QUANTIFYING THE ASYMPTOTIC LINEAR CONVERGENCE SPEED OF ANDERSON ACCELERATION APPLIED TO ADMM

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Abstract. We explain how Anderson Acceleration (AA) speeds up the Alternating Direction Method of Multipliers (ADMM), for the case where ADMM by itself converges linearly. We do so by considering the spectral properties of the Jacobians of ADMM and a stationary version of AA evaluated at the fixed point, where the coefficients of the stationary version are computed such that its asymptotic linear convergence factor is optimal. Numerical tests show that this allows us to quantify the improved linear asymptotic convergence speed of AA-ADMM as compared to the convergence factor of ADMM used by itself. This way of estimating AA-ADMM convergence speed is useful because there are no known convergence bounds for AA with finite window size that would allow quantification of this improvement in asymptotic convergence speed.

Keywords: Anderson acceleration, ADMM, asymptotic linear convergence speed, machine learning

1. Introduction. In this paper, we consider the constrained optimization problem

\[
\min_{x,z} f(x,z) = f_1(x) + f_2(z),
\]

\[
s.t. \quad Ax + Bz = b,
\]

where \(x \in \mathbb{R}^{n_1}, z \in \mathbb{R}^{n_2}\) are optimization variables, \(b \in \mathbb{R}^{n_b}\) is a known vector of data, \(f_1 : \mathbb{R}^{n_1} \to \mathbb{R}, f_2 : \mathbb{R}^{n_2} \to \mathbb{R}\) are the objective functions, and \(A \in \mathbb{R}^{n_b \times n_1}, B \in \mathbb{R}^{n_b \times n_2}\) are linear operators. Many optimization problems in data science and machine learning can be cast into this form.

We consider the well-known Alternating Direction Method of Multipliers (ADMM) [3] for solving problem (1.1), and we apply Anderson Acceleration (AA) [1] to accelerate the convergence of ADMM. In particular, we consider problems where ADMM by itself would converge linearly with a linear asymptotic convergence factor \(\rho_{ADMM}\), and we are interested in explaining and quantifying by how much the combined AA-ADMM method would improve the asymptotic convergence compared to \(\rho_{ADMM}\). In recent papers it has indeed been observed numerically that AA may speed up the convergence of ADMM and related methods substantially [9, 17, 24], but there are no known convergence bounds for AA with finite window size that would allow quantification of this improvement in linear asymptotic convergence speed.

In this paper, we discuss an approach for quantifying the asymptotic convergence improvement provided by applying AA to ADMM. Our approach is based on the methods introduced in [5] for quantifying convergence acceleration by AA and the closely related nonlinear GMRES (NGMRES) method, which were applied in [5] to the acceleration of the Alternating Least Squares (ALS) method to compute canonical tensor decompositions. Our approach for quantifying the convergence improvement considers the stationary version of AA (sAA) where the sAA coefficients are determined in a way that optimizes the asymptotic linear convergence factor of the stationary sAA-ADMM method, given the spectral radius of the Jacobian of the ADMM update at the fixed point. We can use theoretical results from [5] on optimal sAA coefficients...
to compute the optimal sAA-ADMM asymptotic convergence factor, \( \rho^*_{sAA-ADMM} \), and, as in [5] for ALS, numerical results in this paper will show that this leads to a method for estimating the improved asymptotic linear convergence factor of AA with finite window size applied to ADMM, compared to \( \rho_{ADMM} \). We will also demonstrate numerically, as in [5], how the spectral properties of the ADMM and optimal sAA-ADMM Jacobians can be used to explain how and by how much nonlinear acceleration methods like sAA and AA can accelerate ADMM.

1.1. Alternating Direction Method of Multipliers. Recent research has shown that ADMM is an effective tool for solving (1.1), and can be competitive with the best known methods for some problems [3]. To present ADMM for solving (1.1), we first need to define the augmented Lagrangian

\[
L_\rho(x, z, y) = f_1(x) + f_2(z) + y^T(Ax + Bz - b) + \frac{\rho}{2}||Ax + Bz - b||_2^2,
\]

where \( y \) is the Lagrange multiplier, and \( \rho > 0 \) is a penalty parameter. ADMM then solves the original problem by performing alternating minimization of the augmented Lagrangian with respect to variables \( x \) and \( z \) and computes the sub-problems

\[
\begin{align*}
\text{argmin}_x L_\rho(x, z_k, y_k), \\
\text{argmin}_z L_\rho(x_{k+1}, z, y_k), \\
y_{k+1} = y_k + \rho(Ax_{k+1} + Bz_{k+1} - b),
\end{align*}
\]

given initial approximations \( z_0 \) and \( y_0 \). It is often more convenient to write the augmented Lagrangian (1.2) in an equivalent scaled form by replacing \( \frac{1}{\rho}y \) with \( u \)

\[
L_\rho(x, z, u) = f_1(x) + f_2(z) + \frac{\rho}{2}||Ax + Bz - b + u||_2^2 - \frac{\rho}{2}||u||_2^2.
\]

Then the ADMM steps become

\[
\begin{align*}
\text{argmin}_x f_1(x) + \frac{\rho}{2}||Ax + Bz - b + u_k||_2^2, \\
\text{argmin}_z f_2(z) + \frac{\rho}{2}||Ax_{k+1} + Bz - b + u_k||_2^2, \\
u_{k+1} = u_k + Ax_{k+1} + Bz_{k+1} - b,
\end{align*}
\]

given initial approximations \( z_0 \) and \( u_0 \).

The optimality conditions for problem (1.1) using ADMM are the primal feasibility

\[
Ax^* + Bz^* - b = 0,
\]

and dual feasibility

\[
0 \in \partial f_1(x^*) + A^T y^*, \tag{1.5}
\]
\[
0 \in \partial f_2(z^*) + B^T y^*, \tag{1.6}
\]

where \( x^*, z^*, y^* \) are the optimal solutions. It turns out that \( z_{k+1} \) and \( y_{k+1} \) always satisfy dual feasibility (1.6), and the dual feasibility condition (1.5) is equivalent to [3]

\[
\rho A^T B(z_{k+1} - z_k) \in \partial f_1(x_{k+1}) + A^T y_{k+1}.
\]

This means that

\[
r_{k+1}^\rho = Ax_{k+1} + Bz_{k+1} - b \]
can be used as the primal residual at iteration $k + 1$, and

$$r^d_{k+1} = \rho A^T B (z_{k+1} - z_k)$$

can be used as the dual residual at iteration $k + 1$. These two residuals converge to zero as ADMM proceeds [3].

Although there are abundant results on the application of ADMM, studies on ADMM convergence rate are few until recently. When the objective functions $f_1$ and $f_2$ are convex (not requiring strong convexity, and possibly nonsmooth), the work in [4, 12, 13] has shown shown an $O(1/k)$ convergence rate under some additional assumptions. The work in [2, 4, 6, 14, 16, 19] shows linear convergence of ADMM under strong convexity and rank conditions. More specifically, results in [16] show that when $f$ is strongly convex and the composite constraint matrix $[A B]$ is row independent, then ADMM converges linearly to the unique minimizer. More recent work in [2, 6] shows that when at least one of the component functions is strongly convex and has a Lipschitz-continuous gradient, and under certain rank conditions on the constraint matrices, some linear convergence results can be obtained for a subset of primal and dual variables in the ADMM algorithm. The slow convergence of ADMM is one of the reasons that ADMM was not well-known until recently when large-scale distributed optimization became necessary.

1.2. Acceleration methods for ADMM. Results on accelerated versions of ADMM are even fewer. The most widely used acceleration technique is simple over-relaxation, which reliably reduces the total iteration count by a small constant [10]. A GMRES-accelerated ADMM is discussed in [25] for a quadratic problem. For the case of Nesterov acceleration, which is a version of Anderson acceleration with window size one [5, 18], the only papers providing convergence rates for not necessarily differentiable convex functions are [7, 8, 11, 15], among which [11, 15] show that under strong convexity assumptions Nesterov acceleration of ADMM has an optimal global convergence bound of $O(1/k^2)$ in terms of the primal and dual residual norms. In [8] a dynamical system perspective was proposed for understanding ADMM and accelerated ADMM applied to problem

$$\min_x f(x) := f_1(x) + f_2(Ax),$$

and a convergence rate of $O(1/k)$ was proved for ADMM, and a convergence rate of $O(1/k^2)$ for accelerated ADMM, under the assumption that $f_1$ and $f_2$ are both continuously differentiable and convex, and $A$ has full column rank. Following this work, more convergence rates of dynamical systems related to relaxed and accelerated variants of ADMM are given in [7].

Work using Anderson acceleration (AA) can be found in [9, 15, 20, 21, 24], but no convergence rates are given. In this paper, we use AA to accelerate ADMM in the case that ADMM converges linearly, and we estimate an asymptotic convergence rate for the accelerated AA-ADMM algorithm.

1.2.1. Anderson Acceleration for fixed-point iterations. Consider fixed-point iteration (FPI)

$$x_{k+1} = q(x_k).$$
The method of Anderson Acceleration tries to improve convergence by taking
\begin{equation}
    x_{k+1} = \sum_{i=0}^{m_k} \alpha^{(k)}_i q(x_{k-i}),
\end{equation}
where \( m_k = \min\{m, k\} \) with some predefined window size \( m \geq 0 \), and the coefficients \( \alpha^{(k)}_i \) are computed from optimization problem
\begin{equation}
    \{\alpha^{(k)}_i\} = \operatorname{argmin}_{\{\alpha_i\}} \left\| \sum_{i=0}^{m_k} \alpha_i r(x_{k-i}) \right\|, \quad \text{s.t. } \sum_{i=0}^{m_k} \alpha_i = 1,
\end{equation}
where \( r(x_k) = q(x_k) - x_k \) is the residual of FPI (1.7) in iteration \( k \). We refer to Anderson Acceleration with window size \( m \) as AA(\( m \)).

It has been shown that Anderson acceleration is, in the linear case, essentially equivalent to the GMRES method for solving linear systems when \( m = k \) [23]. When \( m = 0 \), the un-accelerated FPI is recovered. The optimization (1.9) is usually solved as a least squares problem. By rearranging the residual terms and eliminating the constraint on weights summing up to 1, we solve
\begin{equation}
    \{\beta^{(k)}_i\} = \operatorname{argmin}_{\{\beta_i\}} \| r(x_k) + \sum_{i=1}^{k-1} \beta_i (r(x_{k-i}) - r(x_{k-i-1})) \|^2,
\end{equation}
which recovers (1.9) if we let \( \beta_j = -\sum_{i=j+1}^{m_k} \alpha_i \), or
\begin{align*}
    \alpha_0 &= 1 + \beta_0, \\
    \alpha_j &= \beta_j - \beta_{j-1}, \quad j = 1, \ldots, k-1, \\
    \alpha_k &= -\beta_{k-1}.
\end{align*}
It turns out this equivalent form usually has good conditioning properties in practice. We then also have
\begin{equation}
    x_{k+1} = q(x_k) + \sum_{i=0}^{m_k-1} \beta^{(k)}_i (q(x_{k-i}) - q(x_{k-i-1})).
\end{equation}
The convergence of Anderson acceleration is not guaranteed, however. The work in [22] shows that for linear problems, if the FPI is a contraction, global convergence can be proved. But for nonlinear problems, only local convergence can be shown under certain conditions. Global convergence properties can be improved by adding a safeguarding step to the algorithm [5,9,18,24]. However, we do not need a safeguarding step for the numerical tests with linear asymptotic convergence that we consider in this paper.

In [5], a stationary variant of AA is considered, which we call sAA, and is given by
\begin{equation}
    x_{k+1} = q(x_k) + \sum_{i=0}^{m_k-1} \beta_i (q(x_{k-i}) - q(x_{k-i-1})).
\end{equation}
We refer to sAA with window size \( m \) as sAA(\( m \)). In [5], the constant sAA coefficients \( \beta_i \) in (1.12) are computed such that the asymptotic linear convergence factor of the
sAA method is optimal, given knowledge of $q'(x)$ evaluated in the fixed point $x^*$. Numerical results in [5] show that this optimal sAA asymptotic convergence factor can be used to estimate the asymptotic convergence speed for AA with finite window size, and we also use this approach in this paper to quantify the asymptotic convergence speed of AA-ADMM, compared to $\rho_{ADMM}$.

1.2.2. Anderson Acceleration applied to ADMM (AA-ADMM). When we use AA to accelerate ADMM, we can treat one iterate of ADMM as a FPI, that is,

\[
\begin{aligned}
    x_{k+1} &= \argmin_x f_1(x) + \frac{\rho}{2}||Ax + Bz_k - b + u_k||_2^2, \\
    z_{k+1} &= \argmin_z f_2(z) + \frac{\rho}{2}||Ax_{k+1} + Bz - b + u_k||_2^2, \\
    u_{k+1} &= u_k + Ax_{k+1} + Bz_{k+1} - b.
\end{aligned}
\]

can be seen as a FPI

\[(z_{k+1}, u_{k+1}) = q(z_k, u_k),\]

given initial approximations $z_0$, $u_0$. Notice that $x_{k+1}$ is only dependent on $z_k$ and $u_k$ and can be recovered from them anytime during the iteration, thus it is included implicitly and can be eliminated when ADMM is seen as a FPI [24]. Moreover, if $B$ is a nonsingular square matrix, since

\[
\nabla f_2(z_{k+1}) + \rho B^T(Ax_{k+1} + Bz_{k+1} - b + u_k) = 0,
\]

from the step of the $z_{k+1}$ update, we get

\[
u_k + Ax_{k+1} + Bz_{k+1} - b = -\frac{1}{\rho}B^{-T}\nabla f_2(z_{k+1}),\]

and thus

\[
    u_{k+1} = -\frac{1}{\rho}B^{-T}\nabla f_2(z_{k+1}).
\]

Then, we can further simplify ADMM as a FPI of variable $z$ only [24], i.e.,

\[z_{k+1} = q(z_k),\]

The other two variables $x_{k+1}$ and $u_{k+1}$ can be recovered from $z_k$. These simplifications are not necessary, but they help avoid computational overhead and simplify implementation. The pseudo-code of AA-ADMM for accelerating the forms (1.13) and (1.14) is given in Algorithms 1.1 and 1.2, respectively.

2. Asymptotic convergence speed of AA-ADMM. As we mentioned earlier, there is a lack of mathematical understanding of the improved convergence speed of AA with finite window size applied to FPI (1.7) [5]. In this section, we discuss quantifying the improved asymptotic convergence speed of AA-ADMM compared to $\rho_{ADMM}$, in the case ADMM by itself converges linearly, following the discussion in [5]. We will focus on the analysis of $m = 1$ here.

2.1. Stationary Anderson Acceleration. The theoretical analysis of the improved convergence speed of AA remains a challenging topic; one of the reasons is that
Algorithm 1.1 Anderson-accelerated ADMM (about $z, u$)

**Input:** $z_0, u_0, m \geq 1$

1. $x_1 = \arg\min_x f_1(x) + \frac{\rho}{2} \|Ax + Bz_0 - b + u_0\|^2$;
2. $z_1 = \arg\min_z f_2(z) + \frac{\rho}{2} \|Ax_1 + Bz - b + u_0\|^2$;
3. $u_1 = \bar{u}_1 = u_0 + Ax_1 + Bz_1 - c$;
4. $Q = \begin{bmatrix} z_1 \\ u_1 \end{bmatrix}$; $R = \begin{bmatrix} \bar{z}_1 - z_0 \\ \bar{u}_1 - u_0 \end{bmatrix}$;

for $k = 1, 2, ..., K - 1$ do

1. $m_k = \min\{m, k\}$;
2. $x_{k+1} = \arg\min_x f_1(x) + \frac{\rho}{2} \|Ax + Bz_k - b + u_k\|^2$;
3. $z_{k+1} = \arg\min_z f_2(z) + \frac{\rho}{2} \|Ax_{k+1} + Bz - b + u_k\|^2$;
4. $\bar{u}_{k+1} = u_0 + Ax_{k+1} + Bz_{k+1} - c$;
5. $Q = \begin{bmatrix} \bar{z}_{k-m_k+1} - z_{k-m_k} \\ \bar{u}_{k-m_k+1} - u_{k-m_k} \\ \vdots \\ \bar{z}_{k+1} - z_k \\ \bar{u}_{k+1} - u_k \end{bmatrix}$;
6. $R = \begin{bmatrix} q_{m_k} - q_{m_k-1} \\ r_{m_k} - r_{m_k-1} \end{bmatrix}$, where $q_j = Q_{:, j}$ is the $j$-th column of $Q$;
7. $H_k = [r_{m_k} - r_{m_k-1}, ..., r_1 - r_0]$, where $r_j = R_{:, j}$ is the $j$-th column of $R$;
8. $\beta(k) = \arg\min_{\|\beta\|_2} \|r_{m_k} + H_k \beta(k)\|^2$;
9. $(\bar{z}_{k+1} - z_k)_{u_k+1} = q_{m_k} + F_k \beta(k)$

end

**Output:** $x_K, z_K$

The AA coefficients $\beta^{(k)}$ in (1.11) change at every iteration and thus are hard to analyze. A recent study by De Sterck and He [5] investigates the asymptotic convergence factor of AA applied to the steepest descent method and Alternating Least Squares by fixing the coefficients $\beta_i$ in stationary variant (1.12) of AA (sAA), and choosing the coefficients to be optimal in terms of the sAA convergence factor. Numerical results in [5] find that the asymptotic convergence speed of AA with finite window size, where the $\beta_i^{(k)}$ are determined in each iteration $k$ in a locally optimal way, is similar to the asymptotic convergence speed of the stationary sAA method with globally optimal coefficients. As such, the sAA convergence factor, which can be computed analytically when the window size $m = 1$, can be used, as we show in this paper, to quantify the AA-ADMM convergence speed. Our numerical tests will also show, as in [5], how the spectral properties of the ADMM and optimal sAA-ADMM Jacobians can be used to give insight into the mechanism by which nonlinear acceleration methods like sAA and AA accelerate ADMM.

We consider sAA with $m = 1$ applied to FPI (1.7):

$$
(2.1) \quad x_{k+1} = \alpha_0 q(x_k) + \alpha_1 q(x_{k-1}) = (1 + \beta)q(x_k) - \beta q(x_{k-1}),
$$

where $\beta$ remains fixed at all iterations. Note that, for $m = 1$, this is a stationary version of Nesterov’s accelerated gradient descent method if $q(x)$ is a gradient descent update.

We will also make the following assumptions:

**Assumption 1.** The iteration operator $q(\cdot)$ is differentiable at $x^*$.

**Assumption 2.** The iteration operator $q(\cdot)$ is a contraction, i.e., there exists a
Algorithm 1.2 Anderson-accelerated ADMM (about $z$)

**input**: $z_0, u_0, m \geq 1$

$\quad x_1 = \arg\min_x f_1(x) + \frac{\rho}{2} ||Ax + Bz_0 - b + u_0||^2$;

$\quad z_1 = z_1 = \arg\min_z f_2(z) + \frac{\rho}{2} ||Ax_1 + Bz - b + u||^2;

$\quad u_1 = u_1 = u_0 + Ax_1 + Bz_1 - c$;

$\quad Q = z_1$; $\bar{z} = z_0$;

**for** $k = 1, 2, ..., K - 1$ **do**

$\quad m_k = \min\{m, k\}$;

$\quad X_{k+1} = \arg\min_x L_\rho(x, z_k, u_k)$;

$\quad \bar{z}, k = \arg\min_z L_\rho(x_{k+1}, z, u_k)$

$\quad Q = [\bar{z}_{k-m_k+1}, ..., \bar{z}_{k+1}]$;

$\quad R = [\bar{z}_{k-m_k+1} - z_{k-m_k}, ..., \bar{z}_{k+1} - z_k]$;

$\quad R_k = [q_{m_k}, q_{m_k-1}, ..., q_1 - q_0]$, where $q_j$ is the $j$-th column of $Q$;

$\quad H_k = [r_{m_k}, r_{m_k-1}, ..., r_1 - r_0]$, where $r_j$ is the $j$-th column of $R$;

$\quad \beta_k = \arg\min_{\beta_k} ||r_{m_k} + H_k \beta_k||^2$;

$\quad z_{k+1} = q_{m_k} + F_k \beta_k$;

$\quad u_{k+1} = -\frac{1}{\rho} A^T f_2(z_{k+1})$;

**end**

**output**: $X, \bar{z}$

---

constant $c \in (0, 1)$ such that $||q(x) - q(y)|| \leq c ||x - y||$ for all $x, y$. This assumption ensures the FPI itself is always convergent.

To study the convergence behaviour and find the optimal choice of $\beta$, we introduce, as in [5],

$$X_k = \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix}$$

and write the sAA iteration as

$$X_{k+1} = \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = \begin{bmatrix} (1 + \beta)q(x_{k}) - \beta x(x_{k-1}) \\ x_k \end{bmatrix} = \Psi(X_k).$$

Because $q(\cdot)$ is assumed to be continuously differentiable, we have by Taylor expansion

$$X_{k+1} - X^* = \Psi(X_k) - X^* = \nabla \Psi(X^*)(X_k - X^*) + o(||X_k - X^*||),$$

where

$$\nabla \Psi(X^*) = \begin{bmatrix} (1 + \beta)q'(x^*) & -\beta q'(x^*) \\ I & O \end{bmatrix}.$$ 

Therefore,

$$\lim_{k \to \infty} \frac{||X_{k+1} - X^*||}{||X_k - X^*||} \leq ||\nabla \Psi(X^*)|| = ||\nabla \Psi||.$$ 

This gives a tight upper bound of the asymptotic convergence factor of sAA(1), $\rho_{sAA(1)} = \rho(\nabla \Psi(X^*))$. We will be interested in finding the optimal asymptotic convergence factor $\rho_{sAA(1)}^*$ of sAA(1) over all possible choices of $\beta$.

$$\rho_{sAA(1)}^* = \min_{\beta} \rho_{sAA(1)}(\beta).$$
By the properties of the Schur complement, we have that
\[
|\lambda I - \nabla \Psi^*| = \begin{vmatrix} \lambda I - (1 + \beta)q'(x^*) & \beta q'(x^*) \\ -I & \lambda I \end{vmatrix} \\
= |\lambda I \cdot (\lambda I - (1 + \beta)q'(x^*)) + \beta q'(x^*)| = |\lambda^2 I - (1 + \beta)\lambda q'(x^*) + \beta q'(x^*)| = 0
\]
where \( \lambda \) is any eigenvalue of \( \nabla \Psi^* \) and \( |M| \) means the determinant of matrix \( M \). Denote the eigenvalues of \( q'(x^*) \) by \( \mu \), then we have
\[
\lambda^2 - (1 + \beta)\mu \lambda + \beta \mu = 0.
\]
Hence, all the eigenvalues of \( \nabla \Psi^* \) are contained in the set
\[
\{ \lambda : \lambda^2 - (1 + \beta)\mu \lambda + \beta \mu = 0, \mu \in \sigma(q'(x^*)) \},
\]
where \( \sigma(M) \) means the spectrum of matrix \( M \). To determine the optimal \( \beta \), we only need to find
\[
\beta^* = \arg \min_{\beta \in \mathbb{R}} \max_{\lambda} \{ |\lambda| : \lambda^2 - (1 + \beta)\mu \lambda + \beta \mu = 0, \mu \in \sigma(q'(x^*)) \}.
\]
To compute \( \beta^* \), we define, for any fixed \( \mu \),
\[
S_\mu(\beta) = \max_{\lambda} \{ |\lambda| : \lambda^2 - (1 + \beta)\mu \lambda + \beta \mu = 0 \}.
\]
We first assume that the spectrum of \( q'(x^*) \) is real. Then the following conclusions hold:

**Proposition 2.1.** [5, Lemmas 3.1,3.2] When \( 0 < \mu < 1 \), \( \min_{\beta} S_\mu(\beta) = 1 - \sqrt{1 - \mu} \), and the optimum is achieved at \( \beta_\mu = \beta^*_\mu = 1 - \sqrt{1 - \mu} \).

When \( \mu \geq 1 \), \( \min_{\beta} S_\mu(\beta) = \sqrt{\mu} \), and the optimum is achieved at \( \beta_\mu = \beta^*_\mu = 1 - \sqrt{1 - \mu} \).

When \( \mu < 0 \), \( \min_{\beta} S_\mu(\beta) = \sqrt{1 - \mu} - 1 \), and the optimum is achieved at \( \beta_\mu = \beta^*_\mu = 1 - \sqrt{1 - \mu} \).

From this proposition and the monotonicity of \( \min_{\beta} S_\mu(\beta) \), still for the case the spectrum of \( q'(x^*) \) is real, we can easily derive the following proposition if we denote
\[
\sigma^q_{\max} = \max(\sigma(q'(x^*))), \quad \sigma^q_{\min} = \min(\sigma(q'(x^*))).
\]

**Proposition 2.2.** [5, Theorem 3.4] When \( \sigma(q'(x^*)) \subset [0,1) \), the optimal weight is
\[
\beta^* = \frac{1 - \sqrt{1 - \sigma^q_{\max}}}{1 + \sqrt{1 - \sigma^q_{\max}}}, \quad \rho^*_A(1) = 1 - \sqrt{1 - \sigma^q_{\max}}.
\]

When \( \sigma(q'(x^*)) \subset (-1,0] \), the optimal weight is
\[
\beta^* = \frac{1 - \sqrt{1 - \sigma^q_{\min}}}{1 + \sqrt{1 - \sigma^q_{\min}}},
\]
and $\rho_{sAA(1)}^* = 1 - \sqrt{1 - \sigma_{q'}^2}$.

When $\sigma(q'(x^*)) \subset (-1, 1)$ and $\sigma_{q'}^{\max} \sigma_{q'}^{\min} < 0$, the optimal weight is

$$\beta^* = \begin{cases} 
1 - \frac{\sqrt{1 - \sigma_{q'}^{\max}}}{1 + \sqrt{1 - \sigma_{q'}^{\max}}}, & \text{if } 1 - \sqrt{1 - \sigma_{q'}^{\max}} > \sqrt{1 - \sigma_{q'}^{\min}} - 1, \\
1 - \frac{\sqrt{1 - \sigma_{q'}^{\min}}}{1 + \sqrt{1 - \sigma_{q'}^{\min}}}, & \text{if } 1 - \sqrt{1 - \sigma_{q'}^{\max}} \leq \sqrt{1 - \sigma_{q'}^{\min}} - 1, 
\end{cases}$$

and

$$\rho_{sAA(1)}^* = \begin{cases} 
1 - \frac{\sqrt{1 - \sigma_{q'}^{\max}}}{1 + \sqrt{1 - \sigma_{q'}^{\max}}}, & \text{if } 1 - \sqrt{1 - \sigma_{q'}^{\max}} > \sqrt{1 - \sigma_{q'}^{\min}} - 1, \\
1 - \frac{\sqrt{1 - \sigma_{q'}^{\min}}}{1 + \sqrt{1 - \sigma_{q'}^{\min}}}, & \text{if } 1 - \sqrt{1 - \sigma_{q'}^{\max}} \leq \sqrt{1 - \sigma_{q'}^{\min}} - 1. 
\end{cases}$$

If the spectrum of $q'(x^*)$ is complex, the following theorem can be used:

**Proposition 2.3.** \([5]\) Let the spectral radius of $q'(x^*)$ be $\rho_{q'}^*$. If there exists at a real eigenvalue $\mu$ of $q'(x^*)$ such that $\rho_{q'}^* = \mu$, then the optimal asymptotic convergence rate of $sAA(1)$, $\rho_{sAA(1)}^*$, is bounded below by

$$\rho_{sAA(1)}^* \geq 1 - \sqrt{1 - \rho_{q'}^*},$$

and if the equality holds,

$$\beta^* = \frac{1 - \sqrt{1 - \rho_{q'}^*}}{1 + \sqrt{1 - \rho_{q'}^*}}.$$

Propositions 2.2 and 2.3 allow us to compute the optimal $sAA(1)$ coefficient $\beta^*$ and the optimal asymptotic convergence factor, $\rho_{sAA}^*$, when $q'(x^*)$ is known. Also, \([5]\) explains how optimal $sAA$ weights and convergence factors $\rho_{sAA}^*$ can be determined for $sAA$ with $m \geq 2$ by optimization, since analytical results are not known in this case.

### 3. Numerical results.

In this section we present numerical results showing how the optimal convergence factor of the stationary AA method with window size $m = 1$, as computed from Propositions 2.2 and 2.3, can be used to quantify the AA-ADMM convergence speed. We also investigate the spectra of the ADMM and optimal $sAA$-ADMM Jacobians to explain the convergence acceleration. In all numerical experiments, we use a zero initial guess unless stated otherwise, and no parameter tuning is applied.

#### 3.1. Ridge regression \([3]\).

**3.1.1. Problem description.** The $l_2$-regularized least squares problem, also called ridge regression, is a common technique in machine learning that reduces model complexity and prevents over-fitting. The optimization problem is

$$\min_x \frac{1}{2} ||Ax - b||_2^2 + \lambda ||x||_2^2,$$

where $(A, b) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m$ is the training set, and $\lambda > 0$ is a regularization parameter.
To use the ADMM method, we write this problem as

\[
\min_{x,z} \frac{1}{2} ||Ax - b||^2_2 + \lambda ||z||^2_2,
\]
\[
s.t. \; x - z = 0.
\]

The scaled augmented Lagrangian is

\[
L_\rho(x, z, u) = \frac{1}{2} ||Ax - b||^2_2 + \lambda ||z||^2_2 + \rho_\frac{1}{2} ||x - z + u||^2_2 - \rho_\frac{1}{2} ||u||^2_2.
\]

The ADMM steps for this problem are:

\[
\begin{align*}
x_{k+1} &= \text{argmin}_{x} \frac{1}{2} ||Ax - b||^2_2 + \rho_\frac{1}{2} ||x - z_k + u_k||^2_2 \\
z_{k+1} &= \text{argmin}_{z} \lambda ||z||^2_2 + \rho_\frac{1}{2} ||x_{k+1} + u_k - z||^2_2 \\
u_{k+1} &= u_k + x_{k+1} - z_{k+1}
\end{align*}
\]

which gives

\[
\begin{align*}
x_{k+1} &= (A^T A + \rho I)^{-1} (A^T b + \rho(z_k - u_k)) \\
z_{k+1} &= \frac{\rho}{2\lambda + \rho} (x_{k+1} + u_k) \\
u_{k+1} &= u_k + x_{k+1} - z_{k+1}
\end{align*}
\]

Since \(u_{k+1}\) can be explicitly obtained from \(z_{k+1}\),

\[
u_{k+1} = \frac{2\lambda}{\rho} z_{k+1},
\]

we can write one iteration of ADMM as one fixed point update of variable \(z\), \(\tilde{z}_{k+1} = q(z_k)\), where

\[
q(z_k) = \left[ \frac{\rho(\rho - 2\lambda)}{\rho + 2\lambda} (A^T A + \rho I)^{-1} + \frac{2\lambda}{\rho + 2\lambda} I \right] z_k + \frac{\rho}{\rho + 2\lambda} (A^T A + \rho I)^{-1} A^T b
\]

\[
= M z_k + \hat{b}.
\]

This problem is so simple that we can directly compute the true solution

\[x^* = z^* = (A^T A + 2\lambda I)^{-1} A^T b.\]

We will use this exact solution to test our theoretical prediction. The ADMM update is simply a stationary iteration. Therefore, \(q' = M\) is independent of \(z\). To determine the optimal sAA acceleration, we can analyze the spectrum of matrix \(M\) to pick the optimal \(\beta^*\).

3.1.2. Parameters for test problem. We implement our algorithms on a randomly generated sparse matrix of size \(m \times n = 150 \times 300\) with density 0.001 sampled from the standard normal distribution. The \(b\) vector is sampled from the standard normal distribution. The regularization parameter is chosen as \(\lambda = 1\), and we pick the penalty parameter \(\rho = 10\).

3.1.3. Convergence results. We obtain convergence plots for the error \(||z - z^*||_2\) as shown in Figure 3.1. We see that ADMM converges linearly. The convergence factor of ADMM is substantially improved by the AA-based methods. AA(2)
and AA(3) converge slightly faster than AA(1), and sAA(1) converges with similar asymptotic speed.

The convergence improvement of the AA-ADMM methods over ADMM can be understood in terms of spectral properties as follows. Figure 3.2 shows the spectrum of the ADMM iteration matrix $M$, $\sigma(M) \in (0, 1)$. The spectrum is real since $M$ is symmetric, and the spectral radius $\rho^* = 0.833$. Therefore, according to Proposition 2.2, the optimal $\beta$ for sAA(1) is

$$
\beta^* = \frac{1 - \sqrt{1 - \rho^*_q}}{1 + \sqrt{1 - \rho^*_q}} = 0.420.
$$
The corresponding optimal sAA(1) linear convergence factor is

$$\rho_{sAA(1)-ADMM}^* = \rho(\nabla \Psi) = 1 - \sqrt{1 - \rho_q^*} = 0.592.$$  

Figure 3.2 also shows the spectrum of the sAA(1)-ADMM iteration matrix, $\nabla \Psi$. The nonlinear acceleration method spreads the ADMM spectrum out in the complex plane in a way that strongly reduces the asymptotic convergence factor: $\rho(\nabla \Psi)$ is much smaller than $\rho_q^*$. Note that stationary iterative method (2.1) maps part of the nonnegative real spectrum of $q'(x^*)$ to a circle. As seen in Figure 3.1, the optimal sAA(1)-ADMM factor, $\rho_{sAA(1)-ADMM}^*$, provides a good prediction of the convergence factors of the AA-ADMM methods. The convergence speed of sAA(1)-ADMM matches our theoretical derivation of $\rho_{sAA(1)-ADMM}^*$.

3.2. Regularized logistic regression [3].

3.2.1. Problem description. We consider a very simple logistic regression model in this section. The objective function of the regularized logistic regression model is

$$\min_{x} \frac{1}{m} \sum_{i=1}^{m} \log(1 + \exp(-y_i(a_i^T w + c))) + \lambda ||x||_2^2,$$

where

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} \in \mathbb{R}^{m \times n},$$

are $m$ data samples, $y_1, \cdots, y_m$ are the corresponding labels, and

$$x = \begin{bmatrix} c \\ w \end{bmatrix}, \quad w \in \mathbb{R}^n, \quad c \in \mathbb{R},$$

are the combination coefficients and bias to be optimized. To apply ADMM, we write this problem as

$$\min_{x,z} \frac{1}{m} \sum_{i=1}^{m} \log(1 + \exp(-y_i(a_i^T w + c))) + \lambda ||z||_2^2,$$

s.t. $x - z = 0$.

This gives the augmented Lagrangian

$$L(x, z, u, \rho) = \frac{1}{m} \sum_{i=1}^{m} \log(1 + \exp(-y_i(a_i^T w + c))) + \lambda ||z||_2^2 + \frac{\rho}{2} ||x - z + u||_2^2 - \frac{\rho}{2} ||u||_2^2.$$

Hence, we get the ADMM steps

$$\begin{align*}
    x_{k+1} &= \arg \min_{x} \frac{1}{m} \sum_{i=1}^{m} \log(1 + \exp(-y_i(a_i^T w + c))) + \frac{\rho}{2} ||x - z_k + u_k||_2^2 \\
    z_{k+1} &= \arg \min_{z} \lambda ||z||_2^2 + \frac{\rho}{2} ||x_{k+1} - z + u_k||_2^2 \\
    u_{k+1} &= u_k + x_{k+1} - z_{k+1}.
\end{align*}$$

To solve for $x_{k+1}$, we use Newton’s method.
3.2.2. Parameters for the test problem. For this problem, we applied our algorithms to the Madelon data set from the UCI machine learning repository\(^1\). To reduce the amount of computation, we only used a portion of the features and examples. The regularization parameter is \(\lambda = 2\), and the augmented Lagrangian penalty parameter is \(\rho = 10\).

3.2.3. Convergence results. Since the FPI representation of ADMM for solving the regularized logistic regression problem is nonlinear, we are now not able to find an explicit expression for \(z_{k+1} = q(z_k)\) like before. To determine the spectrum of \(q'(z^*)\), we use the first-order finite difference method with step size \(h = 1 \times 10^{-4}\) to

\(^1\)https://archive.ics.uci.edu/ml/datasets/Madelon
approximate \( q'(z^*) \) at the approximate true solution solved to \( 10^{-16} \) accuracy.

Figure 3.3 compares the error norm reduction when using ADMM, AA\((m)\)-ADMM and sAA\((1)\)-ADMM. The convergence acceleration seen in the figure can be explained based on the spectra in Figure 3.4. The spectrum of \( q'(z^*) \) has asymptotic convergence factor \( \rho_{q'}^* = 0.714 \). We can choose the optimal \( \beta^* \) the same way as in the ridge regression problem:

\[
\beta^* = \frac{1 - \sqrt{1 - \rho_{q'}^*}}{1 + \sqrt{1 - \rho_{q'}^*}} = 0.303.
\]

The corresponding optimal sAA\((1)\)-ADMM linear convergence factor is

\[
\rho^*_{sAA(1)-ADMM} = \rho(\nabla \Psi) = 1 - \sqrt{1 - \rho_{q'}^*} = 0.465 < (\rho_{q'}^*)^2.
\]

Figure 3.3 shows that \( \rho^*_{sAA(1)-ADMM} \) is a good prediction for the convergence factors of the AA-accelerated ADMM methods.

3.3. Smoothed sparse inverse covariance estimation (adapted from [3]).

3.3.1. Problem description. Let us now look at the sparse inverse covariance estimation problem

\[
\min_{X > 0} - \log \det(X) + \text{tr}(XS) + \lambda \|X\|_1,
\]

where \( \lambda \) is a regularization parameter, and \( S \in \mathbb{R}^{n \times n} \) is the empirical covariance matrix generated from the sample data, which is usually not a good estimate of the actual covariance. This model solves for the maximum likelihood estimate \( X \) of the inverse covariance matrix with sparsity constraint imposed by \( l_1 \) regularization.

In order to test our algorithms on smooth problems, we use the Huber function to approximate the regularization term. Let \( \mu > 0 \) be some small parameter. We change the original problem to

\[
\min_{X > 0} - \log \det(X) + \text{tr}(XS) + \lambda \sum_{i,j} h_\mu(X_{ij}),
\]

where

\[
h_\mu(X_{ij}) = \begin{cases} 
X_{ij}^2 / 2\mu, & |X_{ij}| \leq \mu, \\
|X_{ij}| - \mu / 2, & |X_{ij}| \geq \mu,
\end{cases}
\]

which is differentiable. The derivative of \( h_\mu(X_{ij}) \) is

\[
\nabla h_\mu(X_{ij}) = \begin{cases} 
X_{ij} / \mu, & |X_{ij}| \leq \mu, \\
\text{sign}(X_{ij}), & |X_{ij}| \geq \mu.
\end{cases}
\]

To use ADMM, we solve the following constrained problem

\[
\min_{X > 0, Z > 0} - \log \det(X) + \text{tr}(XS) + \lambda \sum_{i,j} h_\mu(Z_{ij}),
\]

\[
\text{s.t. } X - Z = 0.
\]
which is smooth and convex. The scaled augmented Lagrangian is

\[ L(X, Z, U, \rho) = -\log \det(X) + \text{tr}(XS) + \lambda \sum_{i,j} h_\mu(Z_{ij}) + \frac{\rho}{2} ||X - Z + U||_F^2 - \frac{\rho}{2} ||U||_F^2. \]

It turns out we do not need to include the positive definiteness constraint; this will become clear later. Therefore, we get the ADMM steps as follows

\[
\begin{align*}
X_{k+1} &= \text{argmin}_X \left(-\log \det(X) + \text{tr}(XS) + \rho \frac{1}{2} ||X - Z_k + U_k||_F^2\right) \\
Z_{k+1} &= \text{argmin}_Z \lambda \sum_{i,j} h_\mu(Z_{ij}) + \rho \frac{1}{2} ||X_{k+1} - Z + U||_F^2 \\
U_{k+1} &= U_k + X_{k+1} - Z_{k+1}.
\end{align*}
\]

We can solve for \(X_{k+1}, Z_{k+1}\) analytically by making use of the first-order optimality condition

\[
\frac{\partial}{\partial X} \left(-\log \det(X) + \text{tr}(XS) + \rho \frac{1}{2} ||X - Z_k + U_k||_F^2\right) = -X^{-1} + S^T + \rho(X - Z_k + U_k) = \rho X - \rho(Z_k - U_k) + S - X^{-1} = 0.
\]

Suppose \(X\) is positive definite, then we have

\[
\rho X^2 - (\rho(Z_k - U_k) - S) X - I = 0,
\]

which is a quadratic equation in terms of the matrix \(X\). Suppose \(\rho(Z_k - U_k) - S = QAQ^{-1}\), then it can be shown that \(X_{k+1} = QAQ^{-1}\) solves the quadratic equation. Plugging these two diagonalizations into the equation, it is easy to get

\[
Q \left(\rho \Lambda_x^2 - \Lambda \Lambda_x - I\right) Q^{-1} = 0,
\]

which again gives

\[
\rho \Lambda_x^2 - \Lambda \Lambda_x - I = 0.
\]

Therefore, we have

\[
\Lambda_x = \frac{-\Lambda + \sqrt{\Lambda^2 + 4\rho I}}{2\rho},
\]

and

\[
X_{k+1} = QA_xQ^{-1} = Q \left(\frac{-\Lambda + \sqrt{\Lambda^2 + 4\rho I}}{2\rho}\right) Q^{-1}.
\]

For the \(Z\) update, we have

\[
\lambda \nabla h_\mu(Z) + \rho(Z - X_{k+1} - U_k) = 0.
\]

This gives

\[
[Z_{k+1}]_{ij} = \begin{cases} 
\frac{\beta \rho}{\lambda + \mu \rho} ([X_{k+1}]_{ij} + [U_k]_{ij}), & |[X_{k+1}]_{ij} + [U_k]_{ij}| \leq \mu + \frac{\lambda}{\rho} \\
[X_{k+1}]_{ij} + [U_k]_{ij} - \frac{\lambda}{\rho} \text{sign}(Z_{ij}), & \text{otherwise}.
\end{cases}
\]

Since only element-wise operations are involved, this equation can be solved efficiently just like the proximal operator for the original \(l_1\)-norm.
3.3.2. Parameters for the test problem. We randomly generate a sparse $40 \times 40$ positive definite inverse covariance matrix, and then generate 400 Gaussian samples from the normal distribution with zero mean and the random covariance matrix. After that, we generate the empirical covariance matrix $S$ from the sample data. For the smoothing parameter, we choose $\mu = 0.001$. The regularization parameter is $\lambda = 1$. The augmented Lagrangian penalty parameter is $\rho = 10$.

3.3.3. Convergence results. To determine the spectrum of $q'(z^*)$, we use the first-order finite difference method with the step size $h = 0.005$ to approximate $q'(z^*)$ at the approximate true solution solved to $10^{-16}$ accuracy.

Figure 3.5 compares the error norm reduction when using ADMM, AA$(m)$-ADMM and sAA$(1)$-ADMM. The convergence acceleration seen in the figure can be explained...
based on the spectra in Figure 3.6. The spectrum of $q'(z^*)$ has asymptotic convergence factor $\rho_{Q}^* = 0.829$. We can choose the optimal $\beta^*$ the same way as in the ridge regression problem,

$$\beta^* = \frac{1 - \sqrt{1 - \rho_{Q}^*}}{1 + \sqrt{1 - \rho_{Q}^*}} = 0.415.$$ 

The corresponding optimal sAA(1)-ADMM linear convergence factor is

$$\rho_{sAA(1)-ADMM} = \rho(\nabla \Psi) = 1 - \sqrt{1 - \rho_{Q}^*} = 0.587 < (\rho_{Q}^*)^2.$$ 

3.4. Lasso problem [3].

3.4.1. Problem description. $l_1$-regularized linear regression is also called the lasso problem:

$$\min_x \frac{1}{2} ||Ax - b||_2^2 + \lambda ||x||_1,$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given data, $\lambda > 0$ is a scalar regularization parameter, and $x \in \mathbb{R}^n$ is the optimization variable. In typical applications, there are many more features than training examples, and the goal is to find a parsimonious model for the data [3].

To apply ADMM, we solve the following constrained problem

$$\min_x \frac{1}{2} ||Ax - b||_2^2 + \lambda ||x||_1,$$

s.t. $x - z = 0$.

The scaled augmented Lagrangian is

$$L(x, z, u, \rho) = \frac{1}{2} ||Ax - b||_2^2 + \lambda ||z||_1 + \frac{\rho}{2} ||x - z + u||_2^2 - \frac{\rho}{2} ||u||_2^2.$$ 

Therefore, we get the ADMM steps

$$\begin{cases}
  x_{k+1} = \arg\min_x \frac{1}{2} ||Ax - b||_2^2 + \frac{\rho}{2} ||x - z_k + u_k||_2^2 \\
  z_{k+1} = \arg\min_z \lambda ||z||_1 + \frac{\rho}{2} ||x_{k+1} - z + u_k||_2^2 \\
  u_{k+1} = u_k + x_{k+1} - z_{k+1},
\end{cases}$$

which gives

$$\begin{cases}
  x_{k+1} = (A^T A + \rho I)^{-1} (A^T b + \rho (z_k - u_k)) \\
  z_{k+1} = \text{prox}_{\frac{\lambda}{2} ||\cdot||_1} (x_{k+1} + u_k), \\
  u_{k+1} = u_k + x_{k+1} - z_{k+1},
\end{cases}$$

where $x_{k+1}$ can be solved efficiently as a least squares problem like in ridge regression. Since the update of $z_{k+1}$ is nonsmooth, $u_{k+1}$ can not be expressed explicitly as a function of $z_{k+1}$, and we will treat one ADMM iteration as a FPI about both variables $z$ and $u$ in order to apply Anderson acceleration.

3.4.2. Parameters for the test problem. We test our algorithms on a randomly generate sparse matrix of size $m \times n = 150 \times 300$ with density 0.001, sampled from the uniform distribution on $[0,1)$. The $b$ vector is sampled from the standard normal distribution. The regularization parameter $\lambda = 1$, and we pick the penalty parameter $\rho = 10.$
3.4.3. Convergence results. Since now the FPI is about variables $z$ and $u$, we will accelerate the stacked variable $[z; u]$. The error norm during the iteration is evaluated as

$$e_k = \sqrt{||z_k - z^*||_2^2 + ||u_k - u^*||_2^2}.$$  

We use the first-order finite difference method with step size $h = 0.001$ to approximate $q'([z^*; u^*])$ at the approximate true solution solved to $10^{-16}$ accuracy.

Figure 3.7 compares the error norm reduction when using ADMM, AA($m$)-ADMM, and sAA(1)-ADMM. The convergence acceleration seen in the figure can be explained based on the spectra in Figure 3.8. The spectrum of $q'(z^*)$ has asymptotic convergence factor $\rho^*_q = 0.938$. We can choose the optimal $\beta^*$ the same way as in the ridge
regression problem,
\[
\beta^* = \frac{1 - \sqrt{1 - \rho_{\ell^2}}}{1 + \sqrt{1 - \rho_{\ell^2}}} = 0.601.
\]
The corresponding optimal sAA(1)-ADMM linear convergence factor is
\[
\rho^*_{\text{sAA(1)}-\text{ADMM}} = \rho(\nabla \Psi) = 1 - \sqrt{1 - \rho_{\ell^2}^*} = 0.751 < (\rho_{\ell^2}^*)^2.
\]
3.5. Total variation [3].

3.5.1. Problem description. The total variation model is a widely used method for applications like image denoising. The optimization problem is
\[
\min_x \frac{1}{2} ||y - x||_2^2 + \alpha ||Dx||_1,
\]
where \( x \in \mathbb{R}^n \) is the variable, \( y \in \mathbb{R}^n \) is the problem data (e.g. image pixel values), \( \alpha > 0 \) is a smoothing parameter, and \( D \in \mathbb{R}^{(n-1) \times n} \) is the difference operator
\[
D = \begin{bmatrix}
-1 & 1 \\
-1 & 1 \\
\vdots & \vdots \\
-1 & 1
\end{bmatrix}.
\]
To use ADMM, we write this problem as
\[
\min_{x,z} \frac{1}{2} ||y - x||_2^2 + \alpha ||z||_1^2,
\]
\[
\text{s.t. } Dx - z = 0.
\]
The augmented Lagrangian is
\[
L_\rho(x, z, u) = \frac{1}{2} ||y - x||_2^2 + \alpha ||z||_1^2 + \frac{\rho}{2} ||Dx - z + u||_2^2 - \frac{\rho}{2} ||u||_2^2.
\]
The ADMM steps for this problem are:
\[
\begin{align*}
x_{k+1} &= \arg\min_x \frac{1}{2} ||y - x||_2^2 + \frac{\alpha}{2} ||z_k - u_k||_2^2 + \frac{\rho}{2} ||Dx - z_k + u_k||_2^2 \\
z_{k+1} &= \arg\min_z \alpha ||z||_1^2 + \frac{\rho}{2} ||Dx_{k+1} + u_k - z||_2^2 \\
u_{k+1} &= u_k + Dx_{k+1} - z_{k+1}
\end{align*}
\]
where \( x_{k+1} \) is the proximal operator of the \( l_2 \) norm which can be evaluated from a least squares problem as before,
\[
x_{k+1} = \arg\min_x \left( \frac{1}{\sqrt{\rho}} D \right) x - \left( \frac{1}{\sqrt{\rho}} y \right) \right)^2,
\]
and \( z_{k+1} \) is just the proximal operator of the \( l_1 \)-norm,
\[
z_{k+1} = \text{prox}_{\frac{\rho}{2} \cdot ||\cdot||_1} (Dx_{k+1} + u_k).
\]
3.5.2. Parameters for the test problem. We test our algorithms on randomly generated data $y$ of size 1000 sampled from the standard normal distribution. The smoothing parameter is $\alpha = 0.001 \cdot \|y\|_\infty$. For the penalty parameter, we use $\rho = 10$.

3.5.3. Convergence results. We use the first-order finite difference method with step size $h = 1 \times 10^{-5}$ to approximate $q'(z^*; u^*)$ at the approximate true solution solved to $10^{-16}$ accuracy.

Figure 3.9 compares the error norm reduction when using ADMM, AA($m$)-ADMM and sAA(1)-ADMM. The convergence acceleration seen in the figure can be explained based on the spectra in Figure 3.10. The spectrum of $q'(z^*; u^*)$ has asymptotic convergence factor $\rho_{q}^* = 0.9756092409498504$. The numerically computed spectrum has
some complex eigenvalues. We choose $\beta^*$ according to Proposition 2.3,

$$\beta^* = \frac{1 - \sqrt{1 - \rho_q^*}}{1 + \sqrt{1 - \rho_q^*}} = 0.7298413205294394.$$ 

The corresponding lower bound on the optimal sAA(1)-ADMM linear convergence factor is

$$\rho^*_{sAA(1)-ADMM} \leq 1 - \sqrt{1 - \rho_q^*} = 0.843824588427066.$$ 

The spectral radius of the numerically computed $\nabla \Psi$ using $\beta^*$ is given by

$$\rho_{sAA(1)-ADMM}(\beta^*) = \rho(\nabla \Psi(\beta^*)) = 0.843824588426988 < (\rho_q^*)^2,$$

which is numerically equal to the lower bound. It is interesting to note that it was observed numerically in [5] that, for the case of sAA(1) acceleration of Alternating Least Squares for canonical tensor decomposition, for which $q'(x^*)$ has a complex spectrum, the lower bound in Proposition 2.3 is always achieved.

### 3.6. Nonnegative least squares [9].

#### 3.6.1. Problem description.

The nonnegative least squares problem is

$$\min_x ||Fx - g||_2^2, \quad \text{s.t.} \quad x \geq 0,$$

where $x \in \mathbb{R}^n$ is the variable, and $F \in \mathbb{R}^{m \times n}$ and $g \in \mathbb{R}^m$ are problem data. We can integrate the nonnegativity constraint into the objective function

$$\min_x ||Fx - g||_2^2 + \mathcal{I}_{\mathbb{R}^n_+}(x),$$

and re-write the problem as

$$\min_{x,z} ||Fx - g||_2^2 + \mathcal{I}_{\mathbb{R}^n_+}(z),$$

s.t. $x - z = 0$,

where $\mathcal{I}_{\mathbb{R}^n_+}$ is the indicator function defined as

$$\mathcal{I}_{\mathbb{R}^n_+}(z) = \begin{cases} 0, & z \geq 0 \\ +\infty, & \text{otherwise}. \end{cases}$$

The scaled augmented Lagrangian of this problem is

$$L_\rho(x, z, u) = ||Fx - g||_2^2 + \mathcal{I}_{\mathbb{R}^n_+}(z) + \frac{\rho}{2}||x - z + u||_2^2 - \frac{\rho}{2}||u||_2^2.$$ 

The ADMM steps on this problem are:

$$\begin{align*}
    x_{k+1} &= \arg\min_x \left[ ||Fx - g||_2^2 + \frac{\rho}{2}||x - z_k + u_k||_2^2 \right] \\
    z_{k+1} &= \arg\min_z \left[ \mathcal{I}_{\mathbb{R}^n_+}(z) + \frac{\rho}{2}||x_{k+1} + u_k - z||_2^2 \right] \\
    u_{k+1} &= u_k + x_{k+1} - z_{k+1}
\end{align*}$$
3.6.2. Parameters for the test problem. We test our algorithms on a randomly generate sparse matrix of size $m \times n = 150 \times 300$ with density 0.001, sampled from the standard normal distribution. The $g$ vector is sampled from the standard normal distribution. The augmented Lagrangian penalty parameter is $\rho = 2$. 

where the first step for $x_{k+1}$ is the proximal operator of the $l_2$-norm. The second step is just the proximal operator of the indicator function, which is equivalent to the projection operator

$$z_{k+1} = \frac{1}{\rho} \Pi_{\mathbb{R}^n_+} (x_{k+1} + u_k).$$
3.6.3. **Convergence results.** We use the first-order finite difference method with step size $h = 0.001$ to approximate $q'([z^*; u^*])$ at the approximate true solution solved to $10^{-16}$ accuracy.

Figure 3.11 compares the error norm reduction when using ADMM, AA($m$)-ADMM and sAA(1)-ADMM. The convergence acceleration seen in the figure can be explained based on the spectra in Figure 3.12. The spectrum of $q'([z^*; u^*])$ has asymptotic convergence factor $\rho_{q'}^* = 0.806$. We can choose the optimal $\beta^*$ the same way as in the ridge regression problem

$$\beta^* = \frac{1 - \sqrt{1 - \rho_{q'}^*}}{1 + \sqrt{1 - \rho_{q'}^*}} = 0.389.$$ 

The corresponding optimal sAA(1)-ADMM linear convergence factor is

$$\rho_{sAA(1)-ADMM}^* = \rho(\nabla \Psi) = 1 - \sqrt{1 - \rho_{q'}^*} = 0.560 < (\rho_{q'}^*)^2.$$ 

### 3.7. Regularized nonnegative least squares [17].

#### 3.7.1. Problem description.

The regularized nonnegative least squares problem is

$$\min_x \|Fx - g\|^2_2 + \lambda \|x\|^2_2, \quad \text{s.t.} \quad x \geq 0,$$

where $x \in \mathbb{R}^q$ is the variable, and $F \in \mathbb{R}^{p \times q}$ and $b \in \mathbb{R}^p$ are problem data. As above, we can integrate the nonnegativity constraint into the objective function

$$\min_{x, z} \|Fx - g\|^2_2 + \lambda \|x\|^2_2 + \mathcal{I}_{\mathbb{R}_+^q}(x),$$

and re-write the problem as

$$\min_{x, z} \|Fx - g\|^2_2 + \lambda \|x\|^2_2 + \mathcal{I}_{\mathbb{R}_+^q}(z),$$

s.t. $x - z = 0$.

The scaled augmented Lagrangian of this problem is

$$L_\rho(x, z, u) = \|Fx - g\|^2_2 + \lambda \|x\|^2_2 + \mathcal{I}_{\mathbb{R}_+^q}(z) + \frac{\rho}{2} \|x - z + u\|^2_2 - \frac{\rho}{2} \|u\|^2_2.$$ 

The ADMM steps on this problem are:

$$\begin{align*}
    x_{k+1} &= \arg \min_x \|Fx - g\|^2_2 + \lambda \|x\|^2_2 + \frac{\rho}{2} \|x - z_k + u_k\|^2_2 \\
    z_{k+1} &= \arg \min_z \mathcal{I}_{\mathbb{R}_+^q}(z) + \frac{\rho}{2} \|x_{k+1} + u_k - z\|^2_2 \\
    u_{k+1} &= u_k + x_{k+1} - z_{k+1}
\end{align*}$$

where the first step for $x_{k+1}$ is the proximal operator of the $l_2$-norm. Again, the second step is just the proximal operator of the indicator function.

#### 3.7.2. Parameters for the test problem.

We use the same data and penalty parameter $\rho$ as in the nonnegative least squares problem. For the regularization parameter, we use $\lambda = 1$. 


Fig. 3.13: Regularized nonnegative least squares: comparison of error reduction using ADMM, sAA(1)-ADMM and AA(m)-ADMM.

Fig. 3.14: Regularized nonnegative least squares: spectrum of ADMM iteration matrix $q'(x^*)$ and sAA(1)-ADMM iteration matrix $\nabla \Psi(x^*)$.

3.7.3. Convergence results. We use the first-order finite difference method with step size $h = 0.001$ to approximate $q'(z^*; u^*)$ at the approximate true solution solved to $10^{-16}$ accuracy.

Figure 3.13 compares the error norm reduction when using ADMM, AA(m)-ADMM and sAA(1)-ADMM. The convergence acceleration seen in the figure can be explained based on the spectra in Figure 3.14. The spectrum of $q'(z^*; u^*)$ has asymptotic convergence factor $\rho_q = 0.838$. We can choose the optimal $\beta^*$ the same way as in the ridge regression problem

$$\beta^* = \frac{1 - \sqrt{1 - \rho_q^2}}{1 + \sqrt{1 - \rho_q^2}} = 0.426.$$
The corresponding optimal sAA(1)-ADMM linear convergence factor is
\[ \rho_{sAA(1)-ADMM} = \rho(\nabla \Psi) = 1 - \sqrt{1 - \rho_{q}^*} = 0.597 < (\rho_{q}^*)^2. \]

3.8. Constrained logistic regression [17].

3.8.1. Problem description. The constrained regularized logistic regression adds a constraint on \( ||x||_{\infty} \) to the regularized logistic regression problem that we have already discussed:
\[
\min_{x} \frac{1}{m} \sum_{i=1}^{m} \log(1 + \exp(-y_i(a_i^T w + c))) + \lambda ||x||^2_2,
\]  
\[
\text{s.t. } ||x||_{\infty} \leq 1.
\]
To apply ADMM, we re-write this problem as
\[
\min_{x,z} \frac{1}{m} \sum_{i=1}^{m} \log(1 + \exp(-y_i(a_i^T w + c))) + \lambda ||x||^2_2 + I_\Omega(z),
\]  
\[
\text{s.t. } x - z = 0,
\]
where \( \Omega = \{ x : ||x||_{\infty} \leq 1 \} \). This gives the augmented Lagrangian
\[
L(x, z, u, \rho) = \frac{1}{m} \sum_{i=1}^{m} \log(1+\exp(-y_i(a_i^T w + c))) + \lambda ||x||^2_2 + I_\Omega(z) + \frac{\rho}{2} ||x - z + u||^2_2 - \frac{\rho}{2} ||u||^2_2.
\]
Hence, we get the ADMM steps
\[
\begin{align*}
  x_{k+1} &= \arg\min_x \frac{1}{m} \sum_{i=1}^{m} \log(1 + \exp(-y_i(a_i^T w + c))) + \lambda ||x||^2_2 + \frac{\rho}{2} ||x - z_k + u_k||^2_2, \\
  z_{k+1} &= \arg\min_z I_\Omega(z) + \frac{\rho}{2} ||x_{k+1} - z + u_k||^2_2, \\
  u_{k+1} &= u_k + x_{k+1} - z_{k+1}.
\end{align*}
\]
Like before, we use Newton’s method to solve for \( x_{k+1} \). For \( z_{k+1} \), since the proximal operation of an indicator function is just a projection, we have
\[
z_{k+1} = \frac{1}{\rho} \Pi_\Omega(x_{k+1} + u_k),
\]
which is
\[
[z_{k+1}]_j = \begin{cases} 
  \frac{1}{\rho}, & [x_{k+1} + u_k]_j \in [1, \infty) \\
  \frac{1}{\rho} [x_{k+1} + u_k]_j, & [x_{k+1} + u_k]_j \in (-1, 1) \\
  -\frac{1}{\rho}, & [x_{k+1} + u_k]_j \in (-\infty, -1].
\end{cases}
\]

3.8.2. Parameters for the test problem. We use the same sample data from the Madelon data set as in Section 3.2. The regularization and penalty parameters are \( \lambda = 2 \) and \( \rho = 10 \) respectively, as in Section 3.2.

3.8.3. Convergence results. We use the first-order finite difference method with step size \( h = 0.001 \) to approximate \( q'(z^*; u^*) \) at the approximate true solution solved to \( 10^{-16} \) accuracy.
Figure 3.15 compares the error norm reduction when using ADMM, AA($m$)-ADMM and sAA(1)-ADMM. The convergence acceleration seen in the figure can be explained based on the spectra in Figure 3.16. The spectrum of $q'(z^*; u^*)$ has asymptotic convergence factor $\rho^*_{q'} = 0.900$. We can choose the optimal $\beta^*$ the same way as in the ridge regression problem

$$\beta^* = \frac{1 - \sqrt{1 - \rho^*_{q'}}}{1 + \sqrt{1 - \rho^*_{q'}}} = 0.519.$$ 

The corresponding optimal sAA(1)-ADMM linear convergence factor is

$$\rho^*_{sAA(1)-ADMM} = \rho(\nabla \Psi) = 1 - \sqrt{1 - \rho^*_{q'}} = 0.684 < (\rho^*_{q'})^2.$$ 

4. Conclusions. This paper has discussed a simple strategy for estimating the convergence speed of Anderson Acceleration applied to ADMM, for the case where ADMM by itself converges linearly.

The approach is based on the finding from [5] that convergence factors of the stationary form of Anderson Acceleration with coefficients that are chosen to make the convergence factors optimal, provide a good prediction for the asymptotic convergence speed of AA with finite window size. As discussed in [5], this is intuitively reasonable: the nonstationary AA does not use these globally optimal stationary coefficients, but rather performs a local optimization of the coefficients in every step $k$ by solving least squares problem (1.10). As $x$ approaches $x^*$ in the asymptotic regime and $q'(x)$ approaches $q'(x^*)$, it is not unreasonable to expect the convergence behavior of AA with locally-optimal $\beta^{(k)}$ weights to be similar to the behavior of sAA with weights that are, based on $q'(x^*)$, globally optimal in obtaining the best asymptotic convergence rate. This is indeed what we have observed numerically in this paper for AA applied to ADMM.

The case of sAA with $m = 1$ is easy to analyze and directly leads to the simple analytical prediction formulas of Propositions 2.2 and 2.3 for the optimal convergence
Fig. 3.16: Constrained regularized logistic regression: spectrum of ADMM iteration matrix $q'(x^*)$ and sAA(1)-ADMM iteration matrix $\nabla \Psi(x^*)$. 

factors $\rho^*_sAA(1)$, see [5]. While our numerical results show that $\rho^*_sAA(1)$-ADMM is a useful prediction for $\rho_{AA(m)}$-ADMM also when $m > 1$, it is clear that computing $\rho^*_sAA(m)$-ADMM for $m > 1$ is also of interest. For $m \geq 2$ the optimal $\rho^*_sAA(m)$-ADMM can be obtained by optimization [5], but the lack of analytical results is an interesting avenue for further research, for example, on how the optimal $\rho^*_sAA(m)$-ADMM depends on $m$.

The closely matched asymptotic convergence behavior between AA and optimal sAA allows us to understand the acceleration power of AA in terms of how it reshapes convergence spectra in our numerical tests, in ways that are very similar to how GMRES for linear systems accelerates convergence depending on the spectral and eigenspace properties of the GMRES preconditioner (see [5] for a detailed discussion of this analogy).

The close match between AA and optimal sAA convergence factors also allows us to quantify and predict convergence acceleration by AA, which is especially useful since the quest for linear asymptotic convergence bounds for AA with finite window size has been elusive, due to the AA coefficients changing in every iteration. This close match may also inspire theoretical approaches for finding asymptotic convergence factor bounds for AA with finite window size.

Of course, besides providing useful insight, our approach for estimating AA convergence factors is not really practical, since $\rho(q'(x^*))$ needs to be known or computed to compute the optimal $\rho^*_sAA(1)$-ADMM. However, if an upper bound for $\rho(q'(x^*))$ is known, then an upper bound for the optimal sAA(1)-ADMM convergence factor, $\rho^*_sAA(1)$-ADMM, can directly be obtained from the formulas in Propositions 2.2 and 2.3. In preconditioned GMRES for linear systems, depending on the problem, such upper bounds for $\rho(q'(x^*))$ can often be derived [5]. They may, for example, depend on problem parameters or problem sizes, and for many linear problems GMRES preconditioners have been found that provably lead to favorable convergence bounds independent from, or only weakly dependent on, parameters that characterize the difficulty or conditioning of the problem. Similarly, it may be of practical use to pursue this for various ADMM applications, since it may lead to convergence factor
bound predictions for AA applied to ADMM with favorable dependence on problem parameters.

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