Supplementary Information: On the Role of Longitudinal Currents in Radiating Systems of Charges

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Appendix A: Maxwell’s equations

Here we collect our conventions regarding Maxwell’s equations. We work with harmonic time-dependence, frequency is denoted by \( \omega \), \( k = \omega / c \) is the wavenumber. Maxwell’s equations can be written as

\[
\mu_0 j(r) = -(k^2 + \nabla^2) A(r) + \nabla (\nabla \cdot A(r)) , \tag{A1}
\]

where \( A(r) \) is a vector potential in the Weyl gauge (vanishing scalar potential). In this gauge the electric and magnetic fields are related to the vector potential as follows

\[
E(r) = i k c A(r), \quad H(r) = \nabla \times A(r) . \tag{A2}
\]

Conversely, assuming that the vector potential decays at infinity it can be found by solving (A1)

\[
A(r) = \frac{\mu_0}{4 \pi k^2} \int dr' G(r - r') \left( k^2 j(r') + \nabla \cdot j(r') \right) , \tag{A3}
\]

where \( G(r - r') = \frac{e^{ik|r-r'|}}{|r-r'|} \) is the Green function for Helmholtz equation \((\nabla^2 + k^2)G(r) = -4\pi \delta^{(3)}(r)\). The transverse part of the current satisfies \( \nabla \cdot j_\perp = 0 \) and gives rise to the transverse part of the vector potential

\[
A_\perp(r) = \frac{\mu_0}{4\pi} \int dr' G(r - r') j_\perp(r') . \tag{A4}
\]

The longitudinal part of the current satisfies \( \nabla (\nabla \cdot j_\parallel) = \nabla^2 j_\parallel \), hence

\[
A_\parallel(r) = -\frac{\mu_0}{k^2} j_\parallel(r) . \tag{A5}
\]

Outside of a localized source \( j_\perp = -j_\parallel = 0 \) and, therefore, both \( A_\perp \) and \( A_\parallel \) can be expressed via \( j_\perp \) alone.

Appendix B: Non-radiating charge densities

In the main text we have been careful to make reservations while claiming that any charge density will be consequential for the radiation fields. There is a notable exception. Let us reconsider the Poisson equation (3) and ask whether there exist a charge density \( \rho_s \) which actually does not produce a potential \( \phi_s \) (and hence the electric field) outside its domain of definition. The answer is yes, and it is simple to describe all such densities. Assume that \( \rho_s \) is a Laplacian of some function \( \phi \), i.e. \( \rho_s = \nabla^2 \phi \) and \( \phi \) is zero outside \( R \). Then from (3) it follows that the potential \( \phi_s \) is just equal to \( \phi \) up to constants and hence is itself confined (the argument is almost tautological).

This implies the existence of charge densities that do not produce any electric fields outside. Translated into the language of the Helmholtz decomposition this means that some modifications of the longitudinal current do not affect the transverse part. Namely, one can add to \( j \) a term of the form \( \nabla \phi \) with any confined function \( \phi \).
Appendix C: Volume integral of longitudinal currents

Here we show that for a spatially confined current \( j \) its longitudinal part \( j_\parallel \) satisfies

\[
\int dr \, j_\parallel = 0 ,
\]

which makes it possible to express all multipole terms as integrals of \( j_\perp \) alone. As explained in Section I, relation between \( j \) and \( j_\parallel \) arises through a Poisson equation

\[
\nabla^2 \psi(r) = \nabla \cdot j(r) , \quad j_\parallel(r) = \nabla \psi ,
\]

which is solved by

\[
\psi(r) = -4\pi \int dr' \frac{\nabla \cdot j(r')}{|r - r'|} .
\]

Because the source is spatially confined the range of \( r' \) is bounded and for large enough \( r \) we can write

\[
\psi(r) = -4\pi \frac{r}{r^3} \int dr' \nabla \cdot j(r') + O\left(\frac{1}{r^3}\right) ,
\]

where \( n = r/r \). The first term here is zero by the Gauss theorem, while the second can be rewritten

\[
\psi(r) = \frac{4\pi}{r^2} n \cdot V + O\left(\frac{1}{r^3}\right) , \quad V = \int dr' r' \left(\nabla \cdot j(r')\right) .
\]

Here \( V \) is an analog of the dipole moment, but for our purposes this interpretation is not important. \( V \) is just a constant vector characterizing current distribution \( j \).

Now we are ready to prove (C1). Consider

\[
\int dr \, j_\parallel = \int dr \nabla \psi = \lim_{R \to \infty} \int_{r = R} 4\pi r^2 dn \left(\frac{n \cdot V}{r^2} + O\left(\frac{1}{r^3}\right)\right) .
\]

The leading term vanishes due to identity \( \int dn \, n = 0 \) (the average of the normal vector over a unit sphere is zero), while the subleading terms vanish in the \( R \to \infty \) limit. This establishes (C1).

Appendix D: Multipole form-factors

Functions \( F_{lmk}^{(\lambda)} \) introduced in (24) are regular solutions to the Helmholtz equation

\[
(\nabla^2 + k^2) F_{lmk}^{(\lambda)} = 0
\]

and satisfy orthogonality and completeness relations

\[
\int dr \, F_{lmk}^{(\lambda)}(r) \cdot F_{lm'm'k'}^{(\lambda')}^*(r') = \frac{(2\pi)^3}{k^2} \delta^{\lambda\lambda'} \delta_{ll'} \delta_{mm'} \delta(k - k') ,
\]

\[
\sum_{lmk} \left[F_{lmk}^{(\lambda)}(r)\right]_{ij} \left[F_{lmk}^{(\lambda)}(r')\right]_{ji} = (2\pi)^3 \delta_{ij} \delta(r - r') .
\]

Under parity transformations \( x \to -x \) they behave as

\[
F_{lmk}^{(\lambda)}(-x) = (-1)^{l+\lambda} F_{lmk}^{(\lambda)}(x) .
\]

The scalar functions \( F_{lmk} = j_l(kr)Y_{lm}(n) \) that are used to define \( F_{lmk}^{(\lambda)} \) also solve the Helmholtz equation and have the following \( k \to 0 \) asymptotic

\[
F_{lmk}(r) = \frac{4\pi i^l}{(2l + 1)!!} (kr)^l Y_{lm}(n) (1 + O(k)) ,
\]

from which (28) can be derived.