Stable Degeneracies for Ising Models

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Abstract

We introduce and consider the notion of stable degeneracies of translation invariant energy functions for finite Ising models. By this term we mean the lack of injectivity that cannot be lifted by changing the interaction.

We show that besides the symmetry-induced degeneracies, related to spin flip, translation and reflection, there exist additional stable degeneracies, due to more subtle symmetries. One such symmetry is the one of the Singer group of a finite projective plane. Others are described by combinatorial relations akin to trace identities.

Our results resemble traits of the length spectrum for closed geodesics on a Riemannian surface of constant negative curvature. There stable degeneracy is defined w.r.t. Teichmüller space as parameter space.

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1 Introduction

The energy degeneracies of Ising models are of physical and mathematical relevance. Like in quantum mechanics, they can result from symmetries of the model. If the spins of these models are enumerated by an abelian group $F$, one such symmetry is the translation invariance of the interaction. We assume here that $F$ is finite. Then an Ising model is defined by the choice of the real coefficients $\tilde{j}_{f,g}$ and $\tilde{h}_f$ in the energy function $^1$

$$H : \{-1, 1\}^F \to \mathbb{R}, \quad H(\sigma) = \sum_{f \neq g \in F} \tilde{j}_{f,g} \sigma_f \sigma_g + \sum_{f \in F} \tilde{h}_f \sigma_f.$$  

This absence of $n$-body interactions between $n \geq 3$ spins is physically realistic. Translation invariance means $\tilde{j}_{f,g} \equiv j_{g-f}$ and $\tilde{h}_f \equiv h$. This, too, is a realistic assumption for many physical systems $^2$.

For an Ising model in $d$ spatial dimensions a typical choice for $F$ is $(\mathbb{Z}/N\mathbb{Z})^d$.

1.1 Remark (Nearest neighbour interactions)  An additional feature that leads to large degeneracies is that only nearest neighbours interact $^3$. There are $2^{|F|}$ spin configurations, but in the $d$-dimensional case there are only $d |F|$ edges in the nearest neighbour graph on the vertex set $F = (\mathbb{Z}/N\mathbb{Z})^d$.

So for isotropic such interactions (meaning $j_\ell = j$ for $\|\ell\|_1 = 1$ and $j_\ell = 0$ otherwise) and $h = 0$ there at most $d |F|$ energy values, all multiples of $j$. Their mean degeneracy is thus greater or equal to $2^{|F|}/(d |F|)$, growing exponentially in the thermodynamic limit $N \to \infty$. A similar estimate applies for $h \neq 0$.

For the ‘classical’ $d = 2$ dimensional Ising model $(F = (\mathbb{Z}/N\mathbb{Z})^2)$, $H(\sigma) = -\sum_{f \in F} \sigma_f (\sigma_{f+(1,0)} + \sigma_{f+(0,1)})$ the degeneracies of $H$ were studied in Beale $[\text{Be}]$, based on the celebrated Kaufman-Onsager solution $[\text{Ka}]$.

However, although models with nearest neighbour interactions are a bit easier to analyze than models with general two-body interactions, they are unrealistic. Physical interactions decay as the distance of the spins increases, but there is no reason why they should be of finite range.

Here we consider the degeneracies of translation invariant Ising spin models, but we allow for two-body interactions between all pairs of spins. This realistic

$^1$we omit the conventional negative sign here.

$^2$although it excludes models with frustrated or random interactions, see, e.g. the study of frustrated ground state degeneracy by Loebl and Vondrák $[\text{LV}]$.

$^3$that is, $j_{f,g} = 0$ if $\|f-g\|_1 > 1$ for the norm induced by the Lee norm $\|a\| := \min(a, N-a)$ ($a \in \mathbb{Z}/N\mathbb{Z} \cong \{0,1,\ldots,N-1\}$) on the factors of $F$. 

References
assumption leads to an enormous decrease of degeneracies, independent of the rate of spatial decay of these interactions.

We are particularly interested in stable degeneracies, that is degeneracies that cannot be lifted by changing the translation invariant interaction.

Although typically in the literature one considers $d$–dimensional Ising models with groups $G' = (\mathbb{Z}/N\mathbb{Z})^d$, we just assume that $G$ is finite abelian. Then the configuration space is the multiplicative group $G \equiv G_F := \{-1, 1\}^F$.

The energy function of a translation invariant Ising model has $J \equiv J_F := \mathbb{R}^{F}$ as parameter space (assuming for now vanishing of $h$) and is of the form

$$H \equiv H_F : G_F \times J_F \rightarrow \mathbb{R}, \quad H_F(\sigma, j) = \sum_{f \in F} j_f \sum_{\ell \in F} \sigma_\ell \sigma_{\ell + f}. \quad (1.1)$$

For interaction $j \in J$ the $j$–degeneracy of $\sigma \in G$ is defined as

$$D(\sigma, j) := \left| \{ \tau \in G \mid H(\tau, j) = H(\sigma, j) \} \right|. \quad (1.2)$$

Its stable degeneracy is a lower bound for $D(\sigma, j)$. We define it by

$$D_{\text{stab}}(\sigma) := \left| \{ \tau \in G \mid \forall j \in J : H(\tau, j) = H(\sigma, j) \} \right| = |A_F^{-1}(A_F(\sigma))| \quad (1.3)$$

with the correlation map

$$A \equiv A_F : G_F \rightarrow \mathbb{Z}^F, \quad A_F(\sigma)_f = \sum_{\ell \in F} \sigma_\ell \sigma_{\ell + f}, \quad (1.4)$$

since we have $H(\sigma, j) = \langle j, A(\sigma) \rangle$. This shows that for generic interactions $j$ we have $D(\sigma, j) = D_{\text{stab}}(\sigma)$, see Remark 2.11.

**Results.** In Section 2 we consider a lower bound $D_{\text{sym}}$ of $D_{\text{stab}}$ that is given by the ‘obvious’ symmetries of the energy function (that is, spin flip, translations, and reflection) and thus is of order $O(|F|)$. Actually in the limit of many spins typically there are $4|F|$ configurations related by these symmetries (Prop. 2.1). Product configurations are a simple example of configurations with additional degeneracies (Proposition 2.6).

In Section 3 we address the question by how much stable degeneracy can deviate from $D_{\text{sym}}$. Empirical data are compatible with the supposition that typically there are no additional degeneracies unrelated to the above-mentioned symmetries (Remark 3.2). However, in Section 4 we present an infinite family of configurations $\sigma$ (related to finite projective spaces) where $D_{\text{stab}}(\sigma)$ essentially equals $|F|^2 \gg 4|F|$. Alternatively Proposition 5.1 of Section 5 uses substitution

4 but compare with Lemma 2.5 below.
techniques to generate spin chain configurations with large stable degeneracy. Finally, in Section 6, it is shown that for up to four blocks of equal adjacent spins nontrivial stable degeneracy does not occur (Proposition 6.5), and that under an injectivity condition the same is true for an arbitrary number of blocks (Proposition 6.6).

1.2 Remark (Stable degeneracies for closed geodesics) In [Ra] Randol showed that the length spectrum of the closed geodesics on a Riemann surface of constant negative curvature is of unbounded multiplicity, independent on the point on Teichmüller space fixing the Riemannian metric.

This was based on a result by Horowitz [Ho], who considered the free group generated by two elements of SL(2, \( \mathbb{R} \)). He showed that for any \( N \in \mathbb{N} \) there exist \( N \) non-conjugate words of letters in this generator that encode closed geodesics having the same length. These words are related by substitutions.

In Section 5 we adapt that method to the energy spectrum of the translation invariant Ising model. We show the existence of sequences of spin configurations, whose quotient of stable and symmetry induced degeneracy is unbounded. It should be noted, however that stable degeneracy of closed geodesics and of spin configurations are very different phenomena. Whereas the number of interaction parameters is proportional to the number \( N \) of spins and thus diverges in the thermodynamic limit \( N \to \infty \), the number of parameters for the geodesic problem is bounded by the number of generators of the discrete subgroup \( \Gamma \) of SL(2, \( \mathbb{R} \)) that determines the surface \( \Gamma \backslash \mathbb{H} \).

Thus, in a way, the occurrence of non-trivial stable degeneracies for Ising models is even more astonishing than the one for closed geodesics.

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2 Properties of the Correlation Map \( A_F \)

In order to understand stable degeneracy, we must study the level sets of \( A_F \). There are three obvious types of symmetries, leaving \( A \) invariant:

- The **spin flip** \( G \to G, \sigma \mapsto -\sigma \) induces an action \( S : \{ -1, 1 \} \times G \to G \).

- \( F \) acts on itself by **translations**. This induces the action

  \[
  T : F \times G \to G, \quad (T_t(\sigma))_f = \sigma_{f+t}, \quad \text{and} \quad A \circ T_t = A.
  \]
• Finally, the automorphism group \( \text{Aut}(F) \) of \( F \) acts on \( G \). Whereas this does not in general leave \( A \) invariant, this is the case for the reflection \( F \to F, f \mapsto -f \). This gives rise to an action \( R \) of \( \{-1, 1\} \) on \( G \).

Altogether we obtain an action

\[
\Phi := (S, T, R) : S \times G \to G, \quad (\Phi_{(s,t,r)}(\sigma))_f = s\sigma_{r,f+t}
\]

of the group \( \times \) denoting semidirect product)

\[
S \equiv S_F := \{-1, 1\} \times (F \rtimes \{-1, 1\}).
\]

This action is faithful iff the group exponent of \( F \) is at least 3, that is, unless \( F \cong (\mathbb{Z}/2\mathbb{Z})^d \) for some \( d \in \mathbb{N}_0 \). The \( \Phi \)–orbits have cardinalities dividing the group order \( |S_F| = 4|F| \):

\[
D_{\text{sym}}(\sigma) := |\Phi(S, \sigma)| = |S_F|/|S_F(\sigma)| \quad (\sigma \in G),
\]

with the stabilizer group of \( \sigma \)

\[
S_F(\sigma) := \{(s, t, r) \in S_F \mid \Phi_{(s,t,r)}(\sigma) = \sigma\}.
\]

Since \( A \circ \Phi_s = A \ (s \in S) \), this leads to a lower bound for stable degeneracy:

\[
D_{\text{stab}} \geq D_{\text{sym}}.
\]

As the examples \( \sigma = \pm \mathbb{1}_F \) show, there are \( \Phi \)–orbits of size two. However, we show now that for groups \( F \) of large order typically \( D_{\text{sym}}(\sigma) = |S_F| \). More precisely, convergence in the mean occurs as the group order of \( |F| \) goes to infinity, unless reflection \( R \) acts trivially:

**2.1 Proposition (Average symmetry-induced degeneracy)** For the family of finite abelian groups \( F \) of group exponents \( \geq 3 \), uniformly in the group order

\[
\lim_{|F| \to \infty} \frac{|G_F|^{-1} \sum_{\sigma \in G_F} D_{\text{sym}}(\sigma)}{|S_F|} = 1.
\]

**Proof:** The upper bound \( \sum_{\sigma \in G_F} D_{\text{sym}}(\sigma) \leq |G_F| |S_F| \) being obvious, we need a lower bound. As \( D_{\text{sym}}(\sigma) = |S_F|/|S_F(\sigma)| \), we are to show that typically the stabilizer group \( S_F(\sigma) \) of \( \sigma \in G_F \) is trivial.

1. For all spin flips \( g = (-1, t, r) \in S_F \) a necessary condition for \( \Phi_g(\sigma) = \sigma \)

is that

\[
|\{f \in F \mid \sigma_f = 1\}| = |F|/2.
\]

But by Stirling’s formula, this can only be true for a subset of \( G_F \) which is of order \( O\left(|F|^{-1/2}|G_F|\right) = o(|G_F|) \).
2. A translation \((g = (1, t, 1) \in S_F)\) with \(t \in F \setminus \{0\}\) spans a non-trivial subgroup \(U\) of \(F\), and the set of \(g\)-invariant configurations is isomorphic to \(G_{F/U}\). Thus it is of order \(|G_{F/U}| = O(2^{|F|/2}|, and
\[
|\{\sigma \in G_F \mid \exists t \in F \setminus \{0\} : \Phi_{(1,t,1)}(\sigma) = \sigma\}| = O((|F| - 1)2^{|F|/2}) = o(|G_F|).
\]

3. By assumption \(F \neq (\mathbb{Z}/2\mathbb{Z})^d\). For a reflection \((g = (1, t, -1) \in S_F)\) with \(t \in F\) the fixed point set \(F_g := \{f \in F \mid 2f = t\}\) of the action of \(g\) on \(F\) has thus cardinality \(|F_g| \leq |F|/2\). So the set of \(g\)-invariant configurations is of order \(O(2^{|F_g|} + |F|/2) = O(2^{|F|/4})\). Therefore
\[
|\{\sigma \in G_F \mid \exists t \in F : \Phi_{(1,t,-1)}(\sigma) = \sigma\}| = O(|F|2^{|F|/4}) = o(|G_F|). \quad \Box
\]

2.2 Remark (Group exponent) The condition \(\text{Exp}(F) \geq 3\) in Prop 2.1 is necessary, as otherwise \(F\) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^d\) for some \(d \in \mathbb{N}_0\). Then one has \(\Phi_{(s,t,1)} = \Phi_{(s,t,1)} (s,t) \in S \times F\), so that \(D_{\text{sym}}(\sigma) \leq 2|F| = \frac{1}{2}|S_F|\). \quad \Diamond

2.3 Lemma For all finite abelian groups \(F\), the mean stable degeneracy
\[
\text{MSD}(F) := \frac{|G_F|}{|A_F(G_F)|}
\]
is smaller than the average stable degeneracy:
\[
\text{MSD}(F) \leq |G_F|^{-1} \sum_{\sigma \in G_F} D_{\text{stab}}(\sigma).
\]

Proof: This is an application of the Cauchy-Schwarz inequality:
\[
\text{MSD}(F) = \frac{\sum_{x \in A(G)} |A^{-1}(x)|}{\sum_{x \in A(G)} 1} \leq \frac{\sum_{x \in A(G)} |A^{-1}(x)|^2}{\sum_{x \in A(G)} |A^{-1}(x)|} = |G_F|^{-1} \sum_{\sigma \in G_F} D_{\text{stab}}(\sigma),
\]

since \((\sum_{x \in A(G)} |A^{-1}(x)|)^2 \leq (\sum_{x \in A(G)} 1) (\sum_{x \in A(G)} |A^{-1}(x)|^2)\). \quad \Box

2.4 Remark (Averages of stable degeneracy) By (2.3), the average stable degeneracy meets the estimate
\[
\liminf_{|F| \to \infty} \frac{|G_F|^{-1} \sum_{\sigma \in G_F} D_{\text{stab}}(\sigma)}{|S_F|} \geq 1.
\]
We show in Section 3 that the quotient \( F \to [1, \infty), \sigma \mapsto D_{\text{stab}}(\sigma)/D_{\text{sym}}(\sigma) \) is not bounded as the group order \(|F|\) goes to infinity. Nevertheless we conjecture that for the family of finite abelian groups \( F \) of group exponents \( \geq 3 \)

\[
\lim_{|F| \to \infty} \text{MSD}(F)/|S_F| = 1 \quad \text{and} \quad \lim_{|F| \to \infty} \frac{|G_F|^{-1} \sum_{\sigma \in G_F} D_{\text{stab}}(\sigma)}{|S_F|} = 1, \tag{2.4}
\]

that is, that typically stable degeneracy does not exceed symmetry-induced degeneracy.

\subsection*{2.5 Lemma (Parameter space)} \textit{The number of parameters of the energy function (1.1) is smaller than \( F \), and equals \( \dim(\mathbb{R}^F_{\text{ev}}) \) with}

\[
\text{span}_{\mathbb{R}}(A_F(G_F)) = \mathbb{R}^F_{\text{ev}} := \{ h \in \mathbb{R}^F | \forall f \in F : h_{-f} = h_f \}.
\]

\textbf{Proof:} As \( A_F(\sigma)_{-m} = A_F(\sigma)_m \), \( \text{span}_{\mathbb{R}}(A_F(G_F)) \subseteq \mathbb{R}^F_{\text{ev}} \). Conversely we set

\[
\sigma(f) \in G_F, \quad \sigma(f)_{\ell} := \begin{cases} -1 & \ell = 0 \text{ or } \ell = f \\ 1 & \text{else} \end{cases} \quad (f \in F).
\]

Then, with the characteristic function \( \mathbb{1}_S \) of a subset \( S \subseteq F \),

\[
A_F(\mathbb{1}) = |F| \mathbb{1}_F, \quad A_F(\mathbb{1}) - A_F(\sigma(0)) = 4 \mathbb{1}_{F \setminus \{0\}}, \tag{2.5}
\]

and for \( f \in F \setminus \{0\} \)

\[
A_F(\sigma(f))_m = \begin{cases} |F| & m = 0 \text{ or } (2f = 0 \text{ and } m = f) \\ |F| - 4 & 2f \neq 0 \text{ and } m \in \{-f, f\} \\ |F| - 8 & \text{else} \end{cases}.
\]

Thus for \( f \in F \setminus \{0\} \)

\[
[A_F(\sigma(f)) - A_F(\sigma(0))]_m = \begin{cases} 4 & 2f = 0 \text{ and } m = f \\ 0 & m = 0 \text{ or } (2f \neq 0 \text{ and } m \in \{-f, f\}) \\ -4 & \text{else} \end{cases},
\]

so that with (2.5)

\[
A_F(\sigma(f)) - 2A_F(\sigma(0)) + A_F(\mathbb{1}) = 4(\mathbb{1}_{\{f\}} + \mathbb{1}_{\{-f\}}). \tag{2.6}
\]

By (2.5) and (2.6) the functions \( A_F(\mathbb{1}) \) and \( A_F(\sigma(f)) \ (f \in F) \) span \( \mathbb{R}^F_{\text{ev}} \).

So with the orthogonal projection \( J = \mathbb{R}^F \to \mathbb{R}^F_{\text{ev}}, \ j \mapsto j_{\text{ev}} \) we have

\[
H(\sigma, j) = H(\sigma, j_{\text{ev}}) \quad ((\sigma, j) \in G \times J),
\]
and we can substitute $\mathbb{R}^F_{ev}$ for $\mathbb{R}^F$ in the definition (1.3) of stable degeneracy.

According to the fundamental theorem of finite abelian groups, for (not necessarily distinct) primes $p_i$

$$F \cong \bigoplus_i \mathbb{Z}/p_i^{n_i}\mathbb{Z}$$

(and we henceforth omit the isomorphism). So it is natural to consider for an arbitrary representation

$$F = \bigoplus_{j=1}^d F_j$$

(2.7)

of $F$ as a direct sum of finite non-trivial abelian groups $F_j$ the behavior of the stable degeneracy $D$ and of its lower bound $D_{sym}$ under such a decomposition. The subgroup lattice for a decomposition (2.7) of a group $F$ into its $p$–groups (and more generally if the group orders $|F_j|$ are coprime) is multiplicative. It contains more subgroups (and is quite complicated) for a decomposition (2.7) of a $p$–group $F$ into its factors $\mathbb{Z}/p^{n_j}\mathbb{Z}$, see Călugăreanu [Ca].

Additionally the correlation $A_F$ of the direct sum (2.7) is in general not invariant under reflections $f_j \mapsto -f_j$ of a single group $F_j$. However, this is the case for $A_F(\sigma)$ if $\sigma$ is the product configuration

$$\sigma \in G_F, \quad \sigma := \otimes_{j=1}^d \sigma^{(j)}$$

(2.8)

of $\sigma^{(j)} \in G_{F_j}$ ($j = 1, \ldots, d$).

### 2.6 Proposition (Product configurations)

The correlation of the product configuration $\sigma$ is multiplicative, i.e. $A_F(\sigma) = \otimes_{j=1}^d A_{F_j}(\sigma^{(j)})$ and

1. $4^{1-d} \prod_{j=1}^d D_{sym}(\sigma^{(j)}) \leq D_{sym}(\sigma) \leq \prod_{j=1}^d D_{sym}(\sigma^{(j)})$, and for all $d$ both inequalities cannot be improved.

2. The stable degeneracy obeys the inequality $D_{stab}(\sigma) \geq 2^{1-d} \prod_{j=1}^d D_{stab}(\sigma^{(j)})$.

**Proof:** For all $f = (f_1, \ldots, f_d) \in F$

$$A_F(\sigma)_f = \sum_{\ell \in F} \sigma_{\ell}\sigma_{\ell+f} = \sum_{\ell_1 \in F_1, \ldots, \ell_d \in F_d} \prod_{j=1}^d \left(\sigma^{(j)}_{\ell_j}\sigma^{(j)}_{\ell_j+f_j}\right) = \prod_{j=1}^d A_{F_j}(\sigma^{(j)})_{f_j}.$$

1) The cardinalities are given by $D_{sym}(\sigma) = |S_F|/|S_{F}(\sigma)|$ and $D_{sym}(\sigma^{(j)}) = |S_{F_j}|/|S_{F_j}(\sigma^{(j)})|$, with $|S_F| = 4^{1-d} \prod_{j=1}^d |S_{F_j}|$. So we have to show the inequalities $|S_F(\sigma)| \leq \prod_{j=1}^d |S_{F_j}(\sigma^{(j)})| \leq 4^{d-1}|S_F(\sigma)|$ for the stabilizer groups.

For the lower bound, we observe that for $(s, t, r) \in S_F(\sigma)$ the quotient functions $F_j \to \{-1, 1\}$, $f_j \mapsto \sigma^{(j)}_{r_{f_j}t+f_{j}}/\sigma^{(j)}_{f_j}$ are constant.
The upper bound for $|S_F(\sigma)|$ follows from the observation that as a subgroup of $S_F = \{-1, 1\} \times F \times \{-1, 1\}$ it is isomorphic to $S \times F' \times R$, with $S$ and $R$ trivial or equal to $\{-1, 1\}$, and $F'$ a subgroup of $F$. Similar decompositions with subgroups $F'_j$ of $F_j$ exist for $S_{F_j}(\sigma^{(j)})$, and $F' \cong \bigoplus_{j=1}^d F'_j$.

The left inequality in 1.) is saturated for $\sigma$ with trivial stabilizer groups. The right inequality in 1.) is saturated for $\sigma^{(j)} = (1, -1) \in G_{F_j}$, $F_j = \mathbb{Z}/2\mathbb{Z}$.

2.) Consider $\tau^{(j)} \in A_{F_j}^{-1}(A_{F_j}(\sigma^{(j)}))$ ($j = 1, \ldots, d$). Then by the first statement of the proposition $\tau \in G_F$, $\tau := \bigotimes_{j=1}^d \tau_{F_j}^{(j)}$ is in $A_{F}^{-1}(A_{F}(\sigma))$. Conversely, given such a $\tau$, the $\tau^{(j)}$ can be reconstructed only up to a factor $s^{(j)} \in \{-1, 1\}$, with $\prod_{j=1}^d s^{(j)} = 1$. So the product map is $2^d - 1$ to one. There may be additional elements in $A_{F}^{-1}(A_{F}(\sigma))$, not of product form. This leads to a lower bound for $D_{\text{stab}}(\sigma)$.

\[ \square \]

2.7 Remark (Periodic configurations) For a homomorphism $\pi : F \to F'$ of finite abelian groups, the pull-back $\pi^* : G_{F'} \to G_F$ relates the correlation maps via

\[ A_F \circ \pi^* = |F|/|F'| \pi^* \circ A_{F'} \]

with the subgroup $U := \ker(\pi)$ of $F$. Of course we can also reverse this, setting $F' := F/\ker(\pi)$ for a subgroup $U$ of $F$. The lifted configurations $\pi^* \tau \in G_F$ are $U$-periodic. More precisely, the group homomorphism $\pi_S : S_F \to S_{F'}$, $(s, t, r) \mapsto (s, \pi(t), r)$ relates the actions (2.1): $\pi^* \circ \Phi_{\pi_S(g)} = \Phi_g \circ \pi^*$ ($g \in S_F$).

Thus

\[ D_{\text{sym}}(\sigma) = D_{\text{sym}}(\tau) \quad \text{and} \quad D_{\text{stab}}(\sigma) \geq D_{\text{stab}}(\tau) \quad \text{for} \quad \sigma := \pi^* \tau. \]

Moreover, $A_F(\sigma)$ is $U$-periodic if and only if $\sigma \in G_F$ is $U$-periodic. This follows from (2.9), and conversely since for the $U$-periodic correlations $A_F(\sigma)$ we have $A_F(\sigma)_u = A_F(\sigma)_0 = |F|$ ($u \in U$). That equality implies $\sigma_{k+u} = \sigma_k$ ($k \in F, u \in U$).

\[ \diamond \]

2.8 Remark (Positive Fourier transform) We denote unitary Fourier transform by

\[ \mathcal{F} \equiv \mathcal{F}_F : \ell^2(F) \to \ell^2(F^*), \quad (\mathcal{F} g)_j = |F|^{-1/2} \sum_{m \in F} g_m \chi_m^{(j)}, \]

with the characters $\chi^{(j)} : F \to S^1$, using that the dual group $F^*$ of $F$ is isomorphic to $F$. $\sigma \in G_F$ and $A_F(\sigma)$ can be considered as real-valued functions on $F$, with $A_F(\sigma) = \sigma \ast I(\sigma)$ for $(Ig)_j := g_{-j}$ and $(g \ast h)_k := \sum_{f \in F} g_f h_{k-f}$ convolution. Its Fourier transform is non-negative,

\[ \mathcal{F}(A_F(\sigma)) = |F|^{1/2} |\mathcal{F}(\sigma)|^2 \geq 0 \quad (\sigma \in G_F), \]
\[ |F|^{-1/2} F(A_F(\sigma)) = F(\sigma) F(I(\sigma)) = F(\sigma) I(F(\sigma)) = |F(\sigma)|^2. \]

The (left) action \( \Psi : \text{Aut}(F) \times F \to F \) gives rise to the actions on \( G_F \) and on \( \mathbb{R}^F_{\text{ev}} \)

\[
\Psi^{(G)} : \text{Aut}(F) \times G_F \to G_F, \quad \Psi^{(G)}_g(\sigma) = \sigma \Psi \Psi^{-1}(f)
\]

\[
\Psi^{(\text{ev})} : \text{Aut}(F) \times \mathbb{R}^F_{\text{ev}} \to \mathbb{R}^F_{\text{ev}}, \quad \Psi^{(\text{ev})}_g(h) = h \Psi^{-1}(f).
\]

Together with the \( \Phi \)-action defined in (2.1), we obtain an action of the semidirect product \( S_F \rtimes \text{Aut}(F) \), with

\[
\Psi^{(G)}_g \circ \Phi_{(s,t,r)} = \Phi_{(s,\Psi \Psi^{-1}(s),r)} \circ \Psi^{(G)}_g. \quad ((s, t, r) \in S, \ g \in \text{Aut}(F)); \quad (2.10)
\]

in particular \( \Psi^{(G)}_g \) acts on the \( \Phi \)-orbits.

By Lemma 2.5 the image of the correlation map \( A_F \) spans \( \mathbb{R}^F_{\text{ev}} \). Although \( A_F \) is not invariant with respect to \( \Psi^{(G)} \), it has the following simple properties.

**2.9 Proposition (Image of the correlation map)**

- \( A_F(G_F) \subseteq C_F \) for

\[ C_F := \{ |F| - 4k \mid k = 0, \ldots, \lfloor |F|/2 \rfloor \} \cap \mathbb{R}^F_{\text{ev}}, \]

with \( |F| 1_F \in A_F(G_F) \).

- Its image is in general not a convex subset of \( C_F \).

- \( A_F(G_F) \) is invariant under the action \( \Psi^{(\text{ev})} \) of \( \text{Aut}(F) \) on \( \mathbb{R}^F_{\text{ev}} \), and

\[
A_F \circ \Psi^{(G)}_g = \Psi^{(\text{ev})}_g \circ A_F, \quad (g \in \text{Aut}(F)). \quad (2.11)
\]

**Proof:**

- \( A_F(\sigma) = |F| 1_F \) iff \( \sigma = \pm 1_F \). For all \( k, r \in F \) the spin umklapp \( \sigma_r \mapsto -\sigma_r \) of the \( r \)-th spin, keeping the other spins fixed, changes exactly two terms in the sum \( A_F(\sigma)_k \), by \pm 2. Since \( A_F(-\sigma) = A_F(\sigma) \), we can restrict ourselves to \( \sigma \in G_F \) with \( \{|k \in F \mid \sigma_k = -1\} \leq \lfloor |F|/2 \rfloor \).

- Convexity fails in the example of \( F := \mathbb{Z}/4\mathbb{Z} \equiv \{0, 1, 2, 3\} \), since

\[ A_F((1, 1, 1, 1)) = (4, 4, 4, 4) \quad \text{and} \quad A_F((1, -1, 1, -1)) = (4, -4, 4, -4), \]

but \( (4, 0, 4, 0) \notin A_F(G_F) \).

- The \( A_F \)-equivariance (2.11) of the \( \text{Aut}(F) \)-actions is immediate and implies

\[ \Psi^{(\text{ev})}_g(A_F(G_F)) = A_F(\Psi^{(G)}_g(G_F)) = A_F(G_F). \]

Lack of convexity makes it hard to find a good lower bound on the size \( |A_F(G_F)| \) of the image. Such a bound would be needed to prove conjecture (2.4).

---

5We call a subset \( S \subseteq C_F \) of the discrete cube \( C_F \) convex, if for \( s_0, s_1 \in S \) and \( s_t := (1-t)s_0 + ts_1 \in C_F \) for some \( t \in (0, 1) \) we have \( c_t \in S \), too.
2.10 Example (Nearest neighbour interaction) We consider $F = \mathbb{Z}/N\mathbb{Z}$. If $j(1) = 1$ but $j(d) = 0$ for $d \in F \setminus \{1\}$, then the energy values are $H(G_F, j) = \{N - 4k \mid k = 0, \ldots, \lfloor N/2 \rfloor\}$, and for $h \in H(G_F, j)$ the degeneracy equals $|H(\cdot, j)^{-1}(h)| = 2\left(h + \frac{N}{2}\right)$. Thus the mean degeneracy of the energy levels is asymptotic to $2^{N+1}/N$ in the thermodynamic limit $N \to \infty$, in accordance with Remark 1.1. This is also true for an interaction where $j(d_0) > 0$ for some $d_0$ with $\gcd(d_0, N) = 1$ and $j(d) = 0$ otherwise. ◆

2.11 Remark (Genericity of stable degeneracy) As the maps $j \mapsto H(\sigma, j)$ are linear, $j$-degeneracy $D(\sigma, j)$ (defined in (1.2)) equals $D_{\text{stab}}(\sigma)$ for all $j$ in the complement of a finite union of subspaces in $J$ that are of codimension one. So joint stability of all degeneracies is open and dense and of full Lebesgue measure in the parameter space $J$. ◆

2.12 Remark (Exterior field) An additional translation invariant coupling to the exterior magnetic field $h \in \mathbb{R}$ in the energy function (1.1) leads to

$$\tilde{H} \equiv \tilde{H}_F : G_F \times J_F \times \mathbb{R}_h \to \mathbb{R}, \quad \tilde{H}_F(\sigma, j, h) = \langle (j, h), \tilde{A}(\sigma) \rangle,$$

with the modified correlation map

$$\tilde{A}_F : G_F \to \mathbb{Z}^F \times \mathbb{Z}, \quad \tilde{A}_F(\sigma) = \left(A_F(\sigma), \sum_{f \in F} \sigma_f \right). \quad (2.12)$$

Then $\tilde{A}$ and thus $\tilde{H}$ is still invariant under translations and reflections.

The redefined stable degeneracy (compare with (1.3))

$$\tilde{D}_{\text{stab}}(\sigma) := |\tilde{A}_F^{-1}(\tilde{A}_F(\sigma))|$$

equals $D_{\text{stab}}(\sigma)$ iff $|\{f \in F \mid \sigma_f = 1\}| = |F|/2$, and equals $\frac{1}{2}D_{\text{stab}}(\sigma)$ otherwise. The reason is the equality

$$(\sum_{f \in F} \tau_f)^2 = \sum_{k \in F} A(\tau)_k = \sum_{k \in F} A(\sigma)_k = \left(\sum_{f \in F} \sigma_f\right)^2$$

in the case $A(\tau) = A(\sigma)$. So the last term in (2.12) is determined by the first term up to sign, which equals 0 iff $|\{f \in F \mid \sigma_f = 1\}| = |F|/2$. ◆

3 Configurations With Large Stable Degeneracy

In this section we start our search for spin configurations $\sigma$ where $D_{\text{stab}}(\sigma) > D_{\text{sym}}(\sigma)$. Generally speaking, our examples are based on different kinds of ‘hidden symmetries’ of $\sigma$. 11
3.1 Example (Product configurations) A rather trivial case concerns the degeneracy of a product configuration \((2.8)\), whose factors \(\sigma^{(j)} \in G_{F_j}\) have the maximal symmetry-induced degeneracy, that is \(D_{\text{sym}}(\sigma^{(j)}) = 4|F_j|\).

Then \(D_{\text{sym}}(\sigma) = 4|F|\), but by Prop. 2.6 \(D_{\text{stab}}(\sigma) \geq 2^{1-d} \prod_{j=1}^d D_{\text{stab}}(\sigma^{(j)}) \geq 2^{1-d} \prod_{j=1}^d D_{\text{sym}}(\sigma^{(j)}) = 2^{d+1}|F|\). So in this case \(D_{\text{stab}}(\sigma) \geq 2^{d-1}D_{\text{sym}}(\sigma)\). In other words the quotient \(D_{\text{stab}}(\sigma)/D_{\text{sym}}(\sigma) \geq 1\) is unbounded in general. ◇

One strategy to find more interesting configurations \(\sigma\) with large stable degeneracies is based on the equivariance property \((2.11)\). We are seeking spin configurations \(\sigma \in G_F\) and automorphisms \(g \in \text{Aut}(F)\) so that \(A_F(\sigma)\) is invariant under \(\Psi_{g}^{(\text{ev})}\), but \(\Psi_{g}^{(G)}(\sigma)\) is not in the \(\Phi\)-orbit of \(\sigma\). The correlation \(A_F(\sigma) : F \to \mathbb{Z}\) should not have a large image in \(\mathbb{Z}\) to allow for such \(g \in \text{Aut}(F)\).

Such configurations \(\sigma\) are of some interest, independent of whether they lead to a large stable degeneracy. The only case with \(|A_F(\sigma)| = 1\) is \(\sigma = \pm 1\). In the examples below, \(|A_F(\sigma)| \leq 3\) (Prop. 3.3), respectively \(|A_F(\sigma)| = 2\) (Prop. 4.1).

3.2 Remark (Empirical data) We performed a computer search for degeneracies of the spin configurations \(\sigma \in G_F\) with \(F = \mathbb{Z}/N\mathbb{Z}\) and integers \(N \leq 15\). In analyzing the data, a large variety of phenomena was found. Some examples:

- The first case where the inequality \(D_{\text{sym}}(\sigma) \leq |S| = 4N\) (see \((2.2)\)) is saturated, occurs for \(N = 7\) and \(\sigma = (1, 1, -1, 1, -1, -1, 1)\). It has the unusual property \(AF(\sigma)f = -1\) \((f \in F \setminus \{0\})\), but \(D_{\text{stab}}(\sigma) = D_{\text{sym}}(\sigma) = 4N\). This will be explained in number-theoretic terms (Example 3.5).

- The first \(\sigma\) whose stable degeneracy exceeds its symmetry-induced degeneracy has length \(N = 12\), and \(D_{\text{stab}}(\sigma) = 2D_{\text{sym}}(\sigma) = 8N\). That stable degeneracy follows from the action of \(\text{Aut}(F)\).

- The stable degeneracy \(D_{\text{stab}}(\sigma) = 2|S| = 8N\) is also found for certain configurations \(\sigma\) with \(N = 13\) to 15. We will explain the case \(N = 13\) using the projective plane \(\text{PG}(2, 3)\) (Example 4.4).

- \(N = 14\) is interesting in that

\[
\sigma := (-1, -1, 1, 1, 1, 1, -1, 1, 1, -1, 1, -1, 1, -1, 1) \in F
\]

has stable degeneracy \(D_{\text{stab}}(\sigma) = 2D_{\text{sym}}(\sigma) = 8N\), and

\[
\tau := (-1, -1, 1, -1, 1, 1, 1, -1, 1, 1, -1, 1, -1, 1)
\]

belongs to the same class, but \(\tau\) is not in the orbit of the action \((2.10)\).
Figure 3.1: The quotient \( \text{MSD}(F)/|S_F| \) for \( F = \mathbb{Z}/N\mathbb{Z} \), see Conjecture (2.4).

- For \( N = 16 \), \( \sigma := (-1, -1, 1, -1, 1, 1, -1, 1, -1, 1, -1, 1, 1) \) has stable degeneracy \( D_{\text{stab}}(\sigma) = 3|S| = 12N \). \( \text{Aut}(\mathbb{Z}/16\mathbb{Z}) \) maps \( A(\sigma) \) to a four-element set of correlations, clearly with the same stable degeneracy.

Conjecture (2.4) predicts that the mean stable degeneracy is asymptotic to the average symmetry induced degeneracy. This is compatible with the data of Figure 3.1.

We begin with a number-theoretic construction of certain configurations \( \sigma \in G_F \). This is interesting as the correlation of these \( \sigma \) takes only two or three values, and in fact \( \Psi^G_g(\Phi(S, \sigma)) = A_F(\sigma) \) for all \( g \in \text{Aut}(F) \). But as \( \Psi^G_g(\sigma) \in \Phi(S, \sigma) \), the construction does not lead to large stable degeneracy (see Rem. 3.4).

### 3.3 Proposition (Correlation for Legendre symbols)

For primes \( N \in \mathbb{P} \backslash \{2\} \) and \( F = \mathbb{Z}/N\mathbb{Z} \) the group elements \( \sigma^\pm \in G_F \), given by the values \( \sigma_k^\pm := \left( \frac{k}{N} \right) \) of the Legendre symbol for \( k = 1, \ldots, N-1 \) and \( \sigma_N^\pm := \pm1 \), have correlations

- \( A_F(\sigma^\pm)_f = -1 \) for all \( f \in F \backslash \{0\} \), if \( N \equiv 3 \mod 4 \).
- \( A_F(\sigma^\pm)_f = \left( -1 + 2\sigma_N^\pm \left( \frac{f}{N} \right) \right) \) for all \( f \in F \backslash \{0\} \), if \( N \equiv 1 \mod 4 \).

**Proof:** For \( N \equiv 3 \mod 4 \) and \( f \in F \backslash \{0\} \) we express the correlation entirely in terms of Legendre symbols. This is possible, since \( \left( \frac{0}{N} \right) = 0 \), and the two terms in
\[ A_F(\sigma^\pm)_f \text{ involving } \sigma^\pm_N \text{ cancel, as } (\frac{-f}{N}) = (\frac{-1}{N}) (\frac{f}{N}) \text{ and } (\frac{-1}{N}) = (-1)^{(N-1)/2} = -1. \] So

\[ A_F(\sigma^\pm)_f = \sum_{\ell \in \mathbb{Z}/N\mathbb{Z}} \sigma^\pm_N \sigma^\pm_{f+\ell} = \sum_{\ell \in \mathbb{Z}/N\mathbb{Z}} \left( \frac{\ell}{N} \right) \left( \frac{f+\ell}{N} \right) = \sum_{\ell \in \mathbb{Z}/N\mathbb{Z}} \left( \frac{(f+\ell)\ell}{N} \right). \]

We used here that \( a \mapsto (\frac{a}{b}) \) is completely multiplicative. Now the equation \( x^2 = (f + \ell)\ell \) has \( N - 1 \) solutions \( (x, \ell) \in (\mathbb{Z}/N\mathbb{Z})^2 \). This follows by the substitution \( a := x + \ell + f/2, b := x - \ell - f/2 \), which implies \( ab = x^2 - (f + \ell)\ell - f^2/4 \).

But since then \( \left( \frac{(f+\ell)\ell}{N} \right) = 0 \) if \( x = 0 \) and \( \left( \frac{(f+\ell)\ell}{N} \right) = 1 \) otherwise, this number of solutions also equals \( \sum_{\ell \in \mathbb{Z}/N\mathbb{Z}} \left( 1 + \left( \frac{(f+\ell)\ell}{N} \right) \right) = N + \sum_{\ell \in \mathbb{Z}/N\mathbb{Z}} \left( \frac{(f+\ell)\ell}{N} \right) \).

- For \( N \equiv 1 \mod 4 \) we obtain the additional term \( \sigma^\pm_N (\sigma^+_N + \sigma^-_N) = 2\sigma^\pm_N \left( \frac{f}{N} \right) \) in \( A_F(\sigma^\pm)_f \).

\[ \square \]

### 3.4 Remark (Degeneracy for Legendre symbols)

The automorphism group

\[ \text{Aut}(\mathbb{Z}/N\mathbb{Z}) = \{ a \in \mathbb{Z}/N\mathbb{Z} \mid \gcd(a, N) = 1 \} \]

acts by multiplication on \( \mathbb{Z}/N\mathbb{Z} \). We consider \( \sigma^\pm \) from Proposition 3.3.

- In the case \( N \equiv 3 \mod 4 \) the \( \Phi \)-orbits through \( \sigma^+ \) and \( \sigma^- \) coincide, as \( \sigma^\pm = \Phi_{(-1,0,-1)}(\sigma^\pm) \). That orbit is left invariant by \( \Psi_k^{(G)} \) (since for \( k \in \text{Aut}(\mathbb{Z}/N\mathbb{Z}), \Psi_k^{(G)}(\sigma^\pm) = \sigma^\pm \) for residues \( k \) and \( \Psi_k^{(G)}(\sigma^\pm) = \Phi_{(1,0,-1)}(\sigma^\pm) \) for non-residues \( k \)).

- For \( N \equiv 1 \mod 4 \) the \( \Phi \)-orbits through \( \sigma^+ \) and \( \sigma^- \) are different, since \( A_F(\sigma^+) \neq A_F(\sigma^-) \), see Proposition 3.3. The automorphisms \( \Psi_k^{(G)} \) leave these orbits invariant for residues \( k \) and interchanges them for non-residues \( k \) (as \( \Psi_k^{(G)}(\sigma^\pm) = \sigma^\mp \), respectively \( \Psi_k^{(G)}(\sigma^\pm) = \Phi_{(-1,0,1)}(\sigma^\mp) \)).

So in both cases we cannot conclude that \( D_{\text{stab}}(\sigma^\pm) \) is strictly larger than \( D_{\text{sym}}(\sigma^\pm) \). \[ \diamond \]

### 3.5 Example (Legendre symbols)

For \( F = \mathbb{Z}/7\mathbb{Z} \cong \{1, \ldots, 7\} \), the configuration \( \sigma = (1, 1, -1, 1, -1, -1, 1) \in G_F \) of Remark 3.2 equals \( \sigma^+ \). But \( \sigma \) can also be understood in terms of the Fano plane \( \text{PG}(2,2) \), see Section 4. \[ \diamond \]

### 4 Stable Degeneracy and Singer Sets

We continue our search for spin configurations with large stable degeneracy.
For the prime power $q := p^n$ ($p \in \mathbb{P}$ and $n \in \mathbb{N}$) the Desarguesian projective plane $\text{PG}(2, q)$ consists of $N := q^2 + q + 1$ elements (points). These are the one-dimensional subspaces of the vector space $\mathbb{F}_q^3$ over the Galois field $\mathbb{F}_q$. The lines of $\text{PG}(2, q)$ are the two-dimensional subspaces of $\mathbb{F}_q^3$ and are considered as subsets of $\text{PG}(2, q)$. So there are $N$ lines, too.

As will be shown below, $\text{PG}(2, q)$ allows the construction of configurations $\sigma \in G_F$ for $F = \mathbb{Z}/N\mathbb{Z}$ with constant $A_F(\sigma)_f = 1$ ($f \in F \setminus \{0\}$) and large stable degeneracy $D_{\text{stab}}(\sigma)$.

As will be shown below, $\text{PG}(2, q)$ allows the construction of configurations $\sigma \in G_F$ for $F = \mathbb{Z}/N\mathbb{Z}$ with constant $A_F(\sigma)_f = 1$ ($f \in F \setminus \{0\}$) and large stable degeneracy $D_{\text{stab}}(\sigma)$.

Automorphisms (called collineations) of $\text{PG}(2, q)$ are bijections, mapping lines to lines. According to the fundamental theorem of projective geometry, they are induced by bijective semilinear maps $\phi : \mathbb{F}_q^3 \to \mathbb{F}_q^3$, that is, for some automorphism $\tau = \tau_\phi : \mathbb{F}_q \to \mathbb{F}_q$

$$\phi(v + w) = \phi(v) + \phi(w) \quad \text{and} \quad \phi(\lambda v) = \tau(\lambda)\phi(v) \quad (v, w \in \mathbb{F}_q^3, \lambda \in \mathbb{F}_q).$$

As $\phi$ maps $k$–dimensional subspaces to $k$–dimensional subspaces, it descends to an automorphism $\tilde{\phi} \in \text{Aut}(\text{PG}(2, q))$.

As a vector space over $\mathbb{F}_q$, $\mathbb{F}_q^3 \cong \mathbb{F}_q^3$. The Singer subgroup $\Sigma$ of $\text{Aut}(\text{PG}(2, q))$ consists of the automorphisms induced by multiplication with the non-zero elements of $\mathbb{F}_q^3$. It is thus cyclic and of order $N$ (see Hughes and Piper [HP] and the original article [Si] by Singer). Concretely let the irreducible primitive cubic $X^3 - c_2 X^2 - c_1 X - c_0, c_i \in \mathbb{F}_q$ define multiplication in $\mathbb{F}_q^3$. For a root $\lambda \in \mathbb{F}_q^3$ of that cubic

$$\lambda^3 = c_2 \lambda^2 + c_1 \lambda + c_0,$$

so that multiplication with $\lambda$ corresponds in the ordered basis $(\lambda^2, \lambda, 1)$ of $\mathbb{F}_q^3$ to multiplication with the matrix

$$M := \begin{pmatrix} c_2 & 1 & 0 \\ c_1 & 0 & 1 \\ c_0 & 0 & 0 \end{pmatrix} \in \text{GL}(3, \mathbb{F}_q).$$

$M$ gives rise to a projective collineation $\tilde{M} : \text{PG}(2, q) \to \text{PG}(2, q)$ of period $N$. We use additive notation, identifying $\Sigma$ with $\mathbb{Z}/N\mathbb{Z}$. Starting with a point $A_0 \in \text{PG}(2, q)$ we denote the points of its orbit $\text{PG}(2, q)$ by

$$A_k := \tilde{M}^k(A_0) \quad (k = 0, \ldots, N - 1).$$

For a point $A$ and a projective line $L$ in $\text{PG}(2, q)$ their difference set is defined as

$$\mathcal{D} \equiv \mathcal{D}(A, L) := \{ g \in \Sigma \mid g(A) \in L \}.$$

Its cardinality thus equals the one of the projective line: $|\mathcal{D}| = q + 1$ (see also Golay [Go]).
This set of group elements is perfect: for every $h \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}$ there are unique $d_1, d_2 \in \mathcal{D}$ with $h = d_2 - d_1$ (see Lemma 13.12 of [HP]). If we attribute to a subset $\mathcal{D} \subseteq \mathbb{Z}/N\mathbb{Z}$ the spin configuration

$(-1)^{\mathcal{D}} \in G_F$,

then for a perfect difference set $\mathcal{D}$ it is mapped by (1.4) to

$A((-1)^{\mathcal{D}})_0 = N, \quad A((-1)^{\mathcal{D}})_f = q^2 - 3q + 1 \quad (f \in F \setminus \{0\}). \quad (4.1)$

Assuming $L$ to be the projective line through $A_0$ and $A_1$, $D(A_0, L)$ contains 0 and 1 in $\mathbb{Z}/N\mathbb{Z}$. There is a unique shift that lets a perfect difference set contain 0 and 1, and the corresponding set is called reduced in [Si]. This normalization allows us to discern difference sets not just being mutual translates in $\mathbb{Z}/N\mathbb{Z}$.

4.1 Proposition For the perfect difference sets $D(A, L)$ the stable degeneracy is bounded by

$$D_{\text{stab}}((-1)^{\mathcal{D}}) \geq 2N \varphi(N)/(3n).$$

Proof: A conjecture in [Si] claims that there are exactly $\varphi(N)/(3n)$ reduced perfect difference sets. This conjecture has been proven 25 years later by Halberstam and Laxton in [HL]. For perfect difference sets $\mathcal{D}$, $D_{\text{sym}}((-1)^{\mathcal{D}}) = 4N$, the maximal value possible for an element of $G_F$. This follows, since

- the period of $(-1)^{\mathcal{D}}$ equals $N$, because otherwise $\mathcal{D}$ could not be perfect;
- $(-1)^{\mathcal{D}}$ is not palindromic, for the same reason,
- The number of entries $-1$ in $(-1)^{\mathcal{D}}$ equals $q + 1 < N/2$, since $N = q^2 + q + 1$ and $q = p^n \geq 2$. So $(-1)^{\mathcal{D}}$ is not representable in the form $\Phi_{(1,t,s)}((-1)^{\mathcal{D}})$.

The number of is only half of the product of $|D_{\text{sym}}((-1)^{\mathcal{D}})|$ and $\varphi(N)/(3n)$, since the reflections $\psi_{-1} = \Phi_{(1,0,-1)}$ are counted twice in that product.

Stable degeneracy of $\sigma \in G_F$ can be nearly as large as the square of the group order of $F$, that is, as the square of symmetry-induced degeneracy:

4.2 Corollary For all $q \in \mathbb{P}$, $N := q^2 + q + 1$ there is a configuration $\sigma \in G_{\mathbb{Z}/N\mathbb{Z}}$ with

$$D_{\text{stab}}(\sigma) \geq \frac{N^2}{3 \log(\log N)}. \quad (4.2)$$

Proof: We use the lower bound $2N \varphi(N)/3$, with $N := q^2 + q + 1$ and $q \in \mathbb{P}$. However, we can use that (see Bach and Shallit [BS, Theorem 8.8.7]) for $m \geq 3$ $\varphi(m) \geq m/\left(e^\gamma \log(\log m) + 3/\log(\log m)\right)$. The right hand side is smaller than $m/(2 \log(\log m))$ for all large $m$, and we tested (4.2) for the first 35 000 000 cases $m = N(q)$ with $q \in \mathbb{P}$, which are not covered by that inequality. \qed
4.3 Remark (Stable degeneracy quadratic in the group order) It is unknown (see Baier and Zhao [BZ, Section 1]) whether for any quadratic polynomial $P \in \mathbb{Z}[q]$ there are infinitely many primes in $P(\mathbb{N})$. So it is unclear, whether for some $c > 0$ there are infinitely many prime powers $q$ with $\varphi(N) \geq cN$ for $N = q^2 + q + 1$.

Thus we do not know how to prove that for some $c \in (0, 2/3)$ we have $D_{\text{stab}}(\sigma) \geq cN^2$ for infinitely many $N$ and some $\sigma \in G_{Z/NZ}$.

4.4 Example (Large stable degeneracy by Singer sets, for $F = \mathbb{Z}/13\mathbb{Z}$) We consider the group $F = \mathbb{Z}/N\mathbb{Z} \cong \{0, 1, \ldots, 12\}$ for $N = 13$

\[
\sigma = (1, 1, -1, 1, -1, -1, 1, -1, 1, -1, -1, 1) \in G_F
\]

has the degeneracies $D_{\text{sym}}(\sigma) = |S_F| = 4N$ and $D_{\text{stab}}(\sigma) = 8N$. The explanation is the following: For $q = 3$, $N = q^2 + q + 1$, and $\sigma = (-1)^{3q}$ corresponds to a Singer set $D \subset F$ of the projective plane $PG(2, q)$. By (4.1) its correlation equals

\[
A(\sigma)_0 = 13 \text{ , and } A(\sigma)_f = 1 \quad (f \in (\mathbb{Z}/13\mathbb{Z}) \setminus \{0\}).
\]

So the correlation is constant on the $\Psi$-orbit of $\sigma$. $\Psi(3, \sigma)$ and $\Psi(9, \sigma)$ are translates of $\sigma$, whereas $\Psi(4, \sigma)$, $\Psi(10, \sigma)$ and $\Psi(12, \sigma)$ additionally reflect it. On the other hand, $\Psi(2, \sigma) = (1, 1, -1, 1, -1, 1, -1, -1, -1, -1, 1, -1)$ is not in the $\Phi$-orbit of $\sigma$, leading to $D_{\text{stab}}(\sigma) = 2D_{\text{sym}}(\sigma)$. Here the bound of Proposition 4.1 is sharp, as $2N\varphi(N)/(3n) = 8N = 104$.

Incidentally, this $\sigma \in G_F$ is also defined by $\sigma_k := 1$ if $k = x^4$ for some $x \in \mathbb{Z}/N\mathbb{Z}$ and $\sigma_k := -1$ otherwise.

4.5 Remark (Block designs) The notion of a Singer group is generalized to be an automorphism group acting regularly on the blocks of a symmetric block design. Then every such symmetric block design has a representation by a difference set, see Theorem XI 5.2 of Jacobs and Jungnickel [JJ].

Using this and similar ideas one can probably construct many spin configurations with large stable degeneracy.

5 Substitutions

We now apply a technique, borrowed from trace identities and going back to Horowitz [Ho], to find more configurations of large stable degeneracy.

Over a finite, nonempty set $\mathcal{A}$ (called alphabet with letters $a \in \mathcal{A}$),

\[
\mathcal{A}^* := \epsilon \cup \bigcup_{n \in \mathbb{N}} \mathcal{A}^n
\]
is the set of finite words, with \( w \in \mathcal{A}^n \) a word of length \( |w| := n, \) and \( \epsilon \) the empty word of length \( |\epsilon| := 0. \) With concatenation \( (v_1 \ldots v_m; w_1 \ldots w_n) \mapsto v_1 \ldots v_n w_1 \ldots w_n, \) \( \mathcal{A}^* \) becomes a monoid with identity \( \epsilon. \) For \( a \in \mathcal{A} \) and \( w \in \mathcal{A}^*, \)

\[ |w|_a \in \mathbb{N}_0 \] denotes the number of occurrences of the letter \( a \) in \( w. \) The inverse of \( v := v_1 \ldots v_m \in \mathcal{A}^* \) is \( v^{-1} := v_m \ldots v_1, \) and in the case \( \mathcal{A} = \{-1,1\} \) their correlations coincide:

\[ A_{\mathbb{Z}/m\mathbb{Z}}(v^{-1}) = A_{\mathbb{Z}/m\mathbb{Z}}(v). \]  

(5.1)

Given words \( U, V \in \mathcal{A}^* \) over the alphabet \( \mathcal{A} := \{-1,1\} \) and a word \( W \equiv W(U,V) \in \{U,V\}^*, \) by concatenation one considers \( W \) as a word \( \tilde{W} \in \mathcal{A}^*, \) and thus as a configuration \( \sigma \in G_F \) for some group \( F = \mathbb{Z}/N\mathbb{Z}. \) Here \( |\tilde{W}|_{\pm 1} = |W|_U |U|_{\pm 1} + |W|_V |V|_{\pm 1} \) and therefore \( N = |W|_U |U| + |W|_V |V|. \)

Then \( X := W(U,V)^{-1} \) gives rise to \( \tilde{X} \in \mathcal{A}^*. \) This does not in general coincide with \( (\tilde{W})^{-1}. \)

5.1 Proposition (Correlation map and substitutions) \( A_F(\tilde{X}) = A_F(\tilde{W}). \)

**Proof:** For the word \( W = w_1 \ldots w_\ell \) over the alphabet \( \{U,V\} \) the reversed word equals \( X = w_\ell \ldots w_1. \) We use cyclic indexing, that is \( 1, \ldots, \ell \in F' := \mathbb{Z}/\ell\mathbb{Z}. \)

1. Thus in \( W \) and \( X \) for any \( a, b \in F' \) the letters \( w_a, w_b \) have the same cyclic distance. Moreover, they are separated by the same letters \( w_{a+1}, \ldots, w_{b-1} \) respectively \( w_{b+1}, \ldots, w_{\ell}, w_1, \ldots, w_{a-1}. \)

So if \( w_a = w_b, \) then the contribution of the pair in \( A_F(\tilde{W}) \) coincides with the one in \( A_F(\tilde{X}). \)

2. Consider now pairs \( w_a \neq w_b, \) say, \( w_a = U \) and \( w_b = V. \) Then for all \( k \in \mathbb{Z}/\ell\mathbb{Z} \) there is a bijection \( \rho_k : P_k \to Q_k \) between the two sets

\[ P_k := \{ c \in F' \mid w_c = U, w_{c+k} = V \} \quad Q_k := \{ d \in F' \mid w_d = V, w_{d+k} = U \}, \]

with frequencies of separating letters \( U \)

\[ (w_{\rho_k(c)+1} \ldots w_{\rho_k(c)+k-1})_U = (w_{c+1} \ldots w_{c+k-1})_U \]

(and the same for \( V \)). This follows by induction in the length \( \ell' \) of the subword \( w_1 \ldots w_{\ell'}, \ell' = 1, \ldots, \ell. \) So the contribution of the pairs \( (U,V) \) in \( A_F(\tilde{W}) \) coincides with the one in \( A_F(\tilde{X}), \) and similarly for the pairs \( (V,U). \)

5.2 Example (Substitution) \( W := UVUUVVV, \) with \( U := (1,1,-1) \) and \( V := (-1,1,-1). \) So for \( F := \mathbb{Z}/21\mathbb{Z} \) the configuration equals \( \tilde{W} = \sigma \in G_F, \) with

\[ \sigma := (1,1,-1,-1,1,-1,1, -1,1, -1,1, -1,1, -1,1, -1,1, -1,1, -1,1). \]

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The correlation of $\sigma$ equals $A(\sigma)_0 = 21$, and for $f \in F \setminus \{0\}$: $A(\sigma)_f = 13$ if $f = 0 \pmod{3}$ and $A(\sigma)_f = -7$ otherwise. Thus, unlike $\sigma$, $A(\sigma)$ is a fixed point of the $\text{Aut}(F)$ action.

The inverse of $W$ equals $X = VVVUUUU$. So $\tilde{X} = \tau \in G_F$, with
$$\tau := (-1, 1, -1, 1, -1, 1, 1, 1, -1, -1, -1, 1, -1, 1, -1, 1, -1, 1, -1).$$

$\tau$ is a translate of $\Psi^{(G)}_{10}(\sigma)$. It is not an element of the orbit $\Phi(S, \sigma)$:

- As $|\sigma|_1 = |\tau|_1 = 10 \neq 11 = |\sigma|_{-1} = |\tau|_{-1}$, if $\Phi((s, t, r), \sigma)$ there can be no spin flip $(s = 1)$.
- In addition a pure translation ($r = 1$) is impossible, since the subsequence $(-1, 1, 1, -1, 1, 1, -1, 1, -1, 1, -1, -1)$ of $\sigma$ is not a subsequence of the cyclic word $\tau$.
- It cannot be a reflection ($r = -1$) either, since $\tau$ does not contain the subsequence $(-1, 1, 1, -1, 1, 1, -1, 1, -1, 1, -1, -1)$ of $\Phi((1, 0, -1), \sigma)$.

So $D_{\text{stab}}(\sigma) > D_{\text{sym}}(\sigma)$. $\Box$

The method of Proposition 5.1 can be iterated.

## 6 Block Sizes and Stable Degeneracy

The discrete Laplacian of the correlation reveals some information about the spin configurations sharing that correlation, in particular the structure of blocks of alike spins. This may be useful regarding Conjecture (2.4).

### 6.1 Lemma (Laplacian for correlations)

For the residue class group $F := \mathbb{Z}/N\mathbb{Z}$, $N \in \mathbb{N}$ the Laplacian $\Delta : \mathbb{R}^F \to \mathbb{R}^F$, $(\Delta h)_f = \frac{1}{4}(h_{f-1} - 2h_f + h_{f+1})$ is injective as a map $A_F(G_F) \to \mathbb{Z}^F \cap \mathbb{R}^F_{\text{ev}}$.

**Proof:** The normalization $1/4$ is chosen so that by the first statement in Prop. 2.9 the restriction of $\Delta$ to the image of the correlation map has its image in $\mathbb{Z}^F$. Moreover, $A_F(G_F) \subseteq \mathbb{R}^F_{\text{ev}}$ and $\Delta$ restricts to $\mathbb{R}^F_{\text{ev}}$. Injectivity follows, since $\ker(\Delta) = \mathbb{R}\mathbb{I}_F$, and $A_F(\sigma)_0 = N$ for all $\sigma \in G_F$. $\Box$

So in principle one can calculate the stable degeneracy of a configuration $\sigma \in G_F$ by considering $\Delta A_F(\sigma)$.

The only configurations with constant correlation are $\pm \mathbb{I}_F$ (since then $A_F(\sigma)_f = A_F(\sigma)_0 = N$). For a residue class group $F = \mathbb{Z}/N\mathbb{Z}$ the other configurations are thus translates of $\sigma \in G_F$ which are of the form
$$\sigma = (1)^{m_1}(-1)^{m_2} \ldots (1)^{m_{2k-1}}(-1)^{m_{2k}}$$ (6.1)
with $k \in \mathbb{N}$ and $m_\ell \in \mathbb{N}$ ($\ell = 1, \ldots, 2k$), and the strings $(1)^m$ (respectively $(-1)^m$) of $m$ ones (respectively $-1$). So $\sum_{\ell=1}^{2k} m_\ell = N$. Note that stable and symmetry-induced degeneracy are invariant under translation of a configuration. Thus it is natural to consider the indices $j$ of $m_j$ as elements of $\mathbb{Z}/(2k\mathbb{Z})$.

6.2 Lemma For $F := \mathbb{Z}/N\mathbb{Z}$ and a configuration $\sigma \in G_F$ of the form (6.1),

$$\Delta A_F(\sigma) = \sum_{\ell=1}^{2k} \sum_{n=1}^{k} (\delta_{m_\ell+\ldots+m_\ell+2n-2} - \delta_{m_\ell+\ldots+m_\ell+2n-1}).$$

(6.2)

Proof: For all $f \in F$, $\Delta A_F(\sigma)_f = \frac{1}{4}(A_F(\sigma)_{f-1} - 2A_F(\sigma)_f + A_F(\sigma)_{f+1}) = \frac{1}{4} \sum_{g \in \mathbb{Z}/N\mathbb{Z}} \sigma_g(\sigma_{g+f+1} - 2\sigma_{g+f} + \sigma_{g+f-1}) = \sum_{g \in \mathbb{Z}/N\mathbb{Z}} \frac{1}{4}(\sigma_g - \sigma_{g-1})(\sigma_{g+f-1} - \sigma_{g+f}).$

The terms $\frac{1}{4}(\sigma_g - \sigma_{g-1})(\sigma_{g+f-1} - \sigma_{g+f})$ in the last sum are non-zero iff $\sigma_g \neq \sigma_{g-1}$ and $\sigma_{g+f-1} \neq \sigma_{g+f}$, that is, iff the indices are of the form $g = \sum_{\ell=1}^{r} m_\ell$ and $g + f = \sum_{\ell=1}^{s} m_\ell$ for some $r, s$. The modulus of these terms equals one, and their sign is negative iff $s - r$ is even. This explains the $\delta$-terms in (6.2). $\Box$

6.3 Remarks (Block sizes) 1. Note that Lemma 6.2 implies that we can read off the number $2k$ of blocks in the configuration (6.1) from the Laplacian of its correlation: $(\Delta A_F(\sigma))_0 = -2k$.

2. In the substitution example 5.2 the multisets of block sizes for $\sigma$ and $\tau$ both equal $1^72^7$ ($1^42^3$ for the blocks of ones, and $1^32^4$ for the blocks of minus ones).

However, the Singer set in Example 4.4 shows that the correlation does not in general determine the multiset of block sizes $m_\ell$: $\sigma$ has block sizes $(m_1, \ldots, m_6) = (2, 1, 1, 5, 1, 3)$ and multiset $1^32^31^51$, which differs from the multiset $1^32^26^1$ for the data $(2, 2, 1, 1, 1, 6)$ of $\tau := \Psi(2, \sigma)$ with $A_F(\sigma) = A_F(\tau)$.

3. Nevertheless, the multiplicity of the minimal block size $\min(m_1, \ldots, m_{2k})$ can be deduced from the correlation and equals $(\Delta A_F(\sigma))_{\ell}$ with $\ell > 0$ the smallest index for which $(\Delta A_F(\sigma))_{\ell} \neq 0$. $\Diamond$

From Remark 6.3.2 we see that non-trivial stable degeneracy is possible for $2k = 6$-block configurations. We show now that this is the minimal number. The method is to reconstruct for $k = 1$ and $k = 2$ from $\Delta A_F(\sigma)$ the list $(m_1, \ldots, m_{2k})$ of block sizes, up to cyclic permutations and reflection. It will turn out that already for $k = 2$ the combinatorics is intricate. To simplify the proof, we use the following observation.

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6.4 Lemma ($\ell^1$-norm)  All $2k$–block configurations $\sigma \in G_F$ fulfill the inequalities

$$4k \leq \|\Delta A_F(\sigma)\|_1 \leq 4k^2, \quad (6.3)$$

and $\|\Delta A_F(\sigma)\|_1$ is a multiple of four.

Proof: (6.3) is true for $k = 0$, that is, $\sigma = \pm \mathbb{I}$, since $\Delta A_F(\pm \mathbb{I}_F) = 0$. For $\sigma \in G_F \setminus \{-\mathbb{I}_F, \mathbb{I}_F\}$ we obtain the right inequality in (6.3) by counting the number $4k^2$ of terms in (6.2). The left inequality follows from the observation that $(\Delta A_F(\sigma))_0 = -2k$ and that $\sum_{f \in F}(\Delta A_F(\sigma))_f = 0$. $\|\Delta A_F(\sigma)\|_1$ is a multiple of four, since $\Delta A_F(\sigma)$ is an even function (Lemma 6.1), and since even and odd contributions to (6.2) cancel in pairs. □

On the r.h.s. of (6.3) equality occurs if and only if there is no cancellation between $\delta$–functions with different signs (that is, there are no $1 \leq \ell_1, \ell_2 \leq 2k$ and $1 \leq n_1, n_2 \leq k$ with $m_{\ell_1} + \ldots + m_{\ell_1+2n_1-1} = m_{\ell_2} + \ldots + m_{\ell_2+2n_2}$).

Then given $\Delta A_F(\sigma)$, we know the multisets of lengths $m_{\ell_1} + \ldots + m_{\ell_1+2n_1-1}$ respectively $m_{\ell_2} + \ldots + m_{\ell_2+2n_2}$ of unions of adjacent blocks. In addition we then know whether for a given element of the multiset the number of blocks is even or odd.

6.5 Proposition (Trivial stable degeneracy for at most four blocks) For $\sigma \in F = \mathbb{Z}/N\mathbb{Z}$, written in the form (6.1) with $2k \leq 4$ blocks, one has

$$D_{\text{stab}}(\sigma) = D_{\text{sym}}(\sigma).$$

Proof: • No blocks: $\sigma = \pm \mathbb{I}$ are the only configurations with $\Delta A_F(\sigma) = 0$.
• Two blocks: The configurations (6.1) with $k = 1$, that is

$$\sigma^{(m)} \in G_F, \quad \sigma^{(m)}_\ell = \begin{cases} 1 & \ell \leq m \\ -1 & m < \ell \leq N \end{cases} \quad (m \in \{1, \ldots, N - 1\})$$

have a correlation $A(\sigma^{(m)}) = N - 4 \min(\|m\|_1, \|r\|_1)$ of triangular form, and $\Delta A_F(\sigma^{(m)}) = \delta_m + \delta_{-m} - 2\delta_0$.

$D_{\text{stab}}(\sigma^{(m)}) = D_{\text{sym}}(\sigma^{(m)})$, since by Remark 6.3.1 for any $\tau \in A_F^{-1}(A_F(\sigma^{(m)}))$ there are $\Delta A_F(\tau)_0 = \Delta A_F(\sigma^{(m)})_0 = 2$ blocks, which by Remark 6.3.3 are of sizes $m$ respectively $N - m$. So $\tau \in \Phi(S, \sigma^{(m)})$.

• Four blocks, notation: We consider the function $\Delta A_F(\sigma) : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}$ as a signed multiset

$$M := \prod_{i=1}^{i_{\text{max}}} \ell_i^{c_i} \quad \text{with } 0 < t_i < t_{i+1} \text{ and } c_i \in \mathbb{Z} \setminus \{0\}. \quad (6.4)$$
By Lemma 6.2, the sum of lengths of all four blocks equals \( t_{i_{\text{max}}} = N \), and \( c_{i_{\text{max}}} = -4 \). We denote that contribution to (6.4) by \( M_4^{-1} := N^{-4} \). The other exponents are palindromic:

\[
c_{i_{\text{max}} - i} = c_i \quad (i = 1, \ldots, i_{\text{max}} - 1).
\]

- **Four blocks, equality in (6.3):** For \( \sigma \) of the form (6.1) with \( k = 2 \) we start with the assumption

\[
\| \Delta A_F(\sigma) \|_1 = (\Delta A_F(\sigma))_0^2 \equiv 16. \tag{6.5}
\]

As \( M_4^{-1} = N^{-4} \), the contribution to the signed multiset \( M = M_1 M_2^{-1} M_3 M_4^{-1} \), coming from pairs of neighbouring blocks, equals \( M_2^{-1} = \prod_{t=1, \ldots, N-1} (\Delta A_F(\sigma)_t) \).

Table 1: Reconstruction from the correlations for four blocks, \( \| \Delta A_F(\sigma) \|_1 = 12 \)

| multiset | par further conditions | block lengths | Example | \( t_5 \) in Example |
|----------|------------------------|---------------|---------|---------------------|
| \( F_2^{+1} \) | e | t_1, t_1, t_1, t_2 | 1,1.2 | 1.2.3.4.5 |
| \( F_2^{-1} \) | o | t_1, t_1, t_2 | 1,1.2 | 1,2.3.4.5 |
| \( F_2^{+1} \) | e | t_2 = 2t_1, 3t_1 + 2t_2 = N | t_1, t_1, t_2, t_2 | 2.2.3 | 2.3.4.8.9.10 |
| \( F_2^{+1} \) | o | t_1, t_1, t_1, t_3 | 1,1.3 | 1,2.3.4.5.6 |
| \( F_2^{+1} \) | e | t_3 = 3t_1, 4t_1 + t_2 = N | t_1, t_2, t_1, t_2 | 2.2.4 | 2.5,6.7,8,11 |
| \( F_2^{+1} \) | o | t_1, t_1, 2t_1, t_3 | 1,1.4 | 1,3.4.5.7 |
| \( F_2^{+1} \) | e | t_2 = 2t_1, 3t_1 + 2t_3 = N | t_1, t_1, t_3, t_3 | 1,1.4 | 1,2.3.6.7.8 |
| \( F_2^{+1} \) | o | t_1, t_2, t_2, t_3 | 1,1.2 | 1,2.3.6.7.8 |
| \( F_2^{+1} \) | e | t_3 = 2t_1 + t_2, 2t_1 + 3t_2 = N | t_1, t_2, t_2, t_3 | 1,1.4 | 1,2.3.6.7.8 |
| \( F_2^{+1} \) | o | t_1, t_1, t_3, t_3 | 1,1.3 | 1,2.3.4.6.7.8 |
| \( F_2^{+1} \) | e | t_4 = t_1 + t_3, 3t_1 + 2t_3 = N | t_1, t_2, t_1, t_2 | 1,1.2 | 1,2.3.4.6.7.8 |
| \( F_2^{+1} \) | o | t_1, t_2, t_1, t_3 | 1,1.2 | 1,2.3.4.6.7.8 |
| \( F_2^{+1} \) | e | t_3 = 4t_1 + 2t_2, t_1 + 3t_2 = N | t_1, t_2, t_1, t_3 | 1,1.4 | 1,2.3.4.6.7.8 |
| \( F_2^{+1} \) | o | t_1, t_1, t_3, t_3 | 1,1.3 | 1,2.3.4.6.7.8 |
| \( F_2^{+1} \) | e | t_4 = t_1 + t_3, 3t_1 + 2t_3 = N | t_1, t_2, t_1, t_2 | 1,1.2 | 1,2.3.4.6.7.8 |
| \( F_2^{+1} \) | o | t_1, t_2, t_1, t_3 | 1,1.2 | 1,2.3.4.6.7.8 |
| \( F_2^{+1} \) | e | t_4 = t_1 + t_3, 3t_1 + 2t_3 = N | t_1, t_2, t_1, t_2 | 1,1.2 | 1,2.3.4.6.7.8 |
| \( F_2^{+1} \) | o | t_1, t_2, t_1, t_3 | 1,1.2 | 1,2.3.4.6.7.8 |
| \( F_2^{+1} \) | e | t_3 = 2t_1 + t_2, t_1 + 2t_2 + t_3 = N | t_1, t_2, t_1, t_2 | 1,1.2 | 1,2.3.4.6.7.8 |
| \( F_2^{+1} \) | o | t_1, t_2, t_1, t_3 | 1,1.2 | 1,2.3.4.6.7.8 |

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Table 2: Reconstruction from the correlations for four blocks, $\|\Delta A_F(\sigma)\|_1 = 8$

| multiset | par | further conditions | block lengths | Example | $t_i$ in Example |
|----------|-----|-------------------|--------------|---------|-----------------|
| $t_1 t_2$ | e   | $t_3 = 3t_1 + 3t_2$ | $t_1, t_2, t_1 + t_2, t_1 + 2t_2$ | 1,1,2,3 | 1,6 |
| $t_1 t_2$ | e   | $t_3 = 4t_1 + 2t_2$ | $t_1, t_2, 2t_1 + t_2, t_1 + t_2$ | 1,2,3,5 | 1,2,9,10 |

Conversely, the contribution $M_1 M_3$, coming from single blocks respectively triples of blocks, equals $\prod_{i=1,\ldots,N-1} \Delta A_F(\sigma)_i$. We write the multiset $M_1 M_2 M_3$ in the form $\tilde{t}_1 \cdot \ldots \cdot \tilde{t}_{12}$, with $\tilde{t}_i \leq \tilde{t}_{i+1}$ and $\tilde{t}_{13-i} = N - \tilde{t}_i$. Using a permutation $\rho \in S_4$ so that $m_{\rho(1)} \leq m_{\rho(2)} \leq m_{\rho(3)} \leq m_{\rho(4)}$, $m_{\rho(1)} = \tilde{t}_1$ and $m_{\rho(2)} = \tilde{t}_2$. So in particular $\tilde{t}_1$ and $\tilde{t}_2$ are in $M_1$. By a suitable choice of $\rho$ in case of degeneracy, $\rho(1)$ and $\rho(2)$ are neighbouring indices iff $\tilde{t}_1 + \tilde{t}_2 \in M_2$.

- Assuming this, $m_{\rho(1)} + m_{\rho(2)} \in \{\tilde{t}_3, \tilde{t}_4, \tilde{t}_5\}$. If $m_{\rho(1)} + m_{\rho(2)} = \tilde{t}_3$, then $m_{\rho(3)} = \tilde{t}_4$, and otherwise $m_{\rho(3)} = \tilde{t}_3$.

If $m_{\rho(i)} + m_{\rho(3)} \in M_2$, then $\rho(i)$ and $\rho(3)$ are neighbouring indices ($i = 1, 2$), and by assumption (6.5) exactly one of these alternatives is true.

In both cases, $m_{\rho(4)} = N - m_{\rho(1)} - m_{\rho(2)} - m_{\rho(3)}$.

- Assuming instead that $m_{\rho(1)} + m_{\rho(2)} \notin M_2$, $m_{\rho(3)} = \tilde{t}_3$. Then $M_2$ contains $\tilde{t}_1 + \tilde{t}_3$ and $\tilde{t}_2 + \tilde{t}_3$, the remaining two elements of $M_2$ being of the form $\tilde{t}_1 + m_{\rho(4)} \leq \tilde{t}_2 + m_{\rho(4)}$. By subtracting $\tilde{t}_1$ from $\tilde{t}_1 + m_{\rho(4)}$, we get $m_{\rho(4)}$.

In any case we identified the sequence $(m_1, m_2, m_3, m_4)$ of block lengths, up to the action of the dihedral subgroup $D_4$ of $S_4$.

- **Four blocks, strict inequality in (6.3):** By Lemma 6.4 we are left to consider the values 12 and 8 of $\|\Delta A_F(\sigma)\|_1$. The forms of signed multisets leading to these values are listed in the first column of Table 1 resp. 2. Because their exponents are palindromic, we list only the first half, and (in Column 2) the information whether $i_{\max}$ is even (parity e) or odd (parity o).

In many cases knowledge of the exponents $c_i$ does not suffice to reconstruct the block lengths up to symmetry and thus to show equality of stable and symmetry-induced degeneracy. In these cases the mutually exclusive further conditions in Column 3 allow decoding of the sequence $(m_1, m_2, m_3, m_4)$ of block lengths (Column 4). Finally Column 5 gives examples of realizations $(m_1, m_2, m_3, m_4)$, and their signed multisets (Column 6).
A more conceptual proof would be welcome, since it could help to find general upper bounds for stable degeneracies and to verify Conjecture (2.4).

For some $\sigma \in G_F$ with $F = \mathbb{Z}/N\mathbb{Z}$ their $F$–orbit (see (2.1)) can be reconstructed from the correlation $A_F(\sigma)$ (so that in particular $D_{\text{stab}}(\sigma) = D_{\text{sym}}(\sigma)$). There is an analog of the main theorem of Ginzburg and Rudnick in [GR] valid for Ising chains.

6.6 Proposition (Reconstruction from the correlation) Assume that for $\sigma$ of the form (6.1) the map $\mathcal{P}(\{1, \ldots, 2k\}) \to \mathbb{N}$, $I \mapsto \sum_{i \in I} m_i$ is injective. Then given $\Delta A_F(\sigma)$, the orbit $\Phi(S,\sigma)$ of $\sigma$ can be calculated.

Proof: • We can assume w.l.o.g. that $\sigma \neq \pm \mathcal{I}_F$, that is, $k > 0$.
• If $\Delta A_F(\sigma') = \Delta A_F(\sigma)$ for a $\sigma' \in G_F$, then by Remark 6.3.1 the number of blocks of $\sigma'$ equals $2k$, too. We assume (by applying the $F$ action, if necessary) that $\sigma'$ is of the form (6.1), that is

$$\sigma' = (1)^{m_1'}(-1)^{m_2'} \cdots (1)^{m_{2k-1}'}(-1)^{m_{2k}'}.$$ 

• The next task is to show that $(m'_1, \ldots, m'_{2k})$ is a permutation of $(m_1, \ldots, m_{2k})$. We identify $(\ell, n) \in X := \{1, \ldots, 2k\} \times \{1, \ldots, 2k - 1\}$ with the subinterval $\{\ell, \ldots, \ell + n - 1\} \subset \mathbb{Z}/(2k\mathbb{Z})$. Then there is a unique bijection

$$B : X \to X \text{ with } \sum_{i \in \{\ell, \ldots, \ell + n - 1\}} m_i = \sum_{i \in B(\ell, \ldots, \ell + n - 1)} m'_i \quad ((\ell, n) \in X). \quad (6.6)$$

Let for $\alpha, \beta \in S_{2k}$ the sequences $(m_{\alpha(1)}, \ldots, m_{\alpha(2k)})$ and $(m'_{\beta(1)}, \ldots, m'_{\beta(2k)})$ be weakly ascending. By the assumption of the proposition then $(m_{\alpha(1)}, \ldots, m_{\alpha(2k)})$ and thus also $(m'_{\beta(1)}, \ldots, m'_{\beta(2k)})$ are strictly ascending. Then $m_{\alpha(1)} = m'_{\beta(1)}$. Contradicting our hypothesis that the sets $(m'_1, \ldots, m'_{2k})$ and $(m_1, \ldots, m_{2k})$ coincide, let now $\ell$ be the first integer so that $m_{\alpha(\ell)} \neq m'_{\beta(\ell)}$. Then $m_{\alpha(\ell)}$ is the sum of at least two integers $m'_i$ whose indices $i$ are of the form $\beta(j)$ with $j < \ell$. So $m_{\alpha(\ell)} = \sum_{i \in I} m_{\alpha(i)}$ with index set $I \subseteq \{1, \ldots, \ell - 1\}$. But this contradicts the assumption of the proposition, proving $m'_{\beta(\ell)} = m_{\alpha(\ell)}$ or $m'_{\gamma(\ell)} = m_\ell \quad (\ell = 1, \ldots, 2k)$ with $\gamma := \beta \circ \alpha^{-1} \in S_{2k}$.

• Finally we show that $\sigma' \in \Phi(S,\sigma)$. By again applying the $F$ action, if necessary, we can assume that $\gamma(1) = 1$, so that $m'_1 = m_1$. For all $i \in \{1, \ldots, 2k\}$ the interval $\{i, i + 1\}$ is mapped by $B$ onto an interval which is also of length two, and which is moreover of the form $\{\gamma(i), \gamma(i + 1)\}$, since otherwise (6.6) would contradict the assumption of the proposition.

However this means that $\gamma$ belongs to the dihedral subgroup $D_{2k}$ of the permutation group $S_{2k}$. □
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