Alternating Euler $T$-sums and Euler $\tilde{S}$-sums

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Abstract In this paper, we study the alternating Euler $T$-sums and related sums by using the method of contour integration. We establish the explicit formulas for all linear and quadratic Euler $T$-sums and related sums. Some interesting new consequences and illustrative examples are considered.

Keywords: Multiple zeta values, multiple $t$-values, multiple $T$-values, odd harmonic numbers, Euler $T$-sums.

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1 Introduction and Notations

In our previous paper [7], we introduced and studied the following two variants of the classical Euler sums,

\[ T_{p_1,p_2,\ldots,p_k,q} := \sum_{n=1}^{\infty} \frac{h_{n-1}^{(p_1)} h_{n-1}^{(p_2)} \cdots h_{n-1}^{(p_k)}}{(n-1/2)^q}, \]

\[ \tilde{S}_{p_1,p_2,\ldots,p_k,q} := \sum_{n=1}^{\infty} \frac{h_n^{(p_1)} h_n^{(p_2)} \cdots h_n^{(p_k)}}{n^q}, \]

where $q > 1, p_1 \leq p_2 \leq \cdots \leq p_k$ are positive integers, and the quantity $w := p_1 + \cdots + p_r + q$ is called the weight and the quantity $r$ is called the degree (or order). We often refer these sums as the the Euler $T$-sums and Euler $\tilde{S}$-sums, respectively. Here $h_n^{(p)}$ stands for odd harmonic number of order $p$ defined by

\[ h_n^{(p)} := \sum_{k=1}^{n} \frac{1}{(k-1/2)^p}, \quad h_n \equiv h_n^{(1)} \quad \text{and} \quad h_0^{(p)} := 0. \]

The Euler $T$-sums and Euler $\tilde{S}$-sums can be seen as variants of classical Euler sums [1]

\[ S_{p_1,p_2,\ldots,p_k,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} H_n^{(p_2)} \cdots H_n^{(p_k)}}{n^q}, \]

where $H_n^{(p)}$ stands for the $p$-th generalized harmonic number, which is defined by

\[ H_n^{(p)} := \sum_{k=1}^{n} \frac{1}{k^p}, \quad H_n \equiv H_n^{(1)} \quad \text{and} \quad H_0^{(p)} := 0. \]

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Like classical Euler sums, the Euler $T$-sums and Euler $\tilde{S}$-sums can be evaluated by using the method of contour integration developed by Flajolet and Salvy [1]. In [7], we establish many explicit evaluations of Euler $T$-sums and Euler $\tilde{S}$-sums via $\log(2)$, multiple zeta and $t$-values. Here, for positive integers $p_1, \ldots, p_k$ with $p_1 > 1$, the multiple zeta value (MZV for short) [2, 8] and multiple $t$-values (MtVs for short) [3] are defined by

$$\zeta(p_1, p_2, \ldots, p_k) := \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{p_1} n_2^{p_2} \cdots n_k^{p_k}}$$

and

$$t(p_1, p_2, \ldots, p_k) := \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{p_1} n_2^{p_2} \cdots n_k^{p_k}}$$

$$= \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{(2n_1 - 1)^{p_1}(2n_2 - 1)^{p_2} \cdots (2n_k - 1)^{p_k}}.$$

As it normalized version,

$$\tilde{t}(p_1, p_2, \ldots, p_k) := 2^{p_1 + p_2 + \cdots + p_k} t(p_1, p_2, \ldots, p_k).$$

In above definitions of MZVs and MtVs, we put a bar on top of $p_j$ if there is a sign $(-1)^{n_j}$ appearing in the denominator on the right. Which (one of more the $p_j$ barred) are called the alternating MZVs, alternating multiple $t$-values. For example,

$$\zeta(p_1, \bar{p}_2, p_3, \bar{p}_4) = \sum_{n_1 > n_2 > n_3 > n_4 > 0} \frac{(-1)^{n_2 + n_4}}{n_1^{\bar{p}_2} n_2^{p_3} n_3^{\bar{p}_4} n_4^{p_4}},$$

$$t(\bar{p}_1, \bar{p}_2, p_3, \bar{p}_4) = \sum_{n_1 > n_2 > n_3 > n_4 > 0} \frac{(-1)^{n_1 + n_2}}{(2n_1 - 1)^{p_1}(2n_2 - 1)^{p_2}(2n_3 - 1)^{p_3}(2n_4 - 1)^{p_4}}.$$

In particular, we let

$$\tilde{\zeta}(p) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{kp}, \quad \tilde{t}(p) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1/2)^p}, \quad (p \geq 1).$$

It is clear that the multiple $t$-values can be regard as a level 2 multiple zeta value because of the congruence condition in the summation and of the fact that this value can be written as a linear combination of alternating multiple zeta values. Recently, Kaneko and Tsumura [4, 5] also introduced and studied a new kind of multiple zeta values of level two

$$T(p_1, p_2, \ldots, p_k) := 2^k \sum_{m_1 > m_2 > \cdots > m_k > 0} \frac{1}{m_1^{p_1} m_2^{p_2} \cdots m_k^{p_k}}$$

$$= 2^k \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{1}{(2n_1 - k)^{p_1}(2n_2 - k + 1)^{p_2} \cdots (2n_k - 1)^{p_k}},$$

which was called multiple $T$-values (MTVs).

The subject of this paper are alternating Euler $T$-sums and alternating Euler $\tilde{S}$-sums. First, we give the definitions of alternating harmonic number and odd harmonic number. Let $p$ and $n$
be positive integers, the alternating harmonic number $H_n^{(p)}$ and odd harmonic number $\bar{h}_n^{(p)}$ are defined by

$$H_n^{(p)} := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^p}, \quad H_0^{(p)} := 0, \quad H_n := H_n^{(1)},$$

$$\bar{h}_n^{(p)} := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(k-1/2)^p}, \quad \bar{h}_0^{(p)} := 0, \quad \bar{h}_n := \bar{h}_n^{(1)}.$$  

In the definitions of Euler $T$-sums and Euler $\bar{S}$-sums, if replace $\bar{h}_n^{(p)}$ by $\bar{h}_n^{(p)}$ in the numerator of the summand, we put a “bar” on the top of $p_j$. In particular, we put a bar on the top of $q$ if there is a sign $(-1)^{n-1}$ appearing in the denominator on the right. For example,

$$T_{p_1p_2p_3,q} = \sum_{n=1}^{\infty} \frac{h_{n-1}^{(p_1)} h_{n-1}^{(p_2)} h_{n-1}^{(p_3)}}{(n-1/2)^q} (-1)^{n-1}, \quad T_{p_1p_2p_3,q} = \sum_{n=1}^{\infty} \frac{h_{n-1}^{(p_1)} h_{n-1}^{(p_2)} h_{n-1}^{(p_3)}}{(n-1/2)^q} (-1)^{n-1},$$

$$\bar{S}_{p_1p_2p_3,q} = \sum_{n=1}^{\infty} \frac{\bar{h}_{n-1}^{(p_1)} \bar{h}_{n-1}^{(p_2)} \bar{h}_{n-1}^{(p_3)}}{n^q} (-1)^{n-1}, \quad \bar{S}_{p_1p_2p_3,q} = \sum_{n=1}^{\infty} \frac{\bar{h}_{n-1}^{(p_1)} \bar{h}_{n-1}^{(p_2)} \bar{h}_{n-1}^{(p_3)}}{(n-1/2)^q} (-1)^{n-1}.$$  

The sums of types above (one of more the $p_j$ or $q$ barred) are called the alternating Euler $T$-sums and alternating Euler $\bar{S}$-sums, respectively. In [6], we systematic studied all classical (alternating) Euler sums. In this paper, we study these two above alternating variants of Euler $T$-sums or Euler $\bar{S}$-sums by using the methods of contour integration and residue theorem.

Next, we introduce some basic notations. Let $A := \{a_k\}$, $-\infty < k < \infty$ be a sequence of complex numbers with $a_k = o(k^\alpha)$ ($\alpha < 1$) if $k \to \pm\infty$. For convenience, we let $A_1$ and $A_2$ to denote the constant sequence $\{1^k\}$ and alternating sequence $\{(-1)^k\}$, respectively.

**Definition 1.1** With $A$ defined above, we define the parametric digamma function $\Psi(-s; A)$ by

$$\Psi(-s; A) := \frac{a_0}{s} + \sum_{k=1}^{\infty} \left( \frac{a_k}{k} - \frac{a_k}{k - s} \right). \tag{1.1}$$

Obviously, if $A = A_1$, then the parametric digamma function $\Psi(-s; A)$ becomes the classical digamma function $\psi(-s) + \gamma$.

**Definition 1.2** Define the cotangent function with sequence $A$ by

$$\pi \cot(\pi s; A) = -\frac{a_0}{s} + \Psi(-s; A) - \Psi(s; A) = \frac{a_0}{s} - 2s \sum_{k=1}^{\infty} \frac{a_k}{k^2 - s^2}. \tag{1.2}$$

It is clear that if letting $A = A_1$ or $A_2$ in (1.2), respectively, then it become

$$\cot(\pi s; A_1) = \cot(\pi s), \quad \cot(\pi s; A_2) = \csc(\pi s).$$

The Definitions 1.1 and 1.2 are also introduced in a previous paper [6] of the second named author.
Definition 1.3 For nonnegative integers \( j \geq 1 \) and \( n \), we define

\[
D^{(A)}(j) := \sum_{k=1}^{\infty} \frac{a_k}{k^j}, \quad D^{(A)}(1) := 0, \quad E^{(A)}_n(j) := \sum_{k=1}^{n} \frac{a_{n-k}}{k^j}, \quad E^{(A)}_0(j) := 0, \\
\hat{E}^{(A)}_n(j) := \sum_{k=1}^{n} \frac{a_{n-k}}{(k-1/2)^j}, \quad \hat{E}^{(A)}_0(j) := 0, \quad \hat{E}^{(A)}_n(j) := \sum_{k=1}^{n} \frac{a_{k-n-1}}{(k-1/2)^j}, \quad \hat{E}^{(A)}_0(j) := 0, \\
\tilde{l}^{(A)}(j) := \left\{ \begin{array}{ll}
\sum_{k=1}^{\infty} \left( \frac{a_{k-1}}{k-1/2} - \frac{a_k}{k} \right), & j = 1, \\
\sum_{k=1}^{\infty} \frac{a_{k-1}}{(k-1/2)^j}, & j > 1,
\end{array} \right. \quad \tilde{l}^{(A)}(j) := \left\{ \begin{array}{ll}
\sum_{k=1}^{\infty} \left( \frac{a_k}{k-1/2} - \frac{a_k}{k} \right), & j = 1, \\
\sum_{k=1}^{\infty} \frac{a_k}{(k-1/2)^j}, & j > 1,
\end{array} \right.
\]

\[
F^{(A)}_n(j) = \sum_{k=1}^{\infty} \frac{a_{k+n} - a_k}{k}, \quad F^{(A)}_n(j) = \sum_{k=1}^{\infty} \frac{a_{k+n} - a_k}{k-1/2}, \quad j = 1, \\
\hat{F}^{(A)}_n(j) = \sum_{k=1}^{\infty} \frac{a_{k+n} - a_k}{(k-1/2)^j}, \quad j > 1,
\]

\[
\bar{F}^{(A)}_n(j) = \left\{ \begin{array}{ll}
\sum_{k=1}^{\infty} \left( \frac{a_{k-n}}{k} - \frac{a_k}{k-1/2} \right), & j = 1, \\
\sum_{k=1}^{\infty} \frac{a_{k-n}}{(k-1/2)^j}, & j > 1,
\end{array} \right. \\
M^{(A)}_n(j) := E^{(A)}_n(j) + (-1)^j F^{(A)}_n(j), \quad N^{(A)}_n(j) := \hat{E}^{(A)}_n(j) + (-1)^j \hat{F}^{(A)}_{n-1}(j), \\
\bar{N}^{(A)}_n(j) := \bar{F}^{(A)}_n(j) - \bar{E}^{(A)}_n(j), \quad S^{(A)}_n(j) := N^{(A)}_n(j) - \bar{N}^{(A)}_n(j) - \frac{a_0}{(n-1/2)^j}.
\]

Setting \( A = A_1 \) or \( A_2 \) yield

\[
M^{(A_1)}_n(j) = H^{(j)}_n + (-1)^j \zeta(j), \\
M^{(A_2)}_n(j) = (-1)^{n-1} \bar{H}^{(j)}_n + (-1)^j \left\{ \begin{array}{ll}
(1 - (-1)^n) \log(2), & j = 1, \\
(-1)^{n-1} \bar{\zeta}(j), & j > 1,
\end{array} \right. \\
N^{(A_1)}_n(j) = h^{(j)}_n + (-1)^j \bar{\zeta}(j), \quad \bar{\zeta}(1) := 2 \log(2), \\
N^{(A_2)}_n(j) = (-1)^{n-1} h^{(j)}_n + (-1)^j \left\{ \begin{array}{ll}
(-1)^n \bar{\zeta}(1) + \log(2), & j = 1, \\
(-1)^n \bar{\zeta}(j), & j > 1,
\end{array} \right. \\
\bar{N}^{(A_1)}_n(j) = \bar{\zeta}(j) - h^{(j)}_{n-1}, \quad \bar{\zeta}(1) := 2 \log(2), \\
\bar{N}^{(A_2)}_n(j) = (-1)^{n-1} \bar{h}^{(j)}_{n-1} + \left\{ \begin{array}{ll}
(-1)^{n-1} \bar{\zeta}(1) + \log(2), & j = 1, \\
(-1)^{n-1} \bar{\zeta}(j), & j > 1,
\end{array} \right. \\
S^{(A_1)}_n(j) = (1 + (-1)^j) \bar{\zeta}(j), \quad S^{(A_2)}_n(j) = (-1)^{n-1} (1 - (-1)^j) \bar{\zeta}(j).
\]

2 Lemmas

In this section, we give some power series expansions for parametric digamma function \( \Psi(-s; A) \) and \( \cot(\pi s; A) \).
Lemma 2.1 Let \( p > 0 \) and \( n \) be a non-negative integer, if \( s \in (n - 1, n + 1) \) then
\[
\frac{\Psi(p-1)(1/2 - s; A)}{(p-1)!} = \sum_{j=1}^{\infty} (-1)^{j-1} \binom{j + p - 2}{p - 1} N_n^{(A)}(j + p - 1)(s - n)^{j-1}, \tag{2.1}
\]
and if \( s \in (-n - 1, -n + 1) \) then
\[
\frac{\Psi(p-1)(1/2 - s; A)}{(p-1)!} = \sum_{j=1}^{\infty} (-1)^{j-1} \binom{j + p - 2}{p - 1} N_{n+1}^{(A)}(j + p - 1)(s + n)^{j-1}. \tag{2.2}
\]

Proof. The proofs of this lemma follows the definition of function \( \Psi(-s; A) \).

Letting \( n = 0 \) in (2.1) and (2.2) gives
\[
\frac{\Psi(p-1)(1/2 - s; A)}{(p-1)!} = (-1)^p \sum_{j=1}^{\infty} \binom{j + p - 2}{p - 1} \tilde{t}^{(A)}(j + p - 1)s^{j-1}, \quad (-1 < s < 1). \tag{2.3}
\]

Lemma 2.2 Let \( m > 0 \) and \( n > 1 \) be non-negative integer, if \( s \in (n - 3/2, n + 1/2) \) then
\[
\frac{d^m}{ds^m}(\pi \cot(\pi s; A)) = (-1)^m m! \sum_{j=1}^{\infty} (-1)^{j-1} \binom{j + m - 1}{m} S_n^{(A)}(j + m)(s - n + 1/2)^{j-1}. \tag{2.4}
\]

Proof. Lemma 2.2 follows immediately from Definition 1.2 and Lemma 2.1.

From Lemma 2.2, we have
\[
\lim_{s \to 1/2} \frac{d^m}{ds^m}(\pi \cot(\pi s; A)) = m!((-1)^m \tilde{t}^{(A)}(m + 1) - \tilde{t}^{(A)}(m + 1)), \tag{2.5}
\]
\[
\pi \cot(\pi s; A) = \frac{a_0}{s} - 2 \sum_{j=1}^{\infty} D^{(A)}(2j)s^{2j-1}, \quad (-1 < s < 1). \tag{2.6}
\]

Lemma 2.3 ([6]) Let \( p \) and \( n \) be positive integers, if \( s \in (n - 3/2, n + 1/2) \setminus \{n - 1/2\} \), then
\[
\frac{\Psi(p-1)(1/2 - s; A)}{(p-1)!} = \frac{1}{(s - n + 1/2)^p} \left\{ a_{n-1} - \sum_{j=1}^{\infty} (-1)^j \binom{j + p - 2}{p - 1} M_{n-1}^{(A)}(j + p - 1)(s - n + 1/2)^{j+p-1} \right\}. \tag{2.7}
\]

Proof. This lemma can be immediately obtained from [6, Theorem 2.1].

If \( n = 1 \) then
\[
\frac{\Psi(p-1)(1/2 - s; A)}{(p-1)!} = \frac{a_0}{(s - 1/2)^p} + (-1)^p \sum_{j=1}^{\infty} \binom{j + p - 2}{p - 1} D^{(A)}(j + p - 1)(s - 1/2)^{j-1}. \tag{2.8}
\]

Finally, we give a residue theorem which was given by Flajolet and Salvy.
Lemma 2.4 ([1]) Let $\xi(s)$ be a kernel function and let $r(s)$ be a rational function which is $O(s^{-2})$ at infinity. Then

$$\sum_{\alpha \in O} \text{Res}[r(s)\xi(s), s = \alpha] + \sum_{\beta \in S} \text{Res}[r(s)\xi(s), s = \beta] = 0. \quad (2.9)$$

where $S$ is the set of poles of $r(s)$ and $O$ is the set of poles of $\xi(s)$ that are not poles of $r(s)$. Here $\text{Res}[r(s), s = \alpha]$ denotes the residue of $r(s)$ at $s = \alpha$. The kernel function $\xi(s)$ is defined by the two requirements: 1. $\xi(s)$ is meromorphic in the whole complex plane. 2. $\xi(s)$ satisfies $\xi(s) = o(s)$ over an infinite collection of circles $|s| = \rho_k$ with $\rho_k \to \infty$.

3 Evaluations of Euler T-sums and Euler $\tilde{S}$-sums

Let $B := \{b_k\}$, $-\infty < k < \infty$ be a sequence of complex numbers with $b_k = o(k^\beta)$ ($\beta < 1$) if $k \to \pm \infty$. Flajolet and Salvy [1] applied the kernel function

$$\frac{1}{2} \pi \cot(\pi s) \frac{\psi(p-1)(-s)}{(p-1)!}$$

to the base function $r(s) = s^{-q}$ to prove every linear sum $S_{p,q}$ whose weight $p+q$ is odd is expressible as a polynomial in zeta values. Next, we replace $\cot(\pi s)\psi(p-1)(-s)$ by $\cot(\pi s; A)\psi(p-1)(-s; B)$, and use contour integration to evaluate linear (alternating) Euler $T$-sums and Euler $\tilde{S}$-sums.

Theorem 3.1 Let $p > 0$ and $q > 1$ be positive integers. We have

$$(-1)^{p+q} \sum_{n=1}^{\infty} \frac{\tilde{N}_n^{(B)}(p)}{(n-1/2)^q} \frac{a_{n-1}}{n} + \sum_{n=1}^{\infty} \frac{N_n^{(B)}(p)}{(n-1/2)^q} \frac{a_n}{n}$$

$$- (-1)^p \sum_{k=0}^{p-1} \left( \frac{p+q-k-2}{q-1} \right) \sum_{n=1}^{\infty} \frac{b_{n+1}^{(A)}(k+1)}{n^{p+q-k-1}}$$

$$- b_0 \left( (-1)^{p+q} \tilde{t}^{(A)}(p+q) + \tilde{t}^{(A)}(p+q) \right)$$

$$+ (-1)^p \sum_{k=1}^{q} \left( \frac{k+p-2}{p-1} \right) D^{(B)}(k+p-1) \left( (-1)^{q-k} \tilde{t}^{(A)}(q-k+1) - \tilde{t}^{(A)}(q-k+1) \right)$$

$$= 0. \quad (3.1)$$

Proof. We consider the kernel function

$$\pi \cot(\pi s; A) \frac{\psi(p-1)(1/2-s; B)}{(p-1)!}$$

and base function $r(s) = (s - 1/2)^{-q}$. Clearly, the function $F(s) := \xi(s)r(s)$ only have poles at all integer and $n - 1/2$ ($n$ is a positive integer). The only singularities are poles at the integers. At a negative integer $-n$ and positive integer $n$ these two poles are simple and these residues are

$$\text{Res}[F(s), s = -n] = (-1)^{p+q} \frac{n^{(B)}(p)}{(n+1/2)^q} \frac{a_n}{n} \quad (n \geq 0),$$
\[ \text{Res}[F(s), s = n] = \frac{N_n^{(B)}(p)}{(n - 1/2)^q} a_n \quad (n \geq 1). \]

From (2.7), the pole \( n - 1/2 \) \((n \geq 2)\) has order \( p \) and the residue is
\[
\text{Res}[F(s), s = n - 1/2] = (-1)^p \sum_{k=0}^{p-1} \frac{(p + q - k - 2)}{q - 1} \frac{b_{n-1} S_n^{(A)}(k + 1)}{(n - 1/p + q - k - 1)}. 
\]

From (2.8), the pole 1/2 has order \( p + q \) and the residue is
\[
\text{Res}[F(s), s = 1/2] = -b_0 \left( (-1)^{p+q} \tilde{t}^{(A)}(p + q) + \tilde{t}^{(A)}(p + q) \right) 
+ (-1)^p \sum_{k=1}^{q} \frac{\left( k + p - 2 \right)}{p - 1} D^{(B)}(k + p - 1) \left( (-1)^{q-k} \tilde{t}^{(A)}(q - k + 1) - \tilde{t}^{(A)}(q - k + 1) \right). 
\]

Summing these four contributions yields the statement of the theorem. \(\square\)

**Theorem 3.2** Let \( p > 0 \) and \( q > 1 \) be positive integers. We have
\[
(-1)^{p+q} \sum_{n=1}^{\infty} \frac{N_{n+1}^{(B)}(p)}{n^q} a_n + \sum_{n=1}^{\infty} \frac{N_n^{(B)}(p)}{n^q} a_n 
- (-1)^p \sum_{k=0}^{p-1} \frac{(p + q - k - 2)}{q - 1} \sum_{n=1}^{\infty} \frac{b_{n-1} S_n^{(A)}(k + 1)}{(n - 1/2)^p + q - k - 1} 
+ a_0 (-1)^p \frac{(p + q - 1)}{q} \tilde{t}^{(B)}(p + q) 
- 2(-1)^p \sum_{j=1}^{\left\lfloor q/2 \right\rfloor} \frac{(p + q - 2j - 1)}{p - 1} D^{(A)}(2j) \tilde{t}^{(B)}(p + q - 2j) 
= 0. \quad (3.2)
\]

**Proof.** The proof is similar to the previous proof. We consider the kernel function
\[
\frac{1}{2\pi \cot(\pi s; A)} \frac{\Psi(p-1)(1/2 - s; B)}{(p - 1)!} 
\]
and base function \( r(s) = s^{-q} \). Then, by a similar argument as in the proof of above, we may easily deduce the desired result. \(\square\)

In Theorem 3.1 and 3.2, setting \( A, B \in \{A_1, A_2\} \), by straightforward calculations, we can get the following corollaries.
Corollary 3.3 For positive integers $p$ and $q > 1$,

\[
(1 - (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{h_{n-1}^{(p)}}{(n - 1/2)^q} = (-1)^{p+q} \tilde{t}(p + q) - (-1)^p (1 + (-1)^q) \tilde{t}(p) \tilde{t}(q) 
- (-1)^p \sum_{k=0}^{p-1} ((-1)^k - 1) \binom{p + q - k - 2}{q - 1} \tilde{t}(k + 1) \zeta(p + q - k - 1) 
+ (-1)^p \sum_{k=1}^{q} (1 - (-1)^{q-k}) \binom{k + p - 2}{p - 1} \tilde{t}(q - k + 1) \zeta(k + p - 1),
\]

where $\zeta(1) := 0$ and $\tilde{t}(1) := 2 \log(2)$. 

\[
(1 + (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{\tilde{h}_{n-1}^{(p)}}{(n - 1/2)^q} = (-1)^{p+q} \tilde{t}(p + q) + (-1)^p (1 + (-1)^q) \tilde{t}(p) \tilde{t}(q) 
- (-1)^p \sum_{k=0}^{p-1} ((-1)^k + 1) \binom{p + q - k - 2}{q - 1} \tilde{t}(k + 1) \zeta(p + q - k - 1) 
- (-1)^p \sum_{k=1}^{q} (1 + (-1)^{q-k}) \binom{k + p - 2}{p - 1} \tilde{t}(q - k + 1) \zeta(k + p - 1),
\]

\[
(1 - (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{\tilde{h}_{n-1}^{(p)}}{(n - 1/2)^q} = (-1)^{p+q} \tilde{t}(p + q) - (-1)^p (1 - (-1)^q) \tilde{t}(p) \tilde{t}(q) 
+ (-1)^p \sum_{k=0}^{p-1} ((-1)^k + 1) \binom{p + q - k - 2}{q - 1} \tilde{t}(k + 1) \zeta(p + q - k - 1) 
+ (-1)^p \sum_{k=1}^{q} (1 - (-1)^{q-k}) \binom{k + p - 2}{p - 1} \tilde{t}(q - k + 1) \zeta(p - k - 1),
\]
Corollary 3.4 For positive integers \( p \) and \( q > 1 \),

\[
(1 - (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{h_n^{(p)}}{n^q} = -(-1)^p(1 + (-1)^{q})\tilde{t}(p)\zeta(q) - (-1)^p \left( \frac{p + q - 1}{p - 1} \right)\tilde{t}(p + q) \\
- (-1)^p \sum_{k=0}^{p-1} ((-1)^k - 1) \left( \frac{p + q - k - 2}{q - 1} \right)\tilde{t}(k + 1)\tilde{t}(p + q - k - 1) \\
+ 2(-1)^p \sum_{j=1}^{[q/2]} \left( \frac{p + q - 2j - 1}{p - 1} \right)\zeta(2j)\tilde{t}(p + q - 2j),
\]

\( (1 + (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{\eta_n^{(p)}}{n^q} (-1)^{n-1} \)

\[
= -(-1)^p(1 + (-1)^{q})\tilde{t}(p)\zeta(q) - (-1)^p \left( \frac{p + q - 1}{p - 1} \right)\tilde{t}(p + q) \\
- (-1)^p \sum_{k=0}^{p-1} ((-1)^k - 1) \left( \frac{p + q - k - 2}{q - 1} \right)\tilde{t}(k + 1)\tilde{t}(p + q - k - 1) \\
+ 2(-1)^p \sum_{j=1}^{[q/2]} \left( \frac{p + q - 2j - 1}{p - 1} \right)\zeta(2j)\tilde{t}(p + q - 2j),
\]

\( (1 - (-1)^{p+q}) \sum_{n=1}^{\infty} \frac{h_n^{(p)}}{n^q} (-1)^{n-1} \)

\[
= -(-1)^p(1 + (-1)^{q})\tilde{t}(p)\zeta(q) + (-1)^p \left( \frac{p + q - 1}{p - 1} \right)\tilde{t}(p + q) \\
- (-1)^p \sum_{k=0}^{p-1} ((-1)^k + 1) \left( \frac{p + q - k - 2}{q - 1} \right)\tilde{t}(k + 1)\tilde{t}(p + q - k - 1) \\
+ 2(-1)^p \sum_{j=1}^{[q/2]} \left( \frac{p + q - 2j - 1}{p - 1} \right)\zeta(2j)\tilde{t}(p + q - 2j),
\]

where \( \zeta(1) := 0 \) and \( \tilde{t}(1) := 2 \log(2) \).
Next, we evaluate the quadratic (alternating) Euler $T$-sums and Euler $\tilde{S}$-sums.

**Theorem 3.5** For positive integers $p, m$ and $q > 1$, then

\[
(-1)^{p+q+m} \sum_{n=1}^{\infty} \frac{N_n^{(B)}(m)N_n^{(C)}(p)}{(n-1/2)^q} a_{n-1} + \sum_{n=1}^{\infty} \frac{N_n^{(B)}(m)N_n^{(C)}(p)}{(n-1/2)^q} a_n \\
- (-1)^{p+m} \sum_{k=0}^{p+m} \left(\frac{p+q+m-k-2}{q-1}\right) \sum_{n=1}^{\infty} b_n c_n S_{n+1}(k+1) \frac{n^{p+q+m-k-1}}{n^{m+q-j-1}} \\
- (-1)^m \sum_{j=1}^{m-j} \sum_{k=0}^{j+p-2} \left(\frac{j+p-2}{j-1}\right) \sum_{n=1}^{\infty} \frac{M_n^{(C)}(j+p-1)S_{n+1}(k+1)b_n}{n^{m+q-j-1}} \\
- (-1)^p \sum_{j=1}^{p-j} \sum_{k=0}^{j+m-2} \left(\frac{j+m-2}{j-1}\right) \sum_{n=1}^{\infty} \frac{M_n^{(B)}(j+m-1)S_{n+1}(k+1)c_n}{n^{m+q-j-1}}
\]

+ Res$[F(s), s = 1/2] = 0$, \hspace{1cm}(3.11)

where \[
Res[F(s), s = 1/2] = -b_0 c_0 \left((-1)^{p+q+m}\tilde{A}(p+q+m) + \tilde{A}(p+q+m)\right) \\
+ b_0(-1)^p \sum_{j=1}^{m+q} \left(\frac{j+p-2}{j-1}\right) D^{(C)}(j+p-1) \\
\times \left((-1)^{m+q-j}\tilde{A}(m+q-j+1) - \tilde{A}(m+q-j+1)\right) \\
+ c_0(-1)^m \sum_{j=1}^{p+q} \left(\frac{j+m-2}{j-1}\right) \tilde{A}(j+m-1) \\
\times \left((-1)^{p+q-j}\tilde{A}(p+q-j+1) - \tilde{A}(p+q-j+1)\right) \\
+ (-1)^{p+m} \sum_{j_1+j_2 \leq q+1, j_1, j_2 \geq 1} \left(\frac{j_1+m-2}{j_1-1}\right) \left(\frac{j_2+p-2}{j_2-1}\right) \tilde{A}(j_1+m-1)D^{(B)}(j_2+p-1) \\
\times \left((-1)^{q+1-j_1-j_2}\tilde{A}(q+2-j_1-j_2) - \tilde{A}(q+2-j_1-j_2)\right) \hspace{1cm}(3.12)
\]

**Proof.** We consider the kernel function

\[
cot(\pi s; A) \frac{\Psi^{(m-1)}(1/2-s; B)\Psi^{(p-1)}(1/2-s; C)}{(m-1)!(p-1)!}
\]

and base function $r(s) = (s-1/2)^{-q}$. It is obvious that the function

\[
F(s) := \cot(\pi s; A) \frac{\Psi^{(m-1)}(1/2-s; B)\Psi^{(p-1)}(1/2-s; C)}{(m-1)!(p-1)!} (s-1/2)^q
\]

has simple poles at $s = -n$ ($n \geq 0$) with residues

\[
Res[F(s), s = -n] = (-1)^{p+q+m} \frac{N_{n+1}^{(B)}(m)N_{n+1}^{(C)}(p)}{(n+1/2)^q} a_n,
\]

\]

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and simple poles at \( s = n \) (\( n \geq 1 \)), with residues

\[
\text{Res}[F(s), s = n] = \frac{N_n^{(B)}(m)N_n^{(C)}(p)}{(n - 1/2)^q} a_n,
\]

where we used the identities (2.1) and (2.2). Clearly \( F(s) \) has poles of order \( p + m \) at \( s = n - 1/2 \) (\( n \geq 2 \)). Using (2.4) and (2.7) we find that the residues

\[
\text{Res}[F(s), s = n - 1/2]
\]

\[
= -(-1)^{p+m} \sum_{k=0}^{p+m-1} \frac{(p + q + m - k - 2)}{q - 1} b_{n-1} c_{n-1} S_n^{(A)}(k+1)
\]

\[
- (-1)^m \sum_{j=1}^{m-j} \sum_{k=0}^{m-j} \frac{(j + p - 2)}{p - 1} \frac{(m + q - k - j - 1)}{q - 1} \frac{M_{n-1}^{(C)}(j + p - 1) S_{n}^{(A)}(k+1) b_{n-1}}{(n - 1/2)^{p+q+m-k-1}}
\]

Moreover, \( F(s) \) also has a pole of order \( p + q + m \) at \( s = 1/2 \). Using (2.8) we deduce the (3.12) by a direct calculation. Hence, combining these four residue results, we can obtain the desired evaluation.

**Theorem 3.6** For positive integers \( p, m \) and \( q > 1 \), then

\[
(-1)^{p+q+m} \sum_{n=1}^{\infty} \frac{N_{n+1}^{(B)}(m) N_{n+1}^{(C)}(p)}{n^q} a_n + \sum_{n=1}^{\infty} \frac{N_n^{(B)}(m) N_n^{(C)}(p)}{n^q} a_n
\]

\[
- (-1)^{p+m} \sum_{k=0}^{p+m-1} \frac{(p + q + m - k - 2)}{q - 1} \sum_{n=1}^{\infty} \frac{b_{n-1} c_{n-1} S_n^{(A)}(k+1)}{(n - 1/2)^{p+q+m-k-1}}
\]

\[
- (-1)^m \sum_{j=1}^{m-j} \sum_{k=0}^{m-j} \frac{(j + p - 2)}{p - 1} \frac{(m + q - k - j - 1)}{q - 1} \sum_{n=1}^{\infty} \frac{M_{n-1}^{(C)}(j + p - 1) S_{n}^{(A)}(k+1) b_{n-1}}{(n - 1/2)^{m+q-j-k}}
\]

\[
- (-1)^p \sum_{j=1}^{p-j} \sum_{k=0}^{p-j} \frac{(j + m - 2)}{m - 1} \frac{(p + q - k - j - 1)}{q - 1} \sum_{n=1}^{\infty} \frac{M_{n-1}^{(B)}(j + m - 1) S_{n}^{(A)}(k+1) c_{n-1}}{(n - 1/2)^{p+q-j-k}}
\]

\[+ \text{Res}[F(s), s = 1/2] = 0, \tag{3.13}\]

where

\[
\text{Res}[G(s), s = 0]
\]

\[
= a_0 (-1)^{p+m} \sum_{k_1 + k_2 = q, k_1, k_2 \geq 0} \frac{(m + k_1 - 1)}{k_1} \frac{(p + k_2 - 1)}{k_2} \tilde{t}^{(B)}(m + k_1) \tilde{t}^{(C)}(p + k_2)
\]

\[
- 2(-1)^{m+p} \sum_{j=1}^{[q/2]} \sum_{k_1 + k_2 = q - 2j, k_1, k_2 \geq 0} \frac{(m + k_1 - 1)}{k_1} \frac{(p + k_2 - 1)}{k_2} D^{(A)}(2j) \tilde{t}^{(B)}(m + k_1) \tilde{t}^{(C)}(p + k_2). \tag{3.14}\]
Proof. We consider the kernel function
\[
\pi \cot(\pi s; A) \frac{\Psi^{(m-1)}(1/2-s; B)\Psi^{(p-1)}(1/2-s; C)}{(m-1)!(p-1)!}
\]
and base function \(r(s) = s^{-q}\). By the same calculation as in the proof of Theorem 3.5, we thus immediately deduce (3.13) and (3.14) to complete the proof. \(\square\)

**Theorem 3.7** For positive integers \(p, m\) and \(q > 1\), then
\[
(-1)^{p+q+m} \sum_{n=1}^{\infty} M_n^{(B)}(m) N_n^{(C)}(p) a_n + \sum_{n=1}^{\infty} M_n^{(B)}(m) N_n^{(C)}(p) b_n
\]
\[
+ (-1)^m \sum_{k=0}^{m} \binom{m+q-k-1}{q-1} \binom{p+k-1}{p-1} \sum_{n=1}^{\infty} \frac{N_n^{(C)}(p+k)}{n^{m+q-k}} a_n b_n
\]
\[
- (-1)^m \sum_{k+j \leq m+1, k,j \geq 1} \binom{m-q-k-j}{q-1} \binom{p+k-2}{p-1} \sum_{n=1}^{\infty} \frac{R_n^{(A)}(j) N_n^{(C)}(p+k-1)}{n^{m+q+1-k-j}} b_n
\]
\[
- (-1)^p \sum_{k_1+k_2+k_3=p+1, k_1,k_2,k_3 \geq 0} \binom{m+k_2-1}{m-1} \binom{q+k_3-1}{q-1} \sum_{n=1}^{\infty} \frac{S_n^{(A)}(k_1+1) N_n^{(B)}(m+k_2)}{(n-1/2)^{k_3+q}} c_n
\]
\[
+ \text{Res}[H(s), s = 0] = 0,
\]
where
\[
\text{Res}[H(s), s = 0] = a_0 b_0 (-1)^p \binom{p+q+m-1}{p-1} \hat{r}^{(C)}(p+q+m)
\]
\[
+ a_0 (-1)^{p+m} \sum_{j=1}^{q+1} \binom{j+m-2}{m-1} \binom{p+q-j}{p-1} D^{(B)}(j+m-1) \hat{r}^{(C)}(p+q+1-j)
\]
\[
- 2b_0 (-1)^p \sum_{j=1}^{(m+q)/2} \binom{p+q+m-2j-1}{p-1} D^{(A)}(2j) \hat{r}^{(C)}(p+q+m-2j)
\]
\[
- 2(-1)^{p+m} \sum_{2j_1+j_2 \leq q+1, j_1,j_2 \geq 1} \binom{j_2+m-2}{m-1} \binom{p+q-2j_1-j_2}{p-1}
\]
\[
\times D^{(A)}(2j_1) D^{(B)}(j_2+m-1) \hat{r}^{(C)}(p+q+1-2j_1-j_2).
\]

Proof. We consider the kernel function
\[
\pi \cot(\pi s; A) \frac{\Psi^{(m-1)}(-s; B)\Psi^{(p-1)}(1/2-s; C)}{(m-1)!(p-1)!}
\]
and base function \(r(s) = s^{-q}\). By direct residue computations, we can obtain the desired evaluation. \(\square\)

It is clear that the main results in our pervious paper [7] are immediate corollaries of this paper. Moreover, it is possible that of some other relations involving alternating Euler T-sums
and related sums can be proved by using the techniques of the present paper. For example, let
\( A^{(l)} := \{a^{(l)}_k\}, -\infty < k < \infty \) \((l \text{ is any positive integer})\) be any sequences of complex numbers
with \( a^{(l)}_k = o(k^\alpha) \) \((\alpha < 1)\) if \( k \to \pm\infty\), consider these two function

\[
\cot(\pi s; A) \frac{\Psi(p_{l-1})(1/2 - s; A^{(1)}) \Psi(p_{l-1})(1/2 - s; A^{(2)}) \cdots \Psi(p_{l-1})(1/2 - s; A^{(r)})}{(m-1)!(p-1!)^q(s-1/2)^q}
\]

and

\[
\cot(\pi s; A) \frac{\Psi(p_{l-1})(1/2 - s; A^{(1)}) \Psi(p_{l-1})(1/2 - s; A^{(2)}) \cdots \Psi(p_{l-1})(1/2 - s; A^{(r)})}{(m-1)!(p-1)s^q}
\]

we can deduce the following results

\[
(-1)^{p_1 + \cdots + p_r + q} \sum_{n=1}^{\infty} \frac{N^{(A^{(1)})}_{n}(p_1)N^{(A^{(2)})}_{n}(p_2) \cdots N^{(A^{(r)})}_{n}(p_r)}{(n-1/2)^q} a_{n-1}
\]

\[+ \sum_{n=1}^{\infty} \frac{N^{(A^{(1)})}_{n}(p_1)N^{(A^{(2)})}_{n}(p_2) \cdots N^{(A^{(r)})}_{n}(p_r)}{(n-1/2)^q} a_n
\]

\[+ \sum (\text{sums of degree } \leq r - 1) = 0
\]

and

\[
(-1)^{p_1 + \cdots + p_r + q} \sum_{n=1}^{\infty} \frac{N^{(A^{(1)})}_{n}(p_1)N^{(A^{(2)})}_{n}(p_2) \cdots N^{(A^{(r)})}_{n}(p_r)}{n^q} a_{n-1}
\]

\[+ \sum_{n=1}^{\infty} \frac{N^{(A^{(1)})}_{n}(p_1)N^{(A^{(2)})}_{n}(p_2) \cdots N^{(A^{(r)})}_{n}(p_r)}{n^q} a_n
\]

\[+ \sum (\text{sums of degree } \leq r - 1) = 0,
\]

but we can’t give explicit formulas.

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