A GARDEN OF EDEN THEOREM FOR ANOSOV DIFFEOMORPHISMS ON TORI

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Abstract. Let $f$ be an Anosov diffeomorphism of the $n$-dimensional torus $\mathbb{T}^n$ and $\tau$ a continuous self-mapping of $\mathbb{T}^n$ commuting with $f$. We prove that $\tau$ is surjective if and only if the restriction of $\tau$ to each homoclinicity class of $f$ is injective.

1. Introduction

The Garden of Eden theorem, originally established by Moore [20] and Myhill [21] in the early 1960s, is an important result in symbolic dynamics and coding theory. It provides a necessary and sufficient condition for a cellular automata to be surjective. More specifically, consider a finite set $A$ and the set $A^\mathbb{Z}$ consisting of all bi-infinite sequences $x = (x_i)$ with $x_i \in A$ for all $i \in \mathbb{Z}$. We equip $A^\mathbb{Z}$ with its prodiscrete topology, that is, with the topology of pointwise convergence (this is also the product topology obtained by taking the discrete topology on each factor $A$ of $A^\mathbb{Z}$). A cellular automaton is a continuous map $\tau: A^\mathbb{Z} \to A^\mathbb{Z}$ that commutes with the shift homeomorphism $\sigma: A^\mathbb{Z} \to A^\mathbb{Z}$ given by $\sigma(x) = (x_i-1)$ for all $x = (x_i) \in A^\mathbb{Z}$. Two sequences $x = (x_i), y = (y_i) \in A^\mathbb{Z}$ are said to be almost equal if one has $x_i = y_i$ for all but finitely many $i \in \mathbb{Z}$. A cellular automaton $\tau: A^\mathbb{Z} \to A^\mathbb{Z}$ is called pre-injective if there exist no distinct sequences $x, y \in A^\mathbb{Z}$ that are almost equal and satisfy $\tau(x) = \tau(y)$. The Moore-Myhill Garden of Eden theorem states that a cellular automaton $\tau: A^\mathbb{Z} \to A^\mathbb{Z}$ is surjective if and only if it is pre-injective. The implication surjective $\Rightarrow$ pre-injective was first established by Moore [20], and Myhill [21] proved the converse implication shortly after.

The Moore-Myhill Garden of Eden theorem has been extended in several directions. There are now versions of it for cellular automata over amenable groups [18], [9], cellular automata over subshifts [14], [11], [12], and linear cellular automata over linear shifts and subshifts [6], [8] (the reader is referred to the monograph [7] for a detailed exposition of some of these extensions, as well as historical comments and additional references).

In this note, we present an analogue of the Garden of Eden theorem for Anosov diffeomorphisms on tori. This reveals one more connection between symbolic dynamics and the theory of smooth dynamical systems. Actually our motivation came from a phrase

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of Gromov [14, p. 195] which mentioned the possibility of extending the Garden of Eden theorem to a suitable class of hyperbolic dynamical systems.

Let \((X, f)\) be a dynamical system consisting of a compact metrizable space \(X\) equipped with a homeomorphism \(f: X \to X\). Two points in \(X\) are called \(f\)-homoclinic if their \(f\)-orbits are asymptotic both in the past and the future (see Section 2 for a precise definition). Homoclinicity defines an equivalence relation on \(X\). An endomorphism of the dynamical system \((X, f)\) is a continuous map \(\tau: X \to X\) commuting with \(f\). We say that an endomorphism \(\tau\) of \((X, f)\) is pre-injective (with respect to \(f\)) if the restriction of \(\tau\) to each \(f\)-homoclinicity class is injective (i.e., there is no pair of distinct \(f\)-homoclinic points in \(X\) having the same image under \(\tau\)) (in the particular case when \(X = \mathbb{Z}^d\) and \(f = \sigma\) is the shift homeomorphism, the endomorphisms of \((X, f)\) are precisely the cellular automatons and this definition of pre-injectivity is equivalent to the one given above, see e.g. [5, Proposition 2.5]). We say that the dynamical system \((X, f)\) has the Moore property if every surjective endomorphism of \((X, f)\) is pre-injective and that \((X, f)\) has the Myhill property if every pre-injective endomorphism of \((X, f)\) is surjective. We say that the dynamical system \((X, f)\) has the Moore-Myhill property, or that it satisfies the Garden of Eden theorem, if \((X, f)\) has both the Moore and the Myhill properties.

A \(C^1\)-diffeomorphism \(f\) of a compact \(C^r\)-differentiable \((r \geq 1)\) manifold \(M\) is called an Anosov diffeomorphism if the tangent bundle of \(M\) splits as a direct sum \(TM = E_s \oplus E_u\) of two invariant subbundles \(E_s\) and \(E_u\) such that, with respect to some (or equivalently any) Riemannian metric on \(M\), the differential \(df\) is uniformly contracting on \(E_s\) and uniformly expanding on \(E_u\) (see [23], [4], [10], [15], [22]).

Our main result is the following.

**Theorem 1.1** (Garden of Eden theorem for toral Anosov diffeomorphisms). Let \(f\) be an Anosov diffeomorphism of the \(n\)-dimensional torus \(\mathbb{T}^n\). Then the dynamical system \((\mathbb{T}^n, f)\) has the Moore-Myhill property. In other words, if \(\tau: \mathbb{T}^n \to \mathbb{T}^n\) is a continuous map commuting with \(f\), then \(\tau\) is surjective if and only if the restriction of \(\tau\) to each homoclinicity class of \(f\) is injective.

The paper is organized as follows. In Section 2, we fix notation and present some background material on dynamical systems. In Section 3, we establish Theorem 1.1. The proof uses two classical results in the theory of hyperbolic dynamical systems. The first one is the Franks-Manning theorem [13], [19], which states that any Anosov diffeomorphism on \(\mathbb{T}^n\) is topologically conjugate to a hyperbolic toral automorphism. The second is a theorem due to Walters [24] which asserts that all endomorphisms of a hyperbolic toral endomorphism are affine. This allows us to reduce the proof to an elementary question in linear algebra. In the final section, we discuss some examples and give an extension of the Myhill implication of the Garden of Eden theorem for topologically mixing basic sets of Axiom A diffeomorphisms.

## 2. Background

In this section, we review some basic facts about dynamical systems. For more details, the reader is referred to the monographs [4], [10], [15], [10], and [22].
2.1. Dynamical systems. Throughout this paper, by a dynamical system, we mean a pair \((X, f)\), where \(X\) is a compact metrizable space and \(f : X \to X\) is a homeomorphism. Sometimes, we shall simply write \(f\) or \(X\) instead of \((X, f)\) if there is no risk of confusion. We denote by \(d\) a metric on \(X\) that is compatible with the topology.

The orbit of a point \(x \in X\) is the set \(\{f^n(x) : n \in \mathbb{Z}\} \subset X\). The point \(x\) is called periodic if its orbit is finite. A subset \(Y \subset X\) is said to be invariant if \(f(Y) = Y\). If \(Y \subset X\) is an invariant subset, we denote by \(f|_Y\) the restriction of \(f\) to \(Y\), i.e., the map \(f|_Y : Y \to Y\) given by \(f|_Y(y) := f(y)\) for all \(y \in Y\).

One says that the dynamical systems \((X, f)\) and \((Y, g)\) are topologically conjugate if there exists a homeomorphism \(\varphi : X \to Y\) such that \(\varphi \circ f = g \circ \varphi\).

2.2. Homoclinicity. Two points \(x, y \in X\) are called homoclinic with respect to \(f\) (or \(f\)-homoclinic) if one has \(d(f^n(x), f^n(y)) \to 0\) as \(|n| \to \infty\). Homoclinicity is an equivalence relation on \(X\). By compactness, this equivalence relation is independent of the choice of the metric \(d\).

**Proposition 2.1.** Let \((X, f)\) be a dynamical system. Suppose that \(x\) and \(y\) are periodic points of \(f\). If \(x\) and \(y\) are \(f\)-homoclinic, then \(x = y\).

**Proof.** Since \(x\) and \(y\) are periodic, there are integers \(m, n \geq 1\) such that \(f^n(x) = x\) and \(f^m(y) = y\). If \(x\) and \(y\) are \(f\)-homoclinic, then, given any \(\varepsilon > 0\), we have \(d(x, y) = d(f^{knm}(x), f^{knm}(y)) < \varepsilon\) for \(k\) large enough. This implies \(x = y\). \(\square\)

**Proposition 2.2.** Let \((X, f)\) and \((Y, g)\) be two dynamical systems. Suppose that \(\psi : X \to Y\) is a continuous map such that \(\psi \circ f = g \circ \psi\). If two points in \(X\) are \(f\)-homoclinic, then their images under \(\psi\) are \(g\)-homoclinic.

**Proof.** Suppose that the points \(x, y \in X\) are \(f\)-homoclinic. Let \(d\) (resp. \(d'\)) be a metric on \(X\) (resp. \(Y\)) that is compatible with the topology. Then, we have that
\[
d'(g^n(\psi(x)), g^n(\psi(y))) = d'(\psi(f^n(x)), \psi(f^n(y))) \to 0
\]
as \(|n| \to \infty\) since \(d(f^n(x), f^n(y)) \to 0\) as \(|n| \to \infty\) and \(\psi\) is uniformly continuous. This shows that the points \(\psi(x)\) and \(\psi(y)\) are \(g\)-homoclinic. \(\square\)

2.3. Hyperbolic toral automorphisms. Consider the \(n\)-dimensional torus \(\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n\). For \(x \in \mathbb{R}^n\), we write \(\overline{x} := x + \mathbb{Z}^n \in \mathbb{T}^n\).

Let \(M_n(\mathbb{Z})\) denote the ring of \(n \times n\) matrices with integral entries. Every matrix \(A \in M_n(\mathbb{Z})\) induces a differentiable group endomorphism \(f_A : \mathbb{T}^n \to \mathbb{T}^n\) given by \(f_A(\overline{x}) = \overline{Ax}\) for all \(x \in \mathbb{R}^n\). One says that \(f_A\) is the toral endomorphism associated with \(A\).

The group of invertible elements of \(M_n(\mathbb{Z})\) is the group \(GL_n(\mathbb{Z})\) of \(n \times n\) matrices with integral entries and determinant \(\pm 1\). If \(A \in GL_n(\mathbb{Z})\), then \(f_A\) is a differentiable automorphism of \(\mathbb{T}^n\) and one says that \(f_A\) is the toral automorphism associated with \(A\). If \(f\) is a toral automorphism of \(\mathbb{T}^n\), the homoclinicity class of \(0\) is a subgroup of \(\mathbb{T}^n\), called the homoclinicity group of \(f\), and denoted by \(\Delta(f)\) (cf. [17]). Note that two points \(p, q \in \mathbb{T}^n\) are \(f\)-homoclinic if and only if \(p - q \in \Delta(f)\). Observe also that every point in \(\mathbb{Q}^n / \mathbb{Z}^n\) is \(f\)-periodic so that \(\Delta(f) \cap \mathbb{Q}^n / \mathbb{Z}^n = \{0\}\) by Proposition 2.1.
A matrix $A \in \text{GL}_n(\mathbb{Z})$ is called hyperbolic if its complex spectrum does not meet the unit circle. A diffeomorphism $f$ of $\mathbb{T}^n$ is called a hyperbolic automorphism if there is a hyperbolic matrix $A \in \text{GL}_n(\mathbb{Z})$ such that $f = f_A$.

There are hyperbolic automorphisms on any torus of dimension $n \geq 2$ and every hyperbolic automorphism is Anosov. On the other hand, by the Franks-Manning theorem (cf. [13] and [19]), every Anosov diffeomorphism of $\mathbb{T}^n$ is topologically conjugate to a hyperbolic automorphism.

3. Proof of the main result

Proof of Theorem 1.1. By the Franks-Manning theorem mentioned above, we can assume that $f$ is a hyperbolic automorphism of $\mathbb{T}^n$. Let $A \in \text{GL}_n(\mathbb{Z})$ be a hyperbolic matrix such that $f = f_A$. Let $\Delta(f) \subset \mathbb{T}^n$ denote the homoclinicity group of $f$.

Let $\tau: \mathbb{T}^n \to \mathbb{T}^n$ be a continuous map commuting with $f$. Since $f$ is a hyperbolic automorphism of $\mathbb{T}^n$ and $\tau$ commutes with $f$, it follows from [24, Theorem 2] that $\tau$ is an affine toral endomorphism, that is, there is a matrix $B \in \text{M}_n(\mathbb{Z})$ and $c \in \mathbb{R}^n$ such that $\tau(\mathbf{x}) = B\mathbf{x} + c$ for all $\mathbf{x} \in \mathbb{R}^n$. Note that the map $\mathbf{x} \mapsto \tau(\mathbf{x}) - \mathbf{c}$ is a group endomorphism of $\mathbb{T}^n$.

Suppose first that $\tau$ is surjective. We claim that $\det(B) \neq 0$. Indeed, otherwise, the image of $\mathbb{R}^n$ under the affine map $x \mapsto Bx + c$ would be an affine subspace $L \subset \mathbb{R}^n$ with empty interior and we would deduce from the Baire category theorem that $L + \mathbb{Z}^n \subset \mathbb{R}^n$, which would contradict the surjectivity of $\tau$. Thus, we have $\det(B) \neq 0$ and hence $B \in \text{GL}_n(\mathbb{Q})$.

Let $x, y \in \mathbb{R}^n$ such that the points $\mathbf{x}$ and $\mathbf{y}$ are $f$-homoclinic and satisfy $\tau(\mathbf{x}) = \tau(\mathbf{y})$. We then have $B(x - y) \in \mathbb{Z}^n$ and hence $x - y \in \mathbb{Q}^n$. This implies that the point $\mathbf{x} - \mathbf{y}$ is $f$-periodic. On the other hand, since the points $\mathbf{x}$ and $\mathbf{y}$ are $f$-homoclinic, we have that $\mathbf{x} - \mathbf{y} = \mathbf{x} - \mathbf{y} \in \Delta(f)$. By applying Proposition 2.1, we deduce that $\mathbf{x} - \mathbf{y} = 0$ and therefore $\mathbf{x} = \mathbf{y}$. This shows that $\tau$ is pre-injective and hence that $(\mathbb{T}^n, f)$ has the Moore property.

It remains to show that $\tau$ is surjective. Since $f$ is a hyperbolic automorphism, it is known that the group $\Delta(f)$ is isomorphic to $\mathbb{Z}^n$ (see [17, Example 3.3]). On the other hand, since $\tau(\mathbf{0}) = \mathbf{0}$, we have that $\tau(\Delta(f)) - \mathbf{c} \subset \Delta(f)$ by Proposition 2.1. As the restriction of $\tau$ to $\Delta(f)$ is injective by our pre-injectivity hypothesis, we deduce that $\tau(\Delta(f)) - \mathbf{c}$ is a finite-index subgroup of $\Delta(f)$. It is also known that $\Delta(f)$ is dense in $\mathbb{T}^n$ (see again [17, Example 3.3]). Consider now the closure $C \subset \mathbb{T}^n$ of $\tau(\Delta(f)) - \mathbf{c}$. As $C$ is a closed subgroup of $\mathbb{T}^n$ and hence a torus, we must have $C = \mathbb{T}^n$ since otherwise the group $\Delta(f)$ would be contained in the union of a finite number of translates of a torus of dimension less than $n$ and then could not be dense in $\mathbb{T}^n$. It follows that the closure of $\tau(\Delta(f))$ is also equal to $\mathbb{T}^n$. By continuity, this shows that $\tau$ is surjective. Consequently, $(\mathbb{T}^n, f)$ has the Myhill property. \qed
4. Concluding remarks

4.1. Examples of non-injective pre-injective endomorphisms. Injectivity trivially implies pre-injectivity for endomorphisms of dynamical systems \((X, f)\). However, the converse is false if \(f\) is an Anosov diffeomorphism of \(\mathbb{T}^n\). Indeed, if \(f\) is a hyperbolic automorphism of \(\mathbb{T}^n\) and \(m \in \mathbb{Z}\) satisfies \(|m| \geq 2\), then multiplication by \(m\) on \(\mathbb{T}^n\) is an endomorphism of \((\mathbb{T}^n, f)\) that is surjective and hence pre-injective but not injective (its kernel has cardinality \(|m|^n\)).

This can be generalized in the following way. Let \(f_i\) be a hyperbolic automorphism of the \(n_i\)-dimensional torus \(\mathbb{T}^{n_i}\), where \(n_i \geq 2\) and \(1 \leq i \leq k\). Then \(f := f_1 \times \cdots \times f_k\) is a hyperbolic toral automorphism of the \(N\)-torus

\[\mathbb{T}^N = \mathbb{T}^{n_1} \times \cdots \times \mathbb{T}^{n_k},\]

where \(N := n_1 + \cdots + n_k\). Let us fix some non-zero integers \(m_i \in \mathbb{Z}\), \(1 \leq i \leq k\), with \(|m_i| \geq 2\) for at least one \(i\), and consider the endomorphism \(\tau\) of \((\mathbb{T}^N, f)\) defined by

\[\tau(x) := (m_1 x_1, \ldots, m_k x_k)\]

for all \(x = (x_1, \ldots, x_k) \in \mathbb{T}^N\). Clearly \(\tau\) is surjective and hence pre-injective. On the other hand, the kernel of \(\tau\) has cardinality \(|m_1|^{n_1} \cdots |m_k|^{n_k}\) and therefore \(\tau\) is not injective.

4.2. Ergodic toral automorphisms. Let \(A \in \text{GL}_n(\mathbb{Z})\) and \(f_A : \mathbb{T}^n \to \mathbb{T}^n\) the associated toral automorphism. It is well known (see e.g. [10, Proposition 24.1]) that \(f_A\) is ergodic (with respect to the Lebesgue measure on \(\mathbb{T}^n\)) if and only if \(A\) has no eigenvalues which are roots of unity. This implies in particular that every hyperbolic toral automorphism of \(\mathbb{T}^n\) is ergodic. Observe that the proof of the Moore property for hyperbolic toral automorphisms given in Section 3 applies verbatim to ergodic toral automorphisms. Indeed, a continuous map \(\tau : \mathbb{T}^n \to \mathbb{T}^n\) commuting with an ergodic toral automorphism \(f_A\) is necessarily affine (cf. [24, Theorem 2]). We thus have the following:

**Proposition 4.1.** Let \(f : \mathbb{T}^n \to \mathbb{T}^n\) be an ergodic toral automorphism. Then the dynamical system \((\mathbb{T}^n, f)\) has the Moore property. In other words, if \(\tau : \mathbb{T}^n \to \mathbb{T}^n\) is a surjective continuous map commuting with \(f\), then the restriction of \(\tau\) to each \(f\)-homoclinicity class is injective.

We know that hyperbolic toral automorphisms also satisfy the Myhill property by Theorem 1.1. For ergodic toral endomorphism, however, the Myhill property fails to hold in general. Consider for instance the matrix

\[A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \in \text{GL}_4(\mathbb{Z}).\]
Its eigenvalues are
\[\lambda_1 = \frac{1}{2} - \frac{1}{\sqrt{2}} + i \frac{\sqrt{8} + 1}{2}\]
\[\lambda_2 = \frac{1}{2} - \frac{1}{\sqrt{2}} - i \frac{\sqrt{8} + 1}{2}\]
\[\lambda_3 = \frac{1}{2} + \frac{1}{\sqrt{2}} - \frac{\sqrt{8} - 1}{2}\]
\[\lambda_4 = \frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{\sqrt{8} - 1}{2}\]
and satisfy \(|\lambda_1| = |\lambda_2| = 1\) and \(0 < \lambda_3 < 1 < \lambda_4\). Since none of these eigenvalues is a root of unity, the associated toral automorphism \(f_A: \mathbb{T}^4 \to \mathbb{T}^4\) is ergodic. On the other hand, the characteristic polynomial \(\chi_A(x) = x^4 - 2x^3 + x^2 - 2x + 1\) is irreducible over \(\mathbb{Q}\). It follows that the homoclinicity group \(\Delta(f_A)\) is reduced to 0 (cf. [17, Example 3.4]). Consequently, every endomorphism of the dynamical system \((\mathbb{T}^4, f_A)\) is pre-injective with respect to \(f_A\). Since the zero endomorphism is not surjective, we conclude that \((\mathbb{T}^4, f_A)\) does not have the Myhill property.

4.3. The Myhill property for elementary basic sets. Let us first recall some definitions.

Let \(f\) be a homeomorphism of a compact metrizable space \(X\). One says that the dynamical system \((X, f)\) is expansive if there exists a constant \(\delta > 0\) such that, for every pair of distinct points \(x, y \in X\), there exists \(n = n(x, y) \in \mathbb{Z}\) such that \(d(f^n(x), f^n(y)) \geq \delta\) (here \(d\) denotes any metric on \(X\) that is compatible with the topology). One says that the dynamical system \((X, f)\) is topologically mixing if for any pair of non-empty open subsets \(U, V \subseteq X\), there is an integer \(N = N(U, V) \in \mathbb{Z}\) such that \(f^n(U)\) meets \(V\) for all \(n \geq N\).

One says that the dynamical system \((X, f)\) is a factor of the dynamical system \((Y, g)\) if there exists a continuous surjective map \(\pi: Y \to X\) such that \(\pi \circ g = f \circ \pi\). Such a map \(\pi\) is then called a factor map. A factor map \(\pi: Y \to X\) is said to be uniformly bounded-to-one if there is an integer \(K \geq 1\) such that each \(x \in X\) has at most \(K\) pre-images in \(Y\).

Let \(A\) be a finite set and let \(\sigma\) denote the shift homeomorphism on \(A^\mathbb{Z}\). A \(\sigma\)-invariant closed subset \(\Sigma \subseteq A^\mathbb{Z}\) is called a subshift. A subshift \(\Sigma \subseteq A^\mathbb{Z}\) is said to be of finite type if there is an integer \(n \geq 1\) and a subset \(P \subseteq A^n\) such that \(X\) consists of the sequences \(x = (x_i) \in A^\mathbb{Z}\) that satisfy
\[(x_i, x_{i+1}, \ldots, x_{i+n-1}) \in P\]
for all \(i \in \mathbb{Z}\).

We have the following result.

**Theorem 4.2.** Let \(f: X \to X\) be a homeomorphism of a compact metrizable space \(X\). Suppose that the dynamical system \((X, f)\) is expansive and that there exist a finite set \(A\), a topologically mixing subshift of finite type \(\Sigma \subseteq A^\mathbb{Z}\), and a uniformly bounded-to-one factor map \(\pi: \Sigma \to X\). Then the dynamical system \((X, f)\) has the Myhill property.
Proof. This is a special case of [5] Theorem 1.1] since the group $\mathbb{Z}$ is amenable and every topologically mixing subshift of finite type over $\mathbb{Z}$ is strongly irreducible. \hfill \Box

Note that there exist dynamical systems $(X, f)$ satisfying all the hypotheses of Theorem 4.2 that do not have the Moore property. An example of such a dynamical system is provided by the even subshift, that is, the subshift $X \subset \{0, 1\}^\mathbb{Z}$ consisting of all bi-infinite sequences of 0s and 1s with an even number of 0s between any two 1s. Indeed, if $\Sigma \subset \{0, 1\}^\mathbb{Z}$ denotes the golden subshift, provided by the even subshift $\text{rem } 4.2$ that do not have the Moore property. An example of such a dynamical system is $\Omega(f)$, sequences of 0s and 1s with an even number of 0s between any two 1s. Indeed, if $\Sigma$ finite type and that there is a factor map $\pi : \Sigma \to X$ such that each configuration in $X$ has at most 2 pre-images in $\Sigma$ (see e.g. [16 Example 4.1.6]). Thus, the even subshift satisfies the hypotheses of Theorem 4.2. On the other hand, Fiorenzi [11, Section 3] proved that the even subshift does not have the Moore property.

Let $M$ be a compact $C^r$-differentiable ($r \geq 1$) manifold and $f$ a $C^1$-diffeomorphism of $M$. One says that a closed $f$-invariant subset $\Lambda \subset M$ is a hyperbolic set if the restriction to $\Lambda$ of the tangent bundle of $M$ continuously splits as a direct sum of two invariant subbundles $E_s$ and $E_u$ such that, with respect to some (or equivalently any) Riemannian metric on $M$, the differential $df$ of $f$ is uniformly contracting on $E_s$ and uniformly expanding on $E_u$; i.e., there are constants $C > 0$ and $0 < \lambda < 1$ such that $\|df^n(v)\| \leq C\lambda^n\|v\|$ and $\|df^{-n}(w)\| \leq C\lambda^n\|w\|$ for all $x \in \Lambda$, $v \in E_s(x)$, $w \in E_u(x)$, and $n \geq 0$. Thus, $f$ is an Anosov diffeomorphism if and only if the whole manifold $M$ is a hyperbolic set for $f$. A point $x \in M$ is called non-wandering if for every neighborhood $U$ of $x$, there is an integer $n \geq 1$ such that $f^n(U)$ meets $U$. The set $\Omega(f)$ consisting of all non-wandering points of $f$ is a closed invariant subset of $M$. One says that $f$ satisfies Smale’s Axiom A if the set $\Omega(f)$ is hyperbolic and the periodic points of $f$ are dense in $\Omega(f)$ (cf. [23]). If $f$ is Axiom A, then $\Omega(f)$ can be uniquely written as a disjoint union of closed invariant subsets $\Omega(f) = X_1 \cup \cdots \cup X_k$, such that the restriction of $f$ to each $X_i$ is topologically transitive for $1 \leq i \leq k$ (spectral decomposition theorem). These subsets $X_i$ are called the basic sets of $(M, f)$. A basic set $X_i$ is called elementary if the restriction of $f$ to $X_i$ is topologically mixing.

Corollary 4.3. Let $f$ be a $C^1$-diffeomorphism of a compact $C^r$-differentiable ($r \geq 1$) manifold $M$ that satisfies Axiom A. Suppose that $X$ is an elementary basic set of $(M, f)$ and let $f|_X : X \to X$ denote the restriction of $f$ to $X$. Then the dynamical system $(X, f|_X)$ has the Myhill property.

Proof. The fact that the dynamical system $(X, f|_X)$ satisfies the hypothesis of Theorem 4.2 follows from the classical work of Rufus Bowen on Axiom A diffeomorphisms. The expansivity of $f|_X$ is shown in [2] Lemma 3]. On the other hand, Bowen used a Markov partition to show that one can find a finite set $A$ (the set of rectangles of the Markov partition) and a topologically mixing subshift of finite type $\Sigma \subset A^\mathbb{Z}$ such that there exists a uniformly bounded-to-one factor map $\pi : \Sigma \to X$ (cf. [2] Theorem 28 and Proposition 30] and [11 Proposition 10)). \hfill \Box
It is an open question whether every Anosov diffeomorphism is topologically mixing. However, for an Anosov diffeomorphism \( f : M \to M \), the following conditions are known to be equivalent (see e.g. [4, Theorem 5.10.3]): (a) \( f \) is topologically mixing; (b) \( f \) is topologically transitive; (c) every point in \( M \) is non-wandering; (d) the periodic points of \( f \) are dense in \( M \). This implies in particular that every topologically mixing Anosov diffeomorphism \( f \) of a compact manifold \( M \) is Axiom A with \( \Omega(f) = M \). Thus, as a particular case of Corollary 4.3, we get the following result, which extends the Myhill part of Theorem 1.1 to all topologically mixing Anosov diffeomorphisms.

**Corollary 4.4.** If \( f \) is a topologically mixing Anosov diffeomorphism of a compact \( C^r \)-differentiable \((r \geq 1)\) manifold \( M \), then the dynamical system \((M, f)\) has the Myhill property. \( \square \)

4.4. **Zero-dimensional basic sets.** If \( X \) is a zero-dimensional basic set of an Axiom A diffeomorphism \( f \), it follows from [3, Theorem 6.6] that \((X, f|_X)\) is topologically conjugate to an irreducible subshift of finite type. As every irreducible subshift of finite type has the Moore-Myhill property by [11, Corollary 2.19], we deduce that \((X, f|_X)\) has the Moore-Myhill property.

In view of this observation and of Theorem 1.1 it is very tempting to conjecture that the restriction of an Axiom A diffeomorphism to a (possibly non-elementary) basic set always has the Moore-Myhill property.

**References**

[1] R. Bowen, *Markov partitions and minimal sets for Axiom A diffeomorphisms*, Amer. J. Math., 92 (1970), pp. 907–918.
[2] ———, *Markov partitions for Axiom A diffeomorphisms*, Amer. J. Math., 92 (1970), pp. 725–747.
[3] ———, *Topological entropy and axiom A*, in Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 23–41.
[4] M. Brin and G. Stuck, *Introduction to dynamical systems*, Cambridge University Press, Cambridge, 2002.
[5] T. Ceccherini-Silberstein and M. Coornaert, *Expansive actions of countable amenable groups, homoclinic pairs, and the myhill property*, arXiv:1508.07553, to appear in Illinois Journal of Mathematics.
[6] ———, *The Garden of Eden theorem for linear cellular automata*, Ergodic Theory Dynam. Systems, 26 (2006), pp. 53–68.
[7] ———, *Cellular automata and groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.
[8] ———, *A Garden of Eden theorem for linear subshifts*, Ergodic Theory Dynam. Systems, 32 (2012), pp. 81–102.
[9] T. Ceccherini-Silberstein, A. Machì, and F. Scarabotti, *Amenable groups and cellular automata*, Ann. Inst. Fourier (Grenoble), 49 (1999), pp. 673–685.
[10] M. Denker, C. Grillenberger, and K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Mathematics, Vol. 527, Springer-Verlag, Berlin-New York, 1976.
[11] F. Fiorenzi, *The Garden of Eden theorem for sofic shifts*, Pure Math. Appl., 11 (2000), pp. 471–484.
[12] ———, *Cellular automata and strongly irreducible shifts of finite type*, Theoret. Comput. Sci., 299 (2003), pp. 477–493.
[13] J. Franks, Anosov diffeomorphisms, in Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 61–93.

[14] M. Gromov, Endomorphisms of symbolic algebraic varieties, J. Eur. Math. Soc. (JEMS), 1 (1999), pp. 109–197.

[15] A. Katok and B. Hasselblatt, Introduction to the modern theory of dynamical systems, vol. 54 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[16] D. Lind and B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press, Cambridge, 1995.

[17] D. Lind and K. Schmidt, Homoclinic points of algebraic $\mathbb{Z}^d$-actions, J. Amer. Math. Soc., 12 (1999), pp. 953–980.

[18] A. Machi and F. Mignosi, Garden of Eden configurations for cellular automata on Cayley graphs of groups, SIAM J. Discrete Math., 6 (1993), pp. 44–56.

[19] A. Manning, There are no new Anosov diffeomorphisms on tori, Amer. J. Math., 96 (1974), pp. 422–429.

[20] E. F. Moore, Machine models of self-reproduction, vol. 14 of Proc. Symp. Appl. Math., American Mathematical Society, Providence, 1963, pp. 17–34.

[21] J. Myhill, The converse of Moore’s Garden-of-Eden theorem, Proc. Amer. Math. Soc., 14 (1963), pp. 685–686.

[22] M. Shub, Global stability of dynamical systems, Springer-Verlag, New York, 1987. With the collaboration of A. Fathi and R. Langevin, Translated from the French by Joseph Christy.

[23] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc., 73 (1967), pp. 747–817.

[24] P. Walters, Topological conjugacy of affine transformations of tori, Trans. Amer. Math. Soc., 131 (1968), pp. 40–50.

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