TILTING MODULES AND SUPPORT $\tau$-TILTING MODULES OVER PREPROJECTIVE ALGEBRAS ASSOCIATED WITH SYMMETRIZABLE CARTAN MATRICES

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Abstract. For any given symmetrizable Cartan matrix $C$ with a symmetrizer $D$, Geiß et al. (2016) introduced a generalized preprojective algebra $\Pi(C, D)$. We study tilting modules and support $\tau$-tilting modules for the generalized preprojective algebra $\Pi(C, D)$ and show that there is a bijection between the set of all cofinite tilting ideals of $\Pi(C, D)$ and the corresponding Weyl group $W(C)$ provided that $C$ has no component of Dynkin type. When $C$ is of Dynkin type, we also establish a bijection between the set of all basic support $\tau$-tilting $\Pi(C, D)$-modules and the corresponding Weyl group $W(C)$. These results generalize the classification results of Buan et al. (Compos. Math. 145(4), 1035-1079, 2009) and Mizuno (Math. Zeit. 277(3), 665-690, 2014) over classical preprojective algebras.

1. Introduction

Preprojective algebra associated to a quiver was invented by Gelfand and Ponomarev [11]. Since its appearance, it has been studied extensively due to its relevance to various different parts of mathematics, see [6, 13, 15, 17, 7] for instance. In particular, it has played a key role in Lusztig’s construction [13] of semicanonical bases for the enveloping algebra $U(n)$, where $n$ is a maximal nilpotent subalgebra of a complex symmetric Kac-Moody Lie algebra. The representation theory of preprojective algebras was also the foundation of Nakajima’s construction of quiver varieties [15]. Building on the work of Buan et al. [4] and Geiß et al. [8, 9], preprojective algebras also provided us a large class of 2-Calabi-Yau categories which lead to categorifications of certain important cluster algebras.

Recently, Geiß et al. [10] introduced a class of Iwanaga-Gorenstein algebras via quivers with relations for any symmetrizable Cartan matrices with symmetrizers, which generalizes the path algebras of quivers associated with symmetric Cartan matrices. They also introduced the corresponding generalized preprojective algebras. This new class of preprojective algebras reduces to the classical one provided that the Cartan matrix is symmetric and the symmetrizer is the identity matrix. Surprisingly, the generalized preprojective algebras still share many properties with the classical one. Since the classical preprojective algebras have many important applications in different fields of mathematics, it is an interesting question to find out which results or constructions for classical preprojective algebras can be generalized to the general setting. For example, if one can generalize the constructions of [4, 9] to the new preprojective algebras, then one may obtain new categorifications for certain skew-symmetrizable cluster algebras.

This note gives a first attempt to generalize certain classification results in tilting theory of preprojective algebras to this new setting. For a given algebra, a basic question in tilting theory is to classify all the tilting modules or support $\tau$-tilting modules. For the classical preprojective algebras, the classification has been obtained by Buan et al. [4] for preprojective algebras of non-Dynkin type (cf. 2010 Mathematics Subject Classification. 16G10, 16G20.

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Let $C$ be a symmetrizable Cartan matrix with a symmetrizer $D$ and $\Pi = \Pi(C, D)$ the associated preprojective algebra. Assume that $C$ has no component of Dynkin type, then we have bijections between the following sets:

1. the set of all cofinite tilting $\Pi$-ideals;
2. the ideal semigroup $\langle I_1, I_2, \ldots, I_n \rangle$;
3. the Weyl group $W(C)$.

**Theorem 1.2.** Let $C$ be a symmetrizable Cartan matrix of Dynkin type with a symmetrizer $D$ and $\Pi = \Pi(C, D)$ the associated preprojective algebra, then we have bijections between the following sets:

1. the set of all basic support $\tau$-tilting $\Pi$-modules;
2. the ideal semigroup $\langle I_1, I_2, \ldots, I_n \rangle$;
3. the Weyl group $W(C)$.

Let us mention here that in contrast to the classical cases, we have to pay attention to locally free modules and generalized simple modules in this general setting. After establishing certain necessary properties for locally free modules and generalized simple modules, most of the remain arguments essentially follow the one in [10, 14]. Moreover, for the preprojective algebras of Dynkin type, both the Weyl group and the support $\tau$-tilting $\Pi$-modules admit poset structures. One may adapt the argument in [14] to show that the bijection in Theorem 1.2 is compatible with the poset structures.

The paper is organized as follows. In Section 2, we recall basic definitions and properties of the preprojective algebras associated with symmetrizable Cartan matrices following [10]. In Section 3, we establish the bijection between the set of all cofinite tilting ideals of the preprojective algebras $\Pi(C, D)$ and the ideal semigroup $\langle I_1, I_2, \ldots, I_n \rangle$ provided that $C$ has no component of Dynkin type. Section 4 is devoted to prove that there exists a bijection between the Weyl group $W(C)$ and the ideal semigroup $\langle I_1, I_2, \ldots, I_n \rangle$ of $\Pi(C, D)$ for any symmetrizable Cartan matrix $C$ with a symmetrizer $D$. In Section 5, we study the support $\tau$-tilting $\Pi(C, D)$-modules for Dynkin type symmetrizable Cartan matrix $C$ and establish the bijection between the set of all support $\tau$-tilting $\Pi(C, D)$-modules and the ideal semigroup $\langle I_1, I_2, \ldots, I_n \rangle$.

**Notation.** Let $A$ be an algebra over a field $K$. Denote by $\mathbb{D} = \text{Hom}_K(-, K)$ the usual duality. By a module, we mean a right module unless stated otherwise. For a $A$-module $M$, denote by $|M|$ the number of non-isomorphic indecomposable direct summands of $M$. For an integer $l$, denote by $M^l$ the direct sum of $l$ copies of $M$. Denote by $\text{add} M$ the subcategory of $A$-modules consisting of modules which are finite direct sum of direct summands of $M$.

2. Preliminary

Following [10], we recall basic definitions and properties of the preprojective algebras associated with symmetrizable generalized Cartan matrices. In this section, we denote by $K$ an arbitrary field.
2.1. Symmetrizable Cartan matrix and orientation.

Definition 2.1. A matrix $C = (c_{ij}) \in M_n(\mathbb{Z})$ is a symmetrizable (generalized) Cartan matrix if the following conditions are satisfied:

(C1) $c_{ii} = 2$ for all $i$;
(C2) $c_{ij} \leq 0$ for all $i \neq j$;
(C3) $c_{ij} \neq 0$ if and only if $c_{ji} \neq 0$.

(C4) There is a diagonal integer matrix $D = \text{diag}(c_1, \cdots, c_n)$ with $c_i \geq 1$ for all $i$ such that $DC$ is symmetric.

The matrix $D$ appearing in $(C4)$ is called a symmetrizer of $C$ and is minimal if $c_1 + \cdots + c_n$ is minimal. Denote $g_{ij} := |\gcd(c_{ij}, c_{jj})|$, $f_{ij} := |c_{ij}|/g_{ij}$.

An orientation of $C$ is a subset $\Omega$ of $\{1, 2, \cdots, n\} \times \{1, 2, \cdots, n\}$ such that the following hold:

(A1) $\{(i, j), (j, i)\} \in \Omega \neq \emptyset$ if and only if $c_{ij} < 0$;
(A2) For each sequence $((i_1, i_2), (i_2, i_3), \cdots, (i_r, i_{r+1}))$ with $t \geq 1$ and $(i_s, i_{s+1}) \in \Omega$ for all $1 \leq s \leq t$, we have $i_1 \neq i_{t+1}$.

Given an orientation $\Omega$ of $C$, let $Q := Q(C, \Omega) := (Q_0, Q_1, s, t)$ be the quiver with the set of vertices $Q_0 := \{1, \cdots, n\}$ and with the set of arrows $Q_1 := \{(a_{ij}^{(g)} : j \to i)(i, j) \in \Omega, 1 \leq g \leq g_{ij}\} \cup \{\varepsilon_i : i \to i|1 \leq i \leq n\}. Thus we have $s(a_{ij}^{(g)}) = j$ and $t(a_{ij}^{(g)}) = i$ and $s(\varepsilon_i) = t(\varepsilon_i) = i$, where $s(a)$ and $t(a)$ denote the starting and terminal vertex of an arrow $a$, respectively. If $g_{ij} = 1$, we also write $a_{ij}$ instead of $a_{ij}^{(1)}$.

The quiver $Q$ is called a quiver of type $C$ and we say the generalized Cartan matrix $C$ is connected if $Q$ is connected. Denote by $Q^o$ the quiver obtained from $Q$ by deleting all the loops of $Q$. By the condition (A2), we know that $Q^o$ is an acyclic quiver.

2.2. Preprojective algebras associated to symmetrizable Cartan matrices.

Let $C$ be a symmetrizable Cartan matrix with a symmetrizer $D$. Given an orientation $\Omega$ of $C$, the opposite orientation of $\Omega$ is defined as $\Omega^{op} := \{(j, i)|(i, j) \in \Omega\}$. Denote $\Omega\overline{\Omega} = \Omega \cup \Omega^{op}$. For $(i, j) \in \Omega\overline{\Omega}$, set

$$\text{sgn}(i, j) = \begin{cases} +1 & \text{if } (i, j) \in \Omega; \\ -1 & \text{if } (i, j) \in \Omega^{op}. \end{cases}$$

Let $Q$ be the quiver defined by $(C, \Omega)$. Let $\overline{Q} = \overline{Q}(C, \Omega)$ be the quiver obtained from $Q$ by adding a new arrow $a_{ji}^{(g)} : i \to j$ for each arrow $a_{ij}^{(g)} : j \to i$ of $Q^o$.

Let $\Omega(i, -)$ be the set $\{j|(i, j) \in \Omega\}$. Similar one can define $\Omega(-, i)$, $\overline{\Omega}(-, i)$ and $\overline{\Omega}(i, -)$. Now we can define an algebra $H := H(C, D, \Omega) := KQ/I$, where $KQ$ is the path algebra of $Q$, and $I$ is the ideal of $KQ$ defined by the following relations:

$(H_1)$ For each $i \in Q_0$, we have the nilpotency relation $\varepsilon_i^{c_i} = 0$.

$(H_2)$ For each $(i, j) \in \Omega$ and each $1 \leq g \leq g_{ij}$, we have the commutativity relation

$$\varepsilon_i^{f_{ij}} a_{ij}^{(g)} = a_{ij}^{(g)} \varepsilon_i^{f_{ij}}.$$

Definition 2.2. The preprojective algebra $\Pi := \Pi(C, D, \Omega\overline{\Omega})$ associated to the symmetrizable Cartan matrix $C$ is the quotient algebra $K\overline{Q}/I$ of the path algebra $K\overline{Q}$ by the ideal $I$ generated by the following relations:

$(P_1)$ For each $i \in Q_0$, we have the nilpotency relation $\varepsilon_i^{c_i} = 0$.

$(P_2)$ For each $(i, j) \in \Omega\overline{\Omega}$ and each $1 \leq g \leq g_{ij}$, we have the commutativity relation

$$\varepsilon_i^{f_{ij}} a_{ij}^{(g)} = a_{ij}^{(g)} \varepsilon_i^{f_{ij}}.$$
(P₃) For each i, we have the mesh relation
\[ \sum_{j \in \mathbb{R}(-i)} \sum_{g=1}^{g_{ij}} \sum_{f=0}^{f_{ji}} \text{sgn}(i,j) \epsilon_i^f \ a_{ij}^{(g)} \ a_{ji}^{(g)} \epsilon_j^{f_{ji}^{-1}-f} = 0. \]

When the generalized Cartan matrix C is symmetric and the symmetrizer D is the identity matrix, this definition reduces to one of the classical preprojective algebras for acyclic quivers. As the classical preprojective algebras, the name is also justified by the fact that as an H-module, D ≅ \bigoplus_{j \in Q} \bigoplus_{k \in \mathbb{N}} \tau^{-k}(P_j), where P_j is the projective H-module corresponding the vertex j and \tau is the Auslander-Reiten translation (cf. [10] Thm. 1.7). It is also easy to see that if the symmetrizer D is not the identity matrix, the quiver Q has loops.

By the definition of \(\Pi\), it is clear that \(\Pi\) does not depend on the orientation of C. Hence in the following, we simply write \(\Pi = \Pi(C,D)\).

**Example 2.3.** Let \(C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\), \(D = \text{diag}(2,2)\), \(\Omega = (1,2)\). The preprojective algebra \(\Pi = \Pi(C,D)\) is given by the quiver
\[
\begin{array}{c}
\varepsilon_1 \\
1 \\
\end{array}
\begin{array}{c}
a_{21} \\
a_{12} \\
2 \\
\varepsilon_2
\end{array}
\]
with relations \(\varepsilon_1^2 = 0, \varepsilon_2^2 = 0, \varepsilon_1 a_{12} = a_{12} \varepsilon_2, \varepsilon_2 a_{21} = a_{21} \varepsilon_1, a_{12} a_{21} = a_{21} a_{12} = 0\).

**Example 2.4.** Let \(C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}\), \(D = \text{diag}(2,1)\), \(\Omega = (1,2)\). The preprojective algebra \(\Pi = \Pi(C,D)\) is given by the quiver
\[
\begin{array}{c}
\varepsilon_1 \\
1 \\
\end{array}
\begin{array}{c}
a_{21} \\
a_{12} \\
2 \\
\varepsilon_2
\end{array}
\]
with relations \(\varepsilon_1^2 = 0 = \varepsilon_2, \varepsilon_1 a_{12} = a_{12} \varepsilon_2, \varepsilon_2 a_{21} = a_{21} \varepsilon_1, a_{12} a_{21} \varepsilon_1 + \varepsilon_1 a_{12} a_{21} = 0, a_{21} a_{12} = 0\). Since \(\varepsilon_2^2 = 0 = \varepsilon_2\), we can delete the loop \(\varepsilon_2\) and then \(\Pi(C,D)\) is given by the quiver
\[
\begin{array}{c}
\varepsilon_1 \\
1 \\
\end{array}
\begin{array}{c}
a_{21} \\
a_{12} \\
2
\end{array}
\]
with relations \(\varepsilon_1^2 = 0, a_{12} a_{21} \varepsilon_1 + \varepsilon_1 a_{12} a_{21} = 0, a_{21} a_{12} = 0\).

**2.3. The quadratic form.** Given a symmetrizable Cartan matrix C with a symmetrizer D, the quadratic form \(q_C : \mathbb{Z} \to \mathbb{Z}\) of C is defined as follows
\[ q_C := \sum_{i=1}^{n} \epsilon_i X_i^2 - \sum_{i<j} \epsilon_i \epsilon_j |X_i X_j|. \]
The symmetrizable Cartan matrix C is of Dynkin type if and only if \(q_C\) is positive definite. It is well-known that connected symmetrizable Cartan matrices of Dynkin type can be classified by the Dynkin diagrams of type \(A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2\).

**Lemma 2.5.** [10] Corollary. 12.7 Suppose that the Cartan matrix C is of Dynkin type. Then the preprojective algebra \(\Pi(C,D)\) is a selfinjective algebra for any symmetrizer D of C.

**2.4. Locally free modules.** Let \(e_1, \ldots, e_n\) be the idempotents corresponding to the vertices of \(\overline{Q}\). Let \(H_i\) be the truncated polynomial ring \(K[e_i]/(e_i^2)\). A \(\Pi\)-module M is called locally free if \(Me_i\) is a free \(H_i\)-module for each i. Similarly, we can define locally free \(H\)-modules and we refer to [10] for the precisely definition. It was shown in [10] that the projective \(H\)-modules and the projective \(\Pi\)-module are locally free.

Let \(\text{Rep}_{l.f.}(\Pi)\) be the category of all locally free \(\Pi\)-modules and \(\text{rep}_{l.f.}(\Pi)\) the category of all finitely generated locally free \(\Pi\)-modules. The proof of Lemma 3.8 in [10] also implies the following.
Lemma 2.6. $\text{Rep}_{l,f}(\Pi)$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms.

Note that each projective $\Pi$-module is locally free, we clearly have

Corollary 2.7. Let $M$ be a $\Pi$-module. If proj. dim $M < \infty$, then $M$ is locally free.

For preprojective algebras of non-Dynkin types, the locally free modules are characterized by the finiteness of projective dimension. Namely, we have

Lemma 2.8. [10] Proposition 12.3 Let $C$ be a symmetrizable Cartan matrix with a symmetrizer $D$ which has no component of Dynkin type and $\Pi = \Pi(C, D)$ the associated preprojective algebra. Let $M$ be a $\Pi$-module, then the following statements are equivalent:

1. $M$ is locally free;
2. proj. dim $M \leq 2$;
3. proj. dim $M < \infty$.

The following well-known fact for classical preprojective algebras has been generalized to locally free modules for the preprojective algebras associated with symmetrizable Cartan matrices in [10].

Lemma 2.9. [10] Theorem 12.6 Let $M \in \text{Rep}_{l,f}(\Pi)$, $N \in \text{rep}_{l,f}(\Pi)$. Then
(a) there is a functorial isomorphism
$$\text{Ext}_{\Pi}^{1}(M, N) \cong \text{Ext}_{\Pi}^{1}(N, M).$$
(b) if $C$ has no component of Dynkin type, there are functorial isomorphisms
$$\text{Ext}_{\Pi}^{i-1}(M, N) \cong \text{Ext}_{\Pi}^{i}(N, M)$$ for $i = 0, 1, 2$.

2.5. Generalized simple modules $E_i$. Let $C$ be a symmetrizable Cartan matrix with a symmetrizer $D = \text{diag}(c_1, \cdots , c_n)$. Let $M$ be a locally free $\Pi$-module. For each $i \in Q_0$, let $r_i$ be the rank of the free $H_i$-module $M e_i$. Put $\text{rank}(M) := (r_1, \cdots , r_n)$ and we call $r_1 + \cdots + r_n$ the rank length of $M$.

Let $E_1, \cdots , E_n$ be the indecomposable locally free $\Pi$-modules with $\text{rank}(E_i) = \alpha_i$, where $\alpha_1, \cdots , \alpha_n$ is the standard basis of $\mathbb{Z}_n$. In particular, $E_i$ is the regular representation of $H_i$ and we call $E_i$ the generalized simple $\Pi$-module. Obviously, $E_i$ is also an $H$-module and we also call it a generalized simple $H$-module. If we denote by $S_i$ the simple $\Pi$-module associated to the vertex $i$, then $E_i$ is the uniserial module with composition factor $S_i$ of length $c_i$. For each $1 \leq d \leq c_i$, let $dS_i$ be the uniserial module of length $d$.

Proposition 2.10. For each $j \in Q_0$, denote by $e_j\Pi$ the projective right $\Pi$-module associated to the vertex $j$. We have

(a) $\text{Hom}_{\Pi}(e_i, \Pi, E_i) \cong E_i$ and $\text{Hom}_{\Pi}(e_j, \Pi, E_i) = 0$ for $i \neq j$, where $E_i$ is the generalized simple $\Pi$-module associated to $i$;
(b) $e_i\Pi \otimes_{\Pi} E_i \cong E_i$ and $e_j\Pi \otimes_{\Pi} E_i = 0$ for $i \neq j$, where $E_i$ is the generalized simple $\Pi^{op}$-module associated to $i$.

Proof. Note that $\text{Hom}_{\Pi}(e_j, \Pi, E_i) \cong E_j e_i$ and $e_j\Pi \otimes_{\Pi} E_i \cong e_j E_i$. Now the results follow from the fact that $e_i e_i = 0 = e_i e_j$ for $i \neq j$ and $e_i E_i = E_i = E_i e_i$. \hfill $\Box$

We know that if $C$ is of Dynkin type, by Lemma 2.3 $\Pi$ is a selfinjective algebra. Denote by $\nu = D \text{Hom}_{\Pi}(-, \Pi)$ the Nakayama functor and $\sigma : Q_0 \to Q_0$ the Nakayama permutation of $\Pi$, i.e. $e_i\Pi \cong D(e_{\sigma(i)})$. Then we have
Proposition 2.11. Let $C$ be a symmetrizable Cartan matrix of Dynkin type with a symmetrizer $D$ and $\Pi = \Pi(C, D)$ the associated preprojective algebra. We have $\nu E_{\sigma(i)} \cong E_i$ and $c_i = c_{\sigma(i)}$ for each $i \in Q_0$.

Proof. We show that for each $1 \leq d \leq c_{\sigma(i)}, \nu(dS_{\sigma(i)}) \cong dS_i$ by induction on $d$. For $d = 1$, we clearly have $\nu(S_{\sigma(i)}) = \nu(soc e_i \Pi) = top e_i \Pi = S_i$. Now suppose that we have $\nu(d^{-1}S_{\sigma(i)}) \cong d^{-1}S_i$. Consider the following non-split short exact sequence

$$0 \to S_{\sigma(i)} \to dS_{\sigma(i)} \to d^{-1}S_{\sigma(i)} \to 0.$$ 

Recall that $\Pi$ is selfinjective and hence the functor $\nu$ is exact. Applying $\nu$ to the above short exact sequence yields the following non-split short exact sequence

$$0 \to \nu(S_{\sigma(i)}) \to \nu(dS_{\sigma(i)}) \to \nu(d^{-1}S_{\sigma(i)}) \to 0.$$ 

Now one can deduce $\nu(dS_{\sigma(i)}) \cong dS_i$ from the fact that there is a unique non-split extension of $S_i$ by $d^{-1}S_i$ whose middle term is $dS_i$. In particular, we have $\nu(E_{\sigma(i)}) \cong c_{\sigma(i)}S_i$ which is indecomposable. Consequently, $c_{\sigma(i)} \leq c_i$. Similarly, using the functor $\nu^{-1} = Hom(\Pi, -)$, one can show that $c_i \leq c_{\sigma(i)}$. Hence we have $c_i = c_{\sigma(i)}$ and $\nu(E_{\sigma(i)}) \cong E_i$. □

The following result is an analogue of the classical cases (cf. [3 Prop 4.2]) which plays an important role in our investigation.

Lemma 2.12. Let $\Pi = \Pi(C, D)$ be a preprojective algebra associated to a symmetrizable Cartan matrix $C$ with a symmetrizer $D$. Let $E_i$ be a generalized simple $\Pi$-module.

If $C$ has no component of Dynkin type, then we have an exact sequence

$$0 \to e_i \Pi \to \bigoplus_{j \in I(i,-)} (e_j \Pi)^{c_{ij}} \to e_i \Pi \to E_i \to 0.$$ 

If $C$ is of Dynkin type, then we have an exact sequence

$$0 \to E_{\sigma(i)} \to e_i \Pi \to \bigoplus_{j \in I(i,-)} (e_j \Pi)^{c_{ij}} \to e_i \Pi \to E_i \to 0,$$ 

where $\sigma : Q_0 \to Q_0$ is a Nakayama permutation of $\Pi$.

Proof. Note that the exact sequence (1) is a direct consequence of [10 Prop 12.1]. Namely, by applying the functor $E_i \otimes \Pi$ to the sequence of Proposition 12.1 in [10], we obtain the following sequence

$$e_i \Pi \xrightarrow{f} \bigoplus_{j \in I(i,-)} (e_j \Pi)^{c_{ij}} \to e_i \Pi \to E_i \to 0,$$ 

which is the beginning of a projective resolution of $E_i$.

If $C$ has no component of Dynkin type, then by Lemma 2.8 we deduce that the map $f$ is injective and hence obtain the exact sequence (1).

Now assume that $C$ is of Dynkin type. In this case, the preprojective algebra $\Pi$ is a finite-dimensional selfinjective algebra over $\mathbb{K}$. Denote by $M$ the kernel of $f$. It remains to show that $M \cong E_{\sigma(i)}$. Note that one can easily see that $M$ is locally free by Lemma 2.6. On the other hand, we clearly have $soc M = soc e_i \Pi = S_{\sigma(i)}$. Thus, to show $M \cong E_{\sigma(i)}$, it suffices to prove that $\dim_K M = c_{\sigma(i)}$.

Now choose an orientation $\Omega$ of $C$ such that $i$ is a sink vertex in the quiver $Q^o$. Let $H = H(C, D, \Omega)$ be the algebra defined in Section 2.2. Denote by $P_j = He_j$ the left projective $H$-module corresponding to the vertex $j$. Since $i$ is a sink vertex in $Q^o$, one has $\dim_K Hom_H(P_i, P_j) = c_j$. Consider the following short exact sequence of left $H$-modules

$$0 \to P_i \to \bigoplus_{j \in I(i,-)} e_j ^{c_{ij}} \to \tau^{-1} E_j \to 0,$$ 

where $\tau$ is the shift in the category of $H$-modules, and $\Pi^o$ is the orientation $\Pi$ with all arrows reversed. By [3, Theorem 3.2], we have

$$\bigoplus_{j \in I(i,-)} e_j ^{c_{ij}} \cong e_i \Pi \oplus P_i,$$ 

where $P_i$ is a preprojective algebra. Hence we have $M \cong E_{\sigma(i)}$, as desired.
where $\tau$ is the Auslander-Reiten translation of left $H$-modules. Applying the functor $\text{Hom}_H(\cdot, \Pi)$ to the above exact sequence, we obtain an exact sequence of right $H$-modules

$$0 \to \text{Hom}_H(\tau^{-1}P_i, \Pi) \to \bigoplus_{j \in \Omega(i, -)} (e_j \Pi)^{|\alpha_j|} \to e_i \Pi \to \text{Ext}^1_H(\tau^{-1}P_i, \Pi) \to 0.$$  

Recall that as a left $H$-module, we have

$$\Pi = \bigoplus_{j \in Q_0} \bigoplus_{k \in \mathbb{N}} \tau^{-k}(P_j),$$

which implies that $\text{Ext}^1_H(\tau^{-1}P_i, \Pi) \cong \text{Ext}^1_H(\tau^{-1}P_i, P_j) \cong \mathbb{D}\text{Hom}_H(P_i, P_j)$. In particular, $\dim_K \text{Ext}^1_H(\tau^{-1}P_i, \Pi) = c_i = \dim_K E_i$. Finally, consider the split exact sequence of left $H$-modules

$$0 \to \mathbb{D}H \to \Pi \to \tau \Pi \to 0,$$

where the first map is an inclusion and the second one is the projection of $\Pi$ onto its direct summand $\tau \Pi$. Applying the functor $\text{Hom}_H(P_i, \cdot)$, we obtain a short exact sequence

$$0 \to \text{Hom}_H(P_i, \mathbb{D}H) \to \text{Hom}_H(P_i, \Pi) \to \text{Hom}_H(P_i, \tau \Pi) \to 0.$$  

It is clear that $\text{Hom}_H(P_i, \tau \Pi) \cong \text{Hom}_H(\tau^{-1}P_i, \Pi)$ and $\text{Hom}_H(P_i, \Pi) \cong e_i \Pi$. Note that $i$ is a sink vertex in $Q^\circ$, which implies that $\Omega(i, -) = \Omega(i, -)$. We conclude that $\dim_K M = \dim_K \text{Hom}_H(P_i, \mathbb{D}H) = \dim_K \text{Hom}_H(H, P_i) = c_i = c_{\tau(i)}$ by Proposition 2.11.

2.6. The two-sided ideal $I_i$. Let $\Pi = \Pi(C, D)$ be a preprojective algebra. For each $i \in Q_0$, denote by $I_i$ the two-sided ideal $\Pi(1-e_i)\Pi$, it is easy to see that $e_j I_i = e_j \Pi(1-e_i)\Pi = e_j \Pi$ for $j \neq i$. Thus we obtain the following decomposition of $I_i$ as right $H$-module

$$I_i = \bigoplus_{j \in Q_0} e_j I_i = e_i I_i \oplus \bigoplus_{j \neq i} e_j \Pi.$$  

By the definition of $I_i$, we clearly have the following short exact sequence

$$0 \to I_i \to \Pi \to E_i \to 0,$$

which induces a short exact sequence

$$0 \to e_i I_i \to e_i \Pi \to E_i \to 0.$$  

Now by Lemma 2.12, we obtain a projective presentation of $e_i I_i$:

$$e_i \Pi \to \bigoplus_{j \in \Omega(i, -)} (e_j \Pi)^{|\alpha_j|} \to e_i I_i \to 0.$$  

Since $e_i \Pi$, $\Pi$, $E_i$ are locally free, by Lemma 2.8, both $I_i$ and $e_i I_i$ are locally free. On the other hand, $\text{Hom}_{\Pi}(e_j \Pi, E_i) = 0$ for $j \neq i$, we obtain that $\text{Hom}_{\Pi}(e_i I_i, E_i) = 0$. In particular, we have proved the following result.

**Proposition 2.13.** For any $i \in Q_0$, $I_i$ and $e_i I_i$ are locally free and $\text{Hom}_{\Pi}(I_i, E_i) = 0$.

3. Preprojective algebras of non-Dynkin type

From this section to the end of this paper, we assume moreover that $K$ is an algebraically closed field. The purpose of this section is to generalize the classification results of [4] over classical preprojective algebras of non-Dynkin types. Hence, in this section, we always assume that $C$ is a symmetrizable Cartan matrix with a symmetrizer $D$ which has no component of Dynkin type and $\Pi = \Pi(C, D)$ the associated preprojective algebra. We follow [4, Chapter II].

**Definition 3.1.** For an algebra $\Lambda$, we say that a finitely presented $\Lambda$-module $T$ is a tilting module if:
Lemma 3.3.  

Proof.  

(i) there exists an exact sequence \( 0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow T \rightarrow 0 \) with finitely generated projective \( \Lambda \)-modules \( P_i \);  
(ii) \( \mathbf{Ext}_\Lambda^i(T, T) = 0 \) for any \( i > 0 \);  
(iii) there exists an exact sequence \( 0 \rightarrow \Lambda \rightarrow T_0 \rightarrow \cdots \rightarrow T_n \rightarrow 0 \) with \( T_i \) in \( \text{add} \, T \).  

A partial \( \Lambda \)-tilting module is a direct summand of a \( \Lambda \)-tilting module.  

Since tilting modules have finite projective dimension, the projective dimension of a partial tilting module is also finite. The following result is a direct consequence of Lemma 2.8.

Proposition 3.2. All tilting \( \Pi \)-modules and partial tilting \( \Pi \)-modules are locally free and have projective dimension at most two.

For a partial tilting module of projective dimension at most one, we have

Lemma 3.3. Let \( T \) be a partial tilting \( \Pi \)-module of projective dimension at most one and \( E_i \) a generalized simple \( \Pi^{op} \)-module associated to the vertex \( i \). Then at least one of the statements \( T \otimes \Pi \, E_i = 0 \) and \( \mathbf{Tor}_{\Pi}^1(T, E_i) = 0 \) holds.

Proof. Let \( 0 \rightarrow P_1 \rightarrow P_0 \rightarrow T \rightarrow 0 \) be a minimal projective resolution of \( T \). Applying \( - \otimes \Pi \, E_i \) to this sequence, we get an exact sequence

\[
0 \rightarrow \mathbf{Tor}_{\Pi}^1(T, E_i) \rightarrow P_1 \otimes \Pi \, E_i \rightarrow P_0 \otimes \Pi \, E_i \rightarrow T \otimes \Pi \, E_i \rightarrow 0.
\]

Since \( P_0 \) and \( P_1 \) do not have a common direct summand and \( e_j \Pi \otimes \Pi \, E_i = 0 \) when \( j \neq i \), we have either \( P_1 \otimes \Pi \, E_i = 0 \) or \( P_0 \otimes \Pi \, E_i = 0 \), which implies the desired result. \( \square \)

The following result on (partial) tilting complexes are useful in studying derived equivalences. For the definitions of tilting complex and two-sided tilting complex, we refer to \([16]\) for details.

Lemma 3.4. \([10, 15]\) For rings \( \Lambda \) and \( \Gamma \), let \( T \in D(\text{Mod} \, \Lambda \otimes \mathbb{Z} \, \Gamma^{op}) \) be a two-sided tilting complex.  

(a) For any tilting complex (respectively, partial tilting complex) \( U \) of \( \Gamma \), we have a tilting complex (respectively, partial tilting complex) \( T \odot_U \Pi \) of \( \Lambda \) such that \( \mathbf{End}_{D(\text{Mod} \, \Lambda)}(T \odot_U \Pi) \cong \mathbf{End}_{D(\text{Mod} \, \Gamma)}(U) \);  

(b) \( \mathbf{RHom}_{\Lambda}(T, \Lambda) \) and \( \mathbf{RHom}_{\Gamma^{op}}(T, \Gamma) \) are two-sided tilting complexes which are isomorphic in \( D(\text{Mod} \, \Gamma \otimes \mathbb{Z} \, \Lambda^{op}) \).

We also remark that a tilting module is nothing but a module which is a tilting complex.

Proposition 3.5. The two-side ideal \( I_i \) is a tilting \( \Pi \)-module of projective dimension at most one and \( \mathbf{End}_\Pi(I_i) \cong \Pi \).

Proof. Recall that \( E_i \in \text{Rep}_{\Pi}(\Pi) \) and \( I_i \in \text{Rep}_{\Pi}(\Pi) \), one deduces that

\[
\mathbf{Ext}_\Pi^1(I_i, I_i) \cong \mathbf{Ext}_\Pi^1(E_i, I_i) \cong \mathbb{D} \mathbf{Hom}_\Pi(I_i, E_i) = 0
\]

by the exact sequence \([3]\) and Lemma 2.9. Moreover, we have

\[
\mathbf{Ext}_\Pi^k(E_i, \Pi) \cong \mathbb{D} \mathbf{Ext}_\Pi^{2-k}(\Pi, E_i) = 0
\]

for \( k = 0, 1 \). Now applying the functor \( \mathbf{Hom}_\Pi(\cdot, \Pi) \) to the exact sequence \([3]\), we obtain \( \mathbf{Hom}_\Pi(I_i, \Pi) \cong \Pi \). Recall that \( \mathbf{Hom}_\Pi(I_i, E_i) = 0 \), we obtain \( \mathbf{End}_\Pi(I_i) \cong \Pi \) by applying the functor \( \mathbf{Hom}_\Pi(I_i, \cdot) \) to the exact sequence \([3]\).

It remains to show that \( I_i \) is a tilting \( \Pi \)-module of projective dimension at most one. Note that \( I_i = e_i I_i \oplus (\bigoplus_{j \neq i} e_j \Pi) \) as right \( \Pi \)-module. By the exact sequence \([1]\), we have the following projective resolution of \( e_i I_i \):

\[
0 \rightarrow e_i \Pi \rightarrow \bigoplus_{j \in \Pi(i, \cdot)} (e_j \Pi)^{c_{ij}} \rightarrow e_i I_i \rightarrow 0.
\]
In particular, it implies that $\text{proj. dim } I_i \leq 1$. On the other hand, we also obtain the following exact sequence

$$0 \rightarrow \Pi \rightarrow (\bigoplus_{j \in N(i,-)} (e_j \Pi)^{[e_i, e_j]}) \oplus (1 - e_i)\Pi \rightarrow e_i I_i \rightarrow 0,$$

which implies that $I_i$ is a tilting $\Pi$-module of projective dimension at most one.

**Proposition 3.6.** Let $T$ be a tilting $\Pi$-module of projective dimension at most one and $E_i$ the generalized simple $\Pi^{op}$-module associated to vertex $i$.

(a) If $T \otimes_{\Pi} E_i = 0$, then $TI_i = T$.

(b) If $T \otimes_{\Pi} E_i \neq 0$, then $T \otimes_{\Pi} I_i \cong T \otimes_{\Pi} I_i \cong TI_i$ is also a tilting $\Pi$-module of projective dimension at most one.

**Proof.** Applying $T \otimes_{\Pi} -$ to the exact sequence

$$0 \rightarrow I_i \rightarrow \Pi \rightarrow E_i \rightarrow 0$$

yields an exact sequence

$$\text{Tor}_1^\Pi(T, E_i) \rightarrow T \otimes_{\Pi} I_i \rightarrow T \otimes_{\Pi} \Pi \rightarrow T \otimes_{\Pi} E_i \rightarrow 0.$$

It is clear that $\text{im}(T \otimes_{\Pi} f) = TI_i$.

(a) If $T \otimes_{\Pi} E_i = 0$, then $T \otimes_{\Pi} f$ is surjective and $T \cong T \otimes_{\Pi} \Pi \cong \text{im}(T \otimes_{\Pi} f) = TI_i$.

(b) If $T \otimes_{\Pi} E_i \neq 0$, we have $\text{Tor}_1^\Pi(T, E_i) = 0$ by Lemma 3.3. In particular, the morphism $T \otimes_{\Pi} f$ is injective and consequently $T \otimes_{\Pi} I_i \cong TI_i$. Since $\text{proj. dim } T \leq 1$, then $\text{Tor}_1^\Pi(T, I_i) = \text{Tor}_2^\Pi(T, E_i) = 0$ and $\text{Tor}_2^\Pi(T, I_i) = 0$. Hence $T \otimes_{\Pi} I_i \cong T \otimes_{\Pi} I_i \cong TI_i$. By Lemma 3.4, we know that $T \otimes_{\Pi} I_i = TI_i$ is a tilting $\Pi$-module and $\text{End}_{\Pi}(TI_i) \cong \text{End}_{\Pi}(T)$.

Since $T$ and $TI_i$ are tilting modules, we deduce that $T$ and $TI_i$ are locally free by Proposition 3.2.

Consider the short exact sequence

$$0 \rightarrow TI_i \rightarrow T \rightarrow T/TI_i \rightarrow 0,$$

which implies that $T/TI_i$ is also locally free and $\text{proj. dim } (T/TI_i) \leq 2$ by Lemma 2.8. One can show that $\text{proj. dim } TI_i \leq 1$. This finishes the proof.

**Definition 3.7.** Let $C$ be a symmetrizable Cartan matrix with a symmetrizer $D$ which has no component of Dynkin type and $\Pi = \Pi(C, D)$ the associated preprojective algebra. An ideal $I$ of $\Pi$ is said to be cofinite tilting (resp. partial tilting) if $\Pi/I$ has finite length and $I$ is tilting (resp. partial tilting) as left $\Pi$-module and as right $\Pi$-module.

Denote by $(I_1, I_2, \cdots, I_n)$ the semigroup generated by $I_1, I_2, \cdots, I_n$. The following result is a direct consequence of Proposition 3.5 and Proposition 3.6.

**Proposition 3.8.** Each $T \in \langle I_1, I_2, \cdots, I_n \rangle$ is a cofinite tilting ideal and satisfies $\text{End}_{\Pi}(T) \cong \Pi$.

On the other hand, we have

**Lemma 3.9.** Let $I$ be a cofinite tilting ideal of $\Pi$, then $\text{proj. dim } I \leq 1$ and $\text{Hom}_{\Pi}(I, \Pi) \cong \text{Hom}_{\Pi}(\Pi, \Pi) \cong \Pi$.

**Proof.** Consider the short exact sequence $0 \rightarrow I \rightarrow \Pi \rightarrow \Pi/I \rightarrow 0$. Note that $I$ is an tilting module, which implies that $I$ is locally free by Proposition 3.2. Consequently, $\Pi/I$ is locally free and $\text{proj. dim } (\Pi/I) \leq 2$. Then it is easy to see that $\text{proj. dim } I \leq 1$.

Since $\Pi/I$ is a locally free $\Pi$-module with finite length, we clearly have $\Pi/I \in \text{rep}_{\Pi} (\Pi)$. Therefore, $\text{Ext}_{\Pi}^k(\Pi/I, \Pi) \cong D \text{Ext}_{\Pi}^{k-1}(\Pi, \Pi/I) = 0$ for $k = 0, 1$ by Lemma 2.9. Applying the functor $\text{Hom}_{\Pi}(\cdot, \Pi)$ to the above exact sequence, one obtains that $\text{Hom}_{\Pi}(I, \Pi) \cong \text{Hom}_{\Pi}(\Pi, \Pi) \cong \Pi$. 

\end{proof}
Proposition 3.10. Each cofinite partial tilting left or right ideal of $\Pi$ is a cofinite tilting ideal and any cofinite tilting ideal of $\Pi$ belongs to $(I_1, I_2, \cdots, I_n)$.

Proof. Let $T$ be a cofinite partial tilting right ideal of $\Pi$. By Lemma 3.2, $T$ is locally free. If $T \neq \Pi$, consider the locally free $\Pi$-module $\Pi/T$, which has a generalized simple submodule, say $E_i$. Recall that we have $\text{Hom}_\Pi(E_i, \Pi) = 0 = \text{Ext}^1_\Pi(E_i, \Pi)$. By applying the functor $\text{Hom}_\Pi(E_i, -)$ to the short exact sequence

$$0 \rightarrow T \rightarrow \Pi \rightarrow \Pi/T \rightarrow 0,$$

one obtains $\text{Hom}_\Pi(E_i, T) = 0$ and $\text{Ext}^1_\Pi(E_i, T) \cong \text{Hom}_\Pi(E_i, \Pi/T) \neq 0$. Consequently, $\text{Tor}_1^{\Pi}(T, \Delta E_i) \cong D \text{Ext}^1_\Pi(E_i, T) \neq 0$ and $T \otimes \Pi \Delta E_i = 0$ by Lemma 3.3.

Now put $U = \text{RHom}_{\Pi}(I_i, T)$. By Lemma 3.3, we deduce that $U \cong T \otimes \Pi \text{RHom}_{\Pi}(I_i, T)$ is a partial tilting $\Pi$-module. Moreover, by $U \cong \text{Hom}_\Pi(I_i, T) \subset \text{Hom}_\Pi(I_i, \Pi) \cong \Pi$, we may identify $U$ as a partial tilting right ideal of $\Pi$.

Now applying $\text{Hom}_\Pi(-, T)$ to the exact sequence of $\Pi$-bimodules

$$0 \rightarrow I_i \rightarrow \Pi \rightarrow E_i \rightarrow 0$$

yields the following exact sequence of right $\Pi$-modules

$$0 = \text{Hom}_\Pi(E_i, T) \rightarrow \text{Hom}_\Pi(\Pi, T) \rightarrow \text{Hom}_\Pi(I_i, T) \rightarrow \text{Ext}^1_\Pi(E_i, T) \rightarrow \text{Ext}^1_\Pi(E_i, \Pi) = 0.$$

In particular, we have

$$0 \rightarrow T \rightarrow U \rightarrow \text{Ext}^1_\Pi(E_i, T) \rightarrow 0.$$ 

Hence $U/T \cong \text{Ext}^1_\Pi(E_i, T) \cong \text{Hom}_\Pi(E_i, \Pi/T) \neq 0$. For $j \neq i$, it is not hard to see that $\text{Hom}_\Pi(e_j \Pi, U/T) = 0$. Note that $U/T$ is also locally free, we conclude that $U/T$ is a direct sum copies of $E_i$, say $U/T \cong E_i^l$. We can rewrite the short exact sequence as

$$0 \rightarrow T \rightarrow U \rightarrow E_i^l \rightarrow 0$$

and obtain an exact sequence

$$0 \rightarrow TI_i \rightarrow UI_i \rightarrow E_i^l I_i \rightarrow 0.$$ 

Since $E_i I_i = E_i \Pi(1-e_i)\Pi = E_i(1-e_i)\Pi = 0$ and $TI_i = T$ because of $T \otimes \Pi \Delta E_i = 0$, we have $T = UI_i$.

Thus $T \in \langle I_1, I_2, \cdots, I_n \rangle$ by induction on the rank length of locally free module $\Pi/T$. $\Box$

Proposition 3.11. Let $U, T \in \langle I_1, I_2, \cdots, I_n \rangle$ be two cofinite tilting ideals such that $U$ is isomorphic to $T$ as left modules or as right modules, then $T = U$.

Proof. Assume that $U$ is isomorphic to $T$ as right $\Pi$-modules. Denote the isomorphisms by $f : U \rightarrow T$ and $g = f^{-1} : T \rightarrow U$. By Lemma 3.9, $f, g, fg, gf$ can be extended to morphisms from $\Pi$ to $\Pi$, still denoted by $f, g, fg, gf$. Since $(fg)_T = id_T, (gf)_U = id_U$, then $fg = id_{\Pi} = gf$. Thus there exists an invertible element $x \in \Pi$ such that $f$ is the left multiplication with $x$. Consequently, we have $T = f(U) = xU = U$. $\Box$

In sum, we have proved the main result of this section.

Theorem 3.12. Let $C$ be a symmetrizable Cartan matrix with a symmetrizer $D$ which has no component of Dynkin type and $\Pi = \Pi(C, D)$ the associated preprojective algebra. There is a bijection between the set of all cofinite tilting $\Pi$-ideals and the semigroup $\langle I_1, I_2, \cdots, I_n \rangle$. $\Box$
4. Weyl Group and ideal semigroup

As before, let $C = (c_{ij})$ be a symmetrizable Cartan matrix. Denote by $W = W(C)$ the Coxeter group generated by $\{s_i \mid 1 \leq i \leq n\}$ with the following relations:

1. $s_i^2 = id$;
2. $s_i s_j = s_j s_i$, if $c_{ij} c_{ji} = 0$;
3. $s_i s_j s_i = s_j s_i s_j$, if $c_{ij} c_{ji} = 1$;
4. $s_i s_j s_i s_j = s_j s_i s_j s_i$, if $c_{ij} c_{ji} = 2$;
5. $s_i s_j s_i s_j s_i s_j$, if $c_{ij} c_{ji} = 3$.

By [5, Thm 16.17], $W$ is just the Weyl group of the Kac-Moody Lie algebra associated to the symmetrizable Cartan matrix $C$.

For a word $w = s_{i_1} s_{i_2} \cdots s_{i_l} \in W$, it is obvious that the following operations on $w$ keep the word $w$ unchanged.

(i) remove $s_i s_j$;

(ii) replace $s_i s_j$ by $s_j s_i$, if $c_{ij} c_{ji} = 0$;

(iii) replace $s_i s_j s_i$ by $s_j s_i s_j$, if $c_{ij} c_{ji} = 1$;

(iv) replace $s_i s_j s_i s_j$ by $s_j s_i s_j s_i$, if $c_{ij} c_{ji} = 2$;

(v) replace $s_i s_j s_i s_j s_i s_j$ by $s_j s_i s_j s_i s_j s_i$, if $c_{ij} c_{ji} = 3$.

Here we call the operation (i) nil-move and the operations (ii), (iii), (iv), (v) braid-moves.

The following result for Coxeter group is well-known (cf. Theorem 3.3.1 of [2]).

**Lemma 4.1.** Let $W$ be a Coxeter group and $w \in W$.

(a) Any expression $s_{i_1} s_{i_2} \cdots s_{i_l}$ for $w$ can be transformed into a reduced expression for $w$ by a sequence of nil-moves and braid-moves.

(b) Every two reduced expressions for $w$ can be connected via a sequence of braid-moves.

Let $V^*$ be a vector space with basis $\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*$, define the linear transformation $\sigma_i^* \in GL(V^*)$ by $\sigma_i^*(p) = p - p_i(\sum_{j=1}^{n} c_{ji} \alpha_j^*)$ for $p = \sum_{j=1}^{n} p_j \alpha_j^*$. Obviously, we have

$$\sigma_i^*(\alpha_i^*) = \begin{cases} \alpha_i^* = \sum_{j=1}^{n} c_{ji} \alpha_j^* & \text{if } l = i; \\ \alpha_i^* & \text{if } l \neq i. \end{cases}$$

The following lemma gives a geometric realization of $W$ (cf. Theorem 4.2.7 of [2]).

**Lemma 4.2.** The mapping $s_i \mapsto \sigma_i^*$ for $i = 1, 2, \cdots, n$, extends uniquely to an injective homomorphism $\varphi : W \to GL(V^*)$.

In the following, we are going to discuss the relationship between the Weyl group and the semigroup $\langle I_1, I_2, \cdots, I_n \rangle$. First, let us deal some special cases. Since the preprojective algebras $\Pi(C, D)$ does not depend on the orientation $\Omega$ of $C$, we omit the orientation in the following.

**Proposition 4.3.** Let $C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and $D = \text{diag}(d, d)$ with positive integer $d$. The preprojective algebra $\Pi = \Pi(C, D)$ is given by the quiver

$$\varepsilon_1 \begin{array}{ccc} 1 & \overset{a_{21}}{\longrightarrow} & 2 \end{array} \varepsilon_2$$

with relations $\varepsilon_1^2 = 0$, $\varepsilon_2^2 = 0$, $\varepsilon_1 a_{12} = a_{12} \varepsilon_2$, $\varepsilon_2 a_{21} = a_{21} \varepsilon_1$, $a_{12} a_{21} = a_{21} a_{12} = 0$, then $I_1 I_2 I_1 = 0 = I_2 I_1 I_2$. 

Proposition 4.4. Let $C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, $D = \text{diag}(2d, d)$ with positive integer $d$. The preprojective algebra $\Pi = \Pi(C, D)$ is given by the quiver

\[ \varepsilon_1 \xrightarrow{a_{21}} \varepsilon_1 \xrightarrow{a_{12}} \varepsilon_2 \]

with relations $\varepsilon_1^{2d} = 0, \varepsilon_2^d = 0, \varepsilon_1^2 a_{12} = a_{12} \varepsilon_2, \varepsilon_2 a_{21} = a_{21} \varepsilon_1, a_{12} a_{21} \varepsilon_1 + \varepsilon_1 a_{12} a_{21} = 0, a_{21} a_{12} = 0$, then $I_1 I_2 I_1 I_2 = 0 = I_2 I_1 I_2 I_1$.

Proof. For any non-negative integers $l, k$, using the relations, we have

\[ a_{12} \varepsilon_1^l a_{21} \varepsilon_1^1 a_{12} = a_{12} a_{21} \varepsilon_1^{2k} \varepsilon_1^1 a_{12} = (-1)^{2k+l} \varepsilon_1^{2k+l} a_{12} a_{21} a_{12} = 0, \]

and

\[ a_{21} \varepsilon_1^l a_{21} \varepsilon_1^1 a_{21} = a_{21} \varepsilon_1^l a_{12} a_{21} \varepsilon_1^{2k} = (-1)^{l} a_{21} a_{12} a_{21} \varepsilon_1^{2k+l} = 0. \]

Consequently, $e_2 \Pi e_1 \Pi e_2 \Pi e_1 = 0$ and $e_1 \Pi e_2 \Pi e_1 \Pi e_2 = 0$, which imply the desired equalities $I_1 I_2 I_1 I_2 = \Pi e_2 \Pi e_1 \Pi e_2 \Pi e_1 = 0$ and $I_2 I_1 I_2 I_1 = \Pi e_1 \Pi e_2 \Pi e_1 \Pi e_2 = 0$.

Proposition 4.5. Let $C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, $D = \text{diag}(3d, d)$ with positive integer $d$. The preprojective algebra $\Pi = \Pi(C, D)$ is given by the quiver

\[ \varepsilon_1 \xrightarrow{a_{21}} \varepsilon_1 \xrightarrow{a_{12}} \varepsilon_2 \]

with relations $\varepsilon_1^{3d} = 0, \varepsilon_2^d = 0, \varepsilon_1^3 a_{12} = a_{12} \varepsilon_2, \varepsilon_2 a_{21} = a_{21} \varepsilon_1^3, a_{12} a_{21} \varepsilon_1 = \varepsilon_1 a_{12} a_{21} \varepsilon_1 + \varepsilon_1^2 a_{12} a_{21} = 0, a_{21} a_{12} = 0$, then $I_1 I_2 I_1 I_2 I_1 = 0 = I_2 I_1 I_2 I_1 I_2$.

Proof. The proof involves similar but more complicated calculation as Proposition 4.4 and we left the calculation as an exercise to the reader.

Now we are in a position to consider the general cases.

Proposition 4.6. Let $\Pi = \Pi(C, D)$ be a preprojective algebra, then

(a) $I_i^2 = I_i$;
(b) $I_i I_j I_i = I_i I_i$, if $c_{ij} c_{ji} = 0$;
(c) $I_i I_j I_i = I_j I_i I_i$, if $c_{ij} c_{ji} = 1$;
(d) $I_i I_j I_i I_j = I_j I_i I_i I_i$, if $c_{ij} c_{ji} = 2$;
(e) $I_i I_j I_i I_j I_i = I_i I_j I_j I_i I_i$, if $c_{ij} c_{ji} = 3$.

Proof. (a) is obvious.

Let $I_{i,j} = \Pi(1 - e_i - e_j) \Pi$, then any product of ideals $I_i$ and $I_j$ contains $I_{i,j}$.

If $c_{ij} c_{ji} = 0$, then $\Pi/I_{i,j}$ is semisimple, thus $I_i I_j = I_i J_j = I_i I_i$.

If $c_{ij} c_{ji} = 1$, then $\Pi/I_{i,j}$ is isomorphic to the preprojective algebra in Proposition 4.3. Let $I$ be an ideal of $\Pi$, denote the image of $I$ in $\Pi/I_{i,j}$ by $\overline{I}$. Hence, by Proposition 4.3, we have $\overline{I_i I_j I_i} = \overline{I_i I_j I_i} = 0 = \overline{I_j I_i I_j} = \overline{I_j I_i I_j}$, thus $I_i I_j I_i \subset I_{i,j}$ and $I_j I_i I_j \subset I_{i,j}$. Since any product of ideals $I_i$ and $I_j$ contains $I_{i,j}$, then we have $I_i I_j I_i = I_{i,j} = I_j I_i I_j$.

If $c_{ij} c_{ji} = 2$ or $3$, then $\Pi/I_{i,j}$ is isomorphic to the preprojective algebra in Proposition 4.4 or the preprojective algebra in Proposition 4.5. Similar to the case $c_{ij} c_{ji} = 1$, one can prove the statements (d) and (e).
Theorem 4.7. Let $\Pi = \Pi(C, D)$ be a preprojective algebra. There exists a bijection $\psi : W \to \langle I_1, I_2, \cdots, I_n \rangle$ given by $\psi(w) = I_w = I_{i_1}I_{i_2}\cdots I_{i_k}$ for any reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_k}$.

Proof. We first show that the map $\psi$ is well-defined. For any two reduced expression $w_1, w_2$ of an element in $W$, by Lemma 4.1, $w_1, w_2$ can be connected by a sequence of braid-moves. Then by Proposition 4.6, we have $I_{w_1} = I_{w_2}$.

For any ideal $I \in \langle I_1, I_2, \cdots, I_n \rangle$, take $\tilde{I} = I_{i_1}I_{i_2}\cdots I_{i_k}$ with $k$ minimal, let $w = s_{i_1}s_{i_2}\cdots s_{i_k}$. By Lemma 4.1, $w$ can be transformed into a reduced expression by a sequence of nil-moves and braid-moves. By Proposition 4.6, the nil-moves can not appear since $k$ is minimal. Hence $w$ is a reduced expression and we have $\psi(w) = I_w = I$. In particular, the map is surjective.

It remains to show that the map is injective. We separate the proof into two cases.

Let us assume first that $C$ has no component of Dynkin type. Denote by $\varepsilon = K^b(\text{proj} \Pi)$, i.e., the homotopy category of bounded complexes of projective $\Pi$-modules. For any $i \in Q_0$, since $I_i$ is a $\Pi$-tilting module with $\text{End} \ I_i \cong \Pi$, we have an autoequivalence $- \otimes \Pi I_i$ of $\varepsilon$ which induces an automorphism $- \otimes \Pi I_i$ of the Grothendieck group $K_0(\varepsilon)$. Note that $\{[e_j\Pi] \mid j \in Q_0\}$ is a $\mathbb{Z}$-basis of $K_0(\varepsilon)$. By the projective resolution of $e_iI_i$

$$0 \to e_i\Pi \to \bigoplus_{j \in \Pi(i,-)} (e_j\Pi)[e_{ji}] \to e_iI_i \to 0,$$

it is easy to get that

$$[e_i\Pi \otimes \Pi I_i] = \begin{cases} [e_iI_i] = \bigoplus_{j \in \Pi(i,-)} (e_j\Pi)[e_{ji}] - [e_i\Pi] = [e_i\Pi] - \sum_{j=1}^{n} c_{ji}[e_j\Pi] & \text{if } l = i; \\ [e_iI_i] = [e_i\Pi] & \text{if } l \neq i. \end{cases}$$

In particular, by identifying $- \otimes \Pi I_i$ with $\sigma_\Pi^*$ for $V^* = K_0(\varepsilon) \otimes \mathbb{C}$, we obtain an action $s_i \mapsto [- L_i]$ of $W$ on $V^*$. Note that for any reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_k}$, $I_w = I_{i_1}I_{i_2}\cdots I_{i_k} = I_{i_1} L_i I_{i_2} \otimes \Pi \cdots \otimes \Pi I_{i_k}$. Hence, the action of $[- \otimes \Pi I_w]$ on $V^*$ coincides with the action of $[- \otimes \Pi I_w]$ on $V^*$. Therefore, for any reduced word $w, w'$ such that $I_w = I_{w'}$, we have $[- \otimes \Pi I_w] = [- \otimes \Pi I_{w'}]$. By Lemma 4.2, we deduce that $w = w'$.

Now suppose that $C$ is of Dynkin type and without loss of generality, we may assume that $C$ is connected. Let $\tilde{Q} = \tilde{Q}(\tilde{C}, \tilde{\Omega})$ be an extended Dynkin quiver obtained from $Q = Q(C, \Omega)$ by adding a new vertex (i.e., $\tilde{Q}_0 = \{0\} \cup Q_0$) and the associated arrows, $\tilde{\Pi}$ be the associated preprojective algebra and $\tilde{W}$ the associated Weyl group. Denote $\tilde{I}_i = \tilde{\Pi}(1-e_i)\tilde{\Pi}$ for each $i \in \tilde{Q}_0$. For each $w \in \tilde{W}$, let $w = s_{i_1}\cdots s_{i_k}$ be a reduced expression, denote $\tilde{I}_w = \tilde{I}_{i_1}\cdots \tilde{I}_{i_k}$. Then we have $\Pi = \tilde{\Pi}/\langle e_0 \rangle, I_i = \tilde{I}_i/\langle e_0 \rangle$ for $i \neq 0$. Hence for each reduced word $w = s_{i_1}\cdots s_{i_k} \in W$, we have

$$I_w = (\tilde{I}_{i_1}/\langle e_0 \rangle)\cdots (\tilde{I}_{i_k}/\langle e_0 \rangle) = (\tilde{I}_{i_1}\cdots \tilde{I}_{i_k})/\langle e_0 \rangle = \tilde{I}_w/\langle e_0 \rangle.$$

Suppose $I_w = I_{w'}$ for two reduced words $w, w' \in W$, then $\tilde{I}_w/\langle e_0 \rangle = \tilde{I}_{w'}/\langle e_0 \rangle$. Since $s_0$ does not appear in reduced expression of $w$ and $w'$, we have $e_0 \in \tilde{I}_w$ and $e_0 \in \tilde{I}_{w'}$. Consequently, $\tilde{I}_w = \tilde{I}_{w'}$ and $w = w'$ by the case of non-Dynkin type. \hfill $\square$

5. Preprojective algebras of Dynkin type

In this section, after recall the basic definition and properties of support $\tau$-tilting modules, we generalize the classification results of $[14]$ for classical preprojective algebras of type $A, D, E$ to the preprojective algebras of Dynkin type in the sense of $[10]$. We mainly follow $[14]$. 
5.1. Recollection on support $\tau$-tilting modules. In this subsection, we assume $\Lambda$ to be a finite-dimensional algebra over $K$. Let $\tau$ be the Auslander-Reiten translation of $\Lambda$-modules. We follow [1].

Definition 5.1. Let $M$ be a $\Lambda$-module and $P$ a projective $\Lambda$-module.

(1) $M$ is a $\tau$-rigid $\Lambda$-module if $\text{Hom}_\Lambda(M, \tau M) = 0$.
(2) $M$ is a $\tau$-tilting $\Lambda$-module if it is $\tau$-rigid and $|M| = |\Lambda|$.
(3) $M$ is a support $\tau$-tilting $\Lambda$-module if there exists an idempotent $e \in \Lambda$ such $M$ is a $\tau$-tilting $(\Lambda/e)$-module.
(4) The pair $(M, P)$ is a $\tau$-rigid pair if $M$ is $\tau$-rigid and $\text{Hom}_\Lambda(P, \tau M) = 0$.
(5) A $\tau$-rigid pair $(M, P)$ is a support $\tau$-tilting (respectively, almost complete support $\tau$-tilting) pair if $|M| + |P| = |\Lambda|$ (respectively, $|M| + |P| = |\Lambda| - 1$).

The pair $(M, P)$ is basic if $M$ and $P$ are basic. By [1] Prop 2.3, if $(M, P)$ is a basic support $\tau$-tilting pair, then $P$ is determined by $M$ uniquely, and $(M, P)$ is a support $\tau$-tilting pair if and only if $M$ is a $\tau$-tilting $(\Lambda/e)$-module, where $e$ is an idempotent of $\Lambda$ such that $\text{add} P = \text{add} e\Lambda$. Thus we can identify basic support $\tau$-tilting modules with basic support $\tau$-tilting pairs.

For a $\Lambda$-module $M$, denote by $\text{Fac} M$ the category formed by the quotients of finite direct sum of $M$. Let $\text{sr}$-$\text{tilt} \Lambda$ be the set of isomorphism classes of basic support $\tau$-tilting $\Lambda$-modules. For $T, T' \in \text{sr}$-$\text{tilt} \Lambda$, we write $T \leq T'$ if $\text{Fac} T \subseteq \text{Fac} T'$. This defines a partial order $\leq$ on $\text{sr}$-$\text{tilt} \Lambda$.

The following result has been obtained in [1].

Theorem 5.2. Any basic almost support $\tau$-tilting pair $(U, Q)$ is a direct summand of exactly two basic support $\tau$-tilting pairs $(T, P), (T', P')$. Moreover, we have either $T < T'$ or $T' < T$.

In the situation of the above theorem, we say $(T', P')$ is a left mutation of $(T, P)$ if $T' < T$ and write $\mu_X(T) = T'$, where $X$ is the unique indecomposable direct summand of $(T, P)$ which is different from $(T', P')$. In this case, we also call $(T, P)$ is a right mutation of $(T', P')$ at $X$. Let $T = U \oplus X$ be a support $\tau$-tilting $\Lambda$-module with indecomposable direct summand $X$. It is also known that $T$ has a left mutation at $X$ if and only if $X \not\in \text{Fac} U$. The following result also gives us a method to calculate left mutations of support $\tau$-tilting modules.

Lemma 5.3. [1] Theorem 2.30 Let $X$ be an indecomposable $\Lambda$-module and $T = X \oplus U$ a basic $\tau$-tilting $\Lambda$-module. Assume that $T$ has a left mutation $\mu_X(T)$. Let

$$X \xrightarrow{f} U' \xrightarrow{g} Y \rightarrow 0$$

be an exact sequence, where $f$ is a minimal left $\text{add} U$-approximation. Then one of the following holds:

(1) $Y = 0$. Then $U = \mu_X(T)$ is a basic support $\tau$-tilting module.
(2) $Y \neq 0$ and $Y \cong Y^m$ for some integer $m > 0$, where $Y'$ is an indecomposable direct summand of $Y$. Then $\mu_X(T) = Y' \oplus U$ is a basic $\tau$-tilting module.

Using the left mutation, one may define the support $\tau$-tilting quiver of $\Lambda$.

Definition 5.4. The support $\tau$-tilting quiver $\mathcal{H}(\text{sr}$-$\text{tilt} \Lambda)$ is defined as follows.

(1) The set of vertices is $\text{sr}$-$\text{tilt} \Lambda$;
(2) Draw an arrow from $T$ to $T'$ if $T'$ is a left mutation of $T$.

Lemma 5.5. [1] Corollary 2.38 If $\mathcal{H}(\text{sr}$-$\text{tilt} \Lambda)$ admits a finite connected component $C$, then $\mathcal{H}(\text{sr}$-$\text{tilt} \Lambda) = C$. 
5.2. **Support τ-tilting modules for preprojective algebras of Dynkin type.** In this subsection, we always assume that $C$ is a symmetrizable Cartan matrix of Dynkin type with a symmetrizer $D$ and $\Pi = \Pi(C, D)$ is the associated preprojective algebra. We follow [14]. By Lemma 2.3, we know that $\Pi$ is a selfinjective algebra and the Nakayama functor $\nu = \mathbb{D} \text{Hom}_\Pi(-, \Pi)$ is exact.

**Lemma 5.6.** Let $I$ be a two-sided ideal of $\Pi$. For a primitive idempotent $e_i$ for $i \in Q_0$, $e_i I$ is either indecomposable or zero.

**Proof.** Since $\Pi$ is selfinjective, $e_i \Pi$ has a unique simple socle. If the submodule $e_i I$ is non-zero, then it has a unique simple socle as $e_i \Pi$ and hence indecomposable. □

An easy consequence of the above result is that for $i \neq j$, $e_i I$ and $e_j I$ are non-isomorphic provided that they are not both zero.

**Proposition 5.7.** For each $i \in Q_0$, the generalized simple module $E_i$ and the two-sided ideal $I_i$ are τ-rigid modules.

**Proof.** Recall that $I_i = \bigoplus_{j=1}^n e_j I_i = e_i I_i \oplus (\bigoplus_{j \neq i} e_j \Pi)$. If $e_i I_i = e_i \Pi (1 - e_i) \Pi = 0$, we deduce that $i$ is an isolated vertex. In this case, we have $I_i = \bigoplus_{j \neq i} e_j \Pi$ and $E_i = e_i \Pi$, then the result is obvious.

Now assume that $e_i I_i \neq 0$. By Lemma 2.12 for any $i$, we have an exact sequence

$$0 \to E_{\sigma(i)} \to e_i \Pi \to \bigoplus_{j \in \Pi(i, -)} (e_j \Pi)^{\nu_{ij}} \to e_i \Pi \to E_i \to 0,$$

where $\ker c = e_i I_i$. Applying the exact functor $\nu = \mathbb{D} \text{Hom}_\Pi(-, \Pi)$ to the above exact sequence, one obtains the following exact sequence

$$0 \to \nu(E_{\sigma(i)}) \to \nu(e_i \Pi) \xrightarrow{\nu(a)} \bigoplus_{j \in \Pi(i, -)} \nu(e_j \Pi)^{\nu_{ij}} \xrightarrow{\nu(b)} \nu(e_i \Pi) \xrightarrow{\nu(c)} \nu(E_i) \to 0,$$

By the definition of the functor $\tau$, we know that $\tau(E_i) = \ker \nu(b)$ and $\tau(e_i I_i) = \ker \nu(a) = \nu(E_{\sigma(i)})$.

Recall that we have $\nu(E_{\sigma(i)}) = E_i$ by Proposition 2.11 Therefore, $\tau(I_i) = \tau(e_i I_i \oplus (\bigoplus_{j \neq i} e_j \Pi)) = \tau(e_i I_i) = E_i$. Consequently, $\text{Hom}_\Pi(I_i, \tau(I_i)) = \text{Hom}_\Pi(I_i, E_i) = 0$ and $I_i$ is a τ-rigid module.

To show $\text{Hom}_\Pi(E_i, \tau(E_i)) = 0$, note that

$$\text{Hom}_\Pi(E_i, \bigoplus_{j \in \Pi(i, -)} \nu(e_j \Pi)^{\nu_{ij}}) \cong \text{Hom}_\Pi((\Pi e_j)^{\nu_{ij}}, \mathbb{D}(E_i)) = 0.$$ 

We conclude that $\text{Hom}_\Pi(E_i, \tau(E_i)) = \text{Hom}_\Pi(E_i, \ker \nu(b)) = 0$ and $E_i$ is τ-rigid. □

We have the following easy consequence of Proposition 5.7.

**Proposition 5.8.** For each $i \in Q_0$, the two-sided ideal $I_i$ is a basic support τ-tilting ideal.

**Proof.** If $I_i = 0$, then $I_i$ is obviously a basic support τ-tilting module.

Now suppose that $I_i \neq 0$. If $e_i I_i = 0$, then $i$ is an isolated vertex and $(I_i, e_i \Pi) = (\bigoplus_{j \neq i} e_j \Pi, e_i \Pi)$ is a support τ-tilting pair, hence $I_i$ is a support τ-tilting module. If $e_i I_i \neq 0$, it is clear that $|I_i| = |\Pi|$. Consequently, $I_i$ is a τ-tilting module by Proposition 5.7. □

By Prop 2.5, for each τ-rigid module $X$, we can take a minimal projective presentation $P_1 \to P_0 \to X \to 0$ such that $P_0$ and $P_1$ do not have a common direct summand. Then similar to Lemma 5.8 one can prove

**Lemma 5.9.** Let $T$ be a support τ-tilting $\Pi$-module. For any generalized simple $\Pi^{op}$-module $E_i$, at least one of the statements $T \otimes_\Pi E_i = 0$ and $\text{Tor}_1^\Pi(T, E_i) = 0$ holds.
Similar to Proposition 5.10, applying the functor $T \otimes \Pi$ to the exact sequence
$$0 \to I_i \xrightarrow{f} \Pi \to E_i \to 0,$$
we obtain an exact sequence
$$\text{Tor}_1^\Pi(T, E_i) \to T \otimes \Pi I_i \xrightarrow{T \otimes \Pi f} T \otimes \Pi \Pi \to T \otimes \Pi E_i \to 0.$$ Note that $\text{im}(T \otimes \Pi f) = T I_i$, we have $T \otimes \Pi E_i = 0$ if and only if $T I_i = T$. In particular,

**Lemma 5.10.** Let $T$ be a support $\tau$-tilting $\Pi$-module. If $T I_i \neq T$, then $T \otimes \Pi I_i \cong T I_i$.

For an ideal $T$ of $\Pi$, if $e_i T = 0$, then $I_i T = \Pi(1 - e_i)\Pi T = T$. Thus if $I_i T \neq T$, then $e_i T \neq 0$. The same proof of [14, Lemma 2.8] yields the following.

**Lemma 5.11.** Let $T$ be a support $\tau$-tilting ideal of $\Pi$. If $I_i T \neq T$, then we have $e_i T \notin \text{Fac}((1 - e_i)T)$. In particular, $T$ has a left mutation $\mu_{e_i T}(T)$.

Recall that we have the following two exact sequences
\begin{equation}
0 \to E_{\sigma(i)} \to e_i \Pi \to \bigoplus_{j \in \Pi(i,-)} (e_j \Pi)^{\sim} |_{\text{soc}(i)},
\end{equation}

and
\begin{equation}
e_i \Pi \xrightarrow{\sigma} \bigoplus_{j \in \Pi(i,-)} (e_j \Pi)^{\sim} |_{\text{soc}(i)} \to e_i I_i \to 0,
\end{equation}

which are obtained from the exact sequence (2). The following is an analogue of Lemma 2.9 in [14].

**Lemma 5.12.** Let $T$ be a support $\tau$-tilting ideal of $\Pi$. The map $a \otimes \Pi T$
\begin{equation}e_i \Pi \otimes \Pi T \xrightarrow{a \otimes \Pi T} \bigoplus_{j \in \Pi(i,-)} (e_j \Pi)^{\sim} \otimes \Pi T
\end{equation}
is a left $\text{add}((1 - e_i)T)$-approximation.

**Proof.** Since the socle of $e_i \Pi$ is the simple module $S_{\sigma(i)}$, we have $\text{Hom}_\Pi(E_{\sigma(i)}, (1 - e_i)\Pi) = 0$. By applying the functor $\text{Hom}_\Pi(-, (1 - e_i)\Pi)$ to the exact sequence (5), we obtain a surjective map $\text{Hom}_\Pi\left(\bigoplus_{j \in \Pi(i,-)} (e_j \Pi)^{\sim} |_{\text{soc}(i)} |_{\text{soc}(i)}, (1 - e_i)\Pi\right) \to \text{Hom}_\Pi(e_i \Pi, (1 - e_i)\Pi)$. Then one can apply the same argument of Lemma 2.9 in [14] to obtain the result. \hfill $\square$

**Proposition 5.13.** Assume that $T \in \langle I_1, \ldots, I_n \rangle$ is a basic support $\tau$-tilting ideal of $\Pi$. Then $I_i T$ is a basic support $\tau$-tilting $\Pi$-module.

**Proof.** There is nothing to prove if $I_i T = T$ and we assume that $I_i T \neq T$ in the following. By Lemma 5.11, $T$ has a left mutation $\mu_{e_i T}(T)$. Now let $e$ be an idempotent of $\Pi$ such that $T$ is a $\tau$-tilting $(\Pi/e)$ module. Applying $- \otimes \Pi T$ to the exact sequence (5), we get an exact sequence
\begin{equation}e_i \Pi \otimes \Pi T \xrightarrow{a \otimes \Pi T} \bigoplus_{j \in \Pi(i,-)} (e_j \Pi)^{\sim} \otimes \Pi T \to e_i I_i \otimes \Pi T \to 0.
\end{equation}

and $a \otimes \Pi T$ is a left $\text{add}((1 - e_i)T)$-approximation by Lemma 5.12. On the other hand, by the left module version of Lemma 5.10, we have $e_i I_i \otimes \Pi T = e_i I_i T$. If $e_i I_i T = 0$, then it is clear that $a \otimes \Pi T$ is a minimal left $\text{add}((1 - e_i)T)$-approximation. Now assume that $e_i I_i T \neq 0$. Since $e_j T$ and $e_i I_i T$ have different socles for $j \neq i$, then $e_j T$ and $e_i I_i T$ are non isomorphic for $j \neq i$. Hence the map $a \otimes \Pi T$ is also a minimal left $\text{add}((1 - e_i)T)$-approximation. Consequently, $\mu_{e_i T}(T) = e_i I_i T \otimes (1 - e_i)T = I_i T$ is a $\tau$-tilting $(\Pi/e)$ module by Lemma 5.13 and $I_i T$ is a basic support $\tau$-tilting $\Pi$-module. \hfill $\square$

In particular, we have proved the following result.
Theorem 5.14. Each $T \in \langle I_1, \cdots, I_n \rangle$ is a basic support $\tau$-tilting modules.

Let $W = W(C)$ be the Weyl group associated to the symmetrizable Cartan matrix $C$. By Theorem 5.14 for each reduced expression of $w \in W$, we have a support $\tau$-tilting module $I_w$. Let $(I_w, P_w)$ be the corresponding support $\tau$-tilting pair, where $P_w$ is a projective $\Pi$-module, then we have

Proposition 5.15. Keep notations as above, then

$$P_w = \bigoplus_{i \in Q_0, e_i I_w = 0} e_{\sigma(i)} \Pi,$$

where $\sigma : Q_0 \to Q_0$ is the Nakayama permutation of $\Pi$.

Proof. Suppose $e_i I_w = 0$ for $i \in Q_0$, we need to show $\text{Hom}_\Pi(e_{\sigma(i)} \Pi, I_w) = I_w e_{\sigma(i)} = 0$. Since $I_w$ is an ideal, $I_w e_{\sigma(i)}$ is a left $\Pi$-module. If $I_w e_{\sigma(i)} \neq 0$, then $\text{soc}(I_w e_{\sigma(i)}) = \text{soc}(\Pi e_{\sigma(i)}) = S_i$. Consequently, there is a nonzero morphism $\Pi e_i \to S_i \to I_w e_{\sigma(i)}$. However, $\text{Hom}_\Pi(\Pi e_i, I_w e_{\sigma(i)}) = e_i I_w e_{\sigma(i)} = 0$, a contradiction. Hence $\text{Hom}_\Pi(e_{\sigma(i)} \Pi, I_w) = 0$ and $e_{\sigma(i)} \Pi$ is a direct summand of $P_w$.

On the other hand, suppose we have $\text{Hom}_\Pi(e_{\sigma(i)} \Pi, I_w) = I_w e_{\sigma(i)} = 0$, using similar argument, one can get $e_i I_w = 0$. This completes the proof.

Theorem 5.16. Keep notations as above, for each $i \in Q_0$, the support $\tau$-tilting pairs $(I_w, P_w)$ and $(I_{s_i w}, P_{s_i w})$ are related by a left or right mutation.

Proof. We need to show that the support $\tau$-tilting pairs $(I_w, P_w)$ and $(I_{s_i w}, P_{s_i w})$ have a common almost complete support $\tau$-tilting pair. According to Theorem 5.16 $I_{s_i w}$ and $I_w$ are not isomorphic. Suppose $l(s_i w) > l(w)$, we have $I_{s_i w} = I_i I_w$ and

$$I_{s_i w} = e_i I_i I_w \oplus (1 - e_i) I_i I_w = e_i I_i I_w \oplus (1 - e_i) I_w, \quad P_{s_i w} = \bigoplus_{j \in Q_0, e_j I_i I_w = 0} e_{\sigma(j)} \Pi.$$

Note that $I_w = e_i I_i I_w \oplus (1 - e_i) I_w$ and $P_w = \bigoplus_{j \in Q_0, e_j I_i I_w = 0} e_{\sigma(j)} \Pi$. By $I_i I_w \neq I_w$, it is not hard to see that $e_i I_i I_w \neq 0$. On the other hand, for $j \neq i$, we clearly have $e_j I_i I_w = e_j I_w$ and hence $e_j I_i I_w = 0$ if and only if $e_j I_w = 0$. Then we obtain

$$(I_{s_i w}, P_{s_i w}) = \begin{cases} ((1 - e_i) I_w, P_w \oplus e_{\sigma(i)} \Pi) & \text{if } e_i I_i I_w = 0; \\ (e_i I_i I_w \oplus (1 - e_i) I_w, P_w) & \text{if } e_i I_i I_w \neq 0. \end{cases}$$

In particular, $(I_w, P_w)$ and $(I_{s_i w}, P_{s_i w})$ have a common almost complete support $\tau$-tilting pair $((1 - e_i) I_i I_w, P_w)$ as a direct summand in both cases.

If $l(s_i w) < l(w)$, one applies the same argument to $u := s_i w$ which satisfies $l(s_i u) > l(u)$. \hfill \Box

Theorem 5.17. Each basic support $\tau$-tilting $\Pi$-module is isomorphic to an object of $\langle I_1, I_2, \cdots, I_n \rangle$.

Proof. The same argument of [13] Thm 2.19] applied. Here we provide a sketch of the proof.

Let $\mathcal{C}$ be the connected component of $\mathcal{H}(sr\text{-tilt}\Pi)$ containing $\Pi$. By induction on the length of the element of the Weyl group $W$, it is easy to get that $I_w$ belongs to $\mathcal{C}$ for any reduced word $w \in W$. Then by Theorem 5.10 each neighbor of $I_w$ has the form $I_{s_i w}$ for some $i$, we conclude that the set $\{I_w \mid w \in W\}$ forms the vertices of $\mathcal{C}$ and $\mathcal{C}$ is a finite connected component of $\mathcal{H}(sr\text{-tilt}\Pi)$. Then by Lemma 5.1 we deduce that $\mathcal{H}(sr\text{-tilt}\Pi) = \mathcal{C}$. \hfill \Box

Remark 5.18. An immediately consequence of Theorem 5.17 is that each indecomposable $\tau$-rigid module has the form $e_i I_w$ for some $i \in Q_0$ and some reduced word $w \in W$. On the other hand, for any $i \in Q_0$ and any reduced word $w \in W$, if $e_i I_w \neq 0$, then $e_i I_w$ is a nonzero indecomposable $\tau$-rigid module.
Example 5.19. Let $\Pi = \Pi(C, D)$ be the preprojective algebra in Example 2.3. In this case, $\mathcal{H}(\text{\textit{s}-tilt } \Pi)$ is as follows:

![Diagram]

and all the basic support $\tau$-tilting pairs are

\[
\{(\Pi, 0), (E_2 \oplus e_2 \Pi, 0), (E_1 \oplus e_1 \Pi, 0), (E_1, e_2 \Pi), (E_2, e_1 \Pi), (0, \Pi)\},
\]

where

\[
e_1 \Pi = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, e_2 \Pi = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, E_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.
\]

We also know that all the nonzero indecomposable $\tau$-rigid $\Pi$-modules are $\{e_1 \Pi, e_2 \Pi, E_1, E_2\}$.

Example 5.20. Let $\Pi = \Pi(C, D)$ be the preprojective algebra in Example 2.4. In this case, $\mathcal{H}(\text{\textit{s}-tilt } \Pi)$ is as follows:

![Diagram]

and all the basic support $\tau$-tilting pairs are

\[
\{(\Pi, 0), (e_1 I_1 \oplus e_2 \Pi, 0), (e_2 I_2 \oplus e_1 \Pi, 0), (E_1 \oplus e_2 I_2, 0), (e_1 I_1 \oplus E_2, 0), (E_1, e_2 \Pi), (E_2, e_1 \Pi), (0, \Pi)\},
\]

where

\[
e_1 \Pi = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, e_2 \Pi = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, e_1 I_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, e_2 I_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, E_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.
\]

All the nonzero indecomposable $\tau$-rigid $\Pi$-modules are $\{e_1 \Pi, e_2 \Pi, e_1 I_1, e_2 I_2, E_1, E_2\}$.

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