Upper bound for isometric embeddings $\ell_2^m \to \ell_p^n$

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Abstract

The isometric embeddings $\ell_2^m \to \ell_p^n$ ($m \geq 2$, $p \in 2\mathbb{N}$) over a field $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ are considered, and an upper bound for the minimal $n$ is proved. In the commutative case ($K \neq \mathbb{H}$) the bound was obtained by Delbaen, Jarchow and Pełczyński (1998) in a different way.

Let $K$ be one of three fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (real, complex or quaternion). Let $K^n$ be the $K$-linear space consisting of columns $x = [\xi_i]_1^n$, $\xi_i \in K$, with the right (for definiteness) multiplication by scalars $\alpha \in K$. The normed space $\ell_{p,K}^n$ is $K^n$ provided with the norm

$$||x||_p = \left(\sum_{k=1}^n |\xi_k|^p \right)^{1/p}, \quad 1 \leq p < \infty.$$ 

For $p = 2$ this space is Euclidean, $||x||_2 = \sqrt{\langle x, x \rangle}$, where the inner product $\langle x, y \rangle$ of $x$ and a vector $y = [\eta_i]_1^n$ is

$$\langle x, y \rangle = \sum_{i=1}^n \xi_i \eta_i.$$ 

An isometric embedding $\ell_2^m \to \ell_p^n$, $2 \leq m \leq n$, may exist only if $p \in 2\mathbb{N} = 2, 4, 6, \ldots$, see [3] for $K = \mathbb{R}$ and [4] for any $K$. Conversely, under these conditions for $m$ and $p$, there exists an $n$ such that $\ell_2^m$ can be isometrically embedded into $\ell_p^n$, see [6] (and also [5, 7]) for $K = \mathbb{R}$, [2] for $K = \mathbb{C}$, and [4] for $K = \mathbb{H}, \mathbb{C}$ and $\mathbb{R}$ simultaneously. The proofs of existence in these papers also yield some upper bounds for the minimal $n = N_K(m, p)$. According to [4], these bounds can be joined in the inequality

$$N_K(m, p) \leq \dim \Phi_K(m, p), \quad (1)$$

where $\Phi_K(m, p)$ is the space of homogeneous polynomials (forms) $\phi(x)$ over $\mathbb{R}$ of degree $p$ in real coordinates on $K^m$ such that $\phi(x\alpha) = \phi(x)$ for all $\alpha \in K$, $|\alpha| = 1$. For $K = \mathbb{R}$ the latter condition is fulfilled automatically since $p \in 2\mathbb{N}$, so $\Phi_{\mathbb{R}}(m, p)$ consists of all forms of degree $p$ on $\mathbb{R}^m$. The space $\Phi_{\mathbb{C}}(m, p)$ coincides

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with that which was used in [2]. Note that in all cases \( \dim \Phi_K(m, p) \) can be explicitly expressed through binomial coefficients. (All the formulas are brought together in [4, Theorem 2].)

In the present paper we prove that

\[
N_K(m, p) \leq \dim \Phi_K(m, p) - 1. \tag{2}
\]

For \( K = \mathbb{R} \) and \( \mathbb{C} \) this result (in terms of binomial coefficients) was obtained by Delbaen, Jarchow and Pełczyński [1] as a by-product of the proof of their Theorem B. Their rather complicated technique essentially uses the commutativity of the field \( K \), so it is not applicable to \( K = \mathbb{H} \). Our proof of (2) is general and elementary. Let us start with two lemmas, the first of which is well known.

Lemma 1. A linear mapping \( f : \ell^m_{2,K} \to \ell^n_{p,K} \) is isometric if and only if there is a system of vectors \( u_k \in \ell^m_{2,K}, 1 \leq k \leq n \), such that the identity

\[
\sum_{k=1}^{n} |\langle u_k, x \rangle|^p = \langle x, x \rangle^{p/2} \tag{3}
\]

holds for \( x \in \ell^m_{2,K} \).

Proof. A general form of \( f \) as a linear mapping is \( f x = [\langle u_k, x \rangle]_1^n \), where \( (u_k)_1^n \) is a system of vectors from \( \ell^m_{2,K} \) (called the frame of \( f \) [4, 5]). The identity (3) is nothing but \( \|fx\|_p = \|x\|_2 \).

An isometric embedding \( \ell^m_{2,K} \to \ell^n_{p,K} \) is called minimal if \( n = N_K(m, p) \).

Lemma 2. If \( f \) is minimal and \( (u_k)_1^n \) is its frame then the functions \( |\langle u_k, x \rangle|^p \) are linearly independent.

Proof. Let

\[
\sum_{k=1}^{n} \omega_k |\langle u_k, x \rangle|^p = 0 \tag{4}
\]

with some real \( \omega_k \), \( \max_k \omega_k = 1 \), and let \( \omega_n = 1 \) for definiteness. By subtraction of (4) from (3) we get

\[
\sum_{k=1}^{n-1} (1 - \omega_k) |\langle u_k, x \rangle|^p = \langle x, x \rangle^{p/2},
\]

i.e.

\[
\sum_{k=1}^{n-1} |\langle v_k, x \rangle|^p = \langle x, x \rangle^{p/2},
\]

where \( v_k = u_k (1 - \omega_k)^{1/p} \). This contradicts the minimality of \( f \).

Remark 3. Since all functions \( |\langle \cdot, x \rangle|^p \) belong to \( \Phi_K(m, p) \), the inequality (1) immediately follows from Lemma 2. However, the existence of an isometric embedding \( \ell^m_{2,K} \to \ell^n_{p,K} \) is assumed in this context.
Now we proceed to the proof of (2).

Proof. Let $f : \ell_{2m}^n \to \ell_{p,K}^n$ be a minimal isometric embedding. Then, according to (1), $n \leq \dim \Phi_K(m,p)$. We have to prove that the equality is impossible.

Suppose to the contrary. Then the system $([(u_k, x)]^n_1)$ corresponding to the frame of $f$ is a basis of $\Phi_K(m, p)$ by Lemma 2. In particular, there is an expansion

$$\left(\sum_{i=1}^{m} \lambda_i |\xi_i|^2\right)^{p/2} = \sum_{k=1}^{n} a_k(\lambda_1, \ldots, \lambda_m) |(u_k, x)|^p, \tag{5}$$

where $(\lambda_i)^m \in \mathbb{R}^m$ and $a_k$ are some functions of these parameters.

Now we introduce the inner product

$$(\phi_1, \phi_2) = \int_S \phi_1(x) \phi_2(x) d\sigma(x) \quad (\phi_1, \phi_2 \in \Phi_K(m,p))$$

where $\sigma$ is the standard measure on the unit sphere $S \subset \ell_{2m}^n$. In the Euclidean space $\Phi_K(m, p)$ we have the basis $([\theta_k(x)]^n_1)$ dual to $([(u_k, x)]^n_1)$. This allows us to represent the coefficients $a_k$ as

$$a_k(\lambda_1, \ldots, \lambda_m) = \int_S \left(\sum_{i=1}^{m} \lambda_i |\xi_i|^2\right)^{p/2} \theta_k(x) d\sigma(x).$$

Hence, $a_k(\lambda_1, \ldots, \lambda_m)$ are forms of degree $p/2$, a fortiori, they are continuous.

Denote by $\mathbb{R}^m_+$ the open coordinate cone in $\mathbb{R}^m$, so $\mathbb{R}^m_+ = \{(\lambda_i)^m \subset \mathbb{R}^m : \lambda_1 > 0, \ldots, \lambda_m > 0\}$. We prove that on $\mathbb{R}^m_+$ all $a_k(\lambda_1, \ldots, \lambda_m) > 0$ or equivalently, $\hat{a}(\lambda_1, \ldots, \lambda_m) = \min_k a_k(\lambda_1, \ldots, \lambda_m) > 0$. Suppose to the contrary: let $\hat{a}(\gamma_1, \ldots, \gamma_m) < 0$ for some $(\gamma_i)^m_1 \subset \mathbb{R}^m_+$. On the other hand, $\hat{a}(1, \ldots, 1) = 1$ by comparing (3) to (5) with all $\lambda_i = 1$. Since $\hat{a}$ is continuous, we have $\hat{a}(\mu_1, \ldots, \mu_m) = 0$ for some $(\mu_i)^m_1 \subset \mathbb{R}^m_+$. But the latter means that all $a_k(\mu_1, \ldots, \mu_m) \geq 0$ and, at least one of them is zero, say $a_n(\mu_1, \ldots, \mu_m) = 0$.

Therefore,

$$\left(\sum_{i=1}^{m} \mu_i |\xi_i|^2\right)^{p/2} = \sum_{k=1}^{n-1} a_k(\mu_1, \ldots, \mu_m) |(u_k, x)|^p,$$

whence

$$|z, z|^{p/2} = \sum_{k=1}^{n-1} |(u_k, z)|^p, \tag{6}$$

where

$$z = \mathcal{D} x, \quad v_k = (a_k(\mu_1, \ldots, \mu_m))^{1/p} \mathcal{D}^{-1} u_k$$

and $\mathcal{D}$ is the diagonal matrix with entries $\mu_1^{1/2}, \ldots, \mu_m^{1/2}$. By Lemma 1 the identity (6) means that the system $([v_k])_{n-1}^{p,n-1}$ is the frame of an isometric embedding $\ell_{2m}^n \to \ell_{p,K}^{n-1}$. This contradicts the minimality of $n$. As a result, all $a_k(\lambda_1, \ldots, \lambda_m) \geq 0$ for $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$, i.e. on the closed coordinate cone.
Now we take $\xi_1 = 1$ and $\xi_i = 0$ for all $i \geq 2$, so $x = e_1$, the first vector from the canonical basis of $\ell^m_{2;K}$. In this setting (10) reduces to
\[
\lambda_1^{p/2} = \sum_{k=1}^{n} a_k(\lambda_1, \ldots, \lambda_m)|\langle u_k, e_1 \rangle|^p.
\]
(Recall that $m \geq 2$.) This yields
\[
\sum_{k=1}^{n} a_k(0, \lambda_2, \ldots, \lambda_m)|\langle u_k, e_1 \rangle|^p = 0.
\]
Assume all $\langle u_k, e_1 \rangle \neq 0$. Since for $\lambda_2 > 0, \ldots, \lambda_m > 0$ all $a_k(0, \lambda_2, \ldots, \lambda_m) \geq 0$, all of them are equal to zero. Hence, the right side of the identity (5) vanishes as long as $\lambda_1 = 0$, in contrast to the function on the left side, a contradiction. To finish the proof we only have to show that the assumption $\langle u_k, e_1 \rangle \neq 0$, $1 \leq k \leq n$, is not essential.

First, note that all $u_k \neq 0$, otherwise, the number $n$ in (3) would be reduced. Therefore, the sets $\{x : \langle u_k, x \rangle = 0\}$, $1 \leq k \leq n$, are hyperplanes in $\ell^m_{2;K}$. Their union is different from the whole space. Hence, there is a vector $e$ such that all $\langle u_k, e \rangle \neq 0$, $\|e\|_2 = 1$. This $e$ can be represented as $e = ge_1$ where $g$ is an isometry of the space $\ell^m_{2;K}$. Indeed, this space is Euclidean, so its isometry group is transitive on the unit sphere. Thus, all $\langle g^{-1}u_k, e_1 \rangle \neq 0$. On the other hand, $(g^{-1}u_k)^n_1$ is the frame of the isometric embedding $fg : \ell^m_{2;K} \to \ell^n_{p;K}$.

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