Coherent State Path Integrals
at (Nearly) 40

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Abstract

Coherent states can be used for diverse applications in quantum physics including the construction of coherent state path integrals. Most definitions make use of a lattice regularization; however, recent definitions employ a continuous-time regularization that may involve a Wiener measure concentrated on continuous phase space paths. The introduction of constraints is both natural and economical in coherent state path integrals involving only the dynamical and Lagrange multiplier variables. A preliminary indication of how these procedures may possibly be applied to quantum gravity is briefly discussed.

1 Introduction

Formal expressions for path integrals have been used on a regular basis. While mathematically challenged, such formulas are exceedingly useful for heuristic insight. Behind every formal expression there should stand some well-defined regularization procedure, and we assume that to be the case. In what follows we generally choose units in which $\hbar = 1$.

In a familiar notation, the first path integral [1] was given 50 years ago by

$$\langle x''| e^{-iHT} |x'\rangle = \mathcal{N} \int \exp\{i \int [\frac{1}{2} \dot{x}^2 - V(x)] \, dt\} \, Dx,$$

and it was soon followed by the more general phase space form [2]

$$\langle x''| e^{-iHT} |x'\rangle = \mathcal{M} \int \exp\{i \int [p \dot{x} - H(p, x)] \, dt\} \, Dp \, Dx,$$

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which applies to a wider class of problems. An alternative phase space path integral
based on coherent states appeared subsequently [3]

\[ \langle p'', q'' | e^{-iHT} | p', q' \rangle = \mathcal{M} \int \exp \{ i \int [p \dot{q} - H(p, q)] \, dt \} \, Dp \, Dq , \]

which has an identical formal definition (but a different regularized form). Here,

\[ H(p, q) = \langle p, q | H(P, Q) | p, q \rangle \]

denotes the upper symbol for \( \mathcal{H} \). A different rule of
construction led to another coherent state path integral [4]

\[ \langle p'', q'' | e^{-iHT} | p', q' \rangle = \mathcal{M} \int \exp \{ i \int [p \dot{q} - h(p, q)] \, dt \} \, Dp \, Dq , \]

which involves a different symbol \( h(p, q) \) (defined below) for \( \mathcal{H} \).

In this brief review we focus on the contribution to path integration offered by
coherent states. Let us start with the canonical coherent states.

## 2 Coherent States

The canonical coherent states \(| p, q \rangle\) introduced above are defined by

\[ | p, q \rangle \equiv \exp(-iqP) \exp(ipQ) | \eta \rangle , \]

where \( Q \) and \( P \) denote an irreducible pair of self-adjoint operators that satisfy

\[ [Q, P] = i, \]

along with a normalized fiducial vector \(| \eta \rangle\). Such states are continuously
parametrized and admit a resolution of unity in the form

\[ 1 = \int | p, q \rangle \langle p, q | \, dm(p, q) , \quad dm(p, q) = dp \, dq / 2\pi , \]

integrated over \( \mathbb{R}^2 \), for any choice of \(| \eta \rangle\). This relation leads to a Hilbert space representation
\( \psi(p, q) \equiv \langle p, q | \psi \rangle = \langle \eta | e^{-ipQ} e^{iqP} | \psi \rangle \) by bounded, continuous functions \( \psi \). Each function is a vector in a reproducing kernel Hilbert space with reproducing
kernel \( \langle p'', q'' | p', q' \rangle \).

Operators can be defined to act in two different ways. Operators that act on
the left fulfill

\[ \mathcal{P}'( -p - i\partial / \partial q, i\partial / \partial p ) | p, q \rangle \psi \equiv | \eta | \mathcal{P}( P, Q ) \mathcal{P}' | \psi \rangle = \langle \psi | \mathcal{P}( P, Q ) \mathcal{P}' | \psi \rangle \]

If \( \mathcal{P} | \eta \rangle = 0 \), then \( \mathcal{P} \) defines a "polarization", which can be illustrated by the choice
\( \mathcal{P} = iP + Q \); in turn, this choice implies that \( 0 = \langle x | iP + Q | \eta \rangle = [\partial / \partial x + x] \langle x | \eta \rangle \),
with a solution \( \langle x | \eta \rangle \propto e^{-x^2 / 2} \). If \( \mathcal{P} | \eta \rangle = 0 \), then \( \mathcal{P}' \mathcal{P} | \eta \rangle = 0 \) and conversely.
Indeed, \( \langle \eta | \mathcal{P}' \mathcal{P} | \eta \rangle = \| \mathcal{P} | \eta \rangle \|^2 = 0 \) implies that \( \mathcal{P} | \eta \rangle = 0 \).

Let us consider the expression

\[ K^\nu(p, q; p', q') \equiv \exp[-(\nu / 2) T(\mathcal{P}^n \mathcal{P}')(-p - i\partial / \partial q, i\partial / \partial p)] \times \delta(p - p') \delta(q - q') , \]

where
where \( (\mathcal{P}^{\dagger}\mathcal{P}) = [-iP + Q][iP + Q] \equiv (p + i\partial/\partial q)^2 + (i\partial/\partial p)^2 - 1 \), and reexpress it in a conventional (not coherent state!) phase space path integral as

\[
K^\nu(p'', q''; p', q') = \mathcal{M} \int \exp\{i\int [k\dot{q} - x\dot{p}] dt \}
\]

\[\equiv \mathcal{M} \int \exp\{i\int [p\dot{q} + k\dot{q} - x\dot{p}] dt \}
\]

\[-(\nu/2)\int [(k^2 + x^2) - 1] dt \} \mathcal{D}k \mathcal{D}x \mathcal{D}p \mathcal{D}q .
\]

Carrying out the integrations over \( k \) and \( x \) leads to

\[
K^\nu(p'', q''; p', q') = N \int e^{i\int p\dot{q} dt} e^{-(1/2\nu)\int (\dot{q}^2 + \dot{p}^2) dt} \mathcal{D}p \mathcal{D}q
\]

\[= 2\pi e^{\nu T/2} \int e^{i\int p\dot{q} dt} d\mu^\nu_W (p, q) ,
\]

where we define the stochastic integral \( \int p\dot{q} dt \) by the midpoint (Stratonovich) rule. This final expression exhibits a well-defined path integral over Wiener measure without any ambiguity whatsoever.

The limit of \( K^\nu \) as \( \nu \to \infty \) yields an integral kernel for the projection operator onto the subspace for which \( \mathcal{P}'\psi(p, q) = 0 \), i.e., the subspace of interest. Hence, we conclude that

\[
\langle p'', q'' | p', q' \rangle = \exp\{i \frac{1}{4} (p'' + p')(q'' - q') - \frac{1}{4} [(p'' - p')^2 + (q'' - q')^2] \}
\]

\[= \lim_{\nu \to \infty} 2\pi e^{\nu T/2} \int e^{i\int p\dot{q} dt} d\mu^\nu_V (p, q) .
\]

This a very significant formula.

In evaluating any conditionally convergent integral, such as \( \int e^{-iy^2} dy \), \(-\infty < y < \infty\), it is first necessary to define the formal expression. This may be done, for example, by adopting \( \int e^{-iy^2-\epsilon y^2} dy \) in the limit that \( \epsilon \to 0^+ \), which then gives one possible definition. The path integral regularization given above is fundamentally the same as the one-dimensional example, and it may be given the following interpretation: In defining the coherent state overlap, we have introduced a Brownian motion regularization into the ill-defined, conditionally convergent (at best!) path integral,

\[
\mathcal{M} \int e^{i\int p\dot{q} dt} \mathcal{D}p \mathcal{D}q ,
\]

and instead defined this formal expression by the preceding regularized expression, removing the regularization (i.e., \( \nu \to \infty \)) as a final step. Evidently, the only price to pay to ensure a well-defined path integral expression is the adoption of a metric on phase space to support the Brownian motion, which in the present case corresponds to a flat space expressed in Cartesian coordinates. The resultant
quantization automatically leads to a canonical coherent state representation, which is entirely equivalent to choosing the polarization \( P = Q + iP \).

It is noteworthy that Brownian motion regularization on other geometries, e.g., a sphere or a pseudosphere, leads to alternative quantizations when \( \nu \to \infty \), namely, quantization in terms of the generators of SU(2) or SU(1,1), respectively. \[5, 6, 7\]

3 Dynamics

The second form of introducing operators is those that act on the right. Observe for any \( |\psi\rangle \) and \( |\eta\rangle \) that

\[
\mathcal{H}(-i\partial/\partial q, q + i\partial/\partial p) |p, q\rangle \langle \psi| = |\eta| \exp(-ipQ) \exp(iqP) \mathcal{H}(P, Q) |\psi\rangle .
\]

In canonical coherent states, therefore, Schrödinger’s equation assumes the form

\[
i\partial \psi(p, q, t)/\partial t = \mathcal{H}(-i\partial/\partial q, q + i\partial/\partial p) \psi(p, q, t).
\]

The solution to Schrödinger’s equation has a coherent state path integral given by \[3\]

\[
|p''', q'''\rangle e^{-i\mathcal{H}T} |p', q'\rangle = \lim_{\nu \to \infty} 2\pi e^{\nu T/2} \int e^{i[p dq - h(p, q) dt]} d\mu_{\nu}^W(p, q) ,
\]

where \( h(p, q) \) denotes the lower symbol for the quantum Hamiltonian, i.e.,

\[
\mathcal{H}(P, Q) = \int h(p, q) |p, q\rangle \langle p, q| dp dq/2\pi .
\]

After a canonical coordinate transformation classically generated by \( \overline{p} dq = p dq + dF(p, q) \) and based on the midpoint rule for which the ordinary rules of calculus apply, even for Brownian paths, we determine that

\[
|\overline{p}'', \overline{q}'''\rangle e^{-i\mathcal{H}T} |\overline{p}', \overline{q}'\rangle = \lim_{\nu \to \infty} 2\pi e^{\nu T/2} \int e^{i[\overline{p} dq + \overline{dF}(\overline{p}, \overline{q}) - \overline{h}(\overline{p}, \overline{q}) dt]} d\overline{\mu}_{\nu}^W(\overline{p}, \overline{q}) ,
\]

for some function \( \overline{G}(p, q) \) and where \( |\overline{p}, \overline{q}\rangle \equiv |p(\overline{p}, \overline{q}), q(\overline{p}, \overline{q})\rangle = |p, q\rangle, \overline{h}(\overline{p}, \overline{q}) = h(p(\overline{p}, \overline{q}), q(\overline{p}, \overline{q})) = h(p, q), \) and \( d\overline{\mu}_{\nu}^W \) is Weiner measure on the plane expressed, generally, in curvilinear coordinates. Moreover,

\[
\mathcal{H}(P, Q) = \int \overline{\mu}(\overline{p}, \overline{q}) |\overline{p}, \overline{q}\rangle \langle \overline{p}, \overline{q}| d\overline{p} d\overline{q}/2\pi .
\]

Observe, therefore, that the path integral and resultant propagator are covariant under general canonical coordinate transformations. This fact gives rise to the assertion that we deal here with a coordinate-free formulation of quantization.
4 Metrical Quantization

The preceding analysis suggests a novel two-step quantization procedure. First, add a metric to the classical phase space that can be used to keep track of the physical meaning of mathematical expressions under coordinate transformations. For a flat space expressed in Cartesian coordinates,
\[ \frac{1}{2}(p^2 + q^2) + q^4 \] physically corresponds to a quartic anharmonic oscillator—or, for a space of constant negative curvature,
\[ d\sigma^2 = \beta^{-1}q^2 dp^2 + \beta q^{-2} dq^2, \quad q > 0, \beta \text{ a constant, etc.} \] Second, use the metric in a Wiener measure regularization of an otherwise formal phase space path integral. For example, for the negative curvature case with \( \beta > 1/2 \),
\[ K \equiv \lim_{\nu \to \infty} \mathcal{N} \int e^{i\int [p \dot{q} - h(p,q)] dt} e^{-(1/2\nu)\int [\beta^{-1}q^2 \dot{p}^2 + \beta q^{-2} \dot{q}^2] dt} \mathcal{D}p \mathcal{D}q \]
\[ = \lim_{\nu \to \infty} 2\pi[1 - 1/2\beta] e^{\nu T/2} \int e^{i\int [p dq - h(p,q)] dt} d\tau \mathcal{W}(p,q). \]
This expression leads to a positive-definite function, and that fact alone allows one to conclude that
\[ K = \langle p'', q'' | E \epsilon^{-i\mathcal{H}T} | p', q' \rangle, \quad |p, q\rangle \equiv e^{-ip\theta} e^{ip\theta} e^{-i(lnq)\mathcal{D} |\beta\rangle}, \]
\[ [Q, D] = iQ, \quad Q > 0, \quad (Q - 1 + i\beta^{-1}D) |\beta\rangle = 0, \]
\[ \mathcal{H} = \int h(p,q) |p, q\rangle \langle p, q| dp dq/2\pi[1 - 1/2\beta], \]
\[ 1 = \int |p, q\rangle \langle p, q| dp dq/2\pi[1 - 1/2\beta]. \]

5 Constraints

General constraints may be imposed in the quantum theory by introducing a projection operator \( \mathcal{E} \) from the original Hilbert space \( \mathcal{H} \) onto the physical Hilbert space \( \mathcal{H}_{\text{phys}} \equiv \mathcal{E} \mathcal{H} \). If \( \Phi_\alpha(P,Q), \alpha = 1, \ldots, A \), denote hermitian quantum constraints, it may be argued that it suffices to choose
\[ \mathcal{E} = \mathcal{E} \{ \Phi_\alpha M^{\alpha\beta} \Phi_\beta \leq \delta(h)^2 \} \]
where \( \{ M^{\alpha\beta} \} \) is a symmetric, positive-definite matrix, and \( \delta(h) \) is chosen so that \( \mathcal{E} \) projects onto a suitably small space. By choosing an appropriate measure for the Lagrange multiplier variables, it is possible to generate the expression
\[ \langle p'', q'' | \mathcal{E} \epsilon^{-i\mathcal{E} \mathcal{H}E^T \mathcal{E}} | p', q' \rangle \]
corresponding to temporal evolution entirely within the physical subspace.

In this generality, and by using appropriate integration measures for the Lagrange multiplier variables along with the usual measure for the dynamical variables, coherent state path integrals may be constructed to deal with a general set of constraints.
6 Preliminary Application to Gravity

The classical phase space variables for $3 + 1$ gravity are symmetric $3 \times 3$ matrices \( \{\pi^{jk}\} \) and \( \{g_{jk}\} > 0 \) with the latter being positive definite. A natural metric on phase space is given by

\[
d\sigma^2 = \int_{\Sigma} \left[ g^{-1/2} g_{jk} g_{lm} \pi^{kl} \pi^{jm} + g^{1/2} g^{jk} g^{lm} d\pi_{kl} d\pi_{jm} \right] d^3x,
\]

an expression which is diffeomorphism invariant under coordinate transformations on the space like surface \( \Sigma \). Adopting the viewpoint of metrical quantization, we propose that

\[
\langle \pi'', g'' | \pi', g' \rangle \equiv \lim_{\nu \to \infty} N \int e^{i \int \pi^{jk} \dot{g}_{jk} d^3x \, dt} \times \frac{1}{(2\nu)^{n/2}} \int \left[ g^{-1/2} g_{jk} g_{lm} \pi^{kl} \pi^{jm} + g^{1/2} g^{jk} g^{lm} \dot{g}_{kl} \dot{g}_{jm} \right] d^3x \, dt \prod_{a \leq b} D\pi_{ab} Dg_{ab}
\]

fully determines a reproducing kernel for gravity before any constraints are imposed. It is noteworthy that this formal functional integral is ultralocal, i.e., has no spatial derivatives. Recent advances in the formulation and solution of ultralocal quantum field theories suggest that evaluation of this reproducing kernel may indeed be possible. [10]

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