Abstract

The importance of a rigorous definition of the singular degree of a distribution is demonstrated on the case of two-dimensional QED (Schwinger model). Correct mathematical treatment of second order vacuum polarization in the perturbative approach is crucial in order to obtain the Schwinger mass of the photon by resummation.
1 Introduction

The Schwinger model [1] still serves as a very popular laboratory for quantum field theoretical methods. Although its nonperturbative properties and their relations to confinement [2,3] have always been of greatest interest, it is also possible to discuss the model perturbatively in a straightforward way. The calculation of the vacuum polarization diagram (VP) at second order then turns out to be a delicate task, where a careful discussion of the scaling behaviour of distributions becomes necessary.

We will demonstrate this fact in the framework of causal perturbation theory in the following.

2 The causal approach

In causal perturbation theory, which goes back to a classical paper by H. Epstein and V. Glaser [4], the \(S\)-matrix is constructed inductively order by order as an operator valued functional

\[
S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4 x_n T_n(x_1, \ldots x_n)g(x_1) \ldots g(x_n),
\]

(1)

where \(g(x)\) is a tempered test function that switches the interaction. The first order (e.g. for QED)

\[
T_1(x) = ie : \bar{\Psi}(x) \gamma^\mu \Psi(x) : A_\mu(x)
\]

(2)

must be given in terms of the asymptotic free fields. It is a striking property of the causal approach that no ultraviolet divergences appear, i.e. the \(T_n\)’s are finite and well defined. The only remnant of the ordinary renormalization theory is a non-uniqueness of the \(T_n\)’s due to finite normalization terms. The adiabatic limit \(g(x) \to 1\) has been shown to exist in purely massive theories at each order [4].

To calculate the second order distribution \(T_2\), one proceeds as follows: First one constructs the distribution \(D_2(x, y)\)

\[
D_2(x, y) = [T_1(x), T_1(y)] ,
\]

(3)

\[
supp D_2 = \{(x - y) \mid (x - y)^2 \geq 0\} ,
\]

(4)

which has causal support. Then \(D_2\) is split into a retarded and an advanced part \(D_2 = R_2 - A_2\), with

\[
supp R_2 = \{(x - y) \mid (x - y)^2 \geq 0, (x^0 - y^0) \geq 0\} ,
\]

(5)

\[
supp A_2 = \{(x - y) \mid (x - y)^2 \geq 0, -(x^0 - y^0) \geq 0\} .
\]

(6)

Finally \(T_2\) is given by

\[
T_2(x, y) = R_2(x, y) + T_1(y)T_1(x) = A_2(x, y) - T_1(x)T_1(y) .
\]

(7)

For the massive Schwinger model with fermion mass \(m\), the part in the Wick ordered distribution \(D_2\) corresponding to VP

\[
D_2(x, y) = e^2[d_2^{\mu \nu}(x - y) - d_2^{\nu \mu}(y - x)] : A_\mu(x)A_\nu(y) : + ...
\]

(8)

then becomes

\[
\hat{d}_2^{\mu \nu}(k) = \frac{1}{2\pi} \int d^2z \hat{d}_2^{\mu \nu}(z)e^{ikz}
\]

\[
\hat{d}_2^{\mu \nu}(k) = \left(g_{\mu \nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{4m^2}{2\pi k^2} \frac{1}{\sqrt{1 - 4m^2/k^2}} \text{sgn}(k^0)\Theta(k^2 - 4m^2) .
\]

(9)
3 Power counting degree and singular order

Obviously, $\hat{d}_{2}^{\nu}$ has power counting degree $\omega_{p} = -2$ [6]. But the singular order of the distribution is $\omega = 0$. To show what this means we recall the following definitions [5,8]:

**Definition 1a:** The distribution $\hat{d}(p) \in \mathcal{S}'(\mathbb{R}^{n})$ has quasi-asymptotics $\hat{d}_{0}(p) \not\equiv 0$ at $p = \infty$ with respect to a positive continuous function $\rho(\delta), \delta > 0$, if the limit

$$\lim_{\delta \to 0} \rho(\delta) \langle \hat{d}(\frac{p}{\delta}), \hat{\varphi}(p) \rangle = \langle \hat{d}_{0}, \hat{\varphi} \rangle$$

exists for all $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^{n})$. The Fourier transform of a test function $\varphi(x)$ is defined by

$$\hat{\varphi}(p) = (2\pi)^{-n/2} \int d^{n}x \varphi(x) e^{ipx}$$

By scaling transformation one derives

$$\lim_{\delta \to 0} \frac{\rho(a\delta)}{\rho(\delta)} = a^{\omega} \equiv \rho_{0}(\delta)$$

with some real $\omega$. Thus we call $\rho(\delta)$ the power-counting function. The equivalent definition in $x$-space reads as follows:

**Definition 1b:** The distribution $d(x) \in \mathcal{S}'(\mathbb{R}^{n})$ has quasi-asymptotics $d_{0}(x) \not\equiv 0$ at $x = 0$ with respect to a positive continuous function $\rho(\delta), \delta > 0$, if the limit

$$\lim_{\delta \to 0} \rho(\delta)\delta^{n} d(\delta x) = d_{0}(x)$$

exists in $\mathcal{S}'(\mathbb{R}^{n})$.

**Definition 2:** The distribution $d(x) \in \mathcal{S}'(\mathbb{R}^{n})$ is called singular of order $\omega$, if it has quasi-asymptotics $d_{0}(x)$ at $x = 0$, or its Fourier transform has quasi-asymptotics $\hat{d}_{0}(p)$ at $p = \infty$, respectively, with power-counting function $\rho(\delta)$ satisfying

$$\lim_{\delta \to 0} \frac{\rho(a\delta)}{\rho(\delta)} = a^{\omega} \quad \forall a > 0.$$  \(12\)

Equation \((12)\) implies

$$a^{n} \langle \hat{d}_{0}(p), \hat{\varphi}(ap) \rangle = \langle \hat{d}_{0}(\frac{p}{a}), \hat{\varphi}(p) \rangle = a^{-\omega} \langle \hat{d}_{0}(p), \hat{\varphi}(p) \rangle$$

$$= \langle d_{0}(x), \varphi(\frac{x}{a}) \rangle = a^{n} \langle d_{0}(ax), \varphi(x) \rangle = a^{-\omega} \langle d_{0}(x), \varphi(x) \rangle,$$  \(15\)

i.e. $\hat{d}_{0}$ is homogeneous of degree $\omega$:

$$\hat{d}_{0}(\frac{p}{a}) = a^{-\omega} \hat{d}_{0}(p),$$  \(16\)

$$d_{0}(ax) = a^{-(n+\omega)} d_{0}(x).$$  \(17\)
This implies that \( d_0 \) has power-counting function \( \rho(\delta) = \delta^\omega \) and the singular order \( \omega \), too. In particular, we have the following estimates for \( \rho(\delta) \) [5]: If \( \epsilon > 0 \) is an arbitrarily small number, then there exist constants \( C, C' \) and \( \delta_0 \) such that

\[
C \delta^{\omega+\epsilon} \geq \rho(\delta) \geq C' \delta^{\omega-\epsilon} , \quad \delta < \delta_0.
\]  

(18)

Applying the above definitions to \( \hat{d}_2^{\mu\nu}(k) \), we obtain after a short calculation the quasi-asymptotics

\[
\lim_{\delta \to 0} \hat{d}_2^{\mu\nu}(k/\delta) = \frac{1}{2\pi} \left( g^{\mu\nu} k^2 - k^\mu k^\nu \right) \delta(k^2) \text{sgn}(k^0) ,
\]

(19)

and we have \( \rho(\delta) = 1 \), hence \( \omega = 0 \). Note that the \( g^{\mu\nu} \)-term in (19) does not contribute to the quasi-asymptotics. The reason for the result (19) can be explained by the existence of a sum rule [7]

\[
\int_{4m^2}^{\infty} d(q^2) \frac{\delta^2 m^2}{q^4 \sqrt{1 - 4m^2/q^2}} = \frac{1}{2} ,
\]

(20)

so that the l.h.s. of (19) is weakly convergent to the r.h.s. In spite of \( \text{sgn}(k^0) \), the r.h.s. of (19) is a well-defined tempered distribution due to the factor \( (g^{\mu\nu} k^2 - k^\mu k^\nu) \).

This has the following consequence: The retarded part \( r_2^{\mu\nu} \) of \( \hat{d}_2^{\mu\nu} \) would be given in the case \( \omega < 0 \) by the unsubtracted splitting formula

\[
r_2^{\mu\nu}(k) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{1 - t + i0} d_2^{\mu\nu}(tk)
\]

\[
= \frac{im^2}{\pi^2} \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \frac{1}{k^2 \sqrt{1 - 4m^2/k^2}} \log \frac{\sqrt{1 - 4m^2/k^2} + 1}{\sqrt{1 - 4m^2/k^2} - 1} , \quad k^2 > 4m^2, k^0 > 0.
\]

(21)

This distribution will vanish in the limit \( m \to 0 \), and the photon would remain massless. But since we have \( \omega = 0 \), the subtracted splitting formula [5] must be used:

\[
r_2^{\mu\nu}(k) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{(t - i0)^{\omega+1}} \frac{dt}{1 - t + i0} d_2^{\mu\nu}(tk)
\]

\[
= \frac{im^2}{\pi^2} \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \left( \frac{1}{k^2 \sqrt{1 - 4m^2/k^2}} \log \frac{\sqrt{1 - 4m^2/k^2} + 1}{\sqrt{1 - 4m^2/k^2} - 1} + \frac{1}{2m^2} \right) , \quad k^2 > 4m^2, k^0 > 0.
\]

(22)

The new last term survives in the limit \( m \to 0 \). After resummation of the VP bubbles it gives the well-known Schwinger mass \( m_s^2 = e^2/\pi \) of the photon. Consequently, the difference between simple power-counting and the correct determination of the singular order is by no means a mathematical detail, it is terribly important for the physics.

References

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