REDUCTIONS OF THE UNIVERSAL HIERARCHY AND RDDYM EQUATIONS AND THEIR SYMMETRY PROPERTIES

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ABSTRACT. We consider the equations $u_{yy} = u_y u_{xx} - (u_x + u) u_{xy} + u_x u_y$ and $u_{yy} = (u_x + x) u_{xy} - (u_{xx} + 2) u_y$ that arise as reductions of the universal hierarchy and rdDym equations, respectively, and describe the Lie algebras of nonlocal symmetries in infinite-dimensional coverings naturally associated to these equations.

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INTRODUCTION

In a series of recent papers \[2, 3, 1, 8\] we studied symmetry and integrability properties of the four linearly degenerate 3D equations, \[7\]. In particular, in \[3\] we described 2D reductions of the Pavlov, universal hierarchy and rdDym equations that possess differential coverings of the rational form. In the recent paper \[8\], using the reduction techniques, we showed that for the two of these reductions (one of them being the Gibbons-Tsarev equation) the Lie algebra of nonlocal symmetries is isomorphic to the Witt algebra

\[ \mathfrak{w} = \{ z^{i+1} \frac{\partial}{\partial z} \mid i \in \mathbb{Z} \} \]

of polynomial vector fields.

In the current paper, we prove similar results for the reduction \[ u_{yy} = u_y u_{xx} - (u_x + u)u_{xy} + u_x u_y \]

of the universal hierarchy equation (Section 2) and for the equation \[ u_{yy} = (u_x + x)u_{xy} - (u_{xx} + 2)u_y, \]

which is a reduction of the rdDym equation (Section 3). Namely, we show that in the first case the algebra of nonlocal symmetries is isomorphic to \( \mathfrak{w} \oplus \mathfrak{s}_2 \), where \( \mathfrak{s}_2 \) is the two-dimensional solvable Lie algebra (Theorem 2.1), while in the second case this algebra is \( \mathfrak{w} \oplus \mathfrak{a}_1 \), where \( \mathfrak{a}_1 \) is the one-dimensional Abelian Lie algebra, see Theorem 3.1.

In Section 1 we introduce the necessary definitions and constructions. Section 4 contains a short discussion of the obtained results.

1. DIFFERENTIAL COVERINGS AND NONLOCAL SYMMETRIES

Here we briefly discuss the necessary facts from nonlocal geometry of PDEs. See details in \[6, 9\].

Let \( \mathcal{E} \subset J^\infty(n, m) \) be an infinitely prolonged differential equation (or a system of equations) in unknowns \( u^j(x^1, \ldots, x^n), \ j = 1, \ldots, m \), embedded to the corresponding infinite jet space. Denote by \( u^j_\sigma \) jet coordinates and assume that \( \mathcal{E} \) is defined by a system of relations \( F^\alpha(x^1, \ldots, x^n, u^1_\sigma, \ldots) = 0, \ j = 1, \ldots, l \). Denote by

\[ D_i = \frac{\partial}{\partial x^i} + \sum u^j_\sigma \frac{\partial}{\partial u^j_\sigma} \]

the total derivatives on \( \mathcal{E} \). Let

\[ \ell_\mathcal{E} = (\sum \sigma \frac{\partial F^\alpha}{\partial u^\alpha_\sigma} D_\sigma) \]

be the linearization of \( \mathcal{E} \), where \( D_\sigma \) is the composition of the total derivatives corresponding to the multi-index \( \sigma \).

A symmetry of \( \mathcal{E} \) is an evolutionary vector field

\[ \mathbf{E}_\varphi = \sum D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma} \]

such that \( \ell_\mathcal{E}(\varphi) = 0 \), where \( \varphi = (\varphi^1, \ldots, \varphi^m) \) is a function on \( \mathcal{E} \) which is called the generating section of the symmetry at hand. Symmetries form an \( \mathbb{R} \)-Lie algebra with respect to the commutator. This algebra is denoted by \( \text{sym}(\mathcal{E}) \). The commutator of symmetries induces the Jacobi bracket of their generating sections denoted by \( \{ \cdot, \cdot \} \).

A horizontal \((n-1)\)-form

\[ \omega = \sum A_i \, dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n \]

is a conservation law of \( \mathcal{E} \) if it is closed with respect to the horizontal de Rham differential

\[ d_h = \sum dx^i \wedge D_i. \]
A conservation law is trivial if \( \omega \) is an exact form. Two conservation laws are equivalent if their difference is a trivial conservation law.

Consider another equation \( \tilde{E} \) and a locally trivial bundle \( \tau: \tilde{E} \to E \). It is called a (differential) covering if \( \tau_*(\tilde{D}_i) = D_i \) for any total derivative on \( \tilde{E} \). Two coverings \( \tau_l: \tilde{E}_l \to E \), \( l = 1, 2 \), are equivalent if there exists a diffeomorphism \( F: \tilde{E}_1 \to \tilde{E}_2 \) such that (1) \( \tau_2 \circ F = \tau_1 \) and (2) \( F^*(\tilde{D}_1^l) = \sum \mu^l_j \tilde{D}_j^l \), where \( \mu^l_j \) are smooth functions on \( \tilde{E}_2 \) and \( \tilde{D}_j^l \) are the total derivatives on \( \tilde{E}_l \). Symmetries of \( \tilde{E} \) are said to be nonlocal symmetries of \( E \) and similar for conservation laws.

Denote by \( F \) and \( \tilde{F} \) the rings of smooth functions on \( E \) and \( \tilde{E} \), respectively. Then an \( \mathbb{R} \)-linear derivation \( S: F \to \tilde{F} \) is a nonlocal shadow if

\[
\tilde{D}_i \circ S = S \circ D_i, \quad i = 1, \ldots, n.
\]

In particular, local symmetries can be regarded as shadows in any covering. We say that a shadow \( S \) lifts to \( \tau \) if there exists a nonlocal symmetry \( \tilde{S} \) such that \( \tilde{S} \big|_F = S \). Lifts of the trivial shadow \( S = 0 \) are called invisible symmetries.

Denote by \( \{w^\beta\} \) coordinates in fibers of \( \tau \). They are called nonlocal variables. Using these variables, we can write the fields \( \tilde{D}_i \) as

\[
\tilde{D}_i = D_i + \sum X_i^\beta \frac{\partial}{\partial w^\beta},
\]

where \( X_i^\beta \) are smooth functions on \( \tilde{E} \), while the fact that \( \tau \) is a covering amounts to the compatibility of the system

\[
\dot{w}_x^\beta = X_i^\beta \tag{3}
\]

modulo \( \tilde{E} \). Then nonlocal \( \tau \)-symmetries are vector fields

\[
\tilde{E}_\varphi + \sum \psi^\beta \frac{\partial}{\partial w^\beta},
\]

where \( \tilde{E}_\varphi \) is obtained from (1) by changing \( D_i \) to \( \tilde{D}_i \) and \( \varphi = (\varphi^1, \ldots, \varphi^m) \), \( \psi^\beta \) are functions on \( \tilde{E} \) that enjoy the system

\[
\dot{\ell}_E(\varphi) = 0, \tag{2}
\]

\[
\dot{D}_i(\psi^\beta) = \dot{\ell}_{X_i^\beta}(\varphi) + \sum \frac{\partial X_i^\beta}{\partial w^\gamma} \psi^\gamma, \tag{3}
\]

where “tilde” denotes the natural lift of a differential operator in total derivatives from \( E \) to \( \tilde{E} \). To describe shadows, one must consider Equation (2) only, while invisible symmetries are described by the equation

\[
\dot{D}_i(\psi^\beta) = \sum \frac{\partial X_i^\beta}{\partial w^\gamma} \psi^\gamma.
\]

Let

\[
\omega = (X_1 dx^1 + X_2 dx^2) \wedge dx^3 \wedge \cdots \wedge dx^n
\]

be a two-component conservation law of \( E \). Then one can construct the covering \( \tau_\omega \) with the nonlocal variables \( w^\sigma \), where \( \sigma \) is a symmetrical multi-index containing the integers \( 3, \ldots, m \), and the defining equations

\[
w^\sigma_{x^i} = \tilde{D}_\sigma(X_i), \quad w^\sigma_{x^j} = \tilde{D}_\sigma(X_j), \quad w^\sigma_{x^i} = w^\sigma_i,
\]

for \( i \geq 3 \). This is the Abelian covering associated with \( \omega \); it is one-dimensional for \( n = 2 \) and infinite-dimensional otherwise.
2. The universal hierarchy equation

The universal hierarchy equation is of the form

\[ u_{yy} = u_t u_{xy} - u_y u_{tx}, \]  

see [10, 11].

2.1. Lax pair and the associated covering. Equation (4) admits the following Lax pair

\[ w_t = \lambda^{-2}(\lambda u_t - u_y)w_x, \]
\[ w_y = \lambda^{-1}u_yw_x. \]

Expanding \( w \) in powers of \( \lambda \),

\[ w = \sum_{i \in \mathbb{Z}} w_i \lambda^i, \]

we obtain the infinite-dimensional covering

\[ w_{i,t} = u_t w_{i+1,x} - u_y w_{i+2,x}, \]
\[ w_{i,y} = u_y w_{i+1,x}, \]

\[ i \in \mathbb{Z}, \]

with the additional variables \( w_i^{(j)} \) that satisfy the relations

\[ w_i^{(0)} = w_i, \quad w_i^{(j+1)} = w_i^{(j)} + \lambda. \]

2.2. Symmetries and reductions. The space \( \text{sym}(\mathcal{E}) \) is spanned by the functions \( \theta_0(X) = Xu_x - X'u, \theta_1(X) = X, \varphi_0(T) = Tu_t + T'y_y, \varphi_1(T) = Tu_y, \upsilon = yu_y + u \), where \( X \) is a function in \( x \) and \( T \) is a function in \( t \), while ‘prime’ denotes the corresponding derivatives.

**Lemma 2.1.** The symmetry \( \varphi = \upsilon + \theta_0(1) + \varphi_0(1) \) can be lifted to the covering (5).

**Proof.** Denote the desired lift by

\[ \Phi = \mathcal{E}_\varphi + \sum \varphi^i \frac{\partial}{\partial w_i}, \]

where \( \varphi = yu_y + u + u_x + u_t \), and set

\[ \varphi^i = (-i + 1)w_i + yw_{i,y} + \frac{1}{u_y}w_{i-1,y} + w_{i,t}. \]

Then the result is obtained by the direct check. \( \square \)

Due to Lemma 2.1, we can consider the reduction of Equation (4) together with its covering (5). The resulting objects will be the equation

\[ u_{yy} = u_y u_{xx} - (u_x + u)u_{xy} + u_x u_y \]

and the infinite-dimensional covering

\[ q_{i,y} = (-i + 2)q_{i-1} - \frac{u_x + u}{u_y}q_{i-1,y} + \frac{1}{u_y}q_{i-2,y}, \]
\[ q_{i,x} = q_{i-1,y}/u_y. \]

over this equation. Define the coverings \( \tau^p \) by setting \( q_i = 0 \) for \( i < p, p \in \mathbb{Z} \). Then, setting \( q_i^p = q_{p+i+1} \), we obtain \( q_{-1} = 1, \quad q_0^p = -(p - 1)y \) and

\[ \tau^p: \quad q_{i,y}^p = (-p - i + 1)q_{i-1}^p - \frac{u_x + u}{u_y}q_{i-1,y}^p + \frac{1}{u_y}q_{i-2,y}^p, \]
\[ q_{i,x}^p = \frac{q_{i-1,y}^p}{u_y}, \]

for \( i \geq 1 \). This is an infinite series of nonlocal conservation laws of Equation (4).

**Proposition 2.1.** All the coverings \( \tau^p \) are pair-wise equivalent.
Proof of Proposition 2.1. We shall prove that any \( \tau^p \) considered.

We also assume \( P_{i,j} = 0 \) when at least one of the subscripts is negative.

Proof of Proposition 2.1. We shall prove that any \( \tau^p \) is equivalent to \( \tau^0 \). Two cases are to be considered.

Case \( p \neq 1 \). Let us set

\[
d_i = \sum_{l=0}^{\infty} (-1)^l \frac{(p + l)!}{p!} P_{i,i-l-1}^0.
\]

Then

\[
q_i^p = -(p - 1)(q_i^0 - pd_i), i \geq 1,
\]

is the desired equivalence.

Case \( p = 1 \). This way of proof does not work for \( p = 1 \), but from the defining equations one can easily see that the covering \( \tau^1 \) coincides with \( \tau^2 \).

2.3. Weights. Let us assign to all the local and nonlocal variables the weights

\[
|x| = 0, \quad |y| = 1, \quad |u| = -1, \quad |q_i^p| = i + 1.
\]

We also set \( |u_x| = |u| - |x|, \) \( |u_y| = |u| - |y| \), etc., and assume that the weight of a monomial is the sum of weights of its factor. The weight of a vector field \( Z \partial / \partial z \) is \( |Z| - |z| \). Then all the constructions under consideration become graded with respect to these weights, while the results (provided they are polynomial) split into homogeneous components.

2.4. Nonlocal symmetries of reductions. Let us use the notation

\[
\Phi = (\varphi, \varphi^{p,1}, \ldots, \varphi^{p,i}, \ldots)
\]

for the vector field

\[
S = \tilde{E}_\varphi + \sum \varphi^{p,i} \frac{\partial}{\partial q_i^p}
\]

on \( \tau^p \). Then (9) is a symmetry if and only if

\[
\tilde{D}_y^2(\varphi) = u_y \tilde{D}_x^2(\varphi) - (u_x + u) \tilde{D}_x \tilde{D}_y(\varphi) + (u_y - u_{xy}) \tilde{D}_x(\varphi) + (u_x + u_{xx}) \tilde{D}_y(\varphi) - u_{xy} \varphi
\]

and

\[
\begin{align*}
\tilde{D}_y(\varphi^{p,1}) &= (p - 1) \tilde{\mathcal{L}}_1(\varphi), \\
\tilde{D}_x(\varphi^{p,1}) &= -(p - 1) \tilde{\mathcal{L}}_2(\varphi); \\
\tilde{D}_y(\varphi^{p,2}) &= -(p + 1) \varphi^{p,1} - \frac{u_x + u}{u_y} \tilde{D}_y(\varphi^{p,1}) - q_{i,y}^p \tilde{\mathcal{L}}_1(\varphi) - (p - 1) \tilde{\mathcal{L}}_2(\varphi), \\
\tilde{D}_x(\varphi^{p,2}) &= \frac{1}{u_y} \tilde{D}_y(\varphi^{p,1}) + q_{i,y}^p \tilde{\mathcal{L}}_2(\varphi); \\
\tilde{D}_y(\varphi^{p,i}) &= -(p - i + 1) \varphi^{p,i-1} - \frac{u_x + u}{u_y} \tilde{D}_y(\varphi^{p,i-1}) + \frac{1}{u_y} \tilde{D}_y(\varphi^{p,i-2}) \\
&\quad - q_{i-1,y}^p \tilde{\mathcal{L}}_1(\varphi) + q_{i-2,y}^p \tilde{\mathcal{L}}_2(\varphi), \\
\tilde{D}_x(\varphi^{p,i}) &= \frac{1}{u_y} \tilde{D}_y(\varphi^{p,i-1}) + q_{i-1,y}^p \tilde{\mathcal{L}}_2(\varphi),
\end{align*}
\]

for all \( i > 2 \), where

\[
\tilde{\mathcal{L}}_1 = \frac{1}{u_y} + \frac{1}{u_y} \tilde{D}_x - \frac{u_x + u}{u_y^2} \tilde{D}_y, \quad \tilde{\mathcal{L}}_2 = -\frac{1}{u_y} \tilde{D}_y
\]
are the linearizations of the functions \((u_x + u)/u_y \) and \(1/u_y\), respectively, lifted to \(\tau^p\).

Direct computations show that the functions
\[
\varphi_{-1} = u_y, \quad \varphi_0 = yu_y + u, \quad \psi_0 = u_x, \quad \psi_1 = e^{-x}
\]
constitute a basis of the space \(\text{sym}(E)\). In addition, it can be checked that the function
\[
\varphi_2^p = (2p^2y^2 + py^2 - 3y^2 - 4q_1^p)(u_x + u) - 3py + 3y
\]
\[
+ \frac{1}{3}(5p^3y^3 + 6p^2y^3 - 8py^3 - 3y^3 - 15pyq_1^p - 27yq_1^p - 15q_2^p)u_y
\]
is a shadow in the covering \(\tau^p\). Here the subscripts indicate the weight of the corresponding symmetry.

**Lemma 2.2.** The local symmetries \(\varphi_0, \psi_0, \) and \(\psi_1\) can be lifted to any covering \(\tau^p\).

**Proof.** Let us set
\[
\varphi_0^p,i = -(i + 1)q_i^p + yq_i^p \quad \text{for} \quad \varphi_0 = yu_y + u,
\]
\[
\psi_0^p = q_i^p \quad \text{for} \quad \psi_0 = u_x,
\]
\[
\psi_1^p = 0 \quad \text{for} \quad \psi_1 = e^{-x}.
\]
Then it is an easy exercise to check that \(\text{(10)}\) fulfills. □

**Lemma 2.3.** The symmetry \(\varphi_{-1}\) can be lifted to the covering \(\tau^0\), while the shadow \(\varphi_2^3\) can be lifted to the covering \(\tau^3\).

**Proof.** The lift of \(\varphi_{-1} = u_y\) is given by the formulas
\[
\varphi_{-1}^0 = q_{i,y}^0.
\]
The lift of
\[
\varphi_2^3 = (18y^2 - 4q_1^3)(u_x + u) + (54y^3 - 24yq_1^3 - 5q_2^3)u_y - 6y
\]
is given by
\[
\varphi_{-1}^3 = 2(i + 3)(2q_3^1 - 9y^2)q_3^1 - 6i + 4yq_3^1 + 2(i + 5)q_3^2 + 2(9y^2 - 2q_1^3)q_3^3.
\]
Then \(\text{(10)}\) fulfills identically. □

**Lemma 2.4.** The field \(\Phi_{-2} = \partial/\partial q_1^{-1}\) is an invisible symmetry in \(\tau^{-1}\).

**Proof.** Direct check. □

**Corollary 2.1.** There exist symmetries \(\Phi_{-2}, \Phi_{-1}, \Phi_0, \Phi_1, \Psi_0, \) and \(\Psi_1\) in any covering \(\tau^p\).

**Proof.** The fact follows immediately from Proposition 2.1 and Lemmas 2.2, 2.3. □

**Theorem 2.1.** The Lie algebra of nonlocal symmetries for Equation (6) in \(\tau^p\) is isomorphic to the direct sum
\[
\mathfrak{w} \oplus \mathfrak{s}_2,
\]
where \(\mathfrak{w}\) is the Witt algebra and \(\mathfrak{s}_2\) is the two-dimensional solvable algebra.

**Proof.** Since all the coverings \(\tau^p\) are pair-wise equivalent (Proposition 2.1), we can accomplish the proof in any of them. From the technical viewpoint, \(\tau^0\) is the most convenient one.

Consider the transformation
\[
\tilde{\Phi}_0^0 = -\Phi_0^0 - \Psi_0^0, \quad \tilde{\Phi}_{-1}^0 = -\Phi_{-1}^0, \quad \tilde{\Psi}_0^0 = \Psi_0^0, \quad \tilde{\Psi}_1^0 = \Psi_1^0.
\]
Let us set \(\tilde{\Phi}_1^0 = \frac{1}{3}\{\tilde{\Phi}_{-1}^0, \tilde{\Phi}_2^0\}\) and by induction
\[
\tilde{\Phi}_{k-1}^0 = \frac{1}{k - 1}\{\tilde{\Phi}_{k-2}^0, \tilde{\Phi}_{k}^0\}, \quad \tilde{\Phi}_{k}^0 = \frac{1}{k - 1}\{\tilde{\Phi}_{k-1}^0, \tilde{\Phi}_{k+1}^0\}
\]
for all \(k \geq 2\). Then
\[
\{\tilde{\Phi}_k^0, \tilde{\Phi}_{l}^0\} = (l - k)\tilde{\Phi}_{k+l}^0
\]
for all \( k, l \in \mathbb{Z} \) and the functions \( \tilde{\Phi}_k^0 \) span the algebra \( \mathfrak{w} \). On the other hand, \( \{ \tilde{\Psi}_0^0, \tilde{\Psi}_1^0 \} = \tilde{\Psi}_1^0 \) and \( \{ \tilde{\Psi}_i^0, \tilde{\Phi}_k^0 \} = 0 \) for \( i = 0, 1 \) and \( k \in \mathbb{Z} \).

2.5. **Explicit formulas.** To conclude the discussion of the universal hierarchy equation, we present explicit formulas for the lifts of symmetries \( \Phi_{-2}^p \), \( \Phi_{-1}^p \), \( \Phi_1^p \), and \( \Phi_2^p \) to an arbitrary covering \( \tau^p \), \( p \neq 1 \):

\[
\begin{align*}
\varphi_{-2}^p &= 0, \\
\varphi_{-1}^p &= 1, \\
\varphi_{-2}^{p,i} &= \frac{p + 1}{p - 1} \left( q_{i-2} + \sum_{j=0}^{i-2} \left( \frac{-1}{p - 1} \right)^{j+1} p_{j,i-j-3}^p \prod_{k=0}^{j} (-2 + k(p - 1)) \right); \\
\varphi_{-1}^{p,i} &= q_{i,y}^p + pq_{i-1}^p + p \sum_{j=0}^{i-2} \left( \frac{-1}{p - 1} \right)^{j+1} p_{j,i-2-j}^p \prod_{k=0}^{j} (-1 + k(p - 1)); \\
\varphi_1^{p,i} &= (p - 1)yu_{x} + \frac{1}{4} ((p - 1)(3p + 2)y^2 - 6q_1^p)u_y + (p - 1) \left( yu - \frac{1}{2} \right), \\
\varphi_2^{p,i} &= \frac{1}{4} ((p - 1)(3p + 2)y^2 - 6q_1^p) q_{i,y}^p - \frac{1}{2} (3 + i)(p - 1)q_{i+1}^p \\
&\quad + (p + i)q_{i+1}^p q_i^p - \frac{1}{2} (p - 2) \sum_{j=0}^{i} \left( \frac{-1}{p - 1} \right)^{j} \prod_{k=0}^{j} \left[ 1 + k(p - 1) \right]; \\
\varphi_2^p &= \left( \frac{(2p^2 + p - 3) y^2 - 4q_1^p}{2} \right) u_x + \frac{1}{3} \left( (5p^3 + 6p^2 - 8p - 3)y^3 - (15p - 27)yq_1^p - 15q_2^p \right) u_y \\
&\quad + (2p^2 y^2 + py^2 - 3y^2 - 4q_1^p) u - 3(p - 1)y, \\
\varphi_2^{p,i} &= \left( \frac{(2p^2 + p - 3) y^2 - 4q_1^p}{2} \right) q_{i,x}^p + \frac{1}{3} \left( (5p^3 + 6p^2 - 8p - 3)y^3 - (15p - 27)yq_1^p - 15q_2^p \right) q_{i,y}^p \\
&\quad - (p - 1)(2p + 3)(p + i)y^2 q_{i}^p - 3(p - 1)(p + i + 1)q_{i+1}^p + 4(p + i)q_{i+1}^p q_{i}^p - (5 + i)(p - 1)q_{i+2}^p \\
&\quad - (p - 3) \sum_{j=0}^{i+1} \left( \frac{-1}{p - 1} \right)^{j} \prod_{k=0}^{j} \left( 2 + k(p - 1) \right),
\end{align*}
\]

where the quantities \( P_{i,j}^p \) are described by Equations \( \boxplus \).

3. **The rdDym equation**

The 3D rdDym equation reads

\[
{u}_{ty} = {u}_x {u}_{xy} - {u}_y {u}_{xx},
\]

see \( \boxplus \) \( \boxplus \) \( \boxplus \).

3.1. **Lax pairs and associated coverings.** The following system

\[
\begin{align*}
{w}_t &= (u_{x} - \lambda)w_{x}, \\
{w}_y &= \lambda^{-1}u_{y}w_x
\end{align*}
\]

is a Lax pair for Equation \( \boxplus \). As above, we consider the expansion \( w = \sum_{i \in \mathbb{Z}} w_i x^i \) and obtain the covering

\[
\begin{align*}
{w}_{i,t} &= {u}_x {w}_{i,x} - {w}_{i-1,x}, \\
{w}_{i,y} &= {u}_y {w}_{i+1,x}
\end{align*}
\]

\( i \in \mathbb{Z} \), endowed with the additional nonlocal variables \( w_i^{(j)} \) defines by the relations \( w_i^{(0)} = w_i \), \( w_i^{(j+1)} = w_i^{(j)} \).
3.2. Symmetries and reductions. The space \( \text{sym}(E) \) for Equation (11) is spanned by the functions \( \psi_0 = xu - 2u, \psi_0(Y) = Y_u, \theta_0(T) = Tu + T'(xu - u) + \frac{1}{2} T''x^2, \theta_{-1}(T) = Tu + T'x, \theta_{-2}(T) = T, \) where \( T = T(t), Y = Y(y), \) and the ‘prime’ denotes the derivative with respect to \( t. \)

**Lemma 3.1.** The symmetry \( \varphi = \theta_0(1) - \psi_0(1) + \psi_0 \) can be lifted to the covering (12).

**Proof.** Let \( \varphi^i \) denote the coefficient at \( \partial \varphi \). Then

\[
\varphi^i = w_{i,t} - w_{i,y} - xw_i^{(1)} - (i + 2)w_i
\]

delivers the desired lift. \( \square \)

The reduction with respect to the obtained lift leads to the equation

\[
u_{yy} = (u_x + x)u_{xy} - (u_{xx} + 2)u_y
\] (13)

and the covering

\[
\begin{align*}
 r_{i,x} &= (u_x + x)r_{i-1,x} - r_{i-1,y} - (i + 1)r_{i-1}, \\
 r_{i,y} &= u_yr_{i-1,x}
\end{align*}
\] (14)

over (13). Similar to Subsection 2.2, we fix an integer \( p \) and ‘cut’ this covering at level \( p \), i.e., set \( r_i = 0 \) for all \( i < p \). Then, after relabeling \( r_{p+i} \to r_{p+2}^p \), we obtain that

\[
r_{p+2}^p = 1, \quad r_{p+1}^p = -(p+2)x, \quad r_0^p = -(p+2)u + \frac{1}{2}(p+2)^2x^2
\]

and arrive to the coverings

\[
\begin{align*}
 \rho^p: & \quad r_{i,x}^p = (u_x + x)r_{i-1,x}^p - r_{i-1,y}^p - (p + i + 3)r_{i-1}^p, \\
 r_{i,y}^p &= u_yr_{i-1,x}^p
\end{align*}
\]

\( i \geq 1 \). These are nonlocal conservation laws of Equation (13).

**Proposition 3.1.** All the coverings \( \rho^p \) are pair-wise equivalent.

**Proof.** The proof is very similar to that of Proposition 2.1. Two cases must be considered. \( \square \)

**Case** \( p \neq -2 \). Consider the vector field

\[
Z^p = \sum_{i=-1}^{\infty} (i+2)r_{i+1}^p \frac{\partial}{\partial r_i^p}
\]

and define the quantities \( Q_{i,j} \) by

\[
Q_{i,0}^p = \frac{1}{(i+2)!}(r_{i-1}^{-1})^{i+2}, \quad Q_{i,j}^p = \frac{1}{j}Z(Q_{i,j-1}^p), \quad i = 0, 1, \ldots, \quad j = 1, 2, \ldots
\] (15)

and formally set \( Q_{i,j}^p = 0 \) when at least one of the subscripts is negative. Let

\[
d_i = \sum_{k=0}^{\infty} (-1)^i (p + k + 3)! \frac{Q_{i,i-k}^p}{(p+3)!}
\]

Then

\[
r_i^p = -(p+2)(r_i^{-3} - (p+3)d_i)
\]

is an equivalence between \( \rho^p \) and \( \rho^{-3} \).

**Case** \( p = -2 \). It is easily seen that \( \rho^{-2} \) coincides with \( \rho^{-1}. \) \( \square \)

3.3. Weights. The basic weights assigned in this case are

\[
|x| = 1, \quad |y| = 0, \quad |u| = 2
\]

with the same rules that were described in Subsection 2.3.
### 3.4. Nonlocal symmetries of reductions

Note first that

\[ \Phi = (\varphi, \varphi^1, \ldots, \varphi^i, \ldots) \]

is a symmetry in \( \rho^p \) if and only if

\[ \tilde{D}_y^2(\varphi) = (u_x + x)\tilde{D}_x\tilde{D}_y(\varphi) - u_y\tilde{D}_x^2(\varphi) + u_x\tilde{D}_x(\varphi) - (u_xx + 2)\tilde{D}_y(\varphi) \]

and

\[
\begin{align*}
\tilde{D}_x(\varphi^{p,1}) &= (p + 2)\left(((p + 2)x - 2u_x)\tilde{D}_x(\varphi) + \tilde{D}_y(\varphi)(p + 4)\varphi\right), \\
\tilde{D}_y(\varphi^{p,1}) &= (p + 2)\left(((p + 2)x - u_x)\tilde{D}_y(\varphi) - u_y\tilde{D}_x(\varphi)\right); \\
\tilde{D}_x(\varphi^{p,i}) &= (u_x + x)\tilde{D}_x(\varphi^{p,i-1}) - \tilde{D}_y(\varphi^{p,i+1}) - (p + i + 3)\varphi^{p,i-1} + r^{p,i+1}_1\tilde{D}_x(\varphi), \\
\tilde{D}_y(\varphi^{p,i}) &= u_y\tilde{D}_x(\varphi^{p,i-1}) + r^{p,i+1}_1\tilde{D}_y(\varphi),
\end{align*}
\]  

(16)

where \( i > 1 \).

A basis of \( \text{sym}(E) \) is formed by the functions

\[ \varphi_{-2} = 1, \quad \varphi_{-1} = u_x + x, \quad \varphi_0 = 2u - xu_x, \quad \psi_0 = u_y, \]

where subscripts coincide with weights. In addition, in any covering \( \rho^p \) there exists a shadow of the form

\[
\varphi^i_k = 2r^p_2 + \left((p + 2)\left((5p + 24)xu - \frac{1}{6}(5p^2 + 20p - 18)x^3\right) - 5r^p_1\right)u_x
\]

\[ - (p + 2)(4u + 5x^2)u_y + (6p + 25)xr^p_i
\]

\[ + (p + 2)(2p + 16)u^2 - (p + 4)(3 + 13)x^2u + \frac{1}{12}(9p^3 + 80p^2 + 212p + 168)x^4. \]

**Lemma 3.2.** The symmetries \( \psi_0 \) and \( \varphi_0 \) are lifted to any covering \( \rho^p \).

**Proof.** It is sufficient to set

\[
\begin{align*}
\psi_0^{p,i} &= r^{p}_{i,y} \\
\varphi_0^{p,i} &= -xr^{p}_{i,x} + (i + 2)r^p_i 
\end{align*}
\]

and check that Equations (16) fulfill. \( \square \)

**Lemma 3.3.** The symmetries \( \varphi_{-1} \) and \( \varphi_{-2} \) are lifted to the coverings \( \rho^{-3} \) and \( \rho^{-4} \), respectively, while the shadow \( \varphi^0_2 \) lifts to \( \rho^0 \).

**Proof.** We set

\[
\begin{align*}
\varphi^{-3}_{-1} &= r^{-3}_{-1,x} \quad \text{for} \quad \varphi_{-1} = u_x + x, \\
\varphi^{-4}_{-2} &= 0 \quad \text{for} \quad \varphi_{-2} = 1
\end{align*}
\]

and also

\[
\begin{align*}
\varphi^0_{2,i} &= (48xu + 6x^3 - 5r^0_{i,x})r^0_{i,y} - 2(4u + 5x^2)r^0_{i,y} + 2(i + 6)r^0_{i+2} - 2(36 + 9i)x^2r^0_i \\
&+ 3(5 + i)r^0_{i+1}r^0_{i+1} + 4(i + 4)r^0_{i}r^0_{i}
\end{align*}
\]

for

\[
\varphi^0_2 = 6r^0_2 + (48xu + 6x^3 - 5r^0_{i,x})u_x - 2(4u + 5x^2)u_y + 25xr^0_0 + 32u^2 - 104x^2u + 28x^4.
\]

Then Equations (16) are satisfied for the corresponding values of \( p \). \( \square \)

**Corollary 3.1.** All the symmetries \( \Psi_0^p, \Phi_{-2}^p, \Phi_{-1}^p, \Phi_0^p, \Phi_2^p \) exist in any covering \( \rho^p \).

**Proof.** It immediately follows from Proposition 3.1 and Lemmas 3.2 and 3.3. \( \square \)

We can describe the algebra of nonlocal symmetries for Equation (13) now:

**Theorem 3.1.** The algebra of nonlocal symmetries of Equation (13) in any covering \( \rho^p \) is isomorphic to

\[ \mathfrak{w} \oplus \mathfrak{a}_1, \]

where \( \mathfrak{w} \) is the Witt algebra and \( \mathfrak{a}_1 \) is the one-dimensional Abelian Lie algebra.
Proof. Similar to the proof of Theorem 2.1 we first choose a convenient value of \( p \), which is \( p = -1 \) in our case, and set \( \Phi_2^{-1} = -\Phi_2^{-1} \). The \( \psi \)-component is constructed exactly in the same way as it was done in the proof of Theorem 2.1. The Abelian component is spanned by the symmetry \( \Psi_0^{-1} \) which obviously commutes with all \( \Phi_i^{-1} \). \( \square \)

3.5. Explicit formulas. Let us describe the lifts \( \Phi^p_i \), \( i = -2, -1, 1, 2 \), explicitly:

\[
\varphi_{-2}^{p,i} = (p + 4) \left( -r_{i-2}^p - \sum_{j=1}^{\infty} \left( \left( \frac{-1}{p + 2} \right)^j Q_{j-1,i-1}^p \prod_{l=0}^{j-1} (l(p + 2) - 2) \right) \right),
\]

\[
\varphi_{-1}^{p,i} = r_{i,x}^p + (p + 3) \left( r_{i-1}^p + \sum_{j=1}^{\infty} \left( \left( \frac{-1}{p + 2} \right)^j Q_{j-1,i-j}^p \prod_{l=0}^{j-1} (l(p + 2) - 1) \right) \right),
\]

\[
\varphi_{1}^{p,i} = (p + 2) \left( 3u + \frac{5}{2} x^2 \right) r_{i,x}^p - 2r_{i,y}^p - (i + 4)r_{i+1}^p + 2(p + i + 4) r_{i-1}^p r_i^p
\]

\[
- (p + 1) \left( \sum_{j=0}^{\infty} \left( \frac{-1}{p + 2} \right)^j Q_{j,i}^p \prod_{l=0}^{j} (l(p + 2) + 1) \right) ,
\]

\[
\varphi_{2}^{p,i} = (p + 2) \left( 5p + 24 \right) xu - \frac{1}{6} (5p^2 + 20p - 18)x^3 \right) - 5r_{i,x}^p
\]

\[
- (p + 2) \left( 4u + 5x^2 \right) r_{i,y}^p - (i + 6)r_{i+2}^p - (p + 2) \left( \left( 4ip + 13p + 9i + 36 \right) x^2 + 4pu \right) r_i^p
\]

\[
+ 3(p + i + 5) r_{i-1}^p r_{i+1}^p + 4(i + 4) r_{i}^p r_{i+1}^p - p \sum_{j=0}^{\infty} \left( \left( \frac{-1}{p + 2} \right)^j Q_{j,i}^p \prod_{l=0}^{j} (l(p + 2) + 2) \right).
\]

Here the quantities \( Q_{i,j}^p \) are given by Equations (15).

4. Discussion

Let us conclude with several remarks:

- All the nonlocal symmetry algebras of linear degenerate equations (see [1]) and their reductions (see [3] and the results above) contain the Witt algebra \( \mathfrak{w} \) as their semi-direct (or direct) summand.

- In all the constructions used to describe the symmetry algebras structures the crucial role is played by the operators similar to \( Y \) and \( Z \) from Sections 2.2 and 3.2 and the quantities \( P_{i,j}^p \) and \( Q_{i,j}^p \). It is interesting to understand the geometric origins of these objects.

- It is also interesting to study other Lax integrable equations in dimension \( > 2 \) that are not linear degenerate and compare their nonlocal symmetry structure with the already known results.

We plan to shed the light on the last two items in the forthcoming research.

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