Measuring Black Hole Formations by Entanglement Entropy via Coarse-graining

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Abstract

We argue that the entanglement entropy offers us a useful coarse-grained entropy in time-dependent AdS/CFT. We show that the total von-Neumann entropy remains vanishing even when a black hole is created in a gravity dual, being consistent with the fact that its corresponding CFT is described by a time-dependent pure state. We analytically calculate the time evolution of entanglement entropy for a free Dirac fermion on a circle following a quantum quench. This is interpreted as a toy holographic dual of black hole creations and annihilations. It is manifestly free from the black hole information problem.

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1 Introduction

The concept of holography, especially the AdS/CFT correspondence, has now been the most powerful tool to analyze quantum aspects of gravity [1–3]. Since there have been no satisfactory and non-perturbative formulations of string theory, the direct calculations of quantum gravity effects often turn out to be quite difficult. However, the AdS/CFT equivalently transforms the problems into those in non-gravitational theories, typically large $N$ gauge theories [3]. Since the latter ones are non-perturbatively well defined, we can in principle understand all quantum gravity effects at least in AdS spaces.

In spite of enormous related developments, the AdS/CFT in time-dependent back-grounds have not been well understood at present. One of purposes of this paper is to understand better somewhat puzzling aspects of entropy in time-dependent backgrounds, which is also closely related to the black hole information problem [4]. If we suddenly excite a CFT, it will eventually thermalize after a certain time. In AdS/CFT, this is dual to a black hole formation in AdS spaces [5,6]. In a probe D-brane approximation, a thermalization is dual to a black hole formation in the induced metric on the brane [8].

In such a process, it is clear that the density matrix of the system is described by a time-dependent pure state in the CFT side, assuming that we start with a pure state before the thermalization. Therefore, the von-Neumann entropy is always vanishing. On the other hand, in the AdS gravity, since the black hole will be formed and the spacetime at late time is well approximated by the AdS Schwarzschild black hole, the entropy seems non-vanishing. For example, it is clear that the entropy which appears in the fluid description gets non-vanishing [9]. This might suggest an evolution of a pure state into a mixed state, which contradicts with the principle of quantum mechanics. To resolve this issue, we need to distinguish the microscopic von-Neumann entropy from a coarse-grained entropy. Only the latter can get non-vanishing in this thermalization process. We will argue that the entanglement entropy [10–12] offers us a convenient coarse-grained entropy in holographic setups. At the same time, we will show that the von-Neumann entropy of the total system is always vanishing in the black hole formation process employing the holographic calculation of the entanglement entropy [13–15].

From the CFT viewpoint, the black hole formation process is dual to a problem of non-equilibrium dynamics and therefore is quite complicated to analyze explicitly in general. Fortunately, recently there have been many progresses in understanding of properties of quantum field theories after thermalizations induced by instantaneous excitations called quantum quenches [16–22]. We will consider a two dimensional CFT defined by a free Dirac fermion on a circle and calculate the time-evolution of the entanglement entropy.
and correlation functions after a quantum quench, taking into account the finite size effect. Indeed, the behavior of the entanglement entropy we obtained implies that its gravity dual is supposed to be successive creations and annihilations of black holes in AdS spaces, though a free field theory in general corresponds to an extremely stringy region of the gravity side. This solvable example offers us a CFT description of a black hole formation and decay, which are manifestly free from the black hole information problem, because the system is clearly described by a pure state. Notice that in this problem, the standard thermal entropy is not useful as it is vanishing and the entanglement entropy is an alternative important quantity which can directly probe black holes.

We would also like to mention that quite recently, remarkable numerical calculations of time evolutions of entanglement entropy have been done holographically in [23, 24]. When we essentially finished calculations for this paper, we found the paper [23], which has a partial but important overlap with ours about the argument which shows unitary evolutions of CFTs dual to the black hole formations by using the holographic entanglement entropy. A similar issue has also been discussed independently in the appendix of the paper [24].

The paper is organized as follows: In section two, we will explain a puzzle on the entropy in time-dependent backgrounds and resolve it by employing the entanglement entropy as a coarse-grained entropy. In section three, we will calculate the time evolving entanglement entropy in two dimensional CFT of a free Dirac fermion with finite size effect. We will interpret our results from the gravity dual viewpoint and relate them to the black hole creations and annihilations. In section four, we will study one and two point functions in our time-dependent CFT and confirm that they agree with our holographic interpretation. In section five, we will summarize our conclusion. In the appendix A, we present detailed calculations of two point functions in a free Dirac fermion theory on a cylinder using the boundary state formalism.

2 Holographic Entanglement Entropy as Coarse-grained Entropy

2.1 An Entropy Puzzle about Time-dependent Holography

Consider holography in time-dependent backgrounds, especially the ones in AdS/CFT. We are particularly interested in a situation where a black hole horizon is created by a gravitational collapse. Such a time-dependent process has been considered to be dual to
thermalizations and indeed we can construct explicit examples in AdS/CFT setups \[5,6\], where non-normalizable perturbations lead to a creation of hot plasma fluid in the dual Yang-Mills theory. Similar phenomena also occur in the probe D-brane setups regarding the thermalization as a formation of horizon in the induced metric \[8\].

An important characteristic physical quantity in time-dependent systems will be the entropy. Since the black hole is created after a certain time in our gravity background, it is quite natural that the entropy which is holographically calculated, gets non-vanishing at late time. However, in the dual CFT, the time evolution is described by a time-dependent external force in quantum many-body systems and thus the density matrix $\rho_{tot}(t)$ of the total system will evolve in a unitary way $\rho(t) = U(t,t_0)\rho(t_0)U(t,t_0)^{-1}$. Since the entropy of the total system is defined by the von-Neumann entropy

$$S_{tot}(t) = -\text{Tr}[\rho(t) \log \rho(t)] = S_{tot}(t_0),$$

which is clearly time-independent. Therefore if we start with a pure state, then the entropy is always vanishing at any time, even if the black hole horizon appears in the dual geometry. This observation seems to lead to a contradiction between the gravity and CFT at first sight.

Another confusing point is that there is no known unambiguous definition of entropy $S_{tot}$ in the time-dependent gravity duals. This is partly because there are two different notions of horizons: event horizon and apparent horizon. Currently, there are several evidences that the latter will be more appropriate to define entropy \[14,25\]. However, still we need to specify the choice of time slices to calculate the apparent horizon and thus there are infinitely many different definitions. Remember that in static spacetimes, the time slice is uniquely chosen and there is no ambiguity.

A closely related issue in the dual CFT side is that we can define a non-vanishing entropy even for a pure state if we perform a coarse-graining of the given total system. If the total system consists of a gas and a heat bath, then the entropy of gap is obtained by tracing out the heat bath and by the required coarse-graining due the fact that our observables are quite restricted compared with the microscopic degrees of freedom; see e.g. \[26\] for relevant quantum field theoretic formulations. Again this leads to infinitely many different definitions of (coarse-grained) entropy. Below we would like to resolve the above puzzle by paying attention to this coarse-graining procedure.

The resolution of this puzzle is also closely related to that of the black hole information loss problem. This problem occurs since a massive object described by a pure state collapses into a black hole and finally ends up with a thermal gas generated by Hawking radiations, which looks like a mixed state. If this is true, a pure system has to evolve into
a mixed state after enough time and clearly contradicts with the principle of quantum mechanics.

2.2 Entanglement Entropy as Coarse-grained Entropy

Typically the coarse-graining is done by cutting out higher energy modes or higher order multi-particle interactions. However, this does not seem to allow straightforward calculations in AdS/CFT setup. Therefore, here we would like to perform a coarse-graining by cutting out a spacial part of the total system. In this case, the obtained entropy coincides with the quantity called entanglement entropy.

The entanglement entropy \( S_A(t) \) is defined for each subsystem \( A(t) \) of the total system. In quantum field theories, \( A(t) \) is specified by diving the total space at a fixed time \( t \) into two parts named as \( A(t) \) and \( B(t) \). Thus the number of choices of \( A \) are obviously infinite. The total Hilbert space \( H_{tot} \) now becomes a direct product \( H_{tot} = H_A \otimes H_B \), where the subsystem \( B \) is the complement of \( A \). The reduced density matrix for \( A \) at any time \( t \) is given by

\[
\rho_A(t) = \text{Tr}_{H_B} \rho_{tot}(t). \tag{2.2}
\]

Finally we define the entanglement entropy as follows

\[
S_A(t) = -\text{Tr}[\rho_A(t) \log \rho_A(t)]. \tag{2.3}
\]

This quantity is in general non-vanishing and a non-trivial function of time even for pure states.

We can calculate \( S_A(t) \) via AdS/CFT by applying the holographic formula proposed in [14], which generalizes the holographic formula in static spacetime found in [13]. It is simply given by

\[
S_A(t) = \frac{\text{Area}(\gamma_A(t))}{4G_N}, \tag{2.4}
\]

where \( G_N \) is the Newton constant of the AdS space. The surface \( \gamma_A(t) \) is a codimension two surface in the AdS space which is defined by the extremal surface whose boundary coincides with the boundary of \( A(t) \). We require that \( \gamma_A(t) \) is homotopic to \( A(t) \). If there are more than one such extremal surfaces, we pick up the one with the lowest area, as is required by the strong subadditivity of entanglement entropy [27]. Note that here we need to directly deal with Lorentzian spacetime without its Euclidean counterpart because it is time-dependent. Indeed, in [14], it was confirmed that such extremal surfaces are well-defined in important examples. Recently, this formula is applied to the AdS\(_3\) Vaidya background and the conformal field theory results in [16] have been remarkably reproduced.
in the work [23]. Moreover, this calculation has been extended to the higher dimensional cases in [24].

For our purpose, it is convenient to study a $d + 1$ dimensional time-dependent background which is asymptotically global AdS$_{d+1}$ space, which includes a finite size effect because its boundary is $R \times S^{d-1}$. We assume that a black hole is produced at a certain time via a gravitational collapse and this process lasts only for a finite time. Our argument below is heuristic and does not depend on the detailed form of the time-dependent solution.

We calculate $S_A$ at a time $t$ when the black hole formation has been ended and when the spacetime is well approximated by a static AdS black hole. First we assume the length size of $A$ (denoted by $|A|$ below) is smaller than the inverse temperature $\beta_{BH}$ of the AdS black hole temperature. In this case, the extremal surface $\gamma_A(t)$ is localized near the boundary and surrounds the region $A(t)$ [13]. $S_A$ essentially consists of the area law divergent piece

$$S_A \sim \frac{\text{Area}(\partial A)}{a^{d-2}} + \ldots,$$

which are independent of the temperature. Notice that the condition $|A| < \beta_{BH}$ assumed here physically means that the system is heavily coarse-grained. As $|A|$ exceeds $\beta_{BH}$, $S_A$ gains a finite and extensive contribution, which is essentially the thermal entropy for the subsystem $A$, in addition to the divergent part $S_{\text{div}}$ given by (2.5). This thermal contribution comes from the part of the surface $\gamma_A(t)$ which is wrapped on a part of the apparent horizon of the AdS black hole and thus is proportional to $|A|$ as in Fig.1 see [7][14][23][24] for explicit confirmations. An important fact is that though the apparent horizon requires the choice of the time slice, our extremal surface $\gamma_A(t)$ is uniquely determined by the boundary condition.

When this reaches the point $|A| = |B|$, the situation begins to change importantly. If we naively speculate the surface $\gamma_A(t)$ in a continuous way until $|A|$ exceed $|B|$, we may think that it will wrap on a more than half of the apparent horizon as in the right-up figure of Fig.1. However, since the horizon disappears at an earlier time, we can smoothly deform this surface into the one which wraps the opposite part of the horizon. Therefore the result of $S_A$ as a function of $|A|$ becomes symmetric with respect to the point $|A| = |B|$ as in the right-down figure of Fig.1. In other words, we can conclude that in this time-dependent background the entanglement entropy satisfies

$$S_A = S_B,$$

which is indeed known to be true when the total system is a pure state and thus agrees with our holographic setup. This sudden decreasing of the entanglement entropy occurs
because the subsystem $B$ which we are tracing out gets smaller than the half of the total system and the information lost by the coarse-graining starts recovering.\footnote{A similar behavior has been known in the analysis of quantum information in evaporating black holes \cite{28}. This has been considered to be crucial to resolve the information paradox of quantum black holes because the information may be recovered after more than half of a black hole has been evaporated.}

In the end, we can holographically calculate the von-Neumann entropy for the total system as follows

$$S_{\text{tot}} = \lim_{|B| \to 0} (S_A - S_B) = 0. \quad (2.7)$$

Thus, in the gravity side, we confirmed that the total entropy is always vanishing in spite of the black hole formation. As is clear from the previous argument, the essential point is that the horizon is time-dependent and gets vanishing at earlier time and the minimal area principle (among extremal surfaces) prefers the minimum i.e. $S_{\text{tot}} = 0$ at any time. In summary, in the time-dependent background which describes a thermalization of a pure state, the von-Neumann entropy of the total system is indeed vanishing also in the gravity side. This contrasts strikingly with the setup of an eternal AdS black hole \cite{29} as it is dual to a mixed state in a thermal CFT and has a non-vanishing thermal entropy, which precisely coincides with the value $\lim_{|B| \to 0} (S_A - S_B)$ \cite{30}.

The non-zero entropy is obtained only after a certain coarse-graining. Among various such entropies defined in the dual CFT, only the entanglement entropy has its clear holographic dual in time-dependent backgrounds at present. From this viewpoint, we can define a coarse-grained effective entropy $S_{\text{eff}}$ which looks analogous to the thermal entropy by

$$S_{\text{eff}} = 2 \left( S_A - S_A^{(0)} \right) \bigg|_{|A|=|B|}. \quad (2.8)$$

We subtracted the entanglement entropy $S_A^{(0)}$ before the thermalization. This clearly includes the divergent part $S_{\text{div}}$ given by (2.5) and therefore $S_{\text{eff}}$ is finite. The reason why we put the factor two is $A$ covers only a half of the total system. We would like to argue that (2.8) is a definition of coarse-grained entropy which is uniquely calculable both in gravity and CFT side of the AdS/CFT. Notice that though the calculation of entropy from an apparent horizon requires the choice of time slice and is ambiguous, our entropy $S_{\text{eff}}$ has no ambiguity as the extremal surface condition of $\gamma_A(t)$ determined where the surface wraps the apparent horizon.

Finally, we would like to come back to the relation to the black hole information problem \cite{4}. From the above argument, it is clear that there is no information loss in the gravitational collapsing process because $S_{\text{tot}} = 0$ is always satisfied. However, we cannot study the most important process of the decay due to Hawking radiations because
the absence of the holographic formula of entanglement entropy beyond the supergravity approximation. Therefore, for this purpose, it is a better idea instead to study its CFT dual, directly. We will analytically investigate such an example in the next section.

3 Time-dependent Entanglement Entropy in 2D CFT with finite size via Quantum Quench

In the previous section, we studied the behavior of coarse-grained entropy by using the holographic entanglement entropy. In this section, we would like to complete the argument by presenting an explicit calculations with a finite size effect in two dimensional CFT and by showing that the results agree with the holographic ones. We would like to analyze how the entanglement entropy evolves in time-dependent backgrounds which describe the thermalization phenomena. At the same time, this offers us a toy gravity dual of a black hole formation and evaporation, where there is no information loss problem and where we can solve the model exactly.

We will particularly look at a specific class of time-dependent system which is called quantum quench, whose field theory descriptions have been given in [16–22]. The quantum
quench is a phenomenon when we suddenly change parameters of a given system such as masses and coupling constants. The evolutions of entanglement entropy under quantum quenches have been studied in [16, 19, 20] for infinitely extended systems.

To make an exact calculation possible, we consider a free Dirac fermion theory in two dimension. In free field theories, the thermalization via the quantum quench is not a standard one described by the (grand-)canonical ensemble because the different momentum modes are decoupled and do not mix [18, 21]. Notice also that the free CFT corresponds to string theory with a lot of quantum corrections via the AdS/CFT. Nevertheless, this system is enough for our purpose qualitatively as we will see later. As far as the authors know, our result will be the first analytical calculation of time-dependent entanglement entropy with a finite size effect. The calculation of entanglement entropy at finite temperature and with finite size has been calculated in [30] for a free Dirac fermion.

3.1 How to Calculate Entanglement Entropy under Quantum Quench

The setup of quantum quench that we have in mind is as follows. Start with a Dirac fermion theory with a mass \( m \) in two dimension. Then at the time \( t = 0 \), suddenly we turn off the fermion mass. Then the system after this quantum quench i.e. \( t > 0 \) will be described by a time-dependent background in the \( c = 1 \) 2D CFT defined by a massless Dirac fermion whose Hamiltonian is written as \( H \). If we denote the state just after the quench by \( |\Psi_0\rangle \), then the density matrix at a time \( t > 0 \) is given by

\[
\rho_{\text{tot}}(t) = e^{-itH} |\Psi_0\rangle \langle \Psi_0| e^{itH}.
\]  

(3.9)

To find \( |\Psi_0\rangle \) we would like to follow the prescription in [16]. We expect that in the infrared limit \( |\Psi_0\rangle \) should flow into a boundary fixed point. Therefore we can approximate \( |\Psi_0\rangle \) by a boundary state \( |B\rangle \) with the UV modes filtered out

\[
|\Psi_0\rangle \simeq e^{-\epsilon H} |B\rangle,
\]  

(3.10)

where \( \epsilon \) is the parameter of UV filter and is inversely proportional to the original fermion mass \( \epsilon \sim m^{-1} \). If we denote the UV cut off or lattice spacing of this field theory by \( a_{\text{UV}} \), we need to take \( \epsilon >> a_{\text{UV}} \), which is clearly satisfied in the continuum limit \( a_{\text{UV}} \to 0 \).

In this way we find that the reduced density matrix is eventually written as

\[
\rho_{\text{tot}}(t) = e^{-itH-\epsilon H} |B\rangle \langle B| e^{itH-\epsilon H}.
\]  

(3.11)
In the path-integral description [16], the spacetime where the Dirac fermion lives is the cylinder geometry given by the direct product of an interval (i.e. Euclidean time) and a circle (i.e. space). The length of the interval is $2\epsilon$ and the one of the circle is normalized into $2\pi$. We introduce a complex coordinate $(y, \bar{y})$ for this spacetime such that
\begin{equation}
    y = \tau - i\sigma, \quad \bar{y} = \tau + i\sigma, \quad (0 \leq \tau \leq 2\epsilon, \quad 0 \leq \sigma \leq 2\pi). \tag{3.12}
\end{equation}

The Dirac fermion $(\psi, \bar{\psi})$ on this spacetime is decomposed into the left-moving part $(\psi_L(y), \bar{\psi}_L(y))$ and the right-moving part $(\psi_R(\bar{y}), \bar{\psi}_R(\bar{y}))$.

We define the subsystem $A(t)$ as an interval in the $\text{Im}y$ direction at time $\epsilon + it$. We can parameterize the location of the two ends points of this interval as
\begin{equation}
    (y_1, \bar{y}_1) = (\epsilon + it + i\sigma_1, \epsilon + it - i\sigma_1), \quad \text{and} \quad (y_2, \bar{y}_2) = (\epsilon + it + i\sigma_2, \epsilon + it - i\sigma_2). \tag{3.13}
\end{equation}

In the path-integral description [16, 35], the reduced density matrix $\rho_A(t)$ is obtained from $\rho_{\text{tot}}(t)$ (3.11) by gluing the subsystem $B$ and the two boundaries on the subsystem $A$ are left as they are (see the left figure in Fig.2). Then the important quantity $\text{Tr}[\rho_A(t)^N]$ can be found as the partition function of the free Dirac fermion on a $N$-sheeted Riemann surface $\Sigma_N$ [35] as shown in the right figure in Fig.2. $\Sigma_N$ is defined by the $N$ copies of the cylinders (3.12) which are glued successively along the subsystem $A$. This quantity is used to define so called the Renyi entropy. Recently, remarkable properties of Renyi entropy and their consistency with holographic results have recently been studied in [31].

In this way, the entanglement entropy can be computed via
\begin{equation}
    S_A(t) = -\frac{\partial}{\partial N} \log \text{Tr}[\rho_A(t)^N]|_{N=1}. \tag{3.14}
\end{equation}

By employing the replica trick, we can regard the system is given by $N$ free Dirac fermions $(\psi^{(a)}, \bar{\psi}^{(a)})$ $a = 0, 1, \cdots, N-1 \in Z_N$ on a single cylinder with the twisted boundary conditions at the two points $\psi^{(a)}_L(e^{2\pi i}y_1) = \psi^{(a+1)}_L(y_1)$ and $\psi^{(a)}_L(e^{2\pi i}y_2) = \psi^{(a-1)}_L(y_2)$. Since the lagrangian is gaussian, its form remains unchanged even after we perform the discrete Fourier transformation $\psi^{(a)} \rightarrow \frac{1}{\sqrt{N}} \sum_{b=0}^{N-1} e^{-2\pi iab/N} \psi^{(b)}$ [32]. The interactions between $N$ fermions only appear through the twisted boundary condition which are now diagonalized as follows
\begin{equation}
    \psi^{(a)}_L(e^{2\pi i}y_1) = e^{\frac{2\pi ia}{N}} \psi^{(a)}_L(y_1), \quad \psi^{(a)}_L(e^{2\pi i}y_2) = e^{-\frac{2\pi ia}{N}} \psi^{(a)}_L(y_2). \tag{3.15}
\end{equation}

These boundary conditions can be equally replaced with the twisted sector vertex operators $\sigma^{(a)}$ with the lowest conformal dimensions. Finally, we can calculate the entanglement
The path-integral calculation of the reduced density matrix $[\rho_A]_{\alpha\beta}$ (left) and the trace $\text{Tr}[\rho_A^n]$ (right).

entropy from the formula

$$\text{Tr}[\rho_A(t)^N] = \prod_{a=-N}^{N-1} \langle \sigma^{(a)}(y_1, \bar{y}_1)\sigma^{(-a)}(y_2, \bar{y}_2) \rangle_{\text{cylinder}}$$

$$= \prod_{a=-N}^{N-1} \frac{\langle B|e^{-2\epsilon H}\sigma^{(a)}(y_1, \bar{y}_1)\sigma^{(-a)}(y_2, \bar{y}_2)|B \rangle}{\langle B|e^{-2\epsilon H}|B \rangle}.$$ (3.16)

Here we normalized the two point functions such that $\text{Tr}[\rho_A(t)] = 1$.

### 3.2 Bosonization and Two Point Functions on Cylinder

A useful fact of the free massless Dirac fermion $(\psi, \bar{\psi})$ is that it can be bosonized into a free scalar $X$ via the standard formula

$$\psi_L(y) = e^{iX_L(y)}, \quad \bar{\psi}_L(y) = e^{-iX_L(y)}, \quad \psi_R(\bar{y}) = e^{iX_R(\bar{y})}, \quad \bar{\psi}_R(\bar{y}) = e^{-iX_R(\bar{y})}. \quad (3.17)$$

Our normalization of the two dimensional CFT in this paper is $\alpha' = 2$ in string theory world-sheet and the free Dirac fermion is equivalent to the compact boson $X$ at the radius $R = 1$. If we write the primary fields as $V_{(k_L, k_R)}(y, \bar{y}) = e^{ik_LX_L(y)+ik_RX_R(\bar{y})}$, then there are two candidates of the twisted vertex operators, which lead to the twisted boundary conditions (3.15),

$$\sigma_1^{(a)}(y, \bar{y}) = V_{(a, -a)}^{(a)}(y, \bar{y}) = e^{i\frac{2\pi}{N}(X_L(y)-X_R(\bar{y}))}, \quad (3.18)$$

$$\sigma_2^{(a)}(y, \bar{y}) = V_{(0, a)}^{(a)}(y, \bar{y}) = e^{i\frac{2\pi}{N}(X_L(y)+X_R(\bar{y}))}, \quad (3.19)$$
which have the lowest dimensions if \( a \) runs \( a = -\frac{N-1}{2}, -\frac{N-3}{2}, \ldots, \frac{N-1}{2} \) [30].

What we need to calculate to obtain the entanglement entropy are the two point functions of (3.18) or (3.19) on the cylinder. This can be found by performing calculations using the explicit form of the boundary state \(|\mathcal{B}\rangle\) (see e.g. [43]) as summarized in the appendix A. One may think that there are two choices i.e. the Neumann and Dirichlet boundary condition in the free scalar theory. If we return to the Dirac fermion theory, the Neumann boundary condition relates \( \psi_L \) and \( \bar{\psi}_L \) to \( \psi_R \) and \( \bar{\psi}_R \) at the boundary, while the Dirichlet one does \( \psi_{L,R} \) to \( \psi_{R,L} \), respectively. We can see that the discrete Fourier transformation, employed to map the replicated fermions into the new ones which satisfy (3.15), is consistent only when we choose \( \sigma_1 = \sigma_2 \) or \( \sigma_2 = \sigma_1 \) for the Neumann or Dirichlet boundary condition, respectively.

The final result of the normalized two point functions in both Neumann and Dirichlet is given by the same expression

\[
\langle \sigma^{(a)}(y_1, \bar{y}_1)\sigma^{(-a)}(y_2, \bar{y}_2) \rangle = \left( \eta(\frac{24\pi}{\pi})^6 \cdot \frac{\theta_1\left(\frac{\pi}{\pi} + \frac{\sigma}{2\pi} \frac{2\pi}{\pi}\right)||\theta_1\left(\frac{\pi}{\pi} - \frac{\sigma}{2\pi} \frac{2\pi}{\pi}\right)|}{\theta_1\left(\frac{\pi}{\pi} \frac{2\pi}{\pi}\right)||\theta_1\left(\frac{\pi}{\pi} \frac{2\pi}{\pi}\right)|} \right)^2, \quad (3.20)
\]

where we defined \( \sigma \equiv \sigma_2 - \sigma_1 \). Refer to appendix A for a derivation of this result (3.20). About the definition of the eta function \( \eta(\tau) \) and theta function \( \theta_1(\nu|\tau) \) we followed the convention in [33].

### 3.3 Time Evolution of Entanglement Entropy with Finite Size

By substituting the two point functions (3.20) into (3.16) and (3.14), we can calculate the entanglement entropy \( S_A(t) \). Using the formula

\[
\sum_{a=\frac{-N-1}{2}}^{\frac{N-1}{2}} \frac{a^2}{N^2} = \frac{1}{12} \left( N - \frac{1}{N} \right),
\]

we obtain the entanglement entropy

\[
S_A(t, \sigma) = \frac{1}{6} \log \frac{\theta_1\left(\frac{\pi}{\pi} \frac{2\pi}{\pi}\right)||\theta_1\left(\frac{\pi}{\pi} - \frac{\sigma}{2\pi} \frac{2\pi}{\pi}\right)|}{\theta_1\left(\frac{\pi}{\pi} \frac{2\pi}{\pi}\right)||\theta_1\left(\frac{\pi}{\pi} - \frac{\sigma}{2\pi} \frac{2\pi}{\pi}\right)|} \cdot a_{UV}^2, \quad (3.22)
\]

where we recovered the cut off \( a_{UV} \) dependence in an obvious way. Notice that this is independent from the choice of the boundary condition (Neumann or Dirichlet one). From this it is clear that \( S_A(t) \) satisfies

\[
S_A(t, \sigma) = S_A(t, 2\pi - \sigma) = S_B(t, \sigma). \quad (3.23)
\]
This is consistent with the assumption that the total system is a pure state and the von-Neumann entropy for the total system is vanishing.

After the modular transformation, we can rewrite (3.22) as follows

$$S_A(t, \sigma) = \frac{1}{3} \log \frac{2 \epsilon}{\pi a_{UV}} + \frac{1}{6} \log \frac{\epsilon}{\eta(\frac{\pi t}{2 \epsilon})^2} \cdot \frac{\theta_1(\frac{\pi i |\frac{\pi t}{2 \epsilon}|}{2}) \cdot \theta_1(\frac{\pi i |\frac{\pi t}{2 \epsilon} + \frac{\pi i \sigma}{2}|}{2})}{\theta_1(\frac{\pi i |\frac{\pi t}{2 \epsilon} - \frac{\pi i \sigma}{2}|}{2})}. \quad (3.24)$$

The explicit form of $S_A(t, \sigma)$ is plotted in Fig.3 as a function of $\sigma$ and Fig.4 as a function of $t$. In the limit $\epsilon \to \infty$, where the quench disappears, we reproduce the standard result \[34, 35\]

$$S_A = \frac{1}{3} \log \left( \frac{2}{a_{UV}} \sin \frac{\sigma}{2} \right). \quad (3.25)$$

In the opposite limit $\epsilon \to 0$, which corresponds to the infinitely extended space, we can easily reproduce the known result in [16] at the central charge $c = 1$

$$S_A = S_{div} + \frac{\pi t}{6 \epsilon} \quad (0 < t < \frac{\sigma}{2})$$

$$= S_{div} + \frac{\pi \sigma}{12 \epsilon} \quad (t > \frac{\sigma}{2}), \quad (3.26)$$

where we separate the divergent part $S_{div} \equiv \frac{1}{3} \log \frac{2}{a_{UV}}$. The physical meaning of the constant term $\frac{\pi \sigma}{12 \epsilon}$ can be qualitatively understood as that of a thermal CFT gas with length $\sigma$ and at the effective temperature

$$T_{eff} = \frac{1}{4 \epsilon}. \quad (3.27)$$

We can also calculate the effective coarse-grained entropy introduced in (2.8) as follows

$$S_{eff}(t) = 2 (S_A(t, \pi) - S_A(0, \pi)) = \frac{2}{3} \log \frac{\theta_2(0 | \frac{2i \epsilon \pi}{\pi}) \cdot \theta_1(\frac{\pi i |\frac{2i \epsilon \pi}{\pi}|}{2})}{\theta_1(\frac{\pi i |\frac{2i \epsilon \pi}{\pi} - \frac{\pi i \sigma}{2}|}{2})}. \quad (3.28)$$

which is clearly finite and does not depend on the cut off $a_{UV}$. Then we notice a periodicity in time

$$S_A(t, \sigma) = S_A(t + \pi, \sigma), \quad S_{eff}(t) = S_{eff}(t + \pi), \quad (3.29)$$

as is clear in in Fig.4. This short periodicity is peculiar to our free CFT, where excitations triggered by the quench propagate at the speed of light and return to exactly the same state after it goes half of the circle. This is because the energy of oscillators included in the boundary state is always an even integer. In generic field theories this does not occur. However, when the size of the space manifold is finite, then the Poincare recurrence leads

\[4\] Actually, in this expression, the values of $\theta_1(\frac{\pi i |\frac{\pi t}{2 \epsilon}|}{2})$, $\theta_1(\frac{\pi i |\frac{\pi t}{2 \epsilon} + \frac{\pi i \sigma}{2}|}{2})$ and $\theta_1(\frac{\pi i |\frac{\pi t}{2 \epsilon} - \frac{\pi i \sigma}{2}|}{2})$ are always all real even before we take their absolute values. This justify our analytical continuation $\text{Re}[y] \to \epsilon + it$. \]
to a much longer periodicity of time which is estimated as an exponential of a coarse-grained entropy $\Delta t \sim e^{S_{\text{eff}}}$, which has been the crucial fact to resolve the information problem for eternal AdS black holes [29, 37, 40].

The AdS/CFT implies that a holographic dual of gravity on AdS$_3$ with significant stringy corrections is a weakly (or free) coupled field theory$^5$, which shows the short time periodicity like (3.29). In the gravity side, this oscillating behavior of $S_A(t)$ and $S_{\text{eff}}(t)$ can be interpreted as successive creations and annihilations of black holes as sketched in Fig.5. In this way, our free field theory example offers us a holographic dual of black hole formations and evaporations in an extremely stringy region and this is manifestly free from no information loss problem as the total system is described by a pure state. Even though our argument on the information problem is along the line with [28, 29, 37–40], our approach is new in that we explicitly treat a real time-dependent process of black hole formations by employing its holographic dual, rather than focusing on the static black holes. Notice again that in this example, the size of black holes are probed by the entanglement entropy and the standard total von-Neumann entropy is vanishing and not useful.

4 Time-dependent Correlation Functions

In previous section, we observed that $S_{\text{eff}}$ is oscillating in our time-dependent system. We then identified the dynamics in CFT is dual to successive productions and annihilations

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$^5$Strictly speaking, in the standard examples of AdS$_3$/CFT$_2$, we also need to take into account the free scalar field sector in addition to the fermion sector we analyzed. However, since both are free field theory, we expected that the qualitative result will not change.
Figure 4: The plot of $S_{\text{eff}}(t) \equiv 2(S_A(t, \pi) - S_A(0, \pi))$ as a function of $t$ at $\epsilon = 0.2$.

Figure 5: Quantum black hole creations and annihilations in AdS space by the quantum quench as obtained from Fig.4.
of quantum black holes. We can naturally imagine the mechanism of its dynamics as follows. First, a black hole is created by a quantum quench and later it evaporates via a quantum analogue of Hawking radiations, which are usually semiclassically analyzed. But the radiations are completely reflected by the time-like boundary of the AdS. Therefore they again concentrate at the center of AdS and recreate a black hole. To confirm this idea, we would like to find time-dependent signals of radiations. Though we will perform explicit computations for a free Dirac fermion below, we can equally apply the results in this section to correlation functions in a free scalar theory compactified at $R = 1$.

### 4.1 One Point Functions

Assume that an operator $O$ in CFT is dual to a field $\varphi$ in AdS. The bulk to boundary relation in AdS/CFT \cite{11} argues that the one point function $\langle O(x) \rangle$ in the CFT is proportional to the value of the field $\varphi$ near the AdS boundary. Since we are not deforming the CFT itself, the profile of $\varphi$ is supposed to be normalizable. The most important one point function in time-dependent backgrounds will be that of the energy stress tensor. Due to its conservation law, this clearly becomes time-independent. It is straightforward to calculate the energy density $\mathcal{E}_n$ of the left-moving oscillator $\alpha_n$ from (3.11) as follows

$$\mathcal{E}_n = \frac{1}{2\pi} \langle \alpha_{-n} \alpha_n \rangle = \frac{n}{2\pi (e^{4\epsilon n} - 1)}. \quad (4.30)$$

This indeed obeys the Bose-Einstein distribution with the effective temperature estimated in (3.27). This temperature can also be seen in the reduced density matrix when we trace out the right-moving oscillators $\tilde{\alpha}_n$ (or equally left-moving ones)

$$\rho_L = \text{Tr}_R e^{-\epsilon H} |B \rangle \langle B | e^{-\epsilon H} = \prod_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-4\epsilon mn} \frac{(\alpha_{-m})^n}{\sqrt{n!}} |0 \rangle \langle 0 | \frac{(\alpha_m)^n}{\sqrt{n!}}, \quad (4.31)$$

where we omit the zero modes. The total energy density in the $\epsilon \to 0$ limit can be found as usual $\mathcal{E}_{\text{tot}} \simeq 2 \sum_{n=1}^{\infty} \mathcal{E}_n \simeq \frac{\pi}{6} T_{\text{eff}}^2$.

Even though we confirmed that our system is thermalized, we cannot find signals of black hole creation and annihilation from the time-independent one point function (4.30). Motivated by this we would like to calculate another one point function which is not dual.

\footnote{For asymptotically flat black holes, we can successfully calculate radiations from thermalized D-branes by taking into account the coupling between open string fields and close string fields as in \cite{36}. Below we would like to consider time-dependent radiations from black holes in AdS spacetime rather than flat spacetime.}
to any conserved quantities. Especially we consider the one point function $\langle e^{ikX(t,0)} \rangle$ for the Dirichlet boundary state for the initial state $|\Psi_0\rangle$, where $k$ can take only integer values as we set $R = 1$. After a calculation similar to the appendix A, we finally obtain (we set $\sigma = 0$)

$$\langle e^{ikX(t,0)} \rangle = \frac{\theta_3 \left( \frac{k(\epsilon + it)}{\pi i}, \frac{2\epsilon i}{\pi} \right)}{\theta_3 \left( 0, \frac{2\epsilon i}{\pi} \right)} \cdot |\theta_1 \left( \frac{\epsilon + it}{\pi i}, \frac{2\epsilon i}{\pi} \right) |^{3k^2}.$$  \hspace{1cm} (4.32)

As is clear from the plot in Fig.6, it has peaks at $t = 0, \pi, 2\pi, \ldots$. This behavior is actually natural from our interpretation in terms of black hole creations and annihilations. As the result of the entanglement entropy suggests, a black hole is created at $t = 0$ and reaches its maximal size at $t = \frac{\pi}{2}$. Therefore, we expect that the radiations from the black hole will be the most strong at $t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$, remembering the periodicity $\pi$. They will reach at the AdS boundary after the propagation time $\Delta t = \pi/2$. This explains the peaks of $\langle e^{ikX(t,0)} \rangle$ since the one point function is dual to the value of bulk scalar field near the AdS boundary and its square should be proportional to the strength of radiations. Notice that this non-vanishing one point function is peculiar to time-dependent black holes. This is because the one point function, which is holographically dual to a classical radiation, is vanishing in the static thermal CFT.

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7 This can be found from the metric of the global AdS$_3$: $ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2$ as $\Delta t = \int_0^\infty \frac{d\rho}{\cosh \rho} = \frac{\pi}{2}$. In the presence of black holes one may think that the propagation time of a massless particle will be changed. Indeed, in a semiclassical black hole, this happens and affects the structures of singularities of two point functions [42]. However, we do not consider this modification as we are interested in a region in gravity where quantum corrections are so large that the even horizon does not seem to be defined using a metric.
4.2 Two Point Functions

The another reason that we are interested in time dependent correlation functions is because we would like to discuss the information loss problem of black holes [4]. In [29, 37, 40], it has been argued that information loss problem is dual to the quasinormal behavior (exponential decay) of time dependent two point functions in a large N gauge theory at finite temperature:

\[ \langle O(t_1)O(t_2) \rangle \sim \exp[-c|t_1 - t_2|]. \] (4.33)

where \( c \) is a positive real constant. These behavior breaks the unitarity of the theory because this shows that any fluctuations that we add to the initial state will eventually vanish. In the same papers, it has also been conjectured in a finite \( N \) gauge theory, the decay of the correlation stops at the value of order \( \exp[-O(N^2)] \), then correlation grows, and repeats this behavior because of the Poincare recurrence of the theory.

Indeed, this quasinormal behavior can be seen in our model. If we set \( y_1 = \epsilon + it, y_2 = \epsilon \) and \( k_L = k_R = k \) in the two point functions (A.44) for the Neumann boundary condition, we find

\[
G(t) = \langle \exp i k X(t,0) \exp -i k X(0,0) \rangle_N
\]

\[
= \eta \left( \frac{2i \epsilon}{\pi} \right)^{6k^2} \left| \frac{\theta_1 \left( \frac{\epsilon+i \pi}{\pi} \right) \frac{2i \epsilon}{\pi}}{\theta_1 \left( \frac{\epsilon+i \pi}{2\pi \pi} \right)^2} \right|^{k^2} \left| \frac{\theta_1 \left( \frac{2i \epsilon}{2\pi \pi} \right)}{\theta_1 \left( \frac{2i \epsilon}{2\pi \pi} \right)^2} \right|^{k^2}.
\] (4.34)

In Fig[7] we show the plot of \( G(t) \). It has the periodicity \( 2\pi \) as a signal propagates from \( \sigma = 0 \) and it has to go back to the same point. The divergences at \( t \in 2\pi \mathbb{Z} \) come from the usual short distance behavior of the operator product. If \( \epsilon \ll t \ll 1 \), we can find the following exponential behavior of \( G(t) \)

\[
G(t) \simeq f(\epsilon) \exp \left[ -\frac{\pi k^2 t}{2\epsilon} \right],
\] (4.35)

for a certain time-independent function \( f(\epsilon) \). Due to the \( 2\pi \) periodicity, after \( G(t) \) gets decreased exponentially, it reaches its (non-zero) minimum at \( t = \pi \) and then again begins to increase.

Finally, it is also intriguing to study the time evolution of the two point function with separated points \( \langle \exp i k X(t, \sigma) \exp -i k X(t, 0) \rangle \) for Neumann boundary condition. If we
use the formula (A.44) of Appendix A, and set $y_1 = \epsilon + it + i\sigma$ and $y_2 = \epsilon + it$, we have

$$F(t,\sigma) = \langle \exp ikX(t,\sigma) \exp -ikX(t,0) \rangle_N$$

$$= \eta \left( \frac{2i\epsilon}{\pi} \right)^{6k^2} \frac{1}{|\theta_1 \left( \frac{\sigma}{2\pi}, \frac{2i\epsilon}{\pi} \right)|^{2k^2} |\theta_1 \left( \frac{2(\epsilon + it) + i\sigma}{2\pi}, \frac{2i\epsilon}{\pi} \right)|^{k^2} |\theta_1 \left( \frac{2(\epsilon + it) - i\sigma}{2\pi}, \frac{2i\epsilon}{\pi} \right)|^{k^2}}$$

In Fig 8, we plot $F(t,\sigma)$. The graph shows that the correlation becomes strong when $S_{eff}$ is large (compare with Fig 4). This can be regarded as a further evidence of the successive productions and annihilations of quantum black holes in our system.
5 Conclusions

In the first part of this paper, we pointed out a puzzle. Because a thermalization process in the CFT side should be unitary, the von-Neumann entropy of the final state must remain vanishing. On the other hand, in its gravity dual, we seem to have non-vanishing entropy as the final state includes a black hole. These observations look contradict with each other. However this is not the case. We showed that even in the gravity side, the von-Neumann entropy of the final state is zero by using the property $S_A = S_B$ of holographic entanglement entropy for an arbitrary pure state. We proved this by making use of the fact that the apparent horizon of the black hole vanishes in the past. We then proposed a direct generalization of thermal entropy by using the entanglement entropy. From a holographic point of view, this quantity seems reasonable because $S_{eff}$ is essentially proportional to the area of the apparent horizon of the corresponding black hole.

Nevertheless, if quantum corrections are taken into account, one might think that the entropy of the final state in the gravity dual can be also nonzero due to the Hawking radiation. This observation is closely related to the information loss problem. In the latter part of this paper, we calculated the time evolution of entanglement entropy following a quantum quench. Especially, we considered two dimensional Dirac fermions on a cylinder. The result shows that the von-Neumann entropy is always vanishing and the coarse-grained entropy $S_{eff}$ is oscillating as a function of time. This offers us a clear toy holographic dual of a formation and decay of black holes, which is free from the information problem. We interpreted this fact as the successive productions and annihilations of quantum black holes. We also studied the behavior of one and two point functions and confirmed that they support this interpretation. We found that the two point functions cannot be zero and is periodic in time which shows some recurrence of our free system.

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A Derivation of Two Point Functions on Cylinder

Here we would like to present an explicit derivation of two point functions (3.20) on cylinder. We work in the description of a massless scalar field $X$ via the bosonization of the Dirac fermion. We set $\alpha' = 2$ in string theory convention and follow [33] about the notations of the mode expansions of a scalar field and the definitions of theta functions.

We employ the complex coordinate $(y, \bar{y})$ on the cylinder defined by (3.12). In the operator formalism, the scalar field is quantized as usual

$$X_L = x_L - i p_L y + i \sum_{m \neq 0} \frac{\alpha_m}{m} e^{-my},$$

$$X_R = x_R - i p_R \bar{y} + i \sum_{m \neq 0} \tilde{\alpha}_m e^{-m\bar{y}},$$

(A.36)

where the commutation relations are given by

$$[x_L, p_L] = i, \quad [x_R, p_R] = i, \quad [\alpha_m, \alpha_n] = m\delta_{n+m,0}, \quad [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{n+m,0}.$$  

(A.37)

At the compactification radius $R$, the momenta are quantized as follows

$$p_L = \frac{n}{R} + \frac{wR}{2}, \quad p_R = \frac{n}{R} - \frac{wR}{2},$$

(A.38)

in terms of the integers $n$ (momentum) and $w$ (winding). The (un-normalized) two point function of the normal ordered vertex operator $V_{(k_L, k_R)} = e^{ik_L X_L + ik_R X_R}$ is written as follows

$$\langle V_{(k_L, k_R)}(y_1, \bar{y}_1) V_{(-k_L, -k_R)}(y_2, \bar{y}_2) \rangle_{\text{cylinder}}$$

$$= \langle B| e^{-2\epsilon H} V_{(k_L, k_R)}(y_1, \bar{y}_1) V_{(-k_L, -k_R)}(y_2, \bar{y}_2) |B\rangle.$$  

(A.39)

The Hamiltonian is given in terms of the Virasoro generators as $H = L_0 + \bar{L}_0 - \frac{1}{12}$.

A.1 Neumann Case

The boundary state $|B\rangle$ for the Neumann boundary condition [43] is given by

$$|B\rangle_N = \mathcal{N} e^{-\sum_{n=1}^{\infty} \frac{1}{n} \alpha_n \tilde{\alpha}_n} \sum_{w=-\infty}^{\infty} |w\rangle,$$

(A.40)

where $\mathcal{N}$ is the normalization factor whose explicit value is not necessary for our purpose. For bosonization procedures in the boundary state formalism, refer to [44].
The zero-mode part of (A.39) can be calculated by using the Baker-Campell-Hausdorff (BCH) formula as follows

\[
\sum_{w=-\infty}^{\infty} \langle w | e^{-2kH} e^{ik_L(x_L-iy_1)} e^{-ik_L(x_L-iy_2)} | w \rangle = \sum_{w=-\infty}^{\infty} e^{-R^2w^2/2} e^{R(k_Lw(y_1-y_2) - k_Rw(y_1-y_2))} e^{k^2_2/2(y_1-y_2) + k^2_2/2(y_2-y_1)}.
\]  

(A.41)

The massive modes can be computed by employing the BCH formula, leading to the identity

\[
\langle 0 | e^{-\hat{\alpha}\hat{\beta}z} e^{a_L\hat{\alpha} + a_R\hat{\beta}} e^{(b_L\hat{\alpha} + b_R\hat{\beta})^+} e^{-\hat{\alpha}^+\hat{\beta}^+} | 0 \rangle = \frac{1}{1-z} \cdot e^{a_Lb_L + a_Rb_R - a_La_R - b_Lb_R},
\]

(A.42)

where we assume the commutation relation \([\hat{\alpha}, \hat{\alpha}^+] = [\hat{\beta}, \hat{\beta}^+] = 1\). Especially we can perform the calculations for massive parts by taking \(z = e^{-4n\epsilon}\) and

\[
a_L = -\frac{k_L}{\sqrt{n}}(e^{-ny_1} - e^{-ny_2}), \quad a_R = -\frac{k_R}{\sqrt{n}}(e^{-n\bar{y}_1} - e^{-n\bar{y}_2}),
\]

\[
b_L = \frac{k_L}{\sqrt{n}}(e^{ny_1} - e^{ny_2}), \quad b_R = \frac{k_R}{\sqrt{n}}(e^n\bar{y}_1 - e^n\bar{y}_2),
\]

(A.43)

in (A.42). By expanding \((1 - e^{-4n\epsilon})^{-1} = \sum_{m=0}^{\infty} e^{-4m\epsilon n}\) and changing the order of the summation w.r.t. \(n\) and \(m\), eventually we can factorized the result into \(\theta\) functions. The final result is given by

\[
\langle B | e^{-2kH} V_{(k_L, k_R)} (y_1, \bar{y}_1) V_{(-k_L, -k_R)} (y_2, \bar{y}_2) | B \rangle_N
\]

\[
= \mathcal{N}^2 \left[ \sum_{w=-\infty}^{\infty} e^{-R^2w^2/2} e^{R(k_Lw(y_1-y_2) - k_Rw(y_1-y_2))} \right] \cdot \frac{1}{\eta(2i\pi)} \cdot \left( \frac{\eta(2i\pi)}{\theta_1(2iy_1)} \right)^3 \cdot \left( \frac{\eta(2i\pi)}{\theta_1(2i\bar{y}_1)} \right)^3 \cdot \left( \frac{\theta_1(2iy_2)}{\theta_1(2i\bar{y}_2)} \right) \cdot \left( \frac{\theta_1(2i\bar{y}_1)}{\theta_1(2iy_1)} \right) \cdot \left( \frac{\theta_1(2i\bar{y}_2)}{\theta_1(2iy_2)} \right) \cdot \left( \frac{\theta_1(2iy_1+\bar{y}_1)}{\theta_1(2i\bar{y}_1+iy_1)} \right) \cdot \left( \frac{\theta_1(2i\bar{y}_2+iy_2)}{\theta_1(2iy_2+\bar{y}_2)} \right) \cdot k^2_{k_Lk_R}.
\]

(A.44)

After we substituting the values (3.13) and the twisted vertex operators \((3.18)\) at the free fermion radius \(R = 1\), we find that the second line of the above expression is rewritten as

\[
\sum_{w=-\infty}^{\infty} \frac{e^{-R^2w^2/2}}{\eta(2i\pi)} = \theta_3(0|2i\pi) + \theta_2(0|2i\pi).
\]

(A.45)

where the first and second term are interpreted as the NS and R-sector of the Dirac fermion. Therefore it is equal to \(\langle B | e^{-2kH} | B \rangle\) and can be neglected in order to find the normalized two point functions as we did in (3.16). In this way we obtain the result (3.20).
A.2 Dirichlet Case

For the Dirichlet boundary condition, the boundary state is given by

$$|B\rangle_D = \mathcal{N} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha_n}} \sum_{n=-\infty}^{\infty} |n\rangle.$$  \hfill (A.46)

By repeating similar calculations, in the end, we find the two point functions

$$\langle B| e^{-2\epsilon H} V_{(k_L, k_R)}(y_1, \bar{y}_1) V_{(-k_L, k_R)}(y_2, \bar{y}_2)|B\rangle_D = \mathcal{N}^2 \left[ \sum_{n=-\infty}^{\infty} e^{-2n^2 \epsilon} e^{\eta (k_L (y_1 - y_2) + k_R (y_1 - y_2))} \right] \cdot \frac{1}{\eta \left( \frac{2i\epsilon}{\pi} \right)} \cdot \left( \frac{\eta \left( \frac{2i\epsilon}{\pi} \right)^3}{\theta_1 \left( \frac{y_1 - y_2 + 2i\epsilon}{2\pi} \right)} \right)^{k_L^2} \cdot \left( \frac{\eta \left( \frac{2i\epsilon}{\pi} \right)^3}{\theta_1 \left( \frac{y_2 - y_1 + 2i\epsilon}{2\pi} \right)} \right)^{k_R^2} \cdot \left( \theta_1 \left( \frac{y_1 + y_2 + 2i\epsilon}{2\pi} \right) \theta_1 \left( \frac{y_2 + y_1 + 2i\epsilon}{2\pi} \right) \right)^{-k_L k_R}.$$  

After we substituting the values (3.13) and the twisted vertex operators (3.19) at the free fermion radius $R = 1$, we find that the second line of the above expression is rewritten as

$$\sum_{n=-\infty}^{\infty} e^{-2n^2 \epsilon} \eta \left( \frac{2i\epsilon}{\pi} \right) = \frac{\theta_3 \left( 0 \left| \frac{2i\epsilon}{\pi} \right. \right)}{\eta \left( \frac{2i\epsilon}{\pi} \right)},$$  \hfill (A.47)

which includes only the NS-sector.

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