The Benefit of Thresholding in LP Decoding of LDPC Codes

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Abstract—Consider data transmission over a binary-input additive white Gaussian noise channel using a binary low-density parity-check code. We ask the following question: Given a decoder that takes log-likelihood ratios as input, does it help to modify the log-likelihood ratios before decoding? If we use an optimal decoder then it is clear that modifying the log-likelihoods cannot possibly help the decoder’s performance, and so the answer is “no.” However, for a suboptimal decoder like the linear programming decoder, the answer might be “yes”: In this paper we prove that for certain interesting classes of low-density parity-check codes and large enough SNRs, it is advantageous to truncate the log-likelihood ratios before passing them to the linear programming decoder.

I. INTRODUCTION

While maximum-likelihood (ML) decoding of low-density parity-check (LDPC) codes is reasonably well understood based on the expected weight distribution of the codes, the linear programming (LP) and the related belief propagation (BP) decoding of LDPC codes reveal a number of interesting and unexpected phenomena. The root cause of the difference between these suboptimal decoders and ML decoding is the occurrence of so called pseudo-codewords; from the perspective of an LP or BP decoder, the pseudo-codewords act as attractive solutions to the decoding problem, even though they are not actual codewords in the LDPC code under consideration. In contrast to codewords which, for codes of length $n$ and under antipodal signaling, map to elements of the set $\{+1, -1\}^n$, pseudo-codewords are vectors of length $n$ that map to vectors with entries that lie in the interval $[-1, +1]$. Note that the set of possible pseudo-codewords is a function not only of the code but also of the chosen parity-check matrix.

This paper explores one of the above-mentioned unexpected phenomena of LP decoding and discusses the roots of this behavior. Considering the tight relationship between LP decoding and iterative decoding [1], [2], [3], [4], our observations about LP decoding must also have consequences for iterative decoding. Before we start describing that phenomenon, let us first explain the communication setup (see Fig. 1) that is under consideration.

1. We use a binary channel code of length $n$, dimension $k$, and rate $k/n$.

- The information word $u \in \{0, 1\}^k$ is encoded into the codeword $x \in \{0, 1\}^n$. We assume that all information words are chosen with equal likelihood.
- Let $\theta : \mathbb{R} \rightarrow \mathbb{R}, \omega_i \mapsto 1 - 2\omega_i$, Restricting the domain of $\theta$ to $\{0, 1\}$ we obtain the usual BPSK mapping: $0 \mapsto +1$ and $1 \mapsto -1$. When applying the map $\theta$ to a vector we define the result to be a vector where each component is mapped according to $\theta$. Instead of $\theta(\omega_i)$ and $\theta(\omega)$ we will very often simply write $\bar{\omega}_i$ and $\bar{\omega}$, respectively. For our communication setup this means that the codeword $x \in \{0, 1\}^n$ is mapped to its signal-space point $\bar{x} \triangleq \theta(x) = (\theta(x_1), \ldots, \theta(x_n)) \in \{+1, -1\}^n$.
- For $i = 1, \ldots, n$, the symbols $\bar{x}_i$ are sent over a (binary-input) additive white Gaussian noise channel (AWGNC) with noise power $N_0/2$, i.e. we receive $Y_i \triangleq \bar{x}_i + Z_i$ where $\{Z_i\}_{i=1}^n$ are i.i.d. random variables with $Z_i \sim \mathcal{N}(0, N_0/2)$. Here, $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian random variable with mean $\mu$ and variance $\sigma^2$.
- Based on the observations $Y_i = \bar{y}_i, i = 1, \ldots, n$, we compute the normalized log-likelihood ratios (LLRs)

$$\lambda_i \triangleq \eta \cdot \log \left( \frac{p_{Y_i | X_i}(\bar{y}_i | 1)}{p_{Y_i | X_i}(\bar{y}_i | -1)} \right) = \eta \cdot \log \left( \frac{p_{Y_i | X_i}(\bar{y}_i | 1)}{p_{Y_i | X_i}(\bar{y}_i | 0)} \right),$$

where the normalization constant $\eta \triangleq \eta(N_0)$ is chosen such that $\lambda_i$ equals $+1$ if $\bar{z}_i = 0$.
- A mapping $\mu : \mathbb{R} \rightarrow \mathbb{R}$ is applied to the LLRs and results in the modified LLRs $\lambda_i' \triangleq \mu(\lambda_i), i = 1, \ldots, n$.
- Based on the modified LLR vector $\lambda'$, a decoder $\phi$ tries to make a decision $\bar{x} \triangleq \phi(\lambda')$ about $\bar{x}$. (Or, alternatively, tries to decide on $u$ or $x$.)
- When decoding a code of length $n$, we use the label $P^\phi(n)$ for denoting the block error probability of a decoder $\phi$ which bases its decisions on the modified LLR vector $\lambda' \triangleq \mu(\lambda)$.
Let \( \bar{C} \triangleq \theta(C) \) be the set of points in signal space that correspond to the codewords. Using the (normalized) LLR vector \( \lambda \), the maximum likelihood (ML) decoder \( \phi_{\text{ML}} \) can be cast as

\[
\hat{x} \triangleq \phi_{\text{ML}}(\lambda') \triangleq \arg \max_{x \in \bar{C}} \sum_{i=1}^{n} \bar{x}_i \lambda'_i,
\]

with the trivial mapping \( \lambda'_i \triangleq \mu_{\text{triv}}(\lambda_i) \triangleq \lambda_i, \ i = 1, \ldots, n. \) From this expression it is clear that the LLR vector is also favorable to the trivial map \( \mu_{\text{triv}} \triangleq (\lambda_1, \ldots, \lambda_n) \) for high decoders, e.g. the linear programming (LP) decoder [3], [4].

The situation is not as simple in the case of suboptimal decoders, e.g. the linear programming (LP) decoder [3], [4]. In fact, combining the results in [6] and [1], we show for certain low-density parity-check (LDPC) codes and for high enough SNR it is favorable not to use the trivial map \( \mu_{\text{triv}} \), but to use a two-level quantization map

\[
\lambda'_i \triangleq \begin{cases} 
+L & \text{if } \lambda_i \geq 0 \\
-L & \text{if } \lambda_i < 0 
\end{cases}
\]

before performing the LP decoding.

This seeming paradox is not uncommon for suboptimal algorithms. We cite the following paragraph from Ganti et al. [7, p. 2316] which remarks on a similar phenomenon (albeit in a different context): “... Indeed, in the matched case it is clear that the optimal decoder for the general channel performs at least as well as a decoder that first quantizes the output and then performs optimal processing on the quantized samples. Under mismatched decoding, however, it is unclear how to relate the performance of the mismatched decoder on the original channel to its performance on the output-quantized channel.”

A natural question arises: Is the advantage of using the two-level quantization map the result of a quantization effect, or something else? We show that there are code families such that for any finite \( W \), the thresholding map

\[
\lambda'_i \triangleq \mu_{T,W}(\lambda_i) \triangleq \begin{cases} 
+W & \text{if } \lambda_i \geq +W \\
-W & \text{if } \lambda_i \leq -W \\
\lambda_i & \text{otherwise}
\end{cases}
\]

is also favorable to the trivial map \( \mu_{\text{triv}} \). This suggests that the asymptotic advantage over \( \mu_{\text{triv}} \) is gained not by quantization, but rather by restricting the LLRs to have finite support.

The rest of the paper is structured as follows. We will give a brief introduction to LP decoding and pseudo-codewords in Sec. II. In Sec. III we will talk about pseudo-codewords stemming from the canonical completion and their importance for the asymptotic behavior of the LP decoder. In Secs. IV and V we will discuss the main results of this paper, namely we show examples when thresholding and quantizing of the LLRs can help.

\[1\] For recent work on the notion of pseudo-codewords in decoding we refer to [8], [9], [2], [1], [10], [3], [4].

\[2\] Exceptions to this observation include for example the class of convolutional codes with not too many states.

II. LP Decoding

ML decoding as in (1) can also be formulated as

\[
\hat{x} \triangleq \phi_{\text{ML}}(\lambda') \triangleq \arg \max_{x \in \bar{C}} \sum_{i=1}^{n} \bar{x}_i \lambda'_i.
\]

where \( \text{conv}(\bar{C}) \) is the convex hull of \( \bar{C} \) and where the mapping \( \mu \) is the trivial mapping \( \mu_{\text{triv}} \). Unfortunately, for most codes of interest, the description complexity of \( \text{conv}(\bar{C}) \) grows exponentially in the block length and therefore finding the maximum in (3) with a linear programming solver is highly impractical for reasonably long codes.

A standard approach in optimization in order to simplify the problem, is to replace the maximization over \( \text{conv}(\bar{C}) \) by a maximization over some easily describable polytope \( \bar{P} \) that is a relaxation of \( \text{conv}(\bar{C}) \):

\[
\hat{x} \triangleq \arg \max_{x \in \bar{P}} \sum_{i=1}^{n} \bar{x}_i \lambda'_i. \tag{4}
\]

If \( \bar{P} \) is strictly larger than \( \text{conv}(\bar{C}) \) then the decision rule in (4) obviously represents a sub-optimal decoder. A relaxation which works particularly well for LDPC codes is given by the following approach [3], [4]. Let \( \bar{C} \) be described by an \( m \times n \) parity-check matrix \( \bar{H} \) with rows \( h_1, h_2, \ldots, h_m \). Then the polytopes \( \bar{P} \triangleq \bar{P}(\bar{H}) \) and \( \bar{P} \subseteq \bar{P}(\bar{H}) \triangleq \theta(\bar{P}) \), also called the fundamental polytopes [1], are defined as

\[
\bar{P} \triangleq \bigcap_{i=1}^{m} \text{conv}(\bar{C}_i) \text{ with } \bar{C}_i \triangleq \{ x \in \{0,1\}^n \mid h_i x^T = 0 \mod 2 \}.
\]

\[
\bar{P} \triangleq \bigcap_{i=1}^{m} \text{conv}(\bar{C}_i) \text{ with } \bar{C}_i \triangleq \theta(\bar{C}_i).
\]

Note that \( \bar{P} \) is a convex set within \( 0,1 \) that contains \( \text{conv}(\bar{C}) \) but whose description complexity is much smaller than the description complexity of \( \text{conv}(\bar{C}) \). (A similar comment applies to \( \bar{P} \) which is a convex set within \( [-1,1]^n \) and contains \( \text{conv}(\bar{C}) \).) Points in the set \( \bar{P} \) will be called pseudo-codewords, and since \( \bar{P} \) is a convex polytope, we may restrict our attention to the vertices of \( \bar{P} \) (and \( \bar{P} \)). Because the set \( \bar{P} \) is usually strictly larger than \( \text{conv}(\bar{C}) \), the decoding rule in (4) might deliver a vertex of \( \bar{P} \) that is not the signal-space equivalent of a codeword; these “fractional” vertices are the reason for the sub-optimality of LP decoding (cf. [4], [1]).

For analyzing the above setup it turns out to be useful to define the AWGNC pseudo-weight [11] of a pseudo-codeword \( \omega \in \bar{P} \) to be \( w^\text{AWGNC} \) of \( \omega \) \( = \| \omega \|_2^2 / \| \omega \|_1^2 \). The significance of \( w^\text{AWGNC} \) is the following. The existence of a pseudo-codeword \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \bar{P} \setminus \{0\} \) causes LP decoding to fail to detect the codeword \( 0 \) if the vector of received LLRs \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) satisfies the inequality

\[
\sum_{i=1}^{n} \bar{\omega}_i \lambda'_i > \sum_{i=1}^{n} \bar{0}_i \lambda_i', \quad \text{where } \lambda' = \mu_{\text{triv}}(\lambda).
\]

Then it can be shown that the squared Euclidean distance from \( \bar{0} = +1 \) to the plane \( \{ x' \in \mathbb{R}^n \mid \sum_{i=1}^{n} (\bar{\omega}_i - \bar{0}_i) \lambda_i' = 0 \} \) is

\[
w^\text{AWGNC} \triangleq \| \omega \|_2^2 / \| \omega \|_1^2.
\]
Assuming $\mu$ to be the trivial mapping $\mu_{\text{triv}}$, the above theorem has immediate consequences for the LP decoder: the LP decision region for $\bar{0}$ is constrained by a hyperplane whose squared Euclidean distance from $\bar{0}$ is at most $\beta_{d_\ell,d_\ell}n^{d_\ell d_\ell}$. Because $\beta_{d_\ell,d_\ell} < 1$, this implies that the word error probability \( P_{\mu_{\text{triv}}}^\text{LP}(n) \) of LP decoding is lower bounded: 

\[
P_{\mu_{\text{triv}}}^\text{LP}(n) \geq (1 - 1/(K'n^{\beta_{d_\ell,d_\ell}})) \left(2\pi K'n^{\beta_{d_\ell,d_\ell}}\right)^{-1/2} \exp \left(-\frac{K'}{2}n^{\beta_{d_\ell,d_\ell}}\right)
\]

where $K'$ is positive and a function of the SNR, independent of $n$. This observation implies that the reliability function $\lim_{n \to \infty} \sup \frac{1}{n} \log (P_{\mu_{\text{triv}}}^\text{LP}(n))$ of the AWGN under LP decoding approaches zero for any fixed SNR. This is in stark contrast to ML decoding whose reliability function remains non-zero for large enough signal-to-noise ratios. In this context it is interesting to note that Lentmaier et al. [13] could prove that under some mild technical conditions the block error rate of a $(d_\ell,d_\ell)$-regular code under belief-propagation decoding with a bounded number of iterations is upper bounded by $P_{\text{tree}}(n) \leq n \exp(-K''n^{\beta_{d_\ell,d_\ell}/4})$ for the same constant $\beta_{d_\ell,d_\ell}$, where $P_{\text{tree}}(n)$ refers to the block error rate of a belief propagation decoding algorithm where the number of iterations is one quarter the girth of the Tanner graph.

IV. QUANTIZING AND THRESHOLDING

We still consider the LP decoder, but we want to investigate what happens when $\mu$ is selected to be something other than $\mu_{\text{triv}}$. So, let us consider what happens when $\mu \triangleq \mu_{\text{Q2,L}}$ is selected for some $L > 0$. Actually, it can easily be seen that the combination of the AWGN and this quantization gives (apart from scaling) the same LLR vectors as at the receiver end of a binary symmetric channel (BSC). Recognizing this, we can use the results of [6] which show that there exist families of expander-based $(d_\ell,d_\ell)$-regular LDPC codes which are guaranteed to correct a constant fraction $\tau$ of errors on the BSC. By a simple union bound argument we conclude that for sufficiently large SNR the block error probability is upper bounded by $P_{\mu_{\text{Q2,L}}}^\text{LP}(n) \leq n \exp(-K''n)$ where again $K''$ is positive and independent of $n$. It follows that there exist families of expander-based $(d_\ell,d_\ell)$-regular LDPC codes where $\lim_{n \to \infty} \sup \frac{1}{n} \log (P_{\mu_{\text{Q2,L}}}^\text{LP}(n))$ is strictly larger than zero under LP decoding, for sufficiently large SNR.

What explains this advantage in the asymptotic behavior? Looking at the above results we have to consider two candidates: (i) the quantized values of the modified LLRs or (ii) the finite support of the modified LLRs. It turns out that the answer is given by (ii), namely it is sufficient to threshold the LLRs, whereas quantization as in (i) is not really necessary. As is shown in the Section V one can set $\mu \triangleq \mu_{T,W}$ (see (3)) for any finite $W \geq 1$ and construct classes of $(d_\ell,d_\ell)$-regular expander-based LDPC codes where $\lim_{n \to \infty} \sup \frac{1}{n} \log (P_{\mu_{T,W}}^\text{LP}(n))$ is non-zero under LP decoding.\(^4\)

\(^4\)Note that the result of the LP decoder is independent of the exact choice of $L > 0$.

\(^5\)The constraint $W \geq 1$ is not necessary, but was imposed to simplify the presentation; Th. 4 holds for any $W > 0$.
Theorem 2: Consider the setup as described in Sec. I where we transmit over an AWGNC with noise power $\sigma^2 = N_0/2$. For any finite truncation value $W \geq 1$, any constant rate $0 < r < 1$, and sufficiently small $\sigma^2 > 0$, there exists a family of $(d_v, d_c)$-regular Tanner graphs for low-density parity-check codes of increasing length, each with rate at least $r$, such that $\lim_{n \to \infty} \sup -\frac{1}{n} \log (P_{\mu_{T,W}}(n))$ is strictly larger than zero.

Proof: See Section V.

Putting the above results for the LP decoding with the different mappings $\mu = \mu_{\text{riv}}$ and $\mu = \mu_{T,W}$ in juxtaposition reveals a surprising property of LP decoding. For values of SNR where both the lower bound on $P_{\mu_{\text{riv}}}^\mu$ and the upper bound on $P_{\mu_{T,W}}$ are non-trivial it is actually advantageous for (certain classes of) long codes to threshold the LLRs before attempting to decode. In other words, since there is an $n$ large enough (as a function of $K$ and $K''$) such that $n \exp(-(K'' n))$ is less than $(1-1/(K' n \beta_{d_v} \delta_{d_v}))(2 \pi K' n \beta_{d_v} \delta_{d_v})^{-1/2} \exp(-\frac{d_v \beta_{d_v} \delta_{d_v}}{2})$, operating on the thresholded versions of the LLRs will yield a smaller probability of error than retaining the full information contained in $\lambda$.

What does this mean for a pseudo-codeword $\omega$ associated with a canonical completion? Roughly speaking, the mappings $\mu_{T,W}$ and $\mu_{Q2,L}$ bend the vector $\lambda$ in such a way that the pseudo-codeword $\omega$ is less often the result of the LP decoder. This bending, which for an optimal decoder can only deteriorate its performance, is turned out to be overall helpful for a sub-optimal algorithm like the LP decoder, at least for certain interesting classes of LDPC codes and large enough SNRs.

V. PROOF OF THEOREM 2

This Section is devoted to proving Th. 2. Before we start going through the different steps of the proof, we introduce some useful notation. For an integer $n$, we use $[n]$ to denote the set of integers from 1 to $n$. We use $T(n, m)$ to denote a Tanner graph with $n$ variable nodes and $m$ check nodes. For such a Tanner graph, we will usually identify the set of variable nodes $V$ with $[n]$ and the set of check nodes $C$ with $[m]$. For a set of nodes $S$, let $N(S)$ denote the neighbor set of $S$.

Definition 3: A Tanner graph $T$ with variable node set $V$ of size $n$, is an $(\alpha n, \beta)$-expander if all sets $S \subseteq V$ with $|S| \leq \alpha n$ have $|N(S)| \geq \beta|S|$. □

The following proposition follows from [14] (see also [15]):

Proposition 3: Let $0 < r < 1$, and let $d_v$ and $d_c$ be positive integers such that $r = 1 - \frac{d_v}{d_c}$. Then for any $0 < \delta < 1 - \frac{1}{d_c}$, and sufficiently large $n$, there exists a Tanner graph with $n$ variable nodes, $m = nd_v/d_c$ check nodes, uniform variable node degree $d_v$, and uniform check degree $d_c$, which is an $(\alpha n, \beta)$-expander, where $0 < \alpha < 1$ is a constant that does not depend on $n$. Moreover, a randomly constructed graph has these properties with high probability. □

For the given truncation value $W$ in Th. 2 let $d_v$ be any integer greater than $4(4W + 2)$. Let $\delta$ be any constant where $1 - \frac{d_v}{d_c} > \delta > 1 - \frac{1}{4W + 2}$. Now let $\delta$ be the largest value that is less than or equal to $\delta$ such that $\delta d_v$ is an integer. Note that $\delta - \delta \leq \frac{1}{4W + 2}$. This implies that $\delta > 1 - \frac{1}{4W + 2}$.

From Prop. 3 we obtain a family of Tanner graphs; each graph $T(n, m)$ has uniform variable degree $d_v$, uniform check degree $d_c$, has $r = 1 - \frac{d_v}{d_c}$, and is an $(\alpha n, \beta)$-expander, for some constant $\alpha$ that does not depend on $n$. Fix a particular length $n$, and call $C \triangleq C(n, m)$ the code defined by the Tanner graph $T \triangleq T(n, m)$ from the family.

Suppose the vector $+1 = 0 \in C$ is transmitted over the AWGNC. Define $U \triangleq \{ i \in [n] : \lambda_i' < 1/2 \}$, where $\lambda_i'$ is defined according to (2). This set represents the variable nodes with “high noise.” For one particular $i \in [n]$, define $p(\sigma^2)$ as the probability that $i \in U$. Note that $p(\sigma^2)$ is the same for all $i$, is a function only of the variance $\sigma^2$, and goes to zero as $\sigma^2$ goes to zero.

Define $\gamma \triangleq \frac{(1-\delta)\delta_d}{(1-\delta)\delta_d + 1}$. Note that $0 < \gamma < 1$. Let $\sigma^2$ be sufficiently small so that $p(\sigma^2) < \frac{\alpha}{2(1+\gamma)}$. By a simple Chernoff bound we have that

$$|U| \leq \frac{\alpha n}{2(1+\gamma)} \leq \frac{\alpha n - 1}{1+\gamma}$$

with probability at least $1 - 2^{-\Omega(n)}$. In other words, with high probability, the set of nodes with high noise is “small.”

We let $\delta' \triangleq 2\delta - 1$ and define

$$U' \triangleq \{ i \in V \mid i \notin U \text{ and } |N(i) \cap N(U)| > (1-\delta')d_v \}.$$ 

The set $U'$ represents the variable nodes that do not have high noise, but do have high connectivity to the neighbors of the nodes with high noise.

We appeal to the following, which uses the same argument as a similar theorem in [6]:

Theorem 4: If $T$ is an $(\alpha n, \beta \delta_d)$-expander and $|U| \leq \frac{\alpha n - 1}{1+\gamma}$ then $|U| + |U'| \leq \alpha n$.

□

Using 5 together with this theorem, we have that $|U| + |U'| \leq \alpha n$ with probability at least $1 - 2^{-\Omega(n)}$. At this point we will apply what we know about the expansion of the graph to prove that the LP decoder succeeds. We first need another definition and proposition from [6]:

Definition 4 (6): A $\delta$-matching of $U$ is a subset $M$ of the edges incident to $U' \triangleq U \cup U$ such that (i) every check node incident to at most one edge of $M$, (ii) every node in $U$ is incident to at least $\delta d_v$ edges of $M$, and (iii) every node in $U'$ is incident to at least $\delta' d_v$ edges of $M$.

□

Proposition 5 (6): If $T$ is an $(\alpha n, \beta \delta_d)$-expander with $\delta \delta_d$ an integer, and $|U| + |U'| \leq \alpha n$, then $U$ has a $\delta$-matching. □

It remains to show how the existence of a $\delta$-matching proves that the LP decoder will succeed. To prove that the LP decoder succeeds, we use the method of finding a dual witness. More details, as well as a general treatment of this technique, can be found in [6], [10]. Here, we state the definition and theorem relevant to this application:

4A similar comment can be made about LP decoding with $\mu = \mu_{\text{riv}}$ vs. $\mu = \mu_{Q2,L}$: there is an $n$ from where on it is better to work with the one-bit quantized LLRs than with the original LLRs.

5The value 1/2 in the definition of $U$ was set for simplicity. The main theorem will go through for any $W > 0$, as long as this constant “1/2” is less than 1, greater than zero, and less than or equal to $W$. 


Definition 5 [6]): Given a Tanner graph \( T(n,m) \), and a vector of LLRs \( \lambda'_i \), a setting of weights \( \{\tau_{ij}\} \) to the edges \((i,j)\) in \( T \) is feasible if (i) for all checks \( j \in [m] \) and distinct \( i, i' \in N(j) \), we have \( \tau_{ij} + \tau_{i'j} \geq 0 \), and (ii) for all nodes \( i \in [n] \), we have \( \sum_{j \in N(i)} \tau_{ij} < \lambda'_i \).

Theorem 6 [6]): Under any memoryless binary-input output-symmetric channel, using any binary linear code, under the assumption that \( +1 = \bar{0} \) is transmitted, the LP decoder succeeds with probability \( \geq 0 \) if and only if there exists a feasible weight assignment to the edges of \( T \).

Finally, using a line of reasoning similar to [6], we establish that a \( \delta \)-matching is sufficient to guarantee a feasible edge weight assignment, and thus a proof that the LP decoder succeeds. Here is where we use our bound on \( \delta \) in the theorem:

**Theorem 7:** If \( U \) has a \( \delta \)-matching, and \( \delta > 1 - \frac{1}{4W+2} \), then there exists a feasible edge weight assignment.

**Proof:** Given a \( \delta \)-matching \( \mathcal{M} \), we assign weights \( \tau_{ij} \) to each edge \((i,j)\) in the graph as follows; we later specify the parameter \( \kappa > 0 \).

- For all \( j \) such that \((i,j) \in \mathcal{M}\) for some \( i \in U \), set \( \tau_{ij} = -\kappa \), and set \( \tau_{i'j} = \kappa \) for all \( i' \in N(j) \setminus \{i\} \).
- For all other \( j \), set \( \tau_{ij} = 0 \) for all \( i \in N(j) \).

This weighting clearly satisfies condition (i) of a feasible weight assignment. For the second condition, there are three cases.

1) For a variable node \( i \in U \), we have \(-W \leq \lambda'_i < 1/2 \). By definition of \( \mathcal{M} \), at least \( \delta d_v \) edges incident to \( i \) have \( \tau_{ij} = -\kappa \). All other incident edges have \( \tau_{ij} \in \{0,\kappa\} \), and so the total weight of edges incident to \( i \) is at most \( \delta d_v (-\kappa) + (1 - \delta) d_v \kappa \). If we maintain \( \kappa > \frac{W}{W + \delta W} \), then this total weight less than \(-W \), which is less or equal to \( \lambda'_i \), as required.

2) For a variable node \( i \in \hat{U} \), we have \( \lambda'_i \geq 1/2 \). At least \( \delta' d_v \) edges incident to \( i \), are in \( \mathcal{M} \), and therefore have weight \( 0 \), by definition of \( \mathcal{M} \) and the weight assignment. All other edges have weight \( 0 \) or \( +\kappa \). Therefore the total weight of incident edges is at most \( (1 - \delta') d_v \kappa = (1 - \delta) d_v \kappa \). If we maintain \( \kappa < \frac{1}{\delta(1 - \delta')} \), then this total weight is less than \( 1/2 \), which is less or equal to \( \lambda'_i \), as required.

3) For a variable node \( i \notin (U \cup \hat{U}) \), by definition this variable node has at least \( \delta' d_v \) edges not incident to \( N(U) \). These edges all have weight \( 0 \), and so we get the same condition (b) as in the previous case.

Combining our requirements (a) and (b) on \( \kappa \), we get the overall requirement \( \frac{2(1 - 1/2)}{4(1 - \delta)} > W \), which is equivalent to our assumption on \( \delta \).

Putting it all together, we have shown that for an arbitrary truncation value \( W \), and rate \( r \), there is a sufficiently small \( \sigma^2 \) and a family of \((d_v, d_c)\)-regular graphs on which the LP decoder succeeds with probability \( 1 - 2^{-\Omega(r)} \) when \(+1 = \bar{0}\) is transmitted over an AWGNC with noise power \( \sigma^2 \) and with LLR modification \( \mu = \mu_{T,W} \). The assumption that \(+1 = \bar{0}\) is transmitted is without loss of generality because the polytope is “C-symmetric” (see [4], [3] for details). Thus we have shown that the word error rate of the LP decoder decreases exponentially.

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