FORMALLY INTEGRABLE STRUCTURES
I. RESOLUTION OF SOLUTION SHEAVES

QINGCHUN JI JUN YAO GUANGSHENG YU

Abstract. Since the first exposition of the complex Frobenius structures (\cite{10}), there have been considerable development and many new applications on this subject (see \cite{4}, \cite{5}, \cite{9}, \cite{11}, \cite{16}, \cite{17} and references therein). Inspired by complex Frobenius structures, L. Hörmander introduced a class of first order overdetermined systems of partial differential equations (\cite{5}) and established existence theorems. This paper is devoted to the construction of resolution for the solution sheaves of these overdetermined systems considered in \cite{5}. A sufficient condition for global exactness is obtained, which leads to gluing technique for local solutions of the overdetermined systems by solving Cousin type problems. In addition, we also prove local solvability of the Treves complex without assuming the non-degeneracy of the Levi form.

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1. Introduction

Given a manifold $X$, various extensions to the classical Frobenius theorem and Newlander-Nirenberg theorem concern with the local integrability of involutive structures over $X$, i.e. formally integrable complex subbundles of the complexified tangent bundle $\mathbb{C}TX$. The first important attempt in this direction was made by Nirenberg (\cite{10}), a complex Frobenius theorem was established for arbitrary complex subbundle $E \subseteq \mathbb{C}TX$ under the assumptions that $E + \bar{E}$ is also a subbundle of $\mathbb{C}TX$ and that $E$ and $E + \bar{E}$ are both formally integrable, i.e.

$$\Gamma(X, E), \Gamma(X, E) \subseteq \Gamma(X, E), \Gamma(X, \bar{E}) \subseteq \Gamma(X, E + \bar{E}),$$

(1)

where $\bar{E}$ is the complex-conjugate of $E$ and $\lfloor \cdot, \cdot \rfloor$ denotes the Lie bracket of vector fields. Such a subbundle $E$ is called a complex Frobenius structure.
For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we denote $\partial/\partial x_\nu$ by $\partial_\nu$ for $1 \leq \nu \leq n$.

To extend Nirenberg’s complex Frobenius theorem, Hörmander ([5]) considered, under involutivity conditions inspired by (1), the following general overdetermined system for the unknown $u$

$$P_j u = f_j, 1 \leq j \leq r,$$

where $P_j$’s are first order differential operators in an open subset $\Omega \subseteq \mathbb{R}^n$ given by

$$P_j = \sum_{\nu=1}^n a^{\nu j}_\nu \partial_\nu + a_0^j,$$

where $a^{\nu j}_j \in C^\infty(\Omega)$ for $0 \leq \nu \leq n, 1 \leq j \leq r$.

We consider the sheaf $\mathcal{S}_P$ of germs locally square-integrable solutions of the homogeneous system $Pu = 0$ where $P := (P_1, \cdots, P_r)$ is the $r$-tuple with components given by these differential operators in (3). When the operator $P$ is hypoelliptic, $\mathcal{S}_P$ consists of germs of smooth solutions. Let $\mathcal{L}$ be the sheaf over $\Omega$ consisting of germs of locally square-integrable functions, it’s by definition a sheaf of Abelian groups. In section 2, we will construct a differential complex over $\Omega$

$$\mathcal{L} \xrightarrow{P_1} \mathcal{L}^{\oplus r} \xrightarrow{P_2} \mathcal{L}^{\oplus \binom{r}{2}} \xrightarrow{P_3} \mathcal{L}^{\oplus \binom{r}{3}} \xrightarrow{P_4} \cdots \xrightarrow{P_r} \mathcal{L}^{\oplus \binom{r}{r}} \rightarrow 0,$$

such that

$$P_1 = P,$$

where $\binom{r}{q} = r!/q!(r-q)!$ is the binomial coefficient. In this way, we eventually get a resolution of $\mathcal{S}_P$. Moreover, as $\mathcal{L}$ is a fine sheaf, the above resolution also implies isomorphisms for $1 \leq q \leq r$

$$H^q(\Omega, \mathcal{S}_P) \cong \frac{\text{Ker} \left( L^2_{\text{loc}}(\Omega)^{\oplus \binom{r}{q}} \xrightarrow{P_{q+1}} L^2_{\text{loc}}(\Omega)^{\oplus \binom{r}{q+1}} \right)}{\text{Im} \left( L^2_{\text{loc}}(\Omega)^{\oplus \binom{r}{q-1}} \xrightarrow{P_q} L^2_{\text{loc}}(\Omega)^{\oplus \binom{r}{q}} \right)},$$

where $L^2_{\text{loc}}(\Omega)$ is the space of complex valued functions on $\Omega$ which are locally square-integrable. We will consider global exactness of the previous resolution, i.e. the vanishing of $H^q(\Omega, \mathcal{S}_P)$ in section 3, and the local $L^2$-solvability for the Treves complex follows as a consequence(for formally integrable sub-bundles, not necessarily locally integrable). There have been many deep results on the local solvability for the Treves complex(see [1], [2], [13–15].
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and references therein) under the non-degeneracy assumption of the Levi form which is necessary to make use of the parametrix technique. In section 4, we will formulate additive and multiplicative Cousin problems for $P = (P_1, \cdots, P_r)$, and will give a sheaf-theoretic approach to gluing local solutions of the overdetermined system (2).

We adopt the summation convention over repeated indices. For a multi-index $J = (j_1, \cdots, j_q)$ where $1 \leq j_1, j_2, \cdots, j_q \leq r$, we call $q$ the length of $J$ (denoted by $|J| = q$). A multi-index $J$ is called strictly increasing if $1 \leq j_1 < j_2 < \cdots < j_q \leq r$. Without special explanation, summation is always taken over strictly increasing multi-indices. For a strictly increasing multi-index $J$, $s \in \{1, \cdots, r\}$, $j \in J$ and $k \in \{1, \cdots, r\} \setminus J$, we introduce the following notations:

\[ J \setminus j := \text{the strictly increasing multi-index with components in } J \setminus \{j\}, \]
\[ Jk := \text{the strictly increasing multi-index with components in } J \cup \{k\}, \]
\[ (k, J) := \text{the number of components in } J \text{ which are less than } k, \]
\[ sJ := (s, j_1, \cdots, j_q), \]
\[ (j, sJ \setminus j) := \begin{cases} (j, J \setminus j) + 1, & \text{if } j \neq s, \\ 0, & \text{if } j = s. \end{cases} \]

For $1 \leq q \leq r$, we write $\xi \in \mathbb{C}^{(q)}$ as $\xi = (\xi_J)_{|J|=q}$ where $J$ runs over multi-indices of length $q$ and components $\xi_J \in \mathbb{C}$ are anti-symmetric in $j_1, j_2, \cdots, j_q$. It is clear that $\xi \in \mathbb{C}^{(q)}$ is uniquely determined by these $\xi_J$ with strictly increasing $J$. We will denote the $L^2$-inner product and the usual inner product of the complex Euclidean spaces by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle$ respectively.

2. CONSTRUCTION OF THE DIFFERENTIAL COMPLEX

When an $r$-tuple $P = (P_1, \cdots, P_r)$ of first order partial differential operators is normalized and in the normal Cartan form, there is an explicit procedure for constructing resolutions of the solution sheaf $\mathcal{S}_P$ by using compatibility operators(12). It is known that the non-existence of a compatibility operator is a consequence of the strong degeneracy of $P$.

To construct the desired complex (4), we make the following assumption on the differential operators under consideration $P_1, \cdots, P_r$.

(A1) For all $i, j, k \in \{1, \cdots, r\}$,
\[ [P_j, P_k] := P_j P_k - P_k P_j = c^i_{jk} P_i \quad \text{and} \quad c^i_{jk} = -c^i_{kj}, \quad (7) \]
where \( c_{jk}^1, \ldots, c_{jk}^r \in C^\infty(\Omega) \) are complex valued functions. This is analogous to the first condition in (1). The next assumption will play an important role in our construction compatibility operators.

(A2) For \( 1 \leq k, l, k', l' \leq r \),

\[
  c_{kk'}^l c_{sk'}^l + c_{kk'}^l c_{sk'}^l + c_{skk'}^l c_{sk'}^l = p_k c_{kk'}^l + p_k c_{sk'}^l + p_{kk'} c_{kk'}^l,
\]

where \( p_j \) is the principal part of \( P_j \), i.e.

\[
  p_j = a_j^\nu(x) \partial^\nu.
\]

Remark. We have the following sufficient condition for (8)

\[
  \text{rank}_C (a_j^\nu(x))_{1 \leq j \leq n} = r
\]

holds for all \( x \in \Omega \). By the obvious identity

\[
  [P_k, [P_{k'}, P_l]] + [P_{k'}, [P_l, P_k]] + [P_l, [P_k, P_{k'}]] \equiv 0,
\]

we obtain

\[
  \left( c_{kk'}^l c_{sk'}^l - p_k c_{kk'}^l + c_{kk'}^l c_{sk'}^l - p_k c_{sk'}^l - c_{skk'}^l c_{sk'}^l + p_{kk'} c_{kk'}^l \right) P_l \equiv 0.
\]

Therefore, the condition (8) follows from (10).

Now, we use \( P_1, \ldots, P_r \) and assumptions (A1)-(A2) to construct a sequence of differential operators

\[
  L^2_{\text{loc}}(U) \xrightarrow{P_1} L^2_{\text{loc}}(U)^{\oplus r} \xrightarrow{P_2} L^2_{\text{loc}}(U)^{\oplus (r^2/2)} \xrightarrow{P_3} \cdots \xrightarrow{P_r} L^2_{\text{loc}}(U)^{\oplus (r^r)}
\]

for every open subset \( U \subseteq \Omega \) where each \( P_q : L^2_{\text{loc}}(U)^{\oplus (q^q)} \rightarrow L^2_{\text{loc}}(U)^{\oplus (r^r)} \) is a densely defined operator to be determined \((1 \leq q \leq r)\).

For \( q = 1 \), as indicated by (5), we define \( P_1 u = (P_1 u, \ldots, P_r u) \) for \( u \in L^2_{\text{loc}}(U) \). To find \( P_2 \) such that \( P_2 \circ P_1 = 0 \), let’s first consider \( f = (f_j)_{1 \leq j \leq r} \) given by \( f_j = P_j u \) for some \( u \in L^2_{\text{loc}}(U), 1 \leq j \leq r \). By (7), we have

\[
  P_k f_j = [P_k, P_j] u + P_j P_k u = c_{kj}^l f_j + P_j f_i
\]

which indicates the following compatibility operator \( P_2 \) for \( P_1 \)

\[
  P_2 f = (P_1 f_j - P_j f_i - c_{ij}^l f_j)_{1 \leq i, j \leq r},
\]

where \( c_{mn}^i \)'s are functions appearing in (7), then it is easy to see \( P_2 \circ P_1 = 0 \).
In general, for $1 \leq q \leq r$, we define $\mathcal{P}_q$ as follows

$$
\mathcal{P}_q f = \left( \sum_{j \in J} (-1)^{|j|} \mathcal{P}_j f |_{|j|=q} - \sum_{m<n \in J} \text{sgn}(mn,J\{m,n\}) c_{mn}^q f |_{|J|=q} \right),
$$

(12)

where $f = (f_I)|_{|I|=q-1} \in L^2_{\text{loc}}(U)^{\oplus (r-1)}$. In (12), $\mathcal{P}_q$ is understood as the maximal extension of its restriction to smooth vector-valued functions, and thereby a densely defined operator between Fréchet spaces $L^2_{\text{loc}}(U)^{\oplus (r-1)}$ and $L^2_{\text{loc}}(U)^{\oplus (q-1)}$.

The sequence of differential operators in (11) give the following sequence of sheaf-morphisms over $\Omega$

$$
\mathcal{L} \xrightarrow{\mathcal{P}_1} \mathcal{L}^{\oplus r} \xrightarrow{\mathcal{P}_2} \mathcal{L}^{\oplus (r-1)} \xrightarrow{\mathcal{P}_3} \mathcal{L}^{\oplus (r-2)} \xrightarrow{\mathcal{P}_4} \cdots \xrightarrow{\mathcal{P}_r} \mathcal{L}^{\oplus (r-r)},
$$

which restricts to

$$
\mathcal{A} \xrightarrow{\mathcal{P}_1} \mathcal{A}^{\oplus r} \xrightarrow{\mathcal{P}_2} \mathcal{A}^{\oplus (r-1)} \xrightarrow{\mathcal{P}_3} \mathcal{A}^{\oplus (r-2)} \xrightarrow{\mathcal{P}_4} \cdots \xrightarrow{\mathcal{P}_r} \mathcal{A}^{\oplus (r-r)}, \quad (13)
$$

where $\mathcal{A}$ is the sheaf of germs of smooth functions.

**Remark.** If $r = n$, and $P_j = \partial_j$ for $j \in \{1, \cdots, n\}$, then (13) is the deRham complex. If $n$ is even, $r = n/2$, and $P_j = \partial_{\bar{z}}$ for $j \in \{1, \cdots, r\}$, then the complex (13) is the Dolbeault complex.

First, we need to show that the sequence (4) is actually a complex, i.e. $\mathcal{P}_{q+1} \circ \mathcal{P}_q = 0$.

**Proposition 2.1.** $\mathcal{P}_{q+1} \circ \mathcal{P}_q = 0$ for $1 \leq q \leq r$ where $\mathcal{P}_{r+1} := 0$. 


Proof. Let \( U \subseteq \Omega \) be an open subset and \( f = (f_l)|_\Omega = q-1 \in L^2_{\text{loc}}(U) \oplus (q-1) \), it follows from the definition (12) that

\[
P_{q+1} \circ P_q f = \left( \sum_{k \in K} (-1)^{(k, K \setminus k)} P_k \left( \sum_{m < n \in K} \text{sgn}(m, n) c_{mn}^s \left( P_q f \right)_{s , K \setminus \{ m, n \} } \right) \right)_{|K| = q+1}.
\]

\[
= \left( \sum_{k \in K} \sum_{j \in K \setminus k} (-1)^{(k, K \setminus k) + (j, K \setminus \{ k, j \})} P_k P_j \left( f_{K \setminus \{ k, j \} } \right) \right)_{|K| = q+1}
\]

\[
- \sum_{m < n \in K} \text{sgn}(m, n) \sum_{j \in s K \setminus \{ m, n \} } (-1)^{(j, s K \setminus \{ m, n \} )} P_j f_{s K \setminus \{ m, n \} }.
\]

\[
- \sum_{k \in K} \sum_{m' < n' \in K \setminus k} \text{sgn}(m', n') \sum_{j \in s K \setminus \{ k, m', n' \} } (-1)^{(j, s K \setminus \{ k, m', n' \} )} P_j f_{s K \setminus \{ k, m', n' \} }.
\]

\[
+ \sum_{m < n \in K} \sum_{m'' < n'' \in s K \setminus \{ m, n \} } \text{sgn}(m'', n'') \left( \sum_{k \in K \setminus \{ k, m, n \} } \left. \left[ P_m, P_n \right] f_{K \setminus \{ m, n \} } \right) \right)_{|K| = q+1}.
\]

We will handle these four terms \( I \sim IV \) separately as follows.

\[
I = \sum_{k \in K} \sum_{j \in K \setminus k} \left( (-1)^{(k, K \setminus k) + (j, K \setminus \{ k, j \})} P_k P_j \left( f_{K \setminus \{ k, j \} } \right) \right)_{|K| = q+1}
\]

\[
= \sum_{k < j \in K} \left( (-1)^{(k, K \setminus k) + (j, K \setminus \{ k, j \})} (P_k P_j - P_j P_k) f_{K \setminus \{ k, j \} } \right)_{|K| = q+1}
\]

\[
= \sum_{m < n \in K} \left( \text{sgn}(m, n) \left[ P_m, P_n \right] f_{K \setminus \{ m, n \} } \right)_{|K| = q+1}
\]

\[
= \sum_{m < n \in K} \left( \text{sgn}(m, n) c_{mn}^s f_{K \setminus \{ m, n \} } \right)_{|K| = q+1}.
\]

The second line follows from \((k, K \setminus k) + (j, K \setminus \{ j, k \}) + 1 = (j, K \setminus j) + (k, K \setminus \{ j, k \})\), if \( k < j \in K \). The last line is an application of (7). We rewrite \( II \) as

\[
II = \left( \sum_{m < n \in K} \text{sgn}(m, n) c_{mn}^s f_{K \setminus \{ m, n \} } \right)_{|K| = q+1}
\]

\[
+ \left( \sum_{m < n \in K} \text{sgn}(m, n) c_{mn}^l \sum_{j \in K \setminus \{ m, n \} } (-1)^{(j, K \setminus \{ m, n, j \})} P_j f_{K \setminus \{ m, n, j \} } \right)_{|K| = q+1}
\]

\[
:= II' + II''.
\]
which gives
\[ I = I'. \] (15)

Similarly, we decompose \( III \) into two parts
\[ III = \left( \sum_{k \in K} (-1)^{(k,K \setminus k)} \sum_{m' < n' \in K \setminus k} \text{sgn} \left( K \setminus k \right) c_{m'n'} f_{k} f_{k} \right) \left( \sum_{m' < n' \in K \setminus k} c_{m'n'} P_{k} f_{l} f_{k} \right) |K| = q + 1 \]
\[ + \left( \sum_{k \in K} (-1)^{(k,K \setminus k)} \sum_{m' < n' \in K \setminus k} \text{sgn} \left( K \setminus k \right) c_{m'n'} P_{k} f_{l} f_{k} \right) \left( \sum_{m' < n' \in K \setminus k} c_{m'n'} f_{l} f_{k} \right) |K| = q + 1 \]
\[ := III' + III'' \]

where we denote by \( p_j \) the principal part of \( P_j \). Since
\[ \sum_{k \in K} \sum_{m' < n' \in K \setminus k} c_{m'n'} P_{k} f_{l} f_{k} \left( \sum_{m' < n' \in K \setminus k} c_{m'n'} f_{l} f_{k} \right) \]
\[ = \sum_{m < n} \sum_{j \in K \setminus \{m,n\}} c_{mn} p_{j} f_{l} f_{k} \]
\[ II'' \]
has the same number of terms as \( III' \). From
\[ \binom{K}{mnK \setminus \{k,m,n\}} = \binom{K}{kk \setminus k} \binom{K}{kK \setminus \{k,m,n\}} \binom{mnK \setminus \{k,m,n\}}{mnK \setminus \{m,n\}} \]
\[ = \binom{K}{kk \setminus k} \binom{kK \setminus \{k,m,n\}}{K \setminus \{k,m,n\}} \binom{mnK \setminus \{k,m,n\}}{mnK \setminus \{m,n\}} \]
\[ = \binom{K}{kk \setminus k} \binom{kK \setminus \{k,m,n\}}{K \setminus \{k,m,n\}} \binom{K \setminus \{k,m,n\}}{K \setminus \{m,n\}} \]
and
\[ (-1)^{(k,lK \setminus \{k,m,n\})} = \text{sgn} \left( lK \setminus \{m,n\} \right) = -\text{sgn} \left( K \setminus \{m,n\} \right), \]

it follows that
\[ (-1)^{(k,K \setminus k)} \text{sgn} \left( K \setminus k \right) = -(-1)^{(k,lK \setminus \{m,n,k\})} \text{sgn} \left( mnK \setminus \{m,n\} \right), \]
which indicate that corresponding terms in \( II'' \) and \( III' \) have opposite signs, and therefore
\[ II'' + III' = 0. \] (16)
For the last term $IV$, we have

$$IV = \left( \sum_{m<n \in K} \sum_{m''<n'' \in K \setminus \{m,n\}} \text{sgn} \left( s_{K \setminus \{m,n\}} \right) \right) \times \left( \sum_{m<n \in K} \sum_{m''<n'' \in K \setminus \{m,n\}} \text{sgn} \left( s_{K \setminus \{m,n\}} \right) \right) + \left( \sum_{m<n \in K} \sum_{m''<n'' \in K \setminus \{m,n\}} \text{sgn} \left( s_{K \setminus \{m,n\}} \right) \right)

:= IV' + IV''.

It’s obvious that

$$IV' = 0,$$

indeed,

$$IV' = \left( \sum_{m<n \in K} \sum_{m''<n'' \in K \setminus \{m,n\}} \text{sgn} \left( s_{K \setminus \{m,n\}} \right) \right) \times \left( \sum_{m<n \in K} \sum_{m''<n'' \in K \setminus \{m,n\}} \text{sgn} \left( s_{K \setminus \{m,n\}} \right) \right) = 0.$$

Now, it remains to handle terms $III''$ and $IV''$. For a given multi-index $K$, each $f_{K \setminus \{k,m',n'\}}$ in $III''$ is exactly given by one of the following expressions (without loss of generality, assume $k < m' < n'$)

$$\begin{cases}
(-1)^{(n', K \setminus n')} \text{sgn} \left( k_{m'K \setminus \{k,m',n'\}} \right) p_{n'}(c_{m'K \setminus \{k,m',n'\}}) f_{K \setminus \{k,m',n'\}}, \\
(-1)^{(k,K \setminus k)} \text{sgn} \left( k_{m'K \setminus \{k,m',n'\}} \right) q_{k}(c_{m'K \setminus \{k,m',n'\}}) f_{K \setminus \{k,m',n'\}}, \\
(-1)^{(m',K \setminus m')} \text{sgn} \left( k_{m'K \setminus \{k,m',n'\}} \right) p_{m'}(c_{m'K \setminus \{k,m',n'\}}) f_{K \setminus \{k,m',n'\}}.
\end{cases}$$

(18)

The corresponding terms in $IV''$ are given by

$$\begin{cases}
\text{sgn} \left( k_{m'K \setminus \{k,m'\}} \right) \text{sgn} \left( s_{m'K \setminus \{k,m',n'\}} \right) c_{m'K \setminus \{k,m',n'\}} f_{K \setminus \{k,m',n'\}}, \\
\text{sgn} \left( k_{m'K \setminus \{k,m'\}} \right) \text{sgn} \left( s_{k_{m'K \setminus \{k,m',n'\}} \setminus \{k,m',n'\}} \right) c_{m'K \setminus \{k,m',n'\}} f_{K \setminus \{k,m',n'\}}, \\
\text{sgn} \left( k_{m'K \setminus \{k,m'\}} \right) \text{sgn} \left( s_{m'K \setminus \{k,m',n'\}} \right) c_{m'K \setminus \{k,m',n'\}} f_{K \setminus \{k,m',n'\}}.
\end{cases}$$

(19)
Similarly, we have the following equalities for the other two parts

\[
\begin{align*}
\text{sgn}(K_{km'}K\{k,m'\}) \text{sgn}(sK\{k,m'\}) &= \text{sgn}(K_{n'}{K'\{k,m',n'\}})
\end{align*}
\]

we have

\[
\text{sgn}(K_{km'}K\{k,m'\}) \text{sgn}(sK\{k,m'\}) = \text{sgn}(K_{n'}{K'\{k,m',n'\}})
\]

Similarly, we have the following equalities for the other two parts

\[
\begin{align*}
\text{sgn}(K_{km'}K\{k,m'\}) \text{sgn}(sK\{k,m'\}) &= \text{sgn}(K_{n'}{K'\{k,m',n'\}})
\end{align*}
\]

In the same manner, it is easy to see

\[
\text{sgn}(K_{km'}K\{k,m'\}) \text{sgn}(sK\{k,m'\}) = \text{sgn}(K_{n'}{K'\{k,m',n'\}})
\]

From (18)\~(23) and the assumption (A2), it follows that

\[
III'' = IV''.
\]

Now, \(P_{q+1} \circ P_q f = 0\) is a consequence of (14), (15), (16), (17) and (24). \(\square\)

**Remark.** It seems that the cases of \(q \geq 2\) are different from the case \(q = 1\), our proof depends on the assumption (A2) for \(q < 2\).

We denote by \(tP_q\) the formal adjoint of the differential operator \(P_q\). For later use, we need to compute the expression of \(tP_q f\) in terms of derivatives of its components. For \(\xi, \eta \in \mathbb{C}^{(q)}\), let \(\langle \xi, \eta \rangle := \sum_{|j|=q} \langle \xi_j, \eta_j \rangle\). Given \(f, g \in L^2_{loc}(\Omega) \oplus \mathbb{C}^{(q)}\) one of which has compact support, we denote \((f, g) := \int_{\Omega} (f, g) = \sum_{|j|=q} (f_j, g_j)\). The formal adjoint \(tP_q\) of \(P_q\) is then determined by \((P_q g, f) = (g, tP_q f)\) for all \((f, g) \in L^2_{loc}(\Omega) \oplus \mathbb{C}^{(q)}\). The space of smooth functions with compact supports in \(\Omega\). We have the following formula of \(tP_q\).
Proposition 2.2. For $1 \leq q \leq r$ and $f = (f_I)_{|I| = q} \in L^2_{loc}(\Omega)^{\oplus{\binom{r}{q}}}$,

$$tP_q f = \left( \sum_{j \notin I} tP_j f_I - \sum_{s \in I} \sum_{m < n \notin I \setminus s} (-1)^{|s \setminus I|} \text{sgn}(\binom{I \setminus s}{m, n}) c_{mn}^s f_{|I| \setminus m, n} \right)_{|I| = q-1},$$

where $tP_j$ is the formal adjoint of $P_j (1 \leq j \leq r)$.

Proof. Let $g = (g_I)_{|I| = q-1} \in \mathcal{D}(\Omega)^{\oplus{\binom{r}{q-1}}}$, then we have

$$(P_q g, f) = \sum_{|I| = q} \left( \sum_{j \in J} (-1)^{|j \setminus J|} P_j g_{|J \setminus j|} - \sum_{m < n \in J} \text{sgn}(\binom{J}{m, n}) \sum_{s \in J \setminus (m, n)} c_{mn}^s g_s f_{(\bar{s})} \right)$$

$$= \sum_{|I| = q} \sum_{j \in J} (-1)^{|j \setminus J|} (P_j g_{|J \setminus j|}, f_{|J|}) - \sum_{m < n \in J} \text{sgn}(\binom{J}{m, n}) \left( c_{mn}^s g_s, f_{(\bar{s})} \right)$$

$$= \sum_{|I| = q-1} \sum_{j \notin I} (-1)^{|j |} (g_I, tP_j f_I) - \sum_{s \in I} \sum_{m < n \notin I \setminus s} \text{sgn}(\binom{I \setminus s}{m, n}) (-1)^{|s \setminus I|} c_{mn}^s g_s f_{(\bar{s})}$$

$$= \sum_{|I| = q-1} \sum_{j \notin I} g_I \sum_{s \in I} tP_j f_I - \sum_{s \in I} \sum_{m < n \notin I \setminus s} \text{sgn}(\binom{I \setminus s}{m, n}) (-1)^{|s \setminus I|} c_{mn}^s f_{(\bar{s})}$$

$$= (g, tP_q f).$$

The third line follows from setting $I = J \setminus j$ and $I' = J \setminus \{m, n\}$. The fourth line is valid by letting $I = I's$. The proof is complete. \qed

Remark. When $q = 1, 2$, we have $tP_1 f = \sum_{j=1}^r tP_j f_j$ holds for $f = (f_j)_{1 \leq j \leq r} \in L^2_{loc}(\Omega)^{\oplus{r}}$, and $tP_2 f = \left( \sum_{j \neq i} tP_j f_{ji} + \sum_{m < n} c_{mn}^2 f_{mn} \right)_{1 \leq i \leq r}$ holds for $f = (f_{ij})_{1 \leq i < j \leq r} \in L^2_{loc}(\Omega)^{\oplus{\binom{r}{2}}}$.

We conclude this section by relating (13) to the Treves complex. Let $X$ be a manifold, and $E \subseteq CTX$ be a formally integrable subbundle. For $q \geq 1$ and open subset $\Omega \subseteq X$, we denote by

$$\mathcal{H}_E^q(\Omega) = \{ \omega \in A^q(\Omega) \mid \omega(X_1, \cdots, X_q) = 0 \text{ for } X_1, \cdots, X_q \in \Gamma(\Omega, E) \},$$
where $A^q(\Omega)$ is the space of complex smooth $q$-forms on $\Omega$. By the formal integrability of $E$, it follows from Cartan’s magic formula

\[
d\omega(X_1, \ldots, X_{q+1}) = \sum_{j=1}^{q+1} (-1)^{j+1} X_j \left( \omega(X_1, \ldots, \hat{X}_j, \ldots, X_{q+1}) \right) + \sum_{j<k} (-1)^{j+k} \omega([X_j, X_k], X_1, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_{q+1}),
\]

(26)

that $\mathfrak{N}_E^q(\Omega)$ is preserved by exterior differentiation, i.e.

\[
d\mathfrak{N}_E^q(\Omega) \subseteq \mathfrak{N}_E^{q+1}(\Omega).
\]

Treves ([13]) introduced the following complex as the quotient complex of the de Rahm complex

\[
C^\infty(\Omega) \xrightarrow{d'} U_1 \mathfrak{N}_E^1(\Omega) \xrightarrow{d'} U_2 \mathfrak{N}_E^2(\Omega) \xrightarrow{d'} U_3 \mathfrak{N}_E^3(\Omega) \xrightarrow{d'} \cdots,
\]

(27)

where

\[
\mathfrak{N}_E^q(\Omega) := \frac{A^q(\Omega)}{\mathfrak{N}_E^{q-1}(\Omega)}, \quad q \geq 1.
\]

Let $\{P_1, \ldots, P_r, L_1, \ldots, L_m\}$ be a frame field of $\mathbb{C}TX$ over $\Omega$ such that $E$ is spanned by $P_1, \ldots, P_r$ on $\Omega$, and let $\{\omega_1, \ldots, \omega_r, \theta_1, \ldots, \theta_m\}$ be the dual frame of $\mathbb{C}T^*X$. Given $f = \sum_{|I|=q-1} f_I \omega_I \in A^{q-1}_1(\Omega)$,

\[
df \equiv \sum_{|I|=q} \left( \sum_{j \in J} (-1)^{(j,J \setminus j)} p_j f_{|j|} \right) \omega_I + \sum_{|I|=q-1} f_I d\omega_I \text{ mod } \mathfrak{N}_E^q(\Omega).
\]

(28)

By (26), we also have

\[
d\omega_I(P_{i_1}, \ldots, P_{i_q}) = \sum_{j=1}^{q+1} (-1)^{j+1} P_{i_j} \left( \omega_I \left( P_{i_1}, \ldots, \hat{P}_{i_j}, \ldots, P_{i_{q+1}} \right) \right)

+ \sum_{j<k} (-1)^{j+k} \omega_I \left( [P_{i_j}, P_{i_k}], P_{i_1}, \ldots, \hat{P}_{i_j}, \ldots, \hat{P}_{i_k}, \ldots, P_{i_q} \right)

= \sum_{j<k} (-1)^{j+k} c_{i_j i_k} \omega_I \left( P_{i_s}, P_{i_1}, \ldots, \hat{P}_{i_j}, \ldots, \hat{P}_{i_k}, \ldots, P_{i_q} \right),
\]

which, together with (28), implies

\[
d' \left( f \text{ mod } \mathfrak{N}_E^{q-1}(\Omega) \right) = \sum_{|I|=q} \left( P_q F \right)_I \omega_I \text{ mod } \mathfrak{N}_E^q(\Omega),
\]

(29)
where \( F := (f_t)_{|t|=q-1} \in C^{\infty}(\Omega)_{(q-1)}^r \). In particular,

\[
d'(f \mod N_{q-1}E(\Omega)) = 0 \text{ if and only if } \mathcal{P}_qF = 0.
\]

Hence, the Treves complex (27) can be realized as our previoius complex (13) where \( P_1, \cdots, P_r \) are complex smooth vector fields (i.e. without zero terms). Under the condition that \( E \) is locally integrable, there are deep results on obstructions to the locally solvability of the Treves complex (27) (cf. [1], [2], [3], [15], and references therein). In the next section, we will consider the problem of local \( L^2 \)-solvability for (27) without local integrability assumption for the subbundle under consideration.

### 3. Global exactness of the differential complex

In this section, we will give a criterion for the global exactness of (4), and its local exactness is then obtained as a consequence. Following [5], we make a further assumption on the differential operator \( P = (P_1, \cdots, P_r) \).

\textbf{(A3)} There are functions \( d_{jk}^l \) and \( e_{jk}^l \) in \( C^\infty(\Omega) \) for all \( j, k \in \{1, \cdots, r\} \) such that

\[
[p_j, \bar{p}_k] = d_{jk}^l p_l - e_{jk}^l \bar{p}_l,
\]

where \( p_j \) is the principal part of \( P_j \) defined by (9) and \( \bar{p}_j \) is the operator obtained by conjugating \( p_j \)’s coefficients, i.e.

\[
\bar{p}_j = \bar{a}_j^\nu(x)\partial_\nu \text{ for } 1 \leq j \leq r.
\]

\textbf{Remark.} The coefficients \( d_{jk}^l \) and \( e_{jk}^l \) are not uniquely determined.

By using \( p_j, \bar{p}_j (1 \leq j \leq r) \), we define the following quadratic form for every \( C^2 \)-function on \( \Omega \) which was originally introduced by Hörmander ([6]).

\textbf{Definition 1.} Let \( \varphi \in C^2(\Omega) \) be a real valued function, \( x \in \Omega \), the quadratic form \( Q_{\varphi,x} \) on \( \mathbb{C}^r \) is defined as

\[
Q_{\varphi,x}(\xi, \xi) := \text{Re} \left( p_j\bar{p}_k\varphi(x) + e_{jk}^l \bar{p}_l\varphi(x) \right) \xi_k\bar{\xi}_j
\]

for all \( \xi = (\xi_j)_{1 \leq j \leq r} \in \mathbb{C}^r \).

\textbf{Remark.} (1) Since \( \varphi \) is real valued, it follows from (31) that for every \( \xi = (\xi_j)_{1 \leq j \leq r} \in \mathbb{C}^r \)

\[
0 = \text{Re} \left[ \bar{\xi}_j p_j, \xi_k\bar{p}_k \right] \varphi(x) = \text{Re} d_{jk}^l p_l \xi_k\bar{\xi}_j - \text{Re} e_{jk}^l \bar{p}_l \varphi(x) \xi_k\bar{\xi}_j,
\]
which shows that replacing $e^j_{\bar{j}k}p_l$ by $d^j_{\bar{j}k}p_l$ in Definition 1 results in the same quadratic form $Q_{\varphi,x}$.

(2) Denote by $E \subseteq \mathbb{C}T\Omega$ the subbundle spanned by the principal parts $p_1, \ldots, p_r$, the quadratic form $Q_{\varphi,x}$ only depends on $E$ if $E + \bar{E}$ is also a subbundle (i.e. has constant rank). $Q_{\varphi,x}$ can be viewed as an analogue of the Hessian of the Bott connection of a foliation structure.

For each $1 \leq q \leq r$, $Q_{\varphi,x}$ induces a quadratic form $Q_{q,\varphi,x}$ on $\mathbb{C}^{(r)}$ by

$$Q_{q,\varphi,x}(\xi, \xi) := \sum_{|I| = q - 1} Q_{\varphi,x}(\xi_I, \xi_I)$$

for all $\xi = (\xi_j)_{|j| = q} \in \mathbb{C}^{(q)}$, where $\xi_I := (\xi_{1I}, \cdots, \xi_{rI})^T \in \mathbb{C}^r$ for any multi-index $I$ of length $q - 1$.

**Lemma 3.1.** For a real valued function $\varphi \in C^2(\Omega)$, $x \in \Omega, 1 \leq q \leq r$, denote by $\{\lambda_j\}_{j=1}^r$ the eigenvalues of the quadratic form $Q_{\varphi,x}$ (w.r.t. the standard inner product on $\mathbb{C}^r$), then the eigenvalues (w.r.t. the standard inner product on $\mathbb{C}^{(q)}$) of the quadratic form $Q_{q,\varphi,x}$ are exactly given by $\lambda_J := \sum_{j \in J} \lambda_j$ indexed by strictly increasing multi-indices $J$ of length $q$.

**Proof.** Fix some $x \in \Omega$, find a $r$ by $r$ unitary matrix $A = (a_{ij})_{1 \leq i, j \leq r}$ which diagonalizes $Q_{\varphi,x}$, i.e.

$$Q_{\varphi,x}(A\xi, A\xi) = \sum_{1 \leq j \leq r} \lambda_j |\xi_j|^2, \text{ for } \xi = (\xi_j)_{1 \leq j \leq r} \in \mathbb{C}^r. \quad (32)$$

Let $A_q$ be the unitary transform on $\mathbb{C}^{(q)}$ which is induced by $A$ as follows

$$A_q: \mathbb{C}^{(q)} \longrightarrow \mathbb{C}^{(q)}$$

$$(\xi)_{j_1 \cdots j_q} \mapsto (A_q\xi)_{k_1 \cdots k_q} := \sum_{1 \leq j_1, \cdots, j_q \leq r} a_{k_1j_1} \cdots a_{k_qj_q} \xi_{j_1 \cdots j_q}$$

for all $\xi = (\xi_j)_{|j| = q} \in \mathbb{C}^{(q)}$. Set $\eta = A_q\xi \in \mathbb{C}^{(q)}$, then for every multi-index $I = (i_1, \cdots, i_{q-1})$

$$\eta_I = \sum_{1 \leq l_1, \cdots, l_{q-1} \leq r} a_{i_1l_1} \cdots a_{i_{q-1}l_{q-1}} A_{\xi_L} \in \mathbb{C}^r, \quad (33)$$

or equivalently

$$A_{\xi_I} = \sum_{1 \leq l_1, \cdots, l_{q-1} \leq r} a_{i_1l_1} \cdots a_{i_{q-1}l_{q-1}} \eta_L \in \mathbb{C}^r. \quad (34)$$
From (32)~(33), it follows that

\[
Q_{q,\varphi,x}(\eta,\eta) = \sum_{|I|=q-1} Q_{\varphi,x}(\eta I,\eta I)
\]

\[
= \sum_{|I|=q-1} Q_{\varphi,x}(A\xi L, A\xi M) a_{i_1 j_1} \cdots a_{i_{q-1} j_{q-1}} a_{i_{q-1} m_1} \cdots a_{i_q m_q}
\]

\[
= \sum_{1 \leq i_1, \ldots, i_{q-1} \leq r} \frac{Q_{\varphi,x}(A\xi L, A\xi M)}{(q-1)!} \sum_{1 \leq l_1, \ldots, l_{q-1} \leq r} a_{i_1 l_1} \cdots a_{i_{q-1} l_{q-1}} a_{i_{q-1} m_1} \cdots a_{i_q m_q}
\]

\[
= \sum_{|L|=q-1} Q_{\varphi,x}(A\xi L, A\xi L)
\]

\[
= \sum_{|L|=q-1} \lambda(|(A\xi L)|)^2.
\]

To go back from \(\xi\) to \(\eta\), we substitute (34) into the above identity

\[
Q_{q,\varphi,x}(\eta,\eta) = \sum_{|K|=q-1} \lambda(K) \left( \sum_{1 \leq i_1, \ldots, i_{q-1} \leq r} a_{i_1 j_1} \cdots a_{i_{q-1} j_{q-1}} a_{k_1 l_1} \cdots a_{k_{q-1} l_{q-1}} \eta J^{|J|K} \right)
\]

\[
= \sum_{1 \leq i_1, \ldots, i_{q-1} \leq r} \frac{\lambda|K|}{(q-1)!} \left( \sum_{1 \leq l_1, \ldots, l_{q-1} \leq r} a_{i_1 l_1} \cdots a_{i_{q-1} l_{q-1}} a_{k_1 l_1} \cdots a_{k_{q-1} l_{q-1}} \right)
\]

\[
= \sum_{|K|=q-1} \lambda|\eta K|^2
\]

\[
= \sum_{|J|=q} \lambda_J |\eta J|^2,
\]

where \(\lambda_J = \sum_{j \in J} \lambda_J\). The proof is thus complete. \(\square\)

We will prove the following global exactness result for (4) under the assumptions (A1)~(A3).

**Theorem 3.1.** For \(1 \leq q \leq r\), if \(\Omega\) has an exhaustion function \(\varphi \in C^2(\Omega)\) such that the quadratic form \(Q_{\varphi,x}\) is positive semi-definite and \(\text{rank}Q_{\varphi,x} \geq r - q + 1\) for every \(x \in \Omega\), then for any \(f \in L^2_{\text{loc}}(\Omega)^{\binom{r}{q}}\) satisfying \(\mathcal{P}_{q+1} f = 0\), there is some \(u \in L^2_{\text{loc}}(\Omega)^{\binom{r}{q-1}}\) such that \(\mathcal{P}_q u = f\).

Let’s first recall a variant of the Riesz representation theorem ([7]) which is very useful for solving overdetermined systems.
Lemma 3.2. Let $T : H_1 \to H_2$ and $S : H_2 \to H_3$ be closed, densely defined linear operators such that $\text{Im}T \subseteq \text{Ker}S$. Let $C > 0$ be a constant, if for any $g \in \text{Dom}(T^*)$ we have

$$\|(g, f)_{H_2}\| \leq C\|T^*g\|_{H_1},$$

then the equation $Tu = f$ is solvable and $\|u\|_{H_1} \leq C$. In particular, assume that

$$\|g\|_{H_2}^2 \leq C^2 (\|T^*g\|_{H_1}^2 + \|Sg\|_{H_3}^2), \quad g \in \text{Dom}(T^*) \cap \text{Dom}(S).$$

If $Sf = 0$, then the equation $Tu = f$ is solvable and $\|u\|_{H_1} \leq C\|f\|_{H_2}$.

We will apply the above lemma to Hilbert spaces

$$H_1 = L^2_{\phi_1}(\Omega)^{\oplus (r_{q-1})}, \quad H_2 = L^2_{\phi_2}(\Omega)^{\oplus (r_q)}, \quad H_3 = L^2_{\phi_3}(\Omega)^{\oplus (r_{q+1})}, \quad (35)$$

and

$$S = \text{the maximal extension of } P_{q+1}, \quad T = \text{the maximal extension of } P_q, \quad (36)$$

where $\phi_1, \phi_2, \phi_3 \in C(\Omega)$ are real valued functions. The weighted $L^2$-space $L^2_\phi(\Omega)$ for a real valued function $\phi \in C(\Omega)$ is defined as

$$L^2_\phi(\Omega) = \left\{ f \in L^2_{\text{loc}}(\Omega) \mid \int_\Omega |f(x)|e^{-\phi} < +\infty \right\}.$$ We denote by $(\cdot, \cdot)_\phi$ the inner product of $L^2_\phi(\Omega)$.

Repeating the same argument in [5](Lemma 4 on page 430), we have

Lemma 3.3. Fix a sequence of real valued functions $\{\eta_\nu\}_{\nu=1}^\infty$ in $\mathcal{D}(\Omega)$ such that $0 \leq \eta_\nu \leq 1$ for all $\nu$, and $\eta_\nu = 1$ on any compact subset of $\Omega$ when $\nu$ is large. Then there is a real valued function $\tau \in C^\infty(\Omega)$ such that

$$|\sigma(P_q)(\cdot, d\eta_\nu)|^2 \leq e^\tau, \nu = 1, 2, \ldots \quad (37)$$

hold on $\Omega$ where $\sigma(P_q)$ denotes the principle symbol of $P_q$. Moreover, $\mathcal{D}(\Omega)^{\oplus (r_q)}$ is dense in $\text{Dom}(T^*) \cap \text{Dom}(S)$ for the graph norm

$$f \to \|f\|_{H_2} + \|T^*f\|_{H_1} + \|Sf\|_{H_3},$$

where $H_1, H_2, H_3$ and $S, T$ are as described in (35) and (36) with weight functions $\phi_1, \phi_2, \phi_3 \in C(\Omega)$ satisfying $\phi_3 - \phi_2 = \phi_2 - \phi_1 = \tau$. 


For later use in the proof of Theorem 3.1, we establish the Bochner type formula as follows.

**Proposition 3.1.** For any \( f \in \text{Dom} (T^*) \cap \text{Dom} (S) \) and \( \epsilon > 0 \) we have

\[
\text{Re} \sum_{|I|=q-1} \sum_{j,k
ot \in I} \left( (p_j \bar{p}_k \phi_3 + e_j^k \bar{p}_j \phi_3) f_{kI} , f_j I \right)_{\phi_3} \\
\leq (1 + \epsilon) \left( \|T^* f\|_{\phi_1}^2 + \|S f\|_{\phi_3}^2 \right) + \left( 2 + \frac{2}{\epsilon} \right) \|\sigma (t^s \mathcal{P}_q) (\cdot, d\tau) f\|_{\phi_3}^2 + \frac{1}{\epsilon} \|\kappa f\|_{\phi_3}^2 ,
\]

where \( \tau \) is a function satisfying (37), \( \kappa \) is a polynomial in \( \|e_{ij}^k\|, \|d_{ij}^k\|, \|e_{ij}^k\|, |\nabla e_{ij}^k|, \|\nabla a_{ij}^\nu\| \) and \( |\nabla^2 a_{ij}^\nu| \) with coefficients in \( \mathbb{R}_{\geq 0} \) for all \( 0 \leq \nu \leq n \).

**Proof.** Let \( t^s P_{j,\phi_3} \) be the formal adjoint of \( P_j \) with respect to the inner product \((\cdot, \cdot)_{\phi_3}\), i.e. \( t^s P_{j,\phi_3} = e^{\phi_3} \circ t^s P_j \circ e^{-\phi_3} \). For \( f \in \mathcal{D}(\Omega) \oplus_q (r) \), we know by (12) and (25)

\[
T^* f = e^{\phi_3} \circ t^s P_q \circ e^{-\phi_3} f \\
= e^{-\tau} \left( \sum_{j \not \in I} t^s P_{j,\phi_3} f_{jI} \right)_{|I|=q-1} + e^{-\tau} \sigma (t^s \mathcal{P}_q) (\cdot, d\tau) f + e^{-\tau} \kappa_1 f, \tag{38}
\]

\[
S f = \left( \sum_{k \in K} (-1)^{(k,K\backslash k)} P_k f_{K\backslash k} \right)_{|K|=q+1} + \kappa_1 f, \tag{39}
\]

where we denote by \( \kappa_1 \) various polynomials in \( e_{ij}^k \) with constant coefficients for \( 1 \leq i, j, k \leq r \).

Set

\[
A_1 = \sum_{|I|=q-1} \left\| \sum_{j \not \in I} t^s P_{j,\phi_3} f_{jI} \right\|_{\phi_3}^2,
\]

\[
A_2 = \sum_{|K|=q+1} \left\| \sum_{k \in K} (-1)^{(k,K\backslash k)} P_k f_{K\backslash k} \right\|_{\phi_3}^2.
\]

We will proceed by estimating \( A_1 + A_2 \) from both sides. By (38) and (39), it is easy to see that for any \( \epsilon > 0 \)

\[
A_1 + A_2 \leq (1 + \epsilon) \left( \|T^* f\|_{\phi_1}^2 + \|S f\|_{\phi_3}^2 \right) \\
+ \left( 2 + \frac{2}{\epsilon} \right) \|\sigma (t^s \mathcal{P}_q) (\cdot, d\tau) f\|_{\phi_3}^2 + \frac{1}{\epsilon} \|\kappa f\|_{\phi_3}^2. \tag{40}
\]

Here and hereafter, we denote by \( \kappa \) various polynomials in \( |e_{ij}^k|, |d_{ij}^k|, |e_{ij}^k|, |\nabla e_{ij}^k|, |\nabla a_{ij}^\nu| \) and \( |\nabla^2 a_{ij}^\nu| \) with coefficients in \( \mathbb{R}_{\geq 0} \) for all \( 1 \leq i, j, k \leq r, 0 \leq \nu \leq n \).
\( \nu \leq n \). We rewrite \( A_1 + A_2 \) by integration by parts as follows

\[
A_1 + A_2 = \sum_{|I|=q-1} \sum_{j,k \notin I} (tP_{k,\phi_3} f_{kI}, f_{jI})_{\phi_3} + \sum_{|K|=q+1} \sum_{k \in K} \|P_k f_{K}\|_3^2 + \sum_{|K|=q+1} \sum_{j,k \in K, j \neq k} (-1)^{|(j, K)\cup (j, K)|} (tP_{k,\phi_3} P_j f_{K\setminus j}, f_{K\setminus j})_{\phi_3}
\]

\[
= \sum_{|I|=q-1} \sum_{j,k \notin I} (tP_{k,\phi_3} f_{kI}, f_{jI})_{\phi_3} + \sum_{|J|=q} \sum_{k \notin J} \|P_k f_{J}\|_3^2 - \sum_{|I|=q-1} \sum_{j,k \notin I, j \neq k} (tP_{k,\phi_3} P_j f_{kI}, f_{jI})_{\phi_3}
\]

\[
= \sum_{|I|=q-1} \sum_{j,k \notin I} (tP_{k,\phi_3} f_{kI}, f_{jI})_{\phi_3} + \sum_{|J|=q} \sum_{k \notin J} \|P_k f_{J}\|_3^2 - \sum_{|I|=q-1} \sum_{j \in J} (tP_{j,\phi_3} P_j f_{jI}, f_{jI})_{\phi_3} + \sum_{|I|=q-1} \sum_{j \notin I} (tP_{j,\phi_3} f_{kI}, f_{jI})_{\phi_3}
\]

\[
= \sum_{|I|=q-1} \sum_{j,k \notin I} (tP_{j,\phi_3} f_{kI}, f_{jI})_{\phi_3} + \sum_{|J|=q} \sum_{j \notin J} \|P_j f_{J}\|_3^2.
\]  (41)

In the second equality, we changed the indices of summation by letting \( J = K \setminus k, I = K \setminus \{j, k\} \), and making use of the identity \((-1)^{|(j, I)\cup (k, I)|} = -(-1)^{|(j, I)\cup (k, I)|}\) for \( j, k \notin I, j \neq k \). The equality (41) allows us to estimate \( A_1 + A_2 \) from below

\[
\text{Re} \sum_{|I|=q-1} \sum_{j,k \notin I} (-[p_j, F_k] f_{kI}, f_{jI})_{\phi_3} + B_1 \leq A_1 + A_2 + \|\kappa f\|_3^2,
\]  (42)

where \( F_k := e^{\phi_3} \circ \tilde{p}_k \circ e^{-\phi_3} \) and \( B_1 := \sum_{|I|=q} \sum_{j=1}^r \|p_j f_{J}\|_3^2 \).

By virtue of \( F_k = \tilde{p}_k - \sigma(\tilde{p}_k)(\cdot, d\phi_3) \), we have

\[
- [p_j, F_k] = - [p_j, \tilde{p}_k - \sigma(\tilde{p}_k)(\cdot, d\phi_3)] = - [p_j, \tilde{p}_k] + [p_j, \sigma(\tilde{p}_k)(\cdot, d\phi_3)],
\]

which in turn implies

\[
\text{Re} \sum_{|I|=q-1} \sum_{j,k \notin I} (-[p_j, F_k] f_{kI}, f_{jI})_{\phi_3}
\]

\[
= \text{Re} \sum_{|I|=q-1} \sum_{j,k \notin I} \left( (-[p_j, \tilde{p}_k] f_{kI}, f_{jI})_{\phi_3} + ([p_j, \sigma(\tilde{p}_k)(\cdot, d\phi_3)] f_{kI}, f_{jI})_{\phi_3} \right)
\]

\[
= \text{Re} \sum_{|I|=q-1} \sum_{j,k \notin I} \left( (-[p_j, \tilde{p}_k] f_{kI}, f_{jI})_{\phi_3} + (p_j \tilde{p}_k \phi_3 f_{kI}, f_{jI})_{\phi_3} \right).
\]  (43)
The last equality is valid by the following observation

\[ [p_j, \sigma(\bar{p}_k)(\cdot, d\phi_3)] f = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \left[ a_j^{\mu}(x) \partial_{\mu} \bar{a}_k^{\nu}(x) \partial_{\nu} \phi_3 \right] f \]

\[ = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} a_j^{\mu}(x) \partial_{\mu} (\bar{a}_k^{\nu}(x)) \partial_{\nu} \phi_3 f - \bar{a}_k^{\nu}(x) \partial_{\nu} \phi_3 a_j^{\mu}(x) \partial_{\mu} f \]

\[ = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} a_j^{\mu}(x) \partial_{\mu} (\bar{a}_k^{\nu}(x)) \partial_{\nu} \phi_3 f + a_j^{\mu}(x) \bar{a}_k^{\nu}(x) \partial_{\nu} \phi_3 \partial_{\mu} f \]

\[ = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} a_j^{\mu}(x) \partial_{\mu} (\bar{a}_k^{\nu}(x)) \partial_{\nu} \phi_3 f \]

\[ = p_j \bar{p}_k \phi_3 f. \]

Substituting (31) (43) into (42) gives

\[ \text{Re} \sum_{|I|=q-1, k \notin I} \left( \left( p_j \bar{p}_k \phi_3 + e_{lj}^j \bar{p}_l - d_{lj}^j \bar{p}_l \right) f_{kI}, f_{jI} \right)_{\phi_3} + B_1 \]

\[ \leq A_1 + A_2 + \|\kappa f\|_{\phi_3}^2, \]

which implies by another integration by parts involving \( \bar{p}_k f_{kI} \)

\[ \text{Re} \sum_{|I|=q-1, k \notin I} \left( \left( p_j \bar{p}_k \phi_3 + e_{lj}^j \bar{p}_l \phi_3 \right) f_{kI}, f_{jI} \right)_{\phi_3} + B_1 \]

\[ \leq A_1 + A_2 + \|\kappa f\|_{\phi_3}^2 + B_1^2 \|\kappa f\|_{\phi_3}. \quad (44) \]

Combining (40) and (44), it follows that

\[ \text{Re} \sum_{|I|=q-1, k \notin I} \left( \left( p_j \bar{p}_k \phi_3 + e_{lj}^j \bar{p}_l \phi_3 \right) f_{kI}, f_{jI} \right)_{\phi_3} \]

\[ \leq (1 + \epsilon) \left( \|T^* f\|_{\phi_1}^2 + \|S f\|_{\phi_3}^2 \right) + \left( 2 + \frac{2}{\epsilon} \right) \|\sigma (\mathcal{P}_q) (\cdot, d\tau) f\|_{\phi_3}^2 + \frac{1}{\epsilon} \|\kappa f\|_{\phi_3}^2. \]

According to Lemma 3.3, the desired estimate holds for any \( f \in \text{Dom} (T^*) \cap \text{Dom} (S). \]

We are now in a position to prove Theorem 3.1.

**Proof of Theorem 3.1.** Let \( \tau \in C^\infty(\Omega) \) be a function satisfying (37). We will make use of weight functions \( \phi_1 = \chi(\varphi) - 2\tau, \phi_2 = \chi(\varphi) - \tau, \phi_3 = \chi(\varphi) \)
where \( \chi \in C^\infty(\mathbb{R}) \) is a convex increasing function to be chosen in the sequel. By Proposition 3.1, for any \( f \in \text{Dom} \left( T^* \right) \cap \text{Dom} \left( S \right) \) we have

\[
\text{Re} \sum_{|I|=q-1} \sum_{j,k \notin I} \left( \chi' (\varphi) \left( p_j \vec{p}_k \varphi + e_{jk} \vec{p}_i \varphi \right) f_{kl}, f_{it} \right)_{\phi_3} \leq (1 + \epsilon) \left( \| T^* f \|_{\phi_3}^2 + \| S f \|_{\phi_3}^2 \right) + \left( 2 + \frac{2}{\epsilon} \right) \| \sigma \left( ^t \mathcal{P}_q \right) \left( \cdot, d\tau \right) f \|_{\phi_3}^2 + \frac{1}{\epsilon} \| \kappa f \|_{\phi_3}^2.
\]

Since \( \text{rank}_{\mathbb{C}} \mathcal{Q}_{\varphi, x} \geq r - q + 1 \), Lemma 3.1 shows that there exists some positive \( \sigma \in C(\Omega) \) such that (45) is bounded from below by

\[
\int_{\Omega} \sigma \chi' (\varphi) |f|^2 e^{-\chi(\varphi)}.
\]

As \( \varphi \) is an exhaustion function on \( \Omega \), we can choose \( \chi \) increasing rapidly such that

\[
\sigma \chi' (\varphi) \geq \left( \frac{\kappa^2}{\epsilon} + \left( 2 + \frac{2}{\epsilon} \right) \| \sigma \left( ^t \mathcal{P}_q \right) \left( \cdot, d\tau \right) \|_{\phi_3}^2 + (1 + \epsilon) \right) e^{-\sigma}.
\]

which gives the desired estimate in Lemma 3.2. \( \Box \)

**Definition 2.** \( \Omega \) is said to be \( P \)-convex, if there exists an exhaustion function \( \varphi \in C^2(\Omega) \) such that \( \mathcal{Q}_{\varphi, x} \) is positive definite for every \( x \in \Omega \).

Given a point \( y \in \Omega \), let \( \varphi \in C^2(\Omega) \) be an arbitrary function vanishing at \( y \) up to first order derivatives, then we have

\[
\mathcal{Q}_{\varphi, y}(\xi, \xi) = \sum_{1 \leq \mu, \nu \leq n} a_{\mu}^\nu(y) \xi_\mu \overline{a_{\nu}^\mu(y)} \xi_\nu \partial_\mu \partial_\nu \varphi(y),
\]

which indicates the following assumption.

(\( A2 \))\( ^* \) \( \text{rank}_{\mathbb{C}} \left( a_{\mu}^\nu(x) \right)_{1 \leq \mu, \nu \leq n}^{1 \leq j, k \leq r} = r \) holds for all \( x \in \Omega \).

In view of (10), (\( A2 \))\( ^* \) is stronger than previous assumption (\( A2 \)).

Under the assumption (\( A2 \))\( ^* \), it is clear from (46) that each point \( y \in \Omega \) has a \( P \)-convex neighborhood, and therefore we have the following consequence of Theorem 3.1.

**Corollary 3.1.** Assume that \( P = (P_1, \cdots, P_r) \) satisfies (\( A1 \)), (\( A2 \))\( ^* \) and (\( A3 \)), we have

1. The complex (4) is a resolution of the solution sheaf \( S_P \) and the isomorphisms (6) hold.
2. If there is an exhaustion function \( \varphi \in C^2(\Omega) \) such that \( Q_{\varphi,x} \) is positive semi-definite, rank\(_C Q_{\varphi,x} \geq r - q + 1 \) for every \( x \in \Omega \) and some \( 1 \leq q \leq r \), then \( H^q(\Omega, \mathcal{S}_P) = 0 \). In particular, if \( \Omega \) is \( P \)-convex then \( H^q(\Omega, \mathcal{S}_P) = 0 \) for all \( q \geq 1 \).

In the special case where \( P_j (1 \leq j \leq n) \) are complex vector fields (i.e. \( P_j = p_j \)), since the assumption (A2)* is automatically true for the Treves complex (27), the local \( L^2 \)-solvability of the Treves complex follows directly from (29), (30) and Corollary 3.1. Note that we do not assume the subbundle \( E \) associated to (27) is locally integrable. As we do not make use of the parametrix, we can obtain local exactness of the Treves complex in the Levi flat case. Most of the known results are under the assumption of the non-degeneracy of the Levi form.

In general, there are obstructions to the local solvability of the Treves complex (cf. [15], [8], etc). When the Levi form is non-degenerate, the so-called \( Y(q) \)-condition was shown to be the necessary and sufficient for local solvability of the Treves complex in the frame of distributions (14). It was proven in [3] that the \( Y(q) \)-condition is also necessary for local solvability in the sense of ultra-distributions.

When the system (2) is elliptic (i.e. has empty real characteristic variety), it follows from Corollary 3.1 and standard elliptic regularity that the sub-complex (13) also provides a resolution of the solution sheaf \( \mathcal{S}_P \) of the system of partial differential equations (2).

4. Cousin problems

Throughout this section, we assume that the conditions (A1), (A2)* and (A3) for differential operators \( P_1, \cdots, P_r \) are fulfilled. We will consider both additive and multiplicative Cousin Problems for the system

\[
P_1 u = \cdots = P_r u = 0.
\]

We begin with the formulation of Cousin problems for the system.

**Definition 3.** An additive Cousin datum of the system (47) is a collection \( \{\Omega_\alpha, u_{\alpha\beta}\}_{\alpha, \beta \in \mathbb{Z}} \) where \( \{\Omega_\alpha\}_{\alpha \in \mathbb{Z}} \) is an open covering of \( \Omega \), each \( u_{\alpha\beta} \in L^2_{\text{loc}}(\Omega_\alpha \cap \Omega_\beta) \) satisfies (47) on \( \Omega_\alpha \cap \Omega_\beta(\alpha, \beta \in \mathbb{Z}) \) and

\[
u_{\alpha\beta} + u_{\beta\gamma} + u_{\gamma\alpha} = 0 \text{ on } \Omega_\alpha \cap \Omega_\beta \cap \Omega_\gamma(\alpha, \beta, \gamma \in \mathbb{Z}).
\]
The Cousin problem for an additive Cousin datum \( \{ \Omega_\alpha, u_{\alpha\beta} \}_{\alpha, \beta \in \mathbb{Z}} \) is to find a sequence of solutions \( u_\alpha \in L^2_{\text{loc}}(\Omega_\alpha) \) of (47) such that \( u_{\alpha\beta} = u_\beta - u_\alpha \) hold on \( \Omega_\alpha \cap \Omega_\beta \) for all \( \alpha, \beta \in \mathbb{Z} \).

We can construct global solutions of the overdetermined system (2) from local ones by using solutions of the additive Cousin problem.

**Proposition 4.1.** Assume that the Cousin problem is always solvable for every additive Cousin datum of the system (47). Let \( \{ \Omega_\alpha \}_{\alpha \in \mathbb{Z}} \) be an open covering of \( \Omega \) and \( v_\alpha \in \mathcal{D}'(\Omega_\alpha) \) be a solution of the system (2) (\( \alpha \in \mathbb{Z} \)). If the difference \( v_\alpha - v_\beta \in L^2_{\text{loc}}(\Omega_\alpha \cap \Omega_\beta) \) for all \( \alpha, \beta \in \mathbb{Z} \), there is a global solution \( v \in L^2_{\text{loc}}(\Omega) \) of the system (2) such that \( v|_{\Omega_\alpha} - v_\alpha \in L^2_{\text{loc}}(\Omega_\alpha) \) for each \( \alpha \in \mathbb{Z} \).

**Proof.** According to Definition 3, \( \{ \Omega_\alpha, (v_\alpha - v_\beta)|_{\Omega_\alpha \cap \Omega_\beta} \}_{\alpha, \beta \in \mathbb{Z}} \) is an additive Cousin datum of the system (47). Let \( \{ u_\alpha \}_{\alpha \in \mathbb{Z}} \) be a solution of the above additive Cousin datum, then

\[
    u_\beta - u_\alpha = v_\alpha - v_\beta
\]

hold on \( \Omega_\alpha \cap \Omega_\beta \) for all \( \alpha, \beta \in \mathbb{Z} \). Hence, there is a global solution \( v \in \mathcal{D}'(\Omega) \) of the system (2) such that

\[
    v|_{\Omega_\alpha} = u_\alpha + v_\alpha
\]

for each \( \alpha \in \mathbb{Z} \).

The assumption (A1) is equivalent to

\[
    [p_j, p_k] = c^j_{jk} p_l, \quad p_j a^0_k - p_k a^0_j = c^l_{jk} a^0_l \quad \text{for} \quad 1 \leq i, j \leq r,
\]

and therefore one can find (by Corollary 3.1) a function \( \eta \in L^2_{\text{loc}}(\Omega) \) such that for \( 1 \leq j \leq r \)

\[
    p_j \eta = a^0_j
\]

when \( \Omega \) is assumed to be \( P \)-convex. We will fix such an \( \eta \) satisfying (48) in the sequel and formulate the multiplicative Cousin Problem as below.

**Definition 4.** An multiplicative Cousin datum of the system (47) is a collection \( \{ \Omega_\alpha, u_{\alpha\beta} \}_{\alpha, \beta \in \mathbb{Z}} \) where \( \{ \Omega_\alpha \}_{\alpha \in \mathbb{Z}} \) is an open covering of \( \Omega \), each \( u_{\alpha\beta} \in C^0(\Omega_\alpha \cap \Omega_\beta) \) satisfies (47) on \( \Omega_\alpha \cap \Omega_\beta (\alpha, \beta \in \mathbb{Z}) \) and

\[
    e^{3n} u_{\alpha\beta} u_\beta u_\gamma = 1 \quad \text{on} \quad \Omega_\alpha \cap \Omega_\beta \cap \Omega_\gamma (\alpha, \beta, \gamma \in \mathbb{Z}).
\]
The Cousin problem for a multiplicative Cousin datum \( \{ \Omega_\alpha, u_{\alpha\beta} \}_{\alpha, \beta \in \mathbb{Z}} \) is to find a sequence of solutions \( u_\alpha \in C^0(\Omega_\alpha) \) of (47) such that each \( u_\alpha \) is nowhere zero on \( \Omega_\alpha \) and

\[
e^\eta u_{\alpha\beta} = \frac{u_\beta}{u_\alpha}
\]

hold on \( \Omega_\alpha \cap \Omega_\beta \) for all \( \alpha, \beta \in \mathbb{Z} \), where \( \eta \) is a solution of (48).

The solvability of the multiplicative Cousin problem enables us to patch together local solutions of the system (47) as follows.

**Proposition 4.2.** Assume that the Cousin problem is always solvable for every multiplicative Cousin datum of the system (47) and that the function \( \eta \in C^0(\Omega) \). Let \( \{ \Omega_\alpha \}_{\alpha \in \mathbb{Z}} \) be an open covering of \( \Omega \) and \( v_\alpha \in C^0(\Omega_\alpha) \) be a solution of (47) for each \( \alpha \in \mathbb{Z} \), such that the quotients \( \frac{v_\alpha}{v_\beta} \in C^0(\Omega_\alpha \cap \Omega_\beta) \) are nowhere zero for all \( \alpha, \beta \in \mathbb{Z} \), there is a global solution \( v \in C^0(\Omega) \) of the system (47) such that \( \frac{v_\alpha}{v_\beta} \in C^0(\Omega_\alpha) \) is nowhere zero for each \( \alpha \in \mathbb{Z} \).

**Proof.** We rewrite the identity (48) as

\[
P_j = e^{-\eta} p_j(e^{\eta} \cdot), \quad 1 \leq j \leq r.
\]

Then, it is easy to see that for \( 1 \leq j \leq r \),

\[
P_j\left( \frac{v_\alpha}{e^{\eta}v_\beta} \right) = e^{-\eta} p_j\left( \frac{e^{\eta}v_\alpha}{e^{\eta}v_\beta} \right)
\]

\[
= \frac{P_j v_\alpha}{e^{\eta}v_\beta} - \frac{v_\alpha P_j v_\beta}{e^{\eta}v_\beta^2} = 0
\]

hold on \( \Omega_\alpha \cap \Omega_\beta \) for all \( \alpha, \beta \in \mathbb{Z} \). Consequently, we know that

\[
\{ \Omega_\alpha, \left( \frac{v_\alpha}{e^{\eta}v_\beta} \right)|_{\Omega_\alpha \cap \Omega_\beta} \}_{\alpha, \beta \in \mathbb{Z}}
\]

(50)

is a multiplicative Cousin datum of the system (47). Let \( \{ u_\alpha \}_{\alpha \in \mathbb{Z}} \) be a solution of the above multiplicative Cousin datum (50), then

\[
\frac{u_\beta}{u_\alpha} = \frac{v_\alpha}{v_\beta}
\]

holds on \( \Omega_\alpha \cap \Omega_\beta \) for all \( \alpha, \beta \in \mathbb{Z} \). So, we have a global function \( v \in C^\infty(\Omega) \) satisfying

\[
v|_{\Omega_\alpha} = e^\eta u_\alpha v_\alpha
\]
for each $\alpha \in \mathbb{Z}$. Again, we know by (49) that on each $\Omega_\alpha$

$$P_j v = P_j (e^\eta u_\alpha v_\alpha) = e^{-\eta} p_j (e^{2\eta} u_\alpha v_\alpha)$$

$$= e^\eta (v_\alpha P_j u_\alpha + u_\alpha P_j v_\alpha) = 0$$

for $1 \leq j \leq r$. The proof is thus complete. \hfill \square

Global exactness in section 3(Corollary 3.1) enables us to provide solutions to Cousin problems by a purely sheaf-theoretic approach.

**Theorem 4.1.** When $\Omega$ is $P$-convex, we have

1. The additive Cousin problem is always solvable.
2. Under the condition that the principal part $(p_1, \cdots, p_r)$ of $P$ is $C^0$-hypoelliptic, the multiplicative Cousin problem is always solvable if and only if $H^2(\Omega, \mathbb{Z}) = 0$. Here, the $C^0$-hypoellipticity of $(p_1, \cdots, p_r)$ means that for any $u \in L^2_{\text{loc}}(\Omega)$ if $p_j u \in C^\infty(\Omega)$ for $1 \leq j \leq r$ then $u \in C^0(\Omega)$.

**Proof.** (1) According to Definition 3, we know that an additive Cousin datum $\{\Omega_\alpha, u_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{Z}}$ defines an element

$$[\{u_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{Z}}] \in H^1(\{\Omega_\alpha\}_{\alpha \in \mathbb{Z}}, \mathcal{S}_P),$$

and that the Cousin problem for the datum $\{\Omega_\alpha, u_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{Z}}$ has a solution if and only if $0 = [\{u_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{Z}}] \in H^1(\{\Omega_\alpha\}_{\alpha \in \mathbb{Z}}, \mathcal{S}_P)$. Therefore, (1) follows from Corollary 3.1.

(2) By the hypoellipticity assumption, the function $\eta$ satisfying (48) must be continuous. As a consequence of the identity (49), $C^0$-hypoellipticity of the principal part $(p_1, \cdots, p_r)$ implies

$$\mathcal{S}_P \subseteq \mathcal{C}, \quad (51)$$

where $\mathcal{C}$ is the sheaf of germs of continuous functions over $\Omega$.

We define a multiplication on $\mathcal{C}$ as follows

$$u \star v := e^\eta uv. \quad (52)$$

By virtue of (49), we have for $1 \leq j \leq r$

$$P_j (u \star v) = P_j u \star v + u \star P_j v. \quad (53)$$
Since the multiplicative identity of $\star$ is given by $\frac{1}{e^{\eta}}$ which is annihilated by every $P_j$ (by (49)), it follows from (53) that $\star$ induces a multiplication on $S_P$. In order to deal with the multiplicative Cousin problem, we introduce (by using (51)) the sheaf $S_P^\star$ of invertible germs (according to the multiplication $\star$) of $S_P$. By definition, $S_P^\star$ is a sheaf of Abelian groups w.r.t. the multiplication $\star$ defined by (52). It is clear that $S_P^\star$ consists of germs of $S_P$ which are nowhere zero. Moreover, for any $u \in S_P^\star$, we denote by $u^{-1}$ the inverse of $u$ w.r.t. the multiplication $\star$ in (52). The definition of $\star$ gives

$$u^{-1} = \frac{1}{e^{2\eta}u}.$$  \tag{54}$$

By the definition (52) of the multiplication $\star$, we have the following sheaf-morphism

$$\text{Exp}^\eta : S_P \to S_P^\star$$

$$u \mapsto e^{2\pi \sqrt{-1}u - \eta}$$

whose kernel $K_{\text{Exp}^\eta} \cong \text{the constant sheaf } \mathbb{Z}$ over $\Omega$. Hence, we have the following twisted exponential sequence

$$0 \to \mathbb{Z} \xrightarrow{\frac{\sqrt{-1}}{2\pi}} S_P \xrightarrow{\text{Exp}^\eta} S_P^\star \to 0,$$  \tag{55}$$

which is exact because of (51). The twisted exponential sequence (55) gives the following exact sequence

$$H^1(\Omega, S_P) \to H^1(\Omega, S_P^\star) \to H^2(\Omega, \mathbb{Z}) \to H^2(\Omega, S_P).$$  \tag{56}$$

From Corollary 3.1, it follows that

$$H^1(\Omega, S_P) = H^2(\Omega, S_P) = 0$$

holds on $P$-convex $\Omega$. Hence, we obtain an isomorphism from (56)

$$H^1(\Omega, S_P^\star) \cong H^2(\Omega, \mathbb{Z}).$$  \tag{57}$$

For every multiplicative Cousin datum $\{\Omega_\alpha, u_{\alpha \beta}\}_{\alpha, \beta \in \mathbb{Z}}$,

$$u_{\alpha \beta} \star u_{\beta \gamma} \star u_{\gamma \alpha} = \frac{1}{e^{\eta}} \text{(the multiplicative identity of $\star$)}$$

on $\Omega_\alpha \cap \Omega_\beta \cap \Omega_\gamma (\alpha, \beta, \gamma \in \mathbb{Z})$, i.e. $\{\Omega_\alpha, u_{\alpha \beta}\}_{\alpha, \beta \in \mathbb{Z}}$ defines an element

$$[\{u_{\alpha \beta}\}_{\alpha, \beta \in \mathbb{Z}}] \in H^1(\{\Omega_\alpha\}_{\alpha \in \mathbb{Z}}, S_P^\star).$$  \tag{58}$$
By the identity (54), \( \{ u_\alpha \}_{\alpha \in \mathbb{Z}} \) is a solution for the multiplicative Cousin datum \( \{ \Omega_\alpha, u_{\alpha \beta} \}_{\alpha, \beta \in \mathbb{Z}} \) exactly means that \( u_\alpha \in \Gamma(\Omega_\alpha, S^*_P) \) and
\[
{u_\alpha}_\beta = \frac{u_\beta}{e^{u_\alpha}} = u_\beta \ast (u_\alpha)^{-1}. \tag{59}
\]

The desired conclusion follows from (57), (58) and (59). \( \square \)

Finally, we mention another application of the additive Cousin problem for an elliptic system. By the ellipticity assumption, we can choose some solution \( \eta \in C^\infty(\Omega) \) of the system (48) if \( \Omega \) is assumed to be \( P \)-convex.

Let \( F \in C^\infty(\Omega) \) be a solution of the system \( P_1 F = \cdots = P_r F = 0 \) with the property that
\[
dF \neq 0 \text{ holds on } Z := F^{-1}(0).
\]

Obviously,
\[
d(e^\eta F) = e^\eta dF \neq 0 \text{ on } Z = (e^\eta F)^{-1}(0). \tag{60}
\]

Under the ellipticity assumption, it is known that at each point of \( \Omega \) there is a coordinate chart
\[
(U; x_1, \cdots, x_{r-s}, y_1, \cdots, y_{r-s}, t_1, \cdots, t_s) \text{ (} 2r - s = n \text{)} \tag{61}
\]

such that
\[
\{ p_1, \cdots, p_r \} \text{ and } \{ \partial_{\bar{z}_1}, \cdots, \partial_{\bar{z}_r}, \partial_{t_1}, \cdots, \partial_{t_s} \} \text{ span the same subbundle,} \tag{62}
\]

where \( p_j \) is the principal part of \( P_j \) for \( 1 \leq j \leq r \) and \( z_i = x_i + \sqrt{-1}y_i \) for \( 1 \leq i \leq r - s \) (cf. [10]). Due to the existence of such coordinate charts, it follows from (49) and (60) that \( Z \) is a closed submanifold of \( \Omega \) of real codimension two. Let \( E \subseteq \mathbb{C}T\Omega \) be the subbundle spanned by \( \{ p_1, \cdots, p_r \} \).

Combining (62) and the condition \( P_1 F = \cdots = P_r F = 0 \), we know that
\[
E_Z := E \cap \mathbb{C}TZ \tag{63}
\]
defines a subbundle of \( \mathbb{C}TZ \), i.e. \( Z \) is a compatible submanifold of \( E \).

A smooth function on \( Z \) is said to be a first integral of \( E_Z \) if it is annihilated by sections of \( E_Z \). Taking into account of (49), it is natural to consider the problem of extending a first integral \( g_Z \in C^\infty(Z) \) of \( E_Z \) to a function \( g \in C^\infty(\Omega) \) such that \( P(e^{-\eta}g) = 0 \).
By using a coordinate chart $U$ as described in (61) and (62) which has non-empty intersection with $Z$, $g_z|_{Z \cap U}$ can be extended to a function $g_U \in C^\infty(U)$ such that $P(e^{-\eta}g_U) = 0$ holds on $U$. Hence, we can choose an open covering $\{\Omega_\alpha\}_{\alpha \in Z}$ of $\Omega$ such that for each $\Omega_\alpha$ intersecting $Z$ there is a smooth function $g_\alpha \in C^\infty(\Omega_\alpha)$ such that $g_\alpha|_{\Omega_\alpha \cap Z} = g_z|_{\Omega_\alpha \cap Z}$ and $e^{-\eta}g_\alpha \in \Gamma(\Omega_\alpha, \mathcal{S}_P)$.

For $\alpha, \beta \in Z$, set

$$h_{\alpha\beta} := \frac{g_\alpha - g_\beta}{e^{2\eta F}} \in \Gamma(\Omega_\alpha \cap \Omega_\beta, \mathcal{S}_P)$$

where $g_\alpha := 0$ if $\Omega_\alpha \cap Z = \emptyset$. It is easy to see that $\{\Omega_\alpha, h_{\alpha\beta}\}_{\alpha, \beta \in Z}$ defines an additive Cousin datum. By (1) of Theorem 4.1, one can find some $h_\alpha \in \Gamma(\Omega_\alpha, \mathcal{S}_P)$ for each $\alpha \in Z$ such that

$$h_\beta - h_\alpha = \frac{g_\alpha - g_\beta}{e^{2\eta F}} \text{ on } \Omega_\alpha \cap \Omega_\beta (\alpha, \beta \in Z).$$

Thus, $\{g_\alpha + e^{2\eta}h_\alpha F\}_{\alpha \in Z}$ patches together to give an extension $g \in C^\infty(\Omega)$ of $g_Z$. By the construction of $g$, we know that on each $\Omega_\alpha$

$$P_j (e^{-\eta}g) = P_j (e^{-\eta}g_\alpha + e^{\eta}h_\alpha F)$$

$$= P_j (e^{\eta}F)$$

$$= e^{\eta}FP_j (h_\alpha) + h_\alpha P_j (e^{\eta}F)$$

$$= e^{\eta}h_\alpha P_j (F) = 0, \ 1 \leq j \leq r.$$  

The following proposition records our discussion on the extension of a first integral of $E_z$.

**Proposition 4.3.** Assume that $\Omega$ is $P$-convex, $P = (P_1, \cdots, P_r)$ is elliptic and that $F \in C^\infty(\Omega)$ is a solution of the homogenous system (47) satisfying $dF = 0$ on $Z := F^{-1}(0)$. Then, for each first integral $g_Z \in C^\infty(Z)$ of $E_z$ there exists a $g \in C^\infty(\Omega)$ such that $g|_Z = g_Z$ and $P(e^{-\eta}g) = 0$ on $\Omega$, where $E_z$ is the subbundle of $\mathcal{C}TZ$ defined by (63) and $\eta$ is a solution of (48).

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