ASYMPTOTIC STABILITY FOR A CLASS OF VISCOELASTIC EQUATIONS WITH GENERAL RELAXATION FUNCTIONS AND THE TIME DELAY

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Abstract. The goal of the present paper is to study the viscoelastic wave equation with the time delay

\[ |u_t|^p u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s) \Delta u(s) ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = b |u|^{p-2} u \]

under initial boundary value conditions, where \(\rho, b, \mu_1\) are positive constants, \(\mu_2\) is a real number, \(\tau > 0\) represents the time delay. By using the multiplier method together with some properties of the convex function \(s\), the explicit and general stability results of energy are proved under the general assumption on the relaxation function \(g\). This work generalizes and improves earlier results on the stability of the viscoelastic equations with the time delay in the literature.

1. Introduction

This paper concerns the following initial boundary value problem with the time delay

\[
\begin{aligned}
|u_t|^p u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s) \Delta u(s) ds \\
+ \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = b |u|^{p-2} u \\
(x,t) \in \Omega \times (0, \infty), \\
u_t(x,t-\tau) = f_0(x,t-\tau) \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x) \\
u(x,t) = 0 \\
(x,t) \in \partial \Omega \times [0, \infty),
\end{aligned}
\]

where \(\Omega \subset \mathbb{R}^N (N \geq 1)\) is a bounded domain with a smooth boundary \(\partial \Omega\), the unknown \(u := u(x,t)\) is a real valued function defined on \(\Omega \times (0, \infty)\), \(\rho, b, \mu_1\) are positive constants, \(\mu_2\) is a real number, \(\tau > 0\) represents the time delay, \(g\) is the kernel of the memory term, and the initial data \((u_0, u_1, f_0)\) are given functions belonging to suitable spaces. In addition, the following assumptions are given throughout this paper:

\((H_1)\) The relaxation function \(g : [0, \infty) \to (0, \infty)\) is a differentiable function satisfying

\[ 1 - \int_0^\infty g(s) ds = l > 0, \]
and there exists a $C^1$ function $G : (0, \infty) \to (0, \infty)$ which is strictly increasing and strictly convex $C^2$ function on $(0, r]$, $r \leq g(0)$, with $G(0) = G'(0) = 0$ such that
\[ g'(t) \leq -\zeta(t)G(g(t)) \text{ for } t \geq 0, \tag{1.3} \]
here $\zeta(t)$ is a positive non-increasing differentiable function.

$(H_2)$ $\rho$ and $p$ satisfy
\[ 0 < \rho \leq \frac{2}{N-2} \text{ for } N \geq 3 \text{ and } \rho > 0 \text{ for } N = 1, 2; \]
\[ 2 < p \leq \frac{2(N-1)}{N-2} \text{ for } N \geq 3 \text{ and } p > 2 \text{ for } N = 1, 2. \]

It is well known that time delay effects which often appear in many practical applications may induce some instabilities. Some results on the local existence and blow-up of solutions to a class of equations with delay have been obtained, the interested readers can refer to [4, 12, 13, 14, 23] and the reference therein. Nicaise and Pignotti [20] considered the wave equation with a delay term in the boundary condition as well as the wave equation with a delayed velocity term and mixed Dirichlet-Neumann boundary condition in a bounded and smooth domain, respectively. By introducing suitable energies and by using some observability inequalities, they proved an exponential stability of the solution in both cases under suitable assumptions. Kirane and Said-Houari [11] studied the following viscoelastic wave equation with a delay term in internal feedback
\[ u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0 \tag{1.4} \]
here $\mu_1, \mu_2$ are positive constants, under the initial boundary conditions of problem (1.1). They proved the existence of a unique weak solution for $\mu_2 \leq \mu_1$ by using the Faedo-Galerkin approximations together with some energy estimates. Provided that $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^1$ function satisfying $g(0) > 0$ and (1.2), and there exists a positive non-increasing differentiable function $\zeta(t)$ such that
\[ g'(t) \leq -\zeta(t)g(t) \text{ for } t \geq 0 \text{ and } \int_0^{+\infty} \zeta(t)dt = +\infty, \tag{1.5} \]
by establishing suitable Lyapunov functionals, they also obtained the corresponding exponential stability for $\mu_2 < \mu_1$ and for $\mu_2 = \mu_1$, respectively. Subsequently, Dai and Yang [7] proved an existence result of problem (1.4) without restrictions of $\mu_1, \mu_2 > 0$ and $\mu_2 \leq \mu_1$. Making use of the viscoelasticity term controls the delay term, they also proved an energy decay result for problem (1.4) in the case $\mu_1 = 0$
provided that $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a $C^1$ function satisfying $g(0) > 0$ and (1.2), and there exists a positive constant $\zeta$ such that
\[ g(t) \leq -\zeta g(t) \text{ for } t > 0. \tag{1.6} \]
Liu [15] generalized the results obtained by Kirane and Said-Houari [11]. That is, by the similar method in [11], they established a general energy decay result for problem (1.4) with $\tau(t)$ instead of $\tau$. In the absence of the source term $b|u|^{p-2}u$ in problem (1.1), Wu [21] proved an energy decay by the similar method in [11], and generalized the results to the time-varying delay in [22]. There are many papers concerning with the stability of viscoelastic equations with time delay, the interested readers may refer to [3, 8, 16] and the reference therein. However, the relaxation function $g$ are mainly limited to satisfying among the three conditions, which are
(1.5), (1.6) and that \( g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a differentiable function satisfying \( g(0) > 0 \) and (1.2), and there exists a positive function \( G \in C^1(\mathbb{R}^+) \) and \( G \) is linear or strictly increasing and strictly convex \( C^2 \) function on \( (0, r) \), \( r < 1 \), with \( G(0) = G'(0) = 0 \), such that

\[
g'(t) \leq -G(g(t)) \quad \text{for} \quad t > 0.
\]

Until recently, Chellaoua and Boukhatem [5] generalized the previous conditions that the relaxation function \( g \) satisfied, specifically investigated the following second-order abstract viscoelastic equation in Hilbert spaces

\[
\begin{aligned}
&u_{tt} + Au - \int_0^t g(s)Bu(t-s)ds + \mu_1 \dot{u}(t) + \mu_2 u(t-\tau) = 0 \quad t > 0, \\
&u(t-\tau) = f_0(t-\tau) \quad t \in (0, \tau), \\
&w(-t) = u_0(t), \quad w(0) = u_1 \\
&u(0) = u_0, \quad u_0(0) = u_1
\end{aligned}
\]

where \( A : D(A) \rightarrow H \) and \( B : D(B) \rightarrow H \) are a self-adjoint linear positive operator with domains \( D(A) \subset D(B) \subset H \) such that the embeddings are dense and compact. They established an explicit and general decay results of the energy solution by introducing a suitable Lyapunov functional and some properties of the convex functions under the condition \((H_1)\). Chellaoua and Boukhatem also addressed the stability results for the following second-order abstract viscoelastic equation in Hilbert spaces with time-varying delay in [6]

\[
\begin{aligned}
&u_{tt} + Au - \int_0^t g(t-s)Bu(s)ds + \mu_1 \dot{u}(t) + \mu_2 u(t-\tau(t)) = 0 \quad t > 0, \\
&u(t-\tau(t)) = f_0(t-\tau(t)) \quad t \in (0, \tau(t)), \\
&w(-t) = u_0(t), \quad w(0) = u_1 \\
&u(0) = u_0, \quad u_0(0) = u_1
\end{aligned}
\]

under the condition \((H_1)\). It is worth pointing out that Mustafa [17] first proposed the condition \((H_1)\) to study the decay rates for the following initial boundary value problem

\[
\begin{aligned}
&u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0 \quad \text{in} \quad \Omega \times (0, \infty), \\
&u = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \\
&u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad x \in \Omega.
\end{aligned}
\]

After that, many authors popularized the method used by Mustafa in [17]. The readers may see the references [2, 9, 10, 18, 19] to get more details.

Motivated by the above works, we are committed to considering the stability of problem (1.1) when the relaxation function \( g \) satisfies the condition \((H_1)\). To the best of our knowledge, there is no decay result for problem (1.1) where the relaxation functions satisfy \((H_1)\), although Wu [23] has investigated problem (1.1) and proved the blow-up result with nonpositive and positive initial energy. With minimal conditions on the relaxation function \( g \), we establish a general and optimal energy decay rates of problem (1.1) in Theorem 3.3. Our proof is based on the multiplier method and the similar arguments in [5, 17] but it is different from before since the presence of \( \Delta u_{tt} \), the time delay and the force source \( b|u|^{p-2}u \). The outline of this paper is as follows: In Section 2, we give some preliminary results. Section 3 is used to present the energy decay and its proof. In Section 4, we give the possible generalizations.

2. Preliminaries

Throughout this paper, we denote by \( \|\cdot\|_p \) and \( \|\nabla \cdot\|_2 \) the norm on \( L^p(\Omega) \) with \( 1 \leq p \leq \infty \) and \( H_0^1(\Omega) \), respectively. Let \( \lambda_1 \) be the first eigenvalue of the following
Assume that the solution \( u \) of (1) is a positive constant so that

\[ |z(x, \kappa, t)| \leq \tau \kappa \]

for \( x, \kappa \in (0, 1) \).

The symbol \( c_0 \) is the optimal embedding constant of \( H_0^1(\Omega) \hookrightarrow L^p(\Omega) \).

In order to the completeness of results, in what follows, we state the known results in [23]. Let us introduce the new variable

\[ z(x, \kappa, t) = u_t(x, t - \tau \kappa) \quad \text{for} \ x, \kappa \in (0, 1), \]

then problem (1.1) is equivalent to

\[
\begin{align*}
|u_t|^p u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t - s) \Delta u(s) ds &
+ \mu_1 u_t(x, t) + \mu_2 z(x, 1, t) = b|u|^{p-2} u \\
\tau z_t(x, \kappa, t) + z_\kappa(x, \kappa, t) = 0 & \quad (x, t) \in \Omega \times (0, \infty), \\
z(x, 0, t) = u_t(x, t) & \quad (x, t) \in \Omega \times (0, \infty), \\
z(x, \kappa, 0) = f_0(x, -\tau \kappa) & \quad x \in \Omega, \\
u_t(x, t - \tau) = f_0(x, t - \tau) & \quad (x, t) \in \Omega \times (0, \tau), \\
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) & \quad x \in \Omega, \\
u(x, t) = 0 & \quad (x, t) \in \partial \Omega \times [0, \infty).
\end{align*}
\]  

(2.1)

**Theorem 2.1** (Theorem 2.3 in [23]). Suppose that \(|\mu_2| \leq \mu_1\), (H_1) and (H_2) hold. Assume that \( u_0, u_1 \in H_0^1(\Omega) \) and \( f_0 \in L^2(\Omega \times (0, 1)) \), then there exists a unique solution \((u, z)\) of problem (2.1) satisfying

\[ u, u_t \in C([0, T); H_0^1(\Omega)), \quad z \in C([0, T); L^2(\Omega \times (0, 1))), \]

for \( T > 0 \).

Define the energy functional of problem (2.1) as follows

\[ E(t) = \frac{1}{\rho + 2} \|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2} \left( 1 - \int_0^t g(s) ds \right) \|\nabla u\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) \]

\[ + \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{\xi}{2} \int_\Omega \int_0^1 z^2(x, \kappa, t) d\kappa dx - \frac{b}{p} \|u\|^p_p, \]  

(2.2)

here \( \xi \) is a positive constant so that

\[ \tau |\mu_2| \leq \xi \leq \tau (2 \mu_1 - |\mu_2|) \]  

(2.3)

and

\[ (g \circ u)(t) = \int_0^t g(t - s) \|u(s) - u(t)\|_2^2 ds. \]

**Lemma 2.2** (Lemma 3.1 in [23]). \( E(t) \) is a non-increasing function and

\[ E'(t) \leq -\omega (\|u_t\|_2^2 + \|z(x, 1, t)\|_2^2) + \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) \|\nabla u\|_2^2 \]

\[ \leq -\omega (\|u_t\|_2^2 + \|z(x, 1, t)\|_2^2) \leq 0 \quad \text{for all} \ t \geq 0, \]  

(2.4)

where \( \omega = \min \left\{ \mu_1 - \frac{\xi}{2p}, \frac{\xi}{2p} - \frac{|\mu_2|}{2} \right\} > 0 \).
Lemma 2.3. If \( u \) is a solution for problem (2.1) and
\[
E(0) < E_1 = \frac{p - 2}{2p} \sigma_1^2, \quad l\|\nabla u_0\|_2^2 < \sigma_1^2,
\]
here \( \sigma_1 = b^{\frac{1}{p-2}} B_1^{\frac{p}{p-2}}, \) \( B_1 = \frac{\rho}{l^2} \), then there exists a positive constant \( \sigma_2 \) satisfying
\[ 0 < \sigma_2 < \sigma_1 \] such that
\[
l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \leq \sigma_2^2 \quad \text{for all } t \geq 0.
\]
(2.5)

Proof. Taking the combination of (2.2) and (1.2) with the embedding \( H_0^1(\Omega) \hookrightarrow L^p(\Omega) \), one has
\[
E(t) \geq \frac{l}{2} \|\nabla u\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) - \frac{bB_1^p}{p} \left( \frac{l}{2} \|\nabla u\|_2 \right)^p
\]
\[
\geq F \left( \sqrt{l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)} \right),
\]
(2.6)
where \( F(x) = \frac{1}{2} x^2 - \frac{bB_1^p}{p} x^p \) for \( x > 0 \). We know that \( F \) is strictly increasing in \((0, \sigma_1)\), strictly decreasing in \((\sigma_1, \infty)\), and \( F \) has a maximum at \( \sigma_1 \) with the maximum value \( E_1 \). Since \( E(0) < E_1 \), there exists a \( \sigma_2 < \sigma_1 \) such that \( F(\sigma_2) = E(0) \). Let \( \sigma_0 := \sqrt{l\|\nabla u\|_2^2} \), recall (2.6), then \( F(\sigma_0) \leq E(0) = F(\sigma_2) \), which implies \( \sigma_0 \leq \sigma_2 \) since the given condition \( \sigma_0^2 < \sigma_2^2 \). In order to complete the proof of (2.5), we suppose by contradiction that for some \( t_0 > 0 \),
\[
\sigma(t_0) = \sqrt{l\|\nabla u(t_0)\|_2^2 + (g \circ \nabla u)(t_0)} > \sigma_2.
\]
The continuity of \( \sqrt{l\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)} \) illustrates that we may choose \( t_0 \) such that \( \sigma(t_0) > \sigma(t_0) \), then we have
\[
E(0) = F(\sigma_2) < F(\sigma(t_0)) \leq E(t_0).
\]
This is a contradiction since Lemma 2.2.

Lemma 2.4. Under all the conditions of Lemma 2.3, there exists a positive constant \( D \) such that for all \( t \geq 0 \),
\[
\|u\|_p \leq DE(t) \leq DE(0),
\]
(2.7)
\[
\frac{1}{\rho + 2} \|u_t\|_{\rho+2}^2 + \frac{1}{2} \left( \int_0^t (g(s)ds) \|\nabla u\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) + \frac{1}{2} \|\nabla u_t\|_2^2 \right)
\]
\[
\frac{1}{2} \int_0^1 \int_0^1 z^2(x, \kappa, t)dx \leq DE(t) \leq DE(0).
\]
(2.8)

Proof. Using the embedding \( H_0^1(\Omega) \hookrightarrow L^p(\Omega) \), (2.2) and (2.5), we have
\[
\frac{b}{p} \|u\|_p^p \leq \frac{bB_1^p}{p} \left( \frac{l}{2} \|\nabla u\|_2 \right)^p \leq \frac{bB_1^p}{p} \left( l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \right)^{\frac{p-2}{2}} \left( E(t) + \frac{b}{p} \|u\|_p^p \right)
\]
\[
\leq \frac{2bB_1^p}{p} \left( l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \right)^{\frac{p-2}{2}} \left( E(t) + \frac{b}{p} \|u\|_p^p \right)
\]
\[
\leq \frac{2bB_1^p}{p} \sigma_2^{p-2} \left( E(t) + \frac{b}{p} \|u\|_p^p \right),
\]
which yields (2.7) with
\[
D = \frac{2pB_1^p \sigma_2^{p-2}}{p - 2bB_1^p \sigma_2^{p-2}} > 0.
\]
(2.9)
One has (2.8) by combining (2.7) with (2.2),

Lemma 2.5 (Lemma 4.1 in [2]). For \( u \in H^1_0(\Omega) \), we have for all \( t \geq 0 \),

\[
\int_{\Omega} \left( \int_0^t g(t-s)(\nabla u(s) - \nabla u(t))ds \right)^2 dx \leq C_\alpha(h_\alpha \circ \nabla u)(t)
\]

(2.10)

where, for any \( 0 < \alpha < 1 \),

\[
C_\alpha = \int_0^\infty \frac{g^2(s)}{\alpha g(s) - g'(s)} ds \quad \text{and} \quad h_\alpha(t) = \alpha g(t) - g'(t).
\]

(2.11)

Let us follow from the proof of Lemma 4.1 in [2], we have in fact

\[
\int_{\Omega} \left( \int_0^t g(t-s)(u(s) - u(t))ds \right)^2 dx \leq C_\alpha(h_\alpha \circ u)(t).
\]

(2.12)

Lemma 2.6 (Lemma 2.2 in [6]). There exist positive constants \( \gamma \) and \( t_1 \) such that

\[
g'(t) \leq -\gamma g(t) \quad \text{for} \ t \in [0, t_1].
\]

(2.13)

Lemma 2.7. Let \( u \) be a solution of problem (2.1), then the functional

\[
I_1(t) = \frac{1}{\rho + 1} \int_{\Omega} |u|^\rho u_t u dx + \int_{\Omega} \nabla u_t \nabla u dx,
\]

(2.14)

satisfies, for \( \varepsilon > 0 \) and for all \( t \geq 0 \),

\[
I_1'(t) \leq \frac{1}{\rho + 1} \|u_t\|_{\rho+2}^{\rho+2} - \left[ \frac{1}{\lambda_1} \left( 1 + \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_1} \right) \varepsilon \right] \|\nabla u\|_2^2 + \frac{1}{4\varepsilon} \|g\|_2 \int_{\Omega} \nabla u_t \nabla u dx + \frac{1}{4\varepsilon} \|u_t\|_2^2.
\]

(2.15)

Proof. Multiplying the first identity in problem (2.1) by \( u \), integrating on \( x \) over \( \Omega \), and then using integration by parts, we give

\[
\int_{\Omega} |u|^\rho u_t u dx + \int_{\Omega} \nabla u_t \nabla u dx
\]

\[
= -\left( 1 - \int_0^t g(s)ds \right) \|\nabla u\|_2^2 + \int_{\Omega} \nabla u(t) \int_0^t g(t-s)\nabla(u(s) - u(t))ds \quad (2.16)
\]

\[
- \int_{\Omega} \mu_1 u_t(x,t)u dx - \int_{\Omega} \mu_2 z(x,1,t)u dx + b\|u\|^p_p.
\]

Differentiating (2.14) on \( t \), and using (2.16), one has

\[
I_1'(t) = \frac{1}{\rho + 1} \|u_t\|_{\rho+2}^{\rho+2} + \int_{\Omega} |u|^\rho u_t u_t dx + \int_{\Omega} \nabla u_t \nabla u dx + \|\nabla u_t\|_2^2
\]

\[
= \frac{1}{\rho + 1} \|u_t\|_{\rho+2}^{\rho+2} - \left( 1 - \int_0^t g(s)ds \right) \|\nabla u\|_2^2
\]

\[
+ \int_{\Omega} \nabla u(t) \int_0^t g(t-s)\nabla(u(s) - u(t))ds
\]

\[
- \int_{\Omega} \mu_1 u_t(x,t)u dx - \int_{\Omega} \mu_2 z(x,1,t)u dx + b\|u\|^p_p + \|\nabla u_t\|_2^2.
\]

(2.17)

Applying Cauchy’s inequality with \( \varepsilon > 0 \) and \( \lambda_1 \|u\|_2^2 \leq \|\nabla u\|_2^2 \), it follows that

\[
- \int_{\Omega} \mu_1 u_t(x,t)u dx \leq \frac{\mu_1}{4\varepsilon} \|u_t\|_2^2 + \mu_1 \varepsilon \|u\|_2^2 \leq \frac{\mu_1}{4\varepsilon} \|u_t\|_2^2 + \frac{\mu_1 \varepsilon}{\lambda_1} \|\nabla u\|_2^2, \quad (2.18)
\]
\[- \int_\Omega \mu_2 z(x, 1, t)u dx \leq \frac{\|\mu_2\|}{4\varepsilon} \|z(x, 1, t)\|^2 + \frac{\|\mu_2\|}{4\varepsilon} \|\nabla u\|^2.\]  

(2.19)

It follows from Cauchy’s inequality with \(\varepsilon > 0\) and (2.10) that

\[\int_\Omega \nabla u(t) \int_0^t g(t-s)(\nabla u(s) - \nabla u(t))ds \leq \varepsilon \|\nabla u\|^2 + \frac{1}{4\varepsilon} C_\alpha(h \circ \nabla u)(t).\]  

(2.20)

Inserting (2.18)-(2.20) into (2.17), we obtain (2.15).

\[\square\]

Lemma 2.8. Under all the conditions of Lemma 2.3, let \(u\) be a solution of problem (2.1), then the functional

\[I_2(t) = \int_\Omega \left( \Delta u_t - \frac{1}{\rho + 1} |u_t|^{\rho} u_t \right) \int_0^t g(t-s)(u(t) - u(s))ds dx,\]  

(2.21)

satisfies, for \(\delta > 0\) and for all \(t \geq 0,\)

\[I_2(t) \leq B_1 \|\nabla u\|^2 + B_2 \|h_\alpha \circ \nabla u\| + \left[ B_3 - \int_0^t g(s)ds \right] \|\nabla u\|^2 + \delta \|z(x, 1, t)\|^2 - \int_0^t g(s)ds \cdot \frac{1}{\rho + 1} \|u_t\|^{\rho + 2}.\]  

(2.22)

here \(B_1, B_2\) and \(B_3\) are positive constants depending on \(\delta\) shown in (2.31).

Proof. Differentiating (2.21) on \(t\), and using the first identity in problem (2.1) and integration by parts, we have

\[
I_2'(t) = \int_\Omega \left( \Delta u_{tt} - |u_t|^{\rho} u_{tt} \right) \int_0^t g(t-s)(u(t) - u(s))ds dx \\
+ \int_\Omega \left( \Delta u_t - \frac{1}{\rho + 1} |u_t|^{\rho} u_t \right) \int_0^t g(t-s)(u(t) - u(s))ds dx \\
+ \int_\Omega \left( \Delta u_t - \frac{1}{\rho + 1} |u_t|^{\rho} u_t \right) \int_0^t g(t-s)u_t(t)ds dx \\
= \int_\Omega \nabla u(t) \int_0^t g(t-s)\nabla(u(t) - u(s))ds \\
- \int_\Omega \int_0^t g(t-s)\nabla u(s)ds \int_0^t g(t-s)\nabla(u(t) - u(s))ds dx \\
+ \int_\Omega \mu_1 u_t(x, t) \int_0^t g(t-s)(u(t) - u(s))ds dx \\
+ \int_\Omega \mu_2 z(x, 1, t) \int_0^t g(t-s)(u(t) - u(s))ds dx \\
- b \int_\Omega |u|^{p-2} u \int_0^t g(t-s)(u(t) - u(s))ds dx \\
- \int_\Omega \nabla u_t \int_0^t g_t(t-s)\nabla(u(t) - u(s))ds dx \\
- \int_\Omega \frac{1}{\rho + 1} |u_t|^{\rho} u_t \int_0^t g_t(t-s)(u(t) - u(s))ds dx \\
- \int_0^t g(s)ds \cdot \|\nabla u\|^2 - \int_0^t g(s)ds \cdot \frac{1}{\rho + 1} \|u_t\|^{\rho + 2} \\
= J_1 + J_2 + \cdots + J_8 + J_9.
\]  

(2.23)
It is direct from Cauchy’s inequality with $\delta > 0$ and (2.10) that

\[ J_1 \leq \delta \| \nabla u \|_2^2 + \frac{1}{4\delta} C_\alpha (h_\alpha \circ \nabla u)(t), \quad (2.24) \]

\[ J_2 \leq \delta \int_\Omega \left( \int_0^t g(t-s)|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|ds \right)^2 dx \]
\[ + \frac{1}{4\delta} \int_\Omega \left( \int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds \right)^2 dx \]
\[ \leq 2\delta \int_\Omega \left( \int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds \right)^2 dx + 2\delta(1-l)^2 \| \nabla u(t) \|_2^2 \quad (2.25) \]
\[ + \frac{1}{4\delta} \int_\Omega \left( \int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds \right)^2 dx \]
\[ \leq \left( 2\delta + \frac{1}{4\delta} \right) C_\alpha (h_\alpha \circ \nabla u)(t) + 2\delta(1-l)^2 \| \nabla u \|_2^2. \]

Cauchy’s inequality with $\delta > 0$ and (2.12) yield

\[ J_3 \leq \delta \| u \|_2^2 + \frac{\mu_1^2}{4\delta} C_\alpha (h_\alpha \circ u)(t) \leq \frac{\delta}{\lambda_1} \| u \|_2^2 + \frac{\mu_1^2}{4\delta \lambda_1} C_\alpha (h_\alpha \circ \nabla u)(t), \quad (2.26) \]

\[ J_4 \leq \delta \| z(x,1,t) \|_2^2 + \frac{\mu_2^2}{4\delta \lambda_1} C_\alpha (h_\alpha \circ \nabla u). \quad (2.27) \]

It follows from Cauchy’s inequality with $\delta > 0$, (2.12), the embedding $H^1_0(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$ and (2.8) that

\[ J_5 \leq b\delta \| u \|_{2^{(p-1)}}^{2(p-1)} + \frac{b}{4\delta} \int_\Omega \left( \int_0^t g(t-s)(u(t) - u(s))ds \right)^2 dx \]
\[ \leq b\delta \| u \|_{2^{(p-1)}}^{2(p-1)} + \frac{b}{4\delta} C_\alpha (h_\alpha \circ u)(t) \quad (2.28) \]
\[ \leq b\delta \varepsilon_\alpha^{2(p-1)} \left( \frac{2D}{l} E(0) \right)^{p-2} \| \nabla u \|_2^2 + \frac{b}{4\delta \lambda_1} C_\alpha (h_\alpha \circ \nabla u)(t). \]

Recalling the definition of $g'(t)$ in (2.11), and using Cauchy’s inequality with $\delta > 0$, (2.10) and Hölder’s inequality, one obtains

\[ J_6 = - \int_\Omega \nabla u_t \int_0^t g(t-s)\nabla (u(t) - u(s))ds dx \]
\[ + \int_\Omega \nabla u_t \int_0^t h_\alpha(t-s)\nabla (u(t) - u(s))ds dx \]
\[ \leq \delta \| \nabla u_t \|_2^2 + \frac{\alpha^2}{4\delta} \int_\Omega \left( \int_0^t g(t-s)\nabla (u(t) - u(s))ds \right)^2 dx \]
\[ + \delta \| \nabla u_t \|_2^2 + \frac{1}{4\delta} \int_\Omega \left( \int_0^t h_\alpha(t-s)\nabla (u(t) - u(s))ds \right)^2 dx \quad (2.29) \]
\[ \leq 2\delta \| \nabla u_t \|_2^2 + \frac{\alpha^2}{4\delta} C_\alpha (h_\alpha \circ \nabla u)(t) \]
\[ + \frac{1}{4\delta} \int_0^t h_\alpha(s)ds \int_0^t h_\alpha(t-s)\| \nabla (u(t) - u(s)) \|_2^2 ds \]
\[ \leq 2\delta \| \nabla u_t \|_2^2 + \left( \frac{\alpha^2}{4\delta} C_\alpha + \frac{\alpha(1-l) + g(0)}{4\delta} \right) (h_\alpha \circ \nabla u)(t). \]
Similarly, we get

\[ J_7 = -\int_\Omega \frac{1}{\rho + 1}|u_t|^\rho u_t \int_0^t \alpha g(t - s)(u(t) - u(s))ds dx \]

\[ + \int_\Omega \frac{1}{\rho + 1}|u_t|^\rho u_t \int_0^t h_\alpha(t - s)(u(t) - u(s))ds dx \]

\[ \leq \frac{\delta}{\rho + 1} \|u_t\|_{2(\rho + 1)}^{2(\rho + 1)} + \frac{\alpha^2}{4(\rho + 1)\delta} \int_\Omega \left( \int_0^t g(t - s)(u(t) - u(s))ds \right)^2 dx \]

\[ + \frac{\delta}{\rho + 1} \|u_t\|_{2(\rho + 1)}^{2(\rho + 1)} + \frac{1}{4(\rho + 1)\delta} \int_\Omega \left( \int_0^t h_\alpha(t - s)(u(t) - u(s))ds \right)^2 dx \]

\[ \leq \frac{2\delta}{\rho + 1} \|u_t\|_{2(\rho + 1)}^{2(\rho + 1)} + \frac{\alpha^2}{4(\rho + 1)\delta} C_\alpha (h_\alpha \circ u)(t) \]

\[ + \frac{1}{4(\rho + 1)\delta} \int_0^t h_\alpha(s)ds \int_0^t h_\alpha(t - s)\|u(t) - u(s)\|_2^2 ds \]

\[ \leq \frac{2\delta}{\rho + 1} C_s^{2(\rho + 1)} \left( 2\Delta E(0) \right)^2 \|\nabla u_t\|_2^2 \]

\[ + \left( \frac{\alpha^2}{4(\rho + 1)\delta} C_\alpha + \frac{\alpha(1 - l) + g(0)}{4(\rho + 1)\delta} \right) \frac{1}{\lambda_1} (h_\alpha \circ \nabla u)(t). \]

Inserting (2.25)-(2.30) into (2.23), one has

\[ I_2'(t) \leq B_1 \|\nabla u\|^2_2 + B_2 (h_\alpha \circ \nabla u)(t) + \left[ B_3 - \int_0^t g(s)ds \right] \|\nabla u_t\|^2_2 \]

\[ + \delta \|z(x, 1, t)\|^2_2 - \int_0^t g(s)ds \cdot \frac{1}{\rho + 1} \|u_t\|_{\rho + 2}, \]

with

\[ \begin{cases} 
B_1 = \delta + 2\delta (1 - l)^2 + b\delta C_s^{2(\rho - 1)} \left( 2\Delta E(0) \right)^p; \\
B_2 = \left[ \frac{1}{2\lambda_1} + \delta + \frac{\alpha^2}{4\lambda_1^2} + \frac{\mu^2}{4\lambda_1} + \frac{b}{4\lambda_1} + \frac{\alpha^2}{4} + \frac{\alpha^2}{4(\rho + 1)\delta \lambda_1} \right] C_\alpha \\
B_3 = \frac{\delta}{\lambda_1} + 2\delta + \frac{2\delta}{\rho + 1} C_s^{2(\rho + 1)} \left( 2\Delta E(0) \right)^2. 
\end{cases} \]

\[ \square \]

**Lemma 2.9.** The functional

\[ I_3(t) = \int_0^1 e^{-2\tau} \|z(x, \kappa, t)\|^2_2 d\kappa \]

(2.32)

satisfies for all \( t \geq 0 \),

\[ I_3'(t) = -2I_3(t) + \frac{1}{\tau} \|u_t\|^2_2 - \frac{e^{-2\tau}}{\tau} \|z(x, 1, t)\|^2_2. \]

(2.33)
Proof. Differentiating (2.32) on \( t \), and using the second identity in (2.1), it is direct that
\[
I'_3(t) = \int_0^1 2e^{-2\tau} \int_\Omega z(x, \kappa, t) z_k(x, \kappa, t) dxd\kappa \\
= - \frac{1}{\tau} \int_\Omega \int_0^1 \left( \frac{d}{d\kappa} e^{-2\tau} \kappa^2 (x, \kappa, t) + 2\tau e^{-2\tau} \kappa^2 (x, \kappa, t) \right) dxd\kappa \\
= -2I_3(t) + \frac{1}{\tau} \left[ \frac{\|u_t\|_2^2 - e^{-2\tau} \|z(x, 1, t)\|_2^2}{\tau} \right].
\]
(2.34)

\[
\square
\]

Lemma 2.10 (Lemma 3.4 in [17]). The functional
\[
I_4(t) = \int_0^t f(t - s) \|\nabla u(s)\|_2^2 ds
\]
(2.35)
is satisfies for all \( t \geq 0 \),
\[
I'_4(t) \leq 3(1 - l) \|\nabla u\|_2^2 - \frac{1}{2} [g \circ \nabla u](t),
\]
(2.36)
where \( f(t) = \int_t^\infty g(s) ds. \)

3. Stability results

In this section, we will present and prove the decay results of the energy functional \( E(t) \) based on the lemmas in Section 2. To begin with, we define a functional
\[
L(t) = ME(t) + \sum_{i=1}^3 N_i I_i(t),
\]
(3.1)
where \( M, N_1, N_2, N_3 \) are positive constants. The following lemma is shown to illustrate that \( L(t) \) is equivalent to \( E(t) \).

Lemma 3.1. Under all the conditions of Lemma 2.3, assume that \( M \) is enough large, then there exist two positive constants \( \beta_1 \) and \( \beta_2 \) such that
\[
\beta_1 E(t) \leq L(t) \leq \beta_2 E(t).
\]
(3.2)
Proof. Recalling the definition of \( I_1(t) \) in (2.14), using Young’s inequality and Cauchy’s inequality, and then applying the embedding \( H^1_0(\Omega) \hookrightarrow L^{\rho+2}(\Omega) \) and (2.8), it is not hard to give
\[
|I_1(t)| \leq \frac{1}{\rho + 2} \|u_t\|^{\rho+2}_{\rho+2} + \frac{1}{(\rho + 1)(\rho + 2)} \|u\|^{\rho+2}_{\rho+2} + \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{1}{2} \|\nabla u\|_2^2 \\
\leq \frac{1}{\rho + 2} \|u_t\|^{\rho+2}_{\rho+2} + \left[ \frac{c^{\rho+2}_{\rho+2}}{(\rho + 1)(\rho + 2)} \left( \frac{2D}{2} E(0) \right)^{\frac{\rho+2}{\rho+2}} \right] + \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \|\nabla u_t\|_2^2.
\]
(3.3)
Recalling the definition of $I_2(t)$ in (2.21), using integration by parts, Cauchy’s inequality, Young’s inequality and Hölder’s inequality, we give

\[
|I_2(t)| \leq \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{1}{2} \int_\Omega \left( \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds \right)^2 dx \\
+ \frac{1}{\rho+2} \|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)} \int_\Omega \left( \int_0^t g(t-s)(u(t) - u(s))ds \right)^{\rho+2} dx \\
\leq \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{\rho+2} \|u_t\|_{\rho+2}^{\rho+2} \\
+ \frac{1}{(\rho+1)(\rho+2)}(1-t)^{\rho+1} e_s^{\rho+2} \left( \frac{2D}{1} E(0) \right)^{\frac{s}{\rho}} (g \circ \nabla u)(t),
\]

(3.4)

here we have used

\[
\int_\Omega \left( \int_0^t g(t-s)(u(t) - u(s))ds \right)^{\rho+2} dx \\
\leq \int_\Omega \left( \int_0^t g(t-s)\nabla u_t (g(t-s))\nabla u (u(t) - u(s))ds \right)^{\rho+2} dx \\
\leq \left( \int_0^t g(s)ds \right)^{\rho+1} \int_0^t g(t-s)\|u(t) - u(s)\|_{\rho+2}^{\rho+2} ds \\
\leq (1-t)^{\rho+1} e_s^{\rho+2} \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^{\rho+2} ds \\
\leq (1-t)^{\rho+1} e_s^{\rho+2} \left( \frac{2D}{1} E(0) \right)^{\frac{s}{\rho}} (g \circ \nabla u)(t).
\]

Therefore, it follows from (2.8) that

\[
|L(t) - ME(t)| = \left| \sum_{i=1}^3 N_i I_i(t) \right| \leq CE(t),
\]

here $C$ is some positive constant. \(\square\)

**Lemma 3.2.** Under all the conditions of Lemma 2.3, for $|\mu_2| < \mu_1$, the functional $L(t)$ defined in (3.1) satisfies, for $t \geq t_1$

\[
L'(t) \leq -C_1 \|u_t\|_{\rho+2}^{\rho+2} - C_2 \|\nabla u_t\|_2^2 - 4(1-t)\|\nabla u\|_2^2 \\
+ N_i b \|u\|_p^p + \frac{1}{4} (g \circ \nabla u)(t) - 2N_3 \int_0^1 \|z(x, \kappa, t)\|_2^2 d\kappa.
\]

(3.6)

where $C_1, C_2$ are positive constants given in (3.8).

**Proof.** Taking the combination of (2.4), (2.15) and (2.22) with (2.33), recalling (2.11), and applying $g_1 = \int_0^{t_1} g(s)ds \leq \int_0^t g(s)ds$ for $t \geq t_1$, one has
\[ L'(t) \leq -M\omega(\|u_t\|_2^2 + \|z(x, 1, t)\|_2^2) + \frac{N_1}{\rho + 1}\|u_t\|_{\rho + 2}^{\rho + 2} + N_1 \left[ l - \left( 1 + \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_1} \right) \varepsilon \right] \|\nabla u\|_2^2 + \frac{N_1}{4\varepsilon} C_\alpha (h_\alpha \circ \nabla u)(t) + \frac{N_1}{4\varepsilon} \|z(x, 1, t)\|_2^2 + N_1 b\|u\|_p^p + N_1 \|\nabla u\|_2^2 + N_1 \frac{\mu_1}{\lambda_1} \|u_t\|_2^2 \\
+ N_2 B_1 \|\nabla u\|_2^2 + N_2 B_2 (h_\alpha \circ \nabla u)(t) + N_2 \left[ B_3 - \int_0^t g(s)ds \right] \|\nabla u_t\|_2^2 \\
+ N_2 \varepsilon^2 \|z(x, 1, t)\|_2^2 - N_2 \int_0^t g(s)ds \cdot \frac{1}{\rho + 1} \|u_t\|_{\rho + 2}^{\rho + 2} - 2N_3 l_3(t) + \frac{N_3}{\tau} \|u_t\|_2^2 - \frac{N_3 e^{-2\tau}}{\tau} \|z(x, 1, t)\|_2^2 \leq -C_1 \|u_t\|_{\rho + 2}^2 - C_2 \|\nabla u\|_2^2 - C_3 \|\nabla u\|_2^2 - C_4 \|z(x, 1, t)\|_2^2 - C_5 (h_\alpha \circ \nabla u)(t) + N_1 b\|u\|_p^p + \frac{\alpha M}{2} (g \circ \nabla u)(t) - C_6 \|u_t\|_2^2 - 2N_3 \int_0^1 \|z(x, \kappa, t)\|_2^2 d\kappa \]
\]

with
\[
\begin{align*}
C_1 &= N_2 g_1 \cdot \frac{1}{\rho + 1} - \frac{N_1}{\rho + 1}; \\
C_2 &= N_2 \left[ g_1 \left( \frac{\delta}{\lambda_1} + 2\delta + \frac{2\delta}{\rho + 1} c_{s_2}^{2(\rho + 1)} \left( \frac{2DE(0)}{\tau} \right)^{\frac{p}{2}} \right) \right] - N_1; \\
C_3 &= N_1 \left[ l - \left( 1 + \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_1} \right) \varepsilon \right] - N_2 \left[ \delta + 2\delta(1 - l)^2 + b\varepsilon_0 c_{s_2}^{2(p - 1)} \left( \frac{2DE(0)}{\tau} \right)^{p - 2} \right]; \\
C_4 &= \omega M + N_3 e^{-2\tau} - N_1 \frac{\mu_2}{\lambda_1} - N_2 \delta; \\
C_5 &= \frac{M}{2} - N_1 \frac{1}{\tau} + C_\alpha - N_2 \left[ \left( 1 + \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_1} \right) \varepsilon \right] - \frac{b}{4\delta^{\frac{3}{2}}} \left( \frac{1}{2(1 - l)^2} + b\varepsilon_0 c_{s_2}^{2(p - 1)} \left( \frac{2DE(0)}{\tau} \right)^{\frac{p}{2}} \right) C_\alpha; \\
C_6 &= \omega M - \frac{N_3}{\tau} - N_1 \frac{\mu_2}{\lambda_1} + \frac{\alpha(1 - l) + g(0)}{\mu_2};
\end{align*}
\]

where we have used the values of \(B_1, B_2\) and \(B_3\) defined in (2.31).

Next, we choose \(\delta\) such that
\[
\delta < \left\{ \frac{lg_1}{16 \left[ 1 + 2(1 - l)^2 + b c_{s_2}^{2(p - 1)} \left( \frac{2DE(0)}{\tau} \right)^{p - 2} \right]} , \frac{5g_1}{1024(1 - l)^2} \right\} \left[ \frac{lg_1}{8 \left( \frac{1}{\lambda_1} + 2 + \frac{2\delta}{\rho + 1} c_{s_2}^{2(\rho + 1)} \left( \frac{2DE(0)}{\tau} \right)^{\frac{p}{2}} \right) \right].
\]

Let us choose \(N_1 = \frac{3}{8} g_1 N_2\), then
\[
C_1 = N_2 g_1 \cdot \frac{1}{\rho + 1} - \frac{3}{8} g_1 N_2 \frac{1}{\rho + 1} = \frac{5}{8} g_1 N_2 \frac{1}{\rho + 1} > 0, \quad C_2 > 0.
\]

Let us fix
\[
\varepsilon = \frac{3l}{4} \frac{1}{1 + \frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_1}}.
\]
then
\[ C_3 = \frac{N_1 l}{4} - N_2 \left[ \delta + 2\delta(1-l)^2 + b\delta c_2^{2(p-1)} \left( \frac{2D}{l} E(0) \right)^{p-2} \right] > \frac{l}{32} g_1 N_2 > 0. \]
By taking \( N_2 = \frac{1}{8\delta(1-l)} \), we get
\[ C_3 > \frac{l}{32} g_1 N_2 = \frac{l g_1}{256\delta(1-l)} > 4(1-l). \]
Since \( g'(s) \leq 0 \), one has \( \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} \leq g(s) \), further we get
\[ \lim_{\alpha \to 0^+} \alpha C_\alpha = \lim_{\alpha \to 0^+} \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} ds = 0. \]
Thus, there exists \( 0 < \alpha_0 < 1 \) so that if \( \alpha < \alpha_0 \), then
\[ \alpha C_\alpha < \frac{1}{8 \left[ N_2 \left( \frac{1}{256} + 2\delta + \frac{\mu_2^2 + \mu_3^2}{4\delta\lambda_1} + \frac{b}{4\delta\lambda_1} + \frac{\alpha^2 + \alpha^2}{4(\rho + 1)\delta\lambda_1} \right) + N_1 \frac{l}{4\varepsilon} \right]}. \]
Let us choose \( M \) sufficiently large such that for \( \alpha = \frac{1}{2M} \),
\[ C_5 = \frac{M}{2} - N_2 \left[ \alpha(1-l) + g(0) \right] \frac{\alpha(1-l) + g(0)}{4(\rho + 1)\delta\lambda_1} \]
\[ C_4 = \omega M + N_3 \frac{e^{-2\tau}}{\tau} - N_1 \frac{\mu_2}{4\varepsilon} - N_2 \delta > 0, \]
\[ C_6 = \omega M - \frac{N_3}{\tau} - N_1 \frac{\mu_1}{4\varepsilon} > 0, \]
here we have used \( \omega > 0 \) given in Lemma 2.2 since \( |\mu_2| < \mu_1 \).

Based on the above discussion, one has from (3.7)
\[ L'(t) \leq -C_1 \|u_r\|_p^p - C_2 \|\nabla u_r\|_2^2 - 4(1-l)\|\nabla u\|_2^2 \]
\[ + N_1 b\|u\|_p^p + \frac{1}{4}(g \circ \nabla u)(t) - 2N_3 \int_0^1 \|z(x, \kappa, t)\|_2^2 d\kappa. \]

Now, we give the following stability results.

**Theorem 3.3.** For \( |\mu_2| < \mu_1 \), suppose that (H1), (H2) hold and
\[ E(0) < E_1, \quad l\|\nabla u_0\|_2^2 < \sigma_1^2, \]
then there exist positive constants \( k_1, k_2, k_3 \) and \( k_4 \) such that the solution of problem (1.1) satisfies for all \( t \geq t_1 \),
\[ E(t) \leq \begin{cases} k_1 e^{-k_2 \int_0^t \zeta(s) ds} & \text{for } G \text{ linear}; \\
 k_4 G_1^{-\frac{1}{2}} \left( k_3 \int_0^t \zeta(s) ds \right) & \text{for } G \text{ nonlinear}; \end{cases} \]
where \( E_1 \) and \( \sigma_1 \) are shown in Lemma 2.3, \( G_1(t) = \int_t^1 \frac{1}{sG(s)} ds \) is strictly decreasing and convex in \((0, r] \) with \( \lim_{t \to 0^+} G_1(t) = +\infty. \)
Proof. Using (2.13) and (2.4), one has, for $t \geq t_1$,
\[
\int_0^{t_1} g(s) \|\nabla u(t) - \nabla u(t - s)\|^2 ds \leq -\frac{1}{\gamma} \int_0^{t_1} g'(s) \|\nabla u(t) - \nabla u(t - s)\|^2 ds \leq -cE'(t)
\] (3.9)
here $c$ is used to denote a generic positive constant throughout this proof. Define a functional $F(t)$ that is obviously equivalent to $E(t)$ as follows
\[
F(t) = L(t) + cE(t),
\]
then based on (3.6), (2.2), (3.9), for some $m > 0$ and for any $t \geq t_1$, we have
\[
F'(t) \leq -C_1 \|u_t\|_{p+2}^{p+2} - C_2 \|\nabla u_t\|^2 - 4(1 - t) \|\nabla u\|^2
+ N_1 b \|u\|^p + \frac{1}{4} (g \circ \nabla u)(t) - 2N_3 \int_0^1 \|z(x, \kappa, t)\|^2 d\kappa + cE'(t)
\leq -mE(t) - \left(\frac{bc}{p} - N_1 b\right) \|u\|^p + c(g \circ \nabla u)(t) + cE'(t)
\leq -mE(t) + c \int_{t_1}^t g(s) \|\nabla u(t) - \nabla u(t - s)\|^2 ds,
\] (3.10)
here we have chosen $N_1$ so small that $\frac{bc}{p} - N_1 b > 0$.

In what follows, we will discuss in two cases.

CASE 1: $G$ IS LINEAR Multiplying (3.10) by $\zeta(t)$, using $(H_1)$ and (2.4), one gives
\[
\zeta(t)F'(t) \leq -m\zeta(t)E(t) + c\zeta(t) \int_{t_1}^t g(s) \|\nabla u(t) - \nabla u(t - s)\|^2 ds
\leq -m\zeta(t)E(t) - c \int_{t_1}^t g'(s) \|\nabla u(t) - \nabla u(t - s)\|^2 ds
\leq -m\zeta(t)E(t) - cE'(t),
\]
which implies
\[
(\zeta(t)F(t) + cE(t))' \leq -m\zeta(t)E(t) \quad \text{for } t \geq t_1.
\]
Integrating the above inequality over $(t_1, t)$, and using the fact that $\zeta(t)F(t) + cE(t)$ is equivalent to $E(t)$, one has
\[
E(t) \leq k_1 e^{-k_2 \int_{t_1}^t \zeta(s) ds} \quad \text{for } t \geq t_1,
\]
here $k_1$ and $k_2$ are constants.

CASE 2: $G$ IS NONLINEAR Define a functional
\[
H(t) = L(t) + I_4(t).
\]
Taking the combination of Lemma 3.1 and the non-negativity of $E(t)$ obtained by Lemma 2.4 with the definition of $I_4(t)$ in (2.35), it is not difficult to get the non-negativity of $H(t)$. It follows from (3.6) and (2.36) that for some $m_1 > 0$ and $t \geq t_1$,
\[
H'(t) = L'(t) + I_4'(t)
\leq -C_1 \|u_t\|_{p+2}^{p+2} - C_2 \|\nabla u_t\|^2 - (1 - t) \|\nabla u\|^2
+ N_1 b \|u\|^p - \frac{1}{4} (g \circ \nabla u)(t) - 2N_3 \int_0^1 \|z(x, \kappa, t)\|^2 d\kappa \leq -m_1 E(t).
\]
Integrating the above inequality over \((t_1, t)\) yields

\[
m_1 \int_{t_1}^{t} E(s) \, ds \leq H(t_1) - H(t) \leq H(t_1),
\]

which implies

\[
\int_0^\infty E(s) \, ds < +\infty. \quad (3.11)
\]

Define

\[
\lambda(t) = p \int_{t_1}^{t} \| \nabla u(t) - \nabla u(t - s) \|^2_2 \, ds,
\]

by using (2.8), then we give

\[
\lambda(t) \leq 2p \int_0^t \left( \| \nabla u(t) \|^2_2 + \| \nabla u(t - s) \|^2_2 \right) \, ds \\
\leq 8pD \int_0^t \left( E(t) + E(t - s) \right) \, ds \leq \frac{16pD}{t} \int_0^t E(t - s) \, ds \\
= \frac{16pD}{t} \int_0^t E(s) \, ds \leq \int_0^\infty E(s) \, ds < +\infty.
\]

Thus, we can choose \(p\) so small that for \(t \geq t_1\),

\[
\lambda(t) < 1. \quad (3.12)
\]

It is direct that

\[
G(\theta z) \leq \theta G(z) \quad \text{for } 0 \leq \theta \leq 1 \text{ and } z \in (0, r], \quad (3.13)
\]

since \(G\) is strictly convex on \((0, r]\) and \(G(0) = 0\). Based on (1.3), (3.12), (3.13) and Jensen’s inequality, one gives

\[
I(t) = \frac{1}{p \lambda(t)} \int_{t_1}^{t} \lambda(t)(-g'(s))p\| \nabla u(t) - \nabla u(t - s) \|^2_2 \, ds \\
\geq \frac{1}{p \lambda(t)} \int_{t_1}^{t} \lambda(t)\zeta(s)G(g(s))p\| \nabla u(t) - \nabla u(t - s) \|^2_2 \, ds \\
\geq \frac{\zeta(t)}{p \lambda(t)} \int_{t_1}^{t} \tilde{G}(\lambda(t)g(s))p\| \nabla u(t) - \nabla u(t - s) \|^2_2 \, ds \\
\geq \frac{\zeta(t)}{p} \tilde{G} \left( p \int_{t_1}^{t} g(s)\| \nabla u(t) - \nabla u(t - s) \|^2_2 \, ds \right),
\]

which yields

\[
\int_{t_1}^{t} g(s)\| \nabla u(t) - \nabla u(t - s) \|^2_2 \, ds \leq \frac{1}{p \tilde{G}^{-1}} \left( \frac{pI(t)}{\zeta(t)} \right), \quad (3.14)
\]

where \(G\) has an extension \(\tilde{G}\) which is a strictly increasing and strictly convex \(C^2\) function on \((0, +\infty)\) as in Remark 2.1 [6]. Therefore, (3.10) becomes

\[
F'(t) \leq -mE(t) + \frac{\zeta(t)}{p} \tilde{G}^{-1} \left( \frac{pI(t)}{\zeta(t)} \right). \quad (3.15)
\]

Let us define the functional

\[
F_1(t) = \tilde{G} \left( \frac{E(t)}{E(0)} \right) F(t) + E(t)
\]
with $0 < r_1 < r$, then $F_1$ is equivalent to $E$ and
\[ F_1'(t) = \frac{r_1E'(t)}{E(0)}G' \left( \frac{r_1E(t)}{E(0)} \right) F(t) + G' \left( \frac{r_1E(t)}{E(0)} \right) F'(t) + E'(t) \]
\[
\leq -mE(t)G' \left( \frac{r_1E(t)}{E(0)} \right) + \frac{c}{p}G^{-1} \left( \frac{pI(t)}{\zeta(t)} \right) G' \left( \frac{r_1E(t)}{E(0)} \right) + E'(t) \quad (3.16)
\]

by using (3.15), (2.4), $G' > 0$ and $G'' > 0$. Let $G^*$ be the convex conjugate of $G$ in the sense of Young in [1], which is given by
\[
G^*(s) = s(G^*)^{-1}(s) - G \left[ (G^*)^{-1}(s) \right] \quad (3.17)
\]

and it satisfies the following Young’s inequality
\[
AB \leq G^*(A) + G(B). \quad (3.18)
\]

Choosing
\[
A = G' \left( \frac{r_1E(t)}{E(0)} \right) \quad \text{and} \quad B = G^{-1} \left( \frac{pI(t)}{\zeta(t)} \right),
\]

then using (3.18), (3.17) and the non-negativity of $G$, (3.16) becomes
\[
F_1'(t) \leq -mE(t)G' \left( \frac{r_1E(t)}{E(0)} \right) + \frac{c}{p}G^{-1} \left( \frac{pI(t)}{\zeta(t)} \right) G' \left( \frac{r_1E(t)}{E(0)} \right) + E'(t)
\]
\[
\leq -mE(t)G' \left( \frac{r_1E(t)}{E(0)} \right) + \frac{c}{p}G^* \left( \frac{r_1E(t)}{E(0)} \right) + \frac{I(t)}{\zeta(t)} + E'(t)
\]
\[
\leq -mE(t)G' \left( \frac{r_1E(t)}{E(0)} \right) + \frac{c}{p}r_1E(t)G' \left( \frac{r_1E(t)}{E(0)} \right) + \frac{I(t)}{\zeta(t)} + E'(t). \quad (3.19)
\]

Note that (2.4) implies
\[
I(t) \leq \int_{t_1}^{t} -g'(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \leq -2E'(t),
\]

then one has
\[
\zeta(t)F_1'(t) \leq -m\zeta(t)E(t)G' \left( \frac{r_1E(t)}{E(0)} \right) + \frac{c}{p}r_1E(t)G' \left( \frac{r_1E(t)}{E(0)} \right) - cE'(t) \quad (3.20)
\]

by multiplying (3.19) by $\zeta(t)$ and by using the fact
\[
G' \left( \frac{r_1E(t)}{E(0)} \right) = G' \left( \frac{r_1E(t)}{E(0)} \right).
\]

Define the functional $F_2(t) = \zeta(t)F_1(t) + cE(t)$ which is equivalent to $E(t)$, which means
\[
\gamma_1F_2(t) \leq E(t) \leq \gamma_2F_2(t) \quad (3.21)
\]

for some $\gamma_1$ and $\gamma_2$. Under a suitable choice of $r_1$ and for a positive constant $k$, we have
\[
F_2'(t) \leq -k\zeta(t)E(t)G' \left( \frac{r_1E(t)}{E(0)} \right) = -k\zeta(t)G_2 \left( \frac{E(t)}{E(0)} \right) \quad (3.22)
\]
with $G_2(t) = tG'(r_1t)$. Obviously, $G_2$ and $G_2'$ are positive in $(0,1]$ since

$$G_2'(t) = G'(r_1t) + r_1 t^2 G''(r_1t)$$

and the convexity of $G$ in $(0,r]$, (3.22) and (3.21) imply

$$-k\zeta(t)G_2(E(t)) \leq -k_3\zeta(t)G_2(E(t))$$

(3.23)

with $k_3 = \frac{k_3^2}{E(0)}$. Setting $R(t) = \frac{G_2(E(t))}{E(0)}$, and then integrating (3.23) over $(t_1,t)$, one has

$$\int_{t_1}^{t} -\frac{R'(s)}{G_2(R(s))} ds \geq \int_{t_1}^{t} k_3\zeta(s) ds.$$

Since $r_1 R(t_1) < r$, we have

$$G_1(r_1 R(t)) = \int_{r_1 R(t)}^{r_1 R(t_1)} \frac{1}{sG_2'(s)} ds \geq k_3\int_{t_1}^{t} \zeta(s) ds.$$

It is noted that $G_1$ is strictly decreasing function on $(0,r]$ and $\lim_{t \to 0} G_1(t) = +\infty$ in Theorem 3.3, then

$$R(t) \leq \frac{1}{r_1} G_1^{-1}\left(k_3\int_{t_1}^{t} \zeta(s) ds\right).$$

Since $R(t)$ is equivalent to $E(t)$, further one obtains

$$E(t) \leq k_4 G_1^{-1}\left(k_3\int_{t_1}^{t} \zeta(s) ds\right)$$

with $k_4 = \frac{1}{r_1}$.

This completes the proof of this theorem.  

\[\square\]

4. Possible generalizations

Our results can be generalized to the following initial boundary value problem with the time-varying delay $\tau(t)$

\[
\begin{cases}
|u_t|^p u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds \\
+\mu_1 u_t(x,t) + \mu_2 u_t(x,t - \tau(t)) = b|u|^{p-2} u \\
(x,t) \in \Omega \times (0,\infty), \\
u_t(x,t - \tau(0)) = f_0(x,t - \tau(0)) \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \\
u(x,t) = 0 \\
(x,t) \in \partial\Omega \times [0,\infty),
\end{cases}
\]

equipped with the following assumptions in addition to (H$_1$) and (H$_2$):

(A$_1$) the function $\tau \in W^{2,\infty}([0,T])$, for all $T > 0$ such that

\[
0 < \tau_0 \leq \tau(t) \leq \tau_1 \quad \text{for all } t > 0,
\]

\[
\tau'(t) \leq d < 1 \quad \text{for all } t > 0,
\]

where $\tau_0$ and $\tau_1$ are positive numbers.

(A$_2$) the coefficients of delay and dissipation satisfy

\[
|\mu_2| \leq \frac{2(1-d)}{2-d} |\mu_1|.
\]
The above assumptions in fact are given in [6]. To be more precise, we need to define the new energy functional

\[ E(t) = \frac{1}{\rho + 2} \| u_t \|_{\rho+2}^2 + \frac{1}{2} \left( 1 - \int_0^t g(s) ds \right) \| \nabla u \|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) \]

\[ + \frac{1}{2} \| \nabla u \|_2^2 + \frac{\xi}{2} \tau(t) \int_0^1 z^2(x, \kappa, t) d\kappa dx - \frac{b}{p} \| u \|_p^p, \]

where \( \xi \) satisfies

\[ \frac{|\mu_2|}{1 - d} \leq \xi \leq 2\mu_1 - |\mu_2|. \]

We also need to replace \( I_3(t) \) in Lemma 2.9 by

\[ I_3(t) = \int_0^1 e^{-2\tau(t)z}|z(x, \kappa, t)|^2 d\kappa, \]

further, one has

\[ I_3(t) \leq -2I_3(t) - \frac{(1-d) e^{-2\tau_1}}{\tau_1} \| z(x, 1, t) \|_2^2 + \frac{1}{\tau_0} \| u_t \|_2^2 \]

as shown in Lemma 2.8 of [6]. Based on the above changes, we believe that it is possible to get the similar results in Theorem 3.3 by following the steps in this paper.

In addition, we have a conjecture that our results can also be extended to the following initial boundary value problem:

\[
\begin{align*}
|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s) \Delta u(s) ds \\
+ \mu_1 |u_t(x, t)|^{m-2} u_t(x, t) \\
+ \mu_2 |u_t(x, t-\tau)|^{m-2} u_t(x, t-\tau) = b|u|^{p-2} u \\
(x, t) \in \Omega \times (0, \infty), \\
u_t(x, t-\tau) = f_0(x, t-\tau) \\
u(x, t) = u_0(x), \quad u_t(x, 0) = u_1(x) \\
u(x, t) = 0 \\
(x, t) \in \partial \Omega \times [0, \infty). 
\end{align*}
\]

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ASYMPTOTIC STABILITY FOR A CLASS OF VISCOELASTIC QUATIONS

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