Subconvexity bounds in depth-aspect for automorphic $L$-functions on $GL_2$

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From a spectral identity we obtain asymptotics with error-term for the second integral moments of families of automorphic $L$-functions for $GL_2$ over an arbitrary number field according to twists by idele characters $\chi$ with arbitrary ramification at a fixed finite place. The power-saving in the error term breaks convexity at this non-archimedean place.

1.1 INTRODUCTION

The convexity or trivial bound for the zeta function is

$$|\zeta(\frac{1}{2} + it)| \ll |t|^\frac{1}{4} + \epsilon$$

Any improvement over $\frac{1}{4}$ in this upper bound “breaks convexity”. Various authors have obtained subconvexity bounds in different aspects. [Weyl 1921] gave a subconvex bound

$$|\zeta(\frac{1}{2} + it)| \ll |t|^\frac{1}{6} + \epsilon$$

[Burgess 1962] broke convexity in the conductor aspect for Dirichlet $L$-functions over $\mathbb{Q}$. Subconvexity bounds were also obtained for $GL_2$ $L$-functions in [Good 1982, 1986], [Meurman 1987] and [Duke-Friedlander-Iwaniec 1993, 1994, 2001]. In recent years, subconvexity results were obtained by several authors including Kowalski, Michel, Vanderkam and Venkatesh (see [Kowalski-Michel-Vanderkam 2002] and [Michel-Venkatesh 2006]).

Until recently, all of these results concerned integral moments of automorphic $L$-functions over $\mathbb{Q}$, or over quadratic extensions of $\mathbb{Q}$, and not over an arbitrary number field. In 2006, Diaconu and Goldfeld [Diaconu-Goldfeld 2006] broke convexity in the conductor aspect for Dirichlet $L$-functions over an arbitrary number field...
2006a, 2006b] reconsidered the cases of groundfield \( \mathbb{Q} \) or complex quadratic extensions. Then Diaconu and Garrett [Diaconu-Garrett 2008] obtained asymptotics with error-term for second integral moments of \( GL_2 \) automorphic \( L \)-functions over an arbitrary number field, by a spectral identity. In the relevant spectral identity, Diaconu-Garrett obtained asymptotics with power-saving in the error term for averages not only on the critical line but also over families of twists by grössencharakters:

\[
\sum_{\chi} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 M_\chi(t) \, dt
\]

where \( M_\chi(t) \) are smooth weights. They showed that this breaks convexity in the \( t \)-aspect.

Here we take Diaconu-Garrett’s ideas in a different direction. Fixing a \( GL_2 \) automorphic \( L \)-function over a number field, we arbitrarily deform the data associated with a fixed non-archimedean place \( v_1 \), and allow \( \chi \) to have arbitrary ramification at \( v_1 \). Thus, the weights in the moment expansion are obtained from the archimedean data as well as the data associated with the ramification at the finite prime \( v_1 \). We then obtain asymptotics for that second moment expansion and break convexity in the \( \chi \)-depth-aspect at the non-archimedean place \( v_1 \).

2.1 THE MAIN RESULT

In this paper we break convexity in the \( \chi \)-depth-aspect for a family of \( L \)-functions \( L(\frac{1}{2} + it, f \otimes \chi) \), where \( \chi \) has arbitrary ramification at a fixed finite prime \( v_1 \). For a cuspidal form \( f \) on \( GL_2(k) \), where \( k \) is a number field of degree \( d \) over \( \mathbb{Q} \), the \( \chi \)-depth-aspect convexity bound for the twisted \( L \)-function \( L(\frac{1}{2} + it, f \otimes \chi) \) is

\[
L(\frac{1}{2} + it, f \otimes \chi) \ll q^{N(\frac{d}{2}+\epsilon)}
\]

where \( q^N \), with \( N \geq 1 \), is the conductor of \( \chi \). We break convexity by decreasing the exponent, proving that

\[
L(\frac{1}{2} + it, f \otimes \chi) \ll (q^N)^{\frac{d}{4\vartheta}+\epsilon}
\]

for \( \vartheta < 1 \).

Subconvexity results have been obtained in different aspects by some authors. In particular, for standard \( L \)-functions for \( GL_2 \) over a number field,
Diaconu and Garrett used automorphic spectral theory to break convexity in the $t$-aspect; in a recent preprint, Michel and Venkatesh have claimed a joint convexity bound simultaneously in $t$-aspect, conductor and spectral aspects by using highly non-trivial methods including ergodic theory and regularization of integrals of automorphic forms. Like Diaconu-Garrett, we use a more conceptual argument of spectral identities to break convexity in the $\chi$-depth-aspect at one place. Further research could involve breaking convexity in the depth-aspect at more than one place, and this would involve more careful consideration of analytical details.

A subconvex bound is not merely an improvement of an exponent but also has some important applications. A very concrete example is the sums-of-three-integer-squares problem. An even more striking feature is that many useful corollaries of the Grand Riemann and Lindelöf Hypotheses are also implied by subconvexity results. Thus subconvexity bounds are sufficient for providing solutions to some natural, yet apparently unrelated, questions.

3.1 THE MOMENT EXPANSION
3.1.1 Prologue

In this section, the integral moment expansion is obtained by unwinding the integral representation

$$\int_{Z_k G_k \backslash G_k} P^e \cdot |f|^2$$

where $P^e$ is a Poincaré series and $f$ is a cuspform on $GL_2$. We use Diaconu and Garrett's ideas (see section 2 in [Diaconu-Garrett 2008]) to reformulate the Poincaré series as a single object. The moment expansion is a sum of weighted integrals of $L$-functions $L(s, f \otimes \chi)$ of twists of $f$ by idele class characters $\chi$. The weight functions depend on archimedean data and data associated with the finite place $v_1$ where $\chi$ has arbitrary ramification. We will then obtain asymptotics from the weight functions.

3.1.2 Unwinding to an Euler Product

Define the following subgroups of $G = GL_2$:

$$P = \{(a\ast \ast)\}, \quad N = \{(1\ast 0)\}, \quad H = \{(\ast 0)\}, \quad Z = \text{center of } G, \quad M = ZH = \{(a\ast 0)\}$$
For any place $v$ of $k$, let $K^\text{max}_v$ be the standard maximal compact subgroup. So for finite $v$,

$$K^\text{max}_v = GL_2(o_v)$$

and for infinite $v$,

$$K^\text{max}_v = \begin{cases} O_2 & (v \approx \mathbb{R}) \\ U_2 & (v \approx \mathbb{C}) \end{cases}$$

The Poincaré series $P\acute{e}$ is of the form

$$P\acute{e}(g) = \sum_{\gamma \in M_k \backslash G_k} \varphi(\gamma g)$$

(where $g \in G_k$)

for suitable functions $\varphi$ on $G_k$ defined as follows. Let

$$\varphi = \otimes_v \varphi_v$$

where for finite primes $v \neq v_1$,

$$\varphi_v(g) = \begin{cases} \chi_v(m) = \left| \frac{a}{d} \right|^s_v & (\text{for } g = mk, \ m = (\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}) \in M, \ s \in \mathbb{C}, \ k \in K^\text{max}_v) \\ 0 & (\text{otherwise}) \end{cases}$$

For finite $v = v_1$ (at which $\chi$ is allowed to be ramified)

$$\varphi_v(mg) = \left| \frac{a}{d} \right|^s_v \cdot \varphi_v(g)$$

(m $\in M_v, g \in G_v$)

The data determining $\varphi_v$ for $v = v_1$ consists of its values on $N_v$ where our simple choice is

$$\varphi_v \left( \begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix} \right) = \begin{cases} 1 & (\text{for } x \in o_v) \\ \left| x \right|^{-w'} & (\text{for } w' \in \mathbb{C}, \ x \notin o_v) \end{cases}$$

For infinite $v$ require right $K_v$-invariance and left equivariance:

$$\varphi_v(mg) = \left| \frac{a}{d} \right|^s_v \cdot \varphi_v(g)$$

(m $\in M_v, g \in G_v$)

where

$$\varphi_v \left( \begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix} \right) = \begin{cases} (1 + \left| x \right|^2)^{-\frac{w}{2}} & (\text{for } v \approx \mathbb{R}, \ w \in \mathbb{C}) \\ (1 + x\overline{x})^{-w} & (\text{for } v \approx \mathbb{C}) \end{cases}$$

The Poincaré series $P\acute{e}$ converges absolutely and locally uniformly for $\Re(s') > 1$, $\Re(w) > 1$ for all $v|\infty$, and for $\Re(w') > 1$ (see Proposition 2.6 in [Diaconu-Garrett 2008]).
We want to show that
\[
\int_{Z\backslash G_k \backslash G_{\mathbb{A}}} P\hat{\varepsilon} \cdot |f|^2 \, dg
\]
is an integral of products of local factors of standard $L$-functions. First, the Fourier expansion of a cuspform $f$ on $G_{\mathbb{A}}$ is
\[
f(g) = \sum_{\xi \in Z_k \backslash M_k} W_f(\xi g)
\]
where $W_f$ is the Whittaker function of $f$ and $W_f = \otimes_v W_{f,v}$ is the factorization of $W_f$ into local data. So
\[
\int_{Z\backslash G_k \backslash G_{\mathbb{A}}} P\hat{\varepsilon} \cdot |f|^2 \, dg = \int_{Z\backslash G_k \backslash G_{\mathbb{A}}} \sum_{\gamma \in M_k \backslash G_k} \varphi(\gamma g) |f(g)|^2 \, dg = \int_{Z\backslash M_k \backslash G_{\mathbb{A}}} \varphi(g) |f(g)|^2 \, dg
\]
\[
= \int_{Z\backslash M_k \backslash G_{\mathbb{A}}} \varphi(g) \sum_{\xi \in Z_k \backslash M_k} W_f(\xi g) \overline{\mathcal{F}(g)} \, dg = \int_{Z\backslash G_{\mathbb{A}}} \varphi(g) W_f(g) \overline{\mathcal{F}(g)} \, dg
\]
Let $C$ be the idele class group $GL_1(k) \backslash GL_1(\mathbb{A})$ and $\hat{C}$ its dual. $\hat{C} \approx \mathbb{R} \times \hat{C}_0$ where $\hat{C}_0$ is discrete. The Mellin transform and inversion are
\[
f(x) = \int_C \int_C f(y) \chi^{-1}(y) \, dy \, \chi(x) \, dx = \sum_{\chi' \in \hat{C}_0} \frac{1}{2\pi i} \int_{\mathbb{R}(s)=\sigma} \int_C f(y) \chi'^{-1}(y) |y|^{-s} \, dy \, \chi'(x) |x|^s \, ds
\]
With $Z_{\mathbb{A}} M_k \backslash M_{\mathbb{A}} \approx C$, and for finite $v \neq v_1$,
\[
\int_{Z\backslash G_{\mathbb{A}}} \varphi(g) W_f(g) \overline{\mathcal{F}(g)} \, dg = \int_{Z\backslash G_{\mathbb{A}}} \varphi(g) W_f(g) \left( \int_{\hat{C}} \int_{Z\backslash M_k \backslash M_{\mathbb{A}}} \overline{\mathcal{F}(m'g)} \chi(m') \, dm' \, d\chi \right) \, dg
\]
\[
= \int_{\hat{C}} \left( \int_{Z\backslash G_{\mathbb{A}}} \varphi(g) W_f(g) \int_{Z\backslash M_k \backslash M_{\mathbb{A}}} \sum_{\xi \in Z_k \backslash M_k} W_f(\xi m'g) \chi(m') \, dm' \, d\chi \right) \, d\chi
\]
\[
= \int_{\hat{C}} \left( \int_{Z\backslash G_{\mathbb{A}}} \varphi(g) W_f(g) \int_{Z\backslash M_k \backslash M_{\mathbb{A}}} \overline{\mathcal{W}_f(m'g)} \chi(m') \, dm' \, d\chi \right) \, d\chi
\]
\[
= \int_{\hat{C}} \prod_v \left( \int_{Z\backslash G_v} \int_{Z_v \backslash M_v} \varphi_v(g_v) W_{f,v}(g_v) \overline{\mathcal{W}_{f,v}(m'_v g_v)} \chi_v(m'_v) \, dm'_v \, dg_v \right) \, d\chi
\]
Suppress finite $v \neq v_1$ and write the $v$th local integral as
\[
\int_{Z\backslash M} \varphi(g) W_f(g) \overline{\mathcal{W}_f(m'g)} \chi(m') \, dm' \, dg
\]
Invoke the $v$-adic Iwasawa decomposition $G = MNK$ and rewrite the integral as
\[
\int_{Z\setminus MNK} \int_{Z\setminus M} \varphi(mnk) W_f(mnk) \overline{W}_f(m'mnk) \chi(m') dm' dm dn dk
\]
For simplicity, take $\varphi$ and $f$ to be right $K_v^{\text{max}}$-invariant for finite $v \neq v_1$. This gives
\[
\int_{Z\setminus MN} \int_{Z\setminus M} \varphi(mn) W_f(mn) \overline{W}_f(m'n) \chi(m') dm' dm dn
\]
Replace $m'$ by $m'm^{-1}$ to get
\[
\int_{Z\setminus MN} \int_{Z\setminus M} \varphi(mn) W_f(mn) \overline{W}_f(m'n) \chi(m') \chi^{-1}(m) dm' dm dn
\]
The Whittaker function has the equivariance
\[W_f(ng) = \psi(n) W_f(g) \quad (n \in N_K)\]
Thus,
\[W_f(mn) = W_f(mnm^{-1}m) = \psi(mnm^{-1}) W_f(m) \quad (\text{since } mnm^{-1} \in N)\]
and
\[\overline{W}_f(m'n) = \overline{W}_f(m'nm^{-1}m) = \overline{\psi}(m'nm^{-1}) \overline{W}_f(m')\]
so obtaining
\[
\int_{Z\setminus MN} \int_{Z\setminus M} \varphi(mn) W_f(m) \overline{W}_f(m') \chi(m') \chi^{-1}(m) \psi(mnm^{-1}) \overline{\psi}(m'nm^{-1}) dm' dm dn
\]
Let
\[X(m, m') = \int_N \varphi(n) \psi(mnm^{-1}) \overline{\psi}(m'nm^{-1}) dn\]
We get
\[
\int_{Z\setminus M} \int_{Z\setminus M} \chi_0(m) W_f(m) \overline{W}_f(m') \chi(m') \chi^{-1}(m) X(m, m') dm' dm
\]
Now
\[W_f(mn) = \psi(mnm^{-1}) \cdot W_f(m)\]
and
\[W_f(mn) = W_f(m) \cdot 1\]
by the right $K$-invariance of $W_f$. So for $W_f(m) \neq 0$, $\psi(mm^{-1}) = 1$, and $X(m, m') = 1$ for $m, m'$ in the support of $W_f$. So

$$\int_{Z \setminus M} (\chi_0 \cdot \chi^{-1})(m) W_f(m) \, dm \cdot \int_{Z \setminus M} \chi(m') \overline{W}(m') \, dm'$$

$$= L_v(\chi_{0,v} \cdot \chi_v^{-1}|y|^{-s} , f) \cdot L_v(\chi_v|y'|^{-s} , \overline{f})$$

(where $m = (y_{001}^0, m' = (y_{01}^0)$)

is a product of local factors of $L$-functions at finite primes $v \neq v_1$. Thus the integral can be written as

$$I(\chi_0) = \sum_{\chi \in \mathcal{C}_0} \frac{1}{2\pi i} \int_{\Re(s) = \sigma} L(\chi_0 \cdot \chi^{-1}|y|^{-1-s} , f) \cdot L(\chi|y'|^{-s} , \overline{f}) \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(s, \chi_0, \chi) \, ds$$

where

$$\mathcal{K}_\infty(s, \chi_0, \chi) = \prod_{v|\infty} \mathcal{K}_v(s, \chi_{0,v}, \chi_v)$$

and

$$\mathcal{K}_v(s, \chi_{0,v}, \chi_v) = \int_{Z_v \setminus M_v} \int_{Z_v \setminus M_v} \varphi_v(m_v n_v) W_{f,v}(m_v n_v) \overline{W}_{f,v}(m'_v n_v) \cdot \chi_v(m'_v)^{s} \cdot \chi_v^{-1}(m_v) \cdot |m_v|^{-s} \cdot dm_v \cdot dm_v \cdot dn_v$$

$$\mathcal{K}_{v_1}(w', \chi_{v_1}) =$$

$$\int_{k_{v_1}^5} \int_{k_{v_1}^5} \chi(y) \cdot |y|^{-s} \cdot \chi^{-1}(y') \cdot |y'|^{-s} \cdot W(\frac{y_{001}^0}{001}) \cdot \overline{W}(\frac{y_{01}^0}{01}) \cdot \int_{k_{v_1}} \tilde{\psi}(x \cdot (y - y')) \varphi_{v_1}(\frac{1}{01}) \, dx \, dy'$$

where there is arbitrary ramification of $\chi$ at finite $v = v_1$. The non-decoupled integrals $\mathcal{K}_v(s, \chi_{0,v}, \chi_v)$ and $\mathcal{K}_{v_1}(w', \chi_{v_1})$, which represent the weight functions, will be subsequently computed.

The Poincaré series $P\chi$ has meromorphic continuation to a region in $\mathbb{C}^2$ containing $s' = 0$ and $w' = 1$. As a function of $w'$, for $s' = 0$, it is holomorphic in the half-plane $\Re(w') > \frac{11}{8}$ ([Kim-Shahidi 2002], [Kim 2005]), except for $w' = 1$ where it has a pole of order 1. This can be seen from the spectral decomposition of $P\chi$ (in Chapter 3), and the argument presented by Diaconu-Garrett in the proof of Theorem 4.17 in [Diaconu-Garrett 2008].

For $\Re(s')$ and $\Re(w')$ sufficiently large, the integral $I(\chi_0) = I(s', w')$ is

$$I(s', w') = \sum_{\chi \in \mathcal{C}_0, S} \frac{1}{2\pi i} \int_{\Re(s) = \sigma} L(\chi^{-1} |.|^{s'+1-s} , f) \cdot L(\chi |.|^{s} , \overline{f}) \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(s, s', w, \chi) \, ds$$

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where $S$ is a finite set of places including archimedean places, and the sum is over the set $\hat{C}_{0,S}$ of characters ramified at the finite place $v_1$. $I(s', w')$ has meromorphic continuation to a region in $\mathbb{C}^2$ containing the point $s' = 0$, $w' = 1$, and $I(0, w')$ is holomorphic for $\Re(w') > \frac{11}{18}$ except for $w' = 1$ where it has a pole of order 1.

We will find asymptotics for $K_{v_1}(w', \chi_{v_1})$ and $K_{\infty}(s, s', w, \chi)$ in Section 2.4, shift the line of integration to $\Re(s) = \frac{1}{2}$ and set $s' = 0$. Thus for $\Re(w')$ sufficiently large

$$I(0, w') = \sum_{\chi \in \hat{C}_0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} L(\chi^{-1} \cdot \frac{1}{2} - it, f) \cdot L(\chi \cdot \frac{1}{2} + it, \overline{f}) \cdot K_{v_1}(w', \chi_{v_1}) \cdot K_{\infty}(\frac{1}{2} + it, 0, w, \chi) dt$$

$$= \sum_{\chi} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot K_{v_1}(w', \chi_{v_1}) \cdot K_{\infty}(\frac{1}{2} + it, 0, w, \chi) dt$$

3.1.3 The non-decoupled integrals

The nondecoupled integral $K_{v_1}(w', \chi_{v_1})$ is

$$\int_{k_v} \int_{k_v} \chi(y) |y|_v^s \chi_{v_1}^{-1}(y') |y'|_v^{1-s} W(y_0^0 0 1) \overline{W}(y_0^0 0 1) \cdot \int_{k_v} \overline{\psi}(x(y - y')) \varphi_v(\frac{1}{2} x) dx dy dy'$$

where there is arbitrary ramification of $\chi$ at finite $v = v_1$.

For finite $v = v_1$, define

$$\varphi_v(\frac{1}{2} x) = \begin{cases} 1 & \text{(for } x \in o_v) \\ |x|_v^{-w'} & \text{(for } x \notin o_v) \end{cases}$$

Henceforth, we will suppress the $v$ for ease of notation. $\psi$ is the standard additive character which is trivial on the local integers $o$ and nontrivial on $\omega^{-1} o$. $\chi$ is a ramified multiplicative character, i.e. $\chi$ is non-trivial on $o^\times$. $W$ is a Whittaker function which is invariant on $o^\times$ since it is spherical. The spherical Whittaker function is of the form

$$W(y_0^0 0 1) = \begin{cases} \frac{n+1}{\alpha - \beta}, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha, \beta$ are Satake parameters and $ord(y) = n$.

We will first compute the integral in $y$ and $y'$, and then compute the integral in $x$. Now

$$\overline{\psi}(x(y - y')) = \overline{\psi}(xy - xy') = \overline{\psi}(xy) \cdot \psi(xy')$$

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Thus the integrals in $y$ and $y'$ are as follows:

$$
\int_{k^{\times}} \overline{\psi(xy)} \chi(y) |y|^s W \left( \frac{y}{0 \ 1} \right) dy \cdot \int_{k^{\times}} \psi(xy') \chi^{-1}(y') |y'|^{1-s} \mathbb{W} \left( \frac{y'}{0 \ 1} \right) dy'
$$

Consider the integral in $y$:

$$
\int_{k^{\times}} \overline{\psi(xy)} \chi(y) |y|^s W \left( \frac{y}{0 \ 1} \right) dy
$$

Let $\eta \in \mathfrak{o}^{\times}$. Replace $y$ with $y\eta$ to get

$$
\int_{k^{\times}} \left( \int_{\mathfrak{o}^{\times}} \overline{\psi(xy\eta)} \chi(y\eta) d\eta \right) |y|^s W \left( \frac{y}{0 \ 1} \right) dy
$$

Consider the inner integral:

$$
\int_{\mathfrak{o}^{\times}} \overline{\psi(xy\eta)} \chi(y\eta) d\eta
$$

Recall that $\chi$ is a ramified character. Let $N$ be the conductor of $\chi$. So $\chi$ is trivial on some subgroup $1 + \mathfrak{m}^N$ of $k^{\times}$ and non-trivial on $1 + \mathfrak{m}^{N-1}$ where $N \geq 1$ is the smallest such integer.

As in [Weil 1974], a standard computation shows that

$$
\int_{k^{\times}} \psi(xy) \chi(y) dy = 0 \text{ unless } \text{ord}(x) = -N
$$

So we claim that our inner integral is zero unless \text{ord}(xy) = -N. So \text{ord}(y) = -\text{ord}(x) - N. The integral

$$
\int_{\mathfrak{o}^{\times}} \overline{\psi(xy\eta)} \chi(y\eta) d\eta
$$

is a Gauss sum. A Gauss sum where $\chi$ is a ramified multiplicative character with conductor $N$, is evaluated as follows. Let

$$
\mathbf{g}(\chi, \psi) = \int_{\mathfrak{o}^{\times}} \chi(x) \cdot \overline{\psi \left( \frac{x}{\mathfrak{o} N} \right)} dx
$$

Then

$$
|\mathbf{g}(\chi, \psi)|^2 = \left( \int_{\mathfrak{o}^{\times}} \chi(x) \overline{\psi \left( \frac{x}{\mathfrak{o} N} \right)} dx \right) \cdot \left( \int_{\mathfrak{o}^{\times}} \chi(y) \psi \left( \frac{y}{\mathfrak{o} N} \right) dy \right)
$$

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\[
\int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(xy^{-1}) \psi(y-x) \, dx \, dy = \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(x) \psi(x) \, dx \, dy \quad \text{by replacing } x \text{ with } xy
\]

\[
= \int_{\mathfrak{o}^\times} \chi(x) \int_{\mathfrak{o}} \psi(y(x)) \, dy - \int_{\mathfrak{o}^\times} \int_{\mathfrak{m}} \chi(x) \psi(y(x)) \, dy \, dx
\]

In the first integral, since \( \chi \) is trivial on \( 1 + \varpi^N \mathfrak{o} \), only \( x \in 1 + \varpi^N \mathfrak{o} \) will contribute, and since \( \psi \) is trivial on \( \mathfrak{o} \), the first integral is

\[
\mu(1 + \varpi^N \mathfrak{o}) \cdot \mu(\mathfrak{o}) = \frac{q^{2-N}}{(q-1)^2}
\]

Note that

\[
\mu(1 + \varpi^N \mathfrak{o}) = \mu(\mathfrak{o}^\times) \cdot \frac{\mu(\mathfrak{o})}{|\mathfrak{o}^\times : 1 + \varpi^N \mathfrak{o}|} = \frac{\mu(\mathfrak{o})}{(q-1)q^{N-1}} = \frac{1}{(q-1)q^{N-1}}
\]

where \( \mu(\mathfrak{o}^\times) \) is normalized to 1 and \( \mu(\mathfrak{o}) = \frac{q}{q-1} \). Rewrite the second integral as

\[
\int_{\mathfrak{m}} \psi(y) \int_{\mathfrak{d}} \chi(x) \psi(-x y) \, dx \, dy
\]

This is 0 since

\[
\int_{\mathfrak{o}^\times} \chi(x) \psi(-x y) \, dx = \sum_{a \in \frac{\mathfrak{o}^\times}{1 + \varpi^N \mathfrak{o}}} \chi(a) \int_{1 + \varpi^{N-1} \mathfrak{o}} \psi(-x y) \, dx = 0
\]

because \( \psi(\varpi^N x) \) is trivial on \( \varpi^N \mathfrak{o} \) and non-trivial on \( \varpi^{N-1} \mathfrak{o} \). The integral in \( y' \) is:

\[
\int_{\mathfrak{k}} \left( \int_{\mathfrak{o}^\times} \psi(xy) \chi(y) \, dy \right) |y'|^{1-s} \, \mathcal{W}(y) \, dy', \quad t \in \mathfrak{o}^\times
\]

the conjugate of the integral in \( y \). Thus, by replacing \( y \mathfrak{o} \) with \( u \), and \( x \) with \( m \), the integrals over \( \mathfrak{o}^\times \) in \( y \) and \( y' \) are

\[
\left| \int_{\mathfrak{o}^\times} \psi(xy) \chi(y) \, dy \right|^2 = \left| \int_{\mathfrak{o}^\times} \psi(u) \chi(u) \, du \right|^2 = \frac{q^{2-N}}{(q-1)^2}
\]

So the entire nondecoupled local integral becomes

\[
\frac{q^{2-N}}{(q-1)^2} \left[ \int_{\mathfrak{k}} |y|^s \, \mathcal{W}(y) \, dy \cdot \int_{\mathfrak{k}} |y'|^{1-s} \, \mathcal{W}(y') \, dy' \cdot \int_{\mathfrak{k}} \varphi(x) \, dx \right]
\]

Recall that the integral is zero unless \( \text{ord}(y) = -\text{ord}(x) - N \). So rewrite the integral as:
\[
\frac{q^{2-N}}{(q-1)^2} \cdot \int_k \varphi(\frac{1}{0} \frac{x}{1}) \cdot \int_{\text{ord}(y)=-\text{ord}(x)-N} |y|^s W\left(\frac{y}{0} \frac{0}{1}\right) dy \cdot \int_{\text{ord}(y')=-\text{ord}(x)-N} |y'|^{1-s} W\left(\frac{y'}{0} \frac{0}{1}\right) dy' dx
\]
\[
= \frac{q^{2-N}}{(q-1)^2} \cdot \int_k \varphi(\frac{1}{0} \frac{x}{1}) \cdot \int_{\text{ord}(y)=-\text{ord}(x)-N} |y| |W\left(\frac{y}{0} \frac{0}{1}\right)|^2 dy dx
\]

Since \( \text{ord}(y) = -\text{ord}(x) - N \), then \( y \) can be written as \( y = t^{-N} x, \ t \in o^\times \)

So the entire integral is:

\[
\frac{q^{2-N}}{(q-1)^2} \cdot \int_k \varphi\left(\frac{1}{0} \frac{x}{1}\right) \cdot \left| \frac{1}{\varpi^N x} \right| |W\left(\frac{\varpi}{0} \frac{0}{1}\right)|^2 dx
\]

Now \( y \rightarrow W\left(\frac{y}{0} \frac{0}{1}\right) \) is supported on \( o \cap k^\times \), so \( \text{ord}(x) \leq -N \). Then \( x \notin o \).

Thus, we integrate over \( k^\times \), and by a change in Haar measure, the integral becomes

\[
\frac{q^{1-N}}{q-1} \cdot \int_{k^\times} |x|^{-w'} \cdot \left| \frac{1}{\varpi^N x} \right| |W\left(\frac{1}{0} \frac{0}{1}\right)|^2 dx
\]

Invert \( x \) to get

\[
\frac{q^{1-N}}{q-1} \cdot \int_{k^\times} |x|^{-w'} \cdot \left| \frac{x}{\varpi^N} \right| |W\left(\frac{x}{0} \frac{0}{1}\right)|^2 dx
\]

Replace \( x \) by \( \varpi^N x \) and let \( \text{ord}(x) = \ell \) to get

\[
K_{v_1}(w', \chi_{v_1}) = \frac{q^{1-N}}{q-1} \cdot q^{-Nw'} \cdot q^N \cdot \int_{k^\times} |x|^{w'-1} \cdot |x| \cdot |W\left(\frac{x}{0} \frac{0}{1}\right)|^2 dx
\]
\[
= \frac{q}{q-1} \cdot q^{-Nw'} \cdot \sum_{\ell=0}^{\infty} q^{-\ell w'} \cdot \frac{\alpha^{\ell+1} - \beta^{\ell+1}}{\alpha - \beta} \cdot \frac{\alpha^{\ell+1} - \beta^{\ell+1}}{\alpha x - \beta x}
\]
\[
= \frac{q^{1-Nw'}}{q-1} \cdot \frac{1 - |\alpha|^2 |\beta|^2 q^{-2w'}}{(1 - |\alpha|^2 q^{-w})(1 - |\beta|^2 q^{-w})(1 - \alpha \beta q^{-w})(1 - \alpha \beta q^{-w})}
\]

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3.1.4 Asymptotics

Now
\[ \frac{q}{q-1} \cdot \frac{1 - |\alpha|^2 |\beta|^2 q^{-2w'}}{(1 - |\alpha|^2 q^{-w'})(1 - |\beta|^2 q^{-w'})(1 - \overline{\alpha} \beta q^{-w'})} \]
is independent of the conductor \( q^N \) of \( \chi \), so
\[ K_{v_1}(w', \chi_{v_1}) \ll (q^N)^{-w'} \]

The formulas (5.2) - (5.4) in [Diaconu-Garrett 2008] state that the nonde-
coupled integral
\[ K_\infty(s, \chi_0, \chi) = \prod_v K_v(s, \chi_0, v, \chi_v) = \prod_v K_v(s, s', w, \chi_v) \]
has the following asymptotic formula:

For \( v \) complex,
\[
K_v(s, s', w, \chi_v) = \pi^{-2s'+1} A(s', w, \mu_1, \mu_2) \cdot (1 + \ell_v^2 + 4(t + t_v)^2)^{-w} \cdot \left[ 1 + O\left( \left( \sqrt{1 + \ell_v^2 + 4(t + t_v)^2} \right)^{-1} \right) \right]
\]
where \( A(s', w, \mu_1, \mu_2) \) is the ratio of products of gamma functions
\[
2^{4w - 4s'} 4 \Gamma(w + s' + i\mu_1 + i\mu_2) \Gamma(w + s' - i\mu_1 + i\mu_2) \Gamma(w + s' + i\mu_1 - i\mu_2) \Gamma(w + s' - i\mu_1 - i\mu_2) \Gamma(2w + 2s')
\]
and \( it_v, \ell_v \) are the parameters of the local component \( \chi_v \) of \( \chi \).

For \( v \) real,
\[
K_v(s, s', w, \chi_v) = B(s', w, \mu_1, \mu_2) \cdot (1 + |t + t_v|)^{-w} \cdot \left[ 1 + O\left( (1 + |t + t_v|)^{-\frac{1}{2}} \right) \right]
\]
where \( B(s', w, \mu_1, \mu_2) \) is a similar ratio of products of gamma functions.
4.1 SPECTRAL DECOMPOSITION OF THE POINCARÉ SERIES

4.1.1 Prologue

In this chapter, we spectrally decompose the Poincaré series; this is central to the ideas underlying the integral moments of automorphic $L$-functions on $GL_2$ to prove the meromorphic continuation of the Poincaré series. The decomposition consists of a leading (non-$L^2$) term, cuspidal part and continuous part.

4.1.2 The Cuspidal Part

Let $F$ be a cuspform on $G_A$ generating a spherical representation locally everywhere, and suppose $F$ corresponds to a spherical vector everywhere locally. The $F$th (cuspidal) component of the spectral decomposition of the Poincaré series is $\langle P\acute{e}, F \rangle \cdot F$. So

$$\langle P\acute{e}, F \rangle = \int_{Z \backslash G_A} \varphi(g) \overline{F}(g) \, dg = \sum_{\gamma \in M \backslash G} \varphi(g) \overline{F}(g) \, dg$$

$$= \prod_{v \neq v_1} \int_{Z_v \backslash G_v} \varphi_v(g) \overline{W_F}(g) \, dg \cdot \prod_{v \neq v_1} \int_{Z_v \backslash G_v} \varphi_v(g) \overline{W_F}(g) \, dg$$

Suppress $v$. At finite $v \neq v_1$, by Iwasawa decomposition and right $K_v$-invariance,

$$\int_{Z \backslash M N} \varphi(mn) \overline{W_F}(mn) \, dm \, dn$$

Further, with $Z \backslash M N \approx HN$ and

$$W_F(mn) = \psi(mn^{-1}) W_F(m)$$

we get

$$\int_H \int_N \chi_0(m) \varphi(n) \overline{\psi(mn^{-1})} \overline{W_F}(m) \, dm \, dn$$

Again, for $m$ in the support of $W_F$ and $n \in N \cap K$

$$\int_N \varphi(n) \overline{\psi(mn^{-1})} \, dn = 1$$
So the integral becomes
\[ \int_H \chi_0(m) W_F(m) \, dm = \int_{k \times} |y|^{s'} W(y_{0 1}^0) \, dy = (1-\alpha q^{-s'})^{-1} (1-\beta q^{-s'})^{-1} = L_v(s'+\frac{1}{2}, F) \]

For \( v = v_1 \), \( \langle P', F \rangle \) unwinds to
\[ \int_H \int_N \chi_0(m) \varphi(n) \overline{\psi}(mnm^{-1}) W_F(m) \, dm \, dn \]

Now
\[ mnm^{-1} = \begin{pmatrix} xy & 0 \\ 0 & 1 \end{pmatrix} \]
So the integral becomes
\[ \int_{k \times} \overline{\psi}(xy) |y|^{s'} \overline{W}(y_{0 1}^0) \varphi(x) \, dy \, dx \]

We will first consider the integral in \( y \)
\[ \int_{k \times} \overline{\psi}(xy) |y|^{s'} \overline{W}(y_{0 1}^0) \, dy \]
where \( \psi \) is the standard additive character trivial on \( o \) and non-trivial on \( \varpi^{-1} o \) for absolutely unramified \( v \). The spherical Whittaker function is defined by
\[ W(y_{0 1}^0) = \begin{cases} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} & (n \geq 0) \\ 0 & \text{(otherwise)} \end{cases} \]
where \( \overline{W}_F = W_F = W \), where \( \alpha, \beta \) are Satake parameters of \( F \), and where \( \text{ord}(y) = n \). For \( x \in o \) (i.e. for \( \text{ord}(x) \geq 0 \), since \( \psi \) is trivial on \( o \), the integral is
\[ \sum_{n=0}^{\infty} q^{-ns'} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = (1-\alpha q^{-s'})^{-1} (1-\beta q^{-s'})^{-1} = L_v(s'+\frac{1}{2}, F) \]

For \( x \not\in o \) (i.e. for \( \text{ord}(x) < 0 \), we first evaluate
\[ \int_{o \times} \overline{\psi}(xy) \, dy \]

Let \( \text{ord}(x) = m \). Write
\[ y = \varpi^t, \quad x = \varpi^m \eta, \quad t, \eta \in o^\times \]
Then, replacing $t\eta$ by $u$, the integral becomes

$$\int_{o} \bar{\psi}(\bar{\omega}^{m+n}u) \, du$$

Now

$$\int_{o} \bar{\psi}(\bar{\omega}^{\ell}u) \, du = \int_{o} \bar{\psi}(\bar{\omega}^{\ell}u) \, du - \int_{m} \bar{\psi}(\bar{\omega}^{\ell}u) \, du$$

For $\ell \geq 0$, the integrand is 1 so

$$\int_{o} \bar{\psi}(\bar{\omega}^{\ell}u) \, du = \text{meas}(o^{\times}) = 1$$

For $\ell = -1$, $\bar{\psi}$ is non-trivial on $o$ and trivial on $m$, so

$$\int_{o} \bar{\psi}(\bar{\omega}^{\ell}u) \, du = -\text{meas}(m) = -\frac{1}{q-1}$$

For $\ell \leq -2$, $\bar{\psi}$ is non-trivial on $o$ and on $m$, so

$$\int_{o} \bar{\psi}(\bar{\omega}^{\ell}u) \, du = 0$$

So keeping in mind that $\text{ord}(y) = n$ and $\text{ord}(x) = m$,

$$\int_{o} \bar{\psi}(xy) \, dy = \int_{o} \bar{\psi}(\bar{\omega}^{m+n}u) \, du = \begin{cases} 1 & (\text{for } \text{ord}(y) \geq -\text{ord}(x)) \\ -\frac{1}{q-1} & (\text{for } \text{ord}(y) = -\text{ord}(x) - 1) \\ 0 & (\text{otherwise}) \end{cases}$$

So, for $x \not\in o$,

$$\int_{k} \bar{\psi}(xy) |y|^{s'} W(y \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}) \, dy = \int_{\text{ord}(y) \geq -\text{ord}(x)} |y|^{s'} W(y \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}) \, dy - \frac{1}{q-1} \int_{\text{ord}(y) = -\text{ord}(x) - 1} |y|^{s'} W(y \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}) \, dy$$

Now the whole integral is

$$\int_{k} \int_{k} \bar{\psi}(xy) |y|^{s'} W(y \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}) \varphi(x) \, dy \, dx$$

Again, the sub-integral over $x \in o$ evaluates to

$$(1 - \alpha q^{-s'})^{-1} (1 - \beta q^{-s'})^{-1} = L_v(s' + \frac{1}{2}, f)$$

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The sub-integral over $x \notin o$ becomes
\[
\int_{\text{ord}(x)<0} |x|^{-w'} \int_{\text{ord}(y)\geq-\text{ord}(x)} |y|^s' W\left(\frac{y}{0}^0_1\right) dy dx
\]
\[
-\frac{1}{q-1} \int_{\text{ord}(x)<0} |x|^{-w'} \int_{\text{ord}(y)=\text{ord}(x)-1} |y|^s' W\left(\frac{y}{0}^0_1\right) dy dx
\]

First,
\[
\int_{\text{ord}(x)<0} |x|^{-w'} \int_{\text{ord}(y)\geq-\text{ord}(x)} |y|^s' W\left(\frac{y}{0}^0_1\right) dy dx
\]
\[
= \frac{q-1}{q} \int_{\text{ord}(x)<0} |x|^{-w'} \int_{\text{ord}(y)\geq-\text{ord}(x)} |y|^s' W\left(\frac{y}{0}^0_1\right) dy dx
\]
\[
= \frac{q-1}{q} \sum_{m=1}^{\infty} q^{m(1-w')} \sum_{n=-m}^{\infty} q^{-ns'} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}
\]
\[
= \frac{q-1}{q} \sum_{m=1}^{\infty} q^{m(1-w')} \cdot (\alpha - \beta)^{-1} \left( (1 - \alpha q^{-s'})^{-1} \alpha \sum_{m=1}^{\infty} \left( \alpha^{-1} q^{1-w'+s'} \right)^m - \right)
\]
\[
(1 - \beta q^{-s'})^{-1} \beta \sum_{m=1}^{\infty} \left( \beta^{-1} q^{1-w'+s'} \right)^m
\]
\[
= \frac{q-1}{q} \cdot (\alpha - \beta)^{-1} \left[ \frac{q^{1-w'+s'} \left( \frac{1}{1 - \alpha q^{-s'}} \right)}{(1 - \alpha q^{-s'}) \left( 1 - \alpha^{-1} q^{1-w'+s'} \right)} - \frac{q^{1-w'+s'} \left( \frac{1}{1 - \beta q^{-s'}} \right)}{(1 - \beta q^{-s'}) \left( 1 - \beta^{-1} q^{1-w'+s'} \right)} \right]
\]
\[
= \frac{(1 - \alpha q^{-s'}) \left( 1 - \beta q^{-s'} \right) \left( 1 - \alpha^{-1} q^{1-w'+s'} \right) \left( 1 - \beta^{-1} q^{1-w'+s'} \right)}{(q-1) (q^{-w'} - \frac{q^{1-2w'+2s'}}{\alpha \beta})}
\]
\[
= L_{v_1}(s' + \frac{1}{2}, f) \cdot \frac{(q-1) (q^{-w'} - \frac{q^{1-2w'+2s'}}{\alpha \beta})}{(1 - \alpha^{-1} q^{1-w'+s'}) \left( 1 - \beta^{-1} q^{1-w'+s'} \right)}
\]

For \(\text{ord}(y) = -\text{ord}(x) - 1\), write \(y\) as
\[
y = \frac{t}{\omega x} \quad (t \in o^\infty)
\]
Then
\[
\frac{1}{q-1} \cdot \int_{\text{ord}(x)<0} |x|^{-w'} \cdot \int_{\text{ord}(y)=-\text{ord}(x)-1} |y|^s' W\left(\frac{y}{0}^0_1\right) dy dx
\]
\[
= \frac{1}{q-1} \cdot \int_{\text{ord}(x) < 0} |x|^{-w'} \cdot \frac{1}{\zeta x} |s'| W(\frac{x}{\zeta x} 0) dx
\]

Now \( y \to W(\frac{y}{0} 0) \) is supported on \( \partial \cap k^\infty \). Thus, integrate over \( k^\infty \), and by changing to multiplicative Haar measure, the integral becomes

\[
\frac{q-1}{q} \cdot \frac{1}{q-1} \cdot \int_{\text{ord}(x) < 0} |x|^{-w'} \cdot \frac{1}{\zeta x} |s'| W(\frac{x}{\zeta x} 0) dx
\]

Invert \( x \) to obtain

\[
\frac{1}{q} \cdot \int_{\text{ord}(x) > 0} |x|^{w'-1} \cdot \frac{x}{\zeta x} |s'| W(\frac{x}{\zeta x} 0) dx
\]

Replace \( x \) by \( \zeta x \) and with \( \text{ord}(x) = m \)

\[
\frac{1}{q} \cdot q^{1-w'} \int_{\text{ord}(x) \geq 0} |x|^{w'-1} \cdot |x|^{s'} \cdot W(\frac{x}{\zeta x} 0) dx
\]

\[
= q^{-w'} \cdot \sum_{m=0}^{\infty} q^{-m(w'-1+s')} \cdot \frac{\alpha m+1 - \beta m+1}{\alpha - \beta}
\]

\[
= q^{-w'} \cdot (1 - \alpha q^{1-w'-s'}-1) (1 - \beta q^{1-w'-s'})^{-1} = q^{-w'} L(s' + w' - \frac{1}{2}, F)
\]

So, for \( v = v_1 \), the \( v^{th} \) local factor of \( \langle \dot{P}e, F \rangle \) is

\[
\frac{1}{(1 - \alpha q^{-s'}) (1 - \beta q^{-s'})} + \frac{(q - 1)(q^{-w'} - q^{1-2w'+2s'})}{\alpha \beta}
\]

\[
q^{w'} \left[ \frac{1}{(1 - \alpha q^{1-w'-s'}) (1 - \beta q^{1-w'-s'})} + \frac{(q - 1)(q^{-w'} - q^{1-2w'+2s'})}{\alpha \beta} \right] \cdot L_v(s' + \frac{1}{2}, F) - q^{-w'} L_v(s' + w' - \frac{1}{2}, F)
\]

For infinite \( v \), by formulas (4.2) and (4.3) in [Diaconu-Garrett 2008] the \( v^{th} \) local factor of \( \langle \dot{P}e, F \rangle \) is \( G(\frac{1}{2} + i\mu_{F,v}; s', w) \), where up to a constant, for \( v \approx \mathbb{R} \),

\[
G_v(s; s', w) = \pi^{-s'} \frac{\Gamma(s' + \frac{1}{2} - 2) \Gamma(s' + w - s) \Gamma(s' + s) \Gamma(s' + w + s - 1)}{\Gamma(w/2) \Gamma(s' + w/2)}
\]
and at \( v \approx \mathbb{C}, \)

\[
\mathcal{G}_v(s; s', w) = 2\pi^{-2s'} \frac{\Gamma(s' + 1 - s) \Gamma(s' + w - s) \Gamma(s' + s) \Gamma(s' + w + s - 1)}{\Gamma(w) \Gamma(2s' + w)}
\]

Group the archimedean factors as

\[
\mathcal{G}_{F\infty}(s', w) = \prod_{v|\infty} \mathcal{G}_v\left(\frac{1}{2} + i\mu_{F,s'}; s', w\right)
\]

and let all ambiguous constants be absorbed into \( \rho_F. \) Then, for cuspforms \( F, \) the cuspidal part of the spectral decomposition of the Poincaré series is

\[
\sum_F \langle P\hat{e}, F \rangle \cdot F = \sum_F \mathcal{P}_F \mathcal{G}_{F\infty}(s', w) \cdot \left[ L_v(s' + \frac{1}{2}, F) + L_{v_1}(s' + \frac{1}{2}, F') + \frac{(q - 1)(q^{-u'} - q^{1-2w'+2s'})}{(1 - \alpha^{-1}q^{1-w'+s'}) (1 - \beta^{-1}q^{1-w'+s'})} \cdot L_{v_1}(s' + \frac{1}{2}, F) - q^{-u'} L_{v_1}(s' + w' - \frac{1}{2}, F') \right] \cdot F
\]

There is no residual spectrum since residual automorphic forms on \( GL(2) \) are associated to one-dimensional representations which have no Whittaker models.

### 4.1.3 The continuous part

Subtract an Eisenstein series from the Poincaré series and denote the resulting function by \( P\hat{e}^* \). This function is \( L^2 \) and has sufficient decay so that it can be integrated against an Eisenstein series (see section 4 in [Diaconu-Garrett 2008]). The leading term is:

\[
\int_{N_h} \varphi = \left( \int_{N_{\infty}} \varphi_{\infty} \cdot \left( \int_{N_{v\neq v_1}} \varphi_{v\neq v_1} \cdot \int_{N_{v_1}} \varphi_{v_1} \right) \right) \cdot E_{s'+1,1}
\]

where, as in (4.16) in [Diaconu-Garrett 2008] an elementary computation shows

\[
\int_{N_v} \varphi_v = \begin{cases} \sqrt{\pi} \frac{\Gamma\left(\frac{w+1}{2}\right)}{\Gamma\left(\frac{w+1}{2}\right)} & (v \approx \mathbb{R}) \\ 2\pi(w - 1)^{-1} & (v \approx \mathbb{C}) \end{cases}
\]

Now for \( v \neq v_1, \)

\[
\int_{N_v} \varphi_v \, dn = \int_{k_v} 1 \, dx = 1
\]
and \( \int_{N} \varphi_{v_1} \, dn = \int_{x \in o_v} 1 \, dx + \int_{x \notin o_v} |x|^{-w'} \, dx \)
\[
= 1 + \frac{q - 1}{q} \cdot \sum_{m=1}^{\infty} (q^{m})^{1-w'} = 1 + \frac{q - 1}{q} \cdot \frac{q^{1-w'}}{1-q^{1-w'}} = \frac{1-q^{-w'}}{1-q^{1-w'}}
\]

So the leading term is
\[
\int_{N_{\infty}} \varphi_{\infty} \cdot \frac{1-q^{-w'}}{1-q^{1-w'}} \cdot E_{s'+1,1}
\]

The continuous part of the spectral decomposition of \( P\dot{e} \) is
\[
\frac{1}{4\pi i \kappa} \sum_{\chi} \int_{\Re(s)=\frac{1}{2}} \langle P\dot{e}^{*}, E_{s,\chi} \rangle \cdot E_{s,\chi} \, ds \quad \text{(where } \kappa = \text{meas}(\mathbb{H}/k^{*}))\]

So the spectral decomposition of the Poincaré series is
\[
P\dot{e} = \left( \int_{N_{\infty}} \varphi_{\infty} \right) \cdot \frac{1-q^{-w'}}{1-q^{1-w'}} \cdot E_{s'+1,1} + \sum_{F} \mathcal{F}_{\mathcal{G}_{\infty}}(s', w) \cdot \left[ L_v(s' + \frac{1}{2}, F) + L_{v_1}(s' + \frac{1}{2}, F') + \frac{(q-1)(q^{-w'} - q^{1-2w'+2s'})}{(1-\alpha^{-1}q^{1-w'+s})(1-\beta^{-1}q^{1-w'+s'})} \cdot L_v(s' + \frac{1}{2}, F) - q^{-w'} L_{v_1}(s' + w' - \frac{1}{2}, F) \right] \cdot F + \frac{1}{4\pi i \kappa} \sum_{\chi} \int_{\Re(s)=\frac{1}{2}} \langle P\dot{e}^{*}, E_{s,\chi} \rangle \cdot E_{s,\chi} \, ds
\]

As in section 4 in [Diaconu-Garrett 2008],
\[
\langle P\dot{e}^{*}, E_{s,\chi} \rangle = \left( \int_{Z_{\infty}} \varphi_{\infty} \cdot \overline{W}_{s,\chi,\infty} \right) \cdot \prod_{v<\infty} \int_{Z_{v}} \varphi_{v}(g_v) \cdot \overline{W}_{s,\chi,v}(g_v) \, dg_v
\]

where
\[
\int_{Z_{v}} \varphi_{\infty} \cdot \overline{W}_{s,\chi,v} = \begin{cases} \mathcal{G}_{v}(s, s', w) & (v \approx \mathbb{R}) \\ \frac{\pi^{-s} \Gamma(s)}{\mathcal{G}_{v}(s, s', w)} & (v \approx \mathbb{C}) \\ \frac{2\pi^{-2s-1} \Gamma(2s)}{\mathcal{G}_{v}(s, s', w)} & (v \approx \mathbb{C}) \end{cases}
\]

and for finite \( v \neq v_1 \),
\[
\int_{Z_v \backslash G_v} \varphi_v(g_v) \cdot W_E^{s, \chi, v}(g_v) = |\mathcal{O}_v|^{1/2} \cdot \frac{L_v(s' + \pi, \chi_v) \cdot L_v(s' + 1 - \pi, \chi_v)}{L_v(2\pi, \chi_v^2)} \cdot |\mathcal{O}_v|^{-(s' + 1 - \pi)} \cdot \chi_v(\mathcal{O}_v)
\]

where \(\mathcal{O}\) is the idele with \(v^{th}\) component \(\mathcal{O}_v\) at finite place \(v\) and component 1 at archimedean places.

For finite \(v = v_1\),
\[
\int_{Z_v \backslash G_v} \varphi_v(g_v) \cdot W_E^{s, \chi, v}(g_v) = \int_k \int_k |y|^{s'} \psi(xy) W_E^{s, \chi}(y, 0) \cdot \varphi(x) \, dy \, dx
\]

Define an Eisenstein series by
\[
E(g) = \sum_{\lambda \in P_k \backslash G_k} \eta(\lambda g)
\]
for \(\eta\) left \(P_k\)-invariant, left \(M_k\)-invariant and left \(N_A\)-invariant. Present the vectors \(\eta_v\) in a different form, namely
\[
\eta_v(pk) = \left| \frac{a}{d} \right|^{s} \cdot \chi_v(\frac{a}{d}) \quad \text{(for} \quad p = (\frac{a}{d}) \in P_v, \quad k \in K_v \text{)}
\]

Let \(\phi_v\) be any Schwartz function on \(k_v^2\), invariant under \(k_v\) and put
\[
\eta_v'(g) = \chi_v(\det g) |\det g|^{s} \cdot \int_{k_v^2} \chi_v^2(t) |t|^{2s} \cdot \phi_v(t \cdot e_2 \cdot g) \, dt
\]
where \(e_2 = e_{2,v}\) is the second basis element in \(k_v^2\). \(\eta_v'\) has the same left \(P_v\)-equivariance as \(\eta_v\):
\[
\eta_v'\left( (\frac{a}{d}) \cdot g \right) = \left| \frac{a}{d} \right|^{s} \cdot \chi_v(\frac{a}{d}) \cdot \eta_v'(g)
\]

For \(\phi_v\) invariant under \(K_v\), the function \(\eta_v'\) is right \(K_v\)-invariant. So as in Appendix 2 in [Diaconu-Garrett 2008],
\[
\eta_v'(g) = \eta_v'(1) \cdot \eta_v(g) \quad \text{(since} \quad \eta_v(1) = 1 \text{)}
\]
and
\[
\eta_v'(1) = \int_{k_v^2} \chi_v^2(t) |t|^{2s} \cdot \phi_v(t \cdot e_2 \cdot 1) \, dt = \zeta_v(2s, \chi^2, \phi(0, *))
\]
Thus, it suffices to compute the local Mellin transform of
\[
\eta'_v(1) \cdot W^E_{s,X,v}(m) = \int_{N_v} \overline{\psi(n)} \cdot \eta'_v(w_0 nm) \, dn
\]
\[
= \chi(y)|y|^s \cdot \int_{N_v} \overline{\psi(n)} \int_{k_v^\times} \chi^2_v(t) |t|^{2s} \cdot \phi_v(t \cdot e_2 \cdot w_0 \cdot nm) \, dt \, dn
\]
\[
= \chi(y)|y|^s \cdot \int_{k_v^\times} \overline{\psi(x')} \int_{k_v^\times} \chi^2_v(t) |t|^{2s} \cdot \phi_v(tx', ty) \, dt \, dx' \quad \text{(with } m = (y_0 1)\text{)}
\]

At finite primes, take
\[
\phi(t, x') = ch_{\partial_v}(t) \cdot ch_{\partial_v}(x') \quad \text{(ch}_X \text{ = characteristic function of a set } X)
\]
Then
\[
\eta'_v(1) = \zeta_v(2s, \chi^2, ch_{\partial_v}) = L_v(2s, \chi^2)
\]
and
\[
\eta'_v(1) \cdot W^E_{s,X,v}(y_0 1) = \chi(y)|y|^s \cdot \int_{k_v^\times} \overline{\psi(x')} ch_{\partial_v}(tx') \cdot \int_{k_v^\times} \chi^2_v(t) |t|^{2s} ch_{\partial_v}(ty) \, dt \, dx'
\]
\[
= \chi(y)|y|^s \cdot \text{meas}(\partial_v) \int_{k_v^\times} ch_{\partial_v}(\frac{1}{t}) \chi^2_v(t) |t|^{2s} \cdot ch_{\partial_v}(ty) \, dt
\]
\[
= |\partial_v|^\frac{1}{2} \cdot \chi(y)|y|^s \int_{k_v^\times} ch_{\partial_v}(\frac{1}{t}) \chi^2_v(t) |t|^{2s} \cdot ch_{\partial_v}(ty) \, dt
\]

where \( \partial_v \in k_v^\times \) is such that \((\partial_v)^{-1} = \partial_v \cdot \partial_v\).

So, omitting \(|\partial_v|^\frac{1}{2}\) for now,
\[
\int_k \int_{k^\times} |y|^{s'} \overline{\psi(xy)} \overline{W^E_{s,X,y_0 1}} \cdot \varphi(x) \, dy \, dx
\]
\[
= \int_{k^\times} \int_{k^\times} |y|^{s'} \overline{\psi(xy)} \cdot (\chi(y)|y|^s \int_{k_v^\times} ch_{\partial_v}(\frac{1}{t}) \chi^2_v(t) |t|^{2s} \cdot ch_{\partial_v}(ty) \, dt) \cdot \varphi(x) \, dy \, dx
\]

Consider the integrals in \( y \) and \( t \). Replace \( y \) by \( \frac{y}{t} \) to get
\[
\int_{k^\times} \int_{k^\times} \overline{\psi(xy)} \cdot \chi(y)|y|^{s+s'} \cdot ch_{\partial_v}(y) \cdot \frac{1}{t} \cdot \chi(t) |t|^{s-1-s'} \, dt \, dy
\]
Replace \( t \) by \( \frac{1}{t} \) to get
\[
\int_{k^\times} \int_{k^\times} \overline{\psi(xy)} \cdot \chi(y)|y|^{s+s'} \cdot ch_{\partial_v}(y) \cdot ch_{\partial_v}(t) \cdot \chi(t) |t|^{s'+1-s} \, dt \, dy
\]

First consider the integral in \( y \):
\[
\int_{k^\times} \overline{\psi(xy)} \cdot \chi(y)|y|^{s+s'} \cdot ch_{\partial_v}(y) \, dy
\]
For $x \in o^*$, $\psi$ is trivial on $o$, so we get
\[
\int_{o^*} x(y) |y|^{s+s'} dy \cdot \int_k x \cdot \nu_y(t) \cdot \chi^o(t) \cdot |t|^{s'+1-s} dt
= L_{v_1}(s+s', \chi) \cdot L_{v_1}(s'+1-s, \chi) \cdot |o^*|^{-(s'+1-s)} \chi(o^*)
\]
For $x \not\in o^*$,
\[
\int_{o^*} \overline{\psi}(xty) dy = \begin{cases} 
1 & (\text{ord}(y) \geq -\text{ord}(x) - \text{ord}(t)) \\
-\frac{1}{q-1} & (\text{ord}(y) = -\text{ord}(x) - \text{ord}(t) - 1) \\
0 & (\text{otherwise})
\end{cases}
\]
So
\[
\int_{k^*} \overline{\psi}(xty) \cdot x(y) |y|^{s+s'} ch_o(y) dy = \int_{o^*} \overline{\psi}(xty) \cdot \chi(y) |y|^{s+s'} dy
= \int \chi(y) |y|^{s+s'} dy - \frac{1}{q-1} \int_{\text{ord}(y) = -\text{ord}(x) - \text{ord}(t) - 1} \chi(y) |y|^{s+s'} dy
\]
The entire integral in $t$ and $y$ is:
\[
\int_{x \not\in o} \varphi(x) \int_{k^*} ch_{o^*} t \cdot \chi(t) \cdot |t|^{s'+1-s} dt
\]
\[
[\int_{\text{ord}(y) \geq -\text{ord}(x) - \text{ord}(t)} \chi(y) |y|^{s+s'} dy - \frac{1}{q-1} \int_{\text{ord}(y) = -\text{ord}(x) - \text{ord}(t) - 1} \chi(y) |y|^{s+s'} dy]
\]
First take
\[
\int_{x \not\in o} \varphi(x) \int_{k^*} ch_{o^*} t \cdot \chi(t) \cdot |t|^{s'+1-s} dt \cdot \int_{\text{ord}(y) \geq -\text{ord}(x) - \text{ord}(t)} \chi(y) |y|^{s+s'} dy dx
= q-1 \sum_{m=1}^{\infty} (q^m)^{1-w'} \sum_{n \geq -m-r} (q^{-n})^{s+s'} \int_{k^*} ch_{o^*} t \cdot \chi(t) \cdot |t|^{s'+1-s} dt
\]
(where ord($t$) = $r$ and $\chi(y)$ is omitted for now)
\[
= q-1 \sum_{m=1}^{\infty} (q^m)^{1-w'} \sum_{n \geq -m-r} (q^{-n})^{s+s'} \int_{k^*} ch_{o^*} t \cdot \chi(t) \cdot |t|^{s'+1-s} dt
= \frac{(q-1) q^{-w'+s'}}{q(1-q^{-w'+s'})} \cdot \int_{k^*} ch_{o^*} t \cdot \chi(t) \cdot |t|^{s'+1-s} dt
\]
\[
= \frac{(q-1) q^{-w'+s'}}{(1-q^{-w'+s'})} \cdot L_{v_1} (1-2s, \chi)
\]
Thus adding up we get

\[-\frac{1}{q-1} \int_{x \notin \Phi} \varphi(x) \int_{k \times} ch_{\alpha, \psi}(t) \cdot \overline{\chi}(t) \, |t|^{s'+1-s} \, dt \cdot \int_{ord(y) = -ord(x) - ord(t) - 1} \chi(y) \, |y|^{s'+s'} \, dy \, dx\]

Since \(ord(y) = -ord(x) - ord(t) - 1\), \(y\) can be written as

\[y = \frac{1}{\varpi}tx\]

So the entire integral becomes an integral in \(t\) and \(x\) as follows:

\[-\frac{1}{q-1} \int_{x \notin \Phi} \varphi(x) \int_{k \times} ch_{\alpha, \psi}(t) \cdot \overline{\chi}(t) \, |t|^{s'+1-s} \, dt \cdot \int_{ord(y) = -ord(x) - ord(t) - 1} \chi(y) \, |y|^{s'+s'} \, dy \, dx\]

\[= -\frac{1}{q-1} \cdot \frac{q-1}{q} \int_{x \notin \Phi} |x|^{1-w'} \int_{k \times} ch_{\alpha, \psi}(t) \cdot \overline{\chi}(t) \, |t|^{s'+1-s} \, dt \cdot \frac{1}{\varpi \, xt} \, |s'+s'\, dx\]

\[= -q^{s+s'-1} \int_{|x|>1} |x|^{1-w'-s-s} \int_{k \times} ch_{\alpha, \psi}(t) \cdot \overline{\chi}(t) \, |t|^{1-2s} \, dt \, dx\]

\[= -q^{s+s'-1} \sum_{m=1}^{\infty} (q^m)^{1-w'-s-s} \cdot L_{v_1} (1-2s, \overline{\chi})\]

\[= -q^{s+s'-1} \cdot q^{1-w'-s-s} \cdot L_{v_1} (1-2s, \overline{\chi})\]

\[= -\frac{q^{-w'}}{1-q^{1-w'-s-s}} \cdot L_{v_1} (1-2s, \overline{\chi})\]

Thus adding up we get

\[L_{v_1} (1-2s, \overline{\chi}) \cdot \left[ \frac{(q-1)q^{-w'+s+s'}}{(1-q^{1-w'+s+s})(1-q^{-s-s})} - \frac{q^{-w'}}{1-q^{1-w'-s-s}} \right]\]

Thus at finite primes \(v = v_1\), the integral evaluates to:

\[L_{v_1} (s + s', \chi) \cdot L_{v_1} (s' + 1-s, \overline{\chi}) \cdot |\vartheta_{v_1}|^{-(s'+1-s)} \chi(\vartheta_{v_1}) +\]

\[L_{v_1} (1-2s, \overline{\chi}) \cdot \left[ \frac{(q-1)q^{-w'+s+s'}}{(1-q^{1-w'+s+s})(1-q^{-s-s})} - \frac{q^{-w'}}{1-q^{1-w'-s-s}} \right] \cdot |\vartheta_{v_1}|^{-(1-2s)} \chi(\vartheta_{v_1})\]

Then dividing through by \(\eta_v\) and putting back the measure constant \(|\vartheta_v|^{\frac{1}{2}}\), we get for \(v = v_1\),

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So the spectral decomposition of the Poincaré series is:

\[ |\vartheta_v|^{\frac{1}{2}} \cdot \int_{k } \int_{k } |y|^{s'} \overline{\psi}(xy) W_{s,\chi}^E (y_{01}^{00}) \cdot \varphi(x) \, dy \, dx \]

\[ = \frac{L_v(s' + \chi, s') \cdot L_v(s' + 1 - s, \overline{\chi}) \cdot |\vartheta_v|^{-(s'+1-s)} \cdot |\vartheta_v|^{\frac{1}{2}} \cdot \chi(\vartheta_v)}{L_v(2s, \chi^2)} \]

Replacing \( s \) by \( 1 - s \) and \( \chi \) by \( \overline{\chi} \) we get

\[ |\vartheta_v|^{\frac{1}{2}} \cdot \int_{k } \int_{k } |y|^{s'} \overline{\psi}(xy) W_{1-s,\chi}^E (y_{01}^{00}) \cdot \varphi(x) \, dy \, dx \]

\[ = \frac{L_v(s' + \chi, s') \cdot L_v(s' + 1 - s, \overline{\chi}) \cdot |\vartheta_v|^{-(s'+s-\frac{1}{2})} \cdot \chi(\vartheta_v)}{L_v(2s, \chi^2)} \]

\[ + \frac{L_v(2s - 1, \overline{\chi}) \cdot \left[ \frac{(q - 1) q^{-w' + w' + s + s'}}{(1 - q^{-w' + w' + s + s'})(1 - q^{-1 - s + s'})} - \frac{q^{-w'}}{1 - q^{-w' + s + s'}} \right] \cdot |\vartheta_v|^{\frac{3}{2} - 2s} \cdot \chi(\vartheta_v)}{L_v(2 - 2s, \chi^2)} \]

So the spectral decomposition of the Poincaré series is:

\[
\begin{align*}
&\frac{L_v(s' + s', \chi, s') \cdot L_v(s' + 1 - s, \overline{\chi}) \cdot |\vartheta_v|^{-(s'+s-\frac{1}{2})} \cdot \overline{\chi}(\vartheta_v)}{L_v(2 - 2s, \chi^2)} + \\
&\sum_F \left[ \frac{(q - 1) q^{-w'} - q^{1 - 2w' + 2s'} - q^{-w'} L_v(1 - \alpha^{-1} q^{-1 - w' + s'}) (1 - 1 - \beta^{-1} q^{-1 - w' + s'})}{(1 - q^{-w' + w' + s'})(1 - q^{-1 + s + s'})} \right] \cdot L_v(2s - 1, \overline{\chi}) \cdot F + \\
&\frac{1}{4\pi i} \sum \int_{c(s) = \frac{1}{2}} \left( \int_{G_{s+1/2}} \varphi_{\infty} \cdot W_{s,\chi,\infty}^E \right) \cdot \left( L_v(s' + s, \chi) \cdot L_v(s' + 1 - s, \overline{\chi}) \cdot |\vartheta_v|^{-(s'+s-\frac{1}{2})} \cdot \overline{\chi}(\vartheta_v) \right) \\
&\quad \frac{L_v(s' + s, \chi) \cdot L_v(s' + 1 - s, \overline{\chi}) \cdot |\vartheta_v|^{-(s'+s-\frac{1}{2})} \cdot \overline{\chi}(\vartheta_v)}{L_v(2 - 2s, \chi^2)} + \\
&\frac{L_v(2s - 1, \overline{\chi}) \cdot \left[ \frac{(q - 1) q^{-w' + w' + s + s'}}{(1 - q^{-2 - w' + s'})} - \frac{q^{-w'}}{1 - q^{-w' + s' + s'}} \right] \cdot |\vartheta_v|^{\frac{3}{2} - 2s} \cdot \chi(\vartheta_v)}{L_v(2 - 2s, \chi^2)} \cdot E_{s,\chi} \, ds
\end{align*}
\]
where
\[
\int_{N_v} \varphi_v = \begin{cases} 
\frac{\Gamma\left(\frac{w-1}{2}\right)}{\sqrt{\pi}} & (v \approx \mathbb{R}) \\
\frac{\Gamma\left(\frac{w}{2}\right)}{\Gamma\left(\frac{w}{2}\right)} & (v \approx \mathbb{C})
\end{cases}
\]
and
\[
\int_{Z_v \setminus G_v} \varphi_v \cdot W^E_{s,\chi,v} = \begin{cases} 
\frac{\mathcal{G}_v(s, s', w)}{\pi^{-s} \Gamma(s)} & (v \approx \mathbb{R}) \\
\frac{\mathcal{G}_v(s, s', w)}{2\pi^{-2s-1} \Gamma(2s)} & (v \approx \mathbb{C})
\end{cases}
\]

From the spectral decomposition of the Poincaré series, and the proof of theorem 4.17 in [Diaconu-Garrett 2008], the Poincaré series has meromorphic continuation to a region in $\mathbb{C}^2$ containing $s' = 0$, $w' = 1$. As a function of $w'$, for $s' = 0$, it is holomorphic in the half-plane $\Re(w') = \frac{11}{18}$ ([Kim-Shahidi 2002] and [Kim 2005]), except for $w' = 1$ where it has a pole of order 1.

5.1 PRELIMINARIES TO SUBCONVEXITY

5.1.1 Prologue

Fix a non-archimedean place $v_1$, and take $1 < \beta' < 2$. Recall that
\[
\mathcal{K}_{v_1}(w', \chi_{v_1}) \ll (q^N)^{-w'}
\]
where $\mathcal{K}_{v_1}(w', \chi_{v_1})$ is the non-decoupled integral for finite prime $v_1$ at which $\chi$ has ramification with conductor $q^N$. Define
\[
Z(w') = \sum_{\chi \in \hat{C}_{0,S}} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (q^N)^{-w'} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt
\]
This is a modified function obtained from
\[
I(0, w') = \sum_{\chi} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt
\]
by taking the asymptotic formula for $\mathcal{K}_{v_1}(w', \chi_{v_1})$. $Z(w')$ is absolutely convergent for $\Re(w') > 1$ (see Section 5 in [Diaconu-Garrett 2008]). In this
chapter, we will prove the meromorphic continuation and polynomial growth of $Z(w')$. This will enable us to obtain subconvexity bounds in the $\chi$-depth-aspect.

5.1.2 Meromorphic continuation of $Z(w')$

**Theorem 5.1**

The function

$$Z(w') = \sum_{\chi \in \hat{C}_{0,S}} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (q^N)^{-w'} \cdot K_{\infty}(\frac{1}{2} + it, 0, \beta', \chi) \, dt$$

where the sum is over a set $\hat{C}_{0,S}$ of characters ramified at the finite prime $v_1$ with conductor $q^N$, and $1 < \beta' \leq 2$, $\Re(w') > 1$, has analytic continuation to the half-plane $\Re(w') > \frac{11}{18}$, except for $w' = 1$ where it has a pole of order 1.

**Proof.** Let $w' = \delta + i\eta$. Split $Z$ into $Z_1$ and $Z_2$ as follows:

$$Z(w') = Z_1(w') + Z_2(w')$$

Choose a positive constant $C$ and define

$$Z_1(w') = \sum_{\chi \in \hat{C}_{0,S}, q^N \ll C} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (q^N)^{-w'} \cdot K_{\infty}(\frac{1}{2} + it, 0, \beta', \chi) \, dt$$

We first show that $Z_1(w')$ has analytic continuation by showing that it is holomorphic for $\delta > 0$. Now

$$|Z_1(w')| \leq \sum_{\chi \cdot q^N \ll C} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot |(q^N)^{-w'}| \cdot |K_{\infty}(\frac{1}{2} + it, 0, \beta', \chi)| \, dt$$

$K_{\infty}(\frac{1}{2} + it, 0, \beta', \chi)$ is positive (see Section 4 in [Diaconu-Garrett 2009]). So

$$|Z_1(w')| \leq \sum_{\chi \cdot q^N \ll C} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (q^N)^{-\delta} \cdot K_{\infty}(\frac{1}{2} + it, 0, \beta', \chi) \, dt$$

Since

$$(q^N)^{-\delta} \ll_{\beta', C} (q^N)^{-\beta'}$$
then
\[
|Z_1(w')| \ll \sum_{\chi:q^N \ll C} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (q^N)^{-\beta'} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt
\]
\[
< \sum_{\chi \in \hat{C}_{0,S}} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, \chi)|^2 \cdot (q^N)^{-\beta'} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt
\]
\[
= Z(\beta')
\]
which is convergent for \( \Re(w') < \frac{2}{9} \). Thus, \( Z_1(w') \) is holomorphic for \( \Re(w') = \delta > 0 \) (in particular for \( \Re(w') > \frac{11}{18} \)).

Now we prove that \( Z_2(w') \) has analytic continuation. Consider
\[
I(s', w', \beta') = \sum_{\chi \in \hat{C}_{0,S}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} L(s, f \otimes \chi) \cdot L(s' + 1 - s, f \otimes \chi) \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(s, s', \beta', \chi) \, dt
\]
where
\[
\mathcal{K}_\infty(s, s', \beta', \chi) = \prod_{v|\infty} \mathcal{K}_v(s, s', \beta', \chi_v)
\]
Recall that this expression is obtained from the integral representation
\[
\int_{Z\lambda \mathbb{A} \setminus \mathbb{A}} \text{Pe} |f|^2 \, dg
\]
where \( \text{Pe}(g) \) converges absolutely and locally uniformly for \( \Re(s') > 1 \) and \( \Re(w') > 1 \). Also recall from the spectral decomposition of \( \text{Pe} \), that \( \text{Pe} \) has meromorphic continuation to \( \Re(w') > \frac{11}{18} \) with a pole of order 1 at \( w' = 1 \).

Thus
\[
I(0, w', \beta') = \sum_{\chi \in \hat{C}_{0,S}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt
\]
is holomorphic for \( \Re(w') > \frac{11}{18} \) except at \( w' = 1 \) where there is a pole of order 1.

In the region of absolute convergence for \( \Re(w') = \delta > 1 \), write
\[
I(0, w', \beta') = I_1(0, w', \beta') + I_2(0, w', \beta')
\]
where
\[
I_1(0, w', \beta') = \sum_{\chi:q^N \ll C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt
\]
Now
\[ I(0, w', \beta') = I_1(0, w', \beta') + \sum_{\chi: q^N \gg C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot \mathcal{K}'(q^{-w'}) \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt \]
where the constant
\[ \mathcal{C}' = \frac{q}{q - 1} \cdot \frac{1 - |\alpha|^2 |\beta|^2 q^{-2w'}}{(1 - |\alpha|^2 q^{-w'})(1 - |\beta|^2 q^{-w'})(1 - \overline{\alpha\beta} q^{-w'})(1 - \alpha\beta q^{-w'})} \]
since
\[ \mathcal{K}_{v_1}(w', \chi_{v_1}) = \frac{q^{1-Nw'}}{q - 1} \cdot \frac{1 - |\alpha|^2 |\beta|^2 q^{-2w'}}{(1 - |\alpha|^2 q^{-w'})(1 - |\beta|^2 q^{-w'})(1 - \overline{\alpha\beta} q^{-w'})(1 - \alpha\beta q^{-w'})} \]
So
\[ I(0, w', \beta') = I_1(0, w', \beta') + C' \cdot Z_2(w') \]
Thus, to show that \( Z_2(w') \) has analytic continuation, it suffices to show that \( I_1(0, w', \beta') \) is absolutely convergent for \( \Re(w') > \frac{11}{18} \).

\[ I_1(0, w', \beta') = \sum_{\chi: q^N \ll C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt \]
\[ \ll \sum_{\chi: q^N \ll C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot |(q^N)^{-w'}| \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt \]
\[ \ll \sum_{\chi: q^N \ll C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot (q^N)^{-\beta} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt \]
\[ < \sum_{\chi \in \mathcal{C}_{0,S}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot (q^N)^{-\beta} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt \]
\[ = Z(\beta') \]
which is convergent for \( \Re(w') > \frac{2}{9} \). Thus \( Z_2(w') \) is absolutely convergent for \( \Re(w') > \frac{11}{18} \), proving the theorem. \( \square \)
5.1.3 Polynomial growth of $Z(w')$

**Theorem 5.2**
For every fixed small positive $\epsilon$, the generating function

$$Z(w') = \sum_{\chi \in \hat{C}_{0,S}} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot (q^N)^{-w'} \cdot K_\infty\left(\frac{1}{2} + it, 0, \beta', \chi\right) dt$$

has polynomial growth in the conductor $q^N$ for $\frac{11}{18} + \epsilon \leq \Re(w') \leq 1 + \epsilon$; that is, on the vertical line $\Re(w') = \frac{11}{18} + \epsilon$,

$$Z(w') \ll_{\epsilon, \beta'} (q^N)^\gamma$$

with a computable $\gamma > 0$ independent of $\beta'$.

Define the Poincaré series data at the non-archimedean place $v = v_1$ as earlier, namely:

$$\varphi\left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right) = \begin{cases} 1 & (x \in \mathfrak{a}_v) \\ |x|^{-w'} & (x \notin \mathfrak{a}_v) \end{cases}$$

Denote

$$I(s', w', \beta') = \int_{Z_0G_k \backslash G_\mathbb{A}} P\hat{c}(g) |f(g)|^2 dg$$

For $\Re(s'), \Re(w') > 1$,

$$I(s', w', \beta') = \sum_{\chi \in \hat{C}_{0,S}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} L(s, f \otimes \chi) \cdot L(s' + 1 - s, f \otimes \chi) \cdot K_{v_1}(w', \chi_{v_1}) \cdot K_\infty(s, s', \beta', \chi) dt$$

where

$$K_\infty(s, s', \beta', \chi) = \prod_{v | \infty} K_v(s, s', \beta', \chi_v)$$

In the region of absolute convergence

$$I(0, w', \beta') = I_1(0, w', \beta') + I_2(0, w', \beta')$$

where

$$I_1(0, w', \beta') = \sum_{\chi : q^N \ll C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot K_{v_1}(w', \chi_{v_1}) \cdot K_\infty\left(\frac{1}{2} + it, 0, \beta', \chi\right) dt$$
and
\[ I_2(0, w', \beta') = C' \cdot Z_2(w') \]

The constant
\[ C' = \frac{q}{q-1} \cdot \frac{1 - |\alpha|^2 |\beta|^2 q^{-2w'}}{(1 - |\alpha|^2 q^{-w'})(1 - |\beta|^2 q^{-w'})(1 - \alpha \beta q^{-w'})(1 - \overline{\alpha \beta} q^{-w'})} \]
and
\[ Z_2(w') = \sum_{\chi: q^N \gg C} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (q^N)^{-w'} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt \]
with
\[ Z_2(w') = Z(w') - Z_1(w') \]
so
\[ I_2(0, w', \beta') = C'[Z(w') - Z_1(w')] \]
Recall that \( Z_1(w') \) is holomorphic in the half-plane \( \Re(w') > \frac{11}{18} \), and that \( C' \) is the positive constant where the cutoff of \( I(0, w', \beta') \) was made. So \( Z_1(w') \) has polynomial growth in \( q^N \). Thus, the polynomial bound of \( Z(w') \) will be deduced from that of \( I_2(0, w', \beta') \). That is, we will prove that \( I_2(0, w', \beta') \) has polynomial growth in \( q^N \).

Again recall that
\[ I(w') = I(0, w', \beta') = \sum_{\chi \in \hat{C}_0, S} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) : \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt \]
which is obtained from the integral representation
\[ \int_{Z_\lambda G_k \backslash G_\lambda} P \hat{e} \, |f|^2 \, dg \]

\( P \hat{e} \) admits a spectral decomposition
\[ P \hat{e} = \text{Singular part} + \text{Cuspidal part} + \text{Continuous part} \]
In the spectral decomposition set \( s' = 0 \) and obtain
\[ P \hat{e} = \lim_{s' \to 0} \left( \int_{N_\infty} \varphi_\infty \cdot \frac{1 - q^{-w'}}{1 - q^1 - w'} \cdot E_{s' + 1, 1} + \sum_F \overline{\mathcal{P}_F} G_{F_\infty}(\beta') \cdot \left[ L_v(\frac{1}{2}, \overline{F}) + \right] \right) \]
\[
L_{v_1}(\frac{1}{2}, \overline{\mathcal{F}}) + \frac{(q - 1)(q^{-w'} - q^{\frac{1}{2w'}-2})}{(1 - \alpha^{-1}q^{1-w'})(1 - \beta^{-1}q^{1-w'})} \cdot L_{v_1}(\frac{1}{2}, \overline{\mathcal{F}}) - q^{-w'}L_{v_1}(\frac{2w' - 1}{2}, \overline{\mathcal{F}}) \cdot F + \\
\sum_{\chi} \frac{\chi(\vartheta_v)}{4\pi i k} \int_{\Re(s) = \frac{1}{2}} \left( \int_{Z_\infty \setminus \Gamma_\infty} \varphi_\infty \cdot \prod_{E} \right) \cdot \\
\left( \frac{L_v(s, \chi) \cdot L_v(1 - s, \overline{\chi}) \cdot |\vartheta_v|^{-\frac{s}{2}}}{L_v(2 - 2s, \overline{\chi})} + \frac{L_{v_1}(s, \chi) \cdot L_{v_1}(1 - s, \overline{\chi}) \cdot |\vartheta_v|^{-\frac{s}{2}}}{L_{v_1}(2 - 2s, \overline{\chi})} \right) \cdot (s - \frac{1}{2}) - s, \chi \rangle \cdot |\vartheta_v|^{\frac{3}{2} - 2s} \cdot \langle E_{s, \chi}, |f|^2 \rangle \right) \cdot (s - \frac{1}{2}) - s, \chi \rangle}\cdot |\vartheta_v|^{\frac{3}{2} - 2s} \cdot \langle E_{s, \chi}, |f|^2 \rangle \cdot ds
\]

So

\[I(w') = I_{\text{sing}}(w') + I_{\text{cusp}}(w') + I_{\text{cont}}(w')\]

where

\[I_{\text{sing}}(w') = \lim_{s' \to 0} \left( \int_{\Re(s) = \frac{1}{2}} \varphi_\infty \cdot \frac{1 - q^{-w'}}{1 - q^{1-w'}} \cdot \langle E_{s'+1,1}, |f|^2 \rangle \right)\]

\[I_{\text{cusp}}(w') = \sum_{F} \frac{1}{2} \mathcal{G}_{\mathcal{F}, \infty}(\beta') \cdot \left( 2L_v(\frac{1}{2}, \overline{\mathcal{F}}) + \frac{(q - 1)(q^{-w'} - q^{\frac{1}{2w'}-2})}{(1 - \alpha^{-1}q^{1-w'})(1 - \beta^{-1}q^{1-w'})} \right) \cdot F \cdot |f|^2\]

\[I_{\text{cont}}(w') = \sum_{\chi} \frac{\chi(\vartheta_v)}{4\pi i k} \int_{\Re(s) = \frac{1}{2}} \left( \int_{Z_\infty \setminus \Gamma_\infty} \varphi_\infty \cdot \prod_{E} \right) \cdot \\
\left( \frac{2L_v(s, \chi) \cdot L_v(1 - s, \overline{\chi}) \cdot |\vartheta_v|^{-\frac{s}{2}}}{L_v(2 - 2s, \overline{\chi})} + \frac{L_{v_1}(s, \chi) \cdot L_{v_1}(1 - s, \overline{\chi}) \cdot |\vartheta_v|^{-\frac{s}{2}}}{L_{v_1}(2 - 2s, \overline{\chi})} \right) \cdot (s - \frac{1}{2}) - s, \chi \rangle \cdot |\vartheta_v|^{\frac{3}{2} - 2s} \cdot \langle E_{s, \chi}, |f|^2 \rangle \cdot ds\]

Note that the dependence on \(w'\) is at \(v_1\) only. Let

\[\mathcal{M}_1(w') = \frac{(q - 1)(q^{-w'} - q^{\frac{1}{2w'}-2})}{(1 - \alpha^{-1}q^{1-w'})(1 - \beta^{-1}q^{1-w'})} \cdot L(\frac{1}{2}, \overline{\mathcal{F}}) - q^{-w'}L(\frac{2w' - 1}{2}, \overline{\mathcal{F}})\]

and

\[\mathcal{M}_2(w') = \frac{(q - 1)q^{-w'-s}}{(1 - q^{2w'-s})(1 - q^{s-1})} - \frac{q^{-w'}}{1 - q^{-w'+s}}\]
Then define the auxiliary function $I^{\text{aux}}(w')$ by

$$I^{\text{aux}}(w') = \sum_F \overline{\mathcal{P}_F} \mathcal{G}_{F,\infty}(\beta') \cdot [2L(\frac{1}{2}, F) + \mathcal{M}^{\text{aux}}_1(w')] \cdot \langle F, |f|^2 \rangle +$$

$$\sum_{\chi} \frac{\chi}{4\pi i k} \int_{\Re(s)=\frac{1}{2}} \left( \int_{Z_{\infty} \cap G_{\infty}} \varphi_{\infty} \cdot \overline{\mathcal{W}_{1-s, \infty}} \right) \cdot \frac{2L(s, \chi) \cdot L(1-s, \chi) \cdot |\omega|^{-(s-\frac{1}{2})}}{L(2-2s, \chi^2)} +$$

$$\frac{L(2s-1, \chi) \cdot \mathcal{M}^{\text{aux}}_2(w') \cdot |\omega|^{\frac{3}{2}-2s}}{L(2-2s, \chi^2)} \cdot \langle E_{s, \chi}, |f|^2 \rangle ds$$

where $\mathcal{M}^{\text{aux}}_1(w')$ and $\mathcal{M}^{\text{aux}}_2(w')$ are defined by

$$\mathcal{M}^{\text{aux}}_1(w') = \mathcal{M}_1(w') \cdot (q^N)^{\gamma}$$

$$\mathcal{M}^{\text{aux}}_2(w') = \mathcal{M}_2(w') \cdot (q^N)^{\gamma}$$

where $\gamma > 0$, independent of $\beta'$. Define

$$H(w') = I(w') - I^{\text{aux}}(w') = \lim_{s' \to 0} \left( \int_{N_{\infty}} \varphi_{\infty} \right) \cdot \frac{1-q^{-w'}}{1-q^{-w'}} \cdot \langle E_{s'+1, 1}, |f|^2 \rangle +$$

$$\sum_F \overline{\mathcal{P}_F} \mathcal{G}_{F,\infty}(\beta') \cdot [\mathcal{M}_1(w') - \mathcal{M}^{\text{aux}}_1(w')] \cdot \langle F, |f|^2 \rangle +$$

$$\sum_{\chi} \frac{\chi}{4\pi i k} \int_{\Re(s)=\frac{1}{2}} \left( \int_{Z_{\infty} \cap G_{\infty}} \varphi_{\infty} \cdot \overline{\mathcal{W}_{1-s, \infty}} \right) \cdot \frac{L(2s-1, \chi) \cdot [\mathcal{M}_2(w') - \mathcal{M}^{\text{aux}}_2(w')] \cdot |\omega|^{\frac{3}{2}-2s}}{L(2-2s, \chi^2)} \cdot \langle E_{s, \chi}, |f|^2 \rangle ds$$

**Proposition 5.3**

For $\epsilon$ sufficiently small,

$$H(w') = I(w') - I^{\text{aux}}(w')$$

restricted to $\frac{11}{18} < \Re(w') \leq 1 + \epsilon$, extends holomorphically to the whole vertical strip $-\epsilon \leq \Re(w') \leq 1 + \epsilon$.

**Proof.** The first term in $H(w')$, i.e.

$$\lim_{s' \to 0} \left( \int_{N_{\infty}} \varphi_{\infty} \right) \cdot \frac{1-q^{-w'}}{1-q^{-w'}} \cdot \langle E_{s'+1, 1}, |f|^2 \rangle$$

is holomorphic in the strip $-\epsilon \leq \Re(w') \leq 1 + \epsilon$, except at $w' = 0, 1$ where there are poles.
\[ M_1(w') - M_1^{aux}(w') = M_1(w') - M_1(w') \cdot (q^N)^\gamma = M_1(w')[1 -(q^N)^\gamma] \]
\[ M_2(w') - M_2^{aux}(w') = M_2(w') - M_2(w') \cdot (q^N)^\gamma = M_2(w')[1 -(q^N)^\gamma] \]
Since both \( M_1(w') \) and \( M_2(w') \) are holomorphic in the strip, then \( H(w') \) is also holomorphic in the strip.

**Proposition 5.4**
Fix a small positive \( \epsilon \). For \( \frac{1}{2} + \epsilon \leq \Re(w') \leq 1 + \epsilon \), or \( \Re(w') = -\epsilon \),
\[ I^{aux}(w') \ll_{\epsilon, \beta'} (q^N)^\gamma \]

**Proof.** Again,
\[ I^{aux}(w') = \sum_{F} \overline{\tau}(\beta') \cdot [2L(\frac{1}{2}, F) + M_1^{aux}(w') \cdot \langle F, |f|^2 \rangle + \]
\[ \sum_{\chi} \frac{4\pi i k}{\lambda} \int_{\Re(s) = \frac{1}{2}} \left( \int_{Z_{\chi}} \varphi_{s} \cdot W_{E, s, \chi} \right) \cdot \frac{2L(s, \chi) \cdot L(1 - s, \chi) \cdot |\beta|^{-(s - \frac{1}{2})}}{L(2 - 2s, \chi^2)} + \]
\[ \frac{L(2s - 1, \chi)}{L(2 - 2s, \chi^2)} \cdot M_2^{aux}(w') \cdot |\beta|^{\frac{3}{2} - 2s} \cdot \langle E_{s, \chi}, |f|^2 \rangle \, ds \]
where
\[ M_1^{aux}(w') = M_1(w') \cdot (q^N)^\gamma \]
\[ M_2^{aux}(w') = M_2(w') \cdot (q^N)^\gamma \]
\[ M_1(w') = \frac{(q - 1)(q^{-w} - \frac{q^{1 - 2w'}}{\alpha \beta})}{(1 - \alpha^{-1}q^{-w})(1 - \beta^{-1}q^{-w})} \cdot L(\frac{1}{2}, F) - q^{-w}L\left(\frac{2w' - 1}{2}, F\right) \]
\[ M_2(w') = \frac{(q - 1)q^{1 - w' - s}}{(1 - q^{2 - w' - s})(1 - q^{s - 1})} - \frac{q^{-w'}}{1 - q^{-w' + s}} \]
\[ M_1^{aux}(w') \ll (q^N)^\gamma \]
\[ M_2^{aux}(w') \ll (q^N)^\gamma \]

All other terms in \( I^{aux}(w') \) are independent of the conductor \( q^N \), and have a polynomial bound. Thus
\[ I^{aux}(w') \ll_{\epsilon, \beta'} (q^N)^\gamma \]
Recall we are trying to prove a polynomial bound for $I_2(w')$ in the conductor $q^N$. Now

$$I_2(w') = I_2(w') - I_{aux}(w') + I_{aux}(w')$$

We have proven a polynomial bound for $I_{aux}(w')$, so we now prove a polynomial bound for $I_2(w') - I_{aux}(w')$.

Thus it suffices to prove a polynomial bound for $H(w') - I_1(w')$ on the line $\Re(w') = \frac{11}{18} + \epsilon$.

**Proof.** Recall that $H(w')$ is holomorphic in the strip $-\epsilon < \Re(w') < 1 + \epsilon$. Also recall

$$I_1(w') = \sum_{\chi: q^N \ll C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot K_{v_1}(w', \chi_{v_1}) \cdot K_{\infty}(\frac{1}{2} + it, 0, \beta', \chi) \, dt$$

and

$$I_1(w') \ll Z(\beta') < \infty$$

So $I_1(w')$ converges absolutely throughout the strip; i.e. $I_1(w')$ is holomorphic throughout the strip. Thus $H(w') - I_1(w')$ is also holomorphic throughout the strip. Recall

$$I_2(w') = C'[Z(w') - Z_1(w')]$$

For $\Re(w') = 1 + \epsilon$, since $I_{aux}(w') \ll (q^N)\gamma$, for $\gamma > 0$, $Z(w') = O(1)$ and $Z_1(w')$ already has polynomial growth in $q^N$, we conclude that

$$H(w') - I_1(w') = I_2(w') - I_{aux}(w')$$

has polynomial growth in $q^N$ for $\Re(w') = 1 + \epsilon$.

Now assume $\Re(w') = -\epsilon$.

$$H(w') - I_1(w') = I(w') - I_{aux}(w') - I_1(w')$$

Again, $I_{aux}(w')$ has polynomial growth for $\Re(w') = -\epsilon$, and $I_1(w') \ll Z(\beta')$. The spectral expansion of $I(w')$ and $I_1(w')$ shows that $I(w')$ and $I_1(w')$ also have polynomial growth for $\Re(w') = -\epsilon$. Thus $H(w') - I_1(w')$ has polynomial growth in $q^N$ for $\Re(w') = -\epsilon$. 

\[\square\]
We now apply Phragmen-Lindelöf and conclude that

\[ I_2(w') - I_{aux}(w') \]

has polynomial growth in \( q^N \) within the strip \( \frac{11}{18} + \epsilon \leq \Re(w') \leq 1 + \epsilon \), and hence, so has \( I_2(w') \). Thus, we have proven that \( Z(w') \) has polynomial growth in \( q^N \) within the strip \( \frac{11}{18} + \epsilon \leq \Re(w') \leq 1 + \epsilon \).

### 6.1 SUBCONVEXITY BOUNDS

#### 6.1.1 Prologue

Our goal is to break convexity in the \( \chi \)-depth-aspect for a family of \( L \)-functions \( L(\frac{1}{2} + it, f \otimes \chi) \), where \( \chi \) has arbitrary ramification at a fixed finite prime \( v_1 \). For a cuspform \( f \) on \( GL_2(k) \), the \( \chi \)-depth-aspect convexity bound for the twisted \( L \)-function \( L(\frac{1}{2} + it, f \otimes \chi) \) is

\[
L\left(\frac{1}{2} + it, f \otimes \chi\right) \ll q^{N(\frac{4}{4} + \epsilon)}
\]

where \( q^N \) with \( N \geq 1 \) is the conductor of \( \chi \) allowed to be ramified at the finite place \( v_1 \), and \( d \) is the degree of the number field \( k \) over \( \mathbb{Q} \). Using methods in section 4 in [Diaconu-Garrett 2009], we will break convexity at the finite place \( v_1 \) by decreasing the exponent. So fix a non-archimedean place \( v_1 \), take \( 1 < \beta' < 2 \) and fix \( 0 < t < 1 \) in the nondecoupled integral at the archimedean places. Write

\[
I(0, w') = \sum_{\chi \in \hat{C}_{0,s}} \frac{1}{2 \pi i} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot K_{v_1}(w', \chi_{v_1}) \cdot K_{\infty}(\frac{1}{2} + it, 0, \beta', \chi) \, dt
\]

where

\[
K_{\infty}(\frac{1}{2} + it, 0, \beta', \chi) = \prod_{v \mid \infty} K_v(\frac{1}{2} + it, 0, \beta', \chi_v)
\]

is the nondecoupled integral at the archimedean places, and \( K_{v_1}(w', \chi_{v_1}) \) is the nondecoupled integral at the finite prime \( v_1 \); \( K_{v_1}(w', \chi_{v_1}) \) does not depend on \( t \). We have shown that

\[
K_{v_1}(w', \chi_{v_1}) = \frac{q^{1-Nw'}}{q - 1} \cdot \frac{1 - |\alpha|^2 |\beta|^2 q^{-2w'}}{(1 - |\alpha|^2 q^{-w'})(1 - |\beta|^2 q^{-w'})(1 - \alpha \beta q^{-w'})(1 - \alpha \beta q^{-w'})}
\]

and

\[
K_{v_1}(w', \chi_{v_1}) \ll (q^N)^{-w'}
\]
Define
\[ Z(w') = \sum_{\chi \in \mathcal{C}_{0,S}} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (q^N)^{-w'} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt \]

\( Z(w') \) has analytic continuation to \( \Re(w') > \frac{11}{18} \) with a pole of order 1 at \( w' = 1 \), and has polynomial growth on every vertical strip inside \( \frac{11}{18} + \epsilon \leq \Re(w') \leq 1 + \epsilon \).

Choose \( \frac{11}{18} < \delta_0 < 1 \). From section 4 in [Diaconu-Garrett 2008], for \( \delta_0 \leq \Re(w') \leq 1 + \epsilon \), by Phragmen-Lindelöf, \( Z(\delta_0 + i\eta) \) has polynomial growth of exponent less than \( \frac{1}{2} \).

Consider the rectangle \( R \) with vertices at \( \delta_0 - iS, \beta' - iS, \beta' + iS, \delta_0 + iS \). Recall Perron’s formula: for \( \beta' > 1 \),
\[ \frac{1}{2\pi i} \int_{\beta' - iS}^{\beta' + iS} \frac{x^w}{w} \, dw \]
is given by
\[ \frac{1}{2\pi i} \int_{\beta' - iS}^{\beta' + iS} \frac{Z(w') x^{w'}}{w'} \, dw' \]

\[ = \sum_{\chi \in \mathcal{C}_{0,S}} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot \left( \int_{\beta' - iS}^{\beta' + iS} (x/q^{N})^{w'} \, dw' \right) \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt \]
\[ = \sum_{\chi ; q^N \leq x} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot \left( \frac{x}{q^N} \right)^{\beta'} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \cdot \min\{1, \frac{1}{S|\log(qN)|}\} \, dt \]

where the error term \( E(x, S) \) is
\[ E(x, S) \ll \sum_{\chi} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot \left( \frac{x}{q^N} \right)^{\beta'} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \cdot \min\{1, \frac{1}{S|\log(qN)|}\} \, dt \]

**Theorem 6.1.**
\[ \lim_{S \to \infty} E(x, S) = 0 \quad (\text{for } x > 0) \]

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Proof. We first show that
\[
\lim_{S \to \infty} \int_{\delta_0 + iS}^{\beta' + iS} \frac{Z(w') x^{w'}}{w'} \, dw' = 0 \quad \text{and} \quad \lim_{S \to \infty} \int_{\delta_0 - iS}^{\beta' - iS} \frac{Z(w') x^{w'}}{w'} \, dw' = 0
\]
Let \( w' = \delta + iS \). Then
\[
Z(w') \ll S^m, \quad \text{for} \quad m < \frac{1}{2} \quad \text{and} \quad |w'| = \sqrt{\delta^2 + S^2} \ll S
\]
Thus the integrals above approach 0 as \( S \to \infty \).

Consider the sets:
\[
A = \left\{ N : \frac{1}{S|\log(q^N)|} \leq \frac{1}{\sqrt{S}} \right\}
\]
\[
B = \left\{ N : \frac{1}{S|\log(q^N)|} \geq \frac{1}{\sqrt{S}} \right\}
\]
On \( A \),
\[
E(x, S) \ll \frac{1}{\sqrt{S}} \sum_{\chi} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (\frac{x}{q^N})^{\beta'} \cdot \mathcal{K}_{\infty}(\frac{1}{2} + it, 0, \beta', \chi) \, dt
\]
\[
= \frac{x^{\beta'}}{\sqrt{S}} \sum_{\chi} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (q^N)^{-\beta'} \cdot \mathcal{K}_{\infty}(\frac{1}{2} + it, 0, \beta', \chi) \, dt
\]
\[
= \frac{x^{\beta'}}{\sqrt{S}} Z(\beta') \quad \text{where} \quad Z(\beta') \quad \text{is independent of} \quad S
\]
So
\[
\lim_{S \to \infty} E(x, S) = 0
\]
On \( B \), \( \mathcal{K}_{\infty}(\frac{1}{2} + it, 0, \beta', \chi) \) can be estimated by the analytic conductor:
\[
Q(\chi, t) = \prod_{\nu \in \mathbb{R}} (1 + |t + t_{\nu}|) \cdot \prod_{\nu \in \mathbb{C}} (1 + \ell_{\nu}^2 + 4(t + t_{\nu})^2)
\]
Break up \( E(x, S) \) into two sums over \( q^N \leq \log S \) and \( q^N \geq \log S \). Since \( Z(w') \) converges absolutely for \( \Re(w') > 1 \), the second sum over \( q^N \geq \log S \) approaches 0. So consider
\[
\sum_{\chi \cdot q^N \leq \log S} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (\frac{x}{q^N})^{\beta'} \cdot \mathcal{K}_{\infty}(\frac{1}{2} + it, 0, \beta', \chi) \cdot \min\{1, \frac{1}{S|\log(q^N)|}\} \, dt
\]
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Now in B, 
\[ \sum_{q^N \leq \log S} \int_{-\infty}^{\infty} 1 \ll (\log S)^k, \quad k > 0 \]

The convexity bound in the depth aspect gives 
\[ L(\frac{1}{2} + it, f \otimes \chi) \ll (q^N)^{\frac{1}{2}} \leq (\log S)^{\frac{1}{2}} \]

Fix \( \chi = 1 \) and \( 0 < t < 1 \) for \( v|\infty \). Then 
\[ K_{\infty}(\frac{1}{2} + it, 0, \beta', \chi) \ll 1 \]

Also
\[ \frac{1}{S|\log(q^N)|} \geq \frac{1}{\sqrt{S}} \Rightarrow \frac{1}{\sqrt{S}|\log(q^N)|} \geq 1 \Rightarrow xe^{-\frac{1}{\sqrt{S}}} \leq q^N \leq xe^{\frac{1}{\sqrt{S}}} \]

This restricts \( N \) to a set of measure \( \ll \frac{1}{\sqrt{S}} \). So in the second case 
\[ \lim_{S \to \infty} E(x, S) = 0 \]

By Cauchy’s theorem, 
\[ \frac{1}{2\pi i} \int_R \frac{Z(w') x^{w'}}{w'} dw' = xP(\log x) \]

Indeed, \( Z(w') \) has a pole of order 1 at \( w' = 1 \), so by the residue theorem: 
\[ \frac{1}{2\pi i} \int_R \frac{Z(w') x^{w'}}{w'} dw' = \text{Res}_{w'=1} \left( Z(w') \cdot \frac{x^{w'}}{w'} \right) \]

Suppose the Laurent expansion of \( Z(w') \)
\[ Z(w') = \sum_{n=-\infty}^{\infty} a_n (w' - 1)^n \]
and
\[ x^{w'} = xe^{(w'-1)\log x} = x \sum_{n=0}^{\infty} \frac{(w'-1)^n \log^n x}{n!} \]

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Then the coefficient of \((w' - 1)^{-1}\) in the product \(Z(w') \cdot \frac{x^{w'}}{w'}\) is \(xP(\log x)\), where \(P(\log x)\) is a polynomial in \(\log x\). So

\[
\frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{Z(w') x^{w'}}{w'} dw' = \frac{1}{2\pi i} \int_{\beta-iS}^{\beta+iS} \frac{Z(w') x^{w'}}{w'} dw' - \frac{1}{2\pi i} \int_{\delta_0-i\infty}^{\delta_0+i\infty} \frac{Z(w') x^{w'}}{w'} dw'
\]

\[= xP(\log x)\]

Now Perron’s formula showed that

\[
\frac{1}{2\pi i} \int_{\delta_0-i\infty}^{\delta_0+i\infty} \frac{Z(w') x^{w'}}{w'} dw' = \sum_{q^{N} \leq x} \int_{\delta_0-i\infty}^{\delta_0+i\infty} |L(1/2+it, f\otimes\chi)|^2 \cdot \mathcal{K}_\infty(1/2+it, 0, \beta', \chi) dt
\]

Thus as \(S \to \infty\),

\[
\sum_{q^{N} \leq x} \int_{\delta_0-i\infty}^{\delta_0+i\infty} |L(1/2+it, f\otimes\chi)|^2 \cdot \mathcal{K}_\infty(1/2+it, 0, \beta', \chi) dt = xP(\log x) + \frac{1}{2\pi i} \int_{\delta_0-i\infty}^{\delta_0+i\infty} \frac{Z(w') x^{w'}}{w'} dw'
\]

**Theorem 6.2.**

\[
\frac{1}{2\pi i} \int_{\delta_0-i\infty}^{\delta_0+i\infty} \frac{Z(w') x^{w'}}{w'} dw' \ll x^{2\delta_0+1} \cdot \log x \quad (11/18 < \delta_0 < 1)
\]

**Proof.** By the choice of \(\delta_0\),

\[
\frac{Z(w')}{w'} = \frac{Z(\delta_0 + i\eta)}{\delta_0 + i\eta}
\]

is a square integrable function on \(\mathbb{R}\). Let

\[
E(x) = \frac{1}{2\pi i} \int_{\delta_0-i\infty}^{\delta_0+i\infty} \frac{Z(w') x^{w'}}{w'} dw'
\]

**Lemma 6.3**

\[
\int_0^x |E(t)|^2 dt \ll x^{2\delta_0+1}
\]

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Proof. Let \( x = e^{-2\pi u} \) and again \( u' = \delta + i\eta \). So

\[
E(e^{-2\pi u}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} Z(\delta_0 + i\eta) \cdot e^{-2\pi u(\delta_0 + i\eta)} \cdot i\, d\eta
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\pi i\eta} \cdot f(\eta) \cdot e^{-2\pi u\delta_0} \, d\eta \quad (\text{where} \quad f(\eta) = \frac{Z(\delta_0 + i\eta)}{\delta_0 + i\eta})
\]

Now

\[
\hat{f}(u) = \int_{-\infty}^{\infty} f(\eta) e^{-2\pi i\eta u} \, d\eta
\]

Thus

\[
e^{2\pi u\delta_0} \cdot 2\pi \cdot E(e^{-2\pi u}) = \hat{f}(u)
\]

Using Plancherel’s theorem:

\[
\int_{-\infty}^{\infty} |\hat{f}(u)|^2 \, du = \int_{-\infty}^{\infty} |f(\eta)|^2 \, d\eta \ll 1
\]

So

\[
1 \gg 4\pi^2 \int_{-\infty}^{\infty} |e^{2\pi u\delta_0} \cdot E(e^{-2\pi u})|^2 \, du
\]

Replace \( e^{-2\pi u} \) by \( y \) to get

\[
1 \gg \frac{4\pi^2}{2\pi} \int_{0}^{\infty} y^{-2\delta_0} \cdot |E(y)|^2 \frac{dy}{y} = 2\pi \int_{0}^{\infty} y^{-(2\delta_0+1)} \cdot |E(y)|^2 \, dy
\]

\[
\geq \int_{0}^{\infty} y^{-(2\delta_0+1)} \cdot |E(y)|^2 \, dy \geq x^{-(2\delta_0+1)} \int_{0}^{x} |E(y)|^2 \, dy \quad \text{for} \ 0 \leq y \leq x
\]

Thus

\[
\int_{0}^{x} |E(y)|^2 \, dy \ll x^{2\delta_0+1}, \quad 0 < \delta_0 < 1
\]

We now prove Theorem 6.2, that

\[
E(x) \ll x^{\frac{2\delta_0+1}{3}} \cdot \log x
\]

First note that \( \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \) is positive. Now for \( x \leq y \)

\[
\{N : q^N \leq x\} \subseteq \{N : q^N \leq y\}
\]

Again

\[
E(x) = \sum_{q^N \leq x} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \, dt = xP(\log x)
\]

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So
\[
E(y) - E(x) = \sum_{q \leq y} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot K(\frac{1}{2} + it, 0, \beta', \chi) dt - \sum_{q \leq x} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot K(\frac{1}{2} + it, 0, \beta', \chi) dt - (yP(\log y) - xP(\log x))
\]

Since \(K(\frac{1}{2} + it, 0, \beta', \chi)\) is positive,
\[
E(y) - E(x) \geq -(yP(\log y) - xP(\log x))
\]

Fix \(x \geq 3\)
(a) Replace \(y\) with \(x + u\) for \(0 \leq u \leq x\):
\[
E(x + u) - E(x) \geq -(x + u)P(\log(x + u)) - xP(\log x)
\]
\[
\Rightarrow E(x) \leq E(x + u) + (x + u)P(\log(x + u)) - xP(\log x)
\]

Now \(P\) is a linear polynomial, so rewrite
\[
(x + u)P(\log(x + u)) - xP(\log x) = (x + u)[A \log(x + u) + B] - x[A \log x + B]
\]
\[
= Ax(\log \frac{x + u}{2}) + Au \log(x + u) + Bu = Ax(\log(1 + \frac{u}{2})) + Au \log(x + u) + Bu
\]

Now
\[
\log(1 + h) \leq h \quad \text{for } 0 \leq h \leq 2 \quad \Rightarrow \log(1 + \frac{u}{2}) \leq \frac{u}{2} \leq 1
\]

So
\[
Ax(\log(1 + \frac{u}{2})) + Au \log(x + u) + Bu \leq Ax \cdot \frac{u}{2} + Au \log(x + u) + Bu
\]
\[
= Au + Bu + Au \log(x + u) \leq Du \log x + Au(\log x + \log 2) \quad \text{since } u \leq x
\]

Thus
\[
E(x) \leq E(x + u) + Cu \log x \quad \text{for some constant } C
\]

(b) Replace \(x\) with \(x - u\) and \(y\) with \(x\) for \(0 \leq u < x\). Then
\[
E(x) - E(x - u) \geq -(xP(\log x) - (x - u)P(\log(x - u))]
\]
\[ E(x) \geq E(x - u) - Cu \log x \]

Let \( 0 \leq H \leq x \). Integrate the inequalities over \( 0 \leq u \leq H \):

\[
\int_0^H E(x) \, du \leq \int_0^H (E(x+u)+Cu \log x) \, du = H \cdot E(x) \leq \int_0^H E(x+u) \, du + \frac{C}{2} H^2 \log x
\]

and

\[
H \cdot E(x) \geq \int_0^H E(x-u) \, du - \frac{C}{2} H^2 \log x
\]

So

\[
\int_0^H E(x-u) \, du - \frac{C}{2} H^2 \log x \leq H \cdot E(x) \leq \int_0^H E(x+u) \, du + \frac{C}{2} H^2 \log x
\]

Change variables and replace \( \frac{C}{2} \) with \( C \) to get

\[
\frac{1}{H} \int_{x-H}^x E(t) \, dt - CH \log x \leq E(x) \leq \frac{1}{H} \int_x^{x+H} E(t) \, dt + CH \log x
\]

For \( E(x) \geq 0 \), apply the second inequality, otherwise apply the first. So for \( E(x) \geq 0 \),

\[
E(x)^2 \ll \frac{1}{H^2} \left( \int_x^{x+H} E(t) \, dt \right)^2 + C^2 H^2 \log^2 x
\]

Apply Cauchy-Schwarz:

\[
E(x)^2 \ll \frac{1}{H^2} \int_x^{x+H} |E(t)|^2 \, dt \cdot \int_x^{x+H} 1 \, dt + H^2 \log^2 x
\]

\[
= \frac{1}{H} \int_x^{x+H} |E(t)|^2 \, dt + H^2 \log^2 x \ll \frac{1}{H} \cdot x^{2\delta_0 + 1} + H^2 \log x
\]

since

\[
\int_0^x |E(t)|^2 \, dt \ll x^{2\delta_0 + 1} \quad \text{and} \quad H \leq x
\]

We want \( \frac{1}{H} \cdot x^{2\delta_0 + 1} = H^2 \), so take

\[
H = \frac{x^{2\delta_0 + 1}}{4}
\]

Then

\[
E(x) \ll H \log x = x^{2\delta_0 + 1} \cdot \log x
\]
Recall that the $\chi$-depth-aspect convexity bound for the twisted $L$-function is

$$L\left(\frac{1}{2} + it, f \otimes \chi\right) \ll q^{N(\frac{4}{2} + \epsilon)}$$

Let us now use the results obtained above to break convexity by decreasing the exponent of $q^N$.

Choose $H$ such that

$$x^{2^{\delta_0 + 1} \frac{1}{3}} \ll H \ll x^{2^{\delta_0 + 1} \frac{1}{3}}$$

Let

$$S(x) = \sum_{q^N \leq x} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot \mathcal{K}_\infty\left(\frac{1}{2} + it, 0, \beta', \chi\right) dt$$

$$= xP(\log x) + O\left(x^{2^{\delta_0 + 1} \frac{1}{3}} \log x\right) = xP(\log x) + E(x)$$

Now for $H > 0$, $\{N : q^N \leq x\} \subset \{N : q^N \leq x + H\}$ and $\mathcal{K}_\infty\left(\frac{1}{2} + it, 0, \beta', \chi\right)$ is positive. So for trivial $\chi$,

$$S(x + H + 1) - S(x) \geq \sum_{x \leq q^N \leq x + H} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot \prod_{v \mid \infty} \mathcal{K}_v\left(\frac{1}{2} + it, 0, \beta', 1\right) dt$$

Now

$$S(x + H + 1) - S(x) = (x + H + 1)P(\log(x + H + 1)) - xP(\log x) + E(x + H + 1) - E(x)$$

Since $x^{2^{\delta_0 + 1} \frac{1}{3}} \ll H \ll x^{2^{\delta_0 + 1} \frac{1}{3}}$ and $E(x) \ll x^{\frac{2^{\delta_0 + 1} \frac{1}{3}}{3}} \log x$

and

$$(x + H + 1)P(\log(x + H + 1)) - xP(\log x) \leq C(H + 1) \log x$$

So

$$S(x + H + 1) - S(x) \ll x^{\frac{2^{\delta_0 + 1} \frac{1}{3}}{3}} \cdot \log x$$

$$\Rightarrow \sum_{x \leq q^N \leq x + H} \int_{-\infty}^{\infty} |L\left(\frac{1}{2} + it, f \otimes \chi\right)|^2 \cdot \prod_{v \mid \infty} \mathcal{K}_v\left(\frac{1}{2} + it, 0, \beta', 1\right) dt \ll x^{\frac{2^{\delta_0 + 1} \frac{1}{3}}{3}} \cdot \log x$$

Now

$$Q(\chi, t)^{-\beta'} \ll \mathcal{K}_v\left(\frac{1}{2} + it, 0, w, \chi\right) \ll Q(\chi, t)^{-\beta'} \quad \text{(for } v \mid \infty)$$

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where

\[ Q(\chi, t) = \prod_{v \in \mathbb{R}} (1 + |t + t_v|) \cdot \prod_{v \in \mathbb{C}} (1 + \ell_v^2 + 4(t + t_v)^2) \]

For trivial \( \chi, t_v = l_v = 0 \). Also recall for \( v|\infty, \) fix \( 0 < t < 1 \). Then

\[ K_v(\frac{1}{2} + it, 0, \beta', 1) \gg (1)^{-\frac{(d-1)\beta'}{2}} \]

So

\[ x^{\frac{2h_0 + 1}{3}} \log x \gg \sum_{x \leq q^N \leq x + H} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot \prod_{v|\infty} K_v(\frac{1}{2} + it, 0, \beta', 1) \, dt \]

\[ \gg \sum_{x \leq q^N \leq x + H} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (1)^{-\frac{(d-1)\beta'}{2}} \, dt \]

\[ \gg \sum_{x \leq q^N \leq x + H} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (q^N)^{-\frac{(d-1)\beta'}{2}} \, dt \]

\[ \geq \sum_{x \leq q^N \leq x + H} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (x + H)^{-\frac{(d-1)\beta'}{2}} \]

So

\[ \sum_{x \leq q^N \leq x + H} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \, dt \ll (x + H)^{\frac{(d-1)\beta'}{2}} \cdot x^{\frac{2h_0 + 1}{3}} \cdot \log x \]

\[ \ll x^{d-1+\frac{2h_0 + 1}{3} + \frac{\epsilon}{2}} \cdot \log x \quad \text{where} \quad \beta' = 1 + \frac{\epsilon}{2d - 2} \]

\[ \ll x^{d-1+\frac{2h_0 + 1}{3} + \epsilon} \]

By a standard argument analogous to that in [Good 1982], this short-interval moment bound implies the pointwise bound

\[ L(\frac{1}{2} + it, f \otimes \chi) \ll (q^N)^{d-1+\frac{2h_0 + 1}{3} + \epsilon} \ll (q^N)^{\frac{d+1}{2} + \epsilon} \]

for \( \vartheta < 1 \).
References

[Asai 1977] T. Asai, *On certain Dirichlet series associated with Hilbert modular forms, and Rankin’s method*, Ann. of Math. 226 (1977), 81-94.

[Bernstein-Reznikoff 1999] J. Bernstein, A. Reznikov, *Analytic continuation of representations and estimates of automorphic forms*, Ann. of Math. 150 (1999), 329-352.

[Bump-Duke-Hoffstein-Iwaniec 1992] D. Bump, W. Duke, J. Hoffstein, H. Iwaniec, *An estimate for the Hecke eigenvalues of Maass forms*, Int. Math. Res. Notices 4 (1992) 75-81.

[Burgess 1962] D.A. Burgess, *On character sums and L-series, I*, Proc. London Math. Soc 12 (1962), 193-206.

[Burgess 1963] D.A. Burgess, *On character sums and L-series, II*, Proc. London Math. Soc 313 (1963), 24-36.

[Casselman 1973] W. Casselman, *On some results of Atkin and Lehner*, Ann. of Math. 206 (1973), 311-319.

[Casselman-Shalika 1980] W. Casselman and J. Shalika, *The unramified principal series of p-adic groups, II, the Whittaker function*, Comp. Math. 41 (1980), 207-231.

[Cogdell 2004] J. Cogdell, *Lectures on L-functions, Converse Theorems, and Functoriality for GL_n*, Fields Institute Notes, in *Lectures on Automorphic L-functions*, Fields Institute Monographs 20, AMS, Providence, 2004.

[Cogdell-PS 2005] J. Cogdell, and I. Piatetski-Shapiro, *Remarks on Rankin-Selberg convolutions*, Contributions to Automorphic Forms, Geometry and Number Theory (Shalikafest 2002), (H. Hida, D. Ramakrishnan and F. Shahidi eds.), John Hopkins University Press, Baltimore 2005, 255-278.

[Conrey-Ghosh 1984] J.B. Conrey and A. Ghosh, *Mean values of the Riemann zeta-function*, Mathematika 31 (1984), 159-161.

[Diaconu-Goldfeld-Hoffstein 2003] A. Diaconu, D. Goldfeld and J. Hoffstein, *Multiple Dirichlet series and moments of zeta and L-functions*, Comp. Math 139 (2003), 297-360.
[Diaconu-Goldfeld 2006a] A. Diaconu and D. Goldfeld, *Second moments of GL₂ automorphic L-functions*, Proc. of the Gauss-Dirichlet Conference, Göttingen 2005.

[Diaconu-Goldfeld 2006b] A. Diaconu and D. Goldfeld, *Second moments of Hecke-L-series and multiple Dirichlet series I*, Multiple Dirichlet Series, Automorphic Forms and Analytic Number Theory, Proc. Symp. Pure Math. 75, AMS, Providence, 2006, 59-89.

[Diaconu-Garrett 2008] A. Diaconu and P. Garrett, *Integral moments of automorphic L-functions*, Journal of the Mathematical Institute of Jussieu, 2008 (to appear).

[Diaconu-Garrett 2009] A. Diaconu and P. Garrett, *Subconvexity bounds for automorphic L-functions*, Journal of the Mathematical Institute of Jussieu, 2009 (to appear).

[Duke-Friedlander-Iwaniec 1993] W. Duke, J. Friedlander and H. Iwaniec, *Bounds for Automorphic L-functions*, Inv. Math 112 (1993), 1-18.

[Duke-Friedlander-Iwaniec 1994] W. Duke, J. Friedlander and H. Iwaniec, *A quadratic divisor problem*, Inv. Math 115 (1994), 209-217.

[Duke-Friedlander-Iwaniec 1994] W. Duke, J. Friedlander and H. Iwaniec, *Bounds for Automorphic L-functions, II*, Inv. Math 115 (1994), 219-239.

[Duke-Friedlander-Iwaniec 2002] W. Duke, J. Friedlander and H. Iwaniec, *The subconvexity problem for Artin L-functions*, Inv. Math 149 (2002), 489-577.

[Duke-Friedlander-Iwaniec 2001] W. Duke, J. Friedlander and H. Iwaniec, *Bounds for Automorphic L-functions, III*, Inv. Math 143 (2001), 221-248.

[Gelbart-Shahidi 2001] S. Gelbart and F. Shahidi, *Boundedness of automorphic L-functions in vertical strips*, J. Amer. Math. Soc. 14 (2001), 79-107.

[Godement-Jacquet 1972] R. Godement and H. Jacquet, *Zeta functions of simple algebras*, SLN 260, Springer-Verlag, Berlin, 1972.

[Good 1982] A. Good, *The square mean of Dirichlet series associated with cusp forms*, Mathematika 29 (1982), 278-295.
[Good 1986] A. Good, *The Convolution method for Dirichlet series*, The Selberg trace formula and related topics, (Brunswick, Maine 1984) Contemp. Math. **53**, American Mathematical Society, Providence, RI, 1986, 207-214.

[Hardy-Littlewood 1918] G.H. Hardy and J.E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distributions of primes*, Acta Mathematica **41** (1918), 119-196.

[Hoffstein-Lockhart 1994] J. Hoffstein and P. Lockhart, *Coefficients of Maass forms and the Siegel zero* with appendix *An effective zero-free region*, by D. Goldfeld, J. Hoffstein, D. Lieman, Ann. of Math. **140** (1994), 161-181.

[Ingham 1926] A.E. Ingham, *Mean value theorems in the theory of the Riemann-zeta function*, Proceedings of the London Mathematical Society **27** (1926), 273-300.

[Ivic 1991] A. Ivic, *Lectures on Mean values of the Riemann-zeta function*, Tate Institute of Fundamental Research, Springer-Verlag, Bombay, Berlin, Heidelberg, New York and Tokyo, 1991.

[Iwaniec 2002] Henryk Iwaniec, *Spectral Methods of Automorphic forms 2nd ed.*, The American Mathematical Society, USA, 2002.

[Jacquet-Langlands 1971] H. Jacquet and R.P. Langlands, *Automorphic forms on GL_2*, **14**, Lecture Notes in Mathematics, Springer-Verlag, Berlin and New York, 1971.

[Jacquet-PS-Shalika 1981] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Automorphic forms on GL_3, I*, Ann. of Math. **109** (1979), 169-258.

[Jacquet-Shalika 1981] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations I, II*, Amer. J. Math. **103** (1981), 499-588.

[Jacquet-Shalika 1990] H. Jacquet and J. Shalika, *Rankin-Selberg convolutions: archimedean theory in Festchrift in Honor of I.I. Piatetski-Shapiro, I*, Weizman Science Press, Jerusalem, 1990, 125-207.

[Keating-Snaith 2000] J.P. Keating and N.C. Snaith, *Random matrix theory and L-functions at s = \frac{1}{2}*, Comm. Math. Phys. **214** (2000), 91-110.
[Kim 2005] H. Kim, *On local L-functions and normalized intertwining operators*, Canad. J. Math 57 (2005), 535-597.

[Kim-Shahidi 2002] H. Kim and F. Shahidi, *Cuspidality of symmetric powers with applications*, Duke Math. Journal 112 (2002), 177-197.

[Kowalski-Michel-Vanderkam 2002] E. Kowalski, P. Michel and J. Vanderkam, *Rankin-Selberg L-functions in the level aspect*, Duke Math. Journal 114 (2002), 123-191.

[Langlands 1971] R. Langlands, *Euler Products*, Yale University Press, New Haven, 1971.

[Langlands 1975] R. Langlands, *On the functional equations satisfied by Eisenstein series*, SLN 544, 1975.

[Lune-Riele-Winter 1986] J. van de Lune, J.J. Te Riele and D.T. Winter, *On the zeros of the Riemann zeta function in the critical strip, IV*, Math. of Comp. 46 (1986), 667-681.

[Meurman 1987] T. Meurman, *On the order of Maass L-functions on the critical line*, Number Theory, vol. I, Budapest Colloq. 1987, vol. 51, Math. Soc. Bolyai, Budapest (1990), 325-354.

[Michel-Venkatesh 2006] P. Michel and A. Venkatesh, *Equidistribution, L-functions, and ergodic theory: on some problems of Yu. Linnik*, Int. Cong. Math, vol II, Eur. Math. Soc., Zurich, 2006, pp. 421-457.

[Moeglin-Waldspurger 1989] C. Moeglin and J.L. Waldspurger, *Le spectre résiduel de GL_n with appendix Poles des fonctions L de pairs pour GL_n*, Ann. Sci. École Norm. Sup. 22 (1989), 605-674.

[Moeglin-Waldspurger 1995] C. Moeglin, J.L. Waldspurger, *Spectral Decompositions of Eisenstein series*, Cambridge University Press, Cambridge, 1995.

[Molteni 2000] G. Molteni, *Upper and lower bounds at s = 1 for certain Dirichlet series with Euler product*, Duke Math. J. 111(1) (2000), 133-158.

[Moreno 2005] Carlos J. Moreno, *Advanced Analytic Number Theory: L-functions*, Providence, RI, AMS, 2005.

48
[Odlyzko 1989] A.M. Odlyzko, *Supercomputers and the Riemann zeta function*, Supercomputing 89: Supercomputing Structures and Computations, Proc. 4th Intern. Conf. on Supercomputing, L.P. Kartashev and S.I. Kartashev (eds.), International Computing Institute, pp 348-352, 1989.

[Sarnak 1985] Peter Sarnak, *Fourth moments of Grössencharakteren zeta functions*, Comm. Pure Appl. Math. 38 no. 2 (1985), 167-178.

[Sarnak 1994] Peter Sarnak, *Integrals of products of eigenfunctions*, Int. Math. Res. Notices, 6 (1994), 251-260.

[Shahidi 1978] F. Shahidi, *Functional Equation satisfied by certain L-functions*, Comp. Math. 37 (1978), 171-207.

[Shahidi 1980] F. Shahidi, *On non-vanishing of L-functions*, Bull, AMS 2 (1980), 462-464.

[Shahidi 1983] F. Shahidi, *Local coefficients and normalizations of intertwining operators for GL_n*, Comp. Math. 48 (1983), 271-295.

[Shalika 1974] J. Shalika, *The multiplicity-one theorem for GL_n*, Ann. of Math. 100 (1974), 171-193.

[Shintani 1976] T. Shintani, *On an explicit formula for class-one Whittaker functions on GL_n over p-adic fields*, Proc. Japan Acad. 52 (1976), 180-182.

[Tate 1950] J. Tate, *Fourier Analysis in Number Fields and Hecke’s zeta functions*, Ph.D. Thesis, Princeton University, Princeton, N.J., 1950.

[Titchmarsh 1951] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford University Press, London, 1951.

[Weil 1974] A. Weil, *Basic Number Theory*, Springer-Verlag, New York, 1974.

[Weyl 1921] H. Weyl, *Sur Abschatzung von ζ(1 + it)*, Math. Zeit 10 (1921), 88-101.

[Zhang 2005] Q. Zhang, *Integral Mean values of modular L-functions*, J. No. Th. 115 (2005), 100-112.

[Zhang 2006] Q. Zhang, *Integral Mean values of Maass’ L-functions*, preprint, 2006.