Non-Abelian String-Net Ladders

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We study the string-net model with a string tension for non-Abelian particles in the ladder geometry. Focusing on the simplest non-Abelian theory with a total quantum dimension larger than two, we find evidence for a “russian doll” spectrum featuring size-independent energy levels. Contrary to Fibonacci and Ising ladders, the spectrum is found to be gapped at the self-dual points and low-energy excitations in the ground-state sector consist in two-quasiparticle bound states that are discussed in the weak-tension limit. We argue that such properties are prototypical of theories containing more than one non-Abelian particle.

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Topologically-ordered quantum phases of matter have triggered much attention over the last decades in different domains (see Ref. [1] for a review). In two dimensions, these phases host exotic excitations known as anyons [2] whose experimental observation remains one of the important challenges in condensed matter physics. Anyons are one of the key ingredients of topological quantum computation [3–5] and, in this perspective, it is crucial to understand their behavior as well as transitions they may induce [6–15]. To address these issues, many microscopic models have been proposed among which the string-net model [16] allows to generate a wide class of topological phases [17, 18] and to study their stability [19–25].

This model can be defined on any trivalent graph [16] so that the simplest system that can be studied is the two-leg ladder displayed in Fig. 1.严格 speaking, there is no topological order in gapped one-dimensional systems (due to absence of long-range entanglement [26, 27]). However, some mechanisms driving phase transitions in the so-called string-net ladder are also relevant in two dimensions [19]. So far, the string-net ladder has first been investigated for Fibonacci [19], Ising [20], and Yang-Lee anyons [21]. Surprisingly enough, in the presence of a string tension, these non-Abelian theories display a qualitatively similar phase diagram consisting of two gapped phases separated by a critical point and a critical phase. Nevertheless, one should keep in mind that all these theories contain only one non-Abelian anyon.

In this Letter, we study the string-net ladder for the simplest theory with two non-Abelian anyons and we show that it gives rise to completely different properties. In particular, the critical phase and the critical point observed for Fibonacci, Ising, Yang-Lee anyons are gapped for this theory. Moreover, we find a rich structure of the spectrum with some size-independent energy levels and highly-degenerate zero-energy eigenstates. Interestingly, we also show that non-Abelian bound-states emerge as low-energy excitations in the weak-tension limit. We argue that these features, absent for theories with a unique non-Abelian anyon, are prototypical of large quantum dimension anyonic theories.

Model and symmetries – In the string-net model, microscopic degrees of freedom are defined on edges of a trivalent graph. For a theory with $N$ anyons, each edge can be in $N$ different states. The Hilbert space $\mathcal{H}$ is then defined by the set of states compatible with the fusion rules at each vertex of the graph [16]. For any trivalent graph, the dimension of the Hilbert space can be expressed as a function of the fusion matrices of the theory considered [19].

The string-net Hamiltonian with a string tension is defined as

$$H = -J_p \sum_p \delta_{\Phi(p),0} - J_f \sum_r \delta_{\ell(r),0}. \quad (1)$$

The first term is the string-net Hamiltonian introduced by Levin and Wen [16] defined on a ladder. It involves the projector $\delta_{\Phi(p),0}$ onto states with no flux $\Phi(p)$ through
we set state 0, acting on the rung \( r \) plaquette the link (edge) basis since \( \delta \) basis is given in Ref. [19]. The second term is diagonal in \( \theta = \pi/2 \) and \( 3\pi/2 \) is the same as for \( \theta = 0 \) and \( \pi \), respectively. For periodic boundary conditions, self-duality is broken for finite-size systems but it is restored in the thermodynamical limit up to degeneracies. This discrepancy can readily be observed on the ground-state degeneracy which, for any modular theory with \( N \) anyons and for \( \theta = \pi/2 \), is \( N \)-fold (\( N^2 \)-fold) degenerate for twisted (periodic) boundary conditions. By contrast, for \( \theta = 0 \), the ground state is \( N \)-fold degenerate for twisted and for periodic boundary conditions.

For periodic boundary conditions considered in the following, \( H \) commutes with the translation operator along the ladder direction \( T_x \) and also with the reflection operator in the transverse direction \( T_y \). Interestingly, \( H \) also commutes with less obvious operators that measure the flux “above” and “below” the ladder. Indeed, a simple picture of the string-net model consists in representing each excitation by a flux-line piercing a plaquette (see Fig. 1). These flux lines are endless (no monopoles) so that they have to be connected according to the fusion rules. This physical representation allows one to compute level degeneracies for \( \theta = 0 \) [see Eq. (9)] but it also allows to identify two additional conserved quantities for any \( \theta \). To define the corresponding operators, it is useful to define the projectors onto flux \( \alpha \) “above” \((a)\) and “below” \((b)\) the ladder

\[
P^a(\alpha)|a,b,c\rangle = S_{1a} \sum_{\beta} S_{\alpha \beta} \prod_{i=1}^{N_p} \sum_{a_i^{'} = 1} \left[ F_{a_i^{'} a_i}^{c_i^{'} c_i} \right] |a_i^{'} b, c\rangle, \quad (2)
\]

\[
P^b(\alpha)|a,b,c\rangle = S_{1a} \sum_{\beta} S_{\alpha \beta} \prod_{i=1}^{N_p} \sum_{b_i^{'} = 1} \left[ F_{b_i^{'} b_i}^{c_i^{'} c_i} \right] |a, b_i^{'} c\rangle, \quad (3)
\]

where \( N_p \) is the number of plaquette and \(|a, b, c\rangle\) is a state defined by the set of labels \( \{a_i, b_i, c_i\}_{i=1,\ldots,N_p} \) on all edges (see Fig. 1). The S-matrix and F-symbols are given by the theory of interest (see, e.g., Ref. [28]). From these two sets of (orthogonal) projectors, one can define two operators measuring the flux above and below the ladder as \( Q_i = \sum_\alpha P_i(\alpha) \) with eigenvalues \( q_i = \alpha \) (here, \( i = a, b \)). There are two reasons for which \( P^a(\alpha) \), and hence \( Q_i \), commutes with \( H \): (i) by construction, any closed-string operator commutes with the string-net Hamiltonian [16] and (ii) \( P^a_i(\alpha) \) does not change the rung degrees of freedom \( c_i \) so that it commutes with the second term in \( H \). However, since these fluxes can be different “above” and “below” the ladder (see Fig. 1 for a concrete example), the corresponding operators do not commute with \( T_y \). Thus, there are three conserved quantities: the momentum along the chain direction and fluxes \((q_a, q_b)\). Note that another choice has been proposed in Refs. [19–21].

In the following, we focus on the simplest theory with more than one non-Abelian particle. This theory known as \((A_1, 5)_{1/2} \) [28, 29] contains three anyons 0, 1, and 2, obeying \( SU(2)_5 \) fusion rules restricted to integer labels

\[
0 \times a = a \times 0 = a, \quad \forall a \in \{0, 1, 2\}, \quad (4)
\]

\[
1 \times 1 = 0 + 1 + 2, \quad (5)
\]

\[
1 \times 2 = 2 \times 1 = 1 + 2, \quad (6)
\]

\[
2 \times 2 = 0 + 1. \quad (7)
\]

Setting \( d = 2 \cos(\pi/7) \), quantum dimensions of these anyons are \((d_0, d_1, d_2) = (1, d^2 - 1, d)\) so that the total quantum dimension is \( D = \sqrt{\sum_i d_i^2} = \sqrt{\pi/7} \). For a ladder with \( N_p \) plaquettes, the Hilbert space dimension is given by

\[
\dim \mathcal{H} = (−3d^2 + 2d + 9)^{N_p} + (d^2 - 3d + 4)^{N_p} + (2d^2 + d + 1)^{N_p}, \quad (8)
\]

which grows as \( D^{2N_p} \) in the thermodynamical limit.

By construction, for \( \theta = 0 \), the system is in a doubled \((A_1, 5)_{1/2} \) phase denoted by \( D(A_1, 5)_{1/2} \) in the following. As such, the degeneracy of the eigenstates depends on the graph topology. The degeneracy of the \( k \)-th energy level \( E_k = -N_p + k \) is [30]

\[
\mathcal{D}_k = \left( \binom{N_p}{k} \right) \times \left[ (−3d^2 + 2d + 8)^k + (d^2 - 3d + 3)^k + (2d^2 + d)^k \right], \quad (9)
\]

where the binomial coefficient stems from the different ways to choose \( k \) plaquettes carrying the flux excitations among \( N_p \). In particular, one finds \( \mathcal{D}_0 = 3 \) ground states and one can check that \( \dim \mathcal{H} = \sum_{k=0}^{N_p} \mathcal{D}_k \). For \( \theta = \pi \), the ground-state degeneracy is \( D_{N_p} \simeq (D^2 - 1)^{N_p} \) and the energy of the \( k \)-th energy level is given by \( E_k = k \).

Sketch of the phase diagram – To study the phase diagram we performed exact diagonalizations (ED) of the Hamiltonian for system sizes up to \( N_p = 9 \) for which \( \dim \mathcal{H} \simeq 5 \times 10^6 \). This diagram consists in three gapped phases separated by first-order transitions located at \( \theta = \pi/4, \pi \), and \( 3\pi/2 \) (see Fig. 1 for illustration). For \( \theta \in [3\pi/2, \pi/4] \) the system is in the \( D(A_1, 5)_{1/2} \) phase. Since the spectrum is self-dual, a dual phase \( D(A_1, 5)_{1/2}^* \) is obtained for \( \theta \in [\pi/4, \pi] \). But, the most intriguing result is the existence of a gapped phase (denoted by
Energy levels of the system size. As such, one can compute it exactly
δ of the corresponding plaquette operator (see Supp. Info. in Ref. [19] for details about this basis
switching from the “ladder” basis to the “bubble” basis
\[ M \]
(19) (red=2, green=1, and blue=0).

FIG. 2. (color online). Low-energy spectrum in the \( M \) phase. Size-independent levels are highlighted. The gap is given by
Eq. (10). Inset: a possible zero-energy ground state described in the “bubble” basis [19] (red=2, green=1, and blue=0).

\( M \) in the following) for \( \theta \in [\pi, 3\pi/2] \) instead of critical phases found in this range for other theories [19–21].

The \( M \) phase : \( \theta \in [\pi, 3\pi/2] \) – Unfortunately, the huge ground-state degeneracy at \( \theta = \pi \) and \( 3\pi/2 \) prevents to perform a perturbative analysis of the low-energy spectrum in this parameter range. However, as we shall now see, ED results unveil two striking properties.

\* Despite the fact that \( [\delta_{(r),0}, \delta_{(p),0}] \neq 0 \) if the rung \( r \) belongs to the plaquette \( p \), it is possible to build zero-energy eigenstates of these two non-commuting projectors. Such states are eigenstates for any \( \theta \) but, interestingly, they become ground states for \( \theta \in [\pi, 3\pi/2] \). A simple example of such states can be easily sketched by switching from the “ladder” basis to the “bubble” basis (see Supp. Info. in Ref. [19] for details about this basis transformation). In this representation, a bubble with different inner and outer legs is a zero-energy eigenstate of the corresponding plaquette operator \( \delta_{(p),0} \) whereas two neighboring bubbles of different type ensure a zero-energy eigenstate of the (shared-) link operator \( \delta_{(r),0} \). For the \( (A_1, 5)_{1/2} \) theory, these two constraints can easily be satisfied simultaneously (see Fig. 2). Nevertheless, we checked that such a construction does not exhaust all possible zero-energy states whose degeneracy is numerically found to grow exponentially with the system size. As a side remark, let us mention that we also find evidence for zero-energy eigenstates (stemming from a different construction) in the Ising ladder but with a degeneracy only growing linearly with the system size. Furthermore, such states are absent for Fibonacci anyons.

\* The low-energy gap \( \Delta \) shown in Fig. 2 is numerically found to be dependent of \( \theta \) but surprisingly independent of the system size. As such, one can compute it exactly by solving the \( N_p = 2 \) problem and one gets

\[
\Delta_\theta = -\frac{1}{2} \left[ \cos \theta + \sin \theta + \sqrt{1 + 2 \left( \frac{8}{D^2} - 1 \right) \cos \theta \sin \theta} \right].
\]

(10)

The corresponding eigenstates are all translation invariant and their degeneracy is numerically found to grow exponentially with the system size. Unfortunately, contrary to zero-energy states discussed above, we have not been able to find a simple procedure to build any of these low-energy states. This observation suggests that, for any \( \theta \in [0, 2\pi] \), there might exist other size-independent energy levels and this is indeed the case. For instance, we find that levels with energies \( -\cos \theta \) and \( -\sin \theta \) exist for all \( N_p \). This “russian doll” (Matrosikha) spectrum is an intriguing result that clearly deserves further investigations and a deeper understanding especially since such a structure does not occur for Fibonacci and Ising theories.

First-order transitions – To determine the nature of the transitions between the three phases described above, we computed numerically the derivative of the ground-state energy per plaquette with respect to \( \theta \) (see Ref. [29]). Even at finite sizes, it is discontinuous for \( \theta = \pi \) and \( 3\pi/2 \) indicating a first-order transition as is also the case for Fibonacci and Ising theories.

The analysis of the transition at the self-dual point \( \theta = \pi/4 \) is more subtle. However, at this point, as for Fibonacci and Ising theories, the Hamiltonian can be written as

\[
H = -\sqrt{\frac{7}{2}} D \sum e_i \text{ where } e_i \text{ operators form a representation of the Temperley-Lieb algebra}
\]

\[
e_i^2 = D e_i, e_i e_{i+1} e_i = e_i, [e_i, e_j] = 0 \text{ for } |i-j| > 2. \]

(11)

Derivation of this result is the same as for Fibonacci and Ising theories [19, 20] so that we do not reproduce it here. Using the correspondence between this Temperley-Lieb chain and the antiferromagnetic XXZ chain with an anisotropy parameter \( D/2 \) [31], one can compute the exact value of the gap in the thermodynamical limit [32]. Setting \( \cosh \Phi = D/2 \), one gets

\[
\Delta_{\pi/4} = \frac{\sqrt{2}}{D} \sinh \Phi \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\cosh(n\Phi)} \approx 0.045237. \]

(12)

and one concludes that \( \theta = \pi/4 \) is a first-order transition point. This is in stark contrast with Fibonacci and Ising theories for which \( D \leq 2 \). In this case, the spectrum for \( \theta = \pi/4 \) is critical and described by some conformal field theories [19, 20].

Similarly, the gap for the \( (A_1, 5)_{1/2} \) theory at \( \theta = 5\pi/4 \) is the same as in the ferromagnetic XXZ chain [33]:

\[
\Delta_{5\pi/4} = \frac{\sqrt{2}}{D} \left( \frac{D}{2} - 1 \right) \approx 0.243266. \]

(13)
in agreement with Eq. (10). Once again, this point is part of a gapless phase for Fibonacci and Ising theories.

To observe the gap $\Delta_{\pi/4}$ from ED, one must carefully analyze the spectrum in each topological sector and keep in mind that, with periodic boundary conditions, ground-state topological sectors in the $D(A_1, 5)_{1/2}$ phase are such that $q_a = q_b$. By contrast, in the $D(A_1, 5)_{1/2}$ phase there is one ground state in each sector so that the ground state is expected to be $(9+3)$-fold degenerate at the transition. In Fig. 3, we plot the first excitation energies in sectors $q_a = q_b$. As can be seen, the lowest one goes to zero whereas the second one converges to $\Delta_{\pi/4}$ in the thermodynamical limit.

Non-Abelian bound states – Focusing on the ground state and the first-excited state, we have seen that $(A_1, 5)_{1/2}$ anyons lead to a phase diagram very different from the one obtained for Fibonacci and Ising theories. To conclude this study, we would like to point out that important differences also occur at higher energies. More precisely, we perturbatively compute the low-energy spectrum in the ground-state sector $(q_a, q_b) = (0, 0)$ in the limit $|J_r| \ll J_p$. At order 0, these excitations consist in pairs of particles (1 or 2) and their energy is $\Delta' = 2 J_p$. At order 1, for $J_r > 0$, the low-energy gap in this sector is obtained by diagonalizing the two-quasiparticle problem and one gets

$$\Delta' \simeq 2 J_p - 0.843829 J_r.$$  \hspace{1cm} (14)

It must be compared with the one-quasiparticle (1 or 2) gap that is found in sectors such as $(q_a, q_b) = (1, 1)$ for instance, and reads at order 1

$$\Delta = J_p - \frac{2}{D^2} |J_r| \simeq J_p - 0.215149 |J_r|.$$  \hspace{1cm} (15)

Since $\Delta' < 2 \Delta$, one concludes that, for $J_r > 0$, elementary excitations are bound states made of pairs of quasiparticles (1 and 2). This is no longer the case for $J_r < 0$ where $\Delta' = 2 \Delta$ (see Fig. 4 for illustration). Note that for Fibonacci and Ising theories, no bound states are found near $\theta = 0$. To our knowledge, this is the first example of non-Abelian anyon bound states identified in a microscopic model (for Abelian bound states, see for instance Ref. [34]).

Outlook – To conclude, we would like to emphasize that some of the results obtained for the $(A_1, 5)_{1/2}$ theory are valid for many modular theories with, at least, two non-Abelian particles and $D > 2$. In particular, for anyons satisfying $SU(2)_{k>5}$ fusion rules, one can easily generalize the construction of zero-energy eigenstates and we conjecture that the gap at the self-dual points is always given by Eqs. (12-13). In the limit $|J_r| \ll J_p$, the one-quasiparticle gap expression given in Eq. (15) is valid for any $D$ but we did not find a general condition for the existence of bound states. We hope that the rich structure of the spectrum unveiled in the present work will stimulate further studies of non-Abelian string-nets.

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[1] X.-G. Wen, ISRN Condensed Matter Physics 2013, 198710 (2013).
[2] J. M. Leinaas and J. Myrheim, Il Nuovo Cimento B 37, 1 (1977).
[3] A. Y. Kitaev, Ann. Phys. 303, 2 (2003).
[4] See http://www.theory.caltech.edu/people/preskill/ph219/ for a pedagogical introduction.
[5] Z. Wang, Topological Quantum Computation, CBMS Re-
SUPPLEMENTAL MATERIAL

DATA OF THE \((A_1,5)_{1/2}\) UNITARY MODULAR TENSOR CATEGORY (UMTC)

Thereafter, we give all informations about the \((A_1,5)_{1/2}\) UMTC. Details about their derivation can be found in Ref. [28]. The \((A_1,5)_{1/2}\) UMTC is a rank-3 modular categories which means that it contains three types of anyons 0, 1, 2 (denoted by 1, β, α, respectively, in Ref. [28]).

Fusion rules

These anyons obey the following fusion rules
\begin{align}
0 \times a &= a \times 0 = a, \; \forall a \in \{0,1,2\}, \\
1 \times 1 &= 0 + 1 + 2, \\
1 \times 2 &= 2 \times 1 = 1 + 2, \\
2 \times 2 &= 0 + 1.
\end{align}

S-matrix

The modular S-matrix is given by [28]:
\begin{equation}
S = \frac{1}{D} \begin{pmatrix}
d_0 & d_1 & d_2 \\
d_1 & -d_2 & d_0 \\
d_2 & d_0 & -d_1
\end{pmatrix},
\end{equation}
where \((d_0, d_1, d_2) = (1, d^2 - 1, d)\) are the quantum dimensions of the anyons, and \(D = \sqrt{\sum_i d_i^4} = \frac{2\sin(\pi/7)}{\sqrt{2}}\) is the total quantum dimension. Indices of \(S\) are given by the anyon type and thus run here from 0 to 2.

F-symbols

The key ingredient to build the string-net Hamiltonian is the set of F-symbols [16, 19]. These F-symbols are defined as follows:
\begin{equation}
\alpha \beta \gamma \delta = \sum_f \left[ F_{\alpha \beta \gamma}^{\delta f} \right]_{ef} \alpha \beta \gamma \delta.
\end{equation}

General formula for any SU(2)k theory can be found in Ref. [35] but we give them here explicitly for the case \(k = 5\) which is relevant for our purpose. For \(k = 5\) and considering only integer labels of SU(2)k, there are several nontrivial F- symbols which can be written in the following matrix form
\begin{equation}
F_1^{111} = \frac{1}{d_1^3} \begin{pmatrix}
1 & -\sqrt{d_1} & \sqrt{d_1} \\
-\sqrt{d_1} & 1/d_1 & \sqrt{d_1} \\
\sqrt{d_1} & \sqrt{d_1} & d_1/d_1
\end{pmatrix},
\end{equation}
\[
F_{2}^{111} = \frac{1}{d_1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -d_2 & \sqrt{d_2} \\ 0 & \sqrt{d_2} & d_2 \end{pmatrix},
\]
\[
F_{2}^{112} = \frac{1}{d_1} \frac{1}{d_2} \begin{pmatrix} 0 & \sqrt{d_1} & -\sqrt{d_1} \\ 0 & -\sqrt{d_2} & -\sqrt{d_2} \\ 0 & \sqrt{d_2} & \sqrt{d_2} \end{pmatrix},
\]
\[
F_{2}^{121} = \frac{1}{d_1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sqrt{d_1 d_2} & -1 \end{pmatrix},
\]
\[
F_{1}^{122} = \frac{1}{d_1} \frac{1}{d_2} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{d_1} & 0 & 0 \\ -\sqrt{d_2} & 0 & -\sqrt{d_2} \end{pmatrix},
\]
\[
F_{2}^{222} = \frac{1}{d_2} \begin{pmatrix} 1 & 0 \ -\sqrt{d_1} & 0 \ 0 & 0 \ -\sqrt{d_1} & 0 \ 0 & 1 \ 0 & 0 \ \end{pmatrix}. \]

As for the S-matrix, indices of these matrices run from 0 to 2. Other F-symbols are equal to 1 if fusion channels are allowed and 0 otherwise. Missing symbols can be found by using the following identities:
\[
F_{d}^{abc} = F_{a}^{dcb} = F_{c}^{bad} = F_{b}^{cda}. \tag{28}
\]

**R-symbols**

The R-symbols are defined as follows:
\[
\begin{array}{ccc}
\text{a} & \text{b} & \text{c} \\
\text{b} & \text{c} & \text{a} \\
\text{a} & \text{b} & \text{c} \\
\end{array} = R_{c}^{ab} \begin{array}{ccc}
\text{b} & \text{c} \\
\text{a} & \text{b} \\
\text{c} & \text{a} \\
\end{array}. \tag{29}
\]

Note that the R-symbols are only non-zero for vertices obeying the fusion rules. As braiding with the vacuum is trivial, one has:
\[
R_{a}^{0a} = R_{a}^{a0} = 1, \quad \forall a. \tag{30}
\]

Other R-symbols are given by:
\[
R_{0}^{11} = e^{-\frac{4i\pi}{3}}, R_{0}^{22} = e^{\frac{2i\pi}{3}}, \tag{31}
\]
\[
R_{1}^{11} = e^{\frac{2i\pi}{3}}, R_{1}^{12} = e^{-\frac{i\pi}{3}}, R_{1}^{21} = e^{-\frac{i\pi}{3}}, R_{1}^{22} = e^{-\frac{3i\pi}{2}}, \tag{32}
\]
\[
R_{2}^{11} = e^{\frac{2i\pi}{3}}, R_{2}^{12} = e^{\frac{i\pi}{3}}, R_{2}^{21} = e^{\frac{i\pi}{3}}. \tag{33}
\]

Note that these R-symbols are the complex conjugates of the one given in Ref. [28] where a typo for \(R_{\beta}^{a0}\) must be corrected.

Finally, we would like to mention that this set of F-symbols and R symbols can be changed into a new set \(\tilde{F}\) and \(\tilde{R}\) according to the following gauge transformation (see, e.g., [36])
\[
[\tilde{F}^{abc}]_{ef} = u_{a}^{af} u_{b}^{bf} [F^{abc}]_{ef}, \quad \tilde{R}_{c}^{ab} = u_{c}^{ba} R_{c}^{ab}, \tag{34}
\]
where \(u\)'s are complex numbers. In order to exactly map our Hamiltonian onto a Temperley-Lieb chain Hamiltonian at the self-dual points \(\theta = \pi/4, 5\pi/4\) one needs to choose:
\[
u_{ij}^{11} = u_{ij}^{12} = u_{ij}^{21} = 1, \tag{35}
\]
and \(u_{k}^{ij} = 1\) otherwise.

**EXACT DIAGONALIZATION RESULTS**

To determine the phase diagram, we computed the ground-state energy per plaquette \(e_{0}\) and its derivative with respect to \(\theta\) for various system sizes. As can be seen in Fig. 5, \(\partial_{\theta} e_{0}\) is discontinuous for \(\theta = \pi\) and \(3\pi/2\) indicating a level crossing and hence first-order transitions. For \(\theta = \pi/4\) one clearly sees the emergence of a possible transition but one cannot straightforwardly infer its nature from numerics. However, at this point, the mapping onto the Temperley-Lieb chain allows one to conclude that it is also a first-order transition.