Computational algorithms based on the fundamental solution of task operator

Yu I Skalko¹, S Yu Gridnev²,¹ and N V Minaeva³

¹Computational Physics Department, Moscow Institute of Physics and Technology (State University), 141700, Dolgoprudny, Russia
²Structural Mechanics Department, Voronezh State Technical University, 394026, Voronezh, Russia
³Mechanics and Computer Modeling Department, Voronezh State University, 394018, Voronezh, Russia

E-mail: skalko@mail.mipt.ru, gridnev_s_y@rambler.ru

Abstract An approximation of matrix Green’s function for a hyperbolic system of linear differential equations of the first order has been created. An algorithm for an approximate solution of the generalized Riemann problem of discontinuity decay in the presence of additional conditions at the boundaries has been proposed. A computational algorithm for the approximate solution of boundary value problems for hyperbolic systems of equations has been set up on the basis of these solutions. The algorithm is implemented for a system of equations describing the propagation of elastic waves in a block-fractured medium. The results of the numerical experiments aimed at studying the possible mechanisms of propagation of elastic waves to a considerable depth without scattering have been presented.

1. Introduction

Solving the Riemann problem of discontinuity decay is an important step in the development of computational algorithms for the approximate solution of initial-boundary value problems for hyperbolic systems of linear differential equations, with approximation of the solution by piecewise-smooth functions. A number of methods for solving the Riemann problem were proposed in [1–2] for one spatial variable. In fact, all these methods are based on the presence of hyperbolic systems’ characteristics. Characteristic based methods do not work when there are more than one spatial variable. In these cases, it is usually assumed that in the vicinity of the discontinuity, the solution is a plane wave that moves along the normal to the discontinuity surface [1, 3, 4]. Such an assumption does not always correspond to physical reality.

Further the statement of the generalized Riemann problem of decay of a discontinuity with conjugation conditions at the boundaries for linear hyperbolic systems of partial differential equations of the first order will be formulated. The coefficients of the equations can be piecewise constant. The number of spatial variables can be arbitrary. The proposed algorithm for solving this problem is based on the approximation of a Green’s matrix function. The developed solution of the generalized Riemann problem will form the basis of the computational algorithm for finding an approximate solution of the
initial-boundary value problem for the described class of systems differential equations. The implemented computational algorithm will be used to study the mechanisms of transmission of vibration from a source located on the surface to an oil-containing formation. We will find out in which cases it is possible to propagate elastic waves generated by a vibration source to a substantial depth without significant scattering. The dependence of the nature of elastic wave propagation in a block-fractured medium on the frequency of the source operation will also be investigated.

2. The generalized Riemann problem with conjugation conditions on boundaries

The solution of the generalized Riemann problem on the decay of a discontinuity with conjugation conditions on the boundary will be called the solution of a Cauchy problem for a system of first-order linear differential equations with piecewise constant coefficients:

$$\frac{\partial u(t,x)}{\partial t} + \sum_{i=1}^{N} A_i \frac{\partial u(t,x)}{\partial x_i} = 0, \quad x \in \mathbb{R}^N$$

(1)

$$u(t=0,x) = u_0(x).$$

(2)

The initial data is continuous everywhere except for the hyperplane $\Gamma : x_1 = 0$. The desired solution can have a discontinuity only on the hyperplane $\Gamma$. The system of algebraic equations given in the form must also be fulfilled:

$$L(u(t,x_1 = 0,x_2,...,x_N),u(t,x_1 = +0,x_2,...,x_N)) = 0.$$  

(3)

coupling conditions that link the values of variables on the right and left of the hyperplane $\Gamma$. We assume that the values of the variables at the initial moment of time satisfy the equations (3). Further description will be carried out for the general case $N$ of spatial variables. Separate statements and examples we will formulate for two spatial variables for simplicity and clarity.

3. Fundamental solution

The rest is largely based on the concepts and statements of the theory of generalized functions, the exposition of which can be found in [5-8].

We assume that the number of linearly independent eigenvectors of each matrix $A_i$ is equal to the dimension of the vector $u$. Then these matrices can be represented as

$$A_i = R_i A_i \Omega_i.$$ 

(4)

$A_i$ is the matrix, on the main diagonal of which the eigenvalues of the matrix are arranged in non-decreasing order $A_i$, the remaining elements are zero;

$\Omega_i$ is the matrix of the left eigenvectors-the rows of the matrix $A_i$, ordered as well as the rows of the matrix $A_i$;

$R_i = \Omega_i^{-1}$ is the matrix of right eigenvectors of the matrix $A_i$ columns.

In [9] a fundamental solution of the problem operator is constructed (1). Green's matrix function

$$G(t,x) = \theta(t) \left\{ \prod_{j=1}^{N} R_j \delta(t - \Lambda_j \cdot t) \right\} \Omega_j + \sum_{|\alpha|\geq 2} R_j B_\alpha D^{\alpha} \delta(x)$$ 

(5)

We use the notation:

$\alpha = (\alpha_1,\alpha_2,...,\alpha_N)$ is the vector (multi-index) whose components are integer, non-negative numbers $\alpha_j$, $|\alpha| = (\alpha_1 + \ldots + \alpha_N)$,
\( \mathbf{B}_a \) are the matrices that are polynomials of matrices \( \mathbf{A}_j \) of degree \( |a| \).

\( \delta(\mathbf{I}x - \mathbf{A}_jt) \) are the diagonal matrices, in the \( k \)-th row of which there is a generalized function \( \delta(x_j - \lambda^k_j t) \), \( \lambda^k_j \) - is a \( k \)-th eigenvalue of the \( \mathbf{A}_j \) matrix,

\[
D^a = \frac{\partial^{|a|}}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_N^{a_N}}
\]

is an operator of differentiation with respect to spatial variables

Consider the factor \( \mathbf{R}_j \delta(\mathbf{I}x - \mathbf{A}_jt)\mathbf{\Omega}_j \). If we denote a square matrix, in which the \( k \)-th element on the main diagonal is equal 1 and the other elements are equal to zero as \( \mathbf{D}_k \), then

\[
\mathbf{R}_j \delta(\mathbf{I}x - \mathbf{A}_jt)\mathbf{\Omega}_j = \sum_{k_j=1}^{M} \mathbf{R}_j \mathbf{D}_{k_j} \mathbf{\Omega}_j \delta(x_j - \lambda^k_j t) = \sum_{k_j=1}^{M} \mathbf{C}_{k_j} \delta(x_j - \lambda^k_j t).
\]

Consequently

\[
\mathbf{R}_j \delta(\mathbf{I}x - \mathbf{A}_jt)\mathbf{\Omega}_j = \sum_{k_j=1}^{M} \sum_{k_{j-1}=1}^{M} \cdots \sum_{k_{j-N+1}=1}^{M} \prod_{j=1}^{N} \mathbf{C}_{k_j} \delta(x_j - \lambda^k_j t)
\]

We define:

\( \mathbf{k} = (k_1, k_2, \ldots, k_N) \) - multi-index with integer components \( k_j = 1: M \),

\( \mathbf{C}_k = C_{k_1} C_{k_2} \cdots C_{k_N} \) - multi-index array of matrices,

\( \lambda_k = (\lambda_{k_1}, \lambda_{k_2}, \ldots, \lambda_{k_N}) \) - multi-index array of vectors. Then

\[
\prod_{j=1}^{N} \mathbf{R}_j \delta(\mathbf{I}x - \mathbf{A}_jt)\mathbf{\Omega}_j = \sum_{\mathbf{k}} \mathbf{C}_k \delta(x - \lambda_k t)
\]

and

\[
\mathbf{G}(t, \mathbf{x}) = \theta(t) \sum_{\mathbf{k}} \mathbf{C}_k \delta(x - \lambda_k t) + \theta(t) \sum_{|a| \geq 2} \mathbf{B}_a D^a \delta(x). \quad (6)
\]

The expression (6) can be represented as

\[
\mathbf{G}(t, \mathbf{x}) = \theta(t) \sum_{\mathbf{k}} \mathbf{C}_k \delta(x - \lambda_k t) + \mathbf{O}(t^2). \quad (7)
\]

Note that

\[
\sum_{k_j=1}^{M} \mathbf{C}_{k_j} = \mathbf{R}_j \left( \sum_{k_j=1}^{M} \mathbf{D}_{k_j} \right) \mathbf{\Omega}_j = \mathbf{R}_j \mathbf{I} \mathbf{\Omega}_j = \mathbf{I}. \quad (8)
\]

In the case of two spatial variables

\[
\mathbf{G}(t, \mathbf{x}) = \theta(t) \sum_{\mathbf{k}} C_{k_1} C_{k_2} \delta(x - \lambda_k t) + \frac{\theta(t)}{2} t^2 (\mathbf{A}_2 \mathbf{A}_1 - \mathbf{A}_1 \mathbf{A}_2) \frac{\partial^2 \delta(x)}{\partial x_1 \partial x_2} + \mathbf{O}(t^3). \quad (9)
\]

If we change the numbering of spatial variables, we can write
\[ G(t,x) = \theta(t) \sum_{k} C_k C_{k_1} \delta(x - \lambda_k t) - \frac{\theta(t)}{2} t^2 \left( A_2 A_1 - A_1 A_2 \right) \frac{\partial^2 \delta(x)}{\partial x_1 \partial x_2} + O(t^3). \] (10)

From (9) and (10) it follows that
\[ G(t,x) = \theta(t) \sum_{k} C_{k_1} C_{k_2} + C_{k_2} C_{k_1} \frac{1}{2} \delta(x - \lambda_k t) + O(t^3). \] (11)

Next we will use the notation \( \bar{C}_k = \frac{C_{k_1} C_{k_2} + C_{k_2} C_{k_1}}{2} \).

### 4. The Riemann problem

Let \( u(t,x) \) be the Riemann problem solving (1), (2), (3). Define the functions

\[
\begin{align*}
    u^-(t,x) &= \left\{ 
        \begin{array}{ll}
            u(t,x), & \text{if } t \geq 0, x_1 \leq 0 \\
            0, & \text{with the others } t, x \\
        \end{array} 
    \right.
    \\
    u^+(t,x) &= \left\{ 
        \begin{array}{ll}
            u(t,x), & \text{if } t \geq 0, x_1 \geq 0 \\
            0, & \text{with the others } t, x \\
        \end{array} 
    \right.
    \\
    u_0^-(t,x) &= \left\{ 
        \begin{array}{ll}
            u_0(t,x), & \text{if } x_1 \leq 0 \\
            0, & \text{with the others } x \\
        \end{array} 
    \right.
    \\
    u_0^+(t,x) &= \left\{ 
        \begin{array}{ll}
            u_0(t,x), & \text{if } x_1 \geq 0 \\
            0, & \text{with the others } x \\
        \end{array} 
    \right.
    \\
    v^-(t,x) &= \theta(t) u\left(t,x_1 = -0, x_2, \ldots, x_N\right)
    \\
    v^+(t,x) &= \theta(t) u\left(t,x_1 = +0, x_2, \ldots, x_N\right)
\end{align*}
\]

In [9], it is shown that if we consider \( u^-(t,x) \) as a generalized function of \( \mathcal{S}' \), then it satisfies the equation
\[
\frac{\partial u^-}{\partial t} + \sum_{i=1}^{N} A_i^- \frac{\partial u^-}{\partial x_i} = u_0^- \delta(t) - A_1^- v^- \delta(x_1) \] (12)

Similarly, the generalized function \( u^+(t,x) \), satisfies the equalities
\[
\frac{\partial u^+}{\partial t} + \sum_{i=1}^{N} A_i^+ \frac{\partial u^+}{\partial x_i} = u_0^+ \delta(t) + A_1^+ v^+ \delta(x_1) \] (13)

The solution of the equation (12) can be written as a convolution
\[
u^-(t,x) = G^-(t,x) * \left( u_0^- \delta_{t=0} - A_1^- v^- \delta_{x_1=0} \right). \]

In [9] it is shown that
\[
u^-(t,x) = \sum_{k:x_1 \leq \lambda_k} \bar{C}_k u_0^-(x - \lambda_k t) + \sum_{k:x_1 > \lambda_k} \bar{C}_k A_1^- v^- \left( t - \frac{x_1}{\lambda_k}, x - \frac{x_1}{\lambda_k} - \lambda_k \right) + O(t^3). \] (14)

Passing in (14) to the limit \( x_1 \to -0 \), we get
\[
u^- - \sum_{k: \lambda_k \leq 0} \frac{1}{\lambda_k} \bar{C}_k A_1^- v^- = \sum_{k: \lambda_k \geq 0} \bar{C}_k u_0^\left( -\lambda_k t, x_2 - \lambda_k t \right) + O(t^3). \]

Consider (11), then
\[ v^- - \sum_{k_1, \lambda_{k_1}^- < 0} \frac{1}{\lambda_{k_1}^-} C_{k_1}^- A_{k_1}^- v^- = \sum_{k_1, \lambda_{k_1}^+ \geq 0} \bar{C}_{k_1} u_0^- \left(-\lambda_{k_1}^- t, x_2 - \lambda_{k_2}^- t\right) + O(t^3). \]

as \( A_{k_1}^- = R_1^- k_1 \Omega_{k_1}^- \), to \( \frac{1}{\lambda_{k_1}^-} C_{k_1}^- A_{k_1}^- = C_{k_1}^- \) and

\[ \sum_{k_1, \lambda_{k_1}^- \geq 0} C_{k_1}^- v^- = \sum_{k_1, \lambda_{k_1}^+ \geq 0} \bar{C}_{k_1} u_0^- \left(-\lambda_{k_1}^- t, x_2 - \lambda_{k_2}^- t\right) + O(t^3). \] (15)

Multiply (15) by the rows of the matrix \( \Omega_{k_1}^- \) corresponding to zero or positive eigenvalues. We get that with an accuracy of \( O(t^3) \)

\[ I_{k_1}^- v^- = I_{k_1}^- \sum_{k_1, \lambda_{k_1}^- \geq 0} \bar{C}_{k_1} u_0^- \left(-\lambda_{k_1}^- t, x_2 - \lambda_{k_2}^- t\right), k_1 : \lambda_{k_1}^- \geq 0. \] (16)

In exactly the same way the solution of equation (13) can be represented in the form

\[ u^+(t, x) = \sum_{k_1, \lambda_{k_1}^+ \leq 0} \bar{C}_{k_1} u_0^+ \left(x - \lambda_{k_1}^+ t\right) + \sum_{k_1, \lambda_{k_1}^+ \leq 0} \bar{C}_{k_1} A_{k_1}^+ v^+ \left(t - \frac{x_1}{\lambda_{k_1}^+}, x - \frac{x_1}{\lambda_{k_1}^+} \lambda_{k_1}^+ \right) + O(t^3), \] (17)

and the vector \( v^+ \) with the accuracy to \( O(t^3) \) satisfies the equations

\[ I_{k_1}^+ v^+ = I_{k_1}^+ \sum_{k_1, \lambda_{k_1}^+ \leq 0} \bar{C}_{k_1} u_0^+ \left(-\lambda_{k_1}^+ t, x_2 - \lambda_{k_2}^+ t\right), k_1 : \lambda_{k_1}^+ \leq 0. \] (18)

Note that if the initial data is represented by a polynomial not higher than the second degree, then equations (16) and (18) are exactly fulfilled.

Add to the equalities (16), (18) the conjugation conditions (3). The final system of linear algebraic equations for unknowns \( v^-(t, x) \) and \( v^+(t, x) \) connects the values of the the Riemann problem solution on both sides of the hyperplane \( \Gamma \).

\[
\begin{align*}
L(v^-, v^+) &= 0 \\
I_{k_1}^- v^- &= I_{k_1}^- \sum_{k_1, \lambda_{k_1}^- \geq 0} \bar{C}_{k_1} u_0^- \left(-\lambda_{k_1}^- t, x_2 - \lambda_{k_2}^- t\right), k_1 : \lambda_{k_1}^- \geq 0. \\
I_{k_1}^+ v^+ &= I_{k_1}^+ \sum_{k_1, \lambda_{k_1}^+ \leq 0} \bar{C}_{k_1} u_0^+ \left(-\lambda_{k_1}^+ t, x_2 - \lambda_{k_2}^+ t\right), k_1 : \lambda_{k_1}^+ \leq 0.. 
\end{align*}
\] (19)

Since the initial data satisfy the conjugation conditions at the boundary \( \Gamma \), it is necessary that the number of linearly independent equations of system (19) be equal to the number of unknowns. Let us single out linearly independent equations of system (19). The solution to the resulting joint SLAE will be an approximate solution to the generalized Riemann problem. We solve the system of equations (19) and determine the value of \( v^-(t, x) \) and \( v^+(t, x) \) on both sides of the hyperplane \( x_1 = 0 \).
Formulas (14) and (17) and the obtained dependences $v^-(t,x)$ and $v^+(t,x)$ give, up to $O(t^3)$ a complete solution to the generalized Riemann problem of disintegration of a discontinuity with conjugation conditions at the boundaries for the case of many spatial variables. Again, if the initial data $u_0(x)$ are the polynomials of degree not higher than the second, then the resulting solution is an exact solution to the Riemann problem.

5. Boundary conditions

Consider another problem that will be used later in the computational algorithm development. In the half-space $x_1 \leq 0$ one needs to find a solution to the initial-boundary-value problem (1), with initial data $(2)$. The solution must be continuous in the half-space $x_1 \leq 0$ and satisfy given boundary conditions on the plane $\Gamma : x_1 = 0$

$$L(u(t,x_1 = -0, x_2, \ldots, x_N)) = 0.$$  \hspace{1cm} (20)

We also assume that the initial data satisfy the boundary conditions.

Let $u(t,x)$ be the solution to this problem. Define the functions

$$u^-(t,x) = \begin{cases} u(t,x), & \text{if } t \geq 0, x_1 \leq 0 \\ 0, & \text{with the others} \end{cases}, \quad u_0^-(t,x) = \begin{cases} u_0, & \text{if } x_1 \leq 0 \\ 0, & \text{with the others} \end{cases}$$

$$v^-(t,x) = \theta(t)u(t,x_1 = -0, x_2, \ldots, x_N)$$

As shown above, $u^-(t,x)$, considered as a generalized function of $\mathcal{S}'$, satisfies the equation (12). The solution to this equation is given by formula (14). The vector $v^-$ accurately satisfies the equalities (16).

Combining the equalities (16) and boundary conditions (20), we obtain a system of linear algebraic equations, which must satisfy the values of the solution to the problem on the boundary $\Gamma : x_1 = 0$

$$\begin{cases}
L(v^-) = 0
\end{cases}.$$  \hspace{1cm} (21)

We solve the system of equations (21) and determine the value $v^-(t,x)$ of at $t > 0$ on the hyperplane $x_1 = 0$. The obtained dependences $v^-(t,x)$ and the formulas (14) represent, up to $O(t^3)$, a complete solution to the problem. Again, if the initial data $u_0(x)$ are the polynomials of degree not higher than the second, then the resulting solution is an exact solution to this problem.

In particular, if the boundary conditions have the form

$$\sum_{k: \lambda_{k_1} \geq 0} C_k v^- = 0.$$  \hspace{1cm} (22)

which means that any waves pass through the boundary without being reflected. Such conditions are called "transparent". By multiplying the left boundary condition equations by the left eigenvectors-rows of the matrix $A_1$, the system of equations (21) is given as
As can be seen in (23), since the system of equations (1) is hyperbolic and the matrix $A_1$ has a complete set of linearly independent left eigenvectors, the system of equations is compatible and the problem with boundary conditions has a unique solution.

6. Propagation of elastic waves in block-fractured medium

We will demonstrate the development of a computational algorithm based on the above results on the problem of the elastic waves propagation in an inhomogeneous block-fractured medium.

In the industrial experiments [10, 11], the fact is fixed that if a periodic vibration source (or several sources) is placed on the surface of an oil field, then after a few months a significant, sometimes up to 30-35%, increase in oil recovery is observed. The mechanisms leading to enhanced oil recovery are not clear. First of all, it is not clear how it is possible to deliver vibrational energy to an oil-containing formation avoiding its dispersion. If the depth of the oil reservoir is several kilometers, then even if dissipation in the geological environment is neglected, the energy of the generated elastic wave must propagate over the vast surface and the density of this energy at the depth of the oil-containing reservoir will be negligibly small and not capable of causing the observed effects.

One of the hypotheses under consideration is that the elastic wave propagates to the oil-bearing reservoir without scattering due to the fact that the geological rock consists of blocks, and they can mutually shift along the contact surface. We study this problem by the methods of mathematical modeling. The system of equations [4] describing the propagation of elastic waves

$$
\begin{align*}
\frac{\partial \sigma_{xx}}{\partial t} - (\lambda + 2\mu) \frac{\partial v_x}{\partial x} - \lambda \frac{\partial v_y}{\partial y} &= 0, \\
\frac{\partial \sigma_{yy}}{\partial t} - \lambda \frac{\partial v_x}{\partial x} - (\lambda + 2\mu) \frac{\partial v_y}{\partial y} &= 0, \\
\frac{\partial \sigma_{xy}}{\partial t} - \mu \frac{\partial v_x}{\partial x} - \mu \frac{\partial v_y}{\partial y} &= 0, \\
\frac{\partial v_x}{\partial x} - \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \sigma_{xy}}{\partial y} &= 0, \\
\frac{\partial v_y}{\partial x} - \frac{\partial \sigma_{xy}}{\partial x} - \frac{\partial \sigma_{yy}}{\partial y} &= 0,
\end{align*}
$$

(24)

can be represented in the form (1), where the matrices $A_1$ and $A_2$ have a complete set of linearly independent eigenvectors. Here $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$ are the components of the stress tensor, $v_x$ and $v_y$ are the components of the displacement velocity vector. We set the initial-boundary-value problem in the part $\Omega = [-30 < x < 30, -600 < y < 0]$. At the inner boundaries $\Gamma_1 : x = -15$ and $\Gamma_2 : x = 15$, we set the slippage conditions. That is, when passing through these boundaries, the components of the displacement velocity vector $v_x$ are continuous, the components of the force normal to the boundary on both sides of the boundary are equal in magnitude and oppositely directed, the tangential components of the forces on both sides of the boundary are 0:
\( v_x \left( t, (x, y) \in \Gamma^-_\gamma \right) - v_x \left( t, (x, y) \in \Gamma^+_\gamma \right) = 0 \)
\( \sigma_{xx} \left( t, (x, y) \in \Gamma^-_\gamma \right) - \sigma_{xx} \left( t, (x, y) \in \Gamma^+_\gamma \right) = 0 \)
\( \sigma_{xy} \left( t, (x, y) \in \Gamma^-_\gamma \right) = \sigma_{xy} \left( t, (x, y) \in \Gamma^+_\gamma \right) = 0. \) \hfill (25)

At the external boundaries \( x = -30, y = -600, x = 30 \) we set transparent boundary conditions.

At the boundary there is a vibration source that creates the force \( F \sin \omega t \) directed along the coordinate \( y \). This force is distributed along the boundary with the density \( P(x) \sin \omega t \), so that \( \int P(x) dx = F \). That is, the equalities must be performed at each point
\[ \sigma_{xy} = 0 \]
\[ \sigma_{yy} = P \sin \omega t. \] \hfill (26)

6.1. Numerical algorithm

Let's create in each of the subparts, into which \( \Omega \) is divided, a rectangular grid with sides parallel to the coordinate axes, so that the nodes lying on the internal borders coincide for both adjacent subdomains. Denote by \( h \) the smallest distance between neighboring grid nodes.

We assume that the force density from the vibration source \( P(x) \) is a piecewise linear function, equal to \( 0 \) at all nodes of the boundary \( y = 0, x \neq 0 \), and \( P(0) = \frac{F}{2h} \).

We define a uniform grid in time \( t_m = m \tau \), \( m = 0:1:M \). The grid step \( \tau \) must satisfy the condition \( \tau \leq \frac{h}{c} \), where \( c = \sqrt{\frac{\lambda + 2\mu}{\rho}} \) is the propagation velocity of longitudinal waves in the geological rock.

The quadratic polynomial \( H_p(x, y) \) in each rectangular cell of the grid, linear for each variable, taking a value 1 in this node and a value 0 in the remaining grid nodes, we associate with each grid node.

We approximate the solution \( u(t, x, y) = \left[ \sigma_{xx}, \sigma_{yy}, \sigma_{xy}, v_x, v_y \right]^T \) by a linear combination \( u(t, x, y) = \sum \sigma H_p(x, y) u^p(t) \). Then, to develop a solution to the initial-boundary-value problem for the equations of dynamic elasticity (24) with additional conditions (25) at short boundaries, it is necessary to find values at nodes \( u^p(t_{m+1}) \) on each time layer, with known values on the previous time layer \( u^p(t_m) \).

The solution in the internal nodes on the next time layer is found by the formulas, (14) or (17) respectively
\[ u(t_{m+1}, x) = \sum C_k u(t_m, x - \lambda_k \tau). \] \hfill (27)

On the right side of the formula (27), there are no terms related to values at internal and external boundaries. The time step \( \tau \leq \frac{h}{c} \) is chosen so that these terms are equal to 0.

The values on both sides of the internal boundaries are calculated in accordance with the formulas (19). For nodes at external borders on which “transparent” boundary conditions are set, the values at the next time layer are calculated in accordance with the formulas (23).

For nodes at external borders on which free boundary conditions are imposed, the solution on the next time layer is calculated using the formulas (21). At the boundary, the equalities (26) must be satisfied.
6.2. Computational Experiments
In the first series of experiments, the problem whether the block-fractured structure of the geological environment can lead to the fact that the vibroseismic effect of a source located on the surface reaches depths of the order of 1 km. and more without significant dispersion was examined. First, the dynamics of the propagation of elastic waves in a medium without internal boundaries was calculated. Figure 1 shows the distribution of the kinetic energy density $K = \frac{\rho}{2}(v_1^2 + v_2^2)$ of the elastic wave at time $t = 10$ sec. after the start of the vibration source with a frequency of 10 Hz. The model parameters correspond to the characteristic values for the geological environment. The disturbance created by the vibration source propagates in all directions. Qualitatively, the same energy distribution of the elastic disturbance takes place after the process is established.

![Figure 1. The lack of internal boundaries, $\omega = 10$ Hz (the source is on the right).](image)

The same computational experiments with the same parameters were performed on a model in which slippage conditions were set at the internal boundaries. Figures 2a, 2b, 2c show the distribution of kinetic energy density at time instants $t = 3$ sec., $t = 5$ sec. and $t = 10$ sec after the start of the vibration source. These results show that if slippage conditions are realized at the internal boundaries, then the disturbance created by the vibration source propagates in the channel between these boundaries and beyond these boundaries, practically does not penetrate.

![Figure 2. The distribution profile of the kinetic energy density at different points in time on the internal boundaries of the slip condition, $\omega = 10$ Hz.](image)
In the next series of computational experiments, we studied the change in the propagation of the elastic disturbance created by the vibration source with a change in the frequency of the vibration source operation. Slippage conditions are set at internal borders. Figures 3, 4, 5 show the distribution of the kinetic energy density at the instant of time $t = 10$ sec., at a source frequency of $\omega = 1$ Hz, $\omega = 10$ Hz, and $\omega = 100$ Hz.

**Figure 3.** At the internal boundaries of the slippage condition, $\omega = 1$ Hz (the source is on the right).

**Figure 4.** At the internal boundaries of the slippage condition, $\omega = 10$ Hz (the source is on the right).

**Figure 5.** At the internal boundaries of the slippage condition, $\omega = 100$ Hz (the source is on the right).

At a frequency of $\omega = 1$ Hz, the perturbation created by the vibration source propagates almost completely in the channel between the internal boundaries. With an increase in the frequency of the vibration source operation, a substantially larger part of the energy penetrates beyond the boundaries of the channel. If the frequency of the vibration source is $\omega = 100$ Hz or more, the energy propagation of the elastic disturbance qualitatively resembles the situation of the absence of internal boundaries. The outrage spreads in all directions, almost not noticing the presence of internal boundaries.

7. **Conclusion**

In this paper an approximate solution of the generalized Riemann problem with conjugation conditions on internal boundaries for hyperbolic systems of linear equations with first-order partial derivatives and piecewise constant coefficients, and an arbitrary number of spatial variables has been developed. An approximate solution of the generalized Riemann problem with conditions on external boundaries has also been given.

A computational algorithm for finding an approximate solution of the initial-boundary value problem for hyperbolic systems of linear equations with first-order partial derivatives and piecewise constant coefficients has been developed and implemented on the basis of these Riemann problem solutions. In this case, the statement of the problem allows the existence of internal boundaries on which the solution can have discontinuities in the values of the model variables and the given conditions that link the values of the variables on both sides of these boundaries must be fulfilled.

The constructed computational algorithm has been used to study the nature of the propagation of elastic waves generated by a periodically acting vibration source in a block-fractured geological environment. The existence of cracks is reflected in the model by the presence of internal boundaries at which the “slip conditions without friction” are satisfied. Numerical experiments showed that the elastic waves generated by the vibration source propagate in the channel between the cracks, practically not...
penetrating the boundaries of this channel. That is, the presence of cracks in the geological environment can serve as a mechanism that allows elastic waves to penetrate to a significant depth, avoiding dispersion.

A study of changes in the nature of the propagation of elastic waves in a block-fractured medium with a change in the frequency of the source has been conducted. The performed numerical experiments showed that at low frequencies the disturbance created by the vibration source propagates almost completely in the channel between the internal boundaries. With an increase in the vibration source frequency, a substantially larger part of the elastic disturbance energy penetrates beyond the boundaries of the channel. At high vibration source frequencies, the perturbation propagates in the medium in all directions, almost not noticing the internal boundaries.

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