ISOSPIN SYMMETRY BREAKING
THROUGH $^0 - ^0$ MIXING

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Mixing of the pseudoscalar mesons is discussed in the quark-avoir basis. The divergences of the axial vector currents which embody the axial vector anomaly, combined with the assumption that the decay constants follow the pattern of particle state mixing in that basis, determine the mixing parameters for given masses of the physical mesons. These mixing parameters are compared with results from other work in some detail. Phenomenological applications of the quark-avoir mixing scheme are presented with particular interest focused on isospin symmetry breaking which is generated by means of and $^0$ admixtures to the pion. Consequences of a possible difference in the basis decay constants $f_u$ and $f_d$ for the strength of isospin symmetry breaking are also examined.

Keywords: meson mixing; isospin symmetry breaking.

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1. Introduction

That the character of the approximateavour symmetry of QCD is determined by the pattern of the quark masses is a well-known fact that has been extensively discussed in the literature for decades. Isospin symmetry in particular which would be exact for identical u and d quark masses, $m_u = m_d$, holds to a rather high degree of accuracy empirically, although the ratio $(m_d - m_u)/(m_d + m_u)$ is about 1/3 and not, as one would expect for a true symmetry, much smaller than unity. Violations of isospin symmetry for pseudoscalar mesons within QCD are generated by admixtures of the and $^0$ to the pion. On exploiting the divergences of the axial vector currents but neglecting $^0$ mixing, Gross, Treiman and Wilczek obtained for the mixing angle between the pion and the avoir-octet state

$$\theta = \frac{P}{3} \frac{m_d - m_u}{m_s (m_d + m_u)} = 2 \; ;$$

(1)

a result that also follows from lowest order chiral perturbation theory. We learn from that due to the effect of the $U_A(1)$ anomaly which is embodied in the divergences of the axial vector currents, isospin symmetry breaking (ISB) is of the order of $(m_d - m_u)/m_s$ instead of the expected $(m_d - m_u)/(m_d + m_u)$. Isospin
symmetry is thus partially restored and amounts to only a few percent for pseudoscalar mesons. It is therefore to be interpreted rather as an accidental symmetry which comes about as a consequence of the dynamics. For hadrons other than pseudoscalar mesons the strength of ISB is not necessarily set by the mass ratios for comments on ISB in the vector meson sector see Ref.

Due to a number of recent experiments the interest in ISB has been renewed. It therefore seems opportune to review recent progress in our understanding of mixing. In an analysis of mixing the quark-avor basis has been used and assumed that the decay constants in that basis follow the pattern of particle state mixing. With the help of the divergences of the axial vector currents the basic parameters of that mixing scheme can be determined for given masses of the physical mesons. It has been found in Ref. that this approach leads to consistent results and explains many empirical features of mixing. This mixing scheme will be presented in Sect. 2 and, in Sect. 3, the phenomenology of mixing will be briefly reviewed. Sect. 4 is devoted to a discussion of mixing in the avor octet-singlet basis and to a detailed comparison with other recent work on mixing.

The inclusion of the mixing scheme is straightforward. Here, in this work, mixing will be discussed in detail and, as a new aspect, the role of possible differences in the decay constants and will be investigated. It is important to realize that the decay constants which represent wavefunctions at zero spatial quark-antiquark separation, are functions of the quark masses. Hence, ISB generated through would be related to the u-d quark-mass difference, too. It is only due to our inability of calculating the decay constants to a sufficient degree of accuracy at present that we have to consider them as independent soft parameters. ISB will be discussed in Sect. 5. The vacuum element of the topological charge density is calculated in Sect. 6 and phenomenological consequences are presented. Finally, in Sect. 7, isospin symmetry violating processes, which are not directly controlled by the \( U_A(1) \) anomaly, are examined. The conclusions are provided in Sect. 8.

2. Mixing in the quark-avor basis

The quark-avor basis is constructed by means of the states \( a, a = u, d; s \) which are understood to possess the parton com positions

\[
\begin{align*}
j_a^1 &= a \Delta a_i + a \Delta a_i^+ \\
\end{align*}
\]

in a Fock expansion. Here \( a \) denotes a (light-cone) wavefunction. The higher Fock states, whose presence are indicated by the ellipses, also include two-gluon components. Because of the fact that light-cone wave functions do not depend on the hadron’s momentum, one can define decay constants

\[
f_a = 2^{-1} \frac{1}{6} \int \frac{d^2 k}{16} a(x, k) ;
\]
associated with the states $a$. The variable $x$ denotes the usual (light-cone plus) momenta fraction the quark carries and $k_2$ is transverse on event with respect to its parent meson's $s$-event. The definition is exact, only the $q\bar{q}$ component contributes to the decay constants. The basic decay constants are assumed to possess the property

$$h_0 j_b^a j_b^d (p) i = \sum_{a;b=u;d;s} a,b \quad (4)$$

where $j^a_b = a \otimes a$ denotes the axial-vector current for quarks of flavor $a$. Eq. and form the basis of the quark-avorm mixing scheme proposed in Ref. This mixing scheme holds to the extent that Okubo-Zweig-Iizuka (OZI)-rule violations, except those mediated by the $U_A(1)$ anomaly enhanced $q\bar{q}$ transitions, can be neglected. Flavor symmetry breaking, on the other hand, is large and is to be taken into account in any mixing scheme for the pseudoscalar mesons.

Since mixing of the $0^+$ with the $0^-$ and $0^+$ is weak while $0^0$ mixing is strong it is appropriate to employ isoscalar and isovector combinations

$$= \frac{1}{2} (u\ s\ d); \quad (5)$$

as basis states instead of $u$ and $d$. The unitary matrix $U$ that transforms from the basis $f_1; f_2; s g$ to the physical meson states $f_{P_1} = 0^0; f_{P_2} = 0^0; f_{P_3} = 0^0$ can then be linearized in the $0^+$ and $0^0$ mixing angles. An appropriate parametrization of $U$ reads ($0^0/0^- 0^+$, 1)

$$0^1 + \cos \theta \quad \sin \theta \quad \sin \theta \quad \cos \theta \quad (6)$$

$$\U (; ; ) = \begin{bmatrix} 1 & \cos \sin \theta \sin \cos \theta \cos \sin \theta \end{bmatrix};$$

as the sector of $U$ is identical to the parametrization utilized in Ref. It is also of advantage to introduce isoscalar and isovector axial vector currents

$$J_5 = \frac{1}{2} (u\ s\ u\ d\ s) ; \quad (7)$$

The matrix elements $h_0 j_b^a j_b^d (p) i = (p) i$ then define new decay constants which can be collected in a decay matrix

$$F = \begin{bmatrix} 0 & f_1 & f_2 & 0 & 0 \end{bmatrix}; \quad (8)$$

In contrast to $F$ is non-diagonal. The decay constants $f_i = (f_u + f_d) / 2$ and $f_3$ are the basic decay constants in the $-0^+$ sector while the parameter $z = (f_u + f_d = (f_u + f_d)$ occurs in $0^- ; 0^0$ mixing; it is obviously of order $1$. The decay constants are in principle renormalization scale dependent. Ratios of

$^a$OZI-rule violations are of order $1 - N_c$ and become negligible in the large $N_c$ limit.
decay constants or mixing angles are, on the other hand, scale independent. Since
the anomalous dimension controlling the scale dependence of the decay constants is
of order \( \frac{4}{\alpha} \), this effect is tiny and discarded here. This simplification is consistent
with the neglect of OZI-rule violation.

Taking vacuum-particle matrix elements of the current divergences, one finds
with the help of (7) and (a, b, c, d) = (\( \frac{1}{2} \); \( \frac{1}{2} \); \( \frac{1}{2} \); \( \frac{1}{2} \))

\[ h_0 \mathcal{D}_a^b \mathcal{F}_{a_{imb}} = M^2_{\text{mass}} U_{a_{im}b} F_{b_{mn}} \]  

where \( M^2 = \text{diag}(M^2_{\text{ma}}, M^2_{\text{mb}}, M^2_{\text{mc}}, M^2_{\text{md}}) \) is the particle mass matrix which necessarily quadratic here. Next, I recall the operator relation

\[ \mathcal{D}_a^b = 2m_a i s_{a_{imb}} \]  

which holds as a consequence of the \( U_a \) (1) anomaly. The topological charge density
is given by \( \psi = 4 \mathcal{G} \mathcal{G} \), where \( \mathcal{G} \) denotes the gluon field strength tensor and \( \mathcal{G} \)
its dual. Inserting (9) into (8), and neglecting terms of order \( \frac{4}{\alpha} \), one obtains a set
of equations which can be solved for the mixing parameters

\[
\sin \theta = \frac{\langle H_\alpha^2 - m_{\text{ss}}^2 \rangle \langle H_\beta^2 - m_{\text{ss}}^2 \rangle}{\langle H_\alpha^2 - m_{\text{ss}}^2 \rangle \langle H_\beta^2 - m_{\text{ss}}^2 \rangle};
\]

\[
a^2 = \frac{p_2}{f_+} h_0 \mathcal{G}^j \mathcal{G}^i = \frac{\langle H_\alpha^2 - m_{\text{ss}}^2 \rangle \langle H_\beta^2 - m_{\text{ss}}^2 \rangle}{\langle H_\alpha^2 - m_{\text{ss}}^2 \rangle};
\]

\[
y = 2 - \frac{\langle H_\alpha^2 - m_{\text{ss}}^2 \rangle}{\langle H_\alpha^2 - m_{\text{ss}}^2 \rangle};
\]

and

\[
y = \frac{1}{2} \frac{m_{\text{dd}}^2 - m_{\text{uu}}^2}{M_{\text{ss}}^2 - M_{\text{ss}}^2} + z;
\]

\[
y = \frac{1}{2} \cos \frac{M_{\text{ss}}^2 - M_{\text{ss}}^2}{M_{\text{ss}}^2 - M_{\text{ss}}^2};
\]

The pion's mass and decay constant \( x \) to more parameters

\[
\frac{1}{2} \left( m_{\text{uu}}^2 + m_{\text{dd}}^2 \right) = M_{\text{ss}}^2; \quad f_x = f;
\]

up to corrections of order \( \frac{4}{\alpha} \). Finally, the symmetry of the mass matrix forces
relations between the decay constants and the matrix elements of the topological
charge density

\[
y = \frac{f_x}{f_s} = \frac{p_2}{h_0} \mathcal{G}^j \mathcal{G}^i; \quad z = \frac{f_x}{f_u + f_s} = \frac{h_0 \mathcal{G}^j \mathcal{G}^i}{h_0 \mathcal{G}^j \mathcal{G}^i};
\]

The quark mass terms in the above equations are defined as matrix elements of the pseudoscalar currents

\[
m_{\text{ss}}^2 = h_0 \mathcal{G}^j \mathcal{G}^i = f_x a_{imb}^i a_{imb}^j a_{imb}^i;
\]

The quark-mass mixing scheme can readily be extended to the case of the...
3. **$^0 m$ mixing**

The three relations taken from Ref.9 x the $^0 m$ mixing parameters for given masses of the physical mesons if the current algebra result:

$$m_{ss}^2 = 2M_k^2 M_0^2;$$

is adopted for the strange quark mass term. This theoretical estimate of the $^0 m$ mixing parameters provides the values quoted in Table 1. In order to take full account of flavor symmetry breaking a phenomenological determination of the mixing parameters has also been attempted in Ref.4 by using experimental data instead of the theoretical result. This analysis includes a number of processes like radiative transitions between light vector ($V$) and pseudoscalar ($P$) mesons or scattering processes like $p^0 \rightarrow Pn$, which all rely on the validity of the OZI rule. The $^0 m$ ratios of corresponding processes provide the mixing angle. Other dynamical effects are in general expected to cancel in the ratios, only space corrections have to be considered. An example is set by the radiative decays of the $^0 m$ meson. These decays proceed through the emission of the photon from the strange quark and a subsequent $ss$ transition into the $s$ component, see Fig.1. Hence, the ratio of and $^0 m$ decay widths reads

$$\frac{[!]^0}{[!]^0} = \cot^2 \frac{k_i}{k_i} \frac{k_i}{k_i} \frac{3}{1} \frac{f_2}{f_2} \frac{\tan \frac{1}{2}}{\sin 2};$$

where

$$k_i = \frac{1}{2} \frac{M_i^2 + M_s^2}{M_s^2} \frac{M_i^2 \frac{M_i^2}{M_i^2} \frac{M_i^2}{M_i^2}}{M_i^2};$$

is the three momentum of the final state particles in the rest frame of the decaying meson. A small correction due to vector meson mixing has to be taken into account here; the $^0 m$ mixing angle amounts to $31^\circ$. The PDG average for the ratio leads to a value of $41.5 \pm 2.2$ for the mixing angle. A new preliminary result on that ratio from KLOE provides an angle with an even smaller error namely $41.2 \pm 1.1$. Both these values have not been used in.

$^b$Note that the theoretical estimate presented here differs slightly from the one presented in Ref.4 where an additional constraint on $f_0$ has been employed.
the determination of the phenomenological mixing parameters quoted in Tab. 1. Other reactions like the two-photon decays of the $J=1$ and $J=0$ for the photon-meson transition form factors are sensitive to the decay constants. Thus, one finds for the two-photon width from PCAC

$$
\Gamma_{\gamma\gamma}(J=1) = \frac{2}{144} M^3 \left( \frac{5 P_2 \cos f_s}{f_s} \right) \text{ and } \Gamma_{\gamma\gamma}(J=0) = \frac{2}{144} M^3 \left( \frac{5 P_2 \sin f_s}{f_s} \right)
$$

and for the transition form factor to next-to-leading order (NLO) of perturbative QCD and leading-twist accuracy (in the MS scheme)

$$
F_{\gamma\gamma}^{J=1} = \frac{P_2 f_0}{Q^2} \left( \frac{1}{3} \right) \text{ and } F_{\gamma\gamma}^{J=0} = \frac{P_2 f_0}{Q^2} \left( \frac{1}{3} \right)
$$

where the effective decay constants read ($f_0^e = f_0$)

$$
f_0^e = (5f_s \cos f_0^e) \frac{P_2 f_0}{Z f_0 \sin f_0^e} \text{ and } f_0^e = (5f_s \sin f_0^e + P_2 f_0 \cos f_0^e) = 3.
$$

These processes provide another interesting piece of information. According to Novikov et al., the photon is here emitted from the charm quarks which subsequently annihilate into lighter quark pairs through the effect of the $U_A(1)$ anomaly (see Fig. 1). This mechanism leads to the following result for the ratio of the $J=1$ decay widths

$$
\frac{\Gamma_{\gamma\gamma}(J=1)!}{\Gamma_{\gamma\gamma}(J=0)!} = \frac{h_{\gamma j} j_{1}^{0} k_{j_{1}^{0}}}{h_{\gamma j} j_{1}^{0} k_{j_{1}^{0}}} = \left( \frac{M_2}{M_0} \right)^3
$$

Using Eqs. (22) and (23) one can express the ratio of the vacuum–particle matrix elements of $j_{1}^{0}$ by the mixing angle

$$
\frac{h_{\gamma j} j_{1}^{0}}{h_{\gamma j} j_{1}^{0}} = \tan \frac{M_2}{M_0} \frac{M_2}{M_0}
$$

These are examples of OZI-rule violating decays mediated by the $U_A(1)$ anomaly.
a relation that has been derived by Bal et al. independently on the quark-avoir mixing scheme. From the PDG averages for the radiative $J=0$ decays one obtains $39.0 \pm 1.6$.

The analysis of these reactions and a number of others leads to the set of phenomenological mixing parameters quoted in Table 1. These values absorb corrections from higher orders of avoir symmetry breaking. As the comparison with the theoretical results reveals these effects are not large, they are on the level.

4. The octet-singlet basis

Transforming from the quark-avoir basis to the SU(3)$_F$ octet-singlet one by an appropriate orthogonal matrix, one easily notices that the decay matrix in the octet-singlet basis has a more complex structure than which is diagonal in the sector. In addition to the state mixing angle which is diagonal in the sector, the mixing angle is arctan $\frac{P}{2}$, two more angles, $\theta$ and $\phi$, are needed in order to parameterize the weak decays of a pseudoscalar meson through either the action of a singlet or an octet axialvector current

$$f^0 = f_0 \cos \theta; \quad f^+ = f_0 \sin \theta; \quad f^0 = f_1 \cos \phi; \quad f^+ = f_1 \sin \phi.$$ (24)

The mesons may also decay through strange and non-strange axialvector currents. The corresponding decay constants can be parameterized in a similar fashion

$$f^+ = f_s \cos \theta; \quad f^0 = f_s \sin \theta; \quad f^+ = f_0 \cos \phi; \quad f^0 = f_0 \sin \phi.$$ (25)

In the quark-avoir mixing scheme described in the preceding sections, these angles are assumed fall together, $\theta = \phi$, and, hence, the decay constants follow the pattern of state mixing. For the sake of comparison I will keep the three different angles for the remainder of this section. The two sets of decay constants are related to each other by

$$f_0 = \frac{1}{3} \left( f^0 + 2f^0 \right) = \frac{2f^0}{2f^0 + 2f^0} f_0 \sin (\theta + \phi); \quad f_1 = \frac{1}{3} \left( f^+ + 2f^+ \right) = \frac{2f^+}{2f^+ + 2f^+} f_1 \sin (\theta + \phi);$$ (26)

and

$$\tan \theta = \frac{f_s \sin \theta}{f_s \cos \theta}; \quad \tan \phi = \frac{f_0 \sin \phi}{f_0 \cos \phi};$$ (27)

In the quark-avoir mixing scheme these relations simplify drastically:

$$\theta = \arctan \left( \frac{2f_0}{2f_0}; \quad \phi = \arctan \left( \frac{2f_0}{2f_0}; \right).$$ (28)
m\textit{ixing angle}, \(\theta\), substantially. The angle \(\theta\) is smaller than \(\phi\) if \(\phi > f\). Only in the avor symmetry limit, i.e. if \(f_0 = f\), the three angles fall together.

Evaluating the \(m\textit{ixing} parameters in the octet-singlet basis from the theoretical and phenomenological sets of \(m\textit{ixing} param\text{eters} compiled in Table 1, one obtains the results shown in Table 2. Indeed the three \(m\textit{ixing angles occurring in the} octet-singlet basis, differ markedly. At the best \(m\textit{ixing} is simple in the quark-avor basis which is favored because of the smallness of those OZI-rule violations which are not induced by the \(U_A(1)\) anomaly. On the other hand, \(SU(3)\) breaking is large and cannot be ignored.

It should be clear from the above discussion that any \(m\textit{ixing} scheme which takes into account the \(U_A(1)\) anomaly and includes the proper masses of the physical mesons will lead to similar results for the \(m\textit{ixing} parameters provided different values of the \(m\textit{ixing} angles, \(\theta\), and \(\phi\), are allowed for. This assertion is indeed confirmed by the results found in many papers on this subject, for a detailed comparison of various \(m\textit{ixing} schemes it is referred to the review . With regard to the limited space available for this work I refrain from repeating that and concentrate on the comparison with recent work in which particular attention is paid to the issue of the diverse \(m\textit{ixing} angles.

| \(\theta\) | \(\phi\) | \(f_0-f\) | \(f_1-f\) |
|---|---|---|---|
| 133\(^\circ\) | 194\(^\circ\) | 6\(^\circ\) | 1.19 | 1.10 |
| 153\(^\circ\) | 212\(^\circ\) | 92\(^\circ\) | 126 | 1.17 |
| 1\(^\circ\) | 1\(^\circ\) | 0.06 | 0.04 |
| 20\(^\circ\) | 4 | 1.28 | 1.10 |
| 10\(^\circ\) | 1| 1.31 | 1.24 |
| 23\(^\circ\) | 2\(^\circ\) | 1.51 | 1.29 |

Results from two NLO chiral perturbation theory analyses are listed in Table 2 in which the \(0\) meson acts as a \(n\) Goldstone boson in this limit. For \(J=0\) the experimental widths of the two-photon decays into \(E\) and \(0\) are used as input in addition to the masses and decay constants of the physical mesons, the averages of their three results are quoted. In general fair agreement of the results is to be seen. Only the state \(m\textit{ixing} angle\(,\) quoted in Ref. 16, seems to be too small in magnitude, it corresponds to \(\phi = 442\) with an uncertainty of about 1.5 which is rather large as compared with the above quoted values obtained from radiative and \(J=0\) decays.

DeFelice and Pennington calculated the decay constants \(f_0^{++}\) from QCD sum
rules. The octet-singlet mixing parameters, evaluated through Eqs. (26) and (27), differ quite a bit from the results obtained within the quark-avormixing scheme. The errors in the sum rule analysis are however large. They typically amount to about 4 \% for the mixing angles. Despite the large errors the result on \( \theta \) is in conflict with the experimental value on the ratio \( \theta \). Since the ratio of the matrix elements can also be expressed as

\[
\frac{h_0^j j^0_i}{h_0^j j^i} = \cot \theta; \tag{29}
\]

experiments leads to \( \theta \approx 22^\circ \), 11 directly.

Escrivano and Freire performed a phenomenological analysis of a subset of the data explained in Refs. \( \gamma \), \( \beta \), \( \gamma \), and \( \gamma \), the vector mesons include the \( J=0^+ \) states. They also provide clear evidence for substantially different values of \( \theta \) and \( \phi \), see Table 1. The large value of \( \theta \) along with a large \( \phi \) they obtained, seen \( s \) to be forced by the \( J=0^+ \) with \( \phi \), which has not been included in the analyses of Refs. \( \gamma \), \( \beta \), \( \gamma \), \( \beta \). The angle \( \phi \) exhibits the largest uncertainties of the mixing parameters since none of the present data is really sensitive to it. Helpful in pinning down its value would be more complete data on the \( P \) transition form factors or the \( j^0 g^1 \) ! \( P \) form factors, which may for instance be obtained from the process \( pp \rightarrow jet+jet+P \) in the central rapidity region.

Inspection of Table 1 reveals strong avor symmetry breaking effects as characterized by the large values of \( j(\theta, \phi) = (\theta + \phi) \). In contrast to this the ratio \( \gamma = \phi/j(\theta, \phi) \), which can be evaluated from the information given in Tab. 1 with the help of the inverse of Eqs. (26) and (27) is tiny; the values lie in the range 0.01 - 0.04 and, provided errors are available, are compatible with zero. In this context that the validity of the OZI rule is a prerequisite for the existence of process-independent mixing parameters.

5. Isospin symmetry breaking

Having discussed \( ^0 \) mixing in some detail and shown that \( ^0 \) mixing is well understood, let me now turn to ISB induced by \( ^0 \) mixing. Defining the isospin-zero admixtures to the \( ^0 \) by

\[
j^0_i = j^0 i + j^i + 0^0 i + 0^0 (\theta \); \tag{30}
\]

\( ^0 \) admixture of the mixing angles in the analysis of the P form factor leads to \( j(\theta, \phi) = (\theta + \phi) \).
one finds with the help of the matrix

\[ \begin{align*}
\mathbf{f}_0 &= \mathbf{f}_1 \\
0 &= \sin \frac{1}{2} \left( \frac{m_{dd}^2 - m_{uu}^2}{2 M_{M}^2} + z \right)
\end{align*} \]

The \( f_0 = f_1 \) limits of these results, termed \( ^{\dagger} = (z = 0) \) and \( ^{0} = 0(z = 0) \) in the following, coincide with the results reported in Ref. 3. The numerical value of the quark mass term in the matrix may be estimated from the \( K^{0} - \bar{K}^{0} \) mass difference corrected for mass contributions of electromagnetic origin

\[ m_{dd}^2 - m_{uu}^2 = 2 M_{M}^2 \left( M_{K}^2 - M_{M}^2 \right); \]

A ccording to Dashen’s theorem \( ^{\dagger} \), the electromagnetic correction is given by

\[ \begin{align*}
M_{K}^{2\text{ em}} &= M_{K}^{2} - M_{M}^{2} - \left( M_{K}^{2} - M_{M}^{2} \right) \sin \theta \cos \theta \\
M_{K}^{2\text{ em}} &= M_{K}^{2} - M_{M}^{2} - \left( M_{K}^{2} - M_{M}^{2} \right) \sin \theta \cos \theta ;
\end{align*} \]

in the chiral limit and amounts to \( 1.3 \times 10^{-6} \text{ GeV}^2 \). This leads to \( M_{dd}^2 = 0.0104 \text{ GeV}^2 \). However, quark masses increase \( M_{K}^{2\text{ em}} \) substantially. The exact size of this enhancement is subject to controversy. Different authors have obtained rather different values for the electromagnetic mass splitting of the \( K \) mesons. For an estimate of the \( z = 0 \) values of the \( ^{\dagger} \) and \( ^{0} \) admixtures to the \( ^{0} \), I take the average of the results for \( M_{K}^{2\text{ em}} \) quoted in Ref. 3 (24.07) \( 10^{-6} \text{ GeV}^2 \). This way I obtain

\[ ^{\dagger} = (z = 0) = 0.0170 \quad 0.020; \quad ^{0} = 0(z = 0) = 0.0040 \quad 0.001 ; \]

Due to the electromagnetic mass corrections the value for \( ^{\dagger} \) is larger than that one quoted in Ref. 3.

It is elucidating to turn the masses occurring \( ^{0} \) into quark masses. With the help of the Gell-Mann-Oakes-Renner relation and the use of Dashen’s theorem one nds

\[ \begin{align*}
^{\dagger} &= \frac{p}{3} \cos \theta ;
\end{align*} \]

with \( ^{0} \) needed in \( ^{0} \). The additional factor of \( \frac{p}{3} \cos \theta = \frac{13}{15} \) would be unity if \( ^{0} \) is real, i.e., if the physical and \( ^{0} \) mesons are pure \( \bar{u}u \) and \( \bar{d}d \) states, respectively. In this case there is no \( ^{0} \) mixing, i.e., \( ^{0} = 0 \). We now see that the small value of \( ^{0} \) obtained by Gross, Talmi and Wilczek is a consequence of the disregard of \( ^{0} \) mixing and the use of Dashen’s results for the electromagnetic contribution to the kaon mass splitting.

Chao has also investigated \( ^{0} \) mass mixing on exploiting the axial anomaly, but he has used the PCAC hypothesis instead of diagonalizing the mass matrix. Despite his assumption of equal mixing angles, \( ^{1} = ^{0} = ^{0} \), a supposition that has

\[ \begin{align*}
\text{The QCD contribution to the} \quad ^{0} \text{ mass difference} \quad \text{is} / \quad \text{and amounts to about} \quad 0.2 \text{ MeV} \quad \text{which is much smaller than the experimental value of} \quad 4.6 \text{ MeV}.
\end{align*} \]
been shown to be inadequate and theoretically inconsistent (see Sect. 4), his results on and \( \theta \) agree with the \( \pi \) values within errors. NLO chiral perturbation theory in the large \( N_c \) limit leads to a slightly smaller value for the mixing angle \( \angle \theta \approx 0.1 \) than 

If the decay constants \( f_a \) and \( f_b \) differ from each other the mixing angles may deviate from the values quoted in substantially. This potentially large effect is a source of considerable theoretical uncertainty of our understanding of ISB in the pseudoscalar meson sector. In the absence of theoretical estimates for \( \theta \) one has to rely on phenomenological estimates of its value. Even this difficult will turn out in Sect. 7.

6. The vacuum - \( \theta \) matrix element of the topological charge density

ISB as a consequence of \( \theta \) mixing is accompanied by a non-zero vacuum - \( \theta \) matrix element of\( ! \). From the mixing formulas derived in Sect. 2, one readily finds

\[
\hbar_0^j j^0 i = \frac{\chi}{\cos M^2_s} \frac{M^2}{M^2_s} \hbar_0^j j^0 i \tag{36}
\]

Any \( \pi \) dependence cancels in this matrix element. As an immediate consequence of the pion is contaminated by strange quarks

\[
\hbar_0^j m_\pi s i \bar{s} j^0 i = \frac{m^2}{m^2_s} \hbar_0^j j^0 i \tag{37}
\]

This result implies a tiny violation of the OZI rule through the anomaly. As a consequence the \( \theta \) decay through the strange axial-vector current with a decay constant defined by \( \hbar_0^j J^0 i = f_\pi M^2_s \). Using the operator relation and Eqs. one obtains for this decay constant

\[
f_\pi = \frac{\hbar_0^j j^0 i}{m^2_s M^2} \tag{38}
\]

It is very small, about \( 5 \times 10^{-6} \), and will likely have no experimental consequences. For numerical estimates of the quantities just introduced one may use and the mixing parameters quoted in the preceding sections.

The decays \( (2S)! J = \frac{0}{3} \) allow for an immediate test of the prediction. As has been suggested in Ref. these decays are dominated by a mechanism where the pseudoscalar meson is coupled to the \( \pi \) system through the effect of the anomaly. Hence, the \( \theta = \) ratio of these processes is given by

\[
\left( \frac{(2S)! J = \frac{0}{3}}{(2S)! J = \frac{\pi}{\theta}} \right) = \left( \frac{h_0^j j^0 i}{h_0^j j i} \right)^2 \left( \frac{k_1 (2S)! J = \frac{0}{3}}{k_1 (2S)! J = \frac{\pi}{\theta}} \right)^3 \tag{39}
\]

in analogy to . The ratio of the gluonic \( \theta \) matrix elements follows from and results presented in Sect. 2. It reads

\[
\frac{h_0^j j^0 i}{h_0^j j i} = \frac{\chi}{\cos^2} \tag{40}
\]
It is to be stressed that Eq. (40) includes $^0{}^0$ mixing. It is responsible for the factor $1 = \cos^2 \theta$ which amounts to about 1.7. This is so because the small mixing angle $^0\theta$ is enhanced by the large matrix element $h_0^j j_0^i$, etc. The importance of $^0{}^0$ mixing in Eq. (40) has been pointed out by Genz long time ago. Using the recent, rather accurate BES measurement of the ratio, one extracts the mixing angle

$$\Delta \{ (2S) \uparrow J = P \} = 0.030 \quad 0.002;$$

which is large as compared to the theoretical estimate. The origin of this discrepancy is not understood. Neglected electromagnetic contributions to the transitions $(2S) \downarrow J = P$ are likely not the reason. A possible explanation could be that the coupling of the pseudoscalar mesons to the $cc$ system is not or only partially controlled by the anomaly. This possibility has been suggested in Ref. 25 and discussed within the framework of the heavy quark effective theory. Since the strength of this new mechanism has not been predicted in Ref. 25, its bearing on the determination of $\Delta$ cannot be estimated. A better understanding of the $(2S) \downarrow J = P$ decay may also affect the determination of quark masses.

One may wonder whether the isospin-violating radiative decay of the $J = 1$ into the $^0{}^0$ is also under control of the $U_A (1)$ anomaly. This is, however, not the case in contrast to the $\downarrow$ and $^0$ channels which are well described by the anomaly contribution. An estimate of the $J = \uparrow$ $^0{}^0$ decay width using the analogue of (3) provides values that are too small by more than an order of magnitude. In fact the radiative $J = \uparrow$ transition into the $^0{}^0$ is mediated by the higher order electromagnetic process $cc \uparrow \downarrow$ and by a vector meson dominance contribution $J = \uparrow$ $^0{}^0$.

7. Phenomenological determination of the mixing angles

In this section ISB-violating processes which are not under control of particle-vacuum matrix elements, will be discussed. The full mixing angle and not just its $z = 0$ value, occurs under these circumstances. Care is to be taken that only ISB within QCD are analyzed, eventual electromagnetic effects have to be subtracted.

Clear signals for ISB and/or charge symmetry breaking have been observed in a number of hadronic reactions. The extraction of the mixing angle from these data is however di cult and model dependent. The ratio of the $+d \uparrow N N$ and $-d \uparrow N N$ deviates from unity, the charge symmetry result, experimentally on the basis of a rather simple model that includes $^0{}^0$ mixing and a number of corrections which take into account effects such as differences in the meson-nucleon coupling constants or the proton-neutron mass difference but ignores mixing with the $^0{}^0$, the authors of Ref. 26 extracted a $^0{}^0$ mixing angle of

$$[ -d \uparrow N N ] = 0.026 \quad 0.007;$$

from their data. The non-zero forward-backward asymmetry in np $^0{}^0$ measured at TRIUMF is in conflict with charge symmetry. The phenomenological analy-
sis of this data suffers from large ambiguities. A combination of contributions from $^0$ mixing, from the $u$ and $d$ quark mass difference and from electromagnetic effects in the nucleon mass difference, controls the asymmetry. A further complication arises from the fact that the mixing contribution is in fact given by a product of the $^0$ mixing angle and the badly known nucleon coupling constant $\alpha$. This complicated situation prevents an extraction of the mixing angle with a significant accuracy. But the measured asymmetry in $np!d^0$ is compatible with the present knowledge of the various quantities occurring in the theoretical result. The cross section data for $d!^4He$ have not yet completely been analyzed theoretically, while the COSY measurement of the ratio of the $pd!^3He$ and $pd!^3He^0$ cross sections provides only a very weak signal for ISB. More precise data on the latter cross sections are required before one can draw a definite conclusion here.

The $^0$ and $^0$ decays into $^3S_1$ violate G-parity and hence isospin symmetry. Since electromagnetic contributions are strongly suppressed, ISB is here mainly of QCD origin and can be estimated through $^0$ mixing. Recent analyses of the decays within NLO chiral perturbation theory can be summarized as

$$[^0!^0] = \frac{2}{\alpha};$$

(43)

where $^0$ is not considered as an explicit degree of freedom in the effective Lagrangian. Its effect is, however, partially embodied in the effective coupling constants of the theory. The NLO term of chiral perturbation theory in which a number of low energy processes can be calculated rather precisely, comprises contributions to ISB of various origin. Hence, the comparison with results from the quark–av morning scheme which allows for an investigation of many processes at low and high energies, is not straightforward.

The ratio of the $^0!^3S_1$ and $^0!^0$ decay widths can be written as

$$[^0!^0] = \frac{1}{\alpha};$$

(45)

if electromagnetic contributions can be neglected. Con et al. estimated the factor $R$ relying on PCAC ideas and taking into account the experimentally observed mild dependence of the process amplitudes on kinematics. Their result on $R$ combined with the experimental value of $(7.5 \pm 1.3) \times 10^{-3}$ for the ratio of the decay widths leads to

$$[^0!^0] = 0.021 \pm 0.002;$$

(46)
More accurate experimental data and a revision of the theoretical analysis is advisable. A calculation of the $^6\Omega$ decays within the framework of chiral perturbation theory is not available.

There are several $J^P = 0^-$ decays into vector and pseudoscalar mesons which violate isospin symmetry, e.g., $J^P = 0^-$, $0^+$, $0^-$, $0^+$. As compared to the corresponding isospin allowed decays, $J^P = 0^+$, $0^-$, $0^+$, they are typically suppressed in experiment by a factor of about $10^{1} - 10^{2}$ in accord with expectations for an electromagnetic decay mechanism. Contributions from ISB mechanisms within QCD are negligible small. Indeed, within the $0^-$ mixing scenario, ratios like $[J^P = 0^-] / [J^P = 0^+]$ would be proportional to $10^{-1}$ and would therefore amount to only $10^{-4} - 10^{-3}$. As is the case for the process $J^P = 0^-$ the contribution from $0^-$ mixing to the radiative decay of the $^6\Omega$ meson into the $^6\Omega$ is negligible small; it involves the $s$ component of the $0^-$ (see Fig. 1) which is proportional to $10^{-5}$. The process $J^P = 0^-$ proceeds through mechanisms similar to those occurring in $J^P = 0^+$.

Recently a new $s$-meson, $D_{sJ}^*(2317)$, has been observed. Its experimental properties are consistent with a $J^P = 0^-$ state interpretation. It decays into $D_{sJ}^*0^-$, no other decay channel have been observed as yet. This ISB decay likely proceeds through the production of the $s$ component of the pion and consequently its width is $\Delta$, see Fig. 2. Estimates of the decay width using chiral perturbation theory and the heavy quark effective theory provides values way below the present experimental upper bound for the total $D_{sJ}^*(2317)$ width. A second new $s$-meson which also decays into the isospin symmetry violating channel $D_{sJ}^*(2457)!D_{sJ}^*0^-$ has been observed too. Once the decay widths of these two processes will be measured more precisely the mixing picture of ISB can again be probed.

Isospin violations within QCD are also of relevance in CP-violating processes whose analyses are intricate and do not allow a simple extraction of the mixing angles. An example is set by the $K^0 \rightarrow \pi^0 \nu \bar{\nu}$ decay. In a recent analysis [25] of this process within NLO chiral perturbation theory the values for the mixing angle have been used. The above comments on the comparison of results from this theory with others apply here as well. Another example is the $\pi^- \rightarrow e^- \nu \bar{\nu}$ decay in weak decays is the process $B^+ \rightarrow \pi^- \nu \bar{\nu}$. ISB spoils the triangle relation obeyed by the amplitudes for the three processes $B^+ \rightarrow \pi^- \nu \bar{\nu}$, $B^0 \rightarrow \pi^- \nu \bar{\nu}$ and $B^0 \rightarrow \pi^- \nu \bar{\nu}$, $B^0 \rightarrow \pi^- \nu \bar{\nu}$ in isospin symmetry limit it and consequently affects the determination of the CKM angle $\theta_{13}$ estimated by Gardner [26]. The mixing angles lead to a correction of $\theta_{13}$ which is substantially but still smaller than the error of the present experimental result: $\theta_{13} = 100 - 130$.

8. Summary

A detailed theoretical and phenomenological analysis showed that the quark-avor mixing scheme provides a consistent description of the $0^-$ mixing. On exploiting the

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divergencies of the axial vector currents all basis mixing parameters can be determined for given mass of the physical mesons. It turned out that arrow symmetry breaking manifests itself differently in the mixing properties of states and decay constants, their parametrisation requires three different mixing angles in general. Only in the quark-arrow basis the three angles fall together approximately and mixing is simple. Other approaches to mixing lead to similar results provided the $U_A(1)$ anomaly and the mass of the physical mesons are taken into account.

Inclusion of the $^0$ into the quark-arrow mixing scheme induces ISB of about 2% on the amplitude level. Extraction of the $^0$ mixing angle from experiment, on the other hand, provides values in the range $0.02 - 0.03$. Thus, the exact magnitude of ISB through mixing is not yet determined and more work is clearly needed. There are other sources of ISB within QCD besides mixing. Their estimate is however difficult and often not or only partially taken into account in analyses. One of these sources of ISB is a possible difference between the basic decay constants $f_u$ and $f_d$ or, respectively, the parametrisation in Eq. There is no clear evidence for a non-zero value of $z$. One may only conclude that $z$ is smaller than about 0.015. This entails a difference between $f_u$ and $f_d$ of less than 3%.

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