The Existence of Time

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Abstract

We quote results of investigations showing that of four common gauge theories of gravity – Poincaré, Weyl, Weyl conformal and biconformal – only the Poincaré, Weyl and biconformal theories lead to general relativity, and only the biconformal gauging leads to new structures of interest. The quotient of the conformal group of any pseudo-Euclidean space by its Weyl subgroup always has symplectic and metric structures. Using the metric and symplectic form, we show that there exist canonically conjugate, orthogonal, metric submanifolds if and only if the original gauged space is Euclidean or signature 0. In the Euclidean cases, the resultant configuration space must be Lorentzian. Therefore, in this context, time may be viewed as a derived property of general relativity.

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1 Introduction

1.1 General relativity as a gauge theory

Since Utiyama[33] wrote the first gauge theory of general relativity, numerous authors have streamlined the procedure. Notably, Kibble[34] extended the gauge group to Poincaré, while Ne’eman and Regge[36] (in the context of supergravity) adapted Cartan’s group quotient methods. These methods were applied extensively by Ivanov and Niederle[39, 38] in the early 1980s to Poincaré, de Sitter, conformal and biconformal gaugings. The result is that there are several ways to formulate general relativity as a gauge theory. Presumably, one of these gaugings is relevant to unification, if only as a low energy limit of string or some alternative TOE.

In this paper, we develop some properties of one of these gaugings. To begin, we briefly recall the group quotient method and review certain basic properties of five gauge theories of gravity, four of which lead to general relativity. We then focus on the biconformal gauge theory, which combines the advantages of maximum symmetry with the presence of structures not present in the other four theories. We examine all dimensions and all signatures.

A gravitational gauge theory based on a homogeneous space, \( S \), of dimension \( n \) may be accomplished by taking the quotient of one of its symmetry groups – Poincaré, inhomogeneous Weyl, or conformal – by a subgroup. The Maurer-Cartan connection on the resulting fiber bundle is then generalized so that the subgroup becomes the local symmetry group over the quotient manifold. In order for this quotient to lead to general relativity, the appropriate subgroup must contain the Lorentz group, and because of the balance of units in all physical equations should probably also include dilatations. In models where the Weyl vector is absent or pure gauge, the dilatational symmetry may be broken by fiat. This is generally accomplished by a global choice of the measure of time.\(^1\)

We briefly consider each of the following quotients. The letter \( I \), for Inhomogeneous, preceeding a group name denotes the extension by translations.

\[
\begin{align*}
\text{Poincaré}/\text{Lorentz} \\
I\text{Weyl}/\text{Weyl} \\
\text{Conformal}/I\text{Weyl} \\
\text{Conformal}/\text{Weyl}
\end{align*}
\]

\(^1\)S. Lloyd tells the amusing anecdote[?], “I recently went to the National Institute of Standards and Technology in Boulder. I said something like, ‘Your clocks measure time very accurately.’ They told me, ‘Our clocks do not measure time.’ I thought, Wow, that’s very humble of these guys. But they said, ‘No, time is defined to be what our clocks measure.’” Indeed, the standard second is defined as the duration of 9,192,631,770 oscillations of the radiation from the transition between the two hyperfine levels of the ground state of the cesium 133 atom.
**Poincaré/Lorentz** This is the simplest way to reproduce general relativity as a gauge theory. The quotient gives a 4-dim manifold with local Lorentz symmetry. The Maurer-Cartan connection consists of just the solder form and the spin connection. The torsion is set to zero by hand, and the spacetime metric is assumed. The Poincaré group is not a simple group, and only global scale changes are permitted.

**IWeyl/Weyl** This produces the well-known Weyl geometry\(^8, \, 9, \, 10\). In addition to local Lorentz transformations we have local dilatations. This is the geometry Dirac adopted for his Large Numbers program\[?, \, ?, \, ?\], and is the local symmetry suggested by Ehlers, Pirani and Schild\[7\]. The curvature-linear action assumed by Dirac gives nottrivial results in the second-order formalism. However, when a Palatini variation is used the Weyl vector becomes pure gauge, the geometry becomes Riemannian, and the theory reduces to general relativity with a cosmological constant and local scale invariance. If an action quadratic in the curvature is used, a nontrivial Weyl geometry results, and the gravitational field equation is similar to that of Weyl conformal gravity rather than general relativity.

**Conformal/IWeyl** Alternatively called conformal gravity, Weyl conformal gravity, or the auxiliary conformal gauging, the action is quadratic in the Weyl curvature. Variation of the metric alone leads to fourth-order field equations, raising the spectre of ghosts. However, recent work by Bender and Mannheim\[?\, ?, \, ?\] claims to eliminate ghosts through an alternative definition of the norm. Our investigation, using the Palatini variation, in which all gauge fields are varied, gives the field equation in terms of the divergence of the Weyl curvature. We use the Bianchi identities to show that solutions are given by spacetimes conformal to Ricci flat spacetimes. This explains how the theory can be ghost-free. However, we find that there remain additional constraints on the curvature so that these geometries probably allow only a proper subset of the solutions conformal to solutions to the Einstein equation. This makes the theory distinct from general relativity.

**Conformal/Weyl** First called the special conformal gauging and later biconformal, this quotient was introduced by Ivanov and Niederlie when they realized that the special conformal transformations serve only an auxiliary function in Weyl conformal gravity. Attempting to retain these gauge fields nontrivially in the model, they retained them on the quotient manifold in addition to the usual \( n \) spacetime dimensions. Subsequent work shows that this gauging permits a class of actions linear in the curvature. The field equations on the resulting \( 2n \)-dim manifold describe general relativity on the co-tangent bundle of \( n \)-dim spacetime. The \( 2n \)-dimensional manifold may be given complex structure, resulting in twistor space, \( \mathbb{C}^4 \). However, the gauging described here endows twistor space with additional metric and symplectic structures. These structures are central to the current work.

We see that three of these four gaugings have satisfactory interpretations as formulations of general relativity.

Here we consider the maximal case, in which we take the quotient of the conformal group by its Weyl subgroup. The result is a \( 2n \)-dim symplectic manifold called biconformal space, which becomes the co-tangent bundle of a gravitating \( n \)-dim spacetime in the gravity theory. This symplectic manifold may be given a complex structure. With complex variables we recognize the biconformal space as twistor space modified by additional metric and symplectic structures arising naturally from the conformal group.

It is clear that twistor space, \( \mathbb{C}^4 \), has both Euclidean and Lorentzian submanifolds. However, the symplectic and metric structures of biconformal space tighten the connection between these two, giving a nearly unique correspondence between certain Euclidean and Lorentzian submanifolds. Here we present this result in detail.

### 1.2 General relativistic gauge theory and the existence of time

From the point of view of modern physics, the existence of time is signalled the Lorentzian symmetry of spacetime, and in particular the signature, \( s = 3 - 1 = 2 \), (or, in \( n \) dimensions, \( s = n - 2 \)) of the spacetime metric. This Lorentzian metric with its invariant light cones gives us a universal notion of causality, past, present and future. It also gives us a pseudo-orthogonal symmetry group, \( SO(3, \, 1) \), instead of a neatly Pythagorean \( SO(n) \).

We present a scenario by which Lorentzian signature arises from within a conformal gauge theory of general relativity.

Biconformal gauging of the conformal group gives rise to both a natural metric and a natural symplectic structure. We gauge an arbitrary, \( p + q = n \) dimensional pseudo-Euclidean space \( S_{p, \, q} \) of signature \( s_0 = p - q \), to produce the corresponding biconformal geometry. Our central theorem states that orthogonal, canonically conjugate, metric submanifolds of biconformal space exist if and only if the original \( n \)-dim space is Euclidean \((s_0 = \pm n)\), or of vanishing
signature \((s_0 = 0)\). The signature of the induced metric on the configuration space is \(\pm (n - 2)\) (Lorentzian) or 2, respectively. When \(n = 4\) or \(n = 4m + 2\), the configuration space is always a Lorentzian spacetime.

Our presentation proceeds as follows. In the next Section, we develop the zero-curvature case of biconformal gauge theory. This is sufficient for a general proof. In Section 3, we introduce the conformal Killing metric and its restriction to biconformal space, describe the natural symplectic form, and make the definitions required to state our central theorem. This material is presented in complete detail because it is important to our conclusion that the symplectic and metric structures we discuss are determined uniquely by properties of the conformal group. The Lorentzian structure that emerges does not depend on any outside assumptions.

The subsequent Sections present the various elements of the proof. They will be described once we state the central theorem. The proof is given in detail because it is in the details of the proof that we discover the unique emergence of a Lorentzian metric on configuration space. We conclude, in Section 9, with a brief discussion of our results.

## 2 The Signature Theorem

In this Section, we provide basic definitions required for the statement of our central theorem, followed by a statement of the theorem and an outline of the proof. The remaining Section provide details of the proof.

### 2.1 The construction of biconformal space

Let \(\mathcal{F}^n_{p,q}\) be a pseudo-Euclidean space of signature \(s_0 = p - q\) and dimension \(n = p + q\). Let the \(SO(p,q)\)-invariant metric on this space be

\[
\eta_{ab} = \text{diag}(1, \ldots, 1, -1, \ldots, -1)
\]

with inverse \(\eta^{ab}\). Compactify the space by appending a point at infinity for every null vector from the origin. For Euclidean space, \(p = \pm n\), this will be a single point inverse to the origin; for Minkowski space, \(p = n - 1\), a single light cone is required. With the exception of the Euclidean cases, the null subspace is of dimension \(n - 1\).

We may now define the conformal group of \(\mathcal{F}^n_{p,q}\) as the set of transformations which preserve the metric, or equivalently the infinitesimal line element,

\[
ds^2 = \eta_{ab} dx^a dx^b
\]

up to an overall factor. The conformal transformations include \(SO(p,q)\) transformations, translations, dilatations, and special conformal transformations; together these make up \(SO(p + 1,q + 1)\).

The Lie algebra \(so(p + 1,q + 1)\) may be written in terms of basis 1-forms as the Maurer-Cartan structure equations,

\[
d\omega^\Lambda = -\frac{1}{2} c_{\Sigma\Lambda}^\Delta \omega^\Sigma \omega^\Lambda
\]

where \(c_{\Sigma\Lambda}^\Delta\) are the structure constants of the conformal group (see Appendix 1) and

\[
\omega^\Lambda \in \left\{ \omega^a = -\eta^{ac} \eta_{bd} \omega^c_d, \omega^a, \omega_b, \omega \right\}
\]

where capital Greek indices run over all \(\frac{(n+1)(n+2)}{2}\) dimensions of \(SO(p + 1,q + 1)\), and lowercase Latin indices take values of 1 to \(n\). These forms are dual to the generators of \(SO(p + 1,q + 1)\): \(\omega^a\) are dual to the generators of \(SO(p,q)\), \(\omega^a\) are dual to the generators of translations, \(\omega_b\) are dual to the generators of special conformal transformations, and \(\omega\) is dual to the generator of dilatations. Expanding eq. (2),

\[
d\omega^a_b = \omega^c_b \omega^a_c + 2 \Delta^a_b \omega^c
\]

\[
d\omega^a = \omega^c_a \omega^a_c + \omega \omega^a
\]

\[
d\omega_a = \omega^c_a \omega_c + \omega \omega_a
\]

\[
d\omega = \omega^a \omega_a
\]

where we define \(\Delta^a_b = \frac{1}{2} \left( \delta^a_d \delta^c_b - \eta_{bd} \eta^{ac} \right)\). These are the Maurer-Cartan equations for the conformal group with respect to \(\eta_{ab}\). They are completely equivalent to the Lie algebra commutation relations, with the Jacobi identity following from the integrability condition \(d^2 = 0\).
Definition A flat biconformal space is the quotient of the conformal group by its homothetic Lie subgroup (i.e., \(SO(p,q)\) and dilatations). It is a principal fiber bundle with homothetic local symmetry group and \(2n\)-dim base manifold (\([38]-[40]\)).

The properties we study below, including metric, symplectic structure, and submanifolds, all generalize directly to curved biconformal spaces. Our results therefore apply immediately to this more general class of spaces by continuity. Since biconformal spaces have local dilatational symmetry, definitions and theorems that would refer to a manifold being flat (i.e., vanishing Reimann tensor) are generalized to mean that the Weyl curvature tensor vanishes. Then there exists a conformal gauge in which the space is flat.

Flat biconformal spaces are described by eqs. (3-6), with the connection forms \(\omega_a^b\), \(\omega_a\) taken to be horizontal, i.e., expandible in terms of the basis forms \(\omega^a\) and \(\omega_a\). Solutions to these equations are given elsewhere (\([38]-[29]\)); however we present a new solution satisfying certain conditions described in the next Section. The generality of this new solution is important in establishing our central result.

Now we turn to a discussion of the Killing metric and symplectic structure of the conformal group, and a precise statement of our central theorem.

2.2 Natural metric and symplectic structure

The Killing metric of the conformal group is built from the structure constants \(c_{\Sigma\Lambda}^\Delta\) as the symmetric form

\[
K_{\Sigma\Lambda} = \lambda c_{\Delta\Sigma}^\Theta c_{\Theta\Lambda}^\Delta,
\]

where \(\lambda\) is any convenient constant. Substituting the conformal structure constants and choosing \(\lambda = \frac{1}{2n}\) yields the form,

\[
K_{\Sigma\Lambda} = \begin{pmatrix}
\frac{1}{2}\Delta^{ac}_{db} & 0 & 0 & 0 \\
0 & 0 & \delta^a_b & 0 \\
0 & \delta^b_a & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

where \(\Delta^{ac}_{db}\) corresponds to \(SO(p,q)\) transformations, the middle double block to translations and special conformal transformations, and the final 1 in the lower right to dilatations. Notice that, of the five gauge theories discussed in the introduction, the biconformal case is the only one where the restriction of \(K_{\Sigma\Lambda}\) to the base manifold is non-degenerate. For example, when \(K_{\Sigma\Lambda}\) above is restricted to the Poincaré group it becomes

\[
K_{\mu\nu} = \begin{pmatrix}
\frac{1}{2}\Delta^{ac}_{db} & 0 \\
0 & 0
\end{pmatrix}
\]

and when this is restricted to the Poincaré/Lorentz quotient manifold it vanishes. By contrast, when restricted to the biconformal manifold \(K_{\Sigma\Lambda}\) is the nondegenerate metric

\[
\begin{pmatrix}
0 & \delta^a_b \\
\delta^b_a & 0
\end{pmatrix}
\]

We therefore have a natural metric on both the full group manifold and on the \(2n\)-dim biconformal space.

In addition to the metric, even curved biconformal spaces generically have a natural symplectic structure. Since \((\omega^a, \omega_a)\) together span the base manifold, the dilatational structure equation

\[
d\omega = \omega^a \omega_a
\]

is necessarily a closed, non-degenerate two form.

We now turn to the question of when biconformal spaces coincide with our usual notions of a relativistic phase space. We begin with the definition (\([45]\)).

Definition (Abraham, Marsden) A phase space is a symplectic manifold which is the cotangent bundle of a Riemannian or pseudo-Riemannian manifold.

It would seem that the principal difficulty in identifying a biconformal space with a phase space is that biconformal spaces admit full \(2n\)-dim curvature, and therefore cannot always be identified with cotangent bundles. Remarkably, it has been shown that all torsion-free biconformal spaces solving the curvature-linear field equations are cotangent bundles (\([40]\)). Instead, it turns out that the central issue relating biconformal and phase spaces is the metric. The most
interesting feature of our proof hinges on the signatures of the induced metrics. This is why we only need to consider flat biconformal space. By continuity the results apply to more general spaces.

There is an important difference between the metric of a phase space and the metric of a biconformal space. The use of a Riemannian manifold in the Abraham-Marsden definition implies the presence of a metric on the configuration manifold. This induces an inner product on the cotangent spaces, so the metric on the momentum submanifolds is inverse (since $p_a$ is covariant) to the flat form of the configuration metric. There is not, in general, a single metric on the entire phase space.

By contrast, there is a metric on the entirety of a biconformal space. The Killing metric of the conformal group, restricted to the biconformal submanifold, is nondegenerate. This means that if the biconformal space is identified with a phase space, the configuration and momentum metrics are a priori independent. Nonetheless, the proof below shows that in the majority of cases the induced metric on the configuration space is uniquely Lorentzian, and the metric on the momentum spaces is the negative of the inverse to the corresponding flat Lorentz metric. Thus, the existence of time may be attributed to the necessity for the Lorentz signature of the configuration submanifold. Our concluding remarks focus on this point. It has been argued that the imaginary unit in Dirac’s quantization rule, replacing Poisson brackets by $-\hbar$ times the commutator, may be attributed to the relative minus sign between the configuration and momentum space metrics.

In order to precisely relate the biconformal metric to the configuration space metric, we make the following definition:

**Definition** A **metric phase space** is a phase space with metric, having a basis $(\chi^a, \eta_b)$ such that the following conditions hold:

1. $\chi^a$ and $\eta_b$ are canonically conjugate.
2. $\chi^a$ and $\eta_b$ are orthogonal with respect to the metric,

$$\langle \chi^a, \eta_b \rangle = 0 \quad (8)$$

while the induced configuration space metric,

$$g^{ab} \equiv \langle \chi^a, \chi^b \rangle \quad (9)$$

is non-degenerate. It follows that the momentum space metric is also non-degenerate. Orthogonality is required so that separate configuration and momentum space metrics are well-defined.

3. $\chi^a$ and $\eta_b$ are seperately involute. Thus, the conditions $\chi^a = 0$ and $\eta_b = 0$ each provide a projection to an $n$-dim metric submanifold. The first is called momentum space; the second is called configuration space.

Since a metric phase space is a phase space, it is an even dimensional manifold, $\mathcal{M}$, with symplectic structure. Since it must be a cotangent bundle, the momentum submanifold must be flat. We immediately have the following lemma.

**Lemma** There is a conformal gauge in which flat biconformal space is a metric phase space if there exists a basis, $(\chi^a, \eta_b)$, such that the following conditions hold:

1. In terms of $(\chi^a, \eta_b)$, eq.(6) takes the form $d\omega = \chi^a \eta_a$.
2. $\chi^a$ and $\eta_b$ are orthogonal with respect to the Killing metric of the conformal group, i.e., $\langle \chi^a, \eta_b \rangle_K = 0$.
3. The structure equations for $\chi^a$ and $\eta_b$ are seperately involute.
4. The Weyl curvature of the momentum submanifold vanishes.

**Proof** Since $d\omega$ is the symplectic form, condition 1 holds if and only if $\chi^a$ and $\eta_b$ are canonically conjugate. Since the Killing metric is nondegenerate, and $\langle \chi^a, \eta_b \rangle = 0$, the inner products $g^{ab} \equiv \langle \chi^a, \chi^b \rangle$ and $g'_{ab} \equiv \langle \eta_a, \eta_b \rangle$ are necessarily non-degenerate. Condition 3 is unchanged from the definition of a metric phase space. Finally, condition 4 guarantees the existence of a gauge in which the momentum submanifold is flat.

These conditions on biconformal space produce submanifolds which can be identified with the configuration manifold and cotangent spaces used in the usual construction of phase space. We include the possibility of arbitrary signature for the metric submanifolds, though we will have considerably more to say about this below.
2.3 The Signature Theorem

Now we come to the main theorem of this paper and its corollaries. The full theorem applies to any initial dimension and signature:

**Theorem (Signature Theorem)** Flat 2n-dim (n > 2) biconformal space is a metric phase space if and only if the signature, s, of \( \eta_{ab} \) is \( \pm n \) or 0. These three possibilities lead to the following signatures for the configuration submanifold:

\[
\begin{align*}
\text{configuration space} & : s = n - 2 \text{ (Lorentz)} \\
\text{configuration space} & : s = -n \text{ (Lorentz)} \\
\text{configuration space} & : s = 0
\end{align*}
\]

The signature of the momentum submanifold is always the negative of the signature of the configuration submanifold. Since we can have \( s = 0 \) only if \( n \) is even, configuration space is uniquely Lorentz if \( n \) is odd.

Since, when \( n = 4 \), the signature \(-2\) is also Lorentzian, we have the immediate corollary,

**Corollary** Flat 8-dim biconformal space reduces to a metric phase space if and only if the initial 4-dim space is Euclidean or signature zero, and the resulting configuration space is necessarily Lorentzian.

The details of the proof lead to the following additional conclusions:

1. There exists a basis \((\chi^a, \eta_b)\) such that the connection takes the form

\[
\begin{align*}
\omega^b_a & = 2\Delta^{ab}_{cd} (y_c - d_c) \, dv^d + 2 \frac{y^c}{y^2} \Delta^{ab}_{cd} \eta^{de} y_c \eta_e \\
\chi^a & = dv^a \\
\eta_a & = dy^a + (y_b d_a + y_a d_b - \eta_{ab} (\eta^{ef} y_e d_f)) \, dv^b \\
\omega & = -y_a dv^a - \frac{1}{y^2} \eta^{ab} y_a dy_b
\end{align*}
\]

where \( \eta_{ab} \) is given by eq. (1), and where either

\[
d_a = \frac{\eta_{ab} v^b}{y^2}
\]

or

\[
d_a = \frac{c_a}{a_0 + c_a v^a}
\]

where \( a_0 \) is a constant and \( c_a \) is a constant null vector in the original metric, \( \eta^{ab} c_a c_b = 0 \). The coordinates \( v^a \) and \( y_a \) are defined in the proof.

2. In the \((\chi^a, \eta_a)\) basis, the Killing metric is given by

\[
\begin{align*}
\langle \chi^a, \chi^b \rangle & = -\frac{1}{(y^2)^2} \left( 2y^a y^b - y^2 \eta^{ab} \right) \\
\langle \chi^a, \eta_b \rangle & = 0 \\
\langle \eta_a, \eta_b \rangle & = 2y_a y_b - y^2 \eta_{ab}
\end{align*}
\]

**Proof (outline)** The proof of the Signature Theorem, comprising the bulk of the remaining Sections, is accomplished by imposing the conditions of the Lemma. Here we outline the sequence of demonstrations:

1. In Section 3 we write a general linear transformation between bases \((\chi^a, \eta_b)\) and \((\omega^a, \omega_b)\), then impose condition 2 of the Lemma, using the Killing metric. This gives two important results. First, the metric on the configuration manifold \( g \) and the inverse metric on the tangent space \( g' \) satisfy the relationship \( g' = -g^{-1} \). Second, the exact expansion of the basis \((\chi^a, \eta_b)\) in terms of the original basis \((\omega^a, \omega_b)\) and \( g \) given in eq. (18).
below. This expansion will be used to recast the Maurer-Cartan form and structure equations in the new basis $(\chi^a, \eta_b)$. An arbitrary change of basis within the individual $\chi^a$ and $\eta_b$ submanifolds is allowed by our axioms, so we define another metric $h$ as the metric $g$ simplified by this restricted kind of transformation. We use $h_{ab}$ to denote this metric for the rest of the paper. We conclude Section 3 with the involution conditions required by condition 3 of the Lemma.

2. Section 4 is devoted to proving that the conformal flatness required by the Lemma implies

$$h^{ab} = \frac{(n-2)h}{u^2} \left( -2u^a u^b + u^2 \eta^{ab} \right)$$

for some vector field $u^a$.

3. In Section 5 we solve the structure equations on the momentum submanifold (setting $\chi^a = 0$), and find another distinct form for the metric $h_{ab}$. In a suitable gauge the metric depends on a single function $\sigma$ and its $y_a$ derivatives,

$$h^{ab} = e^{-2\sigma} \left( -2\sigma^{ab} + 2\sigma^a \sigma^b - \sigma^c \sigma^d \eta_{cd} \eta^{ab} \right)$$

We relate the two forms of $h^{ab}$ found in Sections 4 and 5. We find that $u^a$ must be a gradient with respect to $y_a$, $u^a = \frac{\partial u}{\partial y_a}$, for some function $u$, where $u$ and $\sigma$ must satisfy the coupled differential equations:

$$\sigma^{ab} = \sigma^b \sigma^a + \beta u^a u^b - \frac{1}{2} \left( \sigma^2 + \beta u^2 \right) \eta^{ab}$$

$$u^{ab} = u^a \sigma^b + u^b \sigma^a - (u_\sigma \sigma) \eta^{ab}$$

Throughout, we denote derivatives with respect to $\nu^a$ and $y_a$ by $\frac{\partial f}{\partial \nu^a} = f_{\nu^a}$ and $\frac{\partial f}{\partial y_a} = f_{y_a}$, respectively. In Section 7, we find all solutions to these coupled equations, and the corresponding forms for $h_{ab}$.

4. In Section 6, we extend these results back to the full biconformal space and solve the remaining structure equations. The results prove the second and third statements of the theorem.

5. Finally, in Section 7, we look at the signature of the Killing metric on both the configuration manifold and the momentum space. Imposing consistency of the signature across the cotangent bundle proves the last statement in the theorem.

We end with a discussion of the physical meaning of the results.

### 3 Orthogonal symplectic bases

We begin by expressing the biconformal structure equations in an orthogonal symplectic basis. A general basis is given by

$$\chi^a = A^a b \omega^b + B^a \omega_b$$

$$\eta_a = C_a b \omega^b + D_{ab} \omega^b$$

We demand two conditions. First, the inner products must be non-degenerate and orthogonal, according to eqs. (9) and (8). Once we guarantee orthogonality, the nondegeneracy of the Killing metric insures that $g^{ab}$ and $\langle \eta_a, \eta_b \rangle = g'_{ab}$ are non-degenerate. Second, we require the basis to be canonical, $d\omega = \omega^a \omega_a = \chi^a \eta_a$. Substituting, and recasting all three orthogonality and three conjugacy conditions in matrix notation, we find the conditions,

$$g = BA^t + AB$$

$$0 = AC^t + BD^t$$

$$g' = C'D + D'C$$

$$0 = A'D - D'A$$

$$1 = A'C - D'B$$

$$0 = B'C - C'B$$

(10) (11) (12) (13) (14) (15)
where $g$ and $g'$ are non-degenerate and symmetric.

To solve these equations, multiply eq.\ref{eq:10} equation by $D'$, then use eq.\ref{eq:13},

\[
D'g = D'BA' + D'AB' \\
= D'BA' + A'DB'
\]

Now use the transpose of eq.\ref{eq:11} to replace $DB'$,

\[
D'g = (D'B - A'C)A' \\
= -A'
\]

where the last follows from eq.\ref{eq:14}. Therefore,

\[
D = -g^{-1}A 	ag{16}
\]

The solution for $C$ is similar. We multiply eq.\ref{eq:10} on the right by $C$, use eq.\ref{eq:15}, then eq.\ref{eq:11}, and finally eq.\ref{eq:14} to get

\[
C = g^{-1}B. \tag{17}
\]

Substituting eqs.\ref{eq:16} \ref{eq:17} into the conjugate and metric equations, two are satisfied identically while the remaining four simplify to

\[
g = 2BA' = g' \\
g' = -g^{-1} \\
1 = A'g^{-1}B + A'g^{-1}B
\]

The non-degeneracy of $g$ now implies the non-degeneracy of $A$ and $B$. Therefore we may solve for $B$ to get $B = \frac{1}{2}g(A')^{-1}$. The final equation is then identically satisfied.

The most general orthogonal, canonical basis may therefore be written as

\[
\chi^a = A^a \_{\ b} \left( \omega^b + \frac{1}{2} h^{bc} \omega_c \right) \\
\eta_a = (A^{-1})^d \_{\ a} \left( \frac{1}{2} \omega_d - h_{dc} \omega^c \right)
\]

where we have defined

\[
h = A^{-1}g(A')^{-1}
\]

and its inverse, $h_{ab}$. The transformation $A^a \_{\ b}$ is simply a change of basis within the $\chi^a$ and $\eta_a$ submanifolds,

\[
\tilde{\chi}^a = A^a \_{\ b} \chi^b \\
\tilde{\eta}_a = g_{ab}A^b \_{\ c}g^{cd} \eta_d
\]

which our axioms always allow. Therefore, up to changes of basis within the conjugate submanifolds, the most general orthogonal, symplectic basis is:

\[
\chi^a = \omega^b + \frac{1}{2} h^{bc} \omega_c \\
\eta_a = \frac{1}{2} \omega_a - h_{ab} \omega^b \\
\omega^a = \frac{1}{2} (\chi^a - h_{ab} \eta_b) \\
\omega_a = \eta_a + h_{ab} \chi^b
\]
In terms of these, direct substitution into the structure equations, eqs. (3-6), yields

\[ \text{d}\omega_b^a = \omega_b^c \omega_c^a + \Delta_{ab}^{ac} (\eta_c + h_{cd} \chi^d) \left( \chi^d - h^{de} \eta_e \right) \]  

(19)

\[ \text{d}\chi^a = \chi^c \omega_c^a + \omega^a \chi^a + \frac{1}{2} \text{D}h^{ac} \left( \eta_c + h_{cd} \chi^d \right) \]  

(20)

\[ \text{d}\eta_a = \omega^b \eta_b - \omega \eta_a - \frac{1}{2} \text{D}h_{ab} \left( \chi^b - h^{bc} \eta_c \right) \]  

(21)

\[ \text{d}\omega = \chi^a \eta_a \]  

(22)

where we have written the results in terms of the \( \text{SO}(p,q) \) and Weyl-covariant derivatives of \( h_{ab} \) and \( h^{ab} \),

\[ \text{D}h_{ab} = \text{d}h_{ab} + h^{cd} \omega_c \omega_d - h^{ac} \omega_c^b - 2h_{ab} \omega \]  

(23)

\[ \text{D}h^{ac} = \text{d}h^{ac} - h_{cd} \omega_a \omega_d - h^{ad} \omega_c^b + 2h_{ab} \omega \]  

(24)

Eqs. (80-83) describe the spaces we wish to study. Our goal, over the next few Sections, is to solve these equations for the connection 1-forms, \( \omega^a_b \) and \( \omega \), and the basis forms \( \chi^a, \eta_a \), subject to two further conditions required by the Lemma. According to condition 3, both \( \chi^a \) and \( \eta_a \) must be in involution. Also, in order for there to exist a conformal gauge in which the biconformal space is a cotangent bundle, the momentum submanifold must be conformally flat. We end this Section with a discussion of the involution conditions, then take up conformal flatness and related conditions in Section 5.

In order for \( \chi^b = 0 \) and \( \eta_b = 0 \) to specify submanifolds, each of the corresponding structure equations, eqs. (81-82), must be in involution. Therefore, we demand

\[ \text{D}h^{ac} \eta_c \sim \chi^a \]

\[ \text{D}h_{ab} \chi^b \sim \eta_a \]

These conditions will be satisfied by assuming they hold and using the resulting two involutions to study the momentum submanifold separately. Imposing conformal flatness places strong constraints on the corresponding part of the connection. Then, extending back to the full biconformal space in Section 8, we find the full connection. This procedure leads, in Subsection 8.2.2, to final forms for the structure equations, eqs. (80-83), and these final forms are easily checked to be involute, thereby showing that the solution provides necessary and sufficient conditions for involution.

### 4 Conformal flatness of the momentum submanifold

As a first step in solving the structure equations, we study the solution of the reduced set of structure equations describing the momentum submanifold, which arises from the involution of \( \chi^a \).

To start, we assume \( \chi^a \) is in involution. By the Frobenius theorem, there exist \( n \) coordinates \( \nu^a \) such that \( \chi^a = \chi^a_{\beta} \text{d} \nu^\beta \). Holding \( \nu^a \) constant then restricts to submanifolds described by setting \( \chi^a = 0 \) in eqs. (80-83). This gives

\[ \text{d} \beta^a_b = \beta^a_{\beta} \nu^\beta - \Delta^{ac}_{\beta \gamma} \nu^\gamma \eta_a \eta_c \]  

(25)

\[ \text{d} \eta_a = \beta^b_a \eta_b - \tau \eta_a \]  

(26)

\[ \text{d} \tau = 0 \]  

(27)

where \( \beta^a_b \equiv \omega^a_b |_{\chi^a = 0} \) and \( \tau = \omega |_{\chi^a = 0} \) may be expanded in terms of \( \eta_a \) only. We also have the inverse metric, \( h_{ab} = - \langle \eta_a, \eta_b \rangle \) and the constraint condition,

\[ 0 = \text{D}h^{ac} |_{\chi^a = 0} \eta_c = h^{ac}_{\beta \gamma} \eta_b \eta_c \]

Note that on the submanifold the constraint follows automatically from the structure equation for \( \eta_a \), eq. (26), since the covariant constancy of the basis implies the constancy of the associated metric, \( \text{D}_{(\beta)} h^{ac} (\nu) = 0 \). A similar set of equations, found by setting \( \eta_a = 0 \), describes the configuration submanifold, including \( \text{D}_{(\alpha + \beta)} h^{ac} (\nu, \nu) = 0 \). This must not be construed to mean that the full covariant derivative of the metric, \( \text{D}_{(\alpha + \beta)} h^{ac} (\nu, \nu) \) vanishes. It does not.
Eqs. (25-27) may be interpreted as those of a Riemannian geometry if we choose a gauge where $\tau = 0$, or trivial Weyl geometry if $\tau = d\phi$. In either case, the solder form is $\eta_a$, the spin connection is $\beta^a_c$, and the curvature 2-form is

$$R^a_b = d\beta^a_c - \beta^a_d \beta^d_c = -\Delta^a_{de} h^{de} \eta_e \eta_c$$

We also have the inverse metric, $h_{ab} = -\langle \eta_a, \eta_b \rangle$.

However, for this interpretation to be valid, eq. (28) must satisfy the Bianchi identities of eqs. (25-27). In addition, if $h_{ab}$ is to be the metric then it must preserve the antisymmetry of $R^a_b$, that is, $R^a_b = -h_{bc} h_{ad} R^c_d$.

We examine these two conditions in the next two Subsections, using the structure equations on the momentum submanifold. Then, in the final Subsection, we impose conformal flatness.

### 4.1 Bianchi identities

The Bianchi identities on the momentum submanifold follow by taking the exterior derivative of eqs. (25-27) and using the Poincaré lemma, $d^2 = 0$. This guarantees that the equations are integrable. We begin with the trivial case of $\tau$, and work our way up.

#### 4.1.1 Dilatation Bianchi identity

The dilatation Bianchi equation immediately gives

$$d^2 \tau = 0$$

#### 4.1.2 Solder form Bianchi identity

The equation for the basis form $\eta_a$ gives rise to the Bianchi identity:

$$0 \equiv d^2 \eta_a = d\beta_a^b \eta_b - \beta_a^b d\eta_b - d\tau \eta_a + \tau d\eta_a$$

Substituting for $d\beta_a^b$, $d\eta_b$ and $d\tau$ from eqs. (25-27) and simplifying, we find

$$0 = h^{be} \eta_a \eta_c \eta_b$$

which is identically satisfied by the symmetry of the metric.

#### 4.1.3 Spin connection Bianchi identity

Finally, the $SO(p,q)$ equation for the spin connection gives

$$0 \equiv d^2 \beta^a_c = d\beta^a_d \beta^d_c - \Delta^a_{de} d\eta_e \eta_c - \eta_a f \eta_d h^{de} \eta_c$$

We again substitute for $d\beta^a_b$ and $d\eta_b$. Simplifying, and expressing the result in terms of the covariant derivative of the metric, as given by the restriction of eq. (23) to the submanifold,

$$Dh^{ab} \bigg|_{\chi=0} = h^{abc} \eta_c$$

the result is

$$0 = \left( -\delta_b^f h^{ag} e + \delta_b^g h^{af} e - \eta^{af} \eta_{bd} h^{de} + \eta^{ae} \eta_{bd} h^{de} \right) \eta_c \eta_f \eta_g$$

If we define

$$v^c \equiv \eta_{de} \left( h^{dec} - h^{dce} \right) = \eta_{de} h^{d[ce]}$$

then the trace on $fb$ shows that

$$h^{a[b|c]} = \frac{1}{2(n-1)} \left( \eta^{ab} v^c - \eta^{ac} v^b \right)$$

(29)
This has no further nonvanishing trace. Substituting this condition into the full Bianchi identity satisfies the full equation, so eq. (29) is the necessary and sufficient condition on $h_{ab}$ from the $SO(p, q)$ Bianchi identity. The involution condition, however, is $Dh_{ac} \sim \chi^e$ so on the $\chi^a = 0$ submanifold,

$$h^{abc} - h^{acc:b} = 0$$

and eq. (29) is satisfied.

### 4.2 Antisymmetry of the curvature

Next we observe that $h_{ab}$ is constrained by the antisymmetry of the curvature. Demanding

$$h_{bc}h^{ad}R_{cd} = -R_b^a$$

leads to the condition,

$$h_{bf} \eta^f_e (\eta_{bg} h^{ae} h^{de}) - h_{bf} \eta^f_e \eta_{bg} h^{de} = \eta^{ae} \eta_{bd} h^{de} - \eta^{ac} \eta_{bd} h^{de}$$

which, by taking a trace, is easily shown to hold if and only if

$$h_{ab} = \lambda \eta^{ac} \eta_{bd} h^{cd}$$

(30)

where $\lambda = \eta_{cd} h^{cd} / h_{bc} \eta^{bc}$. Thus, the inverse of the original metric, $\eta^{ab}$, must relate the metric $h_{ab}$ to its inverse, $h_{ab}$.

### 4.3 Conformal flatness

To guarantee a gauge in which biconformal space is the cotangent bundle structure of a phase space, we require conformal flatness. This is accomplished by demanding that the conformal curvature tensor vanish,

$$C^a_{bcd} = 0$$

where the traceless part of the Riemann curvature, $C^a_{bcd}$, is given by

$$C^a_{bcd} = R^a_{bcd} - \frac{1}{n-2} \left( \delta^a_b R^{ac} - \delta^a_c R^{bd} + h^{ac} R_b^d - h^{ad} R_b^c \right) + \frac{1}{(n-1)(n-2)} R \left( \delta^a_b h^{cd} - \delta^c_b h^{ad} \right)$$

For $R^a_{bcd}$ as given by eq. (28), the components the Riemann tensor are

$$R^a_{bcd} = \frac{1}{2} \left( \delta^a_b h^{ac} - \delta^c_b h^{ad} + \eta^{ac} \eta_{bd} h^{cd} - \eta^{ad} \eta_{cd} h^{bc} \right)$$

Defining $h = \eta_{cd} h^{cd}$ and, in agreement with eq. (30), $\eta_{cd} h^{ab} = \lambda h$, the vanishing of the conformal curvature implies

$$0 = -\delta^a_b h^{ac} + h^{ad} \delta^c_b + \eta^{ac} \eta_{cd} h^{cd} - \eta^{ad} \eta_{cd} h^{bc}$$

$$- n(n-2) + (h_{ad} \eta^{ad}) \left( \eta_{ef} h^{ef} \right) \left( \delta^a_b h^{cd} - \delta^c_b h^{ad} \right)$$

$$- \eta_{ef} h^{ef} \left( h^{ac} h_{bg} \eta^{bd} - h^{ad} h_{bg} \eta^{cd} \right)$$

$$+ \frac{1}{n-2} \left( \delta^a_b \eta^{ad} - \delta^d_b \eta^{ac} \right) \left( \eta_{ef} h^{ef} \right)$$

(31)

Though all traces of $C^a_{bcd}$ using the metric $h^{ab}$ vanish automatically, we get a nontrivial condition by contracting with $\eta_{ad}$. Contracting, raising the lower index, and collecting terms, yields

$$0 = \left( \lambda h^2 - (n-2)^2 \right) \left( h^{ab} - \frac{1}{n} \lambda h \eta^{ab} \right)$$
so either \( h^{bc} = \frac{\lambda}{n} \eta^{ch} \), or \( \lambda h^2 - (n - 2)^2 = 0 \). The first condition is immediately seen to be sufficient to guarantee vanishing conformal curvature. We seek a sufficient condition when the second condition holds.

Expanding the second condition, we have

\[
\left( \eta_{cd} h^{cd} \right) \left( h_{ab} \eta^{ab} \right) = (n - 2)^2
\]

Substituting this into eq.(31) and raising an index, we find that eq.(31) factors as

\[
0 = k^{ac} k^{bd} - k^{ad} k^{bc}
\]

where

\[
k^{ab} = h^{ab} - \frac{n - 2}{h} \eta^{ab}
\]

Now, since \( k^{ab} \) is symmetric, it may be diagonalized. It will then have at least one nonzero eigenvalue, which we may take to be \( k^{11} \) without loss of generality. Letting \( a = c = 1 \) in eq.(32),

\[
k^{bd} = k_{1d} k_{1b}^{\frac{1}{k^{11}}}
\]

so all elements are determined by a single vector, \( u^a = k^{1a} \),

\[
k^{ab} = \rho u^a u^b
\]

Contracting,

\[
\eta_{ab} k^{ab} = \lambda h - \frac{n (n - 2)}{h} = \rho u^2
\]

Since \( \lambda h^2 = (n - 2)^2 \), we see that \( \rho u^2 = -\frac{2(n - 2)}{h} \), and in particular, \( u^2 \neq 0 \).

Re-expressing this result in terms of the the metric and its inverse yields the forms

\[
h^{ab} = \frac{n - 2}{hu^2} \left( -2u^a u^b + u^2 \eta^{ab} \right)
\]

\[
h_{ab} = \frac{h}{(n - 2)u^2} \left( -2u_a u_b + u^2 \eta_{ab} \right)
\]

and we have determined \( h^{ab} \) in terms of \( u^a \), up to an overall conformal factor. This is the sufficient condition we were seeking.

**Summary:** Let \( h^{ab} \) be symmetric and \( h = \eta^{ab} h_{ab} \) an arbitrary function. Then a space with curvature

\[
R^a_b = -\Delta_{de}^{ac} h_{de} \eta_c \eta_e
\]

is conformally flat if and only if \( h^{ab} \) has one of the forms

\[
h^{ab} = \frac{n}{h} \eta^{ab}
\]

or

\[
h^{ab} = \frac{(n - 2)}{hu^2} \left( -2u^a u^b + u^2 \eta^{ab} \right)
\]

for some vector \( u^a \).

It is important to stress that these are the only allowed forms of the conformally flat submanifold metric. The first possibility, with \( h_{ab} \) conformal to the original metric is the solution we would naively expect. However, it is straightforward to show that the resulting canonical basis,

\[
\chi^a = \omega^b + \frac{n}{2h} \eta^{bc} \omega_c
\]

\[
\eta_a = \frac{1}{2} \omega_a - \frac{h}{n} \eta_{ab} \omega^b
\]
is not involute, for any choice of the conformal factor. This is done by expressing $d\chi^a$ and $d\eta^a$ in terms of $\chi^a, \eta^a$. Therefore, this basis does not define separate configuration and momentum submanifolds. We therefore must restrict our attention to the second solution.

The second form is surprising. It clearly involves a change of signature between the original space and the configuration manifold. We study this case in detail in the remaining Sections.

We have now imposed all conditions of our Lemma. In the next Section, we use the constrained forms of the metric and connection that we have found to solve the structure equations on the momentum submanifold.

5 The connection of momentum space

We now wish to solve for the connection on the momentum submanifold, using eqs. (25-27) when the curvature is conformally flat. First, we find a general form for the solution to the structure equations, then, in the second Subsection, we impose the additional condition of vanishing Weyl curvature. This requires reconciling two different expressions for the metric $h^{ab}$. Equating the two forms of $h^{ab}$ leads to a pair of coupled equations, which we solve completely, completing our solution for the momentum space connection.

5.1 General solution

Note that at the outset, both the conformal gauge and the $SO(p,q)$ gauge are arbitrary, since the form of the structure equations is invariant with respect to these transformations. Therefore, we may solve eq. (27) by choosing the initial conformal gauge to be the gauge (unique up to constant gauge functions, $d\phi = 0$) in which the Weyl vector vanishes.

Since we seek conformally flat solutions, we may also choose the initial $SO(p,q)$ gauge to be the one in which $\eta_a$ is conformal to an exact form. Then we have

$$\eta_a = e^\sigma dy_a$$

$$\tau = 0$$

so that eq. (26) for $\eta_a$ becomes

$$d\eta_a = \beta^a_b \eta_c$$

This is solved by setting

$$\beta^a_b = -2\Delta^{ac}_{db} \sigma^{cd} dy_c$$

Finally, we substitute eqs. (35) and (37) into the remaining structure equation, eq. (25). Notice that $h^{ab}$ is the metric in the $\eta_c$ basis, so we must conformally transform it to have the metric in the exact basis. The equation becomes

$$0 = \frac{1}{2}e^{2\sigma} h^{ab} \eta_{ba} \sigma^{cd} + \Delta^{ac}_{db} \sigma^{cd} \eta_{ba} \sigma^{de} - \Delta^{ac}_{db} \sigma^{de} \eta_{ba} \sigma^{cd} - \Delta^{ac}_{db} \sigma^{de} \eta_{ba} \sigma^{cd} + \eta_{ba} \sigma^{de} \eta_{ba} \sigma^{cd} - \eta_{ba} \sigma^{de} \eta_{ba} \sigma^{cd} + \eta_{ba} \sigma^{de} \eta_{ba} \sigma^{cd}$$

Taking the trace on $eb$, and defining $\sigma^2 = \eta_{ab} \sigma^{ad} \sigma^{bd}$ (not to be confused with the square of the function $\sigma$), together with a second trace on $ac$, allows us to find a form for the metric in terms of the conformal factor, $\sigma$,

$$h^{ac} = e^{-2\sigma} \left( -2\sigma^{ac} + 2\sigma^{ac} \sigma^{ae} - \sigma^2 \eta^{ac} \right)$$

This is the metric given by the inner product of the basis forms $\eta_a$. When the metric $h^{ac}$ is substituted into eq. (38), above the equation is identically satisfied. Therefore, eq. (39) is the necessary and sufficient condition for satisfying eq. (25). The structure equations of the $\chi^a = 0$ submanifold are therefore solved by the connection and the metric in the form

$$\beta^a_b = -2\Delta^{ac}_{db} \sigma^{cd} dy_c$$

$$\eta_a = e^\sigma dy_a$$

$$\omega = 0$$

$$h^{ac} = e^{-2\sigma} \left( -2\sigma^{ac} + 2\sigma^{ac} \sigma^{ae} - \sigma^2 \eta^{ac} \right)$$

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Alternatively, we have the exact basis and corresponding connection,

\[
\begin{align*}
\hat{\rho}_b^a &= -2\Delta_{db}^c \sigma^d dy_c \\
\tilde{\eta}_a &= dy_a \\
\hat{\omega} &= d\sigma
\end{align*}
\]

with the metric

\[
\hat{h}_{ab} = e^{-2\sigma} h_{ab} = -2\sigma^{ac} + 2\sigma^a \sigma^a - \sigma^2 \eta^{ab}
\]

with no conformal factor. This is easily checked by direct substitution.

### 5.2 Conformally flat solution

In Section 4.3, we found that the subspace will be a conformally flat manifold if and only if

\[
h^{ab} = \frac{(n-2)}{hu^2} \left( -2u^a u^b + u^2 \eta^{ab} \right)
\]

for some vector \(u^a\). At the same time, we showed that the metric must be expressible in the form

\[
h^{ab} = -2\sigma^{ab} + 2\sigma^b \sigma^a - \sigma^2 \eta^{ab}
\]

in the exact basis.

To relate the two forms, we first derive the relationship between \(h\) and \(\sigma\). On the submanifold, the integrability condition reduces to

\[
Dh^{ab} = dh^{ab} + h^{ch} \omega^c_a + h^{ac} \omega^b_c - 2h^{ab} \omega = 0
\]

Now, using \(\eta_{ab} h^{ab} = \frac{1}{h} (n-2)^2\), contraction with \(\eta_{ab}\) gives

\[
0 = (n-2)^2 \left( -\frac{1}{h^2} dh + h^{ch} \eta_{ab} \omega^a_c + h^{ac} \eta_{ab} \omega^b_c - \frac{2}{h} \omega \right)
\]

and since, in the exact basis, we have \(\omega = d\sigma\), integration immediately gives

\[
\frac{1}{h} = a e^{2\sigma}
\]

for some constant \(a\). The solution is now

\[
h^{ab} = \frac{(n-2)a}{u^2} e^{2\sigma} \left( -2u^a u^b + u^2 \eta^{ab} \right)
\]

### 5.2.1 Equating the two forms of the metric: Case 2

Now consider the signature changing form of the metric. Replacing \(h\) and equating the two resulting forms of \(h^{ab}\),

\[
-2\sigma^{ab} + 2\sigma^b \sigma^a - \sigma^2 \eta^{ab} = \pm \frac{A}{u^2} (n-2) e^{2\sigma} \left( -2u^a u^b + u^2 \eta^{ab} \right)
\]

We define a new vector,

\[
\hat{u}^a = \sqrt{\frac{(n-2)}{u^2}} e^{\sigma} u^a
\]

with \(\hat{u}^2 = (n-2) e^{2\sigma}\). Then

\[
-2\sigma^{ab} + 2\sigma^b \sigma^a - \sigma^2 \eta^{ab} = \beta \left( -2\hat{u}^a \hat{u}^b + \hat{u}^2 \eta^{ab} \right)
\]

(42)

where \(\beta = \pm 1\). In addition,

\[
\hat{u}_b \hat{u}^{b,a} = \hat{u}^2 \sigma^a \quad \text{with} \quad \hat{u}_b = \eta_{bc} \hat{u}^c.
\]

This will prove to be a useful constraint.

Working directly with eq. (42) does not give enough relations between \(\hat{u}^a\) and \(\sigma^a\) to determine one in terms of the other, because there are only two relevant contractions (with \(\hat{u}_a\) and with \(\sigma_a \equiv \eta_{bc} \sigma^c\)) that leave vector relations. These two contractions let us relate the three vectors \(\sigma^a\), \(\hat{u}^a\), and \((\sigma_a \sigma^a)^b\), but not more. In addition, we need the integrability condition.
5.2.2 The integrability condition

Now, dropping the hat on $\hat{u}^a$, we contract eq.(42) with $d\gamma$ to write
\[
d\sigma^a = \sigma^a \sigma^b \gamma_b - \frac{1}{2} \sigma^2 \gamma^a - \beta u^a \gamma_b \gamma_d + \beta \frac{1}{2} u^2 \gamma^a
\]
where $d\gamma^a \equiv \eta^{ab} \gamma_b$. Using the Poincaré lemma to find the integrability condition,
\[
0 = d^2 \sigma^a
\]
we substitute for the resulting second derivatives, $\sigma^{ab}$. Removing the basis and simplifying, we have,
\[
0 = u^a u^b \sigma^c - u^a u^b \sigma^b + u^2 \eta^{bc} \sigma^c + (\sigma u^d) \eta^{bc} \sigma^d - (u^a \sigma^c) \eta^{ab} + u^a \gamma^b \eta^{ac} - u^a \gamma^b \eta^{ac}
\]
First, contraction with $u^c$, shows that the curl of $u^a$ vanishes so that $u^a = u^a$. Finally, using this, and simplifying with eq.(43), we contract with $\eta_{bc}$, and solve for the second derivative,
\[
\begin{align*}
\sigma^{ab} &= u^a \sigma^b + u^b \sigma^a - (u^a \sigma^c) \eta^{ab} \\
\end{align*}
\]
This necessary condition is readily checked to be sufficient as well.

5.2.3 Solving

We now have a pair of coupled equations,
\[
\begin{align*}
\sigma^{ab} &= \sigma^a \sigma^b + \beta u^a u^b - \frac{1}{2} (\sigma^2 + \beta u^2) \eta^{ab} \quad (44) \\
u^{ab} &= u^a \sigma^b + u^b \sigma^a - (u^a \sigma^c) \eta^{ab} \quad (45)
\end{align*}
\]
with $\hat{u}^{ab}$ given by eq.(41). Adding $\pm \sqrt{\beta}$ times the second equation to the first gives
\[
\begin{align*}
0 &= \kappa^{ab} - \kappa^a \kappa^b + \frac{1}{2} \sqrt{\beta} \kappa^a \eta^{ab} \quad (46) \\
0 &= \tau^{ab} - \tau^a \tau^b + \frac{1}{2} \tau^2 \eta^{ab} \quad (47)
\end{align*}
\]
where
\[
\begin{align*}
\kappa &= \sigma + \sqrt{\beta} u \\
\tau &= \sigma - \sqrt{\beta} u
\end{align*}
\]
Notice that when $\beta = -1$, these variables are complex, though $\sigma$ and $u$ remain real. Observe as well that when $\beta = +1$, we may have either $\kappa = 0$ or $\tau = 0$, though not both at once.

To solve eq.(46), first contract with $\kappa_a = \eta_{ab} \kappa^b$ and integrate, to find $\kappa^2 = Ae^\kappa$. Substituting this for $\kappa^2$ in eq.(46), gives
\[
e^{-\kappa} \left( \kappa^{ab} - \kappa^a \kappa^b \right) = -\frac{1}{2} A \eta^{ab}
\]
which is immediately integrated twice to give
\[
\kappa = -\ln \left( \frac{1}{4} A \right) - \ln \left( y^2 + 2e^a y_a + 2b \right)
\]
Substitution of this result back into eq.(46) shows that $2b = e^a e_a$. Renaming the integration constants, and writing a similar solution for $\tau$, we have
\[
\begin{align*}
\kappa &= a - \ln \left( (y^a + e^a) (y_a + c_a) \right) \\
\tau &= b - \ln \left( (y^a + d^a) (y_a + d_a) \right)
\end{align*}
\]
Recalling the zero solutions, there are therefore four cases:
1. If $\beta = 1$ and $\kappa = 0$, then choosing the $y_a$ origin at $d_a$,

$$-u = \sigma = \frac{b}{2} - \frac{1}{2} \ln |y^2|$$  \hspace{1cm} (48)

2. If $\beta = 1$ and $\tau = 0$, then choosing the $y_a$ origin at $c_a$,

$$u = \sigma = \frac{a}{2} - \frac{1}{2} \ln |y^2|$$  \hspace{1cm} (49)

3. If $\beta = 1$ and neither $\kappa$ nor $\tau$ vanishes, then we may choose the $y_a$ origin so that

$$\sigma = \frac{a+b}{2} - \frac{1}{2} \ln \left( (y+c)^2 (y-c)^2 \right)$$
$$u = \frac{a-b}{2} + \frac{1}{2} \ln \left( \frac{(y-c)^2}{(y+c)^2} \right)$$  \hspace{1cm} (50)

4. If $\beta = -1$, neither $\kappa$ nor $\tau$ may vanish, and we may choose the $y_a$ origin so that

$$\sigma = \frac{a+b}{2} - \frac{1}{2} \ln \left( (y+ia)^2 (y-ia)^2 \right)$$
$$u = \frac{a-b}{2i} + \frac{1}{2i} \ln \left( \frac{(y-ia)^2}{(y+ia)^2} \right)$$  \hspace{1cm} (51)

In each of these cases, we find the metric, $\hat{h}^{ab}$. In the exact basis, we have

$$\hat{h}^{ab} = -2\sigma^{ab} + 2\sigma^b \sigma^a - \sigma^2 \eta^{ab}$$

so cases 1 and 2 both lead to

$$\hat{h}^{ab} = \frac{1}{(y^2)^2} \left( -2y^a y^b + y^2 \eta^{ab} \right)$$

For case 3, define

$$r = \frac{1}{2} \ln (y+c)^2$$
$$s = \frac{1}{2} \ln (y-c)^2$$

Then $\sigma^a = -r^a - s^a$ and

$$\hat{h}^{ab} = -2 (r^a - s^a) \left( r^b - s^b \right) + (r - s)^2 \eta^{ab}$$

Finally, when $\beta = -1$, define

$$r = \frac{1}{2} \ln (y+ia)^2$$

Then $\sigma^a = -r^a - \bar{r}^a$, where $\bar{r}$ is the complex conjugate of $r$. The metric is then

$$\hat{h}^{ab} = 2 (2i mr^a) \left( 2i mr^b \right) + (2i m)^2 \eta^{ab}$$

In every case, the metric is of the general form

$$\hat{h}^{ab} = \beta \left( -2u^a u^b + u^2 \eta^{ab} \right)$$  \hspace{1cm} (52)

This completes the description of the allowed momentum space solutions.
6 Solving for the full biconformal connection

In this Section, we extend the solution for the connection on the momentum submanifold to a form valid on the full biconformal space, then complete our solution by substituting these forms into the structure equations, eqs. (80-83).

On the momentum submanifold, we have the connection in the exact form given in eqs. (40) and the metric given by eq. (52) where \( u^a, a \) is given by any of the solutions in the previous section, eqs. (48-51), and \( \beta = \pm 1 \). Dropping the circumflex on the exact-basis connection and corresponding metric (eq. (41)), we extend eqs. (40) back to the full manifold, by adding arbitrary dependence on \( \chi^a \) to each of the connection forms, \( \omega^a_b = \alpha^a_b - 2\Delta^a_b \sigma^d dy_c \) (53)

\[ \chi^a = \chi^a_{\beta} \, dv^\beta \] (54)

\[ \eta_a = dy_a + b_{a\beta} dv^\beta \] (55)

\[ \omega = W^\beta_{\beta} dv^\beta + \sigma^b dv^b \] (56)

where \( \alpha^a_b = \alpha^a_{\beta\beta} dv^\beta \) and, without loss of generality, we choose the coordinates \( v^\beta \) canonically conjugate to \( y^\alpha \). The coefficients of the new \( dv^\beta \) terms depend arbitrarily on all of the coordinates, \( (v^\beta, y^\alpha) \), and all of the constants (with respect to \( y^a \)) in the expressions for \( u^a, b_{a\beta}, c^a \) are now allowed to depend on \( v^\alpha \).

The form of the connection in eqs. (80-83) satisfies the conditions of the Lemma. In order to provide a description of a biconformal geometry, they must also satisfy the structure equations, eqs. (80-83). We begin, in Subsection 7.1, with the dilatation equation, eq. (83). In the next Subsection, we digress to compute the metric derivatives required for our discussion of the eta and chi equations, eqs. (81) and (82), and check the involution conditions. In Subsections 7.3 and 7.4 we solve the chi and eta equations, respectively. The resulting form of the connection automatically satisfies Eq. (80). This is discussed briefly in Subsection 7.5.

6.1 Dilatation

Since \( v^\beta \) is chosen to be canonically conjugate to \( y^\beta \) we can write the dilatation equation, eq. (83), in two ways, leading to two separate conditions,

\[ dv = dv^\beta dy^\beta \]

\[ dv = \chi^a dy^a \] (57)

With \( \omega \) given by eq. (56) the first becomes

\[ W^{[a, b]} = 0 \]

\[ W^a_{\beta} - \sigma^a_{\beta} = -\delta^a_{\beta} \] (58)

Integrating these,

\[ W^a_{\beta} = -y^a + \theta^a_{\beta}(v) + \sigma^a_{\beta} \]

\[ \rightarrow -y^a + \sigma^a_{\beta} \]

where in the second line we absorb \( \theta^a_{\beta}(v) \) into still undetermined \( v^\beta \)-dependent part of \( \sigma \).

Substituting into the second condition, we see that \( b_{a\beta} = \chi^a_{\beta} \, dv^a + \sigma^a_{\beta} \) must be symmetric and \( \chi^a_{\beta} = \delta^a_{\beta} \). The basis form \( \chi^a = \delta^a_{\beta} dv^\beta = dv^a \) is now exact so there is no longer any need for Greek indices – the coordinate basis is also orthonormal. The Weyl vector and \( \chi^a \) are now

\[ \chi^a = dv^a \]

\[ \omega = -y^a dv^a + d\sigma \] (58)
6.2 Covariant derivative of the metric and the involution conditions

Both the chi and eta structure equations, eqs. (81) and (82), depend on the covariant exterior derivative of $\hat{h}^{ab}$. While this derivative vanishes on the involute submanifolds where (subject to the conditions of Section 8) $\hat{h}^{ab}$ functions as a metric, it does not vanish on the biconformal space as a whole. We therefore compute

$$D\hat{h}^{ab} = d\hat{h}^{ab} + \hat{h}^{cb} \omega^a_a + \hat{h}^{ac} \omega^b_b - 2\hat{h}^{ab} \omega$$

where $\hat{h}^{ab}$ is given by eq.(52) and the allowed forms for $u$ are given by eqs.(48-51). We then verify that the remaining involution condition is satisfied.

6.2.1 The covariant derivative of the metric

Expanding the covariant derivative, and replacing the second $\gamma$-derivatives of $u$ using eqs.(45) and (43), we find that the $d\gamma$ terms components cancel identically, leaving

$$Dh^{ab} = -2\beta \left( u^a_i + u^d \alpha^a_i + u^d \gamma - u^a \sigma_i \right) u^b d\gamma^c$$

Defining the $\alpha$- and Weyl-covariant derivative of $u^a$ by

$$u^a_i \equiv u^a_i + u^d \alpha^a_i - u^a (-\gamma + \sigma_i)$$

we have the relations

$$u^2 \equiv \left( u^2 \right) - 2u^2 \left( -\gamma + \sigma_i \right)$$

$$h^{ab} \equiv h^{ab} + h^{db} \alpha^a_i + h^{ad} \alpha^b_i - 2h^{ab} \left( -\gamma + \sigma_i \right)$$

and the covariant derivative of $h^{ab}$ becomes simply

$$Dh^{ab} = -2\beta \left( -2u^a_i u^b + u^2 \eta^{ab} \right) d\gamma^c$$

Defining $\beta$ such that

$$\beta = \beta \left( -2u^a_i u^b + u^2 \eta^{ab} \right) d\gamma^c$$

we have

$$Dh^{ab} = -2\beta \left( -2u^a_i u^b + u^2 \eta^{ab} \right) d\gamma^c$$

Next, we check the involution conditions.

6.2.2 The involution conditions

This expression for the covariant derivative of the metric must satisfy the two involution conditions,

$$Dh^{ab} \bigg|_{d\gamma} = 0$$

$$h_{bc} Dh^{ab} \bigg|_{d\gamma} = 0$$

The first of these is immediate, since $Dh^{ab} \bigg|_{d\gamma} = 0$. We may write the second as

$$0 = h_{bc} Dh^{ab} \bigg|_{d\gamma} d\gamma^c d\gamma^e$$

Antisymmetrizing, expanding, and dropping an overall factor leads to

$$0 = u_i u^a_i - u_i u^a_i + u^a u_{i\gamma} - u^a u_{i\gamma}$$

$$+ \frac{1}{2} \left( u^2 \right) \delta^a_i - \frac{1}{2} \left( u^2 \right) \delta^a_i$$

(59)
To find the consequences of this condition, contract with $u_a$ to find

$$0 = \left( u^2 \right)_x u_a - \left( u^2 \right)_x u_c + u^2 u_{ac} - u^2 u_{ce}$$  

(60)

Contract again, with $u^c$, to show that

$$u^c u_{ce} = \frac{1}{u^c u^e} \left( u^2 \right)_x u_c - \frac{1}{2} \left( u^2 \right)_x$$  

(61)

A third identity follows from the trace on $ae$ in the original equation, eq. (59). Using eq. (61) to simplify,

$$0 = u_a u^{a\alpha} - u_a u^{a\alpha} + u^{a\alpha} u_{a\alpha} - u^{a\alpha} u_{a\alpha}$$

$$+ \frac{1}{2} \left( u^2 \right)_x \frac{1}{2} n \left( u^2 \right)_x$$

$$= \frac{1}{u^c} \left( u^2 u^{a\alpha} - u^a \left( u^2 \right)_x \right) u_c - \frac{1}{2} \left( n - 2 \right) \left( u^2 \right)_x$$  

(62)

then contracting with $u^c$ gives

$$u^c \left( u^2 \right)_x = \frac{2 u^2}{n} u^{a\alpha}$$  

(63)

and substituting this back into eq. (62) gives

$$\left( u^2 \right)_x = \frac{2}{n} u^{a\alpha} u_c$$  

(64)

Now combine eq. (60) and eq. (64) to show that the curl of $u_a$ vanishes,

$$0 = u_{a;e} - u_{e;a}$$

and this with eq. (59) yields

$$0 = u_c u^{a\alpha} - u_a u^{a\alpha} + \frac{1}{n} u^{b\beta} \delta^{c\beta}_a - \frac{1}{n} u^{b\beta} \delta_{a}^{c\beta}$$

Finally, contract with $u^c$ and use eqs. (61) and (64) to reduce the result to

$$u^{a\alpha} = \frac{1}{n} u^{b\beta} \delta^{c\beta}_a$$  

(65)

Substitution of the result, eq. (65), into eq. (59), shows it to be the necessary and sufficient condition for the involution of $\chi$.

Substituting eq. (65) the derivative of the metric,

$$\mathbf{D} u^{ab} = -\frac{2 \beta}{n} u^{d\beta} \left( \delta^{c\alpha}_a u^b + u^{a\alpha} \delta^{b\beta}_c - u_c \eta^{ab} \right) \mathbf{d} u^c$$  

(66)

In this form, it is easy to confirm that both involution conditions hold.

We now return to impose the remaining structure equations.

### 6.3 Chi equation

Consider eq. (81).

$$\mathbf{d} \chi^a = \chi^c \omega^a_c + \omega^a \chi^c + \frac{1}{2} \mathbf{D} h^{ac} \left( \eta_c + h_{cd} \chi^d \right)$$

Substituting the form of the connection from eqs. (53) (57) and (58), along with eq. (66), yields

$$0 = \sigma^{a\alpha}_{cd} \mathbf{d} u^c \mathbf{d} u^d - 2 \Delta^{a\alpha}_{bf} \sigma^{c\beta} \mathbf{d} u^f \mathbf{d} u^e - y_b \mathbf{d} u^b \mathbf{d} u^a$$

$$+ \sigma^{a\alpha}_{bf} \mathbf{d} u^f + \sigma^{c\beta}_{bf} \mathbf{d} y_b \mathbf{d} u^d$$

$$- \frac{\beta}{n} u^f \left( \delta^{c\alpha}_a u^e + u^{a\alpha} \delta^c_e - u_c \eta^{ac} \right) \mathbf{d} u^c \mathbf{d} y_c$$

$$- \frac{\beta}{n} u^f \left( \delta^{c\alpha}_a u^e + u^{a\alpha} \delta^c_e - u_c \eta^{ac} \right) \left( h_{cd} + h_{cd} \right) \mathbf{d} u^c \mathbf{d} u^d$$  

(67)

Consider the cross-terms, proportional to $\mathbf{d} u^c \mathbf{d} y_c$, first.
6.3.1 Cross-terms

The collected cross-terms may be rearranged as

$$0 = \left( \sigma^a + \frac{\beta}{n} u^b \delta^a_b \right) \delta^c_d - \eta^{ac} \left( \sigma_c + \frac{\beta}{n} u^b \delta^c_b \right)$$

Then, taking the $ec$ trace, shows immediately that

$$\sigma^a = -\frac{\beta}{n} u^d \delta^a_d$$

(68)

This means that $\sigma^a$ and $u^a$ must be parallel, since neither can vanish. However, it is straightforward to show that this can never happen for two of the four solutions for $u$ and $\sigma$ given in Section 7, eqs. (48 - 51), and we discard the solutions in eqs. (50) and (51). Therefore, $u$ and $\sigma$ must be given by one of the remaining cases,

$$\pm u = \sigma = \frac{a(v)}{2} - \frac{1}{2} \ln |y^2|$$

(69)

and $\beta = 1$. The corresponding metric is

$$h_{ab} = -\left( 2y_ay_b - y^2 \eta_{ab} \right)$$

(70)

Also, substituting $\sigma^a$ for $\pm u^a$ in eq. (68), yields $\sigma^d_{cd} = -n$. We may combine this with the result for the covariant derivative of $u^a$, eq. (65), to show that

$$\sigma^a_{cd} = -\delta^a_d$$

These results have some useful consequences. Expanding the covariant derivative, $\sigma^a_{,c} = \sigma^a_{,c} + \sigma^b \alpha^a_{bc} - \sigma^a(-y_c + \sigma_c)$, and substituting, $\sigma^a = -\frac{a}{2}$ and $\sigma_a = \frac{1}{2} a_a$, shows that

$$0 = -y^b \alpha^a_{bc} - y^c y_c + y^d \sigma_c + y^2 \delta^a_c$$

Contracting with $y_a$, produces

$$0 = -y^b y_a \alpha^a_{bc} - y^2 y_c + y^2 \sigma_c + y^2 y_c$$

$$= y^2 \sigma_c$$

thereby showing that $\sigma$ is independent of $v^a$,

$$\sigma_a = 0$$

while we still have the condition

$$0 = -y^b \alpha^a_{bc} - y^2 y_c + y^2 \delta^a_c$$

(71)

We will return to this condition after considering the configuration space terms ($dv^d dv^d$) of eq. (67).

6.3.2 Configuration space terms

For the configuration space terms, we may now set $\beta = 1$, $u^a = \pm \sigma^a$, $\sigma_c = 0$ and $\sigma^d_{,cd} = -n$. Then, antisymmetrizing and dropping the basis forms,

$$0 = \alpha^a_{cd} - \alpha^a_{dc} - y_c \delta^a_d + y_d \delta^a_c$$

$$+ (\sigma^e \delta^a_e + \sigma^e \delta^a_c - \sigma_c \eta^{ac}) (h_{cd} + h_{dc})$$

$$- (\sigma^e \delta^a_e + \sigma^a \delta^e_c - \sigma_e \eta^{ac}) (h_{ce} + h_{ec})$$

First, we note that the parts proportional to the metric vanish identically. Expanding the $h_{bc}$ and $\sigma^a$ parts, with $h_{ab} = -\left( 2y_ay_b - y^2 \eta_{ab} \right)$, and $\sigma_a = -\frac{a}{2}$, all terms cancel so that

$$0 = \alpha^a_{cd} - \alpha^a_{dc} - y_c \delta^a_d + y_d \delta^a_c$$

$$+ (\sigma^e \delta^a_e + \sigma^e \delta^a_c - \sigma_c \eta^{ac}) b_{cd}$$

$$- (\sigma^e \delta^a_e + \sigma^a \delta^e_c - \sigma_e \eta^{ac}) b_{ce}$$
Lowering $a$ and permuting the three free indices, we add the first two and subtract the third. This gives an explicit expression for the $\text{d}v^a$ part of the spin connection. Replacing $\sigma^{a\mu}$ with its explicit form, the result is

$$
\alpha^{bc}_{a} = -y^a\eta_{bc} + y_b\delta^a_c - \frac{1}{y^e}(y_b y^a c - y^a y^b c + y^e b_e \delta^a_c - y^e b^a c \eta_{bc})
$$

(72)

### 6.3.3 Returning to the cross-term

From our analysis of the cross-term, we still have the condition, eq.(71). Combining this with the explicit form for $\alpha^{bc}_{a}$ above, lowering the $a$ index, and defining $b_a = y^b b_{ab}$ gives $b_{ac}$ in terms of $b_a$,

$$
b_{ac} = \frac{1}{y^e} \left( b_{ac} + y_a b_c - \eta_{ac} \left( y^b b_b \right) \right)
$$

(73)

This completes the consequences of the chi structure equation. We now turn to the eta equation.

### 6.4 Eta equation

Consider eq.(82) for $\eta\alpha$,

$$
\text{d}\eta = \omega^b a \eta_b - \omega a \eta - \frac{1}{2} \text{D}h_{ab} \left( \chi^b - h^{bc} \eta_c \right)
$$

where the connection is now given by eqs.(80), (82), (57), and (58), with $\alpha^{bc}_{a}$ given by eq.(72) and $b_{ab}$ given by eq.(73).

The covariant derivative of the metric is found from eq.(66) using the explicit form of $\alpha^{bc}_{a}$, to be

$$
\text{D}h_{ab} = -h_{ac} h_{ba} \text{D}h^{cd}
$$

$$
= 2 \left( 4 y_a y_d y_c - y^2 \left( y_c \eta_{de} + \eta_{ce} y_d + \eta_{cd} y_e \right) \right) \text{d}v^e
$$

Substituting and collecting terms, we have three separate equations,

$$
0 = \left( b_{ac} + \alpha^{bc}_{a} b_{be} + y_e b_{ac} + \left( \eta_{ac} y^a - y_a \delta^a_c - y_c \delta^a_d \right) b_{de} \right) \text{d}v^e \text{d}v^e
$$

$$
0 = \left( b_{ac} \delta^c_e - \alpha^{ac}_{e} + \frac{1}{y^2} \left( y^b \delta^c_a - \eta^{bc} y_a \right) b_{be} - y e \delta^c_a + \frac{1}{y^2} y^2 b_{ae} \right) \text{d}v_e \text{d}v^e
$$

$$
0 = \frac{1}{y^2} \left( y^b \delta^c_a - \eta^{bc} y_a + y^e \delta^b_a \right) \text{d}v_e \text{d}y_b
$$

The $\text{d}v_e \text{d}y_b$ part is quickly seen to vanish identically. We consider each of the remaining two in turn.

#### 6.4.1 The cross-term equation

For the cross term, after substituting for $\alpha^{bc}_{a}$, and simplifying, we are left with

$$
b_{ab} \delta^c_e = \frac{1}{y^2} \left( \delta^c_b b_{a} + b_{b} \delta^c_a - \eta_{ab} b^c \right)
$$

After substituting the expression for $b_{ab}$ on the left and collecting terms this may be cast into the form

$$
0 = \left( y^2 y_b \delta^c_d + y^2 y_a \delta^c_d - y^2 \eta_{ab} \delta^c_d \right) \left( \frac{b_{ab}}{y^2} \right)^c
$$

Contraction with $y^b$ now shows that the final factor must vanish,

$$
\left( \frac{b_{ab}}{y^2} \right)^c = 0
$$
so we may write \( b_a \) as \( y^2 \) times a vector dependent only on \( v^a \),

\[
b_a = y^2 d_a(v)
\]

Then \( b_{ab} \) takes the form

\[
b_{ab} = y_b d_a + y_a d_b - \eta_{ab} y^c d_c
\]

A quick check shows that this result is necessary and sufficient to satisfy the cross-term equation.

### 6.4.2 Configuration space terms

Finally, turn to the configuration space terms. Using the symmetry of \( b_{ab} \), these reduce to

\[
0 = (b_{ab,c} - b_{cb} \alpha_{ac}^{\varepsilon} - \eta_{ac} b_b) \, d^b d^c
\]

Substituting for \( b_{ab} \) and \( \alpha_{ac}^{\varepsilon} \),

\[
0 = \begin{align*}
&\quad y_b d_{a,c} + y_a d_{b,c} - y_{ab} y^f d_{e,c} - y_{ab} y^e d_{e,b} + \eta_{ac} y^f d_{e,b} \\
&\quad - y_b \eta_{ac} (d^f y_e) d_b + \eta_{ac} (d^f y_e) d_b + \eta_{ac} d_b (y^f d_b) - y_c d_a d_b \\
&\quad + \eta_{ac} d_b (y^f d_f) + y_c \eta_{ab} (d^f d_e) + \eta_{ab} (d^f y_e) d_c - \eta_{ab} d_c (y^f d_f) \\
&\quad + y_b d_a d_c - \eta_{ab} d_c y^f d_f
\end{align*}
\]

This expression is linear in \( y_a \). Differentiating with respect to \( y_d \), then taking the \( bd \) trace leaves us with

\[
0 = (n - 1) d_{a,c} - d_{a,a} + \eta_{ac} d^b - b + \eta_{ac} d^f d_e d_c + (n - 2) \eta_{ac} d^f d_e 
\]

Taking the antisymmetric part shows that \( \frac{1}{2} (d_{a,c} - d_{a,a}) = 0 \), and therefore \( d_a \) is a gradient, \( d_{a, \alpha} \), while the contraction with \( \eta_{ac} \) shows that \( d^b \, d^d = \frac{1}{2} (n - 2) d^f d_e d_c \). Substituting and solving gives a differential equation for \( d 
\]

\[
d_{a,c} = -d_{a,d,d} + \frac{1}{2} \eta_{ac} \left( \eta_{bd} d_{b,d} \right)
\]

This relationship is easily shown to be necessary and sufficient for the solution of the configuration space equation.

Eq. (74) is straightforward to integrate. First, contract with \( d^c \). Then if \( d^2 = (\eta_{bd} d_{b,d}) \) is nonzero we may integrate to find \( d^2 = A e^{-d} \). Replacing \( d^2 \) in the original equation allows two immediate integrations, which produce the result

\[
d = \ln \left( \frac{1}{4} v^2 A + A v^a + B \right)
\]

Substituting the solution shows that \( B = \frac{1}{A} A_v A_v \) and the argument of the logarithm factors so that

\[
d = \ln \left( v^a + \frac{2A^a}{A} \right)^2 + \ln \left( \frac{A}{4} \right)
\]

Changing the origin of the \( v^a \) coordinate does not change the conjugacy of \( v^a \) and \( y_a \), so we may shift the origin so that

\[
d = \ln |v^2| + c
\]

where \( c \) is an arbitrary constant.

If \( d^2 = 0 \), then, integrating again, we have the alternative solution

\[
d = \ln (a + b_a v^a)
\]

where \( b_a b^a = 0 \). The special case, \( a = 1 \), \( b_a = 0 \) shows that \( d = 0 \) is a solution.

The \( y_a \) constant subspaces coincide with the \( \eta_a = 0 \) submanifolds if and only if \( d = \) constant (see Appendix 2). In this case, both \( \chi^d \) and \( \eta_a \) are exact. For other solutions for \( d \), the submanifolds are given by solutions of

\[
d y_a + (y_b d_a + y_a d_b - \eta_{ab} (y^f d_c)) d^b = 0
\]
for functions \( y_a (v^b) \). The solutions to this equation are presented in Appendix 3.

The complete conclusion of the eta structure equation is therefore that

\[
\begin{align*}
  b_a &= y^2 d_a \\
  d &= \begin{cases} 
  \ln |v^2| + c_0 & d^2 \neq 0 \\
  \ln |a_0 + c_a v^a| & d^2 = 0 
  \end{cases}
\end{align*}
\]  

(77)

With these results, rewritten in the original \((\chi^a, \eta_a)\) basis, the connection becomes

\[
\begin{align*}
  \omega^a_b &= 2 \Delta^c_{db} (y_c - d_c) \chi^d + \frac{2}{\sqrt{2}} \Delta^c_{db} \eta_c \\
  \chi^a &= dy^a \\
  \eta_a &= d y_a + (y_a d_a + y_d d_b - \eta_a (y^e d_e)) dv^b \\
  \omega &= (-y_a + d_a) \chi^a - \frac{1}{\sqrt{2}} y^d \eta_a
\end{align*}
\]  

(78)

The full \((\chi^a, \eta_a)\) basis is exact if and only if \( d \) is constant.

These results, eqs. (77) and (78), together with the final form of the metric, eq. (70), establish the form of the connection given in the introduction.

Finally, we turn to the spin connection.

6.5 Spin connection

The structure equation for the spin connection, eq. (80),

\[
\frac{d\omega^a_b}{\omega^a_b} = \omega^a_b \omega^c_d + \Delta^c_{db} (\eta_c + h_{cf} \chi^f) \left( \chi^d - h^{da} \eta_e \right)
\]

where the connection now takes the form given in eq. (78) and \( d \) is given by eq. (77). Substituting leads to three extremely long equations, but ultimately we find that each equation is identically satisfied. Eq. (78) is therefore the complete solution for the connection.

We now complete our proof of the Signature Theorem by checking the signature of \( h_{ab} \).

7 The existence of time

Even though our candidate submanifold metric, \( h_{ab} \) or \(-h_{ab}\), is now a uniquely specified, invertible quadratic form, its signature is not consistent for every choice of the original metric \( \eta_{ab} \). The inconsistencies take different forms in the two cases.

For the configuration submanifolds, the signature of \(-h_{ab}\) is consistent on any one submanifold, but may vary as we look at different submanifolds. This means that the signature of spacetime would change for particles of different momentum, which we disallow. For the momentum submanifolds, the situation is even worse – \( h_{ab} \) often has inconsistent signature on each submanifold.

In this Section, we find the conditions under which these metrics are assured to have consistent signature. After writing the full line element for the entire phase space, we consider first the configuration spaces \((\eta_a = 0)\), then the momentum spaces \((\chi^a = 0)\).

7.1 Full line element

The inner products of the basis forms follow from the Killing metric together with eqs. (18) as

\[
\begin{align*}
  \langle \chi^a, \chi^b \rangle &= h^{ab} \\
  \langle \chi^a, \eta_b \rangle &= 0 \\
  \langle \eta_a, \eta_b \rangle &= -h_{ab}
\end{align*}
\]
and the biconformal metric is the inverse of this,
\[
g_{AB} = \begin{pmatrix} h_{ab} & -h_{ab} \end{pmatrix}
\]
The biconformal line element is therefore
\[
ds^2 = h_{ab} \xi^a \xi^b - h_{ab} \eta_a \eta_b
\]
Using the coordinate expressions given for the basis in eqs.(78), the line element becomes
\[
ds^2 = h_{ab} dv^a dv^b - h_{ab} (dy_a + b_{ab} dv^c) \left(dy_b + b_{bd} dv^d\right)
\]
where
\[
b_{ab} = y_b dv_a + y_a dv_b - \eta_{ab} (y^e dv_e)
\]
and \(d_a = \eta_{ab} dv^b\) is determined by eq.(77).

We consider the configuration and momentum submanifolds in turn.

### 7.2 Configuration submanifold, \(\eta_a = 0\)
Setting \(\eta_a = 0\) gives a relationship between the \(y_a\) and \(v^a\) coordinates, and reduces the line element to
\[
ds^2 = y^2 \eta_{ab} dv^a dv^b - 2 (y_a dv^a)^2.
\]
The full solution for \(y_a (v^b)\) is given in Appendix 3. Here we need only the form, \(dv_a = -b_{ab} dv^b\), and hence \(\frac{dv_a}{dv^b} = -b_{ab}\). Then since \(b_{ab}\) is symmetric, we know that \(y_a = t_a\) for some function \(t\) of the coordinates \(v^a\), and we may take \(t\) as a coordinate and write \(y_a dv^a = dt\). The line element becomes:
\[
ds^2 = y^2 \eta_{ab} dv^a dv^b - 2 dt^2.
\]
Project \(dv^a\) into parts parallel and orthogonal to \(dt\), \(dv^a = e^a_\parallel + e^a_\perp\), given by
\[
e^a_\perp = dv^a - \frac{v^a}{y^2} dt
\]
\[
e^a_\parallel = \frac{v^a}{y^2} dt
\]
In terms of these, the line element becomes
\[
ds^2 = y^2 \eta_{ab} \left(e^a_\parallel + e^a_\perp\right) \left(e^b_\parallel + e^b_\perp\right) - 2 dt^2
\]
\[
= y^2 \left(\frac{1}{y^2} dt^2 + \eta_{ab} e^a_\parallel e^b_\perp\right) - 2 dt^2
\]
\[
= y^2 \eta_{ab} e^a_\parallel e^b_\perp - dt^2
\]
We consider the signature of this line element for different signs of \(y^2\), both as \(v^a\) varies over the submanifold and as \((v^a, y_b)\) vary over the full biconformal space. However, since these arguments hold only if both signs of \(y^2\) occur, we first consider when \(y^2\) can change sign.
Comparing the timelike and spacelike cases, we see that they are consistent only if $s = n - 1$ at all points of every submanifold. Then the candidate metric, $h_{ab}$, is consistent and Lorentzian. If $s = -n$, both $\eta_{ab}$ and $\eta_{ab} e^a_\perp e^b_\perp$ are negative definite. The product $\eta_{ab} e^a_\perp e^b_\perp$ is again positive definite so that we again have $s = n - 1$ at all points of every submanifold. The candidate metric is consistent and Lorentzian.

Now suppose $\eta_{ab}$ has signature different from $n$. Then it may be possible to have $\eta^2$ either positive (spacelike) or negative (timelike) on a given submanifold, and as we look at distinct submanifolds at different points of the full biconformal space, $\eta^2$ must change sign. In the next Subsections, we first consider what happens to the signature for spacelike and timelike $y_a$, then check when a change from one to the other may occur consistently.

### 7.2.2 Case 1: Spacelike $y_a$

Let the signature of the full metric be $s = p - q$, and suppose $\eta_{ab} e^a_\perp e^b_\perp$ has signature $s'$. Then when $y_a$ is spacelike, $\eta^2 > 0$,

$$ds^2 = \eta^2 \eta_{ab} e^a_\perp e^b_\perp - dt^2$$

and we have $s = s' - 1$. To find $s'$, note that at a fixed value of $\eta^2$, the differentials $e^a_\perp$ form an orthonormal basis for an $n - 1$ dimensional subspace of the manifold. Furthermore, we know that $\eta_{ab}$ has signature $s_0$, and the basis vectors $e^b_\perp$ are perpendicular to $y_a$. Since $y_a$ is spacelike, the $e^b_\perp$ subspace must therefore have signature $s' = s_0 - 1$, so that

$$s = s_0 - 2$$

### 7.2.3 Case 2: Timelike $y_a$

When $y_a$ is timelike, $\eta^2 < 0$, a similar argument holds. Again, we let the signature of the quadratic form $\eta_{ab} e^a_\perp e^b_\perp$ be $s'$, and the $-dt^2$ term still reduces the signature by one. However, this time the $\eta^2$ factor changes the sign of the contribution of the first term to $-s'$. Combining these, we have $s = -s' - 1$.

Finally, the signature of $\eta_{ab} e^a_\perp e^b_\perp$ is $s' = s_0 + 1$ because all of the differentials $e^a_\perp$ are orthogonal to a timelike direction. Therefore,

$$s = -s' - 1$$

$$= -s_0 - 2$$

### 7.2.4 Changing sign of $\eta^2$

Comparing the timelike and spacelike cases, we see that they are consistent only if

$$s_0 - 2 = -s_0 - 2$$

$$s_0 = 0$$

We must still check whether $\eta^2$ actually does change sign on the configuration submanifold, where $y_a$ is a function of $v^a$. In Appendix 3, we found that when $\eta_{ab} = 0$ we have the generic solution,

$$y_a = \frac{1}{(\eta^2)^2} (\eta^a \delta^c_a - 2v_c v^a) a_c$$

in which case,

$$\eta^2 = \frac{\eta_{ab} a_a a_b}{(\eta^2)^2}$$

Through any given point, $(\eta^a, y_b)$, of the biconformal space there is a configuration submanifold specified by a fixed vector $a_c$, so on these manifolds have consistent signature. However, as we look at distinct configuration submanifolds, we require different vectors $a_c$, in order to get all possible vectors $y_a$. Therefore, for the generic solution for $y_a (\eta^b)$, distinct configuration submanifolds will have different signature unless $s_0 = 0$ or $s_0 = \pm n$. 

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The special solution to the submanifold condition, eq. (87), is
\[
f = A_b v^b + \frac{1}{2} c v^2 + b \left( \frac{A}{A + B v^d} \right)
\]
where \( c = \frac{A_B a}{A} \). Let \( A_b = B_b = b = 0 \). Then
\[
f = \frac{c}{2A} v^2
\]
\[
y_a = f_a = \frac{c}{A} v^a
\]
\[
y^2 = \left( \frac{c}{A} \right)^2 v^2
\]
which has the same sign as \( v^2 \). Therefore, \( y^2 \) may take on both signs on the same configuration submanifold for at least some solutions. On the other hand, suppose \( c = b = B_a = 0 \). Then
\[
f = A_b v^b
\]
so that \( y_a = \frac{A}{A_b} \). Then \( y^2 = \frac{A_A a}{A} \) has a fixed sign for all \( v^a \). In any case, the signature will vary between different submanifolds.

**Conclusion 1** For any signature, \( s_0 \), of \( \eta_{ab} \), there exist solutions for the configuration submanifolds such that each submanifold has consistent signature. However, in all cases, configuration submanifolds at different points of the biconformal space have different signatures unless \( s_0 = \pm n \) or \( s_0 = 0 \). When the dimension is \( n = 4 \) or \( n \) is odd, \( n = 2m + 1 \), the resulting configuration spaces is necessarily Lorentzian. In even dimensions other than 4, the configuration space is Lorentzian unless \( s_0 = 0 \) (in which case, \( s = -2 \)).

We now examine the momentum submanifolds.

7.3 **Momentum submanifold, \( \chi^a = 0 \)**

When \( \chi^a = 0 \), we have \( v^a = \) constant, and the candidate metric is \( -h^{ab} \) so the line element is
\[
ds^2 = -\frac{1}{(y^2)^2} \left( -2 y^a dy_a \left( y^b dy_b \right) + y^2 \eta^{ab} dy_a dy_b \right)
\]
When \( v^a = \) constant, the \( y_a \) coordinates are no longer all independent. The length of \( y_a \) in the induced candidate metric is
\[
|y|^2 = -h^{ab} y_a y_b = \frac{1}{(y^2)^2} \left( 2 y^a y^b - y^2 \eta^{ab} \right) y_a y_b = 1
\]
up to a conformal transformation of the metric. However, notice that \( |y|^2 = 1 \) is distinct from our definition, \( y^2 = \eta^{ab} y_a y_b \), which may take on arbitrary values, and have either sign when \( \eta^{ab} \) is non-Euclidean. We may therefore use \( n - 1 \) independent components of \( y_a \), together with \( y^2 \) to span the space.

We again consider two cases.

7.3.1 **Case 1: Spacelike \( y_a \)**

Define
\[
r^2 = y^2 > 0
\]
Then in terms of $r$, the line element becomes

$$ds^2 = \frac{1}{(y^2)^2} \left( 2y^a y^b - y^2 \eta^{ab} \right) dy_a dy_b$$

$$= \frac{1}{r^2} \left( 2dr^2 - \eta^{ab} dy_a dy_b \right)$$

Now project $dy_a$ into parts along $dr$ and parts perpendicular,

$$e^\parallel_a = \frac{y_a y_b}{y^2} dy_b = \frac{y_a}{r} dr$$

$$e^\perp_a = \left( \delta^b_a - \frac{y_a y^b}{y^2} \right) dy_b$$

Then

$$ds^2 = \frac{1}{r^2} \left( dr^2 - \eta^{ab} e^\perp_a e^\perp_b \right)$$

Now set $t = \ln r$ so that

$$ds^2 = dt^2 - e^{2t} \eta^{ab} e^\perp_a e^\perp_b$$

Since $y^2 > 1$, the quadratic form $e^{2t} \eta^{ab} e^\perp_a e^\perp_b$ has signature $s' = s_0 - 1$, so that the metric of the momentum space is of signature

$$s = 1 - (s_0 - 1)$$

$$= -s_0 + 2$$

### 7.3.2 Case 2: Timelike $y_a$

If $y^2 < 0$ we define

$$r^2 = -y^2 > 0$$

$$rdr = -y^a dy_a$$

This time the line element becomes

$$ds^2 = \frac{1}{r^2} \left( 2dr^2 + \eta^{ab} dy_a dy_b \right)$$

Projecting $dy_a$ into parts along and perpendicular to $dr$,

$$e^\parallel_a = \frac{y_a y_b}{y^2} dy_b = \frac{y_a}{r} dr$$

$$e^\perp_a = \left( \delta^b_a - \frac{y_a y^b}{y^2} \right) dy_b$$

Then

$$ds^2 = \frac{1}{r^2} \left( dr^2 + \eta^{ab} e^\perp_a e^\perp_b \right)$$

and with $t = \ln r$, we have

$$ds^2 = dt^2 + \frac{1}{r^2} \eta^{ab} e^\perp_a e^\perp_b$$

Again consider the signature of the candidate metric. Since the perpendicular part has lost a timelike piece, $\frac{1}{r^2} \eta^{ab} e^\perp_a e^\perp_b$ has signature $s = s_0 + 1$. Including the positive $dt^2$ makes the total signature

$$s = s_0 + 2$$
7.3.3 Changing sign of $\gamma^2$

Since $y_a$ spans the momentum space with $\nu^a$ constant, the sign of $\gamma^2$ will necessarily vary unless $\eta^{ab}$ is Euclidean, $s_0 = \pm 1$. Therefore, when $\eta^{ab}$ is non-Euclidean, the only way to get a consistent signature on any fixed submanifold is to require $s_0 + 2 = -s_0 + 2$ so that

$$s_0 = 0$$

For the Euclidean case, when $s_0 = n$, $\gamma^2$ is always positive and the momentum space signature is Lorentzian, $s = -n + 2$. When the originating space is anti-Euclidean with $s_0 = -n$, $\gamma^2$ is always negative and the momentum space is again anti-Lorentzian with $s = -n + 2$.

We conclude

**Conclusion 2** Unless the signature, $s_0$, of $\eta^{ab}$ is $s_0 = \pm n$ or $s_0 = 0$, there do not exist any momentum submanifolds of consistent signature. When the dimension is $n = 4$ or $n$ is odd, $n = 2m + 1$, the resulting momentum spaces are necessarily Lorentzian. In even dimensions other than 4, the configuration spaces are Lorentzian unless $s_0 = 0$.

8 Discussion

We have carried out a systematic approach to gauge theories of general relativity, including Poincaré and Weyl geometries, and two conformal gauge theories. Details of these investigations will be presented elsewhere. Neither the Poincaré nor the Weyl geometries leads to any new structures beyond general relativity. The Weyl conformal gauging does not lead to general relativity. By contrast, the only conformal gauging leading to general relativity, called biconformal geometry, generically possesses natural metric and symplectic structures arising necessarily from properties of the conformal group.

The biconformal gauging gives a systematic construction of phase spaces associated with pseudo-Euclidean spaces, $\mathcal{S}^{n}_{p,q}$, of arbitrary dimension $n = p + q$ and signature $s_0 = p - q$. Starting from the conformal group, $SO(p + 1, q + 1)$ of $\mathcal{S}^{n}_{p,q}$ and its Weyl subgroup, $W(p,q)$, we form the group quotient

$$SO(p + 1, q + 1)/W(p,q)$$

This leads to a $2n$-dimensional manifold – biconformal spaces, $\mathcal{B}(\mathcal{S}^{n}_{p,q})$ – with both metric and symplectic structures. Using the symplectic form, the biconformal space may be divided into configuration and momentum submanifolds.

We define a metric phase space to be a phase space with metric, having orthogonal configuration and momentum submanifolds. We prove the following theorem:

**Signature Theorem** A flat $2n$-dim ($n > 2$) biconformal space, $B(\mathcal{S}^{n}_{p,q})$, is a metric phase space if and only if the signature, $s_0 = p - q$, of $\mathcal{S}^{n}_{p,q}$ is $\pm n$ or 0. The resulting configuration submanifold is Lorentzian when the dimension of $S_{p,q}$ is $n = 4$ or $n$ odd, or, for even $n$, if $\mathcal{S}^{n}_{p,q}$ is Euclidean. If the signature of $\mathcal{S}^{n}_{p,q}$ is zero, the signature of the configuration space is $s = 2$.

In particular, for the description of any 4-dim geometry we have the immediate special case,

**Signature Theorem (4-dim)** A flat 4-dim biconformal space, $B(\mathcal{S}^{4}_{p,q})$, is a metric phase space if and only if the signature, $s$, of $\mathcal{S}^{4}_{p,q}$ is $\pm 4$ or 0. The resulting configuration space is necessarily Lorentzian.

We conclude with three brief conjectural discussions. First, we show how the difference in sign between the configuration and momentum metrics leads to half of the Dirac quantization procedure. Then we show that the particular form of metric we have found always leads to timelike momenta and the consequent absence of tachyons. Finally, we conclude with some speculations on how we might describe our experience in terms of an underlying Euclidean geometry.

8.1 Elements of quantum physics

A simple way to understand the relative minus sign between the configuration and momentum submanifolds is by absorbing it into the relationship between the geometric variables, $(\nu^a, y_b)$ and the physical coordinates, $x^a$, and the momenta, $p_b$. We may identify $\nu^a$ with $x^a$ directly, but since $p^b$ is tangent to the configuration space, it inherits the metric $h_{ab}$. Then the cotangent space has metric $h^{ab}$. Since the metric is $-h^{ab}$, the relationship between $y_a$ and $p_a$ must be of the form

$$p_a = i\alpha y_a$$

(79)
for some real constant $\alpha$ and $i^2 = -1$ (see [46]). Then, since the canonical bracket between $v^a$ and $y_b$ is

$$\{v^a, y_b\} = \delta^a_b$$

we have

$$\{x^a, p_b\} = i\alpha\delta^a_b$$

and it is not hard to guess that the unit conversion $\alpha$, which takes units of inverse length to units of momentum, should be identified as Planck’s reduced constant, $\hbar$. This suggests that we can derive some properties of quantum mechanics directly from biconformal models, an idea which has been explored in some detail in [7]. One finding of that study is that the usual product of probability amplitudes to compute a measurable probability has its origin in the comparison of lengths in making measurements.

It is possible that the relationship between the underlying Euclidean space, and the emergent spacetime is related to the well-known Wick rotation. This rotation in the complex time plane replaces the usual Minkowski time by a periodic imaginary coordinate, giving a 4-dim Euclidean space on which quantum theory becomes stochastic. The Feynman path integral becomes an ordinary Wiener path integral and the new coordinate acts as a temperature. This approach has allowed a proof of convergence of path integrals with a wide range of potentials. In the approach described here, both Euclidean and Minkowski spaces are present from the start, so it might be possible to realize the Wick rotation as simply the quantization procedure pulled back to the underlying Euclidean space. Alternatively, it may be found to relate distinct submanifolds of the biconformal space.

### 8.2 A tachyon-free world

One consequence of the emergence of the Lorentzian metric of eq.(70), and its negative inverse,

$$-h^{ab} = \frac{1}{(y^2)^2} (2y_ay_b - y^2\eta_{ab})$$

is the absence of tachyons. If we apply $-h^{ab}$ to the momentum coordinate $y_a$, we find

$$-h^{ab}y_a y_b = 1$$

so that the magnitude of the momentum vector is constant, up to a positive conformal factor. If we apply this to the physical momentum, eq.(79), we find that the momentum squared is negative definite,

$$p^2 = -\frac{1}{\alpha^2}$$

over the entirety of the submanifold. The particular, unique form of $h_{ab}$ therefore leads us to the conclusion that momenta are always timelike, and we predict the absence of tachyons in these models.

The mechanism by which tachyons are eliminated here is new. In standard approaches, a given quantum field theory may be shown to be tachyon free by examining its spectrum to see whether all allowed 4-momenta are timelike. In this phase space model, however, the space of all momenta (spanned by $y_a$) actually defines the set of timelike directions. Timelike directions are precisely the set of all possible directions for $y_a$, so the question of tachyons does not arise.

### 8.3 Musings on the Euclidean nature of nature

It is difficult to describe, in everyday language, the physics of spacetime and phase space in Euclidean terms, because so much of our thought and language include elements of time. For example, a careful description of events in a Euclidean world must avoid the use of active verbs! Nonetheless, we venture a few speculative thoughts in that direction.

One unexpected, and somewhat puzzling, thing has emerged with the emergence of time. Our original argument was that the conformal group should underlie our model because it is the symmetry group of measurement, since we only measure by comparison of magnitudes. However, this leads us to the conclusion that the original world we are measuring has signature zero or is Euclidean. In the Euclidean case, we must ask what it means to make a measurement at all. Doesn’t measurement presuppose a time sequence?
We hasten to point out that the problem is no worse here than in any deterministic theory. In general relativity, for example, we have an initial value formulation, but can also find global solutions. In the initial value formulation, we can specify the configuration of the world at a given time, then integrate forward to predict how things will evolve. However, in the case of a global solution such as a cosmological model, we are presented a complete description of past, present and future all at once. In this view, the outcomes of measurements are already fixed. The best we can do is to think of consciousness as sequentially illuminating certain fixed events, then others, with all the events already right there in the solution.

We can certainly think of the Euclidean model in this deterministic way, with the continuity of experience attributed to the continuity of fields in the Euclidean space. If our experience stems from the solution to some field equation in that Euclidean space, then we may expect the events induced on the spacetime submanifold to display that continuity.

We picture ourselves as sub-entities moving within configuration space, with our continuum of experience parameterized by that experience. But there is an underlying deterministic picture in which the whole of our experience exists simultaneously. It is, of course, not yet clear how quantum considerations affect this deterministic picture.

The difficulty we experience at trying to state or digest these ideas may be seen as an indication of just how much our understanding of the world hinges on living in a metric phase space. Once we have that arena, our physical description falls into place. In order to correctly describe the world from a Euclidean, or even a fully biconformal, perspective, we need to map backwards from known processes in metric phase space to a description of the same processes in the underlying Euclidean space. Only then will we have Euclidean objects to discuss in Euclidean terms. At that point, it may be easier to discuss what can now only be conjectural.

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Appendix 1: The conformal Lie algebra

The commutators of the conformal Lie algebra, together with the definitions of the structure constants, $C^{\Theta}_{\Sigma \Delta}$, are

$$[M^a_{\ b}, M^c_{\ d}] = -\frac{1}{2} \left( \delta^e_b \delta^f_d - \eta_{bd} \eta_{ef} \delta^a_c - \delta^a_d \eta_{bf} \delta^e_c + \delta^e_a \eta_{bf} \delta^d_c \right) M^f_{\ e}$$

$$= c^{(a)}_{(b)} (^{(e)} M^f_{\ e})$$

$$[M^a_{\ b}, P_c] = \frac{1}{2} \left( \delta^e_c \delta^d_b - \eta_{bd} \eta_{ec} \right) P_d$$

$$= c^{(a)}_{(b)} (^{(e)} P_d)$$

$$[M^a_{\ b}, K_c] = -\frac{1}{2} \left( \delta^e_b \delta^d_a - \eta_{bd} \eta_{ea} \right) K^d$$

$$= c^{(a)}_{(b)} (^{(e)} K^d)$$

$$[P_a, K^b] = -\delta^b_a D - \left( \eta_{ac} \eta_{bd} - \delta^b_d \delta^c_a \right) M^c_{\ d}$$

$$= c^{(a)}_{(b)} (^{(e)} D + c^{(a)}_{(e)} ^{^{(d)} M^f_{\ e}})$$

$$[D, P_a] = -\delta^b_a P_b$$

$$= c^{(a)}_{(b)} (^{(e)} P_b)$$

$$[D, K^a] = \delta^b_a K^b$$

$$= c^{(a)}_{(b)} (^{(e)} K^b)$$

where $M^a_{\ b}, P_a, K^a$ and $D$ generate rotations, translations, special conformal transformations and dilatations, respectively.

We write the Lie algebra in terms of the dual basis of 1-forms, defined by

$$\langle M^a_{\ b}, \omega^d \rangle = \delta^d_b \delta^a_c - \eta^{cd} \eta_{bc}$$

$$\langle P_b, \omega^a \rangle = \delta^a_b$$

$$\langle K^a, \omega_b \rangle = \delta^a_b$$

$$\langle D, \omega \rangle = 1$$

Then the Maurer-Cartan structure equations take the form

$$d\omega^\Psi = -\frac{1}{2} C_{\Sigma \Delta} \Psi \omega^\Sigma \omega^\Delta$$

where $\Psi, \Sigma, \Delta \in \{^{(a)}, ^{(b)}, ^{(c)}, ^{(d)}\}$ and $a, b, \ldots = 1, 2, \ldots, n$.

Appendix 2: Inner products in the coordinate basis

We compute the inner products of the coordinate differentials. We know that

$$\langle \chi^a, \chi^b \rangle = h^{ab}$$

$$\langle \chi^a, \eta_b \rangle = 0$$

$$\langle \eta_a, \eta_b \rangle = -h_{ab}$$
where
\[
\eta_b = dy_b + (yc_db + yb_d) - \eta_bc (y^c e_c) dv^c
\]
\[
\chi^a = dv^a
\]

Therefore, substituting and solving for the unknown inner products,
\[
\langle dv^a, dv^b \rangle = h^{ab}
\]
\[
\langle dv^a, dy_b \rangle = -(yc_db + yb_d) h_{ac} dy_c + \eta_bc (y^c e_c) dv^c
\]  
\[
\langle dy_a, dy_b \rangle = (yd_ad + ya_da - \eta_ad (y^c e_c)) h^{de} (yc_db + yb_d) - \eta_bc (y^c e_c) - h_{ab}
\]

Orthonormality of the subspaces spanned by \(v^a\) and \(y_a\) requires \(\langle dv^a, dy_b \rangle = 0\), which holds if and only if \(d_a = 0\). In this case, both \(\eta_b\) and \(\chi^a\) are exact and the subspaces coincide with the canonical submanifolds.

For \(y_a\) and \(v^a\) to be canonically conjugate, we require \(dω = dv^a dy_a\).

This is not changed if we replace \(dy_a\) by \(dz_a = dy_a + b_{ab} dv^b\) with \(b_{ab}\) symmetric. This is easily shown to be integrable if and only if \(d_a = d_a = 0\).

**Appendix 3: Non-involuton of the conformally flat spaces**

In Section 4, we showed that the momentum submanifold will be conformally flat if \(h^{ab}\) takes one of the two forms is conformally flat if and only if \(h^{ab}\) has one of the forms
\[
h^{ab} = \frac{1}{n} \eta^{ab}
\]
\[
h^{ab} = \frac{(n-2)}{hu^2} (-2u^a u^b + u^2 \eta^{ab})
\]

In Sections 5 and 6, we show that the second of these provides the unique involute solution we seek. In this Appendix, we establish that the first case, which has the same signature as the original space, is not involute.

Involution of the basis form \(\chi^a\) requires the covariant derivative of \(h^{ab}\), wedged with the solder form, to be proportional to \(\chi^a\). However, computing directly,
\[
Dh^{abc} e^b = \frac{1}{n} dh \eta^{abc} e^b - 2yc_e e^c \frac{1}{n} h \eta^{abc} e^b
\]
\[
= \frac{1}{n} \eta^{abc} \left( h^{c} dy_c e^b - 2yc_h e^c e^b \right)
\]
\[
= \frac{1}{n} \eta^{abc} \left( h^{c} \left( f_c + \left( yc_yd - \frac{1}{2} \eta_{cd} y^2 \right) e^d \right) e^b - 2yc_h e^c e^b \right)
\]
\[
= \frac{1}{n} \eta^{abc} \left( h^{c} \left( f_c + h^{c} \left( yc_yd - \frac{1}{2} \eta_{cd} y^2 \right) e^d e^b - 2yc_h e^c e^b \right) \right)
\]

Now express this in terms of the new basis,
\[
\chi^a = \omega^a + \frac{1}{2} h^{ac} \omega_c
\]
\[
\eta_a = \frac{1}{2} \omega_a - h_{ab} \omega_b
\]
\[
\frac{1}{2} \left( \chi^a - h^{ab} \eta_b \right) = \omega^a
\]
\[
\eta_a + h_{ab} \chi^b = \omega_a
\]
and we have

\[ D_{h_{ab}}e^b = \frac{1}{n} \eta_{ab} \left( \frac{1}{2} h^e \left( \eta_e + h_{eb} \chi^b \right) (\chi^b - h^{bd} \eta_d) + \frac{1}{4} \left( h^{cd} \left( y_c y_d - \frac{1}{2} \eta_c d y^2 \right) - 2 y_c h^d \right) \right) (\chi^e - h^{ef} \eta_f) (\chi^b - h^{bc} \eta_c) \]

Dropping all but the troublesome terms,

\[ D_{h_{ab}}e^b = \frac{1}{4n} h \left( -2 + \frac{h}{n} \left( h^{dy} - \frac{h}{n^2} y^2 - 2 h^2 \right) \right) y^f \eta_f \eta_a \]

This vanishes only if

\[ 4n = 2h \left( h^{dy} \right) - hy^2 - 4h^2 \]

However, if this vanishes, so does the covariant derivative.

If the involution works, \( h \) must be a function of \( y^2 \). Then

\[ D_{h_{ab}}e^b = \frac{2n}{a} (b + cy^2) y^d \eta_{ab} e^b + 2 \left( \frac{n}{a} (b + cy^2)^2 \eta_{ab} \right) \omega e^b \]

\[ = \frac{4cn}{a} (b + cy^2) y^d \eta_{ab} e^b \]

\[ = \frac{4cn}{a} (b + cy^2) y^d \eta_{ab} e^b \]

\[ = \frac{4cn}{a} (b + cy^2) \eta_{ab} \eta^c e^b \]

\[ = \frac{4cn}{a} (b + cy^2) \eta_{ab} \eta^c e^b \]

Collect this as

\[ \dot{\xi}_a = \frac{1}{2a} h_{ab} e^b + \alpha \eta_a \]

\[ d\omega^a = \omega^a \omega^b + \hat{\alpha}_{ab} \left( \eta_e + h_{ef} \chi^f \right) (\chi^d - h^{de} \eta_e) \]

\[ d\chi^a = \chi^e \omega^a + \omega^a \chi^a + \frac{1}{2} D \eta_{ac} \left( \eta_e + h_{ec} \chi^d \right) \]

\[ d\eta_{de} = \omega^a \eta_b - \omega^a \eta_a - \frac{1}{2} D \eta_{ab} \left( \chi^b - h^{bc} \eta_c \right) \]

\[ d\omega = \chi^a \eta_a \]

\[ h_{ab} = \frac{n}{a} \left( \alpha + \beta (y - c)^2 \right)^2 \eta_{ab} \]

satisfies the required conditions of conformal flatness and

\[ 0 = d\omega + h^{cd} \eta_{ab} \omega^a + h^{ac} \eta_{ab} \omega^b - 2 d\omega \]

\[ 0 = d\omega - 2 d\omega \]

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and since, in the exact basis, we have $\omega = d\sigma$, integration immediately gives

$$h = ae^{2\sigma}$$

for some constant $a$. The conformally flat solutions are now

$$h^{ab} = \frac{a}{n} e^{2\sigma} \eta^{ab}$$

or

$$h^{ab} = \frac{(n-2)a}{u^2} e^{2\sigma} \left( -2\omega^a u^b + u^2 \eta^{ab} \right)$$

8.3.1 Equating the two forms of the metric: Case 1

In the first case, eq. (33), equating the two forms,

$$\frac{a}{n} e^{2\sigma} \eta^{ab} = -2\sigma^{ab} + 2\sigma^b \sigma^a - \sigma^2 \eta^{ab}$$

Defining $f = e^{-\sigma}$ this becomes

$$\frac{a}{n} \eta^{ab} = 2ff^{ab} - \eta_{cd} f^c f^d \eta^{ab}$$

and replacing $\frac{a}{n} \eta^{ab}$ with $2f \eta_{ab}$ gives

$$0 = f^{ab} - \frac{1}{n} \left( \eta_{cd} f^c f^d \right) \eta^{ab}$$

This is easily shown (using, for example, a general power series) to have the general solution

$$f = e^{-\sigma} = \alpha + \beta (y - c)^2$$

where $c_a$ is an arbitrary constant vector and $\alpha, \beta$ are constants. Therefore,

$$h^{ab} = \frac{a}{n} \left( \alpha + \beta (y - c)^2 \right)^{-2} \eta^{ab}$$

$$h_{ab} = \frac{n}{a} \left( \alpha + \beta (y - c)^2 \right)^2 \eta_{ab}$$

We may regard this metric as the expected projection back to the space we originally gauged. It is helpful to recall the corresponding Poincaré case.

In the Poincaré gauge theory, we start with the Poincaré symmetry of $n$-dim flat spacetime. Taking the quotient of the Poincaré group by its Lorentz subgroup and generalizing the connection gives a fiber bundle with local Lorentz symmetry, over the quotient manifold. Neighborhoods in the base manifold may be mapped to neighborhoods of the original flat spacetime. Moreover, when the curvature vanishes, the spaces are identical – we may identify the gauged spacetime manifold with the original space.

It seems as if a similar thing is happening here. The metric induced on the configuration space of the symplectic biconformal manifold may be identified with the original flat space. However, the new canonical, orthonormal basis

$$\chi^a = \omega^b + \frac{1}{2} h^{bc} \omega_c$$

$$\eta_a = \frac{1}{2} \omega_a - h_{ab} \omega^b$$

is not involute. Therefore, as we show in Appendix this metric does not satisfy the conditions for a phase space basis. While we can still make local identifications between neighborhoods spanned by $\chi^a$ and neighborhoods in the original space, $\chi^a$ does not span a true configuration space.

The second possible form for $h_{ab}$ does provide a configuration manifold. The unique orthogonal, canonical, metric basis therefore leads to a configuration space of different signature from the original space. The remainder of our discussion will make use of the signature-changing metric only.
Appendix 4: Coordinate specification of the submanifolds

We look at the specification of the momentum and configuration submanifolds in terms of the \((v^a, y_b)\) coordinates.

The momentum submanifold

The result for the momentum submanifold is immediate. The \(\chi^a = 0\) (momentum) manifold is the same as \(v^a = \) constant, so the submanifolds are given by \((v^a_0, y_b)\). Then \(d = d(v^a_0)\) is also constant.

The configuration submanifold

The case is more complicated for the configuration submanifold, since \(\eta_a\) is not, in general, exact. Setting \(\eta_a = 0\), we have the equation

\[ dv_a = -(y_b d_a + y_a d_b - \eta_{ab} (y^e d_e)) dv^b \]  \hspace{1cm} (84)

This is solved by setting \(y_a = f_a(v^b)\) for some functions \(f_a\), and solving for \(f_a\). We first show that this equation is integrable if and only if \(d(v^a)\) satisfies eq.(74). As shown in Section 7, eq.(74) for \(d(v^a)\) has two distinct solutions. For each of these, we find the required form of the functions \(f_a(v^b)\).

Integrability

Eq.(84) is integrable provided \(dv_a = 0\). Therefore, taking the exterior derivative of eq.(84) and carrying out the required antisymmetrization, we require

\[ 0 = 2y_b d_a d_b - \eta_{ab} y_d d^2 + y_b d_a d_a - \eta_{ad} d_b (y^e d_e) \]
\[-2y_b d_a d_a + \eta_{ad} y_b d^2 - y_d d_a d_a + \eta_{ab} d_b (y^e d_e) \]
\[-y_b d_a d_a - y_a d_b d_b + \eta_{ab} y^e d_e d + y_a d_a b + y_d d_b d - \eta_{ad} d_b d_c b \]

Contraction with \(y^a\) shows that, \(0 = d_{a,b} - d_{b,a}\), so that \(d_a\) must be a gradient, \(d_a = d_a\).

Now differentiate eq.(84) with respect to \(y_c\) and take the trace on \(cd\),

\[ 0 = (n - 2) d_{a} d_{b} + \eta_{ab} d^2 \]
\[+ (n - 2) d_{ab} + \eta_{ab} (\eta^{cd} d_{cd}) \]

Contracting this with \(\eta^{ab}\) shows that \(\eta^{cd} d_{cd} = \frac{1}{2} (n - 2) d^2\), and we have recovered eq.(74). Substituting back into eq.(84) shows that it is identically satisfied, so eq.(74) provides the necessary and sufficient condition for integrability of the configuration submanifold. We have two classes of solution for \(d\), and now find the functions \(f_a\) corresponding to each.

First solution for \(f_a(v^b)\)

Now return to eq.(84) and let \(d_a = -\frac{2y_a}{v^2}\). Then

\[ f_{a,b} = -\frac{2}{v^2} (v_b v_a + f_a v_b - \eta_{ab} (v^e f_e)) \]

It is straightforward to check that the integrability condition, \(f_{a,bc} - f_{a,cb} = 0\) is identically satisfied. Also, it is clear that \(f_{[a,b]} = 0\), so \(f_a\) is a gradient, \(f_a = f_a\). Now contract with \(v^a\),

\[ v^a f_{a,b} = -2 f_b \]

Then since \((v^a f_a)_b = f_b + v^a f_{a,b}\) we have \((v^a f_a)_b = -f_b\). Integrating,

\[ f = -v^a f_a + c \]  \hspace{1cm} (85)

Now contract the original equation with \(f^a\). The result is immediately integrable for \(f^2 = \eta^{ab} f_a f_b\) giving

\[ f^2 = \frac{A}{(v^2)^2} \]
Substituting this result and eq. (85) into eq. (84) leads to

\[ 0 = v^2 f_{ab} + 2 \left( f_{b}v_{a} + f_{a}v_{b} - \eta_{ab}v^e f_{e} \right) \]

\[ = \left( v^2 f_{a} \right)_b + 2 \left( f_{a}v_{b} - 2c (v_{a})_b \right) \]

with the immediate integral \( v^2 f_{a} + 2f_{a}v_{b} - 2cv_{a} = a_{a} \). Rewriting this as

\[ \left( v^2 f \right)_a = \left( a_{a}v^e + cv^2 + b \right)_a \]

immediately yields the form \( f = c + \frac{a_{a}v^e}{v^2} \). Substituting into eq. (85) shows that we must have \( b = 0 \), so finally,

\[ f = c + \frac{a_{a}v^e}{v^2} \]

This has the right number of integration constants, and substitution into the original equation satisfies it identically.

We now turn to the second solution for \( d \).

Second solution for \( f (v^e) \)

Substituting the second solution for \( d \),

\[ d = \ln (A + B_{a}v^b) \]

\[ B^2 = 0 \]

into eq. (84) gives

\[ 0 = f_{ab} + \frac{f_{b}B_{a}}{A + B_{a}v^d} + \frac{f_{a}B_{b}}{A + B_{b}v^d} - \eta_{ab}f_{e}B_{e} \]

\[ \left( v^2 f \right)_a = \left( A_{a}v^e + cv^2 + b \right)_a \]

where we have once again used the symmetry of \( f_{a,b} \) to write \( f_{a} = f_{a} \). Contraction with \( f_{b} \) again allows us integrate to find \( f^2 = \eta^{ab}f_{a}f_{b} \),

\[ f^2 = \frac{C}{(A + B_{a}v^a)^2} \]

where \( C \) is constant. The contraction of the original equation with \( v_{b} \) may be put in the form

\[ 0 = \left( v^b f_{b} \right)_a + \left( v^b f_{b} \right) \frac{B_{a}}{A + B_{a}v^d} - \frac{A_{a}f_{a}}{A + B_{a}v^d} - \eta_{ab} \frac{f_{e}B_{e}}{A + B_{a}v^d} \]

Contracting with \( B^{b} \), and using \( B^2 = 0 \) then shows that \( \left( B^{b} f_{b} \right)_a = 0 \). Then setting \( B^{a}f_{a} = c \), eq. (86) may be rewritten as

\[ 0 = \left( A + B_{a}v^d \right) f_{a} + fB_{a} - cv_{a} \]

Two integrations immediately give

\[ f = \frac{A_{a}v^b + \frac{1}{2}cv^2 + b}{A + B_{a}v^d} \]

(87)

Contraction with \( B^{a} \) shows that \( c = \frac{1}{3}A_{a}B^{a} \) while computing \( f^2 \) shows that \( C = A^{a}A_{a} - 2bc \). Again, the number of integration constants is as expected and the solution checks when substituted into the original equation.