Non-Archimedean Scale Invariance and Cantor Sets

Santanu Raut* and Dhurjati Prasad Datta†

Department of Mathematics
University of North Bengal, Siliguri, West Bengal, India, Pin 734013

Abstract

The framework of a new scale invariant analysis on a Cantor set $C \subset I = [0,1]$, presented originally in S. Raut and D. P. Datta, Fractals, 17, 45-52, (2009), is clarified and extended further. For an arbitrarily small $\varepsilon > 0$, elements $\tilde{x}$ in $I \setminus C$ satisfying $0 < \tilde{x} < \varepsilon < x$, $x \in C$ together with an inversion rule are called relative infinitesimals relative to the scale $\varepsilon$. A non-archimedean absolute value $v(\tilde{x}) = \log_{\varepsilon^{-1}} \frac{\varepsilon}{\tilde{x}}$, $\varepsilon \to 0$ is assigned to each such infinitesimal which is then shown to induce a non-archimedean structure in the full Cantor set $C$. A valued measure constructed using the new absolute value is shown to give rise to the finite Hausdorff measure of the set. The definition of differentiability on $C$ in the non-archimedean sense is introduced. The associated Cantor function is shown to relate to the valuation on $C$ which is then reinterpreted as a locally constant function in the extended non-archimedean space. The definitions and the constructions are verified explicitly on a Cantor set which is defined recursively from $I$ deleting $q$ number of open intervals each of length $\frac{1}{r}$ leaving out $p$ numbers of closed intervals so that $p + q = r$.

Key Words: Non-archimedean, scale invariance, Cantor set, Cantor function

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*Chhatgurhati Seva Bhavan Sikshayatan High Scool, New Town, Coochbihar-736101, email: raut_santau@yahoo.com
†corresponding author, email: dp_datta@yahoo.com
1. Introduction

A Cantor set is a totally disconnected compact and perfect subset of the real line. Such a set displays many paradoxical properties. Although the set is uncountable, it’s Lebesgue measure vanishes. The topological dimension of the set is also zero. Cantor set is an example of a self-similar fractal set that arises in various fields of applications. The chaotic attractors of a number of one dimensional maps, such as the logistic maps, turn out to be topologically equivalent to Cantor sets. Cantor set also arises in electrical communications [1], in biological systems [2], and diffusion processes [3]. Recently there have been a lot of interest in developing a framework of analysis on a Cantor like fractal sets [4, 5]. Because of the disconnected nature, methods of ordinary real analysis break down on a Cantor set. Various approaches based on the fractional derivatives [7, 8] and the measure theoretic harmonic analysis [6] have already been considered at length in the literature. However, a simpler intuitively appealing approach is still considered to be welcome.

Recently, we have been developing a non-archimedean framework [9] of a scale invariant analysis which will be naturally relevant on a Cantor set [10, 11]. For definiteness, we consider the Cantor set $C$ to be a subset of the unit interval $I = [0, 1]$. We introduce a non-archimedean absolute value on $C$ exploiting a concept of relative infinitesimals which correspond to the arbitrarily small elements $\tilde{x}$ of $I \setminus C$ satisfying $0 < \tilde{x} < \varepsilon, \varepsilon \to 0^+$ (together with an inversion rule) relative to the scale $\varepsilon$. In ref. [10], we have presented the details of the construction in the light of the middle third Cantor set. Here, we develop the formalism afresh, bringing in more clarity in the approach initiated in ref. [10]. We show that the framework can be extended consistently on a general $(p, q)$ type Cantor set which is derived recursively from the unit interval $I$ first dividing it into $r$ number of equal closed intervals and then deleting $q$ number of open intervals so that $p + q = r$. We show that the non-archimedean valuation is related to the Cantor function $\phi(x) : I \to I$ such that $\phi'(x) = 0$ a.e. on $I$ with discontinuities at the points $x \in C$. In the non-archimedean framework $\phi(x)$ is shown to be extended to a locally constant function for any $x \in I$. Using the non-archimedean valuation we also construct a valued measure on $C$ which is shown to give the finite Hausdorff measure of the set. The variability of the locally constant $\phi(x)$ is reinterpreted in the usual topology as an effect of relative infinitesimals which become dominant by inversion at an appropriate scale.

The paper is organised as follows. In Sec.2, we give a brief sketch of the details of a $(p, q)$ Cantor set and the corresponding Cantor function. In Sec.3, we give an outline of the scale invariant analysis and the valued measure on $C$. The concept of relative infinitesimals
and new absolute values are introduced in Sec.3.1. The valued measure is constructed in Sec.3.2. In Sec.3.3, we define scale invariant differentiability on $C$. In Sec.4, we discuss the example of $(p, q)$ Cantor set and show how the valuation is identified with the Cantor function. We also show explicitly how the variability of a non-archimedean locally constant function $\phi(x)$ is exposed in the usual topology when the ordinary differential equations in $I$ get extended to scale invariant equations in appropriate logarithmic (infinitesimal) variables.

2. $(p, q)$ Cantor Set and Cantor Function

To make the article self-contained we present here a brief review of the Cantor set and Cantor function. We note that the middle third Cantor set and the related Cantor function are well discussed in the literature. The definition of $(p, q)$ Cantor set $C$ and the corresponding Cantor function are analogous to the above case except for minor modifications.

We divide the unit interval $I = [0,1]$ into $r$ number of closed subintervals each of length $\frac{1}{r}$ and delete $q$ number of open subintervals from them so that $p + q = r$. The deletion of $q$ open intervals may be accomplished by an application of an iterated function system (IFS) of similitudes of the form $f = f_i : I \to I$, $i = 1, 2, \ldots, p$, where $f_i(x) = \frac{1}{r}(x + \alpha_i)$ and $\alpha_i$ assumes values from the set $\{0, 1, 2, \ldots, (r-1)\}$, $i = 1, 2, \ldots, p$. We note incidentally that there are, in fact, $r^Cq^p$ distinct IFS each of which has $C$ as the unique limit set, viz., $C = f(C)$.

To construct the limit set $C$ explicitly, we note that the set $I$, after the first iteration, is reduced to $I = \bigcup_{n=1}^{p} F_{1n}$ consisting of $p$ number of closed intervals $F_{1n}$, so that the length of the deleted intervals is $\frac{q}{r}$. Iterating the above steps in each of the closed intervals $F_{1n}$ ad infinitum we get the desired Cantor set $C = \bigcap_{n=0}^{\infty} \bigcup_{m=0}^{p^n} F_{nm}$, $F_{00} = I$. The length of the deleted intervals at the $n$ th iteration is $\frac{q}{r^n} = \frac{q}{r} \left(\frac{r}{r-1}\right)^n = \frac{q}{r} \left[1 - \left(\frac{r}{r-1}\right)^n\right] \to 1$ as $n \to \infty$. Thus the Lebesgue measure of $C$ is zero. However the Hausdorff s-measure of $C$, given by

$$\mu_s[C] = \lim_{\delta \to 0} \inf \Sigma_{i} [d(U_i)]^s$$

where $d(U_i)$ is the diameter of the set $U$ and infimum is taken over all countable $\delta$– covers $I_\delta = \{U_i\}$ such that $C \subset \bigcup U_i$ is finite for the unique value of $s$ satisfying the scaling equation $p = r^s$. The Hausdorff dimension of $C$ thus equals $\frac{\log p}{\log r}$.

Next we define the Cantor function $\phi : [0,1] \to [0,1]$. Let $\phi(0) = 0$, $\phi(1) = 1$. Assign
\[ \phi(x) \] a constant value on each of the deleted open intervals (including the end points of the deleted interval). The constant values are assigned in the following manner.

At the first iteration we set \( \phi(x) = \frac{1}{p} \), \( t = 1, 2, \ldots, q \). At the second step there are \( q(1 + p) \) deleted intervals and so we set \( \phi(x) = \frac{1}{p^2} \), \( t = 1, 2, \ldots, q(1 + p) \) at each of the deleted intervals respectively. The number of deleted intervals at the \( n \) th step is \( q(1 + p + p^2 + \cdots + p^{n-1}) = \frac{q(1-p^n)}{1-p} = N \) (say) so that the value assigned to \( \phi(x) \) at each deleted intervals (including the end points) are \( \phi(x) = \frac{1}{p^t} \), \( t = 1, 2, \ldots, N \). Next, let \( x \in C \). Then for each \( k, x \) belongs to the interior of exactly one of the \( p^n \) remaining closed intervals each of length \( \frac{1}{p^n} \). Let \( [\alpha_k, \beta_k] \) be one such intervals. Then

\[ \beta_k - \alpha_k = \frac{1}{r^k} \quad (2) \]

Further, \( \phi \) is already defined at the \( 2N \) end points of the left over intervals so that

\[ \phi(\beta_{k+j}) = \phi(\alpha_k) + \frac{1}{p^j} \quad (3) \]

where \( 0 \leq j \leq p - q \) and \( \alpha_1 = 0 \). At the next iteration, assuming \( x \in [\alpha_{k+1}, \beta_{k+j+1}] \), \( \alpha_k = \alpha_{k+1} \), say, we have \( \phi(\alpha_{k+j}) \leq \phi(\alpha_{k+1}) \leq \phi(\beta_{k+j+1}) \leq \phi(\beta_{k+1}) \). Define \( \phi(x) = \lim_{k \to \infty} \phi(\alpha_k) = \lim_{k \to \infty} \phi(\beta_{k+j+1}) \). Then \( \phi : [0, 1] \to [0, 1] \) is a continuous, non-decreasing function. Also \( \phi'(x) = 0 \) for \( x \in I \setminus C \) when it is not differentiable at any \( x \in C \).

### 3. Non-archimedean analysis

#### 3.1. Absolute Value

**Definition 1.** Let \( x \in C \subset I \). For an arbitrary small \( x \to 0^+ \), \( \exists \) an \( \varepsilon \in I \) and a 1-parameter family of \( \tilde{x} \) in \( I \setminus C \) such that \( 0 < \tilde{x} < \varepsilon < x \) and

\[ \frac{\tilde{x}}{\varepsilon} = \lambda(\varepsilon) \frac{\varepsilon}{x} \quad (4) \]

where the real constant \( \lambda \) (\( 0 < \lambda \leq 1 \)) may depend on \( \varepsilon \). The set of such \( \tilde{x} \)’s satisfying the inversion law (4) is called the set of relative infinitesimals \( \mathbb{I}_0^+ \) in \( I \) relative to the scale \( \varepsilon \) and is denoted as \( \mathbb{I}_0^+ = \{ \tilde{x} \mid 0 < \tilde{x} < \varepsilon < x, \tilde{x} = \lambda(\varepsilon) \frac{\varepsilon^2}{x} \} \). Two relative infinitesimals \( \tilde{x} \) and \( \tilde{y} \) must satisfy the condition \( 0 < \tilde{x} < \tilde{y} \) if \( \tilde{x} + \tilde{y} < \varepsilon \).

The non-empty set \( \mathbb{I}^+ = \{ \frac{\tilde{x}}{\varepsilon}, \varepsilon \to 0 \} \) is called the set of scale free infinitesimals.

**Definition 2.** Because of the disconnectedness of \( C \), to each \( x \in C \), \( \exists \mathbb{I}_e(x) = (x - \varepsilon, x + \varepsilon) \subset I \), \( \varepsilon > 0 \) such that \( C \cap \mathbb{I}_e(x) = \{ x \} \). Points in \( \mathbb{I}_e(x) \) are called the relative infinitesimal neighbours in \( I \) of \( x \in C \).
Lemma 1. $I_\varepsilon(x) = x + I_0$, $I_0 = I_0^+ \cup I_0^-$, $I_0^- = \{ -\tilde{x} \mid \tilde{x} \in I_0^+ \}$. Further, there exists a bijection between $I_0^+$ and $(0, 1)$ for a given $\varepsilon$.

Proof. Let $y \in I_\varepsilon(x)$. Then $y = x \pm \tilde{x}$, $0 < \tilde{x} < \varepsilon < z$, so that $\tilde{x} = \lambda \frac{\varepsilon}{z}$ for a fixed $z$ and a variable $\lambda$. Thus $y \in x + I_0$. The other inclusion also follows similarly. Finally, the bijection is given by the mapping $\tilde{x} \rightarrow \frac{\tilde{x}}{\varepsilon}$.

Definition 3. Given $\tilde{x} \in I_0$, we define a scale free absolute value of $\tilde{x}$ by $v : I_0 \rightarrow [0, 1]$ where

$$v(\tilde{x}) = \begin{cases} \log_{\varepsilon-1} \frac{\varepsilon}{|\tilde{x}|}, & \tilde{x} \neq 0 \\ 0, & \tilde{x} = 0 \end{cases} \quad (5)$$

as $\varepsilon \rightarrow 0^+$.

Lemma 2. $v$ is a non-archimedean semi-norm over $I_0$.

Note 1. By semi-norm we mean (i) $v(\tilde{x}) > 0$, $\tilde{x} \neq 0$. (ii) $v(-\tilde{x}) = v(\tilde{x})$. (iii) $v(\tilde{x} + \tilde{y}) \leq \max\{ v(\tilde{x}), v(\tilde{y}) \}$. Property (iii) is called the strong (ultrametric) triangle inequality [9].

Proof. The case (i) and (ii) follow from the definition. To prove (iii) let $0 < \tilde{x} \leq \tilde{y} < \tilde{x} + \tilde{y} < \varepsilon$. Then $v(\tilde{y}) \leq v(\tilde{x})$ and hence $v(\tilde{x} + \tilde{y}) = \log_{\varepsilon-1} \frac{\varepsilon}{\tilde{x}+y} \leq \log_{\varepsilon-1} \frac{\varepsilon}{\tilde{x}} = v(\tilde{x}) = \max\{ v(\tilde{x}), v(\tilde{y}) \}$. Moreover, $v(\tilde{x} - \tilde{y}) = v(\tilde{x} + (-\tilde{y})) \leq \max\{ v(\tilde{x}), v(\tilde{y}) \}$. ■

Example 1. Let $\varepsilon = e^{-n}, x = k\varepsilon = e^{-t}\varepsilon$ where $t \rightarrow 0^+$ for an $k \approx 1$. Consider a subset of the open interval $I_\varepsilon = (0, \varepsilon)$ consisting of $q$ open subintervals $I_j, j = 1, 2, \ldots, q$ each of length $\tilde{\varepsilon}$, $r > q$. Let $\tilde{I}_j \subset (0, 1)$ be the image of $I_j$ under rescaling $\tilde{x} \rightarrow \frac{\tilde{x}}{\varepsilon}$. The relative infinitesimals $\tilde{x}_j \in I_j$ are given by $\tilde{x}_j = \lambda_j, k^{-1} := e^{\mu_j t}$ where $\lambda_j \in \tilde{I}_j$ and $\mu_j = 1 + t^{-1} \frac{\log \lambda_j}{\log \varepsilon}$. Then $v(\tilde{x}_j) = \mu_j t$.

Definition 4. The set $B_r(a) = \{ x \mid v(x-a) < r \}$ is called an open ball in $I_0$. The set $\bar{B}_r(a) = \{ x \mid v(x-a) \leq r \}$ is a closed ball in $I_0$.

Lemma 3. (i) Every open ball is closed and vice-versa (clopen ball) (ii) every point $b \in B_r(a)$ is a centre of $B_r(a)$. (iii) Any two balls in $I_0$ are either disjoint or one is contained in another. (iv) $I_0$ is the union of at most a countable family of clopen balls.

Proof follows directly from the ultrametric inequality and the fact that $I_0$ is an open set. It also follows that in the topology determined by the semi-norm, $I_0$ is a totally disconnected set. It is also proved in [10] that a closed ball in $I_0$ is compact. As a result, $I_0$ is the union of countable family of disjoint closed (clopen) balls, in each of which $v(\tilde{x})$ can have a constant value. With this assumption, $v : I_0 \rightarrow [0, 1]$ is discretely valued.
Next, to restore the product rule viz: \( v(\tilde{x}y) = v(\tilde{x}).v(y) \), we note that given \( \tilde{x} \) and \( \varepsilon \), \( 0 < \tilde{x} < \varepsilon \), there exist \( 0 < \sigma(\varepsilon) < 1 \) and \( a : I_0^+ \rightarrow R \) such that

\[
\frac{\tilde{x}}{\varepsilon} = \varepsilon^{\sigma(\tilde{x})}.e^{t(\tilde{x},\varepsilon)}
\]

so that \( v(\tilde{x}) = \sigma^{\sigma(\tilde{x})} \) for an indeterminate vanishingly small \( t : I_0 \rightarrow R \) i.e. \( t(\tilde{x},\varepsilon) \rightarrow 0 \) as \( \varepsilon \rightarrow 0^+ \). For the given Cantor set \( C \) there is a unique (natural) choice of \( \sigma \) dictated by the scale factors of \( C \) viz: \( \sigma = p^{-n} = r^{-ns} \), \( s = \frac{\log p}{\log \tau} \), for some natural number \( n \).

The mapping \( a(\tilde{x}) \) is a valuation and satisfies \( i \) \( a(\tilde{x}y) = a(\tilde{x}) + a(y) \), \( ii \) \( a(\tilde{x} + \tilde{y}) \geq \min \{ a(\tilde{x}), a(\tilde{y}) \} \). Now discreteness of \( v(\tilde{x}) \) implies range \( \{ a(\tilde{x}) \} = \{ a_n \mid n \in \mathbb{Z}^+ \} \). Again for a given scale \( \varepsilon \), \( I_0^+ \) is identified with a copy of \( (0, 1) \) (by Lemma 1) which is clopen in the semi-norm. Thus \( I_0^+ \) is covered by a finite number of disjoint clopen balls \( B(\tilde{x}_n) \) (say), \( \tilde{x}_n \in I_0^+ \). Because of finiteness, values of \( a(\tilde{x}) \) on each of the balls can be ordered \( 0 = a_0 < a_1 < \cdots < a_n = s_0 \) (say). Let \( v_0 = v(B(\tilde{x}_n)) = \sigma^{s_0} \). Then we can write \( v_i = v(B(\tilde{x}_i)) = \alpha_i v_0 = \alpha_i \sigma^{s_0} \) for an ascending sequence \( \alpha_i > 0 \), \( i = 0, 1, \ldots, n \). We also note that \( a_0 = 0 \) corresponding to the unit \( \tilde{x}_0 \) so that \( v(\tilde{x}_0) = 1 \).

From equation (6) we have \( \frac{\tilde{x}_u}{(\tilde{x}_u)} = \varepsilon^{1+t(\tilde{x},\varepsilon)} \) and so it follows that \( \tilde{x} \in I_0^+ \) will admit a factorization

\[
\frac{\tilde{x}}{\varepsilon} = \frac{\tilde{x}_i}{\varepsilon^{\tilde{x}_i}} \cdot \frac{\tilde{x}_u}{\varepsilon^{\tilde{x}_u}}
\]

since \( \tilde{x} \in B(\tilde{x}_i) \) for some \( i \).

Thus

\[
\tilde{x} = \tilde{x}_i (1 + \tilde{x}_\varepsilon)
\]

where \( \tilde{x}_u = \varepsilon^2 (1 + \tilde{x}_\varepsilon) \), \( \tilde{x}_\varepsilon \in I_0 \), so that \( v(\tilde{x}) = v(\tilde{x}_i) \), as \( v(\tilde{x}_\varepsilon) < 1 \).

We thus have,

**Theorem 1.** \( v \) is a discretely valued non-archimedean absolute value on \( I_0^+ \). Any infinitesimal \( \tilde{x} \in I_0^+ \) have the decomposition given by equation (8) so that \( v \) has the canonical form

\[
v(\tilde{x}) = \alpha_i \sigma^{s_0}, \tilde{x} \in B(\tilde{x}_i)
\]

**Definition 5.** The infinitesimals given by equation (8) and having absolute value (9) are called valued infinitesimals.

We now make use these valued infinitesimals to define a non-trivial absolute value on \( C \) in the following steps.

(i) Given \( x \in C \) define a set of multiplicative neighbours of \( x \) which are induced by the valued infinitesimals \( \tau \in I_0^+ \) by

\[
X^\tau_x = x, x^\tau = x
\]

(10)
where \( v(\tau) = \alpha_n \sigma^{s_0} \) and \( \alpha_n = \alpha_n(x) \) may now depend on \( x \). We note that the non-archimedean topology induced by \( v \) makes the infinitesimal neighbourhood of \( 0^+ \) in \( I \) totally disconnected. Equation (10) thus introduces a finer infinitesimal subdivisions in the neighbourhood of \( x \in C \).

(ii) We define the new absolute value of \( x \in C \) by
\[
\| x \| = \inf \log_{x^{-1}} \frac{X_+}{x} = \inf \log_{x^{-1}} \frac{x}{X_-}
\]
so that \( \| x \| = \sigma^s \) where \( \sigma^s = \inf \alpha_n \sigma^{s_0} \) and the infimum is over all \( n \). It thus follows that

**Corollary 1.** \( \| . \| : C \to \mathbb{R}_+ \) is a non-archimedean absolute value.

### 3.2. Valued measure

We define the valued measure \( \mu_v : C \to \mathbb{R}_+ \) by

(a) \( \mu_v(\phi) = 0, \) \( \phi \) the null set.

(b) \( \mu_v[(0, x)] = \| x \| \) when \( x \in C \).

(c) For any \( E \subset C \), we have
\[
\mu_v(E) = \lim \inf_{\delta \to 0} \Sigma \{ d_{na}(I_i) \}
\]
where \( I_i \in \bar{I}_\delta \) and the infimum is over all countable \( \delta - \) covers \( \bar{I}_\delta \) of \( E \) by clopen balls and \( d_{na}(I_i) \) is the non-archimedean “diameter” of \( I_i = \sup \{ \| x - y \| : x, y \in I_i \} \).

It follows that \( \mu_v \) is a metric (Lebesgue) outer measure on \( C \) realized as a non-archimedean space.

Now, denoting the diameter in the usual sense by \( d(I_i) \), one notes that \( d_{na}(I) \leq \{ d(I_i) \}^s \), since \( x, y \in C \) and \( | x - y | = d \) imply \( \| x - y \| = \varepsilon^s \leq d^s \) for a suitable scale \( \varepsilon \leq d \leq \delta \). Using this inequality one can show that
\[
\mu_v(E) = \mu_s(E)
\]
for any subset \( E \subset C \). Finally, for \( s = \dim[C] \), \( \mu_s(C) = 1 \) and so \( \mu_v(C) = 1 \). Thus the valued measure selects naturally the dimension of the Cantor set.

### 3.3. Differentiability

To discuss the formalism of the Calculus on \( C \) we change the notations of section 3.1 a little. Let \( X \) denote a valued infinitesimal while an arbitrarily small real \( x \in I \) denote
the scale $\varepsilon$. The set of infinitesimals is covered by $n$ clopen balls $B_n$ in each of which $v$ is constant. Let

$$\tilde{v}_n(x) = v(X_n(x)) = \log_{x_0^{-1}} \frac{x}{X_n} = \alpha_n x^{s_0}$$

(14)

so that $X_n = x. x^{\tilde{v}_n(x)} \in B_n$. For each $x$, $\tilde{v}_n$ is constant on $B_n$.

**Definition 6.** A function $f : C \to I$ is said to be differentiable at $x_0 \in C$ if $\exists$ a finite $l$ such that $0 < \| x - x_0 \| < \delta \Rightarrow$

$$\left| \frac{\| f(x) - f(x_0) \|}{\| x - x_0 \|} - l \right| < \varepsilon$$

(15)

for $\varepsilon > 0$ and $\delta(\varepsilon) > 0$ and we write $f'(x_0) = l$.

Now $\| x - x_0 \| = \inf \tilde{v}_n(x - x_0) = \log_{x_0^{-1}} \frac{x}{X}$, where the valued infinitesimal $X \in \tilde{B}$, an open sub-interval of $[0,1]$ in the usual topology and $\tilde{B}$ is the ball which corresponds to the infimum of $\tilde{v}_n$. Further $f(x) - f(x_0) = (\log x_0)^{-1} \tilde{f}(X)$, since $x = x_0 x^{\tilde{v}(x)}$, and $\tilde{f}$ is a differentiable function on $\tilde{B}$ in the usual sense. Thus equation (15), viz., the equality $f'(x_0) = l$, extends over $\tilde{B}$ as a scale free differential equation

$$\frac{d\tilde{f}}{d\log X} = l$$

(16)

**Definition 7.** Let $f : C \to C$ be a mapping on a Cantor set $C$ to itself. Then $f$ is differentiable at $x_0 \in C$ if $\exists l$ such that given $\varepsilon > 0, \exists \delta > 0$ so that

$$\left| \frac{\| f(x) - f(x_0) \|}{\| x - x_0 \|} - l \right| < \varepsilon$$

(17)

when $0 < \| x - x_0 \| < \delta$.

As before we write $f'(x_0) = l$ (with an abuse of notation). It follows that the above equality now extends to a scale free equation of the form

$$\frac{d\log \tilde{f}(X)}{d\log X} = l$$

(18)

where notations are analogous to above.

**Remark 1.** The discrete point like structures of $C$ are replaced by infinitesimal open intervals over which the ordinary continuum calculus is carried over on logarithmic variables via the scale invariant non-archimedean metric.

In the next section we show that the Cantor function is a locally constant continuously differentiable function in the new sense. We also give its reinterpretation in the usual topology.
4. Cantor function revisited

We first show that the value \( v(x) \) awarded to the valued infinitesimals \( X \in B_i \), \( i = 1, 2, \ldots, n \) is given by the Cantor function \( \phi : I \to I \) with points of discontinuity in \( \phi'(x) \), in the usual sense, are in \( C \). In the new formalism this discontinuity is removed in a scale invariant way using logarithmic differentiability over (valued) infinitesimal open line segments replacing each \( x \in C \). Our definition of \( v(x) \) is guided by the given Cantor set \( C \) so as to retrieve the finite Hausdorff measure uniquely via the construction of the valued measure.

Let us denote the valued scale free infinitesimals by \([0, 1)\), denoted here by \( \tilde{C} \). The interval \([0, 1)\) here is a copy of the scale free infinitesimals \( I^+ \) for an arbitrary small \( \varepsilon_0 \) (say). The valued infinitesimals in \([0, 1)\) then introduce a new set of scales of the form \( r^{-n} \) (in the unit of \( \varepsilon_0 \)) so that the scales introduced in definition 1 are now parameterized as \( \varepsilon = \varepsilon_0 r^{-n} \). The choice of the ‘secondary’ scales \( r^{-n} \) are motivated by the finite level Cantor set \( C \). At the ordinary level i.e. at the scale 1 (corresponding to \( n = 0 \)), there is no valued infinitesimal (at the level of ordinary real calculus) except the trivial 0. So relative to the finite scale (given by \( \delta = \varepsilon_0 = 1 \) \([0, 1)\) reduces to the singleton \( \{0\} \). At the next level, we choose the smaller scale \( \delta = \varepsilon_0 r^{-1} \). Consequently, elements in \([0, \varepsilon_0 r^{-1})\) are undetectable and identified with 0, again in the usual sense. Presently we have, however, the following.

We assume that the void (emptiness) of 0 reflects in an inverted manner the structure of the Cantor set \( C \) that is available at the finite scale. That is to say, at the first iteration of \( C \) from \( I \), \( q \) open intervals are removed leaving out \( p \) closed intervals \( F_{1n} \), \( n = 1, 2, \ldots, p \).

At the scale \( \delta = \varepsilon_0 r^{-1} \) in the void of \( \tilde{C} \), on the other hand, there now emerges (by “inversion”) \( q \) open islands (intervals) \( I_{1i} \), \( i = 1, 2, \ldots, q \). By definition, \( I_{1i} \) contains, for each \( i \), the so called valued infinitesimals \( X_i \) which are assigned the values \( v(X_i) = \phi(X_i) = i \), \( i = 1, 2, \ldots, q \), \( X_i \in I_{1i} \).

We note that at the scale \( \delta = \varepsilon_0 r^{-1} \), there are \( p \) voids in \( \tilde{C} \). At the next level of the scale \( \delta = \varepsilon_0 r^{-2} \), there emerges again in each void \( q \) islands of open intervals, so that there are now \( pq \) number of total islands \( I_{2i} \), \( i = 1, 2, \ldots, pq \). The value assigned to each of these valued islands of infinitesimals are \( v(X_j) = \phi(X_j) = \frac{j}{p^2} \), \( j = 1, 2, \ldots, pq \), where \( X_j \in I_{2j} \). Continuing this iteration, at the \( n \)th level, the (secondary) scale is \( \delta = \varepsilon_0 r^{-n} \) and the number of open intervals \( I_{nj} \) of infinitesimals are now \( q(1 + p + p^2 + \cdots + p^n) = N \) (say) with corresponding values

\[
v(X_j) = \phi(X_j) = \frac{j}{p^n}, \quad j = 1, 2, \ldots, N
\]
Cantor set \( \hat{C} = \cap \bigcup_{n \ j} I_{nj} \) and is extended to \( \phi : I \to I \) by continuity following equations like equation (3). We note that the absolute value \( \| \| \) awarded to each block of the Cantor intervals \( F_{nk} \) are
\[
\| F_{nk} \| = r^{-ns} \tag{20}
\]
for each \( k = 1, 2, \ldots, p^n \) where \( C = \cap \bigcup F_{nk} \) and so \( s = \frac{\log p}{\log r} \), since the valued set of infinitesimals induces fine structures to an element in \( F_{nk} \) viz. for a \( y \in F_{nk} \), we now have the infinitesimal neighbours \( Y_{\pm}^j = y, y^\mp j p^{-n}, j = 1, 2, \ldots, N. \)

Clearly, the absolute value in equation (20) corresponds to the minimum of \( v(x) \) at the \( n \)th iteration. Thus the valuations defined as the associated Cantor function leads to a valued measure on \( C \) that equals the corresponding Hausdorff measure with \( s = \frac{\log p}{\log r} \).

Let us now recall that the solutions of \( \phi'(x) = 0 \) in a non-archimedean space are locally constant functions \([9]\). To show that Cantor function \( \phi : I \to I \) is a locally constant function, let us recall that the Cantor set \( C \) is constructed recursively as \( C = \cap \bigcup_{n \ k} F_{nk} \). The set \( I \) , on the other hand, is written as \( I = \cap \left[ \bigcup_{n \ k} F_{nk} \right) \cup \left( \bigcup_{j=1}^N I_{nj} \right) \], the open interval \( \tilde{F}_{nk} \) being \( F_{nk} \) with end points removed (recall that \( I_{nj} \) are closed in the ultrametric topology). By definition \( v(I_{nj}) = a_{nj} \) a constant for each \( n \) and \( j \). We set \( v( \tilde{F}_{nk} ) = 0 \) as \( n \to \infty \). This equality is to be understood in the following sense. At an infinitesimal scale \( \epsilon_0 \to 0^+ \) the zero value of \( \tilde{F}_{nk} \) becomes finitely valued recursively for each \( n \) since a Cantor point \( x \in C \) is replaced by a copy of the (inverted) Cantor set \( \hat{C} \) with finite number of closed intervals like \( I_{nj} \). The derivatives of \( \phi \) vanishes not only for each \( n \) and \( j \) but even as \( n \to \infty \) (and \( \epsilon \to 0 \), for each arbitrarily small but fixed \( \epsilon_0 \)). Thus, the equality \( \phi'(x) = 0 \) on \( I/C \), in the ordinary sense, gets extended to every \( x \in C \) when the Cantor set is reinterpreted as a nonarchimedean space. The removal of the usual derivative discontinuities is also explained dynamically as due to the fact that the approach to an actual Cantor set point \( x \) is accomplished in the nonarchimedean setting by inversion. That is to say, as a variable \( X \in I \) approaches \( x \in C \), the usual linear shift in \( I \) is replaced by infinitesimal hoppings between two neigbouring elements of the form \( X_+/x \propto X_-/ \).

The variability of the locally constant function \( \phi : I \to I \) may, however, be captured in the usual topology as follows. Indeed, we show that
\[
\frac{d\phi}{dx} = 0 \tag{21}
\]
for finite values of \( x \in I \) is transformed into
\[
\frac{d\phi}{dv(\tilde{x})} = -O(1)\phi \tag{22}
\]
for an infinitesimal $\tilde{x}$ satisfying $\frac{\tilde{x}}{\varepsilon} = \lambda \frac{\varepsilon}{\tilde{x}} = \varepsilon^{v(\tilde{x})}$, $0 < \tilde{x} < \varepsilon \leq x$, $x \to 0^+$, $x \in I$, $\lambda > 0$, when one interprets $0$ in relation to the scale $\varepsilon$ as $O(\frac{\delta}{x} \log \varepsilon^{-1})$. However, this follows once one notes that eq(21) means, in the ordinary sense, $d\phi = 0 = O(\delta)$, $dx \neq 0$, for a finite $x \in I$. But, as $x \to \varepsilon$, that is, as $dx \to 0 = O(\delta)$, the ordinary variable $x$ is replaced by the ultrametric extension $x = \varepsilon \varepsilon^{-v(\tilde{x})}$ so that $d \log x = d\tilde{v}(\tilde{x}) \log \varepsilon^{-1} = O(\delta)$. On the other hand, the constant function $\phi$ (eq(21)), now, in the presence of smaller scale infinitesimals, has the form $\phi = \phi_0 e^{k_0 v(\tilde{x})}$ for a real constant $k_0$. Eq(22) thus follows. The variability of $\phi(x)$ in the usual topology is thus explained as an effect of the relative infinitesimals which are insignificant relative to the finite scale of $x \in C$, but attain a dominant status in the appropriate logarithmic variable $v(\tilde{x}) = \log \varepsilon^{-1}$. It is also of interest to compare the present case with computation. In the ordinary framework, the scale $\varepsilon$ stands for the level of accuracy in a computational problem. The infinitesimals in $(0, \varepsilon)$ are “valueless” in the sense that these have no effect on the actual computation. The open interval $(0, \varepsilon)$ is thus effectively indentified with $\{0\}$. In the present framework, the zero element $0$ is, however, identified with a smaller interval of the form $(0, \delta)$ where $\delta = \eta \varepsilon \log \varepsilon^{-1}$ and $0 < \eta \lesssim 1$. The valued infinitesimals in the interval $(\delta, \varepsilon)$ are already shown to have significant influence on the structure of the Cantor set. The variability of $\phi(x)$ as given by equation (22) is revealed, on the other hand, in relation to an infinitesimal variable lying in $(0, \delta)$.

Finally, we verify the emergence of equation (22) from the classical Cantor function equation (2) and (3) viz. (we choose $j = 0$ for simplicity)

$$\phi(\beta_k) - \phi(\alpha_k) = \frac{1}{p^k} \quad \text{and} \quad \beta_k - \alpha_k = \frac{1}{r^k}$$

(23)

We have

$$\phi(\beta_k) - \phi(\alpha_k) = \left(\frac{r}{p}\right)^k (\beta_k - \alpha_k)$$

(24)

Let $\phi(\beta_k) = \tilde{\phi}_+$, $\phi(\alpha_k) = \tilde{\phi}_-$, $\beta_k = x_+$, $\alpha_k = x_-$. Suppose also that $r^k(x_+ - x) \to k \log \sigma_+$, $r^k(x - x_-) \to k \log \sigma_-$, $p^k(\tilde{\phi}_+ - \tilde{\phi}_-) \to k \log \phi'_+$ and $p^k(\tilde{\phi}_+ + \tilde{\phi}_-) \to k \log \phi'_-$ as $k \to \infty$.

Equation (24) becomes

$$\log \phi'_+ + \log \phi'_- = \log \sigma_+ + \log \sigma_-$$

(25)

which leads to

$$\frac{\log \phi'_+}{\log \sigma_+} = \frac{\log \phi'_-}{\log \sigma_-} = \frac{\log \phi'_+ + \log \phi'_-}{\log \sigma_+ + \log \sigma_-} = 1$$

(26)

Equation (26) is essentially the left and right brunches of equation (21) at $x \in C$, in appropriate logarithmic variables, where the multiplicative neighbours of $x$, in the present
derivation, is given by the limiting form of the Cantor function defined by

$$\phi'_+ = \sigma^{1+i}, \quad \phi'_- = \sigma^{-(1+i)}, \quad i \geq 0$$  \hspace{1cm} (27)

which follows from the inequality

$$\frac{\alpha + \gamma}{\beta + \delta} \leq \max\left(\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right), \quad \alpha, \gamma \geq 0, \ \beta, \delta > 0$$

so that

$$\left(\frac{\log \phi'_+}{\log \sigma'_+}, \frac{\log \phi'_-}{\log \sigma'_-}\right) \geq 1.$$  \hspace{1cm} (28)

Setting \(\sigma^{-1} \phi'_+ = \left\{\frac{X_x}{x}\right\}^i, \ \sigma \phi'_- = \left\{\frac{X_x}{x}\right\}^i\) and \(\sigma = x^{-v(\tilde{x})}\) the multiplicative neighbours of \(x\) are obtained as

$$X_\pm = x \cdot x^{\pm v(\tilde{x})} \hspace{1cm} (29)$$

The Cantor function \(\phi(\tilde{x})\) over the infinitesimals \(\tilde{x}\) is thus given by

$$\phi(\tilde{x}) = \log_{x^{-1}} \frac{X(\tilde{x})}{x} = v(\tilde{x}) \hspace{1cm} (30)$$

thereby retrieving the variability of \(\phi\) relative to \(v\) trivially viz: \(d\phi = dv\).

We note that this again explains explicitly the removal of derivative discontinuities as encoded in eq(24) in the present formalism. The divergence of either the left or right derivative at an \(x \in C\), that arises due to the divergence of \((r/p)^k, \ k \to \infty\), is smoothed out in the logarithmic variables that replace the ordinary limiting variables as in eqns (25) and (26), which, in fact, correspond to eq(22). We conclude that the multiplicative non-archimedean structure given by (29) induces a smoothening effect on the discontinuity of \(\phi'(x)\) in the usual topology.

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