EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS

PEIYING CHEN

(Communicated by Colding)

ABSTRACT. In this paper, we study the Dirichlet boundary value problem of a class of nonlinear parabolic equations. By a priori estimates, difference and variation techniques, we establish the existence and uniqueness of weak solutions of this problem.

1. Introduction

The purpose of this paper is to investigate the existence and uniqueness of weak solutions to the following parabolic problem

\[
\begin{aligned}
&u_t - \text{div}(a(|\nabla u|)\nabla u) = 0 \quad \text{in } \Omega \times (0, T], \\
&u(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T], \\
&u(x, 0) = u_0(x) \quad \text{in } \Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded, open domain of \( \mathbb{R}^N (N \geq 2) \) with Lipschitz boundary \( \partial \Omega \), \( T \) is a positive number, and \( u_0 \in L^2(\Omega) \). The function \( a \) is such that \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[
\phi(s) = \begin{cases} 
  a(|s|)s & s \neq 0, \\
  0 & s = 0
\end{cases}
\]

is an odd increasing homeomorphism from \( \mathbb{R} \) onto itself.

Problems of type (1.1) have been motivated by a various range of applications, such as motion of non-Newton fluids, image restoration, elastic materials, and mathematical biology. We refer to the bibliography (see, for example, [6, 7, 9, 14, 16, 20]) for more detailed information on the physical situation.

Due to its prominent role in many modeling phenomena, equation (1.1) has been studied extensively. In 1990, Perona and Malik [5] proposed the Malik-Perona model

\[
u_t - \text{div}(c(|\nabla u|^2)\nabla u) = 0 \quad \text{in } \Omega \times (0, T],
\]

where \( \Omega \) is an image domain in \( \mathbb{R}^2 \) and \( c(s) > 0 \). This model is well-known and has been widely used to de-noise and segment images. The equation in (1.2) can
be written as

\[ u_t = c(|\nabla u|^2)u_{TT} + b(|\nabla u|^2)u_{NN}, \]  

with \( b(s) = c(s) + 2sc'(s) \). Thus, the right-hand side of the equation (1.3) may be interpreted as a sum of a diffusion \( u_{TT} \) in the tangent direction (\( T \)) plus a diffusion \( u_{NN} \) in the normal (\( N = \nabla u |\nabla u| \)) direction. Although there are some results about the existence and uniqueness of weak solutions for the Malik-Perona model, the conditions to ensure these results are difficult to check (see [1]).

Therefore, similar to the Perona-Malik model, Wang and Zhou [19] considered the following equation

\[ u_t - \left( \frac{\Phi'(|\nabla u|)}{|\nabla u|} u_{TT} + \Phi''(|\nabla u|)u_{NN} \right) = 0, \]  

with \( \Phi(s) = \text{slog}(1 + s) \). They established the existence and uniqueness of weak solutions of this equation with Neumann boundary.

Later, Feng and Yin [13] investigated the existence and uniqueness of weak solutions of the more general equation (1.4) with \( \Phi(s) = \text{slog}(1 + \beta(s)), s \geq 0 \), where \( \beta(s) \) is a polynomial with the following form: \( \beta(s) = \beta_1 s + \beta_2 s^2 + \cdots + \beta_r s^r \) for some integer \( r \geq 1 \) and \( \beta_1 > 0, \beta_r > 0, \beta_j \geq 0, 1 < j < r \). Obviously the equation considered in [19] is a special case of [13].

In this paper, we study the existence and uniqueness of weak solutions of problem (1.1), where the equation in (1.1) is associated by \( N \)-function \( \Phi(s) := \int_0^s \phi(t)dt \).

These problems arise in the field of physics, e.g.,

(a) nonlinear elasticity: \( \Phi(s) = (1 + s^2)^\gamma - 1, \gamma > \frac{1}{2}; \)

(b) plasticity: \( \Phi(s) = s^\alpha (\text{log}(1 + s))^\beta, \alpha \geq 1, \beta > 0; \)

(c) generalized Newtonian fluids: \( \Phi(s) = \int_0^s t^{1-\alpha}(\sinh^{-1}t)^\beta dt. \)

For details, see [10–12].

We remark that the equation in (1.1) contains the equations proposed in [19] and [13] as particular cases. Here we establish the existence and uniqueness of weak solutions of (1.1) by difference and variation techniques.

The difficulties in this paper are mainly two parts. First, we do not assume polynomial or exponential growth condition as in [19] and [13]. Second, \( C^1(\Omega) \) is not dense in \( W^{1,1}_0(\Omega) \), so we will structure new test functions.

The novelty in this paper is that we provide an approximation argument to study this kind of problems by finding a weak limit for approximation solution sequence with bounded \( L^1 \)-norm under certain conditions and then proving this limit is a weak solution.

We assume that there exist \( l, m > 1 \) such that

\[ l \leq \frac{\phi(s)}{\Phi(s)} \leq m, \quad \text{for any } s > 0. \]  

Denote the cylinder \( Q \equiv \Omega \times (0, T] \) and define weak solutions of problem (1.1) as follows.
Definition 1.1. A function $u : \Omega \rightarrow \mathbb{R}$ is called a weak solution of problem (1.1) if the following conditions are satisfied:

(i) $u \in C \left([0,T]; L^2(\Omega) \right) \cap L^1 \left(0,T; W^{1,1}_0(\Omega) \right)$ with $\int_0^T \int_\Omega a(|\nabla u|) |\nabla u|^2 dx dt < \infty$;

(ii) For any $\varphi \in C^1(\Omega)$ with $\varphi(\cdot,T) = 0$ and $\varphi(\cdot,t)|_{\partial \Omega} = 0$, we have

$$- \int_\Omega u_0(x) \varphi(x,0) dx + \int_0^T \int_\Omega \left[ - u \varphi_t + a(|\nabla u|) \nabla u \cdot \nabla \varphi \right] dx dt = 0.$$ 

Remark 1.1. Let $u$ be a weak solution in Definition 1.1. By using the approximation technique (see [6, Chapter 3]) we have, for all $\varphi \in C^1(\Omega)$ with $\varphi(\cdot,T) = 0$ and $\varphi(\cdot,t)|_{\partial \Omega} = 0$,

$$\int_\Omega u \varphi dx \bigg|_{t=0} + \int_0^T \int_\Omega \left[ - u \varphi_t + a(|\nabla u|) \nabla u \cdot \nabla \varphi \right] dx dt = 0. (1.6)$$

The condition (i) in Definition 1.1 is crucial. It guarantees the uniqueness of weak solutions.

Now we state our main result.

Theorem 1.2. Under assumption (1.5), the initial-boundary value problem (1.1) admits a unique weak solution.

This paper is organized as follows. In Section 2, we will list and prove some useful inequalities and lemmas. In Section 3, the proof of Theorem 1.2 will be given.

2. Preliminaries

In this section, we state some basic results that will be used later and utilize the properties of $N$-function to prove Lemma 2.7.

Definition 2.1 ([3]). $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is called an $N$-function if it has the following properties

(1) $\Phi$ is even, continuous, convex and $\Phi(0) = 0$;

(2) $\Phi(u) > 0$ for all $u \neq 0$;

(3) $\lim_{u \rightarrow 0} \frac{\Phi(u)}{u} = 0$ and $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = \infty$.

Proposition 2.2 ([3]). $\Phi$ is an $N$-function iff $\Phi(u) = \int_0^{|u|} \phi(t) dt$, where the right derivative $\phi$ of $\Phi$ satisfies:

(1) $\phi$ is right-continuous and nondecreasing;

(2) $\phi(t) > 0$ whenever $t > 0$;

(3) $\phi(0) = 0$ and $\lim_{t \rightarrow \infty} \phi(t) = \infty$.

Let $\phi$ satisfy (1)-(3) of Proposition 2.2. Then we call

$$\psi(s) = \sup \{ t \geq 0 : \phi(t) \leq s \} = \inf \{ t \geq 0 : \phi(t) > s \},$$

the right-inverse function of $\phi$. Clearly, $\psi$ also satisfies (1)-(3) of Proposition 2.2.

Definition 2.3 ([3]). Let $\Phi$ be an $N$-function, $\phi$ be the right derivative of $\Phi$, and $\psi$ be the right-inverse function of $\phi$. Then we call

$$\Psi(v) = \int_0^{|v|} \psi(s) ds,$$
the complementary $N$-function of $\Phi$.

Proposition 2.4 ([3]). The relations of $\Phi, \Psi, \phi$ and $\psi$ are as follows:

\[ uv \leq \Phi(u) + \Psi(v) \quad (u, v \in \mathbb{R}). \tag{2.1} \]
\[ uv = \Phi(u) + \Psi(v) \iff u = \psi(|v|) \text{sign} v \text{ or } v = \phi(|u|) \text{sign} u \quad (u, v \in \mathbb{R}). \tag{2.2} \]
\[ \Phi(u) \leq |u|\phi(|u|) \leq \Phi(2u) \quad (u \in \mathbb{R}). \tag{2.3} \]

Inequality 2.1 is called the Young Inequality.

Definition 2.5 ([3]). We say that an $N$-function $\Phi$ satisfies the global $\Delta_2$-condition if there exist $K > 2$ such that

\[ \Phi(2u) \leq K\Phi(u) \quad (u \geq 0). \tag{2.4} \]

In this case, we write $\Phi \in \Delta_2$.

Lemma 2.6 ([15]). The following statements are equivalent:

1. $\Phi \in \Delta_2$.
2. (1.5) is satisfied.
3. There exist $p > 1$ such that for every $u > 0$,
   \[ \frac{u\phi(u)}{\Phi(u)} < p. \]
4. There exist $q = \frac{n}{p-1} > 1$ such that for every $v > 0$,
   \[ \frac{v\psi(v)}{\Psi(v)} > q. \tag{2.5} \]

Lemma 2.7. For all $\xi, \eta \in \mathbb{R}^N$, we have

\[ (a(|\xi|)\xi - a(|\eta|)\eta) \cdot (\xi - \eta) \geq 0. \tag{2.6} \]

Proof. If $\xi = 0, \eta \neq 0$, then

\[ (a(|\xi|)\xi - a(|\eta|)\eta) \cdot (\xi - \eta) = a(|\eta|)\eta \cdot \eta = a(|\eta|)|\eta|^2 = \phi(|\eta|)|\eta| > 0. \]

For the case $\xi \neq 0, \eta = 0$ or $\xi = 0, \eta = 0$, the proof is similar to the case of $\xi = 0, \eta \neq 0$.
If $\xi \neq 0, \eta \neq 0$, then

\[ (a(|\xi|)\xi - a(|\eta|)\eta) \cdot (\xi - \eta) = a(|\xi|)|\xi|^2 + a(|\eta|)|\eta|^2 - a(|\xi|)\xi \cdot \eta - a(|\eta|)\eta \cdot \xi \]
\[ = \phi(|\xi|)|\xi| + \phi(|\eta|)|\eta| - \phi(|\xi|)\frac{\xi \cdot \eta}{|\xi|} - \phi(|\eta|)\frac{\eta \cdot \xi}{|\eta|} \]
\[ \geq \phi(|\xi|)|\xi| + \phi(|\eta|)|\eta| - \phi(|\xi|)|\eta| - \phi(|\eta|)|\xi| \]
\[ = (\phi(|\xi|) - \phi(|\eta|))(|\xi| - |\eta|) \geq 0. \]

Lemma 2.8. Suppose that $\Phi$ is an $N$-function, then $\Phi(|\xi|)$ is a convex function with respect to $\xi \in \mathbb{R}^N$.

Proof. For every pair of $\xi_1, \xi_2 \in \mathbb{R}^N$ and every $\lambda \in [0, 1]$, we have

\[ \Phi(|\lambda \xi_1 + (1 - \lambda)\xi_2|) \leq \Phi(\lambda|\xi_1| + (1 - \lambda)|\xi_2|) \leq \lambda \Phi(|\xi_1|) + (1 - \lambda)\Phi(|\xi_2|). \]

\[ \square \]
Lemma 2.9 (The Biting Lemma [4, 17]). Let Ω ⊂ \( \mathbb{R}^N \) be measurable with finite Lebesgue measure \( \mu_\Omega \) and suppose that \( \{f_n\} \) is a bounded sequence in \( L^1(\Omega; \mathbb{R}^N) \). Then there exist a subsequence \( \{f_{n_j}\} \subset \{f_n\} \), a function \( f \in L^1(\Omega; \mathbb{R}^N) \), and a decreasing family of measurable sets \( E_k \) such that \( \mu_{\Omega \setminus E_k} \to 0 \) as \( k \to \infty \) and for any \( k \), \( f_{n_j} \rightharpoonup f \) weakly in \( L^1(\Omega \setminus E_k; \mathbb{R}^N) \) as \( j \to \infty \).

Lemma 2.10 (De La Vallée Poussin’s Theorem [2]). Suppose \( \Phi \) is an \( N \)-function. Let \( \Omega \subset \mathbb{R}^N \) be measurable with finite Lebesgue measure \( \mu_\Omega \) and suppose that \( \{f_n\} \subset L^1(\Omega; \mathbb{R}^N) \) satisfies \[
\int_\Omega \Phi(|f_n|)dx \leq C,
\] where \( C \) is a positive constant. Then there exist a subsequence \( \{f_{n_i}\} \subset \{f_n\} \), and a function \( f \in L^1(\Omega; \mathbb{R}^N) \) such that \( f_{n_i} \rightharpoonup f \) weakly in \( L^1(\Omega; \mathbb{R}^N) \) as \( i \to \infty \), with \[
\int_\Omega \Phi(|f|)dx \leq C.
\]

3. Existence and Uniqueness

To prove Theorem 1.2, as preparation we first study the existence and uniqueness of weak solutions of the following auxiliary elliptic problems. For \( h > 0 \) and \( u_0 \in L^2(\Omega) \), we consider \[
\begin{cases}
\frac{u - u_0}{h} - \text{div}(a(|\nabla u|)\nabla u) = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{3.1}
\]

Definition 3.1. A function \( u \in L^2(\Omega) \cap W^{1,1}_0(\Omega) \) with \( \int_0^T \int_\Omega a(|\nabla u|)|\nabla u|^2 dx dt < \infty \) is called a weak solution of problem (3.1) if for any \( \varphi \in C^1_c(\Omega) \), we have
\[
\int_\Omega \frac{u - u_0}{h} \varphi dx + \int_\Omega a(|\nabla u|)\nabla u \cdot \nabla \varphi dx = 0.
\]

Theorem 3.2. Assume that \( u_0 \in L^2(\Omega) \), then there exists a unique weak solution for problem (3.1).

Proof. We consider the variational problem
\[
\min \{ J(v) \mid v \in V \},
\]
where
\[
V = \{ v \in L^2(\Omega) \cap W^{1,1}_0(\Omega) \mid \int_\Omega \Phi(|\nabla v|)dx < \infty \},
\]
and the functional \( J \) is
\[
J(v) = \frac{1}{2h} \int_\Omega (v - u_0)^2 dx + \int_\Omega \Phi(|\nabla v|)dx.
\]
We will establish that \( J(v) \) has a minimizer \( u_1 \) in \( V \).
As
\[
0 \leq \inf_{v \in V} J(v) \leq J(0) = \frac{1}{2h} \int_\Omega u_0^2 dx,
\]
then we can find a minimizing sequence \( \{v_m\} \subset V \) such that \( J(v_m) \leq J(0) + 1 \) and

\[
\lim_{m \to \infty} J(v_m) = \inf_{v \in V} J(v).
\]

Since

\[
\int_\Omega \Phi(|\nabla v_m|) dx \leq J(v_m) \leq J(0) + 1,
\]

\[
\int_\Omega v_m^2 dx \leq \int_\Omega (v_m - u_0 + u_0)^2 dx
\]

\[
\leq 2 \int_\Omega (v_m - u_0)^2 dx + 2 \int_\Omega u_0^2 dx
\]

\[
\leq 4h[J(v_m) + J(0)]
\]

\[
\leq 4h[2J(0) + 1],
\]

it follows that

\[
\int_\Omega v_m^2 dx + \int_\Omega \Phi(|\nabla v_m|) dx \leq C.
\]

By Lemma 2.10 and the weak compactness of bounded set in reflexive Banach Space, we can find a subsequence \( \{v_{m_i}\} \) of \( \{v_m\} \) and a function \( u_1 \in L^2(\Omega) \cap W^{1,1}_0(\Omega) \) such that

\[
v_{m_i} \rightharpoonup u_1 \text{ weakly in } L^2(\Omega),
\]

\[
\nabla v_{m_i} \rightharpoonup \nabla u_1 \text{ weakly in } L^1(\Omega; \mathbb{R}^N).
\]

Thus, we have

\[
J(u_1) \leq \lim_{i \to \infty} J(v_{m_i}) = \inf_{v \in V} J(v).
\]

This implies that \( u_1 \in V \) is a minimizer of the functional \( J(u) \) in \( V \), i.e.,

\[
J(u_1) = \inf_{v \in V} J(v).
\]

Furthermore, we have \( J(u_1) \leq J(\lambda u_1) \), \( \lambda \in (0, 1) \). Since \( \Phi \) is an \( \mathcal{N} \)-function, by Lemma 2.8, we know that \( \Phi(|\xi|) \) is a convex function with respect to \( \xi \in \mathbb{R}^N \). Therefore we get

\[
\Phi(|\nabla u_1|) - \Phi(\lambda|\nabla u_1|) \geq (1 - \lambda)\nabla_\xi \Phi(\lambda|\nabla u_1|) \cdot \nabla u_1
\]

\[
= \frac{\Phi'(\lambda|\nabla u_1|)}{|\lambda \nabla u_1|} \lambda \nabla u_1 \cdot (1 - \lambda)\nabla u_1
\]

\[
= \phi(\lambda|\nabla u_1|)(1 - \lambda)|\nabla u_1|
\]

\[
= \lambda(1 - \lambda)a(\lambda|\nabla u_1|)|\nabla u_1|^2,
\]

where

\[
\nabla_\xi \Phi(\lambda|\nabla u_1|) = \left( \frac{\Phi'(|\xi|)\xi_1}{|\xi|}, \frac{\Phi'(|\xi|)\xi_2}{|\xi|}, \ldots, \frac{\Phi'(|\xi|)\xi_N}{|\xi|} \right)_{|\xi| = \lambda \nabla u_1}
\]

\[
= (a(|\xi|)\xi_1, a(|\xi|)\xi_2, \ldots, a(|\xi|)\xi_N)_{|\xi| = \lambda \nabla u_1}.
\]

Then

\[
\frac{1}{2}(1 - \lambda^2) \int_\Omega u_0^2 dx + h\lambda(1 - \lambda) \int_\Omega a(\lambda|\nabla u_1|)|\nabla u_1|^2 dx \leq (1 - \lambda) \int_\Omega u_1 u_0 dx.
\]
Dividing the above inequality by $1 - \lambda$, and passing to limits as $\lambda \to 1$, we have
\[
\int_{\Omega} u_1^2 dx + h \lim_{\lambda \to 1} \lambda (\lambda |\nabla u_1|)|\nabla u_1|^2 dx = \int_{\Omega} u_1^2 dx + h \lim_{\lambda \to 1} \phi(\lambda |\nabla u_1|)|\nabla u_1|dx \\
\leq \int_{\Omega} u_1 u_0 dx.
\]
Since
\[
\phi(\lambda |\nabla u_1|)|\nabla u_1| \leq \phi(|\nabla u_1|)|\nabla u_1| \leq \Phi(2|\nabla u_1|) \leq (K + 1)\Phi(|\nabla u_1|) \in L^1(\Omega),
\]
by Lebesgue Dominated Convergence Theorem,
\[
\int_{\Omega} u_1^2 dx + h \int_{\Omega} a(|\nabla u_1|)|\nabla u_1|^2 dx \leq \int_{\Omega} u_1 u_0 dx.
\]
Thus, we conclude that $a(|\nabla u_1|)|\nabla u_1|^2 \in L^1(\Omega)$.

It follows from (2.2) that
\[
\Psi(a(|\nabla u_1|)|\nabla u_1|) = a(|\nabla u_1|)|\nabla u_1|^2 - \Phi(|\nabla u_1|) \leq a(|\nabla u_1|)|\nabla u_1|^2 + \Phi(|\nabla u_1|),
\]
then we have
\[
\Psi(a(|\nabla u_1|)|\nabla u_1|) \in L^1(\Omega).
\]
For a fixed $\varphi \in C^1_c(\Omega)$, we know that $J(u_1) \leq J(\lambda u_1 + (1 - \lambda)\varphi)$, $\lambda \in (0, 1)$.

Denote $\xi_\lambda = \lambda \nabla u_1 + (1 - \lambda)\nabla \varphi$. Since $\Phi$ is an $N$-function, by Lemma 2.8, we know that $\Phi(|\xi|)$ is a convex function with respect to $\xi \in \mathbb{R}^N$. Therefore we get
\[
\Phi(|\nabla u_1|) - \Phi(|\xi_\lambda|) \geq \frac{\Phi'(||\xi_\lambda||)}{|\xi_\lambda|} \xi_\lambda \cdot (\nabla u_1 - \xi_\lambda) = (1 - \lambda)a(|\xi_\lambda|)\xi_\lambda \cdot (\nabla u_1 - \nabla \varphi),
\]
and deduce as above to have
\[
\int_{\Omega} a(|\xi_\lambda|)\xi_\lambda \cdot (\nabla u_1 - \nabla \varphi)dx \leq \frac{1}{2h} \int_{\Omega} [- (1 + \lambda)(u_1 - u_0)^2 + 2\lambda(u_1 - u_0)(\varphi - u_0) + (1 - \lambda)(\varphi - u_0)^2]dx.
\]
Consider
\[
g(\lambda) = \Phi(|\xi_\lambda|) = \Phi(|\lambda \nabla u_1 + (1 - \lambda)\nabla \varphi|).
\]
It is obvious that $g$ is a convex function in $\mathbb{R}$ and
\[
g'(\lambda) = \frac{\Phi'(||\xi_\lambda||)}{|\xi_\lambda|} \xi_\lambda \cdot (\nabla u_1 - \nabla \varphi) = a(|\xi_\lambda|)\xi_\lambda \cdot (\nabla u_1 - \nabla \varphi).
\]
Then by the monotonicity of a convex function’s derivative, we know
\[
g'(0) \leq g'(\lambda) \leq g'(1), \quad \lambda \in (0, 1),
\]
which implies that
\[
a(|\nabla \varphi|)\nabla \varphi \cdot (\nabla u_1 - \nabla \varphi) \leq a(|\xi_\lambda|)\xi_\lambda \cdot (\nabla u_1 - \nabla \varphi) \leq a(|\nabla u_1|)\nabla u_1 \cdot (\nabla u_1 - \nabla \varphi).
\]
Recalling (2.1), we have
\[
|a(|\nabla u_1|)\nabla u_1 \cdot \nabla \varphi| \leq a(|\nabla u_1|)|\nabla u_1||\nabla \varphi| \leq \Phi(|\nabla \varphi|) + \Psi(a(|\nabla u_1||\nabla u_1|).
\]
As $a(|\nabla u_1|)|\nabla u_1|^2 \in L^1(\Omega)$, $\Psi(a(|\nabla u_1||\nabla u_1|) \in L^1(\Omega)$, and $\varphi \in C^1_c(\Omega)$, it is easy to obtain $a(|\nabla \varphi|)\nabla \varphi \cdot (\nabla u_1 - \nabla \varphi) \in L^1(\Omega)$ and $a(|\xi_\lambda|)\xi_\lambda \cdot (\nabla u_1 - \nabla \varphi) \in L^1(\Omega)$. 

By Lebesgue Dominated Convergence Theorem, we have
\[
\lim_{\lambda \to 1} \int_{\Omega} a(|\xi\lambda|) \xi \cdot (\nabla u_1 - \nabla \varphi) \, dx = \int_{\Omega} \lim_{\lambda \to 1} a(|\xi\lambda|) \xi \cdot (\nabla u_1 - \nabla \varphi) \, dx \\
= \int_{\Omega} a(|\nabla u_1|) \nabla u_1 \cdot (\nabla u_1 - \nabla \varphi) \, dx.
\]
Recalling (3.2), we obtain
\[
\int_{\Omega} a(|\nabla u_1|) \nabla u_1 \cdot (\nabla u_1 - \nabla \varphi) \, dx \leq \frac{1}{h} \int_{\Omega} (u_1 - u_0)(\varphi - u_1) \, dx.
\]
Denote
\[
A = \int_{\Omega} \frac{u_1 - u_0}{h} \, dx + \int_{\Omega} a(|\nabla u_1|) |\nabla u_1|^2 \, dx.
\]
Then we conclude that, for every \( \varphi \in C^1_c(\Omega) \),
\[
\int_{\Omega} \frac{u_1 - u_0}{h} \varphi \, dx + \int_{\Omega} a(|\nabla u_1|) \nabla u_1 \cdot \nabla \varphi \, dx \geq A.
\]
By a scaling argument, it follows that
\[
\int_{\Omega} \frac{u_1 - u_0}{h} \varphi \, dx + \int_{\Omega} a(|\nabla u_1|) \nabla u_1 \cdot \nabla \varphi \, dx = 0.
\]
Therefore, \( u_1 \) is a weak solution of problem (3.1).

Suppose that there exists another weak solution \( v \) of problem (3.1). Then, for every \( \varphi \in C^1_c(\Omega) \), we have
\[
\int_{\Omega} \frac{v - u_0}{h} \varphi \, dx + \int_{\Omega} a(|\nabla v|) \nabla v \cdot \nabla \varphi \, dx = 0,
\]
which follows that
\[
\int_{\Omega} \frac{v - u_1}{h} \varphi \, dx + \int_{\Omega} a(|\nabla v|) \nabla v - a(|\nabla u_1|) \nabla u_1 \cdot \nabla \varphi \, dx = 0. \tag{3.3}
\]
Recalling (2.2), we observe that
\[
|a(|\nabla v|) \nabla v \cdot u_1| \leq a(|\nabla v|) |\nabla v||\nabla u_1| \leq \Phi(|\nabla u_1|) + \Psi(a(|\nabla v|)|\nabla v|) \in L^1(\Omega).
\]
Making use of the approximation argument, we conclude that \( v - u_1 \) can be a test function in (3.3). Therefore,
\[
\int_{\Omega} \frac{(v - u_1)^2}{h} \, dx + \int_{\Omega} [a(|\nabla v|) \nabla v - a(|\nabla u_1|) \nabla u_1] \cdot \nabla (v - u_1) \, dx = 0.
\]
Using inequality (2.6), we have
\[
\int_{\Omega} (v - u_1)^2 \, dx = 0.
\]
which implies \( u_1 = v \) a.e. in \( \Omega \). Thus we complete the proof of the Theorem. \( \square \)

Now we begin to prove Theorem 1.2.

Proof. First we prove the uniqueness of weak solutions. Suppose there exist two weak solutions \( u \) and \( v \) of problem (1.1). Then \( w = u - v \) satisfies the following problem
\[
\begin{cases}
  w_t - \text{div}[a(|\nabla u|) \nabla u - a(|\nabla v|) \nabla v] = 0 & \text{in } \Omega \times (0, T], \\
  w(x, t) = 0 & \text{in } \partial \Omega \times (0, T], \\
  w(x, 0) = 0 & \text{on } \Omega.
\end{cases}
\]
By an approximation argument [8, 18], we first extend solution \( u(x, t) \) to the initial value \( u_0(x) \) when \( t < 0 \). We next mollify \( u \) in the spatial directions to have an approximation \( C^\infty \) sequence \( u_\varepsilon \), then introduce the time average of \( u_\varepsilon(x, t) \),

\[
J_{\varepsilon, h} = \frac{1}{2h} \int_{t-h}^{t+h} u_\varepsilon(x, \tau) d\tau.
\]

As \( u \in C([0, T]; L^2(\Omega)) \cap L^1(0, T; W^{1,1}_0(\Omega)) \), we know that \( J_{\varepsilon, h}(x, t) \in C^1(\overline{Q}) \) with \( J_{\varepsilon, h}(\cdot, t)|_{\partial\Omega} = 0 \). So we choose

\[
w^k_{\varepsilon, h}(x, t) = \frac{1}{2h} \int_{t-h}^{t+h} w^k_\varepsilon(x, \tau) d\tau, w^k = w\chi_{|w| \leq k} - k\chi_{|w| < k} + k\chi_{|w| > k},
\]

as a test function in the above initial-boundary value problem to have

\[
\int_{\Omega} \left[ w w^k_{\varepsilon, h}(x, t) \right] dx - \int_{0}^{t} \int_{\Omega} w [w^k_{\varepsilon, h}] dt
\]

\[
+ \int_{0}^{t} \int_{\Omega \cap \{|u - v| < k\}} \left( a(|\nabla u|) \nabla u - a(|\nabla v|) \nabla v \right) \cdot \nabla (u - v) dxd\tau = 0.
\]

By the approximation argument, we have

\[
\frac{1}{2} \int_{\Omega} (u - v)(t) w^k(t) dx
\]

\[
+ \int_{0}^{t} \int_{\Omega \cap \{|u - v| < k\}} \left( a(|\nabla u|) \nabla u - a(|\nabla v|) \nabla v \right) \cdot \nabla (u - v) dxd\tau = 0.
\]

Sending \( k \to \infty \), we conclude that

\[
\frac{1}{2} \int_{\Omega} (u - v)^2(t) dx + \int_{0}^{t} \int_{\Omega} \left( a(|\nabla u|) \nabla u - a(|\nabla v|) \nabla v \right) \cdot \nabla (u - v) dxd\tau = 0.
\]

which implies \( u = v \) a.e. in \( Q \). Therefore we obtain the uniqueness of weak solutions.

Now we prove the existence of weak solutions. Let \( n \) be a positive integer. Denote \( h = \frac{T}{n} \). In order to construct an approximation solution sequence \( \{u_k\} \) for problem (1.1), we consider the following elliptic problems

\[
\begin{cases}
\frac{u_k - u_{k-1}}{h} - \text{div}(a(|\nabla u_k|) \nabla u_k) = 0 & \text{in } \Omega, \\
u_{k} = 0 & \text{on } \partial\Omega,
\end{cases}
\]

(3.4)

for \( k = 1, 2, \ldots, n \). As \( k = 1 \), it follows from Theorem 3.2 that there is a unique \( u_1 \in V \) satisfying (3.4). Following the same procedures, we can find weak solutions \( u_k \in V \) of (3.4) for \( k = 2, 3, \ldots, n \). Moreover, for every \( \varphi \in C^1_c(\Omega) \), we have

\[
\int_{\Omega} \frac{u_k - u_{k-1}}{h} \varphi dx + \int_{\Omega} a(|\nabla u_k|) \nabla u_k \cdot \nabla \varphi dx = 0,
\]

(3.5)

and

\[
\int_{\Omega} \frac{u_k - u_{k-1}}{h} u_k dx + \int_{\Omega} a(|\nabla u_k|) \nabla u_k \cdot \nabla u_k dx = 0.
\]
Now for every \( h = \frac{T}{n} \), we define
\[
    u_h(x, t) = \begin{cases} 
        u_0(x), & t = 0, \\
        u_1(x), & 0 < t \leq h, \\
        \ldots, & \ldots, \\
        u_j(x), & (j - 1) < t \leq jh, \\
        \ldots, & \ldots, \\
        u_n(x), & (n - 1)h < t \leq nh = T.
    \end{cases}
\] (3.6)

Choosing \( u_k \) as a test function in (3.5), and using \( u_k u_{k-1} \leq \frac{u_k^2 + u_{k-1}^2}{2} \), we have
\[
    \frac{1}{2} \int_{\Omega} u_k^2 dx + h \int_{\Omega} a(|\nabla u_k|)|\nabla u_k|^2 dx \leq \frac{1}{2} \int_{\Omega} u_{k-1}^2 dx. \tag{3.7}
\]

For each \( t \in (0, T] \), there exists some \( j \in \{1, 2, \ldots, n\} \) such that \( t \in ((j - 1)h, jh] \). Adding the inequality (3.7) from \( k = 1 \) to \( k = j \), we get
\[
    \frac{1}{2} \int_{\Omega} u_j^2 dx + h \sum_{k=1}^{j} \int_{\Omega} a(|\nabla u_k|)|\nabla u_k|^2 dx \leq \frac{1}{2} \int_{\Omega} u_0^2 dx. \tag{3.8}
\]

By the definition of \( u_h(x, t) \), we obtain
\[
    \frac{1}{2} \int_{\Omega} u_h^2(x, t) dx + \int_0^t \int_{\Omega} a(|\nabla u_h|)|\nabla u_h|^2 dx \, d\tau \leq \frac{1}{2} \int_{\Omega} u_0^2 dx.
\]

In particular, we get
\[
    \frac{1}{2} \int_{\Omega} u_h^2(x, t) dx + \int_0^t \int_{\Omega} a(|\nabla u_h|)|\nabla u_h|^2 dx \, d\tau \leq \frac{1}{2} \int_{\Omega} u_0^2 dx. \tag{3.8}
\]

Therefore, after taking the supremum over \([0, T] \), we have
\[
    \sup_{0 \leq t \leq T} \int_{\Omega} u_h^2(x, t) dx + \int_0^T \int_{\Omega} a(|\nabla u_h|)|\nabla u_h|^2 dx \, dt \leq \frac{3}{2} \int_{\Omega} u_0^2 dx.
\]

Recalling (2.3), we have
\[
    \sup_{0 \leq t \leq T} \int_{\Omega} u_h^2(x, t) dx + \int_0^T \int_{\Omega} \Phi(|\nabla u_h|) dx \, d\tau \leq \frac{3}{2} \int_{\Omega} u_0^2 dx.
\]

We conclude that
\[
    \sup_{0 \leq t \leq T} \int_{\Omega} u_h^2(x, t) dx \leq C = C(u_0, \Omega),
\]
and
\[
    \int_0^T \int_{\Omega} \Phi(|\nabla u_h|) dx \, d\tau \leq C = C(u_0, \Omega). \tag{3.9}
\]

Thanks to Lemma 2.10, we may choose a subsequence (for simplicity, we also denote it by the original sequence) such that
\[
    u_h \rightharpoonup u \text{ weakly-}^* \text{ in } L^\infty(0, T; L^2(\Omega)),
\]
\[
    u_h \rightharpoonup u \text{ weakly in } L^1(0, T; W_0^{1,1}(\Omega)).
\]

These yield that
\[
    \sup_{0 \leq t \leq T} \int_{\Omega} u^2(x, t) dx \leq C \quad \text{and} \quad \int_0^T \int_{\Omega} \Phi(|\nabla u|) dx \, d\tau \leq C.
\]
Denote
\[ \xi_h = a(\nabla u_h)\nabla u_h. \]

Using (2.2), (2.3), (2.4), Lemma 2.6 and our assumption (1.5), we have
\[
\Psi(|\xi_h|) = \Psi(\xi_h) = \Psi \big( a(\nabla u_h) \big|\nabla u_h \big) = \Psi \big( a(\nabla u_h) \big|\nabla u_h \big)
\leq \Phi(2|\nabla u_h|) - \Phi(|\nabla u_h|)
\leq K\Phi(|\nabla u_h|) - \Phi(|\nabla u_h|) = (K-1)\Phi(|\nabla u_h|). \tag{3.10}
\]

It follows from (3.9) and (3.10) that
\[
\int_0^T \int_\Omega \Psi(|\xi_h|) dx dt \leq \int_0^T \int_\Omega (K-1)\Phi(|\nabla u_h|) dx dt \leq C.
\]

Recalling (2.5), we have that
\[
\int_0^T \int_\Omega |\xi_h|^q dx dt \leq C \quad (q > 1).
\]

Thus we can draw another subsequence \( \{\xi_h\} \) (we also denote it by the original sequence for simplicity) such that
\[
\xi_h \rightharpoonup \xi, \quad \text{weakly in} \quad (L^q(Q))^N \quad (q > 1). \tag{3.11}
\]

We conclude from Lemma 2.10 that
\[
\int_0^T \int_\Omega \Psi(|\xi|) dx dt \leq \lim_{h \to 0} \int_0^T \int_\Omega \Psi(|\xi_h|) dx dt \leq C.
\]

Recalling inequality (2.1), we have
\[
|\xi \cdot \nabla u| \leq |\xi||\nabla u| \leq \Psi(|\xi|) + \Phi(|\nabla u|).
\]

This implies that \( \xi \cdot \nabla u \in L^1(Q) \).

In the following, we prove that the function \( u \) is a weak solution of problem (1.1).

For every \( \varphi \in C_c(Q) \) with \( \varphi(\cdot, 0) = 0 \) and \( \varphi(\cdot, t)|_{\partial\Omega} = 0 \), we take \( \varphi(x, (k-1)h) \) as a test function in (3.5) for every \( k \in \{1, 2, \ldots, n\} \) to have
\[
\int_{\Omega} \frac{u_k - u_{k-1}}{h} \varphi(x, (k-1)h) dx + \int_{\Omega} a(|\nabla u_k|) \nabla u_k \cdot \nabla \varphi(x, (k-1)h) dx = 0.
\]

Summing up all the equalities and recalling \( \varphi(\cdot, T) = \varphi(\cdot, nh) = 0 \), we get
\[
-\frac{1}{h} \int_{\Omega} u_0(x)\varphi(x, 0) dx + \sum_{k=1}^n \int_{\Omega} u_k(x) \frac{\varphi(x, (k-1)h) - \varphi(x, kh)}{h} dx
\]
\[
+ \sum_{k=1}^n \int_{\Omega} a(|\nabla u_k|) \nabla u_k \cdot \nabla \varphi(x, (k-1)h) dx = 0.
\]

In view of the definition of \( u_h(x, t) \) in (3.6), we obtain
\[
\int_0^T \int_{\Omega} u_h(x, t) \varphi_t(x, t) dx dt = \sum_{k=1}^{kh} \int_{(k-1)h}^{kh} \int_{\Omega} u_h(x, t) \varphi_t(x, t) dx dt
\]
\[
= \sum_{k=1}^n \int_{\Omega} u_k(x) \left[ \int_{(k-1)h}^{kh} \varphi(x, t) dt \right] dx = \sum_{k=1}^n \int_{\Omega} u_k(x) \left[ \varphi(x, kh) - \varphi(x, (k-1)h) \right] dx,
\]
and
\[ \int_0^T \int_\Omega a(|\nabla u_h|) \nabla u_h \cdot \nabla \varphi \, dx \, dt = \sum_{k=1}^n \int_{(k-1)h}^{kh} \int_\Omega a(|\nabla u_h|) \nabla u_h \cdot \nabla \varphi \, dx \, dt \]
\[ = \sum_{k=1}^n \int_\Omega a(|\nabla u_k|) \nabla u_k \cdot \left[ \int_{(k-1)h}^{kh} \nabla \varphi(x,t) \, dt \right] \, dx. \]
Thus
\[ - \int_\Omega u_0(x) \varphi(x,0) \, dx - \int_0^T \int_\Omega u_h(x,t) \varphi_t(x,t) \, dx \, dt + \int_0^T \int_\Omega a(|\nabla u_h|) \nabla u_h \cdot \nabla \varphi \, dx \, dt \]
\[ = \sum_{k=1}^n \int_\Omega a(|\nabla u_k|) \nabla u_k \cdot \left[ \int_{(k-1)h}^{kh} \nabla \varphi(x,t) \, dt - h \nabla \varphi(x,(k-1)h) \right] \, dx. \]
Letting \( h \to 0 \), we have
\[ - \int_\Omega u_0(x) \varphi(x,0) \, dx + \int_0^T \int_\Omega \left[ -u \varphi_t + \xi \cdot \nabla \varphi \right] \, dx \, dt = 0. \tag{3.12} \]
Choosing \( \varphi \in C_c^\infty(\Omega) \), we get
\[ \int_0^T \int_\Omega u \varphi_t \, dx \, dt = \int_0^T \int_\Omega \xi \cdot \nabla \varphi \, dx \, dt. \tag{3.13} \]
By (3.11), we know that \( \xi \in (L^2(\Omega))^N \). In view of (3.13), we conclude that \( u_t \in L^1(0,T; H^{-1}(\Omega)) \).
Since
\[ u = \int_0^t u_t \, dt + u_0, \]
and \( u_0 \in L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \), it follows that \( u \in C([0,T]; H^{-1}(\Omega)) \). Here \( H^{-1}(\Omega) \) is the dual space of \( H_0^1(\Omega) = W_0^{1,2}(\Omega) \).
Denote
\[ A_v = a(|\nabla v|) \nabla v, \]
for \( v \in L^1(\Omega) \) with \( \int_\Omega \int_\Omega \Phi(|\nabla v|) \, dx \, dt < \infty \).
Summing up the inequalities (3.7), we get
\[ \frac{1}{2} \int_\Omega u_h^2(T) \, dx + \int_0^T \int_\Omega Au_h \cdot \nabla u_h \, dx \, dt \leq \frac{1}{2} \int_\Omega u_0^2 \, dx. \tag{3.14} \]
Recalling Lemma 2.7, we have
\[ \int_0^T \int_\Omega (Au_h - A_v) \cdot (\nabla u_h - \nabla v) \, dx \, dt \geq 0. \]
Then it follows from (3.14) that
\[ \frac{1}{2} \int_\Omega u_h^2(T) \, dx + \int_0^T \int_\Omega Au_h \cdot \nabla v \, dx \, dt \]
\[ + \int_0^T \int_\Omega A_v \cdot \nabla u_h \, dx \, dt - \int_0^T \int_\Omega A_v \cdot \nabla v \, dx \, dt \leq \frac{1}{2} \int_\Omega u_0^2 \, dx. \]
Letting \( h \to 0 \), and noting
\[
\int_\Omega u^2(T)dx \leq \lim_{h \to 0} \int_\Omega u_h^2(T)dx,
\]
we get
\[
\frac{1}{2} \int_\Omega u^2(T)dx + \int_0^T \int_\Omega \xi \cdot \nabla vdxdt + \int_0^T \int_\Omega A \cdot \nabla vdxdt \leq \frac{1}{2} \int_\Omega u_0^2dx. \tag{3.15}
\]
By an approximation, we may choose the test function \( \varphi = u \) in (3.12) to have
\[
\frac{1}{2} \int_\Omega u^2(T)dx + \int_0^T \int_\Omega \xi \cdot \nabla udxdt = \frac{1}{2} \int_\Omega u_0^2dx. \tag{3.16}
\]
Combining (3.15) with (3.16), we obtain
\[
\int_0^T \int_\Omega (\xi - Av) \cdot (\nabla v - \nabla u)dxdt \leq 0. \tag{3.17}
\]
Set \( v = \lambda u, \lambda \in (0, 1) \). Then
\[
\int_0^T \int_\Omega [\xi - a(\lambda|\nabla u|)\nabla u] \cdot \nabla udxdt \geq 0.
\]
That is
\[
\int_0^T \int_\Omega a(\lambda|\nabla u|)\nabla udxdt \leq \int_0^T \int_\Omega \xi \cdot \nabla udxdt.
\]
Passing to limits as \( \lambda \to 1 \), we conclude that
\[
\int_0^T \int_\Omega a(|\nabla u|)\nabla udxdt \leq \int_0^T \int_\Omega \xi \cdot \nabla udxdt.
\]
Next we choose \( v = \lambda u + (1 - \lambda)w \) for any \( \lambda \in (0, 1), w \in C^1(\overline{Q}) \) in inequality (3.17) to have
\[
\int_0^T \int_\Omega [\xi - A(\lambda u + (1 - \lambda)w)] \cdot (\nabla w - \nabla u)dxdt \leq 0.
\]
Passing to limits as \( \lambda \to 1 \) and using Lesbesgue’s Dominated Convergence Theorem, we obtain
\[
\int_0^T \int_\Omega (\xi - Au) \cdot (\nabla w - \nabla u)dxdt \leq 0.
\]
By a scaling argument again, we have
\[
\int_0^T \int_\Omega (\xi - Au) \cdot \zeta dxdt = 0,
\]
for every \( \zeta \in (L^\infty(\overline{Q}))^N \). It follows that \( \xi = Au \) a.e. in \( Q \).

For every \( 0 < \delta < T \), we denote \( v_\delta(x, t) = u(x, t + \delta) \). By the uniqueness of weak solutions, we conclude that \( v_\delta \) is a weak solution for the following problem
\[
\begin{align*}
\frac{\partial v_\delta}{\partial t} - \text{div} \left( a(|\nabla v_\delta|)\nabla v_\delta \right) &= 0 & \text{in } \Omega \times (0, T - \delta], \\
v_\delta &= 0 & \text{on } \partial \Omega \times (0, T - \delta], \\
v_\delta(x, 0) &= u(x, \delta) & \text{in } \Omega.
\end{align*}
\]
Then it follows that \( \omega_\delta(x, t) = v_\delta(x, t) - u(x, t) = u(x, t + \delta) - u(x, t) \) satisfying
\[
\begin{cases}
\frac{\partial \omega_\delta}{\partial t} - \text{div} \left( a(|\nabla v_\delta|) \nabla v_\delta - a(|\nabla u|) \nabla u \right) = 0 & \text{in } \Omega \times (0, T - \delta], \\
\omega_\delta = 0 & \text{on } \partial \Omega \times (0, T - \delta], \\
\omega_\delta(x, 0) = u(x, \delta) - u_0(x) & \text{in } \Omega.
\end{cases}
\tag{3.18}
\]

For each \( t_0 \in [0, T - \delta] \), we choose a test function \( \omega_\delta \) for equations (3.18) over \([0, t_0]\) to have
\[
\frac{1}{2} \int_\Omega \omega_\delta^2(x, t_0)dx + \int_0^{t_0} \int_\Omega \left[ a(|\nabla v_\delta|) \nabla v_\delta - a(|\nabla u|) \nabla u \right] \cdot (\nabla v_\delta - \nabla u) dx dt \leq \frac{1}{2} \int_\Omega \omega_\delta^2(x, 0)dx.
\]

Thanks to Lemma 2.7, it yields
\[
\int_\Omega \left| u(x, t_0 + \delta) - u(x, t_0) \right|^2 dx \leq \int_\Omega \left| u(x, \delta) - u_0(x) \right|^2 dx.
\]

In order to prove that \( u \in C \left( [0, T]; L^2(\Omega) \right) \), we only need to prove
\[
\lim_{\delta \to 0^+} \int_\Omega \left| u(x, \delta) - u_0(x) \right|^2 dx = 0. \tag{3.19}
\]

Suppose (3.19) is not true. Then there exists a positive number \( \varepsilon_0 \) and a sequence \( \{ \delta_i \} \) with \( \delta_i \to 0 \) as \( i \to \infty \) such that
\[
\lim_{\delta_i \to 0^+} \int_\Omega \left| u(x, \delta_i) - u_0(x) \right|^2 dx \geq \varepsilon_0. \tag{3.20}
\]

By (3.8), we easily see that
\[
\int_\Omega \left| u(x, \delta_i) \right|^2 dx \leq \int_\Omega \left| u_0^2(x) \right| dx. \tag{3.21}
\]

Thus, we have from (3.20) that
\[
\lim_{i \to \infty} \left[ \int_\Omega \left| u_0(x) \right|^2 dx - \int_\Omega u_0(x) u(x, \delta_i) dx \right] \geq \frac{\varepsilon_0}{2}. \tag{3.22}
\]

Hence, it follows from (3.21) that \( \{ u(x, \delta_i) \} \) is bounded sequence in \( L^2(\Omega) \). Moreover, there exist a subsequence (for simplicity, we also denote it by the original sequence) and a \( \tilde{u}_0 \in L^2(\Omega) \) such that
\[
u(x, \delta_i) \rightharpoonup \tilde{u}_0(x) \quad \text{weakly in } L^2(\Omega).
\]

Since \( u \in C \left( [0, T]; H^{-1}(\Omega) \right) \), it follows that
\[
u(x, \delta_i) \rightharpoonup u_0(x) \quad \text{in } H^{-1}(\Omega).
\]

Thus we must have \( \tilde{u}_0(x) = u_0(x) \) and then
\[
u(x, \delta_i) \rightharpoonup u_0(x) \quad \text{weakly in } L^2(\Omega).
\]

The above relation leads to a contradiction with (3.22). Therefore, we conclude that (3.19) is true and \( u \in C \left( [0, T]; L^2(\Omega) \right) \). Thus we complete the proof of the theorem. \( \square \)
References

[1] G. Aubert and P. Kornprobst, *Mathematical Problems in Image Processing*, Springer-Verlag, New York, 2002. MR 1865346

[2] J. Alexopoulos, *de la Vallée Poussin’s theorem and weakly compact sets in Orlicz spaces*, *Quaestiones Math.*, 17 (1994), 231–248. MR 1281594

[3] R. Adams, *Sobolev Spaces*, Academic Press, New York-London, 1975. MR 0450957

[4] J. M. Ball and F. Murat, *Remarks on Chacon’s biting lemma*, *Proc. Amer. Math. Soc.*, 107 (1989), 655–663. MR 984807

[5] P. Clément, M. García-Huidobro, R. Manásevich and K. Schmitt, *Mountain pass type solutions for quasilinear elliptic equations*, *Calc. Var. Partial Differential Equations*, 11 (2000), 33–62. MR 1777463

[6] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration*, *SIAM J. Appl. Math.*, 66 (2006), 1383–1406. MR 2246061

[7] L. Diening, *Theoretical and Numerical Results for Electrorheological Fluids*, Ph.D. Thesis, University of Freiburg, Germany, 2002.

[8] L. C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, Amer. Math. Soc., Providence, RI, 1990. MR 1034481

[9] G. Fragnelli, *Positive periodic solutions for a system of anisotropic parabolic equations*, *J. Math. Anal. Appl.*, 367 (2010), 204–228. MR 2600391

[10] M. Fuchs and L. Gongbao, *Variational inequalities for energy functionals with nonstandard growth conditions*, *Abstr. Appl. Anal.*, 3 (1998), 41–64. MR 1700276

[11] M. Fuchs and V. Osmolovski, *Variational integrals on Orlicz-Sobolev spaces*, *Z. Anal. Anwendungen*, 17 (1998), 393–415. MR 1632563

[12] N. Fukagai and K. Narukawa, Nonlinear eigenvalue problem for a model equation of an elastic surface, *Hiroshima Math. J.*, 25 (1995), 19–41. MR 1326200

[13] Z. Feng and Z. Yin, *On weak solutions for a class of nonlinear parabolic equations related to image analysis*, *Nonlinear Anal.*, 71 (2009), 2506–2517. MR 2532778

[14] P. Gwiazda and A. Świerczewska-Gwiazda, *On non-Newtonian fluids with a property of rapid thickening under different stimulus*, *Math. Models Methods Appl. Sci.*, 18 (2008), 1073–1092. MR 2435185

[15] M. M. Rao and Z. D. Ren, *Applications of Orlicz Spaces*, Marcel Dekker, Inc., New York, 2002. MR 1899178

[16] K. R. Rajagopal and M. Ružička, *Mathematical modelling of electrorheological fluids*, *Continuum Mech. Thermodyn.*, 13 (2001), 59–78.

[17] M. Saadoune and M. Valadier, *Extraction of “good” subsequence from a bounded sequence of integrable functions*, *J. Convex Anal.*, 2 (1995), 345–357. MR 1363378

[18] C. Wu, *Convex Functions and Orlicz Spaces*, Science Press, Beijing, 1961.

[19] L. Wang and S. Zhou, *Existence and uniqueness of weak solutions for a nonlinear parabolic equation related to image analysis*, *J. Partial Differential Equations*, 19 (2006), 97–112. MR 2227688

[20] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, *Math. USSR Izv.*, 9 (1987), 33–66.

Department of Mathematics, Shanghai University, Shanghai 200444, China
E-mail address: peiying0211@163.com