Another smallest part function related to Andrews’ spt function

by

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1. Introduction and main results. In [A2], we find the identity of Andrews

\[
\sum_{n \geq 1} \frac{q^n}{(1 - q^n)(q^n)_\infty} = \sum_{n \geq 1} np(n)q^n + \frac{1}{(q)_\infty} \sum_{n \geq 1} \frac{(-1)^n q^{n(3n+1)/2}(1 + q^n)}{(1 - q^n)^2}.
\]

Here \(p(n)\) is the number of partitions of \(n\), and the last series on the right generates \(N_2(n) = \sum_{m \in \mathbb{Z}} m^2 N(m, n)\), \(N(m, n)\) being the number of partitions of \(n\) with rank \(m\) (see [A1]). The largest part minus the number of parts is defined to be the rank. The function \(\text{spt}(n)\) counts the number of smallest parts among integer partitions of \(n\). For some other functions counting smallest parts among partitions see [P1]. Lastly, we have used the familiar notation \((a)_n = (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\) (see [GR]).

In this note we find a spt-type function that is related to the generating function in (1.1) and falls into the same class of spt-type functions as the one offered in [P2]. However, this note differs from [P2] in that we will find the “crank companion” to create a “full” spt function related to Andrews’ spt function. Here we are also appealing to relations to spt\((n)\) modulo 2, whereas in [P2] we concentrated on relations to spt\((n)\) modulo 3. Lastly, the partitions involved in this study are different, and deserve a separate study.

Let \(M_2(n) = \sum_{m \in \mathbb{Z}} m^2 M(m, n)\), where \(M(m, n)\) is the number of partitions of \(n\) with crank \(m\) (see [AG]).

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Theorem 1.1. We have
\begin{equation}
\sum_{n \geq 1} q^n (q^{2n+1}; q^2)_\infty \frac{1}{(1-q^n)^2 (q^{n+1})_\infty} = \frac{1}{(q^2; q^2)_\infty} \sum_{n \geq 1} nq^n \frac{1}{1-q^n} - \frac{1}{2} \sum_{n \geq 1} N_2(n)q^{2n},
\end{equation}

\begin{equation}
\sum_{n \geq 1} q^{n(n+1)/2} (q^{2n+1}; q^2)_\infty \frac{1}{(1-q^n)^2 (q^{n+1})_\infty} = \frac{1}{(q^2; q^2)_\infty} \sum_{n \geq 1} nq^n \frac{1}{1-q^n} - \frac{1}{2} \sum_{n \geq 1} M_2(n)q^{2n}.
\end{equation}

For our next theorem, which is a number-theoretic interpretation of Theorem 1.1, we will use the following definitions. We define a triangular partition to be of the form $\delta_l = (l-1, l-2, \ldots, 1)$, $l \in \mathbb{N}$. Define the smallest part of a partition $\pi$ to be $s(\pi)$, and the largest part to be $l(\pi)$. We will also consider the partition pair $\sigma = (\pi, \delta_i)$, where we set $i = s(\pi)$. The latter condition yields $s(\pi) - l(\delta_i=\sigma(\pi)) = s(\pi) - (s(\pi) - 1) = 1$. If we include $\delta_i$ in a partition, we are increasing its size by $\binom{i}{2}$ and including the component $q^{1+\cdots+i-1}$ in its generating function. This has the property that all parts from 1 to $i-1$ appear exactly once and are less than $i$.

Theorem 1.2. Let $spt^+\sigma\delta(n)$ count the number of smallest parts among the integer partitions $\pi$ of $n$ where odd parts greater than $2s(\pi)$ do not occur. Let $spt^-\sigma\delta(n)$ count the number of smallest parts among the integer partitions $\sigma = (\pi, \delta(s(\pi))$ of $n$ such that $\pi$ is a partition where odd parts greater than $2s(\pi)$ do not occur. Define $spt_o(n) := spt^+\sigma\delta(n) - spt^-\sigma\delta(n)$. Then $spt_o(2n) = spt(n)$.

With the above definitions, we can write the generating function. We have
\begin{equation}
\sum_{n \geq 1} spt_o(n)q^n = \sum_{n \geq 1} \left(q^n + 2q^{2n} + 3q^{3n} + \cdots\right) \frac{(q^{2n+1}; q^2)_\infty}{(q^{n+1})_\infty} (1-q^{1+2+\cdots+n-1}).
\end{equation}

2. Proof of Theorems 1.1 and 1.2. The proofs require the methods used in [B][C][P2] and a few more observations. A pair of sequences $(\alpha_n, \beta_n)$ is known to be a Bailey pair with respect to $a$ if
\begin{equation}
\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(aq; q)_{n+r}(q; q)_{n-r}}.
\end{equation}

The next result is Bailey’s lemma [B].

Bailey’s Lemma. If $(\alpha_n, \beta_n)$ form a Bailey pair with respect to $a$ then
\begin{equation}
\sum_{n=0}^{\infty} (\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n \beta_n = \frac{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}{(aq)_\infty(aq/\rho_1\rho_2)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\alpha_n}{(aq/\rho_1)_n(aq/\rho_2)_n}.
\end{equation}
The following are known Bailey pairs \((\alpha_n, \beta_n)\) relative to \(a = 1\):

\[
\begin{align*}
\alpha_{2n+1} &= 0, \\
\alpha_{2n} &= (-1)^n q^{n(3n-1)}(1 + q^{2n}), \\
\beta_n &= \frac{1}{(q)_n(q^2;q^2)_n}
\end{align*}
\]

(see [S, C(1)]), and

\[
\begin{align*}
\alpha_{2n+1} &= 0, \\
\alpha_{2n} &= (-1)^n q^{n(n-1)}(1 + q^{2n}), \\
\beta_n &= \frac{q^{n(n-1)/2}}{(q)_n(q^2;q^2)_n}
\end{align*}
\]

(see [S, C(5)]). In both pairs \(\alpha_0 = 1\). Differentiating Bailey’s lemma (putting \(a = 1\)) with respect to both variables \(\rho_1\) and \(\rho_2\) and setting each variable equal to 1 each time gives us (see [P2])

\[
\sum_{n \geq 1} (q)_{n-1}^2 \beta_n q^n = \alpha_0 \sum_{n \geq 1} \frac{nq^n}{1 - q^n} + \sum_{n \geq 1} \frac{\alpha_n q^n}{(1 - q^n)^2}.
\]

Identity (1.2) follows from inserting the Bailey pair (2.3)–(2.5) into (2.9) and then multiplying through by \((q^2; q^2)_\infty^{-1}\). Identity (1.3) follows from inserting the Bailey pair (2.6)–(2.8) into (2.9) and then multiplying through by \((q^2; q^2)_\infty^{-1}\). This proves Theorem 1.1.

To get Theorem 1.2, we subtract (1.3) from (1.2), and note that \(\text{spt}(n) = \frac{1}{2}(M_2(n) - N_2(n))\), after observing that (see [G])

\[
2 \sum_{n \geq 1} np(n)q^n = \sum_{n \geq 1} M_2(n)q^n = \frac{2}{(q)_\infty} \sum_{n \geq 1} \frac{nq^n}{1 - q^n}.
\]

The result follows from equating the coefficients of \(q^{2n}\).

3. More notes and concluding remarks. Naturally, it is of interest to investigate equations (1.2) and (1.3) individually. As we noted previously, the left side of (1.2) generates \(\text{spt}_o^+(n)\), and the left side of (1.3) generates \(\text{spt}_o^-(n)\).

**Theorem 3.1.** We have \(\text{spt}_o^+(2n) \equiv \text{spt}(n) \pmod{2}\).

**Proof.** After noting that \(\sigma(2n) = 3\sigma(n) - 2\sigma(n/2)\), \(\sigma(2n) \equiv \sigma(n) \pmod{2}\), and

\[
\sum_{n \geq 1} \text{spt}_o^+(n)q^n = \frac{1}{(q^2; q^2)_\infty} \sum_{n \geq 1} \sigma(n)q^n - \frac{1}{2} \sum_{n \geq 1} N_2(n)q^{2n},
\]

we have

\[
\sum_{n \geq 1} \text{spt}_o^+(n)q^n = \frac{1}{(q^2; q^2)_\infty} \sum_{n \geq 1} \sigma(n)q^n - \frac{1}{2} \sum_{n \geq 1} N_2(n)q^{2n}.
\]
we can take coefficients of $q^{2n}$ in (1.3) to get
\[ \text{spt}_o^+(2n) = \sum_k p(k)\sigma(2(n-k)) - \frac{1}{2}N_2(n). \]

Hence, combining these, we compute
\[ \text{spt}_o^+(2n) \equiv \sum_k p(k)\sigma(n-k) - \frac{1}{2}N_2(n) \pmod{2} \]
\[ \equiv np(n) - \frac{1}{2}N_2(n) \pmod{2} \equiv \text{spt}(n) \pmod{2}. \]

**Theorem 3.2.** We have $\text{spt}_o^-(2n) \equiv 0 \pmod{2}$.

*Proof.* The computations are similar to Theorem 3.1. Using equation (1.3) we compute
\[ \text{spt}_o^-(2n) \equiv \sum_k p(k)\sigma(n-k) - \frac{1}{2}M_2(n) \pmod{2} \]
\[ \equiv np(n) - \frac{1}{2}M_2(n) \pmod{2} \equiv 0 \pmod{2}. \]

In the last line we have used $2np(n) = M_2(n)$.

For an example illustrating Theorem 3.1, consider partitions of 4: (4), (3, 1), (2, 2), (1, 1, 1, 1). In a partition where odd parts greater than twice the smallest do not occur, we omit (3, 1). Hence $\text{spt}_o^+(4) = 7$, and $\text{spt}(2) = 3$ (counting smallest of (2) and (1, 1, 1)). Hence 2 divides $7 - 3 = 4$.

To illustrate Theorem 3.2, consider the partition pair $\sigma = (\pi^*, \delta_{s(\pi^*)})$ of 6 where $(3, 2) \in \pi^*$, $(1) \in \delta_{s(\pi^*)}$, and $(3) \in \pi^*$, $(2, 1) \in \delta_{s(\pi^*)}$. Hence $\text{spt}_o^-(6)$ is equal to 2 plus the appearances of the smallest parts in $\pi^*$ of those partition pairs $\sigma = (\pi^*, \delta_{s(\pi^*)})$ which have the empty partition $\emptyset \in \delta_i$, that is, $(3, 1, 1, 1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; $(4, 1, 1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; $(2, 1, 1, 1, 1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; $(1, 1, 1, 1, 1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; and finally $(3, 2, 1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$. This gives us $\text{spt}_o^-(6) = 18 = 0 \pmod{2}$.

Equating the coefficients of $q^{2n+1}$ in Theorem 1.1 gives us a nice corollary.

**Theorem 3.3.** We have $\text{spt}_o^-(2n + 1) = \text{spt}_o^+(2n + 1)$.

Let $t_k(n)$ be the number of representations of $n$ as a sum of $k$ triangular numbers. We may use a classical result of Legendre that $\sigma(2n + 1) = t_4(n)$ to see that $\text{spt}_o^-(2n + 1)$ (and therefore also $\text{spt}_o^+(2n + 1)$) is generated by the product expansion
\[ \frac{q^A}{(q^2;q^4)^5}. \]

To see examples for Theorem 3.3, consider first $n = 1$. Then $\text{spt}_o^-(3) = \text{spt}_o^+(3) = 5$. This is because $(2, 1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; $(1, 1, 1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; and $(2) \in \pi^*$, $(1) \in \delta_{s(\pi^*)}$, for $\text{spt}_o^-(3)$. The case of $\text{spt}_o^+(3)$ is clearer.
Another example is $\text{spt}^-_o(5) = \text{spt}^+_o(5) = 12$. We only compute $\text{spt}^-_o(5)$ for the reader: $(2,2) \in \pi^*, (1) \in \delta_s(\pi^*)$; $(2,2,1) \in \pi^*, \emptyset \in \delta_s(\pi^*)$; $(4,1) \in \pi^*, \emptyset \in \delta_s(\pi^*)$; $(1,1,1,1) \in \pi^*, \emptyset \in \delta_s(\pi^*)$; and finally $(2,1,1,1) \in \pi^*, \emptyset \in \delta_s(\pi^*)$.

It is interesting to note that since $\text{spt}(n)$ is even for almost all natural $n$ (see [FO]), the value $\text{spt}^+_o(2n)$ is even for almost all natural $n$ in terms of arithmetic density.

We can also easily obtain congruences for $\text{spt}_o(n)$ using the Ramanujan-type congruences in [A2]:

\[
\begin{align*}
\text{spt}_o(2(5n + 4)) &\equiv 0 \pmod{5}, \\
\text{spt}_o(2(7n + 5)) &\equiv 0 \pmod{7}, \\
\text{spt}_o(2(13n + 6)) &\equiv 0 \pmod{13}.
\end{align*}
\]

It is important to make the observation that the two Bailey pairs (2.3)–(2.5) and (2.6)–(2.8) are key in obtaining the “rank component” (1.2) and the “crank component” (1.3), respectively.

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