Improved Strichartz estimates for a class of dispersive equations in the radial case and their applications to nonlinear Schrödinger and wave equations

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Abstract

We prove some new Strichartz estimates for a class of dispersive equations with radial initial data. In particular, we obtain up to some endpoints the full radial Strichartz estimates for the Schrödinger equation. The ideas of proof are based on Shao’s ideas and some ideas in \cite{15} to treat the non-homogeneous case, while at the endpoint we need to use subtle tools to overcome some logarithmic divergence. We also apply the improved Strichartz estimates to the nonlinear problems. First, we prove the small data scattering and large data LWP for the nonlinear Schrödinger equation with radial critical initial data below \(L^2\); Second, for radial data we improve the results of the \(\dot{H}^s \times \dot{H}^{s-1}\) well-posedness for the nonlinear wave equation in \cite{33}; Finally, we obtain the well-posedness theory for the fractional order Schrödinger equation in the radial case.

Keywords: Strichartz estimates, radial data, nonlinear Schrödinger equation, nonlinear wave equation

1 Introduction

In this paper, we study the Cauchy problems for a class of dispersive equations which are of the following type:

\[
i\partial_t u = -\phi(\sqrt{-\Delta})u + f, \quad u(0, x) = u_0(x),
\]

where \(\phi : \mathbb{R}^+ \to \mathbb{R}\) is smooth away from origin, \(u(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}, \ n \geq 2\) is the unknown function, \(f(t, x)\) is the given function (e.g. \(f = |u|^p u\) in the nonlinear setting) and \(\phi(\sqrt{-\Delta})u = \mathcal{F}^{-1}\phi(|\xi|)\mathcal{F} u\). Here \(\mathcal{F}\) denotes the spatial Fourier transform, and \(\phi(|\xi|)\) is usually referred as the dispersion relation of equation (1.1). Many dispersive equations reduce to this type, for instance, the Schrödinger equation (\(\phi(r) = r^2\)), the wave equation (\(\phi(r) = r\)), the Klein-Gordon equation (\(\phi(r) = \sqrt{1 + r^2}\)), the beam equation (\(\phi(r) = \sqrt{1 + r^4}\)), and the fourth-order Schrödinger equation (\(\phi(r) = r^2 + r^4\)).

In the pioneered work \cite{39}, Strichartz derived the priori estimates of the solution to (1.1) in space-time norm \(L_t^q L_x^r\) by proving some Fourier restriction inequality. Later, his results was improved via a dispersive estimate and duality argument (cf. \cite{20} and references therein). The dispersive estimate

\[
\|e^{it\phi(\sqrt{-\Delta})}u_0\|_{X} \lesssim |t|^{-\theta} \|u_0\|_{X'}
\]
plays a crucial role, where $X'$ is the dual space of $X$. Applying (1.2), together with a standard argument (cf. [20]), we can immediately get the Strichartz estimates. For instance, one can see from the explicit formula of the free Schrödinger solution that
\[
\| e^{it\Delta} u_0 \|_{L^\infty_x} \lesssim |t|^{-n/2} \| u_0 \|_{L^1_x}.
\]
In [15], the authors systematically studied the dispersive estimates for (1.1) by imposing some asymptotic conditions on $\phi$.

As was explained in [20], the full range of the non-retarded Strichartz estimates for the Schrödinger equation were completely known, while that of the retarded estimates remain open. Surprisingly, if the initial data $u_0$ is radial, Shao [27] showed that the frequency localized non-retarded Strichartz estimates for the Schrödinger equation allow a wider range. For example, it was proved that
\[
\| e^{it\Delta} P_k u_0 \|_{L^q_t L^r_x(\mathbb{R}^{n+1})} \leq C 2^{\left(\frac{q}{2} - \frac{n+2}{2}\right)k} \| u_0 \|_2
\]
hold if $q > \frac{4n+2}{2n-1}$ and $u_0$ is radial. The proof relies deeply on the radial assumption which eliminates the bad-type evolution in the non-radial case (e.g. the Knapp counter-example). Similar results hold for the wave equation, see [28]. It is easy to see that equation (1.1) is rotational-invariant, thus it is natural to ask whether one can get better Strichartz estimates for the radial initial data than that derived from the dispersive estimate.

The purposes of this paper are: first, to obtain the sharp range of the type (1.3) for the improved Strichartz estimates for equation (1.1) by using Shao’s ideas [27] and the ideas in [15]. Indeed, we will simplify some proofs and overcome the difficulty caused by the lack of scaling invariance by adapting some ideas in [15], moreover, we will prove that (1.3) actually holds for $q = \frac{4n+2}{2n-1}$ by dealing carefully with some logarithmic divergence; second, to apply the improved Strichartz estimates to the nonlinear equations, including nonlinear Schrödinger equation, nonlinear wave equation, and nonlinear fractional-order Schrödinger equation. In order to apply to the nonlinear problems, we will use the Christ-Kiselev lemma to derive the retarded estimates from the non-retarded estimates. For example, consider the nonlinear Schrödinger equation
\[
iu_t + \Delta u = \mu |u|^p u, \quad u(0, x) = u_0(x),
\]
the well-posedness theory of which were deeply studied during the past decades. We remark that the threshold of the regularity in $H^s$ for the strong well-posedness is $s \geq \max(0, s_c)$, where $s_c$ is the scaling critical regularity, even in the case that $L^2$ is subcritical in the sense of scaling. This can be seen from the Galilean invariance (see [2, 6])
\[
u(t, x) \rightarrow e^{-i|y|^2t+iy\cdot x} u(t, x-2ty), \quad y \in \mathbb{R}^d.
\]
However, it is easy to see that the radial assumption breaks down the Galilean invariance. Thus it is natural to expect that one may go below $L^2$ in the radial case. This is indeed the case, which will be discussed in details in Section 4.

In this paper, we consider the same class of $\phi$ as in [15]. In order to study the non-homogeneous case (e.g. Klein-Gordon equation), we treat the high frequency and the low frequency in different scales. As in [13], we will assume $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ is smooth and satisfies some of the following conditions:

(H1) There exists $m_1 > 0$, such that for any $\alpha \geq 2$ and $\alpha \in \mathbb{N}$,
\[
|\phi'(r)| \sim r^{m_1-1} \text{ and } |\phi^{(\alpha)}(r)| \lesssim r^{m_1-\alpha}, \quad r \geq 1.
\]
Theorem 1.2. Now we are ready to state our first result: if \( \phi \) satisfies (H1) and (H3), then
\[
\| \phi''(r) \| \sim r^{m_2-2}, \quad 0 < r < 1.
\]

(H3) There exists \( \alpha_1 \), such that
\[
|\phi''(r)| \sim r^{\alpha_1-2}, \quad r \geq 1.
\]

(H4) there exists \( \alpha_2 \), such that
\[
|\phi''(r)| \sim r^{\alpha_2-2}, \quad 0 < r < 1.
\]

Remark 1.1. Heuristically, (H1) and (H3) reflect the dispersive effect in high frequency. If \( \phi \) satisfies (H1) and (H2), then \( \alpha_1 \leq m_1 \). Similarly, dispersive effect in low frequency is described by (H2) and (H4). If \( \phi \) satisfies (H2) and (H4), then \( \alpha_2 \geq m_2 \). The special case \( \alpha_2 = m_2 \) happens in the most of time.

For convenience, given \( m_1, m_2, \alpha_1, \alpha_2 \in \mathbb{R} \) as in (H1)-(H4), we denote
\[
m(k) = \begin{cases} m_1, & \text{for } k \geq 0, \\ m_2, & \text{for } k < 0; \end{cases} \quad \text{and} \quad \alpha(k) = \begin{cases} \alpha_1, & \text{for } k \geq 0, \\ \alpha_2, & \text{for } k < 0. \end{cases}
\]

Now we are ready to state our first result:

**Theorem 1.2.** Suppose \( n \geq 2 \), \( k \in \mathbb{Z} \), \( \phi : \mathbb{R}^+ \to \mathbb{R} \) is smooth away from origin, and \( u_0 \) is spherically symmetric. If \( \phi \) satisfies (H1) and (H2), then for \( \frac{2n}{n-1} < q \leq \infty \) we have
\[
\| S_\phi(t)P_k u_0 \|_{L^q_t L^m(x, \mathbb{R}^{n+1})} \lesssim \left( \frac{q}{2} \right)^{\frac{n-k+2}{q}} \| u_0 \|_2,
\]
(1.5)

Furthermore, if \( \phi \) also satisfies (H3) and (H4), then for \( \frac{2n+2}{2n-1} \leq q \leq 6 \) we have
\[
\| S_\phi(t)P_k u_0 \|_{L^q_t L^m(x, \mathbb{R}^{n+1})} \lesssim \left( \frac{q}{2} \right)^{\frac{n-k+2}{q} + \frac{1}{2} (m(k) - \alpha(k))} \| u_0 \|_2,
\]
(1.6)

where \( m(k), \alpha(k) \) are given by (1.4), and \( P_k \) is the Littlewood-Paley projector, \( S_\phi(t) = e^{i\phi(\sqrt{-\Delta})} \) is the dispersive group, which will be defined later. Moreover, the range of \( q \) is optimal in the sense that (1.5) fails if \( q \leq \frac{2n}{n-1} \) and (1.6) fails if \( q < \frac{4n+2}{2n-1} \).

For the Schrödinger equation, \( \phi(r) = r^2 \) and satisfies (H1)-(H4) with \( m(k) = \alpha(k) = 2 \), then it follows immediately from Theorem 1.2 that

**Corollary 1.3.** Assume \( n \geq 2 \), \( k \in \mathbb{Z} \), \( \frac{4n+2}{2n-1} \leq q \leq \infty \). Then there exists \( C > 0 \) such that for \( u_0 \in L^2(\mathbb{R}^n) \) and \( u_0 \) is spherically symmetric, one has
\[
\| e^{it\Delta} P_k u_0 \|_{L^q_t L^m(x, \mathbb{R}^{n+1})} \leq C \left( \frac{q}{2} \right)^{\frac{n-k+2}{q}} \| u_0 \|_2,
\]
(1.7)

and the range of \( q \) is optimal in the sense that (1.7) fails if \( q < \frac{4n+2}{2n-1} \).

Remark 1.4. Shao [27] proved (1.7) for \( q > \frac{4n+2}{2n-1} \). For the wave equation, \( \phi(r) = r \) and satisfies (H1)-(H2) with \( m(k) \equiv 1 \), then (1.5) reduces to the one given in [28]. Interestingly, the range \( q > \frac{2n}{n-1} \) is optimal for the wave equation. It is worth noting that if \( q > \frac{2n}{n-1} \), (1.5) gives better bound than (1.6) since \( k|m(k) - \alpha(k)| \geq 0 \) in view of Remark 1.1.
We will apply Theorem 1.2 to some concrete equations. Then using Christ-Kiselev lemma, we get the retarded Strichartz estimates. In view of the classical Strichartz estimates, it is natural to ask the sharp range of the mixed Strichartz estimates:

$$\|S_\phi(t)P_k u_0\|_{L_t^4 L_x^6(\mathbb{R}^{n+1})} \lesssim C(k)\|u_0\|_2.$$ 

For this purpose, we restrict ourselves to the simple case \(\phi(r) = r^a, a > 0\), namely, we consider the following estimates

$$\|e^{itD_n^a} P_k f\|_{L_t^4 L_x^6(\mathbb{R}^n)} \leq C2^{k(\frac{3}{2} - \frac{n-1}{q})}\|f\|_{L_t^2 L_x^2(\mathbb{R}^n)},$$  \(\text{(1.8)}\)

where \(D = \sqrt{-\Delta}, a > 0\). In this case, we have scaling invariance, thus the proof is less complicated but still can be adapted to the general case. We prove the following:

**Theorem 1.5.** (a) Assume \(a = 1\) and \(n \geq 3\). \(\text{(1.8)}\) hold for all radial functions \(f \in L^2(\mathbb{R}^n)\) if and only if

\[
(q, r) = (\infty, 2) \quad \text{or} \quad 2 \leq q \leq \infty, \quad \frac{1}{q} + \frac{n-1}{r} < \frac{n-1}{2}.
\]

(b) Assume \(0 < a \neq 1\) and \(n \geq 2\). \(\text{(1.8)}\) hold for all radial functions \(f \in L^2(\mathbb{R}^n)\) if

\[
\frac{4n+2}{2n-1} \leq q \leq \infty, \quad \frac{2n-1}{q} + \frac{2n-1}{r} \leq n - \frac{1}{2} \quad \text{or} \quad 2 \leq q < \frac{4n+2}{2n-1}, \quad \frac{2n-1}{q} + \frac{2n-1}{r} < n - \frac{1}{2},
\]

On the other hand, \(\text{(1.8)}\) fail if \(q > 2\) or \(\frac{2}{q} + \frac{2n-1}{r} > n - \frac{1}{2}.

Remark 1.6. The range of \((q, r)\) is indicated in Figure 1, where \(B = (\frac{n-3}{2n-2}, \frac{1}{2}), C = (\frac{n-3}{2n-2}, \frac{1}{2}), D = (\frac{n-1}{2n-1}, \frac{n-1}{2n-1}), B' = (\frac{n-2}{2n-2}, \frac{1}{2}), C' = (\frac{n-2}{2n-2}, \frac{1}{2}), D' = (\frac{n-1}{2n-2}, \frac{2n-1}{2n-2})\). The results for the wave equation \((a = 1)\) are not new. The positive results were due to [24] [34] [38] [11]. The counter-example was given in [18].

On the other hand, for the Schrödinger equation, the results seem to be new. We see that the picture is almost complete, except that the segment \(C'D'\) is unknown. In view of the positive results on the segment \(D'E'\), we conjecture that \((1.8)\) holds on the segment \(C'D'\), which is equivalent to the following

**Conjecture 1.7.** Assume \(n \geq 2\) and \(0 < a \neq 1\). Then

$$\|e^{itD_n^a} P_0 f\|_{L_t^4 L_x^6(\mathbb{R}^{n+1})} \leq C\|f\|_{L_t^2 L_x^2(\mathbb{R}^n)},$$  \(\text{(1.9)}\)

holds for all radial function \(f \in L_x^2(\mathbb{R}^n)\).
This is very similar to the endpoint Strichartz estimates in the non-radial case that was studied in [20]. As is expected, (1.9) is just “logarithmically far” to be proved. Indeed, we have for any $j \in \mathbb{N}$

$$\|e^{itD^s} P_0 f\|_{L^p_t L^{\frac{n+2}{p}}_x (\mathbb{R} \times \{ |x| \sim 2^j \})} \leq C\|f\|_{L^2_0(\mathbb{R}^n)}.$$ 

However, we cannot adapt the method on $D' E'$ to overcome this logarithmical divergence. See Remark 2.13 below for more discussions on (1.9).

Using these Strichartz estimates, we study the nonlinear problems and prove some new results. For example, for the nonlinear Schrödinger equation, we prove the following

**Theorem 1.8.** Assume $n \geq 2$, $0 < p < 4/n$, $s_{sch} = \frac{n}{2} - \frac{2}{p}$, $\frac{1-n}{2n+1} \leq s_{sch} < 0$, and $u_0$ is radial. If $\|u_0\|_{\hat{H}^{s_{sch}}} \leq \delta$ for some $\delta \ll 1$, then there exists a unique global solution $u$ to

$$i u_t + \Delta u = \mu |u|^p u, \quad u(0, x) = u_0(x),$$

where $\mu = \pm 1$, such that $u \in C(\mathbb{R} : \hat{H}^{s_{sch}}) \cap L^{\frac{1(n+2)}{n-2}}_{t,x} (\mathbb{R} \times \mathbb{R}^n)$. Moreover, there exist $u_{\pm} \in \hat{H}^{s_{sch}}$ such that $\|u - S(t)u_\pm\|_{\hat{H}^{s_{sch}}} \to 0$, as $t \to \pm \infty$.

The index $\frac{1-n}{2n+1}$ is sharp for the critical GWP by our methods. We actually obtain more results, see Theorem 4.2 below. For the nonlinear wave equation, we prove the following

**Theorem 1.9.** Assume $n \geq 2$, $0 < p < \frac{4}{n+1}$, $s_w = \frac{n}{2} - \frac{2}{p}$, $\frac{1-n}{2n+1} < s_w < 1/2$, and $u_0$ is radial. If $\|u_0\|_{\hat{H}^{s_w}} + \|u_1\|_{\hat{H}^{s_w-1}} \leq \delta$ for some $\delta \ll 1$, then there exists a unique global solution $u$ to

$$\partial_t u - \Delta u = \mu |u|^p u, \quad (t, x) \in \mathbb{R}^{n+1},$$

$$u(0) = u_0(x), \quad u_t(0) = u_1(x),$$

where $\mu = \pm 1$, such that $u \in C(\mathbb{R} : \hat{H}^{s_w}) \cap C^1(\mathbb{R} : \hat{H}^{s_w-1}) \cap L^{\frac{2n+2}{n-2}}_{t,x} (\mathbb{R} \times \mathbb{R}^n)$, and there exist $(u_{\pm}, v_{\pm}) \in \hat{H}^{s_w} \times \hat{H}^{s_w-1}$ such that $\|u - W(t)u_{\pm}\|_{\hat{H}^{s_w}} + \|u_t - W(t)v_{\pm}\|_{\hat{H}^{s_w-1}} \to 0$, as $t \to \pm \infty$.

Our results also hold for general nonlinearity, e.g. $F(u)$ with $F$ satisfying some conditions such as (4.61). In [20], Lindblad and Sogge studied the semilinear wave equation with the same nonlinearity but with general non-radial initial data. For example, for the nonlinearity $|u|^p$ they proved small data scattering in $\hat{H}^{s_w} \times \hat{H}^{s_w-1}$ with $s = \frac{n}{2} - \frac{2}{p}$ if $p \geq \frac{n+2}{n-1}$, and local well-posedness if $s \geq s(p, n)$ for some $s(p, n)$. Thus, we see that their results covered the case $s_w \geq 1/2$ in Theorem 1.9, which is the main reason why we restrict ourselves to the case $s_w < 1/2$. In the same paper [20], the authors actually showed that their results are sharp by constructing some counter-examples. However, the counter-examples for $s_w < 1/2$ don’t work for the radial case. Our results in Theorem 1.9 showed that in the radial case one can improve their results. Actually, we find a critical regularity in the radial case $s_0(n) < \frac{1}{2}$, which we will discuss in details in Theorem 4.6. In Section 4 we also study nonlinear fractional order Schrödinger equation, and establish the well-posedness theory in the radial case. We do not repeat the theorem here, but refer to Theorem 4.10 below.

The fact that better well-posedness results hold in the radial case was observed before, see [34, 10, 16, 17]. Our results generalize these results. In the non-radial case, with additional angular regularity, one can also go below $L^2$, see [10, 19] and the references therein. Actually, the results in [10] for the Schrödinger equation are more generalized than ours but with different resolution space. Our results for local well-posedness hold without change for the inhomogeneous
data \(u_0 \in H^s\) (see Remark 1.4). It is then natural to ask whether (1.7) and (1.8) hold for non-radial functions with certain angular regularity.

Throughout this paper, \(C > 1\) and \(c < 1\) will denote positive universal constants, which can be different at different places. \(A \lesssim B\) means that \(A \leq CB\), and \(A \sim B\) stands for \(A \lesssim B\) and \(B \lesssim A\). We use \(\hat{f}(\xi)\) and \(\hat{F}(f)\) to denote the spatial Fourier transform of \(f\) on \(\mathbb{R}^n\) defined by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx.
\]

We denote by \(p'\) the dual number of \(p \in [1, \infty]\), i.e., \(1/p + 1/p' = 1\). Let \(\Phi(x) : \mathbb{R} \to [0, 1]\) be a non-negative, smooth even function such that \(\text{supp} \Phi \subseteq \{x : |x| \leq 2\}\), and \(\Phi(x) = 1\), if \(|x| \leq 1\).

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.2. In Section 3 we present the applications of Theorem 1.2 to some concrete equations. In Section 4, we apply the improved Strichartz estimates to the nonlinear problems.

## 2 Proof of Theorem 1.2 and Theorem 1.5

First we prove Theorem 1.2. We will adapt some ideas in [15] and Shao’s ideas [27]. However, there is a new difficulty for the endpoint case \(q = \frac{\frac{4n+2}{n-1}}{2}\) in (1.6) due to some logarithmic divergence. Fortunately enough, this logarithmic divergence can be overcome by using a subtle tool: double weight Hardy-Littlewood-Sobolev inequality. On the other hand, the logarithmic divergence for the endpoint \(q = \frac{2n}{n-1}\) in (1.3) is essential. We present the proof by the following three steps:

### Step 1. Non-endpoint: \(q > \frac{2n}{n-1}\) in (1.3), \(q > \frac{4n+2}{2n-1}\) in (1.6).

For \(j \in \mathbb{Z}\), denote

\[
A_j := \{x \in \mathbb{R}^n : 2^{j-1} \leq |x| < 2^j\}, \quad I_j = [2^{j-1}, 2^j).
\]

Fixing \(k \in \mathbb{Z}\), we decompose \(\|S_\Phi(t) \Delta_k u_0(x)\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^n)}\) and get

\[
\|S_\Phi(t)P_k u_0\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^n+1)} \leq \sum_{j \in \mathbb{Z}} \|S_\Phi(t)P_k u_0\|_{L^q_t L^r_x(\mathbb{R} \times A_j)}
\]

\[
= \sum_{j+k \leq 1} \|S_\Phi(t)P_k u_0\|_{L^q_t L^r_x(\mathbb{R} \times A_j)} + \sum_{j+k \geq 2} \|S_\Phi(t)P_k u_0\|_{L^q_t L^r_x(\mathbb{R} \times A_j)}. \quad (2.10)
\]
The main tasks reduce to estimate $\|S_\phi(t)P_ku_0\|_{L^q_t(L^{r_x}(\mathbb{R}\times\mathbb{R}^n)))}$. It is easy to see that $S_\phi(t)P_ku_0$ is spherically symmetric in space if $u_0$ is radial. Thus we can rewrite it in an integral form related to the Bessel function. The two parts $j+k \leq 1$ and $j+k \geq 2$ exploit different properties of the Bessel function. We give the estimates of the two parts in the following two propositions.

**Proposition 2.1.** Assume $u_0 \in L^2(\mathbb{R}^n)$, $u_0$ is radial, and $\phi$ satisfies (H1) and (H2). Then if $k, j \in \mathbb{Z}$ with $j+k \leq 1$ and $2 \leq q \leq \infty$, we have

$$
\|S_\phi(t)P_ku_0(x)\|_{L^q_t(L^{r_x}(\mathbb{R}\times\mathbb{R}^n)))} \lesssim 2^{\frac{m(j)}{q}} 2^{(\frac{j}{2} + \frac{m(k)}{q})k} \|P_ku_0\|_{L^2},
$$

(2.11)

where $m(k)$ is given by (1.4).

**Proposition 2.2.** Assume $u_0 \in L^2(\mathbb{R}^n)$, $u_0$ is radial, and $\phi$ satisfies (H1) and (H2). Then if $k, j \in \mathbb{Z}$ with $j+k \geq 2$ and $2 \leq q \leq \infty$, we have

$$
\|S_\phi(t)P_ku_0(x)\|_{L^q_t(L^{r_x}(\mathbb{R}\times\mathbb{R}^n)))} \lesssim 2^{(\frac{n+k}{2} - \frac{n-1}{q})j} 2^{(\frac{j}{2} + \frac{m(k)}{q})k} \|P_ku_0\|_{L^2}.
$$

(2.12)

Furthermore, if $\phi$ also satisfies (H3) and (H4), then for $2 \leq q \leq 6$

$$
\|S_\phi(t)P_ku_0(x)\|_{L^q_t(L^{r_x}(\mathbb{R}\times\mathbb{R}^n)))} \lesssim 2^{(\frac{2n+k}{2q} - \frac{2n-1}{q})j} 2^{(\frac{j}{2} + \frac{m(k)}{q})k} \|P_ku_0\|_{L^2},
$$

(2.13)

where $m(k), \alpha(k)$ is given by (1.4).

We postpone the proofs of Proposition 2.1 and Proposition 2.2, and first use them to complete the proof of Theorem 1.2 in the non-endpoint case.

**Proof of Theorem 1.2 (non endpoint).** We may assume $q < \infty$. Assume first that $\phi$ satisfies (H1) and (H2). From (2.10), Proposition 2.1 and Proposition 2.2, we get

$$
\|S_\phi(t)P_ku_0(x)\|_{L^q_t(L^{r_x}(\mathbb{R}\times\mathbb{R}^n)))} \lesssim \sum_{j+k \leq 1} 2^{\frac{m(j)}{q}} 2^{(\frac{j}{2} + \frac{m(k)}{q})k} \|P_ku_0\|_{L^2}
$$

$$
+ \sum_{j+k \geq 2} 2^{(\frac{n+k}{2} - \frac{n-1}{q})j} 2^{(\frac{j}{2} + \frac{m(k)}{q})k} \|P_ku_0\|_{L^2}
$$

$$
\lesssim 2^{\frac{n}{q}} 2^{\frac{n-1}{q}j} 2^{(\frac{j}{2} + \frac{m(k)}{q})k} \|P_ku_0\|_{L^2},
$$

since $q > \frac{2n}{n+1}$ then $\frac{n}{q} - \frac{n-1}{2} < 0$. Thus (1.5) is proved. Now we assume $\phi$ also satisfies (H3) and (H4), then

$$
\|S_\phi(t)P_ku_0(x)\|_{L^q_t(L^{r_x}(\mathbb{R}\times\mathbb{R}^n)))} \lesssim \sum_{j+k \leq 1} \sum_{j+k \geq 2} 2^{\frac{m(j)}{q}} 2^{(\frac{j}{2} + \frac{m(k)}{q})k} \|P_ku_0\|_{L^2}
$$

$$
+ \sum_{j+k \geq 1} 2^{(\frac{2n+k+1}{2q} - \frac{2n-1}{q})j} 2^{(\frac{j}{2} + \frac{m(k)}{q})k} \|P_ku_0\|_{L^2}.
$$

Note that if $q > \frac{4n+2}{2n+1}$, then $\frac{2n+1}{2q} - \frac{2n-1}{q} < 0$. Thus we can sum over $j$ and bound the quantity above by

$$
C \left[ 2^{\frac{n}{q}} 2^{\frac{n+k}{q}j} \right] \|P_ku_0\|_{L^2},
$$

which is sufficient for (1.6) since $\frac{n}{q} - \frac{n-1}{2} > 0$ in view of Remark 1.1 \(\square\)
It remains to prove Proposition 2.1 and Proposition 2.2. The proof relies heavily on the radial properties. In particular, we will use the Fourier-Bessel formula. We denote by $J_m(r)$ the Bessel function:

$$J_m(r) = \frac{(r/2)^m}{\Gamma(m + 1/2)\pi^{1/2}} \int_{-1}^{1} e^{irt} (1 - t^2)^{m-1/2} dt, \quad m > -1/2.$$  

We first list some properties of $J_m(r)$ that will be used in the following lemma. For their proof we refer the readers to [36].

**Lemma 2.3 (Properties of the Bessel function).** We have for $0 < r < \infty$ and $m > -\frac{1}{2}$

(i) $J_m(r) \leq Cr^m$,

(ii) $J_m(r) \leq Cr^{-\frac{1}{2}}$.

It is well known that if $f(x) = g(|x|)$ is radial, then the Fourier transform of $f$ is also radial (cf. [35]), and

$$\hat{f}(\xi) = 2\pi \int_{0}^{\infty} g(s)s^{n-1}(s|\xi|)^{-\frac{n-2}{2}} J_{n-2}(s|\xi|)ds. \quad (2.14)$$

Thus if $\hat{u_0}(\xi) = h(|\xi|)$ is radial, then $S_{\phi}(t)P_k u_0 = F(t, |x|)$, and

$$F(t, |x|) = 2\pi \int_{0}^{\infty} e^{it\phi(s)} \psi_k(s) h(s)s^{n-1}(s|x|)^{-\frac{n-2}{2}} J_{n-2}(s|x|)ds. \quad (2.15)$$

The issues reduce to a one-dimensional problem involving Bessel function. We will use the following local smoothing effect type estimate.

**Lemma 2.4.** Suppose $k \in \mathbb{Z}$, $\varphi \in L^2(\mathbb{R})$ and $\phi$ satisfies (H1) and (H2). Then for $2 \leq q \leq \infty$, we have

$$\left\| \int_{\mathbb{R}} \psi_k(s)\varphi(s)e^{-it\phi(s)} ds \right\|_{L^q_t} \lesssim 2^{\frac{k}{2} - \frac{m(k)}{q}} \|\psi_k\varphi\|_{L^2}$$

where $m(k)$ is defined in [1.4].

**Proof.** It is easy to see that in the support of $\psi_k$, $\phi$ is invertible and we denote $\phi^{-1}$ to be the inverse of $\phi$. By the change of variable $a = \phi(s)$, we get

$$\left\| \int_{\mathbb{R}} \psi_k(s)\varphi(s)e^{-it\phi(s)} ds \right\|_{L^q_t} = \left\| \int_{\mathbb{R}} \psi_k(\phi^{-1}(a))e^{-ita} \frac{\varphi^{-1}(a)}{\phi^{-1'(a))}} da \right\|_{L^q_t}.$$  

Then from the Hausdorff-Young inequality and change of variable $s = \phi(a)$, we get the quantity above is bounded by

$$C\left\| \psi_k(\phi^{-1}(a)) \frac{\varphi^{-1}(a)}{\phi^{-1'(a))}} \right\|_{L^q_t} = C\left\| \psi_k(s) \frac{\varphi(s)}{\phi^{-1'(s))}} \right\|_{L^q_t}.$$  

From the condition we have $\phi'(s) \sim 2^{m(k)-1}k$ in the support of $\psi_k$, and then by Hölder inequality we can bound the quantity above by

$$C'2^{-\frac{m(k)+1}{q}}k^{\frac{1}{2} - \frac{1}{2}k}\|\psi_k(s)\varphi(s)\|_{L^2_t} = C'2^{\frac{k}{2} - \frac{m(k)}{q}}k\|\psi_k\varphi\|_{L^2}$$

Thus we finish the proof. 

□
Lemma 2.5 (Strichartz estimate). Suppose \( \varphi \in L^2(\mathbb{R}) \) and \( \phi \) satisfies one of \( H(3) \) and \( H(4) \). Then for \( k \in \mathbb{Z} \), we have

\[
\left\| \int_{\mathbb{R}} \psi_k(s)\varphi(s)e^{irs-it\phi(s)} \, ds \right\|_{L^1_tL^6_x} \lesssim 2^{\left(\frac{1}{2} - \frac{\eta}{4}\right)k} \|\psi_k\varphi\|_{L^2},
\]

where \( \alpha(k) \) is defined in (1.4).

Proof. Since \( \phi \) satisfies \( H(3) \) and \( H(4) \), then by Theorem 1 in [15], we have the decay estimate

\[
\left\| \int_{\mathbb{R}} \psi_k(s)\varphi(s)e^{irs-it\phi(s)} \, ds \right\|_{L^\infty_r} \lesssim |t|^{-\frac{1}{2}(1 - \frac{\alpha(k)}{2})} \|F\psi_k\varphi\|_{L^1}.
\]

Then Lemma 2 follows immediately from Proposition 1 in [15], also see [20]. \( \square \)

Proof of Proposition 2.1. We get from (2.15) and Lemma 2.3 (i) and Lemma 2.4 that

\[
\left\| S_\phi(t)P_0 u_0(x) \right\|_{L^q_tL^6_x(\mathbb{R} \times A_j)} \lesssim \left\| F_k(t, r)^{\frac{\eta}{4}} \right\|_{L^q_{r,t}L^6_x} \lesssim 2^{\left(\frac{1}{2} - \frac{\eta}{4}\right)k} \|\psi_k\varphi\|_{L^2} \lesssim 2^{\frac{n+1}{4}2\left(1 - \frac{\eta}{4}\right)k} \|\psi_k\varphi\|_{L^2}
\]

which completes the proof of Proposition 2.1 since \( \|\psi_k\varphi\|_{L^2} = \|P_0 u_0\|_{L^2} \).

It remains to prove Proposition 2.2. We will use the decay properties at the infinity of the Bessel function. More precisely,

\[
J_{\frac{n-1}{2}}(r) = \frac{e^{i(r-(n-1)\pi/4)}}{2r^{1/2}} + d_nr^{\frac{n-1}{2}} e^{-ir} E_+(r) - c_n r^{\frac{n-1}{2}} e^{i_r} E_-(r),
\]

where \( E_\pm(r) \lesssim r^{-(n+1)/2} \) if \( r \geq 1 \), \( d_n, c_n \) are constants, see [36]. Inserting (2.16) into (2.15), we then divide \( F(t, |x|) \) into two parts: the main term and the error term, namely

\[
F(t, |x|) = M(t, |x|) + E(t, |x|)
\]

(2.17)

with

\[
M(t, r) = c_n r^{\frac{n-1}{2}} \int_{\mathbb{R}} \psi_k(s)\varphi(s)e^{i(r-(n-1)\pi/4)} \, ds + c_n r^{\frac{n-1}{2}} \int_{\mathbb{R}} \psi_k(s)\varphi(s)e^{-ir} E_+(r) \, ds - c_n r^{\frac{n-1}{2}} e^{i_r} E_-(r).
\]

\[
E(t, r) = c_1 \int_{\mathbb{R}} \psi_k(s)\varphi(s) \, ds + c_1 \int_{\mathbb{R}} \psi_k(s)\varphi(s) E_+(r) \, ds - c_2 \int_{\mathbb{R}} \psi_k(s)\varphi(s) E_-(r) \, ds.
\]

First we estimate the error term \( E(t, |x|) \) in the following Lemma.

Lemma 2.6. Assume \( \varphi \) satisfies \( H(1) \) and \( H(2) \). If \( j + k \geq 2 \) and \( 2 \leq q \leq \infty \), we have

\[
\left\| E(t, |x|) \right\|_{L^q_tL^6_x(\mathbb{R} \times A_j)} \lesssim 2^{-(\frac{n+1}{2} + \frac{\eta}{4})} 2^{-\left(\frac{1}{2} + \frac{\eta}{4}\right)k} \|P_0 u_0\|_{L^2}.
\]

(2.18)

Proof. As in the proof of Proposition 2.1, we have

\[
\left\| E(t, |x|) \right\|_{L^q_tL^6_x(\mathbb{R} \times A_j)} \lesssim \left\| E(t, r)^{\frac{\eta}{4}} \right\|_{L^q_{r,t}L^6_x} \lesssim 2^{\left(\frac{1}{2} - \frac{\eta}{4}\right)k} \|\psi_k\varphi\|_{L^2} \lesssim 2^{\left(\frac{1}{2} + \frac{\eta}{4}\right)k} 2^{\left(\frac{1}{2} + \frac{n+1}{2}\right)k} \|\psi_k\varphi\|_{L^2},
\]

where we used the fact \( |E_\pm(r)| \lesssim r^{-(n+1)/2} \). Thus we complete the proof. \( \square \)
Next we estimate the main term $M(t, |x|)$ in the following Lemma.

**Lemma 2.7.** (a) Assume $\phi$ satisfies (H1) and (H2). If $j + k \geq 2$, we have

$$
\| M(t, |x|) \|_{L^2_{t,x}(\mathbb{R} \times A_j)} \lesssim 2^{j/2} 2^{1 - m(k)/2} \| P_k u_0 \|_{L^2},
$$

(2.19)

$$
\| M(t, |x|) \|_{L^\infty_{t,x}(\mathbb{R} \times A_j)} \lesssim 2^{-j(n-1)/2} 2^{k/2} \| P_k u_0 \|_{L^2}.
$$

(2.20)

(b) Assume $\phi$ satisfies (H3) and (H4). If $j + k \geq 2$, we have

$$
\| M(t, x) \|_{L^6_{t,x}(\mathbb{R} \times A_j)} \lesssim 2^{-n-k/2} 2^{j - n(k)/2} \| P_k u_0 \|_{L^2}.
$$

(2.21)

**Proof.** From symmetry it suffices to estimate the first term in $M(t, |x|)$. We get from Lemma 2.4 with $q = 2$ that

$$
\| M(t, |x|) \|_{L^2_{t,x}(\mathbb{R} \times A_j)} \lesssim \| M(t, r) \|_{L^2_{t,r} L^2_j}
$$

$$
\lesssim \left\| \int_{\mathbb{R}} \psi_k(s) h(s) s^{n-1} e^{i(rs-\hat{\phi}(s))} \, ds \right\|_{L^2_j L^2_t}
$$

$$
\lesssim 2^{j/2} \frac{m(k)-1}{2} \| \psi_k(s) h(s) s^{n-1/2} \|_{L^2_t}
$$

which gives the first inequality, as desired. Similarly,

$$
\| M(t, |x|) \|_{L^\infty_{t,x}(\mathbb{R} \times A_j)} \lesssim \| M(t, r) \|_{L^\infty_{t,r} L^\infty_j}
$$

$$
\lesssim 2^{-j(n-1)/2} \left\| \int_{\mathbb{R}} \psi_k(s) h(s) s^{n-1/2} e^{i(rs-\hat{\phi}(s))} \, ds \right\|_{L^\infty_j L^\infty_t}
$$

$$
\lesssim 2^{-j(n-1)/2} 2^{k/2} \| \psi_k(s) h(s) s^{n-1/2} \|_{L^2_t},
$$

To prove (b), we get from Lemma 2.3 that

$$
\| M(t, |x|) \|_{L^6_{t,x}(\mathbb{R} \times A_j)} \lesssim \| M(t, r) \|_{L^6_{t,r} L^6_j}
$$

$$
\lesssim 2^{-(n-1)j/6} \left\| \int_{\mathbb{R}} \psi_k(s) h(s) s^{n-1/2} e^{i(rs-\hat{\phi}(s))} \, ds \right\|_{L^6_j L^6_t}
$$

$$
\lesssim 2^{-(n-1)j/6} 2^{(n-k)/2} \| \psi_k(s) h(s) s^{n-1/2} \|_{L^2_t},
$$

which completes the proof of the lemma.

Now we are ready to prove Proposition 2.2

**Proof of Proposition 2.2.** If $\phi$ satisfies (H1) and (H2), then by interpolating (2.19) and (2.20) we get that for $2 \leq q \leq \infty$

$$
\| M(t, x) \|_{L^q_{t,x}(\mathbb{R} \times A_j)} \lesssim 2^{(n-q)j/2} 2^{(n-k)/2} \| P_k u_0 \|_{L^2}.
$$

(2.22)

Then from Lemma 2.6 we get for $2 \leq q \leq \infty$

$$
\| S_q(t) P_k u_0(x) \|_{L^q_{t,x}(\mathbb{R} \times A_j)} \lesssim \| E(t, x) \|_{L^q_{t,x}(\mathbb{R} \times A_j)} + \| M(t, x) \|_{L^q_{t,x}(\mathbb{R} \times A_j)}
$$

$$
\lesssim 2^{(n-q)j/2} 2^{(n-k)/2} \| P_k u_0 \|_{L^2}.
$$
Moreover, if $\phi$ also satisfies (H3) and (H4), then by interpolating (2.19) and (2.21) we get that for $2 \leq q \leq 6$

$$\|M(t,x)\|_{L_{t,x}^q(\mathbb{R} \times \mathbb{R}^n)} \leq 2^{\left(\frac{2n+1}{2q} - \frac{2n-1}{4}\right)} \|\phi\|_{L_{t,x}^\infty}^2 \frac{\alpha(k-1)^3}{\alpha + 1} \frac{1}{2q} \|k\|_q^4 \|P \|_{L_{t,x}^1(\mathbb{R} \times \mathbb{R}^n)}.$$  

(2.23)

Thus, in view of Lemma 2.6 and (2.23), the left-hand side of (2.13) can be bounded by

$$\|S_\phi(t)P_ku_0(x)\|_{L_{t,x}^q(\mathbb{R} \times \mathbb{R}^n)} \leq \|E(t,x)\|_{L_{t,x}^\infty(\mathbb{R} \times \mathbb{R}^n)} + \|M(t,x)\|_{L_{t,x}^\infty(\mathbb{R} \times \mathbb{R}^n)}$$

with

$$\|C_1(k,j) + C_2(k,j)\|_{L_{t,x}^1(\mathbb{R} \times \mathbb{R}^n)}$$

where

$$C_1(k,j) = 2^{\left(\frac{2n+1}{2q} - \frac{2n-1}{4}\right)} \|\phi\|_{L_{t,x}^\infty}^2 \frac{1}{2q} \|k\|_q^4 \|P \|_{L_{t,x}^1(\mathbb{R} \times \mathbb{R}^n)},$$

$$C_2(k,j) = 2^{\left(\frac{2n+1}{2q} - \frac{2n-1}{4}\right)} \|\phi\|_{L_{t,x}^\infty}^2 \frac{1}{2q} \|k\|_q^4 \|P \|_{L_{t,x}^1(\mathbb{R} \times \mathbb{R}^n)}.$$  

It remains to prove $C_1(k,j) \leq C_2(k,j)$. Actually, by simple calculation we get

$$\frac{C_2(k,j)}{C_1(k,j)} = 2^{\left(\frac{2n+1}{2q} - \frac{2n-1}{4}\right)} \|\phi\|_{L_{t,x}^\infty}^2 \frac{1}{2q} \|k\|_q^4 \|P \|_{L_{t,x}^1(\mathbb{R} \times \mathbb{R}^n)}.$$  

It is easy to see that

$$(j + k)\left(\frac{1}{2q} + \frac{3}{4}\right) + \left(\frac{1}{4} - \frac{1}{2q}\right)(m(k) - \alpha(k))k \geq 1,$$

since $j + k \geq 2$ and $(m(k) - \alpha(k))k \geq 0$ in view of Remark 1.1. Thus we finish the proof.  

**Step 2.** Endpoint: $q = \frac{4n+2}{2n-1}$ in (1.6).

From step 1 we see that in this case we just fail to sum over $j \geq 2 - k$. To overcome this, we do not decompose for large $j$. The main tools are the Van der Corput Lemma [36] and double weight Hardy-Littlewood-Sobolev inequalities [37].

**Lemma 2.8** (Van der Corput). Assume $\psi \in C_0^\infty(\mathbb{R})$ and $P \in C^2(\mathbb{R})$ is a real-valued function such that $|P'(\xi)| \geq \lambda$ in the support of $\psi$. Then

$$\left|\int e^{ip(\xi)}\psi(\xi)d\xi\right| \leq C\lambda^{-1/2}(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}).$$

**Lemma 2.9.** If $1 \leq r, s < \infty$, $1/r + 1/s \geq 1$, $0 < \lambda \leq d$, $\alpha + \beta \geq 0$ and

$$1 - \frac{1}{r} - \frac{\alpha}{d} < 1 - \frac{1}{r} - \frac{1}{t} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{d} = 2,$$

then

$$\left|\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)g(y)\frac{\left|x^a y^b\right|^d}{|x|^a |y|^b|y|^d}dxdy\right| \leq C_{\alpha,\beta,s,L,d} \|f\|_{L^r} \|g\|_{L^s}.$$  

Now we proceed to prove (1.6) for $q = \frac{4n+2}{2n-1}$. Obviously, we have

$$\|S_\phi(t)P_ku_0\|_{L_{t,x}^q(\mathbb{R} \times \mathbb{R}^n)} \leq \sum_{j \leq 1-k} \|S_\phi(t)P_ku_0\|_{L_{t,x}^1(\mathbb{R} \times \mathbb{R}^n)} + \|S_\phi(t)P_ku_0\|_{L_{t,x}^\infty(\mathbb{R} \times \{|x| \geq 2^{1-k}\})}$$

$$:= I + II.$$
From step 1 we see that the term $I$ is bounded as desired. It remains to bound the term $II$. Using (2.34) we get

$$ II \leq \| M(t, |x|) \|_{L^q_t \cap L^s_x((|x| \geq 2^{1-k}))} + \| E(t, |x|) \|_{L^q_t \cap L^s_x((|x| \geq 2^{1-k}))} := II_1 + II_2. $$

From step 1 we see that the term $I$ is bounded as desired. Thus, it remains to bound the term $II_1$. From symmetry, it suffices to prove

$$ \left\| \frac{1}{2^{1-k,n-1}}(r) r \left( \frac{1}{q} \right)^{(n-1)} \int_{\mathbb{R}} \psi_h(s) h(s) s^{\frac{n-1}{q}} e^{i(r s-t \phi(s))} ds \right\|_{L^q_t \cap L^s_x} \leq 2^{(\frac{1}{q} - \frac{1}{2})^2} \| \frac{2}{k} \| h \|_2. \quad (2.24) $$

which follows from the following estimate

$$ \left\| \left( \frac{1}{q} - \frac{1}{2} \right)^{(n-1)} \int_{\mathbb{R}} \psi_0(s) h(s) e^{i(r s-t \phi(s))} ds \right\|_{L^q_t \cap L^s_x} \leq 2^{(\frac{1}{q} - \frac{1}{2})^2} \| \frac{2}{k} \| h \|_2. \quad (2.24) $$

It remains to prove (2.24). Since $\psi_0(s)$ is supported in $\{ s \sim 1 \}$, then from (H1)-(H4) we get that $\phi_k = 2^{-km(k)} \phi(2^k s)$ has an inverse denoted by $\eta_k = \phi_k^{-1} : \text{range}(\phi_k) \to \{ s \sim 1 \}$, moreover,

$$ |\eta_k| \sim 1, \quad |\eta'| \sim 2^{k(\alpha(k) - m(k))}. \quad (2.25) $$

By a change of variable $s = \eta_k(\mu)$, we get that (2.24) is equivalent to

$$ \left\| \left| \frac{1}{\eta_k} \right|^{(\frac{1}{q} - \frac{1}{2})^2} \int_{\mathbb{R}} \psi_0(\eta_k(\mu)) h(\mu) e^{i(\eta_k s - t \phi(s))} d\mu \right\|_{L^q_t \cap L^s_x} \leq 2^{(\frac{1}{q} - \frac{1}{2})^2} \| \frac{2}{k} \| h \|_2. \quad (2.26) $$

For $f \in L^2(\mathbb{R})$, define operator

$$ Tf(x, t) = \left| x \right|^{(\frac{1}{q} - \frac{1}{2})^2} \int_{\mathbb{R}} \psi_0(\eta_k(\mu)) f(\mu) e^{i(\eta_k s - t \phi(s))} d\mu. $$

It suffices to prove $\| T \|_{L^2 \to L^q_t} \leq 2^{(\frac{1}{q} - \frac{1}{2})^2} \| \frac{2}{k} \| h \|_2$. By duality, we have

$$ T^* g(\mu) = \psi_0(\eta_k(\mu)) \int_{\mathbb{R} \times \mathbb{R}} e^{-i(\eta_k s - t \phi(s))} |x|^{(\frac{1}{q} - \frac{1}{2})^2} g(x, t) dxdt. $$

By the $TT^*$ arguments, it suffices to prove

$$ \| TT^* g \|_{L^q} \leq 2^{(\frac{1}{q} - \frac{1}{2})^2} \| \frac{2}{k} \| h \|_2. $$

From the definition we have

$$ TT^* g(x, t) = |x|^{(\frac{1}{q} - \frac{1}{2})^2} \int \psi_0^2(\eta_k(\mu)) e^{-i(\eta_k s - t \phi(s))} |y|^{(\frac{1}{q} - \frac{1}{2})^2} g(y, t) e^{i(\eta_k s - t \phi(s))} d\mu dy dt = |x|^{(\frac{1}{q} - \frac{1}{2})^2} \int K(x - y, t - \tau) |y|^{(\frac{1}{q} - \frac{1}{2})^2} g(y, t) dy dt, $$

where

$$ K(x - y, t - \tau) = \int \psi_0^2(\eta_k(\mu)) e^{i(x-y) \eta_k(\mu) - (t-\tau) \mu} d\mu. $$
Using Plancherel’s equality, we get
\[
\left\| \int K(x - y, t - \tau)g(y, \tau)d\tau \right\|_{L^2_t} \lesssim \|g(y, \cdot)\|_{L^2}.
\]
On the other hand, it follows from Van der Corput lemma and (2.25) that
\[
|K(x - y, t - \tau)| \lesssim 2^{\frac{k(m(k) - \alpha(k))}{2} |x - y|^{-1/2}}.
\]
Then by interpolation we have
\[
\left\| \int K(x - y, t - \tau)g(y, \tau)d\tau \right\|_{L^q_t} \lesssim 2^{\frac{k(m(k) - \alpha(k))}{2} (\frac{1}{2} - \frac{1}{q}) |x - y|^{-\frac{1}{2}} - \frac{1}{q} \|g(u, \cdot)\|_{L^{q'}_t}.
\]
Using Minkowski inequality we obtain
\[
\|TT^*g\|_{L^q_x} \lesssim 2^{\frac{k(m(k) - \alpha(k))}{2} (\frac{1}{2} - \frac{1}{q}) \int |y|^{-\frac{1}{2}} |g(y, \cdot)\|_{L^{q'}_x} |x - y|^{-\frac{1}{2}} dy
\]
To complete the proof, it suffices to prove
\[
\left\| \int \int \frac{g(y)f(x)}{|x|^{-\frac{1}{2}} |y|^{-\frac{1}{2}} |x - y|^{-\frac{1}{2}}} \right\|_{L^q_x} \lesssim \|g\|_{L^{q'}_x} \|f\|_{L^{q'}_x},
\]
which follows immediately from Lemma 2.9 since it is easy to verify the condition with \( q = \frac{4n+2}{2n-1} \), \( \alpha = \beta = \left( \frac{1}{2} - \frac{1}{q} \right) (n-1) \), \( \lambda = \frac{1}{2} - \frac{1}{q} \), \( r = s = q' \), \( d = 1 \). Therefore, we complete the proof.

**Step 3. Sharpness.**

It remains to prove that the range of \( q \) is optimal. We will prove that \( \|e^{it\sqrt{-\Delta}}P_0u_0\|_{L^q \to \infty} \leq \|u_0\|_2 \) fails if \( q \leq \frac{2n}{n+1} \), and \( \|e^{it\Delta}P_0u_0\|_{L^q \to \infty} \leq \|u_0\|_2 \) fails if \( q \leq \frac{4n+2}{2n-1} \). For the former one, from the proof in step 1 we see that it suffices to disprove: for \( q = \frac{2n}{n+1} \)
\[
\left\| \int \frac{r^{-\frac{n+1}{2}}}{r^{-\frac{n+1}{2}}} \int \psi_0(s)h(s)\cos(rs - \frac{(n-1)\pi}{4})e^{its} ds \right\|_{L^q_{t,r}} \lesssim \|h\|_2.
\]
Indeed, by taking \( h(s) = 1_{[0, 1]}(s) \), and from the fact that for \( r \gg 1 \)
\[
\left\| \int \psi_0(s)\cos(rs - \frac{(n-1)\pi}{4})e^{its} ds \right\|_{L^q_{t,r}} \gtrsim \|\hat{\psi}_{0}(t + r) + \hat{\psi}_{0}(t - r)\|_{L^q_{t,r}} \gtrsim 1,
\]
we obtain that
\[
\left\| \int \frac{r^{-\frac{n+1}{2}}}{r^{-\frac{n+1}{2}}} \int \psi_0(s)h(s)\cos(rs - (n-1)\pi/4)e^{its} ds \right\|_{L^q_{t,r}} = \infty.
\]
Thus (2.28) fails if \( q = \frac{2n}{n+1} \).

To see the latter one, similarly, it suffices to disprove: for \( q < \frac{4n+2}{2n-1} \)
\[
\left\| \int \frac{r^{-\frac{n+1}{2}}}{r^{-\frac{n+1}{2}}} \int \psi_0(s)h(s)\cos(rs - (n-1)\pi/4)e^{its} ds \right\|_{L^q_{t,r}} \lesssim \|h\|_2.
\]
Indeed, fix a \( j \) sufficiently large and take \( h(s) = 2^{j/2}1_{|s-1| \leq 2^{-j}} \). Then \( \|h\|_2 = 1 \). For \( t > 0 \), the main contribution of \( \int \frac{r^{-\frac{n+1}{2}}}{r^{-\frac{n+1}{2}}} \int h(s)\cos(rs - (n-1)\pi/4)e^{its} ds \) is
\[
c_n r^{-\frac{n+1}{2}} \int h(s)e^{-irs}e^{its} ds = c_n 2^{j/2} r^{-\frac{n+1}{2}} \int \int 1_{|s| \leq 2^{-j}}(s)e^{-irs}e^{its} e^{2its} ds.
\]
Thus the left-hand side of (2.29) is larger than
\[ \left\| 2^{j/2} \frac{n-1}{q} - \frac{n-1}{2} \int_{\mathbb{R}} 1_{|s| \leq 2^{-j}}(s)e^{-ir^s}e^{2its}ds \right\|_{L^q_{r \sim 2^{j},|r-2a| \leq 2^j}} \gtrsim 2^{j(2n+1 - 2n-1)} \]
which is unbounded if \( q < \frac{4n+2}{2n-1} \). Therefore, we complete the proof of Theorem 1.2.

Next we prove Theorem 1.3. First we give the following maximal function estimates, which generalize the results in [21] for \( a \geq 2 \) to \( a > 0 \).

**Lemma 2.10.** Assume \( a > 0 \) and \( k \geq 0 \). Then
\[ \left\| \int_{\mathbb{R}} e^{it|\xi|^n}e^{ix\xi} \eta_0(\xi/2^k) f(\xi) d\xi \right\|_{L^2_{|\xi| \leq 1} L^\infty_{|t| \leq 1}} \lesssim B(a,k)\|f\|_2, \]
where
\[ B(a,k) = \begin{cases} 2^{a^2/4}, & a \neq 1, \\ 2^{k/2}, & a = 1. \end{cases} \]
Moreover, the bounds are sharp.

**Proof.** By change of variables: \( \xi = 2^k \eta \) and then \( x = 2^{-k}y \), we get that (2.30) is equivalent to
\[ \left\| \int_{\mathbb{R}} e^{it|\xi|^n}e^{ix\xi} \eta_0(\xi) f(\xi) d\xi \right\|_{L^2_{|\xi| \leq 1} L^\infty_{|t| \leq 2ka}} \lesssim B(a,k)\|f\|_2. \]
By \( TT^* \) methods, (2.31) is equivalent to
\[ \left\| \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}} e^{it(t-t')|\xi|^n}e^{i(x-x')\xi} \eta_0(\xi) d\xi \right] g(t',x') dt' dx' \right\|_{L^2_{|\xi| \leq 1} L^\infty_{|t| \leq 2ka}} \lesssim B(a,k)^2\|g\|_{L^2_{|\xi| \leq 1} L^1_{|t| \leq 2ka}}. \]

Denote \( K_a(x-x',t-t') = \int_{\mathbb{R}} e^{it(t-t')|\xi|^n}e^{i(x-x')\xi} \eta_0(\xi) d\xi \). By Stationary phase and Van der Corput lemma, since \( |t-t'| \lesssim 2^{ka} \), it is easy to see that for \( a \neq 1 \)
\[ |K_a(x-x',t-t')| \lesssim (1 + |x-x'|)^{-1/2} 1_{|x-x'| \leq 2^{ka}} + |x-x'|^{-4} 1_{|x-x'| \gg 2^k} \]
and for \( a = 1 \)
\[ |K_1(x-x',t-t')| \lesssim 1 \cdot 1_{|x-x'| \leq 2^k} + |x-x'|^{-4} 1_{|x-x'| \gg 2^k}. \]

Using bounds above and Young’s inequality, we get
\[ \left\| \int_{\mathbb{R}^2} K_a(x-x',t-t') g(t',x') dt' dx' \right\|_{L^2_{|\xi| \leq 1} L^1_{|t| \leq 2ka}} \lesssim \|K_a\|_{L^1_{|\xi| \leq 1} L^\infty_{|t| \leq 2ka}} \|g\|_{L^2_{|\xi| \leq 1} L^1_{|t| \leq 2ka}} \lesssim B(a,k)^2\|g\|_{L^2_{|\xi| \leq 1} L^1_{|t| \leq 2ka}}. \]

Thus we obtain the bounds as desired.

It remains to show that the bounds are sharp. First we consider \( a = 1 \). For \( f \) supported in \( \{\xi > 0\} \), we have
\[ \text{L.H.S of (2.31)} \gtrsim \left\| \int_{\mathbb{R}} e^{-ix\xi} e^{ix\xi} \eta_0(\xi) f(\xi) d\xi \right\|_{L^2_{|\xi| \leq 2^k}} \gtrsim 2^{k/2} \]

14
which shows the sharpness of the bound $2^{k/2}$. Now we consider $a \neq 1$. Take $f = \theta^{-1/2}1_{|\xi-1| \leq \theta}$, $\theta = 2^{-ka/2}$. Then $\|f\|_2 \sim 1$, and

$$L.H.S\ of\ (2.31) \geq \theta^{-1/2} \left\| \int_{|\xi| \leq \theta} e^{it(\xi+1)\rho} e^{-it\cdot \xi} d\xi \right\|_{L^2_{|s| \leq \theta^{-2}} L_{|t| \leq 2^{ka}}} \geq \theta^{-1/2} \left\| \int_{|\xi| \leq \theta} e^{-it\cdot \xi} d\xi \right\|_{L^2_{|s| \leq \theta^{-2}}} \geq \theta^{-1/2} = 2^{ka/4}$$

where in the last inequality we used the fact that $|(|\xi+1|^2 - 1 - a\xi| \leq \theta^2$. Thus we complete the proof of the lemma.

We present the proof of Theorem 1.2 in the following two cases.

**Case 1:** $a \neq 1$.

First we assume that $a \neq 1$. Since (1.8) is trivial if $(q, r) = (\infty, 2)$, thus by Bernstein’s inequality, Riesz-Thorin interpolation and the classical Strichartz estimates, it suffices to prove (1.8) for $(q, r) = (2, r)$, where $\frac{4n-2}{2n-3} < r < \frac{2n}{n-2}$.

By the scaling transform $(t, x) \rightarrow (\lambda^a t, \lambda x)$, clearly we may assume $k = 0$. By the classical Strichartz estimates (see [20] for $n \geq 3$ and [41] for $n = 2$): $\|e^{itD^a} P_0 f\|_{L^2_t L^{2^*}_{|x| \geq 10}} \leq C\|f\|_{L^2_t}$, then we see that from Hölder’s inequality, it suffices to prove

$$\|e^{itD^a} P_0 f\|_{L^2_t L^{2^*}_{|x| \geq 10}} \leq C\|f\|_{L^2_t}.$$  (2.33)

As before we divide $u_a(t, |x|) = e^{itD^a} P_0 f$ into two parts: the main term and the error term, namely

$$u_a(t, |x|) = M_a(t, |x|) + E_a(t, |x|)$$  (2.34)

with

$$M_a(t, r) = c_n r^{-\frac{n-1}{2}} \int_\mathbb{R} \psi_0(s) g(s) s^{-\frac{n-1}{2}} e^{i(rs-ts^n)} ds + c_n r^{-\frac{n-1}{2}} \int_\mathbb{R} \psi_0(s) g(s) s^{\frac{n-a}{2}} e^{-i(rs+ts^n)} ds,$$

$$E_a(t, r) = c_1 \int_\mathbb{R} \psi_0(s) g(s) s^{n-1} e^{-itis^n - irs} E_{+(rs)} ds - c_2 \int_\mathbb{R} \psi_0(s) g(s) s^{n-1} e^{-itis^n + irs} E_{-(rs)} ds.$$

First we bound the main term. We have

**Lemma 2.11. (a) Assume $a \neq 1$, $a > 0$, $j \geq 2$ and $2 \leq r \leq \infty$. Then**

$$\|M_a(t, |x|)\|_{L^2_t L^r(\mathbb{R} \times A_j)} \lesssim 2^{j(\frac{n-1}{2} - \frac{2n-3}{2})} \|f\|_{L^2_t}.$$  (2.35)

**Proof.** For $r = 2$, it was proven in Lemma 2.7. By Riesz-Thorin interpolation, it suffices to prove for $r = \infty$. By the definition of $M_a$ and symmetry, it suffices to show

$$2^{-\frac{(n-1)j}{2}} \left\| \int_\mathbb{R} \eta(s) g(s) e^{i(rs^{-1} - ts)} ds \right\|_{L^2_t L^\infty(\mathbb{R} \times A_j)} \lesssim \|g\|_2.$$  (2.36)
we see that to prove (1.9), it suffices to prove
\[ 2^{-\left(\frac{n-1}{2}\right)} \left\| \int_{\mathbb{R}} \eta(\xi/2^j)g(\xi)e^{i(\xi_1/a - x_1)}d\xi \right\|_{L^2 L^\infty_{|\eta| \leq 2}} \lesssim 2^{-\left(\frac{(2n-3)}{4}\right)} \|g\|_2 \]  
(2.37)
which reduces to a maximal function estimate associated to the dispersion $\xi_1/a$. Since $a \neq 1$, then (2.37) follows immediately from Lemma 2.10. \[ \square \]

Next, we estimate the error term $E_a(t, |x|)$. This term certainly has better estimates than the main term, but for our purpose, the following rough estimates will be enough.

**Lemma 2.12.** Assume $a \neq 1$, $j \geq 2$ and $2 \leq r \leq \frac{2n}{n-2}$. Then
\[ \| E_a(t, |x|) \|_{L^2_t L^r_x} \lesssim 2^{-\left(\frac{j}{2}(\frac{n}{2} - \frac{n-2}{2})\right)} \|f\|_{L^2}. \]  
(2.38)

**Proof.** For $r = 2$, it was proven in Lemma 2.6. For $r = \frac{2n}{n-2}$ we have
\[ \| E_a(t, |x|) \|_{L^2_t L^4_x (\mathbb{R} \times A_j)} \leq \|u_a(t, |x|)\|_{L^2_t L^8_x (\mathbb{R} \times A_j)} + \|M_a(t, |x|)\|_{L^2_t L^{\frac{2n}{n-2}}_x (\mathbb{R} \times A_j)} \lesssim \|f\|_2 \]
where we used the classical endpoint Strichartz estimates and Lemma 2.7. \[ \square \]

We are ready to prove (2.33). Indeed, since $\frac{4n-2}{2n-3} < r < \frac{2n}{n-2}$, by Lemma 2.7 and Lemma 2.6 we can sum over $j \geq 1$:
\[ \|e^{itD_x}P_0f\|_{L^2_t L^r_x |x| \geq 10} \lesssim \sum_{j=1}^{\infty} \|M_a(t, |x|)\|_{L^2_t L^4_x (\mathbb{R} \times A_j)} + \sum_{j=1}^{\infty} \|E_a(t, |x|)\|_{L^2_t L^4_x (\mathbb{R} \times A_j)} \lesssim \|f\|_2. \]

**Case 2.** $a = 1$ and $n \geq 3$.

As in Case 1, it suffices to prove (1.8) for $(q, r) = (2, r)$, where $\frac{4n-2}{2n-3} < r < \frac{2n}{n-2}$. Using the decomposition (2.34) and the following lemma, we immediately obtain (1.8).

**Lemma 2.13.** Assume $j \geq 2$ and $2 \leq r \leq \infty$, $2 < q \leq \frac{2n-2}{n-3}$. Then
\[ \|M_1(t, |x|)\|_{L^2_t L^q_x (\mathbb{R} \times A_j)} \lesssim 2^{\left(\frac{q}{2} - \frac{n-2}{2}\right)} \|f\|_{L^2}, \quad \|E_1(t, |x|)\|_{L^2_t L^q_x (\mathbb{R} \times A_j)} \lesssim 2^{-\left(\frac{q-1}{2} - \frac{n-3}{2}\right)} \|f\|_{L^2}. \]

**Proof.** The proof follows exactly as the proof of two Lemmas above, thus we omit the details. \[ \square \]

Finally, we show the sharpness. $q \geq 2$ is necessary since (1.8) is time-translation invariant. The same counter-example for (2.29) shows that $\frac{2}{q} + \frac{2n-1}{r} \leq n - \frac{1}{2}$ is necessary.

**Remark 2.14.** We give a remark on Conjecture 1.7 for $a = 2$. From the proof of Theorem 1.5 we see that to prove (1.9), it suffices to prove
\[ \left\| r^{-1/(2n-1)} \int_{\mathbb{R}} \psi_0(s)g(s)e^{i(rs-\frac{1}{2}s^2)}ds \right\|_{L^2 L^\frac{4n-3}{r} (r \geq 1)} \lesssim \|g\|_2. \]
Unfortunately, we are not able to prove this.
3 Strichartz estimates in the radial case

In this section, we will apply Theorem 1.2 to some dispersive equations. Since we do not have the decay estimates, then we use Christ-Kiselev lemma to derive the retarded linear estimates. First we prove a duality property for radial function.

Lemma 3.1. Assume \( 1 \leq p \leq \infty, 1 = 1/p + 1/p', f \in L^p(\mathbb{R}^n) \) and \( f \) is radial. Then

\[
\|f\|_{L^p(\mathbb{R}^n)} = \sup \left\{ \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| : g \in L^{p'}(\mathbb{R}^n), g \text{ is radial and } \|g\|_{L^{p'}} \leq 1 \right\}.
\] (3.39)

Proof. Denote the right-hand side of (3.39) by \( B \). Then it is obviously that \( B \leq \|f\|_{L^p(\mathbb{R}^n)} \), thus it suffices to show \( \|f\|_{L^p(\mathbb{R}^n)} \leq B \). By duality, we have

\[
\|f\|_{L^p(\mathbb{R}^n)} = \sup_{g \in L^{p'}, \|g\|_{L^{p'}} = 1} \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right|
\]

\[
= \sup_{g \in L^{p'}, \|g\|_{L^{p'}} = 1} \left| \int_0^\infty \int_{S^{n-1}} f(r)g(rx')r^{n-1}drd\sigma(x') \right|
\]

\[
= \sup_{g \in L^{p'}, \|g\|_{L^{p'}} = 1} \left| \int_{\mathbb{R}^n} f(x)\tilde{g}(x)dx \right|
\]

where we set \( \tilde{g}(x) = \frac{1}{|x|^{n-1}} \int_{S^{n-1}} g(|x|x')d\sigma(x') \). It’s easy to see from Hölder’s inequality that \( \tilde{g} \) is radial and \( \|\tilde{g}\|_{L^{p'}} \leq 1 \), then we get \( \|f\|_{L^p(\mathbb{R}^n)} \leq B \) as desired.

Obviously, Lemma 3.1 holds similarly for function \( f(t, x) \) spherically symmetric in \( x \), e.g. \( f \in L^p_tL^2_x \). As a corollary, we can apply Lemma 3.1 to get the dual version estimates of the linear estimates in the radial case.

Lemma 3.2. Assume \( 1 \leq q, r \leq \infty, 1/q + 1/q' = 1/r + 1/r' = 1, k \in \mathbb{Z} \). If for all \( u_0 \in L^2(\mathbb{R}^n) \) and \( u_0 \) is radial we have

\[
\|S_\phi(t)P_ku_0\|_{L^q_tL^r_x} \lesssim C(k)\|u_0\|_{L^2},
\]

Then for all \( f \in L^q_tL^{r'}_x \) and \( f \) is spherically symmetric in space we have

\[
\left\| \int_\mathbb{R} S_\phi(-t)[P_kf(t, \cdot)] dt \right\|_{L^2(\mathbb{R}^n)} \lesssim C(k)\|f\|_{L^q_tL^{r'}_x}.
\]

Christ-Kiselev lemma which was obtained by Christ and Kiselev [4] is very useful in deriving the retarded estimates from the non-retarded estimates. The one we need is the following, for its proof we refer the readers to [33].

Lemma 3.3 (Christ-Kiselev). Assume \( 1 \leq p_1, q_1, p_2, q_2 \leq \infty \) with \( p_1 > p_2 \). If for all \( f \in L^{p_2}_tL^{q_2}_x \) spherically symmetric in space

\[
\left\| \int_\mathbb{R} S_\phi(t-s)(P_kf(s))(x)ds \right\|_{L^{p_1}_tL^{q_1}_x} \lesssim C(k)\|f\|_{L^{p_2}_tL^{q_2}_x},
\]

then we have

\[
\left\| \int_0^t S_\phi(t-s)(P_kf(s))(x)ds \right\|_{L^{p_1}_tL^{q_1}_x} \lesssim C(k)\|f\|_{L^{p_2}_tL^{q_2}_x}
\]

holds with the same bound \( C(k) \), for all \( f \in L^{p_2}_tL^{q_2}_x \) spherically symmetric in space.
Now we are ready to give some new Strichartz estimates for some concrete equations. First note that from Minkowski inequality and Littlewood-Paley square function theorem we get if \(2 \leq q,r < \infty\) then
\[
\|f\|_{L_t^qL_x^r} \lesssim \|P_kf\|_{L_t^qL_x^r} \|\hat{f}\|_{\mathcal{F}^k}, \quad \|P_kf\|_{L_t^q' L_x^r'} \lesssim \|f\|_{L_t^q' L_x^r'}. \tag{3.40}
\]
We will apply (3.40) to get the Strichartz estimates on the whole space.

1. Schrödinger equation

\[
\begin{aligned}
&\left\{ \begin{array}{l}
    i\partial_t u + \Delta u = F, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\
    u(0) = u_0(x).
\end{array} \right.
\end{aligned} \tag{3.41}
\]

By Duhamel’s principle, we get \(u = S(t)u_0 - i \int_0^t S(t-\tau)F(\tau) d\tau\), where \(S(t) = e^{-it\Delta}\), which corresponds to \(\phi(r) = r^2\). Then we see that \(\phi\) satisfies (H1), (H2), (H3) and (H4) with \(m_1 = m_2 = \alpha_1 = \alpha_2 = 2\). Thus by Theorem 1.2 we obtain for \(q \geq \frac{2n}{2n-1}\) and if \(u_0\) is radial then
\[
\|S(t)P_k u_0\|_{L_t^qL_x^r} \lesssim 2^{\gamma \frac{n}{q} - \frac{n}{q} \frac{m}{n} + \frac{n}{2}} \|u_0\|_2. \tag{3.42}
\]

**Definition 3.4.** Suppose \(n \geq 2\). The exponent pair \((q,r)\) is said to be n-D radial Schrödinger-admissible if \(q,r \geq 2\), and
\[
\frac{4n+2}{2n-1} \leq q \leq \infty, \quad \frac{2n-1}{r} \leq n - \frac{1}{2} \quad \text{or} \quad 2 \leq q < \frac{4n+2}{2n-1}, \quad \frac{2n-1}{q} + \frac{n}{r} \leq n - \frac{1}{2}. \tag{3.43}
\]

For \(n \geq 3\), the n-D radial Schrödinger-admissible pairs are described in the Figure 1 \(a \neq 1\).

**Proposition 3.5** (Schrödinger Strichartz estimate). Suppose \(n \geq 2\) and \(u, u_0, F\) are spherically symmetric and satisfy equation (3.41). Then
\[
\|u\|_{L_t^qL_x^r} + \|u\|_{C(\mathbb{R};H^\gamma)} \lesssim \|u_0\|_{H^\gamma} + \|F\|_{L_t^q' L_x^r'}. \tag{3.44}
\]

if \(\gamma \in \mathbb{R}, (q,r)\) and \((\tilde{q},\tilde{r})\) are both n-D radial Schrödinger-admissible, either \((\tilde{q},\tilde{r},n) \neq (2,\infty,2)\) or \((q,r,n) \neq (2,\infty,2)\), and satisfy the “gap” condition
\[
\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - \gamma, \quad \frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2} + \gamma.
\]

**Proof.** The case \(F = 0\) follows from Theorem 1.3. Now we assume \(F \neq 0\), \((q,r)\) and \((\tilde{q},\tilde{r})\) are both n-D radial Schrödinger admissible, \((\tilde{q},\tilde{r},n) \neq (2,\infty,2)\) and satisfy the “gap” condition. If \(\gamma = 0\), this is implied by the already known estimates [20]. If \(\gamma \neq 0\), then by scaling it suffices to prove
\[
\left\| \int_0^t S(t-s)P_0 F(s) ds \right\|_{L_t^qL_x^r} \lesssim \|F\|_{L_t^q' L_x^r'}. \tag{3.45}
\]
Since either \(q,r > 2\) or \(\tilde{q},\tilde{r} > 2\), then in view of Christ-Kiselev lemma it suffices to prove
\[
\left\| \int_{\mathbb{R}} S(t-s)P_0 F(s) ds \right\|_{L_t^qL_x^r} \lesssim \|F\|_{L_t^q' L_x^r'}. \tag{3.46}
\]
which follows immediately from the non-retarded linear estimates and Lemma 3.2. Thus we complete the proof of the proposition. \(\square\)
Remark 3.6. We remark that we can take \( \gamma < 0 \), which means there are smoothing effects in the non-retarded Strichartz estimates. This only holds in the radial case. There are also smoothing effects in some retarded estimates, but for our purpose, we only derive the ones without smoothing effect.

2. Wave equation

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t u - \Delta u = F, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
u(0) = u_0(x), \quad u_t(0) = u_1(x).
\end{array} \right.
\tag{3.47}
\end{aligned}
\]

By Duhamel’s principle, we get \( u = W'(t)u_0 + W(t)u_1 - \int_0^t W(t - \tau)F(\tau)d\tau, \) where

\[
W(t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}, \quad W'(t) = \cos(t\sqrt{-\Delta}).
\]

This reduces to \( W_\pm(t) := e^{\pm it(\Delta)^{1/2}} \), which corresponds to \( \phi(r) = r \). Then we see that \( \phi \) satisfies (H1) and (H2) with \( m_1 = m_2 = 1 \). Thus by Theorem 1.2 we obtain for \( q > \frac{2n}{n-1} \) and if \( u_0 \) is radial then

\[
\| W_\pm(t) P_k u_0 \|_{L^q_x((\mathbb{R}^n+1)} \lesssim 2^\left(\frac{\frac{n}{q} - \frac{n}{r}}{2} - 1\right)k \| u_0 \|_2.
\tag{3.48}
\]

Definition 3.7. Suppose \( n \geq 2 \). The exponent pair \((q, r)\) is said to be n-D radial wave-admissible if \( q, r \geq 2 \), and one of the following

1. \( n = 2 \), \( (q, r) \in A_2 = \{(q, r) : \frac{1}{q} + \frac{1}{r} < \frac{1}{2}, q > 4\} \cup \{(4, \infty), (\infty, 2)\}; \)
2. \( n \geq 3 \), \( (q, r) \in A_{\geq 3} = \{(q, r) : q \geq 2, \frac{1}{q} + \frac{n-1}{r} < \frac{n-1}{2}\} \cup \{\infty, 2\} \).

For \( n \geq 4 \), the n-D radial wave-admissible pairs are described in the Figure 1 (\( a = 1 \)).

Proposition 3.8 (Wave Strichartz estimate). Suppose \( n \geq 2 \) and \( u, u_0, u_1, F \) are spherically symmetric and satisfy equation (3.47). Then

\[
\| u \|_{L^q_t L^r_x} + \| u \|_{C([0, T]; H^{\gamma+n})} + \| \partial_t u \|_{C([0, T]; H^{\gamma-1})} \lesssim \| u_0 \|_{H^\gamma} + \| u_1 \|_{H^{\gamma-1}} + \| F \|_{L^q_t L^r_x}. \tag{3.49}
\]

if \( \gamma \in \mathbb{R}, \ (q, r) \) and \( (\bar{q}, \bar{r}, n) \) are both n-D radial wave-admissible, \( (\bar{q}, \bar{r}, n) \neq (2, \infty, 3) \), and satisfy the “gap” condition

\[
\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma, \quad \frac{1}{\bar{q}} + \frac{n}{\bar{r}} = \frac{n}{2} - 1 + \gamma.
\]

Proof. The proof is similar to that of Proposition 3.5. We omit the details. \( \square \)

3. Klein-Gordon equation

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t u - \Delta u + u = F, \\
u(0) = u_0(x), \quad u_t(0) = u_1(x).
\end{array} \right.
\tag{3.50}
\end{aligned}
\]

By Duhamel’s principle, we get \( u = K'(t)u_0 + K(t)u_1 - \int_0^t K(t - \tau)F(\tau)d\tau, \) where

\[
K(t) = \omega^{-1} \sin(t\omega), \quad K'(t) = \cos(t\omega), \quad \omega = \sqrt{1 - \Delta}.
\]

This reduces to the semigroup \( K_\pm(t) := e^{\pm it(1-\Delta)^{1/2}} \), which corresponds to \( \phi(r) = (1 + r^2)^{1/2} \).

By simple calculation,

\[
\phi'(r) = \frac{r}{(1 + r^2)^{1/2}}, \quad \phi''(r) = \frac{1}{(1 + r^2)^{3/2}}.
\]
we see that $\phi$ satisfies (H1), (H2), (H3) and (H4) with $m_1 = 1$, $\alpha_1 = -1$, $m_2 = \alpha_2 = 2$. Thus by Theorem 1.2 we obtain for $q \geq \frac{4n+2}{2n-1}$ and if $u_0$ is radial then

$$\|K_\pm(t)P_ku_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim C(q,k)\|u_0\|_2,$$

(3.51)

where

$$C(q,k) = \begin{cases} 
2\left(\frac{2}{q} - \frac{n+2}{q} \right) k, & k \leq 0; \\
2\left(\frac{2}{q} + \frac{n+1}{q} \right) k, & k \geq 0, \frac{2n}{n-1} < q \leq \infty; \\
2\left(\frac{2}{q} - \frac{n+1}{q} \right) k + \left(\frac{1}{2} - \frac{1}{4} k \right), & k \geq 0, \frac{4n+2}{2n-1} \leq q \leq \frac{2n}{n-1}.
\end{cases}$$

4. Beam equation

By Duhamel’s principle, we have $u = B'(t)u_0 + B(t)u_1 - \int_0^t B(t-\tau)F(\tau)d\tau$, where

$$B(t) = \omega^{-1}(\sin(t\omega), B'(t) = \cos(t\omega), \omega = \sqrt{1+\Delta^2}. $$

This reduces to the semigroup $B_\pm(t) := e^{\pm i(t+\Delta^2)^{1/2}}$, which corresponding to $\phi(r) = (1+r^4)^{1/2}$. By simple calculation,

$$\phi'(r) = 2r^3/(1+r^4)^{3/2}, \quad \phi''(r) = (6r^2 + 2r^6)/(1+r^4)^{3/2},$$

we know that $\phi$ satisfies (H1) and (H2) with $m_1 = \alpha_1 = 2$, $m_2 = \alpha_2 = 4$. Thus by Theorem 1.2 we obtain for $q \geq \frac{4n+2}{2n-1}$ and if $u_0$ is radial then

$$\|B_\pm(t)P_ku_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim B(q,k)\|u_0\|_2,$$

(3.53)

where

$$B(q,k) = \begin{cases} 
2\left(\frac{2}{q} - \frac{n+1}{q} \right) k, & k \leq 0; \\
2\left(\frac{2}{q} - \frac{n+1}{q} \right) k, & k \geq 0.
\end{cases}$$

5. Fractional-order Schrödinger equation

$$\begin{cases}
 i\partial_t u + (-\Delta)^{\frac{\sigma}{2}} u = F, \\
u(0) = u_0(x),
\end{cases}$$

(3.54)

where $1 < \sigma < 2$. By Duhamel’s principle, we have $u = S_\sigma(t)u_0 + \int_0^t S_\sigma(t-\tau)F(\tau)d\tau$, where $S_\sigma(t) = e^{-it\phi(\sqrt{-\Delta})}$ with $\phi(r) = r^\sigma$. By simple calculation, we see that $\phi$ satisfies (H1), (H2), (H3) and (H4) with $m_1 = \alpha_1 = m_2 = \alpha_2 = \sigma$. Thus by Theorem 1.2 we obtain for $q \geq \frac{4n+2}{2n-1}$ and if $u_0$ is radial then

$$\|S_\sigma(t)P_ku_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim 2\left(\frac{2}{q} - \frac{n+2}{q} \right) k\|u_0\|_2.$$
\textbf{Proof.} The proof is similar to that of Proposition 3.9, except \((q, r, n) = (2, \infty, 2)\). This particular case follows similarly as for the Schrödinger equation in [11]. We omit the details. \qed

In particular, taking \(\gamma = 0\) we get a family of Strichartz estimates without loss of regularity.

\textbf{Corollary 3.10.} Suppose \(n \geq 2\), \(\frac{2n}{2n-1} < \sigma \leq 2\) and \(u, u_0, F\) are spherically symmetric in space and satisfy equation (3.54). Then

\[
\|u\|_{L_t^q L_x^r} + \|u\|_{C([t_0; T]; L^2)} \lesssim \|u_0\|_{L^2} + \|F\|_{L_t^q L_x^r},
\]

(3.57) implies if \((q, r)\) and \((\tilde{q}, \tilde{r})\) belong to \(\{(q, r) : q, r \geq 2, \frac{2}{q} + \frac{2}{r} = \frac{2}{2}\}\) and \((\tilde{q}, \tilde{r}, n) \neq (2, \infty, 2)\).

These estimates without loss of derivatives hold only in the radial case. Finally we present the Knapp-counterexample to show that the general non-radial Strichartz estimates have loss of derivative for \(1 < \sigma < 2\).

Assume that the following inequality hold for general non-radial function \(f\):

\[
\left\| \int_{\mathbb{R}^d} e^{it|\xi|^{\sigma}} e^{ix\xi} \eta_0(\xi) \hat{f}(\xi) d\xi \right\|_{L_t^q L_x^r} \lesssim \|f\|_{L^2}.
\]

(3.58)

Take

\(D = \{\xi = (\xi_1, \xi') \in \mathbb{R}^d : |\xi_1| \lesssim \delta, |\xi'| \leq \delta\}\).

Let \(\hat{f} = 1_D(\xi)\). Then \(\|f\|_2 \sim \delta^{d/2}\), and

\[
\int_{\mathbb{R}^d} e^{it|\xi|^{\sigma}} e^{ix\xi} \eta_0(\xi) \hat{f}(\xi) d\xi = e^{it(1 - \sigma)} \int_D e^{it(1 - \sigma) - \sigma(\xi_1 - 1)} e^{it(\sigma + 1)(\xi_1 - 1) - \sigma(\xi_1 - 1)} e^{ix\xi'} d\xi.
\]

Since in \(D\) we have

\[
||\xi|^{\sigma} - \xi_1^{\sigma}| \lesssim |\xi'|^{\sigma} \lesssim \delta^{\sigma}, |\xi_1 - 1 - \sigma(\xi_1 - 1)| \lesssim |\xi_1 - 1| \lesssim \delta^{\sigma},
\]

thus for \(|t| \lesssim \delta^{-2}, |t\sigma + x_1| \lesssim \delta^{-1}, |x'| \lesssim \delta^{-1}\), we have \(\int_{\mathbb{R}^d} e^{it|\xi|^{\sigma}} e^{ix\xi} \eta_0(\xi) \hat{f}(\xi) d\xi \sim |D|\). Therefore, (3.58) implies

\[
\delta^{-\frac{2}{q} - \frac{2}{r} + \frac{2}{2}} \lesssim 1,
\]

which implies immediately that \(\frac{2}{q} + \frac{2}{r} \leq \frac{2}{2}\) by taking \(\delta \ll 1\).

4 Applications to nonlinear equations

In this section, we apply the improved Strichartz estimates to the nonlinear equations, e.g. nonlinear Schrödinger equation, nonlinear wave equation. These equations have been studied extensively.

4.1 Nonlinear Schrödinger equations

First we consider the semi-linear Schrödinger equations:

\[
i\partial_t u + \Delta u = \mu|u|^p u, \quad u(0) = u_0(x),
\]

(4.59)
where \( u(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C} \), \( n \geq 2 \), \( u_0 \in \dot{H}^s \), \( p > 0 \), \( \mu = \pm 1 \). It is easy to see that equation (4.59) is invariant under the following scaling transform: for \( \lambda > 0 \)

\[
    u(t, x) \to \lambda^{2/p} u(\lambda^2 t, \lambda x), \quad u_0(x) \to \lambda^{2/p} u_0(\lambda x).
\]

Then the space \( \dot{H}^{s_{sch}} \), where

\[
    s_{sch} = \frac{n}{2} - \frac{2}{p},
\]

is the critical space to (4.59) in the sense of scaling, namely, \( \|\lambda^{2/p} u_0(\lambda \cdot)\|_{\dot{H}^{s_{sch}}} = \|u_0\|_{\dot{H}^{s_{sch}}} \). In particular, if \( p < 4/n \), then \( s_{sch} < 0 \), which is our main concern.

The well-posedness and scattering for the nonlinear Schrödinger equation (4.59) were extensively studied. We refer the readers to [3, 1, 7, 22, 25, 9, 8] and the reference therein. It is well-known that the threshold of \( \dot{H}^s \)-wellposedness for (4.59) is \( s \geq \max(0, s_{Sch}) \). However, in the radial case we prove the following

**Theorem 4.2.** Assume \( n \geq 2 \), \( 0 < p < 4/n \), \( s_{sch} = \frac{n}{2} - \frac{2}{p}, \frac{1-n}{2n+1} \leq s_{sch} < 0 \), and \( u_0 \) is radial. Then we have

1. **Small data scattering:** If \( \|u_0\|_{\dot{H}^{s_{sch}}} \leq \delta \) for some \( \delta \ll 1 \), then there exist a unique global solution \( u \) to (4.59) such that

\[
    u \in C(\mathbb{R} : \dot{H}^{s_{sch}}) \cap L^{p(n+2)}_{t, x} (\mathbb{R} \times \mathbb{R}^n),
\]

and \( u \pm \dot{H}^{s_{sch}} \) such that \( \|u - S(t)u_\pm\|_{\dot{H}^{s_{sch}}} \to 0 \), as \( t \to \pm \infty \).

2. **Large data local well-posedness:** If \( u_0 \in \dot{H}^{s} \) for some \( s_{sch} \leq s < 0 \), then there exists \( T > 0 \) and a unique solution \( u \in C((-T, T) : \dot{H}^s) \cap L^{n-2s}_{t, x} ((-T, T) \times \mathbb{R}^n) \).

**Proof.** The proof is quite standard. The main point is to choose the resolution space. By Duhamel’s principle, we have

\[
    u = \Phi_{u_0}(u) = S(t)u_0 + \mu \int_0^t S(t-s)(|u|^{4-2s_{sch}} u)(s) \, ds.
\]

First, we show (1). Take\(^1\)

\[
    q = r = \frac{2(n+2)}{n-2s_{sch}}, \quad \tilde{q} = \tilde{r} = \frac{2(n+2)}{n+2s_{sch}}.
\]

It is easy to verify that \((q, r), (\tilde{q}, \tilde{r})\) satisfy the conditions in Proposition 3.5 with \( \gamma = s_{sch} \). Thus by applying Proposition 3.5 we get

\[
    \|\Phi_{u_0}(u)\|_{L^{q}_{t, x}} + \|D^{s_{sch}}\Phi_{u_0}(u)\|_{L^{\infty}_{t}L^{2}_{x}} \lesssim \|S(t)u_0\|_{L^{q}_{t}L^{2}_{x}} + \|u\|_{L^{q}_{t}L^{2}_{x}}^{4-n-2s_{sch}} \|u\|_{L^{q}_{t}L^{2}_{x}}^{1+\frac{4}{n-2s_{sch}}} + \|u\|_{L^{q}_{t}L^{2}_{x}}^{4-n-2s_{sch}+4} \|u\|_{L^{q}_{t}L^{2}_{x}}^{1+\frac{4}{n-2s_{sch}+4}}.
\]

Note that \( \tilde{q}' = \frac{2(n+2)}{n+2s_{sch}+4} \), then \( \frac{(n-2s_{sch}+4)\tilde{q}'}{n-2s_{sch}} = q \). Thus part (1) follows from standard fixed point arguments (3).

---

\(^1\)Indeed, the choice of index was determined by a group of linear equation or inequalities. The choice is not unique, and we choose the simple one here. We will remark more on this for the wave equation.
Next, we show part (2). Local well-posedness for equation (4.59) in $\dot{H}^{s_{sch}}$ follows from the fact that for $q = \frac{2(n+2)}{n-2s_{sch}} < \infty$

$$\lim_{T \to 0} \|S(t)u_0\|_{L^q_t L^q_x} = 0.$$ 

Now we assume $s_{sch} < s < 0$. Take $q = r = \frac{2(n+2)}{n-2s_{sch}}$ and

$$\frac{1}{q} = \frac{n+2s}{2n+4} - \frac{2n(s-s_{sch})}{(n+2)(n-2s_{sch})}, \quad \frac{1}{r} = \frac{n+2s}{2n+4} + \frac{4s-4s_{sch}}{(n+2)(n-2s_{sch})}.$$ 

It is easy to verify that $(q, r), (\tilde{q}, \tilde{r})$ satisfy the conditions in Proposition 3.5 with $\gamma = s$, and $(p+1)\tilde{r}' = q$. Thus by applying Proposition 3.5, we get for some $\theta > 0$

$$\|\Phi u_0(u)\|_{L^q_{t,x}} + \|D^s\Phi u_0(u)\|_{L^\infty L^2} \lesssim \|D^s u_0\|_{L^2} + \|u\|_{\dot{H}^{-s_{sch}} u}\|_{L^\infty_{t \in [-T,T]} L^q_{t,x}} \|D^s u_0\|_{L^2} + T^\theta \|u\|_{\dot{H}^{-s_{sch}} u}\|_{L^\infty_{t \in [-T,T]} L^q_{t,x}}.$$ 

Thus part (2) also follows from standard fixed-point argument.

Remark 4.3. In part (2) of Theorem 4.2, the existence time $T$ depends only on $\|u_0\|_{\dot{H}^s}$ for $s > s_{sch}$, but on the profile of $u_0$ for $s = s_{sch}$.

Actually, we can obtain more conclusions than Theorem 4.2. Using the similar proof, we can obtain if $s_{sch} < \frac{1-n}{2n+1}$, namely $0 < p < \frac{8n+4}{2n^2+3n-2}$, large data local well-posedness for (4.59) hold in $\dot{H}^s$ for $s > s_1$ with

$$s_1 = \begin{cases} \frac{1-n}{2n+1}, & \frac{2n+4}{n-2s_{sch}}, \quad 2 \leq n \leq \frac{8n+4}{2n^2+3n-2}; \\ \frac{1-n}{2n+1} - \frac{p-n}{2n+p}, & \frac{2n+4}{n-2s_{sch}}, \quad p \leq \frac{2}{n}. \end{cases} \quad (4.60)$$

Actually, $s_0$ is determined by the following groups of linear equations:

$$\begin{cases} 2 \leq q, r, \tilde{q}, \tilde{r} \leq \infty, \\ \frac{q}{2} + \frac{2n-1}{r} = n, \\ \frac{q}{2} + \frac{2n-1}{r} = n - \frac{1}{2}, \\ \frac{q}{2} + \frac{2n-1}{r} = n - \frac{3}{2}, \\ \frac{q}{2} + \frac{2n-1}{r} = n - \frac{5}{2}, \\ \frac{q}{2} + \frac{2n-1}{r} = \frac{4}{2}, \\ \gamma, \quad (p+1)\tilde{r}' = r, \tilde{q} = \infty. \end{cases}$$

Then we can also obtain $(q, r), (\tilde{q}, \tilde{r})$ for $s > s_0$, which can be used to prove local well-posedness as in the proof of Theorem 4.2.

Figure 2: $\dot{H}^s$ well-posedness for NLS
The same conclusions obtained above certainly hold for general nonlinear terms \( F(u) \), for example, if \( F \) satisfies
\[
|F(u)| \lesssim |u|^{p+1},
\]
\[
|u| |F'(u)| \sim |F(u)|, \tag{4.61}
\]
We describe the regularity \( s \) for \( \dot{H}^s \) local well-posedness and nonlinear increasing rate \( p+1 \) in Figure 3.

Remark 4.4. Part (2) in Theorem 4.2 also holds for data \( u_0 \in H^s \). Indeed, we simply construct the resolution space as following
\[
\|u\|_{Y_T} = \|P_{\leq 0}u\|_{L^p_t L^2_x} + \|P_{\geq 1}u\|_{L^q_t L^r_x}.
\]

4.5 Nonlinear wave equations

Next, we consider the semi-linear wave equations:
\[
\begin{cases}
\partial_t u - \Delta u = \mu |u|^p u, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
u(0) = u_0(x), \quad u_t(0) = u_1(x). \tag{4.62}
\end{cases}
\]

where \( u(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \ n \geq 2, \mu = \pm 1, u_0 \in \dot{H}^s, u_1 \in \dot{H}^{s-1}. \) It is easy to see that equation (4.62) is invariant under the following scaling transform: for \( \lambda > 0 \)
\[
u(t, x) \to \lambda^{2/p} u(\lambda t, \lambda x), \quad u_0(x) \to \lambda^{2/p} u_0(\lambda x), \quad u_1(x) \to \lambda^{(2+p)/p} u_1(\lambda x).
\]

Then the space \( \dot{H}^{s_w} \times \dot{H}^{s_{w-1}} \), where
\[
s_w = \frac{n}{2} - \frac{2}{p},
\]
is the critical space to (4.62) in the sense of scaling, namely, \( \|\lambda^{2/p} u_0(\lambda \cdot)\|_{\dot{H}^{s_w}} = \|u_0\|_{\dot{H}^{s_w}}. \)

The well-posedness and scattering for equation (4.62) were deeply studied. We refer the readers to [12, 26, 29, 30, 31, 32, 20, 40, 23] and the reference therein. In this section, we study the well-posedness theory for (4.62) in \( \dot{H}^s \times \dot{H}^{s-1} \) with radial initial data. As was indicated in the introduction, the sharp results at the critical regularity were obtained in [26] if \( s_w \geq 1/2 \). Thus we restrict ourselves to the case \( s_w < 1/2 \), and we find an threshold \( s_0(n) \) for the critical GWP in the radial case:
\[
s_0(n) = \begin{cases} 
\frac{5-\sqrt{17}}{4}, & n = 2, \\
\frac{12-\sqrt{125}}{6}, & n = 3, \\
\frac{n^2+3n-3-\sqrt{n^4+6n^3-n^2-12n+9}}{4n^2-4}, & n \geq 4.
\end{cases} \tag{4.63}
\]

It seems that this is the optimal regularity by our methods. We prove the following

Theorem 4.6. Assume \( n \geq 2, \ 0 < p < \frac{4}{n-1}, \ s_w = \frac{n}{2} - \frac{2}{p}, \ s_0(n) < s_w < 1/2 \) with \( s_0(n) \) given by (4.63), and \( u_0 \) is radial. Then

(1) If \( \|u_0\|_{\dot{H}^{s_w}} + \|u_1\|_{\dot{H}^{s_{w-1}}} \leq \delta \) for some \( \delta \ll 1 \), then there exists a unique global solution \( u \) to (4.62) such that
\[
u \in C(\mathbb{R} : \dot{H}^{s_w}) \cap C^1(\mathbb{R} : \dot{H}^{s_{w-1}}) \cap L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^n),
\]

24
where \((q,r)\) are given in the proof, and \((u_\pm,v_\pm)\) are solutions to \(\dot{H}_w^s \times \dot{H}_w^{s-1}\) such that
\[
\|u - W'(t)u_\pm\|_{\dot{H}_w^s} + \|u_t - W(t)v_\pm\|_{\dot{H}_w^{s-1}} \to 0, \text{ as } t \to \pm\infty.
\]

(2) If \(u_0 \in \dot{H}_w^s\) for some \(s_w \leq s < 1/2\), then there exists \(T > 0\) and a unique solution \(u\) defined on \((-T,T)\) such that
\[
u \in C((-T,T) : H^s) \cap C^1((-T,T) : \dot{H}_w^{s-1}) \cap L_t^q L_x^r((-T,T) \times \mathbb{R}^n),
\]
where \((q,r)\) is the index given by part (1) for \(s_w = s\).

**Proof of Theorem 4.6.** By Duhamel’s principle, we have
\[
u = \Phi(u_0,u_1(u)) = W'(t)u_0 + W(t)u_1 + \mu \int_0^t W(t-s)(|u_1|^\frac{q}{n-2s_w}u)(s)ds.
\]

First we show part (1) and explain how \(s_0\) is obtained. The main issue is to choose the admissible pairs \((q,r)\), \((\tilde{q},\tilde{r})\) so that the contraction argument is closed.\(^2\) By the choice of \((q,r)\) and \((\tilde{q},\tilde{r})\), we should have
\[
\|\Phi(u_0,u_1(u))\|_{L_t^q L_x^r} \lesssim \|W'(t)u_0\|_{L_t^q L_x^r} + \|W(t)u_1\|_{L_t^q L_x^r} + \|u\|_{n-2s_w}^\frac{q}{n-2s_w} u\|_{L_t^q L_x^r} \lesssim \|D^s u_0\|_{L^2} + \|\tilde{D}^{s_w-1} u_1\|_{L^2} + \|u\|_{L_t^q L_x^r}^\frac{1}{1} \lesssim \|u\|_{L_t^q L_x^r}^\frac{1}{1}.
\]
The inequalities above hold if \((q,r)\), \((\tilde{q},\tilde{r})\) satisfy
\[
\begin{cases}
(q,r), (\tilde{q},\tilde{r}) \text{ is n-D radial wave admissible,} \\
\frac{1}{q} + \frac{1}{r} = \frac{1}{2} - s_w, \\
\frac{1}{q} + \frac{1}{r} = \frac{1}{2} - 1 + s_w, \\
(p+1)\tilde{r} = r, \quad (p+1)\tilde{q} = q.
\end{cases}
\]
Therefore, once we find solution to \((4.64)\), then part (1) follows from standard arguments. We give a solution to \((4.64)\) case by case:

**Case 1:** \(\frac{1}{2n} < s_w \leq 1/2\).
\(\begin{align*}
(q,r) &= \left(\frac{2n+2}{n-2s_w}, \frac{2n+2}{n-2s_w}\right), \\
(\tilde{q},\tilde{r}) &= \left(\frac{2n+2}{n+2s_w-2}, \frac{2n+2}{n+2s_w-2}\right).
\end{align*}\)

**Case 2:** \(s_0 < s_w \leq \frac{1}{2n}\).

**Case 2a:** \(n = 2\).
\(\begin{align*}
(q,r) &= \left(\frac{3-s_w}{1-s_w}, \frac{3-s_w}{1-s_w}\right), \\
(\tilde{q},\tilde{r}) &= \left(\frac{1}{s_w}, \infty\right).
\end{align*}\)

**Case 2b:** \(n = 3\). For some \(0 < \theta \ll 1\),
\(\begin{align*}
\left(\frac{1}{q}, \frac{1}{r}\right) &= (2s_w - 3\theta, \frac{1}{2} - s_w + \theta), \\
(\tilde{q},\tilde{r}) &= \left(\frac{q}{q-p-1}, \frac{r}{r-p-1}\right).
\end{align*}\)

**Case 2c:** \(n \geq 4\).
\(\begin{align*}
(q,r) &= \left(\frac{2n+8-4s_w}{n-2s_w}, \frac{2n^2+8n-4ns_w}{n^2+3n-4ns_w+4s_w^2-6s_w}\right), \\
(\tilde{q},\tilde{r}) &= \left(\frac{2n}{n+2s_w-3}\right).
\end{align*}\)

\(^2\)The ideas for the Schrödinger equations are the same. However, the choice of the index for the wave equations is more complicated.
Therefore, part (1) is proved.

Next we show part (2). Local well-posedness in $\dot{H}^{s_w}$ follows from the fact that for the choice of $(q,r)$ in the proof of part (1)
\[
\lim_{T \to 0} \|W'(t)u_0\|_{L^q_{t \in [-T,T]}L^2_x} + \|W(t)u_1\|_{L^q_{t \in [-T,T]}L^r_x} = 0.
\]

Now we assume $s_w < s < 1/2$. The proof is very similar to the Schrödinger equations. We take $(\tilde{q}, \tilde{r})$ to be the one corresponding to $s$ in part (1), and then take $(\tilde{q}, \tilde{r})$ to close the argument. We omit the details.

**Remark 4.7.** As the Schrödinger equation, if $s_w \leq s_0(n)$, namely $p \leq \frac{4}{n-2s_0(n)}$, we can’t prove well-posedness in $\dot{H}^s \times \dot{H}^{s-1}$ down to $s = s_w$. However, we can also improve the well-posedness results in [26]. We only mention the case $n \geq 4$, we obtain if $\frac{4}{n} < p \leq \frac{4}{n-2s_0(n)}$, then large data local well-posedness hold in $\dot{H}^s \times \dot{H}^{s-1}$ for $s > s_2$ with
\[
s_2 = \frac{np-3}{2np+2n-2}.
\]
Indeed, take $\tilde{q} = 2, \tilde{r} = \frac{2n}{n-3+2s}$, and $(q,r)$ such that
\[
\frac{1}{q} = \frac{n}{2} - \frac{n}{r} - s, \quad \frac{1}{r} = \frac{1}{p+1} - \frac{1}{(p+1)\tilde{r}}.
\]
Then by this choice we can prove the local well-posedness using the similar arguments as the proof of Theorem 4.6.

The same results hold for general nonlinear terms $F(u)$, e.g. $F$ satisfying (4.61). We describe the regularity $s$ for $\dot{H}^s \times \dot{H}^{s-1}$ local well-posedness and nonlinear increasing rate $p+1$ for (4.62) in Figure 4.

\[
\begin{array}{c}
\text{Figure 3: } \dot{H}^s \times \dot{H}^{s-1} \text{ well-posedness for NLW}
\end{array}
\]

4.8 Nonlinear fractional-order Schrödinger equation

In this section, we apply the improved Strichartz estimates to the nonlinear fractional-order Schrödinger equation:
\[
i\partial_t u + (\sqrt{-\Delta})^\sigma u = \mu |u|^p u, \quad u(0) = u_0(x),
\] (4.65)
where $u(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$, $n \geq 2$, $1 < \sigma < 2$, $\mu = \pm 1$, $u_0 \in \dot{H}^s$. To the best of our knowledge, there are few results concerning the well-posedness for (4.65). The main reason is that the usual Strichartz estimates derived by the decay estimates have a loss in derivatives except the trivial ones $L^\infty_t L^2_x$. Then one may need to use other methods, for example, local smoothing effect methods, and using of the $X^{s,b}$ space. These methods will certainly be able to provide some results at least when $p$ is an even integer.
Then we see the space \( \dot{H}^{s_c} \) of (4.65). For the simplicity of notation, we denote \( S \).

Assume Theorem 4.10.

Proof. Since (4.65) \( \dot{H}^{s_c} \) has the following two symmetries which we will use. One is the scaling invariance: for any \( \lambda > 0 \), (4.65) is invariant under the following transformation

\[
u(t, x) \rightarrow \lambda^{\sigma/p} u(\lambda^{\sigma} t, \lambda x), \quad u_0(x) \rightarrow \lambda^{\sigma/p} u_0(\lambda x).
\]

The others are the conservation laws: if \( u \) is smooth solution to (4.65), then

\[
\frac{d}{dt} \int_{\mathbb{R}^n} |u|^2 dx = 0, \quad (mass)
\]

\[
\frac{d}{dt} \int_{\mathbb{R}^n} |\nabla|^{\sigma/2} u|^2 - \frac{\mu}{p+2} |u|^{p+2} dx = 0. \quad (energy)
\]

Then we see the space \( \dot{H}^{s_c} \), where

\[
s_c = \frac{n}{2} - \frac{\sigma}{p}
\]

is critical in the sense of scaling, and \( \mu = -1 \) is the defocusing case while \( \mu = 1 \) corresponds to the focusing case. We will use the following lemma:

**Lemma 4.9** (Fractional chain rule, [5]). Suppose \( G \in C^1(\mathbb{C}) \), \( s \in (0, 1] \), and \( 1 < p, p_1, p_2 < \infty \) are such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Then

\[
\| |\nabla|^s G(u)\|_{p_2} \lesssim \| G'(u)\|_{p_1} \| |\nabla|^s u\|_{p_2}.
\]

In view of the conservation laws, we only consider the nonlinear terms between mass-critical to energy-critical, namely, \( \frac{2\sigma}{n} \leq p \leq \frac{2\sigma}{n-\sigma} \). First we consider the critical \( \dot{H}^{s_c} \) well-posedness theory of (4.65). For the simplicity of notation, we denote \( S_\sigma(t) = e^{it(\sqrt{-\Delta})^\sigma} \). We prove the following

**Theorem 4.10.** Assume \( n \geq 2 \), \( \frac{2n}{2n-1} \leq \sigma < 2 \), \( p \geq \frac{2\sigma}{n} \) \( s_c = \frac{n}{2} - \frac{\sigma}{p} \), and \( u_0 \in H^{s_c} \) is radial. Then the IVP (4.65) admits

1. Small data scattering: If \( \| u_0 \|_{H^{s_c}} \leq \delta \) for some \( \delta \ll 1 \), then there exists a unique global solution

\[
u \in C(\mathbb{R} : H^{s_c}) \cap L^{p+2}_{t} \frac{2\sigma}{2n-\sigma+np} \dot{L}^{\frac{2n}{n-\sigma+np}}_{t,x}(\mathbb{R} \times \mathbb{R}^n),
\]

and \( u_\pm \in H^{s_c} \) such that \( \| u - S_\sigma(t)u_\pm \|_{H^{s_c}} \to 0 \), as \( t \to \pm \infty \).

2. Large data local well-posedness: There exists \( T = T(u_0) > 0 \) and a unique solution \( u \in C((-T, T) : H^{s_c}) \cap L^{p+2}_{t} \frac{2\sigma}{2n-\sigma+np} \dot{L}^{\frac{2n}{n-\sigma+np}}_{t,x}((-T, T) \times \mathbb{R}^n) \).

Proof. Since \( \sigma \geq \frac{2n}{2n-1} \), then \( \frac{2(n+\sigma)}{n} \geq \frac{2(2n+1)}{2n-1} \). Thus it is easy to see that \( (2 + \frac{2\sigma}{n}, 2 + \frac{2\sigma}{n}) \) is an n-D radial Schrödinger admissible pair and then by Proposition 3.9 we get

\[
\| S_\sigma(t)u_0 \|_{L^{\frac{2n}{n-\sigma+np}}_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| u_0 \|_{L^2_x}.
\]

Then interpolating this with the trivial one \( \| S_\sigma(t)u_0 \|_{L^{p+2}_{t} \dot{L}^{\frac{2n}{n-\sigma+np}}_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| u_0 \|_{L^2_t} \), we get more estimates. The key point is that these Strichartz estimates are without loss of regularity.

With these estimates, the proof is quite standard, for example see [25]. First we show part (1). By Duhamel’s principle, we have

\[
u = \Phi_{u_0}(u) = S_\sigma(t)u_0 + \mu \int_0^t S_\sigma(t-s)(|u|^pu)(s)ds,
\]
Then part (2) follows from standard fixed-point argument too. 

It is easy to verify that \((q, r), (\tilde{q}, \tilde{r})\) satisfy the conditions in Proposition 3.9 with \(\gamma = 0\). Then we define the set \(X = B_1 \cap B_2\) endowed with the metric \(d(u, v) := \|u - v\|_{L^q_t L^r_x}\), where

\[
B_1 = \{u \in L^\infty_t H^{s_c}_x(\mathbb{R} \times \mathbb{R}^n) : \|u\|_{L^\infty_t H^{s_c}_x} \leq 2\|u_0\|_{H^{s_c}_x} + C(n)(2\eta)^{1+p}\}, \\
B_2 = \{u \in L^q_t W^{s_c+r}_x(\mathbb{R} \times \mathbb{R}^n) : \|u\|_{L^q_t W^{s_c+r}_x} \leq 2\eta, \|u\|_{L^q_t L^r_x} \leq 2C(n)\|u_0\|_{L^r_x}\},
\]

with some sufficient small \(\eta > 0\) to be determined latter. It’s easy to see that \((X, d)\) is complete and we will show that the solution map \(\Phi_{u_0}\) is a contraction on \((X, d)\) with the initial data condition

\[
\|u_0\|_{H^{s_c}} \leq \eta \ll 1. 
\] (4.66)

First we show \(\Phi_{u_0} : X \rightarrow X\). Since \(q' = \frac{p(2)}{p+1}, \ r' = \frac{2n(p+2)}{2(n+\sigma)+np}\), then it is easy to see that

\[
\frac{1}{q'} = \frac{1}{q} + \frac{1}{pq}, \quad \frac{1}{r'} = \frac{1}{r} + \frac{2\sigma}{n(p+2)}.
\]

Then by Proposition 3.3 fractional chain rule Lemma 1.9 and Sobolev embedding, we find that for \(u \in X\),

\[
\|\Phi_{u_0}(u)\|_{L^q_t H^{s_c}_x(I \times \mathbb{R}^n)} \leq \|u_0\|_{H^{s_c}_x} + C(n)(\|\nabla\|^{s_c}(|u|^p)u\|_{L^{q'}_t L^{r'}_x} \\
\leq \|u_0\|_{H^{s_c}_x} + C(n)(\|\nabla\|^{s_c}u\|_{L^q_t L^r_x} \{\|p\}^{\frac{n(p+2)}{2(p+2)}} \\
\leq \|u_0\|_{H^{s_c}_x} + C(n)(2\eta + 2C(n))\|u_0\|_{L^r_x}\|\nabla\|^{s_c}u\|_{L^q_t L^r_x} \\
\leq \|u_0\|_{H^{s_c}_x} + C(n)(2\eta + 2C(n))\|u_0\|_{L^r_x}(2\eta)^p
\]

and similarly,

\[
\|\Phi_{u_0}(u)\|_{L^{q'}_t L^{r'}_x} \leq C(d)\|u_0\|_{L^r_x} + C(d)\|(|u|^p)u\|_{L^{q'}_t L^{r'}_x} \\
\leq C(d)\|u_0\|_{L^r_x} + 2C(d^2)\|u_0\|_{L^r_x}(2\eta)^p,
\]

and

\[
\|\nabla\|^{s_c}\Phi_{u_0}(u)\|_{L^q_t L^r_x} \leq \|\nabla\|^{s_c}\nabla\|^{s_c}S_\eta(t)u_0\|_{L^q_t L^r_x} + C(n)(2\eta)^{p+1} \\
\leq C(n)\eta + C(n)(2\eta)^{p+1}.
\]

Thus, choosing \(\eta_0 = \eta_0(n)\) sufficiently small, we see that for \(0 < \eta \leq \eta_0\), the functional \(\Phi_{u_0}\) maps the set \(X\) back to itself. To see that \(\Phi_{u_0}\) is a contraction, we repeat the computations above and get for \(u, v \in X\)

\[
\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{L^q_t L^r_x} \leq C(d)\|(|u|^p)u - (|v|^p)v\|_{L^{q'}_t L^{r'}_x} \\
\leq C(d)(2\eta)^p\|u - v\|_{L^q_t L^r_x}
\]

Thus for \(\eta\) sufficiently, the map \(\Phi_{u_0}\) is a contraction. By the contraction mapping theorem, it follows that \(\Phi_{u_0}\) has a fixed point in \(X\). The rest of part (1) (e.g. the uniqueness) follows from standard arguments [25].

Next, to show part (2), we see that since \(q \neq \infty\), then

\[
\lim_{T \rightarrow 0} \|\nabla\|^{s_c}\nabla\|^{s_c}S_\eta(t)u_0\|_{L^q_{[-T,T]} L^r_x} = 0.
\]

Then part (2) follows from standard fixed-point argument too. □

28
Using the similar arguments above, and in view of the conservation laws, it is not difficult to prove the following corollary for which we do not give the proof.

**Corollary 4.11** (H\(^s\) subcritical). Assume \( n \geq 2, \frac{2n}{2n-1} < \sigma < 2 \) and \( u_0 \) is radial. Then for 
\[
0 < p < \frac{2n}{n-1}, \quad \text{the IVP } (4.65) \text{ is globally well-posed if } u_0 \in L^2; \quad \text{and for } \frac{2n}{n} \leq p < \frac{2n}{n-2}, \quad \text{the IVP } (4.65) \text{ is locally well-posed (globally well-posed in the defocusing case) if } u_0 \in H^{\sigma/2}.
\]

Indeed, we can prove some other subtle well-posedness results. We can also go below \( L^2 \), as long as \( \sigma \) is close to 2. However, we do not pursue this. On the other hand, in the \( H^s \)-critical case, we assumed \( u_0 \in H^{\infty} \) instead of \( u_0 \in H^{\infty} \) as in the work of Cazenave and Weissler [3]. This makes the proof much simpler [25]. We will address this in our consequent works which will concern the large data scattering theory for (4.65).

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