Special polynomials associated with the fourth order analogue to the Painlevé equations

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Abstract

Rational solutions of the fourth order analogue to the Painlevé equations are classified. Special polynomials associated with the rational solutions are introduced. The structure of the polynomials is found. Formulas for their coefficients and degrees are derived. It is shown that special solutions of the Fordy - Gibbons, the Caudrey - Dodd - Gibbon and the Kaup - Kupershmidt equations can be expressed through solutions of the equation studied.

Keywords: Special polynomials, the Painlevé equation, the Painlevé hierarchy, Special solutions, power expansion

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1 Introduction

It is well known that the general solutions of the six Painlevé equations ($P_1 - P_6$) can not be expressed through known elementary or classical special functions in the general case because their solutions determine new transcendental functions. However, the equations $P_2 - P_6$ possess hierarchies of rational and algebraic solutions at certain values of the parameters.

A. I. Yablonskii and A. P. Vorob’ev were first who expressed the rational solutions of $P_2$ via the logarithmic derivative of the polynomials, which now go under the name of the Yablonskii – Vorob’ev polynomials [1, 2]. Later K. Okamoto suggested special polynomials for certain rational solutions of $P_4$ [3]. H. Umemura derived analogues polynomials for some rational and algebraic solutions of $P_3$ and $P_5$ [4]. All these polynomials possess a number of interesting properties. For example, they can be expressed in terms of Schur polynomials. Besides that the polynomials arise as the tau-functions and satisfy recurrence relations of Toda type. Recently these polynomials have been intensively studied [5–10].
Not long ago P. A. Clarkson and E. L. Mansfield suggested special polynomials for the equations of the $P_2$ hierarchy \[11\]. Also they studied the location of their roots in the complex plane and showed that the roots have a very regular structure.

The aim of this work is to introduce special polynomials related to rational solutions of the following equation, which is analogue to the Painlevé equations

\[
w_{zzzz} + 5 w_z w_{zz} - 5 w^2 w_{zzz} - 5 w w_z^2 + w^5 - z w - \beta = 0. \tag{1.1}
\]

Originally this equation was found from the Fordy – Gibbons equation \[12\]

\[
\omega_t + \omega_{xxxx} + 5 \omega_x \omega_{xxx} - 5 \omega^2 \omega_{xx} + 5 \omega_{xx}^2 - 20 \omega \omega_x \omega_{xx} - 5 \omega_x^3 + 5 \omega^4 \omega_x = 0 \tag{1.2}
\]

through the scaling reduction

\[
\omega(x, t) = (5 t)^{-\frac{1}{6}} w(z), \quad z = x (5 t)^{-\frac{1}{6}}. \tag{1.3}
\]

Equation (1.1) was first considered in \[13\] and later in works \[14–20\]. This equation has a number of properties similar to that of the Painlevé equations. More exactly it possesses the Bäcklund transformations, the Lax pair, rational and special solutions at certain values of the parameter $\beta$ \[13,14\]. These special solutions are expressible in terms of the first Painlevé transcendent \[15, 18\]. The Cauchy problem for this equation can be solved by the isomonodromic deformation method.

Let us demonstrate that special solutions of the Caudrey – Dodd – Gibbon equation can be expressed through solutions of (1.1).

The Caudrey – Dodd – Gibbon equation (or Savada - Kotera equation) can be written as \[21–23\]

\[
u_t + u_{xxxx} - 5 u u_{xx} - 5 u_x u_{xx} + 5 u^2 u_x = 0. \tag{1.4}
\]

This equation has the self – similar reduction

\[
u(x, t) = (5 t)^{-\frac{2}{11}} y(z), \quad z = x (5 t)^{-\frac{1}{11}}, \tag{1.5}
\]

where $y(z)$ satisfies the equation

\[
y_{zzzz} - 5 y y_{zzzz} - 5 y_z y_{zz} + 5 y^2 y_z - 2 y - z y_z = 0. \tag{1.6}
\]

The Miura transformation $y(z) = w_z + w^2$ relates solutions of (1.6) to solutions of the equation

\[
\left( \frac{d}{dz} + 2 w \right) \frac{d}{dz} (w_{zzzz} + 5 w_z w_{zz} - 5 w^2 w_{zzz} - 5 w w_z^2 + w^5 - w z - \beta) = 0.
\tag{1.7}
\]

Thus we see that for any solution of (1.1) there exists a solution of (1.4).
The Kaup - Kupershmidt equation \[23, 24\]

\[ v_t + v_{xxxx} + 10v v_{xx} + 25 v_x v_{xx} + 20 v^2 v_x = 0 \] \hspace{1cm} (1.8)

also possesses solutions, which can be expressed via solutions of \(1.1\). Indeed it has the self-similar reduction

\[ v(x, t) = (5t)^{-\frac{2}{5}} y(z), \quad z = x (5t)^{-\frac{1}{5}} \] \hspace{1cm} (1.9)

with \(y(z)\) satisfying the equation

\[ y_{zzzz} + 10 y y_{zz} + 25 y_z y_{zz} + 20 y^2 y_z - z y_z - 2 y = 0. \] \hspace{1cm} (1.10)

After making the Miura transformation \(y(z) = w_z - \frac{1}{2} w^2\) we obtain

\[ \left( \frac{d}{dz} - w \right) \frac{d}{dz} (w_{zzzz} + 5 w_z w_{zz} - 5 w^2 w_{zz} - 5 w w_z^2 + w^5 - w z - \beta) = 0. \] \hspace{1cm} (1.11)

Hence the Fordy - Cibbons equation \(1.2\), the Caudrey – Dodd – Gibbon equation \(1.4\) and the Kaup - Kupershmidt equation \(1.8\) admits solutions in terms of solutions of \(1.1\).

Also we would like to mention that apparently the equation \(1.1\) defines new transcendental functions like the Painlevé equations do.

2 Special polynomials associated with rational solutions of equation \(1.1\)

Let us briefly review some facts concerning the equation \(1.1\), which we will need later. Let \(w \equiv w(z; \beta)\) be a solution of \(1.1\). Then the transformations

\[ T_{2-\beta} : w(z; 2 - \beta) = w + \frac{2\beta - 2}{z - w_{zzz} + w w_{zz} - 3w_z^2 + 4w^2 w_z - w^4} \] \hspace{1cm} (2.1)

\[ T_{-1-\beta} : w(z; -1 - \beta) = w + \frac{2\beta + 1}{z + 2w_{zz} + 4w w_z + 3w_z^2 - 2w^2 w_z - w^4} \]

generate other solutions of \(1.1\), provided that \(\beta \neq 1\) for \(T_{2-\beta}\) and \(\beta \neq -1/2\) for \(T_{-1-\beta}\) \[15\].

Let \(z = z_0\) be a pole of the solution \(w(z; \beta)\), then the Laurent expansion of \(w(z; \beta)\) in a neighborhood of \(z_0\) is the following

\[ w(z; \beta) = e + c^{(e)}(z - z_0) + \phi^{(e)}(z - z_0), \] \hspace{1cm} (2.2)

where \(e\) takes one of the values \(1, 4, -2, -3\) and \(\phi^{(e)}(z - z_0) \sim o(z - z_0)\) is a holomorphic function in a neighborhood of \(z_0\). In fact, there are four types of Laurent expansion around a movable pole.

If now the point \(z = \infty\) is a holomorphic point of \(w(z; \beta)\), in particular it is the case of rational solutions, then the expansion of \(w(z; \beta)\) around infinity is

\[ w(z; \beta) = -\frac{\beta}{z} + \sum_{l=1}^{\infty} c_{\beta, -5l-1} z^{-5l-1}. \] \hspace{1cm} (2.3)
All the coefficients $c_{\beta,-5l-1}$ in (2.3) can be sequently found. The first few of them are given below

$$c_{\beta,-6} = -\beta (\beta + 4) (\beta - 2) (\beta - 3) (\beta + 1),$$
$$c_{\beta,-11} = (5 \beta^4 - 295 \beta^2 + 270 \beta + 3024) c_{\beta,-6},$$
$$c_{\beta,-16} = (35 \beta^8 - 6270 \beta^6 + 6560 \beta^5 + 380055 \beta^4 - 467400 \beta^3 - 9286740 \beta^2 + 8845200 \beta + 72648576) c_{\beta,-6}.\tag{2.4}$$

What is more, the ratio $c_{\beta,-5l-1}/c_{\beta,-6}$ is a polynomial in $\beta$ for all $l > 1$. The expansions (2.2), (2.3) can be obtained with the help of algorithms of power geometry. For more information see [25, 26, 28].

Rational solutions of the equation (1.1) are classified in the following theorem.

**Theorem 2.1.** The equation (1.1) possesses rational solutions if and only if $\beta \in \mathbb{Z}/\{1 \pm 3k, k \in \mathbb{N} \cup 0\}$. They are unique and have the form

$$w(z; \beta_n^{(1)}) = (-1)^n \frac{d}{dz} \ln \left( \frac{Q_{n-1}}{Q_n} \right),$$
$$w(z; \beta_n^{(2)}) = (-1)^n \frac{d}{dz} \ln \left( \frac{R_{n-1}}{R_n} \right),\tag{2.5}$$

where $Q_n(z)$ and $R_n(z)$ are polynomials, $n \in \mathbb{N}$ and

$$\beta_n^{(1)} = (-1)^n \left( 3 \left[ \frac{n+1}{2} \right] - 1 + (-1)^n \right),$$
$$\beta_n^{(2)} = (-1)^n + 1 + (-1)^{n+1} \left( 3 \left[ \frac{n}{2} \right] + 1 + (-1)^{n+1} \right)\tag{2.6}$$

with $[x]$ denoting the integer part of $x$. The only remaining rational solution is the trivial solution $w(z; 0) = 0$.

**Proof.** While proving this theorem we will miss the dependence of $w(z; \beta)$ on $\beta$. Any meromorphic solution of the equation (1.1) can be represented as the ratio of two entire functions, which we are going to construct. With the help of (2.2) we understand that the function

$$f(z) = \exp(-\int z ds_2 \int s_2 w^2(s_1)ds_1)\tag{2.7}$$

is entire and has a zero of multiplicity $e^2$ at $z = z_0$ whenever $w(z)$ has a pole with residue $e$ at the same point $(e = 1, 4, -2, -3)$. The path of integration in (2.7) avoids the poles of $w(z)$. Further the function $g(z) = w(z)f(z)$ is also entire. Choosing in such a way entire functions we obtain the system of equations satisfied by $f(z)$ and $g(z)$

$$ffzz - f_z^2 + g^2 = 0,$$
$$f^4g_{zzzz} - 4f_zf_z^3g_{zzz} + (3g^2f^2 + 5gfz + 6f_z^2f^2 - 5gf_zf^2)g_{zz} + (5gf^2 - 10f_zf^2)g_z^2 + (15gfz - 4f_z^2 + 5gfz - 16g^2f_zf)g_{zz} + g^2 - 5f_zg^4 + 8f_z^2g^3 - 5f_z^3g^2 + (f_z^4 - 4f_z^4)g - \beta f^5 = 0.\tag{2.8}$$
In the case of a rational solution it can be set

\[ f(z) = F(z) \exp(h(z)), \quad g(z) = G(z) \exp(h(z)), \quad (2.9) \]

where \( F(z), G(z) \) are polynomials and \( h(z) \) is an entire function. Moreover it is straightforward to show that \( h_{zz} = 0 \). Substituting (2.9) into (2.8) yields a pair of equations similar to (2.8) with \( f(z) \) replaced by \( F(z) \) and \( g(z) \) replaced by \( G(z) \). As far as a constant term is absent in (2.2), we see that

\[ w^2(z) = 4 \prod_{i=1}^{n} \prod_{k_i=1}^{l_i} \frac{i^2}{(z - z_{k_i})^2}, \quad (2.10) \]

where \( w(z) \) is a rational solution having \( l_i \) poles with residue \( i \) \((i = 1, 4)\) and \( l_j \) poles with residue \(-j\) \((j = 2, 3)\). Thus we easily obtain

\[ F(z) = 4 \prod_{i=1}^{n} \prod_{k_i=1}^{l_i} (z - z_{k_i})^i, \quad (2.11) \]

and

\[ p \overset{\text{def}}{=} \deg(F(z)) = \sum_{i=1}^{4} i^2 l_i + \sum_{j=2}^{3} j^2 l_j, \quad \deg(G(z)) = \deg(F(z)) - 1. \quad (2.12) \]

The second correlation in (2.12) follows from (2.3). The polynomials \( F(z), G(z) \) can be written as

\[ F(z) = \sum_{k=0}^{p} r_k z^{p-k}, \quad G(z) = \sum_{k=0}^{p-1} q_k z^{p-1-k}. \quad (2.13) \]

Substituting (2.13) into (2.8) we find \( q_0^2 = p, \ q_0 = -\beta \). Consequently,

\[ \sum_{i=1}^{4} i^2 l_i + \sum_{j=2}^{3} j^2 l_j = \beta^2. \quad (2.14) \]

Further using the formula for the total sum of the residues of a meromorphic function in the complex plane we get

\[ \sum_{i=1}^{4} i l_i - \sum_{j=2}^{3} j l_j = -\beta. \quad (2.15) \]

The correlations (2.14) and (2.13) should hold for any rational solution. These correlations are not satisfied at \( \beta = 1 \). As a result the equation (1.1) does not have rational solutions at \( \beta = 1 \) and consequently at \( \beta = 1 \pm 3k, \ k \in \mathbb{N} \). Assuming the contrary and applying the Bäcklund transformations \( T_{-\beta} \) to a supposed rational solution sufficient amount of times one can obtain the rational solution at \( \beta = 1 \). What is impossible. Thus the rational solutions of (1.1) necessarily exist at \( \beta \in \mathbb{Z}/\{1 \pm 3k, \ k \in \mathbb{N} \cup \{0\}\} \), what follows from the preceding remarks and from the correlation (2.15), where the left-hand side is an integer. The sufficiency follows from the Bäcklund transformations for (1.1). Sequently applying the Bäcklund transformations \( T_{2-\beta} \) and \( T_{-1-\beta} \) to the "seed" solution \( w(z; 0) = 0 \) we can construct the rational solutions of (1.1) given by (2.5).
Starting with $T_{2-\beta}$ we obtain the rational solutions at $\beta = \beta_{n}^{(1)}$, which we will refer to as the first family. While starting with $T_{-1-\beta}$ we obtain the rational solutions at $\beta = \beta_{n}^{(2)}$, which we will refer to as the second family. Note that $\{\beta_{n}^{(1)}\} \cup \{\beta_{n}^{(2)}\} \cup \{0\} = \mathbb{Z}/\{1 \pm 3k, k \in \mathbb{N} \cup 0\}$.

Remark 1. Expressions (2.6) can be rewritten as

\[
\beta_{n}^{(1)} = (-1)^{n} \frac{3n}{2} + \frac{\delta_{n,odd}}{2}, \\
\beta_{n}^{(2)} = (-1)^{n-1} \frac{3n}{2} + \frac{\delta_{n,odd}}{2}
\]

with $\delta_{n,odd}$ being the Kronecker delta

\[
\delta_{n,odd} = \begin{cases} 
1, & n \pmod{2} = 1; \\
0, & n \pmod{2} = 0.
\end{cases}
\]

Thus we see that the rational solutions of (1.1) can be described with the help of two families of polynomials. The polynomials $\{Q_{n}(z)\}$ we will call the first family and $\{R_{n}(z)\}$ the second. By $p_{n}^{(1)}$ (p_{n}^{(2)}) denote the degree of $Q_{n}(z)$ ($R_{n}(z)$). Analyzing the expression (2.5) we understand that $Q_{n}(z)$ and $R_{n}(z)$ can be defined as monic polynomials. Thus each polynomial can be presented in the form

\[
Q_{n}(z) = \sum_{k=0}^{p_{n}^{(1)}} A_{n,k}^{(1)} z^{p_{n}^{(1)} - k}, \quad A_{n,0}^{(1)} = 1, \\
R_{n}(z) = \sum_{k=0}^{p_{n}^{(2)}} A_{n,k}^{(2)} z^{p_{n}^{(2)} - k}, \quad A_{n,0}^{(2)} = 1.
\]

The first non-trivial solutions of (1.1) are $w(z; -1) = 1/z$ and $w(z; 2) = -2/z$. Hence it can be set $Q_{0}(z) = R_{0}(z) = 1$, $Q_{1}(z) = z$, $R_{1}(z) = z^{2}$. Suppose $a_{n,k}^{(j)}$ ($1 \leq k \leq p_{n}^{(j)}$) are the roots of the polynomial $Q_{n}(z)$ and $R_{n}(z)$, accordingly, then by $s_{n,m}^{(j)}$ we denote the symmetric functions of the roots

\[
s_{n,m}^{(j)} \overset{\text{def}}{=} \sum_{k=1}^{p_{n}^{(j)}} (a_{n,k}^{(j)})^{m}, \quad m \geq 1, \quad j = 1, 2.
\]

Let us show that it is possible to derive the polynomials $Q_{n}(z), R_{n}(z)$ using the power expansion (2.3). For convenience of use let us present this series in the form

\[
w(z; \beta) = \sum_{m=0}^{\infty} c_{\beta,-(m+1)} z^{-m-1},
\]

where $c_{\beta,-(m+1)} = 0$ ($m \geq 1$) unless $m$ is divisible by 5. Our next step is to express $s_{n,m}^{(j)}$ and $p_{n}^{(j)}$ through coefficients of the series (2.20).
\textbf{Theorem 2.2.} Let $c_{-m-1}^{(j)}(i)$ be the coefficient in expansion (2.20) at $\beta = \beta_1^{(j)}$. Then for each $n \geq 2$ and $j = 1, 2$ the following relations hold

\begin{align*}
p_n^{(j)} &= (-1)^{j-1} \sum_{i=1}^{n} (-1)^{i-1} c_{-i}^{(j)}(i), \\
s_{n,m}^{(j)} &= (-1)^{j-1} \sum_{i=2}^{n} (-1)^{i-1} c_{-(m+1)}^{(j)}(i), \quad m \geq 1.
\end{align*}

\textbf{Proof.} Without loss of generality we will prove the theorem for the polynomials \{Q_n(z)\}. Up to the end of the proof the upper index $j$ will be omitted. As far as $Q_n(z)$ is a monic polynomial, then it can be written in the form

\[ Q_n(z) = \prod_{k=1}^{p_n} (z - a_{n,k}). \]

Note that possibly $a_{n,k} = a_{n,l}, k \neq l$. This equality implies that

\[ \frac{Q_n'(z)}{Q_n(z)} = \sum_{k=1}^{p_n} \frac{1}{z - a_{n,k}}. \]

Substituting (2.24) into the expression (2.25) yields

\[ w(z; \beta_n) = (-1)^n \left( \sum_{k=1}^{p_n} \frac{1}{z - a_{n-1,k}} - \sum_{k=1}^{p_n} \frac{1}{z - a_{n,k}} \right). \]

Expanding this function in a neighborhood of infinity we get

\begin{align*}
w(z; \beta_n) &= (-1)^n \left( \frac{b_{n-1}}{z} - \frac{b_n}{z} \right) + (-1)^n \sum_{m=0}^{\infty} \left[ \sum_{k=1+b_{n-1}}^{p_n} (a_{n-1,k})^m \right. \\
&\quad - \left. \sum_{k=1}^{p_n} (a_{n,k})^m \right] z^{-(m+1)} \quad |z| > \max\{\bar{a}_{n-1}, \bar{a}_n\},
\end{align*}

where $\delta_{0,a_{n,k}}$ is the Kronecker delta. The first or the second term in (2.26) are present only if the polynomials $Q_{n-1}(z), Q_n(z)$ have zero roots, accordingly. In our designations the previous expression can be rewritten as

\[ w(z; \beta_n) = (-1)^{n-1} \frac{p_n - p_{n-1}}{z} + (-1)^{n-1} \sum_{m=1}^{\infty} [s_{n,m} - s_{n-1,m}] z^{-(m+1)}, \quad |z| > \max\{\bar{a}_{n-1}, \bar{a}_n\}. \]

The absence of a zero term in sum is essential only at $m = 0$. Comparing expansions (2.21) and (2.24) we obtain the equalities

\begin{align*}
p_n - p_{n-1} &= (-1)^{n-1} c_{-1}(n), \\
s_{n,m} - s_{n-1,m} &= (-1)^{n-1} c_{-(m+1)}(n), \quad m \geq 1.
\end{align*}
In these expressions \( c_{-(m+1)}(n) \) \( \equiv \) \( c_{\beta,(m+1)} \), \( m \geq 0 \). Decreasing the first index by one in (2.28) and adding the result to the original one yields

\[
\begin{align*}
p_n - p_{n-2} &= (-1)^{n-1}[c_{-1}(n) - c_{-1}(n-1)], \\
s_{n,m} - s_{n-2,m} &= (-1)^{n-1}[c_{-(m+1)}(n) - c_{-(m+1)}(n-1)],
\end{align*}
\]

(2.29)

Note that \( c_{\beta,-(m+1)} = 0 \), \( m \geq 1 \) and \( a_{1,1} = 0 \). Then proceeding in such a way we get the required relations (2.21) and (2.22).

Since \( c_{\beta,-1} = -\beta \), we get that the degrees of the polynomials \( Q_n(z) \) and \( R_n(z) \) are

\[
\begin{align*}
p_n^{(1)} &= \frac{1}{2} \sum_{i=1}^{n} (3i - \delta_{i,odd}) = \frac{n(3n + 2) - \delta_{n,odd}}{4}, \\
p_n^{(2)} &= \frac{1}{2} \sum_{i=1}^{n} (3i + \delta_{i,odd}) = \frac{n(3n + 4) + \delta_{n,odd}}{4}.
\end{align*}
\]

(2.30)

We also observe that expression (2.30) can be rewritten in terms of \( \beta_n^{(j)} \)

\[
\begin{align*}
p_n^{(1)} &= \sum_{i=1}^{n} |\beta_n^{(1)}| = \frac{k(k+1)}{2} - \frac{1}{2} \left[ \frac{k+1}{3} \right]^2 - \frac{3}{2} \left[ \frac{k+1}{3} \right]^{2}, \quad k \equiv |\beta_n^{(1)}|, \\
p_n^{(2)} &= \frac{1}{2} \sum_{i=1}^{n} |\beta_n^{(2)}| = \frac{k(k+1)}{2} + \frac{1}{2} \left[ \frac{k+2}{3} \right]^2 - \frac{3}{2} \left[ \frac{k+2}{3} \right]^{2}, \quad k \equiv |\beta_n^{(2)}|.
\end{align*}
\]

(2.31)

The first few \( \beta_n^{(j)} \) and \( p_n^{(j)} \) are given in Table 2.1

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|
| \( \beta_n^{(1)} \) | -1 | 3 | -4 | 6 | -7 | 9 | -10 | 12 | -13 | 15 | -16 | 18 |
| \( p_n^{(1)} \) | 1 | 4 | 8 | 14 | 21 | 30 | 40 | 52 | 65 | 80 | 96 | 114 |
| \( \beta_n^{(2)} \) | 2 | -3 | 5 | -6 | 8 | -9 | 11 | -12 | 14 | -15 | 17 | -18 |
| \( p_n^{(2)} \) | 2 | 5 | 10 | 16 | 24 | 33 | 44 | 56 | 70 | 85 | 102 | 120 |

Theorem 2.2 enables us to prove the following theorem.

**Theorem 2.3.** All the coefficients \( A_{n,m}^{(1)} \) \( (A_{n,m}^{(2)}) \) of the polynomial \( Q_n(z) \) \( (R_n(z)) \) can be obtained with the help of \( p_n^{(1)} \) \( (p_n^{(2)}) \) first coefficients of the expansion (2.20) for the solutions of (1.1).

**Proof.** Again we omit the upper index, when it does not cause any contradiction. For every polynomial there exists a connection between its coefficients and the symmetric functions of its roots \( s_{n,m} \). This connection is the following

\[
mA_{n,m} + s_{n,1}A_{n,m-1} + \ldots + s_{n,m}A_{n,0} = 0, \quad 1 \leq m \leq n,
\]

(2.32)

Taking into account that in our case \( A_{n,0} = 1 \) we get

\[
A_{n,m} = -\frac{s_{n,m} + s_{n,m-1}A_{n,1} + \ldots + s_{n,1}A_{n,m-1}}{m}, \quad 1 \leq m \leq n.
\]

(2.33)
The function \( s_{n,m} \) can be derived using the expression (2.30). Hence recalling the fact that (2.20) is exactly (2.21) we obtain

\[
s_{n,m} = 0, \quad m \in \mathbb{N} / \{5l, \ l \in \mathbb{N}\},
\]

\[
s^{(j)}_{n,5l} = (-1)^{j-1} \sum_{i=2}^{n} (-1)^{i-1} c_{n,(5l+1)}^{(j)}(i), \quad l \in \mathbb{N}.
\] (2.34)

Substituting this into (2.33) yields

\[
A_{n,m} = 0, \quad m \in \{1, 2, \ldots, n(n + 1)/2\} / \{5l, \ l \in \mathbb{N}\},
\]

\[
A_{n,5l} = \frac{1}{5l} \left\{ s_{n,5l} + s_{n,5l-5} A_{n,5} + \ldots + s_{n,5} A_{n,5l-5} \right\}, \quad l \in \mathbb{N}, 5l \leq p_n.
\] (2.35)

Thus we see that the coefficients \( A_{n,k} \) of the polynomial \( Q_n(z) \) \( (R_n(z)) \) are uniquely defined by coefficients \( c_{\beta,-5l-1} \) of the expansion (2.3). This completes the proof.

**Remark 1.** At given \( n \geq 2 \) the functions \( s^{(j)}_{n,m} \) \((m > p_0^{(j)})\) do not contain any new information about the roots of corresponding polynomial \( (Q_n(z) \text{ or } R_n(z)) \) as the following correlation holds

\[
s^{(j)}_{n,m} + s^{(j)}_{n,m-1} A^{(j)}_{n,1} + \ldots + s^{(j)}_{n,m-p_0} A^{(j)}_{n,p_0} = 0, \quad m > p_n, \ m \in \mathbb{N}, \ j = 1, 2. \] (2.36)

**Remark 2.** Expression (2.34) defines the structure of the polynomial \( Q_n(z) \) \( (R_n(z)) \). Namely if \( p_0^{(j)} \) is divisible by 5, then \( Q_n(z) \) \( (R_n(z)) \) is a polynomial in \( z^5 \). Otherwise (if \( p_0^{(j)} \) is not divisible by 5) \( Q_n(z)/z^r \) \( (R_n(z)/z^r) \) is a polynomial in \( z^5 \), where \( r = p_0^{(j)} \mod 5 \).

| Table 2.2: Polynomials \( Q_n(z) \) |
|-------------------------------------|
| \( Q_0 = 1 \)                     |
| \( Q_1 = z \)                     |
| \( Q_2 = z^2 \)                   |
| \( Q_3 = z^3 \)                   |
| \( Q_4 = z^4(5^2 - 504)^2 \)     |
| \( Q_5 = z(5^{20} - 3276 z^{15} + 6604416 z^{10} + 33286256664 z^5 - 119830523904) \) |
| \( Q_6 = (z^{15} - 6552 z^{10} - 13208832 z^5 - 951035904) \) |
| \( Q_7 = z^{40} - 29952 z^{35} + 203793408 z^{30} + 3066139754496 z^{25} + 5234197284126720 z^{20} + 36006491762989203456 z^{15} - 3574462636834928197632 z^{10} - 7206116675859215246426112 z^5 + 129710100165465874435670016 \) |

Calculating the coefficients \( c_{\beta,-5l-1} \) in (2.30) and the symmetric functions \( s^{(j)}_{n,5l} \) one can find the coefficients \( A^{(j)}_{n,5l} \) of the polynomials \( Q_n(z) \) and \( R_n(z) \). It can be proved that \( c_{\beta,-5l-1} \) is a polynomial in \( \beta \). Hence, taking into account (2.16) we see that all the coefficients \( A^{(j)}_{n,5l} \) can be presented in the form

\[
A^{(j)}_{n,5l} = B^{(j)}_{n,5l} + \delta_{n,\text{odd}} C^{(j)}_{n,5l}
\] (2.37)
with $B_{n,5}^{(j)}$ and $C_{n,5}^{(j)}$ being polynomials in $n$. For example,

\[
A_{n,5}^{(1)} = -\frac{3}{320} n (3n + 8) (3n + 2) (3n - 4) (n^2 - 4) + \frac{\delta_{n, \text{odd}}}{320} (2.38)
\]

\[
\times (405 n^4 + 540 n^3 - 495 n^2 - 450 n + 495)
\]

\[
A_{n,5}^{(2)} = -\frac{3}{320} n (3n + 4) (3n - 2) (3n - 8) (n + 4) (n + 2) - \frac{\delta_{n, \text{odd}}}{320} (2.39)
\]

\[
\times (405 n^4 + 1080 n^3 + 315 n^2 - 540 n + 315)
\]

### Table 2.3: Polynomials $R_n(z)$

| $R_n$ | Description |
|-------|-------------|
| $R_0$ | $1$ |
| $R_1$ | $z^2$ |
| $R_2$ | $z^5 + 36$ |
| $R_3$ | $(z^5 - 144)^2$ |
| $R_4$ | $z (z^{15} - 1152 z^{10} + 1824768 z^5 + 131383296)$ |
| $R_5$ | $z^4 (z^{10} - 3168 z^5 - 3193344)^2$ |
| $R_6$ | $z^8 (z^{25} - 15840 z^{20} + 63866880 z^{15} + 708155965440 z^{10} + 192217762806726656)$ |
| $R_7$ | $z^4 (z^{20} - 22176 z^{15} - 95001984 z^{10} - 902898855936 z^5 + 303374015594496)^2$ |

Additionally, the coefficients $A_{2m-1,5}^{(j)}$ and $A_{2m,5}^{(j)}$ are polynomials in $m$. Some of them for the first family of polynomials are the following

\[
A_{2m-1,5}^{(1)} = -\frac{3}{5} m (m - 1) (m - 2) (3m + 4) (3m + 1) (3m - 2) (2.40)
\]

\[
A_{2m,5}^{(1)} = -\frac{3}{5} m (m^2 - 1) (3m + 4) (3m + 1) (3m - 2) (2.41)
\]

\[
A_{2m-1,10}^{(1)} = \frac{9}{50} m (m^2 - 1) (m - 2) (3m + 4) (3m + 1) (3m - 2) (2.42)
\]

\[
\times (3m - 5) (9 m^4 - 12 m^3 - 143 m^2 + 98 m + 1008)
\]

\[
A_{2m,10}^{(1)} = \frac{9}{50} m (m^2 - 1) (3m + 4) (3m + 1) (3m - 2) (3m^2 + m - 9) (2.43)
\]

\[
\times (9 m^4 + 6 m^3 - 125 m^2 - 42 m + 560)
\]

The same coefficients for the second family are given by

\[
A_{2m-1,5}^{(2)} = -\frac{3}{5} m (m^2 - 1) (3m + 2) (3m - 1) (3m - 4) (2.44)
\]

\[
A_{2m,5}^{(2)} = -\frac{3}{5} m (3m + 2) (3m - 1) (3m - 4) (m + 2) (m + 1) (2.45)
\]
\[ A_{2m-1,10}^{(2)} = \frac{9}{50} m (m^2 - 1) (3m + 2) (3m - 1) (3m - 4) (3m^2 - m - 9) \times (9m^4 - 6m^3 - 125m^2 + 42m + 560) \] (2.46)

\[ A_{2m,10}^{(2)} = \frac{9}{50} m (m^2 - 1) (3m + 5) (3m + 2) (m + 2) (3m - 1) (3m - 4) \times (9m^4 + 12m^3 - 143m^2 - 98m + 1008) \] (2.47)

Note that the expressions (2.40), (2.41) are equivalent to (2.38). Similarly, (2.44), (2.45) are equivalent to (2.39). The polynomials \( Q_n(z) \) and \( R_n(z) \) can be written in the form

\[ Q_n(z) = \sum_{l=0}^{[p_n^{(1)}/5]} A_{n,5l}^{(1)} z^{p_n^{(1)}-5l}, \quad A_{n,0}^{(1)} = 1, \] (2.48)

\[ R_n(z) = \sum_{l=0}^{[p_n^{(2)}/5]} A_{n,5l}^{(2)} z^{p_n^{(2)}-5l}, \quad A_{n,0}^{(2)} = 1. \]

The first few of them are gathered in Tables 2.2 and 2.3.

### 3 Rational solutions of the equation studied

Substituting the polynomials \( Q_{n-1}(z), Q_n(z) \) and \( R_{n-1}(z), R_n(z) \) into (2.6) we obtain the rational solutions of the equation (1.1). Some of them are given below.

\[ w(z, -1) = \frac{1}{z}, \quad w(z; 3) = \frac{-3}{z}, \quad w(z, -4) = \frac{4}{z}, \] (3.1)

\[ w(z, 0) = -\frac{6(z^5 + 336)}{z(z^5 - 504)}. \] (3.2)

\[ w(z, 2) = -\frac{2}{z}, \quad w(z, -3) = \frac{3(z^5 - 24)}{z(z^5 + 36)} \] (3.3)

\[ w(z, 5) = -\frac{5z^4(z^5 + 216)}{(z^5 + 36)(z^5 - 144)}, \] (3.4)

\[ w(z, -6) = \frac{6(z^{20} - 576z^{15} - 912384z^{10} - 459841536z^5 - 3153199104)}{z(z^{15} - 1152z^{10} + 1824768z^5 + 131383296)(z^5 - 144)} \] (3.5)

Solutions (3.1), (3.2) belong to the first family, while (3.3) – (3.5) to the second.
4 Conclusion

In this paper we have studied rational solutions of the equation (1.1). We have given necessary and sufficient condition for the existence of the rational solutions. Further we have derived special polynomials associated with these solutions. This property of the equation (1.1) is similar to that of $P_2 - P_6$. The rational solutions have been subdivided into two sequences, each with its own family of polynomials. Using the power expansion in a neighborhood of infinity for the solutions of (1.1) we have found the degrees of the polynomials and formulas for their coefficients. We have shown that special solutions of the Fordy - Gibbons, the Caudrey - Dodd - Gibbon and the Kaup - Kupershmidt equations can be expressed via solutions of (1.1). Consequently, there exist solutions of these equations in terms of the polynomials constructed.

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