Abstract. Involutive Jamesian Functions are functions aimed to predict the outcome of an athletic competition. They were introduced in 1981 by Bill James, but until recently little was known regarding their form. Using methods from quasigroup theory we are able to obtain a complete description of them.

1 Introduction

In 1981, Bill James [4] introduced the notion of a function $P$, now called the James Function, in his effort to answer the following question: Suppose that two baseball teams $A$ and $B$ with winning percentages equal to $a$ and $b$ respectively, play against each other. What is the probability $P(a, b)$ that $A$ beats $B$?

Instead of just providing a formula for $P(a, b)$, James was also interested in what properties such a function should satisfy. In this spirit, he proposed the following self-evident conditions, now called the proto-James conditions:

1. $P(a, a) = \frac{1}{2}$ and $P(a, \frac{1}{2}) = a$, for every $a \in (0, 1)$.
2. If $a > b$, then $P(a, b) > \frac{1}{2}$ and if $a < b$, then $P(a, b) < \frac{1}{2}$.
3. If $b < \frac{1}{2}$, then $P(a, b) > a$ and if $b > \frac{1}{2}$, then $P(a, b) < a$.
4. $0 \leq P(a, b) \leq 1$ and if $a \in (0, 1)$, then $P(a, 0) = 1$ and $P(a, 1) = 0$.
5. $P(a, b) + P(b, a) = 1$.

After introducing the desired conditions, he provided an example of a function satisfying them, the explicit formula of which was given by Dallas Adams:

$$P(a, b) = \frac{a(1 - b)}{a(1 - b) + (1 - a)b}.$$  \hspace{1cm} (1.1)

He then conjectured that this was the only function that satisfied his conditions.
In 2015, Christopher Hammond, Warren Johnson and Steven Miller published a survey [2] on James functions and, among other things, they addressed this conjecture. They discovered a whole class of functions satisfying the proto-James conditions, thus disproving the James conjecture. To avoid working with pathological examples, they proposed the following axioms:

**Definition 1.1.** A function $J : [0, 1] \times [0, 1] \setminus \{(0, 0), (1, 1)\} \to [0, 1]$ is called an **involutive Jamesian function** if

1. $J(a, J(a, b)) = b$ for every $a, b \in (0, 1)$, (involutive property)
2. $J(a, b) + J(b, a) = 1$ for every $a, b \in (0, 1)$ and
3. for a fixed $b_0 \in (0, 1)$, the function $J(\cdot, b_0)$ is strictly increasing and for $b_0 \in [0, 1]$, it is nondecreasing.

Clearly, every involutive Jamesian function also satisfies the proto-James conditions. They showed that even if we restrict ourselves to this class of well behaved functions, we can still find a wealth of counterexamples: For every increasing homeomorphism $f : (0, 1) \to \mathbb{R}$ such that $f(1 - a) = -f(a)$ for every $a \in (0, 1)$, the function

$$J(a, b) = f^{-1}(f(a) - f(b)),$$

is an involutive Jamesian function and James’ original example is just a special case of (1.2). They closed their survey proposing three open problems, one of which was whether the converse were also true:

**Open Problem 1.2.** Is it true that every involutive Jamesian function can be written in the form (1.2) for some function $f$?

In what follows, an involutive Jamesian function which can be written in this form will be called *f-representable*, or just *representable*. The main purpose of this paper is to prove that there exist involutive Jamesian functions which are not $f$-representable. This is achieved in Paragraph 3.1 with the introduction of the notion of Jamesian Loops. Although our proof is existential, we also show how we can adopt it in order to produce explicit examples. We close our paper by providing a complete characterization of involutive Jamesian functions. As it turns out, these functions can indeed be represented in a form very similar to (1.2).

**2 Basic Properties of Involutive Jamesian Functions**

First, we recall a few basic properties. Their proofs can be found in [2].

**Proposition 2.1.** Let $J$ be an involutive Jamesian function. Then

1. $J(a, b) = c$ if and only if $J(a, c) = b$. This is actually equivalent to the involutive property.
2. $J(a, b) = J(1 - b, 1 - a)$ for every $a, b \in (0, 1)$.
3. $J(a, a) = \frac{1}{2}$ and $J(a, \frac{1}{2}) = a$ for every $a \in (0, 1)$.
4. $J$ is continuous.
Although it is not necessary, it will prove useful to change the definition of involutive Jamesian functions slightly. The two definitions are equivalent, so this is more a matter of convenience than of essence.

**Definition 2.2.** A function $J : (0, 1) \times (0, 1) \to (0, 1)$ is called an **involutive Jamesian function** if

1. $J(a, J(a, b)) = b$ for every $a, b \in (0, 1)$, (involutive property)
2. $J(a, b) + J(b, a) = 1$ for every $a, b \in (0, 1)$ and
3. for a fixed $b_0 \in (0, 1)$, the function $J(\cdot, b_0)$ is strictly increasing.

Clearly, for every $J$ satisfying the properties of **Definition 1.1** its restriction on $(0, 1) \times (0, 1)$ also satisfies **Definition 2.2** Conversely, suppose that $J$ satisfies **Definition 2.2** and let $\alpha \in (0, 1)$. We first show that if $(b_n)_{n \in \mathbb{N}}$ is a strictly decreasing sequence in $(0, 1)$ with $b_n \to 0$, then $J(a, b_n) \to 1$.

Suppose not. The sequence $(J(a, b_n))_{n \in \mathbb{N}}$ is strictly increasing and bounded above, so it has to converge to its supremum. Suppose that $c_n = J(a, b_n) \to c_0 < 1$. Then $J(c_n, a) = 1 - b_n$ is a strictly increasing sequence converging to 1. We pick a $c_0' \in (c_0, 1)$. Since $J(\cdot, a)$ is strictly increasing and $c_n < c_0 < c_0'$, we have that

$$J(c_n, a) < J(c_0, a) < J(c_0', a) \quad (2.1)$$

for every $n$ and since $J(c_n, a) \to 1$, this implies that $1 \leq J(c_0, a) < J(c_0', a)$, a contradiction. This argument, which is just an adaptation of the proof of [2, Proposition 7] to our context, implies that $J$ can be extended continuously and uniquely on $[0, 1] \times [0, 1] \setminus \{(0, 0), (1, 1)\}$ and this extension satisfies **Definition 1.1**.

The reason why it is more convenient to work with the second definition will become clear soon, but before that, it is important to begin our study by noting that the $f$-representability of Jamesian functions has a very useful reformulation:

**Proposition 2.3.** Let $J$ be an involutive Jamesian function. The following conditions are equivalent:

1. $J(J(a, c), J(b, c)) = J(a, b)$ for every $a, b, c \in (0, 1)$, (transitivity)
2. $J(J(a, b), J(a, c)) = J(c, b)$ for every $a, b, c \in (0, 1)$,
3. $J(b, J(a, c)) = J(c, J(a, b))$ for every $a, b, c \in (0, 1)$,
4. $J$ is $f$-representable.

**Proof.** The equivalence of (1), (2) and (3) is a straightforward application of the involutive property. For example, to obtain (3) from (2) one needs to plug $b := J(a, b)$ into (2), while for the converse, to pick an $x_0 \in (0, 1)$ such that $b = J(a, x_0)$ and plug it in (3) to obtain $J(J(a, b), J(a, c)) = J(x_0, J(a, c)) = J(c, J(a, x_0)) = J(c, b)$.

If $J$ is $f$-representable, then

$$J(J(a, c), J(b, c)) = f^{-1}(f(J(a, c)) - f(J(b, c)))$$

$$= f^{-1}(f(a) - f(c) - f(b) + f(c))$$

$$= f^{-1}(f(a) - f(b))$$

$$= J(a, b),$$
so it satisfies (1). Only the direction \((1) \Rightarrow (4)\) is non-trivial and it is actually a Theorem by M. Hosszú [3].

**Corollary 2.4.** An involutive Jamesian function \(J\) is \(f\)-representable if and only if it is transitive, that is \(J(J(a, c), J(b, c)) = J(a, b)\) for every \(a, b, c \in (0, 1)\).

Although we will not prove Hosszú’s theorem in detail, it is crucial to understand the main idea behind its proof, as our approach on the representation of Jamesian functions relies, essentially, on the same principles.

Hosszú attempted to solve the functional equation of transitivity

\[
F(F(x, t), F(y, t)) = F(x, y), \quad \forall x, y, t \in (0, 1),
\]

assuming that \(F : (0, 1) \times (0, 1) \to (0, 1)\) was a continuous and strictly monotonic function. He observed that if a function \(F\) satisfied (2.2), then the operation \(x \cdot y = F(x, 1 - y)\) defined a continuous group on \((0, 1)\). A theorem by Brouwer [1] asserts that all one dimensional continuous groups are isomorphic to the additive group of real numbers, so the operation \(\cdot\) on \((0, 1)\) had to be of the form

\[
x \cdot y = f^{-1}(f(x) + f(y)), \quad \forall x, y \in (0, 1),
\]

with \(f : (0, 1) \to \mathbb{R}\) a homeomorphism. As

\[
F(x, y) = x \cdot (1 - y) = f^{-1}(f(x) + f(1 - y)),
\]

he was then able to give a complete description of the solutions of (2.2).

**Theorem 2.5. (M. Hosszú, 1953).** A continuous and strictly monotonic function \(F : (a, b) \times (a, b) \to (a, b)\) satisfies the functional equation of transitivity

\[
F(F(x, t), F(y, t)) = F(x, y), \quad \forall x, y, t \in (a, b),
\]

if and only if \(F\) is written in the form

\[
F(x, y) = f^{-1}(f(x) - f(y)), \quad \text{(quasi-difference)}
\]

where \(f\) is a continuous and strictly monotonic function.

To return to our problem, it is easy to see that an involutive Jamesian function \(J\) is transitive if and only if the induced operation \(a \cdot b = J(a, 1 - b)\) forms a group (Proposition 3.8). Since we do not know yet whether such functions are necessarily transitive, we cannot hope that the corresponding operations will form a group. However, as we will show in the next section, they form the next best structure we could hope for: a loop.

## 3 Loops and Jamesian Loops

Quasigroups are non-associative algebraic structures that arise naturally in many Mathematical fields. Although they first appeared in the literature in the early 1900s, it was not until the 1930s that they were defined and studied systematically. Quasigroups with an identity element are called loops. They emerged by the influential works of Ruth Moufang (1935) and Gerrit Bol (1937) and soon they became
a separate area of research. For a fascinating historical overview of quasigroups and loops, the reader should check [6].

Before we start exploring the connection between loops and Jamesian functions, we need a few basic definitions and properties from loop theory. A standard reference for their algebraic theory is [7], whereas the first chapter of [5] offers an introduction to their topological aspects.

**Definition 3.1.** A loop \((L, \cdot)\) is a set \(L\) equipped with an operation \(\cdot : L \times L \to L\), satisfying the following properties:

1. There exists an element \(e \in L\), such that \(a \cdot e = e \cdot a = a\) for every \(a \in L\). This element is called the identity element, or the unit of \(L\).

2. For every \(a \in L\) there exists an \(a^{-1} \in L\) such that \(a \cdot a^{-1} = a^{-1} \cdot a = e\).

So a loop is a structure that satisfies every group axiom except, possibly, for associativity. A loop which is not a group is called a proper loop.

**Definition 3.2.** A loop \(L\) is said to have the inverse property (or IP) if

\[
x(x^{-1}y) = y = (yx^{-1})x \quad \text{for every } x, y \in L.
\]

(3.1)

It is easy to check that a loop with the inverse property also satisfies \((xy)^{-1} = y^{-1}x^{-1}\) for every \(x, y \in L\): Let \(a, b \in L\). We substitute \(x := (ab)^{-1}\) and \(y = b^{-1}\) into (3.1) to get \((ab)^{-1} ((ab)b^{-1})) = b^{-1}\). By the inverse property, \((ab)b^{-1} = a\), so \((ab)^{-1} \cdot a = b^{-1}\), which implies that \((ab)^{-1} = b^{-1}a^{-1}\).

**Definition 3.3.** A topological loop \((L, \cdot)\) is a loop which is also a topological space, such that the product and the inverse operations are continuous.

**Definition 3.4.** Two topological loops \((L, \cdot), (L', \cdot')\) are said to be isomorphic if there exists a (topological) homeomorphism \(f : L \to L'\) such that \(f(a \cdot b) = f(a) \cdot f(b)\) for every \(a, b \in L\).

**Remark 3.5.** We can easily see that if two loops are isomorphic and one of them is proper, then the other one has to be proper as well: Say that \((L, \cdot)\) is proper. Then there exist \(a, b, c \in L\) such that \(a \cdot (b \cdot c) \neq (a \cdot b) \cdot c\). Since \(f\) is one to one,

\[
f(a \cdot (b \cdot c)) = f(a) * f((b \cdot f(c))) \neq (f(a) * f(b)) \cdot f(c) = f((a \cdot b) \cdot c),
\]

which shows that \(*\) is not associative. Similarly one can check that commutativity and IP are properties preserved by isomorphisms.

An isomorphism from a topological loop \(L\) to itself is called an automorphism. To every loop we associate two important families of automorphisms, its left and right translations:

**Definition 3.6.** Given a loop \((L, \cdot)\) and \(b_0 \in L\), the function \(R_{b_0} : L \to L\), defined as \(R_{b_0}(a) = a \cdot b_0\) for \(a \in L\), is called a right translation. Similarly, given an \(a_0 \in L\), the function \(L_{a_0}(b) = a_0 \cdot b\) is called a left translation. If the loop is commutative, the sets of left and right translations coincide.

The group \(G\) generated by the left translations, equipped with the Arens topology, plays an important role in the classification of loops. In our work we will not need this approach, but the curious reader should consult [5] for more details.
3.1 Jamesian Loops

We now return to our main question:

**Question.** Given an involutive Jamesian function \( J \), what properties does the operation \( a \cdot b = J(a, 1 - b) \) defined on \((0, 1)\) satisfy?

First of all, for every \( a, b, a \cdot b = J(a, 1 - b) = J(b, 1 - a) = b \cdot a \), so it has to be commutative. Additionally, \( a \cdot \frac{1}{2} = J(a, \frac{1}{2}) = a \) and \( a \cdot (1 - a) = J(a, a) = \frac{1}{2} \), so \(((0, 1), \cdot)\) is in fact a loop, having \( e = \frac{1}{2} \) as an identity and such that the inverse of every \( a \) is \( a^{-1} = 1 - a \). The involution property of \( J \) can be rewritten as

\[
 b = J(a, J(a, b)) = J(a, 1 - J(b, a)) = a \cdot J(b, a) = a \cdot (b \cdot (1 - a))
\]

for every \( a, b \), which, because of the commutativity, is just the inverse property of loops. Similarly, the identity \( J(a, b) + J(b, a) = 1 \) implies that \((a \cdot b)^{-1} = b^{-1} \cdot a^{-1}\). This already follows from the inverse property, so we may omit it when we define the notion of Jamesian loops.

Finally, since \( J \) is strictly increasing with respect to the first variable, the left (and right) translations are strictly increasing functions. We collect all these observations into the definition of Jamesian loops:

**Definition 3.7.** A topological loop \(((0, 1), \cdot)\) is called a Jamesian loop if

1. it is commutative,
2. its unit is \( e = \frac{1}{2} \) and the inverse of every \( a \in (0, 1) \) is \( a^{-1} = 1 - a \),
3. \( a(b(a^{-1})) = b \) for every \( a, b \in (0, 1) \) and
4. the right (and left) translations are strictly increasing functions.

As we expected, there is a one to one correspondence between involutive Jamesian functions and Jamesian loops:

**Proposition 3.8.** If \( J \) is an involutive Jamesian function then the operation \( \cdot \) for which \( a \cdot b = J(a, 1 - b) \) for \( a, b \in (0, 1) \), is a Jamesian loop. Conversely, if \(((0, 1), \cdot)\) is a Jamesian loop then the function \( J(a, b) = a \cdot (1 - b) \) is an involutive Jamesian function. In addition, a Jamesian loop is a group if and only if the induced involutive Jamesian function is \( f \)-representable.

**Proof.** From the preceding discussion, if \( J \) is an involutive Jamesian function, then \( \cdot \) is a Jamesian loop. Conversely, if \(((0, 1), \cdot)\) is a Jamesian loop and we define \( J(a, b) = a(1 - b) \), then the inverse property of \( \cdot \) implies that \( J(a, J(a, b)) = b \) for every \( a, b \). Similarly, \((a \cdot b)^{-1} = b^{-1} \cdot a^{-1}\) implies that \( J(a, b) + J(b, a) = 1 \). The function \( J(\cdot, b_0) \) is strictly increasing as \( J(\cdot, b_0) = R_{b_0} \) and every right translation is strictly increasing. So \( J \) satisfies all three axioms of involutive Jamesian functions.

For the additional part, the induced operation is always a loop, so we are in fact seeking an equivalent characterization of associativity. Let \( a, b, c \in (0, 1) \). We compute the expressions

\[
(a \cdot b) \cdot c = J(a, 1 - b) \cdot c = J(J(a, 1 - b), 1 - c)
\]

\[
= J(c, 1 - J(a, 1 - b))
\]

\[
= J(c, J(1 - b, a)) \quad \text{and}
\]

\[
a \cdot (b \cdot c) = a \cdot J(b, 1 - c) = J(a, 1 - J(b, 1 - c))
\]

\[
= J(a, J(1 - c, b))
\]

\[
= J(a, J(1 - b, c)).
\]
The operation is associative if and only if $J(c, J(b, a)) = J(a, J(b, c))$ for every $a, b, c \in (0, 1)$, which is equivalent to the $f$-representability of $J$ due to Proposition 2.3.

The following example was constructed by Salzmann in his attempt to “prove the existence of planar associative division nearings which possess an additive loop with the inverse property and which are homeomorphic to $\mathbb{R}$” [9, pg. 135]. With a slight modification it also proves the existence of proper Jamesian loops and, in view of the previous proposition, of involutive Jamesian functions which are not $f$-representable.

**Theorem 3.9. (H. Salzmann, 1957).** There exists a commutative proper topological loop $(\mathbb{R}, \cdot)$ with the inverse property having $e = 0$ as an identity and such that the inverse of every $x \in \mathbb{R}$ is $x^{-1} = -x$. Additionally, its left (and right) translations are strictly increasing.

**Proof.** The operation, as defined in [8, pg. 459], is:

$$x \ast t = \begin{cases} x + \frac{1}{2} t, & \text{if } \frac{x}{t} \in (-\infty, -\frac{3}{2}] \cup [1, +\infty), \\ \frac{1}{2} x + t, & \text{if } \frac{x}{t} \in [-\frac{2}{3}, 1], \\ 2x + 2t, & \text{if } \frac{x}{t} \in [-\frac{2}{3}, -\frac{3}{2}], \\ x, & \text{if } t = 0. \end{cases}$$

Clearly, $e = 0$ is an identity element for the $\ast$-operation and one can easily check that $x \ast y = y \ast x$ for every $x, y \in \mathbb{R}$. Additionally, since for every $x \neq 0$, $\frac{x}{x} = 1$, then $x \ast (-x) = 2x + (-2x) = 0$ and the pair $(\mathbb{R}, \ast)$ is a commutative loop such that every $x$ has $-x$ as its inverse. With a little more effort we can show that it also possesses the inverse property:

$$(x \ast t) \ast (-t) = \begin{cases} (x \ast t) - \frac{1}{2} t, & \text{if } \frac{x \ast t}{t} \in (-\infty, -\frac{3}{2}] \cup [1, +\infty), \\ \frac{1}{2} (x \ast t) - t, & \text{if } \frac{x \ast t}{t} \in [-\frac{2}{3}, 1], \\ 2(x \ast t) - 2t, & \text{if } \frac{x \ast t}{t} \in [-\frac{2}{3}, -\frac{3}{2}], \\ x, & \text{if } t = 0. \end{cases}$$

- If $\frac{x}{t} \in (-\infty, -\frac{3}{2}] \cup [1, +\infty)$, then $x \ast t = x + \frac{1}{2} t$ and $\frac{x \ast t}{t} = -\frac{3}{2} - \frac{1}{2}$, so $\frac{x \ast t}{t} \leq -\frac{3}{2}$ or $\frac{x \ast t}{t} \geq 1$. This yields that $(x \ast t) \ast (-t) = x \ast t - \frac{1}{2} t = x + \frac{1}{2} t - \frac{1}{2} t = x$.

- If $\frac{x}{t} \in [-\frac{2}{3}, 1)$, then $x \ast t = \frac{1}{2} x + t$ and $\frac{x \ast t}{t} = -\frac{x}{t} - 1$, so $\frac{x \ast t}{t} \in [-\frac{3}{2}, -\frac{2}{3}]$ and $(x \ast t) \ast (-t) = 2(x \ast t) - 2t = 2(\frac{1}{2} x + t) - 2t = x$.

- If $\frac{x}{t} \in [-\frac{3}{2}, -\frac{3}{2}]$, then $x \ast t = 2x + 2t$ and $\frac{x \ast t}{t} = -\frac{2x}{t} - 2$, so $\frac{x \ast t}{t} \in [-\frac{3}{2}, 1]$ and $(x \ast t) \ast (-t) = \frac{1}{2} (x \ast t) - t = \frac{1}{2}(2x + 2t) - t = x$.

In any case $(x \ast t) \ast (-t) = x$.

On the other hand, $(\mathbb{R}, \ast)$ is not associative. We will actually show that it is not even power associative, by proving that $x^2 \ast x^2 \neq x^3 \ast x$ for every $x \neq 0$. Obviously both $x$ and $x^2 = x \ast x$ are uniquely determined and since the operation is commutative, $x^3 = x \ast (x \ast x) = (x \ast x) \ast x$ is also defined. However, this is not true for $x^3$:
\[ x^2 = x \cdot x = x + \frac{1}{2} x = \frac{3}{2} x, \]
\[ x^3 = x \cdot (x^2) = x \cdot \frac{3}{2} x = \frac{1}{2} x + \frac{3}{2} x = 2x, \]
\[ x^2 \cdot x^2 = \frac{3}{2} x \cdot \frac{3}{2} x = \frac{3}{2} x + \frac{3}{4} x = \frac{9}{4} x, \]
\[ x^3 \cdot x = (2x) \cdot x = 2x + \frac{1}{2} x = \frac{5}{2} x. \]

Since
\[ x^2 \cdot x^2 = (x \cdot x) \cdot (x \cdot x) \neq (x \cdot (x \cdot x)) \cdot x = x^3 \cdot x, \]
the operation is not associative, thus \((\mathbb{R}, \cdot)\) is a proper loop.

All that is left now is to transfer Salzmann’s example from \(\mathbb{R}\) to \((0, 1)\) preserving its main properties.

**Lemma 3.10.** Let \((\mathbb{R}, \cdot)\) be a topological loop satisfying all the aforementioned properties of Salzmann’s example. That is, \((\mathbb{R}, \cdot)\) is a commutative proper topological loop with the inverse property, its unit is \(e = 0\), the inverse of each \(x \in \mathbb{R}\) is \(x^{-1} = -x\) and its left translations are strictly increasing. Then for every strictly increasing homeomorphism \(f : (0, 1) \rightarrow \mathbb{R}\) such that \(f(1-x) = -f(x)\), the operation \(\cdot\) for which
\[ a \cdot b = f^{-1}(f(a) \cdot f(b)), \quad a, b \in (0, 1), \tag{3.2} \]
defines a proper Jamesian loop on \((0, 1)\).

**Proof.** The proof follows easily from the fact that the loops \((\mathbb{R}, \cdot)\) and \(((0, 1), \cdot)\) are isomorphic under the function \(g = f^{-1}\). Since \(f(\frac{1}{2}) = 0\), \(g(0) = \frac{1}{2}\) is the unit of \((0, 1)\). Additionally,
\[ x \cdot (1-x) = f^{-1}(f(x) \cdot f(1-x)) = f^{-1}(f(x) \cdot (-f(x))) = f^{-1}(0) = \frac{1}{2}, \]
which shows that the inverse operation on \((0, 1)\) is given by \(x^{-1} = 1-x\). Commutativity, properness and IP are properties preserved by isomorphisms (see Remark 3.5), so they must hold on \(((0, 1), \cdot)\) as well. The left translations being strictly increasing follows from the fact that \(f\) is strictly increasing.

**Corollary 3.11.** There exist involutive Jamesian functions which are not representable.

**Proof.** By Proposition 3.8, every proper Jamesian loop induces such a function and proper Jamesian loops do exist, as was shown in Theorem 3.9 and Lemma 3.10.

**Remark 3.12.** Actually Lemma 3.10 can be used to produce an infinity of counterexamples: Let \(f, g : (0, 1) \rightarrow \mathbb{R}\) be distinct homeomorphisms with the property that \(f(1-x) = -f(x)\) and \(g(1-x) = -g(x)\) for every \(x \in (0, 1)\). Let also \(J\) be Salzmann’s loop on \(\mathbb{R}\) and \(J_f, J_g\) the corresponding proper Jamesian loops on \((0, 1)\), as defined in Lemma 3.10. We will show that \(J_f\) and \(J_g\) are distinct. As \(f\) and \(g\) are continuous and not identically equal, there exists an \(a_0 \in (\frac{1}{2}, \infty)\) and an \(\varepsilon > 0\), such that \(f(x) \neq g(x)\) for every \(x \in I = (a_0 - \varepsilon, a_0 + \varepsilon)\). Without loss of generality we may also assume that \(f(x) > g(x)\) on \(I\). Let \(x_0 \in I\). Then
\[ J_f(x_0, 1-x_0) = f^{-1}(f(x_0) \cdot f(x_0)) > g^{-1}(g(x_0) \cdot g(x_0)) = J_g(x_0, 1-x_0), \]
as all $\ast$, $f$ and $g$ are strictly increasing. In particular, $J_f \not\equiv J_g$.

Similarly, if one begins with two non-isomorphic Jamesian loops, then their corresponding Jamesian functions are distinct. To sum up, there are two different ways of constructing distinct families of involutive Jamesian functions. One is, for a fixed Jamesian loop on $\mathbb{R}$, to let the homeomorphism $f : (0, 1) \to \mathbb{R}$ of Lemma 3.10 vary. The other is to consider Jamesian functions induced by non-isomorphic Jamesian loops.

**Remark 3.13.** Even though there exist non-representable involutive Jamesian functions, it is very natural to ask whether these “pathological” functions can at least be approximated by representable ones. Let us describe the setting first: We denote the classes of representable Jamesian functions and involutive Jamesian functions by $J_{\text{rep}}$ and $J_{\text{inv}}$ respectively and as we have shown, $J_{\text{rep}}$ is a proper subset of $J_{\text{inv}}$.

Let also $S = (0, 1) \times (0, 1)$ be the open unit square of the plane.

Since both $J_{\text{rep}}$ and $J_{\text{inv}}$ are subsets of $(C_b(S), \| \cdot \|_\infty)$, the Banach space of bounded and continuous real valued functions on $S$ equipped with the supremum norm, we can view them as metric spaces with the induced metric. It is easy to check that $J_{\text{rep}}$ is not dense in $J_{\text{inv}}$ and, in fact, a continuity argument and an appeal to Propositions 2.3 and 3.8 shows that none of the elements of $J_{\text{inv}} \setminus J_{\text{rep}}$ can be approximated by elements of $J_{\text{rep}}$.

Before closing this section it is worth to mention a question regarding Jamesian loops and concerns what happens in the case of differentiability. Salzmann’s loop is continuous, but clearly not differentiable everywhere, so the following remains open:

**Open Problem 3.14.** Is it true that every differentiable involutive Jamesian function is representable?

### 3.2 A Specific Example

In the previous paragraph we provided a rule to construct non-representable Jamesian functions, however, until now, we have not been able to give an explicit example of one. In this paragraph we will try to do so.

By Lemma 3.10 in order to find a formula for $J(a, b)$, we need to specify a homeomorphism $f : (0, 1) \to \mathbb{R}$ and then compute

$$J(a, b) = f^{-1}(f(a) \ast f(1 - b)), \quad (3.3)$$

where $(\mathbb{R}, \ast)$ is Salzmann’s loop. If we examine the definition of $\ast$, we see that the plane is partitioned into three regions and $a \ast b$ is defined according to which region the number $ab$ belongs to. So, in order to compute (3.3), first we have to determine where $\frac{f(a)}{f(1 - b)}$ belongs, for every $a, b \in (0, 1)$. Depending on $f$, this can be a difficult to nearly impossible task.

It soon becomes clear that we need to avoid this complexity by picking $f$ appropriately: If $f$ acts as the identity function in a neighborhood around $\frac{1}{2}$, then $\frac{f(a)}{f(1 - b)}$ is easily computed for $a, b$ in this neighborhood and, quite possibly, so is $f^{-1}(f(a) \ast f(1 - b))$. This gives us a simple formula for $J(a, b)$. The downside is that on $(0, 1) \setminus I$, $f$ is no longer the identity, but this is not something we can avoid, as any homeomorphism $f : (0, 1) \to \mathbb{R}$ is bound to deform some regions of $(0, 1)$. 

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In what follows, we construct an example of a non-representable involutive Jamesian function $J$ and we give an explicit formula for $J(a, b)$, but only for $(a, b) \in \Delta \subseteq S = (0, 1) \times (0, 1)$, where $\Delta$ is a region we will specify using the ideas we just discussed.

Let $\varepsilon > 0$ and set $I_\varepsilon = [\varepsilon, 1 - \varepsilon]$. We define $f : I_\varepsilon \to \mathbb{R}$ by $f(x) = x - \frac{1}{2}$ for every $x \in I_\varepsilon$. Clearly $f(1 - x) = -f(x)$ for every $x \in I_\varepsilon$ and $f$ can be extended to a homeomorphism from $(0, 1)$ to $\mathbb{R}$ satisfying the same property. We will denote this extension again by $f$. The inverse of $f$ has the property that $f^{-1}(x) = x + \frac{1}{2}$ for every $x \in I_\varepsilon' = [\varepsilon - \frac{1}{2}, 1 - \varepsilon] \subseteq \mathbb{R}$. If $a, b \in I_\varepsilon$ such that $f(a) * f(b) = (a - \frac{1}{2}) * (\frac{1}{2} - b) \in I_\varepsilon'$, then

$$J(a, b) = f^{-1}(f(a) * f(1 - b)) = f(a) * f(1 - b) + \frac{1}{2},$$

so we have a simple formula for $J(a, b)$. This is possible on the following subsets of $\mathbb{R}^2$:

| $A_1$ | $A_2$ | $A_3$ |
|-------|-------|-------|
| $(a, b) \in S : \frac{a - 1/2}{1/2 - b} \in (-\infty, -\frac{3}{2}] \cup [1, \infty)$ and $a - \frac{1}{2} - \frac{1}{4} \in I_\varepsilon'$ | $(a, b) \in S : \frac{a - 1/2}{1/2 - b} \in [-\frac{2}{3}, 1]$ and $\frac{a}{2} - b + \frac{1}{4} \in I_\varepsilon'$ | $(a, b) \in S : \frac{a - 1/2}{1/2 - b} \in [-\frac{2}{3}, -\frac{2}{5}]$ and $2a - 2b \in I_\varepsilon'$ |

So we can explicitly define $J$ on $\Delta = A_1 \cup A_2 \cup A_3$.

![Figure 3.1: The region $\Delta = A_1 \cup A_2 \cup A_3$ where $J$ can be defined explicitly.](image)

and the formula of $J$ is:

$$J(a, b) = \left( a - \frac{1}{2} \right) * \left( \frac{1}{2} - b \right) + \frac{1}{2} = \begin{cases} a - \frac{b}{2} + \frac{1}{2}, & \text{if } (a, b) \in A_1, \\ a - b + \frac{3}{2}, & \text{if } (a, b) \in A_2, \\ 2a - 2b + \frac{1}{2}, & \text{if } (a, b) \in A_3, \\ f^{-1}(f(a) * f(1 - b)), & \text{everywhere else.} \end{cases}$$

### 4 Conclusions

We proved that an involutive Jamesian function may not necessarily be written in the form

$$J(a, b) = f^{-1}(f(a) + f(1 - b)).$$

(4.1)
Despite this, we can still find a very similar representation by allowing the addition in the expression $f(a) + f(1 - b)$ to be replaced by a more general operation $\ast$:

**Proposition 4.1.** Every involutive Jamesian function $J$ can be written as

$$J(a, b) = f^{-1}(f(a) \ast f(1 - b)),$$

where $f : (0, 1) \to \mathbb{R}$ is a strictly increasing homeomorphism with $f(1 - x) = -f(x)$ for every $x \in (0, 1)$ and $(\mathbb{R}, \ast)$ is a loop.

This gives us a complete description of involutive Jamesian functions, but at a certain cost: The well studied and understood operation of real addition has been replaced by a mysterious and probably complicated real loop operation. There is a deep theory behind the attempts of classification of loops on $\mathbb{R}$ and only certain classes of them have been completely classified. Unfortunately the Jamesian loops do not seem to belong in any of these classes, so the (equivalent) problems of classifying involutive Jamesian functions and classifying Jamesian loops remain open. For an idea of how complicated, but also exciting, this problem can be, the reader should check [5, Chapter 18].

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