A MODIFIED YAMABE INVARIANT AND A HOPF CONJECTURE

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Abstract

In this paper we define the bi-orthogonal sectional curvature and we present two modified Yamabe invariants for compact 4-dimensional manifolds. In particular we obtained a relationship between one of these invariants and a Hopf conjecture.

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1. Introduction and Statements of results

The bi-orthogonal sectional curvature

Let \( M = M^n \) be a compact manifold of dimension \( n \geq 4 \) and denotes by \( \text{Met}(M) \) the set of Riemannian metrics on \( M \). For each metric \( g \in \text{Met}(M) \), let \( s \) be the scalar curvature of \( M \) in metric \( g \) and denotes by \( K \) the sectional curvature of this metric. For each \( x \in M \), let \( P_1, P_2 \) two mutually orthogonal two dimensional subspace of tangent space \( T_x M \). We call of bi-orthogonal sectional curvature \( (K^\perp) \) relative to \( P_1 \) and \( P_2 \) (in \( x \in M \)) the number given by

\[
K^\perp(P_1, P_2) = \frac{K(P_1) + K(P_2)}{2} \quad [1.1]
\]

If \( n = 4 \), we write

\[
K^\perp(P) = \frac{K(P) + K(P^\perp)}{2}. \quad [1.2]
\]

This notion of curvature was used in [7] and [6] by W. Seaman and M. H. Noronha, respectively. In particular, Seaman proved an extension of part of the Sphere Theorem for \( n \geq 5 \) (see Theorem 0.3 in [7]). Now, let \( M \) be a 4-dimensional Riemannian manifold and consider the following functions on \( M \):

\[
k^\perp_1 = \min \{K^\perp(P), \ P \text{ a 2-plan in } T_x M\}, \quad [1.3]
\]

\[
k^\perp_2 = \max \{K^\perp(P), \ P \text{ a 2-plan in } T_x M\}, \quad [1.4]
\]

\[
k^\perp_3 = \frac{s}{4} - k^\perp_1 - k^\perp_2. \quad [1.5]
\]

In four dimension, we have the following features of the bi-orthogonal sectional curvature:

I) \( M \) is an Einstein 4-manifold if and only if \( K^\perp(P) = K(P) \), for each \( x \in M \) and for all 2-plan \( P \subset T_x M \).

II) A 4-manifold \( M \) is conformally flat if and only if \( k^\perp_i(x) = \frac{s(x)}{12} \) \( (i = 1, 2, 3) \), for all \( x \in M \), where \( s \) is the scalar curvature of \( M \).
Remark - Property I is a well known characterization of Einstein 4-manifold. Property II follows from a criterion of Kulkarni.

An extension of the Sphere Theorem in four dimension

If the bi-orthogonal of a manifold $M$ satisfies $1/4 \leq K^\perp \leq 1$ then $M$ has nonnegative isotropic curvature (see [7]). Using the classification of compact manifolds with nonnegative isotropic curvature curvature (see [1] and [8]) we obtained an extension of a theorem of Seaman (see Theorem 0.3 in [7]):

Theorem 1. Let $M$ be a compact oriented 4-manifold such that $1/4 \leq K^\perp \leq 1$ and let the $\tilde{M}$ be the universal covering of $M$.

1. If $M$ is irreducible then $\tilde{M}$ is diffeomorphic to sphere $S^4$, $M$ is biholomorphic to complex projective space $\mathbb{CP}^2$ or $M$ is diffeomorphic to a connected sum $\mathbb{CP}^2\#m\mathbb{RP}^4/(\mathbb{R} \times S^3)/G_2\#2(\mathbb{R} \times S^3)/G_2$, where $m = 0, 1$ and the $G_2$ are discrete subgroup of the isometry group of $\mathbb{R} \times S^3$.

2. If $M$ is reducible then $\tilde{M}$ is isometric to a Riemannian product $\mathbb{R} \times N^2$, where $N^2$ is diffeomorphic to sphere $S^2$.

The bi-orthogonal sectional curvature and the Weyl tensor in four dimension

The Weyl tensor $W$ of an oriented 4-manifold $M$ has the following decomposition: $W = W^+ \oplus W^-$, where $W^\pm$ are the self-dual and anti-self-dual parts of the tensor $W$, respectively. Let $\Lambda^2$ be the space of two-forms $\varphi$ in $M$ and let $\ast : \Lambda^2 \rightarrow \Lambda^2$ be the star operator of Hodge. Then $\Lambda^+ \oplus \Lambda^-$, where $\Lambda^\pm = \{ \varphi \in \Lambda^2; \ast \varphi = \pm \varphi \}$. The Weyl tensor has the decomposition $W = W^+ \oplus W^-$, where $W^\pm : \Lambda^\pm \rightarrow \Lambda^\pm$ are self-adjoint operators with free traces. $W^\pm$ are called the self-dual and anti-self-dual parts of the Weyl tensor $W$ of $M$, respectively. The matrix of the curvature operator $\mathcal{R}$ of $M$ takes the form

$$
\begin{pmatrix}
W^+ + \frac{s}{12}I_{\Lambda^-} & B \\
B^+ & W^- + \frac{s}{12}I_{\Lambda^-}
\end{pmatrix},
$$

where $B : \Lambda^- \rightarrow \Lambda^+$, $|B|^2 = |\text{Ric} - \frac{s}{4}|^2$ and $\text{Ric}$ is the Ricci operator of $M$.

Denote by $w_1^\pm \leq w_2^\pm \leq w_3^\pm$ the eigenvalues of $W^\pm$, respectively.

Let $x \in M$ and consider $X, Y$ orthonormal in tangent space $T_xM$. Then a simple and unitary two-form $\varphi = X \wedge Y$ can be uniquely written as $\varphi = \varphi^+ + \varphi^-$, where $\varphi^\pm \in \Lambda^\pm$ and $|\varphi^\pm|^2 = \frac{1}{2}$. The sectional curvature $K(\varphi)$ is given by

$$
K(\varphi) = \frac{s}{12} + \langle \varphi^+, W^+(\varphi^+) \rangle + \langle \varphi^-, W^-(\varphi^-) \rangle + 2\langle \varphi^+, B\varphi^- \rangle \quad [1.6]
$$

If $\varphi^-$ is replaced by $-\varphi^-$, we have

$$
K(\varphi^+) = \frac{s}{12} + \langle \varphi^+, W^+(\varphi^+) \rangle + \langle \varphi^-, W^-(\varphi^-) \rangle - 2\langle \varphi^+, B\varphi^- \rangle, \quad [1.7]
$$

Where $\varphi^\perp = \varphi^+ - \varphi^-$. Adding [1.6] to [1.7]:

$$
\frac{K(\varphi^+)}{2} + K(\varphi) = \frac{s}{12} + \langle \varphi^+, W^+(\varphi^+) \rangle + \langle \varphi^-, W^-(\varphi^-) \rangle
$$

Using [1.3],

$$
k_1^\perp = \frac{s}{12} + \min \{ \langle \varphi^+, W^+(\varphi^+) \rangle + \langle \varphi^-, W^-(\varphi^-) \rangle, |\varphi^\perp|^2 = 1/2 \} =
$$
\[
\frac{s}{12} + \min \{ (\varphi^+, W^+(\varphi^+)), \ | \varphi^+ |^2 = 1/2 \} + \min \{ (\varphi^-, W^-(\varphi^-)), \ | \varphi^- |^2 = 1/2 \}.
\]

By Propositions 2.1 in [6] there exists an orthonormal basis \( \{P_1, P_2, P_3, P_4\} \) of \( \Lambda^2 \) such that each \( P_i \) is the form \( X_i \wedge Y_i \), where \( X_i, Y_i \in T_x M \). Moreover, in accord with the Proposition 2.5 in [6] is easy see that the eigenvectors of \( W^\pm \) are \( \frac{\sqrt{2}}{2} (P_i \pm P_i^\perp) \), respectively.

Then
\[
k_i^\perp = \frac{s}{12} + \frac{w_i^+ + w_i^-}{2}, \tag{1.8}
\]
where \( w_i^\pm \) are the smallest eigenvalues of \( W^\pm(x) \), respectively.

Similarly and in view of [1.4] we have
\[
k_i^\perp = \frac{s}{12} + \frac{w_i^+ + w_i^-}{2}, \tag{1.9}
\]
where \( w_i^\pm \) are the largest eigenvalues of \( W^\pm(x) \), respectively.

Since that \( w_2^\pm = -w_1^\pm - w_3^\pm \), respectively, we can uses [1.5] and obtain
\[
k_i^\perp = \frac{s}{12} + \frac{w_3^+ + w_3^-}{2}. \tag{1.10}
\]

An extension of the Yamabe Problem was considered by M. Itoh in [4] and more recently, M. Listing (see chapter 2, section 2.2 in [5]) and B-L Chen and X-P Zhu (see [2]) obtained important results on this topic. In our article we use some results of [2] for two modified Yamabe invariants.

**A modified Yamabe invariant and the isotropic curvature.**

Let \( M \) be a compact oriented 4-manifold and let \( \text{Met}(M) \) be the set of Riemannian metrics on \( M \). If \( g \in \text{Met}(M) \), let \( [g] = \{ \bar{g} = u^2 g; u \in C^\infty(M), u > 0 \} \) and let \( dV_{\bar{g}} \) be the volume element in metric \( \bar{g} \).

The Yamabe constant of the metric \( g \) is given by
\[
Y(M, g) := \inf \left\{ \frac{1}{\sqrt{V_{\bar{g}}}} \int_M \bar{s} dV_{\bar{g}}, \bar{g} \in [g] \right\},
\]
where \( \bar{s} \) is the scalar curvature of \( M \) in metric \( \bar{g} \).

The Yamabe invariant of \( M \) is given by
\[
Y(M) := \sup \{ Y(g) \}.
\]

Now let
\[
Y^\perp(M, g) := \inf \left\{ \frac{1}{\sqrt{V_{\bar{g}}}} \int_M [24k_i^\perp - \bar{s}] dV_{\bar{g}}, \bar{g} \in [g] \right\}, \tag{1.11}
\]
where \( \bar{k}_i^\perp \) is the bi-orthogonal sectional curvature of \( M \) given by [1.3] in metric \( \bar{g} \). Consider the following modified Yamabe invariant:
\[
Y^\perp(M) := \sup \{ Y^\perp(M, g), g \in \text{Met}(M) \}. \tag{1.12}
\]

Our next result is the following
Theorem 2. Let $M$ be a compact oriented 4-manifold with Riemannian metric $g$. If $Y^\perp(M, g) \geq 0$ then we have

1. $M$ is diffeomorphic to a connected sum $S^4\sharp_m\mathbb{RP}^4(\mathbb{R} \times S^3)/G_1 \sharp \ldots \sharp (\mathbb{R} \times S^3)/G_n$, where $m = 0$ or $1$, $i \geq 0$ and the $G_i$ are discrete subgroup of the isometry group of $\mathbb{R} \times S^3$ or

2. $(M, g)$ is conformal to a complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric or a finite cover is conformal to a Riemannian product $S^2_c \times T^2$, where $S^2_c$ is a sphere with constant sectional curvature $c$ and $T^2$ is a flat torus.

The conditions of the Theorem 2 imply that $M$ admits a metric with non negative isotropic curvature.

A modified Yamabe invariant and a Hopf conjecture

Let $M$ be a compact oriented 4-manifold and $g \in \text{Met}(M)$ and consider

$$Y^\perp_1(M, g) := \inf \left\{ \frac{1}{\sqrt{\tilde{g}}} \int_M k^\perp_1 dV_{\tilde{g}}, \tilde{g} \in [g] \right\}. \quad [1.13]$$

We have another modified Yamabe invariant:

$$Y^\perp_1(M) := \sup \{ Y^\perp_1(M, g), g \in \text{Met}(M) \}. \quad [1.14]$$

Recall the Hopf conjecture:

$S^2 \times S^2$ no admits a Riemannian metric with positive sectional curvature.

Then we can formulate the following question:

$S^2 \times S^2$ admits a metric with positive bi-orthogonal sectional curvature?

With respect to this question we have:

Theorem 3. Let $M$ be a compact oriented 4-manifold. Then we have

1. $M$ has a metric $g$ with $k^\perp_1 > 0$ if and only if $Y^\perp_1(M) > 0$.

2. $Y^\perp_1(M) \leq Y(M) \leq Y(S^4)$, where $Y(M)$ is the Yamabe invariant of $M$. In particular, if $Y^\perp_1(M) = Y(S^4)$ then $M$ is conformal to the standard sphere $S^4$.

3. If $M$ has a metric $g$ with $k^\perp_1 \geq 0$ and scalar curvature $s > 0$ then

$$8\pi^2 \chi < \max \left\{ \int_M s^2 dV_g + 16\pi^2, \frac{5}{24} \int_M s^2 dV_g \right\},$$

where $\chi$ is the Euler characteristic of $M$.

Corollary 4

1. Let $g_{\text{can}}$ the canonical metric of product of spheres $S^2 \times S^2$ and let $g \in [g_{\text{can}}]$. Then $g$ no has $k^\perp_1 > 0$ on $S^2 \times S^2$.

2. If $Y^\perp_1(S^2 \times S^2) \leq 0$ then the Hopf conjecture is true.

3. Let $g$ a Riemannian metric on $S^2 \times S^2$ with scalar curvature $s$. If $\int s^2 dV_g \leq \frac{256\pi^2}{3}$ then $g$ no has $k^\perp_1 \geq 0$. 
2. Proof of results

Proof of Theorem 1

Let $M$ be a compact oriented 4-manifold such that $1/4 \leq K^+ \leq 1$. In accord with Seaman [7], $M$ has non negative isotropic curvature. Consider $M$ is irreducible. By main result of Seshadri [8] we have only the following possibilities:

i) $M$ admits a metric with positive isotropic curvature. In this case, follow from the main theorem of [1] that $M$ is diffeomorphic to a connected sum $S^4 \# m \# \mathbb{RP}^4 \# (\mathbb{R} \times S^3)/G_1 \# \ldots \# (\mathbb{R} \times S^3)/G_n$, where $m = 0$, $m = 1$ and the $G_i$ are discrete subgroup of the isometry group of $\mathbb{R} \times S^3$.

ii) $M$ is isometric to a irreducible locally symmetric space. Then $M$ is an Einstein space with positive isotropic curvature and so $M$ is isometric to a sphere $\mathbb{S}^4$.

iii) $M$ is biholomorphic to a complex projective space $\mathbb{C}P^2$.

Now, assumes that $M$ is reducible. Since that $M$ has positive scalar curvature, then the universal covering $\widetilde{M}$ is isometric to $\mathbb{R} \times M^2_1$ or isometric to $M^2_1 \times M^2_2$. But $M^2_1 \times M^2_2$ has $k^+_1 = 0$.

So, $\widetilde{M}$ is isometric to a Riemannian product $\mathbb{R} \times N^2$, where $M^2_2$ has positive sectional curvature and in this case $M^2_2$ is diffeomorphic to sphere $\mathbb{S}^3$. This finish the proof of Theorem 1.

For proof of Theorems 2 and 3 we need a lemma. For this let $M$ be a compact oriented 4-manifold $M$ with metric $g$ and scalar curvature $s$ and consider the functions $f(W) = 2s - 24k^+_1 = -12(w^+_1 + w^-_1) \geq 0$ (see (1.8)) and $f_1(W) = s - 12k^+_1 = -6(w^+_1 + w^-_1) \geq 0$, where $w^+_1$ are the smallest eigenvalues of $W^\pm$, respectively. In notation of [2, eq. (2.2) and (2.5)] we have $Y_f(M, [g]) = Y^+(M, g)$ and $Y_{f_1}(M, [g]) = Y^+_1(M, g)$, where $Y^+(M, g)$ and $Y^+_1(M, g)$ are given by [1.11] and [1.13], respectively.

In accord with the lemma 2.1 and 2.2 in [2] we have:

Lemma 5- Let $M$ be a compact oriented 4-manifold with metric $g$.

i) There exists $\bar{g}, \overline{g} \in [g]$ such that $Y^+(M, g) = 24\overline{k}^+ - \overline{s} = \text{constant}$ and $Y^+_1(M, g) = 12\overline{k}^+_1 = \text{constant}$, respectively where $\overline{s}$ is the scalar curvature in metric $\overline{g}$ and $\overline{k}^+_1$ and $\overline{k}^+_1$ are the smallest bi-orthogonal sectional curvatures in metrics $\overline{g}$ and $\overline{g}$, respectively.

ii) If $Y^+(M, g) > 0$ or $Y^+_1(M, g) > 0$ then there exists $\bar{g} \in [g]$ such that $24\bar{k}^+ - \bar{s} > 0$ or $\bar{k}^+_1 > 0$, respectively.

Proof of Theorem 2

Let $M$ be a compact oriented 4-manifold with Riemannian metric $g$ and $Y^+(M, g) \geq 0$. Initially assumes that $Y^+(M, g) > 0$. By Lemma 5) there exists $\bar{g} \in [g]$ such that $24\bar{k}^+ - \bar{s} > 0$. Then (see (1.8)) we have

$$w^+_1 \geq w^+_1 + w^-_1 > -\frac{s}{12}$$

Since that $w^+_3 \leq -2w^+_3$, respectively then $\frac{s}{3} - w^+_3 > 0$ an this proves that $(M, \bar{g})$ has positive isotropic curvature. By main result of [1], $M$ is diffeomorphic to a connected sum $\mathbb{S}^4 \# m \# \mathbb{RP}^4 \# (\mathbb{R} \times S^3)/G_1 \# \ldots \# (\mathbb{R} \times S^3)/G_n$, where $m = 0$ or $1$, $i \geq 0$ and the $G_i$ are discrete.
subgroup of the isometry group of $\mathbb{R} \times S^3$

Now, consider $Y^\perp (M, g) = 0$. By Lemma 5) there exists $\tilde{g}$ such that $Y^\perp (M, \tilde{g}) = 24 \tilde{k}^+_1 - \tilde{s} = 0$. In this case $(M, \tilde{g})$ has nonnegative isotropic curvature. Assumes that $M$ is irreducible. By main result of Seshadri [8] we have only the following possibilities:

i) $M$ admits a metric with positive isotropic curvature. In this case, follow from the main theorem of that $M$ is diffeomorphic to a connected sum $S^4 \sharp m \sharp \mathbb{RP}^4 \sharp (\mathbb{R} \times S^3)/G_1 \sharp \ldots \sharp (\mathbb{R} \times S^3)/G_n$, where $m = 0$, $m = 1$ and the $G_i$ are discrete subgroup of the isometry group of $\mathbb{R} \times S^3$.

ii) $(M, g)$ is conformal to a irreducible locally symmetric space. Then $M$ is an Einstein space with positive isotropic curvature and so $M$ is isometric to a sphere $S^4$.

iii) $(M, g)$ is isometric to a complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric.

Now, assumes that $M$ is reducible. Then $(M, g)$ is either conformal to a finite cover Riemannian product $S^2_{c_1} \times S^2_{c_2}$, $S^2_c \times \mathbb{R}$ or $S^2_{c_1} \times \mathbb{T}^2$, where $S_c$ is a sphere with constant sectional curvature $c$ and $\mathbb{T}^2$ is a flat torus. But $S^2_{c_1} \times S^2_{c_2}$ has $k_1 = 0 \neq s/24$. This finish the proof of Theorem 2.

Proof of Theorem 3

Theorem 3)(1) and 3)(2) are consequences of Corollary 2.2.3 and equation (2.2) in [5], respectively.

Let $M$ with metric $g$, $k_1^+ \geq 0$ and scalar curvature $s > 0$. In accord Theorem 1.1 in [9], if $\chi$ is the Euler characteristic of $M$ then $M$ is isometric to a sphere or

$$8\pi^2(\chi - 2) < \int_M |W|^2 dV_g,$$

where $W$ is the Weyl tensor of $(M, g)$. Note that we have the inequalities:

$$|W|^2 = |W^+|^2 + |W^-|^2 \leq 6 |W^+_1|^2 + 6 |W^-_1|^2 \leq 6(w^+_1 + w^-_1)^2.$$

Using [1,8], we obtain $|W|^2 \leq 6(\tilde{s} - 2k_1)^2 \leq \frac{\tilde{s}^2}{2}$ and so $8\pi^2 \chi < \frac{1}{3} \int_M s^2 dV_g$. In any case have that $8\pi^2 \chi < \frac{1}{3} \int_M s^2 dV_g$ + $16\pi^2$ On the other side, if $(M, g)$ is a compact oriented 4-manifold with metric $g$ and scalar curvature $s$ then the Euler characteristic of $M$ satisfy

$$8\pi^2 \chi = \int_M \left( |W|^2 + \frac{s^2}{24} - \frac{1}{2} |B|^2 \right) dV_g,$$

where $B = 0$ if and only if $g$ is an Einstein metric. Using previous inequalities we have $8\pi^2 \chi \leq \frac{s^2}{24} \int_M s^2 dV_g$. In particular, if $8\pi^2 \chi = \frac{s^2}{24} \int_M s^2 dV_g$ then we can see that $B = 0$ and $g$ is an Einstein metric with nonnegative sectional curvature and positive scalar curvature. By Lemma 2 in [3], $8\pi^2 \chi < \frac{s^2}{24} \int_M s^2 dV_g$. In any case have that $8\pi^2 \chi < \frac{s^2}{24} \int_M s^2 dV_g$. This finish the proof the Theorem 3.

Proof of Corollary 4

(1) Let $g_{can}$ be the canonical metric of product of spheres $S^2 \times S^2$ and let $g \in [g_{can}]$. Consider $k^+_1$ and $k^+_2$ the smallest bi-orthogonal curvatures of the metrics $g$ and $g_{can}$, respectively. Let $g = u^2 g_{can}$. Then $12k^+_1 = u^{-3}[-6\Delta_g u + 12k^+_1]$ (see eq. (2.3) in [2]). Since that $k^+_1 = 0$ we have $\int_M \tilde{k}^+_1 = 0$ and this proves that $g$ no has $\tilde{k}^+_1 > 0$.

(2) Corollary 4)(2) is consequence of Theorem 3)(1).

(3) Let $g$ be a Riemannian metric on $S^2 \times S^2$ with $k^+_1 \geq 0$. Assumes that the scalar curvature
s of $g$ satisfy $\int_M s^2dV_g \leq \frac{7680\pi^2}{5}$. Then by Corollary 3)(3), $\chi < 4$ which contradicts the fact of that $\chi(S^2 \times S^2) = 4$

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