THE LAURENT COEFFICIENTS OF THE HILBERT SERIES OF
A GORENSTEIN ALGEBRA

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Dedicated to the memory of Leslie Tanner Hammontree.

Abstract. By a theorem of R. Stanley, a graded Cohen-Macaulay domain \( A \) is Gorenstein if and only if its Hilbert series satisfies the functional equation

\[
\text{Hilb}_A(t^{-1}) = (-1)^d t^{-a} \text{Hilb}_A(t),
\]

where \( d \) is the Krull dimension and \( a \) is the a-invariant of \( A \). We reformulate this functional equation in terms of an infinite system of linear constraints on the Laurent coefficients of \( \text{Hilb}_A(t) \) at \( t = 1 \). The main idea consists of examining the graded algebra \( F = \bigoplus_{r \in \mathbb{Z}} F_r \) of formal power series in the variable \( x \) that fulfill the condition \( \varphi(x/(x-1)) = (1-x)^r \varphi(x) \). As a byproduct, we derive quadratic and cubic relations for the Bernoulli numbers. The cubic relations have a natural interpretation in terms of coefficients of the Euler polynomials. For the special case of degree \( r = -(a + d) = 0 \), these results have been investigated previously by the authors and involved merely even Euler polynomials. A link to the work of H. W. Gould and L. Carlitz on power sums of symmetric number triangles is established.

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1. Introduction

Let $\mathbb{K}$ be a field. By a positively graded $\mathbb{K}$-algebra we mean a graded $\mathbb{K}$-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ such that $A_0 = \mathbb{K}$ and for all $i$ we have $\dim_{\mathbb{K}}(A_i) < \infty$. To a positively graded $\mathbb{K}$-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ one associates its Hilbert series $\Hilb_A(t) \in \mathbb{Z}[t]$, i.e., the generating function

$$\Hilb_A(t) := \sum_{i=0}^{\infty} \dim_{\mathbb{K}}(A_i) t^i$$

counting the dimensions of the homogeneous components $A_i$. If $A$ is finitely generated, then the Hilbert series is actually a rational function $\Hilb_A(t) =: P(t)/Q(t)$. Moreover, the pole order $d$ in the Laurent expansion

$$\Hilb_A(t) = \sum_{i=0}^{\infty} \frac{\gamma_i}{(1-t)^{d-i}}$$

equals the Krull dimension of $A$. See [3, 6 Section 1.4] or [13 Section 3.10] for more details.

By a theorem of R. Stanley [3, 16], a graded Cohen-Macaulay domain $A$ is Gorenstein if and only if its Hilbert series satisfies the functional equation

$$\Hilb_A(t^{-1}) = (-1)^d t^{-a} \Hilb_A(t).$$

The number $a$ is the so-called $a$-invariant of $A$ and can be understood as $\deg(P) - \deg(Q)$, see [15] Definition 5.1.5. It is well-known that

$$r := 2\gamma_1/\gamma_0 = -(a + d),$$

see [13 Equation (3.32)]. We will call this quantity the degree, as it will play this role for a $\mathbb{Z}$-graded algebra in this work. In the literature (e.g. in [2]), $\gamma_0$ is frequently referred to as the degree, and we hope this will not lead to any confusion. The degree $r$ has a natural interpretation in terms of the canonical module $\omega_A$ of $A$ (see e.g. [2, 3]) as the degree shift in $\omega_A \cong A[r]$, where the shifted module $M[r]$ of a graded module $M = \bigoplus M_i$ is defined via $M[r]_i := M_{r+i}$. In this context, $A$ is Gorenstein if and only if the canonical module $\omega_A$ is one-dimensional.

The starting point and initial motivation of this work is a reformulation of the functional equation (1.3) in terms of an infinite system of linear constraints on the Laurent coefficients $\gamma_i$. In a previous paper [10], we have treated the case of degree $r$ equal zero, i.e. $a + d = 0$. In the general case, the actual shape of the relations depends on the sign and parity of the degree $r$.

**Theorem 1.1.** Let $H(t) = \sum_{i=0}^{\infty} \gamma_i (1 - t)^{i-d}$ be a formal Laurent series around $t = 1$ of pole order at most $d$. Then $H(t)$ satisfies the functional equation (1.3) if and only if the following conditions are fulfilled, where we stipulate $\gamma_i = 0$ for $i < 0$.

If the degree $r$ is even, then for each $m \geq 1$:

$$\sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} \gamma_{m-r+i} = 0. \tag{1.5}$$

If the degree $r > 0$ is odd, then for each $m \geq 1$:

$$\sum_{i=0}^{m} (-1)^i \binom{2m + r - 2}{m - i} \binom{m + i}{i} \gamma_{m+i-1} = 0. \tag{1.6}$$
When \( r < 0 \), then for each \( 1 \leq m \leq \lceil -r/2 \rceil \):

\[
\sum_{i=0}^{m} \binom{m}{i} \left( \frac{1 - r}{m + i} \right)^{-1} \gamma_{m+i-1} = 0,
\]

regardless of the parity of \( r \). Moreover, if \( r < 0 \) is odd, then for each \( m \geq 1 \):

\[
\sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{m + i}{m} \gamma_{m-r+i} = 0.
\]

Note that the relations among the Laurent coefficients are only unique up to scalar factors. Alternate choices of the scaling will arise in the course of Section 4 and be considered as well in Section 8.

The relations given in Theorem 1.1 in the case \( r = 0 \) were first observed in [11, Section 8] in the context of experimental computations of the Laurent coefficients of Hilbert series of a certain class of rings related to symplectic quotients by the circle. An understanding of these relations suggested that these rings are all Gorenstein of degree \( r = 0 \) (sometimes called graded Gorenstein or strongly Gorenstein), which was then verified in [10, Section 4]. The initial purpose of Theorem 1.1 is to demonstrate that analogous relations occur for Gorenstein rings of arbitrary degree \( r \). Hence, one application of Theorem 1.1 is to contexts in which the \( \gamma_i \) are more directly computable or otherwise more accessible than the rational expression of the Hilbert series. The relations for small values of \( m \) can then be used to prove that a ring is not Gorenstein without computing a more complete description of the ring or its Hilbert series. We illustrate an example of such a context in Section 2 for invariants of finite groups.

In addition, the connection between the relations of Theorem 1.1 and the functional equation (1.3) has interesting consequences for some combinatorial number sequences and other combinatorial constructions. After introducing and describing the structure of the \( \mathbb{Z} \)-graded algebra \( \mathcal{F} = \bigoplus_{r \in \mathbb{Z}} \mathcal{F}_r \) in Section 3, the principal tool used in the proof of Theorem 1.1 that is then given in Section 4, we turn our attention to these consequences. In Section 5, we reformulate the constraints of Theorem 1.1 to express the odd coefficients \( \gamma_{2i+1} \) in terms of the even coefficients \( \gamma_{2i} \) and vice versa, see Theorems 5.3 and 5.9. These reformulations are most succinctly stated using the coefficients of Euler polynomials as well as the Bernoulli numbers. Hence, the algebra structure of \( \mathcal{F} \) yields a unified proof of a large collection of quadratic and cubic identities for the Bernoulli numbers in Section 6, see Theorems 6.1 and 6.2.

In Section 7, we consider another application to combinatorics, connecting the functional equation (1.3) to the power sum identities for symmetric number arrays (e.g. Pascal’s triangle) developed by H. W. Gould [9] and L. Carlitz [4]. Specifically, by demonstrating that a generating function for these power sums is an element of \( \mathcal{F}_1 \), we re-derive these power sum identities as corollaries of the case \( r = 1 \) of Theorem 1.1 and its reformulations in Section 6. To complete the paper, we illustrate the coefficient triangles that appear in Theorem 1.1 by displaying specific examples and consider the rescaling of the rows in Section 8.

Note that some of the auxiliary lemmas in this paper can be derived as special cases of known identities. For instance, Lemma 4.5 can be seen to be a consequence of [4, Equation (15.2.5)]. However, for the benefit of the reader, we provide elementary proofs.
Finally, let us observe that there exist Gorenstein algebras for each degree \( r \in \mathbb{Z} \). For example, a hypersurface of degree \( k \) in affine space is Gorenstein of degree \( r = -k \), while invariant rings of unimodular finite group representations with no pseudoreflections are Gorenstein of degree \( r = 0 \) by [19, 20], see Section 2. On the other hand, a polynomial ring \( \mathbb{K}[x_1, x_2, \ldots, x_d] \) in variables of degree \( \deg(x_i) \geq 1 \) has \( r = \sum_i (\deg(x_i) - 1) \).

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2. The \( \gamma_i \) for invariants of finite groups

In this section, we give an example of an application of Theorem 1.1 to computational invariant theory. We refer the reader to [2, Sections 2.5–6], [6, Section 2.6], or [17, Section 2.2] for background on the topic considered here. For simplicity, we work over \( \mathbb{C} \).

Let \( V \) be a vector space over \( \mathbb{C} \) of dimension \( n \), and let \( G \) be a finite subgroup of \( \text{GL}(V) \). By Molien’s formula, the Hilbert series of the ring \( \mathbb{C}[V]^G \) of \( G \)-invariant polynomials is given by

\[
\text{Hilb}_{\mathbb{C}[V]^G}(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(id - tg)},
\]

It is a well-known consequence of this result that the first two coefficients of the Laurent expansion of \( \text{Hilb}_{\mathbb{C}[V]^G}(t) \) at \( t = 1 \) are equal to

\[\gamma_0 = \frac{1}{|G|} \quad \text{and} \quad \gamma_1 = \frac{p}{2|G|},\]

where \( p \) is the number of pseudoreflections in \( G \), elements of \( G \) whose fixed set in \( V \) has codimension 1. By the same method used to determine these coefficients, we now demonstrate that the \( \gamma_k \) for \( k \leq n \) can be computed using only those elements of \( G \) that fix a subset of codimension at most \( k \).

For \( g \in G \), let \( \mu_1(g), \ldots, \mu_n(g) \) denote the eigenvalues of \( g \), where we assume that any eigenvalues with value 1 occur last on this list. Choosing for each \( g \) a basis for \( V \) with respect to which \( g \) is diagonal, Equation (2.1) becomes

\[
\text{Hilb}_{\mathbb{C}[V]^G}(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\prod_{j=1}^n (1 - \mu_j(g)t)}.
\]

Let \( p(g) \) denote the number of \( j \) such that \( \mu_j(g) = 1 \). For \( 0 \leq k \leq n \), let \( G_k = \{ g \in G : p(g) = k \} \), \( G_{\leq k} = \{ g \in G : p(g) \leq k \} \), and \( G_{\geq k} = \{ g \in G : p(g) \geq k \} \). As the term in Equation (2.2) corresponding to an element \( g \in G \) has a pole order equal to \( p(g) \) at \( t = 1 \), the coefficient \( \gamma_k \) of the Laurent series depends only on the
elements of $G_{\geq n-k}$. Specifically, we can express
\[
\text{Hilb}_{[V]}(t) = \frac{1}{|G|} \sum_{k=0}^{n} (1-t)^{k-n} \sum_{g \in G_{\geq n-k}} \frac{1}{\prod_{j=1}^{k} (1 - \mu_j(g)t)}
\]

\[
= \frac{1}{|G|} \sum_{k=0}^{n} (1-t)^{k-n} \prod_{j=1}^{k} \sum_{i=0}^{\infty} \frac{-\mu_j(g)^i}{(\mu_j(g) - 1)^{i+1}(1-t)^i}.
\]

Then as $G_n = \{\text{id}\}$, the sum over $g \in G_n$ is simply equal to 1, yielding $\gamma_0 = 1/|G|$. Similarly,
\[
\text{Hilb}_{[V]}(t) = \frac{1}{|G|} (1-t)^{-n} + \frac{1}{|G|} (1-t)^{1-n} \sum_{g \in G_{\geq n-1}} \frac{1}{1 - \mu_1(g)t}
\]

\[\quad + \frac{1}{|G|} \sum_{g \in G_{\leq n-2}} \prod_{j=1}^{n} \frac{1}{(1 - \mu_j(g)t)},
\]

where the last sum has a pole at $t = 1$ of order at most $n - 2$. In particular,
\[
\gamma_1 = \frac{1}{|G|} \sum_{g \in G_{n-1}} \frac{-1}{\mu_1(g) - 1} = \frac{1}{2|G|} \sum_{g \in G_{n-1}} \left( \frac{-1}{\mu_1(g) - 1} + \frac{-1}{\mu_1(g)^2 - 1} \right)
\]

\[\quad = \frac{1}{2|G|} \sum_{g \in G_{n-1}} \left( \frac{-1}{\mu_1(g) - 1} + \frac{-1}{\mu_1(g)^2 - 1} \right) = \frac{1}{2|G|} \sum_{g \in G_{n-1}} 1 = \frac{|G_{n-1}|}{2|G|} = \frac{p}{2|G|}.
\]

Continuing in this way,
\[
\gamma_2 = \frac{1}{|G|} \left( \sum_{g \in G_{n-1}} \frac{-\mu_1(g)}{(\mu_1(g) - 1)^2} + \sum_{g \in G_{n-2}} \frac{1}{(\mu_1(g) - 1)(\mu_2(g) - 1)} \right),
\]

\[
\gamma_3 = \frac{1}{|G|} \left( \sum_{g \in G_{n-1}} \frac{-\mu_1(g)^2}{(\mu_1(g) - 1)^3} + \sum_{g \in G_{n-2}} \frac{\mu_1(g) + \mu_2(g) - 2\mu_1(g)\mu_2(g)}{(\mu_1(g) - 1)^2(\mu_2(g) - 1)^2} \right)
\]

\[\quad + \sum_{g \in G_{n-3}} \frac{-1}{(\mu_1(g) - 1)(\mu_2(g) - 1)(\mu_3(g) - 1)},
\]

etc.

To apply Theorem 1.1 given a finite group $G$, the value of $r$ is determined from $\gamma_0$ and $\gamma_1$ using Equation (1.4). Note that in this context, $r = p = |G_{n-1}| \geq 0$ is the number of pseudoreflections in $G$. Furthermore, Equation (1.4) is reflected by Theorem 1.1 as Equation (1.5) with $m = \frac{r}{2} + 1$ for $r$ even and as Equation (1.6) with $m = 1$ for $r$ odd. Then, staying for the moment with the case $r$ even, the constraint in Theorem 1.1 corresponding to $m = \frac{r}{2} + 2$ gives a necessary condition for $\mathbb{C}[V]^G$ to be Gorenstein that involves only the eigenvalues of elements of $G_{\geq n-3}$. Similarly, the constraint corresponding to arbitrary $\frac{r}{2} + m$ gives a necessary Gorenstein condition for $\mathbb{C}[V]^G$ involving only $G_{\geq n-2m+1}$. Note that if $G$ contains no pseudoreflections, then $\mathbb{C}[V]^G$ is Gorenstein if and only if $G \leq \text{SL}(V)$ by a Theorem of Watanabe [20, Theorem 1]; hence, the Gorenstein property of $\mathbb{C}[V]^G$ can be established much more easily in this case.

As an explicit example, let $\zeta$ be a primitive 6th root of unity and consider the subgroup $G$ of $\text{GL}_4(\mathbb{C})$ of order 12 generated by $a = \text{diag}(\zeta, \zeta^2, \zeta, 1)$ and $b = \text{diag}(1, \zeta, \zeta^2, \zeta^3)$. Then as $G_n = \{\text{id}\}$, the sum over $g \in G_n$ is simply equal to 1, yielding $\gamma_0 = 1/|G|$. Similarly,
diag(1, 1, 1, −1). Clearly, γ_0 = 1/12, and as G contains the single pseu- 
dreflection b, we have γ_1 = 1/24. It follows that if \( \mathbb{C}[\mathbb{C}^4]^G \) were to be Gorenstein, we must 
have \( r = 1 \) so that \( γ_1 - 3γ_2 + 2γ_3 = 0 \) by Equation (1.6) with \( m = 2 \). However, 
as described above, one may easily compute using only the elements of \( G_4 = \{ \text{id} \} \), 
\( G_3 = \{ b \} \), \( G_2 = \{ a^3 \} \), and \( G_1 = \{ a, a^2, a^4, a^5, a^8b \} \) that \( γ_2 = 1/24 \) and \( γ_3 = 1/72 \) 
so that \( γ_1 - 3γ_2 + 2γ_3 = -1/18 \) and hence \( \mathbb{C}[\mathbb{C}^4]^G \) is not Gorenstein. In this simple 
example, we can conclude that the Gorenstein property fails with a computa-
tion involving only the 8 elements of \( G_{\geq 1} \), and in particular without computing the 
invariant ring or its Hilbert series completely. In larger, less contrived examples, 
\( G_{\geq 1} \) may be much smaller relative to the size of \( G \).

3. Construction of the algebra \( \mathcal{F} \)

For the remainder of this paper, let \( k \) denote one of the fields \( \mathbb{Q}, \mathbb{R}, \) or \( \mathbb{C} \).

Let us assume that \( H(t) \) is a formal Laurent series over the field \( k \) around \( t = 1 \) 
of pole order at most \( d \). Clearly, the substitution \( \varphi(x) := x^d H(1 - x) \) is a formal 
power series in the variable \( x \), i.e., an element of \( k[[x]] \). Conversely, a formal power 
series \( \varphi(x) \) defines a formal Laurent series \( H(t) := (1 - t)^{-d} \varphi(1 - t) \). Assuming 
that \( H(t) \) satisfies the functional equation (1.3), we derive

\[
\frac{(-1)^d t^{-a}}{(1 - t)^d} \varphi(1 - t) = H(t^{-1}) = \frac{1}{(1 - t^{-1})}\varphi(1 - t^{-1}) = \frac{(-t)^d}{(1 - t)^d} \varphi((t - 1)/t).
\]

Multiplying by \( t^{-d}(t - 1)^d \) and substituting \( x = 1 - t \), we find that \( \varphi(x) \) satisfies 
the functional equation

\[
(3.1) \quad \varphi(x/(x - 1)) = (1 - x)^r \varphi(x),
\]

where \( r = -(a + d) \).

**Definition 3.1.** Let \( \mathcal{F}_r \) be the space of formal power series in the variable \( x \) 
satisfying Equation (3.1). We introduce the \( \mathbb{Z} \)-graded vector space \( \mathcal{F} = \bigoplus_{r \in \mathbb{Z}} \mathcal{F}_r \subset k[[x]] \).

Using the above computations, it is an easy task to verify the following state-
ments, which we leave to the reader.

**Proposition 3.2.** The substitution \( H(t) \mapsto \varphi(x) := x^d H(1 - x) \) establishes an 
isomorphism between the space of formal Laurent series of pole order at most \( d \) 
in the variable \( t \) around \( t = 1 \) satisfying Equation (1.3) and \( \mathcal{F}_r \). With respect to the 
Cauchy product of formal power series, \( \mathcal{F} = \bigoplus_{r \in \mathbb{Z}} \mathcal{F}_r \) forms a \( \mathbb{Z} \)-graded algebra.

In [10], the authors have investigated the algebra \( \mathcal{F}_0 \), i.e., the algebra of formal 
power series in the variable \( x \) invariant under the Möbius transformation \( x \mapsto x/(x - 1) \). We recall the follow-

**Theorem 3.3 ([10]).** For a formal power series \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \), the following 
conditions are equivalent:

(i.) \( \varphi(x) \in \mathcal{F}_0 \).

(ii.) \( \varphi(x) \) is a formal composite with \( \lambda_1(x) := x^2/(1 - x) \), i.e., there exists a 
\( \rho(y) \in k[y] \) such that \( \varphi(x) = \rho(x^2/(1 - x)) \).

(iii.) For each \( m \geq 1 \), the relation \( \sum_{i=0}^{m-1} (-1)^i (m-1) \gamma_{m+i} = 0 \) holds.
In fact, in (ii), instead of \( \lambda_1(x) := x^2/(1 - x) \), we could have worked with any \( \lambda(x) \in \mathcal{F}_0 \cap m^2 \) whose class in \( m^2/m^3 \) is nonzero. Here \( m = x\kappa[x] \) is the maximal ideal of \( \kappa[x] \). This was noted in [10] Remark 2.4, where we constructed examples of such series that are related to the Genocchi sequence. Another natural example slipped through our attention, namely:

\[
\lambda_2(x) := \left( \frac{x}{x - 1} \right)^2 \in \mathcal{F}_0 \cap m^2.\]

Moreover, the reader can readily check that

\[
g_{-1}(x) := x - 2 \in \mathcal{F}_{-1}
\]
is invertible. Hence, for any \( \varphi(x) \in \mathcal{F}_r \), it follows that \( \varphi(x)(x - 2)^r \in \mathcal{F}_0 \). Using \( \lambda_2(x) \), we can write \( \varphi(x) \) uniquely in the form

\[
\varphi(x) = \sum_{i=0}^{\infty} \delta_i x^{2i} (x - 2)^{-r - 2i}
\]
for \( \delta_i \in \kappa \). Conversely, every such series is an element of \( \mathcal{F}_r \).

**Theorem 3.4.** The \( \mathbb{Z} \)-graded algebra \( \mathcal{F} = \bigoplus_{r \in \mathbb{Z}} \mathcal{F}_r \) is isomorphic to the algebra of Laurent polynomials \( \mathcal{F}_0(g_{-1}) \) with coefficients in \( \mathcal{F}_0 \) in the variable \( g_{-1} \) of degree \(-1\).

4. **Proof of Theorem 1.1**

Let \( r \in \mathbb{Z} \), and let \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \). In this section, we prove Theorem 1.1 that \( \varphi(x) \in \mathcal{F}_r \) is equivalent to the relations described in Equations (1.5), (1.6), (1.7), and (1.8). In the case \( r < 0 \), Equations (1.5) and (1.8) involve only the coefficients \( \gamma_i \) for \( i \geq 1 - r \), while Equation (1.6) involves the \( \gamma_i \) for \( i \leq r - 1 \) when \( r \) is even and \( i \leq -r \) when \( r \) is odd (in the case of \( r \) even, there is no relation involving \( \gamma_{-r} \)). Hence, we will first assume in this case that \( \gamma_i = 0 \) for \( i \leq -r \) in Subsection 4.1 and deal with Equation (1.7) separately in Subsection 4.2.

4.1. Equations (1.5), (1.6), and (1.8). We first consider the case of \( r \) even with the assumption that the first \( 1 - r \) coefficients vanish.

**Lemma 4.1.** Let \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in \kappa[x] \) and let \( r \) be an even integer. If \( r < 0 \), assume that \( \gamma_i = 0 \) for each \( i \leq -r \). Then \( \varphi(x) \in \mathcal{F}_r \) if and only if \( \varphi(x) := x^r \varphi(x) \in \mathcal{F}_0 \).

**Proof.** Note that when \( r < 0 \), the assumption that the first \( 1 - r \) of the \( \gamma_i \) vanish ensures that \( \varphi(x) \) is a power series. By Equation (3.1), \( \varphi(x) \in \mathcal{F}_0 \) if and only if \( \varphi(x/(x - 1)) = \varphi(x) \), which by a simple computation using \( \varphi(x) = x^r \varphi(x) \) and the fact that \( r \) is even is equivalent to \( \varphi(x/(x - 1)) = (1 - x)^r \varphi(x) \), i.e., \( \varphi(x) \in \mathcal{F}_r \). \( \square \)

**Corollary 4.2.** Let \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in \kappa[x] \) and let \( r \) be an even integer. If \( r < 0 \), assume that \( \gamma_i = 0 \) for each \( i \leq -r \). Then the \( \gamma_i \) satisfy Equation (1.5) for each \( m \geq 1 \) if and only if \( \varphi(x) \in \mathcal{F}_r \).

**Proof.** If \( r = 0 \), then this claim corresponds to the equivalence of (i) and (iii) in Theorem 3.3. If \( r \neq 0 \), set \( \varphi(x) = x^r \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \). Equation (1.5) for all \( m \geq 1 \) is equivalent to

\[
\sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} \varphi_{m+i} = 0,
\]
which is equivalent to \( \mathcal{F}(x) \in \mathcal{F}_0 \) by the above argument. By Lemma 4.1, this is equivalent to \( \varphi(x) \in \mathcal{F}_r \).

To proceed to the odd case, we start with the following useful characterization of solutions for the constraints in Theorem 1.1 when \( r \) is positive and odd, i.e. \( r = 2k + 1 \) for some \( k \geq 0 \). Note that Equation (4.1) simply rewrites Equation (1.6) replacing \( r = 2k + 1 \).

**Lemma 4.3.** Let \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in K[x] \), and let \( k \geq 0 \) be an integer. Then

\[
\sum_{i=0}^{m} (-1)^i \binom{m+i}{m} \gamma_{m+i-1} = 0
\]

for each \( m \geq 1 \) if and only if there is a power series \( \psi(y) \in K[y] \) such that

\[
\sum_{i=0}^{\infty} \gamma_i x^i = \frac{1}{x} \frac{d^{2k-1}}{dx^{2k-1}} \left[ \frac{x^{2k}}{x^2 - 1} \psi \left( \frac{x^2}{1-x} \right) \right],
\]

where, for \( k = 0 \), we interpret \( \frac{d^{-1}}{dx} \) as the indefinite integral (with vanishing constant of integration). If \( k \geq 1 \), then Equation (4.2) is equivalent to

\[
\sum_{i=0}^{\infty} \gamma_i x^i = \frac{1}{x} \frac{d^{2k-1}}{dx^{2k-1}} \left[ x^{2k-2} \rho \left( \frac{x^2}{1-x} \right) \right]
\]

for some \( \rho(y) \in K[y] \).

**Proof.** For each \( m \), multiplying both sides by the scalar \( \frac{(m!)^2}{(2m+2k-1)!} \), we rewrite Equation (4.1) as

\[
\sum_{i=0}^{m} (-1)^i \binom{m+i}{m} \frac{(m+i)!}{(m+2k+i-1)!} \gamma_{m+i-1} = 0.
\]

Define \( \tilde{\varphi}(x) = \sum_{i=0}^{\infty} \tilde{\gamma}_i x^i \) by \( \tilde{\gamma}_0 = \tilde{\gamma}_1 = 0 \) and \( \tilde{\gamma}_{i+2} = \frac{(i+1)!}{(i+2k)!} \gamma_i \), and then for each \( m \), the \( \tilde{\gamma}_i \) satisfy Equation (4.3) if and only if

\[
\sum_{i=0}^{m} (-1)^i \binom{m+i}{m} \tilde{\gamma}_{m+i+1} = 0,
\]

which, along with \( \tilde{\gamma}_1 = 0 \) is equivalent to \( \tilde{\varphi}(x) \in \mathcal{F}_0 \). By Theorem 3.3 and as \( \tilde{\varphi}(x) \) has no constant nor linear terms by definition, this is equivalent to the existence of a \( \psi(y) \in K[y] \) such that

\[
\tilde{\varphi}(x) = \sum_{i=0}^{\infty} \frac{(i+1)!}{(i+2k)!} \gamma_{i+2} x^{i+2} = \left( \frac{x^2}{1-x} \right) \psi \left( \frac{x^2}{1-x} \right).
\]

Multiplying by \( x^{2k-2} \) yields

\[
\sum_{i=0}^{\infty} \frac{(i+1)!}{(i+2k)!} \gamma_{i+2k} x^{i+2k} = \left( \frac{x^{2k}}{1-x} \right) \psi \left( \frac{x^2}{1-x} \right),
\]

i.e.

\[
\sum_{i=0}^{\infty} \gamma_i x^{i+1} = \frac{d^{2k-1}}{dx^{2k-1}} \left[ \left( \frac{x^{2k}}{1-x} \right) \psi \left( \frac{x^2}{1-x} \right) \right].
\]

Dividing both sides by \( x \) yields Equation (4.2).
If \( k \geq 1 \), then we can rewrite Equation (4.2) as
\[
\sum_{i=0}^{\infty} \gamma_i x^i = \frac{1}{x} \frac{d^{2k-1}}{dx^{2k-1}} \left[ x^{2k-2} \left( \frac{x^2}{1-x} \right) \psi \left( \frac{x^2}{1-x} \right) \right],
\]
and Equation (4.3) holds with \( \rho(y) = y \psi(y) \). Note that the derivative maps the constant term \( \rho(0) \) of \( \rho(y) \) to 0, which makes the last step an equivalence. □

We next have the following, which demonstrates Theorem 1.1 in the case \( r = 1 \).

**Lemma 4.4.** A power series \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \) is an element of \( F_1 \) if and only if the \( \gamma_i \) satisfy
\[
(4.5) \quad \sum_{i=0}^{m} (-1)^i \binom{2m-1}{m-i} \binom{m+i}{i} \gamma_m+i-1 = 0
\]
for each \( m \geq 1 \).

**Proof.** As above, by Equation (3.1), \( \rho(x) \in F_0 \) if and only if
\[
(4.6) \quad \rho \left( \frac{x}{x-1} \right) = \rho(x).
\]
Differentiating yields
\[
(4.7) \quad \rho' \left( \frac{x}{x-1} \right) = -(1-x)^2 \rho'(x).
\]

Let \( \varphi(x) \in F_1 \), and then there is a \( \rho(x) \in F_0 \) such that \( \varphi(x) = \rho(x)/(x-2) \) by Theorem 3.3. Define
\[
(4.8) \quad \chi(x) = (1-x) \frac{d}{dx} \left[ x \varphi(x) \right],
\]
and then we may express
\[
\varphi(x) = \frac{1}{x} \int_0^x \frac{1}{1-\xi} \chi(\xi) \, d\xi,
\]
where the integral is the formal integral of power series, interpreted term by term. Substituting \( \varphi(x) = \rho(x)/(x-2) \) into Equation (4.8) and applying the product rule, we have
\[
\chi(x) = (1-x) \left[ \frac{x}{x-2} \rho'(x) - \frac{2}{(x-2)^2} \rho(x) \right],
\]
from which a simple computation using Equations (4.6) and (4.7) demonstrates that \( \chi(x/(x-1)) = \chi(x) \) so that \( \chi(x) \in F_0 \). Then by Theorem 3.3 it follows that there is a formal power series \( \psi(y) \in K[y] \) such that \( \chi(x) = \psi(x^2/(1-x)) \) and hence
\[
\varphi(x) = \frac{1}{x} \int_0^x \frac{1}{1-\xi} \psi \left( \frac{\xi^2}{1-\xi} \right) \, d\xi.
\]
This is precisely Equation (4.2) for the case \( k = 0 \), so by Lemma 4.3 the \( \gamma_i \) satisfy Equation (4.5).
Conversely, suppose the $\gamma_i$ satisfy Equation (4.5) so that by Lemma 4.3 and Theorem 3.3 there is a $\chi(x) \in \mathcal{F}_0$ such that

$$\varphi(x) = \frac{1}{x} \int_0^x \frac{1}{1 - \xi} \chi(\xi) \, d\xi.$$ 

Let

$$\rho(x) = (x - 2) \varphi(x) = \frac{x - 2}{x} \int_0^x \frac{1}{1 - \xi} \chi(\xi) \, d\xi.$$ 

By Theorem 3.3 $\chi(x) = \chi(x/(x - 1))$ so that using the substitution $\zeta = \xi/(1 - \xi)$ we have

$$\rho \left( \frac{x}{x - 1} \right) = -\frac{x - 2}{x} \int_0^x \frac{1}{1 - \xi} \chi(\xi) \, d\xi = -\frac{x - 2}{x} \int_0^x \frac{1}{1 - \xi} \chi \left( \frac{\xi}{\xi - 1} \right) \, d\xi$$

$$= \frac{x - 2}{x} \int_0^x \frac{1}{1 - \zeta} \chi(\zeta) \, d\zeta = \rho(x).$$ 

Hence $\rho(x) \in \mathcal{F}_0$, and hence $\varphi(x) \in \mathcal{F}_1$ by Theorem 3.4. \qed

To proceed, we will need to establish the following identity. We will use $k(r) = k(k + 1)(k + 2) \cdots (k + r - 1)$ to denote the rising Pochhammer symbol and $(k)_r = k(k - 1)(k - 2) \cdots (k - r + 1)$ to denote the falling Pochhammer symbol.

**Lemma 4.5.** Let $r$ be a positive integer and let $k, a \in \mathbb{K}$. Then

$$\frac{d^{r+1}}{dx^{r+1}} \left[ x^r \left( \frac{x}{x-a} \right)^k \right] = (-a)^{r+1} k^{(r+1)} \frac{x^{k-1}}{(x-a)^{k+r+1}}.$$ 

**Proof.** We first use the binomial series to rewrite

$$x^r \left( \frac{x}{x-a} \right)^k = x^r \left( 1 - \frac{a}{x} \right)^{-k}$$

$$= x^r \sum_{i=0}^{\infty} \binom{-k}{i} \left( -\frac{a}{x} \right)^i$$

$$= \sum_{i=0}^{\infty} \binom{-k}{i} (-a)^i x^{r-i}.$$
Differentiating, we express the left-hand side of Equation (4.9) as
\[
\frac{d^{r+1}}{dx^{r+1}} \left[ x^r \left( \frac{x}{x - a} \right)^k \right] = \frac{d^{r+1}}{dx^{r+1}} \left[ \sum_{i=0}^{\infty} \binom{-k}{i} (-a)^i x^{r-i} \right]
\]
\[
= \frac{d^{r+1}}{dx^{r+1}} \left[ \sum_{i=r+1}^{\infty} \binom{-k}{i} (-a)^i x^{r-i} \right]
\]
\[
= \frac{d^{r+1}}{dx^{r+1}} \left[ \sum_{i=0}^{\infty} \binom{-k}{i} (-a)^i x^{r-i-1} \right]
\]
\[
= \sum_{i=0}^{\infty} \binom{-k}{i + r + 1} (-a)^{i+r+1} (1)^{r+1}(i + 1) r! x^{-i-r-2}
\]
\[
= \sum_{i=0}^{\infty} \frac{(-k)^{r+1}}{i!} (-a)^{r+1} x^{-i-r-2}
\]
\[
= (a)^{r+1}k^{r+1}x^{-r-2}\sum_{i=0}^{\infty} \binom{-k-r-1}{i} (-a)^i x^{-i}
\]
\[
= (a)^{r+1}k^{r+1}x^{-r-2}(x - a)^{-k-r-1}
\]
\[
= (a)^{r+1}k^{r+1}x^{-r-2}(x - a)^{-k-r-1}.
\]
\[\square\]

With this, we now prove the following, which demonstrates Theorem 1.1 when \( r > 1 \) is odd, i.e. \( r = 2k + 1 \) for \( k \geq 1 \).

**Lemma 4.6.** Let \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in \mathbb{K}[x] \), and let \( k \geq 1 \) be an integer. Then the \( \gamma_i \) satisfy Equation (4.11), equivalently (1.6) with \( r = 2k + 1 \), if and only if \( \varphi(x) \in \mathcal{F}_{2k+1} \).

**Proof.** As \( k \geq 1 \), we have by Lemma 4.3 that the \( \gamma_i \) satisfy Equation (4.11) if and only if there is a power series \( \rho(y) \in \mathbb{K}[y] \) satisfying Equation (4.13). Using Theorem 4.3 and the generator given in Equation (3.2), given \( \rho(y) \), there is a power series \( \chi(y) = \sum_{i=0}^{\infty} \delta_i y^i \in \mathbb{K}[y] \) such that \( \chi((x/(x - 2))^2) = \rho(x^2/(1 - x)) \in \mathcal{F}_0 \). That is, the \( \gamma_i \) satisfy Equation (4.11) if and only if
\[
\sum_{i=0}^{\infty} \gamma_i x^i = \frac{1}{x} \cdot \frac{d^{2k-1}}{dx^{2k-1}} \left[ x^{2k-2} \chi \left( \frac{x}{x - 2} \right)^2 \right]
\]
\[
= \frac{1}{x} \cdot \frac{d^{2k-1}}{dx^{2k-1}} \left[ x^{2k-2} \sum_{i=0}^{\infty} \delta_i \left( \frac{x}{x - 2} \right)^{2i} \right]
\]
\[
= \frac{1}{x} \sum_{i=0}^{\infty} \delta_i \cdot \frac{d^{2k-1}}{dx^{2k-1}} \left[ x^{2k-2} \left( \frac{x}{x - 2} \right)^{2i} \right].
\]
Applying Lemma 4.7 to each term and noting that the term \( i = 0 \) vanishes, we continue

\[
\frac{1}{x} \sum_{i=1}^{\infty} \delta_i (-2)^{2k-1} (2i)^{(2k-1)} \frac{x^{2i-1}}{(x-2)^{2i+2k-1}} = \frac{1}{(x-2)^{2k+1}} \sum_{i=1}^{\infty} \delta_i (-2)^{2k-1} (2i + 2k - 2)! \left( \frac{x}{x-2} \right)^{2i-2} = \frac{1}{(x-2)^{2k+1}} \sum_{i=0}^{\infty} \delta_{i+1} (-2)^{2k-1} (2i + 2k)! \left( \frac{x}{x-2} \right)^{2i}.
\]

By Theorem 3.3 along with the generator of Equation (3.2), the factor that is a power series in \((x/(x-2))^2\) is an element of \(\mathcal{F}_0\) so that by Theorem 3.4, this expression is an element of \(\mathcal{F}_{2k+1}\), completing the proof.

We next consider the case of \( r < 0 \) odd with the assumption that the first \( 1 - r \) coefficients vanish.

**Lemma 4.7.** Let \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in \mathbb{K}[x] \) and let \( r \) be an odd integer. If \( r < 0 \), assume that \( \gamma_i = 0 \) for each \( i \leq -r \). Then \( \varphi \in \mathcal{F}_r \) if and only if \( \varphi(x) := x^{r-1} \varphi(x) \in \mathcal{F}_1 \).

**Proof.** As in the proof of Lemma 4.1, \( \varphi(x) \) is a power series in each case due to the assumption that the first \( \gamma_i \) vanish. A simple computation using the fact that \( r \) is odd demonstrates that \( \varphi(x/(x-1)) = (1-x)^r \varphi(x) \) if and only if \( \varphi(x/(x-1)) = (1-x)\varphi(x) \). □

**Corollary 4.8.** Let \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in \mathbb{K}[x] \) and let \( r \) be an odd negative integer. Assume that \( \gamma_i = 0 \) for each \( i \leq -r \). Then \( \varphi \in \mathcal{F}_r \) if and only if the \( \gamma_i \) satisfy Equation (1.8).

**Proof.** Define \( \varphi_i(x) = x^{r-1} \varphi_i(x) = \sum_{i=0}^{\infty} \gamma_i x^i \). Specializing Equation (1.6) for the \( \varphi_i \) to the case \( r = 1 \), we have

\[
\sum_{i=0}^{m} (-1)^i \binom{2m-1}{m-i} \binom{m+i}{i} \varphi_{m+i} = 0, \quad m \geq 1.
\]

Using \( \gamma_i = \gamma_{i-r+1} \) and multiplying both sides of the equation by the constant \( m!(m-1)!/2m! \), this is equivalent to Equation (1.8), which along with Lemma 4.7 completes the proof. □

With this, Theorem 4.1 follows when, in the case \( r < 0 \), it is assumed that \( \gamma_i = 0 \) for \( i \leq -r \). However, note that each \( \mathcal{F}_r \) is obviously closed under addition, and that a series \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \) is an element of \( \mathcal{F}_r \) if it is of the form \( \varphi(x) = \sum_{i=0}^{\infty} \delta_i x^{2i} (x-2)^{-r-2i} \) for \( \delta_i \in \mathbb{K} \), cf. Equation (3.4). In particular, \( \varphi(x) = \varphi_1(x) + \varphi_2(x) \) decomposes naturally with

\[
\varphi_1(x) = \sum_{i=0}^{r} \gamma_i x^i = \sum_{i=0}^{[-r/2]} \delta_i x^{2i} (x-2)^{-r-2i} \in \mathcal{F}_r
\]

a polynomial of degree at most \(-r\) and

\[
\varphi_2(x) = \sum_{i=1-r}^{\infty} \gamma_i x^i = \sum_{i=\lceil -r/2 \rceil + 1}^{\infty} \delta_i x^{2i} (x-2)^{-r-2i} \in \mathcal{F}_r.
\]
The same observation holds for power series satisfying the relations given in Theorem 1.1. Thus, for the remainder of the proof, it is sufficient to demonstrate that, when \( r < 0 \), a polynomial of degree at most \(-r\) is an element of \( F_r \) if and only if its coefficients satisfy Equation (1.7) for \( 1 \leq m \leq \lceil -r/2 \rceil \).

4.2. Equation (1.7). We now fix \( r < 0 \) and let \( f(x) = \sum_{i=0}^{-r} \gamma_i x^i \) be a polynomial of degree at most \(-r\). The goal of this section is to demonstrate that \( f(x) \in F_r \) if and only if the \( \gamma_i \) satisfy Equation (1.7) for \( 1 \leq m \leq \lceil -r/2 \rceil \).

Using the description of \( F_r \) given by Theorems 3.3 and 3.4 and the generator given in Equation (3.2), we reformulate this statement into the following.

**Proposition 4.9.** Assume \( r < 0 \) and let \( f(x) = \sum_{i=0}^{-r} \gamma_i x^i \). Then the \( \gamma_i \) satisfy Equation (1.7) for \( 1 \leq m \leq \lceil -r/2 \rceil \) if and only if we may express

\[
(4.10) \quad f(x) = (x - 2)^{-r} \sum_{i=0}^{\lfloor -r/2 \rfloor} \delta_i \left( \frac{x}{x-2} \right)^{2i} = x^{-r} \sum_{i=0}^{\lfloor -r/2 \rfloor} \delta_i \left( \frac{x-2}{x} \right)^{-r-2i}
\]

for \( \delta_i \in K \).

To prove Proposition 4.9, we will first need an auxiliary result.

**Lemma 4.10.** For \( m, n \geq 0 \), we have

\[
(4.11) \quad \sum_{i=2n-m+1}^{m} \binom{m}{i} \frac{(m+i)!}{(m+i-2n-1)!} (-2)^{-i} = 0.
\]

**Proof.** Differentiating

\[
(x^2 + x)^m = (x + 1)^m x^m = \sum_{i=0}^{m} \binom{m}{i} x^{m+i}
\]

yields

\[
\frac{d^{2n+1}}{dx^{2n+1}} [(x^2 + x)^m] = \sum_{i=0}^{m} \binom{m}{i} (m+i)_{2n+1} x^{m-2n+i-1}
\]

\[
= \sum_{i=2n-m+1}^{m} \binom{m}{i} \frac{(m+i)!}{(m+i-2n-1)!} x^{m-2n+i-1}
\]

and

\[
(4.12) \quad \sum_{i=2n-m+1}^{m} \binom{m}{i} \frac{(m+i)!}{(m+i-2n-1)!} x^i = x^{2n-m+1} \frac{d^{2n+1}}{dx^{2n+1}} [(x^2 + x)^m].
\]

Note that

\[
(x^2 + x)^m = \left[ \frac{(2x + 1)^2 - 1}{4} \right]^m
\]

is an even function in \((2x+1)\). Thus, the derivative on the right side of Equation (4.12) is an odd function in \((2x+1)\) and must vanish when substituting \( x = -1/2 \). The result is Equation (4.11). \( \square \)

With this, we are ready to prove the main result of this subsection.
Hence, for \( V \) Equation (1.7) for \( 1 \leq g(x, \gamma) \leq 4.13 \) for each case of the parity and sign of \( \gamma \). Let Corollary 5.1. Proposition 4.9. Substituting Equation (4.14) into Equation (1.7) and applying Lemma 4.10 yields the following corollary to Theorem 1.1, which is easily proven by induction on \( m \) of their coefficients. The initial observation motivating these characterizations is a unique choice of \( V \) vector space \( V \) polynomials with \( \dim V = [-r/2] + 1 \). Note also that Equation (4.10) describes a vector space \( V' \) of polynomials with \( \dim V' = [-r/2] + 1 \) and basis \( g_s(x) = (x - 2)^{-r-2s} x^{2s}, \ 0 \leq s \leq [-r/2] \).

Hence, for \( V = V' \), it is sufficient to show that each of these polynomials satisfies Equation (1.7) for \( 1 \leq m \leq [-r/2] \).

For
\[
g_s(x) = (x - 2)^{-r-2s} x^{2s} = \sum_{j=0}^{r-2s} \binom{r-2s}{j} (-2)^{r-2s-j} x^{2s+j} = \sum_{i=0}^{r} \gamma_i x^i
\]
we have
\[
\gamma_i = \begin{cases} 0, & i < 2s, \\ \frac{(-1)^{i-2s}}{(i-2s)!} (-2)^{-r-i}, & i \geq 2s. \end{cases}
\]
Substituting Equation (4.14) into Equation (1.7) and applying Lemma 4.10 yields
\[
\sum_{i=0}^{m} \binom{m}{i} \binom{1-r}{m+i}^{-1} \gamma_{m+i-1}
= \sum_{i=2s-m+1}^{m} \binom{m}{i} \binom{1-r}{m+i}^{-1} (-2)^{-r-m-i+1}
= \sum_{i=2s-m+1}^{m} \frac{(m+i)!(1-r-m-i)!(-r-2s)!}{(1-r)!(m+i-2s-1)!(r-m-i+1)!} (-2)^{-r-m-i+1}
= \frac{(-r-2s)!}{(1-r)!} (-2)^{-r-m+1} \sum_{i=2s-m+1}^{m} \binom{m}{i} \frac{(m+i)!}{(m+i-2s-1)!} (-2)^{-i} = 0. \quad \square
\]

Theorem 1.1 now follows from Corollaries 4.2 and 4.8, Lemmas 4.4 and 4.6, and Proposition 4.9.

5. Relation to Euler Polynomials and Bernoulli Numbers

In this section, we give two additional characterizations of elements of \( F_r \) in terms of their coefficients. The initial observation motivating these characterizations is the following corollary to Theorem 1.1 which is easily proven by induction on \( m \) for each case of the parity and sign of \( r \).

Corollary 5.1. Let \( r \) be an integer. For each arbitrary choice \( \{\gamma_{2i}\}_{i=0}^{\infty} \subset K \), there is a unique choice of \( \gamma_{2i+1} \in K \) for each \( i \) such that \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in F_r \).

Hence, for an element of \( F_r \), the \( \gamma_{2i+1} \) are completely determined by the \( \gamma_{2i} \). The goal of this section is to indicate explicitly how to compute the odd-degree coefficients from the even, Theorem 5.3 and similarly how to compute the even-degree coefficients from the odd, Theorem 5.9. These pleasant relationships are stated most succinctly in terms of the Bernoulli numbers and the coefficients of the Euler polynomials. First, we recall the definitions of these constants and establish the properties we will need.
For $n = 0, 1, 2, \ldots$, let $E_n(x)$ denote the $n$th Euler polynomial defined by the generating function
\begin{equation}
\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\end{equation}

As an obvious consequence of this equation, we have
\begin{equation}
E_n(x + 1) + E_n(x) = 2x^n.
\end{equation}

Observe that when $n$ is even (respectively odd), the only nonzero even degree (respectively odd degree) term in $E_n(x)$ is the leading monomial $x^n$. We use the notation $\binom{n}{i}$ to denote the (negatives of the) odd degree coefficients of $E_{2n}(x)$ and $\binom{n}{i}$ to denote the (negatives of the) even degree coefficients of $E_{2n+1}(x)$, i.e.
\begin{equation}
\sum_{i=0}^{n} \binom{n}{i} x^{2i-1} = x^{2n} - E_{2n}(x) \quad \text{and} \quad \sum_{i=0}^{n} \binom{n}{i} x^{2i} = x^{2n+1} - E_{2n+1}(x).
\end{equation}

Note that the $\binom{n}{i}$ are integers while the $\binom{n}{i}$ are generally rational, $\binom{n}{i} = 0$ for $i \leq 0$ or $i > n$, and $\binom{n}{i} = 0$ for $i < 0$ or $i > n$. Due to this and other differences between the properties of the $\binom{n}{i}$ and $\binom{n}{i}$, it will usually be simplest to distinguish the even and odd cases with this notation. However, it will sometimes be convenient to use the unified notation
\begin{equation}
\binom{n}{i}_r := \begin{cases} 
\binom{r+2n}{r+i}, & r \text{ is even and } n \geq \max\{0, -r/2\}, \\
-\binom{r-2n}{r-i}, & r \text{ is even and } 0 \leq n < -r/2, \\
\binom{r-1+2n}{r+i}, & r \text{ is odd and } n \geq \max\{0, -(r-1)/2\}, \\
\binom{-r+1+2n}{r+i}, & r \text{ is odd and } 0 \leq n < -(r-1)/2. 
\end{cases}
\end{equation}

We use $B_n$ to denote the $n$th Bernoulli number, given by the generating function
\begin{equation}
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.
\end{equation}

The Bernoulli numbers are related to the even Euler polynomial coefficients via
\begin{equation}
\binom{n}{i} = \frac{4^{n-i+1} - 1}{n-i+1} B_{2(n-i+1)} \left( \frac{2n}{2i-1} \right), \quad 0 \leq i \leq n.
\end{equation}

By the obvious consequence $\frac{d}{dx} E_{n+1}(x) = (n+1)E_n(x)$ of Equation (5.1), we have
\begin{equation}
\frac{2i+1}{2n+2} \left( \begin{array}{c} n+1 \\ i+1 \end{array} \right) = \binom{n}{i}, \quad 0 \leq i \leq n,
\end{equation}

and hence
\begin{equation}
\binom{n}{i} = \frac{4^{n-i+1} - 1}{n-i+1} B_{2(n-i+1)} \left( \frac{2n+1}{2i} \right), \quad 0 \leq i \leq n.
\end{equation}

Differentiating twice yields the useful corollary
\begin{equation}
\binom{n}{i} = \frac{(2n+1)(2n-1)}{(2i-1)(2i-2)} \binom{n-1}{i-1} = \frac{(2n)}{(2i-1)} \binom{n-1}{i-1}, \quad 2 \leq i \leq n.
\end{equation}
In order to express the unified notation of Equation (5.4) with the help of Equations (5.5) and (5.7), we define

\[ f_m := \frac{4^m - 1}{m} B_{2m} \]

and observe

\[
\binom{n}{i}_r = f_{n-i+1} \cdot \begin{cases} \binom{r+2n}{r+2i-1}, & n \geq \max\{0, \lfloor -r/2 \rfloor\}, \\ \binom{-r-2n}{-r-2i}, & 0 \leq n < \lfloor -r/2 \rfloor. \end{cases}
\]

We will also make use of the following.

**Lemma 5.2.** For integers \(0 \leq j \leq n\), we have

\[
(5.10) \sum_{i=j}^{n} \binom{2n}{2i} B_{2n-2i} \begin{bmatrix} i \\ j \end{bmatrix} = \begin{cases} 0, & j < n, \\ n, & j = n. \end{cases}
\]

**Proof.** First note that as \(\begin{bmatrix} i \\ 0 \end{bmatrix} = 0\) for all \(i\), the case \(j = 0\) is clear. Similarly, by Equation (5.5) and the fact that \(B_2 = 1/6\), one checks that \(\binom{n}{n}_n = n\) for any \(n \geq 0\). Then the case \(j = n\) follows from a simple computation.

Now, for \(\ell \geq 2\), we have from [12, Theorem 13.3.6] the identity

\[
(5.11) \sum_{s=1}^{\ell} \binom{2\ell}{2s} (2^{2s} - 1) B_{2s} B_{2(\ell-s)} = 0,
\]

see also [13]. Multiplying both sides by a constant in terms of \(j \geq 1\), we have

\[
0 = \frac{2(2\ell + 2j - 2)!}{(2\ell - 1)! (2\ell)!} \sum_{s=1}^{\ell} \binom{2\ell}{2s} (2^{2s} - 1) B_{2s} B_{2(\ell-s)}
\]

\[
= \sum_{s=1}^{\ell} \frac{(2\ell + 2j - 2)! (2s + 2j - 2)! (4^s - 1)}{(2s + 2j - 2)! (2\ell - 2s)! (2\ell - 1)! (2s - 1)! s} B_{2s} B_{2(\ell-s)}
\]

\[
= \sum_{s=1}^{\ell} \binom{2\ell + 2j - 2}{2s + 2j - 2} \binom{2s + 2j - 2}{2j - 1} \frac{4^s - 1}{s} B_{2s} B_{2(\ell-s)}
\]

Setting \(n = \ell + j - 1 \geq j + 1\) and using Equation (5.5), we continue

\[
= \sum_{s=1}^{n-j+1} \binom{2n}{2s + 2j - 2} \binom{2s + 2j - 2}{2j - 1} \frac{4^s - 1}{s} B_{2s} B_{2(n-s-j+1)}
\]

\[
= \sum_{s=1}^{n-j+1} \binom{2n}{2(s + j - 1)} \binom{s + j - 1}{j} B_{2(n-s-j+1)}
\]

and then setting \(i = s + j - 1\) completes the proof. \(\square\)

### 5.1. Odd coefficients from even: Euler polynomials.

In this subsection, we give explicit formulas for the odd coefficients of an element of \(F_r\) in terms of the even coefficients, yielding another characterization of elements of \(F_r\) and indicating the connection with Euler polynomials. This generalizes [10, Theorem 5.1], which gave the corresponding result for the case \(r = 0\). Specifically, we prove the following.
Theorem 5.3. Let \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \) and let \( r \in \mathbb{Z} \). As in Theorem 1.1, we stipulate that \( \gamma_i = 0 \) for \( i < 0 \). Then \( \varphi(x) \in \mathcal{F}_r \) if and only if the following conditions are fulfilled.

If \( r = 2k \) is even, then for each \( n \geq \max\{0, -k\} \),

\[
\gamma_{2n+1} = \sum_{i=-k}^{n} \binom{n+k}{i+k} \gamma_{2i},
\]

and, if \( r < 0 \) so that \( k < 0 \), then for \( 0 \leq n \leq -k-1 \),

\[
\gamma_{2n+1} = \sum_{i=0}^{n} \left\lfloor \frac{-k-i}{-k-n} \right\rfloor \gamma_{2i}.
\]

If \( r = 2k + 1 \) is odd, then for each \( n \geq \max\{0, -k\} \),

\[
\gamma_{2n+1} = \sum_{i=-k}^{n} \left\lfloor \frac{n+k}{i+k} \right\rfloor \gamma_{2i},
\]

and, if \( r < 0 \) so that \( k < 0 \), then for \( 0 \leq n \leq -k-1 \),

\[
\gamma_{2n+1} = \sum_{i=0}^{n} \left\lfloor \frac{-k-i}{-k-n} \right\rfloor \gamma_{2i}.
\]

Remark 5.4. Note that in Equations (5.12) and (5.14), the sum can be taken over all integers \( i \) as the coefficient vanishes unless \( -k \leq i \leq n \). Hence, using the unified notation of Equation (5.14), the conditions expressed by Equations (5.12), (5.13), (5.14), and (5.16) in Theorem 5.3 can be stated succinctly as: for every \( n \geq 0 \),

\[
\gamma_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} r \gamma_{2i}.
\]

To prove Theorem 5.3, we observe that by Corollary 5.1, for any value of \( r \), either direction of the biconditional in Theorem 5.3 follows from the opposite implication. Note further that when \( r < 0 \), just as in the statement of Theorem 1.1, the \( \gamma_i \) for \( i \leq -r \) do not contribute to Equations (5.12) and (5.14), while the \( \gamma_i \) for \( i > -r \) do not appear in Equations (5.13) and (5.15). Hence, as in Section 4, we first assume that in the case \( r < 0 \), the \( \gamma_i \) vanish for \( i \leq -r \). We begin with the case of \( r \) even.

Lemma 5.5. Let \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in \mathbb{K}[x] \) and let \( r = 2k \) be an even integer. If \( r < 0 \), assume that \( \gamma_i = 0 \) for each \( i \leq -r \). Then the \( \gamma_i \) satisfy Equation (5.12) for each \( n \geq \max\{0, -k\} \) if and only if \( \varphi(x) \in \mathcal{F}_r \).

Proof. If \( r = 0 \), then by an application of Theorem 1.1, this result is precisely Theorem 5.1. If \( r \neq 0 \), define \( \varphi(x) = x^r \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^{i+r} \). By Lemma 4.1, if \( \varphi(x) \in \mathcal{F}_r \), then \( \varphi(x) \in \mathcal{F}_0 \), and hence by the above argument for \( r = 0 \), we have that for each \( n \geq 0 \),

\[
\gamma_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} \gamma_{2i}.
\]

The result then follows from Corollary 5.1 and the fact that \( \gamma_i = \gamma_{i-2k} \) for each \( i \geq 0 \).

Lemma 5.6. Let \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in \mathbb{K}[x] \) and let \( r = 2k + 1 \) be an odd integer. If \( r < 0 \), assume that \( \gamma_i = 0 \) for each \( i \leq -r \). Then the \( \gamma_i \) satisfy Equation (5.14) for each \( n \geq \max\{0, -k\} \) if and only if \( \varphi(x) \in \mathcal{F}_r \).
Proof. We recall [10, Equation (5.7)],

\begin{equation}
\sum_{i=0}^{m} (-1)^i \binom{m}{i} \sum_{j} \left( \frac{m+i}{j} \right) x^{2j-1} + \sum_{i=0}^{m} (-1)^i \binom{m}{i} x^{m+i} = 0,
\end{equation}

which was derived using an application of [8, Theorem 7.4]. For \( r = 1 \), we differentiate Equation (5.17) once and apply Equation (5.6), yielding

\begin{align*}
\sum_{i=0}^{m} (-1)^i \binom{m}{i} (m+i) \sum_{j} \left( \frac{m+i}{j} \right) x^{2j-2} \\
\sum_{i=0}^{m} (-1)^i \binom{m}{i} (m+i)x^{m+i-1} = 0.
\end{align*}

We interpret \( x \) as an umbral variable [8] and define the functional \( \Gamma: \mathbb{K}[x] \rightarrow \mathbb{K} \) by

\[ \Gamma(x^i) = \gamma_i. \]

Applying \( \Gamma \) to the last equation results in

\begin{align*}
\sum_{i=0}^{m} (-1)^i \binom{m}{i} (m+i) \sum_{j} \left( \frac{m+i}{j} \right) \gamma_{2j-2} \\
\sum_{i=0}^{m} (-1)^i \binom{m}{i} (m+i) \gamma_{m+i-1} = 0.
\end{align*}

Using the fact that the \( \gamma_i \) satisfy Equation (5.14) for \( \kappa = 0 \), we rewrite the sum over \( j \) as

\[ \sum_j \left( \frac{m+i}{j} \right) \gamma_{2j-2} = \sum_j \left( \frac{m+i}{j} \right) \gamma_{2j} = \gamma_{m+i-1}, \]

yielding

\[ \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m+i) \gamma_{m+i-1} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{(m+i)!}{(m+i-1)!} \gamma_{m+i-1} = 0. \]

This is Equation (4.4) for \( \kappa = 0 \), which was demonstrated in Lemma 4.3 to be equivalent to Equation (1.6), and hence by Theorem 1.1 and Corollary 5.1, the result holds for \( r = 1 \).

Now assume \( r \neq 1 \). By Lemma 4.7, \( \overline{\varphi}(x) = x^{r-1} \varphi(x) = \sum_{i=0}^{\infty} \overline{\gamma}_i x^i \) is an element of \( \mathcal{F}_1 \) if and only if \( \varphi(x) \in \mathcal{F}_r \). By the above argument for the case \( r = 1 \), we have that for each \( n \geq 0 \)

\[ \overline{\gamma}_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} \overline{\gamma}_{2i}, \]

which, applying \( \overline{\gamma}_i = \gamma_{i-2k} \), is equivalent to Equation (5.14) for the \( \gamma_i \).

We now consider the case of the first \( 1-r \) terms when \( r < 0 \). Similar to the proof of Proposition 4.9 we start with an auxiliary result.

**Lemma 5.7.** For \( n \geq 0 \), we have

\begin{equation}
\sum_{i=0}^{n} \binom{n}{i} (-2)^{-2i} = (-2)^{-2n-1}
\end{equation}

Proof. Substituting \( x = -1/2 \) into Equations (5.2) and (5.3) yields

\[
\sum_{i=0}^{n} \binom{n}{i} (-2)^{-2i} = -\frac{1}{2} \left[ \sum_{i=0}^{n} \binom{n}{i} \left( \frac{1}{2} \right)^{2i} + \sum_{i=0}^{n} \binom{n}{i} \left( -\frac{1}{2} \right)^{2i} \right] \\
= -\frac{1}{2} \left[ \left( \frac{1}{2} \right)^{2n+1} - E_{2n+1} \left( \frac{1}{2} \right) + \left( -\frac{1}{2} \right)^{2n+1} - E_{2n+1} \left( -\frac{1}{2} \right) \right] \\
= \frac{1}{2} \left[ E_{2n+1} \left( -\frac{1}{2} + 1 \right) + E_{2n+1} \left( -\frac{1}{2} \right) \right] = \left( -\frac{1}{2} \right)^{2n+1}.
\]

\( \square \)

With this, we are ready to prove the following.

**Lemma 5.8.** Assume \( r < 0 \) and let \( f(x) = \sum_{i=0}^{-r} \gamma_i x^i \). Then for \( 0 \leq n \leq -k - 1 \), the \( \gamma_i \) satisfy Equation (5.13) when \( r = 2k \) is even and Equation (5.15) when \( r = 2k + 1 \) is odd if and only if \( f(x) \in \mathcal{F}_r \).

**Proof.** Using the description of \( \mathcal{F}_r \) given by Theorems 3.3 and 3.4 and the generator given in Equation (3.2), \( f(x) \in \mathcal{F}_r \) if and only if we can express

\[
(5.19) \quad f(x) = (x - 2)^{-r} \sum_{i=0}^{\lfloor -r/2 \rfloor} \delta_i \left( \frac{x}{x - 2} \right)^{2i} = x^{-r} \sum_{i=0}^{\lfloor -r/2 \rfloor} \delta_i \left( \frac{x - 2}{x} \right)^{r-2i}.
\]

We will follow the proof of Proposition (4.9) and consider two cases.

If \( r = 2k \) is even, it is easy to check that for each arbitrary choice of values \( \gamma_{2i} \) \((0 \leq i \leq -k)\), there is a unique \( f(x) = \sum_{i=0}^{-r} \gamma_i x^i \) satisfying Equation (5.13) for \( 0 \leq n \leq -k - 1 \). In particular, Equation (5.13) describes a vector space \( V \) of polynomials with \( \dim V = -k + 1 \). Note also that Equation (5.19) describes a vector space \( V' \) of polynomials with \( \dim V' = -k + 1 \) and basis

\[
g_s(x) = (x - 2)^{-r-2s} x^{2s}, \quad 0 \leq s \leq -k.
\]

Hence, for \( V = V' \), it is sufficient to show that each of these polynomials satisfies Equation (5.13) for \( 0 \leq n \leq -k - 1 \).

For \( n < s \), Equation (5.13) is trivially satisfied as all entries \( \gamma_i = 0 \), cf. Equation (4.14). For \( s \leq n \leq -k - 1 \), using Equations (4.14), (5.5), (5.7) and (5.18),
and recalling $r = 2k$, the right side of Equation (5.13) can be written

$$
\sum_{i=0}^{n} \left[ \frac{-k-i}{-k-n} \right]_{2i} = \sum_{i=0}^{n} \frac{4^{n-i+1} - 1}{n-i+1} B_{2(n-i+1)} \left( \frac{-2k-2i}{-2k-2n-1} \right)_{2i}
$$

$$
= \sum_{i=s}^{n} \frac{4^{n-i+1} - 1}{n-i+1} B_{2(n-i+1)} \left( \frac{-2k-2i}{-2k-2n-1} \right) \left( \frac{-2k-2s}{2i-2s} \right)(-2)^{-2k-2i}
$$

$$
= \sum_{i=s}^{n} \frac{4^{n-i+1} - 1}{n-i+1} B_{2(n-i+1)} \left( \frac{(-2k-2i)!(-2k-2s)!}{(2n-2i+1)!|(2i-2s)!(-2k-2i)!} \right)
$$

$$
= \frac{(-2k-2s)!(-2)^{-2k-2s}}{(2n-2s+1)!(-2k-2n-1)!} \sum_{i=s}^{n} \frac{4^{n-i+1} - 1}{n-i+1} B_{2(n-i+1)} \left( \frac{2n-2s+1}{2i-2s} \right)(-2)^{2s-2i}
$$

$$
= \frac{(-2k-2s)}{(2n-2s+1)!(-2k-2n-1)!} \sum_{i=s}^{n} \left\{ \frac{n-s}{i-s} \right\}(-2)^{2s-2i}
$$

$$
= \frac{(-2k-2s)}{(2n-2s+1)!} \sum_{i=0}^{n-s} \left\{ \frac{n-s}{i} \right\}(-2)^{-2i}
$$

$$
= \frac{(-2k-2s)}{(2n-2s+1)!} (-2)^{-2k-2s}(-2)^{-2(n-s)-1} = \frac{(-2k-2s)}{(2n-2s+1)}(-2)^{-2k-2n-1}
$$

$$
= \gamma_{2n+1}.
$$

If $r = 2k+1$ is odd, we have $\dim V = \dim V' = -k$ and need to check that

$$
g_s(x) = (x-2)^{-r-2s}x^{2s}, \quad 0 \leq s \leq -k-1
$$

satisfies Equation (5.15) for $0 \leq n \leq -k-1$.

For $s \leq n \leq -k-1$, using Equations (4.14), (5.7) and (5.18), and recalling $r = 2k+1$, the right side of Equation (5.15) can be written

$$
\sum_{i=0}^{n} \left\{ \frac{-k-i-1}{-k-n-1} \right\}_{2i} = \sum_{i=0}^{n} \frac{4^{n-i+1} - 1}{n-i+1} B_{2(n-i+1)} \left( \frac{-2k-2i-1}{-2k-2n-2} \right)_{2i}
$$

$$
= \sum_{i=s}^{n} \frac{4^{n-i+1} - 1}{n-i+1} B_{2(n-i+1)} \left( \frac{-2k-2i-1}{-2k-2n-2} \right) \left( \frac{-2k-2s-1}{2i-2s} \right)(-2)^{-2k-2i-1}
$$

$$
= \frac{(-2k-2s-1)}{(2n-2s+1)} \sum_{i=s}^{n} \frac{4^{n-i+1} - 1}{n-i+1} B_{2(n-i+1)} \left( \frac{2n-2s+1}{2i-2s} \right)(-2)^{2s-2i}
$$

$$
= \frac{(-2k-2s-1)}{(2n-2s+1)} (-2)^{-2k-2s-1}(-2)^{-2(n-s)-1} = \frac{(-2k-2s-1)}{(2n-2s+1)}(-2)^{-2k-2n-2}
$$

$$
= \gamma_{2n+1}.
$$

Theorem 5.3 now follows from Lemmas 5.5, 5.6 and 5.8.

5.2. Even coefficients from odd: Bernoulli numbers. In this subsection, we give another characterization of elements of $F_r$ by exploring whether the even degree coefficients are determined by the odd. The results in this direction are not as straightforward as those of Theorem 5.3; there are for some values of $r$ an additional
constraint and an unconstrained coefficient. However, we again find an appealing characterization, this time most easily expressed in terms of the Bernoulli numbers. Specifically, we have the following.

**Theorem 5.9.** Let \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \) and let \( r \in \mathbb{Z} \). As above, we stipulate that \( \gamma_i = 0 \) for \( i < 0 \). Then \( \varphi(x) \in \mathcal{F}_r \) if and only if the following conditions are fulfilled.

If \( r = 2k \) is even, then for each \( n \geq \max\{0, 1 - k\} \)

\[
\gamma_{2n} = \frac{2}{2n + r} \sum_{i=-k}^{n} \binom{2n + r}{2i + r} B_{2n-2i} \gamma_{2i+1},
\]

and, if \( r \leq 0 \), then \( \gamma_{r} \) is unconstrained, \( \gamma_{1-r} = 0 \), and (when \( r < 0 \))

\[
\gamma_{2n} = \sum_{i=0}^{n} \frac{2}{2i + r} \binom{-2i - r}{-2n - r} B_{2n-2i} \gamma_{2i+1}
\]

holds for \( 0 \leq n \leq -k - 1 \). This completely determines the \( \gamma_{2i} \) in terms of the \( \gamma_{2i+1} \).

If \( r = 2k+1 \) is odd, then Equation (5.20) holds for each \( n \geq \max\{0, -k\} \), and, if \( r < 0 \), then Equation (5.21) holds for \( 0 \leq n \leq -k - 1 \). This completely determines the \( \gamma_{2i} \) in terms of the \( \gamma_{2i+1} \).

**Proof.** We first observe that, by an induction argument similar to that described in Corollary 5.1, an arbitrary choice of \( \{\gamma_{2i+1}\}_{i=0}^{\infty} \subset K \) uniquely determines an element of \( \mathcal{F}_r \) except when \( r = 2k \leq 0 \), in which case any choice of \( \{\gamma_{2i+1}\}_{i=0}^{\infty} \subset K \) such that \( \gamma_{1-r} = 0 \) along with an arbitrary choice of \( \gamma_{-r} \) uniquely determines an element of \( \mathcal{F}_r \). Hence, it is sufficient to show that the coefficients of any element of \( \mathcal{F}_r \) satisfy Equations (5.20) and (5.21) as described in the statement of the theorem.

So assume \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in \mathcal{F}_r \) for some \( r \). The idea of the proof in each case is to use the characterization given in Theorem 5.3 to rewrite the right side of Equation (5.20) or (5.21) in a form that can be simplified using Lemma 5.2.

Assume \( r = 0 \) and then, using \( \gamma_1 = 0 \) and Equation (5.12), the right side of Equation (5.20) with \( n \geq 1 \) becomes

\[
\frac{1}{n} \sum_{i=1}^{n} \binom{2n}{2i} B_{2n-2i} \gamma_{2i+1} = \frac{1}{n} \sum_{i=1}^{n} \binom{2n}{2i} B_{2n-2i} \sum_{j=0}^{i} \binom{i}{j} \gamma_{2j} = \frac{1}{n} \sum_{j=0}^{n} \gamma_{2j} \sum_{i=j}^{n} \binom{2n}{2i} B_{2n-2i} \binom{i}{j}.
\]

Applying Lemma 5.2 to the sum over \( i \), this expression is equal to \( \gamma_{2n} \).

If \( r = 2k > 0 \), then \( \varphi(x) = x^r \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in \mathcal{F}_0 \) by Lemma 4.1. Hence by the above argument, the \( \gamma_i \) satisfy Equation (5.20) for \( \mathcal{F}_0 \) and any \( n \geq 1 \), and hence using \( \gamma_i = \gamma_{i-2k} \), the \( \gamma_i \) satisfy Equation (5.20) for \( \mathcal{F}_r \) and any \( n \geq 0 \). If \( r = 2k < 0 \), then we apply the same argument assuming \( \gamma_0 = \gamma_1 = \ldots = \gamma_{-r} = 0 \) to see that Equation (5.20) for \( \mathcal{F}_r \) is satisfied for any \( n \geq 1 - k \).

Now assume \( r = 1 \). When \( n = 0 \), Equation (5.20) simply states \( \gamma_0 = 2\gamma_1 \), which, as \( \{0\} = 1/2 \), is equivalent to the same case of Equation (5.14). So assume \( n \geq 1 \), and then we use Equation (5.14) to express the right side of Equation (5.20) as

\[
\frac{2}{2n + 1} \sum_{i=0}^{n} \binom{2n+1}{2i+1} B_{2n-2i} \gamma_{2i+1} = \frac{2}{2n + 1} \sum_{i=0}^{n} \binom{2n+1}{2i+1} B_{2n-2i} \sum_{j=0}^{i} \binom{i}{j} \gamma_{2j}.
\]
Applying Equation (5.6) to rewrite \( \{1 \choose j \} \) and exchanging the sums yields

\[
(5.22) \quad \frac{2}{2n+1} \sum_{j=0}^{n} \gamma_{2j} \sum_{i=j}^{n} \left( \frac{2n+1}{2} \right) \frac{2j+1}{2} \frac{2i+1}{2} \left[ \frac{i}{j} + 1 \right]  
\]

The coefficient of \( \gamma_0 \), corresponding to \( j = 0 \), can be rewritten as

\[
2(2n)! \sum_{i=0}^{n} \left( \frac{2(n+1)}{2} \right) B_{2n-2i} \left[ \frac{i}{1} + 1 \right]
\]

and this sum vanishes by Lemma 5.2. Using this fact and Equation (5.8), we rewrite (5.22) as

\[
2 \sum_{j=1}^{n} \gamma_{2j} \sum_{i=j}^{n} \frac{(2i+2)(2n)!}{(2j+1)(2i+2)(2n-2i)!} B_{2n-2i} \left[ \frac{i}{j} \right] = \sum_{j=1}^{n} \gamma_{2j} \sum_{i=j}^{n} \frac{(2n)!}{2i} B_{2n-2i} \left[ \frac{i}{j} \right]
\]

Again by Lemma 5.2, this is equal to \( \gamma_{2n} \).

If \( r = 2k+1 > 1 \), then \( \varphi(x) = x^{r-1} \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \) \( \in F_1 \) by Lemma 4.7. Hence by the above argument, the \( \gamma_i \) satisfy Equation (5.20) for \( F_1 \) and any \( n \geq 0 \), and hence using \( \gamma_i = \gamma_{i-k} \), the \( \gamma_i \) satisfy Equation (5.20) for \( F_r \) and any \( n \geq 0 \). If \( r = 2k+1 < 0 \), then we apply the same argument assuming \( \gamma_0 = \gamma_1 = \ldots = \gamma_{-r} = 0 \) to see that Equation (5.20) for \( F_r \) is satisfied for any \( n \geq -k \).

With this, we have that every element of \( F_r \) satisfies Equation (5.20) for the appropriate values of \( n \).

Now, we turn to Equation (5.21). First assume \( r = 2k < 0 \) is even. For \( 0 \leq n \leq -k-1 \), using Equation (5.12), the right side of Equation (5.21) can be written

\[
\sum_{i=k}^{n} \frac{1}{-2i-2k} B_{2n-2i} \sum_{j=0}^{i} \left[ \frac{-k-j}{-k-i} \right] \gamma_{2j}.
\]

Exchanging the sums and now using Equation (5.5) to rewrite \( \left[ \frac{-k-j}{-k-i} \right] \) yields

\[
- \sum_{j=0}^{n} \gamma_{2j} \sum_{i=j}^{n} \frac{1}{i+k} \left( \frac{-2i-2k}{2n-2k} \right) B_{2n-2i} \sum_{j=0}^{i} \left[ \frac{-k-j}{-k-i} \right] \gamma_{2j}.
\]

Expanding the binomial coefficients, simplifying, setting \( \ell = n-j+1 \) and \( s = i-j+1 \), and recalling \( r = 2k \), this expression is rewritten as

\[
(5.23) \quad \sum_{j=0}^{n} \frac{4(-r-2j)!}{(-r-2n)!(2n-2j+2)!} \sum_{s=1}^{\ell} \left( \frac{2\ell}{2s} \right) (2^{2s} - 1) B_{2(\ell-s)} B_{2s}.
\]

By Equation (5.11), the sum over \( s \) vanishes whenever \( \ell \geq 2 \), implying that the only nonzero term occurs when \( \ell = 1 \), i.e. \( j = n \). Then as \( B_0 = 1 \) and \( B_2 = 1/6 \), this expression then simplifies to \( \gamma_{2n} \), completing the proof of Equation (5.21) when \( r \) is even.

Finally, suppose \( r = 2k+1 < 0 \) is odd. For \( 0 \leq n \leq -k-1 \), the right side of Equation (5.21) can be expressed using Equation (5.15) as

\[
\sum_{i=0}^{n} \frac{2}{2i+2k+1} \left( \frac{-2i-2k-1}{-2n-2k-1} \right) B_{2n-2i} \sum_{j=0}^{i} \left[ \frac{-k-j-1}{-k-i-1} \right] \gamma_{2j}.
\]
Following the same steps as in the case of even $r$ except using Equation (5.7) to rewrite $\{-k-j-1\}$, we again obtain Equation (5.23), and hence that this expression is equal to $\gamma_{2n}$.

### 6. Identities for the Bernoulli numbers

In this section, we use the results of Section 5 to derive two families of combinatorial identities. This is achieved by using the $\mathbb{Z}$-graded algebra structure of $\mathcal{F}$ (cf. Proposition 5.2) combined first with the formulation of Theorem 5.3 given in Equation (5.10), then with Theorem 5.9.

In Subsection 6.1 we derive a family of cubic relations for the coefficients $\binom{n}{r}$ defined in Equation (5.9), see Theorem 6.1 below. In Subsection 6.2 we derive a family of binomial relations for the Bernoulli numbers, see Theorem 6.2.

#### 6.1. Identities derived from Theorem 5.3

Let $\varphi(x) := \sum_{i=0}^{\infty} \varphi_i x^i \in \mathcal{F}_r$ and $\psi(x) := \sum_{j=0}^{\infty} \psi_j x^j \in \mathcal{F}_s$. By Theorem 5.3 we have that for each $n \geq 0$,

$$\varphi_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} \varphi_{2i}, \quad \psi_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} s^i \psi_{2i}.$$  

The product $(\varphi \cdot \psi)(x) := \sum_{i=0}^{\infty} \vartheta_i x^i$ is in $\mathcal{F}_{r+s}$ by Proposition 5.2. On the one hand, we have by the Cauchy product formula that

$$\vartheta_{2n+1} = \sum_{\ell=0}^{n} \binom{n}{\ell} \varphi_{2\ell+1} \psi_{2(n-\ell)} + \varphi_{2(n-\ell)} \psi_{2\ell+1}$$

$$= \sum_{\ell=0}^{n} \sum_{k=0}^{\ell} \binom{\ell}{k} \varphi_{2k} \psi_{2(n-\ell)} + \binom{\ell}{k} \varphi_{2(n-\ell)} \psi_{2k}$$

$$= \sum_{\alpha, \beta \geq 0} \left( \binom{n-\beta}{\alpha} + \binom{n-\alpha}{\beta} \right) \varphi_{2\alpha} \psi_{2\beta}.$$  

On the other hand, $(\varphi \cdot \psi)(x) \in \mathcal{F}_{r+s}$ and Equation (5.10) imply

$$\vartheta_{2n+1} = \sum_{\alpha, \beta \geq 0} \binom{n}{\alpha + \beta} \varphi_{2\alpha} \psi_{2\beta} + \sum_{k=0}^{n} \sum_{\ell \geq 0} \sum_{m \geq 0} \binom{n}{k} \binom{\ell}{\alpha} \binom{m}{\beta} \varphi_{2\alpha} \psi_{2\beta}.$$  

Noticing that we can choose the even coefficients $\varphi_{2\alpha}$ and $\psi_{2\beta}$ freely, we compare coefficients and derive the following.

**Theorem 6.1.** For $n, \alpha, \beta \geq 0$ and $r, s \in \mathbb{Z}$ the following identity holds

$$(\alpha)_{r-s} + (\beta)_{s} = \binom{n}{\alpha + \beta} + \sum_{k=0}^{n} \sum_{\ell, m \geq 0} \binom{n}{k} \binom{\ell}{r+s} (\alpha)_{r} (\beta)_{s}.$$  

In view of Equation (5.20), the above relations can be seen as cubic relations for the even indexed Bernoulli numbers. These generalize relations encountered in [10 Corollary 5.2] for the case $r = s = 0$. 

6.2. Identities derived from Theorem 5.9

In a similar fashion, we now derive identities from Theorem 5.9. To this end we introduce the following notation. For \( n \neq -(r/2) \) and \( 0 \leq i \leq n \), we define

\[
\begin{align*}
[n]_r = & \begin{cases} 
\left(\frac{2}{2n+r} \right) B_{2(n-i)}, & n \geq \max\{0, \lfloor -r/2 \rfloor + 1\}, \\
\frac{1}{2} \left(\frac{2}{-2n+r} \right) B_{2(n-n-i)}, & 0 \leq n \leq \lfloor -(r+1)/2 \rfloor.
\end{cases}
\end{align*}
\]

With this, we can restate Theorem 5.9 as follows: A power series \( \varphi(x) = \sum_{i=0}^{\infty} \gamma_i x^i \) is an element of \( F_r \) if and only if

\[
\gamma_{2n} = \sum_{i=0}^{n} \binom{n}{i}_r \gamma_{2i+1} \quad \text{for all } n \geq 0 \text{ with } n \neq -(r/2)
\]

and, in the case \( r \leq 0 \) is even, \( \gamma_1 = 0 \).

Again we investigate the fact that the product \( (\varphi \cdot \psi)(x) = \sum_{i=0}^{\infty} \varphi_i x^i \in F_r \) and \( \psi(x) := \sum_{j=0}^{\infty} \psi_j x^j \in F_s \) is an element of \( F_{r+s} \). For simplicity, we assume that \( r \) and \( s \) are both elements of the set of positive or odd integers. We observe that

\[
\vartheta_{2n} = \sum_{\ell, m \geq 0, \ell + m = n-1} \varphi_{2\ell+1} \psi_{2m+1} + \sum_{\ell, m \geq 0, \ell + m = n} \varphi_{2\ell} \psi_{2m}
\]

\[
= \sum_{\ell, m \geq 0, \ell + m = n-1} \varphi_{2\ell+1} \psi_{2m+1} + \sum_{\ell, m \geq 0, \ell + m = n} \sum_{i=0}^{m} \binom{\ell}{i}_r \binom{m}{j}_s \varphi_{2i+1} \psi_{2j+1}
\]

\[
= \sum_{\alpha, \beta \geq 0} \varphi_{2\alpha+1} \psi_{2\beta+1} \left( \delta_{\alpha+\beta, n-1} + \sum_{\ell \geq \alpha, m \geq \beta, \ell + m = n} \binom{\ell}{\alpha}_r \binom{m}{\beta}_s \right),
\]

where in the last line, we use the Kronecker delta \( \delta_{i,j} \), i.e. \( \delta_{i,j} = 1 \) if \( i = j \) and 0 otherwise.

On the other hand, for \( n \neq -(r+s)/2 \)

\[
\vartheta_{2n} = \sum_{k=0}^{n} \binom{n}{k}_{r+s} \vartheta_{2k+1}
\]

\[
= \sum_{k=0}^{n} \sum_{\ell=0}^{k} \binom{n}{k}_{r+s} \left( \varphi_{2\ell} \psi_{2(k-\ell)+1} + \varphi_{2(k+\ell)-1} \psi_{2\ell+1} \right)
\]

\[
= \sum_{k=0}^{n} \sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \binom{n}{k}_{r+s} \left( \binom{\ell}{m}_r \varphi_{2m+1} \psi_{2(k-\ell)+1} + \binom{\ell}{m}_s \varphi_{2(k-\ell)+1} \psi_{2m+1} \right)
\]

\[
= \sum_{\alpha, \beta \geq 0} \varphi_{2\alpha+1} \psi_{2\beta+1} \sum_{k=0}^{n} \binom{n}{k}_{r+s} \left( \binom{k-\beta}{\alpha}_r + \binom{k-\alpha}{\beta}_s \right).
\]

Comparing these two expressions yields the following.

**Theorem 6.2.** Let \( r \) and \( s \) be elements of the set of positive or odd integers. Then for every \( n \geq 0 \) with \( n \neq -(r+s)/2 \) and integers \( \alpha, \beta \) such that \( 0 \leq \alpha, \beta \leq n \), we
have the identity

\[ \delta_{\alpha+\beta,m-n} \sum_{\ell \geq \alpha, m \geq \beta} \left( \begin{array}{c} \ell \\ \alpha \end{array} \right) \left( \begin{array}{c} m \\ \beta \end{array} \right) s = \sum_{k=0}^{n} \left( \begin{array}{c} k - \beta \\ \alpha \end{array} \right) r + \left( \begin{array}{c} k - \alpha \\ \beta \end{array} \right) s. \]

In view of Equation (6.2), the above relations can be interpreted as quadratic identities for the even indexed Bernoulli numbers. Note that when either \( r \) or \( s \) is both nonpositive and even, the existence of unconstrained even indexed coefficients in Theorem 5.9 affects carrying out a similar derivation.

7. Power sum formulas revisited

In this section, we indicate connections between Theorems 1.1, 3.4, 5.3, and 5.9 and the work of Gould [9] on power sum identities, clarified by Carlitz [4]. In particular, we demonstrate that many of the main results of [9] can be recovered as special cases of Theorems 5.3 and 5.9.

In [9], Gould investigated power sums of the form

\[ \sum_{k=0}^{n} k^p f_n(k) \]

where \( p \geq 0 \) is an integer and, for integers \( 0 \leq k \leq n \), the \( f_n(k) \in \mathbb{K} \) form a triangular array that is symmetric in the sense that \( f_n(k) = f_n(n-k) \).

To relate Gould’s results to those contained here, define

\[ S_{n,p} := \frac{1}{n^p} \sum_{k=0}^{n} k^p f_n(k) = \frac{n^p}{n^p} f_n(k), \]

and consider the generating function

\[ \varphi_n(x) := \sum_{p=0}^{\infty} S_{n,p} x^p = \sum_{k=0}^{n} f_n(k) \sum_{p=0}^{\infty} \frac{k^p}{n^p} x^p = \sum_{k=0}^{n} \frac{f_n(k)}{1 - \frac{k}{n} x}. \]

We claim that \( \varphi_n(x) \in \mathcal{F}_1 \). To see this, we use the symmetry \( f_n(k) = f_n(n-k) \) to compute

\[ 2\varphi_n(x) = \sum_{k=0}^{n} \left[ \frac{f_n(k)}{1 - \frac{k}{n} x} + \frac{f_n(n-k)}{1 - \frac{n-k}{n} x} \right] \]

\[ = \sum_{k=0}^{n} \left[ \frac{2 - x}{(1 - \frac{k}{n} x)(1 - \frac{n-k}{n} x)} \right] f_n(k) \]

\[ = -\frac{1}{x - 2} \sum_{k=0}^{n} \frac{4 + \frac{x^2}{1 - \frac{n-k}{n} x}}{1 + \frac{x^2}{1 - \frac{k}{n} x}} f_n(k) \]

\[ = \frac{1}{x - 2} \rho \left( \frac{x^2}{1 - x} \right), \]

which is an element of \( \mathcal{F}_1 \) by Theorems 3.3 and 3.4.

As the first consequence of this fact, by Theorem 1.1, we have that for each \( m \geq 1 \),

\[ \sum_{i=0}^{m} (-1)^i \binom{2m-1}{m-i} \binom{m+i}{i} S_{n,m+i-1} = 0. \]
Following [9 Equation (6)], we define
\[ Q_i^m := \binom{m}{i} + 2\binom{m}{i-1} = \frac{m!(m+i+1)}{i!(m-i+1)!} \]
and observe that by a simple computation,
\[ \binom{2m-1}{m-i}\binom{m+i}{i} = \frac{(2m-1)!}{m!(m-1)!}Q_i^{m-1}. \]
Hence, after dividing out the constant \( \frac{(2m-1)!}{m!(m-1)!} \), Equation (7.2) can be rewritten as
\[ \sum_{i=0}^{m} (-1)^i Q_i^{m-1} S_{n,m+i-1} = 0. \]
Substituting Equation (7.1) and multiplying by \( n^{2m-1} \) yields
\[ \sum_{i=0}^{m} (-1)^i n^{m-i} \sum_{k=0}^{n} k^{m+i-1} f_n(k) = 0. \]
This is precisely [9 Equation (5)], where in the reference the term corresponding to \( i = m \) is separated to one side of the equation.

As the second application of the fact that \( \varphi_n(x) \in F_1 \), Theorem 5.3 implies
\[ S_{n,2m+1} = \sum_{i=0}^{m} \binom{m}{i} S_{n,2i}. \]
We claim that up to a constant factor, this is equivalent to [9 Equation (15)], which can be expressed using Equation (7.1) as
\[ 2^{((m+3)/2)} S_{n,2m+1} = \sum_{i=0}^{m} A_i^m S_{n,2i}, \]
where the coefficients \( A_i^m \) are defined recursively in [9 Equation (16)]:
\[ 2^{-(k+2)/2} A_i^{k+1} = \binom{2k+3}{2i} - \sum_{j=1}^{k} \binom{2k+3}{2j+1} 2^{-(j+3)/2} A_j^i, \quad 0 \leq i \leq k \]
with seed values \( A_i^i = (2i+1)2^{[(i+1)/2]} \), \( i \geq 0 \).

To see this, using [9 Equation (18)] as well as [9 Equation (1.5)] to express the \( A_i^m \) in terms of Bernoulli numbers, we have that for integers \( m \) and \( i \) with \( m \geq i \geq 0 \),
\[
A_i^m = \begin{cases} 
\binom{2m+1}{2}2^{[i/2]} A_0^{m-i}, & m - i \text{ odd,} \\
\binom{2m+1}{2i}2^{[(i+1)/2]} A_0^{m-i}, & m - i \text{ even,}
\end{cases}
\]
\[
= \begin{cases} 
\binom{2m+1}{2i}2^{[i/2]}2^{(m-i+3)/2}(4m-i+1 - 1) B_{2(m-i+1)}^{m-i+1}, & m - i \text{ odd,} \\
\binom{2m+1}{2i}2^{[(i+1)/2]}2^{(m-i+2)/2}(4m-i+1 - 1) B_{2(m-i+1)}^{m-i+1}, & m - i \text{ even,}
\end{cases}
\]
\[
= \binom{2m+1}{2i}2^{[(m+3)/2]}(4m-i+1 - 1) B_{2(m-i+1)}^{m-i+1} \frac{m-i+1}{m-i+1}.
\]
With this, Equations (7.3) and (7.4) are clearly equivalent.
As our final application of $\varphi_n(x) \in \mathcal{F}_1$, we apply Theorem 5.9 to obtain that for all $m \geq 0$,

$$S_{n,2m} = \frac{2}{2m+1} \sum_{i=0}^{m} \left(\frac{2m+1}{2i+1}\right) B_{2m-2i} S_{n,2i+1}. \tag{7.5}$$

We set

$$C_m = 1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2m + 1)$$

following [9, page 314] and note that Gould’s $G^m_i$ [9, page 314] are shown in [10, Equation (1.6)] to be given by

$$G^m_i = 2 \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m - 1) \left(\frac{2m+1}{2i+1}\right) B_{2m-2i}.$$

Then Equation (7.5) is evidently equivalent to

$$C_m S_{n,2m} = \sum_{i=0}^{m} C^m_i S_{n,2i+1},$$

which recovers [10, Equation (29)].

8. COEFFICIENT TRIANGLES

In this section, we give a few examples of the first portions of the coefficient triangles resulting from Equations (1.5), (1.6), (1.7), and (1.8) to indicate their structure. Of interest is the appearance of the Lucas triangle, introduced in [7], which satisfies the same recursion as Pascal’s triangle but begins with row $n = 0$ with entry 2 and row $n = 1$ with entries 1, 2, see Table 1. For $n \geq 1$, the $j$th entry of the $n$th row of the Lucas triangle is given by

$$\frac{n+j}{n} \binom{n}{j}, \tag{8.1}$$

see [15, Equation (14)].

Table 1. The Lucas triangle.

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 1 | 2 |   |   |   |   |   |
| 1 | 3 | 2 |   |   |   |   |
| 1 | 4 | 5 | 2 |   |   |   |
| 1 | 5 | 9 | 7 | 2 |   |   |
| 1 | 6 | 14 | 16 | 9 | 2 |   |
| 1 | 7 | 20 | 30 | 25 | 11 | 2 |
| 1 | 8 | 27 | 50 | 55 | 36 | 13 | 2 |
| 1 | 9 | 35 | 77 | 105 | 91 | 49 | 15 | 2 |

For the coefficient triangles given below, the row labeled $i$ lists the coefficients of $\gamma_i, \gamma_{i+1}, \ldots$ in the corresponding constraint.

When $r = 0$, Equation (1.5) evidently yields Pascal’s triangle with an alternating sign. When $r$ is even and positive, Equation (1.5) yields Pascal’s triangle with an upper-left portion removed due to the fact that $\gamma_i = 0$ for $i < 0$. This is illustrated in Table 2 with the case $r = 8$. 
When $r = 1$, the beginning of the coefficient triangle is given in Table 3. Note that many rows have a common factor, and dividing by the gcd of each row yields the Lucas triangle with alternating signs. To see this, note that Equation (1.6) can be rewritten as
\[
\frac{(2m + r - 2)!}{m!(m-1)!} \sum_{i=0}^{m} (-1)^i \frac{(m+i)!}{(m+i+r-2)!m!} \gamma_{m+i-1} = 0.
\]
Dividing out the constant $\frac{(2m+r-2)!}{m!(m-1)!}$ and setting $r = 1$ yields Equation (8.1) with alternating signs.

| 0: | 1 | -2 |
|---|---|---|
| 1: | 3 | -9 | 6 |
| 2: | 35 | -40 | 50 | -20 |
| 3: | 126 | -756 | 1764 | -2016 | 1134 | -252 |
| 4: | 462 | -3234 | 9240 | -13860 | 11550 | -5082 | 924 |
| 5: | 1716 | -13728 | 46332 | -85800 | 94380 | -61776 | 22308 | -3432 |
| ... |

Table 3. The coefficient triangle of Equation (1.6) when $r = 1$.

When $r > 1$ is odd and positive, a similar coefficient triangle is obtained, which is illustrated with the case $r = 9$ in Table 4.

| 0: | 9 | -2 |
|---|---|---|
| 1: | 55 | -33 | 6 |
| 2: | 286 | -312 | 130 | -20 |
| 3: | 1365 | -2275 | 1575 | -525 | 70 |
| 4: | 6188 | -14280 | 14280 | -7616 | 2142 | -252 |
| 5: | 27132 | -81396 | 108528 | -81396 | 35910 | -8778 | 924 |
| 6: | 116280 | -434112 | 732564 | -718200 | 438900 | -166320 | 36036 | -3432 |
| ... |

Table 4. The coefficient triangle of Equation (1.6) when $r = 9$.

When $r$ is even and negative, the result is eventually Pascal’s triangle with alternating signs but begins with a finite positive triangle corresponding to Equation (1.7); see Table 5.
Table 5. The coefficient triangle of Equations (1.5) and (1.7) when \( r = -8 \).

|   | 0    | 1    | 2    | 3    | 9    | 10   | 11   | 12   |
|---|------|------|------|------|------|------|------|------|
| 0 | 1    |      |      |      |      |      |      |      |
| 1 | 1/9  | 1/36 | 1/12 | 1/8  | 1    |      |      |      |
| 2 | 1/126| 2/63 | 1/12 | 1/9  |      |      |      |      |
| 3 |      |      |      |      |      |      |      |      |
| 9 |      |      |      |      | 1    |      |      |      |
| 10|      |      |      |      | 1    | -1   |      |      |
| 11|      |      |      |      | 1    | -2   | 1    |      |
| 12|      |      |      |      | 1    | -3   | 3    | -1   |

Table 6. The coefficient triangle of Equations (1.7) and (1.8) when \( r = -9 \).

|   | 0    | 1    | 2    | 3    | 4    | 9    | 10   | 11   | 12   |
|---|------|------|------|------|------|------|------|------|------|
| 0 | 1    |      |      |      |      |      |      |      |      |
| 1 | 1/10 | 1/45 | 1/10 | 1/45 |      |      |      |      |      |
| 2 | 1/210| 1/63 | 1/35 | 1/42 | 1/210|      |      |      |      |
| 3 | 1/210| 1/63 | 1/35 | 1/42 | 1/210|      |      |      |      |
| 4 |      |      |      |      |      |      |      |      |      |
| 10| 1    |      |      |      |      | 1    | -2   |      |      |
| 11| 1    |      |      |      |      | 1    | -3   | 2    |      |
| 12| 1    |      |      |      |      | 1    | -4   | 5    | -2   |

8.1. Hidden extra symmetries. As a final note, we make a brief observation about an alternative rescaling of Equations (1.7) and (1.5) that yields an additional symmetry.

Suppose \( r = 1 - 2k \) is odd and negative. Then Equation (1.7) can be rewritten as

\[
\frac{m!(k-m)!}{(2k)!} \sum_{i=0}^{m} \frac{(m+i)!(2k-m-i)!}{(m-i)!!(k-m)!} \gamma_{m+i-1} = 0,
\]

yielding a system of equations that is invariant under the substitution \((m, i) \rightarrow (k-i, k-m)\). That is, the corresponding finite positive triangle rescaled in this way has a slanted symmetry where the lower left corner can be treated as the top of a symmetric number array. This is illustrated in Table 7 which gives the rescaled upper triangle of Table 6.
Table 7. The rescaled upper triangle of Table 6, corresponding to Equation (8.2) when \( r = -9 \).

\[
\begin{array}{cccccc}
0: & 15120 & 3360 \\
1: & 6720 & 5040 & 1440 \\
2: & 2520 & 4320 & 3600 & 1440 \\
3: & 720 & 2400 & 4320 & 5040 & 3360 \\
4: & 120 & 720 & 2520 & 6720 & 15120 & 30240
\end{array}
\]

Table 8. Pascal's triangle rescaled with respect to row \( n = 8 \).

\[
\begin{array}{cccccccc}
1 & 8 & 8 & \cdots \\
8 & 56 & 28 & 28 & 28 & 28 & 8 & 1 \\
56 & 280 & 420 & 280 & 70 & 56 & 28 & 8 \\
280 & 560 & 560 & 420 & 168 & 56 & 56 & 8 \\
8 & 280 & 420 & 560 & 420 & 168 & 28 & 8 \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 \\
\end{array}
\]

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