Free Fermions on a Piecewise Linear Four-Manifold. II: Pachner Moves

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Abstract. This is the second in a series of papers where we construct an invariant of a four-dimensional piecewise linear manifold $M$ with a given middle cohomology class $h \in H^2(M, \mathbb{C})$. This invariant is the square root of the torsion of unusual chain complex introduced in Part I of our work, multiplied by a correcting factor. Here we find this factor by studying the behavior of our construction under all four-dimensional Pachner moves, and show that it can be represented in a multiplicative form: a product of same-type multipliers over all 2-faces, multiplied by a product of same-type multipliers over all pentachora.

Keywords. Piecewise linear manifolds, Four-manifolds, Topological quantum field theory, Fermions.

1. Introduction

This is the continuation of paper [7]; we also call this latter ‘Part I’, while the present paper ‘Part II’. For notations, conventions and prior results we refer to Part I, and assume that the reader has digested it. We refer to formulas, definitions, etc. from Part I in the following format: formula (I.5) means formula 5 in Part I, etc.

A standard way to define a piecewise linear (PL) manifold is via its triangulation. In this paper, we will be dealing with four-dimensional manifolds, and a triangulation of such manifold means that it is represented as a union of 4-simplices, also called pentachora, glued together in a proper way that can be described purely combinatorially. An invariant of such manifold is a quantity that may need an actual triangulation for its calculation, but must not depend on this triangulation.
A theorem of Pachner \[9\] states that a triangulation of a PL manifold can be transformed into any other triangulation using a finite sequence of Pachner moves; monograph \[8\] can be recommended as a pedagogical introduction to this subject. And, as indicated in \[8, Sect. 1\], in order to construct invariants of PL manifolds, it makes sense to construct algebraic relations corresponding to Pachner moves, also called their \textit{algebraic realizations}.

It turns out that very interesting mathematical structures appear if we begin constructing a realization of four-dimensional Pachner moves by ascribing a Grassmann–Gaussian weight to each pentachoron. Here ‘Gaussian’ means that this weight is proportional to the exponential of a quadratic form, and ‘Grassmann’ means that this form depends on \textit{anticommuting} Grassmann variables. Each Grassmann variable is supposed to live on a 3-face (tetrahedron) of a pentachoron, and gluing two pentachora along a 3-face corresponds to \textit{Berezin integration} w.r.t. the corresponding variable. A large family of such realizations for Pachner move 3–3 was discovered in paper \[4\], and then a full parameterization for (a Zariski open set of) such relations was found in \[5\]. A beautiful fact is that this nonlinear parameterization goes naturally in terms of a \textit{2-cocycle} given on both initial and final configurations of the Pachner move; we call these respective configurations (clusters of pentachora) the \textit{left-} and \textit{right-hand side} (l.h.s. and r.h.s.) of the move.

There was, however, one unsettled problem with the realization of move 3–3 in \[5\]: not all involved quantities were provided with their explicit expressions in terms of the 2-cocycle. In particular, Theorem 9 in \[5\] was just an existence theorem for the proportionality coefficient between the Berezin integrals representing the l.h.s. and r.h.s. of the move, while this coefficient is crucial for constructing an invariant for a whole ‘big’ manifold. Also, it remained to find realizations for the rest of Pachner moves, namely 2–4 and 1–5.

One possible solution to the problem with the coefficient was proposed in \[6\], in a rather complicated way combining computational commutative algebra with a guess-and-try method, and leaving the feeling that the algebra behind it deserves more investigation.

It turns out that a more transparent way to solving the mentioned problem with the coefficient appears if we consider this problem together with constructing formulas for moves 2–4 and 1–5. This is what we are doing in the present paper: we provide all necessary formulas for coefficients, and in a multiplicative form suitable for ‘globalizing’, that is, transition to a formula for the whole manifold. To be exact, we construct an invariant of a pair \((M, h)\), where \(M\) is a four-dimensional piecewise linear manifold, and \(h \in H^2(M, \mathbb{C})\) is a given middle cohomology class.

One algebraic problem still remains unsolved though, namely, a formal proof of what we have to call ‘Conjecture’, see Sect. 6.4, and what is actually a firmly established mathematical fact. That is simply a formula involving ten indeterminates (over field \(\mathbb{C}\)), whose both sides are composed using the four arithmetic operations and also square root signs and parentheses. The experience gained in the previous work \[3,4\] led the author to the idea that such formula exists—and indeed, the reader can check it on a computer by