ON FUNCTION THEORY IN QUANTUM DISC:
INVARIANT KERNELS

D. Shklyarov    S. Sinel’shchikov    L. Vaksman *

Institute for Low Temperature Physics & Engineering
National Academy of Sciences of Ukraine

In our earlier work [4] some results on integral representations of functions in the quantum disc were formulated without proofs. It was then shown in [5] that the validity of those results is related to the invariance of kernels of integral operators considered in [4]. We introduce in this work a method which allows us to prove the invariance of the above kernels.

Note that the invariant kernels under consideration may be treated as generating functions from the viewpoint of the theory of basic hypergeometric series [2]. A simplest example of relationship between generating functions and q-special functions presents the q-binomial of Newton: let

\[(a + b)^n = \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} b^k a^{n-k}, \quad (q; q)_m = (1 - q)(1 - q^2) \ldots (1 - q^m).\]

Another example is the q-analogue of Van der Vaerden generating function for Clebsch-Gordan coefficients [7].

1 The principal homogeneous space \(\tilde{X}\)

Consider the real Lie group

\[SU(1, 1) = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \in SL_2(\mathbb{C}) \mid \bar{t}_{11} = t_{22}, \bar{t}_{12} = t_{21} \right\}\]

and the element \(w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{C})\). Evidently, \(w \notin SU(1, 1)\).

The subgroup \(SU(1, 1)\) acts on \(SL_2(\mathbb{C})\) via right shifts, and \(\tilde{X} = w^{-1} \cdot SU(1, 1)\) is an orbit of this action. Obviously,

\[\tilde{X} = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \in SL_2(\mathbb{C}) \mid \bar{t}_{11} = -t_{22}, \bar{t}_{12} = -t_{21} \right\}.\]

Now turn to the construction of a q-analogue for the homogeneous space \(\tilde{X}\).

*Partially supported by the grant INTAS-94-4720
Remind first the definition of the algebra $\mathbb{C}[SL_2]_q$ of regular functions on the quantum group $SL_2$ (see [3]). This algebra is determined by its generators $\{t_{ij}\}$, $i, j = 1, 2$, and the relations:

$$
\begin{align*}
  t_{11}t_{12} &= qt_{12}t_{11}, & t_{21}t_{22} &= qt_{22}t_{21}, \\
  t_{11}t_{21} &= qt_{21}t_{11}, & t_{12}t_{22} &= qt_{22}t_{12}, \\
  t_{12}t_{21} &= t_{21}t_{12}, & t_{11}t_{22} - t_{22}t_{11} &= (q - q^{-1})t_{12}t_{21} \\
  t_{11}t_{22} - qt_{12}t_{21} &= 1.
\end{align*}
$$

(1.1)

Equip $\mathbb{C}[SL_2]_q$ with the involution given by

$$
\begin{align*}
  t_{11}^* &= -t_{22}, & t_{12}^* &= -qt_{21}.
\end{align*}
$$

(1.2)

We thus get an involutive algebra which is denoted by $\text{Pol}(\tilde{X})_q$.

The simplest argument in favor of our choice of involution is that at the limit $q \to 1$ one gets the system of equations $t_{11}^* = -t_{22}$, $t_{12}^* = -t_{21}$, which distinguishes the orbit $\tilde{X}$.

A more complete list of arguments and the "algorithm" of producing involutions which leads to the *-algebra $\text{Pol}(\tilde{X})_q$ are expounded in the appendix.

It is well known [1] that $w^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ admits a quantum analogue $w_q^{-1} \in \mathbb{C}[SL_2]_q^*$.

One can deduce from the results of appendix, in particular, that this linear functional is real. Besides that, it is shown in appendix that $\text{Pol}(\tilde{X})_q$ is a covariant *-algebra.

Note that the structure of a $U_q\mathfrak{su}(1, 1)$-module algebra in $\text{Pol}(\tilde{X})_q$ is determined by the relation (1.1) of [5] and the relations

$$
\begin{align*}
  \begin{pmatrix} X^+t_{11} & X^+t_{12} \\ X^+t_{21} & X^+t_{22} \\ X^-t_{11} & X^-t_{12} \\ X^-t_{21} & X^-t_{22} \end{pmatrix} &= \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
  \begin{pmatrix} Ht_{11} & Ht_{12} \\ Ht_{21} & Ht_{22} \end{pmatrix} &= \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
$$

(1.3)

2 A completion

Turn to a construction of the algebra $D(\tilde{X})_q$ of finite functions and the bimodule $D(\tilde{X})_q'$ of distributions on the quantum principal homogeneous space $\tilde{X}$.

Let

$$
M' = \{ (i_1, i_2, j_1, j_2) \in \mathbb{Z}_+^4 | i_2 \cdot j_2 = 0 \}, \quad M'' = \{ (i_1, i_2, j_1, j_2) \in \mathbb{Z}_+^4 | i_1j_1 = i_2j_2 = 0 \}.
$$

It follows from the definition of $\text{Pol}(\tilde{X})_q$ that every element $f \in \text{Pol}(\tilde{X})_q$ admits a unique decomposition

$$
\begin{align*}
  f = \sum_{(i_1, i_2, j_1, j_2) \in M'} a_{i_1i_2j_1j_2}(f)t_{11}^{i_1^*}t_{12}^{i_2^*}t_{21}^{j_1}t_{22}^{j_2}.
\end{align*}
$$

(2.1)

Let $x = t_{12}t_{12}^*$. By a virtue of (2.1), each element $f \in \text{Pol}(\tilde{X})_q$ admits a unique decomposition

$$
\begin{align*}
  f = \sum_{(i_1, i_2, j_1, j_2) \in M''} t_{11}^{i_1}t_{12}^{i_2}t_{21}^{j_1}t_{22}^{j_2}x(x)t_{12}^{j_2}t_{11}^{j_1}.
\end{align*}
$$

(2.2)
REMARK 2.1. In view of the definitions to be imposed below, it should be noted that the sums in (2.1), (2.2) are finite, the coefficients $\psi_{i_1i_2j_1j_2}(x)$ are polynomials, and the functionals $l'_{i_1i_2j_1j_2}: f \mapsto a_{i_1i_2j_1j_2}(f)$, $(i_1, i_2, j_1, j_2) \in M'$, $l''_{i_1i_2j_1j_2}: f \mapsto \psi_{i_1i_2j_1j_2}(q^{2m})$, $(i_1, i_2, j_1, j_2) \in M''$, $m \in \mathbb{Z}_+$, are linear.

Equip the vector space $\text{Pol}(\tilde{X})_q$ with the weakest topology in which all the functionals $l''_{i_1i_2j_1j_2}$, $(i_1, i_2, j_1, j_2) \in M''$, $m \in \mathbb{Z}_+$, are continuous.

A completion of the above Hausdorff topological vector space will be called the space of distributions and denoted by $D(\tilde{X})'_q$.

REMARK 2.2. The vector space $\text{Pol}(\tilde{X})_q$ is equipped with the structure of algebra, and hence with the structure of $\text{Pol}(\tilde{X})_q$-bimodule. This structure is extendable by a continuity onto the completion $D(\tilde{X})'_q \supset \text{Pol}(\tilde{X})_q$.

$D(\tilde{X})'_q$ will be identified with the space of formal series (2.2), whose coefficients are functions on $q^{-2\mathbb{Z}_+}$. The topology in the space of such series is that of pointwise convergence of the coefficients $\psi_{i_1i_2j_1j_2}(x)$.

A distribution $f \in D(\tilde{X})'_q$ is said to be finite if $\#\{(i_1, i_2, j_1, j_2, m) | \psi_{i_1i_2j_1j_2}(q^{-2m}) \neq 0\} < \infty$. The vector space of finite functions will be denoted by $D(\tilde{X})_q$. (It is easy to present a non-degenerate pairing $D(\tilde{X})'_q \times D(\tilde{X})_q \rightarrow \mathbb{C}$ which establishes an isomorphism between $D(\tilde{X})'_q$ and a vector space dual to $D(\tilde{X})_q$).

Proposition 2.3 The structure of a covariant $*$-algebra is transferred by a continuity from $\text{Pol}(\tilde{X})_q$ onto $D(\tilde{X})_q$. The structure of a covariant $D(\tilde{X})_q$-bimodule is transferred by a continuity from $D(\tilde{X})_q$ onto $D(\tilde{X})'_q$.

Proof. A verification of the covariance of the algebra $D(\tilde{X})_q$ and the bimodule $D(\tilde{X})'_q$ reduces to the proof of the identities whose validity is already known in the case when $\psi_{i_1i_2j_1j_2}(x)$ are polynomials. Now what remains is to observe that the polynomials are dense in the space of functions $\psi(x)$ equipped with the topology of pointwise convergence. □

Lemma 2.4 The formal series (2.4) presents a finite function on the quantum principal homogeneous space if the following two conditions are satisfied:

i) $\#\{(i_2, j_2) \in \mathbb{Z}_+^2 | i_2j_2 = 0, \exists (i_1, j_1) \in \mathbb{Z}_+^2 : a_{i_1i_2j_1j_2}(f) \neq 0\} < \infty$,

ii) $\exists N \in \mathbb{N} : f \cdot t_{11}^N = t_{11}^N \cdot f = 0$.

Proposition 2.5 There exists a unique nonzero element $e_0 \in D(\tilde{X})_q$ such that

$$t_{12} e_0 = e_0 t_{12}, \quad t_{12}^* e_0 = e_0 t_{12}^*,$$

$$t_{11}^* e_0 = e_0 t_{11} = 0,$$

$$e_0 \cdot e_0 = e_0.$$
Proof. $e_0$ satisfies (2.3) iff $e_0 = \sum_{i_2j_2=0} t_{12}^{i_2} \psi_{0i_0j_2}(x) t_{12}^{j_2}$. (2.7) means that
\[
e_0 = \psi_{0000}(x),
\]
with $(\psi_{0000}(x))^2 = \psi_{0000}(x)$. Finally, (2.4) is satisfied iff
\[
\psi_{0000}(x) = \begin{cases} 1 & x = 1 \\ 0 & x \in \{q^{-2}, q^{-4}, q^{-6}, \ldots \} \end{cases}
\]
Thus $e_0$ exists, is unique and determined by (2.6), (2.7). \[\square\]

It is easy to prove that the structure of a covariant $\ast$-algebra can be transferred by a continuity from $\text{Pol}(X)_q$ onto $\text{Fun}(\tilde{X})_q \overset{\text{def}}{=} \text{Pol}(\tilde{X})_q + D(\tilde{X})_q$. The structure of a $\ast$-algebra is given by (2.3) - (2.5) and $e'_0 = e_0$, and the $U_q\text{su}(1,1)$-action by
\[
H e_0 = 0, \quad X^+ e_0 = c_+ t_{11} e_0 t_{12}^* , \quad X^- e_0 = c_- t_{12} e_0 t_{11}^*,
\]
with $c_- = -\frac{e^{-3h/4}}{1 - e^{-h}}$, $c_+ = \frac{e^{-5h/4}}{1 - e^{-h}}$. The proof of (2.8) is just the same as that of similar relations (3.5) in [3].

Finally note that for any $f \in D(U)_q$ there exists a unique decomposition
\[
f = \sum_{(i_1,i_2,j_1,j_2) \in M'} c_{i_1i_2j_1j_2} t_{11}^i t_{12}^j t_{12}^0 t_{11}^0,
\]
with $c_{i_1i_2j_1j_2} \in \mathbb{C}$ and only finitely many of those being non-zero.

3 Quantum homogeneous space $X$

In the case $q = 1$ the orbit $X = w^{-1} \cdot SU(1,1)$ can be equipped with a structure of a homogeneous $SU(1,1) \times SU(1,1)$-space:
\[
(g_1, g_2) : x \mapsto (w^{-1} g_1 w)xg_2^{-1}.
\]
In particular, the action of the one-parameter subgroup
\[
i_\varphi : \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mapsto g_1(\varphi) \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}, \quad g_1(\varphi) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}
\]
commutes with the right multiplication by $g_2^{-1} \in SU(1,1)$.

These constructions can be transferred onto the case $0 < q < 1$ via (A.1). Here we restrict ourselves to introducing the one-parameter group $i_\varphi, \varphi \in \mathbb{R}/(2\pi \mathbb{Z})$, of automorphisms of the covariant $\ast$-algebra $\text{Fun}(\tilde{X})_q = \text{Pol}(\tilde{X})_q + D(\tilde{X})_q$,
\[
\begin{pmatrix} i_\varphi(t_{11}) & i_\varphi(t_{12}) \\ i_\varphi(t_{21}) & i_\varphi(t_{22}) \end{pmatrix} = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}, \quad i_\varphi(e_0) = e_0.
\]
Evidently, $i_\varphi \cdot \text{Pol}(\tilde{X})_q = \text{Pol}(\tilde{X})_q$, $i_\varphi \cdot D(\tilde{X})_q = D(\tilde{X})_q$, and the operators $i_\varphi$ are extendable by a continuity onto the entire space $D(\tilde{X})'_q$. 

4
In the case $q = 1$ the orbits of the one-parameter transformation group $i_\varphi : \tilde{X} \to \tilde{X}$ form a homogeneous space $X$ of the group $SU(1,1)$ isomorphic to the unit disc $U \subset \mathbb{C}$. We intend to produce a similar isomorphism in the quantum case $0 < q < 1$. Let us start with introducing the spaces of invariants of the one-parameter group $i_\varphi$:

$$D(X)_q = \{ f \in D(\tilde{X})_q \mid i_\varphi(f) = f, \ \varphi \in \mathbb{R}/(2\pi \mathbb{Z}) \},$$

$$D(X)'_q = \{ f \in D(\tilde{X})'_q \mid i_\varphi(f) = f, \ \varphi \in \mathbb{R}/(2\pi \mathbb{Z}) \}.$$ 

The elements of the covariant $*$-algebra $D(X)_q$ (the covariant $D(X)_q$-bimodule $D(X)'_q$) will be called finite functions (resp. distributions) on the quantum homogeneous space $X$.

In our work [5] a finite function in the quantum disc $f_0$ such that $z^* f_0 = f_0 z = 0$, $f_0 \cdot f_0 = f_0$ is introduced.

**Proposition 3.1** There exists a unique isomorphism $i : D(U)_q \overset{\sim}{\to} D(X)_q$ of covariant $*$-algebras such that $i : f_0 \mapsto e_0$. The operator $i$ admits an extension up to an isomorphism $\tilde{i} : D(U)'_q \overset{\sim}{\to} D(X)'_q$ of $U_q\mathfrak{su}(1,1)$-modules by a continuity, which agrees with the bimodule structures.

**Proof.** The uniqueness of $i$ is obvious since $f_0$ generates $U_q\mathfrak{su}(1,1)$-module $D(U)_q$ (see [5]).

Prove the existence of $i$.

Consider the linear subspace $F$ of those $f \in D(\tilde{X})'_q$ for which only finitely many terms in the formal series (2.2) are non-zero. It is easy to show that the structure of a covariant $*$-algebra is extendable by a continuity from $\text{Pol}(\tilde{X})_q$ onto $F$. It is worthwhile to note that $e_0 \in F$ and the elements $t_{12}, t^\prime_{12}$ are invertible in the algebra $F$:

$$t_{12}^{-1} = x^{-1} \cdot t_{12}, \quad t^\prime_{12}^{-1} = t_{12} \cdot x^{-1}.$$ 

In [5] one can find a description of the covariant $*$-algebra $\text{Fun}(U)_q = \text{Pol}(\mathbb{C})_q + D(U)_q$ in terms of generators and relations. It follows from that description and the results of the previous section that the map

$$i : z \mapsto q t_{11}^{-1} t_{12}^{-1}, \quad i : z^* \mapsto t_{21}^{-1} t_{22}, \quad i : f_0 \mapsto e_0$$

(3.1)
is extendable up to a homomorphism of covariant $*$-algebras $i : \text{Pol}(U)_q \to F$. It remains to note that $D(U)_q \subset \text{Fun}(U)_q$. The possibility of extending by a continuity of $i$ as well as the properties of its extension $\tilde{i}$ are easily derivable from (2.2).

We shall not distinguish in the sequel the functions in the quantum disc and their images under the embedding $i$.

Let $F_m = \{ f \in F \mid i_\varphi(f) = e^{im\varphi} f, \ \varphi \in \mathbb{R}/(2\pi \mathbb{Z}) \}$. Note that for all $m \in \mathbb{Z}$

$$F_m \cap D(\tilde{X})_q = D(U)_q \cdot t_{12}^m.$$  

Finally, we present an explicit form of an invariant integral on the quantum principal homogeneous space. Consider the linear functional $\nu : D(\tilde{X})_q \to \mathbb{C}$,

$$\int_{\tilde{X}} f d\nu = (1 - q^2) \sum_{m=0}^{\infty} \psi_{0000}(q^{-2m})q^{-2m},$$

(3.3)
determined by the coefficient $\psi_{0000}$ in the expansion (2.2).
Proposition 3.2 The linear functional \((\mathcal{X})\) is an invariant integral.

Proof. It is well known (see [5]) that the functional

\[
D(U)q \rightarrow C, \sum_{j>0} z^j \psi_j(y) + \psi_0(y) + \sum_{j>0} \psi_{-j}(y) z^{-j} \rightarrow (1 - q^2) \sum_{j=0}^{\infty} \psi_{0000}(q^{2j})q^{-2j}
\]

is an invariant integral (here \(y = 1 - zz^*\)). Hence, by a virtue of proposition 3.1 and (3.1), the restriction of the linear functional \((\mathcal{X})\) onto the covariant \(\ast\)-subalgebra \(D(U)q \simeq D(X)_q\) is an invariant integral. It remains to elaborate the fact that the averaging operator

\[
j : D(\tilde{X})_q \rightarrow D(X)_q; \quad j : f \mapsto \frac{1}{2\pi} \int_0^{2\pi} i_\varphi(f) d\varphi
\]

is a morphism of \(U_q\mathfrak{su}(1,1)\)-modules. \(\square\)

4 Asymptotic cones \(\tilde{\Xi}\) and \(\Xi\)

The definition of the covariant \(\ast\)-algebra \(\text{Pol}(\tilde{X})_q\) involves the generators \(t_{ij}\), \(i, j = 1, 2\), and the relations \((1.1), (1.2), (1.3)\). All those relations are homogeneous in \(t_{ij}\) except the relation

\[
t_{11}t_{22} - qt_{12}t_{21} = 1. \quad (4.1)
\]

Hence the substitution

\[
t_{ij} = q^{-N} \cdot t^{(N)}_{ij}, \quad N \in \mathbb{N}, \quad (4.2)
\]

together with the succeeding passage to a limit as \(N \rightarrow +\infty\) lead to the same system of relations except \((4.1)\). The latter relation at the limit \(N \rightarrow +\infty\) changes to

\[
t_{11}t_{22} - qt_{12}t_{21} = 0.
\]

The covariant \(\ast\)-algebra given by the initial list of generators and the new list of relations (cf. \((1.1), (1.2), (1.3)\)) is denoted by \(\text{Pol}(\Xi)_q\). It is a q-analogue of the polynomial algebra on the cone \(\Xi = \{(t_{11}, t_{12}) \in \mathbb{C}^2 | |t_{11}| = |t_{12}|\}\).

The automorphisms \(i_\varphi\) are defined in the same way as in section 3. Their invariants constitute a covariant \(\ast\)-algebra which will be denoted in the sequel by \(\text{Pol}(\Xi)_q\). This is a q-analogue of the polynomial algebra on the cone

\[
\Xi = \{(t_{11}, t_{12}) | t_{11} \in \mathbb{C}, t_{12} \in \mathbb{R}, t_{12} = |t_{11}|\}.
\]

Let us pass from polynomials to distributions. As above, set up \(x = t_{12}t_{12}^*\). Each element \(f \in \text{Pol}(\Xi)_q\) admits a unique decomposition

\[
f = \sum_{(i_1, i_2, j_1, j_2) \in M''} t_{i_1}^{i_2} t_{i_2}^{j_1} t_{j_1}^{j_2} (x) \psi_{i_1}^{i_2} \psi_{j_2}^{j_1} t_{11}^{i_1} t_{12}^{j_2} \quad (4.3)
\]

Remind (see section 2) that in the case of \(\text{Pol}(\tilde{X})_q\) a similar decomposition it possible to impose a topology. Specifically, we equipped the vector space \(\text{Pol}(\tilde{X})_q\) with the weakest one
among the topologies in which all the linear functionals $f \mapsto \psi_{i_1i_2j_1j_2}(a)$ with $a \in q^{-2\mathbb{Z}}$, are continuous. The substitution \eqref{1.2} converts $q^{-2\mathbb{Z}}$ into $q^{2N} \cdot q^{-2\mathbb{Z}}$. After a formal passage to a limit as $N \to \infty$ we get $q^{2\mathbb{Z}}$ instead of $q^{-2\mathbb{Z}}$. Equip the vector space $\text{Pol}(\Xi)_q$ with the weakest among the topologies in which all the linear functionals $f \mapsto \psi_{i_1i_2j_1j_2}(a)$ with $a \in q^{2\mathbb{Z}}$, $(i_1, i_2, j_1, j_2) \in M''$, are continuous. A completion $D(\Xi)'_q$ of this topological vector space is the space of formal series \eqref{1.2} whose coefficients are functions $\psi_{i_1i_2j_1j_2}(x)$ on $q^{2\mathbb{Z}}$. The elements $f \in D(\Xi)'_q$ will be called distributions on the quantum cone $\Xi$. The distributions with $\#\{(i_1, i_2, j_1, j_2, m) \mid \psi_{i_1i_2j_1j_2}(q^{-2m}) \neq 0\} < \infty$ constitute the space $D(\Xi)_q$ of finite functions.

Denote by $D(\Xi)_q$, $D(\Xi)'_q$ the subspaces of invariants of the one-parameter operator groups in $D(\Xi)_q$, $D(\Xi)'_q$ which are extensions by a continuity of $i_\phi$, $\phi \in \mathbb{R}/(2\pi\mathbb{Z})$.

**Proposition 4.1** The structure of a covariant $*$-algebra can be transferred by a continuity from $\text{Pol}(\Xi)_q$ onto $D(\Xi)_q$. The structure of a covariant $D(\Xi)_q$-bimodule can be transferred by a continuity from $D(\Xi)_q$ onto $D(\Xi)'_q$.

**Proof.** For any polynomial $\psi$ of one indeterminate one has the following equalities between the elements of the covariant $*$-algebra $\text{Pol}(\Xi)_q$:

$$
t_{11}\psi(x) = \psi(q^2x)t_{11}, \quad t_{12}\psi(x) = \psi(x)t_{12},
$$

$$
t_{21}\psi(x) = \psi(x)t_{21}, \quad t_{22}\psi(x) = \psi(q^{-2}x)t_{22}.
$$

Besides that, $t_{11}t_{22} = -x$, $t_{22}t_{11} = -q^{-2}x$. It follows that the structure of an algebra in $D(\Xi)_q$ and the structure of a $D(\Xi)_q$-bimodule in the space $D(\Xi)'_q$ can be produced via extending by a continuity. It is easy to prove that for any polynomial $\psi$

$$
\begin{align}
X^+\psi(x) &= -q^{1/2}t_{11} \cdot (\psi(x) - \psi(xq^{-2}))/\left(x - xq^{-2}\right) \cdot t_{21}, \\
X^-\psi(x) &= -q^{1/2}t_{12} \cdot (\psi(x) - \psi(q^{-2}x))/\left(x - xq^{-2}\right) \cdot t_{22}, \\
H\psi(x) &= 0.
\end{align}
$$

\eqref{1.4} implies that the structures of a covariant algebra in $D(\Xi)_q$ and that of a covariant $D(\Xi)_q$-bimodule in $D(\Xi)'_q$ can be produced via extending by a continuity. The rest of statements of proposition 4.1 follow from \eqref{1.2} and $\psi(x)^* = \overline{\psi(x)}$. \hfill \Box

Remind that the linear operator $i_\phi : \text{Pol}(\Xi)_q \to \text{Pol}(\Xi)_q$, $\phi \in \mathbb{R}/(2\pi\mathbb{Z})$ is an automorphism of covariant $*$-algebras. Hence the vector space $D(\Xi)_q \subset D(\Xi)'_q$ is a covariant $*$-algebra. One can prove in a similar way that the vector space $D(\Xi)'_q \subset D(\Xi)'_q$ are covariant $D(\Xi)_q$-bimodules.

Consider the linear functional $\nu : D(\Xi)_q \to \mathbb{C}$,

$$
\int_{\Xi_q} fd\nu = (1 - q^2) \sum_{n=-\infty}^{\infty} \psi_{0000}(q^{-2m})q^{-2m},
$$

\eqref{1.3} determined by the coefficient $\psi_{0000}$ in the decomposition \eqref{1.3}.

**Proposition 4.2** The linear functional \eqref{4.5} is an invariant integral.
Proof. Let $\psi(x)$ be a function with finite carrier on $q^{2\mathbb{Z}}$. One can easily deduce from the covariance of the algebra $D(\tilde{\Xi})_q$ and (1.3), (1.4) that

$$\int_{\tilde{\Xi}_q} H\psi(x)d\nu = 0, \quad \int_{\tilde{\Xi}_q} X^+(t_{12}\psi(x)t_{22})d\nu = 0.$$ 

Hence $\int_{\tilde{\Xi}_q} Hf d\nu = \int_{\tilde{\Xi}_q} X^+ f d\nu = 0$ for all $f \in D(\tilde{\Xi})_q$. Thus the linear functional (4.3) is a $U_q\mathfrak{sl}_2$-invariant integral. Now it remains to use the realness of this functional. □

Let $l \in \mathbb{C}$. A distribution $f \in D(\tilde{\Xi})_q'$ is called homogeneous of degree $l$ if all the coefficients $\psi_{i_1i_2j_1j_2}(x)$ in its decomposition (4.3) are homogeneous:

$$\psi_{i_1i_2j_1j_2}(q^2x) = q^{2l-(i_1+i_2+j_1+j_2)}\psi_{i_1i_2j_1j_2}(x).$$

Equivalently, let $\alpha$ be an automorphism of $\text{Pol}(\Xi)_q$ given by $\alpha(t_{ij}) = qt_{ij}, i,j = 1,2,$ and its extension by a continuity onto $D(\tilde{\Xi})_q'$. An element $\psi \in D(\tilde{\Xi})_q'$ is homogeneous of degree $l$ iff $\alpha(\psi) = q^{2l}\psi$.

It follows from the definitions that the action of $H, X^+, X^-$ in $D(\tilde{\Xi})_q'$ preserves the homogeneity degree of a distribution. Hence, the vector space $F(l)$ of distributions of homogeneity degree $l$ is a $U_q\mathfrak{sl}_2$-module.

Consider the linear functional

$$\eta : F(-1) \to \mathbb{C}; \quad \int_{\tilde{\Xi}_q} f d\eta = \psi_{0000}(1),$$

(4.6)
determined by the coefficient $\psi_{0000}$ in the expansion (4.3).

Proposition 4.3 The linear functional (4.6) is an invariant integral.

Proof is completely similar to that of the previous proposition and reduces to a verification of the relations

$$\int_{\tilde{\Xi}_q} Hx^{-1}d\eta = 0, \quad \int_{\tilde{\Xi}_q} X^+(t_{12}x^{-2}t_{22})d\eta = 0.$$ 

(4.7)
The first relation is obvious, and the second one follows from $x = -t_{11}t_{22}, x = -qt_{12}t_{21}, X^+(t_{11}) = X^+(t_{21}) = 0$. □

To conclude let us note that for any function on $q^{2\mathbb{Z}}$ one has $\psi \in D(\tilde{\Xi})_q'$,

$$\Omega\psi(x) = Dx^2D\psi(x)$$

(4.8)
In the special case $\psi(x) = x^l, l \in \mathbb{C}$ the next equality from (1.7) follows:

$$\Omega x^l = \frac{sh((l+1)h/2) \cdot sh(lh/2)}{sh^2(h/2)} x^l$$

(4.9)
One can prove the equality (4.7) by the substitution of variables (4.2) and by the passage to a limit $N \to +\infty$ in the equality (5.7) of 6.
5 Cartesian products

Consider a Hopf algebra $A$ and $A$-module (covariant) Hopf unital algebras $F_1, F_2$. Let $\pi$ be the representation of the algebra $F_1^{\text{op}} \otimes F_2$ in the vector space $F_1 \otimes F_2$ given by

$$\pi(f_1 \otimes f_2)\psi_1 \otimes \psi_2 = \psi_1 f_1 \otimes f_2 \psi_2; \quad f_i, \psi_i \in F_i, \quad i = 1, 2.$$ 

Proposition 5.1 Let $K \in F_1^{\text{op}} \otimes F_2$. The linear operator $\pi(K)$ is a morphism of $A$-modules iff $K$ is an invariant:

$$\forall a \in A \quad a \cdot K = \varepsilon(a) \cdot K. \quad (5.1)$$

Proof. It follows from (5.1) that the linear operator $i : F_1 \otimes F_2 \to F_1^{\otimes 2} \otimes F_2^{\otimes 2}; i : \psi_1 \otimes \psi_2 \mapsto \psi_1 \otimes K \otimes \psi_2$ is a morphism of $A$-modules. Note that $\pi(K) = m_1 \otimes m_2 \cdot i$ with $m_j : F_j \otimes F_j \to F_j, \quad j = 1, 2$, being the multiplication in $F_j$. Hence $\pi(K)$ is a morphism of $A$-modules. Conversely, if $\pi(K)$ is a morphism of $A$-modules, then the elements $1 \otimes 1$ and $K = \pi(K)1 \otimes 1$ of $F_1^{\text{op}} \otimes F_2$ are invariants. \hfill \Box

We present below an evident corollary of proposition 5.1 which justifies the algebra structure in $F_1 \otimes F_2$ introduced in [5].

Corollary 5.2 The invariants form a subalgebra of $F_1^{\text{op}} \otimes F_2$.

Let $\nu_2(f) = \int f \, d\nu_2$ be an invariant integral $F_2 \to \mathbb{C}$. An importance of invariants $K \in F_1 \otimes F_2$ is due to the fact that the associated integral operators $f \mapsto \text{id} \otimes \nu_2(K(1 \otimes f))$ are morphisms of $A$-modules (see [5]).

The previous sections contain the constructions of q-analogues for $SU(1,1)$-spaces $\tilde{X}, X, \tilde{\Xi}, \Xi$. Let $Y \in \{\tilde{X}, X, \tilde{\Xi}, \Xi\}$. It is easy to show that the structure of a covariant *-algebra is extendable by a continuity from $\text{Pol}(Y)_q$ onto $\text{Fun}(Y)_q \overset{\text{def}}{=} \text{Pol}(Y)_q + D(Y)_q$, and the structure of a covariant bimodule is extendable from $\text{Fun}(Y)_q$ onto $D(Y)_q$.

Let $Y_1, Y_2 \in \{\tilde{X}, X, \tilde{\Xi}, \Xi\}$. Introduce the notation

$$\text{Pol}(Y_1 \times Y_2)_q \overset{\text{def}}{=} \text{Pol}(Y_1)_q^{\text{op}} \otimes \text{Pol}(Y_2)_q,$$

$$D(Y_1 \times Y_2)_q \overset{\text{def}}{=} D(Y_1)_q^{\text{op}} \otimes D(Y_2)_q$$

for ‘algebras of functions on Cartesian products of quantum $SU(1,1)$-spaces’. Note that for any of the above algebras the multiplication is not a morphism of $U_q\mathfrak{sl}_2$-modules, that is, the algebras are not covariant. However, by corollary 5.2, the subspaces of invariants are subalgebras.

Associate to each pair of continuous linear functionals $l_j : \text{Pol}(Y_j) \to \mathbb{C}, \quad j = 1, 2$, a linear functional $l_1 \otimes l_2 : \text{Pol}(Y_1 \times Y_2)_q \to \mathbb{C}$. Equip the vector space $\text{Pol}(Y_1 \times Y_2)_q$ with the weakest topology under which all those linear functionals are continuous.

The completion of the Hausdorff topological space $\text{Pol}(Y_1 \times Y_2)_q$ will be denoted by $D(Y_1 \times Y_2)_q$. In the case $Y_1, Y_2 \in \{\tilde{X}, \tilde{\Xi}\}$, $D(Y_1 \times Y_2)_q$ may be identified with the space of formal series

$$f = \sum_{(i', i'' j', j'', i' j' j'' j'') \in M'' \times M''} t_{i' j'} t_{i' j'} t_{i'' j''} t_{i'' j''} \psi_{i' j' j''} \int x' x'' \, t_{i' j'} t_{i' j'} t_{i'' j''} t_{i'' j''} . \quad (5.2)$$
The coefficients of these formal series are functions on the Cartesian product of progressions \((q^{-2n+} \text{ or } q^{+2n})\). the topology in \(D(Y_1 \times Y_2)'_q\) is that of pointwise convergence of the coefficients. It is easy to prove that the structure of \(U_q\)\(\mathfrak{sl}_2\)-module and that of \(\text{Fun}(Y_1 \times Y_2)'_q\)-bimodule are extendable by a continuity from the dense linear subspace \(\text{Fun}(Y_1 \times Y_2)_q \subset D(Y_1 \times Y_2)'_q\) onto the entire space \(D(Y_1 \times Y_2)'_q\).

The following statement justifies calling the bimodules \(D(Y_j)'_q, j=1,2,\) and \(D(Y_1 \times Y_2)'_q\) the bimodules of distributions on the quantum SU(1,1)-spaces \(Y_j, j=1,2,\) and \(Y_1 \times Y_2\).

Let \(\nu_j : D(Y_j)_q \to \mathbb{C}, \nu_j : f \mapsto \int_{Y_{jq}} f d\nu_j, j=1,2,\) be invariant integrals. We follow the conventions of \cite{4} in putting in the integrands into braces the products of elements of the algebra \(F_1^\text{op} \otimes F_2\).

**Proposition 5.3** The bilinear form

\[
D(Y_1 \times Y_2)_q \times D(Y_1 \times Y_2)_q \to \mathbb{C}; \quad \int \int_{Y_{1q}Y_{2q}} \{\psi_1 \cdot \psi_2\} d\nu_1 d\nu_2
\]  

(5.3)

is extendable by a continuity up to a bilinear form \(D(Y_1 \times Y_2)'_q \times D(Y_1 \times Y_2)'_q \to \mathbb{C}\). The associated map from \(D(Y_1 \times Y_2)'_q\) into the dual to \(D(Y_1 \times Y_2)_q\) is an isomorphism.

**Proof.** It suffices to apply the description of the spaces \(D(Y_1 \times Y_2)'_q\) and \(D(Y_1 \times Y_2)_q\) and the invariant integral \(f \mapsto \int \int_{Y_{1q}Y_{2q}} f d\nu_1 d\nu_2\) in terms of the expansion (5.2). \(\square\)

Proposition 5.3 implies

**Corollary 5.4** There exists a unique antilinear map \(# : D(Y_1 \times Y_2)'_q \to D(Y_1 \times Y_2)'_q\) such that for all \(f \in D(Y_2)_q, K \in D(Y_1 \times Y_2)_q\) one has

\[
\int_{Y_{2q}} \{K^# \cdot 1 \otimes f\} d\nu_2 = \int_{Y_{2q}} \{K \cdot 1 \otimes f^*\} d\nu_2
\]

.

Obviously, \(K^^2 = K\). Thus the real kernels \(K = K^#\) generate the ‘real’ integral operators.

6 Examples of invariant kernels

Let \(Y_1, Y_2 \in \{\tilde{X}, \tilde{\Xi}\}\). Consider the algebra \(\text{Pol}(Y_1 \times Y_2)_q\). We omit in the sequel the tensor product sign \(\otimes\) while working with kernels of integral operators. To avoid misunderstanding, we introduce the notation \(t_{ij}, \tau_{ij}\) for the generators \(t_{ij} \otimes 1, 1 \otimes t_{ij}, i, j = 1, 2\).

The definitions imply
Proposition 6.1 The following elements of $\text{Pol}(Y_1 \times Y_2)_q$, $Y_1, Y_2 \in \{\tilde{X}, \tilde{\Xi}\}$, are invariants:

\[
\begin{align*}
  k_{11} &= t_{11} \tau_{22} - q t_{12} \tau_{21}; & k_{12} &= -q^{-1} t_{11} \tau_{12} + t_{12} \tau_{11}; \\
  k_{21} &= t_{21} \tau_{22} - q t_{22} \tau_{21}; & k_{22} &= -q^{-1} t_{21} \tau_{12} + t_{22} \tau_{11}.
\end{align*}
\]

Now prove

Proposition 6.2 The following commutation relations are valid in $\text{Pol}(Y_1 \times Y_2)_q$, $Y_1, Y_2 \in \{\tilde{X}, \tilde{\Xi}\}$:

\[
\begin{align*}
  k_{11} k_{12} &= q^{-1} k_{12} k_{11}, \\
  k_{11} k_{21} &= q^{-1} k_{21} k_{11}, \\
  k_{12} k_{21} &= k_{21} k_{12}, \\
  k_{22} k_{11} - q k_{12} k_{21} &= 1 \quad (Y_2 = \tilde{X}) \\
  k_{22} k_{11} - q k_{12} k_{21} &= 0 \quad (Y_2 = \tilde{\Xi})
\end{align*}
\]

Proof. We start with the special case $Y_1 = Y_2 = \tilde{X}$. One has $\text{Pol}(Y_1 \times Y_2)_q \simeq \mathbb{C}[\text{SL}_2]^\text{op}_q \otimes \mathbb{C}[\text{SL}_2]^\text{op}_q$. Consider the map $\pi : \mathbb{C}[\text{SL}_2]^\text{op}_q \to \mathbb{C}[\text{SL}_2]^\text{op}_q \otimes \mathbb{C}[\text{SL}_2]^\text{op}_q; \pi : f \mapsto (\text{id} \otimes S) \Delta(f)$, with $\Delta$ being the comultiplication and $S$ the antipode of the Hopf algebra $\mathbb{C}[\text{SL}_2]^\text{op}_q$.

The relations to be proved are provided by $\pi$ being an antihomomorphism, and by the relations $\pi(t_{ij}) = k_{ij}$, $i, j = 1, 2$.

The general case reduces to the special case $Y_1 = Y_2 = \tilde{X}$ via introducing a change of generators and a passage to the limit $N \to \infty$ as in section 4. \qed

Remind that the vector space $D(Y_1 \times Y_2)_q^\prime$, $Y_1, Y_2 \in \{\tilde{X}, \tilde{\Xi}\}$, is equipped with the involution $\#$.

Proposition 6.3

\[
\begin{align*}
  t_{11}^\# &= -t_{22}, & t_{21}^\# &= -q^{-1} t_{12}, \\
  \tau_{11}^\# &= -q^{-2} \tau_{22}, & \tau_{21}^\# &= -q^{-1} \tau_{12}.
\end{align*}
\]

Proof. Just as in the proof of proposition 6.2, we restrict ourselves to the special case $Y_1 = Y_2 = \tilde{X}$. The first two relations follow from (1.2).

Now to prove (1.2), consider the linear functional $l : D(\tilde{X})_q \to \mathbb{C}, l : f \mapsto \int_{\tilde{X}_q} f \cdot x^{-1} d\nu$. It follows from the relation (3.3) of [3] that

\[
\forall f_1, f_2 \in D(\tilde{X})_q \quad l(f_1 f_2) = l(f_2 f_1) \quad (6.3)
\]

since $\int_{\tilde{X}_q} f \cdot x^{-1} d\nu = \text{Tr} \tilde{T}(f)$, with $\tilde{T}$ being the representation of the algebra of functions described in that work. The relations (6.2) follow from

\[
\tau_{ij}^\# = \xi^{-1} \tau_{ij}^* \xi, \quad i, j = 1, 2 \quad \text{with} \quad \xi = \tau_{12} \tau_{12}^* \quad (6.4)
\]
Finally, (6.4) follow from (6.3), the realness of the invariant integral $\nu: D(\tilde{X})_q \to \mathbb{C}$, and the definition of the involution $\#$.

Note that, by a virtue of (6.3), the linear functional $l$ may be treated as a quantum analogue of the integral with respect to the Liouville measure associated to the standard symplectic structure on $\tilde{X} \subset SL_2(\mathbb{C})$.

Proposition 6.3 allows one to produce explicit formulae for an involution in the algebra of invariant polynomial kernels. Specifically, one has

**Corollary 6.4** $k_{11}^l = q^2k_{22}^l$, $k_{12}^l = q^{-1}k_{21}^l$.

Our immediate purpose is to study the invariants $k_{22}^l k_{11}^l \in D(Y_1 \times Y_2)_q'$.

Remind the standard notation $(t; q)_n = \prod_{j=0}^{n-1} (1 - tq^j)$. It is easy to prove (see relation(1.3.2)) that

$$(t; q)_n = \sum_{j=0}^{n} (q^{-n}; q)_j q^{j(n+1)}t^j. \quad (6.5)$$

Just as in section 3, introduce the linear subspace $F \subset D(Y_1 \times Y_2)_q'$ of finite sums of the form (6.2). The structure of algebra is transferred by a continuity from $\text{Pol}(Y_1 \times Y_2)_q$ onto $F$. Impose the notation for some special elements of $F$:

$$z = qt_{12}^{-1}t_{11}, \quad z^* = t_{22}t_{21}^{-1}, \quad \zeta = q\tau_{11}\tau_{12}^{-1}, \quad \zeta^* = \tau_{21}\tau_{22}, \quad x = t_{12}t_{11}, \quad \xi = \tau_{12}\tau_{12}^*.$$ Use the commutation relations

$$t_{12}\tau_{21}(z\zeta^*) = q^2(z\zeta^*)t_{12}\tau_{21}; \quad (\zeta z^*)t_{12}\tau_{21} = q^2t_{21}t_{12}(\zeta z^*)$$

to prove following relations in $F$.

Let $l \in \mathbb{Z}_+$. Then

$$k_{11}^l = (-qt_{12}\tau_{21}(1 - q^{-2}z\zeta^*))^l = (-qt_{12}\tau_{21})^l(q^{-2}z\zeta^*; q^{-2})t, \quad k_{22}^l = ((1 - z^*\zeta)(-q^{-1}t_{21}\tau_{12}))^l = (z^*\zeta; q^{-2})t(-q^{-1}t_{21}\tau_{12})^l. \quad (6.6)$$

Hence,

$$k_{22}^l k_{11}^l = (z^*\zeta; q^{-2})_l(t_{12}t_{21})^l(\tau_{12}\tau_{21})^l(q^{-2}z\zeta^*; q^{-2})t; \quad k_{22}^l k_{11}^l = q^{-2l}\xi^l(q^2z^*\zeta; q^2)_l(z\zeta^*; q^2)_lx^l.$$ By a virtue of (6.3) one has

$$(q^2z^*\zeta; q^2)_l = \sum_{n=0}^{\infty} \frac{(q^{-2l}; q^2)_n}{(q^2; q^2)_n} q^{2(l+1)n}(z^*\zeta)^n, \quad (z\zeta^*; q^2)_l = \sum_{n=0}^{\infty} \frac{(q^{-2l}; q^2)_n}{(q^2; q^2)_n} q^{2ln}(z\zeta^*)^n. \quad (6.7)$$

Thus we have proved
Lemma 6.5 For all \( l \in \mathbb{Z}_+ \),
\[
k_{22}^l k_{11}^l = q^{-2l} \sum_{j=0}^{\infty} \frac{\left(q^{-2l}; q^2\right)_j}{(q^2; q^2)_j} \left(q^{2(l+1)} z^* \zeta^j \right) + \sum_{m=0}^{\infty} \frac{\left(q^{-2l}; q^2\right)_m}{(q^2; q^2)_m} q^{2j(l+1)+2ml} \zeta^j z^m x^j l \zeta^j \zeta^m.
\] (6.8)

Remark 6.6. Show that the right hand side of (6.8) is a generalized kernel, that is, it could be written in the form (5.2).
\[
k_{22}^l k_{11}^l = q^{-2l} \sum_{j=0}^{\infty} \frac{\left(q^{-2l}; q^2\right)_j}{(q^2; q^2)_j} q^{2j(l+1)+2ml} \zeta^j z^m x^j l \zeta^j \zeta^m =
\]
\[
= q^{-2l} \sum_{j=0}^{\infty} \frac{\left(q^{-2l}; q^2\right)_j}{(q^2; q^2)_j} q^{2j(l+1)+2ml} \zeta^j z^m x^j l \zeta^j \zeta^m + q^{-2l} \sum_{j=0}^{\infty} \frac{\left(q^{-2l}; q^2\right)_j}{(q^2; q^2)_j} q^{2j+l+2j} \zeta^j z^j x^j l \zeta^j \zeta^j + q^{-2l} \sum_{j=0}^{\infty} \frac{\left(q^{-2l}; q^2\right)_j}{(q^2; q^2)_j} q^{2j(l+1)+2ml} \zeta^j z^m x^j l \zeta^j \zeta^m.
\]

Now we are in a position to make up an analytic continuation of the distribution \( k_{22}^l k_{11}^l \) in \( l \). Consider a vector-function \( f(\lambda) \) of a complex variable \( \lambda \) with values at \( D(Y_1 \times Y_2)_q \), \( Y_1, Y_2 \in \{X, \bar{X}\} \). This vector-function will be called polynomial if for any finite function \( \psi \in D(Y_1 \times Y_2)_q \) the integral \( I(\lambda) = \int \int f(\lambda) \psi d\nu_1 d\nu_2 \) is a polynomial of \( \lambda, \lambda^{-1} \), that is, \( I(\lambda) \in \mathbb{C}[\lambda, \lambda^{-1}] \).

Proposition 6.7 Consider any one among the three spaces of distributions \( D(X \times X)_q \), \( D(\bar{X} \times \bar{X})_q \), \( D(\bar{X} \times \bar{X})_q \). There exists a unique polynomial vector-function \( f(\lambda) \) with values at a selected space of distributions such that \( f(q^{2l}) = k_{22}^l k_{11}^l \) for all \( l \in \mathbb{Z}_+ \).

Proof. The uniqueness of a rational function \( f(\lambda) \) with given (fixed) values in \( \lambda = q^{2l}, l \in \mathbb{Z}_+ \) is evident. Consider a finite function \( \psi \) and replace \( k_{22}^l k_{11}^l \) in the integral \( I(\lambda) = \int \int f(\lambda) \psi d\nu_1 d\nu_2 \) by the right hand side of (6.8). If either \( Y_1 = X \) or \( Y_2 = \bar{X} \), then only finitely many terms of (6.8) contribute to the above integral. So, what remains is to note that each term is a polynomial of \( q^{2l}, q^{-2l} \).

Remark 6.8. A distribution \( f(q^{2l}) \) whose existence and uniqueness was proved in proposition 6.6, will be also denoted by \( k_{22}^l k_{11}^l \). By corollary 5.2, it is invariant for all \( l \in \mathbb{Z}_+ \), and hence for all \( l \in \mathbb{C} \).

To conclude, we prove the following

Proposition 6.9 For any of the spaces \( D(X \times X)_q \), \( D(\bar{X} \times \bar{X})_q \), \( D(\bar{X} \times \bar{X})_q \), there exists a unique polynomial vector-function \( f(\lambda) \) such that for all \( l \in \mathbb{Z}_+ \)
\[
k_{11}^l = (-qt^{12}z^{2l})^{-l} f(q^{2l}).
\]
Proof. The uniqueness is obvious. The existence follows from (6.6), (6.7), and the q-binomial theorem (see [3]).

\[ f = (z\zeta^*; q^2)^{-1} = \sum_{n=0}^{\infty} \frac{(q^2t; q^2)_n}{(q^2; q^2)_n} (z\zeta^*)^n. \] (6.9)

Appendix. On involution in \( \text{Pol}(\tilde{X})_q \)

The standard procedure of constructing an involution in \( \mathbb{C}[SL_2]_q \) is to use the involution \( * : U_q \mathfrak{su}(1, 1) \rightarrow U_q \mathfrak{su}(1, 1) \), the antilinear map

\[ # : \xi \mapsto (S(\xi))^*, \quad \xi \in U_q \mathfrak{su}(1, 1), \]

and the duality argument

\[ \forall \xi \in U_q \mathfrak{su}(1, 1), \ f \in \mathbb{C}[SL_2]_q \quad f^*(\xi) \mathrel{\overset{\text{def}}{=}} \overline{f(\xi^*)}. \]

One gets in this way a Hopf \(*\)-algebra \( \mathbb{C}[SU(1, 1)]_q \) of regular functions on the quantum group \( SU(1, 1) \).

In order to obtain the involution involved in the definition of the principal quantum homogeneous space, one can modify the above procedure. Let \( w_q \in \mathbb{C}[SL_2]_q \) be the element of the quantum Weyl group \([3]\). Consider the linear subspace \( L \mathrel{\overset{\text{def}}{=}} w_q^{-1}U_q \mathfrak{l} \mathfrak{sl}_2 = U_q \mathfrak{l} \mathfrak{sl}_2 w_q^{-1} \) (The last equality follows from \( w_q \cdot K^{\pm} \cdot w_q^{-1} = K^{\pm} \), \( w_q \cdot E \cdot w_q^{-1} = -q^{-1}F \), \( w_q \cdot F \cdot w_q^{-1} = -qE \) (A.1) (see [3])). The ‘involution’ \( # : U_q \mathfrak{su}(1, 1) \rightarrow U_q \mathfrak{su}(1, 1) \) is to be replaced by the ‘involution’

\[ # : L \rightarrow L, \quad # : w_q^{-1} \xi \mapsto w_q^{-1}(S(\xi))^*. \]

Finally, let \( * \) be such an antilinear operator in \( \mathbb{C}[SL_2]_q \) that

\[ \forall \xi \in L, \ f \in \mathbb{C}[SL_2]_q \quad f^*(\xi) = \overline{f(\xi^*)}. \]

Evidently, \( f^{**} = f \) for all \( f \in \mathbb{C}[SL_2]_q \).

Prove that \( * \) is an antihomomorphism of \( \mathbb{C}[SL_2]_q \). This follows from

\[ \Delta w_q^{-1} = w_q^{-1} \otimes w_q^{-1} \cdot R, \quad R^* \otimes R = R_{21}, \quad S \otimes S(R) = R, \]

with \( R \) being a universal R-matrix, and \( R_{21} \) is derived from \( R \) by a permutation of tensor multiples (cf. [8]). It is worthwhile to note that for all \( f \in \mathbb{C}[SL_2]_q \) and all \( \xi \in U_q \mathfrak{su}(1, 1) \) one has \( (\xi f)^* = (S(\xi))^* \cdot f^* \). Thus we get a covariant \(*\)-algebra. Obviously, the linear functional \( w_q \) is real: \( f^*(w_q) = \overline{f(w_q)} \) for all \( f \in \mathbb{C}[SL_2]_q \).

To conclude, let us prove that the above involution coincides with that in \( \text{Pol}(\tilde{X})_q \), i.e. \( t_1^* = -t_2^*, \ t_2^* = -qt_1^* \). It follows from the definition of the involution that the linear span of \( t_{ij} \in \mathbb{C}[SL_2]_q \), \( i, j = 1, 2 \), is an invariant subspace for \(*\). What remains is to apply the realness of the functional \( w_q \in \mathbb{C}[SL_2]_q \) and the relations

\[ \begin{pmatrix} t_{11}(w_q) & t_{12}(w_q) \\ t_{21}(w_q) & t_{22}(w_q) \end{pmatrix} = \text{const}(q) \begin{pmatrix} 0 & -q \\ 1 & 0 \end{pmatrix}, \]

with \( \text{const}(q) \rightarrow 1 \) as \( q \rightarrow 1 \) (see [3]).

14
References

[1] V. Chari, A. Pressley. A Guide to Quantum Groups, Cambridge Univ. Press, 1995.

[2] G. Gasper, M. Rahman. Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[3] A. N. Kirillov, Ya. S. Reshetikhin, *q-Weyl group and a multiplicative formula for universal R-matrices*, Commun. Math. Phys., 134, (1990), 421 – 431.

[4] D. Shklyarov, S. Sinel’shchikov, L. Vaksman. On function theory in quantum disc: integral representations, E-print: math.QA/9808015.

[5] D. Shklyarov, S. Sinel’shchikov, L. Vaksman. On function theory in quantum disc: covariance, E-print: math.QA/9808037.

[6] S. Sinel’shchikov, L. Vaksman. *On q-analogues of Bounded Symmetric Domains and Dolbeault Complexes*, Mathematical Physics, Analysis and Geometry; Kluwer Academic Publishers, V.1, No.1, 1998, 75–100, E-print: q-alg/9703003.

[7] L. Vaksman, *q-analogues of the Clebsch-Gordan coefficients and algebra of functions on the quantum group SU(2)*, Dokl. Acad. Sci. USSR, 306 (1989), No 2, 269 – 271.