Explaining the equality of inertia and gravitational mass

Sanjay M Wagh

Central India Research Institute,
34, Farmland, Ramdaspeth, Nagpur 440 010, India

E-mail: waghsm.ngp@gmail.com

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Abstract

The equality of the inertia and the gravitational mass of a body is explained in a very general manner. We also motivate this explanation by providing analogous examples.
Galileo’s experiments suggested to him the uniform rectilinear motion of a body to be its *natural* or *inertial state of motion*. He had imagined the inertia of a body as its opposition to a change in its natural state of motion, and had conceptualized such a change to be due only to its interaction with another body.

Newton completed Galileo’s conceptual framework with his famous Three Laws of Motion. With an action-at-a-distance interaction of bodies, Newton explained Kepler’s laws of planetary motion by assuming gravitational interaction of a planet and the Sun. Then, the gravitational mass of a body is its source property appearing in the postulated force of gravity, which it can be assumed to exert on all other bodies.

Let us consider a body $A$ of gravitational mass $m$ and inertia $I$. Following Newton, we assume a force $\vec{F}$, of the gravity of another body $B$ and acting on the body $A$, to be proportional to the gravitational mass $m$ of the body $A$, *i.e.* $\vec{F} \propto m$ or $\vec{F} = -mN\hat{r}$. Here, $N$ is a proportionality factor and $\hat{r}$ is a radially outwardly directed unit vector along the line joining the two bodies with the origin at the body $B$.

This force $\vec{F}$ causes the acceleration $\vec{a}$ of the body $A$ towards the body $B$. Newton’s second law of motion now asserts that $\vec{F} = I\vec{a}$. Then, we have $\vec{F} = I\vec{a} = -mN\hat{r}$, and the acceleration of the body $A$ is therefore given as

$$\vec{a} = -\frac{m}{I}N\hat{r}$$

Thus, the magnitude of the acceleration of the body $A$ is proportional to the ratio $m/I$ of its gravitational mass and its inertia. There is nothing within the Newtonian framework of concepts (that is, of inertia, of action-at-a-distance force, of gravitational mass, etc.) that compels this ratio to be unity, *i.e.* $m = I$, for a body.
Consider now two such bodies, say, body 1 and body 2, for which the proportionality factor \( \aleph \) is the same. Then, we have

\[
\vec{a}_1 = -\frac{m_1}{I_1} \aleph {\hat{r}} \quad \text{and} \quad \vec{a}_2 = -\frac{m_2}{I_2} \aleph {\hat{r}}
\]

Therefore, the ratio of the magnitudes of these accelerations is

\[
\frac{a_1}{a_2} = \frac{m_1/I_1}{m_2/I_2}
\]

Clearly, the accelerations of the two bodies would not be same if the ratio \( m/I \) were not the same for them, both. Nothing within the Newtonian framework of concepts compels these ratios to be equal for the two bodies under considerations.

Mach had then stressed the lack of logically compelling reasons for the inertia and the gravitational mass, two conceptually different physical quantities, to be having exactly the same value for a body, let alone for all the bodies of Nature.

But, Galileo’s experiments at the Leaning Tower of Pisa had shown that the inertia and the gravitational mass of a body are equal to a high degree of accuracy. Verifying this, and explaining why this is so, have been certain issues, then.

In the present article, we show that this equality has a remarkably simple and general explanation. To begin with, we consider the following examples to illustrate fundamental principles underlying this simple and very general explanation.

**Example One:** To fix ideas, consider a body of “volume” \( V \) and “surface area” \( A \). Let

\[
V_{eq} = A \times \ell = V \quad \text{where} \quad \ell \text{ is a certain "hypothetical" length. Then, for a sphere of radius } R, \ \ell = R/3; \text{ for a cube of each edge of length } a, \ \ell = a/6; \text{ etc.}
\]
Now, $V_{eq}$ can have the meaning of the volume of “another” body, a *volume-equivalent body*, with a base of area $A$ and a height $\ell$. By definition, $V$ and $V_{eq}$ are numerically equal, then. But, we treat $V$, the “volume of a body”, and $V_{eq}$, the “volume of a volume-equivalent body”, as being two “different” concepts.

When we evaluate another quantity, it makes no difference whether we use the volume $V$ or the volume $V_{eq}$. For example, if $\rho$ is the number density of particles in a body, then we would obtain for the total number $N$ of particles in that body as $N = \rho \times V$ and as $N' = \rho \times V_{eq}$. The values of $N$ and $N'$ obtained from these two expressions are bound to be equal.

Furthermore, if $N = \rho V$ and $N = \rho' V_{eq}$, then no “mystery” is behind the equality of the values of $\rho$ and $\rho'$ obtained from these expressions.

**Example Two:** Consider now another example, that of the specific heat of a substance in different systems of units.

Consider one kilogram of some substance of specific heat of, say, $X$ k-cal/Kg/$^\circ$C in the SI System of Units. Then, $X$ kilo-calories of heat are needed to change its temperature through $1^\circ$C.

In the CGS System of Units, we have 1 Kg of substance to be equal to 1000 gm of that substance, and 1 k-cal is equal to 1000 cal. Then, the heat needed to change the temperature of 1 gm of substance through $1^\circ$C will be

$$X \times 1000 \times \frac{1}{1000} = X \text{ cal}$$

The specific heat of substance is $X$ cal/gm/$^\circ$C in the CGS System of Units.
To any uninitiated mind, the sameness of the value of the specific heat of any substance whatsoever in the aforementioned different systems of units may appear surprising. It is however an artefact of the following.

Notice that the unit of heat is defined in relation to the unit mass of water and the unit change of temperature in all the systems of units.

As has been defined, the specific heat of a substance is the ratio of the “heat needed to cause a unit change in the temperature of the unit mass of the substance” to the “heat needed to cause a unit change in the temperature of the unit mass of water” in any of these systems of units.

The specific heat, being this ratio, then has the same “numerical value” in all the systems of units. This is not any “surprising” fact.

**Example Three:** Now, for Coulomb’s action-at-a-distance force, we have

\[ \vec{F} = I \vec{a} \quad \text{and} \quad \vec{F} = q \left( -\Xi \hat{r} \right) \quad \Rightarrow \quad \vec{a} = \frac{q}{I} \left( -\Xi \hat{r} \right) \]

with the quantity \( -\Xi \hat{r} \), \( \Xi \) as a proportionality factor, not having the same dimensions as those of \( \vec{a} \). No equality of \( q \) and \( I \) is then compelled.

We may now compare the situation of the examples discussed above with that of the inertia and the gravitational mass of a body. Consider then the relevant equations

\[ \vec{F} = I \vec{a} \quad \text{and} \quad \vec{F} = m \left( -\kappa \hat{r} \right) \quad \Rightarrow \quad \vec{a} = \frac{m \kappa}{I} \left( -\kappa \hat{r} \right) \]

Here, the dimensions of \( -\kappa \hat{r} \) and \( \vec{a} \) are the same.
Consequently, the two quantities \((- \Re \hat{r})\) and \(\vec{a}\) “refer” to the same basic concept. This is a case analogous to that of the volume \(V\) and the volume \(V_{eq}\) in the first of the examples considered earlier. The values of \(m\) and \(I\) will be compelled to be equal, whenever the quantity \((- \Re \hat{r})\) is “defined to have” value equal to that of \(\vec{a}\).

Necessarily, the (operational part of the) definition of the gravitational mass is based on, precisely, the equality of the values of \((- \Re \hat{r})\) and of \(\vec{a}\). Then, why the inertia of a body is always equal to its gravitational mass, even when the two are different physical concepts, has the above simple explanation.

Now, the gravitational mass is an artefact of the assumption of the action-at-a-distance force. However, no observation or experiment imposes on us such a character for the force. The concept of the gravitational mass of the body is therefore not an “essential” physical concept in the sense that its “role” is similar to that of the volume \(V_{eq}\) of a volume-equivalent body in the first example considered above.

On the other hand, the concept of the inertia of a body is a deduction, straightforwardly obtainable, from Galileo’s experiments. Therefore, the inertia needs to be considered as an “essential” concept in the sense that its “role” is similar to that of the volume \(V\) of a body in the first of the considered examples.

But, Mach stressed \([3]\) that, within Newton’s theory, there does not exist any principle to say that the inertia of a body equals its gravitational mass, and this had been considered to be one of the lacunae of Newton’s theoretical framework.

In view of the explanation of this equality discussed in the present work, no additional principle is required to explain the equality of the inertia and the gravitational mass of a body. Then, the lack of such a principle is “not a shortcoming” of the Newtonian theory.
But, contrary is what the equality of the inertia and the gravitational mass of a body had us “tricked” into believing for long.

A striking “internal asymmetry” of Newton’s theory based on the postulated action-at-a-distance force but remains that the inertia appears in the expression assumed for the force of gravity, and not in the expressions for the other forces, like Coulomb’s force. This can be considered to be a certain shortcoming of Newton’s theory.

Conceivably, the Fundamental Theory of Physics should then be based only on the “contact” interactions of bodies, and the action-at-a-distance force could “emerge” only as a suitable and sometimes useful approximation within its framework. The mathematical and the conceptual framework of the Universal Theory of Relativity then appears to provide for such a theoretical possibility.

Dedication

This work is dedicated to Professor P C Vaidya who passed away on 12th March 2010.

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[2] Newton I (1999) *The Principia: Mathematical Principles of Natural Philosophy - A New Translation*, by Cohen I and Whitman A, The University of California Press.

[3] Mach E (1960) *The Science of Mechanics* (English translation by Thomas J. McCormack, the 6th Edition with revisions through the 9th German Edition. LaSalle, Illinois: Open Court Publishing Company)

[4] Nagel E (1984) *The Structure of Science: Problems in the Logic of Scientific Explanation*, Routledge & Kegan Paul Ltd, England. Indian Edition by Macmillon India Ltd.
Wagh S M (2007) *Categorical Foundations for Physics - Program at a Glance*, electronic version available as [http://arxiv.org/physics/0708.2468](http://arxiv.org/physics/0708.2468)

Wagh S M (2009) *Emission Origin for the Wave of Quanta*, electronic version available as [http://arxiv.org/physics/0907.1210](http://arxiv.org/physics/0907.1210) [Note a typographical mistake in the expression for Planck’s law: 4 should read $8\pi$ in the relevant equations.]