Picture-Changing Operators and Space-Time Supersymmetry

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Abstract

We explore geometrical properties of fermionic vertex operators for a NSR superstring in order to establish connection between worldsheet and target space supersymmetries. The mechanism of picture-changing is obtained as a result of imposing certain constraints on a world-sheet gauge group of the NSR Superstring Theory. We find that picture-changing operators of different integer ghost numbers form a polynomial ring. By using properties of the picture-changing formalism, we establish a relation between the NSR and GS String theories. We show that, up to picture-changing transformations, the stress-energy tensor of the $N = 1$ NSR superstring theory can be obtained from the stress-energy tensor of the $N = 1$ GS superstring theory in a flat background by a simple field redefinition. The equations of motion of a GS superstring are shown to be fulfilled in the NSR operator formalism; they are also shown to be invariant under $\kappa$-symmetry, in terms of operator products in the NSR theory. This allows us to derive the space-time supersymmetry transformation laws for the NSR String Theory. Then, we explore the properties of the $\kappa$-symmetry in the NSR formalism and find that it leads to some new relations between bosonic and fermionic correlation functions.

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1. Introduction

In Superstring Theory scattering amplitudes are expressed in terms of expectation values of vertex operators. For bosons, these operators are generally of the form \( v = f(X, \partial X, \ldots \partial^n X) \exp[i k X] \), where \( f \) is some function. The typical example of such an operator is given by a vector emission vertex: \( V(k, z) = \exp[ikX]e_\mu(k)(\partial X^\mu + i/2(k\psi)\psi^\mu) \). In case of a closed string this expression must also be multiplied by an antiholomorphic part. Construction of the fermionic counterpart of this vector has been considered in operator formalism in refs \([1], [2], [3]\) and in the framework of covariant theory \([1, 4, 5]\). These remarkable papers succeeded in constructing scattering amplitudes involving bosons and fermions. At the same time we do not have clear geometrical reasons explaining space-time supersymmetry. In this paper we will in particular rederive the results of the above works in a way which may be more appropriate for geometrical interpretation. Our basic starting point is the following. Consider an insertion of a vertex operator describing the emission of a spin 1/2 particle at some point \( z \) on a worldsheet: \( V_{-1/2}(z) = \bar{u}^A(k)\Sigma_A(z)\exp(-\phi/2)\exp(ikX(z)) \), where \( \bar{u}(k) \) is some constant (in a space-time) 10-spinor, \( \phi \) is a bosonized superconformal ghost and \( \Sigma \) is a spin operator for matter fields. The index \( A \) runs from 1 to 32 and the index -1/2 over \( V \) denotes its ghost number. The construction of a fermion emission vertex is described in \([1, 4]\). Now, consider the operator product expansion of \( V_{-1/2}(z) \) with one of 10 fermionic fields \( \psi_\mu(w) \): \( V_{-1/2}(z, k, \bar{u}^A(k))\psi_\mu(w) = (z-w)^{-1/2}V_{-1/2}(w, k, \gamma^{\mu AB}\bar{u}_B) \). It follows from this OPE that the field \( \psi_\mu(w) \) changes sign under the monodromy around \( z \). At the same time the field \( X_\mu(w) \), its superpartner does not change sign under the monodromy. Consequently, the theory does not have a global supersymmetry. The same problem appears if we insert a vector vertex (at zero, for simplicity): \( V_{-1}(0, k, e_\mu(k)) = \exp(-\phi(0))e_\mu(k)\psi_\mu(0)\exp(ikX(0)) \); insertion of this operator causes the field \( \psi_\nu(z) \) to have a singularity: \( \psi_\nu(z) \sim \frac{e_\nu(z)}{z} \); at the same time \( X_\nu(z) \sim k_\mu \log(z) \). One way to cure this problem is to impose certain constraints on gauge parameters. Namely, let us consider the following locally supersymmetric superstring action:

\[
S = 1/2 \int d^2z(-g)^{1/2}g^{ij}(X)\partial_iX_\mu\partial_jX^\mu + \psi_\mu e^{ij}(z)\sigma_j\partial_i\psi^\mu + (\chi_i\sigma^j\sigma^k\psi^{\mu}(\partial_jX_\mu - 1/4(\chi_j\psi_\mu))).
\]

Here \( e^{ij}(z) \) is a zweibein and \( \chi_i(z) \) is a spin 3/2 gravitino field. The symmetry of this action (apart from general covariance) is given by:
\[ \delta X_\mu(z) = \epsilon(z) \psi_\mu(z) \]
\[ \delta \psi_\mu(z) = \sigma^i (\partial_i X_\mu(z) - 1/2(\chi_i(z)\psi_\mu(z))) \epsilon(z) \]
\[ \delta \chi_i(z) = 2 \nabla_i \epsilon(z) \]
\[ \delta g_{ij} = \epsilon(z)(\sigma_i \chi_j(z) + \sigma_j \chi_i(z)) \]

Here \( \sigma_i \) are 2d Dirac matrices; \( \epsilon(z) \) is a local infinitesimal fermionic parameter.

To preserve the local supersymmetry in the above example, we must impose certain constraints on \( \epsilon(z) \). Namely, let \( z_1, z_2, ..., z_N \) be the points where vertices are inserted. The constraints should be

\[ \epsilon(z_1) = 0, ..., \epsilon(z_N) = 0 \] (3)

. Of course, apart from these requirements, the parameter \( \epsilon \) should have a proper behaviour under the monodromy. The nature of the requirement (3) is easy to understand. Let us write down the expression for an N-point scattering amplitude for fermions [6]:

\[ A_N(k_1, ..., k_N) = \int [z_1, z_2, ..., z_N] DX D\psi D\chi \prod_{i=1}^N \int d^2 z_i \exp(ik_i X(z_i)) \exp(-S) \] (4)

Here \([z_1, z_2, ..., z_N]\) denotes anti-periodic boundary conditions that should be imposed on the fermions when we move them around the small contours encircling these points. The measure and the action in (4) are invariant under local supersymmetry, however the factors before \( \exp(-S) \) are not. Therefore, in order for the amplitude to be locally supersymmetric the supersymmetry transformation should vanish at the points \( z_1, ..., z_N \), then the variation of the integral under the local supersymmetry (2) will be zero. From this (3) follows. The imposition of the constraints (3) leads to very important consequences. Namely, these conditions effectively reduce the gauge group (2). As a result it may no longer be possible to access the superconformal gauge where the gravitino field may be gauged away to zero. To investigate this question, it is necessary to consider the Faddeev-Popov procedure modified by the constraints (3).

2. Modified FP Procedure

The usual FP prescription for a gauge group \( g \) is defined by

\[ \Delta(\chi) \int \delta(\varphi(\chi^g))Dg = 1 \] (5)

where \( \varphi \) is a desired gauge condition for the gravitino field. Setting \( g = I + \epsilon \) and imposing the constraints (3) on the gauge group (2) we can rewrite (5) as

\[ \Delta(\chi) \int \delta(\varphi(\chi^g))\delta(\epsilon(z_1))...\delta(\epsilon(z_N)) D\epsilon = 1 \] (6)
or, equivalently,
\[ \Delta(\chi) \int \exp(B \frac{\delta \varphi}{\delta \epsilon}) \epsilon(z_1) ... \epsilon(z_N) D\epsilon DB = 1 \] (7)

where B is some auxiliary fermionic field. In (7) we also used \( \delta(\epsilon) = \epsilon \) since \( \epsilon \) is a fermionic variable. It is helpful to stress that it is possible to rewrite any fermionic functional integral (for instance the one in (7)) as 1 over integral over some bosonic fields.

For instance,
\[ \int \exp(B \frac{\delta \varphi}{\delta \epsilon})(...\epsilon)DBD\epsilon = (\int D\beta D\gamma \exp(\beta \frac{\delta \varphi}{\delta \gamma})(...\gamma))^{-1}, \]

where the integration over the fermions \( B, \epsilon \) is replaced by that over the pair of the bosonic fields \( \beta, \gamma \).

Thus, instead of expressing \( \Delta(\chi) \) in terms of the integral (6) over the pair of fermionic fields \( (\epsilon, B) \) we may choose to write it as the following integral over the corresponding pair of the bosonic fields \( (\beta, \gamma) \):
\[ \Delta(\chi) = \int D\gamma D\beta \exp(\beta \frac{\delta \varphi}{\delta \gamma}) \delta(\gamma(z_1)) \ldots \delta(\gamma(z_N)) \] (8)

Now let us insert (8) into the partition function:
\[ Z = \int D\chi D[...] \Delta(\chi) \exp(-S(\chi, ...) \delta(\varphi(\chi))) = \int D\gamma D\beta D\chi D[...] \exp(\beta \frac{\delta \varphi}{\delta \gamma}) \times \]
\[ \times \delta(\gamma(z_1)) \ldots \delta(\gamma(z_N)) \delta(\varphi(\chi)) \exp(-S(\chi, ...)) \] (9)

For the sake of brevity, we here suppressed the integration over all the matter fields except for the gravitino. The formula (6) shows that, in order to insure the constraints (3) we have to insert the fields \( \epsilon(z_k) \), each field is of a ghost charge 1. Equivalently, in terms of the bosonic integral (9) we should insert \( \delta(\gamma(z_k)) \). Analogously, it is the same way to show that if, instead of imposing constraints (3) we choose to impose the stronger constraints:
\[ \epsilon(z_k) = 0 \]
\[ \partial \epsilon(z_k) = 0 \] (10)

we have to insert the ghost charge 2 operator \( \epsilon(z_k) \partial \epsilon(z_k) \) into the functional integral. It is straightforwardly generalized to the situation when, along with \( \epsilon \) its \( (n - 1) \) derivatives are zero; we would have then to insert the ghost charge \( n \) operators \( \epsilon ... \partial^{n-1} \epsilon(z_k) \). The corresponding bosonic insertion would be \( \delta(\gamma(z_k)) \delta(\partial \gamma(z_k)) \ldots \delta(\partial^{n-1} \gamma(z_k)) \) It is easy now to see why the superconformal gauge where \( \varphi(\chi) = \chi = 0 \) is no longer accessible once the
constraints (3) are imposed. Indeed, the functional integrals (6), (9) would be zero in this case because of the ghost charge conservation. Therefore, if we restrict our gauge group (2) with the constraints (3) we must at the same time relax the gauge condition on \( \chi \). That is, we should fix the gauge as

\[
\chi(z) = \sum_{i=1}^{M} \chi_i(z) \theta^i \tag{11}
\]

, where \( \chi_i \) is a dual basis in the space of 3/2-differentials, and \( \theta^i \) are the grassmanian parameters called supermoduli. The partition function is shown to be independent on small variations of the basis \( \chi_i \). The number \( M \) is equal to the number of independent 3/2-differentials; we will compute this number shortly. A proper choice of the basis in (11) will help us to get the correct ghost number in the integral (9). Thus, for the gauge constraint (3) we may choose

\[
\chi_k(z) = \delta(z - w_k), k = 1, \ldots M \tag{12}
\]

. Then, the Faddeev-Popov insertion (5) will write as follows:

\[
\Delta(\chi) \int DBD\epsilon e^{\exp(B\frac{\delta \varphi}{\delta \chi} \delta(\epsilon(z_1)) \ldots \delta(\epsilon(z_N)) \delta(B(z_1)) \ldots \delta(B(z_N))]} = 1
\]

\[
\Delta(\chi) = \int D\gamma D\beta e^{\exp(\beta \frac{\delta \varphi}{\delta \gamma} \delta(\gamma(z_1)) \ldots \delta(\gamma(z_N)) \delta(\beta(w_1)) \ldots \delta(\beta(w_M))]} \tag{13}
\]

Here, as previously, we put \( \varphi(\chi) = \chi \). Now, let us insert (13) into the partition function. We get:

\[
\int D\chi D[\ldots] D\beta D\gamma \prod_{j=1}^{M} \delta(\beta(w_j)) e^{\exp(-S(\chi, \ldots))} \times \\
\exp(\int \beta \frac{\delta \varphi}{\delta \gamma} \delta(\gamma(z_1)) \ldots \delta(\gamma(z_N)) \delta(\beta(w_1)) \ldots \delta(\beta(w_M))} \tag{14}
\]

Here the field \( \delta(\beta) \) is a ghost number \(-1\) bosonic field which is to compensate the ghost number 1 of \( \delta(\gamma) \). Note that under the change \( \epsilon \rightarrow \gamma \) the SUSY gauge transformations (3) with the fermionic parameter \( \epsilon \) become BRST gauge transformations with the bosonic ghost \( \gamma \) as a parameter (the form of these transformations remains the same as in (3), with \( \epsilon \) replaced by \( \gamma \)). Now it follows from (3) that, since \( \delta_{SUSY} \chi_i(z) = 2\nabla_i \epsilon(z) \), the corresponding BRST gauge transformation is

\[
\delta_{BRST} \chi_i(z) = 2\nabla_i \gamma(z) \tag{15}
\]

and therefore

\[
\frac{\delta \varphi(\chi_i)}{\delta \epsilon} = \frac{\delta \varphi(\chi_i)}{\delta \gamma} = 2\nabla_i \tag{16}
\]
We now have to integrate over the gauge field $\chi$. One problem still remains with the bosonic ghost action in (14). Namely, this action is not BRST invariant. One may check that $Q_{BRST}S_{gh}(\beta, \gamma, b, c) \neq 0$, where

$$S_{gh} = \int d^2 z [-\beta \nabla \gamma + b \nabla c]$$

$$Q_{BRST} = \oint d^2 z [\gamma (G_m + 1/2G_{ghost}) + c (T_{matter} + 1/2T_{ghost})]$$

(17)

Here $G_{matter}, G_{gh}$ are the supercurrents for matter and ghost fields respectively; and $T_m, T_{gh}$ are stress-energy tensors. In order to restore the BRST invariance we must add the term $\sim \chi G_{gh}$ to the ghost action; this term is analogous to $\sim \chi G_{gh}$ we have already got in the matter part in order to insure the local supersymmetry (3). Now we are prepared to perform the integration over the gravitino field. Due to the gauge fixing condition (11) it will simply be an integral over $M$ supermoduli $\theta^i$; the whole dependence of the modified action on the gravirino field in (14) contains in the piece $\sim \int d^2 z \chi (G_m + G_{gh})$; therefore, by using (11), expanding the exponent and performing the integral over $\theta$’s we obtain the following insertion into the partition function (14):

$$\prod_{k=1}^M [G_m(w_k) + G_{gh}(w_k)]$$

(18)

Putting this together with the insertion in (14) that comes from FP determinant with relaxed gauge conditions we find that the total insertion depending on the points $w_k$ chosen for the basic vectors in (11) will be:

$$\prod_{k=1}^M \Gamma_1(w_k)$$

(19)

where

$$\Gamma_1(w_k) = \delta(\beta(w_k))(G_m(w_k) + G_{gh}(w_k))$$

(20)

The operator $\Gamma_1$ is known as a picture-changing operator [5]. Bosonization formulas for $\beta, \gamma$ ghost system are the following:

$$\gamma(z) = \exp(\phi(z) - \kappa(z))$$

$$\beta(z) = \exp(\kappa(z) - \phi(z)) \partial \kappa(z)$$

$$\delta(\beta(z)) = \exp(\phi(z))$$

$$\langle \kappa(z) \kappa(w) \rangle = - \langle \phi(z) \phi(w) \rangle = \log(z - w)$$

(21)
Bosonization rules for $\delta(\beta)$ are explained in [7]. There exists a simple way to rederive them through relations between bosonic and fermionic correlators. The fermionic fields $\epsilon$ and $B$ may be bosonized as $\epsilon = \exp[\hat{\phi}]$; $B = \exp[-i\hat{\phi}]$ with $<\hat{\phi}(z)\hat{\phi}(w)> = \ln(z-w)$. On the other hand, by using $\delta(\epsilon) = \epsilon$, $\delta(B) = B$, we can write:

\[
<\epsilon(z_1) \ldots \epsilon(z_N) B(w_1) \ldots B(w_N)> = <\delta(\epsilon(z_1)) \ldots \delta(\epsilon(z_N)) \delta(B(w_1)) \ldots \delta(B(w_N))> = (\langle \delta(\gamma(z_1)) \ldots \delta(\beta(z_1)) \rangle \ldots \langle \delta(\beta(z_N)) \rangle )^{-1}.
\]

Substituting the bosonization formulas for $\epsilon$ and $B$ and recalling that $\langle \prod_i \exp[\alpha_i \phi(z_i)] \rangle = \exp[\sum_{i,j} \alpha_i \alpha_j \phi(z_i)\phi(z_j)]$ we find that the bosonization formulas (21) are correct with $\phi = i\hat{\phi}$. From here it is clear why the correlator of 2 $\phi$'s should have the "wrong" sign. Note that the conformal dimension of the field $\exp(\alpha\phi)$ is $[8] - \frac{\alpha^2}{2} - \alpha$. Therefore the expression for the picture-changing operator (20) in terms of free fields is:

\[
\Gamma_1(z) = \exp(\phi(z))[-1/2\psi^\mu \partial X_\mu(z) - 1/2b\gamma(z) + c\partial\beta(z) + 3/2\beta\partial c(z)]
\]

This operator has zero conformal dimension. Of course, a proper normal ordering must be done in the formula (22).

3. Counting the number of parameters

Now we have to find the number $M$ of the holomorphic differentials $\eta^\beta(z)$ which is equal to the number of their dual $\chi_\alpha(z)$ in (11). The duality means that

\[
\int d^2z \eta^\beta \chi_\alpha = \delta_{\alpha\beta}
\]

In case of an amplitude on a sphere this number has been already computed in [8] and has been shown to be equal to $\frac{N-4}{2}$ for an N-point fermionic amplitude. The gauge constraints (3) were discussed there. However, we would like to count the number of the basic vectors in a case when other constraints (like (10) or stronger) are imposed. Our approach mostly repeats [8]. Let us consider the most general case of constraining the gauge (2). From now on, unless stated otherwise we will restrict ourselves to the problem on a sphere. Let's consider the points $z_i, i = 1, ... N$ where the vertices are inserted. Suppose that, at the point $z_i$ we have $\epsilon(z_i)$ together with its first $k_i$ derivatives vanishing. Then, the basis in the space of holomorphic $3/2$ differentials may be chosen as

\[
\eta^\beta = z^\beta \prod_{i=1}^{N} (z - z_i)^{-1/2 - k_i}
\]
The number of independent differentials is therefore equal to the number of all admissible \( \beta \)'s. The condition of holomorphy for 3/2 differentials requires that (24) goes to zero at infinity no slower than \( \frac{1}{z^3} \). This leads to the condition

\[
\beta - N/2 - \sum_{i=1}^{N} k_i \leq -3 \tag{25}
\]

Hence it follows that there are \( M = \frac{N-4}{2} + \sum_{i=1}^{N} k_i \) independent holomorphic 3/2 differentials corresponding to (24).

4. Further gauge constraints

All the above arguments may be repeated if we choose to impose the constraints (10) instead of (3) on the gauge group (2). In this case, we will have to make the insertion of a ghost charge 2 FP determinant, as explained previously. In order to compensate this ghost charge we again will have to relax the gauge condition for the gravitino field. The proper choice of the dual basis \( \chi_i \) will now be

\[
\chi_i(z) = \partial [\delta(z - w_i)] \tag{26}
\]

and the gauge fixed gravitino field will be

\[
\sum_{i=1}^{3n-4} \partial [\delta(z - w_i)] \theta^i \tag{27}
\]

. The associated insertion will be

\[
\prod_{j=1}^{3N-4} \Gamma_2(w_j) \tag{28}
\]

where

\[
\Gamma_2(w_k) =: \delta(\beta)\delta(\partial \beta)G\partial G : (w_k) \tag{29}
\]

here \( G \) is a full supercurrent: \( G = G_m + G_{gh} \). Bosonizing according to (21) we get:

\[
\Gamma_2(w_k) =: exp(2\phi)G\partial G : (w_k) \tag{30}
\]

This is another picture-changing operator which has a ghost number 2. Again, it has a zero conformal dimension. One may easily check the following obvious generalization for the last formula. That is, requiring \( \epsilon \) with its \( n-1 \) derivatives to vanish in (2) leads to insertion of the following picture-changing operator of a ghost charge \( n \):

\[
\Gamma_n =: exp(n\phi)G\partial G...\partial^{n-1}G : \tag{31}
\]
All the picture-changing operators (31) have zero conformal dimensions.

5. Properties of the Picture-Changing Operators

First of all, let us write down the normal ordered expression for the picture-changing operator (20). We have:

\[ : \Gamma_1 := \exp(\phi)(G_m + G_{gh}) = -1/2 \exp(\phi)\psi_\mu \partial X^\mu - 1/2 \exp(2\phi - \kappa) \partial \phi b + \exp(\kappa) \partial \kappa c \] (32)

Let us now calculate the zero order term in the operator product of \( \Gamma_1 \) with the fermionic vertex operator \( V_{-1/2} \). We get:

\[ : \exp(\phi(z))\psi_\mu \partial X^\mu(z) :: \bar{u}^A(k) \Sigma_A \exp(-\phi(w)) \exp(ikX(w)) : \sim \]

\[ \sim \cdots + (z - w)^0 \gamma_{AB} \bar{u}^A(k) \Sigma_B [\partial X^\mu + i/4(k\psi)^\mu] \exp(ikX(w)); \]

\[ : \exp(\kappa) \partial \kappa \bar{u} \Sigma_A \exp(-1/2\phi) \exp(ikX) : \sim \]

\[ \sim \exp(\kappa - 1/2\phi) \bar{u} \Sigma_A c \exp(ikX); \] (33)

\[ : \exp(2\phi - \kappa) \partial \phi \bar{u} \Sigma_A \exp(-1/2\phi) \exp[ikX] := \exp(3/2\phi - \kappa) \bar{u} \Sigma_A \exp[ikX]; \]

\[ : \Gamma_1 V_{-1/2}(\bar{u}^A(k), z) : \sim V_{1/2}(\gamma_{AB} \bar{u}^B, w) + c \exp(\kappa) \partial \kappa V_{-1/2}(w) + \]

\[ + 1/2b \exp(3/2\phi - \kappa) \bar{u} \Sigma_A \exp[ikX] \]

Here

\[ V_{1/2} = \gamma_{AB} \bar{u}^A \Sigma^B [\partial X^\mu + i/4(k\psi)^\mu] \exp(ikX) \] (34)

The terms proportional to the ghost fields \( b \) and \( c \) in the product : \( \Gamma_1 V_{-1/2} \) may be omitted since it does not contribute to any correlation function. The operator \( V_{1/2} \) is a well-known fermionic vertex operator in the picture of ghost number 1/2. Now let us consider a construction of the fermionic vertex in the picture of a ghost number -3/2. The obvious candidate (the space-time spinor whose conformal dimension is 1) is:

\[ V_{-3/2} = \bar{u} \Sigma_A \exp(-3/2\phi) \exp[ikX] \] (35)

The normal ordered product of this operator with \( \Gamma_1 \) will give us

\[ : \Gamma_1 V_{-3/2} : \sim (\gamma k)_A B \bar{u}(\bar{k})^A V_{-1/2}^B = 0 \]

because of the on-shell condition.

The reason for that is the BRST non-invariance of \( V_{-3/2} \). In order to make it BRST invariant one should add to it the term equal to

\[ \bar{V}_{-3/2}(k) = 1/2 \bar{u}(k) A \Sigma_A \exp[\kappa - 5/2\phi] \partial^2 c \times \exp[ikX]. \] As one may easily check, \( \bar{V}_{-3/2}(k) \) has the conformal dimension 1, and its BRST variation compensates that of \( V_{-3/2} \); also

\[ : \Gamma_1 \bar{V}_{-3/2}(k) : \sim V_{-1/2}(k), \text{ up to terms that do not contribute to correlation functions.} \]
Similarly, by acting on $V_{1/2}$ with $\Gamma_1$ we obtain fermionic vertex in the $3/2$ picture. However, this procedure contains the subtlety that will be discussed below. The same is true for boson emission vertices. For example, let us consider the emission of a vector boson: $V_{-1}(k) = e_\mu(k)\exp(-\phi)\psi^\mu\exp[ikX]$. The product with $\Gamma_1$ gives

$$\Gamma_1 V_{-1}(k) : \sim e_\mu(k)[\partial X^\mu + i/2(k\psi)\psi^\mu]\exp[ikX] + \beta c_\mu\psi^\mu\exp[ikX]$$

(36)

Again, the term $\sim c$ is irrelevant since not contributing to any correlation function. Therefore

$$\Gamma_1 V_{-1} : \sim V_0$$

(37)

where $V_0 = e_\mu[\partial X^\mu + i/2(k\psi)\psi^\mu]\exp[ikX]$

In general, if we have a vertex operator of a ghost number $m$ ($m$ is integer for bosons and half-integer for fermions), its product with the picture-changing operator (20) will give us the operator describing the emission of the same particle with momentum $k$ but of a ghost number $m-1$. By applying BRST charge (17) to both left and right hand side of (36) it is easy to see that the operator $\Gamma_1$ is BRST invariant since both rhs of (36) and $V_{-1}$ are invariant under BRST. There is another very important property possessed by the operator $\Gamma_1$. That is, if we insert $\Gamma_1(a)$ into any correlation function: $<\Gamma_1(w)V_1(z_1)...V_N(z_N)>$, this correlation function will be independent on the point of insertion $w$. This means that the operator $\Gamma_1$ can be moved freely from $w$ to any other point in a correlator - the correlation function will remain unchanged. This leads to important identities for correlation functions. For example, let us consider the three-point correlation function on a sphere: $<V_0(z_1)V_{-1}(z_2)V_{-1}(z_3)>$. We have:

$$<V_0(z_1)V_{-1}(z_2)V_{-1}(z_3)> = <\Gamma_1(z_1 - \epsilon)V_{-1}(z_1)V_{-1}(z_2)V_{-1}(z_3)>$$

$$= <V_{-1}(z_1)\Gamma_1(z_1 + \epsilon)V_{-1}(z_2)V_{-1}(z_3)> = <V_{-1}(z_1)V_{-1}(z_2)\Gamma_1(z_2 + \epsilon)V_{-1}(z_3)>$$

(38)

$$= <V_{-1}(z_1)V_0(z_2)V_{-1}(z_3)> = <V_0V_{-2}> = <V_1V_{-1}V_{-2}> = ...$$ etc

In the similar way we can show that the operator $\Gamma_2$ increasing the ghost number of vertices by 2 has the properties that are the same as that of $\Gamma_1$. The same is true for any picture-changing operator $\Gamma_n$. However, the separate manipulations with $\Gamma_1$, $\Gamma_2$,... do not give us the full set of existing identities between the correlation functions. We need to establish connection between different $\Gamma$’s. The problem here is the following. There are, for instance, two ways to change the ghost number of $V_m$ by 2. We can make it by acting on it either twice with $\Gamma_1$ or once with $\Gamma_2$. The natural question is whether the results
will be the same (up to maybe BRST trivial terms). For example, acting on $V_{-3/2}$ twice with $\Gamma_1$ is, up to BRST trivial term, the same as acting on it once with $\Gamma_2$ - in both cases we obtain $V_{1/2}$, up to terms that do not contribute to a correlation function. The question is that whether this is true for an arbitrary vertex operator $V_m$ and the picture-changing operators $\Gamma_k$, $\Gamma_l$ and $\Gamma_{k+l}$. Let us check this for two $\Gamma_1$’s. We need to compare the expressions for $\Gamma_1 \Gamma_1$ and $\Gamma_2$ obtained as a result of a normal ordering.

The normal ordering of $\Gamma_1 \Gamma_1$ gives:

$$\Gamma_1 \Gamma_1 := 1/4 \exp(2\phi)[G_m \partial G_m + c_m P^4_\phi + (\partial \phi \partial \phi - \partial^2 \phi) T_m]$$

$$-1/2 \exp(3\phi - \kappa)[\partial G_m \partial \phi b - G_m [\partial(\partial \phi b) + \partial \phi b(\partial \kappa - \partial \phi)]] +$$

$$+1/4 \exp(4\phi - 2\kappa - 2\sigma) \sum_{i,j<4} P^i_\phi P^j_\sigma P^{4-i-j-2\phi-\kappa} +$$

$$+1/4 \exp(2\phi)[c_{gh} P^4_\phi + T_{gh} P^2_\phi] + 1/2 \exp(2\kappa + 2\sigma)$$

(39)

Here $c_m$ and $c_{gh}$ are the matter and ghost central charges, $\sigma$ is the bosonized fermionic ghost : $c = \exp(\sigma), b = \exp(-\sigma)$; $P^i_\phi$ is the polynomial in the derivatives of $\phi$ which has conformal dimension $i$ and is determined by the OPE of $\exp(\phi)$ with itself:

$$\exp(\phi(z)) \exp(\phi(w)) \sim \exp(2\phi(w))\left[\frac{1}{z-w} + \sum_i P^i_\phi (z-w)^{i-1}\right]$$

(40)

For example, $P^2_\phi = \partial \phi \partial \phi - \partial^2 \phi$. The polynomials $P^j_\sigma$ and $P^{4-i-j-2\phi-\kappa}$ are defined in the same way. The normal ordering of $\Gamma_2$ gives:

$$\Gamma_1 \Gamma_1 := 1/4 \exp(2\phi)[G_m \partial G_m + c_m P^4_\phi + (\partial \phi \partial \phi - \partial^2 \phi) T_m]$$

$$-1/2 \exp(3\phi - \kappa)[\partial G_m \partial \phi b - G_m [\partial(\partial \phi b) + \partial \phi b(\partial \kappa - \partial \phi)]] +$$

$$+1/4 \exp(4\phi - 2\kappa - 2\sigma) \sum_{i,j<4} P^i_\phi P^j_\sigma P^{4-i-j-2\phi-\kappa} +$$

$$+1/4 \exp(2\phi)[c_{gh} P^4_\phi + T_{gh} P^2_\phi] + 1/2 \exp(2\kappa + 2\sigma)$$

(41)

Since $c_m + c_{gh} = 0$ we can write the last equation as

$$\Gamma_2 := \Gamma_1 \Gamma_1 :-(\partial \phi \partial \phi - \partial^2 \phi)(T_m + T_{gh}) - 1/4 \exp(2\phi)[(c_m + c_{gh}) P^4_\phi]$$

(42)

Here we used

$$\{Q_{BRST}, (\partial \phi \partial \phi - \partial^2 \phi) \exp(2\phi)\} = 0$$

$$\{Q_{BRST}, b\} = T_m + T_{gh}$$

(43)

Therefore, $\Gamma_2 := \Gamma_1 \Gamma_1$ : up to BRST trivial term. Analogously, one may show that

$$\Gamma_m \Gamma_n := \Gamma_{m+n} : + \{Q_{BRST}, \ldots\};$$

$$\Gamma_{n_1} \ldots \Gamma_{n_k} := \Gamma_{n_1 + \ldots + n_k} + \{Q_{BRST}, \ldots\} :$$

(44)
The straightforward proof of (44) is rather cumbersome; let us first prove it for $\Gamma_m$ and $\Gamma_1$. Rather lengthy calculation shows that in this case

$$
: \Gamma_m \Gamma_1 : =: \Gamma_{m+1} + \{Q_{BRST}, \sum_{l=1}^m \sum_{k=1}^l (-1)^k (l+k-1)! \frac{1}{l!} P_{(m\phi,\phi)}^{m+k} \partial^{l-k} b \} \tag{45}
$$

Here $P_{(m\phi,\phi)}^{m+k}$ is again the polynomial in the derivatives of $\phi$ which has the conformal dimension $m + k$ and is obtained in the process of the OPE of $\exp(m\phi)$ with $\exp(\phi)$. Then, (44) may be proved by induction. The formulas (42)-(44) mean that the picture-changing operators (31) form an infinite-dimensional polynomial ring in the sense described above. In the language of gauge constraints this means that there are many equivalent ways to impose constraints on (2). For example, (42) implies that the constraint that requires $\epsilon$ to vanish at two different points is equivalent to requiring both $\epsilon$ and $\partial \epsilon$ to be zero at the same point on a world sheet.

It follows from (38)-(44) that any correlation function $< V_{i_1}^{k_1}(...V_{i_N}^{k_N} ) >, (i_k$ denotes a ghost number of a vertex $V^{k}(p_k)) is independent upon a distribution of ghost numbers among the vertices, and the only required condition is

$$
\sum_{k=1}^N i_k = 2(g-1) \tag{46}
$$

where $g$ is a genus of the surface on which the correlation function is computed, i.e. the total ghost number must compensate the ghost number anomaly on the surface.

For a sphere

$$
\sum_{k=1}^N i_k = -2 \tag{47}
$$

In other words, due to (44) it does not matter which pictures we choose for vertex operators in the correlator; we only need to fix the total ghost number. Also because of (44) the fermionic vertex operator with an arbitrary ghost number $n - 1/2$ is now defined unambiguously:

$$
V_{n-1/2} =: \Gamma_n V_{-1/2} : \tag{48}
$$

The same is true for the vertex describing the emission of massless vector bosons.

6. Fermionic Background

The relation (44) between the picture-changing operators allows us to construct the consistent perturbation theory for a NSR superstring in order to derive the superstring corrections to the classical equations of motion for the supergravity. The superstring corrections
were discussed in \cite{9,10} by using semi-light-cone gauge fixing \cite{11} though the results were not yet complete. In order to obtain the low-energy effective action in the field theory limit of the superstring theory we need to perturb the superstring action in a flat background

\[
S_0 = \int d^2 z (\partial X^\mu \bar{\partial} X_\mu + \psi^{\mu} \bar{\partial} \psi_{\mu})
\]  

(49)

with the sum of massless background fields multiplied by corresponding vertex operators.

For example, the term describing the perturbation by a graviton is

\[
\sim \int d^2 z G_{\mu\nu}(X) \partial X^\mu \partial X^\nu,
\]

where \( G_{\mu\nu}(X) = \eta_{\mu\nu} + t_{\mu\nu}(X) \), \( \eta_{\mu\nu} \) is a flat space-time metric and \( t_{\mu\nu} \) is its small perturbation. Expanding the partition function of the perturbed action into series in \( t_{\mu\nu} \), extracting the cutoff dependence, we obtain the graviton’s \( \beta \)-function (which is the Ricci tensor in the zero loop approximation). Then, the equations of motion \( \beta(G_{\mu\nu}) = 0 \) define the low-energy gravitational action. In order to obtain the low-energy action for the supergravity (which has been calculated in the first orders in \cite{12} we also need to perturb \( S_0 \) with a 10-dimensional gravitino field times the fermionic vertex operator:

\[
S_1 = \int d^2 z [G_{\mu\nu}(X) \partial X^\mu \bar{\partial} X^\nu + \chi_{\mu}^{(10)}(X) V_{\alpha}^\mu]
\]

(50)

where \( V_{\alpha}^\mu \) is a fermion vertex operator \( V_{\alpha} \) multiplied by \( \bar{\partial} X^\mu \); the index \( \alpha \) depends on which picture we choose for the fermionic vertex. \( V_{\alpha}^\mu \) is the vertex operator describing the emission of a spin 3/2 fermion. However, there are several difficulties with the formula (50). First of all, it is not clear which picture should be chosen for the fermionic vertex operator \( V_{\alpha} \) in (50). If, for instance, we choose \( \alpha = -1/2 \), so that

\[
V_{-1/2}^\mu = \bar{u}^A \Sigma_A \exp(-1/2\phi) \bar{\partial} X^\mu \exp[ikX]
\]

(51)

it is not clear how to develop a perturbation theory for the background (50). Indeed, the only non-vanishing correlation function of \( V_{-1/2}^\mu \)'s is four-point, due to (47). The same problem appears if, instead of \( \alpha = -1/2 \) picture we choose any other half-integer value of \( \alpha \). We need to modify the perturbed partition function in order to insure the correct total ghost number (47) in every term of the expansion. In order to obtain this, the \( n \)th term in the expansion of (50) which is proportional to \( \chi_{\mu}^{10} \) must be multiplied by \( \Gamma_{n-4} \). This goal will be achieved if we write down the perturbed partition function as follows:

\[
Z_{pert} = \int DX D\psi \frac{1}{1 - \frac{1}{\Gamma_{1}}} \exp\left[G_{\mu\nu}(X) \partial X^\mu \bar{\partial} X^\nu + \chi_{\mu}^{10} V_{-1/2}^\mu\right]
\]

(52)
Due to (44) and (47) the factor $\frac{1}{1-\Gamma_1}$ inserted in (52) insures that all the terms in expansion of the exponent in (52) automatically have the correct ghost number. We conclude that, in order to obtain correct perturbed partition function for NSR superstring one must insert the function $f(\Gamma_1) = \frac{1}{1-\Gamma_1}$ of the picture changing operator in the measure of integration.

7. Relation to Green-Schwartz superstring.

The algebra (44) describes the operators that change ghost numbers of bosonic or fermionic vertices by integer values; the question arises whether it is possible to extend $m$ and $n$ in (44) to half-integer values. Operators changing ghost numbers by half-integer values transform bosonic vertices into fermionic and vice versa. Polarisation spinors of the fermions obtained this way will be the expressed in terms of polarisation vectors of bosons multiplied by space-time gamma-matrices or their combination. Therefore, this extension may be justified only when the theory possesses the space-time supersymmetry, i.e. there exists a symmetry between bosonic and fermionic vertex operators. The problem is therefore to find the explicit form of the local "picture-changing" operator $\Gamma_{1/2}$ (other half-integer "picture-changing" operators can be obtained by the multiplication with integer $\Gamma$’s. Given that the expression for $\Gamma_{1/2}$ is found, the question that arises is whether it is possible to "move" it inside the correlator like it is done in (38). If the answer is positive, we can establish the relations between bosonic and fermionic amplitudes that do not follow directly from space-time supersymmetry. The operator $\Gamma_{1/2}$ should be proportional to $exp[1/2\phi]$ and be of zero conformal dimension. The only local operator that satisfies these properties is

$$\Gamma_{1/2} = exp[1/2\phi]\Sigma$$ (53)

However, this operator cannot be the one we are looking for since $\Gamma_{1/2}\Gamma_{1/2}\neq\Gamma_1$ and therefore cannot be used for the extension of the algebra (44); also by direct computation one can show that, for a vertex operator $V_n$ the identity $\Gamma_{1/2}V_n(k) = V_{n+1/2}(k)$ does not hold in general case. Also, this operator is not BRST invariant. However, some of its properties that will be helpful in order to discuss the questions formulated above. We will find out shortly that it has a relevance to the Green-Schwarz (GS) superstring theory. Let us now recall some basic facts from this theory. The action of a Green-Schwarz superstring is given by [8]:

$$S = S_1 + S_2,$$

$$S_1 = -\frac{1}{2\pi} \int d^2z g^{1/2}g^{\alpha\beta}\Pi_\alpha\Pi_\beta,$$

$$S_2 = \frac{1}{\pi} \int d^2z [-i\epsilon^{\alpha\beta}\partial_\alpha X^\mu\theta^\gamma\mu\partial_\beta\theta]$$ (54)
Here
\[ \Pi^\mu_\alpha = \partial_\alpha X^\mu - i\theta\gamma^\mu\partial_\alpha \theta \] (55)

Here \( \theta \) is a 10-dimensional spinor and a world-sheet scalar. For simplicity, we here consider a heterotic string. The global space-time supersymmetry for this theory is written as follows [8]:
\[ \delta \theta^A = \epsilon^A \]
\[ \delta X^\mu = i\epsilon^A\gamma^\mu_{AB}\theta^B \]
\[ \delta e = 0 \] (56)

Here \( A = 1, \ldots, 2^4 \) is a spinor index. Apart from this global space-time supersymmetry the action (54) also possesses the following local fermionic \( \kappa \)-symmetry given by [8]:
\[ \delta \theta^A = 2i\gamma^{AB}\Pi_{KB} \]
\[ \delta X^\mu = i\theta^A\Gamma_{AB}\delta \theta^B \]
\[ \delta (g^{1/2}g^{\alpha\beta}) = -16g^{1/2} (P^\alpha\kappa^-\partial_\gamma \theta) \] (57)

where \( P^\alpha\kappa^- = 1/2 (g^{\alpha\gamma} - g^{-1/2} \epsilon^{\alpha\gamma}) \). \( \kappa \)-symmetry insures that half of the components of \( \theta \) are decoupled from the theory - and therefore the number of these components is proper.

The equations of motion in the conformal gauge are:
\[ \gamma^\mu \Pi^\mu_\bar{\alpha} \bar{\partial} \theta = 0 \]
\[ \bar{\partial} [\partial X^\mu + i\theta\gamma^\mu\partial \theta] = 0; \] (58)

2 other equations are obtained by complex conjugation. The momenta conjugate to the \( X \) and \( \theta \) coordinates are given by \( \pi_X = \Pi, \pi_\theta = i\gamma \Pi \theta \). Because of this phase-space constraint the quantization procedure becomes very cumbersome in covariant gauges and the problem of covariant quantization of the Green-Schwarz superstring has not yet been successfully solved. It was explained in [13,14,15,16] how to perform free field quantization for the \( N = 2 \) GS superstring by using a BRST charge constructed out of a stress-energy tensor with \( c = 6 \). However, in the formalism developed in these papers the amplitudes were not manifestly Lorentz-covariant in 10 dimensions; rather, they only were evidently covariant under \( SO(3,1) \) subgroup of the super-Poincare transformations. Therefore, the problem of the covariant quantization of the 10 dimensional GS superstring in a flat background remains unsolved.

A possible approach to this problem is to establish direct relation between GS and NSR variables; for the latter theory the covariant quantization procedures are well-known. The
idea is [17] to identify $\theta$ and $\Gamma_{1/2} = \exp[1/2\phi]\Sigma$ from the NSR theory by using the fact that
space-time and world-sheet transformation properties of these operators are identical: they both are world-sheet scalars and space-time spinors. Therefore, the identification we have to prove to be correct is

$$\theta = \exp[1/2\phi]\Sigma$$

(59)

We need to prove that, under this identification, the global supersymmetry transformations (56) and the equations of motion (58) are still fulfilled. Let us check the supersymmetry first. In the NSR theory, the space-time supersymmetry generator is given by

$$Q = \epsilon^A \oint \frac{dz}{2i\pi} \exp[-1/2\phi]\Sigma_A.$$  

(60)

Therefore, by using (59), (60) and performing a product expansion, we have

$$\delta_{SUSY} \theta_A = \epsilon^B \oint \frac{dz}{2i\pi} \exp[-1/2\phi(z)]\Sigma_B(z)\exp[1/2\phi(w)]\Sigma_B(w) = $$$$= \epsilon^B \oint \frac{dz}{2i\pi} \left[ \frac{1}{z-w} \epsilon_{BA} + \ldots \right] = \epsilon^B \epsilon_{AB}$$

(61)

. We can redefine the definition (59) of Q by multiplying it by the constant antisymmetric tensor $\epsilon^{AB}$, then we will return to the transformation law (56) for $\theta$. Now, let us do the the same derivation for $\delta_{SUSY} X^\mu$. There is a subtle point containing in this derivation. That is, since the product Q and X is non-singular, the result for $\delta_{SUSY} X^\mu$ should formally be zero. However, let us consider the supersymmetry generator in the different picture:

$$\Gamma_1 Q = \epsilon^A \oint \frac{dz}{2i\pi} \exp[1/2\phi]\Sigma_B \partial X^\nu \gamma^{AB}_\nu.$$  

(62)

Then

$$\Gamma_1 \delta_{SUSY} X^\mu = i\epsilon_A \Sigma_B \gamma^{\mu}_{AB} \exp[1/2\phi] = i\epsilon \Gamma^\mu \theta.$$  

(63)

Therefore, up to picture changing-transformation, the supersymmetry transformation (56) for $X^\mu$ is also fulfilled. Therefore, we find that the space-time supersymmetry (56) admits the substitution (59). Now we have to check the equations of motion (58).

Let us first the quantity $\Pi_\mu$ in terms of NSR variables. Rather then doing it straightforwardly through (55) and (59) we will compute it in the $-1$-picture which will allow us to clearly see the connection with with canonical relations. The conjugate momentum $\pi^B_\theta$ should satisfy the following canonical relation:

$$[\theta^A, \pi^B_\theta] = -\frac{2}{i\pi} \epsilon^{AB},$$

(64)
The operator that satisfies this property is

\[ \pi_A^A = \frac{i}{\pi} \pi \exp[1/2\phi] \Sigma^A. \] (65)

Indeed, by evaluating its O.P.E with \( \theta \) and taking the most singular term we have

\[ -i\pi \exp[1/2\phi] \Sigma^A : (z) : \exp[-1/2\phi] \Sigma^B : (w) \sim \frac{1}{-i\pi} \frac{\epsilon^{AB}}{z-w} + ... \] (66)

On the other hand, by variating the action (54) we have

\[ \pi_A^A = \frac{i}{\pi} \gamma^\mu \Pi^A, \] (67)

therefore we find

\[ \Pi^\mu = 2 \exp[-\phi] \psi^\mu \] (68)

since this operator satisfies (67). This result is, of course, in accordance (up to picture-changing) with the definition (55) of \( \Pi^\mu \) since

\[ \partial X^\mu - i\theta \gamma^\mu \partial \theta = (\Gamma_1 + \Gamma_2) \exp[-\phi] \psi^\mu + \{Q_{BRST}, \ldots \} \] (69)

Given this, let us next calculate the product

\[ T = 1/2 \Gamma_2 \Pi^\mu \Pi_\mu \] (70)

which is the picture-changed stress energy tensor for the GS superstring. It was shown in [18] that there exists a rather complicated field redefinition [18,17] that transforms the N=2 stress-energy tensor of the critical GS supersting in a Calabi-Yau background (with 6 compactified dimensions) into a twisted N=2 tensor which is constructed from a combination of NSR ghosts and a shifted BRST current [19]. We will show how, by using the redefinition (59) to construct (up to picture-changing) an untwisted NSR stress-energy tensor out of the N=1 GS stress-energy tensor of the theory in a flat background (70). The importance of such a construction is that it is relevant to the question of quantization of the GS superstring in a flat background. The difficulties of such the quantization were discussed in [18] and are related to the fact that \( \theta^A = \exp[1/2\phi] \Sigma^A, A = 1, \ldots 16 \) are not free fields. We have:

\[ : \Pi_\mu \Pi^\mu : = \exp[-\phi] \psi_\mu \exp[-\phi] \psi^\mu := \exp[-2\phi] [4\partial \psi_\mu \psi^\mu + \partial^2 \phi - \partial \phi \partial \phi]; \] (71)
then
\[ : \Gamma_1 \Pi_\mu \Pi^\mu := \left[ -1/2 \exp[\phi] \psi_\nu \partial X^\nu - 1/2 \exp[2\phi - \kappa] \partial \phi b + \exp[\kappa] \partial \kappa c \right] \times \exp[-2\phi] (4 \partial \psi_\mu \psi^\mu + \partial^2 \phi - \partial \phi \partial \phi) := \]
\[ = 2 \exp[-\phi] \psi_\nu \partial X^\nu + \exp[\kappa - 2\phi] \partial \kappa c [4 \partial \psi_\mu \psi^\mu + \partial^2 \phi - \partial \phi \partial \phi] \]

Finally, the product of this expression with \( \Gamma_1 \) gives:
\[ : \Gamma_2 \Pi_\mu \Pi^\mu := \Gamma_1 \Gamma_1 \Pi_\mu \Pi^\mu := \]
\[ = \partial X_\mu \partial X^\mu + \partial \psi_\mu \psi^\mu + \partial \sigma \partial \sigma + 3 \partial^2 \sigma + \]
\[ + \partial \kappa \partial \kappa + \partial^2 \kappa - \partial \phi \partial \phi - 2 \partial^2 \phi. \]

Here, as previously, \( \sigma \) is a bosonized fermionic ghost while the fields \( \phi \) and \( \kappa \) come from the bosonization formulas (21) for \( \beta, \gamma \) bosonic ghosts. The \( \sigma \)-term in (73)
\[ 2 T_{b-c} = \partial \sigma \partial \sigma + 3 \partial^2 \sigma \]
is exactly twice the stress-energy tensor of the fermionic ghost system while the term
\[ 2 T_{\beta-\gamma} = -\partial \phi \partial \phi - 2 \partial^2 \phi + \partial \kappa \partial \kappa + \partial^2 \kappa \]
is the stress-energy tensor for the \( \beta - \gamma \) ghost system written in the bosonized form. Therefore we find that
\[ 1/2 : \Gamma_2 \Pi_\mu \Pi^\mu := T_{\text{matter}} + T_{b-c} + T_{\beta-\gamma} \]
But, up to picture-changing transformation, \( 1/2 \Pi_\mu \Pi^\mu \) is a stress-energy tensor for a Green-Schwarz superstring. Therefore by using the equivalence relation (59), we have found the exact relation between GS and NSR stress-energy tensors to be given by
\[ \Gamma_2 T_{GS} = 1/2 (\Gamma_1 + \Gamma_2)^2 T_{NSR} \]
This very important result means that the proposed the equivalence relation (59) between GS and NSR observables transforms, up to picture-changing, \( T_{GS} \)-the stress-energy tensor of Green-Schwarz superstring to \( T_{NSR} \)-of NSR superstring. Let us now check that in NSR formalism the equations of motion (58) are still fulfilled. Let us check the first equation in (58). In the NSR formalism the fulfillment of the equation \( \gamma^\mu \Pi_\mu \partial \theta = 0 \) means that all the terms of the O.P.E. implied in this equation vanish. This is true if and only if the product of \( \Pi(z) \) with \( \theta(w) \) does not contain the term proportional to \( \sim \frac{1}{z-w} \). Since
\[ \gamma^\mu \Pi_\mu(z)\theta(w) = (z - w)^0 \exp[-1/2\phi]\Sigma + \ldots \] we see that the first equation in (58) is indeed fulfilled in the NSR formalism. The second equation of motion in (58) is satisfied because

\[ \bar{\partial}(\partial X^\mu + i\theta\gamma^\mu\partial \theta) = \bar{\partial}(\Gamma_1\Pi^\mu - c\beta\psi^\mu) = 0 \] (78)

since the product \( \Gamma_1(z)\Pi^\mu(w) \) does not contain singular terms. Therefore the equations of motion (58) are satisfied in the NSR formalism. Now, we have to show that the relation (59) leaves the equations of motion (58) invariant under the \( \kappa \)-symmetry transformations (57). We find that the variation of the first equation in (58) under the transformations (57) is given by

\[ \delta_\kappa(\gamma^\mu_{AB}\Pi_\mu\bar{\partial}\theta^B) = -2\partial \theta^C\kappa^C_{AB}\Pi_\mu\bar{\partial}\theta^B + i\kappa_A\Pi^\mu\bar{\partial}\Pi_\mu \] (79)

The first term is zero because of the equations of motion while the second one vanishes because the product expansion of \( \Pi^\mu(z) \) with \( \Pi_\mu(w) \) contains no term proportional to \( \sim 1/ z - w \). The \( \kappa \)-variation of the second equation of motion gives:

\[ \delta_\kappa \bar{\partial}\Pi_z^\mu = \bar{\partial}\delta_\kappa\Pi_z^\mu = -4\bar{\partial}_z\lim_{z \to w}\partial \theta(z)\gamma^\mu\gamma_\nu\Pi_w^\nu\kappa. \] (80)

Again, in order for this variation to vanish, the product \( \partial \theta(z)\Pi_\nu(w) \) should contain no term proportional to \( \sim 1/ z - w \). Writing this product in terms of the NSR formalism (with \( \Pi \) written in the \( -1 \)- picture) and leaving the most singular terms (which are proportional to \( 1/z - w \)):

\[ \partial \theta(z)\Pi_\nu(w) = (1/2\partial \phi e x p[1/2\phi]\Sigma(z) e x p[-\phi]\psi^\nu(w) + e x p[1/2\phi]\partial \Sigma(z) e x p[-\phi]\psi^\nu(w)) = 1/2\frac{1}{z - w}\gamma^\nu e x p[-1/2\phi]\Sigma(w) - 1/2\frac{1}{z - w}\gamma^\nu e x p[-1/2\phi]\Sigma(w) + \ldots = 0 \times \frac{1}{z - w} + \ldots \] (81)

Therefore, we find that, in the NSR formalism the equations of motion of the GS superstring theory do not change under \( \kappa \)-transformation. As a result, we find that the formula (59) that relates GS and NSR parameters indeed establishes the "mapping" between GS and NSR superstring theories, up to picture-changing transformations. The first very important consequence of this fact is that now we can write down the explicit form of the target space supersymmetry for the NSR superstring theory. Repeating the arguments above we have:

\[ \delta_{\text{SUSY}}(\exp[1/2\phi]\Sigma_A) = \epsilon_A \]

\[ \Gamma_1 \delta_{\text{SUSY}} X_\mu = i\epsilon_A\gamma^A_\mu \exp[1/2\phi]\Sigma_B. \] (82)
These are space-time supersymmetry transformations for NS R superstring theory. We have to stress that, while the relation (59) always allows us to express GS variables in terms of NSR variables, the reverse (NSR in terms of GS) is possible only for GSO-projected NSR operators [20]. Let us now return to \( \kappa \)-transformations and their relation to NSR formalism. The physical meaning of these transformations is that they remove 8 spinor components of \( \theta \), thus insuring the supersymmetry. It is therefore reasonable to expect that in NSR formalism they would correspond to BRST symmetry, which has the similar function. In NSR formalism, \( \kappa \)-transformations will look in conformal gauge as follows:

\[
\delta_\kappa (e^{\exp[1/2\phi]}\Sigma) = 2i\gamma^\mu e^{\exp[-\phi]\psi_\mu}\kappa \\
\Gamma_1\delta_\kappa X^\mu = -2\gamma^\mu e^{\exp[-1/2\phi]\Sigma}\kappa
\]

(83)

In order to investigate how \( \kappa \)-transformation acts on an arbitrary operator in NSR formalism it is useful to write down the generator of \( \kappa \)-transformations. From (83) we deduce that in NSR formalism this generator is (up to the picture-changing)

\[
G^A_\theta = e^{\exp[-3/2\phi]\Sigma^A} = V_{-3/2}(0)
\]

(84)

Using this formula, it is now easy to check that \( \kappa \)-transformation of the picture-changing operator \( \Gamma_1 \) give:

\[
\delta_\kappa \Gamma_1 = e^{\exp[\kappa - 3/2\phi(z)]}c(z)\partial_\kappa(z)\Sigma^A\kappa_A
\]

(85)

If case when \( \kappa \)-transformation is applied to vertex operators, the term associated with the variation of \( \Gamma_1 \) gives the contribution proportional to the ghost field \( c(z) \) and therefore can be omitted since it does not contribute to correlation functions.

Let us examine closer how \( \kappa \)-transformation works for vertex operators. Let us first of all consider \( V_{1/2}(k = 0) = \oint \frac{dz}{2i\pi} V_{1/2}(k = 0) \). We have:

\[
\delta_\kappa V_{1/2}(k = 0) = \delta_\kappa \int \frac{dz}{2i\pi} \Gamma_1 \theta \gamma^\mu \Pi_\mu = \Gamma_1 \int \frac{dz}{2i\pi} \gamma^\mu \Pi_\mu \delta_\kappa \theta = 2i\Gamma_1 \int \frac{dz}{2i\pi} \Pi_\mu \Pi^\mu;
\]

(86)

\[
\Gamma_1 \delta_\kappa V_{1/2}(k = 0) = 2i\{Q_{BRST}, \oint \frac{dz}{2i\pi} b(z)\}
\]

The same result can, of course, be obtained by directly applying the generator (84) to \( V_{1/2} \) and making picture-changing manipulations.
We see that $\kappa$-transformation leaves zero momentum fermionic vertices invariant up to BRST trivial terms. It is interesting to notice that the generator (84) resembles the space-time supersymmetry current which is given by $j_{SUSY} = \exp[-1/2\phi]\Sigma = V_{-1/2}(z, k = 0)$. It is because $\Gamma_1V_{-3/2}(k = 0) = 0\neq V_{-1/2}(k = 0)$ that the generator of the $\kappa$-symmetry is not related to the supersymmetry current through picture-changing transformation and therefore the $\kappa$-symmetry leads to some new identities between NSR scattering amplitudes which will be discussed below.

Using (83) let us now compute the $\theta$-transformation of bosonic vertices with zero momenta. To be certain, let us take $V_0^\mu(z, k = 0)$. Omitting terms that depend only on $c(z)$ and do not contribute to correlation functions we have:

$$0 = \delta_\kappa V_0^\mu(z, k = 0) = \delta_\kappa \partial X^\mu(z) = \delta_\kappa(\Gamma_1 \exp[-\phi] \psi^\mu(z) + c(z) \beta \psi^\mu) = \Gamma_1 \delta_\kappa \Pi^\mu = \Gamma_1 2i\partial \theta \gamma^\mu \times 2i\gamma^\nu \Pi_\nu$$

as a consequence of the equations of motion. $\kappa$-transformations for vertices in other pictures is obtained by picture-changing transformation. Therefore we now see that bosonic and fermionic vertex operators are invariant under $\kappa$-transformation. The situation is different for vertex operators with non-zero momenta which are no longer $\kappa$-invariant. Let us compute, for example, $\delta_\kappa V_0^\mu(k, z)$. After rather cumbersome computations we get:

$$\delta_\kappa V_{-1}(k, z) = \Gamma_1 \delta_\kappa e^\mu(k) \Pi_\mu \exp[ikX] = \Gamma_1 e^\mu(k) \Pi_\mu \delta_\kappa \exp[ikX] = \Gamma_1 e^\mu(k) \Pi_\mu \kappa^A \exp[-3/2\phi] \Sigma_A \exp[ikX] = i/2e^\mu(k) \kappa^A k_\alpha [\gamma^\mu, \gamma_\alpha]_{AB} \partial[\exp[-3/2\phi] \Sigma_B \exp[ikX]] = \tilde{u}^B(k) \partial V_{-3/2B}(z, k),$$

where $\tilde{u}^B(k) = i/2e^\mu(k) k_\alpha \kappa^A [\gamma^\mu, \gamma_\alpha]_{AB}$ is the "effective spinor" which determines the spirality. Similarly, evaluation of the $\kappa$-transformation of the fermionic vertex $\delta_\kappa V_{1/2}(k, z)$ gives

$$\delta_\kappa \gamma^\mu_{AB} \exp[1/2\phi] \Sigma^B(\partial X^\mu + i/4(k\psi)\psi^\mu) \exp[ikX] = i/2\partial \tilde{\epsilon}_\mu(k) V_{-1}^\mu(z, k),$$

where

$$\tilde{\epsilon}_\mu(k) = i/2\kappa^A (\gamma^\alpha \gamma^\nu \gamma_\mu \gamma_\nu)_{AB} k_\alpha \tilde{u}_B.$$
\( z_i, i = 1, \ldots N; \) and the integrals of full derivatives would give zero. As of unintegrated correlation functions, \( \kappa \)-symmetry may give new relations between bosonic and fermionic amplitudes additional to those following from supersymmetry. Consider, for example, the correlation function in which all the momenta except for the first two are zero; let us find its theta transformation. For the sake of certainty, let us the first particle be a boson and the second one a fermion. We have:

\[
\delta \kappa < V_B(z_1, k)V_F(z_2, -k)V(z_3, k = 0)\ldots V(z_N, k = 0) >=
\]

\[
= < \delta \kappa V_B(z_1, k)V_F(z_2, -k)V(z_3, k = 0)\ldots V(z_N, k = 0) > +
\]

\[
+ < V_B(z_1, k)\delta \kappa V_F(z_2, -k)V(z_3, k = 0)\ldots V(z_N, k = 0) >=
\]

\[
= \partial_{z_1} < V_FV_F\ldots > + \partial_{z_2} < V_BV_B\ldots >= 0.
\]

The principal distinction of \( \kappa \)-symmetry from the supersymmetry is that the first one does distinguish between zero and non-zero momenta while the latter does not.

8. Conclusion

We have shown that the appearance of picture-changing operators in the String Theory is the consequence of imposing various gauge constraints on the symmetry group (2). The mechanism of the picture-changing is derived from the first principles, i.e. by computing the Faddeev-Popov determinant with the gauge constraints (3). While the similar computations have already been performed in \([5]\) for \( \Gamma_1 \), the computation for higher picture-changing operators has been done for the first time. The exact expression for \( \Gamma_n \) has been obtained. We proved that, up to BRST cohomology various \( \Gamma_n \)’s form the polynomial ring (44). Using this we show that the the correlators are independent on pictures in which the vertex operators are taken. It is not clear yet if it is possible to extend the algebra (44) to half-integer values of \( n \). Using the properties of picture-changing, we have found the correct formula (52) for the fermionic perturbations of a background. This should allow us to compute the \( \beta \)-function of gravitino; hopefully that can be done in future papers. Furthermore, we have established a relation between the \( N = 1 \) GS supersring theory in a flat background and and the \( N=1 \) NSR superstring theory and have shown that the supersymmetry transformations, equations of motion and \( \kappa \)-invariance of the GS superstring theory to be properly fulfilled in the NSR formalism. We have shown that the \( N=1 \) GS stress-energy tensor (of the theory in a flat background) is transformed into the stress-energy tensor of the \( N=1 \) NSR superstring theory. The identities of GS Superstring Theory were shown to be fulfilled in the NSR formalism in terms of operator product expansions. This
connection allows to solve the problem of the covariant quantization of GS Superstring theory by reducing it to NSR formalism for which the covariant quantization is well-known. As a consequence of this relation, we were able to derive the target space supersymmetry transformations (82). These transformation laws include picture-changing operators in a natural way. Many interesting questions still remain in connection with \( \kappa \)-symmetry; establishing its relation to BRST would be especially important. For bosonic particles, BRST symmetry is related to the fact that we can add to the polarization vector the quantity proportional to the momentum of the particle \( \sim k^\mu \). The fact that \( \kappa \)-transformations applied to vertex operators give new effective polarization vectors (spinors for fermions) which are also proportional to the momentum of the particle and the fact that both \( \kappa \)-symmetry and BRST have the similar function of removing redundant states - both these facts show that such kind of relation should exist. Finally, it was shown that the \( \kappa \)-symmetry allows us do derive the relations between bosonic and fermionic correlation functions, additional to the relations following from the space-time supersymmetry.

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