PROPAGATION OF CHAOS FOR THE BOLTZMANN EQUATION WITH MODERATELY SOFT POTENTIALS

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Abstract. We derive the 3D spatially homogeneous Boltzmann’s equation with moderately soft potentials and singular angular interaction, from an interacting particles system. The collision kernel is of the form
\[ B(z,\sigma) = |z|^\gamma b \left( \frac{z}{|z|} \cdot \sigma \right) \]
and for \( K > 0 \), \( \sin(\theta) b(\cos(\theta)) \sim K\theta^{-1-\nu} \), with \( \gamma \in (-2, -1) \) and \( \nu \in (1, 2) \) satisfying \( \gamma + \nu > 0 \). We use at the particle level the regularizing effects of the grazing collisions, in order to control the singularity of the soft potential. This enables to use a classical compactness argument, and provide a qualitative convergence result from the interacting particles system toward the solution of the limit macroscopic equation.

1. Introduction

The Boltzmann equation is a fundamental model of statistical physic. It describes the time evolution of the kinetic distribution of particles in a perfect diluted gas. The particles move with constant velocity until their paths reach another particle, in which case a collision happens and the pre-collisional velocities are changed. We are here interested in the spatially homogeneous case, and if we denote \( g_t(v) \) the particle density at time \( t \) at point \( v \), it solves
\[
\partial_t g_t(v) = \int_{\mathbb{R}^3 \times S^2} B(v-v_*,\sigma)(g_t(v')g_t(v^*_\nu) - g_t(v)g_t(v_*)) \, dv_* d\sigma,
\]
where \( v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma \), \( v^*_\nu = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma \).

The coefficient \( B \) is a nonnegative function on \( \mathbb{R}^3 \times S^2 \) called the collision kernel, which depends on the nature of the interaction between particles. We are interested here in a collision kernel \( B \) satisfying for some \( \gamma \in [-3, 1] \), some \( \nu \in (0, 2) \) and \( K > 0 \)
\[
B(v-v_*,\sigma) = |v-v_*|^\gamma b \left( \frac{v-v_*}{|v-v_*|} \cdot \sigma \right),
\]
and
\[
\cos(\theta) = \frac{v-v_*}{|v-v_*|} \cdot \sigma, \quad \beta(\theta) := \sin(\theta)b(\cos(\theta)) \sim K\theta^{-(1+\nu)}.
\]
The angle \( \theta \) is the angle between the pre and post-collisional relative velocities, and is called the angle of deviation. We thus talk about grazing collisions kernel, since the collisions with small angle of deviation are more weighted by the kernel. When \( \gamma > 0 \), we are in the case of hard potentials, for \( \gamma = 0 \) in the case of Maxwellian molecules and for \( \gamma < 0 \) in the case of soft potentials. In this paper, we focus on the moderately soft potentials, that is the special case \( \gamma \in (-2, 0) \). At the limit \( \nu \to 2^- \), that is the grazing collision limit, equation (1.1) becomes the Landau equation (see for instance on this topic [13], [38]). In the

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In this paper, we address the question of the propagation of chaos for equation \([1.1]\). This question almost goes back to Boltzmann’s historical paper [5], in which he derives the equation which now bares his name, from the dynamic of the atoms composing the gas. To do so, some assumptions coming from the classical mechanics law are used, such as the elasticity of the collisions between atoms, or the reversibility of the atomic dynamic. But he also uses an assumption of statistical nature, namely the \textit{Stosszahlansatz}. Literally, ”assumption about counting of the chocks”, also known as \textit{reversibility of the atomic dynamic}. This assumption means that the correlations between two particles among all the particles composing the gas, are negligible. We refer the reader interested in historical and heuristical considerations about this topic to the french note [6].

In one of the founding papers of the mathematical kinetic theory [26], Kac introduces a probabilistic framework to formalize the Boltzmann’s idea of the molecular chaos, namely the chaos property (we refer to the lecture notes [35]). Defining a component Markov process, he showed that along this process the chaos property is propagated in time. This enables to justify the molecular chaos assumption, and thus to rigorously derive a toy model for the Boltzmann equation.

The rest of the paper is organized as follows. In Section 2, we introduce the particles system from which we derive equation \([1.1]\), make some comments about the more or less recent literature on the topic. We state the main result of the paper Theorem 2.1 and give a sketch of proof. In Section 3, we establish the proof of Theorem 2.1 which contains the bounds, uniform in the number of particles in the system, of the quantities of interest. Finally, in Section 4, we use these bounds to complete the proof of Theorem 2.1 using a classical martingale method (see for instance [34]).

Two appendices are dedicated to gathering some properties of some coefficients of the particles system. Two others, to the proof of the key Propositions 2.1 and 2.2 respectfully.

\textbf{Notation}:

- Lebesgue and Sobolev’s norm: For \(s \in (0,1)\) the fractional Sobolev’s norm as
  \[
  |f|^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} \frac{(f(x) - f(y))^2}{|x-y|^{d+2s}} dxdy.
  \]

  Denoting the Fourier transform as \(\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} e^{-iv \cdot \xi} f(v) dv\), we may also define it as
  \[
  |f|^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\mathcal{F}(f)(\xi)|^2 |\xi|^{2s} d\xi.
  \]

  We denote \(\langle x \rangle = \sqrt{1+|x|^2}\), and for \(k \geq 0\)
  \[
  L_k = \{f \in L^1, \text{s.t. } \langle \cdot \rangle^k f \in L^1\},
  \]
  and
  \[
  L \ln L = \{f \in L^1(\mathbb{R}^d), \text{ s.t. } H(f) := \int_{\mathbb{R}^d} f \ln f < \infty\}.
  \]

- Probability measures: For a functional \(F\) on \(\mathbb{R}^{dN}\), and \(i = 1, \ldots, N\) we note \(\nabla_i F = a_i \cdot \nabla F^N \in \mathbb{R}^d\), where \(a_i = \left(0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0\right) \in \mathbb{R}^{dN}\). The notation \(V\) will stand for the integration variable in \(\mathbb{R}^{dN}\), and for \(V \in \mathbb{R}^{dN}\), \(V^{N-1}\) stands for \((v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_N)\).

  The notation \(\mathcal{P}(E)\) stands for the set of probability measures on \(E\), \(\mathcal{P}_{\text{Sym}}(E^N)\) stands for the set of sequences of symmetric probabilities on \(E^N\), i.e. invariant by permutation. \(W_p\) is the Wasserstein metric on \(\mathcal{P}(\mathbb{R}^d)\) of order \(p \geq 1\). For \(T > 0\), the notation \(\mathbb{D}([0, T]; \mathbb{R}^d)\) stands for the
Define. We refer to \cite[Lemma 1.1]{17} for the implication of this assumption. We also fix the notation throughout all the paper, we work under the assumption
\[
\text{which heuristically values one if}\quad g(\varepsilon)(\varepsilon)\in X\text{ and if it is not the case, if a too large proportion of this mass lies closely to some plane.}
\]

- Miscellaneous: For the sake of simplicity, numerical constants are all denoted $C$, and when constants depend on parameters of the problem, this dependence is expressed in the index. For $\gamma \geq 0$, we denote $K_\gamma : x \in \mathbb{R}^3 \mapsto |x|^\gamma x \in \mathbb{R}^3$. We consider $(\rho_\varepsilon)_{\varepsilon>0}$ a family of even mollifying kernels, such that for any $k \geq 2$
\[
\sup_{\varepsilon \in (0,1)} \int_{\mathbb{R}^3} |v|^k \rho_\varepsilon(dw) \leq C_k, \quad C \in \mathbb{R}^d, \quad \sup_{\varepsilon \in (0,1)} |K_\gamma \ast \rho_\varepsilon(v)| \leq C(1 + |K_\gamma(v)|).
\]
Finally for $A > 0$, we define a smooth function $\chi_A : \mathbb{R}^+ \mapsto [0,1]$ such that $1_{[0,A]} \leq \chi_A \leq 1_{[0,A+(1\vee A)]}$.

2. Preliminaries and main results of the paper

2.1. A perturbed Nanbu particles system. We follow here \cite[Section 1.5]{17}. For $X \in \mathbb{R}^3$ we define $I(X), J(X)$ two unit vectors such that \[
\left( \begin{array}{c} X \vspace{1mm} \\ I(X) \\ J(X) \end{array} \right) \text{ is a direct orthonormal basis of } \mathbb{R}^3, \quad \text{for } \theta \in [0, \pi] \text{ and } \varphi \in [0, 2\pi] \text{ define}\]
\[
\begin{align*}
\Gamma(X, \varphi) & := \cos(\varphi) I(X) + \sin(\varphi) J(X) \\
v' & := v - \frac{1}{2} \frac{\cos(\varphi)}{(v - v_*)} \Gamma(v - v_*, \varphi) \\
v'_* & := v_* + \frac{1}{2} \frac{\cos(\varphi)}{(v - v_*)} \Gamma(v - v_*, \varphi) \\
a(v, v_*, \theta, \varphi) & = v' - v = - (v'_* - v_*).
\end{align*}
\]
Throughout all the paper, we work under the assumption
\[
\text{(H)} \quad \text{There are } K_1 > K_2 > 0 \text{ such that for any } \theta \in (0, \pi), \text{ there holds } K_1 \theta^{-1} \leq \beta(\theta) \leq K_2 \theta^{-1} - \nu.
\]
We refer to \cite[Lemma 1.1]{17} for the implication of this assumption. We also fix the notation
\[
\beta_0 = \pi \int_0^\pi (1 - \cos(\theta)) \beta(\theta)d\theta.
\]
Define
\[
c_{\gamma, \nu}(v, v_*, z, \varphi) = a \left( v, v_*, G_{\nu} \left( \frac{z}{|v - v_*|}, \varphi \right) \right), \quad c_{\gamma, \nu}(z) = \inf \left\{ x \in (0, \pi) , \int_0^\pi \beta(\theta) d\theta \leq z \right\},
\]
and introduce the notations for $\phi \in C^2(\mathbb{R}^3)$ and respectfully $\psi \in C^2(\mathbb{R}^3N)$ and $i = 1, \cdots , N$
\[
\begin{align*}
\Delta_\nu \phi (v, v_*, z, \varphi) & := (v + c_{\gamma, \nu}(v, v_*, z, \varphi)) \cdot \nabla \phi (v) \\
\Delta_\nu \Phi (V, v_*, z, \varphi) & := \Phi (V + c_{\gamma, \nu}(v, v_*, z, \varphi) a_i) - \Phi (V) - c_{\gamma, \nu}(v, v_*, z, \varphi) \cdot \nabla_i \Phi (V).
\end{align*}
\]
Finally for any $R > 1, \delta, \eta \in (0,1)$ we introduce the smooth cut off function on probability measures on $\mathbb{R}^3$
\[
a_{\delta, \eta}^R(g) = (1 - \chi_{\eta}^2) \left( \sup_{\varepsilon \in \mathbb{R}^2} \int_{\mathbb{R}^3} \chi_R(v) \chi_R(v_*) \chi_{2\delta} ((v - v_*) \cdot e) g dv g dv_* \right) + \chi_{4\eta} \left( \int_{\mathbb{R}^3} \chi_{R-1}(v) g dv \right),
\]
which heuristically values one if $g$ the mass of the ball of radius $R$ is too small (w.r.t. to the threshold $4\eta$) and if it is not the case, if a too large proportion of this mass lies closely to some plane.
We can now define the interacting particles system. Consider a sequence \((\varepsilon_N)_{N \geq 2}\) converging to zero. For \(N \geq 2\) we denote \((M^N_i)_{i=1,\ldots,N}\), \(N\) independent Poisson random measures (see for instance Chapter VI) on \(\mathbb{R}^+ \times \mathbb{R}^+ \times [0,2\pi] \times \mathbb{R}^+ \times \{1,\ldots,N\} \setminus \{i\}\) with intensity \(ds \times dz \times d\varphi \times \rho_{\varepsilon_N}(dw) \times \frac{1}{N-1} \sum_{j \neq i} \delta_j\). Let also \((B^i_t)_{t \geq 0, i=1,\ldots,N}\) be \(N\) independent Brownian motions independent of the \((M^N_i)_{i=1,\ldots,N}\). Then we consider the following system of SDEs with jumps

\[
\begin{align*}
\gamma_{t,i}^N &= \nu_{0,i}^N + \int_{[0,t] \times [0,2\pi]} (K^N_{s-} \Phi_{s-}^N + w, z, \varphi) \mathcal{M}_N^i(ds, d\varphi, dz, dw, dj) \\
&+ \frac{1}{N-1} \sum_{j \neq i} \int_0^t \beta_0 K^N_{s-} \rho_{\varepsilon_N} (\gamma_{s-}^N, \gamma_{s-}^N) ds + \int_0^t \sqrt{2} \alpha_{s,H} \left( \frac{1}{N-1} \sum_{j \neq i} \delta_{\gamma_{s-}^N, \rho_{\varepsilon_N}} \right) dB_s^i
\end{align*}
\]

Ito’s rule for jump process yields (see for instance [4, Theorem 4.4.7]) that for any smooth test function \(\Phi \in C^2(\mathbb{R}^{3N})\)

\[
\begin{align*}
\Phi(\gamma_t^N) &= \Phi(\gamma_0^N) + \sum_{i=1}^N \int_0^t \frac{1}{N-1} \sum_{j \neq i} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\infty} (\delta_{\gamma_{s-}^N, \rho_{\varepsilon_N}}(\gamma_{s-}^N, \gamma_{s-}^N) + w, z, \varphi) a_i ds d\varphi dz d\rho_{\varepsilon_N}(dw) ds \\
&+ \sum_{i=1}^N \int_0^t \frac{1}{N-1} \sum_{j \neq i} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\infty} (\delta_{\gamma_{s-}^N, \rho_{\varepsilon_N}}(\gamma_{s-}^N, \gamma_{s-}^N) + w, z, \varphi) a_i ds d\varphi dz d\rho_{\varepsilon_N}(dw) ds \\
&+ \int_0^t \sqrt{2} \alpha_{s,H} \left( \frac{1}{N-1} \sum_{j \neq i} \delta_{\gamma_{s-}^N, \rho_{\varepsilon_N}} \right) \nabla_i \Phi(\gamma_s^N) \cdot dB_s^i + \int_0^t \left( \alpha_{s,H} \left( \frac{1}{N-1} \sum_{j \neq i} \delta_{\gamma_{s-}^N, \rho_{\varepsilon_N}} \right) \right)^2 \Delta_i \Phi(\gamma_s^N) ds.
\end{align*}
\]

Taking the expectation, we find that \((G^N_t)_{t \geq 0}\), where \(G^N_t = \mathcal{L}(\gamma_t^N) \in \mathcal{P}_{sym}(\mathbb{R}^{3N})\), is weak solution to the Master Equation

\[
\partial_t G^N_t = \mathcal{A}^N G^N_t, \tag{2.5}
\]

where the generator \(\mathcal{A}^N\) is defined by duality as

\[
\begin{align*}
\mathcal{A}^N \Phi(V) &= \sum_{i=1}^N \frac{1}{N-1} \int_{\mathbb{R}^3} (K^N_{s-} \rho_{\varepsilon_N})(v_i - v_j) \cdot \nabla_i \Phi(V) + \sum_{i=1}^N \left( \alpha_{s,H} \left( \frac{1}{N-1} \sum_{j \neq i} \delta_{v_{s-}^N, \rho_{\varepsilon_N}} \right) \right)^2 \Delta_i \Phi(V) \\
&+ \sum_{i=1}^N \frac{1}{N-1} \sum_{j \neq i} \int_{\mathbb{R}^3} \rho_{\varepsilon_N}(dw) \int_0^{2\pi} \int_{\mathbb{R}^3} d\varphi dz (\Phi(V + c_{\gamma_{s-}^N, \rho_{\varepsilon_N}}(v_i, v_j + w, z, \varphi)a_i) - \Phi(V) - c_{\gamma_{s-}^N, \rho_{\varepsilon_N}}(v_i, v_j + w, z, \varphi) \cdot \nabla_i \Phi(V))
\end{align*}
\]

We use the change of variables \(w = v_j + w\), and since the mollifiers \(\rho_{\varepsilon}\) are even, we find that

\[
\begin{align*}
&= \int_{\mathbb{R}^3} \frac{1}{N-1} \sum_{j \neq i} \delta_{v_{s-}^N, \rho_{\varepsilon_N}}(dw) \int_0^{2\pi} \int_{\mathbb{R}^3} d\varphi dz (\Phi(V + c_{\gamma_{s-}^N, \rho_{\varepsilon_N}}(v_i, v_j + w, z, \varphi)a_i) - \Phi(V) - c_{\gamma_{s-}^N, \rho_{\varepsilon_N}}(v_i, v_j + w, z, \varphi) \cdot \nabla_i \Phi(V)).
\end{align*}
\]
Then for fixed \( v_i, w \), we use the change of variable \( \theta = G_r \left( \frac{z}{|v_i - w|^\gamma} \right) \), which yields \( dz = -|v_i - w|^{\gamma} \beta(\theta) d\theta \), and then
\[
\int_0^{2\pi} d\varphi \int_{\mathbb{R}^+} dz \left( \Phi(V + c_{\gamma, \nu}(v_i, w, z, \varphi) a_i) - \Phi(V) \right) = \int_0^{2\pi} d\varphi \int_{\mathbb{R}^+} d\theta |v_i - w|^{\gamma} \beta(\cos(\varphi)) \left( \Phi(V + a(v_i, w, \theta, \varphi) a_i) - \Phi(V) \right)
\]
so that, in view of point (i) of Lemma A.1, we can rewrite \( \mathcal{A}^N \) as
\[
\mathcal{A}^N \phi(V) = \sum_{i=1}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v_i - w, \sigma) \left( \phi(V + (v'_i - v_i) a_i) - \phi(V) \right) \left( \frac{1}{N-1} \sum_{j \neq i}^{N} \delta_{v_j} \ast \rho_{\varepsilon N} \right)(w) dwd\sigma
\]

\[\text{(2.6)}\]

2.2. Main results. We begin this section with some comments about the literature concerning the problem of propagation of chaos for the Boltzmann and Landau equations. We start this non exhaustive list with the seminal papers by Tanaka [36], and Sznitman [34], which concern respectively Maxwellian molecules (\( \gamma = 0 \) and \( \int_0^1 \theta \beta(\theta) d\theta < \int_0^1 \beta(\theta) d\theta = +\infty \)) and hard spheres (\( \gamma = 1 \), and \( b(x) = |x| \) in (1.2)). In the later, a convergence result from an interacting particles system to the corresponding limit equation, is qualitatively established thanks to a martingale method, which we are going to use in this paper. Both cases have been quantitatively treated in [29], thanks to functional framework which relies on a semi-group approach. See [29] Theorem 5.1 for the case of true Maxwellian molecules (i.e. \( \gamma = 0 \) and \( \nu = \frac{1}{2} \)), and [29] Theorem 6.1 for the case of hard spheres (i.e. \( \gamma = 1 \) and \( b \equiv 1 \) in (1.2)). A similar semi group approach is applied to the Landau equation for Maxwellian molecules (i.e. \( \gamma = 0 \) and \( \nu \to 2^- \)) in [10]. A probabilistic approach is developed in [24], in the case of hard potentials \( \gamma \in [0, 1] \) and \( \nu \in (0, 1) \) (see [23] Theorem 1.4), which provides a rate of convergence in Wasserstein metric of the empirical measure associated to some interacting particles system (similar to (2.4) modulo the correction coefficient \( \alpha^R_{\delta, \eta} \) and the mollification w.r.t. \( \rho_{\varepsilon, \eta} \)), toward the solution of the corresponding Boltzmann equation. A comparable coupling method is used in [19], for the Landau equation with hard potentials, with uniform in time rate in the special case of Maxwellian molecules (see [19] Theorem 4).

When one turns to the case of soft potentials, difficulties arise from the singularity in the collision kernel, and it is one of the trending topic among the kinetic community to obtain some propagation of chaos result for singular interaction. For the Boltzmann equation, a quantitative result in Wasserstein metric is obtained in [10] in the case of moderately soft potentials \( \gamma \in (-1, 0) \), \( \nu \in (0, 1) \) and \( \gamma + \nu > 0 \) (see [10] Theorem 1.4), and it is to the best of the author’s knowledge, the only propagation of chaos result for the Boltzmann equation with soft potentials. As for the Landau equation, the full range of moderately soft potentials \( \gamma \in (-2, 0) \) is treated in [29]. In the range \( \gamma \in (-1, 0) \), a quantitative result is obtained with a similar technique as [10] (see [20] Theorem 1.6)). For the full range of soft potentials \( \gamma \in (-2, 0) \), a qualitative convergence result is obtained thanks to an information-based approach (see [20] Theorem 1.8). The main result of this paper, consists in extending this approach to the Boltzmann equation, and is stated in the

**Theorem 2.1.** Let \( \gamma \in (-2, -1) \) and \( \nu \in (1, 2) \) be such that \( \gamma + \nu > 0 \), and a collision kernel \( B : \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}^+ \) of the form (1.2)-(1.3) satisfying (H). Let \( p \in \left( \frac{3}{3+\gamma}, \frac{3}{3+\nu} \right) \), \( g_0 \in L \ln L(\mathbb{R}^3) \cap \mathcal{P}_k(\mathbb{R}^3) \) for some \( k \geq |\gamma| p - \frac{3}{3+\nu} - p \), and \( (G^N_0)_{N \geq 1} \in \mathcal{P}_{\text{sym}}(\mathbb{R}^{3N}) \) be a \( g_0 \)-chaotic (in the sense of [35] Definition 2.1)\) sequence satisfying
\[
H_0 := \sup_{N \geq 1} \frac{1}{N} \int_{\mathbb{R}^{3N}} G^N_0(V) \ln(G^N_0(V))dV < \infty, \quad M_0 := \sup_{N \geq 1} \int |v_1|^k G^N_0(V)dV < \infty. \quad (2.7)
\]
Let $T > 0$ and
$$R > 2 + \left( e^{C_k T} (1 + M_0) \right)^{1/k}, \quad \eta < C \left( 1 - \frac{e^{C_k T} (1 + M_0)}{4(R - 2)^k} \right), \quad \delta < \frac{C R^{-5}}{\eta^2} \exp \left( 4 C_k + 2(M_0 + C_k \gamma, \nu(T+1)e^{C_k T} (1+M_0)) \right),$$
for some explicit constants $C, C_k, C_k, \gamma, \nu > 0$. For each $N \geq 2$ consider $(V_N^t)_{t \in [0,T]}$ a $\mathbb{R}^{3N}$-valued process solution to (2.4), for a $G_0^N$-distributed initial condition.

Then $\left( \frac{1}{N} \sum_{i=1}^N \delta_{V_N^t} \right)_{t \in [0,T]}$ converges, as $N$ goes to infinity, weakly in law to the unique solution to the Boltzmann equation (1.1) starting from $g_0$, $(g_t)_{t \in [0,T]} \in L^1(0,T; L^p(\mathbb{R}^3)) \cap L^\infty(0,T; L^1_k(\mathbb{R}^3))$.

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**Figure 1.** Topography of propagation of chaos results for equation (1.1) in the rectangle $(\nu, \gamma) \in (0, 2] \times [-3, 1]$. The case $\nu = 2$ corresponds to the Landau equation. The diagonal corresponds to the inverse power law $\gamma = \frac{s-5}{s-1}, \nu = \frac{2}{s-1}$ for $s \in [2, \infty)$.

The novelty of this result, is that to the best of the author’s knowledge, it deals with a range of softness of potentials which was not covered before, for the Boltzmann equation. The two drawbacks are that it provides only a qualitative convergence, and it misses the physical cases of the inverse power law potentials.

### 2.3. Sketch of the proof.

The strategy we use to treat the propagation of chaos for equation with very singular coefficient is the one which was first established in [21]. The authors treat the 2D Navier-Stokes equation in vortex formulation. Thanks to an entropy dissipation method, they obtain a bound on a key information quantity (namely the Fisher information) uniformly in the number of particles. This
enables to deduce both the existence of a converging subsequence of the empirical measure associated to the interacting diffusions system, and the uniqueness of the limit point. Still in the case of mean-field equation, the same method has been applied to some sub-critical Keller-Segel equation in [24]. Then the author extended this strategy to fractional diffusion in [33].

As for collisional dynamics, the Landau equation with moderately soft potentials \((\gamma \in (-2, 0))\) has been treated in [20, Theorem 1.8], and it is this strategy that we adapt to the Boltzmann equation. Seeing the Landau equation as the grazing collisions limit of the Boltzmann equation, the idea is thus to adapt the techniques used for Brownian-like diffusion to a Lévy flight diffusion (as from [24] to [33]).

In that purpose, the functional which plays an essential role, is the weighted normalized fractional Fisher information \(I^{N}_{H,N} \) defined for \(G^{N} \in \mathcal{P}(\mathbb{R}^{dN})\) as

\[
I^{N}_{\nu,\gamma}(G^{N}) = \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{d(N-1)}} \int_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \frac{\left(\gamma/2 \sqrt{G^{N}(v_{1}, v_{i-1}, v, v_{i+1}, v^{N})} - \langle v_{*} \rangle \sqrt{G^{N}(v_{1}, v_{i-1}, v, v_{i+1}, v^{N})}\right)^{2}}{|v_{*} - v|^{2+\nu}} d\nu dV_{i}^{N-1},
\]

we define it with general dimension, as the results about this quantity, provided in this paper do not depend on the dimension). In view of the different ways to define Sobolev’s semi norm, we may rewrite the fractional Fisher information as

\[
I^{N}_{\nu,\gamma}(G^{N}) = \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{d(N-1)}} \int_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \left| \gamma/2 \sqrt{G^{N}_{i,N-1}(\cdot)} \right|^{2}_{H^{\nu/2}(\mathbb{R}^{d})} dV_{i}^{N-1}.
\]

where for a fixed \(V_{i}^{N-1} \in \mathbb{R}^{d(N-1)}\) we denote

\[
G^{N}_{i,N-1}(v) = G^{N}(v_{1},\cdots,v_{i-1},v,v_{i+1},\cdots,v_{N}).
\]

Fix \(T > 0\), and assume that there is a sequence of solutions \((V_{t}^{N})_{t \in [0,T]} \) and for each \(t \in [0,T]\) denote \((G_{t}^{N},)_{N \geq 2}\), the sequence of the law of these solutions (i.e. \(G_{t}^{N} = \mathcal{L}(V_{t}^{N})\)). The first element of the proof is the control (uniformly w.r.t. the number of particles in the system) of some useful normalized quantities; namely, the entropy, moment and local fractional Fisher information of the law, in the

**Theorem 2.2.** Assume that the collision kernel \(B\) of the form \([1.2]-[1.3]\) satisfies \((H)\) for some \((\gamma, \nu) \in (-2, 0) \times (0, 2)\), with \(\gamma + \nu > 0\). Let \(R > 1, \delta, \eta \in (0, 1)\), and for each \(N \geq 2\), let \((G_{t}^{N})_{t \in [0,T]}\) be a solution to \([2.5]\) starting from \((G_{0}^{N})_{N \geq 1} \in \mathcal{P}_{sym}(\mathbb{R}^{3N})\), satisfying \([2.7]\).

Then for any \(T > 0\), it holds

\[
\sup_{t \in [0,T]} \sup_{N \geq 2} \frac{1}{N} \int_{\mathbb{R}^{3N}} G_{t}^{N}(V) \ln(G_{t}^{N}(V)) dV \leq H_{0} + C_{k,\gamma,\nu} T e^{C_{k} T} (1 + M_{0}),
\]

\[
\sup_{t \in [0,T]} \sup_{N \geq 2} \int_{\mathbb{R}^{3N}} |v_{1}|^{k} G_{t}^{N}(V) dV \leq e^{C_{k} T} (1 + M_{0}),
\]

\[
\sup_{N \geq 2} \int_{0}^{T} I^{N}_{\nu,\gamma}(G_{t}^{N}) dt \leq C_{H_{0},M_{0},T,\gamma,\nu,k,\eta,\delta,\gamma,\nu},
\]

The key estimate \([2.11]\) is obtained using the results of Alexandre et al (see [2], [3]) about the regularizing effects of grazing collisions. It is in order to apply these techniques as black boxes, that we need the Brownian diffusion perturbation in \([2.4]\), which properties are listed in Appendix B. Then we make a first use of this key estimate, thanks to the
Proposition 2.1. For any $d \geq 2$, $\lambda \in (0, \nu)$ and $r \in \left( \frac{d+\nu}{d+\nu} \vee \frac{\nu+\lambda}{2\nu} \vee \frac{2d-(\nu-\lambda)}{2d}, 1 \right)$ there is $c_{\lambda,d,\nu,\gamma,r} > 0$ such that for any $F^N \in \mathcal{P}_\text{sym}(\mathbb{R}^{dN})$ it holds
\[
\sup_{i\neq j} \int_{\mathbb{R}^{dN}} |v_i - v_j|^{-\lambda} F^N (dV) \leq c_{\lambda,d,\nu,\gamma,r} \left( I_{\gamma,\gamma}^N (F^N) + \int_{\mathbb{R}^{dN}} |v_1|^{-\gamma} \pi F^N (dV) + 1 \right).
\]

This result can be seen as some Hardy-Littlewood-Sobolev inequality \cite{25, Theorem 4.3} in the case where the integrating measure is not tensorized. Since the most singular part of the trajectory of one particle, is the drift part in \cite{2,4}, this bound is sufficient to deduce in Proposition 1.1 using some classical stochastic calculus tools, that there exists an accumulation point to the sequence $\left( \frac{1}{N} \sum_{i=1}^N \delta_{v_i} \right)_{N \geq 2}$ in $\mathcal{P}(\mathbb{R}^3)$. We then prove that the support of the law of this accumulation point is included in the set of probability measures with smooth density, thanks to the

Proposition 2.2. The family of functionals $(I_{\gamma,\gamma}^N)_{N \geq 1}$ defined in \cite{2,8} are $\Gamma$-lower semi-continuous, i.e. if $(F_N)_{N \geq 1} \in \mathcal{P}_\text{sym}(\mathbb{R}^{dN})$ converges to $\pi \in \mathcal{P}(\mathbb{P}(\mathbb{R}^d))$ in the sense that for any $j \geq 2$, $F_N \rightharpoonup \int_{\mathbb{P}(\mathbb{R}^d)} \rho^{\otimes j} \pi(d\rho)$, where $F_N^j$ denotes the $j$-particles marginal of $F_N$, then
\[
\liminf_{N \to 1} I_{\gamma,\gamma}^N (F_N) \geq \int_{\mathbb{P}(\mathbb{R}^d)} \left( \langle \cdot \rangle^{\gamma/2} \pi \right)_{H^{\gamma/2}} (d\rho).
\]

Moreover, we show in Proposition 4.2 that the accumulation point solves a martingale problem, as is classical in the context of qualitative propagation of chaos result (see for instance \cite{34,22,30}). Thanks to the above regularity estimate, and a slight modification of \cite{17, Theorem 1.3} we show that this martingale problem admits a unique solution. Finally, choosing the parameters $R, \delta$ and $\eta$ in \cite{2,4}, according to some bounds on the entropy and moment (that is why we specify the independence with respect to these parameters in \cite{2,9} and \cite{2,10}) of the initial condition $g_0$ and the time horizon $T$, this solution actually coincides with the solution to the Boltzmann equation \cite{1,1} starting from $g_0$, on the time interval $[0, T]$.

3. Proof of Theorem 2.2

This section is devoted to the proof of the a priori bounds on the law of solutions to SDE \cite{2,4}. In the first subsection we obtain the moment estimate \cite{2,11,2,10}, mimicking the arguments of \cite{20, Proposition 4.1}. In the second we use some entropy dissipation method to prove the bounds \cite{2,9,2,11}. Before further considerations, we make the following observation. For any functions $h$ and $g$ we have using Young’s and Holder’s inequalities
\[
\frac{1}{2} \int_{\mathbb{R}^d} |h \nabla g|^2 - 4 \int_{\mathbb{R}^d} |g \nabla h|^2 \leq |\nabla (gh)|^2 \leq 2 \int_{\mathbb{R}^d} |h \nabla g|^2 + 2 \int_{\mathbb{R}^d} |g \nabla h|^2. \quad (3.1)
\]

And similarly for any $\nu \in (0, 2)$
\[
\frac{1}{2} \int_{\mathbb{R}^{2d}} \frac{h^2(x) (g(x) - g(y))^2}{|x-y|^d + \nu} \, dx \, dy - 4 \int_{\mathbb{R}^{2d}} g^2(x) \left( \frac{h(x) - h(y))^2}{|x-y|^d + \nu} - h(y) \right) \, dx \, dy \leq |h|^2_{H^{\nu/2}}
\]
\[
\leq 2 \int_{\mathbb{R}^{2d}} h^2(x) \frac{g(x) - g(y))^2}{|x-y|^d + \nu} \, dx \, dy + 2 \int_{\mathbb{R}^{2d}} g^2(x) \frac{h(x) - h(y))^2}{|x-y|^d + \nu} \, dx \, dy. \quad (3.2)
\]

In particular if $h$ is bounded and Lipschitz
\[
\frac{1}{2} \int_{\mathbb{R}^{2d}} h^2(x) \frac{g(x) - g(y))^2}{|x-y|^d + \nu} \, dx \, dy - C_{d,\nu} \|h\|^2_{L^{1,\infty}} \|g\|^2_L \leq |h|^2_{H^{\nu/2}} \leq 2 \int_{\mathbb{R}^{2d}} h^2(x) \frac{g(x) - g(y))^2}{|x-y|^d + \nu} \, dx \, dy + 2C_{d,\nu} \|h\|^2_{W^{1,\infty}} \|g\|^2_L. \quad (3.3)
\]

Finally, let us make some comments about the well posedness of the system of SDE \cite{2,4}. For fixed $N \geq 2$, the maps $v \in \mathbb{R}^d \mapsto K_v * \rho_{\nu, \gamma} (v) \in \mathbb{R}^d$ and $V \in \mathbb{R}^{dN} \mapsto \alpha_{\nu, \gamma} \left( \frac{1}{N} \sum_{j \neq i} \delta_{v_j} * \rho_{\nu, \gamma} \right)$ for each $i = 1, \cdots, N$, are bounded and Lipschitz, so difficulties may arise only from the coefficient $c_{r,\nu}$. But existence
and uniqueness of solution to (2.4) hold when one considers instead this coefficient, a regularization version of it. Therefore using the same tightness-martingale formulation method that we are about to use to treat the limit $N \to \infty$, enables to pass to the limit in the regularized parameter for fixed $N$. This enables to conclude that for each $N \geq 2$, and $\mathcal{Y}^N_0$ a $\mathbb{R}^{3N}$-valued random vector, there exists, on some suitable probability space, a solution to (2.4) starting from $\mathcal{Y}^N_0$. We do not treat the question of uniqueness, but it is not needed in order to state Theorem 2.1

3.1. Moments estimates.

**Lemma 3.1.** For any $k \geq 2$, there is a constant $C_k > 0$ independent of $R, \delta, \eta$ and $N \geq 2$ such that for any $(\mathcal{Y}^N_t)_{t \geq 0}$ solution to the particles system (2.4), it holds for any $T > 0$

$$
\sup_{t \in [0, T]} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ |\mathcal{Y}^N_i|^k \right] \leq e^{C_k T} \left( 1 + \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ |\mathcal{Y}^N_0|^k \right] \right).
$$

**Proof.** First, an application of Ito’s rule yields for any $i = 1, \ldots, N$

$$
|\mathcal{Y}^N_i|^k = |\mathcal{Y}^N_i|^k - k \int_0^t \frac{\beta_0}{N-1} \sum_{j \neq i} \mathcal{K}_\gamma \ast \rho_{\varepsilon_N} (\mathcal{Y}^N_j - \mathcal{Y}^N_i) \cdot |\mathcal{Y}^N_i|^{k-2} \mathcal{Y}^N_i \, ds
$$

$$
+ \int_0^t \frac{1}{N-1} \sum_{j \neq i} \int_{\mathbb{R}^3} \int_{0}^{2\pi} \int_{0}^{\infty} \bar{\Delta} \cdot |\mathcal{Y}^N_i| \left( \mathcal{Y}^N_i \cdot \mathcal{Y}^N_j + w, z, \varphi \right) \, dz \, d\varphi \rho_{\varepsilon_N} (d\omega) ds
$$

$$
+ \int_{[0,t] \times [0,2\pi] \times \mathbb{R}^3 \times \{1, \ldots, N\} \setminus \{i\}} \left( \mathcal{Y}^N_i + c_{ij} \right) \left( \mathcal{Y}^N_j, \mathcal{Y}^N_i \right)^{k-1} \rho_{\varepsilon_N} (ds, dz, d\varphi, dw, dj)
$$

$$
+ \int_0^t \left( \rho_{\delta, \eta} \left( \frac{1}{N} \sum_{j \neq i} |\mathcal{Y}^N_j| * \rho_{\varepsilon_N} \right) \right)^2 |\mathcal{Y}^N_i|^{k-2} ds + \sqrt{2} \int_0^t \alpha_{\delta, \eta} \left( \frac{1}{N} \sum_{j \neq i} \mathcal{D} \rho_{\varepsilon_N} \right) |\mathcal{Y}^N_i|^{k-2} \, dB_t.
$$

We take the expectation, average over $i = 1, \ldots, N$, and denote

$$
Q(t) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ |\mathcal{Y}^N_i|^k \right].
$$

Which yields

$$
Q(t) = Q(0) - k \int_0^t \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \frac{\beta_0}{N-1} \sum_{j \neq i} \mathcal{K}_\gamma \ast \rho_{\varepsilon_N} (\mathcal{Y}^N_j - \mathcal{Y}^N_i) \cdot |\mathcal{Y}^N_i|^{k-2} \mathcal{Y}^N_i \right] ds
$$

$$
+ \frac{1}{N(N-1)} \sum_{i \neq j} \mathbb{E} \left[ \int_{[0,2\pi] \times \mathbb{R}^3} \bar{\Delta} \cdot |\mathcal{Y}^N_i| \left( \mathcal{Y}^N_i \cdot \mathcal{Y}^N_j + w, z, \varphi \right) \, dz \, d\varphi \rho_{\varepsilon_N} (d\omega) \right]
$$

$$
+ \int_0^t \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \left( \rho_{\delta, \eta} \left( \frac{1}{N} \sum_{j \neq i} |\mathcal{Y}^N_j| * \rho_{\varepsilon_N} \right) \right)^2 |\mathcal{Y}^N_i|^{k-2} \right] ds
$$

$$
=: Q(0) + I_t^N + J_t^N + K_t^N.
$$

• Estimate of $I_t^N$:

By evenness of $\rho_{\varepsilon_N}$, it holds $\mathcal{K}_\gamma \ast \rho_{\varepsilon_N}(-x) = -\mathcal{K}_\gamma \ast \rho_{\varepsilon_N}(x)$, and $\mathcal{K}_\gamma \ast \rho_{\varepsilon_N}(0) = 0$. Hence by symmetry we find that for any $(i, j) \in \{1, \ldots, N\}^2$

$$
\mathbb{E} \left[ K_t \ast \rho_{\varepsilon_N} (\mathcal{Y}^N_i - \mathcal{Y}^N_j) \cdot |\mathcal{Y}^N_i|^{k-2} \mathcal{Y}^N_i \right] = \frac{1}{2} \mathbb{E} \left[ K_t \ast \rho_{\varepsilon_N} (\mathcal{Y}^N_i - \mathcal{Y}^N_j) \cdot |\mathcal{Y}^N_i|^{k-2} \mathcal{Y}^N_i - |\mathcal{Y}^N_j|^{k-2} \mathcal{Y}^N_j \right].
$$
Using that there is $C_k > 0$ such that for any $(v, w) \in \mathbb{R}^3$ it holds, $|v|^{k-2}v - |w|^{k-2}w \leq C_k |v-w|(1 + |v|^{k-2} + |w|^{k-2})$ and that for any $\varepsilon \in (0, 1)$, it holds $|K_\gamma * \rho_\varepsilon(v)| \leq C(1 + |v|^\gamma + 1)$, we obtain that

\[
|K_\gamma * \rho_{\varepsilon N}(\gamma^{N,i} \gamma^{N,j}) \cdot (|\gamma^{N,i}|^{k-2} \gamma^{N,i} - |\gamma^{N,j}|^{k-2} \gamma^{N,j})|
\leq C_k (1 + |\gamma^{N,i}|^{\gamma + 1} - |\gamma^{N,j}|^{\gamma + 1})(1 + |\gamma^{N,i}|^{k-2} + |\gamma^{N,j}|^{k-2})
\leq C_k (1 + |\gamma^{N,i}|^{k} + |\gamma^{N,j}|^{k}),
\]

since $\gamma + 2 > 0$. Finally we conclude with

\[
I^N_t \leq C_k \int_0^t (1 + Q(s)) ds.
\]

\[\circ \text{ Estimate of } J^N_t \]

Using point (iii) of Lemma [A.1], we obtain

\[
\int_{[0,2\pi]\times\mathbb{R}^+} |\gamma^{N,i} + c_{\gamma,\nu} (\gamma^{N,i}, \gamma^{N,j} + w, z, \varphi)|^k - |\gamma^{N,i}|^k - kc_{\gamma,\nu} (\gamma^{N,i}, \gamma^{N,j} + w, z, \varphi) \cdot |\gamma^{N,i}|^{k-2} \gamma^{N,i} dz d\varphi
\leq C_k \left( |\gamma^{N,i}|^{k-2} + |\gamma^{N,j}|^{k-2} \right) \leq C_k \left( |\gamma^{N,i}|^k + |\gamma^{N,j}|^k + |w|^k + 1 \right)
\]

Therefore, since there is $C_k > 0$ such that for any $N \geq 2$ it holds $\int_{\mathbb{R}^3} |w|^k \rho_{\varepsilon N}(w) dw \leq C_k$, integrating the above inequality on $\mathbb{R}^3$ w.r.t. the density $\rho_{\varepsilon(N)}$ yields

\[
J^N_t \leq C_k \int_0^t (1 + Q(s)) ds.
\]

\[\circ \text{ Estimate of } K^N_t \]

Since for any $\mu \in \mathcal{P}(\mathbb{R}^3)$ it holds $\alpha^{R,\eta}_{\beta,\eta}(\mu) \leq 2$, we easily get

\[
K^N_t \leq k \int_0^t \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[ |\gamma^{N,i}|^{k-2} \right] ds \leq C_k \int_0^t (Q(s) + 1) ds
\]

Finally, gathering all these estimates yields

\[
Q(t) \leq Q(0) + C_k \int_0^t (Q(s) + 1) ds,
\]

and the conclusion follows by application of Gronwall’s inequality.

\[\square\]

3.2. Entropy dissipation estimates. For each $N \geq 2$, consider $(G^N_t)_{t \in [0,T]}$ solution to (2.5) for the initial condition $G_0^N$. For $V \in \mathbb{R}^{3N}$, we denote

\[
\mu_{i}^{N-1,\varepsilon N} = \frac{1}{N-1} \sum_{j \neq i} \delta_{v_j} * \rho_{\varepsilon N}.
\]

Using the definition (2.6) of $A^N$, and dropping the $t$ in the notations for simplicity yields

\[
\frac{d}{dt} \frac{1}{N} \int_{\mathbb{R}^{3N}} G^N \ln G^N = \frac{1}{N} \int_{\mathbb{R}^{3N}} \partial_i G^N (1 + \ln G^N)
\]

\[
= \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{R}^{3N}} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} B(v_i - w, \sigma) G_N(V) \ln \frac{G^N(V + (v_i - v_i) \mathbf{a}_i)}{G^N(V)} \mu_{i}^{N-1,\varepsilon N}(w) dw d\sigma dV
\]

\[
+ \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{R}^{3N}} \left( \alpha^{R,\eta}_{\beta,\eta} (\mu_{i}^{N-1,\varepsilon N}) \right)^2 \Delta_i (\ln G_N) G_N(V) dV
\]
\[= - \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(v_i - w, \sigma) \ln \left( \frac{G^N(V)}{G^N(V + (v'_i - v_i)a_i)} \right) \mu_i^{N-\varepsilon(N)}(w) dw d\sigma G_N(V) dV \]
\[ - \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^3} \left( \rho_{\beta, \eta} \left( \mu_i^{N-\varepsilon(N)}(w) \right) \right)^2 \nabla_i (\ln G_N) \cdot \nabla_i G_N(V) dV = -D^N(G^N) - C^N(G^N), \]

- Estimate of \(D^N\)

We first rewrite

\[D^N(G^N) = \]
\[\frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(v_i - w, \sigma) \left( G^N(V) \ln \left( \frac{G^N(V)}{G^N(V + (v'_i - v_i)a_i)} \right) - G^N(V) + G^N(V + (v'_i - v_i)a_i) \right) \mu_i^{N-\varepsilon(N)}(w) dw d\sigma dV \]
\[+ \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(v_i - w, \sigma) \left( G^N(V) - G^N(V + (v'_i - v_i)a_i) \right) \mu_i^{N-\varepsilon(N)}(w) dw d\sigma dV \]
\[:= S^N(G^N) + T^N(G^N) \]

- Estimate of \(T^N\):

We use the cancellation Lemma [2, Corollary 1.2, (1)] to obtain

\[
\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(v_i - w, \sigma) \left( G^N(V) - G^N(V + (v'_i - v_i)a_i) \right) \mu_i^{N-\varepsilon(N)}(w) dw d\sigma dV \right| \leq C_{\gamma, \nu} \|G_i^{N-\varepsilon(N)}\|_{L^2_\gamma} \|\mu_i^{N-\varepsilon(N)}\|_{L^2_\gamma}.
\]

Next observing that

\[
\|G_i^{N-\varepsilon(N)}\|_{L^2_\gamma} \|\mu_i^{N-\varepsilon(N)}\|_{L^2_\gamma} = \int_{\mathbb{R}^3} \langle v_i \rangle^2 G_i^{N-\varepsilon(N)}(v_i) dv_i \int_{\mathbb{R}^3} \langle w \rangle^2 \frac{1}{N-1} \sum_{j \neq i} \rho_{\varepsilon}(w - v_j) dw
\]
\[\leq C \left( 1 + \int_{\mathbb{R}^3} |v_i|^2 G_i^{N-\varepsilon(N)}(v_i) dv_i \right) \left( 1 + \int_{\mathbb{R}^3} |w|^2 \rho_{\varepsilon}(w) dw + \frac{1}{N-1} \sum_{j \neq i} |v_j|^2 \right)
\]
\[\leq C \left( 1 + \frac{1}{N-1} \sum_{j \neq i} \int_{\mathbb{R}^3} |v_i|^2 |v_j|^2 G^N(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_N) dv_i \right)
\]
\[\leq C \left( 1 + \frac{1}{N-1} \sum_{j \neq i} \int_{\mathbb{R}^3} |v_i|^4 + |v_j|^4 G^N(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_N) dv_i \right). \]

Integrating w.r.t. \(dv_1 \cdots dv_{i-1}dv_{i+1} \cdots dv_N\) and averaging over \(i = 1, \cdots, N\), yields using the symmetry and the fact that \(k \geq 4\)

\[T^N(G^N) \geq -C_{k, \gamma, \nu} \int_{\mathbb{R}^3} \langle v_1 \rangle^k G^N(dV). \]

- Estimate of \(S^N\)

First we use that for any \(x, y \geq 0\) there holds \(x \ln(x/y) - x + y \geq (\sqrt{x} - \sqrt{y})^2\), to obtain

\[
S^N(G^N) \geq \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} B(v_i - w, \sigma) \left( \sqrt{G^N(V + (v'_i - v_i)a_i)} - \sqrt{G^N(V)} \right)^2 \mu_i^{N-\varepsilon(N)}(w) dw d\sigma dV
\]
\[
\frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{3(N-1)}} dV_i^{N-1} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{6}} |v_i - w|^\gamma b \left( \frac{v_i - w}{|v_i - w|} \cdot \sigma \right) \left( \sqrt{G_i^{N,N-1}(v'_i)} - \sqrt{G_i^{N,N-1}(v_i)} \right)^2 \mu_i^{N-1,\varepsilon N}(w) dw dv_i d\sigma
\]
\[
:= \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{3(N-1)}} dV_i^{N-1} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}^{6}} C_\gamma \left( \mu_i^{N-1,\varepsilon N}, \sqrt{G_i^{N,N-1}} \right),
\]
where we reproduce the notations [3 Proposition 2.1]. Since we assumed \( \gamma + \nu > 0 \), we may also reproduce its arguments.

Let \( \phi_R \) be a smooth function satisfying \( 1_{|\nu| \geq 2\varepsilon} \leq \phi_R(\nu) \leq 1_{|\nu| \geq 4\varepsilon} \). Moreover we choose \( (v_j)_{j \in J} \subset B_{4R} \) with \( \text{card}(J) = n \leq CR^3/\delta^3 \) such that \( B_{4R} \subset \bigcup_{j \in J} B_{\delta}^{v_j} \), and for each \( j \in J \) we choose \( \phi_j \) a smooth function such that \( 1_{B_j^{v_j}} \leq \phi_j \leq 1_{B_j^{v_j}} \) (and for simplicity we set \( \phi_{n+1} = \phi_R, \chi_{n+1} = \chi_{4R} \)). We also introduce for each \( j \in J \) such that \( 1_{B_{4\varepsilon R \setminus B_j^{v_j}} \leq \chi_j \) Following the argument of [3] (2.7) and (2.8) of the proof of Proposition 2.1] we have
\[
C_\gamma \left( \mu_i^{N-1,\varepsilon N}, \sqrt{G_i^{N,N-1}} \right) \geq C_{R,\delta,\gamma} \sum_{j=1}^{n+1} C_0 \left( \chi_j \mu_i^{N-1,\varepsilon N}, \phi_j(\cdot)^{\gamma/2} \sqrt{G_i^{N,N-1}} \right) - C_{R,\delta} \left\| \mu_i^{N-1,\varepsilon N} \right\|_{L^1} \left\| \sqrt{G_i^{N,N-1}} \right\|_{L^2_{\gamma/2}}^2,
\]
First observe that we have
\[
\frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{3(N-1)}} dV_i^{N-1} \left\| \mu_i^{N-1,\varepsilon N} \right\|_{L^1} \left\| \sqrt{G_i^{N,N-1}} \right\|_{L^2_{\gamma/2}}^2 = \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{3(N-1)}} dV_i^{N-1} \int_{\mathbb{R}^3} \langle v_i \rangle \gamma \sqrt{G_i^{N,N-1}} (v_i) dv_i \leq 1,
\]
since \( \mu_i^{N-1,\varepsilon N} \in \mathcal{P}(\mathbb{R}^3) \). Then, due to [2 Corollary 2.1] we have
\[
C_0 \left( \chi \mu_i^{N-1,\varepsilon N}, \phi(\cdot)^{\gamma/2} \sqrt{G_i^{N,N-1}} \right) \geq C \int_{\mathbb{R}^3} \left| \mathcal{F} \left( \phi(\cdot)^{\gamma/2} \sqrt{G_i^{N,N-1}} \right)(\xi) \right|^2 \left( \int_{\mathbb{R}^2} b \left( \frac{\xi}{|\xi|} \right) \sigma \left( \mathcal{F} \left( \chi \mu_i^{N-1,\varepsilon N} \right)(0) - \mathcal{F} \left( \chi \mu_i^{N-1,\varepsilon N} \right)(\xi^-) \right) d\sigma \right) d\xi,
\]
where \( \xi^- = \frac{1}{2}(\xi - |\xi|\sigma) \). This yields integrating over \( \mathbb{R}^{3(N-1)} \) w.r.t the Lebesgue measure
\[
S^N(G^N) + \int_{\mathbb{R}^3} \sum_{j=1}^{n+1} \left| \mathcal{F} \left( \phi_j(\cdot)^{\gamma/2} \sqrt{G_i^{N,N-1}} \right)(\xi) \right|^2 \left( \int_{\mathbb{R}^2} b \left( \frac{\xi}{|\xi|} \right) \sigma \left( \mathcal{F} \left( \chi_j \mu_i^{N-1,\varepsilon N} \right)(0) - \mathcal{F} \left( \chi_j \mu_i^{N-1,\varepsilon N} \right)(\xi^-) \right) d\sigma \right).
\]

**Estimate of \( C^N \)**

First observe that
\[
C^N(G^N) = \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{3(N-1)}} \left( \alpha_R(\mu_i^{N-1,\varepsilon N}) \right)^2 \int_{\mathbb{R}^3} \left| \nabla \sqrt{G_i^{N,N-1}} \right|^2 dv_i dV_i^{N-1}
\]
\[
= \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{3(N-1)}} \left( \alpha_R(\mu_i^{N-1,\varepsilon N}) \right)^2 \int_{\mathbb{R}^3} \left| \left( \cdot \right)^{-\gamma/2} \left( \cdot \right)^{\gamma/2} \sqrt{G_i^{N,N-1}} \right|^2 dv_i dV_i^{N-1}
\]
using [3.1], we obtain since \( \gamma \leq 0 \)
\[
\int_{\mathbb{R}^3} \left| \nabla \left( \left( \cdot \right)^{-\gamma/2} \left( \cdot \right)^{\gamma/2} \sqrt{G_i^{N,N-1}} \right) \right|^2 dv_i \geq \frac{1}{2} \int_{\mathbb{R}^3} \left| \nabla \left( \left( \cdot \right)^{\gamma/2} \sqrt{G_i^{N,N-1}} \right) \right|^2 dv_i - 4 \int_{\mathbb{R}^3} \left| \nabla \left( \left( \cdot \right)^{-\gamma/2} \right) \right|^2 \gamma G_i^{N,N-1} dv_i
\]
Next observe that for each $j = 1 \cdots, n + 1$ since $\phi_j(\cdot)^γ$ (resp. $\phi_R$) is bounded and Lipschitz
\[
\int_{\mathbb{R}^3} \left| \nabla \left( \phi_j(\cdot)^γ/2 \sqrt{G_i^{N,N-1}} \right) \right|^2 \, dv_i \leq 2 \int_{\mathbb{R}^3} \phi_j^2 \left| \nabla (\cdot)^γ/2 \sqrt{G_i^{N,N-1}} \right|^2 \, dv_i + 2 \int_{\mathbb{R}^3} \left| \nabla \phi_j \right|^2 (v_i)^γ G_i^{N,N-1} \, dv_i
\leq 2 \int_{\mathbb{R}^3} \left| \nabla (\cdot)^γ/2 \sqrt{G_i^{N,N-1}} \right|^2 \, dv_i + 2C_3 \int_{\mathbb{R}^3} G_i^{N,N-1} \, dv_i.
\]

Summing the above inequality over $j = 1, \cdots, n + 1$, and since $(v)^γ \left| \nabla (v)^γ/2 \right|^2 = |v|^2 (v)^{-4} \leq 1$, we have
\[
C_{R,δ} \left( \int_{\mathbb{R}^3} \left| \nabla \sqrt{G_i^{N,N-1}} \right|^2 \, dv_i + \int_{\mathbb{R}^3} G_i^{N,N-1} \, dv_i \right) \geq \int_{\mathbb{R}^3} \sum_{j=1}^{n+1} \left| \nabla \left( \phi_j(\cdot)^γ/2 \sqrt{G_i^{N,N-1}} \right) \right|^2 \, dv_i,
\]
Therefore
\[
C^N(G^N) + 1 \geq C_{δ,R,γ,N} \sum_{i=1}^{N} \int_{\mathbb{R}^3(N-1)} \, dV_i^{N-1} \int_{\mathbb{R}^3} \, dξ \left( \sum_{j=1}^{n+1} \left| \mathcal{F} \left( \phi_{A_j}(\cdot)^γ/2 \sqrt{G_i^{N,N-1}} \right) (ξ) \right|^2 \right) \left( \alpha^R_{δ,γ} (μ_i^{N-1,εN}) \right)^2 |ξ|^2
\]
(3.4)

- **Conclusion**

We first prove the bound (2.9). In view of the estimate $S^N$ we have
\[
\frac{d}{dt} N \int_{\mathbb{R}^3 N} G^N \ln G^N = -D^N(G^N) - C^N(G^N) \leq C_{k,γ,ν} \int_{\mathbb{R}^3 N} |v|^4 G^N(V) \, dV,
\]
Hence for any $t \leq T$, using (2.10) we obtain
\[
\sup_{t \in [0,T]} \frac{1}{N} \int_{\mathbb{R}^3 N} G^N \ln G^N \leq \frac{1}{N} \int_{\mathbb{R}^3 N} G^N_0 \ln G^N_0 + C_{k,γ,ν} T \sup_{t \in [0,T]} \int_{\mathbb{R}^3 N} |v|^k G^N(V) \, dV
\leq H_0 + C_{k,γ,ν} T e^{C_4 T} (1 + M_0)
\]
and (2.9) is proved.

Moreover, summing (3.4) and (3.3) and using Lemma B.3, we have
\[
D^N(G^N) + C^N(G^N) + C_{k,γ,ν} \int_{\mathbb{R}^3(N-1)} |v|^k G^N(V) \, dV \geq
\]
\[
\frac{C_{R,δ,γ,ν}}{N} \int_{\mathbb{R}^3(N-1)} \, dV_i^{N-1} \int_{\mathbb{R}^3} \, dξ \left( \sum_{j=1}^{n+1} \left| \mathcal{F} \left( \phi_j(\cdot)^γ/2 \sqrt{G_i^{N,N-1}} \right) (ξ) \right|^2 \right)
\]
\[
\left( \int_{\mathbb{R}^2} b \left( \frac{ξ}{|ξ|} \cdot σ \right) \left( \mathcal{F} \left( \chi_j μ_i^{N-1,εN} \right)(0) - \mathcal{F} \left( \chi_j μ_i^{N-1,εN} \right)(ξ^-) \right) dσ \right)
\]
\[
\geq \kappa \frac{C_{R,δ,γ,ν}}{N} \int_{\mathbb{R}^3(N-1)} \, dV_i^{N-1} \int_{\mathbb{R}^3} \, dξ \left( \sum_{j=1}^{n+1} \left| \mathcal{F} \left( \phi_j(\cdot)^γ/2 \sqrt{G_i^{N,N-1}} \right) (ξ) \right|^2 \right) |ξ|^1 1_{|ξ| \geq πR^{-1}}.
\]

Moreover we have by Parseval’s identity
\[
\frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^3(N-1)} \, dV_i^{N-1} \int_{\mathbb{R}^3} \, dξ \left( \sum_{j=1}^{n+1} \left| \mathcal{F} \left( \phi_j(\cdot)^γ/2 \sqrt{G_i^{N,N-1}} \right) (ξ) \right|^2 \right) |ξ|^ν 1_{|ξ| \leq πR^{-1}}
\]
from which we conclude to
\[ D^N(G^N) + C^N(G^N) + \int_{\mathbb{R}^{3N}} \langle v_1 \rangle^k G^N(V) dV \]
\[ \geq \frac{C_{R,\delta,\gamma,\nu,\delta,\nu}}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{3(N-1)}} dV_i \int_{\mathbb{R}^3} d\xi \left( \sum_{j=1}^{n+1} \left| \phi_j(\xi) \right|^2 \right) \left| \xi \right|^\nu \]
\[ = \frac{C_{R,\delta,\gamma,\nu,\delta,\nu}}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{3(N-1)}} dV_i \int_{\mathbb{R}^3} d\xi \left( \sum_{j=1}^{n+1} \phi_j(\xi) \right)^2 \left| \xi \right|^\nu \]
But using that \( \sum_{j=1}^{n+1} \phi_j^2 \geq 1 \) and observation (3.2) we find that
\[ \sum_{j=1}^{n+1} |\phi_j(\xi)|^2 \leq 2 + \int_{\mathbb{R}^6} (x - y)^2 G_i^{N,N-1}(x) \sum_{j=1}^{n+1} (\phi_j(x) - \phi_j(y))^2 \frac{dxdy}{|x - y|^{3+\nu}} \]
\[ \geq C \int_{\mathbb{R}^{2N}} \left( \sum_{j=1}^{n+1} \phi_j^2(x) \right) \left( (x)^{1/2} G_i^{N,N-1}(x) - (y)^{1/2} G_i^{N,N-1}(y) \right)^2 \frac{dxdy}{|x - y|^{3+\nu}} \]
And since for any \( j = 1, \ldots, n+1, \phi_j \) is bounded and Lipschitz we obtain
\[ C_{R,\delta,\gamma,\nu,\delta,\nu} \int_0^T T^N_{\nu,\gamma}(G^N_i) dt \leq \int_0^T D^N(G^N_i) + C^N(G^N_i) + \int_{\mathbb{R}^{3N}} \langle v_1 \rangle^k G^N_i (dV) dt \]
\[ \leq N^{-1} (H(G^N_i) - H(G^N_i)) + T \sup_{t \in [0,T]} \int_{\mathbb{R}^{3N}} \langle v_1 \rangle^k G^N_i (dV), \]
and we conclude to (2.11), thanks to (2.10) and (2.9).

4. Proof of Theorems 2.1

Thanks to the estimate obtained in the previous section, we may prove our main result.

4.1. Tightness. We begin with a tightness result in the

**Proposition 4.1.** For each \( N \geq 2 \), let \( V_0^N \) be a \( \mathbb{R}^{3N} \)-valued, \( G_0^N \)-distributed random variable, and consider \( (V_t^N)_{t \in [0,T]} \) solution to (2.4) starting from \( V_0^N \). If the sequence \( (G^N_i)_{N \geq 2} \) is chaotic and satisfies [2.7], then the sequence \( \left( \frac{1}{N} \sum_{i=1}^{N} \delta(V_t^N) \right)_{N \geq 2} \) is tight in \( P(\mathbb{D}([0,T], \mathbb{R}^3)) \).

**Proof.** Since \( \mathbb{D}([0,T], \mathbb{R}^3) \) is polish, in view of [35] Proposition 2.2 point (ii), it is enough to show the tightness of the process \( V_t^{N,1} \) \( t \in [0,T] \). By definition, it holds

\[ V_t^{N,1} = V_0^{N,1} + \int_{[0,t] \times [0,2\pi]} c_\epsilon \left( V_{s-}^{N,1}, V_{s-}^{N,2} + w, z, \varphi \right) \mathcal{M}_t^\epsilon(ds, d\varphi, dz, dw, dj) \]
\[ + \frac{\beta_0}{N - 1} \sum_{j=1}^{N} \int_0^t (K_\gamma \ast \rho_\epsilon)(V_{s-}^{N,j} - V_{s-}^{N,j}) ds + \int_0^t \sqrt{2\alpha_\delta^R} \left( \frac{1}{N} \sum_{j \neq 1} \delta(V_{s-}^{N,j} \ast \rho_\epsilon) \right) dB_s^j \]
\[ := \mathcal{V}_0^{N,1} + \mathcal{Z}_t^{1,N} + \mathcal{D}_t^{1,N} + \Lambda_t^{1,N}. \]

It is then enough to show the tightness of each of the term of the sum.

- Tightness of \( \mathcal{V}_0^{N,1} \).

This is a simple consequence of the fact that the sequence of laws of the initial conditions is chaotic.

- Tightness of \( (\mathcal{D}_t^{1,N})_{t \in [0,T]} \).

First recall that for all \( v \in \mathbb{R}^3 \) we have \( |K_\gamma * \rho_{\varepsilon N}(v)| \leq C(1 + |v|^{\gamma+1}) \). For any \( 0 \leq s < t \leq T \) we have

\[
\left| \mathcal{D}_s^{1,N} - \mathcal{D}_t^{1,N} \right| = \left| \frac{\beta_0}{N} \sum_{j=1}^{N} \int_s^t K_\gamma \ast \rho_{\varepsilon N}(\mathcal{V}_{u,j}^{N,1} - \mathcal{V}_{u,j}^{N}) \, du \right| \leq \frac{C}{N} \sum_{j=1}^{N} \int_s^t \left( 1 + |\mathcal{V}_{u,j}^{N,1} - \mathcal{V}_{u,j}^{N}|^{\gamma+1} \right) \, du
\]

\[
\leq |t-s|^{1/q} \frac{C_q}{N} \sum_{j=1}^{N} \left( \int_0^T 1 + |\mathcal{V}_{u,j}^{N,1} - \mathcal{V}_{u,j}^{N}|^{q'(|\gamma|-1)} \, du \right)^{1/q'}
\]

\[
\leq |t-s|^{1/q} \frac{C_q}{N} \sum_{j=1}^{N} \left( 1 + \int_0^T |\mathcal{V}_{u,j}^{N,1} - \mathcal{V}_{u,j}^{N}|^{-q'(|\gamma|-1)} \, du \right) =: |t-s|^{q'} Z_{N,q}^T.
\]

We choose \( q > \frac{\nu}{\nu - |\gamma| - 1} \), so that \( q'(|\gamma|-1) < \nu \). Hence by symmetry, and since \( G_t^N = \mathcal{L}(\mathcal{V}_t^N) \in \mathcal{P}_{\text{sym}}(\mathbb{R}^{dN}) \) it holds

\[
\mathbb{E}\left[ Z_{N,q}^T \right] = \frac{C_q(N-1)}{N} \left( 1 + \int_0^T \mathbb{E}\left[ |\mathcal{V}_{t,j}^{N,1} - \mathcal{V}_{t,j}^{N}|^{q'(|\gamma|+1)} \right] dt \right) = \frac{C_q(N-1)}{N} \left( 1 + \int_0^T \int_{\mathbb{R}^{dN}} |v_1 - v_2|^{q'(|\gamma|+1)} G_t^N(dV) dt \right).
\]

Due to Proposition 2.1 we find that for any \( r \in \left( \frac{d}{d+\nu} \vee \frac{\nu + q(|\gamma|-1)}{2\nu} \vee \frac{2d - (\nu - q'(|\gamma|-1))}{2d} \right) \) there is a constant

\[
\int_{\mathbb{R}^{dN}} |v_1 - v_2|^{-q'(|\gamma|-1)} G_t^N(dV) \leq C_{r,\nu,\gamma,r} \left( \sum_{N=1}^{dN} (G_t^N)^r + \int_{\mathbb{R}^{dN}} |v_1|^{-\gamma} \frac{1}{\gamma} G_t^N(dV) + 1 \right),
\]

and then using the bounds (2.10) and (2.11), we have

\[
\mathbb{E}\left[ Z_{N,q}^T \right] \leq C_{r,\nu,\gamma,r} \left( 1 + \int_0^T \sum_{N=1}^{dN} (G_t^N)^r dt + \int_0^T \int_{\mathbb{R}^{dN}} |v|^r G_t^N(dV) dt \right)
\]

\[
\leq C_{r,\nu,\gamma,r,T,M_0,H_0;k,k,R,\delta,\eta}.
\]

Then for \( A > 0 \) let us denote

\[
\mathcal{K}^A = \left\{ h \in \mathcal{C}([0,T], \mathbb{R}^d), h(0) = 0, \sup_{0 \leq s < t \leq T} \frac{|h(s) - h(t)|}{|s - t|^p} \leq A \right\},
\]

which is compact by Ascoli-Azerla’s Theorem. Then using Markov’s inequality yields

\[
\sup_{N \geq 1} \mathbb{P}\left( (\mathcal{D}_t^{1,N})_{t \in [0,T]} \notin \mathcal{K}^A \right) \leq \sup_{N \geq 1} \mathbb{P}\left( Z_{N,q}^T \geq A \right) \leq A^{-1} \sup_{N \geq 1} \mathbb{E}\left[ Z_{N,q}^T \right],
\]

and the sequence of laws of \( (\mathcal{D}_t^{1,N})_{t \in [0,T]} \) is tight (see Definition before Theorem 5.1 of [7]), since for any \( \varepsilon > 0 \), we can choose \( A \) large enough such that it holds

\[
\sup_{N \geq 1} \mathbb{P}\left( (\mathcal{D}_t^{1,N})_{t \in [0,T]} \notin \mathcal{K}^A \right) \leq \varepsilon.
\]

- Tightness of \( (Z_{t}^{1,N})_{t \in [0,T]} \).
In view of \textbf{[4]} Theorem 4.2.3, and point (ii) of Lemma \textbf{A.1} we have

$$
\mathbb{E} \left[ \mathcal{Z}_t^{1,N} - \mathcal{Z}_s^{1,N} \right]^2 = \mathbb{E} \left[ \int_{[s,t] \times [0,2\pi] \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}} \right. 
\left. c_{\gamma,\nu} \left( \mathcal{V}_{u,1}^{N,1}, \mathcal{V}_{u,1}^{N,j} + w, z, \varphi \right) \mathcal{M}_N^t(du, d\varphi, dz, dw, dj) \right]^2
$$

$$
= \frac{1}{N} \sum_{j=1}^{N} \int_s^t \int_{\mathbb{R}^3} \mathbb{E} \left[ \mathcal{V}_{u,1}^{N,1} - \mathcal{V}_{u,1}^{N,j} + w \right]^2 \rho_{\epsilon,N}(w)du
$$

since $k \geq \gamma + 2 > 0$. The result follows with a similar argument as the one used above, and bound (2.10).

- Tightness of $(\mathcal{X}_t^{1,N})_{t \in [0,T]}$.

By Ito’s isometry, and since for any $\mu \in \mathcal{P}(\mathbb{R}^3)$ it holds $\alpha_{\delta,\eta}(\mu) \leq 2$, for any $N \geq 2$, we have

$$
\mathbb{E} \left[ \mathcal{X}_t^{1,N} - \mathcal{X}_s^{1,N} \right]^2 = \mathbb{E} \left[ \int_s^t \sqrt{2} \alpha_{\delta,\eta}^R \left( \frac{1}{N} \sum_{j \neq i} \delta_{\nu_{u,j}^{N,i}} \ast \rho_{\epsilon,N} \right) dB_{u,i}^2 \right]^2
$$

and the result follows.

\hfill \Box

4.2. \textbf{Martingale problem.} We now show that the accumulation point obtained in the previous section is unique. In that purpose we define the set

$$
\mathcal{S} := \left\{ Q \in \mathcal{P}(\mathbb{D}([0,T], \mathbb{R}^3)) \mid Q \text{ satisfies (a), (b) and (c)} \right\},
$$

where the conditions (a), (b), (c) are defined as

\begin{align*}
(a) \quad & e_0 \# Q = g_0, \\
(b) \quad & Q_t := e_t \# Q, \quad (Q_t)_{t \in [0,T]} \text{ satisfies, } \int_0^T |(\gamma / 2)^{3/2} Q_t^{1/2} \mathbb{R}^{3/2} |^2 dt < \infty, \\
(c) \quad & \forall 0 < t_1 < \cdots < t_m < s < t \leq T, \phi_1, \cdots, \phi_m \in C_b(\mathbb{R}^2), \phi \in C_b^2(\mathbb{R}^2), \text{ it holds}
\end{align*}

$$
\mathcal{M}(Q) = \mathcal{C}(Q)
$$

\begin{align*}
\mathcal{M}(Q) := & \int_{\mathbb{D}([0,T], \mathbb{R}^3)^2} Q(d\varphi)Q(d\check{\varphi}) \prod_{k=1}^m \phi_k(\varphi_k) \left( \phi(\varphi_k) - \phi(\check{\varphi}_k) - \int_s^t \check{\varphi}_k K(\varphi_k - \check{\varphi}_k) \cdot \nabla \phi(\varphi_k) du \\
& - \int_s^t \int_{[0,2\pi] \times \mathbb{R}^+} (\varphi_k(\varphi_k + \check{\varphi}_k, \varphi_k, z)) - \phi(\varphi_k) - c_{\gamma,\nu}(\varphi_k, \check{\varphi}_k, \varphi, z) \cdot \nabla \phi(\varphi_k) dzd\varphi du \right) \\
\mathcal{C}(Q) := & \int Q(d\varphi) \prod_{k=1}^m \phi_k(\varphi_k) \int_s^t \left( \alpha_{\delta,\eta}^R(Q_u) \right)^2 \Delta \phi(\varphi_u) du.
\end{align*}

We show that this accumulation point almost surely belongs to $\mathcal{S}$ in the

\textbf{Proposition 4.2.} For each $N \geq 2$, let $\mathcal{V}_0^N$ be a $G_0^N$-distributed random variable, and consider $(\mathcal{V}_t^N)_{t \in [0,T]}$ $N \geq 2$ solution to (2.4). Assume that $(G_0^N)_{N \geq 2}$ is $g_0$-chaotic, satisfies (2.7) and that there is a subsequence of
A similar argument applies to prove the almost surely bound on the \( k \) moment.

\[
\left( \frac{1}{N} \sum_{i=1}^{N} \delta_{(v_{i,N}^{N,i})_{t \in [0,T]}} \right)_{N \geq 2} \text{ converging in law to some } f \in \mathcal{P}(\mathcal{D}([0,T], \mathbb{R}^3)). \text{ Then } f \text{ almost surely belongs to } S.
\]

\textbf{Proof.} We prove successively that each of the condition is fulfilled.

- \( f \) satisfies (a)

It is a simple consequence form the fact that the law of the initial condition to the particle system (2.4) are \( g_0 \) chaotic.

- \( f \) satisfies (b)

For any \( t \in [0,T] \) we denote \( \pi_t = \mathcal{L}(f_t) \), and for any \( k \geq 2 \), \( \pi_k = \int_{\mathcal{P}(\mathbb{R}^3)} \rho^\otimes k \pi_t(d\rho) \). Recall that \( G_t^N = \mathcal{L}(\mathcal{V}_t^N) \in \mathcal{P}_{sym}(\mathbb{R}^{dN}) \). Since \( \left( \frac{1}{N} \sum_{i=1}^{N} \delta_{v_i^{N,i}} \right)_{N \geq 2} \) converges (up to a subsequence) in law to \( f_t \), for any \( k \geq 2 \), \( (G_t^{N,k})_{N \geq 2} \) (\( k \)-particles marginal of \( G_t^N \)) converges weakly to \( \pi_k \). Using Proposition 2.2 Fatou Lemma and bound 2.11 we obtain

\[
\mathbb{E} \left[ \int_0^T |(\cdot)^{\gamma/2} \sqrt{f_t} |_{H^{\gamma/2}(\mathbb{R}^3)}^2 dt \right] = \int_0^T \int_{\mathcal{P}(\mathbb{R}^3)} |(\cdot)^{\gamma/2} \sqrt{\rho} |_{H^{\gamma/2}(\mathbb{R}^3)}^2 \pi_t(d\rho) dt
\]

\[
\leq \int_0^T \liminf_N T_{\nu^*\gamma}(G_t^N) dt \leq \liminf_N \int_0^T T_{\nu^*\gamma}(G_t^N) dt < +\infty,
\]

and therefore

\[
\int_0^T |(\cdot)^{\gamma/2} \sqrt{f_t} |_{H^{\gamma/2}(\mathbb{R}^3)}^2 dt < +\infty, \quad \text{a.s.}
\]

A similar argument applies to prove the almost surely bound on the \( k \) moment.

- \( f \) satisfies (c)

For any \( i = 1, \cdots, N \) Ito’s rule yields for any test function \( \phi \in C^2(\mathbb{R}^3) \)

\[
\phi(v_{t,N}^{N,i}) = \phi(v_{0,N}^{N,i}) + \int_0^t \frac{\beta_0}{N-1} \sum_{j \neq i} K_\gamma * \rho_{z,N}(v_{s,N}^{N,i} - v_{s,N}^{N,j}) \cdot \nabla \phi(v_{s,N}^{N,i}) + \left( \phi_{\delta,N}^R \left( \frac{1}{N-1} \sum_{j \neq i} \delta_{v_{s,N}^{N,j}} * \rho_{z,N} \right) \right)^2 \Delta \phi(v_{s,N}^{N,i}) ds
\]

\[
+ \frac{1}{N} \sum_{j \neq i} \int_0^t \int_0^{2\pi} d\varphi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Delta \phi(v_{s,N}^{N,i}, v_{s,N}^{N,j} + w, \varphi, z) \rho_{z,N}(dw) ds + \int_0^t \sqrt{2} \phi_{\delta,N}^R \left( \frac{1}{N-1} \sum_{j \neq i} \delta_{v_{s,N}^{N,j}} * \rho_{z,N} \right) \nabla \phi(v_{s,N}^{N,i}) \cdot dB^i_s
\]

\[
+ \int_{[0,t] \times [0,2\pi]} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\{1, \cdots, N\} \setminus \{i\}} \left( \phi(v_{s,N}^{N,i} + c_{\gamma,N}(v_{s,N}^{N,i} - v_{s,N}^{N,j}) + w, \varphi, z) - \phi(v_{s,N}^{N,i}) \right) \mathcal{N}_t^N(ds, dz, d\varphi, dw, dj).
\]

Then we define the processes \( (G_{t,N}^{N,i})_{t \in [0,T]} \) as

\[
G_{t,N}^{N,i} := \phi(v_{t,N}^{N,i}) - \phi(v_{0,N}^{N,i}) - \int_0^t \frac{\beta_0}{N-1} \sum_{j \neq i} K_\gamma * \rho_{z,N}(v_{s,N}^{N,i} - v_{s,N}^{N,j}) \cdot \nabla \phi(v_{s,N}^{N,i}) ds
\]

\[
- \int_0^t \frac{1}{N} \sum_{j \neq i} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_{\mathbb{R}^3} \Delta \phi(v_{s,N}^{N,i}, v_{s,N}^{N,j} + w, z, \varphi, \rho_{z,N}(dw) ds
\]
In view of the above Ito’s expansion, we also have

\[
O_{t}^{N,i} = \int_{0}^{t} \left( \alpha_{\delta,\eta}^{R} \left( \frac{1}{N-1} \sum_{j \neq i}^{N} \delta_{V_{N,j}} * \rho_{\epsilon,N} \right) \right)^{2} \Delta \phi(V_{s}^{N,i})ds + \int_{0}^{t} \sqrt{2} \alpha_{\delta,\eta}^{R} \left( \frac{1}{N-1} \sum_{j \neq i}^{N} \delta_{V_{N,j}} * \rho_{\epsilon,N} \right) \nabla \phi(V_{s}^{N,i}) \cdot dB_{s}^{i} \]

\[
+ \int_{[0,t] \times [0,2\pi] \times \mathbb{R}^{3} \times \{1, \ldots, N\}} \left( \phi(V_{s_{-}}^{N,i} + c_{i},(V_{s_{-}}^{N,i}, V_{s_{-}}^{N,j}, w_{-}, \varphi, z)) + \phi(V_{s_{-}}^{N,i}) \right) dN(dx, dz, d\varphi, dw, dj) \]

We also define

\[
K^{N} := \frac{1}{N} \sum_{i=1}^{N} \prod_{k=1}^{n} \phi_{k}(V_{t_{k}}^{N,i}) \left( O_{t}^{N,i} - O_{s}^{N,i} \right) \]

\[
= \frac{1}{N} \sum_{i=1}^{N} \prod_{k=1}^{n} \phi_{k}(V_{t_{k}}^{N,i}) \int_{s}^{t} \left( \alpha_{\delta,\eta}^{R} \left( \frac{1}{N-1} \sum_{j \neq i}^{N} \delta_{V_{N,j}} * \rho_{\epsilon,N} \right) \right)^{2} \Delta \phi(V_{u}^{N,i})du \]

\[
+ \frac{1}{N} \sum_{i=1}^{N} \prod_{k=1}^{n} \phi_{k}(V_{t_{k}}^{N,i}) \left( \int_{s}^{t} \sqrt{2} \alpha_{\delta,\eta}^{R} \left( \frac{1}{N-1} \sum_{j \neq i}^{N} \delta_{V_{N,j}} * \rho_{\epsilon,N} \right) \nabla \phi(V_{u}^{N,i}) \cdot dB_{u}^{i} \right) \]

\[
+ \int_{[s,t] \times [0,2\pi] \times \mathbb{R}^{3} \times \{1, \ldots, N\}} \left( \phi(V_{u_{-}}^{N,i} + c_{i},(V_{u_{-}}^{N,i}, V_{u_{-}}^{N,j}, w, \varphi, z)) + \phi(V_{u_{-}}^{N,i}) \right) \mathcal{M}_{N}^{i}(du, dz, d\varphi, dw, dj) \right) \]

Finally, observe that by definition, it holds

\[
\mathcal{M}(\mu^{N}) = \frac{1}{N} \sum_{i=1}^{N} \prod_{k=1}^{n} \phi_{k}(V_{t_{k}}^{N,i}) \left( \phi(V_{t_{-}}^{N,i}) - \phi(V_{s_{-}}^{N,i}) \right) - \frac{1}{N} \sum_{j=1}^{N} \int_{s}^{t} \beta_{0} K_{\gamma}^{\epsilon}(V_{u_{-}}^{N,i} - V_{u_{-}}^{N,j}) \cdot \nabla \phi(V_{u}^{N,i}) + \Delta \phi(V_{u}^{N,i}, V_{u}^{N,j}, \varphi, z) \right)du \]

For any $\epsilon > 0$ we define $K^{\epsilon}_{\gamma}$ smooth, bounded, such that $K^{\epsilon}_{\gamma}(x) = K_{\gamma}(x)$ for any $|x| \geq \epsilon$ and $|K^{\epsilon}_{\gamma}(x)| \leq |K_{\gamma}(x)|$ for any $|x| \leq \epsilon$, and

\[
\mathcal{M}_{\epsilon}(Q) := \int_{\mathbb{D}([0,T], \mathbb{R}^{3})^{2}} Q(d\varphi)Q(d\tilde{\varphi}) \prod_{k=1}^{n} \phi_{k}(\vartheta_{k}) \left( \phi(\vartheta_{l}) - \phi(\vartheta_{s}) \right) - \int_{s}^{t} \beta_{0} K^{\epsilon}_{\gamma}(\vartheta_{u} - \tilde{\vartheta}_{u}) \cdot \nabla \phi(\vartheta_{u})du \]

\[
- \int_{s}^{t} \int_{[0,2\pi] \times \mathbb{R}^{3}} \left( \phi(\vartheta_{u} + c_{i,\nu}(\vartheta_{u}, \tilde{\vartheta}_{u}, \varphi, z)) - \phi(\vartheta_{u}) - c_{i,\nu}(\vartheta_{u}, \tilde{\vartheta}_{u}, \varphi, z) \cdot \nabla \phi(\vartheta_{u}) \right) d\vartheta_{u} \varphi \tilde{\vartheta}_{u} \varphi du \right) \]

With this definition, $\mathcal{M}_{\epsilon}$ is a continuous function on $\mathcal{P}(\mathbb{D}([0,T], \mathbb{R}^{3}))$, in view of Lemma A.2 Next introduce the following decomposition

\[
\mathbb{E} \left[ |\mathcal{M}(f) - \mathcal{C}(f) | \right] \leq \mathbb{E} \left[ |\mathcal{M}(f) - \mathcal{M}_{\epsilon}(f) | \right] + \mathbb{E} \left[ |\mathcal{M}_{\epsilon}(f) - \mathcal{M}_{\epsilon}(\mu^{N}) | \right] + \mathbb{E} \left[ |\mathcal{M}_{\epsilon}(\mu^{N}) - \mathcal{M}(\mu^{N}) | \right] + \mathbb{E} \left[ |\mathcal{M}(\mu^{N}) - K^{N} | \right] \]

\[
=: I_{1} + I_{2} + I_{3} + I_{4} + I_{5}. \]

\[\square \text{ Estimate of } I_{1}, I_{3} \]

We first observe that by definition of $K^{\epsilon}_{\gamma}$ we have that

\[|K^{\epsilon}_{\gamma}(x) - K_{\gamma}(x)| \leq 2|x|^{\gamma + 1}1_{|x| \leq \epsilon},\]

so that for any $Q \in \mathcal{P}(\mathbb{D}([0,T], \mathbb{R}^{3}))$ it holds

\[
|\mathcal{M}(Q) - \mathcal{M}_{\epsilon}(Q)| = \int_{\mathbb{D}([0,T], \mathbb{R}^{3})^{2}} Q(d\varphi)Q(d\tilde{\varphi}) \prod_{k=1}^{n} \phi_{k}(\vartheta_{k}) \left( - \int_{s}^{t} \beta_{0}(K^{\epsilon}_{\gamma} - K_{\gamma})(\vartheta_{u} - \tilde{\vartheta}_{u}) \cdot \nabla \phi(\vartheta_{u})du \right) \]
Choosing $p$ \( \leq C_{\phi,n} \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - v_*|^\gamma |1|_{v - v_*} \leq \varepsilon Q_u(dv)Q_u(dv_*) du \)
\( \leq \varepsilon^p C_{\phi,n} \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - v_*|^\gamma |1-\varepsilon|Q_u(dv)Q_u(dv_*) du, \)
for any \( p > 0 \). Hence we have
\[
I_1 \leq \varepsilon^p C_{\phi,n} \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - v_*|^{\gamma+1-p} f_u(dv) f_u(dv_*) du \right]
\]
\[
I_3 \leq \varepsilon^p C_{\phi,n} \frac{1}{N^2} \sum_{i \neq j}^N \int_0^T \mathbb{E} \left[ |\nu_i^{N,i} - \nu_j^{N,j}|^{\gamma+1-p} \right] du.
\]
Choosing \( p \in (0, 1 + \nu + \gamma) \), using Proposition 2.1 and point (i) of Lemma D.1 we obtain
\[
I_1 \leq \varepsilon^p C_{\phi,n,p,\gamma,\nu} \left( 1 + \mathbb{E} \left[ \int_0^T \left| \langle \cdot \rangle \gamma/2 \right| \sqrt{f_u} \right]^2 + \int_{\mathbb{R}^3} |v|^k f_u(dv) du \right]
\]
\[
\leq \varepsilon^p C_{\phi,n,p,\gamma,\nu,H_0,T,M_0,R,\delta,\eta,k}
\]
We obtain a similar estimate for \( I_3 \), using the same argument as in the proof of Proposition 4.1.

□ Estimate of \( I_2 \)

For any \( \varepsilon > 0 \), \( I_2 \) converges to 0 as \( N \) goes to infinity, since \( \mathcal{M}_\varepsilon \) is a continuous function on \( \mathcal{P}(\mathbb{D}([0,T],\mathbb{R}^3)) \), and \( \mu^N \) converges in law to \( f \).

□ Estimate of \( I_4 \)

Observe that
\[
\mathcal{M}(\mu^N) - K^N = \frac{1}{N} \sum_{i=1}^N \prod_{k=1}^n \phi_k(\nu_{i_k}^{N,i}) \left( \frac{1}{N} \sum_{j=1}^N \int_s^t \beta_0 (K_\gamma - K_\gamma * \rho_{\varepsilon N}) (\nu_{u_i}^{N,i} - \nu_{u_j}^{N,j}) \cdot \nabla \phi(\nu_{u_i}^{N,i}) du \right.
\]
\[
+ \int_s^t \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^\infty (\tilde{\Delta} \phi(\nu_{u_i}^{N,i},\nu_{u_i}^{N,j},\varphi,z) - \tilde{\Delta} \phi(\nu_{u_i}^{N,i},\nu_{u_i}^{N,j} + w,\varphi,z)) \rho_{\varepsilon N}(dw) dz d\varphi du \bigg).
\]
Since for any \( v, v_* \in \mathbb{R}^3 \), we have
\[
\lim_{N \to \infty} K_\gamma - K_\gamma * \rho_{\varepsilon N}(v - v_*) = 0,
\]
and for any \( z, \varphi \in \mathbb{R}^4 \times [0, 2\pi] \)
\[
\lim_{N \to \infty} \int_{\mathbb{R}^3} (\tilde{\Delta} \phi(v, v_*, \varphi, z) - \tilde{\Delta} \phi(v, v_* + w, \varphi, z)) \rho_{\varepsilon N}(dw) = 0,
\]
using Proposition 2.1 and bound (2.11), we obtain that \( I_4 \) goes to zero as \( N \) goes to infinity by the Lebesgue dominated convergence Theorem.

□ Estimate of \( I_5 \)

Denoting for \( s \in [0, T] \)
\[
\mu_s^N = \frac{1}{N} \sum_{j=1}^N \delta_{\nu_j^{N,j}} \quad \mu_{s-1}^{N-1,\varepsilon N} = \frac{1}{N-1} \sum_{j \neq i}^{N-1} \delta_{\nu_j^{N,j}} * \rho_{\varepsilon N}.
\]
we rewrite
\[ K^N - C(f) = \frac{1}{N} \sum_{i=1}^{N} \prod_{k=1}^{n} \phi_k(V_{ik}^{N,i}) \int_t^s \left( (\alpha_{\delta,\eta}^R (\mu_u^{N,1})^2 - (\alpha_{\delta,\eta}^R (\mu_u^{N}))^2) \Delta \phi(V_{uk}^{N,i}) du + C(\mu^N) - C(f) \right) \]

Moreover we clearly have
\[ \alpha^R_{\delta,\eta}(\mu^N) - \alpha^R_{\delta,\eta}(\mu^N_{N-1,\varepsilon_N}) \leq \alpha^R_{\delta,\eta}(\mu^N - \mu^N_{N-1,\varepsilon_N}) \]

Using the transport plan which consists in splitting the atom \( \delta_{V_{ij}} \) of mass \( 1/N \) in \( \mu^N_{N-1} \) atoms of mass \( 1/(N(N-1)) \) and transporting each of these atoms onto each of the \( \delta_{V_{ij}} \) for \( j \neq i \), we obtain that

\[ W_1 \left( \mu^N, \frac{1}{N-1} \sum_{j \neq i} \delta_{V_{ij}} \right) \leq \frac{1}{N(N-1)} \sum_{j \neq i} |V_{si}^{N,i} - V_{si}^{N,j}| \leq N^{-1} \frac{1}{(N-1)} \sum_{j \neq i} C_k (|V_{si}^{N,i}|^k + |V_{si}^{N,j}|^k + 1) \]

Moreover we clearly have
\[ W_1 \left( \frac{1}{N-1} \sum_{j \neq i} \delta_{V_{ij}}^{N-1,\varepsilon_N} \right) \leq W_1(\delta_0, \rho_{\varepsilon_N}). \]

Hence using the symmetry bound \[2.10\]

\[ \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \prod_{k=1}^{n} \phi_k(V_{ik}^{N,i}) \int_s^t \left( (\alpha_{\delta,\eta}^R (\mu_u^{N,1,\varepsilon_N})^2 - (\alpha_{\delta,\eta}^R (\mu_u^{N}))^2) \Delta \phi(V_{uk}^{N,i}) du \right) \right] \leq C_{\phi,n,\delta,\eta,M_0,T,k} \left( N^{-1} + W_1(\delta_0, \rho_{\varepsilon_N}) \right) \]

Moreover, since \( C \) is smooth on \( P(\mathbb{D}([0,T],\mathbb{R}^3)) \) (due to Lemma B.1), and \( \mu^N \) converges in law to \( f, \mathbb{E} [|C(\mu^N) - C(f)|] \) converges to 0 as \( N \) goes to infinity. Finally, since the \( N \) independent Brownian motions \( (B_i^i)_{i=1,\ldots,N} \) are independent of the \( N \) independent Poisson random measures \( (\mathcal{M}^t)_{t=1,\ldots,N} \), we deduce from classical stochastic calculus that

\[ \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \prod_{k=1}^{n} \phi_k(V_{ik}^{N,i}) \left( \int_s^t \sqrt{2} \alpha_{\delta,\eta}^R \left( \frac{1}{N-1} \sum_{j \neq i} \delta_{V_{ij}}^{N-1,\varepsilon_N} \right) \Delta \phi(V_{uk}^{N,i}) du \right) + \int_{[s,t] \times [0,2\pi] \times \mathbb{R}^3 \times \{1,\ldots,N\} \setminus \{i\}} \left( \phi(V_{uj}^{N,i} + c_{\gamma,w}(V_{uj}^{N,i},V_{uj}^{N,j} + w, \varphi, z)) - \phi(V_{uj}^{N,i}) \right) d\mathcal{M}_{\gamma,l}^t(du, dz, d\varphi, dw, dj) \right]^2 \]

\[ \leq C_{\phi,k=1,\ldots,n,\delta,\varepsilon,N^{-1}}. \]

Combining all the estimates obtained in this step yields

\[ \lim_{N \to \infty} \mathbb{E} [K^N - C(f)] = 0. \]

\[ \square \] Conclusion.
Gathering all the estimates obtained so far letting \( N \) go to infinity and \( \varepsilon \) to 0, we have
\[
\mathbb{E} \|[\mathcal{M}(f) - \mathcal{C}(f)]\| = 0.
\]
Therefore \( \mathcal{M}(f) = \mathcal{C}(f) \) almost surely, and the result is proved. \( \square \)

**Lemma 4.1.** Whenever \( \gamma + \nu > 0 \), the set \( S \) is a singleton.

**Proof.** Let be \( f \in S \), then \( f \) satisfies condition \((b)\) of \((4.2)\). It follows that \( f \in L^1(0,T;L^p) \) for \( p \in \left( \frac{3}{3+\gamma}, \frac{3}{3-\nu} \right) \). Indeed for \( q \geq \frac{3}{3+\gamma} \) there holds, for any \( h \in L^1 \)
\[
\| h \|_{L^p}^p = \int_{\mathbb{R}^3} h^p = \int_{\mathbb{R}^3} \langle v \rangle^{\gamma(p-1)/q} h^{p-1/q} \langle v \rangle^{-\gamma(p-1)/q} h^{1/q'}
\leq \left( \int_{\mathbb{R}^3} \langle v \rangle^{-\gamma q(p-1)/q} h \right)^{1/q'} \left( \int_{\mathbb{R}^3} \langle v \rangle \gamma h \right)^{1/q}
\leq C_{p,q,r,\gamma} \left( 1 + \int_{\mathbb{R}^3} |v|^{-\gamma q(p-1)/q} h \right)^{1/q} \| \langle \cdot \rangle \gamma h \|_{L^{p(1-1/q)}}^{p(1-1/q)}.
\]
Using successively interpolation between Lebesgue spaces, and Sobolev’s embedding (see for instance [12] Theorem 6.5) we have that
\[
\| \langle \cdot \rangle^{\gamma} h \|_{L^{p'(p-1/q)}} \leq \| \langle \cdot \rangle^{\gamma} h \|_{L^{1}} \left( 1 - \frac{\gamma q(p-1)/q}{\gamma q p/(p-1/q)} \right)^{\frac{2(qp)}{\gamma q p/(p-1/q) - 1}} \| \langle \cdot \rangle \gamma h \|_{L^{p(1-1/q)}}^{\left( 1 - \frac{\gamma q(p-1)/q}{\gamma q p/(p-1/q)} \right)^{\frac{2(qp)}{\gamma q p/(p-1/q) - 1}}}
\leq C_{q,p,d,\nu} \left( \| \langle \cdot \rangle^{\gamma} h \|_{L^{p'/p}} + 1 \right) \leq C_{q,p,d,\nu} \left( \| \sqrt{\langle \cdot \rangle} \gamma h \|_{H^{r/2}}^2 \right) + 1,
\]
Therefore since \( k \geq |\gamma| q(p-1/q) \)
\[
\int_{0}^{T} \| f_t \|_{L^p} dt \leq C_{q,p,d,\nu,r,\gamma} \int_{0}^{T} \left( 1 + \int_{\mathbb{R}^3} |v|^{-\gamma q(p-1)/q} f_t \right)^{1/pq} \left( 1 + \sqrt{\langle \cdot \rangle^{\gamma} f_t \|_{H^{r/2}}}^2 \right)^{1-1/q} dt
\leq C_{q,p,d,\nu,r,\gamma,T} \left( 1 + \int_{0}^{T} \sqrt{\langle \cdot \rangle^{\gamma} f_t \|_{H^{r/2}}^2} dt + \int_{0}^{T} \int_{\mathbb{R}^3} |v|^{-\gamma q(p-1)/q} f_t dt \right).
\]
Then the result is a simple consequence [17] Theorem 1.3. We provide a short sketch of proof, summarizing [17] Section 3.1.

Let \( Q, \tilde{Q} \in \mathcal{P}(\mathcal{D}(0,T;\mathbb{R}^3)) \) be two solutions to the martingale problem. Classical stochastic calculus tools provide the existence, on some suitable probability space, of a Brownian motion \((B_t)_{t \in [0,T]}\) and a Poisson random measure \( M \) on \([0,T] \times \mathbb{R}^3 \times [0,2\pi] \times \mathbb{R}^+ \) with the intensity \( ds \times R_s (dv, dw_s) \times d\varphi \times dz \) where for any \( s \in [0,T]\), \( R_s \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3) \) is a coupling plan between \( Q_s \) and \( \tilde{Q}_s \), and a \( Q \)-distributed process \((V_t)_{t \in [0,T]}\) (resp. a \( \tilde{Q} \)-distributed process \((W_t)_{t \in [0,T]}\) ) such that
\[
V_t = V_0 + \int_{[0,t]} c_{\gamma,\nu}(V_{s-},v,\varphi,z) M(ds, dv, dw_s, d\varphi, dz) + \int_{[0,t]} K_{\gamma}(V_s - v) Q_s (dv) ds + \int_{[0,t]} \alpha^R_{\delta,\eta}(Q_s) dB_s,
\]
\[
W_t = W_0 + \int_{[0,t]} c_{\gamma,\nu}(W_{s-},v_s,\varphi + \varphi_0(V_{s-} - v, W_{s-} - v_s),\varphi) M(ds, dv, dw_s, d\varphi, dz)
+ \int_{[0,t]} K_{\gamma}(W_s - v) \tilde{Q}_s (dv) ds + \int_{[0,t]} \alpha^R_{\delta,\eta}(\tilde{Q}_s) dB_s,
\]
where \( \varphi_0 : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow [0,2\pi) \) is the measurable function of [17] Lemma 3.2. Using some Ito’s expansion and taking the expectation yields
\[
\mathbb{E} \|[V_t - W_t]^2\| = \int_{0}^{t} \left( \alpha^R_{\delta,\eta}(Q_s) - \alpha^R_{\delta,\eta}(\tilde{Q}_s) \right)^2 ds
\]
Finally we obtain

\[W^2_t(Q_t, 
\hat{Q}_t) \leq \mathbb{E} \left[ |V_t - W_t|^2 \right] \leq C_{p, R, \delta, \eta} \int_0^t \left( 1 + \|Q_s\|_{L^p} + \|\hat{Q}_s\|_{L^p} \right) \mathbb{E} \left[ |V_s - W_s|^2 \right] ds,
\]

which yields by application of Gronwall’s inequality, \(W^2_t(Q_t, \hat{Q}_t) = 0\) for any \(t \in [0, T]\) and concludes the proof. \(\square\)

4.3. Conclusion. We are now in position to prove Theorem 2.1. Set \(T > 0\), and

\[R > 2 + \left( e^{C_k T} (1 + M_0) \right)^{1/k}, \quad \eta < C \left( 1 - e^{C_k T} (1 + M_0) \right), \quad \delta < \frac{CR^{-5}}{\eta 4(4R - 2k)}\]

For each \(N \geq 2\), consider \(\mathcal{V}_0^N\), a \(G_0^N\)-distributed random vector, and let \((\mathcal{V}_t^N)_{t \in [0, T]} \in \mathbb{D}([0, T], \mathbb{R}^3)\) be a solution to (2.4) starting from \(\mathcal{V}_0^N\). Due to Proposition 4.1, there is a subsequence of \(\frac{1}{N} \sum_{i=1}^{N} \delta(\mathcal{V}^N_{t, i})_{t \in [0, T]}\) \(N \geq 2\) which goes in law to some random variable \(f \in \mathcal{P}(\mathbb{D}([0, T], \mathbb{R}^3))\), of law \(\pi \in \mathcal{P}(\mathcal{D}([0, T], \mathbb{R}^3))\). We know that this random variable satisfies for any \(t \in [0, T]\) (see for instance [25, Theorem 5.4]),

\[
\mathbb{E} \left[ H(f_t) \right] = \int_{\mathcal{P}(\mathbb{R}^3)} H(\rho) \pi_t(d\rho) \leq \lim_{N \rightarrow \infty} \frac{1}{N} \int_{\mathbb{R}^N} G_t^N \ln(G_t^N) \leq H_0 + C k, \gamma, \nu T e^{C_k T} (1 + M_0) \\
\mathbb{E} \left[ \int_{\mathbb{R}^3} |v|^k f_t(dv) \right] \leq \lim_{N \rightarrow \infty} \frac{1}{N} \int_{\mathbb{R}^N} |v|^k G_t^N dv \leq e^{C_k T} (1 + M_0)
\]

(4.3)

Moreover we know from Proposition 1.2 and Lemma 4.1 that \(f\) almost surely belongs to a singleton of \(\mathcal{P}(\mathcal{D}([0, T], \mathbb{R}^3))\). Hence it follows that the full sequence \(\frac{1}{N} \sum_{i=1}^{N} \delta(\mathcal{V}_t^N, i)_{t \in [0, T]}\) \(N \geq 2\) goes in law to \(f\). Moreover the support of \(\pi\) is reduced to one point, hence it is a Dirac mass of \(\mathcal{P}(\mathbb{D}([0, T], \mathbb{R}^3))\). Therefore the bounds in expectation (4.3) are in fact almost sure bounds. In view of Lemma B.2, for any \(t \in [0, T]\), it holds almost surely

\[\alpha_{\delta, \eta}^R(f_t) = 0.\]

Hence \((f_t)_{t \in [0, T]}\) is a weak solution to the Boltzmann equation, starting from \(g_0\), and satisfying \((f_t)_{t \in [0, T]} \in L^1(0, T; L^p(\mathbb{R}^d))\) for some \(p > \frac{3}{3+\gamma}\). Hence it is the unique solution starting from \(g_0\), in view of [17, Corollary 1.5], which concludes the proof.

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Appendix A. Properties of the collision parametrization

In this appendix, we prove some useful properties of the coefficient \(c_{\gamma, \nu}\) defined in (2.2). We begin with the

Lemma A.1. For any \(k \geq 2\) there are \(C_k, C_{\nu} > 0\) such that for \(v, v_* \in \mathbb{R}^3 \times \mathbb{R}^3\) it holds

(i) \[
\int_{\mathbb{R}^+ \times [0, 2\pi]} c_{\gamma, \nu}(v, v_*, z, \varphi) dz d\varphi = \beta_0 K_\nu (v - v_*).
\]

(ii) \[
\int_{\mathbb{R}^+ \times [0, 2\pi]} |c_{\gamma, \nu}(v, v_*, z, \varphi)|^2 dz d\varphi \leq C_\nu |v - v_*|^{\gamma + 2}.
\]

(iii) \[
\left| \int_{\mathbb{R}^+ \times [0, 2\pi]} \tilde{\Delta}(|\cdot|^k)(v, v_*, z, \varphi) dz d\varphi \right| \leq C_k (|v|^k + |v_*|^k + 1).
\]

Proof. The first two points are taken from [17]. See [17] Lemma 2.1 for the first point and [17] Lemma 2.3 for the second.
For \(s \in (0, 1)\) we set

\[v_* = v + sc_{\gamma, \nu}(v, v_*, z, \varphi).\]

By Taylor’s expansion we find that

\[
\tilde{\Delta}(|\cdot|^k)(v, v_*, z, \varphi) = k \int_0^1 |v_*|^k - (c_{\gamma, \nu}(v, v_*, z, \varphi), I_3 + (k - 2) \frac{v_* \otimes v_*}{|v_*|^2}, c_{\gamma, \nu}(v, v_*, z, \varphi)) ds
\]

\[
\leq k \int_0^1 |v_*|^{k-2} |c_{\gamma, \nu}(v, v_*, z, \varphi)|^2 + (k - 2) \frac{(c_{\gamma, \nu}(v, v_*, z, \varphi) \cdot v_*|^2}{|v_*|^2} ds
\]

\[
\leq 3k(k - 1) \int_0^1 |v_*|^{k-2} |c_{\gamma, \nu}(v, v_*, z, \varphi)|^2 ds
\]

\[
\leq 3k(k - 1) (|v| + |c_{\gamma, \nu}(v, v_*, z, \varphi)|)^{k-2} |c_{\gamma, \nu}(v, v_*, z, \varphi)|^2
\]

\[
\leq C_k \left( |v|^{k-2} |c_{\gamma, \nu}(v, v_*, z, \varphi)|^2 + |c_{\gamma, \nu}(v, v_*, z, \varphi)|^k \right).
\]

Therefore

\[
\int_{\mathbb{R}^+ \times \mathbb{S}^2} \tilde{\Delta}(|\cdot|^k)(v, v_*, z, \varphi) dz d\varphi \leq C_k \left( \int_{\mathbb{R}^+ \times [0, 2\pi]} |v|^{k-2} |c_{\gamma, \nu}(v, v_*, z, \varphi)|^2 + |c_{\gamma, \nu}(v, v_*, z, \varphi)|^k dz d\varphi \right)
\]

\[
\leq C_k \left( |v|^{k-2} |v - v_*|^{\gamma + 2} + |v - v_*|^{\gamma + k} \right) \leq C_k (|v|^k + |v_*|^k + 1).
\]

since \(\gamma + 2 > 0\). Which concludes the proof. \(\Box\)

Then we need the

Lemma A.2. For any \(\phi \in C^2(\mathbb{R}^3)\) the function

\[(v, v_*) \mapsto \int_{[0, 2\pi] \times \mathbb{R}^+} \tilde{\Delta}\phi(v, v_*, z, \varphi) dz d\varphi,\]

is continuous.

Proof. By Taylor’s expansion and change of variables we have

\[
\int_{[0, 2\pi] \times \mathbb{R}^+} \tilde{\Delta}\phi(v, v_*, z, \varphi) dz d\varphi = \int_0^1 \int_{[0, 2\pi] \times \mathbb{R}^+} c_{\gamma, \nu}(v, v_*, \varphi, z) \cdot \nabla^2 \phi(v + sc_{\gamma, \nu}(v, v_*, \varphi, z)) \cdot c_{\gamma, \nu}(v, v_*, \varphi, z) dz d\varphi ds
\]
Then recall from (2.2), that

\[ \frac{v'-v}{|v'-v_*|} = \left( \frac{1-\cos(\theta)}{2}, \frac{\sin(\theta)}{2} \cos(\varphi), \frac{\sin(\theta)}{2} \sin(\varphi) \right) \]

in the orthonormal basis \( \left( \frac{v-v_*}{|v-v_*|}, \frac{f(v-v_*)}{|v-v_*|}, \frac{J(v-v_*)}{|v-v_*|} \right) \).

Therefore, since \( \nabla^2 \phi \) is bounded

\[
\int_{[0,\pi] \times [0,2\pi]} \beta(\theta) \left| \frac{v'-v}{|v-v_*|} \right|^2 \left| \nabla^2 \phi(v + s(v' - v)) \cdot \frac{v'-v}{|v-v_*|} \right| d\varphi d\theta < \infty.
\]

We conclude by observing that for any \( s \in [0,1], \theta \in [0,\pi] \) and \( \varphi \in [0,2\pi] \) the function

\[
(v, v_*) \mapsto |v-v_*|^{\gamma+2} \beta(\theta) \left| \frac{v'-v}{|v-v_*|} \right|^2 \cdot \nabla^2 \phi(v + s(v' - v)) \cdot \frac{v'-v}{|v-v_*|}
\]

is continuous (since \( \phi \in C^2 \) and \( \gamma + 2 > 0 \)).

\[ \square \]

**APPENDIX B. REGULARIZING EFFECTS OF GRAZING COLLISIONS**

In this appendix, we gather some properties of the coefficient \( \alpha_{\delta,\eta}^R \) defined in [2,3]. We begin with the

**Lemma B.1.** For any \( R > \delta, \eta \in (0,1) \), there is \( C_{R,\delta,\eta} \) such that for any \( f, g \in \mathcal{P}_1(\mathbb{R}^3) \)

\[
|\alpha_{\delta,\eta}^R(f) - \alpha_{\delta,\eta}^R(g)| \leq C_{R,\delta,\eta} W_1(f,g).
\]

**Proof.** First observe that for fixed \( e \in \mathbb{S}^2 \), the function

\[
(v, v_*) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \chi_R(v) \chi_R(v_*) \chi_{2\delta}((v-v_*) \cdot e) \in [0,1],
\]

is \( (\|\chi_R\|_{L^p} + \|\chi_{2\delta}\|_{L^p}) \)-Lipschitz. Then using the reversed triangular inequality and classical properties of Wasserstein metric we obtain

\[
\left| \sup_{e \in \mathbb{S}^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi_R(v) \chi_R(v_*) \chi_{2\delta}((v-v_*) \cdot e) f(dv)f(dv_*) - \sup_{e \in \mathbb{S}^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi_R(v) \chi_R(v_*) \chi_{2\delta}((v-v_*) \cdot e) g(dv)g(dv_*) \right| \leq \left( \|\chi_R\|_{L^p} + \|\chi_{2\delta}\|_{L^p} \right) W_1(f,g),
\]

so that

\[
|\alpha_{\delta,\eta}^R(f) - \alpha_{\delta,\eta}^R(g)| \leq \|\chi_{4\delta}\|_{L^p} \left| \int_{\mathbb{R}^3} \chi_{R-1}(v) f(dv) - \int_{\mathbb{R}^3} \chi_{R-1}(v) g(dv) \right| + \|1 - \chi_{\frac{3}{2}\delta}\|_{L^p} \sup_{e \in \mathbb{S}^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi_R(v) \chi_R(v_*) \chi_{\delta}((v-v_*) \cdot e) f(dv)f(dv_*) - \sup_{e \in \mathbb{S}^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi_R(v) \chi_R(v_*) \chi_{\delta}((v-v_*) \cdot e) g(dv)g(dv_*) \right| \leq C_{R,\delta,\eta} W_1(f,g).
\]

\[ \square \]

We then introduce a kind of decomposition of [2] Proposition 3]. We begin with a generalization of some classical equiintegrability result [44] Lemma 6], in the

**Lemma B.2.** Let \( f \in L \ln L(\mathbb{R}^3) \cap \mathcal{P}_k(\mathbb{R}^3) \) and \( H, M \in \mathbb{R} \) be such that \( M \geq \|f\|_{L_1^k} \) and \( H \geq H(f) \). When \( R, \delta, \eta \) satisfy

\[
R > 2 + M^{1/k}, \quad \eta < C \left( 1 - \frac{M}{4(R - 2)^r} \right), \quad \delta < \frac{CR^{-5}}{\exp(\frac{1}{4C_k+2(H+M)})},
\]


for an explicit numerical constant $C > 0$, and a constant $C_k > 0$ depending only on $k$, it holds

$$\alpha_{\delta, \eta}^R(f) = 0.$$  

**Proof.** We first choose $R > 2^k M^{1/k}$ and $\eta \in (0, 1/4)$ such that $1 - \frac{M}{(R - 2)^2} > 8\eta$, so that $\chi\eta (\int_{\mathbb{R}^3} \chi_{R-1}(v)g(dv)) = 0$.

Then define for any $e \in \mathbb{S}^2$

$$K^R_{\delta, \eta}(e) := \{(v, v_*) \text{ s.t. } |v| \leq R + 1, |v_*| \leq R + 1, |(v - v_*) \cdot e| \leq 4\delta \} \subset \mathbb{R}^3 \times \mathbb{R}^3,$$

and observe that it holds

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} 1_{K^R_{\delta, \eta}(e)} dv_* \leq \int_{\mathbb{R}^3} 1_{|v_*| \leq R+1} \left( \int_{\mathbb{R}^3} 1_{|v| \leq R+1, |(v - v_*) \cdot e| \leq 4\delta} dv \right) dv_* \leq CR^5 \delta.$$

Following [20] point (ii) of Lemma C.1 we obtain for some constant $C_k > 1$ depending only on $k$

$$\sup_{e \in \mathbb{S}^2} f \otimes f (K^R_{\delta, \eta}(e)) \leq C_k + H(f \otimes f) + \int_{\mathbb{R}^3} (|v|^k + |v_*|^k) f(dv) f(dv_*),$$

so that for any $e \in \mathbb{S}^2$

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi_R(v)\chi_R(v_*) \chi_\delta((v - v_*) \cdot e) f(dv) f(dv_*) \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} 1_{K^R_{\delta, \eta}(e)} f(dv) f(dv_*) \leq \eta^2/4,$$

and then $(1 - \chi_{\eta^2/2}) \left( \sup_{e \in \mathbb{S}^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi_R(v)\chi_R(v_*) \chi_\delta((v - v_*) \cdot e) f(dv) f(dv_*) \right) = 0$, and the result is proved.

We can now give the result which enables to provide the key estimate (2.11) in the

**Lemma B.3.** Assume that the angular kernel $b$ satisfies (H). For any $R > 1, \delta, \eta \in (0, 1)$, there is $\kappa > 0$ depending on $R, \delta, \eta$ such that for any $f \in \mathcal{P}(\mathbb{R}^3)$, and $|\xi| \geq \pi R^{-1}$ there holds

$$(\alpha_{\delta, \eta}^R(f))^2|\xi|^2 + \int_{\mathbb{S}^2} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left( \mathcal{F}(\chi_{B_{R, \delta}} f)(0) - |\mathcal{F}(\chi_{B_{R, \delta}} f)(\xi_-)| \right) d\sigma \geq \kappa |\xi|'^\nu,$$

with $\xi_- = \frac{1}{2}(\xi - |\xi|\sigma)$, $B_{R, \delta}^* = B_R \setminus B_\delta^*$ and $\chi_{B_{R, \delta}}^*$ smooth and satisfying $1_{B_{R, \delta}^*/2} \leq \chi_{B_{R, \delta}}^* \leq 1_{B_{R+1, \delta}^*}$.  

**Proof.** We proceed by disjunction of the cases.

- **Case 1:** $\int_{B_R} f < 4\eta$.

In this case we clearly have

$$\int_{\mathbb{R}^3} \chi_{R-1}(v) f(dv) < 4\eta$$

and then

$$(\alpha_{\delta, \eta}^R(f))^2|\xi|^2 \geq |\xi|^2 \geq (\pi R^{-1})^2 |\xi|'^\nu.$$  

- **Case 2:** $\int_{B_R} f \geq 4\eta$.

By [2] for any $\xi_- \in \mathbb{R}^3$ there is $\theta_{\xi_-} \in (0, 2\pi)$ such that

$$\mathcal{F} \left( \chi_{B_{R, \delta}} f \right)(0) - |\mathcal{F} \left( \chi_{B_{R, \delta}}^* f \right)(\xi_-)| = 2 \int_{\mathbb{R}^d} \sin^2 \left( \frac{w \cdot \xi_- + \theta_{\xi_-}}{2} \right) \chi_{B_{R, \delta}}(w) f(dw).$$
For fixed $\xi \in \mathbb{R}^d, \sigma \in \mathbb{S}^2$ we denote
\[ K_{\delta, \xi_-} := \left\{ |v| \leq R, \exists \ell \in \mathbb{Z}, \ |v \cdot \xi_- + \theta \xi_- + 2\ell \pi| \leq \delta \right\} \]

Then it holds
\[
\int_{\mathbb{R}^d} \sin^2 \left( \frac{w \cdot \xi_- + \theta \xi_-}{2} \right) \chi_{B_R^{2,\delta}}(w)f(dw) \geq \sin^2 \left( \frac{1}{2} \delta |\xi_-| \right) \int_{(B_R \setminus B_R^2) \setminus K_{\delta, \xi_-}} f(dw)
\]
\[
\geq \sin^2 \left( \frac{1}{2} \delta |\xi_-| \right) \left( \int_{B_R} f(dw) - \int_{K_{\delta, \xi_-}} f(dw) - \int_{B_R^2} f(dw) \right)
\]

\(\circ\) subCase 1 : \(\forall |\xi_-| \leq \pi R^{-1}\) it holds \(\int_{K_{\delta, \xi_-}} f(dw) < \eta\) and \(\int_{B_R^2} f(dw) \leq 2\eta\).

In this case we clearly have for any \(|\xi_-| \leq \pi R^{-1}\)
\[
\int_{\mathbb{R}^3} \sin^2 \left( \frac{w \cdot \xi_- + \theta \xi_-}{2} \right) \chi_R(w)f(dw) \geq \sin^2 \left( \frac{1}{2} \delta |\xi_-| \right) \eta \geq \inf_{u \in (0, \pi R^{-1})} \left( \frac{\sin(x)}{2x} \right)^2 \delta^2 \eta |\xi_-|^2
\]
so that
\[
\int_{\mathbb{S}^2} \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left( \mathcal{F} \left( \chi_{B_R^{2,\delta}} f \right)(0) - |\mathcal{F} \left( \chi_{B_R^{2,\delta}} f \right)(\xi_-)| \right) d\sigma
\]
\[
\geq \int_{\mathbb{S}^2} 1_{|\xi_-| \leq \pi R^{-1}} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \left( \mathcal{F} \left( \chi_{B_R^{2,\delta}} f \right)(0) - |\mathcal{F} \left( \chi_{B_R^{2,\delta}} f \right)(\xi_-)| \right) d\sigma
\]
\[
\geq C_{\eta, \delta, R} \int_{\mathbb{S}^2} 1_{|\xi_-| \leq \pi R^{-1}} b \left( \frac{\xi}{|\xi|} \cdot \sigma \right) |\xi_-|^2 d\sigma
\]
\[
= C_{\eta, \delta, R} \int_0^\pi \beta \cos(\theta) \left( \frac{\xi}{|\xi|} \right) \left( \sqrt{\frac{1 - \cos(\theta)}{2}} - \frac{\pi}{2} \right) d\theta
\]
\[
\geq C_{\eta, \delta, R} K_2 |\xi|^2 \int_0^\pi \left( 1 - \frac{\pi}{2} \right) \frac{d\theta}{\theta^{1+\nu}} = C_{\eta, \delta, R} |\xi|^2 \left[ (2 - \nu)^{-1} \theta^{2-\nu} \right]_0^{\pi R^{-1} |\xi^-|} = C_{\eta, \delta, R, \nu} |\xi|^\nu.
\]

\(\circ\) subCase 2 : \(\exists \xi_- \leq \pi R^{-1}\) such that \(\int_{K_{\delta, \xi_-}} f(dw) \geq \eta\) or \(\int_{B_R^2} f(dw) \geq 2\eta\).

Since \(\frac{2\pi}{|\xi_-|} \geq 2R\) there exists one \(k \in \mathbb{Z}\) such that
\[
K_{\delta, \xi_-} = \left\{ |v| \leq R, \ |v \cdot \xi_- + \theta \xi_- + 2k \pi| \leq \delta \right\},
\]
(or \(K_{\delta, \xi_-} = 0\) which is in contradiction with the assumption). So that for any \(v, v_* \in B_R \setminus K_{\delta, \xi_-}\) it holds
\[
|v - v_*| \cdot \frac{\xi_-}{|\xi_-|} \leq 2\delta,
\]
and then
\[
\eta^2 \leq \left( \int_{K_{\delta, \xi_-} \setminus \xi_-} f(dw) \right)^2 = \int_{\mathbb{R}^3} 1_{K_{\delta, \xi_-} \setminus \xi_-} (v) 1_{K_{\delta, \xi_-} \setminus \xi_-} (v_*) f(dw) f(dw_*)
\]
\[
\leq \int_{\mathbb{R}^3} \chi_R(v) \chi_R(v_*) \chi_2 \delta \left( (v - v_*) \cdot \frac{\xi_-}{|\xi_-|} \right) f(dw) f(dw_*)
\]
\[
\leq \sup_{\mathbb{S}^2} \int_{\mathbb{R}^3} \chi_R(v) \chi_R(v_*) \chi_2 \delta \left( (v - v_*) \cdot \mathbf{e} \right) f(dw) f(dw_*)
\]
and then
\[
\alpha_{\rho, R}(\mu^N) \geq 1.
\]
Or similarly

$$4\eta^2 \leq \left( \int_{B^*_2} f(dv) \right)^2 = \int_{\mathbb{R}^3 \times \mathbb{R}^3} 1_{K_{\delta,\xi,[-\epsilon,\epsilon]}}(v)1_{K_{\delta,\xi,[-\epsilon,\epsilon]}}(v_*) f(dv)f(dv_*)$$

$$\leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi_R(v)\chi_R(v_*)\chi_{2\delta} \left( (v-v_*) \cdot \xi/|\xi| \right) f(dv)f(dv_*)$$

$$\leq \sup_{e \in \mathbb{S}^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi_R(v)\chi_R(v_*)\chi_{2\delta} \left( (v-v_*) \cdot e \right) f(dv)f(dv_*)$$

And the result is proved with $$\kappa = \min \left( \left( \pi R^{-1} \right)^2 - \nu, C_{\eta,\delta,R,\nu} \right)$$. 

\[ \Box \]

**Appendix C. Proof of Proposition 2.1**

In this appendix, we prove Proposition 2.1. Note that in the tensorized case $$F^N = f^\otimes N$$, we directly obtain the result, thanks to the Hardy-Littlewood-Sobolev inequality [28, Theorem 4.3], and point (i) of Lemma [D.1] below. We split the proof in four Steps.

\[ \diamond \] Step one : A preliminary computation.

We denote $$F_2 \in \mathcal{P}_{\text{sym}}(\mathbb{R}^{2d})$$ the two particles marginal of $$F^N$$. By symmetry we have

$$\sup_{i \neq j} \int_{\mathbb{R}^{2N}} |v_i - v_j|^{-\lambda} F^N(dV) = \int_{\mathbb{R}^{2d}} |v_1 - v_2|^{-\lambda} F_2(dv_1, dv_2).$$

Let $$(V_1, V_2)$$ be a couple of symmetric random variables of joint law $$F_2 \in \mathcal{P}_{\text{sym}}(\mathbb{R}^d \times \mathbb{R}^d)$$. We introduce $$\Phi(x,y) = (x - y, y)$$, with $$\Phi^{-1}(x,y) = (x + y, y)$$. Denote $$\tilde{F}_2 = F_2 \circ \Phi^{-1}$$, so that

$$\tilde{F}_2 = \mathcal{L}(V_1 - V_2, V_2).$$

Let us denote $$f$$ the first marginal of $$\tilde{F}_2$$, i.e. $$f = \mathcal{L}(V_1 - V_2)$$. By unitary changes of variables, we obtain

$$\mathcal{I}^2_{\nu,\gamma}(F_2) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^{2d}} \frac{\left( \langle v_* \rangle^{\gamma/2} \sqrt{\tilde{F}_2(v_*, v_1)} - \langle v \rangle^{\gamma/2} \sqrt{\tilde{F}_2(v, v_1)} \right)^2}{|v_* - v|^{d+\nu}} dvdv_* dv_1$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{2d}} \frac{\left( \langle v_* \rangle^{\gamma/2} \sqrt{\tilde{F}_2(v_*, v_1)} - \langle v \rangle^{\gamma/2} \sqrt{\tilde{F}_2(v, v_1)} \right)^2}{|v_* - v - (v_1)|^{d+\nu}} dvdv_* dv_1$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{2d}} \frac{\left( \langle v_* + v_1 \rangle^{\gamma/2} \sqrt{\tilde{F}_2(v_*, v_1)} - \langle v + v_1 \rangle^{\gamma/2} \sqrt{\tilde{F}_2(v, v_1)} \right)^2}{|v_* - v|^{d+\nu}} dvdv_* dv_1$$

$$= \int_{\mathbb{R}^d} \langle v_1 \rangle\int_{\mathbb{R}^{2d}} \frac{\left( a(v, v_1) \sqrt{\tilde{F}_2(v_*, v_1)} - a(v_1, v_1) \sqrt{\tilde{F}_2(v_1, v_1)} \right)^2}{|v_* - v|^{d+\nu}} dvdv_* dv_1 = \int_{\mathbb{R}^d} \langle v_1 \rangle \left| a(\cdot, v_1) \sqrt{\tilde{F}_2(\cdot, v_1)} \right|^{2}_{H^{\nu/2}} dv_1,$$

with

$$a(v, v_1) = \left( \frac{1 + |v + v_1|^2}{1 + |v_1|^2} \right)^\gamma$$

\[ \diamond \] Step two : An intermediary information.

Let be $$S > 0$$ and $$r \in \left( \frac{1}{2}, 1 \right)$$. We define

$$\Psi_r : (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \mapsto (\sqrt{x} - \sqrt{y})^{2r} \left( \frac{x + y}{2} \right)^{1-r},$$

(C.1)
and for \( G^2 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \)

\[
I_{r,S}(G^2) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{S}(v-v_*) \frac{\Psi_r(\chi_S^2(v)G^2(v,v_1),\chi_S^2(v_*)G^2(v_*,v_1))}{|v_*-v|^r} dv_* dv_1
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{S}(v-v_*) \left( \frac{\chi_S(v)\sqrt{G^2(v,v_1)} - \chi_S(v_*)\sqrt{G^2(v_*,v_1)}}{|v_*-v|^{r+d+\nu}} \right)^{2r} \left( \frac{\chi_S^2(v)G^2(v,v_1) + \chi_S^2(v_*)G^2(v_*,v_1)}{2} \right)^{1-r} dv_* dv_1
\]

We claim in this step that there is a constant such that

\[
I_{r,S}(\tilde{F}_2) \leq C_{\gamma,d} (I_{r,S}(F_2) + 1)^r \left( \int_{\mathbb{R}^d} |w|^{-\gamma + \nu} F_2(dw) + 1 \right)^{1-r}.
\]

Indeed by Holder’s inequality we obtain

\[
I_{r,S}(\tilde{F}_2) \leq \left( \int_{\mathbb{R}^d} \langle v_1 \rangle^\gamma \int_{\mathbb{R}^d} \chi_{S}(v-v_*) \left( \frac{\chi_S(v)\sqrt{\tilde{F}_2(v,v_1)} - \chi_S(v_*)\sqrt{\tilde{F}_2(v_*,v_1)}}{|v_*-v|^{d+\nu}} \right)^{2r} dv_* dv_1 \right)^r
\]

\[
\times \left( \int_{\mathbb{R}^d} \langle v_1 \rangle^{-\gamma + \nu} \int_{\mathbb{R}^d} \chi_{S}(v-v_*) \left( \frac{\chi_S(v)\tilde{F}_2(v,v_1) + \chi_S(v_*)\tilde{F}_2(v_*,v_1)}{2} \right) dv_* dv_1 \right)^{1-r}
\]

\[
:= (K_1)^r (K_2)^{1-r}
\]

By symmetry we easily find

\[
K_2 \leq \int_{\mathbb{R}^d} \langle v_1 \rangle^{-\gamma + \nu} \chi_{S}(v)\tilde{F}_2(v,v_1) \left( \int_{\mathbb{R}^d} \chi_{S}(v-v_*) dv_* \right) dv v_1
\]

\[
\leq C_{S,d} \int_{\mathbb{R}^d} \langle v_1 \rangle^{-\gamma + \nu} \tilde{F}_2(v,v_1) dv v_1 = C_{S,d} \int_{\mathbb{R}^d} \langle v_1 \rangle^{-\gamma + \nu} F_2(v,v_1) dv v_1,
\]

since, by definition \( \tilde{F}_2 \) and \( F_2 \) have the same second marginal. Then using (3.2), we have that for any \( v_1 \in \mathbb{R}^d \)

\[
\left| \chi_S(.) \sqrt{\tilde{F}_2(.,v_1)} \right|_{H^{\nu/2}}^2 \leq 2 \int_{\mathbb{R}^d} \chi_{S}^2(v_*) \left( \frac{\sqrt{\tilde{F}_2(v,v_1)} - \sqrt{\tilde{F}_2(v_*,v_1)}}{|v-v_*|^{d+\nu}} \right)^2 dv_* + 2 \int_{\mathbb{R}^d} \frac{\chi_S(v) - \chi_S(v_*)}{|v-v_*|^{d+\nu}} \tilde{F}_2(v,v_1) dv v_*
\]

and similarly

\[
\left| a(.,v_1) \sqrt{\tilde{F}_2(.,v_1)} \right|_{H^{\nu/2}}^2 \geq \frac{1}{2} \int_{\mathbb{R}^d} a^2(v_*,v_1) \left( \frac{\sqrt{\tilde{F}_2(v,v_1)} - \sqrt{\tilde{F}_2(v_*,v_1)}}{|v-v_*|^{d+\nu}} \right)^2 dv_*
\]

\[
- 2 \int_{\mathbb{R}^d} \frac{(a(v,v_1) - a(v_*,v_1))^2}{|v-v_*|^{d+\nu}} \tilde{F}_2(v,v_1) dv v_*
\]

Then observe that for any \( v_1 \in \mathbb{R}^d \) and \( |v| \leq S \), there holds \( a(v,v_1) \geq C_{S,\gamma} \). Indeed for \( |v| \leq 2S \) we obviously have

\[
\frac{1+|v+v_1|^2}{1+|v|^2} \leq 1 + 9S^2 \quad \text{and} \quad a(v,v_1) \geq (1 + 9S^2)^{\gamma/4}.
\]

On the other hand \( |v| \geq 2S \) we have

\[
1 + \frac{2|v+v_1|^2}{1+|v|^2} + \frac{|v|^2}{1+|v|^2} \leq 4 \quad \text{and therefore} \quad a(v,v_1) \geq 4^{\gamma/4}.
\]

Gathering all the above inequalities, and multiplying by \( \langle v_1 \rangle^\gamma \) and integrating w.r.t the Lebesgue measure on \( v_1 \) yields
we obtain that for any $G \mid h \rangle L^2$ and integrating w.r.t. $v,v_*$

and since $\chi_S$ is bounded and Lipschitz, and so is $v \mapsto (v + v_1)^{\gamma/2}$ for any $v_1$ (uniformly in $v_1$), it follows that

$$C_{S,\gamma,d,\nu} \left( T^2_{d,\nu} (F_2) + 1 \right) \geq K_1,$$

which conclude the step.

\begin{itemize}
  \item Step three : Jensen’s inequality
  
  We leave the reader check that the function $\Psi_r$ defined in \textbf{(C.1)} is convex $r \in \left( \frac{1}{2}, 1 \right)$. As a consequence, we obtain that for any $G^2 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, there holds

$$T^2_{d,\nu} (G^2) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{2S}(v - v_*) \frac{\Psi_r(\chi_S^2(v)G^2(v,v_1))}{|v_* - v|^r (d+\nu)} d\nu d\nu_* dv dv_*$$

$$\geq \mathbb{E} \int_{\mathbb{R}^d} \chi_{2S}(v - v_*) \frac{\Psi_r(\chi_S^2(v)G^2(v,v_1))}{|v_* - v|^r (d+\nu)} d\nu d\nu_* dv dv_*.$$ 

Indeed due to Jensen’s inequality we have for any $L > 0$

$$(C_d L^d)^{-1} \int_{\mathbb{R}^d} 1_{|v_*| \leq L} \Psi_r(\chi_S^2(v)G^2(v,v_1)) dv_1$$

$$\geq \Psi_r \left( \chi_S^2(v)(C_d L^d)^{-1} \int_{\mathbb{R}^d} 1_{|v_*| \leq L} G^2(v,v_1) dv_1, \chi_S^2(v_*)(C_d L^d)^{-1} \int_{\mathbb{R}^d} 1_{|v_*| \leq L} G^2(v_*,v_1) dv_1 \right)$$

$$= (C_d L^d)^{-1} \Psi_r \left( \chi_S^2(v) \int_{\mathbb{R}^d} 1_{|v_*| \leq L} G^2(v,v_1) dv_1, \chi_S^2(v_*) \int_{\mathbb{R}^d} 1_{|v_*| \leq L} G^2(v_*,v_1) dv_1 \right),$$

since $\Psi_r$ is 1-homogeneous. Therefore letting $L$ go to infinity, multiplying by $\chi_{2S}(v - v_*)|v_* - v_*|^{-r(d+\nu)}$ and integrating w.r.t. $v,v_*$ yields the desired bound.

\begin{itemize}
  \item Step four : A Sobolev’s like inequality.
  
  For any $\varepsilon \in \left( \frac{1}{2}, 1, \frac{5}{2} \right)$, we set $s = \varepsilon \frac{de}{2}$. In this step we claim that there is a constant $C_{d,s,\nu,\varepsilon,S} > 0$ such that for any $h \in L^1$

$$\|h\|_{L^{\infty}} \leq C_{d,s,\nu,\varepsilon,S} \left( \int_{\mathbb{R}^d} \chi_{2S}(x - y) \frac{\Psi_r(h(v),h(v_*))}{|v_* - v|^r (d+\nu)} dv dv_* \right)^{1/2r} + \|h\|_{L^1}^{1/2r}, \quad (C.4)$$

Indeed, rewriting

$$|h|_{W^{s,p}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \chi_{2S}(x - y) \frac{\Psi_r(h(v),h(v_*))}{|v_* - v|^r (d+sp)} dy + \int_{\mathbb{R}^d} (1 - \chi_{2S}(x - y)) \frac{h(x) - h(y)}{|x - y|^{d+sp}} dy, \quad := K_1 + K_2.$$

First observe that

$$K_2 \leq C_p \int_{\mathbb{R}^d} (1 - \chi_{2S}(x - y)) \frac{h^p(x) + h^p(y)}{|x - y|^{d+sp}} dxdy$$

$$= 2C_p \int_{\mathbb{R}^d} h^p(x) \left( \int_{\mathbb{R}^d} (1 - \chi_{2S}(x - y)) |x - y|^{-(d+sp)} dy \right) d \leq C_{S,d,s,p} \|h\|_{L^p}^p.$$
Then since
\[ d + sp = \frac{p}{2r}r(d + \nu) + \frac{2 - (1 + \varepsilon)p}{2}d, \]
we have
\[
\begin{align*}
K_1 &= \int_{\mathbb{R}^{2d}} \chi_2 s(x - y) \left( \frac{\sqrt{h(x)} - \sqrt{h(y)}}{|x - y|^{d + sp}} \right)^p (\sqrt{h(x)} + \sqrt{h(y)})^p dxy \\
&\leq C \int_{\mathbb{R}^{2d}} \chi_2 s(x - y) \left( \frac{\sqrt{h(x)} - \sqrt{h(y)}}{|x - y|^{d + sp}} \right)^p (h(x) + h(y))^{p/2} dxy \\
&= C \int_{\mathbb{R}^{2d}} \chi_2 s(x - y) \left( \frac{\sqrt{h(x)} - \sqrt{h(y)}}{|x - y|^{d + sp}} \right)^{2r} \left( \frac{h(x) + h(y)}{|x - y|^{r(d + \nu)}} \right)^{1 - r} \frac{(h(x) + h(y))^{p(2r - 1)/2r}}{|x - y|^{\frac{2(1 + \varepsilon)p}{(2r - 1)p}}} dxy \\
&\leq C \left( \int_{\mathbb{R}^{2d}} \chi_2 s(x - y) \frac{\Psi_r(h(x), h(y))}{|x - y|^{r(d + \nu)}} dxy \right)^{p/2r} \left( \int_{\mathbb{R}^{2d}} \chi_2 s(x - y) \frac{(h(x) + h(y))^{p(2r - 1)/2r}}{|x - y|^{\frac{r(2 - (1 + \varepsilon)p)}{2r - 1} - 2r}} dxy \right)^{1/2r}.
\end{align*}
\]

But by symmetry, and since \( r(2 - (1 + \varepsilon)p) < 1 \), we have
\[
\int_{\mathbb{R}^{2d}} \chi_2 s(x - y) \frac{(h(x) + h(y))^{p(2r - 1)/2r}}{|x - y|^{\frac{2(1 + \varepsilon)p}{2r - 1} - 2r}} dxy \leq C_p \int_{\mathbb{R}^d} h^{\frac{2r - 1}{2r - p}}(x) \left( \int_{\mathbb{R}^d} \chi_2 s(x - y)|x - y|^{-r(2 - (1 + \varepsilon)p)/2} dxy \right) dx \\
&\leq C_{p, r, \varepsilon, d, s} \|h\|_{L_p}^{\frac{2r - 1}{2r - p}} + C_{S, d, s, p} \|h\|_{L_p}^p.
\]

Hence
\[
\|h\|_{W^{r, p}(\mathbb{R}^d)} \leq C_{p, r, \varepsilon, d, s} \left( \int_{\mathbb{R}^{2d}} \chi_2 s(x - y) \frac{\Psi_r(h(x), h(y))}{|x - y|^{r(d + \nu)}} dxy \right)^{p/2r} \|h\|_{L_p}^{\frac{2r - 1}{2r - p}} + C_{S, d, s, p} \|h\|_{L_p}^p.
\]

By interpolation between Lebesgue spaces we have
\[
\|h\|_{L_p} \leq \|h\|_{L_1}^{\frac{1}{r}} \|h\|_{L_p}^{\frac{2r - 1}{2r - p}},
\]
therefore we find that
\[
\|h\|_{W^{r, p}(\mathbb{R}^d)} \leq C_{p, r, \varepsilon, d, s} \left( \left( \int_{\mathbb{R}^{2d}} \chi_2 s(x - y) \frac{\Psi_r(h(x), h(y))}{|x - y|^{r(d + \nu)}} dxy \right)^{p/2r} \|h\|_{L_1}^{\frac{1}{r}} \right) \|h\|_{L_p}^{\frac{2r - 1}{2r - p}} + C_{S, d, s, p} \|h\|_{L_p}^p.
\]

Then due to Sobolev’s embedding (see for instance [12 Theorem 6.5])
\[
\|h\|_{L^{p^*}} \leq C_{\text{Sob}} \|h\|_{W^{r, p}(\mathbb{R}^d)},
\]
for \( p^* = \frac{dp}{d - sp} \). Hence
\[
\|h\|_{L^{p^*}} \leq C_{p, r, \varepsilon, d, s} \left( \left( \int_{\mathbb{R}^{2d}} \chi_2 s(x - y) \frac{\Psi_r(h(x), h(y))}{|x - y|^{r(d + \nu)}} dxy \right)^{1/2r} \right)^{1/2r} \|h\|_{L_1}^{\frac{1}{r}} \|h\|_{L_p}^{\frac{2r - 1}{2r - p}}.
\]

We then choose \( p \) such that
\[
\frac{dp}{d - sp} = \frac{2r - 1}{2r - p}, \quad \text{i.e.} \quad p = \frac{d}{d - 2r + p},
\]
which leads to \( p^* = \frac{d}{d - 2r + p} \) to conclude the step.

\circ Final step:

We now fix \( r \in \left( \frac{d + \lambda}{2r} \lor \frac{2d - (\nu - \lambda)}{2r}, 1 \right) \), and \( \varepsilon \in \left( \frac{1}{2} - 1, \frac{\varepsilon}{2r} \wedge \frac{\nu - \lambda}{2d} \right) \) so that \( r(\nu - \varepsilon d) > \lambda \), and set \( p = \frac{d}{d - 2r + p} \). Using the fact that \( F_2 \) is a probability measure, and unitary change of variables, we obtain
\[ \int_{\mathbb{R}^d} |v_1 - v_2|^{-\lambda} F_2(dv_1, dv_2) = \int_{\mathbb{R}^d} |v_1 - v_2|^{-\lambda} \chi_S^2(v_1 - v_2) + |v_1 - v_2|^{-\lambda}(1 - \chi_S^2(v_1 - v_2))F_2(dv_1, dv_2) \]
\[ \leq \int_{\mathbb{R}^d} |v_1 - v_2|^{-\lambda} \chi_S^2(v_1 - v_2)F_2(dv_1, dv_2) + S^{-\lambda} \]
\[ = \int_{\mathbb{R}^d} |v|^{-\lambda} \chi_S^2(v)f(v)dv + S^{-\lambda} \]
\[ \leq \left( \int_{\mathbb{R}^d} 1_{|v| \leq S+1} |v|^{-\lambda'} dv \right)^{1/p'} \| \chi_S^2f \|_{L^p} + S^{-\lambda} \leq C_{d,S,\lambda,r,e,p,} (\| \chi_S^2f \|_{L^p} + 1). \]

Since \( \lambda' = d \frac{\lambda}{r(d-\rho)} < d \). Using successively (C.4), (C.3) and (C.2) we obtain
\[ \| \chi_S^2f \|_{L^p} \leq C_{d,s,v,r,e,s} \left( \left( \int_{\mathbb{R}^d} \chi_2S(x-y) \Psi_r(\frac{\chi_2^2f(v)}{|v-v_*|^r(d+\rho)})dvdv_* \right)^{1/2r} + \| \chi_R^2f \|_{L^2}^{1/2r} \right) \]
\[ \leq C_{d,s,v,r,e,s} \left( \left( \int_{\mathbb{R}^d} \chi_2S(\tilde{F}_2) \right)^{1/2r} + 1 \right) \]
\[ \leq C_{d,s,v,r,e,s} \gamma \left( \left( \int_{\mathbb{R}^d} \chi_2^2(F_2) + 1 \right)^{1/r} \left( \int_{\mathbb{R}^d} |w|^{-\gamma\frac{r}{r-\gamma}}F_2(dw, dw) + 1 \right)^{1-1/r} + 1 \right), \]

which concludes the proof, thanks to point (iii) of Lemma D.1.

APPENDIX D. PROOF OF PROPOSITION 2.2

In this appendix, we prove Proposition 2.2. We begin with stating some properties satisfied by the functionals defined in (2.8), and which allows to call these functionals information, in the

Lemma D.1. For any \( \gamma, \nu \), the family of functionals \( (\mathcal{I}^N_{\nu,\gamma})_{N \geq 1} \) satisfies the following properties.

(i) For any \( N \geq 2 \), \( \mathcal{I}^N_{\nu,\gamma} \) is convex and lower semi continuous w.r.t. the weak convergence on \( \mathcal{P}(\mathbb{R}^d) \) and for any \( f \in \mathcal{P}(\mathbb{R}^d) \) it holds
\[ \mathcal{I}^N_{\nu,\gamma}(f \otimes \mathbb{1}^N) = \left( \sqrt{\langle \cdot \rangle^\gamma f} \right)^2_{H^{\nu/2}((\mathbb{R}^d)^d)}. \]

(ii) For any \( G^N \in \mathcal{P}(\mathbb{R}^d) \), denote \( G^k \) and \( G^{N-k} \) its marginals on \( \mathbb{R}^d \) and \( \mathbb{R}^{d(N-k)} \), then
\[ N \mathcal{I}^N_{\nu,\gamma}(G^N) \geq k \mathcal{I}^k_{\nu,\gamma}(G^k) + (N - k) \mathcal{I}^{N-k}_{\nu,\gamma}(G^{N-k}), \]
with equality if and only if \( G^N = G^k \otimes G^{N-k} \).

(iii) For any \( G^N \in \mathcal{P}_{sym}(\mathbb{R}^d) \) and \( G^k \) its marginals on \( \mathbb{R}^d \), it holds
\[ \mathcal{I}^k_{\nu,\gamma}(G^k) \leq \mathcal{I}^N_{\nu,\gamma}(G^N). \]

Proof: Proof of (i)
We obtain the lower semi continuity, invoking [8 Theorem 3.4.3], following the argument of [15 Lemma 2.4] or [16 Proposition 2.5], since the function \( (x, y) \mapsto (\sqrt{x} - \sqrt{y})^2 \) is jointly convex, continuous and 1-homogeneous.

The last point is straightforward by definition (2.8).

Proof of (ii)
First observe that for any \( \alpha, \beta, a, b \geq 0 \) there holds
\[ (\sqrt{a} - \sqrt{b})^2 = (\sqrt{a} - \sqrt{b})^2 \frac{\alpha + \beta}{2} + (a - b)\frac{(\alpha - \beta)}{2} + 2\sqrt{ab} \left( \frac{\alpha + \beta}{2} - \sqrt{\alpha\beta} \right), \quad (D.1) \]
Let be $G_N \in \mathcal{P}(\mathbb{R}^{dN})$ and $(V_1, \cdots, V_N)$ a random vector of law $G_N$, fix $k = 1, \cdots, N$ and $i \in \{1, \cdots, k\}$ and denote $$g_k(v_1, \cdots, v_k|v_{k+1}, \cdots, v_N) = \mathcal{L}(V_1, \cdots, V_k|V_{k+1}, \cdots, V_N),$$ then $$G_N(v_1, \cdots, v_N) = g_k(v_1, \cdots, v_k|v_{k+1}, \cdots, x_N)G_{N-k}(v_{k+1}, \cdots, v_N) = g_{N-k}(v_{k+1}, \cdots, v_N|v_1, \cdots, v_k)G_k(v_1, \cdots, v_k).$$ Then by definition we have $$NI_{\nu, \gamma}^N(G^N) = \sum_{i=1}^N \int_{\mathbb{R}^d(N-1)} \int_{\mathbb{R}^{2d}} \left( (v)^{\frac{d}{2}} \sqrt{G^N(v_1, v_{i-1}, v, v_{i+1}, v_N)} - (v_{\ast})^{\frac{d}{2}} \sqrt{G^N(v_1, v_{i-1}, v_{\ast}, v_{i+1}, v_N)} \right)^2 \frac{dv_{\ast}dV_{i-1}^N}{|v_{\ast} - v|^{d+\nu}}.$$ By similar considerations, we obtain $$K_1 = \sum_{i=1}^N \int_{\mathbb{R}^d(N-1)} \int_{\mathbb{R}^{2d}} \left( (v)^{\frac{d}{2}} \sqrt{G^k(v^k)} - (v_{\ast})^{\frac{d}{2}} \sqrt{G^k(v^k)} \right)^2 \frac{dv_{\ast}dV_{i-1}^N}{|v_{\ast} - v|^{d+\nu}}.$$ Then due to (D.1), we have $$K_1 \geq (N-k)I_{\nu, \gamma}^{N-k}(G^{N-k}).$$ Moreover, in view of (D.1), equality holds if and only if for any $(v, v_{\ast}) \in \mathbb{R}^d \times \mathbb{R}^d$ and $i = 1, \cdots, k$ it holds $$g_{N-k}(V^{N-k}|v^k) = g_{N-k}(V^{N-k}|v^k),$$ which implies that for any $(v, v_{\ast}) \in \mathbb{R}^d \times \mathbb{R}^d$ and $i = 1, \cdots, k$, $$g_{N-k}(V^{N-k}|v^k) = g_{N-k}(V^{N-k}|v^k) = G^{N-k}(V^{N-k}),$$ and concludes the point.

Proof of point (iii)

By symmetry we can rewrite $$I_{\nu, \gamma}^N(G^N) = \int_{\mathbb{R}^d(N-1)} \int_{\mathbb{R}^{2d}} \left( (v)^{\frac{d}{2}} \sqrt{G^N(v_1, v_2, \cdots, v_N)} - (v_{\ast})^{\frac{d}{2}} \sqrt{G^N(v_{\ast}, v_2, \cdots, v_N)} \right)^2 \frac{dv_{\ast}dV_{i-1}^N}{|v_{\ast} - v|^{d+\nu}}.$$ Using once again (D.1), and integrating over $(v_{k+1}, \cdots, v_N)$ yields $$I_{\nu, \gamma}^N(G^N) \geq$$
Lemma D.2. which is borrowed from [25].

Lemma D.3. For \( \varepsilon > 0 \), define \( \psi_\varepsilon(x) = e^{-2\varepsilon - \varepsilon_0 x^2} \), which is borrowed from [25].

**Lemma D.2.** Let be \( \pi \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d)) \) and for \( N \geq 1 \) define

\[
\pi^{N,\varepsilon} = \int_{\mathcal{P}(\mathbb{R}^d)} (\rho * \psi_\varepsilon)^{\otimes N} \pi(d\rho).
\]

Then for any \( x_2, \cdots, x_N := X^{N-1} \in \mathbb{R}^{d(N-1)} \), define \( p^\varepsilon(|X^{N-1}|) \) the conditional law knowing \( X^{N-1} \) under \( \pi^{N,\varepsilon} \). Then it holds

(i) \[
\|\nabla \ln p^\varepsilon(|X^{N-1}|)\|_{L^\infty} \leq \varepsilon^{-1}.
\]

(ii) For any \( R > 0 \), there exist constants \( C_{\varepsilon,R}, c_{\varepsilon,R} > 0 \) such that for any \( x \in \mathbb{R}^d, e \in \mathbb{S}^d \) and \( u \in [0, R] \)

\[
c_{\varepsilon,R} p^\varepsilon(x | X^{N-1}) \leq p^\varepsilon(x + u e | X^{N-1}) \leq c_{\varepsilon,R} p^\varepsilon(x | X^{N-1}).
\]

The proof of this technical lemma can be found in [33 Lemma Appendix C.1]. We also need an affinity result on the mean fractional Fisher information. The question of affinity of mean information functional, is frequent particles system context (see [31] for the entropy, [27], [25] for the classical Fisher information, and [32] Appendix A for the fractional Fisher information in the case \( \gamma = 0 \)). More precisely, we have the

**Lemma D.3.** Assume that \( \gamma \leq 0 \) and \( \nu \in (0, 2) \). Define \( \tilde{I}_{\nu,\gamma} \), for \( \pi \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d)) \) as

\[
\tilde{I}_{\nu,\gamma}(\pi) = \limsup_{N \to \infty} \int_{\mathcal{P}(\mathbb{R}^d)} I_{\nu,\gamma}(\pi^N), \quad \pi^N = \int_{\mathcal{P}(\mathbb{R}^d)} \rho^{\otimes N} \pi(d\rho).
\]

Let \( \pi \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d)) \) be such that \( \int_{\mathcal{P}(\mathbb{R}^d)} \|\rho\|_{L^1} \pi(d\rho) < \infty \) for some \( \kappa \geq 1 \), and \( f_0 \in \mathcal{P}_\kappa(\mathbb{R}^d) \) and \( \nu > 0 \). Denote \( B_\varepsilon := \{ \rho \in \mathcal{P}_\kappa(\mathbb{R}^d), \text{ s.t. } W_1(\rho, f_0) < \varepsilon \} \), \( F = (\pi(B_\varepsilon))^{-1} \pi_{|B_\varepsilon} \) and \( G = (\pi(\mathcal{B}_{\varepsilon}))^{-1} \pi_{|\mathcal{B}_{\varepsilon}} \). Then for any \( \theta \in (0, 1) \) it holds

\[
\tilde{I}_{\nu,\gamma}(\theta F + (1 - \theta)G) = \theta \tilde{I}_{\nu,\gamma}(F) + (1 - \theta) \tilde{I}_{\nu,\gamma}(G).
\]

**Proof.** We divide the proof in four steps

\( \diamond \) **Step one:** A preliminary computation :

Let \((F^N)_{N \geq 1}, (G^N)_{N \geq 1}\) be two compatible sequences of symmetric probability densities. For \((v_2, \cdots, v_N) \in \mathbb{R}^{d(N-1)}\), fixed we denote \( V^N_N = (v, v_2, \cdots, v_N) \), and define

\[
\mathcal{D}^N(v, v) := \left( (v) \frac{2}{N} \sqrt{\theta F^N(V^N_N) + (1 - \theta) G^N(V^N_N)} - (v) \frac{2}{N} \sqrt{\theta F^N(V^N_N) + (1 - \theta) G^N(V^N_N)} \right)^2
\]

\[
- \theta \left( (v) \frac{2}{N} \sqrt{F^N(V^N_N)} - (v) \frac{2}{N} \sqrt{F^N(V^N_N)} \right)^2 - (1 - \theta) \left( (v) \frac{2}{N} \sqrt{G^N(V^N_N)} - (v) \frac{2}{N} \sqrt{G^N(V^N_N)} \right)^2.
\]
After expanding the squares, we find

\[ D^N(v, v_*) = 2\langle v \rangle \sqrt{\theta F^N(V_N^v)} \sqrt{F^N(V_N^{v_*})} + (1 - \theta) \sqrt{G^N(V_N^v)} \sqrt{G^N(V_N^{v_*})} \]

\[ - \sqrt{\theta F^N(V_N^v)} (1 - \theta) \sqrt{G^N(V_N^v)} \sqrt{G^N(V_N^{v_*})} \]

We first obtain the straightforward upper bound

\[ |D^N(v, v_*)| \leq C_0 \left( F^N(V_N^v) + F^N(V_N^{v_*}) + G^N(V_N^v) + G^N(V_N^{v_*}) \right) \]

Then observe that

\[ (\theta F^N(V_N^v) + (1 - \theta)G^N(V_N^v)) (\theta F^N(V_N^{v_*}) + (1 - \theta)G^N(V_N^{v_*})) = \theta^2 F^N(V_N^v) F^N(V_N^{v_*}) + (1 - \theta)^2 G^N(V_N^v) G^N(V_N^{v_*}) \]

\[ + \theta(1 - \theta) \left( F^N(V_N^v) G^N(V_N^{v_*}) + F^N(V_N^{v_*}) G^N(V_N^v) \right) \]

\[ = \left( \theta \sqrt{F^N(V_N^v)} \sqrt{F^N(V_N^{v_*})} + (1 - \theta) \sqrt{G^N(V_N^v)} \sqrt{G^N(V_N^{v_*})} \right)^2 \]

Therefore

\[ |D^N(v, v_*)| \leq 2 \left( \theta \sqrt{F^N(V_N^v)} \sqrt{F^N(V_N^{v_*})} + (1 - \theta) \sqrt{G^N(V_N^v)} \sqrt{G^N(V_N^{v_*})} \right) \]

\[ \times \left( 1 - \frac{1 + \theta(1 - \theta) F^N(V_N^v) G^N(V_N^{v_*}) + F^N(V_N^{v_*}) G^N(V_N^v) - 2 \sqrt{F^N(V_N^v) G^N(V_N^{v_*})} \sqrt{F^N(V_N^{v_*}) G^N(V_N^v)}}{\left( \theta \sqrt{F^N(V_N^v)} \sqrt{F^N(V_N^{v_*})} + (1 - \theta) \sqrt{G^N(V_N^v)} \sqrt{G^N(V_N^{v_*})} \right)^2} \right) \]

\[ \leq 2\theta(1 - \theta) F^N(V_N^v) G^N(V_N^{v_*}) + F^N(V_N^{v_*}) G^N(V_N^v) - 2 \theta \sqrt{F^N(V_N^v) G^N(V_N^{v_*})} \sqrt{F^N(V_N^{v_*}) G^N(V_N^v)} \]

Then since the sequence \( F^N \) is compatible

\[ F^N(V_N^v) = F^{N-1}(V_{N-1}) f(v|V_{N-1}), \]

with \( f(\cdot|V_{N-1}) \) is the density of the conditional law of the first component of some random vector of law \( F^N \), knowing the \( N - 1 \) last components. We then obtain

\[ |D^N(v, v_*)| \leq 2\theta(1 - \theta) F^{N-1}(V_{N-1}) G^{N-1}(V_{N-1}) \frac{f(v)g(v) + f(v_*)g(v_*) - 2 \sqrt{f(v)g(v) f(v_*)}}{\theta F^{N-1}(V_{N-1}) \sqrt{f(v) f(v_*)} + (1 - \theta) G^{N-1}(V_{N-1}) \sqrt{g(v) g(v_*)} \right)^2. \]

By integration, this yields for any \( L > 0 \)

\[ |\mathcal{I}^{N}_\gamma(\theta F^N - (1 - \theta) G^N) - \theta \mathcal{I}^{N}_\gamma(F^N) - (1 - \theta) \mathcal{I}^{N}_{v_1, \gamma}(G^N)| \leq \int_{\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} dv_{N-1} \int_{\mathbb{R}^{2d}} dv_{L} \frac{|D^N(v, v_*)|}{|v - v_*|^{d+\nu}} \]

\[ = \int_{\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} dv_{N-1} \int_{\mathbb{R}^{2d}} dv_{L} \mathbf{1}_{v - v_* < L} \frac{|D^N(v, v_*)|}{|v - v_*|^{d+\nu}} + \mathbf{1}_{v - v_* \geq L} \frac{|D^N(v, v_*)|}{|v - v_*|^{d+\nu}} := K_1 + K_2. \]

\( \diamond \) Estimate of \( K_1 \)

By symmetry and using bound \( (\text{D.2}) \), we straightforwardly have
\[
K_1 \leq C \int_{\mathbb{R}^d} \left( F^N(V) + G^N(V) \right) \left( \int_{\mathbb{R}^d} |v_1 - v_*|^{-d-\nu} 1_{|v_1 - v_*| < L} dv_* \right) dV \leq C_{d,\nu} L^{-\nu}.
\]

\(\diamond\) Estimate of \(K_2\)

We use bound [D.3] to obtain

\[
K_2 \leq C \int_{\mathbb{R}^{d-1}N} dV_{N-1} \int_{\mathbb{R}^{2d}} 1_{|v_1 - v_*| < L} \frac{\theta(1-\theta)F^{N-1}(V_{N-1})G^{N-1}(V_{N-1})}{\theta F^{N-1}(V_{N-1}) \sqrt{f(v)f(v_*)} + (1-\theta)G^{N-1}(V_{N-1}) \sqrt{g(v)g(v_*)}} \\
g(v_*) \left( \sqrt{f(v)} - \sqrt{f(v_*)} \right)^2 + f(v_*) \left( \sqrt{g(v)} - \sqrt{g(v_*)} \right)^2 \frac{dvdv_*}{|v - v_*|^{d+\nu}}.
\]

\(\diamond\) Step two : Regularization

Fix \(\eta > 0\) and for \(s \in (0, r)\) define

\[
F' = 1_{B_s} F, \quad F'' = F - F' = 1_{B_s \setminus B_s} F.
\]

Let be \(\varepsilon > 0\) fixed and define \(F^\varepsilon\) as the push-forward of \(F\) by the application \(\rho \in \mathcal{P}(\mathbb{R}^d) \mapsto \rho * \psi_\varepsilon \in \mathcal{P}(\mathbb{R}^d)\) (and similarly \(G^\varepsilon\)) the sequence

\[
F_{\varepsilon}^N = \int_{\mathcal{P}(\mathbb{R}^d)} (\rho * \psi_\varepsilon)^\otimes N F'(d\rho),
\]

and similarly \(F_{\varepsilon}^N, F''_{\varepsilon}^N\) and \(G_{\varepsilon}^N\). Since these sequences are compatible we obtain that

\[
F_{\varepsilon}^N(V_N^\varepsilon) = F_{\varepsilon}^{N-1}(V_{N-1}) f_\varepsilon(v|V_{N-1}), \quad G_{\varepsilon}^N(V_N^\varepsilon) = G_{\varepsilon}^{N-1}(V_{N-1}) g_\varepsilon(v|V_{N-1}).
\]

We first fix \(L > 0\) such that \(L^{-\nu} < \eta\), and in view of the previous step, and since \(F_{\varepsilon}^{N,\varepsilon} + F''_{\varepsilon}^{N,\varepsilon} = F_{\varepsilon}^N\) we have

\[
|I_{\nu,\gamma}^N (\theta F_{\varepsilon}^N + (1-\theta)G_{\varepsilon}^N) - \theta I_{\nu,\gamma}^N (F_{\varepsilon}^N) - (1-\theta)I_{\nu,\gamma}^N (G_{\varepsilon}^N)|
\leq \eta + \int_{\mathbb{R}^{d-1}N} dV_{N-1} \int_{\mathbb{R}^{2d}} 1_{|v_1 - v_*| < L} \frac{\theta(1-\theta)F_{\varepsilon}^{N-1}(V_{N-1})G_{\varepsilon}^{N-1}(V_{N-1})}{\theta F_{\varepsilon}^{N-1}(V_{N-1}) \sqrt{f_\varepsilon(v)f_\varepsilon(v_*)} + (1-\theta)G_{\varepsilon}^{N-1}(V_{N-1}) \sqrt{g_\varepsilon(v)g_\varepsilon(v_*)}} \\
g_\varepsilon(v_*) \left( \sqrt{f_\varepsilon(v)} - \sqrt{f_\varepsilon(v_*)} \right)^2 + f_\varepsilon(v_*) \left( \sqrt{g_\varepsilon(v)} - \sqrt{g_\varepsilon(v_*)} \right)^2 \frac{dvdv_*}{|v - v_*|^{d+\nu}}
\]

\[
\leq \eta + C_\theta \int_{\mathbb{R}^{d-1}N} F''_{\varepsilon}^{N-1}(dV_{N-1}) \int_{\mathbb{R}^{2d}} 1_{|v_1 - v_*| < L} \frac{g_\varepsilon(v_*) \left( \sqrt{f_\varepsilon(v)} - \sqrt{f_\varepsilon(v_*)} \right)^2 + f_\varepsilon(v_*) \left( \sqrt{g_\varepsilon(v)} - \sqrt{g_\varepsilon(v_*)} \right)^2}{\sqrt{g_\varepsilon(v)g_\varepsilon(v_*)} |v - v_*|^{d+\nu}} \frac{dvdv_*}{|v - v_*|^{d+\nu}}
\]

Set now \(u = \frac{r+\varepsilon}{2}\) and \(\delta = \frac{r-\varepsilon}{2}\), and denote

\[
\mathbb{B}_u^{N-1} := \left\{ (v_2, \cdots, v_N) \in \mathbb{R}^{d(N-1)} \mid \frac{1}{N-1} \sum_{i=2}^N \delta_{v_i} \in \mathbb{B}_u \right\},
\]
Using that
\[
\frac{F_x^{N-1}(V_N-1) G_y^{N-1}(V_N-1)}{\theta F_x^{N-1}(V_N-1) \sqrt{f_x(v) f_x(v_s)} + (1 - \theta) G_y^{N-1}(V_N-1) \sqrt{g_y(v) g_y(v_s)}} \leq \frac{F_x^{N-1}(V_N-1)}{(1 - \theta) \sqrt{g_y(v) g_y(v_s)}} 1_{1-\varepsilon} + \frac{G_y^{N-1}(V_N-1)}{\theta \sqrt{f_x(v) f_x(v_s)}} 1_{1-\varepsilon},
\]
we deduce
\[
\mathcal{I}_v^{N}(\theta F_x^{N} + (1 - \theta) G_y^{N}) - \mathcal{I}_v^{N}(\theta F_x^{N} - (1 - \theta) G_y^{N})
\]
\[
\leq \eta + C_\theta \int_{B_{(d-1)N}} F_x^{N-1}(dV_N-1) \int_{\mathbb{R}^d} 1_{|v-v_s| < L} g_x(v_s) \left( \frac{f_x(v) - f_x(v_s)}{g_y(v_s)} \right)^2 + f_x(v_s) \left( \frac{g_y(v) - g_y(v_s)}{g_y(v_s)} \right)^2 dv dv_x.
\]
\[
+ \int_{B_{(d-1)N}} F_x^{N-1}(dV_N-1) \int_{\mathbb{R}^d} 1_{|v-v_s| < L} \left( \frac{f_x(v) - f_x(v_s)}{g_y(v_s)} \right)^2 + f_x(v_s) \left( \frac{g_y(v) - g_y(v_s)}{g_y(v_s)} \right)^2 dv dv_x.
\]
\[
+ \int_{B_{(d-1)N}} G_y^{N-1}(dV_N-1) \int_{\mathbb{R}^d} 1_{|v-v_s| < L} \left( \frac{f_x(v) - f_x(v_s)}{g_y(v_s)} \right)^2 + f_x(v_s) \left( \frac{g_y(v) - g_y(v_s)}{g_y(v_s)} \right)^2 dv dv_x.
\]
In view of Lemma D.2, we have for any \(v, v_s\) such that \(|v - v_s| \leq L\)
\[
\frac{g(v_s)}{\sqrt{g(v)g(v_s)}} \leq (c_{x,L})^{-1}
\]
And using Taylor’s expansion we have that
\[
\left( \frac{\sqrt{g(v)} - \sqrt{g(v_s)}}{\sqrt{g(v)g(v_s)}} \right)^2 = \frac{1}{\sqrt{g(v)g(v_s)}} \left( \int_0^1 \nabla \sqrt{g(v + s(v_s - v))} \cdot (v - v_s) ds \right)^2
\]
\[
\leq (c_{x,R})^{-1} \left( \int_0^1 \nabla \sqrt{\frac{g}{g_s}} (v + s(v_s - v)) \cdot (v - v_s) ds \right)^2
\]
\[
= (c_{x,R})^{-2} \left( \int_0^1 \nabla (\ln \sqrt{g}) (v + s(v_s - v)) \cdot (v - v_s) ds \right)^2
\]
\[
\leq (c_{x,R})^{-2} \| \nabla \ln g \|_L^2 \|v - v_s\|^2.
\]
Finally, if \(g \in L^\infty\) and \(\nabla \ln g \in L^\infty\), it holds \(\nabla \sqrt{g} = \frac{1}{2} \sqrt{g} \nabla \ln g \in L^\infty\), and we conclude this step with
\[
\mathcal{I}_v^{N}(\theta F_x^{N} + (1 - \theta) G_y^{N}) - \mathcal{I}_v^{N}(\theta F_x^{N} - (1 - \theta) G_y^{N})
\]
\[
\leq \eta + C_{x,L,\theta} \int_{B_{(d-1)N}} F_x^{N-1}(dV_N-1) + C_{x,\theta} \int_{B_{(d-1)N}} G_y^{N-1}(dV_N-1)
\]
\[
\diamond \text{ Step three : Concentration}
\]
The end of the proof is then exactly taken from [25, Lemma 5.10]. Nevertheless we reproduce it here for the sake of completeness. First we treat the third term in the above r.h.s. by observing that \(F'' = 1_{B_x \setminus B_s} F\). Therefore
\[
\int_{B_{d(N-1)}} F_x^{N-1}(dV_N-1) = \int_{\mathbb{P}(\mathbb{R}^d)} 1_{B_x \setminus B_s}(\rho_x) F(\rho d\rho).
\]
Due to Lebesgue’s dominated convergence Theorem, the r.h.s. in the above identity converges to 0. Therefore one can choose some $s < r$ such that
\[ C_{\epsilon,0} \int_{\mathbb{R}^{d(N-1)}} F_{\epsilon}^{N-1} (dV^{N-1}) < \eta, \]
uniformly in $N$ and $\epsilon$. Then for $V^{N-1} \not\in \mathbb{B}_{\epsilon}^{N-1}$ and $\rho \in \mathbb{B}_{\delta}$ we find that
\[ W_{1} \left( \frac{1}{N-1} \sum_{i=2}^{N} \delta_{\epsilon v_{i}, \rho * \psi_{\epsilon}} \right) \geq W_{1} \left( \frac{1}{N-1} \sum_{i=2}^{N} \delta_{v_{i}, f_{1}} \right) - W_{1} (f_{1}, \rho) - W_{1} (\rho, \rho * \psi_{\epsilon}) \]
\[ \geq u - s - c \epsilon \geq \frac{\delta}{2} \]
for any $\epsilon > 0$ small enough. Therefore using a Chebychev-like argument it holds
\[ \int_{\mathbb{B}_{\epsilon}^{N-1}} F_{\epsilon}^{N-1} (dV^{N-1}) = \int_{\mathbb{P}(\mathbb{R}^{d})} \left( \int_{\mathbb{R}^{d(N-1)}} \mathbb{1}_{\mathbb{B}_{\epsilon}^{N-1},c} (\rho * \psi_{\epsilon})^{\otimes(N-1)} \right) F' (d\rho) \]
\[ \leq \frac{2}{\delta} \int_{\mathbb{P}(\mathbb{R}^{d})} \left( \int_{\mathbb{R}^{d(N-1)}} W_{1} \left( \frac{1}{N-1} \sum_{i=2}^{N} \delta_{x_{i}, \rho * \psi_{\epsilon}} \right) (\rho * \psi_{\epsilon})^{\otimes(N-1)} (dX^{N-1}) \right) F' (d\rho) \]
We claim that there is a constant $C$ depending only on $\kappa$ (see [13, Theorem 1] in case $d = 3, p = 1, q = \kappa < 2$) such that it holds
\[ \int_{\mathbb{R}^{d(N-1)}} W_{1} \left( \frac{1}{N-1} \sum_{i=2}^{N} \delta_{x_{i}, \rho * \psi_{\epsilon}} \right) (\rho * \psi_{\epsilon})^{\otimes(N-1)} (dX^{N-1}) \leq C \| \rho * \psi_{\epsilon} \|_{L_{\kappa}^{\frac{1}{\kappa}}} (N-1)^{-(1-\frac{1}{\kappa})}. \quad (D.4) \]
Note that [25, Remark 2.12] provides the same result with the exponent $1-\frac{1}{\kappa}$ replaced with $\gamma \in \left(0, \frac{1}{3 + \frac{2}{\kappa}} \right)$, but the rate of convergence does not play any role in the proof. Summing up (D.4) w.r.t. $F'$, yields
\[ \int_{\mathbb{B}_{\epsilon}^{N-1}} F_{\epsilon}^{N-1} (dV^{N-1}) \leq \frac{C}{\delta(N-1)^{1-(\frac{1}{\kappa})}} \int_{\mathbb{P}(\mathbb{R}^{d})} \| \rho * \psi_{\epsilon} \|_{L_{\kappa}^{\frac{1}{\kappa}}}^{\frac{1}{\kappa}} F' (d\rho) \]
\[ \leq \frac{C}{\delta(N-1)^{1-(\frac{1}{\kappa})}} \left( \int_{\mathbb{P}(\mathbb{R}^{d})} \| \rho \|_{L_{\kappa}^{\frac{1}{\kappa}}} \pi (d\rho) + \| \psi_{\epsilon} \|_{L_{\kappa}^{\frac{1}{\kappa}}} \right)^{\frac{1}{\kappa}}, \]
which since
\[ \| \rho * \psi_{\epsilon} \|_{L_{\kappa}^{\frac{1}{\kappa}}} = \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \langle x \rangle^{\kappa} \rho (x - y) \psi_{\epsilon} (y) dxdy = \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \langle x + y \rangle^{\kappa} \rho (x) \psi_{\epsilon} (y) dxdy \]
\[ \leq 2^{\kappa} \left( \int_{\mathbb{R}^{d}} \langle x \rangle^{\kappa} \rho (x) dxd + \int_{\mathbb{R}^{d}} \langle y \rangle^{\kappa} \psi_{\epsilon} (y) dy \right) \]
Treating in the exact same fashion the integral w.r.t. $G^{N-1}$ enables to conclude that for any $\epsilon > 0$ we have
\[ \forall \eta > 0, \exists N_{\eta}, \forall N \geq N_{\eta}, \| T_{\nu, \gamma}^{N} (\theta F_{\epsilon}^{N} + (1-\theta) G_{\epsilon}^{N}) - \theta T_{\nu, \gamma}^{N} (F_{\epsilon}^{N}) - (1-\theta) T_{\nu, \gamma}^{N} (G_{\epsilon}^{N}) \| \leq 4 \eta. \]
\( \diamond \) Final step
Gathering all the estimates obtained in the previous steps yields for any $\epsilon > 0$
\[ \lim_{N \to +\infty} \| T_{\nu, \gamma}^{N} (\theta F_{\epsilon}^{N} + (1-\theta) G_{\epsilon}^{N}) - \theta T_{\nu, \gamma}^{N} (F_{\epsilon}^{N}) - (1-\theta) T_{\nu, \gamma}^{N} (G_{\epsilon}^{N}) \| = 0. \]
Hence we deduce
\[ \tilde{I}_{\nu, \gamma}(\theta F^\varepsilon + (1 - \theta)G^\varepsilon) = \sup_{N \geq 1} \tilde{I}^N_{\nu, \gamma}(\theta F^N_\varepsilon + (1 - \theta)G^N_\varepsilon) = \lim_{N \to +\infty} \tilde{I}^N_{\nu, \gamma}(\theta F^N_\varepsilon + (1 - \theta)G^N_\varepsilon) \]
\[ = \theta \lim_{N \to +\infty} \tilde{I}^N_{\nu, \gamma}(F^N_\varepsilon) + (1 - \theta) \lim_{N \to +\infty} \tilde{I}^N_{\nu, \gamma}(G^N_\varepsilon) \]
\[ = \theta \sup_{N \geq 1} \tilde{I}^N_{\nu, \gamma}(F^N_\varepsilon) + (1 - \theta) \sup_{N \geq 1} \tilde{I}^N_{\nu, \gamma}(G^N_\varepsilon) \]
\[ = \theta \tilde{I}_{\nu, \gamma}(F^\varepsilon) + (1 - \theta) \tilde{I}_{\nu, \gamma}(G^\varepsilon). \]

But using the convexity of the functional \((x, y) \mapsto (\sqrt{x} - \sqrt{y})^2\) and Jensen’s inequality yields
\[ \tilde{I}_{\nu, \gamma}(\theta F^\varepsilon + (1 - \theta)G^\varepsilon) \leq \sup_{N \geq 1} \tilde{I}^N_{\nu, \gamma}(\theta F^N_\varepsilon + (1 - \theta)G^N_\varepsilon) = \tilde{I}_{\nu, \gamma}(\theta F + (1 - \theta)G). \]

Moreover it is clear from the fact that for each \(N \geq 2\), the functional \(\tilde{I}^N_{\nu, \gamma}\) is l.s.c. w.r.t. the weak convergence in \(\mathcal{P}(\mathbb{R}^dN)\), that \(\tilde{I}^N_{\nu, \gamma}\) is l.s.c. w.r.t. the weak convergence in \(\mathcal{P}(\mathcal{P}(\mathbb{R}^d))\). Since \(\theta F^\varepsilon + (1 - \theta)G^\varepsilon \rightharpoonup \theta F + (1 - \theta)G\) in \(\mathcal{P}(\mathcal{P}(\mathbb{R}^d))\) we get that
\[ \lim_{\varepsilon \to 0} \tilde{I}_{\nu, \gamma}(\theta F^\varepsilon + (1 - \theta)G^\varepsilon) = \tilde{I}_{\nu, \gamma}(\theta F + (1 - \theta)G). \]

Therefore
\[ \tilde{I}_{\nu, \gamma}(\theta F + (1 - \theta)G) = \theta \tilde{I}_{\nu, \gamma}(F) + (1 - \theta)\tilde{I}_{\nu, \gamma}(G), \]
which concludes the proof. \(\square\)

Proposition 2.2 then follows from [25, Lemma 5.6]. We leave the reader check that points (i)-(ii) of Lemma D.1 and Lemma D.3 consist in checking that the functionals \((\tilde{I}^N_{\nu, \gamma})_{N \geq 2}\) satisfy the technical assumptions of [25, Lemma 5.6].

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