ROBUSTNESS OF GLOBAL ATTRACTORS: ABSTRACT FRAMEWORK AND APPLICATION TO DISSIPATIVE WAVE EQUATIONS

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Abstract. We establish local input-to-state stability and the asymptotic gain property for a class of infinite-dimensional systems with respect to the global attractor of the respective undisturbed system. We apply our results to a large class of dissipative wave equations with nontrivial global attractors.

1. Introduction. Asymptotic stability of an equilibrium is a fundamental property of evolutionary processes and plays important role for many applications. As well its robustness is crucial for a proper operation of practical systems. It is well-known, that a globally asymptotically stable equilibrium of a linear and finite dimensional system is robust in the sense that for any essentially bounded external disturbance entering to the system the corresponding solution remains bounded for all times and that it tends to a ball around the equilibrium, when time goes to infinity. The size of this ball depends on the disturbance norm only. For nonlinear systems this is in general not true and leads to the notion of input-to-state stability (ISS), introduced by [28] for finite dimensional systems and is now recognized to be very fruitful in many applications of control and stability theory due to well elaborated characterizations of the ISS in terms of weaker notions, Lyapunov methods, including small-gain results as well as their extensions to different classes of systems.

This notion is also suitable to study robustness of equilibria in case of infinite dimensional systems [8]. During last decade many authors tried to extend the ISS theory to this class of systems. Many of these extensions were developed for systems given in terms of partial differential equations (PDEs), [22],[7],[15],[16],[17],[27] for 2020 Mathematics Subject Classification. Primary: 35B40, 35B41, 93B35; Secondary: 93D09, 35B35, 35L05.

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parabolic equations and [11], [21] for, in particular, bilinear ones. Several weaker versions of ISS were studied in [20], [26] However, the ISS framework for infinite-dimensional systems is not as well-developed as for the finite-dimensional case. In spite of many resent results a number of important questions remain open, see the survey paper [23]. It should be noted, that almost all ISS-like results for PDEs were developed for the case of single equilibrium point of the unperturbed system. It is well known that many nonlinear systems possess a nontrivial global attractor instead. In this work we study the question of robustness of such an attracting set with respect to external disturbances.

Existence and different properties of global attractors were studied in many books [29], [25], [4], [3] and papers [2], [5], [1], [9], [13], [6], to name a few. We are interested in the following question: given a system like (1) possessing a global attractor, what can we say about attracting sets for solutions if some perturbation $u$ enters to this system like in (2)? The answer depends on the properties of $A$ in (1). For example, if (1) generates a compact semiprocess (as in the parabolic case) an answer can be found in [27]. Here we are interested in wave equations and extend some results of [27] to the case of asymptotically compact semiprocesses.

In this work, we contribute several new results to the development of the ISS framework of infinite dimensional systems. We provide a general scheme to investigate local ISS and asymptotic gain (AG) properties w.r.t. global attractors for dissipative infinite-dimensional systems. We also establish these properties for a wide class of nonlinear hyperbolic equations having a non-trivial global attractor.

2. Setting of the problem. Consider an infinite dimensional system given by

$$\frac{d}{dt} y(t) = Ay(t) + \Phi(y(t)),$$

where $y \in X$, $(X, \| \cdot \|_X)$ is a Banach space, $A$ generates a continuous semigroup on $X$ and $\Phi : X \to X$ is Lipschitz continuous, such that this system possesses a global attractor $\Theta$ in $X$. This property guarantees that any solution to (1) approaches $\Theta$ when $t \to \infty$. We are interested in the long term behavior of the corresponding solutions in the case the system (1) is perturbed by an external signal $u : [0, \infty) \to U$, where $U$ is a Banach space, which in practical situations can enter to the system as a distributed one or through the boundary of the underlying domain. Hence we consider the perturbed system in the form

$$\frac{d}{dt} y(t) = Ay(t) + \Phi(y(t)) + H(u(t)), \quad u \in U := \{u \in L^\infty([0, \infty)) | u(t) \in U, t \geq 0\}$$

where $H : U \mapsto X$ is a bounded linear operator. Given an initial state $y_0 := y(0) \in X$ and a perturbing signal $u \in U$ the corresponding unique solution to (2) is denoted by $y(t, y_0, u)$. Due to the disturbance we have no guarantee in general, that this solution will converge to $\Theta$ when $t \to \infty$. It turns out that for certain classes of systems (2) the global attractor is robust under perturbation, i.e., its attractivity properties are affected only slightly by disturbances of small magnitude. This robustness property can be expressed in the ISS framework as follows: there exist $\beta \in KL$ and $\gamma \in K$ such that for any $y_0 \in X$ and $u \in U$

$$\|y(t, y_0, u)\|_\Theta \leq \beta(\|y_0\|_\Theta, t) + \gamma(\|u\|_\infty), \quad t \geq 0,$$

where well-known classes $K$ stands for the class of continuous strictly increasing functions on $[0, \infty)$ vanishing at the origin, $KL$ is the set of continuous functions
defined on $[0, \infty)^2$ which are of class $\mathcal{K}$ in the first argument and strictly decreasing to zero in the second one, $\|x\|_{\Theta} = \inf_{\theta \in \Theta} \|x - \theta\|_X$, $\|u\|_{\infty} := \sup_{t \geq 0} \|u\|_U$. Unfortunately, this property is in general not guaranteed even for the case $\Theta = \{0\}$, see for example [8, Section 5].

In this paper we will look for conditions that guarantee this property at least locally, namely that (3) holds for $\|y_0\|_{\Theta} \leq r$ and $\|u\|_{\infty} \leq r$ for some fixed $r > 0$. The latter property is called local ISS (LISS). Additionally, under rather general assumptions, we will prove the asymptotic gain (AG) property for (2), that is the existence of some $\gamma \in \mathcal{K}$ such that for any $y_0 \in X$ and $u \in U$ it holds

$$\limsup_{t \to \infty} \|y(t, y_0, u)\|_{\Theta} \leq \gamma(\|u\|_{\infty}).$$

(4)

To prove the LISS property, the Lyapunov techniques will be used. To establish the AG property the uniform attractors theory for non-autonomous systems [4, Chapter IV] will be used. These abstract results will be illustrated on a damped wave equation possessing a nontrivial global attractor in the unperturbed case.

3. Families of semiprocesses and their stability properties. Let $(X, \|\cdot\|_X)$ be a Banach space and $0 \in U \subseteq L^\infty(\mathbb{R}_+)$. We assume additionally that $U$ is translation-invariant, that is

$$\forall \ h \geq 0 \ it \ holds \ that \ T(h)U \subseteq U, \ where \ T(h)u(\cdot) := u(\cdot + h).$$

Let us denote $K := \{(t, s) \mid t \geq s \geq 0\}$.

**Definition 3.1.** A family of maps $\{S_u : K \times X \to X\}_{u \in U}$ is called a semiprocess family, if for all $x \in X$ and all $t \geq s \geq \tau \geq 0$ and all $h \geq 0$ the following three properties hold

$$S_u(t, t, x) = x;$$

(5)

$$S_u(t, s, S_u(s, \tau, x)) = S_u(t, \tau, x);$$

(6)

$$S_u(t + h, s + h, x) = S_{T(h)u}(t, s, x).$$

(7)

**Remark 1.** From this definition follows that any semiprocess family satisfies the cocycle property

$$S_u(t + h, 0, x) = S_{T(h)u}(t, 0, S_u(h, 0, x)).$$

(8)

In particular for the unperturbed case $u = 0$ the semiprocess $S_0$ satisfies the semigroup property

$$S_0(t_1 + t_2, 0, x) = S_0(t_1, 0, S_0(t_2, 0, x)).$$

**Definition 3.2.** A compact set $\Theta \subset X$ is called global attractor of $S_0$ if the following properties hold:

(i) $\Theta = S_0(t, 0, \Theta), \ t \geq 0$,

(ii) for any bounded $B \subset X$

$$\text{dist}(S_0(t, 0, B), \Theta) \to 0 \ as \ t \to \infty,$$

where for given $A, B \subset X$ we denote $\text{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_X$. 

Definition 3.3. We say that the semiprocess family \( \{S_u\}_{u \in U} \)
(i) is locally ISS with respect to \( \Theta \) if there exist \( r > 0, \beta \in KL \) and \( \gamma \in K_\infty \) such that for any \( \|x_0\|_\Theta \leq r \) and \( \|u\|_\infty \leq r \) it holds that
\[
\|S_u(t, 0, x_0)\|_\Theta \leq \beta(\|x_0\|_\Theta, t) + \gamma(\|u\|_\infty), \quad t \geq 0;
\] 
(9)
(ii) satisfies the asymptotic gain property with respect to \( \Theta \) if there exists \( \gamma \in K_\infty \) such that for any \( x_0 \in X \) and \( u \in U \) it holds that
\[
\limsup_{t \to \infty} \|S_u(t, 0, x_0)\|_\Theta \leq \gamma(\|u\|_\infty),
\] 
(10)
Remark 2. As a rule, we do not know the precise formula for the global attractor \( \Theta \). Therefore, it is important to formulate all assumptions imposed on the considered system only in terms of the initial norm \( \| \cdot \|_X \) (and do not use \( \| \cdot \|_\Theta \)).

Our first result establishes a local robustness property of the global attractor. See also a relevant finite dimensional result in [10] (Theorem 3.4.6). Also we note, that in case of \( \Theta = \{0\} \) the LISS property of an abstract semilinear problem with a Lipschitz nonlinear term was established in [19].

Theorem 3.4. Let the semiprocess family \( \{S_u\}_{u \in U} \) be such that \( S_0 \) has a global attractor \( \Theta \), and
(i) there exist \( \sigma \in K \) and \( c_0 > 0 \) such that for any \( x \in X \) we have
\[
\|S_0(t, 0, x)\|_X \leq \sigma(\|x\|_X) + c_0;
\] 
(11)
(ii) there exists a locally bounded function \( c : \mathbb{R}_+ \to \mathbb{R}_+ \) such that for any \( r > 0 \) we have the implication
\[
\|x_1\|_X \leq r, \|x_2\|_X \leq r \implies \|S_0(t, 0, x_1) - S_0(t, 0, x_2)\|_X \leq e^{c(r)t} \|x_1 - x_2\|_X, \quad t \geq 0;
\] 
(12)
(iii) there exist \( \sigma \in K \) and a continuous function \( d : [0, \infty)^2 \to [0, \infty) \) such that for any \( r > 0 \) we have
\[
\lim_{t \to 0+} \frac{d(r, t)}{t} < \infty \quad \text{and the implication}
\]
\[
\|u\|_\infty \leq r, \|x\|_X \leq r \implies \|S_u(t, 0, x) - S_0(t, 0, x)\|_X \leq d(r, t)\sigma(\|u\|_\infty), \quad t \geq 0
\] 
(13)
holds. Then the semiprocess family \( \{S_u\}_{u \in U} \) is locally ISS with respect to \( \Theta \).

Proof. The proof is a slight generalization of results in [7, Theorem 3.3], hence we just indicate the main steps omitting the details.

Step 1. From Lemma 2.6 in [7], replacing (2.27) there with (11) from the above condition (i) it follows that there exists \( \beta_0 \in KL \) such that
\[
\forall x \in X \quad \|S_0(t, 0, x)\|_\Theta \leq \beta(\|x\|_\Theta, t), \quad t \geq 0.
\] 
(14)

Step 2. From Lemma 3.1 in [7] with \( \lambda = c(r) \) it follows that for every \( r > 0 \) there exists a Lipschitz continuous function \( V \) with Lipschitz constant 1 and comparison functions \( \underline{\psi}, \overline{\psi}, \alpha \in K \) such that
\[
\underline{\psi}(\|x\|_\Theta) \leq V(x) \leq \overline{\psi}(\|x\|_\Theta) \quad \forall x \in X, \quad \|x\|_X \leq r,
\] 
(15)
\[
\dot{V}_0(x) := \limsup_{t \to 0+} \frac{1}{t} \left( V(S_0(t, 0, x)) - V(x) \right) \leq -\alpha(\|x\|_\Theta) \quad \forall x \in X, \quad \|x\|_X \leq r.
\] 
(16)
Step 3. Take \( r > 0 \) and \( V \) from the previous step. Then from Lemma 3.2 in [7] and (13) it follows that for some \( \alpha, \sigma \in K \) for any \( u \in U \)
\[
\dot{V}_u(x) := \limsup_{t \to 0+} \frac{1}{t} \left( V(S_u(t, 0, x)) - V(x) \right) \leq -\alpha(\|x\|_0) + \sigma(\|u\|_\infty), \|x\| \leq r. \tag{17}
\]
Now the proof is finished by application of Theorem 3.3 from [7]. \( \square \)

Let \( \Sigma \subset U \) be an arbitrary translation-invariant set. We consider a semiprocess family \( \{S_u\}_{u \in \Sigma} \) and denote \( S_\Sigma := \bigcup_{u \in \Sigma} S_u \).

**Definition 3.5.** A compact set \( \Theta_\Sigma \) is called uniform attractor of \( \{S_u\}_{u \in \Sigma} \) if for every bounded \( B \subset X \) we have
\[
\text{dist}(S_\Sigma(t, 0, B), \Theta_\Sigma) \to 0, \ t \to \infty \tag{18}
\]
and \( \Theta_\Sigma \) is minimal among all closed sets satisfying (18), that is contained in any other closed set satisfying (18).

**Remark 3.** Note that \( \Theta_\Sigma \) is not necessarily invariant (in any sense) w.r.t. \( S_\Sigma \).

The following result is well-known in the theory of non-autonomous systems [4, Chapter VII].

**Lemma 3.6.** Let \( \{S_u\}_{u \in \Sigma} \) be a semiprocess family with sequentially compact \( \Sigma \), that is every sequence in \( \Sigma \) has a convergent subsequence to a point in \( \Sigma \), and (i) there exist a bounded \( B_0 \subset X \) such that for any bounded \( B \subset X \) \( \exists \ T = T(B) \) s.t.
\[
\forall \ t \geq T \ S_\Sigma(t, 0, B) \subset B_0. \tag{19}
\]

(ii) \( \forall \{u_n\} \subset \Sigma \) \( \forall \ t_n \nrightarrow \infty \) \( \forall \) bounded \( \{x_n\} \subset X \) the sequence \( \{S_{u_n}(t_n, 0, x_n)\} \) is precompact in \( X \).

Then \( \{S_u\}_{u \in \Sigma} \) has a uniform attractor \( \Theta_\Sigma \). If, in addition, \( \forall t > 0 \) the map
\[
X \times \Sigma \ni (x, u) \mapsto S_u(t, 0, x) \text{ is continuous}, \tag{21}
\]
then \( \Theta_\Sigma \) is negatively invariant w.r.t. \( S_\Sigma \), that is
\[
\Theta_\Sigma \subset S_\Sigma(t, 0, \Theta_\Sigma) \ \forall t \geq 0. \tag{22}
\]

**Remark 4.** In the unperturbed case \( \Sigma = \{0\} \) conditions (19)-(21) imply the existence of a global attractor \( \Theta \) for the semigroup \( S_0 \).

The next lemma establishes the upper semicontinuity of the attracting set w.r.t. \( \lambda \) at the point \( \lambda = \lambda_0 \) and extends a result from [12, Theorem 1.5] that was proved for autonomous systems. See also a related result in the finite dimensional case in [18].

**Lemma 3.7.** Assume that \( \Sigma \) depends on a parameter \( \lambda \), that is \( \Sigma = \Sigma(\lambda) \), where \( \lambda \) belongs to some metric space \( \Lambda \), \( \lambda_0 \) is an non-isolated point of \( \Lambda \). Assume that
(i) \( \forall \ \lambda \in \Lambda \) the semiprocess family \( \{S_u\}_{u \in \Sigma(\lambda)} \) satisfies (19) for some set \( B_0 \) which does not depend on \( \lambda \);
(ii) \( \forall \ \lambda \in \Lambda \) the semiprocess family \( \{S_u\}_{u \in \Sigma(\lambda)} \) possesses a negatively invariant uniform attractor \( \Theta_{\Sigma(\lambda)} \);
(iii) \( \forall \ \lambda_k \to \lambda_0 \) every sequence \( \{z_k \in \Theta_{\Sigma(\lambda_k)}\} \) contains a convergent subsequence;
(iv) if \( \lambda_k \to \lambda_0 \), \( \xi_k \in S_{\Sigma(\lambda_k)}(t, 0, z_k) \), \( t > 0 \), \( z_k \to z \), then up to subsequence \( \xi_k \to \xi \in S_{\Sigma(\lambda_0)}(t, 0, z) \).

Then
\[
\text{dist}(\Theta_{\Sigma(\lambda)}), \Theta_{\Sigma(\lambda_0)}) \to 0 \ \text{ for } \ \lambda \to \lambda_0. \tag{23}
\]
Assume that a family of sets \( \Sigma_k \) satisfy Assumption (iii) in the last lemma is satisfied if

\[ \forall \lambda_k \to \lambda_0 \forall t_k \nearrow \infty \forall \xi_k \in \Sigma_k(t_k,0,B_0) \{ \xi_k \} \text{ is precompact in } X. \quad (24) \]

**Proof.** Assume by contradiction that (23) is not true. This means \( \exists \lambda_k \to \lambda_0 \exists \epsilon > 0 \exists z_k \in \Theta_{\Sigma_k} \) such that \( z_k \not\in O_{\epsilon}(\lambda_k) \), where \( O_{\epsilon}(A) \) denotes the \( \epsilon \)-neighborhood of \( A \). Due to (iii) up to subsequence we have \( z_k \to z \). Let us chose \( t > 0 \) such that \( S_{\Sigma(\lambda_k)}(t,0,B_0) \subset O_{\epsilon/2}(\Theta_{\Sigma(\lambda_k)}) \). From the semi-invariance property we have that

\[ z_k \in \Theta_{\Sigma(\lambda_k)} \subset S_{\Sigma(\lambda_k)}(t,0,\Theta_{\Sigma(\lambda_k)}) \]

Hence \( z_k \in S_{\Sigma(\lambda_k)}(t,0,\eta_k) \) for some \( \eta_k \in \Theta_{\Sigma(\lambda_k)} \subset B_0 \) and due to (iii) we have \( \eta_k \to \eta \). Therefore, using (iv) we arrive at the following contradiction

\[ z_k \to z \in S_{\Sigma(\lambda_k)}(t,0,\eta) \subset O_{\epsilon/2}(\Theta_{\Sigma(\lambda_k)}) \]

which proves the lemma. \( \square \)

The next result extends Theorem 3.3 in [27] to the case of not necessarily compact semiprocesses.

**Theorem 3.8.** Assume that a family of sets \( \{ \Sigma(u) \}_{u \in U} \) and corresponding semiprocess families satisfy the conditions of Lemma 3.7 with \( \Lambda = U \), \( \lambda_0 = 0 \). Also assume that \( \Sigma(0) = \{ 0 \} \) and

\[ \forall u \in U \ u \in \Sigma(u). \]

Let \( \Theta \) be the global attractor of the semigroup \( S_0 \). Then \( \{ S_u \}_{u \in U} \) has the asymptotic gain property w.r.t. \( \Theta \).

**Proof.** For every \( u \in U \), \( y \in X \) and \( z_u \in \Theta_{\Sigma(u)} \), we have

\[ \| S_u(t,0,y) \|_\Theta = \inf_{\theta \in \Theta} \| S_u(t,0,y) - \theta \|_X \leq \| S_u(t,0,y) - z_u \|_X + \text{dist}(\Theta_{\Sigma(u)}, \Theta). \]

But \( S_u(t,0,y) \in S_{\Sigma(u)}(t,0,y) \), so

\[ \| S_u(t,0,y) \|_\Theta \leq \text{dist}(S_{\Sigma(u)}(t,0,y), \Theta_{\Sigma(u)}) + \text{dist}(\Theta_{\Sigma(u)}, \Theta). \]

From the attraction property of \( \Theta_{\Sigma(u)} \) we get

\[ \limsup_{t \to \infty} \text{dist}(S_{\Sigma(u)}(t,0,y), \Theta_{\Sigma(u)}) = 0. \]

Let us put \( \gamma(s) := \sup_{\| u \|_\infty \leq s} \text{dist}(\Theta_{\Sigma(u)}, \Theta) + s \). Then due to (23) \( \gamma \in \mathcal{K} \) and

\[ \forall u \in U \ \text{dist}(\Theta_{\Sigma(u)}, \Theta) \leq \gamma(\| u \|_\infty) \]

and we obtain the required result. The theorem is proved. \( \square \)

4. Application to wave equations. We consider the following dissipative hyperbolic equation

\[ \begin{cases} y_{tt}(x,t) + ky_t(x,t) - \Delta y(x,t) + f(y(x,t)) = 0, & t > 0, \ x \in \Omega, \\ y|_{\partial \Omega} = 0, \end{cases} \quad (25) \]

where \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \) is a bounded open subset with smooth boundary, \( k > 0 \), \( f \in C^1(\mathbb{R}) \) and the following condition holds:

\[ \exists C > 0 \forall s \in \mathbb{R} \ |f'(s)| \leq C(1 + |s|^r), \ r < \frac{n}{n-2}. \quad (26) \]

Let us introduce the following notation:

\[ \begin{align*} X &= H_0^1(\Omega) \times L_2(\Omega), \ z(\cdot) = \begin{pmatrix} y(\cdot) \\ y_t(\cdot) \end{pmatrix}, \ A = \begin{pmatrix} 0 & 1 \\ \Delta & -k \end{pmatrix}, \ \Phi(z) = \begin{pmatrix} 0 \\ -f(y) \end{pmatrix} \end{align*} \]
Here and after $y_t$ denotes the distributional derivative with respect to $t$ of $y$.

After that we can rewrite \((25)\) in the form \((1)\)

\[
\frac{d}{dt} z = Az + \Phi(z).
\]  

(27)

Under additional assumption

\[
\liminf_{s \to \infty} \frac{f(s)}{s} > -\lambda_1,
\]

(28)

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$, it is known [2, Theorem 3.6] that $\forall T > 0 \ \forall z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in X$ problem \((27)\) has a unique weak solution (see the definition below) $z(\cdot) = \begin{pmatrix} y(\cdot) \\ y_t(\cdot) \end{pmatrix} \in C([0,T];X)$ such that $z(0) = z_0$. All such solutions being extended on $[0, +\infty)$ generates a semigroup $S_0(\cdot, 0, \cdot) : \mathbb{R}_+ \times X \to X$. It is also known [2, Theorem 1.1] that the semigroup $S_0$ possesses the global attractor $\Theta$.

Now let us consider the perturbed system

\[
\begin{cases}
y_{tt}(x, t) + ky_t(x, t) - \Delta y(x, t) + f(y(x, t)) = h(x)u(t), \\
y|_{\partial\Omega} = 0,
\end{cases}
\]  

(29)

where $h \in L^2(\Omega)$, $u \in L^2_{\text{loc}}(\mathbb{R}_+)$.  

**Definition 4.1.** A function $z(\cdot) = \begin{pmatrix} y(\cdot) \\ y_t(\cdot) \end{pmatrix} \in L^\infty(\tau, T; X)$ is called a weak solution of the problem \((29)\) on $(\tau, T)$, if for any $\psi \in H^1_0(\Omega)$ and for all $\eta \in C_0^\infty(\tau, T)$ we have the equality

\[
-\int_\tau^T (y_t, \psi)\eta_t + \int_\tau^T \left( k(y_t, \psi) + (y, \psi)_{H^1_0} + (f(y), \psi) - (h, \psi)u \right) \eta = 0,
\]

(30)

where by $\|\cdot\|$ and $(\cdot, \cdot)$ we denote norm and scalar product in the space $L^2(\Omega)$.

Note that if a function $z(\cdot) \in L^\infty_{\text{loc}}(\tau, \infty; X)$ satisfies \((30)\) for every $T > \tau$, then $z(\cdot)$ is called global weak solution of \((29)\).

To guarantee the existence of weak solutions we need slightly stronger assumptions on the nonlinear term $f$ [4, Chapter VI]. So, we assume that \((26)\) holds and instead of \((28)\) we assume

\[
\exists C_i > 0, \ i = 1, 2, 3, \ \forall s \in \mathbb{R} \ \text{for} \ F(s) := \int_0^s f(p)dp \\
F(s) \geq -ms^2 - C_1, \ f(s)s - C_2F(s) + ms^2 \geq C_3,
\]

(31)

where $m \in (0, \lambda_1)$ is a sufficiently small number.

**Remark 6.** Conditions \((31)\) imply \((28)\) for sufficiently small $m$. In particular, we have existence of global attractor $\Theta$ for the case $u = 0$.

**Remark 7.** Nonlinear hyperbolic equations with nonlinear terms satisfying \((26)\), \((28)\) or \((26),(31)\) are widely used in applications [29, Chapter IV]. For instance, they cover the sine-Gordon equation with $f(s) = b \sin s$, or the relativistic quantum mechanics equation with $f(s) = |s|^r s$.  

It is known [4, Chapter VI] that under conditions (26), (31) \( \forall u \in L^2_{loc}(\mathbb{R}^+) \) \( \forall \tau \geq 0 \) \( \forall z_0 \in X \) problem (29) has a unique global weak solution \( z(\cdot) \in C([\tau, +\infty); X) \) such that \( z(\tau) = z_\tau \).

Therefore, the semiprocess family \( \{S_u : K \times X \to X\}_{u \in U} \),
\( S_u(t, \tau, z_\tau) := z(t) \), \( z(\cdot) \) is a global weak solution of (29), \( z(\tau) = z_\tau \), (32) is well-defined for any translation-invariant set \( U \subseteq L^2_{loc}(\mathbb{R}^+) \).

Let us fix \( u \in L^2_{loc}(\mathbb{R}^+) \), \( z_0 = \left( \begin{array}{c} y_0 \\ y_1 \end{array} \right) \in X \).

For \( z(\cdot) = \left( \begin{array}{c} y(\cdot) \\ y_t(\cdot) \end{array} \right) \in S_u(\cdot, 0, z_0) \) let us put
\[ \Psi(t) := \|\nabla y(t)\|^2 + \|y_t(t) + \alpha y(t)\|^2 + 2F(y(t), 1). \]

Then for sufficiently small \( \alpha, m \in (0, \lambda_1) \) (see Lemma 4.1 in [4]) there exist \( \delta' > 0, C_4 > 0 \) such that
\[ \frac{d}{dt} \Psi(t) + \delta' \Psi(t) \leq C_4(1 + |u(t)|). \]

Due to (26)
\[ |F(s)| \leq C_5 \left( 1 + |s|^\frac{2n-2}{2} \right). \]

Using (34), for sufficiently small \( \alpha \) we have that for some positive \( \delta, C_6 \) and for all \( t \geq s \geq 0 \)
\[ \|y_t(t)\|^2 + \|y(t)\|_{H^1_0}^2 \leq C_6 \left( \|y_t(s)\|_{H^1_0} + \|y(s)\|_{H^1_0} \right)^{\frac{2n-2}{4}} e^{-\delta(t-s)} + \int_s^t |u(p)|^2 e^{-\delta(t-p)} dp + 1 \]
(35)

In particular, for \( u \in L^\infty(\mathbb{R}^+) \) inequality (35) implies that \( \forall t \geq s \geq 0 \)
\[ \|z(t)\| \leq C_7 \left( \|z(s)\|^{\frac{2n-2}{4}} e^{-\delta(t-s)} + \|u\|_{L^\infty}^2 + 1 \right) \]
(36)

**Theorem 4.2.** Under conditions (26), (31) the semiprocess family (32) with \( U = L^\infty(\mathbb{R}^+) \) is locally ISS w.r.t. the global attractor \( \Theta \) of unperturbed system (25).

**Proof.** We are going to prove conditions (i)-(iii) of Theorem 3.4.

Estimate (36) implies condition (i) with \( \sigma(p) = p^{\frac{2n-2}{4}} \).

Due to (26), Hölder inequality and Sobolev embedding \( H^1_0(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega) \) we have that for every \( y_1, y_2 \in H^1_0(\Omega) \), \( \|y_1\|_{H^1_0} \leq r, \|y_2\|_{H^1_0} \leq r \) the following estimate
\[ \int_\Omega |f(y_1) - f(y_2)|^2 dx \leq C_8 \left( 1 + \|y_1\|_{L^{\frac{2n}{n-2}}}^{\frac{n-2}{2}} + \|y_2\|_{L^{\frac{2n}{n-2}}}^{\frac{n-2}{2}} \right) \|y_1 - y_2\|_{L^{\frac{2n}{n-2}}}^2 \leq C(r) \|y_1 - y_2\|_{H^1_0}^2 \]
(37)
holds for some \( C(r) > 0 \).

So for \( w(t) = y_1(t) - y_2(t) \), where \( y_1 \) and \( y_2 \) are solutions of (29) with disturbances \( u_1 \) and \( u_2 \), we have
\[ \frac{1}{2} \frac{d}{dt} (\|w_t(t)\|^2 + \|w(t)\|_{H^1_0}^2) + k\|w_t(t)\|^2 \leq \]
\[ \|f(y_1) - f(y_2)\| + \|h\| \|u_t\| \|u_1(t) - u_2(t)\|. \]
(38)
Let us put \( u_1 \equiv u_2 \equiv 0 \) and let us assume that \( \| z_1(0) \|_X \leq r, \| z_2(0) \|_X \leq r \). Then due to (36)
\[
\forall \ t \geq 0 \ \max \{ \| z_1(t) \|_X, \| z_2(t) \|_X \} \leq \sqrt{\frac{r}{C_T}} \left( \frac{\| z(0) \|_X}{2} + r^2 + 1 \right).
\] (39)
Therefore, from (37),(38) we can find an appropriate positive constant \( c(r) \) such that
\[
\frac{d}{dt} (\| w(t) \|_X^2 + \| w(t) \|_{H_0^1}^2) \leq c(r)(\| w(t) \|_X^2 + \| w(t) \|_{H_0^1}^2).
\] (40)
The application of Gronwall’s Lemma finishes the proof of (ii).

To prove (iii) we put \( u_1 = u, u_2 = 0, z_1(0) = z_2(0), \| z_1(0) \|_X \leq r, \| u \|_\infty \leq r \). Then the inequality (39) holds and for some \( C_9 \) we get for arbitrary \( T \geq t \geq 0 \)
\[
\frac{d}{dt} (\| w(t) \|_X^2 + \| w(t) \|_{H_0^1}^2) \leq c(r)(\| w(t) \|_X^2 + \| w(t) \|_{H_0^1}^2) + C_9 \| u \|_\infty \sup_{t \in [0, T]} (\| w(t) \|_X + \| w(t) \|_{H_0^1}).
\] (41)
Integrating this inequality over \([0, t] \) we get
\[
\| w(t) \|_X^2 + \| w(t) \|_{H_0^1}^2 \leq c(r) \int_0^t (\| w_1 \|_X^2 + \| w \|_{H_0^1}^2) ds + C_9 \| u \|_\infty T \sup_{t \in [0, T]} (\| w(t) \|_X + \| w(t) \|_{H_0^1})
\]
and Gronwall’s Lemma implies
\[
\sup_{t \in [0, T]} (\| w(t) \|_X + \| w(t) \|_{H_0^1}) \leq C_{10} \| u \|_\infty Te^{c(r)T}
\] (42)
hence we obtain the required result. Theorem is proved. \( \square \)

To prove asymptotic gain property we need some additional restrictions on the disturbances.

Let us denote by \( U_1 \) all functions from \( L^\infty (\mathbb{R}^+) \) such that
\[
\sup_{t \geq 0} \int_t^{t+1} |u(s + l) - u(s)|^2 ds \leq \alpha(|l|),
\] (43)
where \( \alpha(\cdot) \) may depend on \( u \) and \( \alpha(p) \to 0, p \to 0+ \).

**Remark 8.** Condition (43) allows for a wide class of disturbance functions. For instance, bounded piecewise continuous functions which are globally Lipschitz between points of discontinuity belong to this class.

It is known [4, Chapter V] that \( U_1 \) is translation-invariant and every \( u \in U_1 \) is translation-compact in \( L^2_{\text{loc}}(\mathbb{R}^+) \), i.e., the set
\[
\Sigma(u) := cl_{L^2_{\text{loc}}} \{ u(\cdot + h) \mid h \geq 0 \}
\]
is compact in \( L^2_{\text{loc}}(\mathbb{R}^+) \)
where \( cl \) stands for the closure of a set in \( L^2_{\text{loc}} \).
Moreover, \( \forall h \geq 0 T(h)\Sigma(u) \subset \Sigma(u) \) and \( \forall v \in \Sigma(u) \)
\[
\sup_{t \geq 0} \int_t^{t+1} |v(s)|^2 ds \leq \sup_{t \geq 0} \int_t^{t+1} |u(s)|^2 ds \leq \| u \|_\infty^2.
\] (44)
Lemma 4.4. from [4, Section 4].

Assumption (ii) is fulfilled for every \( \lambda \) solutions of (29) on \((0, T)\) for arbitrary \( z_0 \in X \).

Using results from Chapter 6 in [4] we can conclude that for every \( u \in U_1 \) the semiprocess family \( \{S_t\}_{t \in \Sigma(u)} \) defined by (32) has a negatively-invariant uniform attractor \( \Theta_{\Sigma(u)} \).

**Theorem 4.3.** Under conditions (26), (31) the semiprocess family (32) with the disturbance set \( U_1 \) satisfies the asymptotic gain property w.r.t. the global attractor \( \Theta \) of the unperturbed system (25).

**Proof.** First, let us prove the asymptotic gain property

\[
\limsup_{t \to \infty} \|S_u(t, 0, z_0)\|_\Theta \leq \gamma(\|u\|_\infty)
\]

for arbitrary \( z_0 \in X \) and for all disturbances with \( \|u\|_\infty \leq r, r > 0 \). For this purpose, let us verify the conditions of Theorem 3.8 (in fact, conditions (i)-(iv) of Lemma 3.7) for \( \Lambda = U_1 \cap \{\|u\|_\infty \leq r\} \) with metric induced from \( L^\infty(R_+) \) and for \( \lambda_0 = 0 \).

Assumption (i) follows from estimates (35), (44) and inequality

\[
\forall t > 0 \int_0^t |v(s)|^2 e^{-\delta(t-s)} ds = (1 - e^{-\delta})^{-1} \sup_{t \in R_+} \int_t^{t+1} |v(s)|^2 ds.
\]

Assumption (ii) is fulfilled for every \( u \in U_1 \) because of condition (43) and Proposition 4.4 from [4].

To prove assumptions (iii) and (iv) we need the following auxiliary result.

**Lemma 4.4.** [9, Section 4] Let \( z_n(\cdot) = \left( \begin{array}{c} y_n(\cdot) \\ y_{nt}(\cdot) \end{array} \right) \) be an arbitrary sequence of weak solutions of (29) on \((0, T)\) with \( u = v_n, z_n(0) = z_0^n, \) and 

\[
v_n \rightarrow v \text{ weakly in } L^2(0, T),
\]

\[
z_0^n \rightarrow z^0 \text{ weakly in } X.
\]

Then

\[
y_n \rightarrow y \text{ in } C([0, T]; L^2(\Omega) \cap (H^1_0(\Omega))_w),
\]

\[
y_{nt} \rightarrow y_t \text{ in } C([0, T]; H^{-1}(\Omega) \cap (L^2(\Omega))_w),
\]

where the subscript \( w \) indicates that we mean convergence w.r.t. weak topology in the corresponding space, \( z(\cdot) = \left( \begin{array}{c} y(\cdot) \\ y_t(\cdot) \end{array} \right) \) is a weak solution of (29) on \((0, T)\) with \( u = v \) and \( z(0) = z^0 \).

If, moreover,

\[
v_n \rightarrow v \text{ in } L^2(0, T),
\]

\[
z_0^n \rightarrow z^0 \text{ in } X,
\]

then

\[
z_n \rightarrow z \text{ in } C([0, T]; X).
\]

Let us take \( v_k \in \Sigma(u_k), u_k \rightarrow 0 \) in \( L^\infty(R_+) \). Then due to (44) we have \( v_k \rightarrow 0 \) in \( L^\infty(R_+) \). So from (47) if \( \xi_k \in S(u_k)(t, 0, z^0_k) \) and \( z^0_k \rightarrow z^0 \) in \( X \), then up to subsequence \( \xi_k \rightarrow \xi \in S_0(t, 0, z^0) \) and we obtain that the assumption (iv) is satisfied.

Let us prove the property (24), which implies that the assumption (iii) is satisfied. To this end we consider

\[
\xi_n \in S_{v_n}(t_n, 0, \eta_n), \ v_n \rightarrow 0 \text{ in } L^\infty(R_+), \ t_n \not\to \infty, \ \eta_n \rightarrow \eta \text{ in } X.
\]

Then
Then $\xi_n = z_n(t_n)$, $z_n(\cdot)$ is a global weak solutions of (29) with $u = v_n$, $z_n(0) = \eta_n$. We want to prove precompactness of $\{\xi_n\}$ in $X$. For this purpose we will use the method proposed by Ball [2, Section 4] for the autonomous case.

Therefore, up to subsequence $\forall M > 0$ there exists $\xi_M$ such that

$$\xi_n \to \xi \text{ weakly in } X, \quad z_n(t_n - M) \to \xi_M \text{ weakly in } X. \quad (48)$$

Moreover, $\forall t \geq 0$

$$z_n(t_n - M + t) \in S_{v_n}(t_n - M + t, t_n - M, z_n(t_n - M)) = S_{\bar{T}(t_n - M)}v_n(t, 0, z_n(t_n - M)).$$

So, $z_n(t_n - M + t) = \bar{z}_n(t)$, where $\bar{z}_n(\cdot) = \left(\bar{y}_n(\cdot) \bar{y}_{nt}(\cdot)\right)$ is a global weak solution of (29) with $u(\cdot) = v_n(\cdot + t_n - M)$, $\bar{z}_n(0) = z_n(t_n - M)$. We also have that

$$\bar{v}_n(\cdot) := v_n(\cdot + t_n - M) \to 0 \text{ in } L^\infty(\mathbb{R}_+).$$

Therefore, from Lemma 4.4 $\forall t \geq 0$

$$\bar{z}_n(t) \to \bar{z}(t) = S_0(t, 0, \xi_M) \text{ weakly in } X.$$

In particular,

$$\bar{z}_n(M) = \xi_n \to \bar{z}(M) = \xi \text{ weakly in } X.$$

It is known [9, Section 4] that every weak solution of (29) satisfies the equality

$$\frac{d}{dt} I(z(t)) = -k I(z(t)) + H_u(t, z(t)), \quad (49)$$

where

$$I(z) = \frac{1}{2} ||y||^2 + \frac{1}{2} ||\nabla y||^2 + (F(y), 1) + \frac{k}{2} (y,t, y),$$

$$H_u(t, z) = k (F(y), 1) - \frac{k}{2} (f(y), y) + \frac{k}{2} u(t)(h, y) + u(t)(h, y).$$

Now we write (49) for $\bar{z}_n$. After integrating over $[0, M]$ we get

$$I(\xi_n) = I(\bar{z}_n(t_n - M))e^{-kM} + \int_0^M e^{k(p-M)} H_0(p, \bar{z}_n(p))dp. \quad (50)$$

Because of Lemma 4.4

$$\bar{y}_n \to \bar{y} \text{ in } C([0, M]; L^2(\Omega)).$$

Therefore,

$$F(\bar{y}_n(t, x)) \to F(\bar{y}(t, x)), \quad f(\bar{y}_n(t, x))\bar{y}_n(t, x) \to f(\bar{y}(t, x)\bar{y}(t, x) \text{ a.e.}$$

Due to (26), the sequences $\{F(\bar{y}_n)\}$ and $\{f(\bar{y}_n)\}$ both are bounded in $L^\infty(\mathbb{R}_+)$. Hence, they weakly converge to $\{F(\bar{y})\}$ and $\{f(\bar{y})\}$ respectively. Using convergence $\bar{v}_n \to 0$ in $L^2(0, M)$, we can get

$$\int_0^M e^{k(p-M)} H_0(p, \bar{z}_n(p))dp \to \int_0^M e^{k(p-M)} H_0(\bar{z}(p))dp, \quad n \to \infty, \quad (51)$$

where $H_0(z) = kF(\bar{y}, 1) - \frac{k}{2} (f(\bar{y}), \bar{y})$.

Using estimate (35), we deduce that $\forall t \geq 0$

$$\limsup_{n \to \infty} |I(z_n(t))| \leq C,$$

where the constant $C$ does not depend on $M$ and $t$.  

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[127x661]
So, from (50) we get
\[
\limsup_{n \to \infty} I(\xi_n) \leq Ce^{-kM} + \int_0^M e^{k(p-M)} H_0(\bar{z}(p))dp
\]
\[
= Ce^{-kM} + I(\xi) - I(\xi_M)e^{-kM} \leq 2Ce^{-kM} + I(\xi).
\]
Using (51), we have
\[
\limsup_{n \to \infty} \frac{1}{2} \|\xi_n\|_X \leq 2Ce^{-kM} + \frac{1}{2} \|\xi\|_X.
\]
Passing to the limit as \(M \to \infty\) we derive the strong convergence \(\xi_n\) to \(\xi\) in \(X\) and, consequently, the assumption (iii) is satisfied.

Now, according to Theorem 3.8 we can claim that for every \(r > 0\) there exists \(\gamma_r \in K\) such that for every \(u \in U_1\) with \(\|u\|_{\infty} \leq r\) and for arbitrary \(z_0 \in X\) the following inequality holds
\[
\limsup_{t \to \infty} \|S_u(t, 0, z_0)\|_\Theta \leq \gamma_r(\|u\|_{\infty}). \quad (52)
\]

5. Conclusions. We have investigated separately the local ISS and the asymptotic gain properties for a class of infinite dimensional systems. It is known that in case of finite dimensional systems, the combination of these properties implies the global ISS. For general infinite dimensional systems such an implication is not true. However it remains an open question, whether for the class of systems considered here this combination yields ISS. This question remains a matter of our future research. Derivation of possibly sharp estimations for gains, also with respect to different norm as in [14], is another important direction of research.

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