Black hole for the Einstein-Chern-Simons gravity

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Abstract

We consider a 5-dimensional action which is composed of a gravitational sector and a sector of matter, where the gravitational sector is given by a Einstein-Chern-Simons gravity action instead of the Einstein-Hilbert action.

We obtain the Einstein-Chern-Simons (EChS) field equations together with its spherically symmetric solution, which lead, in certain limit, to the standard five dimensional solution of the Einstein-Cartan field equations.

It is found the conditions under which the EChS field equations admits black hole type solutions. The maximal extension and conformal compactification are also studied

PACS numbers: 04.50.+h, 04.20.Jb, 04.90.+e

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I. INTRODUCTION

According to the principles of general relativity (GR), the spacetime is a dynamical object which has independent degrees of freedom, and is governed by dynamical equations, namely the Einstein field equations. This means that in GR the geometry is dynamically determined. Therefore, the construction of a gauge theory of gravity requires an action that does not consider a fixed space-time background. An five dimensional action for gravity fulfilling these conditions is the five-dimensional Chern–Simons AdS gravity action, which can be written as

\[ L_{\text{AdS}}^{(5)} = \kappa \left( \frac{1}{5l^5} \epsilon_{a_1 \ldots a_5} e^{a_1} \ldots e^{a_5} + \frac{2}{3l^3} \epsilon_{a_1 \ldots a_3} R^{a_1 a_2} e^{a_3} \ldots e^{a_5} + \frac{1}{l} \epsilon_{a_1 \ldots a_5} R^{a_1 a_2} R^{a_3 a_4} e^{a_5} \right), \]  

(1)

where \( e^a \) corresponds to the 1-form vielbein, and \( R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb} \) to the Riemann curvature in the first order formalism [1], [2], [3].

If Chern-Simons theories are the appropriate gauge-theories to provide a framework for the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to General Relativity.

In ref. [4] was recently shown that the standard, five-dimensional General Relativity (without a cosmological constant) can be obtained from Chern-Simons gravity theory for a certain Lie algebra \( \mathcal{B} \). The Chern-Simons Lagrangian is built from a \( \mathcal{B} \)-valued, one-form gauge connection \( A \) which depends on a scale parameter \( l \) which can be interpreted as a coupling constant that characterizes different regimes within the theory. The \( \mathcal{B} \) algebra, on the other hand, is obtained from the AdS algebra and a particular semigroup \( S \) by means of the S-expansion procedure introduced in refs. [5], [6]. The field content induced by \( \mathcal{B} \) includes the vielbein \( e^a \), the spin connection \( \omega^{ab} \) and two extra bosonic fields \( h^a \) and \( k^{ab} \).

The five dimensional Chern-Simons Lagrangian for the \( \mathcal{B} \) algebra is given by [4]:

\[ L_{\text{ChS}}^{(5)} = \alpha_1 l^2 \zeta_{abcd} R^{ab} R^{cd} e_e + \alpha_3 \zeta_{abcde} \left( \frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e \right), \]  

(2)

where we can see that (i) if one identifies the field \( e^a \) with the vielbein, the system consists of the Einstein-Hilbert action plus nonminimally coupled matter fields given by \( h^a \) and \( k^{ab} \); (ii) it is possible to recover the odd-dimensional Einstein gravity theory from a Chern-Simons
gravity theory in the limit where the coupling constant $l$ equals to zero while keeping the effective Newton’s constant fixed.

It is the purpose of this article to find a spherically symmetric solution for the $EChS$ field equations, which are obtained from the so called Einstein-Chern-Simons action (2) studied in Refs, [4], [7]. It is shown that the standard five dimensional solution of the Einstein-Cartan field equations can be obtained, in a certain limit, from the spherically symmetric solution of $EChS$ field equations. The conditions under which these equations admits black hole type solutions are found and the maximal extension and conformal compactification are also studied.

This paper is organized as follows: In section 2 we find a spherically symmetric solution for the Einstein-Chern-Simons field equations and then it is shown that the standard five dimensional solution of the Einstein-Cartan field equations can be obtained, in a certain limit, from the spherically symmetric solution of $EChS$ field equations. In section 3 we find the conditions under which the field equations admits black hole type solutions and we studied the maximal extension and conformal compactification of such solutions. A brief comment and three appendices conclude this work.

II. EINSTEIN-CHERN-SIMONS FIELD EQUATIONS FOR A SPHERICALLY SYMMETRIC METRIC

In this section we consider the field equations for the lagrangian $L = L_g + L_M$, where $L_g$ is the Chern-Simons gravity lagrangian $L_{ChS}^{(5)}$ and $L_M$ is the corresponding matter lagrangian.

In the presence of matter described by the langragian $L_M = L_M(e^a, h^a, \omega^{ab})$, we have that the field equations obtained from the action (2) are given by [7]:

$$\varepsilon_{abcde} R^{cd} T^e = 0,$$

$$\alpha_3 l^2 \varepsilon_{abcde} R^{bc} R_{de} = -\frac{\delta L_M}{\delta h^a},$$

$$\varepsilon_{abcde} \left( 2\alpha_3 R^{bc} e^d e^e + \alpha_1 l^2 R^{bc} R_{de} + 2\alpha_3 l^2 D_\omega h^{bc} R_{de} \right) = -\frac{\delta L_M}{\delta e^a},$$

$$2\varepsilon_{abcde} \left( \alpha_1 l^2 R^{cd} T^e + \alpha_3 l^2 D_\omega k^{cd} T^e + \alpha_3 e^c e^d T^e + \alpha_3 l^2 R^{cd} D_\omega h^e + \alpha_3 l^2 R^{cd} k^e f^f \right) = -\frac{\delta L_M}{\delta \omega^{ab}}.$$

(3)
If $T^a = 0$ and $k^{ab} = 0$, the equation (3) can be written in the form

\[ de^a + \omega^a e^b = 0, \]
\[ \varepsilon_{abcd} R^{cd} D_\omega h^e = 0, \]
\[ \alpha_3 l^2 Y_a = -\star \left( \frac{\delta L_M}{\delta h^a} \right), \]
\[ \alpha_1 l^2 Y_a + 2 \alpha_3 X_a = \kappa T_{ab} e^b, \]  
\(\text{(4)}\)

where
\[ X_a = \star \left( \varepsilon_{bcde} R^{bc} e^d e^e \right), \quad Y_a = \star \left( \varepsilon_{bcde} R^{bc} R^{de} \right), \quad T_{ab} = -\star \left( \frac{\delta L_M}{\delta e^a} \right) \]  
\(\text{(5)}\)

and where “$\star$” is the Hodge star operator.

$T_{ab}$ is the energy-momentum tensor of matter fields and $\kappa$ is the coupling constant. In the equations (4) are present the fields $e^a$, $\omega^{ab}$ (through $R^{ab}$) and $h^a$. If we wish to find a spherically-and static-symmetric solution, then we must demand that the three fields satisfy this conditions. Since a static space-time is one which possesses a timelike Killing vector orthogonal to the spacelike hypersurfaces. These conditions are satisfied by the metric (6).

### A. Spherically symmetric metric in five dimensions

We consider first the fields $e^a$ and $\omega^{ab}$ (through $R^{ab}$). In five dimensions the static and spherically symmetric metric is given by

\[ ds^2 = -e^{2f(r)} dt^2 + e^{2g(r)} dr^2 + r^2 d\Omega_3^2 = \eta_{ab} e^a e^b \]  
\(\text{(6)}\)

where $d\Omega_3^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2$ and $\eta_{ab} = \text{diag}(-1, +1, +1, +1, +1)$.

Introducing an orthonormal basis, we have

\[ e^T = e^{f(r)} dt, \quad e^R = e^{g(r)} dr, \quad e^1 = r d\theta_1, \quad e^2 = r \sin \theta_1 d\theta_2, \quad e^3 = r \sin \theta_1 \sin \theta_2 d\theta_3. \]  
\(\text{(7)}\)

Taking the exterior derivatives, we get:

\[ de^T = -f' e^{-g} e^T e^R, \quad de^R = 0, \quad de^1 = \frac{e^{-g}}{r} e^R e^1, \]
\[ de^2 = \frac{1}{r \tan \theta_1} e^1 e^2 + \frac{e^{-g}}{r} e^R e^2, \quad de^3 = \frac{1}{r \sin \theta_1 \tan \theta_2} e^1 e^3 + \frac{1}{r \sin \theta_1 \tan \theta_2} e^2 e^3 + \frac{e^{-g}}{r} e^R e^3, \]  
\(\text{(8)}\)
where a prime “′” denotes derivative with respect to \( r \). The next step is to use Cartan’s first structural equation

\[ T^a = de^a + \omega^a_b e^b = 0 \]

and the antisymmetry of the connection forms \((\omega^{ab} = -\omega^{ba})\) to find the non-zero connection forms. The calculations give:

\[
\begin{align*}
\omega_{TR} &= -f' e^{-g} e^T, \\
\omega_{Ri} &= -\frac{e^{-g}}{r} e^i, \\
\omega_{12} &= -\frac{1}{r \tan \theta_1} e^3, \\
\omega_{23} &= -\frac{1}{r \sin \theta_1 \tan \theta_2} e^3; \quad i = 1, 2, 3.
\end{align*}
\]

From Cartan’s second structural equation

\[ R^a_b = d\omega^a_b + \omega^a_c \omega^c_b, \]

we can calculate the curvature matrix. The non-zero components are

\[
\begin{align*}
R^{TR} &= e^{-g} \left( f' g' - f'' - (f')^2 \right) e^T e^R, & R^{Ti} &= -\frac{f' e^{-2g}}{r} e^T e^i \\
R^{Ri} &= \frac{g' e^{-2g}}{r} e^R e^i, & R^{ij} &= \frac{1 - e^{-2g}}{r^2} e^i e^j; \quad i, j = 1, 2, 3.
\end{align*}
\]

Introducing (7), (10) into (5) we find

\[
\begin{align*}
X_T &= 12 e^{-2g} \left( g'r + e^{2g} - 1 \right) e^T, \\
X_R &= 12 e^{-2g} \left( f'r - e^{2g} + 1 \right) e^R, \\
X_i &= 4 e^{-2g} \left( -f' g'r^2 + f'' r^2 + (f')^2 r^2 + 2 f'r - 2 g'r - e^{2g} + 1 \right) e^i, \\
Y_T &= 24 e^{-2g} \frac{g'}{r^3} \left( 1 - e^{-2g} \right) e^T, \\
Y_R &= 24 e^{-2g} \frac{f'}{r^3} \left( 1 - e^{-2g} \right) e^R, \\
Y_i &= 8 e^{-2g} \frac{1}{r^2} \left( f'' + (f')^2 - f' g' - e^{-2g} f'' - e^{-2g} (f')^2 + 3 e^{-2g} f' g' \right) e^i.
\end{align*}
\]

Introducing (7), (11), (12) into the third equation (4) and considering the energy-momentum tensor as the energy-momentum tensor of a perfect fluid at rest, i.e., \( T_{TT} = \rho(r) \) and \( T_{RR} = T_{ii} = P(r) \), where \( \rho(r) \) and \( P(r) \) are the energy density and pressure (for the
perfect fluid), we find
\[\alpha_1 l^2 e^{-2g} \frac{g'}{r^3} (1 - e^{-2g}) + \alpha_3 e^{-2g} \frac{g'}{r^2} (g' r + e^{2g} - 1) = \frac{\kappa}{24} \rho \] (13)

\[\alpha_1 l^2 e^{-2g} \frac{f'}{r^3} (1 - e^{-2g}) + \alpha_3 e^{-2g} \frac{f'}{r^2} (f' r - e^{2g} + 1) = \frac{\kappa}{24} P \] (14)

\[\alpha_1 l^2 e^{-2g} \frac{f''}{r^2} \left( f'' + (f')^2 - f' g' - e^{-2g} f'' - e^{-2g} (f')^2 + 3 e^{-2g} f' g' \right) \]
\[+ \alpha_3 e^{-2g} \frac{1}{r^2} \left( -f' g' r^2 + f'' r^2 + (f')^2 r^2 + 2 f' r - 2 g' r - e^{2g} + 1 \right) = \frac{\kappa}{8} P \] (15)

Now consider the equation (13). After multiplying by \(4r^3\) we find
\[\left\{ (1 - e^{-2g}) \left( \alpha_1 l^2 \left( 1 - e^{-2g} \right) + 2\alpha_3 r^2 \right) \right\}' = \frac{\kappa}{6} \rho r^3. \] (16)

Integrating we have
\[\left( 1 - e^{-2g} \right) \left( \alpha_1 l^2 \left( 1 - e^{-2g} \right) + 2\alpha_3 r^2 \right) = \frac{\kappa}{12\pi^2} \left( M(r) - M_0 \right), \] (17)

where \(M_0\) is an integration constant and \(M(r)\) is the Newtonian mass, which is defined as
\[M(r) = 2\pi^2 \int_0^r \rho(\bar{r}) \bar{r}^3 d\bar{r}. \] (18)

From equation (17) we can see that
\[e^{-2g} = 1 + \alpha \frac{r^2}{l^2} \pm \sqrt{\alpha^2 \frac{r^4}{l^4} + \frac{K}{12\pi^2 l^2} \left( M(r) - M_0 \right)}, \] (19)

where \(\alpha = \alpha_3/\alpha_1, \quad K = \kappa/\alpha_1.\)

In order to make contact with the solutions of the Einstein-Cartan theory, consider the limit \(l \to 0: \)
\[\lim_{l \to 0} e^{-2g} = \lim_{l \to 0} \left( 1 + \alpha \frac{r^2}{l^2} \pm \sqrt{\alpha^2 \frac{r^4}{l^4} + \frac{K}{12\pi^2 l^2} \left( M(r) - M_0 \right)} \right). \] (20)

If we consider the case of small \(l^2\) limit, we can expand the root to first order in \(l^2\). In fact,
\[e^{-2g} \approx 1 + \frac{r^2}{l^2} \left\{ \alpha \pm |\alpha| \left[ 1 + \frac{K l^2}{12\pi^2 l^2 \alpha^2 r^4} \left( M(r) - M_0 \right) + O(l^4) \right] \right\} \]
\[\approx 1 + \frac{r^2}{l^2} (\alpha \pm |\alpha|) \pm \frac{K}{24\pi^2 |\alpha| r^2} \left( M(r) - M_0 \right) + O(l^4). \] (21)

From (21) we can see that for this expression to be finite when \(l \to 0\), is necessary that \((\alpha \pm |\alpha|) = 0.\)

Since \(\alpha = \alpha_3/\alpha_1\) we can distinguish two cases:
(a) If \( \alpha_3 > 0 \) and \( \alpha_1 > 0 \) or if \( \alpha_3 < 0 \) and \( \alpha_1 < 0 \) we have
\[
e^{-2g} = 1 + \alpha \frac{r^2}{l^2} - \sqrt{\alpha^2 \frac{r^4}{l^4} + \frac{K}{12 \pi^2 l^2} (\mathcal{M}(r) - \mathcal{M}_0)}
\]
\[
\approx 1 - \frac{K}{24 \pi^2 |\alpha| r^2} (\mathcal{M}(r) - \mathcal{M}_0)
\]
\[
\approx 1 - \frac{\kappa}{24 \pi^2 \alpha_3 r^2} (\mathcal{M}(r) - \mathcal{M}_0).
\]

(b) If \( \alpha_3 > 0 \) and \( \alpha_1 < 0 \) or if \( \alpha_3 < 0 \) and \( \alpha_1 > 0 \) we have
\[
e^{-2g} = 1 + \alpha \frac{r^2}{l^2} + \sqrt{\alpha^2 \frac{r^4}{l^4} + \frac{K}{12 \pi^2 l^2} (\mathcal{M}(r) - \mathcal{M}_0)}
\]
\[
\approx 1 + \frac{K}{24 \pi^2 |\alpha| r^2} (\mathcal{M}(r) - \mathcal{M}_0)
\]
\[
\approx 1 - \frac{\kappa}{24 \pi^2 \alpha_3 r^2} (\mathcal{M}(r) - \mathcal{M}_0).
\]

This means that whatever the choice of the sign of the constant \( \alpha_1 \) and \( \alpha_3 \) we obtain
\[
\lim_{l \to 0} e^{-2g} = 1 - \frac{\kappa}{24 \pi^2 \alpha_3 r^2} (\mathcal{M}(r) - \mathcal{M}_0).
\]

From (24) we can see that if \( \kappa / 2 \alpha_3 = \kappa_E \) and \( \mathcal{M}_0 = 0 \) we recover the usual 5-dimensional expression for \( e^{-2g} \) (see A17).

\[\text{B. The Exterior Solution}\]

The third equation (4) can be rewritten in the form
\[
\star (\varepsilon_{ab} e_{de} R^{bc} e^{de}) + \frac{1}{2\alpha} \star (\varepsilon_{ab} e_{de} R^{bc} e^{de}) = \kappa_E T_{ab} e^b,
\]
where \( \alpha = \alpha_3 / \alpha_1 \) and \( \kappa_E = \kappa / 2 \alpha_3 \).

Rescaling the parameter \( l \) in the form \( l \to l' = l / \sqrt{|\alpha|} \) we have
\[
\star (\varepsilon_{ab} e_{de} R^{bc} e^{de}) + \sgn(\alpha) \frac{l^2}{2} \star (\varepsilon_{ab} e_{de} R^{bc} e^{de}) = \kappa_E T_{ab} e^b.
\]

If \( \rho(r) = P(r) = 0 \) and \( \delta L_M / \delta h^a \neq 0 \), the field equations are given by:
\[
e^{-2g} \frac{r^3}{r^3} g' (1 - e^{-2g}) + \frac{\sgn(\alpha) e^{-2g}}{l^2} \frac{r^2}{r^2} (g' r + e^{-2g} - 1) = 0,
\]
\[
e^{-2g} \frac{r^3}{r^3} f' (1 - e^{-2g}) + \frac{\sgn(\alpha) e^{-2g}}{l^2} \frac{r^2}{r^2} (f' r - e^{-2g} + 1) = 0,
\]
\[
e^{-2g} \frac{r^2}{r^2} \left( f'' + (f')^2 - f' g' - e^{-2g} f'' - e^{-2g} (f')^2 + 3 e^{-2g} f' g' \right)
\]
\[
+ \frac{\sgn(\alpha) e^{-2g}}{l^2} \frac{r^2}{r^2} \left( -f' g' r^2 + f'' r^2 + (f')^2 r^2 + 2 f' r - 2 g' r - e^{-2g} + 1 \right) = 0.
\]
Following the usual procedure, we find that the equation (27) has the following solution:

\[ e^{-2g} = 1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 l^2} M}, \]  

(30)

where \( M \) is a constant of integration. From (30) is straightforward to see that in the limit \( l \to 0 \) we obtain the solution (A22) to Einstein’s gravity.

Adding equations (27) and (28) we find

\[ e^{2f} = e^{-2g}. \]  

(31)

This solution satisfies the equation (29).

From (30) and (31) we can see that the line element for the outer region is given by

\[ ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega_3^2, \]  

(32)

where

\[ F(r) = 1 + \text{sgn}(\alpha) \frac{r^2}{l^2} - \text{sgn}(\alpha) \sqrt{\frac{r^4}{l^4} + \text{sgn}(\alpha) \frac{\kappa_E}{6\pi^2 l^2} M}. \]  

(33)

III. BLACK-HOLE SOLUTION OF EINSTEIN-CHERN-SIMONS FIELD EQUATIONS

Let us consider now the conditions under which the equation (26) admits black hole type solutions.

A. Case \( \alpha > 0 \): Black Holes

In this case the exterior solution is given by (32) with

\[ F(r) = 1 + \frac{r^2}{l^2} - \sqrt{\frac{r^4}{l^4} + \frac{\kappa_E}{6\pi^2 l^2} M}. \]  

(34)

This solution shows an anomalous behaviour at

\[ F(r_0) = 1 + \frac{r_0^2}{l^2} - \sqrt{\frac{r_0^4}{l^4} + \frac{\kappa_E}{6\pi^2 l^2} M} = 0, \]

i.e., at

\[ r_0 = \sqrt{\frac{\kappa_E}{12\pi^2} M - \frac{l^2}{2}}. \]  

(35)
so that

\[ F(r) = 1 + \frac{r^2}{l^2} - \sqrt{r^4 + 2r^2l^2 + l^4}. \quad (36) \]

From the equations (32) and (35) we can see that if

\[ \frac{\kappa E}{6\pi^2} M > l^2, \quad (37) \]

then the metric (32) shows an anomalous behaviour at \( r = r_0 \). A first elementary anomaly is that we have at \( r = r_0 \)

\[ g_{00} = g^{11} = 0; \quad g^{00} = g_{11} = \infty. \quad (38) \]

A more serious anomaly is the following. One can verify that the parametric lines of the coordinate \( r \), i.e. the lines on which the coordinates \( t, \theta_1, \theta_2, \theta_3 \) have constant values, are geodesics. But these geodesics are space-like for \( r > r_0 \) and time-like for \( r < r_0 \). The tangent vector of a geodesic undergoes parallel transport along the geodesic and consequently it cannot change from a time like to a space-like vector. It follows that the two regions \( r > r_0 \) and \( r < r_0 \) do not join smoothly on the surface \( r = r_0 \).

This can be see in a more striking manner if we consider the radial null directions, on which \( d\theta_1 = d\theta_2 = d\theta_3 = 0 \). We have then

\[ ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} = 0. \quad (39) \]

Consequently the radial null directions satisfies the relations

\[ \frac{dr}{dt} = \pm F(r). \quad (40) \]

If we take into account the fact that the time-like directions are contained in the light-cone, we find that in the region \( r > r_0 \) the light cones have, in the plane \((r, t)\), the orientation shown on the figure 1.

The opening of the light cone, which is nearly equal to \( \pi/4 \) for \( r \gg r_0 \), decreases with \( r \) and tends to zero when \( r \to r_0 \). On the contrary, in the region \( r < r_0 \) the parametric lines of the coordinate \( t \) are space-like and consequently the light cones are oriented as shown on the left-hand side of figure 1, the opening of the cone increasing from the value zero at \( r = 0 \) to \( \pi/2 \) at \( r = r_0 \). Comparing the two different forms of the light cones on figure (1), we see that the regions on either side of the surface \( r = r_0 \) do not join smoothly on this surface.
FIG. 1. Space-time diagram in Schwarzschild-like coordinates for $l^2 = 2$ and $\kappa E_C(6\pi^2)^{-1}M = 20$, so that $r_0 = 3$. Some future light cone has been drawn.

B. Eddington-Finkelstein and Kruskal-Szekeres coordinates

Let us define a radial coordinate

$$r^* = \int \frac{dr}{F(r)},$$

we obtain (see appendix C.2)

$$r^* = \frac{r}{2} + \frac{r_0^2 + l^2}{4r_0} \left( \ln \left( \frac{(r - r_0)^2}{r_0(r + r_0)} \right) + Z_{\alpha > 0}(r) \right)
- \frac{ir_0^2}{2} \sqrt{\frac{2r_0^2l^2 + l^4}{2r_0^2l^2 + l^4}} F \left( \sqrt{\frac{2r_0^2l^2 + l^4}{2r_0^2l^2 + l^4}} r, i \right)
+ \frac{1}{2} \sqrt{\frac{i}{2r_0^2l^2 + l^4}} \left\{ F \left( \sqrt{\frac{i}{2r_0^2l^2 + l^4}} r, i \right) - E \left( \sqrt{\frac{i}{2r_0^2l^2 + l^4}} r, i \right) \right\}.$$  

In these coordinates the equation of the null geodesic (40) takes the form

$$d(t \pm r^*) = 0.$$  

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This means
\[
\frac{dt}{dr} = \pm \frac{dr^*}{dr}, \tag{44}
\]
so that
\[
t = \pm r^* + C_\pm. \tag{45}
\]

The constant $C_+ (C_-)$ uniquely tells us when a photon was sent away (towards) the horizon. We can therefore, consider $v \equiv t + r^*$ as a new time coordinate, which brings the metric on the form
\[
ds^2 = -F(r)dt^2 + 2dvdr + r^2d\Omega_3^2. \tag{46}
\]
We now have a non-singular description of particles falling inwards towards $r = 0$ from spatial infinity ($r = \infty$). These coordinates are called *ingoing Eddington-Finkelstein-coordinates*.

Likewise, if we had chosen $u \equiv t - r^*$ as a new time coordinate we would have gotten the metric
\[
ds^2 = -F(r)dt^2 - 2dudr + r^2d\Omega_3^2. \tag{47}
\]
These coordinates have a non-singular description of particles travelling outwards.

To understand the causal structure in the vicinity of $r = r_0$ is useful to define a new timelike coordinate. In effect, let us define
\[
t^* \equiv v - r, \tag{48}
\]
so that the ingoing null geodesics are given by
\[
t^* = -r + C_. \tag{49}
\]
These are the straight parallel lines shown on figure 2. The outgoing null geodesics are
\[
t^* = 2r^* - r + C_. \tag{50}
\]

We now recall that physical particles move on time-like worldlines or on null-lines, i.e. on lines which lie inside or on the surface of the light cones. It follows then from figure 2 first of all that no particle can cross the surface $r = r_0$ outwards. Moreover, any particle which is at some moment inside the surface $r = r_0$ will necessarily move towards the singularity in $r = 0$, reaching it in finite coordinate time as well as proper time.

The fact that no particle can cross the surface $r = r_0$ outwards means that any observer situated in the region $r > r_0$ cannot receive any information about events occurring inside
FIG. 2. Space-time diagram in advanced Eddington-Finkelstein coordinates for \( l^2 = 2 \) and \( r_0 = 3 \).

Some future light cone has been drawn.

the surface \( r = r_0 \). We say that the surface \( r = r_0 \) is an (event) horizon for all observers in the region \( r > r_0 \).

From the metric (47) we can see that in this case we shall have instead of figure 2 the new figure 3 resulting from the preceding one by reflexion with respect to the axis \( \vec{O} \vec{r} \).

We see from figure 3 that now no particles can cross the horizon inward and that particles situated at some moment in the region \( r < r_0 \) will necessarily move outwards and reach the horizon in finite proper time.

The coordinates used in (46) and (47) have, compared with those used in (32), the advantage that they describe the neighbourhood of the surface \( r = r_0 \) in a satisfactory way. However, the metrics (46) and (47) has still a certain deficiency the same type that appears in the Schwarzschild solution of general relativity. This deficiency is avoided in the Kruskal coordinates, which describe a geodesically complete space. The new coordinates are defined by choosing the combination

\[
    u = t - r^*, \quad v = t + r^*; \quad -\infty < u, v < \infty
\]  

(51)
i.e.,
\[ t = \frac{1}{2} (u + v), \quad r^* = \frac{1}{2} (v - u), \]
so that the metric for \( r > r_0 \) is given by
\[ ds^2 = -F(r) dv du + r^2 d\Omega^2. \tag{53} \]

This does not quite take care of the problem at the horizon. However, introducing the Kruskal-Szekeres coordinates
\[ U = -\exp \left(-\frac{u}{2\beta}\right), \quad V = \exp \left(-\frac{v}{2\beta}\right); \quad r > r_0, \tag{54} \]
where \( \beta \) is a parameter which will be determined, we get the result
\[ ds^2 = -\frac{4\beta F(r)}{r^*} dv du + r^2 d\Omega^2. \tag{55} \]
Using the expression for \( r^* \) given in the equation (42), we obtain

\[
\exp\left(\frac{r^*}{\beta}\right) = \frac{(r - r_0)\frac{r_0^2 + l^2}{2\beta r_0}}{(r_0(r + r_0^2)\frac{r_0^2 + l^2}{2\beta r_0})} \exp\left[\frac{r - r_0^2}{2\beta} + \frac{r_0^2 + l^2}{4\beta r_0}Z_{\alpha > 0}(r)
\right.
\left. - \frac{i\sqrt{r_0^2 + l^2}}{2\beta} \sqrt{\frac{i}{\sqrt{2r_0^2 l^2 + l^4}}} F\left(\sqrt{\frac{i}{\sqrt{2r_0^2 l^2 + l^4}}} r, i\right)
\right.
\left. + \frac{1}{2\beta} \sqrt{i\sqrt{2r_0^2 l^2 + l^4}} \left\{ F\left(\sqrt{\frac{i}{\sqrt{2r_0^2 l^2 + l^4}}} r, i\right) - E\left(\sqrt{\frac{i}{\sqrt{2r_0^2 l^2 + l^4}}} r, i\right)\right\}\right].
\]

(56)

Note that the term \((r - r_0)\frac{r_0^2 + l^2}{2\beta r_0}\) is responsible for the term \(\exp\left(\frac{r^*}{\beta}\right)\) becomes zero or becomes divergent at \(r = r_0\).

Now consider the function \(F(r)\) given in (36)

\[
F(r) = 1 + \frac{r^2}{l^2} - \sqrt{\frac{r^4 + 2r_0^2 l^2 + l^4}{l^4}}.
\]

Expanding \(F(r)\) in power series about \(r = r_0\), we have

\[
F(r) = (r - r_0)\left(\frac{2r_0}{r_0^2 + l^2} + O(r - r_0)\right). 
\]

(57)

From (56) and (57) we can see if \(\beta = \frac{r_0^2 + l^2}{2r_0}\) then the term \(F(r)/\exp\left(\frac{r^*}{\beta}\right)\) is not null or divergent. So that the line element is given by

\[
ds^2 = -\frac{2(r_0^2 + l^2)F(r)}{r_0 \exp\left(\frac{2r_0 r^*}{r_0^2 + l^2}\right)}dUdV + r^2d\Omega_3^2, 
\]

(58)

where \(r > r_0\), \(U < 0\) and \(V > 0\).

We can define

\[
F_{\alpha > 0}(r) = \frac{2(r_0^2 + l^2)F(r)}{r_0 \exp\left(\frac{2r_0 r^*}{r_0^2 + l^2}\right)}, \quad r > 0, 
\]

(59)

therefore we can let that \(U\) and \(V\) take any values

\[
ds^2 = -F_{\alpha}(r)dUdV + r^2d\Omega_3^2, \quad r > 0.
\]

(60)

The curves \(U = constant\) and \(V = constant\) are null geodesics. Introducing the Kruskal coordinates, they are given by

\[
T = \frac{1}{2}(U + V), \quad X = -\frac{1}{2}(U - V), 
\]

(61)
which (when \( r > r_0 \)) are timelike and spacelike respectively.

From (61) we can see that

\[
UV = T^2 - X^2, \quad \frac{V}{U} = \frac{T + X}{T - X}.
\]

(62)

So that the line element takes the form

\[
ds^2 = -F_a(r) \left( -dT^2 + dX^2 \right) + r^2d\Omega_3^2.
\]

(63)

C. Maximal extension and conformal compactification

Now consider the diagram of the solution (32) at coordinates \((X - T)\), holding \(\theta_1, \theta_2\) and \(\theta_3\) fixed, with \(X\) the horizontal axis and \(T\) the vertical axis.

a. Consider the curves characterized by \(r\) constant:

(i) The singularity at \(r = 0\), correspond to \(r^* = 0\), is now two hyperbolas corresponding to the solutions \(UV = T^2 - X^2 = 1\). The manifold is defined only between these two curves.

(ii) The surfaces of \(r = \text{constant} > r_0\) are hyperbolas \(UV = T^2 - X^2 = -b^2\), with \(b = \exp (r^*/2\beta)\). The “asymptotic region”, where \(r\) is very large compared to \(r_0\) is two regions in the Kruskal diagram.

(iii) The surfaces of \(r = \text{constant}\) with \(0 < r < r_0\) are hyperbolas \(UV = T^2 - X^2 = b^2\), with \(0 < b = \exp (r^*/2\beta) < 1\).

(iv) The radius \(r = r_0\) (the event horizon) is at \(UV = T^2 - X^2 = 0\), or \(T = \pm X\).

(v) If \(r \to \infty\), then \(X^2 - T^2 \to \infty\).

b. Consider the curves characterized by \(t\) constant:

(i) If \(r > r_0\) we have \(V/U = -c^2\), so that \(T = eX\) with \(c = \exp (t/2\beta)\) and \(e = (c^2 - 1)/(c^2 + 1) \in [-1, 1]\).

(ii) If \(0 \leq r < r_0\) we have \(V/U = c^2\), so that \(T = e'X\) with \(e' = (c^2 + 1)/(c^2 - 1) \in (-\infty, -1) \cup (1, \infty)\).
FIG. 4. Space-time diagram in Kruskal-Szekeres coordinates that shows its maximal analytic extension.

This means that the surfaces $T = constant$ are straight lines passing through the origin.

These Kruskal-Szekeres coordinates cover the whole spacetime and show explicitly that
the horizon at $r = r_0$ is a mere coordinate singularity in the Schwarzschild coordinates.

In figure 4 we have illustrated the Kruskal-Szekeres diagram for the analytically extended
solution (32). The original metric covers the region $I$, while region $II$ is the interior of the
black hole. Region $IV$ is the interior of a “white hole” while region $III$ is just a copy of
region $I$.

Penrose diagrams, or Penrose-Carter diagrams, are a way to represent the structure of
infinity in different spacetimes. A Penrose diagram is a space-time diagram of a conformally
compactified space-time. The idea is to make a coordinate transformation that brings points
at infinity in to finite values of the coordinates. Since the angular coordinates (in our case
$\theta_1, \theta_2, \theta_3$) have finite ranges anyhow, usually we will ignore them and plot one timelike
coordinate and one radial coordinate, or some combination of these.

Of course it is not possible transform an infinite manifold into a finite region. What
is possible is to find a metric that is not the same as the original one, but related by a
conformal transformation. The essential idea is to start off with a metric $g_{\mu\nu}$, which we call the physical metric, and introduce another metric $\bar{g}_{\mu\nu}$, called the unphysical metric, which is conformally related to $g_{\mu\nu}$, that is $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where $\Omega$ is the conformal factor. Then, by a suitable choice of $\Omega^2$, it may be possible to "bring in" the points at infinity to a finite position and hence study the causal structure of infinity.

It is well known that the null geodesics of conformally related metrics are the same, and that such null geodesics determine the light cones, which in turn define the causal structure. The essential idea for bringing in the points at infinity is to use coordinate transformations involving functions like $\arctan(x)$, which, for example, maps the infinite interval $(-\infty, \infty)$ onto the finite interval $(-\pi/2, \pi/2)$.

We introduce the null coordinate $q$ and $p$ defined from the Kruskal coordinates

$$U = \tan q, \quad V = \tan p.$$  \hspace{1cm} (64)

From (64) we can see that if $U \to \pm \infty$ then $q \to \pm \pi/2$ and if $V \to \pm \infty$ then $p \to \pm \pi/2$.

Now we introduce a timelike and a spacelike coordinates defined by

$$\tau = p + q, \quad x = p - q.$$ \hspace{1cm} (65)

A space-time diagram at $(x - \tau)$ coordinates is shown in figure 5.

c. The curves characterized by $r$ constant are given by

(i) If $r = 0$ we have

$$UV = \tan p \tan q = \frac{\cos x - \cos \tau}{\cos x + \cos \tau} = 1,$$ \hspace{1cm} (66)

so that $\cos \tau = 0$. This means that $\tau = \pm \pi/2$.

(ii) If $0 < r < r_0$ we have $UV = b^2$ with $0 < b < 1$, so that

$$\frac{\cos x - \cos \tau}{\cos x + \cos \tau} = b^2$$

and therefore

$$\cos \tau = \frac{1 - b^2}{1 + b^2} \cos x.$$ \hspace{1cm} (67)

(iii) If $r = r_0$ we have $UV = 0$, so that $\cos x - \cos \tau = 0$ and therefore $\tau = \pm x$.

(iv) If $r > r_0$ we have $UV = -b^2$ with $b > 0$, so that

$$\frac{\cos x - \cos \tau}{\cos x + \cos \tau} = -b^2$$
and therefore

\[ \cos \tau = \frac{1 + b^2}{1 - b^2} \cos x. \]  

(68)

(v) If \( r \to \infty \) we have \( UV \to -\infty \), so that

\[ \tau = \pm \pi - x \quad \text{or} \quad \tau = \pm \pi + x. \]  

(69)

The curves characterized by \( t \) constant are given by

(i) Since \( V/U = c \), where

\[ c = \begin{cases} 
\text{exp} \left( \frac{t}{\beta} \right), & 0 \leq r \leq r_0 \\
-\text{exp} \left( \frac{t}{\beta} \right), & r > r_0 
\end{cases} \]  

we have

\[ \frac{V}{U} = \tan p = \frac{\sin \tau + \sin x}{\sin \tau - \sin x}, \]  

so that

\[ \sin \tau = \frac{c + 1}{c - 1} \sin x. \]  

(72)

(ii) If \( t = 0 \) we have \( \tau = 0 \) or \( x = 0 \).

(iii) If \( t \to -\infty \) we have \( \tau = -x \).

(iv) If \( t \to \infty \) we have \( \tau = x \).

The null geodesics are given by \( U = \text{constant} \) and \( V = \text{constant} \), so that \( q = \text{constant} \) and \( p = \text{constant} \). This means that

\[ \tau = \pm x + \text{constant}. \]
D. Case $\alpha > 0$: Naked singularity

From the equations (32) and (35) we can see that if

$$\frac{\kappa_E}{6\pi^2} M \leq l^2,$$

(73)

then

$$F(r) = 1 + \frac{r^2}{l^2} - \sqrt{\frac{r^4}{l^4} + \frac{\kappa_E}{6\pi^2} l^2 M}$$

(74)

has no real roots.

Defining a radial coordinate

$$r^* = \int \frac{dr}{F(r)}$$

(75)

Setting the integration constant so that $r^*(r = 0) = 0$, we obtain (see appendix C 3)

$$r^*(r) = \frac{r}{2} + \frac{m + l^2}{2\sqrt{2}(l^2 - m)} \arctan \left( \sqrt{\frac{2}{l^2 - m}} \frac{r}{l^2} \right)$$

$$+ \frac{1}{2} \sqrt{i l \sqrt{m}} \left\{ F\left( \sqrt{\frac{i}{l \sqrt{m}}} r, i \right) - E\left( \sqrt{\frac{i}{l \sqrt{m}}} r, i \right) \right\}$$

$$+ \frac{i(l^2 - m)}{4l} \sqrt{i l \sqrt{m}} F\left( \sqrt{\frac{i}{l \sqrt{m}}} r, i \right) - \frac{i(m + l^2)^2}{4l(l^2 - m)} \sqrt{i l \sqrt{m}} \Pi\left( \sqrt{\frac{i}{l \sqrt{m}}} r, \frac{2il \sqrt{m}}{l^2 - m}, i \right)$$

(76)
FIG. 7. Penrose diagram for the case $\alpha > 0$ with $\frac{\kappa_E M}{6\pi^2} \leq l^2$.

where $m = \kappa_E M/6\pi^2$. (see Figure 6)

The corresponding radial null geodesic, incoming and outgoing, are given respectively by

$$t = -r^*(r) + C, \quad t = r^*(r) + C.$$  \hfill (77)

Since no singularities can be removed, we have no maximal extensions for this solution.

Consider then the corresponding conformal compactification. Let’s start by defining the radial null coordinates, incoming and outgoing, as

$$u = t + r^*, \quad v = t - r^*; \quad -\infty < u, v < \infty.$$  \hfill (78)

The corresponding coordinates, type Kruskal, are defined by

$$U = -\exp\left(-\frac{u}{2}\right), \quad V = \exp\left(\frac{v}{2}\right),$$  \hfill (79)

so that

$$UV = -\exp\left(r^*(r)\right) \quad \text{and} \quad \frac{V}{U} = -\exp(t).$$  \hfill (80)

Then the line element is given by

$$ds^2 = -\frac{4F(r)}{\exp\left(r^*\right)} dU dV + r^2 d\Omega^2_3.$$  \hfill (81)

We conducted the compactification, defining the following null coordinate $q$ and $p$

$$U = \tan q, \quad -\frac{\pi}{2} < q \leq 0;$$

$$V = \tan p, \quad 0 < p < \frac{\pi}{2}.$$  \hfill (82)
Now we introduce the coordinates defined by

$$\tau = p + q \quad \text{and} \quad x = p - q.$$  \hspace{1cm} (83)

The figure 7 shows the corresponding Penrose diagram.

E. Case $\alpha < 0$: Black Holes

In this case the exterior solution is given by (32) with

$$F(r) = 1 - \frac{r^2}{l^2} + \sqrt{\frac{r^4}{l^4} - \frac{\kappa E}{6\pi^2 l^2} M}. \hspace{1cm} (84)$$

From (84) we can see that there is a minimum value of $r$,

$$r_m = \sqrt[4]{\frac{\kappa E M l^2}{6\pi^2}} \hspace{1cm} (85)$$

for which the function $F(r)$ is well defined. However, it is straightforward to see that

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{4}{l^4} \left\{ \frac{4r^{12}}{(r^4 - r_m^4)^3} - \frac{12r^8}{(r^4 - r_m^4)^2} + \frac{4r^6}{(r^4 - r_m^4)^2} \right. \right.$$ 

$$\left. + \frac{15r^4}{r^4 - r_m^4} - \frac{18r^2}{\sqrt{r^4 - r_m^4}} - \frac{6\sqrt{r^4 - r_m^4}}{r^2} - \frac{3r^4}{r^4} + 13 \right\}, \hspace{1cm} (86)$$

where we see that at $r = r_m$, the invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges. This means that the 3-sphere defined by $r = r_m$ is a space-time singularity.

From the equations (32) and (35) we can see that if

$$\frac{\kappa E}{6\pi^2 M} > l^2, \hspace{1cm} (87)$$

then the metric (32) shows an anomalous behaviour at

$$r = r_0 = \sqrt{\frac{\kappa E}{12\pi^2} M + \frac{l^2}{2}} = \sqrt{\frac{r_m^4 + l^4}{2l^2}}. \hspace{1cm} (88)$$

Analogously to the case $\alpha > 0$, we consider the analysis of causal structure of spacetime in the vicinity of $r = r_0$.

Let us define a radial coordinate

$$r^* = \int \frac{dr}{F(r)} \hspace{1cm} (89)$$
FIG. 8. Space-time diagram in Schwarzschild-like coordinates for $r_m = 2$ and $r_0 = 6$.

In these coordinates, the null geodesics are given by

$$ t = \pm r^* + C. \quad (90) $$

The figure 8 shows a space-time diagram with this geodesics.

We can therefore consider $v \equiv t + r^*$ as a new time coordinate, which brings the metric on the form

$$ ds^2 = -F(r)dt^2 + 2dvdr + r^2d\Omega_3^2. \quad (91) $$

We now have a non-singular description of particles falling inwards.

Likewise, if we had chosen $u \equiv t - r^*$ as a new time coordinate we would have gotten the metric

$$ ds^2 = -F(r)dt^2 - 2dudr + r^2d\Omega_3^2. \quad (92) $$

These coordinates have a non-singular description of particles travelling outwards.

To understand the causal structure in the vicinity of $r = r_0$ is useful to define a new timelike coordinate. In effect let us define

$$ t^* \equiv v - r, \quad (93) $$

so that the ingoing null geodesics are given by

$$ t^* = -r + C. \quad (94) $$
These are the straight parallel lines shown on figure 9.

On the other hand, it is useful define another timelike coordinate

\[ \tilde{t} \equiv u + r, \]  

so that the outgoing null geodesics are given by

\[ \tilde{t} = r + C. \]  

The figure 10 show a spacetime diagram in these coordinates.

In the coordinate system defined by

\[ u = t - r^*, \quad v = t + r^*; \quad -\infty < u, v < \infty, \]  

the metric is given by

\[ ds^2 = -F(r)dudv + r^2d\Omega_3^2. \]  

Introducing the Kruskal-Szekeres coordinates

\[ U = -\exp\left(-\frac{r_0}{\sqrt{r_0^4 - r_m^4}}u\right), \quad V = \exp\left(\frac{r_0}{\sqrt{r_0^4 - r_m^4}}v\right), \]  

we get the result

\[ ds^2 = -\frac{2\sqrt{r_0^4 - r_m^4}}{r_0 \exp\left(\frac{2r_0}{\sqrt{r_0^4 - r_m^4}}r^*(r)\right)}dUdV + r^2d\Omega_3^2, \quad r > r_0. \]
Defining the function

\[ F_{α<0}(r) = \frac{2\sqrt{r_0^4 - r_m^4}F(r)}{r_0 \exp \left( \frac{2r_0}{\sqrt{r_0^4 - r_m^4}}r^*(r) \right)}, \quad r > r_m, \quad (101) \]

whose graph is shown in figure 11.

Therefore, we have

\[ ds^2 = -F_{α<0}(r)dUdV + r^2dΩ_3^2, \quad r > r_m. \quad (102) \]

Using the Kruskal coordinates

\[ T = \frac{1}{2}(U + V), \quad X = -\frac{1}{2}(U - V), \quad (103) \]

we can see that the line element takes the form

\[ ds^2 = -F_{α}(r) \left( -dT^2 + dX^2 \right) + r^2dΩ_3^2. \quad (104) \]

In figure 12 we have illustrated the corresponding Kruskal-Szekeres diagram.

We conducted the compactification, defining the following null coordinate \( q \) and \( p \)

\[ U = \tan q, \quad V = \tan p. \quad (105) \]
Now we introduce the coordinates defined by

\[ \tau = p + q \quad \text{and} \quad x = p - q. \]  \hspace{1cm} (106)

The figure 13 shows the corresponding Penrose diagram.
F. Case $\alpha < 0$: Naked singularity

In this section we considerer that

$$\frac{\kappa_E C}{6\pi^2} < l^2,$$

or equivalently

$$r_m < l.$$  \hfill (107)

Therefore we have

$$F = 1 - \frac{r^2}{l^2} + \sqrt{\frac{r^4 - r_m^4}{l^4}},$$  \hfill (108)

this function have no real roots. This mean that the metric is not singular at $r \neq r_m$.

Let us define a new radial coordinate

$$r^*(r) = \int \frac{dr}{F(r)},$$  \hfill (109)

with $r^*(r_m) = 0$, we obtain (see appendix C5)

$$r^* = \frac{r - r_m}{2} + \frac{\sqrt{r_0^4 - r_m^4}}{4r_0} \left( \ln \left( \frac{r + r_0}{r_m + r_0} \right) + Z_{\alpha<0}(r) \right)$$

$$+ \frac{r_m}{2} \left\{ F \left( \frac{r}{r_m}, i \right) - E \left( \frac{r}{r_m}, i \right) - F \left( i, i \right) + E \left( i, i \right) \right\}$$

$$+ \frac{r_0^2}{2r_m} \left\{ F \left( \frac{i}{r_m}, i \right) - F \left( i, i \right) \right\},$$  \hfill (110)

where $r_0 = \sqrt{\frac{\kappa_E}{12\pi^2} M + \frac{l^2}{2}}$. A graph for $r^*(r)$ is shown in the figure 14.
FIG. 14. Graph for $r^*(r)$ with $\frac{\kappa_{EC}}{6\pi^2} M = 1$ and $l = 4$, so that $r_m = 2$ and $r_0 = \sqrt{\frac{17}{2}} \approx 2.92$.

In these coordinates, the null geodesics are given by

$$t = \pm r^*(r) + C$$

(112)

Since no singularities can be removed ($r = r_m$ is a singularity of the space-time), we have no maximal extensions for this solution. Consider then the corresponding conformal compactification. For this purpose we can define the radial null coordinates, incoming and outgoing

$$u = t - r^* \quad \text{and} \quad v = t + r^.$$  

(113)

The Kruskal-type coordinates are defined by

$$U = -\exp\left(-\frac{u}{2}\right), \quad V = \exp\left(\frac{u}{2}\right).$$

(114)

Then, we conducted the compactification defining another radial null coordinates given by

$$U = \tan q, \quad V = \tan q,$$

(115)

so that we introduce a time-like and a space-like coordinates ($\tau - x$) with which we construct the Penrose diagram given in figure 15.
IV. SUMMARY AND OUTLOOK

We have considered a 5-dimensional action $S = S_g + S_M$ which is composed of a gravitational sector and a sector of matter, where the gravitational sector is given by a Chern-Simons gravity action instead of the Einstein-Hilbert action. We studied the implications that has on the Black Holes solutions, the fact of replacing the Einstein-Hilbert lagrangian by the Chern-Simons lagrangian in the gravitational sector of the action.

We have found some solutions for the Einstein-Chern-Simons field equations, which were obtained from the action $S = S_g + S_M$ where $S_g$ is the action for the Einstein-Chern-Simons gravity theory, studied in Ref. [4].

ACKNOWLEDGMENTS

This work was supported in part by Dirección de Investigación, Universidad de Concepción through Grant # 210.011.053-1.0 and in part by FONDECYT through Grants # 1080530.
Appendix A: Static and spherically symmetric solutions for Einstein-Cartan gravity in 5D

We consider the Einstein field equation

\[ \varepsilon_{abce} R^e_b e^d e^e = -\frac{\delta L_M}{\delta e^a}, \quad T^a = 0 \]

we take the Hodge dual on first equation

\[ X_a = \kappa E T_{ab} e^b \] (A1)

where

\[ X_a = \star (\varepsilon_{abce} R^e_b e^d e^e) \] (A2)

and \( T_{ab} \) is the energy-momentum tensor of the matter.

In five dimensions the spherically-and static- symmetric metric is given by

\[ ds^2 = -e^{2f(r)} dt^2 + e^{2g(r)} dr^2 + r^2 d\Omega_3^2 \] (A3)

where

\[ d\Omega_3^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2 \] (A4)

Introducing an orthonormal basis (\( ds^2 = \eta_{ab} e^a e^b \)):

\[ \eta_{ab} = \text{diag}(-1, +1, +1, +1, +1), \]

\[ e^T = e^{f(r)} dt, \quad e^R = e^{g(r)} dr, \quad e^1 = r d\theta_1, \quad e^2 = r \sin \theta_1 d\theta_2, \quad e^3 = r \sin \theta_1 \sin \theta_2 d\theta_3. \] (A5)

We can use Cartan’s first structural equation (\( T^a = d\omega^a + \omega^a e^b \)) and the antisymmetry of the connection forms, \( \omega^a = -\omega^b \), to find the non-zero connection forms. The calculations give:

\[ \omega_{TR} = -f' e^{-g} e^T, \quad \omega_{Ri} = -\frac{e^{-g}}{r} e^i, \quad \omega_{12} = -\frac{1}{r \tan \theta_1} e^2, \]

\[ \omega_{13} = -\frac{1}{r \tan \theta_1} e^3, \quad \omega_{23} = -\frac{1}{r \sin \theta_1 \tan \theta_2} e^3; \quad i = 1, 2, 3. \] (A6)

From Cartan’s second structural equation (\( R^a_b = d\omega^a + \omega^a \omega^b \)) we can calculate the curvature matrix. The non-zero components are

\[ R^{TR} = e^{-g} \left( f' g' - f'' \right) e^T e^R, \quad R^{Ti} = -\frac{f' e^{-2g}}{r} e^T e^i \]

\[ R^{Ri} = \frac{g' e^{-2g}}{r} e^R e^i, \quad R^{ij} = \frac{1 - e^{-2g}}{r^2} e^i e^j; \quad i, j = 1, 2, 3. \] (A7)
Introducing (A5), (A7) into (A2)

\[ X_T = 12 \frac{e^{-2g}}{r^2} \left( g' r + e^{2g} - 1 \right) e^T, \]
\[ X_R = 12 \frac{e^{-2g}}{r^2} \left( f' r - e^{2g} + 1 \right) e^R, \]
\[ X_i = 4 \frac{e^{-2g}}{r^2} \left( -f' g' r^2 + f'' r^2 + (f')^2 r^2 + 2 f' r - 2 g' r - e^{2g} + 1 \right) e^i; \quad i = 1, 2, 3. \]

Introducing (A8) into (A1) and considering the energy-momentum tensor as the energy-momentum tensor of a perfect fluid at rest, i.e., \( T_{TT} = \rho(r) \) and \( T_{RR} = T_{ii} = P(r) \), where \( \rho(r) \) and \( P(r) \) are the energy density and pressure we find

\[ 12 \frac{e^{-2g}}{r^2} \left( r g' + e^{2g} - 1 \right) = \kappa_E \rho, \quad (A9) \]
\[ 12 \frac{e^{-2g}}{r^2} \left( r f' - e^{2g} + 1 \right) = \kappa_E P, \quad (A10) \]
\[ 4 \frac{e^{-2g}}{r^2} \left( -f' g' r^2 + f'' r^2 + (f')^2 r^2 + 2 f' r - 2 g' r - e^{2g} + 1 \right) = \kappa_E P. \quad (A11) \]

1. The Exterior Solution

If \( \rho(r) = P(r) = 0 \) the field equation are given by

\[ 12 \frac{e^{-2g}}{r^2} \left( r g' + e^{2g} - 1 \right) = 0, \quad (A12) \]
\[ 12 \frac{e^{-2g}}{r^2} \left( r f' - e^{2g} + 1 \right) = 0, \quad (A13) \]
\[ 4 \frac{e^{-2g}}{r^2} \left( -f' g' r^2 + f'' r^2 + (f')^2 r^2 + 2 f' r - 2 g' r - e^{2g} + 1 \right) = 0. \quad (A14) \]

Consider the equation (A12). After multiplying (A12) by \( r^3/6 \) we find

\[ \left( r^2 \left( 1 - e^{-2g} \right) \right)' = 0. \quad (A15) \]

Integrating we have

\[ e^{-2g} = 1 - \frac{\kappa_E}{12\pi^2 r^2} M. \quad (A16) \]

Adding equations (A13) and (A14) we find

\[ e^{2f} = e^{-2g} = 1 - \frac{\kappa_E M}{12\pi^2 r^2}, \quad (A17) \]

and equation (A14) is satisfied.
2. The Interior Solution

Now consider the equation (A9). After multiplying by $r^3/6$ we find

$$\left( r^2 \left( 1 - e^{-2g} \right) \right)' = \frac{k}{6} \rho r^3. \quad (A18)$$

Integrating we have

$$r^2 \left( 1 - e^{-2g} \right) = \frac{k}{12 \pi^2} \left( M(r) - M_0 \right), \quad (A19)$$

where $M_0$ is an integration constant and $M(r)$ is the Newtonian mass, which is defined as

$$M(r) = 2 \pi^2 \int_0^r \rho(\bar{r}) \bar{r}^3 d\bar{r}, \quad (A20)$$

so that

$$e^{-2g} = 1 - \frac{k}{12 \pi^2 r^2} \left( M(r) - M_0 \right). \quad (A21)$$

To eliminate the singularity at $r = 0$ put $M_0 = 0$, then

$$e^{-2g} = 1 - \frac{k}{12 \pi^2 r^2} M(r). \quad (A22)$$

3. The Tolman-Oppenheimer-Volkoff equation in 5D

Our interest is to compute the pressure and density of matter in a spherically symmetric, static star. Since we are assuming spherical symmetry the metric will be of the form (A3). Let us recall that the energy-momentum tensor satisfies the condition

$$\nabla_\mu T^{\mu \nu} = 0. \quad (A23)$$

If $T_{TT} = \rho(r)$ and $T_{RR} = T_{ii} = P(r)$ we find

$$\nabla_\mu T^{\mu r} = \frac{f' \left( \rho(r) + P(r) \right) + P'(r)}{e^{2g}} = 0,$$

so that

$$f' = - \frac{P'}{\rho + P}, \quad (A24)$$

expression known as hydrostatic equilibrium equation.

From (A10) and (A22) we find

$$f'(r) = \frac{k E M(r)}{12 \pi^2 r^3} \left( 1 + \frac{\pi^2 r^4 P(r)}{M(r)} \right) \left( 1 - \frac{k}{12 \pi^2 r^2 M(r)} \right)^{-1}. \quad (A25)$$
Introducing (A24) into (A25) we obtain the following equation

\[ P'(r) = -\frac{\kappa EM(r)}{12\pi^2 r^3} \left( 1 + \frac{P(r)}{\rho(r)} \right) \left( 1 + \frac{\pi^2 r^4 P(r)}{M(r)} \right) \left( 1 - \frac{\kappa}{12\pi^2 r^2} M(r) \right)^{-1}, \quad \text{(A26)} \]

which is the five-dimensional Tolman-Oppenheimer-Volkoff equation. Compare with the 4-dimensional case shown in equation (1.11.13) of the reference [9].

We can resolve this equation for objects that are isentropic, that is, in which the entropy per nucleon does not vary throughout the it. For example, we have two very different kinds of star that satisfies this condition: (i) stars at absolute zero. According to Nernst’s theorem, the entropy per nucleon will then be zero throughout the star and (ii) stars in convective equilibrium. If the most efficient mechanism for energy transfer within the star is convection, then in equilibrium the entropy per nucleon must be nearly constant throughout the star. We also assume that the stars we consider have a chemical composition that is constant throughout.

With preceding assumptions, the pressure \( P \) may be expressed as a function of the density \( \rho \), the entropy per nucleon \( s \), and the chemical composition. So, with \( s \) and the chemical composition constant throughout the star, \( P(r) \) may be regarded as a function of \( \rho(r) \).

Given an equation of state \( P(\rho) \), we now formulate our problem as a pair of first-order differential equations for \( P(r) \), \( M(r) \) and \( \rho(r) \), the equation (A26) and

\[ M'(r) = 2\pi^2 r^3 \rho(r), \quad \text{(A27)} \]

with an initial condition \( M(0) = 0 \). In addition, it is necessary to provide other initial condition, that is, the value \( \rho(0) = \rho_0 \).

The differential equations must be integrated out from the center of the star, until \( P(\rho(r)) \) drops to zero at some point \( r = R \), which we then interpret as the radius of star.

Let us return to the problem of calculating the metric. Once we compute \( \rho(r) \), \( M(r) \) and \( P(r) \), we can immediately obtain \( g(r) \) from equation (A22) and \( f(r) \) from the equation (A25)

\[ f(r) = -\int_r^\infty \frac{\kappa EM(\bar{r})}{12\pi^2 \bar{r}^3} \left( 1 + \frac{\pi^2 \bar{r}^4 P(\bar{r})}{M(\bar{r})} \right) \left( 1 - \frac{\kappa}{12\pi^2 \bar{r}^2} M(\bar{r}) \right)^{-1} d\bar{r}, \quad \text{(A28)} \]

where we have set \( f(\infty) = 0 \), condition consistent with the asymptotic limit from the exterior solution.
Appendix B: Dynamics of the field $h^a$

So far we have interpreted the field $h^a$ as a field of matter whose nature has not been specified.

We consider now the field $h^a$. Expanding the field $h^a$ in their holonomic index we have

$$ h^a = h_{b
u} \eta^{ab} dx^\nu = h_{\mu\nu} \eta^{ab} e^\nu_\mu dx^\nu. \quad (B1) $$

Whether the space-time is static and spherically symmetric, the field $h_{\mu\nu}$ therefore must satisfy the Killing equation $\mathcal{L}_\xi h_{\mu\nu} = 0$ for $\xi_0 = \partial_t$ (stationary) and the six generators of the sphere $S^3$

$$ \begin{align*}
\xi_1 &= \partial_{\theta_3}, \\
\xi_2 &= \sin \theta_3 \partial_{\theta_2} + \cot \theta_2 \cos \theta_3 \partial_{\theta_3}, \\
\xi_3 &= \cos \theta_3 \partial_{\theta_2} - \cot \theta_2 \sin \theta_3 \partial_{\theta_3}, \\
\xi_4 &= \cos \theta_2 \partial_{\theta_1} - \cot \theta_1 \sin \theta_2 \partial_{\theta_2}, \\
\xi_5 &= \sin \theta_2 \sin \theta_3 \partial_{\theta_1} + \cot \theta_1 \cos \theta_2 \sin \theta_3 \partial_{\theta_2} + \cot \theta_1 \csc \theta_2 \cos \theta_3 \partial_{\theta_3}, \\
\xi_6 &= \sin \theta_2 \cos \theta_3 \partial_{\theta_1} + \cot \theta_1 \cos \theta_2 \cos \theta_3 \partial_{\theta_2} - \cot \theta_1 \csc \theta_2 \sin \theta_3 \partial_{\theta_3}. \quad (B2)
\end{align*} $$

Then, we have

$$ \begin{align*}
h^T &= h_t(r) e^T + h_{tr}(r) e^R \\
h^R &= h_{rt}(r) e^T + h_r(r) e^R \\
h^i &= h(r) e^i \quad (B3)
\end{align*} $$

From (9) and (10) and replacing in the second field equation from (4), we can see that

$$ h_{tr} = h_{rt} = 0, \quad (B4) $$

and

$$ h_r = (rh)_r', \quad h_t' = f'(h_r - h_t). \quad (B5) $$

Appendix C: Integrals

1. Elliptic Integrals

The incomplete elliptic integral of the first kind is defined as
The incomplete elliptic integral of the second kind is defined as
\[
E(z, k) := \int_0^z \frac{\sqrt{1 - k^2t^2}}{\sqrt{1 - t^2}} dt. \tag{C2}
\]

The incomplete elliptic integral of the third kind is defined as
\[
\Pi(z, v, k) := \int_0^z \frac{dt}{(1 - vt^2)\sqrt{1 - t^2} \sqrt{1 - k^2t^2}}. \tag{C3}
\]

For more information you can see ref. [10].

2. Case \(\alpha > 0\) and \(\frac{\kappa E}{6\pi^2}M > l^2\)

\[
r^*(r) = \frac{dr}{1 + \frac{r^2}{l^2} + \sqrt{\frac{r^4 + 2r_0^2l^2 + l^4}{l^4}}}, \tag{C4}
\]

where \(r_0 = \sqrt{\frac{\kappa E}{12\pi^2}M - \frac{l^2}{2}}\) is that
\[
1 + \frac{r^2}{l^2} + \sqrt{\frac{r^4 + 2r_0^2l^2 + l^4}{l^4}} \bigg|_{r=r_0} = 0. \tag{C5}
\]

We can separate \(r^*\) in the following way
\[
r^* = r^2 + \frac{r_0^2 + l^2}{2} I_{11} + \frac{1}{2} I_{21} + \frac{r_0^2}{2} I_{22} + \frac{(r_0^2 + l^2)^2}{2} I_{23}, \tag{C6}
\]

where
\[
I_{11}(r) = \int \frac{dr}{r^2 - r_0^2}, \quad I_{21}(r) = \int \frac{r^2}{\sqrt{r^4 + 2r_0^2l^2 + l^4}} dr,
\]
\[
I_{22}(r) = \int \frac{dr}{\sqrt{r^4 + 2r_0^2l^2 + l^4}}, \quad I_{23}(r) = \int \frac{dr}{(r^2 - r_0^2)\sqrt{r^4 + 2r_0^2l^2 + l^4}}. \tag{C7}
\]
the computations are
\[
I_{11}(r) = \frac{1}{2r_0} \ln \left| \frac{r - r_0}{r + r_0} \right|
\]
\[
I_{21}(r) = \sqrt{il} \sqrt{2r_0^2 + l^2} \left\{ F \left( \sqrt{\frac{i}{l \sqrt{2r_0^2 + l^2}}} r, i \right) - E \left( \sqrt{\frac{i}{l \sqrt{2r_0^2 + l^2}}} r, i \right) \right\}
\]
\[
I_{22}(r) = -i \sqrt{\frac{i}{l \sqrt{2r_0^2 + l^2}}} F \left( \sqrt{\frac{i}{l \sqrt{2r_0^2 + l^2}}} r, i \right)
\]
\[
I_{23}(r) = \frac{i}{r_0^2 l} \frac{il}{\sqrt{2r_0^2 + l^2}} \Pi \left( \sqrt{\frac{i}{l \sqrt{2r_0^2 + l^2}}} r, -\frac{il \sqrt{2r_0^2 + l^2}}{r_0^2}, i \right)
\]

However \( I_{23} \) has been computed with help from the incomplete elliptic integral of the third kind. Sadly, this way cannot get the correct result for input data provided. To solve this problem, we can separate the non finite part from the integrand, we obtain
\[
\frac{1}{(r^2 - r_0^2) \sqrt{r^4 + 2r_0^2l^2 + l^4}} = \frac{1}{2r_0(r_0^2 + l^2)(r - r_0)} - \frac{2r_0(r_0^2 + l^2)}{2r_0(r_0^2 + l^2)(r^2 - r_0^2) \sqrt{r^4 + 2r_0^2l^2 + l^4}},
\]
so that we can immediately integrate to obtain
\[
I_{23} = \frac{1}{2r_0(r_0^2 + l^2)} \left( \ln \frac{r - r_0}{r_0} + Z_{\alpha > 0}(r) \right),
\]
where we define the following smooth function to be computed through numerical methods
\[
Z_{\alpha > 0}(r) = \int_0^r \frac{2r_0(r_0^2 + l^2) - (t + r_0) \sqrt{t^4 + 2r_0^2l^2 + l^4}}{(t^2 - r_0^2) \sqrt{t^4 + 2r_0^2l^2 + l^4}} \, dr.
\]

Collecting all contributions and setting \( r^*(0) = 0 \)
\[
r^*(r) = \frac{r}{2} + \frac{r_0^2 + l^2}{4r_0} \left\{ \ln \left( \frac{(r - r_0)^2}{r_0(r + r_0)} \right) + Z_{\alpha > 0}(r) \right\}
\]
\[
- \frac{ir_0^2}{2} \sqrt{\frac{i}{l \sqrt{2r_0^2 + l^2}}} F \left( \sqrt{\frac{i}{l \sqrt{2r_0^2 + l^2}}} r, i \right)
\]
\[
+ \frac{1}{2} \sqrt{il \sqrt{2r_0^2 + l^2}} \left\{ F \left( \sqrt{\frac{i}{l \sqrt{2r_0^2 + l^2}}} r, i \right) - E \left( \sqrt{\frac{i}{l \sqrt{2r_0^2 + l^2}}} r, i \right) \right\}.
\]

3. Case \( \alpha > 0 \) and \( \frac{\kappa E}{6\alpha^s} M < l^2 \)

Now we will compute
\[
r^*(r) = \int \frac{1}{1 + \frac{r^2}{l^2} - \sqrt{\frac{r^4 + ml^2}{l^4}}} \, dr,
\]

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where $m = \frac{\kappa_E}{6\pi^2} M$ and $m < l^2$. It is important to note that

$$1 + \frac{r^2}{l^2} - \sqrt{\frac{r^4 + ml^2}{l^4}} \neq 0, \quad \forall r > 0. \quad (C14)$$

We can separate $r$ in the following way

$$r^* = \frac{r}{2} + \frac{l^2 + m}{4} I_{11} + \frac{1}{2} I_{21} - \frac{l^2 - m}{4} I_{22} + \frac{(l^2 + m)^2}{8} I_{23}, \quad (C15)$$

where

$$I_{11} = \int \frac{dr}{r^2 + \frac{l^2 - m}{2}}, \quad I_{21} = \int \frac{r^2}{\sqrt{r^4 + ml^2}} dr \quad (C16)$$

$$I_{22} = \int \frac{dr}{\sqrt{r^4 + ml^2}}, \quad I_{23} = \int \frac{dr}{(r^2 - \frac{l^2 - m}{2}) \sqrt{r^4 + ml^2}},$$

whose results are

$$I_{11}(r) = \sqrt{\frac{2}{l^2 - m}} \arctan \left( \sqrt{\frac{2}{l^2 - m}} \right)$$

$$I_{21}(r) = \sqrt{il\sqrt{m}} \left\{ F \left( \sqrt{\frac{i}{l\sqrt{m}} r, i} \right) - E \left( \sqrt{\frac{i}{l\sqrt{m}} r, i} \right) \right\}$$

$$I_{22}(r) = -i \sqrt{\frac{i}{l\sqrt{m}}} F \left( \sqrt{\frac{i}{l\sqrt{m}} r, i} \right)$$

$$I_{23}(r) = -\frac{2i}{l^2 - m} \sqrt{\frac{i}{l\sqrt{m}}} \Pi \left( \sqrt{\frac{i}{l\sqrt{m}}} r, \frac{2i l\sqrt{m}}{l^2 - m}, i \right). \quad (C17)$$

This time, the incomplete elliptic integral of the third kind have no problem.

Collecting all contributions and setting $r^*(0) = 0$.

$$r^*(r) = \frac{r}{2} + \frac{\sqrt{2}}{4} \frac{l^2 + m}{\sqrt{(l^2 - m)}} \arctan \left( \sqrt{\frac{2}{l^2 - m}} \right)$$

$$+ \frac{1}{2} \sqrt{il\sqrt{m}} \left\{ F \left( \sqrt{\frac{i}{l\sqrt{m}} r, i} \right) - E \left( \sqrt{\frac{i}{l\sqrt{m}} r, i} \right) \right\}$$

$$+ \frac{i(l^2 - m)}{4} \sqrt{\frac{i}{l\sqrt{m}}} F \left( \sqrt{\frac{i}{l\sqrt{m}} r, i} \right) - \frac{i(l^2 + m)^2}{4(l^2 - m)} \sqrt{\frac{i}{l\sqrt{m}}} \Pi \left( \sqrt{\frac{i}{l\sqrt{m}}} r, \frac{2i l\sqrt{m}}{l^2 - m}, i \right) \quad (C18)$$

4. **Case $\alpha < 0$ and $\frac{\kappa_E}{6\pi^2} M > l^2$**

The next integral to be computing is

$$r^*(r) = \int \frac{1}{1 - \frac{r^2}{l^2} - \sqrt{\frac{r^4 - 2l^2r^2 + l^4}{l^4}}} dr, \quad (C19)$$
where \( r_0 = \sqrt{\frac{\kappa E}{12\pi^2}} M + \frac{l^2}{2} \) is that
\[
1 - \frac{r^2}{l^2} + \sqrt{\frac{r^4 - 2l^2 r_0^2 + l^4}{l^4}} \bigg|_{r=r_0} = 0
\]  

(C20)

You can note there is a minimum value for \( r \) given by \( r_m = \sqrt{l^2(2r_0^2 - l^2)} = \sqrt{\frac{\kappa E}{6\pi^2}} M \) that satisfies \( l < r_m < r_0 \). So, we can write
\[
r^*(r) = \int \frac{1}{1 - \frac{r^2}{l^2} + \sqrt{\frac{r^4 - r_m^4}{l^4}}} dr,
\]  

(C21)

and separate this way
\[
r^* = \frac{r}{2} + \frac{r_0^2 - l^2}{2} I_{11} + \frac{1}{2} I_{21} + \frac{r_0^2}{2} I_{22} + \frac{(r^2 - l^2)^2}{2} I_{23},
\]  

(C22)

where
\[
I_{11} = \int \frac{dr}{r^2 - r_0^2}, \quad I_{21} = \int \frac{r^2}{\sqrt{r^4 - r_m^4}} dr,
\]  

(C23)

\[
I_{22} = \int \frac{dr}{\sqrt{r^4 - r_m^4}}, \quad I_{23} = \int \frac{dr}{(r^2 - r_0^2) \sqrt{r^4 - r_m^4}}.
\]  

The calculations gives this results
\[
I_{11}(r) = \frac{1}{2r_0} \ln \left| \frac{(r_m + r_0)(r - r_0)}{(r_m - r_0)(r + r_0)} \right|
\]
\[
I_{21}(r) = r_m \left\{ \text{F} \left( i \frac{r}{r_m}, i \right) - \text{E} \left( i \frac{r}{r_m}, i \right) - \text{F} (i, i) + \text{E} (i, i) \right\},
\]  

(C24)

\[
I_{22}(r) = \frac{1}{r_m} \left\{ \text{F} \left( i \frac{r}{r_m}, i \right) - \text{F} (i, i) \right\}.
\]

Again, we have troubles with the computation of \( I_{23} \) through incomplete elliptic integral of the third kind. We can use the same procedure from the preceding section. We separate the non finite part from the integrand,
\[
\frac{1}{(r^2 - r_0^2) \sqrt{r^4 - r_m^4}} = \frac{1}{2r_0 \sqrt{r_0^4 - r_m^4}} \left( \frac{1}{r - r_0} + \frac{2r_0 \sqrt{r_0^4 - r_m^4} - (r + r_0) \sqrt{r^4 - r_m^4}}{(r^2 - r_0^2) \sqrt{r^4 - r_m^4}} \right).
\]  

(C25)

Then, we integrate to obtain
\[
I_{23} = \frac{1}{2r_0 \sqrt{r_0^4 - r_m^4}} \left( \ln \left| \frac{r - r_0}{r_0 - r_m} \right| + Z_{\alpha<0}(r) \right),
\]  

(C26)
where we define

\[ Z_{\alpha<0}(r) = \int_{r_m}^{r} \frac{2r_0 \sqrt{r_0^4 - r_m^4} - (t + r_0) \sqrt{t^4 - r_m^4}}{(t^2 - r_0^2) \sqrt{t^4 - r_m^4}} \, dt \]  

(C27)

to be integrate through numerical methods.

Collecting all contributions and setting \( r^*(r_m) = 0 \)

\[
\begin{align*}
    r^* &= \frac{r - r_m}{2} + \frac{\sqrt{r_0^4 - r_m^4}}{4r_0} \left\{ \ln \left( \frac{(r_0 + r_m)(r - r_0)^2}{(r_0 - r_m)^2(r + r_0)} \right) + Z_{\alpha<0}(r) \right\} \\
    &+ \frac{r_m}{2} \left\{ F \left( i \frac{r}{r_m}, i \right) - E \left( i \frac{r}{r_m}, i \right) - F \left( i, i \right) + E \left( i, i \right) \right\} \\
    &+ \frac{r_0^2}{2r_m} \left\{ F \left( i \frac{r}{r_m}, i \right) - F \left( i, i \right) \right\}.
\end{align*}
\]

(C28)

5. Case \( \alpha < 0 \) and \( \frac{\kappa E}{6\pi^2} M < l^2 \)

The last integral is

\[ r^* = \int \frac{dr}{1 - \frac{r^2}{l^2} + \sqrt{\frac{r^4 - r_m^4}{l^2}}} \]  

(C29)

where \( r_m = \sqrt{\frac{\kappa E}{6\pi^2} M l^2} \). It is useful to define

\[ r_0 = \sqrt{\frac{r_m^4 + l^4}{2l^2}}. \]  

(C30)

Note that \( r_m < r_0 < l \).

So, \( r^* \) is given by

\[ r^* = \frac{r}{2} - \frac{l^2 - r_0^2}{2} I_{11} + \frac{1}{2} I_{21} + \frac{r_0^2}{2} I_{22} + \frac{(l^2 - r_0^2)^2}{2} I_{23}, \]  

(C31)

where

\[
\begin{align*}
    I_{11} &= \int \frac{dr}{r^2 - r_0^2}, \quad I_{21} = \int \frac{r^2}{\sqrt{r^4 - r_m^4}} \, dr \\
    I_{22} &= \int \frac{dr}{\sqrt{r^4 - r_m^4}}, \quad I_{23} = \int \frac{dr}{(r^2 - r_0^2) \sqrt{r^4 - r_m^4}}.
\end{align*}
\]

(C32)
The computation gives the following results

\[ I_{11}(r) = \frac{1}{2r_0} \ln \left| \frac{(r_m + r_0)(r - r_0)}{(r_m - r_0)(r + r_0)} \right| \]

\[ I_{21}(r) = r_m \left\{ F \left( i \frac{r}{r_m}, i \right) - E \left( i \frac{r}{r_m}, i \right) - F(i, i) + E(i, i) \right\} \]

\[ I_{22}(r) = \frac{1}{r_m} \left\{ F \left( i \frac{r}{r_m}, i \right) - F(i, i) \right\} \]

\[ I_{23}(r) = \frac{1}{2r_0 \sqrt{r_0^4 - r_m^4}} \left( \ln \left| \frac{r - r_0}{r_0 - r_m} \right| + Z_{\alpha<0}(r) \right), \]

with \( Z_{\alpha<0}(r) \) given in eq. C27.

Collecting all contributions and setting \( r^*(r_m) = 0 \)

\[ r^* = \frac{r - r_m}{2} + \frac{\sqrt{r_0^4 - r_m^4}}{4r_0} \left\{ \ln \left( \frac{r + r_0}{r_m + r_0} \right) + Z_{\alpha<0}(r) \right\} \]

\[ + \frac{r_m}{2} \left\{ F \left( i \frac{r}{r_m}, i \right) - E \left( i \frac{r}{r_m}, i \right) - F(i, i) + E(i, i) \right\} \]

\[ + \frac{r_0^2}{2r_m} \left\{ F \left( i \frac{r}{r_m}, i \right) - F(i, i) \right\}. \]

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