TESTING TREE-LEVEL PERTURBATION THEORY FOR LARGE-SCALE STRUCTURE WITH THE LOCAL LAGRANGIAN APPROXIMATION

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ABSTRACT

We test tree-level perturbation theory for Gaussian initial conditions with power spectra $P(k) \propto k^n$ by comparing the probability distribution function (PDF) for the density predicted by the Local Lagrangian Approximation (LLA) with the results of numerical gravitational clustering simulations. Our results indicate that our approximation correctly reproduces the evolved density PDF for $-3 \leq n \leq -1$ power spectra up to the weakly nonlinear regime, while it shows marginal agreement for power indices $n = 0$ and $+1$ in the linear regime and poor agreement beyond this point. This suggests that tree-level perturbation theory (as realized in the Local Lagrangian Approximation) can accurately predict the density distribution function for $-3 \leq n \leq -1$ but fails for $n \geq 0$.

Subject headings: galaxies: clustering, large-scale structure of universe

1. Introduction

One of the focal points in the study of large scale structure has been the evolution of $P(\rho)$, the one-point probability distribution (PDF) of the density field (Kofman et al. 1994, Juszkiewicz et al. 1994, Bernardeau and Kofman 1995, Protogeros and Scherrer 1996). Whereas in the linear regime, assuming Gaussian initial conditions, $P(\rho)$ scales up self-similarly by $D(t)$, the growing
mode solution, this is not the case once entering the weakly non-linear regime, defined by density contrast $\sigma = \left( \frac{\bar{\delta}}{\bar{\rho}} \right)^2 \approx 1$. This behavior may be attributed to coupling of different Fourier modes, which in the linear regime evolved independently, due to effects of non-local interactions in the density field evolution. Furthermore, multistreaming is expected to become important at this stage, contributing to the non-linear evolution of the PDF.

Major progress has been made in the past few years on understanding the quasi-linear evolution of the PDF, i.e., the evolution for $\sigma \lesssim 1$. In particular, a formalism has been developed by Bernardeau (1992) to calculate, to lowest order, the hierarchical amplitudes of the evolved density field (see the next section for a discussion). More recently, Protogeros and Scherrer (1996, PS hereafter) have derived an approximate method (the Local Lagrangian Approximation or LLA), which provides a simple analytic expression for the evolved PDF which reproduces, nearly exactly, the tree-level hierarchical amplitudes of the "true" evolution. In PS, the predictions of the Local Lagrangian Approximation were compared with the results of the "exact" gravitational evolution as calculated numerical in a gravitational clustering code. For Gaussian initial conditions with tophat smoothing and a scale free $n = -1$ initial power spectrum, the LLA predictions were found to be in excellent agreement with the "true" evolution. In this paper, we extend this comparison to a range of power spectra: $-3 \leq n \leq +1$.

Our motivation for undertaking this study is two-fold: i) On a practical level, the Local Lagrangian Approximation appears to provide an amazingly simple description of the evolution of the PDF for Gaussian initial conditions. We wish to determine if this method is accurate for all Gaussian initial conditions, or only for certain power spectra. ii) From a theoretical point of view, the accuracy of the Local Lagrangian Approximation can also be considered a test of tree-level perturbation theory, since the LLA reproduces, nearly exactly, all of the hierarchical amplitudes of the "true" final density field. Previous studies (references) have examined the validity of tree-level perturbation theory and the importance of higher-order contributions for a few low-order cumulants of the final density field, such as the variance and the skewness. However, the LLA provides a method to test all of the hierarchical amplitudes at once.

We present a short review of the LLA scheme in Section 2 and a description of the N-body simulations and our comparative results in Section 3. Our conclusions are presented in Section 4. We find that the Local Lagrangian Approximation (and, therefore, tree-level perturbation theory) accurately predicts the evolution of the density PDF for initial power spectra with $n \leq -1$, but fails for $n \geq 0$.

2. The Local Lagrangian Approximation
The Local Lagrangian Approximation is based on the idea that the density at a Lagrangian point $q$ at a time $t$ can be approximated as a function only of $t$ and the initial value of $\rho(q)$:

$$\rho(q, t) = f(\rho(q, t_0), t).$$  \hfill (1)

If we define $\eta = \rho/\bar{\rho}$, where $\bar{\rho}$ is the mean density, then we choose a mapping of the form:

$$\eta(q, t) = \frac{\eta_0(q)}{[1 - D(t)\delta_0(q)/\alpha]^{\alpha}}.$$  \hfill (2)

The mapping in equation (2) is related to the Zeldovich approximation in the sense that $\alpha = 1$ corresponds to the limit of planar collapse (in which the Zeldovich approximation gives an exact description of the evolution of the PDF) and $\alpha = 3$ corresponds to the spherical collapse in the Zeldovich approximation.

In PS, it was shown that the generating function for the hierarchical amplitudes is related in a trivial way to the mapping given in equation (2). In particular, if we take $\alpha = 3/2$, the hierarchical amplitudes for the final density field produced by the mapping in equation (2) will be almost exactly equal to the true amplitudes produced by exact evolution (see PS and earlier work by Bernardeau 1992 and Bernardeau & Kofman 1995). The resulting PDF, $P(\eta)$, which is derived by applying equation (2) to an initially Gaussian density distribution, automatically satisfies the normalization condition $\int P(\eta)d\eta = 1$, but it fails to satisfy $\int P(\eta)d\eta = 1$; this is related, at some level, to the problem of multistreaming (see PS for the details). To correct this problem, we multiply equation (2) by a time-dependent normalization factor $N(t)$:

$$N(t) = \left\langle \frac{1}{\eta(q, t)} \right\rangle,$$  \hfill (3)

leading to the LLA mapping:

$$\eta(q, t) = \frac{(1 - 2\delta_l(q, t)/3)^{3/2}}{[1 - 2\delta_l(q, t)/3]^{3/2}},$$  \hfill (4)

where $\delta_l$, the linear-evolved initial density fluctuation, is simply $\delta_l = D(t)\delta_0$.

This expression gives the final density for an unsmoothed density field, and so cannot be compared directly with either observations or gravitational clustering simulations. If we smooth the final density field with a spherical tophat window function, then we obtain a new “smoothed” local Lagrangian mapping, $f_s$, given by (Bernardeau 1994; PS):

$$\eta_s = f_s(\delta_l) = f\left[\delta_l f_s(\delta_l)^{-\frac{(n+3)}{6}}\right],$$  \hfill (5)

where we have assumed a power-law power spectrum $P(k) \propto k^n$, and the smoothed mapping $f_s$ must then be multiplied by the normalization factor given in equation (3). Note that the mapping given in equation (2) is not the smoothed density field; rather, it is a density field which is guaranteed to give the same hierarchical amplitudes as the “true” evolved density field.
Combining equations (4) and (5), we obtain, for Gaussian initial conditions with spherical tophat smoothing, the final PDF:

\[
P(\eta)d\eta = \frac{1}{N^2}g(\frac{\eta}{N})d\eta
\]

where the function \( g \) is given by

\[
g(x) = \frac{\alpha}{\sqrt{2\pi}\sigma_l} \left[ e^{-\frac{x^2}{2\sigma_l^2} \left( x^{\beta} - x^{\frac{1}{\alpha} + \beta} \right)^2} \left[ \beta x^{\beta - 2} - \left( \beta - \frac{1}{\alpha} \right) x^{\frac{1}{\alpha} + \beta - 2} \right] \right]
\]

\[
+ e^{-\frac{x^2}{2\sigma_l^2} \left( x^{\beta} + x^{\frac{1}{\alpha} + \beta} \right)^2} \left[ \beta x^{\beta - 2} + \left( \beta - \frac{1}{\alpha} \right) x^{\frac{1}{\alpha} + \beta - 2} \right].
\]

Here \( \alpha = 3/2, \sigma_l \) is the linear-evolved rms fluctuation: \( \sigma_l = D(t)\sigma_0 \), and we have defined

\[
\beta \equiv \frac{n + 3}{6}
\]

The normalization factor \( N \) is given by:

\[
N = \int_0^\infty g(x)dx.
\]

Note that the second term in equation (7) corresponds to the case where the argument of the absolute value in equation (4) is negative; this term is negligible compared with the first term as long as \( \sigma_l \ll 1 \). Dropping this second term, we can express the PDF in our Local Lagrangian Approximation in the particularly simple form:

\[
P(\eta)d\eta = \frac{1}{\eta \sqrt{2\pi}\sigma_l} e^{-\frac{\delta_l^2(\eta)}{2\sigma_l^2}} d\delta_l(\eta),
\]

where \( \delta_l(\eta) \) is given by

\[
\delta_l = \frac{3}{2} \left[ \left( \frac{\eta}{N} \right)^{\beta} - \left( \frac{\eta}{N} \right)^{-2/3 + \beta} \right],
\]

and the normalization factor \( N \) is:

\[
N = \frac{1}{\sqrt{2\pi}\sigma_l} \int_0^\infty \left( \frac{N}{\eta} \right) e^{-\frac{\delta_l^2(\eta)}{2\sigma_l^2}} d\delta_l(\eta).
\]

Note that equation (12) can be integrated by making the change of variables \( x = \eta/N \), which eliminates \( N \) from the integral.

3. N-body Simulations and Results
We calculate the time evolved density PDF in collisionless (Melott et al. 1996) gravitational clustering simulations of a pressureless dust, $\Omega = 1$ universe described more fully elsewhere (Melott and Shandarin 1993). We use $128^3$ particles on a $128^3$ grid and assign initial spectra to have the form $P(k) \equiv \langle |\delta_k|^2 \rangle \propto k^n$, with $n = -3, -2, -1, 0, +1$ respectively. The initial spectrum high frequency cutoff is given by the Nyquist frequency of the simulation cube $k_{Ny} = 64k_f$, where $k_f = 2\pi/L$ is the fundamental mode of the cube and the simulations are terminated at a scale of nonlinearity $k_{nl} = 16k_f$. The initial density contrast variance $\sigma_0$ is calculated in tophat smoothed spheres of radii $\lambda = 1, 2, 4, 8, 16$ cells and the linear variance corresponding to the final field is obtained from $\sigma_l = (a_{nl}/a_0)\sigma_0$, where $a_{nl}/a_0$ is the ratio of the expansion factors at the end and at the beginning of the simulation respectively. We use a cloud-in-cell binning for the final densities and consider only $\eta$ in the range $[0..4]$. The simulation PDF $P_s(\eta)$ is defined as the fraction of the number of volumes of a specific $\Delta \eta$ range, $N_B$, over the total number of grid volumes $N_T$ in the simulation,

$$P_s(\eta) = \frac{N_B}{N_T \Delta \eta},$$

(13)

We expect the rms fluctuation associated with each measured $P_s(\eta)$ value to scale as the square root of $N_B$, provided we take into account only the independent number of volumes arising after smoothing, $N_f = 3L^3/4\pi \lambda^3$, where the $L$ is the side of the simulation box and $\lambda$ is the radius of the tophat smoothing sphere. One can then express the rms fluctuation in $P_s(\eta)$ as:

$$\sigma_{P_s(\eta)} = \frac{\sigma_{N_B}}{N_f \Delta \eta},$$

(14)

where $\sigma_{N_B} = \sqrt{N_f P_s(\eta) \Delta \eta}$. To avoid discreteness effects we only use results corresponding to smoothing lengths $\lambda \geq 2$ in our comparisons, keeping in mind though that the longer smoothing lengths lead to a smaller number of independent volumes within the simulation cube and therefore to larger error bars in the evaluation of $P_s(\eta)$. We examine the evolved PDF results at $\sigma_l$ values selected in the range $0.1 < \sigma_l < 2.0$ so as to extend our investigation from the linear ($\sigma_l < 0.5$), through the weakly nonlinear ($\sigma_l < 1.0$), to the nonlinear ($\sigma_l > 1.0$) regimes.

Our results are presented in Figure 1, for negative power indices, and Figure 2 for $n = 0, +1$. The solid lines correspond to the Local Lagrangian prediction, while the results of the gravitational clustering simulations are given as points with $1 - \sigma$ error bars. In each graph we quote the extrapolated linear variance $\sigma_l$ and the corresponding smoothing scale $\lambda$. For the $n = -3, -2$ models we use a smoothing length $4 \leq \lambda \leq 16$ whereas for the $n = -1, 0, +1$ spectral indices we use $8 \leq \lambda \leq 2$ in order to examine the PDF behavior in approximately the same $\sigma_l$ range for both the LLA predictions and the N-body simulation results. For all the negative power models, as easily seen from our results in Figure 1, the agreement between the Local Lagrangian predicted $P(\eta)$ and the N-body simulation $P_s(\eta)$ is reasonably good in the range $0.35 < \sigma_l < 0.72$ and it starts breaking down beyond that point, the Local Lagrangian $P(\eta)$ systematically overestimating the peak of $P_s(\eta)$ and underestimating the tail. However, the picture is quite different for the $n = 0$
and $n = +1$ power spectra. Here we have agreement between $P(\eta)$ and $P_s(\eta)$ only at very low $\sigma_l$ values. Our results indicate not only that the Local Lagrangian Approximation breaks down very early in the linear regime for these values of $n$, but also that its range of validity decreases as the power index moves toward more positive values.

In order to examine the behavior of the higher moments of the density field, we calculated the quantity

$$<\delta_s^2> - \sigma_s^2 \sigma_l^2,$$

where the nonlinear $<\delta_s^2>$ was obtained from our simulations. We observed a decrease with increasing $n$, a result in agreement with Lokas et al. (1996) and explained as the effect of previrialization. Furthermore, our N-body simulations indicate that a similar behavior is also exhibited by the skewness-related quantity

$$<\delta_s^3> - \frac{34\sigma_s^4}{\sigma_l^6}.$$

4. Conclusions

Despite its simple form, the Local Lagrangian Approximation yields the right predictions for the evolved $P(\eta)$ in the weakly nonlinear regime ($\sigma_l < 0.72$), but only for the cases where $n \leq -1$. For $n > 0$, the LLA fails to agree with the simulations except at very early stages in the evolution. Our results are in rough agreement with those of Scoccimarro & Frieman (1996), who examined the contribution of next-to-leading order terms in the perturbative expansion of $\langle \sigma^2 \rangle$. They found that these terms diverge for $n \geq -1$ and converge for $n < -1$. Our results support the conclusion that tree-level perturbation theory fails for $n \geq 0$. However, we find that tree-level theory, as expressed in the LLA, can be applied in the case $n = -1$. This is not totally contradicted by the results of Scoccimarro & Frieman (1996), because in this case the divergence of the next-to-leading order term in their calculations is only logarithmic.

To the degree that $P(\eta)$ carries all the information about its higher order moments, one would expect the predictive power of the LLA to extend to the calculation of such moments in its range of validity. Calculation of the skewness based on the LLA and including one loop corrections (Scoccimarro 1996) leads to reasonable agreement as expected. However, the smallness-of-$\delta$ condition is probably not satisfied for the power spectra $n = 0, +1$ we examined. Possible reasons for the breakdown of the perturbative approach may be the strong coupling of the long $k$ modes due to non-local interactions, as well as multistreaming which may alter the picture even at the weakly nonlinear stage (Bharadwaj 1996).
¿From a practical point of view, the Local Lagrangian Approximation seems to provide an excellent prediction for the evolution of the density PDF in the range \( n \leq -1 \). Since the power spectrum in the quasilinear regime is close to \( n = -1 \) (Klypin and Melott 1992), the LLA may be useful in comparing with observations. In particular, it may be possible to invert our mapping to go from the evolved (observed) PDF backwards to the initial conditions. Since the LLA provides better agreement with the evolved PDF than the Zeldovich approximation, it is also worthwhile to determine whether a modification of the Zeldovich approximation can be derived which corresponds to the mapping in equation (2).

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Fig. 1.— Comparison of the Local Lagrangian Approximation top-hat smoothed density $P(\eta)$, shown as a continuous line, to the N-body simulation results, shown as points with $1-\sigma$ error bars, for $\eta = \rho/\bar{\rho} \leq 4$ and $-3 \leq n \leq -1$. For each power index $n$ we show three different $\sigma_{l(\text{linear})}$ regimes corresponding to smoothing lengths $\lambda$. 
Fig. 2.— Same as in Fig. 1 but for spectral indices of $n = 0, +1$ respectively.
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