Statistical estimation requires unbounded memory

Leonid (Aryeh) Kontorovich

today

Abstract

We investigate the existence of bounded-memory consistent estimators of various statistical functionals. This question is resolved in the negative in a rather strong sense. We propose various bounded-memory approximations, using techniques from automata theory and stochastic processes. Some questions of potential interest are raised for future work.

1 Introduction

It is well-known that the empirical average of independent, identically distributed (iid) random variables rapidly converges to their expectation. For concreteness, suppose that $X_i$, $i = 1, \ldots, n$ are iid random variables with mean $\theta$ taking values in $[0, 1]$. Then Hoeffding’s bound (Hoeffding, 1963) states that for all $\varepsilon > 0$, we have

$$P\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i > \theta + \varepsilon \right\} \leq \exp(-2n\varepsilon^2),$$

with an analogous bound for the left tail. The inequality (1) guarantees the convergence of $A : (X_1, \ldots, X_n) \mapsto n^{-1} \sum X_i$ to $\theta$ in probability, implying that $A$ is a consistent estimator for $\theta$.

In this note we address the following question: can statistical estimation be performed with bounded memory? The answer, of course, depends on the particular memory model we have in mind. A vast literature has dealt with analyzing the space complexity of various computations on data streams under different notions of memory (see the discussion in Section [3]).

*Supported in part by the Israel Science Foundation
For the purpose of this paper, *memory* will be measured in the number of bits stored (or equivalently, in the number of tape squares written on by a Turing machine). This stringent definition precludes processing even a single infinite-precision real number, so without loss of generality we assume henceforth that the $X_i$ are \{0,1\}-valued. Assuming, as we do for now, that the $X_i$ are iid, we have that $X$ is a Bernoulli process, parametrized entirely by its mean $\theta = \mathbb{E}X_i = \mathbb{P}\{X_i = 1\} = 1 - \mathbb{P}\{X_i = 0\}$.

The empirical average in (1) may be computed naively by summing the $n$ bits $X_1, \ldots, X_n$ and dividing by $n$. A standard trick circumvents storing the entire bit sequence, by initializing $A_1 := X_1$ and updating

$$A_{n+1} := \frac{nA_n + X_{n+1}}{n+1}. \quad (2)$$

This trick is infeasible under our model of memory since storing the integer $n$ requires $\Omega(\log n)$ bits. There remains, however, the possibility that some other scheme performs consistent estimation in bounded memory. Thus, rather than analyzing the behavior of a particular function, such as the empirical mean, on the data stream, we ask whether any function computable in bounded memory can be a consistent estimator of some distribution parameter.

A *consistent estimator* of the Bernoulli parameter $\theta$ is a function $A : \{0,1\}^* \rightarrow \mathbb{R}$ such that $A(X_1, \ldots, X_n)$ converges in probability to $\theta$:

$$\lim_{n \rightarrow \infty} \mathbb{P}\{|A(X_1, \ldots, X_n) - \theta| > \varepsilon\} = 0 \quad (3)$$

for all $\varepsilon > 0$. An obvious obstacle to achieving (3) is again the issue of precision: if $\theta$ takes values in some infinite set $\Theta$, then clearly (3) is impossible for any function $A$ computable with $O(1)$ bits of memory (as the latter can only distinguish among finitely many possibilities).

The main result of this paper is that even when the precision obstacle is removed, statistical estimation with bounded memory remains impossible. In particular, we prove

**Theorem 1.1.** Suppose that $X_i \in \{0,1\}$ are Bernoulli random variables with parameter $\theta$ taking values in a fixed finite set $\Theta$ which contains distinct $\theta_0, \theta_1 \in (0,1)$. Then there is no consistent estimator for $\theta$ computable using $f(\Theta)$ bits of memory, for any function $f : \Theta \mapsto \mathbb{N}$.\footnote{This question was posed to us by Ronen Brafman, motivated by problems in Reinforcement Learning \cite{Brafman and Tennenholtz 2002}.}
This claim follows directly from a much more general result on the im-
possibility of estimating any nontrivial statistical functional of a Bernoulli
process in bounded memory, proved in Theorem 5.1. In Theorem 6.1 this
result is generalized further to a much broader class of random processes,
including stationary ones with full support.

We also investigate a partial converse to Theorem 1.1, where we con-
struct \( \varepsilon \)-consistent estimators as DFAs with \( O(\log(1/\varepsilon)) \) states (Theorem
8.1). This initiates the study of approximate statistical estimation by finite-
state automata.

2 Outline of paper

This paper is organized as follows. We briefly review streaming algorithms
and their connection to regular approximations in Section 3 and set down
the terminology used throughout the paper in Section 4. The main negative
results are proved in Section 6 for the Bernoulli case and generalized to all
stationary processes with full support in Section 5. Counterexamples to the
main theorem are given in Section 7 for some pathological random processes.
In Section 8 we give some results on approximate statistical estimation with
DFAs. Finally, we give a brief recap and suggest some future directions in
Section 9.

3 Background and related work

The problem of efficiently (in time and space) extracting relevant infor-
mation from a long sequence of data goes under the general heading of
streaming algorithms. It appears that the earliest results along these lines
were the papers of Morris (1978) and Flajolet and Martin (1985), in which
(sub)logarithmic space was shown to suffice for approximating, respectively,
the count and the number of unique items in a stream of length \( n \). The field
saw a surge of activity starting from the 1990’s, including the seminal papers
of Alon et al. (1999), Henzinger et al. (1999), and Feigenbaum et al. (2002),
among many others. See Guha and McGregor (2009) for a recent result con-
cerning quantile estimation and Muthukrishnan (2005) for a comprehensive
survey of the subject.

A different line of research investigates approximations of non-regular
languages by finite automata; see Eisman and Ravikumar (2005), Cordy and Salomaa
(2007) and the references therein. Eisman and Ravikumar (2005) apparently made the first explicit connection between streaming algorithms and
regular approximations. Indeed, since any bounded-memory algorithm can be implemented as a finite automaton, Theorem 1.1 may be recast as the following claim: if \( A : \{0, 1\}^* \to \Theta \) is a consistent estimator for the Bernoulli parameter \( \theta \in \Theta \), then the language \( L_t \subset \{0, 1\}^* \) given by

\[
L_t = A^{-1}(\{t\}) = \{x \in \{0, 1\}^* : A(x) = t\}
\]

is not regular for \( 0 < t < 1 \).

Theorem 1.1 is also equivalent to the statement that any consistent estimator for \( 1_{\{\theta > a\}} \), \( a \in (0, 1) \) cannot have a regular support set. For \( a = \frac{1}{2} \), this last formulation is deceptively similar to Theorem 5 of Eisman and Ravikumar (2005), which states, roughly, that no regular language can approximate the majority language on a set of uniform measure more than one-half. However, as we show in Section 8, this is not true for biased Bernoulli measures. Nevertheless, the main ingredient in our proofs — Markov chain analysis on the states of the DFA — was largely inspired by the technique of Eisman and Ravikumar (2005).

4 Notation

We follow the standard conventions for sets, languages, probability and automata. Thus, \( \{0, 1\}^* \) is the set of all finite bit strings and a language is any \( L \subset \{0, 1\}^* \). String length is denoted by \( |x| \) and the notation

\[
\{0, 1\} \leq k := \bigcup_{i=0}^k \{0, 1\}^i = \{x \in \{0, 1\}^* : |x| \leq k\}
\]

is used. For any \( f : \{0, 1\}^* \to \{0, 1\} \), its support set is

\[
supp(f) = f^{-1}(\{1\}) = \{x \in \{0, 1\}^* : f(x) = 1\}.
\]

A Deterministic Finite-state Automaton (DFA) over the alphabet \( \{0, 1\} \) is defined as the tuple \( A = (Q, q_0, F, \delta) \) where

- \( Q = \{1, 2, \ldots, n\} \) is a finite set of states
- \( q_0 \in Q \) is the starting state
- \( F \subset Q \) is the set of the accepting states
- \( \delta : Q \times \{0, 1\} \to Q \) is the deterministic transition function
among the standard introductory texts on automata are Lewis and Papadimitriou (1981) and Sipser (2005)). The transition function $\delta$ may be extended to $Q \times \{0, 1\}^*$ via the recursion

$$\delta(q, (u_1, u_2, \ldots, u_n)) = \delta(\delta(q, (u_1, u_2, \ldots, u_{n-1})), u_n).$$

We regularly blur the distinction between an automaton and its underlying (multi)graph, whose vertex set is $Q$ and whose edges are induced by $\delta$. We also abuse the notation slightly by identifying languages $L \subseteq \{0, 1\}^*$ and automata $A$ with their characteristic functions $L, A : \{0, 1\}^* \rightarrow \{0, 1\}$, defined by $L(x) = 1_{\{x \in L\}}$ (resp., $A(x) = 1_{\{\delta(q_0, x) \in F\}}$). The probability $P\{\cdot\}$ is always defined with respect to the random process $X$ specified in context, and we often use the shorthand $X^n \equiv (X_1, X_2, \ldots, X_n)$. We say that the Bernoulli process $X$ has parameter $\theta$ if the $X_i$ are iid with

$$P\{X_i = 1\} = \theta = 1 - P\{X_i = 0\}.$$

5 The Bernoulli case

We begin with the problem of estimating the Bernoulli parameter $\theta$. Although the result in this section is subsumed by the more general (and arguably simpler) Theorem 6.1, we present Theorem 5.1 here for expositional clarity.

In this section, a statistical functional $T : [0, 1] \rightarrow \{0, 1\}$ is any binary map acting on the Bernoulli parameter (e.g., $T(\theta) = 1_{\{\theta > 1/2\}}$). We say that $T$ is nontrivial if it is not identically 0 or 1. As before, a consistent estimator for $T$ is any $A : \{0, 1\}^* \rightarrow \{0, 1\}$ such that $A(X)$ converges in probability to $T(\theta)$:

$$\lim_{n \rightarrow \infty} P\{A(X^n) \neq T(\theta)\} = 0. \quad (4)$$

We prove that finite automata cannot be consistent estimators, which in particular implies Theorem 5.1.

**Theorem 5.1.** Suppose $X$ is a Bernoulli process with parameter $\theta \in (0, 1)$ and $T$ is a nontrivial statistical functional. If $A$ is a consistent estimator for $T$, then its support language, $L_A = \text{supp}(A)$, is not regular.

Remark: this theorem was proved together with Gerald Eisman.

**Proof.** Suppose to the contrary that the language $L_A \subseteq \{0, 1\}^*$ is regular. Then it is recognized by some DFA $A = (Q, \delta, q_0, F)$. We may take $A$ to be
a minimal DFA, and in particular, every state is reachable from the starting state \( q_0 \):

\[
\forall q \in Q \exists u \in \{0,1\}^* : q = \delta(q_0, u) .
\] (5)

The Bernoulli process \( X \) together with the transition function \( \delta \) define a Markovian dynamics on \( Q \) as follows. Defining the random variable \( \xi_n \in Q \) to be the state of the automaton after reading the random string \((X_1, \ldots, X_n)\), we have \( \mathbf{P}\{\xi_0 = q_0\} = 1 \) and

\[
\mathbf{P}\{\xi_{n+1} = q' | \xi_n = q\} = \begin{cases} 
\theta, & \delta(q, 1) = q' \\
1 - \theta, & \delta(q, 0) = q' \\
0, & \text{otherwise}
\end{cases}.
\]

Since \( T \) is nontrivial, there are \( \theta_0, \theta_1 \in (0,1) \) such that \( T(\theta_0) = 0 \) and \( T(\theta_1) = 1 \). Thus the property in (4) may be restated as

\[
\lim_{n \to \infty} \mathbf{P}\{\xi_n \in F\} = T(\theta), \quad \theta \in \{\theta_0, \theta_1\},
\] (6)

which says that the probability that the DFA \( A \) is in an accepting state approaches either 0 or 1, depending on whether \( \theta = \theta_0 \) or \( \theta = \theta_1 \).

Since \( Q \) is finite, the Markov chain has at least one ergodic component (see Kemeny and Snell (1976) for general facts about finite Markov chains). Furthermore, with probability 1, the chain enters an ergodic component after finitely many steps.

Let \( \omega = (\omega_1, \omega_2, \ldots) \in \{0,1\}^\mathbb{N} \) be a particular realization of the Bernoulli process and let \( \xi(\omega) = (\xi_1, \xi_2, \ldots) \) be the induced sequence of Markov states. Elementary theory of finite Markov chains implies the following:

(i) there is an ergodic component \( E \subseteq Q \) such that

\[
\mathbf{P}\{\xi_n \in E \text{ for all but finitely many } n \in \mathbb{N}\} = 1
\]

(ii) for each \( q \in E \) there is a \( \pi_q > 0 \) such that

\[
\mathbf{P}\{\omega : \xi_n = q\} = \pi_q
\] (7)

for infinitely many \( n \in \mathbb{N} \).

Our final observation is that any ergodic component must contain both accepting and non-accepting states:

\[ \emptyset \neq E \cap F \subseteq E \].

(8)
This holds because the ergodic components depend only on the connective properties of the Markov chain and not on the actual value of the Bernoulli parameter $\theta$ — as long as it is nontrivial (i.e., not 0 or 1). If there were a “homogeneous” ergodic component $E$, in the sense that $E \subseteq F$ or $E \cap F = \emptyset$, condition (3) would be violated since for all $\theta \in (0,1)$, the Markov chain becomes trapped in $E$ with positive probability, as per (5).

But the conjunction of (7) and (8) contradicts (6), since the latter requires that $P\{\xi_n \in S\} \to 0$ for $S = E \cap F$ or $S = E \setminus F$. Thus, we conclude that no DFA $A$ satisfying (4) exists.

6 Stationary processes with full support

We refer the reader to Kallenberg (2002) for the relevant background on random processes. The $\{0,1\}$-valued process $X = (X_1, X_2, \ldots)$ is said to be stationary if

$$P\{(X_{t_1}, X_{t_2}, \ldots, X_{t_m}) = x\} = P\{(X_{t_1+k}, X_{t_2+k}, \ldots, X_{t_m+k}) = x\}$$

for all $k, m \geq 1$, all $0 < t_1 < \ldots < t_k$, and all $x \in \{0, 1\}^m$. We say that $X$ has full support if every realization occurs with positive probability:

$$P\{(X_1, \ldots, X_n) = x\} > 0$$

(9)

for all $n \geq 1$ and all $x \in \{0, 1\}^n$.

The extension of Theorem 5.1 to the much broader class of stationary processes with full support is quite straightforward, requiring only minor additional abstraction. Thus, if $\mathcal{M}$ is any collection of discrete-time, $\{0, 1\}$-valued random processes, then each process is determined by some measure $\mu$ on $\{0, 1\}^\mathbb{N}$. A statistical functional is any mapping $T : \mathcal{M} \to \{0, 1\}$, and $T$ is nontrivial if there are $\mu_0, \mu_1 \in \mathcal{M}$ such that $T(\mu_0) \neq T(\mu_1)$. Finally, $A : \{0, 1\}^\mathbb{N} \to \{0, 1\}$ is a consistent estimator for $T$ if

$$\lim_{n \to \infty} \mu\{x \in \{0, 1\}^n : A(x) \neq T(\mu)\} = 0$$

(10)

for all $\mu \in \mathcal{M}$; a shorthand way of writing the above is $P\{A(X^n) \neq T(\mu)\} \to 0$.

We are ready to proceed with the generalization:

**Theorem 6.1.** Let $\mathcal{M}$ be a collection of stationary $\{0, 1\}$-valued processes with full support and suppose $T : \mathcal{M} \to \{0, 1\}$ is a nontrivial functional. Then a consistent estimator for $T$ cannot have regular support.
Proof. Assume to the contrary that $T$ has an estimator with a regular support set, and that the latter is recognized by the minimal DFA $A = (Q, \delta, q_0, F)$. Let $\omega = (\omega_1, \omega_2, \ldots) \in \{0, 1\}^\mathbb{N}$ be a particular realization of the random process $X$ and let $\xi(\omega) = (\xi_1, \xi_2, \ldots)$ be the induced sequence of states $\xi_i \in Q$ traversed by the automaton when reading $X$. We claim that there is a strongly connected component $E \subseteq Q$ such that

(i) $P\{\xi_n \in E \text{ for all but finitely many } n \in \mathbb{N}\} = 1$

(ii) there is a $\pi_0 > 0$ such that for all $q \in E$ we have $P\{\xi_n = q\} \geq \pi_0$ for infinitely many $n \in \mathbb{N}$.

Part (i) follows by elementary graph theory since any directed graph decomposes into transient and strongly connected (SC) components, and $E \subseteq Q$ is simply the SC component in which $\xi_n$ eventually becomes trapped. The process will not become trapped in a transient component $H$ because by the full support property (9), there is a positive escape probability from $H$ and by stationarity, this escape probability cannot decrease with $n$. To prove (ii), let $y \in Q^m$ be any path that traverses all the elements of $E$ (which must exist since $E$ is SC) and let $x \in \{0, 1\}^m$ be the corresponding underlying bit sequence. By stationarity, $\pi_0 = P\{X_n^m = x\}$ is a well-defined quantity independent of $n$, and by (9) it must be positive.

Finally, we argue that

$\emptyset \neq E \cap F \subsetneq E$,

since for all $\mu \in \mathcal{M}$, the process $\xi_n$ becomes trapped in the SC component $E$ with positive probability and in order for $A$ to satisfy (10), $E$ must contain both accepting and non-accepting states. The remainder of the proof proceeds analogously to that of Theorem 5.1.

7 Counterexamples

Theorem 6.1 gives sufficient conditions for a random process not to admit consistent finite-state statistical estimators. In this section, we give examples of such estimators for processes violating the conditions of stationarity and full-support. The basic intuition is that a DFA cannot accumulate statistical information; it can only be driven into a certain state by the process (as in Section 7.1), or exploit forbidden patterns (as in Section 7.2).
7.1 Non-stationary process

Let $\mathcal{M}$ be the collection of $\{0, 1\}$-valued processes $X = (X_1, X_2, \ldots)$ where for each sample path we have

$$\lim_{n \to \infty} P\{X_n = 0\} \in \{0, 1\}.$$ 

In other words, any realization of $X$ eventually becomes dominated entirely by 0s or 1s. Processes of this type are clearly not stationary, and a simple automaton (Figure 1) distinguishes the 0-dominant processes from the 1-dominant ones. It is not difficult to verify that this automaton will occupy state $q_j$ with probability approaching 1 when reading the realization of a $j$-dominant process, for $j \in \{0, 1\}$.

7.2 Process without full support

Let $\mathcal{M}$ be the collection of $\{0, 1\}$-valued iid Bernoulli processes $X = (X_1, X_2, \ldots)$ with parameter $\theta \in [0, 1]$. We call the processes with $\theta \in \{0, 1\}$ degenerate and those with $\theta \in (0, 1)$ nondegenerate. The processes comprising $\mathcal{M}$ are stationary but do not all have full support, and a simple automaton (Figure 2) distinguishes degenerate processes from nondegenerate ones. It is easily verified that this automaton will occupy state $q_{01}$ with probability approaching 1 when reading a nondegenerate process and will become trapped either in state $q_0$ or $q_1$ when reading a degenerate process.

8 Approximate statistics with a DFA

We revisit the problem of approximating the Bernoulli parameter $\theta$ with a DFA. As discussed in Section 1, this question is only meaningful if $\theta$ is allowed to take values in some finite set $\Theta$. Suppose for concreteness that $\Theta = \{0 < \theta_0 < \theta_1 < \ldots < \theta_k < 1\}$. Then the problem of determining whether $\theta = \theta_j \in \Theta$ is reduced to deciding whether $\theta > \theta_{j-1}$ and $\theta < \theta_{j+1}$. Of course, by Theorem 5.1 a consistent estimator for $T_a(\theta) = 1_{\{\theta > a\}}$ cannot
be realized by any DFA. Consider, however, relaxing the requirement of consistency in (4) to \( \varepsilon \)-consistency:

\[
\lim_{n \to \infty} \sup \mathbb{P}\{A(X^n) \neq T(\theta)\} < \varepsilon.
\]  

(11)

We shall examine case of \( T_{1/2} \) in some detail. To this end, recall the majority function \( \text{MAJ} : \{0,1\}^* \to \{0,1\} \), defined by

\[
\text{MAJ}(x) = \mathbb{1}\{\sum_{i=1}^{|x|} x_i > \frac{|x|}{2}\}.
\]

We observe that any consistent estimator of \( \mathbb{1}_{\{\theta > 1/2\}} \) must asymptotically agree with MAJ:

**Theorem 8.1.** Let \( X \) be a Bernoulli process with parameter \( \theta \neq \frac{1}{2} \) and suppose that \( A : \{0,1\}^* \to \{0,1\} \) is a consistent estimator of the functional \( T_{1/2} : \theta \mapsto \mathbb{1}_{\{\theta > 1/2\}} \). Then

\[
\lim_{n \to \infty} \mathbb{P}\{A(X^n) \neq \text{MAJ}(X^n)\} = 0.
\]

**Proof.** Assume without loss of generality that \( \theta > \frac{1}{2} \), so \( T_{1/2}(\theta) = 1 \). Then,

\[
\mathbb{P}\{\text{MAJ}(X^n) \neq 1\} = \mathbb{P}\left\{\sum_{i=1}^n X_i \leq \frac{n}{2}\right\}
\]

\[= \mathbb{P}\left\{\frac{1}{n} \sum_{i=1}^n X_i \leq \theta - (\theta - \frac{1}{2})\right\}
\]

\[\leq \exp(-2n(\theta - \frac{1}{2})^2), \quad (12)\]
Figure 3: The automaton $M(5)$ agrees with MAJ on all $x \in \{0, 1\}^\leq 4$. The general $M(k) = (Q, q_0, F, \delta)$ is constructed as follows: $Q = \{1, \ldots, k\}$, $q_0 = \left\lfloor \frac{k+1}{2} \right\rfloor$, $F = \{\left\lfloor k/2 \right\rfloor + 1, \ldots, k\}$ and $\delta(i, 0) = i - 1 + 1_{\{i=1\}}$, $\delta(i, 1) = i + 1 - 1_{\{i=k\}}$ for $1 \leq i \leq k$.

where the last inequality is Hoeffding’s; a similar analysis holds for $\theta < \frac{1}{2}$.

Thus,

$$
P\{A(X^n) \neq \text{MAJ}(X^n)\} \leq P\{A(X^n) \neq T_{1/2}(\theta)\} + P\{\text{MAJ}(X^n) \neq T_{1/2}(\theta)\} \to 0$$

where the first term goes to 0 because $A$ is a consistent estimator for $T_{1/2}$ and the second term vanishes by (12).

Let $M(k)$ be the minimal DFA\(^2\) which agrees with MAJ on all binary strings of length less than $k$; these are illustrated in Figure 3. One might inquire how well $M(k)$ approximates $1_{\{\theta > 1/2\}}$ on long Bernoulli sequences, and the following theorem provides an answer:

Theorem 8.2. Let $X$ be a Bernoulli process with parameter $\theta$ let $M(k)$ be defined as above. Then

$$\eta \equiv \lim_{n \to \infty} P\{X^n \notin M(k)\} = \frac{\theta^{-1} - 1 - (\theta^{-1} - 1)^{\left\lfloor k/2 \right\rfloor + 1}}{\theta^{-1} - 1 - (\theta^{-1} - 1)^{k+1}}. \quad (13)$$

For $k$ even and $1/2 \leq \theta < 1$, we have

$$\eta \leq \frac{1}{2}(2 - 2\theta)^{k/2}. \quad (14)$$

Remark: we thank Daniel Dadush for help with this calculation.

\(^2\) One may take the family of automata constructed in Figure 3 as the definition of $M(k)$ and prove as a simple exercise that this is indeed the smallest DFA agreeing with MAJ on all of $\{0, 1\}^\leq k$. For small values of $k$, the techniques of Trakhtenbrot and Barzdin\(^{1973}\) or Angluin\(^{1987}\) may be used to construct the minimal DFA agreeing with a given membership oracle on $\{0, 1\}^\leq k$. 

11
Proof. The Bernoulli process $X$ induces the Markov chain $\xi = (\xi_1, \ldots)$, $\xi_i \in Q$, on the states of $M(k)$ as described in the proof of Theorem 5.1. By construction of the DFA $M(k)$, the induced Markov chain is ergodic (for a visual illustration, relabel every “1” edge in Figure 3 with $\theta$ and every “0” edge with $1 - \theta$). Its unique stationary distribution $\pi \in \mathbb{R}^k$ has the interpretation

$$\pi_q = \lim_{n \to \infty} \mathbb{P}\{\xi_n = q\}$$

and obeys the recurrence

$$\pi_i = \theta \pi_{i-1} + (1 - \theta) \pi_{i+1}, \quad 1 \leq i \leq k.$$ 

This relation is satisfied by the vector

$$\tilde{\pi}_i = \theta^i (1 - \theta)^{k-i}, \quad 1 \leq i \leq k,$$

which must be normalized to make it into a probability distribution. The accepting states $F \subset Q = \{1, \ldots, k\}$ of $M(k)$ are all $q > \lceil k/2 \rceil$, and so the limiting probability of being in a rejecting state is given by

$$\sum_{q \notin F} \pi_q = \frac{\sum_{i=1}^{\lceil k/2 \rceil + 1} \theta^i (1 - \theta)^{k-i}}{\sum_{i=1}^{k} \theta^i (1 - \theta)^{k-i}} = \frac{(1 - \theta)^k \sum_{i=1}^{\lceil k/2 \rceil + 1} \left(\frac{\theta}{1 - \theta}\right)^i}{(1 - \theta)^k \sum_{i=1}^{k} \left(\frac{\theta}{1 - \theta}\right)^i}$$

and the latter sum up as geometric series to yield (13). From there, obtaining (14) is a matter of simple calculus. □

It follows from Theorem 8.2 that the DFA $M(2k)$ will disagree with the majority function on long runs of Bernoulli processes with parameter $\theta$ with probability at most

$$R(\theta, 2k) = \frac{1}{2} \left(1 - 2 \left|\theta - \frac{1}{2}\right|\right)^k.$$ 

Note that

$$\lim_{k \to \infty} R(\theta, k) = 0 \quad (15)$$
Figure 4: The automaton $M_{1/3}(7)$ for approximately estimating $T_{1/3}$.

Figure 5: The limiting behavior of $M_{1/3}(7)$ and $M_{1/3}(24)$, whose sizes are 9 and 29, respectively. The automata were obtained by Angluin’s algorithm [Angluin, 1987].
exponentially fast, while

\[ \lim_{\theta \to 1/2} R(\theta, k) = \frac{1}{2}. \]  

(16)

This approach can be generalized to obtain approximate finite-state estimators for the Bernoulli statistical functional \( T_a(\theta) = \mathbb{1}_{\{\theta > a\}} \) for \( a \in (0, 1) \). For \( k \in \mathbb{N} \) and \( a \in (0, 1) \), define \( M_a(k) \) be the (unique) smallest DFA which agrees on all \( x \in \{0, 1\}^k \) with the function \( \text{MAJ}_a : \{0, 1\}^* \to \{0, 1\} \), defined by

\[ \text{MAJ}_a(x) = \mathbb{1}_{\{\sum_{i=1}^{\left| x \right|} x_i > a \left| x \right| \}} \]

(\( M_{1/3}(7) \) is illustrated in Figure 1). We can associate to each \( M_a(k) \) an ergodic Markov chain with a unique stationary distribution, as done in the proof of Theorem 5.1. Thus, each \( M_a(k) \) has a well-defined limiting acceptance probability

\[ \rho_a(k, \theta) = \lim_{n \to \infty} \mathbb{P}\{X^n \in M_a(k)\} \]

as well as a limiting probability of error

\[ R_a(k, \theta) = \lim_{n \to \infty} \mathbb{P}\{M_a(k)(X^n) \neq T_a(\theta)\} \]

(the curves of \( \rho_{1/3}(7, \cdot) \) and \( \rho_{1/3}(24, \cdot) \) are plotted in Figure 5). It is not difficult to show, using arguments analogous to those in Theorem 8.1, that

\[ \lim_{k \to \infty} R_a(k, \theta) = \begin{cases} 0, & \theta \neq a \\ \frac{1}{2}, & \theta = a \end{cases} \]

This is a natural generalization of (15) and (16) for \( a \neq \frac{1}{2} \); we leave the analysis of the convergence rates for future work.

Contrast these results with a theorem of Eisman and Ravikumar (2005), which may be stated as follows.

**Theorem 8.3** (Eisman and Ravikumar (2005)). Let \( X \) be a Bernoulli process with parameter \( \theta = \frac{1}{2} \) and suppose that \( L \subseteq \{0, 1\}^* \) is a regular language. Then

\[ \limsup_{n \to \infty} \mathbb{P}\{L(X^n) \neq \text{MAJ}_{1/2}(X^n)\} \geq \frac{1}{2}. \]
The moral is that for a given Bernoulli process with parameter \( \theta \), the majority function can be \( \varepsilon \)-approximated (in the sense of (11)) by a DFA with \( O(\log(1/\varepsilon)) \) states, and the approximation gets progressively worse as \( \theta \) approaches \( \frac{1}{2} \). Thus, (15) and (16) provide a converse of sorts to Theorem 8.3 which eliminates the possibility of a better than \( \frac{1}{2} \) approximation to MAJ by any DFA under the unbiased Bernoulli process.

See Cordy and Salomaa (2007) for other results on approximating non-regular languages by DFAs.

9 Discussion and future work

We have shown that consistent statistical estimation is not realizable by finite-state automata, but if the consistency requirement is relaxed, efficient \( \varepsilon \)-approximations exist. The negative result holds for the broad class of stationary processes with full support.

Along the way, we encountered several insights. In Section 7, we saw that although a DFA cannot accumulate statistical information, it can exploit a time drift or forbidden patterns in the random process. It would be interesting to make this intuition more rigorous — for example, by giving a full characterization of the random processes that do not admit consistent finite-state estimators of nontrivial statistical functionals.

The observations in Section 8 raise a number of interesting questions. We conjecture the family of DFAs we constructed to approximate the majority function in Theorem 8.2 is optimal in the sense that any finite-state \( \varepsilon \)-approximation (see (11)) to MAJ must use \( \Omega(\log(1/\varepsilon)) \) states.

A general method for approximating the Bernoulli parameter \( \theta \) is suggested in Section 8: use \( M_a(k) \) and \( M_b(k) \) with large \( k \) to pinpoint \( \theta \), with high probability, to the interval \((a, b)\). It would be interesting to analyze the asymptotic error \( R_a(k, \theta) \) and the size of \( M_a(k) \) as functions of \((k, a, \theta)\) and perhaps establish an optimality property of some sort for this class of estimators.

Acknowledgements

Ronen Brafman posed the question that motivated this whole paper. Many thanks to Gerald Eisman for debugging the numerous faulty versions of Theorem 5.1 and to Daniel Dadush for help with stationary distributions. This work was done at the Weizmann Institute under the kind hosting and guidance of Gideon Schechtman.
References

Noga Alon, Yossi Matias, and Mario Szegedy. The space complexity of approximating the frequency moments. *Journal of Computer and System Sciences*, 58(1):137 – 147, 1999.

Dana Angluin. Learning regular sets from queries and counterexamples. *Inf. Comput.*, 75(2):87–106, 1987.

Ronen I. Brafman and Moshe Tennenholtz. R-max - a general polynomial time algorithm for near-optimal reinforcement learning. *Journal of Machine Learning Research*, 3:213–231, 2002.

Amit Chakrabarti, Graham Cormode, and Andrew McGregor. Robust lower bounds for communication and stream computation. In *STOC ’08: Proceedings of the 40th annual ACM symposium on Theory of computing*, pages 641–650, New York, NY, USA, 2008.

Brendan Cordy and Kai Salomaa. On the existence of regular approximations. *Theor. Comput. Sci.*, 387(2):125–135, 2007.

Gerald Eisman and Bala Ravikumar. Approximate recognition of non-regular languages by finite automata. In *ACSC ’05: Proceedings of the Twenty-eighth Australasian conference on Computer Science*, pages 219–227, Darlinghurst, Australia, 2005.

Joan Feigenbaum, Sampath Kannan, Martin Strauss, and Mahesh Viswanathan. An approximate l1-difference algorithm for massive data streams. *SIAM J. Comput.*, 32(1):131–151, 2002.

Philippe Flajolet and G. Nigel Martin. Probabilistic counting algorithms for data base applications. *J. Comput. System Sci.*, 31(2):182–209, 1985.

Sudipto Guha and Andrew McGregor. Stream order and order statistics: Quantile estimation in random-order streams. *SIAM J. Comput.*, 38(5):2044–2059, 2009.

Monika R. Henzinger, Prabhakar Raghavan, and Sridhar Rajagopalan. Computing on data streams. pages 107–118, 1999.

Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *American Statistical Association Journal*, 58:13–30, 1963.
Piotr Indyk and David Woodruff. Optimal approximations of the frequency moments of data streams. In STOC ’05: Proceedings of the thirty-seventh annual ACM symposium on Theory of computing, pages 202–208, New York, NY, USA, 2005.

Olav Kallenberg. Foundations of modern probability. Second edition. Probability and its Applications. Springer-Verlag, 2002.

John G. Kemeny and J. Laurie Snell. Finite Markov chains. Springer-Verlag, New York, 1976.

Harry R. Lewis and Christos H. Papadimitriou. Elements of the Theory of Computation. Prentice Hall, 1981.

Robert Morris. Counting large numbers of events in small registers. Commun. ACM, 21(10):840–842, 1978.

S. Muthukrishnan. Data streams: algorithms and applications. Found. Trends Theor. Comput. Sci., 1(2):117–236, 2005.

Michael Sipser. Introduction to the Theory of Computation. Course Technology, 2005.

Boris A. Trakhtenbrot and Janis M. Barzdin’. Finite Automata: Behavior and Synthesis, volume 1 of Fundamental Studies in Computer Science. North-Holland, Amsterdam, 1973.