ON GLOBAL ASYMPTOTIC STABILITY FOR THE LSW MODEL WITH SUBCRITICAL INITIAL DATA

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Abstract. The main result of the paper is a global asymptotic stability result for solutions to the Lifschitz-Slyozov-Wagner (LSW) system of equations. This extends some local asymptotic stability results of Niethammer-Velázquez (2006). The method of proof is along similar lines to the one used in a previous paper of the authors. This previous paper proves global asymptotic stability for a class of infinite dimensional dynamical systems for which no Lyapunov function is (apparently) available.

1. Introduction.

In this paper we shall be principally concerned with studying the large time behavior of solutions to the Lifschitz-Slyozov-Wagner (LSW) equations [7, 14]. The LSW equations occur in a variety of contexts [12, 13] as a mean field approximation for the evolution of particle clusters of various volumes. Clusters of volume $x > 0$ have density $c(x, t) \geq 0$ at time $t > 0$. The density evolves according to a linear law, subject to the linear mass conservation constraint as follows:

$$\frac{\partial c(x,t)}{\partial t} = \frac{\partial}{\partial x} \left( (1 - (x L^{-1}(t))^\alpha) c(x,t) \right), \quad x > 0,$$

(1.1)

$$\int_0^\infty x c(x,t) dx = 1.$$  

(1.2)

We shall require the parameter $\alpha$ in (1.1) to lie in the interval $0 < \alpha \leq 1$, whence the system (1.1), (1.2) includes the standard LSW model corresponding to $\alpha = 1/3$, and the much simpler Carr-Penrose model [1] corresponding to $\alpha = 1$.

One wishes then to solve (1.1) for $t > 0$ and initial condition $c(x,0) = c_0(x) \geq 0, \quad x > 0$, subject to the constraint (1.2). The parameter $L(t) > 0$ is determined by the constraint (1.2) and is therefore given by the formula,

$$L(t)^\alpha = \int_0^\infty x^\alpha c(x,t) dx / \int_0^\infty c(x,t)dx.$$  

(1.3)

Existence and uniqueness of solutions to (1.1), (1.2) with given initial data $c_0(\cdot)$ satisfying the constraint has been proven in [9] for integrable functions $c_0(\cdot)$, and in [8] for initial data such that $c_0(x)dx$ is an arbitrary Borel probability measure with compact support. In [10] the methods of [9] are further developed to prove existence and uniqueness for initial data such that $c_0(x)dx$ is a Borel probability measure with finite first moment.

The system (1.1), (1.2) can be interpreted as an evolution equation for the probability density function (pdf) of random variables. Thus let us assume that the

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initial data \( c_0(x) \geq 0, \ x \geq 0 \), for \( (1.1), \ (1.2) \) satisfies \( \int_0^\infty c_0(x) \, dx < \infty \). The conservation law \( (1.2) \) implies that the mean \( \langle X_0 \rangle \) of \( X_0 \) is finite, and this is the only absolute requirement on the variable \( X_0 \). If for \( t > 0 \) the variable \( X_t \) has pdf \( c(\cdot, t) / \int_0^\infty c(x, t) \, dx \) then \( (1.1) \) with \( L(t) = \langle X_t^\alpha \rangle \) is an evolution equation for the pdf of \( X_t \). We can also see that

\[
\frac{d}{dt}(X_t) = c(0, t) \left[ \int_0^\infty c(x, t) \, dx \right]^2,
\]

whence the function \( t \to \langle X_t \rangle \) is increasing. In [2] it was shown that for a wide range of initial data \( c_0(\cdot) \), there exists positive constants \( C_1, C_2 \), depending only on the initial data, such that

\[
C_1 T \leq \langle X_T \rangle \leq C_2 T \quad \text{for} \ T \geq 1.
\]

We shall show here that for initial data variables \( X_0 \) which are subcritical [4], the limit \( \lim_{T \to \infty} \langle X_T \rangle / T \) exists, and may be computed in terms of the corresponding self-similar solution to \( (1.1), \ (1.2) \).

Let \( X \) be a non-negative random variable. With \( X \) we may associate a beta function [2] with domain \([0, \|X\|_\infty)\) defined by

\[
\beta_X(x) = \frac{d}{dx} E[X - x \mid X > x] + 1 = \frac{c_X(x) h_X(x)}{w_X(x)^2},
\]

where the functions \( c_X(\cdot), w_X(\cdot), h_X(\cdot) \) are given by

\[
c_X(\cdot) = \text{pdf of } X, \quad w_X(x) = P(X > x), \quad h_X(x) = E[X - x; X > x].
\]

The variable \( X \) is said to be subcritical if \( \lim_{x \to \|X\|_\infty} \beta_X(x) = \beta_0 \) exists and \( 0 < \beta_0 < 1 \). It is easy to see from \( (1.6) \) that if \( X \) is subcritical then \( \|X\|_\infty < \infty \).

Furthermore, \( w_X(x) \approx (\|X\|_\infty - x)^p \) as \( x \to \|X\|_\infty \), where \( \beta_0 = p/(p + 1) \).

For each \( \beta \) in the interval \( 0 < \beta \leq 1 \) the system \( (1.1), \ (1.2) \) has a unique self-similar solution. The corresponding random variable \( X_\beta \), normalized to have mean 1, has the property \( \lim_{x \to \|X_\beta\|_\infty} \beta_{X_\beta}(x) = \beta \). The solution to \( (1.1), \ (1.2) \) with initial data variable \( X_0 \) a constant times \( X_\beta \) is then given from \( (1.6) \), \( (1.7) \) by

\[
X_t = \langle X_t \rangle \beta_\beta, \quad \frac{d}{dt} \langle X_t \rangle = \beta_{X_\beta}(0).
\]

Our main result is that \( (1.8) \) holds approximately at large time if the initial data for \( (1.1), \ (1.2) \) is subcritical.

**Theorem 1.1.** Assume the initial data for \( (1.1), \ (1.2) \) has compact support, and that the corresponding random variable \( X_0 \) has continuous beta function \( \beta_{X_0} : [0, \|X_0\|_\infty) \to \mathbb{R} \) which satisfies \( \lim_{x \to \|X_0\|_\infty} \beta_{X_0}(x) = \beta \) with \( 0 < \beta < 1 \). Let \( X_t \) be the random variable corresponding to the solution \( c(\cdot, t) \) of \( (1.1), \ (1.2) \) at time \( t > 0 \). Then

\[
\frac{X_t}{\langle X_t \rangle} \overset{D}{\to} \mathcal{X}_\beta \quad \text{as} \ t \to \infty, \quad \lim_{t \to \infty} \frac{d}{dt} \langle X_t \rangle = \beta_{X_\beta}(0),
\]

where \( \overset{D}{\to} \) denotes convergence in distribution.

If the function \( x \to \beta_{X_0}(x) \) is Hölder continuous at \( x = \|X_0\|_\infty \), then there exist constants \( C, \nu > 0 \) such that

\[
\left| \frac{d}{dt} \langle X_t \rangle - \beta_{X_\beta}(0) \right| \leq \frac{C}{1 + t^\nu}, \quad t \geq 0.
\]
The cdf of $X_t / \langle X_t \rangle$ has a corresponding rate of convergence to the cdf of $X_\beta$. In particular, for any $\delta$ with $0 < \delta < 1$, one has

$$
(1.11) \quad P \left( X_\beta > x \left[ 1 + C_\delta / (1 + t^\nu) \right] \right) \leq P \left( \frac{X_t}{\langle X_t \rangle} > x \right) 
\leq P \left( X_\beta > \frac{x}{1 + C_\delta / (1 + t^\nu)} \right), \quad \text{for} \ 0 \leq x \leq (1 - \delta) \|X_\beta\|_\infty, \ t \geq 0, 
$$

where $C_\delta > 0$ is a constant depending on $\delta$.

Previous results imply that (1.9) holds in certain cases. In particular, the results of [1] imply that (1.9) holds when $\alpha = 1$. For $\alpha = 1/3$ Theorem 4.1 of [11] implies that (1.9) holds if $\beta < 1$ is sufficiently small. Theorem 3.2 of [11] implies that (1.9) holds for any $\beta < 1$, provided the initial data random variable $X_0$ for (1.1), (1.2) satisfies the additional condition $\|\beta X_0 (\|X_\beta\|^{-1}_\infty X_0) - \beta X_0 (\cdot)\|_\infty < \delta(\beta)$, where $\delta(\beta) > 0$ depends on $\beta$ (see Remark 7 in §8).

The first step in proving Theorem 1.1 is to observe following [8] that if the initial data $c_0(\cdot)$ for (1.1), (1.2) has compact support then $c(\cdot, t)$ has compact support for all $t > 0$. Then by means of a transformation (see §8) one can normalize the support for all time to the interval $[0, 1]$. In that case the LSW equations may be written as

$$
(1.12) \quad \partial w(x, t) / \partial t + [\phi(x) - \kappa(t) \psi(x)] \partial w(x, t) / \partial x = w(x, t), \quad 0 \leq x < 1, \ t \geq 0, 
$$

with the mass conservation law

$$
(1.13) \quad \int_0^1 w(x, t) \ dx = 1, \quad t \geq 0. 
$$

The functions $\phi(\cdot), \ \psi(\cdot)$ are given by the formulæ

$$
(1.14) \quad \phi(x) = x^\alpha - x, \quad \psi(x) = 1 - x^\alpha, \quad 0 \leq x \leq 1. 
$$

We shall study solutions to (1.12), (1.13) for more general $\phi(\cdot), \psi(\cdot)$ than (1.14) with $0 < \alpha \leq 1$. In particular, we require $\phi(\cdot), \psi(\cdot)$ to satisfy the conditions

$$
(1.15) \phi(\cdot) \text{ is concave and satisfies } \phi(0) = \phi(1) = 0, \ -1 < \phi'(1) < 0, 
$$

$$
(1.16) \psi(\cdot) \text{ is convex and satisfies } \psi(1) = 0, \ \psi'(1) < 0, \ \psi''(1) - \phi''(1) > 0. 
$$

The fundamental quantity of study in the generalized LSW model (1.12), (1.13) is the (renormalized) cluster density $c(x, t) \geq 0$ at volume $x \in [0, 1]$ and time $t > 0$. The function $w(\cdot, t)$ in (1.12), (1.13) is the integral of $c(\cdot, t)$ and is therefore non-negative decreasing with $\lim_{x \to 1} w(x, t) = 0$. We define $w(\cdot, t)$ in terms of $c(\cdot, t)$, and in addition the function $h(\cdot, t)$ by

$$
(1.17) \quad w(x, t) = \int_x^1 c(x', t) \ dx', \quad h(x, t) = \int_x^1 w(x', t) \ dx', \quad 0 \leq x < 1. 
$$

The conservation law (1.13) is evidently the same as $h(0, t) = 1$. Conditions on the initial data for (1.12), (1.13) are given in terms of the beta function for $c(\cdot, t)$ defined by

$$
(1.18) \quad \beta(x, t) = \frac{c(x, t) h(x, t)}{w(x, t)^2}, \quad 0 \leq x \leq 1. 
$$
We assume that \( \beta(\cdot, 0) \) has the property
\[
\lim_{x \to 1} \beta(x, 0) = \beta_0 > 0.
\]
It is easy to see that \( \beta_0 \leq 1 \), and we say that the initial data for (1.12), (1.13) is subcritical if \( \beta_0 < 1 \). The main result of [4] is a weak asymptotic stability result for solutions to (1.12), (1.13) with initial data satisfying (1.19). Further assumptions on the functions \( \phi(\cdot), \psi(\cdot) \) beyond (1.15), (1.16) are required to prove this. In the case of subcritical initial data these are as follows:
\[
\phi(x), \psi(x) = C^3 \text{ on } (0, 1] \text{ and } \phi'''(x) \geq 0, \psi''(x) \leq 0 \text{ for } 0 < x \leq 1.
\]
Evidently (1.20) holds for the functions (1.14) with \( 0 < \alpha \leq 1 \).

**Theorem 1.2.** Let \( w(x,t), x,t \geq 0 \), be the solution to (1.12), (1.13) with coefficients satisfying (1.15), (1.16) and assume that the initial data \( w(\cdot,0) \) has beta function \( \beta(\cdot, 0) \) satisfying (1.19) with \( 0 < \beta_0 < 1 \). Then there is a positive constant \( C_1 \) depending only on the initial data such that \( \kappa(t) \geq C_1 \) for all \( t \geq 0 \). If in addition (1.20) holds, then there is a positive constant \( C_2 \) depending only on the initial data such that \( \kappa(t) \leq C_2 \) for all \( t \geq 0 \) and
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \kappa(t) \, dt = \frac{1}{1/\beta_0 - \phi'(1) - 1/|\psi'(1)|}.
\]

In [4] we were also able to prove a strong asymptotic stability result for solutions to (1.12), (1.13) in the case when \( \phi(\cdot), \psi(\cdot) \) are quadratic:

**Theorem 1.3.** Assume that the functions \( \phi(\cdot), \psi(\cdot) \) are quadratic, and that the initial data \( w(\cdot,0) \) for (1.12), (1.13) has beta function \( \beta(\cdot, 0) \) satisfying (1.19). Then setting \( \kappa = [1/\beta_0 - \phi'(1) - 1/|\psi'(1)|] \), one has for \( \beta_0 < 1 \),
\[
\lim_{t \to \infty} \kappa(t) = \kappa, \quad \lim_{t \to \infty} \|\beta(\cdot, t) - \beta_{\kappa}(\cdot)\|_{\infty} = 0,
\]
where \( \beta_{\kappa}(\cdot) \) is the beta function of the time independent solution \( w_{\kappa}(\cdot) \) of (1.12).

Evidently Theorem 1.3 applies to the Carr-Penrose model where \( \alpha = 1 \). Our main goal here will be to show that the conclusions of Theorem 1.3 also hold for a class of non-quadratic functions \( \phi(\cdot), \psi(\cdot) \), which includes the functions (1.14) with \( 0 < \alpha < 1 \). Theorem 1.1 is then an easy consequence.

We accomplish this by establishing asymptotic stability results for a class of PDE similar to the one studied in [3]. In particular, we consider for some \( \varepsilon_0 > 0 \) the evolution PDE
\[
\frac{\partial \xi(y,t)}{\partial t} - h(y) - h(y) \frac{\partial \xi(y,t)}{\partial y} + \rho(\xi(t), t) \left( \xi(y,t) - y \frac{\partial \xi(y,t)}{\partial y} \right) = 0, \quad y > \varepsilon_0, \ t > 0,
\]
with given non-negative initial data \( \xi(y,0), y > \varepsilon_0 \). We assume as in [3] that the function \( h : [\varepsilon_0, \infty) \to \mathbb{R} \) has the properties:
\[
h(\cdot) \text{ is continuous positive and } \lim_{y \to \infty} h(y) = h_\infty > 0.
\]
For an equilibrium of (1.23) to exist corresponding to \( \rho = 1/p > 0 \) we need the equilibrium function \( \xi_p(y), y > \varepsilon_0 \), to satisfy
\[
- h(y) - h(y) \frac{d\xi_p(y)}{dy} + \frac{1}{p} \left( \xi_p(y) - y \frac{d\xi_p(y)}{dy} \right) = 0, \quad \lim_{y \to \infty} \xi_p(y) = ph_\infty.
\]
We can easily solve (1.25) by using the integrating factor

\[(1.26) \quad \tilde{h}(y) = K \exp \left[ \int_{y_0}^{y} \frac{dy'}{ph(y') + y'} \right] \quad \text{for some constant } K,\]

where \(K\) is chosen so that \(\lim_{y \to \infty} \tilde{h}(y)/y = 1\). Then we have that

\[(1.27) \quad \xi_p(y) = \tilde{h}(y) \int_{y_0}^{y} \frac{ph(y')}{(ph(y') + y') h(y')} \, dy'.\]

It follows from (1.24), (1.27) that \(\lim_{y \to \infty} \xi_p(y) = ph_\infty\), whence (1.26) implies that \(\lim_{y \to \infty} y d\xi_p(y)/dy = 0\). Furthermore, if the function \(h(\cdot)\) is decreasing then \(ph_\infty \leq \xi_p(y) \leq ph(y)\) for all \(y > \varepsilon_0\).

Following [3], we impose conditions on the functional \(\rho(\cdot)\) so that the equilibrium \(\xi_\varepsilon\) is a global attractor for (1.23). To do this we assume there is a positive functional \(I(\cdot)\) on continuous positive functions \(\zeta : [\varepsilon_0, \infty) \to \mathbb{R}\) with the property that

\[(1.28) \quad \frac{1}{p} \frac{d}{dt} I(\xi(\cdot, t)) = \left[ \rho(\xi(\cdot, t)) - \frac{1}{p} \right] I(\xi(\cdot, t)) \quad \text{for solutions } \xi(\cdot, t) \text{ of (1.23)}.\]

From (1.28) it follows that if we can show that \(\log I(\xi(\cdot, t))\) remains bounded as \(t \to \infty\) then \(\rho(\xi(\cdot, t))\) converges as \(t \to \infty\) to \(1/p\) in the averaged sense

\[(1.29) \quad \lim_{T \to \infty} \frac{1}{T} \int_0^T \rho(\xi(\cdot, t)) \, dt = \frac{1}{p}.\]

Let \([\cdot, \cdot]\) denote the Euclidean inner product on \(L^2([\varepsilon_0, \infty))\) and \(dI(\zeta(\cdot)) : [\varepsilon_0, \infty) \to \mathbb{R}\) the gradient of the functional \(I(\cdot)\) at \(\zeta(\cdot)\). Then from (1.29), (1.28) we have that

\[(1.30) \quad p \left[ \rho(\zeta(\cdot)) - \frac{1}{p} \right] I(\zeta(\cdot)) = [dI(\zeta(\cdot)), h + hD\zeta] + \rho(\zeta(\cdot)) [dI(\zeta(\cdot)), yD\zeta - \zeta].\]

We conclude from (1.30) that

\[(1.31) \quad \rho(\zeta(\cdot)) = \frac{I(\zeta(\cdot)) + [dI(\zeta(\cdot)), h + hD\zeta]}{pI(\zeta(\cdot)) + [dI(\zeta(\cdot)), \zeta - yD\zeta]}.\]

In §2 we shall use a transformation introduced in [4] to relate (1.12), (1.13) to (1.23). The limit (1.24) of Theorem 1.2 then follows from (1.29) by showing the boundedness of \(\log I(\xi(\cdot, t))\) as \(t \to \infty\).

### 2. Derivation of the PDE (1.23) from the LSW Model

We shall use the transformation introduced in §6 of [4] to obtain (1.23) from (1.12). We first recall that the solution \(w(x, t)\) of (1.12) is given by \(w(x, t) = e^t w(F(x, t), 0)\) where \(F(x, t)\) is the solution to the initial value problem

\[(2.1) \quad \frac{\partial F(x, t)}{\partial t} + [\phi(x) - \kappa(t)] \psi(x) \frac{\partial F(x, t)}{\partial x} = 0, \quad 0 \leq x < 1, \ t \geq 0, \quad F(x, 0) = x, \quad 0 \leq x < 1.\]

For \(t \geq 0\) let \(u(t)\) be the function

\[(2.2) \quad u(t) = \exp \left[ \int_0^t \{ \phi'(s) - \psi'(s) \kappa(s) \} \, ds \right],\]
and \( f(x), 0 \leq x < 1, \) be the function defined by

\[
(2.3) \quad \frac{d}{dx} \log f(x) = -\frac{\psi'(1)}{\psi(x)}, \quad 0 \leq x < 1, \quad \lim_{x \to 1^-} (1 - x) f(x) = 1.
\]

Evidently \( f : [0, 1) \to \mathbb{R} \) is a strictly increasing function satisfying \( f(0) > 0 \) and \( \lim_{x \to 1^-} f(x) = \infty. \) We define now the domains \( \mathcal{D} = \{(x, u) \in \mathbb{R}^2 : 0 < x < 1, \ u > 0\} \) and \( \hat{\mathcal{D}} = \{(z, u) \in \mathbb{R}^2 : z > f(0), \ u > 0\}. \) Then the transformation \((z, u) = (f(x)u, u)\) maps \( \mathcal{D} \) to \( \hat{\mathcal{D}}. \) Let \( g : \hat{\mathcal{D}} \to \mathbb{R} \) be the function

\[
(2.4) \quad g(z, u) = uf(x) \left[ \phi'(1) - \frac{\psi'(1) \phi(x)}{\psi(x)} \right], \quad z = f(x)u, \ 0 < x < 1, \ u > 0,
\]

and \( \hat{F}(z, t) \) the solution to the initial value problem

\[
(2.5) \quad \frac{\partial \hat{F}(z, t)}{\partial t} + g(z, u(t)) \frac{\partial \hat{F}(z, t)}{\partial z} = 0, \quad z > f(0)u(t), \ t \geq 0, \quad \hat{F}(z, 0) = z, \quad z > f(0).
\]

The solutions \( F(x, t) \) of (2.1) and \( \hat{F}(z, t) \) of (2.5) are related by the identity

\[
(2.6) \quad f(F(x, t)) = \hat{F}(f(x)u(t), t), \quad 0 < x < 1, \ t > 0.
\]

It follows from (1.15), (1.16) (2.3), (2.4) that the function \( g(\cdot, u) \) is negative and

\[
(2.7) \quad -\lim_{z \to \infty} g(z, u) \cdot u = \frac{\psi''(1) \phi'(1) - \psi'(1) \phi''(1)}{2 \psi'(1)} = \alpha_0 > 0.
\]

In the case of \( \phi(\cdot), \ \psi(\cdot) \) being quadratic functions then \( -g(\cdot, u)/u = \alpha_0. \)

We write

\[
(2.8) \quad \hat{F}(z, t) = z + \alpha_0 u(t) \xi \left( \frac{z}{\alpha_0 u(t)}, t \right), \quad z > f(0)u(t).
\]

Then setting \( y = z/\alpha_0 u(t) = f(x)/\alpha_0 \) we see from (2.5) that \( \xi(y, t) \) satisfies the PDE (2.3) in the domain \( \{ (y, t) : y > f(0)/\alpha_0, \ t > 0 \} \), where the functions \( h(\cdot), \rho(\cdot) \) are given by the formulae

\[
(2.9) \quad h(y) = \frac{f(x)}{\alpha_0} \left[ \frac{\psi'(1) \phi(y)}{\psi(y)} - \phi'(1) \right], \quad \rho(\xi(\cdot, t)) = \frac{1}{u(t)} \frac{du(t)}{dt}.
\]

Evidently for the function \( h(\cdot) \) of (2.9) one has \( h(\infty) = 1 \) in (1.24). Next for any \( p > 0 \) we consider (1.12), (1.13) with initial data \( w(\cdot, 0) \) given by

\[
(2.10) \quad w(x, 0) = \frac{K_p}{f(x)^p}, \quad \text{where} \quad K_p \int_0^1 \frac{dx}{f(x)^p} = 1.
\]

It is easy to see that the \( \beta_0 \) of (1.19) corresponding to (2.10) is given by \( \beta_0 = p/(1 + p). \) The conservation law for solutions to (1.12), (1.13) becomes then

\[
(2.11) \quad 1 = e^t \int_0^1 w(F(x, t), 0) \ dx = e^t K_p \int_0^1 \frac{dx}{f(F(x, t))^p} = e^t K_p \int_0^1 \frac{dx}{F(f(x)u(t), t)^p},
\]

where we have used (2.6). Let \( I(\cdot) \) be the functional defined by

\[
(2.12) \quad I(\xi(\cdot)) = \frac{K_p}{\alpha_0} \int_{y_0}^\infty \frac{a(y)}{b(y) + \xi(y)^p} \ dy,
\]
where \(a(\cdot), b(\cdot), \varepsilon_0\) are given by the formulae
\[
a(y) = \frac{dx}{dy} = -\frac{\psi(x)}{y\psi'(1)}, \quad b(y) = y, \quad \varepsilon_0 = f(0)/\alpha_0.
\]

Observe that \(\lim_{y \to \infty} y^2 a(y) = 1/\alpha_0\), whence the integral in (2.12) is finite for all non-negative \(\zeta : (0, \infty) \to \mathbb{R}\). Evidently (2.11) is the same as
\[
(2.14) \quad \frac{e^t}{u(t)^p} I(\zeta(\cdot), t) = 1.
\]

Differentiating (2.14), we see using the formula for \(\rho(\cdot)\) in (2.9) that (1.25) holds. We conclude that the function \(\zeta(\cdot), t > 0\), defined in (2.8) is a solution to the evolution equation (1.23) with \(h(\cdot)\) given by (2.9), \(\rho(\cdot)\) by (1.31), and \(I(\cdot)\) by (2.12), (2.13). The initial data for (1.23) is \(\zeta(\cdot,0) = 0\).

We have a similar development when the initial data for (1.12), (1.13) satisfies (1.19) with \(\beta_0 = p/(1 + p)\). Let the function \(\tilde{w}_0 : (0, 1) \to \mathbb{R}\) be defined by
\[
(2.15) \quad \tilde{w}_0 \left( \frac{f(0)}{f(x)} \right) = w(x, 0), \quad 0 \leq x < 1,
\]
and the functional \(I(\cdot)\) by
\[
(2.16) \quad I(\zeta(\cdot), \eta) = \int_0^\infty \eta^{-p} \tilde{w}_0 \left( \frac{f(0)}{\varepsilon_0 f(0) [y + \zeta(y)]} \right) a(y) \, dy,
\]
where \(a(\cdot)\) is given by (2.13). The conservation law (1.13) is then expressed in terms of the functional \(I(\cdot)\) by
\[
(2.17) \quad \frac{e^t}{u(t)^p} I(\zeta(\cdot), t, \eta(t)) = 1, \quad \text{where } \eta(t) = u(t)^{-1}.
\]

Differentiating (2.17), we have from (2.9) that
\[
(2.18) \quad \frac{1}{p} \frac{d}{dt} I(\zeta(\cdot), t, \eta(t)) = \left[ \rho(\zeta(\cdot), t, \eta(t)) - \frac{1}{p} \right] I(\zeta(\cdot), t).
\]

We can obtain from (2.18) a formula for \(\rho(\cdot)\) similar to (1.31). Thus we have from (1.23), (2.9), (2.18) that
\[
(2.19) \quad p \left[ \rho(\zeta(\cdot), \eta) - \frac{1}{p} \right] I(\zeta(\cdot), \eta) =
\]
\[
[d\zeta I(\zeta(\cdot), \eta), h + hD\zeta]\rho(\zeta(\cdot), \eta)[d\zeta I(\zeta(\cdot), \eta), yD\zeta - \eta - \eta\rho(\zeta(\cdot), \eta)]/\partial\eta,
\]
where \(d\zeta I(\zeta(\cdot), \eta)\) denotes gradient with respect to the function \(\zeta(\cdot)\). We conclude from (2.19) that \(\rho(\cdot)\) is given by the formula
\[
(2.20) \quad \rho(\zeta(\cdot), \eta) = \frac{I(\zeta(\cdot), \eta) + [d\zeta I(\zeta(\cdot), \eta), h + hD\zeta]}{p I(\zeta(\cdot), \eta) + [d\zeta I(\zeta(\cdot), \eta), \zeta - yD\zeta] + \eta\partial I(\zeta(\cdot), \eta)/\partial\eta}.
\]

In the case \(h(\cdot) \equiv 1\) and \(\zeta(\cdot)\) a constant function, (2.9), (2.20) with \(I(\cdot) = G_1(\cdot)\) yield equation (3.23) of [11]. Similarly (1.23), (2.20) yield equation (3.22) of [11].

Note also from (1.21), (2.2) that \(\eta(t)\) converges to zero as \(t \to \infty\) at exponential rate \(1/p\). Letting \(\eta \to 0\) in (2.10), (2.21) we obtain the situation of the previous paragraph.
We shall obtain some boundedness and regularity properties of the function $I(\cdot)$ given by (2.16), assuming only that $\beta(\cdot, 0)$ is continuous, satisfies (1.19) and in addition the inequality
\begin{equation}
0 < \inf \beta(\cdot, 0) \leq \sup \beta(\cdot, 0) < 1 .
\end{equation}
Observe that
\begin{equation}
0 < \frac{f(0)\eta}{\alpha_0[y + \zeta(y)]} \leq 1 \text{ if } 0 < \eta \leq 1, \quad \zeta(y) \geq 0, \quad y \geq \varepsilon_0 .
\end{equation}
Hence $I(\zeta(\cdot), \eta)$ is well defined by (2.16) provided $\zeta(\cdot)$ is a non-negative function and $0 < \eta \leq 1$. In order to use our assumptions (1.19), (2.21) on the beta function, we make a change of variable in the integral (2.16). Let $\Gamma : \mathbb{R}^+ \times (0, 1] \times [0, 1) \to [0, 1)$ be the function defined by
\begin{equation}
f(\Gamma(\zeta, \eta, x)) = f(x) + \alpha_0 \zeta, \quad \zeta \geq 0, \quad 0 < \eta \leq 1, \quad 0 \leq x < 1 .
\end{equation}
From (2.3), (2.23) we see that
\begin{equation}
limit_{\eta \to 0} \eta^{-1}[1 - \Gamma(\zeta, \eta, x)] = \frac{1}{f(x) + \alpha_0 \zeta} .
\end{equation}
Setting $y = f(x)/\alpha_0$ in (2.20) we have that
\begin{equation}
I(\zeta(\cdot), \eta) = \eta^{-p} \int_0^1 w(\Gamma(\zeta(f(x)/\alpha_0), \eta, x), 0) \, dx .
\end{equation}
We recall from [2] how $\beta(\cdot, 0)$ is related to the function $w(\cdot, 0)$. Let $X$ be the random variable with cdf determined by $w(\cdot, 0)$, so
\begin{equation}
P(X > x) = \frac{w(x, 0)}{w(0, 0)}, \quad 0 < x < 1 .
\end{equation}
The beta function is related to the cdf of $X$ by the formula
\begin{equation}
\beta(x, 0) = 1 + \frac{d}{dx}E[X - x \mid X > x], \quad 0 \leq x < 1 .
\end{equation}
The function $h(\cdot, 0)$ defined by (1.17) can be expressed as the expectation value
\begin{equation}
h(x, 0) = w(0, 0)E[X - x; X > x] .
\end{equation}
Hence on integration of (2.27) we obtain the identity
\begin{equation}
E[X - x \mid X > x] = \frac{h(x, 0)}{w(x, 0)} = \int_x^1 [1 - \beta(x', 0)] \, dx' .
\end{equation}
Integrating (2.29) yields the formula
\begin{equation}
h(x, 0) = h(0, 0) \exp \left[ -\int_0^x \frac{dx'}{\int_{x'}^1 [1 - \beta(x'', 0)] \, dx''} \right] .
\end{equation}

\textbf{Lemma 2.1.} Assume $\beta(\cdot, 0)$ is continuous and satisfies (1.19), (2.24). Then for any $M, \gamma > 0$ there exist positive constants $C_\gamma$, depending only on $\gamma$, and $c_{M, \gamma}$, depending only on $M, \gamma$, such that
\begin{equation}
c_{M, \gamma} \eta^{\gamma} \leq I(\zeta(\cdot), \eta) \leq C_\gamma \eta^{-\gamma} \text{ for } 0 \leq \zeta(\cdot) \leq M, \quad 0 < \eta \leq 1 .
\end{equation}
If $\beta(x, 0), \quad 0 \leq x < 1$, is Hölder continuous at $x = 1$ then the inequality (2.31) also holds with $\gamma = 0$. 
Proof. It follows from (1.13) with $t = 0$ that for $0 < \delta \leq \eta \leq 1$ the inequality (2.31) holds with constants depending only on $\delta, M$. Therefore we may restrict ourselves to the situation where $0 < \eta < \delta$ and $\delta > 0$ can be arbitrarily small. Next observe from (1.19) that for any $\gamma$, $0 < \gamma < 1$, there exists $\varepsilon_\gamma > 0$ such that

\[(2.32) \quad \frac{1}{p + 1 + \gamma} [1 - z] \leq \int_z^1 [1 - \beta(x', 0)] \, dx' \leq \frac{1}{p + 1 - \gamma} [1 - z] \quad \text{for } 1 - \varepsilon_\gamma \leq z < 1.\]

Hence from (2.23), (2.30), (2.32) there exists for any $0 < \gamma \leq 1$ constants $\delta_\gamma > 0$ and $c_\gamma, C_\gamma > 0$ such that

\[(2.33) \quad c_\gamma [1 - \Gamma(\zeta, \eta, x)]^{p+1+\gamma} \leq h(\Gamma(\zeta, \eta, x), 0) \leq C_\gamma [1 - \Gamma(\zeta, \eta, x)]^{p+1-\gamma} \quad \text{for } \zeta \geq 0, 0 < \eta \leq \delta_\gamma, 0 \leq x < 1.\]

The inequality (2.33) follows now from (2.24), (2.25), (2.29), (2.33).

If $\beta(x, 0)$, $0 \leq x < 1$, is H"older continuous at $x = 1$ with exponent $q$, $0 < q \leq 1$, then there exist constants $C_1, C_2 > 0$ such that

\[(2.34) \quad \frac{1}{p + 1} [1 - z] \frac{1}{1 + C_1 (1 - z)^q} \leq \int_z^1 [1 - \beta(x', 0)] \, dx' \leq \frac{1}{p + 1} [1 - z] \{1 + C_2 (1 - z)^q\} \quad \text{for } 0 \leq z < 1.\]

One easily sees that the inequality (2.33) with $\gamma = 0$ follows from (2.34), whence (2.31) with $\gamma = 0$ also holds. \[\square\]

We obtain an expression for $d_\zeta I(\zeta(\cdot), \eta)$ by differentiating (2.25). To do this we use the formula

\[(2.35) \quad - \frac{\partial w(x, 0)}{\partial x} \frac{\partial w(x, 0)}{\partial x} = \beta(x, 0) \int_x^1 [1 - \beta(x', 0)] \, dx'.\]

We have also from (2.23), (2.25) that

\[(2.36) \quad \frac{\partial \Gamma(\zeta, \eta, x)}{\partial \zeta} = - \frac{\alpha_0 \psi(\Gamma(\zeta, \eta, x))}{\eta \psi'(1)f(\Gamma(\zeta, \eta, x))}.\]

It follows from (2.24), (2.26) that

\[(2.37) \quad \frac{\partial \Gamma(\zeta, \eta, x)}{\partial \zeta} \approx \alpha_0 \eta(1 - x)^2 \quad \text{as } \eta \to 0.\]

Differentiating (2.25) we conclude from (2.23), (2.33), (2.36) that

\[(2.38) \quad d_\zeta I(\zeta(\cdot), \eta; f(x)/\alpha_0) = -\eta^{-p} \frac{\alpha_0 g(\Gamma(\zeta(f(x)/\alpha_0), \eta, x))}{f(x) + \alpha_0 \zeta(f(x)/\alpha_0)} \psi(\Gamma(\zeta(f(x)/\alpha_0), \eta, x), 0)a(f(x)/\alpha_0), \quad 0 \leq x < 1,\]

where the function $g(\cdot)$ is given by the formula

\[(2.39) \quad g(x) = -\psi(x) \beta(x, 0) \int_x^1 [1 - \beta(x', 0)] \, dx'.\]
Observe from (2.21), (2.39) that
\begin{equation}
\inf g(\cdot) > 0 \quad \text{and} \quad \lim_{x \to 1} g(x) = p.
\end{equation}

We see from (2.38), (2.39) that \( d_\zeta I(\zeta, \eta) \) is a negative function for all non-negative \( \zeta(\cdot) \) and \( 0 < \eta \leq 1 \).

We can obtain similar formulas for \( \partial I(\zeta(\cdot), \eta) / \partial \eta \) and for the denominator in the expression (2.20) for \( \rho(\zeta(\cdot), \eta). \) First observe from (2.3), (2.23) that
\begin{equation}
\frac{\partial \Gamma(\zeta, \eta, x)}{\partial \eta} = \frac{\psi(\Gamma(\zeta, \eta, x))}{\eta^\psi(1)}.
\end{equation}

Now differentiating (2.25) with respect to \( \eta \) we have from (2.35), (2.39), (2.41) that
\begin{equation}
pI(\zeta(\cdot), \eta) + \eta \frac{\partial I(\zeta(\cdot), \eta)}{\partial \eta} = \eta^{-p} \int_0^1 g(\Gamma(\zeta(f(x)/\alpha_0), \eta, x)) w(\Gamma(\zeta(f(x)/\alpha_0), \eta, x), 0) \, dx.
\end{equation}

Note that the RHS of (2.42) is positive. From (2.38), (2.42) we conclude that the denominator in the expression (2.20) is given by the formula
\begin{equation}
p(\zeta(\cdot), \eta) + \eta \frac{\partial I(\zeta(\cdot), \eta)}{\partial \eta} + [{d_\zeta I(\zeta(\cdot), \eta), \zeta - yD\zeta}] = \eta^{-p} \int_0^1 g(\Gamma(\zeta(f(x)/\alpha_0), \eta, x)) \left[ f(x)[1 + D\zeta(f(x)/\alpha_0)] \right. \\
x \left. w(\Gamma(\zeta(f(x)/\alpha_0), \eta, x), 0) \right] dx = -[d_\zeta I(\zeta(\cdot), \eta), y(1 + D\zeta)].
\end{equation}

**Lemma 2.2.** Assume \( \beta(\cdot, 0) \) is continuous and satisfies (1.14), (2.21). Let \( \zeta : [\varepsilon_0, \infty) \to \mathbb{R} \) be a \( C^1 \) non-negative function such that \( \inf D\zeta(\cdot) \geq -1 \), and \( 0 < \eta \leq 1 \). Then the function \( \rho(\cdot, \cdot) \) of (2.20) satisfies the inequality
\begin{equation}
\rho(\zeta(\cdot), \eta) \geq -\sup_{y > \varepsilon_0} \frac{h(y)}{y}.
\end{equation}

Suppose there is a constant \( M \) such that \( \sup D\zeta(\cdot) \leq M \). Then there exists \( \delta_M > 0 \), depending only on \( M \), such that (2.44) can be improved to
\begin{equation}
\rho(\zeta(\cdot), \eta) \geq \delta_M - \sup_{y > \varepsilon_0} \frac{h(y)}{y}.
\end{equation}

Furthermore, there exists \( \delta_M, K_M > 0 \), depending only on \( M \), such that
\begin{equation}
\rho(\zeta(\cdot), \eta) \geq \delta_M \quad \text{provided} \quad \inf \zeta(\cdot) \geq K_M.
\end{equation}

Suppose there is a constant \( M > 0 \) such that \( \inf [1 + D\zeta(\cdot)] \geq m \). For any \( M > 0 \) there is a constant \( C_{m,M} \), depending only on \( m, M \), such that
\begin{equation}
\rho(\zeta(\cdot), \eta) \leq C_{m,M} \quad \text{provided} \quad \sup \zeta(\cdot) \leq M.
\end{equation}

**Proof.** The inequality (2.44) follows immediately from (2.20), (2.43) on using the positivity of \( I(\zeta(\cdot), \eta) \) and the negativity of the function \( d_\zeta I(\zeta(\cdot), \eta) : [\varepsilon_0, \infty) \to \mathbb{R} \). To prove the remaining inequalities we observe from (2.25), (2.38), (2.40) there are constants \( C, c > 0 \) such that
\begin{equation}
\int_{\varepsilon_0}^{\infty} |d_\zeta I(\zeta(\cdot), \eta; y)| \, dy \leq \frac{C}{\inf_{y > \varepsilon_0} [y + \zeta(y)]} I(\zeta(\cdot), \eta).
\end{equation}
and also
\begin{equation}
(2.49) \quad c \inf_{y \geq \epsilon_0} \frac{y}{y + \zeta(y)} I(\zeta(\cdot), \eta) \leq \int_{\epsilon_0}^{\infty} |d_{\zeta} I(\zeta(\cdot), \eta; y)| \, dy \leq CI(\zeta(\cdot), \eta) .
\end{equation}

Then (2.45) follows from the upper bound in (2.49). We see from (2.48) that it is possible to choose $K_M$ sufficiently large, depending on $M$, so that $0 \geq |d_{\zeta} I(\zeta(\cdot), \eta; h + hD\zeta)| \geq -I(\zeta(\cdot), \eta)/2$ when $\inf(\zeta(\cdot)) \geq K_M$. From the upper bound in (2.49) it follows that $0 \leq |d_{\zeta} I(\zeta(\cdot), \eta; y(1 + D\zeta))| \leq C[1 + M]I(\zeta(\cdot), \eta)$. Evidently we may choose $\delta_M = 1/2C[1 + M]$ in (2.46). The inequality (2.47) is a consequence of the lower bound in (2.49).

Next we estimate some second derivatives of $I(\zeta(\cdot), \eta)$ with respect to $\zeta(\cdot)$. To carry this out we first observe from (2.23), (2.23) that
\begin{equation}
(2.50) \quad \frac{\partial}{\partial x} \Gamma(\zeta(f(x)/\alpha_0), \eta, x) = \frac{f(x) \psi(\Gamma(\zeta, \eta, x))}{\eta \psi(x) f(\Gamma(\zeta, \eta, x))} [1 + D\zeta(f(x)/\alpha_0)] .
\end{equation}

We conclude from (2.50) that
\begin{equation}
(2.51) \quad d_{\zeta} I(\zeta(\cdot), \eta; f(x)/\alpha_0) [1 + D\zeta(f(x)/\alpha_0)] = -\eta^{-p} \psi(x) a(x) \frac{\partial}{\partial x} \Gamma(\zeta(f(x)/\alpha_0), \eta, x, 0) .
\end{equation}

Suppose now that $\zeta, G : [\epsilon_0, \infty) \to \mathbb{R}$ are non-negative $C^1$ functions with bounded first derivatives. We have from (2.51) upon integration by parts that
\begin{equation}
(2.52) \quad |d_{\zeta} I(\zeta(\cdot), \eta; G(1 + D\zeta))| = \eta^{-p} \psi(0) G(0)/\alpha_0 \int_0^1 \left( \frac{\psi' \cdot \psi(1) f(x)}{\eta \psi(1) f(x)} - \frac{G'(f(x)/\alpha_0)}{\alpha_0} \right) w(\Gamma(\zeta(f(x)/\alpha_0), \eta, x, 0) \, dx .
\end{equation}

**Lemma 2.3.** Assume $\beta(\cdot, 0)$ is continuous and satisfies (1.19), (2.21). Let $G : [\epsilon_0, \infty) \to \mathbb{R}$ be a $C^1$ non-negative function with bounded first derivative. Define $H_G(\zeta(\cdot), \eta)$ for non-negative $C^1$ functions $\zeta : [\epsilon_0, \infty) \to \mathbb{R}^+$ with bounded derivative and $0 < \eta \leq 1$ by $H_G(\zeta(\cdot), \eta) = -d_{\zeta} I(\zeta(\cdot), \eta; G(1 + D\zeta))$. Given any $\gamma > 0$ there is a constant $C_\gamma$ such that
\begin{equation}
(2.53) \quad |H_G(\zeta(\cdot), \eta) - H_G(\zeta(\cdot), \eta)| \leq C_\gamma \eta^{-\gamma} \| \zeta_1(\cdot) - \zeta_2(\cdot) \|_{\infty} ,
\end{equation}
for all non-negative $\zeta_j(\cdot)$, $j = 1, 2$, and also
\begin{equation}
(2.54) \quad |H_G(\zeta(\cdot), \eta_1) - H_G(\zeta(\cdot), \eta_2)| \leq C_\gamma \eta_1^{-(\gamma + 1)}|\eta_1 - \eta_2| ,
\end{equation}
for all $\zeta(\cdot) \geq 0$, $0 < \eta_1 < \eta_2 \leq 1$.

**Proof.** We use the representation (2.52) and the fundamental theorem of calculus to prove (2.53). For the first term on the RHS of (2.52) we have
\begin{equation}
(2.55) \quad w(\Gamma(\zeta_1(f(0)/\alpha_0), \eta, 0, 0)) - w(\Gamma(\zeta_2(f(0)/\alpha_0), \eta, 0, 0)) =
\end{equation}

\begin{equation}
- \int_0^1 d\lambda x_0 \lambda \zeta_1(x_0) + (1 - \lambda)\zeta_2(x_0) \, w(X(\lambda) \cdot 0)[\zeta_1(x_0) - \zeta_2(x_0)] ,
\end{equation}

where $X(\lambda) = \Gamma(\lambda \zeta_1(\epsilon_0) + (1 - \lambda)\zeta_2(\epsilon_0), \eta, 0)$. It follows from (2.23), (2.24), (2.25), (2.56) there is for any $\gamma > 0$ a constant $C_1, \gamma$ such that
\begin{equation}
(2.56) \quad |w(\Gamma(\zeta_1(x_0), \eta, 0, 0)) - w(\Gamma(\zeta_2(x_0), \eta, 0, 0))| \leq C_1, \gamma \eta^{p - \gamma} |\zeta_1(x_0) - \zeta_2(x_0)| .
\end{equation}
We can make a similar argument to estimate the integral on the RHS of (2.52), whence (2.53) follows. We prove (2.54) in the same way as (2.53) by using (2.41).

**Remark 1.** Note that we only use the integrability of \( \psi'(\cdot) \) in our estimate, so \( \psi'(x) \) can diverge as \( x \to 0 \), as in the case of the LSW functions \((1.14)\). Similarly we only need the function \( y \to G'(y) \) to be integrable close to \( y = \varepsilon_0 \). Hence the result of Lemma 2.3 applies to the function \( G(\cdot) = h(\cdot) \).

In Lemma 2.2 we obtained bounds on \( \rho(\zeta(\cdot), \eta) \) which are uniform for \( 0 < \eta \leq 1 \). Next we show that \( \lim_{\eta \to 0} \rho(\zeta(\cdot), \eta) \) exists.

**Lemma 2.4.** Assume \( \beta(\cdot, 0) \) is continuous and satisfies \((1.14), (2.27)\). For \( m, M \geq 0 \), let \( S_{m,M} \) be the set of non-negative \( C^1 \) functions \( \zeta : [\varepsilon_0, \infty) \to \mathbb{R} \) such that \( 1 + \inf D\zeta(\cdot) \geq m \), and \( ||\zeta(\cdot)||_{1,\infty} \leq M \), where \( ||\cdot||_{1,\infty} \) is defined by \((2.57)\). Then the function \( \rho(\cdot, \cdot) \) of \((2.26)\) satisfies the limit

\[
\lim_{\eta \to 0} \sup_{\zeta(\cdot) \in S_{m,M}} \{ |\rho(\zeta(\cdot), \eta) - \rho_p(\zeta(\cdot))| : \zeta(\cdot) \in S_{m,M} \} = 0 ,
\]

where \( \rho_p(\zeta(\cdot)) \) is given by the RHS of \((1.37)\) with \( I(\zeta(\cdot)) \) as in \((2.13), (2.15)\). If the function \( x \to \beta(x,0) \) is Hölder continuous at \( x = 1 \) of order \( \gamma > 0 \) then there exists a constant \( C_{m,M} \), depending only on \( m, M \), such that

\[
\sup_{\zeta(\cdot) \in S_{m,M}} \{ |\rho(\zeta(\cdot), \eta) - \rho_p(\zeta(\cdot))| : \zeta(\cdot) \in S_{m,M} \} \leq C_{m,M} \eta^\gamma \quad \text{for } 0 < \eta \leq 1 .
\]

**Proof.** We already saw in Lemma 2.1 that the function \( I(\zeta(\cdot), \eta) \) can diverge as \( \eta \to 0 \). We can however obtain a finite limit by introducing a suitable normalization. Thus we show that

\[
\lim_{\eta \to 0} \sup_{\zeta(\cdot) \in S_{m,M}} \left\{ \frac{\eta^p}{w(\Gamma(\zeta(f(0)/\alpha_0), \eta,0),0)} I(\zeta(\cdot), \eta) - [\varepsilon_0 + \zeta(\varepsilon_0)]^p \frac{\rho_p(\zeta(\cdot))}{K_p} : 0 \leq \zeta(\cdot) \leq M \right\} = 0 ,
\]

where \( I_p(\zeta(\cdot)) \) is the function \( I(\zeta(\cdot)) \) of \((2.12), (2.13)\). To see \((2.59)\) we observe from \((2.26)\) that

\[
\frac{\eta^p}{w(\Gamma(\zeta(f(0)/\alpha_0), \eta,0),0)} I(\zeta(\cdot), \eta) = \int_0^1 \frac{w(\Gamma(\zeta(f(x)/\alpha_0), \eta,0),0)}{w(\Gamma(\zeta(f(0)/\alpha_0), \eta,0),0)} \, dx = \int_0^1 \alpha(\zeta(\cdot), \eta, x) \, dx .
\]

From \((2.29), (2.30)\) we have that

\[
\alpha(\zeta(\cdot), \eta, x) = \exp \left[ -\int_{\Gamma(\zeta(f(x)/\alpha_0), \eta,0)}^{\Gamma(\zeta(f(x)/\alpha_0), \eta, x)} dx' / \int_{\zeta(\cdot)}^{1} \Gamma(\zeta(f(x)/\alpha_0), \eta,0) dx'' \times \int_{\zeta(\cdot)}^{1} [1 - \beta(x'', 0)] \, dx'' \right] 
\times \int_{\zeta(\cdot)}^{1} \frac{I_{\Gamma(\zeta(f(x)/\alpha_0), \eta,0)[1 - \beta(x', 0)]} \, dx'}{\int_{\zeta(\cdot)}^{1} \Gamma(\zeta(f(x)/\alpha_0), \eta, x)[1 - \beta(x', 0)] \, dx'} .
\]

From \((1.19), (2.24), (2.61)\) we see that

\[
\lim_{\eta \to 0} \alpha(\zeta(\cdot), \eta, x) = \left( \frac{f(0) + \alpha_0 \zeta(f(0)/\alpha_0)}{f(x) + \alpha_0 \zeta(f(x)/\alpha_0)} \right)^p .
\]
Note that for any $M > 0$ and $\delta$ with $0 < \delta < 1$, the limit in \((2.62)\) is uniform for $\{[\zeta(\cdot), x] : 0 \leq \zeta(\cdot) \leq M, 0 \leq x \leq 1 - \delta\}$. We also have that for any $\gamma$ with $0 < \gamma < p$ there exists a constant $C_{M, \gamma}$ such that
\[(2.63) \quad 0 \leq \alpha(\zeta(\cdot), \eta, x) \leq C_{M, \gamma}(1 - x)^{p - \gamma} \quad \text{for } 0 < \eta \leq 1, \ 0 \leq \zeta(\cdot) \leq M .\]
We conclude from \((2.12), (2.14), (2.60), (2.62), (2.63)\) that
\[(2.64) \quad \lim_{\eta \to 0} \frac{\eta^p}{w(\Gamma(\zeta(f(0)/\alpha_0), \eta, 0), 0)} I_1(\zeta(\cdot), \eta) = \frac{[f(0)/\alpha_0 + \zeta(f(0)/\alpha_0)]^p}{[f(x)/\alpha_0 + \zeta(f(x)/\alpha_0)]^p} \int_0^1 \frac{dx}{f(x)/\alpha_0 + \zeta(f(x)/\alpha_0)} = \frac{[f(0)/\alpha_0 + \zeta(f(0)/\alpha_0)]^p}{[f(x)/\alpha_0 + \zeta(f(x)/\alpha_0)]^p} \int_{\varepsilon_0}^\infty \frac{a(y)}{b(y) + \zeta(y)} dy = \frac{[\varepsilon_0 + \zeta(\varepsilon_0)]^p}{K_0} .\]
The limit in \((2.64)\) is uniform for $\zeta(\cdot)$ in the set $0 \leq \zeta(\cdot) \leq M$, whence \((2.60)\) follows.
To show convergence of the terms involving $d_\zeta I$ in \((2.20)\) we observe from \((2.38)\) that
\[(2.65) \quad \frac{\eta^p}{w(\Gamma(\zeta(f(0)/\alpha_0), \eta, 0), 0)} d_\zeta I_1(\zeta(\cdot), \eta; f(x)/\alpha_0) = - \frac{a(f(x)/\alpha_0)}{f(x) + \alpha_0 \zeta(f(x)/\alpha_0)} \alpha(\zeta(\cdot), \eta, x) a(f(x)/\alpha_0) , \ 0 \leq x < 1 .\]
We conclude from \((2.11), (2.40), (2.43)\) that
\[(2.66) \quad \lim_{\eta \to 0} \frac{\eta^p}{w(\Gamma(\zeta(f(0)/\alpha_0), \eta, 0), 0)} d_\zeta I_1(\zeta(\cdot), \eta; f(x)/\alpha_0) = - \frac{p [f(0)/\alpha_0 + \zeta(f(0)/\alpha_0)]^p}{[f(x)/\alpha_0 + \zeta(f(x)/\alpha_0)]^p} a(f(x)/\alpha_0) = \frac{[\varepsilon_0 + \zeta(\varepsilon_0)]^p}{K_0} .\]
Let $G : [\varepsilon_0, \infty) \to \mathbb{R}$ be a $C^1$ function with bounded first derivative. It follows from \((2.26)\) that
\[(2.67) \quad \lim_{\eta \to 0} \frac{\eta^p}{w(\Gamma(\zeta(f(0)/\alpha_0), \eta, 0), 0)} [d_\zeta I(\zeta(\cdot), \eta), G(1 + D_\zeta)] = \frac{[\varepsilon_0 + \zeta(\varepsilon_0)]^p}{K_0} .\]
The limit in \((2.67)\) is uniform for non-negative functions $\zeta(\cdot)$ satisfying $||\zeta(\cdot)||_{1, \infty} \leq M$. Now \((2.57)\) follows from the lower bound in \((2.49)\) and \((2.59), (2.67)\) upon using the formula \((2.48)\) for the denominator of \((2.20)\).
To prove \((2.58)\) we observe that if the function $x \to \beta(x, 0)$ is Hölder continuous at $x = 1$ of order $\gamma > 0$ then there is a constant $C_1 > 0$ such that
\[(2.68) \quad \frac{1}{[1 + C_1 \eta^\gamma]} \left( \frac{f(0) + \alpha_0 \zeta(f(0)/\alpha_0)}{f(x) + \alpha_0 \zeta(f(x)/\alpha_0)} \right)^p \leq \alpha(\zeta(\cdot), \eta, x) \leq \frac{1}{[1 + C_1 \eta^\gamma]} \left( \frac{f(0) + \alpha_0 \zeta(f(0)/\alpha_0)}{f(x) + \alpha_0 \zeta(f(x)/\alpha_0)} \right)^p \quad \text{for } \zeta(\cdot) \geq 0, \ 0 < \eta \leq 1, \ 0 \leq x < 1 .\]
By carrying through the argument for the proof of (2.57) and using (2.68) we obtain (2.69).

We can use the method in the proof of Lemma 2.4 to improve on the result of Lemma 2.1.

**Lemma 2.5.** Assume $\beta(\cdot,0)$ is continuous and satisfies (1.19), (2.21). Then for any $M, \gamma > 0$ there exist positive constants $C_{M,\gamma}$, $c_{M,\gamma}$, depending only on $M, \gamma$, such that

$$
c_{M,\gamma} \left( \frac{\eta_1}{\eta_2} \right)^\gamma \leq \frac{I(\zeta_1(\cdot),\eta_1)}{I(\zeta_2(\cdot),\eta_2)} \leq C_{M,\gamma} \left( \frac{\eta_2}{\eta_1} \right)^\gamma \quad \text{for } 0 \leq \zeta_1(\cdot), \zeta_2(\cdot) \leq M, \ 0 \leq \eta_1 \leq \eta_2 \leq 1.
$$

**Proof.** From (2.60) we have that

$$
I(\zeta_1(\cdot),\eta_1) = \left( \frac{\eta_2}{\eta_1} \right)^\gamma \frac{w(\Gamma(\zeta_1(0)/\alpha_0), \eta_1, 0)}{w(\Gamma(\zeta_2(0)/\alpha_0), \eta_2, 0)} \int_0^1 \alpha(\zeta_1(\cdot), \eta_1, x) \, dx.
$$

Using the representation analogous to (2.61) for $\alpha(\zeta(\cdot),\eta, x)$, we see that

$$
ce \left( \frac{\eta_1}{\eta_2} \right)^{p+\gamma} \leq \frac{w(\Gamma(\zeta_1(0)/\alpha_0), \eta_1, 0)}{w(\Gamma(\zeta_2(0)/\alpha_0), \eta_2, 0)} \leq C \left( \frac{\eta_2}{\eta_1} \right)^{p-\gamma},
$$

where $c, C > 0$ depend only on $M, \gamma > 0$. \qed

### 3. Proof of Theorem 1.2

We shall prove Theorem 1.2 using the setup developed in §2. In order to carry this out we shall need to assume that the function $u(\cdot)$ of (2.2) satisfies $\inf u(\cdot) \geq 1$ and also that $\beta(\cdot,0)$ satisfies (2.21) as well as (1.19). The lower bound on $\inf u(\cdot)$ and the requirement that (2.21) holds are not actually additional constraints beyond (1.19) on the initial data. We can see this from the arguments in §2 of [1]. In fact there exists $T_0 > 0$ such that $\inf_{t \geq T_0} u(t) \geq 1$ and $\beta(\cdot, T_0)$ satisfies (2.21). Hence the formula (2.60) for $I(\xi(\cdot), \eta(\cdot))$ with $\eta(t) = u(t)^{-1}$ is well-defined provided $t \geq T_0$ and $\xi(\cdot, t)$ is a non-negative function.

Let $\eta : [0, \infty) \to (0, 1]$ be a continuous strictly positive function. We study solutions to (2.23) with $\rho(\xi(\cdot, t)) = \rho(\xi(\cdot, t), \eta(t))$, $t \geq 0$, where $\rho(\zeta(\cdot), \eta)$ is given by the formula (2.20) and $I(\xi(\cdot), \eta)$ by (2.10). We first consider the linear PDE

$$
\frac{\partial \xi(y, t)}{\partial t} - h(y) - h(y) \frac{\partial \xi(y, t)}{\partial y} + \rho(t) \left[ \xi(y, t) - y \frac{\partial \xi(y, t)}{\partial y} \right] = 0, \quad y > \varepsilon_0, \ t > 0,
$$

where $\rho : [0, \infty) \to \mathbb{R}$ is assumed to be a known function. The initial value problem for (3.1) can be solved globally in time by the method of characteristics provided

$$
\inf \rho(\cdot) \geq - \frac{h(\varepsilon_0)}{\varepsilon_0}.
$$

If (3.2) holds the characteristic $y(\cdot)$, defined as the solution to the terminal value problem

$$
\frac{dy(s)}{ds} = -h(y(s)) - \rho(s) y(s), \quad 0 \leq s < t, \ y(t) = y,
$$

the function $u(\cdot, t)$ of (2.2) must satisfy $\inf u(\cdot, t) \geq 1$ for $t \geq T_0$.
has the property that for \( y \geq \varepsilon_0 \) then \( y(s) \geq \varepsilon_0, \, 0 \leq s < t \). The solution to (3.1) with given initial data \( \xi(y, 0), \, y \geq \varepsilon_0 \), is then expressed in terms of the characteristic (3.4)

\[
\xi(y, t) = \exp \left[ - \int_0^t \rho(s) \, ds \right] \xi(y(0), 0) + \int_0^t ds \, h(y(s)) \exp \left[ - \int_s^t \rho(s') \, ds' \right].
\]

It follows from (2.24), (3.3) that if the initial data \( \xi(\cdot, 0) \) is non-negative, then the function \( \xi(\cdot, t) \) is non-negative for all \( t > 0 \).

We prove a local existence and uniqueness theorem for the initial value problem for (1.23), where the function \( t \to \rho(\xi(\cdot, t)) \) in (1.23) is replaced by the function \( t \to \rho(\xi(\cdot, t), \eta(t)) \), with \( \rho(\xi(\cdot, \eta), \eta) \) given by (2.20). We shall follow the same line of argument as in Lemma 2.1 of [3]. Thus following (1.9) of [3], we define for \( m = 1, 2, \ldots, \) norms on \( C^m \) functions \( \xi : [\varepsilon_0, \infty) \to \mathbb{R} \) by

\[
\|\xi(\cdot)\|_{m, \infty} = \sup_{\varepsilon_0 \leq y < \infty} \sum_{k=0}^m y^k \left| \frac{d^k \xi(y)}{dy^k} \right|.
\]

**Lemma 3.1.** Let \( h : [\varepsilon_0, \infty) \to \mathbb{R} \) be a continuous positive function, which is \( C^1 \) on \( (\varepsilon_0, \infty) \), and satisfies \( \|h(\cdot)\|_\infty < \infty, \int_{\varepsilon_0}^{\varepsilon_0+1} |h'(y)| \, dy + \sup_{\varepsilon_0 \leq y \leq \varepsilon_0+1} y|h'(y)| < \infty, \sup_{y \geq \varepsilon_0} h(y)/y = h(\varepsilon_0)/\varepsilon_0 \). Assume \( \beta(\cdot, 0) \) is continuous and satisfies (1.19), (2.21), \( I(\xi(\cdot), \eta) \) is given by (2.22), \( I(\eta) \) is given by (2.21), and \( \eta : [0, \infty) \to (0, 1] \) is a continuous strictly positive function. Let \( \rho(\xi(\cdot), \eta) \) be defined by (2.20), and consider the initial value problem for (3.1) with \( \rho(t) = \rho(\xi(\cdot, \eta), \eta(t)) \). Given \( C^1 \) non-negative initial data \( \xi(y, 0), \, y \geq \varepsilon_0, \) such that \( \|\xi(\cdot, 0)\|_{1, \infty} < \infty \) and \( \inf D\xi(\cdot, 0) \geq -1 \), there exists for some \( T > 0 \) a unique non-negative solution \( \xi(y, t), \, y \geq \varepsilon_0, \, 0 \leq t \leq T, \) such that \( \|\xi(\cdot, t)\|_{1, \infty} < \infty \) and \( \inf D\xi(\cdot, t) \geq -1, \, 0 \leq t \leq T \).

**Proof.** Since \( \xi(\cdot, 0) \) is a non-negative function we have that \( 1 + D\xi(\cdot, 0) \geq 0 \), but is not identically zero. Hence the denominator \( [d\xi I(\xi(\cdot, 0), \eta(0)), y(1+D\xi(\cdot, 0))] \) in the formula (2.21) is positive. Since \( \|\xi(\cdot, 0)\|_{1, \infty} < \infty \) we have from (2.21) that \( \rho_0 = \rho(\xi(\cdot, 0), \eta(0)) \geq -h(\varepsilon_0)/\varepsilon_0 \). For \( T > 0 \) and \( 0 < \varepsilon < [\rho_0 + h(\varepsilon_0)/\varepsilon_0]/2 \), let \( \varepsilon, T \) be the metric space of continuous functions \( \rho : [0, T] \to \mathbb{R} \) such that \( \rho(0) = \rho_0 \) and \( \|\rho(\cdot) - \rho_0\|_\infty < \varepsilon \). If \( \rho(\cdot) \in \varepsilon, T \), then \( \inf \rho(\cdot) \geq -h(\varepsilon_0)/\varepsilon_0 + \varepsilon \) and the function \( \xi(\cdot, t) \) is well defined by (2.24). We define the function \( K\rho : [0, T] \to \mathbb{R} \) by

\[
K\rho(t) = \rho(\xi(\cdot, t), \eta(t)), \quad 0 \leq t \leq T, \quad \text{where } \xi(\cdot, t) \text{ is given by (3.1)}.
\]

Evidently Lemma 2.2 implies that \( \inf K\rho(\cdot) \geq -h(\varepsilon_0)/\varepsilon_0 \), and fixed points of \( K \) correspond to solutions \( \xi(\cdot, \cdot) \) of (3.1) with \( \rho(t) = \rho(\xi(\cdot, t), \eta(t)) \).

We first show that \( K \) maps \( \varepsilon, T \) to itself provided \( \varepsilon, T > 0 \) are sufficiently small. To do this we use the representation

\[
I(\xi(\cdot), t, \eta(t)) - I(\xi(\cdot, 0), \eta(0)) = \int_0^1 d\lambda \frac{\partial I(\xi(\cdot, t), \lambda\eta(t) + (1 - \lambda)\eta(0))}{\partial \eta} [\eta(t) - \eta(0)] + \int_0^1 d\lambda [d\xi(\lambda\xi(\cdot, t) + (1 - \lambda)\xi(\cdot, 0), \eta(0)), \xi(\cdot, t) - \xi(\cdot, 0)].
\]
We have from Lemma 2.1, (2.25), (2.42) there is a constant $C_1$ such that

\[(3.8) \quad \eta \frac{\partial I(\zeta(\cdot), \eta)}{\partial \eta} \leq C_1 I(\zeta(\cdot), \eta) \leq C_1 C_\gamma \eta^{-\gamma}.
\]

It follows then from (2.49), (2.77), (3.8) there is for any $\gamma > 0$ a constant $C_\gamma$ such that

\[(3.9) \quad |I(\xi(\cdot, t), \eta(t)) - I(\xi(\cdot, 0), \eta(0))| \leq \frac{C_\gamma}{\min|\eta(0), \eta(t)|} (|\eta(t) - \eta(0)| + \|\xi(\cdot, t) - \xi(\cdot, 0)\|_\infty),
\]

Since the function $\eta(\cdot)$ is continuous and $\eta(0) > 0$ the first term on the RHS of (3.9) can be made arbitrarily small for $0 \leq t \leq T$ by choosing $T > 0$ sufficiently small. We estimate the second term on the RHS of (3.9) by using (3.3). Thus from (3.4) we have that

\[(3.10) \quad \xi(0, t, 0)(0) = -\int_0^t ds [h(y(s)) + \rho(s)y(s)]D_\xi(y(s), 0),
\]

whence we conclude that for all $\rho(\cdot)$ such that $\|\rho(\cdot) - \rho_0\|_\infty \leq \varepsilon$ there is a constant $C_{1,\varepsilon}$ such that $\|\xi(0, t, 0)(0)\|_\infty \leq C_{1,\varepsilon} \varepsilon_{1,\infty}$ Now from (3.10) and the boundedness of $h(\cdot)$ it follows there is a constant $C_{2,\varepsilon}$ such that

\[(3.11) \quad \|\xi(\cdot, t) - \xi(\cdot, 0)\|_\infty \leq C_{2,\varepsilon} \|\|\xi(\cdot, 0)\|_\infty + 1\|, \quad 0 \leq t \leq T \leq 1.
\]

It follows from (3.11) that the second term on the RHS of (3.9) can be made arbitrarily small for $0 \leq t \leq T$ by choosing $T > 0$ sufficiently small. In order to estimate $|K \rho(t) - \rho_0|$ we also need to estimate differences for the function $H_G$ of Lemma 2.3 when $G(y) = h(y)$ and $G(y) = y, y \geq \varepsilon_0$. The inequalities (2.65), (2.66) hold for both of these functions (see especially the remark following Lemma 2.3).

We have therefore for any $\gamma > 0$ there exists a constant $C_\gamma$ such that

\[(3.12) \quad |H_G(\xi(\cdot, t), \eta(t)) - H_G(\xi(\cdot, 0), \eta(0))| \leq \frac{C_\gamma}{\min|\eta(0), \eta(t)|} (|\eta(t) - \eta(0)| + \|\xi(\cdot, t) - \xi(\cdot, 0)\|_\infty).
\]

From (3.9), (3.11), (3.12) we conclude for given $\varepsilon > 0$, there exists $T > 0$ such that if $\rho(\cdot) \in \mathcal{E}_{\varepsilon,T}$ then $K \rho(\cdot) \in \mathcal{E}_{\varepsilon,T}$.

We may make a similar argument to show that $K : \mathcal{E}_{\varepsilon,T} \to \mathcal{E}_{\varepsilon,T}$ is a contraction mapping provided $T > 0$ is sufficiently small. Let $\xi_j(\cdot, t), j = 1, 2, 0 \leq t \leq T$, denote the functions (3.2) with $\rho(\cdot) = \rho_j(\cdot) \in \mathcal{E}_{\varepsilon,T}$, $j = 1, 2$. We shall show that

\[(3.13) \quad \|\xi_1(\cdot, t) - \xi_2(\cdot, t)\|_\infty \leq C_{3,\varepsilon} \|\rho_1(\cdot) - \rho_2(\cdot)\|_\infty (\|\xi(\cdot, 0)\|_\infty + 1), \quad 0 \leq t \leq T,
\]

for a constant $C_{3,\varepsilon}$ depending only on $\rho_0, \varepsilon$, provided $T \leq 1$. To see this first observe from (3.3) the identity

\[(3.14) \quad y(s) = \exp \left[ \int_s^t \rho(s') \, ds' \right] y + \int_s^t \exp \left[ \int_s^{s'} \rho(s'') \, ds'' \right] h(y(s')) \, ds's'.
\]

Since $h(\cdot)$ is non-negative and bounded there exist constants $C_\varepsilon, c_\varepsilon > 0$ depending only on $\varepsilon, \rho_0$ such that for $0 < T \leq 1$,

\[(3.15) \quad c_\varepsilon y \leq y(s) \leq C_\varepsilon y \quad \text{for} \quad y \geq \varepsilon_0, 0 \leq s \leq t \leq T, \quad \rho(\cdot) \in \mathcal{E}_{\varepsilon,T}.
\]
Next we estimate the difference of the characteristics \( y_j(\cdot), \ j = 1, 2, \) which are the solutions to (3.3) corresponding to \( \rho_j(\cdot) \) \( j = 1, 2 \) respectively. Setting \( z(s) = y_1(s) - y_2(s), \ s \leq t, \) we have from (3.3) that

\[
(3.16) \quad \frac{dz(s)}{ds} = -[g(s) + \rho_1(s)]z(s) - [\rho_1(s) - \rho_2(s)]y_2(s), \quad 0 \leq s < t, \ z(t) = 0,
\]

where the function \( g(\cdot) \) is given by

\[
(3.17) \quad g(s) = \int_0^1 d\lambda \ h'(\lambda y_1(s) + (1 - \lambda)y_2(s)), \quad 0 \leq s \leq t.
\]

Evidently \( g(s) = g(s, y) \) depends on the terminal condition \( y_1(t) = y_2(t) = y > \varepsilon_0 \) as well as on \( s, \) and it is possible that \( g(s, y) \) becomes unbounded as \( y \to \varepsilon_0. \) However, in view of our assumptions on \( h'(\cdot) \) and the inequality \( \inf \rho_j(\cdot) \geq -h(\varepsilon_0)/\varepsilon_0 + \varepsilon, \ j = 1, 2, \) there is a constant \( M_\varepsilon \) independent of \( T \) such that

\[
(3.18) \quad \int_0^t |g(s, y)| \ ds \leq M_\varepsilon \quad \text{for} \ y > \varepsilon_0, \ 0 < t \leq T.
\]

The solution to (3.18) has the representation

\[
(3.19) \quad z(s) = \int_s^t \exp \left[ \int_s^{s'} \{g(s'') + \rho_1(s'')\} \ ds'' \right] [\rho_1(s') - \rho_2(s')]y_2(s') \ ds'.
\]

We now can estimate the LHS of (3.13) from (3.4) by using the fundamental theorem of calculus and (3.15), (3.18), (3.19). We conclude from (3.13) that \( K \) is a contraction if \( T > 0 \) is sufficiently small, by using the inequality (2.53) and the inequality

\[
(3.20) \quad |I(\zeta_1(\cdot), \eta) - I(\zeta_2(\cdot), \eta)| \leq C_\gamma \eta^{-7}\|\zeta_1(\cdot) - \zeta_2(\cdot)\|_\infty,
\]

which follows from (2.45).

It follows from the contraction mapping theorem there exists for sufficiently small \( T > 0 \) a unique \( \rho(\cdot) \in E_{\varepsilon,T} \) such that \( K\rho(\cdot) = \rho(\cdot). \) Setting this \( \rho(\cdot) \) into the expressions (3.3), (3.4) we obtain a solution \( \xi(y, t), \ y \geq \varepsilon_0, \ 0 \leq t \leq T, \) to the initial value problem. We see from (3.4), using the boundedness of \( h(\cdot), \) that \( \sup_{y \geq \varepsilon_0} \xi(y, t) < \infty. \) We can obtain a formula for \( D\xi(y, t) \) from the first variation equation for (3.3). Thus letting \( Y(s, y), \ s \leq t, \) denote the solution to (3.3) so as to indicate the dependence on the variable \( y, \) we have that \( DY(s, y) \) satisfies the linear variation equation

\[
(3.21) \quad \frac{d}{ds} DY(s, y) = -[h'(y(s)) + \rho(s)]DY(s, y), \quad 0 \leq s \leq t, \ DY(t, y) = 1.
\]

Integrating (3.21) we obtain the formula

\[
(3.22) \quad DY(s, y) = \exp \left[ \int_s^t \{h'(y(s')) + \rho(s')\} \ ds' \right].
\]

Differentiating (3.4) we have from (3.22) the expression

\[
(3.23) \quad D\xi(y, t) = \exp \left[ \int_0^t h'(y(s)) \ ds \right] D\xi(y(0), 0)
\]

\[
+ \int_0^t ds \ h'(y(s)) \exp \left[ \int_s^t h'(y(s')) \ ds' \right]
\]
Assume that $\beta$ with $\lim_{\parallel \cdot \parallel}$ Proposition 3.1. that $D\xi$ the RHS of (3.23) that $1 + \delta, M > (3.25)$ and the first expression on the RHS of $y > \epsilon$ (3.24) $\sup_{y > \epsilon_0} |D\xi(y, t)| < \infty$ for $0 \leq t \leq T$. We have therefore shown that the solution to the initial value problem satisfies $\parallel \xi(\cdot, t) \parallel_{1, \infty} < \infty$ when $0 < t \leq T$. We see from the second expression on the RHS of (3.23) that $1 + D\xi(\cdot, t) \geq 0$ for $0 < t \leq T$. □

The local existence result of Lemma 3.1 can be extended to global existence by using Lemma 2.2.

**Proposition 3.1.** Let $h : [\epsilon_0, \infty) \to \mathbb{R}$ be a continuous positive decreasing function with $\lim_{y \to \infty} h(y) = h_\infty > 0$, which is $C^1$ on $(\epsilon_0, \infty)$ and $\sup_{y \geq \epsilon_0 + 1} y |h'(y)| < \infty$.

Assume that $\beta(\cdot, 0), \eta(\cdot, 0)$ satisfy the conditions of Lemma 3.1, and in addition that $\inf[1 + D\xi(\cdot, 0)] \geq m$ for some $m > 0$. Then with $I(\xi(\cdot), \eta)$ given by (2.25) and $\rho(\xi(\cdot), \eta)$ by (2.27), there exists for $0 < t < \infty$ a unique non-negative solution $\xi(y, t), y \geq \epsilon_0$, to the initial value problem for (3.7) with $\rho(t) = \rho(\xi(\cdot, t), \eta(t))$ such that $\parallel \xi(\cdot, t) \parallel_{1, \infty} < \infty$ and $\inf D\xi(\cdot, t) \geq -1$.

The solution $\xi(\cdot, t), t > 0$, has the property

$$\sup_{y \geq \epsilon_0} \parallel \xi(\cdot, t) \parallel_{1, \infty} + \parallel D\xi(\cdot, t) \parallel_{1, \infty} < \infty, \quad \inf_{y \geq \epsilon_0, t > 0} [1 + D\xi(y, t)] > 0.$$  

Furthermore there exist constants $\delta, M > 0$, depending only on $m$ and $\parallel \xi(\cdot, 0) \parallel_{1, \infty}$, such that

$$\delta - \frac{h(\epsilon_0)}{\epsilon_0} \leq \rho(\xi(\cdot, t), \eta(t)) \leq M \quad \text{for} \quad t > 0.$$  

**Proof.** Note that $h(\cdot)$ satisfies the assumptions of Lemma 3.1. To prove existence and uniqueness of the solution $\xi(\cdot, t)$ for all $t > 0$ it is sufficient to show for any $T > 0$ there exists $A(T) > 0$, depending only on $\parallel \xi(\cdot, 0) \parallel_{1, \infty}$, $m$ and $T$, such that $\parallel \xi(\cdot, t) \parallel_{1, \infty} \leq A(T)$ for all $0 < t < T$. If that is the case then by Lemma 3.1 we may extend the solution of the initial value problem beyond $T$, whence the solution exists globally in time.

We see from (3.24) that $\inf[1 + D\xi(\cdot, t)] \geq 0$ for $0 < t < T$ and there exists $A_1 > 0$, depending only on $\parallel D\xi(\cdot, 0) \parallel_{1, \infty}$, such that $\sup_{0 < t < T} \parallel D\xi(\cdot, t) \parallel_{1, \infty} \leq A_1$.

The lower bound in (3.24) follows from this and (2.15) of Lemma 2.2. It also follows from the lower bound that for any $T > 0$ there exists $A(T)$ such that $\sup_{y \geq \epsilon_0} |y D\xi(y, t)| \leq A(T)$ if $0 < t < T$. We see this by using (3.4) and the first expression on the RHS of (3.23).

Next observe from (3.3) that for any positive $\nu < \min\{1, T\}$, there exists a constant $C_\nu$, depending only on $\nu$ and $\parallel \xi(\cdot, 0) \parallel_{1, \infty}$, such that

$$0 < \sup_{y \geq \epsilon_0} \xi(y, t) \leq C_\nu \inf_{y \geq \epsilon_0} \xi(y, t), \quad \nu < t < T.$$  

From (2.46) of Lemma 2.2 there exists $\delta, K > 0$, depending only on $\parallel D\xi(\cdot, 0) \parallel_{1, \infty}$, such that for $0 < t < T$, if $\inf \xi(\cdot, t) \geq K$ then $\rho(\xi(\cdot, t), \eta(t)) \geq \delta$. We choose now $K$ sufficiently large so there exists $T_\nu > \nu > 0$ such that $\sup_{y \geq \epsilon_0} \xi(y, t) < C_\nu K$ for $0 < t < T_\nu$. Letting $(0, T_\nu)$ be the maximal such interval, we either have that $T_\nu = \infty$ or $\sup_{y \geq \epsilon_0} \xi(y, t) = C_\nu K$. Let us suppose that $T_\nu < \infty$ and $\sup_{y \geq \epsilon_0} \xi(y, t) > C_\nu K$
for $T_\nu < t < T_\nu'$. From $3.20$ we have that $\inf_{y \geq \epsilon_0} \xi(y,t) > K$ for $T_\nu < t < T_\nu'$. It follows now from $3.4$ that

$$
\sup_{y \geq \epsilon_0} \xi(y,t) \leq e^{-\delta(t-T_\nu)}C_\nu K + \frac{1-e^{-\delta(t-T_\nu)}}{\delta}\|h(\cdot)\|_\infty \text{, } T_\nu < t < T_\nu'.
$$

Evidently the RHS of $3.24$ is bounded independent of $T_\nu' > T_\nu$. Iterating this argument we conclude there exists $A_2 > 0$, depending only on $\|\xi(\cdot,0)\|_{1,\infty}$, such that $\sup_{0<t<T} \|\xi(\cdot,t)\|_\infty \leq A_2$. We have established the first inequality of $3.24$.

To prove the second inequality of $3.24$, let $y_\delta > \epsilon_0$ be such that

$$
y \left[ \delta - \frac{h(\epsilon_0)}{\epsilon_0} \right] + h(y) \geq \frac{\epsilon_0 \delta}{2} \text{ for } \epsilon_0 \leq y \leq y_\delta.
$$

From $3.3$, the lower bound in $3.29$, and $3.28$ it follows that the characteristic $y(s), 0 \leq s \leq t$, with $y(t) = y$ satisfies $y(s) \geq y_\delta, 0 \leq s \leq t$, if $y \geq y_\delta$. If $\epsilon_0 \leq y < y_\delta$ then $y'(s) \leq -\epsilon_0 \delta/2$ when $y(s) < y_\delta$. It follows that

$$
\int_0^t |h'(y(s))| \, ds \leq \frac{2}{\epsilon_0 \delta} \int_{\epsilon_0}^{y_\delta} |h'(y)| \, dy + \sup_{y \geq y_\delta} [y|h'(y)|] \int_0^t \frac{ds}{y(s)}.
$$

From $3.4$, $3.14$ we have that

$$
\int_0^t \frac{ds}{y(s)} \leq \frac{1}{y} \int_0^t ds \exp \left[ -\int_s^t \rho(s') \, ds' \right] \leq \frac{1}{yh_\infty} \xi(y,t).
$$

Now the first inequality of $3.24$ and $3.24$, $3.30$ imply there is a constant $C$ such that

$$
\int_0^t |h'(y(s))| \, ds \leq C \text{ for all } t > 0, y(t) = y \geq \epsilon_0.
$$

We conclude from $3.28$, $3.31$ that $\inf_{[0,\infty)} [1 + D\xi(\cdot,t)] \geq e^{-Cm}$ for $t > 0$, whence the second inequality of $3.24$ holds. The upper bound in $3.29$ follows from $3.24$ and $3.47$ of Lemma 2.2.

In order to show the solution $\xi(\cdot,t)$ of Proposition 3.1 satisfies $\sup_{t>0} \|\xi(\cdot,t)\|_{1,\infty} < \infty$, we need a strict positivity assumption on the function $\rho(\cdot)$ at large time.

**Lemma 3.2.** Let $\rho : [0,\infty) \to \mathbb{R}$ be a continuous function satisfying $\inf \rho(\cdot) > -h(\epsilon_0)/\epsilon_0$ i.e. a strict version of $3.24$, and $\delta_0, \tau_0 > 0$ have the property that

$$
\int_s^t \rho(s') \, ds' \geq \delta_0(t-s) \text{ for } t \geq \tau_0, 0 \leq s \leq t - \tau_0.
$$

Assume the function $h : [\epsilon_0,\infty) \to \mathbb{R}$ is positive continuous decreasing, $C^1$ on $(\epsilon_0, \infty)$ and satisfies the inequality

$$
\|h(\cdot)\|_{\infty} + \sup_{y \geq \epsilon_0 + 1} y|h'(y)| < \infty.
$$

Then there are constants $C_1, C_2$, independent of $\xi(\cdot,0)$, such that the solution $\xi(\cdot,t), t \geq 0$, to the initial value problem for $3.7$ satisfies the inequality

$$
\|\xi(\cdot,t)\|_{1,\infty} \leq C_1 e^{-\delta_0 t} \|\xi(\cdot,0)\|_{1,\infty} + C_2, \text{ } t \geq 0.
$$

Assume in addition to $3.32$ that $h(\cdot)$ is convex, $C^2$ on $(\epsilon_0, \infty)$ and

$$
\sup_{y \geq \epsilon_0 + 1} y^2 h''(y) < \infty.
$$
If \( \xi(\cdot, 0) \) is \( C^2 \) on \( (\varepsilon_0, \infty) \) then \( \xi(\cdot, t) \) is \( C^2 \) on \( (\varepsilon_0, \infty) \), and there are positive constants \( \nu_0, C_3, C_4 \), independent of \( \xi(\cdot, 0) \), such that

\[
(3.36) \quad \int_{\varepsilon_0}^{\varepsilon_0 + \nu_0} |D^2\xi(y, t)| \, dy + \sup_{y \geq \varepsilon_0 + \nu_0} y^2 |D^2\xi(y, t)| \leq C_3 e^{-\delta_0 t} \left\{ \|\xi(\cdot, 0)\|_{1, \infty} + \int_{\varepsilon_0}^{\varepsilon_0 + \nu_0} |D^2\xi(y, 0)| \, dy + \sup_{y \geq \varepsilon_0 + \nu_0} y^2 |D^2\xi(y, 0)| \right\} + C_4, \quad t \geq 0.
\]

Proof. Multiplying (3.33) by \( y \) we have from (3.14), upon using the assumption that \( h(\cdot) \) is decreasing, the inequality

\[
(3.37) \quad y|D\xi(y, t)| \leq \exp \left[ -\int_0^t \rho(s) \, ds \right] y(0)|D\xi(y(0), 0)| + \int_0^t ds \, y(s)|h'(y(s))| \exp \left[ -\int_s^t \rho(s') \, ds' \right], \quad y \geq \varepsilon_0.
\]

The bound (3.34) follows now from (3.32) by using (3.41) to estimate \( \|\xi(\cdot, t)\|_{1, \infty} \) and (3.37) to estimate \( \sup_{y \geq \varepsilon_0} y|D\xi(y, t)| \). Note that we need to use the assumption \( \inf \rho(\cdot) > -h(\varepsilon_0)/\varepsilon_0 \) so as to estimate the second term on the RHS of (3.37) when \( y \) is close to \( \varepsilon_0 \).

Differentiating (3.33) with respect to \( y \), we have from (3.22) that

\[
(3.38) \quad D^2\xi(y, t) = \exp \left[ \int_0^t \left\{ \rho(s) + 2h'(y(s)) \right\} ds \right] D^2\xi(y(0), 0)
+ \exp \left[ \int_0^t h'(y(s)) \, ds \right] \{1 + D\xi(y(0), 0)\} \times
\int_0^t ds \, h''(y(s)) \exp \left[ \int_s^t \left\{ \rho(s') + h'(y(s')) \right\} ds' \right].
\]

Multiplying (3.38) by \( y^2 \) we have from (3.14) the inequality

\[
(3.39) \quad y^2|D^2\xi(y, t)| \leq \exp \left[ -\int_0^t \rho(s) \, ds \right] y(0)^2|D^2\xi(y(0), 0)|
+ \{1 + |D\xi(y(0), 0)|\} \int_0^t ds \, y(s)^2h''(y(s)) \exp \left[ -\int_s^t \rho(s') \, ds' \right].
\]

We choose \( \nu_0 > 0 \) so that for some \( \delta > 0 \),

\[
(3.40) \quad \inf \rho(\cdot) \geq -\frac{h(\varepsilon_0 + \nu_0)}{\varepsilon_0 + \nu_0} + \delta.
\]

We see from (3.3), (3.40) that

\[
(3.41) \quad \frac{dy(s)}{ds} \leq -\min\{\delta\varepsilon_0, h(\varepsilon_0 + \nu_0)\} \quad \text{if} \quad y(s) \leq \varepsilon_0 + \nu_0,
\]

\[
\text{if} \quad y(s) \geq \varepsilon_0 + \nu_0 \quad \text{for} \quad s \leq t \quad \text{if} \quad y(t) = y \geq \varepsilon_0 + \nu_0.
\]

The inequality (3.36) follows from (3.32), (3.39), (3.41) by observing that

\[
(3.42) \quad \int_{\varepsilon_0 < y < y' < \varepsilon_0 + 1} h''(y') \, dy' \, dy < \infty.
\]

\[\square\]
Theorem 3.1. In addition to the assumptions of Proposition 3.1, suppose the function \( \eta : [0, \infty) \to (0, 1) \) is \( C^1 \) and satisfies

\[
|\eta'(t)| \leq C \eta(t), \quad \lim_{t \to \infty} \eta(t) = 0,
\]

for some positive constant \( C \). Let \( \xi(\cdot, t) \), \( t > 0 \), be the solution to the initial value problem for (3.1) with \( \rho(t) = \rho(\xi(\cdot, t), \eta(t)) \), \( t \geq 0 \). Then one has

\[
\sup_{t > 0} \|\xi(\cdot, t)\|_{1, \infty} < \infty, \quad \lim_{T \to \infty} \frac{1}{T} \int_0^T \rho(\xi(\cdot, t), \eta(t)) \, dt = \frac{1}{p}.
\]

Proof. We observe from (2.20), (3.1) that

\[
\frac{1}{p} \frac{d}{dt} \log I(\xi(\cdot, t), \eta(t)) = \rho(t) - \frac{1}{p}
\]

\[
+ \frac{1}{p} \left[ \frac{d\eta(t)}{dt} + \rho(t)\eta(t) \right] \frac{1}{I(\xi(\cdot, t), \eta(t))} \frac{\partial I(\xi(\cdot, t), \eta(t))}{\partial \eta}, \quad \text{where } \rho(t) = \rho(\xi(\cdot, t), \eta(t)).
\]

We have now that

\[
\lim_{\eta \to 0} \sup_{\zeta > 0} \frac{1}{I(\zeta, \eta)} |\eta| \frac{\partial I(\zeta, \eta)}{\partial \eta} = 0.
\]

The limit (3.46) follows from (2.24), (2.25), (4.40), (4.42). The limit in (3.44) can now be obtained as a consequence of Lemma 2.1 and (3.43), (3.46) by integrating (3.45). In fact from (3.43) there is a constant \( c > 0 \) such that

\[
1 \geq \eta(t) \geq ce^{-Ct}, \quad t \geq 0.
\]

In view of (3.24) we may apply Lemma 2.1 to conclude that for any \( \nu > 0 \) the integral of the LHS of (3.45) over the interval \([0, T]\) is bounded by \( \nu T \) at large \( T \). The inequality in (3.44) follows from (3.24) of Lemma 3.2 once we show that (3.24) holds. We prove (3.24) in the same way we proved the limit in (3.44) by integrating (3.45) over the interval \([s, t]\). The inequality follows upon using the differential inequality in (3.24) and Lemma 2.5.

Proof of Theorem 1.2. We first show that the function \( u(\cdot) \) defined by (2.2) satisfies \( \lim_{t \to \infty} u(t)^{-1} = 0 \). To see this we use the identity from (2.2) of \[4\] that \( u(t)^{-1} = \partial F(1, t) / \partial x \). Now the function \( x \to F(x, t) \), \( 0 \leq x < 1 \), is increasing and convex for all \( t > 0 \). From (2.2) and (2.44) of \[4\] we see there exists a constant \( C \) such that

\[
\frac{\partial F(1, t)}{\partial x} \leq C \frac{\partial F(0, t)}{\partial x}, \quad t \geq 0.
\]

We also have from Lemma 2.1 of \[4\] that \( \lim_{t \to \infty} F(0, t) = 1 = F(1, t) \), whence (3.47) implies that \( \lim_{t \to \infty} \partial F(1, t) / \partial x = 0 \). We may now choose \( T_0 > 0 \) such that the function \( \eta(t) = u(t)^{-1} \) satisfies \( 0 < \eta(t) \leq 1 \) for \( t \geq T_0 \) and \( \lim_{t \to \infty} \eta(t) = 0 \). Since \( \lim_{t \to \infty} F(0, t) = 1 \), we may also choose \( T_0 \) sufficiently large so that the function \( x \to \beta(x, T_0) \), \( 0 \leq x < 1 \), satisfies (2.21) and \( \lim_{x \to 1} \beta(x, T_0) = p/(p + 1) \).

Next we show that the function \( \xi(\cdot, T_0) \) defined by (2.8) satisfies the assumptions of Theorem 3.1 for the initial data of \( \xi(\cdot, t) \). Since \( \phi(\cdot) \), \( \psi(\cdot) \) are assumed to satisfy (1.15), (1.16), (1.20) it follows from Lemma 6.1 of \[4\] that the function \( h : [\varepsilon_0, \infty) \to \mathbb{R} \) defined by (2.9) is continuous, positive decreasing, \( \lim_{y \to \infty} h(y) = h_\infty = 1 \) and \( \sup_{y > \varepsilon_0} |h'(y)| < \infty \). Furthermore, the function \( g(\cdot, \cdot) \) in (2.4) is related to \( h(\cdot) \) by \( g(z, u) = -\alpha_0 u h(z/\alpha_0 u) \). We then infer from (2.5) that

\[
z + \alpha_0 h_\infty \int_0^t u(s) \, ds \leq \hat{F}(z, t) \leq z + \alpha_0 h(\varepsilon_0) \int_0^t u(s) \, ds.
\]
We conclude from (2.8) and (3.48) that
\[
(3.49) \quad \frac{h_{\infty}}{u(T_0)} \int_0^{T_0} u(s) \, ds \leq \inf \xi(\cdot; T_0) \leq \sup \xi(\cdot; T_0) \leq \frac{h(\varepsilon_0)}{u(T_0)} \int_0^{T_0} u(s) \, ds .
\]
To obtain bounds on \( D\xi(\cdot; T_0) \) we differentiate (2.8) to obtain the identity
\[
(3.50) \quad 1 + D\xi \left( \frac{z}{\alpha_0 u(t)}, t \right) = \frac{\partial \hat{F}(z,t)}{\partial z} .
\]
Differentiating (2.8) we have that
\[
(3.51) \quad \frac{\partial \hat{F}(f(x)u(t), t)}{\partial z} = \frac{f(F(x,t))\psi(x)}{f(x)\psi(F(x,t))u(t)} \frac{\partial F(x,t)}{\partial x} .
\]
From (2.2) of \([4]\) we see that \( \alpha_0 \) is the same formula as (3.23) in the case \( \beta = 0 \). Applying Taylor’s theorem to (3.55), we conclude from (3.54) and the inequality \( \sup_{y \geq \varepsilon_0} |y D\xi(y,T_0)| < \infty \) that \( \sup_{y \geq \varepsilon_0} |y D\xi(y,t)| < \infty \) for all \( t \geq 0 \).

To obtain Theorem 1.2 as a consequence of Theorem 3.1 we observe from (2.2), (2.8) that
\[
(3.56) \quad \rho(\xi(\cdot), t) = \phi'(1) - \psi'(1)\kappa(t), \quad t \geq T_0 .
\]
From (2.28) we see that \( h(\varepsilon_0)/\varepsilon_0 = -\phi'(1) \), whence (3.25) implies that \( \kappa(t) \geq \delta/|\psi'(1)| > 0 \) for \( t \geq T_0 \). The upper bound in (3.25) implies that \( \kappa(t) \leq [M - \phi'(1)]/|\psi'(1)|, \quad t \geq T_0 \). The limit (1.21) is a consequence of (3.44) upon using the relations (3.55) and \( \beta_0 = p/(p+1) \). Note for \( u(\cdot) \) given by (2.22) that \( \eta(t) = u(t)^{-1} \) satisfies
\[
(3.57) \quad \frac{d\eta(t)}{dt} + \rho(t)\eta(t) = 0 , \quad \rho(t) = \rho(\xi(\cdot), t) .
\]
It follows from the already established properties of \( \eta(\cdot) \) and (3.25), (3.57) we see that \( \eta(\cdot) \) is only required to satisfy (3.44). The system (3.44), (3.47) with \( \rho(t) = \rho(\xi(\cdot), t), \eta(t) \) given by (2.22) and \( f(\xi(\cdot), \eta) \) by (2.26) is equivalent to the LSW dynamics (1.13), (1.15).

Remark 2. Observe that Theorem 3.1 yields a reduction of dimension in dynamics from the original LSW problem, since the function \( \eta(\cdot) \) is only required to satisfy (3.44).
4. Local Asymptotic Stability

The limit \((3.44)\) of Theorem 3.1 is a \textit{weak global} asymptotic stability result for solutions to the PDE \((3.1)\) with \(\rho(t) = \rho(\xi(\cdot, t), \eta(t))\), \(t \geq 0\), where \(\rho(\cdot, \eta)\) is defined by \((2.20)\) and \(I(\cdot, \eta)\) by \((2.25)\). In this section we will prove a \textit{strong local} asymptotic result, showing that \(\xi(\cdot, t)\) converges as \(t \to \infty\) to the equilibrium solution \(\xi_p(\cdot)\) defined in \((1.27)\). In order to do this we shall need to impose further assumptions on the function \(h(\cdot)\), beyond those required for Theorem 3.1.

We proceed in parallel to the argument followed in §3 of [3]. We first linearize \((3.1)\) with \(\rho(\xi(\cdot, \eta))\) given by \((2.20)\), \((2.25)\) about the equilibrium \(\xi_p(\cdot)\) and study its stability. To do this we denote by \(A, B\) the operators
\[
(4.1)\quad A\zeta(y) = \zeta(y) - y \frac{d\zeta(y)}{dy}, \quad B\zeta(y) = \frac{1}{p} \zeta(y) - \left[ h(y) + \frac{y}{p} \right] \frac{d\zeta(y)}{dy}. 
\]
Observe now that the functional \(\rho(\cdot)\) of \((1.31)\) satisfies the identity
\[
(4.2)\quad \rho(\xi(\cdot)) - \frac{1}{p} = - \frac{[dI(\xi(\cdot)), B[\xi(\cdot) - \xi_p(\cdot)]]}{pI(\xi(\cdot)) + [dI(\xi(\cdot)), A\xi(\cdot)]}. 
\]
Let \(\xi(\cdot, t), \ t \geq 0, \) be the solution of \((4.1)\) with \(\rho(t) = \rho(\xi(\cdot, t), \eta(t))\), \(t \geq 0\), where \(\rho(\xi(\cdot, \eta))\) is defined by \((2.20)\), \((2.25)\), which is constructed in Proposition 3.1. We denote by \(\gamma: [0, \infty) \to \mathbb{R}\) the function
\[
(4.3)\quad \gamma(t) = \rho(\xi(\cdot, t), \eta(t)) - \rho_p(\xi(\cdot, t)), \quad t \geq 0, 
\]
where \(\rho_p(\xi(\cdot))\) is given by the RHS of \((1.31)\) with \(I(\cdot)\) as in \((2.12), (2.13)\). We shall regard \(\gamma(\cdot)\) as a \textit{given} function, for which we can derive some properties using Lemma 2.4 and Theorem 3.1. We may rewrite \((3.1)\) with \(\rho(t) = \rho(\xi(\cdot, t), \eta(t))\) as
\[
(4.4)\quad \frac{\partial \xi(\cdot, t)}{\partial t} + [B + \gamma(t)A]\{\xi(\cdot, t) - \xi_p(\cdot)\} \\
- \frac{[dI_p(\xi(\cdot, t)), B[\xi(\cdot, t) - \xi_p(\cdot)]]}{pI_p(\xi(\cdot, t)) + [dI_p(\xi(\cdot, t)), A\xi(\cdot)]} A\xi(\cdot, t) + \gamma(t)A\xi_p(\cdot) = 0, 
\]
where \(I_p(\cdot)\) is the functional \(I(\cdot)\) defined by \((2.12), (2.13)\). Setting \(\tilde{\xi}(\cdot, t) = \xi(\cdot, t) - \xi_p(\cdot)\), we see that the linearization of \((4.1)\) about \(\xi_p(\cdot)\) is given by
\[
(4.5)\quad \frac{d\tilde{\xi}(t)}{dt} + [B + \gamma(t)A]\tilde{\xi}(t) - [dI_p(\xi_p), B\tilde{\xi}(t)] \frac{A\xi_p}{pI_p(\xi_p) + [dI_p(\xi_p), A\xi_p]} + \gamma(t)A\xi_p = 0. 
\]
If we set \(\gamma(\cdot) \equiv 0\) in \((4.3)\) then the solution satisfies the equation
\[
(4.6)\quad \tilde{\xi}(t) = e^{-B_t}\tilde{\xi}(0) + \int_0^t ds \left[ dI_p(\xi_p), B\tilde{\xi}(s) \right] \frac{e^{-B(t-s)A\xi_p}}{pI_p(\xi_p) + [dI_p(\xi_p), A\xi_p]}.
\]
Letting \(u(t) = [dI_p(\xi_p), B\tilde{\xi}(t)]\), we see from \((4.6)\) that \(u(\cdot)\) is the solution to the Volterra integral equation
\[
(4.7)\quad u(t) + \int_0^t K(t-s)u(s) \ ds = g(t), \quad t > 0, 
\]
where the functions \(K, g\) are given by
\[
(4.8)\quad K(t) = - \frac{[dI_p(\xi_p), e^{-Bt}BA\xi_p]}{pI_p(\xi_p) + [dI_p(\xi_p), A\xi_p]} \quad g(t) = [dI_p(\xi_p), e^{-Bt}B\tilde{\xi}(0)] \quad t \geq 0. 
\]
We proceed now as in [3] to obtain properties of the function $h(\cdot)$, which will imply that the solution to (4.7) satisfies $\lim_{t \to \infty} u(t) = 0$.

**Lemma 4.1.** Assume that the function $h : [\varepsilon_0, \infty) \to \mathbb{R}$ is positive, continuous, decreasing, convex, and $C^2$ on $(\varepsilon_0, \infty)$. Assume further that $h(\cdot)$ satisfies the inequalities
\begin{equation}
\sup_{y \geq \varepsilon_0 + 1} \{ y|h'(y)| \} < \infty, \quad \left[ \frac{y}{p} + h(y) \right] y h''(y) + h'(y)[h(y) - y h'(y)] \geq 0, \quad y \geq \varepsilon_0.
\end{equation}

Then $K(\cdot)$ defined by (4.8) has the property that the function $t \to e^{t/p} K(t)$, $t \geq 0$, is positive and decreasing.

**Proof.** We first show that the denominator in the formula (4.8) for $K(\cdot)$ is positive. To see this we use the identity
\begin{equation}
pI_p(\xi_p) + [dI_p(\xi_p), A\xi_p] = -[dI_p(\xi_p), y(1 + D\xi_p)] ,
\end{equation}
which is a particular case of (2.43). Next we have from (1.25) that
\begin{equation}
1 + D\xi_p(y) = \frac{\xi_p(y) + y}{ph(y) + y}, \quad y \geq \varepsilon_0.
\end{equation}

Since $ph \to \infty \leq \xi_p(y) \leq ph(y)$, $y \geq \varepsilon_0$, it follows from (4.11) that there are positive upper and lower bounds on $1 + D\xi_p(y)$, which are uniform for $y \geq \varepsilon_0$. Now from (4.10) we infer that the denominator is finite and positive.

Next we show that $K(\cdot)$ is a positive function. It is evident from (1.25), (4.11) that $A\xi_p(\cdot)$ is positive and $\|A\xi_p\|_{\infty} < \infty$. We can also see that the function $BA\xi_p(\cdot)$ is positive. To show this we use the commutation relation
\begin{equation}
BA - AB = [h(y) - y h'(y)] \frac{d}{dy}.
\end{equation}

From (1.25), (4.12) we have that
\begin{equation}
BA\xi_p(y) = Ah(y) + [h(y) - y h'(y)] \frac{d\xi_p(y)}{dy} = [h(y) - y h'(y)][1 + D\xi_p(y)],
\end{equation}
whence the positivity of $BA\xi_p(\cdot)$ follows from (4.11), (4.13). Since $dI_p(\xi_p)$ is a negative function, the positivity of $BA\xi_p(\cdot)$ implies the positivity of $K(\cdot)$. Note that the first inequality of (4.9) is needed in order to guarantee that $K(t)$ is finite for $t \geq 0$.

To show that the function $t \to e^{t/p} K(t)$ is decreasing we need to show that
\begin{equation}
(B - 1/p)BA\xi_p(y) = B \left[ \frac{1}{p} A\xi_p(y) - y h'(y) \frac{d\xi_p(y)}{dy} - y h'(y) \right] - \frac{1}{p} BA\xi_p(y)
\end{equation}
\begin{equation}
= \left[ \frac{y}{p} + h(y) \right] [y h''(y) + h'(y)] \frac{d\xi_p(y)}{dy} - y h'(y) B \frac{d\xi_p(y)}{dy} - B[yh'(y)].
\end{equation}
is positive. Using the commutator relation
\begin{equation}
B \frac{d}{dy} - \frac{d}{dy} B = \left[ \frac{1}{p} + h'(y) \right] \frac{d}{dy},
\end{equation}
we have from (4.14) that
with respect to the space variable, we have from (4.19) that
\[ C \]
where
\[ C \]
the function
\[ t \]
u
solution
\[ We conclude that the formula (4.8) for
\[ (4.21) \]
\[ \| \]
It follows now from (4.18), (4.20) that
\[ \| \]
\[ Now the second inequality of (4.9) and (4.11) imply that the expression in (4.16) is non-negative.

Similarly to Proposition 3.1 of [3] we have the following:

**Proposition 4.1.** Assume \( h(\cdot) \) satisfies the conditions of Lemma 4.1. Then the linear evolution equation (4.6) is asymptotically stable in the following sense: Let the initial data \( \tilde{\xi}_0 : [\varepsilon_0, \infty) \to \mathbb{R} \) satisfy \( \| \tilde{\xi}_0(\cdot) \|_{1, \infty} < \infty \). Then there is for any \( q > p \) a constant \( C_q \) depending on \( q \) such that
\[ (4.17) \]
\[ \| \tilde{\xi}(\cdot, t) \|_{1, \infty} \leq C_q e^{-t/q} \| \tilde{\xi}_0(\cdot) \|_{1, \infty} \quad \text{when } t \geq 0. \]

**Proof.** Observe that for a function \( \zeta : [\varepsilon_0, \infty) \to \mathbb{R} \) we have
\[ (4.18) \]
\[ e^{-Bt} \zeta(y) = e^{-t/p} \zeta(y_p(0)), \quad y \geq \varepsilon_0, \]
where \( y_p(\cdot) \) is the solution of (3.3) with \( \rho(\cdot) \equiv 1/p \). Evidently we have from (4.18) that \( \| e^{-Bt} \zeta(\cdot) \|_{1, \infty} \leq e^{-t/p} \| \zeta(\cdot) \|_{1, \infty} \). From (3.22) we have that
\[ (4.19) \]
\[ \frac{\partial}{\partial y} e^{-Bt} \zeta(y) = \exp \left[ \int_0^t h'(y_p(s)) \, ds \right] D\zeta(y_p(0)). \]

From (3.14) we see that \( y_p(0) \geq e^{t/p} y, \quad y \geq \varepsilon_0, \) whence we conclude from (4.19) that
\[ (4.20) \]
\[ \sup_{y \geq \varepsilon_0} \left| \frac{\partial}{\partial y} e^{-Bt} \zeta(y) \right| \leq e^{-t/p} \sup_{y \geq \varepsilon_0} |y D\zeta(y)|. \]

It follows now from (4.18), (4.20) that \( \| e^{-Bt} \zeta(\cdot) \|_{1, \infty} \leq e^{-t/p} \| \zeta(\cdot) \|_{1, \infty} \) for \( t \geq 0 \).
We conclude that the formula (4.13) for \( g(t) \) is bounded as
\[ (4.21) \]
\[ \left| [dI_p(\xi_p), e^{-Bt} B \tilde{\xi}(0)] \right| \leq C e^{-t/p} \| \tilde{\xi}_0(\cdot) \|_{1, \infty} \quad \text{for } t \geq 0, \]
where \( C \) is a constant.

From Lemma 4.1 and results on Volterra integral equations [5], we see that the solution \( u(t), \ t \geq 0, \) of (4.7) with \( K(\cdot), \ g(\cdot) \) given by (4.8), has the property that the function \( t \to e^{t/q} u(t) \) is bounded by a constant times \( \| \tilde{\xi}_0(\cdot) \|_{1, \infty} \) for any \( q > p \). It follows now, upon using (4.18) to estimate the RHS of (4.16), that \( \| \tilde{\xi}(\cdot, t) \|_{1, \infty} \leq C_q e^{-t/q} \| \tilde{\xi}_0(\cdot) \|_{1, \infty} \) for some constant \( C_q \) depending on \( q > p \). Differentiating (4.6) with respect to the space variable, we have from (4.11) that
\[ (4.22) \]
\[ D\tilde{\xi}(y, t) = \exp \left[ \int_0^t h'(y_p(s)) \, ds \right] D\tilde{\xi}_0(y_p(0)) + \int_0^t ds \exp \left[ \int_s^t h'(y_p(s')) \, ds' \right] [dI_p(\xi_p), B\tilde{\xi}(s)] \frac{DA\xi_p(y_p(s))}{pI_p(\xi_p) + [dI_p(\xi_p), A\xi_p]} . \]
To estimate the RHS of (4.22) we note on differentiating (4.11) that
where
\[(4.28)\]
\[yDAξ_p(y) = -y^2D^2ξ_p(y)\]
\[= \frac{[ph'(y) + 1]y^2ξ_p(y)}{|ph(y) + y|^2} - \frac{y^2Dξ_p(y)}{ph(y) + y} - \frac{py^2[h(y) - yh'(y)]}{|ph(y) + y|^2}.
\]
We may bound the RHS of \(4.28\) using \(3.33\) to obtain the inequality
\[(4.24)\]
\[
\int_{ε_0}^{ε_0 + 1} |yDAξ_p(y)| \, dy + \sup_{y ≥ ε_0 + 1} |yDAξ_p(y)| < ∞.
\]
Using \(4.24\) to bound the RHS of \(4.22\), we conclude that
\[
(4.25)\]
\[
\sup_{y > ε} \left[\frac{y}{dI} \right] \leq C_ε e^{-1/ε} \|ξ_0(·)\|_{∞} \quad \text{for} \ t ≥ 0,
\]
where \(C_ε\) is a constant depending only on \(q > p\).

We generalize the result of Proposition 4.1 to apply to the non-linear PDE \(1.4\) by considering \(4.1\) as a perturbation of \(4.5\) with \(γ(·) \equiv 0\) of the form
\[(4.26)\]
\[
\frac{d\bar{ξ}(t)}{dt} + [B + \{γ(t) + δ_1(\bar{ξ}(t))\}A]\bar{ξ}(t)
+ \left\{γ(t) - \frac{[dI(ξ_p), B\bar{ξ}(t)] + δ_2(\bar{ξ}(t))}{pI(ξ_p) + |dI(ξ_p), Aξ_p|} \right\} Aξ_p = 0,
\]
where \(δ_1(·), δ_2(·)\) are real valued functionals of \(C^1\) functions \(\tilde{ζ} : [ε_0, ∞) → R\). If we take
\[(4.27)\]
\[
δ_1(\tilde{ζ}(·)) = -\frac{[dI_p(ξ_p + \tilde{ζ}), B\tilde{ζ}]}{pI_p(ξ_p + \tilde{ζ}) + [dI_p(ξ_p + \tilde{ζ}), A\{ξ_p + \tilde{ζ}\}]},
\]
and
\[(4.28)\]
\[
δ_2(\tilde{ζ}(·)) = \frac{pI_p(ξ_p + \tilde{ζ}) + [dI_p(ξ_p), Aξ_p]}{pI_p(ξ_p + \tilde{ζ}) + [dI_p(ξ_p + \tilde{ζ}), A\{ξ_p + \tilde{ζ}\}]} - \frac{[dI_p(ξ_p), B\tilde{ζ}]}{[dI_p(ξ_p + \tilde{ζ}), B\tilde{ζ}]},
\]
then \(4.3, 4.24\) are equivalent.

**Lemma 4.2.** Assume \(h(·)\) satisfies \(1.24\). Then the functionals \(δ_1(·), δ_2(·)\) defined by \(4.27\), \(4.28\) are Lipschitz continuous close to \(ζ(·) \equiv 0\) in the \(m = 1\) norm \(3.7a\).

In particular, there exist constants \(C, ν > 0\) such that
\[(4.29)\]
\[
|δ_1(\tilde{ζ}_1) - δ_1(\tilde{ζ}_2)| \leq C\|\tilde{ζ}_1 - \tilde{ζ}_2\|_{1, ∞},
\]
\[
|δ_2(\tilde{ζ}_1) - δ_2(\tilde{ζ}_2)| \leq C\{\|\tilde{ζ}_1\|_{1, ∞} + \|\tilde{ζ}_2\|_{1, ∞}\} \|\tilde{ζ}_1 - \tilde{ζ}_2\|_{1, ∞},
\]
provided \(\|\tilde{ζ}_j\|_{1, ∞} < ν, \ j = 1, 2\).

**Proof.** The function \(I_p(ζ(·))\) defined by \(2.12\) is infinitely differentiable with respect to \(ζ(·)\), and has the property there exist constants \(C_1, ν_1 > 0\) such that
\[(4.30)\]
\[
\left[I_p(ζ_p + \tilde{ζ}_1) - I_p(ζ_p + \tilde{ζ}_2)\right] + \|I_p(ζ_p + \tilde{ζ}_1) - dI_p(ζ_p + \tilde{ζ}_1)\|_{L^1(ε_0, ∞)}
\leq C_1\|\tilde{ζ}_1 - \tilde{ζ}_2\|_{∞} \quad \text{for} \ \|\tilde{ζ}_j\|_{∞} < ν_1, \ j = 1, 2.
\]

The result follows from \(4.30\) and the inequality
\[
\sup_{y > ε_0} |Bζ(y)| \leq \left[1/p + \sup_{y ≥ ε_0} h(y)/y\right] \|ζ(·)\|_{1, ∞}.
\]

□
Let $\delta : [0, \infty) \to \mathbb{R}$ be a continuous function and consider the linear PDE
\begin{equation}
(4.31) \quad \frac{d\tilde{\xi}(t)}{dt} + [B + \delta(t)A] \tilde{\xi}(t) - [dI_p(\xi_p), B\tilde{\xi}(t)] \frac{A\tilde{\xi}_p}{pI_p(\xi_p) + [dI_p(\xi_p), A\tilde{\xi}_p]} = 0.
\end{equation}

The results of Proposition 4.1 extend to solutions of (4.31) provided $\|\delta(\cdot)\|_\infty$ is sufficiently small. Parallel to Lemma 3.3 of [3] we have the following:

**Lemma 4.3.** Assume $h(\cdot)$ satisfies the conditions of Lemma 4.1. Assume further that $\delta : [0, \infty) \to \mathbb{R}$ is a continuous function and $\|\delta(\cdot)\|_\infty \leq 1/p$. Then the linear evolution equation (4.31) with initial data $\tilde{\xi}_0 : [\varepsilon_0, \infty) \to \mathbb{R}$ satisfying $\|\tilde{\xi}_0(\cdot)\|_{1, \infty} < \infty$ has a unique solution globally in time, $\tilde{\xi}(y, t; \delta(\cdot))$, $y \geq \varepsilon_0, t \geq 0$, which satisfies $\|\tilde{\xi}(\cdot, t; \delta(\cdot))\|_{1, \infty} < \infty$ for all $t \geq 0$. For any $q > p$ there exists $C_q, \varepsilon_q > 0$ such that if $\|\delta(\cdot)\|_\infty < \varepsilon_q$ then
\begin{equation}
(4.32) \quad \|\tilde{\xi}(\cdot, t; \delta(\cdot))\|_{1, \infty} \leq C_q e^{-t/q} \|\tilde{\xi}_0(\cdot)\|_{1, \infty} \text{ when } t \geq 0.
\end{equation}

We apply Lemma 4.3 as in [3] to obtain bounds on solutions to the non-linear PDE (4.26).

**Theorem 4.1.** Assume $h(\cdot)$ satisfies the conditions of Lemma 4.1 and $\gamma : [0, \infty) \to \mathbb{R}$ is a continuous function satisfying $\lim_{t \to \infty} \gamma(t) = 0$. Let $\delta_1(\cdot), \delta_2(\cdot)$ be real valued functionals of $C^1$ functions $\tilde{\xi} : [\varepsilon_0, \infty) \to \mathbb{R}$, which satisfy $\delta_1(0) = \delta_2(0) = 0$ and the local Lipschitz conditions (4.26). Then there exists $\varepsilon > 0$ such that the solution $\tilde{\xi}(\cdot, t)$, $t \geq 0$, to the nonlinear evolution equation (4.26) with initial data $\tilde{\xi}_0 : [\varepsilon_0, \infty) \to \mathbb{R}$ satisfying $\|\tilde{\xi}_0(\cdot)\|_{1, \infty} + \|\gamma(\cdot)\|_\infty < \varepsilon$ has the property $\lim_{t \to \infty} \|\tilde{\xi}(\cdot, t)\|_{1, \infty} = 0$. If $|\gamma(t)| \leq C e^{-\nu t}$, $t \geq 0$, for some constants $C, \nu > 0$ with $\nu < 1/p$ then $\varepsilon = \varepsilon_\nu$ can be chosen depending on $\nu$ so that $\|\tilde{\xi}(\cdot, t)\|_{1, \infty} \leq C_\nu e^{-\nu t}$, $t \geq 0$, for some constant $C_\nu$.

**Proof.** We define the Green’s function for (4.31) as $G(t, s; \delta(\cdot))$, $0 \leq s \leq t$, so $\tilde{\xi}(t) = G(t, s; \delta(\cdot))\tilde{\xi}_0, t \geq s$, is the solution to (4.31) with initial data $\tilde{\xi}(s) = \tilde{\xi}_0$. The solution to (4.26) then satisfies the identity
\begin{equation}
(4.33) \quad \tilde{\xi}(t) = G(t, 0; \gamma(\cdot) + \delta_1(\cdot))\tilde{\xi}_0 + \int_0^t \frac{\delta_2(\tilde{\xi}(s))}{pI(\xi_p) + [dI_p(\xi_p), A\xi_p]} - \gamma(s) G(t, s; \gamma(\cdot) + \delta_1(\cdot))A\xi_p.
\end{equation}

We estimate $\|\tilde{\xi}(t)\|_{1, \infty}$ just as in the proof of Theorem 3.1 of [3]. \hfill \Box

5. A Differential Delay Equation

We shall formulate the problem of solving the initial value problem for (5.1), with $\rho(t) = \rho(\xi(\cdot, t), \eta(t))$ given by (2.20), as an initial value problem for a differential delay equation (DDE). To do this we define a function $I : [0, \infty) \to \mathbb{R}^+$ by integration of the function $\rho(\cdot)$. Thus solving the initial value problem for the equation
\begin{equation}
(5.1) \quad \frac{1}{p} \frac{d}{dt} \log I(t) = \rho(t) - \frac{1}{p}, \quad t > 0, \ I(0) > 0,
\end{equation}
we obtain a formula for \( I(\cdot) \) in terms of \( \rho(\cdot) \) as

\[
\exp \left[ \int_s^t \rho\left(s^'\right) \, ds^' \right] = e^{\frac{(t-s)}{p} \left( \frac{I(t)}{I(s)} \right)^{1/p}}.
\]

We rewrite the characteristic equation (5.3) using the function \( I(\cdot) \) by setting

\[
y(s) = \frac{z(s)}{v_t(s)}, \quad v_t(s) = \left( \frac{I(s)}{I(t)} \right)^{1/p}, \quad 0 \leq s \leq t.
\]

From (5.1), (5.3) we see that (3.3) is equivalent to

\[
\frac{dz(s)}{ds} = \frac{z(s)}{p} - v_t(s)h(s), \quad s < t, \quad z(t) = y.
\]

The DDE is now obtained by rewriting (5.1) with \( \rho(t) = \rho(\xi(t), \eta(t)) \) in the form

\[
\frac{1}{p} \frac{d}{dt} \left[ \log I(t) \right] = \rho_p(\xi(t), t) - \frac{1}{p} + \gamma(t),
\]

where \( \gamma(t) = \rho(\xi(t), \eta(t)) - \rho_p(\xi(t), t), \ t \geq 0, \) is assumed to be a known continuous function satisfying \( t \to \infty \gamma(t) = 0. \) As with (4.4), equation (5.3) can be seen to be equivalent to

\[
\frac{1}{p} \frac{d}{dt} \log I(t) = -\left[ \frac{dI_p(\xi(t), t)}{I_p(\xi(t), t)} + |dI_p(\xi(t), t)| + A\xi(t) \right] + \gamma(t).
\]

We see from the representations (5.4) for \( \xi(\cdot, t) \) and (5.22) for \( D\xi(\cdot, t) \) that the first term on the RHS of (5.6) is a function of \( v_t(\cdot) \), whence (5.6) is a DDE.

We define a function \( F(t, y, v_t(\cdot)) \) by

\[
F(t, y, v_t(\cdot)) = \frac{1}{p} \int_0^t h(s) \left( \frac{z(s)}{v_t(s)} \right) e^{-\frac{(t-s)}{p}v_t(s)} ds
\]

\[
- \left[ h(y) + \frac{y}{p} \left\{ \exp \left[ \int_0^t h(s) \left( \frac{z(s)}{v_t(s)} \right) ds \right] - 1 \right\} \right.
\]

We wish to estimate \( F(t, y, 1(\cdot)) - h(y) \). In order to do this we denote by \( z_p(s), \ s \leq t \), the solution to (5.4) when \( v_t(\cdot) \equiv 1 \). From (5.22) we see that \( F(t, y, 1(\cdot)) = B\tilde{F}(t, y) \) where

\[
\tilde{F}(t, y) = \int_0^t ds \ h(z_p(s)) e^{-\frac{(t-s)}{p}}.
\]

We compare \( \tilde{F}(t, y) \) to the formula (1.27) for \( \xi_p(y) \) by making the change of variable \( y' = z_p(s) \). Then (5.3) is the same as

\[
\tilde{F}(t, y) = \int_y^{z_p(0)} dy' \frac{ph(y')}{ph(y') + y'} e^{-\frac{(t-s)}{p}}.
\]

Next from (1.26) and (5.4) with \( v_t(\cdot) \equiv 1 \) we have that

\[
e^{-\frac{(t-s)}{p}} = \frac{\tilde{h}(y)}{\tilde{h}(y')}.
\]

We conclude from (1.27), (5.9), (5.10) that

\[
\tilde{F}(t, y) - \xi_p(y) = -\tilde{h}(y) \int_{z_p(0)}^\infty \frac{ph(y')}{(ph(y') + y') \tilde{h}(y')} dy'.
\]
Applying $B$ to (5.11) we obtain the formula

$$
(5.12) \quad F(t, y, 1(\cdot)) - h(y) = \frac{e^{t/p} [ph(y) + y h(y) h(z_p(0))]}{[ph(z_p(0)) + z_p(0)] h(z_p(0))} \exp \left[ \int_0^t h'(z_p(s)) \, ds \right]
$$

$$
= - \frac{[ph(y) + y h(z_p(0))]}{[ph(z_p(0)) + z_p(0)]} \exp \left[ \int_0^t h'(z_p(s)) \, ds \right].
$$

Since $z_p(0) \simeq e^{t/p}$ at large $t$, it is clear from (5.12) that $F(t, y, 1(\cdot)) - h(y)$ decays like $e^{-t/p}$ as $t \to \infty$. We conclude from (5.4), (5.23) (5.7), (5.12) that

$$
(5.13) \quad B \xi(y, t) - B_z p(y) = F(t, y, v_1(\cdot)) - F(t, y, 1(\cdot)) + G(t, y, v_1(\cdot)),
$$

where $G$ is given by the formula

$$
(5.14) \quad G(t, y, v_1(\cdot)) = \frac{1}{p} e^{-t/p} v_1(0) \xi \left( \frac{z(0)}{v_1(0)} \right) \left[ \int_0^t h'(z(s)/v_1(s)) \, ds \right] \frac{D_z \xi \left( \frac{z(0)}{v_1(0)} \right)}{v_1(0)}
$$

$$
- \left( h(y) + \frac{y}{p} \right) \exp \left[ \int_0^t h'(z(s)/v_1(s)) \, ds \right] \frac{D_z \xi \left( \frac{z(0)}{v_1(0)} \right)}{v_1(0)}
$$

$$
- \frac{[ph(y) + y h(z_p(0))]}{[ph(z_p(0)) + z_p(0)]} \exp \left[ \int_0^t h'(z_p(s)) \, ds \right].
$$

From (5.13) we may rewrite the DDE equation (5.6) as

$$
(5.15) \quad \frac{1}{p} \frac{d}{dt} \log I(t) + f(t, v_1(\cdot)) = g(t, v_1(\cdot)) + \gamma(t),
$$

where the functions $f, g$ are given by the formulae

$$
(5.16) \quad f(t, v_1(\cdot)) = \frac{[dI_p(\xi(\cdot), t), F(t, v_1(\cdot)), F(t, , 1(\cdot))] + [dI_p(\xi(\cdot), t), A(\cdot, t)]}{pI_p(\xi(\cdot), t) + [dI_p(\xi(\cdot), t), A(\cdot, t)]},
$$

$$
g(t, v_1(\cdot)) = - \frac{[dI_p(\xi(\cdot), t), G(t, v_1(\cdot))]}{pI_p(\xi(\cdot), t) + [dI_p(\xi(\cdot), t), A(\cdot, t)]}.
$$

In the proof of global asymptotic stability for the DDE (5.13) the key property which needs to be established is monotonicity of the function $f(t, v_1(\cdot))$ in the following sense: $f(t, v_1(\cdot)) \geq 0$ on the set $0 < v_1(\cdot) \leq 1$, and $f(t, v_1(\cdot)) \leq 0$ on the set $v_1(\cdot) \geq 1$. This property of $f(t, v_1(\cdot))$ holds to first order in $v_1(\cdot) - 1$ provided the gradient $dF(t, y, v_1(\cdot), \tau), 0 < \tau < t$, of $F(t, y, v_1(\cdot))$ with respect to $v_1(\cdot)$ at $v_1(\cdot) \equiv 1$ is non-negative for $y \geq \varepsilon_0$. It holds to all orders if we can show that

$$
(5.17) \quad \sup_{0 < v_1(\cdot) < 1} F(t, y, v_1(\cdot)) = F(t, y, 1(\cdot)) \quad \text{for } y \geq \varepsilon_0,
$$

$$
(5.18) \quad \inf_{1 < v_1(\cdot) < \infty} F(t, y, v_1(\cdot)) = F(t, y, 1(\cdot)) \quad \text{for } y \geq \varepsilon_0.
$$

Using the results of §6 we have the following:

**Proposition 5.1.** Assume the function $h(\cdot)$ satisfies the assumptions of Lemma 4.1, in particular that (4.9) holds. Then $dF(t, y, 1(\cdot), \cdot)$ is a non-negative function for all $t > 0$, $y \geq \varepsilon_0$. If in addition the second inequality of (4.9) holds for all $p > 0$ then (5.17) is true. If the function

$$
(5.19) \quad y \to \frac{y^2 h''(y)}{h(y) - y h'(y)} \text{ decreases for } y > \varepsilon_0
$$

then (5.18) is true.
Proof. We see that the first integral in the formula (6.7) for \( F(t, y, v(\cdot)) \) is the same as (6.1) with \( q(\cdot) \equiv h(\cdot)/p \). The condition (6.13) for non-negativity of the gradient is trivial in this case. To show non-negativity of the gradient of the second integral in (6.7) at \( v(\cdot) \equiv 1 \), one considers the functional (6.29) with \( g(\cdot) \equiv -h(\cdot) \). In this case the second inequality of (6.30) and (6.31) are equivalent. Hence \( dF(t, y, 1(\cdot); \cdot) \) is non-negative if (6.11) holds.

The identity (6.17) follows from propositions 6.1, 6.2 and the associated remarks. The identity (5.18) is a consequence of Proposition 6.3 and associated remarks. Note that the function \( h(\cdot) \) has domain \([\varepsilon_0, \infty)\) and we wish to allow \( h'(y) \) to diverge as \( y \to \varepsilon_0 \). We can still apply propositions 6.1-6.3 in this case by approximating \( h(\cdot) \) with functions which have domain \((0, \infty)\) and also preserve the monotonicity and convexity properties required for the propositions. □

### 6. Some Optimal Control Problems

Let \( y > 0, T \in \mathbb{R} \), and consider the dynamics

\[
\frac{dx(s)}{ds} = -\frac{1}{p} x(s) - v(s)h\left(\frac{x(s)}{v(s)}\right), \quad s < T, \quad x(T) = y,
\]

with terminal condition \( x(T) = y \) and controller \( v(\cdot) \). The function \( h(\cdot) \) is assumed to be positive and decreasing. Let \( g : (0, \infty) \to \mathbb{R}^+ \) be a positive decreasing function and for \( y > 0, t < T \) define the function

\[
q(x, y, t, T) = \max_{0 < v(\cdot) \leq 1} \left[ \int_t^T ds \ g \left( \frac{x(s)}{v(s)} \right) e^{-p(T-s)/p} \right] \bigg| x(t) = x.
\]

The reachable set for the control problem (6.2) is the set of \((x, t)\) which satisfy

\[
\frac{e^{(T-t)/p}}{x} \leq \frac{x(t)}{v(t)} \quad \text{for all } t < T.
\]

where \( x_p(\cdot) \) is the solution to (6.1) when \( v(\cdot) \equiv 1 \). Letting

\[
q(y, v(\cdot), t, T) = \int_t^T ds \ g \left( \frac{x(s)}{v(s)} \right) e^{-p(T-s)/p} v(s),
\]

we have that the gradient \( dq \) of \( q \) with respect to \( v(\cdot) \) is given by

\[
e^{-p(T-\tau)/p} dq(y, v(\cdot), t; \tau) = -\frac{x(\tau)}{v(\tau)} g' \left( \frac{x(\tau)}{v(\tau)} \right) + g \left( \frac{x(\tau)}{v(\tau)} \right) + \int_t^\tau ds \ g' \left( \frac{x(s)}{v(s)} \right) e^{-p(T-s)/p} dx(s)(v(\cdot); \tau),
\]

where \( dx(s)(v(\cdot); \tau), s < \tau < T \), is the gradient of \( x(s) \) with respect to \( v(\cdot) \). We obtain a formula for \( dx(s)(v(\cdot); \cdot) \) by observing that for any function \( \phi : (-\infty, T) \) the inner product

\[
u_{\phi}(s) = [dx(s)(v(\cdot)), \phi] = \int_{-\infty}^T dx(s)(v(\cdot); \tau) \phi(\tau) \, d\tau,
\]

is a solution to the terminal value problem,

\[
\frac{d u_{\phi}(s)}{ds} = -\frac{u_{\phi}(s)}{p} - \phi(s)h\left(\frac{x(s)}{v(s)}\right)
\]
Evidently $dx(s)v(s) - x(s)\phi(s) v(s)^2$ for $s < T$, $u_\phi(T) = 0$.

Integrating (6.5) we obtain the formula

$$u_\phi(s) = \int_s^T \tau = \frac{s}{p} + \int_s^T \tau h'(x(s')) v(s') \: ds' \right] \times$$

$$\left\{h \left( \frac{x(\tau)}{v(\tau)} \right) - \frac{x(\tau)}{v(\tau)} h' \left( \frac{x(\tau)}{v(\tau)} \right) \right\} \phi(\tau).$$

Evidently $dx(s)v(\cdot;\tau)$ is the coefficient of $\phi(\tau)$ in the integral on the RHS of (6.8).

We conclude then from (6.5), (6.8) that

$$e^{(T-\tau)/p} dq(y, v(\cdot), t, T; \tau) = -x_p(\tau)g'(x_p(\tau)) + g(x_p(\tau))$$

$$+ \int_t^\tau ds.g'(x_p(s)) \exp \left[ \int_s^\tau h'(x(s')) v(s') \: ds' \right] \{h(\tau) - x_p(\tau)h'(x_p(\tau))\}.$$ 

The first two terms on the RHS of (6.9) are positive but the third term is negative. Setting $v(\cdot) \equiv 1$ in (6.9) we have that

$$e^{(T-\tau)/p} dq(y, 1(\cdot), t, T; \tau) = -x_p(\tau)g'(x_p(\tau)) + g(x_p(\tau))$$

$$+ \int_t^\tau ds.g'(x_p(s)) \exp \left[ \int_s^\tau h'(x(s')) v(s') \: ds' \right] \{h(\tau) - x_p(\tau)h'(x_p(\tau))\}.$$ 

We have now on integration by parts and using the ODE (6.1) with $v(\cdot) \equiv 1$ which $x_p(\cdot)$ satisfies that

$$\int_t^\tau ds.g'(x_p(s)) \exp \left[ \int_s^\tau h'(x(s')) v(s') \: ds' \right] =$$

$$\frac{p}{x_p(t)+ph(x_p(t))} \exp \left[ \int_t^\tau h'(x_p(s')) v(s') \: ds' \right] g(x_p(t)) - \frac{p}{x_p(\tau)+ph(x_p(\tau))} g(x_p(\tau))$$

$$+ \int_t^\tau ds.\frac{1}{x_p(s)+ph(x_p(s))} g(x_p(s)) \exp \left[ \int_s^\tau h'(x(s')) v(s') \: ds' \right].$$

It follows that $dq(y, 1(\cdot), t, T; \cdot)$ is non-negative provided

$$-xg'(x) + g(x) - \frac{g(x)}{x/p + h(x)} [h(x) - xh'(x)] \geq 0 \text{ for } x > 0.$$ 

Evidently (6.12) holds for all $p > 0$ provided

$$\frac{g'(x)h(x) - g(x)h'(x)}{x} \leq 0 \text{ for } x > 0.$$ 

The Hamilton-Jacobi (HJ) equation associated with (6.2) is given by

$$\frac{\partial^2 (x, y, t)}{\partial t} - \frac{x}{p} \frac{\partial q(x, y, t)}{\partial x}$$

$$+ \sup_{0 < v < 1} \left[ -vh \frac{1}{v} \frac{\partial q(x, y, t, T)}{\partial x} + e^{-(T-t)/p} g \left( \frac{x}{v} \right) v \right] = 0.$$ 

We shall obtain the solution to the variational problem (6.2) by producing a $C^1$ solution to the HJ equation (6.13). The solution is obtained by using bang-bang control settings. Thus for $(x, t)$ in the reachable set (6.3) we define $\tau_{x,t}$ as the time
at which the trajectory for (6.1) with \( v(\cdot) \equiv 0 \) and \( x(t) = x \) reaches the curve \( x_p(\cdot) \). Hence \( \tau_{x,t} \) satisfies the identity

\[
e^{-\tau_{x,t}/p_x} = x_p(\tau_{x,t}).
\]

Then we set

\[
q(x,y,t,T) = \int_{\tau_{x,t}}^{T} ds \ g(x_p(s)) e^{-\left(T-s\right)/p} \ ds.
\]

**Proposition 6.1.** Assume \( g(\cdot) \), \( h(\cdot) \) are \( C^1 \) non-negative decreasing functions such that (6.13) holds. For any \( y > 0 \), \( t < T \), let \( (x,t) \) satisfy (6.3) and \( \tau_{x,t} \) be defined by (6.15). Then \( t < \tau_{x,t} < T \) and the function \( q(x,y,t,T) \) of (6.3) is given by the formula (6.16).

**Proof.** Differentiating (6.15) with respect to \( x \) we have that

\[
\frac{\partial \tau_{x,t}}{\partial x} = - \frac{x_p(\tau_{x,t})}{x h(x_p(\tau_{x,t}))}, \quad \frac{\partial \tau_{x,t}}{\partial t} = - \frac{x_p(\tau_{x,t})}{p h(x_p(\tau_{x,t}))}.
\]

We have from (6.16) that

\[
\frac{\partial q(x,y,t,T)}{\partial x} = - \exp\left[-\frac{(T-\tau_{x,t})}{p}\right] g(x_p(\tau_{x,t})) \frac{\partial \tau_{x,t}}{\partial x},
\]

and similarly that

\[
\frac{\partial q(x,y,t,T)}{\partial t} = - \exp\left[-\frac{(T-\tau_{x,t})}{p}\right] g(x_p(\tau_{x,t})) \frac{\partial \tau_{x,t}}{\partial t}.
\]

It follows from (6.17), (6.19) that \( q \) is a solution to the PDE

\[
\frac{\partial q(x,y,t,T)}{\partial t} - \frac{x}{p} \frac{\partial q(x,y,t,T)}{\partial x} = 0.
\]

Since \( q \) is a solution to (6.20) we need only show that

\[
e^{-\left(T-t\right)/p} g\left(\frac{x}{v}\right) \leq h\left(\frac{x}{v}\right) \frac{\partial q(x,y,t,T)}{\partial x} \quad \text{for } 0 < v < 1, \ x > 0,
\]

in order to prove that \( q \) is a solution to the HJ equation (6.13). Observe now that since (6.13) holds the function \( x \to g(x)/h(x), \ x > 0 \), is decreasing. Hence to prove (6.21) it is sufficient to show that

\[
u(x,t) = \frac{\partial q(x,y,t,T)}{\partial x} - e^{-\left(T-t\right)/p} \frac{g(x)}{h(x)} \geq 0 \quad x > 0.
\]

Observe next from (6.17), (6.18) that \( \frac{\partial q(x,y,t,T)}{\partial x} \) approaches \( e^{-\left(T-t\right)/p} g(x_p(t))/h(x_p(t)) \) as \( x \to x_p(t) \). It follows from (6.22) that \( u(x,t) = 0 \) if \( x = x_p(t) \). We have now on differentiating (6.20) with respect to \( x \) that

\[
\frac{\partial u(x,t)}{\partial t} - \frac{x}{p} \frac{\partial u(x,t)}{\partial x} - \frac{1}{p} u(x,t) = \frac{x}{p} e^{-\left(T-t\right)/p} \frac{d \ g(x)}{dx} h(x) \leq 0.
\]

It follows by the method of characteristics that \( u(x,t) \geq 0 \) for all \( (x,t) \) satisfying (6.3). Hence the function \( q \) of (6.16) is a solution to the HJ equation (6.13). Since \( q \) is a \( C^1 \) solution to the HJ equation for \( (x,t) \) in the reachable set we see that the solution to the variational problem (6.2) is given by (6.10).

**Remark 3.** Since the function (6.10) satisfies \( \frac{\partial q(x,y,t,T)}{\partial x} \geq 0 \), it follows that the solution of the variational problem \( \max_{0 < v < 1} q(y,v(\cdot),t,T) \), with \( q \) as in (6.4), is given by \( v(\cdot) \equiv 1 \).
Next we consider the variational problem analogous to (6.2) given by

\[ q(x, y, t, T) = \min_{0 < v(\cdot) \leq 1} \left[ \int_t^T ds \ g \left( \frac{x(s)}{v(s)} \right) e^{-(T-s)/p v(s)} \right] \left. \right| x(t) = x . \]  

The reachable set for the control problem (6.24) is the set of \((x, t)\) which satisfy

\[ x_p(t) < x < x_p(t) \quad \text{for} \quad t < T . \]

The HJ equation corresponding to (6.24) is given by

\[ \frac{\partial q(x, y, t, T)}{\partial t} - \frac{x}{p} \frac{\partial q(x, y, t, T)}{\partial x} + \inf_{1 < v < \infty} \left[ -v h(x, t, p, g) \right] = = 0 . \]

The minimization problem (6.24) is trivial in the case \(g(\cdot) \equiv C_0 h(\cdot)\) for some constant \(C_0\), since then the function \(q(y, v(\cdot), t, T)\) of (6.4) is independent of \(v(\cdot)\). In fact if \(g(\cdot) \equiv h(\cdot)\) we have from (6.4) that

\[ q(y, v(\cdot), t, T) = - \int_t^T ds e^{-(T-s)/p v} \left[ \frac{dx(s)}{v(s)} + x(s) \right] . \]

Evaluating the integral on the RHS of (6.22), we conclude that \(q(x, y, t, T) = = e^{-(T-t)/p} x - y\) for \((x, t)\) in the reachable set (6.25). Note that since \(\partial q(x, y, t, T)/\partial x = e^{-(T-t)/p}\), the infimum in (6.20) is now simply zero.

Next we consider the optimization problem

\[ q(x, y, t, T) = \max_{0 < v(\cdot) \leq 1} \left[ \int_t^T ds \ g \left( \frac{x(s)}{v(s)} \right) \right] \left. \right| x(t) = x . \]

where \(x(\cdot)\) has the dynamics (6.2) and \(g(\cdot)\) is assumed positive decreasing with \(\lim_{x \to \infty} g(x) = 0\). The reachable set for the control problem (6.24) is given by (6.3). Letting

\[ q(y, v(\cdot), t, T) = \int_t^T g \left( \frac{x(s)}{v(s)} \right) ds , \]

we have that \(dq\) is given by the formula

\[ dq(y, v(\cdot), t, T; \tau) = \]

\[ - \frac{x(\tau)}{v(\tau)^2} g' \left( \frac{x(\tau)}{v(\tau)} \right) + \int_t^\tau ds g' \left( \frac{x(s)}{v(s)} \right) \frac{dx(s)/v(\cdot; \tau)}{v(s)} . \]

Observe now that similarly to (6.11) we have

\[ \int_t^\tau ds g' \left( x_p(s) \right) \exp \left[ (\tau - s)/p + \int_s^\tau h' \left( x_p(s') \right) ds' \right] = \frac{g \left( x_p(t) \right)}{x_p(t)/p + h \left( x_p(t) \right)} \exp \left[ (\tau - t)/p + \int_t^\tau h' \left( x_p(s') \right) ds' \right] - \frac{g \left( x_p(\tau) \right)}{x_p(\tau)/p + h \left( x_p(\tau) \right)} . \]

From (6.8), (6.31) we then see on setting \(v(\cdot) \equiv 1\) in (6.30) that

\[ dq(y, 1(\cdot), t, T; \tau) \geq \]
\(-x_p(\tau)g'(x_p(\tau)) - \frac{g(x_p(\tau))}{x_p(\tau)/p + h(x_p(\tau))} \{h(x_p(\tau)) - x_p(\tau)h'(x_p(\tau))\}\).

It follows from (6.32) that \(dq(y,1(\cdot),t,T;\cdot)\) is positive provided

\[
(6.33) \quad \left[\frac{x}{p} + h(x)\right]g'(x) + g(x)[h(x) - xh'(x)] \leq 0 \quad \text{for } x > 0.
\]

The HJ equation for the variational problem (6.28) is given by

\[
(6.34) \quad \frac{\partial q(x,y,t,T)}{\partial t} - \frac{x}{p} \frac{\partial q(x,y,t,T)}{\partial x} + \sup_{0 < v < 1} \left[-vh\left(\frac{x}{v}\right) \frac{\partial q(x,y,t,T)}{\partial x} + g\left(\frac{x}{v}\right)\right] = 0.
\]

We seek a solution to (6.34) by using bang-bang control. Thus we consider similarly to (6.16) the function

\[
(6.35) \quad q(x,y,t,T) = \int_{\tau x,t}^{T} ds g(x_p(s)) \ ds.
\]

**Proposition 6.2.** Assume \(g(\cdot), h(\cdot)\) are \(C^1\) non-negative decreasing functions such that (6.33) holds for all \(p > 0\). For any \(y > 0\), \(t < T\), let \((x,t)\) satisfy (6.3) and \(\tau_{x,t}\) be defined by (6.17). Then \(t < \tau_{x,t} < T\) and the function \(q(x,y,t,T)\) of (6.28) is given by the formula (6.35).

**Proof.** From (6.41) we see that the function (6.35) satisfies (6.20). Hence \(q\) is a solution to (6.34) provided

\[
(6.36) \quad g\left(\frac{x}{v}\right) \leq vh\left(\frac{x}{v}\right) \frac{\partial q(x,y,t,T)}{\partial x} \quad \text{for } 0 < v < 1, \ x > 0.
\]

Letting \(p \to \infty\) in (6.33) we see that the function \(x \to xg(x)/h(x)\) is decreasing, (whence \(\lim_{x \to \infty} g(x) = 0\). The inequality (6.36) holds therefore if

\[
(6.37) \quad u(x,t) = \frac{\partial q(x,y,t,T)}{\partial x} - \frac{g(x)}{h(x)} \geq 0 \quad \text{for } x > 0.
\]

Observe next from (6.17), on differentiating (6.35) with respect to \(x\), that \(\partial q(x,y,t,T)/\partial x\) approaches \(g(x_p(t))/h(x_p(t))\) as \(x \to x_p(t)\). It follows from (6.37) that \(u(x,t) = 0\) if \(x = x_p(t)\). We see now on differentiating (6.20) with respect to \(x\) that

\[
(6.38) \quad \frac{\partial u(x,t)}{\partial t} - \frac{u(x,t)}{p} \frac{\partial u(x,t)}{\partial x} - \frac{1}{p} u(x,t) = \frac{1}{p} \frac{d}{dx} \left[\frac{xg(x)}{h(x)}\right] \leq 0.
\]

It follows by the method of characteristics that \(u(x,t) \geq 0\) for all \((x,t)\) satisfying (6.3). Hence the function \(q\) of (6.35) is a solution to the HJ equation (6.31). Since \(q\) is also a \(C^1\) solution to (6.34), the solution to the variational problem (6.28) is given by (6.35). \(\square\)

**Remark 4.** Since the function (6.35) satisfies \(\partial q(x,y,t,T)/\partial x \geq 0\), it follows that the solution of the variational problem \(\max_{0 < v(\cdot) < 1} q(y,v(\cdot),t,T)\), with \(q\) as in (6.34), is given by \(v(\cdot) \equiv 1\).

Next we consider the optimization problem

\[
(6.39) \quad q(x,y,t,T) = \min_{1 < v(\cdot) < \infty} \left[\int_{t}^{T} ds g\left(\frac{x(s)}{v(s)}\right) \mid x(t) = x\right],
\]
where \( x(\cdot) \) has the dynamics \( (x(t)) \) and \((x, t)\) belongs to the reachable set \( \text{of the control system} \). The HJ equation for \( (x, t) \) is given by

\[
\frac{\partial q(x, y, t, T)}{\partial t} - \frac{x}{p} \frac{\partial q(x, y, t, T)}{\partial x} + \inf_{1 < v < \infty} \left[ -vh \left( \frac{x}{v} \right) \frac{\partial q(x, y, t, T)}{\partial x} + g \left( \frac{x}{v} \right) \right] = 0 .
\]

Letting \( G(x, \xi, v) \), \( \tilde{G}(x, \xi) \) be defined by

\[
G(x, \xi, v) = -vh \left( \frac{x}{v} \right) \xi + g \left( \frac{x}{v} \right), \quad \tilde{G}(x, \xi) = \inf_{1 < v < \infty} G(x, \xi, v),
\]

we see that the function \( \xi \to \tilde{G}(x, \xi) \) is concave. We can make a change of variable \( v \to w = vh(x/v) \), so

\[
\frac{dw}{dv} = h \left( \frac{x}{v} \right) - \frac{x}{v} h' \left( \frac{x}{v} \right) > 0 .
\]

Hence the function \( w \to G(x, \xi, v(w)) \) is convex provided

\[
\frac{d}{dw} g \left( \frac{x}{v} \right) = \frac{dv}{dw} \left[ -\frac{x}{v^2} g' \left( \frac{x}{v} \right) \right]
\]

increases as a function of \( w \). From \( (6.42) \) we see that this is equivalent to the function

\[
x \to -\frac{x^2 g'(x)}{h(x) - x h'(x)} = m(x)
\]

decreases.

Observe that if \( (6.43) \) holds for \( p = \infty \) then

\[
\frac{x}{m(x)} < \frac{x^2 g'(x)}{h(x) - x h'(x)} \geq \frac{xg(x)}{h(x)}, \quad x > 0 .
\]

By \( (6.33) \) with \( p = \infty \) the function on the RHS of \( (6.45) \) is decreasing, so \( (6.44) \) is an extra condition that the function on the LHS of \( (6.45) \) also decreases.

We assume now that \( (6.44) \) holds. Then the minimum of \( G(x, \xi, v) \) on the interval \( 1 < v < \infty \) is attained at \( v = 1 \) if \( \xi \leq -hxg'(x)/[h(x) - x h'(x)] \). If \( \xi > -hxg'(x)/[h(x) - x h'(x)] \) the minimizer of \( \min_{v \geq 1} G(x, \xi, v) \) is the solution to the equation

\[
-x \frac{h(x) - x h'(x)}{h(x) - x h'(x)} \xi - \frac{x g(x)}{h(x)} = 0 , \quad \text{whence } m \left( \frac{x}{v} \right) = \frac{x}{v} \xi = \zeta.
\]

A solution to \( (6.46) \) exists for all \( \xi > m(x) \) provided \( \lim_{z \to 0} m(z) = \infty \). From \( (6.46) \) it follows that the minimizing \( v = v_{\min}(x, \xi) = x/m^{-1}(\xi) \), where \( m^{-1}(\cdot) \) is the inverse function for \( m \). The corresponding HJ equation has therefore the form

\[
\frac{\partial q(x, y, t, T)}{\partial t} + H \left( x \frac{\partial q(x, y, t, T)}{\partial x} \right) = 0 ,
\]

where

\[
H(\zeta) = -\frac{\zeta}{p} - \frac{\zeta h(m^{-1}(\zeta))}{m^{-1}(\zeta)} + g \left( m^{-1}(\zeta) \right) .
\]

Note that \( \zeta = x \partial q(x, y, t, T)/\partial x \) is constant along characteristics for the HJ equation \( (6.47) \), whence it follows from \( (6.46) \) that \( x(\cdot)/v(\cdot) \) is also constant along characteristics.
The considerations of the previous paragraph lead us to propose a solution to (6.40). For \( s < t < T \) let \( x_p(s, t) \) be the solution to the terminal value problem

\[
\frac{dx_p(s, t)}{ds} = - \left[ \frac{1}{p} + \frac{h(x_p(t))}{x_p(t)} \right] x_p(s, t), \quad s < t < T, \quad x_p(t, t) = x_p(t).
\]

Setting \( x(s) = x_p(s, t) - x_p(s) \), we see from (6.11) with \( v(\cdot) \equiv 1 \) and (6.49) that

\[
\frac{dx(s)}{ds} = - \left[ \frac{1}{p} + \frac{h(x_p(t))}{x_p(t)} \right] x(s) - x_p(s) \left( \frac{h(x_p(t))}{x_p(t)} - \frac{h(x_p(s))}{x_p(s)} \right), \quad s < t, \quad x(t) = 0.
\]

Observe that the function \( x \rightarrow h(x)/x \) is decreasing and also the function \( s \rightarrow x_p(s) \). It follows then from (6.50) that \( x(s) > 0 \) for \( s < t \). Hence the trajectory \( x_p(s, t) \), \( s < t \), lies in the reachable set (6.25) for the variational problem (6.39).

We can show similarly that the trajectories \( x_p(\cdot, t) \), \( t < T \), do not intersect. Thus for \( t_1 < t_2 < T \) let \( x(s) = x_p(s, t_2) - x_p(s, t_1) \), \( s < t_1 \). We have already seen that \( x(t_1) > 0 \), and from (6.49) we also have that

\[
\frac{dx(s)}{ds} = - \left[ \frac{1}{p} + \frac{h(x_p(t_1))}{x_p(t_1)} \right] x(s) - x_p(s) \left( \frac{h(x_p(t_2))}{x_p(t_2)} - \frac{h(x_p(t_1))}{x_p(t_1)} \right), \quad s < t_1.
\]

Since \( x_p(t_1) > x_p(t_2) \) we conclude from (6.51) that \( x_p(s, t_2) > x_p(s, t_1) \), \( s < t_1 \).

Since the trajectories \( x_p(\cdot, t) \), \( t < T \), do not entirely cover the reachable set we complement them with a set of trajectories with terminal point \( y \) at time \( T \). Thus for \( s < T, 0 < \lambda < y \) we define \( y_p(s, \lambda) \) as the solution to

\[
\frac{dy_p(s, \lambda)}{ds} = - \left[ \frac{1}{p} + \frac{h(\lambda)}{\lambda} \right] y_p(s, \lambda), \quad s < T, \quad y_p(T, \lambda) = y.
\]

If \( t < T \) and \( x_p(t) < x < x_p(t, t) \) then there exists unique \( \tau = \tau_{x,t} \) such that \( t < \tau < T \) and \( x_p(t, \tau) = x \). If \( x > x_p(t, T) \) then there exists unique \( \lambda = \lambda_{x,t} \) such that \( 0 < \lambda < y \) and \( y_p(t, \lambda) = x \). We define now a function \( q(x, y, t, T) \) for \( t < T \) and \( (x, t) \) satisfying (6.25) by

\[
q(x, y, t, T) = (\tau_{x,t} - t)g(x_p(\tau_{x,t})) + \int_{\tau_{x,t}}^{T} g(x_p(s)) ds \quad \text{if} \quad x_p(t) < x < x_p(t, T),
\]

\[
q(x, y, t, T) = (T - t)g(\lambda_{x,t}) \quad \text{if} \quad x > x_p(t, T).
\]

Proposition 6.3. Assume \( g(\cdot), h(\cdot) \) are \( C^1 \) non-negative decreasing, and also that (6.44) holds. For any \( y > 0 \), \( t < T \), let \( (x, t) \) satisfy (6.25). Then the function \( q(x, y, t, T) \) of (6.39) is given by the formula (6.53).

Proof. We first consider the case \( x > x_p(t, T) \). The partial derivatives of \( \lambda_{x,t} \) can be computed by using the formula

\[
\exp \left( \frac{1}{p} \frac{h(\lambda_{x,t})}{\lambda_{x,t}} \right) (T - t) = x.
\]

Thus we have that

\[
\frac{\partial \lambda_{x,t}}{\partial x} = - \frac{\lambda_{x,t}^2}{(T - t)x[h(\lambda_{x,t}) - \lambda_{x,t}h'(\lambda_{x,t})]}.
\]
Similarly we have that
\[
\frac{\partial \lambda_{x,t}}{\partial t} = -\frac{\lambda_{x,t}^2}{(T-t)[h(\lambda_{x,t}) - \lambda_{x,t}h'(\lambda_{x,t})]} \left( \frac{1}{p} + \frac{h(\lambda_{x,t})}{\lambda_{x,t}} \right).
\]

It follows from (6.53), (6.55) that
\[
(6.56) \quad \frac{x}{p} \frac{\partial q(x,y,t,T)}{\partial x} = -\frac{g'(\lambda_{x,t})\lambda_{x,t}^2}{[h(\lambda_{x,t}) - \lambda_{x,t}h'(\lambda_{x,t})]},
\]
\[
\frac{\partial q(x,y,t,T)}{\partial t} = -\frac{g(\lambda_{x,t})}{[h(\lambda_{x,t}) - \lambda_{x,t}h'(\lambda_{x,t})]} \left( \frac{1}{p} + \frac{h(\lambda_{x,t})}{\lambda_{x,t}} \right).
\]

Hence \( q \) is a solution to the PDE
\[
\frac{\partial q(x,y,t,T)}{\partial t} - \frac{x}{p} \frac{\partial q(x,y,t,T)}{\partial x} - v(x,t)h \left( \frac{x}{v(x,t)} \right) \frac{\partial q(x,y,t,T)}{\partial x} + g \left( \frac{x}{v(x,t)} \right) = 0,
\]
where \( \frac{x}{v(x,t)} = \lambda_{x,t} \).

Note that \( v(x,t) > 1 \) since \( \lambda_{x,t} < y < x \). We also have that
\[
(6.58) \quad \frac{\partial}{\partial v} \left[ -v h \left( \frac{x}{v} \right) \frac{\partial q(x,y,t,T)}{\partial x} + g \left( \frac{x}{v(x,t)} \right) \right] = 0 \quad \text{at} \ v = v(x,t).
\]

Hence, in view of (6.54), we conclude that \( q(x,y,t,T) \) satisfies the HJ equation (6.40) in the region \( \{(x,t) : t < T, x > x_p(t,T)\} \).

Next we consider the region \( \{(x,t) : t < T, x_p(t) < x < x_p(p,t,T)\} \). In that case we have
\[
(6.59) \quad \exp \left\{ \left\{ \frac{1}{p} + \frac{h(x_p(\tau_{x,t}))}{x_p(\tau_{x,t})} \right\} (\tau_{x,t} - t) \right\} x_p(\tau_{x,t}) = x.
\]

Differentiating (6.59) with respect to \( x \) gives
\[
(6.60) \quad \frac{\partial \tau_{x,t}}{\partial x} = \frac{x_p(\tau_{x,t})^2}{[x_p(\tau_{x,t})/p + h(x_p(\tau_{x,t}))][\tau_{x,t} - t]\cdot[\{h(x_p(\tau_{x,t})) - x_p(\tau_{x,t})h'(x_p(\tau_{x,t}))\}]^{-1}.
\]

Similarly we have that
\[
(6.61) \quad \frac{\partial \tau_{x,t}}{\partial t} = \frac{x_p(\tau_{x,t})}{[\tau_{x,t} - t][h(x_p(\tau_{x,t})) - x_p(\tau_{x,t})h'(x_p(\tau_{x,t}))]}.
\]

From (6.53), (6.60) we have that
\[
(6.62) \quad \frac{x}{p} \frac{\partial q(x,y,t,T)}{\partial x} = -g(x_p(\tau_{x,t}))(x_p(\tau_{x,t})) \left[ \frac{x_p(\tau_{x,t})}{p} + h(x_p(\tau_{x,t})) \right] \frac{\partial \tau_{x,t}}{\partial x} = -\frac{x_p(\tau_{x,t})^2 g'(x_p(\tau_{x,t}))}{[h(x_p(\tau_{x,t})) - x_p(\tau_{x,t})h'(x_p(\tau_{x,t}))]},
\]
and also from (6.61) that
\[
(6.63) \quad \frac{\partial q(x,y,t,T)}{\partial t} = -g(x_p(\tau_{x,t}))(x_p(\tau_{x,t})) \left[ \frac{x_p(\tau_{x,t})}{p} + h(x_p(\tau_{x,t})) \right] \frac{\partial \tau_{x,t}}{\partial t} = -\frac{x_p(\tau_{x,t}) g'(x_p(\tau_{x,t}))}{[h(x_p(\tau_{x,t})) - x_p(\tau_{x,t})h'(x_p(\tau_{x,t}))]} \left[ \frac{x_p(\tau_{x,t})}{p} + h(x_p(\tau_{x,t})) \right] .
\]
It follows from (6.62), (6.63) that \( q \) is a solution to the PDE

\[
\frac{\partial q}{\partial t} - \frac{x}{p} \frac{\partial q}{\partial x} - v(x,t)h\left(\frac{x}{v(x,t)}\right) \frac{\partial q}{\partial x} + g\left(\frac{x}{v(x,t)}\right) = 0,
\]

where \( \frac{x}{v(x,t)} = x_p(\tau_x,t) \).

Note that since \( x > x_p(\tau_x,t) \) we have \( v(x,t) > 1 \) in (6.64). Furthermore, the identity (6.58) also holds. We therefore conclude that \( q \) is a solution to the HJ equation (6.40). Since \( q \) is a \( C^1 \) solution to the HJ equation for \((x,t)\) in the reachable set (6.25) it follows that the solution to the variational problem (6.39) is given by (6.53).

\[\square\]

Remark 5. Since the function (6.53) satisfies \( \frac{\partial q}{\partial x} \geq 0 \), it follows that the solution of the variational problem \( \min_{1 < v(\cdot) < \infty} q(y,v(\cdot),t,T) \) is given by \( v(\cdot) \equiv 1 \).

We wish to relate the condition (6.33) with \( p = \infty \), which insures a local extremum at \( v(\cdot) \equiv 1 \), to the condition (6.44). We have already observed that (6.33) with \( p = \infty \) is equivalent to the function on the RHS of (6.44) decreasing. Our goal is to show that (6.44) is a convexity condition on this function. To see this we set \( z(x) = x/h(x) \), whence (6.33) with \( p = \infty \) implies that

\[
\frac{dz(x)}{dz} [zg(x)] = g(x) + z \frac{dx}{dz} g'(x) \leq 0.
\]

We also have that the function \( m(\cdot) \) of (6.44) is given by

\[
m(x) = g'(x) \left\{ \frac{d}{dx} \left[ \frac{h(x)}{x} \right] \right\} = -z^2 \frac{dx}{dz} g'(x).
\]

Hence the condition \( m(\cdot) \) decreasing is equivalent to

\[
\frac{dz}{dz} \left[ z^2 \frac{dx}{dz} g'(x) \right] \geq 0.
\]

Observe from (6.65) that (6.67) is the same as

\[
\frac{d^2}{dz^2} [zg(x)] = \left[ 2 \frac{dx}{dz} + z \frac{d^2x}{dz^2} \right] g'(x) + z \left( \frac{dx}{dz} \right)^2 g''(x) \geq 0.
\]

Hence (6.33) is equivalent to the function \( z \rightarrow zg(x(z)) \) decreasing, while (6.44) is equivalent to convexity of the function \( z \rightarrow zg(x(z)) \). Note that convexity of a function implies that it is decreasing, provided the function is bounded at infinity.

### 7. Global Asymptotic Stability

In this section we prove a global asymptotic stability result for solutions of (3.1) with \( \rho(t) = \rho(\xi(\cdot,t),\eta(t)) \) given by (2.20), which extends the local asymptotic stability result Theorem 4.1. As in [3], the key to proving this is to establish global asymptotic stability for the DDE (5.15), (5.16) using the monotonicity properties of the function \( f(t,v(\cdot)) \) implied by Proposition 5.1. Adapting the argument of Proposition 8.1 of [3], we obtain the following:
Proposition 7.1. Let \( \xi(\cdot, t) \), \( t > 0 \), be the solution of \( (7.1) \) with \( \rho(t) = \rho(\xi(\cdot, t), \eta(t)) \), which is considered in Proposition 3.1. In addition to the assumptions \( (3.32), (3.33) \) on \( h(\cdot) \) required for Lemma 3.2, assume \( h(\cdot) \) satisfies \( (4.9) \) for all \( p > 0 \) and also that \( (5.14) \) holds. Let \( \eta(\cdot) \) satisfy \( (3.43) \) and the inequality \( \eta(t) \leq Ce^{-\delta t}, \ t > 0 \), for some constants \( C, \delta > 0 \). Then if the function \( x \to \beta(x, 0) \) is Hölder continuous at \( x = 1 \), there exists \( I_\infty > 0 \) such that the function \( I(\cdot) \) defined by \( (7.7) \) satisfies \( \lim_{t \to \infty} I(t) = I_\infty \).

Proof. It follows from \( (2.43) \) and \( (3.24) \) of Proposition 3.1 that
\[
\inf_{t > 0} \{ pI_p(\xi(\cdot, t)) + |dI_p(\xi(\cdot, t)), A\xi(\cdot, t)| \} > 0 .
\]
We note that the function \( I(\cdot) \) defined by \( (6.1) \) has the property
\[
(7.2) \quad c_0 I(0) \leq I(t) \leq C_0 I(0), \ t > 0, \ for \ some \ constants \ C_0, c_0 > 0 .
\]
This is a consequence of the Hölder assumption on \( \beta(\cdot, 0) \) using Lemma 2.1, Lemma 2.2, the identity \( (3.45) \), and our assumptions on the function \( \eta(\cdot) \). It further follows from the inequality in \( (3.44), (5.4) \) and \( (7.2) \) that the function \( G(\cdot) \) satisfies an inequality \( |G(t, y, v_1(\cdot))| \leq Ce^{-\delta t}, \ y \geq \epsilon, \ t \geq 0, \ for \ some \ constant \ C \). We conclude then from \( (4.4) \) that the function \( g(\cdot) \) satisfies an inequality \( |g(t, v_1(\cdot))| \leq Ce^{-\delta t/p} \) for some constant \( C \). We observe also from \( (2.45) \) of Lemma 2.4 that the function \( \gamma(\cdot) \) on the RHS of \( (6.15) \) satisfies \( |\gamma(t)| \leq C_1 e^{-\delta t}, \ t > 0, \ for \ some \ constants \ C_1, \delta_1 > 0 \).

To prove convergence of \( I(t) \) as \( t \to \infty \), we first assume that for any \( \epsilon, \tau > 0 \), \( \tau' > \tau \) and \( \epsilon < 1/2 \), there exists \( T_{\epsilon, \tau, \tau'} > \tau' \) such that \( |I(t)|^{1/p} |I(s)|^{1/p} - 1| < \epsilon \) for \( t, s \in [T_{\epsilon, \tau, \tau'} - \tau, T_{\epsilon, \tau, \tau'}] \). For \( t > T_{\epsilon, \tau, \tau'} \) we set \( I_{\max}(t) = \sup_{T_{\epsilon, \tau, \tau'} < s \leq t} I(s) \) and consider \( T > T_{\epsilon, \tau, \tau'} \) such that \( I(T) = I_{\max}(T) \). Integrating \( (5.15) \) and using the exponential decay of the RHS of \( (6.10) \) we have that
\[
(7.3) \quad \frac{1}{p} \log \left[ \frac{I(T)}{I(T_{\epsilon, \tau, \tau'})} \right] + \int_{(T_{\epsilon, \tau, \tau'}, T) - \{ T_{\epsilon, \tau, \tau'} < t \leq T: I_{\max}(t) > I(t) \}} f(t, v_1(\cdot)) \, dt \leq C_2 e^{-\delta_2 T_{\epsilon, \tau, \tau'}} ,
\]
for some constants \( C_2, \delta_2 > 0 \). We can estimate the second term on the LHS of \( (6.9) \) by using Proposition 5.1. First we write the function \( F(\cdot) \) of \( (5.7) \) as \( F = F_1 + F_2 \), where
\[
(7.4) \quad F_1(t, y, v_1(\cdot)) = \frac{1}{p} \int_{T_{\epsilon, \tau, \tau'} - \tau}^{t} h \left( \frac{z(s)}{v_1(s)} \right) e^{-(t-s)/p} v_1(s) \, ds \\
\quad - \left[ h(y) + \frac{y}{p} \right] \left\{ \exp \left[ \int_{T_{\epsilon, \tau, \tau'} - \tau}^{t} h' \left( \frac{z(s)}{v_1(s)} \right) \, ds \right] - 1 \right\} .
\]
Note that from \( (5.4) \) the first term on the RHS of \( (7.3) \) has the simplification
\[
(7.5) \quad \frac{1}{p} \int_{T_{\epsilon, \tau, \tau'} - \tau}^{t} h \left( \frac{z(s)}{v_1(s)} \right) e^{-(t-s)/p} v_1(s) \, ds \\
= \frac{1}{p} \left[ \exp \left\{ -(t + \tau - T_{\epsilon, \tau, \tau'})/p \right\} z(T_{\epsilon, \tau, \tau'} - \tau) - y \right] .
\]
We see from (5.4), (7.2) and the assumptions on \( \varepsilon, \tau, \tau, T \) that
\[
|F_0(t, y, v_0(t))| \leq C_3 e^{-\tau/p} e^{-(t - T_{\varepsilon, \tau'})/p}, \quad y \geq \varepsilon_0, \quad t \geq T_{\varepsilon, \tau', \tau'}, \quad \text{for some constant } C_3.
\]
Next we define the function \( \tilde{v}_t(\cdot) \) as
\[
\begin{align*}
\tilde{v}_t(s) &= v_t(T_{\varepsilon, \tau}, \cdot), \quad T_{\varepsilon, \tau} - \tau < s < T_{\varepsilon, \tau}, \\
\tilde{v}_t(s) &= v_t(s), \quad T_{\varepsilon, \tau} < s < t.
\end{align*}
\]

We define also the function \( \tilde{z}(s) \), \( T_{\varepsilon, \tau} - \tau < s < t \), as the solution to (5.4) with \( \tilde{v}_t(\cdot) \) in place of \( v_t(\cdot) \). Evidently \( z(s) = \tilde{z}(s) \) for \( s \in [T_{\varepsilon, \tau}, t] \). For \( s \in [T_{\varepsilon, \tau}, \cdot] \) we note that \( \tilde{z}(s)/\tilde{v}_t(s) = \tilde{z}(s)/v_t(T_{\varepsilon, \tau}) \geq z(T_{\varepsilon, \tau'})/v_t(T_{\varepsilon, \tau'}) \geq \varepsilon_0 \).

Hence the equation (5.3) is well-defined for \( \tilde{v}_t(\cdot), \tilde{z}(\cdot) \). We define the function \( \tilde{F}_1 \) by (7.4) with \( \tilde{v}_t(\cdot), \tilde{z}(\cdot) \) in place of \( v_t(\cdot), z(\cdot) \). We see from (7.4) that
\[
(7.7) \quad \left| F_1(t, y, v_t(\cdot)) - \tilde{F}_1(t, y, \tilde{v}_t(\cdot)) \right| \leq
\]
\[
\frac{1}{p} \exp\{-(t + \tau - T_{\varepsilon, \tau'})/p\} \left| z(T_{\varepsilon, \tau}, \cdot) - \tilde{z}(T_{\varepsilon, \tau}, \cdot) \right|
\]
\[
\left| -\left( h(y) + \frac{y}{p} \right) \int_{T_{\varepsilon, \tau}, \cdot} \int_{T_{\varepsilon, \tau}, \cdot} ds \right| \left| h'(\tilde{z}(\cdot)) - h'(\tilde{z}(\cdot)) \right|.
\]

Since we have by assumption that the fluctuation of \( I(\cdot) \) in the interval \( [T_{\varepsilon, \tau} - \tau, T_{\varepsilon, \tau}] \) is small, we should be able to estimate the RHS of (7.7) by a small constant. However since \( h'(y) \) may diverge as \( y \to \varepsilon_0 \) we cannot simply apply Taylor’s theorem to estimate the RHS of (7.7). We can directly apply Taylor’s theorem if \( t \geq T_{\varepsilon, \tau'} \), \( y \geq \varepsilon_0 + \nu_0 \) or if \( t \geq T_{\varepsilon, \tau} + \nu_0 \), \( y \geq \varepsilon_0 \), for any \( \nu_0 > 0 \). To do this we first observe by integrating (5.4) that
\[
(7.8) \quad z(s) = e^{(T_{\varepsilon, \tau}, \cdot - s)/p} \int_{s}^{T_{\varepsilon, \tau}, \cdot} ds e^{(s') - s)/p} v_t(s') h\left( \frac{z(s')}{v_t(s')} \right), \quad s < T_{\varepsilon, \tau}, \cdot,
\]
with a similar formula for \( \tilde{z}(s) \). Since \( z(T_{\varepsilon, \tau}) = \tilde{z}(T_{\varepsilon, \tau}) \) we have from (7.8) that
\[
(7.9) \quad \frac{1}{(1 + \varepsilon)} \frac{h_{\infty}}{h(\varepsilon_0)} \leq \frac{\tilde{z}(s)}{z(s)} \leq (1 + \varepsilon) \frac{h(\varepsilon_0)}{h_{\infty}}, \quad T_{\varepsilon, \tau}, \cdot - \tau < s < T_{\varepsilon, \tau}, \cdot.
\]

Taylor’s theorem implies then upon using (7.9) and our assumptions on derivatives of the function \( h(\cdot) \), there is a constant \( C_4 \) such that
\[
(7.10) \quad \left| F_1(t, y, v_t(\cdot)) - \tilde{F}_1(t, y, \tilde{v}_t(\cdot)) \right| \leq C_4 \tau e^{-(t - T_{\varepsilon, \tau})/p} \sup_{T_{\varepsilon, \tau}, \cdot - \tau < s < T_{\varepsilon, \tau}, \cdot} \left[ \left| v_t(s) - \tilde{v}_t(s) \right| + \frac{v_t(s)}{z(s)} \left| \frac{z(s) - \tilde{z}(s)}{v_t(s)} \right| \right].
\]

From our assumptions on \( I(\cdot) \) in the interval \( [T_{\varepsilon, \tau}, \cdot - \tau, T_{\varepsilon, \tau}, \cdot] \), we see that the first term in the supremum on the RHS of (7.10) is bounded above by \( \varepsilon \). In order to estimate \( |1 - \tilde{z}(s)/z(s)| \) for \( s \in [T_{\varepsilon, \tau}, \cdot - \tau, T_{\varepsilon, \tau}, \cdot] \), we observe that the function \( w(s) = z(s) - \tilde{z}(s) \) is a solution to the terminal value problem
\[
(7.11) \quad \frac{dw(s)}{ds} = -a(s) w(s) - b(s) [v_t(s) - \tilde{v}_t(s)], \quad s < T_{\varepsilon, \tau}, \cdot, \quad w(T_{\varepsilon, \tau}, \cdot) = 0,
\]
where the functions \( a(\cdot), b(\cdot) \) are given by the formulae
Integrating (7.11) we have that

\[ w(s) = \int_s^{T_\varepsilon,T_\tau,T_\tau'} ds' \exp \left[ \int_s^{s'} a(s'') ds'' \right] b(s') \left[ v_1(s') - \hat{v}_1(s') \right]. \]  

Observe now that \( a(\cdot) \leq 1/p \), and in view of (7.12) that \( b(\cdot) \) is bounded in the interval \([T_\varepsilon,T_\tau',-\tau,\tau,T_\tau',\tau']\). We conclude from (7.8), (7.13) that \( |w(s)/z(s)| \leq C_4 \varepsilon \) for \( s \in [T_\varepsilon,T_\tau',-\tau,\tau,T_\tau',\tau'] \), where \( C_4 \) is independent of \( \varepsilon, \tau, \tau' \). We have shown that if \( t \geq T_\varepsilon,T_\tau,T_\tau', \) \( y \geq \varepsilon_0 + \epsilon_0 \) or if \( t \geq T_\varepsilon,T_\tau', \) \( y \geq \varepsilon_0 \), then

\[ |F_1(t,y,v_1(\cdot)) - \tilde{F}_1(t,y,v_1(\cdot))| \leq C_5 \varepsilon \tau e^{-(t-T_\varepsilon,T_\tau')/p} \]

for some constant \( C_5 \) independent of \( \varepsilon, \tau, \tau' \).

We estimate the expression on the RHS of (7.7) when \( T_\varepsilon,T_\tau,T_\tau' \leq t \leq T_\varepsilon,T_\tau', \) \( \nu_0, \varepsilon_0 \leq y \leq \varepsilon_0 + \nu_0 \). Since we can apply the argument of the previous paragraph to integration on the RHS of (7.12) over the interval \([T_\varepsilon,T_\tau',-\tau,\tau,T_\tau',\tau']\), we restrict ourselves to the integral over the interval \([T_\varepsilon,T_\tau',-\nu_0,\tau,T_\tau',\tau']\). Observe from (8.28), (8.29), (8.31) that

\[ \frac{dy_\nu(s)}{ds} = -\mu \left[ \rho(s)y_1(s) + h(y_1(s)) \right] - (1 - \mu) \left[ y_0(s)/p + h(y_0(s)) \right]. \]

Upon choosing \( \nu_0 > 0 \) sufficiently small, we see from (7.14) and the inequality \( \inf \rho(s) > -h(\varepsilon_0)/\varepsilon_0 \) there exists \( c_0 > 0 \) such that \( y'_\mu(s) \leq -c_0 \) for all \( 0 < \mu < 1 \) and \( s \in [T_\varepsilon,T_\tau',-\nu_0,T_\tau',T_\tau'] \). It follows that the function \( b(\cdot) \) of (7.12) satisfies the inequality

\[ \int_{T_\varepsilon,T_\tau',-\nu_0}^{T_\varepsilon,T_\tau',-\nu_0} |b(s)| ds \leq C_0, \]

where the constant \( C_0 \) is inversely proportional to \( c_0 \). We conclude from (7.8), (7.16) that the function \( w(\cdot) \) of (7.13) satisfies the inequality \( |w(s)/z(s)| \leq C_5 \varepsilon, \) \( s \in [T_\varepsilon,T_\tau',-\nu_0,T_\tau',T_\tau'] \), for some constant \( C_5 \). It follows there are constants \( C_6, C'_6 \) such that

\[ \left| \frac{z(s)}{v_1(s)} - \frac{\hat{z}(s)}{\hat{v}_1(s)} \right| \leq C_6 \varepsilon \quad \text{for} \quad T_\varepsilon,T_\tau',-\nu_0 < s < T_\varepsilon,T_\tau', \]

\[ \min \left[ \frac{z(s)}{v_1(s)}, \frac{\hat{z}(s)}{\hat{v}_1(s)} \right] \geq \varepsilon_0 + 2C_6 \varepsilon \quad \text{for} \quad T_\varepsilon,T_\tau',-\nu_0 < s < T_\varepsilon,T_\tau',-C'_6 \varepsilon. \]

We estimate now

\[ \int_{T_\varepsilon,T_\tau',-\nu_0}^{T_\varepsilon,T_\tau',-\nu_0} ds \left| h' \left( \frac{z(s)}{v_1(s)} \right) - h' \left( \frac{\hat{z}(s)}{\hat{v}_1(s)} \right) \right| \leq \int_{T_\varepsilon,T_\tau',-\nu_0}^{T_\varepsilon,T_\tau',-\nu_0} + \int_{T_\varepsilon,T_\tau',-\nu_0}^{T_\varepsilon,T_\tau',-C'_6 \varepsilon}.
\]

Since \( h'(\cdot) \) is integrable the first integral on the right of (7.18) converges to 0 as \( \varepsilon \to 0 \). Using (7.17) we see that the second integral on the RHS of (7.18) also
converges to 0 as $\varepsilon \to 0$. We have therefore shown that the integral on the RHS of (7.7) converges to 0 as $\varepsilon \to 0$. Evidently the first term on the RHS of (7.7) also converges to 0.

To estimate the second term on the LHS of (5.13) we write $f(t, v_t(\cdot)) = f_1(t, v_t(\cdot)) + f_2(t, v_t(\cdot))$, corresponding to the decomposition $F = F_1 + F_2$. From our bound on $F_2$ we see there is a constant $C_7$ such that

$$
\int_{T_{\varepsilon, \tau, \tau^\prime}} |f_2(t, v_t(\cdot))| \, dt \leq C_7 e^{-\tau/p} .
$$

Letting $\tilde{f}_1(t, \tilde{v}_t(\cdot))$ be the function (5.10) corresponding to $\tilde{F}_1$ in place of $F$, we have from (7.19) and the argument of the previous paragraph that

$$
\int_{T_{\varepsilon, \tau, \tau^\prime}} |f_1(t, v_t(\cdot)) - \tilde{f}_1(t, \tilde{v}_t(\cdot))| \, dt \leq C_8 \tau \varepsilon + C_9(\varepsilon) ,
$$

for a constant $C_8$ independent of $\varepsilon$, and a constant $C_9(\varepsilon)$ which has the property $\lim_{\varepsilon \to 0} C_9(\varepsilon) = 0$. Observe next that by Proposition 5.1 one has $\tilde{F}_1(t, y, \tilde{v}_t(\cdot)) \leq \tilde{F}_1(t, y, 1(\cdot))$ for $t > T_{\varepsilon, \tau, \tau^\prime}$ such that $I(t) = I_{\text{max}}(t)$, whence $\tilde{F}_1(t, \tilde{v}_t(\cdot)) \geq 0$. We note also from (2.58) of Lemma 2.4 that the function $\gamma(\cdot)$ on the RHS of (5.13) satisfies an inequality $|\gamma(t)| \leq C e^{-\delta_1 t}$, $t \geq 0$, for some constants $C, \delta_1 > 0$. We conclude then from (5.13), (7.19), (7.20) and the bounds we have on the functions $(t, y) \to G(t, y, v_t(\cdot))$ and $t \to \gamma(t)$ that

$$
\frac{1}{p} \log \left[ \frac{I(T)}{I(T_{\varepsilon, \tau, \tau^\prime})} \right] \leq C_{10} \left[ e^{-\tau/p} + e^{-\delta_1 \tau} \right] + C_7 e^{-\tau/p} + C_8 \tau \varepsilon + C_9(\varepsilon) .
$$

Since the constants $C_7, C_8, C_9, C_{10}$ in (7.21) are independent of $\varepsilon, \tau, \tau^\prime$, we conclude that for any $\delta > 0$ there exists $T_\delta > 0$ such that $\sup_{t > T_\delta} |I(t)/I(T_\delta) - 1| < \delta$. Since we can make an exactly analogous argument with the function $I_{\text{min}}(t) = \inf_{T_{\varepsilon, \tau, \tau^\prime}} I(s)$, we conclude that $\lim_{t \to \infty} I(t) = I_\infty > 0$ exists.

Alternatively there exists $\varepsilon_0, \tau_0 > 0$, $\tau_1 > \tau_0$ such that $\sup_{s, t \in [T_{\tau_0}, T]} |I(t)/I(s)|^{1/p} - 1 \geq \varepsilon_0$ for all $T \geq \tau_1$. Letting $I_{\infty}^* = \limsup_{t \to \infty} I(t)$, there exists for any $\delta > 0$, $N = 1, 2, \ldots$, a time $T_{\delta, N} > \max[\tau_1, N]$ such that $I(T_{\delta, N}) \geq I_{\infty}^* - \delta$ and $I(t) \leq I_{\infty}^* + \delta$ for $T_{\delta, N} - N \leq t \leq T_{\delta, N}$. Since the oscillation of $I(\cdot)$ in the interval $[T_{\delta, N} - \tau_0, T_{\delta, N}]$ exceeds $\varepsilon_0$, there exists $\tau_0, N \in [T_{\delta, N} - \tau_0, T_{\delta, N}]$ such that $I(T_{\delta, N})^{1/p} \leq (I_{\infty}^* + \delta)^{1/p}/(1 + \varepsilon_0)$. We proceed similarly to before by writing the function $F$ of (5.7) as $F = F_1 + F_2$, where $F_1$ is given by (7.4), but with the interval of integration now $[T_{\delta, N} - N, t]$ in place of $[T_{\varepsilon, \tau, \tau^\prime}, \tau, t]$. As previously, one has the bound $|F_2(t, y, v_t(\cdot))| \leq C_{10} e^{(\tau_0 - N)/p}$, $y \geq \varepsilon_0$, $t \in [T_{\delta, N} - \tau_0, T_{\delta, N}]$. Evidently $F_1(t, y, v_t(\cdot))$ depends only on the values of $I(\cdot)$ for $s \in [T_{\delta, N} - N, t]$. We define $\tilde{v}_t(s) = I(s)^{1/p}/(I_{\infty}^* + \delta)^{1/p}$ for $s \in [T_{\delta, N} - N, t]$, and $\tilde{z}(\cdot), s \in [T_{\delta, N} - N, t]$ as the solution to (5.2) with $\tilde{v}_t(\cdot)$ replacing $v_t(\cdot)$. Since $\tilde{v}_t(\cdot) \leq 1$ we have that $\tilde{z}(s)/\tilde{v}_t(s) \geq \tilde{z}(t) \geq \varepsilon_0$ for $s \in [T_{\delta, N} - N, \tau_0]$. Hence (5.4) is well-defined for $\tilde{v}_t(\cdot), \tilde{z}(\cdot)$. Similarly to (7.9), it is easy to see there are positive constants independent of $\delta, N, t$ such that

$$
c_{12} \leq \frac{\tilde{z}(s)}{z(s)} \leq C_{12} \text{ for } T_{\delta, N} - N < s < t .
$$

Setting $w(s) = z(s) - \tilde{z}(s)$ we see using a representation analogous to (7.13), that for some constant $C_{13}$ one has $|w(s)/z(s)| \leq C_{13} J(t)$, where $J(t) = \log \left[ (I_{\infty}^* + \delta)^{1/p} / I(t)^{1/p} \right] \geq 0$. 

Let \( \hat{F}_1(t, y, \tilde{v}_t(\cdot)) \) be defined in the same way as \( F_1(t, y, v_t(\cdot)) \), but with \( \tilde{v}_t(\cdot), \tilde{z}(\cdot) \) replacing \( v_t(\cdot), z(\cdot) \). The difference \(|F_1(t, y, v_t(\cdot)) - \hat{F}_1(t, y, \tilde{v}_t(\cdot))|\) is bounded by the RHS of (7.27), with the interval of integration now \([T_{\delta, N} - N, t]\) in place of \([T_{\varepsilon, r, \tau} - \tau, T_{\varepsilon, r, \tau}]\). Instead of (7.10) we have if \( y \geq \varepsilon_0 + \nu_0 \) the estimate

\[
(7.23) \quad |F_1(t, y, v_t(\cdot)) - \hat{F}_1(t, y, \tilde{v}_t(\cdot))| \leq \int_{T_{\delta, N} - N}^{t} ds \, e^{- (t-s)/p} \left| \frac{v_t(s) - \tilde{v}_t(s)}{z(s)} + \frac{\dot{v}_t(s) - \dot{\tilde{v}}_t(s)}{\tilde{v}_t(s)} \right| ,
\]

for some constant \( C_{14} \). It follows from (7.23) there is a constant \( C_{15} \) such that \(|F_1(t, y, v_t(\cdot)) - \hat{F}_1(t, y, \tilde{v}_t(\cdot))| \leq C_{15}J(t)\) for \( t \in [T_{\delta, N} - \tau_0, T_{\delta, N}]\), provided \( y \geq \varepsilon_0 + \nu_0 \). For the case \( \varepsilon_0 \leq y \leq \varepsilon_0 + \nu_0 \) we need only estimate the RHS of (7.27) with the interval of integration \([t - \nu_0, t]\), since the previous argument applies to the integral over the interval \([T_{\delta, N} - N, t - \nu_0]\).

We shall show that

\[
(7.24) \quad \int_{\varepsilon_0}^{\varepsilon_0 + \nu_0} dy \int_{t-\nu_0}^{t} ds \, |h'(\frac{z(s, y)}{v_t(s)}) - h'(\frac{\tilde{z}(s, y)}{\tilde{v}_t(s)})| \leq C_{15}J(t)
\]

for some constant \( C_{15} \). First observe that since the LHS of (7.24) is bounded, we need only consider the situation when \( J(t) << 1 \). We proceed similarly to the method used in (7.18). Thus we first observe there are constants \( C_{16}, C'_{16} \) such that

\[
(7.25) \quad \left| \frac{z(s, y)}{v_t(s)} - \frac{\tilde{z}(s, y)}{\tilde{v}_t(s)} \right| \leq C_{16}J(t) \quad \text{for} \quad t - \nu_0 < s < t, \quad \varepsilon_0 < y < \varepsilon_0 + \nu_0,
\]

\[
\min \left[ \frac{z(s, y)}{v_t(s)}, \frac{\tilde{z}(s, y)}{\tilde{v}_t(s)} \right] \geq y + 2C_{16}J(t) \quad \text{for} \quad t - \nu_0 < s < t - C'_{16}J(t) .
\]

We have now that

\[
(7.26) \quad \int_{t-C'_{16}J(t)}^{t} |h'(\frac{z(s, y)}{v_t(s)})| ds \leq C_{17} \int_{y}^{y+C_{16}J(t)} |h'(y')| dy' = C_{17}[h(y) - h(y + C_{18}J(t))] \quad \text{for some constants} \quad C_{17}, C_{18} .
\]

It follows from (7.26) that

\[
(7.27) \quad \int_{\varepsilon_0}^{\varepsilon_0 + \nu_0} dy \int_{t-C'_{16}J(t)}^{t} ds \left| h'(\frac{z(s, y)}{v_t(s)}) - h'(\frac{\tilde{z}(s, y)}{\tilde{v}_t(s)}) \right| \leq C_{17}C_{18}h(\varepsilon_0)J(t) .
\]

Since we can obtain a similar estimate to (7.27) when \( z(s, y)/v_t(s) \) is replaced by \( \tilde{z}(s, y)/\tilde{v}_t(s) \), we need only estimate the integral in (7.22) for \( t - \nu_0 < s < t - C'_{16}J(t) \).

To do this we note from the convexity of the function \( h(\cdot) \) and (7.25) the inequality

\[
(7.28) \quad h'(\frac{z(s, y)}{v_t(s)}) - h'(\frac{\tilde{z}(s, y)}{\tilde{v}_t(s)}) \leq h'(y') + C_{16}J(t)) - h'(y' - C_{16}J(t)) ,
\]

where \( y' = \frac{z(s, y)}{v_t(s)} \) and \( t - \nu_0 < s < t - C'_{16}J(t) \).

Integrating (7.25) we obtain the inequality

\[
(7.29) \quad \int_{t-C'_{16}J(t)}^{t} ds \, \left| h'(\frac{z(s, y)}{v_t(s)}) - h'(\frac{\tilde{z}(s, y)}{\tilde{v}_t(s)}) \right| \leq C_{19}[h(y + C_{16}J(t)) - h(y + 3C_{16}J(t))] \quad \text{for some constant} \quad C_{19} .
\]
Integrating (7.29) with respect to \( y \) then yields the inequality

\[
\int_{c_0}^{c_0 + n_0} dy \int_{t - n_0}^{t} C_{19}(t) \left| h'(\frac{z(s, y)}{\nu_0(s)}) - h'(\frac{\hat{z}(s, y)}{\hat{\nu}_0(s)}) \right| \leq 2 C_{19} C_{19} h(\varepsilon_0) J(t) .
\]

Now (7.30) follows from (7.27), (7.30).

We estimate the second term on the LHS of (5.15) by writing \( f(t, v_1(\cdot)) = f_1(t, v_1(\cdot)) + f_2(t, v_1(\cdot)) \), corresponding to the decomposition \( F = F_1 + F_2 \). From our bound on \( F_2 \) we see there is a constant \( C_{20} \) such that \( |f_2(t, v_1(\cdot))| \leq C_{20} e^{-N/p} \) for \( t \in [T_{\delta,N} - \tau_0, T_{\delta,N}] \). Letting \( \hat{f}_1(t, \nu_1(\cdot)) \) be the function corresponding to \( \hat{F}_1 \) in place of \( F \), we also have from the previous paragraph that \( |f_1(t, v_1(\cdot)) - \hat{f}_1(t, \hat{\nu}_1(\cdot))| \leq C_{21} J(t) \) for some constant \( C_{21} \) if \( t \in [T_{\delta,N} - \tau_0, T_{\delta,N}] \). Furthermore, Proposition 5.1 implies that \( \hat{F}_1(t, y, \hat{\nu}_1(\cdot)) \leq \hat{F}_1(t, y, 1(\cdot)) \) for \( t \in [T_{\delta,N} - \tau_0, T_{\delta,N}] \), whence \( \hat{f}_1(t, \hat{\nu}_1(\cdot)) \geq 0 \) if \( t \in [T_{\delta,N} - \tau_0, T_{\delta,N}] \). It follows now from (5.15) that

\[
\frac{dJ(t)}{dt} + C_{22} J(t) \geq -C_{20} e^{-N/p} - C_{23} [e^{-t/p} + e^{-\delta_1 t}], \quad t \in [T_{\delta,N} - \tau_0, T_{\delta,N}],
\]

for some positive constants \( C_{22}, C_{23} \). In deriving (7.31), we have used the fact that the function \( t \rightarrow J(t) \) is non-negative. Integrating (7.31) over the interval \([T_{\delta,N}, T_{\delta,N}]\), we obtain the inequality

\[
J(T_{\delta,N}) \geq e^{-C_{23} \tau_0} J(T_{\delta,N}) - C_{20} e^{-N/p} - C_{23} [e^{-(T_{\delta,N} - \tau_0)/p} + e^{-\delta_1 (T_{\delta,N} - \tau_0)}].
\]

Observe that \( J(T_{\delta,N}) \leq 2\delta/p (I_{\infty}^+ - \delta) \) and \( J(T_{\delta,N}) \geq \log(1 + \varepsilon_0) \). Since \( T_{\delta,N} \geq N \), the inequality (7.32) yields a contradiction if \( \delta > 0 \) is sufficiently small and \( N \) sufficiently large.

**Lemma 7.1.** Assume \( \rho(\cdot), h(\cdot) \) satisfy the assumptions of Lemma 3.2, and let \( \xi(\cdot, t), t > 0 \) be the solution to (3.1) with initial condition \( \xi(\cdot, 0) \) satisfying \( \|\xi(\cdot, 0)\|_{1,\infty} < \infty \). Assume further that

\[
\lim_{t \to \infty} \sup_{1 - \tau < s < t} \left| \int_{s}^{t} \rho(s') - \frac{1}{p} \right| ds' = 0 \quad \text{for all} \quad \tau > 0 .
\]

Then \( \lim_{t \to \infty} \|\xi(\cdot, t) - \hat{\xi}(\cdot)\|_{1,\infty} = 0 \).

**Proof.** Let \( \xi_0(\cdot, \cdot) \) be defined by

\[
\xi_0(y, t) = \int_{0}^{t} ds \ h(y(s)) \exp \left[ - \int_{s}^{t} \rho(s') \, ds' \right], \quad y \geq \varepsilon_0, t > 0 ,
\]

where \( y(\cdot) \) is the solution to (3.3). We see from (3.14), (3.22) upon using the representations (3.24), (3.23) for \( \xi(\cdot, t), D\xi(\cdot, t) \) that \( \|\xi(\cdot, t) - \xi(\cdot, 0)\|_{1,\infty} < C_0 e^{-\delta_0 t}, t > 0 \) for some constant \( C_0 \). For any \( \tau, \nu > 0 \) with \( 0 < \nu \leq 1 \), let \( T_{\tau,\nu} > \tau \) have the property that

\[
\int_{s}^{t} \left| \rho(s') - \frac{1}{p} \right| ds' \leq \nu \quad \text{for} \quad t - \tau < s < t, \quad t > T_{\tau,\nu} .
\]

Letting \( \xi(\cdot, t) \) be defined as in (7.34), but with the interval of integration \([t - \tau, t]\) instead of \([0, t]\), then we have that \( \|\xi(\cdot, t) - \xi(\cdot, 0, \cdot)\|_{1,\infty} \leq C e^{-\delta_0 \tau} \) for some constant \( C \) independent of \( \tau > 0 \).

Let \( \xi(\cdot, t) \) be the function \( \xi_0(\cdot, t) \) in the case \( \rho(\cdot) \equiv 1/p \). We wish to estimate \( \|\xi_0(\cdot, t) - \hat{\xi}_0(\cdot, t)\|_{1,\infty} \). In order to do this we first need an estimate on the
difference $g(s) - y_p(s), \ t - \tau < s < t$, where $y_p(\cdot)$ is the solution to (3.14) with $\rho(\cdot) \equiv 1/p$. Observe from (3.14) that $u(s) = e^{-(t-s)/p}[y(s) - y_p(s)]$ satisfies the integral equation

$$u(s) + \int_s^t K(s')u(s') \, ds' = g(s), \ s < t,$$

where

$$K(s) = -\int_0^1 h'(\lambda y(s) + (1-\lambda)y_p(s)) \, ds \geq 0,$$

$$g(s) = \left\{ \exp\left[ \int_s^t \rho(s') - \frac{1}{p} \, ds' \right] - 1 \right\} y + \int_s^t ds' \exp\left[ \int_s^{s'} \rho(s'') - \frac{1}{p} \, ds'' \right] - 1 \right\} h(y(s')) .$$

Observing that $g(t) = 0$, we see on differentiating (7.36) that $u(s), \ s < t$, is the solution to the terminal value problem

$$\frac{du(s)}{ds} - K(s)u(s) = g'(s), \ s < t, \ u(t) = 0 .$$

The integral representation for the solution to (7.38) is given by

$$u(s) = -\int_s^t \exp\left[ -\int_s^{s'} K(s'') \, ds'' \right] g'(s') \, ds' = g(s) - \int_s^t \exp\left[ -\int_s^{s'} K(s'') \, ds'' \right] K(s')g(s') \, ds' .$$

Since $K(\cdot)$ is non-negative we conclude from (7.39) that

$$|u(s)| \leq 2 \sup_{s \leq s' \leq t} |g(s')|, \ s < t .$$

It follows from (7.35), (7.37) that

$$|g(s)| \leq |e^{e^r} - 1| y_p(s) + |e^{2e^r} - 1| h(\varepsilon_0), \ t - \tau < s < t, \ t > T_{\tau,\nu} .$$

We conclude from (3.14), (7.40), (7.41) there is a constant $C$ such that

$$|\xi(s) - y_p(s)| \leq C\nu y_p(s) \quad 0 < \nu \leq 1, \ t - \tau < s < t, \ t > T_{\tau,\nu} .$$

It follows easily from (3.32), (7.42) that $||\xi\bar{\tau}(\cdot,t) - \bar{\xi}\bar{\tau}(\cdot,t)||_{\infty} \leq C\nu$ for some constant $C$. To bound the derivative $D\bar{\xi}(\cdot,t) - D\bar{\xi}(\cdot,t)$ we observe as in (3.26) that

$$D\bar{\xi}(y,t) = \exp\left[ \int_{t-\tau}^t h'(y(s)) \, ds \right] - 1 ,$$

with a similar representation for $D\bar{\xi}(\cdot,t)$. Hence using the negativity of $h'(\cdot)$ we have from (7.43) and Taylor’s theorem that

$$|D\bar{\xi}(y,t) - D\bar{\xi}(y,t)| \leq \left| \int_{t-\tau}^t [h'(y(s)) - h'(y_p(s))] \, ds \right| .$$
We can estimate the RHS of (7.44) by applying Taylor’s theorem and using (3.14), (8.35) to conclude that for any $\nu > 0$,
\begin{equation}
\sup_{y \geq \epsilon_0 + \nu_0} y |D\xi_{0,\tau}(y,t) - D\tilde{\xi}_{0,\tau}(y,t)| \leq C(\nu_0)\nu , \quad t > T_{\tau,\nu} ,
\end{equation}
where the constant $C(\nu_0)$ may depend on $\nu_0 > 0$. We could extend the estimate of (7.45) to the supremum over $y \geq \epsilon_0$ if we were to replace the interval of integration $[t - \tau, t]$ on the RHS of (7.44) by $[t - \tau, t - \nu_0]$ for any $\nu_0 > 0$. Hence if we can show that
\begin{equation}
\lim_{t \to \infty} \sup_{t \geq \epsilon_0} \int_{t - \nu_0}^t \int_{t - \nu_0}^t |h'(y(s)) - h'(y_p(s))| ds = 0 ,
\end{equation}
it follows that $\lim_{t \to \infty} \|\xi_{0,\tau}(\cdot, t) - \tilde{\xi}_{0,\tau}(\cdot, t)\|_{1,\infty} = 0$. The limit (7.46) is a consequence of the dominated convergence theorem.

To complete the proof we observe there is a constant $C$ such that
\begin{equation}
\|\tilde{\xi}_{0,\tau}(\cdot, t) - \xi_p(\cdot)\|_{1,\infty} \leq Ce^{-\tau/p} \quad \text{for } 0 < \tau < t .
\end{equation}
This follows from the identity
\begin{equation}
\xi_p(y) = e^{-t/p}\xi_p(y_p(0)) + \int_0^t e^{-(t-s)/p}h(y_p(s)) ds .
\end{equation}
\[\square\]

Under the assumptions of Proposition 7.1, we see from (5.2) of Lemma 7.1 holds. Thus we obtain a global asymptotic stability theorem in the case when the function $x \to \beta(x,0)$ is Hölder continuous at $x = 1$. To prove global asymptotic stability under just a continuity assumption on the function $x \to \beta(x,0)$ at $x = 1$, we need to proceed somewhat differently.

Recall that Proposition 8.1 of [3] is a non-linear generalization of Proposition 6.2 of [3]. This is a stability result for solutions to the linear differential delay equation (DDE)
\begin{equation}
\frac{dI(t)}{dt} + \int_0^t k(t,s)[I(t) - I(s)] ds = f(t) , \quad t > 0 ,
\end{equation}
where $k(\cdot, \cdot)$ is non-negative and $f \in L^1(\mathbb{R}^+)$. We first prove a result for solutions to (7.49) when $f \in L^\infty(\mathbb{R}^+)$, and then generalize it to the non-linear case.

**Lemma 7.2.** Assume the function $k(\cdot, \cdot)$ of (7.49) is non-negative, and the function $b(t) = \int_0^t k(t,s) ds$, $t \geq 0$, is bounded. Assume further there exists $\tau > 0$ such that $k(t,s) = 0$ for $t - s > \tau$. Then there exists a constant $C$ such that the solution to (7.49) satisfies the inequality $\|I'(\cdot)\| \leq C\|f(\cdot)\|_{\infty}$.

**Proof.** Assuming $0 \leq T_1 < T_2$, we integrate (7.49) to obtain the formula
\begin{equation}
I(T_2) = I(T_1) \exp \left[ - \int_{T_1}^{T_2} b(t) \ dt \right] + \int_{T_1}^{T_2} \exp \left[ - \int_t^{T_2} b(s) \ ds \right] \left\{ f(t) + \int_t^T k(t,s)I(s) \ ds \right\} dt .
\end{equation}
We rewrite (7.51) as
\[
I(T_2) - I(T_1) = \int_{T_1}^{T_2} \exp \left[ - \int_t^{T_2} b(s) \, ds \right] f(t) \, dt + \delta(T_1, T_2) E[I(\mathcal{T}) - I(T_1)] ,
\]
where
\[
\delta(T_1, T_2) = 1 - \exp \left[ - \int_{T_1}^{T_2} b(t) \, dt \right] ,
\]
and \( \mathcal{T} \) is a random variable with distribution in the interval \([T_1 - \gamma, T_2] \).

For \( n = 1, 2, \ldots \), we may use (7.51) to estimate the oscillation of \( I(\cdot) \) on the interval \([n\tau, (n+1)\tau] \) in terms of the oscillation of \( I(\cdot) \) on the interval \([(n-1)\tau, n\tau] \) and \( \sup |f(\cdot)| \) on \([n\tau, (n+1)\tau] \). We set \( T_1 = n\tau \) and choose \( T_2 \in [n\tau, (n+1)\tau] \) such that \( I(T_2) = \sup_{n\tau < t < (n+1)\tau} I(t) \). Then (7.51) yields the inequality
\[
(7.53) \quad \sup_{n\tau < t < (n+1)\tau} |I(t) - I(n\tau)| \leq \tau \sup_{n\tau < t < (n+1)\tau} |f(t)| + \delta \alpha \left\{ \sup_{n\tau < t < (n+1)\tau} |I(t) - I(n\tau)| \right\} ,
\]
where \( \delta, \alpha \) are given by
\[
(7.54) \quad \delta = 1 - \exp [ -\tau \sup b(\cdot) ], \quad \alpha = P(\mathcal{T} > n\tau) .
\]
It follows from (7.53) that
\[
(7.55) \quad \sup_{n\tau < t < (n+1)\tau} |I(t) - I(n\tau)| \leq \frac{\tau}{1 - \delta \alpha} \sup_{n\tau < t < (n+1)\tau} |f(t)|
\]
\[
+ \frac{\delta (1 - \alpha)}{1 - \delta \alpha} \left\{ \sup_{(n-1)\tau < t < n\tau} |I(t) - I(n\tau)| \right\} .
\]
We may apply a similar argument using (7.51) with \( T_1 = n\tau \) and \( T_2 \in [n\tau, (n+1)\tau] \) such that \( I(T_2) = \inf_{n\tau < t < (n+1)\tau} I(t) \). This yields the inequality
\[
(7.56) \quad \sup_{n\tau < t < (n+1)\tau} |I(n\tau) - I(t)| \leq \frac{\tau}{1 - \delta \alpha} \sup_{n\tau < t < (n+1)\tau} |f(t)|
\]
\[
+ \frac{\delta (1 - \alpha)}{1 - \delta \alpha} \left\{ \sup_{(n-1)\tau < t < n\tau} |I(n\tau) - I(t)| \right\} .
\]
Adding (7.55) and (7.56) we obtain the estimate
\[
(7.57) \quad \sup_{n\tau < s, t < (n+1)\tau} |I(t) - I(s)| \leq \frac{2\tau}{1 - \delta} \sup_{n\tau < t < (n+1)\tau} |f(t)|
\]
\[
+ \delta \left\{ \sup_{(n-1)\tau < s, t < n\tau} |I(t) - I(s)| \right\} .
\]
We conclude from (7.57) that
\[
(7.58) \quad \sup_{n\tau < s, t < (n+1)\tau} |I(t) - I(s)| \leq \frac{2\tau (1 - \delta^n)}{(1 - \delta)^2} \|f(\cdot)\|_\infty
\]
\[
+ \delta^n \sup_{0 < s, t < \tau} |I(t) - I(s)| , \quad \text{for} \ n = 1, 2, \ldots
Following the same argument as before, we have from (7.51) that

\[ (7.59) \quad \sup_{0 < s, t < \tau} |I(t) - I(s)| \leq \frac{2\tau}{(1 - \delta)^2} \|f(\cdot)\|_\infty. \]

We conclude from (7.58), (7.59) that

\[ (7.60) \quad \sup_{n\tau < s, t < (n+1)\tau} |I(t) - I(s)| \leq \frac{2\tau}{(1 - \delta)^2} \|f(\cdot)\|_\infty \quad \text{for } n = 0, 1, 2, \ldots \]

It follows now from (7.60) that

\[ (7.61) \quad \|I(\cdot)\|_\infty \leq \left[ \frac{4\tau \sup_{s} b(\cdot)}{(1 - \delta)^2} + 1 \right] \|f(\cdot)\|_\infty. \]

\[ \square \]

**Remark 6.** Lemma 7.2 implies a result for Volterra integral equations. Thus consider the integral equation

\[ (7.62) \quad u(t) + \int_0^t K(t, s)u(s)\,ds = f(t), \quad t > 0, \]

with continuous kernel \( K(t, s), \quad 0 \leq s \leq t < \infty \). Assume the functions \( s \to K(t, s), \quad 0 \leq s \leq t \), are increasing for all \( t > 0 \), there exists \( \tau > 0 \) such that \( K(t, s) = 0 \) for \( t - s > \tau \), and that \( \sup_{t \geq 0} K(t, t) < \infty \). Then there is a constant \( C \) such that the solution to (7.62) satisfies \( \|u(\cdot)\|_\infty \leq C\|f(\cdot)\|_\infty \).

One should compare this result to the analogous result of Gripenberg (Theorem 9.1 of Chapter 9 of [3]), given as Proposition 6.1 of [3]. The monotonicity assumption on \( K(\cdot, \cdot) \) in this case is that the functions \( t \to K(t, s) \) are decreasing on \([s, \infty)\) for all \( s \geq 0 \).

**Proposition 7.2.** Let \( h(\cdot), \xi(\cdot, \cdot) \) satisfy the assumptions of Proposition 7.1, \( \eta(\cdot) \) satisfy (3.43), and the function \( x \to \beta(x, 0) \) be continuous at \( x = 1 \). Then (7.33) holds.

**Proof.** We first observe that (7.1) holds, but not (7.2) in general. We replace (7.2) by the inequality

\[ (7.63) \quad c_\alpha e^{-\alpha(t-s)} \leq \left( \frac{I(t)}{I(s)} \right)^{1/p} \leq C_\alpha e^{\alpha(t-s)} \quad 0 \leq s \leq t < \infty, \]

which is valid for any \( \alpha > 0 \), where \( C_\alpha, c_\alpha \) are positive constants depending on \( \alpha \). This follows from (3.49), (3.50) and (2.68) of Lemma 2.5 upon integrating (3.55) over the interval \([s, t]\). It follows from (7.63) that the function \( G \) of (7.12) satisfies an inequality \( |G(t, y, v_\ell(\cdot))| \leq C_\alpha e^{-t(1/p - \alpha)}, \quad y \geq \varepsilon_0, \quad t \geq 0 \), and hence that \( |g(t, v_\ell(\cdot))| \leq C_\alpha e^{-t(1/p - \alpha)}, \quad t \geq 0 \), where \( \alpha > 0 \) can be arbitrarily small, and the constant \( C_\alpha \) depends on \( \alpha > 0 \). We also have from (2.67) of Lemma 2.4 that the function \( \gamma(\cdot) \) on the RHS of (4.15) is bounded and \( \lim_{t \to \infty} \gamma(t) = 0 \).

We proceed now as in the proof of Proposition 7.1 by writing the function \( F \) of (6.4) as \( F = F_1 + F_2 \), where \( F_1 \) is given by (7.3), but with the interval of integration \([\max\{t - \tau, 0\}, t]\) in place of \([\varepsilon_r, \tau', \tau, t]\). From (7.63) we see that \( |F_2(t, y, v_\ell(\cdot))| \leq C_\alpha e^{-t(1/p - \alpha)}, \quad y \geq \varepsilon_0, \quad t \geq \tau \), where \( \alpha > 0 \) can be arbitrarily small. Evidently \( F_1(t, y, v_\ell(\cdot)) \) depends only on the values of \( I(s) \) for \( s \in [t - \tau, \tau] \). Let \( I^* > 0 \) be a constant and define \( \bar{v}_\ell(s) = I(s)^{1/p}/(I^*)^{1/p} \) for \( \max\{t - \tau, 0\} < s < t \), and \( \bar{z}(s), \quad s \in [\max\{t - \tau, 0\}, t] \) as the solution to (5.4) with \( \bar{v}_\ell(\cdot) \) replacing \( v_\ell(\cdot) \).
Let \( \tilde{F}_1(t, y, \tilde{v}_t(\cdot)) \) be defined in the same way as \( F_1(t, y, v_t(\cdot)) \), but with \( v_t(\cdot) \), \( \tilde{z}(\cdot) \) replacing \( v_t(\cdot) \), \( \tilde{z}(\cdot) \). We write the second term on the LHS of (5.16) as \( f(t, v_t(\cdot)) = f_1(t, v_t(\cdot)) + f_2(t, v_t(\cdot)) \), corresponding to the decomposition \( F = F_1 + F_2 \). We also denote by \( \tilde{f}_1(t, \tilde{v}_t(\cdot)) \) the function (5.16) corresponding to \( \tilde{F}_1 \) in place of \( F \). On setting \( J(t) = \log \left[ (I^*)^{1/p} / I(t)^{1/p} \right] \), we see that (5.15) is equivalent to the equation

\[
(7.64) \quad \frac{dJ(t)}{dt} + \tilde{f}_1(t, \tilde{v}_t(\cdot)) - f_1(t, v_t(\cdot)) = \tilde{f}_1(t, \tilde{v}_t(\cdot)) + \Gamma(t),
\]

where \( \Gamma(t) = f_2(t, v_t(\cdot)) - g(t, v_t(\cdot)) - \gamma(t) \).

Assume now that for some given \( T_1 \geq \tau \), the constant \( I^* \) has been chosen so that \( J(\cdot) \) is non-negative in the interval \([T_1 - \tau, T_1]\). Let \( T_2 > T_1 \) be such that \( J(s) \geq 0 \) for \( T_1 \leq s \leq T_2 \). Then \( \tilde{f}_1(t, \tilde{v}_t(\cdot)) \geq 0 \) for \( T_1 \leq s \leq T_2 \). Note from (7.63) that

\[
(7.65) \quad \frac{1}{C} e^{-\alpha \tau} \leq \frac{1}{v_t} e^{\alpha \tau}, \quad \max\{t - \tau, 0\} \leq s \leq t.
\]

We also have that \( \tilde{v}_t(\cdot) \leq 1 \), so \( \tilde{z}(s)/\tilde{v}_t(s) > \varepsilon_0 \) for \( s < t \). Hence we may argue as in the proof of Proposition 7.1 that

\[
(7.66) \quad |f_1(t, v_t(\cdot)) - \tilde{f}_1(t, \tilde{v}_t(\cdot))| \leq C(\tau) J(t), \quad T_1 < t < T_2,
\]

where the constant \( C(\tau) \) depends on \( \tau \) since we need to use the inequality (7.65). Integrating (7.64), using (7.66) and the non-negativity of \( \tilde{f}_1 \), we obtain the inequality

\[
(7.67) \quad J(T_2) \geq \exp \left[ -C(\tau) \{T_2 - T_1\} \right] J(T_1) - \int_{T_1}^{T_2} \exp \left[ -C(\tau) \{T_2 - t\} \right] \left| \Gamma(t) \right| dt.
\]

We can use (7.67) to estimate the oscillation of \( I(\cdot) \) in the interval \([n\tau, (n + 1)\tau]\) in terms of the oscillation of \( I(\cdot) \) in the interval \([(n - 1)\tau, n\tau]\), \( n = 1, 2, \ldots \). We set \( T_1 = n\tau \) and define \( I^* \) in such a way that the RHS of (7.67) is non-negative for \( T_1 \leq t \leq T_2 \leq (n + 1)\tau \). Thus we require \( I^* \) to satisfy the inequality

\[
(7.68) \quad \log \left( \frac{I^*}{I(n\tau)} \right)^{1/p} \geq C(\tau)^{-1} \left[ e^{\tau C(\tau)} - 1 \right] \sup_{n\tau < t < (n + 1)\tau} \left| \Gamma(t) \right|.
\]

We also require \( I^* \geq \sup_{(n - 1)\tau < t < n\tau} I(t) \) so that \( J(t) \geq 0 \) for \((n - 1)\tau < t < n\tau\). Hence we define \( I^* \) as

\[
(6.99) \quad \log \left( \frac{I^*}{\sup_{(n - 1)\tau < t < n\tau} I(t)} \right)^{1/p} = C(\tau)^{-1} \left[ e^{\tau C(\tau)} - 1 \right] \sup_{n\tau < t < (n + 1)\tau} \left| \Gamma(t) \right|.
\]

In that case (7.64) holds for all \( T_2 \) such that \( n\tau < T_2 < (n + 1)\tau \), whence we obtain the inequality

\[
(7.70) \quad \log \left[ \left( \frac{I^*}{I(t)} \right)^{1/p} \right] \geq e^{-\tau C(\tau)} \log \left[ \left( \frac{I^*}{I(n\tau)} \right)^{1/p} \right] - C(\tau)^{-1} \left[ 1 - e^{-\tau C(\tau)} \right] \sup_{n\tau < t < (n + 1)\tau} \left| \Gamma(t) \right| \quad \text{for} \quad n\tau < t < (n + 1)\tau.
\]

Analogously to the previous paragraphs, we let \( I_0 \) be a constant and set \( J(t) = \log \left[ (I(t))^{1/p} / I_0^{1/p} \right] \). Defining \( \tilde{v}_t(\cdot) = I(s)^{1/p} / I_0^{1/p} \), we see as in (7.64) that \( J(t) \) is a
integrate (7.71) the inequality (7.67) again. We choose

\[ T_{\tau} \]

given by

\[ f \]

Assume now that (7.72) $\log \frac{1}{I_{\tau}}$ holds for all $T_{\tau} > T_{1}$. If $T_{\tau} > T_{1} > T_{\tau}$ is such that $J(t) = 0$ for $T_{\tau} < t < T_{\tau}$, then the function $f(t, v_{1}(\cdot))$ is negative for $T_{\tau} < t < T_{\tau}$. Hence if (7.66) holds, we obtain upon integrating (7.71) the inequality (7.67) again. We choose $T_{1} = n\tau$ and $I_{\tau}$ to be given by

\[ (7.72) \]

\[ \log \left( \inf_{t \leq n\tau} f(t, v_{1}(\cdot)) \right)^{1/p} = C(\tau)^{-1} \left[ e^{-\tau C(\tau)} - 1 \right] \sup_{n\tau < t < (n+1)\tau} |\Gamma(t)|. \]

In that case (7.70) holds for all $T_{\tau} > T_{2} < (n+1)\tau$, whence we obtain the inequality

\[ (7.73) \]

\[ \log \left( \frac{I(t)}{I_{\tau}} \right)^{1/p} \geq e^{-\tau C(\tau)} \left( \frac{I(n\tau)}{I_{\tau}} \right)^{1/p} \sup_{n\tau < t < (n+1)\tau} |\Gamma(t)| \text{ for } n\tau < t < (n+1)\tau. \]

\[ (7.74) \]

\[ \log \left( \frac{I(s)}{I(t)} \right)^{1/p} \geq e^{-\tau C(\tau)} \left( \frac{I(n\tau)}{I_{\tau}} \right)^{1/p} \sup_{n\tau < t < (n+1)\tau} |\Gamma(t)| \text{ for } n\tau < s, t < (n+1)\tau. \]

Upon taking the infimum of the LHS of (7.73) over $n\tau < s, t < (n+1)\tau$, and using the formulae (7.69), (7.72) for $I_{\tau}, I_{\tau}$, we obtain the estimate

\[ (7.75) \]

\[ \left\{ 1 - e^{-\tau C(\tau)} \right\} \left[ \log \left( \frac{\sup_{n\tau < t < (n+1)\tau} f(t, v_{1}(\cdot))}{\inf_{n\tau < t < (n+1)\tau} f(t, v_{1}(\cdot))} \right)^{1/p} \right] \leq \left[ 2 \exp[\tau C(\tau)] \sup_{n\tau < t < (n+1)\tau} |\Gamma(t)| \right]. \]

Arguing now as in the proof of Lemma 7.2 we see from (7.70) that for any integers $n \geq N \geq 1$ there is the inequality

\[ (7.76) \]

\[ \left\{ 1 - e^{-\tau C(\tau)} \right\} \left[ \log \left( \frac{\sup_{n\tau < t < (n+1)\tau} f(t, v_{1}(\cdot))}{\inf_{n\tau < t < (n+1)\tau} f(t, v_{1}(\cdot))} \right)^{1/p} \right] \leq \left[ 2 \exp[\tau C(\tau)] \sup_{t > N\tau} |\Gamma(t)| \right]. \]

The property (7.33) follows now from (7.69), (7.70) upon using the fact that $\lim_{t \to \infty} \Gamma(t) = 0$. 

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We are left to establish that \( \gamma(s) = I(s)^{1/p}/I^{1/p} \), where \( I \) is given by \( \int_{0}^{1} \). To see this we first observe that \( y(s) = \tilde{y}(s)/\tilde{v}(s) \), \( s < t \), is a solution to (5.16) with terminal data \( y(t) = \tilde{z}(t)/\tilde{v}(t) \leq z(t) = y \), where \( z(\cdot) \) is the solution to the terminal value problem \( \text{Theorem 8.1.} \)

Theorem 8.1. Let \( h(\cdot) \) satisfy the assumptions of Lemma 7.1 and \( \xi(\cdot), t > 0 \), be the solution of \( \text{Proposition 7.1.} \) If \( \eta(\cdot) \) satisfies \( \text{Lemma 7.1} \) then

\[
\lim_{t \to \infty} \|\xi(\cdot), t - \xi_p(\cdot)\|_{1, \infty} = 0, \quad \lim_{t \to \infty} \left[ \rho(\xi(\cdot), t), \eta(\cdot) \right] = 0.
\]

Suppose in addition the inequality \( \eta(t) \leq C_1 e^{-\nu_1 t} \), \( t > 0 \), holds for some constants \( C_1, \nu_1 > 0 \), and the function \( x \to \beta(x, 0) \) is Hölder continuous at \( x = 1 \). Then there are constants \( C_2 > 0 \) and \( \nu_2 \), \( 0 < \nu_2 \leq 1/p \), such that

\[
\|\xi(\cdot), t - \xi_p(\cdot)\|_{1, \infty} \leq C_2 e^{-\nu_2 t}, \quad \left| \rho(\xi(\cdot), t), \eta(\cdot) \right| = C_2 e^{-\nu_2 t}, \quad t \geq 0.
\]

Proof. From \( \text{Proposition 7.2} \) we see that the condition \( 7.33 \) of Lemma 7.1 holds. The convergence of \( \xi(\cdot), t \) to \( \xi_p(\cdot) \) then follows from Lemma 7.1. The convergence of \( \rho(\xi(\cdot), t), \eta(\cdot) \) follows now from \( 4.1, 1 \) upon writing \( \rho(\xi(\cdot), t), \eta(\cdot) = \rho_p(\xi(\cdot), t) + \gamma(t) \), noting we have already shown that \( \gamma(t) \) converges to 0. The exponential convergence \( 7.79 \) can already be obtained from Proposition 7.1 since the exponential decay of \( \eta(\cdot) \) implies the exponential decay of \( \gamma(\cdot) \).

8. Asymptotic Stability for the LSW Model

Just as we obtained the proof of Theorem 1.2 from Theorem 3.1, we show here that Theorem 7.1 enables us to generalize Theorem 1.3 beyond the case of quadratic \( \phi(\cdot) \) and \( \psi(\cdot) \).

Theorem 8.1. Let \( w(x, t), x, t \geq 0 \), be the solution to \( 1.12, 1.13 \) with coefficients satisfying \( 1.13, 1.14, 1.27 \), and assume that the initial data \( w(\cdot, 0) \) has beta function \( \beta(\cdot, 0) \) satisfying \( 1.19 \) with \( 0 < \beta_0 < 1 \). Assume also that the function \( h : [\varepsilon_0, \infty) \to \mathbb{R} \) defined by \( 2.3 \) is convex, that \( 4.9 \) holds for all \( p > 0 \) and
also (8.7). Then setting \( \kappa = [1/\beta_0 - \phi'(1) - 1]/|\psi'(1)| \) one has

\[
\lim_{t \to \infty} \kappa(t) = \kappa, \quad \lim_{t \to \infty} \|\beta(\cdot, t) - \beta_\kappa(\cdot)\|_\infty = 0,
\]

where \( \beta_\kappa(\cdot) \) is the beta function of the time independent solution \( w_\kappa(\cdot) \) of (1.12). If the function \( x \to \beta(x, 0) \) is Hölder continuous at \( x = 1 \) then the convergence in (8.2) is exponential:

\[
|\kappa(t) - \kappa| \leq Ce^{-\nu t}, \quad \|\beta(\cdot, t) - \beta_\kappa(\cdot)\|_\infty \leq Ce^{-\nu t}, \quad t \geq 0,
\]

for some constants \( C, \nu > 0 \).

Proof. We show that (8.4) holds for the function \( h(\cdot) \) defined by (2.3). Given the further properties of \( h(\cdot) \) established in the proof of Theorem 1.2 in §3, we conclude that \( h(\cdot) \) satisfies all the assumptions of Theorem 7.1. To prove (8.4) we use the formula

\[
h'(y) = \frac{\phi(x)}{\psi(x)}|\psi'(x) + \psi'(1)| - [\phi'(x) + \phi'(1)],
\]

which is equivalent to (6.9) of [4]. One easily sees from (2.3), (8.3) that \( \lim_{y \to \infty} yh'(y) = 0 \). It follows from this, upon using Taylor’s theorem about \( y \) and \( h(\cdot) \), also (5.19). Then setting \( \nu h''(y) = 0 \), we obtain more precise information on the behavior of \( h''(y) \) as \( y \to \infty \) by differentiating (8.3) with respect to \( x \). Upon using (2.3) we then obtain the formula

\[
yh''(y) = \frac{\phi(x)\psi'(x) - \phi'(x)\psi(x)}{\psi'(1)} [\psi'(x) + \psi'(1)] + \frac{\phi''(x)\psi(x) - \phi(x)\psi''(x)}{\psi'(1)}.
\]

We see from (2.3), (8.4) that

\[
\lim_{y \to \infty} y^3h''(y) = \frac{1}{3\omega_0^2\psi'(1)}[\phi''(1)\psi'(1) - \phi'(1)\psi''(1)].
\]

Note that (1.15), (1.16), (1.20) imply the RHS of (8.5) is non-negative, whence the function \( y \to h(y) \) is convex for large \( y \).

In order to apply Theorem 7.1 we need to show that \( \|\xi(\cdot, T_0)\|_{1, \infty} < \infty \) for all sufficiently large \( T_0 \). We have already shown this in the proof of Theorem 1.2 in §3. Applying now Theorem 7.1 and using the identity (5.56) we have that \( \lim_{t \to \infty} \kappa(t) = \kappa \). To show that \( \beta(\cdot, t) \) converges to \( \beta_\kappa(\cdot) \) we argue as in the proof of Proposition 3.1 of [4]. Thus it is sufficient to show that the function \( g(x, t) = w(x, t)/w_\kappa(x) \) satisfies an inequality

\[
(1 - x) \frac{\partial}{\partial x} \log g(x, t) \leq \delta(t), \quad 0 \leq x < 1, \quad t \geq 0,
\]

where \( \lim_{t \to \infty} \delta(t) = 0 \). In the case of \( \beta(x, 0) \) being Hölder continuous at \( x = 1 \) then \( |\delta(t)| \leq Ce^{-\nu t}, \quad t \geq 0 \), for some positive constants \( C, \nu \). To prove (8.6) we observe from (1.12), (2.35), upon using the identity \( w(x, t) = e^{\Phi t}F(x, t, 0) \), that

\[
\frac{\partial}{\partial x} \log g(x, t) = \frac{1}{\kappa\psi(x) - \phi(x)} - \frac{\partial F(x, t)}{\partial x} \int_0^1 \frac{\beta(F(x, s), 0)}{1 - F(x, s)} ds - \frac{\beta(F(x, t), 0)}{\partial F(t)}.
\]

From (5.50), (5.51) we have that

\[
\frac{1}{1 - F(x, t)} \frac{\partial F(x, t)}{\partial x} = \frac{f(x)\psi(F(x, t))w(t)}{1 - F(x, t)\psi(x)f(F(x, t))} [1 + D\xi(f(x)/\alpha_0, t)]
\]

for some \( D \).
Remark 7. Theorem 3.1 of [11] establishes a necessary condition for local asymptotic stability of the LSW model in the case when the functions \( \alpha \) given by (1.14) with asymptotic stability of the LSW model in the case when the functions

\[
\text{(8.13)} \lim_{t \to \infty} E \quad \text{Evidently (8.14) holds if}
\]

From (2.3), (8.12) we have that

\[
\text{(8.10)} \lim_{t \to \infty} \psi(x) \frac{\partial F(x, t)}{\partial x} \frac{\beta(F(x, t), 0)}{\int_{F(x, t)} 1 - \beta(x', 0) \, dx'}
\]

It follows from (2.9), (8.11) that the expression on the RHS of (8.10) is the same as \( 1 \). Hence the inequality \( (8.6) \) with \( \lim_{t \to \infty} \delta(t) = 0 \) holds. It was already shown in the proof of Proposition 3.1 of [4] that \( (8.6) \) implies \( \lim_{t \to \infty} \|\beta(-, t) - \beta_{\alpha}(\cdot)\|_{\infty} = 0 \). We have established (8.1). A similar argument gives \( (8.2) \) in the case when the function \( x \to \beta(x, 0) \) is Hölder continuous at \( x = 1 \).

Remark 7. Theorem 3.1 of [11] establishes a necessary condition for local asymptotic stability of the LSW model in the case when the functions \( \phi(\cdot), \psi(\cdot) \) are given by (1.14) with \( \alpha = 1/3 \). The condition is given in terms of the function

\[
\text{(8.12)} \quad z = \int_{0}^{x} \frac{dx'}{x' - x'^{1/3} + \kappa[1 - x'^{1/3}]} = \langle |\psi'(1)| \int_{0}^{x} \frac{y \, dx'}{\psi(x') [h(y) + y/p]} \rangle, \quad \text{with } y = y(x').
\]

From (2.3), (8.12) we have that

\[
\text{(8.13)} \quad \frac{dy}{dx} = -\frac{w \psi'(1)}{\psi(x)}, \quad \frac{dz}{dx} = -\frac{1}{\kappa \psi(x) - \phi(x)} = \frac{|\psi'(1)|y}{\psi(x) [h(y) + y/p]}.
\]

The necessary condition on \( S_{0}(\cdot) \) is that

\[
\text{(8.14)} \quad \lim_{z \to \infty} \sup \frac{|S_{0}(z) - S_{0}(y)|}{S_{0}(z)} = 0.
\]

Evidently \( (8.14) \) holds if \( S_{0}(\cdot) \) is \( C^{1} \) and

\[
\text{(8.15)} \quad \lim_{z \to \infty} \frac{S_{0}'(z)}{S_{0}(z)} = \lim_{z \to \infty} \frac{d}{dz} \log S_{0}(z) = 0.
\]

From (2.3), (8.13) we have that

\[
\text{(8.16)} \quad -\frac{d}{dz} \log w(x, 0) = \frac{\psi(x)}{|\psi'(1)|} \left[ \frac{h(y)}{y} + \frac{1}{p} \right] \beta(x, 0) / \int_{x}^{1} [1 - \beta(x', 0)] \, dx'.
\]
Taking $z \to \infty$ in (8.10) and using (1.12) we have that

$$\lim_{z \to \infty} \frac{d}{dz} \log w(x, 0) = -\frac{\beta_0}{p(1 - \beta_0)} = -1.$$  \hspace{1cm} (8.17)

Evidently (8.17) implies (8.15).

Theorem 3.2 of [11] proves local asymptotic stability. The condition on the initial data is that

$$\text{sup}_{z \geq 0} \text{sup}_{z \leq y \leq z + 1} \frac{|S_0(z) - S_0(y)|}{S_0(z)} < \varepsilon \text{ for sufficiently small } \varepsilon,$$

and also that (8.14) holds. Observe that if we set $\beta = \beta_\kappa$, then the LHS is equal to $-1$. It follows there exists $\delta > 0$, depending on $\varepsilon$, such that if $\|\beta(\cdot, 0) - \beta(\cdot, \cdot)\|_\infty < \delta$ then (8.18) holds.

Next we obtain conditions on the functions $\phi(\cdot), \psi(\cdot)$ which imply that (1.9) and (5.19) hold.

**Lemma 8.1.** Assume that $\phi(\cdot), \psi(\cdot)$ satisfy (1.12), (1.10), (1.20) and $h(\cdot)$ is defined by (2.9). Then (4.9) holds for all $p > 0$ if and only if the function

$$x \to \frac{\phi'(x) + \phi'(1)\psi(x) - \{\psi'(x) + \psi'(1)\} \phi(x)}{\psi'(1)\phi(x) - \phi'(1)\psi(x)}$$

is decreasing.

The inequality (5.19) holds if and only if the function

$$x \to \psi(x)\phi'(x) + \psi'(x)\phi(x)$$

is decreasing.

**Proof.** We first reformulate the condition (4.9) at $p = \infty$. Setting $z = \log y$, $y \geq \varepsilon_0$, and $G(z) = h(y)$, then we have that

$$G'(z) = yh'(y), \quad G''(z) = y^2h''(y) + yh'(y).$$

From (8.21) we see that (4.9) at $p = \infty$ is equivalent to the inequality

$$G(z)G''(z) \geq G'(z)^2 \quad \text{or} \quad \frac{d^2}{dz^2} \log G(z) \geq 0.$$  \hspace{1cm} (8.22)

Since $z$ is an increasing function of $x$ the inequality (8.22) is equivalent to showing that

$$\frac{d}{dx} \frac{G'(z)}{G(z)} \geq 0.$$  \hspace{1cm} (8.23)

From (2.9) we have that

$$G(z) = \frac{f(x)}{\alpha_0 \psi(x)} \left[\psi'(1)\phi(x) - \phi'(1)\psi(x)\right].$$  \hspace{1cm} (8.24)

Recalling that $y = f(x)/\alpha_0$ we have upon differentiating (8.24) using (2.3) that

$$G'(z) = \frac{f(x)}{\alpha_0 \psi(x)} \left[\{\psi'(x) + \psi'(1)\} \phi(x) - \{\phi'(x) + \phi'(1)\} \psi(x)\right].$$  \hspace{1cm} (8.25)

It follows from (8.24), (8.25) that (8.23) is equivalent to (8.19).

To see (8.19) we observe from (8.21) that

$$\frac{y^2h''(y)}{h(y) - yh'(y)} = \frac{G''(z) - G'(z)}{G(z) - G'(z)}.$$  \hspace{1cm} (8.26)
It follows from (8.26) that (5.19) is equivalent to
\[
(8.27) \quad \frac{d}{dx} \left[ \frac{G''(z) - G'(z)}{G(z) - G'(z)} \right] \leq 0.
\]
From (8.24), (8.25) we have that
\[
(8.28) \quad G(z) - G'(z) = \frac{f(x)}{\alpha \psi(x)} [\phi'(x)\psi(x) - \phi(x)\psi'(x)].
\]
Differentiating (8.28) using (2.3) again we have that
\[
(8.29) \quad |\psi'(1)| |G''(z) - G'(z)| = \frac{f(x)}{\alpha \psi(x)} \left[ \psi(x) \{ \phi(x)\psi''(x) - \phi''(x)\psi(x) \right)
+ \{ \phi'(x)\psi(x) - \phi'(x)\phi(x) \} [\psi'(x) + \psi'(1)].
\]
One sees from (8.28), (8.29) that (8.27) and (8.24) are equivalent. \(\square\)

Unlike (1.15), (1.16), (1.20) the conditions (8.19), (8.20) are not immediately checkable for the functions \(\phi(\cdot), \psi(\cdot)\) of (1.14). However they do hold for these functions provided \(0 < \alpha < 1\).

**Lemma 8.2.** Let \(\phi(\cdot), \psi(\cdot)\) be the functions (1.14) for some \(\alpha\) with \(0 < \alpha < 1\). Then (8.19), (8.20) hold.

**Proof.** We have that
\[
(8.30) \quad \psi'(1)\phi(x) - \phi'(1)\psi(x) = (1 - \alpha) - x^\alpha + \alpha x,
\]
and also
\[
(8.31) \quad |\phi'(x) + \phi'(1)|\psi(x) - [\psi'(x) + \psi'(1)]\phi(x)
= \alpha x^{\alpha-1} - (2 - \alpha) + (2 - \alpha)x^\alpha - \alpha x.
\]
Hence (8.19) becomes the function
\[
(8.32) \quad x \rightarrow \frac{\alpha x^{\alpha-1} - (2 - \alpha) + (2 - \alpha)x^\alpha - \alpha x}{(1 - \alpha) - x^\alpha + \alpha x}
\]
is decreasing.

In order to prove (8.32) we need to show that the function
\[
(8.33) \quad g(x) = [\alpha x^{\alpha-1} - (2 - \alpha) + (2 - \alpha)x^\alpha - \alpha x][-x^{\alpha-1} + 1]
- [(1 - \alpha) - x^\alpha + \alpha x][-\alpha x^\alpha + (2 - \alpha)x^\alpha - 1] \quad \text{is positive.}
\]
We have that
\[
(8.34) \quad g(x) = (1 - \alpha)^2 x^{\alpha-2} - x^{2\alpha-2} + 2\alpha(2 - \alpha)x^{\alpha-1} - 1 + (1 - \alpha)^2 x^\alpha.
\]
Note that \(g(1) = 0\), whence to prove (8.34) it is sufficient to show that \(g'(x)\) is negative. We have that \(g'(x) = -(1 - \alpha)x^{\alpha-1} g_1(x)\) where
\[
(8.35) \quad g_1(x) = (1 - \alpha)(2 - \alpha)x^\alpha - 2x^{\alpha-2} + 2\alpha(2 - \alpha)x^\alpha - x^\alpha - 1 - (1 - \alpha).
\]
Since \(g_1(1) = 0\) it is sufficient for us to show that \(g_1'(x)\) is negative. Setting \(g_1'(x) = -2(2 - \alpha)x^{-2} g_2(x)\), we need to show that the function
\[
(8.36) \quad g_2(x) = (1 - \alpha)x^\alpha - x^\alpha - x^{\alpha-1} + \alpha
\]
is positive. Observe that \(g_2(1) = 0\), whence it is sufficient for us to show that \(g_2'(x)\) is negative. This is clear since \(g_2'(x) = -(1 - \alpha)x^{-2}[1 - x^\alpha].\)
To prove (8.20) we note that

\[ (8.37) \quad\frac{\phi'(x)\psi(x) - \phi(x)\psi'(x)}{x} = \alpha x^{\alpha - 1} - 1 + (1 - \alpha)x^\alpha, \]
\[ \phi(x)\psi''(x) - \phi''(x)\psi(x) = \alpha(1 - \alpha)[x^{\alpha - 2} - x^{\alpha - 1}] \, . \]

From (8.37) we see that the function in (8.20) is given by

\[ (8.38) \quad \frac{\alpha(1 - \alpha)x^{\alpha - 2} - x^{2\alpha - 2} - x^{\alpha - 1} + x^{2\alpha - 1}}{\alpha x^{\alpha - 1} - 1 + (1 - \alpha)x^\alpha} - \alpha[x^{\alpha - 1} + 1] \, . \]

We need therefore to show that the function (8.39) is given by

\[ (8.39) \quad x \to (1 - \alpha)x^{\alpha - 2} - x^{\alpha - 1} + 2(2 - \alpha)x^{\alpha - 2} - [x^{\alpha - 1} + 1] \text{ is decreasing} \, . \]

The derivative of the function (8.39) is given by

\[ (8.40) \quad (1 - \alpha)\left[ \frac{g(x)}{\alpha x^{\alpha - 1} - x^{-\alpha} + (1 - \alpha)]x^{\alpha - 2} \right] \, , \]

where

\[ (8.41) \quad g(x) = -\alpha x^{-4} - 4(1 - \alpha)x^{-3} + (2 - 3\alpha)x^{-2} + \alpha(1 - \alpha)x^{-4} + (2 - 3\alpha + 2\alpha^2)x^{\alpha - 3} - (1 - \alpha)^2x^{\alpha - 2} + (2 - \alpha)x^{-3} - (1 - \alpha)x^{-2} \, . \]

We have that

\[ (8.42) \quad g(x) + x^{\alpha - 2}[\alpha x^{\alpha - 1} - x^{-\alpha} + (1 - \alpha)] = -x^{\alpha - 3}g_1(x) \, , \]

where

\[ (8.43) \quad g_1(x) = \alpha x^{\alpha - 1} + 2(2 - \alpha)x^\alpha + \alpha x^{\alpha + 1} - \alpha x^{2\alpha - 1} - (2 - \alpha)x^{2\alpha} - (2 - \alpha) - \alpha x \, . \]

We need to show that \( g_1(\cdot) \) is positive. Since \( g_1(1) = 0 \) we consider the derivative \( g'_1(x) = -\alpha g_2(x) \), where

\[ (8.44) \quad g_2(x) = (1 - \alpha)x^{\alpha - 2} - 2(2 - \alpha)x^{\alpha - 1} - (\alpha + 1)x^\alpha + (2\alpha - 1)x^{2\alpha - 2} + 2(2 - \alpha)x^{2\alpha - 1} + 1 \, . \]

It is sufficient then to show that \( g_2(\cdot) \) is positive. Since \( g_2(1) = 0 \) we may consider the derivative \( g'_2(x) = -x^{\alpha - 2}g_3(x) \), where

\[ (8.45) \quad g_3(x) = (1 - \alpha)(2 - \alpha)x^{\alpha - 1} - 2(1 - \alpha)(2 - \alpha) + \alpha(\alpha + 1)x^\alpha \]

\[ + (2 - (\alpha - 1)x^{\alpha - 1} - 2(2 - \alpha)(2\alpha - 1)x^\alpha \, . \]

Again it is sufficient to show that \( g_3(\cdot) \) is positive. Since \( g_3(1) = 0 \) we consider \( g'_3(x) \) given by the formula

\[ (8.46) \quad g'_3(x) = -(1 - \alpha)(2 - \alpha)x^{-2} + \alpha(\alpha + 1) \]

\[ - 2(1 - \alpha)^2(2\alpha - 1)x^{\alpha - 2} - 2(2 - \alpha)(2\alpha - 1)x^{\alpha - 1} \, . \]

Since \( g'_3(1) = 0 \) we evaluate \( g''_3(x) \), which is given by the formula

\[ (8.47) \quad g''_3(x) = 2(1 - \alpha)(2 - \alpha)x^{-3} + 2(2 - \alpha)(1 - \alpha)^2(2\alpha - 1)x^{\alpha - 3} + 2(1 - \alpha)(2 - \alpha)(2\alpha - 1)x^{\alpha - 2} \]

\[ = 2(1 - \alpha)(2 - \alpha)\left[ x^{-3} + (2\alpha - 1)(1 - \alpha)x^{\alpha - 3} + \alpha x^{\alpha - 2} \right] \, . \]
It is evident the RHS of (8.47) is non-negative provided 0 < α < 1, whence we conclude that $g_3(\cdot)$ is positive in this case.

**Remark 8.** We wish to relate our computations for local asymptotic stability to those of [11]. They define a function

$$H(z) = \frac{1 - x^{1/3}}{x - x^{1/3} + \kappa(1 - x^{1/3})} = \frac{\left|\psi'(1)\right|y}{h(y) + y/p},$$

where $z$ is given by (8.12). We have that

$$H'(z) = \left|\psi'(1)\right| \frac{[h(y) - yh'(y)]}{[h(y) + y/p]^2} \frac{dy}{dz}.$$

It follows from (8.13) that

$$dy/dz = h(y) + y/p.$$

We then have from (8.49), (8.50) that

$$H'(z) = \left|\psi'(1)\right| \frac{[h(y) - yh'(y)]}{h(y) + y/p}.$$

Differentiating (8.51) with respect to $y$ yields the formula

$$\frac{d}{dy}H'(z) = -\left|\psi'(1)\right| \frac{[h(y) + y/p]yh''(y) + [h'(y) + 1/p][h(y) - yh'(y)]}{[h(y) + y/p]^2}.$$

Since $y$ is an increasing function of $z$ we conclude from (8.52) that $H''(\cdot) < 0$ if (4.9) holds. The condition $H''(\cdot) < 0$ is required in the proof of local stability for the LSW model in §3 of [11]. Note that (4.9) is a slightly stronger condition than the condition $H''(\cdot) < 0$. This is not surprising since Lemma 4.1 proves that the kernel of the Volterra integral equation decays exponentially at rate $1/p$. For asymptotic stability all one needs is some exponential decay of the kernel.

**Proof of Theorem 1.1.** We first recall the transformation which converts the system (1.1), (1.2) to the system (1.12), (1.13). We define the function $w : [0, \infty) \times \mathbb{R}^+ \rightarrow \mathbb{R}$ in terms of the solution $c(\cdot, \cdot)$ to (1.1), (1.2) by

$$w(x,t) = \int_x^\infty c(x',t) \, dx', \quad x,t \geq 0.$$

Evidently $w(\cdot, \cdot)$ is the solution to the system

$$\frac{\partial w(x,t)}{\partial t} = \left(1 - (xL^{-1}(t))^\alpha\right) \frac{\partial w(x,t)}{\partial x}, \quad x > 0,$$

$$\int_0^\infty w(x,t) \, dx = 1.$$

Letting $[0, \Gamma(t)]$ be the support of $w(\cdot, t)$, $t \geq 0$, we have from (8.54) that

$$\frac{d\Gamma(t)}{dt} = \left(\frac{\Gamma(t)}{L(t)}\right)^\alpha - 1.$$

Note that $\Gamma(t) > L(t)$ for all $t \geq 0$, so the function $t \rightarrow \Gamma(t)$ is increasing. We normalize the support of $w(\cdot, t)$ to the interval $[0, 1]$ by making the variable change

$$y = \frac{x}{\Gamma(t)}, \quad w(x,t) = \frac{1}{\Gamma(t)} \tilde{w}(y,s(t)),$$
where the function $t \to s(t)$ is to be determined. It follows from (8.55) that
\begin{equation}
(8.58) \quad \int_0^1 \tilde{w}(y, s) \, dy = 1.
\end{equation}
From (8.57) we have that
\begin{equation}
(8.59) \quad \frac{\partial w(x, t)}{\partial x} = \frac{1}{\Gamma(t)^2} \frac{\partial \tilde{w}(y, s(t))}{\partial y}.
\end{equation}
We also have that
\begin{equation}
(8.60) \quad \frac{\partial w(x, t)}{\partial t} = \frac{1}{\Gamma(t)} \frac{\partial \tilde{w}(y, s(t))}{\partial t} - \frac{1}{\Gamma(t)^2} \left[ \frac{\partial \tilde{w}(y, s(t))}{\partial y} + \tilde{w}(y, s(t)) \right] y \frac{\partial \tilde{w}(y, s(t))}{\partial y}.
\end{equation}
From (8.59), (8.60) we see that (8.54) becomes
\begin{equation}
(8.61) \quad \Gamma(t) \frac{\Gamma'(t)}{\Gamma(t)^2} \frac{\partial \tilde{w}(y, s(t))}{\partial t} - \frac{(\Gamma(t)/L(t))^\alpha - 1}{(\Gamma(t)/L(t))(1-\alpha)} \frac{\partial \tilde{w}(y, s(t))}{\partial y} = 0.
\end{equation}
If we set
\begin{equation}
(8.62) \quad \kappa(t) = \frac{1}{(\Gamma(t)/L(t))^\alpha - 1},
\end{equation}
then (8.61) becomes
\begin{equation}
(8.63) \quad \frac{\Gamma(t)}{\Gamma'(t)} \frac{\partial \tilde{w}(y, s(t))}{\partial t} + \left[ (y^\alpha - y) - \kappa(t)(1-y^\alpha) \right] \frac{\partial \tilde{w}(y, s(t))}{\partial y} = \tilde{w}(y, s(t)).
\end{equation}
Now by a change of time variable $s(t) = \log \Gamma(t)$ we can normalize the coefficient of the time derivative in (8.63) to 1. Evidently the system (8.58), (8.63) with the time variable $s$ is the same as (1.12), (1.13) with $\phi(\cdot)$, $\psi(\cdot)$ given by (1.14).
For $s \geq 0$ let $\tilde{\beta}(\cdot, s)$ be the beta function corresponding to $\tilde{w}(\cdot, s)$, as given by the formula (1.18). Then from (8.57) we have that
\begin{equation}
(8.64) \quad \beta_X(x) = \tilde{\beta} \left( \frac{x}{\Gamma(t)}, \log \Gamma(t) \right), \quad 0 \leq x < \Gamma(t), \quad t \geq 0.
\end{equation}
It follows from (1.2), (8.64) that
\begin{equation}
(8.65) \quad \frac{d}{dt} \langle X_t \rangle = \tilde{\beta} (0, \log \Gamma(t)), \quad t \geq 0.
\end{equation}
We conclude from (8.1) of Theorem 8.1 that
\begin{equation}
(8.66) \quad \lim_{t \to \infty} \frac{d}{dt} \langle X_t \rangle = \beta_X(0).
\end{equation}
Next observe that
\begin{equation}
(8.67) \quad \frac{\langle X_t^\alpha \rangle}{\langle X_t \rangle^\alpha} = \alpha \int_0^1 y^{\alpha-1} \tilde{w}(y, \log \Gamma(t)) \, dy / \tilde{w}(0, \log \Gamma(t))^{(1-\alpha)}.
\end{equation}
It follows from (2.35), (8.1), (8.58), (8.67) that
\begin{equation}
(8.68) \quad \lim_{t \to \infty} \frac{\langle X_t^\alpha \rangle}{\langle X_t \rangle^\alpha} = \frac{\langle X_0^\alpha \rangle}{\langle X_0 \rangle^\alpha}.
\end{equation}
We also have from \[8.1\], \[8.02\] that
\[
\lim_{t \to \infty} \Gamma(t)^\alpha \langle X_t \rangle_t \alpha = 1 + \frac{1}{\kappa} = \frac{\|X_\alpha\|_\infty}{\langle X_\beta \rangle_\infty}.
\]
Evidently \[2.35\], \[8.1\] imply that
\[
\lim_{s \to \infty} \tilde{w}(y, s) \tilde{w}(0, s) = P(X_{t, \Gamma(t)} > y), 0 \leq y < 1.
\]
We also have that
\[
P(X_{t, \Gamma(t)} > x) = P(X_{t, \Gamma(t)} > x, \log \Gamma(t)) = \tilde{w}(x, \langle X_t \rangle / \Gamma(t), \log \Gamma(t)) / \tilde{w}(0, \log \Gamma(t)).
\]
We conclude from \[8.68\]-\[8.71\] that
\[
\lim_{t \to \infty} P(X_{t, \Gamma(t)} > x) = P(X_\beta > x), 0 \leq x < \infty.
\]
Evidently \[1.10\] follows from \[8.68\], \[8.72\].

We obtain the rate of convergence results in Theorem 1.1 in the case when \[x \to \beta X_0(x)\] is Hölder continuous at \[x = \|X_0\|_\infty\] by applying \[8.02\]. From \[8.02\], \[8.65\] we have that
\[
\left| \frac{d}{dt} (X_t) - \beta X_0(0) \right| \leq C \exp[-\nu \log \Gamma(t)] \leq C', \ t \geq 0.
\]
for some constant \(C'\), whence \[1.10\] holds. Similarly we have from \[2.35\], \[8.2\] that for any \(\delta > 0\), \(0 < \delta < 1\), there is a constant \(C_\delta > 0\) such that
\[
P(X_{t, \Gamma(t)} > x) \leq P(X_\beta > x, 1 + C_\delta/(1 + t^\nu)),
\]
for \(0 \leq x \leq (1 - \delta)\|X_\beta\|_\infty, \ t \geq 0\).

Now \[8.73\] and \[8.74\] (with the corresponding lower bound in addition) proves \[1.11\].

\[\square\]

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