(\lambda, \mu) \textbf{Hesitant fuzzy subalgebras of Boolean algebras}

Jiang Man

Public Course Department, Xi’an traffic engineering institute, Xi’an 710300, China
Email: jiangman2008@yeah.net

Abstract. In this paper, the concept of (\lambda, \mu)-hesitant fuzzy subalgebras is introduced in Boolean algebra. Some properties of (\lambda, \mu)-hesitant fuzzy subalgebras are discussed. Finally, we proved that the intersection and direct product of two (\lambda, \mu)-hesitant fuzzy subalgebras are also (\lambda, \mu)-hesitant fuzzy subalgebras in Boolean algebra.

1. Introduction

After Zadeh put forward fuzzy set [1], many scholars combine fuzzy set with algebraic system and draw important conclusions. Later, Spanish scholar torra put forward the concept of hesitant fuzzy set [2]. Hesitant fuzzy number is more comprehensive than traditional fuzzy element. It contains different groups with a certain degree of hesitation. It has been applied in many mathematical models. Liu pengde et al. Generalized hesitant fuzzy sets and proposed the concepts of interval valued intuitionistic hesitant fuzzy sets [3] and interval valued hesitant fuzzy sets [4-5]. Combining algebraic structure with the concept of fuzzy set is an important method to study algebraic structure. In 1984, Liu Xuhua put forward fuzzy Boolean algebra [6]. The research ideas and methods of fuzzy sets were applied to Boolean algebra, and many meaningful conclusions were obtained. For example, sun Shaoquan proposed the concepts of fuzzy subalgebras, fuzzy ideals and fuzzy congruence relations of Boolean algebras in references [7] and [8], gave the isomorphism theorem of quotient Boolean algebras, and obtained the basic theorem of homomorphism of fuzzy subalgebras of Boolean algebras. Sun Zhongpin proposed (\lambda, \mu)-fuzzy subalgebras and (\lambda, \mu)-fuzzy ideals of Boolean algebras in references [9], and discussed some properties of them in [10]. Wang fengxiao studied I-V fuzzy ideals of Boolean algebras, and concluded that the intersection, homomorphic image and direct product of I-V fuzzy ideals of Boolean algebras are also I-V fuzzy ideals. In this paper, on the basis of references [11]-[14], we study Boolean algebras by using the research ideas and methods of hesitant fuzzy sets, propose the concept of (\lambda, \mu) hesitant fuzzy subalgebras of Boolean algebras, obtain some equivalent characterizations, discuss the relationship between the image and the original image of hesitant fuzzy subalgebras under homomorphic mapping, give the definition of direct product under hesitant fuzzy sets, and study the correlation of (\lambda, \mu) hesitant fuzzy direct product. The related results further enrich and perfect the theoretical research of fuzzy algebra and Boolean algebra.

2. Preliminaries

Definition 2.1[6] An algebraic system \( < R;+,\cdot,0,1 > \) with two binary algebraic operations \( +,\cdot \) is called Boolean algebra. If there are at least two different elements, and \( \forall a,b,c \in R \), the following
axiom holds:
(1) \( a + b = b + a, ab = ba \) where \( ab \) is \( a \circ b \);
(2) \( (a+b)+c = a+(b+c), \quad a + (bc) = (a + b)(a + c) \);
(3) \((a\circ b)+c = a\circ(b+c), \quad (ab)c = a(bc)\);
(4) \( \exists 0.1 \in R, a + 0 = a, \quad a1 = a \);
(5) \( \forall a, \in R, \quad \exists \overline{a} \in R, \text{satisfies } a + \overline{a} = 1, \quad a\overline{a} = 0 \).

Boolean algebra \(< R; +, \cdot, 0, 1 >\) is also recorded as \(< R; +, \cdot, 0, 1, 1 >\).

If there is no special explanation, Boolean algebra \(< R; +, \cdot, 0, 1 >\) is abbreviated as \(R\), Boolean algebra \(< R; +, \cdot, 0, 1 >\) is abbreviated as \(R\).

Definition 2.2[7] let \( \mu : \{0, 1\} \to [0, 1] \) be a fuzzy subset of \(R\), if for any \(a, b \in R\), the following condition holds:
(1) \( A(a + b) \geq A(a) \land A(b) \);
(2) \( A(ab) \geq A(a) \land A(b) \);
(3) \( A(\overline{a}) \geq A(a) \).

Then \(A\) is called a fuzzy subalgebra of \(R\).

Definition 2.3[6] let a mapping \( f : R \to R'\) be called a homomorphic mapping from an algebraic system \(R\) to an algebraic system \(R'\). If all the algebraic operations of the algebraic system are preserved, the mapping \(f\) satisfies the following conditions:
(1) \( \forall a, b \in R, \quad f(a + b) = f(a) + f(b) \);
(2) \( \forall a, b \in R, \quad f(ab) = f(a) \bullet f(b) \);
(3) \( \forall a \in R, \quad f(\overline{a}) = \overline{f(a)} \).

Definition 2.4[8] let to be a Boolean algebra, a mapping \( f : R \to R'\) is called homomorphic mapping of \(R\) to \(R'\) if and only if:
(1) \( \forall a, b \in R, \quad f(a + b) = f(a) + f(b) \);
(2) \( \forall a \in R, \quad f(\overline{a}) = \overline{f(a)} \).

Definition 2.5[2] Let \(X\) be a reference set. Then a hesitant fuzzy set \( \tilde{F} \) in \(X\) is represented mathematically as:
\[
\tilde{F} : \{(x, h_F(x)) : h_F(x) \in p([0, 1]), x \in X\}
\]
where \(p([0, 1])\) is the power set of \([0, 1]\).

So, we can define a set of fuzzy sets an HFS by union of their membership functions.

Let \( \tilde{F} \) be the hesitation fuzzy set in \(X\), \(p([0, 1])\) is a Power set of interval\([0,1]\),the set \(X(\tilde{F}, \gamma) = \{x \in X : \gamma \subseteq h_F(x)\}\) is called a hesitation level set of \(\tilde{F}\),and \(\gamma \in p([0, 1])\).

Definition 2.6[2] Let \(X\) be a reference set, \(\tilde{F}, \tilde{G} \in \text{HFE}(X)\). Then the operations complement, union and intersection are defined as follows:
(1) \( h_{\tilde{F}}(x) = \neg h_F(x) = \bigcup_{\gamma \subseteq h_F(x)} \{1 - \gamma\} \);
(2) \( h_{\tilde{F} \cup \tilde{G}}(x) = h_F(x) \cup h_G(x) = \{h \in h_F(x) \cup h_G(x) | h \geq \max\{h_F(x), h_G(x)\}\} \);
(3) \( h_{\tilde{F} \cap \tilde{G}}(x) = h_F(x) \cap h_G(x) = \{h \in h_F(x) \cap h_G(x) | h \geq \min\{h_F(x), h_G(x)\}\} \).

3. Hesitant fuzzy subalgebras of Boolean algebras

In this section, we give the definition of \((\lambda, \mu)\) hesitant fuzzy subalgebras and discuss their properties.

The following general assumptions.
Definition 3.1 let \( \tilde{A} \) be a hesitant fuzzy set on \( R \), if \( \forall a, b \in R \), the following conditions are true:

1. \( h_{\tilde{A}}(a + b) \supseteq h_{\tilde{A}}(a) \cap h_{\tilde{A}}(b) \);
2. \( h_{\tilde{A}}(ab) \supseteq h_{\tilde{A}}(a) \cap h_{\tilde{A}}(b) \);
3. \( h_{\tilde{A}}(\bar{a}) \supseteq h_{\tilde{A}}(a) \).

Then \( \tilde{A} \) is a hesitant fuzzy subalgebra on \( R \).

Definition 3.2 let \( \tilde{A} \) be a hesitant fuzzy set on \( R \), if \( \forall a, b \in R \), the following conditions are true:

1. \( h_{\tilde{A}}(a + b) \cup \lambda \supseteq h_{\tilde{A}}(a) \cap h_{\tilde{A}}(b) \cap \mu \);
2. \( h_{\tilde{A}}(ab) \cup \lambda \supseteq h_{\tilde{A}}(a) \cap h_{\tilde{A}}(b) \cap \mu \);
3. \( h_{\tilde{A}}(\bar{a}) \cup \lambda \supseteq h_{\tilde{A}}(a) \cap \mu \).

Then \( \tilde{A} \) is a \((\lambda, \mu)\) hesitant fuzzy subalgebra on \( R \). Let \((\lambda, \mu)HFS[R]\) be all the hesitant fuzzy sets on \( R \).

Theorem 3.1 Let \( \tilde{A} \in HFS[R] \), then \( \tilde{A} \in (\lambda, \mu)HFS[R] \) if and only if the conditions (1) and (3) in definition 3.2 holds and:

1. \( h_{\tilde{A}}(a + b) \cup \lambda \supseteq h_{\tilde{A}}(a) \cap h_{\tilde{A}}(b) \cap \mu \).

Proof. The necessity is obvious. To prove the sufficiency, we only need to prove the condition (2) in definition 3.1. In fact, for \( \forall a, b \in R \),

\[ h_{\tilde{A}}(ab) \cup \lambda = h_{\tilde{A}}(\bar{a} + \bar{b}) \cup \lambda \supseteq h_{\tilde{A}}(a) \cap h_{\tilde{A}}(b) \cap \mu \supseteq h_{\tilde{A}}(a) \cap h_{\tilde{A}}(b) \cap \mu. \]

Therefore, condition (2) in definition 3.2 holds.

Theorem 3.2 A hesitant fuzzy subalgebra on \( R \) must be a \((\lambda, \mu)\) hesitant fuzzy subalgebra.

Proof. Suppose that \( \tilde{A} \in HFS[R] \). Let \( 0 \leq \lambda < \mu \leq 1 \), then \( h_{\tilde{A}}(a + b) \cup \lambda \supseteq h_{\tilde{A}}(a) \cap h_{\tilde{A}}(b) \supseteq h_{\tilde{A}}(a) \cap h_{\tilde{A}}(b) \cup \lambda \)

\[ \supseteq h_{\tilde{A}}(\bar{a}) \supseteq h_{\tilde{A}}(a) \cap \mu. \]

Then according to definition 3.2, \( \tilde{A} \in (\lambda, \mu)HFS[R] \).

Theorem 3.3 Let \( \tilde{A} \in (\lambda, \mu)HFS[R] \), then \( h_{\tilde{A}}(0) \cup \lambda \supseteq h_{\tilde{A}}(a) \cap \mu, \forall a \in R \).

Proof. For all \( a \in R \), \( h_{\tilde{A}}(0) \cup \lambda = h_{\tilde{A}}(a) \supseteq h_{\tilde{A}}(a) \cap \mu \).

Theorem 3.4 Let \( \tilde{A} \in HFS[R] \), then \( \tilde{A} \in (\lambda, \mu)HFS[R] \) if and only if \( \forall \gamma \in R(P([0,1]), R(\tilde{A}, \gamma) \neq \phi \), then \( R(\tilde{A}, \gamma) \) is a subalgebra of \( R \). Among them \( R(\tilde{A}, \gamma) = \{ x \in R | \gamma \subseteq h_{\tilde{A}}(x) \} \).

Proof. Let \( \tilde{A} \in (\lambda, \mu)HFS[R], \forall \gamma \in (\lambda, \mu) \), if \( R(\tilde{A}, \gamma) \neq \phi \), \( \forall a, b \in R(\tilde{A}, \gamma) \), then \( h_{\tilde{A}}(a) \supseteq \gamma \), \( h_{\tilde{A}}(b) \supseteq \gamma \). Therefore \( h_{\tilde{A}}(a + b) \supseteq h_{\tilde{A}}(a) \cap h_{\tilde{A}}(b) \supseteq \gamma \), that is \( h_{\tilde{A}}(a + b) \supseteq \gamma \), so \( a + b \in R(\tilde{A}, \gamma) \).

Similarly \( ab \in R(\tilde{A}, \gamma) \), \( \tilde{A} \in R(\tilde{A}, \gamma) \). So \( R(\tilde{A}, \gamma) \) is a subalgebra of \( R \).

Theorem 3.5 Let \( \tilde{A} \in (\lambda, \mu)HFS[R], \tilde{B} \in (\lambda, \mu)HFS[R] \), then \( \tilde{A} \cap \tilde{B} \in (\lambda, \mu)HFS[R] \).

Proof. Assume that \( \tilde{A} \in (\lambda, \mu)HFS[R] \) and \( \tilde{B} \in (\lambda, \mu)HFS[R] \), for \( a, b \in R \), then \( h_{\tilde{A} \cap \tilde{B}}(a + b) \cup \lambda \).
\[(h_\lambda(a + b) \cap h_{\lambda}(a + b)) \cup \lambda = (h_\lambda(a + b) \cup \lambda) \cap (h_{\lambda}(a + b) \cup \lambda) \supseteq (h_\lambda(a) \cap h_{\lambda}(b) \cap \mu) \cup (h_{\lambda}(a) \cap h_{\lambda}(b) \cap \mu) \cap \mu = h_{\lambda}(a) \cap h_{\lambda}(b) \cap \mu \]. Similarly available: \[h_{\lambda}(a) \cap h_{\lambda}(b) \cap \mu, h_{\lambda}(a) \cap h_{\lambda}(b) \cap \mu \] and \[\lambda \supseteq h_{\lambda}(a) \cap h_{\lambda}(b) \cup \mu \]. According to definition 3.2 that \([A] \cap [B] \in (\lambda, \mu)HFS[R]\).

**Definition 3.3** [13] Let \(R_1\) and \(R_2\) be Boolean algebras, \(f : R_1 \rightarrow R_2\) is homomorphic mapping,

\(A \in HF[R_1], B \in HF[R_2]\), then two hesitant fuzzy sets \(f(A)\) and \(f^{-1}(B)\) can be induced from \(f: h_{f(A)}(y) = \bigcup_{f(x) = y} h_{\lambda}(x), y \in R_2, h_{f^{-1}(B)}(x) = h_{\lambda}(f(x)), x \in R_2\).

Note: the definition of \(h_{f(A)}\) is equivalent to \(\forall y \in R_2, when\)

\[f^{-1}(y) \neq \emptyset, h_{f(A)}(y) = \bigcup_{f(x) = y} h_{\lambda}(x); f^{-1}(y) = \emptyset, h_{f(A)}(y) = \emptyset.\]

**Theorem 3.6** \(f: R \rightarrow R_2\) is the homomorphic surjective of Boolean algebra \(R_1\) to \(R_2, A \in HF[R_1],\) if \(\tilde{A} \in (\lambda, \mu)HFS[R_1], then f(\tilde{A}) \in (\lambda, \mu)HFS[R_2]\).

Proof. If \(\tilde{A} \in (\lambda, \mu)HFS[R_1], \forall y_1, y_2 \in R_2, for f : R_1 \rightarrow R_2\) is the homomorphic surjective of Boolean algebra, so there must be \(x_1, x_2 \in R_1\) such that \(f(x_1) = y_1, f(x_2) = y_2\).

\[h_{f(A)}(y_1) \cup y_2 \cup \lambda = (\bigcup_{f(x_1) = y_1, y_2} h_\lambda(x_1 + x_2)) \cup \lambda = (\bigcup_{f(x_1) = y_1, y_2} (h_\lambda(x_1 + x_2) \cup \lambda)) \cong h_{f(A)}(y_1) \cap h_{f(A)}(y_2) \cap \mu.

h_{f(A)}(y_1) \cap y_2 \cup \lambda = (\bigcup_{f(x_1) = y_1} h_\lambda(x_1 + x_2)) \cup \lambda = (\bigcup_{f(x_1) = y_1} (h_\lambda(x_1 + x_2) \cup \lambda)) \cong \bigcup_{f(x_1) = y_1} h_\lambda(x_1) \cap \mu = h_{f(A)}(y_1) \cap \mu.

According to definition 3.2 that \(f(\tilde{A}) \in (\lambda, \mu)HFS[R_2].\)

**Theorem 3.7** \(f: R \rightarrow R_2\) is the homomorphic surjective of Boolean algebra \(R_1\) to \(R_2, \tilde{B} \in HF[R_2],\) if \(\tilde{B} \in (\lambda, \mu)HFS[R_2], then f^{-1}(\tilde{B}) \in (\lambda, \mu)HFS[R_1].\)

Proof. Necessity If \(\tilde{B} \in (\lambda, \mu)HFS[R_2], \forall y_1, x_2 \in R_1, then h_{f^{-1}(\tilde{B})}(x_1 + x_2) \cup \lambda \equiv h_\lambda(f(x_1 + x_2)) \cup \lambda \equiv h_\lambda(f(x_1)) \cap h_{f^{-1}(\tilde{B})}(x_2) \cap \mu.

h_{f^{-1}(\tilde{B})}(x_1 + x_2) \cup \lambda = h_\lambda(f(x_1)) \cap h_{f^{-1}(\tilde{B})}(x_2) \cap \mu = h_{f^{-1}(\tilde{B})}(x_1) \cap h_{f^{-1}(\tilde{B})}(x_2) \cap \mu.

\[\forall y_1, x_2 \in R_2, we have f(x_1) = y_1, so then f(\tilde{B}) = \tilde{y}_1, and h_{f^{-1}(\tilde{B})}(x_1) \cup \lambda = h_\lambda(f(x_1)) \cup \lambda = h_\lambda(y_1) \cup \lambda \equiv h_\lambda(y_1) \cap \mu = h_\lambda(y_1) \cap \mu.

Therefore \(f^{-1}(\tilde{B}) \in (\lambda, \mu)HFS[R_1].\)

Adequacy \(\forall y_1, y_2 \in R_2, for f : R_1 \rightarrow R_2\) is the homomorphic surjective of Boolean algebra \(R_1\) to \(R_2\), so there must be \(x_1, x_2 \in R_1\) such that \(f(x_1) = y_1, f(x_2) = y_2\).

If \(f^{-1}(\tilde{B}) \in (\lambda, \mu)HFS[R_1], then h_{f^{-1}(\tilde{B})}(x_1) \cap h_{f^{-1}(\tilde{B})}(x_2) \cap \mu = h_\lambda(f(x_1)) \cap h_{f^{-1}(\tilde{B})}(x_2) \cap \mu = h_\lambda(y_1) \cap h_{f^{-1}(\tilde{B})}(x_2) \cap \mu.

h_{f^{-1}(\tilde{B})}(x_1) \cap h_{f^{-1}(\tilde{B})}(x_2) \cap \mu = h_\lambda(f(x_1)) \cap h_{f^{-1}(\tilde{B})}(x_2) \cap \mu = h_\lambda(y_1) \cap h_{f^{-1}(\tilde{B})}(y_2) \cap \mu.

h_{f^{-1}(\tilde{B})}(x_1) \cap h_{f^{-1}(\tilde{B})}(x_2) \cap \mu = h_\lambda(f(x_1)) \cap h_{f^{-1}(\tilde{B})}(x_2) \cap \mu = h_\lambda(y_1) \cap h_{f^{-1}(\tilde{B})}(y_2) \cap \mu.

According to definition 3.2 that \(f^{-1}(\tilde{B}) \in (\lambda, \mu)HFS[R_1].\)
4. Direct product of $(\lambda, \mu)$ hesitant fuzzy subalgebras

**Definition 4.1** Let $A \in HF[X]$ and $B \in HF[Y]$ be hesitant fuzzy subsets of nonempty set sum, respectively: $\forall (x, y) \in X \times Y, A \times B : X \times Y \rightarrow [0,1]$ $h_{A \times B}(x, y) = h_{A}(x) \cap h_{B}(y)$. The $\tilde{A} \times \tilde{B}$ is called the hesitant fuzzy subset of $X \times Y$, and it is called the direct product of $\tilde{A}$ and $\tilde{B}$.

**Theorem 4.1** Let $\tilde{A} \in (\lambda, \mu) HF[R_1], \tilde{B} \in (\lambda, \mu) HF[R_2]$, then $\tilde{A} \times \tilde{B} \in (\lambda, \mu) HF[R_1 \times R_2]$.

**Proof** Let $A \in (\lambda, \mu) HF[R_1], \tilde{B} \in (\lambda, \mu) HF[R_2], \forall (x_1, y_1), (x_2, y_2) \in R_1 \times R_2$, and so

$h_{A \times B}((x_1, y_1) + (x_2, y_2)) \cup \tilde{\lambda} = h_{A \times B}(x_1 + x_2, y_1 + y_2) \cup \tilde{\lambda} = (h_{A}(x_1 + x_2) \cap h_{B}(y_1 + y_2)) \cup \tilde{\lambda} \supseteq (h_{A}(x_1) \cap h_{A}(x_2) \cap \mu) \cup (h_{B}(y_1) \cap h_{B}(y_2) \cap \mu)$

Define $h_{A \times B}(x_1, y_1) \cap \tilde{h}_{A \times B}(x_2, y_2) \cap \mu$.

$h_{A \times B}((x_1, y_1) + (x_2, y_2)) \cup \tilde{\lambda} = h_{A \times B}(x_1 + x_2, y_1 + y_2) \cup \tilde{\lambda} = (h_{A}(x_1 + x_2) \cap h_{B}(y_1 + y_2)) \cup \tilde{\lambda}$

$= (h_{A}(x_1) \cup \lambda) \cap (h_{B}(y_1) \cup \lambda) \supseteq (h_{A}(x_1) \cap h_{A}(x_2) \cap \mu) \cup (h_{B}(y_1) \cap h_{B}(y_2) \cap \mu)$

Therefore $\tilde{A} \times \tilde{B} \in (\lambda, \mu) HF[R_1 \times R_2]$.

**Definition 4.2** $\tilde{A} \times \tilde{B} \in HF[R_1 \times R_2]$, the hesitant fuzzy subsets of $R_1$ and $R_2$ are defined as follows: $h_{A}(x) = \cup_{z \in R_1} h_{A \times B}(x, z), h_{B}(y) = \cup_{z \in R_1} h_{A \times B}(z, y)$;

$h_{A \times B}(x) = h_{A \times B}(x, 0) \cap h_{A \times B}(0, y) = h_{A \times B}(x, y) = h_{A \times B}(0, y) \cap h_{A \times B}(1, y)$.

**Theorem 4.2** Let $\tilde{A} \times \tilde{B} \in (\lambda, \mu) HF[R_1 \times R_2]$, if $\tilde{A}_{1}(x), \tilde{A}_{2}(x) \in (\lambda, \mu) HF[R_1], \tilde{B}_{1}(x), \tilde{B}_{2}(x)$

$\in (\lambda, \mu) HF[R_1 \times R_2]$.

**Proof** First we prove that $\tilde{A}_{1}(x) \in (\lambda, \mu) HF[R_1], \forall x_1, x_2 \in R_1$,

$h_{A_{1}}(x_1 + x_2) \cup \lambda = \cup_{z \in R_1} h_{A_{1}}(x_1 + x_2, z) \cup \lambda \supseteq \cup_{z \in R_1} h_{A_{1}}(x_1 + x_2, z_1 + z_2) \cup \lambda$

$= \cup_{z_1, z_2 \in R_1} (h_{A_{1}}(x_1 + x_2, z_1 + z_2) \cup \lambda) \supseteq \cup_{z_1, z_2 \in R_1} h_{A_{1}}(x_1 + x_2, z_1 + z_2) \cup \lambda$

$\supseteq \cup_{z_1, z_2 \in R_1} h_{A_{1}}(x_1, z_1) \cap h_{A_{1}}(x_2, z_2) \cap \mu \supseteq \cup_{z_1, z_2 \in R_1} h_{A_{1}}(x_1, z_1) \cap h_{A_{1}}(x_2, z_2) \cap \mu$

Similarly, we can get $h_{A_{1}}(x_1, x_2) \cup \lambda \supseteq h_{A_{1}}(x_1) \cap h_{A_{1}}(x_2) \cap \mu$.

$h_{A_{1}}(x_1 + x_2) \cup \lambda = \cup_{z \in R_1} h_{A_{1}}(x_1 + x_2, z) \cup \lambda = \cup_{z \in R_1} h_{A_{1}}(x_1, z) \cup \lambda = \cup_{z \in R_1} (h_{A_{1}}(x_1, z) \cup \lambda) \supseteq \cup_{z \in R_1} h_{A_{1}}(x_1, z) \cup \lambda$

$= \cup_{z \in R_1} h_{A_{1}}(x_1, z) \cup \mu = h_{A_{1}}(x_1) \cap h_{A_{1}}(x_2) \cup \mu$. Therefore, $\tilde{A}_{1}(x) \in (\lambda, \mu) HF[R_1]$.

Next, let's prove that $\tilde{A}_{1}(x) \in (\lambda, \mu) HF[R_1], \forall x_1, x_2 \in R_1$,

$h_{A_{1}}(x_1 + x_2) \cup \lambda = h_{A_{1}}(x_1 + x_2, 0) \cup \lambda = h_{A_{1}}(x_1 + x_2, 0') \cup \lambda = h_{A_{1}}(x_1 + x_2, 0' + 0') \cup \lambda$.

$h_{A_{1}}(x_1) \cup \lambda = h_{A_{1}}(x_1 + x_2, 0') \cup \lambda = h_{A_{1}}(x_1 + x_2, 0' + 0') \cup \lambda$. Therefore, $\tilde{A}_{1}(x) \in (\lambda, \mu) HF[R_1]$. 

$h_{A_{1}}(x_1 + x_2) \cup \lambda = (h_{A_{1}}(x_1 + x_2, 0') \cup \lambda) \cup \lambda = (h_{A_{1}}(x_1 + x_2, 0' + 0') \cup \lambda) \cup \lambda$.
$$= (h_{\Delta \mathcal{B}}(x_1,0') + (x_2,0')) \cup \lambda \cup (h_{\Delta \mathcal{B}}(x_1,1') + (x_2,1')) \cup \lambda \supseteq h_{\Delta \mathcal{B}}(x_1,0') \cap h_{\Delta \mathcal{B}}(x_1,1') \cap \mu$$

Similarly, we can get:

$$h_{\Delta \mathcal{B}}(x_1,1') \cup \lambda = h_{\Delta \mathcal{B}}(x_1,0') \cap h_{\Delta \mathcal{B}}(x_1,1') \cup \lambda = (h_{\Delta \mathcal{B}}(x_1,0') \cup \lambda \cap (h_{\Delta \mathcal{B}}(x_1,0') \cup \lambda)$$

5. Conclusion

On the basis of fuzzy set, hesitant fuzzy set is introduced. Hesitant fuzzy set replaces a number of $[0,1]$ by a subset of $[0,1]$, which changes the membership degree of interval. This expression is closer to human thinking. If we use a numerical value to describe an event, the result is too absolute, and the hesitant fuzzy set can better reflect the hesitant degree of certain elements in membership. In this paper, we combine hesitant fuzzy set with Boolean algebra, introduce the concept of $(\lambda, \mu)$ hesitant fuzzy subalgebra by using the idea and method of hesitant fuzzy set, we study its properties and equivalent characterization, and get some meaningful conclusions. These conclusions enrich Boolean algebra and hesitant fuzzy set theory.

Acknowledgement

This work is supported by the natural science basic research program of Shaanxi Province (2021JQ-893).

References:

[1] L.A. Zadeh. Fuzzy sets[J]. Information and Control, 1965, 8(3):338-353.
[2] Torra V. Hesitant fuzzy sets[J]. International Journal of Intelligent Systems, 2010, 25(6):529-539.
[3] Zhang Zhiming, Luis Javier Herrera. Interval-valued Intuitionistic Hesitant Fuzzy Aggregation operators and their application in Group Decision Making[J]. Journal of Applied Mathematics, 2013.
[4] Liu Peide, Shufeng Cheng. Interval-Valued Probabilistic Dual Hesitant Fuzzy Sets for Multi-Criteria Group Decision-Making[J]. International Journal of Computational Intelligence Systems, 2019, 12(2): 1393-1411.
[5] R. Krishankumar, K.S.Ravichandran, Samarjit Kar, etal. Interval-valued probabilistic hesitant fuzzy set for multi-criteria group decision-making[J]. Soft Computing, 2019, 23(21): 10853-10879.
[6] Liu Xuhua. Fuzzy Boolean algebra [J]. Science Bulletin, 1984, 35 (14): 843-845.
[7] Sun Shaoquan. Fuzzy subalgebras and fuzzy ideals of Boolean algebras [J]. Fuzzy systems and mathematics, 2006, 20 (1): 90-94.
[8] Sun Shaoquan, Gu Wenxiang. Fuzzy congruence relations on Boolean algebra [J]. Fuzzy systems and mathematics, 2005, 19 (4): 34-38.
[9] Sun Zhongpin, sun Shaoquan. $(\epsilon, \in \vee q)$-Fuzzy subalgebras and $(\epsilon, \in \vee q)$-fuzzy ideals of Boolean algebras [J]. Journal of Qingdao University of science and Technology (Natural Science Edition), 2006, 27 (3): 275-278.
[10] Wang fengxiao. I-V fuzzy ideals of Boolean algebra [J]. fuzzy systems and mathematics, 2017, 31 (3): 49-54.
[11] Peng Jiayin. Hesitant fuzzy filter theory of FI algebras [J]. Fuzzy systems and mathematics, 2018, 32 (03): 1-15.
[12] Peng Jiayin. Hesitant fuzzy filters and ideals of $BR_0$ algebra [J]. Computer engineering and applications, 2018, 54 (11): 62-66.
[13] Fu Xiaobo, Zhang Jianzhong. Hesitant fuzzy Li ideals of lattice implication algebras [J]. Fuzzy systems and mathematics, 2019,33 (02): 36-44.
[14] Liu Chunhui. Hesitant fuzzy filters and hesitant fuzzy congruence relations of be algebras [J]. Journal of Ningxia University (Natural Science Edition), 2019,40 (1): 20-26.