Practical Finite-Time Event-Triggered Trajectory Tracking Control for Underactuated Surface Vessels With Error Constraints

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This work was supported in part by the 2019 “Chong First-Class” Provincial Financial Special Funds Construction Project under Grant 231419019, in part by the Key Project of Department of Education of Guangdong Province under Grant 2021ZDZX1041, and in part by the Science and Technology Planning Project of Zhanjiang City under Grant 2020B01267 and Grant 2021E05012.

ABSTRACT This paper proposes a robust trajectory tracking control method for underactuated unmanned surface vessels (USVs) with input saturation based on a relative threshold event-triggered mechanism (ETM), prescribed performance and finite-time convergence. First, to address the challenges posed to controller design by the nondiagonal inertia matrix, coordinate transformation technology is adapted to change the vessel position. An asymmetric smooth saturation function is employed to address the input saturation problem. Second, an adaptive neural network (NN) is developed to approximate the uncertain nonlinear dynamics and external environmental disturbances. Third, the performance function constraints are integrated into the tan-type barrier Lyapunov function (BLF) to greatly enhance the robustness of the system by combining with the finite-time convergence property. In addition, the dynamic surface control (DSC) technique is employed to address the differential explosion problem. Subsequently, a relative threshold ETM is incorporated into the controller to reduce the communication burden. The rationality and feasibility of the proposed algorithm are confirmed by theoretical analysis and numerical simulations.

INDEX TERMS Finite-time control, barrier Lyapunov function, event-triggered, neural networks, underactuated surface vessel.

I. INTRODUCTION
With the surge in research on marine environments, investigations of marine surface vessels have aroused great concern in the control community [1]–[4]. Trajectory tracking control of underactuated unmanned surface vessels (USVs) is both an important and challenging problem [5]–[7]. Compared with path following, in trajectory tracking, a vessel tracks a time-parameterized desired trajectory at a specified time, while the path following problem without any temporal specification [8]. Moreover, underactuated vessels are different from fully actuated vessels, and trajectory tracking control of USVs is more difficult to achieve. This difficulty arises because the movements of the USVs is achieved by only one propeller and one rudder acting on the wave motion and yaw motion, respectively, but without an actuator directly controlling the sway motion [9].

Fortunately, in previous studies, many positive results have been achieved on the trajectory tracking control of USVs [10]–[12]. In [10], a sliding mode control method was constructed, and based on these control laws, an experimental test was carried out. In [11], actor-critic neural network (NN) were utilized to design a tracking controller. To reduce the tuning parameters, [12] employed the minimum learning parameter algorithm in the control system. However, in the aforementioned works in [10]–[12], not only the mass matrix but also the damping matrix of the vessels are assumed to be in diagonal form, that is, the shape of the vessel is identified as a sphere. Notably, the above assumptions are impractical. Consider that the nondiagonal inertia matrix introduces nonintegrable dynamics into dynamics in nonlinear systems and complicates the controller design. To solve this problem, Do and Pan used the
coordinate transformation technique to adjust the positions of vessels [13].

To guarantee the performance of tracking control, error constraint methods have been introduced in many studies [14]–[16]. In [14], by the characteristics of the ln(·) function, the trajectory tracking errors of USVs are constrained. In [15], to address the output constraints in the tracking control system of vessels, the asymmetric barrier Lyapunov function (BLF) is utilized. In [16], Li et al. introduced full-state constraints in the controller via a tan-type BLF to ensure the stability of the tracking system for USVs. In addition to the above error-constrained methods, in [17], Bu and Qi selected a concise, single-network, adaptive critic design that can also guarantee the real-time performance of tracking control for hypersonic flight vehicles. The controllers in [14]–[17] were designed based on the backstepping technique but did not consider the problem of differential explosion in the controller. When the control laws need to use the derivative of the virtual control laws, the differential explosion problem may appear in the control system. Thereby, the dynamic surface control (DSC) technique is adopted to prevent the differential explosion problem from occurring in the tracking control system in [18].

However, the aforementioned method can only ensure that the system is asymptotically stable but cannot guarantee that the system converges in a finite time. Considering that there are merits of fast convergence, better robustness and strong anti-interference ability in the finite-time convergence system [19]–[23] introduce finite-time convergence control into the control system. In [20], a finite-time control method is introduced into the fuzzy neural controller of waverider aircraft such that the system can reach a steady state in a finite time. For the vessel tracking control system, a finite-time control method with error constraints is proposed in [21]. Reference [22] investigated a formation control for multi-USVs and ensured that the errors of formation tracking can converge within a finite time. Aiming at multiple flight vehicle trajectory tracking control with actuator saturation, a second-order finite-time formation control strategy was developed in [23].

It is noteworthy that the trajectory tracking control methods of the abovementioned USVs do not consider the communication constraints caused by the limited network bandwidth. In the traditional control method, the actuator is updated at each sampling moment, and high-frequency updates will cause wasted resources and even communication constraints. To realize control system communication only when needed, the relative threshold event-triggered mechanism (ETM) has attracted the attention of scholars in research. For path-following control of USVs, Li et al. in [24] introduced a fixed threshold ETM into the NN control system to reduce the number of communications. Reference [25] proposed a new control method that combines the ETM and consensus control and employed it in high-order multiagent systems. In [26], to reduce the burden of signal transmission, a periodic ETM is proposed for linear systems. In [27], a static ETM is constructed in a finite-time controller of Hamiltonian systems. However, in the previously described system, the control input signal will be small when the system states are near the origin and will be large when the system states are far from the origin. To not only reduce the communication burden but also guarantee control performance, a relative threshold ETM is proposed in [28]. Unlike the above ETM, the threshold size of the relative threshold ETM can be changed with the control input signal.

Nevertheless, in all previous published studies on the trajectory tracking control of USVs, few scholars consider the relative threshold ETM, finite-time convergence and prescribed performance simultaneously. Inspired by the abovementioned observations, a strong robustness trajectory tracking control with prescribed performance, finite-time convergence and relative threshold ETM for USVs is proposed. By using the BLF, the tracking errors of USVs are constrained in the prescribed bounds. To address the difficulty of controller design caused by the inertia matrix with nondiagonal terms, coordinate transformation technology is employed to change the vessel position. To obtain an unknown nonlinear term, the radial basis function neural network (RBFNN) is selected. To obtain fast convergence, finite-time control is investigated in the systems. To reduce the burden of communication, the relative threshold ETM is proposed. The main contributions of this paper are generalized as follows:

1. A novel trajectory tracking control method of USVs with a nondiagonal inertial matrix is addressed. Moreover, input saturation and differential explosion problems are considered.
2. The tan-type BLF is employed to constrain the tracking errors within the range of prescribed performance constraints based on the backstepping technique, and the tracking performance of the systems is guaranteed.
3. Finite-time convergence is achieved in the controller of USVs. Compared with traditional asymptotically stable systems, the system has better robustness and strong anti-interference ability.
4. The relative threshold ETM is incorporated into the control method to ensure that the actuator is updated only when the set trigger conditions are met, which effectively reduces the communication and computation burden.

II. PRELIMINARIES

A. USV DYNAMICS MODEL DESCRIPTION

Since the vessel described in this paper is a surface vessel, this paper only takes into account three motions of surge, sway and yaw. A nonlinear mathematical model of USVs is shown here [29]:

\[
\dot{\eta} = R(\psi) \nu
\]

\[
M\dot{\nu} = -C(\nu) \nu - D(\nu) \nu + N + T
\]

where \( \eta = [x, y, \psi]^T \) represents the location in the earth-fixed frame, \( \nu = [u, v, r]^T \) represents the velocity in the body-fixed frame, and \( N = [N_x, N_y, N_z]^T \) represents the
and erf actuator, respectively, which will be given later in this paper; the highest output of the actuator and lowest output of the saturation limitation in a smooth form.

The asymmetric smooth saturation functions, which is the estimation of \(W\), \(\hat{W}\) and \(\tilde{W}\) is bounded, namely, there exist two unknown positive constants \(\mu_\tilde{W}\) and \(\mu_\hat{W}\) such that \(\|\hat{W}\| \leq \mu_\hat{W}\) and \(\|\tilde{W}\| \leq \mu_\tilde{W}\).

### C. FINITE-TIME STABILITY THEOREM

**Lemma 1 [33]:** For any \(\delta > 0\) and \(H \in \mathbb{R}\), the following inequality given holds.

\[
0 \leq |H| - H \cos \left( \frac{H}{\delta} \right) \leq 0.2785\delta
\]

**Lemma 2 [34]:** Consider the system \(\dot{x} = f(x)\). For any continuously differentiable function \(v(x)\), if there exist parameters that satisfy \(a, b > 0\), \(0 < l < 1\) and \(0 < \varepsilon < \infty\) such that the system satisfies

\[
\dot{V}_3 \leq -aV_3 - bV_3^l + \varepsilon
\]

then the system is practical finite-time stable (PFS), and its residual set of the solution is \(\Omega = \min \left\{ \left\frac{a}{(1-l)^b} \left\frac{1}{(1+b)} \right\right\ \right\} \)

with \(0 < \Pi < 1\). The settling time is \(t_s = \max \left\{ \frac{1}{a \Pi (1-l)} \ln \frac{a \Pi (1-l) (k_0+b)}{b}, \frac{1}{a \Pi (1-l)} \ln \frac{a \Pi (1-l) (k_0+b)}{b} \right\} \).

### D. TAN-TYPE BARRIER LYAPUNOV FUNCTIONS

To ensure that the tracking errors are constrained within the range of the prescribed performance function, a tan-type BLF is introduced from [22]:

\[
\begin{align*}
V_u &= \frac{\rho_1^2}{\pi} \tan \left( \frac{\pi Z_u^2}{2 \rho_1^2} \right), \quad |Z_u(0)| < \rho_1(0) \\
V_r &= \frac{\rho_2^2}{\pi} \tan \left( \frac{\pi \psi_r^2}{2 \rho_2^2} \right), \quad |\psi_r(0)| < \rho_2(0)
\end{align*}
\]
where \( Z_e \) and \( \psi_e \) denote tracking errors of vessel, which will be given in Section III.

When there is no constraint, we will have \( \rho_{1,2} \to \infty \).

By L’Hospital’s rule, we have

\[
\begin{align*}
\lim_{\rho_1 \to \infty} \frac{\rho_1^2}{\pi} \tan \left( \frac{\pi Z_e^2}{2\rho_1^2} \right) &= \frac{1}{2} Z_e^2 \\
\lim_{\rho_2 \to \infty} \frac{\rho_2^2}{\pi} \tan \left( \frac{\pi \psi_e^2}{2\rho_2^2} \right) &= \frac{1}{2} \psi_e^2
\end{align*}
\]

(12)

Assumption 2: The unknown environmental disturbances \( N = [N_u, N_v, N_r]^T \) are bounded, i.e., there exist unknown constants \( N_{i\text{max}} \) such that \( |N_i| \leq N_{i\text{max}} \) \( i = u, v, r \).

Assumption 3: The first derivatives of \( x_d, y_d, \psi_d \) in the desired trajectory vector \( \eta_d = [x_d, y_d, \psi_d]^T \) are bounded.

Assumption 4 [1]: In system (14), the sway velocity \( v_h \) is passively bounded.

Control objective: Based on assumptions 1-4, design a practical, finite-time, event-triggered trajectory tracking controller with strong robustness such that the vessel can complete the tracking of the desired trajectory within a finite time and guarantee that the tracking errors are constrained in (24).

USVs can be rewritten as:

\[
\begin{align*}
\dot{x}_h &= u \cos \psi - v_h \sin \psi \\
\dot{y}_h &= u \sin \psi + v_h \sin \psi \\
\dot{\psi} &= r \\
\dot{u} &= f_u + \frac{1}{m_{11}} \tau_u \\
\dot{v}_h &= f_r \\
\dot{r} &= f_r + \frac{m_{22}}{\Lambda} \tau_r
\end{align*}
\]

(14)

where

\[
\begin{align*}
f_u &= \frac{1}{m_{11}} \left( m_{22}(v_h - hr) + m_{23}r^2 - d_{11}u \right) + \frac{1}{m_{11}} N_u - \frac{1}{m_{12}} \Delta u \\
f_r &= \frac{1}{m_{22}} \left( -m_{11}ur - d_{22}(v_h - hr) - d_{23}r \right) + \frac{1}{m_{22}} N_r \\
f_r &= \frac{1}{\Lambda} \left( (m_{11}m_{22} - m_{22}^2)u(v_h - hr) + (m_{11}m_{23} - m_{23}m_{22})ur - \left( d_{13}r + d_{32}(v_h - hr) \right) \right) m_{22} + \left( d_{23}r + d_{22}(v_h - hr) \right) m_{23} \\
\Lambda &= m_{22}m_{33} - m_{23}^2
\end{align*}
\]

It is clearly seen from (14) that after using coordinate transformation technology, the \( \tau_r \) on the \( \dot{r} \) dynamics has been successfully eliminated.

B. ERROR TRANSFORMATION

A schematic of trajectory tracking and a description of error variables for USVs are shown in Fig. 2. The error variables are defined as follows:

\[
\begin{align*}
x_e &= x_d - x_h \\
y_e &= y_d - y_h \\
\tilde{x}_e &= \cos(\psi)x_e + \sin(\psi)y_e \\
\tilde{y}_e &= -\sin(\psi)x_e + \cos(\psi)y_e \\
E &= \sqrt{\xi_e^2 + \eta_e^2} \\
\psi_e &= \text{atan} \, 2(\tilde{y}_e, \tilde{x}_e)
\end{align*}
\]

(15)

(16)

(17)

(18)

where \((x_d, y_d)\) is the point on the desired trajectory, and \(E\) and \(\psi_e\) are the position tracking error and bearing angle tracking error, respectively. The specific definition of \(\text{atan} \, 2(\tilde{y}_e, \tilde{x}_e)\) is given as:

\[
\text{atan} \, 2(\tilde{y}_e, \tilde{x}_e) = \begin{cases} \arctan \left( \frac{\tilde{y}_e}{\tilde{x}_e} \right) & \text{if } \tilde{x}_e > 0 \\
\arctan \left( \frac{\tilde{y}_e}{\tilde{x}_e} \right) + \pi & \text{if } \tilde{x}_e < 0 \text{ and } \tilde{y}_e \geq 0 \\
\arctan \left( \frac{\tilde{y}_e}{\tilde{x}_e} \right) - \pi & \text{if } \tilde{x}_e < 0 \text{ and } \tilde{y}_e < 0 \\
\frac{\pi}{2} & \text{if } \tilde{x}_e = 0 \text{ and } \tilde{y}_e > 0 \\
-\frac{\pi}{2} & \text{if } \tilde{x}_e = 0 \text{ and } \tilde{y}_e < 0 \\
\text{undefined} & \text{if } \tilde{x}_e = 0 \text{ and } \tilde{y}_e = 0
\end{cases}
\]

(19)
To prepare for the controller design in Section III, the tracking errors of USA are given as follows:

\[ Z_e = E - \sigma \]  
\[ \psi_e = \psi_d - \psi \]  

where \( Z_e \) denotes the position tracking error, \( \psi_d \) denotes the desired bearing angle, and \( \sigma \) is a positive threshold parameter, which will be given later in this paper.

**Remark 1:** In order to avoid the singularity of the nonlinear system due to tracking errors near zero, the error variable \( E \) in (17) is adjusted as \( Z_e \): the threshold parameter \( \sigma \) is introduced into the position error such that the vessel tracks the desired trajectory with an error of \( \sigma \), as shown in (20). Note that to prevent adverse effects on the tracking performance, the parameter \( \sigma \) should be a smaller positive constant.

Taking the derivative of (20) and (21), and considering (17) and (18), we obtain

\[
\dot{Z}_e = \dot{x}_d \cos(\psi_e + \psi) + \dot{y}_d \sin(\psi_e + \psi) \\
- u \cos \psi_e - v_b \sin \psi_e
\]

\[
\dot{\psi}_e = \frac{1}{E} [-\dot{x}_d \sin(\psi_e + \psi) + \dot{y}_d \cos(\psi_e + \psi)] \\
+ u \sin \psi_e - v_b \cos \psi_e - r
\]  

To ensure the transient and stable performance of the overall tracking errors, we introduce the following constraints into the controller design such that the errors \( Z_e \) and \( \psi_e \) satisfy the following inequalities:

\[
\begin{align*}
\rho_1 &< Z_e < \rho_1 \\
\rho_2 &< \psi_e < \rho_2
\end{align*}
\]  

where \( \rho_1 \) and \( \rho_2 \) are the performance functions, and the definition of the performance function is given as follows [35], [36]:

\[
\begin{align*}
\rho_1 &= (\rho_{10} - \rho_{1\infty})e^{-\sigma_1 t} + \rho_{1\infty} \\
\rho_2 &= (\rho_{20} - \rho_{2\infty})e^{-\sigma_2 t} + \rho_{2\infty}
\end{align*}
\]  

where \( \rho_{10}, \rho_{1\infty}, \rho_{20}, \rho_{2\infty}, \sigma_1 \) and \( \sigma_2 \) are positive constants that will be subsequently designed.

**C. CONTROLLER DESIGN**

Herein, a novel algorithm is designed by employing the tan-type BLF to guarantee optimal tracking performance, by introducing fractional power into the controller to obtain finite-time convergence, and by using a relative threshold ETM to reduce the system communication burden. The design process of the controller is detailed as follows:

**Step 1:** Design the virtual control laws as follows:

\[
\alpha_u = \frac{1}{\cos \psi_e} \left[ \frac{\rho_1^2}{2 \pi Z_e} \left( \frac{2 \dot{\rho}_1}{\rho_1} + k_1 \right) \sin \left( \frac{\pi Z_e^2}{\rho_1^2} \right) \right. \\
+ \frac{1}{2 Z_e} \sin \left( \frac{\pi Z_e^2}{\rho_1^2} \right) k_1 \left( \frac{\rho_1^2}{\pi} \right)^\gamma \tan^{-1} \left( \frac{\pi Z_e^2}{2 \rho_1^2} \right) \\
- \dot{\rho}_1 Z_e - v_b \sin \psi_e + \dot{x}_d \cos(\psi_e + \psi) + \dot{y}_d \sin(\psi_e + \psi) \left. \right]
\]  

\[
\alpha_r = \frac{\rho_2^2}{2 \pi \psi_e} \left( \frac{2 \rho_2}{\rho_2} + k_2 \right) \sin \left( \frac{\pi \psi_e^2}{\rho_2^2} \right) - \frac{\rho_2}{\rho_2} \left. \right]
\]

\[
+ \frac{1}{2 \psi_e} \sin \left( \frac{\pi \psi_e^2}{\rho_2^2} \right) k_2 \left( \frac{\rho_2^2}{\pi} \right)^\gamma \tan^{-1} \left( \frac{\pi \psi_e^2}{2 \rho_2^2} \right) \\
+ \frac{1}{E} [u \sin \psi_e - v_b \cos(\psi_e + \psi) \\
+ \dot{y}_d \cos(\psi_e + \psi)]
\]  

where \( k_1, k_2, k_{10} \) and \( k_{20} \) are positive constants that will be subsequently designed.

**Step 2:** Define the coordinate transformation errors as follows:

\[
e_u = u - \alpha_u
\]

\[
e_r = r - \alpha_r
\]

Define the boundary layer errors as:

\[
e_{au} = \alpha_u - \alpha_u
\]

\[
e_{ar} = \alpha_r - \alpha_r
\]

where \( \alpha_u \) and \( \alpha_r \) are the filtered virtual control laws, \( \alpha_u \) and \( \alpha_r \) are the virtual control laws.

To address the differential explosion problem, a first-order low-pass filter is employed to estimate the virtual control laws \( \alpha_u \) and \( \alpha_r \) with the help of DSC technology [18], [37].

\[
\lambda_u \dot{\alpha}_u + \alpha_u = \alpha_u + \lambda_u Z_e sec^2 \left( \frac{\pi Z_e^2}{2 \rho_1^2} \right) \cos \psi_e
\]

\[
\lambda_r \dot{\alpha}_r + \alpha_r = \alpha_r + \lambda_r \psi_e sec^2 \left( \frac{\pi \psi_e^2}{2 \rho_2^2} \right)
\]

where \( \alpha_u(0) = \alpha_u(0), \alpha_r(0) = \alpha_r(0), \lambda_u \) and \( \lambda_r \) are filter time constants.

Differentiating (28), (29) and considering system (14), we have

\[
\dot{e}_u = f_u + \frac{1}{m_1} \tau_u + \mu u^{-1} e_{au} - Z_e sec^2 \left( \frac{\pi Z_e^2}{2 \rho_1^2} \right) \cos \psi_e
\]

\[
\dot{e}_r = f_r + \frac{m_2}{\Lambda} \tau_r + \lambda_r e_{ar} - \psi_e sec^2 \left( \frac{\pi \psi_e^2}{2 \rho_2^2} \right)
\]

where \( f_u \) and \( f_r \) contain the uncertain nonlinear terms, saturation error and unknown external environmental disturbances. Thus, the RBFNN is utilized to estimate \( f_u \) and \( f_r \):

\[
f_u = \tilde{W}_u^T \chi_u (\omega_u) + \tilde{e}_u, \quad \forall \omega_u \in \Omega_u
\]

\[
f_r = \tilde{W}_r^T \chi_r (\omega_r) + \tilde{e}_r, \quad \forall \omega_r \in \Omega_r
\]

The input vectors \( \omega_u = u \) and \( \omega_r = [v \ r]^T \); \( \tilde{e}_u \) and \( \tilde{e}_r \) denote the total approximation errors of NN which are bounded, i.e., \( |\tilde{e}_u| < \theta_u \) and \( |\tilde{e}_r| < \theta_r \), where both \( \theta_u \) and \( \theta_r \) are unknown positive constants. The NN weight errors are defined as \( \hat{W}_u = \tilde{W}_u - W_u \) and \( \hat{W}_r = \tilde{W}_r - W_r \).
\[ \hat{W}_u = \hat{W}_u - W_u \text{ and } \hat{W}_r = \hat{W}_r - W_r, \text{ that is, } \| \hat{W}_u \| \leq \kappa_u \text{ and } \| \hat{W}_r \| \leq \kappa_r, \text{ where both } \kappa_u \text{ and } \kappa_r \text{ are unknown positive constants.} \]

Substituting (36) and (37) into (34) and (35), respectively (34) and (35) are rewritten as follows:

\[ \dot{e}_u = \hat{W}_u^T \chi_u (\omega_u) + \frac{1}{m_{11}} \tau_u + \lambda_u^{-1} e_{au} - Z_e \sec^2 \left( \frac{\sigma Z_e}{2 \rho_u^2} \right) \cos \psi_e + \hat{e}_u \]

\[ \dot{e}_r = \hat{W}_r^T \chi_r (\omega_r) + \frac{m_{22}}{2} \tau_r + \lambda_r^{-1} e_{ar} - \psi_e \sec^2 \left( \frac{\sigma \psi_e}{2 \rho_r^2} \right) + \hat{e}_r \]

Considering the derivative of \( \alpha_u \) and \( \alpha_r \) given in (26) and (27) could be written as \( \hat{\alpha}_u (\eta, \bar{\eta}, \eta_d, \bar{\eta}_d, t, \bar{v}, \bar{v}, \rho_u, \bar{\rho}_u, \bar{\beta}_u, \bar{\beta}_r, \hat{\sigma}_u, \psi_e, \hat{\sigma}_r, \hat{\beta}_r, \hat{\rho}_r, \hat{\rho}_u, \hat{\psi}_e, \hat{\sigma}_e, \hat{\rho}_e, \hat{\sigma}_e) \) and \( \hat{\alpha}_r (\eta, \bar{\eta}, \eta_d, \bar{\eta}_d, t, \bar{v}, \bar{v}, \rho_u, \bar{\rho}_u, \bar{\beta}_u, \bar{\beta}_r, \hat{\sigma}_u, \psi_e, \hat{\sigma}_r, \hat{\beta}_r, \hat{\rho}_r, \hat{\rho}_u, \hat{\psi}_e, \hat{\sigma}_e, \hat{\rho}_e, \hat{\sigma}_e) \) respectively, with \( \eta_d = [x_d, y_d, \psi_d] \). Next, differentiating (30) and (31), and using (26), (27), (32) and (33), it is not difficult to obtain

\[ \dot{e}_{au} = -\lambda_u^{-1} e_{au} + Z_e \sec^2 \left( \frac{\sigma Z_e}{2 \rho_u^2} \right) \cos \psi_e - Q_u (\cdot) \]

\[ \dot{e}_{ar} = -\lambda_r^{-1} e_{ar} + \psi_e \sec^2 \left( \frac{\sigma \psi_e}{2 \rho_r^2} \right) - Q_r (\cdot) \]

where \( Q_u (\cdot) \hat{\alpha}_u \) and \( Q_r (\cdot) \hat{\alpha}_r \) are continuous functions.

The control laws \( \varphi_u \) and \( \varphi_r \) are designed as follows:

\[ \varphi_u = m_{11} \left[ -k_u e_u - \lambda_u^{-1} e_{au} + 2 Z_e \sec^2 \left( \frac{\sigma Z_e}{2 \rho_u^2} \right) \cos \psi_e \right] \]

\[ \varphi_r = m_{22} \left[ -k_r e_r - \lambda_r^{-1} e_{ar} + 2 \psi_e \sec^2 \left( \frac{\sigma \psi_e}{2 \rho_r^2} \right) \cos \psi_e \right] \]

with the adaptive laws

\[ \dot{\hat{W}}_u = \gamma_u \left[ e_u \chi_u (\omega_u) - \hat{\xi}_u \hat{W}_u \right] \]

\[ \dot{\hat{W}}_r = \gamma_r \left[ e_r \chi_r (\omega_r) - \hat{\xi}_r \hat{W}_r \right] \]

in which \( k_u, k_r, k_{0u}, k_{0r}, \kappa_u, \kappa_r, \hat{\eta}_u, \hat{\eta}_r, \gamma_u \) and \( \gamma_r \) are positive control parameters, which will be designed later in this paper.

For the proposed system, a relative threshold ETM is defined as [28]:

\[ \begin{cases} \tau_u (t) = -\left(1 + \beta_u\right) \\
[\varphi_u \tanh \left( \frac{e_u \psi_e}{\hat{\xi}_u} \right) + \Xi_u \tanh \left( \frac{e_u \hat{\Xi}_u}{\hat{\xi}_u} \right)] \end{cases} \]

\[ \begin{cases} \tau_r (t) = -\left(1 + \beta_r\right) \\
[\varphi_r \tanh \left( \frac{e_r \psi_e}{\hat{\xi}_r} \right) + \Xi_r \tanh \left( \frac{e_r \hat{\Xi}_r}{\hat{\xi}_r} \right)] \end{cases} \]

\[ \tau_k \in [t_k^u, t_{k+1}^u) \quad \forall t \in [t_k^u, t_{k+1}^u) \]

\[ \tau_k \in [t_k^r, t_{k+1}^r) \quad \forall t \in [t_k^r, t_{k+1}^r) \]

\[ t_k^u + 1 = \inf \left\{ t \in R^+ \mid \left| e_{ru} (t) \right| \geq \beta_u \left| \tau_u (t) \right| + \xi_u \right\} \]

\[ t_k^r + 1 = \inf \left\{ t \in R^+ \mid \left| e_{re} (t) \right| \geq \beta_r \left| \tau_r (t) \right| + \xi_r \right\} \]

where \( \partial_u, \partial_r, \Xi_u, \Xi_r, \xi_u \) and \( \xi_r \) are constants, and \( 0 < \beta_u < 1, 0 < \beta_r < 1, e_{ru} (t) = \tau_u (t_k) - \tau_u (t) \) and \( e_{re} (t) = \tau_r (t_k) - \tau_r (t) \) denotes the measurement error. Note that the design parameters \( \Xi_u \) and \( \Xi_r \) should meet \( \Xi_u > \frac{1}{1 - \beta_u} \) and \( \Xi_r > \frac{1}{1 - \beta_r} \), respectively. When inequalities \( \left| e_{ru} (t) \right| \geq \beta_u \left| \tau_u (t) \right| + \xi_u \) and \( \left| e_{re} (t) \right| \geq \beta_r \left| \tau_r (t) \right| + \xi_r \) are true at \( t_k + 1 \) in (48), the control input \( \tau_r (t) \) is updated to \( \tau_r (t_k + 1) \), and \( \tau_r (t) \) holds constant when \( t \in [t_k, t_{k+1}) \).

**Theorem 1:** Consider the uncertain nonlinear system (14), with the control laws (42) and (43), NN adaptive laws (44) and (45), relative threshold ETM (46), (47), (48) and saturation function (3) with assumptions 1-4. All the variables are PFS, and the tracking errors converge to a small neighborhood of zero after the finite time \( t_f \).

**Proof:** The above theorem is proven by the following three steps.

**Step 1:** Choose the following Lyapunov function:

\[ V_1 = V_u + V_r \]

Differentiating \( V_1 \) with respect to time, we have

\[ \dot{V}_1 = \frac{2 \rho u \rho_1}{\rho_1^{\gamma}} \tan \left( \frac{\pi Z_e}{2 \rho_1^{\gamma}} \right) + 2 \rho u \rho_2 \tan \left( \frac{\pi \psi_e}{2 \rho_2} \right) \]

\[ \begin{align*}
&+ \sec^2 \left( \frac{\pi Z_e}{2 \rho_1^{\gamma}} \right) \left( Z_e \dot{Z}_e - \frac{Z_e^2 \rho_1}{\rho_1} \right) \\
&+ \sec^2 \left( \frac{\pi \psi_e}{2 \rho_2} \right) \left( \psi_e \dot{\psi}_e - \frac{\psi_e^2 \rho_2}{\rho_2} \right)
\end{align*} \]

Substituting (22), (23), (26), (27), (28), (29), (30) and (31) into (50), we obtain

\[ \dot{V}_1 = -k_1 \rho u^{\gamma} \tan \left( \frac{\pi Z_e}{2 \rho_1^{\gamma}} \right) - k_{10} \left( \frac{\rho_1}{\rho_1^{\gamma}} \right)^{\gamma} \tan \left( \frac{\pi Z_e}{2 \rho_1^{\gamma}} \right) \]

\[ -k_2 \rho_2^{\gamma} \tan \left( \frac{\pi \psi_e}{2 \rho_2} \right) - k_{20} \left( \frac{\rho_2}{\rho_2^{\gamma}} \right)^{\gamma} \tan \left( \frac{\pi \psi_e}{2 \rho_2} \right) \]

\[ -Z_e \rho u \sec^2 \left( \frac{\pi Z_e}{2 \rho_1^{\gamma}} \right) \cos \psi_e - \psi_e \psi_e \sec^2 \left( \frac{\pi \psi_e}{2 \rho_2} \right) \]

\[ -Z_e \rho u \sec^2 \left( \frac{\pi Z_e}{2 \rho_1^{\gamma}} \right) \cos \psi_e - \psi_e \psi_e \sec^2 \left( \frac{\pi \psi_e}{2 \rho_2} \right) \]

**Step 2:** Choose the Lyapunov function as

\[ V_2 = \frac{1}{2} e_u^2 + \frac{1}{2} e_r^2 + \frac{1}{2} e_{au}^2 + \frac{1}{2} e_{ar}^2 \]
Differentiating $V_2$ with respect to time and substituting (38), (39), (40) and (41) yields

$$
\dot{V}_2 = e_u \left[ \dot{W}_u^T \Theta_u (\chi_u) + \frac{1}{m_{11}} \tau_u + \lambda_u^{-1} e_{au} \right] - Z_e \sec^2 \left( \frac{\pi Z_e}{2 \rho_1^2} \right) \cos \psi_e + \bar{e}_u \\
+ e_r \left[ \dot{W}_r^T \Theta_r (\chi_r) + \frac{m_{22}}{\Lambda} \tau_r \right] + \lambda_r^{-1} e_{ar} - \psi_e \sec^2 \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) + \bar{e}_r \\
+ e_{au} \left[ -\lambda_u^{-1} e_{au} + Z_e \sec^2 \left( \frac{\pi Z_e}{2 \rho_1^2} \right) \cos \psi_e - Q_u (\cdot) \right] \\
+ e_{ar} \left[ -\lambda_r^{-1} e_{ar} + \psi_e \sec^2 \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) - Q_r (\cdot) \right]
$$

(53)

Step 3: Choose the Lyapunov function as

$$
V_3 = V_1 + V_2 + \frac{1}{2} \dot{W}_u^T Y_{11}^{-1} \dot{W}_u + \frac{1}{2} \dot{W}_r^T Y_{11}^{-1} \dot{W}_r
$$

(54)

Taking the time derivative of $V_3$ and substituting (51) and (53), we have

$$
\dot{V}_3 = -k_1 \beta_1^2 \tan \left( \frac{\pi Z_e}{2 \rho_1^2} \right) - k_{10} \left( \frac{\beta_1^2}{\pi} \right)^\gamma \tan^\gamma \left( \frac{\pi Z_e}{2 \rho_1^2} \right) \\
- k_2 \beta_2^2 \tan \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) - k_{20} \left( \frac{\beta_2^2}{\pi} \right)^\gamma \tan^\gamma \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) \\
- 2Z_e e_u \sec^2 \left( \frac{\pi Z_e}{2 \rho_1^2} \right) \cos \psi_e - 2 \psi_e e_r \sec^2 \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) \\
- \lambda_u^{-1} e_{au} - \lambda_r^{-1} e_{ar} - e_{au} Q_u (\cdot) - e_{ar} Q_r (\cdot) \\
+ e_u \left[ \dot{W}_u^T \chi_u ( \omega_u ) + \frac{1}{m_{11}} \tau_u + \lambda_u^{-1} e_{au} + \bar{e}_u \right] \\
+ e_r \left[ \dot{W}_r^T \chi_r ( \omega_r ) + \frac{m_{22}}{\Lambda} \tau_r + \lambda_r^{-1} e_{ar} + \bar{e}_r \right] \\
+ \dot{W}_u^T \chi_{1u}^{-1} \dot{W}_u + \dot{W}_r^T \chi_{1r}^{-1} \dot{W}_r
$$

(55)

From (48), we have

$$
\begin{align*}
\tilde{e}_u ( t ) &= [1 + \gamma_{1u} ( t ) \beta_u] \tau_u ( t ) + \gamma_{2u} ( t ) \bar{e}_u \\
\tilde{e}_r ( t ) &= [1 + \gamma_{1r} ( t ) \beta_r] \tau_r ( t ) + \gamma_{2r} ( t ) \bar{e}_r 
\end{align*}
$$

(56)

where $\gamma_{1i} ( t ) \in [-1, 1)$ and $\gamma_{2i} ( t ) \in [-1, 1)$ are time-dependent variables. Therefore, we obtain

$$
\begin{align*}
\tau_u &= \frac{1}{[1 + \gamma_{1u} ( t ) \beta_u]} \left[ \tilde{e}_u ( t ) - \gamma_{2u} ( t ) \bar{e}_u \right] \\
\tau_r &= \frac{1}{[1 + \gamma_{1r} ( t ) \beta_r]} \left[ \tilde{e}_r ( t ) - \gamma_{2r} ( t ) \bar{e}_r \right]
\end{align*}
$$

(57)

Substituting (44), (45) and (57) into (55) yields

$$
\dot{V}_3 = -k_1 \beta_1^2 \tan \left( \frac{\pi Z_e}{2 \rho_1^2} \right) - k_{10} \left( \frac{\beta_1^2}{\pi} \right)^\gamma \tan^\gamma \left( \frac{\pi Z_e}{2 \rho_1^2} \right) \\
- k_2 \beta_2^2 \tan \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) - k_{20} \left( \frac{\beta_2^2}{\pi} \right)^\gamma \tan^\gamma \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) \\
- 2Z_e e_u \sec^2 \left( \frac{\pi Z_e}{2 \rho_1^2} \right) \cos \psi_e - 2 \psi_e e_r \sec^2 \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) \\
- \lambda_u^{-1} e_{au} - \lambda_r^{-1} e_{ar} - e_{au} Q_u ( \cdot ) - e_{ar} Q_r ( \cdot ) \\
+ e_u \left[ \dot{W}_u^T \chi_u ( \omega_u ) + \frac{1}{m_{11}} \tau_u + \lambda_u^{-1} e_{au} + \bar{e}_u \right] \\
+ e_r \left[ \dot{W}_r^T \chi_r ( \omega_r ) + \frac{m_{22}}{\Lambda} \tau_r + \lambda_r^{-1} e_{ar} + \bar{e}_r \right] \\
+ \dot{W}_u^T \chi_{1u}^{-1} \dot{W}_u + \dot{W}_r^T \chi_{1r}^{-1} \dot{W}_r
$$

(58)

Owing to $\gamma_{1u} ( t ) \in [-1, 1)$ and $\gamma_{2u} ( t ) \in [-1, 1)$, we have the following inequalities:

$$
\begin{align*}
\frac{e_u \tilde{e}_u}{1 + \gamma_{1u} ( t ) \beta_u} &\leq \frac{e_u \tilde{e}_u}{1 + \beta_u} \\
\frac{1 + \lambda_u^{-1} e_{au}}{e_r \tilde{e}_r} &\leq \frac{1 + \beta_r}{e_r \tilde{e}_r} \\
\frac{1 + \gamma_{1r} ( t ) \beta_r}{e_r \tilde{e}_r} &\leq \frac{1 - \beta_r}{e_r \tilde{e}_r}
\end{align*}
$$

(59)

To facilitate the following proof, according to Lemma 1, we can obtain the following inequality:

$$
- |H| \leq - H \tan \left( \frac{H}{\delta} \right) \leq 0.2785 \delta - |H|
$$

(60)

Substituting (59) and (46) into (58) in sequence, and using (60), we have

$$
\dot{V}_3 \leq -k_1 \beta_1^2 \tan \left( \frac{\pi Z_e}{2 \rho_1^2} \right) - k_{10} \left( \frac{\beta_1^2}{\pi} \right)^\gamma \tan^\gamma \left( \frac{\pi Z_e}{2 \rho_1^2} \right) \\
- k_2 \beta_2^2 \tan \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) - k_{20} \left( \frac{\beta_2^2}{\pi} \right)^\gamma \tan^\gamma \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) \\
- 2Z_e e_u \sec^2 \left( \frac{\pi Z_e}{2 \rho_1^2} \right) \cos \psi_e - 2 \psi_e e_r \sec^2 \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) \\
- \lambda_u^{-1} e_{au} - \lambda_r^{-1} e_{ar} - e_{au} Q_u ( \cdot ) - e_{ar} Q_r ( \cdot ) \\
+ e_u \left[ \dot{W}_u^T \chi_u ( \omega_u ) + \frac{1}{m_{11}} \tau_u + \lambda_u^{-1} e_{au} + \bar{e}_u \right] \\
+ e_r \left[ \dot{W}_r^T \chi_r ( \omega_r ) + \frac{m_{22}}{\Lambda} \tau_r + \lambda_r^{-1} e_{ar} + \bar{e}_r \right] \\
+ \dot{W}_u^T \chi_{1u}^{-1} \dot{W}_u + \dot{W}_r^T \chi_{1r}^{-1} \dot{W}_r
$$

(61)

With $-|e_u \tilde{e}_u| \leq e_u \tilde{e}_u$ and $-e_r \tilde{e}_r \leq e_r \tilde{e}_r$, then substituting them into (61), we obtain

$$
\dot{V}_3 \leq -k_1 \beta_1^2 \tan \left( \frac{\pi Z_e}{2 \rho_1^2} \right) - k_{10} \left( \frac{\beta_1^2}{\pi} \right)^\gamma \tan^\gamma \left( \frac{\pi Z_e}{2 \rho_1^2} \right) \\
- k_2 \beta_2^2 \tan \left( \frac{\pi \psi_e}{2 \rho_2^2} \right) - k_{20} \left( \frac{\beta_2^2}{\pi} \right)^\gamma \tan^\gamma \left( \frac{\pi \psi_e}{2 \rho_2^2} \right)
$$

(62)
\[-2Z_u e_u \sec^2 \left( \frac{\pi Z_u^2}{2 \rho_i^2} \right) \cos \psi_e - 2 \psi_e e_r \sec^2 \left( \frac{\pi \psi_e}{2 \rho_i^2} \right)\]
\[-\lambda_u^{-1} e_{uu} - \lambda_r^{-1} e_{rr} - e_{ar} Q_u (\cdot) - e_{ar} Q_r (\cdot)\]
\[+ \frac{1 + m_{11}}{m_{11}} e_u \varphi_u + \frac{1 + m_{12}}{\varphi_r} \left( -e_{rr} + \frac{\varphi_r}{1 - \beta_u} \right) + \frac{m_{22}}{\Lambda} e_r \varphi_r\]
\[+ \frac{m_{22}}{\Lambda} \left( -e_r \varphi_r + \frac{e_r \varphi_r}{1 - \beta_r} \right) + e_u W_u^T \chi_u (o_u)\]
\[+ e_r W_r^T \chi_r (o_r) + e_u W_u^T \chi_u (o_u) + e_r W_r^T \chi_r (o_r)\]
\[+ \lambda_u^{-1} e_u e_u + \lambda_r^{-1} e_r e_r - \xi_r W_r^T W_r - \xi_e W_r^T W_r + \xi_u e_u + \xi_r e_r + 1.1148\]

(62)

Substituting (42) and (43) into (62) and considering \(\xi > \frac{1}{2} \pi\), we can obtain
\[\dot{V}_3 \leq -k_1 \left( \frac{\rho_i^2}{\pi} \right) \tan \left( \frac{\pi Z_u^2}{2 \rho_i^2} \right) - k_{10} \left( \frac{\rho_i^2}{\pi} \right)^\gamma \tan^\gamma \left( \frac{\pi Z_u^2}{2 \rho_i^2} \right)\]
\[+ \left( \lambda_u^{-1} e_{uu} - \lambda_r^{-1} e_{rr} - k_u e_{ar} - km_e e_{ar} - e_{ar} Q_u (\cdot)\right)\]
\[+ k_{10} e_{uu} - k_r e_{rr} - k_m e_{ar} - e_{ar} Q_r (\cdot)\]
\[+ e_u W_u^T \chi_u (o_u) + e_r W_r^T \chi_r (o_r) - \xi u W_u^T W_u\]
\[+ \xi_r W_r^T W_r + \xi_u e_u + \xi_r e_r + 1.1148\]

(63)

Consider the sets \(M_{b1} \left( \|u\|_2 + \|\dot{u}\|_2 + \|u_d\|_2 + \|\dot{u}_d\|_2 + \|v\|_2 + \|\dot{v}\|_2 + \|\rho_u\|_2 + \|\rho_r\|_2 + \|\dot{\rho}_u\|_2 + \|\dot{\rho}_r\|_2 \right)\leq A \) and\( M \left( Z_u^2 + \psi_e^2 + e_u^2 + e_r^2 + e_{uu}^2 + e_{rr}^2 \leq 2 \sigma \right) \) with \(A > 0, \sigma > 0\). Since all the parameters in continuous functions \(Q_u (\cdot)\) and \(Q_r (\cdot)\) are bounded in the compact set \(M = \{M_{b1}, M\} \) and \(|Q_u (\cdot)| \leq q_u \) and \(|Q_r (\cdot)| \leq q_r \) hold, where \(q_u > 0\) and \(q_r > 0\). By Young’s inequality \([22, 24, 30]\), we have
\[-e_u Q_u (\cdot) \leq -\left( \frac{1}{2} e_{uu} + \frac{1}{2} \rho_u \right)\]
\[-e_r Q_r (\cdot) \leq -\left( \frac{1}{2} e_{rr} + \frac{1}{2} \rho_r \right)\]
\[e_u W_u^T \Theta_u (\chi_u) \leq \frac{1}{2} e_{uu} + \frac{1}{2} \rho_u W_u^T W_u\]
\[e_r W_r^T \Theta_r (\chi_r) \leq \frac{1}{2} e_{rr} + \frac{1}{2} \rho_r W_r^T W_r\]
\[\xi_u W_u^T W_u \leq -\xi_u W_u^T W_u + \frac{\xi_u}{2} W_u^T W_u\]
\[\xi_r W_r^T W_r \leq -\xi_r W_r^T W_r + \frac{\xi_r}{2} W_r^T W_r\]
\[\xi_u \leq \frac{\xi_u}{4} \left( \|W_u\|_2 - \|W_u\|_2^{2\gamma - 1} \right)^2 + \frac{\xi_u}{4} \|W_u\|_2^{2(\gamma - 1)} - \frac{\xi_u}{2} W_u^T W_u \]
\[\xi_r \leq \frac{\xi_r}{4} \left( \|W_r\|_2 - \|W_r\|_2^{2\gamma - 1} \right)^2 + \frac{\xi_r}{4} \|W_r\|_2^{2(\gamma - 1)} - \frac{\xi_r}{2} W_r^T W_r\]
\[\xi_u \leq \frac{\xi_u}{4} \left( \|W_u\|_2 - \|W_u\|_2^{2\gamma - 1} \right)^2 + \frac{\xi_u}{4} \|W_u\|_2^{2(\gamma - 1)} - \frac{\xi_u}{2} W_u^T W_u \]
\[\xi_r \leq \frac{\xi_r}{4} \left( \|W_r\|_2 - \|W_r\|_2^{2\gamma - 1} \right)^2 + \frac{\xi_r}{4} \|W_r\|_2^{2(\gamma - 1)} - \frac{\xi_r}{2} W_r^T W_r\]

(64)

Substituting (64) into (63), we have
\[\dot{V}_3 \leq -k_1 \left( \frac{\rho_i^2}{\pi} \right) \tan \left( \frac{\pi Z_u^2}{2 \rho_i^2} \right) - k_{10} \left( \frac{\rho_i^2}{\pi} \right)^\gamma \tan^\gamma \left( \frac{\pi Z_u^2}{2 \rho_i^2} \right)\]
\[+ \left( \lambda_u^{-1} e_{uu} - \lambda_r^{-1} e_{rr} - k_u e_{ar} - k_m e_{ar} - k_o e_{ar} - e_{ar} Q_u (\cdot)\right)\]
\[+ k_{10} e_{uu} - k_r e_{rr} - k_m e_{ar} - k_o e_{ar} - e_{ar} Q_r (\cdot)\]
\[+ e_u W_u^T \chi_u (o_u) + e_r W_r^T \chi_r (o_r) - \xi u W_u^T W_u - \xi_r W_r^T W_r + \xi_u e_u + \xi_r e_r + 1.1148\]

(65)

Define \(\tilde{a} = \min \left\{ k_1, k_2, \left( \lambda_u^{-1} + \frac{1}{\lambda_r} \right), \left( \lambda_r^{-1} + \frac{1}{\lambda_u} \right), (k_u - \frac{3}{4}), (k_r - \frac{3}{4}), (\epsilon_u, \epsilon_r), (\epsilon_r, \epsilon_u) \right\}\), \(\tilde{b} = \min \left\{ k_{10}, k_{20}, k_m, k_r, k_o \right\}\), \(\epsilon u, \epsilon r \), and \(\delta = \left\{ W_u^T \chi_u (o_u) \right\} + \left\{ W_r^T \chi_r (o_r) \right\} + \left\{ W_u^T \chi_u (o_u) \right\} + \left\{ W_r^T \chi_r (o_r) \right\} + \xi u W_u^T W_u + \xi r W_r^T W_r + \xi u e_u + \xi r e_r + 1.1148\), and then (65) can be abbreviated as:
\[\dot{V}_3 \leq -\tilde{a} V_3 - \tilde{b} V_3^\gamma + \delta^*\]

(66)

Remark 3: To enable the proposed control system to better achieve finite time convergence, the parameters \(k_u \) and \(k_r \) should meet \(k_u > \frac{3}{4}\) and \(k_r > \frac{3}{4}\), respectively.

According to Lemma 2, all the signals are stable at the region \(\Omega = \left\{ \frac{1}{(1 - \Pi) \alpha}, \frac{1}{(1 - \Pi) \beta} \right\} \) within finite time
\[T \leq \frac{1}{\alpha \Pi (1 - \gamma)} \ln \left( \frac{\tilde{a} \Pi^{1 - \gamma} + \tilde{b}}{b} \right) + \alpha \Pi (1 - \gamma) \ln \left( \frac{\tilde{a} \Pi^{1 - \gamma} + \tilde{b}}{b} \right)\]
where \(0 < \Pi < 1\) and \(0 < \gamma < 1\). All the signals in the nonlinear trajectory tracking control system are PFS. End of proof.

Remark 4: Inspired by [38], to avoid the chattering caused by unmodeled, high-frequency dynamics due to the low value of \(\gamma\), the fractional power \(\gamma\) should be \(0.7 < \gamma < 0.9\).

Theorem 2: The nonlinear control system with the relative threshold ETM for USVs can guarantee that the lower bound of the minimum intersample time \(\{ T^u, T^r_{u+1} \} \) and \(\{ T^u, T^r_{r+1} \} \) are nonzero positive constants, that is, the proposed controller does not exhibit Zeno behavior.
Proof [28]: From \( e_{tu}(t) = \tilde{r}_u(t_k) - r_u(t) \) and \( e_{tr}(t) = \tilde{r}_r(t_k) - r_r(t) \ \forall t \in [t_k, t_{k+1}] \), we obtain

\[
\begin{align*}
\frac{d}{dt} |e_{tu}| &= \frac{d}{dt} (e_{tu} \cdot e_{tu}) = \text{sign}(e_{tu}) \dot{e}_{tu} \leq |\dot{r}_u(t)| \\
\frac{d}{dt} |e_{tr}| &= \frac{d}{dt} (e_{tr} \cdot e_{tr}) = \text{sign}(e_{tr}) \dot{e}_{tr} \leq |\dot{r}_r(t)|
\end{align*}
\]

Differentiating \( \tilde{r}_u(t) \) and \( \tilde{r}_r(t) \) given in (46) yields

\[
\begin{align*}
\dot{\tilde{r}}_u(t) &= \left( \frac{\dot{r}_u \psi_u}{\partial u} + \frac{e_{tu} \psi_u}{\partial u} \right) \left[ 1 - \tanh^2 \left( \frac{e_{tu}}{\partial u} \right) \right] \\
\dot{\tilde{r}}_r(t) &= \left( \frac{\dot{r}_r \psi_r}{\partial r} + \frac{e_{tr} \psi_r}{\partial r} \right) \left[ 1 - \tanh^2 \left( \frac{e_{tr}}{\partial r} \right) \right]
\end{align*}
\]

(67)

(68)

Since all the signals are PFS, constants \( \Gamma_u > 0 \) and \( \Gamma_r > 0 \) must exist such that \( |\tilde{r}_u(t)| \leq \Gamma_u \) and \( |\tilde{r}_r(t)| \leq \Gamma_r \). From (48), we have \( e_{tu}(t_k) = 0 \), \( e_{tr}(t_k) = 0 \), \( \lim_{t \to t_{k+1}} e_{tu}(t_k) = \beta_u |\tau_u(t_k)| + \zeta_u \) and \( \lim_{t \to t_{k+1}} e_{tr}(t_k) = \beta_r |\tau_r(t_k)| + \zeta_r \). Thus, the communication intervals time \( t_u^* \) and \( t_r^* \) satisfy \( t_u^* \geq \frac{\lambda_u |\tau_u(t_k)| + \zeta_u}{\beta_u} \) and \( t_r^* \geq \frac{\lambda_r |\tau_r(t_k)| + \zeta_r}{\beta_r} \), which means that the Zeno behavior did not occur in the system. End of proof.

The framework of the proposed control method is shown in Fig. 3.

FIGURE 3. The framework of the proposed control method.

IV. SIMULATIONS

Some numerical simulations are performed to demonstrate the effectiveness of the proposed method in this section. The parameters of the USV model are obtained from [39] and given in Table 1.

The desired trajectory is an Archimedes spiral with a large turning angle, whose expression is presented as follows:

\[
\begin{align*}
x_d &= 0.2 t \cos (0.04t) \\
y_d &= 0.2 t \sin (0.04t)
\end{align*}
\]

(69)

The initial states of the system initialization in the control system are expressed as \( \eta (0) = [-0.8, 0, 0.3]^T \) and \( \nu (0) = [0, 0, 0]^T \), and the time parameterized environment disturbances are simulated as:

\[
\begin{align*}
N_u &= -4 + 6 \sin (0.4t) \cos (0.5t) + 4 \sin (0.1t) \\
N_r &= 0.2 \sin (0.1t) \\
N_r &= 8 \sin (0.3t) \cos (0.2t) + 6 \sin (0.2t)
\end{align*}
\]

(70)

The design parameters are chosen as \( \sigma = 0.05 \), \( \rho_{10} = 1 \), \( \rho_{1\infty} = 0.05 \), \( \rho_{20} = \pi \sqrt{4} \), \( \rho_{2\infty} = 0.05 \),

\[
\begin{align*}
m_{11} &= 25.8 \\
m_{32} &= 33.8 \\
m_{33} &= 2.76 \\
m_{33} &= 1.0948 \\
d_{11} &= 0.7225 + 1.3274 |u| \\
d_{22} &= 0.8612 + 36.2823 |v| + 0.805 |r| \\
d_{33} &= 1.9 - 0.08 |v| + 0.75 |r| \\
d_{33} &= -0.1079 + 0.845^{|v|}
\end{align*}
\]

\( \sigma_1 = 0.08, \ \sigma_2 = 0.1, \ k_1 = 10, \ k_2 = 17, \ k_{10} = 1, \ k_{20} = 1, \ k_u = 10, \ k_r = 52, \ k_{u0} = 5, \ k_{r0} = 5, \ k_{fu} = 1, \ k_{fr} = 1, \ \gamma = 7/8, \ \tau_u = 0.01, \ \lambda_u = 0.01, \ \lambda_r = 0.01, \ \bar{\alpha}_u = 25, \ \bar{\alpha}_r = 25, \ \bar{\zeta}_u = 0.2, \ \bar{\zeta}_r = 0.1, \ \zeta_u = 0.1, \ \zeta_r = 0.8. \)

TABLE 1. Parameters of the USV model.

![FIGURE 4. Trajectory tracking result.](image)

![FIGURE 5. Convergence of error variables x_e and y_e.](image)

In the simulation results, Fig. 4 describes the results after the USVs track the Archimedes spiral. Fig. 5 describes the convergence of the error variables \( x_e \) and \( y_e \) after coordinate transformation. Fig. 6 shows that the position tracking error is constrained within the range of prescribed performance.
Fig. 7 shows that the bearing angle tracking error is constrained within the range of prescribed performance functions $\bar{\rho}_2$ and $\bar{\rho}_2$. Figs. 4-7 clearly shows that the threshold parameter $\sigma$ in (20) does not affect the tracking performance that the steady-state position tracking error is within 0.06 m, including the threshold parameter $\sigma$ and that the response time of the position tracking is within 2 seconds. Likewise, the steady-state bearing angle tracking error is within 0.02 rad, and the response time of bearing angle tracking is within 3 seconds. These results show that the proposed trajectory tracking control has strong robustness.

Fig. 8 and Fig. 9 show the control inputs $T_u$ and $T_r$ with input saturation and event-triggered, respectively. As shown in Figs. 8-9, both $T_u$ and $T_r$ are fed to the system in stepped form.
V. CONCLUSION

This paper investigates robust trajectory tracking problems for USVs with model uncertainties, unknown time parameterized environmental disturbances and input saturation. A robust practical finite-time event-triggered tracking control algorithm is proposed based on coordinate transformation technology, the tan-type BLF, the RBFFNN and the relative threshold ETM. The tracking errors are constrained within prescribed performance constraints by the tan-type BLF, and the differential explosion problem is solved by DSC technology. The RBFFNN can effectively approximate the nonlinear terms and compensate for the saturation error. With the help of the relative threshold ETM, the system communication burden is significantly reduced. By appropriately selecting the fractional power, finite-time convergence is achieved. Through the simulation results of the Archimedes spiral as the reference trajectory, the effectiveness of the proposed controller is verified. Future research will focus on actual experimental research for the proposed algorithm and applications of the research results in the multi-USVs system.

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