Existence and regularity results for the penalized thin obstacle problem with variable coefficients

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Abstract

In this paper we give a comprehensive treatment of a two-penalty boundary obstacle problem for a divergence form elliptic operator, motivated by applications to fluid dynamics and thermics. Specifically, we prove existence, uniqueness and optimal regularity of solutions, and establish structural properties of the free boundary. The proofs are based on tailor-made monotonicity formulas of Almgren, Weiss, and Monneau-type, combined with the classical theory of oblique derivative problems.

1 Introduction

The focus of this paper is the study of regularity of solutions and the structure of the free boundary in a penalized boundary obstacle problem of interest in therms, fluid mechanics, and electricity.

To set the stage, we let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and $\Gamma$ be a relatively open subset of $\partial \Omega$. We will denote by $A = [a^{ij}]_{i,j=1,...,n}$, with $a^{ij} \in L^\infty(\Omega)$, a symmetric matrix satisfying the uniform ellipticity condition

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad (1.1)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, and some constants $0 < \lambda \leq \Lambda < \infty$. Finally, we let $1 < p < \infty$ be a constant, $k_+, k_- \in L^1(\Gamma)$ with $k_+, k_- \geq 0$ on $\Gamma$, and $h \in L^\infty(\Gamma)$. We are interested in studying the oblique derivative problem

$$D_i(a^{ij}D_ju) = 0 \quad \text{in } \Omega, \quad a^{ij}D_ju \nu_i = -k_+(u-h)^{p-1} + k_-(u-h)^{-p-1} \quad \text{on } \Gamma. \quad (1.2)$$

Here $v^+ = \max\{v, 0\}$, $v^- = -\min\{v, 0\} \geq 0$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the outward unit normal to $\Omega$ with respect to the standard Euclidian metric. We say that $u \in W^{1,2}(\Omega)$ is a weak solution to (1.2) if

$$\int_{\Omega} a^{ij}D_ju D_i\zeta + \int_{\Gamma}(k_+(u-h)^{p-1} - k_-(u-h)^{-p-1}) \zeta = 0 \quad (1.3)$$

for all $\zeta \in W^{1,2}(\Omega)$ with $\int_{\Gamma}(k_+ + k_-)|\zeta|^p < \infty$ and $\zeta = 0$ near $\partial \Omega \setminus \Gamma$.

The study of the model (1.2) is motivated by applications to fluid dynamics and temperature control problems, which we now briefly describe. Following [DL76 Section 2.2.2], when considering the process of osmosis through semi-permeable walls the region $\Omega$ consists of a porous medium

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occupied by a viscous fluid which is only slightly compressible, with its field of pressure denoted by \( u(x) \). We assume that the portion \( \Gamma \) of \( \partial \Omega \) consists of a semi-permeable membrane of finite thickness, i.e. the fluid can freely enter in \( \Omega \), but the outflow of fluid is prevented. Combining the law of conservation of mass with Darcy’s law, one finds that the equilibrium configuration for \( u \) satisfies the equation

\[
\Delta u = f \text{ in } \Omega,
\]

where \( f = f(x) \) is a given function. When a fluid pressure \( h(x) \), for \( x \in \Gamma \), is applied to \( \Gamma \) on the outside of \( \Omega \), one of two cases holds: \( h(x) < u(x,t) \), or \( h(x) \geq u(x,t) \). In the former, the semi-permeable wall prevents the fluid from leaving \( \Omega \), so that the flux is null and thus

\[
\frac{\partial u}{\partial \nu} = 0. \tag{1.4}
\]

In the latter case, the fluid enters \( \Omega \). It is reasonable to assume the outflow to be proportional to the difference in pressure, so that

\[
-\frac{\partial u}{\partial \nu} = k(u-h), \tag{1.5}
\]

where \( k > 0 \) measures the conductivity of the wall. Combining (1.4) and (1.5), we obtain the boundary condition

\[
\frac{\partial u}{\partial \nu} = k(u-h)^- \text{ on } \Gamma. \tag{1.6}
\]

In our problem (1.2), we allow for fluid flow to occur both into and out of \( \Omega \) with different permeability functions \( k_+ \) and \( k_- \) (not necessarily constant), under the assumption that the flux in each direction is proportional to a power of the pressure.

An alternate interpretation of the model (1.2) is as a boundary temperature control problem. We assume that a continuous medium occupies the region \( \Omega \) in \( \mathbb{R}^n \), with boundary \( \Gamma \) and outer unit normal \( \nu \). Given a reference temperature \( h(x) \), for \( x \in \Gamma \), it is required that the temperature at the boundary \( u(x) \) deviates as little as possible from \( h(x) \). To this end, thermostatic controls are placed on the boundary to inject an appropriate heat flux when necessary. The controls are regulated as follows:

(i) If \( u(x) = h(x) \), no correction is needed and therefore the heat flux is null (i.e., (1.4) holds).

(ii) If \( u(x) \neq h(x) \), a quantity of heat proportional to the difference between \( u(x,t) \) and \( h(x) \) is injected.

We can thus write the boundary condition as

\[
-\frac{\partial u}{\partial \nu} = \Phi(u),
\]

where

\[
\Phi(u) = \begin{cases} 
  k_-(u-h) & \text{if } u < h \\
  0 & \text{if } u = h \\
  k_+(u-h) & \text{if } u > h 
\end{cases}
\]

which is precisely the condition in (1.2) when \( p = 2 \). For further details, we refer to [DL76, Section 2.3.1], see also [AC04] for the limiting case \( k_- = 0 \) and \( k_+ = +\infty \), and [ALP15] for the case \( a^{ij} = \delta^{ij}, \ p = 1, \) and \( h = 0 \) in (2.1) below.

The study of (1.2) was initiated in [DJ], where the constant coefficient case \( A = I \) was considered, together with the simplifying assumptions \( h \equiv 0 \) and \( k_+, \ k_- \) being constant. In [DJ] the authors establish regularity properties of the solution and describe the structure of the free boundary. From the point of view of applications, however, it is important to allow for variable coefficients in (1.2). In fact, if \( \Gamma \) is sufficiently smooth, a standard flattening procedure leads to
the analysis of a similar model for a flat thin manifold, where the relevant operator has, however, variable coefficients.

Quite relevant to our analysis, as observed in [DJ], are the two limiting cases $k_+ = k_- = 0$, and $k_+ = 0$ and $k_- = +\infty$ (or equivalently $k_+ = +\infty$ and $k_- = 0$). In the first one $u$ is clearly the solution of a classical Neumann problem. The other one is more interesting, since the boundary condition becomes

\[(u - h)\frac{\partial u}{\partial \nu} = 0,\]

and $u$ is a solution of the Signorini problem, also known as the thin obstacle problem. The Signorini problem has received a resurgence of attention in the last decade, due to the discovery of several families of powerful monotonicity formulas, which in turn have allowed to establish the optimal regularity of the solution, a full classification of free boundary points, smoothness of the free boundary at regular points, and the structure of the free boundary at singular points. We refer the interested reader to [AC04], [ACS08], [CSS08], [GP09], [GS14], [GPS18], [DSS16], [KPS15], see also the survey [DS18] and the references therein.

The general scheme of a solution to the variable-coefficient Signorini problem provides a blueprint for the solution of problem (1.2), but there are two new substantial difficulties. Due to the non-homogeneous nature of the boundary condition in (1.2), this problem does not admit global homogeneous solutions of any degree. This is in stark contrast with the thin obstacle problem, where the existence and classification of such solutions play a fundamental role. Moreover, in the Signorini problem it is easily verified that continuity arguments force $u$ to be always above $h$ (that is, $h$ plays the role of an obstacle), whereas the case $h(x) > u(x)$ is no longer ruled out in (1.2). Allowing for both constants $k_+, k_-$ to be finite (even when one of the two vanishes) essentially destroys the one-phase character of the problem. Finally, we notice that allowing for variable coefficients and removing the hypothesis $h = 0$ introduces a host of new technical challenges not present in [DJ].

We begin our study by showing existence and uniqueness of solutions to (1.2) using variational techniques (see Theorem 3.1 and Lemma 2.2). We then proceed to establish the Hölder regularity of solutions in Theorem 4.1. More precisely, we show that if $\kappa \geq 1$ is an integer and $\alpha \in (0, 1)$ with $\kappa + \alpha \leq p$, $\Gamma$ is a relatively open $C^{\kappa, \alpha}$-portion of $\partial \Omega$, $a^{ij} \in C^{\kappa-1, \alpha}(\Omega \cup \Gamma)$, $k_+, k_- \in C^{\kappa-1, \alpha}(\Gamma)$, and $h \in C^{\kappa-1, 1}(\Gamma)$, then

\[u \in C^{\kappa, \alpha}(\Omega \cup \Gamma).\]

Our approach is centered on a bootstrapping argument, which in turn is based on the Caccioppoli inequality, an initial modulus of continuity for the solution, and variational methods, combined with the classical regularity theory for the oblique derivative problem. Furthermore, we prove that such regularity is optimal (see Theorem 4.2), by means of a careful construction of a suitable family of rescalings, and of an explicit solution to the ensuing limiting problem. As an immediate consequence of the regularity of the solution and of the implicit function theorem, we obtain the following result concerning the regularity of the free boundary.

**Definition 1.1.** The regular set of the free boundary is defined as

\[\mathcal{R}(u) = \{(x', 0) \in B'_1(0) \mid u(x', 0) = h(x'), \nabla_{x'}u(x', 0) \neq \nabla_{x'}h(x')\}\]

**Theorem 1.1.** If $x_0 \in \mathcal{R}(u)$, then in a neighborhood of $x_0$ the free boundary $\{u(x', 0) = 0\}$ is a $C^{1, \alpha}$-graph for all $0 < \alpha < 1$.

The remainder of the paper is devoted to the study of rectifiability properties of the free boundary $\Sigma(u) = \partial_f \{x \in \Gamma : u(x) = h(x)\}$, inspired by the approach introduced in [GP09], [GS14], [DSS16].
\[ \Sigma \]

\[ \begin{align*}
  D_i(a^{ij}D_ju) &= 0 \quad \text{in } B_1^+(0), \\
  a^{nn}D_nu &= k_+((u - h)^+)^{p-1} - k_-((u - h)^-)^{p-1} \quad \text{on } B_1'(0).
\end{align*} \]

Here

\[ B_\rho^+(y) = B_\rho(y) \cap \{ x_n > 0 \}, \]
\[ B_\rho^-(y) = B_\rho(y) \cap \{ x_n = 0 \} \]

for each \( y \in \mathbb{R}^{n-1} \times \{0\} \) and \( \rho > 0 \). We have used the notation, for points \( x \in \mathbb{R}^n, x = (x', x_n) \), with \( x' = (x_1, \ldots, x_{n-1}) \). For future reference, we will denote

\[ (\partial B_\rho(y))^+ = \partial B_\rho(y) \cap \{ x_n > 0 \}. \]

The first step in our program consists in essentially subtracting the \( h \) (or better, its Taylor polynomial of order \( \kappa \)) from the solution, so that we can rewrite our problem as

\[ \begin{align*}
  D_i(a^{ij}D_jv) &= f \quad \text{in } B_1^+(0), \\
  a^{nn}D_nv &= k_+(v^+)^{p-1} - k_-(v^-)^{p-1} \quad \text{on } B_1'(0),
\end{align*} \]

with the regularity of the right-hand side \( f \) being correlated to the one of \( h \). Next, we establish one of the main ingredients in our analysis in Theorem 6.1, namely a monotonicity formula of Almgren’s type for solutions to (1.8). We were inspired by some of the ideas first introduced in [GS14] for unconstrained divergence form equations \( \div(A\nabla u) = 0 \), and then in [GSI18] for the Signorini problem with variable coefficients, but we had to tackle some delicate differences due to the nature of the boundary conditions. With this tool at our disposal, we proceed to establish optimal growth rates of the solution at free boundary points (Lemma 7.1), which in turn allow us to control the convergence of a suitable family of rescalings to a homogeneous harmonic polynomial, the so-called blow-up (see Theorem 8.1). At this point, we introduce two families of functionals of Weiss’ and Monneau’s type, and we prove their almost-monotonicity in Theorem 9.1 and Theorem 10.1, respectively. These results yield the nondegeneracy of the solution (Lemma 11.1) and the continuous dependance of the blow-up limits on the free boundary point (Theorem 11.2). We explicitly observe that, to the best of our knowledge, this is the first result of this kind for variable coefficients and non-zero obstacles. Our final main result asserts the rectifiability of the free boundary.

**Theorem 1.2.** Let \( 2 \leq p < \infty \), and let \( \kappa, \nu \) be positive integers with \( 2 \leq \nu < \kappa \). Let \( a^{ij} \in C^{\kappa-1,1}(B_1^+(0) \cup B_1'(0)) \) such that \( a^{ij} = a^{ji} \) on \( B_1^+(0) \), satisfying (1.1) for some constants \( 0 < \lambda \leq \Lambda < \infty \), and such that (1.2) holds true on \( B_1'(0) \). Let \( k_+, k_- \in C^{0,1}(B_1'(0)) \) with \( k_+, k_- \geq 0 \) and \( h \in C^{\kappa,1}(B_1'(0)) \). Let \( u \in W^{1,2}(B_1^+(0)) \) be a solution to (1.8). Then for each \( d = 0, 1, 2, \ldots, n-2 \), \( \Sigma^d_\nu \) is contained in a countable union of \( d \)-dimensional \( C^1 \)-submanifolds of \( B_1'(0) \).

The set \( \Sigma^d_\nu \) consists of free boundary points with Almgren’s frequency \( \nu \) and degree \( d \). We refer to Section 11 for its precise definition. The proof of Theorem 1.2 hinges on Theorem 11.2 combined with Whitney’s extension and the implicit function theorem.
1.1 Structure of the paper

The paper is organized as follows. In Section 2 we introduce the variational formulation of the problem. In Section 3 we prove existence and uniqueness of solutions. In Section 4 we establish the optimal regularity of solutions. In Section 5 we show how to transform problem (1.7) into (1.8). Section 6 is devoted to proof of the almost-monotonicity of a truncated functional of Almgren type, and in Section 7 we infer growth properties of the solution near free boundary points as a consequence. In Section 8 we introduce the Almgren rescalings, and discuss their blow-up limits. In Sections 9 and 10 we prove the almost-monotonicity of Weiss-type and Monneau-type functionals, respectively. Finally, in Section 11 we establish non-degeneracy of solutions and continuous dependance of the blow-up limits on the free boundary point, and prove Theorem 1.2.

2 Preliminaries

Our goal is to understand the existence and regularity of solutions to the penalized thin obstacle problem, following some of the ideas introduced in [17]. Throughout this section,

**Definition 2.1.** $J$ is a functional of $v \in W^{1,2}(\Omega)$ defined by

$$J(v) = \frac{1}{2} \int_{\Omega} a^{ij} D_i v D_j v + \frac{1}{p} \int_{\Gamma} (k_+(v-h)^+)^p + k_-(v-h)^-)^p,$$

where $v$ takes the values of its trace on $\Gamma$ and we let $(v-h)^+ = \max\{v-h,0\}$ and $(v-h)^- = -\min\{v-h,0\}$.

The functional $J$ satisfies the following transformation formula under $C^1$-diffeomorphisms, which follows by a straight-forward application of change of variables.

**Lemma 2.1.** Let $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ be bounded Lipschitz domains, $\Gamma$ be a smooth relatively open subset of $\partial \Omega$, $\tilde{\Gamma}$ be a smooth relatively open subset of $\partial \tilde{\Omega}$, and $\varphi: \Omega \cup \Gamma \to \tilde{\Omega} \cup \tilde{\Gamma}$ be a $C^1$-diffeomorphism. Let $J$ be the functional defined by (2.1), where $1 < p < \infty$ is a constant, $a^{ij} \in L^\infty(\Omega)$ such that $a^{ij} = a^{ji}$ on $\Omega$, $k_+, k_- \in L^1(\Gamma)$ with $k_+, k_- \geq 0$ on $\Gamma$, and $h \in L^\infty(\Gamma)$. Then for each $v \in W^{1,2}(\Omega)$,

$$\int_{\Omega} a^{ij} D_i \tilde{v} D_j \tilde{v} + \frac{1}{p} \int_{\tilde{\Gamma}} (\tilde{k}_+(\tilde{v}-\tilde{h})^+)^p + \tilde{k}_-(\tilde{v}-\tilde{h})^-)^p$$

where $\tilde{v} = v \circ \varphi^{-1}$, $\tilde{a}^{ij} = (a^{kl} D_k \varphi \cdot D_l \varphi) \circ \varphi^{-1}$, $\tilde{k}_+ = (k_+ \circ \varphi^{-1}) J_{\tilde{\Gamma}} \varphi^{-1}$, and $\tilde{h} = h \circ \varphi^{-1}$ where $J_{\varphi^{-1}}$ is the Jacobian of $\varphi^{-1}: \Omega \to \tilde{\Omega}$ and $J_{\tilde{\Gamma}} \varphi^{-1}$ is the Jacobian of $\varphi^{-1} |_{\tilde{\Gamma}}: \tilde{\Gamma} \to \Gamma$.

Let $J$ be as in Definition 2.1. We say that $u \in W^{1,2}(\Omega)$ is $J$-minimizing if $J(u) < \infty$ and

$$J(u) \leq J(v)$$

for all $v \in W^{1,2}(\Omega)$ with $u = v$ on $\partial \Omega \setminus \Gamma$, where $u, v$ take the values of their trace on $\partial \Omega \setminus \Gamma$. Recall that $u \in W^{1,2}(\Omega)$ is a weak solution to (1.2) if (1.1) holds true for all $\zeta \in W^{1,2}(\Omega)$ with $\int_{\Gamma} (k_+ + k_-) |\zeta|^p < \infty$ and $\zeta = 0$ near $\partial \Omega \setminus \Gamma$. The assumption that $\int_{\Gamma} (k_+ + k_-) |\zeta|^p < \infty$ guarantees that if $u \in W^{1,2}(\Omega)$ with $J(u) < \infty$, then by Young’s inequality

$$\int_{\Gamma} k_+ ((u+t\zeta-h)^+)^p \leq \int_{\Gamma} k_+ ((u-h)^+)^p + |t||\zeta|^p$$

$$\leq 2p^1 \int_{\Gamma} k_+ ((u-h)^+)^p + 2p^1 \int_{\Gamma} k_+ |\zeta|^p < \infty,$$

$$\int_{\Gamma} k_+ ((u-h)^+)^{p-1} |\zeta| \leq \frac{p-1}{p} \int_{\Gamma} k_+ ((u-h)^+)^p + \frac{1}{p} \int_{\Gamma} k_+ |\zeta|^p < \infty.$$
Hence \( J(u + t\zeta) < \infty \) for all \(-1 \leq t \leq 1\) and the second integral on the right-hand side of (1.3) is also finite. Moreover, \( k_+, k_- \in L^1(\Gamma) \) guarantees that \( \int_\Gamma (k_+ + k_-)|\zeta|^p < \infty \) for all \( \zeta \in C^1_c(\Omega \cup \Gamma) \).

**Lemma 2.2.** Let \( J \) be as in Definition 2.1. For each \( u \in W^{1,2}(\Omega) \) with \( J(u) < \infty \), \( u \) is \( J \)-minimizing if and only if \( u \) is a weak solution to (1.2).

**Proof.** If \( u \) is \( J \)-minimizing, then by a standard variational argument \( u \) is a weak solution to (1.2); see for instance the proof of Lemma 4.1 of [ALP15]. To see the converse, suppose that \( u \) is a weak solution to (1.2) and fix any function \( \zeta \in W^{1,2}(\Omega) \) with \( \int_\Gamma (k_+ + k_-)|\zeta|^p < \infty \) and \( \zeta = 0 \) near \( \partial\Omega \setminus \Gamma \). Define \( F : \Gamma \times \mathbb{R} \to [0, \infty) \) by

\[
F(x, z) = \frac{1}{p} k_+(x) (z^+)^p + \frac{1}{p} k_-(x) (z^-)^p
\]

for \( x \in \Gamma \) and \( z \in \mathbb{R} \). Since \( F(x, \cdot) \) is convex for each \( x \in \Gamma \),

\[
F(x, u + \zeta - h) - F(x, u - h) - D_z F(x, u - h) \zeta
= \int_0^1 (D_z F(x, u + t\zeta - h) - D_z F(x, u - h)) \zeta \, dt \geq 0
\]

on \( \Gamma \). Hence by (2.3)

\[
J(u + \zeta) = \frac{1}{2} \int_\Omega a^{ij} D_i(u + \zeta) D_j(u + \zeta) + \int_\Gamma F(x, u + \zeta - h)
\geq \frac{1}{2} \int_\Omega (a^{ij} D_i u D_j u + 2 a^{ij} D_i u D_j \zeta + a^{ij} D_i \zeta D_j \zeta)
+ \int_\Gamma (F(x, u - h) + D_z F(x, u - h)) \zeta.
\]

Note that we have equality in (2.4) if and only if we have \( D_z F(x, u + t\zeta - h) = D_z F(x, u - h) \) on \( \Gamma \) for all \( 0 \leq t \leq 1 \), so that we have equality in (2.3). The assumption that \( u \) is a weak solution to (1.2) means that (1.3) holds true. We can rewrite (1.3) as

\[
\int_\Omega a^{ij} D_j u D_i \zeta + \int_\Gamma D_z F(x, u - h) \zeta = 0.
\]

Combining (2.4), (2.5) and (1.1), we obtain

\[
J(u + \zeta) \geq \frac{1}{2} \int_\Omega a^{ij} D_i u D_j u + \frac{1}{2} \int_\Omega a^{ij} D_i \zeta D_j \zeta + \int_\Gamma F(x, u - h) \geq J(u)
\]

(with equality if and only if \( \zeta \) is constant and \( D_z F(x, u + t\zeta - h) = D_z F(x, u - h) \) on \( \Gamma \) for all \( 0 \leq t \leq 1 \)).

\[
3 \quad \text{Existence and uniqueness of } J\text{-minimizers}
\]

Theorem 3.1 below is the general existence and uniqueness theorem for \( J \)-minimizing functions on bounded domains \( \Omega \). We distinguish between the cases \( H^{n-1}(\partial\Omega \setminus \Gamma) > 0 \) and \( H^{n-1}(\partial\Omega \setminus \Gamma) = 0 \) (i.e. \( \Gamma = \partial\Omega \) up to a set of \( H^{n-1} \)-measure zero). For the case \( H^{n-1}(\partial\Omega \setminus \Gamma) > 0 \) we include the Dirichlet boundary condition \( u = \varphi \) on \( \partial\Omega \setminus \Gamma \). In the case \( \Gamma = \partial\Omega \) uniqueness is determined by the values of \( k_+, k_- \) and \( h \), and one might have infinitely many \( J \)-minimizing constant functions \( c \) with \( J(c) = 0 \). We show existence by the direct method, and uniqueness using the equality case of (2.4).
Theorem 3.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\Gamma$ be a relatively open subset of $\partial \Omega$. Let $J$ be as in Definition 2.1 where $a^{ij} \in L^\infty(\Omega)$ are such that $a^{ij} = a^{ji}$ on $\Omega$ and (1.1) hold true for some constants $0 < \lambda \leq \Lambda < \infty$, $1 < p < \infty$ is a constant, $k_+ , k_- \in L^1(\Gamma)$ with $k_+ , k_- \geq 0$ on $\Gamma$, and $h \in L^\infty(\Gamma)$.

(i) If $H^{n-1}(\partial \Omega \setminus \Gamma) > 0$, for each $\varphi \in W^{1,2}(\Omega)$ with $J(\varphi) < \infty$ there exists a unique $J$-minimizing function $u \in W^{1,2}(\Omega)$ with $u = \varphi$ on $\partial \Omega \setminus \Gamma$.

Suppose instead that $H^{n-1}(\partial \Omega \setminus \Gamma) = 0$ and set

$$M_+ = \inf_{\Gamma \cap \{k_+ > 0\}} h, \quad M_- = \sup_{\Gamma \cap \{k_- > 0\}} h$$

(with the usual convention that $M_+ = +\infty$ if $k_+ = 0$ on $\Gamma$ and $M_- = -\infty$ if $k_- = 0$ on $\Gamma$).

(ii) If $H^{n-1}(\partial \Omega \setminus \Gamma) = 0$ and $M_+ \leq M_-$, then there exists a unique $J$-minimizing function $u \in W^{1,2}(\Omega)$.

(iii) If $H^{n-1}(\partial \Omega \setminus \Gamma) = 0$ and $M_- \leq M_+$, then the $J$-minimizing functions $u \in W^{1,2}(\Omega)$ are precisely the constant functions $u$ with $M_- \leq u \leq M_+$.

To prove Theorem 3.1(i), we will need the following variant of the Poincaré inequality.

Lemma 3.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\Gamma$ be a relatively open subset of $\partial \Omega$ with $H^{n-1}(\partial \Omega \setminus \Gamma) > 0$. Suppose that $v \in W^{1,2}(\Omega)$ with $v = 0$ on $\partial \Omega \setminus \Gamma$. Then

$$\int_\Omega v^2 \leq C \int_\Omega |Dv|^2$$

for some constant $C = C(n, \Omega, \Gamma) \in (0, \infty)$.

Proof. Suppose to the contrary that for $\ell = 1, 2, 3, \ldots$ there exist non-zero functions $v_\ell \in W^{1,2}(\Omega)$ such that $v_\ell = 0$ on $\partial \Omega \setminus \Gamma$ and $|Dv_\ell|_{L^2(\Omega)} \leq 1/\ell$. By scaling we may also assume that $\|v_\ell\|_{L^2(\Omega)} = 1$ and thus $\|Dv_\ell\|_{L^2(\Omega)} \leq 1/\ell$. By the Rellich compactness theorem, there exists a constant function $v$ on $\Omega$ such that, after passing to a subsequence, $v_\ell \rightharpoonup v$ strongly in $L^2(\Omega)$ and $Dv_\ell \rightharpoonup 0$ weakly in $L^2(\Omega)$. In particular, $v$ is a non-zero constant function with $\|v\|_{L^2(\Omega)} = 1$. Since the trace operator $W^{1,2}(\Omega) \to L^2(\partial \Omega)$ is a bounded compact linear operator, after passing to a further sequence, $v_\ell \to v$ strongly in $L^2(\partial \Omega)$ and pointwise $H^{n-1}$-a.e. on $\partial \Omega$. In particular, since $v_\ell \to 0$ on $\partial \Omega \setminus \Gamma$, $v = 0$ on $\partial \Omega \setminus \Gamma$. Since $v$ is a constant function on $\Omega$, the trace of $v$ on $\partial \Omega$ is equal to the constant value of $v$. Therefore, $v = 0$ on $\Omega$, contradicting $v$ being non-zero. \qed

Proof of Theorem 3.1(i). To prove the existence of a $J$-minimizing function, for $\ell = 1, 2, 3, \ldots$ let $u_\ell \in W^{1,2}(\Omega)$ such that $u_\ell = \varphi$ on $\partial \Omega$ and

$$\lim_{\ell \to \infty} J(u_\ell) = \inf\{J(v) : v \in W^{1,2}(\Omega), v = \varphi \text{ on } \partial \Omega\}.$$

By Lemma 3.1 applied to $v = u_\ell - \varphi$ and (1.1),

$$\|u_\ell\|_{W^{1,2}(\Omega)} \leq \|u_\ell - \varphi\|_{W^{1,2}(\Omega)} + \|\varphi\|_{W^{1,2}(\Omega)} \leq C\|Du_\ell - D\varphi\|_{L^2(\Omega)} + \|\varphi\|_{W^{1,2}(\Omega)}$$

$$\leq C\|Du_\ell\|_{L^2(\Omega)} + (C + 1) \|\varphi\|_{W^{1,2}(\Omega)} \leq \frac{C}{\sqrt{\lambda}} J(u_\ell)^{1/2} + (C + 1) \|\varphi\|_{W^{1,2}(\Omega)}$$

$$\leq \frac{C}{\sqrt{\lambda}} (J(\varphi)^{1/2} + 1) + (C + 1) \|\varphi\|_{W^{1,2}(\Omega)}.$$
for all sufficiently large $\ell$, where $C = C(n, \Omega, \Gamma) \in (0, \infty)$ is a constant. Thus by the Rellich compactness theorem, there exists a function $u \in W^{1,2}(\Omega)$ such that, after passing to a subsequence, $u_\ell \to u$ strongly in $L^2(\Omega)$ and $Du_\ell \to Du$ weakly in $L^2(\Omega, \mathbb{R}^n)$. Since the trace operator $W^{1,2}(\Omega) \to L^2(\partial\Omega)$ is a bounded compact linear operator, after passing to a further sequence, $u_\ell \to u$ strongly in $L^2(\partial\Omega)$ and pointwise $H^{n-1}$-a.e. on $\partial\Omega$. Since $u_\ell = \varphi$ on $\partial\Omega \setminus \Gamma$, $u = \varphi$ on $\partial\Omega \setminus \Gamma$. The weak convergence $Du_\ell \to Du$ in $L^2(\Omega, \mathbb{R}^n)$ implies that

$$\lim_{\ell \to \infty} \int_\Omega a^{ij} D_i u_\ell \zeta_j = \lim_{\ell \to \infty} \int_\Omega a^{ij} D_i u \zeta_j$$

for all $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in C^0_c(\Omega, \mathbb{R}^n)$ (as $a^{ij} \zeta_j \in C^0_c(\Omega)$); that is, $Du_\ell \to Du$ weakly in $L^2(\Omega, \mathbb{R}^n)$ equipped with the inner product $\langle \xi, \zeta \rangle = \int_\Omega a^{ij} \xi_i \zeta_j$. Hence

$$\int_\Omega a^{ij} D_i u D_j u \leq \liminf_{\ell \to \infty} \int_\Omega a^{ij} D_i u_\ell D_j u_\ell. \quad (3.1)$$

Recalling from above that $u_\ell \to u$ pointwise $H^{n-1}$-a.e. on $\partial\Omega$, $(u_\ell - h)^+ \to (u - h)^+$ pointwise $H^{n-1}$-a.e. on $\Gamma$. By Fatou’s lemma,

$$\int_\Gamma k_+(u_\ell - h)^+ p \leq \liminf_{\ell \to \infty} \int_\Gamma k_+(u_\ell - h)^+ p. \quad (3.2)$$

Similarly,

$$\int_\Gamma k_-(u_\ell - h)^- p \leq \liminf_{\ell \to \infty} \int_\Gamma k_-(u_\ell - h)^- p. \quad (3.3)$$

By (3.1), (3.2), and (3.3),

$$J(u) \leq \liminf_{\ell \to \infty} J(u_\ell)$$

and it follows that $u$ is $J$-minimizing.

To show uniqueness of $J$-minimizing functions, suppose that $u, v \in W^{1,2}(\Omega)$ are both $J$-minimizing functions with $u = v = \varphi$ on $\partial\Omega \setminus \Gamma$. Set $\zeta = v - u$ so that $\zeta \in W^{1,2}(\Omega)$ with $\int_\Gamma (k_+ + k_-) |\zeta|^p < \infty$ (by Minkowski’s inequality) and $\zeta = 0$ on $\partial\Omega \setminus \Gamma$. Since $J(u + \zeta) = J(u)$, i.e. equality holds true in (2.6), $\zeta$ must be a constant function on $\Omega$ and in particular $\zeta = 0$ on $\Omega$. Therefore $u = v$ on $\Omega$.

Proof of Theorem 3.7(iii). To prove the existence of a $J$-minimizing function, for $\ell = 1, 2, 3, \ldots$ let $u_\ell \in W^{1,2}(\Omega)$ such that

$$\lim_{\ell \to \infty} J(u_\ell) = \inf\{J(v) : v \in W^{1,2}(\Omega)\}.$$

Note that since $M_+ \leq M_-^1$ we have $M_+ > -\infty$ and $M_- < \infty$. Truncate $u_\ell$ by letting $\overline{u}_\ell \in W^{1,2}(\Omega)$ be given by

$$\overline{u}_\ell = \begin{cases} M_+ & \text{if } u_\ell < M_+ \\ u_\ell & \text{if } M_+ \leq u_\ell \leq M_- \\ M_- & \text{if } u_\ell > M_- \end{cases}$$

Clearly $M_+ \leq \overline{u}_\ell \leq M_- \text{ on } \Omega$. Moreover,

$$J(\overline{u}_\ell) = \frac{1}{2} \int_{\Omega \cap \{M_+ < u_\ell < M_-\}} |Du_\ell|^2 + \frac{1}{p} \int_{\Gamma \cap \{\overline{u}_\ell > h \geq M_+\}} k_+(\overline{u}_\ell - h)^p + \frac{1}{p} \int_{\Gamma \cap \{\overline{u}_\ell < h \leq M_-\}} k_- (h - \overline{u}_\ell)^p \leq J(u_\ell),$$

8
where the first step uses $k_+ = 0$ on $\Gamma \cap \{ h < M_+ \}$ and $k_- = 0$ on $\Gamma \cap \{ h > M_- \}$ and the last step uses $\bar{\pi}_t \leq u_\ell$ on $\Gamma \cap \{ \pi_\ell > h \geq M_+ \}$ and $\pi_\ell \geq u_\ell$ on $\Gamma \cap \{ \bar{\pi}_\ell < h \leq M_- \}$. Thus, using (1.1),

$$\limsup_{\ell \to \infty} \lambda \|D\bar{\pi}_\ell\|_{L^2(\Omega)}^2 \leq \lim_{\ell \to \infty} J(\bar{\pi}_\ell) \leq \lim_{\ell \to \infty} J(u_\ell) = \inf \{ J(v) : v \in W^{1,2}(\Omega) \}.$$

Thus by the Rellich compactness theorem, there exists a function $u \in W^{1,2}(\Omega)$ such that, after passing to a subsequence, $\bar{\pi}_\ell \to u$ strongly in $L^2(\Omega)$ and $D\bar{\pi}_\ell \to Du$ weakly in $L^2(\Omega, \mathbb{R}^n)$. Moreover, by the compactness of the trace operator $W^{1,2}(\Omega) \to L^2(\partial\Omega)$, after passing to a further sequence, $u_\ell \to u$ strongly in $L^2(\partial\Omega)$ and pointwise $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. Clearly $M_+ \leq u \leq M_-$ on $\Omega$. Arguing as we did for (i), $u$ is $J$-minimizing.

To show uniqueness of $J$-minimizing functions, suppose that $u, v \in W^{1,2}(\Omega)$ are both $J$-minimizing functions. Without loss of generality assume that $M_+ \leq u \leq M_-$ on $\Omega$. Set $\zeta = v - u$ so that $\zeta \in W^{1,2}(\Omega)$ with $\int_{\Gamma}(k_++k_-)|\zeta|^p < \infty$ (by Minkowski’s inequality). Since $J(u+\zeta) = J(u)$, i.e. equality holds true in (2.6), $\zeta$ must be a constant function on $\Omega$ and

$$0 = \frac{d}{dt}D_xF(x, u + t\zeta - h) = \begin{cases} (p-1)k_+(u + t\zeta - h)^{p-2}\zeta & \text{if } u + t\zeta > h \\ -(p-1)k_-(h - u - t\zeta)^{p-2}\zeta & \text{if } u + t\zeta < h \end{cases}$$

(3.4)

on $\Gamma \cap \{ u + t\zeta \neq h \}$ for all $0 \leq t \leq 1$, where $F$ is as in (2.2). If $\zeta > 0$, then by (3.1) we have $k_+ = 0$ on $\Gamma \cap \{ u + \zeta > h \}$. However, recalling that $M_+ > -\infty$ and applying the definition of $M_+$, the set $S = \Gamma \cap \{ k_+ > 0 \} \cap \{ h < M_+ + \zeta \} \in \mathcal{H}^{n-1}(S) > 0$ and we have $k_+ > 0$ and $u + \zeta \geq M_+ + \zeta > h$ on $S$, giving us a contradiction. Hence we cannot have $\zeta > 0$. By similar reasoning we cannot have $\zeta < 0$. Therefore, $\zeta = 0$ on $\Omega$, that is $u = v$ on $\Omega$.

Proof of Theorem 3.7(iii). We observe that $J(v) \geq 0$ for all $v \in W^{1,2}(\Omega)$ and $J(v) = 0$ if and only if $v$ is a constant function on $\Omega$, $v \leq h$ on $\Gamma \cap \{ k_+ > 0 \}$, and $v \geq h$ on $\Gamma \cap \{ k_- > 0 \}$, i.e. $v$ is a constant function with $M_- \leq v \leq M_+$.

4 Optimal regularity near points of $\Gamma$

The goal of this section is to investigate the local regularity of a weak solution $u \in W^{1,2}(\Omega)$ to (1.2). Suppose that $\Gamma$ is a $C^{1,1}$-hypersurface and $x_0 \in \Gamma$. After a $C^{1,1}$-change of coordinates we may assume that $x_0 = 0$ and that $\Omega = B_1^+(0)$ and $\Gamma = B_1'(0)$, see Lemma 2.4. Without loss of generality, throughout the remainder of the paper, we will assume

$$a^{ij}(0) = \delta_{ij}, \hspace{1cm} i,j = 1, \ldots, n. \hspace{2cm} (4.1)$$

Moreover, by [GS14] Theorem B.1 after a further $C^{1,1}$-change of coordinates we may assume that

$$a^{in} = a^{ni} = 0 \text{ on } B_1'(0), \hspace{1cm} i = 1, 2, \ldots, n-1. \hspace{2cm} (4.2)$$

We begin by establishing a Caccioppoli-type inequality.

**Lemma 4.1.** Let $a^{ij} \in L^\infty(B_1'(0))$ such that $a^{ij} = a^{ji}$ on $B_1^+(0)$, (4.1) holds true for some constants $0 < \lambda \leq \Lambda < \infty$, and (4.2) holds true on $B_1'(0)$. Let $1 < p < \infty$ be a constant, $k_+, k_- \in L^1(B_1'(0))$ with $k_+, k_- \geq 0$ on $B_1'(0)$, and $h \in L^\infty(B_1'(0))$. If $u \in W^{1,2}(B_1^+(0))$ is a weak...
solution to \((1.2)\), then
\[
\int_{B_{\rho/2}^+(0)} |Du|^2 + \frac{1}{p} \int_{B_{\rho/2}^+(0)} (k_+((u-h)^+)^p + k_-((u-h)^-)^p) \\
\leq C \int_{B_{\rho}^+(0)} u^2 + C \int_{B_{\rho}^+(0)} (k_+ + k_-) |h|^p
\]
\hspace{1cm} (4.3)
for all \(0 < \rho \leq 1\) and some constant \(C = C(p, \lambda, \Lambda) \in (0, \infty)\).

**Proof.** Let \(\zeta = \eta^2 u\) in \((1.3)\), where \(\eta \in C^1(B_1^+ \cup B_1')\) with \(\eta = 0\) on \((\partial B_1)^+\) to obtain
\[
\int_{B_1^+} a^{ij} D_i u D_j u \eta^2 + \int_{B_1'} (k_+((u-h)^+)^{p-1} - k_-((u-h)^-)^{p-1}) u \eta^2 \\
= -2 \int_{B_1^+} a^{ij} D_i u D_j \eta, \\
\]
Using \((1.1)\) and Cauchy’s inequality,
\[
\frac{\lambda}{2} \int_{B_1^+} |Du|^2 \eta^2 + \int_{B_1'} (k_+((u-h)^+)^{p-1} - k_-((u-h)^-)^{p-1}) u \eta^2 \leq \frac{2\Lambda^2}{\lambda} \int_{B_1^+} u^2 |D\eta|^2.
\]
Since \((u-h)^\pm)^{p-1}u = \pm((u-h)^\pm)^p + ((u-h)^\pm)^{p-1}h\ on B_1',
\[
\frac{\lambda}{2} \int_{B_1^+} |Du|^2 \eta^2 + \int_{B_1'} (k_+((u-h)^+)^p + k_-((u-h)^-)p) \eta^2 \leq \frac{2\Lambda^2}{\lambda} \int_{B_1^+} u^2 |D\eta|^2 - \int_{B_1'} (k_+((u-h)^+)^{p-1} - k_-((u-h)^-)^{p-1}) h \eta^2
\]
Hence by Young’s inequality \(ab \leq \frac{p-1}{p} a^{p-1} + \frac{1}{p} b^p\ with a = (u-h)^\pm and b = |h|,
\[
\frac{\lambda}{2} \int_{B_1^+} |Du|^2 \eta^2 + \frac{1}{p} \int_{B_1'} (k_+((u-h)^+)^p + k_-((u-h)^-)p) \eta^2 \leq \frac{2\Lambda^2}{\lambda} \int_{B_1^+} u^2 |D\eta|^2 + \frac{1}{p} \int_{B_1'} (k_+ + k_-) |h|^p \eta^2.
\]
Now to obtain \((1.3)\) choose \(\eta\) so that \(0 \leq \eta \leq 1, \eta = 1\ on B_{\rho/2}^+, \eta = 0\ on B_1^+ \backslash B_{\rho}^+, \) and \(|D\eta| \leq 3/\rho\).

Next, we prove the boundedness of solutions to \((1.2)\).

**Lemma 4.2.** Let \(a^{ij} \in L^\infty(B_1^+(0))\) such that \(a^{ij} = a^{ji} on B_1^+(0)\), \((1.1)\) holds true for some constants \(0 < \lambda \leq \Lambda < \infty\ and \((1.2)\) holds true on \(B_1'(0)\). Let \(1 < p < \infty\ be a constant, k_+, k_- \in L^1(B_1'(0)) with k_+, k_- \geq 0\ on \(B_1'(0)\), and \(h \in L^\infty(B_1'(0))\). If \(u \in W^{1,2}(B_1^+(0))\) is a weak solution to \((1.2)\), then \(u \in L^\infty(B_{1/2}^+(0))\) and
\[
\sup_{B_{1/2}^+(0)} |u| \leq C\|u\|_{L^2(B_1^+(0))} + C\|h\|_{L^\infty(B_1'(0))}
\]
\hspace{1cm} (4.4)
for some constant \(C = C(n, \lambda, \Lambda) \in (0, \infty)\).
Proof. Set

$$M_- = \sup_{B_1 \cap \{k_- > 0 \}} h, \quad M_+ = \inf_{B_1 \cap \{k_+ > 0 \}} h.$$  

Extend \(a^{ij}\) to an \(L^\infty\)-function on \(B_1\) by even reflection, i.e. \(a^{ij}(x', x_n) = a^{ij}(x', -x_n)\), if \(i, j < n\) and if \(i = j = n\) and by odd reflection, i.e. \(a^{ij}(x', -x_n) = -a^{ij}(x', x_n)\), if \(i < n\) and \(j = n\) and if \(i = n\) and \(j < n\). Similarly, extend \(u\) to a \(W^{1, 2}\)-function on \(B_1\) by even reflection, i.e. \(u(x', x_n) = u(x', -x_n)\). First we claim that

$$D_i(a^{ij}D_j(u - M_-)^+) \geq 0 \text{ weakly in } B_1. \quad (4.5)$$

Let \(t > 0\) and \(\zeta \in W^{1, 2}_0(B_1)\) be an arbitrary nonnegative function. Let’s compare \(u\) to \(u_t \in W^{1, 2}(B_1)\), where

$$u_t = \begin{cases} 
  u & \text{if } u \leq M_- \\
  M_- & \text{if } M_- < u \leq M_- + t\zeta \\
  u - t\zeta & \text{if } u > M_- + t\zeta.
\end{cases}$$

Notice that since \(\zeta = 0\) on \(\partial B_1\), \(u_t = u\) on \(\partial B_1\). Since \(u_t \leq u\) on \(B_1^t\), \((u_t - h)^+ \leq (u - h)^+\) on \(B_1^t\). Moreover, we have \(u \geq u_t \geq M_- \geq h\) on \(B_1^t \cap \{k_- > 0\} \text{ and } u > M_-=\}\) and \(u_t = u\) on \(B_1^t \cap \{u \leq M_-\}\), and therefore \(k_-(u - h)^-) = k_-(u - h)^-\) on \(B_1^t\). Hence, recalling that \(J(u) \leq J(u_t)\) for all \(t > 0\) because \(u\) is \(J\)-minimizing, and utilizing the symmetry properties, we infer

$$\int_{B_1 \cap \{u > M_-\}} a^{ij}D_iu D_ju \leq \int_{B_1 \cap \{u > M_- + t\zeta\}} a^{ij}D_i(u - t\zeta)^+ D_j(u - t\zeta)^+$$

$$\leq \int_{B_1 \cap \{u > M_-\}} a^{ij}D_i(u - t\zeta) D_j(u - t\zeta)$$

for all \(t > 0\). Thus by differentiating \(\int_{B_1^t \cap \{u > M_-\}} a^{ij}D_i(u - t\zeta) D_j(u - t\zeta)\) at \(t = 0^+\),

$$\int_{B_1^t \cap \{u > M_-\}} a^{ij}D_iu D_j\zeta \leq 0$$

for all nonnegative functions \(\zeta \in W^{1, 2}_0(B_1^+).\) Therefore (4.5) holds true. By (4.5) and [GT01, Theorem 8.17],

$$\sup_{B_{1/2}^+} (u - M_-) \leq C \|(u - M_-)^+\|_{L^2(B_1^+)} \quad (4.6)$$

for some constant \(C = C(n, \lambda, \Lambda) \in (0, \infty).\)

By instead comparing \(u\) to

$$u_t = \begin{cases} 
  u + t\zeta & \text{if } u \leq M_+ - t\zeta \\
  M_+ & \text{if } M_+ - t\zeta < u \leq M_+ \\
  u & \text{if } u \geq M_+
\end{cases}$$

where \(t > 0\) and \(\zeta \in W^{1, 2}_0(B_1^+),\) we can show that

$$D_i(a^{ij}D_j(u - M_+)^-) \geq 0 \text{ weakly in } B_1.$$  

Then by applying [GT01, Theorem 8.17] to \((u - M_+)\)

$$\sup_{B_{1/2}^+} (M_+ - u) \leq C \|(u - M_+)^-\|_{L^2(B_1^+)} \quad (4.7)$$

for some constant \(C = C(n, \lambda, \Lambda) \in (0, \infty).\) It follows from (4.6) and (4.7) that (4.4) holds true.
Our next step consists in proving an initial Hölder modulus of continuity for the solution.

**Lemma 4.3.** Let \( a^{ij} \in C^0(B_1^+(0) \cup B_1'(0)) \) such that \( a^{ij} = a^{ji} \) on \( B_1^+(0) \cup B_1'(0) \), \( \| \nabla u \| \leq 1 \) holds true for some constants \( 0 < \lambda \leq \Lambda < \infty \), and \( \| \nabla v \| \leq 1 \) holds true on \( B_1'(0) \). Let \( 1 < p < \infty \) be a constant and \( k_+, k_- \in L^\infty(B_1'(0)) \) with \( k_+, k_- \geq 0 \) on \( B_1'(0) \). If \( u \in W^{1,2}(B_1^+(0)) \) is a weak solution to \( (1.2) \), then \( u \in C^{0,1/2}(B_{1/8}(0)) \) with

\[
[u]_{1/2,B_{1/8}^+(0)}^2 \leq C \| u \|^2_{L^2(B_1^+(0))} + C \| k_+ + k_- \|_{L^\infty(B_1'(0))} \left( \| u \|^p_{L^p(B_1^+(0))} + \| h \|^p_{L^p(B_1'(0))} \right)
\]

for some constant \( C \in (0, \infty) \) depending on \( n, p, \lambda, \Lambda \), and the modulus of continuity of \( a^{ij} \).

**Proof.** Let \( y \in B_1'(0), 0 < \rho \leq 1/4 \), and let \( v \in W^{1,2}(B_1^+(0)) \) be the weak solution to the constant coefficient problem

\[
\begin{align*}
D_i (a^{ij}(y) D_j v) &= 0 \text{ in } B_\rho^+(y), \\
D_n v &= 0 \text{ on } B_\rho'(y), \\
v &= u \text{ on } \partial B_\rho(y)^+. 
\end{align*}
\]

Extending \( v \) by even reflection \( v(x', x_n) = v(x', -x_n), v \) is a solution to the constant coefficient elliptic equation \( D_i (a^{ij}(y) D_j v) = 0 \) in \( B_\rho(y) \) with \( v = u \) on \( \partial B_\rho(y) \). Thus by the maximum principle, \( \sup_{B_\rho^+(y)} |v| \leq \sup_{\partial B_\rho^+(y)} |u| \). Using the \( J \)-minimality of \( u \) and Lemma 4.2,

\[
\int_{B_\rho^+(y)} (a^{ij} D_i u D_j u - a^{ij} D_i v D_j v) \leq \frac{2}{p} \int_{B_\rho^+(y)} \left( k_+ ((v - h)^+)^p - k_- ((u - h)^+)^p \right) + \frac{2}{p} \int_{B_\rho^+(y)} \left( k_- ((v - h)^-)^p - k_- ((u - h)^-)^p \right)
\]

\[
\leq CK \rho^{n-1},
\]

where \( C = C(n, p) \in (0, \infty) \) is a constant and

\[
K = \| k_+ + k_- \|_{L^\infty(B_1'(0))} \left( \| u \|^p_{L^p(B_1^+(0))} + \| h \|^p_{L^p(B_1'(0))} \right).
\]

Since \( v \) is the solution to \( (4.3) \),

\[
\int_{B_\rho^+(y)} a^{ij}(y) D_i (u - v) D_j v = 0
\]

and thus

\[
\int_{B_\rho^+(y)} a^{ij}(y) D_i (u - v) D_j (u - v) = \int_{B_\rho^+(y)} (a^{ij}(y) D_i u D_j u - a^{ij}(y) D_i v D_j v)
\]

\[
(4.11)
\]

Hence by \( (4.11) \) and \( (4.10) \),

\[
\int_{B_\rho^+(y)} |D u - D v|^2 \leq \frac{1}{\lambda} \int_{B_\rho^+(y)} a^{ij}(y) D_i (u - v) D_j (u - v)
\]

\[
= \frac{1}{\lambda} \int_{B_\rho^+(y)} (a^{ij}(y) D_i u D_j u - a^{ij}(y) D_i v D_j v)
\]

\[
\leq -\frac{1}{\lambda} \int_{B_\rho^+(y)} (a^{ij} - a^{ji}(y)) (D_i u D_j u - D_i v D_j v) + CK \rho^{n-1}
\]

\[
(4.12)
\]
for some constant $C = C(n, p, \lambda) \in (0, \infty)$. Thus there exists $\rho_0 \in (0, 1/4]$ depending on $\lambda$ and the
modulus of continuity of $a$ such that if $0 < \rho \leq \rho_0$ then

$$
\int_{B_\rho^+(y)} |Du - Dv|^2 \leq \frac{1}{4} \int_{B_\rho^+(y)} (|Du|^2 + |Dv|^2) + CK \rho^{n-1}
$$

for some constant $C = C(n, p, \lambda) \in (0, \infty)$. Thus, by the triangle inequality we have, if $0 < \rho \leq \rho_0$

$$
\int_{B_\rho^+(y)} |Du - Dv|^2 \leq \frac{3}{2} \int_{B_\rho^+(y)} |Dv|^2 + 2CK \rho^{n-1}, \quad (4.12)
$$

$$
\int_{B_\rho^+(y)} |Du - Dv|^2 \leq \frac{3}{2} \int_{B_\rho^+(y)} |Du|^2 + 2CK \rho^{n-1} \quad (4.13)
$$

for some constant $C = C(n, p, \lambda) \in (0, \infty)$.

Let $0 < \sigma < \rho \leq \rho_0$. Using the triangle inequality and (4.12),

$$
\sigma^{1-n} \int_{B_\sigma^+(y)} |Du|^2 \leq 2\sigma^{1-n} \int_{B_\sigma^+(y)} |Du - Dv|^2 + 2\sigma^{1-n} \int_{B_\rho^+(y)} |Dv|^2
$$

$$
\leq 5\sigma^{1-n} \int_{B_\rho^+(y)} |Dv|^2 + 4CK.
$$

for some constant $C = C(n, p, \lambda) \in (0, \infty)$. By [L13, Equation (5.82)],

$$
\sigma^{-n} \int_{B_\rho^+(y)} |Dv|^2 \leq C \rho^{-n} \int_{B_\rho^+(y)} |Dv|^2
$$

for some constant $C = C(n, \lambda, \Lambda) \in (0, \infty)$ and thus

$$
\sigma^{1-n} \int_{B_\sigma^+(y)} |Du|^2 \leq C \left(\frac{\sigma}{\rho}\right) \rho^{1-n} \int_{B_\rho^+(y)} |Dv|^2 + CK.
$$

for some constant $C = C(n, p, \lambda, \Lambda) \in (0, \infty)$. By the triangle inequality and (4.13),

$$
\sigma^{1-n} \int_{B_\sigma^+(y)} |Du|^2 \leq 2C \left(\frac{\sigma}{\rho}\right) \rho^{1-n} \int_{B_\rho^+(y)} |Du - Dv|^2 + 2C \left(\frac{\sigma}{\rho}\right) \rho^{1-n} \int_{B_\rho^+(y)} |Dv|^2 + CK
$$

$$
\leq C \left(\frac{\sigma}{\rho}\right) \rho^{1-n} \int_{B_\rho^+(y)} |Dv|^2 + CK,
$$

where $C = C(n, p, \lambda, \Lambda) \in (0, \infty)$ are constants. Hence there exists $\theta = \theta(n, p, \lambda, \Lambda) \in (0, 1)$ such
that, setting $\sigma = \theta \rho$,

$$
(\theta \rho)^{1-n} \int_{B_{\theta \rho}^+(y)} |Du|^2 \leq \frac{1}{2} \rho^{1-n} \int_{B_\rho^+(y)} |Dv|^2 + CK \quad (4.14)
$$

for all $0 < \rho \leq \rho_0$ and some constant $C = C(n, p, \lambda, \Lambda) \in (0, \infty)$. Iteratively applying (4.14) with $\rho = \theta^{-1} \rho_0$ for $j = 1, 2, 3, \ldots, m$,

$$
(\theta^n \rho_0)^{1-n} \int_{B_{\theta^m \rho_0}^+(y)} |Du|^2 \leq \frac{1}{2^m} \rho_0^{1-n} \int_{B_{\rho_0}^+(y)} |Dv|^2 + CK \sum_{j=0}^{m-1} \frac{1}{2^j}
$$

$$
\leq \frac{1}{2^m} \rho_0^{1-n} \int_{B_{\rho_0}^+(y)} |Dv|^2 + 2CK \quad (4.15)
$$
for each integer $m = 1, 2, 3, \ldots$. Take $0 < \rho \leq \rho_0$ and choose $m$ to be the nonnegative integer such that $\theta^{m+1}\rho_0 < \rho \leq \theta^m\rho_0$. Then (4.15) gives us
\[\rho^{1-n} \int_{B^n_\rho(y)} |Du|^2 \leq C \rho^{1-n} \int_{B^n_\rho(y)} |Du|^2 + CK\]
for some constant $C = C(n, p, \lambda, \Lambda) \in (0, \infty)$. Therefore, using Lemma 4.1,
\[\int_{B^n_\rho(y)} |Du|^2 \leq C \rho^{n-1} \left(\|u\|^2_{L^2(B^n_\rho(0))} + K\right)\]
(4.16) for all $y \in B^n_1(0)$ and $0 < \rho \leq 1/4$, and for some constant $C \in (0, \infty)$ depending only on $n, p, \lambda, \Lambda$, and the modulus of continuity of $a^{ij}$. By (4.16),
\[\int_{B^n_\rho(y)} |Du|^2 \leq \int_{B^n_\rho(y',0)} |Du|^2 \leq C \rho^{n-1} \left(\|u\|^2_{L^2(B^n_\rho(0))} + K\right)\]
(4.17) for all $y = (y', y_n) \in B^n_1(0)$ and $|y_n| \leq 1/8$, where $C \in (0, \infty)$ is a constant depending only on $n, p, \lambda, \Lambda$, and the modulus of continuity of $a^{ij}$. By the argument leading to (4.16) with obvious modifications,
\[\int_{B^n_\rho(y)} |Du|^2 \leq C \left(\frac{\rho}{|y_n|}\right)^{n-1} \|u\|^2_{L^2(B^n_\rho(y_n))}\]
(4.18) for all $y = (y', y_n) \in B^n_1(0)$ and $0 < \rho \leq |y_n|$ and some constant $C \in (0, \infty)$ depending only on $n, p, \lambda, \Lambda$, and the modulus of continuity of $a^{ij}$. By (4.17) and (4.18), (4.16) holds true for all $y = (y', y_n) \in B^n_1(0)$ and $0 < \rho \leq |y_n|$. In other words, (4.16) holds true for all $y \in B^n_1(0)$ and $0 < \rho \leq 1/8$. Hence we may apply Morrey’s Dirichlet Growth Theorem using (4.16) to conclude that $u \in C^{0,1/2}(B^n_1(0))$ and (4.18) holds true.

We are now ready to prove our main result in this section.

**Theorem 4.1.** Let $\kappa \geq 1$ be an integer and $\alpha \in (0, 1)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset and $\Gamma$ be a relatively open $C^{\kappa,}\alpha$-portion of $\partial\Omega$. Let $a^{ij} \in C^{\kappa-1,\alpha}(\Omega \cup \Gamma)$ such that $a^{ij} = a^{ji}$ on $\Omega \cup \Gamma$ and (1.1) holds true for some constants $0 < \lambda \leq \Lambda < \infty$. Let $1 < p < \infty$ be a constant, $k_+, k_- \in C^{\kappa-1,\alpha}(\Omega)$ with $k_+, k_- \geq 0$ on $\Gamma$, and $h \in C^{\kappa-1}(\Gamma)$. Let $u \in W^{1,2}(\Omega)$ be a weak solution to (1.2).

(a) If $\kappa + \alpha \leq p$, then $u \in C^{\kappa,\alpha}(\Omega \cup \Gamma)$.

(b) Let $x_0 \in \Gamma$ and $0 < \delta < \text{dist}(x_0, \partial\Omega \setminus \Gamma)$. Suppose that either

(i) $u \neq h$ on $\Gamma \cap B_\delta(x_0)$ or

(ii) $p$ is an even integer and $k_+ = k_-$ on $\Gamma \cap B_\delta(x_0)$ or

(iii) $p$ is an odd integer and $k_+ = k_- = 0$ on $\Gamma \cap B_\delta(x_0)$.

Then $u \in C^{\kappa,\alpha}(\Omega \cup \Gamma \cap B_\delta(x_0))$.

**Remark 4.1.** In the special case that $a^{ij} \in C^\infty(\Omega \cup \Gamma)$ and $k_+, k_-, h \in C^\infty(\Gamma)$, Theorem 4.1 gives us that if $u \in W^{1,2}(\Omega)$ is a weak solution to (1.2), then $u \in C^{p,\kappa,\alpha}(\Omega \cup \Gamma)$ if $p$ is not an integer and $u \in C^{p-1,\alpha}(\Omega \cup \Gamma)$ for all $\alpha \in (0, 1)$ if $p$ is an integer. Moreover, given $x_0 \in \Gamma$ and $0 < \delta < \text{dist}(x_0, \partial\Omega \setminus \Gamma)$, if either $u \neq h$ in $\Lambda \cap B_\delta(x_0)$ or $p$ is an even integer and $k_+ = k_- = 0$ on $\Gamma \cap B_\delta(x_0)$, then $u \in C^\infty(\Omega \cup \Gamma \cap B_\delta(x_0))$. This is consistent with what was found in [4.1] in the special case that $\Omega = B^+_1(0)$, $\Lambda = B^+_1(0)$, $a^{ij} = \delta^{ij}$ on $B^+_1(0) \cup B^+_1(0)$, $h = 0$ on $B^+_1(0)$, and $k_+, k_-$ are constants on $B^+_1(0)$. 14
Proof of Theorem 4.1. By elliptic regularity \cite{GT01}, Corollary 8.36 and Theorem 6.17, \( u \in C^{\kappa,\alpha}(\Omega) \). Thus to show (a), it suffices to show that for each \( x_0 \in \Gamma \) and some \( 0 < \delta < \text{dist}(x_0, \partial \Omega \setminus \Gamma) \) (depending on \( x_0 \)) that \( u \in C^{\kappa,\alpha}((\Omega \cup \Gamma) \cap B_\delta(x_0)) \). After a \( C^{\kappa,\alpha} \)-change of coordinates we may assume that \( x_0 = 0, \delta = 1, \Omega \cap B_\delta(x_0) = B_1^+(0) \), and \( \Gamma \cap B_\delta(x_0) = B_1^-(0) \). Moreover, by \cite{GST14} Theorem B.1] after a further \( C^{\kappa,\alpha} \)-change of coordinates we may assume that (4.2) holds true on \( B_1^-(0) \). (Note that if \( a^{ij} \) is in \( C^{\kappa-1,\alpha}(\Omega \cup \Gamma) \) rather than in \( C^{1,1}(\Omega \cup \Gamma) \) then the change of coordinates constructed in \cite{GST14} Theorem B.1] is in fact \( C^{\kappa,\alpha} \).

Let’s suppose that \( \kappa + \alpha \leq p \) and show that \( u \in C^{\kappa,\alpha}(B_1^+ \cup B_1^-) \). By Lemma 4.3, \( u \in C^{0,1/2}(B_1^+ \cup B_1^-) \). Notice that since \( \kappa + \alpha \leq p \), \( z \mapsto (z^+)^{p-1} \) and \( z \mapsto (z^-)^{p-1} \) are \( C^{\kappa-1,\alpha} \)-functions of \( z \in \mathbb{R} \). All this together with \( k_+, k_- \in C^{\kappa-1,\alpha}(B_1^-) \) and \( h \in C^{\kappa-1,1}(B_1^-) \), gives us

\[
 k_+((u-h)^+)^{p-1} - k_-((u-h)^-)^{p-1} \in \begin{cases} 
 C^{0,\alpha/2}(B_1^-) & \text{if } \kappa = 1 \\
 C^{0,1/2}(B_1^-) & \text{if } \kappa \geq 2. 
\end{cases}
\]

By the regularity theory for weak solutions to oblique derivative problems (see e.g. \cite{Li13} Proposition 5.53),

\[
u \in \begin{cases} 
 C^{1,\alpha/2}(B_1^+ \cup B_1^-) & \text{if } \kappa = 1 \\
 C^{1,1/2}(B_1^+ \cup B_1^-) & \text{if } \kappa \geq 2. 
\end{cases}
\]

But then

\[
 k_+((u-h)^+)^{p-1} - k_-((u-h)^-)^{p-1} \in \begin{cases} 
 C^{0,\alpha}(B_1^-) & \text{if } \kappa = 1 \\
 C^{1,\alpha/2}(B_1^-) & \text{if } \kappa = 2 \\
 C^{1,1/2}(B_1^-) & \text{if } \kappa \geq 3 
\end{cases}
\]

and so applying the regularity theory for solutions to oblique derivative problems again,

\[
u \in \begin{cases} 
 C^{1,\alpha}(B_1^+ \cup B_1^-) & \text{if } \kappa = 1 \\
 C^{2,\alpha/2}(B_1^+ \cup B_1^-) & \text{if } \kappa = 2 \\
 C^{2,1/2}(B_1^+ \cup B_1^-) & \text{if } \kappa \geq 3 
\end{cases}
\]

arriving at the desired conclusion \( u \in C^{1,\alpha}(B_1^+ \cup B_1^-) \) if \( \kappa = 1 \). If instead \( \kappa \geq 2 \), for each positive integer \( \ell \leq \kappa + 1 \) by inductively applying the regularity theory solutions to oblique derivative problems a total of \( \ell \) times we obtain

\[
u \in \begin{cases} 
 C^{\ell-1,\alpha}(B_1^+ \cup B_1^-) & \text{if } \kappa = \ell - 1 \\
 C^{\ell,\alpha/2}(B_1^+ \cup B_1^-) & \text{if } \kappa = \ell \\
 C^{\ell,1/2}(B_1^+ \cup B_1^-) & \text{if } \kappa \geq \ell + 1. 
\end{cases}
\]

Indeed, by (4.19) and (4.20), (4.21) holds true if \( \ell = 1, 2 \). If (4.21) holds true for some integer \( \ell \leq \kappa \), then (4.21) implies that

\[
k_+((u-h)^+)^{p-1} - k_-((u-h)^-)^{p-1} \in \begin{cases} 
 C^{\ell-1,\alpha}(B_1^-) & \text{if } \kappa = \ell \\
 C^{\ell,\alpha/2}(B_1^-) & \text{if } \kappa = \ell + 1 \\
 C^{\ell,1/2}(B_1^-) & \text{if } \kappa \geq \ell + 2 
\end{cases}
\]

and so applying the regularity theory one more time gives us (4.21) with \( \ell + 1 \) in place of \( \ell \). Taking \( \ell = \kappa + 1 \) in (4.21) gives us the desired conclusion that \( u \in C^{\kappa,\alpha}(B_1^+ \cup B_1^-) \).

(b) follows by a similar bootstrap argument, noting for (i) that \( z \mapsto (z^+)^{p-1} \) and \( z \mapsto (z^-)^{p-1} \) are smooth functions of \( z \in \mathbb{R} \setminus \{0\} \). Also, assuming (ii) or (iii),

\[
k_+(x)(z^+)^{p-1} - k_-(x)(z^-)^{p-1} = k_+(x)z^{p-1}
\]

for each \( x \in \Gamma \cap B_\delta(x_0) \) and \( z \in \mathbb{R} \) and \( z \mapsto z^{p-1} \) is a smooth function of \( z \in \mathbb{R} \). \qed
In Theorem 4.2 below, we show that the regularity in Theorem 4.1 is optimal in the sense that if $p$ is an integer, $u$ is not locally $C^{p-1,1}$ near free boundary points $x_0 \in \Gamma$ at which $u(x_0) = h(x_0)$, $\nabla^1 u(x_0) \neq \nabla^1 h(x_0)$, and either $p$ is even and $k_+(x_0) \neq k_-(x_0)$ or $p$ is odd and $k_+(x_0) 
eq 0$ or $k_-(x_0) \neq 0$. This was previously shown in [DJ, Theorem 2] in the special case $p = 2$, $a^{ij} = \delta_{ij}$, $k_+$ and $k_-$ are constants, and $h = 0$ on $\Gamma$. The key obstruction is that after an orthogonal change of coordinates, we expect $u$ to be asymptotic to a degree $(p-1)$ polynomial plus a constant multiple of $\text{Re}((x_1 - ix_n)^p \log(x_n + ix_1))$.

**Theorem 4.2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset and $\Gamma$ be a relatively open $C^p$-portion of $\partial \Omega$. Let $a^{ij} \in C^{p-1}(\Omega \cup \Gamma)$ such that $a^{ij} = a^{ij}_0$ on $\Omega$ and (4.11) holds true for some constants $0 < \lambda \leq \Lambda < \infty$. Let $p \geq 2$ be an integer, $k_+, k_- \in C^0(\Gamma)$ such that $k_+, k_- \geq 0$ on $\Gamma$, and $h \in C^1(\Gamma)$. Let $u \in W^{1,2}(\Omega)$ be a weak solution to (1.2). Moreover, let $x_0 \in \Gamma$ be such that $u(x_0) = h(x_0)$ and $\nabla^1 u(x_0) \neq \nabla^1 h(x_0)$, where $\nabla^1$ denotes tangential gradient of $\Gamma$. In addition, assume that either

(i) $p$ is even and $k_+(x_0) \neq k_-(x_0)$, or

(ii) $p$ is odd and $k_+(x_0) \neq 0$ or $k_-(x_0) \neq 0$.

Then there is no $\delta > 0$ such that $u \in C^{p-1,1}(\Omega \cap \Gamma) \cap B_\delta(x_0)$.

**Proof.** After a change of coordinates assume that $x_0 = 0$, $\delta = 1$, $\Omega = B_1^+(0)$, $\Gamma = B_1'(0)$, and (4.11) and (4.12) hold true. Suppose to the contrary that $u \in W^{1,2}(B_1^+(0))$ is a weak solution to (1.2) such that either $p$ is even and $k_+(0) = k_-(0)$, or $p$ is odd and $k_+(0) = k_-(0) = 0$, $u(0) = h(0)$, $\nabla^1 u(0) \neq \nabla^1 h(0)$, and $u \in C^{p-1,1}(B_1^+(0) \cup B_1'(0))$ with $\|D^p u\|_{L^\infty(B_1^+(0))} < \infty$. Without loss of generality assume that

$$D_1(u - h)(0) = b, \quad D_i(u - h)(0) = 0 \quad \text{for all} \quad i = 2, 3, \ldots, n - 1$$

(4.22)

for some constant $b > 0$. Notice that by (1.2) and $u(0) = h(0)$, $D_n u(0) = 0$. Let $P(x) = \sum_{|\alpha| \leq p-1} \frac{1}{\alpha!} D^\alpha u(0) x^\alpha$ be the degree $(p-1)$ Taylor polynomial of $u$ at the origin. For each $\rho > 0$ define $u_\rho \in C^{p-1,1}(B_1^+/\rho(0) \cup B_1'/\rho(0))$ by

$$u_\rho(x) = \frac{u(\rho x) - P(\rho x)}{\rho^p}.$$

By (1.2) and $u(0) = h(0)$, $|D_n u| \leq C|x|^{p-1}$ for all $x \in B_1'(0)$ and some constant $C \in (0, \infty)$. Since $P$ is the degree $(p-1)$ Taylor polynomial of $u$ at the origin, $D_n P = 0$ on $\mathbb{R}^{n-1} \times \{0\}$. Hence

$$D_i(a^{ij}(\rho x) D_j u_\rho) = -\frac{(D_i(a^{ij} D_j P)(\rho x))}{\rho^p} D_i^+ B_1^+/\rho(0),$$

$$a^{ij}(\rho x) D_n u_\rho = k_+(\rho x) \left( \left( \frac{P(\rho x) - h(\rho x)}{\rho} + \rho^{p-1} u_\rho(x) \right)^{p-1} + \left( \frac{P(\rho x) - h(\rho x)}{\rho} + \rho^{p-1} u_\rho(x) \right)^{-p-1} \right) \text{ on } B_1'/\rho(0).$$

(4.23)

Since $u \in C^{p-1,1}(B_1^+(0) \cup B_1'(0))$ with $\|D^p u\|_{L^\infty(B_1^+(0))} < \infty$,

$$\|u_\rho\|_{C^{p-1,1}(B_1^+/\rho(0))} \leq C(n, p) \|D^p u\|_{L^\infty(B_1^+(0))}.$$
Thus, by the Arzelà-Ascoli theorem, there exists a sequence \( \rho_n \to 0^+ \) and function \( v \in C^{p-1,1}(\mathbb{R}_+^n) \) such that \( u_{\rho_n} \to v \) in the \( C^{p-1,1} \)-topology on compact subsets of \( \mathbb{R}_+^n \). By continuity, \( a^{ij}(\rho x) \to a^{ij}(0) = \delta_{ij} \) uniformly on compact subsets of \( \mathbb{R}_+^n \) as \( \rho \to 0^+ \) and \( k_{\pm}(\rho x) \to k_{\pm} \) uniformly on compact subsets of \( \mathbb{R}^{n-1} \times \{0\} \) as \( \rho \to 0^+ \), where \( k_{\pm} = k_{\pm}(0) \). Notice that \( D_i(a^{ij} D_j P) \in C^{p-2}(B_1^\pm(0) \cup B_1^\mp(0)) \). Keeping in mind that \( u \) satisfies \( (1.2) \) and \( P \) is the degree \((p-1)\) Taylor polynomial of \( u \) at the origin, the degree \((p-3)\) Taylor polynomial of \( D_i(a^{ij} D_j P) \) is zero. Thus,

\[
\frac{(D_i(a^{ij} D_j P))(\rho x)}{\rho^{p-2}} \to Q(x)
\]

uniformly on compact subsets of \( \mathbb{R}^n \) as \( \rho \to 0^+ \), where \( Q(x) = \sum_{|\alpha|=p-2} \sum_{i,j=1}^n \frac{1}{\alpha!} D^\alpha D_i(a^{ij} D_j P)(0) x^\alpha \) is a homogeneous degree \((p-2)\) polynomial. Since \( P \) is the degree \((p-1)\) Taylor polynomial of \( u \) at the origin, \( u(0) = h(0) \), and \((4.22)\) holds true,

\[
P(\rho x) - h(\rho x) \to (\nabla^\Gamma u(0) - \nabla^\Gamma h(0)) \cdot x = bx_1
\]

uniformly on compact subsets of \( \mathbb{R}^{n-1} \times \{0\} \) as \( \rho \to 0^+ \). Therefore, by letting \( \rho = \rho_\ell \to 0^+ \) in \((4.23)\), \( v \in C^{p-1,1}(\mathbb{R}_+^n) \) is a solution to

\[
\Delta v = Q \text{ in } \mathbb{R}_+^n, \\
D_n v = \kappa_+ (bx_1^+)^{p-1} - \kappa_- (bx_1^-)^{p-1} \text{ on } \mathbb{R}^{n-1} \times \{0\}, \tag{4.24}
\]

where we recall that \( Q \) is a homogeneous polynomial of degree \( p-2 \) on \( \mathbb{R}^n \) and \( b > 0 \) is a constant. Now it suffices to show that:

Claim. Let \( p \geq 0 \) be a positive integer and let \( b > 0 \) and \( \kappa_+, \kappa_- \geq 0 \) be constants such that either \( p \) is even and \( \kappa_+ \neq \kappa_- \) or \( p \) is odd and \( \kappa_+ \neq 0 \) or \( \kappa_- \neq 0 \). Let \( Q \) be a homogeneous degree \( p-2 \) polynomial. Then there are no weak solutions \( v \in C^{p-1,1}(\mathbb{R}_+^n) \) to \((4.24)\).

To see this, first we make the following simplifying assumptions. Notice that we can find a homogeneous polynomial \( R \) of degree \( p \) such that

\[
\Delta R = Q \text{ in } \mathbb{R}_+^n, \\
D_n R = -\kappa_- (-bx_1)^{p-1} \text{ on } \mathbb{R}^{n-1} \times \{0\}.
\]

To see this, suppose that

\[
R(x', x_n) = \sum_{j=0}^{p} \alpha_j(x') x_j^n, \quad Q(x', x_n) = \sum_{j=0}^{p-2} \beta_j(x') x_j^n
\]

for all \( x' \in \mathbb{R}^{n-1} \) and \( x_n \in \mathbb{R} \), where \( \alpha_j(x') \) are homogeneous degree \((p-j)\) polynomials on \( \mathbb{R}^{n-1} \) and \( \beta_j(x') \) are homogeneous degree \((p-2-j)\) polynomials on \( \mathbb{R}^{n-1} \). Let’s solve for \( \alpha_j(x') \) for \( j = 0, 1, 2, \ldots, p \). We compute that

\[
\Delta R(x', x_n) = \sum_{j=0}^{p-2} ((j+1)(j+2)\alpha_{j+2}(x') + \Delta \alpha_j(x')) x_j^n.
\]

Setting \( \Delta R = Q \) gives us that \( \alpha_j(x') \) are defined by the recurrence relation

\[
\alpha_1(x') = -\kappa_- (-bx_1)^{p-1}, \quad \alpha_{j+2}(x') = \frac{\beta_j(x') - \Delta \alpha_j(x')}{(j+1)(j+2)} \text{ for } j = 0, 1, 2, \ldots, p-2,
\]

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with $a_0(x')$ to be chosen freely (as $R$ is uniquely defined up at adding a homogeneous degree $p$ harmonic polynomial which is even in $x_n$). By replacing $v$ with $v - R$, we may assume that $Q = 0$, $\kappa_+ \neq 0$, and $\kappa_- = 0$ and show that there are no weak solutions $v \in C^{p-1,1}(\mathbb{R}^n_+)$ to

$$
\Delta v = 0 \text{ in } \mathbb{R}^n_+,
$$

$$
D_n v = \kappa_+(x_+)^{p-1} \text{ on } \mathbb{R}^{n-1} \times \{0\}.
$$

(4.25)

Consider the function $w : \mathbb{R}^n_+ \rightarrow \mathbb{R}$ defined by

$$
w(x) = \text{Re} \left( (-i)^p \kappa_+(x_n + ix_1)^p \left( \frac{1}{\pi p} \log(x_n + ix_1) - \frac{1}{\pi p^2} + \frac{i}{2p} \right) \right),
$$

(4.26)

where $i = \sqrt{-1}$ and we use the branch of $\log(x_n + ix_1)$ on $\{x_n + ix_1 : x_n \geq 0\}$ for which $\log(x_n + i0) = \log|x_n|$. One readily verifies that $w \in C^{p-1}(\mathbb{R}^n_+)$ but $w \notin C^{p-1,1}(\mathbb{R}^n_+)$. We claim that (4.25) holds true with $w$ in place of $v$. Clearly $w \in W^{2,2}(\mathbb{R}^n_+)$ and $\Delta w = 0$ in $\mathbb{R}^n_+$. By differentiating (4.26),

$$
D_n w(x',0) = \text{Re} \left( \kappa_+ x_1^{p-1} \left( \frac{1}{\pi} \log(ix_1) + \frac{1}{2} \right) \right) = \kappa_+(x_1)^{p-1},
$$

(4.27)

for all $(x',0) \in \mathbb{R}^{n-1} \times \{0\}$, where the last step follows from $\text{Im} \log(ix_1) = \pi/2$ if $x_1 > 0$ and $\text{Im} \log(ix_1) = -\pi/2$ if $x_1 < 0$.

Now let $v \in C^{p-1,1}(\mathbb{R}^n_+) \text{ solve (4.25)}$ and $w$ be as in (4.26). $\Delta (v - w) = 0$ in $\mathbb{R}^n_+$ and $D_n (v - w) = 0$ on $\mathbb{R}^{n-1} \times \{0\}$. Thus the extension of $v - w$ across $\mathbb{R}^{n-1} \times \{0\}$ via odd reflection is a harmonic function on $\mathbb{R}^n$. Hence $v - w$ is smooth on $\mathbb{R}^n_+$. Since $w \notin C^{p-1,1}(\mathbb{R}^n_+)$, we have $v \notin C^{p-1,1}(\mathbb{R}^n_+)$. \square

5 Normalized solutions

Let $u \in W^{1,2}(\Omega)$ be a solution to (1.2). We define the contact set $\Lambda(u)$ and the free boundary $\Sigma(u)$ of $u$, respectively, by

$$
\Lambda(u) = \{x \in \Gamma : u(x) = h(x)\}, \quad \Sigma(u) = \partial \Gamma \setminus \Lambda(u),
$$

where $\partial \Gamma$ denotes the boundary (frontier) with respect to the topology of $\Gamma$. The remainder of the paper will be focused on studying the regularity of the free boundary $\Sigma(u)$. We will use Almgren’s frequency function to determine the local asymptotic behavior of $u$ at a free boundary, using a combination of approaches of [GP09], [GS14], and [DJ].

We shall assume the following. Let $2 \leq p < \infty$. Let $\Omega = B^+_1(0)$ and $\Gamma = B'_1(0)$. For a fixed positive integer $\kappa$, let $a^{ij} \in C^{\kappa-1,1}(B^+_1(0))$ be such that $a^{ij} = a^{ji}$ on $B^+_1(0)$, the ellipticity condition (1.1) holds for some constants $0 < \lambda \leq \Lambda < \infty$, and the normalization assumptions (1.1) and (1.2) are satisfied. Let $k_+, k_- \in C^{0,1}(B^+_1(0))$ with $k_+, k_- \geq 0$, and let $h \in C^{\kappa,1}(B^+_1(0))$. We explicitly observe that the regularity assumptions on $a^{ij}$ and $h$ will be used in Lemma 5.1 below to construct the extension $\hat{h}$ of a Taylor polynomial of $h$. In the special case that $h = 0$ on $B'_1(0)$, it suffices to assume that $a^{ij} \in C^{0,1}(B^+_1(0) \cup B'_1(0))$. Finally, let $u \in W^{1,2}(B^+_1(0))$ be a weak solution to (1.2) and $0 \in \Sigma(u)$.

Following the approach in [GP09], we will extend the degree $\kappa$ Taylor polynomial of $h$ to a polynomial solution $\hat{h}$ to $|D_i(a^{ij} D_j \hat{h})| \leq C|x|^{\kappa-1}$ in $B^+_1(0)$ and $D_n \hat{h} = 0$ on $B'_1(0)$, which we will then subtract from $u$ (see (5.3) below).
Lemma 5.1. Let $\kappa$ be a positive integer. Let $a^{ij} \in C^{\kappa-1,1}(B^+_1(0))$ such that $a^{ij} = a^{ji}$ on $B^+_1(0)$ and (1.1) holds true for some constants $0 < \lambda \leq \Lambda < \infty$. Let $h \in C^{\kappa,1}(B'_1(0))$. There exists a unique polynomial $\overline{h} : \mathbb{R}^n \to \mathbb{R}$ of degree at most $\kappa$ such that

$$|D_{i}(a^{ij} D_{j}\overline{h})| \leq C|x|^{{\kappa-1}} \text{ in } B^+_1(0), \quad D_{n}\overline{h} = 0 \text{ on } B'_1(0),$$

$$|\overline{h}(x',0) - h(x')| + \sum_{i=1}^{n-1} |x'| |D_{i}\overline{h}(x',0) - D_{i}h(x')|$$

$$+ \sum_{i,j=1}^{n-1} |x'|^2 |D_{ij}\overline{h}(x',0) - D_{ij}h(x')| \leq C|x'|^{\kappa+1}, \tag{5.1}$$

for some constant $C = C(n, \lambda, \|a^{ij}\|_{C^{\kappa-1,1}(B^+_1(0))}, \|h\|_{C^{\kappa,1}(B'_1(0))}) \in (0, \infty)$.

Proof. We will prove the result only in the case $\kappa \geq 2$, and leave the easy modifications for the case $\kappa = 1$ to the reader. Express the Taylor polynomial of $\kappa$ of degree $(\kappa - 1)$ as

$$\sum_{k=0}^{\kappa-1} a_{k}^{ij}(x') x_{n}^{k}$$

for some polynomials $a_{k}^{ij}(x')$ on $\mathbb{R}^{n-1}$ of degree at most $\kappa - 1 - k$. Also, let

$$\overline{h}(x', x_{n}) = \sum_{k=0}^{\kappa} h_{k}(x') x_{n}^{k}$$

for some polynomials $h_{k}(x')$ on $\mathbb{R}^{n-1}$ of degree at most $\kappa - k$. We have that

$$D_{i}(a^{ij} D_{j}\overline{h}) = \sum_{k=0}^{\kappa-2} \sum_{\ell=0}^{k} \sum_{i,j=1}^{n-1} D_{i}(a_{k-\ell}^{ij}(x') D_{j}h_{\ell}(x')) x_{n}^{k} + \sum_{k=0}^{\kappa-2} \sum_{\ell=0}^{k} \sum_{i=1}^{n-1} (\ell + 1) D_{i}(a_{k-\ell}^{n}(x') h_{\ell+1}(x')) x_{n}^{k}$$

$$+ \sum_{k=0}^{\kappa-2} \sum_{\ell=0}^{k} (k + 1) a_{k-\ell+1}^{n}(x') D_{j}h_{\ell}(x') x_{n}^{k} + \sum_{k=0}^{\kappa-2} \sum_{\ell=0}^{k} (\ell + 1)(k + 1) a_{k-\ell+1}^{n}(x') h_{\ell+1}(x') x_{n}^{k}$$

$$+ \sum_{k=0}^{\kappa-2} (k + 2)(k + 1) a_{k}^{n}(x') h_{k+2}(x') x_{n}^{k} + E(x),$$

where $E(x)$ is an error term such that $|E(x)| \leq C|x|^{{\kappa-1}}$ for some constant $C = C(n, \|a^{ij}\|_{C^{\kappa-1,1}(B^+_1(0))}, \|h\|_{C^{\kappa,1}(B'_1(0))}) \in (0, \infty)$. Therefore (5.1) holds true if and only if $h_{0}(x')$ is the degree $\kappa$ Taylor polynomial of $h(x')$, $h_{1}(x') = 0$ on $\mathbb{R}^{n-1}$, and $h_{k+2}(x')$ satisfies the recurrence relation

$$h_{k+2}(x') = \frac{-1}{(k+1)(k+2)} a_{k}^{n}(x') \left( \sum_{\ell=0}^{k} \sum_{i,j=1}^{n-1} D_{i}(a_{k-\ell}^{ij}(x') D_{j}h_{\ell}(x'))ight.$$

$$+ \sum_{\ell=0}^{k} \sum_{i=1}^{n-1} (\ell + 1) D_{i}(a_{k-\ell}^{n}(x') h_{\ell+1}(x')) + \sum_{\ell=0}^{k+1} (k + 1) a_{k-\ell+1}^{n}(x') D_{j}h_{\ell}(x')$$

$$+ \sum_{\ell=0}^{k} (\ell + 1)(k + 1) a_{k-\ell+1}^{n}(x') h_{\ell+1}(x')) + E(x) \tag{5.2}$$

for $k = 0, 1, 2, \ldots, \kappa - 2$, where $E(x)$ is a polynomial such that $|E(x)| \leq C|x|^{\kappa-k-1}$ for some constant $C \in (0, \infty)$ (independent of $|x|$). In particular, observe that the recurrence relation (5.2) uniquely defines $h_{k+2}(x')$. \qed
Now recall that \( u \in W^{1,2}(B_1^+(0)) \) is a solution to (5.2) with \( \Omega = B_1^+(0) \) and \( \Gamma = B_1'(0) \), and that \( 0 \in \Sigma(u) \). Following [GP09], take an integer \( \kappa \geq 2 \) and let \( \tilde{h} = \tilde{h}_\kappa \) be as in Lemma 5.1. Set
\[
\tilde{h}(x', x_n) = \tilde{h}_\kappa(x', x_n) = h(x', x_n) - \overline{h}(x', 0) + h(x'),
\]
\[
v(x', x_n) = v_\kappa(x', x_n) = u(x', x_n) - \tilde{h}(x', x_n),
\]
\[
f = f_\kappa = - D_i(a^{ij} D_j \tilde{h})
\]
(5.3)
for each \((x', x_n) \in B_1^+(0)\). By (5.1),
\[
|f| \leq \left| \sum_{i,j=1}^n D_i (a^{ij} D_j \overline{h}(x', x_n)) \right| + \left| \sum_{i=1}^n \sum_{j=1}^{n-1} D_i (a^{ij} D_j (h(x') - \overline{h}(x', 0))) \right| \leq C |x|^{\kappa-1}
\]
(5.4)
in \( B_1^+(0) \) for some constant \( C = C(n, \lambda, \|a^{ij}\|_{C^{\kappa-1,1}(B_1^+(0))}, \|h\|_{C^{\kappa,1}(B_1'(0))}) \in (0, \infty) \). Since \( u \) is \( J \)-minimizing, \( v \) is a minimizer for the functional in \( w \in W^{1,2}(B_1^+(0)) \)
\[
J(w + \tilde{h}) = \frac{1}{2} \int_{B_1^+(0)} a^{ij} D_i (w + \tilde{h}) D_j (w + \tilde{h}) + \frac{1}{p} \int_{B_1'(0)} (k_+(w^+)^p + k_-(w^-)^p).
\]
Using (5.2), and the facts that \( D_i (a^{ij} D_j \tilde{h}) = -f \) weakly in \( B_1^+(0) \) and \( D_n \tilde{h} = 0 \) on \( B_1'(0) \) (since \( h_1(x') = 0 \) on \( \mathbb{R}^{n-1} \)), we obtain
\[
\frac{1}{2} \int_{B_1^+(0)} (a^{ij} D_i (w + \tilde{h}) D_j (w + \tilde{h}) - a^{ij} D_i (v + \tilde{h}) D_j (v + \tilde{h})) = \frac{1}{2} \int_{B_1^+(0)} a^{ij} D_i (w - v) D_j (w + v + 2 \tilde{h}) = \frac{1}{2} \int_{B_1^+(0)} (a^{ij} D_i w D_j w - a^{ij} D_i v D_j v + 2 (w - v) f)
\]
for all \( w \in W^{1,2}(B_1^+(0)) \) with \( w = v \) on \( (\partial B_1(0))^+ \). Hence \( v \) is a minimizer for the functional in \( w \in W^{1,2}(B_1^+(0)) \)
\[
\tilde{J}(w) = \frac{1}{2} \int_{B_1^+(0)} (a^{ij} D_i w D_j w + 2 w f) + \frac{1}{p} \int_{B_1'(0)} (k_+(w^+)^p + k_-(w^-)^p).
\]
By a standard variational argument (see for instance the proof of Lemma 4.1 of [ALP15]),
\[
\int_{B_1^+(0)} (a^{ij} D_j v D_i \zeta + f \zeta) + \int_{B_1'(0)} (k_+(v^+)^{p-1} - k_-(v^-)^{p-1}) \zeta = 0
\]
(5.6)
for each \( \zeta \in C^1_c(B_1^+(0) \cup B_1'(0)) \). That is, \( v \) is a weak solution to
\[
D_i (a^{ij} D_j v) = f \text{ in } B_1^+(0),
\]
\[
a^{mn} D_n v = k_+(v^+)^{p-1} - k_-(v^-)^{p-1} \text{ on } B_1'(0).
\]
(5.7)

6 Almgren’s monotonicity formula

Throughout this section, we shall assume the following:
Hypothesis 6.1. Let $2 \leq p < \infty$ and $\kappa$ be a positive integer. Let $a^{ij} \in C^{0,1}(B_1^+(0) \cup B'_1(0))$ such that $a^{ij} = a^{ij}$ on $B_1^+(0)$, \((1.1)\) holds true for some constants $0 < \lambda \leq \Lambda < \infty$, and satisfying \((1.1)\) and \((1.2)\). Let $f \in L^\infty(B_1^+(0))$ such that
\[
\sup_{x \in B_1^+} |x|^{1-\kappa} |f(x)| < \infty.
\] (6.1)
Let $k_+, k_- \in C^{0,1}(B_1^+(0))$ such that $k_+, k_- \geq 0$ on $B'_1(0)$.

We will show that a frequency function of Almgren’s type for $v$ is monotone nondecreasing by suitably modifying an approach from [GSL1]. It will be convenient to define $\mu$ for each $\zeta$ by
\[
\mu(x) = \sum_{i,j=1}^n a^{ij}(x) \frac{x_i x_j}{|x|^2} = 1 + \sum_{i,j=1}^n b^{ij}(x) \frac{x_i x_j}{|x|^2}
\] (6.3)
for all $x \in B_1^+(0) \cup B'_1(0)$. By \((1.1)\),
\[
\lambda \leq \mu(x) \leq \Lambda
\] (6.4)
on $B_1^+(0) \cup B'_1(0)$. Recalling that $a^{ij}(0) = \delta_{ij}$,
\[
|b^{ij}(x)| \leq C|x|, \quad |Db^{ij}(x)| \leq C, \quad |\mu(x) - 1| \leq C|x|, \quad |D\mu(x)| \leq C
\] (6.5)
on $B_1^+(0) \cup B'_1(0)$, for some constant $C = C(n, \|Da^{ij}\|_{L^\infty(B_1^+)}) \in (0, \infty)$. The relevant Almgren’s frequency function is defined as follows.

Definition 6.1. Assume Hypothesis \([6.1]\) and let $v \in C^1(B_1^+(0) \cup B'_1(0))$ be a solution to \((5.7)\). We let
\[
H(\rho) = H_v(\rho) = \rho^{1-n} \int_{(\partial B_1^+(0))^+} \mu v^2,
\] (6.6)
\[
I(\rho) = I_v(\rho) = \rho^{-2-n} \int_{(\partial B_1^+(0))^+} a^{ij} v D_i v \frac{x_j}{\rho},
\] (6.7)
\[
D(\rho) = D_v(\rho) = \rho^{-2-n} \int_{B_1^+(0)} a^{ij} D_i v D_j v + \frac{2}{p} \rho^{-2-n} \int_{B'_1(0)} (k_+(v^+)^p + k_-(v^-)^p)
\] (6.8)
for each $0 < \rho < 1$, where $\mu$ is as in \((6.3)\).

Lemma 6.1. Assume Hypothesis \([6.1]\) and let $v \in C^1(B_1^+(0) \cup B'_1(0))$ be a solution to \((5.7)\). Then
\[
\int_{B_1^+(0)} (a^{ij} D_i v D_j v \zeta + a^{ij} v D_j v D_i \zeta + v f \zeta) + \int_{B_1^+(0)} (k_+(v^+)^p + k_-(v^-)^p) \zeta = 0
\] (6.9)
for each $\zeta \in C_c^1(B_1^+ \cup B'_1)$. Moreover,
\[
\int_{B_1^+(0)} (a^{ij} D_i v D_j v \text{div} \zeta - 2 a^{ij} D_j v D_k v D_i \zeta^k + D_k a^{ij} D_i v D_j v \zeta^k - 2 D_i v f \zeta) + \frac{2}{p} \int_{B_1^+(0)} \text{div}_{B_1^+(0)} (k_+ \zeta) (v^+)^p + \text{div}_{B'_1(0)} (k_- \zeta) (v^-)^p = 0
\] (6.10)
for each $\zeta = (\zeta^1, \ldots, \zeta^n) \in C_c^1(B_1^+ \cup B'_1, \mathbb{R}^n)$ with $\zeta^n = 0$ on $B'_1$, where $\text{div}$ and $\text{div}_{B_1^+}$ denote the divergence operators on $B_1^+$ and $B'_1$ respectively.
Proof. Replacing \( \zeta \) with \( v \zeta \) in (5.6) and noting that \( \pm(v^\pm)^{p-1}v = (v^\pm)^p \) on \( B'_1 \) gives us (5.9). To obtain (6.10), notice that since \( v \) is \( \tilde{J} \)-minimizing (where \( \tilde{J} \) is as in (5.3)),

\[
\frac{d}{dt} \tilde{J}(v(x + t\zeta(x))) \bigg|_{t=0} = \frac{d}{dt} \left( \frac{1}{2} \int_{B'_1} a^{ij}(x) D_k v(x + t\zeta(x)) (\delta_{ik} + tD_i \zeta^k(x)) D_l v(x + t\zeta(x)) (\delta_{jl} + tD_j \zeta^l(x)) \, dx \right)
\]

\[
+ \int_{B'_1} v(x + t\zeta(x)) f(x) + \frac{1}{p} \int_{B'_1} k_+(x) (v(x + t\zeta(x))^+)^p \, dx
\]

\[
+ \frac{1}{p} \int_{B'_1} k_-(x) (v(x + t\zeta(x))^-)^p \, dx \bigg|_{t=0}
\]

(6.11)

for all \( \zeta = (\zeta^1, \ldots, \zeta^n) \in C^1_c(B'_1 \cup B'_2, \mathbb{R}^n) \). We have that

\[
\frac{d}{dt} \int_{B'_1} v(x + t\zeta(x)) f(x) \, dx \bigg|_{t=0} = \int_{B'_1} D_t v f \zeta^i.
\]

For the remaining terms in (6.11), we use the change of variables \( y = x + t\zeta(x) \) and note that \( x = y - t\zeta(y) + O(t^2) \). By the dominated convergence theorem,

\[
\frac{d}{dt} \int_{B'_1} a^{ij}(x) D_k v(x + t\zeta(x)) (\delta_{ik} + tD_i \zeta^k(x)) D_l v(x + t\zeta(x)) (\delta_{jl} + tD_j \zeta^l(x)) \, dx \bigg|_{t=0}
\]

\[
= \int_{B'_1} (a^{ij} D_i v D_j v \div \zeta - 2 a^{ij} D_j v D_k v D_i \zeta^k + D_k a^{ij} D_i v D_j v \zeta^k)
\]

and

\[
\frac{d}{dt} \int_{B'_1} (k_+(x) (v(x + t\zeta(x))^+)^p + k_-(x) (v(x + t\zeta(x))^-)^p) \, dx \bigg|_{t=0}
\]

\[
= \int_{B'_1} (k_+(y - t\zeta(y))^+(y)^+ + k_-(y - t\zeta(y))^-(y)^-)(1 - t \div B'_1 \zeta(y)) \, dy \bigg|_{t=0}
\]

\[
= - \int_{B'_1} \left( (k_+ \div B'_1 \zeta + \nabla B'_1 k_+ \cdot \zeta) (v^+)^p + (k_- \div B'_1 \zeta + \nabla B'_1 k_- \cdot \zeta) (v^-)^p \right)
\]

\[
= - \int_{B'_1} (\div B'_1 (k_+ \zeta) (v^+)^p + \div B'_1 (k_- \zeta) (v^-)^p),
\]

where \( \nabla B'_1 \) denotes the gradient on \( B'_1 \), and thus (6.10) holds true. \( \square \)

Lemma 6.2. Let \( a^{ij} \in C^{0,1}(B'_1(0) \cup B'_2(0)) \) such that \( a^{ij} = a^{ji} \) on \( B'_1(0) \). (1.1) holds true for some constants \( 0 < \lambda \leq \Lambda < \infty \), and satisfying (4.1) and (4.2). Let \( w \in W^{1,2}(B'_1) \) and

\[
H_w(\rho) = \rho^{1-n} \int_{(\partial B'_0)^+} \mu w^2
\]

(6.12)

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for each \(0 < \rho < 1\), where \(\mu\) is as in \((6.3)\). Then \(H_w\) is absolutely continuous as a function of \(\rho \in (0,1)\) and

\[
H'_w(\rho) = \frac{2}{\rho} I(\rho) + \rho^{1-n} \int_{(\partial B_\rho)^+} w^2 \left( (1 - n) \frac{\mu - 1}{\rho} + D_i \left( \frac{b^{ij} x_j}{|x|} \right) \right) \tag{6.13}
\]

for \(L^1\)-a.e. \(0 < \rho < 1\), where \(b^{ij}\) is as in \((6.3)\).

**Proof.** By the divergence theorem, \((5.2)\), \((6.3)\), and \((6.2)\),

\[
\int_{(\partial B_\rho)^+} \mu w^2 = \int_{(\partial B_\rho)^+} a^{ij} x_i x_j \frac{|x|^2}{|x|^2} w^2 = \int_{B_\rho} a^{ij} x_j |x|^2 w^2 + \int_{B_\rho^*} \left( 2a^{ij} w D_i w \frac{x_j}{|x|} + w^2 D_i \left( \frac{a^{ij} x_j}{|x|} \right) \right)
\]

\[
= \int_{B_\rho^*} \left( 2a^{ij} w D_i w \frac{x_j}{|x|} + w^2 \frac{n-1}{|x|^2} + w^2 D_i \left( \frac{b^{ij} x_j}{|x|} \right) \right).
\]

Hence by the coarea formula, \(H_w(\rho)\) is absolutely continuous and

\[
\frac{\partial}{\partial \rho} \left( \int_{(\partial B_\rho)^+} \mu w^2 \right) = (1 - n) \rho^{-n} \int_{(\partial B_\rho)^+} \mu w^2 + \rho^{1-n} \frac{d}{d\rho} \left( \int_{(\partial B_\rho)^+} \mu w^2 \right)
\]

\[
= 2\rho^{1-n} \int_{(\partial B_\rho)^+} a^{ij} w D_i w \frac{x_j}{\rho} + \rho^{1-n} \int_{(\partial B_\rho)^+} w^2 \left( (1 - n) \frac{\mu - 1}{\rho} + D_i \left( \frac{b^{ij} x_j}{|x|} \right) \right)
\]

for \(L^1\)-a.e. \(0 < \rho < 1\).

**Lemma 6.3.** Assume Hypothesis \((6.7)\) and let \(v \in C^1(B_1^+ \cup B_1^-)\) be a solution to \((5.7)\). Let \(I(\rho)\) and \(D(\rho)\) be as in Definition \((6.7)\). Then \(D(\rho)\) and \(I(\rho)\) are locally absolutely continuous functions of \(\rho \in (0,1)\), with

\[
D(\rho) = I(\rho) - \rho^{2-n} \int_{B_\rho^*} v f - \frac{p-2}{p} \rho^{2-n} \int_{B_\rho^-} (k_+(v^+)^p + k_-(v^-)^p) \tag{6.14}
\]

for all \(0 < \rho \leq 1\) and

\[
D'(\rho) \geq 2\rho^{2-n} \int_{(\partial B_\rho)^+} \frac{1}{\mu} \left( \sum_{i,j=1}^n a^{ij} D_j v \frac{x_i}{\rho} \right)^2 - 2\rho^{1-n} \int_{B_\rho^*} \frac{1}{\mu} a^{ij} D_j v f x_i
\]

\[
+ \frac{2}{p} \rho^{1-n} \int_{B_\rho^-} \frac{1}{\mu} a^{ij} x_i (D_j k_+(v^+)^p + D_j k_-(v^-)^p) - CD(\rho) \tag{6.15}
\]

for \(L^1\)-a.e. \(0 < \rho < 1\), where \(\mu\) is as in \((6.3)\) and \(C = C(n, \lambda, \|Da^{ij}\|_{L^\infty(B_1^+)}) \in (0,\infty)\) is a constant.

**Proof.** By \((6.9)\) with \(\zeta\) approximating the characteristic function \(1_{B_\rho^+}\) on \(B_\rho^+\),

\[
\int_{B_\rho^+} (a^{ij} D_i v D_j v + v f) - \int_{(\partial B_\rho)^+} a^{ij} v D_j v \frac{x_i}{|x|} + \int_{B_\rho^-} (k_+(v^+)^p + k_-(v^-)^p) = 0,
\]

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which after multiplying by \( \rho^{2-n} \) and rearranging terms gives us \( 6.14 \). Let \( Z = (Z^1, Z^2, \ldots, Z^n) \in C^{0,1}(B^+_1 \cup B^+_1, \mathbb{R}^n) \) be the vector field defined by

\[
Z^i(x) = \frac{1}{\mu(x)} \sum_{j=1}^{n} a^i_j(x) x_j.
\]

By the definition of \( \mu \) in \( 6.3 \) and by \( 6.4 \) and \( 6.5 \),

\[
\sum_{i=1}^{n} Z^i(x) x_i = \frac{1}{\mu} \sum_{i,j=1}^{n} a^i_j(x) x_i x_j = r^2,
\]

(6.16)

| \( D_i Z^k(x) - \delta_{ik} \) | \( \leq C|x|, \quad |\text{div} Z(x) - \mu| \leq C|x| \) (6.17)

on \( B^+_1 \cup B^+_1 \), where \( C = C(n, \lambda, \|Da^{ij}\|_{L^\infty(B^+_1)}) \in (0, \infty) \) is a constant. By \( 6.10 \) with \( \zeta(x) \) approximating \( 1_{B^+_1}(x)Z(x) \) and using \( 6.16 \), see also [GST14 LemmaA.9],

\[
\int_{B^+_1} (a^{ij} D_i v D_j v \text{ div } Z - 2 a^{ij} D_i v D_k v D_j v Z^k + D_k a^{ij} D_i v D_j v Z^k - 2 D_i v f Z^i) + \frac{2}{p} \int_{B^+_1} (k_+(v^+)p + k_-(v^-)p) \text{ div } Z
\]

\[
+ \frac{2}{p} \int_{B^+_1} (Z \cdot Dk_+(v^+)p + Z \cdot Dk_+(v^-)p) = \rho \int_{(\partial B^+_1)^+} a^{ij} D_i v D_j v - 2 \rho \int_{(\partial B^+_1)^+} \frac{1}{\mu} \left( \sum_{i,j=1}^{n} a^{ij} D_i v \frac{x_i}{\rho} \right)^2 + \frac{2}{p} \rho \int_{(\partial B^+_1)^+} (k_+(v^+)p + k_-(v^-)p).
\]

Hence by \( 6.17 \), together with \( \lambda |Dv|^2 \leq a^{ij} D_i v D_j v \) on \( B^+_1 \) by \( 1.1 \), we have that

(\text{some constant } C \in (0, \infty) \text{ depending only on } n, \lambda, \text{ and } \|Da^{ij}\|_{L^\infty(B^+_1)}). \text{ On the other hand, by the coarea formula } D(\rho) \text{ is locally absolutely continuous on } (0, 1) \text{ and}

\[
D'(\rho) = (2 - n) \rho^{1-n} \int_{B^+_1} a^{ij} D_i v D_j v + \rho^{2-n} \int_{\partial B^+_1} a^{ij} D_i v D_j v
\]

(6.18)

\[
+ \frac{2(2-n)}{p} \rho^{1-n} \int_{B^+_1} (k_+(v^+)p + k_-(v^-)p) + \frac{2}{p} \rho^{2-n} \int_{\partial B^+_1} (k_+(v^+)p + k_-(v^-)p)
\]

(6.19)

for \( L^1 \)-a.e. \( 0 < \rho < 1 \). Combining \( 6.18 \) and \( 6.19 \) gives us \( 6.13 \). Note that by \( 6.14 \), since \( D(\rho) \) is absolutely continuous on \((0, 1)\), \( I(\rho) \) must also be absolutely continuous on \((0, 1)\). \( \square \)

**Lemma 6.4.** Assume Hypothesis \( 6.7 \) and let \( v \in C^1(B^+_1 \cup B^+_1) \) be a solution to \( 5.7 \). Then

\[
\rho^{-n} \int_{B^+_1} v^2 \leq CH(\rho) + C\rho^{2n+2}
\]

(6.20)
for all $0 < \rho < 1$, where $H(\rho)$ is as in (6.6) and $C \in (0, \infty)$ is a constant depending only on $n$, $\lambda$, $\|D_{\rho}^{ij}\|_{L^\infty(B_1^+)}$, and $\sup_{B_1^+} |x|^1 |f|$.

**Proof.** We compute that

$$
\frac{d}{d\rho} \left( \int_{\partial B_\rho^+} \mu v^2 \right) = \frac{d}{d\rho} (\rho^{n-1} H(\rho)) = (n - 1) \rho^{n-2} H(\rho) + \rho^{n-1} H'(\rho) \geq \rho^{n-1} H'(\rho)
$$

(6.21) for $\mathcal{L}^1$-a.e. $0 < \rho < 1$. By (6.13) with $w = v$, (6.9), and (6.4),

$$
\frac{d}{d\rho} \left( \int_{\partial B_\rho^+} \mu v^2 \right) \geq 2 \rho^{n-2} I(\rho) - C \int_{\partial B_\rho^+} \mu v^2.
$$

(6.22) for $\mathcal{L}^1$-a.e. $0 < \rho < 1$, where $C \in (0, \infty)$ is a constant depending only on $n$, $\lambda$, and $\|D_{\rho}^{ij}\|_{L^\infty(B_1^+)}$. By (6.13),

$$
\frac{d}{d\rho} \left( \int_{\partial B_\rho^+} \mu v^2 \right) \geq 2 \int_{B_\rho^+} f v - C \int_{\partial B_\rho^+} \mu v^2
$$

for $\mathcal{L}^1$-a.e. $0 < \rho < 1$, where $C \in (0, \infty)$ is a constant depending only on $n$, $\lambda$, and $\|D_{\rho}^{ij}\|_{L^\infty(B_1^+)}$. Multiplying by sides of (6.22) by $e^{C\rho}$ and rearranging terms,

$$
\frac{d}{d\rho} \left( e^{C\rho} \int_{\partial B_\rho^+} \mu v^2 \right) \geq 2 e^{C\rho} \int_{B_\rho^+} f v
$$

for $\mathcal{L}^1$-a.e. $0 < \rho < 1$. Integrating over $[\sigma, \rho]$ gives us

$$
e^{C\rho} \int_{\partial B_\rho^+} \mu v^2 \geq \int_{\partial B_\rho^+} \mu v^2 - 2 e^{C\rho} \rho \int_{B_\rho^+} |f| |v|
$$

for all $0 < \sigma \leq \rho < 1$. Integrating over $\sigma \in (0, \rho]$ gives us

$$
e^{C\rho} \rho \int_{\partial B_\rho^+} \mu v^2 \geq \int_{B_\rho^+} \mu v^2 - 2 e^{C\rho} \rho^2 \int_{B_\rho^+} |f| |v|
$$

for all $0 < \rho \leq 1$. By applying the Cauchy-Schwarz inequality,

$$
\int_{B_\rho^+} \mu v^2 \leq 2 e^{C\rho} \rho \int_{\partial B_\rho^+} \mu v^2 + 4 e^{2C\rho} \rho^4 \int_{B_\rho^+} \frac{f^2}{\mu}.
$$

Multiplying by $\rho^{-n}$ and applying (6.1) and (6.4) gives us (6.20). \qed

We now follow an approach introduced in [GS14] to define the relevant Almgren frequency function. Recalling (6.13) (with $w = v$), for each $0 < \rho < 1$ we define $G(\rho) = G_v(\rho)$ by

$$
G(\rho) = \rho^{1-n} H(\rho) \int_{\partial B_\rho^+} v^2 \left( (1 - n) \frac{\mu - 1}{\rho} + D_i \left[ b^{ij} \frac{f_j}{x_j} \frac{x_j}{|x|} \right] \right)
$$

(6.23)

if $H(\rho) > 0$ and $G(\rho) = 0$ if $H(\rho) = 0$. By (6.4) and (6.5), $G(\rho)$ is a bounded function on $(0, 1)$ with

$$
|G(\rho)| \leq C
$$

(6.24)
for some constant $C = C(n, \lambda, \|Da^{ij}\|_{L^\infty(B_1^+)} \in (0, \infty)$. Thus, we may define $K(\rho) = K_v(\rho)$ by

$$K(\rho) = H(\rho) \exp \left(-\int_0^\rho G(\tau) \, d\tau \right)$$  \hspace{1cm} \text{(6.25)}

for all $0 < \rho \leq 1$. By (6.24),

$$\frac{1}{C} H(\rho) \leq K(\rho) \leq C H(\rho)$$  \hspace{1cm} \text{(6.26)}

for all $0 < \rho < 1$ and some constant $C = C(n, \lambda, \|Da^{ij}\|_{L^\infty(B_1^+)} \in [1, \infty)$. By (6.13) with $w = v$, we have that

$$\frac{\rho K'(\rho)}{2K(\rho)} = \frac{\rho H'(\rho)}{2H(\rho)} - \frac{\rho}{2} G(\rho) = \frac{I(\rho)}{H(\rho)}$$  \hspace{1cm} \text{(6.27)}

for $\mathcal{L}^1$-a.e. $0 < \rho < 1$ with $H(\rho) > 0$. We introduce the $\mathcal{L}^1$-measurable function $\Phi(\rho) = \Phi_v(\rho)$, called the Almgren frequency function, as

$$\Phi(\rho) = \frac{\rho}{2} \frac{d}{d\rho} \log \max \{K(\rho), \rho^{2\kappa}\}$$  \hspace{1cm} \text{(6.28)}

for $\mathcal{L}^1$-a.e. $0 < \rho < 1$.

We are now ready to prove the main result in this section, i.e. the almost-monotonicity of the Almgren frequency function.

**Theorem 6.1.** Assume Hypothesis 6.1 and let $v \in C^1(B_1^+ \cup B'_1)$ be a solution to (5.7). There exists constants $\rho_0 \in (0, 1/2)$ and $C \in (0, \infty)$ depending only on $n$, $p$, $\lambda$, $\|a^{ij}\|_{C^{0,1}(B_1^+)}$, $\sup_{B_1^+} |x|^{-\kappa}|f|$, $\|k_+\|_{C^{0,1}(B'_1)}$, $\|v\|_{L^\infty(B_1)}$ such that the function

$$e^{C\sigma} \Phi(\sigma) + Ce^C \sigma \leq e^{C\rho} \Phi(\rho) + Ce^C \rho$$  \hspace{1cm} \text{(6.29)}

is nondecreasing for all $0 < \rho \leq \rho_0$.

**Remark 6.1.** Notice that $\Phi$ is a measurable function defined only up to a set of $\mathcal{L}^1$-measure zero. We take Theorem 6.1 to assert that there is a nondecreasing function which equals $\Phi$ $\mathcal{L}^1$-a.e. on $(0, 1)$. To be more precise, by (6.27),

$$\Phi(\rho) = \frac{I(\rho)}{H(\rho)}$$  \hspace{1cm} \text{(6.30)}

on the open set $\{\rho \in (0, 1) : K(\rho) > \rho^{2\kappa}\}$, whereas $\Phi(\rho) = \kappa$ for $\mathcal{L}^1$-a.e. $\rho \in (0, 1)$ with $K(\rho) \leq \rho^{2\kappa}$. Thus $\Phi(\rho) = \overline{\Phi}(\rho)$ for $\mathcal{L}^1$-a.e. $\rho \in (0, 1)$ where $\overline{\Phi} : (0, 1) \to [0, \infty)$ is the function defined by

$$\overline{\Phi}(\rho) = \begin{cases} I(\rho)/H(\rho) & \text{if } K(\rho) > \rho^{2\kappa} \\ \kappa & \text{if } K(\rho) \leq \rho^{2\kappa} \end{cases}$$

for each $\rho \in (0, 1)$. We can take Theorem 6.1 to mean that $\rho \mapsto e^{C\overline{\Phi}(\rho)} + Ce^C \rho$ is monotone nondecreasing on $(0, 1)$.

**Proof of Theorem 6.1.** Let's take any open interval $(\alpha, \beta) \subset \{\rho \in (0, \rho_0) : K(\rho) > \rho^{2\kappa}\}$ and show that $e^{C\overline{\Phi}(\rho)} + Ce^C \rho$ is monotone nondecreasing on $(\alpha, \beta)$. Unless otherwise stated, throughout the proof we will let $C$ denote positive constants depending only on $n$, $p$, $\lambda$, $\|a^{ij}\|_{C^{0,1}(B_1^+)}$, $\sup_{B_1^+} |x|^{-\kappa+1}|f|$, $\|k_+\|_{C^{0,1}(B'_1)}$, $\|v\|_{L^\infty(B_1)}$. Note that by (5.1), $\sup_{B_1^+} |x|^{-\kappa+1}|f| < \infty$. By (6.26),

$$\rho^{2\kappa} < K(\rho) \leq CH(\rho)$$  \hspace{1cm} \text{(6.31)}
for each $\alpha < \rho < \beta$. By (6.20) and (6.31),

$$
\rho^{-n} \int_{B_\rho^+} v^2 \leq CH(\rho) + C\rho^{2n+2} \leq CH(\rho)
$$

(6.32)

for all $\alpha < \rho < \beta$. Using (1.1), (6.4), (6.31), (6.32), and the Cauchy-Schwarz inequality,

$$
\left| \rho^{-n} \int_{B_\rho^+} f v \right| \leq \sup_{B_\rho^+} |f| \left( \rho^{-n} \int_{B_\rho^+} v^2 \right)^{1/2} \leq C\rho^{\kappa-1} H(\rho)^{1/2} \leq \frac{C}{\rho} H(\rho),
$$

(6.33)

$$
\left| \rho^{-n} \int_{B_\rho^+} \frac{1}{\mu} a^{ij} D_j f x_i \right| \leq C \sup_{B_\rho^+} |f| \left( \rho^{2n-2} \int_{B_\rho^+} |Dv|^2 \right)^{1/2} \leq C\rho^{\kappa-1} D(\rho)^{1/2}
$$

(6.34)

$$
\left| \rho^{1-n} \int_{(\partial B_\rho)^+} f v \right| \leq C \sup_{B_\rho^+} |f| \left( \rho^{1-n} \int_{(\partial B_\rho)^+} v^2 \right)^{1/2} \leq C\rho^{\kappa-1} H(\rho)^{1/2} \leq \frac{C}{\rho} H(\rho),
$$

(6.35)

for all $\alpha < \rho < \beta$. By [DJ] Lemma 4.2,

$$
\rho^{1-n} \int_{B_\rho} v^2 \leq CH(\rho) + CD(\rho)
$$

for all $\alpha < \rho < \beta$ and some constant $C = C(n) \in (0, \infty)$. Moreover, since $p \geq 2$ we have

$$
\rho^{-n} \int_{B_\rho^+} (k_+ (v^+) + k_- (v^{-})) + \rho^{2-n} \int_{B_\rho^+} \frac{1}{\mu} a^{ij} x_i (D_j k_+ (v^+)) (D_j k_- (v^{-}))^p \leq CH(\rho) + CD(\rho)
$$

(6.36)

for all $\alpha < \rho < \beta$. By (6.24) and (6.27),

$$
\left| H'(\rho) - \frac{2}{\rho} I(\rho) \right| \leq CH(\rho)
$$

(6.37)

for all $\alpha < \rho < \beta$ and some constant $C \in (0, \infty)$ depending only on $n$, $\lambda$, and $\|D a^{ij}\|_{L^\infty(B_1)}$. By (6.14) and (6.15),

$$
I'(\rho) \geq 2\rho^{2-n} \int_{(\partial B_\rho)^+} \frac{1}{\mu} \left( a^{ij} D_j v x_i \rho \right)^2 - 2\rho^{1-n} \int_{B_\rho^+} \frac{1}{\mu} a^{ij} D_j v f x_i + (2-n)\rho^{1-n} \int_{B_\rho^+} v f
$$

$$
+ \rho^{2-n} \int_{(\partial B_\rho)^+} v f + \frac{2}{p} \rho^{1-n} \int_{B_\rho^+} \frac{1}{\mu} a^{ij} x_i (D_j k_+ (v^+)) (D_j k_- (v^{-}))^p
$$

$$
- \frac{(p-2)(n-2)}{p} \rho^{1-n} \int_{B_\rho^+} (k_+ (v^+) + k_- (v^{-})) + \rho^{2-n} \int_{(\partial B_\rho)^+} (k_+ (v^+) + k_- (v^{-})) - CD(\rho)
$$

(6.38)

for all $0 < \rho < 1$. Thus, by keeping in mind that $p \geq 2$ and applying (6.33)-(6.36), we obtain

$$
I'(\rho) \geq 2\rho^{2-n} \int_{(\partial B_\rho)^+} \frac{1}{\mu} \left( a^{ij} D_j v x_i \rho \right)^2 - CH(\rho) - CD(\rho)
$$

(6.38)
for $\mathcal{L}^1$-a.e. $\alpha < \rho < \beta$. By (6.33) and (6.37),
\[
\Phi'\rho = \frac{H\rho I'(\rho) - H'(\rho) I\rho}{H\rho^2}
\geq 2\rho^{\beta-2n}\frac{1}{H\rho^2} \left( \int_{\partial B_\rho} \mu v^2 \right) \left( \int_{\partial B_\rho} \frac{1}{\rho} \left( a^{ij} D_j v \frac{x_i}{\rho} \right)^2 \right) - C \frac{H\rho + D\rho + |I\rho|}{H\rho}.
\]
(6.39)

Using (6.33) and (6.36), (6.14) gives us
\[
(1 - C \rho) D\rho - C \rho H\rho \leq I\rho \leq (1 + C \rho) D\rho + C \rho H\rho
\]
(6.40)
for all $\alpha < \rho < \beta$. Thus provided $\rho \leq \rho_0$ for $\rho_0 < 1/(2C)$, $D\rho \leq 2I\rho + 2C \rho H\rho$. Thus if $\alpha < \rho < \beta$ is such that $I\rho \geq 0$, then (6.40) gives us
\[
\Phi'\rho \geq -C (\Phi\rho + 1).
\]
(6.41)

If instead $I\rho < 0$, then $|I\rho| = -I\rho \leq C \rho H\rho$ and $D\rho \leq C \rho H\rho$ for all $\alpha < \rho < \beta$. Thus (6.39) gives us $\Phi\rho \geq -C \rho$ and $\Phi'\rho \geq -C$. In either case we conclude that (6.41) holds true for all $\alpha < \rho < \beta$. Hence, letting $C$ be a constant such that (6.41) holds true and $\Phi\rho \geq -C \rho$ for all $\alpha < \rho < \beta$, we have that
\[
\frac{d}{d\rho} \left( e^{2C \rho} \Phi\rho + 2C e^{2C \rho} \right) \geq C e^{2C \rho} (\Phi\rho - 1) + 2C e^{2C} \geq -C e^{2C \rho} (\Phi\rho + 1) + 2C e^{2C} \geq 0
\]
(6.42)
for all $\alpha < \rho < \beta$ provided $\rho \leq \rho_0$ for $\rho_0 < 1/C$. This gives us (6.29) with $2C$ in place of $C$.

Now let $0 < \sigma < \rho \leq \rho_0$. We want to show that (6.29) holds true, where we use the convention that $\Phi\tau = \kappa$ if $K\tau \leq \tau^{2\kappa}$ for $0 < \tau \leq \rho_0$ (see Remark 6.1). In the case that $\sigma$ and $\rho$ belong to the same connected component of the open set $\{ \tau \in (0, \rho_0) : K\tau > \tau^{2\kappa} \}$, we showed above that (6.29) holds true. Suppose that $\sigma$ belongs to the connected component $(\alpha, \beta)$ of $\{ \tau \in (0, \rho_0) : K\tau > \tau^{2\kappa} \}$. Then $K\tau > \tau^{2\kappa}$ for all $\alpha < \tau < \beta$ and $K\tau = \tau^{2\kappa}$ for $\tau = \alpha, \beta$. Hence,
\[
\Phi\alpha^+ \geq \kappa, \quad \Phi\beta^- \leq \kappa.
\]
(6.43)

If $\rho$ belongs to the connected component $(\gamma, \delta)$ of $\{ \tau \in (0, \rho_0) : K\tau > \tau^{2\kappa} \}$ we have $\alpha < \sigma < \beta \leq \gamma < \rho < \delta$ and in addition to (6.43) we have $\Phi\gamma^+ \geq \kappa$. Thus
\[
e^{C\sigma} \Phi\sigma + C e^{C\sigma} \sigma \leq e^{C\beta} \Phi\beta^- + C e^{C\beta} \beta \leq e^{C\beta} \kappa + C e^{C\beta} \gamma \leq e^{C\rho} \Phi\rho + C e^{C\rho} \rho,
\]
proving (6.29). If instead $K\rho \leq \rho^{2\kappa}$, then we have $\alpha < \sigma < \beta \leq \rho$ and thus using (6.43)
\[
e^{C\sigma} \Phi\sigma + C e^{C\sigma} \sigma \leq e^{C\beta} \Phi\beta^- + C e^{C\beta} \beta \leq e^{C\beta} \kappa + C e^{C\beta} \gamma = e^{C\rho} \Phi\rho + C e^{C\rho} \rho,
\]
again proving (6.29). Similarly, if $K\sigma \leq \sigma^{2\kappa}$ and $K\rho > \rho^{2\kappa}$, (6.29) holds true. In the case that $K\sigma \leq \sigma^{2\kappa}$ and $K\rho \leq \rho^{2\kappa}$, then $\Phi\sigma = \Phi\rho = \kappa$ and thus (6.29) holds true.

7 Growth rates at free boundary points

Assume that $2 \leq \rho < \infty$. Let $u \in W^{1,2}(B^+_1(0))$ be a weak solution to (1.2) and $0 \in \Sigma(u)$. For each positive integer $\kappa$, let $v_\kappa$ be as in (5.3) and $\Phi v_\kappa$ be as in (6.28). By Theorem 6.1 for each integer $\kappa$ the limit
\[
\Phi v_\kappa(0^+) = \lim_{\rho \downarrow 0} \Phi v_\kappa(\rho)
\]
28
exists. However, $\Phi_{\nu}(0^+)$ might depend on $\kappa$. In Lemma 6.2, we show that $\Phi_{\nu}(0^+) = \min\{\Phi_{v}(0^+), \kappa\}$ for integers $\kappa < \lambda$. It follows in Definition 7.1 that we can define the Almgren frequency $N_u(0)$ of $u$ at the origin which is independent of $\kappa$, though possibility infinite.

**Lemma 7.1.** Assume Hypothesis 6.2 and let $v \in C^1(B_1^+ \cup B_1^-)$ be a solution to (5.7). Let $H_\nu$ be as in (6.6) and $K_\nu$ be as in (6.25). There exists $\rho_0 > 0$ depending only on $n$, $p$, $\lambda$, $\|a^{ij}\|_{C^{0,1}(B_1)}$, $\|k_+\|_{C^{0,1}(B_1^+)}$, $\|k_-\|_{C^{0,1}(B_1^-)}$, and $\|v\|_{L^\infty(B_1)}$ such that the following holds true.

(a) If $0 < \alpha < \sigma < \rho \leq \rho_0$ such that $K_\nu(\tau) \geq \tau^{2\kappa}$ for all $\alpha \leq \tau \leq \rho$, then

$$
\left(\frac{\sigma}{\rho}\right)^{2c^{\kappa} \Phi_\nu(\rho) + 2C^{\kappa}} \Phi_\nu(\rho) \leq K_\nu(\sigma) \leq e^{2c^{\kappa} \Phi_\nu(\alpha) + e^C} \left(\frac{\sigma}{\rho}\right)^{2\Phi_\nu(\alpha)} K_\nu(\rho), \tag{7.1}
$$

$$
\frac{1}{C} \left(\frac{\sigma}{\rho}\right)^{2c^{\kappa} \Phi_\nu(\rho) + 2C^{\kappa}} \Phi_\nu(\rho) \leq H_\nu(\rho) \leq Ce^{2c^{\kappa} \Phi_\nu(\alpha) + e^C} \left(\frac{\sigma}{\rho}\right)^{2\Phi_\nu(\alpha)} H_\nu(\rho), \tag{7.2}
$$

where $C \in (0, \infty)$ is a constant depending only on $n$, $p$, $\lambda$, $\|a^{ij}\|_{C^{0,1}(B_1^+)}$, $\|k_+\|_{C^{0,1}(B_1^+)}$, $\|k_-\|_{C^{0,1}(B_1^-)}$, and $\|v\|_{L^\infty(B_1)}$.

(b) If $0 < \sigma < \rho \leq \rho_0$ such that $K_\nu(\tau) \geq \tau^{2\kappa}$ for all $0 < \tau \leq \rho$, then

$$
\frac{1}{C} \left(\frac{\sigma}{\rho}\right)^{2c^{\kappa} \Phi_\nu(\rho) + 2C^{\kappa}} \Phi_\nu(\rho) \leq \rho^{-n} \int_{B_1^+} \frac{v^2}{\rho^2} \leq \sigma^{-n} \int_{B_1^+} \frac{v^2}{\rho^2} \leq Ce^{2c^{\kappa} \Phi_\nu(0^+) + e^C} \left(\frac{\sigma}{\rho}\right)^{2\Phi_\nu(0^+)} \rho^{-n} \int_{B_1^+} \frac{v^2}{\rho^2}, \tag{7.3}
$$

where $C \in (0, \infty)$ is a constant depending only on $n$, $p$, $\lambda$, $\|a^{ij}\|_{C^{0,1}(B_1^+)}$, $\|k_+\|_{C^{0,1}(B_1^+)}$, $\|k_-\|_{C^{0,1}(B_1^-)}$, and $\|v\|_{L^\infty(B_1)}$.

**Proof.** Throughout this proof we let $C \in (0, \infty)$ denote constants depending only on $n$, $p$, $\lambda$, $\|a^{ij}\|_{C^{0,1}(B_1^+)}$, $\|k_+\|_{C^{0,1}(B_1^+)}$, $\|k_-\|_{C^{0,1}(B_1^-)}$, and $\|v\|_{L^\infty(B_1)}$. Let $\rho_0 \in (0, 1/2)$ be a small constant to be later determined. To see (a), suppose that $0 < \alpha < \sigma < \rho \leq \rho_0$ such that $K_\nu(\tau) > \tau^{2\kappa}$ for all $\alpha \leq \tau \leq \rho$. Then $\Phi(\tau) = \tau K_\nu(\tau)/(2K_\nu(\tau))$ for all $\alpha \leq \tau \leq \rho$. Hence by Theorem 6.1

$$
e^{C\alpha} \Phi_\nu(\alpha) + Ce^C \alpha \leq e^{C\tau} \frac{\tau K_\nu(\tau)}{2K_\nu(\tau)} + Ce^C \tau \leq e^{C\rho} \Phi_\nu(\rho) + Ce^C \rho \tag{7.4}
$$

for all $\alpha \leq \tau \leq \rho$. Note that by (6.30) and (6.40) we have $\Phi_\nu(\alpha) \geq -C\alpha$ for some constant $C \in (0, \infty)$. Thus, assuming $\rho_0$ is sufficiently small, $e^{C\alpha} \Phi_\nu(\alpha) + Ce^C \alpha \geq 0$ and thus $e^{C\rho} \Phi_\nu(\rho) + Ce^C \rho \geq 0$ (for $C$ as in (7.4)). If $\Phi_\nu(\alpha) \geq 0$, then by rearranging terms in (7.4) and using $1 - C\tau \leq e^{-C\tau} \leq 1$ yield

$$
(1 - C\tau) \Phi_\nu(\alpha) - Ce^C \tau \leq \frac{\tau K_\nu(\tau)}{2K_\nu(\tau)} \leq e^{C\rho} \Phi_\nu(\rho) + Ce^C \rho \tag{7.5}
$$

for all $\alpha \leq \tau \leq \rho \leq \rho_0$, provided $\rho_0$ is small enough that $1 - C\rho_0 \geq 0$, where $C$ is as in (7.5). If instead $\Phi_\nu(\alpha) < 0$, then rearranging terms in (7.4) gives us $\tau K_\nu(\tau)/(2K_\nu(\tau)) \geq -Ce^C \tau$ for some constant $C \in (0, \infty)$. Hence (7.5) holds true for all $\alpha \leq \tau \leq \rho \leq \rho_0$ provided $\rho_0$ is small enough that $1 - C\rho_0 \geq 0$, where $C$ is as in (7.5). Multiplying (7.5) by $2/\tau$ and integrating over $\tau \in [\sigma, \rho]$ gives us (7.1). By using (6.26) in (7.1) gives us (7.2). Integrating (7.2) with $\alpha = 0^+$ and using (6.4) gives us (7.3). \qed
Thus we can apply (7.2) to obtain

\[ \Phi_{v_{\kappa}}(0^+) \leq \kappa. \]  

(7.6)

We want to show (7.7). We will separately consider the cases \( \Phi_{v_{\kappa}}(7.6) \) must hold true.

Let’s first take a positive integer \( \kappa \) and show (7.6). Let \( v = v_{\kappa} \) be as in (5.3) and \( K_{v_{\kappa}} \) be as in (6.25). Notice that if there exists \( \rho_j \to 0^+ \) such that \( K_{v_{\kappa}}(\rho_j) < \rho^{2\kappa}_j \), then \( \Phi_{v_{\kappa}}(\rho_j) = \kappa \) and thus letting \( j \to \infty \) gives us \( \Phi_{v_{\kappa}}(0^+) = \kappa \). Hence we may assume that there exists \( \delta > 0 \) such that

\[ K_{v_{\kappa}}(\rho) \geq \rho^{2\kappa} \]  

(7.7)

By (7.8) we can apply (7.1) to obtain

\[ \sigma^{2\kappa} \leq K_{v_{\kappa}}(\sigma) \leq C \left( \frac{\sigma}{\rho} \right)^{2\Phi_{v_{\kappa}}(0^+)} K_{v_{\kappa}}(\rho) \]

for all \( 0 < \sigma < \rho \) sufficiently small and some constant \( C \in (0, \infty) \) independent of \( \sigma \) and \( \rho \). Hence (7.6) must hold true.

Now take positive integers \( \kappa < \ell \). Let \( v_{\kappa} \) be as in (5.3) and \( v_{\ell} \) is as in (5.3) with \( \ell \) in place of \( \kappa \). We want to show (7.7). We will separately consider the cases \( \Phi_{v_{\ell}}(0^+) < \kappa \) and \( \kappa \leq \Phi_{v_{\ell}}(0^+) \leq \ell \).

First suppose that \( \Phi_{v_{\ell}}(0^+) < \kappa \). Let \( 0 < \varepsilon < \kappa - \Phi_{v_{\ell}}(0^+) \). As we discussed above, since \( \Phi_{v_{\ell}}(0^+) < \ell \) there exists \( \delta > 0 \) such that

\[ K_{v_{\ell}}(\rho) \geq \rho^{2\ell} \]

(7.9)

Thus we can apply (7.2) to obtain

\[ \frac{1}{C} \left( \frac{\sigma}{\rho} \right)^{2\Phi_{v_{\ell}}(0^+) + 2\varepsilon} H_{v_{\ell}}(\rho) \leq H_{v_{\ell}}(\sigma) \leq C \left( \frac{\sigma}{\rho} \right)^{2\Phi_{v_{\ell}}(0^+)} H_{v_{\ell}}(\rho) \]  

(7.10)

for all sufficiently small \( 0 < \sigma < \rho \) and some constant \( C \in (0, \infty) \) independent of \( \sigma \) and \( \rho \). Let \( \overline{h}_{\kappa} \) be as in Lemma (5.1) and \( \overline{h}_{\ell} \) be as in Lemma (5.1) with \( \ell \) in place of \( \kappa \). Since \( \overline{h}_{\kappa} \) is the unique degree \( \kappa \) polynomial such that \( (5.1) \) holds true and \( (5.1) \) holds true with the polynomial \( \overline{h}_{\ell} \) in place of \( \overline{h}_{\kappa} \), \( \overline{h}_{\kappa} \) and \( \overline{h}_{\ell} \) must be equal up to terms of degree \( \leq \kappa \). In particular,

\[ |\overline{h}_{\kappa}(x) - \overline{h}_{\ell}(x)| \leq C|x|^{\kappa + 1} \]

for each \( x \in B_1^+ \), where \( C \in (0, \infty) \) is a constant depending only on \( n, \ell, \|a^{ij}\|_{C^{\ell - 1,1}(B_1^+)} \), and \( \|h\|_{C^{\ell,1}(B_1^+)} \). It follows from the definition of \( v_{\kappa} \) and \( v_{\ell} \) in (5.3) that

\[ |v_{\kappa}(x) - v_{\ell}(x)| \leq C|x|^{\kappa + 1} \]  

(7.11)
for all $x \in B_1^+$, where $C \in (0, \infty)$ is a constant depending only on $n$, $\ell$, $\|a^{ij}\|_{C^{\alpha,1}(B_1^+)}$, and $\|h\|_{C^{\alpha,1}(B_1^+)}$. Hence by (7.10),

$$
\frac{1}{2C} \left( \frac{\sigma}{\rho} \right)^{2\Phi_{\nu}(0^+) + 2\epsilon} H_{\nu}(\rho) - C\sigma^{2(\kappa+1)} \leq H_{\nu}(\sigma) \leq 2C \left( \frac{\sigma}{\rho} \right)^{2\Phi_{\nu}(0^+)} H_{\nu}(\rho) + 2C\sigma^{2(\kappa+1)} \quad (7.12)
$$

for all sufficiently small $0 < \sigma < \rho$ sufficiently small and some constant $C \in (1, \infty)$ independent of $\sigma$ and $\rho$. Note that by (6.20) and (7.9), $H_{\nu}(\rho) > 0$. Thus by fixing $\rho$ and applying (6.20) and (7.12),

$$
K_{\nu}(\sigma) \geq \frac{1}{C} H_{\nu}(\sigma) > \frac{1}{2C^2} \left( \frac{\sigma}{\rho} \right)^{2\Phi_{\nu}(0^+) + 2\epsilon} H_{\nu}(\rho) - C\sigma^{2(\kappa+1)} > \sigma^{2\kappa}
$$

for all sufficiently small $\sigma > 0$, where $C \in (1, \infty)$ denotes constants independent of $\sigma$ and $\rho$. Thus we can apply (7.2) to obtain

$$
\frac{1}{C} \left( \frac{\sigma}{\rho} \right)^{2\Phi_{\nu}(0^+) + 2\epsilon} H_{\nu}(\rho) \leq H_{\nu}(\sigma) \leq C \left( \frac{\sigma}{\rho} \right)^{2\Phi_{\nu}(0^+)} H_{\nu}(\rho). \quad (7.13)
$$

for all sufficiently small $0 < \sigma < \rho$ and some constant $C \in (1, \infty)$ independent of $\sigma$ and $\rho$. By (7.12) and (7.13),

$$
\frac{1}{C} \left( \frac{\sigma}{\rho} \right)^{2\Phi_{\nu}(0^+) + 2\epsilon} H_{\nu}(\rho) - C\sigma^{2(\kappa+1)} \leq H_{\nu}(\sigma) \leq C \left( \frac{\sigma}{\rho} \right)^{2\Phi_{\nu}(0^+)} H_{\nu}(\rho)
$$

for all sufficiently small $0 < \sigma < \rho$ sufficiently small and thus $\Phi_{\nu}(0^+) \leq \Phi_{\nu}(0^+) + \epsilon$. On the other hand, by (7.12) and (7.13),

$$
\frac{1}{C} \left( \frac{\sigma}{\rho} \right)^{2\Phi_{\nu}(0^+) + 2\epsilon} H_{\nu}(\rho) \leq H_{\nu}(\sigma) \leq C \left( \frac{\sigma}{\rho} \right)^{2\Phi_{\nu}(0^+)} H_{\nu}(\rho) + C\sigma^{2(\kappa+1)}
$$

for all sufficiently small $0 < \sigma < \rho$ sufficiently small and thus $\Phi_{\nu}(0^+) \leq \Phi_{\nu}(0^+) + \epsilon$. In other words,

$$
\Phi_{\nu}(0^+) - \epsilon \leq \Phi_{\nu}(0^+) \leq \Phi_{\nu}(0^+) + \epsilon. \quad (7.14)
$$

Letting $\epsilon \to 0^+$ gives us $\Phi_{\nu}(0^+) = \Phi_{\nu}(0^+)$. Suppose instead that $\kappa \leq \Phi_{\nu}(0^+) \leq \ell$. Note that if $\Phi_{\nu}(0^+) = \kappa$, then there is nothing to show. Assume thus $\Phi_{\nu}(0^+) < \kappa$, and let $0 < \epsilon < \kappa - \Phi_{\nu}(0^+)$. We can then repeat the above arguments, with the roles of $\kappa$ and $\ell$ interchanged, to show $\Phi_{\nu}(0^+) = \Phi_{\nu}(0^+)$. However, since $\Phi_{\nu}(0^+) < \kappa \leq \Phi_{\nu}(0^+)$, we have a reached a contradiction and the proof is complete.

In light of Lemma 7.2, we can define the Almgren frequency $\mathcal{N}_u(0)$ of $u$ at the origin as follows.

**Definition 7.1.** Let $u \in W^{1/2}(B_1^+)$ be a solution to (1.1) and $x_0 \in \Sigma(u)$ as above. Translate $x_0$ to the origin and then apply a linear change of variables so that $a^{ij}(0) = \delta_{ij}$. Then for each positive integer $\kappa$ define $v_\kappa$ as in (5.3). If there exists a positive integer $\kappa$ such that $\Phi_{v_\kappa}(0^+) < \kappa$, where $v_\kappa$ is as in (5.3), then we define $\mathcal{N}_u(x_0) = \Phi_{v_\kappa}(0^+)$. Otherwise, we define $\mathcal{N}_u(x_0) = +\infty$.

It follows from Lemma 7.2 that $\mathcal{N}_u(x_0)$ is well-defined and $\Phi_{v_\kappa}(0^+) = \min\{\mathcal{N}_u(x_0), \kappa\}$ for each positive integer $\kappa$. 

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8 Blow-ups

Assume Hypothesis [6.1] and let \( v \in C^1(B_1^+(0) \cup B_1'(0)) \) be a solution to (5.7). Suppose \( \Phi_v(0^+) < \infty \).

We want to determine the asymptotic behavior of \( v \) at the origin. Let

\[
v_\rho(x) = \frac{v(\rho x)}{H_\rho(\rho)^{1/2}} \tag{8.1}
\]

for each \( 0 < \rho \leq 1 \) and \( x \in B_{1/(2\rho)}^+ \cup B_{1/(2\rho)}' \), where \( H_\rho \) is as in (6.6). A blow-up of \( v \) at the origin is a function \( v^* \in C^1(\mathbb{R}^n_+) \) such that if \( v_\rho \to v^* \) in the \( C^1 \)-topology on compact subsets of \( \mathbb{R}^n_+ \).

**Theorem 8.1.** Assume Hypothesis [6.1] and let \( v \in C^1(B_1^+(0) \cup B_1'(0)) \) be a solution to (5.7) with \( \Phi_v(0^+) < \infty \). For each sequence \( \rho_\ell \to 0^+ \), there exists a subsequence \( \rho_\ell \to \rho_\ell' \) and a non-zero function \( v^* \in C^1(\mathbb{R}^n_+) \) such that

\[
v_{\rho_{\ell'}} \to v^* \tag{8.2}
\]

in \( C^1(B_\sigma^+ \cup B_\sigma') \) for all \( \sigma \in (0, \infty) \), where \( v_{\rho_{\ell'}} \) is as in (8.1) with \( \rho = \rho_{\ell'} \). Furthermore, the even reflection of \( v^* \) across \( \{x_n = 0\} \) (also denoted by \( v^* \)) is a homogeneous harmonic polynomial of degree \( \Phi_v(0^+) \).

Before proving Theorem 8.1 let’s first prove the following estimate:

**Lemma 8.1.** Assume Hypothesis [6.1] and let \( v \in C^1(B_1^+(0) \cup B_1'(0)) \) be a solution to (5.7). Then

\[
\sup_{B_{\sigma/2}^+(0)} |v| + \sigma \sup_{B_{\sigma/2}^+(0)} |Dv| + \sigma^{3/2} [Dv]_{1/2,B_{\sigma/2}^+(0)} \leq C \left( \sigma^{-n/2} \|v\|_{L^2(B_1^+(0))} + \sigma^2 \sup_{B_1^+(0)} |f| \right) \tag{8.3}
\]

for each \( \sigma \in (0,1) \) and some constant \( C \in (0,\infty) \) depending only on \( n, p, \lambda, \|a^{ij}\|_{C^{0,1}(B_1^+)}, \sup_{B_1^+} |x|^{-\kappa} |f|, \|k_+\|_{C^{0,1}(B_1^+)}, \|k_-\|_{C^{0,1}(B_1^+)}, \|\|v\|_{L^\infty(B_1)} \).

**Proof.** Throughout the proof we will denote by \( C \in (0,\infty) \) constants depending on \( n, p, \lambda, \|a^{ij}\|_{C^{0,1}(B_1^+)}, \sup_{B_1^+} |x|^{-\kappa} |f|, \|k_+\|_{C^{0,1}(B_1^+)}, \|k_-\|_{C^{0,1}(B_1^+)}, \|\|v\|_{L^\infty(B_1)} \). Without loss of generality assume \( \sigma = 1 \). First we claim that for each \( \delta > 0 \) there exists \( \tau_0 = \tau_0(\rho, \|v\|_{C^0(B_1)}) \in (0,1/2) \) such that if \( B_\tau(z) \subset B_1(0) \) with \( 0 < \tau \leq \tau_0 \) then

\[
\left|v_\rho + \tau \sup_{B_{\tau/2}(z) \cap B_1^+(0)} |Dv| + \tau^{3/2} |Dv|_{1/2,B_{\tau/2}(z) \cap B_1^+(0)} \right| \leq C \delta \left( \sup_{B_{\tau}(z) \cap B_1^+(0)} |v| + \tau \sup_{B_{\tau}(z) \cap B_1^+(0)}^1 [Dv]_{1/2,B_{\tau/2}(z) \cap B_1^+(0)} \right) + C \left( \tau^{-n/2} \|v\|_{L^2(B_\tau(z) \cap B_1^+(0))} + \tau^2 \sup_{B_{\tau}(z) \cap B_1^+(0)} |f| \right). \tag{8.4}
\]

To see this, fix \( B_\tau(z) \subset B_1(0) \). Notice that if \( B_{\tau/8}(z) \subset B_1^+(0) \), then (8.3) holds true by standard elliptic estimates [GT01, Theorem 8.32]. Otherwise, \((z',0) \in B_{\tau/8}(z) \cap B_1^+(0)\), where \( z = (z',z_n) \). By \( C^1,\alpha \)-estimates for weak solutions to the oblique derivative problems [BBY Proposition 5.53]
applied to the solution $v$ to (8.1),

$$
\sup_{B_{r/4}(z',0) \cap B_1^+} |v| + \tau \sup_{B_{r/4}(z',0) \cap B_1^+} |Dv| + \tau^{3/2} [Dv]_{1/2,B_{r/4}(z',0) \cap B_1^+} \leq C \left( \tau^{-n/2} \|v\|_{L^2(B_{r/2}(z',0) \cap B_1^+)} + \tau \sup_{B_{r/2}(z',0) \cap B_1^+} |v|^{p-1} + \tau^{3/2} \sup_{B_{r/2}(z',0) \cap B_1^+} |v|^{p-2} \right) \right).
\tag{8.5}
$$

Notice that $B_{r/16}(z) \subset B_{r/4}(z',0)$ and $B_{r/2}(z',0) \subset B_r(z)$. Hence if we require that $0 < \tau \leq \tau_0$ where

$$
\tau_0 \sup_{B_1} |v|^{p-2} \leq \delta,
$$

then (8.5) gives us (8.4). Now having shown (8.4), it follows by interpolation that

$$
\sup_{B_{r/16}(z) \cap B_1^+} |v| + \tau \sup_{B_{r/16}(z) \cap B_1^+} |Dv| + \tau^{3/2} [Dv]_{1/2,B_{r/16}(z) \cap B_1^+} \leq C \delta \tau^{3/2} [Dv]_{1/2,B_r(z) \cap B_1^+} + C \left( \tau^{-n/2} \|v\|_{L^2(B_r(z) \cap B_1^+)} + \tau^2 \sup_{B_r(z) \cap B_1^+} |f| \right).
\tag{8.6}
$$

Then using Lemma 2 of Section 2.8 of [S96], it follows that (8.3) holds true. \hfill \Box

Proof of Theorem 8.1. Fix $1 < \sigma < \infty$. First we will show that $v_\rho$ is uniformly bounded in $C^{1,1/2}(B^+_\rho)$. First let $\kappa$ be an integer such that $\Phi(0^+) < \kappa$, and notice that we thus have $K_v(\tau) > \tau^{2\kappa}$ for all sufficiently small $\tau > 0$. Thus we may apply (6.3) and (7.2) to obtain

$$
\lambda \int_{B^+_{2\rho\sigma}} v^2 \leq \int_{B^+_{2\rho\sigma}} v^2 \mu = \int_0^{2\rho\sigma} \int_{(\partial B_{\rho\sigma})^+} v^2 \mu = C \int_0^{2\rho\sigma} \int_{(\partial B_{\rho\sigma})^+} \tau^{-n-1+2\Phi_v(0^+)} v^2 \mu \, d\tau \leq C \rho \int_{(\partial B_{\rho\sigma})^+} v^2 \mu \tag{8.6}
$$

for all sufficiently small $\rho > 0$ and some constant $C \in (0, \infty)$ independent of $\rho$. By multiplying both sides by $(\rho\sigma)^{-n}$,

$$
\sigma^{-n} \int_{B^+_{2\rho\sigma}} v^2 \rho = \frac{1}{(\rho\sigma)^n H_v(\rho)} \int_{B^+_{2\rho\sigma}} v^2 \leq C \tag{8.6}
$$

for all sufficiently small $\rho > 0$ and some constant $C \in (0, \infty)$ independent of $\rho$. Also by (7.2),

$$
H_v(\rho) \geq c \rho^{2\kappa} \tag{8.7}
$$

for all sufficiently small $\rho > 0$ and some constant $c \in (0, \infty)$ independent of $\rho$.

By Lemma 8.1 with $\rho\sigma$ in place of $\sigma$,

$$
\sup_{B^+_{\rho\sigma}} |v| + \rho \sigma \sup_{B^+_{\rho\sigma}} |Dv| + (\rho\sigma)^{3/2} [Dv]_{1/2,B^+_{\rho\sigma}} \leq C \left( (\rho\sigma)^{-n/2} \|v\|_{L^2(B^+_{2\rho\sigma})} + (\rho\sigma)^2 \sup_{B^+_{\rho\sigma}} |f| \right), \tag{8.8}
$$

for all sufficiently small $\rho > 0$ and some constant $C \in (0, \infty)$ independent of $\rho$. The proof is now complete.
where $f$ is as in (5.3) and $C \in (0, \infty)$ is a constant. By dividing (8.8) by $H_v(\rho)^{1/2}$ and rescaling,

$$
\sup_{B^+_{\rho}} |v_{\rho}| + \sigma \sup_{B^+_{\rho}} |Dv_{\rho}| + \sigma^{3/2} |Dv_{\rho}|_{1/2, B^+_{\rho}} \leq C \left( \sigma^{-n/2} \|v_{\rho}\|_{L^2(B^+_{\rho})} + \sigma^2 \sup_{B^+_{\rho}} |f_{\rho}| \right),
$$

(8.9)

where $f_{\rho} \in L^{\infty}(B^+_{\rho})$ is defined by

$$
f_{\rho}(x) = \frac{\sigma^2 f(\rho x)}{H_v(\rho)^{1/2}}
$$

(8.10)

for each $x \in B^+_{\rho}$. By (6.1) and (8.7),

$$
\sigma^2 \sup_{B^+_{\rho}} |f_{\rho}| = \frac{(\rho \sigma)^2 \sup_{B^+_{\rho}} |f|}{H_v(\rho)^{1/2}} \leq \frac{C (\rho \sigma)^{k+1}}{H_v(\rho)^{1/2}} \leq C \rho
$$

(8.11)

for all sufficiently small $\rho$, where $C \in (0, \infty)$ denotes constants independent of $\rho$. By bounding the right-hand side of (8.9) using (8.6) and (8.11),

$$
\sup_{B^+_{\rho}} |v_{\rho}| + \sigma \sup_{B^+_{\rho}} |Dv_{\rho}| + \sigma^{3/2} |Dv_{\rho}|_{1/2, B^+_{\rho}} \leq C
$$

(8.12)

for all sufficiently small $\rho$ and some constant $C \in (0, \infty)$ independent of $\rho$. It now follows from (8.12) and the Arzela-Ascoli theorem that for each sequence $\rho_\ell \to 0^+$ there exists a subsequence $\{\rho_{\ell'}\} \subset \{\rho_\ell\}$ and a function $v^* \in C^1(\mathbb{R}^n_+)$ such that (8.2) holds true with convergence in the $C^1(B^+_{\rho} \cup B'_{\rho})$ topology for all $\sigma \in (0, \infty)$. Note that since

$$
\int_{(\partial B_1)^+} (v^*)^2 = \lim_{\ell \to \infty} \int_{(\partial B_1)^+} (v_{\rho_\ell}(x))^2 \mu(\rho_\ell x) \, dx
$$

$$
= \lim_{\ell \to \infty} \rho_\ell^{n-1} H_v(\rho_\ell) \int_{(\partial B_{\rho_\ell})^+} (v(\rho_\ell x))^2 \mu(\rho_\ell x) \, dx = 1,
$$

$v^*$ is a non-zero function.

Again fix $1 < \sigma < \infty$. Notice that by (6.7), for each sufficiently small $\rho > 0$, $v_{\rho}$ satisfies

$$
D_i(a^{ij}(\rho x) D_j v_{\rho}) = f_{\rho} \text{ in } B^+_{\sigma},
$$

$$
D_n v_{\rho} = \psi_{\rho} \text{ on } B'_{\sigma}
$$

(8.13)

where $f_{\rho}$ is as in (8.10) and $\psi_{\rho} \in C^{0,1}(B'_{\sigma})$ is defined by

$$
\psi_{\rho}(x) = \frac{\rho}{H_v(\rho)^{1/2}} \left( k_+(\rho x) (v^+(\rho x))^{p-1} + k_-(\rho x) (v^-(\rho x))^{p-1} \right)
$$

for each $x \in B'_{\rho}$. Since $a^{ij}$ is Lipschitz continuous, $a^{ij}(\rho x) \to a^{ij}(0) = \delta_{ij}$ uniformly on $B_{\sigma}$ as $\rho \to 0^+$. By (8.11), $f_{\rho} \to 0$ uniformly on $B_{\sigma}$ as $\rho \to 0^+$. By (8.12),

$$
\sup_{B^+_{\rho}} |\psi_{\rho}| \leq C \rho \sup_{B^+_{\sigma}} (k_+ + k_-) |v|^{p-2}.
$$

Therefore, $\psi_{\rho} \to 0$ uniformly on $B_{\sigma}$ as $\rho \to 0^+$. Therefore, by letting $\rho = \rho \epsilon \to 0^+$ in (8.13) and noting that $\sigma$ is arbitrary,

$$
\Delta v^* = 0 \text{ in } \mathbb{R}^n_+,
$$

$$
D_n v^* = 0 \text{ on } \mathbb{R}^{n-1} \times \{0\}.
$$
Hence the even extension of $v^*$ across $\{x_n = 0\}$ is a harmonic function in $\mathbb{R}^n$.

Let $0 < \sigma < \tau < \infty$. By rescaling (7.3), for each sufficiently small $\rho > 0$
$$
\frac{1}{\rho^2} \left( \frac{\sigma}{\tau} \right)^{2\Phi_v(\rho^+)} \frac{2C\sigma \rho^{2\Phi_v(\rho^+)}}{\tau - n} \int_{B_+^\tau} |v_\rho|^2 \leq \sigma^{-n} \int_{B_+^\tau} |v_\rho|^2 \leq C \left( \frac{\sigma}{\tau} \right)^{2\Phi_v(\rho^+)} \tau^{-n} \int_{B_+^\tau} |v_\rho|^2.
$$

Letting $\rho = \rho_0 \to 0^+$,
$$
\frac{1}{\rho^2} \left( \frac{\sigma}{\tau} \right)^{2\Phi_v(\rho^+)} \tau^{-n} \int_{B_+^\tau} |v^*|^2 \leq \sigma^{-n} \int_{B_+^\tau} |v^*|^2 \leq C \left( \frac{\sigma}{\tau} \right)^{2\Phi_v(\rho^+)} \tau^{-n} \int_{B_+^\tau} |v^*|^2
$$
for all $0 < \sigma < \tau < \infty$. Recalling that the even extension of $v^*$ across $\{x_n = 0\}$ is a harmonic function in $\mathbb{R}^n$, it follows from (8.14) using the Liouville Theorem that the even extension of $v^*$ across $\{x_n = 0\}$ is also a homogeneous polynomial of degree $\Phi_v(\rho^+)$. 

\section{Weiss’ monotonicity formula}

Assume Hypothesis \[6.1\] and let $v \in C^1(B_1^+(0) \cup B_1'(0))$ be a solution to \[6.7\]. For $\nu \in \mathbb{R}$, we introduce the Weiss function $W(\rho) = W_{v,\nu}(\rho)$ as

$$
W(\rho) = \frac{1}{\rho^2} (I_\nu(\rho) - \nu H_v(\rho)) = \rho^{1-n-2\nu} \int_{(\partial B_\rho)^+} (a^{ij} v D_j v x_i - \mu \nu v^2)
$$

for each $0 < \rho < 1$, where $I_\nu(\rho)$ and $H_v(\rho)$ are as in \[5.7\] and \[6.6\] respectively.

\begin{theorem}
Let $\nu \in \mathbb{R}$ and $\kappa \in \mathbb{N}$, with $\nu \leq \kappa$. Assume Hypothesis \[6.1\] and let $v \in C^1(B_1^+(0) \cup B_1'(0))$ be a solution to \[5.7\] with $\Phi_v(\rho^+) \geq \nu$. Let $\rho_0$ be as in Lemma \[7.7\]. For each $0 < \sigma < \rho \leq \rho_0/2$ such that $K_v(\tau) > \tau^{2\kappa}$ for all $0 < \tau \leq 2\rho$,

$$
W(\sigma) + C \sigma \leq W(\rho) + C \rho
$$

for some constant $C \in (0, \infty)$ depending only on $n, \rho, \lambda, \|a^{ij}\|_{C^{0,1}(B_1^+)}$, $\|B_+\|_{C^{0,1}(B_1^+)}$, $\|k_+\|_{C^{0,1}(B_1^+)}$, $\|k_-\|_{C^{0,1}(B_1^+)}$, and $\|v\|_{L^\infty(B_1)}$.

\end{theorem}

\begin{proof}
Throughout the proof we will denote by $C \in (0, \infty)$ constants depending only on $n, \rho, \lambda, \|a^{ij}\|_{C^{0,1}(B_1^+)}$, $\sup_{B_1^+} |x|^{1-\kappa} |f|$, $\|k_+\|_{C^{0,1}(B_1^+)}$, $\|k_-\|_{C^{0,1}(B_1^+)}$, and $\|v\|_{L^\infty(B_1)}$. We compute

$$
\frac{d}{d\rho} W(\rho) = \frac{1}{\rho^2} \left( I'(\rho) - \nu H'(\rho) \right) - \frac{2\nu}{\rho^{2\nu+1}} (I(\rho) - \nu H(\rho))
$$

for $L^1$-a.e. $0 < \rho < \rho_0/2$. Taking into account \[6.37\] and \[6.38\],

$$
\frac{d}{d\rho} W(\rho) \geq 2 \rho^{-n-2\nu} \int_{(\partial B_\rho)^+} \frac{1}{\mu} \left( \sum_{i,j=1}^n a^{ij} D_j v x_i \right)^2 - \frac{4\nu}{\rho^{2\nu+1}} I(\rho) + \frac{2\nu^2}{\rho^{2\nu+1}} H(\rho)
$$

for $L^1$-a.e. $0 < \rho < \rho_0/2$. Taking into account \[6.37\] and \[6.38\],

$$
\frac{d}{d\rho} W(\rho) \geq 2 \rho^{-n-2\nu} \left\{ \rho^2 \int_{(\partial B_\rho)^+} \frac{1}{\mu} \left( \sum_{i,j=1}^n a^{ij} D_j v x_i \right)^2 - 2\nu \rho \left( \int_{(\partial B_\rho)^+} a^{ij} v D_j v x_i \rho^2 + \nu \rho \int_{(\partial B_\rho)^+} \mu v^2 \right) \right\} - \frac{C}{\rho^{2\nu}} (D(\rho) + H(\rho))
$$

for $L^1$-a.e. $0 < \rho < \rho_0/2$. Taking into account \[6.37\] and \[6.38\],

\end{proof}
for $L^1$-a.e. $0 < \rho < \rho_0/2$. We now observe

$$\rho^2 \int_{(\partial B_\rho)^+} \frac{1}{\mu} \left( \sum_{i,j=1}^n a^{ij} D_j v \frac{x_i}{\rho} \right)^2 - 2 \rho \nu \int_{(\partial B_\rho)^+} a^{ij} v D_j v \frac{x_i}{\rho} + \nu^2 \int_{(\partial B_\rho)^+} \mu v^2$$

$$= \int_{(\partial B_\rho)^+} \frac{1}{\mu} \left( \sum_{i,j=1}^n a^{ij} D_j v x_i - \nu \mu v \right)^2$$

for each $0 < \rho < \rho_0/2$. Combining (9.3) and (9.4), we infer

$$\frac{d}{d\rho} W(\rho) \geq 2\rho^{-n-2\nu} \int_{(\partial B_\rho)^+} \frac{1}{\mu} \left( \sum_{i,j=1}^n a^{ij} D_j v x_i - \nu \mu v \right)^2 - C.$$ (9.5)

for $L^1$-a.e. $0 < \rho < \rho_0/2$. Since $p \geq 2$, an application of Lemma 8.1

$$H(\rho) + D(\rho) \leq \sup_{B_{\rho}^n} v^2 + \rho^2 \sup_{B_{\rho}^{2a}} |Dv|^2 + \sup_{B_{\rho}^n} (k_+ + k_-) |v|^p$$

$$\leq C \left( \rho^{-n} \|v\|^2 + \rho^4 \sup_{B_{\rho}^{2a}} |f|^2 \right).$$

Since by assumption $K(\tau) > \tau^{2\kappa}$ for all $0 < \tau \leq 2\rho$, we can use (7.3), (6.1), and the fact that $\nu \leq \kappa + 1$ to obtain

$$H(\rho) + D(\rho) \leq C \left( \rho^{2\nu} + \rho^{2\nu+2} \right) \leq C \rho^{2\nu}.$$ (9.6)

Hence (9.5) gives us

$$\frac{d}{d\rho} W(\rho) \geq 2\rho^{-n-2\nu} \int_{(\partial B_\rho)^+} \frac{1}{\mu} \left( \sum_{i,j=1}^n a^{ij} D_j v x_i - \nu \mu v \right)^2 - C.$$ (9.6)

for each $0 < \rho < 1/2$. Integrating (9.6) gives us (9.2).

10 **Monneau’s monotonicity formula**

Assume Hypothesis 6.1 and let $v \in C^1(B_1^+(0) \cup B_1'(0))$ be a solution to (5.7). Let $v$ be any positive integer and $p_{\nu}$ be any homogeneous degree $\nu$ harmonic polynomial on $\mathbb{R}^n$ such that $p_{\nu}(x', x_n) = p_{\nu}(x', -x_n)$ for each $x' \in \mathbb{R}^{n-1}$ and $x_{n+1} \in \mathbb{R}$. We introduce the **Monneau function** $M(\rho) = M_{\nu, p_{\nu}}(\rho)$ as

$$M(\rho) = \frac{1}{\rho^{n-1+2\nu}} \int_{(\partial B_\rho)^+} (v - p_{\nu})^2 \mu.$$ (10.1)

for each $0 < \rho < 1$.

**Theorem 10.1.** Let $v \in \mathbb{R}$ and $\kappa \in \mathbb{N}$, with $\nu \leq \kappa$. Assume Hypothesis 6.1 and let $v \in C^1(B_1^+(0) \cup B_1'(0))$ be a solution to (5.7) with $\Phi_\nu(0^+) \geq \nu$. Let $p_{\nu}$ be a harmonic polynomial on $\mathbb{R}^n$, homogeneous of degree $\nu$, such that $p_{\nu}(x', x_n) = p_{\nu}(x', -x_n)$ for each $x' \in \mathbb{R}^{n-1}$ and $x_{n+1} \in \mathbb{R}$. Let $\rho_0$ be as in Lemma 7.7. For each $0 < \sigma < \rho \leq \rho_0/2$ such that $K_{\nu}(\tau) > \tau^{2\kappa}$ for all $0 < \tau \leq 2\rho$,

$$M(\sigma) + C \sigma \leq M(\rho) + C \rho.$$ (10.2)

for some constant $C \in (0, \infty)$ depending only on $n, p, \lambda, \|a^{ij}\|_{C^{0,1}(B_1^+)}, \sup_{B_1^+} |x|^{1-\kappa} |f|$, $\|k_+\|_{C^{0,1}(B_1^+)}$, $\|k_-\|_{C^{0,1}(B_1^+)}$, $\|v\|_{L^\infty(B_1^+)}$, and $\|p_{\nu}\|_{L^\infty(B_1^+)}$.

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Proof. Throughout the proof we will denote by $C \in (0, \infty)$ constants depending only on $n$, $p$, $\lambda$, $\|a^{ij}\|_{C^{0,1}(B^+_1)}$, $\sup_{B^+_1} |x|^{-\kappa}|f|$, $\|k_+\|_{C^{0,1}(B^+_1)}$, $\|k_-\|_{C^{0,1}(B^+_1)}$, $\|v\|_{L^\infty(B_1)}$, and $\|p_v\|_{L^\infty(B_1)}$. Set $w = v - p_v$ so that

$$
\mathcal{M}(\rho) = \frac{1}{\rho^{n-1+2\nu}} \int_{(\partial B^+_1)^+} \mu w^2 = \frac{1}{\rho^{2\nu}} \mathcal{H}_w(\rho),
$$

(10.3)

where $\mathcal{H}_w(\rho)$ is given by (5.12). Let $\mathcal{W}_w(\rho)$ and $\mathcal{W}_{p_v}(\rho)$ be given by (9.1). Using $v = p_v + w$ and recalling the definitions of the Weiss function in (9.1) and of $\mu$ in (6.3), we have

$$
\mathcal{W}_w(\rho) = \mathcal{W}_{p_v}(\rho) + \mathcal{W}_{w}(\rho) + \rho^{1-n-2\nu} \int_{(\partial B^+_1)^+} (a^{ij} p_v D_j w x_i + a^{ij} w D_j p_v x_i - 2\nu \mu p_v v).
$$

(10.4)

In the last term, using $w = v - p_v$ we obtain

$$
\mathcal{W}_w(\rho) = \mathcal{W}_{w}(\rho) - \mathcal{W}_{p_v}(\rho) + \rho^{1-n-2\nu} \int_{(\partial B^+_1)^+} (a^{ij} p_v D_j v x_i + a^{ij} v D_j p_v x_i - 2\nu \mu p_v v).
$$

By the homogeneity of $p_v$, $x \cdot \nabla p_v = \nu p_v$, which together with (6.2) gives us

$$
\mathcal{W}_w(\rho) = \mathcal{W}_{w}(\rho) - \mathcal{W}_{p_v}(\rho)
$$

(10.4)

$$
+ \rho^{1-n-2\nu} \int_{(\partial B^+_1)^+} (a^{ij} p_v D_j v x_i + v x \cdot \nabla p_v + b^{ij} v D_j p_v x_i - 2\nu \mu p_v v)
$$

$$
= \mathcal{W}_{w}(\rho) - \mathcal{W}_{p_v}(\rho) + \rho^{1-n-2\nu} \int_{(\partial B^+_1)^+} (a^{ij} p_v D_j v x_i - v x \cdot \nabla p_v)
$$

$$
+ \rho^{1-n-2\nu} \int_{(\partial B^+_1)^+} (b^{ij} p_v D_j p_v x_i - 2\nu (\mu - 1) p_v v).
$$

(10.4)

Now we want to bound the terms $\mathcal{W}_{p_v}(\rho)$, $I_1$, and $I_2$ on the right-hand side of (10.4). By the homogeneity of $p_v$ and (6.2),

$$
\mathcal{W}_{p_v}(\rho) = \rho^{1-n-2\nu} \int_{(\partial B^+_1)^+} (a^{ij} p_v D_j p_v x_i - \nu p_v^2)
$$

$$
= \rho^{1-n-2\nu} \int_{(\partial B^+_1)^+} p_v (x \cdot \nabla p_v - \nu p_v)
$$

$$
+ \rho^{1-n-2\nu} \int_{(\partial B^+_1)^+} (b^{ij} p_v D_j p_v x_i - \nu (\mu - 1) p_v^2)
$$

$$
= \rho^{1-n-2\nu} \int_{(\partial B^+_1)^+} (b^{ij} p_v D_j p_v x_i - \nu (\mu - 1) p_v^2).
$$

Hence by (6.3) and the homogeneity of $p_v$,

$$
|\mathcal{W}_{p_v}(\rho)| \leq C \rho
$$

(10.5)

for some constant $C \in (0, \infty)$ depending on $n$ and $\|p_v\|_{L^\infty(B^+_1)}$. To bound $I_1$, notice that by the divergence theorem and using (5.4) with $\zeta = p_v 1_{B^+_\rho}$, where $1_{B^+_\rho}$ is the characteristic function of $B^+_\rho$,

$$
\int_{(\partial B^+_\rho)^+} a^{ij} p_v D_j v x_i = \int_{B^+_\rho} (a^{ij} D_j v D_i p_v + f p_v) + \int_{B^+_\rho} p_v (k_+(v^+)p_{-1} - k_-(v^-)p_{-1}).
$$

(10.6)
By again applying the divergence theorem, using the homogeneity of $p_\nu$, $\Delta p_\nu = 0$ in $B_\rho^+$, and $D_n p_\nu = 0$ on $B_\rho^+$,
\[
\int_{(\partial B_\rho^+)} \frac{v}{\rho} \cdot \nabla p_\nu = \int_{B_\rho^+} v D_n p_\nu + \int_{B_\rho^+} (v \Delta p_\nu + \nabla v \cdot \nabla p_\nu) = \int_{B_\rho^+} \nabla v \cdot \nabla p_\nu. \tag{10.7}
\]
Therefore, by subtracting (10.7) from (10.6) and using (6.2),
\[
I_1 = \rho^{2-n-2\nu} \int_{B_\rho^+} b^{ij} D_j p_\nu x_i + \rho^{2-n-2\nu} \int_{B_\rho^+} f p_\nu + \rho^{2-n-2\nu} \int_{B_\rho^+} p_\nu (k_+(v^+)^{p-1} - k_-(v^-)^{p-1}). \tag{10.8}
\]
We can bound the right-hand side of (10.8) using elliptic estimates. By Lemma 8.1,
\[
\sup_{B_\rho^+} |v| + \rho \sup_{B_\rho^+} |Dv| \leq C (\rho^{-n/2} \|v\|_{L^2(B_\rho^+)} + \rho^\kappa \sup_{B_\rho^+} |f|).
\]
Since by assumption $K_\nu(\tau) > \tau^{2\kappa}$ for all $0 < \tau \leq 2\rho$, we can use (10.3) and (6.1) to obtain
\[
\sup_{B_\rho^+} |v| + \rho \sup_{B_\rho^+} |Dv| \leq C \rho^\nu. \tag{10.9}
\]
By the homogeneity of $p_\nu$, (6.13), (6.5), (10.9), and the facts $\kappa \geq \nu$ and $p \geq 2$, we can bound the right-hand side of (10.8) by
\[
|I_1| \leq \rho^{2-\nu} \sup_{B_\rho^+} |Dv| + \rho^{p-\nu + 1} + \rho^{1-\nu} \sup_{B_\rho^+} |v|^{p-1} \leq C \rho. \tag{10.10}
\]
To bound $I_2$, by the homogeneity of $p_\nu$, (10.9), and (6.5) we have that
\[
|I_2| = \left| \rho^{1-n-2\nu} \int_{(\partial B_\rho^+)} (b^{ij} D_j p_\nu x_i - 2\nu (\mu - 1) p_\nu v) \right| \leq \rho^{1-\nu} \sup_{(\partial B_\rho^+)} |v| \leq C \rho. \tag{10.11}
\]
Combining (10.4) with (10.5), (10.10), and (10.11) we infer
\[
|W_\nu(\rho) - W_\nu(\rho)| \leq C \rho. \tag{10.12}
\]
On the other hand, by differentiating (10.3),
\[
\mathcal{M}'(\rho) = \frac{1}{\rho^{2\nu}} H_w'(\rho) - \frac{2\nu}{\rho^{2\nu+1}} H_w(\rho).
\]
Applying (6.13) yields
\[
\mathcal{M}'(\rho) = \frac{2}{\rho^{2\nu+1}} I_w(\rho) + \rho^{1-n-2\nu} \int_{(\partial B_\rho^+)} w^2 \left( (1-n) \frac{\mu}{\rho} + D_i \left( b^{ij} x_i \right) \right) - \frac{2\nu}{\rho^{2\nu+1}} H_w(\rho)
\]
\[
= \frac{2}{\rho} W_w(\rho) + \rho^{1-n-2\nu} \int_{(\partial B_\rho^+)} w^2 \left( (1-n) \frac{\mu}{\rho} + D_i \left( b^{ij} x_i \right) \right)
\]
By (10.12),
\[
\mathcal{M}'(\rho) \geq \frac{2}{\rho} W_w(\rho) + \rho^{1-n-2\nu} \int_{(\partial B_\rho^+)} w^2 \left( (1-n) \frac{\mu}{\rho} + D_i \left( b^{ij} x_i \right) \right) - C. \tag{10.13}
\]
Recalling that \( w = v - p_n \), and applying (5.3) and (10.9), we can bound the integral on the right-hand side of (10.13) by
\[
\left| \rho^{-n-2\nu} \int_{(\partial B_{\rho})^+} w^2 \left( (1-n) \frac{\mu-1}{\rho} + D_1(b^j x_j) \right) \right| \\
\leq C \rho^{-2\nu} \sup_{(\partial B_{\rho})^+} w^2 \leq C \rho^{-2\nu} \left( \sup_{(\partial B_{\rho})^+} v^2 + \rho^2 \right) \leq C.
\]
Therefore, (10.13) gives us
\[
\mathcal{M}'(\rho) \geq \frac{2}{\rho} W_\nu(\rho) - C.
\]
Notice that by Theorem 9.1, \( W_\nu(0^+) = \lim_{\rho \to 0^+} W_\nu(\rho) \) exists. Since \( \Phi_\nu(0^+) \geq \nu, \) \( W_\nu(0^+) \geq 0. \) Thus by again using Theorem 9.1 we obtain \( W_\nu(\rho) \geq -C\rho, \) which by (10.14) gives us \( \mathcal{M}'(\rho) \geq -C. \) The desired conclusion follows by integration.

11 Rectifiability of the free boundary

Let \( 2 \leq p < \infty, \) \( u \in W^{1,2}(B_1^+(0)) \) be a solution to (1.2), and \( 0 \in \Sigma(u) \) with \( \Phi_u(0) < \infty. \) We want to show that the free boundary \( \Sigma(u) \) is countably \((n-2)\)-rectifiable in an open neighborhood of the origin. We will follow the approach introduced in [GP09], based on the Weiss’ and Monneau’s monotonicity formulas proved in Sections 9 and 10.

Let \( \Omega = B_1^+(0) \) and \( \Gamma = B_1^-(0). \) Fix a positive integer \( \kappa, \) and let \( a^{ij} \in C^{\kappa-1,1}(B_1^+(0) \cup B_1^-(0)) \) such that \( a^{ij} = a^{ji} \) on \( B_1^+(0), \) satisfying (4.4) for some constants \( 0 < \lambda \leq \Lambda < \infty, \) and that (4.2) holds true on \( B_1^+(0). \) Let \( k_+, k_- \in C^{1,1}(B_1^+(0)) \) with \( k_+, k_- \geq 0 \) and \( h \in C^{1,1}(B_1^+(0)). \) Let \( u \in W^{1,2}(B_1^+(0)) \) be a weak solution to (1.2), and fix \( x_0 \in \Sigma(u). \) We begin by observing that, as a consequence of Theorem 1.1, we may assume \( \nabla_{x'} u(x_0) = \nabla_{x'} h(x_0). \) By Lemma 5.1 there exists a polynomial \( \overline{h}_{x_0,\kappa} \) of degree at most \( \kappa \) such that \( \overline{h}_{x_0,\kappa}(x',0) \) is the Taylor polynomial of degree \( \kappa, \) of \( h(x') \) at \( x_0 \) and
\[
D_i(a^{ij}D_j \overline{h}_{x_0,\kappa}) \leq C |x - x_0|^\kappa \text{ in } B_1^+(0), \quad D_n \overline{h}_{x_0,\kappa} = 0 \text{ on } B_1^+(0),
\]
for some constant \( C = C(n, \lambda, \|a^{ij}\|_{C^{\kappa-1,1}(B_1^+(0))}, \|h\|_{C^{1,1}(B_1^+(0))}) \in (0, \infty). \) Using the notations introduced in (5.3), for fixed \( x_0 \in \Sigma(u) \) let
\[
v_{x_0,\kappa}(x) = u(x + x_0) - \overline{h}_{x_0,\kappa}(x + x_0) + \overline{h}_{x_0,\kappa}(x + x_0)'(0) - h((x + x_0)'(0)
\]
for each \( x = (x', x_n) \in B_1^+(0) \cup B_1^-(0). \) For each positive integer \( \nu \) define
\[
\Sigma_\nu(u) = \{ x \in \Sigma(u) : \mathcal{N}_u(x_0) = \nu \},
\]
where \( \mathcal{N}_u(x_0) \) is as in Definition 7.1. Since we are assuming \( \nabla_{x'} u(x_0) = \nabla_{x'} h(x_0), \) by virtue of Theorem 8.1 in the sequel we can restrict our attention to the case \( \nu \geq 2. \)

We begin by proving the non-degeneracy property of the solution.

**Lemma 11.1.** Let \( 2 \leq p < \infty, \) and let \( \kappa, \nu \) be positive integers with \( \nu < \kappa. \) Let \( a^{ij} \in C^{\kappa-1,1}(B_1^+(0) \cup B_1^-(0)) \) such that \( a^{ij} = a^{ji} \) on \( B_1^+(0), \) satisfying (1.1) for some constants \( 0 < \lambda \leq \Lambda < \infty, \) and that (1.2) holds true on \( B_1^+(0). \) Let \( k_+, k_- \in C^{0,1}(B_1^+(0)) \) with \( k_+, k_- \geq 0 \) and \( h \in C^{0,1}(B_1^+(0)). \) Let \( u \in W^{1,2}(B_1^+(0)) \) be a solution to (1.2) and \( x_0 \in \Sigma_\nu(u). \) Let \( v = v_{x_0,\kappa} \) be as in (11.1). Then there exists constants \( \delta \in (0, 1) \) and \( c > 0 \) such that for each \( \rho \in (0, \delta] \)
\[
\sup_{(\partial B_{\rho}(x_0))^+} |v| \geq c \rho^\nu.
\]
Proof. After translating $x_0$ to the origin and applying a linear change of variables, assume that $x_0 = 0$ and $a^{ij}(0) = \delta_{ij}$. Suppose there exists a sequence $\rho_\ell \to 0^+$ such that

$$
\lim_{\ell \to \infty} \sup_{(\partial B_{\rho_\ell})^+} \frac{|v|}{\rho_\ell^p} = 0. \tag{11.2}
$$

Set $h_\ell = H_v(\rho_\ell)^{1/2}$, where $H_v$ is as in (6.6). By Theorem 8.1 there exists an harmonic polynomial $v^*$, homogeneous of degree $\nu$ and even in $x_0$, such that after passing to a subsequence

$$
v_\ell(x) = \frac{v(\rho_\ell x)}{h_\ell} \to v^*(x) \tag{11.3}
$$
in the $C^1$-topology on compact subsets of $\mathbb{R}^n \cap \{x_n \geq 0\}$ as $\ell \to \infty$. The monotonicity of $M_{v,v^*}$, established in Theorem 10.1, implies that $M_{v,v^*}(0^+) = \lim_{\rho \to 0^+} M_{v,v^*}(\rho)$ exists. Moreover, by (11.2) and the homogeneity of $v^*$,

$$
M_{v,v^*}(0^+) = \lim_{\ell \to \infty} M_{v,v^*}(\rho_\ell) = \lim_{\ell \to \infty} \rho_\ell^{1-n-2\nu} \int_{(\partial B_{\rho_\ell})^+} (v - v^*)^2 \mu = \lim_{\ell \to \infty} \int_{(\partial B_{\rho_\ell})^+} \left( \frac{v(\rho_\ell x)}{\rho_\ell^p} - v^*(x) \right)^2 \mu_{\rho_\ell}(x) \, dx = \lim_{\ell \to \infty} \int_{(\partial B_{\rho_\ell})^+} (v^*)^2 \mu_{\rho_\ell}
$$

where we let $\mu_\rho(x) = \mu(\rho x)$ for each $x \in (\partial B_1)^+$ and we note that $\mu_{\rho_\ell} \to 1$ uniformly on $(\partial B_1)^+$ by (6.3). Using again the monotonicity of $M_{v,v^*}$ and the homogeneity of $v^*$, we infer

$$
\rho_\ell^{1-n-2\nu} \int_{(\partial B_{\rho_\ell})^+} (v^*)^2 = \int_{(\partial B_1)^+} (v^*)^2 = M_{v,v^*}(0^+) \leq M_{v,v^*}(\rho_\ell) + C \rho_\ell
$$

where $C \in (0, \infty)$ is a constant depending only on $n, p, \lambda, \|a^{ij}\|_{C^{\lambda-1,1}(B_1^+)}$, $\|h\|_{C^{\nu,1}(B_1^+)}$, $\|k^+\|_{C^{\nu,1}(B_1^+)}$, $\|k_-\|_{C^{\nu,1}(B_1^+)}$, $\|v\|_{L^\infty(B_1^+)}$, and $\|v^*\|_{L^\infty(B_1^+)}$. By rescaling and recalling that $v_\ell(x) = v(\rho_\ell x)/h_\ell$ and $\mu_{\rho_\ell}(x) = \mu(\rho_\ell x)$,

$$
0 \leq \rho_\ell^{1-n-2\nu} \int_{(\partial B_{\rho_\ell})^+} (v^2 - 2vv^*) \mu + C \rho_\ell
$$

Dividing by $\rho_\ell^{-\nu} h_\ell$,

$$
0 \leq \int_{(\partial B_1)^+} \left( \frac{h_\ell^2}{\rho_\ell^p} v_\ell^2 - 2h_\ell v_\ell v^* \right) \mu_{\rho_\ell} + \frac{C \rho_\ell^{\nu+1}}{h_\ell}. \tag{11.4}
$$

Note that, since $\nu < \kappa$, we have $K(\tau) > \tau^{2\kappa}$ for all $\tau > 0$ sufficiently small. We can then apply (7.2) to get $h_\ell = H_v(\rho_\ell)^{1/2} \geq c \rho_\ell^{\nu+1/2}$ for some constant $c \in (0, \infty)$ independent of $\ell$. Hence letting $\ell \to \infty$ in (11.4) and using $\mu_{\rho_\ell} \to 1$ and $\rho_\ell^{-\nu} h_\ell \to 0$ by (11.2) and (11.3),

$$
0 \leq -2 \int_{(\partial B_1)^+} (v^*)^2.
$$

This fact, together with the homogeneity of $v^*$, implies that $v^*$ is identically zero on $\mathbb{R}^n_+$, contradicting $v^*$ being a non-zero polynomial. \qed
Theorem 11.1. Let $2 \leq p < \infty$, and let $\kappa, \nu$ be positive integers with $2 \leq \nu < \kappa$. Let $a^{ij} \in C^{\kappa-1,1}(B_1^+(0) \cup B_1^-(0))$ such that $a^{ij} = a^{ji}$ on $B_1^+(0)$, satisfying (11.1) for some constants $0 < \lambda \leq \Lambda < \infty$, and such that (11.2) holds true on $B_1^-(0)$. Let $k_+, k_- \in C^0,1(B_1(0))$ with $k_+, k_- \geq 0$ and $h \in C^{\kappa-1}(B_1^-(0))$. Let $u \in W^{1,2}(B_1^+(0))$ be a solution to (12.1) and $x_0 \in \Sigma(U)$. Let $v = v_{x_0, \kappa}$ be as in (11.1). Then there exists a unique function $v^* = v_{x_0}^* \in C^1(\mathbb{R}^n)$ such that

$$
\frac{v(\rho x)}{\rho^\nu} \to v_{x_0}^*(x) \text{ as } \rho \to 0^+
$$

in $C^1(B_\sigma^+ \cup B_\sigma^-)$ for every $\sigma \in (0, \infty)$. Furthermore, the even reflection of $v_{x_0}^*$ across $\{x_n = 0\}$ (also denoted by $v^* = v_{x_0}^*$) is homogeneous degree $\nu$ and is a smooth solution to

$$
a^{ij}(x) D_{ij} v_{x_0}^* = 0 \in \mathbb{R}^n.
$$

Remark 11.1. By (11.1), the limit $v^*$ in Theorem 11.1 is independent of the choice of integer $\kappa$.

Proof of Theorem 11.1. We begin by noticing that $H_{\kappa}(\rho)^{1/2} \approx \rho^\nu$ by Lemmas 7.1 and 11.1. We can thus follow the lines of the proof of Theorem 8.1 to show the existence of a sequence $\rho_\ell \to 0^+$ and of a harmonic polynomial $v^* \in C^1(\mathbb{R}^n)$, homogeneous of degree $\nu$ and even in $x_n$, such that for each $\sigma \in (0, \infty)$

$$
v_{\rho_\ell}(x) = \frac{v(\rho_\ell x)}{\rho_\ell^\nu} \to v^*(x)
$$

in $C^1(B_\sigma^+ \cup B_\sigma^-)$ as $\ell \to \infty$. Moreover, an application of Theorem 10.1 yields

$$
\mathcal{M}_{\nu, v^*}(0^+) = \lim_{\ell \to \infty} \mathcal{M}_{\nu, v^*}(\rho_\ell) = \lim_{\ell \to \infty} \rho_\ell^{1-\nu} \int_{(\partial B_{\rho_\ell})^+} (v - v^*)^2 \mu_\ell = \lim_{\ell \to \infty} \int_{(\partial B_{\rho_\ell})^+} (v_{\rho_\ell} - v^*)^2 \mu_\ell = 0,
$$

where $\mu_\rho(x) = \mu(\rho x)$ for each $x \in (\partial B_1)^+$ and $\rho > 0$ and in the last equality we have used 11.7. Therefore, by Theorem 10.1 again,

$$
\lim_{\rho \to 0^+} \int_{(\partial B_1)^+} (v_{\rho_\ell} - v^*)^2 \mu_\rho = \lim_{\rho \to 0^+} \rho^{1-\nu} \int_{(\partial B_{\rho_\ell})^+} (v - v^*)^2 \mu_\ell = \lim_{\rho \to 0^+} \mathcal{M}_{\nu, v^*}(\rho) = \mathcal{M}_{\nu, v^*}(0^+) = 0.
$$

If follows that if for a different sequence of radii $(\rho'_\ell)$ with $\rho'_\ell \to 0^+$ and some function $v' \in C^1(\mathbb{R}^n)$ we had

$$
v_{\rho'_\ell}(x) = \frac{v(\rho'_\ell x)}{(\rho'_\ell)^\nu} \to v'(x)
$$

in $C^1(B_\sigma^+ \cup B_\sigma^-)$ as $\ell \to \infty$, then

$$
\int_{(\partial B_1)^+} (v' - v^*)^2 = \lim_{\ell \to \infty} \int_{(\partial B_{\rho'_\ell})^+} (v_{\rho'_\ell} - v^*)^2 \mu_{\rho'_\ell} = 0,
$$

and thus $v' = v^*$. We leave the verification of (11.6) to the reader. \hfill \Box

Theorem 11.2. Let $2 \leq p < \infty$, and let $\kappa, \nu$ be positive integers with $2 \leq \nu < \kappa$. Let $a^{ij} \in C^{\kappa-1,1}(B_1^+(0) \cup B_1^-(0))$ such that $a^{ij} = a^{ji}$ on $B_1^+(0)$, satisfying (11.1) for some constants $0 < \lambda \leq \Lambda < \infty$, and such that (11.2) holds true on $B_1^-(0)$. Let $k_+, k_- \in C^0,1(B_1(0))$ with $k_+, k_- \geq 0$ and $h \in C^{\kappa-1}(B_1^-(0))$. Let $u \in W^{1,2}(B_1^+(0))$ be a solution to (12.1). For each $x_0 \in \Sigma(U)$, let $v_{x_0}^*$ be as in Theorem 11.1. Then the mapping $x_0 \mapsto v_{x_0}^*$ from $\Sigma(U)$ to $C^\nu(B_1(0))$ is continuous. Moreover, for each compact set $K \subseteq \Sigma(U)$ there exists a modulus of continuity $\omega = \omega^K$ such that $\omega(0^+) = 0$ and

$$
|v_{x_0, \kappa}(x) - v_{x_0, \kappa}^*(x)| \leq \omega(|x - x_0|)|x - x_0|^{\nu}
$$

for all $x_0 \in K$, where $v_{x_0, \kappa}$ is as in (11.1).
Proof. Assume first that $0 \in \Sigma_\nu(u)$. We will show that the map $x_0 \mapsto v^*_0$ is continuous at the origin using Monneau’s monotonicity formula, Theorem 10.1. Recall that Theorem 10.1 applies for $x_0 \in \Sigma_\nu(u)$ after an affine change of variables $x \mapsto L(x_0)^{-1}(x - x_0)$ translates $x_0$ to the origin and transforms $a^{ij}(x_0)$ to $\delta_{ij}$. In particular, by the continuity of $a^{ij}$, for each $x_0 \in \Sigma_\nu(u)$ there exists an invertible $n \times n$ matrix $L(x_0) = (L_{ij}(x_0))$, with inverse $L(x_0)^{-1} = (L^{ij}(x_0))$ such that $x_0 \mapsto L(x_0)$ is a continuous mapping on $\Sigma_\nu(u)$, $L_0(0) = \delta_{ij}$, and

$$
\sum_{k,l=1}^n \sqrt{\det L(x_0)} a^{kl}(x_0 + L(x_0)x)L^{ik}(x_0)L^{kj}(x_0) = \delta_{ij}
$$

for all $x_0 \in \Sigma_\nu(u)$ and $i, j = 1, 2, \ldots, n$. For each $x_0 \in \Sigma_\nu(u) \cap B^\nu_{1/4}(0)$ let

$$\tilde{v}_{x_0,\nu}(x) = v_{x_0,\nu}(L(x_0)x)$$

for all $x \in B^\nu_{1/4}(0) \cup B^\nu_{1/4}(0)$. Note that since $L_{ij}(0) = \delta_{ij}$, $\tilde{v}_{0,\nu} = v_{0,\nu}$. Fix $\varepsilon \in (0, 1)$. Select $\rho_\varepsilon > 0$ such that

$$\mathcal{M}_{\tilde{v}_{0,\nu}, v^*_0}(\rho_\varepsilon) = \rho^{1-n-2\nu}_{\varepsilon} \int_{(\partial B_{\rho_\varepsilon})^+} (\tilde{v}_{0,\nu} - v^*_0)^2 \mu < \varepsilon.
$$

By the continuous dependence of $L(x_0)$ and $\tilde{v}_{x_0,\nu}$ on $x_0 \in \Sigma_\nu(u)$, there exists $\delta_\varepsilon > 0$ such that for each $x_0 \in \Sigma_\nu(u) \cap B^\nu_\delta(0)$ we have

$$|L_{ij}(x_0) - \delta_{ij}| < \varepsilon,
$$

$$\mathcal{M}_{\tilde{v}_{x_0,\nu}, v^*_0}(\rho_\varepsilon) = \rho^{1-n-2\nu}_{\varepsilon} \int_{(\partial B_{\rho_\varepsilon})^+} (\tilde{v}_{x_0,\nu} - v^*_0)^2 \mu < 2\varepsilon.
$$

By Theorem 10.1 and (11.10),

$$\mathcal{M}_{\tilde{v}_{x_0,\nu}, v^*_0}(\rho) = \rho^{1-n-2\nu} \int_{(\partial B_{\rho})^+} (\tilde{v}_{x_0,\nu} - v^*_0)^2 \mu < C \rho + \rho_\varepsilon
$$

for all $x_0 \in \Sigma_\nu(u) \cap B^\nu_{\delta_\varepsilon}(0)$ and $0 < \rho \leq \rho_\varepsilon$, where $C \in (0, \infty)$ is a constant. Integrating (11.11),

$$\rho^{-n-2\nu} \int_{B^\nu_{\rho}} (\tilde{v}_{x_0,\nu} - v^*_0)^2 \mu < C(\varepsilon + \rho_\varepsilon)
$$

for all $x_0 \in \Sigma_\nu(u) \cap B^\nu_{\delta_\varepsilon}(0)$ and $0 < \rho \leq \rho_\varepsilon$, where $C \in (0, \infty)$ is a constant. Taking (11.9) and (11.12) into account, for sufficiently small $\delta_\varepsilon > 0$ we obtain

$$\rho^{-n-2\nu} \int_{B^\nu_{\rho}} (v_{x_0,\nu} - v^*_0)^2 \mu \leq C \rho^{-n-2\nu} \int_{B^\nu_{\rho}} (v_{x_0,\nu}(L(x_0)x) - v^*_0(L(x_0)x))^2 \mu dx
$$

$$= C \rho^{-n-2\nu} \int_{B^\nu_{\rho}} (\tilde{v}_{x_0,\nu}(x) - v^*_0(L(x_0)x))^2 \mu dx
$$

$$\leq 2C \rho^{-n-2\nu} \int_{B^\nu_{\rho}} (\tilde{v}_{x_0,\nu}(x) - v^*_0(x))^2 \mu dx
$$

$$+ 2C \rho^{-n-2\nu} \int_{B^\nu_{\rho}} (v^*_0(L(x_0)x) - v^*_0(x))^2 \mu dx
$$

$$< C(\varepsilon + \rho_\varepsilon) + C \max_{1 \leq i,j \leq n} \|L_{ij}(x_0) - \delta_{ij}\|
$$

$$< C(\varepsilon + \rho_\varepsilon)
$$

(11.13)
for all \( x_0 \in \Sigma_\nu(u) \cap B_{\delta_\epsilon}(0) \) and \( 0 < \rho \leq \rho_\epsilon / 2 \), where \( C \in (0, \infty) \) denotes constants. After rescaling,

\[
\int_{B_1^+} \left( \frac{v_{x_0,\kappa}(\rho x)}{\rho^p} - v_0^*(x) \right)^2 \mu(\rho x) \, dx < C(\epsilon + \rho_\epsilon)
\]

for all \( x_0' \in \Sigma_\nu(u) \cap B_{\delta_\epsilon}^c(0) \) and \( 0 < \rho \leq \rho_\epsilon / 2 \). Thus letting \( \rho \to 0^+ \) using Theorem [11.1] and [6.5],

\[
\int_{B_1^+} (v_{x_0}^* - v_0^*)^2 \mu(x) < C(\epsilon + \rho_\epsilon)
\]  \hspace{1cm} (11.14)

for all \( x_0 \in \Sigma_\nu(u) \cap B_{\delta_\epsilon}'(0) \). Therefore, \( x_0 \mapsto v_{x_0}^* \) is continuous at the origin as a mapping from \( \Sigma_\nu(u) \) to \( L^2(B_1^+) \). Since the space of all homogeneous degree \( \nu \) polynomials on \( \mathbb{R}^n \) is a finite dimensional vector space, \( x_0 \mapsto v_{x_0}^* \) is continuous at the origin as a mapping from \( \Sigma_\nu(u) \) to \( C^0((B_1^+)) \).

Now let \( K \subset \Sigma_\nu(u) \) be a compact set. By [11.13] and [11.14], for each \( \epsilon > 0 \) and \( x_0 \in \Sigma_\nu(u) \) there exists \( \rho_\epsilon(x_0) > 0 \) such that

\[
\rho^{-n-2\nu} \int_{B_1^+} (v_{x_0,\kappa} - v_{x_0}^*)^2 \mu(x) \leq 2\rho^{-n-2\nu} \int_{B_1^+} (v_{x_0,\kappa} - v_{x_0}^*)^2 \mu(x) + 2 \rho^{-n-2\nu} \int_{B_1^+} (v_{x_0}^* - v_{x_0}^*)^2 \mu(x) \leq C(\epsilon + \rho_\epsilon)
\]

for all \( x_0' \in \Sigma_\nu(u) \cap B_{\delta_\epsilon}'(x_0) \) and \( 0 < \rho \leq \rho_\epsilon(x_0) / 2 \), where \( v_{x_0,\kappa} \) is as in [11.1] (with \( x_0' \) replacing \( x_0 \)), \( \mu(x_0) = \sum_{i=1}^n a^{ij}(x + x_0)(x + x_0)_i(x + x_0)_j / |x + x_0|^2 \), and \( C = C(x_0) \in (0, \infty) \) is a constant. Since \( K \) is compact, we can cover \( K \) by open balls \( B_{\delta_\epsilon}(x_0') \), where \( x_0' \in K \) for \( i = 1, 2, \ldots, N \). Then setting

\[
\rho^K_\epsilon = \min\{\rho_\epsilon(x_0') : i = 1, 2, \ldots, N\}, \quad C^K = \max\{C(x_0') : i = 1, 2, \ldots, N\},
\]

we have that

\[
\rho^{-n-2\nu} \int_{B_1^+} (v_{x_0,\kappa} - v_{x_0}^*)^2 \mu(x) \leq C^K(\epsilon + \rho^K_\epsilon)
\]  \hspace{1cm} (11.15)

for all \( x_0' \in K \) and \( 0 < \rho \leq \rho^K_\epsilon / 2 \). At this point we introduce the rescalings

\[
w^{(\rho)}(x) = w^{(\rho)}_{x_0,\kappa}(x) = \frac{v_{x_0,\kappa}(\rho x)}{\rho^\nu} \quad \text{and} \quad f^{(\rho)}(x) = f^{(\rho)}_{x_0,\kappa}(x) = \frac{f_{x_0,\kappa}(\rho x)}{\rho^{p-2}},
\]

where \( f_{x_0,\kappa} \) as in [5.3], but corresponding to the free boundary point \( x_0' \in \Sigma_\nu(u) \). Since \( v_{x_0,\kappa} \) as in [11.1] satisfies [5.7] (with \( x_0' \) replacing the origin), and \( v_{x_0}^* \) satisfies

\[
a^{ij}(x_0') D_{ij} v_{x_0}^* = 0 \quad \text{in} \quad \mathbb{R}^n_+ \quad \text{and} \quad D_n v_{x_0}^* = 0 \quad \text{on} \quad \mathbb{R}^{n-1} \times \{0\},
\]

we have

\[
D_i(\tilde{a}^{ij} D_j(w^{(\rho)} - v_{x_0}^*)) = f^{(\rho)} - D_i((\tilde{a}^{ij} - \tilde{a}^{ij}(0)) D_j v_{x_0}^*) \quad \text{in} \quad B_1^+,
\]

\[
a^{mn} D_n(w^{(\rho)} - v_{x_0}^*) = \rho^{1+(p-2)\nu}\left(\tilde{k}_+((w^{(\rho)})^+) - \tilde{k}_-((w^{(\rho)})^-)\right) \quad \text{on} \quad B_1^+,
\]

where \( \tilde{a}^{ij}(x) = a^{ij}(x' + \rho x) \) and \( \tilde{k}_\pm(x) = k_\pm(x' + \rho x) \). Using [6.1], \( a^{ij} \in C^{0,1}(B_1^+ \cup B_1') \), and \( 2 \leq \nu < \kappa \), we infer

\[
|f^{(\rho)}(x) - D_i((\tilde{a}^{ij}(x) - \tilde{a}^{ij}(0)) D_j v_{x_0}^*(x))| \leq C(\rho^{\kappa-\nu+1} + \rho) \leq C\rho.
\]  \hspace{1cm} (11.16)
on $B^+_i$ for some constant $C \in (0, \infty)$ independent of $\rho$. Observing that $(w^{(\rho)})^{p-1}$ is bounded, taking (11.15) and (11.16) into account, and applying $L^2 - L^\infty$ estimates for weak solutions to the oblique derivative problems (see for instance [L13, Theorem 5.36]) we obtain

$$|w^{(\rho)}(x) - v_{x_0}^*(x)| \leq C \left( (\varepsilon + \rho_\varepsilon^K)^{1/2} + \rho_\varepsilon^K \right) \leq C \varepsilon$$

for all $x_0^i \in K$ and $x \in B_{1/2}(x_0^i)$, where $C \in (0, \infty)$ is a constant. It is easy to see that this implies the second part of the theorem, and the proof is thus complete.

For each $x_0 \in \Sigma_\nu(u)$, since $v_{x_0}^*$ is a homogeneous degree $\nu$ polynomial, we can express $v_{x_0}^*$ as

$$v_{x_0}^*(x) = \sum_{|\alpha| = \nu} \frac{F_\alpha(x_0)}{\alpha!} x_\alpha,$$

where $F_\alpha(x_0) \in \mathbb{R}$ for all $|\alpha| = \nu$ and $x_0 \in \Sigma_\nu(u)$. Define $F_\alpha(x_0) = 0$ for all $|\alpha| < \nu$ and $x_0 \in \Sigma_\nu(u)$.

**Lemma 11.2.** For each compact set $K \subset \Sigma_\nu(u)$,

$$F_\alpha(x) = \sum_{|\beta| \leq \nu - |\alpha|} \frac{F_{\alpha+\beta}(x_0)}{\beta!} (x - x_0)^{\alpha+\beta} + R_\alpha(x, x_0)$$

for all $|\alpha| \leq \nu$ and $x, x_0 \in K$, where

$$|R_\alpha(x, x_0)| \leq \omega_\alpha(|x - x_0|) |x - x_0|^{\nu - |\alpha|}$$

for all $|\alpha| \leq \nu$ and $x, x_0 \in K$ and some modulus of continuity $\omega_\alpha$ with $\omega_\alpha(0^+) = 0$.

**Proof.** In the case that $|\alpha| = \nu$, then

$$R_\alpha(x, x_0) = F_\alpha(x) - F_\alpha(x_0) = D^\nu v_{x_0}^*(x) - D^\nu v_{x_0}^*(x_0)$$

for each $x, x_0 \in K$. Thus (11.18) holds true by the continuity of $x_0 \mapsto v_{x_0}^*$. In the case that $|\alpha| < \nu$, then

$$R_\alpha(x, x_0) = -\sum_{|\beta| = \nu - |\alpha|} \frac{F_{\alpha+\beta}(x_0)}{\beta!} (x - x_0)^{\alpha+\beta} = -D^\nu v_{x_0}^*(x - x_0)$$

for each $x, x_0 \in K$. To show (11.18), suppose to the contrary that there exists $\varepsilon > 0$ and $x_i, x_0^i \in K$ such that $\rho_i = |x_i - x_0^i| \to 0$ and

$$|R_\alpha(x_i, x_0^i)| = |D^\nu v_{x_0^i}^*(x_i - x_0^i)| \geq \varepsilon |x_i - x_0^i|^{\nu - |\alpha|}.$$

(11.19)

Set

$$w_i(x) = \frac{v_{x_i,0^i}(\rho_i x)}{\rho_i^\nu}, \quad \tilde{w}_i(x) = \frac{v_{x_i,0^i}(\rho_i x)}{\rho_i^\nu}, \quad \xi_i = \frac{x_i - x_0^i}{\rho_i}.$$

(11.20)

After passing to a subsequence there exists $x_0 \in K$ such that $x_i \to x_0$ and $x_0^i \to x_0$ as $i \to \infty$, and there exists $\xi$ such that $|\xi| = 1$ and $\xi_i \to \xi$ as $i \to \infty$. By Theorem 11.2

$$|w_i(x) - v_{x_0}^*(x)| \leq |w_i(x) - v_{x_0}^*(x)| + |v_{x_0}^*(x) - v_{x_0}^*(x)| \leq \omega(\rho_i |x|) |x|^\nu + \|v_{x_0}^* - v_{x_0}^*\|_{L^2(B^+_i)} |x|^\nu$$

for each $x \in B_{1/(2\rho_i)}(0) \cup B_{1/(2\rho_i)}'(0)$, where $\omega$ is a modulus of continuity with $\omega(0^+) = 0$ and $\|v_{x_0}^* - v_{x_0}^*\|_{L^2(B^+_i)} \to 0$ as $i \to \infty$. Hence

$$w_i(x) \to v_{x_0}^*(x)$$

(11.21)
uniformly on each compact subset of \( \mathbb{R}^n_+ \) as \( i \to \infty \). Similarly,
\[
\tilde{w}^i(x) \to v_{x_0}^*(x) \tag{11.22}
\]
uniformly on each compact subset of \( \mathbb{R}^n_+ \) as \( i \to \infty \). By (11.20) and (11.1),
\[
w^i(\xi_i + x) = \frac{u(x^i + \rho x) - h((x^i + \rho x)')}{\rho^\nu_i} = \tilde{w}^i(x)
\tag{11.23}
\]
for each \( x \in B_{1/(4\rho_i)}(0) \). Letting \( i \to \infty \) in (11.23) using \( \xi_i \to \xi \), (11.21), and (11.22), we obtain
\[
v_{x_0}^*(\xi + x) = v_{x_0}^*(x)
\]
for each \( x \in \mathbb{R}^{n-1} \times \{0\} \). Hence, recalling that \( v_{x_0}^* \) satisfies (11.16),
\[
d^{\alpha}(x_0) D_{i\nu}v_{x_0}^*(\xi + x) = 0 = d^{\alpha}(x_0) D_{i\nu}v_{x_0}^*(x) \text{ in } \mathbb{R}^n_+,
\]
\[
v_{x_0}^*(\xi + x) = v_{x_0}^*(x) \text{ on } \mathbb{R}^{n-1} \times \{0\},
\]
\[
D_n v_{x_0}^*(\xi + x) = 0 = D_n v_{x_0}^*(x) \text{ on } \mathbb{R}^{n-1} \times \{0\},
\]
and thus we must have that
\[
v_{x_0}^*(\xi + x) = v_{x_0}^*(x)
\]
for each \( x \in \mathbb{R}^n_+ \). Recalling that \( |\alpha| < \nu \), it follows that \( D^\alpha v_{x_0}^*(\xi) = D^\alpha v_{x_0}^*(0) = 0 \). However, by dividing both sides of (11.19) by \( \rho_i^{\nu - |\alpha|} \) and letting \( i \to \infty \), we must have that \( |D^\alpha v_{x_0}^*(\xi)| \geq \varepsilon \), yielding a contradiction. \( \square \)

At this point, we are ready to give the proof of Theorem 1.2. For each \( x_0 \in \Sigma_{\nu}(u) \), there exists a largest linear subspace \( S(v_{x_0}^*) \subseteq \mathbb{R}^{n-1} \times \{0\} \) (with respect to set inclusion) such that \( v_{x_0}^*(z + x) = v_{x_0}^*(x) \) for all \( x \in \mathbb{R}^n \) and \( z \in S(v_{x_0}^*) \). Note that if \( \dim S(v_{x_0}^*) = n - 1 \), then \( v_{x_0}^* \) must be a non-zero linear function of \( x_n \), contradicting \( D_n v_{x_0}^* = 0 \) on \( \mathbb{R}^{n-1} \times \{0\} \). Therefore, \( \dim S(v_{x_0}^*) \leq n - 2 \) for each \( x_0 \in \Sigma_{\nu}(u) \). For each positive integer \( d \) we define
\[
\Sigma_d^\nu(u) = \{ x \in \Sigma_{\nu}(u) : \dim S(v_{x_0}^*) = d \}.
\]

**Proof of Theorem 1.2.** Let \( K \subseteq \Sigma_{\nu} \) be a compact set. By Lemma 11.2 and the Whitney extension theorem, there exists a function \( F \in C^\nu(\mathbb{R}^{n-1} \times \{0\}) \) such that
\[
D^\alpha F(x_0) = F_\alpha(x_0)
\tag{11.24}
\]
for all \( x_0 \in K \) and \( |\alpha| \leq \nu \), where \( F_\alpha(x_0) \) is as in (11.17).

Let \( x_0 \in \Sigma^\nu_d(u) \cap K \). Since \( \dim S(v_{x_0}^*) = d \), there exists a set \( \{ \xi_1, \xi_2, \ldots, \xi_{n-1-d} \} \subseteq \mathbb{R}^{n-1} \times \{0\} \) of \( (n - 1 - d) \) linearly independent vectors which are orthogonal to \( S(v_{x_0}^*) \). Hence \( \xi_i \cdot \nabla v_{x_0}^* \) is not identically zero on \( \mathbb{R}^n \) for \( i = 1, 2, \ldots, n - 1 - d \). Thus for each \( i = 1, 2, \ldots, n - 1 - d \) there exists a multi-index \( \beta_i \) such that \( |\beta_i| = \nu - 1 \)
\[
\xi_i \cdot \nabla D^\beta_i v_{x_0}^*(0) \neq 0.
\]
By (11.17) and (11.24), \( \xi_i \cdot \nabla D^\beta_i v_{x_0}^*(0) = \xi_i \cdot \nabla D^\beta_i F(x_0) \) and thus
\[
\xi_i \cdot \nabla D^\beta_i F(x_0) \neq 0.
\]
However,
\[ \Sigma^d_u(\omega) \cap K \subseteq \bigcap_{i=1}^{n-1-d} \{ D^i F = 0 \}. \]

Using the implicit function theorem it follows that \( \Sigma^d_u(\omega) \cap K \) is contained in a \( d \)-dimensional submanifold in some open neighborhood of \( x_0 \). Hence by a standard covering argument, \( \Sigma^d_u \) is contained in a countable union of \( d \)-dimensional \( C^1 \)-submanifolds of \( B'_1(0) \).

We conclude by noting that \( N_u(x_0) < \infty \) for all \( x_0 \in \Sigma(\omega) \) in some special cases.

**Corollary 11.1.** Let \( 2 \leq p < \infty \) and \( v \) be a positive integer. Let \( a^{ij} \in C^{0,1}(B^+_1(0) \cup B'_1(0)) \) such that \( a^{ij} = a^{ji} \) on \( B^+_1(0) \), satisfying (1.1) holds true for some constants \( 0 < \lambda \leq \Lambda < \infty \), and such that (1.2) holds true on \( B'_1(0) \). Let \( k_+, k_- \in C^{0,1}(B'_1(0)) \) with \( k_+, k_- \geq 0 \) and \( h : B'_1(0) \to \mathbb{R} \) be a function. Let \( u \in W^{1,2}(B^+_1(0)) \) be a solution to (1.2) with \( h = 0 \) on \( B'_1(0) \). If either

(i) \( h = 0 \) on \( B'_1(0) \), or

(ii) \( a^{ij} \) and \( h \) are both locally real-analytic at each point of \( B'_1(0) \),

then \( N_u(x_0) < \infty \) for all \( x_0 \in \Sigma(\omega) \) and thus \( \Sigma(\omega) \) is contained in a countable union of \((n - 2)\)-dimensional \( C^1 \)-submanifolds of \( B'_1(0) \).

**Proof.** First we show that case (ii) reduces to case (i). Suppose that \( a^{ij} \) and \( h \) are both locally real-analytic at each point of \( B'_1(0) \). After rescaling, assume that the radii of convergence of \( a^{ij} \) and \( h \) at the origin are greater than one. Then by the Cauchy-Kowalevskiy theorem there exists a locally real-analytic function \( \overline{h} : B^+_1(0) \cup B'_1(0) \to \mathbb{R} \) such that

\[ D_i(a^{ij} D_j \overline{h}) = 0 \text{ on } B^+_1(0), \]
\[ \overline{h}(x',0) = h(x'), \quad D_n \overline{h}(x',0) = 0 \text{ on } B'_1(0). \]  
(11.25)

Thus we can set \( v = u - \overline{h} \) so that

\[ D_i(a^{ij} D_j v) = 0 \text{ on } B^+_1(0), \]
\[ a^{mn} D_n v = k_+(v^+)^{p-1} - k_-(v^-)^{p-1} \text{ on } B'_1(0). \]

In other words, \( v \) solves the boundary value obstacle problem with obstacle zero.

Now let’s consider the case (i) where \( h = 0 \) on \( B'_1(0) \). Let \( x_0 \in \Sigma(\omega) \). After translating \( x_0 \) to the origin and applying a linear change of variables, assume that \( x_0 = 0 \) and \( a^{ij}(x_0) = \delta_{ij} \). Since \( h = 0 \) on \( B'_1(0) \), we can omit the truncation step in Section 5 and simply set \( v = u \). Then we simply define Almgren frequency function \( \Phi(\rho) \) by

\[ \Phi(\rho) = \frac{I(\rho)}{H(\rho)} \]

for all \( 0 < \rho < 1 \) with \( H(\rho) > 0 \). Theorem 6.1] e^{C\rho} \Phi(\rho) + Ce^C \rho \) is monotone nondecreasing for all \( 0 < \rho \leq \rho_0 \) with \( H(\rho) > 0 \), where \( \rho_0 \in (0, 1/2) \) and \( C \in (0, \infty) \) are constants as in Theorem 6.1. By Lemma 7.1] either \( u = 0 \) in \( B^+_1(0) \cup B'_1(0) \) or \( H(\rho) > 0 \) for all \( 0 < \rho \leq \rho_0 \). Hence if \( u \) is not identically zero on \( B^+_1(0) \cup B'_1(0) \), \( e^{C\rho} \Phi(\rho) + Ce^C \rho \) is monotone nondecreasing for all \( 0 < \rho \leq \rho_0 \). In particular,

\[ \Phi(\rho) \leq e^{C/2} \Phi(1/2) + Ce^C/2 < \infty \]

for all \( 0 < \rho \leq \rho_0 \) and we can define the Almgren’s frequency of \( u \) at the origin to be equal to \( \Phi(0^+) < \infty \). The rest of the argument proceeds exactly like before with obvious modifications. \( \square \)
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