Abstract

It is shown that in the passage of a short burst of non-linear plane gravitational wave, the kinetic energy of free particles may either decrease or increase. The decreasing or increasing of the kinetic energy depends crucially on the initial conditions (position and velocity) of the free particle. Therefore a plane gravitational wave may extract energy from a physical system.
1 Introduction

The memory effect is a very interesting feature of gravitational waves. It expresses the change in a physical system (a detector), between the final and initial state of the system, after the passage of a gravitational wave. It is an effect that might be observable in the future LISA operations, and also raises quite interesting theoretical issues. If the physical system is composed of free test particles, the memory effect is determined by the permanent displacement of the particles, caused by the gravitational wave. The memory effect due to bursts of plane gravitational waves has been recently considered by Zhang, Duval, Gibbons and Horvathy [1,2]. Zeldovich and Polnarev [3] first considered the effect of linearised gravitational waves on non-interacting bodies, such as satellites, but the memory effect seems to be first proposed by Braginsky and Grishchuk [4], who were really interested in the motion of free particles in the space-time of a gravitational wave. Soon after, a distinction was made between gravitational wave bursts with and without memory [5]. This distinction was already considered in Ref. [6], but not in the context of the memory effect. A non-linear form of memory effect had been discovered independently by Blanchet and Damour [7], and by Christodoulou [8]. A thorough mathematical treatment of the memory effect has been recently made by Favata [9].

One interesting manifestation of the memory effect is the velocity memory effect, that is characterized by a permanent change of the velocity of the free particle after the passage of the wave. This effect was considered by Souriau [10], Braginsky and Thorne, [5], Grishchuk and Polnarev [11], Bondi and Pirani [12], and more recently by Zhang, Duval and Horvathy [13].

In this article, we consider the approach to plane gravitational waves given in Ref. [2], and address the memory effect from the point of view of the velocities of free particles, after the passage of a wave. The wave burst is modelled by Gaussian functions of the retarded time. If the free particles are initially at rest in the space-time, then they acquire velocity after the passage of the wave. Physically, we expect a tiny variation of the velocity of the particles, because the gravitational wave is supposed to be very weak. However, we have found that for certain initial conditions of the free particles, with non-vanishing initial velocities, the final kinetic energy of the particles is smaller than the initial energy. Therefore, contrary to the common expectation, the gravitational wave may absorb energy of the physical system. We argue that such extraction of energy is consistent with the expression
of the gravitational energy-momentum of plane-fronted gravitational waves calculated in Ref. [14], and might explain the propagation of waves with very slow dissipation in time.

2 Free particles and gravitational waves

A non-linear plane gravitational wave may be written in several different forms [15]. One possible form is the plane-fronted gravitational wave that travels in the $z$ direction, and which is given by [16, 17, 18]

$$ds^2 = dx^2 + dy^2 + 2du dv + H(x, y, u)du^2.$$  

The function $H(x, y, u)$ satisfies

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) H(x, y, u) = 0.$$  

Transforming $(u, v)$ to $(t, z)$ coordinates, where

$$u = \frac{1}{\sqrt{2}}(z - t), \quad v = \frac{1}{\sqrt{2}}(z + t),$$  

we find

$$ds^2 = \left( \frac{H}{2} - 1 \right) dt^2 + dx^2 + dy^2 + \left( \frac{H}{2} + 1 \right) dz^2 - H dt dz.$$  

We are assuming $c = 1$. The function $H$ must satisfy only Eq. (2). Otherwise it is arbitrary, specially regarding the dependence on the retarded time ($-u$). The geodesic equations in terms of the $t, x, y, z$ coordinates are, respectively [19]

$$2\ddot{t} + \sqrt{2}H\ddot{u} + \sqrt{2}\dot{H}\dot{u} - \frac{1}{\sqrt{2}} \frac{\partial H}{\partial u} \dot{u}^2 = 0,$$  

$$2\ddot{x} - \frac{\partial H}{\partial x} \dot{u}^2 = 0,$$  

$$2\ddot{y} - \frac{\partial H}{\partial y} \dot{u}^2 = 0,$$  

$$2\ddot{z} + \sqrt{2}H\ddot{u} + \sqrt{2}\dot{H}\dot{u} - \frac{1}{\sqrt{2}} \frac{\partial H}{\partial u} \dot{u}^2 = 0.$$  


where the dot represents derivative with respect to \( s \). From the first and fourth equations above we get

\[
\ddot{z} - \ddot{t} = 0 \rightarrow \dot{u} = 0 \rightarrow \dot{u} = \frac{1}{\sqrt{2}}(\ddot{t} - \ddot{z}) = \text{constant.} \tag{9}
\]

The line element for the gravitational wave considered in Refs. \[1\] \[2\] is presented in Brinkmann coordinates \[20\], and is similar to Eq. (1). In terms of the notation above, the line element for a wave propagating in the \( z \) direction is given by

\[
ds^2 = dx^2 + dy^2 + 2 du dv + K_{ij}(u)x^i x^j du^2, \tag{10}
\]

where

\[
K_{ij}(u)x^i x^j = \frac{1}{2} A_+(u)(x^2 - y^2) + A_x(u)xy. \tag{11}
\]

In the expression above, \((+, \times)\) represent the two polarization states. There are several possible choices for the amplitude \( A_{(+, \times)} \), as discussed in Ref. \[2\]. These choices are given by a Gaussian and by derivatives of the Gaussian. We will choose to work with a simple Gaussian,

\[
A_{(+, \times)}(u) = \frac{1}{L^2} e^{-u^2/\lambda^2}, \tag{12}
\]

where \( L \) and \( \lambda \) are constants with dimension of length: \( \lambda \) is related to the width of the Gaussian, and \( L^2 \) could be interpreted as the size of the transversal area of the wave (we are requiring \( g_{00} < 0 \) in Eq. (4)). These constants are necessary, so that the line element (4) has dimension of \((\text{length})^2\). The results to be presented below do not depend on whether we use a Gaussian or derivatives of the Gaussian, as in Ref. \[2\].

In what follows, we will consider three simple choices for \( A_+ \): \( A_+ = e^{-u^2} \), \( A_+ = (1/4)e^{-u^2/2} \) and \( A_+ = (1/8)e^{-u^2/3} \). We must remember that the constants \( L \) and \( \lambda \) are present in the expression of \( A_+ \) and, for our purposes, it makes no difference whether we use centimetres or metres.

Free particles follow geodesics in space-time. The geodesics in the space-time of a wave determined by

\[
K_{ij}(u)x^i x^j = \frac{1}{2} A_+(u)(x^2 - y^2), \tag{13}
\]

satisfy the equations
\[
\begin{align*}
\frac{d^2x}{du^2} - \frac{1}{2}A_+ x &= 0, \\
\frac{d^2y}{du^2} + \frac{1}{2}A_+ y &= 0, \\
\frac{d^2v}{du^2} + \frac{1}{4} \frac{dA_+}{du} (x^2 - y^2) + A_\left( \frac{dx}{du} - y \frac{dy}{du} \right) &= 0,
\end{align*}
\]

which are strictly equivalent to eqs. (5-8) provided we make \( s = u \) and identify

\[ H = \frac{1}{2} A_+ (u) (x^2 - y^2). \]

By using the program MAPLE 12 we have specified initial conditions for a free particle and solved the equations above numerically for the velocities \( V_x = dx/du, V_y = dy/du \) and \( V_z = dz/du \). Then we have evaluated

\[ 2K = \frac{1}{2} (V_x^2 + V_y^2 + V_z^2), \]

before and after the passage of the wave. \( K \) is the kinetic energy per unit mass, given by

\[ K = \frac{1}{2} \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right]. \]

The results are displayed in the figures below.

The initial conditions for all geodesic curves, at \( u = 0 \), are \( x(0) = 1, \ y(0) = 1, \ z(0) = 0 \) and \( V_z(0) = 0 \). The \textbf{increasing} of the kinetic energy per unit mass occurs with the initial conditions

\[ V_x(0) = 0.4, \quad V_y(0) = 0, \quad (17) \]

and the \textbf{decreasing} of the kinetic energy takes place for the initial conditions

\[ V_x(0) = 0, \quad V_y(0) = 0.4. \quad (18) \]

The three cases we will consider are

\begin{itemize}
    \item Case 1: \( A_+ = e^{-u^2} \)
    \item Case 2: \( A_+ = \frac{1}{4} e^{-u^2/2} \)
\end{itemize}
Case 3: $A_+ = \frac{4}{8} e^{-u^2/3}$

It is important to note that, in view of the definition $u = \frac{1}{\sqrt{2}}(z - t)$, the time coordinate $t$ runs in the opposite direction relatively to $u$. Therefore, since the coordinate $u$ runs from left to right in the figures, $t$ runs from right to left.

![Figure 1: Case 1 - velocities before and after the passage of the wave for the initial conditions (17)](image)

Figures (4), (8) and (12) clearly show the decreasing of the kinetic energy per unit mass of the particle, after the passage of the gravitational wave. The local space-time is flat before and after the passage of the wave.

Other initial conditions on the free particle lead to similar results, including the situation in which the kinetic energy is not changed. We have not found a criterium for choosing general initial conditions such that the kinetic energy always increases or always decreases. The result also does not depend on using the polarization $A_\times$. Taking into account the latter, the qualitative results are the same, and the kinetic energy might likewise increase or decrease.
The gravitational energy momentum of the plane-fronted gravitational wave described by Eq. (4) has been calculated in Ref. [14], in the context of the teleparallel equivalent of general relativity (TEGR) [21]. The latter is a tetrad description of the gravitational field, and the gravitational energy-momentum has been evaluated in the frame of a static observer in space-time. In the TEGR, the gravitational energy-momentum and the 4-angular momentum satisfy the algebra of the Poincaré group. We will not repeat the details of the calculations here, but just present the final result. We have found that the non-vanishing components of the gravitational energy-momentum $P^a$ for the wave that travels in the $z$ direction, given by Eq. (4), are given by

$$P^{(0)} = P^{(3)} = -\frac{k}{8} \int_V d^3 x \frac{\left(\partial_i H\right)^2}{(-g^{00})^{3/2}} \leq 0 ,$$

where $k = c^3/16\pi G = 1/16\pi$, $g^{00} = -1 + H/2$ and $i = (x, y)$. The expression of $P^a$ satisfies $P^a P^b \eta_{ab} = 0$, where $\eta_{ab} = (-1, +1, +1, +1)$. $P^{(0)}$ and $P^{(3)}$ are evaluated over an arbitrary volume $V$ of the three-dimensional space.
The gravitational energy $P^{(0)}$ is non-positive. Therefore we see that in order for a gravitational wave given by Eqs. (1) or (4) to dissipate in space, it must absorb positive energy from the medium where it travels. Of course, the wave also transfers positive energy to the physical system, as we have seen, and the occurrence of both processes might explain why a gravitational wave travels in space for periods of time such as billions of years, without dissipating.

4 Final Comments

It is reasonable to consider that gravitational waves interact with the detector, which is inextricably linked with the space-time geometry. The effect of this interaction results in some form of the memory effect as, for instance, in the velocity memory effect. The absorption of energy by a system of free particles, after the passage of a gravitational wave, was already envisaged by Bondi [22]. This is the physical feature one would expect from our experience with electromagnetic waves and charged classical particles.

The absorption of energy from a system of free particles, by a non-linear gravitational wave, is an unexpected feature. In view of this result, the vari-
Figure 4: Case 1- decreasing of the kinetic energy - initial conditions

The variation of kinetic energy of free particles is not, in principle, a useful criterion for investigating the memory effect, since both kinds of variation occur for the same wave. However, we have not been able to classify the conditions under which the kinetic energy of the particles increases or decreases. This issue will be investigated elsewhere.

References

[1] P.-M. Zhang, C. Duval, G. W. Gibbons and P. A. Hovarthy, “The Memory Effect for Plane Gravitational Waves”, arXiv:1704.05997.

[2] P.-M. Zhang, C. Duval, G. W. Gibbons and P. A. Hovarthy, Phys. Rev. D 96, 064013 (2017).

[3] Y. B. Zeldovich and A. G. Polnarev, Sov. Astron. 18, 17 (1974).

[4] V. B. Braginsky and L. P. Grishchuk, Zh. Eksp. Teor. Fiz. 89, 744 (1985) [Sov. Phys. JETP 62, 427 (1985)].

[5] V. B. Braginsky and K. S. Thorne, “Gravitational-wave burst with memory and experimental prospects”, Nature (London) 327, 123 (1987).
Figure 5: Case 2 - velocities before and after the passage of the wave for the initial conditions [17]

[6] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 4, 2192 (1971).
[7] L. Blanchet and T. Damour, Phys. Rev. D 46, 4302 (1992).
[8] D. Christodoulou, Phys. Rev. Lett. 67, 1486 (1992).
[9] M. Favata, Class. Quantum Grav. 27, 084036 (2010).
[10] J.-M. Souriau, “Ondes et radiations gravitationnelles”, Colloques Internationaux du CNRS No 220, 243. Paris (1973).
[11] L. P. Grishchuk and A. G. Polnarev, “Gravitational wave pulses with ‘velocity coded memory’ ”, Sov. Phys. JETP 69, 653 (1989) [Zh. Eksp. Teor. Fiz. 96, 1153 (1989)].
[12] H. Bondi and F. A. E. Pirani, “Gravitational Waves in General Relativity XIII: Caustic Property of Plane Waves,” Proc. Roy. Soc. Lond. A 421, 395 (1989).
[13] P. M. Zhang, C. Duval and P. A. Horvathy, “Memory Effect for Impulsive Gravitational Waves,” arXiv:1709.02299 [gr-qc], to appear in Class. Quantum Grav.
Figure 6: Case 2 - increasing of the kinetic energy - initial conditions (17)

[14] J. W. Maluf and S. C. Ulhoa, Phys. Rev. D 78, 047502 (2008); 78, 069901(E) (2008).

[15] D. Kramer, H. Stephani, M. A. H. MacCallum and H. Herlt, “Exact Solutions of the Einstein’s Fields Equations”, Cambridge University Press, Cambridge, (1980).

[16] J. Ehlers and W. Kundt, in “Gravitation: an Introduction to Current Research”, edited by L. Witten (Wiley, New York, 1962).

[17] P. Jordan, J. Ehlers and W. Kundt, “Republication of: Exact solutions of the field equations of the general theory of relativity”, Gen. Relativ. Grav. 41, 2191-2280 (2009).

[18] H. Stephani, “Relativity: an Introduction to Special and General Relativity”, third edition (Cambridge Univ. Press, 2004).

[19] J. F. da Rocha-Neto and J. W. Maluf, Gen. Relativ. Grav. 46, 1667 (2014).

[20] M. W. Brinkmann, “On Riemann spaces conformal to Euclidean spaces”, Proc. Natl. Acad. Sci. U.S. 9, 1 (1923).
Figure 7: Case 2 - velocities before and after the passage of the wave for the initial conditions [18]

[21] J. W. Maluf, Ann. Phys. (Berlin) 525, 339 (2013).

[22] H. Bondi, Nature 179, 1072 (1957).
Figure 8: Case 2 - decreasing of the kinetic energy - initial conditions \((18)\)

Figure 9: Case 3 - velocities before and after the passage of the wave for the initial conditions \((17)\)
Figure 10: Case 3 - increasing of the kinetic energy - initial conditions (17)

Figure 11: Case 3 - velocities before and after the passage of the wave for the initial conditions (18)
Figure 12: Case 3- decreasing of the kinetic energy - initial conditions (18)