COMPLETELY INTEGRABLE TORUS ACTIONS ON COMPLEX MANIFOLDS WITH FIXED POINTS

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Abstract. We show that if a holomorphic $n$ dimensional compact torus action on a compact connected complex manifold of complex dimension $n$ has a fixed point then the manifold is equivariantly biholomorphic to a smooth toric variety.

1. Introduction

We begin by recalling some notions from the theory of toric varieties. We work in the vector space $\text{Lie}(S^1)^n \cong \mathbb{R}^n$ with the lattice $\text{Hom}(S^1, (S^1)^n) \cong \mathbb{Z}^n$. Here, we identify $\text{Lie}(S^1)$ with $\mathbb{R}$ such that the exponential map $\exp: \mathbb{R} \to S^1$ is $t \mapsto e^{2\pi it}$.

A unimodular fan is a finite set $\Delta$ of convex polyhedral cones with the following properties.

1. A face of a cone in $\Delta$ is also a cone in $\Delta$.
2. The intersection of two cones in $\Delta$ is a common face.
3. Every cone in $\Delta$ is unimodular, i.e., it has the form $\text{pos}(\lambda_1, \ldots, \lambda_k)$ where $\lambda_1, \ldots, \lambda_k$ is part of a $\mathbb{Z}$-basis of the lattice. Here, pos denotes the positive span: the set of linear combinations with non-negative coefficients.

A fan $\Delta$ is complete if the union of the cones in $\Delta$ is all of $\text{Lie}(S^1)^n$.

The theory of toric varieties associates to a unimodular fan $\Delta$ a complex manifold $M_\Delta$ with a holomorphic $(\mathbb{C}^*)^n$-action with the following properties.

1. The fixed points in $M_\Delta$ are in bijection with the $n$-dimensional cones in $\Delta$.
2. Let $p$ be a fixed point in $M_\Delta$. Then the isotropy weights at $p$ are a $\mathbb{Z}$-basis to the lattice $\text{Hom}((S^1)^n, S^1) \subset (\text{Lie}(S^1)^n)^*$. Moreover, let $\lambda_1, \ldots, \lambda_n$ be the dual basis; then the cone in $\Delta$ that corresponds to $p$ is $\text{pos}(\lambda_1, \ldots, \lambda_n)$.
3. The manifold $M_\Delta$ is compact if and only if the fan $\Delta$ is complete.
Explicitly, let \( \lambda_1, \ldots, \lambda_m \in \mathbb{Z}^n \) be the primitive generators of the one dimensional cones in \( \Delta \). Each \( \lambda_i \) encodes a homomorphism \( a \mapsto a^{\lambda_i} \) from \( \mathbb{C}^* \) to \( (\mathbb{C}^*)^n \); together they give a homomorphism \( \pi : (a_1, \ldots, a_m) \mapsto \prod_{j=1}^{m} a_j^{\lambda_j} \) from \( (\mathbb{C}^*)^m \) to \( (\mathbb{C}^*)^n \). Then \( M_\Delta = U_\Delta / K_\Delta \), where \( U_\Delta = \{ z \in \mathbb{C}^m \mid \text{pos}(\lambda_i | z_i = 0) \in \Delta \} \) and \( K_\Delta = \ker \pi \). For the details of the construction and the proof of its properties, we refer the reader to the book [3] by Cox, Little, and Schenck and to the book [1] by Audin.

In fact, \( M_\Delta \) is an algebraic variety. Moreover, every smooth complex algebraic variety that is equipped with an algebraic \( (\mathbb{C}^*)^n \)-action with an open dense free orbit is isomorphic to some \( M_\Delta \). (The proof of this fact appeared in the book [11] by Kempf, Knudsen, Mumford, and Saint-Donat and in the article [15] by Miyake and Oda and relies on a lemma of Sumihiro [16]; see Corollary 3.1.8 in [3].) Our main theorem is a complex analytic variant of this result:

**Theorem 1.** Let \( M \) be a connected complex manifold of complex dimension \( n \), equipped with a faithful action of the torus \( (S^1)^n \) by biholomorphisms. If \( M \) is compact and the action has fixed points, then there exists a unimodular fan \( \Delta \) and an \( (S^1)^n \)-equivariant biholomorphism of \( M \) with \( M_\Delta \).

**Remark 2.**

1. Our theorem gives a negative answer to a question that was raised by Buchstaber and Panov in [2, Problem 5.23]. Let \( M \) be a closed \( 2n \) dimensional manifold with an \( (S^1)^n \)-action that is locally standard: every orbit has a neighbourhood that is equivariantly diffeomorphic, up to an automorphism of \( (S^1)^n \), to an invariant open subset of \( \mathbb{C}^n \) with the standard \( (S^1)^n \)-action. Also assume that the quotient \( M/(S^1)^n \) is diffeomorphic, as a manifold with corners, to a simple convex polytope \( P \) in \( \mathbb{R}^n \). Such manifolds, introduced in [4] and studied in the toric topology community, are called quasi-toric manifolds.

The question of Buchstaber and Panov is whether there exists a non-toric quasitoric manifold that admits an \( (S^1)^n \)-invariant complex structure.

2. Our theorem strengthens an earlier result of Ishida and Masuda, that if a closed complex manifold of complex dimension \( n \) admits an \( (S^1)^n \)-action, and if its odd-degree cohomology groups vanish, then the Todd genus of the manifold is equal to one. See [9, Theorem 1.1 and Remark 1.2].

3. In Theorem 1 the assumption “complex” cannot be weakened to “almost complex”. For example, for every two complex toric manifolds of complex dimension 2, their equivariant connected sum along a free orbit supports an invariant almost complex structure, has fixed points, but is not (equivariantly) diffeomorphic.

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2 A map from \( M/(S^1)^n \) to \( P \) is a diffeomorphism of manifolds with corners if and only if it is a homeomorphism and, for every real valued function on \( P \), the function extends to a smooth function on \( \mathbb{R}^n \) if and only if its pullback to \( M \) is smooth. For every \( k \in \{0, \ldots, n\} \), a diffeomorphism carries the \( k \) dimensional orbits in \( M \) to the relative interiors of the \( k \) dimensional faces of \( P \).

3 Davis-Januszkiewicz [4] used the term toric manifold, but this term was already used in the literature to mean a smooth toric variety, so Buchstaber-Panov [2] introduced instead the term quasitoric manifold.
to) a toric manifold; see [10, §11.2]. For higher dimensional analogues, see [6, §13]; for more interesting four dimensional examples, see [14] Theorem 5.1]. A necessary and sufficient condition for a quasitoric manifold to admit an invariant almost complex structure was given in [13, Theorem 1].

(4) The symplectic analogue of Theorem 1 is also true: a closed symplectic manifold of dimension $2n$ with a faithful $(S^1)^n$ action with at least one fixed point is a symplectic toric manifold. To see this, it is enough to show that such an action is Hamiltonian; being a toric manifold then follows from Delzant’s theorem [5, Théorème 2.1]. Let $p$ a fixed point. There exist $n$ subcircles of $(S^1)^n$ that span $(S^1)^n$ and whose isotropy weights are all positive. In order to show that the $(S^1)^n$ action is Hamiltonian, it is enough to show that each of these $S^1$ actions has a momentum map. Fix one of these $S^1$ actions. Because there is a fixed point, the $S^1$ orbits are null-homotopic, so the $S^1$ action lifts to an $S^1$ action on the universal bundle, $\tilde{M}$. Because $H^1(\tilde{M}) = 0$, this lifted action is Hamiltonian. By Morse theory, at most one point of $\tilde{M}$ can be a strict local minimum for the momentum map (see, e.g., [7]). So the fibre of $\tilde{M}$ over the fixed point $p$ can contain only one point. So $\tilde{M} = M$, and so there is a momentum map on $M$.

(5) It is necessary to assume that the action has fixed points: the complex torus $\mathbb{C}^*/(z \sim 2z)$ has a holomorphic $S^1$-action, induced from multiplication on $\mathbb{C}^*$, but it is not a toric variety: the $\mathbb{C}^*$-action is not faithful.

(6) It is necessary to assume that the manifold is compact: the open unit disc in $\mathbb{C}$ with the natural circle action has a fixed point, but it is not a toric variety: the circle action does not extend to a $\mathbb{C}^*$-action.

2. The complexified action

Let the torus $(S^1)^n$ act on a complex manifold $M$ by biholomorphisms. If the manifold $M$ is compact, then the $(S^1)^n$-action extends to a $(\mathbb{C}^*)^n$-action that is holomorphic not only in the sense that each element of $(\mathbb{C}^*)^n$ acts by a biholomorphism but also in the sense that the action map $(\mathbb{C}^*)^n \times M \to M$ is holomorphic. See, e.g., [8, Theorem 4.4]. For the convenience of the reader, we briefly recall here some of the details of this standard construction.

Let $\xi_1, \ldots, \xi_n$ be the fundamental vector fields of the $(S^1)^n$-action with respect to the coordinate one-dimensional subtori. Let $J: TM \to TM$ be the multiplication by $\sqrt{-1}$. We claim that the vector fields $-J\xi_1, \ldots, -J\xi_n$ are holomorphic (in the sense that their flows preserve the complex structure) and commute with each other and with the vector fields $\xi_i$.

Because the $(S^1)^n$-action preserves $J$ and $\xi_j$, it preserves $-J\xi_j$, for each $j$. So the vector fields $-J\xi_j$ commute with the vector fields $\xi_i$ that generate this action. Because $J$ is a complex structure, its Nijenhuis tensor, $N(Z, W) := 2([JZ, JW] - J[Z, JW] - J[JZ, W] - [Z, W])$, vanishes. Setting $Z = \xi_i$ and $W = \xi_j$, we get that $[J\xi_i, J\xi_j] = [J^2 \xi_i, \xi_j] + J[J\xi_i, \xi_j] + [\xi_i, J\xi_j], and each of the three terms on the right hand side is zero. So the vector fields $-J\xi_j$ commute with each other. A vector field $Y$ is holomorphic if and only if $[Y, JW] = J[Y, W]$ for
each vector \( W \); see [12] Proposition 2.10 in Chapter IX. Set \( Y := -J\xi \) and \( W \) arbitrary; because \( JY(\xi) \) is holomorphic, \([JY,JW] = J[JY,W] \); by the vanishing of the Nijenhuis tensor,

\[
0 = N(JY,W) = 2 \left( [-Y,JW] - J[JY,JW] - J[-Y,W] - [JY,W] \right)
\]

so \( Y \) is holomorphic.

If \( M \) is compact, the vector fields \(-J\xi_1, \ldots, -J\xi_n\) are complete, and we get an \( \mathbb{R}^{2n} \)-action, \( \mathbb{R}^{2n} \times M \rightarrow M \), via

\[
\left( \sum_{i=1}^{2n} a_i e_i, x \right) \mapsto c(x),
\]

where \( c_x(r) \) is the integral curve of the vector field \( \sum_{i=1}^n -a_i J\xi_i + a_{n+i} \xi_i \) with \( c_x(0) = x \). This action descends to a \((\mathbb{C}^*)^n\)-action by biholomorphisms that extends the given \((S^1)^n\)-action. Finally, the action map \((\mathbb{C}^*)^n \times M \rightarrow M \) is holomorphic, because its differential, which at the point \((z,m)\) is the map \( \mathbb{C}^n \times T_m M \rightarrow T_{z_m} M \) that takes \((2\pi r_1 + i\theta_1, \ldots, r_n + i\theta_n, v)\) to \( \sum_j r_j J\xi_j |_{z_m} + \theta_j \xi_j |_{z_m} + z_j v \), is complex linear.

**Remark 3.** In the next section we will see that if there exists a fixed point then the extended \((\mathbb{C}^*)^n\)-action is faithful. In general, the extended \((\mathbb{C}^*)^n\)-action might not be faithful.

**Example 4.** Let \((S^1)^n\) act on \( \mathbb{C}^n \) with weights \( \alpha_1, \ldots, \alpha_n \):

\[
g \cdot (z_1, \ldots, z_n) = (g^{\alpha_1} z_1, \ldots, g^{\alpha_n} z_n),
\]

where \( g^{\alpha_i} = g_1^{\alpha_i} \cdots g_n^{\alpha_i} \) for \( g = (g_1, \ldots, g_n) \in (S^1)^n \) and for the isotropy weight \( \alpha_i = (\alpha_{i1}, \ldots, \alpha_{in}) \in \mathbb{Z}^n \). Then the complexified action is given by the same formula applied to \( g = (g_1, \ldots, g_n) \in (\mathbb{C}^*)^n \).

3. **Structures near fixed points**

Let \( M \) be a connected complex manifold of complex dimension \( n \). Let the torus \((S^1)^n\) act on \( M \) faithfully by biholomorphisms. Let \( p \) be a point in \( M \) that is fixed by the \((S^1)^n\)-action. Let \( \alpha_1, \ldots, \alpha_p \) be the isotropy weights at \( p \).

Let \( \mathbb{C}_{\alpha_i} \) denote the one dimensional complex vector space \( \mathbb{C} \) with the \((S^1)^n\)-action that is obtained by composing the homomorphism \((S^1)^n \rightarrow S^1 \) that is encoded by the weight \( \alpha_i \) with the standard action of \( S^1 \) on \( \mathbb{C} \) by scalar multiplication.

We begin with a local result:

**Lemma 5.** There exists an \((S^1)^n\)-invariant neighbourhood \( U_p \) of \( p \) in \( M \), an \((S^1)^n\)-invariant neighbourhood \( \tilde{U}_p \) of the origin in \( T_p M \), and an \((S^1)^n\)-equivariant biholomorphism \( \varphi_p : U_p \rightarrow \tilde{U}_p \) whose differential at \( p \) is the identity map on \( T_p M \).

**Proof.** Let \( \varphi : U \rightarrow \tilde{U} \subseteq \mathbb{C}^n \) be a local holomorphic chart near \( p \) with \( \varphi(p) = 0 \). Identifying \( \mathbb{C}^n \) with \( T_p M \) via the differential

\[
(d\varphi)_p : T_p M \rightarrow T_0 \mathbb{C}^n \cong \mathbb{C}^n,
\]
we get a biholomorphism
\[ \varphi' : U \to \tilde{U}' \subseteq T_pM \]
whose differential at \( p \) is the identity map on \( T_pM \). We want to obtain such a biholomorphism that is also equivariant.

Set
\[ U' := \bigcap_{g \in (S^1)^n} gU. \]

Clearly, \( U' \) is invariant and contains \( p \). We now show that \( U' \) is open. The complement of \( U' \) is the image of the closed subset \((S^1)^n \times (M \setminus U)\) of \((S^1)^n \times M \to M\). Because \((S^1)^n\) is compact, the action map is proper. Being proper means that the preimage of every compact set is compact; when the target space \( M \) is a manifold it implies that the map is closed. Thus, the complement \( M \setminus U' \) is closed, and so \( U' \) is open.

To obtain an equivariant chart, we average \( \varphi' \): let
\[ \tilde{\varphi} := \int_{g \in (S^1)^n} (g \circ \varphi' \circ g^{-1}) \, dg : U' \to T_pM, \]
where \( dg \) is Haar measure on \((S^1)^n\). The map \( \tilde{\varphi} \) is holomorphic and \((S^1)^n\)-equivariant. Moreover, its differential at \( p \) is the identity map on \( T_pM \). By the implicit function theorem, \( \tilde{\varphi} \) restricts to a biholomorphism from some smaller open neighbourhood \( U'' \) of \( p \) in \( M \) to an open neighbourhood of the origin in \( T_pM \). The restriction of \( \tilde{\varphi} \) to the invariant neighbourhood \( U_p := \bigcap_{g \in (S^1)^n} g \cdot U'' \) of \( p \) in \( M \) satisfies the requirements of the lemma. \( \square \)

**Corollary 6.** There exists an \((S^1)^n\)-equivariant local holomorphic chart
\[ \varphi_p : U_p \to \mathbb{D}^n \]
from an invariant open neighbourhood \( U_p \) of \( p \) to a polydisc \( \mathbb{D}^n \) in \( \mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n} \).

**Proof.** By the definition of the isotropy weights, there exists a complex linear \((S^1)^n\)-equivariant isomorphism between the tangent space \( T_pM \) and the representation \( \mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n} \). Corollary\( \square \) then follows from Lemma\( \square \) by restricting the chart to the preimage of a polydisc. \( \square \)

We would like to extend the chart of Corollary\( \square \) to a chart whose image is all of \( \mathbb{C}^n \). We can do this when the \((S^1)^n\) extends to a \((\mathbb{C}^\ast)^n\)-action; for example, if the manifold is compact; by “sweeping” by the \((\mathbb{C}^\ast)^n\)-action.

**Lemma 7.** Suppose that the \((S^1)^n\)-action extends to a \((\mathbb{C}^\ast)^n\)-action. Then there exists an invariant open neighbourhood \( V_p \) of \( p \) in \( M \) and an \((S^1)^n\)-equivariant biholomorphism of \( V_p \) with \( \mathbb{C}_{\alpha_1} \oplus \ldots \oplus \mathbb{C}_{\alpha_n} \).

\( ^4 \) In fact, it is enough to assume that the target space is Hausdorff and compactly generated. Compactly generated means that a subset is closed if and only if its intersection with every compact set \( K \) is closed in \( K \); this property holds if the space is locally compact or if the space is metrizable.
Proof. Let \( \varphi_p : U_p \to \mathbb{D}^n \) be an \((S^1)^n\)-equivariant holomorphic local chart, as in Corollary 6. Because \( \varphi_p \) is \((S^1)^n\)-equivariant and holomorphic, it intertwines the restriction to \( U_p \) of the vector fields that generate the complexified \((\mathbb{C}^*)^n\)-action on \( M \) with the restriction to \( \mathbb{D}^n \) of the vector fields that generate the complexified \((\mathbb{C}^*)^n\)-action on \( \mathbb{C}^n = C_{\alpha_1} \oplus \cdots \oplus C_{\alpha_n} \). This, and the fact that \( \varphi_p \) is a diffeomorphism between \( U_p \) and \( \mathbb{D}^n \), implies that \( \varphi_p \) also intertwines the partial flows on \( U_p \) and on \( \mathbb{D}^n \) that are generated by these vector fields; in particular it intertwines the domains of definition of these partial flows.

For each \( t \in \mathbb{R} \), let \( g_t \) be the element of \((\mathbb{C}^*)^n\) that acts on \( \mathbb{C}^n \) as scalar multiplication by \( e^{-t} \), and let \( \eta \in \text{Lie}(\mathbb{C}^*)^n \) be the generator of the one-parameter subgroup \( t \mapsto g_t \). Because \( e^{-t}\mathbb{D}^n \subset \mathbb{D}^n \) for all \( t \geq 0 \), and because \( \varphi_p \) intertwines the domains of definition of the partial flows on \( U_p \) and on \( \mathbb{D}^n \) that correspond to \( \eta \), we get that \( g_t U_p \subset U_p \) for all \( t \geq 0 \). So, for every \( t \geq 0 \), the domain of definition of the \((S^1)^n\)-equivariant biholomorphism \( \varphi_p^{(t)} := (g_t)^{-1} \circ \varphi_p \circ g_t : g_{-t}U_p \to e^t\mathbb{D}^n \) contains \( U_p \). Here, \( g_{-t} : g_{-t}U_p \to U_p \) and \((g_t)^{-1} : \mathbb{D}^n \to e^t\mathbb{D}^n \) are given by the complexified actions on \( M \) and on \( \mathbb{C}^n \). By the choice of \( g_t \), the latter map is multiplication by \( e^t \).

Moreover, because \( \varphi_p \) intertwines the partial flows that correspond to \( \eta \) and these partial flows are defined for all \( t \geq 0 \), the restriction to \( U_p \) of \( \varphi_p^{(t)} \) coincides with \( \varphi_p \) for all \( t \geq 0 \). Substituting \( t - s \) instead of \( t \), we get that the maps \( \varphi_p^{(t)} \) and \( \varphi_p^{(s)} \) agree whenever they are both defined. Thus, all these maps fit together into a map

\[
\bigcup_{t \geq 0} \varphi_p^{(t)} : V_p \to C_{\alpha_1} \oplus \cdots \oplus C_{\alpha_n},
\]

where \( V_p = \bigcup_{t \geq 0} g_{-t}U_p \). This map is onto, because its image is the union of the sets \( e^t\mathbb{D}^n \) over all \( t \geq 0 \). The map is one to one, because it is one to one on each \( g_{-t}U_p \), and for every two points in the domain there exists a \( t \geq 0 \) such that the points are both in \( g_{-t}U_p \). Because \( V_p \) is covered by \((S^1)^n\)-invariant open sets \( g_{-t}U_p \) on which the map is an \((S^1)^n\)-equivariant biholomorphism, we deduce that the map is itself an \((S^1)^n\)-equivariant biholomorphism, as required. \(\square\)

4. Obtaining a fan

Let \( M \) be a connected complex manifold of complex dimension \( n \), let the torus \((S^1)^n\) act on \( M \) faithfully by biholomorphisms, and assume that this action extends to a holomorphic \((\mathbb{C}^*)^n\)-action. The set of fixed points is discrete; assume that it is nonempty and finite.

In Lemma 7 we assigned to every fixed point \( p \) in \( M \) an open subset \( V_p \) that is biholomorphic to \( \mathbb{C}^n \). By assumption, there exists at least one fixed point. So the union \( X \) of the sets \( V_p \) over these fixed points,

\[
X := \bigcup_{p \in M/S^1} V_p,
\]

is nonempty.
Remark 8. In Section 6 we show that if $M$ is compact and connected then the union $X$ of the sets $V_p$ is all of $M$. The proof relies on the results of Sections 4 and 5.

By its definition, $X$ is a $(\mathbb{C}^*)^n$-invariant open submanifold of $M$. Moreover, we claim that there exists a unique open $(\mathbb{C}^*)^n$ orbit in $M$, this orbit and free and is dense in $M$, and it coincides with the free $(\mathbb{C}^*)^n$ orbit in $V_p$ for each $p$. To see this, we consider the fundamental vector fields $\xi_1, \ldots, \xi_n$ of the $(S^1)^n$-action with respect to the coordinate one-dimensional subtori. We think of them as holomorphic sections $M \rightarrow T_{1,0}M \cong TM$ of the holomorphic tangent bundle $T_{1,0}M$ of $M$. The $n$-th exterior product $\bigwedge^n T_{1,0}M \rightarrow M$ is a holomorphic line bundle and $\xi_1 \wedge \cdots \wedge \xi_n$ is a holomorphic section of this line bundle.

A point $x \in M$ belongs to an open $(\mathbb{C}^*)^n$ orbit if and only if $(\xi_1 \wedge \cdots \wedge \xi_n)(x)$ is not zero. This means that the union of the open $(\mathbb{C}^*)^n$ orbits is the complement of the zero locus of a holomorphic section. Because the zero locus is a complex analytic subvariety of $M$ and $M$ is connected, the union of the open $(\mathbb{C}^*)^n$ orbits is either empty, or it is open, dense, and connected. The claim then follows from the facts that there exists at least one $V_p$, it contains a free and open $(\mathbb{C}^*)^n$ orbit, and every two distinct orbits are disjoint.

In particular, $X$ is connected and dense in $M$.

The connected components of the fixed point sets of the circle subgroups of $(S^1)^n$ are closed complex submanifolds of $X$. If such a submanifold has complex codimension one, then, in analogy with the toric topology literature, we call it a characteristic submanifold of $X$ (cf. [14, p. 240]).

Because $X$ is a union of finitely many $V_p$s and each $V_p$ has only finitely many characteristic submanifolds, there are only finitely many characteristic submanifolds in $X$. Denote them $X_1, \ldots, X_m$.

Let $T_i$ be the subgroup of $T$ that fixes $X_i$. If a compact group acts faithfully on a connected manifold then at every fixed point the linear isotropy representation is faithful. Therefore, the linear isotropy representation of $T_i$ at any point $q$ of $X_i$ is faithful. Because $T_i$ acts holomorphically and fixes $X_i$, we get a faithful representation of $T_i$ on the one dimensional complex space $T_qX_i/T_qX_i$. This gives an injection $T_i \rightarrow S^1$, where $S^1$ acts on $T_qX_i/T_qX_i$ by scalar multiplication. By continuity, this injection is independent of the choice of point $q$ in $X_i$. Because, by assumption, $T_i$ contains a circle subgroup of $T$, this injection is an isomorphism. Let

$$\lambda_i : S^1 \rightarrow T_i \subset (S^1)^n$$

be the inverse of this isomorphism, composed with the inclusion map into $(S^1)^n$.

We define an abstract simplicial complex:

$$\Sigma := \left\{ I \subseteq \{1, \ldots, m\} \mid \bigcap_{i \in I} X_i \neq \emptyset \right\}.$$
To each simplex $I \in \Sigma$ we assign the cone

$$C_I := \text{pos}(\lambda_i \mid i \in I) := \left\{ \sum_{i \in I} a_i \lambda_i \mid a_i \geq 0 \right\}$$

in $\text{Lie}(S^1)^n$.

**Example 9.** Take $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$. Let $(S^1)^n$ act on it with weights $\alpha_1, \ldots, \alpha_n \in \text{Hom}((S^1)^n, S^1) \subset (\text{Lie}(S^1)^n)^*$. Suppose that the action is faithful; then $\alpha_1, \ldots, \alpha_n$ are a $\mathbb{Z}$-basis of $\text{Hom}((S^1)^n, S^1)$. The characteristic submanifolds are the coordinate hyperplanes $\{z_i = 0\}$ for $i = 1, \ldots, n$. The homomorphisms $\lambda_1, \ldots, \lambda_n$ are the basis to $\text{Hom}(S^1, (S^1)^n) \subset \text{Lie}(S^1)^n$ that is dual to $\alpha_1, \ldots, \alpha_n$.

Recall that a cone in $\text{Lie}(S^1)^n$ is **unimodular** if it is generated by part of a $\mathbb{Z}$-basis of $\text{Hom}(S^1, (S^1)^n)$.

Returning to our general case –

**Lemma 10.** The cones $C_I$, for $I \in \Sigma$, are unimodular.

*Proof.* Let $I \in \Sigma$. By the definition of $\Sigma$, this means that the intersection $\bigcap_{i \in I} X_i$ is nonempty. Let $q$ be a point in this intersection. Let $p$ be a fixed point such that $q \in V_p$. Because $V_p$ is isomorphic to some $\mathbb{C}_{\alpha_1} \oplus \cdots \oplus \mathbb{C}_{\alpha_n}$ on which the action is faithful, the lemma follows from Example 9. \hfill \Box

Fix a point $q$ in the free $(\mathbb{C}^*)^n$ orbit in $X$. For any $\xi \in \text{Lie}(S^1)^n$, consider the curve

$$c^\xi_q : \mathbb{R} \rightarrow X$$

that is given by

$$c^\xi_q(r) := \exp(-r J \xi) \cdot q \quad \text{for } r \in \mathbb{R}$$

where $\exp : \text{Lie}(\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n$ is the exponential map and where $J$ denotes multiplication by $i$ in $\text{Lie}(\mathbb{C}^*)^n$.

Denote by $C^0_I$ the relative interior of the cone $C_I$. Denote

$$X_I = \bigcap_{i \in I} X_i \quad \text{and} \quad X^0_I = \bigcap_{i \in I} X_i \setminus \bigcup_{j \notin I} X_j.$$ 

**Lemma 11.** Let $\xi \in \text{Lie}(S^1)^n$ and $I \in \Sigma$. Then $\xi \in C^0_I$ if and only if the curve $c^\xi_q(r)$ converges as $r \rightarrow -\infty$ to a point $q'$ in $X^0_I$. Moreover, in this case the limit point $q'$ belongs to $V_p$ for every $p$ such that $V_p \cap X_I \neq \emptyset$.

*Proof.* Suppose that $\xi \in C^0_I$. By the definition of $\Sigma$, $X_I$ is nonempty. Let $p$ be such that $V_p$ meets $X_I$. Without loss of generality assume that $I = \{1, \ldots, k\}$ and that the characteristic submanifolds that meet $V_p$ are $X_1, \ldots, X_n$. Let $\alpha_1, \ldots, \alpha_n$ denote the isotropy weights at $p$. The assumption that $\xi \in C^0_I$ exactly means that $\langle \xi, \alpha_i \rangle$ is positive for $i = 1, \ldots, k$ and zero for $i = k+1, \ldots, n$. Fix an isomorphism $(z_1, \ldots, z_n) : V_p \rightarrow \mathbb{C}^n = \mathbb{C}_{\alpha_1} \oplus \cdots \oplus \mathbb{C}_{\alpha_n}$ such that $z_i(q) = 1$ for all $i$. In these coordinates, the curve $c^\xi_q(r)$ is represented as

$$(z_1, \ldots, z_n)(c^\xi_q(r)) = (e^{2\pi r \langle \xi, \alpha_1 \rangle}, \ldots, e^{2\pi r \langle \xi, \alpha_n \rangle}).$$
As $r$ approaches $-\infty$, the curve in $\mathbb{C}^n$ approaches the point $(0, \ldots, 0, 1, \ldots, 1)$. On the other hand, the coordinates take each intersection $V_p \cap X_i$ to the coordinate hyperplane ${\{ (z_1, \ldots, z_n) \mid z_i = 0 \}}$, and they take the intersection $V_p \cap X_i^0$ to the set ${\{ (z_1, \ldots, z_n) \mid z_i = 0 \text{ iff } 1 \leq i \leq k \}}$. So the curve approaches a point in $V_p \cap X_i^0$, as required.

Now suppose that the curve $c^\xi_q(r)$ converges as $r \to -\infty$ to a point in $X_i^0$. Let $p$ be such that this limit point is contained in $V_p$. As before, without loss of generality assume that $I = \{1, \ldots, k\}$ and that the characteristic submanifolds that meet $V_p$ are exactly $X_1, \ldots, X_n$; fix an isomorphism $(z_1, \ldots, z_n) : V_p \to \mathbb{C}^n = \mathbb{C}_{a_1} \oplus \cdots \oplus \mathbb{C}_{a_n}$ such that $z_i(q) = 1$ for all $i$; the curve $c^\xi_q(r)$ is represented as $(z_1, \ldots, z_n)(c^\xi_q(r)) = (e^{2\pi i \langle \xi, a_1 \rangle}, \ldots, e^{2\pi i \langle \xi, a_n \rangle})$. Because the curve approaches a limit as $r \to -\infty$, the pairings $\langle \xi, a_i \rangle$ are nonnegative for all $i = 1, \ldots, n$.

Because this limit is in $X_i^0$, the pairings are positive for every $i \in I$ and they vanish for every $i \in \{1, \ldots, n\} \setminus I$. Thus, $\xi \in C^0_I$ as required. 

**Corollary 12.**

(1) For every $I, J \in \Sigma$, if $I \neq J$, then $C^0_I \cap C^0_J = \emptyset$.

(2) For every $I, J \in \Sigma$,

$$C_I \cap C_J = C_{I \cap J}.$$

(3) The collection of cones

$$\Delta := \left\{ C_I \mid I \in \Sigma \right\}$$

is a fan, that is, every face of every cone in $\Delta$ is itself in $\Delta$, and the intersection of every two cones in $\Delta$ is a common face.

**Proof.** Part (1) follows from Lemma 7 because the sets $X_I^0$ are disjoint. Part (3) follows from Part (2).

For Part (2), we only need to show the inclusion $C_I \cap C_J \subseteq C_{I \cap J}$, because the opposite inclusion is trivial. Let $\xi \in C_I \cap C_J$. Let $I' \subset I$ and $J' \subset J$ be the subsets such that $\xi \in C^0_{I'}$ and $\xi \in C^0_{J'}$. Then $C^0_{I'} \cap C^0_{J'} \neq \emptyset$. By Part (1), $I' = J'$. Let $L = I' = J'$. Then $L \subset I \cap J$, and $\xi \in C^0_L \subset C_{I \cap J}$. 

**Lemma 13.** For every $I \in \Sigma$, the set $X_I$ is an $(S^1)^n$-invariant smooth closed complex submanifold of $X$ of complex codimension $|I|$, it is connected, and it contains a fixed point.

**Proof.** Fix $I \in \Sigma$.

Because each of the sets $X_i$, for $i \in I$, is closed in $X$, so is the intersection $X_I$ of these sets.

Because $X$ is the union of open subsets $V_p$, and because every intersection $V_p \cap X_I$ is an $(S^1)^n$-invariant complex submanifold of codimension $|I|$ in $V_p$, we deduce that $X_I$ is itself an $(S^1)^n$-invariant complex submanifold of codimension $|I|$ in $X$. It remains to show that $X_I$ is connected and contains a fixed point.

Choose any $\xi \in C_I^0$ (for example, we may take $\xi = \sum_{i \in I} \lambda_i$), and choose any $q$ in the free $(\mathbb{C}^*)^n$ orbit in $X$. By Lemma 7, the curve $c^\xi_q(r)$ converges as $r \to -\infty$; let $q'$ be its limit. Also by Lemma 7, for every $p$ such that $V_p \cap X_I \neq \emptyset$, the limit point $q'$ belongs to $V_p$. Because $X_I$ is the union over such $p$ of the subsets $V_p \cap X_I$, and because each of these
subsets is connected and contains \( q' \), the union \( X_I \) is connected. Also, every \( p \) such that \( V_p \cap X_I \neq \emptyset \) belongs to \( V_p \cap X_I \); because the set of such \( ps \) is nonempty, \( X_I \) contains a fixed point. □

**Corollary 14.** In the fan \( \Delta \), every cone is contained in an \( n \) dimensional cone.

**Proof.** Every cone in the fan has the form \( C_I \) for some \( I \in \Sigma \). By Lemma 13, the set \( X_I \) contains a fixed point; let \( p \) be such a fixed point. Since \( V_p \) was chosen as in Lemma 7, by Example 9 there exist exactly \( n \) characteristic submanifolds, say, \( X_j \) for \( j \in J \subset \{1, \ldots, m\} \) with \( |J| = n \), that pass through \( p \). Then \( J \in \Sigma \), and \( C_J \) is an \( n \) dimensional cone in \( \Delta \) that contains \( C_I \). □

5. **Isomorphism of the Subset \( X \) with a Toric Manifold**

Let \( M \) be a connected complex manifold of complex dimension \( n \), let the torus \( (S^1)^n \) act on \( M \) faithfully by biholomorphisms, and assume that this action extends to a holomorphic \((\mathbb{C}^*)^n\)-action. The set of fixed points is discrete; assume that it is nonempty and finite.

In Section 4 we described an open subset \( X \) of \( M \) and a unimodular fan \( \Delta \). Let \( M_\Delta \) be the toric variety that is associated to the fan \( \Delta \).

**Lemma 15.** There exists an \((S^1)^n\)-equivariant biholomorphism between \( M_\Delta \) and \( X \).

We recall some properties of the set \( X \) and the fan \( \Delta \). Let \( F = M^{(S^1)^n} \) denote the fixed point set. For every fixed point \( p \in F \), let \( \alpha_{p,1}, \ldots, \alpha_{p,n} \) denote the isotropy weights of the torus action at \( p \).

1. The set \( X \) is the union over \( p \in F \) of subsets \( V_p \), such that each \( V_p \) is an invariant open neighbourhood of \( p \) that is equivariantly biholomorphic to the linear representation \( \mathbb{C}_{\alpha_{p,1}} \oplus \ldots \oplus \mathbb{C}_{\alpha_{p,n}} \).

2. The \( n \)-dimensional cones in \( \Delta \) are in bijection with the fixed point sets \( p \in F \), and the cone corresponding to the fixed point \( p \) is \( \text{pos}(\lambda_{i_1}, \ldots, \lambda_{i_n}) \), where \( \lambda_{i_1}, \ldots, \lambda_{i_n} \) is a basis of \( \text{Lie}(S^1)^n \) that is dual to the basis \( \alpha_{p,1}, \ldots, \alpha_{p,n} \) of \( \text{Lie}(S^1)^n \).

The toric variety \( M_\Delta \) that is associated to the fan \( \Delta \) has similar properties: it is the union over \( p \in F \) of invariant subsets \( V'_p \), and every \( V'_p \) is equivariantly biholomorphic to \( \mathbb{C}_{\alpha_{p,1}} \oplus \ldots \oplus \mathbb{C}_{\alpha_{p,n}} \).

Lemma 15 follows immediately from these properties of \( X \) and \( M_\Delta \), by the following lemma.

**Lemma 16.** Let \( X \) and \( X' \) be complex manifolds of complex dimension \( n \), equipped with holomorphic \((\mathbb{C}^*)^n\)-actions. Suppose that there exist open dense \((\mathbb{C}^*)^n\) orbits \( \mathcal{O} \) in \( X \) and \( \mathcal{O}' \) in \( X' \). Suppose that there exist invariant open subsets \( V_p \) in \( X \) and \( V'_p \) in \( X' \), both indexed by \( p \in F \), such that \( X \) is the union of the sets \( V_p \) and \( X' \) is the union of the sets \( V'_p \); and that for each \( p \in F \) there exists an equivariant biholomorphism \( \varphi_p : V_p \to V'_p \). Then \( X \) is equivariantly biholomorphic to \( X' \).
Proof. Necessarily, $O$ is contained in each $V_p$ and $O'$ is contained in each $V'_p$. Fix a point $q$ in $O$ and a point $q'$ in $O'$. After possibly composing each $\varphi_p$ by the action of an element of $(\mathbb{C}^*)^n$, we may assume that $\varphi_p(q) = q'$ for each $p \in F$. So, for each $p$ and $p' \in F$, the maps $\varphi_p$ and $\varphi_{p'}$ coincide at the point $q$. By equivariance, $\varphi_p$ and $\varphi_{p'}$ coincide on all of $O$; by continuity, they coincide on the entire overlap $V_p \cap V'_{p'}$. Thus, the $\varphi_p$ fit together into a map

$$\varphi = \bigcup_p \varphi_p : X \to X'.$$

This map is holomorphic, equivariant, and onto. Similarly, the inverses $\psi_p := \varphi_p^{-1}$ fit together into a map

$$\psi = \bigcup_p \psi_p : X' \to X.$$

We have that $\psi \circ \varphi = \text{id}_X$ and $\varphi \circ \psi = \text{id}_{X'}$; thus, $\varphi : X \to X'$ is an equivariant biholomorphism, as required. □

6. The compact case

Let $M$ be a connected complex manifold of complex dimension $n$, with a faithful $(S^1)^n$-action, with fixed points.

Suppose that $M$ is compact. In Section 2 we extended the $(S^1)^n$-action to a holomorphic $(\mathbb{C}^*)^n$-action. In Section 4 we described an open subset $X$ of $M$ and we associated to it a fan $\Delta$.

Lemma 17. The fan $\Delta$ is complete.

We begin by proving a special case:

Lemma 18. Let $M'$ be a complex manifold of complex dimension one, equipped with a faithful holomorphic action of $S^1$ with at least one fixed point. Suppose that $M'$ is compact and connected. Then $M'$ is equivariantly biholomorphic to $\mathbb{C}P^1$ with a standard $\mathbb{C}^*$-action.

Proof. Consider the $S^1$-action on $M'$. Near a fixed point, it is isomorphic to the restriction of either the standard $S^1$-action on $\mathbb{C}$ or the opposite $S^1$-action on $\mathbb{C}$ to an invariant neighbourhood of the origin in $\mathbb{C}$.

Consider the flow that is generated by $-J\xi$, where $\xi$ generates the $S^1$-action. If the $S^1$-action near a fixed point is standard, then the trajectories of this flow converge to the fixed point as their parameter approaches $-\infty$. If the $S^1$-action near a fixed point is opposite from standard, then the trajectories of this flow converge to the fixed point as their parameter approaches $\infty$. 
Outside the fixed point set, the action is free. The quotient $M'/S^1$ is a real one-manifold with boundary; its boundary is exactly the image of the fixed point set. Because $M'$ is compact and connected and contains a fixed point, and by the classification of one-manifolds, the quotient $M'/S^1$ must be a closed segment.

The flow on $M'$ that is generated by $-J\xi$ descends to a flow on the interior of $M'/S^1$ that does not have fixed points. For each boundary component, the flow approaches that component either as its parameter approaches $\infty$ or as the parameter approaches $-\infty$. Necessarily, it approaches one boundary component when the parameter approaches $\infty$ and it approaches the other boundary component when the parameter approaches $-\infty$.

The corresponding fan must then be equal to the fan of $\mathbb{C}P^1$, and the manifold is equivariantly biholomorphic to $\mathbb{C}P^1$ by Lemma 16. \hfill \Box

We now return to the setup of Lemma 17. We have a connected complex manifold $M$ of complex dimension $n$, with a faithful $(S^1)^n$-action, with fixed points. We assume that $M$ is compact. We consider the open subset $X$ of $M$ and the associated fan $\Delta$ as described in Section 4.

**Lemma 19.** Every $n-1$ dimensional cone in $\Delta$ is a common face of two $n$ dimensional cones in $\Delta$.

**Proof.** Let $C_I$ be an $n-1$ dimensional cone in $\Delta$, corresponding to the subset $I = \{i_1, \ldots, i_{n-1}\}$ of $\{1, \ldots, m\}$.

Let $T_I$ be the codimension one subtorus of $(S^1)^n$ that is generated by the circles $T_i$ for $i \in I$. By Lemma 13, $X_I$ is a connected complex manifold of dimension one, equipped with a faithful holomorphic action of the circle $(S^1)^n/T_I$ with at least one fixed point. We will now show that $X_I$ is compact, and will deduce Lemma 19 from Lemma 18.

First note that $X_I$ is a connected component of the fixed point set of $T_I$ in $X$. This follows from the facts that $X_I$ is connected (by Lemma 13) and that, for each of the subsets $V_p$, if the intersection $V_p \cap X_I$ is nonempty then it is a connected component of the fixed point set of $T_I$ in $V_p$. Let $N$ denote the connected component of the fixed point set of $T_I$ in $M$ that contains $X_I$. As in any holomorphic torus action on a complex manifold, $N$ is an $(S^1)^n$-invariant closed complex submanifold of $M$. By examining $N$ near a point of $X_I$, we deduce that $N$ has complex dimension one. Because $N$ is closed in $M$ and $M$ is compact, $N$ is compact. By Lemma 18, $N$ is equivariantly biholomorphic to $\mathbb{C}P^1$ with a standard action of the circle $(S^1)^n/T_I$. In particular, $N$ contains two fixed points; denote them $p'$ and $p''$. The intersection $V_{p'} \cap N$, being a $(\mathbb{C}^*)^n$-invariant neighbourhood of $p'$ in $N$, must be all of $N \setminus \{p''\}$. Similarly, the intersection $V_{p''} \cap N$ is all of $N \setminus \{p'\}$. So $N$ is contained in the union $X$ of the sets $V_p$, and so $N$ must be equal to $X_I$. Thus, $X_I$ is equivariantly biholomorphic to $\mathbb{C}P^1$ with a standard action of the circle $(S^1)^n/T_I$. This implies the result of Lemma 19. \hfill \Box

We are now ready to prove Lemma 17.

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Here, “is” means that there exists a unique manifold-with-boundary structure on $M'/S^1$ such that a function is smooth if and only if its pullback to $M'$ is smooth.
Proof of Lemma 17. Let $|\Delta|$ denote the union of the cones in $\Delta$, and let $|\Delta^{n-2}|$ denote the union of the cones in $\Delta$ that have codimension $\geq 2$. The complement $\text{Lie}(S^1)^n \setminus |\Delta^{n-2}|$ is connected, open, and dense in $\text{Lie}(S^1)^n$.

By Lemma 19, the union of the relative interiors of the faces of $\Delta$ of dimension $(n-1)$ and of dimension $n$ is open in $\text{Lie}(S^1)^n$. This union is $|\Delta| \setminus |\Delta^{n-2}|$. Thus, $|\Delta| \setminus |\Delta^{n-2}|$ is also open in $\text{Lie}(S^1)^n \setminus |\Delta^{n-2}|$.

But because $|\Delta|$ is closed in $\text{Lie}(S^1)^n$, we also have that $|\Delta| \setminus |\Delta^{n-2}|$ is closed in $\text{Lie}(S^1)^n \setminus |\Delta^{n-2}|$.

Because $|\Delta| \setminus |\Delta^{n-2}|$ is open and closed in $\text{Lie}(S^1)^n \setminus |\Delta^{n-2}|$ and $\text{Lie}(S^1)^n \setminus |\Delta^{n-2}|$ is connected, we deduce that $|\Delta| \setminus |\Delta^{n-2}|$ is either empty or is equal to all of $\text{Lie}(S^1)^n \setminus |\Delta^{n-2}|$.

Because, by assumption, $M$ has a fixed point, $\Delta$ has at least one $n$ dimensional cone, so $|\Delta| \setminus |\Delta^{n-2}|$ is not empty. So $|\Delta| \setminus |\Delta^{n-2}|$ is equal to all of $\text{Lie}(S^1)^n \setminus |\Delta^{n-2}|$. Taking the closures, we deduce that $|\Delta| = \text{Lie}(S^1)^n$, as required. □

We are now ready to prove our main theorem.

Proof of Theorem 7. Lemma 16 gives an equivariant biholomorphism

$$\varphi : M_\Delta \to X.$$ 

By Lemma 17, the fan $\Delta$ is complete. This implies that the toric variety $M_\Delta$ is compact. So $X$ must be compact. Because $M$ is Hausdorff and connected, and $X$ is a subset that is both compact and open, $X$ is all of $M$. So $\varphi$ defines an equivariant biholomorphism from $M_\Delta$ to $M$, as required. □

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