Wiring up single electron traps to perform quantum gates

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\textbf{Abstract.} Two electrons in planar Penning traps can be coupled by connecting the trap electrodes with a metallic wire. The wire provides a capacitive and resistive link between the image charges that the electron oscillation (axial motion) is inducing in the electrodes. This can be operated as a quantum network to transfer excitation quanta between the traps and to entangle the two electrons. We give a detailed analysis of this system in the dispersive limit and assess the impact of resistive thermal noise. The latter is found to be remarkably small at temperatures below 4 K in miniaturized planar traps (sizes below 1 mm). The master equation for the two-electron system in the presence of resistive noise is solved exactly. We compute fidelities for basic quantum gates needed in quantum information processing.
1. Introduction

The manipulation and control of quantum wavefunctions are, for many physical systems, very active fields of research. One of the foreseen applications is the development of computing architectures, where scalability is a desired property. To conform with scalability, the elemental quantum units constituting the whole system must be interconnected. This interconnection is needed to perform quantum computing operations and to transport information. Some of the physical implementations that promise to fulfill this requirement are: traps for ions, atoms, quantum dots and Josephson junctions. Besides these systems, electrons are candidates to implement a quantum architecture. Charged particles confined in Penning traps behave like atoms with typical characteristic frequencies lying in the radio and microwave spectra [1, 2]. These traps have been used, for example, to measure with high precision the electron’s g-factor [2]. Due to the atom-like properties of the trapped charge, it has been proposed that confined electrons in Penning traps can implement quantum gates [3]–[6]. This is the scope of our paper: the quantum manipulation of electrons in Penning traps. Several configurations of electron trap arrays have been proposed such as a cylindrical chain of electrodes [3, 4], a set of wires acting as an electrode [7], or an array of planar ring electrodes [5]. In the two former configurations, the neighboring electrons can be coupled by direct Coulomb interaction. In the planar trap, an electrical transmission line connecting two traps serves as the interaction channel between the confined electrons (see figure 1). Among the advantages of these ‘wired’ planar traps are the ease of fabrication and the possibility that the transmission line provides a stronger electron–electron interaction than the direct Coulomb interaction [8]. This leads to a faster operation of quantum gates. Also, the interaction can be set up between any two traps in the planar array, not necessarily between
neighboring traps. Herein, we focus on this scheme and consider in particular the impact of losses on the coherency of the interaction.

Josephson junctions and Rydberg atoms also exhibit transition frequencies lying in the microwave spectrum. The use of electrical transmission lines as interconnecting channels for these quantum systems has been proposed previously [8]–[11], assuming ideal and lossless connections between the qubit carrier, typically superconducting lines or cavities. Recently, transfer of quantum signals between solid-state qubits via a resonant superconducting transmission line has been achieved experimentally [12, 13]. In this paper, we consider that the transmission line is made of a conventional conductor and the characteristic frequencies of the trapped electrons are far from the resonant frequencies of the line, so the electrons are weakly coupled to each other. This setting is similar to having two atoms close to each other in free space. In principle, a regime of coherent interaction can be attained (dipole–dipole coupling), but due to the existence of many channels, dissipation takes over at interatomic separations larger than a few nanometres (for atomic transitions in the visible range). For our wired traps, this is overcome since the interaction is restricted to only one channel. Herein, we show that in the dispersive regime, conventional conductors, despite their intrinsic thermal resistive noise, do not necessarily degrade the coherent dynamics of the confined electrons on the timescale relevant for typical quantum manipulations. The relevant timescales are identified by solving exactly the master equation for the damped two-electron system, which also gives the fidelity for two generic quantum gates. Our approach can be applied as well to other ‘wired’ physical systems.

The paper is organized as follows. In section 2, we present the Hamiltonian that describes the interaction of the axial motion of the confined electron in the traps which are connected by a lossy transmission line. Each of the electrons couples capacitively to the line. Section 3 calculates the strength of the different couplings for realistic parameters of the line. The strength of these coupling parameters determines the reduced density matrix for the two harmonic oscillators for which a master equation is presented in section 4. The fidelity for a swap gate and a square-root swap gate using the Fock states of the two-oscillator system is calculated in section 5. This section also discusses gate speed and decoherence rate. Finally, conclusions are presented in section 6. The appendices contain details of the image-charge coupling between the axial electron motion and the transmission line (appendix A), the line’s Green function (appendix B), and the solution of the master equation (appendix C).

2. Hamiltonian

In a Penning trap, charged particles are confined by static magnetic and electric fields. A homogeneous magnetic field provides lateral confinement, whereas a spatially inhomogeneous electric field provides the axial confinement. This electric field is generated by static potentials applied to the electrodes. The motion of a single electron in a Penning trap can be described in terms of three harmonic oscillators [1]. These harmonic motions are referred as magnetronic, cyclotronic and axial. In this paper, we restrict ourselves to the axial motion.

We consider two traps connected by a transmission line as depicted in figure 1. Each trap stores a single electron. The Hamiltonian that describes the dynamics of the electrons interacting via the transmission line is

\[ \hat{H} = \hat{H}_a + \hat{H}_l + \hat{H}_{\text{int}}. \]

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\(\hat{H}_a\) represents the axial harmonic motion of the two trapped electrons with (angular) trap frequencies \(\omega_1\) and \(\omega_2\). The operator for the displacement along the axial direction is given by

\[
\hat{y}_m = \sqrt{\frac{\hbar}{2\omega_m m_e}} (\hat{b}_m + \hat{b}_m^\dagger), \quad m = 1, 2,
\]

(2)

where \(\hbar\) is the reduced Planck’s constant, \(m_e\) the electron mass and \(\hat{b}_m\) (\(\hat{b}_m^\dagger\)) the annihilation (creation) operator for the \(m\)th electron. \(\hat{H}_{tl}\) is the Hamiltonian of the transmission line. We take the dissipation from the non-perfect conductivity of the wires into account and get a collection of harmonic oscillators:

\[
\hat{H}_{tl} \equiv \int_0^d \int_0^\infty \hbar \omega \hat{\phi}^\dagger(z, \omega) \hat{\phi}(z, \omega) \, dz \, d\omega.
\]

(3)

Here, \(z\) is the coordinate along the line, and \(\hat{\phi}(z, \omega)\) and \(\hat{\phi}^\dagger(z, \omega)\) are bosonic fields that describe the electrical signals at frequency \(\omega\) and position \(z\) along the lossy transmission line [14]. They satisfy the commutation rule

\[
\left[\hat{\phi}(z, \omega), \hat{\phi}^\dagger(z', \omega')\right] = \delta(z - z') \delta(\omega - \omega').
\]

(4)

In both traps, the electron motion induces a charge \(Q_{1,2}\) on the electrode. This charge induces an electrical signal that propagates through the transmission line and influences the motion of the electron at the other end and vice versa. The explicit expression on \(H_{int}\) depends on the particular trap configuration, and we outline a derivation in appendix A for the planar Penning trap described in [5]. In this configuration and by assuming small-amplitude oscillations around the trap center, the interaction Hamiltonian \(\hat{H}_{int}\) takes the form

\[
\hat{H}_{int} = -\sum_{m=1}^2 \frac{\alpha(\tilde{y}_m)}{a_m C_m} e \hat{y}_m \hat{Q}_m, \quad m = 1, 2,
\]

(5)

where \(e\) is the charge of the electron, \(C_m\) the capacitance for the \(m\)th planar trap, \(\tilde{y}_m\) the equilibrium position of the \(m\)th trapped electron and \(\alpha(\tilde{y}_m)\) a geometrical factor defined.
Typical values are of the order of \( \alpha(\bar{y}_m) \lesssim 1 \) if the distance \( \bar{y}_m \) is comparable with the radius \( a_m \) of the central electrode. The charges on the planar electrodes belong to the attached transmission line and can therefore be expressed as \( \hat{Q}_{1,2} = \hat{Q}(z_{1,2}) \) [\( z_1 = 0 \) and \( z_2 = d \)]. In terms of the fundamental operator \( \hat{\phi}(x, \omega) \), the charge operator \( \hat{Q}(z) \) at the line is

\[
\hat{Q}(z) = \int_0^\infty \hat{q}(z, \omega) \, d\omega + \text{h.c.},
\]

with the definition

\[
\hat{q}(z, \omega) \equiv \sqrt{\frac{\hbar \omega}{\pi}} C \int_0^d G(z, x, \omega) \hat{\phi}(x, \omega) \, dx.
\]

Here, ‘h.c.’ denotes the Hermitian conjugate, \( C \) \(( R \) is the capacitance \(( \text{resistance} \) per unit length of the transmission line. \( G(z, x, \omega) \) is the Green function of the line that fulfills the boundary conditions at the ends of the line and the equation

\[
\frac{\partial^2}{\partial z^2} G(z, x, \omega) + \beta^2(\omega) G(z, x, \omega) = -\delta(z - x).
\]

Here, \( \beta^2(\omega) \equiv \omega C (L + i\omega R) \) where \( L \) is the inductance per unit length. The explicit expression for the Green function is given in appendix \( B \). Note that the electrode capacitances \( C_m \) enter the equations of motion of the transmission line via the load impedances \( Z_m(\omega) = i/(\omega C_m) \). (Here and in the following, we assume a time dependence \( \exp(-i\omega t) \).)

3. Coupling two traps

3.1. Typical parameters

Herein, we consider electrical and geometric parameters of the traps and the transmission line that can be realistically attained. The quantities of interest, such as the effective coupling strength between the two electrons and the dissipation rate, depend on these global parameters as we will discuss later. A prototype of an array of planar traps is sketched in figure 2.

The planar trap has a circular electrode with radius \( a \) surrounded by coplanar concentric rings (figure 2). We consider that the electrodes have a thickness \( h \) and the gaps between them (defined by etchings) have width \( s \). Using the formula for a planar capacitor, the capacitance of the single trap can be estimated as

\[
C_o = \varepsilon_o \frac{2\pi ah}{s},
\]

where \( \varepsilon_o \) is the vacuum permittivity. Taking the values given in the figure caption for the electrode thickness and the gap width, the typical capacitance for the circular electrode of the \( n \)th planar trap (radius \( a_m \)) is \( C_m = 0.225 \, (a_m \, \text{mm}^{-1}) \) pF. Hereafter, we assume for simplicity that the two traps are constructed and biased equally. Thus, \( a_1 = a_2 = a \), and \( C_1 = C_2 = C_o \) and we write \( \omega_o, \bar{y}, \alpha(\bar{y}) \) for trap frequency, equilibrium position and geometrical factor, respectively.

To connect any two traps in the array of figure 2, we consider a pair of gold wires bonded to pads that lead to the trap electrodes. With radius \( r_o = 5 \mu m \) and separation \( b = 0.5 \text{mm} \), the wire pair has a capacitance per unit length \( C = \pi \varepsilon_o / \ln[b/r_o] \approx 6.04 \text{pF mm}^{-1} \) and an inductance per length \( L = \mu_o \ln[b/r_o]/\pi \approx 1.84 \mu \text{H mm}^{-1} \) (\( \mu_o \) being the vacuum permeability) [15]. Since
Figure 2. Layout of an electrode pattern for nine planar traps. The electrons are trapped above the concentric ring electrodes. The electrodes are made of a 40 $\mu$m thick gold layer, separated by etched gaps (width of the order of 10 $\mu$m). Figure courtesy of Michael Hellwig (Universität Ulm, Germany).

$LC = 1/c^2$ ($c$ being the vacuum speed of light), the line’s propagation constant in (8) becomes $\beta^2 = \omega^2/c^2 + i\omega RC$. The resistivity $\varrho_{Au}$ of gold changes drastically with temperature, and we get a resistance per length $R = 2\varrho_{Au}((\pi r_o)^2) \approx 5.6 \Omega \text{m}^{-1}$ below 10 K and $R \approx 122.5 \Omega \text{m}^{-1}$ at 80 K [16].

All the information needed for the coupling of the two-electron system via the transmission line is encoded in the Green function $G(z, x, \omega)$. For the parameters mentioned above (and lengths $d < 10$ mm), we find that the line shows resonances at frequencies of the order of $\omega/2\pi = c/(2d) > 15$ GHz. This is well above the typical axial oscillation frequencies of trapped electrons, and we are in the dispersive regime where the transmission line is only virtually excited. The low-frequency expansion of the Green function is listed in appendix B.2. These expressions only involve the line’s total capacitance $C_w = Cd$ and total resistance $R_w = Rd$. To this, we can add the capacitance of the connection pads leading from the bonding points for the gold wires to the trap electrodes, using a formula similar to equation (9).

3.2. Frequency shift on the oscillators

By coupling the electron to the transmission line, its natural oscillation frequency in the trap is modified. The new oscillation frequency becomes $\tilde{\omega}_o = \omega_o - \Delta \omega_o$. We follow the technique
of [17] to eliminate the off-resonant modes of the transmission line and find a frequency shift

$$\Delta \omega_o = \frac{g^2 C}{2m_c \omega_o} \left[ \text{Re}[G(0, 0, \omega_o)] - \frac{2}{\pi} \int_0^\infty \frac{\text{Im}[G(0, 0, \omega)]}{\omega + \omega_o} d\omega \right],$$  \hspace{1cm} (10)

where

$$g \equiv \frac{e\alpha(\tilde{y})}{aC_o}. \hspace{1cm} (11)$$

Note that this shift is actually independent of temperature. The integral on the right-hand side of equation (10) can be written as the imaginary part of a complex integral that we evaluate by shifting the contour to the imaginary axis (where $\omega = i\xi$)

$$\text{Im} \left[ \int_0^\infty \frac{G(0, 0, \omega)}{\omega + \omega_o} d\omega \right] = \text{Im} \left[ \int_0^\infty \frac{iG(0, 0, i\xi)}{i\xi + \omega_o} d\xi \right] = \int_0^\infty \omega_o G(0, 0, i\xi) \frac{d\xi}{\xi^2 + \omega_o^2}. \hspace{1cm} (12)$$

In the last line of equation (12), we have used the fact that the Green function at imaginary frequencies is real. In addition, the function is monotonous, so the resonances are washed out. It turns out that $G(0, 0, i\xi) = A$ ($A$ is a constant) up to a cut-off frequency $\omega_{cut}$ for the geometrical and electrical parameters of the traps and the line that are considered herein. The cut-off frequency is determined by the line’s resonant modes, $\omega_{cut} \sim \pi c/d$, and typically falls in the range 10–150 GHz. Thus, the integral equation (12) can be approximated by changing the upper limit to $\omega_{cut}$. On the other hand, $A$ can be found as

$$A = \lim_{\xi \to 0} G(0, 0, i\xi) = \text{Re}[G_{low}(0, 0, 0)] = \text{Re}[G_{low}(0, 0, \omega_o)]. \hspace{1cm} (13)$$

Here, $G_{low}(0, 0, \omega_o)$ refers to the low-frequency limit of the Green function whose real part is independent of frequency (see appendix B.2). With these considerations, it is straightforward to get

$$\frac{2}{\pi} \int_0^\infty \frac{\text{Im}[G(0, 0, \omega)]}{\omega + \omega_o} d\omega \approx \frac{2}{\pi} \text{Re}[G_{low}(0, 0, \omega_o)] \frac{d\omega}{\omega_o} \arctan \left( \frac{\omega_{cut}}{\omega_o} \right). \hspace{1cm} (14)$$

Now for the transmission line discussed above, the cut-off frequency is much larger than the natural frequency of the electrons, $\omega_{cut} \gg \omega_o$. The frequency shift $\Delta \omega_o$ then almost vanishes, since $\arctan[\omega_{cut}/\omega_o] \to \pi/2$ and the two terms in equation (10) nearly cancel out. Hence, the frequency shift due to the coupling to the transmission line is negligible.

The direct Coulomb interaction can, however, perturb the trapping potentials of the electrons. In order that the electron is kept in the trap, it is necessary that $\omega_o \gg \Omega_c$, where $\Omega_c \equiv \sqrt{\epsilon^2/(4\pi \epsilon_0 m_e d^3)} \approx 5 \times 10^5 (d/\text{mm})^{-3/2} \text{rad s}^{-1}$ [4]. The Coulomb repulsion thus defines a minimum confinement strength in the Penning trap (that determines $\omega_o$) for a given intertrap distance $d$. In the following, we always assume that $\omega_o > \Omega_c$.

### 3.3. Coherent coupling

The coherent coupling between the electrons arises in our approach from the off-resonant response of the charges on the transmission line [17]. We adiabatically eliminate these modes and promote the resulting change in the average energy of the electron+line system to an
effective (or adiabatic) Hamiltonian. This gives an effective dipole–dipole coupling of the form [17]

\[ \hat{H}_{\text{dip–dip}} = -\hbar \Omega_{12} \left( b_1^\dagger b_2 + b_2^\dagger b_1 \right). \]  

(15)

At this step, we have made the rotating-wave approximation, neglecting terms that rotate at twice the trap frequency \( \omega_o \). The Rabi frequency \( \Omega_{12} \) is determined by the real part of the Green function as

\[ \Omega_{12} = \frac{g^2 C}{2m^*_c \omega_o} \text{Re}[G(d, 0, \omega_o)]. \]  

(16)

In the context of quantum computing architectures, this interaction allows the implementation of quantum gates [4, 5]. The speed of the gate is proportional to \( \Omega_{12} \). In our case, the low-frequency approximation for \( \text{Re}[G(d, 0, \omega_o)] \) (see equation (B.11)) can be applied, and we get

\[ \Omega_{12} = \frac{g^2 \alpha^2(\bar{y})}{2a^2 m^*_c \omega_o} \frac{1}{2C_w + C_o}. \]  

(17)

Note that the coherent coupling scales with the trap frequency as \( \omega_o^{-1} \), which is due to the matrix elements of the quantized displacement operator. On the other hand, the scaling with trap separation \( d \) and trap size is \( \Omega_{12} \propto a^{-2} d^{-1} \) if the wire capacitance is large, \( C_w = C d \gg 2C_o \). In the other limit, \( C_o \ll 2C_w \), we get a scaling \( \Omega_{12} \propto a^{-3} \), independent of \( d \). This case coincides with the expression found in [5]. Both estimates illustrate the advantages of miniaturizing the trap to speed up the intertrap coupling.

We recall that for the direct Coulomb interaction, the Rabi frequency is \( \Omega_{12} = \Omega_C^2 / \omega_o \) [4]. We tabulate the ratio between the two Rabi frequencies in table 1 for a few intertrap separations and electrode sizes. Note that this ratio does not depend on the trap frequency \( \omega_o \). The main result is that the dipole–dipole coupling strength for a transmission line is larger than that for the direct Coulomb interaction [8]. The column featuring \( \Omega_c \) gives the lower limit on \( \omega_o \), and the third column gives the product \( \omega_o \Omega_{12} \) that is actually independent of \( \omega_o \) (see equation (17)). We have used the transmission line parameters that were mentioned in section 3.1.

### Table 1. Ratio \( \Omega_{12} / \Omega_c^2 \), \( \Omega_c \) and \( \omega_o \Omega_{12} \) for \( d = 1 \) and 10 mm and trap radius \( a = 0.1 \) and 1 mm. \( \alpha^2(\bar{y}) = 0.512 \) (\( \bar{y} = a/2 \)).

| \( \Omega_{12} / \Omega_c^2 \) | \( \Omega_c \) rad s\(^{-1} \) | \( \omega_o \Omega_{12} \) (rad s\(^{-1} \)) \(^2 \) |
|-----------------|-----------------|-----------------|
| \( a = 1 \) mm  | \( d = 10 \) mm | 5.64 \times 10\(^4 \) | 1.59 \times 10\(^4 \) |
| \( d = 1 \) mm  | —               | 5.64 \times 10\(^1 \) | —               |
| \( a = 0.1 \) mm | \( d = 1 \) mm  | 2.72 \times 10\(^4 \) | 1.43 \times 10\(^5 \) |
| \( d = 10 \) mm | —               | 5.03 \times 10\(^5 \) | —               |
| \( a = 0.1 \) mm | \( d = 1 \) mm  | 1.43 \times 10\(^3 \) | —               |

### 3.4. Dissipation and diffusion

The spectrum of charge fluctuations near the frequency \( \omega_o \) is determined by the imaginary part of the Green function. The dissipation of the energy of each electron into the lossy transmission line is quantified by the rate

\[ \Gamma = \frac{g^2 C}{2m^*_c \omega_o} \text{Im}[G(0, 0, \omega_o)]. \]  

(18)
Table 2. $\Gamma$ and $\Gamma_{12}$ for $d = 1$ and 10 mm and trap radius $a = 0.1$ and 1 mm.

| $\alpha^2(\bar{y})$ = 0.512 ($\bar{y} = a/2$) | $\Gamma$ rad s$^{-1}$ | $\Gamma_{12}$ rad s$^{-1}$ |
|-----------------------------------------------|----------------------|----------------------|
| $a = 1$ mm | $a = 0.1$ mm | $a = 1$ mm | $a = 100$ $\mu$m |
| $d = 10$ mm | $1.02 \times 10^{-4}$ | $1.12 \times 10^{-2}$ | $1.01 \times 10^{-4}$ | $8.99 \times 10^{-3}$ | $T < 10$ K |
| $d = 1$ mm | — | $1.02 \times 10^{-3}$ | — | $1.01 \times 10^{-3}$ |
| $d = 10$ mm | $2.22 \times 10^{-3}$ | $2.45 \times 10^{-1}$ | $2.20 \times 10^{-3}$ | $1.96 \times 10^{-1}$ | $T = 80$ K |
| $d = 1$ mm | — | $2.22 \times 10^{-2}$ | — | $2.20 \times 10^{-2}$ |

The dynamics of the electron–electron system (master equation) is determined by the shape of $\text{Im}[G(0, 0, \omega)]$ around $\omega_o$ and by the relative strength of $\Gamma$ to $\omega_o$ and $\omega_T = k_B T / \hbar = 1.31 \times 10^{11}$ (T/K) rad s$^{-1}$ (see [18, 19]; $k_B$ is the Boltzmann constant). We focus on a weak coupling regime $\Gamma < \omega_o$ in section 4 below. The energy that one electron has dissipated into the reservoir can also be absorbed by the other electron. This process is characterized by the rate

$$
\Gamma_{12} = \frac{g^2 C}{2 m_e \omega_o} \text{Im}[G(0, d, \omega_o)]
$$

(19)

that characterizes the correlation between the thermal fluctuations on both ends of the transmission line. We shall see below that the dissipation rates of the two-electron system are given by $\Gamma \pm \Gamma_{12}$, depending on the relative phase of the axial oscillations.

In the low-frequency regime, the imaginary part of the Green function is proportional to $\omega$ (see equation (B.9)). Thus, equation (18) becomes independent of the frequency of oscillation (ohmic bath); explicitly, we find

$$
\Gamma = \frac{e^2 \alpha^2(\bar{y})}{6 \alpha^2 m_e} \eta_1 \left( \frac{C_o}{C_w} \right) \bar{R}_w.
$$

(20)

The dimensionless function $\eta_1(u)$ is defined in equation (B.12) and satisfies $3/4 < \eta_1(u) < 1$. The cross-dissipation rate $\Gamma_{12}$ is also independent of the natural frequency and reduces to an expression identical to equation (20), with $\eta_1(u)$ replaced by $\eta_2(u)$ defined in equation (B.13). Since $1/2 < \eta_2(u) < 3/4$, $\Gamma_{12}$ is slightly smaller than $\Gamma$. This feature is connected to the one-dimensionality of the transmission line. Table 2 shows $\Gamma$ and $\Gamma_{12}$ for $d = 1$ and 10 mm, $a = 0.1$ and 1 mm and $\alpha^2(\bar{y}) = 0.512$. Note that the dissipation rates are extremely small compared with both the natural trap frequency and the coherent coupling.

We anticipate from the following section that the temperature of the trap components enters into the rate at which non-equilibrium states start to relax towards equilibrium. The master equation (22) contains rates for absorption and emission of one excitation quantum $\hbar \omega_o$ with the transmission line that scale with the product of $\Gamma$ and $\Gamma_{12}$ with $\bar{n}$ and $\bar{n} + 1$ where we denote

$$
\bar{n} = 1 / (e^{\hbar \omega_o / (k_B T)} - 1).
$$

(21)

We call these rates ‘diffusion rates’ as becomes clear when the master equation is written in the Fokker–Planck form (see equation (23)).
4. Weak coupling master equation

In the previous section, we estimated the order of magnitude of the interaction terms that define the dynamics of the two electrons coupled by the transmission line. This estimation yields the hierarchy $\Gamma, \Gamma_{12} \ll \Omega_{12} \ll \omega_o < \omega_r, \omega_{cu}$ for temperatures $T \geq 1$ K. In this regime, $\text{Im}[G(0, 0, \omega)]$ is a smooth function around $\omega_o$, and the thermal fluctuation spectrum is approximately constant there. These properties define a Markovian, weak-coupling regime in which the rotating wave approximation is applicable. Following the standard projector method under these conditions, the reduced density operator $\hat{\rho}$ that describes the two oscillating electrons satisfies the master equation [20]–[22]

$$\frac{\partial \hat{\rho}}{\partial t} = -i \sum_{m=1,2} \omega_m \left[ \hat{b}_m^+ \hat{b}_m, \hat{\rho} \right] + i\Omega_{12} \left[ \hat{b}_2^+ \hat{b}_1 + \hat{b}_1^+ \hat{b}_2, \hat{\rho} \right] + \sum_{m,m'=1,2} \Gamma_{mm} \left[ (\tilde{n} + 1) \hat{b}_m \hat{b}_m^+ \hat{\rho} - \hat{b}_m^+ \hat{b}_m \hat{\rho} - \hat{b}_m \hat{b}_m^+ \hat{\rho} + \hat{b}_m^+ \hat{b}_m \hat{\rho} \right].$$

where $\Gamma_{11} = \Gamma_{22} = \Gamma$ and $\Gamma_{12} = \Gamma_{21}$. We have used the thermal equilibrium occupation number defined in equation (21). We solve equation (22) in the phase-space domain, where an operator $\hat{O}(b_1, b_1^+, b_2, b_2^+)$ is mapped to a complex variable function: $\hat{O}(z_1, z_1^+, z_2, z_2^+)$ using an operator ordering rule. Adopting normal ordering, we obtain from equation (22) the following master equation for the so-called $P$-function:

$$\frac{\partial}{\partial t} P(z, t) = \sum_{m=1,2} \left\{ \left[ \Gamma + i\omega_m \right] \frac{\partial}{\partial z_m} \left[ z_m P(z, t) \right] + \left[ \Gamma - i\omega_m \right] \frac{\partial}{\partial z_m^*} \left[ z_m^* P(z, t) \right] + 2\bar{n}\Gamma \frac{\partial^2}{\partial z_m \partial z_m^*} P(z, t) \right\}$$

$$+ \left[ \Gamma_{12} - i\Omega_{12} \right] z_1 \frac{\partial}{\partial z_2} P(z, t) + \left[ \Gamma_{12} + i\Omega_{12} \right] z_1^+ \frac{\partial}{\partial z_2^*} P(z, t)$$

$$+ \left[ \Gamma_{12} - i\Omega_{12} \right] z_2 \frac{\partial}{\partial z_1} P(z, t) + \left[ \Gamma_{12} + i\Omega_{12} \right] z_2^+ \frac{\partial}{\partial z_1^*} P(z, t)$$

$$+ 2\bar{n}\Gamma_{12} \frac{\partial^2}{\partial z_1 \partial z_2^*} P(z, t) + 2\bar{n}\Gamma_{12} \frac{\partial^2}{\partial z_2 \partial z_1^*} P(z, t).$$

(23)

Here, $z = (z_1, z_1^+, z_2, z_2^+)$. Recall that the $P$-function provides an expansion of the density matrix in terms of coherent states $|z_1 z_2\rangle$ as

$$\hat{\rho}(t) = \int |z_1 z_2\rangle \langle z_1 z_2| P(z, t) \, d^4z.$$  

(24)

Here, $d^4z \equiv dx_1 \, dy_1 \, dx_2 \, dy_2$, where $z_1 \equiv x_1 + iy_1$ and $z_2 \equiv x_2 + iy_2$. Appendix C gives details of how to solve the master equation analytically. Let us simply mention here that the equation is in the Fokker–Planck form and that the ‘diffusion coefficients’ $\bar{n}\Gamma$ and $\bar{n}\Gamma_{12}$ provide the leading order timescale for the impact of dissipation.

5. Fidelity

The use of the axial degrees of freedom for implementing quantum gates has been proposed previously [3, 5]. The two-qubit is encoded in the first two Fock states of each of the oscillators. We mention that for the readout of the computation in an experimental setting, the states need to be distinguishable. This can be achieved by applying a small anharmonic perturbation
in the trapping potential for the measurement [4]. If we consider an ideal system (without imperfections or losses) and prepare an initial state, then after a certain time the system will evolve to a desired target state. In the case where the system is not ideal, there is uncertainty in obtaining the target state. The fidelity is a measure of this uncertainty.

One could also ask for the entanglement between the two electrons that is generated by the interaction and likely to be spoiled by the losses in the transmission line. This entanglement is recognized as an enabling resource for quantum computation. There are sufficient and necessary conditions of entanglement of a bipartite system for certain classes of states, in particular Gaussian mixed states like coherent or thermal states. But in our case, we deal with an initial Fock state (non-Gaussian) that evolves into a non-Gaussian mixed state, and entanglement conditions for this class are still an open problem [23].

We consider that the system of two harmonic oscillators is prepared, at $t = 0$, in a Fock state with $l$ ($p$) excitation quanta for the first (second) electron. Given this initial condition, the evolution of the $P$-distribution $P(z, t)$ is

$$P(z, t) = \int P_3(z, z_0, t) P_{lp}(z_0) \, dz_0.$$  \hspace{1cm} (25)

Here, $P_3(z, z_0, t)$ is the propagator for an initial Dirac-$\delta$ function in the phase-space centered at $z_0$ and $P_{lp}(z_0)$ is the $P$-representation of the initial Fock state $|lp\rangle$ which takes the explicit form [24]

$$P_{lp}(z) = \frac{1}{l!p!} e^{[z_1^2 + z_2^2]} \frac{\partial^{2l}}{\partial z_1^{l} \partial z_2^{l}} \delta^2(z_1) \frac{\partial^{2p}}{\partial z_1^{p} \partial z_2^{p}} \delta^2(z_2).$$  \hspace{1cm} (26)

Here, $\delta^2(z_m) \equiv \delta(x_m) \delta(y_m)$ ($m = 1, 2$). The density matrix elements in the Fock state basis are then calculated from

$$\langle ns| \hat{\rho}(t)|mr\rangle = \frac{1}{\sqrt{m!r!n!s!}} \int e^{-[(z_1^2 + z_2^2)z_1^n z_2^m z_1^r z_2^r]} P(z, t) \, dz.$$  \hspace{1cm} (27)

These matrix elements provide the gate fidelity, as we illustrate now for two simple gates: the ‘swap’ protocol $|01\rangle \rightarrow |10\rangle$, where the two electrons exchange one excitation quantum, and the ‘$\sqrt{\text{swap}}$’ that aims at entangling the two systems, $|01\rangle \rightarrow (|01\rangle + |10\rangle)/\sqrt{2}$. We mention that the subspace spanned by $|01\rangle$ and $|10\rangle$ is not closed under the dissipative evolution because of thermal absorption from the bath (the last line of equation (22)).

### 5.1. Swap gate

A swap gate is implemented by considering the initial state $|01\rangle$ and target state $|10\rangle$. Then, the fidelity of the gate is given by

$$F_{\text{swap}}(t) = \langle 10| \hat{\rho}(t)|10\rangle.$$  \hspace{1cm} (28)

In the absence of losses, we get by solving the Schrödinger equation: $F_{\text{swap}}(t) = (1/2) (1 - \cos 2\Omega_{12}t)$. Consequently, the gate time, for which the fidelity is maximum, is $\pi/2|\Omega_{12}|$ (inversely proportional to the coherent coupling). Finite resistivity and finite temperature reduce the maximum fidelity. An unwieldy, exact expression for equation (28) can be obtained and is presented in appendix C.

The analysis of this formula reveals the characteristic dissipation rates

$$\Gamma_\pm = \Gamma \pm \Gamma_{12}$$  \hspace{1cm} (29)
that arise from the eigenvalues $\pm i\Omega_{12} - \Gamma_{\pm}$ of the drift matrix $A$ (equation (C.3)). They characterize the dissipation of oscillation modes $\hat{b}_1 = \pm \hat{b}_2$ where the two electrons move in phase (symmetric mode) or in phase opposition (antisymmetric). The antisymmetric mode can show a much smaller dissipation, providing an example of a decoherence-‘free’ subspace.

We analyze two regimes: (1) weak damping where $\Gamma \sim \bar{n}\Gamma \ll \Omega_{12}$ (low temperature) or $\Gamma < \bar{n}\Gamma \ll \Omega_{12}$ (high temperature) and (2) strong damping where $\Gamma \ll \bar{n}\Gamma \sim \Omega_{12}$ (high temperature). In all cases, we assume that the relevant frequencies are small compared with the bare trap frequency $\omega_o$, validating the use of our weak coupling master equation.

For the weak damping regime, the fidelity shows oscillations. For $t \ll 1/\Gamma(1 + 4\bar{n})$, the fidelity can be approximated as

$$F_{\text{swap}} \approx \exp[-2(1 + 4\bar{n})\Gamma t] \sin^2[\Omega_{12} t]. \quad (30)$$

Notice that the envelope of the oscillations decays with a rate $2(1 + 4\bar{n})\Gamma$ rather than simply $\Gamma$. Due to its similarity to the diffusion terms in the Fokker–Planck equation (23), we call $(1 + 4\bar{n})\Gamma$ the diffusion rate. It actually interpolates between $4\bar{n}\Gamma$ at high temperatures ($\bar{n} \gg 1$) and $\Gamma$ at low temperatures ($\bar{n} \ll 1$). In figure 3, we show a generic plot of the fidelity for a diffusion rate small or large compared with the coherent coupling $\Omega_{12}$. In both cases, we consider high temperatures. The fidelity is seen to decay on a timescale set by $\bar{n}\Gamma$: it is significantly reduced well before the system thermalizes with its surroundings (timescale $\Gamma$). This is a typical manifestation of ‘fast decoherence’. In the strong damping regime, the fidelity collapses nearly immediately (see figure 3).

In the limit $\Gamma t \to \infty$, regardless of the regime, the two electrons get thermalized. As a consequence, the fidelity becomes

$$\lim_{\Gamma t \to \infty} F_{\text{swap}}(t) = \frac{\bar{n}}{(1 + \bar{n})^3}. \quad (31)$$

Figure 3. Fidelity of a swap gate for the weak and strong damping regimes. For the weak damping regime, $\Gamma_+/\Omega_{12} = 10^{-5}$, $\Gamma_-/\Omega_{12} = 10^{-7}$ and $\bar{n}\Gamma_+ / \Omega_{12} = 10^{-2}$. For the strong damping regime, $\Gamma_+ / \Omega_{12} = 10^{-4}$, $\Gamma_- / \Omega_{12} = 10^{-6}$ and $\bar{n}\Gamma_+ / \Omega_{12} = 1$. The dashed curve is the short-time approximation $\exp[-2(1 + 4\bar{n})\Gamma t]$ for the envelope at weak damping (see equation (30)). The inset shows the fidelity for weak damping over an extended timescale.
As seen in figure 3, this value is not approached in an exponential way (equation (30) becomes invalid at large times). Equation (31) is the probability of the system being in state $|10\rangle$ at thermal equilibrium, and can be calculated straightforwardly from equation (C.15). We use the fact that the functions $f_1(\tilde{x}, t)$ (see equation (C.11)) and $f_2(t)$ (see equation (C.12)) vanish at large times and that, from (C.3), the exponentiated drift matrix $\exp[\mathcal{A}t] \to 0$.

From the above discussion, it follows that the ratio $R \equiv \Omega_{12}/[(1 + 4\bar{n})\Gamma]$ gives an estimate of the number of gate operations that can be performed during the coherence time. Also, this ratio determines the performance of the gate—the closer the fidelity is to one, the larger is $R$.

In the low-frequency range (see equations (17) and (20)), we have

$$R = \frac{3}{\eta_1(C_o/C_w)(2C_o + C_w)R_w\omega_o(1 + 4\bar{n})}. \quad (32)$$

Notice that equation (32) depends on: the temperature, the geometries of the trap and the transmission line and the resistance of the wires. If $k_B T \gg \hbar \omega_o$ ($\bar{n} \gg 1$), then $1 + 4\bar{n} \approx 4k_B T/\hbar \omega_o$. In this case, the trap frequency $\omega_o$ remarkably drops out from the ratio $R$.

Up to now, we have described generically the fidelity for different damping regimes. Now we calculate it for realistic settings of the trap and line. In figure 4, we plot the fidelity $F_{\text{swap}}(t)$ for temperatures $T = 4.0$ and $80 \text{ K}$, natural frequency of the oscillator $\omega_o = 10^8 \text{ rad s}^{-1}$, transmission line lengths $d = 1.0$ and $10.0 \text{ mm}$ and trap radii $a = 0.1$ and $1.0 \text{ mm}$. The strength of the coupling parameters can be found in tables 1 and 2. The damping of the fidelity oscillations shows the impact of temperature: at $T = 4 \text{ K}$, the fidelity can become as high as 0.955, 0.989 and 0.999 for the curves of figures 4(a)–(c), respectively. Setting the temperature to $T = 80 \text{ K}$ greatly deteriorates the performance of the gate.

The dependence of $R$ (see equation (32)) on $T$ and $R_w$ provides an explanation for the strongly reduced fidelity at 80 K that is due to both a larger thermal occupation $\bar{n}$ and a larger resistivity. In table 3, the values of $R$ for the considered particular cases are tabulated. Comparison with figure 4 shows that the performance of the gate scales with the figure of merit $R$. Also, it can be seen that the largest $R$ occurs for the most miniaturized configuration. At $T = 0.1 \text{ K}$, for the configurations of table 3, $R$ increases roughly 40 times compared with $T = 4 \text{ K}$. In this range, if the wire material becomes superconducting, the resistance $R_w$ in equation (32) is limited by the contacts only, and $R$ improves even beyond the estimates of table 3.

If the natural frequency $\omega_o$ of the oscillator is increased by one order of magnitude, the fidelity curves would be nearly identical to those of figure 4, except for a slower time scale (the gate speed scales like $1/\omega_o$). Alternatively, one can increase the gate speed by lowering $\omega_o$, without changing the gate performance. However, the Coulomb frequency $\Omega_c$ provides a lower limit for $\omega_o$.

To summarize, it is quite remarkable that quantum gates with high fidelity can be implemented with liquid helium cooling ($T = 4 \text{ K}$) and conventional ohmic wires.

5.2. Square-root swap gate

A square-root swap gate is attained by considering an initial state $|01\rangle$ and target state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|01\rangle + e^{i\delta}|10\rangle), \quad (33)$$
Figure 4. Fidelity of a swap gate as a function of time for $T = 4$ and 80 K. The angular frequency of the oscillators is $\omega_0 = 10^8$ rad s$^{-1}$. (a) $d = 10$ mm and $a = 1$ mm. (b) $d = 10$ mm and $a = 0.1$ mm. (c) $d = 1$ mm and $a = 0.1$ mm. The dashed curves give the envelope $\exp[-2(1+4\bar{n})\Gamma t]$ when the weak damping regime applies.

where $\delta$ is the relative phase. The fidelity for this gate is calculated as

$$F_{sq} (t, \delta) = \langle \psi | \hat{\rho}(t) | \psi \rangle.$$

Without losses, $F_{sq} (t, \delta) = (1/2)(1 + \sin \delta \sin 2\Omega_{12} t)$. In the presence of losses, the analytical expression of the fidelity is as cumbersome as for the swap gate (see appendix C). In the weak damping regime, for $t \ll 1 / \Gamma (1 + 4\bar{n})$, it can be approximated as

$$F_{sq} (t, \delta) \approx \frac{1}{2} e^{-2(1+4\bar{n})\Gamma t} \left[ e^{-2 \cos \delta (1+2\bar{n}) \Gamma_{12} t} + \sin \delta \sin (2\Omega t) \right].$$

Equation (35) shows that the maximum fidelity is achieved for a relative phase $\delta = \pi/2$, as in the non-dissipative case. If $\delta = 0, \pi$, a monotonous decay is observed that actually probes how the symmetric and antisymmetric two-electron oscillation modes are affected by dissipation. Indeed, equation (35) gives in that case the diffusion rates $\Gamma \pm \Gamma_{12} + 2\bar{n}(2\Gamma \pm \Gamma_{12})$. In figure 5,
Table 3. Figure of merit $R$ (number of gate operation cycles per decoherence time). Parameters: wire length $d = 1$ and 10 mm and trap radius $a = 0.1$ and 1 mm. $\alpha^2(\bar{y}) = 0.512$ ($\bar{y} = a/2$) and $T = 0.1$, 4 and 80 K. Between $T = 0.1$ K and $T = 4$ K, $R$ is approximately inversely proportional to $T$.

| $d$ = 10 mm | 2689.85 | 11726.25 | $T = 0.1$ K |
| $d$ = 1 mm | 268985.11 |
| $d$ = 10 mm | 67.12 | 292.61 | $T = 4$ K |
| $d$ = 1 mm | 6712.14 |
| $d$ = 10 mm | 0.15 | 0.67 | $T = 80$ K |
| $d$ = 1 mm | 15.35 |

Figure 5. Fidelity of a square-root swap gate as a function of time for phase factors $\delta = 0, \pi/2$ and $\pi$. The angular frequency of the oscillators is $\omega_0 = 10^8$ rad s$^{-1}$; $T = 80$ K, $d = 1$ mm, and $a = 0.1$ mm. The dashed curves are related to the short-time approximation of the fidelity (see equation (35)). The red and blue dashed curves are $0.5 \exp[-2(1 + 4\bar{n})\Gamma t]$; respectively. The green dashed curve is the envelope $\exp[-2(1 + 4\bar{n})\Gamma t]$. We illustrate these behaviors for the parameters of figure 4(c) and $T = 80$ K. For a lower temperature, of course, the diffusion rate is decreased; therefore the fidelity decays slower. The fidelity oscillations are washed out in the strong damping regime before reaching equilibrium. This is shown in figure 6 for the parameters of figure 4(b) and $T = 80$ K. In comparison to the previous case, the line is 10 times longer; thus the gate speed is slower and the damping is higher. This translates into a much smaller figure of merit $R$, as can be seen in table 3.

In the limit $t \rightarrow \infty$, the fidelity becomes

$$\lim_{t \rightarrow \infty} F_{sq}(t, \delta) = \frac{\bar{n}}{(1 + \bar{n})^3}.$$
This is the same result as for the swap gate, but can be interpreted in a slightly different way: it represents the sum of half the probability of being in state $|10\rangle$ and half the probability of being in state $|01\rangle$. In this limit, the initial coherent superposition has been destroyed through the action of the environment; thus the fidelity becomes independent of the relative phase $\delta$.

6. Conclusions

We have presented a theoretical framework to study the axial dynamics of two confined electrons in Penning traps joined by a lossy electrical transmission line. A master equation in the weak coupling regime for the axial motion of the electrons is derived. In addition to the temperature $T$, three parameters ($\Omega_{12}$, $\Gamma$ and $\Gamma_{12}$) characterize the dynamics of the two-electron system; they can be extracted from the Green function $G(z, x, \omega)$ for the transmission line. The parameter $\Omega_{12}$, on the one hand, characterizes the coherent, capacitive coupling between the Penning traps created by the line, which enhances the free-space dipole–dipole coupling. Thus, $\Omega_{12}$ determines the rate with which energy is exchanged between the electrons; it defines how fast a quantum gate operation can be realized. On the other hand, the parameter $\Gamma$ gives the dissipation rate for one electron due to the presence of the lossy transmission line. Finally, there is the possibility that the energy dissipated by one electron is absorbed by the other, characterized by the rate $\Gamma_{12}$. This ‘crossed dissipation’ arises from the fact that both electrons couple to the same bath.

We have calculated analytically the fidelity for the swap gate and the square-root swap gate for all temperatures and parameters $\Omega_{12}$, $\Gamma$, $\Gamma_{12}$, using the weak coupling master equation. Considering realistic parameters for the transmission line, we have given explicit results for $T = 4$ and $80\,$K (roughly the temperatures of liquid helium and nitrogen, respectively), at different intertrap separations and trap sizes. The fidelity is damped with approximately a rate $\Gamma \bar{n}$ (diffusion rate). In the case of a swap gate, for an intertrap separation of $d = 10\,$mm, the fidelity reaches as high a value as 0.989 for the temperature $T = 4\,$K, whereas the fidelity is
strongly deteriorated at \( T = 80 \) K. For the same gate, but an intertrap separation \( d = 1 \) mm, the fidelity becomes as high as 0.829 at \( T = 80 \) K and the performance is nearly perfect at \( T = 4 \) K. It is realistic that quantum computing applications can be implemented with a normal cryogenic system. It is also possible to implement an entire array of electron traps with arbitrary interconnections based on conventional metallic wires.

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**Appendix A. Image-charge interaction**

We consider a thin disk of radius \( a \) and a concentric screen with a hole of radius \( a \) lying on the plane \( y = 0 \) (the gap between the disk and the concentric screen is infinitesimal). The \( y \)-axis crosses the center of the disk. If a charge \( q \) is located at position \( r_s = y_s n_y \) (\( n_y \) is the unit vector along the \( y \)-axis), then the potential obeys

\[
\nabla^2 \phi(r) = -\frac{q}{\varepsilon_0} \delta(r - r_s),
\]

(A.1)

with boundary conditions: \( \phi (r \in \text{disk}) = V \) and \( \phi (r \in \text{screen}) = 0 \). To solve equation (A.1) with the given boundary conditions, we can apply Green’s theorem

\[
\oint_{\partial V} \left[ G(r', r) \frac{\partial \phi(r')}{\partial n'} - \phi(r') \frac{\partial}{\partial n'} G(r', r) \right] d\alpha' = \int \left[ G(r', r) \nabla^2 \phi(r') - \phi(r') \nabla^2 G(r', r) \right] d^3 r',
\]

(A.2)

where \( G(r', r) \) is Green’s function that satisfies

\[
\nabla^2 G(r', r) = -\delta(r' - r).
\]

(A.3)

The normal derivative \( \partial / \partial n' \) is taken with respect to the outside normal vector of the integration volume. We shall take \( G(r', r) \) as the potential of a point charge at \( r \) with the boundary condition that \( G(r', r) = 0 \) for \( r' \) on the plane containing the disk and the screen. We shall assume that the Green function and the potential decay sufficiently fast at infinity so that the surface integral runs only over the plane. In this case, Green’s theorem (A.2) gives

\[
\phi(r) = \phi_1(r) + \phi_2(r),
\]

(A.4)

with the definition

\[
\phi_1(r) \equiv V \int_{\text{disk}} \frac{\partial}{\partial y'} G(r', r) d\alpha',
\]

(A.5)

\[
\phi_2(r) \equiv \frac{q}{\varepsilon_0} G(r_s, r).
\]

(A.6)

Notice that the potential is the sum of two terms. The potential \( \phi_1(r) \), involving the value \( V \) on the disk, is the result of a pure boundary value problem without sources \( [\nabla^2 \phi_1(r) = 0] \). The solution for the potential along the \( y \)-axis is [5]

\[
\phi_1(y n_y) = V \tilde{\alpha}(y/a),
\]

(A.7)

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where $\tilde{\alpha}(x) \equiv 1 - 1/\sqrt{1 + 1/x^2}$. The second term $\phi_2(\mathbf{r})$ is the image potential in front of a grounded screen (segmented or not); thus $\nabla^2 \phi_2(\mathbf{r}) = -q \delta(\mathbf{r} - \mathbf{r}_s)/\varepsilon_o$.

The charge on the disk is

$$Q = \int_{\text{disk}} \sigma(\mathbf{r}) \, d\mathbf{a} = -\varepsilon_o \int_{\text{disk}} \lim_{y \to 0} \frac{\partial}{\partial y} \left[ (\phi_1(\mathbf{r}) + \phi_2(\mathbf{r})) \right] \, d\mathbf{a}, \quad (A.8)$$

where $\sigma(\mathbf{r})$ is the surface charge density. For an on-axis trapped particle, the charge $Q$ reduces to

$$Q = C_{\text{disk}} V - q \tilde{\alpha}(y_s/a). \quad (A.9)$$

The first term arises from the boundary value problem. The second term is the charge induced. For $a \to \infty$, we have $\tilde{\alpha}(y_s/a) \to 1$, and the induced charge exactly compensates the electron charge $q$. The same geometrical factor $\tilde{\alpha}(y_s/a)$ applies here as in equation (A.7), because the charge due to the image potential $\phi_2$ in (A.8) involves the same integral as the potential $\phi_1$ involving $V$.

In equation (A.9), the voltage $V$ on the disk can be chosen arbitrarily; we can take any solution to the homogeneous equation (given voltages on the plane, but no outside charges) and add it to the image-charge solution generated by the electron. Hence, $V$ is only specified if we impose extra boundary conditions, for example the value of the charge $Q$ on the disk. The total charge on disk plus screen is fixed at $q$ and cannot be set independently. Then equation (A.9) can be understood as defining the voltage induced by the electron

$$V = \frac{1}{C_{\text{disk}}} \left[ Q + q \tilde{\alpha}(y_s/a) \right] = \frac{\partial H}{\partial Q}, \quad (A.10)$$

where the potential energy can be chosen as

$$H = \frac{Q^2}{2C_{\text{disk}}} + \frac{\tilde{\alpha}(y_s/a)}{C_{\text{disk}}} Qq. \quad (A.11)$$

Expanding $\tilde{\alpha}$ around the equilibrium trapping position $\bar{y}$, we get the interaction Hamiltonian

$$H_{\text{int}} = -\frac{\alpha(\bar{y})}{a C_{\text{disk}}} Qq (y - \bar{y}), \quad (A.12)$$

where $y - \bar{y}$ becomes the quantized displacement operator of equation (2) and

$$\alpha(\bar{y}) \equiv \frac{a^3}{(\bar{y}^2 + a^2)^{3/2}}. \quad (A.13)$$

We mention that for an infinitesimal gap between disk and concentric screen, the total charge in the disk diverges. Consequently, the capacitance of the disk diverges as well. In a realistic experiment, the aforementioned gap is finite and the disk has a finite thickness. This leads to a finite capacitance that can be estimated by the approach of section 3.1.

**Appendix B. The Green function**

**B.1. Finite transmission line**

The Green function of a finite transmission line with length $d$ can be expressed as [14]

$$G(z, x) = G_o(z, x, \omega) + G_d(z, x, \omega) + G_b(z, x, \omega). \quad (B.1)$$
The ends of the line \( z = 0 \) and \( z = d \) are loaded with impedances \( Z_1(\omega) \) and \( Z_2(\omega) \), respectively. \( G_s(z, x, \omega) \) is the direct path propagator from the point source \( x \) to the field point \( z \). \( G_s(z, x, \omega) \) and \( G_b(z, x, \omega) \) represent the field at \( z \) that is reflected at the boundaries from a source at \( x \). Explicitly, the three terms of equation (B.1) are

\[
G_o(z, x, \omega) \equiv \frac{i}{2\beta} e^{i\beta|z-x|}, \tag{B.2}
\]

\[
G_a(z, x, \omega) \equiv \frac{i}{2\beta} \left[ \rho_2(\omega) e^{i\beta[(d-z)+(d-x)]} + r_1(\omega) \rho_1(\omega) e^{i\beta[(d-z)+(d+x)]} \right], \tag{B.3}
\]

\[
G_b(z, x, \omega) \equiv \frac{i}{2\beta} \left[ \rho_1(\omega) e^{i\beta(z+x)} + r_2(\omega) \rho_1(\omega) e^{i\beta[(d+z)+(d-x)]} \right]. \tag{B.4}
\]

Here, the complex wavenumber is \( \beta = \sqrt{LC\omega^2 + i\omega RC} \); and \( r_1(\omega) \) and \( r_2(\omega) \) are the reflection coefficients at the end points \( z = 0 \) and \( z = d \), respectively. \( \rho_1(\omega) \) and \( \rho_2(\omega) \) take into account multiple reflections. They are defined as \((m = 1, 2)\)

\[
r_m(\omega) \equiv \frac{Z_o(\omega) - Z_m(\omega)}{Z_o(\omega) + Z_m(\omega)}, \tag{B.5}
\]

\[
\rho_m(\omega) \equiv \frac{r_m(\omega)}{1 - r_1(\omega)r_2(\omega) \exp[2i\beta d]}, \tag{B.6}
\]

Here, \( Z_o(\omega) \) is the characteristic impedance of the transmission line given by

\[
Z_o(\omega) = \sqrt{\frac{L}{C} + \frac{i}{\omega C}}. \tag{B.7}
\]

The resonances of the line correspond to the poles of equation (B.6) that occur near \( \text{Re}[\beta_o d] = \pi, 2\pi, \ldots \).

### B.2. Low-frequency approximations

When the transmission line of length \( d \) is loaded at the ends with the same capacitive impedance \( Z_{1,2}(\omega) = i/(\omega C_o) \), the low-frequency limit of \( G(0, 0, \omega) \) and \( G(0, d, \omega) \) becomes

\[
\text{Re}[G(0, 0, \omega)] \approx \frac{C_o}{C} \left[ \frac{C_o + C_w}{2C_o + C_w} \right], \tag{B.8}
\]

\[
\text{Im}[G(0, 0, \omega)] \approx \frac{1}{3} \frac{R_w C_o^2}{C} \omega \eta_1 \left( \frac{C_o}{C_w} \right), \tag{B.9}
\]

\[
\text{Re}[G(0, d, \omega)] \approx \frac{C_o^2}{C} \left[ \frac{1}{2C_o + C_w} \right], \tag{B.10}
\]

\[
\text{Im}[G(0, d, \omega)] \approx \frac{1}{3} \frac{R_w C_o^2}{C} \omega \eta_2 \left( \frac{C_o}{C_w} \right), \tag{B.11}
\]

where

\[
\eta_1(u) = 1 - \frac{u(1+u)}{4u^2 + 4u + 1}. \tag{B.12}
\]
\[ \eta_2(u) = \frac{1}{2} + \frac{u(1 + u)}{4u^2 + 4u + 1}. \]  

(B.13)

Here, \( C_w \equiv C_d \) and \( R_w \equiv R_d \). Notice that \( 3/4 < \eta_1(u) < 1 \) and \( 1/2 < \eta_2(u) < 3/4 \). The above conditions require that \( \sqrt{R \omega / (2C)} C_o \ll 1 \) and \( 2d \sqrt{R \omega / 2} \ll 1 \). Notice that the expressions in this frequency range are independent of \( L \).

Appendix C. Solving the master equation and the fidelity

C.1. The propagator

The master equation is solved in terms of the quadratures \( \mathbf{x} = (x_1, y_1, x_2, y_2) \) instead of \( \mathbf{z} \). We define a rotating frame with angular frequency \( \omega_o \). The rotating variables \( \bar{\mathbf{x}} = (\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2) \) and the laboratory frame variables \( \mathbf{x} = (x_1, y_1, x_2, y_2) \) are connected by the orthogonal transformation \( \bar{\mathbf{x}} = B(t)\mathbf{x} \), where

\[
B(t) = \begin{bmatrix}
\cos(\omega_o t) & -\sin(\omega_o t) & 0 & 0 \\
\sin(\omega_o t) & \cos(\omega_o t) & 0 & 0 \\
0 & 0 & \cos(\omega_o t) & -\sin(\omega_o t) \\
0 & 0 & \sin(\omega_o t) & \cos(\omega_o t)
\end{bmatrix}.
\]  

(C.1)

In the rotating frame, the master equation becomes

\[
\frac{\partial}{\partial t} P(\bar{\mathbf{x}}, t) = \left[ -(\partial \bar{\mathbf{x}})^T A \bar{\mathbf{x}} + \frac{1}{2} (\partial \bar{\mathbf{x}})^T D \partial \bar{\mathbf{x}} \right] P(\bar{\mathbf{x}}, t).
\]  

(C.2)

Here, \( \partial \bar{\mathbf{x}} = (\partial / \partial \bar{x}_1, \partial / \partial \bar{y}_1, \partial / \partial \bar{x}_2, \partial / \partial \bar{y}_2) \), and \( T \) denotes the transpose. Notice that (C.2) is a Fokker–Planck equation. The drift matrix \( A \) is

\[
A = \begin{bmatrix}
-\Gamma & 0 & -\Gamma_{12} & -\Omega_{12} \\
0 & -\Gamma & \Omega_{12} & -\Gamma_{12} \\
-\Gamma_{12} & -\Omega_{12} & -\Gamma & 0 \\
\Omega_{12} & -\Gamma_{12} & 0 & -\Gamma
\end{bmatrix}.
\]  

(C.3)

The diffusion matrix \( D \) is

\[
D = \bar{n} \begin{bmatrix}
\Gamma & 0 & \Gamma_{12} & 0 \\
0 & \Gamma & 0 & \Gamma_{12} \\
\Gamma_{12} & 0 & \Gamma & 0 \\
0 & \Gamma_{12} & 0 & \Gamma
\end{bmatrix}.
\]  

(C.4)

The solution for an initial \( \delta \)-function distribution \( [P(\bar{\mathbf{x}}; 0) = \delta^4(\bar{\mathbf{x}} - \bar{\mathbf{x}}_0)] \), which defines a propagator for any initial distribution, is given by [24]

\[
P_\delta(\bar{\mathbf{x}}, t) = N[\bar{\mathbf{x}}, e^{A t} \bar{\mathbf{x}}_o, Q(t)].
\]  

(C.5)

Here, \( N[\mathbf{y}, \mathbf{y}_o, \mathcal{J}] \) is a four-variable normalized Gaussian distribution:

\[
N[\mathbf{y}, \mathbf{y}_o, \mathcal{J}] \equiv \frac{1}{4\pi^2 \sqrt{\text{Det}[\mathcal{J}]}} \exp \left[ -\frac{1}{2} (\mathbf{y} - \mathbf{y}_o)^T \mathcal{J}^{-1} (\mathbf{y} - \mathbf{y}_o) \right].
\]  

(C.6)

where \( \text{Det}[\cdots] \) denotes the determinant, \( \mathcal{J} \) is the covariance matrix and \( \mathbf{y}_o \) is the mean vector. In our particular case, \( Q(t) \) is defined by

\[
Q(t) = C^{-1} \tilde{Q}(t) \left( C^{-1} \right)^T.
\]  

(C.7)
The integral equation (C.3.1) is solved by partial integration, using the particular $P$-distribution for the Fock state $|01\rangle$ (see equation (C.10)). In the rotating frame, it turns out that

$$ P(\tilde{x}, t) = \left[ P_\delta(\tilde{x}, \tilde{x}_o, t) + \frac{1}{4} \frac{\partial^2}{\partial \tilde{x}_o^2} P_{10}(\tilde{x}, \tilde{x}_o, t) + \frac{1}{4} \frac{\partial^2}{\partial \tilde{y}_o^2} P_{10}(\tilde{x}, \tilde{x}_o, t) \right]_{\tilde{x}_o=0}, $$

(C.10)

where $f_1(\tilde{x}, t)$ and $f_2(t)$ are given by

$$ f_1(\tilde{x}, t) \equiv \frac{1}{4} \left( (\tilde{x}^T Q^{-1}(t) e^{i \Delta t} e_3)^2 + (\tilde{x}^T Q^{-1}(t) e^{i \Delta t} e_4)^2 \right), $$

(C.11)

$$ f_2(t) \equiv \frac{1}{4} \left( [e^{i \Delta t} e_3]^T Q^{-1}(t) [e^{i \Delta t} e_3] + [e^{i \Delta t} e_4]^T Q^{-1}(t) [e^{i \Delta t} e_4] \right). $$

(C.12)

Here, $e_i$ is a four-dimensional unit vector whose $i$th component is one and the remaining components are zero.

### C.2. Solving the fidelity

**C.3.1. Swap gate.** The starting point is equation (C.3.1) for the density matrix elements. We switch to the quadrature variables $x = (x_1, y_1, x_2, y_2)$ and get a fidelity

$$ F_{\text{swap}}(t) \equiv \int e^{-|\tilde{x}|^2} \left( x_1^2 + y_1^2 \right) P(x, t) \, d^4x. $$

(C.13)

The integral equation (C.13) takes the same form in the rotating frame of $\tilde{x}$ where the propagator can be better solved. Thus, the transformation to this frame is achieved just by replacing $x$ by $\tilde{x}$. After the variable transformation and using the explicit expression of $P(\tilde{x}, t)$ (see equation (C.10)), equation (C.13) becomes

$$ F_{\text{swap}}(t) = 4\pi^2 \sqrt{\text{Det}[L]} \int N[\tilde{x}, 0, L] N[\tilde{x}, 0, \mathbb{Q}(t)] \left( \tilde{x}_1^2 + \tilde{y}_1^2 \right) \left[ 1 + f_1(\tilde{x}, t) - f_2(t) \right] \, d^4\tilde{x}, $$

(C.14)

where $L = I/2$ ($I$ being a 4 x 4 unit matrix) and $f_1$ and $f_2$ are defined in equations (C.11) and (C.12), respectively. Since the multiplication of two Gaussian functions yields another Gaussian, equation (C.14) can be further simplified to

$$ F_{\text{swap}}(t) = \frac{\sqrt{\text{Det}[\mathbb{H}(t)]}}{\text{Det}[\mathbb{Q}(t)]} \int N[\tilde{x}, 0, \mathbb{H}(t)] f_3(\tilde{x}, t) \, d^4\tilde{x}, $$

(C.15)

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where
\[ f_3(\mathbf{x}, t) \equiv (\tilde{x}_1^2 + \tilde{y}_1^2) \left[ 1 + f_1(\mathbf{x}, t) - f_2(t) \right] \]  
\[ (C.16) \]
and \( \mathbb{H}^{-1}(t) \equiv \mathbb{L}^{-1} + Q^{-1}(t) \). The integral in equation (C.15) can be calculated straightforwardly, since it is the expectation of \( f_3(\mathbf{x}, t) \) for a normal distribution with covariance matrix \( \mathbb{H}(t) \).

Moments up to the fourth order are involved that can be expressed in terms of the matrix elements \( H_{ij}(t) \) as (odd moments are vanishing)
\[ \langle \mathbf{x}_i \mathbf{x}_j \rangle = H_{ij}(t), \]  
\[ (C.17) \]
\[ \langle \mathbf{x}_i \mathbf{x}_j \mathbf{x}_k \mathbf{x}_l \rangle = \langle \mathbf{x}_i \mathbf{x}_j \rangle \langle \mathbf{x}_k \mathbf{x}_l \rangle + \langle \mathbf{x}_i \mathbf{x}_k \rangle \langle \mathbf{x}_j \mathbf{x}_l \rangle + \langle \mathbf{x}_i \mathbf{x}_l \rangle \langle \mathbf{x}_j \mathbf{x}_k \rangle. \]  
\[ (C.18) \]

Here, \( i = 1, 2, 3, 4 \) and \( \mathbf{x}_j \) is the \( j \)th component of \( \mathbf{x} \) (the same for the other indices \( j, k, l \)).

The matrix \( \mathbb{H}(t) \) can be computed analytically. It is built from exponentials of \( \Gamma_{\pm t} \equiv (\Gamma \pm \Gamma_{12})t \) and characterizes the broadening in phase space of an initial vacuum state. Explicitly,
\[ \mathbb{H}(t) = \begin{bmatrix} h_1(t) & 0 & h_2(t) & 0 \\ 0 & h_1(t) & 0 & h_2(t) \\ h_2(t) & 0 & h_1(t) & 0 \\ 0 & h_2(t) & 0 & h_1(t) \end{bmatrix}, \]  
\[ (C.19) \]

where \( h_1(t) \) and \( h_2(t) \) are
\[ h_1(t) \equiv \tilde{n} \left[ 2 - e^{-2\Gamma_{-t}} - e^{-2\Gamma_{+t}} + 2\tilde{n} \left( 1 - e^{-2\Gamma_{-t}} - e^{-2\Gamma_{+t}} + e^{-4\Gamma_{+t}} \right) \right] [4 h(t)]^{-1}, \]  
\[ (C.20) \]
\[ h_2(t) \equiv \tilde{n} \left[ e^{-2\Gamma_{-t}} - e^{-2\Gamma_{+t}} \right] /[4 h(t)]. \]  
\[ (C.21) \]

The factor \( h(t) \) is
\[ h(t) \equiv \sqrt{\frac{\det[Q(t)]}{\det[\mathbb{H}(t)]}} = (\tilde{n} + 1 - \tilde{n} e^{-2\Gamma_{-t}})(\tilde{n} + 1 - \tilde{n} e^{-2\Gamma_{+t}}). \]  
\[ (C.22) \]

After some lengthy algebra, the fidelity can be expressed as
\[ \mathcal{F}_{\text{swap}}(t) = \frac{\xi_1(t) + \xi_2(t)}{h(t)}, \]  
\[ (C.23) \]

where
\[ \xi_1(t) \equiv h_1(t) \left[ 2 - \frac{1}{2} r_1(t) [g_1^2(t) + g_2^2(t)] + \frac{1}{2} r_2(t) g_1(t) g_2(t) \right], \]  
\[ (C.24) \]
\[ \xi_2(t) \equiv \frac{1}{2} h_1^2(t) [k_1^2(t) + k_2^2(t)] + \frac{1}{4} \cos^2(\Omega_{12} t) \left[ k_1^2(t) k_2^2(t) + h_2^2(t) k_3^2(t) \right] \\ + \frac{1}{4} \sin^2(\Omega_{12} t) \left[ h_1^2(t) k_3^2(t) + h_2^2(t) k_1^2(t) \right] + h_1(t) h_2(t) k_1(t) k_2(t). \]  
\[ (C.25) \]

Here, \( \xi_1(t) \) is the expectation \( \langle (\tilde{x}_1^2 + \tilde{y}_1^2)(1 - f_2(t)) \rangle \), while \( \xi_2(t) \) is the expectation \( \langle (\tilde{x}_1^2 + \tilde{y}_1^2) f_1(\mathbf{x}, t) \rangle \). On the other hand, \( r_1(t) \) and \( r_2(t) \) are the diagonal and off-diagonal elements, respectively, of the inverse covariance matrix \( \mathbb{Q}^{-1}(t) \) (same structure as \( \mathbb{H}(t) \)). These functions are given by
\[ r_1(t) = \left[ 2 - e^{-2\Gamma_{-t}} - e^{-2\Gamma_{+t}} \right] /[\tilde{n} r(t)], \]  
\[ (C.26) \]
\[ r_2(t) = -\left[ e^{-2\Gamma_{-t}} - e^{-2\Gamma_{+t}} \right] /[\tilde{n} r(t)]. \]  
\[ (C.27) \]
with \( r(t) \) defined as
\[
 r(t) = (1 - e^{-2\Gamma t})(1 - e^{-2\Gamma t}).
\]
We also need the exponential of the drift matrix:
\[
 \exp[A(t)] = \frac{1}{2} \begin{bmatrix}
 \beta_{1+}(t) & \beta_{2+}(t) & -\beta_{1-}(t) & -\beta_{2-}(t) \\
 -\beta_{2-}(t) & \beta_{1+}(t) & \beta_{2+}(t) & -\beta_{1-}(t) \\
 -\beta_{1-}(t) & -\beta_{2+}(t) & \beta_{1+}(t) & \beta_{2+}(t) \\
 \beta_{2+}(t) & -\beta_{1-}(t) & -\beta_{2-}(t) & \beta_{1+}(t)
\end{bmatrix},
\]
where \( \beta_{1\pm}(t) \) and \( \beta_{2\pm}(t) \) are defined as
\[
 \beta_{1\pm}(t) = g_{\pm}(t) \cos(\Omega_{12} t),
\]
\[
 \beta_{2\pm}(t) = g_{\pm}(t) \sin(\Omega_{12} t),
\]
and \( g_{\pm}(t) \) is defined as
\[
 g_{\pm}(t) = \exp(\Gamma_- t) \pm \exp(\Gamma_+ t).
\]
Finally, \( k_1(t) \) and \( k_2(t) \) are
\[
 k_1(t) = -r_1(t)g_+(t) + r_2(t)g_-(t),
\]
\[
 k_2(t) = r_1(t)g_+(t) - r_2(t)g_-(t).
\]

### C.3.2. Square-root swap gate.

We apply the same procedure as for the swap gate. Working out the relevant density matrix elements, we get
\[
 F_{\text{sqr}}(\delta) = \frac{1}{2} \int e^{-|\textbf{y}|^2} \left[ |\textbf{x}|^2 + 2 \text{Re}[e^{-i\delta}(x_1 + iy_1)(x_2 - iy_2)] \right] P(\textbf{x}, t) \, d^4\textbf{x}.
\]
This can be brought into the form
\[
 F_{\text{sqr}}(\delta) = \sqrt{\frac{\text{Det}[\underline{H}(t)]}{\text{Det}[\underline{Q}(t)]}} \int N[\textbf{h}, \textbf{0}, \underline{H}(t)] \, f_4(\textbf{h}, t) \, d^4\textbf{h},
\]
where
\[
 f_4(\textbf{h}, t) \equiv \frac{1}{2} \left[ |\textbf{h}|^2 + 2 \text{Re}[e^{-i\delta}(\textbf{h}_1 + i\textbf{h}_1)(\textbf{h}_2 - i\textbf{h}_2)] \right] \left[ 1 + f_1(\textbf{h}, t) - f_2(\textbf{h}, t) \right].
\]
After carrying out the algebra, the square-root swap fidelity is
\[
 F_{\text{sqr}}(\delta, \delta) = \frac{1}{h(t)} \left\{ \xi_1(t) + \xi_3(t, \delta, \delta) + \xi_5(t, \delta) \right\}.
\]
Here, \( \xi_3(t) \) is the expectation \( \langle |\textbf{h}|^2 f_1(\textbf{h}, t) \rangle / 2 \), \( \xi_4(t, \delta) \) is the expectation \( \langle \text{Re}[e^{-i\delta}(\textbf{h}_1 + i\textbf{h}_1)(\textbf{h}_2 - i\textbf{h}_2)] \rangle / 2 \), and \( \xi_5(t) \) is the expectation \( \langle \text{Re}[e^{-i\delta}(\textbf{h}_1 + i\textbf{h}_1)(\textbf{h}_2 - i\textbf{h}_2)] f_1(\textbf{h}, t) \rangle \). These functions are
\[
 \xi_3(t) \equiv \frac{1}{4} h_1(t)[k_1(t) + k_2(t)] + \frac{1}{8} \left[ h_1(t)k_1(t) + h_2(t)k_2(t) \right] + \frac{1}{8} \left[ h_1(t)k_2(t) + h_2(t)k_1(t) \right] + h_1(t)h_2(t)k_1(t)k_2(t),
\]
\[
 \xi_4(t, \delta) \equiv h_2(t) \cos\delta \left[ 2 - \frac{1}{4} r_1(t)[g_2(t) + g_2(t)] + \frac{1}{4} r_2(t)g_+(t)g_-(t) \right],
\]
\[
 \xi_5(t, \delta) \equiv \cos\delta \left[ \frac{1}{2} h_1(t)h_2(t)[k_1(t) + k_2(t)] + \frac{1}{4} k_1(t)k_2(t)[h_1(t) + 3h_2(t)] \right] + \frac{1}{4} \sin\delta \cos[\Omega_{12} t] \sin[\Omega_{12} t][k_1(t) - k_2(t)][h_2(t) - h_1(t)].
\]
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