1. Introduction

One of the possible ways to create a universal quantum computer is by one-way quantum computation [1]. Its implementation requires a kind of ‘resource’—a physical system in a quantum cluster state. Every single element of such a system is connected to one or more others via quantum entanglement so that all together they form a complex physical structure. Here, the entanglement is a multipartite-entanglement or inseparability in terms of [2]. Mathematically, such a structure can be described by an undirected graph whose nodes are the elements of the physical system, and the edges are the quantum entanglements between them. The type of logic gate, implemented by sequential local measurements over single nodes, depends on the configuration (or topology) of the graph. Since the quantum entanglement between nodes reduces in the process of local measurements, the cluster state dimension (‘the amount of resource’) gradually decreases. This makes the entire process irreversible or one-way. However, it was shown that such an approach to building a universal quantum computer is effective and in no way inferior to quantum computing on reversible quantum logic gates [3].

To date, various one-way quantum computation schemes have been proposed for both discrete [1] and continuous variables [4]. Some of them have been successfully realized experimentally [5, 6]. As such, the main obstacle to their implementation in effective information applications, as well as in the case of quantum computations on reversible logic gates, is the low degree of scalability. At the material level, the scalability problem relates to limitations on the size and topology of the cluster states, which in turn lead to limitations on the number of logical operations and the volume of processed data. The type of variable that describes the physical system plays a significant role since it determines the nature of the limitations.

In the case of discrete variables, one generates a cluster state on the bases of single independent qubits (or qudits [7]). Single-photon sequences from quantum dots [8] or single atoms located at the nodes of an optical lattice can be used [9]. As such, the probabilistic nature of the operations being performed (e.g. the obtaining of single photons, qubit entangling, etc) restricts the generation of a large-scale cluster state. Due to the low efficiency of these processes, the generation of a large-scale cluster state can take an extremely long time in practice, substantially exceeding the decoherence time of individual qubits. It proves to be difficult to scale such systems.

For continuous variables, all operations on systems (oscillators) in the Gaussian quadrature squeezed state that have
been used to generate a cluster state are deterministic. For their realization, schemes based on light pulse trains [10], spin waves within atomic ensembles [11], and the eigenmodes of optomechanical systems [12] are proposed. Moreover, methods for generating ‘hybrid’ cluster states based on matter-field oscillators have been put forward [11]. In this case, the limitations in generating a large-scale cluster state relate to a finite degree of quadrature squeezing of the oscillators. By analogy with the Duan criterion [13], characterizing the measure of the entanglement of two quantum oscillators, the van Loock–Furusawa criterion [2] for cluster states shows that the entanglement of several ($N \geq 2$) quantum oscillators depends directly on their initial degree of squeezing. Since a large number of squeezed oscillators is required to generate a large-scale cluster state, it is clear that the squeezing degree should be high. Most of the early works on continuous variable cluster states aimed at the demonstration of the fundamental possibility of performing one-way quantum computations estimating the degree of squeezing of individual oscillators by their limiting value—infinity. In practice, it turns out to be challenging to get oscillators with a high degree of squeezing. To date light pulses with quadrature squeezing of 15 dB have been obtained experimentally [14]. As such, the question of the minimum squeezing degree of the quantum oscillator required to generate the cluster states of different topologies based therein has been little studied [15].

In this paper, we will obtain a criterion determining the minimum degree of quadrature squeezing of the initial oscillators needed to generate a cluster state with a given topology. Using this criterion, we will find which of the cluster state nodes require the highest degree of squeezing for its generation. Then, in the case of a given degree of squeezing, we estimate the maximum number of adjacent nodes entangled with a selected one. This will enable us to evaluate quantitatively the various quantum cluster topologies and formulate optimal generation strategies, based on these estimates.

## 2. Continuous variable cluster state

Cluster states are a type of quantum multipartite entangled state characterized by an undirected graph. To define graph $G$ of the cluster state, one needs to specify the set of nodes and the set of edges that reveal the interconnections between the nodes. Examples are shown in figure 1. Every edge connecting the $i$th node with the $j$th one can be characterized by a real number $a_{ij} \in [-1, 1]$, called the weight of the edge. The set of these weights defines an adjacency matrix $A$, which in turn completely defines graph $G$. Thus, the graphs in figures 1(a) and (b) correspond to the adjacency matrices $A^{(1)}$ and $A^{(2)}$, respectively:

\[
A^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 0 & -1/2 & 0 & 0 & -1/2 & 0 \\ -1/2 & 0 & -1/2 & 1/2 & 0 & -1/2 \\ 0 & -1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 1/2 & 0 & 1/2 \\ 0 & -1/2 & 0 & 0 & 1/2 & 0 \end{pmatrix}.
\] (1)

A mathematical object—a graph—can be put in correspondence with a physical one—a quantum cluster state. As for cluster state generation, the following procedure is usually discussed: one considers $n$ independent quantum harmonic oscillators with squeezed quadratures, and every oscillator is assigned by the quadrature operators $\hat{x}$ and $\hat{y}$, which obey the canonical commutation relations

\[
[\hat{x}_j, \hat{y}_k] = i \delta_{jk},
\] (2)

where $j$ and $k$ are the numbers of the oscillators, and $\delta_{jk}$ is the Kronecker delta. All the oscillators are assumed to be squeezed in $\hat{y}$-quadratures [16], i.e. their variances are less than those of the vacuum state

\[
\langle \delta \hat{y}_j^2 \rangle < \frac{1}{4}, \quad j = 1, \ldots, n.
\] (3)

One entangles these subsystems in pairs so that the coupling force between the $i$th and $j$th oscillators corresponds to the element $a_{ij}$ of the adjacency matrix $A$ to satisfy equations (5) and (6); see below. This results in a cluster state of the physical system, corresponding to the graph $G$ defined by the adjacency matrix $A$. Note that the nodes of graph $G$ correspond to the quantum harmonic oscillators entangled in a given way. Such an entanglement can be described by Bogolyubov’s transformation [17] of the initial set of independent quadrature-squeezed oscillators:

\[
\hat{X}_j + i \hat{Y}_j = \sum_{k=1}^n u_{jk} (\hat{x}_k + i \hat{y}_k), \quad j = 1, \ldots, n.
\] (4)

where $U = \{u_{jk}\}_{j,k=1}^n$ is a unitary matrix that specifies the set of transformations of the subsystems, so that the result corresponds to the adjacency matrix $A$ (the connection of these matrices will be discussed below). $\hat{X}_j$ and $\hat{Y}_j$ are quadrature operators of the $j$th node of the cluster state.

To describe the quantum statistical properties of discrete variable cluster states, stabilizers are usually used [1]. However, for continuous variables, the most natural way to describe the statistics of a cluster state is to use the nullifier operators defined for every node of graph $G$.

\[
\hat{N}_j = \hat{Y}_j - \sum_{i=1}^n a_{ij} \hat{X}_i, \quad j = 1, \ldots, n.
\] (5)

By definition, the physical system is in a quantum cluster state if the variances of all of its nullifiers tend to zero in the limit of infinite squeezing of the quantum harmonic oscillators used to generate it [16]:

\[
\forall j = 1, \ldots, n, \quad \lim \langle \delta \hat{N}_j^2 \rangle = 0, \quad \text{for} \quad \langle \delta \hat{x}_j^2 \rangle \to 0, \quad \langle \delta \hat{y}_j^2 \rangle \to 0.
\] (6)
3. Bogoliubov transformation for cluster states

To obtain the desired condition on a minimal squeezing of the initial oscillators required for cluster state generation, we need to rewrite the Bogoliubov transformation $U$ given by the adjacency matrix $A$ in explicit form. Let us derive equation (4) in a vector form

$$\mathbf{\hat{X}} + i\mathbf{\hat{Y}} = U (\mathbf{\hat{x}} + i\mathbf{\hat{y}}) = (\text{Re } U\mathbf{\hat{X}} - \text{Im } U\mathbf{\hat{Y}}) + i (\text{Re } U\mathbf{\hat{Y}} + \text{Im } U\mathbf{\hat{X}}),$$

where $\mathbf{\hat{Y}} = (\mathbf{\hat{Y}}_1, \mathbf{\hat{Y}}_2, \ldots, \mathbf{\hat{Y}}_n)^T$, $\mathbf{\hat{X}} = (\mathbf{\hat{X}}_1, \mathbf{\hat{X}}_2, \ldots, \mathbf{\hat{X}}_n)^T$ are column vectors formed by the quadratures of the nodes of the cluster state. We introduce vector $\mathbf{\hat{N}}$, whose elements are the nullifiers (5), by the adjacency matrix $A$ as follows:

$$\mathbf{\hat{N}} = \mathbf{\hat{Y}} - A\mathbf{\hat{X}}.$$  

(8)

By equation (7), we express the quadratures of the nodes of the cluster state through the variables of independent quadrature-squeezed oscillators that were used to generate the cluster, and substitute the result in equation (8)

$$\mathbf{\hat{N}} = \text{Re } U\mathbf{\hat{Y}} + \text{Im } U\mathbf{\hat{X}} - A (\text{Re } U\mathbf{\hat{X}} - \text{Im } U\mathbf{\hat{Y}})$$

$$= (\text{Im } U - A \cdot \text{Re } U)\mathbf{\hat{x}} + (\text{Re } U + A \cdot \text{Im } U)\mathbf{\hat{y}}.$$  

(9)

In order for operators to be nullifiers, their variances have to tend to zero. We assume that all $\mathbf{\hat{y}}$-quadratures are squeezed; hence, according to the Heisenberg’ uncertainty principle, all the $\mathbf{\hat{X}}$-quadratures are stretched. For this reason, the variances of the nullifiers (9) tend to zero only if the coefficients in front of the stretched quadratures are zero. Thus, the set of transformations $U$ has to satisfy the equality

$$\text{Im } U = A \cdot \text{Re } U.$$  

(10)

Hence, the nullifier vector $\mathbf{\hat{N}}$ and the Bogoliubov transformation matrix $U$ are related as follows:

$$\mathbf{\hat{N}} = (I + A^2) \text{ Re } U \mathbf{\hat{y}},$$  

(11)

$$U = \text{Re } U + i \text{Im } U = (I + iA) \text{ Re } U.$$  

(12)

Here, matrix $\text{Re } U$ remains unknown. In order to specify it, we use the condition of its unitarity $U^\dagger U = ((I + iA) \text{ Re } U)^\dagger (I + iA) \text{ Re } U$

$$= (\text{Re } U)^\dagger (I + A^2) \text{ Re } U = I,$$  

(13)

$$(I + A^2) = ((\text{Re } U)^\dagger)^{-1} (\text{Re } U)^{-1} = (\text{Re } U (\text{Re } U)^\dagger)^{-1}$$

$$\Rightarrow \text{Re } U (\text{Re } U)^\dagger = (I + A^2)^{-1}.$$  

(14)

The resulting expressions arise from the fact that matrix $\text{Re } U$ is real, and matrix $A$ is Hermitian.

For further analysis, we employ the polar decomposition of matrix $\text{Re } U$. A polar decomposition is a representation of an arbitrary square matrix $M$ as a product of a Hermitian $H (H = H^*)$ and a unitary $U (U^\dagger U = UU^\dagger = I)$ matrix, so $M = HU$. Since the matrix $\text{Re } U$ is real, the polar decomposition for it turns into a product of the symmetric matrix $S (S = S^\dagger)$ and the arbitrary orthogonal matrix $Q (QQ^\dagger = Q^\dagger Q = I)$. Substituting this expansion into equation (14), we obtain

$$SQ (SQ)^\dagger = SQQ^\dagger S = S^2 = (I + A^2)^{-1}$$

$$\Rightarrow S = (I + A^2)^{-1/2}$$ 

(15)

$$\Rightarrow \text{Re } U = (I + A^2)^{-1/2} Q.$$  

(16)

Thus, equations (11) and (12) can be derived as

$$\mathbf{\hat{N}} = (I + A^2)^{1/2} Q \mathbf{\hat{y}},$$  

(18)

$$U = (I + iA)(I + A^2)^{-1/2} Q.$$  

(19)

Let us analyze the result. We have obtained the dependence of the nullifiers and transformation matrix $U$ on the configuration of graph $G$. The dependences (18) and (19) are defined within an arbitrary orthogonal matrix $Q$. The question that arises is what this matrix influences. Assuming that one-way quantum computations depend only on a cluster state topology, the authors in [15] showed the influence of this matrix on the generation procedure. In addition, it can be assumed that via this matrix it is possible to minimize the errors that appear in the generation process due to the non-ideality of physical systems.

Thus, for the cluster state characterized by graph $G$ we have indicated the relation between the explicit form of the Bogoliubov transformation $U$ and the adjacency matrix $A$. 

Figure 1. An example of two graphs with different topologies. Graph (a): edges connecting adjacent nodes have the same weight equal to 1. Graph (b): double and single lines are the edges with weights $-1/2$ and $1/2$, respectively.
Furthermore, based on equations (18) and (19), we will prove the theorem on the relation between the nullifiers’ variances and the variances $\langle \delta y_j^2 \rangle$ in the case of the identical initial oscillators used to generate the cluster state.

4. Criterion on minimal squeezing degree

To generate a quantum physical system in a cluster state, it is important to know the dependence of nullifier variances on the quadrature variances of the independent quantum harmonic oscillators used for cluster generation. In general, if the quadratures of the oscillators are squeezed differently, one cannot exhibit the explicit formula. However, if the oscillators are squeezed identically, this dependence will only be characterized by the adjacency matrix $A$. Let us prove the following theorem.

**Theorem.** Suppose a cluster state that corresponds to graph $G$ with the adjacency matrix $A$, given by the weight coefficients $a_{ij}$. Let $Q \in \mathbb{M}^{n \times n}$ be the orthogonal matrix in a polar decomposition of the real part of the Bogoliubov transformation $U (\operatorname{Re} U = (I + A^2)^{-1/2} Q)$. If the quantum oscillators used to generate the cluster state have the same statistically independent squeezing degree, then the variances of the cluster state nullifiers can be expressed in terms of the variances of the initially squeezed oscillators as follows:

$$\langle \delta N_j^2 \rangle = 1 + \sum_{i=1}^{n} a_{ij}^2 \langle \delta y_i^2 \rangle, \quad j = 1, \ldots, n. \quad (20)$$

**Proof.** To prove the theorem, let us use equation (18)

$$\tilde{N} = (I + A^2)^{1/2} Q \tilde{y}. \quad (21)$$

We introduce notation $V = (I + A^2)^{1/2}$ and derive the $j$th nullifier as

$$\tilde{N}_j = \sum_{k=1}^{n} v_{jk} \sum_{p=1}^{n} q_{kp} \tilde{y}_p. \quad (22)$$

where $v_{jk}$ are the matrix elements $V$, and $q_{kp}$ are the matrix elements $Q$. Let us consider the nullifier variance

$$\langle \delta N_j^2 \rangle = \langle \delta \left( \sum_{k=1}^{n} v_{jk} \sum_{p=1}^{n} q_{kp} \tilde{y}_p \right)^2 \rangle = \langle \delta \left( \sum_{p=1}^{n} \left( \sum_{k=1}^{n} v_{jk} q_{kp} \right) \tilde{y}_p \right)^2 \rangle = \sum_{p=1}^{n} \left( \sum_{k=1}^{n} v_{jk} q_{kp} \right)^2 \langle \delta y_p^2 \rangle. \quad (23)$$

Here, we employ the statistical independence of the quantum oscillators, i.e. $\langle \delta y_i \delta y_j \rangle = \delta_{ij} \langle \delta y_i^2 \rangle$, where $\delta_{ij}$ is the Kronecker delta. The first and the second terms of the multiplier on the right-hand side of equation (23) can be derived as:

$$\sum_{p=1}^{n} \sum_{k=1}^{n} v_{jk}^2 q_{kp}^2 = \sum_{p=1}^{n} \sum_{k=1}^{n} v_{jk}^2 \delta_{ip} = \sum_{k=1}^{n} v_{jk}^2 \langle \delta y_i^2 \rangle, \quad (24)$$

$$\sum_{p=1}^{n} \left( \sum_{w=1}^{n} \sum_{r=w+1}^{n} (v_{jw} q_{wp}) (v_{jr} q_{rp}) \right) = \sum_{p=1}^{n} \left( \sum_{w=1}^{n} \sum_{r=w+1}^{n} (v_{jw} v_{jr}) \langle \delta y_w \delta y_r \rangle \right). \quad (25)$$

$\tilde{q}_i = (q_{i1}, q_{i2}, \ldots, q_{in})^T$ is a row vector corresponding to the $i$th row of matrix $Q$. By definition, an orthogonal matrix is a matrix with columns and rows that are all orthonormal, i.e. $\langle \tilde{q}_i \tilde{q}_j \rangle = \delta_{ij}$. Taking into account equations (24) and (25), the equation (23) for nullifier variance can be derived as:

$$\langle \delta N_j^2 \rangle = \sum_{k=1}^{n} v_{jk}^2 \langle \delta y_i^2 \rangle. \quad (26)$$

Let us consider the matrix $V = (I + A^2)^{1/2}$. Since the adjacency matrix for the cluster state is symmetric ($A = A^T$), it follows that

$$(A^2)^T = (AA)^T = A^TA^T = AA = A^2. \quad (27)$$

This means $(I + A^2)^T = (I + A^2)$, so $V$ is also symmetric ($V = V^T$) since

$$(VV)^T = (I + A^2)^T = (I + A^2) = VV. \quad (28)$$

Thus, for the diagonal elements $[V^2]_{jj}$, on the one hand, we have

$$[V^2]_{jj} = \sum_{k=1}^{n} v_{jk}^2, \quad (29)$$

on the other hand,

$$[V^2]_{jj} = [I + A^2]_{jj} = 1 + [A^2]_{jj}. \quad (30)$$

Since $A$ is an adjacency matrix, its symmetry implies that $[A^2]_{jj} = \sum_{i=1}^{n} a_{ij}^2$. Summarizing the outcome, we derive the resulting expression for nullifiers

$$\langle \delta y_j^2 \rangle = 1 + \sum_{i=1}^{n} a_{ij}^2 \langle \delta y_i^2 \rangle, \quad j = 1, \ldots, n. \quad (31)$$
The proven theorem provides us with a tool for the easy calculation of the nullifiers’ variances, which in turn allows one to answer the question of the separability of the cluster.

**Example 1.** Let us consider the cluster state corresponding to the graph in figure 1(b), where double and single lines denote edges with weights \(-1/2\) and \(1/2\), respectively. According to the theorem proved above, the nullifier variances are equal to

\[
\langle \delta \hat{N}_i^2 \rangle = \frac{3}{2} \langle \delta y^2 \rangle, \quad \langle \delta \hat{N}_j^2 \rangle = 2 \langle \delta y^2 \rangle, \quad \langle \delta \hat{N}_k^2 \rangle = \frac{3}{2} \langle \delta y^2 \rangle, \quad (32)
\]

\[
\langle \delta \hat{N}_l^2 \rangle = \frac{3}{2} \langle \delta y^2 \rangle, \quad \langle \delta \hat{N}_m^2 \rangle = 2 \langle \delta y^2 \rangle, \quad \langle \delta \hat{N}_n^2 \rangle = \frac{3}{2} \langle \delta y^2 \rangle. \quad (33)
\]

**Corollary 1.** In the case of the ‘unweighted’ cluster state, i.e. when all its non-zero weight coefficients are equal to one, we have

\[
\langle \delta \hat{N}_i^2 \rangle = (1 + \dim \{Nb [j]\}) \langle \delta y^2 \rangle, \quad j = 1, \ldots, n. \quad (34)
\]

where \(\dim \{Nb [j]\}\) is the dimension of the set of adjacent nodes (the number of neighbors) for the \(j\)th node of graph \(G\).

**Proof.** From equation (31) it follows that if all non-zero weight coefficients \(a_j\) are equal to one, then the sum on the right-hand side of equation (31) becomes the dimension of the set of adjacent nodes for the \(j\)th node (\(\dim \{Nb [j]\}\)):

\[
\langle \delta \hat{N}_i^2 \rangle = (1 + \dim \{Nb [j]\}) \langle \delta y^2 \rangle, \quad j = 1, \ldots, n. \quad (35)
\]

Thus, in the case of an ‘unweighted’ graph the variance of the \(j\)th nullifier is determined only by the number of nodes of graph \(G\) connected with the \(j\)th node.

**Example 2.** Let us consider an ‘unweighted’ linear cluster state that corresponds to the 4-node linear graph in figure 1(a). According to corollary 1, the nullifier variances of this state are equal to

\[
\langle \delta \hat{N}_1^2 \rangle = 2 \langle \delta y^2 \rangle, \quad \langle \delta \hat{N}_2^2 \rangle = 3 \langle \delta y^2 \rangle, \quad \langle \delta \hat{N}_3^2 \rangle = 3 \langle \delta y^2 \rangle, \quad \langle \delta \hat{N}_4^2 \rangle = 2 \langle \delta y^2 \rangle. \quad (36)
\]

Having proved the theorem on the cluster state nullifier variances in the case of the identical independent quantum oscillators used for its generation, we can analyze the cluster via the van Loock–Furusawa separability criterion [2]. This criterion allows one to specify the minimum squeezing of oscillators required to generate a cluster of a given topology based therein. Violation of this condition means that the state is separable.

**Corollary 2.** To generate a cluster state corresponding to graph \(G\) with the adjacency matrix \(A\) given by the weight coefficients \(a_{ij}\), the squeezing of every initial quantum oscillator has to satisfy the inequality

\[
\langle \delta y_i^2 \rangle < \min_{(i,j)} \left[\frac{|a_{ij}|}{2 + \sum_{k=1}^{n} a_{ik}^2 + \sum_{l=1}^{n} a_{lj}^2}\right]. \quad (37)
\]

where the minimum on the right-hand side is taken over all the pairs of the adjacent nodes \(i\) and \(j\).

**Proof.** Let us apply the van Loock–Furusawa separability criterion to the cluster state corresponding to graph \(G\). In general, this criterion determines the possibility of separating the set of \(n\) elements described by the canonical variables \(\{\hat{X}_k, \hat{Y}_k\}_{k=1}^{n}\) by \(M\)-independent subsets \(S_r (r = 1, \ldots, M)\). Mathematically, it can be derived as inequality [18, 19]

\[
\langle \delta \hat{b}_i^2 \rangle + \langle \delta \hat{c}_i^2 \rangle \geq \frac{1}{2} \sum_{r=1}^{M} \sum_{k \in S_r} \left| \hat{h}_k \hat{g}_k - \hat{h}_k \hat{g}_k \right|. \quad (38)
\]

where \(\hat{b} = \sum_{k=1}^{n} \left[ a_k \hat{X}_k + b_k \hat{Y}_k \right] \hat{c} = \sum_{k=1}^{n} \left[ h_k \hat{X}_k + g_k \hat{Y}_k \right] \) are auxiliary Hermitian operators that are linear combinations of all the canonical variables, and \(h_k, h_k, g_k, g_k\) are real constants. In our case, the quadratures of the cluster state are taken as canonical variables. We choose the constants \(\{h_k, h_k, g_k, g_k\}_{k=1}^{n}\) so that the operators \(\hat{b} \) and \(\hat{c}\) turn into the nullifiers of adjacent nodes of the cluster state, i.e \(\hat{b} = \hat{N}_i = \hat{Y}_i = \sum_{k=1}^{n} a_{ik} \hat{X}_k \) and \(\hat{c} = \hat{N}_j = \hat{Y}_j = \sum_{k=1}^{n} a_{jk} \hat{X}_k\). Thus, we obtain an inequality

\[
\langle \delta \hat{N}_i^2 \rangle + \langle \delta \hat{N}_j^2 \rangle \geq \begin{cases} 0, & \text{при } i \in S_r, \ j \in S_r, \\ |a_{ij}|, & \text{при } i \in S_r, \ j \in S_{r'}. \end{cases} \quad (39)
\]

where the nodes \(i\) and \(j\) of graph \(G\) are connected by an edge with the weight coefficient \(a_{ij}\). The first condition in equation (39) is trivial since it is always met and carries no information about the connection between the nodes. The second one is met only if nodes \(i\) and \(j\) belong to different independent subsystems. This condition determines the separability criterion for two adjacent nodes of the cluster state. The nodes will be inseparable if the inequality is violated, i.e.

\[
\langle \delta \hat{N}_i^2 \rangle + \langle \delta \hat{N}_j^2 \rangle < |a_{ij}|. \quad (40)
\]

Let us substitute the nullifier variances (31) in the inequality (40)

\[
\langle \delta \hat{N}_i^2 \rangle + \langle \delta \hat{N}_j^2 \rangle = \left(2 + \sum_{k=1}^{n} a_{ik}^2 + \sum_{l=1}^{n} a_{lj}^2\right) \langle \delta y^2 \rangle < |a_{ij}|. \quad (41)
\]

For the variance of the \(y\)-quadrature, we obtain

\[
\langle \delta y_i^2 \rangle < \frac{|a_{ij}|}{2 + \sum_{k=1}^{n} a_{ik}^2 + \sum_{l=1}^{n} a_{lj}^2}. \quad (42)
\]
Depending on the topology of the graph, the right-hand side of this equation may vary. Since the variances of the nullifiers should tend to zero to generate the cluster state, we minimize the value on the right-hand side of equation (42) by the numbers \(i\) and \(j\) of the adjacent nodes of graph \(G\) as

\[
\langle \delta y^2 \rangle < \min_{(i,j)} \left[ \frac{|a_{ij}|}{2 + \sum_{k=1}^{n} a_{ik}^2 + \sum_{l=1}^{n} a_{lj}^2} \right]. \tag{43}
\]

Thus, we have obtained an estimation criterion on the squeezing of the \(\hat{y}\)-quadratures of the initial quantum oscillators used to generate a cluster state.

Let us consider an application of corollary 2.

**Example 3.** Let us take an ‘unweighted’ linear cluster state that corresponds to a 4-node linear graph (see figure 1(a)). For this cluster state, there are two types of condition

\[
\langle \delta y^2 \rangle < \frac{1}{5}, \quad \langle \delta y^2 \rangle < \frac{1}{6}. \tag{44}
\]

Since we should choose the minimum, the condition for the cluster to be inseparable is

\[
\langle \delta y^2 \rangle < \frac{1}{6}. \tag{45}
\]

This condition indicates that to generate a cluster with the given topology it is sufficient to have 4 oscillators with quadratures squeezed more than \(-1, 77\) dB.

**Example 4.** Let us consider the cluster state corresponding to the graph in figure 1(b), where double and single lines denote edges with weights equal to \(-1/2\) and \(1/2\), respectively. For this state, the inseparability condition is

\[
\langle \delta y^2 \rangle < \frac{1}{7}. \tag{46}
\]

The required squeezing, in this case, should exceed \(-2.43\) dB.

The considered examples demonstrate that finite, experimentally realizable squeezing is sufficient to generate a cluster state. Note that a more complex and branched structure of the cluster requires more stringent conditions for its generation. In this respect, the use of simple (e.g. linear) clusters is preferable if they satisfy computational needs.

The question arises of whether such squeezing is sufficient to implement quantum computations on the cluster state. The answer is ambiguous. On the one hand, this squeezing would be sufficient for single quantum gate realization. On the other hand, when considering quantum computations with a large number of gates, the errors will accumulate and could eventually abolish all the computation results. It was long believed that an arbitrarily large number of logic gates could only be performed in the case of infinite squeezing of the oscillators used to generate the cluster states. However, it was shown in [20] that any errors can be corrected at an initial squeezing of 20, 5 dB. Unfortunately, at the moment such squeezing has not yet been experimentally implemented. Thus, even low experimentally realizable squeezing can be a resource for cluster generation and the performing of limited quantum computations. In this case, the ‘redundant data’ procedure [21] can be used for error correction.

The statement above also allows us to consider the inverse problem: what kind of cluster structure can be generated with a given resource?

**Corollary 3.** When generating an ‘unweighted’ cluster state on the base of statistically independent oscillators with a given squeezing degree \(\langle \delta y_{\text{fix}}^2 \rangle\), the maximum number of neighbors for two adjacent nodes in the cluster is estimated by an inequality

\[
\max_{(i,j)} \left[ \dim (\text{Nb} [j]) + \dim (\text{Nb} [i]) \right] < \frac{1}{\langle \delta y_{\text{fix}}^2 \rangle} - 2, \tag{47}
\]

where the maximum on the left-hand side is taken all over the pairs of adjacent nodes \(i\) and \(j\).

**Proof.** Let us rewrite the inequality (43) as

\[
\frac{2 + \max_{(i,j)} \left( \sum_{k=1}^{n} a_{ik}^2 + \sum_{l=1}^{n} a_{lj}^2 \right)}{\min_{(i,j)} \left( |a_{ij}| \right)} < \frac{1}{\langle \delta y^2 \rangle}. \tag{48}
\]

Since we are considering an unweighted cluster state, we replace all non-zero weight coefficients \(a_{ij}\) by 1. Due to the fixed squeezing for all oscillators we put \(\langle \delta y^2 \rangle = \langle \delta y_{\text{fix}}^2 \rangle\); this results in equation (47).

This corollary shows the maximum number of edges coming from two adjacent nodes of the graph at a given squeezing degree \(\langle \delta y_{\text{fix}}^2 \rangle\). Let us consider an example of the application of corollary 3.

**Example 5.** Let us suppose that we have oscillators with \(\hat{y}\)-quadrature squeezing of 6 dB, i.e. \(\langle \delta y^2 \rangle \approx 0.06\), and we want to generate an unweighted cluster state. Via corollary 3 we obtain a condition on the maximum number of edges of two adjacent nodes in the cluster state

\[
\max_{(i,j)} \left[ \dim (\text{Nb} [j]) + \dim (\text{Nb} [i]) \right] < 14. \tag{49}
\]

Hence, the maximum number of edges of two adjacent nodes should not exceed 13. Next, we should distribute these 13 edges between two nodes so that the designed cluster state can perform a certain quantum computation.

5. Conclusion

In this paper, we discussed the generation of cluster states based on identical independent quadrature-squeezed oscillators. We have shown that with the help adjacency matrix elements, the variances of the cluster state nullifiers can be
expressed through the variances of the quadratures (or the squeezing degree) of the oscillators. This allows us to formulate the criterion of the minimum degree of quadrature squeezing required to generate a cluster state with a given topology. With this criterion it was shown that the minimum squeezing degree is determined by the nodes of the cluster that are connected with the largest number of adjacent nodes. We estimated the maximum possible number of adjacent nodes on the graph of a cluster, depending on the squeezing degree of the initial oscillators.

The study of cluster state topology is interesting from two perspectives: for generating a cluster with a given topology on the basis of the available resource and for organizing computation in limited conditions. The first issue has been discussed in this paper, and the second one points the direction for further research.

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