ON THE PAIR CORRELATION DENSITY FOR HYPERBOLIC ANGLES

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Abstract. Let $\Gamma \leq \text{PSL}_2(\mathbb{R})$ be a lattice and $\omega \in \mathbb{H}$ a point in the upper half plane. We prove the existence and give an explicit formula for the pair correlation density function for the set of angles between geodesic rays of the lattice $\Gamma \omega$ intersected with increasingly large balls centered at $\omega$, thus proving a conjecture of Boca-Popa-Zaharescu.

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1. Introduction

Let $\Gamma < G := \text{PSL}_2(\mathbb{R})$ be a lattice (i.e., a co-finite Fuchsian group) acting by isometries on the upper half plane $\mathbb{H}$ via fractional linear transformations. Given a fixed base point $\omega \in \mathbb{H}$, consider the set of directions of geodesic rays connecting $\omega$ to points $\gamma \omega$ lying in a growing hyperbolic norm ball in $\mathbb{H}$ (see Figure 1 for an illustration).

It is classical that these angles become equidistributed in $\mathbb{R}/(2\pi \mathbb{Z})$; see, e.g., [Boc07, Nic83, Goo83, RT10]. Going beyond equidistribution, the finer structure of a set of real numbers can be measured by

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Figure 1. Geodesic rays connecting $\omega = i$ to points $\gamma \omega$ for the uniform lattice $\Gamma$ corresponding to the spin cover of the special orthogonal group preserving the (Q-anisotropic, ternary, indefinite) form $x^2 + y^2 - 3z^2$. A fundamental domain for $\Gamma$ is shaded.

(among other statistics) its pair correlation, measuring the distribution of spacings at distances of mean order. Such spacing statistics have been studied by many authors for many naturally occurring sequences arising in mathematical physics, analysis, and number theory, both experimentally and theoretically. In particular, for the Euclidean analogue of this problem, the pair correlation, as well as other spacing statistics, were studied in [BCZ00, BZ05, BZ06, MS10, EMV13]. It is thus very surprising that the question of spacing statistics for the well-studied set of geodesic ray angles in the hyperbolic plane was considered for the first time only recently in [BPPZ12, BPZ13]. To state the results, we introduce some notation.

Fix a point $(\omega, \nu) \in T^1\mathbb{H}$ in the unit tangent bundle. For any $g \in G$ with $g\omega \neq \omega$, let

$$\theta_g = \theta_g(\omega, \nu) \in \mathbb{R}/(2\pi\mathbb{Z}) \quad (1.1)$$

be the signed angle between the vector $\nu$ and the tangent vector at $\omega$ of the directed geodesic connecting $\omega \rightarrow g\omega$, see Figure 2. When $g\omega = \omega$, set $\theta_g = [0]$.

Define

$$\|g\|^2 := 2 \cosh d(\omega, g\omega), \quad (1.2)$$
Figure 2. The angle $\theta_g$ is the directed angle between $\nu$ and the tangent vector at $\omega$ of the geodesic connecting $\omega$ to $g\omega$.

where $d(\cdot, \cdot)$ is the hyperbolic distance. Setting

$$B_Q := \{ g \in G : \|g\| < Q \},$$

and normalizing $dg$ so that

$$\text{vol}(B_Q) \sim \pi Q^2, \quad (Q \to \infty)$$

it is well-known that

$$\#\Gamma \cap B_Q \sim \frac{\text{vol}(B_Q)}{V_\Gamma} \sim \frac{\pi Q^2}{V_\Gamma}, \quad (Q \to \infty)$$

where

$$V_\Gamma := \text{vol}(G/\Gamma).$$

Hence for $\gamma \in \Gamma \cap B_Q$, the average spacing of the angles $\theta_\gamma$ is

$$\frac{2\pi}{\pi Q^2} V_\Gamma,$$

and we should consider, for fixed $\xi > 0$ and $Q \to \infty$, the correlation of pairs of angles by defining

$$N_Q(\xi) := \frac{1}{2} \left| \left\{ (\gamma, \gamma') \in \Gamma^2 : \gamma \omega \neq \gamma' \omega, \ \|\gamma\|, \|\gamma'\| < Q, \ |\theta_\gamma - \theta_{\gamma'}| < \frac{2V_\Gamma}{\pi Q^2} \xi \right\} \right|.$$

The constant $\frac{1}{2}$ in front is to account for the symmetry in $\gamma, \gamma'$. Note that while $\nu$ is needed to define $\theta_\nu$, the difference $|\theta_\gamma - \theta_{\gamma'}|$, defined as the distance to $2\pi \mathbb{Z}$, is independent of $\nu$.

The pair correlation distribution function is then defined as

$$R_2(\xi) := \lim_{Q \to \infty} \frac{V_\Gamma}{\pi Q^2} \cdot N_Q(\xi),$$

if the limit exists. If $R_2$ is moreover differentiable, then its derivative defines the pair correlation density,

$$g_2(\xi) := \frac{d}{d\xi} R_2(\xi).$$
Figure 3. The empirical pair correlation density \( g_2(\xi) \) for \( \omega \) and \( \Gamma \) as in Figure 1 (with \( Q = 2000 \)), plotted against the function given on the right side of (1.10). The fundamental domain in Figure 1 is a hyperbolic octagon with all right angles, hence \( V_\Gamma = 2\pi \) by Gauss-Bonnet. The cardinality of \( \Gamma \cap B_Q \) is 2000.914, which beautifully matches (1.5).

The difficulty in this setting of determining the pair correlation density is illustrated in Figure 3. The main goal of this paper is to explain this picture.

**Theorem 1.9.** Let \( \Gamma < G \) be any lattice and \( \omega \in \mathbb{H} \) be any fixed base point. The limit defining the pair correlation function \( R_2(\xi) \) in (1.7) exists, and is moreover differentiable. The density function defined in (1.8) is given by the formula

\[
g_2 \left( \frac{\xi}{V_\Gamma} \right) = \frac{V_\Gamma}{2\pi} \sum_{M \in \Gamma} f_\xi(\ell(M)). \tag{1.10}
\]

Here

\[
\ell(M) := d(\omega, M\omega) = \operatorname{arccosh} \left( \frac{\|M\|^2}{2} \right), \tag{1.11}
\]

and, on writing

\[
A = \cosh \ell, \quad B = \sinh \ell, \quad C = 2 \sinh(\ell/2) = \sqrt{2(A - 1)}, \tag{1.12}
\]
the function $f_\xi$ is given by

$$f_\xi(\ell) := \frac{2}{\xi^2} \begin{cases} 
\ell, & \text{if } B \leq \xi, \\
\ell + \log(1 + \xi^2) - 2\log \left( A + \sqrt{B^2 - \xi^2} \right), & \text{if } B \geq \xi \geq C, \\
\ell - \log \left( A + \sqrt{B^2 - \xi^2} \right), & \text{if } C \geq \xi.
\end{cases}$$  \hspace{1cm} (1.13)

Furthermore, there exists some $\delta > 0$, depending only on the spectral gap for $\Gamma$, so that, for fixed $\xi > 0$ and $Q \to \infty$, we have

$$N_Q(\xi) = \frac{\pi Q^2}{V_\Gamma} R_2(\xi) + O(\xi^{2-\delta}).$$  \hspace{1cm} (1.14)

Remark 1.15. This confirms a conjecture due to Boca, Popa, and Zaharescu [BPZ13, Conjecture 1], who proved Theorem 1.9 for the special case $\Gamma = \text{SL}_2(\mathbb{Z})$ and $\omega$ an elliptic point. (They did not give a rate, though their proof is in principle effective.) It is remarkable that their proof does not use any spectral theory (and is based instead on repulsion arguments for Farey tessellations and Weil’s bound for Kloosterman sums), the downside being that it does not apply to general lattices $\Gamma$, or base points $\omega$ other than $i$ or $e^{i\pi/3}$. It is not surprising, then, that our approach is completely different from theirs.

Remark 1.16. Note that $f_\xi(\ell)$ is continuous, see Figure 4 for a plot. For given $\xi_0 > 0$, we have uniformly over $\xi \in (0, \xi_0]$ the bound

$$f_\xi(\ell) \ll_{\xi_0} \frac{1}{e^{2\ell}}, \quad \ell \to \infty,$$  \hspace{1cm} (1.17)

whence the sum defining (1.10) easily converges, since $e^{2t(M)} \asymp \|M\|^4$.

Remark 1.18. The sum on $M \in \Gamma$ given in (1.10) can be reformulated via the trace formula in terms of the spectral expansion of $L^2(\Gamma \backslash \mathbb{H})$. 

Figure 4. Typical plots of the function $f_\xi(\ell)$ in (1.13).
but it does not seem to be expressible in a more intrinsic way that does not depend so explicitly on either the norm or Laplace spectrum of $\Gamma$. This explains the complicated function whose graph is illustrated in Figure 3: each $M \in \Gamma$ contributes a peak from $f_\xi$ in Figure 4b to the sum in (1.10).

Remark 1.19. While the limiting pair-correlation is highly non-universal, depending critically on both $\Gamma$ and $\omega$, it does have (at least) one trait of universality: the method of proof of Theorem 1.9 easily extends to give the following more general statement. Let $\mathcal{I} \subset \mathbb{R}/(2\pi\mathbb{Z})$ be any fixed subinterval, and consider the pair-correlation restricted to $\mathcal{I}$, that is, define

$$\mathcal{N}_Q^{(\mathcal{I})}(\xi) := \frac{1}{2} \left| \left\{ (\gamma, \gamma') \in \Gamma^2 : \gamma \omega \neq \gamma' \omega, \theta_\gamma, \theta_{\gamma'} \in \mathcal{I}, \|\gamma\|, \|\gamma'\| < Q, |\theta_\gamma - \theta_{\gamma'}| < \frac{2V_{\Gamma}^{Q^2} \xi}{\pi} \right\} \right|.$$ 

Then, in analogy with (1.7), the limit as $Q \to \infty$ of $\frac{2\pi}{|\mathcal{I}|} \cdot \frac{V_{\Gamma}}{\pi Q^2} \cdot \mathcal{N}_Q^{(\mathcal{I})}(\xi)$ exists, and is also equal to the same function $R_2(\xi)$, independently of the choice of $\mathcal{I}$. This generalization, kindly suggested to us by both Jens Marklof and a referee, has the following interpretation: the pair correlation is the same regardless of which part of the “sky” we observe from our “planet” $\omega$.

Remark 1.20. In [BPPZ12], another expression for $g_2$ is given, again for the case $\Gamma = \text{SL}_2(\mathbb{Z})$ and $\omega = i$, in terms of lengths of reciprocal geodesics on the modular surface. More generally, keeping the base point $\omega = i$, if we assume that $\Gamma$ is invariant under transpose and there is another lattice $\Gamma'$ such that the matrices $A = M'M$ with $M \in \Gamma$ are all the symmetric matrices in $\Gamma'$, then any sum over any function $f(\ell(M))$ with $M \in \Gamma$ can be written as a sum over $f(\ell(C)/2)$, where $C$ runs over closed geodesics in $\Gamma' \setminus \mathbb{H}$ passing through $i$. Explicitly:

$$\sum_{M \in \Gamma} f(\ell(M)) = |\Gamma_\omega| \sum_{C} f(\ell(C)/2).$$

Here $\Gamma_\omega$ is the subgroup of $\Gamma$ which stabilizes $\omega$. Note that on the right we divide $\ell$ by 2, which compensates for the fact that the sum is over a much smaller set. For a different base point $\omega$, the same is true but the requirement that $\Gamma$ be transpose-invariant is replaced by a different involution corresponding to $\omega$.

Remark 1.21. We make no attempt to optimize the rate in (1.14), as can surely be done with some effort. Our point is simply that the method is completely effective, with a power gain. The value of $\delta$ coming from
our proof is given as follows. Let

\[ \Theta \in (0, \frac{1}{2}) \]

be a spectral gap for \( \Gamma \), that is, a number so that the first non-zero eigenvalue \( \lambda_1 \) of the hyperbolic Laplacian on \( L^2(\Gamma \backslash \mathbb{H}) \) satisfies

\[ \lambda_1 > \frac{1}{4} - \Theta^2. \]  

(1.22)

(If \( \Gamma \) is arithmetic, then \( \Theta = 7/64 \) is known [KS03].) Then (1.14) holds with any

\[ \delta < (1 - 2\Theta)/26. \]  

(1.23)

Remark 1.24. It is interesting to compare our result to analogous results in the Euclidian setting. In this case one fixes a Euclidian lattice \( \Lambda \subseteq \mathbb{R}^2 \) and studies the distribution of angles between line segments connecting the origin (or a different point \( \alpha \in \mathbb{R}^2/\Lambda \)) to lattice points contained in increasing domains of \( \mathbb{R}^2 \). Here the angles become equidistributed on the circle (independently on the choice of \( \alpha \)) but the fine scale statistics depend on the choice of \( \alpha \). In [EMV13], for \( \alpha \) satisfying certain diophantine properties the pair correlation was shown to be that of a Poisson process, in agreement with the average pair correlation previously computed in [BZ06]. On the other hand, for \( \alpha = 0 \), it is natural to consider primitive vectors in \( \mathbb{Z}^2 \), in which case the pair correlation density was explicitly computed in [BZ05] and is far from Poisson.

Remark 1.25. Returning to hyperbolic space, one can formulate an alternate version of the problem, similar to the Euclidian setting above with \( \alpha \neq 0 \). Fixing two base points \( \omega_1 \) and \( \omega_2 \), and \( g \in G \), consider the angle \( \theta_g(\omega_1, \omega_2) \) between some fixed direction \( \nu \) and the tangent vector at \( \omega_1 \) of the geodesic ray connecting \( \omega_1 \) to \( g\omega_2 \). The distribution of the angles \( \theta_\gamma(\omega_1, \omega_2) \) with \( \gamma \in \Gamma \) was studied in [Boc07] when the angles are ordered by \( d(\omega_1, \gamma\omega_1) \), and again in [RT10] when ordered by \( d(\omega_1, \gamma\omega_2) \). In the second ordering these angles become equidistributed with respect to Lebesgue measure (in fact, by conjugating the lattice this is reduced to the case of \( \omega_1 = \omega_2 \)). However, in the first ordering they become equidistributed with respect to a different measure, \( \rho_{\omega_1, \omega_2}(\theta) \text{d}\theta \), depending on the base points. It would thus be interesting to study the pair correlation also for the first ordering. (Note that when the angles themselves are not uniformly distributed, one must “renormalize” the pair correlation function to have mean spacing one everywhere.)
It is interesting to examine the boundary behavior of the pair correlation density function \( g_2(\xi) \). For \( \xi \to 0 \), it follows immediately from (1.10) and l'Hopital's rule that

\[
g_2(0) = \frac{V_\Gamma}{\pi} \sum_{M \in \Gamma} \frac{1}{e^{2\ell(M)} - 1},
\]

as observed in [BPZ13, (1.3)]. In particular, \( g_2(0) > 0 \) which is in contrast to the result in the Euclidean setting studied in [BZ05], where the pair correlation vanishes near zero. For the other extreme, \( \xi \to \infty \), in many natural settings, the pair correlation density function approaches 1; see Figure 3. We confirm the conjecture in [BPZ13, (1.4)] that this case is no different.

**Theorem 1.26.** As \( \xi \to \infty \),

\[
g_2(\xi) = 1 + O\left( \frac{1}{(1-2\Theta)^3} \right),
\]

where \( \Theta \in (0, \frac{1}{2}) \) is a spectral gap for \( \Gamma \).

1.1. Outline.

The method of proof, and the rest of the paper, proceed as follows. Following Boca-Pasol-Popa-Zaharescu [BPPZ12], we first replace \( \gamma' \) in \( N_Q(\xi) \) by the variable \( M = \gamma^{-1}\gamma' \), to measure how far \( (\gamma, \gamma') \) is "off-diagonal". Let the stabilizer of \( \omega \) be denoted by

\[
K_\omega = \text{Stab}_G(\omega) \cong \text{PSO}(2),
\]

which is a maximal compact subgroup of \( G \).

Then switching to the more convenient variable, \( \frac{\xi}{V_\Gamma} \), we may write

\[
N_Q(\frac{\xi}{V_\Gamma}) = \frac{1}{2} \sum_{M \in \Gamma, M \notin K} \left| \left\{ \gamma \in \Gamma : \|\gamma\|, \|\gamma M\| < Q, |\theta_\gamma - \theta_{\gamma M}| < \frac{2\xi}{Q^2} \right\} \right|.
\]

After conjugating \( \Gamma \), we may assume that \( (\omega, \nu) = (i, \uparrow) \), dropping all subscript \( \omega \)'s.

For each \( M \), let

\[
\mathcal{R}_M(Q, \xi) := \left\{ g \in G : \|g\|, \|g M\| < Q, |\theta_g - \theta_{g M}| < \frac{2\xi}{Q^2} \right\}, \quad (1.27)
\]

be the region of interest, so that

\[
N_Q(\frac{\xi}{V_\Gamma}) = \frac{1}{2} \sum_{M \in \Gamma, M \notin K} \#\Gamma \cap \mathcal{R}_M(Q, \xi).
\]
One can hope that
\[ \#\Gamma \cap R_M(Q, \xi) \sim \frac{\text{vol}(R_M(Q, \xi))}{V_\Gamma} \]
can be proved using automorphic tools (spectral theory and dynamics). This is indeed the case for \(|M|\) small, but for larger \(M\), this volume can be of such small size that spectral methods are hopeless; the error term dominates the volume (see Proposition 1.29 below). Instead, once we are far enough off-diagonal, the entire contribution should be treated as a remainder.

To this end, we introduce another parameter \(T = T(Q) \to \infty\), and break \(N_Q\) into “main” and “error” terms according to whether or not \(|M| < T\), writing
\[ N_Q(\frac{\xi}{V_\Gamma}) = \frac{1}{2} \sum_{\|M\| < T, \#\Gamma \cap R_M(Q, \xi)} + E_{Q,T}(\xi), \]
say.

After a few preliminary computations in §2, we turn our attention in §3 to individual \(M\)’s with \(|M| < T\), analyzing the volumes of \(R_M(Q, \xi)\). These are the two most technically challenging sections, and we use §3 to prove the following

**Proposition 1.28.** As \(Q \to \infty\), we have that
\[ \text{vol}(R_M(Q, \xi)) = Q^2 \int_0^\xi f_\xi(\ell(M))d\xi + O_\xi(|M|^2 Q^{2/3}), \]
where \(\ell(M)\) is defined in (1.11), and \(f_\xi(\ell)\) is given in (1.13).

The proposition is proved by a direct and delicate analysis of the region in \(G\) corresponding to \(R_M\). The estimate is evidently non-trivial only when \(|M| = o(Q^{2/3})\).

In section §4, equipped with our understanding of these volumes, we prove the following

**Proposition 1.29.** Recalling from (1.22) that \(\Theta \in (0, \frac{1}{2})\) is a spectral gap for \(\Gamma\), we have
\[ \#\Gamma \cap R_M(Q, \xi) = \frac{\text{vol}(R_M(Q, \xi))}{V_\Gamma} + O_\xi(Q^{17+\Theta} |M|^{16/9}), \]
as \(Q \to \infty\).

We are quite crude here in the estimation of the error, but it more than suffices for our applications, so we do not pursue the issue. The idea of the proof is a more-or-less standard smoothing and un-smoothing argument, though the execution of the method requires a bit of care.
Next we spend §5 disposing of the error, by proving the following

**Proposition 1.30.** For fixed $\xi > 0$ and $T < Q$,

$$E_{Q,T}(\xi) \ll_\xi Q^2 \cdot \frac{\log Q}{T^2},$$

as $T, Q \to \infty$.

This is proved by returning to $\gamma'$ and the double sum, estimating directly the number of $(\gamma, \gamma') \in \Gamma^2$ with

$$\|\gamma\|, \|\gamma'\| < Q, \quad \|\gamma^{-1} \gamma'\| \geq T, \quad \text{and} \quad |\theta_\gamma - \theta_{\gamma'}| < \frac{2\xi}{Q^2}.$$

Combining these ingredients, we prove Theorem 1.9 in §6. Theorem 1.26 is proved in §7 by observing that $g_2(\xi_{V\Gamma})$ is a multiple of an automorphic kernel

$$\sum_{M \in \Gamma} f_\xi(d(\omega, M\omega)),$$

and then using a standard argument to show that, in the limit $\xi \to \infty$, this kernel is asymptotic to

$$\frac{1}{V_T} \int_G f_\xi(d(\omega, g\omega))dg.$$

1.2. **Notation.**

We use the following standard notation. The symbol $f \sim g$ means $f/g \to 1$, and the notations $f \ll g$ and $f = O(g)$ are synonymous; moreover $f \asymp g$ means $f \ll g \ll f$. Unless otherwise specified, the implied constants may depend at most on $\Gamma$, which is treated as fixed. The letter $c$ is a positive constant, not necessarily the same at each occurrence. The symbol $1_{\{\cdot\}}$ is the indicator function of the event $\{\cdot\}$. The cardinality of a finite set $S$ is denoted $|S|$ or $\#S$.

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2. Preliminary Computations

In this section, we record a number of computations which will be useful in the sequel. Recall that \( \Gamma \) is an arbitrary lattice in \( G = \text{PSL}_2(\mathbb{R}) \), and we may assume \((\omega,\nu) = (i,\uparrow)\).

Define the (semi)groups

\[
K := \text{PSO}(2) = \left\{ k_\theta := \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} : \theta \in \mathbb{R}/(2\pi \mathbb{Z}) \right\},
\]

\[
A^+ := \left\{ a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \geq 0 \right\}.
\]

Taking a Cartan decomposition of \( G = KA^+K \) and writing (uniquely for \( g \notin K \))

\[
g = k_{\theta(g)}a_{t(g)}k_{\varphi(g)},
\]

we see that the angle in (1.1) is given simply by

\[
\theta_g(\omega,\nu) = \theta(g).
\]

Next, for \( \xi > 0, Q \to \infty \) and \( M \in \Gamma \), we study the region \( R_M(Q,\xi) \) given in (1.27). It will be convenient to parametrize these conditions with explicit coordinates.

**Lemma 2.3.** Fix \( M \in G, M \notin K \), and write \( M \) in the Cartan decomposition

\[
M = k_m a_\ell k_s,
\]

so that \( \ell = \ell(M) \) in the notation of (1.11). For \( g \in G, g \notin K \), take the decomposition \( G = KA^+Kk_m \), writing

\[
g = k_{\theta(g)}a_{t(g)}k_{\varphi(g)}k_{-m}.
\]

Then recalling the notation (1.12), we have

\[
\|g\|^2 = 2 \cosh t,
\]

\[
\|gM\|^2 = 2(A \cosh t + B \cos(\varphi) \sinh t),
\]

and

\[
\tan(\theta_gM - \theta_g) = \frac{B \sin(\varphi)}{A \sinh t + B \cos(\varphi) \cosh t}.
\]

**Proof.** The equation (2.6) is immediate from the definition (1.2). For (2.7), we observe that

\[
gM = (k_{\theta(g)}a_t k_{\varphi}k_{-m})(k_m a_\ell k_s) = k_{\theta(g)}(a_t k_\varphi a_\ell)k_s,
\]
and take norms:
\[ \|g_M\|^2 = \| a_t k_\varphi a_\ell \|^2 \]
\[ = 2(\cosh(\ell + t) \cos^2 \frac{\varphi}{2} + \cosh(\ell - t) \sin^2 \frac{\varphi}{2}) \]
\[ = 2(\cosh \ell \cosh t + \cos \varphi \sinh \ell \sinh t), \]
as claimed.

To recover \( \theta_{gM} = \theta_{gM}(\omega, \nu) \), or rather \( \theta_{gM} - \theta_g \), we need to compute the left “\( K \)” in the \( K A^+ K \) decomposition of \( (k_{\theta_g})^{-1} gM \). To accomplish this, we act on \( i \), letting
\[ z := (k_{\theta_g})^{-1} gM \cdot i = a_t k_\varphi a_\ell \cdot i, \]
and send \( \mathbb{H} \to \mathbb{D} \) via \( z \mapsto w = \frac{z - i}{z + i} \). Then the left hand side of (2.8) is equal to \( \Im(w) / \Re(w) \), and a computation shows that
\[ \tan(\theta_{gM} - \theta_g) = \frac{(e^{2\ell} - 1) e^t \sin(\varphi)}{(e^{2(\ell+t)} - 1) \cos^2(\frac{\varphi}{2}) - (e^{2\ell} - e^{2t}) \sin^2(\frac{\varphi}{2})} \]
\[ = \frac{\sinh \ell \sin(\varphi)}{\sinh t \cosh \ell + \cos(\varphi) \cosh t \sinh \ell^i}, \]
whence the claim follows. \( \square \)

Next we record an estimate of how much \( \theta(g), \varphi(g) \), and \( \|g\| \) change when \( g \) is multiplied by an element from \( KA_\delta K \), where
\[ A_\delta = \{ a_\ell : |\ell| < \delta \}. \]
Recalling the notation (1.2)–(1.3), observe that this set is the same as \( B_{\delta_1} = KA_\delta K \), (2.10)
where
\[ \delta_1^2 = 2 \cosh \delta. \] (2.11)

**Lemma 2.12.** Let \( g \in G \) with \( \|g\| > 3 \) and \( h \in B_{\delta_1} \). For small \( \delta > 0 \), when multiplying from the right we have
\[ \|gh\| = \|g\|(1 + O(\delta)), \quad t(gh) = t(g) + O(\delta), \] (2.13)
\[ |\theta_g - \theta_{gh}| = O(\frac{\delta}{\|g\|^2}), \] (2.14)
and when multiplying from the left,
\[ \|hg\| = \|g\|(1 + O(\delta)), \quad t(hg) = t(g) + O(\delta), \] (2.15)
\[ |\varphi(g) - \varphi(hg)| = O(\frac{\delta}{\|g\|^2}). \] (2.16)

**Proof.** First note that \( \theta(g^{-1}) = \pi - \varphi(g) \), since to remain in \( A^+ \) we must write
\[ (k_\theta a_t k_\varphi)^{-1} = k_{-\varphi}a_{-t}k_{-\theta} = k_{-\varphi}a_t k_{-\pi} k_{-\theta}. \]

Then using this and \( \|g\| = \|g^{-1}\| \), it is enough to prove the results when \( h \) acts on the right.

Now, write \( g = k_\theta a_t k_\varphi \) with \( t > 1 \) and \( h = k_s a_s k_\varphi \) with \( |s| < \delta \) so that, say, \( gh = k_\theta a_t k_\varphi a_s k_\varphi \). Using (2.7) we have
\[ \|gh\|^2 = \cosh(t) \cosh(s) + 2 \cos(\varphi) \sinh(t) \sinh(s) \]
\[ = \|g\|^2(1 + O(\delta^2)) + O(\delta\|g\|^2) \]
\[ = \|g\|^2(1 + O(\delta)), \]
which, after taking square roots, implies the first equality in (2.13). For the second, we have from the above that
\[ 2 \cosh t(gh) = 2 \cosh t(g) + O(\delta\|g\|^2), \]
and hence
\[ t(gh) = \arccosh(\cosh t(g) + O(\delta\|g\|^2)) = t(g) + O\left(\frac{\delta\|g\|^2}{\sqrt{\cosh^2 t(g) - 1}}\right). \]

This implies (2.13) since \( t(g) > 1 \).

Finally, we see from (2.8) that
\[ |\tan(\theta_g - \theta_{gh})| = \frac{|\sinh(s) \sin(\varphi)|}{|\sinh t \cosh s + \cos(\varphi) \cosh t \sinh s|} < \frac{\delta}{2 \sinh t - \delta \cosh(t)} < \frac{\delta}{\|g\|^2 \tanh(t) - \delta/2} \ll \frac{\delta}{\|g\|^2}. \]
This implies (2.14) by Lemma 2.9.

Note that if we multiply \( g \) from the right by an element of \( B_\delta \), we clearly have no control over how much \( \varphi(g) \) changes. Instead, for \( \delta > 0 \) small, we define a small \( \delta \)-ball

\[
D_\delta := K_\delta A_\delta K_\delta,
\]

where

\[
K_\delta := \{ k \theta \in K : |\theta| < \delta \}.
\]

The next lemma gives the desired control.

**Lemma 2.18.** Let \( g \in G \) with \( \|g\| > 3 \) and \( h \in D_\delta \). For small \( \delta > 0 \), when multiplying from the right we have

\[
\varphi(gh) = \varphi(g) + O(\delta).
\]

**Proof.** It is clearly enough to consider \( h = a_s \in A_\delta \). We will show that \( \theta(gh) = \theta(g) + O(\delta) \), from which (2.19) follows on taking inverses. Writing \( g = k_\theta a_t k_* \), we again study the angle in the disk model \( \mathbb{D} \), setting \( z = hg \cdot i \) and \( w = (z - i)/(z + i) \). As before, a calculation shows that

\[
\cos(\theta(gh)) = \cos \theta + \frac{\tan s \coth t}{\sqrt{(\cos \theta + \tanh s \coth t)^2 + \sin^2 \theta}} = \cos \theta + O(\delta),
\]

where we used that \( t > 1 \). A similar identity holds for \( \sin(\theta(gh)) \), whence we are done. \( \square \)

Next we estimate how equations (2.6)–(2.8) are affected under simultaneous left-\( B_\delta \) and right-\( D_\delta \) perturbations.

**Lemma 2.20.** Fix \( g, M \in G \), \( M \notin K \), with \( \|g\| \geq 10\|M\| \). Then, for any \( g_1 \in B_\delta \), \( g D_\delta \), with sufficiently small \( \delta > 0 \), we have

\[
\|g_1M\|^2 = \|gM\|^2 + O(\|g\|^2\|M\|^2), \quad (2.21)
\]

and

\[
\tan(\theta_{g_1} - \theta_{g_1M}) = \tan(\theta_g - \theta_{gM}) + O\left( \frac{\delta\|M\|^4}{\|g\|^2} \right). \quad (2.22)
\]

**Proof.** First we note that, on writing \( g_1 = h_1gh_2 \) with \( h_1 \in B_\delta \) and \( h_2 \in D_\delta \), we have

\[
\varphi(g_1) = \varphi(h_1gh_2) = \varphi(gh_2) + O(\delta/\|gh_2\|^2) = \varphi(g) + O(\delta), \quad (2.23)
\]

where we used (2.16), (2.13), (2.19), and \( \|g\| \gg 1 \).

Next to deal with the norm of \( gM \), recall again the notation (1.12) and, in light of (2.7), consider the function

\[
F_1(\varphi, t) = 2(A \cosh t + B \cos(\varphi) \sinh t).
\]
Then writing \( g_M = k_0 a_\varphi k_\ell k_* \) and \( g_1 M = k_0 a_\varphi k_\ell k_* \), we have
\[
\|gM\|^2 = F_1(\varphi, t), \quad \text{and} \quad \|g_1 M\|^2 = F_1(\varphi, t_1).
\]
The partial derivatives \( |\frac{\partial F_1}{\partial \varphi}(\varphi, t)| \) and \( |\frac{\partial F_1}{\partial t}(\varphi, t)| \) are easily seen to be bounded by \( O(A \cosh(t)) \). So using (2.13) and (2.15) that \( t_1 = t + O(\delta) \), together with (2.23), we see that
\[
\|g_1 M\|^2 - \|gM\|^2 = |F_1(\varphi_1, t_1) - F_1(\varphi, t)| \ll \delta \|M\|^2 \|g\|^2,
\]
giving (2.21).

Similarly, to deal with \( \tan(\theta gM - \theta g) \) we consider from (2.8) the function
\[
F_2(\varphi, t) = \frac{B \sin(\varphi)}{A \sinh t + B \cos(\varphi) \cosh t},
\]
so that
\[
\tan(\theta gM - \theta g) = F_2(\varphi, t).
\]
We will show that both partial derivatives \( \frac{\partial F_2}{\partial \varphi}(\varphi, t) \), \( \frac{\partial F_2}{\partial t}(\varphi, t) \) are bounded by \( O(\frac{e^\ell}{\cosh(t)}) \) implying that
\[
F_2(\varphi_1, t_1) = F_2(\varphi, t) + O(\delta \|M\|^4 \|g\|^2).
\]

To bound the partial derivatives, a simple calculation gives
\[
\left| \frac{\partial F_2}{\partial \varphi}(\varphi, t) \right| = \frac{1}{\cosh(t)} \cdot \frac{\tanh^2 \ell}{\tanh^2 t} \cdot \frac{|1 + \frac{\tanh t}{\tanh \ell} \cos(\varphi)|}{|1 + \frac{\tanh \ell}{\tanh t} \cos(\varphi)|^2}.
\]
We have assumed that \( \|g\| > \|M\| > 1 \); hence we have \( t > \ell > 0 \), and
\( 0 < \frac{\tanh \ell}{\tanh t} < 1 \). For \( X \in (0, 1) \), the function
\[
\frac{|1 + \frac{1}{X} \cos(\varphi)|}{|1 + X \cos(\varphi)|^2}
\]
is maximized at \( \cos(\varphi) = -1 \), with maximum value \( \frac{1}{X(1-X)} \). Hence
\[
\left| \frac{\partial F_2}{\partial \varphi}(\varphi, t) \right| \leq \frac{1}{\cosh(t)} \cdot \frac{\tanh \ell}{\tanh t - \tanh \ell}.
\]
Assuming further that \( \|g\| \geq 10 \|M\| \) gives \( t > \ell + 1 \), and
\[
\tanh t - \tanh \ell > \tanh(\ell + 1) - \tanh \ell > \frac{1}{2} e^{-2\ell}, \quad (2.24)
\]
whence
\[
\left| \frac{\partial F_2}{\partial \varphi}(\varphi, t) \right| < \frac{2e^{2\ell}}{\cosh(t)},
\]
as desired.
For the second partial derivative, we have
\[
\left| \frac{\partial F_2}{\partial t} (\varphi, t) \right| = \frac{|\sin(\varphi)|}{\sinh t} \cdot \frac{\tanh \ell}{\tanh t} \cdot \frac{|1 + \tanh^2 \ell \tanh t \cos(\varphi)|}{|1 + \tanh \ell \tanh t \cos(\varphi)|^2}.
\]

It is easy to see that if \(0 < Y < X < 1\) with
\[
1 < \frac{2X^2}{Y(1+X)}, \tag{2.25}
\]
then the function
\[
\frac{|1 + \frac{Y}{X} \cos(\varphi)|}{|1 + X \cos(\varphi)|^2}
\]
attains its maximum value of
\[
\frac{1 - \frac{Y}{X}}{(1 - X)^2}
\]
at \(\cos(\varphi) = -1\). Letting \(Y = \tanh^2 \ell\) and \(X = \tanh \ell \tanh t\), it is clear that \(0 < Y < X < 1\). We also have
\[
1 < \frac{2}{\tanh t (\tanh t + 1)} < \frac{2}{\tanh t (\tanh t + \tanh \ell)} = \frac{2 \left( \frac{\tanh \ell}{\tanh t} \right)^2}{\tanh^2 \ell (1 + \frac{\tanh \ell}{\tanh t})},
\]
whence (2.25) is satisfied. We can thus estimate
\[
\left| \frac{\partial F_2}{\partial t} (\varphi, t) \right| \leq \tanh \ell \cdot \frac{1 - \tanh \ell \tanh t}{\cosh t} \cdot \frac{1}{(\tanh t - \tanh \ell)^2}.
\]

Now we use (2.24) and
\[
1 - \tanh \ell \tanh t < 1 - \tanh \ell \tanh(\ell + 1) < 3e^{-2\ell}
\]
to finish the proof of (2.22).

We conclude this section with a few computations regarding the function \(f_\xi\) in (1.13); these are needed for the proof of Theorem 1.26.

**Lemma 2.26.** As \(\xi \to \infty\) we have
\[
\int_G f_\xi(\ell(g)) \, dg = 2\pi + O \left( \frac{1}{\xi^2} \right), \tag{2.27}
\]
and for any fixed \(\alpha \in (0, 1)\),
\[
\int_G f_\xi(\ell(g)) \|g\|^{-\alpha} \, dg \ll_{\alpha} \frac{1}{\xi^\alpha}. \tag{2.28}
\]
Proof. In the $KA^+K$ coordinates (2.1), we have $\ell(g) = t(g)$ and a computation shows that the assumption (1.4) forces the normalization
\[ dg = \frac{1}{2\pi} d\theta \sinh t \, dt \, d\varphi. \]
We thus have that
\[ \int_G f_\xi(\ell(g)) dg = 2\pi \int_0^\infty f_\xi(t) \sinh(t) \, dt \]
\[ = -2\pi \int_0^\infty f_\xi'(t) \cosh(t) \, dt \]
The derivative, $f_\xi'(\ell)$, is given by
\[ f_\xi'(\ell) = \frac{2}{\xi^2} \times \begin{cases} 
1, & \text{if } \ell < \ell_1(\xi), \\
1 - \frac{2 \sinh(\ell)}{\sqrt{\sinh^2(\ell) - \xi^2}}, & \text{if } \ell_1(\xi) < \ell < \ell_2(\xi), \\
1 - \frac{\sinh(\ell)}{\sqrt{\sinh^2(\ell) - \xi^2}}, & \text{if } \ell > \ell_2(\xi),
\end{cases} \]
where the two points of discontinuity $\ell_1(\xi)$ and $\ell_2(\xi)$ satisfy
\[ \sinh(\ell_1(\xi)) = 2 \sinh(\frac{\ell_2(\xi)}{2}) = \xi. \]
Plugging this into the integral, and bounding $f_\xi'(t) = O(\frac{1}{\sinh^2(t)})$ for $t > \ell_2(\xi)$, we obtain
\[ \int_G f_\xi(\ell(g)) \, dg = -\frac{4\pi}{\xi^2} \int_0^{\ell_2(\xi)} \cosh(t) \, dt \]
\[ + \frac{4\pi}{\xi^2} \int_{\ell_1(\xi)}^{\ell_2(\xi)} 2 \sinh(t) \cosh(t) \, dt + O\left(\frac{1}{\sinh(\ell_2(\xi))}\right) \]
\[ = \frac{4\pi}{\xi^2} \left(2 \sqrt{\sinh^2(\ell_2(\xi)) - \xi^2} - \sinh(\ell_2(\xi))\right) + O\left(\frac{1}{\sinh(\ell_2(\xi))}\right) \]
\[ = \frac{4\pi \sinh(\ell_2(\xi))}{\xi^2} + O\left(\frac{1}{\sinh(\ell_2(\xi))}\right) = 2\pi + O\left(\frac{1}{\xi^2}\right). \]
For the second statement, similarly
\[ \int_G f_\xi(\ell(g)) \|g\|^{-\alpha} \, dg = 2\pi \int_0^\infty f_\xi(t) \frac{\sinh(t)}{\cosh^{\alpha/2}(t)} \, dt \]
\[ \ll_{\alpha} \int_0^{\ell_2(\xi)} f_\xi'(t) \cosh^{1-\alpha/2}(t) \, dt + \frac{1}{\xi^2} \]
\[ \ll \frac{e^{(1-\alpha/2)\ell_2(\xi)}}{\xi^2} + \frac{1}{\xi^2} \ll \xi^{-\alpha} \]
as claimed. \qed
We will also require the following estimate, recording how much $f_\xi$ changes under a small perturbation in $\ell$.

**Lemma 2.32.** Let $\delta \in (0,1]$ and let $\ell, \ell' > 0$ satisfy $|\ell - \ell'| < \delta$. Then for $\xi > 1$, we have

$$|f_\xi(\ell) - f_\xi(\ell')| \ll \begin{cases} \frac{\delta}{\xi^2}, & \ell < \ell_1(\xi) - \delta, \\ \frac{\sqrt{\delta}}{\xi^2}, & \ell_1(\xi) - \delta \leq \ell \leq \ell_1(\xi) + 1, \\ \frac{\delta}{\xi^2}, & \ell_1(\xi) + 1 \leq \ell \leq \ell_2(\xi) + 1, \\ \frac{\delta}{\sinh^2(\ell)}, & \ell \geq \ell_2(\xi) + 1. \end{cases}$$

(2.33)

**Proof.** The derivative $f_\xi'(\ell)$ blows up at $\ell_1(\xi)$, but away from that point, a simple estimate using (2.30) gives

$$|f_\xi'(\ell)| \ll \begin{cases} \frac{1}{\xi^2}, & \ell < \ell_1, \\ \frac{\sqrt{\delta}}{\xi^2}, & \ell_1 + \delta < \ell < \ell_1 + 1, \\ \frac{1}{\xi^2}, & \ell_1 + 1 < \ell < \ell_2, \\ \frac{1}{\sinh^2(\ell)}, & \ell > \ell_2. \end{cases}$$

(2.34)

When $\ell \in (\ell_1 - 2\delta, \ell_1 + 2\delta)$ is close to the singular point, we use the crude bound

$$|f_\xi(\ell) - f_\xi(\ell')| \leq |f_\xi(\ell_1) - f_\xi(\ell_1 - 3\delta)| + |f_\xi(\ell_1 + 3\delta) - f_\xi(\ell_1)|.$$

The first term is bounded by $O(\frac{\delta}{\xi^2})$. For the second term, using (1.13) together with the estimates

$$\cosh(\ell_1 + \delta) = \sqrt{\xi^2 + 1} + O(\delta \xi), \quad \sinh(\ell_1 + \delta) = \xi + O(\delta \xi),$$

we get that

$$|f_\xi(\ell_1 + 3\delta) - f_\xi(\ell_1)| \ll \frac{\delta}{\xi^2}.$$

For the remaining cases, $|\ell - \ell_1| > 2\delta$, hence $|\ell' - \ell_1| > \delta$, and we can bound $|f_\xi(\ell) - f_\xi(\ell')| \leq \delta |f_\xi'(t)|$ with $t$ between $\ell$ and $\ell'$. The result follows immediately from (2.34). \qed
ON THE PAIR CORRELATION DENSITY FOR HYPERBOLIC ANGLES

3. Computing Volumes

In this section we asymptotically compute the volume of the region $R_M(Q, \xi)$ in (1.27) to prove Proposition 1.28. Recall the notation in (1.12) that $A = \cosh(\ell)$, $B = \sinh(\ell)$, and $C = 2\sinh(\ell/2)$, where $\ell = \ell(M)$ as in (1.11). We first prove the following asymptotic formula.

Proposition 3.1. For $M \in \Gamma, M \not\in K$, we have

$$\text{vol}(R_M(Q, \xi)) = Q^2 \int_{-1}^{1} \frac{|J_\xi(y)|}{\sqrt{1-y^2}} dy + O \left( \|M\|^2 Q^{2/3} \left( 1 + \frac{1}{\xi} \right) + \frac{\xi^2}{Q^2} \right),$$

as $Q \to \infty$, where $J_\xi(y) \subseteq [0, 1]$ is the interval defined by

$$J_\xi(y) := \left\{ x \in [0, 1] : \frac{B\sqrt{1-y^2}}{\xi(A+By)} \leq x \leq \frac{1}{A+By} \right\}.$$

Proof. We may assume that

$$\|M\| < Q^{2/3},$$

for otherwise the estimate is trivial. We introduce a large truncation parameter $X$ in the range

$$5\|M\|^2 = 10A < X < Q^2,$$

(3.5)

to be chosen later in (3.17), and let

$${\mathcal{R}}_M(Q, \xi) := R_M(Q, \xi) \cap \{ g \in G : \|g\|^2 > X \},$$

where $R_M$ is the region in (1.27). We then clearly have

$$\text{vol}(R_M(Q, \xi)) = \text{vol}(\mathcal{R}_M(Q, \xi)) + O(X).$$

(3.6)

Observe that in the $KA^+K$ coordinates (2.1), $\mathcal{R}_M$ is left $K$-invariant, that is, poses no restriction on $\theta$. In the other coordinates, we use (2.6)–(2.8) and Lemma 2.9 to parametrize the region as

$$X < 2 \cosh t < Q^2,$$

$$2(A \cosh t + B \cos(\varphi) \sinh t) < Q^2,$$

$$\left| \frac{B \sin(\varphi)}{A \sinh t + B \cos(\varphi) \cosh t} \right| < \tan \left( \frac{2\xi}{Q^2} \right).$$

(3.7)

(3.8)

(3.9)

Thus, recalling our normalization (2.29), we have

$$\text{vol}(\mathcal{R}_M(Q, \xi)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{\infty} \int_{-\pi}^{\pi} \chi_{Q, \xi}(t, \varphi) \sinh(t) dt \, d\varphi \, d\theta = 2 \int_{0}^{\pi} \int_{0}^{\infty} \chi_{Q, \xi}(t, \varphi) \sinh(t) dt \, d\varphi$$

where $\chi_{Q, \xi}(t, \varphi)$ is the characteristic function of the region $\mathcal{R}_M(Q, \xi)$. This completes the proof of the proposition.
where \( \chi_{Q,\xi} \) denotes the indicator function of the set of
\[
(t, \varphi) \in [0, \infty) \times [-\pi, \pi)
\]
satisfying (3.7)–(3.9). (The condition in \( \varphi \) is invariant under \( \varphi \mapsto -\varphi \) so we may multiply by 2 and integrate over \( \varphi \in [0, \pi) \).

Define new coordinates given by
\[
\begin{align*}
x &= \frac{2 \cosh(t)}{Q^2}, \\
y &= \cos(\varphi).
\end{align*}
\]
(3.10)

In these coordinates, the conditions (3.7)–(3.9) are equivalent to
\[
\frac{X}{Q^2} < x < 1,
\]
\[
x(A + yBz) < 1,
\]
and
\[
\frac{B \sqrt{1 - y^2}}{|Az + yB|} < \frac{Q^2}{2} \tan\left(\frac{2\xi}{Q^2}\right),
\]
(3.12)

where
\[
z := \sqrt{1 - \frac{4}{Q^4 x^2}} \in (0, 1).
\]

For any \( y \in [-1, 1] \) let \( \tilde{J}_{Q,\xi}(y) \) denote the set of all \( x \in [\frac{X}{Q^2}, 1] \) satisfying (3.11) and (3.12); then
\[
\text{vol}(\tilde{R}_M(Q, \xi)) = Q^2 \int_{-1}^{1} |\tilde{J}_{Q,\xi}(y)| \frac{dy}{\sqrt{1 - y^2}}.
\]
(3.13)

Now for \( x \in \tilde{J}_{Q,\xi}(y) \), we estimate
\[
1 - z \leq \frac{4}{X^2},
\]
(3.14)

whence (3.11) can be replaced by
\[
x < \frac{1}{A + yBz} = \frac{1}{A + yB} + \frac{yB(1 - z)}{(A + yBz)(A + yB)}
\]
\[
= \frac{1}{A + yB} + O\left(\frac{\|M\|_6^6}{X^2}\right),
\]
(3.15)

where we estimated
\[
A + yBz \geq A - B \geq \frac{1}{2A} = \|M\|^{-2}.
\]

Next to replace (3.12), we use (3.5) and (3.14) to estimate
\[
Az + By \geq A - B - A(1 - z) \geq \frac{1}{2A} - \frac{4A}{X^2} \gg \frac{1}{A}.
\]
In particular, \( Az + By > 0 \), so we do not need the absolute values in (3.12). Now start with the estimate

\[
\frac{Q^2}{2} \tan \frac{2\xi}{Q^2} = \xi + O \left( \frac{\xi^3}{Q^4} \right),
\]

for the right hand side of (3.12), multiply both sides by \( x \), divide by \( \xi \), and use

\[
\frac{\sqrt{1 - y^2}}{A + By} \leq 1,
\]

to arrive at

\[
x > \frac{B \sqrt{1 - y^2}}{\xi (Az + yB)} + O \left( \frac{\xi^2}{Q^4} \right) = \frac{B \sqrt{1 - y^2}}{\xi (A + yB)} + O \left( \frac{\xi^2}{Q^4} + \frac{\|M\|^6}{\xi X^2} \right). \tag{3.16}
\]

Finally, combining (3.15) and (3.16), we obtain

\[
|\tilde{J}_{Q,\xi}(y)| = |J_{\xi}(y)| + O \left( \frac{X}{Q^2} + \frac{\xi^2}{Q^4} + \frac{\|M\|^6}{\xi X^2} \right),
\]

where \( J_{\xi}(y) \) is as defined in (3.3). Now to balance the error terms, we take

\[
X = \|M\|^2 Q^{2/3}, \tag{3.17}
\]

whence (3.5) is satisfied by (3.4) and taking \( Q \) large. Inserting these estimates into (3.13) and (3.6) gives (3.2), as claimed. \( \square \)

To prove Proposition 1.28, it remains to analyze the integral in the main term of (3.2). (Note that \( \|M\| = \sqrt{2 \cosh \ell(M)} \geq \sqrt{2} \), so the \( \xi^2/Q^2 \) error term in (3.2) may be dropped.)

The length \( |J_{\xi}(y)| \) is given explicitly by

\[
|J_{\xi}(y)| = \begin{cases} 
1 - \frac{B \sqrt{1 - y^2}}{\xi (A + By)}, & \text{if } y \in I_1(\xi), \\
\frac{1}{A + By} - \frac{B \sqrt{1 - y^2}}{\xi (A + By)}, & \text{if } y \in I_2(\xi), \\
0, & \text{if } y \in I_3(\xi),
\end{cases}
\]

where

\[
I_1(\xi) = \left\{ y \in [-1, 1] : \frac{B \sqrt{1 - y^2}}{\xi (A + By)} < 1 \leq \frac{1}{A + By} \right\},
\]

\[
I_2(\xi) = \left\{ y \in [-1, 1] : \frac{B \sqrt{1 - y^2}}{\xi (A + By)} < \frac{1}{A + By} \leq 1 \right\},
\]

and \( I_3(\xi) = [-1, 1] \setminus (I_1 \cup I_2) \).
Solving these inequalities, we get an explicit description for the intervals $I_1(\xi)$ and $I_2(\xi)$.

**Lemma 3.18.** Let $y = \lambda_{\pm}(\xi)$ denote the roots of the quadratic equation
\[ B^2(\xi^2 + 1)y^2 + 2AB\xi y + A^2\xi^2 - B^2 = 0, \] (3.19)
and if $\xi \leq B$, then set $y = \alpha(\xi)$ to be the non-negative solution to $B\sqrt{1 - y^2} = \xi$. We then have that
\[
I_1(\xi) = \begin{cases} 
[-1, \frac{1-A}{B}], & \text{if } B < \xi, \\
[-1, \lambda_-(\xi)) \cup (\lambda_+(\xi), \frac{1-A}{B}], & \text{if } C < \xi \leq B, \\
[-1, \lambda_-(\xi)), & \text{if } \xi \leq C,
\end{cases}
\] (3.20)
and
\[
I_2(\xi) = \begin{cases} 
[\frac{1-A}{B}, 1], & \text{if } B < \xi, \\
[\frac{1-A}{B}, -\alpha(\xi)) \cup (\alpha(\xi), 1], & \text{if } C < \xi \leq B, \\
(\alpha(\xi), 1], & \text{if } \xi \leq C.
\end{cases}
\] (3.21)

**Proof.** We analyze $I_1(\xi)$, the analysis of $I_2(\xi)$ being similar. The equation (3.19) corresponds to the boundary condition $\frac{B\sqrt{1 - y^2}}{\xi(A + By)} = 1$. This quadratic equation in $y$ is asymptotically positive, and has discriminant
\[ 4B^2(B^2 - \xi^2), \]
so the roots $\lambda_{\pm}(\xi)$ are only real if $B \geq \xi$. Thus if $B < \xi$, we only have the restriction $y \leq \frac{1-A}{B}$, giving the top line of (3.20).

When $\lambda_+(\xi) < \frac{1-A}{B}$, we obtain the middle line of (3.20), with the boundary condition $\lambda_+(\xi) = \frac{1-A}{B}$ being equivalent to
\[ \xi^2 = 2A - 2 = C^2. \]

The bottom line of (3.20) is then immediate. \qed

We may now complete the

**Proof of Proposition 1.28.** We define
\[ F_M(\xi) := \int_{-1}^{1} |J_\xi(y)| \frac{dy}{\sqrt{1 - y^2}}, \]
so that the main term of (3.2) is $Q^2F_M(\xi)$. It is easy to see that $F_M(\xi) \to 0$ as $\xi \to 0$, so it remains now to prove that
\[ \frac{d}{d\xi}F_M(\xi) = f_\xi(\ell(M)). \]
We write explicitly

\[
F(\xi) = \int_{I_1(\xi)} \left( 1 - \frac{B\sqrt{1-y^2}}{\xi(A+By)} \right) \frac{dy}{\sqrt{1-y^2}} \\
+ \int_{I_2(\xi)} \left( \frac{1}{A+By} - \frac{B\sqrt{1-y^2}}{\xi(A+By)} \right) \frac{dy}{\sqrt{1-y^2}},
\]

and compute the derivative in the three cases separately.

**Case I: \( \xi < C \).** Here we have

\[
F(\xi) = \int_{-1}^{\lambda_-(\xi)} \left( 1 - \frac{B\sqrt{1-y^2}}{\xi(A+By)} \right) \frac{dy}{\sqrt{1-y^2}} \\
+ \int_{\alpha(\xi)}^{1} \left( \frac{1}{A+By} - \frac{B\sqrt{1-y^2}}{\xi(A+By)} \right) \frac{dy}{\sqrt{1-y^2}},
\]

and hence

\[
F'(\xi) = \left( 1 - \frac{B\sqrt{1-(\lambda_-(\xi))^2}}{\xi(A+B\lambda_-(\xi))} \right) \frac{\lambda'_-(\xi)}{\sqrt{1-(\lambda_-(\xi))^2}} \\
- \left( \frac{1}{A+Ba(\xi)} - \frac{B\sqrt{1-\alpha(\xi)^2}}{\xi(A+Ba(\xi))} \right) \frac{\alpha'(\xi)}{\sqrt{1-\alpha(\xi)^2}} \\
+ \frac{1}{\xi^2} \left( \ln(A+B\lambda_-(\xi)) - \ln(A-B) - \ln(A+Ba(\xi)) + \ln(A+B) \right).
\]

Note that \( \lambda_\pm(\xi) \) and \( \alpha(\xi) \) are the precise points where \( \frac{B\sqrt{1-(\lambda_\pm(\xi))^2}}{\xi(A+B\lambda_\pm(\xi))} = 1 \) and \( \frac{1}{A+Ba(\xi)} = \frac{B\sqrt{1-\alpha(\xi)^2}}{\xi(A+Ba(\xi))} \), respectively. Thus the first two terms vanish, giving

\[
F'(\xi) = \frac{2}{\xi^2} \ln\left( \frac{A+B}{A+\sqrt{B^2-\xi^2}} \right),
\]

as claimed.
Case II: \( C < \xi < B \). Here we have
\[
F(\xi) = \int_{-1}^{\lambda_-} \left( 1 - \frac{B\sqrt{1 - y^2}}{\xi(A + By)} \right) \frac{dy}{\sqrt{1 - y^2}} \\
+ \int_{\lambda_+}^{1} \left( 1 - \frac{B\sqrt{1 - y^2}}{\xi(A + By)} \right) \frac{dy}{\sqrt{1 - y^2}} \\
+ \int_{-\alpha}^{\alpha} \left( \frac{1}{A + By} - \frac{B\sqrt{1 - y^2}}{\xi(A + By)} \right) \frac{dy}{\sqrt{1 - y^2}} \\
+ \int_{\alpha}^{1} \left( \frac{1}{A + By} - \frac{B\sqrt{1 - y^2}}{\xi(A + By)} \right) \frac{dy}{\sqrt{1 - y^2}}.
\]

When taking derivatives, the contribution of the end points cancel out as before, and we obtain
\[
F'(\xi) = \frac{1}{\xi^2} \left( \ln(A + B\lambda_-) - \ln(A - B) - \ln(A + B\lambda_+) \right) \\
+ \ln(A - B\alpha) - \ln(A + B\alpha) + \ln(A + B) \\
= \frac{2}{\xi^2} \ln\left( \frac{(A + B)(1 + \xi^2)}{(A + \sqrt{B^2 - \xi^2})^2} \right),
\]
as desired.

Case III: \( B < \xi \). Now we have
\[
F(\xi) = \int_{-1}^{1} \left( 1 - \frac{B\sqrt{1 - y^2}}{\xi(A + By)} \right) \frac{dy}{\sqrt{1 - y^2}} \\
+ \int_{-\alpha}^{\alpha} \left( \frac{1}{A + By} - \frac{B\sqrt{1 - y^2}}{\xi(A + By)} \right) \frac{dy}{\sqrt{1 - y^2}}.
\]

Here the only dependence on \( \xi \) is in the integral and we get
\[
F'(\xi) = \frac{1}{\xi^2} (\ln(A + B) - \ln(A - B)) = \frac{2\ell}{\xi^2}.
\]

We have now verified that the derivative \( F' \) agrees with \( f_\xi(\ell) \) away from the potential points of discontinuity, but \( f_\xi \) is continuous (cf. Remark 1.16) so we are done. This completes the proof of Proposition 1.28. \( \square \)
4. Relating the Counts to Volumes

The purpose of this section is to prove Proposition 1.29. While the idea of the proof is more-or-less standard, the region $R_M(Q, \xi)$ in (1.27) is not exactly well-rounded, resulting in a number of technical obstructions which must be overcome before the method works.

First some preliminaries. We may certainly assume that
\[ \|M\| < Q^{(1-2\Theta)/16}, \tag{4.1} \]
or else Proposition 1.29 is trivial.

Next, to address the issue of well-roundedness, we introduce a truncation parameter
\[ 1 < X < \frac{Q}{20\|M\|}, \tag{4.2} \]
and define
\[ R_M(Q, \xi; X) := \{ g \in R_M(Q, \xi) : \|g\| > \frac{Q}{X} \}. \tag{4.3} \]
The volume of the complement is clearly $O\left( \frac{Q^2}{X^2} \right)$.

Let $\delta > 0$ be some small parameter and let $\delta_1$ be related to $\delta$ by (2.11). Recall that the regions $B_\delta$ and $D_\delta$ are defined in (2.10) and (2.17), respectively. Let $R_M^+$ be the $B_{\delta_1} \times D_\delta$-thickening of $R_M$, that is,
\[ R_M^+(Q, \xi; X) := B_{\delta_1} \cdot R_M(Q, \xi; X) \cdot D_\delta. \tag{4.4} \]
Similarly, let $R_M^-$ be the set whose thickening is in $R_M$,
\[ R_M^-(Q, \xi; X) := \bigcap_{(h_1, h_2) \in B_{\delta_1} \times D_\delta} h_1 \cdot R_M(Q, \xi; X) \cdot h_2 \tag{4.5} \]
With the region $R_M$ slightly truncated, we can now prove the following well-roundedness statement.

**Lemma 4.6.** With $\xi > 0$ fixed, there are constants $c, c' > 0$, depending on $\xi$ and $\Gamma$, such that for
\[ \delta < c'\|M\|^{-2}, \tag{4.7} \]
we have
\[ R_M^+(Q, \xi; X) \subseteq R_M(Q(1 + c\delta\|M\|^2), \xi(1 + c\delta X^2\|M\|^4); 2X), \tag{4.8} \]
and
\[ R_M(Q, \xi; X) \subseteq R_M^-(Q(1 + c\delta\|M\|^2), \xi(1 + c\delta X^2\|M\|^4); 2X). \tag{4.9} \]
Proof. Let \( g_1 \in R_M^+(Q, \xi; X) \); then \( g_1 = h_1^{-1}g_2 \) with \( g \in R_M(Q, \xi; X) \), \( h_1 \in B_{\delta_1} \) and \( h_2 \in D_{\delta} \subset B_{\delta_1} \). The assumptions \( \|g\| > Q/X \) and (4.2) ensure that \( \|g\| > 20\|M\| \), so we are in position to use Lemmata 2.12 and 2.20. Recall that \( c \asymp 1 \) is a constant which can change from line to line, or even in the same line. Applying (2.21) gives
\[
\|g_1M\|^2 < \|gM\|^2 + c\delta\|g\|^2\|M\|^2 < Q^2(1+c\delta\|M\|^2)^2,
\]
so the thickening replaces \( Q \) by \( Q(1+c\delta\|M\|^2) \).

By (2.13) and (2.15), we have,
\[
\|g_1\|^2 = \|g\|^2(1 + O(\delta))^2,
\]
and hence (4.7) gives
\[
\frac{Q^2(1+c\delta\|M\|^2)^2}{(2X)^2} < \frac{Q^2(1-c\delta)^2}{X^2} < \|g_1\|^2 < Q^2(1+c\delta)^2 < Q^2(1+c\delta\|M\|^2)^2.
\]
Thus in thickening we may replace \( X \) by \( 2X \).

Lastly, (4.2) and (4.7) ensure that
\[
\frac{\delta\|M\|^4}{\|g\|^2} < c.
\]
Then by (2.22), together with Lemma 2.9 (note that \( \|g\| > 20\|M\| \) by (4.2)–(4.3)), we have
\[
|\theta_{g_1} - \theta_{g_1M}| \leq |\theta_g - \theta_{gM}| + c\delta\|M\|^4X^2/Q^2 < \frac{2\xi(1+c\xi\delta\|M\|^4X^2)}{Q^2}
\]
\[
< \frac{2\xi}{Q^2(1+c\delta\|M\|^2)^2(1 + c\delta\|M\|^4X^2)}.
\]
This proves (4.8), and the proof of (4.9) is similar. \( \square \)

Now we follow a standard procedure to compute the cardinality of \( \Gamma \cap R_M(Q, \xi; X) \).

**Proposition 4.10.** Let \( \xi > 0 \) be fixed. Recall from (1.22) that \( \Theta \in (0, \frac{1}{2}) \) is a spectral gap for \( \Gamma \), and from (1.6) that \( V_{1\Gamma} \) is the co-volume of \( \Gamma \). Then for any \( M \in \Gamma, \ M \notin \hat{K} \), assuming (4.2) we have
\[
\left| \#\Gamma \cap R_M(Q, \xi; X) - \frac{\text{vol}(R_M(Q, \xi))}{V_{1\Gamma}} \right| 
\]
\[
\ll \xi \frac{Q^2}{X^2} + Q^{(9+2\delta)/5}X^{8/5}\|M\|^{16/5} + \|M\|^2Q^{2/3},
\]
as \( Q \to \infty \).
Sketch of the proof. Let \( \delta > 0 \) be small enough that (4.7) is satisfied. Let \( \psi_1 = \psi_{1,\delta} \) be a spherical \( \delta \)-bump function about the origin in \( G \), that is, \( \psi_1 \) is smooth, non-negative, \( \int_G \psi_1 = 1 \), \( \psi_1(kgk') = \psi_1(g) \) and \( \text{supp} \psi_1 \subset B_\delta \). Also let \( \psi_2 = \psi_{2,\delta} \) denote a (non-spherical) \( \delta \)-bump function supported on \( D_\delta \).

For \( j = 1, 2 \), let

\[
\Psi_j(g) := \sum_{\gamma \in \Gamma} \psi_j(g\gamma),
\]

so that \( \Psi_1 \in L^2(K \backslash G/\Gamma) \) and \( \Psi_2 \in L^2(G/\Gamma) \). We can choose the bump functions so that

\[
\|\Psi_1\| \approx \frac{1}{\text{vol}(B_\delta)^{1/2}} \approx \frac{1}{\delta}, \quad \text{and} \quad S\Psi_2 \approx \frac{1}{\delta} \cdot \frac{1}{\text{vol}(D_\delta)^{1/2}} \approx \frac{1}{\delta^3}. \tag{4.12}
\]

Here \( S \) is a first-order Sobolev norm, defined as follows. Fix a basis \( X_1, X_2, X_3 \) for the Lie algebra \( g = \mathfrak{sl}_2(\mathbb{R}) \); then \( S\Psi = \max_{j=1,2,3} \|X_j.\Psi\| \).

Given \( c > 0 \) from Lemma 4.6 with \( c' \) small enough, let \( \delta_2 := 1 + c\delta\|M\|^2 \approx 1 \), and \( \delta_3 := 1 + c\delta X^2\|M\|^4 \ll X^2\|M\|^2 \). \tag{4.13}

Let \( \mathcal{F} = \mathcal{F}_{Q,\xi,X,M} \in L^2(K \backslash G/\Gamma \times G/\Gamma) \) be defined by

\[
\mathcal{F}(g, h) := \sum_{\gamma \in \Gamma} 1_{R_M(Q,\xi,X,M)}(g\gamma h^{-1}).
\]

We will prove an upper bound on the cardinality of \( \Gamma \cap R_M(Q,\xi;X) \), the lower bound being similar.

Using (4.9), an easy calculation shows that

\[
\# \Gamma \cap R_M(Q,\xi;X) \leq \# \Gamma \cap R_M^{-}(Q,\xi,\delta_3;2X) \leq \langle \mathcal{F}, \Psi_1 \otimes \Psi_2 \rangle, \tag{4.14}
\]

and that

\[
\langle \mathcal{F}, \Psi_1 \otimes \Psi_2 \rangle = \int_{R_M(Q,\xi,\delta_3;2X)} \langle \pi(g).\Psi_2, \Psi_1 \rangle \, dg, \tag{4.15}
\]

where \( \pi \) is the left-regular representation on \( G \).

To estimate the last integral, decompose each function as

\[
\Psi_j = \frac{1}{V_\Gamma} + \Psi_j^\perp,
\]

where \( \Psi_j^\perp \) is orthogonal to constants. Recall the well-known decay of matrix coefficients (see, e.g., [War72, CHH88, Sha99]; and in particular, [Ven10, §9.1.2]): for mean-zero functions \( F_1, F_2 \in L^2_0(G/\Gamma) \), we have

\[
\langle \pi(g).F_1, F_2 \rangle \ll \|g\|^{-1+2\Theta} S F_1 \cdot S F_2, \tag{4.16}
\]
where $\Theta$ is a spectral gap for $\Gamma$. We are being a bit crude here with the $F_j$ dependence in the error; as we are not trying to optimize exponents, we opt for a cleaner statement than best known. That said, when an $F_j$ is $K$-fixed, then its Sobolev norm can be replaced by its $L^2$-norm.

Applying (4.16), we have

$$\langle \pi(g) \Psi_2, \Psi_1 \rangle = \frac{1}{V\Gamma} + O(\|g\|^{1+2\Theta} \|\Psi_1\|_{S\Psi_2}) = \frac{1}{V\Gamma} + O\left(\|g\|^{1+2\Theta} \frac{1}{\delta^4}\right),$$

by (4.12). Inserting this into (4.15) and (4.14) gives

$$\#\Gamma \cap R_M(Q,\xi;X) \leq \text{vol}(R_M(Q\delta_2,\xi\delta_3;2X)) + O\left(\frac{1}{\delta^4} Q^{1+2\Theta}\right),$$

where we crudely estimated $R_M(Q\delta_2,\xi\delta_3;2X) \subset B_{2Q}$.

Using the estimate in Proposition 1.28 for the volumes, with the uniform in $\xi$ error terms given in (3.2), we see that

$$|\text{vol}(R_M(Q\delta_2,\xi\delta_3)) - \text{vol}(R_M(Q,\xi))|$$

$$= Q^2 \delta_2^2 \int_{\xi}^{\xi\delta_3} f_\xi(\ell(M))d\xi + Q^2(\delta_2 - 1) \int_{0}^{\xi} f_\xi(\ell(M))d\xi$$

$$+ O_\xi(\|M\|^2 Q^{2/3}) + O(\|M\|^2 (Q\delta_2)^{2/3}(1 + 1/(\xi\delta_3)) + (\xi\delta_3/Q)^2)$$

$$\ll \xi Q^2 (\delta_3-1) + Q^2 (\delta_2 - 1) + \|M\|^2 Q^{2/3} + \frac{X^2\|M\|^2}{Q^2},$$

where we used (4.7), (4.13), and (4.2).

Combining (4.17) with (4.18), we choose the optimal value

$$\delta = \frac{1}{Q^{(1-2\Theta)/5} X^{2/5} \|M\|^{4/5}}.$$ 

It is easy to see from (4.1) that (4.7) is satisfied.

We thus obtain

$$\#\Gamma \cap R_M(Q,\xi;X) \leq \frac{\text{vol}(R_M(Q,\xi))}{V\Gamma}$$

$$+ O_\xi \left(\frac{Q^2}{X^2} + \|M\|^2 Q^{2/3} + Q^{(9+2\Theta)/5} X^{8/5} \|M\|^{16/5}\right).$$

The lower bound is proved similarly, concluding the proof of (4.11). \qed

We are finally in position to give a
Proof of Proposition 1.29. We easily see that
\[ \#\Gamma \cap \mathcal{R}_M(Q, \xi) = \#\Gamma \cap \mathcal{R}_M(Q, \xi; X) + O \left( \frac{Q^2}{X^2} \right). \]
Combined with (4.11), we choose the optimal value
\[ X = \frac{Q^{(1-2\Theta)/18}}{\|M\|^{8/9}}. \]
Then (4.2) is satisfied by (4.1). We can also use (4.1) to bound
\[ \|M\|^{2}Q^{2/3} \leq \|M\|^{16/9}Q^{(17+2\Theta)/9}, \]
and the claim follows immediately.

Remark 4.19. For the generalization described in Remark 1.19 we need to replace the regions \( \mathcal{R}_M(Q, \xi) \) by \( \mathcal{R}_N(Q, \xi) = \{ g \in \mathcal{R}_M(Q, \xi) | \theta_g \in \mathcal{I} \} \), and similarly \( \mathcal{R}_M(Q, \xi; X) \) by \( \mathcal{R}_N(Q, \xi; X) \); when doing this the volumes change by a factor of \( |\mathcal{I}| \). These new regions are no longer left \( K \)-invariant, which requires a few small modifications to the proofs. In particular, in the smoothing process, (4.5) and (4.4), we need to replace \( B_\delta \) by \( D_\delta \), and in Lemma 4.6 we also need to enlarge the interval \( \mathcal{I} \) on the left hand side by \( O(\delta) \). Next, in the proof of Proposition 4.10 we need both \( \delta \)-bump functions to be non-spherical, resulting in a slightly worse power saving for the error term. After these modification the rest of the proof follows without a change.
5. Bounding the Error $\mathcal{E}$

Recall the notation in §1. In this section, we prove Proposition 1.30, estimating the “error” term $\mathcal{E}_{Q,T}(\xi)$. Recall that this is half the cardinality of the set

$$S = \left\{ (\gamma, M) \in \Gamma^2 : \|\gamma\|, \|\gamma M\| \leq Q, \|M\| \geq T, |\theta_\gamma - \theta_{\gamma M}| < \frac{2\xi}{Q^2} \right\}.$$

It will be more convenient to return to $\gamma' = \gamma M$, renaming

$$S = \left\{ (\gamma, \gamma') \in \Gamma^2 : \|\gamma\|, \|\gamma'\| \leq Q, |\theta_\gamma - \theta_{\gamma'}| < \frac{2\xi}{Q^2}, \|\gamma^{-1}\gamma'\| \geq T \right\}.$$

We first prove the following

**Lemma 5.1.** For $10 < Q_1 < Q$, and $\theta' \in [-\pi, \pi)$ fixed,

$$\# \left\{ \gamma \in \Gamma : Q_1 \leq \|\gamma\| < 2Q_1, |\theta_\gamma - \theta'| < \frac{2\xi}{Q^2} \right\} \ll \xi 1. \tag{5.2}$$

Recall that the implied constant above, as throughout, may depend on $\Gamma$ without further notice.

**Proof.** Fix $\delta = \delta(\Gamma) \asymp 1$ sufficiently small that

$$(K A_{2\delta} K) \cap \Gamma = \{1\}.$$

Letting $\delta_1$ be related to $\delta$ by (2.11), recall the region $B_{\delta_1} \subset G$ defined in (2.10).

Writing $\mathcal{L}$ for the left hand side of (5.2), we thicken each $\gamma$ on the right by $B_{\delta_1}$, giving

$$\mathcal{L} = \frac{1}{\text{vol}(B_{\delta_1})} \sum_{\gamma \in \Gamma, Q_1 \leq \|\gamma\| < 2Q_1, |\theta_\gamma - \theta'| < \frac{2\xi}{Q^2}} \text{vol}(\gamma B_{\delta_1}) \leq \frac{1}{\text{vol}(B_{\delta_1})} \text{vol} \left\{ g \in G : \|g\| \leq 3Q_1, |\theta_g - \theta'| \leq \frac{c\delta + 2\xi}{Q_1^2} \right\},$$

where we used (2.13) and (2.14). It is elementary to compute that this volume is $\ll \xi 1$, giving the claim (since $\delta \asymp 1$). \qed

We can now give a

**Proof of Proposition 1.30.**

Let $(\gamma, \gamma') \in S$ and write $\gamma = k_\theta a_t k$ and $\gamma' = k_{\theta'} a_{t'} k'$, so that, as in (2.7), we have

$$\|\gamma^{-1}\gamma'\|^2 = \|a_{-t} k_{\theta'} a_{t'}\|^2 = 2(\cosh(t - t') \cos^2 \left(\frac{\theta - \theta'}{2}\right) + \cosh(t + t') \sin^2 \left(\frac{\theta - \theta'}{2}\right)).$$
Since \((\gamma, \gamma') \in S\), we have \(|\theta - \theta'| \leq \frac{2\xi}{Q^2}\) and \(2 \cosh t, 2 \cosh t' \leq Q^2\), whence
\[
\|\gamma^{-1} \gamma'\|^2 = 2 \cosh(t - t') + O_\xi(1).
\]
Then for \(T \gg \xi\), the condition \(\|\gamma^{-1} \gamma'\| \geq T\) implies
\[
2 \cosh(t - t') > \frac{1}{2} T^2.
\]
Assuming \(t' \leq t\), we can relax this even further to \(t' < 2 \log(Q/T) + c\) and \(t < 2 \log Q\).

Hence \(E_{Q,T}(\xi) = \frac{1}{2} \#S \ll \#S', \text{ where}
\[
S' := \left\{ (\gamma, \gamma') \in \Gamma^2 : \|\gamma\| \leq Q, \|\gamma'\| \ll \frac{Q}{T}, |\theta_\gamma - \theta_{\gamma'}| < \frac{2\xi}{Q^2} \right\}.
\]
We sum \(\gamma'\) on the outside, and break the \(\gamma\) sum dyadically:
\[
\#S' \ll \sum_{\gamma' \in \Gamma} \left( 1 + \sum_{10 < Q_1 < Q \text{ dyadic}} \# \left\{ \gamma \in \Gamma : \|\gamma\| \asymp Q_1, |\theta_\gamma - \theta_{\gamma'}| < \frac{2\xi}{Q_1^2} \right\} \right)
\ll \log Q \left( \frac{Q}{T} \right)^2,
\]
where we used (5.2). This completes the proof. \(\square\)
6. Proof of Theorem 1.9

The purpose of this section is to combine the ingredients in §§3–5 to prove Theorem 1.9. Recall that
\[ \mathcal{N}_Q(\xi_{\Gamma}) = \frac{1}{2} \sum_{M \in \Gamma \setminus K} |\Gamma \cap R_M(Q, \xi)|. \]

For a parameter $T$ to be chosen later, we use Proposition 1.30 to write
\[ \mathcal{N}_Q(\xi_{\Gamma}) = \frac{1}{2} \sum_{M \in \Gamma \setminus K, \|M\| < T} |\Gamma \cap R_M(Q, \xi)| + O_\xi \left( Q^2 \frac{\log Q}{T^2} \right). \]

Applying Proposition 1.29 gives
\[ \mathcal{N}_Q(\xi_{\Gamma}) = \frac{1}{2} \sum_{M \in \Gamma \setminus K, \|M\| < T} \frac{\text{vol}(R_M(Q, \xi))}{V_{\Gamma}} + O_\xi \left( T^{34/9} Q^{(17+2\Theta)/9} + Q^2 \frac{\log Q}{T^2} \right). \]

Next apply Proposition 1.28 to obtain
\[ \mathcal{N}_Q(\xi_{\Gamma}) = \frac{Q^2}{2V_{\Gamma}} \sum_{M \in \Gamma \setminus K, \|M\| < T} \int_0^\xi f_\xi(\ell(M)) d\zeta + O_\xi \left( T^{34/9} Q^{(17+2\Theta)/9} + Q^2 \frac{\log Q}{T^2} \right). \]

Here we have dropped the error term $O(T^4 Q^{2/3})$ from Proposition 1.28, which will be of lower order. Finally, use (1.17) to estimate
\[ \sum_{M \in \Gamma \setminus K} \int_0^\xi f_\xi(\ell(M)) d\zeta = \sum_{M \in \Gamma} \int_0^\xi f_\xi(\ell(M)) d\zeta + O_\xi \left( T^2 \frac{1}{T^4} \right), \]
whence
\[ \mathcal{N}_Q(\xi_{\Gamma}) = \frac{Q^2}{2V_{\Gamma}} \sum_{M \in \Gamma} \int_0^\xi f_\xi(\ell(M)) d\zeta + O_\xi \left( T^{34/9} Q^{(17+2\Theta)/9} + Q^2 \frac{\log Q}{T^2} \right). \]

Setting optimally
\[ T = Q^{(1-2\Theta)/52}, \]
we obtain
\[ \mathcal{N}_Q(\xi_{\Gamma}) = \frac{Q^2}{2V_{\Gamma}} \sum_{M \in \Gamma} \int_0^\xi f_\xi(\ell(M)) d\zeta + O_\xi \left( Q^{2-\frac{(1-2\Theta)}{26}+\epsilon} \right), \]
thereby confirming (1.14) with the rate claimed in (1.23). The rest of Theorem 1.9 follows immediately.
7. Proof of Theorem 1.26

The purpose of this section is to prove that the pair correlation density $g_2(\xi)$ approaches 1 in the limit as $\xi \to \infty$. To do this we observe that $g_2(\frac{\xi}{V})$ is a multiple of an automorphic kernel

$$g_2(\frac{\xi}{V}) = \frac{V}{2\pi} K_\xi(1, 1),$$

where

$$K_\xi(g, h) = \sum_{M \in \Gamma} f_\xi(\ell(g M h^{-1})).$$

(7.1)

As the authors of [BPZ13] observed, had the function $f_\xi$ been smooth, the Selberg (pre) trace formula would imply that

$$K_\xi(1, 1) \sim \frac{1}{V} \int_G f_\xi(\ell(g)) dg,$$

and the result would follow by (2.27). The only difficulty is that the function $f_\xi$ in (1.13) is not differentiable at two points, so one cannot apply the trace formula directly. Nevertheless, using a standard smoothing and unsmoothing argument (similar to the proof of Proposition 4.10), we show the following

**Proposition 7.2.** As $\xi \to \infty$ we have

$$K_\xi(1, 1) = \frac{1}{V} \int_G f_\xi(\ell(g)) dg + O\left(\frac{1}{\xi^{(1-2\Theta)/3}}\right).$$

**Proof.** As in the proof of Proposition 4.10, let $\delta > 0$ be a small parameter to be chosen later, and let $\delta_1$ be related to $\delta$ by (2.11). Let $\psi = \psi_\delta$ denote a spherical bump function supported on $B_{\delta_1}$, and let

$$\Psi(g) = \sum_{\gamma \in \Gamma} \psi(g \gamma),$$

where, as in (4.12), we choose the bump function so that

$$\|\Psi\| \approx \frac{1}{\sqrt{\text{vol}(B_{\delta_1})}} \approx \frac{1}{\delta}.$$
Note that for $g, h \in B_{\delta_1}$ we have $\ell(gMh^{-1}) = \ell(M) + O(\delta)$ and using Lemma 2.32 we see that

$$|K_\xi(g, h) - K_\xi(1, 1)| \leq \sum_{M \in \mathcal{F}} |f_{\xi}(\ell(M)) - f_{\xi}(\ell(gMh^{-1}))|$$

$$\ll \sum_{\|M\| \ll \sqrt{\xi}} \frac{\sqrt{\delta}}{\xi^2} + \sum_{\|M\| \ll \xi} \frac{\delta}{\xi^2} + \sum_{\|M\| \gg \xi} \frac{\delta}{\|M\|^4}$$

$$\ll \frac{\sqrt{\delta}}{\xi} + \delta + \frac{\delta}{\xi^2} \ll \frac{\sqrt{\delta}}{\xi} + \delta.$$

Since $\psi$ is supported on $B_{\delta_1}$ and has integral one, we have that

$$K_\xi(1, 1) = \langle K_\xi, \Psi \otimes \Psi \rangle + O\left(\frac{\sqrt{\delta}}{\xi} + \delta\right). \quad (7.3)$$

On the other hand, unfolding as in (4.15), we obtain

$$\langle K_\xi, \Psi \otimes \Psi \rangle = \int_G f_\xi(\ell(g)) \langle \pi(g)\Psi, \Psi \rangle \, dg.$$ 

Decomposing $\Psi = \frac{1}{V_{\Gamma}} + \Psi^\perp$ together with (4.16), namely that

$$\langle \pi(g)\Psi^\perp, \Psi^\perp \rangle \ll \|\Psi\|^2\|g\|^{-1+2\Theta} \ll \delta^{-2}\|g\|^{-1+2\Theta},$$

gives

$$\langle K_\xi, \Psi \otimes \Psi \rangle = \frac{1}{V_{\Gamma}} \int_G f_\xi(\ell(g)) \, dg + O\left(\delta^{-2} \int_G f_\xi(\ell(g)) \|g\|^{-1+2\Theta} \, dg\right)$$

$$= \frac{1}{V_{\Gamma}} \int_G f_\xi(\ell(g)) \, dg + O\left(\frac{1}{\delta^2 \xi^{1-2\Theta}}\right) \quad (7.4)$$

where we used (2.28).

Finally, combining (7.3) and (7.4), with an optimal choice of

$$\delta = \frac{1}{\xi^{(1-2\Theta)/3}},$$

gives

$$K_\xi(1, 1) = \frac{1}{V_{\Gamma}} \int_G f_\xi(\ell(g)) \, dg + O(\xi^{(2\Theta-1)/3}),$$

as claimed. \qed

Theorem 1.26 now follows immediately from Proposition 7.2 together with (2.27).
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