Application of Conformal Gauge Theories
Derived from Field-String Duality

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Abstract. In this article I first give an abbreviated history of string theory and then describe the recently-conjectured field-string duality. This suggests a class of nonsupersymmetric gauge theories which are conformal (CGT) to leading order of 1/N and some of which may be conformal for finite N. These models are very rigid since the gauge group representations of not only the chiral fermions but also the Higgs scalars are prescribed by the construction. If the standard model becomes conformal at TeV scales the GUT hierarchy is nullified, and model-building on this basis is an interesting direction. Some comments are added about the dual relationship to gravity which is absent in the CGT description.

1. Abbreviated History of String Theory

The recent development of field string duality possesses some quality of déja vu and yet seems the most promising development in the theory in terms of its most optimistic prognosis that it may provide a successful connection between string theory and the real world, and in doing so necessarily a first connection between gravity and the other interactions.

The initial seed of string theory was the Veneziano model in 1968[1]. At the time, finite energy and superconvergence sum rules for hadron scattering (the subject of my DPhil thesis) posed a "duality" of descriptions generally similar to the now-proposed one between a ten-dimensional superstring (or 11 dimensional M theory) and a conformal gauge theory. The

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hadron sum rules equated quantities of quite different functional dependences on the Mandelstam variables $s$ and $t$, seemingly an impossibility until the Veneziano model showed an explicit realization.

From 1968 to 1973 the resultant dual resonance models were leading candidates to describe the strong interactions. In 1973 they were, however, elbowed aside by an alternative theory, quantum chromodynamics (QCD). Now the discarded theory is dual to the QCD which replaced it. This is what I mean by *déjà vu*.

The decade 1974-1984 saw a hiatus in string theory. In 1984-85 the First Superstring Revolution included a stab at nearly a Theory of Everything, the perturbative $E(8) \times E(8)$ heterotic string. But its apparent uniqueness turned out to be illusory (as was its perturbativity), and another decade 1985-95 of quiescence followed.

In 1995 came the Second Superstring evolution, and in 1997 the 2\(\frac{1}{2}\) revolution with AdS/CFT duality. Understanding of duality between weak and strong coupling of supersymmetric field theories led to a corresponding breakthrough in string theory culminating with the idea of M theory as a more basic theory which unified all of the five known ten-dimensional superstrings (Types I, IIA, IIB and the $O_{32}$ and $E_8 \times E_8$ heterotic strings) as well as eleven-dimensional supergravity by duality transformations.

One of the most important realizations of the recent period is that string solitons, or D branes, play a dynamical role in the theory equally as important as do the superstrings themselves. D branes are crucial for the field-string duality which is our principal subject. The string duality has also led to a better understanding of the quantum mechanics of black holes.

Our starting point here is the duality between string theory in 10 dimensions (or of M theory in 11 dimensions) and gauge field theory in four dimensions. As already mentioned, this in a real sense closes a 25-year cycle in the history of strings.

Certainly one of the major changes in string theory in the recent years is the appreciation of the role of D branes which are topological defects on which open strings can end. Their necessity in string theory was realized only in 1989 and particularly in 1995. Their presence follows from considering the $R \rightarrow 0$ limit of a bosonic string compactified on a circle of radius $R$. Open strings, unlike the closed strings which are necessarily contained in the same theory, cannot wrap around the compactified dimension in the $R \rightarrow 0$ limit. Hence for a consistent theory the open strings do not simply end, but are attached to D branes.

These D branes have their own dynamics and play a central role in the full non-perturbative theory. D branes have provided insight into (1) Black hole quantum mechanics; (2) Large N gauge field theory (discussed here).
In general, the term duality applies to a situation where two quite different descriptions are available for the same physics.

The difference can be very striking. For example, in 1997 Maldacena proposed the duality between $d = 4$ $SU(N)$ gauge field theory (GFT) and a $d = 10$ superstring. In the perturbative regime of the GFT this duality cannot hold just because the degrees of freedom are missing, but non-perturbatively the GFT contains sufficient additional states at strong coupling for the duality to be indeed possible.

Take a Type IIB superstring (closed, chiral) in $d = 10$ and compactify it on the manifold:

$$\text{(AdS)}_5 \times S^5$$

Here $(\text{AdS})_5$ is a 5-dimensional Anti De Sitter space whose four dimensional surface $M_4$ is the $d = 4$ spacetime in which the $SU(N)$ GFT occurs. Note that the isometry group of $(\text{AdS})_5$ is $SO(4,2)$, the conformal group for four dimensional spacetime. The $S^5$ is a five-sphere with isometry $SU(4)$ which is the R symmetry of the resultant $\mathcal{N} = 4$ supersymmetric $SU(N)$ GFT. The $S^5$ can be regarded as a surface in a $C_3$ three-dimensional space in which $N$ D3 branes are coincident.

The D branes each carry an associated $U(1)$ gauge symmetry. This is understandable as a correct generalization of the Chan-Paton factors which were once used to attach charges to the ends of open strings. $N$ parallel D branes with vanishing separation yield a $U(N)$ gauge group where the additional $N^2 - N$ gauge bosons arise from connecting open strings which become massless in the zero-length limit. This $U(N)$ turns out to be broken to $SU(N)$ by the brane dynamics. The resultant $\mathcal{N} = 4$ SUSY Yang-Mills theory is well-known to be a very well-behaved, finite field theory. It is conformally invariant even for finite $N$ with all RG $\beta$–functions (gauge, Yukawa and quartic Higgs) vanishing.

This perturbative finiteness was proved in 1983 by Mandelstam. The Maldacena conjecture is primarily aimed at the $N \to \infty$ limit with the ’t Hooft parameter of $N$ times the squared gauge coupling fixed, and makes no claim concerning conformality for finite $N$. But since the $\mathcal{N} = 4$ case is known to be conformal even for finite $N$ one is tempted to extend the conjecture to finite $N$ cases even where all supersymmetry is broken. In that case the standard model can be a part of a conformal nonsupersymmetric gauge theory where the $\beta$–functions become zero at a TeV scale. Then the coupling constants cease to run and there is no grand unification. This nullifies the gauge hierarchy problem between the weak scale and the GUT scale, and yet it is still possible to derive the correct electroweak mixing angle. In particular there is no reason to invoke low-energy supersymmetry either. Gravity is itself non-conformal (it necessitates the dimensionful Newton constant). We shall address this at the end of the article.
2. Breaking Supersymmetries.

To approach the real world one needs less supersymmetry than $\mathcal{N} = 4$, in fact the empirical data presently suggest no supersymmetry at all, $\mathcal{N} = 0$.

By factoring out a discrete group (we shall assume it is an abelian discrete group, only because that case has been most fully investigated; it is possible that a non-abelian discrete group can work as well) and composing the orbifold:

$$S^5/\Gamma$$

one may break $\mathcal{N} = 4$ supersymmetry to $\mathcal{N} = 2, 1$ or 0. Of special interest is the $\mathcal{N} = 0$ case.

We may take $\Gamma = \mathbb{Z}_p$ which identifies $p$ points in $C_3$.

The rule for breaking the $\mathcal{N} = 4$ supersymmetry is:

$$\Gamma \subset SU(2) \implies \mathcal{N} = 2 \quad (3)$$

$$\Gamma \subset SU(3) \implies \mathcal{N} = 1 \quad (4)$$

$$\Gamma \not\subset SU(3) \implies \mathcal{N} = 0 \quad (5)$$

In fact, to specify the embedding of $\Gamma = \mathbb{Z}_p$, we need to identify three integers $a_i = (a_1, a_2, a_3)$ such that the action of $\mathbb{Z}_p$ on $C_3$ is:

$$C_3 : (X_1, X_2, X_3) \overset{\mathbb{Z}_p}{\longrightarrow} (\alpha^{a_1}X_1, \alpha^{a_2}X_2, \alpha^{a_3}X_3) \quad (6)$$

with

$$\alpha = exp \left( \frac{2\pi i}{p} \right) \quad (7)$$

The scalar multiplet is in the 6 of $SU(4)$ R symmetry and is transformed by the $\mathbb{Z}_p$ transformation:

$$diag(\alpha^{a_1}, \alpha^{a_2}, \alpha^{a_3}, \alpha^{-a_1}, \alpha^{-a_2}, \alpha^{-a_3}) \quad (8)$$

together with the gauge transformation

$$diag(\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5) \times \alpha^i \quad (9)$$

for the different $SU(N)_i$ of the gauge group $SU(N)^p$.

What will be relevant are states invariant under a combination of these two transformations, as discussed in the next subsection.

If $a_1 + a_2 + a_3 = 0 \pmod{p}$ then the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & 4 \end{pmatrix} \quad (10)$$
is in $SU(3)$ and hence $\mathcal{N} \geq 1$ is unbroken and this condition must therefore be avoided if we want $\mathcal{N} = 0$.

If we examine the 4 of $SU(4)$, we find that the matter which is invariant under the combination of the $Z_p$ and an $SU(N)^p$ gauge transformation can be deduced similarly.

It is worth defining the spinor 4 explicitly by $A_q = (A_1, A_2, A_3, A_4)$ with the $A_q$, like the $a_i$, defined only mod $p$. Explicitly we may define $A_1 = A_1 + A_2$, $A_2 = A_2 + A_3$, $A_3 = A_3 + A_1$ and $A_4 = -(A_1 + A_2 + A_3)$. In other words, $A_1 = \frac{1}{2}(a_1 - a_2 + a_3)$, $A_2 = \frac{1}{2}(a_1 + a_2 - a_3)$, $A_3 = \frac{1}{2}(-a_1 + a_2 + a_3)$, and $A_4 = -\frac{1}{2}(a_1 + a_2 + a_3)$. To leave no unbroken supersymmetry we must obviously require that all $A_q$ are non-vanishing. In terms of the $a_i$ this condition which we shall impose is:

$$\sum_{i=1}^{i=3} \pm (a_i) \neq 0 \pmod{p} \quad (11)$$

The question at issue is whether the gauge theories derived in this way are conformal for finite $N$. What is known is that at leading order in $1/N$ the $\beta-$ functions vanish to all orders in perturbation theory\cite{13}. This is already remarkable from the field theory point of view because without the stimulus of the AdS/CFT duality it would be difficult to guess any $\mathcal{N} = 0$ theory with all $\beta-$ functions zero to leading order in $1/N$ and all orders in the GFT coupling. Without non-renormalization theorems this imposes an infinite number of constraints on a finite number of choices of the fermion and scalar representations of $SU(N)^p$.

Nevertheless, since $\mathcal{N} = 4$ is conformal (all $\beta-$ functions vanish) we can be more ambitious and ask\cite{11} that all $\beta-$ functions vanish even for finite $N$, at least for some fixed point in coupling constant space, and use the construction to motivate phenomenological model-building.

3. Matter Representations.

The $Z_p$ group identifies $p$ points in $C_3$. The $N$ converging D branes approach all $p$ such points giving a gauge group with $p$ factors:

$$SU(N) \times SU(N) \times SU(N) \times \ldots \times SU(N) \quad (12)$$

The matter which survives is invariant under a product of a gauge transformation and a $Z_p$ transformation.

For the covering gauge group $SU(pN)$, the transformation is:

$$(1, 1, \ldots, 1; \alpha, \alpha, \ldots \alpha; \alpha^2, \alpha^2, \ldots \alpha^2; \ldots; \alpha^{p-1}, \alpha^{p-1}, \ldots \alpha^{p-1}) \quad (13)$$
with each entry occurring $N$ times.

Under the $Z_p$ transformation for the scalar fields, the $6$ of $SU(4)$, the transformation is

$$\sim X \Rightarrow (\alpha^{a_1}, \alpha^{a_2}, \alpha^{a_3})$$  \hspace{1cm} (14)

The result can conveniently be summarized by a quiver diagram. One draws $p$ points and for each $a_k$ one draws a non-directed arrow between all modes $i$ and $i + a_k$. Each arrow denotes a bi-fundamental representation such that the resultant scalar representation is:

$$\sum_{k=1}^{p} \sum_{i=1}^{i+p} (N_i, \bar{N}_{i+a_k})$$  \hspace{1cm} (15)

If $a_k = 0$ the bifundamental is to be reinterpreted as an adjoint representation plus a singlet representation.

For the chiral fermions one must construct the spinor $4$ of $SU(4)$. The components are the $A_q$ given above. The resultant fermion representation follows from a different quiver diagram. One draws $p$ points and connects with a directed arrow the node $i$ to the node $i + A_q$. The fermion representation is then:

$$\sum_{q=4}^{i+p} \sum_{i=1}^{i+p} (N_i, \bar{N}_{i+A_q})$$  \hspace{1cm} (16)

Since all $A_q \neq 0$, there are no adjoint representations for fermions. This completes the matter representation of $SU(N)^p$.

4. Two-Loop $\beta$- Functions.

We know that if $\Gamma$ is absent the resultant $N = 4$ SUSY SU(N) GFT has $\beta_g = \beta_Y = \beta_H = 0$ to all orders of perturbation theory.

When supersymmetries are broken one must check in more detail:

$$\beta_g = \beta_g^{(1)} + \beta_g^{(2)}$$  \hspace{1cm} (17)

where

$$\beta_g^{(1)} = -\frac{g^3}{(4\pi)^2} \left[ \frac{11}{3} C_2(G) - \frac{4}{3} \kappa S_2(F) - \frac{1}{6} S_2(S) \right]$$  \hspace{1cm} (18)

Here the quadratic Casimir is $C_2(G) = N$. The Dynkin indices are $S_2(F) = 4N$ and $S_2(S) = 6N$ for the fermion ($\kappa = 1/2$ for Weyl spinors) and scalar representations respectively.

Thus $\beta_g^{(1)} = 0$.  

The general expression for \( \beta_g^{(2)} \) has six terms. See [15]. The 1st, 3rd and 5th of the six terms are the same in all theories, namely:

\[
\frac{34N^2}{3} - \frac{40N^2}{3} - 2N^2 = -4N^2
\]  

(19)

In the 2nd and 4th terms there is an implicit sum over irreducible representations. In the 6th term are Yukawa couplings \( Y_4(F) \) which are included in this order because \( Y_4(F) \sim g^2 \).

It is amusing to obtain an idea of how many candidates there are for \( \mathcal{N} = 0 \) \( d = 4 \) conformal theories following these rules.

Each \( a_i \) can, without loss of generality, be in the range \( 0 \leq a_i \leq p - 1 \). Further we may set \( a_1 \leq a_2 \leq a_3 \) since permutations of the \( a_i \) are equivalent. Let us define \( \nu_k(p) \) to be the number of possible \( \mathcal{N} = 0 \) theories with \( k \) non-zero \( a_i \) \( (1 \leq k \leq 3) \).

Since \( a_i = (0, 0, a_3) \) is clearly equivalent to \( a_i = (0, 0, p - a_3) \) the value of \( \nu_1(p) \) is

\[
\nu_1(p) = \lfloor p/2 \rfloor
\]  

(20)

where \( \lfloor x \rfloor \) is the largest integer not greater than \( x \).

For \( \nu_2(p) \) we observe that \( a_i = (0, a_2, a_3) \) is equivalent to \( a_i = (0, p - a_3, p - a_2) \). Then we may derive, taking into account Eq.(11), that, for \( p \) even

\[
\nu_2(p) = 2 \sum_{r=1}^{\lfloor (p-2)/2 \rfloor} r = \frac{1}{4} p(p - 2)
\]  

(21)

while, for \( p \) odd

\[
\nu_2(p) = 2 \sum_{r=1}^{\lfloor p/2 \rfloor} r + \lfloor p/2 \rfloor = \frac{1}{4} (p - 1)^2
\]  

(22)

For \( \nu_3(p) \), the counting is only slightly more intricate. There is the equivalence of \( a_i = (a_1, a_2, a_3) \) with \( (p - a_3, p - a_2, p - a_1) \) as well as Eq.(11) to contend with.

In particular the theory \( a_i = (a_1, p/2, p - a_1) \) is a self-equivalent (SE) one; let the number of such theories be \( \nu_{SE}(p) \). Then it can be seen that \( \nu_{SE}(p) = p/2 \) for \( p \) even, and \( \nu_{SE}(p) = 0 \) for \( p \) odd. With regard to Eq.(11), let \( \nu_p(p) \) be the number of theories with \( \sum a_i = p \) and \( \nu_{2p}(p) \) be the number with \( \sum a_i = 2p \). Then because of the equivalence of \( (a_1, a_2, a_3) \)
with \((p - a_3, p - a_2, p - a_1)\), it follows that \(\nu_p(p) = \nu_{2p}(p)\). The value will be calculated below; in terms of it \(\nu_3(p)\) is given by

\[
\nu_3(p) = \frac{1}{2} [\bar{\nu}(p) - 2\nu_p(p) + \nu_{SE}(p)]
\]  

(23)

where \(\bar{\nu}(p)\) is the number of unrestricted \((a_1, a_2, a_3)\) satisfying \(1 \leq a_i \leq (p - 1)\) and \(a_1 \leq a_2 \leq a_3\). Its value is given by

\[
\bar{\nu}(p) = \frac{p - 1}{6} \left( \sum_{a_1=1}^{p-1} a_2 \right) = \frac{1}{6} p(p^2 - 1)
\]  

(24)

It remains only to calculate \(\nu_p(p)\) given by

\[
\nu_p(p) = \sum_{a_1=1}^{\left\lfloor \frac{p}{3} \right\rfloor} \left( \left\lfloor \frac{p - a_1}{2} \right\rfloor - a_1 + 1 \right)
\]  

(25)

The value of \(\nu_p(p)\) depends on the remainder when \(p\) is divided by 6. To show one case in detail consider \(p = 6k\) where \(k\) is an integer. Then

\[
\nu_p(p) = \sum_{a_1=odd}^{2k-1} \left( 3k + \frac{1}{2} - \frac{3a_1}{2} \right) + \sum_{a_1=even}^{2k} \left( 3k + \frac{3a_1}{2} + 1 \right) = 3k^2 = \frac{1}{12} p^2
\]  

(26)

Hence from Eq.(23)

\[
\nu_3(p) = \frac{1}{2} \left[ \frac{1}{6} p(p^2 - 1) - \frac{1}{6} p^2 + \frac{p}{2} \right] = \frac{p}{12} (p^2 - p + 2)
\]  

(27)

Taking \(\nu_1(p)\) from Eq.(20) and \(\nu_2(p)\) from Eq.(22) we find for \(p = 6k\)

\[
\nu_{TOTAL}(p) = \nu_1(p) + \nu_2(p) + \nu_3(p) = \frac{P}{12} (p^2 + 2p + 2)
\]  

(28)

For \(p = 6k + 1\) or \(p = 6k + 5\) one finds similarly

\[
\nu_3(p) = \frac{1}{12} (p - 1)^2 (p + 1) \quad (p = 6k + 1 \ or \ 6k + 5)
\]  

(29)

\[
\nu_{TOTAL} = \frac{1}{12} (p - 1)(p + 1)(p + 2) \quad (p = 6k + 1 \ or \ 6k + 5)
\]  

(30)

For \(p = 6k + 2\) or \(p = 6k + 4\)

\[
\nu_3(p) = \frac{1}{12} (p + 1)(p^2 - 2p + 4) \quad (p = 6k + 2 \ or \ 6k + 4)
\]  

(31)

\[
\nu_{TOTAL} = \frac{1}{12} (p^3 + 2p^2 + 2p + 4) \quad (p = 6k + 2 \ or \ 6k + 4)
\]  

(32)
and finally for \( p = 6k + 3 \)

\[
\nu_1(p) = \frac{1}{12}(p^3 - p^2 - p - 3) \quad (p = 6k + 3) \tag{33}
\]

\[
\nu_{\text{TOTAL}} = \frac{1}{12}(p^3 + 2p^2 - p - 6) \quad (p = 6k + 3) \tag{34}
\]

The values of \( \nu_1(p), \nu_2(p), \nu_3(p), \nu_{\text{TOTAL}}(p) \) and \( \sum_{p'=2}^{p} \nu_{\text{TOTAL}}(p') \) for \( 2 \leq p \leq 41 \) are listed in Table 1.

Table 1 (next page) gives values of \( \nu_1(p), \nu_2(p), \nu_3(p), \nu_{\text{TOTAL}}(p), \sum_{p'=1}^{p} \nu_{\text{TOTAL}}(p'), \nu_{\text{alive}}(p) \) and \( \sum_{p'=2}^{p} \nu_{\text{alive}}(p') \) for \( 2 \leq p \leq 41 \).

The next question is: of all these candidates for conformal \( \mathcal{N} = 0 \) theories, how many if any are conformal? As a first sifting we can apply the criterion found in [11] from vanishing of the two-loop RGE \( \beta \)-function, \( \beta_\alpha^{(2)} = 0 \), for the gauge coupling. The criterion is that \( a_1 + a_2 = a_3 \). Let us denote the number of theories fulfilling this by \( \nu_{\text{alive}}(p) \).

If \( p \) is odd there is no contamination by self-equivalent possibilities and the result is

\[
\nu_{\text{alive}} = \sum_{r=1}^{\frac{p-1}{2}} (p - 2r) = \frac{1}{4}(p - 1)^2 \quad (p = \text{odd}) \tag{35}
\]

For \( p \) even some self equivalent cases must be subtracted. The sum in Eq.\((35)\) is \( \frac{1}{4}p(p - 2) \) and the number of self-equivalent cases to remove is \( \lfloor p/4 \rfloor \) with the results

\[
\nu_{\text{alive}} = \frac{1}{4}p(p - 3) \quad (p = 4k) \tag{36}
\]

\[
\nu_{\text{alive}} = \frac{1}{4}(p - 1)(p - 2) \quad (p = 4k + 2) \tag{37}
\]

In the last two columns of Table 1 are the values of \( \nu_{\text{alive}}(p) \) and \( \sum_{p'=2}^{p} \nu_{\text{alive}}(p') \).

Asymptotically for large \( p \) the ratio \( \nu_{\text{alive}}(p)/\nu_{\text{TOTAL}}(p) \sim 3/p \) and hence vanishes although \( \nu_{\text{alive}}(p) \) diverges; the value of the ratio is \( \approx 0.28 \) at \( p = 5 \) and at \( p = 41 \) is 0.066. It is being studied how the two-loop requirements \( \beta_\alpha^{(2)} = 0 \) and \( \beta_\beta^{(2)} = 0 \) select from such theories. That result will further indicate whether any \( \nu_{\text{alive}}(p) \) can survive to all orders.

5. Directions.

We have begun the selection process by looking at one and two loop. At one loop we are still at leading order in \( N \) at least for \( \beta_\alpha \) so there is coincidence with the \( \mathcal{N} = 4 \) case. At 2 loops we found already that only 8%
of a sample satisfy one criterion, the fraction remaining alive diminishing like $3/p$ for large $p$.

Checking the Yukawa and Higgs running for 2 loops needs more calculation of couplings and is underway.

Beyond that:

- If all 2-loop tests are satisfied, what about 3 or more? It rapidly becomes impractical to take the approach of direct calculation.
- There is the question of uniqueness of any surviving $\mathcal{N} = 0$ CGT.
- The CGT may be inspirational in model building, to be discussed below.

*Why $\mathcal{N} = 0$?*

$\mathcal{N} = 1$ is motivated by accommodation of the gauge hierarchy $M_{\text{GUT}}/M_{\text{Weak}}$.

In a conformal gauge theory the gauge couplings cease to run and the GUT scale does not exist; this hierarchy is therefore nullified.

Low-scale Kaluza-Klein is similar to the conformality approach in this particular regard, although the idea is quite different.

More philosophically, we may recall the over 50 years ago the infinite renormalization of QED was greeted with much skepticism. If the conformality of even $\mathcal{N} = 4$ CFT had been already discovered, surely the skepticism would have been far greater?
Table 1

| p   | ν1(p) | ν2(p) | ν3(p) | νTOTAL(p) | \( \sum \nu_{TOTAL} \) | \( \sum \nu_{alive} \) | \( \sum \nu_{alive} \) |
|-----|-------|-------|-------|-----------|----------------|----------------|----------------|
| 2   | 1     | 0     | 1     | 2         | 2             | 0              | 0              |
| 3   | 1     | 1     | 1     | 3         | 5             | 1              | 1              |
| 4   | 2     | 2     | 5     | 9         | 14            | 1              | 2              |
| 5   | 2     | 4     | 8     | 14        | 28            | 4              | 6              |
| 6   | 3     | 6     | 16    | 25        | 53            | 5              | 11             |
| 7   | 3     | 9     | 24    | 36        | 89            | 9              | 20             |
| 8   | 4     | 12    | 39    | 55        | 144           | 10             | 30             |
| 9   | 4     | 16    | 53    | 73        | 217           | 16             | 46             |
| 10  | 5     | 20    | 77    | 102       | 319           | 18             | 64             |
| 11  | 5     | 25    | 100   | 130       | 449           | 25             | 89             |
| 12  | 6     | 30    | 134   | 170       | 619           | 27             | 116            |
| 13  | 6     | 36    | 168   | 210       | 829           | 36             | 152            |
| 14  | 7     | 42    | 215   | 264       | 1093          | 39             | 191            |
| 15  | 7     | 49    | 261   | 317       | 1410          | 49             | 240            |
| 16  | 8     | 56    | 323   | 387       | 1797          | 52             | 292            |
| 17  | 8     | 64    | 384   | 456       | 2253          | 64             | 356            |
| 18  | 9     | 72    | 462   | 543       | 2796          | 68             | 424            |
| 19  | 9     | 81    | 540   | 630       | 3426          | 81             | 505            |
| 20  | 10    | 90    | 637   | 737       | 4163          | 85             | 590            |
| 21  | 10    | 100   | 733   | 843       | 5006          | 100            | 690            |
| 22  | 11    | 110   | 851   | 972       | 5978          | 105            | 795            |
| 23  | 11    | 121   | 968   | 1100      | 7078          | 121            | 916            |
| 24  | 12    | 132   | 1108  | 1252      | 8330          | 126            | 1042           |
| 25  | 12    | 144   | 1248  | 1404      | 9734          | 144            | 1186           |
| 26  | 13    | 156   | 1413  | 1582      | 11316         | 150            | 1336           |
| 27  | 13    | 169   | 1577  | 1759      | 13075         | 169            | 1505           |
| 28  | 14    | 182   | 1769  | 1965      | 15040         | 175            | 1680           |
| 29  | 14    | 196   | 1960  | 2170      | 17210         | 196            | 1876           |
| 30  | 15    | 210   | 2180  | 2405      | 19615         | 203            | 2079           |
| 31  | 15    | 225   | 2400  | 2640      | 22255         | 225            | 2304           |
| 32  | 16    | 240   | 2651  | 2907      | 25162         | 232            | 2536           |
| 33  | 16    | 256   | 2901  | 3173      | 28335         | 256            | 2792           |
| 34  | 17    | 272   | 3185  | 3474      | 31809         | 264            | 3056           |
| 35  | 17    | 289   | 3468  | 3774      | 35583         | 289            | 3345           |
| 36  | 18    | 306   | 3796  | 4110      | 39693         | 297            | 3642           |
| 37  | 18    | 324   | 4104  | 4446      | 44139         | 324            | 3966           |
| 38  | 19    | 342   | 4459  | 4820      | 48959         | 333            | 4299           |
| 39  | 19    | 361   | 4813  | 5193      | 54152         | 361            | 4660           |
| 40  | 20    | 380   | 5207  | 5607      | 59759         | 370            | 5030           |
| 41  | 20    | 400   | 5600  | 6020      | 65779         | 400            | 5430           |
6. Conformality and Particle Phenomenology.

Let us itemize the following points:

- The hierarchy between the GUT and weak scales is 14 orders of magnitude.
- Why do the two very different scales exist?
- How are the scales stabilized under quantum corrections?
- Supersymmetry solves the second problem but not the first.

*Successes of supersymmetry.*

- Cancellations of UV infinities.
- Technical naturalness of hierarchy.
- Unification of gauge couplings.
- Natural appearance in string theory.

*Puzzles about supersymmetry.*

- The “\( \mu \) problem” - why is the Higgs mass at the weak scale and not at the Planck scale (hierarchy).
- Breaking supersymmetry leads to too large a cosmological constant.
- Is supersymmetry fundamental to string theory?
- There are solutions of string theory without supersymmetry.

*Supersymmetry replaced by conformality at TeV scale.*

The following aspects of the idea are discussed:

- The idea is possible.
- Explicit examples containing standard model states.
- Finiteness as a more rigid constraint than supersymmetry.
- Predicts additional states for finiteness/conformality.
- Rich inter-family structure of Yukawa couplings.
7. Conformality as Hierarchy Solution.

The quark and lepton masses, the QCD scale and the weak scale are extremely small compared to a TeV scale. They may all be put to zero suggesting: add degrees of freedom to yield GFT with conformal invariance (CGT). ’t Hooft’s naturalness condition holds since zero mass increases the symmetry.

The theory is assumed to be given by the action\[17\]

\[ S = S_0 + \int d^4 x \alpha_i O_i \] (38)

where \( S_0 \) is the action for the conformal theory and the \( O_i \) are operators with dimension below four which break conformal invariance softly.

The mass parameters \( \alpha_i \) have mass dimension \( 4 - \Delta_i \) where \( \Delta_i \) is the dimension of \( O_i \) at the conformal point.

Let \( M \) be the scale set by the parameters \( \alpha_i \) and hence the scale at which conformal invariance is broken. Then for \( E \gg M \) the couplings will not run while they start running for \( E < M \). To solve the hierarchy problem we assume \( M \) is close to the TeV scale.

8. Large Class of d=4 QFTs - Each SU(4) Subgroup.

There is first the choice of \( N \). One knows that for leading \( 1/N \) the theory is conformal. What about finite \( N \)? One expects at least a conformal fixed point in some cases. One starts from \( \mathcal{N} = 4 \) GFT, eliminates fields and re-identifies others such that conformality results.

It is important to realize that, even without supersymmetry, boson-fermion number equality holds, and underlies the finiteness.

Let \( \Gamma \subset SU(4) \) denote a discrete subgroup of \( SU(4) \). Consider irreducible representations of \( \Gamma \). Suppose there are \( k \) irreducible representations \( R_i \), with dimensions \( d_i \) with \( i = 1, \ldots, k \). The gauge theory in question has gauge symmetry

\[ SU(N d_1) \times SU(N d_2) \times \ldots SU(N d_k) \]

The fermions in the theory are given as follows. Consider the 4 dimensional representation of \( \Gamma \) induced from its embedding in \( SU(4) \). It may or may not be an irreducible representation of \( \Gamma \). We consider the tensor product of \( 4 \) with the representations \( R_i \):

\[ 4 \otimes R_i = \oplus_j n^j_i R_j \]
The chiral fermions are in bifundamental representations

$$(1, 1, ..., N_{d_i}, 1, ..., N_{d_j}, 1, ...)$$

with multiplicity $n^j_i$ defined above. For $i = j$ the above is understood in
the sense that we obtain $n^i_i$ adjoint fields plus $n^i_i$ neutral fields of $SU(N_{d_i})$.
Note that we can equivalently view $n^j_i$ as the number of trivial representations in the tensor product

$$(4 \otimes R_i \otimes R_j^*)_{\text{trivial}} = n^j_i$$

The asymmetry between $i$ and $j$ is manifest in the above formula. Thus in general we have

$$n^j_i \neq n^i_j$$

and so the theory in question is in general a chiral theory. However if $\Gamma$
is a real subgroup of $SU(4)$, i.e. if $4 = 4^*$ as far as $\Gamma$ representations are
concerned, then we have by taking the complex conjugate:

$$n^j_i = (4 \otimes R_i \otimes R_j^*)_{\text{trivial}} = (4 \otimes R_i^* \otimes R_j)_{\text{trivial}} = n^i_j.$$

So the theory is chiral if and only if $4$ is a complex representation of $\Gamma$,
i.e. if and only if $4 \neq 4^*$ as a representation of $\Gamma$. If $\Gamma$ were a real subgroup
of $SU(4)$ then $n^j_i = n^i_j$.

If $\Gamma$ is a complex subgroup, the theory is chiral, but it is free of gauge
anomalies. To see this, note that the number of chiral fermions in the
fundamental representation of each group $SU(N_{d_i})$ plus $N_{d_i}$ times the
number of chiral fermions in the adjoint representation is given by

$$\sum_j n^j_i N_{d_j} = 4N_{d_i}$$

(where the number of adjoints is given by $n^i_i$). Similarly the number of
anti-fundamentals plus $N_{d_i}$ times the number of adjoints is given by

$$\sum_j n^j_i N_{d_j} = \sum N_{d_j}(4 \otimes R_j \otimes R_i^*)_{\text{trivial}} = \sum N_{d_j}(4^* \otimes R_j^* \otimes R_i)_{\text{trivial}} = 4N_{d_i}$$

Thus we see that the difference of the number of chiral fermions in the
fundamental and the anti-fundamental representation is zero (note that the
adjoint representation is real and does not contribute to anomaly). Thus
each gauge group is anomaly free.

In addition to fermions we also have bosons in bi-fundamental represen-
tations. The number of bosons $M^j_i$ in the bi-fundamental representation
of $SU(N_{d_i}) \otimes SU(N_{d_j})$ is given by the number of $R_j$ representations in
the tensor product of the representation $6$ of $SU(4)$ restricted to $\Gamma$ with the $R_i$ representation. Note that since $6$ is a real representation we have

$$M_i^j = (6 \otimes R_i \otimes R_j^*)_{\text{trivial}} = (6 \otimes R_i^* \otimes R_j)_{\text{trivial}} = M_j^i$$

In other words for each $M_i^j$ we have a complex scalar field in the corresponding bi-fundamental representation.

**Interactions.** The interactions of the gauge fields with the matter is fixed by the gauge coupling constants for each gauge group. The inverse coupling constant squared for the $i$-th group combined with the theta angle for the $i$-th gauge group is

$$\tau_i = \theta_i + \frac{i}{4\pi g_i^2} = \frac{d_i \tau}{|\Gamma|}$$

where $\tau = \theta + \frac{i}{4\pi g^2}$ is an arbitrary complex parameter independent of the gauge group and $|\Gamma|$ denotes the number of elements in $\Gamma$.

There are two other kinds of interactions: Yukawa interactions and quartic scalar field interactions. The Yukawa interactions are in 1-1 correspondence with triangles in the quiver diagram with two directed fermionic edges and one undirected scalar edge, with compatible directions of the fermionic edges:

$$S_{\text{Yukawa}} = \frac{1}{g^2} \sum_{\text{directed triangles}} d^{abc} \text{Tr} \psi^a_{ij} \phi^b_{jk} \psi^c_{ki}$$

where $a, b, c$ denote a degeneracy label of the corresponding fields. $d^{abc}$ are flavor dependent numbers determined by Clebsch-Gordon coefficients as follows: $a, b, c$ determine elements $u, v, w$ (the corresponding trivial representation) in $4 \otimes R_i \otimes R_j^*, 6 \otimes R_j^* \otimes R_k^*$ and $4 \otimes R_k \otimes R_i^*$. Then

$$d^{abc} = u \cdot v \cdot w$$

where the product on the right-hand side corresponds to contracting the corresponding representation indices for $R_m$'s with $R_m^*$'s as well as contracting the $(4 \otimes 6 \otimes 4)$ according to the unique $SU(4)$ trivial representation in this tensor product.

Similarly the quartic scalar interactions are in 1-1 correspondence with the 4-sided polygons in the quiver diagram, with each edge corresponding to an undirected line. We have

$$S_{\text{Quartic}} = \frac{1}{g^2} \sum_{4\text{-gons}} f^{abcd} \Phi^a_{ij} \Phi^b_{jk} \Phi^c_{kl} \Phi^d_{li}$$

where again the fields correspond to lines $a, b, c, d$ which in turn determine an element in the tensor products of the form $6 \otimes R_m \otimes R_n^*$. $f^{abcd}$ is
obtained by contraction of the corresponding element as in the case for Yukawa coupling and also using a $[\mu, \nu][\mu, \nu]$ contraction in the $6 \otimes 6 \otimes 6 \otimes 6$ part of the product.

*Conformal Theories in 4 Dimensions.* There follows a large list of quantum field theories in 4 dimensions, one for each discrete subgroup of $SU(4)$ and each choice of integer $N$, motivated from string theory considerations which has been proven to have vanishing beta function to leading order in $N$. Below we argue for the existence of at least one fixed point even for finite $N$ (under some technical assumptions). The vanishing of the beta function at large $N$ can also be argued using AdS/CFT correspondence.

Consider strong-weak duality. This duality exchanges $8\pi g^2 \leftrightarrow 1/8\pi g^2$ (at $\theta = 0$). This follows from their embedding in the type IIB string theory which enjoys the same symmetry. In fact, this gauge theory defines a particular type IIB string theory background and so this symmetry must be true for the gauge theory as well. In the leading order in $N$ the beta function vanishes. Let us assume at the next order there is a negative beta function, i.e., that we have an asymptotically free theory. Then the flow towards the infrared increases the value of the coupling constant. Similarly, by the strong-weak duality, the flow towards the infrared at large values of the coupling constant must decrease the value of the coupling constant. Therefore we conclude that the beta function must have at least one zero for a finite value of $g$.

This argument is not rigorous for three reasons: One is that we ignored the flow for the $\theta$ angle. This can be remedied by using the fact that the moduli space is the upper half-plane modulo $SL(2, Z)$ which gives rise to a sphere topology and using the fact that any vector field has a zero on the sphere ("it is impossible to comb the hair on a sphere"). The second reason is that we assumed asymptotic freedom at the first non-vanishing order in the large $N$ expansion. This can in principle be checked by perturbative techniques and at least it is not a far-fetched assumption. More serious, however, is the assumption that there is effectively one coupling constant. It would be interesting to see if one can relax this assumption, which is valid at large $N$.

9. Comments

The most exciting aspect of the conformality approach is in model building beyond the standard model. The reason the model building is so interesting is that not only the fermion but also the scalar representations are prescribed by the construction. Thus one may not simply add whatever Higgs scalars are required for the appropriate symmetry breaking. This rigidity is
actually helpful. Lack of adequate space precludes including details of the model building described in \[12, 13\]. Clearly a simple model would encourage support for this approach. The simplest model using abelian orbifolds and found in \[13\] is based on the gauge group \(SU(3)^7\). This has less generators than the \(E(6)\) gauge group and may therefore be of considerable interest. Non-abelian orbifolds are currently under examination.

The final issue concerns gravity. In the CGT for strong and electroweak interactions there is no manifest gravity. One may say there is no evidence for a graviton and that one is concerned only with observable physics. Nevertheless if one extrapolates to extremely high energy gravity should enter and it is \textit{not} conformally invariant because, of course, the Newton constant is dimensional. It would be attractive to understand the incorporation of gravitation while staying in only four spacetime dimensions but this possibility remains elusive. The CGT itself stands on its own without need of the string from which its construction was inferred. But to describe gravity the most promising idea seems to be to add an extra dimension and consider \((AdS)_5\). Keeping the full range of the fifth coordinate leads one back to the absence of gravity on the surface. But, as pointed out in \[19\], truncating the range of the fifth coordinate leads to a metric field on the surface and hence to a graviton.

As a final speculation, is it possible that conformality is related to the vanishing cosmological constant? Until conformal invariance is broken the vacuum energy is zero. It then depends on how softly conformal invariance can be broken if a \((TeV)^4\) contribution is to be avoided. Clearly, the breaking of conformal invariance needs to be studied, not only for this reason, but also to allow predictions for dimensionless quantities like mass ratios and mixing angles in the low-energy theory.

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