Fast Operations on Linearized Polynomials and their Applications in Coding Theory

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Abstract

This paper considers fast algorithms for operations on linearized polynomials. We propose a new multiplication algorithm for skew polynomials (a generalization of linearized polynomials) which has sub-quadratic complexity in the polynomial degree \( s \), independent of the underlying field extension degree \( m \). We show that our multiplication algorithm is faster than all known ones when \( s \leq m \). Using a result by Caruso and Le Borgne (2017), this immediately implies a sub-quadratic division algorithm for linearized polynomials for arbitrary polynomial degree \( s \). Also, we propose algorithms with sub-quadratic complexity for the \( q \)-transform, multi-point evaluation, computing minimal subspace polynomials, and interpolation, whose implementations were at least quadratic before. Using the new fast algorithm for the \( q \)-transform, we show how matrix multiplication over a finite field can be implemented by multiplying linearized polynomials of degrees at most \( s = m \) if an elliptic normal basis of extension degree \( m \) exists, providing a lower bound on the cost of the latter problem. Finally, it is shown how the new fast operations on linearized polynomials lead to the first error and erasure decoding algorithm for Gabidulin codes with sub-quadratic complexity.

Key words: linearized polynomials, skew polynomials, fast multiplication, fast multi-point evaluation, fast minimal subspace polynomial, fast decoding

1. Introduction

Linearized polynomials (Ore, 1933a) are polynomials of the form

\[
a = \sum_k a_k x^{q^k}, \quad a_k \in \mathbb{F}_{q^m} \text{ (finite field)},
\]

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which possess a ring structure with ordinary addition and polynomial composition. They are an important class of polynomials which are of theoretical interest (Evans et al., 1992; Wu and Liu, 2013) and have many applications in coding theory (Gabidulin, 1985; Silva et al., 2008), dynamical systems (Cohen and Hachenberger, 2000) and cryptography (Gabidulin et al., 1991). Especially in coding theory, designing fast algorithms for certain operations on these polynomials is crucial since it directly determines the complexity of decoding Gabidulin codes, an important class of rank-metric codes.

The operations that we consider in this paper are multiplication, division, $q$-transform, computing minimal subspace polynomials, multi-point evaluation, and interpolation of linearized polynomials of degree at most $s$ over $\mathbb{F}_{q^m}$.

In this section, we omit log factors using the $O^\ast (\cdot)$ notation. These factors can be found in the respective theorems or references. By $\omega \leq 3$, we denote the matrix multiplication exponent.

1.1. Related Work

For $s \leq m$ and $m$ admitting a low-complexity normal basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, Silva and Kschischang (2009) and Wachter-Zeh et al. (2013) presented algorithms for the $q$-transform with respect to a basis of $\mathbb{F}_{q^m} \subseteq \mathbb{F}_{q^s}$ ($\mathcal{O}(ms^2)$ over $\mathbb{F}_q$), multi-point evaluation ($\mathcal{O}(m^2s)$ over $\mathbb{F}_q$), and multiplication of linearized polynomials modulo $x^{m} - x$ ($\mathcal{O}(m^2s)$ over $\mathbb{F}_q$), where the complexity bottleneck of the latter two methods is the so-called $q$-transform with respect to a basis of $\mathbb{F}_{q^m}$ with complexity $O(m^2s)$.

For arbitrary $s > 0$, Wachter-Zeh (2013, Sec. 3.1.2) presented an algorithm for multiplying two linearized polynomials of degree at most $s$ with complexity $\mathcal{O}(s^{\min\{\frac{2s}{m+1}, 1.635\}})$ over $\mathbb{F}_{q^m}$, where $\omega$ is the matrix multiplication exponent. Finding a minimal subspace polynomial and performing a multi-point evaluation are both known to be implementable with $\mathcal{O}(s^2)$ operations in $\mathbb{F}_{q^m}$, see (Li et al., 2014). Similarly, the known implementations of the $q$-transform require $\mathcal{O}(s^2)$ operations over $\mathbb{F}_{q^m}$ (Wachter-Zeh, 2013), and the interpolation $\mathcal{O}(s^3)$ over $\mathbb{F}_{q^m}$ (Silva and Kschischang, 2007).

Recently, Caruso and Le Borgne (2017) proposed algorithms for multiplication and division of skew polynomials (a generalization of linearized polynomials) that have complexity $O^\ast (sm)$. If $m \in o(s)$, then these algorithms are sub-quadratic in $s$. Further, they presented two Karatsuba-based algorithms where the so-called Karatsuba method has complexity $\mathcal{O}(s^{1.58m^{1.41}})$ over $\mathbb{F}_q$, if $s > m$ and the so-called matrix method has complexity $\mathcal{O}(s^{1.58m^{0.2}})$ over $\mathbb{F}_q$, if $s > m^2/2$. For $m \in \Omega(s)$, it has been an open problem if division algorithms of sub-quadratic complexity exist.

Since operations over $\mathbb{F}_{q^m}$ can be performed in $O^\ast (m)$ operations over $\mathbb{F}_q$ (cf. (Couveignes and Lercier, 2009)), a quadratic complexity $\mathcal{O}(s^2)$ over $\mathbb{F}_{q^m}$ corresponds to $O^\ast (ms^2)$ over $\mathbb{F}_q$. Hence, all the mentioned results over $\mathbb{F}_{q^m}$ are not slower than the ones over $\mathbb{F}_q$ from (Silva and Kschischang, 2009) and (Wachter-Zeh et al., 2013) and it suffices to compare our results to the cost bounds over $\mathbb{F}_{q^m}$.

The results of this paper were partly presented at the IEEE International Symposium on Information Theory (Puchinger and Wachter-Zeh, 2016), with an emphasis on the implications for coding theory and omitting many proofs and comparisons.

1.2. Our Contribution

In this paper, we present algorithms for the above operations that are sub-quadratic in the polynomial degree $s$ of the involved polynomials, independent of the field extension degree $m$.

First, we generalize the multiplication algorithm for linearized polynomials from (Wachter-Zeh, 2013), which is based on a fragmentation of the involved polynomials similar to (Brent et al.,
1980a), to the more general class of skew polynomials. We also analyze the resulting cost bounds in more details than in (Wachter-Zeh, 2013). This algorithm has complexity $O(q^{\min\{\frac{s}{m+1},1.635\}})$ and, together with a result of Caruso and Le Borgne (2017), implies a division algorithm in $O(q^{\max\{\frac{s}{max(3, \min(\frac{37}{7}, 1.635))}\}})

We show that computing the $q$-transform and its inverse can be reduced to a matrix-vector multiplication and solving a system of equations, respectively, where in both cases the involved matrix has Toeplitz form. Thus, it can be computed in $O^*(s)$ operations over $\mathbb{F}_{q^m}$.

Our fast algorithms for multi-point evaluation and computing minimal subspace polynomials are divide-&-conquer methods that call each other recursively. These convoluted calls enable us to circumvent problems that arise from the non-commutativity of the linearized polynomial multiplication. We also propose a divide-&-conquer interpolation algorithm that uses the new multi-point evaluation and minimal subspace polynomial routines. All three methods use ideas from well-known algorithms from (Gathen and Gerhard, 1999, Section 10.1-10.2) and can be implemented in $O(n^{\min\{\frac{s}{m+1},1.635\}})$ operations over $\mathbb{F}_{q^m}$.

Table 1 summarizes the new cost bounds of operations that we prove in this paper.

| Operation (Source)                                      | New                                      | Before                                      |
|--------------------------------------------------------|------------------------------------------|---------------------------------------------|
| Division ((Caruso and Le Borgne, 2017) and Theorem 6)  | $O^*(\min\{sm, q^{\min\{\frac{s}{m+1},1.635\}}\})$ | $O^*(sm)$                                  |
| (Inverse) $q$-Transform (Theorem 12)                    | $O^*(s)$                                 | $O(s^3)$                                   |
| Minimal Subspace Polynomial Computation (Theorem 15)   | $O^*(q^{\max\{\log(3), \min(\frac{37}{7}, 1.635)\}})$ | $O(s^3)$                                   |
| Multi-point Evaluation (Theorem 15)                    | $O^*(q^{\max\{\log(3), \min(\frac{37}{7}, 1.635)\}})$ | $O(s^3)$                                  |
| Interpolation (Theorem 17)                             | $O^*(q^{\max\{\log(3), \min(\frac{37}{7}, 1.635)\}})$ | $O(s^3)$                                  |

1.3. Structure of the Paper

This paper is structured as follows. In Section 2, we give definitions and formally introduce the operations that are considered in this paper. Section 3 contains the main results of the paper: we present fast algorithms for division, $q$-transform, calculation of minimal subspace polynomials, multi-point evaluation, and interpolation. Using these new algorithms, we accelerate a known linearized polynomial multiplication algorithm and prove its optimality for the case $s = m$ in Section 4. In Section 5, we show how our fast algorithms for linearized polynomials imply sub-quadratic decoding algorithms for a special class of rank-metric codes, Gabidulin codes, and Section 6 concludes this paper.
2. Preliminaries

Let $q$ be a prime power, $\mathbb{F}_q$ be a finite field with $q$ elements and $\mathbb{F}_{q^r}$ an extension field of $\mathbb{F}_q$. The field $\mathbb{F}_{q^r}$ can be seen as an $m$-dimensional vector space over $\mathbb{F}_q$. A subspace of $\mathbb{F}_{q^r}$ is always meant w.r.t. $\mathbb{F}_q$ as the base field. For a given subset $A \subseteq \mathbb{F}_{q^r}$, the subspace $\langle A \rangle$ is the $\mathbb{F}_q$-span of $A$.

2.1. Normal Bases

Normal bases facilitate calculations in finite fields and can therefore be used to reduce the computational complexity. We shortly summarize important properties of normal bases in the following, cf. (Gao, 1993; Lidl and Niederreiter, 1997; Menezes et al., 1993). A basis $B = \{\beta_0, \beta_1, \ldots, \beta_{m-1}\}$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ is a normal basis if $\beta_i = \beta^i$ for all $i$, where $\beta \in \mathbb{F}_{q^m}$ is called normal element. As shown in (Lidl and Niederreiter, 1997, Thm. 2.35), there is a normal basis for any finite extension field $\mathbb{F}_{q^r}$ over $\mathbb{F}_q$.

The dual basis $B^\perp$ of a basis $B$ is needed to switch between a polynomial and its $q$-transform (cf. Section 3.3). For a given basis $B$ of $\mathbb{F}_{q^r}$ over $\mathbb{F}_q$, there is a unique dual basis $B^\perp$. The dual basis of a normal basis is also a normal basis, cf. (Menezes et al., 1993, Thm. 1.1).

If we represent elements of $\mathbb{F}_{q^m}$ in a normal basis over $\mathbb{F}_q$, applying the Frobenius automorphism $^q$ to an element can be accomplished in $O(1)$ operations over $\mathbb{F}_{q^m}$ as follows. Let $[A_1, \ldots, A_m]^T \in \mathbb{F}_q^{m \times 1}$ be the vector representation of $a \in \mathbb{F}_{q^m}$ in a normal basis. Then, for any $j$, the vector representation of $a^j$ is given by $[A_{m-j}, A_{m-j+1}, \ldots, A_0, A_1, \ldots, A_{m-1}]^T$, which is just a cyclic shift of the representation of $a$. The same holds for an arbitrary automorphism $\sigma \in \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$ since it is of the form $\sigma(\cdot) = \cdot^q$ for $i < m$.

2.2. Linearized Polynomials

In this paper, we present operations with linearized polynomials, also called $q$-polynomials and defined as follows. A linearized polynomial (Ore, 1933a) is a polynomial of the form

$$ a = \sum_{k=0}^t a_k x^q^k = \sum_{k=0}^t a_k x^{[k]}, \quad a_k \in \mathbb{F}_{q^m}, \quad t \in \mathbb{N}, $$

where we use the notation $[i] := q^i$. The set of all linearized polynomials for given $q$ and $m$ is denoted by $\mathcal{L}_{q^m}$. We define the addition $+$ of $a, b \in \mathcal{L}_{q^m}$ as for ordinary polynomials

$$ a + b = \sum_i (a_i + b_i)x^{[i]} $$

and the multiplication $\cdot$ as

$$ a \cdot b = \sum_i \left( \sum_{j=0}^i a_j b_{i-j}^{[j]} \right)x^{[i]} , \quad (1) $$

Note that if $\mathcal{L}_{q^m}$ is seen as a subset of $\mathbb{F}_{q^m}[x]$, the multiplication $\cdot$ equals the composition $a(b(x))$. Using these operations, $(\mathcal{L}_{q^m}, +, \cdot)$ is a (non-commutative) ring (Ore, 1933a). The identity element of $(\mathcal{L}_{q^m}, +, \cdot)$ is $x^{[0]} = x$. In the following, all polynomials are linearized polynomials.

We say that $a \in \mathcal{L}_{q^m}$ has $q$-degree $\deg_q a = \max\{k \in \mathbb{N} : a_k \neq 0\}$, where $\max \emptyset := -\infty$. For $s \in \mathbb{N}$, we define the set $\mathcal{L}_{q^m}^s := \{ a \in \mathcal{L}_{q^m} : \deg_q a \leq s \}$, and $\mathcal{L}_{q^m}^\leq$ analogously. A polynomial $a$ is called monic if $a_{\deg_q a} = 1$. Further, $\deg_q(a \cdot b) = \deg_q a + \deg_q b$ and $\deg_q(a + b) \leq \max\{\deg_q a, \deg_q b\}$. 

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For $a \in \mathcal{L}_{q^m}$, the evaluation (Boucher and Ulmer, 2014, Operator Evaluation) is defined by

$$a(\cdot) : \mathbb{F}_{q^m} \to \mathbb{F}_{q^m}, \ a \mapsto a(\alpha) = \sum_i a_i \alpha^{|i|}.$$  

Since $\sigma(\alpha) = \alpha^q$ is the Frobenius automorphism, $a^{(\cdot)} = \sigma^i(\alpha)$ is also an automorphism and it can be shown that $a(\cdot)$ is an $\mathbb{F}_q$-linear map for any $a \in \mathcal{L}_{q^m}$. It follows that the root space $\ker(a) = \{\alpha \in \mathbb{F}_{q^m} : a(\alpha) = 0\}$ is a subspace of $\mathbb{F}_{q^m}$. It is also clear that $(a \cdot b)(\alpha) = a(b(\alpha))$.

2.2.1. Division

The ring of linearized polynomials is a left and right Euclidean domain and therefore admits a left and right division.

**Lemma 1** (Ore (1933a)). For all $a, b \in \mathcal{L}_{q^m}, b \neq 0$, there are unique $\chi_R, \chi_L \in \mathcal{L}_{q^m}$ (quotients) and $\varrho_R, \varrho_L \in \mathcal{L}_{q^m}$ (remainders) such that $\deg_q \varrho_R < \deg_q b, \deg_q \varrho_L < \deg_q b$, and

$$a = \chi_R \cdot b + \varrho_R \quad \text{(right division)},$$

$$a = b \cdot \chi_L + \varrho_L \quad \text{(left division)}.$$  

Lemma 1 allows us to define a (right) modulo operation on $\mathcal{L}_{q^m}$ such that $a \equiv b \mod c$ if there is a $d \in \mathcal{L}_{q^m}$ such that $a = b + d \cdot c$. In the following, we use this definition of “mod”.

2.2.2. Minimal Subspace Polynomials

Subspace polynomials are special linearized polynomials, with the property that their $q$-degree equals their number of linearly independent roots.

**Lemma 2** (Lidl and Niederreiter (1997)). Let $\mathcal{U}$ be a subspace of $\mathbb{F}_{q^m}$. Then there exists a unique nonzero monic polynomial $M_{\mathcal{U}} \in \mathcal{L}_{q^m}$ of minimal degree such that $\ker(M_{\mathcal{U}}) = \mathcal{U}$. Its degree is $\deg_q M_{\mathcal{U}} = \dim \mathcal{U}$.

The polynomial $M_{\mathcal{U}}$ in Lemma 2 is called minimal subspace polynomial (MSP) of $\mathcal{U}$.

2.2.3. Multi-point Evaluation

Multi-point evaluation (MPE) is the process of evaluating a polynomial $a \in \mathcal{L}_{q^m}$ at multiple points $\alpha_1, \ldots, \alpha_s \in \mathbb{F}_{q^m}$, i.e. computing the vector $[a(\alpha_1), \ldots, a(\alpha_s)] \in \mathbb{F}_{q^m}$.

Notice that for linearized polynomials $a(\beta_2 \alpha_1 + \beta_2 \alpha_2) = \beta_2 a(\alpha_1) + \beta_2 a(\alpha_2)$ for any $\beta_1, \beta_2 \in \mathbb{F}_q$ and $\alpha_1, \alpha_2 \in \mathbb{F}_{q^m}$. If we have therefore evaluated $a(\alpha)$ at a few linearly independent points, the evaluation of any $\mathbb{F}_q$-linear combination of these points can be calculated by simple additions with almost no cost.

2.2.4. Interpolation

The dual problem of MPE is to find a polynomial of bounded degree that evaluates at given distinct points to certain values and is called interpolation. It is based on the following lemma.

**Lemma 3** (Silva and Kschischang (2007, Sec. III-A)). Let $(x_1, y_1), \ldots, (x_s, y_s) \in \mathbb{F}_{q^m}^2$ such that $x_1, \ldots, x_s$ are linearly independent over $\mathbb{F}_q$. Then there exists a unique interpolation polynomial $I_{[(x_i, y_i)]_{i=1}^s} \in \mathcal{L}_{q^m}$ such that

$$I_{[(x_i, y_i)]_{i=1}^s}(x_i) = y_i, \quad \forall i = 1, \ldots, s.$$
2.2.5. The q-transform
Let $s$ divide $m$ and let $\mathcal{B}_N = \{\beta^{(0)}, \ldots, \beta^{(s-1)}\}$ be a normal basis of $\mathbb{F}_{q^s} \subseteq \mathbb{F}_{q^n}$ over $\mathbb{F}_q$.

**Definition 4.** The $q$-transform (w.r.t. $s$ and $\mathcal{B}_N$) is a mapping $\hat{\cdot} : \mathcal{L}_{q^s}^{\mathcal{B}_N} \rightarrow \mathcal{L}_{q^s}^{\mathcal{B}_N}$, $a \mapsto \hat{a}$ with

$$\hat{a}_j = a(\beta^{(j)}) = \sum_{i=0}^{s-1} a_i \beta^{i+j}, \quad \forall j = 0, \ldots, s-1. \quad (2)$$

Given a dual normal basis $\mathcal{B}_N^\perp = \{\beta^{s(0)}, \ldots, \beta^{s(s-1)}\}$ of $\mathcal{B}_N$, the inverse $q$-transform can be computed by $a_i = \hat{a}(\beta^{i(0)} + i) = \sum_{j=0}^{s-1} \hat{a}_j \beta^{i+j}$ for all $i = 0, \ldots, s-1$, cf. (Silva and Kschischang, 2009). Thus, the $q$-transform is bijective.

2.3. Skew Polynomials

Let $K$ be a field. The ring of skew polynomials $K[x; \sigma, \delta]$ over $K$ with automorphism $\sigma \in \text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ and derivation $\delta$, satisfying $\delta(a + b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for all $a, b \in K$, is defined as the set of polynomials $\sum_i a_i x^i$, $a_i \in K$, with the multiplication rule

$$xa = \sigma(a)x + \delta(a) \quad \forall a \in K$$

and ordinary component-wise addition. The degree of a skew polynomial is defined as usual. $K[x; \sigma, \delta]$ is left and right Euclidean, i.e., Lemma 1 also holds for skew polynomials. A comprehensive description of skew polynomial rings and their properties can be found in (Ore, 1933b).

In this paper, we only consider the special case $\delta = 0$, in which we abbreviate the ring by $K[x; \sigma]$. Also, we restrict ourselves to finite fields $K = \mathbb{F}_{q^n}$. Note that there is a ring isomorphism $\varphi : \mathcal{L}_{q^n} \rightarrow \mathbb{F}_{q^n}[x; \sigma]$, $\sum_i a_i x^i \mapsto \sum_i a_i x^i$, where $\sigma(\cdot) = \sigma$ is the Frobenius automorphism. Although some of our results might extend to a broader class of skew polynomials, we consider $\mathbb{F}_{q^n}[x; \sigma]$ only as an auxiliary tool for obtaining fast algorithms for linearized polynomials.

3. Fast Algorithms

This section presents the main results of this paper: new fast algorithms for division (Section 3.2), $q$-transform (Section 3.3), calculation of the MSP for a given subspace (Section 3.4), multi-point evaluation (also in Section 3.4), and interpolation (Section 3.5).

3.1. Notations

We count complexities in terms of operations in the field $\mathbb{F}_{q^n}$. For convenience, we use the following notations.

**Definition 5.** Let $s \in \mathbb{N}$. We define the (worst-case) complexity measures, i.e., the infimum of the worst-case complexities of algorithms that solve the given problem.

i) Complexity of left- or right-dividing $a \in \mathcal{L}_{q^n}^{\leq s}$ by $b \in \mathcal{L}_{q^n}^{\leq s}$:

$$D_{q^n}(s).$$

ii) Complexity of computing the MSP $\mathcal{M}_U(U)$ for a generating set $U = \{u_1, \ldots, u_s\}$ of a subspace of $\mathbb{F}_{q^n}$:

$$\text{MSP}_{q^n}(s).$$
iii) Complexity of a multi-point evaluation of \( a \in \mathcal{L}_{q^m}^{\leq s} \) at the points \( \alpha_1, \ldots, \alpha_s \in \mathbb{F}_{q^m} \):
\[
\text{MP}E_{q^m}(s).
\]
iv) Complexity of computing the \( q \)-transform or its inverse of a polynomial \( a \in \mathcal{L}_{q^m}^{\leq s} \), given a normal basis \( B_N = \{\beta^0, \ldots, \beta^{s-1}\} \) of \( \mathbb{F}_{q^m} \subseteq \mathbb{F}_{q^n} \):
\[
\text{QT}_{q^m}(s).
\]
v) Complexity of finding the interpolation polynomial of \( s \) point tuples
\[
I_{q^m}(s).
\]

Table 2 summarizes best known cost bounds for these operations with linearized polynomials.

| Operation  | Cost Bound                  | Source                                           |
|------------|------------------------------|--------------------------------------------------|
| \( D_{q^m}(s) \) | \( O((sm) \)  | (Caruso and Le Borgne, 2017)                     |
| \( MSP_{q^m}(s) \) | \( O(x^s) \)  | (Silva et al., 2008)                           |
| \( MP\mathcal{E}_{q^m}(s) \) | \( O(x^s) \)  | “naive” (s ordinary polynomial evaluations)     |
| \( QT_{q^m}(s) \) | \( O(x^s) \)  | (Silva et al., 2008)                           |
| \( I_{q^m}(s) \) | \( O(x^s) \)  | “naive” (using linearized Lagrange bases, cf. (Silva and Kschischang, 2007)) |

3.2. Fast Division

In this section, we present a division algorithm that has sub-quadratic complexity in the polynomial degree \( s \) for arbitrary \( s \). The currently best-known division algorithm has complexity \( O((sm) \) (Caruso and Le Borgne, 2017), which is quasi-linear for \( s \gg m \), but quadratic if \( s \in \Theta(m) \). We improve upon this algorithm in the latter case.

Our method is based on the following result by Caruso and Le Borgne (2017), which states that skew polynomial division can be reduced to multiplication in another skew polynomial ring. We denote by \( \mathcal{M}_{q^m}(s) \) the complexity multiplying two skew polynomials in \( \mathbb{F}_{q^m}[x; \sigma]_{\leq s} \).

**Theorem 6** (Caruso and Le Borgne (2017)). \( D_{q^m}(s) \in O(\mathcal{M}_{q^m}(s) \log s) \).

We generalize the fast multiplication algorithm for linearized polynomials from (Wachter-Zeh, 2013, Theorem 3.1) to arbitrary skew polynomial rings with derivation \( \delta = 0 \) in order to obtain a sub-quadratic cost bound on \( \mathcal{M}_{q^m}(s) \). The algorithm is based on a fragmentation of polynomials, which was used for calculating power series expansions in (Brent and Kung, 1978; Paterson and Stockmeyer, 1973) and is related to the baby-steps giant-steps method.
We say that polynomials $a, b \in \mathbb{F}_{q^s}[x; \sigma]$ overlap at $k$ positions if the intersection of their supports $\text{supp}(a) := \{ i : a_i \neq 0 \}$ and $\text{supp}(b) := \{ i : b_i \neq 0 \}$ has cardinality $|\text{supp}(a) \cap \text{supp}(b)| = k$.

If $k = 0$, we say that $a$ and $b$ are non-overlapping. Obviously, the sum of two polynomials overlapping at $k$ (known) positions can be calculated with $k$ additions in $\mathbb{F}_{q^s}$ when the overlapping positions are known.

**Theorem 7.** Let $a, b \in \mathbb{F}_{q^s}[x; \sigma]_{\leq s^*}$, $s^* := \lceil \sqrt{s + 1} \rceil$. Then $c = a \cdot b$ can be calculated in $O(s^2)$ field operations, plus the cost of multiplying an $s^* \times s^*$ matrix with an $s^* \times (s + s^*)$ matrix, using Algorithm 1.

**Proof.** We can fragment $a$ into $s^*$ non-overlapping polynomials $a^{(i)}$ as

$$a = \sum_{i=0}^{s^*-1} a^{(i)} = \sum_{i=0}^{s^*-1} \left( \sum_{j=0}^{s-1} a_{i+j} x^{i+j} \right)$$

and the result $c$ of the multiplication $c = a \cdot b$ can also be fragmented as

$$c = a \cdot b = \left( \sum_{i=0}^{s^*-1} a^{(i)} \cdot b \right) = \sum_{i=0}^{s^*-1} c^{(i)}$$

with

$$c^{(i)} = \left( \sum_{j=0}^{s-1} b_j x^j \right) \cdot \left( \sum_{k=0}^{s-1} \left( \sum_{j=0}^{s^*-1} a_{i+j} x^{i+j} \cdot b_k x^k \right) = \sum_{j=0}^{s^*-1} \left( a_{i+j} \sigma^{is^*+i} \cdot b_k x^k \right) \cdot x^{i^*+h} = \sum_{j=0}^{s^*-1} c^{(i)} j x^{i^*+h}. $$

Thus the $c^{(i)}$‘s pairwise overlap at not more than $s$ positions, which we know. In order to obtain the polynomials $c^{(i)}$, we can use

$$\sigma^{-is^*}(c_h^{(i)}) = \sigma^{-is^*} \left( \sum_{j=0}^{h} a_{i+j} \sigma^{is^*+i}(b_{h-j}) \right) = \sum_{j=0}^{h} \sigma^{-is^*} (a_{i+j}) \sigma^{-is^*+i}(b_{h-j}) = \sum_{j=0}^{h} \sigma^{-is^*} (a_{i+j}) \sigma^{j}(b_{h-j}).$$

We can write this expression as a vector multiplication

$$\begin{bmatrix} \sigma^{-is^*}(a_{i^*}) & \ldots & \sigma^{-is^*}(a_{i^*+s^*-1}) \end{bmatrix} \cdot \begin{bmatrix} \sigma^0(b_h) & \ldots & \sigma^h(b_0) & 0 & \ldots & 0 \end{bmatrix}^T,$$

where the left vector does not depend on $h$ and the right side is independent of $i$. Thus, we can write the computation of $\sigma^{-is^*}(c_h^{(i)})$ as a matrix multiplication $C = A \cdot B$ with

$$C = \begin{bmatrix} C_{ij} \end{bmatrix}_{i,j=0, \ldots, s^*-1}, \quad C_{ij} = \sigma^{-is^*}(c_j^{(i)}),$$

$$A = \begin{bmatrix} A_{ij} \end{bmatrix}_{i,j=0, \ldots, s^*-1}, \quad A_{ij} = \sigma^{-is^*}(a_{i+j}),$$

$$B = \begin{bmatrix} B_{ij} \end{bmatrix}_{i,j=0, \ldots, s^*-1}, \quad B_{ij} = \begin{cases} \sigma^j(b_{i-j}), & 0 \leq i-j \leq s, \\ 0, & \text{else}. \end{cases}$$
Setting up the matrices $A$ and $B$ costs $s^2 \cdot s + s^2 \cdot s \approx s^3$ many computations of automorphisms to $\mathbb{F}_{q^s}$ elements. Computing the matrix $C$ from $A$ and $B$ requires a multiplication of an $s^r \times s^r$ with an $s^r \times (s + s^r)$ matrix. Extracting $c^{(i)}_j = \sigma^{i^2}(C_{ij})$ from $C$ costs a computation of an automorphism each, thus $\approx s^2$ computations in total. In order to obtain the skew polynomial $c$, we need to add up the $c^{(i)}$s. For some $k < s^r$, the polynomials $\sum_{i=0}^{k-1} c^{(i)}$ and $c^{(k)}$ overlap at not more than $s$ positions. Since we know these overlapping positions, we can compute the sum of all $c^{(i)}$s in $O(s^r \cdot s) = O(s^2)$ time. In a finite field, the computation of an automorphism can be done in $O(1)$, so Algorithm 1 costs $O(s^2)$, plus the matrix multiplication.

Corollary 8. Different techniques for the multiplication of the $s^r \times s^r$ with the $s^r \times (s + s^r)$ matrices in Theorem 7 result in the following cost bounds on the multiplication of skew polynomials:

i) Using $s^r + 1$ many multiplications of $s^r \times s^r$ with $s^r \times s^r$ matrices, we obtain

$$\bar{M}_{q^s}(s) \in O(s^r \cdot (s^r)^{\omega}) \subseteq O\left(s^{\frac{29}{11}}\right).$$

For instance, we get

$$\bar{M}_{q^s}(s) \in \begin{cases} O\left(s^{1.91}\right), & \omega \approx 2.8074 \text{ (Strassen, 1969)}, \\ O\left(s^{1.69}\right), & \omega \approx 2.376 \text{ (Coppersmith and Winograd, 1990)}. \end{cases}$$

ii) Direct multiplication algorithms for rectangular matrices (cf. (Huang and Pan, 1998; Ke et al., 2008)) result in

$$\bar{M}_{q^s}(s) \in O\left((s^r)^{3.2699}\right) \subseteq O\left(s^{1.635}\right),$$

where the power $3.2699$ (Ke et al., 2008, Example 1) holds for multiplying an $s^r \times s^r$ with an $\approx s^r \times (s^r)^2$ matrix.

Remark 9. Naive skew/linearized polynomial multiplication using the definition from (1) uses approximately $2s^2$ many field operations. For comparison, if we use case (i) of Corollary 8 with naive matrix multiplication, where each multiplication uses approximately $2(s^r)^3$ operations, skew polynomial multiplication takes $\approx s^r \cdot (2(s^r)^3) = 2(s^r)^4 = 2s^2$ operations in total. Thus, we improve upon the naive case as soon as the algorithm for multiplying two matrices of dimension $s^r \times s^r$ is faster than $2(s^r)^3$.

For instance, the algorithm of Strassen (1969) uses $\approx 4.7(s^r)^{\log_2(7)}$ field operations, which is smaller than $2(s^r)^3$ for $s^r \geq 85$, or in other words $s \geq 7225$. The algorithm by Coppersmith and Winograd (1990) has a much larger “hidden constant” and improves upon the naive case only for huge values of $s$. 

Algorithm 1: Multiplication

\begin{verbatim}
Input: $a, b \in \mathbb{F}_{q^s}[x; \sigma]_{\leq s}$
Output: $c = a \cdot b$
1 Set up matrices $A$ and $B$ as in (3)  // $s^2 \cdot O(1)$
2 $C \leftarrow A \cdot B$  // $s^r \cdot O((s^r)^{\omega})$ or $(s^r)^{3.2699}$
3 Extract the $c^{(i)}$s from $C$ as in (3)  // $s^2 \cdot O(1)$
4 return $c \leftarrow \sum_{i=0}^{r-1} c^{(i)}$  // $O(s^2)$
\end{verbatim}
Besides the asymptotic improvement, Algorithm 1 can yield a practical speed-up, compared to a naive implementation that does not use linear-algebraic operations, by relying on efficiently implemented linear algebra libraries that are optimized for the used programming language or hardware.

Using Theorem 6 and Corollary 8, we obtain the following new cost bound on the division of linearized polynomials.

**Corollary 10.** \( D_{q^m}(s) \in O \left( s^{\min \left\{ \frac{\omega}{2}, 1.635 \right\}} \log s \right). \)

As a direct consequence of the result above, we obtain a fast (half) linearized extended Euclidean algorithm (LEEA) (cf. (Wachter-Zeh, 2013, Corollary 3.2)).

**Corollary 11.** The fast (half) LEEA from (Wachter-Zeh, 2013, Algorithm 3.4) for two input polynomials \( a, b \), where \( s := \deg_q a \geq \deg_q b \) can be implemented in \( O \left( \max \left\{ D_{q^m}(s), M_{q^m}(s) \right\} \log s \right) \subseteq O \left( s^{\min \left\{ \frac{\omega}{2}, 1.635 \right\}} \log^2(s) \right) \) operations over \( \mathbb{F}_{q^m} \).

### 3.3. Fast \( q \)-Transform

The following theorem states that both the \( q \)-transform and its inverse can be obtained in quasi-linear time over \( \mathbb{F}_{q^m} \). Recall that \( s \) must divide \( m \) in order for the \( q \)-transform to be well-defined. The idea of the fast \( q \)-transform is based on the fact that the \( q \)-transform is basically the multiplication of the vector with a Toeplitz matrix. Since Toeplitz matrix multiplication can be reduced to multiplication of polynomials in \( \mathbb{F}_{q^m}[x] \) (cf. (Gathen and Gerhard, 1999)), also the \( q \)-transform can be implemented in quasi-linear time over \( \mathbb{F}_{q^m} \).

**Theorem 12.** \( QT_{q^m}(s) \in O \left( s \log^2(s) \log(\log(s)) \right) \).

**Proof.** Let \( a \in L_{q^m}^{\leq t} \). From (2) we know that

\[
(\hat{a}_0, \ldots, \hat{a}_{t-1}) = (a_{t-1}, \ldots, a_0) \cdot \begin{bmatrix}
\beta^{t-1} & \beta^s & \beta^{s+1} & \cdots & \beta^{2t-1} \\
\beta^{t-2} & \beta^{s-1} & \beta^s & \cdots & \beta^{2t-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta^0 & \beta^1 & \beta^2 & \cdots & \beta^{s-1}
\end{bmatrix} = B,
\]

where the matrix \( B \) is an \( s \times s \) Toeplitz matrix over \( \mathbb{F}_{q^m} \). At the same time, it is a \( q \)-Vandermonde matrix which is invertible (see (Lidl and Niederreiter, 1997, Lemma 3.5.1)).

As described in (Bini and Pan, 2012, Problems 2.5.1), Toeplitz matrix vector multiplication can be reduced to multiplication of \( \mathbb{F}_{q^m}[x] \) polynomials with degree \( \leq s \), which has complexity \( O \left( s \log(s) \log(\log(s)) \right) \), cf. (Gathen and Gerhard, 1999).

The inverse \( q \)-transform consists of solving a Toeplitz linear system, which is reducible to a Padé approximation problem, that again can be solved using the extended Euclidean algorithm over \( \mathbb{F}_{q^m}[x] \), cf. (Brent et al., 1980a). A fast Extended Euclidean Algorithm with stopping condition was introduced by Aho and Hopcroft in (Aho and Hopcroft, 1974). Its complexity is shown...
to be $O(\tilde{M}(s) \log(s))$, where $\tilde{M}(s)$ is the complexity of multiplying two polynomials of degree $s$ in $\mathbb{F}_{q^r}[x]$. However, for some special cases the algorithm does not work properly and therefore the improvements from (Gustavson and Yun, 1979), (Brent et al., 1980b) have to be considered. This fast EEA was summarized and proven in (Blahut, 1985). The resulting complexity of solving the Toeplitz linear system is thus $O\left( s \log^2(s) \log(\log(s)) \right)$. \hfill \Box

3.4. Fast Computation of MSP and MPE

In this subsection, we consider an efficient way to calculate the minimal subspace polynomial and the multi-point evaluation. Fast algorithms for multi-point evaluation at a set $\mathcal{S} \subseteq \mathbb{F}_{q^r}$ over $\mathbb{F}_{q^r}[x]$ typically pre-compute a sub-product tree (consisting of polynomials of the form $M_U = \prod_{u \in U} (x - u)$ for $U \subseteq S$) and then use divide-and-conquer methods for fast MPE. Such a sub-product tree can only be computed fast since in the commutative case, the polynomial $M_U$ can be written as the product of two such polynomials of a partition $U = A \cup B$, $A \cap B = \emptyset$,

$$M_U = M_A \cdot M_B.$$  

The equivalent statement for linearized polynomials, using minimal subspace polynomials, is given in Lemma 13 (see below). In contrast to the commutative case, one of the factors depends on a multi-point evaluation of the other factor. Hence, we cannot immediately apply the known methods.

The following two lemmas lay the foundation to algorithms that compute MSPs and MPEs by convoluted recursive calls of each other. Thus, we need to analyze their complexities jointly.

**Lemma 13 (Li et al. (2014)).** Let $U = \{u_1, \ldots, u_s\}$ be a generating set of a subspace $\mathcal{U} \subseteq \mathbb{F}_{q^r}$, $A, B \subseteq \mathbb{F}_{q^r}$ such that $U = A \cup B$. Then,

$$M_{\mathcal{U}} = M_{\{U\}} = M_{(M_{A(B)})} \cdot M_{(A)}$$  

and

$$M_{(u_i)} = \begin{cases} x[0], & \text{if } u_i = 0, \\ x[1] - u_i^{q - 1} x[0], & \text{else.} \end{cases}$$  

(4)

**Lemma 14.** Let $a \in L_{q^r}$ and let $U, A, B \subseteq \mathbb{F}_{q^r}$ where $A, B \subseteq \mathbb{F}_{q^r}$ are disjoint and $U = A \cup B$. Let $q_A, q_B$ be the remainders of the right divisions of $a$ by $M_{(A)}$ and $M_{(B)}$ respectively. Then, the multi-point evaluation of $a$ at the set $U$ is

$$a(U) = q_A(A) \cup q_B(B).$$

If $U = \{u\}$ and $\deg_a a \leq 1$,

$$a(U) = \{a(u) = a_1 u[1] + a_0 u[0]\}.$$  

(5)

**Proof.** Let $u \in U$. If $u \in A$,

$$a(u) = (\chi_{A} \cdot M_{(A)}) + q_A(u) = \chi_{A}(M_{(A)}(u)) + q_A(u) = \chi_{A}(0) + q_A(u) = q_A(u).$$

Otherwise, $u \in B$ and

$$a(u) = (\chi_{B} \cdot M_{(B)}) + q_B(u) = \chi_{B}(M_{(B)}(u)) + q_B(u) = \chi_{B}(0) + q_B(u) = q_B(u).$$
Thus, \( a(U) = \varrho_s(A) \cup \varrho_B(B) \). Equation (5) follows directly from the definition of the evaluation map. □

This yields the main statement of this subsection.

**Theorem 15.** Finding the MSP of an \( \mathbb{F}_{q^p} \)-subspace spanned by \( s \) elements of \( \mathbb{F}_{q^p} \) and computing the MPE of a polynomial of \( q \)-degree at most \( s \) at \( s \) many points can be implemented in

\[
\text{MSP}_{q^p}(s), \text{MPE}_{q^p}(s) \in O \left( \max \{ \log_2(3) \log(s), \overline{M}_{q^p}(s), D_{q^p}(s) \} \right) \\
\subseteq O \left( s^{\max \{ \log_3(3), \min \{ \frac{16}{21}, 1.633 \} \} \log(s) \right).
\]

operations over \( \mathbb{F}_{q^p} \) using Algorithms 2 and 3 respectively.

**Proof.** We prove that Algorithm 2 for computing the MSP and Algorithm 3 for MPE are correct and have the desired complexity.

**Correctness:** Since the algorithms call each other recursively, we need to prove their correctness jointly by induction.

For \( s = 1 \), Algorithm 2 returns the base case of Lemma 13 and Algorithm 3 uses Equation (5) of Lemma 14 to compute the evaluation of a polynomial of deg \( q^p \alpha \leq 1 \) at one point.

Now suppose that both algorithms work up to an input size of \( s - 1 \) for some \( s \geq 2 \). Then, Algorithm 2 works for an input of size \( s \) because it uses the recursion formula of Lemma 13 to reduce the problem to two MSP computations and a multi-point evaluation, each of input size \( \approx s/2 \leq s - 1 \). Algorithm 3 works for an input of size \( s \) due to Lemma 14 and a similar argument.

Hence, both algorithms are correct.

**Complexity:**

The lines of Algorithm 2 have the following complexities:

- The complexities of Lines 2 (base case) and 4 (partitioning of \( U \)) are negligible.
- Lines 5 and 7 both have complexity \( \text{MSP}_{q^p}(\frac{s}{2}) \) because \( |A| \approx |B| \approx \frac{|U|}{2} = \frac{s}{2} \).
- Line 6 computes the result in \( \text{MPE}_{q^p}(\frac{s}{2}) \) time because \( \deg_{q^p} M(A) \leq |B| \approx \frac{|U|}{2} = \frac{s}{2} \).
- Line 8 performs a multiplication of two polynomials of \( q \)-degree \( \leq \frac{s}{2} \leq s \) and has time complexity \( \text{MPE}_{q^p}(s) \).

In total, we obtain

\[
\text{MSP}_{q^p}(s) = 2 \cdot \text{MSP}_{q^p}(\frac{s}{2}) + \text{MPE}_{q^p}(\frac{s}{2}) + \overline{M}_{q^p}(s). \tag{6}
\]

Algorithm 3 consists of the following steps:

- Again, the complexities of Lines 2 (base case) and 4 (partitioning of \( U \)) are negligible.
- Lines 5 and 6 compute the MSP of bases with input size \( |A| \approx |B| \approx \frac{|U|}{2} = \frac{s}{2} \), so both have complexity \( \text{MSP}_{q^p}(\frac{s}{2}) \).
- Lines 7 and 8 divide polynomials from \( L_{q^p}^s \) and therefore have complexity \( D_{q^p}(s) \) each.
- Line 9 performs two multi-point evaluations of polynomials with \( q \)-degree \( < |A| \approx |B| \approx \frac{s}{2} \) (cf. Lemma 1, deg \( q \) of remainder) at \( |A| \approx |B| \approx \frac{s}{2} \) positions. Thus, the line has complexity \( 2 \cdot \text{MPE}_{q^p}(\frac{s}{2}) \).

Summarized, we get

\[
\text{MPE}_{q^p}(s) = 2 \cdot \text{MSP}_{q^p}(\frac{s}{2}) + 2 \cdot \text{MPE}_{q^p}(\frac{s}{2}) + 2 \cdot D_{q^p}(s). \tag{7}
\]

In fact, the MSP computed in Line 5 of Algorithm 3 is the same as the MSP which was computed in Line 5 of Algorithm 2 at the same recursion depth before (note that \( \text{MSP}(U) \) first calls \( \text{MSP}(A) \)
and then $\text{MPE}(\mathcal{M}(A), B)$. This means that we can store this polynomial instead of recomputing it and can reduce (7) to
\[
\text{MPE}_{q^m}(s) = \text{MSP}_{q^m}\left(\frac{s}{2}\right) + 2 \cdot \text{MPE}_{q^m}\left(\frac{s}{2}\right) + 2 \cdot \mathcal{D}_{q^m}(s). \tag{8}
\]
We define $C(s) := \max\left\{\text{MPE}_{q^m}(s), \text{MSP}_{q^m}(s)\right\}$ and derive an upper bound on $C(s)$. Using (6) and (8), we obtain
\[
C(s) \leq 3 \cdot C\left(\frac{s}{2}\right) + \max\left\{\overline{\mathcal{M}}_{q^m}(s), 2 \cdot \mathcal{D}_{q^m}(s)\right\}.
\]
We distinguish three cases and use the master theorem:
- If $\max\left\{\overline{\mathcal{M}}_{q^m}(s), \mathcal{D}_{q^m}(s)\right\} \in O\left(s^{\log_2(3) - \varepsilon}\right)$ for some $\varepsilon > 0$, then $C(s) \in O\left(s^{\log_2(3)}\right)$.
- If $\max\left\{\overline{\mathcal{M}}_{q^m}(s), \mathcal{D}_{q^m}(s)\right\} \in \Theta\left(s^{\log_2(3)}\right)$, then $C(s) \in O\left(s^{\log_2(3) \log(s)}\right)$.
- If $\max\left\{\overline{\mathcal{M}}_{q^m}(s), \mathcal{D}_{q^m}(s)\right\} \in \Omega\left(s^{\log_2(3) + \varepsilon}\right)$ for some $\varepsilon > 0$, then $C(s) \in O\left(\max\left\{\overline{\mathcal{M}}_{q^m}(s), \mathcal{D}_{q^m}(s)\right\}\right)$.

In summary, we obtain
\[
\text{MSP}_{q^m}(s), \text{MPE}_{q^m}(s) \in O\left(\max\left\{s^{\log_2(3) \log(s)}, \overline{\mathcal{M}}_{q^m}(s), \mathcal{D}_{q^m}(s)\right\}\right)
\leq O\left(s^{\max\{\log_2(3), \min\left\{\frac{\log_2(3)}{\log_2(3)}, 1.635\}\}\log(s)}\right).
\]

\[\square\]

Algorithm 2: MSP($U$)

\begin{algorithmic}[1]
  \State \textbf{Input:} Generating set $U = [u_1, \ldots, u_s]$ of a subspace $\mathcal{U} \subseteq \mathbb{F}_{q^m}$.
  \State \textbf{Output:} MSP $\mathcal{M}(U)$.
  \If{$s = 1$}
    \State \textbf{return} $M_{u_1}(s)$ according to (4) \hfill // $O(1)$
  \Else
    \State $A \leftarrow [u_1, \ldots, u_{\lfloor s/2 \rfloor}]$, $B \leftarrow [u_{\lfloor s/2 \rfloor + 1}, \ldots, u_s] \hfill // O(1)$
    \State $\mathcal{M}(A) \leftarrow \text{MSP}(A) \hfill // \text{MSP}_{q^m}\left(\frac{s}{2}\right)$
    \State $\mathcal{M}(A)(B) \leftarrow \text{MPE}(\mathcal{M}(A), B) \hfill // \text{MPE}_{q^m}\left(\frac{s}{2}\right)$
    \State $\mathcal{M}(A)(B) \leftarrow \text{MSP}(\mathcal{M}(A)(B)) \hfill // \text{MSP}_{q^m}\left(\frac{s}{2}\right)$
    \State \textbf{return} $M_{\mathcal{M}(A)(B)} \cdot M_{A} \hfill // \overline{\mathcal{M}}_{q^m}(s)$
\end{algorithmic}
3.5. **Fast Interpolation**

In this section, we present a fast divide-\&-conquer interpolation algorithm for linearized polynomials that relies on fast algorithms for both computing MSPs and MPEs. The idea resembles the fast interpolation algorithm in $\mathbb{F}_q[x]$ from (Gathen and Gerhard, 1999, Section 10.2) with additional considerations for the non-commutativity, and is based on the following lemma.

**Lemma 16.** Let $(x_i, y_i)$ be as in Lemma 3. The interpolation polynomial fulfills

$$I_{(x_i, y_i)}^{[\ell]}(x) = I_{(\bar{x}_i, y_i)}^{[\ell]}(x) + M_{(x_i, y_i)}(\bar{x}_i)$$

with

$$\bar{x}_i := \begin{cases} M_{(x_i, y_i)}(x), & \text{if } i = 1, \ldots, \frac{\ell}{2} \\ M_{(x_i, y_i)}(x), & \text{otherwise} \end{cases}$$

and $I_{(x_0, y_0)}(x) = \frac{y_0}{x_0}$ (base case $s = 1$).

**Proof.** For $i = 1, \ldots, \frac{\ell}{2}$, the $\bar{x}_i$ are linearly independent since the $x_i$ are linearly independent and $M_{(x_i, y_i)}(\bar{x}_i)$ is a linear map whose kernel is spanned by $x_i[1, \ldots, x_s]$, and therefore does not include any $x_i$ for $i = 1, \ldots, \frac{\ell}{2}$. Furthermore,

$$I_{(x_i, y_i)}^{[\ell]}(x) = \begin{cases} M_{(x_i, y_i)}(\bar{x}_i) + I_{(\bar{x}_i, y_i)}^{[\ell]}(0) = y_i + 0 = y_i \\ \frac{y_j}{x_j} \\
\end{cases}$$

By the same argument, also $\bar{x}_i[1, \ldots, \bar{x}_i]$ are linearly independent and $I_{(x_i, y_i)}^{[\ell]}(x) = y_i$ for all $i = \frac{\ell}{2} + 1, \ldots, s$. Since in addition,

$$\deg_q I_{(x_i, y_i)}^{[\ell]}(x) \leq \max \{ \deg_q I_{(\bar{x}_i, y_i)}^{[\ell]}(x), \deg_q M_{(x_i, y_i)}(\bar{x}_i), \deg_q I_{(\bar{x}_i, y_i)}^{[\ell]}(0) \} < s,$$
Theorem 17. Computing the interpolation polynomial of \( s \) point tuples can be implemented in
\[
\mathcal{I}_{q^e}(s) \in O(MSP_{q^e}(s)) \subseteq O(s^{\max\{\log_3(s),\min\{\log_2(1.635)\}\}} \log(s))
\]
operations over \( \mathbb{F}_{q^e} \) using Algorithm 4.

Proof. Algorithm 4 computes the correct interpolation polynomial due to Lemma 16. Its lines have the following complexities:

- Lines 2 and 4 are again negligible.
- The complexities of Lines 5 and 6 are \( MSP_{q^e}(\frac{1}{2}) \).
- Lines 7 and 8 take \( MPE_{q^e}(\frac{1}{2}) \) time each.
- The algorithm calls itself recursively with input size \( \frac{s}{2} \) in Lines 9 and 10.
- Finally, the result is reassembled in line 11 using two multiplications in \( 2 \cdot \mathcal{M}_{q^e}(\frac{1}{2}) \) time.

Overall, we have
\[
\mathcal{I}_{q^e}(s) = 2 \cdot \mathcal{I}_{q^e}(\frac{s}{2}) + 2 \cdot (MSP_{q^e}(\frac{s}{2}) + MPE_{q^e}(\frac{s}{2}) + \mathcal{M}_{q^e}(\frac{s}{2})) = 2 \cdot \mathcal{I}_{q^e}(\frac{s}{2}) + O(MSP_{q^e}(s))
\]

By the master theorem, we obtain the desired complexity \( \mathcal{I}_{q^e}(s) \in O(MSP_{q^e}(s)) \). \( \square \)

---

Algorithm 4: \( \text{IP}\left(\{(x_i, y_i)\}_{i=1}^{s}\right)\)

Input: \((x_1, y_1), \ldots, (x_s, y_s) \in \mathbb{F}_{q^e}^2, \ x_i \text{ linearly independent}\)

Output: Interpolation polynomial \( \mathcal{I}_{\{(x_i, y_i)\}_{i=1}^{s}} \)

If \( s = 1 \) then

\[ \text{return } \{ \frac{x_1}{y_1}, \ 0 \} \]  // \( O(1) \)

Else

\[ A \leftarrow \{x_1, \ldots, x_{\left\lfloor \frac{s}{2} \right\rfloor} \}, \ B \leftarrow \{x_{\left\lfloor \frac{s}{2} \right\rfloor + 1}, \ldots, x_s \} \]  // \( O(1) \)

\[ \mathcal{M}(A) \leftarrow \text{MSP}(A) \]  // \( MSP_{q^e}(\frac{1}{2}) \)

\[ \mathcal{M}(B) \leftarrow \text{MSP}(B) \]  // \( MSP_{q^e}(\frac{1}{2}) \)

\[ [\overline{x}_1, \ldots, \overline{x}_{\left\lfloor \frac{s}{2} \right\rfloor}] \leftarrow \text{MPE}(\mathcal{M}(B), A) \]  // \( MPE_{q^e}(\frac{1}{2}) \)

\[ [\overline{x}_{\left\lfloor \frac{s}{2} \right\rfloor + 1}, \ldots, \overline{x}_s] \leftarrow \text{MPE}(\mathcal{M}(A), B) \]  // \( MPE_{q^e}(\frac{1}{2}) \)

\[ \mathcal{I}_1 \leftarrow \text{IP}\left(\{(\overline{x}_i, y_i)\}_{i=1}^{\left\lfloor \frac{s}{2} \right\rfloor}\right) \]  // \( \mathcal{I}_{q^e}(\frac{1}{2}) \)

\[ \mathcal{I}_2 \leftarrow \text{IP}\left(\{(\overline{x}_i, y_i)\}_{i=\left\lfloor \frac{s}{2} \right\rfloor + 1}^{s}\right) \]  // \( \mathcal{I}_{q^e}(\frac{1}{2}) \)

\[ \text{return } \mathcal{I}_1 \cdot \mathcal{M}(B) + \mathcal{I}_2 \cdot \mathcal{M}(A) \]  // \( 2 \cdot \mathcal{M}_{q^e}(\frac{1}{2}) \)
3.6. Concluding Remarks

In this section, we have presented fast algorithms for the division, \( q \)-transform, MSP, MPE and interpolation with subquadratic complexity in \( s \) over \( \mathbb{F}_{q^m} \), independent of \( m \) (cf. Table 1 on page 3). Our fast algorithms are faster than all previously known algorithms when \( s \leq m \) (see previous work in Section 1.1).

After the initial submission of this paper, the preprint (Caruso and Borgne, 2017) proposed a fast algorithm for multiplication of skew polynomials of complexity \( O(s^{\omega^2 - 2}m^2) \) over \( \mathbb{F}_{q} \), which improves upon the multiplication algorithm in Section 3.2 for \( m^2/(5 - \omega) \leq s \leq m \). As an immediate consequence, also the cost bound for division is improved to \( O(s^{\omega^2 - 2}m^2) \) over \( \mathbb{F}_{q} \) in this range. However, the result does not improve our cost bounds for the \( q \)-transform, MSP, MPE, and interpolation since the first is already quasi-linear and the other algorithms’ complexity would be dominated by the \( s \log_2(3) \) factor, cf. Table 1 on page 3.

4. An Optimal Multiplication Algorithm for \( s = m \)

Since the algorithms in Section 3 rely on fast multiplication of linearized polynomials, we would like to know a lower bound on the cost of it. In this section, we therefore show that \( m \times m \) matrix multiplication can be reduced to multiplication of linearized polynomial of degree at most \( s = m \) if an elliptic normal basis of \( \mathbb{F}_{q^m} \) exists. This gives a lower bound on the cost of solving the latter problem.

We also speed up the algorithm for linearized polynomial multiplication modulo \( x^m - x \) from (Wachter-Zeh, 2013, Section 3.1.3) and show that it achieves this optimal complexity for \( s = m \) and is faster than the fragmentation-based multiplication algorithm from (Wachter-Zeh, 2013, Section 3.1.2) for \( m^2/(5 - \omega) \leq s < m/2 \). As a by-product, we show that the MPE at a basis of \( \mathbb{F}_{q^m} \) from (Wachter-Zeh, 2013, Section 3.1.3) can be implemented in sub-cubic time using our fast \( q \)-transform algorithm from Section 3.3.

4.1. Relation of Linearized Polynomial Multiplication and Matrix Multiplication

As a first step, we summarize and slightly reformulate the statements of (Wachter-Zeh, 2013, Section 3.1.3) which imply that matrix multiplication and linearized polynomial multiplication are closely connected.

Lemma 18. The evaluation maps \( \text{ev}_a, \text{ev}_b : \mathbb{F}_{q^m} \rightarrow \mathbb{F}_{q^m} \) of \( a, b \in L_{q^m} \) are the same if and only if \( a \equiv b \mod (x^m - x) \).

Proof. Let \( B = \{\beta_1, \ldots, \beta_m\} \) be an \( \mathbb{F}_q \)-basis of \( \mathbb{F}_{q^m} \) and suppose \( \text{ev}_a = \text{ev}_b \). Then, the remainder of \( a - b \) right-divided by \( x^m - x \) must be zero because it vanishes on the basis \( B \) and has degree smaller than \( m \). The other direction is clear due to \( \text{ev}_{x^m - x} = 0 \). \( \square \)

Lemma 18 implies that the evaluation map provides a bijection between \( L_{q^m}^{cm} \) and the set \( \text{End}_{\mathbb{F}_q}(\mathbb{F}_{q^m}) \) of \( \mathbb{F}_q \)-linear maps \( \mathbb{F}_{q^m} \rightarrow \mathbb{F}_{q^m} \). Furthermore, the multiplication modulo \( x^m - x \), denoted by \( \circ \mod (x^m - x) \) of two polynomials \( a, b \in L_{q^m}^{cm} \) corresponds to the composition \( \circ \) of their outputs might differ due to the modulo operation, so they cannot be compared
evaluation maps since $\text{ev}_{a,b} = \text{ev}_a \circ \text{ev}_b$. Using the matrix representation $[\psi]^B_q$ of a linear map $\psi \in \text{End}_{\mathbb{F}_q}(\mathbb{F}_{q^m})$, we obtain a monoid isomorphism

$$\varphi_B : (L_{q^m}^{\mathbb{F}_q}, \cdot \mod (x^{m-1} - 1)) \rightarrow (\mathbb{F}_{q^m}^{\mathbb{F}_q}, \cdot), \quad a \mapsto [\text{ev}_a]^B_q.$$  

Thus, multiplication of matrices in $\mathbb{F}_{q^m}^{\mathbb{F}_q}$ is equivalent to multiplication modulo $x^m - x$ in $L_{q^m}$ and either operation can be efficiently reduced to the other, given that $\varphi_B$ and its inverse can be computed fast.

Note that $\varphi_B(a)$ can be computed by evaluating $a$ at the elements of $\mathcal{B}$ and representing the result in the basis $\mathcal{B}$. The inverse mapping $\varphi_B^{-1}(A)$ corresponds to finding the polynomial that evaluates to the values represented by the columns of the matrix $A$ (in the basis $\mathcal{B}$) at the elements of $\mathcal{B}$, i.e., an interpolation. Both maps can be efficiently computed as follows.

**Lemma 19.** Let $\mathcal{B}$ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, $a \in L_{q^m}$, and $A \in \mathbb{F}_{q^m}^{\mathbb{F}_q}$.

- If $\mathcal{B}$ is a normal basis, then $\varphi_B(a)$ (or $\varphi_B^{-1}(A)$) can be computed by a $q$-transform (or an inverse $q$-transform).
- Otherwise, $\varphi_B(a)$ (or $\varphi_B^{-1}(A)$) can be computed by a $q$-transform (or an inverse $q$-transform), plus two matrix multiplications.

**Proof.** Recall that $\varphi_B(a)$ can be obtained by a multi-point evaluation at the basis $\mathcal{B}$ and by representing the result in the basis $\mathcal{B}$. If $\mathcal{B}$ is a normal basis, this corresponds to a $q$-transform. In the other cases, we can choose a normal basis $\mathcal{B}'$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ and first compute $\varphi_{\mathcal{B}'}(a)$. Then, we use two matrix multiplications by the change of basis matrices $T_{\mathcal{B}'}^B$ (from $\mathcal{B}'$ to $\mathcal{B}$) and $T_{\mathcal{B}'}^B$ to obtain

$$\varphi_B(a) = [\text{ev}_a]^B_q = T_{\mathcal{B}'}^B \cdot [\text{ev}_a]^B_{\mathcal{B}'} \cdot T_{\mathcal{B}'}^B = T_{\mathcal{B}'}^B \cdot \varphi_{\mathcal{B}'}(a) \cdot T_{\mathcal{B}'}^B.$$

Analogously, we can compute $\varphi_B^{-1}(A)$ by two matrix multiplications and an inverse $q$-transform instead of an interpolation. \hfill $\square$

Lemma 19 implies that any MPE and interpolation w.r.t. a basis of $\mathbb{F}_{q^m}$ can be computed in $O(\text{m}^3\text{m})$ operations over $\mathbb{F}_q$. In the following two subsections, we use this observation to speed up the multiplication algorithm modulo $x^m - x$ from Wachter-Zeh (2013) and show that it has optimal complexity.

**4.2. Faster Implementation of an Existing Multiplication Algorithm**

The following theorem shows how to speed up the algorithm for linearized polynomial multiplication modulo $x^m - x$ from (Wachter-Zeh, 2013, Section 3.1.3) using our fast $q$-transform algorithm from Theorem 12. The resulting complexity bottleneck is thus a matrix multiplication instead of a $q$-transform.

**Theorem 20.** Using the $q$-transform as described in Theorem 12, multiplication of $a, b \in L_{q^m}$ modulo $x^m - x$ can be implemented in $O(m^3\text{m})$ operations over $\mathbb{F}_q$.

**Proof.** By the properties of $\varphi_B$, we can compute $c = a \cdot b \mod (x^m - x)$ by

$$c = \varphi_B^{-1}((\varphi_B(a) \cdot \varphi_B(b))).$$
The computations of the two $\varphi_B$ and one $\varphi_B^{-1}$ cost six matrix multiplications, a $q$-transform and an inverse $q$-transform. In addition, we need to perform a matrix multiplication. Using the algorithm for $q$-transform described in Theorem 12 together with the bases from (Couveignes and Lercier, 2009), the (inverse) $q$-transform costs $O^*(m^2)$ operations over $\mathbb{F}_q$. Hence, the matrix multiplications with complexity $O(m^\omega)$ over $\mathbb{F}_q$ are dominant. $\square$

Remark 21. For polynomials of $q$-degree $s < m/2$, the algorithm described above is a linearized polynomial multiplication algorithm since the result has degree $< m$ and is not affected by the modulo $x[m] - x$ reduction. It is possible to extend the algorithm to polynomials of $q$-degree $s \geq m/2$ as follows.

Let $s = \mu \cdot m/2$ for $\mu \geq 1$. Then, we can fragment $a, b$ into $\mu$ polynomials of degree $< m/2$. We then pairwise multiply the fragments of $a$ and $b$ respectively (costs $\mu^2$ many multiplications of degree $< m/2$ polynomials: In total, $O(\mu^2 m^\omega)$). Addition of the fragments is negligible since we know the overlapping positions. Hence, we obtain a complexity of

$$O\left(\max\left\{s^2 m^{\omega-2}, m^\omega\right\}\right)$$

in operations over $\mathbb{F}_q$. For $m^2/4 < s < m^2$, this multiplication algorithm is faster than the one of (Wachter-Zeh, 2013, Section 3.1.2) (see Section 3.2), which has complexity $O^*(\min\{s^{2/3}, 1.635\})$ over $\mathbb{F}_q$ when using the bases of Couveignes and Lercier (2009). In addition, the constant hidden by the $O$-notation is smaller since the matrix multiplication is with respect to $m$, which is much larger than $\sqrt{s}$ in the case of the algorithm in Section 3.2 (cf. Remark 9) for $s \approx m$.

4.3. Optimal Multiplication Algorithm for $s = m$

We prove the optimality of the algorithm described in Theorem 20 by reducing matrix multiplication to polynomial multiplication. Lemma 19 implies that if the basis in which we represent elements of $\mathbb{F}_{q^m}$ is a normal basis, we can reduce matrix multiplication to a $q$-transform, two inverse $q$-transforms and a multiplication of two linearized polynomials modulo $x[m] - x$ (note that the modulo reduction only requires Frobenius automorphisms and $O(m)$ many additions in $\mathbb{F}_{q^m}$, so in total $O(m^2)$ operations in $\mathbb{F}_q$).

In addition, when the bases admit quasi-linear multiplication as the so-called normal elliptic bases from (Couveignes and Lercier, 2009), the $q$-transform only costs $O^*(m^2)$ operations over $\mathbb{F}_q$ by Theorem 12, and the complexity bottleneck becomes the multiplication of two linearized polynomials of degree $m$. This can be summarized in the following statement. Let $M_{q^m}(m)$ denote the worst-case cost of multiplying two polynomials from $L_{q^m} < m$ (note that the polynomial degrees are smaller than $s = m$).

Lemma 22. Let $q, m$ be such that there is an elliptic normal basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Then, the multiplication of two matrices from $\mathbb{F}_{q^m}^{m \times m}$ can be implemented in

$$O(m^\omega) \subseteq O\left(M_{q^m}(m)\right)$$

operations over $\mathbb{F}_q$.

Lemma 22 states that if a normal elliptic basis exists for $q, m$, then matrix multiplication can be efficiently reduced to linearized polynomial multiplication in $L_{q^m}$. Such bases do not exist for all pairs $(q, m)$. However, we can give the following statement.
Lemma 23. Let \((m_i)_{i \in \mathbb{N}}\) be a sequence of \(m_i \in \mathbb{N}\) with \(m_i \to \infty (i \to \infty)\). Then, there is a sequence \((q_i)_{i \in \mathbb{N}}\), where the \(q_i\) are prime powers, such that there is an elliptic normal basis of \(\mathbb{F}_{\bar{q}_i}^{m_i}\) over \(\mathbb{F}_q^m\).

Proof. Let \(\bar{q}_i\) be any sequence of prime powers. Due to (Couveignes and Lercier, 2009, Section 5.2), there is a positive integer \(f_i \in O(\log^2(m_i)\log \log(m_i)^2)\) such that \(q_i = \bar{q}_i^{f_i}\) admits an elliptic normal basis as desired. \(\square\)

Suppose that \(M_{q,m} (m) \in \Theta(m^\gamma)\) for some \(\gamma \geq 2\), independent of the ground field \(q\). Let \((q_i, m_i)_{i \in \mathbb{N}}\) be a sequence of pairs \(q_i\) and \(m_i\) as in Lemma 23. Then, by Lemma 22 there must be a constant \(C \in \mathbb{R}_{>0}\) and an index \(j \in \mathbb{N}\) such that \(m_i^{\omega} \leq C \cdot m_i^\gamma\) for all \(i \geq j\). Hence, \(\omega \leq \gamma\), which provides a lower bound for the linearized polynomial multiplication exponent \(\gamma\). Note that the multiplication algorithm in Section 4.2 achieves this bound and therefore has optimal complexity.

The fragmentation algorithm described in (Wachter-Zeh, 2013) (see also Section 3.2), in combination with the bases of (Couveignes and Lercier, 2009) achieves \(\gamma = \omega + \frac{3}{2}\), so its complexity differs from an optimal solution by a factor \(m^{\frac{1}{s}}\) (note that this only holds for \(s = m\)).

Note that our argumentation also implies that the existence of a linearized polynomial multiplication algorithm with quasi-linear complexity in \(s\) over \(\mathbb{F}_{q^m}\), independent of \(s\), would give a quasi-quadratic matrix multiplication algorithm in the cases where an elliptic normal basis of \(\mathbb{F}_{q^m}\) exists. Hence, proving that a quasi-linear linearized polynomial multiplication algorithm exists is at least as hard as proving that matrix multiplication can be implemented in quasi-quadratic time.

5. Decoding Gabidulin Codes in Sub-quadratic Time

Gabidulin codes are rank-metric codes that can be found in a wide range of applications, including network coding (Silva et al., 2008), code-based cryptosystems (Gabidulin et al., 1991), and distributed storage systems (Silberstein et al., 2012).

In this section, we show that two algorithms for decoding Gabidulin codes from (Wachter-Zeh, 2013), one for only errors and one including generalized row and column erasures, can be implemented in \(O(\sim (n_{\max} \log^2(3) \min(\omega, 1))^{1.635})\) operations over \(\mathbb{F}_{q^m}\) using the methods presented in this paper. This yields the first decoding algorithms for Gabidulin codes with sub-quadratic complexity.

5.1. Notation

A rank-metric code \(C \subseteq \mathbb{F}_{q^m}^{n \times k}\) is a set of matrices over a finite field \(\mathbb{F}_q\), where the distance of two codewords is measured w.r.t. the rank distance
\[
d_R : \mathbb{F}_{q^m}^{n \times k} \times \mathbb{F}_{q^m}^{n \times k} \to \mathbb{N}_{\geq 0}, \quad (C_1, C_2) \mapsto \text{rank}(C_1 - C_2).
\]
Since, for a fixed \(\mathbb{F}_q\)-basis of \(\mathbb{F}_{q^m}\), elements in \(\mathbb{F}_{q^m}\) can be expanded into matrices in \(\mathbb{F}_{q^m}^{n \times k}\), the rank distance is also well-defined over \(\mathbb{F}_{q^m}\). A linear rank-metric code of length \(n\), dimension \(k\), and minimum rank distance \(d_R\) is a \(k\)-dimensional \(\mathbb{F}_{q^m}\)-subspace of \(\mathbb{F}_{q^m}\), whose elements have pairwise rank distance at least \(d_R\). It was shown in (Delsarte, 1978; Gabidulin, 1985; Roth, 1991) that any
such code with \( n \leq m \) fulfills the rank-metric Singleton bound \( d_R \leq n - k + 1 \). Codes achieving this bound with equality are called maximum rank distance (MRD) codes.

**Gabidulin codes** (Delsarte, 1978; Gabidulin, 1985; Roth, 1991) are a special class of MRD codes and are often considered as the analogs of Reed–Solomon codes in rank metric. They can be defined by the evaluation of degree-restricted linearized polynomials as follows.

**Definition 24** (Gabidulin (1985)). A linear Gabidulin code \( \mathcal{G}[n, k] \) over \( \mathbb{F}_{q^n} \) of length \( n \leq m \) and dimension \( k \leq n \) is the set

\[
\mathcal{G}[n, k] = \left\{ \left[ f(g_1) \ f(g_2) \ldots \ f(g_n) \right] : f \in \mathcal{L}_k^{m} \right\},
\]

where the fixed elements \( g_1, g_2, \ldots, g_n \in \mathbb{F}_{q^n} \) are linearly independent over \( \mathbb{F}_q \).

Note that the encoding of Gabidulin codes, see Definition 24, is equivalent to the calculation of one MPE and can therefore be accomplished with complexity \( O(s^{\min\left(\frac{\omega}{\log(q)}, 1.635\right)} \log(s)) \). If \( \{g_1, \ldots, g_n\} \) is a normal basis, it can be computed as a \( q \)-transform in \( O(s \log^2(s) \log(\log(s))) \).

In this section, we assume that a word \( r = c + e \) is received, where \( d_R(r, c) = \text{rk}(e) \) denotes the number of rank errors, and the decoder wants to retrieve \( c \) from \( r \).

### 5.2. Error-Only Decoding of Gabidulin Codes

Algorithm 5 shows (Wachter-Zeh, 2013, Algorithm 3.6) for decoding Gabidulin codes up to \( \lfloor (d - 1)/2 \rfloor \) rank errors. This algorithm can be seen as the rank-metric equivalent of the Reed–Solomon decoding algorithms from (Sugiyama et al., 1975; Welch and Berlekamp, 1986; Sudan, 1997; Gao, 2003). Its correctness was proven in (Wachter-Zeh, 2013, Theorem 3.7) and its complexity was shown to be in \( O(n^2) \) over \( \mathbb{F}_{q^n} \). Since all steps of Algorithm 5 can be performed by algorithms with sub-quadratic complexity from this paper, the following corollary holds.

**Corollary 25.** Algorithm 5 can be implemented in \( O\left(n^{\max\left(\log_2(3), \min\left(\frac{\omega}{\log(q)}, 1.635\right)\right)} \log^2(n)\right) \) operations over \( \mathbb{F}_{q^n} \).

---

**Algorithm 5: DecodeGabidulin(\( r, \{g_1, g_2, \ldots, g_n\} \))**

**Input:** Received word \( r \in \mathbb{F}_{q^n} \) and \( g_1, g_2, \ldots, g_n \in \mathbb{F}_{q^n} \), linearly independent over \( \mathbb{F}_q \).

**Output:** Estimated evaluation polynomial \( f \) with \( \deg q f < k \) and error span polynomial \( \Lambda \) or “decoding failure”.

1. \( \hat{f} \leftarrow I_{\left\{ (g_i, g_{i+n}) \right\}_{i=1}^{n}} \quad \text{// } I_q(\hat{n}) \)
2. \( M \leftarrow M_{\left\{ (g_i, g_{i+n}) \right\}_{i=1}^{n}} \quad \text{// } M_{\mathbb{F}_q}(\hat{n}) \)
3. \([r_{\text{out}}, u_{\text{out}}, v_{\text{out}}] \leftarrow \text{RankLEEA}(M, \hat{f}, [(n + k)/2]) \quad \text{// } M_{\mathbb{F}_q}(\hat{n}) \log^2(n) \text{ (Corollary 11)}
4. \([x_L, q_L] \leftarrow \text{Left-divide } r_{\text{out}} \text{ by } u_{\text{out}} \quad \text{// } D_{\mathbb{F}_q}(n) \)
5. **if** \( q_L = 0 \text{ then} \)
6. **then** \[ \text{return } [f, \Lambda] \leftarrow [x_L, u_{\text{out}}] \]
7. **else** \[ \text{return } \text{ "decoding failure"} \]
5.3. Error-Erasure Decoding

Algorithm 6 shows (Wachter-Zeh, 2013, Algorithm 3.7) for decoding Gabidulin codes with $t$ errors, $\rho$ generalized row erasures and $\gamma$ generalized column erasures if

$$2t + \rho + \gamma \leq d - 1,$$

where $d$ is the minimum rank distance of the Gabidulin code. The correctness of Algorithm 6 was proven in (Wachter-Zeh, 2013, Theorem 3.9).

**Theorem 26.** Algorithm 6 can be implemented in $O\left(n^{\max\{\log_2(3), \min\{\omega+1/2, 1.635\}\}} \log^2(n)\right)$ operations over $\mathbb{F}_{q^m}$.

**Proof.** Its lines have the following complexities:

- Line 1: $d^{(C)}_i \in \mathbb{F}_{q^m}$ and if $\mathbb{F}_{q^m}$ elements are represented in the normal basis generated by $\beta$, the $B^{(C)}_{ij}$’s are already the representation of $d^{(C)}_i$, and thus no computation is needed.
- Lines 2 and 3 calculate MSPs whose cost is in $\text{MSP}_{q^m}(n) \subseteq O\left(n^{\max\{\log_2(3), \min\{\omega+1/2, 1.635\}\}} \log(n)\right)$.
- The cost of Line 4 is negligible.
- Line 5 finds the interpolation polynomial of $n$ point tuples, implying a cost of $\mathcal{I}_{q^m}(n) \subseteq O\left(n^{\max\{\log_2(3), \min\{\omega+1/2, 1.635\}\}} \log(n)\right)$.
- Line 6 requires three multiplications of linearized polynomials of degree $\leq 3n$ plus the modulo operation which requires $O(m) \subseteq O(n)$ additions because $x^m - x$ has only two non-zero coefficients. Hence, its complexity lies in $O(\mathcal{M}_{q^m}(n)) \subseteq O(n^{\min\{\omega+1/2, 1.635\}})$.
- Line 7 has complexity $O\left(n^{\max\{\log_2(3), \min\{\omega+1/2, 1.635\}\}} \log^2(n)\right)$ by Corollary 11.
- Lines 8 and 9 compute a multiplication of polynomials of degree $\leq n$ and a left division, yielding a complexity of $O(\mathcal{D}_{q^m}(n)) \subseteq O(n^{\min\{\omega+1/2, 1.635\}} \log(n))$.

Thus, using the results of Section 3, the overall complexity is as stated. □
Algorithm 6: DecodeErrorErasureGabidulin(r, \{g_1, g_2, \ldots, g_n\}, a^{(R)}, B^{(C)})

Input: Received word \( r \in \mathbb{F}_{q^n}^{m} \),
\[ g_i = \beta^{i-1} \in \mathbb{F}_{q^n}, \ i = 1, \ldots, n, \] normal basis of \( \mathbb{F}_{q^n} \) over \( \mathbb{F}_q \),
\[ a^{(R)} = [a_1^{(R)}, a_2^{(R)}, \ldots, a_N^{(R)}] \in \mathbb{F}_q^N, \]
\[ B^{(C)} = [B_{ij}]_{i,j \in [1,n]} \in \mathbb{F}_q^{n \times n} \]

Output: Estimated evaluation polynomial \( f \) with \( \text{deg}_q f < k \) or “decoding failure”.

1. \( d_i^{(C)} = \sum_{j=0}^{n-1} B_{ij} \beta^{j-1} \) for all \( i = 1, \ldots, \gamma \) \hspace{1cm} // negligible
2. \( \Gamma^{(C)} = M_{\{d_1^{(C)}, d_2^{(C)}, \ldots, d_n^{(C)}\}} \hspace{1cm} // \text{MSP}_{\mathbb{F}_q^n} (\gamma) \subseteq \text{MSP}_{\mathbb{F}_q^n} (n) \)
3. \( \Lambda^{(R)} = M_{\{d_1^{(R)}, d_2^{(R)}, \ldots, d_n^{(R)}\}} \hspace{1cm} // \text{MSP}_{\mathbb{F}_q^n} (\gamma) \subseteq \text{MSP}_{\mathbb{F}_q^n} (n) \)
4. \( \tilde{\Gamma}^{(C)} = \sum_{i=0}^{m-1} \Gamma_i^{(C)} x_i \) with \( \Gamma_i^{(C)} := \Gamma^{(C)} \times_i \text{mod} \ m \) for all \( i = 0, \ldots, m \) \hspace{1cm} // \( O(m) \subseteq O(n) \)
5. \( \hat{r} = I_{\{(\mu, r)|\mu|\leq n\}} \hspace{1cm} // I_{\mathbb{F}_q^n} (n) \)
6. \( \hat{y} = \Lambda^{(R)} \cdot \hat{r} \cdot \tilde{\Gamma}^{(C)} \cdot x^{(\gamma)} \) \hspace{1cm} mod \( (x^m - x) \) \hspace{1cm} // \( \text{MSP}_{\mathbb{F}_q^n} (m) \) \( \log_\gamma (m) \) \( n \)
7. \( [r_{\text{out}}, u_{\text{out}}, v_{\text{out}}] \leftarrow \text{HalfLLEA} (x^{(m)} - x, \hat{y}, \left\lceil \frac{n + \gamma + \gamma + \gamma}{2} \right\rceil) \hspace{1cm} // \text{MSP}_{\mathbb{F}_q^n} (m) \) \( \log_\gamma (m) \) \( n \)
8. \( [\chi_L, \rho_L] \leftarrow \text{LeftDiv} (r_{\text{out}}, u_{\text{out}} \cdot \Lambda^{(R)}) \hspace{1cm} // \text{D}_{\mathbb{F}_q^n} (m) \)
9. \( [\chi_R, \rho_R] \leftarrow \text{RightDiv} (\chi_L, \hat{\Gamma}^{(C)} \cdot x^{(\gamma)} \) \hspace{1cm} \( \chi^{(m)} - x) \) \hspace{1cm} // \text{D}_{\mathbb{F}_q^n} (m) \)

10. if \( \rho_L = 0 \) and \( \rho_R = 0 \) then
11. return \( f \leftarrow \chi_L \)
12. else
13. return “decoding failure”

Remark 27. In both Algorithm 5 and 6, the involved polynomials have \( q \)-degree at most \( n \leq m \).
Hence, the new algorithms in this paper are asymptotically faster than the ones from (Caruso and Le Borgne, 2017) in this case.

6. Conclusion

In this paper, we have reduced the complexity of several operations with linearized polynomials. Table 1 on page 3 summarizes the new complexity bounds on operations with linearized polynomials. In particular, we have generalized a fast algorithm for linearized polynomial multiplication to skew polynomials, implying the first sub-quadratic algorithm for division of two linearized polynomials of degree \( s \leq m \). We have also presented new algorithms with sub-quadratic complexity for calculating the \( q \)-transform, minimal subspace polynomial computation, multipoint evaluation and interpolation. For the sake \( s = m \), we have presented a lower bound on the cost of linearized polynomial multiplication and an algorithm that achieves it. Further, we have shown how to apply these algorithms when decoding Gabidulin codes. This yields the first decoding algorithm of Gabidulin codes which has, over \( \mathbb{F}_{q^n} \), sub-quadratic complexity in the code-length.

For future work, it is interesting to include our new algorithms in the study from (Bohaczuk Venturelli and Silva, 2014) on fast erasure decoding of Gabidulin codes.
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References

Aho, A. V., Hopcroft, J. E., 1974. The Design and Analysis of Computer Algorithms, 1st Edition. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA.
Bini, D., Pan, V., 2012. Polynomial and matrix computations: fundamental algorithms. Springer Science & Business Media.
Blahut, R. E., 1985. Fast Algorithms for Digital Signal Processing. Addison-Wesley.
Bohaczuk Venturelli, R., Silva, D., Aug. 2014. An evaluation of erasure decoding algorithms for Gabidulin codes. In: Int. Telecommunications Symposium (ITS). pp. 1–5.
Boucher, D., Ulmer, F., 2014. Linear codes using skew polynomials with automorphisms and derivations. Designs, codes and cryptography 70 (3), 405–431.
Brent, R. P., Gustavson, F. G., Yun, D. Y., 1980a. Fast solution of toeplitz systems of equations and computation of padé approximants. Journal of Algorithms 1 (3), 259–295.
Brent, R. P., Gustavson, F. G., Yun, D. Y. Y., 1980b. Fast Solution of Toeplitz Systems of Equations and Computation of Padé Approximants. J. Algorithms 1 (3), 259–295.
Brent, R. P., Kung, H. T., Oct. 1978. Fast Algorithms for Manipulating Formal Power Series. J. ACM 25 (4), 581–595.
Caruso, X., Borgne, J. L., 2017. Fast Multiplication for Skew Polynomials. arXiv preprint arXiv:1702.01665.
Caruso, X., Le Borgne, J., 2017. A new faster algorithm for factoring skew polynomials over finite fields. Journal of Symbolic Computation 79, 411–443.
Cohen, S. D., Hachenberger, D., 2000. The Dynamics of Linearized Polynomials. Proceedings of the Edinburgh Mathematical Society (Series 2) 43 (01), 113–128.
Coppersmith, D., Winograd, S., 1990. Matrix Multiplication via Arithmetic Progressions. Journal of symbolic computation 9 (3), 251–280.
Couveignes, J.-M., Lercier, R., 2009. Elliptic Periods for Finite Fields. Finite Fields and Their Applications 15 (1), 1–22.
Delsarte, P., 1978. Bilinear Forms over a Finite Field with Applications to Coding Theory. J. Combin. Theory Ser. A 25 (3), 226–241.
Evans, R. J., Greene, J., Niederreiter, H., et al., 1992. Linearized Polynomials and Permutation Polynomials of Finite Fields. Michigan Math. J 39 (3), 405–413.

Gabidulin, E. M., 1985. Theory of Codes with Maximum Rank Distance. Probl. Inf. Transm. 21 (1), 3–16.

Gabidulin, E. M., Paramonov, A., Tretjakov, O., 1991. Ideals Over a Non-Commutative Ring and Their Application in Cryptology. In: Advances in Cryptology—EUROCRYPT’91. Springer, pp. 482–489.

Gao, S., 1993. Normal Bases over Finite Fields. Ph.D. thesis, University of Waterloo, Waterloo, Canada.

Gao, S., 2003. A New Algorithm for Decoding Reed–Solomon Codes. Commun. Inform. Network Sec. 712, 55–68.

Gathen, J., Gerhard, J., 1999. Modern Computer Algebra. Cambridge university press.

Gustavson, F., Yun, D., September 1979. Fast Algorithms for rational Hermite approximation and solution of Toeplitz systems. IEEE Transactions on Circuits and Systems 26-9, 750–755.

Huang, X., Pan, V. Y., 1998. Fast Rectangular Matrix Multiplication and Applications. Journal of complexity 14 (2), 257–299.

Ke, S., Zeng, B., Han, W., Pan, V. Y., 2008. Fast Rectangular Matrix Multiplication and Some Applications. Science in China Series A: Mathematics 51 (3), 389–406.

Li, W., Sidorenko, V., Silva, D., 2014. On Transform-Domain Error and Erasure Correction by Gabidulin Codes. Designs, Codes and Cryptography 73 (2), 571–586.

Lidl, R., Niederreiter, H., 1997. Finite Fields: Encyclopedia of Mathematics and Its Applications. Computers & Mathematics with Applications 33 (7), 136–136.

Menezes, A. J., Blake, I. F., Gao, X., Mullin, R. C., Vanstone, S. A., Yaghoobian, T., 1993. Applications of Finite Fields, 1st Edition. Springer.

Ore, O., 1933a. On a Special Class of Polynomials. Transactions of the American Mathematical Society 35 (3), 559–584.

Ore, O., Jul. 1933b. Theory of Non-Commutative Polynomials. Annals of Mathematics 34 (3), 480–508.

Paterson, M. S., Stockmeyer, L. J., 1973. On the Number of Nonscalar Multiplications Necessary to Evaluate Polynomials. SIAM Journal on Computing 2 (1), 60–66.

Puchinger, S., Wachter-Zeh, A., Jul. 2016. Sub-quadratic Decoding of Gabidulin Codes. In: IEEE Int. Symp. Inf. Theory (ISIT).

Roth, R. M., 1991. Maximum-Rank Array Codes and their Application to Crisscross Error Correction. IEEE Trans. Inform. Theory 37 (2), 328–336.

Silberstein, N., Rawat, A. S., Vishwanath, S., Oct. 2012. Error Resilience in Distributed Storage via Rank-Metric Codes. In: Allerton Conf. Communication, Control, Computing (Allerton). pp. 1150–1157.

Silva, D., Kschischang, F. R., 2007. Rank-Metric Codes for Priority Encoding Transmission with Network Coding. In: Canadian Workshop on Information Theory. IEEE, pp. 81–84.

Silva, D., Kschischang, F. R., Jun. 2009. Fast Encoding and Decoding of Gabidulin Codes. In: IEEE Int. Symp. Inf. Theory (ISIT). pp. 2858–2862.

Silva, D., Kschischang, F. R., Koetter, R., 2008. A Rank-Metric Approach to CError Control in Random Network Coding. IEEE Transactions on Information Theory 54 (9), 3951–3967.

Strassen, V., 1969. Gaussian Elimination is not Optimal. Numerische Mathematik 13 (4), 354–356.

Sudan, M., Mar. 1997. Decoding of Reed–Solomon Codes beyond the Error-Correction Bound. J. Complexity 13 (1), 180–193.
Sugiyama, Y., Kasahara, M., Hirasawa, S., Namekawa, T., 1975. A Method for Solving Key Equation for Decoding Goppa Codes. Information and Control 27 (1), 87–99.

Wachter-Zeh, A., 2013. Decoding of Block and Convolutional Codes in Rank Metric. Ph.D. thesis, Ulm University and University of Rennes.

Wachter-Zeh, A., Afanassiev, V., Sidorenko, V., 2013. Fast Decoding of Gabidulin Codes. Designs, Codes and Cryptography 66 (1), 57–73.

Welch, L. R., Berlekamp, E. R., 1986. Error Correction for Algebraic Block Codes.

Wu, B., Liu, Z., 2013. Linearized Polynomials over Finite Fields Revisited. Finite Fields and Their Applications 22, 79–100.