POSITIVE RADIAL SOLUTIONS INVOLVING NONLINEARITIES WITH ZEROS

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Abstract. In this paper we consider the non-autonomous quasilinear elliptic problem
\[
\begin{aligned}
\begin{cases}
-\Delta_p u = \lambda |x|^{\delta} f(u) & \text{in } B_1(0) \\
u = 0 & \text{in } \partial B_1(0),
\end{cases}
\end{aligned}
\]
where \( f : \mathbb{R} \to [0, \infty) \) is a nonnegative \( C^1 \)-function with \( f(0) = 0 \), \( f(U) = 0 \) for some \( U > 0 \), and \( f \) is superlinear in \( 0 \) and in \( U \). Assuming subcriticality either in \( U \) or at infinity we study existence and multiplicity of positive radial solutions with respect to the parameter \( \lambda \). In addition, we study the bifurcation diagrams with respect to the maximum over the eventual solutions as the parameter \( \lambda \) varies in the positive halfline.

1. Introduction. This article is devoted to the study of positive radial solutions of the following semilinear elliptic problem
\[
\begin{aligned}
\begin{cases}
-\Delta u = \lambda |x|^{\delta} f(u) & \text{in } B_1(0) \\
u = 0 & \text{in } \partial B_1(0),
\end{cases}
\end{aligned}
\]  
where \(| \cdot |\) denotes the usual norm in \( \mathbb{R}^n \), \( n > 2 \), \( \delta > -2 \), \( B_1 \) is the unit ball, and \( f \) is a \( C^1 \) function such that \( f(0) = 0 \), \( f(U) = 0 \) for some \( U > 0 \), and it is positive and either linear or super-linear in \( u \) for small values. In the case \( f(u) = |u|^{q-2} u \) where \( q > 2 \), equation (1) is traditionally called the Hénon (respectively, Hardy or Lane-Emden) equation for \( \delta > 0 \) (resp., \( \delta < 0 \), \( \delta = 0 \)).

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Concerning the $\delta > 0$ case, in 1973, M. Hénon [19] introduced the equation (1) in the context of a concentric shell model, used to investigate numerically the stability of spherical steady states of stellar systems with respect to spherical disturbances. Since then the equation is called in the literature as Hénon equation. Later, in 1982, W. M. Ni [28] wrote the first rigorous study, where he showed that the presence of the radial weight $|x|^\delta$ affects the critical exponent. As a matter of fact, it modifies the Pohozaev identity and produces the new critical exponent $2^*(\delta) := \frac{2(n+\delta)}{n-2}$.

Problems with superlinear nonlinearities which have different behaviors at the origin and at infinity have been extensively studied. For the Laplacian, see for example [1, 7, 8, 23, 24, 6, 16]. For the $p-$Laplacian, see for example [2, 30, 17, 10, 15, 14]. In most of these works, the nonlinearity is strictly positive for $u > 0$; however, the characteristics of the problem are quite different when the nonlinearity has a positive zero. In the nice work [26], this type of problems is considered for the Laplacian operator and a nonlinearity $f$ that is independent of $x$, satisfying $f(0) \geq 0$, $f(U) = 0$, and assuming that $f$ is positive and superlinear for $u > U > 0$. Using topological degree arguments and under additional technical conditions which ensure a priori bounds, it is shown that there exist two positive solutions of Problem (1). It is further shown that one solution lies strictly below $U$, while the other has a maximum greater than $U$. This type of problems was also studied in [27], where again the existence of two positive solutions of Problem (1) was shown. One solution was obtained as a minimal positive solution, while the other was obtained as the limit of a gradient flow whose starting point is properly chosen. This strategy allows to show that certain technical hypotheses given in [26] can be weakened; moreover, a better insight on the behavior with respect to $\lambda$ of the minimal solution of [26] can be obtained. In this direction, some progress has been made: we cite the works [20, 22] for the $p-$Laplacian case, and [3, 18] for the semilinear case, who have observed that the behavior near the zeros of the nonlinearity is relevant to construct solutions for large $\lambda$.

Here, by considering a dynamical system approach, we study the existence and multiplicity of radial solutions in the non-homogeneous case and when the nonlinearity $f$ is positive, but it is null in 0 and $U$. In addition, we study the asymptotic behavior of the solutions with respect to the parameter $\lambda$. As far as we know, this is the first attempt to obtain multiplicity results in the weighted case, with a nonlinearity which has a positive zero, compare with [3, 18, 20, 21] and references therein.

Since we just deal with radial solutions we will indeed consider the following singular O.D.E.

\[ (u'(r)r^{n-1})' + \lambda f(u)r^\delta r^{n-1} = 0, \]

where, abusing the notation, we have set $u(r) = u(x)$ for $|x| = r$, and $'$ denotes differentiation with respect to $r$. We are interested in regular solutions, i.e. in solutions $u(r, d)$ of (2) satisfying the following initial condition:

\[ u(0, d) = d \geq 0 \quad u'(0, d) = 0 \]

together with the border condition $u(1, d) = 0$. The prototype of non-linearity $f$ we are interested in is

\[ f(u) = u^{q-1}|1 - u|^a \]

where $q > 2$, and $2 \leq a + 1 < 2^*(\delta) = \frac{2(n+\delta)}{n-2}$.

We collect here the main assumptions used in the paper:
**A:** There are $\sigma > 0$, $U > 0$ such that $f(U) = 0$, and $f(u) > 0$ for $u \in (0, U + \sigma) \setminus \{U\}$.

**B:** There is $U > 0$ such that $f(u) > 0$ for $0 < u < U$.

**F0:** There are $q_s > 2$, $h_0 > 0$, $c_s > 0$ such that

$$f'(u) = (q_s - 1) c_s u^{q_s - 2} + o(u^{q_s - 2 + h_0}), \quad \text{as } u \to 0. \quad (5)$$

**F1:** There are $2 \leq Q_s < 2^*(\delta)$, $h_1 > 0$, $C_s > 0$ such that

$$f'(U + u) = (Q_s - 1) C_s u^{Q_s - 2} + o(u^{Q_s - 2 + h_1}), \quad \text{as } u \to 0. \quad (6)$$

**F2:** There are $2 < q_s < 2^*(\delta)$, $h_2 > 0$, $c_s > 0$ such that

$$f'(u) = (q_s - 1) c_s u^{q_s - 2} + o(u^{q_s - 2 - h_2}), \quad \text{as } u \to \infty. \quad (7)$$

As a consequence of our main results we obtain the following.

**Theorem 1.1.** Assume $A, F0, F1$, then there is $\lambda^* > 0$ such that (1) admits at least 3 radial positive solutions for $\lambda > \lambda^*$.

**Theorem 1.2.** Assume $B, F0, F2$, then there is $\lambda^* > 0$ such that (1) admits at least 3 radial positive solutions for $\lambda > \lambda^*$, at least 2 radial positive solutions for $\lambda = \lambda^*$, at least 1 radial positive solution for $0 < \lambda < \lambda^*$.

**Theorem 1.3.** Assume $f$ satisfies (5) with $q_s = 2$, $A, F1$. Then there is $\lambda^* > 0$ such that (1) admits at least 2 radial positive solutions for $\lambda > \lambda^*$.

Assume $f$ satisfies (5) with $q_s = 2$, $B, F2$. Then there is $\lambda^* > 0$ such that (1) admits at least 2 radial positive solution for $\lambda > \lambda^*$ and at least 1 radial positive solution for $0 < \lambda < \lambda^*$.

**Remark 1.** Notice that if $f$ is of type (4) then $A$ is satisfied and consequently $B$ too. Further $F0, F1$, and $F2$ are satisfied respectively if $q_s = q > 2$, $Q_s = a + 1 \in [2, 2^*(\delta)]$, and $q_s = q + a \in [2, 2^*(\delta)]$.

**Corollary 1.** The solutions found by Theorems 1.1 and 1.2 have the following properties. For $\lambda$ large (i.e $\lambda > \lambda^*$) we have 3 positive solutions say $u(r, d_i)$ for $i = 1, 2, 3$. We have $d_i = d_i(\lambda)$, $d_i(\lambda) < d_2(\lambda) < U < d_3(\lambda)$ and $d_2(\lambda) \to U^-$, $d_1(\lambda) \to m^+$ as $\lambda \to +\infty$, where $m = 0$ if $2 < q_s < 2^*(\delta)$ and $m \in [0, U]$ if $q_s \geq 2^*(\delta)$. Further $\frac{\partial}{\partial r} u(r, d_i) < 0$, for any $0 < r \leq 1$, for $i = 1, 2$.

Moreover if we are in the assumptions of Theorem 1.1 then $d_3(\lambda) \to U^+$ as $\lambda \to +\infty$ and $\frac{\partial}{\partial r} u(r, d_3) < 0$ for $0 < r \leq 1$ too.

If we are in the assumptions of Theorem 1.2 we have $0 < d_1 < d_2 < U$ when $\lambda > \lambda^*$, $d_1 = d_2$ when $\lambda = \lambda^*$, and they do not exist for $0 < \lambda < \lambda^*$.

Via Theorem 1.3 we just find the solutions $u(r, d_i)$ for $i = 2, 3$, where $d_2(\lambda) < U < d_3(\lambda)$, and they have the properties described above.

In fact the whole discussion is generalized to embrace the more general case of $p$-Laplace equation, i.e.

$$\text{div}(r^\ell \nabla u|\nabla u|^{p - 2}) + \lambda r^\ell + \delta f(u) = 0$$

where $n + \ell > p$, $\delta > -p$, and we also need to assume $1 < p \leq 2$ in order to avoid cumbersome technicalities. Notice that for $\ell = 0$ we obtain the $\Delta_p$ operator. Again we are interested in radial positive solutions of (8) of the Dirichlet problem in the ball of radius 1. So in fact we consider the following ODE

$$(u'(r)|u'(r)|^{p-2}r^{\ell+n-1})' + \lambda f(u)r^{\ell+n-1} = 0$$

(9)
Using the concept of natural dimension introduced in [31] and performing the change of variables introduced in [31, §2, Remark (i)], see also [14, Appendix B] and in particular Remark B.1, we pass from (9) to the following

\[(u'(r)|u'(r)|^{p-2}r^{N-1})' + \lambda f(u)r^{N-1} = 0\]  \hspace{1cm} (10)

where \(N = p^{\frac{2+\ell+n}{\delta+p}} > 1\) is not anymore an integer and is called natural dimension. Obviously (9) can be regarded as the equation for radial solutions of \(\Delta_p u + \lambda f(u) = 0\), but asking for the Prototypical setting too with trivial adaptation.

Finally we conclude with a classical scaling argument: the problem of finding intersections between such a graph and the level line \(R \geq 0\) we simply have a shift in the values of the critical exponents so that the Sobolev critical exponent \(p^\ast\), as in the previous section, is replaced by \(p^\ast(\delta) := \frac{p(n+\ell+2)}{n+\ell+p}\), which reduces to usual one if \(\delta = \ell = 0\).

The prototypical \(f\) we are interested in is again (4), where \(p^\ast(\delta)\) replaces \(2^\ast(\delta)\), namely we rephrase \(F_0, F_1, F_2\) as follows.

**F0**': There is \(q_0 \geq 2, h_0 > 0, c_s > 0\) such that (5) holds, but \((p, q_s) \neq (2, 2)\).

**F1**': There are \(2 \leq Q_s < p^\ast(\delta), h_1 > 0, C_s > 0\) such that (6) holds.

**F2**': There are \(2 \leq q_u < p^\ast(\delta), (p, q_u) \neq (2, 2), h_2 > 0, c_u > 0\) such that (7) holds.

**Theorem 1.4.** Assume \(1 < p \leq 2, F_0', F_1', A\), then there is \(\lambda^\ast > 0\) such that the Dirichlet problem in the ball of radius 1 associated to (8) admits at least 3 radial positive solutions for \(\lambda \geq \lambda^\ast\).

**Theorem 1.5.** Assume \(1 < p \leq 2, F_0', F_2', B\), then there is \(\lambda^\ast > 0\) such that the Dirichlet problem in the ball of radius 1 associated to (8) admits at least 3 radial positive solutions for \(\lambda > \lambda^\ast\), at least 2 radial positive solutions for \(\lambda = \lambda^\ast\), at least 1 radial positive solution for \(0 < \lambda < \lambda^\ast\).

**Corollary 2.** Corollary 1 holds in this \(p\)-Laplace setting too.

**Remark 2.** Remark 1 and Corollary 1 holds in this context too with trivial adaptation. In particular if \(f\) is of type (4) again we have \(q_s = \gamma, Q_s = a + 1\) and \(q_u = q + a\).

The proofs are developed directly in the more general \(p\)-Laplace case, apart from the case where \(p = q_s = 2\) which needs a separate discussion.

The outline of the proof goes as follows. In section 2 we introduce Fowler transformation, one of the main tools used in the proofs. In section 3 we turn to consider (2) assuming \(\lambda = 1\), and we study the dependence on \(d\) of the first zero \(R(d)\) of the solution \(u(r, d)\) of (9), (3). In fact we aim to prove that \(R(d)\) is a graph as sketched in figures 3, 4, 5. Then we study the asymptotic properties of the function \(R(d)\) and we look for intersections between such a graph and the level line \(R = K > 0\). Finally we conclude with a classical scaling argument: the problem of finding intersections between the graph \(R(d)\) and the line \(R = K\) for \(K > 0\) large is then shown to be equivalent to find positive radial solutions of (9) in the ball of radius 1 for \(\lambda\) large.

2. **Fowler transformation.** The main tool of investigation is the Fowler transformation, developed by Fowler in the 30s and extended to the \(p\)-Laplace case by Bidaut-Veron in [4] and independently by Franca in [12]. Let us set

\[\alpha_l = \frac{p+\ell+n}{\ell+p}, \quad \beta_l = \frac{p+\ell}{\ell+p}(p-1), \quad \gamma_l = \beta_l - (n-1), \quad x_l = u(r)r^{\alpha_l}, \quad y_l = u'(r)r^{p-2}p^{\beta_l}, \quad r = e^t\]

\[g_l(x_l, t) = f(x_l e^{-\alpha_l} e^{\alpha_l(n-1)t})\]  \hspace{1cm} (11)

\[y_l = u'(r)r^{p-2}p^{\beta_l}, \quad r = e^t\]
The new variables \( \xi, \eta \) differ from the given ones \( u, u' \) in the presence of weight terms, which will help us to determine the asymptotic behaviors. Using (11), we pass from (9) to the following system

\[
\begin{pmatrix}
    \dot{x} \\
    \dot{y}
\end{pmatrix} = \begin{pmatrix}
    \alpha_l & 0 \\
    0 & \gamma_l
\end{pmatrix} \begin{pmatrix}
    x \\
    y
\end{pmatrix} + \begin{pmatrix}
    \text{sign}(y)|y|^{1/(p-1)} \\
    -g_l(x,t)
\end{pmatrix}
\] (12)

In the whole paper the dot indicates differentiation with respect to \( t \), and we write \( \phi_l(t, \tau; Q) = (x_l(t, \tau; Q), y_l(t, \tau; Q)) \) for a trajectory of (12), evaluated at \( t \) and departing from \( Q \in \mathbb{R}^2 \) at \( t = \tau \).

The behavior of positive solutions of equation (9) undergoes to several bifurcations due to the presence of critical exponents, such as the Sobolev critical exponent \( p^*(\delta) \) introduced above. A further important critical value is given by

\[
p_* := \frac{p(n-1)+\delta p^*}{n-p} \quad \text{for } \delta = 0.
\]

**Remark 3.** The change of variables (11) is particularly useful when \( f(u, r) = cv^p - v^{q-1} \); in this case, setting \( l = \frac{p(q+\delta)}{p+\delta} \), we find \( g_l(x_l, t) = cx_l^{q-1} \) so (12) is autonomous. Observe that if \( \delta = 0 \) then \( l = q \).

Further notice that \( p_* < q < p^*(\delta) \) if and only if \( p_* < l < p^* \).

We denote by \( M^u \) the set

\[
M^u := \left\{ Q \mid \lim_{t \to -\infty} \|\phi_l(t, 0, Q)\|e^{-\alpha_l t} = c \in \mathbb{R} \right\}
\]

From now on, we assume \( 1 < p \leq 2 \) so that (12) is \( C^1 \). In fact this assumption may be relaxed but paying the prize of cumbersome technical difficulties, see e.g. [14].

From standard facts in ODE theory we see that \( M^u \) is a 1-dimensional \( C^1 \) manifold, see e.g. [5, §13].

If \( l > p_* \) the origin is a saddle, so it admits an unstable manifold, \( M^u \), and a stable manifold, \( M^s \), i.e. \( Q \in M^s \) if \( \phi_l(t, 0, Q) \to (0, 0) \) as \( t \to +\infty \). If \( l = p_* \), the origin has a central direction but \( M^u \) is anyway the unstable manifold. If \( p < l < p_* \) and \( q \geq 2 \) the origin is an unstable node and \( M^u \) is the strongly unstable manifold, see again [5, §13], and figures 1, 2. Using elementary facts from ODE theory we find that trajectories of (12) correspond to regular solutions of (2), see e.g. [13].

**Remark 4.** Assume \( f(u) = cv^p - v^{q-1} \), \( c > 0 \), so that setting \( l = \frac{p(q+\delta)}{p+\delta} \) we get \( g_l(x_l, t) \equiv cx_l^{q-1} \); let \( q > p \) (and consequently \( l > p \), \( q \geq 2 \)). Regular solutions \( u(r) \) of Eq. (9) correspond to trajectories \( \phi_l(t) \) of system (12) departing from points in \( M^u \) and vice versa.

Further, using the invariance for \( t \)-translations of (12) and the fact that \( M^u \) is the graph of three trajectories (one corresponding to \( u(r, d) \) where \( d > 0 \), one corresponding to \( u(r, d) \) where \( d < 0 \) and the origin), we get the following known result.

**Remark 5.** Assume \( g_l(x_l, t) \equiv cx_l^{q-1} \), where \( q > p \), \( q \geq 2 \), \( c > 0 \). Fix \( Q \in M^u \) and let \( u(r, d(\tau)) \) be the regular solution of (2) corresponding to \( \phi_l(t, \tau; Q) \) of (12). Then \( d(\tau) \) is continuous, \( d(\tau) \to +\infty \) as \( \tau \to 0 \) and \( d(\tau) \to 0 \) as \( \tau \to -\infty \).

**Proof.** Let \( Q \in M^u \), \( \tau \in \mathbb{R} \) and let \( u(r, d(0)) \), \( u(r, d(\tau)) \) be the solution of (9) corresponding to \( \phi_l(t, 0, Q) \) and \( \phi_l(t, \tau, Q) \) respectively. Notice that \( \phi_l(t, 0, Q) = \)
we get the following well known result.

So the remark follows.

Figure 1. Sketch of the phase portrait of the autonomous system (12) where \( g_l(x,t) \equiv x|x|^{q-2}, q > 2, \) when \( p < l \leq p^* \) on the left (a), and when \( p_* < l < p^* \) on the right (b). The manifold \( M^u \) is the solid (black) curve; in fig. (a) the dotted (magenta) curve denotes a trajectory \( \chi(t) \) converging to the origin as \( t \to -\infty \) but not staying in the strongly unstable manifold \( M^u \), in fig (b) the dashed (blue) curve denotes the stable manifold.

\[
\phi_l(t + \tau, \tau, Q) \text{ for any } t \in \mathbb{R}, \text{ therefore }
\]
\[
d(\tau) = \lim_{r \to 0} u(r, d(\tau)) = \lim_{t \to -\infty} x(t + \tau, \tau, Q)e^{-\alpha_l(t+\tau)}
\]
\[
= \lim_{t \to -\infty} x(t, 0, Q)e^{-\alpha_l(t+\tau)} = \lim_{r \to 0} u(r, d(0))e^{-\alpha_l \tau} = d(0)e^{-\alpha_l \tau}.
\]

Using the Pohozaev identity it is easy to show that the phase portrait is as depicted in figures 1, 2, see e.g. [13, Theorem 1], [25] see also [11, 29].

Thus in particular if \( Q \in M^u \), there is \( T(Q) \) such that \( y_l(T(Q), 0; Q) < 0 \). Using Remark 4 and 5 we get the following well known result.

**Remark 6.** Assume \( g_l(x, t) \equiv cx^{q-1}, q \geq 2, c > 0 \) and \( p < l < p^* \); then all the regular solutions \( u(r, d) \) are crossing solutions, i.e. there is \( R(d) > 0 \) such that \( u(R(d), d) = 0 \) and \( u'(R(d), d) < 0 \). Further \( R(d) \to 0 \) as \( d \to +\infty \) and \( R(d) \to +\infty \) as \( d \to 0 \).

We briefly recall the well known results which enable us to draw pictures 1, 2, and to deduce the structure of (9) when \( f(u) = u^{q-1} \).

We emphasize that when \( g_l(x, t) = cx^{q-1} \) and \( p_* < q < p^* \), then (12) admits a further critical point in \( x > 0 \), say \( P = (P_x, P_y) \) where \( P_x = \gamma_l(\alpha_l)^{p-1}/c^{1/(q-p)} \) and \( P_y = -(\alpha_l P_x)^{p-1} \), which converges to the origin as \( l \to p_* \). When \( p_* < l < p^* \) trajectories of (12) converging to \( P \) as \( t \to -\infty \) correspond to singular solutions of (9). When \( l = p_* \) singular solutions of (9) exist, but correspond to trajectories of the central manifold of (12). When \( p < l < p_* \) singular solutions of (9) again exist and correspond to trajectories converging to the origin as \( t \to -\infty \) but not belonging to the strongly unstable manifold \( M^u \), see e.g. [16, §2] for a proof in the Laplace context. However this fact will not be used in this article.

**Remark 7.** Assume \( g_l(x, t) \equiv cx^{q-1}, q > p, q \geq 2, c > 0 \). We can find \( K > 0 \) such that if \( \|Q\| \geq K \) then there is \( T(Q) > 0 \) such that \( \phi_l(t, 0; Q) \) crosses
transversally the $y$ negative semi-axis at $t = \bar{T}(Q)$. Further $\bar{T}(Q) \to 0$ as $\|Q\| \to +\infty$

Proof. The result is a consequence of the superlinearity of $g_l$ and it is borrowed from [15]: we sketch the proof for completeness.

Let us set

$$x_i |x_i|^{p-2} = \rho_l \cos(\theta_i) \quad y = \rho_l \sin(\theta_i) \quad (13)$$

From a straightforward computation we see that

$$\dot{\theta}_l(t, Q) = (p - n) \sin(\theta_l) \cos(\theta_l) - (p - 1) \sin(\theta_l) ||T^\theta_t|| \cos(\theta_l) ||x||^{\frac{p-2}{2}} - c \text{ sign}[\cos(\theta_l)] |\cos(\theta_l)|^{\theta p^{\theta - \theta}} \quad (14)$$

Hence $\dot{\theta_l}(t, Q) \to -\infty$ as $x \to +\infty$ and it is negative if $x = 0$. So the Remark follows.

Let $k > 0$; we introduce the following set:

$$\mathcal{T}(k) := \{(x, y) \mid 0 < kx < |y|\}, \quad (15)$$

We emphasize the following facts, which follow easily from some standard phase plane analysis and from Remark 7, see e.g. [16] for a full fledged proof in the Laplace context, or again [13, Theorem 1].

**Remark 8.** Assume $g_l(x, t) = cx^{q-1}$, $c > 0$, $q \geq 2$, $p < l \leq p_*$; then the origin is the unique critical point of the system and it is unstable. Hence all the trajectories rotate clockwise and cross the coordinate axes indefinitely as $t \to +\infty$.

**Remark 9.** Assume $g_l(x, t) = cx^{q-1}$, $c > 0$, $q \geq 2$, and $p_* < l < p^*$; then the origin admits a stable manifold which is a heteroclinic connection between the origin and $P$. However there is $k > 0$ such that $\mathcal{T}(k)$ does not intersect $M^s$ and $P \not\in \mathcal{T}(k)$.

**Remark 10.** Assume $g_l(x, t) = cx^{q-1}$, $c > 0$, $q \geq 2$.

Let $l = p^*$ (i.e. $q = p^*(\theta)$); then the origin is a saddle and $M^u = M^s$ is the union of the origin and of the graphs of two homoclinic trajectories converging to the origin and surrounding respectively the critical points $P$ and $-P$, see figure 2a.

Let $l > p^*$; then the origin is a saddle and $M^u$ is made up by two heteroclinic connections between the origin and $P$, the origin and $-P$. The stable manifold is a double spiral intersecting the coordinate axes indefinitely, see figure 2b.

Let us denote by $B(K) = \{\phi \mid |\phi| \leq K\}$

**Remark 11.** Assume $g_l(x, t) = cx^{q-1}$, $c > 0$, $l \geq p^*$ and consider a solution $u(r)$ of (9) and the corresponding trajectory $\tilde{\phi}_l(t)$ of (12). Then there is $\bar{K} > 0$ such that if $|\tilde{\phi}_l(T)| > \bar{K}$ and $x_i(T) > 0$ for a certain $T \in \mathbb{R}$ then $u(r)$ becomes negative for some $R > e^T$.

Proof. Let us choose $\bar{K}$ as $\bar{K} = \sup\{\|Q\| \mid Q \in M^s\}$. Assume by contradiction that $u(r) > 0$ is definitively positive when $r > e^T$: it easily follows that $u(r) \to 0$ as $r \to +\infty$.

If $l > p^*$ then either $\tilde{\phi}_l(t) \in M^s$ for any $t \in \mathbb{R}$ or $\tilde{\phi}_l(t)$ converges to $P$ as $t \to +\infty$; hence we get $|\tilde{\phi}_l(T)| \leq \bar{K}$.

Similarly if $l = p^*$ then $\tilde{\phi}_l(t)$ is contained in the compact set enclosed by $M^u = M^s$, see figure 2, and we conclude as above.
2.1. **Unstable leaves for non-autonomous systems.** In this subsection, following [10], we combine the results of [5, §13] and of [23] to construct the unstable manifolds for (12) when \( g_l \) depends on \( t \). Assume \( F_0' \) and \( F_2' \) and set \( \omega_u = h_0/(2\alpha_{l_u}) \), \( \omega_u = h_2/(2\alpha_{l_u}) \). We introduce the following notation that will be in force in the whole paper. Set
\[
\omega_s := \frac{p(q_s + \delta)}{p + \delta}, \quad \omega_u := \frac{p(q_u + \delta)}{p + \delta} \tag{16}
\]

Remark 12. In the whole paper we just give conditions on \( q_u \) and \( q_s \). We emphasize that this is equivalent to ask for conditions for \( l_u \) and \( l_s \); namely \( p < l_s < p^* \), or \( p^* < l_s < p^* \), \( l_s > p^* \) if and only if \( p < q_s < p_u(\delta) \), or respectively \( p_u(\delta) < q_s < p^*(\delta) \), \( q_s > p^*(\delta) \); analogously \( p < l_u < p_u \), \( p_u < l_u < p^* \), \( l_u > p^* \) respectively if and only if \( p < q_u < p_u(\delta) \), \( p_u(\delta) < q_u < p^*(\delta) \), \( q_u > p^*(\delta) \).

Let us consider (12) where we have added the extra variable \( z(t) = e^{\omega_u t} \) in order to deal with the following 3-dimensional autonomous system:
\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{pmatrix} = \begin{pmatrix}
\alpha_{l_u} & 0 & 0 \\
0 & \gamma_{l_u} & 0 \\
0 & 0 & \omega_u
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} + \begin{pmatrix}
\text{sign}(y)|y|^{1/(p-1)} \\
-g_u(\omega_u, \ln(\omega_u)) \\
0
\end{pmatrix} \tag{17}
\]
System (17) is useful to discuss the behavior of trajectories of (12) as \( t \to -\infty \), and correspondingly the behavior of (9) for \( r \) small (and \( u \) large as we will see below). Similarly if we replace \( z(t) \) by \( \zeta(t) = e^{-\sigma t} \) we get a system which is useful to detect the behavior of trajectories of (12) in the future and correspondingly to analyze the behavior of (9) for \( r \) large (and \( u \) small, see below), i.e.:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{\zeta}
\end{pmatrix} = \begin{pmatrix}
\alpha_t & 0 & 0 \\
0 & \gamma_t & 0 \\
0 & 0 & -\omega_t
\end{pmatrix} \begin{pmatrix}
x \\
y \\
\zeta
\end{pmatrix} + \begin{pmatrix}
\text{sign}(y)|y|^{1/(\sigma - 1)} \\
-g_t(x, \frac{\ln(y)}{-\omega_t})
\end{pmatrix}
\]  

(18)

**Remark 13.** It is straightforward to check that (17) is \( C^1 \) also for \( z = 0 \) if \( f \) satisfies (7) (in particular if \( F' \) holds), while (18) is \( C^1 \) also for \( \zeta = 0 \) if \( f \) satisfies (5) (in particular if \( F' \) holds).

**Proof.** Consider (18) and assume (7); notice that

\[
g_t(x, t) = x^{q_t - 1}\{c_s + o((xe^{-\alpha_t,t})^h)\} = c_x x^{q_t - 1} + x^{q_t - 1 + 2\omega_t} o(\zeta(t)^2\omega_t)
\]  

(19)
as \( t \to +\infty \), i.e. as \( \zeta(t) \to 0 \). Further

\[
\frac{\partial g_t}{\partial x}(x, t) = x^{q_t - 2}\{(q_t - 1)c_s + o((xe^{-\alpha_t,t})^h)\} = (q_t - 1)c_x x^{q_t - 1} + x^{q_t - 1 + 2\omega_t} o(\zeta(t)^2\omega_t)
\]  

and it is continuous as \( \zeta \to 0 \). Moreover, from (19), we see that the derivative with respect to \( \zeta \) of the second equation in (18) is continuous and converges uniformly to 0 as \( \zeta \to 0 \) when \( x \) is in a compact set. Then it is easy to check that (18) is in fact \( C^1 \).

Similarly consider (17) and assume (5); notice that

\[
g_u(x, t) = x^{q_u - 1}\{c_u + o((xe^{-\alpha_u,t})^h)\} = c_u x^{q_u - 1} + x^{q_u - 1 - 2\omega_u} o(\zeta(t)^2\omega_u)
\]  
as \( z(t) \to 0 \), i.e. as \( t \to -\infty \). Then, reasoning as above, we see that (17) is \( C^1 \) too.

From [6, Remark 2.5] we know that all the solutions of (12) may be continued for any \( t \in \mathbb{R} \). Let \((q_{t_u}(t), z(t))\) be a trajectory of (17); we recall that the \( \alpha \)-limit set of \((q_{t_u}(t), z(t))\) is defined as

\[
\{(Q, \bar{z}) | \exists t_t \searrow -\infty : (q_{t_t}(t_t), z(t_t)) \to (Q, \bar{z})\}.
\]

It is easy to check that the \( \alpha \)-limit set of a bounded trajectory of (17) is contained in the \( z = 0 \) plane; moreover such a plane is invariant and the dynamics reduced to the \( z = 0 \) plane coincides with the one of the autonomous system (12) where \( g_u(x, t) = g_u(x, -\infty) \). Assume first that \( q_u \in [p_u(\delta); p^*(\delta)] \); then the origin of (17) admits a 2-dimensional unstable manifold \( W^u_{t_u} \) which is transversal to \( z = 0 \). Following [15], see also [23, 24], [16, §6], we see that, for any \( \tau \in \mathbb{R} \), the sets \( W^u_{t_u}(\tau) \) and \( W^u_{t_u}(-\infty) \) defined as

\[
W^u_{t_u}(\tau) = (W^u_{t_u} \cap \{z = e^{\omega_u \tau}\}), \quad W^u_{t_u}(-\infty) = (W^u_{t_u} \cap \{z = 0\})
\]  

(20)
are \( C^1 \) immersed 1-dimensional manifolds, i.e. the graph of \( C^1 \) regular curves.

Further notice that \( W^u_{t_u}(\tau) \) can be characterized as follows:

\[
W^u_{t_u}(\tau) := \{Q \mid q_{t_u}(t, \tau; Q) \to (0, 0) \text{ as } t \to -\infty\}
\]  

(21)
Moreover it depends continuously on \( \tau \). More precisely we have the following see e.g. [24], see also [5, §13.4].
Remark 14. Let either \( \tau_0 \in \mathbb{R} \) or \( \tau_0 = -\infty \), and assume that \( W^u_{\tau_0}(\tau_0) \) intersects transversally a segment \( L \) in a point denoted by \( Q(\tau_0) \). Then there is a neighborhood \( J \) of \( \tau_0 \) such that \( W^u_{\tau_0}(\tau) \) still intersects \( L \) transversally in a point \( Q(\tau) \) for \( \tau \in J \); moreover \( Q(\tau) \) is as smooth as \( (17) \).

Assume \( F2' \); using standard tools of invariant manifold theory, see e.g. \([5, \S 13.4]\), we see that if \( q_u \in [p_u(\delta); p^*(\delta)) \), then \( W^u_{\tau_0}(\tau) \) can be equivalently characterized as follows:

\[
W^u_{\tau_0}(\tau) := \{ Q \mid \| \phi_{\tau_0}(t, \tau, Q) \| e^{-\alpha_1 u} t \to k \in \mathbb{R} \text{ as } t \to -\infty \} \tag{22}
\]

If \( p < q_u < p_u(\delta) \) the origin has a 3 dimensional unstable manifold (an open set), but we can define a 2-dimensional strongly unstable manifold. Set

\[
W^u_{\tau_0} = \{ (Q, e^{\alpha_1 r}) \mid \| \phi_{\tau_0}(t, \tau, Q) \| e^{-\alpha_1 u} t \to k \in \mathbb{R} \text{ as } t \to -\infty \}
\]

then \( W^u_{\tau_0} \) is an invariant manifold and the manifolds \( W^u_{\tau_0}(\tau) \) and \( W^u_{\tau_0}(-\infty) \) defined as in \((20)\) or as in \((22)\) (but not as in \((21)\)) satisfy Remark 14, see again \([5, \S 13.4]\). In this case we have also trajectories converging to the origin at a slower exponential rate, corresponding to singular solutions of \((9)\); however this fact will not be used in this article.

It is easy to check that, if \( Q \in W^u_{\tau_0}(\tau) \), there is \( d > 0 \) such that \( \phi_{\tau_0}(t, \tau, Q) e^{-\alpha_1 u} t \to (d, 0) \). Hence the corresponding solution of \((2)\) is a regular solution, i.e. we have the following.

Remark 15. Assume \( F2' \) with \( q_u > p_u \), and fix \( \tau \in \mathbb{R} \). Then regular solutions \( u(r, d) \) of Eq. \((9)\) correspond to trajectories \( \phi_{\tau_0}(t) \) of system \((12)\) departing from points in \( W^u_{\tau_0}(\tau) \) and vice versa: i.e. Remark 4 still holds true.

Further, we get the following generalization of the second part of Remark 5, see \([6, \text{Lemma 2.10}]\) for a detailed proof.

Remark 16. Fix \( \tau \in \mathbb{R} \) and consider the manifold \( W^u_{\tau_0}(\tau) \), the trajectory \( \phi_{\tau_0}(t, \tau, Q) \) of \((12)\), where \( Q \in W^u_{\tau_0}(\tau) \) and the corresponding solution \( u(r, d) \) of \((2)\), so that \( d(Q) \) is an invertible function. Follow \( W^u_{\tau_0}(\tau) \) from the origin towards \( x > 0 \), then \( d \) increases as we go further from the origin, and \( d(Q) \to 0 \) as \( Q \to (0, 0) \).

We observe that for any fixed \( \tau \in \mathbb{R} \) we can parameterize \( W^u_{\tau_0}(\tau) \) by \( d > 0 \). Assume \( B \); for later purposes we denote by \( \tilde{W}^u_{\tau_0}(\tau) \) the branch of \( W^u_{\tau_0}(\tau) \) between the origin and \( \psi_{\tau_0}(\tau) := (U e^{\alpha_1 u}, 0) \), i.e. the branch corresponding to the values \( d \in [0, U] \) in a parametrization by \( d > 0 \).

With analogous reasoning if \( F0' \) holds and \( q_u > p_u(\delta) \), we see that \((18)\) admits a two dimensional invariant stable manifold \( W^s_{\tau} \). Further

\[
W^s_{\tau}(\tau) = (W^s_{\tau} \cap \{ \zeta = e^{-\alpha_1 r}\}) \quad \text{and} \quad W^s_{\tau}(+\infty) = (W^s_{\tau} \cap \{ \zeta = 0\}) \tag{23}
\]

are \( C^1 \) immersed 1-dimensional manifold depending in a \( C^1 \) way from \( \tau \), and \( W^s_{\tau}(\tau) \to W^s_{\tau}(+\infty) \) as \( \tau \to +\infty \) in the sense specified in Remark 14. Moreover

\[
W^s_{\tau}(\tau) := \{ Q \mid \phi_{\tau}(t, \tau, Q) \to (0, 0) \} \quad \text{as} \quad t \to +\infty,
\]

and \( W^s_{\tau}(+\infty) \) coincide with the stable manifold \( M^s \) of the autonomous system \((12)\) where \( g_l = c e^{\alpha_1 r} \). From now on, since we are just interested in positive solutions, abusing the notation, for \( W^u_{\tau_0}(\tau) \) (respectively \( W^s_{\tau}(\tau) \)) we mean just the branch of the manifold leaving from the origin towards \( x > 0 \) and corresponding to solutions \( u(r) \) of \((9)\) which are positive for \( r \) small (respectively for \( r \) large).
Sometimes it will be useful to switch between different values of \( l \) in system (12), e.g. to pass from \( l = l_u \) to \( l = l_s \). It is straightforward to notice that if \( \phi_{l_u}(t, \tau, Q) \) and \( \phi_{l_s}(t, \tau, R) \) correspond to the same solution \( u(r) \) of (2), then

\[
\begin{align*}
\frac{x_{l_u}(t, \tau, R)}{x_{l_u}(t, \tau, Q)} &= e^{(\alpha_{l_s} - \alpha_{l_u})t}, \\
\frac{y_{l_u}(t, \tau, R)}{y_{l_u}(t, \tau, Q)} &= e^{(\alpha_{l_s} - \alpha_{l_u})t} \frac{y_{l_u}}{y_{l_u}}, \\
R_x &= Q_x e^{(\alpha_{l_s} - \alpha_{l_u})\tau}, \quad R_y = Q_y e^{(\alpha_{l_s} - \alpha_{l_u})\tau}
\end{align*}
\]

(24)

where we used (11) and the fact that \( \beta_{l_u} - \beta_{l_u} = \frac{\alpha_{l_s} - \alpha_{l_u}}{p-1} \). It follows that the curves of the form \( y = k|x|^p \) remain invariant as we pass from \( l = l_u \) to \( l = l_s \) at a fixed \( \tau \in \mathbb{R} \), for any fixed \( k \in \mathbb{R} \). In fact the whole portrait is subject either to a dilatation if \( (\alpha_{l_u} - \alpha_{l_s})\tau > 0 \) or to a contraction if \( (\alpha_{l_u} - \alpha_{l_s})\tau < 0 \).

Let \( \tau \in \mathbb{R} \) and denote by

\[ W^u_{l_u}(\tau) := \{(R_x e^{(\alpha_{l_s} - \alpha_{l_u})\tau}, R_y e^{(\alpha_{l_s} - \alpha_{l_u})\tau}) \mid (R_x, R_y) \in W^u_{l_u}(\tau)\} \]

Observe that \( W^u_{l_u}(\tau) \) is diffeomorphic to \( W^u_{l_u}(\tau) \), hence \( W^u_{l_u}(\tau) \) is 1 dimensional too, and inherits the transversal smoothness property described in Remark 14.

We denote by \( \tilde{W}^u_{l_u}(\tau) \) the branch of \( W^u_{l_u}(\tau) \) between the origin and \( \psi_{l_u}(t) := (Ue^{\alpha_{l_u}\tau}, 0) \), i.e. the branch corresponding to the values \( d \in [0, U] \) in a parametrization by \( d > 0 \).

In fact the unstable leaves \( W^u_{l_u}(\tau) \) may be constructed through the invariant manifold theory for non-autonomous systems, simply requiring that for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( \frac{\partial u}{\partial x}(x, t) < \varepsilon \) for \( |x| \leq \delta \) for any \( t \leq \tau \) (we stress that such an assumption is satisfied if \( F2' \) holds), cf [5, §13.4] and in particular Theorems 4.1, 4.3, 4.4. Since the linearization of (12) in the origin has \( t \) independent eigenvalues and eigenvectors, from [5, Theorem 4.2, §13.4] we get the following.

**Remark 17.** Assume \( F2' \), then for any \( \tau \in \mathbb{R} \) the manifold \( W^u_{l_u}(\tau) \) is tangent in the origin to the \( y = 0 \) axis. So, thanks to (24), \( W^u_{l_u}(\tau) \) is tangent in the origin to the \( y = 0 \) axis, too. Similarly, if \( F'0 \) holds then \( W^u_{l_u}(\tau) \) is tangent to the \( y \)-negative semi-axis if \( 1 < p < 2 \) and to the line \( y = -(n-2)x \) if \( p = 2 \).

3. **Proofs.** Let us observe that the flow of (12) on the \( y \) axis rotates clockwise for any \( t \in \mathbb{R} \) and the origin is a critical point. Hence for the corresponding solutions \( u(r) \) of (9) we see that all the zeroes are non-degenerate.

Let us introduce the following set

\[ I := \{ d > 0 \mid u(r, d) \text{ is a crossing solution} \} \]

(25)

where \( u(r, d) \) has a (non-degenerate) zero for a certain \( r = R(d) > 0 \) iff \( d \in I \). The scheme of the proof is the following: in the first two subsections we set \( \lambda = 1 \) and we consider (12); first we show that \( I \) is open, \( R(d) \) is continuous. In § 3.1 we fix \( \lambda = 1 \), and we consider the setting of Theorem 1.5 and we aim to draw the graph of \( R(d) \) i.e. pictures 3b, 4b: this is the content of Propositions 1, 2. Then in § 3.2 we use the information of § 3.1 to draw the graph of \( R(d) \) with \( \lambda = 1 \) in the setting of Theorem 1.4, see pictures 3a, 4a and Propositions 4, 6. Then we adapt the argument to draw the graph of \( R(d) \) in the setting of Theorem 1.3. Finally in § 3.3 we perform a scaling argument to obtain the proof of Theorems 1.5 and 1.4.

Let us begin from the following results developed directly on (9).
Remark 18. Assume \( f \) satisfies \( A \) and (5) with \( q_s \geq 2 \), i.e. we assume \( F0' \) but we allow \( p = q_s = 2 \). If \( d \not\in I, d \leq U + \sigma, d \not\in U \), then \( u(r, d) > 0 \) for any \( r > 0 \) and either \( \lim_{r \to \infty} u(r, d) = 0 \) or \( \lim_{r \to \infty} u(r, d) = U \).

Proof. First of all observe that \( u(r, d) \) is positive and strictly decreasing for \( r > 0 \) small enough, if \( d < U + \sigma \) and \( d \not\in U \), see e.g. [14, Lemma 2.1]. Set

\[
\rho_d = \sup \left\{ R \mid u(r, d) > 0, \frac{\partial u}{\partial r}(r, d) < 0, \text{ for any } 0 < r < R \right\},
\]

and \( L(d) = \lim_{r \to \rho_d} u(r, d) \); hence if \( d < U + \sigma \) then \( u(r, d) \) is positive and decreasing when \( 0 \leq r < \rho_d \).

It is easy to check that if \( \rho_d = +\infty \) then \( f(L(d)) = 0 \) so the Remark is proved, so we just need to discuss the case \( \rho_d < \infty \).

We claim that if \( L(d) = 0 \), \( L'(d) := \lim_{r \to \rho_d} u'(r, d) = 0 \) then \( \rho_d = +\infty \). In fact let \( \phi(t) \) be the trajectory of (12) corresponding to \( u(r) \), then \( \phi(t) \to (0, 0) \) as \( t \to \ln(\rho_d) \), and the origin is a critical point of (12) so the claim follows.

Analogously if \( L(d) = U \), \( L'(d) = \lim_{r \to \rho_d} u'(r, d) = 0 \) then \( \rho_d = +\infty \). In fact let us consider the modified system (12) where \( g \) is obtained via (11) but replacing \( f \) by \( \tilde{f}(u) = f(u - U) \). Let \( \tilde{\phi}(t) \) be the trajectory of the modified system (12) corresponding to \( u(r, d) \); it follows that \( \tilde{\phi}(t) \to (0, 0) \) as \( t \to \ln(\rho_d) \), and the origin is again a critical point so \( \rho_d = +\infty \), and the claim in proved.

So we can assume \( \rho_d < \infty \), \( f(L(d)) > 0 \), and \( L'(d) = u'(\rho_d, d) = 0 \); but from (9) we get

\[(p - 1)u''(r, d) = |u'(r, d)| \left[ \frac{n - 1}{r} - r^\delta f(u(r, d))|u'(r, d)|^{-(1-p)} \right]. \tag{26}\]

Since \( u'(\rho_d, d) = 0 \), if \( p = 2 \) we get \( u''(\rho_d, d) = -\rho_d^\delta f(u(\rho_d, d)) < 0 \). In general, if \( p > 1 \), then the term in the brackets in (26) is negative in a left neighborhood of \( r = \rho_d \). In both the cases this contradicts the fact that \( u'(r, d) \) is negative for \( r < \rho_d \) and null for \( r = \rho_d \), so the Remark is proved.

Using the part of the proof concerning \( \rho_d < \infty \) we easily get the following.

Remark 19. Assume \( f \) satisfies \( B \) and (5) with \( q_s \geq 2 \). Then \( u(r, d) \) is decreasing as long as it is positive for any \( 0 < d < U \).

Remark 20. Assume \( f \) satisfies \( B \) and (5) with \( q_s \geq 2 \). If there is \( R \geq 0 \) such that \( u(R) < U \) and \( u'(R) \leq 0 \) then either \( u(r) \) has a zero for \( r > R \), or \( u(r) > 0 \) for any \( r > 0 \) and \( \lim_{r \to +\infty} u(r) = 0 \). In particular if \( u(r) = u(r, d) \) is a regular solution and \( d < U \), then either \( d \in I \) or \( u(r, d) > 0 \) for any \( r > 0 \) and \( \lim_{r \to +\infty} u(r, d) = 0 \).

We give a result inspired by [14, Lemma 3.16].

Lemma 3.1. Assume \( f \) satisfies (5) with \( q_s \geq 2 \); then \( I \) is open and the function \( R(d) : I \to (0, +\infty) \) is continuous. Moreover if \( u(r, d) > 0 \) for any \( r \geq 0 \), and \((d_-, d_+ + \varepsilon) \subset I \) and/or \((d_- - \varepsilon, d_+) \subset I \), then \( \lim_{d \to d_-} R(d) = +\infty \) and/or \( \lim_{d \to d_+} R(d) = +\infty \).

Proof. First of all we prove that \( I \) is open. Let \( D \in I \) we show that there is \( \nu > 0 \) such that \((D - \nu, D + \nu) \subset I \). Fix \( \tau \in \mathbb{R} \) and consider the trajectories \( \phi_{\nu_1}(t, \tau, Q(D)) \) and \( \phi_{\nu_2}(t, \tau, Q(d)) \) of (12) corresponding respectively to \( u(r, D) \) and \( u(r, d) \), where \( |D - d| < \nu \). From Remarks 15 and 16 we see that for any \( \sigma > 0 \) we can find \( \nu(\sigma) > 0 \) such that \( ||Q(D) - Q(d)|| < \sigma \). Observe that the flow of (12) on the y negative
semi-axis is transversal. Therefore we can find $T$ slightly larger than $\ln(R(D))$ such that $x_I(T, \tau, Q(D)) < 0$. Using continuous dependence on initial data we see that we can find $\sigma > 0$ small enough so that $x_I(T, \tau, Q(d)) < 0$. Consequently we see that $u(r, d)$ is a crossing solution if $\nu > 0$ is small enough; further its first zero $R(d)$ satisfies $R(d) < e^T$. Using a continuity argument, see Remark 14, and the transversality of the flow of (12) on the negative semi-axis, it is easy to see that $R(d) \to R(D)$ as $d \to D$.

Let $d_\ast \notin I$, but $(d_\ast - \varepsilon, d_\ast) \subset I$ as above, so that $x_I(t, \tau, Q(d_\ast)) > 0$ for any $t \in \mathbb{R}$. Let $d \in (d_\ast - \varepsilon, d_\ast)$: since $u(r, d)$ is a regular solution we can assume that there is $K$ small enough so that $u(r, d)$ is positive for $0 \leq r \leq K$ (see e.g. [14, Lemma 2.1]). In fact we can choose the same $K > 0$ for any $d \in [d_\ast - \varepsilon, d_\ast]$.

Let us choose $\tau$ such that $e^\tau < K$. Choose $\|Q - Q(d_\ast)\| < \sigma$; from a continuity argument we see that for any $\bar{T} > \tau$ we can find $\sigma > 0$ such that $\phi_{\bar{u}}(t, \tau, Q)$ lies in $x > 0$ for any $\tau \leq t \leq \bar{T}$. Consequently we can find $\nu \in (0, \varepsilon)$ such that $u(r, d)$ is positive and decreasing for any $e^\tau < r \leq e^T$, whenever $0 < d_\ast - d < \nu$. Since $e^\tau < K$ we see that $u(r, d)$ is positive for $0 \leq r \leq e^T$. From the arbitrariness of $\bar{T}$ we easily conclude that $R(d) \to +\infty$ as $d \to d_\ast$.

Now we show two results analogous to Remark 11, for this non-autonomous context.

**Lemma 3.2.** Assume $F_0'$ and let $\tilde{u}(r)$ be a solution of (9) such that $\tilde{u}(r) > 0$ for any $r > 0$ and $\lim_{r \to \infty} \tilde{u}(r) = 0$. Then the corresponding trajectory $\tilde{\phi}_{\tilde{u}}(t)$ of (12) is bounded.

**Proof.** From $F_0'$ we see that for any $\varepsilon > 0$ there is $\bar{R} > 0$ such that

$$(c_\ast - \varepsilon)|\tilde{u}(r)|^{q_\ast - 1} < f(\tilde{u}(r)) < (c_\ast + \varepsilon)|\tilde{u}(r)|^{q_\ast - 1}$$

for any $r \geq \bar{R}$. Correspondingly we find that

$$(c_\ast - \varepsilon)|\tilde{\phi}_{\tilde{u}}(t)|^{q_\ast - 1} < q_\ast(\tilde{\phi}_{\tilde{u}}(t), t) < (c_\ast + \varepsilon)|\tilde{\phi}_{\tilde{u}}(t)|^{q_\ast - 1}$$

for any $t \geq \bar{T} = \ln(\bar{R})$. Let us introduce polar coordinates as in (13). Assume for contradiction that $\tilde{\phi}_{\tilde{u}}(t)$ becomes unbounded as $t \to +\infty$. Then repeating the computation in (14) we see that the angular coordinate $\tilde{\theta}(t)$ of $\tilde{\phi}_{\tilde{u}}(t)$ is such that $\tilde{\theta}(t)$ is bounded above from a negative constant, i.e. there is $K > 0$ such that $\tilde{\theta}(t) \leq -K$ for $t \geq \bar{T}$. Hence $\tilde{\phi}_{\tilde{u}}(t)$ has to cross the coordinate axes indefinitely and $\tilde{u}(r)$ changes sign, but this is a contradiction, so $\tilde{\phi}_{\tilde{u}}(t)$ is bounded and the Lemma is proved.

Now we show that we can find a uniform bound for all the trajectories of Remark 3.2.

**Lemma 3.3.** Assume $F_0'$ with $q_\ast > p_\ast(\delta)$, and let $\tilde{u}(r)$ and $\tilde{\phi}_{\tilde{u}}(t)$ be as in Remark 3.2. Then there are $K_0 > 0$ and $T = T(\tilde{u})$, such that $|\tilde{\phi}_{\tilde{u}}(t)| \leq K_0$ for any $t > T(\tilde{u})$.

**Proof.** From Lemma 3.2 we know that $\tilde{\phi}_{\tilde{u}}(t)$ is bounded. Let us consider the trajectory $(\tilde{\phi}_{\tilde{u}}(t), \zeta(t))$ of (18): it is easy to check that it must converge to the union of invariant sets of the plane $\zeta = 0$.

Assume first $q_\ast \neq p_\ast(\delta)$ (i.e. $l_\ast \neq p_\ast$), then $(\tilde{\phi}_{\tilde{u}}(t), 0)$ must converge either to the origin or to the critical point $P$, which are the unique invariant sets in the $x \geq 0$ semi-plane, so the Lemma follows easily.
Assume now \( q_s = p^*(\delta) \); denote by \( G(x, +\infty) = \int_0^x g_{l_s}(x, +\infty) \) and set
\[
H(x, y) = \frac{n - p}{p} xy + \frac{p - 1}{p} |y| \int x + G(x, +\infty)
\]
\[
D_c := \{(x, y) \mid H(x, y) \leq c, x \geq 0\}
\]

Notice that \( D_0 \) is a compact connected set enclosed in the 4th quadrant and that \( D_{c_1} \subset D_{c_2} \) if \( c_1 < c_2 \). Further observe that \( P \in D_0 \) and that the origin lies on the border of \( D_0 \).

It is well known, see e.g. [12], that if \( l_s = p^* \) then \( H \) is a first integral for the autonomous system (12) with \( l = l_s \) and \( g_1(x, t) \equiv g_{l_s}(x, +\infty) \) (this fact may be verified by a straightforward computation), and this allows to draw the phase portrait, see figure 2. Notice that all the trajectories lying in \( x \geq 0 \) for any \( t \in \mathbb{R} \) for such an autonomous system are in fact contained in \( D_0 \). Now turn to consider the original non-autonomous system and the trajectory \( \phi_{l_s}(t) \) and the solution \( \hat{u}(r) \). From a standard continuity argument we see that there is \( T = T(\hat{u}) \) such that \( \phi_{l_s}(t) \in D_1 \) for any \( t \geq T \), so the Lemma follows.

3.1. Theorem 1.5: The graph of \( R(d) \) for \( \lambda = 1 \) fixed.

Lemma 3.4. Assume \( F2' \); then there is \( M > 0 \) large such that \( (M, +\infty) \subset I \) and \( R(d) \rightarrow 0 \) as \( d \rightarrow +\infty \).

Proof. The proof is based on the comparison between the original non-autonomous system (12) and the autonomous system obtained at \( \tau = -\infty \), i.e. by setting
$g_{l_u}(x,t) \equiv c_u x^{\alpha_u - 1}$. Let $u(r,d)$, $\phi_{l_u}(t,r,Q)$, $W_{l_u}^u(\tau)$ denote a regular solution of (9), the corresponding trajectory of (12) and the unstable manifold of the original non-autonomous system, and let $\bar{u}(r,d)$, $\bar{\phi}_{l_u}(t,r,\bar{Q})$, $W_{l_u}^u(-\infty)$ be the ones of the autonomous system where $g_{l_u}(x,t) \equiv c_u x^{\alpha_u - 1}$.

Follow $W_{l_u}^u(-\infty)$ from the origin towards $x > 0$: it intersects the $\dot{x} = 0$ isocline in $\tilde{R} = (\tilde{R}_x, \tilde{R}_y)$, where $\tilde{R}_y < 0 < \tilde{R}_x$ and then the $y$-negative semi-axis transversally in $\tilde{Q} = (0, \tilde{Q}_y)$, $\tilde{Q}_y < 0$, see figure 1. From Remark 14 it follows that there is $K > 0$ such that for any $\tau < -K$ the manifold $W_{l_u}^u(\tau)$ intersects the $y$-negative semi-axis transversally in $Q(\tau) = (0, \tilde{Q}_y(\tau))$, where $\tilde{Q}_y(\tau) < 0$ approaches $\tilde{Q}_y$ as $\tau \to -\infty$.

Consider the regular solution $u(r,d(\tau))$ corresponding to the trajectory $\phi_{l_u}(t,r,\bar{Q}(\tau))$. Notice that $Q(\tau)$ and $d(\tau)$ are well defined and continuous for $\tau < -K$ since the crossing between $W^u(\tau)$ and the $y$ negative semi-axis is transversal, see Remark 14.

Using a continuity argument we see that

$$\sup_l\{\|\phi_{l_u}(t,r;Q(\tau)) - \bar{\phi}_{l_u}(t,r,\bar{Q})\| \mid t \leq \tau\} \to 0 \quad \text{as} \quad \tau \to -\infty. \quad (27)$$

Hence there is $\tilde{T}_1 = \tilde{T}_1(\tau) > 0$ such that $\tilde{z}_{l_u}(\tilde{T}_1(\tau), \tau; Q(\tau)) = 0$ and $\tilde{z}_{l_u}(\tilde{T}_1, \tau; Q(\tau)) > \tilde{R}_x/2$, for $\tau < -K$. Thus we can choose $K$ large enough so that $u(e^\tilde{T}_1, d) > \tilde{R}_x/2 e^{-\alpha_u \tau} = C > 0$ is large enough so that $f(u) > 0$ for $u \geq C$. Then, using the fact that $u'(e^\tilde{T}_1, d) > 0$ and arguing as in the proof of Remark 18, we see that $u'(r,d) < 0$ for any $0 < r \leq e^\tilde{T}_1$.

So from (27) we see that $u(r,d(\tau))$ is positive and decreasing for $0 < r < R(d(\tau)) = e^c$, it becomes null with negative slope for $r = R(d(\tau)) = e^c$ and from the above argument we infer that

$$d(\tau) = u(0, d(\tau)) > u(e^{\tilde{T}_1}, d(\tau)) = \frac{\tilde{R}_x}{2} e^{-\alpha_u \tau} > M := \frac{\tilde{R}_x}{2} e^{\alpha_u K}. \quad (28)$$

Hence, for $\tau < -K$, $u(r,d(\tau))$ is a crossing solution, i.e. $d(\tau) \in I$, and its first zero is $R(d(\tau)) = e^c$.

We claim that $d(\tau)$ is invertible for $\tau < -K$; then from (28) we see that $\left|M, +\infty\right| \subseteq I$ and $R(d) = e^c \to 0$ as $d \to +\infty$, i.e. as $\tau \to -\infty$, and the Lemma is proved. We prove the claim: assume by contradiction that $d(\tau_1) = d(\tau_2) = D$ for some $\tau_1 \leq \tau_2 < -K$. Then the regular solution $u(r,D)$ has its first zero for $r = e^c$, and for $r = e^{c^2}$; hence $\tau_1 = \tau_2$ i.e. $d(\tau)$ is injective. Recalling that $d(\tau)$ is continuous for $\tau < -K$ the claim easily follows.

Lemma 3.5. Assume $F_0'$, $F_2'$ with $2 \leq q_s \leq p^*(\delta)$ and $B$. Let $u(r)$ be such that there is $\tilde{R} > 0$ at which $u(\tilde{R}) < U$ and $\frac{\partial u}{\partial r} (\tilde{R}) \leq 0$. Then $u(r)$ has a non-degenerate zero at some $R > \tilde{R}$.

Proof. From Remark 20 we see that either $u(r) \to 0$ as $r \to +\infty$ or it is a crossing solution and we are done. Assume the former; then from Lemma 3.2 we see that the trajectory $\phi_{l_u}(t)$ corresponding to $u(r)$ is bounded. Let us recall that the autonomous system (12) obtained by setting $g_{l_u}(x,t) \equiv g_{l_u}(x, +\infty)$ has the origin as unique critical point and that all its trajectories cross the coordinates axes indefinitely, see Remark 8 and fig. 1a. Then, from a standard continuity argument we find that $\phi_{l_u}(t)$ crosses the coordinate axes indefinitely so the Lemma is proved.

We recall that (18) admits a critical point $P^* = (P_x^*, P_y^*, 0)$ such that $P_x^* > 0$ iff $l_s > p_s$ (i.e. $q_s > p^*(\delta)$), where $P_x^* = [\gamma - (\alpha_1)^{p-1} / c_s]^{1/(q_1 - p)}$ and $P_y^* = -\alpha_1 P_x^*$. If $p_s < l_s < p^*$ (i.e. $p_s(\delta) < q_s < p^*(\delta)$) there is a unique trajectory,
and a standard continuity argument, we obtain the following.

Lemma 3.6. Assume $F^{0'}$, $p_*(\delta) < q_s < p^*(\delta)$. Then we can find $\tau_0 > 0$ and $k_0 > 0$ such that $T(k_0)$ does not intersect $W^u_l(t)$ for any $\tau \geq \tau_0$, and $\phi^u_l(t) \notin T(\infty)$ for any $t \geq \tau_0$.

Now we are ready to prove the following.

Lemma 3.7. Assume $B$, $F^{0'}$ with $2 \leq q_s < p^*(\delta)$, $F^{2'}$; then there is $D > 0$ such that $(0, D) \subset I$ and $R(d) \rightarrow +\infty$ as $d \rightarrow 0$. 

Proof. From Remark 20 we know that, if $0 < d < U$ then either $d \in I$ or $u(r, d) \rightarrow 0$ as $r \rightarrow +\infty$.

Assume first $p_*(\delta) < q_s < p^*(\delta)$. Let $\tau > \tau_0$ where $\tau_0$ is defined in Lemma 3.6; choose $\rho > 0$ small enough so that $W^u_l(\tau)$ intersects the line $x = \rho$ for any $\tau \geq \tau_0$. Follow $W^u_l(\tau)$ from the origin towards $x > 0$ and denote by $W^u_l(\tau, \rho)$ the branch of $W^u_l(\tau)$ between the origin and the first intersection with $x = \rho$. Since $W^u_l(\tau)$ is tangent to the $x$ axis in the origin, possibly choosing a smaller $\rho > 0$, we can assume that $W^u_l(\tau, \rho)$ is a graph on the $x$ axis and $W^u_l(\tau, \rho) \subset T(\infty)$. Further we can find $D \in (0, U)$ such that, if $\phi^u_l(t, \tau, Q)$ corresponds to a solution $u(r, d)$ of (9) then $Q \in W^u_l(\tau, \rho)$, for any $0 < d < D$: this is an easy consequence of Remarks 5 and 14. Assume for contradiction that there is $0 < d < D$ such that $d \notin I$ and let $\phi^u_l(t, \tau, Q(d))$ be the corresponding trajectory of (12). Then $u(r, d) \rightarrow 0$ as $r \rightarrow +\infty$, so, from Lemma 3.2, $\phi^u_l(t, \tau, Q(d))$ is bounded: hence either $\phi^u_l(t, \tau, Q(d))$ converges to the origin or it coincides with $\phi^u_l(t)$ and converges to $(P^u_l, P^u_l)$ as $t \rightarrow +\infty$. In the former case we have $Q(d) \in W^u_l(\tau)$, in the latter $Q(d) = \phi^u_l(\tau)$: in either case we are in contradiction with Lemma 3.6. So $d \in I$ and there is $T(Q(d)) \rightarrow \tau$ such that $\phi^u_l(t, \tau, Q(d))$ intersects transversally the $y$ negative semi-axis at $t = T(Q(d))$.

Since we can choose $\tau$ arbitrarily large the Lemma is proved by observing that $R(d) = \exp[T(Q(d))] > e^r$.

Assume now $2 \leq q_s \leq p_*(\delta)$ (but $(p, q_s) \neq (2, 2)$). In this case from Lemma 3.5 we see directly that $u(r, d)$ is a crossing solution for any $0 < d < U$. Notice that in this case $W^u_l(\tau)$ is the strongly unstable manifold and it is still tangent to the $x$ axis in the origin (for any $\tau$), see Remark 17. So we can construct $W^u_l(\tau, \rho)$, and we can assume that it is a graph on $\{(x, 0) | 0 < x <= \rho\}$. Hence repeating the last lines of the previous case we easily see that $R(d) \rightarrow +\infty$ as $d \rightarrow 0$.

We emphasize that if $(p, q_s) = (2, 2)$, so that $l_s = q_s = p$, the argument fails because we cannot apply anymore Fowler transformation which indeed requires $l = l_s > p$.

Now, putting together Lemmas 3.1, 3.4, 3.7 and the fact that $U \notin I$ if $B$ holds, we get the following result, which is summarized by figure 3b.

Proposition 1. Consider (9) where $\lambda = 1$. Assume $F^{2'}$, $F^{0'}$ with $2 \leq q_s < p^*(\delta)$ and $B$, then $I$ is open. Further there is $M \geq U$ such that $(0, U) \cup (M, +\infty) \subset I$, but $U, M \notin I$. Let $R(d)$ be the first zero of $u(r, d)$, then $R(d)$ is continuous in $I$, $\lim_{d \rightarrow +\infty} R(d) = 0$, while $\lim_{d \rightarrow 0^+} R(d) = \lim_{d \rightarrow -U^{-}} R(d) = \lim_{d \rightarrow M^+} R(d) = +\infty$.

Now we turn to consider the case in which $F^{0'}$ holds but $q_s \geq p^*(\delta)$: in this case we cannot expect for a result analogous to Lemma 3.7, but we aim to prove that
there is some \( m \in [0, U] \) such that \((m, U) \subset I\) and \(\lim_{d \to m^+} R(d) = +\infty\), see figure 4b.

For any \( C_2 > C_1 > 0 \), we denote by \( B(C_1) := \{ \phi = (x, y) \mid |\phi_1| \leq C_1 \} \) and

\[
B^*(C_1, C_2) := \{ \phi = (x, y) \mid C_1 \leq |\phi_1| \leq C_2, \; x \geq 0 \}
\]

We recall that there is \( K_0 > 0 \) such that if \( u(r) \) is definitively positive then it must stay in \( B(K_0) \) for \( t \) large, see Lemma 3.3. Now we need to show that there are \( K_2 > K_1 \) such that if \( Q \in B^*(K_1, K_2) \) we can choose \( \tau \) large enough so that the trajectory \( \phi_{t\tau}(t, \tau; Q) \) has to cross the \( y \) negative semi-axis at some \( T > \tau \).

**Lemma 3.8.** Assume \( B, F^2', \) and \( F^0' \) with \( q_s \geq p^*(\delta) \). Then there are \( \tau_1 > 0, K_1 > 0 \) such that for any \( K_2 > K_1 \) we can find \( \tau_2 = \tau_2(K_2) \geq \tau_1 \) with the following property: if \( Q \in B^*(K_1, K_2) \) and \( \tau \geq \tau_2 \) the trajectory \( \phi_{t\tau}(t, \tau; Q) \) crosses the \( y \) negative semi-axis at some \( T(Q) > \tau \).

**Proof.** Let us consider the autonomous system (12) where \( l = l_s \) and \( g_l(x, t) \equiv g_{l_s}(x, +\infty) \), and let \( K > 0 \) be as in Remark 11. Then any trajectory \( \phi_{t\tau}(t, \tau; Q) \) of such an autonomous system is a crossing trajectory if \( Q \in B(K) \); in particular if \( Q \in B^*(2K, 5K) \) then \( \phi_{t\tau}(t, \tau; Q) \) is a crossing trajectory.

Let us set \( K_1 = 2K \). Now we turn to consider the original non-autonomous system (12), and its trajectory \( \phi_{t\tau}(t, \tau; Q) \), where \( Q \in B^*(K_1, 2K_1) \). Using a standard continuous dependence argument and \( F^0' \), see also Remark 14, we can find \( \tau_1 > 0 \) such that \( \phi_{t\tau}(t, \tau; Q) \) is a crossing trajectory whenever \( Q \in B^*(K_1, 2K_1) \) and \( \tau \geq \tau_1 \).

Now choose \( K_2 > K_1 \) (possibly very large) and let \( Q \in B^*(K_1, K_2) \). With the same argument we can find \( \tau_2 = \tau_2(K_2) \geq \tau_1 \) so that the trajectory \( \phi_{t\tau}(t, \tau; Q) \) of the original non-autonomous system is a crossing trajectory too, for any \( Q \in B^*(K_1, K_2) \) and any \( \tau \geq \tau_2 \), so the Lemma is proved.

Notice that \( \tau_2(K_2) \) may and (in our context will) go to infinity as \( K_2 \to +\infty \).

Let us spend some further lines on Lemma 3.8.

Let us consider the trajectories \( \phi_{t\tau}(t, \tau; Q) \) of the non-autonomous system (12). Assume to fix the ideas \( q_s > p^*(\delta) \). Roughly speaking if \( Q \in B(K_0) \) (see Lemma 3.3), then \( \phi_{t\tau}(t, \tau; Q) \) may converge to the origin or to \( P \) without crossing the \( y \) axis. Further, the continuous dependence argument used in the proof of Lemma 3.8 just works in compact sets such as \( B^*(K_1, K_2) \). In fact, from assumption \( B \), we see that the trajectory \( \psi_{t\tau}(t) \equiv (Ue^{\alpha t}, 0) \) corresponding to the constant solution \( u(r) \equiv U \) of (9) becomes unbounded as \( t \to +\infty \). Hence it is not “close” to any trajectory of the autonomous system at \( +\infty \) for \( t \) large, and it does not cross the \( y \) negative axis nor converge to the origin. However for any \( K_2 > 0 \) we can choose \( \tau_2 \) large enough so that \( \psi_{t\tau}(t) \not\in B^*(K_1, K_2) \) for any \( t \geq \tau_2 \); so the existence of \( \psi_{t\tau}(t) \) is not in contradiction with Lemma 3.8.

We also stress that if \( q_s > p^*(\delta) \), using Remark 14 we can show that \( W^*(\tau) \) is “close” to \( M^* \cap \{ x \geq 0 \} \) for \( \tau \geq \tau_0 \) and \( \tau_0 \) large enough.

**Remark 21.** In the continuous time case we have preferred to restrict to consider the case \( q_s \geq p^*(\delta) \) instead of \( q_s > p \), since the case \( p < q_s < p^*(\delta) \) has already been analyzed. However we think it is worthwhile to mention that Lemma 3.8 holds if \( p < q_s < p^*(\delta) \), too.

**Lemma 3.9.** Assume \( B, F^0', F^2' \); then there is \( m \in (0, U) \) such that \((m, U) \subset I \).
Proof. Let us recall that $\hat{W}^u_\ell(\tau)$ is the branch of $W^u_\ell(\tau)$ between the origin and $\psi_\ell(\tau) = (Ue^{\omega_\ell R}, 0)$, i.e. corresponding to the regular solutions $u(r, d)$ with $d \in [0, U]$, see Remark 16.

Observe that $\hat{W}^u_\ell(\tau)$ is a $C^1$ manifold having the origin and $\psi_\ell(\tau)$ as endpoints. From Lemma 3.8 we see that we can choose $K_2 > K_1$ and $\tau$ large enough so that $Ue^{\omega_\ell R} > K_2$, hence $[\hat{W}^u_\ell(\tau) \cap B^*(K_1, K_2)] \neq \emptyset$ and we can choose $\tilde{Q} \in [\hat{W}^u_\ell(\tau) \cap B^*(K_1, K_2)]$. From Lemma 3.8 it follows that there is $T(\tilde{Q}) > \tau$ such that $\phi_{\ell}(t, \tau; \tilde{Q})$ intersects transversally the $y$ negative semi-axis at $t = T(\tilde{Q})$. So the corresponding solution $u(r, \bar{d})$ of (9) is a crossing solution, i.e. there is $\bar{d} \in (0, U) \cap I$.

Further, let $Q(d)$ be such that the trajectory $\phi_{\ell}(t, \tau; Q(d))$ corresponds to the regular solution $u(r, d)$, i.e. $Q(d)$ gives a parametrization of $W^u_\ell(\tau)$. We denote by $\hat{W}^u_\ell(\tau) := \{Q(d) | d \in (m, U)\} where m is chosen in such a way that $[\hat{W}^u_\ell(\tau) \cap B(K_1)] = \emptyset$ (i.e. $m$ is close to $U$). Assume by contradiction that there is $\bar{d} \in (m, U) \setminus I$, i.e. $u(r, \bar{d}) > 0$ for any $r > 0$ and $\lim_{r \to \infty} u(r, \bar{d}) = 0$, cf. Remark 18. If $Q(d) \in B^*(K_1, K_2)$ then we get a contradiction by Lemma 3.8, so assume $Q(d) > K_2$. Then from Lemma 3.3 we see that there is $T > \tau$ such that $\phi_{\ell}(t, \tau; Q(d)) \in K_0 < K_1$ for any $t > T$. Hence, from a continuity argument, we see that there is $T \in [\tau, T)$ such that $\phi_{\ell}(T, \tau; Q(d)) \in B^*(K_1, K_2)$. Then, from Lemma 3.8 it follows that $u(r, \bar{d})$ is a crossing solution, and we have found a contradiction and the Lemma is proved.

Now we show that if $F0'$ holds and $q_s \geq p^*(\delta)$ then the graph of $R(d)$ is as in figure 4b.

**Proposition 2.** Assume $B$, $F0'$, $F2'$; then there are $m \in [0, U)$ and $M \geq U$ such that $(m, U) \cup (M, \infty) \subset I$ and $m, M \not\in I$. Let $R(d)$ be the first zero of $u(r, d)$, then $R(d)$ is continuous in $I$, and $\lim_{d \to m^+} R(d) = \lim_{d \to U^-} R(d) = \lim_{d \to M^+} R(d) = +\infty$, and $\lim_{d \to +\infty} R(d) = 0$. 

![Figure 4. Sketch of the graph of the function $R(d)$ in the setting of Proposition 6 on the left, and in the setting of Proposition 2 on the right.](image-url)
Proof. From Lemmas 3.9 and 3.1 we immediately see that \(|m, U| \subset I\), then we conclude with Lemma 3.1 that \(R(d)\) is continuous in \(I\) and that \(\lim_{d \to m^+} R(d) = \lim_{d \to m^-} R(d) = +\infty\). The claims concerning \(d > M\) follows from Lemma 3.4. □

Remark 22. We observe that if \(B, F_0', F_2'\) are assumed and \(2 \leq q_s < p^*(\delta)\), then Lemmas 3.7 and 3.9 both hold; hence we find \(m = 0\), and there are no Ground States (i.e. solutions positive for any \(r \geq 0\)) for any \(d \in [0, U + \epsilon)\).

If \(q_s \geq p^*(\delta)\) we just have Lemma 3.9, and we may find \(0 < m < U\). In fact, if \(q_s > p^*(\delta)\), following [32] we may show that \(m > 0\) and that \(u(r, m)\) is a Ground State with fast decay, i.e. such that \(u(r, m) \sim r^{-(n-p)/(p-1)}\) as \(r \to +\infty\). Further, following again [32], we may observe that if \(d \in [0, m]\), then \(u(r, d)\) is a Ground State with slow decay, i.e. \(u(r, d) \sim P_x r^{-\alpha_x}\) as \(r \to +\infty\). We think that something similar might happen in the critical case \(q_s = p^*(\delta)\) too.

However these results are a bit beyond the purposes of this paper, so we do not give a full fledged argument remanding to [32] for details.

Remark 23. We observe that Proposition 2 is enough to prove Theorem 1.5, via the scaling argument performed in Section 3.3. Proposition 1 is added here for completeness, since the range \(p < q_s < p^*(\delta)\) is the main object of our interest, and also because it was already present in the original version of the paper. Proposition 2 was in fact suggested to us by a clever question posed by the referee, and added in a second time.

Now we turn to consider the case where (5) holds but \(p = q_s = 2\): some changes are needed. First of all notice that we cannot set \(l = l_u = q_s\) in (11), so we cannot construct system (18). On the other hand if \(F_2\) holds we can set \(l = l_u > 2\) to construct the unstable manifolds \(W^u_{l_u}(-\infty)\) and \(W^u_{l_u}(\tau)\), and Remarks 14, 15, 16 hold with no changes. However, since the linearization of (12) in the origin does not have constant eigenvalues and eigenvectors, we can just say that \(W^u_{l_u}(-\infty)\) is tangent to the \(x\) axis in the origin, but the tangent to \(W^u_{l_u}(\tau)\) may (and usually will) change with \(\tau\). Hence Remark 17 does not hold in this context.

Further notice that Remark 18 and Lemma 3.1 hold (with no changes in the proof). However we have to replace Lemma 3.5 by the following.

Lemma 3.10. Assume \(F_0, F_2\) with \(p = q_s = 2, \delta > -2\) and \(B\); then we get the same conclusions as in Lemma 3.5. In particular \((0, U) \subset I\).

Proof. This result follows from some standard results in oscillation theory, see, e.g., [9, Theorem 3.1.4]; however we give a full fledged proof for completeness.

Let \(u(r)\) be such that \(u(\tilde{R}) < U\) and \(\frac{d}{dr} u(\tilde{R}) \leq 0\). Again from Remark 20 we see that either \(u(r) \to 0\) as \(r \to +\infty\) or it is a crossing solution and we are done, so we assume the former. Hence for any \(\varepsilon > 0\) we can find \(\tilde{R}_1 > \tilde{R}\) such that \(u(r) < \varepsilon\) and \((n-1)/r < \varepsilon\) if \(r \geq \tilde{R}_1\). Then we can choose \(\varepsilon > 0\) small enough and \(\tilde{R}_1\) large enough so that \(\overline{\varpi}(r) < f(u(r)) < 2c_t u(r)\) for \(r \geq \tilde{R}_1\). Let us consider the equations

\[
\begin{align*}
    u'' + \varepsilon u' + \frac{c_t}{r} u &= 0 \\
    u'' + 2c_t u &= 0
\end{align*}
\]

and denote respectively by \(\overline{\varpi}(r)\) and \(\underline{\varpi}(r)\) the solutions of the former and the latter equation in (29) such that \(\overline{\varpi}(\tilde{R}_1) = u(\tilde{R}_1) = \underline{\varpi}(\tilde{R}_1) = u'(\tilde{R}_1)\). Notice that there are \(\overline{R} > \overline{\varpi} > \tilde{R}_1\) such that \(u(r)\) and \(\varpi(r)\) are positive and decreasing respectively for \(\tilde{R}_1 \leq r < \overline{R}\), and for \(\tilde{R}_1 \leq r < \overline{\varpi}\) and they become null with nonnegative slope at \(r = \overline{R}\) and at \(r = \overline{\varpi}\).
We claim that there is $R \in (\widetilde{R}, R)$ such that $u'(r)$ is positive and decreasing for $\tilde{R}_1 < r < R$ and $u(R) = 0 > u'(R)$, so that the Lemma is proved. To prove the claim consider the phase plane $u, u'$ and draw the curves

$$\Gamma := \{(u(r), u'(r)) \mid \tilde{R}_1 \leq r \leq \tilde{R}\}, \quad \Gamma := \{(u(r), u'(r)) \mid \tilde{R}_1 \leq r \leq R\},$$

and denote by $E$ the compact set enclosed by $\Gamma, \Gamma$ and the line $u = 0$. Notice that the flow of $(2)$ on $\Gamma \cup \Gamma$ points towards the interior of $E$ for any $r \geq \tilde{R}_1$. So there is $R \in (\tilde{R}, R)$ such that $(u(r), u'(r)) \in E$ for any $\tilde{R}_1 \leq r \leq R$ and it crosses transversally the line $u = 0$ for $r = R$, and the claim is proved. 

Remark 24. We emphasize that in the assumption of Lemma 3.10 we lose control of $R(d)$ as $d \to 0^+$. In fact it might be shown that $R(d) \to R(0) := \lambda_1/c_s > 0$ as $d \to 0^+$ where $\lambda_1$ is the first eigenvalue of the Laplacian in the ball of radius 1. We do not give a proof of the result which is beyond the purpose of this paper, however see [26].

Now we are ready to state the counterpart of Proposition 1 for the case $p = q = 2$, see figure 5.

Proposition 3. Consider $(9)$ where $\lambda = 1$. Assume $F0$ and $B$, and that $f$ satisfies $(5)$ with $p = q = 2$. Then $I$ is open and there is $M \geq U$ such that $(0, U) \cup (M, +\infty) \subset I$, but $U, M \notin I$. Further $R(d)$ is continuous in $I$, and $\lim_{d \to U^-} R(d) = \lim_{d \to M^+} R(d) = +\infty$.

![Figure 5](image)

**Figure 5.** Sketch of the graph of the function $R(d)$ in the setting of Proposition 3 on the left, and in the setting of Proposition 5 on the right. We have drawn with solid lines the part of the graph which has been constructed through the Propositions and with dotted lines the part of the graph constructed through modified problem (for $d$ large in fig. (a)) or which are just conjectured (for $d$ small in both fig. (a) and (b)). In both graphs (a) and (b) we have at least 2 values $d_i$ such that $R(d_i) = R$ when $R > R_2$; further $0 < d_2 < U < d_3$. Moreover in picture (a) we have $d_2 \to U^-$ and $d_3 \to U^+$ as $R \to +\infty$, while in picture (b) we have $d_2 \to U^-$ and $d_3 \to M^+$ as $R \to +\infty$. 


3.2. **Theorem 1.4:** The graph of $R(d)$ for $\lambda = 1$ fixed. In this subsection we assume the hypotheses of Theorem 1.4.

Let us introduce now some auxiliary functions which will allow us to construct the unstable manifold. Let us choose $2 < q_u < p^*(\delta)$ and set

$$f^m(u) = \begin{cases} 
 f(u) & \text{if } 0 \leq u \leq U + \sigma \\
 \eta(u) & \text{if } U + \sigma \leq u \leq U + \sigma + 1 \\
 u^{\beta - 1} & \text{if } u \geq U + \sigma + 1 
\end{cases}$$

and $f^h(u) = f^m(u + U)$; here $\eta(u)$ is a positive function such that $f^m(u)$ is $C^1$.

We introduce the following notation: we denote by $g^m_t(x, t)$ and by $g^h_t(x, t)$ the functions $g_t$ of (11) where $f$ is replaced by $f^m$ and $f^h$ respectively. Similarly we refer to system (12) where $g_t$ is replaced by $g^m_t$ and $g^h_t$ as to (12m) and (12h). Analogously we denote with a $m^-$ and $m^+$ all the quantities of (12m) and (12h) respectively to distinguish them from the ones of (12). We use the same notation for $u^m(r, d)$, $u^h(r, d)$ and $u(r, d)$. Notice in particular that $u^h(r, d) = u^m(r, d + U)$, for any $r \geq 0$.

**Remark 25.** Observe that by construction both $f^m(u)$ and $f^h(u)$ satisfy $F^2'$ for any possible choice of $f$, since $F^2'$ just regards the behavior for $u$ large. So, for any $\tau \in \mathbb{R}$, (12m) and (12h) admit an unstable manifold denoted by $W^u_{\tau^m}(\tau)$ and by $W^u_{\tau^h}(\tau)$ respectively.

Further $f(u)$ satisfies $F^0'$ or $F^1'$ if and only if $f^m(u)$ satisfies $F^0'$ or $F^1'$ too. Similarly if $f(u)$ satisfies $F^1'$ and $(p, Q_s) \neq (2, 2)$ then $f^h(u)$ satisfies $F^0'$.

The key idea in this subsection is to show that the graph of $R(d)$ is as sketched in figure 3a or as in 4a for $f^m$. Then we conclude by observing that the solution $u(r, d)$ we consider are always smaller than $U + \sigma$ (for $0 \leq r \leq R(d)$) so we find the same picture for the original $f$ (at least for $0 < d < U + \sigma$).

**Lemma 3.11.** Assume $F^1$, $A$, and $(p, Q_s) = (2, 2)$. Then there is $\epsilon \in (0, \sigma)$, such that for any $U < d < U + \epsilon$ there is $R_1(d) > 0$ such that $u(R_1(d), d) = U$ and $\frac{\partial u}{\partial r}(R_1(d), d) < 0$.

**Proof.** We recall that for $d \in (0, U + \epsilon) \setminus \{U\}$, $u(r, d)$ is decreasing as long as it is positive; therefore $u(r, d) = u^m(r, d) = u^h(r, d - U)$ for any $U < d < U + \epsilon < U + \sigma$ and any $0 \leq \tau < R(d)$. So the Lemma is equivalent to say that $u^h(r, d)$ is a crossing solution for $0 < d < \epsilon$. But from Remark 25 we know that $f^h$ satisfies $F^2'$; further, again by Remark 25, if $(p, Q_s) \neq (2, 2)$ then $f^h$ satisfies $F^0'$ too and we conclude by applying Lemma 3.7. Similarly if $(p, Q_s) = (2, 2)$ we see that $f^h$ satisfies (5) with $Q_s = 2$ and $C_s$ replacing respectively $q_u$ and $c_s$; so we conclude by applying Lemma 3.10.

Via Lemma 3.11 we have shown that if $d \in (U, U + \epsilon)$, then $u(r, d)$ cannot converge to $U$, so either it is positive for any $r > 0$ and converges to 0 or it is a crossing solution, i.e. $d \in I$. Now we show that indeed $(U, U + \epsilon) \subset I$. For this purpose we consider the solutions $u(r, d) = u^m(r, d)$ for $d \in (U, U + \epsilon)$ in the interval $r \in (R_1(d), R(d))$, and the corresponding trajectories of (12m) with $l = l_s$.

**Lemma 3.12.** Assume $F^1$, $A$, and $(p, Q_s) = (2, 2)$ with $2 \leq q_u < p^*(\delta)$. Then there is $\epsilon \in (0, \sigma)$, such that $u(r, d)$ is a crossing solution for any $U < d \leq U + \epsilon$, and it is decreasing as long as it is positive.

**Proof.** Observe that $u(r, U) \equiv u^m(r, U) \equiv U$ for any $r \geq 0$. Let $\tau_0 > 0$ be the constant defined in Lemma 3.6; it follows that $\psi_{l_s}(\tau_0) = (Ue^{\alpha_s \tau_0}, 0) \in W^{u^m}_{l_s}(\tau_0)$
and \( \psi_\ast(t) = (ue^t, t) \in W^{-1}_{\ast}(t) \) for any \( t \in \mathbb{R} \). Further \( \psi_\ast(t) \in T(0) \) for any \( t \in \mathbb{R} \), where \( T(0) \) is defined in Lemma 3.6. Since \( W^{-1}_{\ast}(\tau_0) \) is a connected manifold there is a small connected branch of \( W^{-1}_{\ast}(\tau_0) \) containing \( \psi_\ast(\tau_0) \) which is contained in \( T(0) \), say \( \omega^{\ast}(\tau_0) \subset T(0) \). Let \( \phi^{\ast}_\omega(t) \) be the unique trajectory of (12) converging to the critical point \( P = (P_x, P_y) \) as \( t \to +\infty \). From Lemma 3.6 we see that \( W^{+}_{\ast}(\tau_0) \) does not intersect \( T(0) \) and \( \phi^{\ast}_\omega(\tau_0) \not\in T(0) \). Hence if \( Q \in \omega^{\ast}(\tau_0) \) then \( \phi^{\ast}_\omega(t, \tau_0, \psi_\ast(\tau_0)) \) cannot converge neither to the origin nor to \( P \) as \( t \to +\infty \). Therefore, using also Lemma 3.2, we see that the corresponding solutions \( u^{\ast}(r, d) \) do not converge to 0. Hence, possibly choosing a smaller \( \varepsilon > 0 \), we can assume that \( u^{\ast}(r, d) \) does not converge to 0 for any \( U < d < U + \varepsilon \).

Further, reasoning as in the proof of Remark 18, it is easy to check that \( u^{\ast}(r, d) \) is decreasing as long as it is positive. Hence from Lemma 3.11 and Remark 18 we see that, for any \( U < d < U + \varepsilon \), there is \( R(d) > 0 \) such that \( u^{\ast}(r, d) \) is positive and decreasing for \( 0 \leq r < R(d) \) and becomes null with non-negative slope at \( r = R(d) \). Further \( u^{\ast}(r, d) \) is actually smaller than \( U + \sigma \) for any \( 0 \leq r \leq R(d) \); so it solves the original equation (2), and we have \( u^{\ast}(r, d) = u(r, d) \) for \( 0 \leq r \leq R(d) \). \( \square \)

**Proposition 4.** Consider (9) where \( \lambda = 1 \). Assume \( F_0' \) with \( 2 \leq q_s < p^*(\delta), F_1' \) and \( B \), then \( I \) is open and there is \( \varepsilon > 0 \) such that \( (0, U + \varepsilon) \setminus U \subset I \). Further \( \lim_{d \to U^+} R(d) = \lim_{d \to U^-} R(d) = \lim_{d \to 0} R(d) = +\infty \).

**Proof.** From Lemma 3.12 we see that in the setting of Theorem 1.4 we can still apply Lemma 3.7 to \( f^m(u) \), see Remark 25. So using also Lemma 3.1 we prove Proposition 4 for eq. (9m). Then we observe that the solutions \( u^{\ast}(r, d) \) of this equation are decreasing as long as they are positive, whenever \( 0 < d < U + \varepsilon \), so they also satisfy the original equation (9). \( \square \)

Similarly if \( p = q_s = 2 \), combining Lemma 3.12, Lemma 3.7 we get a result analogous to Proposition 3.

**Proposition 5.** Consider (9) where \( \lambda = 1 \). Assume \( f \) satisfies (5) with \( p = q_s = 2 \), \( F_1 \) and \( B \), then \( I \) is open and there is \( \varepsilon > 0 \) such that \( (0, U + \varepsilon) \setminus U \subset I \). Further \( \lim_{d \to U^+} R(d) = \lim_{d \to U^-} R(d) = +\infty \).

Now we turn to consider \( F_0' \) with \( q_s \geq p^*(\delta) \) and we reprove Lemma 3.12 in this context too.

**Lemma 3.13.** Assume \( F_1', A \) and \( F_0' \) with \( q_s \geq p^*(\delta) \). Then there are \( m \in [0, U) \) and \( \varepsilon \in (0, \sigma) \), such that \( u(r, d) \) is a crossing solution for any \( d \in (m, U) \cup (U, U + \varepsilon) \), and it is decreasing as long as it is positive.

**Proof.** Let us consider \( u^{\ast}(r, d) \) where \( 0 < d < U + \varepsilon \), and \( \varepsilon > 0 \) is as in Lemma 3.11. Observe first that \( u^{\ast}(r, d) \) is decreasing as long as it is positive, so in fact \( u^{\ast}(r, d) = u(r, d) \), i.e. it solves the original problem (9). Since (9m) satisfies the assumptions of Lemma 3.9 (cf. Remark 25), we deduce the existence of \( m \in [0, U) \) such that \( (m, U) \subset I \). To prove that \( (U, U + \varepsilon) \subset I \) we reason again as in Lemma 3.9.

Let \( K_1 > 0 \), be as in Lemma 3.8, and set \( K_2 = 2K_1 \), via Lemma 3.8 we find \( \tau_2 \) such that \( B^{\ast}(K_1, K_2) \) has the following property: if \( \tau \geq \tau_2 \) and \( Q \in B^{\ast}(K_1, K_2) \) then \( \phi_{\ast}(t, \tau; Q) \) has to cross the \( y \) negative semi-axis at some \( T > \tau \). From Lemma 3.11 we know that there is \( R_1(U + \varepsilon) \) such that \( u(r, U + \varepsilon) = U \) and \( \frac{\partial u}{\partial r}(r, U + \varepsilon) < 0 \) at \( r = R_1(U + \varepsilon) \). Let us choose \( \tau \geq \max\{\ln[R_1(U + \varepsilon)], \tau_2\} \). We introduce the following notation: we denote by \( Q(d) \) the point in \( W^{+}_{\ast}(\tau) \) such that \( \phi_{\ast}(t, \tau; Q(d)) \)
corresponds to the regular solution $u(r, d)$ of (9m). Hence $Q(d)$ gives a parametrization of $W_{\tilde{I}}^u(\tau)$. We denote by $W_{\tilde{I}}^u(\tau) := \{Q(d) \mid U \leq d \leq d + \varepsilon\}$.

Possibly choosing a larger $\tau$ we can assume $Ue^{\alpha + \tau} > K_2$. Without loss of generality, possibly choosing a smaller $\varepsilon > 0$ we can assume that $W_{\tilde{I}}^u(\tau) \cap B(K_2) = \emptyset$.

Assume by contradiction that there is $d \in (U, U + \varepsilon)$ such that $d \notin I$; then from Lemma 3.11 and Remark 18 we see that $u(r, d) > 0$ for any $r > 0$ and $u(r, d) \to 0$ as $r \to +\infty$. Hence, by Lemma 3.3, there is $T(u(r, d))$ such that $\phi_{\tilde{I}}(t, \tau, Q(d)) \in B(K_0)$ for any $t \geq T(u(r, d))$; moreover recall that $K_2 > K_1 > K_0$. Further, by construction $Q(d) = \phi_{\tilde{I}}(\tau, \tau, Q(d)) \notin B(K_2)$. So there is $\bar{\tau} \in (\tau, T(u(r, d)))$ such that $\phi_{\tilde{I}}(\bar{\tau}, \tau, Q(d)) \in B^*(K_1, K_2)$. Then from Lemma 3.8 we see that $\phi_{\tilde{I}}(\bar{\tau}, \tau; Q(d))$ has to cross the y negative semi-axis and $d \in I$: a contradiction. So $(U, U + \varepsilon) \subset I$ and the Lemma is proved.

Now, from Lemma 3.13, Lemma 3.7 and Lemma 3.1 we get the following result which is summarized by figure 4b.

**Proposition 6.** Consider (9) where $\lambda = 1$. Assume $F0'$ with $q_0 \geq p^*(\delta)$, $F1'$ and $B$, then $I$ is open and there are $m \in [0, U)$, $m \notin I$, $\varepsilon > 0$ such that $(m, U + \varepsilon] \subset I$. Further $\lim_{d \to U^+} R(d) = \lim_{d \to U^-} R(d) = \lim_{m \to m^+} R(d) = +\infty$.

**Remark 26.** In the assumptions either of Propositions 4, 5, 6 $u(r, d)$ is decreasing as long as it is positive for any $0 < d < U + \varepsilon$, $d \neq U$.

**Proof.** The proof for $0 < d < U$ follows from Remark 19, the case $U < d < U + \varepsilon$ follows from the argument of the proof of Lemma 3.12.

### 3.3. Scaling argument.

**Corollary 3.** Assume that we are either in the assumptions of Propositions 1, 2, 4, 6 and consider the equation in $d$, $R(d) = K$. Then there is $K_0 > 0$ such that $R(d) = K$ has 3 solutions, say $d_1 < d_2 < d_3$, for any $K \geq K_0$. Further $d_2 < U < d_3$ and $d_1 \to m^-$, $d_2 \to U^-$, $d_3 \to M^+$, where $0 \leq m < U \leq M$ (notice that we can set $M = U$ if $A$ holds). Finally, if $p < q_0 < p^*(\delta)$, $(p, q_0) \neq (2, 2)$ (i.e. we are in the assumptions either of Proposition 1 or of Proposition 6) then $m = 0$, i.e. $d_1 \to 0^+$ as $K \to +\infty$.

Now the proof of Theorems 1.5, 1.4 easily follows from a standard scaling argument.

**Proof of Theorem 1.5 and Theorem 1.4.** Assume that we are in the hypotheses either of Theorem 1.5 or of Theorem 1.4. We reformulate our problem in the setting of (10); let us recall that starting from (9) where $q < p^*(\delta)$ (resp. $q > p^*(\delta)$) we get (9) where $q < \frac{Np}{N-p}$ (resp. $q > \frac{Np}{N-p}$). Then the following Dirichlet problem

\[
\begin{align*}
(w' | w^{p-2})_r^{N-1} + r^{N-1}f(w) &= 0, \\
w(0, d) &= d > 0, \\
w(1, d) &= 0, \\
w(r) &= 0, \text{ for } 0 < r < R
\end{align*}
\]

admits at least 3 solutions for any $R \geq K_0$.

Now we turn to consider the original equation (10) where $\lambda \neq 1$. Set $u(r) = w(r^\lambda)$, $a = \frac{1}{\lambda^p > 0}$, $\rho = R\lambda^{-a}$, $\rho^*(\lambda) = K_0\lambda^{-a}$. If $w(r)$ satisfies (31) then $u(r)$ satisfies the following:

\[
\begin{align*}
(u' | u^{p-2})_r^{N-1} + \lambda r^{N-1}f(u) &= 0, \\
u(0, d) &= d > 0, \\
u(\rho, d) &= 0, \\
u(r) &= 0, \text{ for } 0 < r < \rho
\end{align*}
\]
Set $\lambda_0^K = K_0^{1/a}$, then the Dirichlet problem (9) in the ball of radius 1 admits at least 3 positive solutions for $\lambda \geq \lambda_0^K$. So the proof of Theorem 1.4 is concluded.

Now restrict to Theorem 1.5, so that Proposition 1 holds. Then denote by $\mathcal{M}^* = \min\{R(d) \mid 0 < d < U\}$, and set $\lambda^* = [\mathcal{M}^*]^{1/a}$, then we easily see that (32) admits at least 3 solutions for $\lambda > \lambda^*$ at least 2 solutions for $\lambda = \lambda^*$, and at least 1 solution for $0 < \lambda < \lambda^*$, so the proof of Theorem 1.5 is concluded. 

With the same scaling argument from Propositions 3, 5 we obtain Theorem 1.3.

We conclude by noticing that Corollaries 1 and 2 follow from Corollary 3, Remarks 19 and 26.

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