ESTIMATES FOR A GEOMETRIC FLOW FOR THE TYPE IIB STRING

Teng Fei, Duong H. Phong, Sebastien Picard, and Xiangwen Zhang

Abstract

It is shown that bounds of all orders of derivative would follow from uniform bounds for the metric and the torsion 1-form, for a flow in non-Kähler geometry which can be interpreted as either a flow for the Type IIB string or the Anomaly flow with source term and zero slope parameter. A key ingredient in the proof is a formulation of this flow unifying it with the Ricci flow, which was recently found.

1 Introduction

Anomaly flows were introduced in [18] as a way of enforcing the conformally balanced condition in the Hull-Strominger system for supersymmetric compactifications of the heterotic string [4, 14, 28]. Their effectiveness can be inferred from their having produced an alternative and unified approach [19] to the solutions found by Fu and Yau [10, 11], which had originally required solving a complicated Monge-Ampère type equation with gradients and very different estimates. Since then, they have also revealed themselves to be at the interface of many well-known questions in complex geometry: the special case with zero slope and no source can give another proof of the Calabi conjecture as well as a test of the Kähler property [20]; it also turns out to be a natural generalization of the Ricci flow to the complex and non-Kähler setting [8], which had been considered independently in [27, 33]; and we shall see shortly below that the version with zero slope but with source terms provides a natural geometric flow for supersymmetric compactifications of the Type IIB string, which had been formulated in [13, 29] and [30, 31, 32]. Anomaly flows also fit naturally in the broad theme of curvature flows in non-Kähler geometry, to which also belong many flows of great current interest [1, 2, 3, 16, 17, 27, 33].

While Anomaly flows have been worked out in a number of examples (see e.g. [6, 8, 9, 19, 21] and the recent case of nilmanifolds [26]), their understanding is still very incomplete. In particular, no general criterion for their long-time existence or convergence is as yet known. For this, it would be valuable to know what minimum set of estimates would imply estimates to all orders. This issue is of particular interest for flows in non-Kähler geometry, as the precise role of torsion terms is not yet fully understood. Even for scalar equations, $C^1$ estimates often stand apart and require different tools (see e.g. [5, 22]).

---

1Work supported in part by the National Science Foundation Grants DMS-1855947 and DMS-1809582.
The goal of the present paper is to address this issue for the following flow. Let $X$ be a compact $n$-dimensional complex manifold, equipped with a nowhere vanishing holomorphic $(n, 0)$-form $\Omega$, and a conformally balanced Hermitian metric $\omega_0$, i.e., $d(||\Omega||_{\omega_0} \omega_0^{n-1}) = 0$ where $||\Omega||_{\omega_0}$ is the norm of $\Omega$ with respect to $\omega_0$, that is, $i^n \Omega \wedge \bar{\Omega} = ||\Omega||^2_{\omega_0}(\omega^n/n!)$. Let $\Psi$ be a smooth given real and closed $(n - 1, n - 1)$-form. We consider the flow $t \to \omega(t)$ defined by

$$\partial_t(||\Omega||_{\omega^{n-1}}) = i\partial \bar{\partial} \omega^{n-2} - \Psi.$$ (1.1)

A fundamental property of this flow is that it preserves the conformally balanced condition $d(||\Omega||_{\omega^{n-1}}) = 0$. If we specialize to $n = 3$, we recognize this flow as the flow of the Hermitian metric in the Anomaly flow defined in [18], with slope $\alpha' = 0$ and source $\Psi$. Also, recall that the equations for supersymmetric compactifications for the Type IIB string have been worked out in [13] and [29]. In the simplest form studied in [30, 31, 32], they can be expressed as follows. In this case, one sets again $n = 3$, and looks for a Hermitian metric $\hat{\omega}$ on $X$ satisfying the following equations

$$d\hat{\omega}^2 = 0, \quad i\partial \bar{\partial}(||\Omega||_{\hat{\omega}}^2 \hat{\omega}) = \frac{1}{2} \rho_B$$ (1.2)

where $\rho_B$ is the Poincaré dual to a given linear combination of holomorphic 2-cycles. If we set $\omega = ||\Omega||_{\hat{\omega}}^2 \hat{\omega}$, then $||\Omega||_{\omega} = ||\Omega||_{\hat{\omega}}^{\frac{2}{3}}$, then the preceding equations can be rewritten as

$$d(||\Omega||_{\omega} \omega^2) = 0, \quad i\partial \bar{\partial} \omega = \frac{1}{2} \rho_B.$$ (1.3)

This just means that $\omega$ is a stationary point of the flow (1.1) with source $\Psi = \frac{1}{2} \rho_B$, and the flow provides a natural parabolic approach to solving the equations of the Type IIB compactifications (see also [12]). Henceforth, we shall refer to (1.1) flow as the Type IIB flow, reserving the name “Anomaly flow” for the cases involving a non-zero parameter $\alpha'$.

To state precisely our main results, we describe our conventions. If we write $\omega = ig_{kj} dz^j \wedge d\bar{z}^k$, the torsion of $\omega$ is

$$T^{m}_{jp} = g^{m\bar{q}}(\partial_j g_{\bar{q}p} - \partial_p g_{\bar{q}j}).$$ (1.4)

The torsion 1-form $\tau$ has components

$$\tau_i = T^p_{pi}.$$ (1.5)

It is well-known (e.g. [23]) that for conformally balanced metrics, the identity

$$\tau_i = \partial_i \log ||\Omega||_\omega$$ (1.6)

holds. Our main theorem is then
Theorem 1. Let $(X, \omega_0)$ be a compact Hermitian manifold with nowhere vanishing holomorphic $(n, 0)$ form $\Omega$ satisfying $d(\|\Omega\|_{\omega_0} \omega_0^{-1}) = 0$ and given closed $\Psi \in \Lambda^{n-1,n-1}(X)$. Let $\omega(t)$ evolve by the Type IIB flow
\[ \partial_t \left( \|\Omega\|_{\omega_0} \omega_0^{n-1} \right) = i\bar{\partial}\partial \omega^{n-2} - \Psi, \quad \omega(0) = \omega_0. \] (1.7)
Suppose on $[0, T_0]$, we have the estimate
\[ K_1^{-1} \omega_0 \leq \omega(t) \leq K_1 \omega_0, \quad |\tau| \leq K_2 \] (1.8)
for $K_1, K_2 > 0$. Then, there exist constants $C > 0$ and $0 < \alpha < 1$ depending on $K_1, K_2, (X, \omega_0, \Omega)$ and $\Psi$ such that
\[ \|g\|_{C^{2+\alpha,1+\alpha/2}(X \times (0, T_0])} \leq C. \] (1.9)

As an intermediate consequence of the above theorem, we can provide a criterion for the long-time existence of the flow (1.7)

Corollary 1. Under the same hypotheses as in the above theorem, the flow can be extended to $[0, T_0 + \varepsilon]$ for some $\varepsilon > 0$.

In the case with no source, we can weaken our assumptions:

Theorem 2. Let $(X, \omega_0)$ be a compact Hermitian manifold with nowhere vanishing holomorphic $(n, 0)$ form $\Omega$ satisfying $d(\|\Omega\|_{\omega_0} \omega_0^{-1}) = 0$. Let $\omega(t)$ evolve by the Anomaly flow
\[ \partial_t \left( \|\Omega\|_{\omega_0} \omega_0^{n-1} \right) = i\bar{\partial}\partial \omega^{n-2}, \quad \omega(0) = \omega_0. \] (1.10)
Suppose on $[0, T_0]$, we have the estimate
\[ \omega(t) \leq K_1 \omega_0, \quad |\tau| \leq K_2 \] (1.11)
for $K_1, K_2 > 0$. Then, there exist constants $C > 0$ and $0 < \alpha < 1$ depending on $K_1, K_2, (X, \omega_0, \Omega)$ and $\Psi$ such that
\[ \|g\|_{C^{2+\alpha,1+\alpha/2}(X \times (0, T_0])} \leq C. \] (1.12)
Moreover, there exists $\varepsilon > 0$ such that the flow can be extended to $[0, T_0 + \varepsilon]$.

We stress several aspects of the estimates obtained in the above theorems. First, it is remarkable that only a bound for $\tau$ was needed, and not for the whole torsion tensor $T$. Second, unlike the Shi-type estimates obtained in [23] assuming bounds for $|Rm|_{\omega} + |T|_{\omega} + |DT|_{\omega}$, they are independent of the time interval $T_0$. Third, they allow for a non-trivial source term $\Psi$. The incorporation of sources had turned out to be surprisingly difficult in the approach of [23], and a key catalyst for the approach in the present paper is the new formulation of the Type IIB flow found in [8] which unifies it with the Ricci flow. This allows in particular the application of Calabi-type identities for the connection, as introduced in [24]. Once the $C^1$ estimates for the metric have been obtained, we can apply the general Schauder theory for systems of quasilinear parabolic system to obtain all the higher order estimates [15].
2 First Order Estimate of the Metric

The bulk of the paper is devoted to the proof of Theorem 1.

To prove Theorem 1, following [8], we consider the flow of

\[ \eta_{kj} = \|\Omega\|_\omega g_{kj}. \]  

(2.1)

The form \( \eta = i\eta_{kj}dz^j \wedge d\bar{z}^k \) satisfies

\[ d(\|\Omega\|_\eta^2 \eta^{n-1}) = 0. \]  

(2.2)

In [8], it is shown that \( \eta \) evolves by

\[ \partial_t \eta_{kj} = -\tilde{R}_{kj}(\eta) - \frac{1}{2} T_{kpq}(\eta) \tilde{T}_{j}^{pq}(\eta) - \Phi_{kj}. \]  

(2.3)

where \( \tilde{R}_{kj}(\eta) \) is the second Chern-Ricci curvature of the metric \( \eta \) and \( \Phi(z, \eta_{kj}) \in \Lambda^{n-1,n-1}(X) \) involves combinations of the given \( \Psi \in \Lambda^{n-1}(X) \) and \( \eta_{kj} \).

This evolution equation for \( \eta \) is simpler than the one for \( \omega \), and we will use it to obtain estimates on \( \eta \) and then deduce estimates on \( \omega \). For this, we note a relation between their torsion 1-forms. Taking the determinant of the defining relation, we see that

\[ \|\Omega\|_\eta^2 = \|\Omega\|_g^{2-n}. \]  

(2.4)

Therefore

\[ \partial_t \log \|\Omega\|_\eta = \partial_t \log \|\Omega\|_g^{1-(n/2)} = \frac{2-n}{2} \tau_i. \]  

(2.5)

It follows that if

\[ K^{-1} \omega_0 \leq \omega(t) \leq K \omega_0, \quad |\tau| \leq K \]  

(2.6)

then

\[ C^{-1} \eta_0 \leq \eta(t) \leq C \eta_0, \quad |\nabla \log \|\Omega\|_\eta| \leq C, \]  

(2.7)

and for the remainder of the paper, we will assume (2.7) and work exclusively with the metrics \( \eta(t) \). Note that if \( \eta \) is bounded in the \( C^k \) norm and \( \eta \geq C^{-1} \eta_0 \), then

\[ g_{kj} = \|\Omega\|_\eta^{2/(n-2)} \eta_{kj}. \]  

(2.8)

is also bounded in the \( C^k \) norm.

In the remaining part of this section, we follow the calculation in [24] (see also [25, 35]) to derive a gradient estimate for the metric. Let \((X,g)\) be a compact Hermitian manifold. We will work with the flow

\[ \partial_t g_{kj} = -\tilde{R}_{kj} - \frac{1}{2} T_{kpq} \tilde{T}_{j}^{pq} - \Phi(z, g(t))_{kj}. \]  

(2.9)
In comparison to the flow (2.3), here we write \( g_{kj} \) instead of \( \eta_{kj} \). For abbreviation, we will denote \( \hat{g} = g(0) = g_0 \) and \( g = g(t) \). Let

\[
h^\alpha_\beta = \hat{g}^{\alpha\gamma} g_{\gamma\beta}
\]  
(2.10)

In the following computation, we will use \( \nabla \) and \( \hat{\nabla} \) to denote the Chern connections, \( \theta \) and \( \hat{\theta} \) to denote the connection 1-forms, \( R \) and \( \hat{R} \) to denote the curvatures, with respect to the metrics \( g \) and \( \hat{g} \). We write

\[
\Delta = g^{pq} \nabla_p \nabla_q,
\]  
(2.11)

for the Laplacian, and

\[
i\Lambda_\omega \Phi = g^{jk} \Phi_{kj}, \quad \omega = ig_{kj} dz^j \wedge dz^k
\]  
(2.12)

for the contraction operator. Our curvature conventions are

\[
R_{kj}^p_q = -\partial_k \Gamma_{jq}^p, \quad \Gamma_{jq}^p = g^{\alpha\beta} \partial_j g_{\alpha q},
\]  
(2.13)

and

\[
\tilde{R}_{pq} = g_{\mu\nu} g^{jk} R_{kj}^\mu^\nu
\]  
(2.14)

for the second Chern-Ricci curvature.

### 2.1 Evolution of relative endomorphism

In this section, we derive the following evolution equation.

**Proposition 1** Along the flow (2.9), we have

\[
(\partial_t - \Delta) \mathrm{Tr} h = -g^{qp} h^{-1\gamma}_\mu \hat{\nabla}_q h^\mu_j \hat{\nabla}_p h^j_\gamma - g^{qp} \hat{R}_{pq}^\alpha h^j_\alpha - \frac{1}{2} \hat{g}^{jk} T_{kqp} \bar{T}_{j}^{pq} - i\Lambda_\omega \Phi.
\]  
(2.15)

**Proof:** By definition,

\[
\partial_t \mathrm{Tr} h = \hat{g}^{pq} \partial_t g_{qp},
\]  
(2.16)

which by the equation for the flow is

\[
\partial_t \mathrm{Tr} h = -\hat{g}^{jk} \hat{R}_{kj} - \frac{1}{2} \hat{g}^{jk} T_{kqp} \bar{T}_{j}^{pq} - i\Lambda_\omega \Phi.
\]  
(2.17)

In general, we can compute

\[
\theta = g^{-1} \partial g = h^{-1} \hat{g}^{-1} \partial(\hat{g} h) = h^{-1} \partial h + h^{-1} \hat{\theta} h
\]  
(2.18)

\[
= h^{-1} \partial h + h^{-1} \hat{\theta} h - h^{-1} h \hat{\theta} + \hat{\theta}
\]  
\[
= \hat{\theta} + h^{-1} \hat{\nabla} h.
\]
The definition of the curvature of the Chern connection (2.13) implies

\[ R_{\bar{p}q}^{\alpha} \beta = \hat{R}_{\bar{p}q}^{\alpha} \beta - \partial_{\bar{p}}(h^{-1}\hat{\nabla}_q h)^{\alpha} \beta \]  

(2.19)

and

\[ \hat{R}_{kj} = g_{k\alpha}g^{q\bar{p}}\hat{R}_{\bar{p}q}^{\alpha} j - g_{k\alpha}g^{q\bar{p}}\hat{\nabla}_p (h^{-1}\hat{\nabla}_q h)^{\alpha} j. \]  

(2.20)

Substituting this identity into (2.17), we obtain

\[ \partial_t \text{Tr } h = g^{q\bar{p}}\hat{\nabla}_\bar{p} (h^{-1}\hat{\nabla}_q h)^{\alpha} j h^j_\alpha - g^{q\bar{p}}\hat{R}_{\bar{p}q}^{\alpha} j h^j_\alpha - \frac{1}{2} \hat{g}^{j\bar{k}}T_{k\bar{p}q} \bar{T}_{j}^{\bar{p}q} - i\Lambda_\omega \Phi. \]  

(2.21)

Expanding the first term gives

\[ \partial_t \text{Tr } h = g^{q\bar{p}}(h^{-1})^{\alpha} \mu \hat{\nabla}_\bar{p} \hat{\nabla}_q h^\mu j h^j_\alpha + g^{q\bar{p}}(\hat{\nabla}_\bar{p} h^{-1})^{\alpha} \mu (\hat{\nabla}_q h)^{\mu} j h^j_\alpha - g^{q\bar{p}}\hat{R}_{\bar{p}q}^{\alpha} j h^j_\alpha - \frac{1}{2} \hat{g}^{j\bar{k}}T_{k\bar{p}q} \bar{T}_{j}^{\bar{p}q} - i\Lambda_\omega \Phi. \]  

(2.22)

Since

\[ \hat{\nabla}_\bar{p} (h^{-1})^{\alpha} \mu = -(h^{-1})^{\alpha} \nu \hat{\nabla}_\bar{p} h^\nu \gamma (h^{-1})^{\gamma} \mu, \]  

(2.23)

we obtain

\[ \partial_t \text{Tr } h = g^{q\bar{p}}\partial_{\bar{p}}\partial_q \text{Tr } h - g^{q\bar{p}}(h^{-1})^{\gamma} \mu \hat{\nabla}_\bar{p} h^\mu j \hat{\nabla}_q h^j_\gamma \] 

\[ - g^{q\bar{p}}\hat{R}_{\bar{p}q}^{\alpha} j h^j_\alpha - \frac{1}{2} \hat{g}^{j\bar{k}}T_{k\bar{p}q} \bar{T}_{j}^{\bar{p}q} - i\Lambda_\omega \Phi. \]  

(2.24)

Here we used that

\[ \hat{\nabla}_\bar{p} \hat{\nabla}_q h^i_j = \partial_{\bar{p}}(\partial_q h^i_j + \hat{\Gamma}^i_{aj} h^j_\alpha - h^j_\alpha \hat{\Gamma}^j_{qi}) = \partial_{\bar{p}}\partial_q \text{Tr } h. \]  

(2.25)

This proves the identity. Q.E.D.

This identity allows us to use the quantity Tr \( h \) to produce a negative term involving the \( C^1 \) norm of the metric.

**Proposition 2** Let \( g(t) \) evolve by Type IIB flow (2.9). Suppose on \([0, T]\), we have the estimate

\[ K^{-1}\hat{g} \leq g(t) \leq K\hat{g}, \]  

(2.26)

for \( K > 0 \). Then

\[ (\partial_t - \Delta)\text{Tr } h \leq -\frac{1}{K^2} |\nabla g|^2 + C, \]  

(2.27)

on \([0, T]\), where \( C \) depends on \( K \), \((X, \hat{g}), \Phi\).
Proof: We can estimate
\[ g^{pq} (h^{-1})^{\gamma} \nu_q h^{\mu} \nu^\rho h^j_{\gamma} = g^{pq} g^{i\bar{k}} \hat{\nabla}_q g_{ki} \hat{\nabla}_q g_{\bar{k}i} \geq K^{-2}|\hat{\nabla} g|^2. \] (2.28)

We also note that
\[ -\frac{1}{2} \hat{g}^{j\bar{k}} T_{kpq} T_{j}^{pq} \leq 0. \] (2.29)

Altogether, we obtain
\[ (\partial_t - \Delta) \text{Tr } h \leq -K^{-2}|\hat{\nabla} g|^2 - g^{pq} \hat{R}_{p\bar{q}} \alpha_j h^j_{\alpha} - i\Lambda_\omega \Phi, \] (2.30)

which implies the estimate. Q.E.D.

2.2 \( C^1 \) estimate for the metric

As in [34] and [24], we consider
\[ S = |\nabla h h^{-1}|^2_g = g^{m\bar{n}} g_{\beta\bar{\alpha}} g^{\ell\bar{\alpha}} (\nabla_m h h^{-1})^\beta_{\ell} (\nabla_\gamma h h^{-1})^{\bar{\alpha}}. \] (2.31)

The guiding principle of the computations below is that \( \nabla h h^{-1} \) is essentially a connection. Note that by the identity (which can be derived by a computation similar to (2.18))
\[ \hat{\theta} = \theta - \nabla h h^{-1}, \] (2.32)

we can write
\[ S = |\theta - \hat{\theta}|^2_g. \] (2.33)

By the identity
\[ \Gamma^k_{ij} - \hat{\Gamma}^k_{ij} = g^{k\bar{s}} \partial_i g_{\bar{s}j} - \hat{\Gamma}^k_{ij} = g^{k\bar{s}} \hat{\nabla}_i g_{\bar{s}j}, \] (2.34)

we can also write
\[ S = |\hat{\nabla} g|^2_g. \] (2.35)

In this section, we will show the following estimate.

Proposition 3 Let \( g(t) \) evolve by Type IIB flow (2.9). Suppose on \([0, t_0]\), we have the estimate
\[ K^{-1} \hat{g} \leq g(t) \leq K \hat{g}, \quad |T|^2 \leq K \] (2.36)

for \( K > 0 \). Then
\[ (\partial_t - \Delta) S \leq CS + C, \] (2.37)

on \([0, t_0]\), where \( C \) depends on \( K, (X, \hat{g}), \Phi \).

Given Proposition 3 and (2.27), we conclude
Theorem 3 Let $g(t)$ evolve by Type IIB flow \((2.9)\). Suppose on $[0, t_0]$, we have the estimate
\[
K^{-1} \hat g \leq g(t) \leq K \hat g, \quad |T|^2 \leq K
\]
for $K > 0$. Then
\[
|\hat \nabla g|^2 \leq C
\]
on $[0, t_0]$, where $C$ depends on $K$, $(X, \hat g)$, $\Phi$, $g(0)$.

Proof: Let
\[
G = S - A Tr h,
\]
where $A \gg 1$ will be chosen depending on $K$, $(X, \hat g)$, $\Phi$. Combining Proposition 3 and (2.27), we obtain
\[
(\partial_t - \Delta) G \leq C_1 S + C_2 A K^2 S + AC.
\]
Choosing $A = K^2 C_1 + K^2$,
\[
(\partial_t - \Delta) G \leq AC - S.
\]
Let $(p, t) \in X \times [0, T]$, be a point where $G$ attains its maximum with $t > 0$. Then
\[
0 \leq AC - S(p, t),
\]
which implies $S(p, t) \leq AC$, and hence
\[
S = G + A Tr h \leq G(p, t) + C \leq C.
\]
Since $S = |\hat \nabla g|^2$, this proves the desired estimate. Q.E.D.

2.2.1 General formula for the evolution of $S$

From [24], under any flow, we have the general formula
\[
(\partial_t - \Delta) S = -|\hat \nabla (\nabla h h^{-1})|^2 - |\nabla (\nabla h h^{-1})|^2 + g^{\alpha \beta} \left\{ (\partial_t - \Delta) (\nabla_m h h^{-1}, \nabla_\alpha h h^{-1}) + g^{\gamma \delta} (\nabla_m h h^{-1}, (\partial_t - \Delta) (\nabla_\gamma h h^{-1})) \right\}
\]
\[
- (\nabla_m h h^{-1})_\beta (\nabla_\beta h h^{-1})_{\mu \alpha} \cdot \left\{ (h^{-1} \dot h + \tilde R)^{m \gamma} g_{\mu \beta} g_{\ell \alpha} \\ - g^{\gamma \delta} (h^{-1} \dot h + \tilde R)_{\mu \beta} g_{\ell \alpha} + g^{\gamma \delta} g_{\mu \beta} (h^{-1} \dot h + \tilde R)_{\ell \alpha} \right\}
\]
We provide the derivation for completeness. We have
\[
\nabla_q S = \langle \nabla_q \nabla h h^{-1}, \nabla h h^{-1} \rangle + \langle \nabla h h^{-1}, \nabla_q \nabla h h^{-1} \rangle,
\]
and
\[
\Delta S = g^{pq} \langle \nabla_p \nabla_q \nabla h h^{-1}, \nabla h h^{-1} \rangle + g^{pq} \langle \nabla_q \nabla h h^{-1}, \nabla_p \nabla h h^{-1} \rangle \\
+ g^{pq} \langle \nabla_p \nabla h h^{-1}, \nabla_q \nabla h h^{-1} \rangle + g^{pq} \langle \nabla h h^{-1}, \nabla_p \nabla_q \nabla h h^{-1} \rangle.
\]
Therefore

\[
\Delta S = |\nabla (\nabla hh^{-1})|^2 + |\nabla (\nabla hh^{-1})|^2 + \langle \Delta (\nabla hh^{-1}), \nabla hh^{-1} \rangle + g^{pq}(\nabla hh^{-1}, \nabla p \nabla q (\nabla hh^{-1})) \quad (2.48)
\]

Our curvature convention is

\[
[\nabla_j, \nabla_k]W_i = -R_{kj}^{\quad j} W_p, \quad [\nabla_j, \nabla_k]W_i = R_{kji}^{\quad j} W_{\bar{p}}. \quad (2.49)
\]

Therefore

\[
\nabla_p \nabla_q (\nabla_i hh^{-1})^\alpha_\beta = \nabla_q \nabla_p (\nabla_i hh^{-1})^\alpha_\beta + R_{pq}^{\quad j}(\nabla_j hh^{-1})^\alpha_\beta - R_{pq}^{\quad \alpha}(\nabla_i hh^{-1})^\gamma_\beta + R_{pq}^{\quad \gamma}(\nabla_i hh^{-1})^\alpha_\gamma. \quad (2.50)
\]

It follows that

\[
\Delta S = |\nabla (\nabla hh^{-1})|^2 + |\nabla (\nabla hh^{-1})|^2 + \langle \Delta (\nabla hh^{-1}), \nabla hh^{-1} \rangle + g^{kj}(\nabla_k hh^{-1}, \tilde{R}_j(\nabla_j hh^{-1})^\alpha_\beta) - g^{ki}(\nabla_k hh^{-1}, \tilde{R}^\alpha_{\gamma_\gamma}(\nabla_i hh^{-1})^\gamma_\beta) + g^{ki}(\nabla_k hh^{-1}, \tilde{R}^\alpha_{\beta}(\nabla_i hh^{-1})^\alpha_\gamma). \quad (2.51)
\]

On the other hand,

\[
\partial_t S = \partial_t \left[ g^{\bar{k}i} g_{\bar{\mu} \alpha} g^{\beta \bar{\rho}} (\nabla_k hh^{-1})^\alpha_\beta (\nabla_i hh^{-1})^{\bar{\mu} \bar{\rho}} \right]. \quad (2.52)
\]

We note

\[
\dot{h}^\alpha_\beta = \dot{g}^{\alpha \bar{\mu}} \dot{g}_{\bar{\mu} \beta}, \quad (h^{-1} \dot{h})^\alpha_\beta = g^{\alpha \bar{\mu}} \dot{g}_{\bar{\mu} \beta}. \quad (2.53)
\]

Therefore

\[
\partial_t S = \langle \partial_t (\nabla hh^{-1}), \nabla hh^{-1} \rangle + \langle \partial_t (\nabla hh^{-1}), \partial_t (\nabla hh^{-1}) \rangle - (h^{-1} \dot{h})^{\bar{k}i} g^{\bar{k}i} g_{\bar{\mu} \alpha} g^{\beta \bar{\rho}} (\nabla_k hh^{-1})^\alpha_\beta (\nabla_i hh^{-1})^{\bar{\mu} \bar{\rho}} + g^{\bar{k}i} g_{\bar{\mu} \alpha} (h^{-1} \dot{h})^{\bar{\beta}} g^{\beta \bar{\rho}} (\nabla_k hh^{-1})^\alpha_\beta (\nabla_i hh^{-1})^{\bar{\mu} \bar{\rho}} - g^{\bar{k}i} g_{\bar{\mu} \alpha} (h^{-1} \dot{h})^{\bar{\beta}} g^{\beta \bar{\rho}} (\nabla_k hh^{-1})^\alpha_\beta (\nabla_i hh^{-1})^{\bar{\mu} \bar{\rho}} \quad (2.54)
\]

Putting this together, we obtain (2.45).

### 2.2.2 Evolution of S along Type IIB flow

In this section, we derive the expression for the evolution of $S$ along the flow (2.9). For this, we start by using

\[
(h^{-1} \dot{h})^{\gamma_\beta} = g^{\alpha \bar{\mu}} \dot{g}_{\bar{\mu} \beta}. \quad (2.55)
\]
to rewrite the flow as
\[ (h^{-1} \dot{h})^{\gamma}_{\beta} + \dot{R}^{\gamma}_{\beta} = Q^{\gamma}_{\beta} - \Phi^{\gamma}_{\beta}. \] (2.56)

Here \( Q^{\gamma}_{\beta} = -\frac{1}{2} T_{\gamma pq} \tilde{T}^{pq}. \) Next, we compute \( \partial_t (\nabla m h h^{-1})^\alpha_{\beta}. \) Similarly as (2.18), we have
\[ \dot{\theta} = \theta - \nabla h h^{-1}. \] (2.57)

Therefore
\[ \partial_t (\nabla m h h^{-1})^\alpha_{\beta} = \partial_t (\Gamma_{m\beta}^{\alpha} - \hat{\Gamma}_{m\beta}^{\alpha}) = \partial_t (g^{\alpha\gamma} \partial_m g_{\gamma\beta}). \] (2.58)

It follows that
\[ \partial_t (\nabla m h h^{-1})^\alpha_{\beta} = g^{\alpha\gamma} (\partial_m \dot{g}_{\gamma\beta} - \dot{g}_{\gamma q} \Gamma_{m\beta}^{q}) = g^{\alpha\gamma} \nabla_m \dot{g}_{\gamma\beta}. \] (2.59)

Thus
\[ \partial_t (\nabla m h h^{-1})^\alpha_{\beta} = \nabla M (h^{-1} \dot{h})^\alpha_{\beta} = g^{\alpha\gamma} \nabla_m \left( -\dot{R}^{\gamma}_{\beta} + Q^{\gamma}_{\beta} - \Phi^{\gamma}_{\beta} \right) \] (2.60)
\[ = -\nabla M \dot{R}^{\alpha}_{\beta} + \nabla M \Phi^{\alpha}_{\beta} - \nabla M \Phi^{\alpha}_{\beta}. \]

We need to compare this expression with \( \Delta (\nabla M h h^{-1}). \) Using the relation between connection 1-forms (2.57), we obtain
\[ \dot{R}_{\gamma m}^{\alpha}_{\beta} = R_{\gamma m}^{\alpha}_{\beta} + \partial_q \left( \nabla M h h^{-1} \right)^\alpha_{\beta}. \] (2.61)

It follows that
\[ \Delta (\nabla M h h^{-1})^\alpha_{\beta} = g^{\rho q} \nabla_p \nabla_q (\nabla M h h^{-1})^\alpha_{\beta} = \nabla_q \dot{R}_{\gamma m}^{\alpha}_{\beta} - \nabla q \dot{R}_{\gamma m}^{\alpha}_{\beta}. \] (2.62)

Recall that, for the Chern connections of general Hermitian metrics, we have the following first Bianchi identities
\[ R_{\ell m k j} = R_{\ell j k m} + \nabla_{\ell} T_{k j m}; \quad R_{\ell m k j} = R_{k m \ell j} + \nabla_{m} \tilde{T}_{j k \ell}. \] (2.63)

and also the second Bianchi identities
\[ \nabla_{m} R_{k j}^{\alpha}_{\beta} = \nabla_{j} R_{k m}^{\alpha}_{\beta} + T^{\rho}_{j m} R_{k \rho}^{\alpha}_{\beta}, \quad \nabla_{m} R_{k j}^{\alpha}_{\beta} = \nabla_{j} R_{k m}^{\alpha}_{\beta} + T^{\rho}_{j m} R_{k \rho}^{\alpha}_{\beta}; \] (2.64)
\[ \nabla_{m} R_{k j}^{\alpha}_{\beta} = \nabla_{k} R_{m j}^{\alpha}_{\beta} + \tilde{T}^{\rho}_{k m} R_{\rho j}^{\alpha}_{\beta}, \quad \nabla_{m} R_{k j}^{\alpha}_{\beta} = \nabla_{k} R_{m j}^{\alpha}_{\beta} + \tilde{T}^{\rho}_{k m} R_{\rho j}^{\alpha}_{\beta}. \] (2.65)

Using (2.64), we can compute
\[ \nabla^{q} R_{q m}^{\alpha}_{\beta} = g^{pq} \nabla_p R_{q m}^{\alpha}_{\beta} = g^{pq} (\nabla_{m} R_{q p}^{\alpha}_{\beta} + T^{\rho}_{m p} R_{q \rho}^{\alpha}_{\beta}) \] (2.66)
\[ = \nabla_{m} \dot{R}^{\alpha}_{\beta} + g^{pq} T^{\rho}_{m p} R_{q \rho}^{\alpha}_{\beta}. \]

It follows that
\[ \Delta (\nabla M h h^{-1})^\alpha_{\beta} = -\nabla M \dot{R}^{\alpha}_{\beta} + \nabla^{q} \dot{R}_{q m}^{\alpha}_{\beta} - g^{pq} T^{\rho}_{m p} R_{q \rho}^{\alpha}_{\beta}. \] (2.67)
Thus, combining this with (2.60), we obtain
\[
(\partial_t - \Delta)(\nabla_m h^{-1})^{\alpha}_{\beta} = -\nabla^q \hat{R}_{qm}{}^\alpha_{\beta} + g^{pq} T_{mp} R_{qr}{}^\alpha_{\beta} + \nabla_m Q^\alpha_{\beta} - \nabla_m \Phi^\alpha_{\beta}. \tag{2.68}
\]
Substituting this expression and (2.56) into the general formula (2.45) for \( S \), we obtain
\[
(\partial_t - \Delta) S = -|\nabla (\nabla h^{-1})|^2 - |\nabla (\nabla h^{-1})|^2 + (I) + (II) + (III) + (IV) + (V) + (VI)
\]
where
\[
(I) = -g^{m\xi} \nabla^q \hat{R}_{qm}{}^\alpha_{\beta} (\nabla_m h^{-1})^\alpha_{\beta} - g^{m\xi} (\nabla_m h^{-1})^\alpha_{\beta} \hat{\nabla}^\xi \hat{R}^\beta_{\gamma}^\alpha \\
(II) = g^{m\xi} g^{pq} T_{mp} R_{qr}{}^\alpha_{\beta} (\nabla_m h^{-1})^\alpha_{\beta} + g^{m\xi} (\nabla_m h^{-1})^\alpha_{\beta} \nabla_m Q_h \Phi^\beta_{\alpha} \\
(III) = -g^{m\xi} \nabla_m \Phi^\alpha_{\beta} (\nabla_m h^{-1})^\alpha_{\beta} + g^{m\xi} (\nabla_m h^{-1})^\alpha_{\beta} \nabla_m \Phi^\alpha_{\beta} \\
(IV) = -g^{m\xi} \nabla_m \Phi^\alpha_{\beta} (\nabla_m h^{-1})^\alpha_{\beta} + g^{m\xi} (\nabla_m h^{-1})^\alpha_{\beta} \nabla_m \Phi^\alpha_{\beta} \\
(V) = - (\nabla_m h^{-1})^\beta_{\alpha} (\nabla_m h^{-1})^\alpha_{\beta} \left\{ Q^m_{\nu} g_{\mu\beta} g^{\nu\alpha} - g^m_{\nu} Q_{\mu\beta} g^{\nu\alpha} + g^{m\xi} g_{\mu\beta} Q_{\xi}^{\nu} \right\} \\
(VI) = - (\nabla_m h^{-1})^\beta_{\alpha} (\nabla_m h^{-1})^\alpha_{\beta} \left\{ -\Phi^m_{\nu} g_{\mu\beta} g^{\nu\alpha} + g^m_{\nu} \Phi_{\mu\beta} g^{\nu\alpha} - g^m_{\nu} g_{\mu\beta} \Phi^{\nu\alpha} \right\}.
\]

2.2.3 Estimate of \( S \)

In this section, we estimate terms in the previous expression. The final result will be
\[
(\partial_t - \Delta) S \leq -\frac{1}{2} \left( |\nabla (\nabla h^{-1})|^2 + |\nabla (\nabla h^{-1})|^2 \right) + C(1 + S + |T|^2 S), \tag{2.70}
\]
where \( C \) depends on \( (X, \hat{g}), \Phi, \) and bounds for the metric \( g \) above and below in terms of the reference \( \hat{g} \). To prove this, we will need to estimate terms one by one.

For (I): Because of the presence of connection in e.g. \( \nabla^q \hat{R}_{qm}{}^\alpha_{\beta} \), the first covariant derivatives are of the order \( O(S^{1/2}) \). Therefore,
\[
(I) \leq C_1 (S + S^{1/2}). \tag{2.71}
\]

For (II): Recall that
\[
\hat{R}_{qm}{}^\alpha_{\beta} = R_{qm}{}^\alpha_{\beta} + \nabla_q (\nabla_m h^{-1})^\alpha_{\beta}. \tag{2.72}
\]
Using this, we can estimate (II) as
\[
(II) \leq C_2 S^{1/2} |T| + C_2 S^{1/2} |T| \left\{ |\nabla (\nabla h^{-1})| + |\nabla (\nabla h^{-1})| \right\}. \tag{2.73}
\]
For (III): We have

\[ \nabla_m Q^\alpha_\beta = \frac{1}{2} \nabla_m T^\alpha_{pq} \bar{T}_\beta^{pq} - \frac{1}{2} T^\alpha_{pq} \nabla_m \bar{T}_\beta^{pq}. \]  

(2.74)

By definition

\[ \nabla_m T^\alpha_{pq} = \nabla_m (\Gamma^\alpha_{pq} - \Gamma^\alpha_{qp}). \]  

(2.75)

By the relation between the reference and evolving connections (2.57), we have

\[ \nabla_m T^\alpha_{pq} = \nabla_m \bar{T}^\alpha_{pq} + \nabla_m (\nabla_p hh^{-1})^\alpha_q - (\nabla_q hh^{-1})^\alpha_p. \]  

(2.76)

Terms \( \nabla_m \bar{T}_\beta^{pq} = g^{\alpha p} g^{\beta q} \nabla_m T^{\alpha \beta}_{ij} \) can be analyzed similarly. Altogether, we have

\[ (III) \leq C_3 (S + S^{1/2})|T| + C_3 S^{1/2}|T| \left\{ |\nabla(\nabla hh^{-1})| + |\nabla(\nabla h h^{-1})| \right\}. \]  

(2.77)

For (IV): Similarly as (I), we have

\[ (IV) \leq C_4 (S + S^{1/2}), \]  

(2.78)

where the term \( \nabla \Phi(z, g(t)) \) contributes an order of \( S^{1/2} \).

For (V): We directly estimate

\[ (V) \leq C_5 S|T|^2. \]  

(2.79)

For (VI): We directly estimate

\[ (VI) \leq C_6 S. \]  

(2.80)

Altogether, we obtain

\[ (\partial_t - \Delta)S \leq -|\nabla(\nabla h h^{-1})|^2 - |\nabla(\nabla h h^{-1})|^2 \]  

\[ + C S^{1/2} |T| \left\{ |\nabla(\nabla hh^{-1})| + |\nabla(\nabla h h^{-1})| \right\} 
\]  

\[ + CS|T|^2 + C(S + S^{1/2})(1 + |T|). \]  

(2.81)

From here, we can use \( 2ab \leq a^2 + b^2 \) to obtain (2.70) and hence prove Proposition 3.

### 3 Improved \( C^1 \) estimate

We now assume that \( X \) admits a nowhere vanishing holomorphic \((n, 0)\) form \( \Omega \). This extra structure will allow us to improve the estimate of the previous section.
**Theorem 4** Let \((X, \hat{g})\) be a compact Hermitian manifold with nowhere vanishing holomorphic \((n, 0)\) form \(\Omega\). Let \(g(t)\) evolve by Type IIB flow (2.9). Suppose on \([0, t_0]\), we have the estimate
\[
K_1^{-1}\hat{g} \leq g(t) \leq K_1\hat{g}, \quad |\nabla \log \|\Omega\|_g| \leq K_2
\]
for \(K_1, K_2 > 0\). Then
\[
|\hat{\nabla} g|_g^2 \leq C
\]
on \([0, t_0]\), where \(C\) depends on \(K_1, K_2, (X, \hat{g}, \Omega), \Phi, g(0)\).

This theorem is the \(C^1\) estimate stated in Theorem 1. Before giving the proof, we compute the evolution of the dilaton function \(\log \|\Omega\|_g\).

**3.1 Evolution of the dilaton**

Recall that locally, \(\Omega = \Omega(z)dz^1 \wedge \ldots \wedge dz^n\) for a local holomorphic function \(\Omega(z)\), and
\[
\|\Omega\|^2_g = \frac{\Omega(z)\overline{\Omega(z)}}{\det g}.
\]

We compute
\[
\partial_t \log \|\Omega\|_g = -\frac{1}{2} \partial_p \log \det g = -\frac{1}{2} g^{pq} \hat{g}_{qp}.
\]

By the equation of the flow (2.9), we have
\[
\partial_t \log \|\Omega\|_g = \frac{1}{2} R + \frac{1}{4} |T|^2 + \frac{1}{2} i \Lambda_\omega \Phi,
\]
where \(\Lambda_\omega\) is defined in (2.12). On the other hand,
\[
R = -g^{pq} \partial_p \partial_q \log \det g = \Delta \log \|\Omega\|^2_g.
\]

Therefore
\[
(\partial_t - \Delta) \log \|\Omega\|_g = \frac{1}{4} |T|^2 + \frac{1}{2} i \Lambda_\omega \Phi.
\]

**3.2 Maximum principle**

Consider the test function
\[
G = \log S + \epsilon \text{Tr} h - A \log \|\Omega\|_g,
\]
for \(\epsilon, A > 0\) to be determined. The evolution of \(G\) is given by
\[
(\partial_t - \Delta)G = \frac{(\partial_t - \Delta)S}{S} + \frac{|\nabla S|^2}{S^2} + \epsilon(\partial_t - \Delta)\text{Tr} h - A(\partial_t - \Delta) \log \|\Omega\|_g.
\]
By (2.27), (2.70), (3.7),

\[(\partial_t - \Delta) G \leq \left[ \frac{C}{S} + C + C|T|^2 \right] + \frac{|\nabla S|^2}{S^2} + \varepsilon \left[ - \frac{1}{K_1^2} S + C \right] - A \left[ \frac{1}{4} |T|^2 + \frac{1}{2} \Lambda \Phi \right]. \quad (3.10)\]

Suppose \(G\) attains a maximum at a point \((\hat{x}, \hat{t})\) with \(\hat{t} > 0\) and \(S(\hat{x}, \hat{t}) > 1\). Then

\[0 \leq \left( \frac{1}{K_1^2} - \frac{\varepsilon^2}{2} \right) S - \left( \frac{A}{4} - C \right) |T|^2 + C(\varepsilon, A) \quad (3.11)\]

at \((\hat{x}, \hat{t})\). We also have the critical equation \(\nabla G(\hat{x}, \hat{t}) = 0\) which implies the relation

\[\frac{\nabla S}{S} = -\varepsilon \nabla \text{Tr} h + A \nabla \log \|\Omega\|_g. \quad (3.12)\]

Therefore

\[\frac{|\nabla S|^2}{S^2} \leq 2\varepsilon^2 |\nabla \text{Tr} h|^2 + 2A^2 |\nabla \log \|\Omega\|_g|^2. \quad (3.13)\]

Since

\[|\nabla \text{Tr} h|^2 = |\text{Tr} \hat{\nabla} h|^2 = g^{ij} g^{pq} g^{rs} \hat{\nabla}_i g_{qp} \hat{\nabla}_j g_{rs} \leq nK_1^2 |\hat{\nabla} g|^2, \quad (3.14)\]

we conclude

\[\frac{|\nabla S|^2}{S^2} \leq 2nK_1^2 \varepsilon^2 S + 2A^2 K_2^2. \quad (3.15)\]

Substituting this inequality in (3.11), we obtain

\[0 \leq - \left( \frac{\varepsilon}{K_1^2} - 2nK_1^2 \varepsilon^2 \right) S - \left( \frac{A}{4} - C_0 \right) |T|^2 + C(\varepsilon, A) + 2A^2 K_2^2. \quad (3.16)\]

Let \(\varepsilon = 1/(4nK_1^4)\) and \(A = 4C_0\). Then

\[0 \leq - \frac{1}{8nK_1^q} S + C(\varepsilon, A) + 2A^2 K_2^2. \quad (3.17)\]

It follows that

\[S \leq C(K_1, K_2), \quad (3.18)\]

at \((\hat{x}, \hat{t})\), and hence \(S\) is bounded uniformly at all points \((x, t) \in X \times [0, t_0]\). This completes the proof of Theorem 4.

**4 Higher Order Estimates**

We now establish higher order estimates for the Type IIB flow, using the theory of quasi-linear parabolic systems.
4.1 Higher order estimates for quasi-linear parabolic system

We recall the following theorem about the higher order estimates for general linear and quasi-linear parabolic system from the book by Ladyzenskaja et al. ([15] Theorem 5.1 in Chapter 7, page 586). Consider the following quasi-linear parabolic systems of the form

\[ u_t = a^{ij}(x, t, u) u_{ij} + a(x, t, u, u_x), \quad (4.1) \]

in which \( u(x, t) = (u^1(x, t), u^2(x, t), \cdots, u^N(x, t)) \) is an unknown vector function defined in \( Q_T = \Omega \times (0, T) \) with \( \Omega \subset \mathbb{R}^n \) and \( a(x, t, u, p) \) is a given \( N \)-dimensional vector-valued function with components \( a^\ell(x, t, u(x, t), u_x(x, t)) \); \( a^{ij}(x, t, u) \) is an \( n \times n \) matrix function satisfying

\[ \lambda |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq \Lambda |\xi|^2 \quad (4.2) \]

for any real \( \xi = (\xi_1, \cdots, \xi_n) \).

The theorem states the following

**Theorem 5** Let \( u(x, t) \in C^{2,1}(\bar{Q}_T) \) be a solution of the quasi-linear system (4.1). Suppose that

1) The functions \( a^{ij}(x, t, u) \) and their derivatives with respect to the \( x_k \) and \( u^\ell \) are all continuous in the domain

\[ D = \{(x, t) \in \bar{Q}_T : |u| \leq M_0, |p| \leq M_1 \} \]

where \( M_0 = \max_{Q_T} |u(x, t)| \) and \( M_1 = \max_{Q_T} |u_x(x, t)| \);

2) The functions \( a^\ell(x, t, u, p) \) are continuous in \( D \).

Then for any \( Q' \subset Q_T \), there exist two positive constants \( C \) and \( \alpha \) such that

\[ \|u\|_{C^{1+\alpha, \alpha/2}(Q')} < C. \]

(4.4)

Here \( \alpha \) and \( C \) depend on \( M_0, M_1, \lambda, \) the distance from \( Q' \) to \( \partial \Omega \times [0, T], \|u(x, 0)\|_{C^{1+\alpha}(\Omega)}, \) and the moduli of the continuity in 1) and 2).

If, in addition, the functions \( a^\ell(x, t, u, p) \) and \( a^{ij}(x, t, u) \) together with its derivatives satisfy a Hölder condition in \( D \) in the arguments \( x, t, u, p \) with exponents \( \beta, \beta/2, \beta, \beta \) respectively, then

\[ \|u\|_{C^{2+\beta, 1+\beta/2}(Q')} < C \]

(4.5)

for some constant \( C \) depending on \( M_0, M_1, \lambda, \) \( \text{dist}(Q', \partial \Omega \times [0, T]) \), the Hölder continuity of \( a^{ij} \) and \( a^\ell \), and \( \|u(x, 0)\|_{C^{2+\beta}(\Omega)} \).

The requirement about the derivatives of \( a^{ij}(x, t, u) \) in condition 1) is used to re-write the system of equations (4.1) into a quasi-linear system of divergence form.
4.2 Higher order estimates for the Type II B flow

We will use the above estimates for general quasi-linear parabolic systems to derive the higher order estimates for our flow (2.9)

\[ \partial_t \bar{g}_{kj} = -\bar{R}_{kj} - \frac{1}{2} T_{kpq} \bar{T}_j^{pq} - \Phi(z, \bar{g}(t))_{kj}. \]  

(4.6)

From the definition, we know

\[ \bar{R}_{kj} = -g^{\bar{q}\bar{p}} \partial_{\bar{p}} \partial_q g_{kj} + g^{\bar{q}\bar{p}} g^{\bar{u}\bar{v}} \partial_{\bar{q}} g_{kn} \partial_{\bar{p}} g_{\bar{v}j} \]  

(4.7)

and

\[ T_{kjm} = \partial_j g_{km} - \partial_m g_{kj}. \]  

(4.8)

Therefore, we can treat flow (2.9) as a quasi-linear parabolic system of the form (4.1) by taking \( u = g \),

\[ a^{ij}(x, t, u) = g^{-1} \]  

(4.9)

and

\[ a(x, t, u, p) = g^{\bar{q}\bar{p}} g^{\bar{u}\bar{v}} \partial_{\bar{q}} g_{kn} \partial_{\bar{p}} g_{\bar{v}j} - \frac{1}{2} T_{kpq} \bar{T}_j^{pq} - \Phi(z, \bar{g}(t))_{kj} \]  

(4.10)

Using the \( C^1 \) estimate on \( g \), we can check that \( a^{ij} \) and \( a \) satisfy the conditions 1) and 2) in the general theorem. Then, the higher order estimates for \( g \) follows directly.

**Theorem 6** Let \((X, \hat{g})\) be a compact Hermitian manifold with nowhere vanishing holomorphic \((n, 0)\) form \( \Omega \). Let \( g(t) \) evolve by Type IIB flow (2.9). Suppose on \([0, t_0]\), we have the estimate

\[ K^{-1} \hat{g} \leq g(t) \leq K \hat{g}, \quad |\nabla \log \| \Omega \|_g| \leq K \]  

(4.11)

for \( K > 0 \). Then

\[ \| g \|_{C^{2+\alpha, 1+\alpha/2}(X \times [0, t_0])} \leq C, \]  

(4.12)

where \( C \) depends on \( K, (X, \hat{g}, \Omega), \Phi, g(0) \).

Higher order estimates on \( g \) follow from the parabolic Schauder estimates. Once \( g(t) \) is uniformly bounded in all \( C^k \) norms on \( X \times [0, t_0] \), a standard compactness argument using the Arzela-Ascoli theorem and the short-time existence theorem gives the extension of the flow to \([0, t_0 + \epsilon] \) for some \( \epsilon > 0 \).
5 Proof of Theorem 2

In comparison with the estimates in Theorem 1, the assumption on the lower bound of the evolving metric $\omega(t)$ is removed when $\Psi = 0$. Indeed, it was noted in [7] that the following bound holds along the Anomaly flow:

$$\|\Omega\|_{\omega(t)} \leq \sup_X \|\Omega\|_{\omega_0}. \quad (5.1)$$

This can also be seen from (3.7), which shows that

$$(\partial_t - \Delta) \log \|\Omega\|_{\eta(t)} \geq 0,$$

where $\eta = \|\Omega\|_\omega$. The bound (5.1) implies

$$\frac{\det g_0}{\det g} \|\Omega\|^2_{g_0} \leq C. \quad (5.3)$$

If we let $h^i_j = (g_0)^{ik}g_{kj}(t)$, this implies

$$\det h \geq C_0^{-1}. \quad (5.4)$$

If $0 < \lambda_n \leq \ldots \leq \lambda_1$ are the eigenvalues of $h$, by assumption we have that

$$\lambda_1 \leq K_1. \quad (5.5)$$

It follows that

$$\frac{1}{\lambda_n} = \frac{\lambda_{n-1} \cdots \lambda_1}{\det h} \leq C_0 K_1^{n-1}.$$

Therefore

$$\omega(t) \geq C^{-1} \omega_0, \quad (5.7)$$

and we may apply Theorem 1. Q.E.D.

References

[1] Bedulli, L. and L. Vezzoni, A parabolic flow of balanced metrics, J. Reine Angew. Math. 723 (2017) 79-99.

[2] Bedulli, L. and L. Vezzoni, Stability of geometric flows of closed forms, arXiv:1811.09416, to appear in Adv. Math.

[3] Bryant, R. and F. Xu, Laplacian flow for closed $G_2$ structures: Short-time Behavior, arXiv:1101.2004.

[4] Candelas, P., G. Horowitz, A. Strominger, and E. Witten, Vacuum configurations for superstrings, Nuclear Phys. B 258 (1985), no. 1, 46-74.
[5] Dinew, S. and S. Kolodziej, *Liouville and Calabi-Yau type theorems for complex Hessian equations*, Amer. J. Math. 139 (2017), no. 2, 403-415.

[6] Fei, T., Z. Huang, and S. Picard, *A construction of infinitely many solutions to the Strominger system*, arXiv:1703.10067, Int. Math. Res. Not. rnz076.

[7] Fei, T. and S. Picard, *Anomaly Flow and T-Duality*, arXiv:1903.08768, to appear in Pure Appl. Math. Q..

[8] Fei, T. and D.H. Phong, *Unification of the Kähler-Ricci and Anomaly flows*, arXiv:1905.02274, to appear in Surv. Differ. Geom.

[9] Fei, T. and S.T. Yau, *Invariant solutions to the Strominger system on complex Lie groups and their quotients*, Comm. Math. Phys., 338 (2015), no. 3, 1183-1195.

[10] Fu, J.X. and S.T. Yau, *The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation*, J. Differential Geom., 78 (2008), no. 3, 369-428.

[11] Fu, J.X. and S.T. Yau, *A Monge-Ampère type equation motivated by string theory*, Comm. Anal. Geom. 15 (2007), no. 1, 29-76.

[12] Garcia-Fernandez, M., *Lectures on the Strominger system*, Travaux mathématiques Vol. XXIV, 7-61, Luxembourg, 2016.

[13] Graña, M., Minasian, R., Petrini, M. and Tomasiello A., *Generalized structures of N = 1 vacua*, arXiv:hep-th/0505212, JHEP 11 (2005) 020.

[14] Hull, C., *Compactifications of the Heterotic Superstring*, Phys. Lett. B 178 (1986), no. 4, 357-364.

[15] Ladyzenskaja, O.A., Solonnikov, V. A. and Uraltseva, N.N., *Linear and Quasi-linear Equations of Parabolic Type*, Nauka, Moscow, 1967 [Russian]; English transl., Translations of Mathematical Monographs Vol. 23, AMS, Providence, RI, 1968.

[16] Lotay, J., *Geometric flows of G₂ Structures*, arXiv: 1810.13417.

[17] Lotay, J. and Y. Wei, *Laplacian flow for closed G₂ structures: real analyticity*, Comm. Anal. Geom. 27 (2019) 73-109.

[18] Phong, D.H., S. Picard, and X.W. Zhang, *Geometric flows and Strominger systems*, Math. Z. 288 (2018), no. 1-2, 101-113.

[19] Phong, D.H., S. Picard, and X.W. Zhang, *The Anomaly flow and the Fu-Yau equation*, Ann. PDE 4 (2018), no. 2, Paper No. 13, 60 pp.

[20] Phong, D.H., S. Picard, and X.W. Zhang, *A flow of conformally balanced metrics with Kähler fixed points*, Math. Ann. 374 (2019), no. 3-4, 2005-2040.
[21] Phong, D.H., S. Picard, and X.W. Zhang, *The Anomaly flow on unimodular Lie groups*, Advances in Complex Geometry, 217-237, Contemporary Mathematics Vol. 735, AMS, Providence, RI, 2019.

[22] Phong, D.H., S. Picard, and X.W. Zhang, *The Fu-Yau equation with negative slope parameter*, Invent. Math. 209 (2017), no. 2, 541-576.

[23] Phong, D.H., S. Picard, and X.W. Zhang, *Anomaly flows*, Comm. Anal. Geom. 26 (2018), no. 4, 955-1008.

[24] Phong, D.H., N. Sesum, and J. Sturm, *Multiplier ideal sheaves and the Kähler-Ricci flow*, Comm. Anal. Geom. 15 (2007), no. 3, 613-632.

[25] Phong, D.H., J. Song, J. Sturm, and B. Weinkove, *On the convergence of the modified Kähler-Ricci flow and solitons*, Comment. Math. Helv. 86 (2011), no. 1, 91-112.

[26] Pujia, M. and L. Ugarte, *The Anomaly flow on nilmanifolds*, arXiv: 2004.06744.

[27] Streets, J. and G. Tian, *Hermitian curvature flow*, J. Eur. Math. Soc. 13 (2011), no. 3, 601-634.

[28] Strominger, A., *Superstrings with torsion*, Nuclear Phys. B 274 (1986), no. 2, 253-284.

[29] Tomasiello, A., *Generalized structures of ten-dimensional supersymmetric solutions*, arXiv:1109.2603, JHEP 03 (2012) 073.

[30] Tseng, L.S. and S.T. Yau, *Cohomology and Hodge theory on symplectic manifolds: I*, J. Differential Geom. 91 (2012), no. 3, 383-416

[31] Tseng, L.S. and S.T. Yau, *Cohomology and Hodge theory on symplectic manifolds: II*, J. Differential Geom. 91 (2012), no. 3, 417-443.

[32] Tseng, L.S. and S.T. Yau, *Generalized cohomologies and supersymmetry*, Comm. Math. Phys. 326 (2014), no. 3, 875-885.

[33] Ustinovskiy, Y., *Hermitian curvature flow and curvature positivity conditions*, Princeton University, PhD Thesis, June 2018.

[34] Yau, S.T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339-411.

[35] Zhang, X. and Zhang, X.W., *Regularity estimates for solutions to complex Monge-Ampère equations on Hermitian manifolds*, J. Funct. Anal. 260 (2011), no. 7, 2004-2026.
Department of Mathematics & Computer Science, Rutgers, Newark, NJ 07102, USA
teng.fe@rutgers.edu

Department of Mathematics, Columbia University, New York, NY 10027, USA
phong@math.columbia.edu

Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
spicard@math.harvard.edu

Department of Mathematics, University of California, Irvine, CA 92697, USA
xiangwen@math.uci.edu