Admissible pair of spaces for not correctly solvable linear differential equations

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Abstract

We consider the differential equation

\[-y'(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R}, \tag{0.1}\]

where \(f \in L_p(\mathbb{R}), \, p \in [1, \infty), \) and \(0 \leq q \in L^\text{loc}_1(\mathbb{R}), \) \(\int_{-\infty}^{0} q(t) \, dt = \int_{0}^{\infty} q(t) \, dt = \infty,\)

\[q_0(a) = \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} q(t) \, dt = 0 \quad \text{for any} \quad a \in (0, \infty).\]

Under these conditions, the equation (0.1) is not correctly solvable in \(L_p(\mathbb{R})\) for any \(p \in [1, \infty).\) Let \(q^*(x)\) be the Otelbaev-type average of the function \(q(t), \, t \in \mathbb{R},\) at the point \(t = x; \) \(\theta(x)\) be a continuous positive function for \(x \in \mathbb{R},\) and

\[L_{p,\theta}(\mathbb{R}) = \{ f \in L^\text{loc}_p(\mathbb{R}) : \int_{-\infty}^{\infty} |\theta(x) f(x)|^p \, dx < \infty \},\]

\[\|f\|_{L_{p,\theta}(\mathbb{R})} = \left( \int_{-\infty}^{\infty} |\theta(x) f(x)|^p \, dx \right)^{1/p}\]

We show that if there exists a constant \(c \in [1, \infty),\) such that the inequality

\[c^{-1} q^*(x) \leq \theta(x) \leq c q^*(x)\]

holds for all \(x \in \mathbb{R},\) then under some additional conditions for \(q\) the pair of spaces \(\{L_{p,\theta}(\mathbb{R}); L_p(\mathbb{R})\}\) is admissible for the equation (0.1).

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1 Introduction

In the present paper we consider the equation
\[ -y'(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R}, \]  
where \( f \in L_p(L_p := L_p(\mathbb{R})), p \in [1, \infty), \) and
\[ 0 \leq q \in L^1_{\text{loc}}(\mathbb{R}). \]  
By a solution of (1.1) we mean any absolutely continuous function \( y(x) \) that satisfies (1.1) almost everywhere on \( \mathbb{R} \). By \( \theta \) we denote a continuous positive function of \( x \in \mathbb{R} \) and, we set
\[ L^p_{\theta}(\mathbb{R}) = \{ f \in L^1_{\text{loc}}(\mathbb{R}) : \int_{-\infty}^{\infty} |\theta(x)f(x)|^p \, dx < \infty \}, \]  
with
\[ \| f \|_{L^p_{\theta}(\mathbb{R})} = \left( \int_{-\infty}^{\infty} |\theta(x)f(x)|^p \, dx \right)^{1/p}. \]
Furthermore, for \( \theta \equiv 1, x \in \mathbb{R} \), we write \( L^p, \| \cdot \|_p \) in place of \( L^p_{\theta} \) and \( \| \cdot \|_{p,\theta} \), respectively.

**Definition 1.1** We say that the pair of spaces \( \{ L^p_{\theta}(\mathbb{R}); L^p(\mathbb{R}) \} \) (hence \( \{ L^p_{\theta}; L^p \} \)) is admissible for equation (1.1) if
(i) for any \( f \in L^p(\mathbb{R}) \), the equation (1.1) has a unique solution \( y \in L^p_{\theta} \);
(ii) there exists an absolute positive constant \( c(p) \in (0, \infty) \) such that the solution of (1.1) \( y \in L^p_{\theta} \) satisfies the inequality
\[ \| y \|_{p,\theta} \leq c(p)\| f \|_p, \quad \text{for any} \quad f \in L^p. \]  
Moreover, if \( \theta \equiv 1 \) and conditions (i)–(ii) are satisfied, we say that equation (1.1) is correctly solvable in \( L^p \).

The problem of finding minimal requirements for \( q(\cdot) \) under which the conditions (i)–(ii) are satisfied is studied in the case \( \theta \equiv 1 \) in [2, 3]. The main result of these works can be summarized as follows.

**Theorem 1.2** [2] Let \( p \in [1, \infty) \). Equation (1.1) is correctly solvable in \( L^p \) if and only if there is an \( a \in (0, \infty) \) such that
\[ q_0(a) = \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} q(t) \, dt > 0. \]  
Thus, if conversely
\[ q_0(a) = 0 \quad \text{for all} \quad a \in (0, \infty), \]  
then for any \( p \in [1, \infty) \) equation (1.1) is not correctly solvable in \( L^p \). In this paper we continue to study the problems from [2, 3]. Our aim is to find an analogue of Theorem 1.2 under the condition (1.6). To formulate the question precisely, we need a new statement, which is
equivalent to Theorem 1.2 (and easily follows from it, see Section 4). To this end, we introduce the auxiliary function \( d(x), \ x \in \mathbb{R} \) (see [3]). Assume that in addition to (1.2) the following condition is satisfied:

\[
\int_{-\infty}^{\infty} q(t) \, dt = \infty. \tag{1.7}
\]

Then, for a fixed \( x \in \mathbb{R} \), define

\[
d(x) \overset{\text{def}}{=} \inf_{d \geq 0} \{ d : \int_{x-d}^{x+d} q(\xi) \, d\xi = 2 \}. \tag{1.8}
\]

We note that functions of type \( d(x), \ x \in \mathbb{R} \), have been introduced and systematically used by M. Otelbaev (see [4]).

It follows immediately from (1.8) that the function

\[
q^*(x) = \frac{1}{d(x)}, \ x \in \mathbb{R}, \tag{1.9}
\]

can be interpreted as the Steklov-type average \( Q(x, h) \) of the function \( q(t) \) at the point \( t = x \) with particular averaging step \( h = d(x) \):

\[
Q(x, h) \overset{\text{def}}{=} \frac{1}{2h} \int_{x-h}^{x+h} q(t) \, dt, \ h > 0 \implies \quad Q(x, d(x)) = \frac{1}{2d(x)} \int_{x-d(x)}^{x+d(x)} q(\xi) \, d\xi = \frac{1}{d(x)} = q^*(x).
\]

We call the function \( q^*(x) \), which plays a significant role in this work, the Otelbaev-type average of the function \( q(x), \ x \in \mathbb{R} \).

**Theorem 1.3** (see Section 4). Let \( p \in [1, \infty) \) and suppose (1.2) holds. Then equation (1.1) is correctly solvable in \( L_p \) if and only if (1.7) holds and \( d_0 < \infty \) (equivalently, \( q_0^* > 0 \)). Here

\[
d_0 = \sup_{x \in \mathbb{R}} d(x) \left( q_0^* = \inf_{x \in \mathbb{R}} q^*(x) \right). \tag{1.10}
\]

**Corollary 1.4** (see Section 4). Assume the conditions (1.2) and (1.7) are satisfied. Then \( d_0 < \infty \) if and only if \( q_0(a) > 0 \) for some \( a \in (0, \infty) \).

Turning back to our problem, we conclude (see Theorem 1.3) that the equation (1.1) is not correctly solvable in \( L_p \) in two typical cases:

1) (1.7) holds, but \( d_0 = \infty \),
2) (1.7) does not hold.

In the present paper we study the first case. The second case will be considered in a forthcoming paper.

We are now in a position to formulate our problem in precise terms. Consider the equation (1.1), with a function \( q \) that satisfies the conditions (1.2), (1.6), (1.7) and \( p \in [1, \infty) \). The task is to find a positive continuous function \( \theta(x), \ x \in \mathbb{R} \) such that the pair \( \{ L_p, \theta \}; L_p \) is admissible
for equation (1.1). The solution of this problem (under some additional requirements on \( q \)) is presented in this paper.

The paper is divided into Sections 2 through 5. In Section 2 we present preliminaries, Section 3 lists all our results, in Section 4 all proofs are presented, and, finally, Section 5 is devoted to examples.

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2 Preliminaries

In the sequel (without special mention), we assume that the conditions (1.2), (1.7) are satisfied. The symbols \( c, \cdots, c_1, c_2, \cdots \) stand for absolute positive constants, the values of which are not essential for the exposition, and can even change within a single chain of calculations. One more definition is in order.

Definition 2.1 Let \( \varphi(x) \) and \( \psi(x) \), \( x \in \mathbb{R} \), be continuous positive functions defined on interval \((a, b)\), \(-\infty \leq a < b \leq \infty\). We say that \( \varphi \) and \( \psi \) are weakly equivalent on \((a, b)\), and write \( \varphi(x) \sim \psi(x) \), \( x \in (a, b) \), if there exists an absolute constant \( c \geq 1 \) such that
\[
    c^{-1} \varphi(x) \leq \psi(x) \leq c \varphi(x) \quad \text{for all} \quad x \in (a, b).
\]

Theorem 2.2 [5] The function \( d(x) \) is finite and continuous on \( \mathbb{R} \), and possesses the following properties:

1. it satisfies the inequality
\[
    |d(x + h) - d(x)| \leq |h|, \quad \text{whenever} \quad |h| \leq d(x);
\]

2. it is differentiable almost everywhere on \( \mathbb{R} \);

3. for any \( x \in \mathbb{R}, \varepsilon \in [0, 1], \) and \( t \in [x - \varepsilon d(x), x + \varepsilon d(x)] \),
\[
    (1 - \varepsilon)d(x) \leq d(t) \leq (1 + \varepsilon)d(x) < (1 - \varepsilon)^{-1}d(x). \tag{2.2}
\]

Note also that we have the following implications:

\[
    \int_{-\infty}^{0} q(t) \, dt = \infty \implies \lim_{x \to -\infty} (x + d(x)) = -\infty, \tag{2.3}
\]
and
\[
    \int_{0}^{\infty} q(t) \, dt = \infty \implies \lim_{x \to \infty} (x - d(x)) = \infty. \tag{2.4}
\]

Definition 2.3 [5] Let \( x \in \mathbb{R} \), \( \varkappa(t) \) be a positive continuous function defined on \( \mathbb{R} \), and \( \{x_n\}_{n=-\infty}^{\infty} \) (resp., \( \{x_n\}_{n=-\infty}^{1} \)) be a sequence of points. Consider the intervals
\[
    \Delta_n = [\Delta_n^-, \Delta_n^+] = x_n \pm \varkappa(x_n), \quad n \geq 1 \quad (n \leq -1).
\]
We say that the sequence of intervals \( \{\Delta_n\}_{n=1}^{\infty} \) (resp., \( \{\Delta_n\}_{n=-\infty}^{1} \)) forms an \( R(x, \varkappa) \)-covering of \([x, \infty)\) (resp., \((-\infty, x])\) if the following conditions are satisfied:
1. \( \Delta_n^+ = \Delta_{n+1}^-, n \geq 1 \) (resp., \( \Delta_n^- = \Delta_n^+, n \leq -1 \));
2. \( \Delta_1 \equiv x \) (resp., \( \Delta_1^+ = x \));
3. \( \bigcup_{n \geq 1} \Delta_n = [x, \infty) \) (resp., \( \bigcup_{n \leq -1} \Delta_n = (-\infty, x) \)).

**Theorem 2.4** [5]. Suppose that the positive continuous function \( \varkappa \) defined on \( \mathbb{R} \) satisfies the condition
\[
\lim_{t \to \infty} (t - \varkappa(t)) = \infty \quad \text{(resp.,} \lim_{t \to -\infty} (t + \varkappa(t)) = -\infty) .
\] (2.5)
Then for every \( x \in \mathbb{R} \), there exists an \( R(x, \varkappa) \)-covering of \([x, \infty) \) (resp., of \((-\infty, x]\)).

**Corollary 2.5** [5] Suppose that conditions (2.4) (2.3) are satisfied. Then for every \( x \in \mathbb{R} \) there exists an \( R(x, d) \)-covering of \([x, \infty) \) (resp., of \((-\infty, x]\)).

**Definition 2.6** [6] Let \( q \) be a function such that for some \( a \geq 1 \), \( b > 0 \) and \( x_0 \geq 1 \), for all \( |x| \geq x_0 \), we have the inequalities
\[
a^{-1}d(x) \leq d(t) \leq ad(x) \quad \text{for} \quad |t - x| \leq bd(x).
\] (2.6)
Then we say that the function \( q \) belongs to the class \( K(\gamma) \) and write \( q \in K(\gamma) \), where
\[
\gamma = \gamma(a, b) = a \exp(-b/a^2)
\] (2.7)
According to Definition 2.6 and Theorem 26 (see [9]), we have the following:

**Theorem 2.7** [6] \( q \in K(\gamma) \) for any \( \gamma > 1 \).

**Lemma 2.1** [6] Let \( a \geq 1 \), \( b > 0 \) and \( \gamma = \gamma(a, b) \leq e^{-1} \). Then \( b \geq 1 \).

The problem of studying whether \( q \in K(\gamma) \) for a given value of parameter \( \gamma \) is considered in the following two theorems.

**Theorem 2.8** [6] Suppose a function \( q \) can be decomposed into a sum
\[
q(x) = q_1(x) + q_2(x), \quad x \in \mathbb{R},
\] (2.8)
where the function \( q_1(x) \) is positive on \( \mathbb{R} \) and absolutely continuous along with its derivative, and \( q_2 \in L^1_{\text{loc}}(\mathbb{R}) \). Suppose further that
\[
\varkappa_1(x) \to 0 \quad \text{and} \quad \varkappa_2(x) \to 0 \quad \text{as} \quad |x| \to \infty,
\]
where
\[
\varkappa_1(x) = \frac{1}{q_1(x)^2} \sup_{|s| \leq 2/q_1(x)} \left| \int_{x-s}^{x+s} q_1''(s) \, ds \right|, \quad x \in \mathbb{R}
\] (2.9)
and
\[
\varkappa_2(x) = \sup_{|s| \leq 2/q_1(x)} \left| \int_{x-s}^{x+s} q_2(s) \, ds \right|, \quad x \in \mathbb{R}.
\] (2.10)
Then the following relations hold:
\[
q_1(x)d(x) = 1 + \varepsilon(x), \quad |\varepsilon(x)| \leq \varkappa_1(x) + \varkappa_2(x), \quad |x| \gg 1,
\] (2.11)
\[
q^*(x) \asymp q_1(x), \quad x \in \mathbb{R}.
\] (2.12)
Theorem 2.9 \[6\] Let the hypotheses of Theorem 2.8 hold. Suppose in addition that

\[
\lim_{|x| \to \infty} \frac{q_1'(x)}{q_2(x)} = 0
\]

(2.13)

\[
\lim_{|x| \to \infty} |x| q_1(x) = \infty
\]

(2.14)

Then for any \(\gamma_0 \in (0, 1)\),

\[
q \in K(\gamma), \text{ where } \gamma = \gamma(a, b) \leq \gamma_0.
\]

(2.15)

3 Results

The next statement is the main result of this work.

Theorem 3.1 Consider the equation (1.1). Suppose the coefficient \(q\) satisfies the conditions (1.2) (1.6), and

\[
\int_{-\infty}^{0} q(t) \, dt = \int_{0}^{\infty} q(t) \, dt = \infty,
\]

(3.1)

and, in addition, \(q \in K(\gamma), \gamma \leq e^{-1}\). Then, if the function \(\theta\) satisfies the condition

\[
\theta(x) \asymp q^*(x), \quad x \in \mathbb{R},
\]

(3.2)

the pair of spaces \(\{L_p, \theta; L_p\}\) is admissible for the equation (1.1) for any \(p \in [1, \infty)\).

Remark 3.2 We will check below the hypotheses of Theorem 3.1 (in view of theorems 2.8 and 2.9) for a particular equation (see Section 5).

Theorem 3.3 Suppose conditions (1.2), (3.1), and \(q \in K(\gamma), \gamma \leq e^{-1}\) are satisfied. Then

\[
J(x) \asymp I(x) \asymp S(x) \asymp d(x) = \frac{1}{q^*(x)}, \quad x \in \mathbb{R}.
\]

(3.5)
Remark 3.4 For \( d_0 < \infty \) (see (1.10)), the relations (3.5) were proven in [6]. It follows from Corollary [1.4] that the inequality \( d_0 < \infty \) is incompatible with the requirement (1.6). Thus, we need to derive (3.5) in such a way that \( d_0 < \infty \) is not used (see Section 4).

Below we give an example of an application of (3.5). Consider the integral

\[
F(x) = \int_{-\infty}^{\infty} G(x, t) \, dt, \quad x \in \mathbb{R},
\]

where

\[
G(x, t) = \begin{cases} 
  u(x)v(t), & x \geq t, \\
  u(t)v(x), & x \leq t,
\end{cases}
\]

and \( u(x), v(x) \) are absolutely continuous positive functions defined on \( \mathbb{R} \), with the properties

\[
u'(t) \leq 0, \quad v'(t) \geq 0, \quad t \in \mathbb{R}
\]

and

\[
\lim_{t \to -\infty} v(t) = \lim_{t \to +\infty} u(t) = 0.
\]

Note that by virtue of (3.8) and (3.9), \( G \) is a unimodal function for any fixed \( x \in \mathbb{R} \) [7]. Therefore, it is natural to say that (3.7) is a uniformly unimodal function of \( x \in \mathbb{R} \) when (3.8) and (3.9) hold. Our goal is to find estimates of \( F(x) \) for \( x \in \mathbb{R} \). To begin, we introduce the functions

\[
q_1(t) = -\frac{u'(t)}{u(t)}, \quad q_2(t) = \frac{v'(t)}{v(t)}, \quad t \in \mathbb{R}.
\]

The following relations are obvious consequences of (3.8) and (3.9):

\[
0 \leq q_1(t), \quad q_2(t) \in L^1_{\text{loc}}(\mathbb{R}),
\]

\[
\int_{0}^{\infty} q_1(t) \, dt = \int_{-\infty}^{0} q_2(t) \, dt = \infty.
\]

Thus, we can define the functions

\[
d_1(x) = \inf_{d > 0} \left\{ d : \int_{x-d}^{x+d} q_1(t) \, dt = 2 \right\}, \quad x \in \mathbb{R}
\]

and

\[
d_2(x) = \inf_{d > 0} \left\{ d : \int_{x-d}^{x+d} q_2(t) \, dt = 2 \right\}, \quad x \in \mathbb{R}.
\]

**Theorem 3.5** Suppose the relations (3.11) and (3.12) hold, and, in addition, \( q_k \in K(\gamma_k) \), \( \gamma_k \leq e^{-1} \) for \( k = 1, 2 \). Then

\[
F(x) \asymp u(x) v(x) [d_1(x) + d_2(x)], \quad x \in \mathbb{R}.
\]

An example of an application of Theorem [3.5] is given in Section 5.
4 Proofs

In what follows we will assume that conditions (1.2) and (1.7) are always satisfied, and do not mention them any more.

Proof of Theorem 3.3

We first need to establish the following lemma.

Lemma 4.2 Suppose the equalities (2.6) hold for some $a \geq 1$, $b > 0$, and $x_0 \gg 1$ such that $|x| \geq x_0$. Then if $q \notin L_1(0, \infty)$ ($q \notin L_1(-\infty, 0)$), an $R(x, bd(x))$-covering of $[x, \infty)$ (resp., $(-\infty, x]$) exists for any $x \in \mathbb{R}$.

We prove the statement of Lemma 4.2 for the $[x, \infty)$ semi-axis only (the case $(-\infty, x)$ is dealt with in a similar manner). By Theorems 2.2 and 2.4 we need to show that (2.4) holds when $x(t) = bd(t)$. For $b \in (0, 1]$ we have $x - bd(x) \rightarrow \infty$ as $x \rightarrow \infty$, and the equality is an obvious consequence of Theorem 2.4. Now let $b > 1$. Let $x \geq x_0$ and $\{\triangle_n\}_{n=1}^{\infty}$ be an $R(x - bd(x), d)$-covering of $[x - bd(x), \infty)$. Clearly, there are two possibilities:

1. $[x - bd(x), x + bd(x)] \subseteq \triangle_1 = [x_1 - d(x_1), x_1 + d(x_1)];$
2. $\triangle_1 \subset [x - bd(x), x + bd(x)].$

In the case 1) we have (see (1.8)):

$$
\int_{x - bd(x)}^{x + bd(x)} q(t) \, dt \leq \int_{x_1 - d(x_1)}^{x_1 + d(x_1)} q(t) \, dt = 2. \quad (4.1)
$$

Consider now the case 2). It is obvious that there exists $n > 1$ such that $\triangle_n^+ < x + bd(x)$ and $\triangle_{n+1}^+ > x + bd(x)$. For such $n$, (2.6) yields

$$
2bd(x) \geq \sum_{k=1}^{n} 2d(x_k) = \sum_{k=1}^{n} \frac{2d(x_k)}{d(x)}d(x) \geq \sum_{k=1}^{n} \frac{2}{a}d(x) = \frac{2}{a}nd(x) \quad \Longrightarrow \quad n \leq ab. \quad (4.2)
$$

Consequently,

$$
\int_{x - bd(x)}^{x + bd(x)} q(t) \, dt \leq \sum_{k=1}^{n+1} \int_{\triangle_k} q(t) \, dt = 2(n + 1) \leq 2(ab + 1). \quad (4.3)
$$

Thus, in both the cases we have:

$$
\int_{x - bd(x)}^{x + bd(x)} q(t) \, dt \leq c < \infty, \quad \text{for any } |x| \geq x_0. \quad (4.4)
$$

Now assume that $x - bd(x)$ does not tend to infinity when $x \rightarrow \infty$. Then we can find a number $c$ and a sequence $\{x_l\}_{l=1}^{\infty}$ such that $x_l \rightarrow \infty$ as $l \rightarrow \infty$, and $x_l - bd(x_l) \leq c < \infty$, $l = 1, 2, \ldots,$ which in conjunction with (4.4) implies that

$$
\infty > c \geq \int_{x_l - bd(x_l)}^{x_l + bd(x_l)} q(t) \, dt \geq \int_{c}^{x_l} q(t) \, dt \rightarrow \infty \quad \text{as } \quad l \rightarrow \infty,
$$

which is a contradiction. Thus, $x - bd(x) \rightarrow \infty$ as $x \rightarrow \infty$. To finish the proof, it remains to refer to Theorems 2.2 and 2.4.
Proof of Theorem 3.4. Let us check the inequality \((3.4)\) for \(J(x)\) (for the function \(I(x)\), \((3.3)\) can be verified in a similar way). In turn, for \(S(x)\) the inequality is easily deduced from a combination of the ones for \(I(x)\) and \(J(x)\).

The lower estimate of \(J(x)\) follows immediately from \((1.8)\):

\[
J(x) = \int_x^\infty \exp \left\{ - \int_x^t q(\xi) \, d\xi \right\} \, dt \geq \int_x^{x+d(x)} \exp \left\{ - \int_x^{x+d(x)} q(\xi) \, d\xi \right\} \, dt
\]

\[
\geq \int_x^{x+d(x)} \exp \left\{ - \int_x^{x+d(x)} q(\xi) \, d\xi \right\} \, dt = e^{-2x} = e^{-2 \frac{cJ^*(x)}{\lambda}}.
\]

To estimate \(J(x)\), \(x \in \mathbb{R}\), from above, we distinguish three cases:

1) \(x \geq x_0\), 2) \(x \leq -x_0 - 1\), 3) \(x \in [-x_0 - 1, x_0]\).

The hypotheses of Theorem 3.5 are assumed to be satisfied and will not be mentioned in the following statements. We begin with case 1)

Lemma 4.3 Let \(x \geq x_0\). Then \(J(x) \leq cJ^*_a(x)\), where

\[
J^*_a(x) = \int_x^\infty \exp \left\{ - \frac{1}{a} \int_x^t q^*(\xi) \, d\xi \right\} \, dt
\]

**Proof.** Let \(\{\triangle_n\}_{n=1}^\infty\) be an \(R(x, d)\)-covering of \([x, \infty)\), \(x \geq x_0\). Since \(b \geq 1\) (see Lemma 2.1), the inequalities

\[
a^{-1}d(t) \leq d(x(t)) - d(x(t)), \quad t \in [x - d(x), x + d(x)],
\]

hold for \(t \geq x_0\). According to Definition 2.3, \((1.8)\), and \((4.6)\), for any \(n \geq 1\) we have

\[
\int_{\triangle_n} q(\xi) \, d\xi = \int_{\triangle_n^-} q(\xi) \, d\xi - \int_{\triangle_n} q(\xi) \, d\xi = \sum_{k=1}^{n} \int_{\triangle_k} q(\xi) \, d\xi - \int_{\triangle_n} q(\xi) \, d\xi
\]

\[
= \sum_{k=1}^{n} 1 - \frac{1}{d(x(k))} \int_{\triangle_k} 1 \, d\xi - 2 = \sum_{k=1}^{n} \frac{d(\xi)}{d(x(k))} \, d\xi) - 2
\]

\[
\geq \sum_{k=1}^{n} \frac{1}{a} \int_{\triangle_k} \frac{d\xi}{d(\xi)} - 2 = \frac{1}{a} \int_{\triangle} \frac{d\xi}{d(\xi)} - 2.
\]

Using Definition 2.3 and relating to \((1.8)\) and \((4.7)\), we get:

\[
J(x) = \int_x^\infty \exp \left\{ - \int_x^t q(\xi) \, d\xi \right\} \, dt = \sum_{n=1}^\infty \int_{\triangle_n} \exp \left\{ - \int_{\triangle_n} q(\xi) \, d\xi \right\} \, dt
\]

\[
\leq \sum_{n=1}^\infty 2d(x_n) \exp \left\{ - \int_{\triangle_n} q(\xi) \, d\xi \right\} \leq e^2 \sum_{n=1}^\infty 2d(x_n) \exp \left\{ - \frac{1}{a} \int_{\triangle_n} \frac{d\xi}{d(\xi)} \right\}
\]

\[
c \sum_{n=1}^\infty \int_{\triangle_n} \exp \left\{ - \frac{1}{a} \int_{\triangle_n} \frac{d\xi}{d(\xi)} \right\} \, dt \leq c \sum_{n=1}^\infty \int_{\triangle_n} \exp \left\{ - \frac{1}{a} \int_{\triangle_n} \frac{d\xi}{d(\xi)} \right\} \, dt
\]

\[
c \int_x^\infty \exp \left\{ - \frac{1}{a} \int_{\triangle_n} \frac{d\xi}{d(\xi)} \right\} \, dt = c \int_x^\infty \exp \left\{ - \frac{1}{a} \int_{\triangle_n} q^*(\xi) \, d\xi \right\} \, dt = cJ^*_a(x).
\]
Lemma 4.4 Let \( x \geq x_0 \). Then

\[
J(x) \leq cd(x) = \frac{c}{q^*(x)}. \tag{4.8}
\]

Proof. We use the assumptions of Lemma 4.2 to deduce that an \( R(x, bd) \)-covering of \([x, \infty), x \geq x_0\) exists. Let \( \{ \omega_n \}_{n=1}^\infty \) be the system of segments forming the \( R(x, bd) \)-covering. Then for \( n \geq 1 \) we have (see the notation of Lemma 4.3)

\[
\int_{\omega_n}^{\omega_{n+1}} \frac{d\xi}{d(\xi)} = \int_{\omega_n}^{\omega_{n+1}} \frac{d\xi}{d(\xi)} - \int_{\omega_n}^{\omega_{n+1}} \frac{d\xi}{d(\xi)} = \sum_{k=1}^{n} \int_{\omega_k}^{\omega_{k+1}} \frac{d\xi}{d(\xi)} = \sum_{k=1}^{n} \frac{1}{a} \int_{\omega_k}^{\omega_{k+1}} \frac{d\xi}{d(\xi)} - \int_{\omega_n}^{\omega_{n+1}} \frac{d\xi}{d(\xi)} = \frac{2b}{a} n - 2ab. \tag{4.9}
\]

Note that the inequalities (see Definition 2.3 and Definition 2.6)

\[
\frac{1}{a} \leq \frac{d(x_{n+1})}{d(x_n)} \leq a, \quad \frac{1}{a} \leq \frac{d(\omega_{n+1}^+)}{d(\omega_n)} \leq a,
\]

hold for \( n \geq 1 \). Since the above relations imply

\[
\frac{1}{a^2} \leq \frac{d(x_{n+1})}{d(x_n)} \leq a^2,
\]

we obtain

\[
\frac{1}{a^{2n-2}} \leq \frac{d(x_n)}{d(x_1)} \leq a^{2n-2}, \quad n \geq 1. \tag{4.10}
\]

Thus, using Definition 2.3 and the inequalities (4.5), (4.9) and (4.10) we get

\[
J(x) \leq cJ^*_{a^*}(x) = c \int_x^\infty \exp \left\{ - \frac{1}{a} \int_{x}^{t} \frac{d\xi}{d(\xi)} \right\} dt = c \sum_{n=1}^{\infty} \int_{\omega_n}^{\omega_{n+1}} \exp \left\{ - \frac{1}{a} \int_{\omega_n}^{\omega_{n+1}} \frac{d\xi}{d(\xi)} \right\} dt \\
\leq c \sum_{n=1}^{\infty} d(x_n) \exp \left\{ - \frac{1}{a} \int_{\omega_n}^{\omega_{n+1}} \frac{d\xi}{d(\xi)} \right\} \leq c \sum_{n=1}^{\infty} d(x_n) \exp \left\{ - \frac{2b}{a^2} n \right\} \\
\leq \frac{d(x_1)}{d(x)} \cdot d(x) \sum_{n=1}^{\infty} \frac{d(x_n)}{d(x_1)} \exp \left\{ - \frac{2b}{a^2} n \right\} \leq cd(x) \sum_{n=1}^{\infty} \gamma^{2n} = cd(x) = \frac{c}{q^*(x)}.
\]

Now we consider case 2).
Lemma 4.5 Let $x \leq -x_0$. Then
\[ \tilde{J}(x) \leq c \tilde{J}_a^*(x), \tag{4.11} \]
where
\[ \tilde{J}(x) = \int_x^{-x_0} \exp \left\{ - \int_x^t q(\xi) \, d\xi \right\} \, dt, \quad \text{if} \quad x \leq -x_0 \tag{4.12} \]
\[ \tilde{J}_a^*(x) = \int_x^{-x_0} \exp \left\{ - \frac{1}{a} \int_x^t \frac{d\xi}{d(\xi)} \right\} \, dt, \quad \text{if} \quad x \leq -x_0 \tag{4.13} \]

Proof. Let $\{\triangle_n\}_{n=-\infty}^{-1}$ be an $\mathbb{R}(\mathbb{R}, d)$-covering of $(-\infty, x_0]$. The following estimate is obtained similarly to (4.7), therefore we omit its derivation:
\[ \int_{\triangle_n^-} \tilde{J}_a(\xi) \, d\xi \geq \frac{1}{a} \int_{\triangle_n^-} \frac{d\xi}{d(\xi)} - 2, \quad n \leq k \leq -1. \tag{4.14} \]

By the inequalities (4.12), (4.13), and (4.14),
\[ \tilde{J}(\triangle_n^-) = \int_{\triangle_n^-} \exp \left\{ - \int_{\triangle_n^-} q(\xi) \, d\xi \right\} \, dt = \sum_{k=n}^{1} \int_{\triangle_k} \exp \left\{ - \int_{\triangle_k} q(\xi) \, d\xi \right\} \, dt \leq \sum_{k=n}^{1} 2d(x_k) \exp \left\{ - \frac{1}{a} \int_{\triangle_k} \frac{d\xi}{d(\xi)} \right\} \]
\[ = c \sum_{k=n}^{1} \int_{\triangle_k} \exp \left\{ - \frac{1}{a} \int_{t}^{t} \frac{d\xi}{d(\xi)} - \frac{1}{a} \int_{t}^{t} \frac{d\xi}{d(\xi)} \right\} \, dt \leq c \sum_{k=n}^{1} \int_{\triangle_k} \exp \left\{ - \frac{1}{a} \int_{t}^{t} \frac{d\xi}{d(\xi)} \right\} \, dt = c \tilde{J}_a^*(\triangle_n^-). \tag{4.15} \]

Below we derive (4.11) from (4.15). To this end we use the inequalities
\[ \frac{2}{a} \leq \int_{\Delta(x)} \frac{d\xi}{d(\xi)} \leq 2a, \quad \Delta(x) = [x - d(x), x + d(x)], \quad x \leq -x_0, \tag{4.16} \]
and
\[ \int_{\Delta_n^+} \frac{d\xi}{d(\xi)} \geq \int_x^t \frac{d\xi}{d(\xi)} - 2a, \quad x \in \Delta_n, \quad t \in [\Delta_n^+, -x_0], \quad n \leq -1. \tag{4.17} \]

Inequality (4.16) follows from (2.6) (in view of Lemma 2.1 $b \geq 1$):
\[ \frac{2}{a} \leq \int_{\Delta(x)} \frac{d\xi}{d(\xi)} = \int_{\Delta(x)} \frac{d(x)}{d(\xi)} \frac{d(\xi)}{d(x)} \leq 2a, \]
while (4.17) is easily obtained from (4.16):
\[ \int_{\Delta_n^+} \frac{d\xi}{d(\xi)} = \int_x^t \frac{d\xi}{d(\xi)} - \int_x^t \frac{d\xi}{d(\xi)} \geq \int_x^t \frac{d\xi}{d(\xi)} - \int_x^t \frac{d\xi}{d(\xi)} \geq \int_x^t \frac{d\xi}{d(\xi)} - 2a. \]
Assume now that $x \in \Delta_n$, $n \leq -1$. In the next chain of calculations, we use (4.16), (4.17), and (4.14):

\[
\begin{split}
\widetilde{J}(x) &= \int_{x}^{-x_0} \exp \left\{ - \int_{x}^{t} \frac{d\xi}{d(\xi)} \right\} dt = \int_{x}^{\Delta_n^+} \exp \left\{ - \int_{x}^{t} \frac{d\xi}{d(\xi)} \right\} dt \\
&\quad + \int_{\Delta_n^+}^{x} \exp \left\{ - \int_{x}^{t} \frac{d\xi}{d(\xi)} \right\} dt \leq c^2 \int_{x}^{\Delta_n^+} \exp \left\{- \int_{x}^{t} \frac{d\xi}{d(\xi)} \right\} dt \\
&\quad \leq c \int_{x}^{\Delta_n^+} \exp \left\{ - \int_{x}^{t} \frac{d\xi}{d(\xi)} \right\} dt + c \int_{\Delta_n^+}^{x} \exp \left\{ - \int_{x}^{t} \frac{d\xi}{d(\xi)} \right\} dt \\
&\quad \leq c \int_{x}^{\Delta_n^+} \exp \left\{ - \int_{x}^{t} \frac{d\xi}{d(\xi)} \right\} dt + c \int_{\Delta_n^+}^{x} \exp \left\{ - \int_{x}^{t} \frac{d\xi}{d(\xi)} + 2 \right\} dt \\
&\quad \leq c \int_{x}^{\Delta_n^+} \exp \left\{ - \int_{x}^{t} \frac{d\xi}{d(\xi)} \right\} dt = c \tilde{J}_a^*(x).
\end{split}
\]

\[\square\]

**Lemma 4.6** Let $x \leq -x_0$. Then

\[
\widetilde{J}(x) \leq c d(x) = \frac{c}{q^*(x)}.
\]

**Proof.** Let $\{\omega_n\}_{n=-\infty}^{1}$ be an $R(x, bd)$-covering of $(-\infty, -x_0]$. Then

\[
\int_{\omega_n^-}^{\omega_0^-} \frac{d\xi}{d(\xi)} \geq \frac{2b}{a} |n - k| - 2ab, \quad n \leq k \leq -1,
\]

(4.19)

\[
\frac{1}{a^{2|n-k|-2}} \leq \frac{d(x_k)}{d(x_n)} \leq a^{2|n-k|-2}, \quad n \leq k \leq -1.
\]

(4.20)

This implies (similarly to Lemma 4.3)

\[
\widetilde{J}(\omega_n^-) \leq c \tilde{J}_a^*(\omega_n^-) \leq c d(\omega_n^-), \quad n \leq -1.
\]

(4.21)

The inequalities (4.19), (4.20), (4.21) are checked in the same way as (4.9), (4.10), (4.8) respectively, therefore we drop their derivation.

Assume now that $x \in \omega_n$, $n \leq -1$. Then, using the above relations, we have

\[
\begin{split}
\widetilde{J}(x) &\leq c \tilde{J}_a^*(x) = c \int_{x}^{\omega_n^+} \exp \left\{ - \int_{x}^{t} \frac{d\xi}{d(\xi)} \right\} dt \\
&\quad + c \int_{\omega_n^+}^{x} \exp \left\{ - \int_{x}^{t} \frac{d\xi}{d(\xi)} \right\} dt \cdot \exp \left\{ - \int_{x}^{\omega_n^+} \frac{d\xi}{d(\xi)} \right\} \\
&\quad \leq c(\omega_n^+ - x) + c \tilde{J}_a^*(\omega_n^+) \leq c \left( d(x_n) + d(\omega_n^+) \right) \leq c d(x) = \frac{c}{q^*(x)}.
\end{split}
\]
Let us continue with the case 2). To estimate \( J(x) \) for \( x \leq -x_0 - 1 \), first note that the integral \( J(x_0) \) converges. Indeed, from the inequality \( J(x) \leq cd(x) \) it follows that

\[
J(-x_0) = \int_{-x_0}^{\infty} \exp \left\{ - \int_{-x_0}^{t} q(\xi) \, d\xi \right\} dt = \int_{-x_0}^{x_0} \exp \left\{ - \int_{-x_0}^{t} q(\xi) \, d\xi \right\} dt + \int_{x_0}^{\infty} \exp \left\{ - \int_{-x_0}^{t} q(\xi) \, d\xi \right\} dt \leq 2x_0 + J(x_0) \leq c(1 + d(x_0)) < \infty.
\]

Thus, for \( x \leq -x_0 - 1 \) we get

\[
J(x) = \int_{x}^{\infty} \exp \left\{ - \int_{x}^{t} q(\xi) \, d\xi \right\} dt = \tilde{J}(x) + \int_{x}^{x_0} \exp \left\{ - \int_{x}^{t} q(\xi) \, d\xi \right\} dt \leq \tilde{J}(x)
\]

\[
= \tilde{J}(x)\left\{ 1 + \frac{1}{x_0 - x} \right\} \leq \frac{c}{q^{\ast}(x)}.
\]

Hence, in both the cases 1) and 2), inequality (3.5) is true. Let us now consider the case 3). Let

\[
f(x) = \frac{J(x) \cdot \frac{1}{d(x)}}{x \in [-x_0 - 1, x_0].
\]

From Theorem (2.2) it follows that \( f(x) \) is a continuous function, and, as the interval \([-x_0 - 1, x_0]\) is finite and closed, \( f(x) \) is bounded. The statement is proved.

### Proof of Theorem 3.1

In what follows, we suppose that the assumptions of Theorem 3.1 are in force. Let \( p \in [1, \infty) \) and \( f \in L_p \). Consider the function

\[
y(x) := (Gf)(x) = \int_{x}^{\infty} \exp \left\{ - \int_{x}^{t} q(\xi) \, d\xi \right\} f(t) \, dt, \quad x \in \mathbb{R}. \tag{4.22}
\]

We claim that the integral in (4.22) converges for all \( x \in \mathbb{R} \), i.e., the function \( y(x) \) is defined for any \( x \in \mathbb{R} \). Indeed, by H"{o}lder’s inequality and (3.5) we have

\[
|y(x)| \leq \int_{x}^{\infty} \exp \left\{ - \int_{x}^{t} q(\xi) \, d\xi \right\} |f(t)| \, dt \leq \left( \int_{x}^{\infty} \exp \left\{ - \int_{x}^{t} q(\xi) \, d\xi \right\} dt \right)^{1/p'} \cdot \left( \int_{x}^{\infty} \exp \left\{ - \int_{x}^{t} q(\xi) \, d\xi \right\} dt \right)^{1/p} \leq c(p)d(x)^{1/p'} ||f||_p < \infty, \quad x \in \mathbb{R}.
\]
Obviously, \( y(x) \) is an absolutely continuous function for \( x \in \mathbb{R} \) and a solution of (1.1). Let us check the inclusion \( y \in L_{p,\theta} \), and inequality (1.4). To this end, we formulate one more lemma.

**Lemma 4.7** Let

\[
M(x) = \int_{-\infty}^{x} \theta(t) \exp \left\{ - \int_{t}^{x} q(\xi) \, d\xi \right\} \, dt. \tag{4.23}
\]

Then \( M = \sup_{x \in \mathbb{R}} M(x) < \infty \).

**Proof** Let \( \{\triangle_n\}_{n=-\infty}^{-1} \) be an \( R(x, bd) \)-covering of \((-\infty, x]\) (see Definition 2.2 and Corollary 2.4). Then, by (1.8), for \( n \leq -1 \) we have

\[
\int_{\triangle_n} q(\xi) \, d\xi = \int_{\triangle_n} q(\xi) \, d\xi - \sum_{k=-n}^{-1} \int_{\triangle_k} q(\xi) \, d\xi = -2 - 2 = 2(|n| - 1). \tag{4.24}
\]

Further, since \( b \geq 1 \) (see Lemma 2.7):

\[
\frac{1}{a} \leq \frac{d(t)}{d(x)} \leq a \quad \text{for} \quad |t-x| \leq d(x), \quad |x| \geq x_0. \tag{4.25}
\]

But \( d(x) \) is a positive continuous function for all \( x \in \mathbb{R} \) (see Theorem 2.2). Therefore, there exists a number \( \alpha \in [1, \infty) \), such that

\[
\alpha^{-1} \leq \frac{d(t)}{d(x)} \leq \alpha \quad \text{for} \quad |t| \leq |x_0|, \quad |x| \leq x_0. \tag{4.26}
\]

Denote \( c = \max\{a, \alpha\} \). It follows from (4.25) and (4.26) that

\[
c^{-1} \leq \frac{d(t)}{d(x)} \leq c \quad \text{for} \quad |t-x| \leq d(x), \quad x \in \mathbb{R}. \tag{4.27}
\]

Let us return to our Lemma. According to (4.24), (4.27), Definition 2.2, and (3.2), we get:

\[
M(x) = \int_{-\infty}^{x} \theta(t) \exp \left\{ - \int_{t}^{x} q(\xi) \, d\xi \right\} \, dt \leq c \int_{-\infty}^{x} \frac{1}{d(t)} \exp \left\{ - \int_{t}^{x} q(\xi) \, d\xi \right\} \, dt
\]

\[
= c \sum_{n=-\infty}^{-1} \int_{\triangle_n} \frac{1}{d(t)} \exp \left\{ - \int_{t}^{x} q(\xi) \, d\xi \right\} \, dt
\]

\[
\leq c \sum_{n=-\infty}^{-1} \left\{ \int_{\triangle_n} \frac{d(x_n)}{d(\xi)} \frac{d\xi}{d(x_n)} \right\} \exp \left\{ - \int_{\triangle_n} q(\xi) \, d\xi \right\}
\]

\[
\leq c \sum_{n=-\infty}^{-1} \exp \left\{ - 2(|n| - 1) \right\} = c < \infty
\]
Below we obtain inequality (1.4). Let \( f \in L^p \), \( p \in [1, \infty) \). Using Hölder’s inequality, (3.5), (3.2), Lemma 4.7, and Fubini’s theorem we get

\[
\|y(x)\|_{p, \theta}^p = \int_{-\infty}^{\infty} \theta(x)^p \left( \int_{-\infty}^{\infty} \exp \left\{ -\int_{x}^{t} q(\xi) \, d\xi \right\} |f(t)|^p \, dt \right)^p \, dx \\
\leq \int_{-\infty}^{\infty} \theta(x)^p \left( \int_{-\infty}^{\infty} \exp \left\{ -\int_{x}^{t} q(\xi) \, d\xi \right\} |f(t)|^p \, dt \right)^{p/p'} \left( \int_{-\infty}^{\infty} \exp \left\{ -\int_{x}^{t} q(\xi) \, d\xi \right\} |f(t)|^p \, dt \right) \, dx \\
\leq c \int_{-\infty}^{\infty} \theta(x)^p d(x)^{p-1} \left( \int_{-\infty}^{\infty} \exp \left\{ -\int_{x}^{t} q(\xi) \, d\xi \right\} |f(t)|^p \, dt \right) \, dx \\
\leq c \int_{-\infty}^{\infty} \theta(x) \left( \int_{x}^{\infty} \exp \left\{ -\int_{x}^{t} q(\xi) \, d\xi \right\} |f(t)|^p \, dt \right) \, dx = c \int_{-\infty}^{\infty} M(t) |f(t)|^p \, dt \leq c \|f\|_p^p.
\]

Hence, it remains to check that a solution of the homogenous equation

\[-z'(x) + q(x)z(x) = 0, \quad x \in \mathbb{R}\]  \hspace{1cm} (4.28)

belongs to the space \( L_{p, \theta} \) only if \( z \equiv 0 \). Clearly, we can write the solution of (4.28) as

\[z(x) = \alpha \exp \left\{ \int_{x_0}^{x} q(\xi) \, d\xi \right\}, \quad x \in \mathbb{R}\]  \hspace{1cm} (4.29)

where \( \alpha \) is constant and \( x_0 \) is as in Definition 2.4. Let \( \alpha \neq 0 \) and, with this, \( z \in L_{p, \theta} \). The following estimates are based on (4.29), (2.6), (3.2), (4.7), (4.9); also, \( R[x_0, d) \) and \( R[x_0, bd) \) –
coverings of \([x_0, \infty)\) are used.

\[
\infty > \|z(x)\|_{p, \theta}^p \geq \int_{-\infty}^{\infty} |\theta(t)z(t)|^p \, dt \geq (|\alpha|c^{-1})^p \int_{x_0}^{\infty} \frac{1}{d(t)^p} \exp \left\{ p \int_{\Delta_n^-}^t q(\xi) \, d\xi \right\} \, dt
\]

\[
= (|\alpha|c^{-1})^p \sum_{n=1}^{\infty} \int_{\Delta_n^-}^{\Delta_n^+} \frac{1}{d(t)^p} \exp \left\{ \frac{p}{a} \int_{\Delta_n^-}^{\Delta_n^+} \frac{d\xi}{d(\xi)} \right\} \, dt \geq (|\alpha|c^{-1})^p \sum_{n=1}^{\infty} \int_{\omega_n}^{\infty} \frac{1}{d(t)^p} \exp \left\{ \frac{p}{a} \int_{\omega_n}^{\infty} \frac{d\xi}{d(\xi)} \right\} \, dt
\]

\[
= (|\alpha|c^{-1})^p \sum_{n=1}^{\infty} \int_{\omega_n}^{\infty} \left( \frac{d(x_n)}{d(t)} \right)^p \frac{dt}{d(x_n)^p} \exp \left\{ \frac{p}{a} \int_{\omega_n}^{\infty} \frac{d\xi}{d(\xi)} \right\} \geq (|\alpha|c^{-1})^p \sum_{n=1}^{\infty} \frac{1}{d(x_n)^{p-1}} \exp \left\{ \frac{2b}{a^2} np \right\}
\]

\[
= (|\alpha|c^{-1})^p \sum_{n=1}^{\infty} u_n, \quad (4.30)
\]

Using d’Alambert criterion for the convergence of series \((4.30)\) we get (see \((4.10)\))

\[
\frac{u_{n+1}}{u_n} = \left( \frac{d(x_n)}{d(x_{n+1})} \right)^{p-1} \exp \left\{ \frac{2b}{a^2} p \right\} \geq \frac{a^2}{2^p a^{2p}} \exp \left\{ \frac{2b}{a^2} p \right\} \geq \frac{1}{2} 2^p \geq e^p > 1.
\]

This is a contradiction. Hence, \(\alpha = 0\) and \(z \equiv 0\). The statement is proved.

\[\blacksquare\]

**Proof of Theorem 3.5** The following relations are obvious consequences of \((3.10)\), \((3.11)\), \((3.12)\), and Theorem 3.3:

\[
F(x) = \int_{-\infty}^{\infty} G(x, t) \, dt = u(x) \int_{-\infty}^{x} v(t) \, dt + v(x) \int_{x}^{\infty} u(t) \, dt
\]

\[
= u(x)v(x) \left( \int_{-\infty}^{x} \frac{v(t)}{u(x)} \, dt + \int_{x}^{\infty} \frac{u(t)}{v(x)} \, dt \right)
\]

\[
= u(x)v(x) \left( \int_{-\infty}^{x} \exp \left\{ - \int_{-\infty}^{t} q_2(\xi) \, d\xi \right\} \, dt + \int_{x}^{\infty} \exp \left\{ - \int_{t}^{\infty} q_1(\xi) \, d\xi \right\} \, dt \right)
\]

\[
\asymp u(x)v(x) (d_1(x) + d_2(x)), \quad x \in \mathbb{R}.
\]

\[\blacksquare\]
**Proof of Theorem 1.3** *Necessity.* Suppose equation (1.1) is correctly solvable in $L_p$ for $p \in [1, \infty)$. Then, according to Theorem 1.2, $q_0(a) > 0$ for some $a \in (0, \infty)$, and (1.7) holds. Let $n$ be the smallest integer such that $nq_0(a) \geq 1$. Then for any $x \in \mathbb{R}$ we have

$$\int_{x-2na}^{x+2na} q(\xi) \xi = \int_{x-2na}^{x+2na} q(\xi) d\xi \geq \int_{x}^{x+2na} q(\xi) d\xi \geq nq_0(a) + nq_0(a) \geq 2 \Rightarrow$$

$$d(x) \leq 2na \Rightarrow d_0 < \infty$$

* Sufficiency. Since (1.7) holds, $d(x)$ is defined for any $x \in \mathbb{R}$, and

$$\int_{x-d(x)}^{x+d_0} q(\xi) d\xi \geq \int_{x-d(x)}^{x+d_0} q(\xi) d\xi = 2.$$

Hence, (1.5) holds.

\[\blacksquare\]

5 Examples

The following examples are applications of Theorems 3.1 and 3.5.

*Example 1.* Consider the equation

$$-y'(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R}, \quad (5.1)$$

where $f \in L_p$, $p \in [1, \infty)$,

$$q(x) = \frac{1}{(1 + x^2)^\alpha} + \frac{\cos(1 + x^2)^\beta}{(1 + x^2)^\alpha}, \quad x \in \mathbb{R}, \quad (5.2)$$

and the parameters $\alpha, \beta$ satisfy the conditions

$$0 < \alpha < \frac{1}{2}, \quad \alpha + \beta > \frac{1}{2}. \quad (5.3)$$

We formulate our results in the following theorem.

**Theorem 5.1** Let $\theta(x)$ be a positive continuous function for $x \in \mathbb{R}$ which satisfies the relation

$$\theta(x) \asymp (1 + x^2)^{-\alpha}, \quad x \in \mathbb{R}.$$

Then the pair $\{L_p, \theta; L_p\}$ is admissible for equation (5.1) for any $p \in [1, \infty)$. 

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Proof. Clearly, the requirements (1.2), (1.6) hold. It remains to check (3.1). Since (5.2) is an even function, we investigate it on the interval $[0, \infty)$ half-axis only. We introduce the sequences

$$x_k = \sqrt{(2\pi k)^{1/\beta} - 1} \quad \text{and} \quad z_k = \sqrt{(2\pi k + \frac{\pi}{4})^{1/\beta} - 1}, \quad k \geq 1.$$  

Let $\delta = \frac{1}{2} - \alpha$ and $x_0 \gg 1$. In the following estimates we use the binomial series:

$$\int_0^\infty q(t) \, dt \geq \sum_{k=k_0}^\infty \int_{x_k}^{x_k} \left( \frac{1}{(1 + x^2)^\alpha} + \frac{\cos(1 + x^2)^\beta}{(1 + x^2)^\alpha} \right) \, dx$$

$$\geq \sum_{k=k_0}^\infty \int_{x_k}^{x_k} \frac{dx}{(1 + x^2)^\alpha} \geq \sum_{k=k_0}^\infty z_k - x_k = \sum_{k=k_0}^\infty \frac{z_k^2 - x_k^2}{1 + z_k^2} \frac{1}{\alpha} \quad \text{and} \quad \sigma(x) = \frac{(2\pi k + \frac{\pi}{4})^{1/\beta} - 1}{1 + (\frac{2\pi k}{4})^{1/\beta}} \cdot k^{1/\beta}$$

$$\geq c^{-1} \sum_{k=k_0}^\infty \frac{1}{1 + (\frac{2\pi k}{4})^{1/\beta}}$$

Thus, (3.1) holds, and $d(x)$ is defined for all $x \in \mathbb{R}$ (see Theorem 2.2). Let us now find the exact order two-sided estimates of $d(x)$. To this end we use Theorem 2.8. Let

$$q_1(x) = \frac{1}{(1 + x^2)^\alpha}, \quad q_2(x) = \frac{\cos(1 + x^2)^\beta}{(1 + x^2)^\alpha}, \quad x \in \mathbb{R},$$

in (2.8). In view of the obvious relations

$$[x - 2(1 + x^2)^\alpha, x + 2(1 + x^2)^\alpha] \subseteq \sigma(x) = [x - 4x^{2\alpha}, x + 4x^{2\alpha}]$$

$$\lim_{x \to -\infty} (x + 4x^{2\alpha}) = -\infty, \quad \lim_{x \to \infty} (x - 4x^{2\alpha}) = \infty,$$

which hold for $|x| \geq x_0 \gg 1$, we have

$$\frac{1}{2} \leq 1 - \frac{4}{|x|^{1-2\alpha}} \leq |\frac{s}{x}| \leq 1 + \frac{4}{|x|^{1-2\alpha}}, \quad s \in \sigma(x),$$

and

$$\frac{1}{16} \leq \frac{1}{4} \left( \frac{s}{x} \right)^2 \leq \frac{1 + s^2}{1 + x^2} \leq 4 \left( \frac{s}{x} \right)^2 \leq 16, \quad s \in \sigma(x).$$
Below to estimate \( \kappa_1(x) \) when \( |x| \to \infty \) we use relations (5.6) and (5.7):

\[
\kappa_1(x) = (1 + x^2)^{2\alpha} \sup_{|\xi| \leq 2(1 + x^2)^\alpha} \left| \int_{x-\xi}^{x+\xi} \frac{1}{(1 + s^2)^{\alpha}} \right| ds \leq c|x|^{4\alpha} \sup_{|\xi| \leq 2} \left| \int_{x-\xi}^{x+\xi} \frac{ds}{(1 + s^2)^{\alpha+1}} \right|
\]

\[
\leq \frac{c}{(1 + x^2)^{\alpha+1}} \leq \frac{c}{|x|^{2-4\alpha}} \to 0 \text{ when } |x| \to \infty.
\]

Let us now estimate \( \kappa_2(x) \) for \( |x| \gg 1 \). Note that \( f(s) = s(1 + s^2)^{\beta+\alpha-1} \) is a monotonically increasing function when \( s \gg 1 \) (see 5.3). Hence, using (5.4), (5.5), (5.6), (5.7) and the second mean value theorem (see [8]), we get:

\[
\kappa_2(x) = \sup_{|\xi| \leq 2(1 + x^2)^\alpha} \left| \int_{x-\xi}^{x+\xi} \cos \left( \frac{1}{(1 + s^2)^{\alpha}} \right) ds \right| \leq \sup_{|\xi| \leq 2(1 + x^2)^\alpha} \left| \int_{x-\xi}^{x+\xi} \frac{(\sin(1 + s^2)^{\beta})'}{2\beta s(1 + s^2)^{\alpha+\beta-1}} ds \right|
\]

\[
\leq \frac{c}{|x|^{2(\beta+\alpha)+1}} \sup_{|\xi| \leq 2(1 + x^2)^\alpha} \left| \sin (1 + s^2)^{\beta} \right|, \quad \text{where } |x| \gg 1.
\]

Thus, by Theorem 2.8 we have

\[
c^{-1}(1 + x^2)^{\alpha} \leq d(x) \leq c(1 + x^2)^{\alpha}, \quad x \in \mathbb{R},
\]

and

\[
d(x) = (1 + \varepsilon(x))(1 + x^2)^{\alpha}, \quad |\varepsilon(x)| \leq \frac{c}{|x|^\nu}, \quad |x| \gg 1.
\]

Here \( \nu = \min\{4\alpha, 2\beta + 2\alpha - 1\} \). Further, conditions (2.13) and (2.14) are obviously satisfied. So by Theorem 2.9 \( q \in K(\gamma), \gamma \leq e^{-1} \). Now Theorem 5.1 is seen to be a consequence of Theorem 3.1, and the proof is finished.

\[\blacksquare\]

**Example 2.** Consider the integral \( F_1(x) = \int_{-\infty}^{\infty} G_1(x, t) \, dt, \ x \in \mathbb{R}, \) where

\[
G_1(x, t) = \begin{cases} 
\exp \{ t^3 + t \cos t - x^3 \}, & t \leq x \\
\exp \{ x^3 - t \cos t - t^3 \}, & t \geq x.
\end{cases}
\]

(5.8)
Our goal is to find the exact order two-sided estimates of $F_1(x)$ for all $x \in \mathbb{R}$. The question is reduced to estimation of the integral of the type (3.4). The following relations are obvious:

\[
F_1(x) = \int_{-\infty}^{x} G_1(x, t) \, dt + \int_{x}^{\infty} G_1(x, t) \, dt
\]

\[
= e^x \cos x \int_{-\infty}^{x} e^{-(x^3-t^3+x \cos t \cos x)} \, dt
\]

\[
+ e^{-x} \cos x \int_{x}^{\infty} e^{-(t^3-x^3+t \cos t-x \cos x)} \, dt
\]

\[
= e^x \cos x \int_{-\infty}^{x} e^{-\int_x^t (3\xi^2+(\xi \cos \xi)) \, d\xi} \, dt
\]

\[
+ e^{-x} \cos x \int_{x}^{\infty} e^{-\int_x^t (3\xi^2-(\xi \cos \xi)) \, d\xi} \, dt
\]

\[
= e^x \cos x \int_{-\infty}^{x} e^{-\int_x^t (3\xi^2-\xi \sin \xi) \, d\xi} \cdot e^{(\sin t-\sin x)} \, dt
\]

\[
+ e^{-x} \cos x \int_{x}^{\infty} e^{-\int_x^t (3\xi^2-\xi \sin \xi) \, d\xi} \cdot e^{(\sin x-\sin t)} \, dt.
\]

Hence,

\[
e^{-2}h(x, t) < F_1(x) < e^2h(x, t),
\]

where

\[
h(x, t) = e^x \cos x \int_{-\infty}^{x} e^{-\int_x^t (3\xi^2-\xi \sin \xi) \, d\xi} \, dt + e^{-x} \cos x \int_{x}^{\infty} e^{-\int_x^t (3\xi^2-\xi \sin \xi) \, d\xi} \, dt. \tag{5.9}
\]

It remains to estimate the integrals

\[
I_1(x) = \int_{-\infty}^{x} e^{-\int_x^t q(\xi) \, d\xi} \, dt \quad \text{and} \quad J_1(x) = \int_{x}^{\infty} e^{-\int_x^t q(\xi) \, d\xi} \, dt \tag{5.10}
\]

for all $x \in \mathbb{R}$, where $q(t) = 3t^2 - t \sin t$, $t \in \mathbb{R}$. In order to apply Theorem 3.3, let us introduce the functions (see Theorem 2.8)

\[
q_1 = 3t^2 + 1, \quad q_2 = -1 - t \sin t, \quad t \in \mathbb{R}, \tag{5.11}
\]

and check that $q \in K(\gamma)$, $\gamma \leq e^{-1}$. Now for $\varkappa_1(x)$, $x \gg 1$, we have

\[
\varkappa_1(x) = \frac{1}{q_1(x)^2} \sup_{|\xi| \leq 2/q_1(x)} \left| \int_{x-\xi}^{x+\xi} q''_1(s) \, ds \right| \leq \frac{c}{|x|^6} \to 0
\]
as $|x| \to \infty$. To estimate $\kappa_2(x)$ when $|x| \to \infty$, we use the second mean value theorem (see [8]):

$$
\kappa_2(x) = \sup_{|\xi| \leq x^{-1/3}} \left| \int_{x-\xi}^{x+\xi} q_2(s) \, ds \right| = \sup_{|\xi| \leq 2/3x^2 + 1} \left| \int_{x-\xi}^{x+\xi} (1 + t \sin t) \, dt \right|
\leq \frac{c}{x^2} + \sup_{|\xi| \leq 2/3x^2 + 1} \left| \int_{x-\xi}^{x+\xi} t \sin t \, dt \right| \leq \frac{c}{x^2} + \sup_{|\xi| \leq 2/3x^2 + 1} \sup_{\theta \in [x-\xi, x+\xi]} |x + \xi| \left| \int_{\theta}^{x+\xi} \sin t \, dt \right|
\leq \frac{c}{x^2} + c|x| \sup_{|\xi| \leq 2/3x^2 + 1} \sup_{\theta \in [x-\xi, x+\xi]} \left| \cos(x + \xi) - \cos \theta \right|
\leq \frac{c}{x^2} + c|x| \sup_{|\xi| \leq 2/3x^2 + 1} \sup_{\theta \in [x-\xi, x+\xi]} \left| \frac{\sin(x + \xi - \theta)}{2} \right|
\leq \frac{c}{x^2} + c|x| \frac{1}{3x^2 + 1} \to 0, \quad \text{as} \quad |x| \to \infty.
$$

Hence,

$$
d(x) = \frac{1 + \varepsilon(x)}{3x^2 + 1}, \quad |\varepsilon(x)| \leq \frac{c}{|x|}, \quad |x| \gg 1,
$$

so

$$
d(x) \asymp \frac{1}{3x^2 + 1}, \quad x \in \mathbb{R}.
$$

The hypotheses of Theorem 2.9 are obviously satisfied, hence, $q \in K(\gamma), \gamma \leq e^{-1}$. Finally, using Theorem 3.3 we get

$$
I_1 \asymp J_1 \asymp \frac{1}{3x^2 + 1} \asymp \frac{1}{x^2 + 1}, \quad x \in \mathbb{R}
$$

and

$$
F_1(x) \asymp \frac{\cosh(x \cos x)}{x^2 + 1}, \quad x \in \mathbb{R}.
$$

**References**

[1] Massera J.L., Schaffer J.J., *Linear Differential Equations and Function Spaces*, Pure and Applied Mathematics, Vol. 21, Academic Press, New York-London, 1966.

[2] Lukachev M., Shuster L., *On uniqueness of the solution of a linear differential equation without boundary conditions*, Functional Differential Equations 14 (2007), no 2, 337–346.

[3] Chernyavskaya N., *Conditions for correct solvability of a simplest singular boundary value problem*, Math.Nachr., 243 (2002), no. 1, 5–18.

[4] Mynbaev K. T., Otelbaev M. O., *Weighted Function Spaces and the Spectrum of Differential Operators*, Nauka, Moscow, (1988).
[5] Chernyavskaya N., Shuster L., *Conditions for correct solvability of a simplest singular boundary value problem of general form*, 1, Zeitschrift für Analysis und ihre Anwendungen 25, (2006), no. 2, 205–235.

[6] Chernyavskaya N., Shuster L., *Weight estimates for solutions of linear singular differential equations of the first order and the Everitt-Giertz problem*, Differential Integral Equations, 25 (2012), no. 5/6, 467–504.

[7] Cottle R. W., *Mathematical Programming Essays in Honor of George B. Dantzig, Part II*, Springer Berlin Heidelberg, 1985

[8] Titchmarsh E. C., *The Theory of Functions*, Oxford University Press, 1939.

[9] Otelbaev M., *The smoothness of the solution of differential equations*, Izv. Acad. Nauk Kazak. SSR, 5 (1977), 45–48.