Toward the correlated analysis of perturbative QCD observables
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Abstract

We establish direct connection between ghost–free formulations of RG–invariant perturbation theory in the both Euclidean and Minkowskian regions.

By combining the trick of resummation of the $\pi^2$–terms for the invariant QCD coupling and observables in the time-like region with fresh results on the “analyticized” coupling $\alpha_{an}(Q^2)$ and observables in the space-like domain we formulate a self–consistent scheme, free of ghost troubles. The basic point of this joint construction is the “dipole spectral relation” $[\Box]$ emerging from axioms of local QFT.

Then we consider the issue of the heavy quark thresholds and devise a global scheme for the data analysis in the whole accessible space-like and time-like domain with various numbers of active quarks. Observables in both the regions are presented in a form of non-power perturbation series with improved convergence properties.

Preliminary estimates indicate that this global scheme produces results a bit different – on a few per cent level for $\bar{\alpha}_s$ – from the usual one, thus influencing the total picture of the QCD parameters correlation.

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1 Introduction

1.1 Preamble

The item of the low energy behavior of a strong interaction attracts more and more interest along with the further experimental data accumulation. In the perturbative quantum chromodynamics (pQCD) this behavior is spoiled by unphysical singularities lying at the three flavour region \( f = 3 \) and associated with the scale parameter \( \Lambda_{f=3} \simeq 350 \text{ MeV} \). In the “small energy” and “small momentum transfer” regions \( \sqrt{s}, Q \equiv \sqrt{Q^2} \lesssim 3\Lambda \) these singularities complicate theoretical interpretation of data. On the other hand, their existence contradicts some general statements of the local QFT.

Meanwhile, this issue has a rather elegant solution. As it has been shown \([1, 2] \) (see, also fresh review \([3] \)), by combining three elements:

1. Usual Feynman perturbation theory for effective coupling(s) and observables,
2. Renormalizability, i.e., renormalization–group (RG) invariance, and
3. General principles of local QFT — like causality, unitarity, Poincaré invariance and spectrality — in the form of spectral representations of Källen–Lehmann and Jost–Lehmann–Dyson type

it turns out to be possible to formulate an Invariant Analytic Approach (IAA) for the pQCD invariant coupling and observables in which the central theoretical object is a spectral density. Being calculated by usual, RG–improved, perturbation theory it defines and relates \( Q^2 \)-analytic, RG-invariant expressions in the both Euclidean and Minkowskian channels.

The IAA obeys several remarkable properties:
— It enables one to obtain modified perturbation expressions for observables, free of unphysical singularities, poles and cuts, with behavior correlated in both space-like and time-like domains.
— In particular, the IAA results in modified ghost-free expressions for invariant QCD coupling \( \alpha_{\text{an}}(Q^2; f) \) and \( \tilde{\alpha}(s; f) \) which obey reduced higher–loops and renormalization–scheme sensitivity \([2] – [4] \). See, Fig.1.
— Then, it yields changing the structure of perturbation expansion for observables: instead of common power series, as a result of its integral transformation, there appear non-power asymptotic series \([4] \) à la Erdélyi over the sets of specific functions \( A_k(Q^2; f) \) and \( \tilde{A}_k(s; f) \). These functions are defined via integral transformations of related powers \( \alpha_{\text{an}}^k(Q^2; f) \) in terms of relevant spectral densities. At small and moderate argument values, they diminish with the \( k \) growth much quicker than the corresponding powers \( \alpha_{\text{an}}^k(Q^2; f) \) and \( \tilde{\alpha}_k(s; f) \), (and even oscillate in the region \( \sqrt{s}, Q \simeq \Lambda \)) thus improving essentially the convergence of perturbation expansion for observables.

We review all these IAA features, important for our further developments, in the second part of Section 1.

The first purpose of this work is to elucidate relation between the Radyushkin–Krasnikov–Pivovarov “pipization” trick\([1, 2] \) and the Solovtsov–Milton\([3] \) construction of effective \( s \)-channel QCD coupling within the IAA scheme.
In the course of this analysis — see Section 2 — we reveal a spectacular “distorting mirror” correlation between analyticized and pipizated invariant QCD coupling in space-like $\alpha_{an}(Q^2; f)$ and time-like $\tilde{\alpha}(s; f)$ regions as well as between corresponding expansion functions $A_k(Q^2; f)$ and $\mathfrak{A}_k(s; f)$. See Fig. 2.

Then, in Section 2.3, we consider an issue of transition across the heavy quark thresholds, for constructing a “global” picture valid in the whole physical region $M_r \lesssim \sqrt{s}, Q \lesssim M_Z$.

It should be noted, that all precedent papers Refs. [1] – [12] dealt only with the massless quarks at fixed flavour number, $f$, case. This can be justified, to some extent, for analysis inside a narrow interval of relevant energy $\sqrt{s}$ or momentum transfer $Q$ values. Meanwhile, the ultimate goal of all the pQCD business is a correlation of QCD effective coupling values extracted from different experiments.

To construct the global invariant analytic couplings, one needs recipe of relating expressions with different $f$ values. For this goal, we use the same guideline as previously, at the “fixed $f$ case”, that is we start with adequately defined “global” spectral functions — see expression (14). Here, an essential point is the matching condition that relates $\Lambda_f$ parameters for different fixed flavor number $f$ values. We use standard $\overline{\text{MS}}$ prescription ascending to early 80s. This results in a smooth global Euclidean $\alpha_{an}(Q)$, $A_k(Q^2)$ and spline–continuous Minkowskian $\tilde{\alpha}(s)$, $\mathfrak{A}_k(s)$ expressions.

In the concluding Section, using examples of inclusive $\tau$ decay, $e^+e^- \rightarrow$ hadrons an- nihilization and sum rules, we shortly comment the possible implication of our new global scheme on perturbative analysis of QCD processes.

The main results of this work are summed concisely in the Subsection 3.3.

1.2 The $s$–channel: early attempts

As it is well known, the notion of invariant (or effective) coupling originally was introduced in the RG treatment[13] of renormalizable QFT. In the RG formalism, invariant coupling function $\overline{\alpha}$ was defined as a product of propagator and vertex amplitudes initially related with a product of real finite Dyson’s renormalization constants. This construction is valid only in the space-like domain $[1]$ and can be directly used for analysis of corresponding observables. However, the RG formalism does not provide us with analogous object in time-like region.

It is worth noting that sporadic attempts to define the effective coupling $\alpha(s)$ in the Minkowskian, time-like, domain were made in late 70s. Omitting an early simple–minded trick with “mirror reflection” of singular function

$$\alpha_s(Q^2; f) \rightarrow \alpha(s; f) \equiv |\alpha_s(-s; f)|,$$

we mention here the practically simultaneous results of Radyushkin [11] and Krasnikov and Pivovarov [12]. In both the papers, the integral transformation $\tilde{\alpha}(s; f) = R[\tilde{\alpha}_s(Q^2; f)]$

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1 Physically, in QED it can be considered as a Fourier transform of spatial electron charge distribution first discussed by Dirac [14].
reverse $\bf{R} = \bf{D}^{-1}$ to “dipole representation” for the Adler function

$$D(Q^2) = \frac{Q^2}{\pi} \int_0^{\infty} \frac{ds}{(s + Q^2)^2} R(s) \equiv \bf{D} \{ R(s) \}$$  \hspace{1cm} (1)$$

in terms of an observable $R(s)$ in the time-like region, has been used.

In [11, 12], as a starting point for observables in the Euclidean, i.e., space-like domain $Q^2 > 0$, the perturbation series

$$D_{pt}(Q^2) = 1 + \sum_{k \geq 1} d_k \bar{\alpha}^k_s(Q^2; f)$$  \hspace{1cm} (2)$$

has been assumed. It contains powers of usual, RG summed, invariant coupling $\bar{\alpha}_s(Q^2; f)$ that obeys unphysical singularities in the infrared (IR) region around $Q^2 \approx \Lambda^2$.

By using the reverse transformation

$$R(s) = \frac{i}{2\pi} \int_{s - i\varepsilon}^{s + i\varepsilon} \frac{dz}{z} D_{pt}(-z) \equiv \bf{R} \{ D_{pt}(Q^2) \}$$  \hspace{1cm} (3)$$

these authors arrived at the “$\bf{R}$–transformed” expansion that, in our notation, reads

$$R_\pi(s) = 1 + \sum_{k \geq 1} d_k \mathfrak{A}_k(s; f); \quad \mathfrak{A}_k(s; f) = \bf{R} \{ \alpha^k_s(Q^2; f) \}.$$  \hspace{1cm} (4)$$

For example

$$\bf{R} \left[ \frac{1}{l} \right] = \frac{1}{2} - \frac{1}{\pi} \arctan \frac{L}{\pi}; \quad \text{with} \quad l = \ln \frac{Q^2}{\Lambda^2}; \quad L = \ln \frac{s}{\Lambda^2},$$

$$\bf{R} \left[ \frac{\ln l}{l^2} \right] = \frac{\ln \left[ \sqrt{L^2 + \pi^2} + 1 - L \bf{R} \left[ 1/l \right] \right]}{L^2 + \pi^2}; \quad \bf{R} \left[ \frac{1}{l^2} \right] = \frac{1}{L^2 + \pi^2}.$$ 

This yields

$$\bar{\alpha}^{(1)}(s; f) = \frac{1}{\beta_0} \left[ \frac{1}{2} - \frac{1}{\pi} \arctan \frac{L}{\pi} \right]; \quad \beta_0 = \frac{33 - 2f}{12\pi}. \hspace{1cm} (5)$$

At the two–loop iterative case with

$$\beta_{[f]} \bar{\alpha}^{(2)}_s(Q^2; f) = \frac{1}{l} - b_f \frac{\ln l}{l^2}, \quad \beta_{[f]} \equiv \beta_0; \quad \beta_1 = \frac{102 - 38f}{12\pi}, \quad b_f = \frac{\beta_1}{\beta_0^2},$$

by combining $\bf{R} \left[ 1/l \right] - b_f \bf{R} \left[ \ln l/l^2 \right]$ one obtains explicit expression for the “iterative” two-loop effective $s$–channel coupling $\bar{\alpha}^{(2)}(s; f) = \mathfrak{A}^{(2)}(s; f),$

$$\bar{\alpha}^{\text{iter}}_s(s; f) = \left( 1 - \frac{b_f L}{L^2 + \pi^2} \right) \bar{\alpha}^{(1)}(s; f) + \frac{b_f \ln \left[ \sqrt{L^2 + \pi^2} + 1 \right]}{L^2 + \pi^2}.$$  

\[\text{This expression we give in the form equivalent to that one used in }[4]. \text{ In papers }[11, 13] \text{ it was given in an another form, non-adequate at } L \leq 0. \text{ See also }[16].\]
Obtained $\tilde{\alpha}^{(1)}$ and $\tilde{\alpha}^{(2)}_{\text{iter}}$ are monotonous functions with finite IR limit, free of $\Lambda$-singularity which is “screened” by resummed “$\pi^2$-terms”. Non-singular expressions for higher functions $\mathfrak{A}_k$ could be constructed in the same way.

The positive feature of this construction was an automatic summation of the so-called “$\pi^2$ - terms” that “screen” unphysical singularities and observed\cite{11} property

$$
(R \left[ \tilde{\alpha}^{k+1}_s \right])^{1/(k+1)} \prec (R \left[ \tilde{\alpha}^k_s \right])^{1/k}
$$

that improves the convergence of perturbation series.

However, there was one essential drawback. The dipole transformation (1), that is supposed to be reverse to $R$, being applied to (4) does not return us to the input (2)

$$
D \{ R \pi(s) \} = D \{ R [D_{pt}] \} \neq D_{pt}(Q^2) \Rightarrow D \cdot R \neq I,
$$

as far as the unphysical singularities of $\tilde{\alpha}_s(Q^2; f)$ and of its powers are incompatible with analytic properties in the complex $Q^2$ plane of the integral in the r.h.s. of (1).

Resolution of this issue came 15 years later with the IAA. The “missing link” is the analyticization transformation.

1.3 Analyticization in the $Q^2$–channel

Operation

$$F(Q^2) \rightarrow F_{\text{an}}(Q^2) = A \cdot F(Q^2)$$

has been introduced\cite{11} in terms of the Källén–Lehmann representation and correlates with analytic properties of the Adler function contained in eq. (1).

Generally, this transformation is defined for a function $F$ that should be analytic in the $Q^2$ plane with a cut along the negative part of the real axis. In our case, this function could be either invariant coupling $\tilde{\alpha}_s$ itself\cite{11} or its power, or some series in its powers.

Operation $A$ consists of two elements:

– use the Källén–Lehmann representation

$$F_{\text{an}}(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + Q^2} \rho_{pt}(\sigma) \quad \text{with}
$$

– the spectral density defined via straightforward continuation of $F$ on the cut

$$\rho_{pt}(\sigma) = \Im F(-\sigma).$$

A couple of comments are in order.

\footnote{As it has been explained in detail in the first papers\cite{11, 12} on the IAA, the QCD invariant coupling, according to general properties of local QFT, should satisfy the Källén–Lehmann spectral representation. For the original analysis of this issue see Ref.\cite{17}.}
• Operation $A$, being applied to the usual coupling $F = \alpha_s(Q^2; f)$, results in the analyticized coupling

$$\alpha_{\text{an}}(Q^2; f) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + Q^2} \rho(\sigma; f) ; \quad \rho(\sigma; f) = \Im \bar{\alpha}_s(-\sigma; f)$$

(6)

which is a smooth monotonic function free of unphysical singularities, with a finite value at the origin

$$\alpha_{\text{an}}(0; f) = 1/\beta_0 \approx 1.4$$

which is remarkably independent (see, e.g., [3]) of higher loop contributions.

Here, $\rho$ is defined as an imaginary part of the usual, RG invariant, effective coupling $\bar{\alpha}_s$ continued on the physical cut.

• Operation $A$, applied to power perturbation series (2) for an observable $D_{\text{pt}}(Q^2)$, produces a non-power series

$$D_{\text{an}}(Q^2; f) = 1 + \sum_{k \geq 1} d_k A_k(Q^2; f) ; \quad \alpha_{\text{an}}(Q^2; f) = A_1(Q^2; f)$$

(7)

with

$$A_k(x; f) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + x} \rho_k(\sigma; f) ; \quad \rho_k(\sigma; f) = \Im \left[ \bar{\alpha}_s^k(-\sigma; f) \right].$$

(8)

For example,

$$A \cdot \left[ \frac{1}{l} \right] = \frac{1}{l} + \frac{1}{1 - e^l} ; \quad A \cdot \left[ \frac{1}{l^2} \right] = \frac{1}{l^2} + \frac{e^l}{(1 - e^l)^2}, \ldots$$

that is

$$A \cdot \alpha_s^{[1]}(Q^2; f) = \alpha_{\text{an}}^{[1]}(Q^2; f) = \frac{1}{\beta_0} \left[ \frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} \right]$$

(9)

and so on.

Here, in the invariant analytic coupling $\alpha_{\text{an}}$, the $\Lambda$-pole is compensated by power term containing the non-perturbative $Q^2/\Lambda^2 = (Q^2/\mu^2) \exp(1/\beta_0 \alpha_\mu)$ structure.

Properties of the invariant analytic functions $A_k$, free of ghost troubles, and non-power expansion (7) have been discussed in papers [10]. They are quite similar to those for $\mathfrak{A}_k$ and expansion (4) — see below.

\footnote{For the time being, we consider the massless case with a fixed number $f$ of effective quark flavors in the $\overline{\text{MS}}$ scheme. For the transition between the regions with different $f$ values, see Section 2.3.}

5
1.4 Summary of the IAA

Here, we repeat in brief basic definitions of the Invariant Analytic Approach. First, one has to transform the usual singular invariant coupling

\[ \bar{\alpha}_s(Q^2; f) \rightarrow A \cdot \bar{\alpha}_s(Q^2; f) = \alpha_{an}(Q^2; f) \]

into the analyticized one, free of ghost singularities in the space-like region.

Second, with the help of the operation \( \mathbf{R} \), one defines\(^4\) invariant coupling \( \tilde{\alpha}(\tilde{s}; f) \) in the time-like domain

\[ \alpha_{an}(Q^2; f) \rightarrow \tilde{\alpha}(\tilde{s}; f) = \mathbf{R} [\alpha_{an}] = \int_\infty^\infty \frac{d\sigma}{\sigma} \rho(\sigma; f) \]

with spectral density \( \rho \) defined in (6).

Here, we have a possibility of reconstructing the Euclidean, \( Q^2 \)–channel, invariant coupling \( \alpha_{an}(Q^2; f) \) from the Minkowskian, \( s \)–channel, one \( \tilde{\alpha}(s; f) \) by the dipole transformation

\[ \alpha_{an}(Q^2; f) = \frac{Q^2}{\pi} \int_0^\infty \frac{ds}{(s + Q^2)^2} \tilde{\alpha}(s; f) \equiv \mathbf{D} \{\tilde{\alpha}(s; f)\} . \]

For instance, substituting \( \tilde{\alpha}^{(1)}(s; f) \) into the integrand, one obtains after integration by parts

\[ \frac{Q^2}{\pi \beta_0} \int_0^\infty \frac{d\sigma}{(\sigma + Q^2)^2} \left( \frac{1}{2} - \frac{1}{\pi} \arctan \frac{\ln(\sigma/\Lambda^2)}{\pi} \right) = \frac{Q^2}{\pi \beta_0} \int_0^\infty \frac{d\sigma}{(\sigma + Q^2)^2} \frac{1}{\ln^2(\sigma/\Lambda^2) + \pi^2} = \alpha_{an}^{(1)}(Q^2, f) \]

precisely in the form (9). This simple calculation elucidates the connection between the ghost–free expressions both in the \( s \)– and \( Q^2 \)–channels. They are connected also by the reverse transformation \( \tilde{\alpha}^{(1)}(s; f) = \mathbf{R} [\alpha_{an}^{(1)}(Q^2; f)] \).

On the Fig. 1 we give a concise summary of the IAA results for invariant analytic couplings \( \alpha_{an}(Q^2, 3) \) and \( \tilde{\alpha}(s, 3) \) calculated for one–, two– and three–loop cases in both the Euclidean and Minkowskian domains.

Here, dash–dotted curves represent one–loop IAA approximations\(^5\) and (9). Solid IAA curves are based on exact two–loop solutions of RG equations\(^6\) and approximate three–loop solutions in the \( \overline{\text{MS}} \) scheme. Their remarkable coincidence (within the 1–2 per cent limit) demonstrates reduced sensitivity of the IAA with respect to the higher–loops effects in the whole Euclidean and Minkowskian regions from IR till UV limits.

\(^5\) As it has been recently established, the exact solution to two–loop RG differential equation for the invariant coupling can be expressed in terms of a special function \( W \), the Lambert function, defined by relation \( W(z)e^{W(z)} = z \), with an infinite number of branches \( W_n(z) \). For some detail of analyticized solutions expressed in terms of Lambert function — see Refs.\(^{18, 19, 20, 21}\).
Figure 1: Space-like and time-like invariant analytic couplings in a few GeV domain

For comparison, by dotted line we also give usual $\tilde{\alpha}_s(Q^2)$ two-loop effective QCD coupling with a pole at $Q^2 = \Lambda^2$.

As it has been shown in [2, 3, 5], relations parallel to eqs.(10) and (11) are valid for powers of the pQCD invariant coupling. This can be resumed in the form of a self-consistent scheme.

2 Self-consistent scheme for observables

2.1 Relations between Euclidean and Minkowskian

First, one has to transform usual power perturbation series (3) of the $Q^2$ domain into the non-power one (7).

\[ D_{pt}(Q^2) \rightarrow D_{an}(Q^2) = A \cdot D_{pt}(Q^2) \]

Second, with the help of the operation $R$, one introduces

\[ D_{an}(Q^2) \rightarrow R_\pi(s) = R \left[ D_{an}(Q^2) \right] \]
the s–channel non-power expansion \( R_\pi(s) \) with

\[
\mathcal{A}_k(s) = \int_s^\infty \frac{d\sigma}{\sigma} \rho_k(\sigma); \quad \rho_k(\sigma) = \Im [\alpha^k_s(-\sigma)].
\]  

(12)

The third element is the closure of the scheme that is provided by the operation \( [III] \)

\[
\text{III.} \quad R_\pi(s) \to D_{an}(Q^2) = D \{ R_\pi(s) \}
\]

reverse to \( II. \)

In the other words, to enjoy self-consistency \( R \cdot D = D \cdot R = 1 \), one should abandon completely the usual effective coupling \( \alpha_s(Q^2) \) and power series \( D_{pt} \), eq.(2), applying operations \( R \) and \( D = R^{-1} \) only to IAA invariant couplings \( \alpha_{an} \), \( \tilde{\alpha} \) and to non-power expansions \( D_{an} \) and \( R_\pi \).

2.2 Expansion of observables over non-power sets \( \{A\} \) and \( \{\mathcal{A}\} \)

To realize the effect of transition from expansion over the “traditional” power set

\[
\{ \tilde{\alpha}^k_s(Q^2, f) \} = \tilde{\alpha}_s(Q^2), \tilde{\alpha}^2_s, \ldots \tilde{\alpha}_s^k \ldots
\]

to expansions over non–power sets in the space-like and time-like domains

\[
\{ A_k(Q^2, f) \} = \alpha_{an}(Q^2, f), A_2(Q^2, f), A_3 \ldots ; \quad \{ \mathcal{A}_k(s, f) \} = \tilde{\alpha}(s, f), \mathcal{A}_2(s, f), \mathcal{A}_3 \ldots,
\]

it is instructive to learn properties of the latters.

In a sense, both non-power sets are similar:

— They consist of functions that are free of unphysical singularities.

— First functions, the new effective couplings, \( \tilde{A}_1 = \alpha_{an} \) and \( \tilde{\mathcal{A}}_1 = \tilde{\alpha} \) are monotonically decreasing. In the IR limit, they are finite and equal \( \alpha_{an}(0, 3) = \tilde{\alpha}(0, 3) \approx 1.4 \) with the same infinite derivatives. Both have the same leading term \( \sim 1/\ln x \) in the UV limit.

— All other functions (“effective coupling powers”) of both the sets start from the zero IR values \( A_{k \geq 2}(0, f) = \mathcal{A}_{k \geq 2}(0, f) = 0 \) and obey the UV behavior \( \sim 1/(\ln x)^k \) corresponding to \( \tilde{\alpha}_s^k(x) \). They are no longer monotonous. The second functions \( \tilde{A}_2 \) and \( \tilde{\mathcal{A}}_2 \) are positive with maximum around \( s, Q^2 \sim \Lambda^2 \). Higher functions \( \tilde{A}_{k \geq 3} \) and \( \tilde{\mathcal{A}}_{k \geq 3} \) oscillate in the region of low argument values and obey \( k - 2 \) zeroes.

Remarkably enough, the mechanism of liberation of unphysical singularities is quite different. While in the space-like domain it involves non-perturbative, power in \( Q^2 \), structures, in the time-like region it is based only upon resummation of the “\( \pi^2 \) terms”. Figuratively, (non-perturbative !) analyticization in the \( Q^2 \)-channel can be treated as a quantitatively distorted reflection (under \( Q^2 \to s = -Q^2 \)) of (perturbative) “pipization” in the \( s \)-channel. This effect of “distorting mirror” first discussed in \( [I] \) is illustrated on figures 1 and 2.
Summarize the main results essential for data analysis. Instead of power perturbative series in the space-like

$$D_{pt}(Q^2) = 1 + d_{pt}(Q^2) ; \quad d_{pt}(Q^2) = \sum_{k \geq 1} d_k \bar{\alpha}^k_s(Q^2; f)$$  \hfill (24)

and time-like regions

$$R_{pt}(s) = 1 + r_{pt}(s) ; \quad r_{pt}(s) = \sum_{k \geq 1} r_k \bar{\alpha}^k(s; f) ; \quad (r_{1,2} = d_{1,2} = d_3 - d_1 \frac{\pi^2 \beta^2}{3},)$$

one has to use asymptotic expansions (7) and (4)

$$d_{an}(Q^2) = \sum_{k \geq 1} d_k \mathcal{A}_k(Q^2, f) ; \quad r_{\pi}(s) = \sum_{k \geq 1} d_k \mathcal{A}_k(s, f)$$

with the same coefficients $d_k$ over non-power sets of functions \{A\} and \{\mathcal{A}\}.

### 2.3 Global formulation

To apply the new scheme for analysis of QCD processes, one has to formulate it “globally”, in the whole experimental domain, i.e., for regions with different values of a number $f$ of active quarks. For this goal, we revise the issue of the threshold crossing.

**Threshold matching** In a real calculation, the procedure of the threshold matching is in use. One of the simplest is the matching condition in the massless $\overline{\text{MS}}$ scheme[22]

$$\bar{\alpha}_s(Q^2 = M^2_f; f - 1) = \bar{\alpha}_s(Q^2 = M^2_f; f)$$  \hfill (13)

related to the mass squared $M^2_f$ of the $f$-th quark.

This condition allows one to define a “global” function $\bar{\alpha}_s(Q^2)$ consisting of the smooth parts

$$\bar{\alpha}_s(Q^2) = \bar{\alpha}_s(Q^2; f) \quad \text{at} \quad M^2_{f-1} \leq Q^2 \leq M^2_f$$

s and continuous in the whole space-like interval of positive $Q^2$ values with discontinuity of derivatives at the matching points. We call such a functions as the spline–continuous ones.

At the first sight, any massless matching, yielding the spline–type function, violates the analyticity in the $Q^2$ variable, thus disturbing the relation between the $s$– and $Q^2$–channels[6].

However, in the IAA, the original power perturbation series (2) with its unphysical singularities and possible threshold non-analyticity has no direct relation to data, being

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[6] Any massless scheme is an approximation that can be controlled by the related mass–dependent scheme [23]. Using such a scheme, one can devise [24] a smooth transition across the heavy quark threshold. Nevertheless, from the practical point of view, it is sufficient (besides the case of data lying in close vicinity of the threshold) to use the spline–type matching (13) and forget about the smooth threshold crossing.
a sort of a “raw material” for defining spectral density. Meanwhile, the discontinuous
density is not dangerous. Indeed, expression of the form

\[ \rho_k(\sigma) = \rho_k(\sigma; 3) + \sum_{f \geq 4} \theta(\sigma - M_f^2) \{ \rho_k(\sigma; f) - \rho_k(\sigma; f - 1) \} \]  

(14)

where \( \rho_k(\sigma; f) = \Im \alpha^k_s(-\sigma, f) \) defines, according to (8) and (12), the smooth global

\[ A_k(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + x} \rho_k(\sigma) \]  

(15)

and spline-continuous global

\[ A_k(s) = \int_s^\infty \frac{d\sigma}{\sigma} \rho_k(\sigma) \]  

(16)

functions\footnote{Here, by eqs. (15), (16) and (14) we introduced new “global” effective invariant couplings and higher expansion functions different from the previous ones with fixed \( f \) value.}

We see that in this construction the role of the input perturbative invariant coupling \( \alpha_s(Q^2) \) is twofold. It provides us not only with spectral density (14) but with matching conditions (13) relating \( \Lambda_f \) with \( \Lambda_{f+1} \) as well.

Note that the matching condition (13) is tightly related \cite{22,24} to the renormalization procedure. Just for this profound reason we keep it untouched (compare with Ref. [7]).

The \textit{s-channel: shift constants} As a practical result, we now observe that the “global” \( s \)-channel coupling \( \tilde{\alpha}(s) \) and other functions \( \mathcal{A}_k(s) \), generally, differs of effective coupling with fixed flavor number \( f \) value \( \tilde{\alpha}(s; f) \) and \( \mathcal{A}_k(s; f) \) by a constants. For example, at \( M_5^2 \leq s \leq M_6^2 \)

\[ \tilde{\alpha}(s) = \int_s^\infty \frac{d\sigma}{\sigma} \rho(\sigma) = \int_s^{M_5^2} \frac{d\sigma}{\sigma} \rho(\sigma; 5) + \int_{M_5^2}^\infty \frac{d\sigma}{\sigma} \rho(\sigma; 6) = \tilde{\alpha}(s; 5) + c(5) . \]

Generally,

\[ \tilde{\alpha}(s) = \tilde{\alpha}(s; f) + c(f) \quad \text{at} \quad M_f^2 \leq s \leq M_{f+1}^2 \]  

(17)

with \textit{shift constants} \( c(f) \) that can be calculated in terms of integrals over \( \rho(\sigma; f + n) \) \( n \geq 1 \) with additional reservation \( c(6) = 0 \) related to the asymptotic freedom condition.

More specifically,

\[ c(f - 1) = \tilde{\alpha}(M_f^2; f) - \tilde{\alpha}(M_f^2; f - 1) + c(f) , \quad c(6) = 0 . \]

These \( c(f) \) reflect the \( \tilde{\alpha}(s) \) continuity at the matching points \( M_f^2 \).
Analogous shift constants

$$ \mathcal{A}_k(s) = \mathcal{A}_k(s; f) + c_k(f) \quad \text{at} \quad M^2_f \leq s \leq M^2_{f+1}$$  \hspace{1cm} (18)

are responsible for continuity of higher expansion functions. Meanwhile, \( c_2(f) \) relates to discontinuities of the “main” spectral function (14).

The one-loop estimate with \( \beta(f) \rho(\sigma; f) = \{ \ln^2(\sigma/\Lambda^2_f) + \pi^2 \}^{-1} \),

$$c(f - 1) - c(f) = \frac{1}{\pi \beta[f]} \arctan \frac{\pi}{\ln \frac{M^2_f}{\Lambda^2_f}} - \frac{1}{\pi \beta[f-1]} \arctan \frac{\pi}{\ln \frac{M^2_{f-1}}{\Lambda^2_{f-1}}} \simeq \frac{17 - f}{54} \alpha^3_s(M^2_f)$$  \hspace{1cm} (19)

and traditional values of the scale parameter \( \Lambda_3, \Lambda_4 \sim 350 - 250 \text{ MeV} \) reveals that these constants

$$c(5) \simeq 3.10^{-4}; \ c(4) \simeq 3.10^{-3}; \ c(3) \sim 0.01, \ c_2(f) \simeq 3 \alpha(M^2_f) c(f)$$

are essential at a few per cent level for \( \tilde{\alpha} \) and at ca 10% level for the \( \mathcal{A}_2 \).

This means that the quantitative analysis of some \( s \)-channel events like, e.g., \( e^+e^- \) annihilation [3], \( \tau \)-lepton decay [5] and charmonium width [12] at the \( f = 3 \) region should be influenced by these constants.

**Global Euclidean functions** On the other hand, in the Euclidean, instead of the spline-type function \( \bar{\alpha}_s \), we have now continuous, analytic in the whole \( Q^2 > 0 \) domain, invariant coupling defined, along with (15), via the spectral integral

$$\alpha_{an}(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + Q^2} \rho(\sigma)$$  \hspace{1cm} (20)

with the discontinuous density \( \rho(\sigma) \) (14).

Unhappily, here, unlike for the time-like region, there is no possibility of enjoying any more explicit expression for \( \alpha_{an}(Q^2) \) even in the one-loop case. Moreover, the Euclidean functions \( \alpha_{an} \) and \( \mathcal{A}_k \), being considered in a particular \( f \)-flavour region \( M^2_f \leq Q^2 \leq M^2_{f+1} \), do depend on all \( \Lambda_3, \ldots, \Lambda_6 \) values simultaneously.

Nevertheless, the real difference from the \( f = 3 \) case, numerically, is not big at small \( Q^2 \) and in the “few GeV region”, for practical reasons, it could be of importance.

This situation is illustrated by Fig. 2. Here, by thick solid curves with maxima around \( \sqrt{s}, Q \equiv \Lambda \), we draw expansion functions \( \mathcal{A}_2 \) and \( \mathcal{A}_2 \) in a few GeV region. Thin solid lines zeroes around \( \Lambda \) and negative values below, represent \( \mathcal{A}_3 \) and \( \mathcal{A}_3 \). For comparison, we give also second and third powers of relevant analytic couplings \( \alpha_{an} \) and \( \tilde{\alpha} \).

All these functions correspond to exact two-loop solutions expressed in terms of Lambert function [1].

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8Details of these calculations will be published elsewhere. The assistance of D.S. Kurashev and B.A. Magradze in calculation curves with Lambert functions is gratefully acknowledged.
3 Correlation of experiments

Another quantitative effect stems from the non-power structure of the IAA perturbative expansion. It is also emphasized at the few GeV region.

3.1 The $s$–channel

To illustrate the qualitative difference between our global scheme and other practice of data analysis, we first consider the $f = 3$ region.

Inclusive $\tau$ decay. The IAA scheme with fixed $f = 3$ was used in Ref. [4] for analysis of the inclusive $\tau$–decay. Here, the observed quantity, the $\tau$ lepton time of half–decay, depends on the integral of the $s$–channel matrix element over the region $0 < s < M_\tau^2$. As a result of the 2–loop IAA analysis of the experimental input $R_\tau = 3.633$ [25], the value $\tilde{\alpha}^{(2)}(M_\tau^2) = 0.378$ has been obtained that has to be compared with related result of usual analysis $\tilde{\alpha}_s^{(3)}(M_\tau^2) = 0.337$. This shift $\Delta \alpha \simeq 0.04$ resulted in a rather big change in the extracted $\Lambda$ value. Meanwhile, some part of this shift can be “absorbed” by the shift constant $c(3)$.

The process of Inclusive $e^+e^-$ hadron annihilation provides us with an important piece of information on the QCD parameters. In the usual treatment, (see, e.g., Refs.[25, 26])
the basic relation looks like

\[ \frac{R(s)}{R_0} = 1 + r(s); \quad r(s) = \frac{\tilde{\alpha}_s(s)}{\pi} + r_2 \tilde{\alpha}_s^2(s) + r_3 \tilde{\alpha}_s^3(s). \] (21)

Here, the numerical coefficients \( r_1 = 1/\pi = 0.318, \ r_2 = 0.142, \ r_3 = -0.413 \) (related to the \( f = 5 \) case) are not diminishing. However, a rather big negative \( r_3 \) value comes mainly from the \(-r_1 \pi^2 \beta_5^2/3\) contribution equal to \(-0.456\). Instead of (21), with due account of (4), we now have

\[ r(s) = 1 + \frac{\tilde{\alpha}(s)}{\pi} + d_2 \mathcal{A}_2(s) + d_3 \mathcal{A}_3(s); \] (22)

with reasonably decreasing coefficients \( d_1 = 0.318; \ d_2 = 0.142; \ d_3 = 0.043 \), the mentioned \( \pi^2 \) term of \( r_3 \) being “swallowed” by \( \tilde{\alpha}(s) \).

Now, the main difference between (22) and (21) is due to the term \( d_2 \mathcal{A}_2 \) standing in the place of \( d_2 \alpha^2 \). The difference can be estimated by adding into (21) the structure \( r_4 \alpha^4 \) with \( r_4 = d_2 \beta_5^2 \pi^2 \simeq -0.62 \). This effect could be essential in the region of \( \tilde{\alpha}(s) \simeq 0.20 - 0.25 \).

### 3.2 The \( Q^2 \)-channel

The \( Q^2 \)-channel: Bjorken and GLS sum rules. In the paper [6], the IAA has been applied to the Bjorken sum rules. Here, one has to deal with the \( Q^2 \)-channel at small transfer momentum squared \( Q^2 \lesssim 10 \text{GeV}^2 \). Due to some controversy of experimental data, we give here only a part of the results of [6]. For instance, using data of the SMC Collaboration [27] for \( Q_0^2 = 10 \text{GeV}^2 \) the authors obtained \( \alpha_{an}(Q_0^2) = 0.301 \) instead of \( \alpha_{pt}(Q_0^2) = 0.275 \).

In the Euclidean channel, instead of power expansion like (2), we typically have

\[ d(Q^2) = \frac{\alpha_{an}(Q^2)}{\pi} + d_2 \mathcal{A}_2(Q^2) + d_3 \mathcal{A}_3(Q^2). \] (23)

Here, the modification is related to non-perturbative power structures behaving like \( \Lambda^2/Q^2 \) at \( Q^2 \gg \Lambda^2 \). As it has been estimated above, these corrections could be essential in a few GeV region.

The same remark could be made with respect to analysis of the Gross–Llywellin-Smith sum rules of [8].

**Some comments** are in order:

— We see that, generally, the extracted values of \( \alpha_{an} \) and of \( \tilde{\alpha} \) are both slightly greater in a few GeV region than the relevant values of \( \tilde{\alpha}_s \) for the same experimental input. This term contributes about \( 8.10^{-4} \) into the \( r(M_Z^2) \) and, correspondingly, \( 0.0025 \) into the extracted \( \tilde{\alpha}_s(M_Z^2) \) value. This means, that the main part of the “traditional three-loop term” \( r_3 \tilde{\alpha}_s^3 \) in the r.h.s. of (21) being of the one–loop origin is essential for the modern quantitative analysis of the data. In particular, it should be taken into the account even in the so-called NLLA which is a common approximation for the analysis of events at \( \sqrt{s} = M_Z \).
corresponds to the above-mentioned non-power character of new asymptotic expansions with a suppressed higher-loop contribution.

— At the same time, for equal values of $\alpha_{an}(x_*) = \tilde{\alpha}(x_*) = \bar{\alpha}_s(x_*)$, the analytic scale parameter $\Lambda_{an}$ values extracted from $\alpha_{an}$ and $\tilde{\alpha}$ are a bit greater than that $\Lambda_{\overline{MS}}$ taken from $\bar{\alpha}_s$. This feature is related to a “smoother” behavior of both the regular functions $\alpha_{an}$ and $\tilde{\alpha}$ as compared to the singular $\bar{\alpha}_s$.

### 3.3 Conclusion

To summarize, we repeat once more our main points.

1. We have formulated the self-consistent scheme for analyzing data both in the space-like and time-like regions.

   The fundamental equation connecting these regions is the dipole spectral relation \([1]\) between renormalization–group invariant non-power expansions $D_{an}(Q^2)$ and $R_\pi(s)$.

   Just this equation, equivalent to the Källen–Lehmann representation, is responsible for non-perturbative terms in the $Q^2$–channel involved into $\alpha_{an}(Q^2)$ and non-power expansion functions $\{A_k(Q^2)\}$. These terms, non-analytic in the coupling constant $\alpha$, are a counterpart to the perfectly perturbative $\pi^2$–terms effectively summed in the $s$–channel expressions $\bar{\alpha}(s)$ and $\{\bar{A}_k(s)\}$.

2. As a by-product, we ascertain a new qualitative feature of the IAA, relating to its non-perturbativity in the $Q^2$–domain. It can be considered as a minimal non-perturbativity or minimal non-analyticity\(^{10}\) in $\alpha$ as far as it corresponds to perturbativity in the $s$–channel.

   Physically, it implies that minimal non-perturbativity cannot be referred to any mechanism producing effect in the $s$–channel.

3. The next result relates to the correlation between regions with different values of the effective flavor number $f$. Dealing with the massless $\overline{MS}$ renormalization scheme, we argue that the usual perturbative QCD expansion provides our scheme only with step–discontinuous spectral density \([13]\) depending simultaneously on different scale parameters $\Lambda_f$; $f = 3, \ldots, 6$ connected by usual matching relations.

   This step–discontinuous spectral density yields, on the one hand, smooth analytic coupling $\alpha_{an}(Q^2)$ and higher functions $\{A_k(Q^2)\}$ in the space-like region — eq.\((13)\).

   On the other hand, it produces the spline–continuous invariant coupling $\bar{\alpha}(s)$ and functions $\{\bar{A}_k(s)\}$ in the time-like region — eq.\((13)\).

   As a result, the global expansion functions $\{\bar{A}_k(Q^2)\}$ and $\{\bar{A}_k(s)\}$ differ both from the that ones $\{A_k(Q^2; f)\}$ and $\{\bar{A}_k(s; f)\}$ with a fixed value of a flavour number.

4. Thus, our global IAA scheme uses common invariant coupling $\bar{\alpha}(Q^2, f)$ and matching relations, only as an input. Practical calculation for an observable now involves expansions over the sets $\{A_k(Q^2)\}$ and $\{\bar{A}_k(s)\}$, that is non-power series with usual numerical coefficients $d_k$ obtained by calculation of the relevant Feynman diagrams.

\(^{10}\)Compatible with the RG invariance and the $Q^2$ analyticity — compare with \([28]\).
This means that, generally, one should check the accuracy of the bulk of extractions of the QCD parameters from diverse “low energy” experimental data. Our preliminary estimate shows that such a revision could influence the rate of their correlation.

5. Last but not least. As it has been mentioned in our recent publications [2, 3], the IAA obeys an immunity with respect to higher loop and renormalization scheme effects. Now, we got an additional insight into this item related to observables and can state that the perturbation series for an observable in the IAA have better convergence properties (than in usual RG–summed perturbation theory) in both the $s$– and $Q^2$ – channels.

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