There have been a number of mathematical results recently identifying algebras over certain operads [1, 25, 17, 28, 16, 10, 14, 11]. See [4, 26] for expository surveys of the basics of operad theory. Before citing any of these results, let us mention some trivial classical examples. Let $A$ denote one of the three words: “commutative”, “associative” and “Lie”. In each of these cases, consider the corresponding operad $O(n) = \langle \text{words in the free $A$ algebra on $n$ generators having at most one occurrence of each generator $x_i$} \rangle_k$, $n \geq 1$, $\langle \rangle_k$ meaning the linear span over the ground field $k$. When $A$ is commutative, associative or Lie, we will denote the corresponding operad $O$ by $C$, $A$s and $L$e, respectively. Note that $C(n) = \langle x_1 \ldots x_n \rangle_k = k$ and $A_s(n) = \langle x_{\sigma(1)} \ldots x_{\sigma(n)} \mid \sigma \in S_n \rangle_k = k[S_n]$. The main feature of these three operads is that they describe algebras of the corresponding types. More precisely, algebras over an operad $O$, $O = C$, $A$s or $L$e], (or simply, $O$-algebras) are exactly $A$ algebras.

A hint of another relation between operads and algebras may be given by the formula:

$$\bigoplus_{n \geq 1} (O(n) \otimes V^\otimes n)_{S_n} \text{ is the free } O\text{-algebra generated by a vector space } V.$$  

Here $S_n$ in the subscript denotes coinvariants of the symmetric group, that is to say, the quotient by the diagonal action of the symmetric group.

Several recent, less obvious examples were largely inspired by the work of physicists, who came up with new and not so new kinds of
algebras, surprisingly appearing in certain models of quantum field theory. An essential feature of these algebras was that they did not satisfy the classical identities strictly, but rather satisfied them ‘up to homotopy’ in a strong sense. Mathematicians, in their attempt to describe the algebras geometrically, found operads responsible for the types of algebras produced by the physical theories, see Table 1.

This paper proposes a theory with a physics flavor, filtered topological gravity, whose state space is endowed with the structure of a commutative homotopy algebra or $C_\infty$-algebra. The core of the paper is the following theorem. Consider the operad of moduli spaces $\mathcal{M}(n)$, the Deligne-Knudsen-Mumford compactification of the moduli space of $(n + 1)$-punctured Riemann spheres. Let $C(\mathcal{M}(n))$ be the corresponding operad of singular chains. This is an operad of complexes. Hence, an algebra over it is a complex $V$ with a morphism of operads $C(\mathcal{M}(n)) \to \text{Hom}(V\otimes^n V, V)$.

**Theorem 0.1.** Let $V$ be an algebra over $C(\mathcal{M}(n))$ such that the structure morphism $\mu : C(\mathcal{M}(n)) \to \text{Hom}(V\otimes V, V)$ vanishes on all elements in $C_p(\mathcal{M}(n))$ for all $p > n - 2$, then $V$ has the structure of a $C_\infty$-algebra.

This result may be regarded as a partial answer to a question of Lian and Zuckerman [30] of lifting the BV algebra structure on BRST cohomology to the cochain level, more precisely, providing the BV dot product with higher homotopies. In fact, in [24], it was shown that the BRST complex formed a homotopy associative algebra which arose from an operad which is closely related to the commutative homotopy associative operad above.

Another result, which is a byproduct of our work, is our description of a spectral sequence which converges to $H^\bullet(\mathcal{M}_{g,n})$, whose $E_1$ term consists of tensor products of the cohomology groups of compactified moduli spaces. Our result can be stated succinctly by using the generalization of the operad cobar construction which includes graphs as well as trees, called the Feynman transform, a notion due to Getzler and Kapranov [15].

**Theorem 0.2.** There exists a spectral sequence converging to $H^\bullet(\mathcal{M}_{g,n})$ and degenerating at the $E_2$ term, such that the term $E_1$ is the cobar construction of the modular co-operad $H^\bullet(\mathcal{M}_{g,n})$.

The proof of this theorem is based on purity of the Hodge structure on $H^\bullet(\mathcal{M}_{g,n})$. We also present a “dual” version of this theorem, where the spectral sequence converges to $H_\ast(\mathcal{M}_{g,n})$ and does not necessarily degenerate at $E^2$, unless $g = 0$. 
1. \( C_\infty \) OPERAD AND \( C_\infty \)-ALGEBRAS

Let us begin with a definition of an operad.

**Definition 1.1.** An operad \( \mathcal{O} = \{ \mathcal{O}(n) \}_{n \geq 0} \) with unit is a collection of objects (topological spaces, complexes, etc.) such that each \( \mathcal{O}(n) \) has an action of \( S_n \), the permutation group on \( n \) elements (\( S_0 \) is contains only the identity), and a collection of operations for \( n \geq 1 \) and \( 1 \leq i \leq n \), \( \mathcal{O}(n) \times \mathcal{O}(n') \to \mathcal{O}(n + n' - 1) \) given by \( (f, f') \mapsto f \circ_i f' \) satisfying

1. if \( f \in \mathcal{O}(n) \), \( f' \in \mathcal{O}(n') \), and \( f'' \in \mathcal{O}(n'') \) where \( 1 \leq i < j \leq n \), \( n', n'' \geq 0 \), and \( n \geq 2 \) then
   \[
   (f \circ_i f') \circ_{j+n'-1} f'' = (-1)^{|f'||f''|}(f \circ_j f'') \circ_i f'
   \]
   where signs on the right hand side should be ignored if \( \mathcal{O} \) is not an operad of (graded) vector spaces,
2. if \( f \in \mathcal{O}(n) \), \( f' \in \mathcal{O}(n') \), and \( f'' \in \mathcal{O}(n'') \) where \( n, n' \geq 1 \), \( n'' \geq 0 \), and \( i = 1, \ldots, n \) and \( j = 1, \ldots, n' \) then
   \[
   (f \circ_i f') \circ_{i+j-1} f'' = f \circ_i (f' \circ_j f''),
   \]
3. the composition maps are equivariant under the action of the permutation groups,
4. there exists an element \( I \) in \( \mathcal{O}(1) \) called the unit such that for all \( f \) in \( \mathcal{O}(n) \) and \( i = 1, \ldots, n \),
   \[
   I \circ_1 f = f = f \circ_1 I
   \]

By iterating \( k \) composition maps, one obtains the more common form of the operad composition, \( \gamma : \mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k) \).

**Example 1.1.** The endomorphism operad \( \text{End}_V := \{ \text{End}_V(n) \} \) of a differential graded \( k \)-vector space \((V, Q)\) is defined to be \( \text{End}_V(n) := \text{Hom}_k(V^{\otimes n}, V) \) where \( S_n \) acts naturally upon \( V^{\otimes n} \) and the composition maps are given by

\[
(f \circ_i f')(v_1 \otimes \cdots \otimes v_{n+n'-1}) = \pm f(v_1 \otimes \cdots \otimes v_{i-1} \otimes f'(v_i \otimes \cdots v_{i+n'}) \otimes v_{i+n'+1} \otimes v_{n+n'-1})
\]

for all \( f \) in \( \text{End}_V(n) \), \( f' \) in \( \text{End}_V(n') \) and \( i = 1, \ldots, n \) for all \( n, n' \), and the unit \( e \) is just the identity map from \( V \) to itself. The \( \pm \) is the sign which is obtained by sliding \( f' \) through \( v_1, \ldots, v_{i-1} \). Let us denote the component of \( \text{End}_V(n) \) with degree \( g \) by \( \text{End}_V^g(n) \).

Algebraic structures on a differential graded vector space \( V \) are often parametrized by an operad through the following notion.
Definition 1.2. Let $O$ be an operad. A differential graded $k$-vector space, $(V, Q)$, is said to be an $O$-algebra if there is a morphism of operads $O \to \mathcal{E}nd_V$.

Notice that given an operad of topological spaces $O$, the singular chains on $O$, $C_\bullet(O)$ := \{ $C_\bullet(O(n))$ \}, naturally forms an operad of differential graded vector spaces and, consequently, so do the homology groups $H_\bullet(O)$ := \{ $H_\bullet(O(n))$ \}. Furthermore, if $(V, Q)$ is an algebra over $C_\bullet(O)$, then $H_\bullet(V)$ is naturally an algebra over $H_\bullet(O)$. However, there is more information in the $C_\bullet(O)$-algebra, $(V, Q)$, than at the cohomology level. This provides motivation for the study of algebraic structures up to homotopy, the first of which we now recall.

Definition 1.3. An $A_\infty$ (or strongly homotopy associative) algebra, $V$, is a complex $Q : V^g \to V^{g-1}$ endowed with a collection of $n$-ary (linear) operations \{ $m_n : V^\otimes n \to V$ \}_{n\geq 2} with $m_n$ having degree $n - 2$ satisfying

\[
Q(m_n(v_1, \ldots, v_n)) = (-1)^n \sum_{k=1}^n (-1)^{\epsilon(k)} m_n(v_1, \ldots, Qv_k, \ldots, v_i)
\]

\[
= \sum_{r,s,k} (-1)^{k(s-1)+sn} (m_r \circ_k m_s)(v_1, \ldots, v_n) \tag{3}
\]

where the summation runs over $r, s, k$ satisfying $r+s = n+1$, $1 \leq k \leq r$, $2 \leq r < n$ for all $v_1, \ldots, v_n$ in $V$ and $n \geq 2$, and $\epsilon(k)$ denotes the sign obtained by sliding $Q$ through $v_1, \ldots, v_{k-1}$.

We note that $m_2$ is a multiplication, $m_3$ is an associating homotopy and the further $m_k$'s are “higher associating homotopies”.

In the context of (strong) homotopy algebras, the simplest definition of a $C_\infty$-algebra, first appeared in work of Kadeishvili [20, 21] and then in that of Smirnov [41] (both of whom called them commutative $A_\infty$-algebras), is:

Definition 1.4. A $C_\infty$-algebra is an $A_\infty$-algebra $(A, \{m_n\})$ such that each map $m_n : A^\otimes n \to A$ is a Harrison cochain, i.e. $m_n$ vanishes on the sum of all $(p, q)$-shuffles for $p + q = n$, the sign of the shuffle coming from the grading of $A$ shifted by 1.

The single object equivalent definition (cf. [39, 35, 34]) is:

Definition 1.5. A $C_\infty$-algebra is a graded vector space $A$ together with a codifferential on the free Lie coalgebra cogenerated by $sA$, which is isomorphic to $A$ with the grading shifted by 1. (The convention is that the shift is opposite to the degree of the differential; since we will be working homologically at the operad level, $d$ is of degree $-1$ while $s$ is of degree $+1$.)
This definition hints of the ancient Koszul duality (or adjointness) between commutative algebras and Lie coalgebras (or Lie algebras and commutative coalgebras) \cite{8, 35, 34, 39}, which carries over to the operad level, \cite{16}.

Getzler and Jones \cite{14} established the equivalence of the above with the operadic definition of a $C_\infty$-algebra implicit in \cite{18}.

**Definition 1.6.** A $C_\infty$-algebra is an algebra over the operad $\mathcal{O}_\infty[\lrcorner \nabla \mathcal{L}]$. 

The concept of a co-operad is defined dually to that of an operad \cite{14}, so that if $\mathcal{K} = \{\mathcal{K}(n)\}$ is a co-operad over a field $k$, then the linear duals $\text{Hom}(\mathcal{K}(n), k)$ form an operad. Conversely, if $\mathcal{O} = \{\mathcal{O}(n)\}$ is an operad with each $\mathcal{O}(n)$ finite dimensional (or of finite type, i.e. graded and finite dimensional in each grading), then the linear duals $\text{Hom}(\mathcal{O}(n), k)$ form a co-operad. (A co-operad is “an operad with the arrows reversed”.)

The operad $\mathcal{L}_\infty[\lrcorner \nabla \mathcal{L}]$ is a functor from co-operads to operads. (In \cite{16}, the use of co-operads is avoided by assuming finite type for operads and defining $\mathcal{O}_\infty[\lrcorner \nabla \mathcal{L}]$ only for the linear duals of operads.) All we need to know is:

For any co-operad $\mathcal{K}$, $\mathcal{O}_\infty[\lrcorner \nabla \mathcal{K}]$ is an operad with pieces indexed by trees, constructed as products of various $\mathcal{K}(n)$’s according to a prescription determined by the tree. Our convention is that trees have vertices all of valence $> 1$, e.g. the corolla with $n$ leaves, denoted by $\delta_n$, and one root has just one vertex. The piece of $\mathcal{O}_\infty[\lrcorner \nabla \mathcal{K}]$ indexed by this corolla is just $\mathcal{K}(n)$ with a shift in grading.

To make a comparison between $C_\infty$-algebras and $A_\infty$-algebras at the operad level, recall that the free Lie algebra $\mathcal{L}(x_1, \ldots, x_n)$ can be realized as the primitive subspace of $T(x_1, \ldots, x_n)$ with respect to the unshuffle coproduct \cite{36}. Dually, the free Lie coalgebra $\mathcal{L}^c(x_1, \ldots, x_n)$ can then be identified with the space of indecomposables of the free associative coalgebra $T^c(x_1, \ldots, x_n)$, i.e. the quotient by the image of the shuffle product. (See \cite{38} for definitions which do not rely on finite type and duality.) Then $\mathcal{L}_\infty^c(n)$ is defined as the quotient of the free Lie coalgebra $\mathcal{L}_\infty^c(x_1, \ldots, x_n)$ by those multilinear functions which vanish whenever two arguments are equal.

An $A_\infty$-algebra is an algebra over the operad $\mathcal{O}_\infty[\lrcorner \nabla \mathcal{A}_\infty^c]$ where $\mathcal{A}_\infty^c$ is the co-operad for associative coalgebras. $\mathcal{A}_\infty^c(n)$ can be realized as the quotient of the tensor coalgebra $T^c(x_1, \ldots, x_n)$ by those multilinear functions which vanish whenever two arguments are equal.
Since $C \langle -\nabla L \rangle \rangle (n)$ has its corolla component equal to $L \rangle ) \rangle c(n)$ with a shift in grading, the structure map $m_n$ and its permutations are given by $L \rangle ) \rangle c(n) \rightarrow \text{Hom}(A^{\otimes n}, A)$. Interpretation of $m_n$ as the structure map for an $A_\infty$-algebra means pulling back the map $L \rangle ) \rangle c(n) \rightarrow \text{Hom}(A^{\otimes n}, A)$ to $T^{c}(n) \rightarrow \text{Hom}(A^{\otimes n}, A)$, which guarantees that $m_n$ vanishes on the image of the shuffle product.

Strictly speaking, an algebra over $A \int$, i.e. a morphism $A \int \rightarrow \text{End}(V)$, does not determine a unique associative multiplication on $V$, but rather two of them, since $A \int (2) = k[S_2]$. We make the obvious choice corresponding to the identity element of $S_2$ to determine an associative multiplication on $V$. For $C \langle -\nabla A \rangle \rangle$, we must make choices for each $m_n$, but we make the same choice, corresponding to the identity element of $S_n$. (This choice is implicit in the result of [16] that an $A_\infty$-algebra is the same as an algebra over $C \langle -\nabla A \rangle \rangle$.) For $C \langle -\nabla L \rangle \rangle$, we just take the equivalence class (mod the shuffle product) of the above choice for $C \langle -\nabla A \rangle \rangle$.

### 2. Moduli Spaces of Punctured Spheres

Let $\mathcal{M}_n$ be the moduli space of Riemann spheres with $n$ punctures. That is, points in $\mathcal{M}_n$ consist of configurations of $n$ ordered points on $\mathbb{CP}^k$ with any two such configurations being identified if they are related by a biholomorphic map. In other words,

$$\mathcal{M}_n := ((\mathbb{CP}^k)^\times \setminus \Delta)/\text{PSL(}\not=, \mathbb{C})$$

where $\Delta = \{(z_1, \ldots, z_n) \in (\mathbb{CP}^k)^\times \mid F_{\not=} = F_{\not=} \text{ for some } \not= \not= \}$, the set of diagonals. There is a compactification of $\mathcal{M}_n$ when $n \geq 3$ due to Deligne-Knudsen-Mumford [6, 22, 23, 27] which is the moduli space of stable genus 0 curves with $n$ punctures, $\overline{\mathcal{M}}_n$.

Recall that a stable $n$ punctured complex curve of genus 0 is a connected compact complex curve $C$ of genus 0 with $n$ punctures, such that it may have ordinary double points away from the punctures, each irreducible component of the curve $C$ is a projective line and the total number of punctures and double points on each component of $C$ is at least 3. Both $\mathcal{M}_n$ and $\overline{\mathcal{M}}_n$ are smooth complex algebraic manifolds of complex dimension $n - 3$. The moduli space $\mathcal{M}_n$ of nonsingular curves is an open submanifold in the projective manifold $\overline{\mathcal{M}}_n$. The complement is a divisor, formed by all degenerate curves.

Let $\overline{\mathcal{M}}(1) := \{e_C\}$ and $\overline{\mathcal{M}}(n) := \overline{\mathcal{M}}_{n+1}$ for $n \geq 2$, then the set $\overline{\mathcal{M}} := \{ e_C \} \overline{\mathcal{M}}(n)$ is naturally an operad of algebraic varieties where the element $e_C$ is defined to be a unit with respect to the operad composition.\footnote{We include a unit in this manner for convenience.}
permutation group on \( n \) elements, \( S_n \), acts on \( \mathcal{M}(n) \) by reordering the first \( n \)-punctures. The composition maps \( \gamma_i : \mathcal{M}(n) \times \mathcal{M}(n') \to \mathcal{M}(n + n' - 1) \) for all \( i = 1, \ldots, n \) and for all \( n, n' \) are defined by
\[
(\Sigma, \Sigma') \mapsto \gamma_i(\Sigma, \Sigma') := \Sigma \circ_i \Sigma'
\]
where \( \Sigma \circ_i \Sigma' \) is obtained by attaching the \((n' + 1)\)st puncture of \( \Sigma' \) to the \( i \)th puncture of \( \Sigma \) thereby creating a curve with a new double point and the \((n + n' - 1)\) punctures are ordered in the natural way.

Each space \( \mathcal{M}(n) \) is naturally stratified by smooth, connected locally closed algebraic subvarieties. Each stratum of \( \mathcal{M}(n) \) consists of those points arising from \( n \)-punctured stable curves of a given topological type. Since any stable curve can be obtained by attaching spheres together, the combinatorics of this attaching process can be encoded in a tree. Therefore, each stratum of \( \mathcal{M}(n) \) is naturally indexed by a tree (see Figure 2). This stratification gives rise to a filtration of \( \mathcal{M}(n) \):
\[
F_{-1}(n) = \emptyset \subseteq F_0(n) \subseteq \cdots \subseteq F_{n-2}(n) = \mathcal{M}(n)
\]
where \( F_p(n) \) is the disjoint union of all the strata of \( \mathcal{M}(n) \) with complex dimension less than or equal to \( p \). Furthermore, this filtration is invariant under the action of the permutation group and it behaves nicely with respect to operad composition, i.e. for all \( i = 1, \ldots, n \) and for all \( n, n', p, p' \), we have
\[
\gamma_i : F_p(n) \times F_{p'}(n') \to F_{p+p'}(n + n' - 1).
\]

The filtration gives rise to a spectral sequence associated to each \( \mathcal{M}(n) \) which converges finitely to \( H_\bullet(\mathcal{M}(n)) \). This spectral sequence is known to degenerate at the \( E^2 \)-term \([3]\). We show in Section 3 that any filtration which respects the operad structure induces an operad structure on the \( E^r \) term in its associated spectral sequence for all \( r \geq 0 \). In particular, each \( E^r \) term contains a collection of suboperads. In our case, the only nontrivial suboperad is the \( q = 0 \) (“middle”) row of the \( E^1 \) term. The main result for our purposes, due to Beilinson-Ginzburg \([4]\), cf. F. Cohen \([5]\) and Schechtman-Varchenko \([38]\), is the following:

**Theorem 2.1.** The “middle” row of the \( E^1 \) term of the spectral sequence associated to the canonical filtration of \( \mathcal{M}(n) \)
\[
0 \to H_{n-2}(F_{n-2}(n), F_{n-3}(n)) \to \cdots \to H_p(F_p(n), F_{p-1}(n)) \to \cdots \to H_0(F_0(n)) \to 0,
\]
is an operad isomorphic to \( \mathcal{C}[\nabla]^1(n) \), the \( C_\infty \) operad.

This identification is quite explicit. Lefshetz duality gives an isomorphism \( H_p(F_p(n), F_{p-1}(n), \mathbb{C}) \cong H^p(F_p(n) \setminus F_{p-1}(n)) \) but \( F_p(n) \setminus F_{p-1}(n) \) is the disjoint union of those strata in \( \mathcal{M}(n) \) with complex dimension \( p \) each of which is indexed by an \( n \)-tree with \( n - 2 - p \) internal edges.
If $T$ is a tree, then let $S_T$ denote the strata associated to this tree, e.g. see Figure 2. Each such tree $T$ is the iterated composition of corollas, say $\delta_{n_1}, \ldots, \delta_{n_k}$, and we have the isomorphism $S_T \simeq S_{\delta_{n_1}} \times \cdots \times S_{\delta_{n_k}}$. The stratum associated to any corolla, $\delta_n$, can be identified with $\mathcal{M}(n)$. Therefore, $S_T \simeq \mathcal{M}(n_1) \times \cdots \times \mathcal{M}(n_k)$. Using the fact that the cohomology of $\mathcal{M}(n)$ vanishes in degree above $n-2$, which follows from a similar vanishing of homology of configuration spaces by Arnold [2], we obtain the isomorphism $H^p(S_T) \simeq H^{n_1-2}(\mathcal{M}(n_1)) \otimes \cdots \otimes H^{n_k-2}(\mathcal{M}(n_k))$. However, $L \cap (n)$ can be identified with $H^{n-2}(\mathcal{M}(n))$ (with a shift in degree) [3] and therefore, $H^{n-2}(\mathcal{M}(n))$ can be identified with $\mathcal{L} \cap (n)$. Finally, we obtain $H^p(S_T) \simeq \mathcal{L} \cap (n_1) \otimes \cdots \otimes \mathcal{L} \cap (n_k)$ which is an element of $\mathcal{C}^{(-\nabla \mathcal{L}) \cap (n)}$ associated to the tree $T$ with the proper element in $\mathcal{L} \cap (n)$ decorating the corresponding vertex of $T$.

3. Moduli Spaces of Punctured Riemann Surfaces

Here we make a generalization of the results of the previous section; we give a complete description of the analogous spectral sequence in the case of moduli spaces of Riemann surfaces of higher genera. In particular, we identify the other rows of the $E^1$ term of the spectral sequence of the previous section.

Let us restrict ourselves to those Riemann surfaces with genus $g$ and $n$ ordered punctures which have negative Euler characteristic, i.e. $g \geq 2$ or $g = 1, n \geq 1$ or $g = 0, n \geq 3$. Let $\mathcal{M}_{g,n}$ be the moduli space of genus $g$ Riemann surfaces, that is, complex algebraic curves, with $n$ punctures and let $\overline{\mathcal{M}}_{g,n}$ be its compactification due to Deligne-Knudsen-Mumford. This is a smooth, complete stack of dimension $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g-3+n$. The compactified moduli parameterize isomorphism classes of stable curves, ones which have a finite number of singularities, which are double points, and such that each irreducible component of genus 1 has at least one puncture or double point and each irreducible component of genus 0 has at least three punctures or double points. These conditions insure that each component of the complement of the punctures and double points has negative Euler characteristic.

The spaces $\overline{\mathcal{M}}_{g,n}$ form a modular operad, see Getzler-Kapranov [15], that is, two kinds of operations are defined: attaching two curves at punctures: $\overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g',n'} \to \overline{\mathcal{M}}_{g+g',n+n'-2}$ and gluing two punctures together on a single curve $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g+1,n-2}$. Furthermore, there is the action of the symmetric group $S_n$ on $\overline{\mathcal{M}}_{g,n}$ which reorders the punctures.

Let $F_p = F_p(g,n) \subset \overline{\mathcal{M}}_{g,n}$ be the closed subspace (substack, in fact) of dimension $p$ formed by stable curves with at least $\dim_{\mathbb{C}} (\overline{\mathcal{M}}_{g,n}) - p =$
3g − 3 + n − p double points. We obtain an ascending filtration of the moduli space \( \mathcal{M}_{g,n} \):

\[
F_{-1} = \emptyset \subset F_0 \subset \cdots \subset F_{3g-3+n} = \overline{\mathcal{M}}_{g,n}.
\]

As in the genus zero case, this filtration behaves nicely with respect to the modular operad operations:

\[
F_p(g, n) \times F_{p'}(g', n') \to F_{p+p'}(g + g', n + n' - 2)
\]
corresponding to attaching two curves at punctures and

\[
F_p(g, n) \to F_p(g + 1, n - 2)
\]
corresponding to glueing together two punctures on a single curve.

Irreducible components (strata) \( S_G \) of \( F_p \) are indexed by stable labeled \( n \)-graphs \( G \) with \( 3g - 3 + n - p + 1 \) vertices and with the invariant \( g(G) \), defined below, equal to the genus \( g \). Stable refers to graphs of the following kind. Each graph is connected, has a root vertex and \( n \) enumerated exterior edges, edges which are coincident with only one vertex of the graph. Each vertex \( v \) of the graph is labeled by a nonegative integer \( g(v) \), called the genus of a vertex. The stability condition means that any vertex \( v \) labeled by \( g(v) = 1 \) should be coincident with at least one edge (i.e., be of at least valence one) and each vertex \( v \) with \( g(v) = 0 \) should be of at least valence three. The invariant \( g(G) \) is given by the formula

\[
g(G) = b_1(G) + \sum_{v} g(v),
\]

where \( b_1(G) \) is the first Betti number of the graph. Each component stratum \( S_G \) is a quotient of the product of uncompactified moduli spaces (via the modular operad structure), the combinatorics of which are neatly encoded in the graph:

\[
S_G = \left( \prod_{v \in G} \mathcal{M}_{g(v), n(v)} \right) / \text{Aut}(G)
\]

where the product is over all vertices of \( G \) and \( n(v) \) is the valence of the vertex \( v \) and \( \text{Aut}(G) \) is the automorphism group of a graph \( G \) (a bijection on vertices and edges, preserving the exterior edges, the labels of vertices and the incidence relation).

Thus, we get a modular operad in the category of filtered varieties (stacks, in fact) and therefore, applying the spectral sequence functor, we obtain a modular operad of spectral sequences, see Section 3 for related formalism. Its \((g, n)\)-component can be described, as usual, in the following way

\[
E^1_{p,q} = H_{p+q}(F_p, F_{p-1}, \mathbb{C}) = H^{p-q}(F_p \setminus F_{p-1}, \mathbb{C}),
\]
due to Poincaré-Lefschetz duality, with the differential $d^1 : H^{p-q}(F_p \setminus F_{p-1}) \to H^{p-q-1}(F_{p-1} \setminus F_{p-2})$ being the Poincaré residue, and

$$\bigoplus_{p+q=k} E^\infty_{p,q} = H_k(\mathcal{M}_{g,n}, \mathbb{C}).$$

Theorem 3.1. 1. The $E^1$ term of the spectral sequence is naturally isomorphic to the Feynman transform, see [15], of the modular co-operad $H^\bullet(M_{g,n})$. Namely, $E^1_{p,q} = 0$, unless $-p \leq q \leq p \leq 3g - 3 + n$, when

$$E^1_{p,q} = \bigoplus_G \left( \bigoplus_{k(v) = p-q} \bigotimes_{v \in G} H^k(v)(\mathcal{M}_{g(v),n(v)}) \right)^{\text{Aut}(G)},$$

the first summation running over all stable labeled $n$-graphs $G$ with $g(G) = g$ and $3g - 3 + n - p + 1$ vertices, the second over all functions $k(v) \in \mathbb{Z}$ of vertices $v \in G$ summing up to $p - q$. The differential is induced by contracting internal edges in $G$, which corresponds to forming new double points on a curve, and taking the Poincaré residue.

2. When $g = 0$, the graphs $G$ have no loops and are labeled with zeroes, i.e., just trees where all vertices have valence $\geq 3$. Then the $E^1$ term is nothing but the cobar construction of the co-operad $H^\bullet(M_{0,n})$. Moreover, $E^1_{p,q} = 0$ unless $0 \leq q \leq p \leq n - 3$, and $E^2_{p,q} = E^\infty_{p,q} = 0$, except

$$E^2_{p,p} = E^\infty_{p,p} = H_2p(\mathcal{M}_{0,n}).$$

Remark 1. The second part of the theorem, as well as of Theorem 3.3 below, is proved independently by Getzler. Just after we proved it, we came across his two-day old preprint [13] dedicated to this kind of duality between the operads $H_\bullet(M_{0,n})$ and $H_\bullet(\mathcal{M}_{0,n})$.

In contrast to Getzler’s proof, our proof below of the statement about $E^2$ being concentrated on the diagonal uses the operad structure and known results on the cohomology of $\mathcal{M}_{0,n}$.

Getzler [13], on the other hand, uses the fact that the mixed Hodge structure on the cohomology of $M_{0,n}$ is pure and that the operad $H_\bullet(\mathcal{M}_{0,n})$ is Koszul. The Hodge structure on the cohomology of $\mathcal{M}_{g,n}$ is no longer pure for higher genera, but Mumford’s conjecture for the stable cohomology implies that the Hodge structure on the stable cohomology of $\mathcal{M}_{g,n}$ should be pure.
Question 3.2. For $g > 0$, describe the locus of the $E^\infty$ term on the $(p, q)$ plane. Is there a stable version of this theorem, that is, for large $g > 0$, where the sequence degenerates at $E^2$ and the $E^2$ term is located on the diagonal $p = q$? It would be interesting to find the proper notion of Koszulness for the stable homology of the modular operad $H^\ast_s(\mathcal{M}_{g,n})$.

Proof. The first part of the theorem is obvious after the description of the strata $S G$ above. In the second part, to show that $E^1_{p,q} = 0$ for $q < 0$ notice that $H^{k(v)}(\mathcal{M}_{0,n(v)}) = 0$ for $k(v) > n(v) - 3$. Denote by $ed(G)$ the number of edges of the graph, including the $n$ exterior edges and by $v(G)$ the number of vertices. Adding the above inequalities together, we get $E^1_{p,q} = 0$ for $p - q > \sum_v (n(v) - 3) = 2ed(G) - n - 3v(G) = 2(n + v(G) - 1) - n - 3v(G) = n - 2 - v(G) = p$, that is, for $q < 0$.

The degeneration $E^2_{p,q} = E^\infty_{p,q}$ of the spectral sequence for $g = 0$ follows from the purity of the Hodge structure on $E^1$, which is the sum of tensor products of the cohomologies of $\mathcal{M}_{0,n}$, where the Hodge structure is pure, see [4]. We will show the vanishing $E^2_{p,q} = 0$ for $g = 0$ and $p \neq q$ by using Keel’s description [23] of the homology of $\overline{\mathcal{M}}_{0,n}$, the operad structure and induction on $n$. $\overline{\mathcal{M}}_{0,3}$ is a point and the statement is trivial. Assuming it is proved for $k \leq n$, let us prove it for $k = n + 1$. Keel’s description says that $H_*(\overline{\mathcal{M}}_{0,n})$ is generated as an intersection algebra by $H_2$ so $H^1$ is zero. Thus the term $E^2_{1,0}$ must be zero and hence the statement is true for $\overline{\mathcal{M}}_{0,4}$. Now let $E^r_{p,q}(n)$ refer to the spectral sequence for $\overline{\mathcal{M}}_{0,n}$. Except for the fundamental class of $\overline{\mathcal{M}}_{0,n+1}$, which is in $E^2_{n-2,n-2}(n + 1)$, the rest of the terms $E^2_{p,q}(n + 1)$ are the operad compositions of $E^2_{p_i,q_i}(n_i)$’s for $n_i \leq n$, where we know $p_i = q_i$ by the induction assumption.

The situation with the “Koszul dual” spectral sequence, the one which converges to $H^\ast(\mathcal{M}_{g,n}, \mathbb{C})$ and whose $E_1$ term is formed by the cohomology of closed strata is somewhat better. A similar theorem holds and moreover, the spectral sequence degenerates at $E_2$ even for $g > 0$, due to the purity of the Hodge structure on $H^\ast(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$, see Deligne [6]. Let $X = \overline{\mathcal{M}}_{g,n}$, $U = \mathcal{M}_{g,n}$ and $D = X \setminus U$. Following Deligne [6], consider the double complex $\Omega^\ast_X(\log D)$, the smooth global $(p, q)$-forms on $X$ with at most logarithmic singularities $\partial f/f$ along $D$, where $f = 0$ is a local equation of $D$. Its hyper(=total)cohomology is equal to $H^{p+q}(U, \mathbb{C})$. Let $W_m$ be the subcomplex of the total complex generated by products $\alpha \land \partial f_{i_1}/f_{i_1} \land \cdots \land \partial f_{i_s}/f_{i_s}$ for $s \leq m$ and smooth $\alpha$, where each $f_i$ is a local equation of an irreducible
component of $D$. The $W_m^m = W_{-m}$'s for $m \leq 0$ define a decreasing filtration of the logarithmic double complex. Consider the spectral sequence $E_r$ associated with this filtration. With respect to this filtration, we have $E_1^{p,q} = H^{2p+q}(\tilde{F}_{3g-3+n+p})$, where $\tilde{F}_s$ is the disjoint union of irreducible components of $F_s$. The sequence degenerates: $E_2 = E_\infty = H^\bullet(\mathcal{M}_{g,n}, \mathbb{C})$, see [7].

In addition, for $g = 0$, it is known from Arnold’s explicit description of cohomology classes of configuration spaces that all nonzero cohomology classes in $H^m(\mathcal{M}_{0,n}, \mathbb{C})$ are represented by $(m,0)$-forms with exactly $m$ logarithmic singularities. Thus, $E_2^{p,q} = E_\infty^{p,q}$ vanishes unless $p = -m$ and $2p + q = 0$. It follows that $H^m(\mathcal{M}_{0,n}, \mathbb{C}) = E_{-m,2m}^\infty$, where the Hodge structure is pure of weight $2m$, see [7].

From these descriptions, we have the following theorem.

**Theorem 3.3.**
1. The $E_1$ term of the spectral sequence is dual to the Feynman transform of the modular co-operad $H^\bullet(\mathcal{M}_{g,n})$. Namely, $E_1^{p,q} = 0$, unless $-3g+3-n \leq p \leq 0$ and $-2p \leq q \leq 6g-6+2n$, when

$$E_1^{p,q} = \bigoplus_G \left( \bigoplus_{k(v)=2p+q} \bigotimes_{v \in G} H^k(v)(\mathcal{M}_{g(v),n(v)}) \right),$$

the first summation running over all stable labeled $n$-graphs $G$ with $g(G) = g$ and $-p+1$ vertices, the second over all functions $k(v) \in \mathbb{Z}$ of vertices $v \in G$ summing up to $2p+q$. The differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is induced by contracting internal edges in $G$, which corresponds to forming new double points on a curve. The $E_2$ term is equal to $E_\infty = H^\bullet(\mathcal{M}_{g,n}, \mathbb{C})$.

2. When $g = 0$, the graphs $G$ have no loops and are labeled with zeroes, i.e., just trees where all vertices have valence $\geq 3$. Then the $E_1$ term is nothing but the dual of the cobar construction of the co-operad $H^\bullet(\mathcal{M}_{0,n})$. Moreover, $E_2^{p,q} = E_\infty^{p,q} = 0$, except

$$E_2^{-2p} = E_\infty^{-2p} = H^{-p}(\mathcal{M}_{0,n}), \quad 0 \leq -p \leq n-3.$$

4. **Filterd Topological Gravity and $C_\infty$-algebras**

**Definition 4.1.** Let $\overline{\mathcal{M}}$ be the operad of the moduli space of stable curves of genus zero. A topological gravity consists of a differential graded complex vector space $Q : V_g \to V_{g-1}$ and a collection $\{\omega_n\}$ such that

1. $\omega_n = \sum_{r=0}^{2n-4} \omega_n^r$, where $\omega_n^r$ belongs to $\Omega(\overline{\mathcal{M}}(n)) \otimes \operatorname{End} V(n)$ with bidegree $(r,-r)$.
2. $d\omega_n = Q\omega_n$.
3. \( \sigma^* \omega_n = \omega_n \circ \sigma \) for all \( \sigma \) in \( S_n \) where \( \sigma^* \) denotes the pullback of the action of \( S_n \) on \( \mathcal{M}(n) \) and \( \sigma \) on the right hand side indicates the action of \( S_n \) on \( V^\otimes_n \) and

4. \( \gamma_i^* \omega_{n+n'-1} = \omega_n \circ_i \omega_{n'} \) for all \( i = 1, \ldots, n \) and all \( n, n' \) where \( \circ_i \) denotes composition in the endomorphism operad and \( \gamma_i^* \) is the pullback of the composition map \( \gamma_i : \mathcal{M}(n) \times \mathcal{M}(n') \to \mathcal{M}(n+n' - 1) \).

A filtered topological gravity is a topological gravity in which the differential forms \( \omega_n^p \) vanish for all \( p > \dim \mathcal{M}(n) = n - 2 \).

Since \( \mathcal{M} \) is an operad of topological spaces, \( (C_*(\mathcal{M}), \partial) \), the singular chains on \( \mathcal{M} \), inherits the structure of an operad. If \((V, Q)\) is a topological gravity with differential forms \( \{ \omega_n \} \) then \( V \) is an algebra over \( C_*(\mathcal{M}) \) where the morphism \( C_*(\mathcal{M}(n)) \to \mathcal{E}nd_V n \) is given by \( c \mapsto \int_c \omega_n \). The axioms of a topological gravity are formulated precisely so as to insure that this map is a morphism of operads. This definition of topological gravity was motivated by its origins in the physics literature [17].

Let \((V, Q)\) be a filtered topological gravity with differential forms \( \{ \omega_n \} \). Integration of \( \{ \omega_n \} \) makes \((V, Q)\) into a \( C_*(\mathcal{M}) \) algebra with a morphism \( \mu : C_*(\mathcal{M}) \to \mathcal{E}nd_V \) such that \( \mu(c) = 0 \) for all \( p \)-chains \( c \) on \( \mathcal{M}(n) \) with \( p > \dim \mathcal{M}(n) = n - 2 \).

**Theorem 4.1.** Let \((V, Q)\) be a filtered topological gravity, let \( V \) be filtered by the degree in the complex and let \( F = \{ F(n) \} \) be the filtration of \( \mathcal{M} \) arising from the canonical stratification of \( \mathcal{M} \), then \( V \) is a \( C_*(\mathcal{M}) \)-algebra preserving the filtration.

**Proof.** The filtration of \( \mathcal{M} \) by \( F \) induces a filtration on its singular chains, i.e. \( \cdots \subseteq C_{p+q}(F_p(n)) \subseteq C_{p+q}(F_{p+1}(n)) \subseteq \cdots \). Similarly, \( \mathcal{E}nd_V \) is filtered by \( \cdots \subseteq F_p \mathcal{E}nd_V^{p+q}(n) \subseteq F_p \mathcal{E}nd_V^{p+q}(n) \subseteq \cdots \). However, with the given choice of filtration degree, we see that

\[
F_p \mathcal{E}nd_V^{p+q}(n) = \begin{cases} 0, & \text{if } q \geq 1 \\ \mathcal{E}nd_V^{p+q}(n), & \text{if } q \leq 0. \end{cases}
\]

Therefore, the morphism \( \mu : C_*(\mathcal{M}) \to \mathcal{E}nd_V \) is an algebra preserving the filtrations if and only if \( \mu(c) = 0 \) when acting upon elements in \( C_{p+q}(F_p(n)) \) for all \( q \geq 1 \). This is precisely the case when \((V, Q)\) is a filtered topological gravity.

**Corollary 4.2.** A filtered topological gravity is a \( C_\infty \)-algebra.

**Proof.** Let the \( E^r \) terms in the spectral sequences associated to \( C_*(\mathcal{M}) \) and \( \mathcal{E}nd_V \) be denoted by \( E^r \) and \( E^r[V] \), respectively. Since \( V \) is a
filtered $\mathcal{C}(\overline{\mathcal{M}})$-algebra, $E^1[V]$ is an algebra over the operad $E^1$. However, since $E^1[V] \simeq \mathcal{E}nd_V$ and $E^1[V]$ contains a suboperad $D^1_0 = \oplus_p H_p(F_p(n), F_{p-1}(n))$ which is precisely the $C_\infty$ operad, $V$ is a $C_\infty$-algebra.

Notice that we only needed the fact that $\mu$ vanished when acting upon chains that are greater than half the dimension of the corresponding moduli space, thereby proving Theorem 0.1.

5. Filtrations of Operads and Algebras

We will now study properties of operads with a filtration and algebras which respect this filtration. We shall see that a filtered operad makes each term in its associated spectral sequence into an operad. Furthermore, there are natural suboperads in each term of the spectral sequence, one of which is the $C_\infty$-algebra in the $E^1$ term of the spectral sequence to the moduli space of stable curves. All of the results in this section can be naturally extended to the moduli spaces of higher genus curves in a straightforward way using the formalism of modular operads.

**Definition 5.1.** Let $\mathcal{O} = \{ O(n) \}$ be an operad of complexes with differentials $\partial : O_p(n) \to O_{p-1}(n)$. Let $F_p = \{ F_p(n) \}_{p \geq 1}$ be a filtration of $O(n)$ as complexes such that for all $i = 1, \ldots, n$, the composition maps $\circ_i$ take $F_p,q(n) \otimes F_{p',q'}(n')$ to $F_{p+p',q+q'}(n + n' - 1)$ for all $p, p', q, q'$, and the filtration degree $q$ part of $F_p(n)$ is stable under the action of the permutation group which commutes with the differential. The collection $F$ is said to be a filtration of the operad $\mathcal{O}$.

It is clear that a filtered operad can be defined in the category of topological spaces as well and that such a filtration induces a filtration on the associated operad of singular chains.

**Proposition 5.1.** Let $\mathcal{O}$ be an operad with filtration $F$ and let $E^r = \{ E^r(n) \}$ be the $E^r$ term in the associated spectral sequence then for all $r$, $E^r$ inherits the structure of an operad of complexes with a differential $\partial^r : E^r_{p,q}(n) \to E^r_{p-r,q+r-1}(n)$ and composition maps satisfying

$$\circ_i : E^r_{p,q}(n) \otimes E^r_{p',q'}(n) \to E^r_{p+p',q+q'}(n + n' - 1)$$

for all $i = 1, \ldots, n$. In particular, $E^0_{p,q}(n) = F_{p+p-q}(n)/F_{p-1,p+q}(n)$ and $E^1_{p,q}(n) = H_{p+q}(F_p(n), F_{p-1}(n))$. 
Proof. Let $Z^r_{p,q}(n) = \{ x \in F_{p,q}(n) \mid \partial x \in F_{p-r,q+r-1}(n) \}$ then we can write (see, for example, [29]):

$$E^r_{p,q}(n) = \frac{Z^r_{p,q}(n)}{\partial Z^r_{p+(r-1),q-(r-1)+1}(n) + Z^r_{p-1,q+1}(n)}$$

with the differential $\partial^r : E^r_{p,q}(n) \to E^r_{p-r,q+r-1}(n)$ induced from $\partial$. Keeping track of the composition maps, a computation shows that the numerators assemble into an operad and the denominators into an operad ideal so their quotient is an operad.

Given a filtered operad, there is a natural collection of suboperads of each $E^r$ term as shown by the following result.

**Proposition 5.2.** Let $O$ be an operad with filtration $F$. For each $r$ and $k$, there exists a suboperad of $E^r$, $D^r_k = \{ D^r_k(n) \}_{n \geq 1}$ such that

$$D^r_k(n) = \bigoplus_{(r-1)p+rq=k(n-1)} E^r_{p,q}(n).$$

In particular, $D^1_0 = \{ D^1_0(n) \}$ is a suboperad of $E^1$ with $n$th component $D^1_0(n) = \bigoplus_p H_p(F_p(n), F_{p-1}(n))$.

Proof. Each $E^r(n)$ is the union of subcomplexes $K^r_s(n) = \bigoplus_{(r-1)p+rq=s} E^r_{p,q}(n)$ since the differential maps $\partial^r : E^r_{p,q}(n) \to E^r_{p-r,q+r-1}(n)$. The composition maps take $\circ : K^r_s(n) \otimes K^r_{s'}(n') \to K^r_{s+s'}(n+n'-1)$. Therefore, $D^r_k(n) = \bigoplus_{(r-1)p+rq=k(n-1)} K^r_{k(n-1)}(n)$ assemble to form an operad of complexes.

In the case of $\overline{M}$ with its canonical filtration, the suboperads $D^1_k$ are all trivial for dimensonal reasons except when $k$ is 0.

We now introduce a filtration on the endomorphism operad of $V$ by endowing $V$ with an additional grading. Let $V$ be a complex with differential $Q : V^q \to V^{q-1}$ and a second grading called the filtration degree denoted by $V_p$ such that $V$ is now a filtered complex. The endomorphism operad $\mathcal{E}nd_V$ inherits a natural filtration $F_p \mathcal{E}nd_V = \{ F_p \mathcal{E}nd_V(n) \}$ where $F_p \mathcal{E}nd_V(n)$ is the space of maps in $\mathcal{E}nd_V(n)$ with filtration degree less than or equal to $p$. In this case, the spectral sequence associated to this filtration makes each $E^r$ term into an operad as well as the suboperads indicated above. Perhaps the simplest filtration of $V$ is when the filtration degree is exactly the degree in the complex. In this case, the associated spectral sequence degenerates at the $E^2$ term, where $E^0 \simeq \mathcal{E}nd_V$ with a zero differential, $E^1 \simeq \mathcal{E}nd_V$ with the differential $Q$, and $E^2 \simeq H(\mathcal{E}nd_V)$. 

\[\text{MODULI SPACES AND COMMUTATIVE HOMOTOPY ALGEBRAS 15}\]
Definition 5.2. Let $V$ be a vector space graded by a filtration degree as above. We say that an $O$-algebra, $V$, is a filtered $O$-algebra if the morphism of operads $O \to \text{End}_V$ preserves the filtrations.

If $O$ is an operad with filtration $F$ and $V$ a filtered $O$-algebra, then the $E^r$ term of $\text{End}_V$ is an algebra over the $E^r$ term of $O$ for all $r \geq 0$.

In the case of the operad $\overline{\mathcal{M}}$ with its canonical filtration where $V$ is a filtered $C_*(\overline{\mathcal{M}})$-algebra with the filtration degree on $V$ equal to the degree of the complex, then $V$ is an algebra (with zero differential) over the operad $E^0$, $V$ is an algebra (with $Q$ differential), over the operad $E^1$, and $H(V)$ is an algebra over $E^2$ (which is isomorphic to the operad $H_*(\overline{\mathcal{M}})$) where we have used the canonical morphism $H(\text{End}_V) \to \text{End}_{H(V)}$ in the last step.

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| Physical Label | Type of Algebra | Phys Reference | Operad | Math Reference |
|---------------|----------------|---------------|--------|----------------|
| TFT           | Commutative algebra | folklore | The space of diffeomorphism classes of $n$-holed Riemann spheres, the same as $\{C(n)\}$ | folklore |
| CFT           | (Nonholomorphic) VOA | Belavin-Polyakov-Zamolodchikov (BPZ) | The space $P(n)$ of conformal classes of $n$-holed spheres | Segal |
| Chiral CFT    | VOA | Same | Same | Huang-Lepowsky |
| N/A           | Gerstenhaber algebra | N/A | Homology of the little disks operad | F. Cohen |
| String background/TCFT | TVOA | BPZ, Dijkgraaf | Chain operad $C_\bullet(\mathcal{P}(n))$ | Lian-Zuckerman, KSV, Voronov |
| Antibracket formalism/TCFT on shell | BV algebra | Batalin-Vilkovisky, Witten, A. S. Schwarz | Homology framed little disks operad | Penkava-Schwarz, Getzler, Huang, Lian-Zuckerman, KSV |
| Topological gravity | WDVV algebra | WDVV, Dubrovin | Homology operad of the moduli spaces $\mathcal{M}_{g+1}$ | Kontsevich-Manin, Getzler |
| Topological sigma-model | Quantum cohomology ring | Gepner, Witten, Vafa | Same | Same + Ruan-Tian |
| N/A           | Gravity algebra | N/A | Homology operad of the moduli spaces $\mathcal{M}_{g+1}$ | Getzler |
| CSFT          | Homotopy Lie algebra | Witten-Zwiebach, Zwiebach | Homotopy Lie operad | Beilinson-Ginzburg, Ginzburg-Kapranov, Stasheff, Kimura-Stasheff-Voronov |
| Filtered topological gravity | Commutative homotopy $(C_\bullet)$ algebra | N/A | Commutative homotopy operad $\text{CobarLie}^c$ | Present paper |

Table 1. The references above may not be complete as they were chosen according to our tastes and knowledge of the subject.
Figure 1. An $N$-corolla, $\delta_N$

Figure 2. The composition map $\circ_1 : \overline{M}(3) \times \overline{M}(2) \to \overline{M}(4)$ and the trees indexing the strata to which they belong