On the diameter for various types of domination vertex critical graphs

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Abstract

In this paper, we consider various types of domination vertex critical graphs, including total domination vertex critical graphs and independent domination vertex critical graphs and connected domination vertex critical graphs. We provide upper bounds on the diameter of them, two of which are sharp.

1 Introduction

All graphs considered here are finite, undirected, and simple. Let $G$ be a graph with vertex set $V$ and edge set $E$. The neighborhood of a vertex $v$ in a graph $G$, denoted by $N_G(v)$, is the set of all the vertices adjacent to the vertex $v$, i.e., $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the closed neighborhood of a vertex $v$ in $G$, denoted by $N_G[v]$, is defined by $N_G[v] = N_G(v) \cup \{v\}$. A vertex of degree one is called a leaf vertex, the edge connected to that vertex is called a pendant edge and the only neighbor of a leaf vertex is called a support vertex. We denote the distance between $u$ and $v$ in $G$ by $\text{dist}_G(u, v)$, and denote the diameter of $G$ by $\text{diam}(G)$. The degree of a vertex $v$ in $G$, denoted by $\deg(v)$, is the number of incident edges of $G$. A vertex of degree $k$ is called a $k$-vertex, and a vertex of degree at most or at least $k$ is call a $k^-$- or $k^+$-vertex, respectively.

A vertex subset $S \subseteq V$ is called a dominating set of a graph $G$ if every vertex in $V$ is an element of $S$ or is adjacent to a vertex in $S$. The domination number of a graph $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A graph is domination vertex critical if the removal of any vertex decreases its domination number. If $G$ is domination vertex critical and $\gamma(G) = k$, we say that $G$ is a $k$-vertex-critical graph.

A vertex subset $S \subseteq V$ is a total dominating set of a graph $G$ if every vertex in $V$ is adjacent to a vertex in $S$. Every graph without isolated vertices has a total dominating set, since $V$ is such a set. The total domination number of a graph $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. A graph is total domination vertex critical if the removal of any vertex that is not adjacent to a vertex of degree one decreases its total domination number. If $G$ is total domination vertex critical and $\gamma_t(G) = k$, we say that $G$ is a $k$-vertex-critical graph.

A vertex subset $S \subseteq V$ is an independent dominating set of a graph $G$ if it is a dominating set and it is also an independent set in $G$. Equivalently, an independent dominating set is a maximal independent set. The independent domination number of a graph $G$, denoted by $i(G)$, is the minimum cardinality of an independent dominating set of $G$. A graph is independent domination vertex critical if the removal of any vertex decreases its independent domination number. If $G$ is independent domination vertex critical and $i(G) = k$, we say that $G$ is a $k$-vertex-critical graph.

A vertex subset $S \subseteq V$ is a connected dominating set of a graph $G$ if it is a dominating set of $G$ and the subgraph induced by $S$ is connected. Every connected graph has a connected dominating set, since $V$ is such a set. The connected domination number of a graph $G$, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected

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Theorem 2.1. Let \( \gamma_c(G) = k \) be a minimum total dominating set of a connected domination vertex critical graph \( G \). A graph is connected domination vertex critical if the removal of any vertex decreases its connected domination number. If \( G \) is connected domination vertex critical and \( \gamma_c(G) = k \), we say that \( G \) is a \( k \)-\( \gamma_c \)-vertex-critical graph. A necessary condition for a graph to be \( k \)-\( \gamma_c \)-vertex-critical is 2-connected. For more details on connected domination vertex critical graphs, see [6].

Goddard et al. [4] characterized the class of \( k \)-\( \gamma_c \)-vertex-critical graphs with leaf vertices.

**Theorem 1.1.** Let \( G \) be a connected graph of order at least three with at least one leaf vertex. Then \( G \) is a \( k \)-\( \gamma_c \)-vertex-critical graph if and only if \( \delta(H) \geq 2 \) for some connected graph \( H \) of order \( k \). In this paper, we also adopt the same details on connected domination vertex critical graphs, see [6].

**Observation 1.** If \( D \) is a total dominating set of a graph \( G \), then for every vertex \( v \) in \( G \), the set \( D \) contains a neighbor of \( v \).

**Lemma 1.** If \( G \) is a \( k \)-\( \gamma_c \)-vertex-critical graph without leaf vertices, then for any vertex \( w \), there exists a minimum total dominating set of \( G \) containing \( w \), and \( \gamma_c(G - w) = \gamma_c(G) - 1 \).

**Proof.** Let \( v \) be a neighbor of \( w \) in \( G \), and let \( D \) be a minimum total dominating set of \( G - v \). It follows that \( w \notin D \) and \( D \cap N_G(w) \neq \emptyset \), thus \( D \cup \{w\} \) is a total dominating set of \( G \). Furthermore, we have that \( |D \cup \{w\}| = |D| + 1 \leq k \), and then \( D \cup \{w\} \) is a minimum total dominating set of \( G \) containing \( w \) and \( \gamma_c(G - w) = |D| = \gamma_c(G) - 1 \).

**Lemma 2.** If \( G \) is a \( k \)-\( v \)-vertex-critical graph, then for any vertex \( v \), there exists a minimum total dominating set of \( G \) containing \( v \), and \( i(G - v) = i(G) - 1 \).

The method developed in [3] is a powerful technique to obtain sharp upper bounds on various types of domination vertex critical graphs, it has been used for the \( k \)-\( \gamma \)-vertex-critical graphs [3] and paired domination vertex critical graphs [5].

Edward and MacGillivray [2] presented better upper bounds on the diameter of total domination and independent domination vertex critical graphs, but the proofs have big gaps. In this paper, we also adopt the same technique to obtain sharp upper bounds on the diameter, one of which is a slightly improvement on a result in [2].

## 2 Upper bounds on the diameter

**Theorem 2.1.** If \( G \) is a connected \( k \)-\( \gamma_c \)-vertex-critical graph without leaf vertices and \( k \geq 4 \), then \( \text{diam}(G) \leq \frac{5k-2}{3} \).

**Proof.** Let \( x \) and \( x_n \) be vertices such that \( \text{dist}(x, x_n) = \text{diam}(G) = n \). If \( n \leq 4 \), then we are done. So we may assume that \( n \geq 5 \). Let \( x_1, \ldots, x_{n-1}, x_n \) be a shortest path between \( x \) and \( x_n \). Define \( L_0, L_1, \ldots, L_n \) by \( L_i = \{v \in V(G) \mid \text{dist}_G(x, v) = i\} \) for \( 0 \leq i \leq n \). In particular, \( L_0 = \{x\} \) and \( L_1 = N_G(x) \). Let \( R_i = L_i \cup L_{i+1} \cup \cdots \cup L_n \) for \( 0 \leq i \leq n \). Let \( D \) be a minimum total dominating set in \( G \). If \( |D \cap R_i| \geq \frac{3j+10}{5} \), then we say that \( R_j \) is sufficient with respect to \( D \).

Let \( m \) be the maximum integer \( j \) such that \( |D \cap R_j| \geq \frac{3j+10}{5} \). Notice that the value of \( m \) depends on the minimum total dominating set \( D \), we may assume that \( D \) is chosen such that \( m \) is maximum among all the minimum total dominating set.

Firstly, we must show the existence of \( m \). Let \( D_1 \) be a minimum total dominating set of \( G - x_1 \). It is obvious that \( x \notin D_1 \) and \( D_1 \cap L_1 \neq \emptyset \) and \( |D_1 \cap (L_1 \cup L_2)| \geq 2 \). Suppose that the value of \( m \) does not exist, it follows that \( 1 + |D_1 \cap R_1| < \frac{3j+10}{5} \), otherwise \( R_j \) is sufficient with respect to \( D_1 \cup \{x\} \). Hence, we have that \( |D_1 \cap L_1| = 1 \) and \( |D_1 \cap (L_1 \cup L_2)| < 2.2 \). In fact \( |D_1 \cap L_2| = 1 \). From the fact that \( |D_1 \cap R_1| < 2.8 \), we have that \( D_1 \cap L_3 = \emptyset \). If \( D_1 \cap L_4 \neq \emptyset \), then we can conclude that \( |D_1 \cap (L_4 \cup L_5)| \geq 2 \) from the fact that \( D_1 \) is a total dominating set of \( G - x_1 \), and then \( R_3 \) is sufficient with respect to \( D_1 \cup \{x\} \), a contradiction. Hence we may assume that \( D_1 \cap L_4 = \emptyset \). Let \( D_0 \) be a minimum total dominating set of \( G - x_4 \). If \( |D_0 \cap R_3| \geq 3 \), then \( R_3 \) is sufficient with respect to \( D_0 \cup \{x_3\} \), a
contradiction. Hence $|D_0 \cap R_3| = 2$ and $D_0 \cap L_3 = \emptyset$. If $D_0 \cap L_4 = \emptyset$, then $D_1 \cap R_3$ totally dominates $R_3$, and then $(D_0 \cap R_3) \cup (D_1 \setminus R_3)$ is a smaller total dominating set of $G$, a contradiction. Hence we have that $D_0 \cap L_4 \neq \emptyset$ and $|D_0 \cap (L_4 \cup L_5)| \geq 2$. Therefore, we have that $|D_0 \cap R_3| \geq 4$ and the set $R_5$ is sufficient with respect to $D_0 \cup \{x_3\}$, which leads to a contradiction.

Now, we know that the value of $m$ must exist. If $m = n$, then $n = m \leq \frac{5k-10}{4} \leq \frac{5k-7}{3}$, we are done. So we may assume that $m < n$.

If $m = 5t + 2$, then $|D \cap R_m| \geq 3t + 3.2$ and $|D \cap R_{m+1}| < 3t + 3.8$, which is a contradiction. If $m = 5t + 4$, then $|D \cap R_m| \geq 3t + 3.4$ and $|D \cap R_{m+1}| < 3t + 5$, which is also a contradiction. So we have that $m = 5t, 5t + 1$ or $5t + 3$. We further assume that $m \leq 2 \leq n$.

If $m = 5t$, then $|D \cap R_m| \geq 3t + 2$ and $|D \cap R_{m+1}| < 3t + 2.6$, which implies that $|D \cap R_m| = 3t + 2$ and $D \cap L_{m+1} = \emptyset$. From the fact that $|D \cap R_{m+2}| < 3t + 3.2$ and $D$ is a total dominating set and $|D \cap R_{m+1}| < 3t + 3.8$ (if $L_{m+3}$ exists), we can conclude that $D \cap L_{m+2} = \emptyset$ and $D \cap L_{m+3}$ exists. Consequently, the set $L_{m+4}$ exists and $D \cap L_{m+3}$ dominates $L_{m+2}$. But $|D \cap R_{m+4}| < 3t + 4.4$, so we have that $|D \cap L_{m+4}| = 1$.

If $m = 5t + 1$, then $|D \cap R_m| \geq 3t + 2.6$ and $|D \cap R_{m+1}| < 3t + 3.2$ and $|D \cap R_{m+2}| < 3t + 3.8$, which implies that $|D \cap R_m| = 3t + 3$ and $D \cap L_{m+1} = D \cap L_{m+2} = \emptyset$. In order to dominate $L_{m+2}$, the set $L_{m+3}$ exists and $D \cap L_{m+3}$ dominates $L_{m+2}$. But $|D \cap R_{m+2}| < 3t + 4.4$, so we have that $|D \cap L_{m+3}| = 1$. The set $D$ totally dominates $G$, it follows that $L_{m+4}$ exists and $D \cap L_{m+4} = \emptyset$. Hence $|D \cap R_{m+4}| \geq 3t + 5$ and $R_{m+4}$ is sufficient with respect to $D$, a contradiction to the maximality of $m$.

If $m = 5t + 3$, then $|D \cap R_m| \geq 3t + 3.8$ and $|D \cap R_{m+1}| < 3t + 4.4$ and $|D \cap R_{m+2}| < 3t + 5$, which implies that $|D \cap R_m| = 3t + 4$ and $D \cap L_{m+1} = D \cap L_{m+2} = \emptyset$. In order to dominate $L_{m+2}$, the set $L_{m+3}$ exists and $D \cap L_{m+3}$ dominates $L_{m+2}$. But $|D \cap R_{m+3}| < 3t + 5.6$, so we have that $|D \cap R_{m+3}| = 1$. Since $D$ is a total dominating set in $G$, it follows that $L_{m+4}$ exists and $D \cap L_{m+4} \neq \emptyset$, but with $|D \cap R_{m+4}| < 3t + 6.2$, we have that $|D \cap L_{m+4}| = 1$.

By the above arguments, we may assume that $D \cap L_{m+1} = D \cap L_{m+2} = \emptyset$ and $D \cap L_{m+3} = D \cap L_{m+4} = 1$, where $m = 5t$ or $m = 5t + 3$. Without loss of generality, we assume that $D \cap L_{m+3} = \{x_{m+3}\}$ and $D \cap L_{m+4} = \{x_{m+4}\}$.

Let $D_3$ and $D_4$ be a minimum total dominating set of $G - x_{m+3}$ and $G - x_{m+4}$, respectively.

Recall that the vertex $x_{m+3}$ dominates $L_{m+3}$, hence $D_3 \cap L_{m+3} = \emptyset$ and $D_3 \cap R_{m+1}$ totally dominates $R_{m+1}$. If $D_3 \cap R_{m+1} < |D \cap R_{m+1}|$, then $(D_3 \cap R_{m+1}) \cup (D \setminus R_{m+1})$ is a smaller total dominating set in $G$, which leads to a contradiction. If $D_3 \cap R_{m+1} > |D \cap R_{m+1}|$, then $R_{m+1}$ is sufficient with respect to the minimum total dominating set $D_3 \cup \{x_{m+3}\}$. Hence we have that $|D_3 \cap R_{m+1}| = |D \cap R_{m+1}|$. Notice that maybe $L_{m+5}$ does not exist, if this happens, then we view $L_{m+5}$ as an empty set. If $|D_3 \cap (L_{m+3} \cup L_{m+4} \cup L_{m+5})| \geq 2, then |D_3 \cup \{x_{m+4}\} \cap R_{m+5}| \geq |D \cap R_{m+5}| + 3$, and then $R_{m+5}$ (or $L_{m+5}$) does not exist) is sufficient with respect to $D_3 \cup \{x_{m+4}\}$, which contradicts the maximality of $m$. Hence, we have that $|D_3 \cap (L_{m+3} \cup L_{m+4} \cup L_{m+5})| \leq 1$, which implies that $L_{m+5}$ exists and $D_3 \cap L_{m+4} = \emptyset$ and $L_{m+3} = \{x_{m+3}\}$.

Notice that $D_4 \cap L_{m+3} = \emptyset$ and $D_4 \cap R_{m+2}$ totally dominates $R_{m+2}$. If $|D_4 \cap R_{m+2}| < |D \cap R_{m+2}|$, then $(D_4 \cap R_{m+2}) \cup (D \setminus R_{m+2})$ is a smaller total dominating set of $G$, which leads to a contradiction. If $|D_4 \cap R_{m+2}| > |D \cap R_{m+2}|$, then $(D_4 \cup \{x_{m+3}\}) \cap R_{m+2} \geq |D \cap R_{m+2}| + 2$, and then $R_{m+3}$ is sufficient with respect to $D_4 \cup \{x_{m+3}\}$, which leads to a contradiction. Hence, we have that $|D_4 \cap R_{m+2}| = |D \cap R_{m+2}|$.

If $D_4 \cap L_{m+2} \neq \emptyset$, then $(D_4 \cap R_{m+2}) \cup (D_4 \setminus R_{m+2})$ is a smaller total dominating set of $G$, a contradiction. It follows that $D_4 \cap L_{m+2} = \emptyset$. In order to dominate the vertex $x_{m+3}$, we must have that $D_4 \cap L_{m+4} \neq \emptyset$. Hence, we can conclude that $|D_4 \cap (L_{m+4} \cup L_{m+5})| \geq 2$ and $R_{m+5}$ is sufficient with respect to $D_4 \cup \{x_{m+3}\}$, a contradiction.

Finally, we have to deal with the case that $m = n - 1$. Recall that $m$ is the maximum integer $j$ such that $|D \cap R_j| \geq \frac{j+10}{4}$. It follows that $D \cap L_{m+1} = D \cap L_0 = \emptyset$, and $|D \cap R_m| = k$ and $n = m + 1 \leq \frac{5k-10}{3} + 1 = \frac{5k-7}{3}$. □

The coalescence of two graphs $G_1$ and $G_2$ with respect to a vertex $x$ in $G_1$ and a vertex $y$ in $G_2$, is the graph $G_1(x \ast y)G_2$ obtained by identifying $x$ and $y$; in other words, replacing the vertices $x$ and $y$ by a new vertex $w$ adjacent to the same vertices in $G_1$ as $x$ and the same vertices in $G_2$ as $y$. If there is no confusion, then we write $G_1 \ast G_2$ instead of $G_1(x \ast y)G_2$.

**Theorem 2.2.** If $G$ is a connected $k$-i-critical graph, then $\text{diam}(G) \leq 2(k-1)$.

**Proof.** Let $x$ and $x_0$ be vertices such that $\text{dist}(x, x_0) = \text{diam}(G) = n$. Let $x_{01} \ldots x_{0m} x_0$ be a shortest path between $x$ and $x_0$. Define $L_0, L_1, \ldots, L_n$ by $L_i = \{v \in V(G) \mid \text{dist}_G(x, v) = i\}$ for $0 \leq i \leq n$. In particular, $L_0 = \{x\}$ and $L_1 = N_G(x)$. Let $R_i = L_0 \cup L_1 \cup \cdots \cup L_i$ for $0 \leq i \leq n$. Let $D$ be a minimum independent dominating set in $G$. If $|D \cap R_i| \geq \frac{i+2}{2}$, then we say that $R_i$ is sufficient with respect to $D$. □
Let $m$ be the maximum integer $j$ such that $|D \cap R_j| \geq \frac{j^2}{2}$. The value of $m$ depends on the minimum independent dominating set $D$, we may assume that $D$ is chosen such that $m$ is maximum among all the minimum independent dominating set. Let $D_1$ be a minimum independent dominating set of $G - x_1$. It is obvious that $x \notin D_1$ and $D_1 \cap L_1 \neq \emptyset$. Thus $D_1 \cup \{x_1\}$ is a minimum independent dominating set of $G$ with $(\{D_1 \cup \{x_1\} \cap R_1| \geq 2$, and then the value of $m$ exists and $m \geq 1$. If $m = n$, then $n = m \leq 2(k - 1)$, we are done. So we may assume that $m < n$.

If $m = 2t + 1$, then $|D \cap R_m| \geq t + 1.5$ and $|D \cap R_{m+1}| < t + 2$, which is a contradiction. So we have that $m = 2t$.

We further assume that $m = 2t$. It follows that $|D \cap R_m| \geq t + 1$ and $|D \cap R_{m+1}| < t + 1.5$ and $|D \cap R_{m+2}| < t + 2$, and then $|D \cap R_m| = t + 1$ and $D \cap L_{m+1} = D \cap L_{m+2} = \emptyset$. In order to dominate $L_{m+2}$, the set $L_{m+3}$ must exist and $D \cap L_{m+3}$ dominates $L_{m+2}$. The fact that $|D \cap R_{m+1}| < t + 2.5$ implies that $|D \cap L_{m+3}| = 1$. Let $D \cap L_{m+3} = \{w\}$. Notice that if $L_{m+4}$ exists, we can conclude that $D \cap L_{m+4} = \emptyset$ from the fact that $|D \cap R_{m+4}| < t + 3$. Hence, the vertex $w$ dominates $L_{m+2} \cup L_{m+3}$.

Let $D_2$ be a minimum independent dominating set of $G - w$. Notice that $D_2 \cap (L_{m+2} \cup L_{m+3}) = \emptyset$. If $|D_3 \cap R_{m+1}| > |D \cap L_{m+3}|$, then $R_{m+1}$ is sufficient with respect to $D_2 \cup \{w\}$. If $|D_3 \cap R_{m+1}| < |D \cap R_{m+1}|$, then $(D_3 \cap R_{m+1}) \cup (D \cap R_{m+1})$ is a smaller independent dominating set of $G$, which is a contradiction. Hence, we have that $|D_3 \cap R_{m+1}| = |D \cap R_{m+1}|$.

Suppose that the set $L_{m+4}$ does not exist. It implies that $|D \cap R_m| = k - 1 = t + 1$. Recall that $w$ dominates $L_{m+2} \cup L_{m+3}$, it follows that $D_3 \subseteq L_{m+1}$, and thus $L_{m+3} = \{w\}$. Let $D_2$ be a minimum independent dominating set of $G - x_{m+2}$. Therefore, the set $D_2 \cup \{x_{m+2}\}$ is a minimum independent dominating set with $|D_2 \cup \{x_{m+2}\} \cap R_{m+2} = k + 1.5$, which is a contradiction. So we may assume that $L_{m+4}$ exists.

If $|D_3 \cap (L_{m+2} \cup L_{m+3})| \geq 1$, then $R_{m+4}$ is sufficient with respect to $D_3 \cup \{w\}$, which leads to a contradiction. So we have that $D_3 \cap (L_{m+2} \cup L_{m+3}) = \emptyset$ and $L_{m+3} = \{w\}$. Let $D_4$ be a minimum independent dominating set of $G - x_{m+4}$. Notice that $D_4 \cap L_{m+3} = \emptyset$ and $D_4 \cap R_{m+2}$ totally dominates $R_{m+2}$. If $|D_4 \cap R_{m+2}| = |D \cap R_{m+2}|$, then $R_{m+2}$ is sufficient with respect to $D_4 \cup \{x_{m+4}\}$.

If $|D_4 \cap R_{m+2}| \leq |D \cap R_{m+2}|$, and $D_3 \cap L_{m+2} \neq \emptyset$, then $(D_4 \cap R_{m+2}) \cup (D_3 \setminus R_{m+3})$ is a smaller independent dominating set of $G$, a contradiction.

If $|D_4 \cap R_{m+2}| = |D \cap R_{m+2}|$ and $D_3 \cap L_{m+2} = \emptyset$, then $D_4 \cap L_{m+4} \neq \emptyset$ in order to dominates $w$, and then $R_{m+4}$ is sufficient with respect to $D_4 \cup \{x_{m+4}\}$.

If $|D_4 \cap R_{m+2}| < |D \cap R_{m+2}|$ and $D_4 \cap L_{m+2} = \emptyset$, then $(D_4 \cap R_{m+2}) \cup (D \setminus R_{m+2})$ is a smaller independent dominating set of $G$, a contradiction.

By the above arguments, the theorem is true except the case that $m = 2t = n - 1$. Notice that $G(x \times x)G$ is a $(2k - 1)$-vertex-critical graph with diameter $2n$. The theorem is true for the graph $G(x \times x)G$, it implies that $2n \leq 2(2k - 1) - 1$, thus $n \leq 2(k - 1)$.

**Theorem 2.3.** If $G$ is a $k$-$\gamma_\ell$-vertex-critical graph, then $\text{diam}(G) \leq k$.

**Proof.** Let $x$ and $x_0$ be vertices such that $\text{dist}(x, x_0) = \text{diam}(G) = n$. Let $x_1, \ldots, x_{n-1}, x_0$ be a shortest path between $x$ and $x_0$. Define $L_0, L_1, \ldots, L_t$ by $L_i = \{v \in V(G) \mid \text{dist}_G(x, v) = i\}$ for $0 \leq i \leq n$. In particular, $L_0 = \{x\}$ and $L_1 = N_G(x)$. Let $D_1$ be a minimum connected dominating set of $G - x_1$. It is obviously that $x \notin D_1$ and $D_1 \cap L_1 = \emptyset$. Since $D_1$ is a connected dominating set of $G$, it follows that $D_1 \cap L_i \neq \emptyset$ for every $1 \leq i \leq n - 1$. Hence we have that $|D_1| = k - 1 \geq n - 1$, which implies that $\text{diam}(G) = n \leq k$.

3 Sharpness of the upper bounds

We characterize when the coalescence of two total domination vertex critical graphs is still a total domination vertex graph.

**Theorem 3.1.** Let $G_1$ and $G_2$ be $k_1$-$\gamma_\ell$-vertex-critical and $k_1$-$\gamma_\ell$-vertex-critical graphs without leaf vertices, respectively. Let $x$ and $y$ be two vertices in $G_1$ and $G_2$, respectively. Then $G_1(x \times y)G_2$ is $(k_1 + k_2 - 1)$-$\gamma_\ell$-vertex-critical if and only if $\gamma_\ell(G_2 - N_{G_2}[y]) \geq k_2 - 1$ and $\gamma_\ell(G_1 - N_{G_1}[x]) \geq k_1 - 1$.

**Proof.** Denote the graph $G_1(x \times y)G_2$ by $G$ for short. Let $D$ be a minimum total dominating set of $G$ and $w$ be the new created vertex in $G$. Let $D_1$ and $D_2$ be a minimum total dominating set of $G_1 - x$ and $G_2 - y$, respectively. Then $|D_1| \geq k_1 - 1$ and $|D_2| \geq k_2 - 1$. It is obvious that $\gamma_\ell(G - w) = k_1 + k_2 - 2$. For any vertex $v \in V(G_1) \setminus \{x\}$, the union of $D_2$ and a minimum total dominating set of $G_1 - v$ is a total dominating set of $G - v$, thus $\gamma_\ell(G - v) \leq k_1 + k_2 - 2$.
Similarly, for any vertex \( v \in V(G_2) \setminus \{y\} \), the union of \( D_1 \) and a minimum total dominating set of \( G_2 - v \) is a total dominating set of \( G - v \), and then \( \gamma_t(G - v) \leq k_1 + k_2 - 2 \). Hence, for any vertex \( v \in V(G) \), we have that
\[
\gamma_t(G - v) \leq k_1 + k_2 - 2.
\]

\((\Leftarrow)\) Suppose that \( \gamma_t(G_2 - N_{G_2}[y]) \geq k_2 - 1 \) and \( \gamma_t(G_1 - N_{G_1}[x]) \geq k_1 - 1 \). We want to prove \( \gamma_t(G) \geq k_1 + k_2 - 1 \).

Notice that either \( D \cap V(G_1) \) totally dominates \( G_1 \) or \( D \cap V(G_2) \) totally dominates \( G_2 \). By symmetry, we may assume that \( D \cap V(G_1) \) totally dominates \( G_1 \) and \( |D \cap V(G_2)| \geq k_1 \). If \( w \notin D \), then \( D \cap V(G_2) \) dominates \( G_2 - w \) and \( |D \cap V(G_2)| \geq k_2 - 1 \), and then \( |D| \geq k_1 + k_2 - 2 \). So we may assume that \( w \in D \). If \( D \cap N_{G_1}(w) \neq \emptyset \), then \( D \cap V(G_2) \) is a total dominating set of \( G_2 \) and \( |D \cap V(G_1)| \geq k_1 \) and \( |D \cap V(G_2)| \geq k_2 \). Hence \( |D| \geq k_1 + k_2 - 1 \).

\((\Rightarrow)\) Suppose that \( |D| = k_1 + k_2 - 1 \). We want to prove that \( \gamma_t(G_2 - N_{G_2}[y]) \geq k_2 - 1 \) and \( \gamma_t(G_1 - N_{G_1}[x]) \geq k_1 - 1 \).

By Lemma 1, let \( D_i \) be a minimum total dominating set of \( G_i \) containing \( x \). It follows that \( \gamma_t(G_2 - N_{G_2}[y]) \geq k_2 - 1 \); otherwise, the union of \( D_i \) and a minimum total dominating set of \( G_2 - N_{G_2}[y] \) is a smaller total dominating set of \( G \), a contradiction. Similarly, we can prove that \( \gamma_t(G_1 - N_{G_1}[x]) \geq k_1 - 1 \).

\[\square\]

**Remark 1.** From the characterization, the graph \( C_6 \ast C_6 \) is not a total domination vertex critical graph as mentioned in [2].

A **pointed graph** is a graph with two assigned diametrical vertices called **Left** and **Right**. For a pointed graph \( G \), we define \( L_k(G) \) and \( R_k(G) \) as the sets of vertices which are distance \( k \) from the **Left-vertex** and **Right-vertex**, respectively.

For two pointed graphs \( G_1 \) and \( G_2 \), we define \( G_1 \bullet G_2 \) as the pointed graph obtained by identifying and unassigning the **Right-vertex** from \( G_1 \) and the **Left-vertex** from \( G_2 \).

Let \( K_{m,m} \) be a complete bipartite graph with bipartition \( \{y_1, y_2, \ldots, y_{2m-1}\} \) and \( \{y_2, y_4, \ldots, y_{2m}\} \), where \( m \geq 2 \). Let \( F \) be the graph obtained from \( K_{m,m} \) by removing one edge \( y_1y_{2m} \), and let \( \tilde{F} \) be the complement of \( F \) with \( y_1 \) corresponding to \( y \). Notice that \( \gamma_t(F) = \gamma_t(\tilde{F}) = 2 \) and \( \{x_1, x_{2m}\} \) totally dominates \( F \) and every pair of adjacent vertices in \( K_{m,m} \) totally dominates \( K_{m,m} \). Let \( R \) be the pointed graph obtained from the disjoint union of \( \tilde{F} \) and \( K_{m,m} \), by joining every vertex of \( \tilde{F} \) to every vertex of \( K_{m,m} \) except edges between the corresponding vertices, and adding five new vertices \( z_1, z_2, z_3 \). **Left** and **Right** such that **Left** is adjacent to every vertex in \( \tilde{F} \), the vertex \( z_1 \) is adjacent to \( \{x_1, x_2, \ldots, x_{2m-1}\} \cup \{y_2, y_3, \ldots, y_{2m-1}\} \), the vertex \( z_3 \) is adjacent to \( \{x_2, x_3, \ldots, x_{2m}\} \cup \{y_2, y_3, \ldots, y_{2m-1}\} \), and **Right** is adjacent to every vertex in \( K_{m,m} \) and \( z_1 \), while **Right** is adjacent to every vertex in \( K_{m,m} \) and \( z_2 \).

**Theorem 3.2.** The graph \( R \) is a 3-\( \gamma_t \)-vertex-critical graph with diameter three.

Let \( H \) be a graph with at least four vertices. Let \( V(H) = \{x_1, \ldots, x_t\} \) and \( V(\bar{H}) = \{y_1, \ldots, y_t\} \) with \( x_t \) corresponding to \( y_t \). Let \( A \) be the pointed graph obtained by joining every vertex of \( H \) to every vertex of \( \bar{H} \) except edges between the corresponding vertices, and adding two new vertices **Left** and **Right** such that **Left** is adjacent to every vertex in \( H \) and **Right** is adjacent to every vertex in \( \bar{H} \). It can be shown that \( A \) is a 3-\( \gamma_t \)-vertex-critical graph if and only if \( \gamma_t(H) = \gamma_t(\bar{H}) = 2 \). Simply write the **Left-vertex** as \( x \) and **Right-vertex** as \( y \). Suppose that \( \gamma_t(H) = \gamma_t(\bar{H}) = 2 \). A minimum total dominating set of \( H \) totally dominates \( A - y \) and a minimum total dominating set of \( \bar{H} \) totally dominates \( A - x \). For any vertex \( x_i \), the two vertices \( y_i \) and a nonadjacent vertex \( x_j \) of \( x_i \) totally dominates \( A - x_i \); similarly, for any vertex \( y_i \), the two vertices \( x_i \) and a nonadjacent vertex \( y_j \) of \( y_i \) totally dominates \( A - y_i \). But \( \gamma_t(A) > 2 \), thus \( A \) is a 3-\( \gamma_t \)-vertex-critical graph. Conversely, if \( G \) is a 3-\( \gamma_t \)-vertex-critical graph, then a minimum total dominating set of \( A - y \) is also a minimum total dominating set of \( H \) and a minimum total dominating set of \( A - x \) is also a minimum total dominating set of \( \bar{H} \), and then \( \gamma_t(H) = \gamma_t(\bar{H}) = 2 \). In what follows, we assume that \( \gamma_t(H) = \gamma_t(\bar{H}) = 2 \). Notice that \( \text{diam}(A) = 3 \).

**Remark 2.** For every \( t \geq 4 \), we can find at least one graph \( H \) on \( t \) vertices with \( \gamma_t(H) = 2 \) and \( \gamma_t(\bar{H}) = 2 \). For instance, let \( K_{t,2} \) be a complete graph on \( t - 2 \) vertices, and let \( H \) be the graph on \( t \) vertices obtained from \( K_{t,2} \) by attaching a path \( x_1x_2x_3 \). It is easy to check that \( \gamma_t(H) = 2 \) and \( \gamma_t(\bar{H}) = 2 \).

Let \( Q \) be the pointed graph obtained from two copies of \( A \), called \( A_1 \) and \( A_2 \), by deleting the **Right-vertex** \( y \) from \( A_1 \) and the **Left-vertex** \( x \) from \( A_2 \), and joining every neighbor of \( y \) in \( A_1 \) to every neighbor of \( x \) in \( A_2 \). Notice that \( \text{diam}(Q) = 5 \) and \( \gamma_t(Q) = 4 \). By Theorem 2.1, the graph \( Q \) is not a \( 4-\gamma_t \)-vertex-critical graph. Let \( Q^{(1)} = Q \) and \( Q^{(m)} = Q^{(m-1)} \ast Q \). We simply denote \( R \ast Q^{(m)} \) by \( C_5 \).

Let \( J_1 \) and \( J_2 \) be disjoint union of \( tK_2 \) and let \( J_2 \) be \( tK_2 \), where \( t \geq 2 \). Let \( J \) be the pointed graph obtained from \( J_1 \cup J_2 \cup J_3 \) by joining every vertex of \( J_1 \) to every vertex of \( J_2 \) except the edges corresponding vertices in \( J_1 \) and
Theorem 3.3. (a) \( \gamma_j(R \cdot Q^{(n)}) \geq 3n + 3 \); (b) \( \gamma_j(R \cdot Q^{(n)} - y) \geq 3n + 2 \); (c) \( \gamma_j(R \cdot Q^{(n)} - N[y]) \geq 3n + 2 \).

Proof. We prove the results by mathematical induction.

Basis step: If \( n = 0 \), then the results are trivially true.

Inductive step: Suppose that the results are true for all values less than \( n \). Let \( D, D_1 \), and \( D_2 \) be a minimum total dominating set of \( \xi_0, \xi_n - y \), and \( \xi_n - N[y] \), respectively. Denote the left vertex of \( Q_n \) by \( x \) and the right vertex of \( Q_n \) by \( y \). If \( D \cap V(\xi_{n-1}) \) totally dominates \( \xi_{n-1} \), then \( |D \cap V(\xi_{n-1})| \geq 3n \), but \( |D \cap V(\xi_{n-1})| \geq 3 \), so we may assume that \( D \cap V(\xi_{n-1}) \) does not totally dominates \( \xi_{n-1} \). Notice that \( D \cap V(Q_n) \) must totally dominate \( Q_n \) and \( D \cap V(Q_n) \geq 3n \). If \( x \in D \), then \( D \cap R_1(Q_n) = \emptyset \) and \( D \cap V(\xi_{n-1}) \) totally dominates \( \xi_{n-1} - x \) and \( |D \cap V(\xi_{n-1})| \geq 3n - 1 \), thus \( |D| \geq 3n - 1 + 3 = 3n + 3 \). If \( x \notin D \), then \( D \cap R_1(Q_n) = \emptyset \) and \( D \cap V(\xi_{n-1} - x) \) totally dominates \( \xi_{n-1} - N[x] \) and \( |D \cap V(\xi_{n-1} - x)| \geq 3n - 1 \). Since \( D \cap V(Q_n) \) totally dominates \( Q_n \) and \( x \in D \), it follows that \( |D \cap V(Q_n)| \geq 5 \), and then \( |D| \geq 3n - 1 + 5 = 3n + 4 \). Hence, we have that \( \gamma_j(R \cdot Q^{(n)}) \geq 3n + 3 \).

If \( D_1 \cap V(\xi_{n-1}) \) totally dominates \( \xi_{n-1} \), then \( D_1 \cap V(V(\xi_{n-1})) \geq 3n \), but \( |D_1 \cap V(V(\xi_{n-1}))| \geq 2 \), thus \( |D_1| \geq 3n + 2 \). So we may assume that \( D_1 \cap V(\xi_{n-1}) \) does not totally dominates \( \xi_{n-1} \). If \( x \notin D_1 \), then \( D_1 \cap R_1(Q_n) = \emptyset \) and \( D_1 \cap V(\xi_{n-1}) \) totally dominates \( \xi_{n-1} - x \) and \( |D_1 \cap V(\xi_{n-1})| \geq 3n - 1 \). Notice that \( D_1 \cap V(Q_n) \) totally dominates \( Q_n - y \) and \( D_1 \cap L(Q_n) \neq \emptyset \) and \( |D_1 \cap V(Q_n)| \geq 4 \), thus \( |D_1| \geq 3n - 1 + 4 = 3n + 3 \). If \( x \in D_1 \), then \( D_1 \cap R_1(Q_n) = \emptyset \) and \( D_1 \cap V(\xi_{n-1} - x) \) totally dominates \( \xi_{n-1} - N[x] \) and \( |D_1 \cap V(\xi_{n-1} - N[x])| \geq 3n - 1 \). Notice that \( x \in D_1 \) and \( D_1 \cap L(Q_n) \neq \emptyset \), and then \( |D_1 \cap (Q_n - y)| \geq 4 \), thus \( |D_1| \geq 3n - 1 + 4 = 3n + 3 \). Hence, we have that \( \gamma_j(R \cdot Q^{(n)} - y) \geq 3n + 2 \).

If \( D_2 \cap V(\xi_{n-1}) \) totally dominates \( \xi_{n-1} \), then \( D_2 \cap V(V(\xi_{n-1})) \geq 3n \), but \( |D_2 \cap V(V(\xi_{n-1}))| \geq 2 \), thus \( |D_2| \geq 3n + 2 \). So we may assume that \( D_2 \cap V(\xi_{n-1}) \) does not totally dominates \( \xi_{n-1} \). If \( x \notin D_2 \), then \( D_2 \cap R_1(Q_n) = \emptyset \) and \( D_2 \cap V(\xi_{n-1}) \) totally dominates \( \xi_{n-1} - x \) and \( |D_2 \cap V(\xi_{n-1})| \geq 3n - 1 \). Notice that \( D_2 \cap V(Q_n) \) totally dominates \( Q_n - N[y] \) and \( D_2 \cap L(Q_n) \neq \emptyset \) and \( |D_2 \cap V(Q_n)| \geq 3 \), thus \( |D_2| \geq 3n - 1 + 3 = 3n + 2 \). If \( x \in D_2 \), then \( D_2 \cap R_1(Q_n) = \emptyset \) and \( D_2 \cap V(\xi_{n-1} - x) \) totally dominates \( \xi_{n-1} - N[x] \) and \( |D_2 \cap V(\xi_{n-1} - N[x])| \geq 3n - 1 \). Notice that \( x \in D_2 \) and \( D_2 \cap L(Q_n) \neq \emptyset \), and then \( |D_2 \cap (Q_n - N[y])| \geq 3 \), thus \( |D_2| \geq 3n - 1 + 3 = 3n + 2 \). Hence, we have that \( \gamma_j(R \cdot Q^{(n)} - N[y]) \geq 3n + 2 \).

Corollary 1. (a) \( \gamma_j(R \cdot Q^{(n)}) = 3n + 3 \); (b) \( \gamma_j(R \cdot Q^{(n)} - y) = 3n + 2 \); (c) \( \gamma_j(R \cdot Q^{(n)} - N[y]) = 3n + 2 \).

Theorem 3.4. The graph \( R \cdot Q^{(n)} \cdot J \) is \( (3n + 6) \cdot \gamma_j \)-vertex-critical graph with diameter \( 5n + 7 \).

Proof. If \( n = 0 \), then the statement follows by Theorem 3.1. So we may assume that \( n \geq 1 \). Denote the graph \( R \cdot Q^{(n)} \cdot J \) by \( G \) and denote the ith copy of \( Q \) by \( Q_i \), with \( L = x_i \) and \( R = y_i \). Denote the left vertex of \( J \) by \( x \) and the right vertex of \( Q_n \). Let \( D \) be a minimum total dominating set of \( G \). Notice that there exists a minimum total dominating set \( D_{i,j} \) of \( Q_i - N[y] \) containing \( x_i \), that is, a vertex from each of \( Q_i(Q_0), L_i(Q_1) \) and \( L_i(Q_2) \) totally dominates \( L_i(Q_0) \cup L_i(Q_1) \cup L_i(Q_2) \cup L_i(Q_3) \); by symmetry, there exists a minimum total dominating set \( D_{i,x} \) of \( Q_i - N[x] \) containing \( x_i \), that is, a vertex from each of \( R_i(Q_0), R_i(Q_1) \) and \( R_i(Q_2) \) totally dominates \( R_i(Q_0) \cup R_i(Q_1) \cup R_i(Q_2) \cup R_i(Q_3) \). For the graph \( G \), there exists a minimum total dominating set \( D_{i,j} \) of \( R \cdot Q_n \), and a minimum total dominating set \( D_{i,j} \) of \( R \cdot Q_n \) containing \( x_i \), and a minimum total dominating set of \( D_{i,j} \). Thus \( D_{i,j} \cup D_{i,x} \cup D_{i,y} \cup \ldots \cup D_{i,z} \cup D_{i,1,j} \cup D_{i,1,x} \cup D_{i,1,y} \) is a minimum dominating set of \( G - v \) with \( 3n + 5 \) vertices.

Let \( v \) be an arbitrary vertex. If \( v \in J \), then \( D_{i,j} \cup D_{i,x} \cup \ldots \cup D_{i,1,j} \) and \( D_{i,1,x} \cup D_{i,1,y} \cup D_{i,1,z} \) form a dominating set of \( G - v \) with \( 3n + 5 \) vertices.

If \( v \in L_i(Q_1) \cup L_i(Q_3) \), then \( D_{i,j} \cup D_{i,x} \cup \ldots \cup D_{i,1,j} \) and \( D_{i,1,x} \cup D_{i,1,y} \cup D_{i,1,z} \) form a minimum total dominating set of \( L_i(Q_2) - v \). Thus \( D_{i,j} \cup D_{i,x} \cup D_{i,1,x} \cup \ldots \cup D_{i,n+1,j} \cup D_{i,n+1,x} \cup D_{i,n+1,y} \) is a minimum dominating set of \( G - v \) with \( 3n + 5 \) vertices.
Remark 3. As in [2], the upper bound in Theorem 2.2 is sharp. We provide infinitely many

Proof.

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We can adapt the similar technique to prove that \( R \bullet Q^{(t)} \bullet R \bullet R \) is \((3n + 7)\)-\( \gamma_t \)-vertex-critical, so we omit the details of the proof.

Theorem 3.5. The graph \( R \bullet Q^{(t)} \bullet R \bullet R \) is a \((3n + 7)\)-\( \gamma_t \)-vertex-critical graph with diameter \(5n + 9\).

Proof. We divide the graphs into four classes according to the value of \(k\).

(1) Suppose that \(k \equiv 2 \pmod{3}\) and \(k = 3n + 5\). Notice that the graph \(A \bullet Q^{(t)} \bullet A\) is a \((3n + 5)\)-\( \gamma_t \)-vertex-critical graph and \(\text{diam}(A \bullet Q^{(t)} \bullet A) = 5n + 6 = \frac{3k-2}{3}\), which has been proved in [4, Theorem 13].

(2) Suppose that \(k \equiv 0 \pmod{3}\) and \(k = 3n + 6\). Notice that the graph \(R \bullet Q^{(t)} \bullet J\) is a \((3n + 6)\)-\( \gamma_t \)-vertex-critical graph and \(\text{diam}(R \bullet Q^{(t)} \bullet J) = 5n + 7 = \frac{3k-2}{3}\).

(3) Suppose that \(k \equiv 1 \pmod{3}\) and \(k = 3n + 7\). Notice that the graph \(R \bullet Q^{(t)} \bullet R \bullet R\) is a \((3n + 7)\)-\( \gamma_t \)-vertex-critical graph and \(\text{diam}(R \bullet Q^{(t)} \bullet R \bullet R) = 5n + 9 = \frac{3k-2}{3}\).

(4) If \(k = 4\), then the graph \(J\) meet the requirement by Theorem 3.1.

Remark 3. As in [2], the upper bound in Theorem 2.2 is sharp. We provide infinitely many \(k-i\)-vertex-critical graphs with diameter \(2(k-1)\) for each \(k \geq 2\). For instance, let \(B\) be the complete graph on \(2t\) vertices with a perfect matching removed, and let \(G\) be the graph whose block graph is a path on \(k - 1\) vertices and every block is a copy of \(B\); notice that \(i(G) = k\) and \(\text{diam}(G) = 2(k - 1)\).

Remark 4. So far, we don’t know if the given upper bound on the \(k-\gamma_t\)-vertex-critical graphs is the best possible.

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