General response theory of topologically stable Fermi points in the presence of disorders

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We develop a general response theory of gapless Fermi points with nontrivial topological charges, which asserts that the topological character of the Fermi points is embodied as the terms with discrete coefficients proportional to the corresponding topological charges. Applying the theory to the effective non-linear sigma models for topological Fermi points with disorders in the framework of replica approach, we derive rigorously the Wess-Zumino terms with the topological charge being their levels in the two complex symmetry classes of A and AIII. In particular, a Wess-Zumino-Witten model, as a conformal field theory, is shown to be an exact result for a two-dimensional Dirac point under disorders respecting the chiral symmetry. We also address a qualitative connection of topological charges of Fermi points in the real symmetry classes to the topological terms in the non-linear sigma models, based on the one-to-one classification correspondence.

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Introduction Recently, topological semimetals have been among the hottest topics in condensed matter physics, including graphene, Weyl semimetals, and the surface states of topological insulators and superconductors [15]. This may be attributed to not only their practical applications based on their exotic transport properties, but also broad interests of anomalous currents in condensed matter physics, quantum field theory anomalies, and topological characters of these gapless modes [8–11]. As is known, gapless Fermi points in topological semimetals can be classified by their topological charges with respect to their symmetries [12–16]. Although various investigations have been made on exploring phenomena and classifications of these topological points, implications of the topological charges to quantum field theories are still badly awaited to be explored, which is obviously of fundamental importance and interdisciplinary interest. Mainly motivated by this, we here first establish a general connection between the topological character of the topological Fermi points and topological terms with discrete coefficients in its effective response theory, being coupled to external sources. These topological terms correspond usually to anomalous transport properties of these Fermi points, and are related to quantum field theory anomalies, namely classical symmetries being broken by quantization [17]. For instance, in Weyl semimetals, the A phase of Helium-3 or the surface of a four-dimensional topological insulator, there are Weyl points with nontrivial Chern numbers, as their topological charges, for the Berry bundle on a two-dimensional sphere enclosing them in $k$ space, which leads to the abelian chiral anomaly in the $U(1)$-response theory with anomalous currents [10, 17, 22]. Then we apply our general theory to the response theories risen up in disordered Fermi points with nontrivial topological charges, by using the replica approach [23, 25], which is of both practical and theoretical meaningfulness. On one hand, a variety of disorders are ubiquitous in most real materials, while, on the other hand, as we will see, anomalies of non-abelian gauge theories are naturally related to such condensed matter problems [24, 55]. As our main results, the integer topological charge $\nu$ of Fermi points in the complex symmetry classes of A and AIII in the Altland-Zirnbauer (AZ) classification [36, 37] is rigorously shown to lead to the Wess-Zumino terms (WZ terms) at level $\nu$ [26, 28]. For the class A, the emergence of the WZ terms is related to the parity anomaly in odd dimensions [38]; while, for the class AIII, it is associated with the non-abelian anomaly in even dimensions [26, 27, 51]. Finally, we also address the qualitative relationship of topological charges in the eight real AZ classes to the topological terms, which is strongly supported by the one-to-one correspondence of their classifications.

Stable equivalence and universal responses Let $\mathcal{H}(k)$ be a Hamiltonians in the momentum space, which can be regarded as a mapping from $k$ space to hermitian matrices, and assume that there are gapless points inside a finite region with the chemical potential being set as zero. Note that the energy gap is open far away from the origin of the $k$ space. $\mathcal{H}_1(k)$ and $\mathcal{H}_2(k)$ are two such Hamiltonians, whose dimensions may be different. The stable equivalence between $\mathcal{H}_1$ and $\mathcal{H}_2$ may be defined in the following way. After adding an arbitrary number of pairs of valence and empty bands to $\mathcal{H}_j$, namely $\tilde{\mathcal{H}}_j(k) = \mathcal{H}_j(k) \oplus \sigma_3 \cdots \oplus \sigma_3$ with $\sigma_3$ being the third Pauli matrix, if $\mathcal{H}_1(k)$ and $\mathcal{H}_2(k)$ can be smoothly deformed to each other without closing the gap far away from the origin of the $k$ space, $\mathcal{H}_1(k)$ and $\mathcal{H}_2(k)$ are stably equivalent, $\mathcal{H}_1(k) \approx \mathcal{H}_2(k)$ [39, 41]. If a set of symmetries of the Hamiltonians is required for the smooth deformations, then the two Hamiltonians are said to be stably equivalent under the symmetries. For the ten AZ symmetry
classes, distinct collections of stably equivalent gapless Hamiltonians in every AZ class are identified by their topological charges $\nu$ that are symmetry-related topological invariants formulated on the gapped spheres enclosing from transverse dimensions in the gapped regions in $k$ space $\mathbf{12, 13}$. In short two gapless Hamiltonians in a symmetry class with the same topological charge are stably equivalent in the symmetry class.

We consider in general a response theory of such a gapless Hamiltonian $H = \int dk \psi(k) H(k) \psi(k)$ with a nontrivial topological charge $\nu$, which is coupled to a sigma field $Q(x)$, describing low-energy degrees of freedom, by $-g \int \psi Q \psi$. At a point $x$ in the real space, $Q(x)$ is in a given target manifold, for instance, which is $U(2N)/U(N) \times U(N)$ for the non-linear model (NLσM) of disordered gapless modes in class A, and $U(N)$ for class AIII. The low-energy effective theory is given by integrating over the fermionic fields,

$$- \ln \frac{\text{Det}(-H(k) + gQ(x))}{\text{Det}H(k)} = \sum_j \lambda_j S^{(j)}[Q],$$

namely $S_{eff}[Q] = \sum_j \lambda_j S^{(j)}[Q]$, where $\lambda_j$ is the coefficient of the term $S^{(j)}$, which is a product of $Q$s and derivatives of $Q$. We consider that one of the terms, $S_{top}[Q] = \lambda_1 S^{(1)}[Q]$, is originated from the stable topological property of $H(k)$, and thus is invariant under the smooth deformations of $H(k)$ specified by the stable equivalence in Eq. $\mathbf{1}$. Accordingly, it is expected that $\lambda_1$ is a function of $\nu$, i.e., $\lambda_1(\nu)$, which is actually proportional to $\nu$: $\lambda_1(\nu) = a \nu$ with $a$ being a constant. This is because that the $H(k)$ can always be smoothly deformed to be a multiple of $\nu$ identical gapless Hamiltonians with unit topological charge, $H(k) \approx \oplus_{n=1}^{\nu} H_1(k)$, after adding sufficient number of trivial bands. Applying the deformations to Eq. $\mathbf{1}$, we find that $S_{top}[Q] = \nu S^{(1)}[Q]$, as $S_{top}[Q]$ is invariant during the whole deformation process and the diagonalization of $\oplus_{n=1}^{\nu} H_1(k)$ on the left hand of Eq. $\mathbf{1}$ is translated to the summation on the right. As a result, it is found that

$$S_{top}[Q] = \nu S^{(1)}[Q],$$

where $\nu$ has been absorbed into $S^{(1)}$ for convenience.

Wess-Zumino terms of the class A Now we apply Eq. $\mathbf{2}$ to the NLσMs of topological Fermi points with disorders through the replica trick, starting with the AZ class A that possesses no any discrete symmetry. As is known, nontrivial topological points in the class A can exist only in odd dimensions, $d = 2n + 1$, due to the Bott periodicity $\mathbf{20}$. The corresponding topological charge $\nu_A$ of a Fermi point is given as the Chern number of the Berry bundle of occupied bands on the gapped $2n$-dimensional sphere enclosing the Fermi point in $k$ space $\mathbf{21}$. For example, a formula for calculating the topological charge of the Weyl point $\mathcal{H}_W = \sigma \cdot \mathbf{k}$ is given by

$$\nu_A[G_W] = \frac{1}{24\pi^2} \int_{S^3} \text{tr}(G_W dG_W^{-1}(\omega, k))^3,$$

where $G_W = 1/(i\omega - \mathcal{H}_W)$ is the imaginary Green’s function, and the $S^3$ is a three-dimensional sphere chosen in $(\omega, k)$ space enclosing the gapless points. In the class A, the sigma field

$$Q \in BU = \frac{U(2N)}{U(N) \times U(N)}$$

with $N$ being the number of replicated systems, describing low-energy degrees of freedom near a saddle point after applying the mean-field theory in the standard replica method. The topological term in Eq. $\mathbf{2}$ for a gapless $\mathcal{H}_A(k)$ with an integer topological charge $\nu_A \in \mathbb{Z}$ is the WZ term at level $\nu_A$,

$$S^A_{WZ}[Q] = \nu_A C_d \int_{D^{d+1}} \text{tr} \tilde{Q}(dQ)^{d+1},$$

where $C_d = \frac{2 \pi^2}{(2\pi)^{d+2-1}}$, and $\tilde{Q}(x, \tau)$ is a continuous extension of $Q(x)$ along the parameter $\tau \in [0, 1]$ with $\tilde{Q}(x, 1) = Q(x)$ and $Q(x, 0)$ being constant. Since the homotopy group $\pi_{2n+1}(BU) = 0$, the extension is always possible. Accordingly, the original real space is extended to the $(d + 1)$-dimensional disk $D^{d+1}$, whose boundary $S^d$ is assumed to be the original real space after compactification. The difference of the values of Eq. $\mathbf{5}$ for two extensions is $2\pi i \nu_A N$ with integer $N$ being the winding number difference of the two extensions recalling that $\pi_{2n}(BU) \cong \mathbb{Z}$, which justifies that the WZ-term is well defined $\mathbf{20}$. In particular, the coefficient of a WZ term can only take discrete values labelled by its level $m$, which is perfectly in consistence with a fact that the topological charge as a topological invariant is an integer, considering $m = \nu_A$ in Eq. $\mathbf{5}$. It is noted that Eq. $\mathbf{4}$ can be argued from the boundary-bulk correspondence of a disordered $(2n + 2)$-dimensional Chern insulator $\mathbf{11}$.

To prove Eq. $\mathbf{5}$, as we discussed above Eq. $\mathbf{2}$, it is sufficient to consider merely the case for unit topological charge $\nu_A = 1$, which can be realized by Dirac type Hamiltonian, $H(k) = \sum_{j=1}^{2n+1} k_j \Gamma_j (2n+1)$ with $\Gamma_j (2n+1)$ being $2n \times 2n$ Dirac matrices, satisfying $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$. We adopt the Hamiltonian of a Weyl point $\mathcal{H}(k) = \mathbf{k} \cdot \sigma$ with $d = 3$ to exemplify the proof, and it is straightforward to see its validity in any odd dimensions. The sigma field can be explicitly expressed as $Q = T \tau_5 T^{-1}$, where $T(x) \in U(2N)$ and $\tau_3$ is the third Pauli matrix acting on the retarded-advanced space. Before evaluating the functional determinant in Eq. $\mathbf{1}$, it is helpful to express the WZ action in terms of $T$. It is straightforward, although tedious, to check that

$$\text{tr} \tilde{Q}(dQ)^4 = \frac{8}{3} \text{dtr}(T^{-1} dT \tau_3)^3 - 8 \text{dtr} \tau_3 (T^{-1} dT)^3,$$
which implies that the WZ action for unit $\nu_A$, in terms of $T$, can be written explicitly in the original real space $S^3$ as

$$S_{WZ}^A = \frac{i\nu_A}{48\pi} \int_{S^3} \text{tr}(T^{-1}dT\tau_3)^3 - 3\text{tr}T_3(T^{-1}dT)^3.$$ 

For brevity, we define the projectors $P_{\pm} = (1 \pm \tau_3)/2$ into advanced and retarded spaces and $A = T^{-1}dT$, accordingly the action is translated to be

$$S_{WZ}^A = \frac{i\nu_A}{2}(S_{CS}[AP_+] - S_{CS}[AP_-]),$$

where $S_{CS}[A] = \frac{i}{4\pi} \int_{S^3} [\text{tr}A dA + \frac{2}{3} (A)^3]$ is the Chern-Simons (CS) term. The Chern-Simons expression has been derived in studying a single disordered Weyl point in a Weyl semimetal, without recognizing it is actually a WZ term [12].

Now our aim is to deduce Eq. (5) from the functional determinant $\text{Det}(-\tilde{k} + i\Delta Q)$ with $\tilde{k} = \sigma \cdot k$, recalling Eq. (1) with $g = \Delta A$ in this case. After a unitary transformation we have $\text{Det}T^{-1}(-\tilde{k} + i\Delta Q)T = \text{Det}(-\tilde{k} + i\Delta A)$, which is equal to $\text{Det}(1 + G(\Delta,k)i\Delta A)$ with $G(\Delta,k) = 1/(i\Delta A - \tilde{k})$ as a propagator and $i\Delta A$ as a vertex. So the effective theory is given by a summation of one-loop Feynman diagrams, $S_{eff}[Q] = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \text{Tr}(G(\Delta,k)n\Delta A)^n$. It is well-known that the CS terms, Eq. (6), that we are looking for are related to the parity anomaly in odd dimensions [38], which provides us a clue to derive them from the Feynman diagrams, analogous to the derivation of the CS term for electromagnetic response of a $(2 + 1)$-dimensional Chern insulator [16, 17, 21], but with additional complications and new interpretations. Considering that $A(q)$ encodes low-energy freedoms of small momentum $q$, a term with the same form of Eq. (6) can be collected from the two-vertex loop and three-vertex one,

$$S = iN[G_+]S_{CS}[AP_+] + iN[G_-]S_{CS}[AP_-],$$

where $G_\pm = 1/(\pm i\Delta - \tilde{k})$, and $N[g] = \frac{i}{4\pi} \int_{M} \text{tr}(gdg^{-1})^3$ is an integration over the whole momentum space $M$.

To see the topological origin of Eq. (7), let us regard $\pm \Delta$ as the given values of $\omega$, and therefore viewed in $(\omega,k)$ space the momentum spaces of $G_\pm$ are two parallel three-dimensional surfaces $M_\pm$ sandwiching the Weyl point as shown in Fig. (1). Since $M_\pm$ are two infinite parallel surfaces, the union of them are equivalent to a three-dimensional sphere $S^3$ enclosing the gapless point as shown in Fig. (1), namely $\nu_A[G_W] = N[G_+] - N[G_-]$ with orientations of surfaces being considered. Since $|N[G_+]| = |N[G_-]|$ due to $\omega = \pm \Delta$, we find

$$N[G_+] = -N[G_-] = \nu_A[G_W]/2,$$

and it follows that Eq. (7) is just Eq. (6). In conclusion we have identified the topological terms of Eq. (2) for class $A$ is the WZ terms, Eq. (5), as the generalization to other odd dimensions is obvious.

**Wess-Zumino terms of the class AIII** A Hamiltonian $H_c(k)$ in class AIII has a chiral symmetry, namely there is a unitary matrix $\Gamma$ anti-commuting with $H_c(k)$, $\{H_c, \Gamma\} = 0$. Without loss of generality, we assume that $\Gamma = \text{diag}(1,-1)$, and accordingly $H_c = \text{antidiag}(H_-, H_+)$, $\psi = (\psi_-, \psi_+)^T$ and $\bar{\psi} = (\bar{\psi}_-, \bar{\psi}_+)$. Randomness respecting this symmetry leads to $Q = \text{antidiag}(M^{-1}, M)$ with $M \in U(N)$, where $N$ is the number of the replicated systems. Fermi points of $H_c(k)$ in the class AIII of nontrivial topological charge $\nu_c$ can exist only in even dimensions $d = 2n$ [12], and the topological term in Eq. (2) for $H_c$ is the WZ term of $U(N)$ at level $\nu_c$,

$$S_{WZ}^{\text{AIII}}[M] = \nu_c C_d' \int_{D^{d+1}} \text{tr}(\tilde{M}^{-1}d\tilde{M})^{2n+1},$$

where $C_d' = \frac{(-1)^{n}}{2^{(2n+1)/2}}$, and $\tilde{M}(x,\tau)$ is the continuous extension of $M$ through $\tau \in [0,1]$ with $M(x,1) = M(x)$ and $M(x,0)$ being constant. The extension independence of Eq. (2) has the same reasons as that of Eq. (5).

As discussed above Eq. (2), to derive the WZ terms of Eq. (9), it is sufficient to work out $H^D = \sum_{j=1}^{2n} \Gamma_j k_j$ ($\Gamma_j = \Gamma_j^{(2n+1)}/2$) with unit topological charge $[13]$. Let us use the four-dimensional case $H^D(k) = \tilde{k}$ with $\tilde{k} = \sum_{j=1}^{4} \Gamma_j^{(5)} k_j$ to exemplify the proof. First the Lagrangian is

$$\mathcal{L} = -\bar{\psi}_+ \tilde{k} \psi_+ - \bar{\psi}_- \tilde{k}_- \psi_- + g\bar{\psi}_- M \psi_+ + g\bar{\psi}_+ M^{-1} \psi_-, $$

with $\tilde{k}_\pm = \tilde{k}(1 \pm \Gamma_5)/2$. Since $\pi_4(U(N)) = 0$, $M(x) = e^{im(x)}$, where $m(x) \in u(N)$ with $u(N)$ being the Lie algebra of $U(N)$. Let us introduce a series of fields parametrized by $\tau \in [0,1]$, $\tilde{M}(\tau) = e^{i\tau m(x)}$, $\psi_+(\tau) = \tilde{M}(\tau) \psi_+$ and $\bar{\psi}_-(\tau) = \tilde{M}(\tau) \bar{\psi}_-$. And we further introduce

$$\mathcal{L}(\tau) = -\bar{\psi}_+(\tau)[\tilde{k}_+ - i\tilde{A}_+(\tau)]\psi_+(\tau) - \bar{\psi}_- \tilde{k}_- \psi_- + g\bar{\psi}_- \tilde{M}(1-\tau) \psi_+(\tau) + g\bar{\psi}_+(\tau) \tilde{M}^{-1}(1-\tau) \psi_-, $$

FIG. 1: The $(\omega,k)$ space with a Weyl point at the centre. $M_\pm$ is the $k$ space of $G_\pm$ and $S^3$ is chosen to enclose the Weyl point.
where \( A(\tau) = \tilde{M}(\tau) d\tilde{M}(-\tau) \), and the partition function \( Z(\tau) = \int D\psi(\tau) D\bar{\psi}(\tau) e^{-\int L(\tau)} \). Let \( \tau \) vary from \( \tau = 0 \) with \( L(0) = \mathcal{L} \) to \( \tau = 1 \) where the dependence of \( \mathcal{L}(1) \) on \( M \) is entirely encoded in \( A(1) = M dM^{-1} \). Under field variable transformations, \( \psi_+(\tau) \rightarrow \psi_+(\tau + d\tau) \) and \( \psi_-(\tau) \rightarrow \psi_-(\tau + d\tau) \), although \( L(\tau + d\tau) = \mathcal{L}(\tau) \), there exists a nontrivial Jacobian determinant \( J(\alpha, \tau) = 1 + \frac{dJ}{d\alpha} |_{\alpha = 0} \alpha \), noting that \( \delta \psi \psi \). So \( Z(\tau) \) satisfies the differential equation, \( dZ(\tau) = Z(\tau)(f_+d\tau + f_-d\tau) \) with \( f(\tau) = \frac{\partial J}{\partial \alpha} \), which implies \( Z(0) = Z(1)J \) with \( J = \exp(-\int_0^1 f(\tau)d\tau) \). The Jacobian determinant \( J(\alpha, \tau) \) depends on regularization schemes, and under the commonly adopted one for non-abelian anomalies, we have \( f(\tau) = \frac{1}{2\pi^2} \int S^4 tr[\text{mald}(DA + A^3/2)] \). Plugging in \( A(\tau) = \tilde{M}(\tau)d\tilde{M}(\tau) \), we have \( f(\tau) = \frac{1}{48\pi^2} \int S^4 tr[\tilde{M}(\tau)\frac{\partial \tilde{M}(\tau)}{\partial \tau}(\tilde{M}(\tau)d\tilde{M}(\tau))] \). Then in the effective action of \( M \), there exists the term \( \int_0^1 ds f(s) \) which is explicitly written as

\[
S[M] = \frac{1}{240\pi^2} \int D^5 \text{tr}(\tilde{M}^{-1}d\tilde{M})^5,
\]

where \( D^5 \) is naturally given by \( S^4 \times [0,1] \) with \( \tilde{M}(0) = 1 \) and \( \tilde{M}(1) = M \). Eq. (10) is exactly Eq. (9) when \( d = 4 \).

### 2d All III

For two dimensions, the Dirac matrices are just Pauli matrices and \( \mathcal{H}_{2d} = \sigma_1 k_x + \sigma_2 k_y \) with unit topological charge. The existence of WZ terms for \( \mathcal{H}_{2d} \) has been argued by Fendley when studying a p-wave triplet superconductor model in the same spirit of the above non-abelian anomaly [31]. In this case, the above regularization scheme suffers from a fact that coupling \( \psi_\pm \) to gauge fields independently in two dimensions is not well-defined. To see this, we represent the gauge field as \( A_\mu = (g^{-1} \partial_+ g, h^{-1} \partial_- h) \), where \( g, h \in U(N) \) and \( x_\pm = x_1 \pm x_2 \), or equivalently denote it concisely as \( (g, h) \). After a gauge transformation given by \( t(x) \in U(N) \), it is found that \( (g, h) \rightarrow (gt^{-1}, ht^{-1}) \), which means \( (g, h) \) is gauge equivalent to \( (gh^{-1}, 1) \) and \( (1, hg^{-1}) \) that couple to \( \psi_+ \) and \( \psi_- \), respectively. Thus in two dimensions it is plausible to adopt a gauge invariant regularization scheme, which, however, is not apparently equivalent to the one we used above. Furthermore there is an extra advantage working with this regularization, namely all the renormalizable terms are contained in \( J'[M] \) that can be calculated exactly. To evaluate \( Z[M] \), we construct a series of infinitesimal axial gauge transformations still parametrized by \( \tau \in [0,1] \), and accordingly define \( \psi_+(\tau) = \tilde{M}(\tau/2)\psi_+ + \tilde{M}(\tau/2)\psi_- \), \( \psi_-(\tau) = \tilde{M}(\tau/2)\psi_- \), \( \psi_+(\tau) = \tilde{M}(\tau/2)\psi_+ \), and \( \psi_-\psi_+ \)

\[
\mathcal{L}(\tau) = -\bar{\psi}_+ [k_+ \text{e}^{iA_+(\tau)}] \psi_+(\tau) - \bar{\psi}_- [k_- \text{e}^{iA_-\tau}] \psi_-(\tau),
\]

Now similar to the previous case, accumulating infinitesimal axial gauge transformations given by \( M(d\tau/2) \), we obtain the Jacobian determinant \( J'[M] \) for the finite transformation, \( M(1/2) \). In stead of calculating \( J'[M] \) directly, we note that \( J'[M] = Z_0^{-1} \int D\psi D\bar{\psi} \exp(-\int \bar{\psi} D\bar{\psi}) = e^{-S[A]} \), where \( D\mu \) is the covariant derivative with \( A_+ = M^{-1} A_+ M \) and \( A_- = 0 \). And the explicit expression of \( S[A] \) has been derived in Ref. [24] by solving the equations, \( D_\mu J_\mu = 0 \) and \( \epsilon_{\mu \nu} D_\mu J_\nu = \frac{1}{4} \epsilon_{\mu \nu} F_{\mu \nu} \), coming from the gauge invariant regularization scheme. It is exactly the well-known Wess-Zumino-Witten model in terms of \( M_{42} \),

\[
S = \frac{1}{8\pi} \int S^2 d^2 x tr[(M^{-1} d\mu M) + i \frac{1}{12\pi} \int D^3 tr(\tilde{M}^{-1} d\tilde{M})^3].
\]

Actually all the renormalizable terms in the effective action have been contained in Eq. (11) as an extra advantage of the two-dimensional model of \( \mathcal{H}_{2d} \). Recalling that \( Z = e^{-S}(Z(1)) \) and \( Z(1) = \int D\psi D\bar{\psi} \exp(-\int \bar{\psi} D\bar{\psi}) \) with \( A_+ = M\partial_+ M^{-1} \) and \( A_- = 0 \), the gauge invariance of \( Z(1) \) implies that terms contained in \( Z(1) \) are polynomials of \( F \), which begins with \( F^2 \) due to the spatial invariance symmetry. Since \( F_{\mu \nu} = \partial_-(M \partial_+ M^{-1}) \), the remaining terms in \( Z(1) \) are all higher order terms that are not renormalizable. It is concluded that the Wess-Zumino-Witten model of Eq. (11), as a conformal field theory, is an exact result for \( \mathcal{H}_{2d} \). However, it should be noted that only the coefficient of WZ term (the second one) in Eq. (11) has a topological origin, and therefore is determined by the topological charge of \( \mathcal{H}_{2d} \) in accord with Eq. (2) \((\nu = 1)\).

### Real AZ classes

We have above considered the two complex symmetry classes of the ten AZ classes in the framework of our general theory. Now let us address qualitatively the remaining eight real symmetry classes with a variety set of anti-unitary symmetries including the time-reversal and particle-hole symmetries [12]. Let \( g = 0, \ldots, 7 \) be the index of the eight classes. For the class \( q \), due to the constraints from the corresponding symmetries, the one-point Hamiltonian is in the manifold \( R_q \) called the qth classifying space, noting that the subscript \( n \) of \( R_q \) is an integer modulo 8. Under the symmetry-preserving randomness, the target space of the NL\( \sigma \)M is \( T_q \approx R_{4-q} \), and if the dimension is \( d \), the homotopy group is given by \( \pi_d(T_q) = \pi_0(R_{d-q+d}) \) with \( \pi_0(R_q) \approx Z, Z_2, Z_2, 0, 0, 0, 0, 0 \) for \( q = 0, \ldots, 7 \). There are two kinds of topological terms with discrete coefficients, consisting of WZ terms and \( Z_2 \)-\( \theta \) terms [44]. The WZ terms are possible when \( \pi_d(T_q) \approx Z_2, \pi_{d+1}(T_q) \approx Z, \) which are satisfied if \( d + 1 - q = 0 \) mod 4, while \( Z_2 \)-\( \theta \) terms exist when \( \pi_d(T_q) \approx Z_2 \), correspondingly \( 4 - q + d = 1 \) or 2 mod 8. On the other hand, the classification of a Fermi point in the q-th class in a d-dimensional k space is given by \( \pi_0(R_{q-(d+1)}) \approx Z_2 \approx Z \). Thus the Z topological charges
appear when \( q - (d+1) \equiv 0 \mod 4 \), which is just the condition for WZ terms. And \( \mathbb{Z}_2 \) topological charges exist when \( q - (d+1) \equiv 1 \mod 8 \) or \( q - (d+1) \equiv 2 \mod 8 \), exactly the conditions for \( \mathbb{Z}_2 - \theta \) terms, which is easy to check. To conclude, this one-to-one correspondence of topological charges to WZ and \( \mathbb{Z}_2 - \theta \) terms strongly suggest that the topological term in Eq. (2) for a Fermi point with topological charge \( \nu \in \mathbb{Z} \) is just a WZ term at level \( \nu \), and that for a Fermi point with nontrivial \( \mathbb{Z}_2 \) topological charge is just a \( \mathbb{Z}_2 - \theta \) term, even though it is extremely hard to analytically derive these topological terms due to the lack of unified expressions for these terms and complicated structures of the classifying spaces \( R_q \).

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