Symmetric Teleparallel Gravity: Some exact solutions
and spinor couplings

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06 November 2013, file CosmicSpeedUpInSTPG21.tex

Abstract

In this paper we elaborate on the symmetric teleparallel gravity (STPG) written in a non-Riemannian spacetime with nonzero nonmetricity, but zero torsion and zero curvature. Firstly we give a prescription for obtaining the nonmetricity from the metric in a peculiar gauge. Then we state that under a novel prescription of parallel transportation of a tangent vector in this non-Riemannian geometry the autoparallel curves coincides with those of the Riemannian spacetimes. Subsequently we represent the symmetric teleparallel theory of gravity by the most general quadratic and parity conserving lagrangian with lagrange multipliers for vanishing torsion and curvature. We show that our lagrangian is equivalent to the Einstein-Hilbert lagrangian for certain values of coupling coefficients. Thus we arrive at calculating the field equations via independent variations. Then we obtain in turn conformal, spherically symmetric static, cosmological and pp-wave solutions exactly. Finally we discuss a minimal coupling of a spin-1/2 field to STPG.

PACS numbers: 04.50.Kd, 98.80.Jk, 02.40.Yy

Keywords: Non-Riemannian geometry, Modified theories of gravity, Lagrange formulation, Dirac equation

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1 Introduction

Einstein’s theory of gravity, the so-called general relativity (GR), is formulated in a Riemannian spacetime. But there are convincing reasons to go beyond the Riemannian spacetime coming from both mathematical and physical points of view. Mathematically both group theoretical \[1,2\] and gauge theoretical \[3,4\] approaches predict non-Riemannian geometries with especially nonmetricity. Physically unsuccessful affords to quantize gravity with standard field theoretical methods are longstanding reasons for modification of GR. Furthermore the recent observational data received from type-Ia supernovae have showed that the expansion of the Universe is currently undergoing a period of acceleration \[5,6\]. Cosmic speed-up can be explained within general relativity by invoking a mysterious cosmic fluid with large negative pressure, the so-called dark energy. Although the simplest possibility for dark energy is a cosmological constant, unfortunately the estimated value for it is too large compared with today’s value. Consequently GR must be modified or replaced with new one.

Einstein’s equations

\[
(Ric)_a - \frac{1}{2}R e_a + \Lambda e_a = \kappa \, t_a \tag{1}
\]

are obtained as local extremum of the action \( I = \int_M L_0 \) where

\[
L_0 = -\frac{1}{2\kappa} R^a b \wedge^* e_a^b + \frac{\Lambda}{\kappa} *1 + \lambda_a \wedge T^a + \mu_{ab} \wedge Q^{ab} - K \tag{2}
\]

is the Einstein-Hilbert lagrangian 4-form with the cosmological constant \( \Lambda \). Here \( \kappa \) is a coupling constant, \( t_a = \delta K/\delta e^a \) are the matter energy-momentum 3-forms, \( \lambda_a \) are the Lagrange multiplier 2-forms constraining torsion to zero, \( \mu_{ab} \) are the Lagrange multiplier 3-forms constraining nonmetricity to zero. Other notations will be introduced in the next section. In one approach of modified gravity models one passes to the non-Riemannian geometries; \( R^a b \neq 0, \, T^a \neq 0 \) and \( Q_{ab} \neq 0 \). A gravity model developed in such a spacetime is often called the metric-affine gauge theory of gravity (MAG) \[4\]. Despite a big degrees of freedom of those models it is not an easy matter to handle the most general case. Therefore people put some restrictions on their gravity models, see for example \[4,7,8,9,10,11\] and the references therein. In this work we adhere the Yang-Mills manner for a gravitational lagrangian, that is, we consider the most general nonmetricity quadratic and parity conserving lagrangian 4-form

\[
L = \sum_{I=1}^{4} k_I Q_{ab} \wedge^* (I) Q^{ab} + k_5 \left( (3) Q_{ab} \wedge e^b \right) \wedge^* \left( (4) Q^{ac} \wedge e_c \right) + \Lambda *1 + \lambda_a \wedge T^a + P_{ab} \wedge \rho^b a - K \tag{3}
\]

where \( k_I \)s are coupling constants, \( (I) Q^{ab} \) are irreducible decompositions of nonmetricity \[8,12\], \( \Lambda \) is the cosmological constant, \( K \) is the matter Lagrangian 4-form, \( \lambda_a \) are the Lagrange multiplier 2-forms for zero-torsion, \( \rho^b a \) are the Lagrange multiplier 2-forms for teleparallelism. In other
words, we have simply replaced the left hand side of (1) with a new one written in terms of only nonmetricity. This model is called symmetric teleparallel gravity (STPG) [13]. Thereby metric $g_{\alpha\beta}$ is seen as gauge potential and the corresponding gauge field strength is nonmetricity $Q_{\alpha\beta} \sim D g_{\alpha\beta}$ that satisfies the Bianchi identity $D Q_{\alpha\beta} = 0$.

There is another teleparallel approach to gravity, the so-called teleparallel gravity (TPG), formulated only with torsion. Although TPG has attracted a lot of attention in the literature, STPG is infancy. One can consult, for example, the references [14], [15] to start reading the wide literature on TPG. On the other hand, a few physically interesting analysis and solutions of STPG can be found in [12], [13], [16], [17]. Our wish here is to fill in a bit this big gap at STPG. Our other motivation is to gain new insights on the nometricity. Recently it has been related with the breaking of the Weyl symmetry [7], with the spin-3 gravity field [18] and used in the solar neutrino mixing problem [19].

The outline of the paper is as follows. In Section 2 after introducing the non-Riemannian geometry and then symmetric teleparallel geometry we define a gauge fixing procedure and a deliberative parallel transportation rule for a tangent vector. From now on we use the computer algebra system REDUCE and its package EXCALC intensively [20], [21]. In Section 3 we present a systematic treatment of STPG through a lagrange 4-form quadratic in the nonmetricity. Here we show explicitly that for certain values of coupling coefficients our lagrangian corresponds to the Einstein-Hilbert lagrangian in Riemannian spacetime. Then the field equations are obtained consistently by independent variations. By solving $D \lambda_a$ from the connection varied equation and then by inserting it into the coframe varied equation we write down the dynamical field equation (39) in a naive way. In the next four sections we obtain some classes of exact solutions in the type of conformal, spherically symmetric static, cosmological and pp-wave, respectively. Finally we analyze a minimal coupling of a Dirac spinor to STPG. Section 9 is devoted for the concluding remarks.

## 2 Mathematical preliminaries

Spacetime is denoted by the triple $\{ M, g, \nabla \}$ where $M$ is differentiable and orientable 4-dimensional manifold, $g = g_{\alpha\beta} e^\alpha \otimes e^\beta$ is the metric tensor, and $\nabla$ represents connection. $\{ X_\alpha \}$ denotes the basis vector set of tangent space $T_p(M)$ at the point $p$ of manifold. Similarly, $\{ e^\alpha \}$ denotes the basis co-vector set of cotangent space $T^*_p(M)$ at the point $p$ of manifold. We show duality as

$$e^\alpha(X_\beta) \equiv \iota_{X_\beta}(e^\alpha) = \delta_\beta^\alpha \quad (4)$$

where $\delta_\beta^\alpha$ is Kronecker symbol and $\iota_{X_\beta}$ is the interior product. Full connection is determined by local 1-forms, $\omega^{\alpha\beta}$ as follows:

$$\nabla_{X_\alpha} X_\beta = \omega^{\gamma\beta}(X_\alpha) X_\gamma. \quad (5)$$
From now on we call $e^\alpha$ basis 1-forms and $\omega^\alpha{}_{\beta}$ full connection 1-forms. Thus, under a general coordinate transformation while basis 1-forms transform as

$$e'^\alpha = h'^\alpha{}_\alpha e^\alpha$$

full connection 1-forms transform in the following manner

$$\omega'^\alpha{}_{\beta'} = h'^\alpha{}_{\alpha} \omega^\alpha{}_{\beta} h^\beta{}_{\beta'} + h'^\alpha{}_{\alpha} d h^\alpha{}_{\beta'}$$

where $h'^\alpha{}_{\alpha}$ are transformation elements. The following shorthand notation for exterior multiplications is adhered $e^\alpha \wedge e^\beta \cdots \equiv e^{\alpha\beta\cdots}$. A geometry is determined by the Cartan structure equations

$$Q_{\alpha\beta} := -\frac{1}{2} D g_{\alpha\beta} = \frac{1}{2} (-d g_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha}),$$
$$T^\alpha := D e^\alpha = d e^\alpha + \omega^\alpha{}_{\beta} \wedge e^\beta,$$
$$R^\alpha{}_{\beta} := D \omega^\alpha{}_{\beta} := d \omega^\alpha{}_{\beta} + \omega^\alpha{}_{\gamma} \wedge \omega^\gamma{}_{\beta},$$

where $Q_{\alpha\beta}$ are the nonmetricity 1-forms, $T^\alpha$ are the torsion 2-forms and $R^\alpha{}_{\beta}$ are the curvature 2-forms. $d$ and $D$ denote the exterior derivative and the covariant exterior derivative, respectively. These tensors satisfy the Bianchi identities

$$D Q_{\alpha\beta} = \frac{1}{2} (R_{\alpha\beta} + R_{\beta\alpha}),$$
$$D T^\alpha = R^\alpha{}_{\beta} \wedge e^\beta,$$
$$D R^\alpha{}_{\beta} = 0.$$

In general, full connection 1-forms can be decomposed as follows [4,7,8]:

$$\omega^\alpha{}_{\beta} = (g^\alpha{}_{\gamma} d g_{\beta\gamma} + p^\alpha{}_{\beta})/2 + \tilde{\omega}^\alpha{}_{\beta} + K^\alpha{}_{\beta} + Q^\alpha{}_{\beta}$$

Riemannian Torsional Nonmetrical

where Levi-Civita connection 1-forms $\tilde{\omega}^\alpha{}_{\beta}$

$$\tilde{\omega}^\alpha{}_{\beta} \wedge e^\beta = -d e^\alpha,$$

contortion 1-forms $K^\alpha{}_{\beta}$

$$K^\alpha{}_{\beta} \wedge e^\beta = T^\alpha,$$

and anti-symmetric 1-forms

$$p_{\alpha\beta} = -(t_{\alpha} d g_{\beta\gamma}) e^\gamma + (t_{\beta} d g_{\alpha\gamma}) e^\gamma,$$
$$q_{\alpha\beta} = -(t_{\alpha} Q_{\beta\gamma}) e^\gamma + (t_{\beta} Q_{\alpha\gamma}) e^\gamma.$$
This decomposition is self-consistent. To see that one just has to multiply (14) by $\wedge e^\beta$ and to use above definitions, where special care is needed to raise and lower an index in front of both $d$ and $D$. Symmetric part of (14) comes from (8)

$$\omega_{(\alpha\beta)} = Q_{\alpha\beta} + \frac{1}{2} d g_{\alpha\beta}$$

and the remaining is anti-symmetric

$$\omega_{[\alpha\beta]} = \frac{1}{2} p_{\alpha\beta} + \tilde{\omega}_{\alpha\beta} + K_{\alpha\beta} + q_{\alpha\beta}.$$

We denote symmetrization and anti-symmetrization by $\omega_{(\alpha\beta)} := \frac{1}{2} (\omega_{\alpha\beta} + \omega_{\beta\alpha})$ and $\omega_{[\alpha\beta]} := \frac{1}{2} (\omega_{\alpha\beta} - \omega_{\beta\alpha})$, respectively.

In general, the triple of nonmetricity, torsion and curvature classifies a geometry. If $Q_{\alpha\beta} = 0$, $T^\alpha \neq 0$ and $R^\alpha_{\beta} \neq 0$, we have a metric compatible connection in the Riemann-Cartan (RC) geometry. If $Q_{\alpha\beta} = 0$, $T^\alpha = 0$ and $R^\alpha_{\beta} \neq 0$, then we have the Levi-Civita connection of a Riemann geometry. If $Q_{\alpha\beta} = 0$, $T^\alpha \neq 0$ and $R^\alpha_{\beta} = 0$, it is called the Weitzenb"{o}ck geometry.

Here we work in the symmetric teleparallel geometry defined by $Q_{\alpha\beta} \neq 0$, $T^\alpha = 0$ and $R^\alpha_{\beta} = 0$. It is always possible to choose orthonormal basis 1-forms which we denote $\{ e^a(x) \}$. Then the metric turns out to be $g = \eta_{ab} e^a(x) \otimes e^b(x)$ where $\eta_{ab} = \text{diag}(-1, +1, +1, +1)$. In this case, the orientation of the manifold can be fixed by the Hodge star $*1 = e^{0123}$ and also the connection 1-forms decompose according to

$$\omega^a_b = \bar{\omega}^a_b + K^a_b + q^a_b + Q^a_b.$$

After this point while the Greek indices figure holonomic (coordinate) indices, the Latin indices are anholonomic (orthonormal) ones.

A gauge fixing

We will be working in the orthonormal frame through the paper. But we can not solve equations $T^a = de^a + \omega^a_b \wedge e^b = 0$ and $R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = 0$ together for $\omega^a_b$. This is opposed to the Riemannian case in which one can solve $\omega^a_b$ uniquely from $Q_{ab} = (\omega_{ab} + \omega_{ba})/2 = 0$ and $T^a = de^a + \omega^a_b \wedge e^b = 0$. Thus one method may be to guess suitable coframe and connection such that zero-torsion and zero-curvature are satisfied. In this work, instead of that approach, we prefer to develop a general recipe for treating the problem. In our recipe we start calculation in the coordinate frame in which the metric is $g = g_{\alpha\beta}(x) dx^\alpha \otimes dx^\beta$ and the coframe is $e^\alpha = dx^\alpha$ where $\{x^\alpha\}$ is a local coordinate system. Now we choose a special coordinate system where $\omega^\alpha_{\beta} = 0$. This may be seen as a gauge fixing as well. In this gauge (or coordinate), two conditions, $T^\alpha = 0$ and $R^\alpha_{\beta} = 0$, are met automatically and we obtain non-zero nonmetricity $Q_{\alpha\beta} = -\frac{1}{2} d g_{\alpha\beta} \neq 0$, see the equations (8)-(10). Then we need to go back the orthonormal frame via local vierbein $h^a_{\alpha}$ where $e^a = h^a_{\alpha} dx^\alpha$. 

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Before diving transformation details, it is worthwhile to remind a well-known fact. Since both curvature and torsion are tensors, they are invariant under coordinate transformations by definition. That is, it is not possible to make them nonzero by gauge transformation or fixing if they are zero in some other basis. The story, however, is completely different for connection since it is not tensor by definition. In other words, it is possible to make it nonzero by a gauge transformation even if it is zero in some other gauge. Accordingly, in the orthonormal frame, one calculates full connection

$$\omega_{ab} = h_{\alpha a} \omega_{\alpha \beta} h_{\beta b} + h_{\alpha a} dh_{\alpha b} = h_{\alpha a} dh_{\alpha b} \neq 0,$$

full curvature

$$R_{ab} = h_{\alpha a} R_{\alpha \beta} h_{\beta b} = 0,$$

torsion

$$T_a = h_{\alpha a} T_{\alpha} = 0$$

and nonmetricity

$$Q_{ab} = h_{\alpha a} Q_{\alpha \beta} h_{\beta b} \neq 0.$$

Finally, we have checked that this result corresponds to

$$\tilde{\omega}_{ab} + q_{ab} = 0$$

(together with

$$K_{ab} = 0$$

since

$$T_a = 0$$

which means $\omega_{ab} = Q_{ab}$ via (21).

Here since our method crucially relies on a very strong condition for the full connection $\omega_{\alpha \beta}^\prime$ to vanish in the coordinate basis which we describe as a gauge fixing, it is natural to ask; does such a gauge always exist or what are the conditions for it? According to (7), the components of the full connection transform inhomogeneously under a local linear gauge transformation. So, even when one starts with a non-zero full connection, $\omega_{\alpha \beta}^\prime$, a vanishing full connection, $\omega_{\alpha \beta}$, is reached under the condition

$$h_{\alpha a}^\prime dh_{\alpha \beta}^\prime + h_{\alpha a}^\prime \omega_{\alpha \beta}^\prime h_{\beta b}^\prime = 0$$

where the transformation elements $h_{\alpha a}^\prime$ define a diffeomorphism. When one considers a coordinate transformation $x^{\alpha\prime} = x^{\alpha}(x^\alpha)$, it is realized that $h_{\alpha a}^\alpha = \partial x^{\alpha\prime}/\partial x^\alpha$ and $h_{\alpha a}^\alpha = \partial x^\alpha/\partial x^{\alpha\prime}$, and then $h_{\alpha a}^\prime h_{\alpha a}^\beta = \delta_\beta^\alpha$. Consequently, the condition above contains a system of the second order partial differential equations as the following

$$\frac{\partial^2 x^\alpha}{\partial x^{\alpha\prime} \partial x^{\beta\prime}} + \omega_{\beta\alpha}^\prime \frac{\partial x^\beta}{\partial x^{\beta\prime}} = 0.$$ 

Thus we can conclude that, in the context of STPG-TPG, such a gauge always exists, since the integrability conditions of the above set of equations imply that all components of the curvature two-form must vanish.

**Autoparallel curves in the symmetric teleparallel geometry**

Here it seems that we have a problem concerning the trajectory of a test particle. In this geometry trajectories are given by autoparallel curves. We know that they are basically obtained by the notion of parallel propagation of tangent vectors to them. Now we want to review very basic concepts on those issues.

In the Riemann geometry one requires that the tangent vector, $V^\alpha = dx^\alpha/d\tau$, to the autoparallel curve, $x^\alpha(\tau)$ with the parameter $\tau$, points in the same direction as itself when parallel propagated, and demands that it maintains the same length,

$$DV^\alpha = 0.$$ 

(22)
When we adhere this definition directly in symmetric teleparallel geometry together with the
gauge \( \omega^{\alpha \beta} = 0 \) we obtain an autoparallel equation

\[
dV^\alpha + \omega^{\alpha \beta} V^\beta = 0 \quad \Rightarrow \quad dV^\alpha = 0 \quad \Rightarrow \quad \frac{d^2 x^\alpha}{d\tau^2} = 0
\]

whose solution is simply a straight line. This means that all free test particles, e.g. planets
around the Sun, move on a straight line. Of course, this is not the case. Therefore, we can
not use this rule of parallel translation. This is the problem mentioned above and has to be
amended.

Thus, since intuitively autoparallel curves are "those as straight as possible" we may not
demand the vector to keep the same length \[22\] during parallel propagation in STPG. Besides,
it is known that nonmetricity is related to length and angle standards at parallel transportation
\[4\],\[8\]. Therefore we prescribe a novel rule for the parallel propagation of a tangent vector \[23\]

\[
DV^\alpha = (aQ^\alpha_\beta + bq^\alpha_\beta) V^\beta + cQV^\alpha
\]

where \( a, b, c \) are arbitrary constants and \( Q = Q^\alpha_\alpha \). Now all trajectories do not need to be
straight lines. Since it is a new problem to determine \( a, b, c \), for example, for our solar system,
we leave that for our future project. Nevertheless, we want to remark one point. When it is
chosen \( a = b = 1 \) and \( c = 0 \) in the gauge \( e^\alpha = dx^\alpha \) and \( \omega^{\alpha \beta} = 0 \), the autoparallel curve equation
is obtained as

\[
\frac{d^2 x^\alpha}{d\tau^2} + \frac{1}{2} g^{\alpha \beta} (\partial_\gamma g_{\beta \delta} + \partial_\delta g_{\beta \gamma} - \partial_\beta g_{\gamma \delta}) \frac{dx^\gamma}{d\tau} \frac{dx^\delta}{d\tau} = 0.
\]

This is the same as the geodesic equation of the Riemann geometry.

### 3 Field equations of STPG

With the assignments

\[
\alpha_a = Q_{ab} \wedge e^b, \quad \beta_a = e^b Q_{ab}, \quad Q = n_{ab} Q^{ab}, \quad \gamma_a = \iota_a Q
\]

the lagrangian \[3\] is equivalent to the following

\[
L = c_1 Q_{ab} \wedge^* Q^{ab} + c_2 \alpha_a \wedge^* \alpha^a + c_3 \beta_a \wedge^* \beta^a + c_4 Q \wedge^* Q + c_5 \beta_a \wedge^* \gamma^a + \lambda_1 + \lambda_a \wedge T^a + R^a_{\beta b} \wedge \rho^b_a - K
\]

under the redefinitions of the coupling coefficients, see \[12\]

\[
c_1 = k_1, \\
c_2 = 2(-k_1 + k_2)/3, \\
c_3 = 2(-k_1 - k_2 + 2k_3)/9, \\
c_4 = (-8k_1 - 32k_2 + 4k_3 + 36k_4 + 9k_5)/144, \\
c_5 = (-8k_1 + 16k_2 - 8k_3 - 9k_5)/36.
\]
Since an odd number of Hodge stars occurring in a Lagrangian warrants its parity conservation, the Lagrangian (3) or (26) is the most general nonmetricity quadratic and parity conserving lagrangian. Although in the first order formalism we are using, higher order terms in lagrangian would preserve the second order of the field equations, the quasilinearity of the field equations would be damaged and then the Cauchy problem may be ill-posed. Therefore we consider a lagrangian not higher than nonmetricity squared.

Certain values of \( c_i \)s correspond to GR. For determining these values firstly the full curvature 2-form is decomposed as Riemannian curvature plus non-Riemannian terms by substituting (21) (with \( T^a = 0 \)) into
\[
R_{ab} = \tilde{R}_{ab} + \tilde{D}\Omega_{ab} + \Omega^c e_c \wedge \Omega^e_b \quad (28)
\]
where \( \Omega_{ab} = Q_{ab} + g_{ab} \). Riemannian quantities are marked by a tilde. Then the Einstein-Hilbert lagrangian 4-form is formed
\[
R_{ab} \wedge {}^* e_{ab} = \tilde{R}_{ab} \wedge {}^* e_{ab} + (\tilde{D}\Omega_{ab}) \wedge {}^* e_{ab} + \Omega^c \wedge \Omega^e_b \wedge {}^* e_{ab} \quad (29)
\]
Here since \( \tilde{D}e_a = 0 \), the second term is an exact form, i.e. \( (\tilde{D}\Omega_{ab}) \wedge {}^* e_{ab} = d(\Omega^c \wedge {}^* e_{ab}) \). Finally, because of vanishing of the full curvature 2-form in this geometry the left hand is zero. Correspondingly, it may be written
\[
-\frac{1}{2} \tilde{R}_{ab} \wedge {}^* e_{ab} = \frac{1}{2} \Omega^e_c \wedge \Omega^e_b \wedge {}^* e_{ab} \quad (30)
\]
from which one computes the GR-equivalent values of \( c_i \)s by comparing with (26)
\[
c_1 = -1/2, \quad c_2 = 1, \quad c_3 = 0, \quad c_4 = -1/2, \quad c_5 = 1. \quad (31)
\]
The advantage of this result is that we can classify the solutions to be obtained as Einsteinian or not.

Now we vary the lagrangian (26) to obtain the field equations by noticing \( \delta Q_{ab} = \delta \omega_{(ab)} \) and \( (\delta e^a) e^b = -e^a (\delta e^b) \), and by using the general formula
\[
\delta (A \wedge {}^* B) = \delta A \wedge {}^* B + \delta B \wedge {}^* A - \delta e^a \wedge [(\iota_a B) \wedge {}^* A - (-1)^p A \wedge (\iota_a {}^* B)] \quad (32)
\]
where \( A \) and \( B \) are any \( p \)-forms on \( n \)-dimensional manifold. Thus the variations with respect to \( \lambda_a, \rho^a_b, e^a \) and \( \omega^b_a \) yield the following field equations, respectively
\[
T^a = 0, \quad R^a_b = 0, \quad (33)
\]
\[
\tau_a + \Lambda^e \iota_a + D\lambda_a = t_a, \quad (34)
\]
\[
\Sigma^a_b + e^a \wedge \lambda_b + D\rho^a_b = \ell^a_b, \quad (35)
\]
where \( t_a = \frac{\delta K}{\delta e_a} \) are the energy-momentum 3-forms of matter, \( \ell^a_b = \frac{\delta K}{\delta \omega^a_{bc}} \) are the angular momentum 3-forms of matter, \( \tau_a \) are the energy-momentum 3-forms of nonmetricity

\[
\tau_a = c_1[t_a(Q_{bc} \wedge \gamma^{bc}) - 2(t_a Q_{bc}) \wedge (\gamma^{bc})] + c_2[t_a(\omega^a_{bc} \wedge \gamma^{bc})] + c_3[t_a(\beta_b \wedge \gamma^{bc}) - 2(t_a Q_{bc} \wedge (\gamma^{bc})] + c_4[t_a(Q \wedge \gamma^Q) - 2(t_a Q_{bc} \wedge (\gamma^{bc})] + c_5[t_a(\beta_b \wedge \gamma^{bc}) - (t_a Q_{bc}) \wedge (\gamma^{bc})]
\]

(36)

and \( \Sigma_{ab} \) are the angular momentum 3-forms of nonmetricity

\[
\Sigma_{ab} = 2c_1 \epsilon_{abc}^2 + 2c_2 e_{abc} + 2c_3 e_{abc}^2 + 2c_4 \eta_{abc}^2 + c_5 \eta_{abc}^2
eq \delta_{abc}^2
\]

(37)

Let \( t_a \) and \( \ell^a_b \) be known. Then we count the number of the field equations: 24 from \( T^a = 0 \) plus 96 from \( R_{ab} = 0 \) plus 16 from (34) plus 64 from (35) that add to 200. On the other hand, the number of the unknowns is 200 as well: 16 for \( e_a \) plus 64 for \( \omega_{ab} \) plus 24 for \( \lambda_a \) plus 96 for \( \rho_{ab} \).

Thus the system is closed. In principle, the Lagrangian multipliers are solved from (35) and the result is substituted into (34). We notice that what we need in the equation (34) is only \( D\lambda_a \) rather than \( \lambda_a \) or \( \rho_{ab} \). Thus, by multiplying (35) first by \( D \) and then by \( \eta_a \) we obtain

\[
D\lambda_a = \eta_b D(\Sigma_{ba} - \ell^b_a)
\]

(38)

where we used \( De^a = T^a = 0 \) and \( D^2 \rho_{ab} = R^a_c \wedge \rho^c_b - R^c_b \wedge \rho^a_c = 0 \). Substitution of this into (34) yields the dynamical field equation of STPG

\[
t_b D\Sigma_{ba} + \tau_a + \Lambda^e e_a = t_a + t_b D\ell^b_a.
\]

(39)

4 Conformal solution

We consider the metric

\[
g = \exp(2\Phi)\eta_{\alpha\beta} dx^\alpha \otimes dx^\beta
\]

(40)

where \( \eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1) \) and \( \Phi = \Phi(t, x, y, z) \) is the conformal function in the cartesian coordinates, \( x^\alpha = (t, x, y, z) \). In the coordinate frame, i.e. \( e^a = dx^a \), we choose such a gauge that \( \omega^\alpha_{\beta} = 0 \) and then calculate the nonmetricity via \( Q_{\alpha\beta} = -\frac{1}{2} d g_{\alpha\beta} \) as

\[
Q_{\alpha\beta} = -\eta_{\alpha\beta} \exp(2\Phi) d \Phi.
\]

(41)

Now we pass to the orthonormal frame

\[
g = \eta_{ab} e^a \otimes e^b \quad \text{with} \quad e^a = \exp(\Phi) dx^a.
\]

(42)

Accordingly the nonmetricity in the orthonormal frame can be computed

\[
Q_{ab} = -\eta_{ab} d \Phi.
\]

(43)
Here we checked that \( \tilde{\omega}_{ab} + q_{ab} = 0 \). After substituting this into the equation \( 39 \) with \( t_a = 0 \) and \( \ell^a b = 0 \) we see that the conformal function is arbitrary under the constraints
\[
4c_1 + 5c_2 - c_3 + 16c_4 = 0, \quad 2(c_1 + c_2 + 4c_4) + c_5 = 0
\]
with \( \Lambda = 0 \). We also remark that GR-values of \( c_i \)s \( 31 \) do not satisfy these constraints.

5 Spherically symmetric static solutions

We write the following metric in spherical polar coordinates \( (r, \theta, \phi) \)
\[
g = -F(r)^2 dt^2 + G(r)^2 dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).
\]
(45)

Now in the coordinate frame and in the gauge \( \omega^\alpha_\beta = 0 \) we calculate the nonmetricity, \( Q^{\alpha\beta} = -\frac{1}{2}d_{\alpha\beta} \),
\[
Q_{tt} = FF' dr, \quad Q_{rr} = -GG' dr, \quad Q_{\theta\theta} = -r dr, \\
Q_{\phi\phi} = -r \sin^2 \theta dr - r^2 \sin \theta \cos \theta d\theta, \quad \text{others} = 0,
\]
(46)
where prime denotes derivative with respect to \( r \). Then we pass to the orthonormal frame
\[
e^0 = Fdt, \quad e^1 = Gdr, \quad e^2 = r d\theta, \quad e^3 = r \sin \theta d\phi.
\]
(47)

That gives the chance of computing the nonmetricity in the orthonormal frame
\[
Q_{00} = \frac{F'}{FG} e^1, \quad Q_{11} = -\frac{G'}{G^2} e^1, \quad Q_{22} = -\frac{1}{rG} e^1, \quad Q_{33} = -\frac{1}{rG} e^1 - \frac{\cot \theta}{r} e^2, \quad \text{others} = 0.
\]
(48)

When we substitute these findings to the field equation \( 39 \) in vacuum, that is \( t_a = 0 \) and \( \ell^a b = 0 \), we obtain a very complicated set of equations. Therefore we try simplifying them firstly by choosing \( G = 1/F \). Then under the case of \( c_1 \neq 0 \), \( c_2 = c_5 = -2c_1, c_3 = 0, c_4 = c_1 \) a dynamical solution is obtained
\[
F^2 = 1 - \frac{M}{r} + \frac{\Lambda}{6c_1} r^2
\]
(49)
where \( M \) is an integration constant. For \( c_1 = -1/2 \) this is exactly the Schwarzschild-de Sitter solution of GR. This result is also a crosscheck of GR-values of \( c_i \)s \( 31 \). If we set \( c_1 = c_2 = c_4 = c_5 = 0 \), but \( c_3 \neq 0 \) we find another solution
\[
F = a + br, \quad G = 1
\]
(50)
where \( a \) and \( b \) are constants. These solutions are not conformally invariant. One class of conformally invariant solution is obtained as
\[
F = \exp(-m/r), \quad G = kr
\]
(51)
with \( \Lambda = 0 \) and the constraints \( c_1 \neq 0, c_2 = c_3 = -c_1, c_4 = c_5 = 0 \), where \( m \) and \( k \) are arbitrary constants.
6 Homogeneous and isotropic cosmology

In spite of the fact that if a model more general than the general relativity is investigated, principally a complete model with any kind matter such as hyperfluid, spinorial etc has to be considered, we assume at this point that the matter lagrangian does not depend explicitly on the connection, i.e. $\ell^a_b = 0$. This assumption is because we will be searching cosmological solutions with a matter source in the form of non-spinning perfect fluid. Otherwise, one can always drop this restriction for spinning sources. Thus we consider the non-spinning perfect fluid energy-momentum 3-form

$$t_a = [(\rho + p)\eta_{0a}\eta_{0b} + \eta_{ab}p]^*e^b$$

where $\rho$ is matter density, $p$ is pressure. We consider the Robertson-Walker metric in the isotropic coordinates $x^\alpha = (t, x, y, z)$

$$g = -dt^2 + S^2(t)\frac{dx^2 + dy^2 + dz^2}{(1 + kr^2/4)^2}$$

where $k$ is a constant, $r^2 = x^2 + y^2 + z^2$ and $S(t)$ is the expansion function. First, we choose such a frame and a connection that $e^\alpha = dx^\alpha$ and $\omega^\alpha{}_\beta = 0$, then torsion and curvature are both zero, but $Q_{\alpha\beta} = -\frac{1}{2}dg_{\alpha\beta} \neq 0$,

$$Q_{xx} = Q_{yy} = Q_{zz} = -\frac{S\dot{S}dt}{(1 + kr^2/4)^2} + \frac{kS^2(xdx + ydy + zdz)}{2(1 + kr^2/4)^3}, \text{ others } = 0$$

where dot denotes derivative with respect to $t$. Now we pass to the orthonormal frame

$$e^0 = dt, \quad e^1 = \frac{S}{1 + kr^2/4}dx, \quad e^2 = \frac{S}{1 + kr^2/4}dy, \quad e^3 = \frac{S}{1 + kr^2/4}dz$$

and correspondingly calculate

$$Q_{11} = Q_{22} = Q_{33} = -\frac{\dot{S}}{S}e^0 + \frac{k}{2S}(xe^1 + ye^2 + ze^3), \text{ others } = 0.$$ 

Now we saw again that $\omega_{ab} = Q_{ab}$. We also crosschecked GR-values of $c_i$s given by (31). Then we deal with the case with $k = 0$, because cosmic microwave background experiments have indicated that the geometry of the Universe is spatially flat [24], [25]. Thus by setting $k = 0$ from beginning we arrive at the equations

$$3a\frac{\dot{S}}{S} + 3b\left(\frac{\dot{S}}{S}\right)^2 = -(\Lambda + \rho)$$

$$(b - 2a)\left[2\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S}\right)^2\right] = -(\Lambda - p)$$

where $a = 2c_4 + c_5$ and $b = c_1 + c_2 + 7c_4 + 2c_5$. Here note that because of the assumed homogeneity, $\rho$ and $p$ are functions of $t$ alone. Here there are two equations: (57), (58) but three unknowns:
So by postulating an equation of state \( p = w\rho + B \) we close the system where \( w \) and \( B \) are constants. This equation of state is a special case of the modified Chaplygin gas defined as \( p = w\rho + B/\rho^\alpha \) where \( B < 0 \) and \( 0 \leq \alpha \leq 1 \). As a result we obtain a solution

\[
S = \left[ K_2 e^{-nt} \left( e^{nt} - K_1 \right)^2 \right]^{m/n}
\]

where \( K_1 \) and \( K_2 \) are integration constants, and

\[
m = \sqrt{-B + (1 + w)\Lambda} / 3(2a - b - w(a + b)), \quad n = 6m(2a - b - w(a + b)) / 4a - 2b - 3aw.
\]

Here we notice that since \( a = 0 \), but \( b \neq 0 \) for GR-values of \( c_i \), although our solution is non-Einsteinian it behaves qualitatively like Einsteinian one.

### 7 pp-Wave Solution

We start with the metric

\[
g = -dt^2 + p^2 dx^2 + q^2 dy^2 + dz^2
\]

where \( p = p(u) \) and \( q = q(u) \) are functions of the null coordinate \( u = z - t \). This metric represents the plane fronted the gravitational waves propagating in the \( z \)-direction with the speed of light.

In the gauge \( e^\alpha = dx^\alpha \) and \( \omega^\alpha{}_\beta = 0 \) we calculate the nonmetricity, \( Q_{\alpha\beta} = -\frac{1}{2}dg_{\alpha\beta} \)

\[
Q_{xx} = -pdp, \quad Q_{yy} = -qdq, \quad \text{others} = 0.
\]

Now we pass to orthonormal frame

\[
e^0 = dt, \quad e^1 = pdx, \quad e^2 = dy, \quad e^3 = dz,
Q_{11} = -\frac{p'}{p}(e^3 - e^0), \quad Q_{22} = -\frac{q'}{q}(e^3 - e^0), \quad \text{others} = 0,
\]

where prime denotes derivative with respect to \( u \). When we insert these components of nonmetricity to the Eqn(39) in the vacuum we obtain \( \Lambda = 0 \). Then the set of equations turns out to be

\[
c_5pq(pq'' + qp'') + 2(c_1 + c_2 + c_4)[(pq')^2 + (qp')^2] + 2(2c_4 + c_5)pqp'q' = 0
\]

Here we checked that GR-equivalent values of \( c_i \)S gives the Einstein equation, namely \( p''/p + q''/q = 0 \). There are various solutions to this equation in the literature, for example \( p = \cos(a_1u + a_2) \) and \( q = \cosh(a_1u + a_3) \) where \( a_i \) are arbitrary constants. On the other hand if \( c_1 + c_2 = 0 \), then \( pq = \text{constant} \) is a non-Einsteinian solution. Consequently we see that STPG predicts gravitational waves as well.
A Dirac field coupling to STPG

Ordinary matter is made up by fermionic particles. Therefore, we investigate a spin-1/2 particle in STPG. We, however, emphasise that since we are in a symmetric teleparallel geometry, not in the RC, the common knowledge of spinor being the source of torsion is not valid here. Instead, all kinds of matter source nonmetricity and also the gravitation is ascribed to the nonmetricity of the spacetime.

We are using the formalism of Clifford algebra $C\ell_{1,3}$-valued exterior forms. The $C\ell_{1,3}$ algebra is generated by the relation among the orthonormal basis $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$

$$\gamma(a\gamma_b) = \eta_{ab}.$$ (65)

One particular representation of the $\gamma^a$'s is given by the following Dirac matrices

$$\gamma_0 = i \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \quad \gamma_1 = i \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \quad \gamma_2 = i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \gamma_3 = i \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$ (66)

where $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices. In this case a Dirac spinor $\Psi$ can be represented by a 4-component column matrix. Thus we write explicitly the covariant exterior derivative of $\Psi$ via the Kosmann lift [8],[12],[26]

$$D \Psi = d \Psi + \frac{1}{4} \omega^{ab}(\eta_{ab} + 2\sigma_{ab})\Psi$$ (67)

where $\sigma_{ab} := \frac{1}{2}\gamma[a\gamma_b]$ are the generators of the Lorentz group. Correspondingly we calculate $D^2 \Psi$

$$D(D \Psi) = \frac{1}{4} R^{ab}(\eta_{ab} + 2\sigma_{ab})\Psi.$$ (68)

Here let us note that in teleparallel geometries as the curvature of cotangent bundle, i.e. $R^a{}_b$, is zero, the curvature of the spinor bundle, i.e. $\frac{1}{4} R^{ab}(\eta_{ab} + 2\sigma_{ab})$, is zero as well. Some relations of the Dirac matrices are

$$\sigma_{ab}\gamma_c = \gamma_c\sigma_{ab} + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b,$$
$$\sigma_{ab}\gamma_c + \gamma_c\sigma_{ab} = -\epsilon_{abcd}\gamma_d\gamma_5,$$
$$\gamma_c\sigma_{ab} = \frac{1}{2}\eta_{ac}\gamma_b - \frac{1}{2}\eta_{bc}\gamma_a - \frac{1}{2}\epsilon_{abcd}\gamma_d\gamma_5,$$
$$[\sigma_{ab}, \sigma_{cd}] = -\eta_{ac}\sigma_{bd} - \eta_{bd}\sigma_{ac} + \eta_{ad}\sigma_{bc} + \eta_{bc}\sigma_{ad},$$ (69)

where $\gamma_5 := \gamma_0\gamma_1\gamma_2\gamma_3$. In order to obtain the Bjorken-Drell conventions [27] one has to replace $\gamma^a \rightarrow -i\gamma^a$ and $\gamma_5 \rightarrow i\gamma_5$. Now we take the minimally coupled Dirac lagrangian given by the hermitian 4-form

$$K = \frac{i}{2} \left( \overline{\Psi} \gamma \wedge D\Psi + D\overline{\Psi} \wedge \gamma \Psi \right) + im \overline{\Psi}\gamma_5 \Psi^1,$$ (70)

where $\gamma := \gamma_0 e^a$ is $C\ell_{1,3}$-valued 1-form and $\overline{\Psi}$ is the Dirac adjoint spinor, $\overline{\Psi} := \Psi^\dagger \gamma_0$. The coframe $e^a$ necessarily occurs in the Dirac Lagrangian, even in special relativity. The hermiticity of the
lagrangian (70) leads to a charge current which admits the usual probabilistic interpretation. Then for a Dirac field we calculate the energy-momentum 3-form

$$t_a = \frac{i}{2} \left[ \overline{\Psi}^\dagger (\gamma \wedge e_a) \wedge D\Psi - D\overline{\Psi} \wedge \gamma^\dagger (\gamma \wedge e_a) \Psi \right] + \text{i} m \overline{\Psi} \Psi^* e_a,$$

the angular momentum 3-form

$$\ell^a_{\, b} = \frac{i}{4} (\overline{\Psi} \gamma_5 \Psi) \wedge e^a_{\, b}.$$ (72)

The matter field equation $\delta K/\delta \Psi = 0$ yields the Dirac equation in the most general non-Riemannian spacetime with $Q_{ab} \neq 0$, $T^a \neq 0$, $R^a_{\, b} \neq 0$;

$$^b \gamma \wedge \left( D - \frac{1}{2} T - Q \right) \Psi + \gamma_{(a} \gamma_{b)} \wedge Q^{ab} \Psi + m \Psi \gamma 1 = 0$$ (73)

where $T = \epsilon_a T^a$ which is zero in STPG. So, in the gauge, $e^\alpha = dx^\alpha$ and $\omega^\alpha_{\, \beta} = 0$, the Dirac equation turns out to be

$$^b \gamma \wedge \left( d - \frac{1}{2} Q \right) \Psi + \gamma_{(a} \gamma_{b)} \wedge Q^{ab} \Psi + m \Psi \gamma 1 = 0.$$ (74)

Before closing the section we want to remark on the world line of a spinning test particle. Although the Mathisson-Papapetrou (MP) equations [28],[29] are the first and successful attempts on this route, it is of interest to obtain alternative formulations for various reasons, see [30],[31] and references therein. For example, as in Ref.[30] the authors apply the MP method based crucially on the matter current densities to spinning test particles moving in the RC spacetime, in [31] the author shows that, starting with a simple lagrangian, one can derive equations of world line of spinning test particles without explicitly referring to spin and stress energy densities in the RC geometry. The case with nonmetricity has not been investigated in the literature, at least to our knowledge. Nevertheless since we know that point particles can be obtained by composing spinor particles, we have to note that there may be inconsistency between definitions (23) and (67). The first simple suggestion to remedy contradictory may be to add a linear combination of $(\epsilon_a Q^b_{\, a}) e^b \Psi$ and $Q \Psi$ to the exterior covariant derivative of the spinor (67). Thus, the equations of motion for spinning particles in a non-Riemannian spacetime with nonmetricity is an original problem which is postponed as a future research project.

9 Concluding Remarks

In this paper we studied the general symmetric teleparallel gravity model within the framework of the generic non-Riemannian formulation. A similar but narrow-scoped work was performed in [12]. Generalizing the previous work in which we had guessed convenient coframe and full connection, we here prescribed a gauge $\omega^\alpha_{\, \beta} = 0$ in the coordinate frame $e^\alpha = dx^\alpha$ for writing
down the nonmetricity directly from a metric ansatz, then obtained new exact solutions with
the help of computer and finally considered the minimal Dirac spinor coupling to STPG in a
consistent way.

We described a peculiar rule for parallel transportation of a tangent vector, see equation
\textbf{(23)}. Thus with a special choice of parameters in our gauge, i.e. \( e^\alpha = dx^\alpha \) and \( \omega^\alpha{}_{\beta} = 0 \), the
autoparallel curves of the symmetric teleparallel geometry coincide with those of the Riemannian
geometry. We also showed that certain values of the parameters \( c_i \) correspond GR at the level
of lagrangian and solutions. It is important to know them because the GR-equivalent STPG,
for obvious reasons, is satisfactorily supported by observations. Since the first most important
motivation for a new theory of gravity was the quantization problems of GR, and since the
conformally invariant theories predict better UV behavior \cite{32}, we investigated that type of
solution which is non-Einsteinian. But whether our scale-invariant solution really makes sense
at the quantum level is an open issue. We could find three classes of the spherically symmetric
static solutions. Two of them are non-Einsteinian. Nevertheless there may be more solutions of
this type. The second most important motivation for looking for a new theory of gravity was
the observational data on cosmic speed-up. Correspondingly, we investigated a cosmological
solution for a non-spinning perfect fluid source with assumption of the modified Chaplygin
gas. Although it is non-Einsteinian (since \( b \neq 0 \) in equation \textbf{(60)}) , it qualitatively behaves like
Einsteinian solution with a cosmological constant. Thus, our cosmological solution can predict
accelerating universe by some suitable adjustments of coupling parameters. Furthermore since
apart from Einsteinian solution we found a non-Einsteinian pp-wave solution, we conclude that
STPG predicts the existence of gravitational waves. In this context, we want to remark that
an intriguing physical quantity that may be associated with such waves is the helicity of the
gravitational field. We postpone a detailed analysis on this issue for a future work. Finally
because of the fact that matter is mainly made up by fermions it was indispensable to elaborate
consistently on a minimal Dirac spinor coupling to STPG in order to make our model self-
contained as much as possible. Thus we wrote down naively the Dirac equation in the most
general non-Riemannian geometry and then reduced it to the symmetric teleparallel geometry
in our gauge.

\textbf{Acknowledgment}

MA thanks T Dereli for the stimulating discussions, F W Hehl for the fruitful correspondences
and B Tekin for the valuable conversations especially on Section \textbf{4}.
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