Measurement, information, and disturbance in Hamiltonian mechanics

David Theurel
Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139
(Dated: April 7, 2021)

Measurement in classical physics is examined here as a process involving the joint evolution of object-system and measuring apparatus. For this, a model of measurement is proposed which lends itself to theoretical analysis using Hamiltonian mechanics and Bayesian probability. At odds with a widely-held intuition, it is found that the ideal measurement capable of extracting finite information without disturbing the system is ruled out (by the third law of thermodynamics). And in its place a Heisenberg-like precision-disturbance relation is found, with the role of $\hbar/2$ played by $k_B T/\Omega$; where $T$ and $\Omega$ are a certain temperature and frequency characterizing the ready-state of the apparatus. The proposed model is argued to be maximally efficient, in that it saturates this Heisenberg-like inequality, while various modifications of the model fail to saturate it. The process of continuous measurement is then examined; yielding a novel pair of Liouville-like master equations—according to whether the measurement record is read or discarded—describing the dynamics of (a rational agent’s knowledge of) a system under continuous measurement. The master equation corresponding to discarded record doubles as a description of an open thermodynamic system. The fine-grained Shannon entropy is found to be a Lyapunov function (i.e. $\dot{S} \geq 0$) of the dynamics when the record is discarded, providing a novel H-theorem suitable for studying the second law and non-equilibrium statistical physics. These findings may also be of interest to those working on the foundations of quantum mechanics, in particular along the lines of attempting to identify and unmix a possible epistemic component of quantum theory from its ontic content. More practically, these results may find applications in the fields of precision measurement, nanoengineering and molecular machines.

I. INTRODUCTION

It is commonly held among the wider physics community that the topic of classical measurement is essentially trivial. I don’t mean the modeling in physical detail of any one laboratory setup, which of course can get very complicated, but just the examination of “measurement” as a bare-bones physical process, idealized away from as many complications as possible; a theoretical physicist’s model of measurement. One way of stating the widespread intuition is that there is in principle no obstruction in classical physics to measuring any observable of a system with arbitrary precision while disturbing the system arbitrarily little. This intuition is in sharp contrast to the situation in quantum physics, where the Heisenberg uncertainty principle (specifically the Ozawa inequality [1]) asserts just such a limit. Surely influenced by this attitude, there is a correspondingly sharp contrast between the little attention ever paid to the measurement process in classical physics, and the large attention paid over the decades (deservedly) to that same process in quantum physics. To the best of my knowledge, only a handful of examples can be associated with the first category: Heisenberg’s own thought experiments in the late 1920’s [2] (particularly Heisenberg’s microscope); although they served as the motivation for his quantum uncertainty principle, they were essentially classical arguments, augmented only by Einstein’s theory of the photon. In 1996 Lamb and Fearn [3] set up the problem of a classical point particle (the system) in interaction with a second point particle (the “apparatus”) subject to noise. They stopped short of a thorough analysis; their primary interest being the quantum case. Recently Morgan [4] and Katagiri [5] made use of KvN formalism in separate attempts to use quantum measurement theory to examine measurement in classical mechanics.

The only long-lasting foray into classical measurement seems to be the body of work surrounding Maxwell’s demon and the foundations of thermodynamics. The demon was first conceptualized by Maxwell in 1867 [6] as a “very observant and neat-fingered being” capable of monitoring the molecules of a gas, and, by opening and closing a small door without exerting any work, of sorting the high-energy molecules from the low, thus creating a temperature gradient. This amplifier of fluctuations, if it existed, could then be used to run a perpetual motion machine of the second kind, violating the second law. Writing in 1929 Szilárd [7] realized that, to save the second law, somewhere in the measurement process entropy had to be produced. Soon afterwards von Neumann [8], in his reading of Szilárd, pointed to information acquisition as the key step incurring entropy cost—a claim that would later be developed by Brillouin [9,10]. Building on work by Landauer [11], Bennett in 1982 [12,13] argued against Brillouin, pointing instead to erasure of the measurement record as the key step incurring entropy cost. This 150-year-long inquiry seems to have finally near a close in recent years, with the realization that the entropy cost of measurement can in fact be traded between the acquisition and erasure steps, as reviewed in [14]. Crucially, in the absence of a mature theory of classi-
cal measurement, the latter rigorous analyses had to rely on quantum measurement theory to settle the problem. Earlier attempts had it worse: with neither a mature classical nor quantum theory of measurement available, most of them proceeded by contradiction; requiring that measurement (whatever its mechanism may be) not be incompatible with the laws of thermodynamics.

The above illustrates three points which I would like to contend: (i) despite the wide-held intuition, measurement in classical physics is far from trivial; (ii) it is a woe-fully underdeveloped subject; and (iii) unacknowledged, it is a subject whose absence has long held back progress in physics. To address the issue, a reasonable aim would be a theory of measurement in the context of Hamiltonian mechanics, which is the mathematical framework at the foundation of classical physics. The research program I’m suggesting can be summarized as: to bring Bayesian probability to bear on an ontology governed by Hamiltonian mechanics, with the full strength, and no more, that is permitted by the geometro-algebraic structure of the ontology.

The present paper aims to kickstart this program. We begin by noting that the assumption of perfect information regarding the initial state of the measuring apparatus is unrealistic. In fact it is ruled out as a matter of principle by the third law of thermodynamics; initial uncertainty must be present if for nothing other than for finite-temperature thermal noise. Next we posit a model of the measurement as a physical process. While some assumptions are made concerning the systems that can be used as measuring apparatuses, no restrictions are placed on the system under measurement. This model enjoys substantial generality while at the same time lending itself to Bayesian analysis. We then show that, in the process of measurement, the uncertainty in the state of the apparatus propagates into two uncertainties regarding the object-system: one is the precision of the measurement; and the other an uncertainty in the magnitude of the disturbance caused upon the system—that is, an observer effect. And we find that these two are bound by a Heisenberg-like precision-disturbance relation. In particular, while we find no obstacle in principle to making a measurement arbitrarily precise, we do find an obstruction to realizing such a measurement without disturbance. Thus our findings are at odds with the wide-held intuition. We argue that our measurement model is maximally efficient, in the sense that it saturates the precision-disturbance relation, while deviations from it decrease efficiency. Next, we derive a novel pair of Liouville-like master equations describing the dynamics of (a rational agent’s knowledge of) a system under continuous measurement, according to whether the measurement record is read or discarded. And we show that the fine-grained Shannon entropy is a Lyapunov function (i.e. $S \geq 0$) of the dynamics when the record is discarded. Finally, we suggest that the master equation for the case of discarded record doubles as a description of an open thermodynamic system. In this case the above result constitutes a novel H-theorem detailing entropy increase in non-equilibrium thermodynamics. I hope our topic will be of interest to several fields of physics, particularly to (non-equilibrium) statistical physics and to the foundations of quantum mechanics.

The rest of the paper is organized as follows. We begin in Section II by reminding the reader of the basic concepts and equations of Hamiltonian mechanics. Section III does the conceptual heavy lifting; there we construct our measurement model and obtain the basic results on which the rest of the paper is based. In Section IV we arrive at the precision-disturbance relation. Section V considers the problem of continuous weak measurement over time. The method of analysis there is drawn directly from the field of continuous quantum measurement. In Section VI we discuss a number of relevant topics in the new light of our results: the similarities, and likely coexistence in the real world, of the classical and quantum uncertainty relations; the approach to thermal equilibrium in statistical physics; the subtle interplay between ontology and epistemology in a theory of measurement; and the epistemic limitations inherent to classical Hamiltonian ontology. The paper ends by contemplating some of the many possibilities ahead; both the concrete and the speculative.

II. BRIEF RECAP OF HAMILTONIAN MECHANICS

Hamiltonian mechanics is a confluence of differential, algebraic and symplectic geometry, Lie algebra and Lie groups. A wonderful resource for the topic is [15]. We consider a continuous-time dynamical system over a $2n$-dimensional symplectic manifold, called phase space. The observables of the system (e.g. position, momentum, angular momentum, etc) are the smooth, single-valued, real-valued functions defined globally over phase space. By convention we take observables to not depend explicitly on time. (With this convention, any explicit time-dependence is regarded as specifying a different observable at each moment in time.) The points in phase space can be expressed in local canonical coordinates $(q,p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ (Darboux’s theorem). The state of the system evolves over time according to Hamilton’s equations,

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t); t), \quad \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t); t),$$

where at each moment the system’s Hamiltonian, $H$, is an observable. Notice that “Hamiltonian” and “H” are indexical terms; they don’t specify any concrete function over phase space, but refer to whichever observable happens to serve as the generator of time-evolution (as in [1]) for a given system at a given time. At each moment Hamilton’s equations describe a flow $\Phi^H_t$ on phase space. Along the integral curves of this flow the value of
any observable \(A(q,p)\) changes as
\[
\dot{A} = \{A, H\},
\]
where \(\{A, H\}\) denotes the Poisson bracket,
\[
\{A, H\} = \sum_{j=1}^{n} \left( \frac{\partial A}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial H}{\partial q_j} \right).
\]
(Note that (2) follows from (1) after application of the chain rule to \(\frac{d}{dt} A(q,p)\); but also contains (1) as special cases when \(A\) equals one of the canonical coordinates.)

Two observables \(A, B\) for which \(\{A, B\}\) is identically zero are said to be in involution with each other. In this case, by (2), the value of \(A\) remains constant along the integral curves of the flow \(\Phi^t\) (and vice versa). It follows that any observable in involution with the Hamiltonian is a constant of the motion. In particular, if \(H\) is not explicitly time-dependent then it is itself a constant of the motion (conservation of energy). Including itself, a given observable can be in involution with as few as one and as many as \(2n-1\) independent observables, but only as many as \(n\) can be all in involution with one another.

On the other hand, if \(\{A, B\} = 1\) identically then \(A, B\) are said to be conjugate to each other. In this case \(B\) is also said to be “the” generator of translations in \(A\) (and vice versa); because, by (2), the value of \(A\) changes monotonically at unit rate along the integral curves of the flow \(\Phi^t\). A given observable, \(A\), may fail to have a conjugate observable. In this case it is still possible to speak of a locally-defined conjugate “quantity”, \(B\), which satisfies \(\{A, B\} = 1\) but fails to satisfy the stringent definition of a bona fide observable. This is illustrated on the 2D phase space by the observable \(I = \frac{1}{2}(q^2 + p^2)\) (the Hamiltonian for the simple harmonic oscillator); whose conjugate quantity \(\phi = \text{arg}(q+ip)\) (the phase of oscillation for the sho) either fails to be globally continuous, or else fails to be single-valued (depending on one’s choice of definition).

Notice that the components of \((q,p)\) satisfy the canonical relations
\[
\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij},
\]
so each canonical coordinate is in involution with all other coordinates but one, to which it is conjugate. A diffeomorphism of phase space, \((q,p) \mapsto (q',p')\), such that \((q',p')\) again satisfy these canonical relations is said to be a canonical transformation. Canonical transformations have Jacobian determinant equal to 1, so they preserve the Liouville measure of phase space volume, \(d^nq d^n p = d^n q' d^n p'\). For any flow parameter, \(\tau\), the Hamiltonian flow \(\Phi^\tau\) is an example of an (active) canonical transformation; in particular, Hamiltonian flow preserves the Liouville measure (Liouville’s theorem). Changes of coordinates implemented by (passive) canonical transformations are particularly convenient since they preserve the simple form of the Liouville measure, the equations of motion (1), and the Poisson bracket (3).

### III. A MODEL OF MEASUREMENT IN A HAMILTONIAN WORLD

Suppose we wished to measure an observable \(A(q,p)\) of the system (1) at time \(t_0\). In the world of Hamiltonian mechanics this can only be done by coupling the system to a measuring apparatus, where the joint system (= object-system + apparatus) is itself a Hamiltonian system, with
\[
H_{\text{joint}}(q,p,x,y,t) = H(q,p,t) + H_{\text{app}}(x,y,t) + H_{\text{int}}(q,p,x,y,t).
\]
Here \((x,y)\) are canonical coordinates on the \(2m\)-dimensional phase space of the apparatus; \(H_{\text{app}}\) is the apparatus’ Hamiltonian; and \(H_{\text{int}}\) is the interaction between system and apparatus, which we will assume to be switched on only briefly around \(t = t_0\). We now stipulate a model for the measurement.

#### A. System-apparatus coupling

Consider the “gauge”, or “pointer display”, of the apparatus; by which I mean the observable of the apparatus which, after interaction with the system, we want to reflect the sought-after value of \(A\) at time \(t_0\). Denote this observable of the apparatus by \(Q(x,y)\). Suppose \(Q\) has a conjugate observable, \(P(x,y)\). For the interaction to imprint the value of \(A\) on \(Q\), the interaction Hamiltonian must involve \(A\) and the conjugate quantity to \(Q\), namely \(P\); because this is the generator of translations in \(Q\) (10).

The simplest interaction of this form is the product
\[
H_{\text{int}}(q,p,x,y,t) = \sqrt{k} \delta(t-t_0) A(q,p) P(x,y),
\]
where the constant \(k > 0\) represents the strength of the measurement (see Section III C), and \(\delta(t-t_0)\) is the Dirac delta function indicating that the measurement is idealized as taking place instantaneously at \(t_0\). Without loss of generality we will assume that \(P^2\) has physical dimensions of energy. (Otherwise this can be achieved by scaling \((Q,P) \mapsto (aQ, P/a)\) for some suitable choice of constant \(a\).) The physical dimensions of \(k\) are then \([k] = \text{energy} \cdot \text{time}^2/|A|^2\).

#### B. Readying the apparatus

Let us take a step back to consider how to initialize the apparatus into its “ready state” prior to interaction at \(t_0\). Being, as we are, in the process of defining what we mean by “measurement”, on pain of circularity we shouldn’t appeal to measurement to assess the state of the apparatus, as might be needed to actively manipulate it into a state ready for measurement of the system. This difficulty can be circumvented by letting low-temperature thermalization take care of confining the state of the apparatus to a narrow region of its phase space. The region
in question can be specified experimentally by setting up
a deep energetic well there—a “trap”. This trap could be
due to a confining gravitational or electrostatic potential;
a combination of near-field electric and magnetic fields;
a light field; atomic chemical bonds; etc. We write
\[ H_{\text{app}}(x, y; t) = H_{\text{app}}^{\text{own}}(x, y) + \Pi(t) H_{\text{trap}}(x, y), \]
where \( H_{\text{app}}^{\text{own}} \) is the apparatus’ own, or internal, Hamiltonian,
which we take to be time-independent; and \( \Pi(t) \) is a rectangular step-function taking only the values 1/0,
describing the on/off switch of the trap. The trap will be
switched off for all \( t > t_0 \); it is only switched on in the time
leaving up to \( t_0 \), to help bring the apparatus into its ready
state, as we will now describe. The trap consists of a deep
energetic well which, when switched on (\( \Pi(t) = 1 \)), sets
the ground state of the apparatus at some point \((x^*, y^*)\).
Without loss of generality we may set our coordinates
such that \((x^*, y^*) = (0, 0)\), and we may assume that
the corresponding energy is \( H_{\text{app}}^{\text{own}}(x^*, y^*) \).\( H_{\text{trap}} = 0 \). (Otherwise
these conditions can be met by shifted redefinitions of \( x, y, H_{\text{app}} \).) We Taylor-expand \( H_{\text{app}}(x, y) \) on
around the ground state, obtaining a positive-definite quadratic
form:
\[ H_{\text{app}}(x, y) |_{\text{trap on}} = H_{\text{app}}^{\text{own}}(x, y) + H_{\text{trap}}(x, y) \]
\[ = \frac{1}{2} (x^* y^* \mathcal{M}(x^* y^*) + O(3)), \]
where \( \mathcal{M} \) is a symmetric positive-definite 2m-by-2m ma-
trix of coefficients, and \( O(3) \) denotes all higher-degree
terms in the series. As shown by Whittaker [17] (see
also theorem by Williamson [18], explained in [15, ap-
pendix 6]), there exists a local linear canonical transforma-
tion \((x, y) \mapsto (z, w)\) which reduces \(8\) to the normal
form
\[ H_{\text{app}}(z, w) |_{\text{trap on}} = \frac{1}{2} \sum_{i=1}^{m} (b_i^2 z_i^2 + w_i^2) + O(3). \]
Here \( b_1 \geq b_2 \geq \cdots \geq b_m > 0 \) are constants with physi-
cal dimensions of angular frequency; they are the natu-
ral frequencies of oscillation of the apparatus around its
trapped ground state.

Now to ready the apparatus: while the trap is on, the
apparatus is brought into contact with a thermal bath at
some temperature \( T = 1/\beta k_B \), allowed to equilibrate, and
then isolated again [19]. After this knowledge about
the state of the apparatus is given by the Boltzmann
probability distribution
\[ \rho(z, w) d^m z d^m w \propto e^{-\beta H_{\text{app}}(z, w) |_{\text{trap on}}} d^m z d^m w. \]

At this point we make three requirements that con-
strain the apparatuses, traps, and temperatures allowed
by our model. (i) We require that the trap be har-
monic enough, or the temperature be low enough, that in
the Boltzmann distribution [10] the higher-degree terms
in [9] can be neglected. (ii) We require that at least
one of the coordinates \( z_i \) be in involution with \( H_{\text{app}}^{\text{own}} \).
Let \( i = i^* \) be the index of this special coordinate. (If
given a choice, we want the associated frequency \( b_{i^*} \)
to be as large as possible, for a reason to be seen in Sec-
tion IV.) The condition means that \( z_i \) will be a constant
of the motion of the apparatus when the trap is switched
off—a desirable property for the pointer \( Q \) (introduced in
Section III A); so that the measurement record is sta-
ble after the interaction has past. We thus identify the
pointer \( Q = z_{i^*} \), and its conjugate \( P = w_{i^*} \). We denote
the corresponding frequency by \( \Omega \equiv b_{i^*} \). Note the physical
interpretation of \( \Omega \) as the natural frequency of oscilla-
tion of the pointer around its trapped state. Since \( Q, P \) are ob-
servables, in making these identifications we’re implicitly
assuming that (iii) the pair of conjugate local quantities
\( (z_{i^*}, w_{i^*}) \) are globally extendable to smooth single-valued
functions on phase space. From now on \( Q, P \) are the only ob-
servables of the apparatus with which we will be con-
cerned. With the above requirements met, we can easily
marginalize over all other variables in [10] to find the
probability distribution over the pointer and its conju-
gate:
\[ \rho(Q, P) dQ dP = \frac{\beta \Omega}{2\pi} e^{-\frac{m^2}{2} Q^2 - \frac{\beta}{2} \Omega^2 P^2} dQ dP. \]

This is the apparatus ready state. It describes a prepara-
tion in which the pointer and its conjugate have been set
independently to zero, but there remains some uncer-
tainty on their exact values.

C. Integrating Hamilton’s equations

Integrating Hamilton’s equations for the joint system,
the effect of the interaction \[0\] is to instantaneously
change the state of both object-system and apparatus
as [20]
\[ \Phi^A_{\sqrt{\kappa} P} \left( q \right)_{t_0^*} = \Phi^A_{\sqrt{\kappa} P} \left( q \right)_{t_0} \] \[ \left( Q \right)_{t_0^*} = \left( Q + \sqrt{\kappa} A(q, p) \right)_{t_0} \]
where \( \Phi^A_{\sqrt{\kappa} P} \) is the transformation on the system’s phase
space that implements flowing for a “time” \( \tau \) under the
Hamiltonian flow generated by \( A \). Having initialized the
apparatus to its ready state \[11\] prior to the interaction,
then, in view of \[12\], after the interaction our state of
knowledge of the apparatus, conditional on a given state
of the system at the time of measurement, is
\[
\rho(Q, P|q, p) dQ dP = \frac{\beta \Omega}{2\pi} e^{-\frac{2\beta^2}{\pi} (q^2 + \pi A(q, p))^2} dQ dP.
\]
(13)

Note that the dependence on \((q, p)\) is only through \(A(q, p)\).

The trap on the apparatus is released at the moment of measurement \((\Pi(t) = 0 \text{ for } t > t_0)\), so that the apparatus Hamiltonian returns to its internal setting \(H^{\text{app}}\). By construction the pointer \(Q\) is in involution with this Hamiltonian, so it constitutes a stable record of the measurement. At this time (i.e. any time after \(t_0\)) we read the pointer on the apparatus, yielding some definite value \(Q^*\), or equivalently
\[
A^* = \frac{Q^*}{\sqrt{k}}.
\]
(14)

\((A^* \text{ is just the reading on the pointer with the scale set appropriately.})\) Note that this does not mean that the value of \(A\) at the time of measurement is \(A^*\). Rather, given this datum, the likelihood function for the value of \(A\) at the time of measurement is, from (13),
\[
\rho(A'|A) dA^* = \sqrt{\frac{\beta k \Omega^2}{2\pi}} e^{-\frac{\beta^2 k}{\pi} (A^*-A)^2} dA^*.
\]
(15)

This completes our model of measurement. The measurement record \(A^*\), or equivalently the likelihood function (15) (with \(A^*\) specified), constitutes the outcome of the measurement.

D. Consuming the measurement

There are two operations that one, as a recipient, should perform to consume the information of the measurement. The first is triggered by the information that the observable \(A\) of the system was measured at time \(t_0\) by the stipulated procedure, with specified settings \((\beta, k, \Omega)\). As seen in (12a), the interaction involved in this measurement affects the state of the system by causing it to move along the flow generated by \(A\) for some unknown “time” \(\sqrt{k} P\). If one knew the value of \(P\) then one should change their probability distribution about the state of the system at time \(t_0\) according to
\[
\rho(q, p; t_0^*) \to \rho(q, p; t_0) = \left(\Phi_{A^*} p\right)^* (q, p; t_0^*),
\]
where \(\Phi_{A^*}\) denotes the push-forward of the transformation \(\Phi_{A^*}\), defined as \(\left(\Phi_{A^*} p\right)(q, p) = \rho(\Phi_{A^*}^{-1}(q, p))\); and the +/- superscripts on \(t_0^*\) are meant as a reminder that this update reflects a physical transition of the system that took place in a short time interval around \(t_0\). But one does not know the value of \(P\); all that is know about it is expressed by the probability distribution (11). One folds this in by marginalizing over \(P\):
\[
\rho(q, p; t_0^*) = \sqrt{\frac{\beta}{2\pi}} \int dP e^{-\frac{\beta^2}{\pi} P^2} \left(\Phi_{A^*}^* \rho\right)(q, p; t_0^*).
\]
(16)

The second operation is triggered by the information of the measurement outcome [15]. One assimilates this by performing the Bayesian update \(\rho_{\text{prior}}(q, p; t_0) \to \rho_{\text{posterior}}(q, p; t_0)\), with
\[
\rho_{\text{posterior}}(q, p; t_0) \propto \rho_{\text{prior}}(q, p; t_0) \rho(A^*|A(q, p))
\]
\[
\propto \rho_{\text{prior}}(q, p; t_0) e^{-\frac{\beta^2 k}{2} (A^*-A(q, p))^2},
\]
(17)

where the omitted factor of proportionality is just the normalization, obtained by integrating the expression shown over the system’s phase space \(\int d^q q d^p p\). Since multiplication by a function of \(A\) commutes with the push-forward \(\Phi_{A^*}^*\), operations (16) (17) can be performed in either order to the same effect. If (17) is performed first, it corresponds to updating one’s knowledge about the state the system was in before the measurement was made (i.e. at \(t_0^*\)); if second, about the state the system was left in by the measurement. Notice that if only the fact of the measurement is revealed but not the outcome (in this case we say the outcome was discarded), then one should only perform operation (16), not (17).

Finally, if a single number is desired as an objective quantification of the measured observable (i.e. not biased by anyone’s prior), the maximum-likelihood estimate can be given, from (15):
\[
A^*|_{t_0} = A^* \pm \frac{1}{\sqrt{\beta k \Omega^2}}
\]
(18)

(mean ± standard deviation) [21]. We will refer to
\[
\epsilon_A \pm \frac{1}{\sqrt{\beta k \Omega^2}}
\]
(19)
as the precision of the measurement, or of the measurement’ outcome. (But notice that to translate this to an uncertainty in a given agent’s knowledge of \(A\) we must first combine the likelihood function with the agent’s prior, as in [17].) At this point it is clear why \(k\) represents the strength of the measurement: the larger \(k\) the higher the measurement’s precision [22].

IV. A HEISENBERG-LIKE PRECISION-DISTURBANCE RELATION IN HAMILTONIAN MECHANICS

The measurement model we’ve just stipulated is characterized by the triad \((\beta, k, \Omega)\); respectively the (inverse) temperature of the thermal bath (= temperature of apparatus), the measurement strength, and the frequency of oscillation of the apparatus’ pointer around its trapped ground state. Notice, from (19), that for a given bath
and trap (i.e. fixed $\beta, \Omega$), we can make our measurement of $A$ more precise by cranking up the strength, $k$, which we might think of as a knob on our experimental setup. However, notice that the more precisely $A$ is measured (i.e. the larger $k$), the more the system is affected by the measurement (i.e. the longer the “flow time” $\sqrt{kP}$ in [12a]), and more importantly, the more uncertain we are about the magnitude of said observer effect (since the uncertainty in $P$, $1/\sqrt{\beta}$ from [11], translates into uncertainty $\sqrt{k/\beta}$ in the “flow time”). It’s worth emphasizing that this disturbance of the system is not arbitrary, but has the form of time-evolution along the Hamiltonian flow generated by the measured observable, $A$: the only thing uncertain is how much “time” the system flowed.

We can see that this disturbance will affect some observables of the system more than others: in particular, any observable $B$ in involution with $A$ will emerge undisturbed in the immediate aftermath of the measurement; although subsequent time-evolution under the system’s own dynamics will cause the initial disturbance to “leak into” such a $B$, unless $B$ is also in involution with $H$.

Concretely, we find that the precision of a measurement [19], and the disturbance caused by the measurement upon the system,

$$\eta_A \doteq \sqrt{k/\beta},$$

obey the inverse relation

$$\epsilon_A \eta_A = \frac{1}{\beta \Omega},$$

which is fixed for a given bath and trap; independent of the identity of the system measured, of that of the system used as measuring apparatus, of the measurement strength and of the choice of observable measured. The product on the left-hand side can easily be made larger (see discussion in Section [VIA]) but not smaller, as far as I can tell. This Heisenberg-like precision-disturbance relation (or “uncertainty relation” for short) suggests an obstruction to how close we can come in a world governed by Hamiltonian mechanics to the idealization of measurement without disturbance. To the extent that our model of measurement has a claim to generality, relation [21] will be a general principle. Note that this relation is softer than the Heisenberg uncertainty principle of quantum mechanics: for any given bath and trap one will have a finite obstruction on the right-hand side of [21], but one can always endeavor to make the obstruction smaller by cooling the apparatus further or tightening the trap. Instead this obstruction is of a kind with the third law, to which it is clearly related: it suggests that it is impossible by any procedure, no matter how idealized, to reduce the observer effect of measurement to zero in a finite number of operations.

V. CONTINUOUS MEASUREMENT OVER TIME: NEW MASTER EQUATIONS AND H-THEOREM

Extracting information about the system by measurement increases our knowledge about some aspect of it. However, we’ve seen that any such measurement according to our model will disturb the system to an extent that we cannot monitor; and this decreases our knowledge about some other aspect of the system. For a single measurement this tradeoff is expressed by the precision-disturbance relation [21], or in more detail by the updates [16] [17]. In this section we explore the compound effect of such tradeoff due to multiple measurements; specifically, a continuous succession of vanishingly-weak measurements. The method of analysis we follow is drawn from the field of continuous quantum measurement, which addresses the corresponding problem in that setting. (See for example [23].)

Subdivide a finite interval of time $[0, T]$ into $N$ equal subintervals demarcated by $t_0 = 0 < t_1 < t_2 < \cdots < t_N = T$, with $t_j = j\Delta t$. For each $j \in \{1, \ldots, N\}$, select an observable $A_j = A_j(q,p)$ of the system, and prepare for it a measurement $(\beta_j, k_j, \Delta t, \Omega_j)$ to be carried out at time $t_j$. Notice that we’ve scaled the strength according to the size of the subintervals; smaller $\Delta t$ means each individual measurement is weaker, but a greater number of them fit into $[0, T]$. We will see that this is the right scaling for the effects to converge when we take the limit of smaller and smaller $\Delta t$. (Note that this changes the physical dimensions of $k_j$; they are now $[k_j] = \text{energy} \cdot \text{time}/[A_j]^2$.) The resulting tuple of pointer readings $A^* \doteq (A_1^*, A_2^*, \ldots, A_N^*)$ constitutes the measurement record for the entire succession of measurements. To assimilate the $j$-th measurement we perform the two operations (16, 17), resulting in the update

$$\rho(q,p;t_{j+1}) \propto e^{-\frac{\beta_j}{2} \frac{\Delta t_j}{P_j^2} [(A_j^*, A_j) - (A_j(q,p))^2]}
\cdot \frac{\beta_j}{\sqrt{2\pi}} \int dP_j e^{-\frac{\beta_j}{2} P_j^2} \left[ \left( \Phi_{A_j} \right)_{\sqrt{k_j} \Delta t_j} \right] (q,p,t_j).$$

For small $\tau$, the push-forward

$$[(\Phi^A_A)_{\tau}] (q_0,p_0) = \rho(\Phi^A_A, (q_0,p_0)) = \rho(q(-\tau), p(-\tau))$$

(23)

can be calculated by Taylor-expanding the function $\tau \mapsto \rho(q(-\tau), p(-\tau))$ around $\tau = 0$; using the chain rule to pass all time-derivatives onto $q, p$, and calculating the latter from Hamilton’s equations with Hamiltonian $A$. The result is

$$(\Phi^A_A)_\tau \rho = \rho + \tau \{A, \rho\} + \frac{\tau^2}{2} \{A, \{A, \rho\}\} + O(\tau^3).$$

(24)

Putting this into (22), the integral over $P_j$ can then be done order-by-order. The odd-order terms all vanish by
symmetry, leaving us with
\[
\rho(q,p;t_{j+1}) \propto e^{-\frac{\beta_j k_j \beta_k}{2} (A_j^* - A_j)^2} \cdot \left( 1 + \frac{k_j \Delta t}{2 \beta_j} \{A_j, \{A_j, \rho\}\} + O(\Delta t^2) \right) \bigg|_{(q,p;t_j)}.
\] (25)

A. Case of discarded measurement record

Let’s pause to consider the case in which the measurement record \( A^* \) is discarded. In this case we should skip update [17], which amounts to dropping the exponential factor and the omitted proportionality factor in [25]. Taking then the limit \( \Delta t \to dt \) describing a continuous succession of vanishingly-weak measurements, we arrive in this case (discarded measurement record) at
\[
\frac{\partial \rho}{\partial t} = \left( \frac{\{H,\rho\}}{\text{internal dynamics}} + \frac{k}{2\beta} \{A, \{A, \rho\}\} \right) \text{Hamiltonian flow info preserved}
\]
\[
+ \left( \frac{\{A, \log \rho\}}{\text{observer effect diffusion along flow } \Phi_A^t} \right) \text{info about } A \text{ preserved}
\]
\[
+ \left( \frac{\beta_j k_j \beta_k}{2} \{A_j, \{A_j, \rho\}\} \right) \text{other info lost}
\] (26)

where we’ve introduced the well-known Liouville term \( \{H,\rho\} \) accounting for the internal dynamics of the system under \( H \) [24], which we had been ignoring until now; and all quantities shown may be explicit functions of time. This is a Liouville-like master equation, with an additional second-order term due to the observer effect of measurement. We can get some sense for the effect of this new term as follows. Let \( B(q,p;t) \) denote any function over phase space, possibly explicitly time-dependent. Here and throughout let’s use \( \langle \cdot \rangle \) to denote the phase-space average:
\[
\langle B \rangle \equiv \int dq dp \rho(q,p;t) B(q,p;t).
\] (27)

In Appendix A we prove that under master equation (26) any such phase-space average evolves as
\[
\frac{d}{dt} \langle B \rangle = \left\langle \left\{ B, H \right\} \right\rangle + \left\langle \left\{ \frac{\partial B}{\partial t} \right\} \right\rangle + \frac{k}{2\beta} \left\langle \left\{ A, \log \rho \right\} \{ A, B \} \right\rangle.
\] (28)

The first term on the right-hand side of this equation is due to the Liouville term in (26); the second term is due to any explicit time-dependence of \( B \); and the third term is due to the second-order term in (26). As a special application of this equation consider \( B = -\log \rho \), in which case the phase-space average is the Shannon entropy:
\[
S(t) \equiv \langle -\log \rho \rangle.
\] (29)

It is not hard to show that the first two terms on the right-hand side of (28) vanish in this case [25]. Thus we find that under dynamics (26),
\[
\dot{S} = \frac{k}{2\beta} \left\langle \left\{ A, \log \rho \right\}^2 \right\rangle \geq 0.
\] (30)

It is a well-known result that \( S(t) \) remains constant \((\dot{S} = 0)\) under the Liouville equation \( \partial \rho / \partial t = \{H, \rho\} \) (see example in Figure 1b). In breaking with that, we have just found that entropy generally increases over time under (26) on account of the new term. Thus the Liouville term preserves information, while the second-order term causes information loss. Indeed, in accordance with our discussion in Section XIV concerning the nature of the observer effect, this term describes diffusion along the flow lines generated by the instantaneous observable \( A(q,p;t) \) (see example in Figure 1c). This diffusion preserves-instant-to-instant information pertaining to that instant’s observable \( A(q,p;t) \), while it erases information pertaining to observables not in involution with it.

We note here, in passing, that (26) may alternatively be construed as an H-theorem detailing entropy increase in an open thermodynamic system in contact with a thermal bath (described here as the collection of apparatuses). In this light (26, 28, 30) may help further our understanding of non-equilibrium statistical physics and the approach to thermal equilibrium. We discuss this topic further in Section VIB.

B. Case of simulated measurement record

Returning now to (25), suppose instead that the measurement record is not discarded but that we have only yet read up to the \( (j-1) \)-th entry; i.e. \( A_j^* \) through \( A_{j-1}^* \) are known while \( A_j^* \) onward are not. We would like to simulate ahead of time (say, on a computer) how our state of knowledge will evolve as we continue to read more of the record. However, without the benefit of hindsight the upcoming record entries appear to us as random variables. The language for this kind of simulation is stochastic calculus. (See tutorial on stochastic calculus in [23].) Let us first ask: what should be our probability distribution for the \( j \)-th outcome, \( A_j^* \)? Making use of the likelihood function (15), this question can be answered in terms of our current knowledge of the value of \( A_j \):
\[
\rho(A_j^*; t_j) = \int dA \rho(A_j; t_j) \rho(A_j^* | A_j)
\]
\[
\propto \int dA \rho(A_j; t_j) e^{-\frac{\beta_j k_j \beta_k}{2} \{A_j^* - A_j\}^2}.
\] (31)

As \( \Delta t \to dt \), the Gaussian in this expression becomes very wide and spread out as a function of \( A_j \). The distribution \( \rho(A_j; t_j) \) becomes very narrow by comparison, and can be replaced by a Dirac delta, which must be centered at \( \langle A_j \rangle \) for the means to match. Using the delta to do the integral over \( A_j \) we have, up to a normalization factor,
\[
\rho(A_j^*; t_j) \propto e^{-\frac{\beta_j k_j \beta_k}{2} \{A_j^* - \langle A_j \rangle\}^2}.
\] (32)

By a simple change of variables we introduce \( \Delta W_j^* \), our probability distribution of which is a zero-mean Gaussian
FIG. 1. Master equation dynamics in various measurement regimes. Evolution of the state of knowledge $\rho(q,p;t)$ of a rational agent under master equation (38) is illustrated in a simple example: the system under measurement is a 1D simple harmonic oscillator (sho); the measurement is characterized by constant $\beta, k, \Omega$ and fixed $A$; the measured observable is the energy $A = H \pm \frac{1}{2}(\omega^2 q^2 + p^2)$; and the initial distribution over phase space is unimodal. Although not proven here, three timescales are involved: that of internal dynamics, $\tau_{\text{dyn}} \sim \frac{1}{\omega}$; that of diffusion due to observer effect, $\tau_{\text{dif}} \sim \frac{\beta}{k \omega^2}$; and that of collapse due to Bayesian update on the measurement record, $\tau_{\text{col}} \sim \frac{1}{\beta k \Omega^2 \Delta E^2}$, where $\Delta E$ is the target certainty on $H$ (i.e. $\tau_{\text{col}}$ is the characteristic timescale for the variance of $\rho(H;t)$ to fall below $\Delta E^2$). (a) Phase portrait showing level sets of the sho Hamiltonian. (b–e) From left to right, snapshots of $\rho(q,p;t)$ at successive times, indicated at top in units of the sho period, for four different measurement regimes (rows). For ease of visualization the color scheme (left) is normalized anew for each plot. (b) Regime $\tau_{\text{syn}} \ll \tau_{\text{dif}}, \tau_{\text{col}}$: describes an isolated system; (c) reduces to the Liouville equation $\frac{\partial \rho}{\partial t} = \{ H, \rho \}$. (c) Regime $\tau_{\text{syn}} \sim \tau_{\text{dif}} \ll \tau_{\text{col}}$: describes case of discarded measurement record; (d) reduces to (26). Notice entropy increase, in accordance with (26), due to diffusion along the flow generated by $A$. (d) Regime $\tau_{\text{syn}} \sim \tau_{\text{col}} \ll \tau_{\text{dif}}$: describes an approximation to ideal classical measurement with minimal disturbance. Notice the trend of decreasing entropy, in accordance with (41), due to collapse towards the measurement outcome. (e) Regime $\tau_{\text{syn}} \sim \tau_{\text{col}} \sim \tau_{\text{dif}}$: describes the three processes (dynamics, diffusion and collapse) happening together. Notice the tradeoff between information about $A$ and information about the conjugate quantity (sho phase). (f) Evolution of the first four cumulants of $\rho(H;t)$ in regime (equivalently regime (e)). For ease of visualization each cumulant is rescaled to 1 at $t = 0$. Note qualitative agreement with (43).

\[
A_j^* = \langle A_j \rangle + \frac{1}{\sqrt{\beta_j k_j \Omega_j^2}} \frac{\Delta W_j}{\Delta t}, \tag{33}
\]

The value of expressing (32) this way is two-fold. From a simulation standpoint, we can use a random number generator to sample $\Delta W_j$ from its Gaussian distribution, and (33) then tells us how to convert this into a sample of $A_j^*$. And from an analysis standpoint, this expression enables a very convenient form of calculation: in the limit $\Delta t \to dt$ we write

\[
A^* = \langle A \rangle + \frac{1}{\sqrt{\beta k \Omega^2}} \frac{dW}{dt}, \tag{34}
\]

where $W(t) \equiv \int_0^t dW$ is a standard Wiener process, with $dW$ obeying the basic rule of Itô calculus $dW^2 = dt$. Notice also that $\Delta W_j$ is statistically-independent from $A_j$. Using $\langle \cdot \rangle$ to denote averaging over the Wiener process we thus have, for any function $f(A)$:

\[
\langle f(A) dW \rangle = f(A) \langle dW \rangle = 0. \tag{35}
\]
Taking stock: given $\rho(q,p; t_k)$ for a given time $t_k$ we can use it to calculate $\langle A_k \rangle$, and combine this with the output of a random number generator as in (33) to simulate the upcoming entry of the measurement record $A_{k+1}$. We can then use (25) to calculate what our updated state of knowledge $\rho(q,p; t_{k+1})$ would be upon reading that entry, and iterate the process. Analytically we proceed as follows. Substitute (33) into (25); expand the square in the exponent, discarding the overall factor $\exp\{-\Delta W_j^2/2\Delta t\}$ which is independent of $(q,p)$; and Taylor-expand the exponential, keeping in mind that powers of $\Delta t$ count for "half an order", to obtain

$$\rho(q,p; t_{j+1}) \propto \left(1 - \beta_j k_j \Omega_{j}^2 (A_j - \langle A_j \rangle)^2 \Delta t + \sqrt{\beta_j k_j \Omega_{j}^2 (A_j - \langle A_j \rangle) \Delta W_j } + \beta_j k_j \Omega_{j}^2 (A_j - \langle A_j \rangle)^2 \Delta W_j + O(\Delta t \Delta W_j) \right) \cdot \left(\rho + \frac{k_j}{2\beta_j} \langle A_j, \{A_j, \rho\} \rangle + O(\Delta t^2) \right) \bigg|_{(q,p; t_j)}.$$  

In the limit of continuous measurement $\Delta t \to dt$, $\Delta W_j \to dW$, $\Delta W_j^2 \to dt$ this reduces to

$$\rho(q,p; t + dt) \propto \rho + \frac{k}{2\beta} \langle A, \{A, \rho\} \rangle dt \left( \rho + \sqrt{\beta k \Omega^2 (A - \langle A \rangle) \rho} dW \right) \bigg|_{(q,p; t)},$$

where again all quantities shown may be explicit functions of time. One can check that the right-hand side is already normalized, so the omitted factor of proportionality is 1. We arrive in this case (simulated measurement record) at

$$\frac{\partial \rho}{\partial t} = \{H, \rho\} + \frac{k}{2\beta} \langle A, \{A, \rho\} \rangle.$$  

where again we’ve re-introduced the Liouville term $\{H, \rho\}$ accounting for the internal dynamics of the system. Compared to (24) we now have a new stochastic term appearing, which is due to assimilation of the simulated measurement record via Bayesian update. To get some sense for the effect of this new term, in Appendix [C] we prove that under master equation (38) the Shannon entropy (29) evolves as

$$\dot{S} = \frac{k}{2\beta} \langle \{A, \log \rho\}^2 \rangle$$

Bayesian update

$$-\beta k \Omega^2 \sigma_A^2 - \sqrt{\beta k \Omega^2} \langle (A - \langle A \rangle) \log \rho \rangle \frac{dW}{dt},$$

where

$$\sigma_A^2 \equiv \langle (A - \langle A \rangle)^2 \rangle$$

is the variance in our knowledge of $A$. The first term on the r.h.s. of (39) is familiar from (35); it describes increasing entropy due to the observer effect of measurement. The remaining two terms are due to the stochastic term in (38); these two together may be positive for particular measurement outcomes, but they are non-positive on average, as can be seen by invoking (35):

$$\langle \dot{S} \rangle = \frac{k}{2\beta} \langle \{A, \log \rho\}^2 \rangle - \beta k \Omega^2 \sigma_A^2.$$  

Thus the stochastic term in (38) leads, on average, to increasing information (see example in Figure 11). It is also interesting to note that this term is both non-linear and non-local in $\rho$, since $\langle A \rangle$ depends on the value of $\rho$ everywhere on phase space. To gain further insight into the effects of this term, suppose the measured observable is fixed $A = A(q,p)$, and consider our PDF over this observable, $\rho(A; t)$, which is just the marginal

$$\rho(A'; t) = \int d^q q d^p p \delta(A(q,p) - A') \rho(q,p; t).$$

Let $\kappa_i$ denote the $i$-th cumulant of this distribution. In Appendix [C] we prove the following hierarchy of equations describing the contribution of the stochastic term in (38) to the evolution of these cumulants:

$$dk_1 = \sqrt{\beta k \Omega^2} \kappa_2 dW,$$

$$dk_2 = \sqrt{\beta k \Omega^2} \kappa_3 dW - \beta k \Omega^2 \frac{k^2}{2}(2\kappa_3^2) dt,$$

$$dk_3 = \sqrt{\beta k \Omega^2} \kappa_4 dW - \beta k \Omega^2 \frac{k^2}{2}(6\kappa_2 \kappa_3) dt,$$

$$dk_4 = \sqrt{\beta k \Omega^2} \kappa_5 dW - \beta k \Omega^2 \frac{k^2}{2}(8\kappa_2 \kappa_4 + 6\kappa_3^2) dt,$$

$$\cdots$$

Notice in particular the trends $\langle \kappa_1 \rangle = 0$, $\langle \kappa_2 \rangle \sim -\langle \kappa_3 \rangle^2$, $\langle \kappa_3 \rangle \sim -\langle \kappa_4 \rangle$, $\langle \kappa_4 \rangle \sim -\langle \kappa_5 \rangle$, . . . . These trends tell us that (supposing $A$ is not explicitly time-dependent and the Liouville term does not intervene too strongly) the stochastic term in (38) causes all cumulants of $\rho(A; t)$
higher than second to vanish exponentially fast, leaving \( \rho(A; t) \) a Gaussian; it then causes the variance to vanish like \( \sim 1/t \), while the mean jiggles around in a random walk of zero drift and volatility decaying with the variance. In the limit in which the measurement process is complete, \( \rho(A; t) \) converges to a delta distribution centered at the simulation’s putative true value of \( A \). (See example in Figure 1d–f.)

C. Simultaneous measurements

Simultaneous weak measurement of multiple observables \( A_j(q, p), \ldots, A_k(q, p) \), whether these are in involution or not, can be handled by letting \( A(q, p; t) \) switch between these observables on a fast time scale. By averaging (38) over this fast time scale this equation then becomes

\[
\frac{\partial \rho}{\partial t} = \{H, \rho\} + \sum_{j=1}^{s} \frac{k_j}{2\beta_j} \{A_j, \{A_j, \rho\}\}
+ \sum_{j=1}^{s} \sqrt{\beta_j k_j \Omega_j^2 (A_j - \{A_j\})} \frac{dW_j}{dt}, \tag{44}
\]

where \( (\beta_j, k_j, \Omega_j) \) describes the measurement setup for the \( j \)-th observable, and the \( W_j(t) = \int_0^t dW_j \) are independent Wiener processes for \( j \neq j' \). The analogues of (39) and (41) for this equation are

\[
\dot{S} = \sum_{j=1}^{s} \frac{k_j}{2\beta_j} \langle \{A_j, \log \rho\}^2 \rangle
- \sum_{j=1}^{s} \left( \frac{\beta_j k_j \Omega_j^2 \sigma_j^2}{2} + \sqrt{\beta_j k_j \Omega_j^2 \langle \{A_j - \{A_j\}\} \log \rho \rangle} \frac{dW_j}{dt} \right), \tag{45}
\]

and

\[
\langle \dot{S} \rangle = \sum_{j=1}^{s} \frac{k_j}{2\beta_j} \langle \{A_j, \log \rho\}^2 \rangle - \sum_{j=1}^{s} \frac{\beta_j k_j \Omega_j^2 \sigma_j^2}{2}. \tag{46}
\]

If some of the measurement outcomes are discarded the corresponding terms should be dropped from the rightmost sums in (44–46). If all the outcomes are discarded we are left with

\[
\frac{\partial \rho}{\partial t} = \{H, \rho\} + \sum_{j=1}^{s} \frac{k_j}{2\beta_j} \{A_j, \{A_j, \rho\}\}, \tag{47}
\]

which is linear, local, and deterministic; and

\[
\dot{S} = \sum_{j=1}^{s} \frac{k_j}{2\beta_j} \langle \{A_j, \log \rho\}^2 \rangle \geq 0. \tag{48}
\]

VI. DISCUSSION

A. Comparing the quantum and classical uncertainty relations

How does our classical precision-disturbance relation (21) compare to the Heisenberg uncertainty principle of quantum mechanics? The latter can be stated in a few different forms. We will consider the Kennard-Weyl-Robertson form in section VI D, where we discuss the epistemology of Hamiltonian ontology. Here we consider the “joint measurement form” (26–29), pertaining to simultaneous measurement of two observables, \( A \) and \( B \). When \( A, B \) are conjugate to each other this reads:

\[
\epsilon_A \epsilon_B \geq \frac{\hbar}{2}, \tag{49}
\]

where \( \epsilon_A \) and \( \epsilon_B \) denote the precisions in the measurements of \( A \) and \( B \), respectively (30); and \( \hbar \) is the reduced Planck constant.

One superficial difference between (21) and (49) is that one is an equality while the other an inequality. However, this difference is illusory. The product on the left-hand side of (21) can easily be made larger than the right-hand side, so that for a more general class of measurement models I indeed expect us to have

\[
\epsilon_A \eta_A \geq \frac{1}{\beta \Omega}. \tag{50}
\]

Let us call a measurement that saturates this bound maximally efficient, while one failing to saturate it, inefficient (or not as efficient). In these terms any measurement following our model is maximally efficient, as we’ve seen. The most obvious modification to our model that leads to inefficiency is if the measurement outcome is discarded (\( \epsilon_A \to \infty; \eta_A \) unchanged). Another way to reduce efficiency is if the apparatus’ pointer fails to be in involution with \( H_{\text{app}} \), so that some amount of “deterioration” of the measurement record can happen between the time of the system-apparatus interaction and whenever the record is read. In the opposite direction, one might ask: could not a higher efficiency than (21) be reached, say, by using a pointer and coupling, \( (Q, P) \), that are correlated in the apparatus ready state? To achieve the latter, one would need \( (Q, P) \) to not diagonalize the quadratic form \( (9) \); namely, instead of choosing \( (Q, P) = (z_\nu, w_\nu) \), one would choose \( (Q, P) \) related to \( (z_\nu, w_\nu) \) by some linear canonical transformation. In fact, although not proven here, I find that this approach leads to the same precision-disturbance relation (21); the only difference (aside from the Bayesian analysis becoming more involved) is that a systematic component is added to the observer effect. This component can be corrected given the measurement outcome \( A^* \); so it doesn’t count towards the disturbance \( \eta_A \) (31). In summary, these remarks suggest that, while it is easy to do worse than (21), it may not be possible to do better; i.e. they suggest inequality (50) to be the general principle.
A second difference, which remains between (49) and (50), is that one involves the product of two precisions, while the other the product of a precision with a disturbance. This difference can be bridged as well. Recall that the disturbance in question amounts to flowing along \( \Phi^A \) for an unknown “time” \( \tau \) whose uncertainty is \( \eta_A \). Under this flow the “rate” of change of any observable \( B \) is given by \( \frac{d}{d\tau} B = \{ B, A \} \). In particular, if \( B \) is the conjugate to \( A \), so that \( \{ B, A \} = 1 \), then \( B \) increases monotonically at the steady rate of 1; and the net effect of the flow on \( B \) is simply to displace its value by \( \tau \). This final step can fail if \( B \) has a discontinuity somewhere; so it is important that \( B \) be a bona fide observable, not just a local quantity such as the phase \( \phi \) of an oscillator. So the uncertainty in the “flow time”, \( \eta_A \), translates directly into a disturbance in the value of the conjugate observable, \( B \). This places a bound on the precision, \( \epsilon_B \), with which any subsequent measurement can hope to determine the original value of \( B \): \( \epsilon_B \geq \eta_A \), with equality holding only if the measurement of \( B \) is done at full strength. Thus we have

\[
\epsilon_A \epsilon_B \geq \frac{1}{\beta \Omega}, \tag{51}
\]

and the parallel with (49) becomes apparent. Historically it seems that Heisenberg’s own interpretation of the uncertainty principle was as a precision-disturbance relation [2], not very different in spirit from (50). And in recent years work in quantum mechanics has paid considerable attention to precision-disturbance relations, yielding formulas similar to (50) but with \( h/2 \) on the right-hand side [11, 92,93].

The real world is no doubt quantum mechanical, and so the Heisenberg uncertainty principle is fundamental. But as we know, as one “zooms out” to larger scales somehow an approximately Hamiltonian world effectively emerges (Bohr’s correspondence principle and the quantum-to-classical transition). Hand in hand with the emergence of this effective Hamiltonian world I expect our classical uncertainty relation to gain traction. Figure 2 illustrates how the classical and quantum relations then must coexist. For a tight enough trap and/or cold enough bath (below the dashed diagonal), the obstruction in (51) is brought below \( h/2 \) and becomes unreachable; the quantum obstruction acts like rock bottom. For less tight traps and/or less cold baths the obstruction in (51) becomes larger than \( h/2 \) and begins to dominate. Taken together, one may expect to have in the real world an obstruction that interpolates between these two; something along the lines of

\[
\epsilon_{AB} \geq \frac{h}{2} + \frac{1}{\beta \Omega} \quad \text{or perhaps} \quad \frac{h/2}{1 - e^{-\frac{h}{\beta \Omega}}} \tag{52}
\]

it will take a detailed quantum calculation to work out the precise formula (see Section VII for a germ of how this might be done). To gain some perspective for the scales involved, note from Figure 2 that, at room temperature, trap frequencies any lower than about 12 THz (corresponding to light wave-lengths \( \gtrsim 25 \mu \text{m} \)) are already enough to put us in the classical regime. At the same time, even for the highest temperatures and lowest frequencies shown in the top-left of Figure 2 the classical obstruction hardly becomes larger than \( 10^{-17} \text{J.s} \); an extremely small quantity by macroscopic standards. And yet, even in more moderate regimes towards the center of Figure 2, the classical obstruction may be relevant in the contexts of precision measurement, nanoeengineering and molecular machines.

**B. On the foundations of statistical physics**

As obtained here, master equation (26) describes a rational agent’s knowledge of a Hamiltonian system under continuous measurement when the measurement record is discarded. As briefly mentioned in Section V A this equation can be repurposed to describe (a rational agent’s knowledge of) a Hamiltonian system \( (q,p) \) in interaction with a thermal bath at (inverse) temperature \( \beta \). Seen in this light, corollary (30) constitutes a novel H-theorem detailing how the Shannon entropy increases over time for an open thermodynamic system. Note that this H-theorem deals directly with the fine-grained entropy, without coarse-graining phase space as in Gibbs’ H-theorem [36]. We should summarize the inputs that go into justifying this application of (26): (i) equilibrium thermodynamics to give us prior knowledge about the

![FIG. 2. Coexistence of quantum and classical uncertainty relations in the real world.](image)
state of the bath, in (10); (ii) Hamiltonian mechanics to describe the system-bath dynamics, in (12); (iii) Bayesian probability to translate ontic into epistemic dynamics, in (16) [37]. A fourth and final ingredient perhaps deserves the most scrutiny in future work: (iv) modeling of the bath as a collection of temporarily-uncorrelated systems, each initially in thermal equilibrium, each momentarily “minimally coupled” to the object-system, as in (9) [38].

There is ongoing debate among different programs of statistical mechanics for understanding the approach to thermal equilibrium [39]. Equation (26), and its corollaries [28, 30], seem ideally poised for informing this debate. For instance, consider the ergodic program [39], which posits that thermal equilibrium corresponds to a steady state of \( \rho(q,p;t) \). A criticism levied against this program is that, even if the ergodic hypothesis is granted—according to which the integral curves of the Hamiltonian flow \( \Phi^H_t \) will typically fill the level sets of \( H \) densely—it still remains the case that under the Liouville equation it is impossible for a non-steady state to evolve into a steady state (because this equation implies \( S = 0 \)); which seems to lead to the absurd conclusion that thermal equilibrium could never follow from non-equilibrium. I think this objection can be entirely resolved by the use of master equation (26) in place of Liouville’s equation. To see how this comes about consider the special case in which \( A = A(q,p) \) is not explicitly time-dependent, and the system Hamiltonian is trivial, \( H(q,p) \equiv 0 \). In this case [24] simply describes diffusion along the integral curves of the flow \( \Phi^A_t \). Clearly, from any given initial condition a steady state will be reached in which probability is distributed uniformly along each integral curve. If we suppose further, as in the ergodic hypothesis, that such integral curves typically fill the level sets of \( A \), then we see that an initially-localized distribution (Figure 3a) will converge as time passes to a distribution uniform on the level sets of \( A \) (Figure 3b). In the general case, the features of the steady state (including whether or not there is one) will depend on the interplay between \( H \) and \( A(t) \) (alternatively, between \( H,A_1, \ldots, A_n \) in (17)). I expect that settings can be found corresponding to the various ensembles of statistical mechanics. For example, Figure 1 illustrates a situation in which the coupling to the bath does not disturb the energy of the system. As shown, the steady state is uniformly distributed on the energy level sets. If the value of the energy is precisely known (Figure 1, late time), this is just the micro-canonical ensemble.

We should noted that, despite the time-irreversibility present in (26), it would be mistaken to suggest this equation as an answer to Loschmidt’s reversibility objection [39] and the issue of the arrow of time. Rather, I think the arrow of time has been baked into (26) by our model of the thermal bath; according to which in each infinitesimal interval of time the system comes into contact with a fresh uncorrelated thermal system and leaves it a little bit correlated. This is not unlike the effect that Boltzmann’s molecular chaos hypothesis has for his H-theorem [24, 39]. Personally I’m not too troubled by this, since I consider the “past hypothesis” [39] a better avenue for pursuing this issue.

Finally, I would emphasize that the equations we have derived are valid for systems of any size, not just for the large-size limit; they are well suited for studying fluctuations from the mean in finite-size thermodynamic systems, both in and out of equilibrium.

C. Reading the measurement record; is it turtles all the way down?

We return to a complication that was tactically overlooked in Section III. In our measurement model, after the apparatus had interacted with the system—let’s call that step “pre-measurement”—, we stipulated that the pointer on the apparatus should be read, which would yield some definite value \( Q^* \). But what could it mean to “read \( Q \)” if not to measure this observable of the apparatus? This seems to lead us down an infinite regression in which the system is pre-measured by an apparatus, which must then be pre-measured by an apparatus, which must then be… the passage from “systems interacting” to “agent being informed” never quite taking place. In a sense this predicament is similar to the quantum measurement problem, particularly as articulated by the Wigner’s friend thought experiment [40]. In both settings, the paradoxical step is the passage from what seems to be best regarded as an ontic-level description (Hamiltonian or unitary dynamics) to what seems to be best regarded as an epistemic-level operation (Bayesian updating or collapse of the wave function). At the epistemic level we speak freely of agents, observers, measurements, observations, reading the measurement record, information, probability, Bayesian updating, collapse of the wave function. But at the ontic level all of these are complicated phenomena, resisting precise characterization. Until such characterizations are available [41] we are stuck with the shifty split—to use a term coined by Bell [42].

In our model, the shifty split was introduced at one degree of separation from the system under study: we described the system-apparatus interaction at the ontic level; then we described the apparatus-agent interaction at the epistemic level. Such a once-removed approach can be very useful, as demonstrated by the example of the theory of general quantum measurement [43]. However, there is a pair of consistency tests that should be checked of such a theory. Notice that the once-removed theory contains the twice-removed and higher theories. To see this, simply let what we have been calling the object-system instead be used as an apparatus of some kind to (pre-)measure another system. (Now we have an object-system which is pre-measured by an apparatus, which is pre-measured by a second apparatus, whose pointer is “read” by an agent.) This maneuver uses only the
FIG. 3. Thermalization under master equation according to the ergodic program. Cartoon of the 2n-dimensional phase space of system \((q,p)\), showing a level set of \(H(q,p)\) ((2n − 1)-dimensional), and a single integral curve of the flow \(\Phi^t\) on this level set (1-dimensional). Note that the apparent self-intersections of the curve are an artifact of our low-dimensional cartoon. For \(A\) ergodic, almost all its level sets are densely filled by almost all integral curves on them. Assuming \(H(q,p) \geq 0\), dynamics \((26)\) just describes diffusion along these integral curves. (a–c) The value of \(\rho(q,p,t)\) along one integral curve is illustrated at three different times. For ease of visualization the color scheme (below) is normalized anew for each plot. The depicted process happens in parallel along each integral curve (only one shown). (a) An initial state of knowledge is localized in a small neighborhood on the level set. (b) As time passes probability diffuses out along the integral curve. Because the curve is ergodic, this can lead to rapid spread of probability on the level set. Taking all integral curves into account amplifies the speed of spreading on the level set. (c) At late times the distribution converges to a steady state which is uniform all along the curve. Taking all integral curves into account, the distribution is now uniform on the level set.

rules of the ontology, yet succeeds in shifting away the split by one degree. In light of this feature of the theory, for a first test (T1) we ask: is there any such way to shift away the split that increases the efficiency of our measurement (i.e. decreases the obstruction in \((50)\))? If the answer is yes then test (T1) is failed; it is a sign that the way we are bridging the shifty split does not fully exploit the possibilities allowed by the ontology, and we should strengthen it. (This strengthening is necessary. So long as test (T1) is not passed, bounds such as \((50)\) derived from once-removed measurement schemes cannot be taken seriously, since they can be circumvented by better use of the operations allowed by the ontology.) On the other hand, for a second test (T2) we ask: is it the case that when we shift away the split, no matter how we do it, we find that it always decreases the efficiency of our measurement (i.e. increases the obstruction in \((50)\))? If the answer is yes then test (T2) is failed; it is a sign that the way we are bridging the shifty split requires an operation that is not allowed by the ontology, and must be revised.

How does our model fair on these tests? Consider test (T2) first. We have bridged the shifty split by stipulating: (i) “read” the pointer on the apparatus, yielding some definite value \(Q^*\). (This is after the apparatus has pre-measured the object-system.) Here’s a way to shift away the split without reducing efficiency. Instead of (i) do: (ii) use a second apparatus, operated according to our same model with parameters \((\beta(2),k(2),\Omega(2))\), to pre-measure at full strength \((k(2) \rightarrow \infty)\) the value of \(Q\), recording it on its own pointer \(Q(2)^*\) which we then “read”, yielding some definite value \(Q(2)^*\). To see that procedure (ii) is just as efficient as (i) notice, first, that neither of the two involve further disturbance of the object-system, so \(\eta_A\) is the same for both. Second, since the final measurement in procedure (ii) is at full strength, we have \(\epsilon_Q(2)^* \approx 1/\sqrt{k(2)} \rightarrow 0\), so this measurement reveals the exact value of \(Q\): \(Q(2)^* = Q^*\). Thus procedure (ii) leads to the same likelihood function \((15)\) as (i), and hence to the same \(\epsilon_A\). This gives us proof-by-example that it is possible to shift away the split without reducing our model’s efficiency, so test (T2) is passed. It is harder to prove that test (T1) is passed since this requires proving that efficiency remains unimproved for all ways of shifting away the split. I don’t know how to do that, but I conjecture that our model passes this test too. In support of this conjecture, note one way in which efficiency might have been increased by shifting away the split, but isn’t. Turn again to procedure (ii) just discussed. Imagine that, after having measured the pointer of the first apparatus (\(Q\)) as described, it were possible to do another measurement on this apparatus to determine \(P\). If we could gain even a little bit of information about the value that \(P\) had at the time of interaction with the object-system (\(t_0\)), we could combine it with what we know of \(P(t_0)\) from the thermal distribution \((11)\) to reduce our uncertainty \(\sigma_P\), and hence reduce \(\eta_A = \sqrt{k} \sigma_P\). If this were possible our model would fail test (T2). That it is impossible follows from the fact that the measurement of \(Q\) in procedure (ii) had to be done at full strength \((k(2) \rightarrow \infty)\), which leads to an infinite disturbance of the first apparatus \((\eta_Q(2)^* \approx 1/\epsilon_Q(2)^* \rightarrow \infty)\); and this infinitely disturbs the value of \(P\) (since \(\{Q,P\} = 1\)). So we see
that in the course of carrying out procedure (ii) all information about $P(t_0)$ is lost beyond recovery.

D. On the epistemology of Hamiltonian ontology

Consider the Kennard-Weyl-Robertson (KWR) form of the Heisenberg uncertainty principle of quantum mechanics [33]. For a pair of conjugate observables, $A$ and $B$, it reads:

$$\sigma_A \sigma_B \geq \frac{\hbar}{2},$$

(53)

where $\sigma_A$ and $\sigma_B$ denote the standard deviations at a given time in our knowledge of $A$ and $B$, respectively [39]. This form of the uncertainty principle speaks directly to the epistemology of quantum ontology. In this section we ask whether the present developments allow us to establish an analogous result about the epistemology of Hamiltonian ontology.

Notice, first of all, the sense in which we must understand such a question. Unlike in the quantum formalism, there is nothing in our classical formalism that rules out the possibility of starting with perfect information about conjugate observables:

$$\rho(A', B'; t) = \delta(A(q(t), p(t)) - A')\delta(B(q(t), p(t)) - B'),$$

(54)

where

$$\rho(A', B'; t) \equiv \int dq dp \rho(A(q, p) - A') \cdot \delta(B(q, p) - B') \rho(q, p; t).$$

(55)

Rather, the question is whether it is at all possible to arrive at such a state of perfect information from a state of less information. In particular: suppose we were handed a Hamiltonian system of which we knew nothing at all, so that $\rho$ were initially uniform on phase space. Does there exist a sequence of measurements on the system that would take $\rho$ into the perfect-information state? [44]

Consider the direct approach of performing simultaneous measurement of $A$ and $B$, with respective measurement settings ($\beta_A, k_A, \Omega_A$) and ($\beta_B, k_B, \Omega_B$). The evolution of $\rho$ is as given by master equation [44]. Suppose that the measurements are strong enough that they come to completion on a much faster timescale than that of the system’s dynamics, so that the Liouville term in [44] can be neglected. It is a bit tricky, because one must be mindful of the rules of Itô calculus, but one can check that, starting from an uncorrelated Gaussian distribution in $A$ and $B$ (of which the uniform distribution is the special case of infinite variances), the general solution to [44] is

$$\rho(A, B; t) = \frac{1}{2\pi \sigma_A \sigma_B} e^{-\frac{(A-\mu_A)^2}{2\sigma_A^2} - \frac{(B-\mu_B)^2}{2\sigma_B^2}},$$

(56)

where the means $\mu_A, \mu_B$ are stochastic functions of time evolving as

$$d\mu_A = \sqrt{\beta_A k_A \Omega_A^2} \sigma_A(t) dW_A,$$

(57a)

$$d\mu_B = \sqrt{\beta_B k_B \Omega_B^2} \sigma_B(t) dW_B,$$

(57b)

while the variances $\sigma_A^2, \sigma_B^2$ are the deterministic functions of time

$$\sigma_A(t)^2 = \frac{\coth(\beta_A |B| \beta_B k_A k_B \Omega_A^2(t - t_0))}{\sqrt{\beta_A \beta_B (k_A k_B \Omega_A^2 \Omega_B^2)}},$$

(58a)

$$\sigma_B(t)^2 = \frac{\coth(\beta_B |A| \beta_A k_A k_B \Omega_B^2(t - t_0))}{\sqrt{\beta_A \beta_B (k_A k_B \Omega_A^2 \Omega_B^2)}},$$

(58b)

where $t_0 < t$ is a constant of integration, and $l_A, l_B \in \{+1, -1\}$. We see that, under simultaneous measurement of conjugate observables, an initially-Gaussian PDF remains Gaussian for all time. Also, much like we saw in [43], the mean of the distribution executes a random walk (this time in two dimensions) of volatilities proportional to the variances. However, unlike in [43], now the variances converge to non-zero values as the measurement runs to completion:

$$\sigma_A^2 \to \frac{1}{\sqrt{\beta_A \beta_B (k_A k_B \Omega_A^2 \Omega_B^2)}},$$

(59a)

$$\sigma_B^2 \to \frac{1}{\sqrt{\beta_A \beta_B (k_A k_B \Omega_A^2 \Omega_B^2)}},$$

(59b)

This comes about because the measurement of $A$ causes collapse “along the $A$-direction” (along the integral curves of $\Phi_A$) and diffusion “perpendicular to the $A$-direction” (along the integral curves of $\Phi_A$); while the simultaneous measurement of $B$ causes the converse; and at completion of the measurement the effects precisely cancel out. Notice that [59] gives

$$\sigma_A \sigma_B = \frac{1}{\sqrt{\beta_A \mu_A \beta_B \mu_B}},$$

(60)

which begins to resemble [53]. Is it the case that the product $\sigma_A(t) \sigma_B(t)$ remains above this limit at all times? That depends on the exponents $l_A, l_B$. The case $l_A = +1$ gives $\sigma_A(t_0) \to \infty$; it describes complete ignorance about $A$ at some past time $t_0$. On the other hand, the case $l_A = -1$ gives $\sigma_A(t_0) = 0$; it describes perfect information about $A$ at the past time $t_0$. Likewise for $l_B$. Since we are interested in beginning from a state of ignorance, the relevant solution for us has $l_A = l_B = +1$. It then follows from [58] that the inequality

$$\sigma_A \sigma_B \geq \frac{1}{\sqrt{\beta_A \Omega_A} \sqrt{\beta_B \Omega_B}}$$

(61)

holds for all times. If both measurements are characterized by the same (inverse) temperature $\beta$ and trap
frequency $\Omega$, this further reduces to

$$\sigma_A \sigma_B \geq \frac{1}{\beta \Omega}.$$  \hspace{1cm} (62)

We have derived this uncertainty-uncertainty relation by considering simultaneous measurement of the pair of conjugate observables $A, B$. Could this be a general epistemic obstruction, or is there some different sequence of measurements that fares better? We leave the question open for future investigation.

E. Future directions

In closing we look to some of the many questions and possibilities ahead. We have argued that our model of measurement is maximally efficient; i.e. that it is impossible to do better than (50) in the way of measuring without disturbing. It would be of fundamental interest to have a proof of this claim at the level of rigor of mathematical physics. Complementing this, it would be valuable to have experimental tests of (50) and of the bigger picture outlined in Figure 8. Moving forward it will be worth honing our intuition about the range of possible dynamics of a Hamiltonian system under measurement (or more accurately, of the epistemic state of a rational agent about a Hamiltonian system under measurement). For this it would be good to see numerical studies of equations (26, 38) in more interesting scenarios than the one-dimensional simple harmonic oscillator explored in Figure 1. For this task it might be useful if somebody reproduced in the classical setting the steps, discovered already in the quantum setting, to transform a non-linear stochastic master equation like (38) into an equivalent linear equation (23, 45). Another calculation that I think should be attempted is the derivation of the precise version of (72), which I suspect can be obtained by reproducing our measurement model from Section III in the quantum setting.

As discussed in Section VI B, equations (26, 28, 30) seem like a promising basis for a better understanding of equilibrium and non-equilibrium statistical physics. I think this research program deserves much attention. To name just two topics of inquiry in this direction: (i) Can the various ensembles of equilibrium statistical mechanics indeed be obtained as the equilibrium solutions to (26) under suitable choices of the coupling to the bath, $A$? And if so, what does this teach us about the approach to thermal equilibrium in each case? (ii) How do the present results relate to the body of work on Maxwell’s demon reviewed in Section VI? Can the old debate now be resolved within classical physics, without recourse to quantum physics?

In connection with the quantum measurement problem and the interpretation of quantum mechanics, there is a program dating back to Einstein [46, 47] of attempting to identify and unmix a possible epistemic component of quantum theory from its ontic content. In recent times this program has made promising progress at the hands of Caves, Fuchs, and others [47,19]. In particular Spekkens [50, 51], and Bartlett, Rudolph, and Spekkens [52], have illustrated how an uncircumventable epistemic limitation in an otherwise classical world, much like what is suggested by our discussion in Section VI B, can lead to several of the phenomena usually regarded as characteristic of quantum mechanics. It will be interesting to see what these two programs can contribute to each other.

Finally I would like to venture the following speculative suggestions. (i) As we know from general relativity, gravity couples directly to energy. Perhaps a system subject to a strong external gravitational field can, in certain cases, be reasonably modeled by (26) with $A = H$. If so, could this tell us something interesting about the entropy of a system falling onto the event horizon of a black hole? Could this be a useful tool for studying black hole thermodynamics? (A quantum version of this master equation (c.f. earlier comments in connection to (52)) might be an even better tool.) (ii) To the best of my knowledge, theoretical computer science grounds its notions of computability and complexity in concrete (if highly abstracted and idealized) physical models. Does the existence of an obstruction to ideal measurement without disturbance in Hamiltonian mechanics have some bearing on those notions of computer science grounded in the world of classical physics? (iii) Hamilton’s equations and their underlying geometric-geometric structure are not unique to physics; they emerge wherever the equations of a theory can be gotten out of a variational principle. Indeed, in classical physics they emerge in just this way from Hamilton’s principle of stationary action. In particular, optimal control theory uses essentially the same equations under the name of Pontryagin’s minimum principle [51]. Could the present results have consequences for aspects of optimal control under partial information and, by extension, for artificial intelligence? At least, these musings illustrate the breadth of potential implications of our subject.

ACKNOWLEDGMENTS

It is a pleasure to thank Omar Eulogio López, Kurt Jacobs, and Pavel Chytkov for reading versions of the typescript and providing many useful suggestions. I’m particularly grateful to Matthew A. Wilson, for believing in me when I most needed it and for his continued mentorship and patience. While conducting this research I was supported by the Picower Neurological Disorder Research Fund.

Appendix A: Derivation of equation (28)

Our objective is to derive equation (28). For brevity of notation we will omit the integration measure $d\gamma dq dp$
in integrals over phase space. We will make use of the identity

\[ \int A\{B,C\} = \int B\{C,A\} = \int C\{A,B\}, \tag{A1} \]

which is valid for any smooth functions \(A(q,p;t), B(q,p;t), C(q,p;t)\) as long as their product decays to zero as \(\|(q,p)\| \to \infty\), so that boundary terms from integration by parts can be discarded. This identity is readily verified:

\[
\int A\{B,C\} = \int A \sum_{i=1}^{\rho} \left( \frac{\partial B}{\partial \rho_i} \frac{\partial C}{\partial t} - \frac{\partial B}{\partial t} \frac{\partial C}{\partial \rho_i} \right)
= \int C \sum_{i=1}^{\rho} \left( -\frac{\partial}{\partial \rho_i} \left( A \frac{\partial B}{\partial \rho_i} \right) + \frac{\partial}{\partial \rho_i} \left( A \frac{\partial B}{\partial t} \right) \right)
= \int C\{A,B\}. \tag{A2} 
\]

In our applications of the identity one of the factors will always be homogeneous in \(\rho\), which it is safe to assume decays fast enough for the identity to hold (e.g. for each \(t\), \(r(q,p;t)\) can be assumed to have compact support over phase space without any loss of physical generality.)

Now, the phase-space average of \(B\) is \(\langle B \rangle = \int \rho B\), and the time derivative of this is

\[
\frac{d}{dt} \langle B \rangle = \int \left( B \frac{\partial \rho}{\partial t} + \rho \frac{\partial B}{\partial t} \right) = \int B \frac{\partial \rho}{\partial t} + \left( \frac{\partial B}{\partial t} \right). \tag{A3} 
\]

Working with the first term on the r.h.s. here, we substitute into it from (26):

\[
\int B \frac{\partial \rho}{\partial t} = \int B \left( \{H,\rho\} + \frac{k}{2\beta} \{A,\{A,\rho\}\} \right). \tag{A4} 
\]

Using identity (A1), the first term on the r.h.s. here can be written as \(\int \rho \{B,H\} = \langle B,H \rangle\). Turning to the remaining term on the r.h.s. of (A4), we let \(C = \{A,\rho\}\) and again use identity (A1), so that the integral in this term can be written as

\[
\int B\{A,C\} = \int C\{B,A\} = -\int \{A,\rho\} \{A,B\}
= -\int \rho \{A,\log \rho\} \{A,B\}
= -\langle \{A,\log \rho\} \{A,B\} \rangle. \tag{A5} 
\]

All together we have

\[
\frac{d}{dt} \langle B \rangle = \langle \{B,H\} \rangle + \frac{\partial B}{\partial t} - \frac{k}{2\beta} \{A,\{A,\log \rho\} \{A,B\} \}, \tag{A6} 
\]

which is (28), as desired.

**Appendix B: Derivation of equation (39)**

Our objective is to derive equation (39). For brevity of notation we will omit the integration measure \(d^n q d^n p\) in integrals over phase space. Expanding the differential of \(S\) (from (29)) to second order in \(d\rho\):

\[
\frac{dS}{dt} = -\int \frac{d(d(\rho \log \rho) - \rho \log \rho)}{dt}
= -\int \left( (\rho + d\rho) \log(\rho + d\rho) - \rho \log \rho \right) - \frac{1}{2} \frac{d^2 \rho}{\rho} dt^2
= -\int \left( \log(\rho + 1) + \frac{1}{2} \frac{d^2 \rho}{\rho} \right)
= -\int \left( \log \rho + \frac{1}{2} \frac{d^2 \rho}{\rho} \right). \tag{B1} 
\]

(In the last step we used the fact that \(\int d\rho = d\int \rho = d1 = 0\). We will now substitute into here for \(d\rho\) from (38); however, notice that the non-stochastic terms from that equation will only contribute linearly (since terms of order \(dt^2\) and \(dW^2\) are negligible), so their final contribution to \(dS\) will be same as already deduced in connection to master equation (26) (c.f. (31)). We therefore need only consider here the contribution to \(dS\) of the stochastic term in (38), that is \(d\rho = \sqrt{\beta k \Omega^2} (A - \{A\}) dW\). Substituting this into (B1), and in the following step using the rule of Itô calculus \(dW^2 = dt\):

\[
\frac{dS}{dt} = -\int \left( \log \rho \sqrt{\beta k \Omega^2} (A - \{A\}) dW \right)
= -\int \left( \frac{1}{2} \frac{\sqrt{\beta k \Omega^2} (A - \{A\})^2 dW}{\rho} \right)
= -\sqrt{\beta k \Omega^2} dt \int (A - \{A\})^2 \rho dW
= -\frac{\sqrt{\beta k \Omega^2}}{2} dW (A - \{A\}) \log \rho
= -\frac{\sqrt{\beta k \Omega^2}}{2} dt ((A - \{A\})^2). \tag{B2} 
\]

This, together with the contribution (30) due to the non-stochastic terms from (38), gives us (39), as desired.

**Appendix C: Derivation of the hierarchy of equations (43)**

Our objective is to derive the hierarchy of equations (43), which describes the contribution of the stochastic term in (38) to the evolution of the cumulants of \(\rho(A,t)\) when \(A = A(q,p)\) is not explicitly time-dependent. For brevity of notation we will omit the integration measure \(d^n q d^n p\) in integrals over phase space.
Consider the cumulant-generating function for $\rho(A; t)$:

$$f(z; t) = \log \{e^{zA}\} \equiv \kappa_1(t) \frac{z}{1!} + \kappa_2(t) \frac{z^2}{2!} + \kappa_3(t) \frac{z^3}{3!} + \ldots$$

(1)

Let $df$ denote the differential of this function with respect to time, and $f'$ denote its derivative with respect to the dummy variable $z$. Expanding the differential of $f$ to second order in $df$:

$$df = d\left(\log \int \rho e^{zA}\right) = \log \int (\rho + df)e^{zA} - \log \int \rho e^{zA}$$

$$= \left(\frac{d\rho e^{zA}}{\int \rho e^{zA}}\right) dt - \frac{1}{2} \left(\frac{d\rho e^{zA}}{\int \rho e^{zA}}\right)^2. \quad (C2)$$

Substituting into here the stochastic term from (38) (that is $d\rho = \sqrt{\beta k \Omega^2} (A - \langle A \rangle) \rho dW$, and using the rule of Itô calculus $dW^2 = dt$:

$$df = \sqrt{\beta k \Omega^2} dW \left(\frac{\int (A - \langle A \rangle) \rho e^{zA}}{\int \rho e^{zA}}\right)$$

$$- \frac{\beta k \Omega^2 dt}{2} \left(\frac{\int (A - \langle A \rangle) \rho e^{zA}}{\int \rho e^{zA}}\right)^2$$

$$= \sqrt{\beta k \Omega^2} dW (f' - \langle A \rangle) - \frac{\beta k \Omega^2 dt}{2} (f' - \langle A \rangle)^2. \quad (C3)$$

Writing $f$ in terms of its cumulant expansion (C1), and noting that $\langle A \rangle = \kappa_1$:

$$d\kappa_1 \frac{z}{1!} + d\kappa_2 \frac{z^2}{2!} + d\kappa_3 \frac{z^3}{3!} + \ldots = \sqrt{\beta k \Omega^2} dW \left(\kappa_2 \frac{z^2}{2!} + \kappa_3 \frac{z^3}{3!} + \ldots\right) - \frac{\beta k \Omega^2 dt}{2} \left(\kappa_2 \frac{z^2}{2!} + \kappa_3 \frac{z^3}{3!} + \ldots\right)^2. \quad (C4)$$

Expanding the square on the r.h.s. and equating coefficients of corresponding powers of $z$ yields the hierarchy of equations (43), as desired.

---

[1] M. Ozawa, Universally valid reformulation of the Heisenberg uncertainty principle on noise and disturbance in measurement, Phys. Rev. A 67, 042105 (2003).
[2] W. Heisenberg, The physical principles of the quantum theory (Dover, 1949).
[3] W. E. Lamb and H. Fearn, Classical theory of measurement: A big step towards the quantum theory of measurement, in Amazing Light (Springer, 1996) pp. 373–389.
[4] P. Morgan, An algebraic approach to Koopman classical mechanics, Ann. Phys. 414, 168090 (2020).
[5] S. Katagiri, Measurement theory in classical mechanics, Prog. Theor. Exp. Phys. 2020, 063A02 (2020).
[6] J. Collier, Two faces of Maxwell’s demon reveal the nature of irreversibility, Stud. Hist. Philos. Sci. 21, 22J (1990).
[7] L. Szilard, Über die entropieverminderung in einem thermodynamischen system bei eingriffen intelligenter wesen, Z. Phys. 53, 840 (1929).
[8] J. Von Neumann, Mathematical foundations of quantum mechanics (Princeton University Press, 1955) Original work published 1932.
[9] L. Brillouin, Maxwell’s demon cannot operate: Information and entropy. I, J. Appl. Phys. 22, 334 (1951).
[10] L. Brillouin, The negentropy principle of information, J. Appl. Phys. 24, 1152 (1953).
[11] R. Landauer, Irreversibility and heat generation in the computing process, IBM J. Res. Dev. 5, 183 (1961).
[12] C. H. Bennett, The thermodynamics of computation—a review, Int. J. Theor. Phys. 21, 905 (1982).
[13] C. H. Bennett, Demons, engines and the second law, Sci. Am. 257, 108 (1987).
[14] T. Sagawa, Thermodynamics of information processing in small systems, Prog. Theor. Phys. 127, 1 (2012).
[15] V. I. Arnol’d, Mathematical methods of classical mechanics, 2nd ed. (Springer-Verlag, 1982).
[16] To be more precise: for any specified pointer $Q(x, y)$, by the Carathéodory-Jacobi-Lie (CJL) theorem there exists, in a neighborhood of any point $(x, y)$, a canonical coordinate system for the apparatus in which $Q$ is one of the coordinates. By $P$ we mean the coordinate conjugate to $Q$ in this system. The requirement that $P$ be a bona fide observable amounts to the non-trivial assumption that this coordinate can be extended to a smooth single-valued global function on phase space. As seen in [1], $Q$ is in involution with all other coordinates of this system but $P$. It follows that if, upon expressing $H_{\text{int}}$ in these coordinates, $P$ did not appear, then we would have $\{H_{\text{int}}, Q\} = 0$; and by [2], the interaction would have no immediate effect on the pointer $Q$. Since this is the opposite of what we want, we see that $H_{\text{int}}$ should depend on $P$.
[17] E. T. Whittaker, A treatise on the analytical dynamics of particles and rigid bodies, 4th ed. (Cambridge University Press, London, 1959) section 192.
[18] J. Williamson, On the algebraic problem concerning the normal forms of linear dynamical systems, Am. J. Math. 58, 141 (1936).
[19] Instead of removing the bath, we might require just that its coupling to the apparatus be weak enough that it doesn’t spoil the measurement record, $Q$, on the timescales of interest.
[20] To do this calculation it helps to approximate the $\delta$ by a square impulse of width $\Delta t$ and height $1/\Delta t$. As $\Delta t$
is taken smaller and smaller, the joint Hamiltonian becomes dominated by $H_{int}$ during the interaction, so that $H$ and $H_{app}$ can be neglected during the brief time $\Delta t$. Noting that both $A(q,p)$ and $P$ are constant under the flow generated by the interaction Hamiltonian, both parts of (12) then follow readily.

[21] The justification for calling this number a “mean ± standard deviation” is that that is what it corresponds to in the posterior (17) when the marginalized prior $\rho_{\text{prior}}(A'; t_0) \propto \int d^m q d^m p \delta(A' - A(q,p)) \rho_{\text{prior}}(q,p; t_0)$ is sufficiently flat.

[22] Given the definition of $A^*$ in (14), one might worry that $A^*$ also scales with $1/\sqrt{k}$, but that’s not the case: notice, from (13), that $Q^*$ is drawn from a gaussian centered at $\sqrt{k}A$. Obviously the value of $A$ is independent of our decision to measure it, and a fortiori of our setting of $k$. Hence it’s the reading on the dial, $Q^*$, that scales with $\sqrt{k}$, so $A^*$ is unaffected (in expectation) by the strength of the measurement.

[23] K. Jacobs and D. A. Steck, A straightforward introduction to continuous quantum measurement, Contemp. Phys. 47, 279 (2006).

[24] M. Kardar, Statistical physics of particles (Cambridge University Press, 2007).

[25] Proof: by identity (A1) from Appendix A, the first term on the right-hand side of (25) can be written as $\int H(\rho, -\log \rho)$, which is zero because the bracket vanishes. The second term on the right-hand side of (25) is $\int \rho \frac{d}{dt} (\log \rho) = -\int \frac{\partial}{\partial t} \log \rho = -\frac{\partial}{\partial t} \log \rho^{\text{ext}} = 0$.

[26] E. Arthurs and J. L. Kelly, BSTJ Briefs: On the simultaneous measurement of a pair of conjugate observables, Bell Syst. Tech. J. 44, 725 (1965).

[27] E. Arthurs and M. Goodman, Quantum correlations: A generalized Heisenberg uncertainty relation, Phys. Rev. Lett. 60, 2447 (1988).

[28] M. Ozawa, Quantum limits of measurements and uncertainty principle, in Quantum Aspects of Optical Communications, edited by C. Bendjaballah, O. Hirota, and S. Reynaud (Springer, 1991) pp. 1–17.

[29] S. Ishikawa, Uncertainty relations in simultaneous measurements for arbitrary observables, Rep. Math. Phys. 29, 257 (1991).

[30] To facilitate comparison between the quantum and classical ontologies, it is convenient to speak of quantum mechanics from a realist/hidden-variable interpretation, in which measurement outcomes are outcomes about an underlying unknown state. We won’t concern ourselves here with the ongoing debate about the plausibility of such an interpretation.

[31] Remember: $\eta$ is our uncertainty in the “flow time” $\tau$ for which the measurement caused the system to move along $\Phi^x$.

[32] J. Erhart, S. Sponar, G. Sulyok, G. Badurek, M. Ozawa, and Y. Hasegawa, Experimental demonstration of a universally valid error-disturbance uncertainty relation in spin measurements, Nat. Phys. 8, 185 (2012).

[33] K. Fujikawa, Universally valid Heisenberg uncertainty relation, Phys. Rev. A 85, 062117 (2012).

[34] S. -Y. Baek, F. Kaneda, M. Ozawa, and K. Edamatsu, Experimental violation and reformulation of the Heisenberg’s error-disturbance uncertainty relation, Sci. Rep. 3, 2221 (2013).

[35] P. Busch, P. Lahti, and R. F. Werner, Proof of Heisenberg’s error-disturbance relation, Phys. Rev. Lett. 111, 160405 (2013).

[36] R. C. Tolman, The principles of statistical mechanics (Courier Corporation, 1979).

[37] Bayesian probability may be replaced here by the reader’s favored interpretation of probability.

[38] In this case we don’t need the coupling $P$ to have a conjugate observable $Q$, let alone that $Q$ be conserved; since there is no need for a record of the interaction. The only requirement is that the interaction Hamiltonian factors: $H_{int} \propto AP$ for some $A = A(q,p,t), P = P(x,y,t)$.

[39] R. Frigg, A field guide to recent work on the foundations of statistical mechanics (2008), arXiv:0804.0399.

[40] E. P. Wigner, Remarks on the mind body question, in The Scientist Speculates, edited by I. J. Good (Heinemann, London, 1961).

[41] Of course, such characterizations may end up being different in the classical and quantum ontologies.

[42] J. Bell, Against ‘measurement’, Phys. World 3, 33 (1990).

[43] M. A. Nielsen and I. Chuang, Quantum computation and quantum information (American Association of Physics Teachers, 2002).

[44] D. J. Griffiths, Introduction to quantum mechanics, 2nd ed. (Pearson Prentice Hall, 2005).

[45] H. Wiseman, Quantum trajectories and quantum measurement theory, Quantum Semic. Opt. 8, 205 (1996).

[46] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, Phys. Rev. 47, 777 (1935).

[47] N. Harrigan and R. W. Spekkens, Einstein, incompleteness, and the epistemic view of quantum states, Found. Phys. 40, 125 (2010).

[48] C. A. Fuchs, Quantum mechanics as quantum information (and only a little more) (2002), arXiv:quant-ph/0205039.

[49] C. M. Caves, C. A. Fuchs, and R. Schack, Unknown quantum states: the quantum de Finetti representation, J. Math. Phys. 43, 4537 (2002).

[50] R. W. Spekkens, Evidence for the epistemic view of quantum states: A toy theory, Phys. Rev. A 75, 032110 (2007).

[51] R. W. Spekkens, Quasi-quantization: classical statistical theories with an epistemic restriction, in Quantum Theory: Informational Foundations and Foils (Springer, 2016) pp. 83–135.

[52] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Reconstruction of Gaussian quantum mechanics from Liouville mechanics with an epistemic restriction, Phys. Rev. A 86, 012103 (2012).

[53] Landauer’s work establishing the thermodynamic irreversibility of certain computing processes has certainly had such an impact; launching the field of reversible computing.

[54] V. I. Arnol’d and A. B. Givental’, Symplectic geometry, in Dynamical Systems IV: Symplectic geometry and its applications (Berlin: Springer-Verlag, 1990) pp. 1–136.

[55] D. E. Kirk, Optimal control theory: an introduction (Prentice-Hall, Inc., 1970).

[56] P. Libermann and C.-M. Marle, Symplectic geometry and analytical mechanics, Vol. 35 (Springer Science & Business Media, 2012).

[57] M. P. Frank, Foundations of generalized reversible computing, in International Conference on Reversible Computation (Springer, 2017) pp. 19–34.