Higher Apéry-like numbers arising from special values of the spectral zeta function for the non-commutative harmonic oscillator

Kazufumi Kimoto

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Abstract

A generalization of the Apéry-like numbers, which is used to describe the special values \( \zeta_Q(2) \) and \( \zeta_Q(3) \) of the spectral zeta function for the non-commutative harmonic oscillator, are introduced and studied. In fact, we give a recurrence relation for them, which shows a ladder structure among them. Further, we consider the 'rational part' of the higher Apéry-like numbers. We discuss several kinds of congruence relations among them, which are regarded as an analogue of the ones among Apéry numbers.

1 Introduction

The non-commutative harmonic oscillator is the system of differential equations defined by the operator

\[
Q = Q_{\alpha, \beta} := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \left( -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 \right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left( x \frac{d}{dx} + \frac{1}{2} \right),
\]

(1.1)

where \( \alpha \) and \( \beta \) are real parameters. In this paper, we always assume that \( \alpha > 0, \beta > 0 \) and \( \alpha \beta > 1 \). Under these conditions, one can show that the operator \( Q \) defines an unbounded, positive, self-adjoint operator on the space \( L^2(\mathbb{R}; \mathbb{C}^2) \) of \( \mathbb{C}^2 \)-valued square integrable functions which has only a discrete spectrum, and the multiplicities \( m(\lambda) \) of the eigenvalues \( \lambda \in \text{Spec}(Q) \) are uniformly bounded [22]. Hence, in this case, it is meaningful to define its spectral zeta function \( \zeta_Q(s) = \text{Tr} Q^{-s} = \sum_{\lambda \in \text{Spec}(Q)} m(\lambda) \lambda^{-s} \). This series converges absolutely if \( \Re s > 1 \), and hence defines a holomorphic function on the half plane \( \Re s > 1 \). Further, \( \zeta_Q(s) \) is meromorphically continued to the whole complex plane \( \mathbb{C} \) which has 'trivial zeros' at \( s = 0, -2, -4, \ldots \) (see [7], [21]).

The aim of this paper is to study the higher Apéry-like numbers \( J_k(n) \) defined by

\[
J_k(n) := 2^k \int_{[0,1]^k} \left( \frac{(1 - x_1^4)(1 - x_2^4 \cdots x_k^4)}{(1 - x_1^2 \cdots x_k^2)^2} \right)^n dx_1 dx_2 \cdots dx_k
\]

for \( k \geq 2 \) and \( n \geq 0 \), which are a generalization of the Apéry-like numbers \( J_2(n) \) and \( J_3(n) \) studied in [11]. This object arises from the special values of the spectral zeta function \( \zeta_Q(s) \): In [8], the generating functions of the numbers \( J_2(n) \) and \( J_3(n) \) are used to describe the special values \( \zeta_Q(2) \) and \( \zeta_Q(3) \) of the spectral zeta function \( \zeta_Q(s) \). Similarly, the higher Apéry-like numbers \( J_k(n) \) are closely related to the special values \( \zeta_Q(k) \) (see §3.3).

We first show that \( J_k(n) \) satisfy three-term (inhomogeneous) recurrence relations, which is translated to (inhomogeneous) singly confluent Heun differential equations for their generating functions. The point is that these relations or differential equations are connecting \( J_k(n) \)'s and \( J_{k-2}(n) \)'s. This fact implies that there could be a certain relation between \( \zeta_Q(k) \) and \( \zeta_Q(k-2) \). It would be very interesting if one can utilize these relations to understand a modular interpretation of \( \zeta_Q(4), \zeta_Q(6), \ldots \) based on that of \( \zeta_Q(2) \) (see [12]). We also notice that these recurrence relations quite resemble to those for Apéry numbers used to prove the irrationality of \( \zeta(2) \) and \( \zeta(3) \) (see [27]), and this is why we call \( J_k(n) \) the (higher) Apéry-like numbers.
By a suitable change of variable in the differential equation, we also obtain another kind of recurrence relations, which allow us to define the rational part of the higher Apéry-like numbers (or normalized higher Apéry-like numbers) \( \tilde{J}_k(n) \). In fact, each \( J_k(n) \) is a linear combination of the Riemann zeta values \( \zeta(k), \zeta(k-2), \ldots \) and the coefficients are given by \( \tilde{J}_m(n) \)'s. Since there are various kind of congruence relations satisfied by Apéry numbers (see, e.g. [4], [5], [1]), it would be natural and interesting to find an analogue for our higher Apéry-like numbers. Actually, we give several congruence relations among \( \tilde{J}_2(n) \) and \( \tilde{J}_3(n) \) in [12]. We add such congruence relations among \( \tilde{J}_k(n) \), and give some conjectural congruences.

2 Apéry numbers for \( \zeta(2) \) and \( \zeta(3) \)

As a quick reference for the readers, we recall the definitions and several properties on the original Apéry numbers.

2.1 Apéry numbers for \( \zeta(2) \)

Apéry numbers for \( \zeta(2) \) are given by

\[
A_2(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \quad B_2(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n} \left( 2 \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m^2} + \sum_{m=1}^{k} \frac{(-1)^{n+m-1}}{m} \frac{n!}{(n+m)!} \right).
\]

These numbers satisfy a recurrence relation of the same form

\[
n^2u(n) - (11n^2 - 11n + 3)u(n-1) - (n-1)^2u(n-2) = 0 \quad (n \geq 2)
\]

with initial conditions \( A_2(0) = 1, A_2(1) = 3 \) and \( B_2(0) = 0, B_2(1) = 5 \). The ratio \( B_2(n)/A_2(n) \) converges to \( \zeta(2) \), and this convergence is rapid enough to prove the irrationality of \( \zeta(2) \). Consider the generating functions

\[
A_2(t) = \sum_{n=0}^{\infty} A_2(n)t^n, \quad B_2(t) = \sum_{n=0}^{\infty} B_2(n)t^n, \quad R_2(t) = A_2(t)\zeta(2) - B_2(t).
\]

It is proved that

\[
L_2A_2(t) = 0, \quad L_2B_2(t) = -5, \quad L_2R_2(t) = 5,
\]

where \( L_2 \) is a differential operator given by

\[
L_2 = t(t^2 + 11t - 1) \frac{d^2}{dt^2} + (3t^2 + 22t - 1) \frac{d}{dt} + (t + 3).
\]

The function \( R_2(t) \) is also expressed as follows:

\[
R_2(t) = \int_0^1 \int_0^1 \frac{dxdy}{1 - xy + txy(1-x)(1-y)}.
\]

The family \( Q_t^2 : 1 - xy + txy(1-x)(1-y) = 0 \) of algebraic curves, which comes from the denominator of the integrand, is birationally equivalent to the universal family \( C_t^2 \) of elliptic curves having rational 5-torsion. Moreover, the differential equation \( L_2A_2(t) = 0 \) is regarded as a Picard-Fuchs equation for this family, and \( A_2(t) \) is interpreted as a period of \( C_t^2 \) (see [2]).
2.2 Apéry numbers for $\zeta(3)$

Apéry numbers for $\zeta(3)$ are given by

$$A_3(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad B_3(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3(m)^{n+m}} \right)$$

These numbers satisfy a recurrence relation of the same form

$$n^3u(n) - (34n^3 - 51n^2 + 27n - 5)u(n-1) + (n-1)^3u(n-2) = 0 \quad (n \geq 2)$$

with initial conditions $A_3(0) = 1, A_3(1) = 5$ and $B_3(0) = 0, B_3(1) = 6$. The ratio $B_3(n)/A_3(n)$ converges to $\zeta(3)$ rapidly enough to allow us to prove the irrationality of $\zeta(3)$. Consider the generating functions

$$A_3(t) = \sum_{n=0}^{\infty} A_3(n)t^n, \quad B_3(t) = \sum_{n=0}^{\infty} B_3(n)t^n, \quad R_3(t) = A_3(t)\zeta(3) - B_3(t).$$

It is proved that

$$L_3A_3(t) = 0, \quad L_3B_3(t) = 5, \quad L_3R_3(t) = -5,$$

where $L_3$ is a differential operator given by

$$L_3 = t^2(t^2 - 34t^2 + 1)\frac{d^3}{dt^3} + t(6t^2 - 153t + 3)\frac{d^2}{dt^2} + (7t^2 - 112t + 1)\frac{d}{dt} + (t - 5).$$

The function $R_3(t)$ is also expressed as follows:

$$R_3(t) = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{dxdydz}{1 - (1 - xy)z - txyz(1-x)(1-y)(1-z)}.$$

The family $Q_1^3 : 1 - (1 - xy)z - txyz(1-x)(1-y)(1-z) = 0$ of algebraic surfaces coming from the denominator of the integrand is birationally equivalent to a certain family $C_3^3$ of $K3$ surfaces with Picard number 19. Furthermore, the differential equation $L_3A_3(t) = 0$ is regarded as a Picard-Fuchs equation for this family, and $A_3(t)$ is interpreted as a period of $C_3^3$ (see [3]).

2.3 Congruence relations for Apéry numbers

Apéry numbers $A_2(n)$ and $A_3(n)$ have various kind of congruence properties. Here we pick up several of them, for which we will discuss an Apéry-like analogue later.

**Proposition 2.1.** Let $p$ be a prime and $n = n_0 + n_1p + \cdots + n_kp^k$ be the $p$-ary expansion of $n \in \mathbb{Z}_{\geq 0}$ ($0 \leq n_j < p$). Then it holds that

$$A_2(n) \equiv \prod_{j=0}^{k} A_2(n_j) \pmod{p}, \quad A_3(n) \equiv \prod_{j=0}^{k} A_3(n_j) \pmod{p}.$$

**Proposition 2.2** ([4, Theorems 1 and 2]). For all odd prime $p$, it holds that

$$A_2(mp^r - 1) \equiv A_2(mp^r - 1) \pmod{p^r},$$

$$A_3(mp^r - 1) \equiv A_3(mp^r - 1) \pmod{p^r}$$

for any $m, r \in \mathbb{Z}_{\geq 0}$. These congruence relations hold modulo $p^{3r}$ if $p \geq 5$ (known and referred to as a super-congruence).
We denote by $\eta(\tau)$ the Dedekind eta function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau} \quad (3\tau > 0).$$

(2.2)

**Proposition 2.3** ([28, Theorem 13.1]). For any odd prime $p$ and any $m, r \in \mathbb{Z}_{>0}$ with $m$ odd, it holds that

$$A_2(\frac{mp^r-1}{2}) - \lambda_r A_2(\frac{mp^r-1}{2}) + (-1)^{(p-1)/2}p^2 A_2(\frac{mp^r-2}{2}) \equiv 0 \pmod{p^r}. \quad (2.3)$$

Here $\lambda_r$ is defined by

$$\sum_{n=1}^{\infty} \lambda_n q^n = \eta(4\tau)^6 = q \prod_{n=1}^{\infty} (1 - q^{4n})^6.$$

**Proposition 2.4** ([5, Theorem 4]). For any odd prime $p$ and any $m, r \in \mathbb{Z}_{>0}$ with $m$ odd, it holds that

$$A_3(\frac{mp^r-1}{2}) - \gamma_r A_3(\frac{mp^r-1}{2}) + p^3 A_3(\frac{mp^r-2}{2}) \equiv 0 \pmod{p^r}. \quad (2.4)$$

Here $\gamma_n$ is defined by

$$\sum_{n=1}^{\infty} \gamma_n q^n = \eta(2\tau)^4 \eta(4\tau)^4 = q \prod_{n=1}^{\infty} (1 - q^{2n})^4(1 - q^{4n})^4.$$

### 3 Apéry-like numbers for $\zeta_Q(2)$ and $\zeta_Q(3)$

We introduce the Apéry like numbers $J_2(n)$ and $J_3(n)$, and give a brief explanation on their basic properties and the connection between the special values $\zeta_Q(2), \zeta_Q(3)$ of the spectral zeta function $\zeta_Q(s)$.

#### 3.1 Definition

We define the *Apéry-like numbers for $\zeta_Q(2)$ and $\zeta_Q(3)$* by

$$J_2(n) := 4 \int_{0}^{1} \int_{0}^{1} \frac{(1 - x_1)(1 - x_2)^n}{(1 - x_1^2 x_2^2)^2} \frac{dx_1 dx_2}{1 - x_1^2 x_2^2},$$

$$J_3(n) := 8 \int_{0}^{1} \int_{0}^{1} \frac{(1 - x_1^4)(1 - x_2^4 x_3^4)^n}{(1 - x_1^2 x_2^2 x_3^2)^2} \frac{dx_1 dx_2 dx_3}{1 - x_1^2 x_2^2 x_3^2}.$$  

The sequences $\{J_2(n)\}$ and $\{J_3(n)\}$ satisfy the recurrence formula (Propositions 4.11 and 6.4 in [8])

$$4n^2 J_2(n) - (8n^2 - 8n + 3)J_2(n - 1) + 4(n - 1)^2 J_2(n - 2) = 0, \quad (3.1)$$

$$4n^2 J_3(n) - (8n^2 - 8n + 3)J_3(n - 1) + 4(n - 1)^2 J_3(n - 2) = \frac{2^n(n - 1)!}{(2n - 1)!!}, \quad (3.2)$$

with the initial conditions

$$J_2(0) = 3\zeta(2), \quad J_2(1) = \frac{9}{4} \zeta(2); \quad J_3(0) = 7\zeta(3), \quad J_3(1) = \frac{21}{4} \zeta(3) + \frac{1}{2}.$$  

It is notable that the left-hand sides of these relations have the same shape. Since the relations (3.1),(3.2) and the one (2.1) for $A_2(n)$ have quite close shapes, we call the numbers $J_2(n)$ and $J_3(n)$ the *Apéry-like numbers*.
3.2 Generating functions and their differential equations

The generating functions for $J_2(n)$ and $J_3(n)$ are defined by

$$w_2(t) := \sum_{n=0}^{\infty} J_2(n) t^n = 4 \int_{0}^{1} \int_{0}^{1} \frac{1-x_1^2 x_2^2}{(1-x_1^2 x_2^2)^2 - t(1-x_1^2)(1-x_2^2)} \, dx_1 dx_2,$$

$$w_3(t) := \sum_{n=0}^{\infty} J_3(n) t^n = 8 \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{1-x_1^2 x_2^2 x_3^2}{(1-x_1^2 x_2^2 x_3^2)^2 - t(1-x_1^2)(1-x_2^2 x_3^2)(1-x_2^2 x_3^2)} \, dx_1 dx_2 dx_3. \tag{3.4}$$

By the recurrence relations (3.1) and (3.2), we get the differential equations

$$D_t w_2(t) = 0, \tag{3.5}$$

$$D_t w_3(t) = \frac{1}{2} {}_2 F_1 \left( 1, 1; \frac{3}{2}; t \right), \tag{3.6}$$

where $D_t$ denotes the singly confluent Heun differential operator given by

$$D_t = t(1-t)^2 \frac{d^2}{dt^2} + (1-3t)(1-t) \frac{d}{dt} + t - \frac{3}{4}. \tag{3.7}$$

(3.5) is solved in [18] as

$$w_2(t) = \frac{3 \zeta(2)}{1-t} {}_2 F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{t}{t-1} \right).$$

Here $_2 F_1 (a, b; c; z)$ is the Gaussian hypergeometric function. Now it is immediate that

$$J_2(n) = 3 \zeta(2) \sum_{j=0}^{n} (-1)^j \left( -\frac{1}{2} \right)^2 \binom{n}{j}. \tag{3.8}$$

Similarly, (3.6) is solved in [11] as

$$w_3(t) = \frac{7 \zeta(3)}{1-t} {}_2 F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{t}{t-1} \right) - 2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \left( -\frac{1}{2} \right)^2 \binom{n}{k} \frac{1}{(2k+1)^2} \left( -\frac{1}{2} \right)^{-2} t^n. \tag{3.9}$$

Therefore it follows that

$$J_3(n) = 7 \zeta(3) \sum_{j=0}^{n} (-1)^j \left( -\frac{1}{2} \right)^2 \binom{n}{j} - 2 \sum_{j=0}^{n} (-1)^j \left( -\frac{1}{2} \right)^2 \binom{n}{j} \sum_{k=0}^{j} \frac{1}{(2k+1)^2} \left( -\frac{1}{2} \right)^{-2}. \tag{3.9}$$

Remark 3.1. The function

$$W_2(T) = {}_2 F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; T^2 \right) = \frac{1}{3 \zeta(2)} (1-t)w_2(t) \quad \left( T^2 = \frac{t}{t-1} \right)$$

satisfies the differential equation

$$\left( T(T^2 - 1) \frac{d^2}{dT^2} + (3T^2 - 1) \frac{d}{dT} + T \right) W_2(T) = 0,$$

which can be regarded as a Picard-Fuchs equation for the universal family of elliptic curves having rational 4-torsion [12]. This is an analogue of the result [2] for the Apéry numbers for $\zeta(2)$ (see also Section 2.1). It is natural to ask whether there is such a modular interpretation for $w_3(t)$ (or “$W_3(T)$”). We have not obtained an answer to this question so far. \hfill \Diamond
3.3 Connection to the special values of $\zeta_Q(s)$

We also introduce another kind of generating functions for $J_k(n)$ as

$$g_2(z) := \sum_{n=0}^{\infty} \left( -\frac{3}{n} \right) J_2(n) z^n = 4 \int_0^1 \int_0^1 \frac{dx_1 dx_2}{\sqrt{(1-x_1^2 x_2^2)^2 + z(1-x_1^4)(1-x_2^4)}},$$

$$g_3(z) := \sum_{n=0}^{\infty} \left( -\frac{3}{n} \right) J_3(n) z^n = 8 \int_0^1 \int_0^1 \int_0^1 \frac{dx_1 dx_2 dx_3}{\sqrt{(1-x_1^2 x_2^2 x_3^2)^2 + z(1-x_1^4)(1-x_2^4)(1-x_3^4)}}.$$

The special values of $\zeta_Q(s)$ at $s = 2, 3$ are given as follows.

**Theorem 3.2** (Ichinose-Wakayama [8]). If $\alpha \beta > 2$ (i.e. $0 < 1/(1-\alpha \beta) < 1$), then

$$\zeta_Q(2) = 2 \left( \frac{\alpha + \beta}{2 \sqrt{\alpha \beta(\alpha \beta - 1)}} \right)^2 \left( \zeta(2, \frac{1}{2}) + \frac{(\alpha - \beta)}{\alpha + \beta} g_2 \left( \frac{1}{\alpha \beta - 1} \right) \right),$$

$$\zeta_Q(3) = 2 \left( \frac{\alpha + \beta}{2 \sqrt{\alpha \beta(\alpha \beta - 1)}} \right)^3 \left( \zeta(3, \frac{1}{2}) + 3 \frac{(\alpha - \beta)}{\alpha + \beta} g_3 \left( \frac{1}{\alpha \beta - 1} \right) \right),$$

where $\zeta(s, x) = \sum_{n=0}^{\infty} (n + x)^{-s}$ is the Hurwitz zeta function.

**Remark 3.3.** We can determine the functions $g_2(x)$ and $g_3(x)$ as follows:

$$g_2(x) = J_2(0) \bar{g}_2(x), \quad g_3(x) = J_3(0) \bar{g}_2(x) + \bar{g}_3(x),$$

where

$$\bar{g}_2(x) := \frac{1}{\sqrt{1+x}} 2F_1 \left( \frac{1}{4}, \frac{3}{4}; \frac{1}{1+x} \right) = 2F_1 \left( \frac{1}{4}, \frac{3}{4}; 1; -x \right),$$

$$\bar{g}_3(x) := \frac{2}{\sqrt{1+x}} \sum_{n=1}^{\infty} (-1)^n \left( -\frac{1}{n} \right)^3 \left( \frac{x}{1+x} \right)^n \frac{1}{(2j+1)^3} \left( \frac{1}{2} \right)^{-2}. $$

See [18] and [11] for detailed calculation. □

4 Higher Apéry-like numbers

Looking at the definition of $J_2(n)$ and $J_3(n)$, it is natural to introduce the numbers $J_k(n)$ by

$$J_k(n) := 2^k \int_{[0,1]^k} \left( \frac{1-x_1^4(1-x_2^4 \cdots x_k^4)}{1-x_1^2 \cdots x_k^2} \right)^n dx_1 dx_2 \cdots dx_k.$$

We refer to $J_k(n)$ as higher Apéry-like numbers. In fact, the generating function

$$g_k(z) := \sum_{n=0}^{\infty} \left( -\frac{3}{n} \right) J_k(n) z^n = 2^k \int_{[0,1]^k} \frac{dx_1 dx_2 \cdots dx_k}{\sqrt{(1-x_1^2 x_2^2 \cdots x_k^2)^2 + z(1-x_1^4)(1-x_2^4)(1-x_3^4)(1-x_3^4)}}$$

and its further generalizations are used to describe the ‘higher’ special values $\zeta_Q(k)$ ($k \geq 4$) like Theorem 3.2 (see Remark 4.1 below).

It is immediate that $J_k(0) = (2^k - 1) \zeta(k)$. Further, as we mentioned in [11], the formula

$$J_k(1) = \frac{3}{4} \sum_{m=0}^{\left\lfloor k/2 \right\rfloor} \frac{1}{4^m} \zeta \left( k - 2m, \frac{1}{2} \right) + \frac{1 - (-1)^k}{2^{k-1}}$$

(4.2)
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Example 4.3. First several terms of $J_k$ and shape with those of (3.1) and (3.2), and (4.3) gives a ‘vertical’ relation among $n$ for Theorem 4.2.

We also see that $J_4 (k) = 2 J_k (0) = J_{k-2} (1)$ $(k \geq 4)$.

Remark 4.1. We can calculate that

$$\zeta_2 (4) = 2 \left( \frac{\alpha + \beta}{2 \sqrt{\alpha \beta (\alpha \beta - 1)}} \right)^4 (\zeta (4, 1/2) + 4 \left( \frac{\alpha - \beta}{\alpha + \beta} \right)^2 g_4 \left( \frac{1}{\alpha \beta - 1} \right)$$

$$+ 2 \left( \frac{\alpha - \beta}{\alpha + \beta} \right)^4 \int_{[0, 1]^4} \frac{16 dx_1 dx_2 dx_3 dx_4}{\sqrt{(1 - x_1^2 x_2^2 x_3^2 x_4^2)^2 + 1 (1 - x_1^2 x_2^2)(1 - x_3^2 x_4^2)}}$$

$$+ \left( \frac{\alpha - \beta}{\alpha + \beta} \right)^4 \int_{[0, 1]^4} \frac{16 dx_1 dx_2 dx_3 dx_4}{\sqrt{(1 - x_1^2 x_2^2 x_3^2 x_4^2)^2 + 1 (1 - x_1^2 x_2^2)(1 - x_3^2 x_4^2)}}$$

where $\gamma_1 = 1/(\alpha \beta - 1)$ and $\gamma_2 = \alpha \beta/(\alpha \beta - 1)^2$.

Similar to the case of $J_2 (n)$ and $J_3 (n)$, the higher Apéry-like numbers $J_k (n)$ also satisfy a three-term recurrence relation as follows.

Theorem 4.2. The numbers $J_k (n)$ satisfy the recurrence relations

$$4 n^2 J_k (n) - (8 n^2 - 8 n + 3) J_k (n-1) + 4 (n-1)^2 J_k (n-2) = J_k_-2 (n)$$

for $n \geq 2$ and $k \geq 4$.

We give the proof of Theorem 4.2 in §5. It is remarkable that the left-hand side of (4.3) has a common shape with those of (3.1) and (3.2), and (4.3) gives a ‘vertical’ relation among $J_k (n)$’s, i.e. it connects $J_k (n)$’s and $J_k_-2 (n)$’s.

Example 4.3. First several terms of $J_4 (n)$ are given by

$$J_4 (0) = 15 \zeta (4), \quad J_4 (1) = \frac{45}{4} \zeta (4) + \frac{9}{16} \zeta (2), \quad J_4 (2) = \frac{615}{64} \zeta (4) + \frac{807}{1024} \zeta (2), \quad J_4 (3) = \frac{2205}{256} \zeta (4) + \frac{3745}{4096} \zeta (2), \quad J_4 (4) = \frac{129735}{16384} \zeta (4) + \frac{1044135}{1048576} \zeta (2), \ldots$$

We also see that

$$4 J_4 (1) - 3 J_4 (0) = \frac{9}{4} \zeta (2) = J_2 (1),$$

$$16 J_4 (2) - 19 J_4 (1) + 4 J_4 (0) = \frac{123}{64} \zeta (2) = J_2 (2),$$

$$36 J_4 (3) - 51 J_4 (2) + 16 J_4 (1) = \frac{441}{256} \zeta (2) = J_2 (3),$$

$$64 J_4 (4) - 99 J_4 (3) + 36 J_4 (2) = \frac{25947}{16384} \zeta (2) = J_2 (4).$$

Define another kind of generating function for $J_k (n)$ by

$$w_k (t) := \sum_{n=0}^{\infty} J_k (n) t^n = 2^k \int_0^1 \cdots \int_0^1 \frac{1 - x_1^2 \cdot x_2^2 \cdots x_k^2}{(1 - x_1^2 \cdot x_2^2 \cdots x_k^2) - (1 - x_1^2)(1 - x_2^2 \cdots x_k^2) t} dx_1 dx_2 \cdots dx_k. \quad (4.4)$$

Theorem 4.2 readily implies the
Corollary 4.4. The differential equation

\[ D_h w_k(t) = \frac{w_{k-2}(t) - w_{k-2}(0)}{4t} \]  

holds for \( k \geq 4 \). Here \( D_h \) is the differential operator given in (3.7).

Put

\[ J_0(n) := 0, \quad J_1(n) := \frac{(-1)^n}{n}, \quad J_k(n) := \left( \frac{-1/2}{n} \right) J_k(n) \quad (k \geq 2). \]

By Theorem 4.2, we have

\[ 8n^3 J_k(n) - (1 - 2n)(8n^2 - 8n + 3)J_k(n - 1) + 2(n - 1)(1 - 2n)(3 - 2n)J_k(n - 2) = 2nJ_{k-2}(n) \]

for \( k \geq 2 \) and \( n \geq 1 \). Hence, if we put

\[ D_W := 8z^2(1+z)^2 \frac{d^3}{dz^3} + 24z(1+z)(1+2z) \frac{d^2}{dz^2} + 2(4+27z+27z^2) \frac{d}{dz} + 3(1+2z), \]

then we have the following (See also [11, Proposition A.3]).

Corollary 4.5. The differential equations

\[ D_W g_2(z) = 0, \]

\[ D_W g_3(z) = -\frac{2}{1+z}, \]

\[ D_W g_k(z) = 2z \frac{d}{dz} \left( \frac{g_{k-2}(z) - g_{k-2}(0)}{z} \right) \quad (k \geq 4) \]

hold.

\[ \square \]

5 Proof of Theorem 4.2

5.1 Setting the stage

Assume \( k \geq 2 \). We notice that

\[ J_k(n) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-(t_1+\cdots+t_k)/2} (1 - e^{-2t_1})^n (1 - e^{-2(t_2+\cdots+t_k)})^n \frac{dt_1 \cdots dt_k}{(1 - e^{-(t_1+\cdots+t_k)})^{2n+1}} \]

\[ = \int_0^\infty \frac{e^{-u/2}}{(1 - e^{-u})^{2n+1}} du \int_0^u \frac{t^{k-2}}{(k-2)!}(1 - e^{-2t})^n (1 - e^{-2u+2t})^n \ dt \]

for each \( n \geq 0 \). Let us introduce

\[ I_{n,m}^{(k)}(u) := \int_0^u \frac{t^{k-2}}{(k-2)!}(1 - e^{-2t})^n (1 - e^{-2u+2t})^m \ dt \]

for \( n, m \geq 0 \). We also put

\[ \tilde{I}_{n,m}^{(k)}(u) := \frac{1}{2}(I_{n,m}^{(k)}(u) + I_{m,n}^{(k)}(u)), \quad \bar{I}_{n,m}^{(k)}(u) := \frac{1}{2}(I_{n,m}^{(k)}(u) - I_{m,n}^{(k)}(u)). \]

\( I_{n,m}^{(k)}(u) \) is symmetric in \( n \) and \( m \) if \( k = 2 \) so that \( \tilde{I}_{n,m}^{(2)}(u) = 0 \), but \( \bar{I}_{n,m}^{(k)}(u) \neq 0 \) in general.
Higher Apéry-like numbers arising from NCHO

It is convenient to set \( I_{n,m}^{(k)}(u) = 0 \) when \( k < 2 \). We see that

\[
J_k(n) = \frac{1}{2^{2n+1}} \int_0^\infty \frac{e^{nu}}{(\sinh \frac{u}{2})^{2n+1}} I_{n,m}^{(k)}(u) \, du.
\]

Thus we also set \( J_k(n) = 0 \) if \( k < 2 \). Under these convention, the following discussion for \( J_k(n) \) is reduced to the one given by Ichinose and Wakayama [8] when \( k = 2, 3 \).

For later use, we define

\[
a_n^{(k)}(u) := I_{n,n}^{(k)}(u) = \mathbb{I}^{(k)}_{n,n}(u) \quad (n \geq 0),
\]

\[
b_n^{(k)}(u) := \frac{1}{2} \left( I_{n,n-1}^{(k)}(u) + I_{n-1,n}^{(k)}(u) \right) = \mathbb{I}^{(k)}_{n,n-1} \quad (n \geq 1),
\]

\[
c_n^{(k)}(u) := \frac{1}{2} \left( I_{n,n-1}^{(k)}(u) - I_{n-1,n}^{(k)}(u) \right) = \mathbb{I}^{(k)}_{n,n-1} \quad (n \geq 1),
\]

\[
A_n^{(k)}(u) := e^{nu}a_n^{(k)}(u), \quad B_n^{(k)}(u) := \frac{A_n^{(k)}(u)}{(\sinh \frac{u}{2})^{2n+1}} \quad (n \geq 0),
\]

so that

\[
J_k(n) = \frac{1}{2^{2n+1}} \int_0^\infty B_n^{(k)}(u) \, du.
\]

### 5.2 Recurrence formulas for \( I_{n,m}^{(k)}(u) \)

Integration by parts implies

\[
I_{n,m}^{(k-1)} = \int_0^u \left( \frac{d}{dt} t^{k-2} \right) (1 - e^{-2t})^n (1 - e^{-2u+2t})^m \, dt
\]

\[
= - \int_0^u \frac{t^{k-2}}{(k-2)!} \left( \frac{d}{dt} e^{-2t} \right) (1 - e^{-2u+2t})^m \, dt
\]

\[
- \int_0^u \frac{t^{k-2}}{(k-2)!} (1 - e^{-2t})^n \left( \frac{d}{dt} e^{-2u+2t} \right) \, dt.
\]

when \( n, m \geq 1 \). Since

\[
\frac{d}{dt} (1 - e^{-2t})^n = 2ne^{-2t} (1 - e^{-2t})^{n-1}
\]

\[
= 2n \left( (1 - e^{-2t})^{n-1} - (1 - e^{-2t})^n \right),
\]

\[
\frac{d}{dt} (1 - e^{-2u+2t})^m = -2me^{-2u+2t} (1 - e^{-2u+2t})^{m-1}
\]

\[
= -2m \left( (1 - e^{-2u+2t})^{m-1} - (1 - e^{-2u+2t})^m \right)
\]

for \( n, m \geq 1 \), we obtain the

**Lemma 5.1.** The following three relations hold:

\[
\frac{1}{2} I_{n,m}^{(k-1)} = (n - m) I_{n,m}^{(k)} - nI_{n-1,m}^{(k)} + mI_{n,m-1}^{(k)} \quad (n, m \geq 1), \tag{5.2}
\]

\[
nI_{n,m}^{(k)} - (2n - 1)I_{n-1,m}^{(k)} + (n - 1)I_{n-2,m}^{(k)} - me^{-2u}I_{n-1,m-1}^{(k)} = \frac{1}{2} \left( I_{n,m}^{(k-1)} - I_{n-1,m}^{(k-1)} \right) \quad (n \geq 2, m \geq 1), \tag{5.3}
\]

\[
mI_{n,m}^{(k)} - (2m - 1)I_{n,m-1}^{(k)} + (m - 1)I_{n,m-2}^{(k)} - ne^{-2u}I_{n-1,m-1}^{(k)} = \frac{1}{2} \left( I_{n,m}^{(k-1)} - I_{n-1,m}^{(k-1)} \right) \quad (n \geq 1, m \geq 2). \tag{5.4}
\]
Letting $m \geq 2$. Similarly, specializing $n$ in (5.5), then we have
\[ I_{n,m}^{(k)} - 2I_{n,m-1}^{(k)} + (1 - e^{-2u})I_{n-1,m-1}^{(k)} = 0 \quad (n \geq 1, m \geq 1), \tag{5.5} \]
which is a generalization of (4.14) in [8]. In particular, if we let $n = m$ in (5.5), then we have
\[ I_{n,n}^{(k)} - 2I_{n,n-1}^{(k)} + (1 - e^{-2u})I_{n-1,n-1}^{(k)} = 0. \tag{5.6} \]
Letting $m = n - 1$ (or $n = m - 1$ and exchanging $m$ by $n$) in (5.5), we also have another specialization
\[ I_{n,n-1}^{(k)} - (I_{n-1,n}^{(k)} + I_{n,n-2}^{(k)}) + (1 - e^{-2u})I_{n-1,n-2}^{(k)} = 0 \quad (n \geq 2), \]
\[ I_{n-1,n}^{(k)} - (I_{n-2,n}^{(k)} + I_{n-1,n-1}^{(k)}) + (1 - e^{-2u})I_{n-2,n-1}^{(k)} = 0 \quad (n \geq 2). \]
Adding these equations, we get
\[ I_{n,n-2}^{(k)} = b_n^{(k)}(u) - a_{n-1}^{(k)}(u) + (1 - e^{-2u})b_{n-1}^{(k)}(u) \quad (n \geq 2). \tag{5.7} \]
By specializing $m = n$ in (5.3) and (5.4), we have
\[ nI_{n,n-1}^{(k)} - (2n - 1)I_{n-1,n}^{(k)} + (n - 1)I_{n-2,n}^{(k)} - 2ne^{-2u}I_{n-1,n-1}^{(k)} = \frac{1}{2}(I_{n,n-1}^{(k-1)} - I_{n-1,n}^{(k-1)}), \tag{5.8} \]
\[ nI_{n,n-2}^{(k)} - (2n - 1)I_{n-1,n-1}^{(k)} + (n - 1)I_{n-2,n-1}^{(k)} - 2ne^{-2u}I_{n-1,n-2}^{(k)} = \frac{1}{2}(I_{n,n-1}^{(k-1)} - I_{n-1,n}^{(k-1)}). \tag{5.9} \]
for $n \geq 2$. Similarly, specializing $m = n - 1$ in (5.3) and $n = m - 1$ in (5.4) (and exchanging $m$ by $n$), we have
\[ nI_{n,n-1}^{(k)} - (2n - 1)I_{n-1,n-1}^{(k)} + (n - 1)I_{n-2,n-1}^{(k)} - (n - 1)e^{-2u}I_{n-1,n-2}^{(k)} = \frac{1}{2}(I_{n,n-1}^{(k-1)} - I_{n-1,n}^{(k-1)}), \]
\[ nI_{n-1,n}^{(k)} - (2n - 1)I_{n-1,n-1}^{(k)} + (n - 1)I_{n-2,n}^{(k)} - (n - 1)e^{-2u}I_{n-1,n-2}^{(k)} = \frac{1}{2}(I_{n,n-1}^{(k-1)} - I_{n-1,n}^{(k-1)}). \]
for $n \geq 2$. Adding each pair of relations, we obtain
\[ 2n^{(k)}_{n,n-1} - 2(2n - 1)I_{n-1,n-1}^{(k)} + 2(n - 1)I_{n-2,n}^{(k)} - 2ne^{-2u}I_{n-1,n-1}^{(k)} = \frac{1}{2}(k_{n,n-1}^{(k-1)}), \tag{5.10} \]
\[ 2n^{(k)}_{n,n-2} - 2(2n - 1)I_{n-1,n-2}^{(k)} + 2(n - 1)(1 - e^{-2u})I_{n-1,n-2}^{(k)} = \frac{1}{2}(k_{n,n-1}^{(k-1)}). \tag{5.11} \]
The formulas (5.6), (5.10) and (5.11) are rewritten as follows.

**Lemma 5.2.** The equations
\[ a_n^{(k)}(u) + (1 - e^{-2u})a_{n-1}^{(k)}(u) = 2b_n^{(k)}(u), \tag{5.12} \]
\[ na_n^{(k)}(u) - (2n - 1)b_n^{(k)}(u) + (n - 1)I_{n,n-2}^{(k)} - ne^{-2u}a_{n-1}^{(k)}(u) = \frac{1}{2}I_n^{(k-1)}(u), \tag{5.13} \]
\[ nb_n^{(k)}(u) - (2n - 1)a_{n-1}^{(k)}(u) + (n - 1)(1 - e^{-2u})b_{n-1}^{(k)}(u) = \frac{1}{2}b_n^{(k-1)}(u). \tag{5.14} \]
hold.

As a corollary, we also get

**Lemma 5.3.** The equation
\[ na_n^{(k)}(u) - (2n - 1)(1 + e^{-2u})a_{n-1}^{(k)}(u) + (n - 1)(1 - e^{-2u})a_{n-2}^{(k)}(u) = b_n^{(k-1)}(u) \tag{5.15} \]
holds.
Thus we have

\[ \begin{align*}
\tilde{b}^{(k-1)}_n(u) &= 2nb_n^{(k)}(u) - 2(2n-1)a^{(k)}_{n-1}(u) + 2(n-1)(1-e^{-2u})b^{(k)}_{n-1}(u) \\
&= n\left(a^{(k)}_n(u) + (1-e^{-2u})a^{(k)}_{n-1}(u)\right) - 2(2n-1)a^{(k)}_{n-1}(u) \\
&\quad + (n-1)(1-e^{-2u})\left(a^{(k)}_{n-1}(u) + (1-e^{-2u})a^{(k)}_{n-2}(u)\right) \\
&= na^{(k)}_n(u) - (2n-1)(1+e^{-2u})a^{(k)}_{n-1}(u) + (n-1)(1-e^{-2u})^2a^{(k)}_{n-2}(u),
\end{align*} \]

which is the desired formula. \( \square \)

Here we give one more useful relation. Using (5.2) twice, we see that

\[ \begin{align*}
\frac{1}{4}I^{(k-2)}_{n,n} &= \frac{1}{2}\left(-nI^{(k)}_{n-1,n} + nI^{(k)}_{n,n-1}\right) \\
&= -n\left(I^{(k)}_{n-1,n} - (n-1)I^{(k)}_{n-2,n} + nI^{(k)}_{n-1,n-1}\right) + n\left(I^{(k)}_{n,n-1} - nI^{(k)}_{n-1,n-1} + (n-1)I^{(k)}_{n,n-2}\right) \\
&= n\left(2b^{(k)}_n(u) - 2na^{(k)}_{n-1}(u) + 2(n-1)I^{(k)}_{n,n-2}\right).
\end{align*} \]

Thus we have

\[ a^{(k-2)}_n(u) = 8n\left(b^{(k-1)}_n(u) - na^{(k)}_{n-1}(u) + (n-1)I^{(k)}_{n,n-2}\right). \tag{5.16} \]

Combining (5.7), (5.16) and (5.14), we obtain

**Lemma 5.4.** The equation

\[ a^{(k-2)}_n(u) = 4nb^{(k-1)}_n(u) \tag{5.17} \]

holds. \( \square \)

In particular, the formula (5.15) is rewritten as

\[ na^{(k)}_n(u) - (2n-1)(1+e^{-2u})a^{(k)}_{n-1}(u) + (n-1)(1-e^{-2u})^2a^{(k)}_{n-2}(u) = \frac{1}{4n}I^{(k-2)}_n(u). \tag{5.18} \]

### 5.3 Relations for \( B^{(k)}_n(u) \)

In view of (5.1), the differential

\[ \frac{d}{du}a^{(k)}_n(u) = 2n \int_0^u \frac{I^{(k-2)}_n}{(k-2)!}(1-e^{-2t})^n e^{-2u+2t}(1-e^{-2u+2t})^{n-1} dt \]

is written in two ways as

\[ \frac{d}{du}a^{(k)}_n(u) = 2n\left(I^{(k)}_{n,n-1} - I^{(k)}_{n,n}\right) = -2n\left(I^{(k)}_{n,n} - I^{(k)}_{n-1,n}\right) + I^{(k-1)}_{n,n} \]

for \( n \geq 1 \). Hence it follows that

\[ \frac{d}{du}a^{(k)}_n(u) = n\left(I^{(k)}_{n,n-1} - I^{(k)}_{n,n}\right) - n\left(I^{(k)}_{n,n} - I^{(k)}_{n-1,n}\right) + \frac{1}{2}I^{(k-1)}_{n,n} \]

\[ = -na^{(k)}_n(u) + n(1-e^{-2u})a^{(k)}_{n-1}(u) + \frac{1}{2}a^{(k-1)}_n(u). \tag{5.19} \]
Using this formula, we have
\[
\frac{d}{du} A_n^{(k)}(u) - 2n \sinh u A_n^{(k)}(u) = e^{nu} \left( \frac{d}{du} a_n^{(k)}(u) + na_n^{(k)}(u) - n(1 - e^{-2u}) a_{n-1}^{(k)}(u) \right) = \frac{1}{2} A_n^{(k-1)}(u) \tag{5.20}
\]
Thus we obtain the

**Lemma 5.5.** The equation
\[
2 \tanh \frac{u}{2} \frac{d}{du} B_n^{(k)}(u) = 8n B_n^{(k)}(u) - (2n + 1) B_n^{(k)}(u) + \tanh \frac{u}{2} B_n^{(k-1)}(u) \tag{5.21}
\]
holds for \( n \geq 1 \).

**Remark 5.6.** The differential of \( a_0^{(k)}(u) \) is given by
\[
\frac{d}{du} a_0^{(k)}(u) = \frac{u^{k-2}}{(k-2)!}
\]
when \( k \geq 2 \). If \( k \geq 3 \), this is equal to \( a_0^{(k-1)}(u) \).

We also see from (5.18) that
\[
n A_n^{(k)}(u) - 2(2n - 1) \cosh u A_n^{(k-1)}(u) + 4(n - 1) \sinh u A_{n-2}^{(k)}(u) = e^{nu} A_n^{(k-2)}(u) = \frac{1}{4n} A_n^{(k-2)}(u) \tag{5.22}
\]
This implies the

**Lemma 5.7.** The equation
\[
n \left( 1 - \frac{1}{\cosh^2 \frac{u}{2}} \right) B_n^{(k)}(u) = 4(2n - 1) B_{n-1}^{(k)}(u) - \frac{2(2n - 1)}{\cosh^2 \frac{u}{2}} B_{n-1}^{(k)}(u) - 16(n - 1) B_{n-2}^{(k)}(u) + \frac{1}{4n} \left( 1 - \frac{1}{\cosh^2 \frac{u}{2}} \right) B_{n-2}^{(k-2)}(u) \tag{5.23}
\]
holds for \( n \geq 2 \).

5.4 Recurrence formula for \( J_k(n) \)

Define
\[
K_k(n) = \frac{1}{2^{2n+1}} \int_0^\infty \frac{B_n^{(k)}(u)}{\cosh^2 \frac{u}{2}} du, \quad M_k(n) = \frac{1}{2^{2n+1}} \int_0^\infty \tanh \frac{u}{2} B_n^{(k-1)}(u) du. \tag{5.24}
\]
By integrating (5.21) and (5.23), we have
\[
K_k(n) = (2n + 1) J_k(n) - 2n J_k(n - 1) - M_k(n), \tag{5.25}
2n(J_k(n) - K_k(n)) = (2n - 1)(2J_k(n - 1) - K_k(n - 1)) - 2(n - 1) J_k(n - 2) + \frac{1}{2n} (J_{k-2}(n) - K_{k-2}(n)). \tag{5.26}
\]
Plugging these equations, we obtain
Lemma 5.8. Put
\[ L_k(n) := J_{k-2}(n) - J_{k-2}(n-1) + 2nM_k(n) - (2n-1)M_k(n-1) - \frac{1}{2n}M_{k-2}(n). \] (5.27)

The recurrence formula
\[ 4n^2J_k(n) - (8n^2 - 8n + 3)J_k(n-1) + 4(n-1)^2J_k(n-2) = L_k(n) \] (5.28)
holds for \( k \geq 2 \) and \( n \geq 2 \).

When \( k = 2 \), the inhomogeneous term \( L_2(n) \) in (5.28) vanishes and we get (3.1). When \( k = 3 \), we see that \( L_3(n) = 2nM_3(n) - (2n-1)M_3(n-1) \), which is equal to \( \frac{2^{2n}(n-1)!}{(2n-1)!} \) (Lemma 6.3 in [8]), so we have (3.2).

5.5 Calculation of the inhomogeneous terms

Let us put
\[ Q_k(n) := \frac{1}{2^{2n+1}} \int_0^\infty \frac{B_n^{(k)}(u)}{\tanh \frac{u}{2}} du. \] (5.29)

This definite integral converges if \( k \geq 3 \).

From (5.21), we have
\[ 2 \frac{d}{du} B_n^{(k)}(u) = 8n B_{n-1}^{(k)}(u) - (2n + 1) \frac{B_n^{(k)}(u)}{\tanh \frac{u}{2}} + B_n^{(k-1)}(u). \]

It follows then
\[ 0 = 8n \cdot 2^{2n-1} Q_k(n-1) - (2n + 1) 2^{2n+1} Q_k(n) + 2^{2n+1} J_{k-1}(n), \]
and hence
\[ J_{k-1}(n) = (2n + 1)Q_k(n) - 2nQ_k(n-1) \] (5.30)
for \( k \geq 3 \) and \( n \geq 1 \).

From (5.22), we also see that
\[ n \tanh \frac{u}{2} B_n^{(k)}(u) - 2(2n-1) \left( \frac{1}{\tanh \frac{u}{2}} + \tanh \frac{u}{2} \right) B_n^{(k-1)}(u) + 16(n-1) \frac{B_n^{(k)}(u)}{\tanh \frac{u}{2}} = \frac{1}{4n} \tanh \frac{u}{2} B_n^{(k-2)}(u). \]

Thus we have
\[ n 2^{2n+1} M_{k+1}(n) - 2(2n-1) 2^{2n-1} (Q_k(n-1) + M_k(n-1)) + 16(n-1) 2^{2n-3} Q_k(n-2) = \frac{1}{4n} 2^{2n+1} M_{k-1}(n), \]
which implies
\[ 2nM_{k+1}(n) - (2n-1)M_{k+1}(n-1) - \frac{1}{2n}M_{k-1}(n) \]
\[ = (2n-1)Q_k(n-1) - 2(n-1)Q_k(n-2) \] (5.31)
for \( k \geq 3 \) and \( n \geq 2 \).
Using (5.30) and (5.31), we obtain

\[ 2nM_k(n) - (2n - 1)M_k(n - 1) - \frac{1}{2n}M_{k-2}(n) \]

\[ = (2n - 1)Q_{k-1}(n - 1) - 2(n - 1)Q_{k-1}(n - 2) = J_{k-2}(n - 1) \]

for \( k \geq 4 \) and \( n \geq 2 \). Hence the inhomogeneous term is computed as

\[ L_k(n) = J_{k-2}(n) - J_{k-2}(n - 1) + J_{k-2}(n - 1) = J_{k-2}(n) \]

(5.32)

for \( k \geq 4 \) and \( n \geq 2 \). This completes the proof of Theorem 4.2.

**Remark 5.9.** It may be “natural” to assume (or interpret) that

\[ J_0(n) = 0, \quad J_1(n) = 2 \int_0^1 (1 - x^2)^{n-1} dx = \frac{2^n(n-1)!}{(2n-1)!!} \]

and

\[ w_0(t) = 0, \quad "w_1(t) - w_1(0)" = \sum_{n=1}^{\infty} \frac{2^n(n-1)!}{(2n-1)!!} t^n = 2t_2F_1\left(1, 1; \frac{3}{2}; t\right). \]

Under this convention, Theorem 4.2 and Corollary 4.4 would include the case where \( k = 2, 3 \). 

\[ \diamond \]

## 6 Infinite series expression

We give an infinite series expression of \( J_k(n) \). Using it, we prove the equation (4.2).

### 6.1 Infinite series expression of \( J_k(n) \)

Let us put

\[ f_n(s, t) := \frac{1}{(1 - s^2 t^2)^2} \left( \frac{(1 - s^4)(1 - t^4)}{1 - s^2 t^2} \right)^n = (1 - s^4)^n (1 - t^4)^n (1 - s^2 t^2)^{-2n-1}. \]

Then we have

\[ J_k(n) = 2^k \int_0^1 \int_0^1 \cdots \int_0^1 f_n(x_1, x_2, \ldots, x_k) dx_1 \cdots dx_k. \]

Since

\[ f_n(s, t) = (1 - s^4)^n (1 - t^4)^n \sum_{l=0}^{\infty} \left( \frac{-2n-1}{l} \right) (-s^2 t^2)^l = \frac{1}{(2n)!} \sum_{l=0}^{\infty} (l + 1)_{2n} s^{2l} (1 - s^4)^n t^{2l} (1 - t^4)^n, \]

it follows that

\[ J_k(n) = \frac{2^k}{(2n)!} \sum_{l=0}^{\infty} (l + 1)_{2n} I_1(l, n) I_{k-1}(l, n). \]

Here \( I_p(l, n) \) is given by

\[ I_p(l, n) := \int_0^1 \int_0^1 \cdots \int_0^1 (u_1 \cdots u_p)^{2l} (1 - (u_1 \cdots u_p)^4)^n du_1 \cdots du_p. \]
Notice that

\[
I_1(l, n) = \int_0^1 u^{2l} (1 - u^4)^n du = \frac{4^n n!}{(2l + 1)(2l + 5) \cdots (2l + 4n + 1)},
\]

\[
I_p(l, n) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \int_0^1 \cdots \int_0^1 (u_1 \cdots u_p)^{2l+4j} du_1 \cdots du_p = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{(2l + 4j + 1)^{p}}.
\]

Thus we obtain the expression

\[
J_k(n) = 2^{k+1} \frac{4^n n!}{(2n)!} \sum_{l=0}^{\infty} \frac{(l + 1)(l + 2)}{(2l + 1)(2l + 5) \cdots (2l + 4n + 1)} \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j} (2l + 4j + 1)^{-k}}{(2l + 4j + 1)^{k-1}}.
\]

### 6.2 Example: calculation of \( J_k(1) \)

When \( n = 1 \), we see that

\[
J_k(1) = 2 \cdot 2^k \sum_{l=0}^{\infty} \frac{(l + 1)(l + 2)}{(2l + 1)(2l + 5) \cdots (2l + 4n + 1)} \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j} (2l + 4j + 1)^{-k}}{(2l + 4j + 1)^{k-1}}
\]

Using the identity

\[
(l + 1)(l + 2) \left( (2l + 5)^{-k-1} - (2l + 1)^{-k-1} \right)
\]

\[
= (2l + 1)(2l + 5) - (2l + 1) + (2l + 5) - 1 \sum_{j=0}^{k-2} (2l + 1)^j (2l + 5)^{-k-2-j},
\]

we have

\[
J_k(1) = 2 \cdot 2^k \sum_{j=0}^{k-2} \left\{ S(k - j - 1, j + 1) - S(k - j - 1, j + 2) + S(k - j, j + 1) - S(k - j, j + 2) \right\},
\]

where

\[
S(\alpha, \beta) := \sum_{l=0}^{\infty} (2l + 1)^{-\alpha} (2l + 5)^{-\beta}.
\]

Since

\[
\sum_{j=1}^{k-1} S(j, k - j) = \sum_{l=0}^{\infty} \sum_{j=1}^{k-1} (2l + 1)^{-j} (2l + 5)^{j-k}
\]

\[
= \sum_{l=0}^{\infty} \frac{1}{(2l + 5)^k} \frac{2l + 5}{2l + 1} \frac{1 - \left( \frac{2l + 5}{2l + 1} \right)^{k-1}}{1 - \left( \frac{2l + 5}{2l + 1} \right)}
\]

\[
= \frac{1}{4} \sum_{l=0}^{\infty} \left( \frac{1}{(2l + 1)^{k-1}} - \frac{1}{(2l + 5)^{k-1}} \right) = \frac{1 + 3^{1-k}}{4},
\]

we have

\[
J_k(1) = 2^{k+1} \left( \frac{2}{3^{k+1}} + S(1, k) - S(1, k) + S(k + 1, 1) + S(1, k + 1) \right).
\]
Let us calculate $S(k,1)$ and $S(1,k)$. By the partial fraction expansion

$$\frac{1}{x(x + \alpha)^k} = \frac{1}{\alpha^k} \left( \frac{1}{x} - \frac{1}{x + \alpha} \right) - \sum_{m=2}^{k} \frac{1}{\alpha^{k-m+1}(x + \alpha)^m},$$

we see that

$$\frac{1}{(2l + 1)(2l + 5)} = -\left( 1 \right)^k \left( \frac{1}{2l + 1} - \frac{1}{2l + 5} \right) + \frac{1}{2} \sum_{m=2}^{k} \left( \frac{1}{2} \right)^{k-m+2} \frac{1}{(l + \frac{1}{2})^m},$$

$$\frac{1}{(2l + 1)^k(2l + 5)^k} = \left( \frac{1}{4} \right)^k \left( \frac{1}{2l + 1} - \frac{1}{2l + 5} \right) + \frac{1}{2} \sum_{m=2}^{k} \left( \frac{1}{2} \right)^{k-m+2} \frac{1}{(l + 2 + \frac{1}{2})^m}.$$}

Thus it follows that

$$S(k,1) = \frac{1}{2k} \sum_{m=2}^{k} \left( \frac{1}{2} \right)^{k-m+2} \zeta(m, \frac{1}{2}) + \frac{1}{3} \left( \frac{1}{4} \right)^{k-1},$$

$$S(1,k) = -\frac{1}{2k} \sum_{m=2}^{k} \left( \frac{1}{2} \right)^{k-m+2} \zeta(m, \frac{1}{2}) + \frac{1}{3} \left( \frac{1}{4} \right)^{k-1} + \frac{1}{3}.$$

If we substitute these to (6.2), then we have

$$J_k(1) = 2^{k+1} \left( \frac{2}{3^{k+1}} + \frac{1}{2} \sum_{m=2}^{k} \left( \frac{1}{2} \right)^{k-m+2} \zeta(m, \frac{1}{2}) + \frac{1}{3} \left( \frac{1}{4} \right)^{k-1} \right)$$

$$+ \frac{1}{2} \sum_{m=2}^{k} \left( \frac{1}{2} \right)^{k-m+2} \zeta(m, \frac{1}{2}) + \frac{1}{3} \left( \frac{1}{4} \right)^{k-1} - \frac{1}{3} - \frac{1}{3^k}$$

$$+ \frac{1}{2} \sum_{m=2}^{k+1} \left( \frac{1}{2} \right)^{k-m+3} \zeta(m, \frac{1}{2}) + \frac{1}{3} \left( \frac{1}{4} \right)^{k}$$

$$- \frac{1}{2} \sum_{m=2}^{k+1} \left( \frac{1}{2} \right)^{k-m+3} \zeta(m, \frac{1}{2}) - \frac{1}{3} \left( \frac{1}{4} \right)^{k} + \frac{1}{3} + \frac{1}{3^{k+1}}.$$}

Now it is straightforward to see that

$$J_k(1) = 3 \sum_{m=2}^{k} \frac{1 + (-1)^{k-m}}{2k-m+3} \zeta(m, \frac{1}{2}) + \frac{1 + (-1)^{k-1}}{2k-1}$$

$$= 3 \sum_{2 \leq m \leq k} \frac{2^m - 1}{2k-m} \zeta(m) + \frac{1 + (-1)^{k-1}}{2k-1}$$

$$= 3 \sum_{m=0}^{\lfloor k/2 \rfloor - 1} 2^{-2m} \zeta(k - 2m, \frac{1}{2}) + \frac{1 - (-1)^k}{2k-1}.$$}

### 7 Differential equations for generating functions

Utilizing the differential equations for the generating functions $w_k(t)$, we give another kind of relations among the higher Apéry-like numbers $J_k(n)$. 
7.1 Equivalent differential equations

Consider the inhomogeneous (singly confluent) Heun differential equation

\[
D_u w(t) = u(t)
\]

for a given function \(u(t)\). Put \(z = \frac{t}{1-t}\) and \(v(z) = (1-t)w(t)\). Then we have

\[
D_o v(z) = \frac{1}{z-1} u \left( \frac{z}{z-1} \right).
\]

Here \(D_o\) is the \textit{hypergeometric} differential operator given by

\[
D_o = z(1-z) \frac{d^2}{dz^2} + (1-2z) \frac{d}{dz} - \frac{1}{4}.
\]

We also remark that this is also the Picard-Fuchs differential operator for the family \(y^2 = x(x-1)(x-z)\) of elliptic curves.

7.2 Recurrence formula for \(J_k(n)\)

Put \(z = \frac{t}{1-t}\) and \(v_k(z) = (1-t)w_k(t)\). By Theorem 4.2, \(v_k(z)\) satisfies the differential equation

\[
(D_o v)(z) = \frac{1}{4(z-1)} \sum_{j=0}^{\infty} J_{k-2}(j+1) \left( \frac{z}{z-1} \right)^j = \sum_{n=0}^{\infty} \left( \frac{1}{4} \right) \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{j} J_{k-2}(j+1) z^n.
\] (7.1)

The polynomial functions

\[
p_n(z) := -\frac{4}{(2n+1)^2} \left( -\frac{1}{2} \right)^n \sum_{k=0}^{n} \left( -\frac{1}{2} \right)^k z^k
\]

satisfy the equation

\[
(D_o p_n)(z) = z^n.
\] (7.3)

Hence we can construct a local holomorphic solution to (7.1) as

\[
v(z) = \sum_{n=0}^{\infty} \left( \frac{1}{4} \right) \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{j} J_{k-2}(j+1) p_n(z).
\] (7.4)

Notice that the difference \(v_k(z) - v(z)\) satisfies the homogeneous differential equation

\[
(D_o(v_k - v))(z) = 0.
\] (7.5)

Thus it follows that

\[
v_k(z) - v(z) = C_k v_2(z),
\] (7.6)

where the constant \(C_k\) is determined by

\[
C_k = \frac{v_k(0) - v(0)}{v_2(0)} = \frac{(2^k - 1)\zeta(k) - v(0)}{3\zeta(2)},
\] (7.7)
and \( v(0) \) is given by
\[
v(0) = -\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \left( \frac{-1/2}{n} \right)^2 \left( \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{j} J_{k-2}(j+1) \right).
\] (7.8)

Therefore we have
\[
v_k(z) = (2^k - 1)\zeta(k) \binom{1/2}{1} F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; z \right) + \left( v(z) - v(0) \right) \binom{1/2}{1} F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; z \right).
\] (7.9)

Consequently, we obtain the

**Theorem 7.1.** When \( k \geq 4 \), the equation
\[
J_k(n) = \sum_{p=0}^{n} (-1)^p \binom{1/2}{p} \left( \binom{1/2}{n} \right)^2 \left( \sum_{i=0}^{p-1} \frac{1}{2i+1} \right) \left( \sum_{j=0}^{i} (-1)^{j+1} \binom{i}{j} J_{k-2}(j+1) \right)
\] (7.10)
holds.

**Remark 7.2.** If we formally put \( J_1(n) = \frac{2^n(n-1)!}{(2n-1)!} \) in (7.10), then we have
\[
J_3(n) = \sum_{p=0}^{n} (-1)^p \binom{1/2}{p} \left( \binom{1/2}{n} \right)^2 \left( 7\zeta(3) - 2 \sum_{i=0}^{p-1} \frac{1}{2i+1} \right)
\]

since
\[
\sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{2^{j+1}j!}{(2j+1)!!} = \frac{2}{2i+1}.
\]

This is nothing but the explicit formula (3.9) for \( J_3(n) \).

**Example 7.3.** Since
\[
\sum_{j=0}^{i} (-1)^j \binom{i}{j} J_2(j+1) = 3\zeta(2) \left( \binom{1/2}{i}^2 - \binom{1/2}{i+1}^2 \right) = 3\zeta(2) \binom{1/2}{i}^2 \left( 1 - \frac{(2i+1)^2}{(2i+2)^2} \right),
\]
we have
\[
J_4(n) = \sum_{p=0}^{n} (-1)^p \binom{1/2}{p} \left( \binom{1/2}{n} \right)^2 \left( 15\zeta(4) - 3\zeta(2) \sum_{i=1}^{2p} \frac{(-1)^{i-1}}{i^2} \right).
\]

### 7.3 Ascent operation and normalized higher Apéry-like numbers

For a given sequence \( \{J(n)\}_{n \geq 0} \), we associate a new sequence
\[
J(n)^\#: = \sum_{p=0}^{n} (-1)^p \binom{1/2}{p} \left( \binom{1/2}{n} \right)^2 \left\{ \sum_{i=0}^{p-1} \frac{-1}{(2i+1)^2} \binom{1/2}{i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} J(j+1) \right\}.
\] (7.11)

Notice that \( J(0)^\# = 0 \). It would be natural to extend \( J(n)^\# = 0 \) if \( n < 0 \). By the discussion in the previous subsection, we have the
Lemma 7.4. Let \( \{J(n)\} \) be a given sequence and \( \{J(n)^p\} \) the one defined by (7.11). Then the equation
\[
4n^2J(n)^2 - (8n^2 - 8n + 3)J(n - 1)^2 + 4(n - 1)^2J(n - 2)^2 = J(n)
\]
holds for \( n \geq 1 \).

Let us introduce the rational sequences \( \tilde{J}_k(n) \) by
\[
\tilde{J}_1(n) := \frac{2^n(n-1)!}{(2n-1)!!} \quad (n \geq 1), \quad \tilde{J}_2(n) := J_2(n)/J_2(0) \quad (n \geq 0),
\]
\[
\tilde{J}_k(n) := \tilde{J}_{k-2}(n)^4 \quad (k \geq 3, \ n \geq 0).
\]
We see that
\[
\tilde{J}_{2k}(1) = \frac{3}{4^k}, \quad \tilde{J}_{2k+1}(1) = \frac{2}{4^k}.
\]
It is immediate to verify the

**Proposition 7.5.**

\[
J_k(n) = \sum_{m=0}^{\lfloor k/2 \rfloor - 1} \zeta(k-2m, 1/2)\tilde{J}_{2m+2}(n) + \frac{1 - (-1)^k}{2}\tilde{J}_k(n).
\]

After this fact, we call \( \tilde{J}_k(n) \) the normalized (higher) Apéry-like numbers. By definition, \( \tilde{J}_k(n) \) for \( k \geq 2 \) are written in the form
\[
\tilde{J}_k(n) = \sum_{p=0}^{n} (-1)^p \left( \frac{1}{p} \right)^2 \left( \frac{n}{p} \right) S_k(p),
\]
where
\[
S_2(p) = 1, \quad S_3(p) = -2 \sum_{i=0}^{p-1} \frac{1}{(2i+1)^3} \left( \frac{-1}{i} \right)^2 = -2 \sum_{i=0}^{p-1} \frac{(1/2)_i(1/2)_{i+1}^2}{(3/2)_i^3},
\]
\[
S_k(p) = \sum_{i=0}^{p-1} \frac{-1}{(2i+1)^2} \left( \frac{-1}{i} \right)^2 \sum_{j=0}^{i} (-1)^j \left( \frac{i}{j} \right) \tilde{J}_{k-2}(j+1) \quad (k \geq 4).
\]
Thus it is enough to investigate \( S_k(p) \) to obtain an explicit expression for normalized Apéry-like numbers.

**Lemma 7.6.**

\[
S_{k+2}(p+1) - S_{k+2}(p) = \frac{S_k(p+1)}{(2p+2)^2} - \frac{S_k(p)}{(2p+1)^2}.
\]

**Proof.** By definition, we have
\[
S_{k+2}(p+1) - S_{k+2}(p) = \frac{-1}{(2p+1)^2} \left( \frac{-1/2}{p} \right)^2 \sum_{j=0}^{p} (-1)^j \left( \frac{p}{j} \right) \tilde{J}_k(j+1).
\]
The sum in the right hand side is calculated as
\[
\sum_{j=0}^{p} (-1)^j \left( \frac{p}{j} \right) \tilde{J}_k(j+1)
\]
\[
= \sum_{j=0}^{p} (-1)^j \left( \frac{p}{j} \right) \left( \sum_{q=0}^{j} (-1)^q \left( \frac{-1/2}{q} \right)^2 \left( \frac{j+1}{q} \right) S_k(q) + (-1)^{j+1} \left( \frac{-1/2}{j+1} \right)^2 S_k(j+1) \right)
\]
\[
= \sum_{q=0}^{p} (-1)^q \left( \frac{-1/2}{q} \right)^2 S_k(q) \sum_{j=q}^{p} (-1)^j \left( \frac{p}{j} \right) \left( \frac{j+1}{q} \right) - \sum_{j=0}^{p} (-1)^j \left( \frac{-1/2}{j+1} \right)^2 S_k(j+1).
\]
By the elementary identity
\[ \sum_{j=p}^{n} (-1)^j \binom{n}{j} \binom{j}{p} = (-1)^p \delta_{np} \quad (n, p \in \mathbb{Z}_{\geq 0}), \]
we get
\[ \sum_{j=q}^{p} (-1)^j \binom{p}{j} \left( \frac{j+1}{q} \right) = \sum_{j=q}^{p} (-1)^j \binom{p}{j} \left( \frac{j}{q-1} \right) = (-1)^q \left( \delta_{pq} + \binom{p}{q-1} \right). \]

Thus it follows that
\[ \sum_{j=0}^{p} (-1)^j \binom{p}{j} \tilde{J}_k(j+1) = \left( -\frac{1}{2} \right)^2 \binom{p}{2} S_k(p) + \sum_{q=0}^{p} \left( -\frac{1}{2} \right)^2 \binom{p}{q} S_k(q) \left( \frac{p}{q-1} \right) - \sum_{j=0}^{p} \binom{p}{j} \left( -\frac{1}{2} \right)^2 \binom{p}{j+1} S_k(j+1) \]
\[ = \left( -\frac{1}{2} \right)^2 \binom{p}{2} S_k(p) - \left( -\frac{1}{2} \right)^2 \binom{p+1}{p+1} S_k(p+1) \]
\[ = (2p+1)^2 \left( -\frac{1}{2} \right)^2 \left( \frac{S_k(p)}{(2p+1)^2} - \frac{S_k(p+1)}{(2p+2)^2} \right). \]

Therefore we obtain
\[ S_{k+2}(p+1) - S_{k+2}(p) = \frac{S_k(p+1)}{(2p+2)^2} - \frac{S_k(p)}{(2p+1)^2} \]
as we desired.

As a corollary, we readily have the

**Lemma 7.7.**
\[ S_{k+2}(p) = \sum_{q=1}^{p} \left( \frac{S_k(q)}{(2q)^2} - \frac{S_k(q-1)}{(2q-1)^2} \right). \] (7.18)

Using this lemma repeatedly, we obtain the

**Proposition 7.8.** For each \( r \geq 1 \),
\[ S_{2r+2}(p) = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq 2p} \frac{(-1)^{i_1+\cdots+i_r}}{i_1^2 \cdots i_r^2} \varepsilon_{i_1, \ldots, i_r}, \] (7.19)
\[ S_{2r+3}(p) = \sum_{1 \leq 2j-1 \leq \cdots \leq i_r \leq 2p} \frac{1}{(2j-1)^3} \left( -\frac{1}{2} \right)^2 \binom{i_1+\cdots+i_r}{j-1} \frac{(-1)^{i_1+\cdots+i_r}}{i_1^2 \cdots i_r^2} \varepsilon_{i_1, \ldots, i_r}, \] (7.20)

where
\[ \varepsilon_{i_1, \ldots, i_r} := \begin{cases} 0 & 1 \leq \exists j < r \text{ s.t. } i_j = i_{j+1} \equiv 1 \pmod{2}, \\ 1 & \text{otherwise.} \end{cases} \] (7.21)
Example 7.9. We have
\[
S_4(p) = \sum_{j=1}^{2p} \frac{(-1)^j}{j^2}, \\
S_6(p) = \sum_{1 \leq i \leq j \leq 2p} \frac{(-1)^{i+j}}{i^2 j^2} \xi_{i,j} = \sum_{1 \leq i < j \leq 2p} \frac{(-1)^{i+j}}{i^2 j^2} + \sum_{i=1}^{p} \frac{1}{(2i)^4}, \\
S_8(p) = \sum_{1 \leq i \leq j \leq k \leq 2p} \frac{(-1)^{i+j+k}}{i^2 j^2 k^2} \varepsilon_{i,j,k} = \sum_{1 \leq i < j < k \leq 2p} \frac{(-1)^{i+j+k}}{i^2 j^2 k^2} + \left( \sum_{1 \leq i \leq 2} \frac{1}{(2i)^4} \right) \left( \sum_{1 \leq k \leq 2p} \frac{(-1)^k}{k^2} \right).
\]

\[\diamond\]

Remark 7.10. We see that
\[
\lim_{p \to \infty} S_{2r}(p) = 1, \quad \lim_{p \to \infty} S_4(p) = -\frac{\pi^2}{12}, \quad \lim_{p \to \infty} S_6(p) = -\frac{\pi^4}{720}.
\] (7.22)

In general, we can prove that
\[
\lim_{p \to \infty} S_{2r+2}(p) = \frac{\zeta(2r)}{2^{2r-1}}.
\] (7.23)

See [13] for the proof. \[\diamond\]

8 Congruence relations among Apéry-like numbers

In this section, we study the congruence relation among the normalized Apéry-like numbers introduced in the previous section.

8.1 Congruence relations for Apéry-like numbers

We give several congruence relations among Apéry-like numbers.

Proposition 8.1 ([11, Proposition 6.1]). Let \( p \) be a prime and \( n = n_0 + n_1 p + \cdots + n_k p^k \) be the \( p \)-ary expansion of \( n \in \mathbb{Z}_{\geq 0} \) \((0 \leq n_j < p)\). Then it holds that
\[
\tilde{J}_2(n) \equiv \prod_{j=0}^{k} \tilde{J}_2(n_j) \pmod{p}.
\]

The following claim is regarded as an analogue of Proposition 2.2.

Proposition 8.2 ([11, Theorem 6.2]). For any odd prime \( p \) and positive integers \( m, r \), the congruence relation
\[
\tilde{J}_2(mp^r) \equiv \tilde{J}_2(mp^{r-1}) \pmod{p^r}, \\
\tilde{J}_3(p^r)p^{3r} \equiv \tilde{J}_3(p^{r-1})p^{3(r-1)} \pmod{p^r},
\]
holds.
Proposition 8.3. For any odd prime $p$, the congruence relation

$$\sum_{n=0}^{p-1} \tilde{J}_2(n) \equiv 0 \pmod{p^2} \tag{8.1}$$

holds.

Proof. We see that

$$\sum_{n=0}^{p-1} \tilde{J}_2(n) = \sum_{n=0}^{p-1} \sum_{j=0}^{n} (-1)^j 16^{-j} \binom{2j}{j} \binom{n}{j} = \sum_{j=0}^{p-1} (-1)^j 16^{-j} \binom{2j}{j} \sum_{n=j}^{p-1} \binom{n}{j}$$

$$= \sum_{j=0}^{p-1} (-1)^j 16^{-j} \binom{2j}{j} \binom{p}{j+1} \equiv p \sum_{j=0}^{p-1} 16^{-j} \binom{2j}{j} \binom{p-1}{j+1} (-1)^j$$

$$\equiv p \sum_{j=0}^{p-1} 16^{-j} \binom{2j}{j} \frac{1}{j+1} \pmod{p^2}$$

since $\binom{2j}{j}$ is divisible by $p^2$ if $\frac{p-1}{2} < j < p$. Notice that

$$16^{-j} \binom{2j}{j} \equiv (-1)^j \left( \frac{p-1}{j} + j \right) \left( \frac{p-1}{2} \right) \pmod{p}$$

for $0 \leq j < p$. Hence we have

$$\sum_{n=0}^{p-1} \tilde{J}_2(n) \equiv p \sum_{j=0}^{p-1} \left( \frac{p-1}{j} + j \right) \left( \frac{p-1}{2} \right) \frac{(-1)^j}{j+1} \pmod{p^2}.$$

By putting $n = \frac{p-1}{2}$ and $m = 0$ in the identity (see [6, Chapter 5.3])

$$\sum_{k \geq 0} \binom{n+k}{k} \binom{n}{k} \frac{(-1)^k}{k+1+m} = (-1)^n \frac{m!n!}{(m+n+1)!} \binom{m}{n}, \tag{8.2}$$

we have

$$\sum_{j=0}^{p-1} \left( \frac{p-1}{j} + j \right) \left( \frac{p-1}{2} \right) \frac{(-1)^j}{j+1} = \frac{(-1)^{p-1}}{(p-1)!} \binom{0}{\frac{p-1}{2}} = 0. \tag{8.3}$$

Hence we obtain the desired conclusion.

Proposition 8.4. For each odd prime $p$, it holds that

$$\tilde{J}_2\left(\frac{p-1}{2}\right) \equiv A_2\left(\frac{p-1}{2}\right) \pmod{p^2}. \tag{8.4}$$

Here $A_2(n)$ is the Apéry number for $\zeta(2)$.

Proof. It is elementary to check that

$$\binom{(p-1)/2}{k} \equiv \left( \frac{-1/2}{k} \right) \left\{ 1 - p \sum_{j=1}^{k} \frac{1}{2j-1} \right\} \pmod{p^2},$$

$$\binom{(p-1)/2 + k}{k} \equiv (-1)^k \left( \frac{-1/2}{k} \right) \left\{ 1 + p \sum_{j=1}^{k} \frac{1}{2j-1} \right\} \pmod{p^2}.$$
for \( k = 0, 1, \ldots, (p - 1)/2 \). Using these equations, we easily see that both \( A_2 \left( \frac{p-1}{2} \right) \) and \( \tilde{J}_2 \left( \frac{p-1}{2} \right) \) are congruent to
\[
\sum_{k=0}^{(p-1)/2} (-1)^k \binom{-1/2}{k} \left\{ 1 - p \sum_{j=1}^{k} \frac{1}{2j-1} \right\}
\]
modulo \( p^2 \).

\textbf{Remark 8.5.} The following supercongruence
\[
A_2 \left( \frac{p-1}{2} \right) \equiv \lambda_p \pmod{p^2}
\]
holds if \( p \) is a prime larger than 3 (see [9]; see also [15, 29]).

\section{Conjectures}
In the final position, we give several conjectures on congruence relations among normalized (higher) Apéry-like numbers.

\textbf{Conjecture 8.6} (Remark 6.3 in [11]). For any odd prime \( p \), the congruence relation
\[
\sum_{n=0}^{p-1} \tilde{J}_2(n)^2 \equiv \left( \frac{-1}{p} \right) \pmod{p^3}
\]
holds.

\textbf{Remark 8.7}. The conjecture above is quite similar to the Rodriguez-Villegas-type congruence due to Mortenson [14]
\[
\sum_{n=0}^{p-1} \left( \frac{2n}{n} \right) 16^{-n} \equiv \left( \frac{-4}{p} \right) \pmod{p^2}.
\]

We also remark that the following “very similar” congruence relation is obtained in [19]:
\[
\sum_{n=0}^{(p-1)/2} \left( \frac{2n}{n} \right) 16^{-n} + \frac{3}{8} (-1)^{(p-1)/2} \sum_{i=1}^{(p-1)/2} \left( \frac{2i}{i} \right) \frac{1}{i} \equiv \left( \frac{-1}{p} \right) \pmod{p^3},
\]
where \( p \) is an arbitrary odd prime number.

The following conjecture is regarded as a “true” analogue of Proposition 2.2:

\textbf{Conjecture 8.8} (Kimoto-Osburn [10]). For any odd prime \( p \), the congruence relation
\[
\tilde{J}_2(mp^r - 1) \equiv \left( \frac{-1}{p} \right) \tilde{J}_2(mp^r - 1) \pmod{p^r}
\]
holds for any \( m, r \geq 1 \).

\textbf{Remark 8.9}. When \( r = 1 \), the conjecture (8.8) is obtained by using the elementary formulas
\[
\left( \frac{-1/2}{kp+j} \right) \equiv \left( \frac{-1/2}{k} \right) \left( \frac{-1/2}{j} \right) \pmod{p}, \quad \left( \frac{mp-1}{n} \right) \equiv (-1)^{n-[n/p]} \left( \frac{m-1}{[n/p]} \right) \pmod{p},
\]
and Mortenson’s result (8.6).
Conjecture 8.10. For any odd prime $p$ and $m, r \in \mathbb{Z}_{>0}$ with $m$ odd, it holds that
\[
\tilde{J}_2\left(\frac{mp^r-1}{2}\right) - \lambda_p \tilde{J}_2\left(\frac{mp^{r-1}-1}{2}\right) + (-1)^{p(p-1)/2} \frac{p^2}{2} \tilde{J}_2\left(\frac{mp^{r-2}-1}{2}\right) \equiv 0 \pmod{p^r},
\]
where $\lambda_n$ is given by
\[
\sum_{n=1}^{\infty} \lambda_n q^n = q \prod_{n=1}^{\infty} (1 - q^{4n})^6 = \eta(4\tau)^6.
\]
Further, the congruence (8.9) holds modulo $p^{2r}$ if $p \geq 5$.

Conjecture 8.11. For any odd prime $p$, the congruence relation
\[
\sum_{n=0}^{p-1} J_{2k}(n) \equiv -1 \pmod{p^2}
\]
holds for any $k \geq 2$.

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Department of Mathematical Sciences, University of the Ryukyus
Nishihara, Okinawa 903-0231 Japan
kimoto@math.u-ryukyu.ac.jp