A Single-Letter Upper Bound on the Mismatch Capacity via Multicast Transmission

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Abstract

We derive a single-letter upper bound on the mismatch capacity of a stationary memoryless channel with decoding metric $q$. Our bound is obtained by considering a multicast transmission over a 2-user broadcast channel with decoding metrics $q$ and $\rho$ at the receivers, referred to as $(q, \rho)$-surely degraded. The channel has the property that the intersection event of successful $q$-decoding of receiver 1 and erroneous $\rho$-decoding of receiver 2 has zero probability for any codebook of a certain composition $P$. Our bound holds in the strong converse sense of exponential decay of the probability of correct decoding at rates above the bound. Several examples which demonstrate the strict improvement of our bound compared to previous results are analyzed.

Further, we detect equivalence classes of isomorphic channel-metric pairs $(W, q)$ that share the same mismatch capacity. We prove that if the class contains a matched pair, then our bound is tight and the mismatch capacity of the entire class is fully characterized and can be achieved by random coding.

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I. INTRODUCTION

One of the most intriguing open problems in Information Theory concerns the fundamental limits of channel coding with a fixed and possibly suboptimal decoder, where only the codebook can be optimized. This problem, termed mismatched decoding, is closely related to other fundamental information theoretic problems such as the zero error capacity of the discrete memoryless channel (DMC). The question of characterizing the mismatch capacity of a stationary memoryless channel by a single-letter expression (if there is one) is a long standing open problem.

Achievable rates for channels with mismatched decoding have been studied extensively, especially for DMCs. The simplest lower bound called the GMI [1] is achievable by i.i.d. random coding, and is given by:

\[
R_{q,GMI}(W) = \max_{Q_X} \min_{\tilde{P}_{XY}} D(\tilde{P}_{XY}||Q_X \times P_Y),
\]

where \(P_{XY} = Q_X \times W\), and \(P_Y\) is the marginal \(Y\) distribution. Csiszár and Körner [2] and Hui [3] derived the following formula for the achievable rate using random constant composition coding for the DMC \(W\) from \(X\) to \(Y\) and decoding metric \(q\):

\[
R_{q,LM}(W) = \max_{Q_X} \min_{\tilde{P}_{XY}} I_{\tilde{P}}(X;Y),
\]

The rate \(R_{q,LM}(W)\) is called the LM rate, and its multi-letter extension to the channel \(W^k\) from \(X^k\) to \(Y^k\), is also achievable, and in certain cases can exceed the LM rate [4].

Lapidoth [5] introduced an improved lower bound on the mismatch capacity of the DMC by studying the achievable sum-rate of an appropriately chosen mismatched multiple access channel (MAC), whose codebook was obtained by expurgating codewords from the product of the codebooks of the two users. In [6], [7] the achievable region and error exponents of a cognitive MAC were considered using superposition coding or random binning whose sum-rate serves as a lower bound on the capacity of the single-user channel. An improved bound was presented by Scarlett et al. [8] using a refinement of the superposition coding ensemble. For given auxiliary random variables, the results of [6]–[8] may yield improvement of the achievable rates of [5] for the DMC. For other related works and extensions see the survey on Information-Theoretic foundations of mismatched decoding [9] and references therein, such as [1], [10]–[18].

While there have been quite a few works on achievable rates, and some works on multi-letter expressions and upper bounds on the mismatch capacity [19] [20], much less has been known
about single-letter upper bounds. Csiszár and Narayan [4] proved that a necessary condition for the positivity of the mismatch capacity is the positivity of the LM rate. For the binary input binary output case, the mismatch capacity was fully characterized in [4]. They showed that the mismatch capacity $C_q(W)$ is equal to the Shannon capacity, $C(W)$, if $W(0|1) + W(1|0) - 1$ and $q(0,1) + q(1,0) - q(0,0) - q(1,1)$ have the same sign, and otherwise $C_q(W) = 0$. The single-letter converse result reported in [21] for binary-input DMCs was disproved in [22]. Specifically, a rate based on superposition coding was shown to exceed the claimed mismatch capacity of [21].

In a recent work, [23], [24], Kangarshahi and Guillén i Fàbregas presented a single-letter upper bound on $C_q(W)$, denoted $\bar{R}_q(W)$, for a general DMC $W$ with an additive metric $q$. They showed that in certain cases, this bound is strictly lower than the matched capacity, and in the binary input binary output case gives Csiszár and Narayan’s above mentioned capacity formula. They also proved that if $R > \bar{R}_q(W)$, then the maximal error probability converges to 1 exponentially fast. Other properties of the bound were studied in [25], and the proofs are given in the full version paper [24], where also the convergence of a numeric algorithm to calculate $\bar{R}_q(W)$ is proved and analyzed. It is also proved that the multi-letter form of the bound is equal to the single-letter bound. The proof of the bound of [23], [24] uses the method of types and graph theory, constructing a graph in the $Y^n$ space such that if maximum likelihood (ML) decoding at the output of another channel $P_{Y'|X}$ makes a type conflict error for some $y \in Y^n$, then, the $q$-decoder makes an error for some $y' \in Y^n$ connected to $y$ in the graph. The bound is expressed as the mutual information of a transformation of the channel, such that a maximum-likelihood decoding error on the translated channel $P_{Y'|X}$ implies a mismatched-decoding error in the original channel $P_{Y|X}$.

As we shall see, the class of transformations that was considered in [23], [24] includes only channels $P_{Y'|X}$ such that $q$-decoding at their output is at least as successful as it is for the original channel $P_{Y|X}$, for every possible codebook.

In this paper we derive a single-letter upper bound on $C_q(W)$. Our bound is based on considering a set of broadcast channels that assign zero probability to the intersecting event of successful $q$-decoding by the $Y$-receiver and erroneous $\rho$-decoding by the $Z$-receiver for every codebook of a certain composition of the input distribution $P$. Here, $q$ is the decoding metric of interest, and $\rho$ is some metric which can be optimized to yield the tightest bound, including for example the ML metric w.r.t. the marginal channel to the $Z$-receiver.
While our bound is always at least as tight as that of [23, 24] (for appropriate choices of \( \rho \)), we show that there are many cases in which our bound is strictly tighter. For example, in the particular case of a 5-letter noiseless channel with the pentagon connectivity graph metric, it turns out that
\[
\tilde{R}_q(W) = C(W) = \log_2(5) \text{[bits/channel use]}
\]
and we show that our bound \( \overline{C}_q(W) \) satisfies
\[
\overline{C}_q(W) \leq \log_2(5/2) \text{[bits/channel use]}.
\]
Note that in this case, Lovász [26] established that \( C_q(W) = \log_2 \sqrt{5} \) [bits/channel use] (the zero error capacity of the 5-letter typewriter channel).

Our bounding technique also generalizes Csiszár and Narayan’s observation that the zero-error capacity \( C_0(W) \) of the DMC \( W \) is equal to the mismatch capacity of the noiseless channel with input and output alphabets \( X \) and the decoding metric \( q_0 \) induced by the connectivity graph associated with the channel. This metric is given by \( q_0(x, x') = 1 \) iff \( W(y|x) \cdot W(y|x') > 0 \) for some \( y \in Y \) and \( q_0(x, x') = 0 \) otherwise. This enables to restate the obvious inequality \( C_0(W) \leq C(W) \) as an inequality between two mismatch capacities of two different channels.

Finally, we introduce a relation of superiority between channel-metric pairs, and we show that it is a transitive relation. We detect an isomorphism between channel-metric pairs superior w.r.t. one another, and we define equivalence classes of isomorphic pairs. We show that if there exists a matched channel-metric pair \((\tilde{W}, \tilde{q}_{ML})\), where \( \tilde{q}_{ML} \) is the maximum likelihood metric w.r.t. \( \tilde{W} \) which is isomorphic to \((W, q)\), then \( C_q(W) = R_{q,LM}(W) = R_{q,GMI}(W) = C(\tilde{W}) \), i.e., the LM rate is equal to the mismatch capacity and it is also equal to the matched capacity of \( \tilde{W} \).

The existence of an isomorphic matched channel-metric pair is thus a sufficient condition for the tightness of our bound. This also yields, as a special case, a sufficient condition for a metric to be capacity achieving for a certain channel. We further extend this notion to isomorphism for a given codebook composition.

This paper is organized as follows. After a short presentation of notational conventions in Section II, we present the mismatch decoding problem formally in Section III. In Section IV we present our main results: Section IV-A is devoted to a simple bound, looser than our main result, which holds for additive metrics and stationary memoryless channels. Section IV-B presents our main result which is a bound for type-dependent metrics. Section IV-C introduces equivalence classes of channel-metric pairs and a sufficient condition for the tightness of our bound, and
Section IV-D present how to adapt our second bound to continuous input alphabet channels with a cost constraint. Section IV-E presents some examples, and in Section V we state some concluding remarks.

II. Notation

Throughout this paper, scalar random variables are denoted by capital letters, their sample values are denoted by their respective lower case letters, and their alphabets are denoted by their respective calligraphic letters, e.g. \( X, x, \) and \( \mathcal{X} \), respectively. A similar convention applies to random vectors of dimension \( n \) and their sample values, which are denoted in boldface; e.g., \( x \). The set of all \( n \)-vectors with components taking values in a certain finite alphabet are denoted by the same alphabet superscripted by \( n \), e.g., \( \mathcal{X}^n \). Logarithms are taken to the natural base \( e \), unless stated otherwise.

For a given sequence \( y \in \mathcal{Y}^n \), where \( \mathcal{Y} \) is a finite alphabet, \( \hat{P}_y \) denotes the empirical distribution on \( \mathcal{Y} \) extracted from \( y \); in other words, \( \hat{P}_y \) is the vector \( \{ \hat{P}_y(y), y \in \mathcal{Y} \} \), where \( \hat{P}_y(y) \) is the relative frequency of the symbol \( y \) in the vector \( y \). The type-class of \( x \) is the set of \( x' \in \mathcal{X}^n \) such that \( \hat{P}_{x'} = \hat{P}_x \), which is denoted \( \mathcal{T}(\hat{P}_x) \). The set of all probability distributions on \( \mathcal{X} \) is denoted by \( \mathcal{P}(\mathcal{X}) \), and the set of empirical distributions of order \( n \) on alphabet \( \mathcal{X} \) is denoted \( \mathcal{P}_n(\mathcal{X}) \).

Information theoretic quantities such as entropy, conditional entropy, and mutual information are denoted following the usual conventions in the information theory literature, e.g., \( H(X) \), \( H(X|Y) \), \( I(X;Y) \) and so on. To emphasize the dependence of the quantity on a certain underlying probability distribution, say \( \mu \), we may use notations such as \( H(\mu) \), \( H(\mu_{X|Y}) \), \( I(\mu_{XY}) \), etc. The expectation operator is denoted by \( \mathbb{E}(\cdot) \), and to make the dependence on the underlying distribution \( \mu \) clear, it is denoted by \( \mathbb{E}_{\mu}(\cdot) \). The cardinality of a finite set \( A \) is denoted by \( |A| \).

The indicator function of an event \( \mathcal{E} \) is denoted by \( 1\{\mathcal{E}\} \).

For two measures \( P, Q \) defined on the same measurable space \( (\Omega, \mathcal{F}) \) the measure \( P \) is said to be absolutely continuous w.r.t. \( Q \) if for every \( \mathcal{E} \in \mathcal{F} \) such that \( Q(\mathcal{E}) = 0 \) it also holds that \( P(\mathcal{E}) = 0 \), this is denoted \( P \ll Q \).

The empty set will be denoted \( \phi \).

III. Problem Setup

Consider transmission over a stationary memoryless channel defined by a conditional probability distribution \( W \) from \( \mathcal{X} \) to \( \mathcal{Y} \), which are not necessarily finite sets. The input-output
probabilistic relation is given by:

\[ W^n(y|x) = \prod_{k=1}^{n} W(y_k|x_k) \]  (3)

where \( x = (x_1, \ldots, x_n) \in \mathcal{X}^n \) and \( y = (y_1, \ldots, y_n) \in \mathcal{Y}^n \) are input and output sequences of length \( n \), respectively. Our notation is such that in the finite case \( W(y|x) \) stands for the conditional p.m.f. of \( Y \) given \( X \), and in the infinite case, \( W(y|x) \) signifies the respective conditional p.d.f.

An encoder maps a message \( m \in \{1, \ldots, M_n\} \) to a channel input sequence \( x_m \in \mathcal{X}^n \), creating an \((n, M_n)\)-codebook \( C_n = \{x_1, \ldots, x_{M_n}\} \) of rate \( R_n = \frac{1}{n} \log M_n \). The message is a random variable \( M \), which is uniformly distributed on \( \{1, \ldots, M_n\} \).

The decoder’s role is to provide an estimate \( \hat{m} \in \{1, \ldots, M_n\} \) of the transmitted message. A maximum metric decoder is defined by a function, \( q(x, y) : \mathcal{X}^n \times \mathcal{Y}^n \rightarrow \mathbb{R} \), referred to as “metric” yielding

\[ \hat{m} = \arg \max_{i \in \{1, \ldots, M_n\}} q(x_i, y). \]  (4)

If \( \hat{m} \neq m \) an error occurs, and the event of having several maximizers is also considered as an error\(^1\). The decoder’s output, being a function of \( Y \) (the channel output vector of length \( n \)) is denoted \( \hat{M}_q(Y) \). The resulting average probability of error is given by

\[ P_e(W, C_n, q) = \sum_{m=1}^{M_n} \frac{1}{M_n} W^n(\hat{M}_q(Y) \neq m|X = x_m). \]  (5)

In this paper we assume that in the finite alphabet case, the decoding metric \( q(x, y) \) depends on \( x, y \) only via their joint empirical distribution, i.e., \( q(x, y) = q(\hat{P}_{x,y}) \), so \( q \) can be viewed as a mapping from the empirical distributions to the real numbers \( q : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \rightarrow \mathbb{R} \). Or more generally, in order not to restrict attention to a specific block-length \( n \), we assume that it maps the simplex to real number, i.e.,

\[ q : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \rightarrow \mathbb{R}. \]  (6)

In the case of type-dependent metrics (4) becomes:

\[ \hat{m} = \arg \max_{i \in \{1, \ldots, M_n\}} q(\hat{P}_{x_i,y}). \]  (7)

\(^1\)Similar to classical channel decoding, breaking ties arbitrarily and declaring an error are equivalent capacity-wise.
An important sub-class of type-dependent metrics is that of additive metrics for which there exists a single-letter mapping $q : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ such that

$$q(\hat{P}_{xy}) = \frac{1}{n} \sum_{i=1}^{n} q(x_i, y_i) = \mathbb{E}_{\hat{P}_{xy}}[q(X, Y)],$$

(8)

where for convenience we slightly abuse notation using $q$ for both the per-letter metric $q(x, y)$ and the $n$-letter metric $q(\hat{P}_{xy})$, as the intention is made clear by the argument of $q(\cdot)$. Note that in the special case of equiprobable codewords, the ML decoder, which minimizes the average probability of error, reduces to the additive metric $q(x, y) = \log W(y|x)$. Otherwise, the decoder is said to be mismatched [1], [4].

A rate $R > 0$ is said to be an achievable rate for the channel $W$ with decoding metric $q$ if for all $\epsilon > 0$ there exists a sequence of codes $\{C_n\}_{n \in \mathbb{N}}$ such that $|C_n| > e^{n(R-\epsilon)}$ and the average probability of error vanishes; i.e., $\lim_{n \to \infty} P_e(W, C_n, q) = 0$.

The mismatch capacity of channel $W$ with an additive decoding metric $q$, denoted $C_q(W)$, is the supremum of all achievable rates. For brevity we shall use the term $q$-mismatch capacity of $W$. The Shannon (matched) capacity of $W$ will be denoted $C(W)$.

We next describe the single-letter bound on $C_q(W)$ of [23], [24] that was mentioned in the introduction. Let $\mathcal{M}_{\text{max}}(q)$ stand for the following set of joint conditional distributions from $\mathcal{X}$ to $\mathcal{Y}^2$:

$$\mathcal{M}_{\text{max}}(q) = \left\{ P_{Y'Y|X}(y, y'|x) = 0 \text{ if } x \notin S_q(y, y') \right\},$$

(9)

where $S_q(y, y') = \{ x : x = \arg \max_{x'} [q(x', y') - q(x', y)] \}$.

**Theorem 1.** ([23], [24]) The mismatch capacity $C_q(W)$ of the DMC $W$ with additive metric $q$ is upper bounded as follows

$$C_q(W) \leq \tilde{R}_q(W) \triangleq \min_{P_{Y'Y|X} \in \mathcal{M}_{\text{max}}(q)} C(P_{Y'|X}),$$

(10)

**IV. MAIN RESULTS**

In this section we derive a single-letter upper bound on $C_q(W)$. For the simplicity of the presentation, we begin by presenting a simpler bound in Section IV-A, and in Section IV-B we proceed to our main result. Our bounding technique relies on multicast transmission over a broadcast channel $P_{YZ|X}$ from $\mathcal{X}$ to $\mathcal{Y} \times \mathcal{Z}$ with the marginal conditional distribution $P_{Y|X} = W$. 
A. Surely Degraded Broadcast Channels

As mentioned before, we begin by describing a simple bound which is in fact a corollary of our main result of Theorem 3, and is looser compared to it. It holds for additive metrics only.

Let $Z$ be a given set (either finite, countably infinite, or continuous), and let $q : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ and $\rho : \mathcal{X} \times Z \rightarrow \mathbb{R}$ be two additive metrics. Define

$$
\tau_{q,\rho}(y, z) = \max_{x' \in \mathcal{X}} [\rho(x', z) - q(x', y)].
$$

Consider the following set of broadcast channels

$$
\Gamma(q, \rho) \triangleq \{ V_{YZ|X} \mid V_{Y|X} = W_{C\rho}(P_{Z|X}) \land \forall (x, y, z) : \rho(x, z) - q(x, y) < \tau_{q,\rho}(y, z) \}.
$$

Note that the set $\Gamma(q, \rho)$ may be empty, but at least when $Z = \mathcal{Y}$ and $\rho = q$ it contains the channels of the form $V_{YZ|X} = V_{Y|X} \cdot 1\{Z = Y\}$.

For reasons that will be clarified later, we refer to channels in $\Gamma(q, \rho)$ as follows.

**Definition 1.** We say that the broadcast channel $P_{YZ|X}$ is $(q,\rho)$-surely degraded if $P_{YZ|X} \in \Gamma(q, \rho)$.

The upper bound is given in the following theorem.

**Theorem 2.** For all $Z$, additive metrics $q, \rho$, and a stationary memoryless channel $W$

$$
C_q(W) \leq \min_{P_{YZ|X} \in \Gamma(q, \rho) : P_{Y|X} = W} C_{\rho}(P_{Z|X}) \quad (13)
$$

Further, for all $\epsilon > 0$, the average probability of correct decoding of any sequence of codes of rate $R > \min_{P_{YZ|X} \in \Gamma(q, \rho) : P_{Y|X} = W} C(P_{Z|X}) + \epsilon$ vanishes exponentially fast with $n$.

**Proof of Theorem 2:** Consider a multicast transmission of a single message $M$ over the broadcast channel $P_{YZ|X} \in \Gamma(q, \rho)$ which satisfies $P_{Y|X} = W$.

By definition of $\Gamma(q, \rho)$, if $P_{YZ|X}(y_i, z_i|x_i) > 0$ then for all $x' \in \mathcal{X}$, $\rho(x_i, z_i) - q(x_i, y_i) \geq \rho(x', z_i) - q(x', y_i)$. By additivity of the metrics, and memorylessness of the channel, it follows that

$^2$If the sets $\mathcal{X}, \mathcal{Y}, Z$ are continuous, $\mathcal{P}(\mathcal{Y} \times Z|\mathcal{X})$ should be understood as the set of conditional p.d.f.’s rather than conditional p.m.f.’s, and the metrics $\rho$ and $q$ should be such that the resulting support of the distribution $P_{YZ|X}$ is measurable w.r.t. the Lesbegue measure.
if $P^n_{Y|Z,X}(y, z|x) > 0$, then for every $x' \in \mathcal{X}^n$ it holds that $\rho(x, z) - q(x, y) \geq \rho(x', z) - q(x', y)$. Rearranging the inequality we get

$$\rho(x, z) - \rho(x', z) \geq q(x, y) - q(x', y), \quad \forall (x, y, z, x') : P^n_{Y|Z,X}(y, z|x) > 0,$$  \hspace{1cm} (15)

where $x' \in \mathcal{X}^n$, thus, in particular, letting $C_n = \{x_j\}, j = 1, ..., e^{nR}$ be a given codebook, we have that if $P^n_{Y|Z,X}(y, z|x_m) > 0$, then for all $j$,

$$\rho(x_m, z) - \rho(x_j, z) \geq q(x_m, y) - q(x_j, y).$$  \hspace{1cm} (16)

Taking the minimum over $j \neq m$ on both sides of the inequality we get that if $P^n_{Y|Z,X}(y, z|x_m) > 0$ then

$$\rho(x_m, z) - \max_{j \neq m} \rho(x_j, z) \geq q(x_m, y) - \max_{j \neq m} q(x_j, y).$$  \hspace{1cm} (17)

This implies that given that $x_m$ is transmitted, if the received $y$ is such that $q(x_m, y) > \max_{j \neq m} q(x_j, y)$ then necessarily also $z$ is such that $\rho(x_m, z) > \max_{j \neq m} q(x_j, z)$. In words, the error event of the $\rho$-decoder applied to the channel output $Z$ is contained in the error event of the $q$-decoder applied to the channel output $Y$. This yields

$$\forall n, \Pr\left(\hat{M}_q(Y) = M \cap \hat{M}_\rho(Z) \neq M\right) = 0$$  \hspace{1cm} (18)

and consequently

$$C_q(W) \leq \min_{P_{Y|Z,X} \in \Gamma(q, \rho): P_{Y|X} = w} C_\rho(P_{Z|X}).$$  \hspace{1cm} (19)

Note that for rates exceeding the (potentially) looser upper bound $\min_{P_{Y|Z,X} \in \Gamma(q, \rho): P_{Y|X} = w} C(P_{Z|X})$, Eq. (18) also straightforwardly implies the exponential decay of the probability of correct $q$-decoding at the $Y$ output, from the strong converse property for the stationary memoryless channel $P_{Z|X}$.

Remarks:

- Inspecting (10), it is easy to see that Theorem 1 ([23], [24]) follows from Theorem 2 by taking $Z = Y$ and choosing the suboptimal degenerate case of $\rho = q$, in which case $\mathcal{M}_{\text{max}}(q) = \Gamma(q, q)$, and by noting that $C_\rho(P_{Z|X}) \leq C(P_{Z|X})$. This is because, as mentioned before, the class of transformations $\mathcal{M}_{\text{max}}(q)$ that was considered in the derivation of $\tilde{R}_q(W)$ includes only channels $P_{Y'|X}$ such that $q$-decoding at their output is at least as successful as it is for the original channel $P_{Y'|X}$, for every possible codebook.
In Section IV-E we provide examples for which the choice $\rho = q$ in (14) is strictly suboptimal; that is, the bound of Theorem 2 is strictly tighter than that of Theorem 1.

In addition to providing a tighter bound, Theorem 2 has the following advantages over [23], [24]:

- Our proof is significantly simpler and follows from an observation about multicast transmission over a $(q, \rho)$ surely degraded memoryless broadcast channel.
- Our proof holds as is for continuous alphabet stationary memoryless channels (with or without cost constraints), whereas the proof of [23], [24] relies on the method of types and results from graph theory and holds for the discrete memoryless case only.

The term $(q, \rho)$-sure degradedness of Definition 1 comes from (18), i.e., the fact that the error event of the $\rho$-decoder applied to the channel output $Z$ is contained in the error event of the $q$-decoder applied to the channel output $Y$.

Note that any choice of $Z$ and channel $P_{YZ|X} \in \Gamma(q, \rho)$ with marginal $P_{Y|X} = W$ leads to a valid bound, so there are many different (possibly looser) bounds that are implied by Theorem 2, without necessarily solving the minimization problem.

Further, any additive $\rho$ is valid, and in particular, $\rho$ that depends on the channel; e.g., the matched metric with respect to the marginal $P_{Z|X}$ of $P_{YZ|X}$, in which case $C_\rho(P_{Z|X})$ becomes $C(P_{Z|X})$.

In many cases one can prove that the choice $Y = Z$ suffices in the sense of providing the tightest bound. Nevertheless, we provide the bound in more generality, since this bound allows us to compare between the mismatch capacities corresponding to various channel-metric pairs (see Section IV-C).

As mentioned before, Theorem 2 is valid for continuous alphabet channels, in which case, if there is any input cost constraint, it should be understood that $C(P_{Z|X})$ and $C_\rho(P_{Z|X})$ in (14) are the corresponding capacity and $\rho$-capacity w.r.t the cost constraint.

Being tighter than $\bar{R}_q(W)$, our bound is clearly tight (and recovers the mismatch capacity formula) for the binary input binary output channel. In other words, in this case the choice $\rho = q$ produces a tight result.

We next show that the multi-letter version of the bound (14) cannot improve on the single-letter version. In [23], [24] such claim is proved for $\bar{R}_q(W)$. We present a different proof for our bound, which does not explicitly rely on KKT conditions (and holds also for $\bar{R}_q(W)$ as a
special case). Let

\[
\Gamma^{(k)}(q, \rho) = \left\{ V_{Y^kZ^k|X^k} : V(y^k, z^k|x^k) = 0 \forall x^k, y^k, z^k : \sum_{i=1}^k \rho(x_i, z_i) - q(x_i, y_i) < \sum_{i=1}^k \tau_{q,\rho}(y_i, z_i) \right\}.
\]

Lemma 1.

\[
\frac{1}{L} \min_{P_{Y|Z}|X^L \in \Gamma^{(L)}(\rho, q)} \min_{P_{Y|X^L} = W^L} C(P_{Z^L|X^L}) \geq \min_{P_{Y|X} \in \Gamma(\rho, q)} C(P_Z). \tag{21}
\]

Proof.

\[
\frac{1}{L} C(P_{Z^L|X^L}) = \frac{1}{L} \max_{P_{X^L}} I_{P_{X^L} \times P_{Z^L|X^L}}(X^L; Z^L) \geq \frac{1}{L} \max_{P_X} I_{P_X \times P_{Z^L|X^L}}(X^L; Z^L) \tag{22}
\]

\[
= \frac{1}{L} \max_{P_X} \sum_{\ell=1}^L I_{P_X \times P_{Z^L|X^L}}(X^{\ell}; Z^L, X^{\ell-1}) \tag{23}
\]

\[
\geq \max_{P_X} \sum_{\ell=1}^L \frac{1}{L} I_{P_X \times P_{Z^{\ell}|X^{\ell}}}(X^{\ell}; Z^{\ell}) \tag{24}
\]

\[
\geq \max_{P_X} \min_{\ell} I_{P_X \times P_{Z^{\ell}|X^{\ell}}}(X^{\ell}; Z^{\ell}) \tag{25}
\]

where (24) follows from the chain rule for mutual information, \( P_X \times P_{Z^L|X^L} \) is the marginal distribution of \((X^{\ell}, Z^{\ell})\) resulting from \( P_X^L \times P_{Z^{L|X^L}} \).

Next, we argue that

\[
\max_{P_X} \min_{\ell} I_{P_X \times P_{Z^{\ell}|X^{\ell}}}(X^{\ell}; Z^{\ell}) \geq \max_{P_X} \min_{P_{Y|X} \in \Gamma(\rho, q)} C(P_Z). \tag{27}
\]

This is because by definition of \( \Gamma^{(L)}(q, \rho) \), if \( P_{Z^L|X^L} \) is the marginal of \( P_{Y^{L|Z^L|X^L}} \in \Gamma^{(L)}(q, \rho) \) such that \( P_{Y^{L|X^L}} = W^L \), it must hold that the marginal \( P_{Y^{L|Z^L|X^L}} \) resulting from \( P_X^L \times P_{Y^{L|Z^L|X^L}} \) satisfies \( P_{Y^{L|X^L}} = W \) and lies in \( \Gamma(q, \rho) \). To realize this, note that if \( P_{Y^{L|Z^L|X^L}} \in \Gamma^{(L)}(q, \rho) \), then \( P_{Y^{L|Z^L|X^L}}(y^L, z^L|x^L) \) can be positive only if

\[
\rho(x^L, z^L) - q(x^L, y^L) = \max_{\bar{x}^L} [\rho(\bar{x}^L, z^L) - q(\bar{x}^L, y^L)], \quad \rho(x, z) - q(x, y) = \max_{\bar{x}} [\rho(\bar{x}, z) - q(\bar{x}, y)]
\]

which implies, by the additivity of \( q \) and \( \rho \), that \( P_{Y^{L|Z^L|X^L}}(y^L, z^L|x^L) \) can be positive only if for all \( \ell \)

\[
\rho(x, z) - q(x, y) = \max_{\bar{x}} [\rho(\bar{x}, z) - q(\bar{x}, y)]. \tag{29}
\]
Now, denote \( x^{-\ell} = (x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_L) \), and note that
\[
P_{Y|Z|X}(y|z|x) = \sum_{y^{\ell},z^{\ell},x^{\ell}} P_{X}^{L-1}(x^{-\ell}) P_{Y|Z|X^{L}}(y^{L},z^{L}|x^{L}). \tag{30}
\]
This implies that if \( P_{Y|Z|X}(y|z|x,y) > 0 \) there must be at least one triplet \((x', y', z')\) having \((x, y, z)\) as its \( \ell \)-th entry satisfying \( P_{Y|Z|X^{L}}(y^{L},z^{L}|x^{L}) > 0 \). Therefore, if that entry satisfies \( \rho(x, z) - q(x, y) < \max_{\tilde{x}} \rho(\tilde{x}, z) - q(\tilde{x}, y) \), then necessarily \( \rho(x^{L}, z^{L}) - q(x^{L}, y^{L}) < \max_{z^{L}} [\rho(\tilde{x}^{L}, z^{L}) - q(\tilde{x}^{L}, y^{L})] \) and consequently (28) cannot hold and therefore \( P_{Y|Z|X^{L}} \) cannot be a member of \( \Gamma^{(L)}(q, \rho) \).

To conclude, we have \( \max_{P_{X}} \min_{P_{Y|Z|X} \in \Gamma(q, \rho); P_{Y|Z|=W}} I(X; Z) = \min_{P_{Y|Z|X} \in \Gamma(q, \rho); P_{Y|Z|=W}} C(P_{Z|X}) \), which is due to the minimax theorem, which holds since \( \{ \Gamma(q, \rho) : P_{Y|Z|=W} \} \) is a convex set and since \( I(X; Z) \) is concave in \( P_{X} \) for fixed \( P_{Z|X} \) and convex in \( P_{Z|X} \) for fixed \( P_{X} \).

Therefore, we obtain (21). \( \square \)

Finally note that in the DMC case, the algorithm of [24] for computing \( R_{q}(W) \) can be adapted to compute the bound (14) which is equal to \( \max_{P_{X}} \min_{P_{Y|Z|X} \in \Gamma(q, \rho); P_{Y|X}=W} I(P_{X} \times P_{Z|X}) \), by replacing \( S_{q}(y, y') \) (see definition following eq. (9)) by
\[
S_{q,\rho}(y, z) = \left\{ x : x = \arg \max_{x'} [\rho(x', z) - q(x', y)] \right\}, \tag{31}
\]
and the proof of its convergence is essentially the same.

B. Surely Degraded Broadcast Channels - Non Rectangular Sets

In this section we consider a larger set of broadcast channels compared to \( \Gamma(q, \rho) \) that may depend not only on \( q \) and \( \rho \), but also on the composition of the input distribution \( P \in \mathcal{P}(X) \):
\[
\Gamma(q, \rho, P) = \left\{ V_{Y|X|Z} : \max_{U_{XY} \in P \times V_{Y|Z}|X} \left[ q(U_{XY}) - q(U_{\tilde{X}Y}) \right] \leq 0 \right\} \tag{32}
\]
\[
= \left\{ V_{Y|X|Z} : \forall U_{XY} \in P \times V_{Y|Z}|X, \ U_{X} = U_{\tilde{X}} \right\}.
\tag{33}
\]
Note that in addition to considering a larger set for additive metrics, here we also widen the scope to include \( q \) and \( \rho \) which are type-dependent metrics, and not necessarily additive, as Theorem 2 holds for additive metrics only. An important example for a useful type-dependent metric which is not additive is the MMI metric:
\[
q_{\text{MMI}}(\tilde{P}_{xy}) = I(\tilde{P}_{xy}), \tag{34}
\]
where $I(\hat{P}_{xy})$ is the mutual information induced by the joint distribution $\hat{P}_{xy}$.

The following theorem holds in the discrete alphabets case.

**Theorem 3.** For all finite $Z$, type-dependent metrics $q, \rho$, and a DMC $W$

$$C_q(W) \leq \max_P \min_{P_{Y|X} \in \Gamma(q, \rho, P)} I(X; Z). \tag{35}$$

Further, for all $\epsilon > 0$, the average probability of correct decoding of any sequence of codes of rate $R > \max_P \min_{P_{Y|X} \in \Gamma(q, \rho, P)} I(X; Z) + \epsilon$ vanishes exponentially fast with $n$.

One can easily realize that for any $P$, 

$$\Gamma(q, \rho) \subseteq \Gamma(q, \rho, P), \tag{36}$$

hence, the bound (35) is tighter than (14) since $C(P_{Z|X}) = \max_P I(X; Z)$, and as we shall see in Section IV-E, it can be strictly tighter than that of Theorem 2. This can happen when the maximizing $P$ in (35) is such that $\Gamma(q, \rho) \subset \Gamma(q, \rho, P)$ (with strict inclusion) and the minimizer $P_{Y|X}$ in (35) belongs to $\Gamma(q, \rho, P) \setminus \Gamma(q, \rho)$, in which case the order of the maximization and minimization cannot be swapped.

**Proof of Theorem 3:** Consider transmission of a single message over the stationary memoryless channel $W$. Let $\{C_n\}$ be a sequence of codebooks of rates $\{R_n\}$, where $R_n > R$ and with vanishing average probability of error $\epsilon_n$. Let $P_n$ be a constant composition of a sub-codebook $\tilde{C}_n \subseteq C_n$ of rate at least $R' = R - O(\frac{1}{n} \log n)$. Since $\tilde{C}_n \subseteq C_n$, the average probability of error of the sequence of sub-codebooks does not exceed $\epsilon_n$. Now, let $P_{Z|XY}$ be a conditional distribution such that the broadcast channel $P_{Y|X} = W \times P_{Z|XY}$ satisfies $P_{Y|X} \in \Gamma(q, \rho, P_n)$. By definition, since $P_{Y|X} \in \Gamma(q, \rho, P_n)$, if $x \in \mathcal{T}(P_n)$ and $P_{Y|X}(y, z|x) > 0$, then for every $x' \in \mathcal{T}(P_n)$ it holds that $\rho(x', z) \geq \rho(x, z) \Rightarrow q(x', y) \geq q(x, y)$. Similar to (15)-(17) we obtain

$$\rho(x_m, z) \leq \max_{j \neq m} \rho(x_j, z) \Rightarrow q(x_m, y) \leq \max_{j \neq m} q(x_j, y), \tag{37}$$

and thus for $Y, Z$ the output of the channel $P_{Y|Z|X}^n$ whose input is uniform over $\tilde{C}_n$, we have

$$\forall n, \Pr(\tilde{M}_q(Y) = M \cap \tilde{M}_\rho(Z) \neq M) = 0. \tag{38}$$

$^3$Such sub-codebook always exists because the number of compositions (type classes) grows polynomially with $n$, and the codebook grows exponentially with $n$. 
Let $T$ be a random variable uniformly distributed over $\{1, \ldots, n\}$, independent of $M$, and let $X_T$ be the channel input symbol at time $T$. Since $\mathcal{C}_n \subseteq \mathcal{T}(P_n)$, we have $X_T \sim P_n$. Therefore, a standard application of Fano’s inequality to the channel $P_{Z|X}$ gives

$$R' \leq I(P_n \times P_{Z|X}) + \epsilon_n \cdot R' + 1/n.$$  \hfill (39)

Now, since (39) holds for all $P_{YZ|X} \in \Gamma(q, \rho, P_n)$ such that $P_{Y|X} = W$ this gives

$$R' \leq \min_{P_{YZ|X} \in \Gamma(q, \rho, P_n)} I(P_n \times P_{Z|X}) + \epsilon_n \cdot R' + 1/n$$  \hfill (40)

and since $P_n \in \mathcal{P}_n(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$

$$C_q(W) \leq \max_{P \in \mathcal{P}(\mathcal{X})} \min_{P_{YZ|X} \in \Gamma(q, \rho, P_n)} I(P \times P_{Z|X}).$$  \hfill (41)

Note that here too, for rates exceeding $R > \max_{P} \min_{P_{YZ|X} \in \Gamma(q, \rho, P_n)} P_{Y|X} = W I(P \times P_{Z|X}) + \epsilon$, Eq. (18) also straightforwardly implies the exponential decay of the probability of correct $q$-decoding at the $Y$ output, from the strong converse property for every stationary memoryless $P_{Z|X}$.

While the proof of Theorem 2 holds as is for continuous input alphabet channels, the proof Theorem 3 applied to additive metrics needs to be slightly adapted to the continuous alphabet case. The first reason for that is that constant composition codebooks are defined for finite alphabets, the second reason is that the absolute continuity operator $\ll$ needs to be slightly changed. We demonstrate such application in Section IV-D.

### C. Equivalence Classes of Channel-Metric Pairs

In this section we introduce equivalence classes of isomorphic channel-metric pairs $(W, q)$ that share the same mismatch capacity for additive metrics $q$. We prove that if one of the pairs in the class is matched, then the mismatch capacity of the entire class is fully characterized and equal to the LM rate and to the GMI. This gives a sufficient condition for the tightness of our bound. In particular, it gives a sufficient condition for a metric to be capacity achieving.

Subsequently, we extend this notion to isomorphic channel-metric-composition triplets $(P, W, q)$ where here $q$ can be type-dependent and $P \in \mathcal{P}_n(\mathcal{X})$.

We next introduce useful notation and definitions.

**Definition 2.** We say that a channel-metric pair $(P_{Z|X}, \rho)$ is superior to the channel-metric pair $(P_{Y|X}, q)$ if there exists a joint conditional distribution $P_{YZ|X} \in \Gamma(q, \rho)$, whose marginal
conditional distributions are \( P_{Y|X} \) and \( P_{Z|X} \). We say that this channel is surely degraded w.r.t. \((q, \rho)\), and we denote the superiority relation by

\[
(P_{Y|X}, q) \rightarrow (P_{Z|X}, \rho).
\]

If both \((P_{Y|X}, q) \rightarrow (P_{Z|X}, \rho)\) and \((P_{Z|X}, \rho) \rightarrow (P_{Y|X}, q)\) we say that the pairs are isomorphic and denote this isomorphism relation by

\[
(P_{Y|X}, q) \leftrightarrowtriangle (P_{Z|X}, \rho),
\]

The following lemma holds trivially

**Lemma 2.** (a) The relation \( \rightarrowtriangle \) is transitive; i.e., if \((W_1, q_1) \rightarrowtriangle (W_2, q_2)\) and \((W_2, q_2) \rightarrowtriangle (W_3, q_3)\) then \((W_1, q_1) \rightarrowtriangle (W_3, q_3)\).

(b) The relation \( \leftrightarrowtriangle \) is an equivalence relation: it is reflexive, symmetric, and transitive.

**Proof.** The proof of part (b) is trivial by definition of \( \leftrightarrowtriangle \). Hence we only prove (a). First note that by definition of \( \tau_{q, \rho}(y, z) \) (11)

\[
\tau_{q_1, q_3}(y_1, y_3) = \max_{x' \in \mathcal{X}} [q_3(x', y_3) - q_1(x', y_1)] \tag{44}
\]

\[
= \max_{x' \in \mathcal{X}} [q_3(x', y_3) - q_2(x', y_2) + q_2(x', y_2) - q_1(x', y_1)] \tag{45}
\]

\[
\leq \tau_{q_2, q_3}(y_2, y_3) + \tau_{q_1, q_2}(y_1, y_2). \tag{46}
\]

Next, let \( P_{Y_1Y_2|X} \in \Gamma(q_1, q_2) \) with marginals \( W_1 \) and \( W_2 \) be given, and let \( P_{Y_2Y_3|X} \in \Gamma(q_2, q_3) \) with marginals \( W_2 \) and \( W_3 \) be given. We need to show that there exists \( P_{Y_1Y_3|X} \in \Gamma(q_1, q_3) \) with marginals \( W_1 \) and \( W_3 \).

Let \( P_{Y_3|XY_2} \) be the conditional distribution induced by \( P_{Y_2Y_3|X} \), and consider the channel

\[
P_{Y_1Y_3|X}(y_1, y_3|x) = \sum_{y_2} P_{Y_1Y_2|X}(y_1, y_2|x) \cdot P_{Y_3|XY_2}(y_3|x, y_2)
\]

which obviously has marginals \( W_1 \) and \( W_3 \).

Now, if \( q_3(x, y_3) - q_1(x, y_1) < \tau_{q_1, q_3}(y_1, y_3) \) then from (44), for all \( y_2 \in \mathcal{Y}_2 \), \( q_3(x, y_3) - q_2(x, y_2) + q_2(x, y_2) - q_1(x, y_1) < \tau_{q_1, q_3}(y_1, y_3) \leq \tau_{q_2, q_3}(y_2, y_3) + \tau_{q_1, q_2}(y_1, y_2) \) which implies that either \( q_2(x, y_2) - q_1(x, y_1) < \tau_{q_1, q_3}(y_1, y_2) \) or \( q_3(x, y_3) - q_2(x, y_2) < \tau_{q_2, q_3}(y_2, y_3) \) which yields that either \( P_{Y_1Y_2|X}(y_1, y_2|x) = 0 \) or \( P_{Y_2Y_3|X}(y_2, y_3|x) = 0 \) for all \( y_2 \in \mathcal{Y}_2 \), and consequently \( P_{Y_1Y_3|X}(y_1, y_3|x) = 0 \).

The following theorem provides (among other things) a sufficient condition for the tightness of our bound.
**Theorem 4.** If \((P_{Y|X}, q) \rightarrow (P_{Z|X}, \rho)\) then

\[ C_q(P_{Y|X}) \leq C_\rho(P_{Z|X}). \]  
(47)

and consequently, if \((P_{Y|X}, q) \leftrightarrow (P_{Z|X}, \rho)\) then

\[ C_q(P_{Y|X}) = C_\rho(P_{Z|X}). \]  
(48)

If there exists a matched channel-metric pair \((\tilde{P}_{Z|X}, \tilde{q}_{ML})\) where \(\tilde{q}_{ML} = \log \tilde{P}_{Z|X}\) is the maximum likelihood metric w.r.t. \(\tilde{P}_{Z|X}\) such that \((P_{Y|X}, q) \leftrightarrow (\tilde{P}_{Z|X}, \tilde{q}_{ML})\) then

\[ C_q(P_{Y|X}) = R_{q,GMI}(P_{Y|X}) = R_{q,LM}(P_{Y|X}) = C(\tilde{P}_{Z|X}), \]  
(49)

where \(R_{q,LM}(P_{Y|X})\) and \(R_{q,GMI}(P_{Y|X})\) are the LM and GMI rates of channel \(P_{Y|X}\) with decoding metric \(q\) (see (2) and (1)).

**Proof.** The statement (47) follows trivially from Theorem 2 see (14), and (48) follows from (47). The equality \(C_q(P_{Y|X}) = C(\tilde{P}_{Z|X})\) is a special case of (48). It remains to prove \(C_q(P_{Y|X}) = R_{q,LM}(P_{Y|X}) = R_{q,GMI}(P_{Y|X})\).

Since \((P_{Y|X}, q) \leftrightarrow (\tilde{P}_{Z|X}, \tilde{q}_{ML})\), similarly to (15)-(18) it follows that there exists a channel \(P_{ZY|X}\) with marginals \(\tilde{P}_{Z|X}\) and \(P_{Y|X}\) such that one has \(\Pr(\hat{M}_{q,ML}(Z) = M, \hat{M}_q(Y) \neq M) = 0\) for every codebook. Since we also have \(C_q(P_{Y|X}) = C(\tilde{P}_{Z|X})\), this yields that the random coding scheme which is capacity achieving for channel \(\tilde{P}_{Z|X}\) must be capacity achieving for \(P_{Y|X}\) with decoding metric \(q\). Since random coding (either i.i.d. or constant composition) is capacity achieving for \(\tilde{P}_{Z|X}\) with \(\tilde{q}_{ML}\), it is also capacity achieving for \(P_{Y|X}\) with \(q\)-decoding, and thus \(C_q(P_{Y|X}) = R_{q,LM}(P_{Y|X}) = R_{q,GMI}(P_{Y|X})\). \(\square\)

Note that Theorem 4 implies that if \(R_{q,GMI}(W) < R_{q,LM}(W)\) then, there exists no matched pair \((\tilde{P}_{Z|X}, \tilde{q}_{ML})\) such that \((P_{Y|X}, q) \leftrightarrow (\tilde{P}_{Z|X}, \tilde{q}_{ML})\). But, this does not necessarily imply that the bound of Theorem 2 is loose, since there are channel-metric pairs which are not isomorphic but have the same capacity.

The following corollary gives a sufficient condition for a metric \(q\) to be capacity achieving for the channel \(W\).

**Corollary 1.** If \((W, \log W) \rightarrow (W, q)\) then

\[ C_q(W) = C(W). \]  
(50)
Next we extend the notion of isomorphism to type-dependent metrics w.r.t. the codebook composition $P$.

**Definition 3.** We say that a composition-channel-metric triplet $(P, P_{Z|X}, \rho)$ is superior to the composition-channel-metric triplet $(P, P_{Y|X}, q)$ if there exists a joint conditional distribution $P_{YZ|X} \in \Gamma(q, \rho, P)$, whose marginal conditional distributions are $P_{Y|X}$ and $P_{Z|X}$. We say that this channel is surely degraded w.r.t. $(P, q, \rho)$, and we denote the superiority relation by

$$(P, P_{Y|X}, q) \rightarrowtriangle (P, P_{Z|X}, \rho).$$

(51)

If both $(P, P_{Y|X}, q) \rightarrowtriangle (P, P_{Z|X}, \rho)$ and $(P, P_{Z|X}, \rho) \rightarrowtriangle (P, P_{Y|X}, q)$ we say that the triplets are isomorphic and denote this isomorphism relation by

$$(P, P_{Y|X}, q) \leftrightarrowtriangle (P, P_{Z|X}, \rho),$$

(52)

The following corollary follows by definition of $\Gamma(q, \rho, P)$ from the proof of Theorem 3 (see (37)-(38)).

**Corollary 2.** If there exists a sequence of empirical distributions $P_n \in \mathcal{P}_n(\mathcal{X})$ converging to $P^*$ which is the maximizer of $C(W) = \max_P I(P \times P_{Y|X})$ such that for all $n$ sufficiently large $(P_n, W, \log W) \rightarrowtriangle (P_n, P_{Z|X}, q)$, then

$$C(W) \leq C_q(P_{Z|X}).$$

(53)

In particular, if for all $n$ sufficiently large $(P_n, W, \log W) \rightarrowtriangle (P_n, W, q)$ then

$$C_q(W) = C(W).$$

(54)

Corollary 2 gives a sufficient condition (54) which is less strict than that of Corollary 1 for a metric to be capacity achieving.

**D. Surely Degraded Broadcast Channels w.r.t. Spherical Codes**

We next demonstrate how to obtain a tighter bound (compared to that of Theorem 2) to the continuous input alphabet case with a cost constraint using an approach that is similar to Theorem 3. For simplicity of the presentation we consider a power constraint setup in which all
codewords are required to have a fixed (identical) energy\(^4\) equal to \(\sigma^2\); i.e., they all lie in the \(L_2\) sphere of constant norm \(\sqrt{n}\sigma\) in \(\mathbb{R}^n\), i.e., \(C_n \subseteq T(\sigma^2)\) where \(T(\sigma^2) \triangleq \{ x \in \mathbb{R}^n : \| x \|_2 = \sqrt{n}\sigma \} \). (55)

Let \(C_q(\sigma^2, P_{Y|X})\) stand for the \(q\)-mismatch capacity of channel \(P_{Y|X}\) using codebooks which satisfy \(C_n \subseteq T(\sigma^2)\).

Theorem 2 tells us that \(C_q(\sigma^2, P_{Y|X}) \leq \min_{P_{Y|X} \in \Gamma(q, \rho)} P_{Y|X} = W C_{\rho}(\sigma^2, P_{Z|X}).\) (56)

Our next theorem shows how this bound can be improved in the spirit of Theorem 3.

For a distribution \(P\) we define \(\text{supp}(P)\) to be the support of \(P\), in the sense that the probability density function (p.d.f.) corresponding to \(P\) is non-zero.

We consider the set:
\[
\Lambda(q, \rho, \sigma) = \left\{ V_{Y|X} : \forall U_{XY} \ s.t. \supp(U_{XY,Z}) \subseteq \supp(U_{X \times Y|Z}), \ \rho(U_{XZ}) \leq \rho(U_{X\tilde{Z}}) \Rightarrow q(U_{XY}) \leq q(U_{X\tilde{Y}}) \right\}.
\] (57)

**Theorem 5.** For all \(Z\), additive metrics \(q, \rho\), and a stationary memoryless channel \(W\)
\[
C_q(\sigma^2, W) \leq \min_{P_{Y|X} \in \Lambda(q, \rho, \sigma)} \min_{P_{Y|X} = W} C_{\rho}(\sigma^2, P_{Z|X}).
\] (59)

Further, for all \(\epsilon > 0\), the average probability of correct decoding of any sequence of codes of rate \(R > \min_{P_{Y|X} \in \Lambda(q, \rho, \sigma)} P_{Y|X} = W \max_{P_{X}: \mathbb{E}(X^2) = \sigma^2} I(X; Z) + \epsilon\) vanishes exponentially fast with \(n\).

**Proof of Theorem 5:** Consider transmission of a single message over the stationary memoryless channel \(W\). Let \(C_n = \{ x_i \}_{i=1}^n \) be a given codebook where for all \(i\), \(x_i \in T(\sigma^2)\).

Now, let \(P_{Z|XY}\) be a conditional distribution such that the broadcast channel \(P_{Y|Z,X} = W \times P_{Z|XY}\) satisfies \(P_{Y|Z,X} \in \Lambda(q, \rho, \sigma)\).

\(^4\)Shannon [27] showed that in the matched decoding case, considering codebooks of signals of constant energy \(n\sigma^2\) does not reduce the achievable rate compared to \(n\sigma^2\) expected energy constraint.
Let \( x, y, z, \bar{x} \) be given. Consider the discrete distribution induced by \( x, y, z, \bar{x} \):

\[
U_{XYZX}(x, y, z, \bar{x}) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{(x_i, y_i, z_i, \bar{x}_i) = (x, y, z, \bar{x})\},
\]

(61)

where \( x_i, y_i, z_i, \bar{x}_i \) are the \( i \)-th entries of the vectors \( x, y, z, \bar{x} \), respectively. Note that

- \( x \in \mathcal{T}(\sigma^2) \), \( \bar{x} \in \mathcal{T}(\sigma^2) \) can be expressed as \( \mathbb{E}_U(X^2) = \mathbb{E}_U(\bar{X}^2) = \sigma^2 \)
- \( (y, z) \in \text{supp}(P_{YZ|X}^n(\cdot|x)) \) can be expressed as \( \text{supp}(U_{XYZ}) \subseteq \text{supp}(U_X \times P_{YZ|X}) \)
- \( \rho(\bar{x}, z) \geq \rho(x, z) \) can be expressed as \( \rho(U_{\bar{X}Z}) \geq \rho(U_{XZ}) \)
- \( q(\bar{x}, y) \geq q(x, y) \) can be expressed as \( q(U_{\bar{X}Y}) \geq q(U_{XY}) \).

Thus, the condition appearing in \( \Lambda(q, \rho, \sigma) \) (see (58)) is merely a single-letter formulation guaranteeing that if \( x \in \mathcal{T}(\sigma^2) \) and \( (y, z) \) is a possible channel output in the sense of having positive p.d.f., i.e., \( (y, z) \in \text{supp}(P_{YZ|X}^n(\cdot|x)) \), then for every \( \bar{x} \in \mathcal{T}(\sigma^2) \) it holds that

\[
\rho(\bar{x}, z) \geq \rho(x, z) \Rightarrow q(\bar{x}, y) \geq q(x, y).
\]

Since if \( x_m \) is transmitted, the received signals \( (y, z) \) must satisfy \( (y, z) \in \text{supp}(P_{YZ|X}^n(\cdot|x_m)) \), we always have

\[
\rho(x_m, z) \leq \max_{j \neq m} \rho(x_j, z) \Rightarrow q(x_m, y) \leq \max_{j \neq m} q(x_j, y),
\]

(62)

and thus we obtain (for \( Y, Z \) the output of the channel \( P_{YZ|X}^n \) whose input is uniform over \( C_n \)),

\[
\Pr\left(\hat{M}_q(Y) = M \cap \hat{M}_\rho(Z) \neq M\right) = 0
\]

(63)

and consequently

\[
R \leq C_\rho(\sigma, P_{Z|X}),
\]

(64)

and since this is true for all \( P_{YZ|X} \in \Lambda(q, \rho, \sigma) : P_{Y|X} = W \) we get

\[
C_q(W) \leq \min_{P_{YZ|X} \in \Lambda(q, \rho, \sigma) : P_{Y|X} = W} C_\rho(\sigma, P_{Z|X}).
\]

(65)

Since \( C_\rho(\sigma, P_{Z|X}) \leq \max_{P_X: \mathbb{E}(X^2) = \sigma^2} I(P_X \times P_{Z|X}) \) we obtain the possibly looser upper bound of (60). And, again, the last assertion of Theorem 5 follows from the strong converse for the stationary memoryless channel \( P_{Z|X} \).

\[\square\]

\[E. \text{ Examples:}\]

We next present examples for strict improvements of Theorem 2 over Theorem 1, and of Theorem 3 over Theorem 2:
1) Example 1) A Noiseless Channel with the Pentagon Graph Connectivity Metric: Consider the noiseless channel \( W_r(y|x) = \mathbb{1}\{y = x\} \) with \( r \triangleq |\mathcal{X}| \) and the following additive decoding metric

\[
q(x, y) = 1\{(y - x) \mod r \in \{0, 1, r - 1\}\}. \tag{66}
\]

It is easily verified that the only channel in \( \mathcal{M}_{\max}(q) \) which satisfies \( P_{Y|x} = W \) is the noiseless channel for which \( P_{Y|XY}(y'|xy) = \mathbb{1}\{y' = x\} \). We demonstrate this for the noiseless channel with the pentagon channel adjacency graph metric; i.e, for \( r = 5 \) with alphabets \( \mathcal{X} = \mathcal{Y} = \{0, 1, 2, 3, 4\} \) and with the decoding metric matrix \( q(x, y) = A_{x+1,y+1} \) where

\[
\{A_{ij}\} = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
\end{pmatrix}, \tag{67}
\]

Now, recall the definition of \( S_q(y, y') \) which appears after (9). We have:

\[
S_q(y, y') = \begin{cases}
\{0, 1, 2, 3, 4\} & y = y' \\
\{2\} & (y, y') = (0, 1), (y, y') = (4, 3) \\
\{2, 3\} & (y, y') = (0, 2), (y, y') = (0, 3) \\
\{3\} & (y, y') = (0, 4), (y, y') = (1, 2) \\
\{4\} & (y, y') = (1, 0), (y, y') = (2, 3) \\
\{3, 4\} & (y, y') = (1, 3), (y, y') = (1, 4) \\
\{0, 4\} & (y, y') = (2, 0), (y, y') = (2, 4) \\
\{0\} & (y, y') = (2, 1), (y, y') = (3, 4) \\
\{0, 1\} & (y, y') = (3, 0), (y, y') = (3, 1) \\
\{1\} & (y, y') = (3, 2), (y, y') = (4, 0) \\
\{1, 2\} & (y, y') = (4, 1), (y, y') = (4, 2) \\
\end{cases}, \tag{68}
\]

and since the channel dictates \( x = y \) we get that \( P_{Y|x}(y'|x) > 0 \) only for \( y' = x \) and thus, Theorem 1 gives

\[
C_q(W) \leq \bar{R}_q(W) = \log_2 5 \text{ [bits/channel use]}. \tag{69}
\]

It is easy to verify that the typewriter channel

\[
P_{Z|XY} = P_{Z|X} = W_{C_5} \triangleq \frac{1}{2} \cdot \mathbb{1}\{(z - x) \mod 5 \in \{0, 1\}\} \tag{70}
\]
with the metric

$$\rho(x, y) = \mathbb{I}\{(y - x) \mod 5 \in \{0, 1\}\} = B_{x+1,y+1}, \quad (71)$$

where

$$\{B_{ij}\} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (72)$$

satisfies the condition that if $P_{YZ|X}(y, z|x) > 0$, then $\forall x', q(x, y) - q(x', y) \leq \rho(x, z) - \rho(x', z)$; that is

$$(W_5, q) \rightarrowtriangle (W_{C_5}, \rho), \quad (73)$$

where, as mentioned before $W_5$ is the noiseless channel from $\{0, 1, 2, 3, 4\}$ to itself. This is because whenever $P_{YZ|X}(y, z|x) > 0$, we have $q(x, y) = \rho(x, y)$ and also $(y, z) = (y, y)$ or $(y, z) = (y, y + 1 \mod 5)$. Hence, the condition becomes $\forall x', q(x', y) \geq \rho(x', y)$ and $q(x', y) \geq \rho(x', y + 1 \mod 5)$ which is always satisfied, and therefore

$$C_q(W) \leq C_{\rho}(W_{C_5}) = \log_2(5) - \log_2 2 \text{ [bits/channel use]} \quad (74)$$

2) An Example with a Non-Symmetric Metric: Consider the noiseless channel $W_5$ and $q(x, y) = q_{xy}, \rho(x, y) = \rho_{xy}$ where

$$\{q_{ij}\} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad (75)$$

$$\{\rho_{ij}\} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad (76)$$
it is easy to verify (e.g. by a computer program) that any channel \( P_{Z|X} \) whose support is given in the following matrix

\[
\{L_{ij}\} = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\] (77)

belongs to \( \Gamma(q, \rho) \), and thus satisfies

\[
(W_5, q) \rightarrow (W_{Z|X}, \rho), \quad C_q(W_5) \leq C_\rho(P_{Z|X}).
\] (78)

3) A metric with input score: Consider additive metrics of the form:

\[
q(x, y) = a(x, y) + b(x) \quad (79)
\]

\[
\rho(x, y) = a(x, y) \quad (80)
\]

The metrics \( \rho \) and \( q \) are obviously equivalent for constant composition codebooks since their codewords have equal \( \sum_{i=1}^n b(x_i) \) value. And indeed, this is reflected in Theorem 3 which gives \( W_{Z|X} \{Z = Y\} \in \Gamma(q, \rho, P) \), for all \( P \), and yields \( C_q(W) \leq C_\rho(W) \) (and vice versa, by changing the roles of \( \rho \) and \( q \)). On the other hand, one can easily find examples for which \( \Gamma(a(x, y) + b(x), a(x, y)) = \phi \), so Theorem 2 gives meaningless bounds in these cases.

Note that in [24] it was shown that the bound of Theorem 1, \( \bar{R}_q(W) \), is insensitive to metrics differences of the form \( b(x) \). This is indeed true for \( \mathcal{M}_{\text{max}}(q) = \Gamma(q, q) \), but it is no longer true for \( \Gamma(q, \rho) \) in general, and affects the bound of Theorem 3. Therefore, Theorem 3 gives a strict improvement over Theorems 1 and 2 in certain cases.

4) AWGN Channel with a mismatched metric: Consider an additive white Gaussian noise (AWGN) channel

\[
Y = X + N,
\] (81)

where \( N \sim \mathcal{N}(0, \sigma_n^2) \) for some noise power \( \sigma_n^2 > 0 \), and \( X \) and \( N \) are independent. We consider a power constraint corresponding to \( x_i \in \mathcal{T}(\sigma^2) \); that is \( \|x_i\|^2 = \sigma^2 \) for every codewords \( x_i \). Note that when matched decoding is concerned, this is equivalent capacity-wise to require \( \|x_i\|^2 \leq \sigma^2 \) [27] (and this may be the case with mismatched decoding as well).

Clearly, the maximum likelihood decoding rule is the nearest neighbor which minimizes \( \|y - x\|^2 \). Consider a decoder that erroneously thinks that there is a scaling factor \( \beta \) (i.e., that the
channel is \( Y = \beta X + N \), and decodes as output accordingly: \( \hat{m} = \arg \min_{j=1, \ldots, M} \| y - \beta x_j \|_2^2 \). These decoders correspond to the additive decoding metrics \(-(y - x)^2\) and \(-(y - \beta x)^2\), respectively. Since \( \|y\|_2^2 \) does not affect the decision, we obtain for \( \beta > 0 \)

\[
q_1(x, y) = 2 < x, y > - \|x\|_2^2 \tag{82}
\]

\[
q_2(x, y) = 2 < x, y > - \beta \|x\|_2^2. \tag{83}
\]

Now, the mismatched and maximum-likelihood decoding rules are equivalent for codebooks in which all codewords have the same energy \( \|x\|_2^2 \). Since it is assumed that the codebooks belong to the sphere \( T(\sigma^2) \) this is indeed the case. Nevertheless, in Theorem 2 we have \( \Gamma(q_1, q_2) = \emptyset \) and no meaningful result follows, whereas the set \( \Lambda(q_1, q_2, \sigma) \) of Theorem 5 is not empty, and in particular \( P_{YZ|X} = P_{Y|X} \cdot \mathbb{1}\{Z = Y\} \in \Lambda(q_1, q_2, \sigma) \), so Theorem 5 applied to \( (q_1, q_2) = (q, \rho) \) and vice versa implies that \( C_{q_1}(\sigma, W_{Y|X}) = C_{q_2}(\sigma, W_{Y|X}) \).

5) **Example 5: same channel, different metrics**: ([5, Example 2]) Consider Example 2 of Lapidoth with \( p' = 0.3, p'' = 0.4 \), and input alphabet \( \{0, 1, 2, 3\} \). The channel transition matrix is given by

\[
W_{Y|X} = \begin{pmatrix}
(1-p')(1-p'') & (1-p')p'' & p'(1-p'') & p'p'' \\
(1-p')p'' & (1-p')(1-p'') & p'p'' & p'(1-p'') \\
p'(1-p'') & p'p'' & (1-p')(1-p'') & (1-p')p'' \\
p'p'' & p'(1-p'') & (1-p')p'' & (1-p')(1-p'')
\end{pmatrix} \tag{84}
\]

and \( q(x, y) = q_{ML}(x, y) = \log_2 W_{Y|X}(y|x) \), and \( \rho(x, y) = r_{x+y+1} \) where

\[
r = \begin{pmatrix}
0 & -1 & -1 & -2 \\
-1 & 0 & -2 & -1 \\
-1 & -2 & 0 & -1 \\
-2 & -1 & -1 & 0
\end{pmatrix}. \tag{85}
\]

Lapidoth showed that \( C_{q_{ML}}(W) = C_{\rho}(W) \) and that \( C_{\rho}(W) > R_{LM,\rho}(W) \) with strict inequality.

Clearly \( C_{q_{ML}}(W) \) is obviously known to equal \( C(W) \) because the metric is matched. Nevertheless, it is interesting to see if we can show that \( C_{q_{ML}}(W) \leq C_{\rho}(W) \) by treating \( q_{ML} \) and \( \rho \) in the roles of \( q \) and \( \rho \) of Theorem 2.

A scan of the possibilities (using a simple computer program) reveals that one has
so evidently $\Gamma(q, \rho)$ is empty because there is no channel that satisfies this (for example, there is no $z$ such that $P(z|xy) > 0$ for $(x, y) = (0, 0)$). Hence, Theorem 2 does not imply that $C_{q_{ML}}(W) \leq C_{\rho}(W)$ although there is an equality. Nevertheless, this is not surprising and is consistent with the fact that $C_{\rho}(W) > R_{LM, \rho}(W)$; had Theorem 2 been tight in this case, this would have been in contradiction to Theorem 4, which would imply an equality between $C_{\rho}(W)$ and the LM rate $R_{LM, \rho}(W)$.

V. Conclusion

In this paper we presented a single-letter upper bound on the mismatch capacity, which is based on multicast transmission over a broadcast channel.

The introduction of this multicast transmission setup essentially enables to derive quite straightforwardly upper bounds on the $q$-mismatch capacity of the channel to the first receiver (Channel 1) by the $\rho$-mismatch capacity of the channel to the second receiver (Channel 2). While the latter upper bounds the former, the matched capacity of Channel 2 can be strictly lower than the matched capacity of Channel 1, thereby yielding a tighter bound compared to the trivial matched capacity of Channel 1. This setup can also be viewed as a generalization of the notion of degradedness of broadcast channels to the mismatched case. We further analyzed several examples of channels with mismatched decoding, and demonstrated a strict improvement of our bound compared to previous results in the DMC case, and presented a few examples for continuous alphabet channels as well.

In addition to providing tighter bounds, our method of proof via multicast transmission over a broadcast channel places error events in the same probability space induced by the broadcast channel, yielding a considerably simpler bounding technique compared to that of [23], [24] (which in turn can be viewed as constructing a graph between two separate probability spaces). Another significant advantage of our approach, is that it holds for continuous alphabet channels as is. Moreover, our bound holds in greater generality as it encompasses also $q$ and $\rho$ which are type-dependent metrics, not necessarily additive, such as the MMI metric.
The introduction of equivalence classes of channel-metric pairs \((W, q)\), which is important in itself, enabled us to derive a sufficient condition for the tightness of our bound. This condition states that if the equivalence class includes a matched channel-metric pair, then all the members of that class share the same mismatch capacity, the same LM rate, and the same GMI, which are all equal to the Shannon capacity of the matched pair. This is important since indeed a numerical computation of the LM rate and our upper bound can indicate that equality holds but does not form a rigorous proof of equality.

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