More on the identity of Chaundy and Bullard

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In [2] Chaundy and Bullard proved the identity

\[1 = x^{k+1} \sum_{i=0}^{m} \binom{k+i}{k} \frac{(1-x)^i x^i}{(x+y)^{i+1}} \]

(1)

for integers \(k, m \geq 0\). Many different proofs of (1) are known. See [5] for a detailed account. For the case \(m = k\), (1) is frequently called the Daubechies identity. See [8]. As described in [6], the Chaundy and Bullard inequality has roots going back three centuries. In what follows we present some ramifications of (1). In the first part we discuss extensions to several variables and relations with other identities. In the second part we obtain additional identities with more parameters.

1 The homogeneous form of (1)

The homogeneous identity

\[x^{m+1} y^{k+1} = \sum_{i=0}^{m} \binom{k+i}{k} x^{m-i+1} \left( \frac{xy}{x+y} \right)^{k+i+1} + \sum_{i=0}^{k} \binom{m+i}{m} y^{k-i+1} \left( \frac{xy}{x+y} \right)^{m+i+1} \]

(2)

deserves attention for its own sake and has several interesting conclusions:

(a) If we divide (2) by \(x^{m+1} y^{k+1}\) and choose \(x + y = 1\), we get identity (1).

(b) Another conclusion of (2) is an identity given by Graham, Knuth and Patashnik [4, p. 246]: If \(xy = x + y\) then

\[x^{m+1} y^{k+1} = \sum_{i=0}^{m} \binom{k+i}{k} x^{m-i+1} + \sum_{i=0}^{k} \binom{m+i}{m} y^{k-i+1}. \]

(3)

It is not difficult to see that (1)–(3) are all equivalent.

We suggest a proof of (2) which conveniently generalizes to more than two variables. Let us apply \((-\partial/\partial x)^m (-\partial/\partial y)^k\) to the identity

\[\frac{1}{xy} = \frac{1}{x(x+y)} + \frac{1}{y(x+y)}.\]

(4)

First we apply \((-\partial/\partial x)^m\). To the term \(\frac{1}{x(x+y)}\) we use the Leibnitz formula \((fg)^{(m)} = \sum_{i=0}^{m} \binom{m}{i} f^{(m-i)} g^{(i)}\):

\[\frac{m!}{x^{m+1} y} = \sum_{i=0}^{m} \binom{m}{i} \frac{(m-i)!}{i! x^{m-i+1}} \frac{y^{i+1}}{(x+y)^{i+1}} + \frac{m!}{y^{x+1} y^{x+1}} = m! \sum_{i=0}^{m} \frac{1}{i! x^{m-i+1+1}} \frac{y^{i+1}}{(x+y)^{i+1}} + \frac{m!}{y^{x+1} y^{x+1}}.\]

Next, applying \((-\partial/\partial y)^k\):

\[\frac{m! \cdot k!}{x^{m+1} y^{k+1}} = m! \sum_{i=0}^{m} \frac{1}{i! x^{m-i+1+1}} \frac{y^{i+1}}{(x+y)^{i+k+1}} + \frac{k!}{j! y^{k-j+1}} \frac{y^{j+k+1}}{(x+y)^{j+1}} \]

\[= m! \sum_{i=0}^{m} \frac{1}{i! x^{m-i+1+1}} \frac{y^{i+k+1}}{(x+y)^{i+k+1}} + \frac{1}{j! m!} \sum_{j=0}^{k} \frac{m+j}{m} \frac{y^{j+k+1}}{(x+y)^{m+j+1}}.\]

(5)
Replacing \( x, y \) by \( x^{-1}, y^{-1} \), respectively, one gets (2).

Note that the case \( m = k = 1 \) of (5), namely
\[
\frac{1}{x^2y^2} = \left( \frac{1}{x^2} + \frac{1}{y^2} \right) \frac{1}{(x+y)^2} + \frac{1}{x+y} \frac{2}{(x+y)^3}
\]
plays a central role in the development of the theory of G. Eisenstein about periodic functions. See [3, p. 252].

## 2 A generalization for \( n \) variables

We propose a homogeneous identity with \( n \) variables which generalizes both (2) and (3). Our method uses only very elementary tools of analysis. We apply the differential operator \((-\partial/\partial x_1)^{m_1} \cdots (-\partial/\partial x_n)^{m_n}\) to the elementary identity
\[
\frac{1}{x_1x_2 \cdots x_n} = \sum_{t=1}^{n} \frac{1}{x_1 \cdots \langle x_t \text{ skipped} \rangle \cdots x_n(x_1 + \cdots + x_n)},
\]
(6)

which generalizes (4). The result of applying the operator to the left hand side of (6) is
\[
\frac{m_1!m_2! \cdots m_n!}{x_1^{m_1+1}x_2^{m_2+1} \cdots x_n^{m_n+1}}.
\]
(7)

On the right hand side of (6) we differentiate each term separately, i.e., take a fixed \( t, 1 \leq t \leq n \), and calculate
\[
(-\partial/\partial x_1)^{m_1} \cdots (-\partial/\partial x_n)^{m_n} \frac{1}{x_1 \cdots \langle x_t \text{ skipped} \rangle \cdots x_n(x_1 + \cdots + x_n)}.
\]
(8)

Since the variable \( x_t \) appears only in one factor of the denominator of (8), while each other \( x_j, j \neq t \) appears in two factors, we apply \((-\partial/\partial x_t)^{m_t}\) first and get
\[
\frac{m_t!}{x_1 \cdots \langle x_t \text{ skipped} \rangle \cdots x_n(x_1 + \cdots + x_n)^{m_t+1}}.
\]
(9)

Next we apply \( \prod_{j \neq t} (-\partial/\partial x_j)^{m_j} \) to (9). By the Leibnitz formula we get that (8) equals
\[
\prod_{j \neq t} \left( -\partial/\partial x_j \right)^{m_j} \frac{m_t!}{x_1 \cdots \langle x_t \text{ skipped} \rangle \cdots x_n(x_1 + \cdots + x_n)^{m_t+1}}
= \sum_{0 \leq i_j \leq m_j \atop j \neq t} \left( \sum_{t=1}^{n} \frac{m_1! - i_1!}{x_1^{m_1-i_1+1}} \cdots \frac{i_1x_t}{\text{skipped}} \cdots \frac{m_n! - i_n!}{x_n^{m_n-i_n+1}} \frac{(m_t + i_1 + \cdots + i_{t-1} + i_{t+1} + \cdots + i_n)!}{(x_1 + \cdots + x_n)^{m_t+i_1+i_{t-1}+i_{t+1}+\cdots+i_n+1}} \right)
\]
(10)

Summing (10) for \( t = 1, \ldots, n \) and comparing with (7), results
\[
\frac{1}{x_1^{m_1+1}x_2^{m_2+1} \cdots x_n^{m_n+1}} = \sum_{t=1}^{n} \left[ \sum_{0 \leq i_j \leq m_j \atop j \neq t} \frac{(i_1 + \cdots + i_{t-1} + i_t + i_{t+1} + \cdots + i_n)!}{i_1! \cdots i_{t-1}!m_t!i_{t+1}! \cdots i_n!} \right]
\times \frac{1}{x_1^{m_1-i_1+1} \cdots \langle x_t \text{ skipped} \rangle \cdots x_n^{m_n-i_n+1}(x_1 + \cdots + x_n)^{i_1+i_{t-1}+i_{t+1}+i_{t+1}+\cdots+i_n+1}}.
\]
(11)

Finally we replace \( x_t \) by \( x_t^{-1} \) and use for two of the basic symmetric polynomials in \( n \) variables the notation
\[
S_{n,n}(x_1, \ldots, x_n) = x_1 \cdots x_n, \quad S_{n-1,n}(x_1, \ldots, x_n) = \sum_{t=1}^{n} x_1 \cdots \langle x_t \text{ skipped} \rangle \cdots x_n.
\]
Then \((11)\) becomes our main homogeneous identity

\[
x_1^{m_1+1}x_2^{m_2+1} \cdots x_n^{m_n+1} = \sum_{i=1}^{n} \left[ \sum_{0 \leq i_j \leq m_j} \frac{(i_1 + \cdots + i_{t-1} + m_t + i_{t+1} + \cdots + i_n)!}{i_1! \cdots i_{t-1}! m_t! i_{t+1}! \cdots i_n!} \right] \times x_1^{m_1-i_1+1} \cdots x_t \left( \sum_{i=1}^{S_{n,n}} \left( S_{n,n} \right)^{-1} \right) \times x_n^{m_n-i_n+1}
\]

(12)

Examples. For \(n = 2\), (12) reduces to

\[
x_1^{m_1+1}x_2^{m_2+1} = \sum_{i_2=0}^{m_2} \left( m_1 + i_2 \right) \frac{x_1^{m_1}x_2^{m_2-i_2+1}}{m_1!} + \sum_{i_1=0}^{m_1} \left( m_2 + i_1 \right) \frac{x_1^{m_1-i_1+1}x_2^{m_2}}{m_1!}
\]

(13)

i.e., \((2)\).

Assuming the equality \(S_{n,n} = S_{n-1,n}\), identity (12) implies a \(n\)-variable analogue to (3). For \(n = 3\) it is: If \(xyz = xy + yz + zx\), then

\[
x_1^{m_1+1}y^{m_2+1}z^{m_3+1} = \sum_{j \leq m_2, k \leq m_3} \frac{(m_1 + j + k)!}{m_1! j! k!} y^{m_2-j+1}z^{m_3-k+1} + \sum_{k \leq m_3, i \leq m_1} \frac{(i + m_2 + k)!}{i! m_2! k!} z^{m_3-k+1}x^{m_1-i+1} + \sum_{i \leq m_1, j \leq m_2} \frac{(i + j + m_3)!}{i! j! m_3!} x^{m_1-i+1}y^{m_2-j+1}.
\]

(14)

If we divide (12) by \(x_1^{m_1+1} \cdots x_n^{m_n+1}\) and take \(S_{n-1,n} = 1\), we get a \(n\)-variable analogue of the identity of Chaundy and Bullard. For \(n = 3\) it is: If \(xyz = xy + yz + zx = 1\), then

\[
(yz)^{m_1+1} + \sum_{j \leq m_2, k \leq m_3} \frac{(m_1 + j + k)!}{m_1! j! k!} y^j x^j + \sum_{k \leq m_3, i \leq m_1} \frac{(i + m_2 + k)!}{i! m_2! k!} z^i x^j y^j + \sum_{i \leq m_1, j \leq m_2} \frac{(i + j + m_3)!}{i! j! m_3!} x^i y^j z^j = 1.
\]

□

The change of variables \(u_t = x_1 \cdots x_{t-1}x_{t+1} \cdots x_n, \ t = 1, \ldots, n,\) and the inverse transformation

\[
x_t = \frac{(u_1 \cdots u_n)^{(n-1)}}{u_t}, \quad t = 1, \ldots, n,
\]

yield \(S_{n-1,n}(x_1, \ldots, x_n) = u_1 + \cdots + u_n, \ S_{n,n}(x_1, \ldots, x_n) = (u_1 \cdots u_n)^{(n-1)}\). After some elementary calculations this transforms identity (12) into

\[
(u_1 + \cdots + u_n)^{m_1+\cdots+m_n+1} = \sum_{i=1}^{n} \left[ \sum_{0 \leq i_j \leq m_j} \frac{(i_1 + \cdots + i_{t-1} + m_t + i_{t+1} + \cdots + i_n)!}{i_1! \cdots i_{t-1}! m_t! i_{t+1}! \cdots i_n!} \right] \times u_1^{i_1} \cdots \left( \sum_{i=1}^{S_{n,n}} \left( S_{n,n} \right)^{-1} \right) \times u_n^{i_n} (u_1 + \cdots + u_n)^{\sum_{i \neq t}(m_j-i_j)}
\]

(16)

is precisely equation (10.2) of [5]. Two proofs of this result are given in [7], one by a probabilistic argument and the other by using generating functions.
3 Another generalization of CB

The next identity is another generalization of (1) which depends on three independent parameters:

Let \( m - r + k - \ell = 0, m, r, k, \ell \) positive integers. Then

\[
(1-x)^{r+1} \sum_{i=0}^{k} \binom{m+i}{i} x^{i+m-r} + x^{r+1} \sum_{i=0}^{m} \binom{k+i}{i} (1-x)^{i+k-\ell} = \begin{cases} 
1 - \sum_{i=0}^{m-r-1} \binom{m}{i} x^i (1-x)^{m-i} & \text{if } m-r > 0, \\
1 - \sum_{i=0}^{k-\ell-1} \binom{k}{i} (1-x)^i x^{k-i} & \text{if } k-\ell > 0, \\
1 & \text{if } m = r, k = \ell.
\end{cases}
\]

(17)

In the previous version of this manuscript (17) was proved by elementary methods using ideas presented in [1]. Professor T. Koornwinder kindly brought to our attention a shorter proof of (17), which follows hereby:

Let us verify the case \( m - r = \ell - k > 0 \). In the second sum on the left hand side the terms are nonzero only when \( k + i \geq \ell \), hence it is sufficient to sum only for \( i \geq \ell - k = m - r \). We change the summation index in the first sum on the left to \( j = i + (m-r) \) and in the second sum to \( j = i - (m-r) \). By repeated use of \( m-r = \ell-k \), the left side becomes

\[
(1-x)^{r+1} \sum_{j=m-r}^{k+m-r} \binom{j+r}{r} x^j + x^{r+1} \sum_{j=0}^{r} \binom{j+\ell}{\ell} (1-x)^j.
\]

Let us rewrite this as

\[
(1-x)^{r+1} \left[ \sum_{j=0}^{k+m-r} \binom{j+r}{r} x^j - \sum_{j=0}^{m-r-1} \binom{j+r}{r} x^j \right] + x^{r+1} \sum_{j=0}^{r} \binom{j+\ell}{\ell} (1-x)^j,
\]

and rearrange it to

\[
\left[ (1-x)^{r+1} \sum_{j=0}^{\ell} \binom{j+r}{r} x^j + x^{r+1} \sum_{j=0}^{r} \binom{j+\ell}{\ell} (1-x)^j \right] - (1-x)^{r+1} \sum_{j=0}^{m-r-1} \binom{j+r}{r} x^j.
\]

The first two sums total to 1 by the original Chaundy-Bullard identity, so (17) will follow if one shows that

\[
1 - (1-x)^{r+1} \sum_{j=0}^{m-r-1} \binom{j+r}{r} x^j = 1 - \sum_{i=0}^{m-r-1} \binom{m}{i} x^i (1-x)^{m-i},
\]

i.e.,

\[
1 - (1-x)^{r+1} \sum_{j=0}^{m-r-1} \binom{j+r}{r} x^j = 1 - (1-x)^{r+1} \sum_{i=0}^{m-r-1} \binom{m}{i} x^i (1-x)^{m-r-i-1},
\]

But the remaining

\[
\sum_{j=0}^{m-r-1} \binom{j+r}{r} x^j = \sum_{i=0}^{m-r-1} \binom{m}{i} x^i (1-x)^{m-r-i-1}
\]

is precisely equation (2.7) of [5], hence (17) follows.

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