Integrability of twist-three evolution equations in QCD.

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Abstract

We describe a recent progress in finding solutions to three-particle evolution equations at leading order in the QCD coupling constant for multiparton correlation functions based on the integrability of corresponding interaction Hamiltonians.

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INTEGRABILITY OF TWIST-THREE EVOLUTION EQUATIONS IN QCD

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1 Multiparton correlations in hard scattering.

A typical lepton-hadron cross section at high momentum transfer \( Q \) is given by a series in the latter

\[
\sigma = \sum_{\tau = 2}^{\infty} \left( \frac{A}{Q} \right)^{\tau-2} \int \{ dx_{\tau} \} C \left( x, \{ x_{\tau} \} | \alpha_s \right) f \left( \{ x_{\tau} \}, Q^2 \right),
\]

where \( \tau \) stands for the twist of contributing operators which parametrize the physics at soft scales. The first term in the expansion (1) corresponds to the Feynman model picture. Consequent terms, standing for multi-parton correlations in hadron, reveal QCD dynamics not encoded into the conventional parton densities. They manifest genuine quantum mechanical effects being an interference of hadron wave functions with different number of particles, see Fig. [1]. Twist-three correlations are unique among other higher-twist effects by their feature to appear as a leading effect in certain spin asymmetries. The most familiar examples are the transverse spin structure function \( g_2 \) measured in deep inelastic scattering. If a cross section is measured in a rather wide range of hard momentum \( Q \), the logarithmic scaling violation in the functions \( f \) in Eq. (1) becomes an issue since hadron substructure is seen by a probe with different resolutions. In QCD it is formalized by means of renormalization group equations with kernels which are derived by standard methods of QCD perturbation theory (see e.g. [1-3]). The specifics of twist-three sector as compared to the twist-two case is that the afore mentioned structure functions being defined as Fourier transform (\( FT \)) of two-particle hadron-hadron matrix elements \( f \sim FT \langle h|\phi_1\phi_2|h \rangle \) receive contributions from three-parton
The first term on the r.h.s. of the equality stands for leading twist-two contributions. The second one (plus its complex conjugate), to the twist-three effects, etc.

Figure 1: Hadronic structure as a series in a number of Fock components participating in a hard rescattering. The quark-gluon correlators, plus a kinematical piece from twist-two operators going under the name of Wandzura-Wilczek part. The original operators, not possessing a definite twist, mix with two- as well as three-particle operators and thus evolution equation will have an extremely complicated form. Resolving this complication in favour of an independent study of the renormalization group evolutions of separate twist components one finds an autonomous equations for two- and three-particle sector, respectively. The twist-two sector is simple and is well-studied in the literature. It is the latter which is the subject of this presentation. In spite of simplifications due to reduction alluded to before, anyway, one ends up with a very complicated problem of working out the mixing of three-particle local operators.

2 Leading order evolution.

As we have stated above reducing the twist-three two-particle operators to the three-particle ones $\phi_1\phi_2\phi_3$ allows to easily find corresponding evolution equation since the latter operators fall into a class of the so-called quasi-partonic ones and thus have a bunch of remarkable properties which simplifies their renormalization properties. Namely, at leading order in the QCD coupling constant the total kernel $K_{123}$ reduces to the sum of conventional pair-wise twist-two non-forward kernels $K_{ab}$ which includes the momentum conservation delta function $\delta(k_a + k_b - k'_a - k'_b)$. Thus, the leading order evolution equation is of the form

$$\frac{d}{d\ln Q^2} F_{123} = -\frac{\alpha_s}{\pi} K_{123} * F_{123}, \quad \text{with} \quad K_{123} = K_{12} + K_{23} + K_{13}. \quad (2)$$

Here $F_{123} \equiv F(x_1, x_2, x_3)$ and $* \equiv \int dx_1 dx_2 dx_3 \delta (x_1 + x_2 + x_3 - \eta)$ and variable $\eta$ which encodes the skewedness of the matrix element: $\eta = 0$ for the forward scattering and $\eta = 1$ for totally exclusive kinematics. The solution to Eq. (2) is expected to be of the form

$$F_{123}(Q^2) = \sum_{\{\alpha\}} \Psi_{\{\alpha\}} 123 \left( \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{f_{\{\alpha\}} \mathcal{E}_{\{\alpha\}} / \beta_0} \mathcal{F}_{\{\alpha\}}(Q_0^2), \quad (3)$$
where $E_{\{\alpha\}}$ are the eigenvalues of evolution kernels (and $f_{(c)}$ is an extracted colour factor) and $\Psi_{\{\alpha\}}(x,x')$ are the corresponding eigenfunctions parametrized by a set quantum numbers $\{\alpha\}$. As usual $\beta_0 = \frac{4}{3}T_F N_f - \frac{44}{3}C_A$ is the leading term of the QCD $\beta$-function and $\langle\langle F_{\{\alpha\}}(Q_0^2)\rangle\rangle$ stand for reduced matrix elements of local operators at a low normalization point $Q_0$. Therefore the evolution equation (2) can be reduced to a stationary Schrödinger equation

$$K_{123} \Psi_{\{\alpha\}123} = f_{(c)} E_{\{\alpha\}} \Psi_{\{\alpha\}123}. \tag{4}$$

Simplifications in diagonalization of this equation occur due to use of QCD symmetries which we now use in turn.

### 3 Conformal symmetry.

The classical QCD Lagrangian enjoys the property of invariance under the 15-parameter group of conformal transformations which consists of the Lorentz group, $P_\mu$ and $M_{\mu\nu}$, and generators of dilatation $D$ and special conformal $K_{\mu}$ transformations. At quantum level the trace of the energy momentum tensor develops an anomaly $\Theta_{\mu\mu} \neq 0$ and the generators $D$ and $K_{\mu}$ cease to be symmetries of the theory. The invariance of Lagrangian under a given transformation imposes severe restrictions on the form of eigenfunctions of composite operators entering the evolution. For illustration purposes let us consider the situation with two-particle operators since they enter as a building block in $(ab)$-subchannel for our general consideration of three-particle problem. It was known for a long time that these are conformal operators which diagonalize the leading order renormalization group equation for local operators with total derivatives, e.g.

$$\phi_b(\partial_a + \partial_b)\left((\partial_a - \partial_b)^+/(\partial_a + \partial_b)^+\right)\phi_a,$$

where $P^{(\nu_b,\nu_a)}$ are Jacobi polynomials with $\nu_a = d_a + s_a - 1$, and $d_a$ and $s_a$ being dimension and spin of a field $\phi_a$. Since we deal with operators on the light-cone it is enough to consider a collinear sub-group of the conformal group, which consists of projections $P_+ = P_\mu n_\mu$, $M_{\pm} = M_{\mu\nu} n_\mu n_\nu$, $D$ and $K_- = K_{\mu} n_\mu$, forming an $so(2,1)$ algebra, $[J^3, J^\pm]_\pm = \pm J^\pm$, $[J^+, J^-]_- = -2J^3$, defined by generators of the momentum $J^+ = iP_+$, special conformal transformation $J^- = \frac{i}{2}K_-$ and a particular combination of dilatation and angular momentum $J^3 = \frac{1}{2}(D + M_{--})$. The Casimir operator is $J^2 = J^3(J^3 - 1) - J^+ J^-$. 


Simple calculation shows that the operators (5) transform covariantly under dilatation and special conformal transformation, namely,

\[ i\{\mathcal{O}_{jl}, \mathcal{P}_+\} = i\mathcal{O}_{jl+1}, \quad i\{\mathcal{O}_{jl}, \mathcal{M}_-\} = -(l + s_a + s_b)\mathcal{O}_{jl}, \]
\[ i\{\mathcal{O}_{jl}, \mathcal{D}\} = -(l + d_a + d_b)\mathcal{O}_{jl}, \quad i\{\mathcal{O}_{jl}, \mathcal{K}_-\} = ia(j,l)\mathcal{O}_{jl-1}, \quad (6) \]

with \(a_{jk}(l) = \delta_{jk} \cdot a(j,l)\) and \(a(j,l) = 2(j - l)(j + l + \nu_a + \nu_b)\). From these equations one easily sees the meaning for \(J^\pm\) to be step-up and -down operators as \(\mathcal{P}_+\) adds one unit of spin to the operator in a conformal tower while \(\mathcal{K}_-\) reduces it by one. These operators possess conformal spin \(J^{ab} = j + \frac{1}{2}(\nu_a + \nu_b + 2)\), \([\mathcal{J}_2^{ab}, \mathcal{O}_{jk}]_\pm = J_{ab}(J_{ab} - 1)\mathcal{O}_{jk}\), spin \(l + s_a + s_b\) and scale dimension \(l + d_a + d_b\). Thus conformal operators form an irreducible infinite dimensional representation of the collinear conformal group spanned by bilocale operators.

The renormalization group equation for conformal operators have the form

\[ \frac{d}{d\ln Q} \mathcal{O}_{jl} = -\sum_{k=0}^j \gamma_{jk} \mathcal{O}_{kl}, \quad (7) \]

with triangular matrix of anomalous dimensions \(\gamma_{jk}, \quad j \geq k\). To find a constraint on the form of the anomalous dimension matrix of \(\mathcal{O}_{jk}\) stemmed from the special conformal symmetry, we use Ward identities for \(\mathcal{D}\) and \(\mathcal{K}_-\) transformations. Since the dilatation WI is equivalent to the renormalization group equation (7) we get an equation on \(\gamma_{jk}\). Explicit analysis gives \(7\gamma_{jj'}a_{jj'}(l) - a_{jj'}(l)\gamma_{jj'} = 0, \quad (8)\)

at leading order in coupling constant, \(\gamma \sim \mathcal{O}(\alpha_s)\). Since the matrix \(a\) in (6) is diagonal it forces \(\gamma\) to be diagonal as well at lowest order in strong constant, \(\gamma_{jk} = \gamma_{jj}\delta_{jk}\). Beyond this order conformal symmetry is violated and the diagonal matrix \(a\) is promoted to a non-diagonal one \(a_{jk} \rightarrow a_{jk} + \gamma^e + 2\frac{\delta}{\xi}b\), with new objects which are special conformal anomaly \(\gamma^e\) appearing from the renormalization of the trace anomaly and a conformal operator \(\mathcal{O}_{jl} \int dx x^- \Theta_{\mu \nu} \propto \sum_{k=0}^j \gamma_{jk}^e \mathcal{O}_{kl}\), and an \(\alpha_s\)-independent matrix \(b\). These extra terms induce the non-diagonal part of the anomalous dimensions beyond leading order.

Therefore, the net product of analysis is that due to preservation of conformal invariance for the leading order anomalous dimensions (pair-wise kernels of the previous section 2), one immediately concludes that the former are the function of eigenvalues of the Casimir operator \(\mathcal{J}^2\). Thus the interaction kernels can be expressed as functions of Casimir operator in a given subchannel, e.g. \(K_{ab} = h(\mathcal{J}_2^{ab})\).
Let us give an explicit example of a basis, convenient for present studies. Choose a space \( V = \{ \theta^k | k = 0, 1, \ldots, \infty \} \) spanned by elements \( \theta^k \equiv \partial_{x^k} \theta \Gamma(k + \nu + 1) \). In this representation \( [\hat{J}^{\pm,3}, \chi(\theta)] = \hat{J}^{\pm,3} \chi(\theta) \) the generators take the following form

\[
\hat{J}^+ = (\nu + 1) \theta + \theta^2 \partial_\theta, \quad \hat{J}^- = \partial_\theta, \quad \hat{J}^3 = \frac{1}{2} (\nu + 1) + \theta \partial_\theta,
\]

where \( \partial_\theta \equiv \frac{\partial}{\partial \theta} \) and the quadratic Casimir operator reads \( \hat{J}^2 = \hat{J}^3 (\hat{J}^3 - 1) - \hat{J}^+ \hat{J}^- \). The advantage of this basis lies in the fact that conformal operator has the form of translation, so that the highest weight vector condition is easier to solve. For a multi-variable function \( \chi(\theta) \equiv \chi(\theta_1, \theta_2, \ldots, \theta_n) \) the operators are defined by the sum of single particle ones as \( J^{\pm,3} = \sum_{i=1}^n J_i^{\pm,3} \) and they obey the usual commutation rules \( [\hat{J}^3, \hat{J}^\pm] = \pm \hat{J}^\pm, [\hat{J}^+, \hat{J}^-] = -2 \hat{J}^3 \). Obviously, a single particle state \( \theta_i \) is an eigenstate of the Casimir operator \( \hat{J}_a^2 \theta_a = \bar{\hat{J}}_a^2 \theta_a \), with \( \bar{\hat{J}}_a^2 \equiv \frac{1}{2} (\nu_a^2 - 1) \). The eigenstates of two-particle Casimir operator \( \hat{J}_a^2 \) in \( (ab) \)-subchannel are \( \hat{J}_a^2 \theta_a \equiv \bar{\hat{J}}_a^2 \theta_a \), and coincide with the bilinear conformal operator \( \hat{F}_1 \) and possess the same eigenvalues. Now, the three-particle basis efficient for present applications has to be chosen so that it diagonalizes the total three-particle Casimir operator \( \hat{J}^2 = \hat{J}_{12}^2 + \hat{J}_{23}^2 + \hat{J}_{13}^2 - \sum_{\ell=1}^3 \hat{J}_\ell^2 \) and one in a subchannel \( \hat{J}_a^2 \), say \( a = 1 \) and \( b = 2 \),

\[
\hat{J}^2 \mathcal{P}_{J,j} = (J + 1)(\nu_1 + \nu_2 + \nu_3 + 3) (J + 1)(\nu_1 + \nu_2 + \nu_3 + 1) \mathcal{P}_{J,j},
\]

\[
\hat{J}_{12}^2 \mathcal{P}_{J,j} = (j + 1)(\nu_1 + \nu_2) (j + 2)(\nu_1 + \nu_2 + 2) \mathcal{P}_{J,j}.
\]

The solution is given in terms of hypergeometric function \( \mathcal{P}_{J,j}(\theta_1, \theta_2 | \theta_3) \propto \theta_1^{\nu_1} \theta_2^{\nu_2} F_1 (j - J, j + \nu_1 + 1, 2j + \nu_1 + \nu_2 + 2 | \theta) \) with \( \theta \equiv \theta_{12} / \theta_{32} \).

Since the physical anomalous dimensions have to be real we have to define an appropriate scalar product resulting in selfadjoint Hamiltonian, namely

\[
\langle \chi(\tilde{\theta}) | \chi(\theta) \rangle = \int \prod_{n_{1=1}}^n d\theta_i d\tilde{\theta}_i \frac{1}{2\pi i} (1 - \theta_i \tilde{\theta}_i)^{\nu_i - 1} \chi(\tilde{\theta}) \chi(\theta),
\]

and \( \tilde{\theta} = \theta^* \). Then it can be seen that the Casimir operator and, as a consequence, the Hamiltonian are selfadjoint operators w.r.t. such defined inner product \( \langle \hat{F}_1 \rangle = \hat{J}^2, \hat{h}^\dagger = h, \) and therefore \( \text{Im} \mathcal{E} = 0 \).

Obviously, to solve the three particle problem \( \hat{A} \) one has to find an extra integral of motion in addition to the one provided by the conformal symmetry. As we show in the next sections it turns out that this is the case of antiquark-gluon-quark and three-gluon systems in some limits.
4 Antiquark-gluon-quark correlators.

The antiquark-gluon-quark correlations contribute to a number of single spin asymmetries and enter as a three-particle part in the twist-3 (skewed) parton distributions. For instance, the transverse spin structure function $g_2$ admits the following three-parton piece

$$g_2^{tw-3}(x) = \int dx_1 dx_3 \mathcal{J}(x_1, x_3) Y(x_1, x_3),$$

where $\mathcal{J}$ is a differential operator acting on the momentum fractions $x_i$ of a two-argument function $Y(x_1, x_3)$ which is defined as a light-cone Fourier transform of a hadron matrix element of a $C$-even combination $\langle p| S^+(\kappa_1, \kappa_3) + S^-(-\kappa_3, -\kappa_1)|p \rangle$ of nonlocal operators $S^\pm_\mu(\kappa_1, \kappa_3) = ig \bar{\psi}(\kappa_3) \gamma_\mu \psi(\kappa_1) + \gamma_5 G^\pm_\mu(0) \bar{\psi}(\kappa_1)$. This will be the case of study in this section, in particular we elaborate on operator $S^+$, since $S^-$ is related to it by charge conjugation. Similar considerations apply to chiral odd sector.

In view of our discussion in Section 2 the total $K_{qg}$ evolution kernel splits into a sum $K_{\bar{q}g} + K_{qg} + K_{\bar{q}q} - \frac{1}{3} \delta_0$, where the $\delta$-function term is due to presence of the coupling constant in the definition of the operators $S$. The leading twist QCD interaction kernels $K_{ab}$ with non-zero momentum transfer in $t$-channel are known for a long time, see e.g.\cite{16}. One of the complications to solve the three-particle system is due to non-trivial colour structures of the latter, e.g. $K_{\bar{q}g}$ depends on $C_F - \frac{2}{3} C_A$, while $K_{(\bar{q}g)q}$ on both $C_F - \frac{2}{3} C_A$ and $C_F$. Obviously, the kernel simplifies drastically in multicolour limit $N \rightarrow \infty$, since the interaction of end point quarks effectively vanishes, $C_F - \frac{2}{3} C_A \sim O(N_c^{-1})$. In this case the total interaction kernel takes a form $K_{\bar{q}g} = \frac{2}{3} N_c \mathcal{H}$ where

$$\mathcal{H} = h_{\bar{q}g} \left( \frac{3}{2} + h_{gq} \left( \frac{3}{2} - \frac{1}{2} \right) - \frac{1}{2} \right), \quad \text{with} \quad h_{ab}(\delta) = \psi(\tilde{J}_{ab} + \delta) + \psi(\tilde{J}_{ab} - \delta) - 2\psi(1),$$

and $\tilde{J}^2 = \tilde{J}(\tilde{J} - 1)$. Thus we have reduced the original problem to an open chain of three particles on a line with conformal invariant interaction of neighbors (quarks and gluons) and no interaction of end points (quarks). This observation turns out to be fruitful since one can find the pair-wise Hamiltonians among the series of integrable ones for inhomogeneous spin chains. It can be easily generated from the Yang-Baxter bundle, which is a solution to the Yang-Baxter equation

$$R_{ab}(\lambda) = f(\nu, \lambda) P_{ab} \frac{\Gamma(\tilde{J}_{ab} + \lambda) \Gamma(\nu + 1 - \lambda)}{\Gamma(\tilde{J}_{ab} - \lambda) \Gamma(\nu + 1 + \lambda)} \quad (14)$$

as follows $h_{ab}(\delta) = R_{ab}(-\delta) R_{ab}'(-\delta)$, provided the function $f$ is chosen like this $f(\nu, \lambda) = \frac{\Gamma(1 - \lambda) \Gamma(\nu + 1 + \lambda)}{\Gamma(1 + \lambda) \Gamma(\nu + 1 - \lambda)}$. The total $\mathcal{H}$ can be produced from the open spin
Inhomogeneities as \( \delta_1 = \frac{3}{2} \) and \( \delta_3 = \frac{1}{2} \) we immediately find Eq. (13). This tells us that the antiquark-gluon-quark system is exactly integrable in multicolour limit. This fact implies that there exist a family of commuting integrals of motion. The first of them is provided by the conformal symmetry of interaction and is given by Casimir operator \( \hat{J}^2 \). Thus, we have to find only one ‘hidden’ conserved charge. This follows from the standard formalism of integrable spin chain models and forces us to calculate transfer matrix with auxiliary space being two-dimensional. With inhomogeneity parameters taken as above it gives an expansion in rapidity \( \lambda \):

\[
 t_{\frac{1}{2}}(\lambda) = \Omega(\lambda) - (4\lambda^2 - 1)(\lambda^2 - \hat{J}^2_3)\hat{J}^2 - \frac{1}{2}(4\lambda^2 - 1)Q(\delta_1, \delta_3),
\]

with constant function \( \Omega \) and the ‘hidden’ charge

\[
 Q(\delta_1, \delta_3) = [\hat{J}^2_{12}, \hat{J}^2_{23}]_+ - 2\delta_1^2 \hat{J}^2_{23} - 2\delta_3^2 \hat{J}^2_{12}.
\]

Below we will find eigenfunctions of this much more simple (compared to original Hamiltonian) operator \( Q \) and use them to find the eigenvalues of the Hamiltonian (13). Let us note in passing that integrable open spin chain models have been encountered as well in the solution of BFKL-type equation for quark-gluon reggeons.

To find the eigenfunctions of the Hamiltonian (13) we solve the equation for the charge with eigenfunctions \( \Psi \) expanded in the conformal basis discussed in Section 3

\[
 Q \left( \frac{1}{2}, \frac{1}{2} \right) \Psi = gS \Psi, \quad \Psi = \sum_{j=0}^{J} \varnothing_j \mathcal{T}_j \mathcal{P}_{J,j}(\theta_1, \theta_2|\theta_3),
\]

with \( g^{-1} = [ (J - j + 1)(J + j + 5)(j + 1)^3(j + 3)^3/(j + 2)^3]^2 \) being normalization coefficients. The main advantage of the basis is a three-diagonal form of the matrix elements of non-proper two-particle Casimir operators \( \hat{J}^{(1,2,3)} \). Then one establishes that Eq. (17) is equivalent to the three-term recursion relation

\[
 (2j+3)\mathcal{T}_{j-1} + (2j+5)\mathcal{T}_j - g_j^2 (2j+3)(2j+5) \left( Q \left( \frac{1}{2}, \frac{1}{2} \right) \right)_{j,j} - gS \right) \mathcal{T}_j = 0,
\]

where \( [Q]_{j,j} \) are the diagonal elements of the charge \( \langle \mathcal{P}_{j,j} | Q | \mathcal{P}_{j,j} \rangle \) in the basis \( \mathcal{P}_{J,j}(\theta_1, \theta_2|\theta_3) \). Polynomials require the expansion coefficients, \( T_j \), to satisfy the boundary conditions \( T_{-1} = T_{J+1} = 0 \).
The knowledge of eigenfunctions allows to find the energy of the antiquark-gluon-quark system via the formula

$$E(J,q_S) = \left( \sum_{j=0}^{J} (-1)^j \frac{(j + 2)^3}{(j + 1)(j + 3)} \right)^{-1} \sum_{j=0}^{J} (-1)^j \epsilon(j) \frac{(j + 2)^3}{(j + 1)(j + 3)} Y_j + \frac{1}{3},$$

(19)

with $\epsilon(j) = \psi(j + 1) + \psi(j + 4) - 2\psi(1)$. Unfortunately, the exact solution to (18), and thus (19), can hardly be found. Therefore, we restrict ourselves to WKB approximation for large conformal spins $J$. For the levels behaving as $E \propto 2 \ln J$ we can find the exact lowest trajectory

$$E(J) = \psi(J + 3) + \psi(J + 4) - 2\psi(1) - \frac{1}{2}.$$  

(20)

The remainder of the spectrum is described by the formula

$$E(J, q_S) = 2 \ln J - 4\psi(1) + 2 \Re \psi \left( \frac{3}{2} + i\eta_S \right) - \frac{4}{3},$$  

(21)

with $\eta_S = \frac{1}{2} \sqrt{2q_S/J^2 - 3}$. The conserved charge is quantized, with WKB quantization condition arisen from the matching of the WKB solution with exact ones at ‘turning’ points, and gives quantized values of energy via (21). It compares well with an explicit numerical diagonalization of the anomalous dimension matrix as shown in Figs. 2. There is an alternative description of the spectrum by trajectories which behave as $E \propto 4 \ln J$ at large $J$. The dependence on the conserved charge reads asymptotically $E(J, q_S) = \ln \frac{1}{2} q_S - 4\psi(1) - \frac{3}{2} + O(J^{-1})$. The first few expansion coefficients of energy in the series in $J^{-1}$ was found in (24). Corrections in $N_{c}^{-1}$ have been discussed recently in Ref. 23.

5 Three-gluon correlators.

Three-gluon correlators enter the $g_2$ via the quark loop coupling shown in Fig. 3 and might be responsible for its small-$x$ behaviour,

$$g_2(x) = \int dx_1 dx_3 C_{g\bar{g}g}(x_1, x_3) G(x_1, x_3).$$  

(22)

Here $C_{g\bar{g}g} = C^{(a)}_{g\bar{g}g} + C^{(b)}_{g\bar{g}g}$, with $C^{(a)}_{g\bar{g}g}$ calculated in Ref. 24 but contact-type contributions (b) not accounted for. The function $G(x_1, x_3)$ is a Fourier transform of non-local operators, e.g. $G_\mu(\kappa_1, \kappa_3) = g f_{abc} G^{a+\mu}_+(k_3 n) G^{b+\mu}_+(0) G^{c+\mu}_+(k_1 n)$.

\footnote{For the chiral odd distribution similar result has been found in Refs. 25 and 26 and for twist-three fragmentation functions in 27.}
The analysis, equivalent to the one done for the antiquark-gluon-quark correlator, gives the evolution kernel \( K_{\hat{g}\tilde{g}} \) which can be expressed up to corrections playing negligible role in the generation of the spectrum of anomalous dimensions as a sum \( K_{\hat{g}\tilde{g}} = \frac{1}{2} C_A H + \frac{1}{2} \beta_0 \) where \( H \) is split into two pieces,

\[
H = H_0 + V, \quad \text{where} \quad H_0 = h_{12} + h_{23} + h_{31}, \quad V = v_{12} + v_{23}, \quad (23)
\]

and pair-wise interactions given by

\[
h_{ab} = 2\psi(J_{ab}) - 2\psi(1), \quad v_{ab} = -4\hat{J}_{ab}^{-2}. \quad (24)
\]

Obviously, the separation into \( H_0 \) and \( V \) is not accidental, but reflect the fact that \( H_0 \), which describes the gluon interaction with aligned helicities, coincides with the Hamiltonian of XXX closed magnet of noncompact spin \(-\frac{3}{2}\). This can be easily found by means of the same \( R \) matrix \([14]\) from the logarithmic derivative, \( H_0 = \frac{d}{d\lambda}\ln t_b(\lambda) \), of the transfer matrix of an open spin chain \( t_b(\lambda) = \text{tr}_b R_{a,b}(\lambda)R_{a,b}(\lambda)R_{a,b}(\lambda) \). As a result the system described by \( H_0 \) contains a family of mutually commuting conserved charges which arise from the expansion of the auxiliary transfer matrix

\[
t_b(\lambda) = 2\lambda^3 + \lambda \left( \hat{J}^2 - 3\hat{j}^2 \right) + i\mathcal{Q}, \quad \text{with} \quad \mathcal{Q} = i\frac{1}{2}[\hat{J}_{12}, \hat{J}_{23}]. \quad (25)
\]

Note that the same Hamiltonian, however of different conformal spin, has been encountered in interaction of QCD reggeons \([24]\) and Brodsky-Lepage evolution.
Figure 3: Coefficient function of three-gluon state in the structure function \( g_2 \).

equation for baryons in the case of the same helicities of participating quarks. The coincidence of quark and gluon kernels for the same helicity states is a mere consequence of \( N = 1 \) supersymmetry which relates corresponding kernels,

\[
h^T_{qq}(2n+1) = h^T_{qg}(2n+1) = h^T_{gg}(2n+1) \quad \text{where} \quad h^T_{qq}(J) = 2\psi(J + 2) + 2\psi(J + 3) - 4\psi(1) + 2(-1)^J/(J + 2) - 3.
\]

So our strategy will be, first, to diagonalize the exactly solvable interaction and then to consider the effect of \( \mathcal{V} \) interaction on the spectrum. Thus in parallel to previous section, we diagonalize first the charge \( Q \) and find its eigenfunctions \( \Upsilon_j \)

\[
h_T^{qg}(J) = 2\psi(J + 3) - 2\psi(1).
\]

The lowest trajectory can be found exactly

\[
\mathcal{E}_0(J, 0) = 2\psi(J + 3) + 2\psi(J + 2) - 4\psi(1) + 4.
\]

While the rest of the spectrum is described by WKB formula

\[
\mathcal{E}_0(J, q) = 4\ln(q) - 6\psi(1) + 2\operatorname{Re}(\psi(\frac{q}{2} - iq^*)),
\]

where \( q^* = q/\eta^2 \) and is valid with \( \mathcal{O}(\eta^{-1}) \). We use the convention \( \eta^2 = (J + \frac{3}{2}(\nu + 1)) (J + \frac{3}{2}(\nu + 1) - 1) \) for the eigenvalues of the quadratic Casimir operator of the chain which to the stated accuracy reads for gluons \( \eta = J + 4 + \mathcal{O}(J^{-1}) \). Matching of the WKB and exact solutions to \( (26) \) gives a quantization conditions for the charge, \( q^* \ln \eta = \arg \Gamma(\frac{3}{2} + iq^*) + \frac{2m + \frac{1}{2}}{\pi}(2n + \frac{1}{2}[1 - (-1)^J]). \)

An alternative set of trajectories behaves like \( \mathcal{E}(J, q) = 2\ln q - 6\psi(1) + \mathcal{O}(J^{-1}) \).
The perturbation $V$ does not affect a bulk of the spectrum except for the lowest few levels. To analyze the situation one considers Hamiltonian $H_\alpha = H + \alpha V$ specified by a coupling constant $\alpha$ and study the level flow as a function of $\alpha$. Making use of these results we design the following formulae which describe well the spectrum of the perturbed Hamiltonian, see Fig. 2.

$$E(J, m) = \mathcal{E}_0(J, 0)(\delta_{m, 0} + \delta_{m, 1}) + \mathcal{E}_0(J, q(m))\theta(m - 2) - \Delta(m).$$ (30)

Here $\mathcal{E}_0(J, 0)$ is the ground state energy (28) and $\mathcal{E}_0(J, q(m))$ is Eq. (29) with the $q(m)$ trajectories deduced from the quantization condition. Finally,

$$\Delta(m) = 0.54 \delta_{m, 0} + 0.08 \delta_{m, 1} + (\delta_0 - \delta(m - 2)) \theta(\delta_0 - \delta(m - 2)),$$ (31)

is the shift-function with $\delta_0 = 0.15$ and $\delta = 0.01$.

6 Conclusions

To conclude, it remains a challenge for QCD to understand the appearance of integrable structures for light-cone and transverse momentum dynamics and unravel their striking similarity.

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