Estimating Relaxation Time and Fractionality Order Parameters in Fractional Non-Fourier Heat Conduction Using Conjugate Gradient Inverse Approach in Single and Three-Layer Skin Tissues

Piran Goudarzi, Awatef Abidi, Seyed Abdollah Mansouri Mehryan, Mohammad Ghalambaz, and Mikhail A. Sheremet

Abstract: In this work, the relaxation parameter ($\tau$) and fractionality order ($\alpha$) in the fractional single phase lag (FSPL) non-Fourier heat conduction model are estimated by employing the conjugate gradient inverse method (CGIM). Two different physics of skin tissue are chosen as the studied cases; single-layer skin is exposed to laser radiation having the constant heat flux of $Q_0$. However, a heat pulse with constant temperature is imposed on the three-layer skin. The required inputs for the inverse problem in the fractional diffusion equation are chosen from the outcomes of the dual phase lag (DPL) theory. The governing equations are solved numerically by utilizing implicit approaches. The results of this study showed the efficiency of the CGIM to estimate the unknown parameters in the FSPL model. In fact, obtained numerical results of the CGIM are in excellent compatibility with the FSPL model.

Keywords: inverse fractional non-Fourier; fractional heat conduction; parameter estimations; tissues

1. Introduction

Inverse analysis has received more attention recently due to its wide applications in engineering and industry. Inverse problems are often used in engineering problems where direct measurements are difficult in the body. An inverse problem in heat transfer is important and includes obtaining surface temperature, diffuse heat flux, heat source, conductivity and displacement coefficients, and so on. The available literature shows the use of inverse analysis in the non-Fourier heat conduction problem is novel. The unknown or non-measurable parameters in the problems can be estimated using inverse analysis methods such as the conjugate gradients method with/without adjoint problem and the Levenberg–Marquardt algorithm.

The dual-phase-lag (DPL) non-Fourier technique base on the Levenberg–Marquardt non-linear parameter estimation (LMNPE) approach is utilized to predict the thermal diffusivity and the time lags at the presence of a pulse heating [1]. Hsu and Chu [2] studied a non-Fourier heat conduction electronic device to obtain the temperature of...
the surface. They used the linear least-squares method to obtain the solution. Yang [3] estimated boundary conditions in the 2D field of hyperbolic heat conduction problems. The modified Newton–Raphson technique is employed for the inverse analysis and it is observed that this method leads to simpler expressions compared to the non-linear least-squares technique. Hsu [4] provided a linear least-squares inverse technique to estimate the unknown temperature on the boundary in a 3D non-Fourier heat conduction problem. Moreover, various types of heat transfer Fourier [5,6] and non-Fourier [7,8] and bioheat transfer [9–11] problems have been investigated in the literature.

Liu and Lin [12] obtained the phase lag times of tissue by employing the DPL model, utilizing the experimental input. In this work, a hybrid scheme of the least-squares technique, change of variables for a direct problem, and Laplace transform are applied. They also investigated the impact of measurement locality on the computed results. Azimi et al. [13] employed the ACGM in an inverse non-Fourier heat transfer problem to compute the root temperature of a fin having diverse profiles. They used the function-estimation form of ACGM, utilizing the border temperature evaluation. They found that ACGM can be used to analyze the non-Fourier inverse heat transfer of fins in various conditions.

Das et al. [14] estimated the coefficients of extinction and conduction-radiation by minimizing the objective function in a non-Fourier conduction-radiation heat transfer problem. The genetic algorithm (GA) is used for this purpose. Ghazizadeh et al. [15] estimated the relaxation time and fractionality order in the fractional single-phase lag (FSPL) heat model for two different physics. The LMNPE method is used to solve the inverse FSPL heat conduction. Their results illustrated that the LMNPE technique can be successfully used to solve the problem of inverse fractional heat transfer. Azimi et al. [16] estimated root temperature distribution in several fins having non-Fourier behavior. This study considered the function-estimation form of the ACGM applying boundary temperature valuations to solve the inverse problem. The results showed that the ACGM method can be recognized as a stable and reliable method for determining temperature boundary conditions in the non-Fourier problems.

Wu et al. [17] employed a conjugate gradient inverse method (CGIM) to estimate the unknown boundary pulse heat flux in a limitless-length cylinder. They solved the problem with the hyperbolic heat conduction and DPL heat transfer theory. Mozafarifard et al. [18] employed FSPL and DPL techniques to investigate transient non-Fourier heat transfer in an upstanding expanded surface with an energy source at the presence of a periodic temperature imposed on the expanded surface root. This study, for the first time, used LMNPE to obtain the heat flux relaxation time and fractional derivation orders for an upstanding fin with the mentioned conditions.

Ali et al. [19] used an inverse method to determine the time-dependent source term for the space-time fractional differential equation based on Caputo derivative in two problems. They investigated the stability and well pose of the inverse problem. Cheng et al. [20] obtained the space-dependent source term in the time-fractional diffusion equation based on Caputo derivative. They used the CGM method to solve the inverse problem. Their results show the effectiveness of the CGM method to obtain unknown functions. Sun and Liu [21] applied the CGM method to obtain a time-dependent source in the time-fractional diffusion equation. Their results were validated by several numerical examples. Tuan et al. [22] determined an unknown source term for fractional diffusion equation based on the Riemann-Liouville derivative. They employed the quasi-boundary value approach to arrange the unstable inverse problem. Their results show convergence of the method.

In the present research, for the first time, the CGIM algorithm is utilized to estimate the undetermined parameters of τ and α of the fractional non-Fourier model in two different physics of skin tissues with single or three-layer tissues. The single-layer tissue is affected by laser radiation having the constant energy of $Q_{in}$. However, three-layer skin tissue is heated by a source having a constant temperature. These two physics with different conditions were studied by Goudarzi and Azimi [23] to develop the capability of FSPL. The
relaxation and fractionality order parameters in a try and error manner are investigated to capture FSPL results by employing DPL parameters as inputs.

2. Conjugate Gradient Inverse Method

The CGIM is a powerful iterative technique for solving linear and nonlinear inverse problems of parameter estimation. In the iterative procedure of the CGIM, at each iteration, appropriate step size is chosen along a descanting direction to minimize the objective function [24]. Herein, the fractionality order, i.e., \( \alpha \), and the relaxation parameter, i.e., \( \tau \), in the FSPL model are unknown. This inverse problem is solved by utilizing the measured outcomes of the case studies presented in [25,26]. This method minimizes the least-squares norm of the evaluated temperatures resulted from DPL, i.e., \( \theta_{DPL} \), and FSPL, i.e., \( \theta_{C} \), as expressed below [24]:

\[
S(P) = [\theta_{C}(P) - \theta_{DPL}]^{T} [\theta_{C}(P) - \theta_{DPL}] \quad (1)
\]

where \( P_{Tr} = [\tau, \alpha] \). In the current work, an alternative technique is used for the estimation of unknown parameters in the CGIM based on [20]. The CGIM algorithm can be found in [24] in detail. By initial guess for unknown parameters, i.e., \( \alpha_{0} \) and \( \tau_{0} \), the iteration process starts. Then, the governing equations are solved and follow the iterative procedure until the stopping criterion. The stopping criterion is as follows:

\[
|P_{m+1} - P_{m}| < \varepsilon \quad (2)
\]

where \( \varepsilon \) is a suitable tolerance and \( m \) is the iterations number. It is worth mentioning that the discretized equations are implemented using FORTRAN programming language. The governing equations of each case study are separately given below.

3. Governing Equations and Discretization

In this research, two different cases are studied. One is related to single-layer skin tissue and the other one is for three-layer tissue. As previously discussed, these cases were investigated by Goudarzi and Azimi [23]. They used the numerical FSPL method to simulate non-Fourier heat conduction in the skin tissue to calculate undetermined the parameters by try and error approach. In the present research, inverse analysis based on the conjugate gradient method is used.

3.1. Test Case 1

3.1.1. Direct Problem

Figure 1 depicts a schematic view of the single-layer skin. The thickness, i.e., \( h \), and initial temperature, i.e., \( \theta_{0} \), of the skin are 1 mm and 37 °C, respectively. When \( t = 0^+ \), laser radiation having an energy of \( Q_{in} \) is radiated on the left side of the domain for 5 s, then it is stopped. The governing equation of this case is as the following [23]:

\[
\partial q \partial t + \tau q \partial^2 q \partial^2 t + D \frac{\partial^2 q}{\partial x^2} + Dw_{b} \rho_{b} c_{b} \frac{\partial \theta}{\partial x} = 0 \quad (3)
\]

The controlling boundary conditions can be defined as the following:

\[
\forall x, t \mid x = 0, \quad 0 < t < 5 \text{ s} \Rightarrow q = Q_{in}(1 - R_d) \\
\forall x, t \mid x = h, \quad 0 < t < 40 \text{ s} \Rightarrow q = 0 
\]

The initial conditions are as follows:

\[
\forall x, t \mid 0 < x < h, \quad t = 0 \Rightarrow q(x) = 0, \quad \frac{\partial q}{\partial t} = 0
\]
The controlling equation of temperature field is:
\[ \rho c \frac{\partial \theta}{\partial t} = -\frac{\partial q}{\partial x} + w_b \rho_b c_b (\theta_b - \theta) + q_{net} + q_{ext} \] (6)

The discretization of the governing equations and the numerical method can be found in [23]. In Figure 1 one can find the following parameters, like \( P \) is a general node, \( E \) and \( W \) are its neighboring nodes to the east and west. The east and west sides of the control volume are identified as \( e \) and \( w \). \( \Delta x \) and \( h \) are the size of control volume and the thickness of the skin tissue, respectively.

![Figure 1. A schematic view of the physical model.](image)

3.1.2. Inverse Problem

The conjugate gradient parameter estimation method is used to evaluate the underdetermined parameters in the FSPL model, described in Section 2 in detail. There are two undefined parameters, namely \( \tau \) and \( \alpha \). Hence, two sensitivity equations along with the initial and boundary conditions are needed to be solved: one is for \( \tau \) and the other one is for \( \alpha \). To obtain the sensitivity equation for \( \tau \), Equations (3)–(5) should be derived with respect to \( \tau \).

\[ \frac{\partial}{\partial \tau} \left( D \frac{\partial q}{\partial t} \right) + \frac{\partial}{\partial \tau} \left( \tau \frac{\partial q}{\partial x} \right) = \frac{\partial}{\partial \tau} \left( D \frac{\partial^2 q}{\partial x^2} \right) + \frac{\partial}{\partial \tau} \left( D w_b \rho_b c_b \frac{\partial \theta}{\partial x} \right) \] (7)

\[ \frac{\partial}{\partial \tau} \left( \rho \frac{\partial T}{\partial t} \right) = \frac{\partial}{\partial \tau} \left( -\frac{\partial q}{\partial x} \right) + \frac{\partial}{\partial \tau} \left( D w_b \rho_b c_b (\theta_b - \theta) \right) + \frac{\partial q_{net}}{\partial \tau} + \frac{\partial q_{ext}}{\partial \tau} \] (8)

with the following boundary conditions:

\[ \forall x, t \mid x = 0, 0 < t < 5s \Rightarrow \frac{\partial q}{\partial \tau} = \frac{\partial}{\partial \tau} (Q_{in}(1 - R_d)) \] (9)

\[ \forall x, t \mid x = h, 0 < t < 40s \Rightarrow \frac{\partial q}{\partial \tau} = 0 \]

The initial condition is

\[ \forall x, t \mid 0 < x < h, t = 0 \Rightarrow \frac{\partial \theta(x)}{\partial \tau} = \frac{\partial \theta_b}{\partial \tau} = 0, \quad \frac{\partial q}{\partial \tau} = 0, \quad \frac{\partial}{\partial \tau} \left( \frac{\partial q}{\partial t} \right) = 0 \] (10)

As presented in Appendix A, the sensitivity coefficients equation of the relaxation time is expressed as the following:

\[ \rho \frac{\partial l_t}{\partial \tau} + \tau_q \alpha^\alpha \frac{\partial^2 l_t}{\partial x^2} = D \frac{\partial^2 l_t}{\partial x^2} + D w_b \rho_b c_b \frac{\partial l_t}{\partial x} - \alpha^\alpha \tau_q \alpha^\alpha \frac{\partial l_t}{\partial x} \] (11)

with the following boundary conditions:

\[ \forall x, t \mid x = 0, 0 < t < 5s \Rightarrow l_t = 0 \]

\[ \forall x, t \mid x = h, 0 < t < 40s \Rightarrow l_t = 0 \] (12)
Initial condition is:

\[ \forall x, t | 0 < x < h, \ t = 0 \Rightarrow f_\tau = 0, \quad J_{\alpha} = 0, \quad \frac{\partial J_{\alpha}}{\partial t} = 0 \]  
(13)

where \( J_{\alpha} \) and \( f_\tau \) are the heat flux and temperature sensitivities with respect to \( \tau \), respectively. Transferring Equations (3)–(5) to \( \alpha \) space leads to the following equations:

\[
\frac{\partial}{\partial \alpha} \left( \frac{\partial q}{\partial t} \right) + \frac{\partial}{\partial \alpha} \left( \tau_\alpha \frac{\alpha^{2}+\alpha_{q}}{\alpha t^{2}} \right) = \frac{\partial}{\partial \alpha} \left( D \frac{\partial^2 q}{\partial x^2} \right) + \frac{\partial}{\partial \alpha} \left( D \frac{\partial \theta}{\partial x} \right)  
(14)
\]

\[
\frac{\partial}{\partial \alpha} \left( \frac{\partial \theta}{\partial t} \right) = \frac{\partial}{\partial \alpha} \left( - \alpha^{\frac{\alpha}{\alpha}} \frac{\partial q}{\partial x} \right) + \frac{\partial q_{\text{met}}}{\partial \alpha} + \frac{\partial q_{\text{ext}}}{\partial \alpha}  
(15)
\]

According to the Appendix A, the sensitivity coefficients equations of fractional order can be obtained as:

\[
\frac{\partial J_{\alpha}}{\partial t} + \tau_\alpha \frac{\partial}{\partial \alpha} \left( \frac{\alpha^{2}+\alpha_{q}}{\alpha t^{2}} \right) = D \frac{\partial^2 q}{\partial x^2} + D \frac{\partial \theta}{\partial \alpha} - \tau_\alpha \ln(\tau) \frac{\alpha^{2}+\alpha_{q}}{\alpha t^{2}}  
(16)
\]

with the boundary and initial conditions expressed below:

\[ \forall x, t | x = 0, \ 0 < t < 5s \Rightarrow J_{\alpha} = 0 \]
\[ \forall x, t | x = h, \ 0 < t < 40s \Rightarrow J_{\alpha} = 0 \]
\[ \forall x, t | 0 < x < h, \ t = 0 \Rightarrow f_\tau = 0, \quad J_{\alpha} = 0, \quad \frac{\partial J_{\alpha}}{\partial t} = 0 \]  
(17)

where \( J_{\alpha} \) and \( f_\tau \) are the heat flux and temperature sensitivities with respect to \( \alpha \), respectively. The finite volume method is employed to discretize Equations (11)–(13), (16), and (17) as presented in Appendix B. Then, the tridiagonal matrix algorithm is applied to solve the algebraic form of controlling equations.

3.2. Test Case 2

3.2.1. Direct Problem

Initially, the skin is exposed to a heat pulse with a constant temperature of 100 °C for 15 s. The heat flux applied to the skin can be resulted from immediate contact with the hot water. After heating, cooling the skin surface is done by a water-ice mixture of 0 °C for 30 s. Figure 2 depicts a schematic view of physics.

The governing equations can be formulated as follows:

\[
\frac{\partial \theta}{\partial t} + \tau_\alpha \frac{\alpha^{2}+\alpha_{q}}{\alpha t^{2}} \frac{\partial q_{\text{ext}}}{\partial \alpha} + \tau_\alpha \frac{\partial q_{\text{met}}}{\partial \alpha} = D \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial q_{\text{ext}}}{\partial \alpha} \left( \frac{\partial q_{\text{ext}}}{\partial \alpha} \right) + \frac{\partial q_{\text{met}}}{\partial \alpha} \left( \frac{\partial q_{\text{met}}}{\partial \alpha} \right) + \frac{\tau_\alpha}{\rho c_p} \frac{\partial \theta}{\partial \alpha} \left( \frac{\partial \theta}{\partial \alpha} \right)  
(18)
\]

Here \( i \) is the number of layers.

The boundary conditions are as follows:

\[ \forall x, t | x = 0, \ 0 < t \leq 45 \Rightarrow \theta(t) = 100(1 - u(t - 15)) \]
\[ \forall x, t | x = h, \ 0 < t \Rightarrow \theta(t) = \theta_{b} = 37 \ ^{\circ}C \]  
(19)

Moreover, the initial conditions are as the following:

\[ \forall x, t | 0 < x < h, \ t = 0 \Rightarrow \theta(x) = \theta_{b} = 37 \ ^{\circ}C, \quad \frac{\partial \theta(x)}{\partial t} = \frac{\partial^2 \theta(x)}{\partial t^2} = 0 \]  
(20)

The discretization of the governing equations and the utilized numerical method are discussed in [23] in detail.
3.2.2. Inverse Problem

Similar to the previous one, for using the conjugate gradient method, two sensitivity equations should be obtained with their initial and boundary conditions: one is for \( \tau \) and the other one is for \( \alpha \). Equations (18)–(20) can be derived with respect to \( \tau \) as the following:

\[
\frac{\partial}{\partial \tau} \left( \frac{\partial \alpha}{\partial t} + \tau q^d \frac{\partial \alpha}{\partial t^{1+\alpha}} + \frac{w_b \rho_b c_b}{\rho_i c_i} \tau q^c \frac{\partial \alpha}{\partial t^{1+\alpha}} \right) = \frac{\partial}{\partial \tau} \left( D_i \frac{\partial^2 \alpha}{\partial x^2} + \frac{w_b \rho_b c_b}{\rho_i c_i} (\theta_b - \theta) + \frac{\rho_e + \rho_m}{\rho_i c_i} \right)
\]

(21)

with the boundary and initial conditions:

\[
\begin{align*}
\forall x, t | x = 0, 0 < t \leq 45, & \quad \frac{\partial \alpha(t)}{\partial \tau} = \frac{\partial \alpha(t)}{\partial \tau} (100 - 100u(t - 15)) \\
\forall x, t | x = h, 0 < t, & \quad \frac{\rho_b}{\rho_i c_i} = \frac{\rho_b}{\rho_i c_i} \\
\forall x, t | 0 < x < h, t = 0, & \quad \frac{\partial \alpha(x)}{\partial \tau} = \frac{\partial \alpha(x)}{\partial \tau} \\
\forall x, t | x = h, 0 < t \Rightarrow \frac{\partial \alpha(x)}{\partial \tau} = 0
\end{align*}
\]

(22)

The equation expressed below is the sensitivity coefficients equation of the relaxation time:

\[
\frac{\partial f_{\tau}}{\partial \tau} + \tau q^d \frac{\partial f_{\tau}}{\partial t^{1+\alpha}} + \frac{w_b \rho_b c_b}{\rho_i c_i} \tau q^c \frac{\partial f_{\tau}}{\partial t^{1+\alpha}} = D_i \frac{\partial^2 f_{\tau}}{\partial x^2} - \frac{w_b \rho_b c_b}{\rho_i c_i} f_{\tau} - \alpha q^c \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}
\]

(23)

with the initial and boundary conditions expressed below:

\[
\begin{align*}
\forall x, t | 0 < t < 45, t = 0 & \Rightarrow f_{\tau}(x) = 0 \quad \text{and} \quad \frac{\partial f_{\tau}(x)}{\partial t} = \frac{\partial f_{\tau}(x)}{\partial t} = 0 \\
\forall x, t | x = 0, 0 < t < 45 & \Rightarrow f_{\tau} = 0 \\
\forall x, t | x = h, 0 < t \Rightarrow f_{\tau} = 0
\end{align*}
\]

(24)

where \( f_{\tau} \) is the temperature sensitivity to \( \tau \).

The sensitivity equations of the layers for \( \alpha \) can be obtained by transferring Equations (18)–(20) to the \( \alpha \) space:

\[
\frac{\partial}{\partial \alpha} \left( \frac{\partial \theta}{\partial t} + \tau q^d \frac{\partial \alpha}{\partial t^{1+\alpha}} + \frac{w_b \rho_b c_b}{\rho_i c_i} \tau q^c \frac{\partial \alpha}{\partial t^{1+\alpha}} \right) = \frac{\partial}{\partial \alpha} \left( D_i \frac{\partial^2 \theta}{\partial x^2} + \frac{w_b \rho_b c_b}{\rho_i c_i} (\theta_b - \theta) + \frac{\rho_e + \rho_m}{\rho_i c_i} \right)
\]

(25)

The controlling boundary and initial conditions are:

\[
\begin{align*}
\forall x, t | x = 0, 0 < t \leq 45, & \quad \frac{\partial \alpha(t)}{\partial \alpha} = 100 \frac{\partial \alpha(t)}{\partial \alpha} (1 - u(t - 15)) \\
\forall x, t | x = h, 0 < t, & \quad \frac{\partial \alpha(x)}{\partial \alpha} = \frac{\partial \alpha(x)}{\partial \alpha} \\
\forall x, t | 0 < x < h, t = 0, & \quad \frac{\partial \alpha(x)}{\partial \alpha} = \frac{\partial \alpha(x)}{\partial \alpha} \quad \text{and} \quad \frac{\partial \alpha(x)}{\partial \alpha} = \frac{\partial \alpha(x)}{\partial \alpha} = 0
\end{align*}
\]

(26)
As shown in Appendix A, the sensitivity coefficients equation for the fractionality order, i.e., $\alpha$, is expressed below.

$$\frac{\partial J_\alpha}{\partial t} + \tau_q \alpha \frac{\partial \alpha}{\partial t} + \frac{w_0 \rho \nu c_b \tau_q \alpha}{\rho_c} \frac{\partial J_\alpha}{\partial x} = D_i \frac{\partial^2 J_\alpha}{\partial x^2} - \frac{w_0 \rho \nu c_b}{\rho_c} J_\alpha - \tau_q \alpha \ln \tau_q \frac{\partial J_\alpha}{\partial x}$$

$$-\tau_q \alpha \frac{\partial J_\alpha}{\partial x} \sum_{j=1}^{n} W^{j}\alpha \left(\theta_i^{n-j+1} - 2\theta_i^{n-j} + \theta_i^{n-j-1}\right) - \tau_q \alpha \sigma_1 \sum_{j=1}^{n} \frac{\partial w_0}{\partial x} \left(\theta_i^{n-j+1} - 2\theta_i^{n-j} + \theta_i^{n-j-1}\right)$$

(27)

The initial and boundary conditions in the $\alpha$ space are defined as the following:

$$\forall x, t \mid 0 < x \leq h, t = 0 \Rightarrow J_\alpha(x) = 0 \text{ and } \frac{\partial J_\alpha(x)}{\partial t} = \frac{\partial^2 J_\alpha(x)}{\partial t^2} = 0$$

(28)

From the above equations, $J_\alpha$ is the sensitivity equation to $\alpha$. The tridiagonal matrix algorithm is utilized to solve the governing algebraic system of equations resulting from the finite difference method. It is worth noting that the values of relevant parameters used in the present work are for real skin tissue as presented in [25].

4. Results and Discussion

In this section, the calculated numerical results, including the time lag and fractionality order in the FSPL non-Fourier model are presented. In both cases, the initial values of $\alpha$ and $\tau$ are accidentally chosen. Then, the direct and inverse problems, as well as the sensitive equations, are solved by using the inverse conjugate method introduced above. The solution process continues to iterate until satisfying the convergence criteria according to Equation (2). The tolerance in the stopping criterion, i.e., $\varepsilon = 10^{-4}$, is considered for both cases in Equation (2).

4.1. Test Case 1

For test case 1, the initial guesses of $\alpha_0 = 0.9$ and $\tau_0 = 10$ are used. The direct problem with Equations (3)–(6), and the sensitivity problem with Equations (11)–(13), (16), and (17) are solved and the convergence occurs after 38 iterations. The time-lag and fractionality as two unknown parameters are obtained as the follows:

$$\tau = 16 \text{ s}, \alpha = 0.9985068$$

Figure 3 depicts the temperature history on the skin surface obtained by conjugate gradient parameter estimation inverse analysis. As shown, the CGIM in the estimation of unknown parameters in the non-Fourier heat conduction fractional single-phase lag model is accurate and reliable. Figure 4 illustrates the Jacobian coefficient of the order of fractionality and the relaxation time. To obtain the correct estimation, the Jacobian coefficients should not be linearly related to each other. As can be seen in Figure 4, there is no linear dependence between Jacobian coefficients.
4.1. Test Case 1

For test case 1, the initial guesses are $\alpha_0 = 0.9$ and $\tau_0 = 10$. The direct problem with the governing Equations (18)–(20), and the sensitivity problem with Equations (23)–(28) are solved and the convergence occurs after 52 iterations. The two unknown parameters are estimated as the following:

$$\tau = 9.888 \text{ s}, \quad \alpha = 0.986$$

The temperature history resulted from conjugate gradient parameter estimation inverse analysis is depicted in Figure 5. Figure 6 shows the Jacobian coefficient of the fractionality order and the relaxation time. As can be observed, the Jacobian coefficients are non-zero and non-linearly related to each other. Therefore, the required conditions for obtaining the unknown parameters are provided. Once again, the accuracy of the CGIM for estimating unknown parameters in the non-Fourier heat conduction FSPL model is proved. For other values of temperature phase lag in the DPL method, the CGIM is utilized to determine the fractionality and time-lag as the unknown parameters. The results are tabulated in Table 1. As $\tau_T$ tends to zero, the fractionality, i.e., $\alpha$, approaches one. This means that the fractional non-Fourier model is approaching the single phase non-Fourier model.

Figure 3. Comparison of the estimation temperature history from the FSPL model with the measured temperature from the DPL model for test case 1.

Figure 4. Jacobian coefficient for test case 1; (a) time-lag Jacobian coefficient and (b) fractionality Jacobian coefficient.

4.2. Test Case 2

Herein, the initial guesses are $\alpha_0 = 0.9$ and $\tau_0 = 15$. The direct problem with the governing Equations (18)–(20), and the sensitivity problem with Equations (23)–(28) are solved and the convergence occurs after 38 iterations. The time-lag and fractionality are solved and the convergence occurs after 52 iterations. The two unknown parameters are estimated as the following:

$$\tau = 16 \text{ s}, \quad \alpha = 0.9985$$

The temperature history resulted from conjugate gradient parameter estimation inverse analysis is depicted in Figure 5. Figure 6 shows the Jacobian coefficient of the fractionality order and the relaxation time. As can be observed, the Jacobian coefficients are non-zero and non-linearly related to each other. Therefore, the required conditions for obtaining the unknown parameters are provided. Once again, the accuracy of the CGIM for estimating unknown parameters in the non-Fourier heat conduction FSPL model is proved. For other values of temperature phase lag in the DPL method, the CGIM is utilized to determine the fractionality and time-lag as the unknown parameters. The results are tabulated in Table 1. As $\tau_T$ tends to zero, the fractionality, i.e., $\alpha$, approaches one. This means that the fractional non-Fourier model is approaching the single phase non-Fourier model.
thermal wave model. It is also observed that an increment in $\tau_q$ leads to increasing time-lag, meaning an increase in non-Fourier effects.

![Figure 5. Comparison of the estimation temperature history from the FSPL model with the measured temperature from the DPL model for test case 2 (a) at the ED interface and (b) at the DF interface in skin tissue.](image)

![Figure 6. Jacobian coefficient for test case 2 (a) time-lag Jacobian coefficient and (b) fractionality Jacobian coefficient.](image)

**Table 1.** Estimation values of fractionality and time-lag in the conjugate gradient method in three samples for test case 2.

| Sample | $\alpha$   | $\tau$          | $\tau_f$ | $\tau_q$ |
|--------|------------|----------------|----------|----------|
| A      | 0.98570786 | 9.88794652387 | 0.005    | 10       |
| B      | 0.98259319 | 9.82287275801 | 0.05     | 10       |
| C      | 0.95851395 | 9.87599109128 | 0.1      | 10       |
| D      | 0.93391580 | 0.5792302938   | 0.05     | 1        |
| E      | 0.98259319 | 9.82287275801 | 0.05     | 10       |
| F      | 0.98976278 | 14.831300395   | 0.05     | 15       |
5. Conclusions

In this paper, conjugate gradient parameter estimation inverse analysis is employed to determine the parameters of time-lag and fractionality in the fractional non-Fourier heat conduction model for two different skin tissue cases. The finite volume and difference numerical methods are used for solving the direct and sensitivity problems of test cases 1 and 2, respectively. The results show the ability and precision of the CGIM analysis for parameter estimation in the FSPL heat conduction model. This investigation also expresses that the CGIM analysis can be successfully applied for the parameter estimation of the fractional heat equation. Moreover, it is concluded that the CGIM application can be expanded for parameter estimation in fractional calculus.

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Nomenclature

Latin symbols
- \( c \): tissue heat capacity, \( \text{Jkg}^{-1} \text{K}^{-1} \)
- \( c_b \): blood heat capacity, \( \text{Jkg}^{-1} \text{K}^{-1} \)
- \( D \): coefficient of thermal diffusion, \( \text{WJ}^{-1} \text{m}^{-3} \)
- \( D^p_t \): time derivative order
- \( f(t) \): continuous function
- \( h \): skin thickness (mm)
- \( I_{\alpha} \): sensitivity coefficient of heat respect to time lag
- \( I_{\tau} \): sensitivity coefficient of heat respect to order of fractionality
- \( I_{\alpha} \): sensitivity coefficient of heat respect to time lag
- \( k \): tissue thermal conductivity, \( \text{Wm}^{-1} \text{K}^{-1} \)
- \( m \): number of iterations
- \( P \): unknown parameters in inverse problem
- \( q_{\text{gen}} \): generated heat in tissue, \( \text{Wm}^{-3} \)
- \( q_{\text{met}} \): metabolic heating source, \( \text{Wm}^{-3} \)
- \( Q_{\text{in}} \): laser intensity, \( \text{Wcm}^{-2} \)
- \( R_d \): diffusion reflection
- \( t \): time, s
- \( t_f \): time duration from the onset to the end, s
- \( t_{\tau} \): time period of laser radiation on the skin, s
- \( u(t) \): unit step function
- \( w_b \): blood perfusion rate, \( \text{m}^3 \text{m}^{-3} \text{ tissue} \)
- \( w_j \): average of weighted arithmetic
- \( w_r \): weight function
\[\frac{\partial}{\partial \alpha} \left( \frac{d^{n+1} \theta}{d\tau^{n+1}} \right) = \frac{\partial \sigma_{1+\alpha}}{\partial \alpha} \sum_{j=1}^{n} w_{j}^{1+\alpha} \left( -2\theta_{i}^{n-j} + \theta_{i}^{n-j+1} + \theta_{i}^{n-j-1} \right) + \sigma_{1+\alpha} \sum_{j=1}^{n} w_{j}^{1+\alpha} \frac{\partial}{\partial \alpha} \left( -2\theta_{i}^{n-j} + \theta_{i}^{n-j+1} + \theta_{i}^{n-j-1} \right) \]

\[\frac{\partial \sigma_{1+\alpha}}{\partial \alpha} = \sigma_{1+\alpha} \left[ \frac{-\Psi(1-\alpha)}{\Gamma(1-\alpha)} + \frac{1}{1-\alpha} - \ln(\Delta t) \right] \]

\[\frac{\partial \sigma_{\alpha}}{\partial \alpha} = \sigma_{\alpha} \left[ \frac{-\Psi(1-\alpha)}{\Gamma(1-\alpha)} + \frac{1}{1-\alpha} - \ln(\Delta t) \right] \]

\[\frac{\partial \sigma_{1-\alpha}}{\partial \alpha} = -\Psi(1-\alpha) \Gamma(1-\alpha) \]

\[\frac{\partial w_{j}^{1+\alpha}}{\partial \alpha} = \left( j - 1 \right)^{1-\alpha} \ln(j - 1) - j^{1-\alpha} \ln(j) \]
Appendix B

The finite volume approach is employed to discretize the controlling equations. The discretization of the sensitivity equation for \( \tau \), discussed in [23] in detail, can be obtained as the following:

\[
\begin{align*}
    a_p f^{+\Delta t}_{q+\rho} &= a_E f^{+\Delta t}_{q+E} + a_W f^{+\Delta t}_{q+W} + b \\
    a_E &= D \frac{\Delta t}{\Delta x}, \quad a_W = D \frac{\Delta t}{\Delta x} \\
    a_p &= a_E + a_W + \Delta x \tau_q^\alpha \sigma_\alpha \\
    b &= [\Delta x + 2\Delta x \tau_q^\alpha \sigma_\alpha] f^{+\Delta t}_{q+\rho} - \Delta x \tau_q^\alpha \sigma_\alpha f_{q+\rho}^{+\Delta t} + \Delta x D w_b \rho \sigma_\alpha \left( \frac{f^{+\Delta t}_{E} - f^{+\Delta t}_{W}}{2} \right)
\end{align*}
\]  

(A5)

where

\[
\begin{align*}
    \text{source 1} &= \sum_{j=2}^{l} w_j^\alpha \left( f_{q+\rho}^{t+j+\Delta t} - f_{q+\rho}^{t+j-\Delta t} \right) \\
    \text{source 2} &= \sum_{j=2}^{l} w_j^\alpha \left( f_{q+E}^{t+j+\Delta t} - f_{q+E}^{t+j-\Delta t} \right) \\
    \text{source 3} &= \sum_{j=1}^{l} w_j^\alpha \left( q_{p}^{t+j+\Delta t} - q_{p}^{t+j-\Delta t} - \Delta x \tau_q^\alpha \sigma_\alpha \right) \\
    \text{source 4} &= \sum_{j=1}^{l} w_j^\alpha \left( q_{p}^{t+j+1} - q_{p}^{t+j-1} \right)
\end{align*}
\]

(A6)

where \( w \) is the weighted arithmetic mean [27],

\[
\begin{align*}
    \sigma_\alpha &= \frac{1}{\left( \frac{1}{2-\alpha} \right)} \cdot \frac{1}{1-\alpha} \cdot \frac{1}{\Delta x} \\
    w_j^\alpha &= \left( j^{2-\alpha} - (j-1)^{2-\alpha} \right)
\end{align*}
\]

(A7)

the sensitivity of temperature, i.e., \( f_{\tau} \), is reached as the following:

\[
f_{\tau}^{t+\Delta t} = f_{\tau}^{t} + \Delta t \rho c \left[ - f_{q+\rho}^{t+\Delta t} f_{q+E}^{t+\Delta t} - w_b \rho c f_{\tau}^{t+\Delta t} \right]
\]

(A8)

The discretization of the sensitivity equation for \( \alpha \) is also obtained as the following:

\[
\begin{align*}
    a_p f^{+\Delta t}_{q+\alpha} &= a_E f^{+\Delta t}_{q+E} + a_W f^{+\Delta t}_{q+W} + b \\
    a_E &= D \frac{\Delta t}{\Delta x}, \quad a_W = D \frac{\Delta t}{\Delta x} \\
    a_p &= a_E + a_W + \Delta x \tau_q^\alpha \sigma_1^{\alpha+1} \\
    b &= [\Delta x + 3\Delta x \tau_q^\alpha \sigma_1^{\alpha+1}] f^{+\Delta t}_{q+\alpha} - 3\Delta x \tau_q^\alpha \sigma_1^{\alpha+1} f_{q+\alpha}^{+\Delta t} + 3\Delta x \tau_q^\alpha \sigma_1^{\alpha+1} f_{q+\alpha}^{t-\Delta t} + \Delta x D w_b \rho \sigma_\alpha \left( \frac{f^{+\Delta t}_{E} - f^{+\Delta t}_{W}}{2} \right)
\end{align*}
\]

(A9)
where

\[
\begin{align*}
\text{source 1} & = \sum_{j=1}^{t+\Delta t} w_j^\alpha \left( q_p^{t-j+\Delta t+1} - q_p^{t-j+\Delta t} \right) \\
\text{source 2} & = \sum_{j=1}^{t} w_j^\alpha \left( q_p^{t-j+1} - q_p^{t-j} \right) \\
\text{source 3} & = \sum_{j=2}^{t+\Delta t} w_j^{1+\alpha} \left( q_p^{t+j+\Delta t+1} - 2q_p^{t+j+\Delta t} + q_p^{t+j+\Delta t-1} \right) \\
\text{source 4} & = \sum_{j=2}^{t} w_j^{1+\alpha} \left( q_p^{t+j+1} - 2q_p^{t+j} + q_p^{t+j-1} \right) \\
\text{source 5} & = \sum_{j=1}^{t+\Delta t} \frac{\partial q_p^{1+\alpha}}{\partial \alpha} \left( q_p^{t+j+\Delta t+1} - 2q_p^{t+j+\Delta t} + q_p^{t+j+\Delta t-1} \right) \\
\text{source 6} & = \sum_{j=1}^{t} \frac{\partial q_p^{1+\alpha}}{\partial \alpha} \left( q_p^{t+j+1} - 2q_p^{t+j} + q_p^{t+j-1} \right) \\
\text{source 7} & = \sum_{j=2}^{t+\Delta t} w_j^{1+\alpha} \left( q_{q,p}^{t-j+\Delta t+1} - 2q_{q,p}^{t-j+\Delta t} + q_{q,p}^{t-j+\Delta t-1} \right) \\
\text{source 8} & = \sum_{j=2}^{t} w_j^{1+\alpha} \left( q_{q,p}^{t-j+1} - 2q_{q,p}^{t-j} + q_{q,p}^{t-j-1} \right)
\end{align*}
\]

(A10)

Finally, the temperature sensitivity with respect to \( \alpha \), i.e., \( f_\alpha \), is obtained as follows:

\[
f_\alpha^{t+\Delta t} = f_\alpha^t + \frac{\Delta t}{\rho c} \left[ - \frac{q^{t+\Delta t}}{q_{q,p}^{t+\Delta t}} - \frac{q^{t+\Delta t}}{q_{q,p}^{t+\Delta t}} \right]
\]

(A12)

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