POTENTIAL–DENSITY PAIRS FOR A FAMILY OF FINITE DISKS

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ABSTRACT

Exact analytical solutions are given for the three finite disks with surface density \( \Sigma_n = \sigma_0 (1 - R^2/\alpha^2)^{n-1/2} \) with \( n = 0, 1, 2 \). Closed-form solutions in cylindrical coordinates are given using only elementary functions for the potential and for the gravitational field of each of the disks. The \( n = 0 \) disk is the flattened homeoid for which \( \Sigma_{\text{hom}} = \sigma_0 \sqrt{1 - R^2/\alpha^2} \). Improved results are presented for this disk. The \( n = 1 \) disk is the Maclaurin disk for which \( \Sigma_{\text{mac}} = \sigma_0 \sqrt{1 - R^2/\alpha^2} \). The Maclaurin disk is a limiting case of the Maclaurin spheroid. The potential of the Maclaurin disk is found here by integrating the potential of the \( n = 0 \) disk over \( \alpha \), exploiting the linearity of Poisson’s equation. The \( n = 2 \) disk has the surface density \( \Sigma_{D2} = \sigma_0 (1 - R^2/\alpha^2)^{3/2} \). The potential is found by integrating the potential of the \( n = 1 \) disk.

Key words: celestial mechanics – galaxies: individual (NGC 891) – galaxies: kinematics and dynamics – gravitation – methods: analytical

1. INTRODUCTION

Observations of edge-on galaxies now provide detailed structural and kinematic three-dimensional information. However, attempts to use this data to generate mass distribution have not been completely successful. One problem is that it is difficult to calculate the potential and force vectors for finite disks. There is a need for fully solved finite disks which can be used for theoretical studies, for benchmarking computer programs, and as basis functions to directly model three-dimensional observations.

Mass modeling commonly assumes that the disk mass is contained in an infinite disk. These include the exponential disk (Freeman 1970), the Mestel disk (Mestel 1963), the Kuzmin–Toomre disk (Toomre 1963; Binney & Tremaine 2008; Evans & de Zeeuw 1992; Conway 2000), and the Rybicki disk (Evans & Collett 1993).

Only a few finite disks have been solved analytically for all \((R, z)\). Lass & Blitzer (1983) and Vokrouhlický & Karas (1998) give a solution for all \((R, z)\) for a finite thin disk with constant density. The gravitational attraction approaches infinity at the edge of this disk. The gravitational attraction of the finite Mestel disk (Mestel 1963; Lynden-Bell & Pineault 1978; Hunter et al. 1984) and the truncated exponential disk (Casertano 1983) are finite at the edge of the disk but these disks have not been solved in closed form for points off the disk. Huré et al. (2008) and Huré (2005) describe a method to approximate the potential of a power-law disk for points both on and off the disk.

The family of finite disks with surface density \( \Sigma_n(R; \alpha) = (1 - R^2/\alpha^2)^{n-1/2} \) is related to the Maclaurin spheroid which has been studied since the time of Newton. This family has recently been studied by González & Reina (2006) and Pedraza et al. (2008). González & Reina (2006) use the method of Hunter (1963) to obtain the general solution as a sum of Legendre polynomials in elliptical coordinates and give evaluated expressions for the potentials for the disks 1, 2, and 3. Here, we derive complete closed-form solutions in cylindrical coordinates for the potential and the gravitational fields of the \( n = 0, 1, 2 \) disks. We simplify the integration by moving to the imaginary domain in a way which is similar to the complex-shift method introduced by Appell. See Ciotti & Marinacci (2008); Ciotti & Giampieri (2007) and references therein.
\[ \Sigma_{\text{hom}}(R; \alpha) = \begin{cases} \sigma_0/\sqrt{1 - R^2/\alpha^2} & \text{for } R < \alpha \\ 0 & \text{for } R > \alpha \end{cases} \] (1)

Lynden-Bell (1989) gives a formula for the external potential of the thin homeoidal shell. Taking the limit \( e = (1 - \alpha^2)^{1/2} \rightarrow 0 \) gives a solution for the disk which is valid at all \( R \) and \( z \).

Cuddeford (1993) gives an expression for the potential of this disk which is simpler than previous solutions. Equation (4) gives a solution for the disk which is valid at all \( R \).

\[ \Phi_{\text{hom}}(R, z; \alpha) = \frac{2\alpha}{\sqrt{z^2 + (R + \alpha)^2}} \frac{\alpha - 1z}{R} \frac{\alpha + 1z}{R} \] (2)

Equation (2) can be further simplified. First, make the trivial transformation

\[ \Phi_{\text{hom}}(R, z; \alpha) = -2\pi\alpha\sigma_0 G \sin^{-1} \left[ \frac{\alpha - 1z}{\sqrt{R^2 - (\alpha - 1z)^2}} + \frac{\alpha + 1z}{\sqrt{R^2 + (\alpha - 1z)^2}} \right] \] (3)

Now use the identity A1 to obtain

\[ \Phi_{\text{hom}}(R, z; \alpha) = -\pi\alpha\sigma_0 G \left[ \sin^{-1} \left( \frac{\alpha - 1z}{R} \right) + \sin^{-1} \left( \frac{\alpha + 1z}{R} \right) \right] \] (4)

Equation (4) can be integrated over \( \alpha \) whereas Equation (3) yields an impossible integral. Equation (4) must be real valued on physical grounds. This is easy to prove by noting that \( \sin^{-1}(x) \) is an odd function of \( x \) and so the odd powers of \( I_z \) in the series expansion of Equation (4) cancel, leaving a real-valued result.

The gravitational field for the collapsed homeoid is obtained from the potential. Equation (4) yields particularly simple expressions for the field vectors \( F_{R, \text{hom}} \) and \( F_{z, \text{hom}} \)

\[ F_{R, \text{hom}}(R, z; \alpha) = -\pi\alpha\sigma_0 G \left[ \frac{\alpha - 1z}{\sqrt{R^2 - (\alpha - 1z)^2}} + \frac{\alpha + 1z}{\sqrt{R^2 + (\alpha - 1z)^2}} \right] \] (5)

\[ F_{z, \text{hom}}(R, z; \alpha) = -\pi\alpha\sigma_0 G \left[ \frac{1}{\sqrt{R^2 - (\alpha - 1z)^2}} - \frac{1}{\sqrt{R^2 - (\alpha + 1z)^2}} \right] \] (6)

These force vectors can be expressed as entirely real functions using Identities A8 and A9

\[ F_{R, \text{hom}}(R, z; \alpha) = -\sqrt{2}\pi\alpha\sigma_0 G \frac{\alpha\sqrt{f_1 f_2 - f_3 - |z|} \sqrt{f_1 f_2 + f_3}}{R f_1 f_2}, \] (7)

\[ F_{z, \text{hom}}(R, z; \alpha) = -\sqrt{2}\pi\alpha\sigma_0 G \frac{\text{sgn}(z)\sqrt{f_1 f_2 + f_3}}{f_1 f_2}, \] (8)

where

\[ f_1 = \sqrt{\alpha^2 + (R + \alpha)^2} \]
\[ f_2 = \sqrt{\alpha^2 + (R - \alpha)^2} \]
\[ f_3 = \alpha^2 - R^2 - z^2. \] (9)

3. THE \( n = 0 \) DISK: THE MACLAURIN DISK

Beginning in the early 18th century Colin Maclaurin, along with James Ivory and many others, studied the properties of elliptical bodies. Chandrasekhar (1987) includes a very good historical summary. See also Binney & Tremaine (2008); Bertin (2000); Schmidt (1956); Mihalas & Routly (1968); Kalnajs (1971, 1972).

The homogeneous oblate spheroid is the simplest case of a spinning body for which the gravitational attraction balances the centrifugal force. The Maclaurin disk, also known as the Kalgren disk (Kalnajs 1972), is a limiting case for which minor axis is 0. The Maclaurin disk is defined by the surface density

\[ \Sigma_{\text{Mac}}(R; \alpha) = \begin{cases} \sigma_0/\sqrt{1 - R^2/\alpha^2} & \text{for } R < \alpha \\ 0 & \text{for } R > \alpha \end{cases} \] (10)

There are a few solutions for the potential of the Maclaurin disk in the literature. Mihalas & Routly (1968) gives an expression for the potential of an oblate homogeneous spheroid based on the derivation in Schmidt (1956). The potential of the Maclaurin disk can be found by letting the eccentricity \( e \rightarrow 1 \) while holding the mass constant. Hunter (1963) gives the solution for the Maclaurin disk as a series of Legendre polynomials in elliptic coordinates. Neugebauer & Meinel (1995), Meinel (2001), and González & Reina (2006) give a closed form solution for the potential of the Maclaurin disk in elliptic coordinates.

The starting point here is the potential–density pair for the \( n = 0 \) disk, the flattened homeoid for which \( \Sigma_{\text{hom}}(R; \alpha) = \sigma_0/\sqrt{1 - R^2/\alpha^2} \). The surface density of the Maclaurin disk is found from the transformation

\[ \Sigma_{\text{Mac}}(R; \alpha) = \frac{1}{\alpha} \int_0^\alpha \Sigma_{\text{hom}}(R; \hat{\alpha}) d\hat{\alpha} \]
\[ = \frac{1}{\alpha} \int_0^\alpha \sigma_0 \frac{\sqrt{1 - \hat{\alpha}^2/\alpha^2}}{\hat{\alpha}} d\hat{\alpha} \]
\[ = \sigma_0 \sqrt{1 - R^2/\alpha^2}. \] (11)

The corresponding potential is

\[ \Phi_{\text{Mac}}(R, z; \alpha) = \frac{1}{\alpha} \int_0^\alpha \Phi_{\text{hom}}(R; \hat{\alpha}) d\hat{\alpha} \]
\[ = -\pi\sigma_0 G \int_0^\alpha \hat{\alpha} \left[ \sin^{-1} \left( \frac{\hat{\alpha} - 1z}{\hat{\alpha} - 1z} \right) \right] d\hat{\alpha}, \] (12)

where the expression for \( \Phi_{\text{hom}} \) is given by Equation (4). Use integral 2.813 and 2.833 from Gradshteyn & Ryzhik (1994) to obtain
The radial force vector in the $z = 0$ plane is found by taking the limit of Equation (15) or by differentiating Equation (18) with respect to $R$.

$$F_{R,Mac}(R, 0; \alpha) = \begin{cases} -\frac{\pi \sigma_0 G}{2\alpha} \left[ R \sin^{-1}(\frac{\alpha}{\sqrt{\alpha^2 + z^2}}) - \alpha \sqrt{1 - \frac{\alpha^2}{R^2}} \right] & \text{for } R \leq \alpha, \\ -\frac{\pi \sigma_0 G}{2\alpha} (2\alpha^2 - R^2) \sin^{-1} \left( 1 + \frac{1}{R} \right) - \alpha & \text{for } R > \alpha. \end{cases}$$

The axial force vector on the $z$ axis is found by taking the limit of Equation (15) or differentiating Equation (17) with respect to $z$.

$$F_{z,Mac}(0, z; \alpha) = -2\pi \sigma_0 G \left[ z \sin^{-1} \left( \frac{\alpha}{\sqrt{\alpha^2 + z^2}} \right) - \alpha \text{sgn}(z) \right]$$

4. THE $n = 0$ DISK

González & Reina (2006) give a closed-form solution for the potential of the $n = 2$ disk in elliptic coordinates.

The disk surface density of the $n = 2$ disk is

$$\Sigma_{D2}(R; \alpha) = \begin{cases} \sigma_0(1 - R^2/\alpha^2)^{3/2} & \text{for } R < \alpha, \\ 0 & \text{for } R > \alpha. \end{cases}$$

This mass distribution can be obtained from Equation (10), the disk surface density of the $n = 1$ disk, with the transformation

$$\Sigma_{D2}(R; \alpha) = \frac{3}{\alpha^3} \int_0^R \hat{\alpha}^2 \Phi_{Mac}(R, \hat{\alpha}) d\hat{\alpha}$$

The corresponding potential is

$$\Phi_{Mac}(R, \hat{\alpha}) = \frac{3}{\alpha^3} \int_0^R \hat{\alpha}^2 \Phi_{Mac}(R, \hat{\alpha}) d\hat{\alpha}.$$
Expressions were expressed as entirely real functions by using

$$\Phi_{D2}(R, z; \alpha) = -\frac{\pi \sigma_0 G}{64\alpha^3} \left[ 6(8\alpha^4 - 8\alpha^2 R^2 + 16\alpha^2 z^2 + 3R^4)
- 24R^2 z^2 + 8z^4) \sin^{-1}\left(\frac{f_1 - f_2}{2R}\right)
+ \sqrt{2}\alpha(18\alpha^2 - 9R^2 + 26z^2) \sqrt{f_1 f_2 - f_3}
- \sqrt{2} | z | (58\alpha^2 - 55R^2 + 50z^2) \sqrt{f_1 f_2 + f_3},
\right]$$

(25)

where \(f_1, f_2, f_3\) are given by Equation (9).

The gravitational field of the \(n = 2\) disk is the gradient of the potential using \(\Phi\) as given by Equation (24). The resulting expressions were expressed as entirely real functions by using the Identities A1, A8, and A9

$$F_{R,D2}(R, z; \alpha) = -\frac{3\pi \sigma_0 G}{16\alpha^3} \left[ 2R^2(4\alpha^2 R^2 - 3R^4)
+ 12R^2 z^2 \sin^{-1}\left(\frac{f_1 - f_2}{2R}\right)
+ \sqrt{2}\alpha(2\alpha^4 - 5\alpha^2 R^2 + 4\alpha^2 z^2 + 3R^4)
- 25R^2 z^2 + 2z^4) \sqrt{f_1 f_2 - f_3}
+ \sqrt{2} | z | (2\alpha^4 + 9\alpha^2 R^2 + 4\alpha^2 z^2 - 13R^4)
- 11R^2 z^2 + 2z^4) \sqrt{f_1 f_2 + f_3},
\right]$$

(26)

$$F_{z,D2}(R, z; \alpha) = -\frac{\pi \sigma_0 G}{4\alpha^3} \left[ -6z(2\alpha^2 - 3R^2 + 2z^2) \sin^{-1}\left(\frac{f_1 - f_2}{2R}\right)
+ \sqrt{2}\alpha z(13\alpha^2 - 13R^2 + 17z^2) \sqrt{f_1 f_2 - f_3}
\right],$$

where \(f_1, f_2, f_3\) are given by Equation (9).

The radial force vector in the \(z = 0\) plane is found by taking the limit of Equation (26) or differentiating Equation (29) with respect to \(R\).

$$F_{R,D2}(R, 0; \alpha) = \begin{cases} 
-\frac{3\pi \sigma_0 G}{64\alpha^3} (8\alpha^4 - 8\alpha^2 R^2 + 3R^4) & \text{for } R \leq \alpha \\
-\frac{3\pi \sigma_0 G}{32\alpha} (8\alpha^4 - 8\alpha^2 R^2 + 3R^4) & \text{for } R \geq \alpha 
\end{cases}$$

(29)

$$= \left\{-\frac{3\pi \sigma_0 G}{64\alpha^3} (8\alpha^4 - 8\alpha^2 R^2 + 3R^4) \right\} \sin^{-1}\left(\frac{\alpha}{R}\right) + 3\alpha(2\alpha^2 - R^2)
\times \sqrt{R^2 - \alpha^2}$$

for \(R \geq \alpha\).

The radial force vector in the \(z = 0\) plane is found by taking the limit of Equation (26) or differentiating Equation (29) with respect to \(R\).

$$F_{R,D2}(R, 0; \alpha) = \begin{cases} 
-\frac{3\pi \sigma_0 G}{16\alpha^3} (4\alpha^2 - 3R^2) & \text{for } R \leq \alpha \\
-\frac{3\pi \sigma_0 G}{8\alpha^3} [R(4\alpha^2 - 3R^2) \sin^{-1}(\alpha/R)] & \text{for } R \geq \alpha 
\end{cases}$$

(30)
Table 1: Comparison of the Maclaurin and the $n = 2$ Disks

| Property | Maclaurin Disk | $n = 2$ Disk |
|----------|---------------|-------------|
| Surface density $\Sigma(R) = \sigma_0 \sqrt{1 - R^2/\alpha^2}$ | $\sigma_0(1 - R^2/\alpha)^{3/2}$ |
| Total mass $M = \frac{1}{2} \pi a^2 \sigma_0$ | $\frac{7}{2} \pi a^2 \sigma_0$ |
| Circular velocity $V_c^2(R, 0) = \frac{\pi R^2 \sigma_0 G}{2a}$ | $\frac{3\pi R^2 \sigma_0 G(4a^2 - 3R^2)}{16\alpha^3}$ |
| Disk edge velocity $V_e^2(\alpha, 0) = \frac{3\piMG}{4\alpha}$ | $\frac{3\pi^2 \sigma_0 G}{32\alpha^3}$ |

The axial force vector on the $z$ axis is found by taking the limit of Equation (27) or differentiating Equation (28) with respect to $z$,

$$F_z = \frac{\pi \sigma_0 G}{2a^3(a^2 + z^2)} \left[ -6(a^2 + z^2)^2 z \sin^{-1} \left( \frac{a}{\sqrt{a^2 + z^2}} \right) + \alpha z^2(13a^2 + 17z^2) + \alpha \text{sgn}(z) \right] \times (4a^4 - 3a^2 z^2 - 11z^4). \quad (31)$$

4.1. Comparison of the Maclaurin Disk and the $n = 2$ Disk

Table 1 compares important properties of the two disks. Figure 1(a) compares the surface mass density and potential. Figure 1(b) compares rotational velocity in the disks. The $n = 2$ disk is more centrally concentrated than the Maclaurin disk. The rotational velocity increases more quickly in the inner disk and begins to fall before reaching the edge of the disk. As is apparent from Figure 1(b), the derivative of the circular velocity of the Maclaurin disk is discontinuous at the edge of the disk whereas the $n = 2$ disk is better behaved.

5. EXAMPLE: THE FORCE FIELD OF A SIMPLE GALAXY MODEL

Three-dimensional problems such as that of the structure and kinematics of the extraplanar gas will benefit from the use of the new density–potential pairs. A simple galaxy model was constructed for illustration. The model consists of an $n = 2$ disk and a core/bulge region modeled as a point mass. The model is defined by three parameters: the mass of the disk, the mass of the core/bulge region, and the diameter of the disk. As shown in Figure 2, the circular velocity, calculated as $V_c = \sqrt{-V_F^2(R, z)}$, is nearly constant over much of the disk. Also, the derivative of the circular velocity with $z$ is nearly linear over a wide range of both $R$ and $z$.

Figure 2 agrees surprisingly well with Figure 5 of Fraternali et al. (2005) which shows that the measured velocity of HI for NGC891 decreases linearly with the height above the disk. See also Rand (1997); Swaters et al. (1997); Kamphuis et al. (2007); Oosterloo et al. (2007); Fraternali & Binney (2006); Barnabe et al. (2006). Further work is planned on this topic.

6. SUMMARY AND CONCLUSION

We have presented new solutions for a family of finite disks. Closed-form expressions in cylindrical coordinates using elementary functions are given for the potential and gravitational force for the disks with surface density $\Sigma_n = \sigma_0(1 - R^2/\alpha^2)^{n-1/2}$ with $n = 0, 1, 2$. Expressions are also given for the limiting cases of $R = 0$ and $z = 0$.

These solutions fill a need and should make it easier to model three-dimensional gravitational phenomenon involving disk galaxies. This is particularly important due to the recent availability of detailed kinematic data above the plane of the disk.

I am grateful to the anonymous referee for useful suggestions and comments which improved the presentation.

APPENDIX A

SOME SIMPLE IDENTITIES

A number of identities are gathered here. In all cases $x, y \in \mathbb{R}$. The ranges of validity must avoid the discontinuity of the principal value of the square root function on the negative real axis.

Equation (A1) can be proved by taking the sine of both sides; reducing the terms using the identities $\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)$ and $\sin(2a) = 2 \sin(a) \cos(b)$; and substituting $\cos = \sqrt{1 - \sin^2}$. The other identities can be found by substituting into the relation

$$\sqrt{x + iy} = \sqrt{\sqrt{x^2 + y^2} + x + \text{sgn}(y)\sqrt{\sqrt{x^2 + y^2} - x}}$$

and into the expression found by taking the reciprocal of both sides.

$$\sin^{-1}(x - iy) + \sin^{-1}(x + iy) = 2 \sin^{-1}\left[ \frac{1}{2} \sqrt{(x + 1)^2 + y^2} \right]$$

$$- \frac{1}{2} \sqrt{(x - 1)^2 + y^2} \quad (A1)$$
$$\sqrt{x - 1} y + \sqrt{x + 1} y = 2 \sqrt{x^2 + y^2 + x}$$  \hspace{1cm} (A2)$$

$$I \sqrt{x - 1} y - I \sqrt{x + 1} y = \text{sgn}(y) 2 \sqrt{x^2 + y^2 - x}$$  \hspace{1cm} (A3)$$

$$\frac{1}{\sqrt{x + 1} y} + \frac{1}{\sqrt{x - 1} y} = 2 \sqrt{x^2 + y^2 + x}$$  \hspace{1cm} (A4)$$

$$\frac{1}{\sqrt{x + 1} y} - \frac{1}{\sqrt{x - 1} y} = \text{sgn}(y) 2 \sqrt{x^2 + y^2 - x}$$  \hspace{1cm} (A5)$$

$$\sqrt{1 - (x + 1)^2} + \sqrt{1 - (x - 1)^2} = 2 \sqrt{1 - 2x^2 + 2y^2 + x^4 + 2x^2y^2 + y^4 + 1 - x^2 + y^2}$$  \hspace{1cm} (A6)$$

$$I \sqrt{1 - (x + 1)^2} - I \sqrt{1 - (x - 1)^2} = \text{sgn}(xy) \times 2 \sqrt{1 - 2x^2 + 2y^2 + x^4 + 2x^2y^2 + y^4 - 1 + x^2 - y^2}$$  \hspace{1cm} (A7)$$

$$\frac{1}{\sqrt{1 - (x - 1)^2}} + \frac{1}{\sqrt{1 - (x + 1)^2}} = 2 \sqrt{1 - 2x^2 + 2y^2 + x^4 + 2x^2y^2 + y^4 - 1 - x^2 + y^2}$$  \hspace{1cm} (A8)$$

$$\frac{1}{\sqrt{1 - (x - 1)^2}} - \frac{1}{\sqrt{1 - (x + 1)^2}} = \text{sgn}(xy) \times 2 \sqrt{1 - 2x^2 + 2y^2 + x^4 + 2x^2y^2 + y^4 - 1 + x^2 - y^2}$$  \hspace{1cm} (A9)$$

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1. Discussion

Review of the published article found errors that affect Equations (5), (13), (20), (26), and (29). These are purely typographical errors that do not affect the other equations, numerical results, or supporting calculations.

In Equation (5), the operand of the second square root term should be \((R^2 - (\alpha + I_z)^2)\), not \((R^2 + (\alpha - I_z)^2)\). In Equation (13), the denominator of the lead term, not the numerator, should include the term \(\alpha\); and the sign of last term is incorrect. In Equation (20), the leading sign is incorrect. In Equation (26), the first term includes an extra factor of \(R^2\); and the sign of the last term is incorrect. In Equation (29), for \(R > \alpha\), the denominator of the lead term should include the term \(\alpha^3\), not \(\alpha\).

The corrected equations are as follows:

\[
F_{R,\text{hom}}(R, z; \alpha) = -\frac{\pi \sigma G}{R} \left[ \frac{\alpha - I_z}{\sqrt{R^2 - (\alpha - I_z)^2}} + \frac{\alpha + I_z}{\sqrt{R^2 - (\alpha + I_z)^2}} \right] \tag{5}
\]

\[
\Phi_{\text{Mac}}(R, z; \alpha) = -\frac{\pi \sigma G}{4\alpha} \left[ \left( 2\alpha^2 - R^2 + 2z^2 \right) \sin^{-1} \left( \frac{\alpha + I_z}{R} \right) + \sin^{-1} \left( \frac{\alpha - I_z}{R} \right) \right] + \alpha \left( \sqrt{R^2 - (\alpha - I_z)^2} + \sqrt{R^2 - (\alpha + I_z)^2} \right) + 3I_z \left( \sqrt{R^2 - (\alpha - I_z)^2} - \sqrt{R^2 - (\alpha + I_z)^2} \right) \tag{13}
\]

\[
F_{c,\text{Mac}}(0, z; \alpha) = \frac{2\pi \sigma G}{R} \left[ z \sin^{-1} \left( \frac{\alpha}{\sqrt{\alpha^2 + z^2}} \right) - \alpha \text{sgn}(z) \right] \tag{20}
\]

\[
F_{R,\text{DD}}(R, z; \alpha) = -\frac{3\pi \sigma G}{16R\alpha^3} \left[ 2R^2 (4\alpha^2 - 3R^2 + 12z^2) \sin^{-1} \left( \frac{f_1 - f_2}{2R} \right) \right.
\]
\[
+ \sqrt{2} \alpha (2\alpha^4 - 5\alpha^2 R^2 + 4\alpha^2 z^2 + 3R^4 - 25R^2 z^2 + 2z^4) \sqrt{f_1 f_2 - f_3}
\]
\[
- \sqrt{2} |z|(2\alpha^4 + 9\alpha^2 R^2 + 4\alpha^2 z^2 - 13R^4 - 11R^2 z^2 + 2z^4) \sqrt{f_1 f_2 + f_3} \right] \tag{26}
\]

\[
\Phi_{\text{DD}}(R, 0; \alpha) = -\frac{3\pi \sigma G}{32\alpha^3} \left[ (8\alpha^4 - 8\alpha^2 R^2 + 3R^4) \sin^{-1} \left( \frac{\alpha}{R} \right) + 3\alpha (2\alpha^2 - R^2) \sqrt{R^2 - \alpha^2} \right] \text{ for } R \geq \alpha. \tag{29b}
\]

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