DERIVED IDENTITIES OF DIFFERENTIAL ALGEBRAS

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Abstract. Suppose $A$ is a not necessarily associative algebra with a derivation $d$. Then $A$ may be considered as a system with two binary operations $\succ$ and $\prec$ defined by $x \succ y = d(x)y$, $x \prec y = xd(y)$, $x, y \in A$. Suppose $A$ satisfies some multi-linear polynomial identities. We show how to find the identities that hold for operations $\succ$ and $\prec$. It turns out that if $A$ belongs to a variety governed by an operad $\text{Var}$ then $\succ$ and $\prec$ satisfy the defining relations of the operad $\text{Var} \circ \text{Nov}$, where $\circ$ is the Manin white product of operads, $\text{Nov}$ is the operad of Novikov algebras. Moreover, there are no other independent identities that hold for operations $\succ, \prec$ on a differential $\text{Var}$-algebra.

1. Introduction

Let $A$ be an algebra over a field $k$, i.e., a linear space equipped with a binary linear operation (multiplication) $\cdot : A \otimes A \to A$. Suppose $T$ is a linear operator on $A$, and let $\prec$ and $\succ$ be two new linear maps $A \otimes A \to A$ defined by

$$a \prec b = a \cdot T(b), \quad a \succ b = T(a) \cdot b, \quad a, b \in A.$$  

Denote the system $(A, \prec, \succ)$ by $A^{(T)}$. If the initial algebra $A$ satisfies a polynomial identity then what could be said about $A^{(T)}$? The answer is known if $T$ is a Rota—Baxter operator [1], [4], [2] or averaging operator [5]. In these cases, the identities of $A^{(T)}$ may be obtained by means of categoric procedures (black and white Manin products of operads [3]).

The purpose of this note is to consider the case when $T$ is a derivation (or generalized derivation). It is well-known [7] that a commutative (and associative) algebra $A$ with a derivation $d$ induces Novikov algebra structure on $A^{(d)}$, assuming $(a \succ b) = (b \prec a)$. Conversely, if an identity holds on $A^{(d)}$ for an arbitrary commutative algebra $A$ with a derivation $d$ then this identity is a consequence of Novikov identities.

In this paper, we generalize this observation for an arbitrary variety $\text{Var}$ of algebras. Namely, if an identity $f$ holds on $A^{(d)}$ for every $\text{Var}$-algebra $A$ with a derivation $d$ then we say $f$ is a derived identity of $\text{Var}$. For $\text{Var} = \text{As}$, some derived identities were found in [8]. We show that for a multi-linear variety $\text{Var}$ the set of derived identities coincides with the set of relations on the operad $\text{Var} \circ \text{Nov}$, where $\circ$ is the Manin white product of operads.

Calculation of Manin products is a relatively simple linear algebra problem, it is based on finding intersections of vector spaces. Therefore, our result provides an easy way for finding a complete list of derived identities for an arbitrary binary operad.

2. White product of operads

Suppose $\text{Var}$ is a multi-linear variety of algebras, i.e., a class of all algebras that satisfy a given family of multi-linear identities (over a field $k$ of characteristic zero,
every variety is multi-linear). Fix a countable set of variables \(X = \{x_1, x_2, \ldots\}\) and denote

\[ \text{Var}(n) = M_n(X)/M_n(X) \cap T_{\text{Var}}(X), \]

where \(M_n(X)\) is the space of all (non-associative) multi-linear polynomials of degree \(n\) in \(x_1, \ldots, x_n\), \(T_{\text{Var}}(X)\) is the T-ideal of all identities that hold in \(\text{Var} \).

The collection of spaces \((\text{Var}(n))_{n \geq 1}\) forms a symmetric operad relative to the natural composition rule and symmetric group action (see, e.g., [9]). We will denote it as \(\text{Var}\), where \(\text{Var}(n)\) is the space of all identities that hold in \(\text{Var}\).

Example 1. The operad \(\text{Lie}\) governing the variety of Lie algebras is generated by 1-dimensional space 

\[ \text{Lie}(1) = \mathbb{R}, \mu((12)) = -\mu. \]

The space \(\text{Lie}(1)\) is also 1-dimensional, it is spanned by the identity \(\text{id}\) of the operad. If we identify \(\text{id}\) with the Hadamard product \([1] \otimes [1] = 1\), then

\[ \mu(\text{id}, \text{id}) = [x_1[x_2 x_3]], \quad \mu(\text{id}, \text{id}) = [[x_1 x_2] x_3], \]

so the Jacobi identity may be expressed as

\[ \mu(\text{id}, \text{id}) - \mu(\text{id}, \text{id}) = \mu(\text{id}, \text{id}). \]

Example 2. The operad \(\text{As}\) governing the variety of associative algebras is generated by 2-dimensional space \(\text{As}(2)\) spanned by 

\[ \nu((12)) = (x_2 x_1). \]

Associativity relations form an \(S_3\)-submodule in \(M_3(X)\) spanned by

\[ \nu(\nu, \text{id}) = \nu(\text{id}, \nu). \]

Example 3. Novikov algebra is a linear space with a multiplication satisfying the following axioms:

\[ (x_1 x_2) x_3 - x_1 (x_2 x_3) = (x_2 x_1) x_3 - x_2 (x_1 x_3), \quad (x_1 x_2) x_3 = (x_1 x_3) x_2. \]

The corresponding operad is generated by 2-dimensional \(\text{Nov}(2)\) spanned by \(\nu = (x_1 x_2)\) and \(\nu((12)) = (x_2 x_1)\). Defining identities of the variety Nov may be expressed as

\[ \nu(\nu, \text{id}) - \nu(\text{id}, \nu) = \nu(\nu(12), \text{id}) - \nu(12) \nu(\text{id}), \nu(\nu, \text{id}) = \nu(\nu, \text{id}). \]

Let \(\text{Var}_1\) and \(\text{Var}_2\) be two operads. Then the family of spaces \((\text{Var}_1(n) \otimes \text{Var}_2(n))_{n \geq 1}\) is an operad relative to the natural (componentwise) composition and symmetric group action (known as the Hadamard product of \(\text{Var}_1\) and \(\text{Var}_2\)). Even if \(\text{Var}_1\) and \(\text{Var}_2\) were binary operads, their Hadamard product \(\text{Var}_1 \otimes \text{Var}_2\) may be non-binary. The sub-operad of \(\text{Var}_1 \otimes \text{Var}_2\) generated by \(\text{Var}_1(2) \otimes \text{Var}_2(2)\) is known as Manin white product of \(\text{Var}_1\) and \(\text{Var}_2\), it is denoted \(\text{Var}_1 \circ \text{Var}_2\).

Example 4. The operad \(\text{Lie} \circ \text{Nov}\) is isomorphic to the operad governing the class of all algebras (magmatic operad).

Indeed, both \(\text{Lie}\) and \(\text{Nov}\) are quadratic operad, and so is \(\text{Lie} \circ \text{Nov}\). Let \(\mu\) and \(\nu\) be the generators of \(\text{Lie}\) and \(\text{Nov}\). It is enough to find the defining identities of \(\text{Lie} \circ \text{Nov}\) that are quadratic with respect to \(\mu\) and \(\nu\).

Identify \(\mu \circ \nu\) with \([x_1 \prec x_2]\), then \(\mu \circ \nu((12)) = -((\mu \circ \nu)(12)\) corresponds to \([-x_2 \prec x_1]\). Hence, \((\text{Lie} \circ \text{Nov})(n)\) is an image of the space \(M_n(X)\) of all multi-linear non-associative polynomials of degree \(n\) in \(X = \{x_1, x_2, \ldots\}\) relative to the operation \([- \prec \cdot]\). Calculating the compositions \(\mu(\mu, \text{id}) \otimes \nu(\nu, \text{id})\) and \(\mu(\text{id}, \nu) \otimes \nu(\text{id}, \nu)\), we obtain

\[ m_1 = [[x_1 \prec x_2] \prec x_3] = [[x_1 x_2] x_3] \otimes (x_1 x_2) x_3, \]

\[ m_2 = [x_1 \prec [x_2 \prec x_3]] = [x_1 [x_2 x_3]] \otimes x_1 (x_2 x_3), \]
It remains to find the intersection of the \( S_3 \)-submodule generated by \( m_1 \) and \( m_2 \) in \( M_3(X) \otimes M_3(X) \) with the kernel of the projection \( M_3(X) \otimes M_3(X) \rightarrow \text{Lie}(3) \otimes \text{Nov}(3) \). Straightforward calculation shows the intersection is zero. Hence, the operation \( [\cdot, \cdot] \) satisfies no identities.

**Example 5.** The operad \( \text{As} \circ \text{Nov} \) generated by 4-dimensional space \( (\text{As} \circ \text{Nov})(2) \) spanned by \( (x_1 \triangleright x_2), (x_2 \triangleright x_1), (x_1 \triangleright x_2), (x_2 \prec x_1) \) relative to the following identities:

\[
(x_1 \triangleright x_2) \prec x_3 - x_1 \triangleright (x_2 \prec x_3) = 0, \\
(x_1 \prec x_2) \prec x_3 - x_1 \prec (x_2 \triangleright x_3) + (x_1 \prec x_2) \triangleright x_3 - x_1 \triangleright (x_2 \prec x_3) = 0.
\]

This result may be checked with a straightforward computation, as in Example 4. Namely, one has to find the intersection of the \( S_3 \)-submodule in \( M_3(X) \otimes M_3(X) \) generated by

\[
(x_1 \prec x_2) < x_3 = (x_1 x_2 x_3) \otimes (x_1 x_2) x_3, \\
x_1 \triangleright (x_2 \triangleright x_3) = x_1 (x_2 x_3) \otimes (x_3 x_2) x_1, \\
(x_1 \triangleright x_2) \prec x_3 = (x_1 x_2 x_3) \otimes (x_2 x_1) x_3, \\
x_1 \triangleright (x_2 \prec x_3) = x_1 (x_2 x_3) \otimes (x_2 x_3) x_1, \\
(x_1 \triangleright x_2) \triangleright x_3 = (x_1 x_2 x_3) \otimes x_3 (x_1 x_2), \\
x_1 \prec (x_2 \triangleright x_3) = x_1 (x_2 x_3) \otimes x_1 (x_2 x_3), \\
x_1 \prec (x_2 \prec x_3) = x_1 (x_2 x_3) \otimes x_1 (x_2 x_3), \\
(x_1 \triangleright x_2) \triangleright x_3 = (x_1 x_2 x_3) \otimes x_3 (x_2 x_1)
\]

with the kernel of the projection \( M_3(X) \otimes M_3(X) \rightarrow \text{As}(3) \otimes \text{Nov}(3) \).

### 3. Derived identities

Given a (non-associative) algebra \( A \) with a derivation \( d \), denote by \( A^{(d)} \) the same linear space \( A \) considered as a system with two binary linear operations of multiplication \( (\cdot, \cdot) \) and \( (\cdot \triangleright \cdot) \) defined by

\[
x \prec y = xd(y), \quad x \triangleright y = d(x)y, \quad x, y \in A.
\]

Let \( \text{Var} \) be a multi-linear variety of algebras. As above, denote by \( T_{\text{Var}}(X) \) the \( T \)-ideal of all identities in a set of variables \( X \) that hold on \( \text{Var} \). A non-associative polynomial \( f(x_1, \ldots, x_n) \) in two operations of multiplication \( < \) and \( \triangleright \) is called a derived identity of \( \text{Var} \) if for every \( A \in \text{Var} \) and for every derivation \( d \in \text{Der}(A) \) the algebra \( A^{(d)} \) satisfies the identity \( f(x_1, \ldots, x_n) = 0 \). Obviously, the set of all derived identities is a \( T \)-ideal of the algebra of non-associative polynomials in two operations.

For example, \( (x \triangleright y) = (y \prec x) \) is a derived identity of \( \text{Com} \). Moreover, the operation \( (\cdot, \cdot) \) satisfies the axioms of Novikov algebras \( \text{As} \) and \( \text{Nov} \). It was actually shown in [2] that the entire \( T \)-ideal of derived identities of \( \text{Com} \) is generated by these identities.

For the variety of associative algebras, it was mentioned in [8] that \( \text{As} \) and \( \text{Nov} \) are derived identities of \( \text{As} \).

**Remark 1** (c.f. [3]). For a multi-linear variety \( \text{Var} \), the free differential \( \text{Var} \)-algebra generated by a set \( X \) with one derivation is nothing but \( \text{Var}(d^\omega X) \), where \( d^\omega X = \{ d^s(x) \mid s \geq 0, x \in X \} \).

Indeed, consider the free magmatic algebra \( F(X; d) = M(d^\omega X) \) and define a linear map \( d : F(X; d) \rightarrow F(X; d) \) in such a way that \( d(d^s(x)) = d^{s+1}(x) \), \( d(uv) = d(u)v + ud(v) \). Since \( Var \) is defined by multi-linear relations, the \( T \)-ideal \( T_{\text{Var}}(d^\omega X) \) is \( d \)-invariant. Hence, \( U = F(X; d)/T_{\text{Var}}(d^\omega X) \) is a \( \text{Var} \)-algebra with a derivation \( d \). Since all relations from \( T_{\text{Var}}(d^\omega X) \) hold in every differential \( \text{Var} \)-algebra, \( U \) is free.
Let $\mathcal{N}_{\text{Var}}$ be the class of differential Var-algebras with one locally nilpotent derivation, i.e., for every $A \in \mathcal{N}_{\text{Var}}$ and for every $a \in A$ there exists $n \geq 1$ such that $d^n(a) = 0$.

The following statement shows why the variety generated by $\mathcal{N}_{\text{Var}}$ coincides with the class of all differential Var-algebras.

**Lemma 1.** Suppose $f = f(x_1, \ldots, x_n)$ is a multi-linear identity that holds on $A^{(d)}$ for all $A \in \mathcal{N}_{\text{Var}}$. Then $f$ is a derived identity of Var.

**Proof.** Consider the free differential Var-algebra $U_n$ generated by the set $X = \{x_1, x_2, \ldots\}$ with one derivation $d$ modulo defining relations $d^n(x) = 0$, $x \in X$.

Then $U_n \in \mathcal{N}_{\text{Var}}$. As a differential Var-algebra, $U_n$ is a homomorphic image of the free magma $M(d^s X)$. Denote by $I_n$ the kernel of the homomorphism $M(d^s X) \to U_n$. As an ideal of $M(d^s X)$, $I_n$ is a sum of two ideals: $I_n = T_{\text{Var}}(d^s X) + N_n$, where $N$ is generated by $d^{n+1}(x)$, $x \in X$, $t \geq 0$. Note that the last relations form a $d$-invariant subset of $M(d^s X)$, so the ideal $N_n$ is $d$-invariant.

If a polynomial $F \in M(d^s X)$ belongs to $I_n$ and its degree in $d$ is less than $n$ then $F$ belongs to $T_{\text{Var}}(d^s X)$. If $f = f(x_1, \ldots, x_n)$ is a multi-linear identity in $U_n^{(d)}$ then the image $F$ of $f$ in $M(d^s X)$ has degree $n - 1$ in $d$, so $F \in T_{\text{Var}}(d^s X)$. Hence, $f$ is a derived identity of Var. \hfill \square

**Theorem 1.** If a multi-linear identity $f$ holds on the variety governed by the operad $\text{Var} \circ \text{Nov}$ then $f$ is a derived identity of Var.

**Proof.** For every $A \in \mathcal{N}_{\text{Var}}$ we may construct an algebra in the variety $\text{Var} \circ \text{Nov}$ as follows. Consider the linear space $H$ spanned by elements $x^{(n)}$, $n \geq 0$, with multiplication

$$x^{(n)} \cdot x^{(m)} = \left(\frac{n + m - 1}{n}\right)x^{(n+m-1)}.$$

This is a Novikov algebra (in characteristic 0, this is just the ordinary polynomial algebra). Denote $A = A \otimes k[x]$ and define operations $\prec$ and $\succ$ on $\hat{A}$ by

$$(a \otimes f) \prec (b \otimes g) = ab \otimes f \cdot g, \quad (a \otimes f) \succ (b \otimes g) = ab \otimes g \cdot f.$$

The algebra $\hat{A}$ obtained belongs to the variety governed by the operad $\text{Var} \circ \text{Nov}$. There is an injective map

$$A \to \hat{A},$$

$$a \mapsto \sum_{s \geq 0} d^s(a) \otimes x^{(s)},$$

(5)

which preserves operations $\prec$ and $\succ$. Indeed, apply (5) to $a \prec b$, $a, b \in A$:

$$(a \prec b) = ad(b) \mapsto \sum_{s \geq 0} d^s(ad(b)) \otimes x^{(s)} = \sum_{s, t \geq 0} \left(\frac{s + t}{s}\right)d^s(a)d^{t+1}(b) \otimes x^{(s+t)}$$

$$= \sum_{s, t \geq 0} d^s(a)d^{t+1}(b) \otimes x^{(s)} \cdot x^{(t+1)} = \left(\sum_{s \geq 0} d^s(a) \otimes x^{(s)}\right) \prec \left(\sum_{t \geq 0} d^t(b) \otimes x^{(t)}\right).$$

Therefore, $A^{(d)}$ is a subalgebra of a $(\text{Var} \circ \text{Nov})$-algebra, so all defining identities of $\text{Var} \circ \text{Nov}$ hold on $A^{(d)}$. Lemma 1 completes the proof. \hfill \square

**Theorem 2.** Let $f$ be a multi-linear derived identity of Var. Then $f$ holds on all $(\text{Var} \circ \text{Nov})$-algebras.

**Proof.** Let $X = \{x_1, x_2, \ldots\}$, and let $f(x_1, \ldots, x_n)$ be a non-associative multi-linear polynomial in two operations of multiplication $\prec$ and $\succ$. By definition, $f$ is an identity of the algebra $\text{Var}(X) \otimes \text{Nov}(X)$ with operations

$$(u \otimes f) \prec (v \otimes g) = (uv \otimes fg), \quad (u \otimes f) \succ (v \otimes g) = (uv \otimes gf)$$


if and only if $f$ is a defining identity of $\text{Var} \circ \text{Nov}$.

It is well-known that $\text{Nov}(X)$ is a subalgebra of $C^{(d)}$ for the free differential commutative algebra $C$ generated by $X$ with a derivation $d$ [7]. Therefore, $\text{Var}(X) \otimes \text{Nov}(X)$ is a subalgebra of $(\text{Var}(X) \otimes C)^{(d)}$, where $d(u \otimes f) = u \otimes d(f)$ for $u \in \text{Var}(X)$, $f \in C$. Hence, $\text{Var}(X) \otimes \text{Nov}(X)$ is embedded into a differential Var-algebra. □

**Remark 2.** In this paper, we consider algebras with one binary operation, so Var has only one generator. However, Theorems 1, 2 are easy to generalize for algebras with multiple binary operations, i.e., for an arbitrary binary operad Var.

**Remark 3.** In the definition of a derived identity, a derivation $d$ of an algebra $A$ may be replaced with a generalized derivation, i.e., a linear map $D : A \rightarrow A$ such that

$$D(ab) = D(a)b + aD(b) + \lambda ab, \quad a, b \in A,$$

where $\lambda$ is a fixed scalar in $\mathbb{k}$. Indeed, it is enough to note that $D - \lambda \text{id}$ is an ordinary derivation.

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