–228. MR1501013
20. D. Clarke, *Integrability in submanifold geometry*, Ph.D. Thesis, University of Bath, 2012. MR3389373
21. G. Darboux, *Sur les surfaces isothermiques*, Ann. Sci. École Norm. Sup. (3) 16 (1899), 491–508. MR1508975
22. A. Demoulin, *Sur les surfaces R et les surfaces Ω*, C. R. Acad. Sci. Paris 153 (1911), 590–593.
23. A. Demoulin, *Sur les surfaces R et les surfaces Ω*, C. R. Acad. Sci. Paris 153 (1911), 705–707.
24. A. Demoulin, *Sur les surfaces Ω*, C. R. Acad. Sci. Paris 153 (1911), 927–929.
25. M. Desbrun, A. N. Hirani, M. Leok, and J. E. Marsden, *Discrete exterior calculus* (2005).
26. A. Doliwa, P. M. Santini, and M. Mañas, *Transformations of quadrilateral lattices*, J. Math. Phys. 41 (2000), no. 2, 944–990. MR1737004
27. V. V. Dolotin, A. Yu. Morozov, and Sh. R. Shakirov, *The A∞-structure on simplicial complexes*, Teoret. Mat. Fiz. 156 (2008), no. 1, 3.37. MR2488210
28. L. P. Eisenhart, *Surfaces with isothermal representation of their lines of curvature and their transformations*, Trans. Amer. Math. Soc. 9 (1908), no. 2, 149–177. MR1500806
29. L. P. Eisenhart, *Transformations of surfaces of Guichard and surfaces applicable to quadrics*, Ann. di Mat. 22 (1914), no. 3, 191–247.
30. L. P. Eisenhart, *Surfaces Ω and their transformations*, Trans. Am. Math. Soc. 16 (1915), no. 3, 275–310. MR151013
31. L. P. Eisenhart, *Transformations of surfaces Ω. II*, Trans. Am. Math. Soc. 17 (1916), no. 1, 53–99. MR1510130
32. D. Ferus and F. Pedit, *Curved flats in symmetric spaces*, Manuscripta Math. 91 (1996), no. 4, 445–454. MR1421284 (97k:53074)
33. C. Guichard, *Sur les surfaces isothermiques*, C. R. Math. Acad. Sci. Paris 130 (1900), 159–162.
34. U. Hertrich-Jeromin, *Introduction to Möbius differential geometry*, London Mathematical Society Lecture Note Series, vol. 300, Cambridge University Press, Cambridge, 2003. MR2002458
35. A. N. Hirani, *Discrete exterior calculus*, Ph. D. Thesis, California Institute of Technology, 2003. MR2704508
36. P. E. Hydon and E. L. Mansfield, *A variational complex for difference equations*, Found. Comput. Math. 4 (2004), no. 2, 187–217. MR2049870
37. E. Musso and L. Nicolodi, *The Bianchi-Darboux transform of L-isothermic surfaces*, Internat. J. Math. 11 (2000), no. 7, 911–924. MR1792958 (2002d:53012)
38. E. Musso and L. Nicolodi, *Deformation and applicability of surfaces in Lie sphere geometry*, Tohoku Math. J. **58** (2006), no. 2, 161–187. MR2248428

39. M. Pember, *Lie applicable surfaces*, Comm. Anal. Geom. **28** (2020), no. 6, 1407–1450. MR4184823

40. W. K. Schief, *On the unification of classical and novel integrable surfaces. II. Difference geometry*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **459** (2003), no. 2030, 373–391. MR1997461

41. W. K. Schief and B. G. Konopelchenko, *On the unification of classical and novel integrable surfaces. I. Differential geometry*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **459** (2003), no. 2029, 67–84. MR1993345

42. W. K. Schief, A. Szereszewski, and C. Rogers, *On shell membranes of Enneper type: generalized Dupin cyclides*, J. Phys. A **42** (2009), no. 40, 404016–404033. MR2544280