Schwartz families in tempered distribution spaces

David Ćarfi

Abstract

In this paper we define Schwartz families in tempered distribution spaces and prove many their properties. Schwartz families are the analogous of infinite dimensional matrices of separable Hilbert spaces, but for the Schwartz test function spaces, having elements (functions) realizable as vectors indexed by real Euclidean spaces (ordered families of scalars indexed by real Euclidean spaces). In the paper, indeed, one of the consequences of the principal result (the characterization of summability for Schwartz families) is that the space of linear continuous operators among Schwartz test function spaces is linearly isomorphic with the space of Schwartz families. It should be noticed that this theorem is possible because of the very good properties of Schwartz test function spaces and because of the particular structures of the Schwartz families viewed as generalized matrices; in fact, any family of tempered distribution, regarded as generalized matrix, has one index belonging to a Euclidean space and one belonging to a test function space, so that any Schwartz family is a matrix of type $\left(\mathbb{R}^m, \mathbb{R}^n\right)$ in the sense of distributions. Another motivation for the introduction and study of these families is that these are the families which are summable with respect to every tempered system of coefficients, in the sense of superpositions. The Schwartz families we present in this paper are one possible rigorous and simply manageable mathematical model for the infinite matrices used frequently in Quantum Mechanics.
1 Families of distributions

Let $n$ be a positive natural number; by $\mathcal{S}'_n$ we denote the space of complex (or real) tempered distributions defined on the $n$-dimensional real Euclidean space. Let $I$ be a non-empty set, we shall denote by $(\mathcal{S}'_n)^I$ the space of all the families in the space of tempered distributions $\mathcal{S}'_n$ indexed by the set $I$, i.e., the set of all the surjective maps from the set $I$ onto a subset of the space $\mathcal{S}'_n$. Moreover, as usual, if $v$ is any family in the space of tempered distributions $\mathcal{S}'_n$ and indexed by the set $I$, for each index $p \in I$, the distribution $v(p)$ (corresponding to the index $p$ in the map $v$) is denoted by $v_p$, and the family $v$ itself is also denoted by the expressive notation $(v_p)_{p \in I}$.

Dangerous bend. Note that it is not correct and strongly misleading to consider the family $v$ coincident with its trace (trajectory, underlying set) $v(I)$, i.e. its image in $\mathcal{S}'_n$: the family $v$ is a function, an ordered set, and not a simple set of objects.

Linear operations for families. The set $(\mathcal{S}'_n)^I$ of all families in the space $\mathcal{S}'_n$ indexed by a non-empty set $I$ is a vector space with respect to the following standard binary operations:

- the componentwise addition
  \[ + : (\mathcal{S}'_n)^I \times (\mathcal{S}'_n)^I \to (\mathcal{S}'_n)^I, \]
  defined pointwise by
  \[ v + w := (v_p + w_p)_{p \in I}, \]
  for any two families $v, w$;

- the componentwise multiplication by scalars
  \[ \cdot : \mathbb{K} \times (\mathcal{S}'_n)^I \to (\mathcal{S}'_n)^I \]
  defined pointwise by
  \[ av := (av_p)_{p \in I}, \]
  for any family $v$ and any scalar $a$.  

In other words, the sum family $v + w$ is defined by
\[(v + w)_p = v_p + w_p,\]
for every index $p$ in $I$, and the product family $av$ is defined by
\[(av)_p = av_p,\]
for every $p$ in $I$.

**Interpretations.** We are already familiar with ordered systems of vectors in linear spaces: the ordered bases, which permit to define the systems of components; the Hilbert bases of a Hilbert space, which permit to define the abstract Fourier transforms; the Schauder bases of topological vector spaces; and so on. But there is another natural way to think at a (ordered) family of vectors: it can be viewed as a matrix, the unique matrix having the ordered family as ordered systems of its rows (or columns). This last interpretation is the useful way to think at families of distributions, when such families should be summed or multiplied by scalars.

## 2 Action of a family on test functions

The basic important consideration for our purposes is the observation that a family of tempered distributions can act naturally on test functions, as the following definition specifies. This capacity of action is not surprising, if we think to the families as matrices and to the test functions as vectors.

**Definition (image of a test function by a family of distributions).** Let $v$ be a family of tempered distributions in the space $S'_n$ indexed by a non-empty set $I$ and let $\phi \in S_n$ be any test function. The mapping
\[v(\phi) : I \to \mathbb{K}\]
defined by
\[v(\phi)(p) := v_p(\phi),\]
for each index $p \in I$, is called the **image of the test function $\phi$ under the family of tempered distributions $v$.**
So, in the conditions of the above definition, the function \( v(\phi) \) resulting from the action of the family \( v \) upon the test function \( \phi \) is a function belonging to the function space \( \mathcal{F}(I, \mathbb{K}) \) (following Bourbaki, this is the set of all scalar function on \( I \)).

Thus, with any family \( v \) belonging to the vector space \( (\mathcal{S}_n')^I \) we can associate a mapping from the space of test functions \( \mathcal{S}_n \) into the function space \( \mathcal{F}(I, \mathbb{K}) \).

**Matrix interpretation.** Note that if we think to a family of distribution \( v \) belonging to the space \( \mathcal{F}(I, \mathcal{S}_n') \) as a generalized matrix with \( I \) rows and \( \mathbb{R}^n \) columns (actually the second index is belonging to the test function space \( \mathcal{S}_n \), but we can think, in the sense of distributions, that it is belonging to \( \mathbb{R}^n \)) and if we think to any test function \( f \) in \( \mathcal{S}_n \) as to a scalar vector with \( \mathbb{R}^n \) components, it is natural to understand that the result \( v(f) \) of the action of the family \( v \) on the function \( f \) is a scalar vector with \( I \) components. We so have build up a multiplication
\[
\mathcal{F}(I, \mathcal{S}_n') \times \mathcal{S}(\mathbb{R}^n, \mathbb{K}) \to \mathcal{F}(I, \mathbb{K})
\]

analogous to that defined for finite matrices.

**Remark.** Equivalently, for every test function \( \phi \), we have a “projection” \( \pi_\phi \) sending any family of \( (\mathcal{S}_n')^I \) into a scalar family of the product \( (\mathbb{K})^I \):
\[
\pi_\phi(v) = (v_p(\phi))_{p \in I},
\]
for every family \( v \) of the space \( (\mathcal{S}_n')^I \).

## 3 \( \mathcal{S} \) Families

In the Theory of Superpositions on the space of tempered distributions \( \mathcal{S}_n' \) the below class of \( \mathcal{S} \) families plays a basic role.

**Definition (family of tempered distributions of class \( \mathcal{S} \)).** Let \( v \) be a family of distributions in the space \( \mathcal{S}_n' \) indexed by the Euclidean space \( \mathbb{R}^m \). The family \( v \) is called a **Schwartz family** or **family of class \( \mathcal{S} \)** or even
**$S$ family** if, for each test function $\phi \in S_n$, the image of the test function $\phi$ by the family $v$ - that is the function $v(\phi) : \mathbb{R}^m \to \mathbb{K}$ defined by

$$v(\phi)(p) := v_p(\phi),$$

for each index $p \in \mathbb{R}^m$ - belongs to the space of test functions $S_m$. We shall denote the set of all $S$ families by $S(\mathbb{R}^m, S'_n)$.

**Example (the Dirac family in $S'_n$).** The **Dirac family in $S'_n$**, i.e., the family $\delta := (\delta_x)_{x \in \mathbb{R}^n}$, where $\delta_x$ is the Dirac (tempered) distribution centered at the point $x$ of $\mathbb{R}^n$, is a Schwartz family.

**Proof.** Indeed, for each test function $\phi \in S_n$ and for each index (point) $x$ in $\mathbb{R}^n$, we have

$$\delta(\phi)(x) = \delta_x(\phi) = \phi(x),$$

and hence the image of any test function $\phi$ by the Dirac family is the function $\delta(\phi) = \phi$. So the image of the test function $\phi$ under the family $\delta$ is the function $\phi$ itself, which (in particular) lies in the Schwartz space $S_n$. ■

It is clear that the space of $S$ families in $S'_n$, indexed by some Euclidean space $I$, is a subspace of the vector space $(S'_n)^I$ of all families in $S'_n$ indexed by the same index set $I$.

### 4 $S$ Family generated by an operator

In this section we introduce a wide class of $S$ families. We will see later that this class is indeed the entire class of Schwartz families. We recall that by $\sigma(S_n)$ we denote the weak topology $\sigma(S_n, S'_n)$.

**Theorem (on the $S$ family generated by a linear and continuous operator).** Let $A : S_n \to S_m$ be a linear and continuous operator with respect to the natural topologies of $S_n$ and $S_m$ (or equivalently, continuous with respect to the weak topologies $\sigma(S_n)$ and $\sigma(S_m)$) and let $\delta$ be the Dirac family in $S'_m$. Then, the family of functionals

$$A^\vee := (\delta_p \circ A)_{p \in \mathbb{R}^m}$$

is a family of distribution and it is an $S$ family.

Proof. Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a linear and continuous operator with respect to the natural topologies of $\mathcal{S}_n$ and $\mathcal{S}_m$ (since these topologies are Fréchet-topologies, this is equivalent to assume the operator $A$ be linear and continuous with respect to the weak topologies $\sigma(\mathcal{S}_n)$ and $\sigma(\mathcal{S}_m)$). Let $\delta$ be the Dirac family in $\mathcal{S}_m'$ and consider the family

$$A^\vee := (\delta_p \circ A)_{p \in \mathbb{R}^m}.$$ 

The family $A^\vee$ is a family in $\mathcal{S}_n'$, since each functional $A^\vee_p$ is the composition of two linear and continuous mappings. Moreover, the family $A^\vee$ is of class $\mathcal{S}$, in fact, for every test function $\phi$ in $\mathcal{S}_n$ and for every index $p$ in $\mathbb{R}^m$, we have

$$A^\vee(\phi)(p) = A^\vee_p(\phi) = (\delta_p \circ A)(\phi) = \delta_p(A(\phi)) = A(\phi)(p),$$

so that the image of the test function by the family $A^\vee$ is nothing but the image of the test function under the operator $A$, i.e.

$$A^\vee(\phi) = A(\phi),$$

and this image belongs to the space $\mathcal{S}_m$ by the choice of the operator $A$ itself.

So we can give the following definition.

**Definition (of $S$ family generated by a linear and continuous operator).** Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a linear and continuous operator with respect to the natural topologies of $\mathcal{S}_n$ and $\mathcal{S}_m$ (or equivalently, continuous with respect to the weak topologies $\sigma(\mathcal{S}_n)$ and $\sigma(\mathcal{S}_m)$) and let $\delta$ be the Dirac family in $\mathcal{S}_m'$. The family

$$A^\vee := (\delta_p \circ A)_{p \in \mathbb{R}^m}$$

is called the Schwartz family generated by the operator $A$.  


Remark. We have so constructed the mapping
\[(\cdot)^\vee : \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) \to \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) : A \mapsto (\delta_x \circ A)_{x \in \mathbb{R}^m},\]
which we shall call the canonical representation of the operator space \(\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)\) into the family space \(\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)\). It is quite simple to prove that this mapping is a linear injection.

We shall see, as we already said, that every Schwartz family is generated by a linear and continuous operator as specified above, or, in other terms, that the canonical representation \((\cdot)^\vee\) is a linear isomorphism.

Matrix interpretation. As in the finite dimensional case, recalling our interpretation of the families of distributions as generalized matrices, we have so associated with every linear and continuous operator a generalized matrix which is of Schwartz class.

5 The operator generated by an \(\mathcal{S}\) family

Definition (operator generated by an \(\mathcal{S}\) family). Let \(v\) be a family of class \(\mathcal{S}\) belonging to the space \(\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)\). We call operator generated by the family \(v\) (or associated with the family \(v\)) the operator
\[
\hat{v} : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto v(\phi),
\]
sending every test function \(\phi\) of \(\mathcal{S}_n\) into its image \(v(\phi)\) under the family \(v\).

Example (on the Dirac family). The operator (on \(\mathcal{S}_n\)) generated by the Dirac family, i.e., by the ordered family \(\delta = (\delta_y)_{y \in \mathbb{R}^n}\) of Dirac distributions is the identity operator of the space \(\mathcal{S}_n\).

Proof. In fact, for each \(y \in \mathbb{R}^n\), we have
\[
\hat{\delta}(\phi)(y) = \delta_y(\phi) = \\
= \phi(y) = \\
= \mathbb{I}_{\mathcal{S}_n}(\phi)(y),
\]
for any test function \( \phi \) in \( S_n \). ■

We recall that the set \( S(\mathbb{R}^m, S'_n) \), of \( S \) families indexed by \( \mathbb{R}^m \), is a subspace of the vector space \( (S'_n)^{\mathbb{R}^m} \), of all families in \( S'_n \) indexed by \( \mathbb{R}^m \).

Moreover, we immediately can prove the following obvious proposition.

**Proposition.** For each family \( v \in S(\mathbb{R}^m, S'_n) \), the operator \( \hat{v} \) associated with the family \( v \) is linear and the map

\[
S(\mathbb{R}^m, S'_n) \to \text{Hom} (S_n, S_m) : v \mapsto \hat{v}
\]

is an injective linear operator.

**Example (on the family generated by an operator).** The operator associated with the family \( A^v \) generated by a linear and continuous operator \( A \) in \( \mathcal{L} (S_n, S_m) \) is the operator \( A \) itself, as can be immediately proved. In other terms we can write

\[
(A^v)^\wedge = A.
\]

## 6 Summability of \( S \) families

Here we give one of the fundamental definitions of Superposition Theory: the definition of summability of a Schwartz family with respect to linear functionals.

**Definition (summability of \( S \) families).** Let \( v \) be a Schwartz family of tempered distributions belonging to the space \( S(\mathbb{R}^m, S'_n) \). The family \( v \) is said to be **summable with respect to a linear functional** \( a \) on the space \( S_m \) if the composition \( u = a \circ \hat{v} \), i.e., the linear functional

\[
u : S_n \to \mathbb{K} : \phi \mapsto a(\hat{v}(\phi)),
\]

is a tempered distribution (in the space \( S'_n \)).

It happens that a Schwartz family is summable with respect to any tempered distribution on its index Euclidean space.
Theorem (summability of $\mathcal{S}$ families). Let $\nu$ be a Schwartz family of tempered distributions belonging to the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, the family is summable with respect to every distribution in $\mathcal{S}'_m$, in other terms:

- for every tempered distribution $a \in \mathcal{S}'_m$, the composition $u = a \circ \hat{\nu}$, i.e., the linear functional
  \[ u : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto a(\hat{\nu}(\phi)), \]
  is a tempered distribution in the space $\mathcal{S}'_n$.

Proof. Let $a \in \mathcal{S}'_m$ and let $\delta$ be the Dirac family of the space $\mathcal{S}'_m$. Since the linear hull $\text{span}(\delta)$ of the Dirac family is $\sigma(\mathcal{S}'_m)$-sequentially dense in the space $\mathcal{S}'_m$ (see, for example, Boccara, page 205), there is a sequence of distributions $\alpha = (\alpha_k)_{k \in \mathbb{N}}$ in the linear hull $\text{span}(\delta)$ of the Dirac family, converging to the distribution $a$ with respect to the weak* topology $\sigma(\mathcal{S}'_m)$; that is we have
  \[ \sigma(\mathcal{S}'_m) \lim_{k \to +\infty} \alpha_k = a. \]

Now, since for any natural $k$, the distribution $\alpha_k$ belongs to the linear hull $\text{span}(\delta)$, there exists a finite family $(y_i)_{i=1}^h$ of points in $\mathbb{R}^m$ and there is a finite family of scalars $(\lambda_i)_{i=1}^h$ in the field $\mathbb{K}$ such that
  \[ \alpha_k = \sum_{i=1}^h \lambda_i \delta_{y_i}. \]

Consequently, by obvious calculations, we have
  \[ \alpha_k \circ \hat{\nu} = \sum_{i=1}^h \lambda_i (\delta_{y_i} \circ \hat{\nu}) = \sum_{i=1}^h \lambda_i \nu_{y_i}. \]

Hence, for every index $k \in \mathbb{N}$, the linear functional $\alpha_k \circ \hat{\nu}$ belongs to the space $\mathcal{S}'_n$. Now, let $s$ be the topology of pointwise convergence on the algebraic dual $(\mathcal{S}_n)^*$, we claim that
  \[ \mathcal{s} \lim_{k \to +\infty} (\alpha_k \circ \hat{\nu}) = a \circ \hat{\nu}. \]
In fact, for every test function $\phi$ in $\mathcal{S}_n$, we obtain
\[
\lim_{k \to +\infty} (\alpha_k \circ \hat{v})(\phi) = \lim_{k \to +\infty} \alpha_k (\hat{v}(\phi)) = a(\hat{v}(\phi)).
\]
So we have proved that the sequence of continuous linear functionals $(\alpha_k \circ \hat{v})_{k \in \mathbb{N}}$ is pointwise convergent to the linear functional $a \circ \hat{v}$. Hence, by the Banach-Steinhaus theorem (that is applicable since $\mathcal{S}_n$ is a barreled space), the linear functional $a \circ \hat{v}$ must be continuous too, i.e. $a \circ \hat{v}$ should be a tempered distribution in $\mathcal{S}'_n$. So summability of the family $v$ holds true. ■

7 Characterization of summability

In the following we shall denote by $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ the set of all the linear and continuous operators among the two topological vector spaces $(\mathcal{S}_n)$ and $(\mathcal{S}_m)$.

Moreover, let consider a linear operator $A : \mathcal{S}_n \to \mathcal{S}_m$, we say that $A$ is (topologically) transposable if its algebraic transpose (adjoint) $^*A : \mathcal{S}_m^* \to \mathcal{S}_n^*$ ($X^*$ denotes the algebraic dual of a topological vector space $X$), defined by
\[
^*A(a) = a \circ A,
\]
maps the distribution space $\mathcal{S}_m'$ into the distribution space $\mathcal{S}_n'$.

**Theorem (characterization of summability of $S$ families).** Let $v$ be a Schwartz family of tempered distributions belonging to the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}_n')$. Then, the following assertions hold and they are equivalent:

1) the family $v$ is summable with respect to every tempered distribution $a \in \mathcal{S}_m'$;

2) the operator $\hat{v}$ is transposable;

3) the operator $\hat{v}$ is weakly continuous, i.e. continuous from $\mathcal{S}_n$ to $\mathcal{S}_m$ with respect to the pair of weak topologies $(\sigma(\mathcal{S}_n), \sigma(\mathcal{S}_m))$;

4) the operator $\hat{v}$ is continuous from the space $(\mathcal{S}_n)$ to the space $(\mathcal{S}_m)$. 

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Proof. Note that, after the proof of the summability property (1) in the preceding section, we have to prove only that the last three properties are equivalent to property (1). Property (1) is equivalent to property (2) by definition of transposable operator. Property (2) is equivalent to property (3) because the space of linear continuous operators $\mathcal{L}((S_n)_\sigma, (S_m)_\sigma)$ is also the space of all the transposable linear operators from the space $(S_n)$ to $(S_m)$ (see Horváth, chap. 3, § 12, Proposition 1, page 254). Property (3) is equivalent to property (4). In fact, since the space $(S_n)$ is a Fréchet space (and then its topology coincides with the Mackey topology $\tau(S_n, S'_n)$), the space $\mathcal{L}(S_n, S_m)$ contains the above space $\mathcal{L}((S_n)_\sigma, (S_m)_\sigma)$, of all weakly linear and continuous operators from $S_n$ to $S_m$ (i.e. with respect to the pair of topologies $(\sigma(S_n), \sigma(S_m))$, see for this result Dieudonné Schwartz, page 91, Corollary or Horváth, page 258, Corollary). Moreover, the space $\mathcal{L}(S_n, S_m)$ is contained in the space $\mathcal{L}((S_n)_\sigma, (S_m)_\sigma)$, since every continuous linear operator among two Hausdorff locally convex topological vector spaces is weakly continuous (see proposition 3, page 256 of J. Horváth), so the two spaces must coincide. ■

8 Isomorphic realization of $\mathcal{L}(S_n, S_m)$

By the characterization of summability for Schwartz families we deduce the following important corollary, which gives the canonical isomorphic realization of the space of linear continuous operators among Schwartz test function spaces $\mathcal{L}(S_n, S_m)$ as a space of Schwartz families $S(\mathbb{R}^m, S'_n)$.

Corollary (of isomorphism). The vector spaces $S(\mathbb{R}^m, S'_n)$ and $\mathcal{L}(S_n, S_m)$ are isomorphic. Namely, the map $(\cdot)^\wedge$ from the space of family $S(\mathbb{R}^m, S'_n)$ into the space of operators $\mathcal{L}(S_n, S_m)$, associating with each family $v$ its operator $\hat{v}$, is a vector space isomorphism. Moreover, the inverse of the above isomorphism is the linear mapping

$$(\cdot)^\vee : \mathcal{L}(S_n, S_m) \rightarrow S(\mathbb{R}^m, S'_n)$$

defined by

$$A \mapsto A^\vee := (\delta_p \circ A)_{p \in \mathbb{R}^m},$$
i.e. the canonical representation of the operator space $L(S_n, S_m)$ into the family space $S(\mathbb{R}^m, S'_n)$, which, as a consequence, is an isomorphism too.

We can give also the following definition.

**Definition (canonical representation of the space $S(\mathbb{R}^m, S'_n)$).** The mapping $(\cdot)^\wedge : S(\mathbb{R}^m, S'_n) \to L(S_n, S_m) : v \mapsto \hat{v}$

is called the canonical representation of the Schwartz family space $S(\mathbb{R}^m, S'_n)$ onto the operator space $L(S_n, S_m)$.

## 9 Characterization of transposability

A way to see that an operator is transposable is given by the following characterization. It is an immediate consequence of the characterization of the $S$ families but we want to prove it independently.

**Theorem.** Let $A : S_n \to S_m$ be a linear operator and let $\delta$ be the Dirac family of the space $S'_m$. Then, the operator $A$ is (topologically) transposable if and only if, for every point $p \in \mathbb{R}^m$, the composition $\delta_p \circ A$ is a tempered distribution in $S'_n$.

**Proof.** ($\Rightarrow$) The necessity of the condition is obvious. In fact, we have

$$\delta_p \circ A = *A(\delta_p),$$

and so if $A$ is topologically transposable, the composition $\delta_p \circ A$ is continuous. ($\Leftarrow$) Let $a \in S'_m$ be a tempered distribution; we should prove that the composition $a \circ A$ is continuous. Since the linear hull $\text{span}(\delta)$ is sequentially dense in the space $S'_m$ (see [Bo] page 205), there is a sequence of distributions $(\alpha_k)_{k \in \mathbb{N}}$ in the hull $\text{span}(\delta)$ such that

$$\sigma(S'_m) \lim_{k \to +\infty} \alpha_k = a.$$
Now, since any distribution $\alpha_k$ lives in the hull $\operatorname{span}(\delta)$ there exist a finite family $(y_i)_i$ in $\mathbb{R}^m$ and a finite sequence $(\lambda_i)_i$ in $\mathbb{K}$ such that

$$\alpha_k = \sum_{i=1}^{h} \lambda_i \delta_{y_i},$$

thus we have

$$\alpha_k \circ A = \sum_{i=1}^{h} (\lambda_i \delta_{y_i}) \circ A =$$

$$= \sum_{i=1}^{h} \lambda_i (\delta_{y_i} \circ A);$$

hence, for every number $k \in \mathbb{N}$, the composition $\alpha_k \circ A$ belongs to $S'_n$. Let now $s$ be the topology of pointwise convergence in the algebraic dual $S^*_n$; we have

$$^s \lim_{k \to +\infty} (\alpha_k \circ A) = a \circ A,$$

in fact

$$\lim_{k \to +\infty} (\alpha_k \circ A) (\phi) = \lim_{k \to +\infty} \alpha_k (A (\phi)) =$$

$$= a (A (\phi)),$$

so we have that the sequence (in $S'_n$) of continuous linear form $(\alpha_k \circ A)_{k \in \mathbb{N}}$ converges pointwise to the linear form $a \circ A$, then, by the Banach-Steinhaus theorem, we conclude that the composition $a \circ A$ lives also in the space $S'_n$.

$\blacksquare$

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David Carfì  
Faculty of Economics  
University of Messina  
davidcarfi71@yahoo.it