STABILITY OF DING PROJECTIVE MODULES*

ZHANPING WANG ZHONGKUI LIU

Abstract

A left $R$-module $M$ is called two-degree Ding projective if there exists an exact sequence $\cdots \rightarrow D_1 \rightarrow D_0 \rightarrow D_{-1} \rightarrow D_{-2} \rightarrow \cdots$ of Ding projective left $R$-modules such that $M \cong \ker(D_0 \rightarrow D_{-1})$ and $\text{Hom}_R(-, F)$ leaves the sequence exact for any flat (or Ding flat) left $R$-module $F$. In this paper, we show that the two-degree Ding projective modules are nothing more than the Ding projective modules.

2000 Mathematics Subject Classification: 16D40, 16D50, 16E05, 16E30

Keywords and phrases: Ding projective modules, Ding injective modules, two-degree Ding projective modules, two-degree Ding injective modules, Gorenstein projective modules.

1. Introduction

Throughout this paper, $R$ denotes an associative ring with unity, and all modules are assumed to be a left $R$-module. Denote by $P(R)$, $I(R)$ and $F(R)$ the class of all projective, injective and flat left $R$-modules respectively.

The development of the Gorenstein homological algebra has reached an advanced level since the pioneering works of Auslander and Bridger([1]). One of the key points of this theory is its ability to identify Gorenstein rings. In the Gorenstein homological algebra one replaces projective, injective and flat modules, the elementary entities on which the classical homological algebra is based, with the Gorenstein projective, Gorenstein injective and Gorenstein flat modules. Recall from [5] that a left $R$-module $M$ is called Gorenstein projective if there is an exact sequence

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \cdots$$

of projective left $R$-modules such that $M \cong \ker(P_0 \rightarrow P_{-1})$ and $\text{Hom}_R(-, P(R))$ leaves the sequence exact. Dually, The Gorenstein injective modules are defined. In [6], the Gorenstein flat modules are defined in terms of the tensor product.

Recently, Sather-Wagstaff et al. [11] introduced modules that we call two-degree Gorenstein projective modules: an module $M$ is two-degree Gorenstein projective if there exists an exact sequence

$$\cdots \rightarrow G_1 \rightarrow G_0 \rightarrow G_{-1} \rightarrow G_{-2} \rightarrow \cdots$$

of the Gorenstein projective modules with $M \cong \ker(G_0 \rightarrow G_{-1})$ such that the functor $\text{Hom}_R(-, G)$ and $\text{Hom}_R(G, -)$ leave the sequence exact for any Gorenstein projective module $G$. They proved that any two-degree Gorenstein projective module is nothing but a Gorenstein projective module (Theorem A in [11]). Later, similar notions were introduced and studied in [3, 12, 13].

---

*Supported by National Natural Science Foundation of China (Grant No. 11201377, 11261050) and Program of Science and Technique of Gansu Province (Grant No. 1208RJZA145).

Address correspondence to Zhanping Wang, Department of Mathematics, Northwest Normal University, Lanzhou 730070, PR China.

E-mail: wangzp@nwnu.edu.cn (Z.P. Wang), liuzk@nwnu.edu.cn (Z.K. Liu).
As a special case of Gorenstein projective module, Ding, Li and Mao introduced and studied in [4] strongly Gorenstein flat module, and several well-known classes of rings are characterized in terms of these modules. A left \( R \)-module \( M \) is called strongly Gorenstein flat if there is an exact sequence
\[
\cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots
\]
of projective left \( R \)-modules with \( M \cong \text{Ker}(P_0 \to P_{-1}) \) such that \( \text{Hom}_R(-, F(R)) \) leaves the sequence exact. Dually, Mao and Ding introduced and studied in [10] Gorenstein FP-injective modules, and showed that there is a very close relationship between Gorenstein FP-injective modules and Gorenstein flat modules. Since over a Ding-Chen ring the strongly Gorenstein flat modules and Gorenstein FP-injective modules have many nice properties analogous to Gorenstein projective modules and Gorenstein injective modules, Gillespie [8] renamed these modules as Ding projective modules and Ding injective modules, respectively. At the same time, Gillespie introduced the Ding flat modules but it turns out that they are nothing more than the Gorenstein flat modules by [10, Lemma 2.8].

The main purpose of this paper is to establish the stability of the Ding projective modules under the very process used to define these entities.

2. Main results

According to [4], a left \( R \)-module \( M \) is called Ding projective if there is an exact sequence
\[
\cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots
\]
of projective left \( R \)-modules such that \( M \cong \text{Ker}(P_0 \to P_{-1}) \) and \( \text{Hom}_R(-, F(R)) \) leaves the sequence exact. We use \( \text{DP}(R) \) to denote the class of all Ding projective modules.

Note that every projective module is Ding projective, and every Ding projective module is Gorenstein projective. For a left coherent ring \( R \), it follows from [7, Proposition 10.2.6] that a finitely presented module is Ding projective if and only if it is Gorenstein projective. Clearly, every Gorenstein projective module over a left perfect ring is Ding projective. Also it follows easily from [8, Corollary 4.6] that every Gorenstein projective module over a Gorenstein ring is Ding projective.

Recall that a class of modules is called projectively resolving (injective coresolving) if it is closed under extensions and kernels of surjections (cokernels of injections), and it contains all projective (injective) modules.

Ding projective modules have the following properties.

Lemma 2.1. The following assertions hold.

1. If \( M \in \text{DP}(R) \), then \( \text{Ext}_R^i(M, L) = 0 \) for all \( i > 0 \) and all module \( L \) of finite flat dimension.
2. \( \text{DP}(R) \) is a projectively resolving class, and closed under direct sums and direct summands.

Proof. (1) It is trivial. 
(2) It follows by analogy with the proof of Theorem 2.5 in [9].

Definition 2.2. A module \( M \) is called two-degree Ding projective if there exists an exact sequence
\[
\cdots \to D_1 \to D_0 \to D_{-1} \to D_{-2} \to \cdots
\]
of Ding projective modules such that \( M \cong \text{Ker}(D_0 \to D_{-1}) \) and \( \text{Hom}_R(-, F(R)) \) leaves the sequence exact.

We use \( \text{D}^2\text{P}(R) \) to denote the class of all two-degree Ding projective modules. Clearly, \( \text{DP}(R) \subseteq \text{D}^2\text{P}(R) \).
Proposition 2.3. If \( M \in D^2P(R) \), then \( \text{Ext}_R^i(M, L) = 0 \) for each module \( L \) with finite flat dimension and each integer \( i \geq 1 \).

**Proof.** We proceed by induction on \( n := fd_R(L) < \infty \). Suppose \( M \) is a two-degree Ding projective module. Then there exists a short exact sequence \( 0 \to K \to D \to M \to 0 \) such that \( D \in \text{DP}(R) \), \( K \in D^2P(R) \) and \( \text{Hom}_R(-, F) \) leaves the sequence exact for each flat module \( F \). Thus \( \text{Ext}_R^i(M, F) = 0 \) for each flat module \( F \). Applying the functor \( \text{Hom}_R(-, F) \) to the above sequence, we get the following exact sequence

\[
0 = \text{Ext}_R^1(D, F) \to \text{Ext}_R^1(K, F) \to \text{Ext}_R^2(M, F) \to \text{Ext}_R^2(D, F) = 0,
\]

which yield \( \text{Ext}_R^1(K, F) \cong \text{Ext}_R^2(M, F) \). By the above proof for \( M \), we have \( \text{Ext}_R^1(K, F) = 0 \), and so \( \text{Ext}_R^2(M, F) = 0 \). Reiterating this process, we get \( \text{Ext}_R^i(M, F) = 0 \). Then the case \( n = 0 \) holds. Now suppose \( n \geq 1 \) and \( L \) is a module of flat dimension \( n \). Let \( 0 \to L' \to F \to L \to 0 \) be an exact sequence such that \( F \) is flat. Applying the functor \( \text{Hom}_R(M, -) \) to it, we get the following exact sequence

\[
0 = \text{Ext}_R^i(M, F) \to \text{Ext}_R^i(M, L) \to \text{Ext}_R^{i+1}(M, L') \to \text{Ext}_R^{i+1}(M, F) = 0.
\]

By inductive assumptions, \( \text{Ext}_R^i(M, L) \cong \text{Ext}_R^{i+1}(M, L') = 0 \) for each integer \( i \geq 1 \), as desired. \( \square \)

Recall from Definition 2.1 in [2] that a module \( M \) is called strongly Gorenstein projective if there exists an exact sequence

\[
\cdots \to P \to P \to P \to P \to \cdots
\]

of projective modules such that \( M \cong \text{Ker}(f) \) and \( \text{Hom}_R(-, P(R)) \) leaves the sequence exact. It is proved that each Gorenstein projective module is a direct summand of a strongly Gorenstein projective module (Theorem 2.7 in [2]). Inspired by it, we introduce the notion of strongly two-degree Ding projective modules. The notion play a crucial role in the proof of main theorem (see Theorem 2.7).

**Definition 2.4.** A module \( M \) is called strongly two-degree Ding projective if there exists an exact sequence

\[
\cdots \to D \to D \to D \to D \to \cdots
\]

of Ding projective modules such that \( M \cong \text{Ker}(f) \) and \( \text{Hom}_R(-, F(R)) \) leaves the sequence exact.

We use SD\(^2\)P(R) to denote the class of all strongly two-degree Ding projective modules. Clearly, SD\(^2\)P(R) \( \subseteq \) D\(^2\)P(R).

**Proposition 2.5.** For any module \( M \), the following statements are equivalent.

1. \( M \in \text{SD}^2\text{P}(R) \).
2. There is a short exact sequence \( 0 \to M \to D \to M \to 0 \) such that \( D \in \text{DP}(R) \) and \( \text{Ext}_R^1(M, F) = 0 \) for each flat module \( F \).
3. There is a short exact sequence \( 0 \to M \to D \to M \to 0 \) such that \( D \in \text{DP}(R) \) and \( \text{Ext}_R^1(M, L) = 0 \) for each module \( L \) with finite flat dimension.
4. There is a short exact sequence \( 0 \to M \to D \to M \to 0 \) such that \( D \in \text{DP}(R) \) and \( \text{Hom}_R(-, F(R)) \) leaves the sequence exact.
5. There is a short exact sequence \( 0 \to M \to D \to M \to 0 \) such that \( D \in \text{DP}(R) \) and \( \text{Hom}_R(-, L) \) leaves the sequence exact for each module \( L \) with finite flat dimension.

**Proof.** Using standard argument, it follows immediately from the definition of strongly two-degree Ding projective modules. \( \square \)

**Proposition 2.6.** Let \( M \) be a two-degree Ding projective module. Then \( M \) is a direct summand of a strongly two-degree Ding projective module.
Proof. Let $M$ be a two-degree Ding projective module. Then there exists an exact sequence

$$\cdots \xrightarrow{\delta_2} D_1 \xrightarrow{\delta_1} D_0 \xrightarrow{\delta_0} D_{-1} \xrightarrow{\delta_{-1}} D_{-2} \xrightarrow{\delta_{-2}} \cdots$$

of Ding projective modules such that $M \cong \text{Ker}(\delta_0)$ and $\text{Hom}_R(\cdot, F(R))$ leaves the sequence exact. Consider the exact sequence

$$\cdots \xrightarrow{\oplus \delta_i} \bigoplus D_i \xrightarrow{\oplus \delta_i} \bigoplus D_i \xrightarrow{\oplus \delta_i} \bigoplus D_i \xrightarrow{\oplus \delta_i} \cdots.$$

Since $\text{Ker}(\bigoplus \delta_i) \cong \bigoplus \text{Ker}(\delta_i)$, $M$ is a direct summand of $\text{Ker}(\bigoplus \delta_i)$. By Lemma 2.1 and $\text{Hom}_R(\bigoplus, D_i, F) \cong \prod_i \text{Hom}_R(D_i, F)$ for each flat module $F$, we get $\text{Ker}(\bigoplus \delta_i)$ is a strongly two-degree Ding projective module. □

**Theorem 2.7.** $\text{DP}(R) = D^2\text{P}(R)$.

Proof. Clearly, $\text{DP}(R) \subseteq D^2\text{P}(R)$. It suffices to prove that $D^2\text{P}(R) \subseteq \text{DP}(R)$. Since $\text{DP}(R)$ is closed under direct summands. Thus it suffices to prove that any strongly two-degree Ding projective module is Ding projective by Proposition 2.6. Suppose $M$ is a strongly two-degree Ding projective module. Then there exists a short exact sequence $0 \to M \to D \to M \to 0$ such that $D \in \text{DP}(R)$ and $\text{Ext}^1_R(M, F) = 0$ for each flat module $F$ by Proposition 2.5. As $D$ is Ding projective, there is a short exact sequence $0 \to D \to P \to D_1 \to 0$ such that $P \in \text{P}(R)$ and $D_1 \in \text{DP}(R)$. Then we get the following pushout diagram:

```
\[
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & M & D & M & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & M & P & N & 0 \\
\downarrow & & \downarrow & & \downarrow \\
D_1 & D_1 & D_1 & D_1 & \ldots
\end{array}
\]
```

For any flat module $F$, applying the functor $\text{Hom}_R(\cdot, F)$ to the exact sequence $0 \to M \to N \to D_1 \to 0$, we get the following exact sequence

$$0 = \text{Ext}^i_R(D_1, F) \to \text{Ext}^i_R(N, F) \to \text{Ext}^i_R(M, F) \to \text{Ext}^{i+1}_R(D_1, F) = 0$$

for each integer $i \geq 1$. This yield that $\text{Ext}^i_R(N, F) \cong \text{Ext}^i_R(M, F)$. By Proposition 2.3, we have $\text{Ext}^1_R(M, F) = 0$, and so $\text{Ext}^i_R(N, F) = 0$. Consider the following pushout diagram:
Because both $D$ and $D_1$ are Ding projective, $D_2$ is also Ding projective by Lemma 2.1. Then there exists a short exact sequence $0 \to D_2 \to P_0 \to W \to 0$ with $P_0$ projective and $W$ Ding projective. Consider the following pushout diagram:

Applying the functor $\text{Hom}_R(-, F)$ to the exact sequence $0 \to M \to G \to W \to 0$, we get $\text{Ext}_R^1(G, F) = 0$. On the other hand, applying the functor $\text{Hom}_R(-, F)$ to the exact sequence $0 \to N \to P_0 \to G \to 0$, we get the following exact sequence

$$0 \to \text{Hom}_R(G, F) \to \text{Hom}_R(P_0, F) \to \text{Hom}_R(N, F) \to 0.$$ 

Thus we obtain that an $\text{Hom}_R(-, F)$ exact exact sequence $0 \to N \to P_0 \to G \to 0$ where $P_0$ is projective and $G$ is a module with the same property as $N$. Recursively, we get an exact sequence

$$0 \to N \to P_0 \to P_{-1} \to \cdots$$

of projective modules, which remains exact after applying the functor $\text{Hom}_R(-, F(R))$. Thus $N$ is Ding projective. Because both $N$ and $D_1$ are Ding projective, $M$ is also Ding projective by Lemma 2.1. □

**Remark 2.8.** Denote by $D^2_P(R)$ the subcategory of all modules for which there exists an exact sequence

$$\cdots \to D_1 \to D_0 \to D_{-1} \to D_{-2} \to \cdots$$

of Ding projective modules such that $M \cong \text{Ker}(D_0 \to D_{-1})$ and $\text{Hom}_R(-, H)$ leaves the sequence exact for each Ding flat module (i.e. Gorenstein flat module) $H$. It is routine to check that $D_P(R) \subseteq D^2_P(R) \subseteq D^3_P(R)$. By Theorem 2.7, $D_P(R) = D^2_P(R) = D^3_P(R)$. 
Remark 2.9. Denote by $\text{GP}(R)$ and $G^2 \text{P}(R)$ the subcategories of all Gorenstein projective and two-degree Gorenstein projective modules, respectively. Denote by $G^2 \text{P}(R)$ the subcategory of all modules for which there exists an exact sequence

$$\cdots \to G_1 \to G_0 \to G_{-1} \to G_{-2} \to \cdots$$

of Gorenstein projective modules such that $M \cong \text{Ker}(G_0 \to G_{-1})$ and $\text{Hom}_R(\cdot, G)$ leaves the sequence exact for each Gorenstein projective module $G$. Denote by $G^2_p \text{P}(R)$ the subcategory of all modules for which there exists an exact sequence

$$\cdots \to G_1 \to G_0 \to G_{-1} \to G_{-2} \to \cdots$$

of Gorenstein projective modules such that $M \cong \text{Ker}(G_0 \to G_{-1})$ and $\text{Hom}_R(\cdot, P)$ leaves the sequence exact for each projective module $P$. It is routine to check that $\text{GP}(R) \subseteq G^2_p \text{P}(R) \subseteq G^2 \text{P}(R) \subseteq G^2 \text{P}(R)$. According to [10], a left $R$-module $M$ is called Ding injective if there is an exact sequence

$$\cdots \to E_1 \to E_0 \to E_{-1} \to E_{-2} \to \cdots$$

of injective left $R$-modules with $M \cong \text{Ker}(E_0 \to E_{-1})$ such that $\text{Hom}_R(E, \cdot)$ leaves the sequence exact whenever $E$ an FP-injective $R$-module. We use $\text{DI}(R)$ to denote the class of all Ding injective modules.

By definitions, every injective module is Ding injective, and every Ding injective module is Gorenstein injective. If $R$ is left Noetherian, then every Gorenstein injective module is Ding injective.

Dual arguments to the above give the following assertions concerning the Ding injective modules.

Definition 2.10. A module $M$ is called two-degree Ding injective if there exists an exact sequence

$$\cdots \to D_1 \to D_0 \to D_{-1} \to D_{-2} \to \cdots$$

of Ding injective modules such that $M \cong \text{Ker}(D_0 \to D_{-1})$ and $\text{Hom}_R(H, \cdot)$ leaves the sequence exact for each FP-injective module $H$.

We use $\text{D}^2 \text{I}(R)$ to denote the class of all two-degree Ding injective modules.

Theorem 2.11. $\text{DI}(R) = \text{D}^2 \text{I}(R)$.

References

[1] Auslander, M., Bridger, M.: Stable module theory. In Memoirs of the American mathematical society, vol. 94 (American Mathematical Society, Providence, RI) (1969)
[2] Bennis, D., Mahdou, N.: Strongly Gorenstein projective, injective, and flat modules. J. Pure Appl. Algebra, 210, 437–445 (2007)
[3] Bouchiba, S., Khaloui, M.: Stability of Gorenstein flat modules. Glasgow Math. J., 54, 169–175 (2012)
[4] Ding, N.Q., Li, Y.L., Mao, L.X.: Strongly Gorenstein flat modules. J. Aust. Math. Soc., 86, 323–338 (2009)
[5] Enochs, E.E., Jenda, O.M.G.: Gorenstein injective and projective modules. Math.Z., 220, 611–633 (1995)
[6] Enochs, E.E., Jenda, O.M.G., Torrecillas, B.: Gorenstein flat modules. Nanjing Daxue Xuebao Shuxue Bannian Kan, 10, 1–9 (1993)
[7] Enochs, E.E., Jenda, O.M.G.: Relative Homological Algebra. de Gruyter Exp. Math., vol. 30, de Gruyter, Berlin (2000)
[8] Gillespie, J.: Model structures on modules over Ding-Chen rings. *Homology, Homotopy and Applications*, **12**, 61–73 (2010)

[9] Holm, H.: Gorenstein homological dimensions. *J. Pure Appl. Algebra*, **189**, 167–193 (2004)

[10] Mao, L.X., Ding, N.Q.: Gorenstein FP-injective and Gorenstein flat modules. *J. Algebra Appl.*, **7**, 491–506 (2008)

[11] Sather-Wagstaff, S., Sharif, T., White, D.: Stability of Gorenstein categories. *J. Lond. Math. Soc.*, **77**, 481–502 (2008)

[12] Sather-Wagstaff, S., Sharif, T., White, D.: AB-contexts and stability for Gorenstein flat modules with respect to semi-dualizing modules. *Algebr. Represent. Theory*, **14**, 403–428 (2011)

[13] Yang, G., Liu, Z.K.: Stability of Gorenstein flat categories. *Glasgow Math. J.*, **54**, 177–191 (2012)