Improved RIP-Based Bounds for Guaranteed Performance of Several Compressed Sensing Algorithms

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Abstract

Iterative hard thresholding (IHT), compressive sampling matching pursuit (CoSaMP), and subspace pursuit (SP) are three types of mainstream compressed sensing algorithms using hard thresholding operators for signal recovery and approximation. The guaranteed performance for signal recovery via these algorithms has mainly been analyzed under the condition that the restricted isometry constant of a sensing matrix, denoted by $\delta_K$ (where $K$ is an integer number), is smaller than a certain threshold value in the interval $(0, 1)$. The condition $\delta_K < \delta^*$ for some number $\delta^* \leq 1$ ensuring the success of signal recovery with a specific compressed sensing algorithm is called the restricted-isometry-property-based (RIP-based) bound for guaranteed performance of the algorithm.

At the moment the best known RIP-based bound for the guaranteed recovery of $k$-sparse signals via IHT is $\delta_{3k} < 1/\sqrt{3} \approx 0.5773$, the bound for guaranteed recovery via CoSaMP is $\delta_{4k} < 0.4782$, and the bound via SP is $\delta_{3k} < 0.4859$. A fundamental question in this area is whether such theoretical results can be further improved. The purpose of this paper is to affirmatively answer this question and rigorously prove that the RIP-based bounds for guaranteed performance of IHT can be significantly improved to $\delta_{3k} < (\sqrt{5} - 1)/2 \approx 0.618$, the bound for CoSaMP can be improved and pushed to $\delta_{4k} < 0.5593$, and the bound for SP can be improved to $\delta_{3k} < 0.5108$. These improvements are far from being trivial and are achieved through establishing some deep properties of the hard thresholding operator and certain tight error estimations by efficiently exploiting the structure of the underlying algorithms.

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I. INTRODUCTION

One of the important tasks in signal processing is to recover (reconstruct) an unknown signal from the linear and nonadaptive measurements acquired for the signal. The compressed sensing algorithms have been developed for signal recovery when the number of measurements is much less than the signal dimension and when the signal is sparse or can be sparsely approximated [1]–[4]. The sparse or nearly sparse signals arise in many scenarios especially when the signal is represented in certain transformed domains or over redundant bases [5]–[10]. The recovery of a sparse signal or the significant information of a signal usually amounts to solving a sparse optimization model and the numerical methods for solving such a model are often called compressed sensing algorithms (see, e.g., [3], [8], [9], [11], [12]). Denote by \( \|z\|_0 \) the '\( \ell_0 \)-norm' counting the number of nonzero entries of the \( n \)-dimensional vector \( z \in \mathbb{R}^n \).

Let \( A \) be an \( m \times n \) sensing (or measurement) matrix with \( m < n \). The typical model for sparse signal recovery can be formulated as the \( \ell_0 \)-minimization problem

\[
\min \{\|z\|_0 : \|Az - y\|_2 \leq \epsilon\},
\]

where \( \epsilon \geq 0 \) is a given parameter and \( y := Ax + \nu \) are the measurements of the target signal \( x \in \mathbb{R}^n \) with measurement errors \( \nu \in \mathbb{R}^m \) bounded as \( \|\nu\|_2 \leq \epsilon \). The model above aims at finding the sparsest data \( z^* \) that can best fit the linear measurements of \( x \) and thus under suitable assumption the recovery \( z^* = x \) can be achieved. In many practical situations, however, one is interested in recovering only the significant information of a signal which usually is interpreted as a small number of the largest absolute coefficients over the redundant bases for the signal (such as the redundant wavelet bases for a natural image). Based on this consideration, the sparse recovery model can be formulated as the following minimization problem with a sparsity constraint:

\[
\min_{z} \{\|Az - y\|_2^2 : \|z\|_0 \leq k\},
\]

where \( k \) is a given integer number, the estimation of the sparsity level of the signal. The purpose of the model (1) is to find the \( k \)-term approximation of the target signal such that the selected \( k \) terms can best fit the acquired measurements compared to other \( k \) terms. The model (1) is not only an essential model for sparse signal recovery to which several compressed sensing algorithms have been developed (see, e.g., [8], [9], [11]–[13]), but also an important model closely related to the low-rank matrix recovery [14]–[16], variable selections in statistics [17]–[19], and sparse optimization and its various applications [12], [20]–[22].
For the model (1), the basic algorithm using the hard thresholding operator is called the iterative hard thresholding (IHT) [23]–[25] which admits several modifications such as the hard thresholding pursuit (HTP) [26], the IHT with a fixed stepsize [27], the normalized iterative hard thresholding (NIHT) [28], the graded IHT [29], [30], and the recent Newton-step-based hard thresholding algorithms [31], [32]. The more sophisticated methods using the hard thresholding operator include the well known compressive sampling matching pursuit (CoSaMP) [33] and subspace pursuit (SP) [34]. This study is focused on the analysis of the above-mentioned IHT, CoSaMP and SP, three well known compressed sensing algorithms. Our purpose is to achieve remarkable improvement on the existing theoretical results concerning the guaranteed success in signal recovery/approximation with these algorithms.

To describe these algorithms, let us first introduce a few notations that are used throughout the paper. We use \( \mathbb{R}^n \) to denote the \( n \)-dimensional Euclidean space and all vectors are understood as column vectors unless otherwise specified. Given a vector \( z \in \mathbb{R}^n \), the operator \( \mathcal{H}_k(z) \in \mathbb{R}^n \) called the hard thresholding operator retains the \( k \) largest absolute entries of \( z \) and sets other entries to zeros. We use \( L_k(z) \) to denote the index set of the \( k \) largest absolute entries of the vector \( z \), and we use \( \text{supp}(z) = \{ i : z_i \neq 0 \} \) to denote the support of the vector \( z \), i.e., the index set of nonzero entries of \( z \). Clearly, for a given vector \( z \), both \( L_k(z) \) and \( \mathcal{H}_k(z) \) may not be uniquely determined. As a result, we will point out a predefined rule for the selection of the largest entries and their index sets when such a selection matters. A vector \( x \) is said to be \( k \)-sparse if \( \|x\|_0 \leq k \).

We now recall a few algorithms concerned about in this paper. The IHT [23]–[25] is a simple iterative scheme for the model (1).

\[ \text{Algorithm 1 Iterative Hard Thresholding (IHT)} \]
Input the measurement matrix \( A \), measurement vector \( y \), and sparsity level \( k \). Perform the following steps:

S1 Choose an initial \( k \)-sparse vector \( x^0 \), typically \( x^0 = 0 \);
S2 Repeat

\[ x^{p+1} = \mathcal{H}_k(x^p + A^T(y - Ax^p)) \]

until a stopping criterion is met.
Output: the \( k \)-sparse vector \( \hat{x} \).

More efficient algorithms than the IHT can be obtained by integrating an orthogonal projection into the algorithm (also called a pursuit step) (see, e.g., [11], [26]). Using both hard thresholding operator and orthogonal projection, the next algorithm (i.e., Algorithm 1) is referred to as compressive sampling matching pursuit (CoSaMP) which was introduced by Needell and Tropp [33]. The CoSaMP was closely related to an earlier greedy method called regularized orthogonal matching pursuit proposed
by Needell and Vershynin [35], [36]. The step (CP2) in CoSaMP is an orthogonal projection which

**Algorithm 2** Compressive Sampling Matching Pursuit (CoSaMP)

**Input** the measurement matrix $A$, measurement vector $y$, and sparsity level $k$. Perform the steps below:

**S1** Choose an initial $k$-sparse vector $x^0$, typically $x^0 = 0$;

**S2** Repeat

$$
U^{p+1} = \text{supp}(x^p) \cup L_{2k}(A^T(y - Ax^p)),
$$

(CP1)

$$
z^{p+1} = \arg \min_{z \in \mathbb{R}^n} \{ \| y - Az \|_2 : \text{supp}(z) \subseteq U^{p+1} \},
$$

(CP2)

$$
x^{p+1} = H_k(z^{p+1})
$$

(CP3)

until a stopping criterion is met.

**Output:** the $k$-sparse vector $\hat{x}$.

seeks a vector with a prescribed support and it also best fits the measurements over the given support. As pointed out in [37], [38], the orthogonal projection may generally stabilize or speed up the IHT framework.

The subspace pursuit (SP) (i.e., Algorithm 3) introduced by Dai and Milenkovic [34] admits a similar structure to CoSaMP but with a different selection of the index set for the orthogonal projection which is performed twice in each loop instead of only once in CoSaMP.

**Algorithm 3** Subspace Pursuit (SP)

**Given** $A, y, k$ and the initial iterate $x^0 = 0$ with $S^0 = \text{supp}(x^0) = \emptyset$. At the $k$-th step, set $S^p = \text{supp}(x^p)$ and perform the following steps to generate the next iterate $x^{p+1}$ :

$$
\Lambda^{p+1} = S^p \cup L_k(A^T(y - Ax^p)),
$$

$$
z^{p+1} = \arg \min_{z \in \mathbb{R}^n} \{ \| y - Az \|_2 : \text{supp}(z) \subseteq \Lambda^{p+1} \},
$$

$$
S^{p+1} = L_k(z^{p+1}),
$$

$$
x^{p+1} = \arg \min_{z \in \mathbb{R}^n} \{ \| y - Az \|_2 : \text{supp}(z) \subseteq S^{p+1} \}.
$$

Repeat the above steps until a certain stopping criterion is met.

The analyses for the guaranteed performance (including stability and convergence) of these algorithms were carried out widely in terms of the restricted isometry property (RIP) of the sensing matrix. The concept of RIP and the associated restricted isometry constant (RIC) of order $K$, denoted by $\delta_K$, were first introduced by Candès and Tao [3], [4]. The RIP tool is quite natural for the analysis of various compressed sensing algorithms. The IHT for compressive sensing was initiated
by Blumensath and Davies in \[23\] and was shown convergent under the condition $\delta_{3k} < 1/\sqrt{32}$. Stability and guaranteed performance for this method were established in \[24\] under the condition $\delta_{3k} < 1/\sqrt{8}$, which is still rather restrictive. This result was improved to $\delta_{3k} < 1/\sqrt{3}$ by Foucaut in \[26\] (see also in \[11\]). This bound remains the best bound for the hard thresholding pursuit (HTP) algorithm which is a simple combination of IHT and orthogonal projection \[26\]. In this paper, we will show that the current RIP-based bound for IHT is definitely not tight, and it can be improved to $\delta_{3k} < (\sqrt{3} - 1)/2 \approx 0.618$. Certain evidences point to the conjecture that this new bound is optimal, i.e., the tightest one.

In \[24\], some theoretical results (stability and robustness) for CoSaMP were established under the condition $\delta_{4k} \leq 0.1$. (Their proof actually implies that their results are valid under the bound $\delta_{4k} < 0.17157$.) This initial result was significantly improved to $\delta_{4k} < 0.4782$ by Foucart and Rauhut in \[11\]. In this paper, we will further improve this result to $\delta_{4k} < 0.5593$. As seen later, such an improvement is far from being trivial and is achieved by establishing some deep properties of the hard thresholding operator and CoSaMP.

The initial analysis for the SP algorithm was carried out by Dai and Milenkovic \[34\] who showed that the SP converges under the condition $\delta_{3k} < 0.205$. The RIP-based performance analysis of SP has also been done by other researchers. The relatively earlier results were established by Giryes and Elad \[39\] who showed that the stable signal recovery via SP can be ensured under the condition $\delta_{3k} < 0.139$, and by Chang and Wu \[40\] who established their results under the condition $\delta_{3k} < 0.2412$. Recent remarkable improvements of these results have been achieved in \[41\], \[42\]. In \[41\], Lee, Bresler and Junge proved that the bound for SP can be improved to $\delta_{3k} < 0.325$, and in \[42\], Song, Xia and Liu showed that this bound can be significantly improved to $\delta_{3k} < 0.4859$. In this paper, we will further improve these existing results for SP and prove that the signal recovery via SP can be guaranteed under the condition $\delta_{3k} < 0.5108$. We show such a result for a more general algorithmic framework called general subspace pursuit (GSP) which encompasses the SP as a special case. The bound $\delta_{3k} < 0.5108$ is the first convergent result rigorously shown for GSP in this paper.

The main contribution of the paper is summarized in the table below:

| Algorithms | Existing results | New results |
|------------|------------------|-------------|
| IHT        | $\delta_{3k} < 0.5573$ | $\delta_{3k} < 0.618$ |
| CoSaMP     | $\delta_{4k} < 0.4782$ | $\delta_{4k} < 0.5593$ |
| GSP        | $\delta_{4k} < 0.5108$ | $\delta_{4k} < 0.5108$ |
| SP         | $\delta_{4k} < 0.4859$ | $\delta_{4k} < 0.5108$ |

It is worth mentioning that an open question for CoSaMP and SP remains standing: What is the optimal (i.e., the tightest) RIP-based bound for these algorithm? Any improvement on the known
RIP-based bounds for guaranteed success of signal recovery via these algorithms allows us to move closer to the unknown optimal bound which clearly exists in the interval (0,1) for every individual compressed sensing algorithm. While at the moment it remains unclear whether the new results established in this paper for IHT, CoSaMP and SP are optimal or not, from the analysis in this paper it seems that the room for a further improvement of our results is somewhat limited.

The paper is organized as follows. In section II we prove some basic and deep properties of the hard thresholding operator as well as the orthogonal projection. In section III we show an improved RIP-based bound for the guaranteed performance of signal recovery via IHT and CoSaMP algorithms, respectively. In section IV, we show the first convergence result for GSP and obtain an improved RIP-based bound for the SP algorithm.

II. PROPERTIES OF HARD THRESHOLDING AND ORTHOGONAL PROJECTION

For a given set \( S \subseteq \{1,2,\ldots,n\} \), we use \(|S|\) to denote the cardinality of \( S \), and \( \overline{S} = \{1,2,\ldots,n\}\setminus S \) to denote the complement set of \( S \). The set difference of sets \( S \) and \( U \) is denoted by \( S \setminus U = \{i : i \in S, i \notin U\} \). Given \( S \subseteq \{1,\ldots,n\} \), a vector \( x \in \mathbb{R}^n \) and an \( m \times n \) matrix \( A \), the vector \( x_S \in \mathbb{R}^n \) is obtained by retaining the entries of \( x \) indexed by \( S \) and zeroing out other components of \( x \), and the \( m \times |S| \) submatrix \( A_S \) is extracted from \( A \) by retaining only those columns indexed by \( S \) (i.e., by deleting all columns of \( A \) indexed by \( \overline{S} \)).

To establish improved convergence results for the algorithms IHT, CoSaMP and SP, we need to characterize some deep properties of the hard thresholding operator \( H_k(\cdot) \) and orthogonal projection.

**Lemma 2.1:** For any vector \( z \in \mathbb{R}^n \) and any \( k \)-sparse vector \( x \in \mathbb{R}^n \), one has

\[
\| (x - H_k(z))_{S\setminus S^*} \|_2 \leq \| (x - z)_{S\setminus S^*} \|_2 + \| (x - z)_{S^*} \|_2,
\]

where \( S = \text{supp}(x) \) and \( S^* = \text{supp}(H_k(z)) \).

**Proof.** By the definition of \( H_k(\cdot) \), we immediately see that \( \| z - H_k(z) \|_2^2 \leq \| z - d \|_2^2 \) for any \( k \)-sparse vector \( d \). In particular, setting \( d = z_S \), where \( S = \text{supp}(x) \), yields

\[
\| z - H_k(z) \|_2^2 \leq \| z - z_S \|_2^2 = \| z_S \|_2^2 = \| (z - x)_{S^*} \|_2^2,
\]

where the last equality follows from \( x_{\overline{S}} = 0 \). Note that

\[
\| z - H_k(z) \|_2^2 = \| (z - x) + (x - H_k(z)) \|_2^2 \\
= \| z - x \|_2^2 + \| x - H_k(z) \|_2^2 - 2(z - H_k(z))^T(x - z).
\]
Therefore,

\[
\|x - \mathcal{H}_k(z)\|^2_2 \leq -\|(z - x)\S\|^2_2 + 2(x - \mathcal{H}_k(z))^T(x - z). \tag{3}
\]

Note that \(\text{supp}(x - \mathcal{H}_k(z)) \subseteq S \cup S^\ast\) which can be decomposed into three disjoint sets \(S \setminus S^\ast, S^\ast \setminus S\) and \(S^\ast \cap S\). We also note that \((\mathcal{H}_k(z))_i = z_i\) for every \(i \in S^\ast\), and thus \((\mathcal{H}_k(z))_{S \setminus S} = z_{S \setminus S}\) and \((\mathcal{H}_k(z))_{S \cap S^\ast} = z_{S \cap S^\ast}\). The left-hand side of (3) can be written as

\[
\|x - \mathcal{H}_k(z)\|^2_2 = \|[x - \mathcal{H}_k(z)]_{S \setminus S^\ast}\|^2_2 + \|[x - \mathcal{H}_k(z)]_{S^\ast \setminus S}\|^2_2 + \|[x - \mathcal{H}_k(z)]_{S \cap S^\ast}\|^2_2.
\]

The right-hand side of (3) is bounded as

\[
-\|(z - x)\S\|^2_2 + 2(x - \mathcal{H}_k(z))^T(x - z)
= -\|(z - x)\S\|^2_2 + 2((x - \mathcal{H}_k(z))_{S \setminus S^\ast})^T(x - z)_{S \setminus S^\ast} + 2\|(x - z)_{S \setminus S^\ast}\|^2_2 + 2\|(z - x)_{S \setminus S^\ast}\|^2_2
\leq -\|(z - x)\S\|^2_2 + 2\|[x - \mathcal{H}_k(z)]_{S \setminus S^\ast}\|_2\|(x - z)_{S \setminus S^\ast}\|_2 + 2\|(x - z)_{S \setminus S^\ast}\|^2_2
+ \|(x - z)_{S \setminus S^\ast}\|^2_2.
\]

Therefore, by substituting the above two relations into (3) and cancelling and rearranging terms, we obtain that

\[
\|[x - \mathcal{H}_k(z)]_{S \setminus S^\ast}\|^2_2 \leq -\|(z - x)\S\|^2_2 + \|(x - z)_{S \setminus S^\ast}\|^2_2 + 2\|(x - \mathcal{H}_k(z))_{S \setminus S^\ast}\|_2\|(x - z)_{S \setminus S^\ast}\|_2 + 2\|(x - z)_{S \setminus S^\ast}\|^2_2.
\]

Thus \(\|(x - \mathcal{H}_k(z))_{S \setminus S^\ast}\|_2\) is smaller than or equal to the largest real root of the quadratic equation

\[
Q(r) := r^2 - 2r\|(z - x)\S\|_2 + \|(x - z)_{S \setminus S^\ast}\|^2_2 - \|(x - z)_{S \setminus S}\|^2_2 = 0,
\]

to which the largest real root is given by

\[
r^* = \|(x - z)_{S \setminus S^\ast}\|_2 + \|(x - z)_{S \setminus S}\|_2.
\]

Thus we immediately obtain the inequality (2). \(\Box\)

The next useful result is key to our later analysis.

**Lemma 2.2:** For any vector \(z \in \mathbb{R}^n\) and for any \(k\)-sparse vector \(x \in \mathbb{R}^n\) (i.e., \(\|x\|_0 \leq k\)), one has

\[
\|x - \mathcal{H}_k(z)\|_2 \leq \frac{\sqrt{5} + 1}{2}\|(x - z)_{S \cup S^\ast}\|_2, \tag{4}
\]

where \(S = \text{supp}(x)\) and \(S^\ast = \text{supp}(\mathcal{H}_k(z))\).

**Proof.** By Lemma 2.1, \(\|(x - \mathcal{H}_k(z))_{S \setminus S^\ast}\|_2 \leq \Delta_1 + \Delta_2\), where \(\Delta_1\) and \(\Delta_2\) are defined as

\[
\Delta_1 = \|(x - z)_{S \setminus S^\ast}\|_2, \quad \Delta_2 = \|(x - z)_{S \setminus S^\ast}\|_2.
\]
Thus,

\[
\|x - \mathcal{H}_k(z)\|_2^2 = \|(x - \mathcal{H}_k(z))_{S \cup S'}\|_2^2 \\
= \|(x - \mathcal{H}_k(z))_{S'}\|_2^2 + \|(x - \mathcal{H}_k(z))_{S \setminus S'}\|_2^2 \\
\leq \|(x - \mathcal{H}_k(z))_{S'}\|_2^2 + (\Delta_1 + \Delta_2)^2 \\
= \|(x - \mathcal{H}_k(z))_{S \setminus S'}\|_2^2 + \|(x - \mathcal{H}_k(z))_{S \cap S'}\|_2^2 + (\Delta_1 + \Delta_2)^2 \\
= \|(x - z)_{S \setminus S'}\|_2^2 + \|(x - z)_{S \cap S'}\|_2^2 + (\Delta_1 + \Delta_2)^2. \quad (5)
\]

Let \( C := \|(x - z)_{S \cup S'}\|_2 \). We see that \( C^2 = \|(x - z)_{S \cap S'}\|_2^2 + \Delta_1^2 + \Delta_2^2 \), and thus

\[
\|(x - z)_{S \setminus S'}\|_2^2 + \|(x - z)_{S \cap S'}\|_2^2 = C^2 - \Delta_2^2.
\]

Substituting this relation into (5) yields

\[
\|x - \mathcal{H}_k(z)\|_2^2 \leq C^2 + \Delta_1^2 + 2\Delta_1\Delta_2.
\]

When \( \Delta_1 = 0 \), then the above inequality immediately implies the bound (4). Thus, without loss of generality, we assume that \( \Delta_1 \neq 0 \). Denote by \( r = \Delta_2 / \Delta_1 \). Substituting \( \Delta_2 = r \Delta_1 \) into the above inequality, we have

\[
\|x - \mathcal{H}_k(z)\|_2^2 \leq (1 + 2r)\Delta_1^2 + C^2. \quad (6)
\]

We also note that \( \Delta_1^2 + \Delta_2^2 \leq C^2 \) which together with \( \Delta_2 = r \Delta_1 \) implies that \( \Delta_1^2 \leq \frac{C^2}{1 + r^2} \). Thus it follows from (6) that

\[
\|x - \mathcal{H}_k(z)\|_2^2 \leq \left( 1 + \frac{1 + 2r}{1 + r^2} \right) C^2 = g(r)C^2, \quad (7)
\]

where

\[
g(r) := 1 + \frac{1 + 2r}{1 + r^2} = \frac{2(1 + r) + r^2}{1 + r^2}.
\]

Consider the maximum of \( g(r) \) over the interval \([0, \infty)\). If \( r = 0 \), then \( g(0) = 2 \). When \( r \to \infty \), we see that \( g(r) \to 1 \). Note that \( g(r) \) has a unique stationary point in \([0, \infty)\), i.e., the equation

\[
0 = g'(r) = \frac{2(1-r-r^2)}{(1+r^2)^2}
\]

has a unique solution given by \( r^* = \frac{\sqrt{5} - 1}{2} \approx 0.618 \) at which

\[
g(r^*) = 1 + \frac{1 + 2r^*}{1 + (r^*)^2} = \frac{5 + \sqrt{5}}{5 - \sqrt{5}} = \left( \frac{\sqrt{5} + 1}{2} \right)^2.
\]

Thus the maximum value of \( g(r) \) over the interval \([0, \infty)\) is given by

\[
\max\{g(0), g(r^*), g(\infty)\} = g(r^*) = \left( \frac{\sqrt{5} + 1}{2} \right)^2.
\]

Therefore it follows from (7) that

\[
\|x - \mathcal{H}_k(z)\|_2 \leq \sqrt{g(r^*)}C = \frac{\sqrt{5} + 1}{2}C,
\]

which is the desired relation (4). \( \square \)
Note: After the first version of the manuscript appeared in arXiv, J. Shen communicated to us to point out that Lemma 2.2 above can actually follow from Shen and Li’s Theorem 1 in [45] which claims that for any vector $b \in \mathbb{R}^n$ and $k$-sparse vector $x \in \mathbb{R}^n$ and for any $q \geq k$, one has
\[
\|H_q(b) - x\|_2 \geq \sqrt{\mu}\|b - x\|_2, \quad \mu = 1 + \frac{\rho + \sqrt{(4 + \rho)\rho}}{2}, \quad \rho = \frac{\min\{k, n - q\}}{q - k + \min\{k, n - q\}}.
\]

From such a result, it is not difficult to verify that the bound in Lemma 2.2 can be also obtained. Therefore, Lemma 2.2 is a special case of Theorem 1 in [45].

Later we will find that the estimation (4) is very useful to establish the improved performance results for IHT, CoSaMP and SP algorithms. We now point out that the bound (4) is the tightest of its kind (and hence it cannot be improved further). It is this tightness that makes it possible to improve the RIP-based bounds for the guaranteed performance of signal recovery with the above-mentioned algorithms.

**Example 2.3:** [Tightness of (4)]. Let $0 < \tau < k$ be two given integer numbers. Consider two vectors in $\mathbb{R}^n$ ($n > k + \tau$) of the following form:
\[
z = (1, \ldots, 1, \varepsilon, \ldots, \varepsilon, 1/2, \ldots, 1/2)^T \in \mathbb{R}^n,\\
x = (0, \ldots, 0, 1, \ldots, 1, \alpha + \varepsilon, \ldots, \alpha + \varepsilon, 0, \ldots, 0)^T \in \mathbb{R}^n,
\]
where $\alpha \geq 0$ and $0 < \varepsilon \leq 1$ are two parameters.

For the vectors $x$ and $z$ given above, we see that $x$ is $k$-sparse and we may take
\[
H_k(z) = (1, \ldots, 1, 0, \ldots, 0)^T \in \mathbb{R}^n.
\]

Clearly, $S^* = \text{supp}(H_k(z)) = \{1, \ldots, k\}$ and $S = \text{supp}(x) = \{\tau + 1, \ldots, \tau + k\}$. Thus
\[
S^* \cup S = \{1, 2, \ldots, k + \tau\}, \quad S^* \cap S = \{\tau + 1, \ldots, k\},
\]
and hence
\[
\|(x - z)_{S^* \cup S}\|_2^2 = \tau(1 + \alpha^2), \quad \|x - H_k(z)\|_2^2 = \tau \left[1 + (\alpha + \varepsilon)^2\right].
\]

Consider the ratio
\[
\frac{\|x - H_k(z)\|_2^2}{\|(x - z)_{S^* \cup S}\|_2^2} = \frac{1 + (\alpha + \varepsilon)^2}{1 + \alpha^2} =: g(\alpha, \varepsilon).
\]

We now find the maximum value of the function $g(\alpha, \varepsilon)$ with respect to $\alpha \in [0, \infty)$. It is easy to check that there exists a unique stationary point of $g(\alpha, \varepsilon)$ with respect to $\alpha \in [0, \infty)$. In fact, let \[
\frac{\partial g(\alpha, \varepsilon)}{\partial \alpha} = 0
\]
which leads to $\alpha^2 + \alpha \varepsilon - 1 = 0$. Thus the unique stationary point of $g(\alpha, \varepsilon)$ in $[0, \infty)$ is
\[
\alpha^* = \frac{-\varepsilon + \sqrt{\varepsilon^2 + 4}}{2},
\]
at which
\[
g(\alpha^*, \varepsilon) = \frac{1 + (\alpha^* + \varepsilon)^2}{1 + (\alpha^*)^2} = 1 + \frac{(\sqrt{\frac{4 + \varepsilon^2}{2} + \varepsilon})^2}{1 + (\sqrt{\frac{4 + \varepsilon^2}{2} - \varepsilon})^2} = \frac{1 + \sqrt{\frac{\varepsilon^2}{4 + \varepsilon^2}}}{1 - \sqrt{\frac{\varepsilon^2}{4 + \varepsilon^2}}} = g_1(g_2(\varepsilon)),
\]
where
where the functions \( g_1 \) and \( g_2 \) are defined as follows:

\[
g_2(\varepsilon) = \sqrt{\frac{\varepsilon^2}{4 + \varepsilon^2}}, \quad g_1(t) = \frac{1 + t}{1 - t},
\]

where \( 0 \leq t < 1 \). Clearly, \( g_1 \) and \( g_2 \) are increasing functions and \( g_2(\varepsilon) < 1 \). Thus \( g(\alpha^*, \varepsilon) \) is an increasing function of \( \varepsilon \) over \((0, 1]\). Therefore, as \( \varepsilon \) takes a value close to 1, the maximum of the function is achieved at \( \varepsilon = 1 \). Note that \( g(\alpha^*, \varepsilon) = \frac{\sqrt{5 + 1}}{\sqrt{5 - 1}} \). Thus

\[
\lim_{\varepsilon \to 1} g(\alpha^*, \varepsilon) = \frac{\sqrt{5 + 1}}{\sqrt{5 - 1}} = \left(\frac{\sqrt{5 + 1}}{2}\right)^2 \geq 1 + \varepsilon^2
\]

for any \( \varepsilon \in (0, 1] \). As \( g(0, \varepsilon) = 1 + \varepsilon^2 \) and \( g(\infty, \varepsilon) := \lim_{\alpha \to \infty} g(\alpha, \varepsilon) = 1 \), the maximum of \( g(\alpha, \varepsilon) \) in \([0, \infty)\) is determined as follows:

\[
\max_{\alpha \in [0, \infty)} g(\alpha, \varepsilon) = \max\{g(0, \varepsilon), g(\infty, \varepsilon), g(\alpha^*, \varepsilon)\} = \max\{1 + \varepsilon^2, 1, g(\alpha^*, \varepsilon)\} = g(\alpha^*, \varepsilon),
\]

which tends to \( (\frac{\sqrt{5 + 1}}{2})^2 \) as \( \varepsilon \to 1 \). This means the bound (4) is tight since the ratio \( g(\alpha, \varepsilon) \) can approach to \( (\frac{\sqrt{5 + 1}}{2})^2 \) for any level of accuracy provided that \( \alpha \) and \( \varepsilon \) are suitably chosen. In particular, this ratio can achieve the exact value \( (\frac{\sqrt{5 + 1}}{2})^2 \) by taking \( \varepsilon = 1 \) and \( \alpha = \frac{\sqrt{5} - 1}{2} \). In other words, the equality in (4) can be achieved at the vectors

\[
z = (1, \ldots, 1, 1/2, \ldots, 1/2)^T \in \mathbb{R}^n,
\]

\[
x = (0, \ldots, 0, 1, \ldots, 1, \eta, \ldots, \eta, 0, \ldots, 0)^T \in \mathbb{R}^n,
\]

where \( \eta = (\sqrt{5} + 1)/2 \).

In the rest of the paper, we will use the following concept of restricted isometry constant (RIC) and its several useful properties listed in Lemma 2.5 below.

**Definition 2.4:**[3] Let \( A \) be a given \( m \times n \) matrix with \( m < n \). The restricted isometry constant (RIC), denoted \( \delta_q := \delta_q(A) \), is the smallest number \( \delta \geq 0 \) such that

\[
(1 - \delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta)\|x\|_2^2
\]

holds for all \( q \)-sparse vectors \( x \in \mathbb{R}^n \). If \( \delta_q < 1 \), then \( A \) is said to satisfy the restricted isometry property (RIP) of order \( q \).

From the definition, we see that \( \delta_{q_1} \leq \delta_{q_2} \) if \( q_1 \leq q_2 \). Implied directly from the above definition are the following properties which are widely utilized in the compressed sensing literature and in this paper.

**Lemma 2.5:**[3, 26, 33] (i) Let \( u, v \in \mathbb{R}^n \) be \( s \)-sparse and \( t \)-sparse vectors, respectively. If \( \text{supp}(u) \cap \text{supp}(v) = \emptyset \), then

\[
|u^TAv| \leq \delta_{s+t}\|u\|_2\|v\|_2.
\]
(ii) Let \( v \in \mathbb{R}^n \) be a vector and \( S \subseteq \{1, 2, \ldots, n\} \) be an index set. If \( |S \cup \text{supp}(v)| \leq t \), then
\[
\|[(I - A^T A)v]_S\|_2 \leq \delta_t \|v\|_2.
\]

(iii) Let \( \Lambda \subseteq \{1, \ldots, n\} \) be an index set satisfying that \( \Lambda \cap \text{supp}(u) = \emptyset \) and \( |\Lambda \cup \text{supp}(u)| \leq t \). Then
\[
\|(A_{\Lambda})^T Au\|_2 = \|(A^T Au)_{\Lambda}\|_2 \leq \delta_t \|u\|_2.
\]

Item (iii) follows from (ii). In fact, when \( \Lambda \cap \text{supp}(u) = \emptyset \) which means \( u_\Lambda = 0 \), one has
\[
\|(A_{\Lambda})^T Au\|_2 = \|(A^T Au)_{\Lambda}\|_2 = \|[(I - A^T A)u]_\Lambda\|_2 \leq \delta_t \|u\|_2.
\]

As pointed out in [37], [38], the hard thresholding operator may cause numerical oscillation, and thus the IHT may fail to consistently reduce the objective value of the model (1) during the course of iterations. The orthogonal projection is one of the techniques which may alleviate the oscillation problem. Thus it is widely used in hard-thresholding-based algorithms including CoSaMP, SP, the latest Newton-step-based thresholding [31], and optimal \( k \)-thresholding algorithms [37], [38]. We now state a property of orthogonal projection that will be used in the analysis of CoSaMP and SP later. We first point out the following technical result.

**Lemma 2.6:** Given three constants \( \alpha_1, \alpha_2, \alpha_3 \geq 0 \) where \( \alpha_1 < 1 \), if \( t \) satisfies the condition
\[
0 \leq t - \alpha_3 \leq \alpha_1 \sqrt{t^2 + \alpha^2_2},
\]
then
\[
t \leq \frac{\alpha_1}{\sqrt{1 - \alpha_1^2}} \alpha_2 + \frac{1}{1 - \alpha_1} \alpha_3.
\]

**Proof.** Under the conditions of the Lemma, \( t \) satisfies the condition \((t - \alpha_3)^2 \leq \alpha_1^2 (t^2 + \alpha_2^2)\), i.e.,
\[
\phi(t) := (1 - \alpha_1^2)t^2 - 2t\alpha_3 + \alpha_3^2 - \alpha_1^2\alpha_2^2 \leq 0.
\]
Thus \( t \) is less than or equal to the largest real root of the quadratic equation \( \phi(t) = 0 \). That is,
\[
t \leq \frac{2\alpha_3 + \sqrt{4\alpha_3^2 - 4(1 - \alpha_1^2)(\alpha_3^2 - \alpha_1^2\alpha_2^2)}}{2(1 - \alpha_1^2)}
\]
\[
= \frac{\alpha_3 + \sqrt{\alpha_1^2\alpha_3^2 + (1 - \alpha_1^2)\alpha_1^2\alpha_2^2}}{1 - \alpha_1^2}
\]
\[
\leq \frac{\alpha_3 + \alpha_1\alpha_3 + \alpha_1\alpha_2 \sqrt{1 - \alpha_1^2}}{1 - \alpha_1^2}
\]
\[
= \frac{\alpha_1}{\sqrt{1 - \alpha_1^2}} \alpha_2 + \frac{\alpha_3}{1 - \alpha_1},
\]
as desired. \( \Box \)

The following is a fundamental property of the orthogonal projection. A similar property can be found in the literature, however, the following one is more general than the existing ones.
Lemma 2.7: Let \( y = Ax + \nu \) be the measurements of the signal \( x \) where \( \nu \) is a noisy vector. Let \( S, \Lambda \subseteq \{1, \ldots, n\} \) be two nonempty index sets and \( |S| \leq \tau \) where \( \tau \) is an integer number. Let \( x^* \) be the solution to the orthogonal projection problem

\[
x^* = \arg \min_{z \in \mathbb{R}^n} \{ \| y - Az \|_2 : \text{supp}(z) \subseteq \Lambda \}.
\]  

Let \( \Gamma \) be any given index set satisfying \( \Lambda \subseteq \Gamma \subseteq \{ i : [A^T(y - Ax^*)]_i = 0 \} \). If \( \delta_{|\Gamma| + \tau} < 1 \), then

\[
\|(x_S - x^*)_\Gamma\|_2 \leq \frac{\delta_{|\Gamma| + \tau} \|(x_S - x^*)_\Gamma\|_2}{\sqrt{1 - \delta_{|\Gamma| + \tau}^2}} + \frac{\|A^T\nu'\|_2}{1 - \delta_{|\Gamma| + \tau}}.
\]  

and hence

\[
\|x_S - x^*\|_2 \leq \frac{\|(x_S - x^*)_\Gamma\|_2}{\sqrt{1 - \delta_{|\Gamma| + \tau}^2}} + \frac{\|A^T\nu'\|_2}{1 - \delta_{|\Gamma| + \tau}},
\]  

where \( \nu' = Ax_S + \nu \).

Proof. Since \( x^* \) is the optimal solution to the problem (9), by optimality, we immediately see that \( [A^T(y - Ax^*)]_\Lambda = 0 \) and thus the set \( \{ i : [A^T(y - Ax^*)]_i = 0 \} \) is nonempty since it contains \( \Lambda \) as a subset. By the definition of \( \Gamma \), we have \( [A^T(y - Ax^*)]_\Gamma = 0 \) which, together with \( y = Ax_S + \nu' \) where \( \nu' = Ax_S + \nu \), implies that

\[
0 = [A^T A(x_S - x^*) + A^T \nu']_\Gamma = [(A^T A - I)(x_S - x^*)]_\Gamma + (x_S - x^*)_\Gamma + [A^T \nu']_\Gamma.
\]  

As \( \text{supp}(x_S - x^*) \subseteq S \cup \Lambda \) and \( \Lambda \subseteq \Gamma \), we see that

\[
|\text{supp}(x_S - x^*) \cup \Gamma| \leq |(S \cup \Lambda) \cup \Gamma| = |S \cup \Gamma| \leq |\Gamma| + \tau.
\]  

Thus it follows from (12) and Lemma 2.5(ii) that

\[
\|(x_S - x^*)_\Gamma\|_2 \leq \|(A^T A - I)(x_S - x^*)\|_\Gamma + \|A^T \nu'\|_2
\leq \delta_{|\Gamma| + \tau} \|x_S - x^*\|_2 + \|A^T \nu'\|_2
= \delta_{|\Gamma| + \tau} \sqrt{\|(x_S - x^*)_\Gamma\|_2^2 + \|(x_S - x^*)_\Gamma\|_2^2} + \|A^T \nu'\|_2.
\]  

If \( \|(x_S - x^*)_\Gamma\|_2 \leq \|A^T \nu'\|_2 \), the desired relations (10) and (11) hold trivially. Otherwise if \( \|(x_S - x^*)_\Gamma\|_2 > \|A^T \nu'\|_2 \), then by setting \( \alpha_1 = \delta_{|\Gamma| + \tau} < 1, \alpha_2 = \|(x_S - x^*)_\Gamma\|_2, \alpha_3 = \|A^T \nu'\|_2 \), and \( t = \|(x_S - x^*)_\Gamma\|_2 \), it follows from (13) and Lemma 2.6 that

\[
\|(x_S - x^*)_\Gamma\|_2 \leq \frac{\delta_{|\Gamma| + \tau} \||(x_S - x^*)_\Gamma\|_2}{\sqrt{1 - \delta_{|\Gamma| + \tau}^2}} + \frac{\|A^T \nu'\|_2}{1 - \delta_{|\Gamma| + \tau}}.
\]  

Note that

\[
\sqrt{(a + b)^2 + c^2} \leq \sqrt{a^2 + c^2} + b
\]  

(15)
for any \(a, b, c \geq 0\). It follows from (14) and (15) that
\[
\|x_S - x^*\|_2^2 = \|(x_S - x^*)_\Gamma\|_2^2 + \|(x_S - x^*)_{\overline{\Gamma}}\|_2^2 \\
\leq \left( \frac{\delta_{|\Gamma|+r} \|(x_S - x^*)_{\overline{\Gamma}}\|_2}{1 - \delta_{|\Gamma|+r}^2} + \frac{\|A^T \nu\|_2}{1 - \delta_{|\Gamma|+r}} \right)^2 + \|(x_S - x^*)_{\overline{\Gamma}}\|_2^2 \\
\leq \left( \frac{\delta_{|\Gamma|+r}^2 \|(x_S - x^*)_{\overline{\Gamma}}\|_2^2}{1 - \delta_{|\Gamma|+r}^2} + \|(x_S - x^*)_{\overline{\Gamma}}\|_2^2 + \frac{\|A^T \nu\|_2^2}{1 - \delta_{|\Gamma|+r}} \right)^2 \\
= \left( \frac{1}{1 - \delta_{|\Gamma|+r}^2} \|(x_S - x^*)_{\overline{\Gamma}}\|_2^2 + \frac{\|A^T \nu\|_2^2}{1 - \delta_{|\Gamma|+r}} \right)^2,
\]
as desired. \(\square\)

\section{III. RIP-based bounds for IHT and CoSAMP}

In this section, we establish improved results on the guaranteed performance for signal recovery via IHT and CoSaMP. The result for SP will be given separately in section \[IV\]. Such improvements in terms of RIP are vital for both compressed sensing theory and algorithms. The RIP-based bound directly clarifies the scenarios in which the algorithms are guaranteed to be successful for signal recovery. Thus a more relaxed RIP condition is imposed, the broader the class of signal recovery problems can be identified to be solved successfully by the algorithms. Moreover, the relaxed RIP-based bound can also dramatically impact on the number of measurements required for signal recovery.

As shown in \[4\], \[43\], \[44\], for Gaussian random sensing matrix \(A\) of size \(m \times n\) \((m \ll n)\), there is a universal constant \(C^* > 0\) such that the RIC of \(A/\sqrt{m}\) satisfies \(\delta_{2k} \leq \delta^* < 1\) with probability at least \(1 - \xi\) provided that
\[
m \geq C^*(\delta^*)^{-2}(k(1 + \ln(n/k)) + \ln(2\xi - 1)).
\]
From this result, it can be seen that the higher the bound \(\delta^*\), the less number of measurements is required.

It is worth mentioning that the practical signal \(x\) may not necessarily be \(k\)-sparse. Let \(S\) be the index set for the largest \(k\) absolute entries of the signal \(x\), i.e., \(S = L_k(x)\). Then the \(k\)-sparse vector \(x_S\) is the best \(k\)-terms approximation of \(x\). In terms of \(x_S\), \(y = Ax + \nu = Ax_S + \nu'\) with \(\nu' = Ax_{\overline{S}} + \nu\). This means the measurements \(y\) of the original signal \(x\) can be seen as the measurements of the \(k\)-sparse signal \(x_S\) with error \(\nu'\). Thus when the signal is not \(k\)-sparse, the recovery can be made for a smaller number of significant components of the target signal, i.e., only the \(k\) terms of the signal are recovered.

\subsection{A. Improved result for IHT}

We are ready to show the following main result for IHT.

\[
\|x_S - x^*\|_2^2 = \|(x_S - x^*)_\Gamma\|_2^2 + \|(x_S - x^*)_{\overline{\Gamma}}\|_2^2 \\
\leq \left( \frac{\delta_{|\Gamma|+r} \|(x_S - x^*)_{\overline{\Gamma}}\|_2}{1 - \delta_{|\Gamma|+r}^2} + \frac{\|A^T \nu\|_2}{1 - \delta_{|\Gamma|+r}} \right)^2 + \|(x_S - x^*)_{\overline{\Gamma}}\|_2^2 \\
\leq \left( \frac{\delta_{|\Gamma|+r}^2 \|(x_S - x^*)_{\overline{\Gamma}}\|_2^2}{1 - \delta_{|\Gamma|+r}^2} + \|(x_S - x^*)_{\overline{\Gamma}}\|_2^2 + \frac{\|A^T \nu\|_2^2}{1 - \delta_{|\Gamma|+r}} \right)^2 \\
= \left( \frac{1}{1 - \delta_{|\Gamma|+r}^2} \|(x_S - x^*)_{\overline{\Gamma}}\|_2^2 + \frac{\|A^T \nu\|_2^2}{1 - \delta_{|\Gamma|+r}} \right)^2,
\]
as desired. \(\square\)
\textbf{Theorem 3.1:} Suppose that the sensing matrix $A$ satisfies
\[ \delta_{3k} < \frac{\sqrt{5} - 1}{2} \approx 0.618. \]

Let $y = Ax + \nu$ be the measurements of $x$ with error $\nu$ and $S = L_k(x)$. Then the iterates $x^p$, generated by the IHT, approximate $x$ with error
\[ \|x^p - x_S\|_2 \leq \rho^p \|x^0 - x_S\|_2 + \frac{\sqrt{5} + 1}{2(1 - \rho)} \|A^T \nu'\|_2, \quad (16) \]
where $\nu' = Ax_S + \nu$, and the constants $\rho$ is given as
\[ \rho = \frac{\sqrt{5} + 1}{2}\delta_{3k} < 1. \quad (17) \]

\textbf{Proof.} Denote by $u^p := x^p + A^T(y - Ax^p)$. By the structure of the IHT, $S^{p+1} := \text{supp}(x^{p+1}) = \text{supp}(\mathcal{H}_k(u^p))$. By Lemma 2.2, one has
\[ \|x_S - x^{p+1}\|_2 = \|x_S - \mathcal{H}_k(u^p)\|_2 \leq \frac{\sqrt{5} + 1}{2}\|A^p - u^p\|_{S^{p+1} \cup S}\|_2. \quad (18) \]

We now estimate the term $\|A^p - u^p\|_{S^{p+1} \cup S}$ which can be bounded as follows:
\[
\|A^p - u^p\|_{S^{p+1} \cup S} = \|A^p - \mathcal{H}_k(u^p)\|_{S^{p+1} \cup S} = \|A^p - \mathcal{H}_k(x^p)\|_{S^{p+1} \cup S} \\
\leq \|A^p - \mathcal{H}_k(x^p)\|_{S^{p+1} \cup S} \leq \delta_{3k}\|x^p - x_S\|_2 + \|A^T \nu'\|_2,
\]
where the last inequality follows from Lemma 2.5 (ii) with $|\text{supp}(x_S - x^p) \cup (S^{p+1} \cup S)| \leq 3k$. Substituting this into (18) yields
\[ \|x_S - x^{p+1}\|_2 \leq \frac{\sqrt{5} + 1}{2}(\delta_{3k}\|x_S - x^p\|_2 + \|A^T \nu'\|_2) = \rho\|x_S - x^p\|_2 + \frac{\sqrt{5} + 1}{2}\|A^T \nu'\|_2, \quad (19) \]
where the constant
\[ \rho = \frac{\sqrt{5} + 1}{2}\delta_{3k} < 1, \]
provided that
\[ \delta_{3k} < \frac{2}{\sqrt{5} + 1} = \frac{\sqrt{5} - 1}{2} \approx 0.618. \]
The error bound (16) immediately follows from (19).

\textbf{Remark 3.2:} The above result improves the current best known bound $\delta_{3k} < \frac{1}{\sqrt{3}} \approx 0.5773$ for IHT established by Foucart and Rauhut [11], [26]. The truly tightness of the relation (4) (as indicated by Example 2.3) is essential to the improvement of the RIP-based bound for IHT. The tightness of the bound (4) and the simple argument in the proof of Theorem 3.1 point to the conjecture that
the new bound \( \delta_{3k} < (\sqrt{5} - 1)/2 \) for IHT is optimal (cannot be improved further). This, however, is an interesting conjecture requiring a further investigation. It is also worth mentioning that some researchers developed the RIP-based bounds for the performance of compressed sensing algorithms according to the geometric rate \( \rho \leq 0.5 \) instead of \( \rho < 1 \). From the analysis above, if we require the geometric rate \( \rho \) given in (17) be less than 0.5, namely, \( \rho = \frac{\sqrt{\delta + 1}}{2} \delta_{3k} \leq 0.5 \), which is guaranteed by \( \delta_{3k} \leq (\sqrt{5} - 1)/4 \approx 0.309 \), then we immediately obtain from (19) the following recovery error:

\[
\|x^p - x_S\|_2 \leq 0.5^p \|x^0 - x_S\|_2 + (\sqrt{5} + 1) \|A^T \nu\|_2.
\]

We clearly see that our result for IHT also remarkably improves the existing result \( \delta_{3k} \leq 0.22 \) established by Shen and Li [45] for IHT in terms of geometric rate 0.5. From the proof of Theorem 6.18 in [11], it is easy to verify that the RIP-based bound obtained by Foucart and Rauhut is \( \delta_{3k} \leq \frac{1}{2 \sqrt{3}} \approx 0.2886 \) in terms of geometric rate 0.5.

The estimation (16) implies the finite termination and stability of IHT through a standard lemma such as Lemma 6.23 in [11]. We don’t state the stability results and the ones concerning the number of iterations required to achieve the desired recovery accuracy (the interested reader can see the statement of such results in [11], [30] for details). In this paper, we only focus on the establishment of the estimation like (16) which ensures the convergence of an algorithm and the success of signal recovery/approximation with the algorithm.

**B. Improved result for CoSaMP**

We now focus on the analysis of CoSaMP which was described as Algorithm 2 in Section I of this paper. We first establish the following helpful technical result which together with Lemma 2.2 eventually yields an improved RIP-based bound for the guaranteed success of CoSaMP. The idea of this technical result will also come into play in the analysis of GSP and SP that is presented separately in section IV. We define the univariate function

\[
\phi(t) = \frac{t + \sqrt{t(4 - t)}}{4 - 2t}, \quad 0 \leq t \leq 1
\]

which is used frequently in this and next section. It is easy to show that the function satisfies the following properties: \( t \leq \phi(t) \) for any \( 0 \leq t \leq 1 \), and \( \phi \) is increasing in the sense that \( \phi(t_1) \leq \phi(t_2) \) for any \( 0 \leq t_1 \leq t_2 \leq 1 \).

**Lemma 3.3:** Let \( y = Ax + \nu \) be the measurements of \( x \) with error \( \nu \). Let \( S = L_k(x) \). Given a \( k \)-sparse vector \( x^p \) with support \( S^p = \text{supp}(x^p) \) and the index set

\[
T = \text{supp}[\mathcal{H}_\beta(A^T(y - Ax^p))],
\]
where $\beta \geq 2k$ is an integer number, if $\delta_{2k+\beta} < 1$ one has

$$
\| (x^p - x_S)^T \|_2 \leq \phi(\delta_{2k+\beta}) \| x^p - x_S \|_2 + \frac{2}{2 - \delta_{2k}} \| A^T \nu' \|_2,
$$

where $\nu' = Ax_S + \nu$ and

$$
\phi(\delta_{2k+\beta}) = \frac{\delta_{2k+\beta} + \sqrt{\delta_{2k+\beta}(4 - \delta_{2k+\beta})}}{4 - 2\delta_{2k+\beta}}.
$$

Proof. Let $S, S^p$, and $T$ be defined as above. If $S \cup S^p \subseteq T$, then $\| (x^p - x_S)^T \|_2 = 0$, and hence the relation (21) holds trivially. Thus we only need to consider the case $S \cup S^p \not\subseteq T$. Denote by

$$
\Omega := \| [x_S - (x^p + A^T(y - Ax^p))]_{(S \cup S^p) \setminus T} \|_2.
$$

We distinguish two cases: $\Omega = 0$ and $\Omega \neq 0$. For the first case, i.e., $\Omega = 0$, we have

$$
(x_S - x^{p})_{(S \cup S^p) \setminus T} = [A^T(y - Ax^p)]_{(S \cup S^p) \setminus T},
$$

As the cardinality $|T| = \beta \geq 2k \geq |S \cup S^p|$, we see that

$$
|(S \cup S^p) \setminus T| = |S \cup S^p| - |(S \cup S^p) \cap T| \leq |T| - |(S \cup S^p) \cap T| = |T| \setminus (S \cup S^p)|.
$$

By the definition of $T$, the entries of the vector $A^T(y - Ax^p)$ supported on $(S \cup S^p) \setminus T$ are not among the $\beta$ largest absolute entries of the vector. This observation, together with (23), implies that

$$
\| [A^T(y - Ax^p)]_{(S \cup S^p) \setminus T} \|_2 \leq \| [A^T(y - Ax^p)]_{T \setminus (S \cup S^p)} \|_2.
$$

(The logic here will be used again when we analyze the GSP algorithm in section [IV].)

Combining (22) and (24) yields

$$
\| (x_S - x^p)^T \|_2 = \| (x_S - x^{p})_{(S \cup S^p) \setminus T} \|_2
$$

$$
\leq \| [A^T(y - Ax^p)]_{T \setminus (S \cup S^p)} \|_2
$$

$$
\leq \| [A^T A(x_S - x^p)]_{T \setminus (S \cup S^p)} \|_2 + \| (A^T \nu')_{T \setminus (S \cup S^p)} \|_2
$$

$$
\leq \delta_{2k+\beta} \| x_S - x^p \|_2 + \| A^T \nu' \|_2
$$

$$
\leq \phi(\delta_{2k+\beta}) \| x^p - x_S \|_2 + \frac{2}{2 - \delta_{2k}} \| A^T \nu' \|_2,
$$

where the last second inequality follows from Lemma 2.5(iii) as $T \setminus (S \cup S^p)$ and $S \cup S^p$ are disjoint, and $|T \setminus (S \cup S^p)| + |S \cup S^p| \leq |T| \geq |T| \geq 2k = 2k + \beta$. The final inequality follows from the fact $2/(2 - \delta_{2k}) \geq 1$ and $\delta_{2k+\beta} \leq \phi(\delta_{2k+\beta})$ since $t \leq \phi(t)$ for any $0 \leq t \leq 1$, where $\phi(\cdot)$ is defined as (20). Thus the bound (21) is valid when $\Omega = 0$.

It is sufficient to show that the bound (21) remains valid when $\Omega \neq 0$. For this case, we use $r$ to denote the ratio of $\| (x_S - x^p)^T \|_2$ and $\Omega$, i.e.,

$$
\| (x_S - x^p)^T \|_2 = r\Omega.
$$
To show (21), it suffices to bound $r\Omega$. As $y = Ax_S + \nu'$, we first note that

$$\Omega = \|[(I - AT)A(x_S - x^p) + AT\nu']_{(S \cup S^p) \setminus T}\|_2$$

$$\leq \|[(I - AT)A(x_S - x^p)]_{(S \cup S^p) \setminus T}\|_2 + \|AT\nu'\|_2$$

$$\leq \delta_{2k}\|x_S - x^p\|_2 + \|AT\nu'\|_2,$$

(25)

where the last inequality follows from Lemma 2.5 (ii). For convenience, denote by $\Theta = (S \cup S^p) \setminus T$. Note that $\|(x_S - x^p)\|_2 = \|(x_S - x^p)\|_T$. Then $\Omega^2$ can be written as

$$\Omega^2 = \|(x_S - x^p)\|_\Theta^2 + \|[AT(y - Ax^p)]\|_\Theta^2$$

$$= \|(x_S - x^p)\|_\Theta^2 + 2(x_S - x^p)^T[AT(y - Ax^p)]\|_\Theta + \|[AT(y - Ax^p)]\|_\Theta^2$$

$$= \|(x_S - x^p)\|_T^2 + 2F_1 + F_2,$$

(26)

where

$$F_1 = (x_S - x^p)^T[AT(y - Ax^p)]\|_\Theta, \quad F_2 = \|[AT(y - Ax^p)]\|_\Theta^2.$$

We now find the lower bounds for $F_1$ and $F_2$. Note that $y = Ax_S + \nu'$ where $\nu' = Ax_S + \nu$. We have

$$F_1 = (x_S - x^p)^T[AT(x_S - x^p) + AT\nu']\|_\Theta$$

$$= (x_S - x^p)^T(A\Theta)^T(x_S - x^p) + (x_S - x^p)^T[AT\nu']\|_\Theta$$

$$= [x_S - x^p]_\Theta^T(A\Theta)^T(x_S - x^p) + [x_S - x^p]_\Theta^T[AT\nu']\|_\Theta + (x_S - x^p)\|_\Theta^T(A\Theta)^T A\Theta(x_S - x^p)\|_\Theta$$

$$\geq (1 - \delta_{2k})\|(x_S - x^p)\|_\Theta^2 - \|(x_S - x^p)\|_T^2 + \|AT\nu'\|_2 - \|(x_S - x^p)\|_\Theta^T(A\Theta)^T A\Theta(x_S - x^p)\|_\Theta$$

$$\geq (1 - \delta_{2k})\|(x_S - x^p)\|_T^2 - \|(x_S - x^p)\|_T^2 + \|AT\nu'\|_2 - \|(x_S - x^p)\|_\Theta^T(A\Theta)^T A\Theta(x_S - x^p)\|_\Theta,$$

where the first inequality follows from the definition of $\delta_{2k}$. Note that

$$\|(x_S - x^p)\|_\Theta^T(A\Theta)^T A\Theta(x_S - x^p)\|_\Theta \leq \delta_{2k}\|(x_S - x^p)\|_\Theta^2 \leq 1/2 \delta_{2k}\|x_S - x^p\|^2_2,$$

where the first inequality follows from Lemma 2.5 (i) due to the fact that the supports of $(x_S - x^p)\|_\Theta$ and $(x_S - x^p)\|_\Theta$ are disjoint and the cardinality of the union of their supports is at most $2k$. The last inequality above follows from the fact $ab \leq (a^2 + b^2)/2$. Thus,

$$F_1 \geq (1 - \delta_{2k})\|(x_S - x^p)\|_T^2 - \|(x_S - x^p)\|_T^2 + \|AT\nu'\|_2 - \delta_{2k}\|x_S - x^p\|^2_2$$

$$= (1 - \delta_{2k})(r\Omega)^2 - (r\Omega)\|AT\nu'\|_2 - \delta_{2k}\|x_S - x^p\|^2_2.$$

(27)
We now bound $F_2$ from below. By the triangle inequality, we obtain

$$F_2 = \|(x_S - x^p)\| - \|(x_S - x^p) - A^T(y - Ax^p)\|_2^2$$

where $\alpha$ which is simplified to

$$\|(x_S - x^p)\|_2 - \|(x_S - x^p) - A^T(y - Ax^p)\|_2^2$$

$= (r - 1)^2\Omega^2.$ \hfill (28)

Combining (26), (27) and (28) yields

$$\Omega^2 \geq (3 - 2\delta_2k)(r\Omega)^2 - 2(\Omega + \|A^T\|_2) - \|x_S - x^p\|^2_2 + (r - 1)^2\Omega^2,$$

which is simplified to

$$(4 - 2\delta_2k)(r\Omega)^2 - 2(r\Omega)(\Omega + \|A^T\|_2) - \|x_S - x^p\|^2_2 \leq 0.$$  Thus $r\Omega$ is smaller than or equal to the largest real root of the quadratic equation $\alpha_1t^2 - \alpha_2t - \alpha_3 = 0,$ where $\alpha_1 = 4 - 2\delta_2k$, $\alpha_2 = 2(\Omega + \|A^T\|_2)$, and $\alpha_3 = \delta_2k\|x_S - x^p\|^2_2$. Thus

$$\|(x_S - x^p) \|_2 = r\Omega \leq \frac{\alpha_2 + \sqrt{\alpha_2^2 + 4\alpha_1\alpha_3}}{2\alpha_1} \leq \frac{2(\Omega + \|A^T\|_2) + \vartheta}{2(4 - 2\delta_2k)},$$

(29)

where

$$\vartheta = \sqrt{4(\Omega + \|A^T\|_2)^2 + 4(4 - 2\delta_2k)\delta_2k\|x_S - x^p\|^2_2}.$$  

By using (15) and (25), we see that

$$\vartheta = 2\sqrt{\frac{\Omega + \|A^T\|_2^2 + (4 - 2\delta_2k)\delta_2k\|x_S - x^p\|^2_2}{2(4 - 2\delta_2k)}}$$

$$\leq \sqrt{\Omega^2 + (4 - 2\delta_2k)\delta_2k\|x_S - x^p\|^2_2 + 2\|A^T\|_2}$$

$$\leq 2\left(\sqrt{\delta_2k\|x_S - x^p\|^2_2 + 2(4 - 2\delta_2k)\delta_2k\|x_S - x^p\|^2_2 + \|A^T\|_2^2} + 2\|A^T\|_2 \right)$$

$$= 2\sqrt{\delta_2k(4 - \delta_2k)\|x_S - x^p\|_2^2 + 4\|A^T\|_2^2}.$$  \hfill (30)

where the first inequality follows from (15), and the second inequality follows from (25) and (15). By using (30) and (25), we conclude from (29) that

$$\|(x_S - x^p) \|_2 \leq \frac{2\delta_2k\|x_S - x^p\|_2 + 4\|A^T\|_2 + \vartheta}{2(4 - 2\delta_2k)}$$

$$\leq \frac{2\left(\delta_2k + \sqrt{\delta_2k(4 - \delta_2k)}\right)\|x_S - x^p\|_2 + 8\|A^T\|_2}{2(4 - 2\delta_2k)}$$

$$= \phi(\delta_2k)\|x_S - x^p\|_2 + \frac{2}{2 - \delta_2k}\|A^T\|_2$$

$$= \phi(\delta_2k + \beta)\|x_S - x^p\|_2 + \frac{2}{2 - \delta_2k}\|A^T\|_2,$$  \hfill (31)
where the function $\phi$ is defined as (20), which satisfies that $\phi(\delta_{2k}) \leq \phi(\delta_{2k+1})$ due to $\delta_{2k} \leq \delta_{2k+1}$. Thus the relation (31) implies that the inequality (21) remains valid for the case $\Omega \neq 0$. The proof is complete.

The main result for CoSaMP is summarized as follows.

**Theorem 3.4:** If the restricted isometry constant of the sensing matrix $A$ satisfies that

$$\delta_{4k} < 0.5593,$$

then the iterates $\{x^p\}$, generated by the CoSaMP, satisfy that

$$\|x_S - x^p\|_2 \leq \rho^p \|x_S - x^0\|_2 + \frac{C}{1 - \rho} \|A^T \nu'\|_2,$$

where the constants $\rho$ and $C$ are given as

$$\rho = \left(\frac{\delta_{4k} + \sqrt{\delta_{4k}(4 - \delta_{4k})}}{4 - 2\delta_{4k}}\right) \sqrt{1 + \frac{\sqrt{5} + 1}{4\delta_{4k}}} < 1$$

and

$$C = \sqrt{\frac{1 + \frac{\sqrt{5} + 1}{4\delta_{4k}}}{2 - \delta_{2k}}} \left(\frac{2}{1 - \delta_{2k}}\right) + \frac{\sqrt{5} + 1}{2(1 - \delta_{4k})}.$$

**Proof.** Let $U^{p+1}$, $z^{p+1}$ and $x^{p+1}$ are given, respectively, by steps (CP1)–(CP3) of CoSaMP (Algorithm 2 in section I). From the structure of CoSaMP, we see that $S^p = \text{supp}(x^p) \subseteq U^{p+1}$ and $S^{p+1} = \text{supp}(x^{p+1}) = \text{supp}(H_k(z^{p+1})) \subseteq U^{p+1}$ (so $(x^{p+1})_{U^{p+1}} = x^{p+1}$). By Lemma 2.2, we have

$$\|(x_S - x^{p+1})_{U^{p+1}}\|_2 = \|x_{S \cap U^{p+1}} - x^{p+1}\|_2$$

$$= \|x_{S \cap U^{p+1}} - H_k(z^{p+1})\|_2$$

$$\leq \frac{\sqrt{5} + 1}{2} \|(x_{S \cap U^{p+1}} - z^{p+1})_{(S \cap U^{p+1}) \cup S^{p+1}}\|_2$$

$$\leq \frac{\sqrt{5} + 1}{2} \|(x_{S \cap U^{p+1}} - z^{p+1})_{U^{p+1}}\|_2 \quad \text{ (since } S \cap U^{p+1} \cup S^{p+1} \subseteq U^{p+1} \text{) }$$

$$= \eta \|(x_S - z^{p+1})_{U^{p+1}}\|_2,$$

where $\eta = (\sqrt{5} + 1)/2$. Also, since $\text{supp}(x^{p+1}) \subseteq U^{p+1}$, we see that $(x^{p+1})_{U^{p+1}} = 0 = (z^{p+1})_{U^{p+1}}$. This together with the above relation implies that

$$\|x_S - x^{p+1}\|_2^2 = \|(x_S - x^{p+1})_{U^{p+1}}\|_2^2 + \|(x_S - x^{p+1})_{U^{p+1}}\|_2^2$$

$$= \|(x_S - z^{p+1})_{U^{p+1}}\|_2^2 + \|(x_S - z^{p+1})_{U^{p+1}}\|_2^2$$

$$\leq \|(x_S - z^{p+1})_{U^{p+1}}\|_2^2 + [\eta \|(x_S - z^{p+1})_{U^{p+1}}\|_2]^2. \tag{32}$$
Consider the step (CP2) of the CoSaMP. Applying Lemma 2.7 with \( \Gamma = \Lambda = U^{p+1} \), \( S = L_k(x) \), \( x^* = z^{p+1} \), \( \tau = k \) and \( |\Gamma| = 3k \), we conclude that
\[
\| (x_S - z^{p+1})_{U^{p+1}} \|_2 \leq \frac{\delta_{4k} \| (x_S - z^{p+1})_{U^{p+1}} \|_2 + \eta \| A^T \nu' \|_2}{1 - \delta_{4k}}.
\] (33)

Then combining the two relations (32) and (33) and using the inequality (15), we have
\[
\| x_S - x^{p+1} \|_2^2 \leq \| (x_S - z^{p+1})_{U^{p+1}} \|_2^2 + \left( \frac{\eta \delta_{4k} \| (x_S - z^{p+1})_{U^{p+1}} \|_2 + \eta \| A^T \nu' \|_2}{1 - \delta_{4k}} \right)^2
\]
\[
\leq \left( \sqrt{1 + \frac{\eta^2 \delta_{4k}^2}{1 - \delta_{4k}}} \| (x_S - z^{p+1})_{U^{p+1}} \|_2 + \frac{\eta \| A^T \nu' \|_2}{1 - \delta_{4k}} \right)^2,
\] (34)

where the equality follows from the fact \( \eta^2 - 1 = \eta \). In the remainder of this proof, it is sufficient to bound the term \( \| (x_S - z^{p+1})_{U^{p+1}} \|_2 \) in terms of \( \| x_S - x^p \|_2 \). Note that \( (z^{p+1})_{U^{p+1}} = 0 \) and \( S^p = \text{supp}(x^p) \subseteq U^{p+1} \) which implies that \( (x^p)_{U^{p+1}} = 0 \). Thus
\[
\| (x_S - z^{p+1})_{U^{p+1}} \|_2 = \| (x_S)_{U^{p+1}} \|_2 = \| (x_S - x^p)_{U^{p+1}} \|_2.
\] (35)

Setting \( \beta = 2k \) and \( T = \text{supp}[H_\beta(A^T(y - Ax^p))] \), it follows from Lemma 3.3 that the CoSaMP satisfies the relation
\[
\| (x_S - x^p)_T \|_2 \leq \phi(\delta_{4k}) \| x^p - x_S \|_2 + \frac{2}{2 - \delta_{2k}} \| A^T \nu' \|_2,
\] (36)

where the function \( \phi(\cdot) \) is defined as (20). Note that \( T \subseteq U^{p+1} \) which implies that \( U^{p+1} \subseteq \overline{T} \). Thus
\[
\| (x_S - x^p)_{U^{p+1}} \|_2 \leq \| (x_S - x^p)_T \|_2.
\] (37)

Merging the above three relations (35)–(37) leads to
\[
\| (x_S - z^{p+1})_{U^{p+1}} \|_2 \leq \phi(\delta_{4k}) \| x^p - x_S \|_2 + \frac{2}{2 - \delta_{2k}} \| A^T \nu' \|_2.
\]

Therefore, it follows from (33) that
\[
\| x_S - x^{p+1} \|_2 \leq \sqrt{\frac{1 + \eta \delta_{4k}^2}{1 - \delta_{4k}} \left( \phi(\delta_{4k}) \| x_S - x^p \|_2 + \frac{2}{2 - \delta_{2k}} \| A^T \nu' \|_2 \right) + \frac{\eta}{1 - \delta_{4k}} \| A^T \nu' \|_2}
\]
\[
= \rho \| x_S - x^p \|_2 + C \| A^T \nu' \|_2,
\]

where the constants \( \rho \) and \( C \) are given by
\[
\rho = \phi(\delta_{4k}) \sqrt{\frac{1 + \eta \delta_{4k}^2}{1 - \delta_{4k}}} = \left[ \frac{\delta_{4k} + \sqrt{\delta_{4k}(4 - \delta_{4k})}}{4 - 2 \delta_{4k}} \right] \sqrt{\frac{1 + \eta \delta_{4k}^2}{1 - \delta_{4k}}},
\]
\[
C = \left( \frac{2}{2 - \delta_{2k}} \right) \sqrt{\frac{1 + \eta \delta_{4k}^2}{1 - \delta_{4k}}} + \frac{\eta}{1 - \delta_{4k}}.
\]
We now prove that if \( \delta_{4k} < 0.5593 \) then the constant \( \rho < 1 \). Let us focus our attention to the region \( \delta_{4k} < 0.7 \). It is not difficult to verify that \( \delta_{4k} < 0.7 \) guarantees the following two inequalities:

\[
(4 - 2\delta_{4k}) \sqrt{1 - \frac{\delta_{4k}^2}{1 + \eta\delta_{4k}^2}} - \delta_{4k} > 0,
\]

(38)

\[
(1 - \delta_{4k}^2)(4 - 2\delta_{4k}) - \delta_{4k} (1 + \eta\delta_{4k}^2) > 0.
\]

(39)

The condition \( \rho < 1 \) can be written as

\[
\sqrt{\delta_{4k}(4 - \delta_{4k})} \leq (4 - 2\delta_{4k}) \sqrt{1 - \frac{\delta_{4k}^2}{1 + \eta\delta_{4k}^2}} - \delta_{4k}.
\]

(40)

For \( \delta_{4k} < 0.7 \) which guarantees (38), the inequality (40) is equivalent to its squared version, i.e.,

\[
\delta_{4k}(4 - \delta_{4k}) \leq \left[ (4 - 2\delta_{4k}) \sqrt{1 - \frac{\delta_{4k}^2}{1 + \eta\delta_{4k}^2}} - \delta_{4k} \right]^2.
\]

Thus,

\[
\delta_{4k}(4 - \delta_{4k}) < \delta_{4k}^2 + \frac{(4 - 2\delta_{4k})^2(1 - \delta_{4k}^2)}{1 + \eta\delta_{4k}^2} - 2\delta_{4k}(4 - 2\delta_{4k}) \sqrt{1 - \frac{\delta_{4k}^2}{1 + \eta\delta_{4k}^2}}.
\]

Combining two terms leads to

\[
\delta_{4k}(4 - 2\delta_{4k}) < \frac{(4 - 2\delta_{4k})^2(1 - \delta_{4k}^2)}{1 + \eta\delta_{4k}^2} - 2\delta_{4k}(4 - 2\delta_{4k}) \sqrt{1 - \frac{\delta_{4k}^2}{1 + \eta\delta_{4k}^2}}.
\]

The term \( (4 - 2\delta_{4k}) \) is positive. By cancelling it from the above inequality and rearranging terms, we get

\[
2\delta_{4k} \sqrt{1 - \frac{\delta_{4k}^2}{1 + \eta\delta_{4k}^2}} < (4 - 2\delta_{4k}) \frac{1 - \delta_{4k}^2}{1 + \eta\delta_{4k}^2} - \delta_{4k}.
\]

Multiplying by \( (1 + \eta\delta_{4k}^2) \) leads to

\[
2\delta_{4k} \left( 1 - \frac{\delta_{4k}^2}{1 + \eta\delta_{4k}^2} \right)^{1/2} < (1 - \delta_{4k}^2)(4 - 2\delta_{4k}) - \delta_{4k} (1 + \eta\delta_{4k}^2).
\]

The condition \( \delta_{4k} < 0.7 \) also guarantees (39). Thus the above inequality is equivalent to its squared version below

\[
\left( 2\delta_{4k} \left( 1 - \frac{\delta_{4k}^2}{1 + \eta\delta_{4k}^2} \right)^{1/2} \right)^2 < \left[ (1 - \delta_{4k}^2)(4 - 2\delta_{4k}) - \delta_{4k} (1 + \eta\delta_{4k}^2) \right]^2.
\]

Simplifying and combining two terms yields

\[
8\delta_{4k}(1 - \delta_{4k}^2)(1 + \eta\delta_{4k}^2) < (1 - \delta_{4k}^2)^2(4 - 2\delta_{4k})^2 + \delta_{4k}^2 (1 + \eta\delta_{4k}^2)^2.
\]

By expanding, combining and rearranging the terms, the above inequality is eventually written as

\[
-(\frac{11 + \sqrt{5}}{2})\delta_{4k}^6 + (12 - 4\sqrt{5})\delta_{4k}^5 - (9 + \sqrt{5})\delta_{4k}^4 - (36 - 4\sqrt{5})\delta_{4k}^3 + 27\delta_{4k}^2 + 24\delta_{4k} - 16 < 0.
\]

(41)
Fig. 1. The graph of the univariate function $h(t)$ and its unique real root $t^*$ in interval $[0,0.7]$. $h(t)$ is strictly increasing and $h(t) < 0$ in the interval $[0,t^*)$.

It is not difficult to show that the polynomial function

$$h(t) := -(\frac{11 + \sqrt{5}}{2})t^6 + (12 - 4\sqrt{5})t^5 - (9 + \sqrt{5})t^4 - (36 - 4\sqrt{5})t^3 + 27t^2 + 24t - 16$$

has a unique positive root in the interval $0 \leq t \leq 0.7$, denoted by $t^*$, which is approximately equal to 0.5593. The function $h(t)$ is negative (i.e., $h(t) < 0$) in the interval $0 \leq t < t^*$. Thus the inequality (41) is satisfied when $\delta_{4k} < t^* \approx 0.5593$, which is smaller than 0.7. Therefore, $\rho < 1$ is guaranteed if $\delta_{4k} < 0.5593$. The proof of the theorem is completed. □

Remark 3.5: The graph of the univariate function $h(t)$ defined in (42) over the interval $[0,1]$ is plotted in Figure 1 from which it can be clearly seen that $h(t) < 0$ for $0 \leq t < t^*$. The best known bound $\delta_{4k} < 0.4782$ for CoSaMP was shown by Foucart and Rauhut (see Theorem 6.18 in [11]). Our result improves their result to $\delta_{4k} < 0.5593$. In terms of geometric rate 0.5, Foucart and Rauhut’s result is equivalent to $\delta_{4k} < 0.299$, and their bound was slightly improved to $\delta_{4k} < 0.301$ by Shen and Li [45]. Let us find out our RIP-based bound if the geometric rate $\rho < 0.5$ is required. To see this, let

$$\rho = \left(\frac{\delta_{4k} + \sqrt{\delta_{4k}(4 - \delta_{4k})}}{4 - 2\delta_{4k}}\right) \sqrt{\frac{1 + \eta\delta_{4k}^2}{1 - \delta_{4k}^2}} \leq 0.5.$$  (43)

Following the same way as we did in proof Theorem 3.4. It is not difficult to show that (43) is satisfied when $\delta_{4k} < 0.3307$, which guarantees the following estimation of the error for signal recovery with CoSaMP:

$$\|x^p - x_S\|_2 \leq 0.5^p\|x^0 - x_S\|_2 + \gamma\|A^T\nu'\|_2$$  (44)

where $\gamma$ is a certain univariate constant. Clearly, our result for CoSaMP in terms of geometric rate 0.5 also remarkably improves the existing result established by Shen and Li [45].
Remark 3.6: It is worth mentioning that there are two major reasons for why the analysis in this paper can improve the existing RIP-based performance results for CoSaMP. The first reason is the tightness of the estimation in Lemma 2.2 and the second reason lies in the novel bound established in Lemma 3.3. The similar reasons cause an improved result for SP algorithm in section IV. From our analysis, the geometric rate $\rho$ for CoSaMP is given as

$$\rho = \left( \frac{\delta_{4k} + \sqrt{\delta_{4k}(4 - \delta_{4k})}}{4 - 2\delta_{4k}} \right)^\frac{1 + \eta\delta_{4k}^2}{1 - \delta_{4k}^2}.$$ 

Lemma 3.3 contributes to the first term above in the bracket which is $\phi(\delta_{4k})$, and Lemma 2.2 contributes to the second term $\sqrt{\frac{1 + \eta\delta_{4k}^2}{1 - \delta_{4k}^2}}$. Each of these lemmas alone can improve the existing RIP-based bounds for CoSaMP to a certain degree, but all together yield a remarkable improvement of the existing results as claimed in Theorem 3.4. The traditional estimation similar to (21) in Lemma 3.3 (see, e.g., Foucart and Rauhut [11]) is

$$\| (x_S - x^p)^T \|_2 \leq \sqrt{2\delta_{4k}} \| x^p - x_S \|_2 + \gamma \| A^T \nu' \|_2,$$

where $\gamma$ is a certain constant. The clear difference between the traditional estimation and ours is the coefficient for the first term on the right-hand side. The traditional one is $\hat{\phi}(\delta) := \sqrt{2}\delta_{4k}$ which is linear in $\delta_{4k}$. When $\delta_{4k}$ is relatively low, for instance, $\delta_{4k} < 0.2367$, the difference between our estimation and the above-mentioned existing one is unnoticeable, and both the value of $\phi(\delta_{4k})$ and $\hat{\phi}(\delta_{4k})$ are small enough to ensure the geometric rate $\rho$ to be smaller than 1, and hence the success of signal recovery is always guaranteed. However, for relatively large value of $\delta_{4k}$, namely, $0.2326 < \delta_{4k} < 1$, our estimation in Lemma 3.3 is strictly tighter than the traditional one since $\phi(\delta_{4k}) < \hat{\phi}(\delta_{4k})$. This can be seen from Figure 2 which demonstrates the graph of the difference $D(t) = \phi(t) - \hat{\phi}(t)$ over the interval $[0, 1]$. Clearly, we see that $D(t) < 0$ over the region $0.2326 < t < 1$. This can also be
Fig. 3. The graphs of the univariate functions $\phi(t)$ and $\hat{\phi}(t)$. $\phi(t) < \hat{\phi}(t)$ for $0.2326 < t < 1$.

seen directly from their individual graphs plotted in Figure 3 in which the curve is the graph of $\phi(t)$ and straight line is for $\hat{\phi}(t)$. This means our estimation in Lemma 3.3 is better than the traditional one in the sense that it allows a wider range of $\delta_{4k}$ to guarantee $\rho < 1$.

IV. GUARANTEED PERFORMANCE FOR (GENERAL) SUBSPACE PURSUITS

In this section, we prove an improved RIP-based bound for the guaranteed performance of signal recovery via the SP algorithm. We show this result for a more general version of the algorithm described as Algorithm 4.

**Algorithm 4 General Subspace Pursuit (GSP)**

Given $A, y, k, \sigma$ (where $\sigma \geq k$ is an integer number) and an initial iterate $x^0 = 0$ with $S^0 = \text{supp}(x^0) = \emptyset$. At the $p$-th step, set $S^p = \text{supp}(x^p)$ and perform the following steps to generate the next iterate $x^{p+1}$:

1. $\Lambda^{p+1} = S^p \cup L_\sigma(A^T(y - Ax^p))$ (GSP1)
2. $z^{p+1} = \arg\min_{z \in \mathbb{R}^n} \{\|y - Az\|_2 : \text{supp}(z) \subseteq \Lambda^{p+1}\}$ (GSP2)
3. $S^{p+1} = L_k(z^{p+1})$ (GSP3)
4. $x^{p+1} = \arg\min_{z \in \mathbb{R}^n} \{\|y - Az\| : \text{supp}(z) \subseteq S^{p+1}\}$ (GSP4)

Repeat the above steps until a certain stopping criterion is met.

Output: $k$-sparse signal $\hat{x}$.

The algorithm is referred to as the general subspace pursuit (GSP) since it is more general than SP in the sense that the operator $L_\sigma$ at the first step (GSP1) allows $\sigma$ to be any given positive integer.
number larger than or equal to $k$. When $\sigma = k$, the GSP immediately becomes the SP. In this section, we first establish the convergent result concerning the guaranteed success of signal recovery via GSP. Reduced to the case $\sigma = k$, an improved result for the SP algorithm is immediately obtained.

By the structure of GSP algorithm, we see that $[A^T(y - Ax^p)]_{S^p} = 0$ at each iterate $x^p$ for $p \geq 1$. This follows directly from the optimality of $x^p$ which is an optimal solution to the orthogonal projection at the step (GSP4). When $\|A^T(y - Ax^p)\|_0 \geq \sigma$, the size $|\Lambda^{p+1}| = k + \sigma$ is maximal since in this case

$$S^p \cap L_\sigma(A^T(y - Ax^p)) = \emptyset.$$  \hfill (45)

When $\|A^T(y - Ax^p)\|_0 < \sigma$, the set $L_\sigma(A^T(y - Ax^p))$ is not uniquely determined. To maintain the size $|\Lambda^{p+1}| = k + \sigma$, we adopt the following predefined selection rule for $L_\sigma(\cdot)$: When $\|A^T(y - Ax^p)\|_0 < \sigma$, the indices in $L_\sigma(A^T(y - Ax^p))$ are taken outside the set $S^p = \text{supp}(x^p)$ so that (45) is maintained during the course of iterations. The analysis of GSP and SP is carried out under the above prescribed selection rule for the selection $\Lambda^{p+1}$.

Similar to CoSaMP, the following lemma plays a vital role in our analysis. The main idea of the lemma is similar to that of Lemma 3.3. However, the proof of Lemma 3.3 does not directly apply to the lemma below. Thus we provide a detainted proof here.

**Lemma 4.1:** Let $y = Ax + \nu$ be the measurements of $x$ with error $\nu$. Let $x^p$ be the iterate generated by the GSP algorithm. Let $S = L_k(x)$, $S^p = \text{supp}(x^p)$, and $T = L_\sigma(A^T(y - Ax^p))$ where $\sigma \geq k$ is an integer number. If $\delta_{2k+\sigma} < 1$, then

$$\| (x^p - x^p)_{T \cup S^p} \|_2 \leq \phi(\delta_{2k+\sigma}) \| x^p - x^p \|_2 + \frac{2}{2 - \delta_{2k}} \| A^T \nu' \|_2,$$  \hfill (46)

where $\nu' = Ax^p + \nu$ and

$$\phi(\delta_{2k+\sigma}) = \frac{\delta_{2k+\sigma} + \sqrt{\delta_{2k+\sigma}(4 - \delta_{2k+\sigma})}}{4 - 2\delta_{2k+\sigma}}.$$

**Proof.** Let $S, S^p$ and $T$ be defined as the lemma, and define

$$\tilde{\Omega} := \|[x^p - (x^p + A^T(y - Ax^p))]_{(S \cup S^p) \setminus (T \cup S^p)}\|_2.$$

Note that if $(S \cup S^p) \subseteq (T \cup S^p)$, which implies that $T \cup S^p \subseteq S \cup S^p$, then $\| (x^p - x^p)_{T \cup S^p} \|_2 = 0$, and thus (46) holds trivially. Therefore, without loss of generality, in the remainder of the proof, we only consider the case $(S \cup S^p) \nsubseteq (T \cup S^p)$. So $(S \cup S^p) \setminus (T \cup S^p) \neq \emptyset$, and thus $\tilde{\Omega}$ is well defined. It is easy to see that

$$(S \cup S^p) \setminus (T \cup S^p) = [S \setminus (T \cup S^p)] \cup [S^p \setminus (T \cup S^p)] = S \setminus (T \cup S^p) = (S \setminus S^p) \setminus T.$$  \hfill (47)
By the predefined rule for the selection of $\Lambda^{p+1}$, we have $S^p \cap T = \emptyset$ which implies that $T \subseteq S^\complement$. Thus

$$T \setminus (S \cup S^p) = T \setminus (S^p \cup (S \setminus S^p)) = T \cap S^\complement \cup (S \setminus S^p) = T \cap (S^\complement \cap S^\complement) = T \cap S^\complement S^\complement = T \setminus (S \setminus S^p).$$

We now distinguish two cases: $\tilde{\Omega} = 0$ and $\tilde{\Omega} \neq 0$. For the first case, i.e., $\tilde{\Omega} = 0$, one has

$$(x_S - x^p)_{(S \cup S^p) \setminus (T \cup S^\complement)} = [A^T(y - Ax^p)]_{(S \cup S^p) \setminus (T \cup S^\complement)} = [A^T(y - Ax^p)]_{(S \setminus S^p) \setminus T},$$

where the last relation follows from (47). Note that $|T| = \sigma \geq k \geq |S \setminus S^p|$. We have

$$|(S \setminus S^p) \setminus T| = |S \setminus S^p| - |T \cap (S \setminus S^p)| \leq |T| - |T \cap (S \setminus S^p)| = |T \setminus (S \setminus S^p)|$$

which means the number of elements in $(S \setminus S^p) \setminus T$ is no more than those in $T \setminus (S \setminus S^p)$. By the definition of $T$, the entries of the vectors $A^T(x - Ax^p)$ supported on $(S \setminus S^p) \setminus T$ are outside the support of its $\sigma$ largest absolute entries indexed by $T$. This together with (50) implies that

$$
\| [A^T(x - Ax^p)]_{(S \setminus S^p) \setminus T} \|_2 \leq \| [A^T(y - Ax^p)]_{T \setminus (S \setminus S^p)} \|_2.
$$

It follows from (49) and (51) that

$$
\| (x_S - x^p)_{T \setminus (S \setminus S^p)} \|_2 = \| (x_S - x^p)_{(S \cup S^p) \setminus (T \cup S^\complement)} \|_2
= \| (x_S - x^p)_{(S \cup S^p) \setminus (T \cup S^\complement)} \|_2
= \| [A^T(y - Ax^p)]_{(S \setminus S^p) \setminus T} \|_2
\leq \| [A^T(y - Ax^p)]_{T \setminus (S \setminus S^p)} \|_2
\leq \| [A^T A(x_S - x^p)]_{T \setminus (S \setminus S^p)} \|_2 + \| A^T \nu' \|_2
= \| [A^T A(x_S - x^p)]_{T \setminus (S \setminus S^p)} \|_2 + \| A^T \nu' \|_2
\leq \delta_{2k+\sigma} \| x_S - x^p \|_2 + \| A^T \nu' \|_2,
$$

where the last equality follows from (48) and the last inequality follows from Lemma 2.5(iii) since $|\mathrm{supp}(x_S - x^p) \cup (T \setminus (S \cup S^p))| \leq 2k + \sigma$ and the two sets $\mathrm{supp}(x_S - x^p)$ and $T \setminus (S \cup S^p)$ are disjoint.

By the property of the function $\phi$, we see that $\delta_{2k+\sigma} \leq \phi(\delta_{2k+\sigma})$ for $\delta_{2k+\sigma} < 1$. Thus the bound (46) holds for the case $\tilde{\Omega} = 0$.

In the remainder of the proof, we assume $\tilde{\Omega} \neq 0$ and use $\tau$ to denote the ratio of $\| (x_S - x^p)_{T \setminus (S \setminus S^p)} \|_2$ and $\tilde{\Omega}$, i.e.,

$$\| (x_S - x^p)_{T \setminus (S \setminus S^p)} \|_2 = \tau \tilde{\Omega}.$$

To show (46), it suffices to bound $\tau \tilde{\Omega}$. We start with the upper estimation of $\tilde{\Omega}$.

$$
\tilde{\Omega} \leq \| [(I - A^T A)(x_S - x^p) + A^T \nu']_{(S \cup S^p) \setminus (T \cup S^\complement)} \|_2 \leq \delta_{2k} \| x_S - x^p \|_2 + \| A^T \nu' \|_2.
$$

(52)
Denote by $D := (S \cup S^p) \setminus (T \cup S^p)$. Note that $\| (x_S - x^p)_D \|_2 = \| (x_S - x^p)_{\overline{T \cup S^p}} \|_2$. By the definition of $\tilde{\Omega}$, we see that

$$\tilde{\Omega}^2 = \| (x_S - x^p)_D + [A^T(y - Ax^p)]_D \|_2^2$$

$$= \| (x_S - x^p)_D \|_2^2 + \|[A^T(y - Ax^p)]_D \|_2^2 + 2(x_S - x^p)_D[A^T(y - Ax^p)]_D$$

$$= \| (x_S - x^p)_{\overline{T \cup S^p}} \|_2^2 + \beta_1 + 2\beta_2,$$  \hspace{1cm} (53)

where

$$\beta_1 = \|[A^T(y - Ax^p)]_D \|_2^2, \hspace{0.5cm} \beta_2 = (x_S - x^p)_D[A^T(y - Ax^p)]_D.$$

The lower bounds for $\beta_1$ and $\beta_2$ can be estimated as follows. First, by triangle inequality and definition of $\tilde{\Omega}$, we have

$$\beta_1 = \| (x_S - x^p)_D - [(x_S - x^p) - A^T(y - Ax^p)]_D \|_2^2$$

$$\geq (\| (x_S - x^p)_D \|_2 - \| [(x_S - x^p) - A^T(y - Ax^p)]_D \|_2)^2 \hspace{1cm} (54)$$

By Lemma \ref{lem:28} (i) and the fact that $ab \leq (a^2 + b^2)/2$, we have

$$|(x_S - x^p)_D^T(A_D)^T(A_T(x_S - x^p))_D| \leq \delta_{2k}\| (x_S - x^p)_D \|_2\| (x_S - x^p)_{\overline{T \cup S^p}} \|_2 \leq \frac{\delta_{2k}}{2}\| x_S - x^p \|_2^2.$$

Thus, by the definition of $\delta_{2k}$ and the inequality above, we obtain

$$\beta_2 = (x_S - x^p)_D^T(A_D)^T A(x_S - x^p) + (x_S - x^p)_D^T[A^T\nu']_D$$

$$= (x_S - x^p)_D^T(A_D)^T A(x_S - x^p)_D + (x_S - x^p)_D^T[A^T\nu']_D + (x_S - x^p)_D^T(A_D)^T A(x_S - x^p)_{\overline{D}}$$

$$\geq (1 - \delta_{2k})\| (x_S - x^p)_D \|_2^2 - \| (x_S - x^p)_D \|_2\| A^T\nu' \|_2 - \| (x_S - x^p)_D^T(A_D)^T A(x_S - x^p)_{\overline{D}}\|_2$$

$$\geq (1 - \delta_{2k})\| (x_S - x^p)_{\overline{T \cup S^p}} \|_2^2 - \frac{\delta_{2k}}{2}\| x_S - x^p \|_2^2 - \| (x_S - x^p)_{\overline{T \cup S^p}} \|_2\| A^T\nu' \|_2.$$  \hspace{1cm} (55)

Combining (53)–(55) yields

$$\tilde{\Omega}^2 \geq (3 - 2\delta_{2k})\| (x_S - x^p)_{\overline{T \cup S^p}} \|_2^2 - 2\delta_{2k}\| x_S - x^p \|_2^2 - 2\| (x_S - x^p)_{\overline{T \cup S^p}} \|_2\| A^T\nu' \|_2 + (r - 1)^2\tilde{\Omega}^2$$

$$= (3 - 2\delta_{2k})\tau^2\tilde{\Omega}^2 + (r - 1)^2\tilde{\Omega}^2 - 2(\tilde{\Omega})\| A^T\nu' \|_2 - \delta_{2k}\| x_S - x^p \|_2^2.$$

which can be simplified to

$$\begin{align*}
(4 - 2\delta_{2k})(\tau\tilde{\Omega})^2 - 2(\tilde{\Omega})\| A^T\nu' \|_2 - \delta_{2k}\| x_S - x^p \|_2^2 & \leq 0. \hspace{1cm} (56)
\end{align*}$$
Applying the inequality (15) twice, we see that
\[
\tilde{\omega} := \sqrt{4(\tilde{\Omega} + \|A^T\nu\|_2^2) + 4(4 - 2\delta_{2k})\delta_{2k}\|x_S - x^p\|_2^2}
\]
\[
= 2\sqrt{(\tilde{\Omega} + \|A^T\nu\|_2^2) + (4 - 2\delta_{2k})\delta_{2k}\|x_S - x^p\|_2^2}
\]
\[
\leq 2\sqrt{\tilde{\Omega}^2 + (4 - 2\delta_{2k})\delta_{2k}\|x_S - x^p\|_2^2 + 2\|A^T\nu\|_2^2}
\]
\[
\leq 2\left(\delta_{2k}\|x_S - x^p\|_2^2 + (4 - 2\delta_{2k})\delta_{2k}\|x_S - x^p\|_2^2 + \|A^T\nu\|_2^2\right) + 2\|A^T\nu\|_2^2
\]
\[
= 2\sqrt{\delta_{2k}(4 - \delta_{2k})}\|x_S - x^p\|_2 + 4\|A^T\nu\|_2.
\]
Thus treating the inequality (56) as a quadratic inequality of $\tau\tilde{\Omega}$, then (56) implies that $\tau\tilde{\Omega}$ is less than or equal to the largest real root of the underlying quadratic equation. So we immediately deduce that
\[
\|(x_S - x^p)\|_{\text{GSP}}^2 = \tau\tilde{\Omega} \leq \frac{2(\tilde{\Omega} + \|A^T\nu\|_2^2) + \tilde{\omega}}{2(4 - 2\delta_{2k})}
\]
\[
\leq \frac{\tilde{\Omega} + \sqrt{\delta_{2k}(4 - \delta_{2k})}\|x_S - x^p\|_2 + 3\|A^T\nu\|_2}{4 - 2\delta_{2k}}.
\]
(57)

From (52) and (57), we obtain
\[
\|(x_S - x^p)\|_{\text{GSP}}^2 \leq \frac{\delta_{2k} + \sqrt{\delta_{2k}(4 - \delta_{2k})}}{4 - 2\delta_{2k}}\|x_S - x^p\|_2 + \frac{4\|A^T\nu\|_2}{4 - 2\delta_{2k}}
\]
\[
= \phi(\delta_{2k})\|x_S - x^p\|_2 + \frac{2}{2 - \delta_{2k}}\|A^T\nu\|_2.
\]
Since $\phi(\delta_{2k}) \leq \phi(\delta_{2k+\sigma})$, the inequality (46) is valid for the case $\tilde{\Omega} \neq 0$. The proof of this lemma is complete.

The main result for GSP is given as follows.

**Theorem 4.2:** If the restricted isometry constant of the sensing matrix $A$ satisfies that
\[
\delta_{2k+\sigma} < 0.5108,
\]
then the iterates $\{x^p\}$, generated by the GSP, satisfy that
\[
\|x_S - x^p\|_2 \leq \rho^p\|x_S - x^0\|_2 + \frac{C}{1 - \rho}\|A^T\nu\|_2,
\]
where the constants $\rho$ and $C$ are given by
\[
\rho = \frac{\left(\delta_{2k+\sigma} + \sqrt{\delta_{2k+\sigma}(4 - \delta_{2k+\sigma})}\right)}{4 - 2\delta_{2k+\sigma}} \sqrt{1 + \eta\delta_{2k+\sigma}^2} < 1,
\]
\[
C = \frac{2\sqrt{1 + \eta\delta_{2k+\sigma}^2}}{(2 - \delta_{2k})(1 - \delta_{2k+\sigma}^2)} + \frac{\eta}{(1 - \delta_{2k+\sigma})\sqrt{1 - \delta_{2k}^2}} + \frac{1}{1 - \delta_{2k}},
\]
where $\eta = (\sqrt{5} + 1)/2$. 

Proof. To analyze the GSP, we start with its step (GSP3) together with (GSP4) which is an orthogonal projection satisfying the property stated in Lemma 2.2. By setting \( x^* = x^{p+1}, S = L_k(x^p), \Gamma = \Lambda = S^{p+1}, \tau = k \) and \(|\Gamma| = k\), by Lemma 2.7, the vector \( x^{p+1} \) produced at (GSP3) and (SGP4) satisfies that
\[
\|x_S - x^{p+1}\|_2 \leq \frac{\|x_S - x^{p+1}\|_2}{\sqrt{1 - \delta_{2k}^2}} + \frac{\|A^T \nu\|_2}{1 - \delta_{2k}},
\]
where \( \nu' = Ax_S + \nu \). Let \( z^{p+1} \) be generated at (GSP2) of the GSP. Throughout this section, we define
\[
\tilde{x}^{p+1} = H_k(z^{p+1}) = (z^{p+1})_{S^{p+1}}.
\]
This means \( \text{supp}(\tilde{x}^{p+1}) \subseteq S^{p+1} \). By (GSP4), we also have \( \text{supp}(x^{p+1}) \subseteq S^{p+1} \). Therefore, \( (x^{p+1})_{S^{p+1}} = 0 = (\tilde{x}^{p+1})_{S^{p+1}} \), and hence
\[
\|x_S - x^{p+1}\|_2 = \|x_S - \tilde{x}^{p+1}\|_2 = \|x_S - x^{p+1}\|_2.
\]
Combining the two relations (58) and (59) yields
\[
\|x_S - x^{p+1}\|_2 \leq \frac{\|x_S - \tilde{x}^{p+1}\|_2}{\sqrt{1 - \delta_{2k}^2}} + \frac{\|A^T \nu\|_2}{1 - \delta_{2k}}.
\]
The remainder of this proof is devoted to the estimation of the upper bound for the term \( \|x_S - \tilde{x}^{p+1}\|_2 \) in terms of RIC and \( \|x_S - x^p\|_2 \). By the definition of \( \Lambda^{p+1} \) in (GSP1), it follows from (GSP2) that \( \text{supp}(z^{p+1}) \subseteq \Lambda^{p+1} \). Note that \( \text{supp}(\tilde{x}^{p+1}) \subseteq S^{p+1} \subseteq \Lambda^{p+1} \) and that \( x_{S \cap \Lambda^{p+1}} \) is \( k \)-sparse, it follows from Lemma 2.7 that
\[
\|x_S - \tilde{x}^{p+1}\|_{\Lambda^{p+1}} \leq \frac{\|x_{S \cap \Lambda^{p+1}} - x^{p+1}\|}{2} = \frac{\|x_{S \cap \Lambda^{p+1}} - H_k(z^{p+1})\|}{2} \leq \frac{\sqrt{5} + 1}{2} \|x_{S \cap \Lambda^{p+1}} - z^{p+1}(S \cap \Lambda^{p+1}) \cup S^{p+1}\|_2 \leq \frac{\sqrt{5} + 1}{2} \|x_{S \cap \Lambda^{p+1}} - z^{p+1}(S \cap \Lambda^{p+1}) \cup \Lambda^{p+1}\|_2 = \eta \|x_S - z^{p+1}\|_{\Lambda^{p+1}}_2.
\]
where \( \eta = (\sqrt{5} + 1)/2 \). Consider the orthogonal projection (GSP2) which satisfies the property stated in Lemma 2.7 By setting \( x^* = z^{p+1}, \Gamma = \Lambda = \Lambda^{p+1}, \tau = k \) and noting that \(|\Gamma| = k + \sigma\), it follows from Lemma 2.7 that
\[
\|x_S - z^{p+1}\|_{\Lambda^{p+1}}_2 \leq \frac{\delta_{2k+\sigma} \|x_S - z^{p+1}\|_{\Lambda^{p+1}}_2}{\sqrt{1 - \delta_{2k+\sigma}^2}} + \frac{\|A^T \nu\|_2}{1 - \delta_{2k+\sigma}}.
\]
Merging the two relations above yields
\[
\|x_S - \tilde{x}^{p+1}\|_{\Lambda^{p+1}}_2 \leq \frac{\eta \delta_{2k+\sigma} \|x_S - z^{p+1}\|_{\Lambda^{p+1}}_2}{\sqrt{1 - \delta_{2k+\sigma}^2}} + \frac{\eta \|A^T \nu\|_2}{1 - \delta_{2k+\sigma}}.
\]

As \( S^{p+1} \subseteq \Lambda^{p+1} \) implies \( \overline{\Lambda^{p+1}} \subseteq \overline{S^{p+1}} \), we have

\[
||(x_S - \tilde{x}^{p+1})_{\overline{S^{p+1}}}||_2^2 = ||(x_S - \tilde{x}^{p+1})_{\overline{S^{p+1}} \setminus \overline{\Lambda^{p+1}}}||_2^2 + ||(x_S - \tilde{x}^{p+1})_{\overline{\Lambda^{p+1}} \setminus \overline{S^{p+1}}}||_2^2
\]

\[
\leq ||(x_S - \tilde{x}^{p+1})_{\overline{\Lambda^{p+1}}}||_2^2 + ||(x_S - \tilde{x}^{p+1})_{\overline{\Lambda^{p+1}} \setminus \overline{S^{p+1}}}||_2^2
\]

where the last equality follows from \( (\tilde{x}^{p+1})_{\overline{\Lambda^{p+1}}} = 0 = (z^{p+1})_{\overline{\Lambda^{p+1}}} \). Combining \( (61) \) and \( (62) \) leads to

\[
||(x_S - \tilde{x}^{p+1})_{\overline{S^{p+1}}}||_2 \leq \left[ \frac{\eta \delta_{2k+\sigma} ||(x_S - z^{p+1})_{\overline{S^{p+1}}}||_2 + \eta \|A'T'\|_2}{\sqrt{1 - \delta_{2k+\sigma}}} \right]^2 + ||(x_S - z^{p+1})_{\overline{\Lambda^{p+1}}}||_2^2.
\]

Taking square root and using \( (15) \) yields

\[
||(x_S - \tilde{x}^{p+1})_{\overline{S^{p+1}}}||_2 \leq \sqrt{\left( \frac{\eta \delta_{2k+\sigma} ||(x_S - z^{p+1})_{\overline{S^{p+1}}}||_2 + \eta \|A'T'\|_2}{\sqrt{1 - \delta_{2k+\sigma}}} \right)^2 + \eta \|A'T'\|_2 (1 - \delta_{2k+\sigma})}
\]

where the last equality follows from the fact \( \eta^2 - 1 = \eta \). Note that \( S^p = \text{supp}(x^p) \subseteq \Lambda^{p+1} \) which implies that \( (x_S - z^{p+1})_{\overline{\Lambda^{p+1}}} = 0 \). Thus

\[
(x_S - z^{p+1})_{\overline{\Lambda^{p+1}}} = (x_S)_{\overline{\Lambda^{p+1}}} = (x_S - x^p)_{\overline{\Lambda^{p+1}}}.
\]

Substituting this into the above inequality leads to

\[
||(x_S - \tilde{x}^{p+1})_{\overline{S^{p+1}}}||_2 \leq \sqrt{\left( \frac{1 + \eta \delta_{2k+\sigma} ||(x_S - x^p)_{\overline{\Lambda^{p+1}}}||_2}{1 - \delta_{2k+\sigma}} \right)^2 + \eta \|A'T'\|_2 (1 - \delta_{2k+\sigma})}
\]

Therefore, by \( (60) \) and by noting that \( \delta_{2k} \leq \delta_{2k+\sigma} \) which implies that \( \frac{1}{\sqrt{1 - \delta_{2k}}} \leq \frac{1}{\sqrt{1 - \delta_{2k+\sigma}}} \), we obtain the following estimation:

\[
\|x_S - x^{p+1}\|_2 \leq \frac{1}{\sqrt{1 - \delta_{2k}^2}} \sqrt{\left( \frac{1 + \eta \delta_{2k+\sigma} ||(x_S - x^p)_{\overline{\Lambda^{p+1}}}||_2}{1 - \delta_{2k+\sigma}} \right)^2 + \eta \|A'T'\|_2 (1 - \delta_{2k+\sigma})}
\]

\[
+ \left( \frac{\eta}{1 - \delta_{2k+\sigma}} + \frac{1}{1 - \delta_{2k}} \right) \|A'T'\|_2
\]

\[
\leq \phi(\delta_{2k+\sigma}) \sqrt{\left( \frac{1 + \eta \delta_{2k+\sigma} ||(x_S - x^p)_{\overline{\Lambda^{p+1}}}||_2}{1 - \delta_{2k+\sigma}} \right)^2 + \eta \|A'T'\|_2 (1 - \delta_{2k+\sigma})}
\]

\[
+ \left( \frac{\eta}{1 - \delta_{2k+\sigma}} + \frac{1}{1 - \delta_{2k}} \right) \|A'T'\|_2
\]

\[
\leq \rho \|x_S - x^p\|_2 + C \|A'T'\|_2 \quad \text{(By Lemma 4.1)}
\]

\[
= \rho \|x_S - x^p\|_2 + C \|A'T'\|_2,
\]
where the last inequality follows from Lemma 4.1 by noting that $\Lambda^{p+1} = T \cup S^n$, and the constants $\rho$ and $C$ are given as

$$\rho = \frac{\delta_{2k+\sigma} + \sqrt{\delta_{2k+\sigma}(4 - \delta_{2k+\sigma})}}{(4 - 2\delta_{2k+\sigma})(1 - \delta_{2k+\sigma})} \sqrt{1 + \eta \delta^2_{2k+\sigma}},$$

$$C = \frac{2\sqrt{1 + \eta \delta^2_{2k+\sigma}}}{(2 - \delta_{2k})(1 - \delta_{2k+\sigma})} + \frac{\eta}{[1 - \delta_{2k+\sigma}]\sqrt{1 - \delta^2_{2k}}} + \frac{1}{1 - \delta_{2k}}.$$

We now derive the upper bound for $\delta_{2k+\sigma}$ to ensure $\rho < 1$. Consider the range $\delta_{2k+\sigma} < 0.6$ which guarantees that the following two inequalities hold:

$$\frac{(4 - 2\delta_{2k+\sigma})(1 - \delta_{2k+\sigma})}{\sqrt{1 + \eta \delta^2_{2k+\sigma}}} - \delta_{2k+\sigma} > 0,$$

(63)

$$(1 - \delta^2_{2k+\sigma})^2(4 - 2\delta_{2k+\sigma}) - \delta_{2k+\sigma}(1 + \eta \delta^2_{2k+\sigma}) > 0.$$

(64)

We first see that the condition $\rho < 1$ is equivalent to

$$\sqrt{\delta_{2k+\sigma}(4 - \delta_{2k+\sigma})} \leq \frac{(4 - 2\delta_{2k+\sigma})(1 - \delta_{2k+\sigma})}{\sqrt{1 + \eta \delta^2_{2k+\sigma}}} - \delta_{2k+\sigma}.$$

The range $\delta_{2k+\sigma} < 0.6$ ensures (63). Thus the inequality above is equivalent to its squared version, i.e.,

$$\delta_{2k+\sigma}(4 - \delta_{2k+\sigma}) \leq \left[\frac{(4 - 2\delta_{2k+\sigma})(1 - \delta_{2k+\sigma})}{\sqrt{1 + \eta \delta^2_{2k+\sigma}}} - \delta_{2k+\sigma}\right]^2.$$

Thus,

$$\delta_{2k+\sigma}(4 - \delta_{2k+\sigma}) < \delta^2_{2k+\sigma} + \frac{(4 - 2\delta_{2k+\sigma})(1 - \delta^2_{2k+\sigma})}{1 + \eta \delta^2_{2k+\sigma}} - 2\delta_{2k+\sigma} \frac{(4 - 2\delta_{2k+\sigma})(1 - \delta^2_{2k+\sigma})}{\sqrt{1 + \eta \delta^2_{2k+\sigma}}}.$$

Combining the left term and the first term on the right hand side leads to

$$\delta_{2k+\sigma}(4 - 2\delta_{2k+\sigma}) < \frac{(4 - 2\delta_{2k+\sigma})(1 - \delta^2_{2k+\sigma})^2}{1 + \eta \delta^2_{2k+\sigma}} - 2\delta_{2k+\sigma} \frac{(4 - 2\delta_{2k+\sigma})(1 - \delta^2_{2k+\sigma})}{\sqrt{1 + \eta \delta^2_{2k+\sigma}}}.$$

Cancelling $(4 - 2\delta_{2k+\sigma})$ and rearranging terms produces

$$\frac{2\delta_{2k+\sigma}(1 - \delta^2_{2k+\sigma})}{\sqrt{1 + \eta \delta^2_{2k+\sigma}}} < \frac{(4 - 2\delta_{2k+\sigma})(1 - \delta^2_{2k+\sigma})^2}{1 + \eta \delta^2_{2k+\sigma}} - \delta_{2k+\sigma}.$$

Multiplying the inequality by $(1 + \eta \delta^2_{2k+\sigma})$ leads to

$$2\delta_{2k+\sigma}(1 - \delta^2_{2k+\sigma}) < (1 - \delta^2_{2k+\sigma})^2(4 - 2\delta_{2k+\sigma}) - \delta_{2k+\sigma}(1 + \eta \delta^2_{2k+\sigma}).$$

The region $\delta_{2k+\sigma} < 0.6$ ensures (64). Thus the inequality above is equivalent to its squared version. Therefore, by squaring both sides of the above inequality and combining terms, one has that

$$8\delta_{2k+\sigma}(1 - \delta^2_{2k+\sigma})^2(1 + \eta \delta^2_{2k+\sigma}) < [1 - \delta^2_{2k+\sigma}]^4[4 - 2\delta_{2k+\sigma}] + \delta^2_{2k+\sigma}[1 + \eta \delta^2_{2k+\sigma}]^2.$$

(65)
The left-hand side of (65) is equal to
\[
8\delta_{2k+\sigma}^2 + 4(\sqrt{5} - 3)\delta_{2k+\sigma}^3 - 8\sqrt{5}\delta_{2k+\sigma}^5 + 4(\sqrt{5} + 1)\delta_{2k+\sigma}^7.
\]
The first term on the right-hand side of (65) is equal to
\[
4\delta_{2k+\sigma}^{10} - 16\delta_{2k+\sigma}^9 + 64\delta_{2k+\sigma}^7 - 40\delta_{2k+\sigma}^6 - 96\delta_{2k+\sigma}^5 + 80\delta_{2k+\sigma}^4 + 64\delta_{2k+\sigma}^3 - 60\delta_{2k+\sigma}^2 - 16\delta_{2k+\sigma} + 16.
\]
and the second term on the right-hand side of (65) is equal to
\[
\delta_{2k+\sigma}^2 + (\sqrt{5} + 1)\delta_{2k+\sigma}^4 + (\frac{\sqrt{5} + 3}{2})\delta_{2k+\sigma}^6.
\]
Substituting these terms into (65) and combining terms, the inequality (65) is eventually written as
\[
-4\delta_{2k+\sigma}^{10} - 16\delta_{2k+\sigma}^9 + (60 - 4\sqrt{5})\delta_{2k+\sigma}^7 + \frac{77 - \sqrt{5}}{2}\delta_{2k+\sigma}^5 + (96 - 8\sqrt{5})\delta_{2k+\sigma}^3 - (81 + \sqrt{5})\delta_{2k+\sigma} - (76 - 4\sqrt{5})\delta_{2k+\sigma}^3 + 59\delta_{2k+\sigma}^2 + 24\delta_{2k+\sigma} - 16 < 0.
\]
(66)

It is not difficult to show that the polynomial function
\[
d(t) := -4t^{10} + 16t^9 - (60 - 4\sqrt{5})t^7 + \frac{77 - \sqrt{5}}{2}t^6 + (96 - 8\sqrt{5})t^5 - (81 + \sqrt{5})t^4
\]
\[- (76 - 4\sqrt{5})t^3 + 59t^2 + 24t - 16
\]
is strictly increasing in the interval \([0, t^\ast]\), where \(t^\ast \approx 0.5108\) is the unique real root of \(d(t) = 0\) in the interval \([0, 0.6]\). This means the inequality (66) holds when \(\delta_{2k+\sigma} < t^\ast \approx 0.5108\). Therefore we conclude that \(\rho < 1\) if \(\delta_{2k+\sigma} < 0.5108\). The property of the function \(d(t)\) can also be seen from its graph over the interval \([0, 1]\) which is demonstrated in Figure 4.

Taking \(\sigma = k\), the GSP algorithm is immediately reduced to the standard SP algorithm (i.e., Algorithm 3 in section I). From Theorem 4.2, we immediately obtain the following corollary.
Corollary 4.3: If the restricted isometry constant of the sensing matrix $A$ satisfies that

$$\delta_{3k} < 0.5108,$$

then the iterates $\{x^p\}$ generated by the SP satisfy that

$$\|x_S - x^p\|_2 \leq \rho^p\|x_S - x^0\|_2 + \frac{C}{1 - \rho}\|A_\nu\|_2,$$

where the constants $\rho$ and $C$ are given as

$$\rho = \left(\delta_{3k} + \sqrt{\delta_{3k}(4 - \delta_{3k})}\right) \sqrt{1 + \eta\delta^2_{3k}} < 1,$$

$$C = \frac{2\sqrt{1 + \eta\delta^2_{3k}}}{(2 - \delta_{2k})(1 - \delta^2_{3k})} + \frac{\eta}{(1 - \delta^2_{3k})\sqrt{1 - \delta^2_{2k}}} + \frac{1}{1 - \delta_{2k}},$$

where $\eta = (\sqrt{5} + 1)/2$.

The above result improves the existing recovery bound $\delta_{3k} \leq 0.4859$ for SP established recently by Song, Xia and Liu [42].

Remark 4.4: Theorem 4.2 and Corollary 4.3 are established according to the geometric rate $\rho < 1$.

Similar to the discussion in Remark 3.5, we may also state the results in terms of geometric rate 0.5 in order to establish the error bound in the form [44] for GSP and SP algorithms. Let

$$\rho = \left(\delta_{2k+\sigma} + \sqrt{\delta_{2k+\sigma}(4 - \delta_{2k+\sigma})}\right) \sqrt{1 + \eta\delta^2_{2k+\sigma}} < \frac{1}{2},$$

By a similar treatment in the proof of Theorem 4.2, we may rewrite (the details were omitted here) the inequality above as the following polynomial inequality of $\delta_{2k+\sigma}$:

$$- \delta_{2k+\sigma}^{10} + 4\delta_{2k+\sigma}^{9} + (8\eta - 16)\delta_{2k+\sigma}^{7} - (4\eta^2 - 10)\delta_{2k+\sigma}^{6} + (32 - 16\eta)\delta_{2k+\sigma}^{5} - (20 + 8\eta)\delta_{2k+\sigma}^{4} + (8\eta - 32)\delta_{2k+\sigma}^{3} + 11\delta_{2k+\sigma}^{2} + 12\delta_{2k+\sigma} - 4 < 0.$$

It is not difficult to verify that this inequality holds if $\delta_{2k+\sigma} < 0.3112$. Thus we conclude that the iterates, generated by GSP (or SP), converge geometrically in rate 0.5 to the $k$ largest absolute entries of the original signal if $\delta_{2k+\sigma} < 0.3112$ (or $\delta_{3k} < 0.3112$ for SP). At the moment the guaranteed success of compressed sensing algorithms for signal recovery was usually established under the RIP.

To obtain more relaxed conditions for the guaranteed success of a compressed sensing algorithm, improving RIP-based bounds is not the only channel to achieve this goal. Other important tools such as the null space property (NSP) and the range space property (RSP) of transposed sensing matrices might also be useful tools that can be employed to achieve such a goal. Unlike the RIP which involves the constant RIC, the NSP and RSP are constant-free in the sense that their definitions do not involve any constant, and they are generally less restrictive than the RIP assumption [11], [12].
It is well known that the performance of $\ell_1$-minimization method has been analyzed under all these assumptions (see, e.g., [11], [12], [47]). For thresholding algorithms including IHT, CoSaMP and (G)SP, however, there is no convergence results at the moment which are shown under an NSP or RSP condition. To develop a performance theory for signal recovery under the NSP or RSP might bring a further improvement on those RIP-based bounds for these algorithms.

V. CONCLUSIONS

The RIP-based bounds that guarantee the success of signal recovery/approximation via three types of compressed sensing algorithms (IHT, CoSaMP, and SP) have been improved in this paper. A common feature of these algorithms is using hard-thresholding operators to produce a sparse approximation of the unknown signal. A fundamental property of the hard thresholding operator was shown in Lemma 2.2 which provides a useful basis to the improvement of the current performance theory for these algorithms. This lemma, together with the new and tighter error estimations gave in Lemmas 3.3 and 4.1 makes it possible to significantly improve the existing RIP-based performance results for signal recovery via these algorithms. The new RIP-based bound $\delta_{2k} < (\sqrt{5} - 1)/2$ shown for IHT in this paper remarkably improves the current best known bound for this algorithm. The improvement of the performance theory for CoSaMP and SP are much more challenging. For the former, the existing RIP-based bound was improved to $\delta_{4k} < 0.5593$, and for the latter the existing bound was improved to $\delta_{3k} < 0.5108$. The general subspace pursuit (GSP) was also introduced and analyzed for the first time in this paper. However, the question for the optimal (or the tightest) RIP-based bounds for the guaranteed performance of these algorithms remains open at the moment.

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