GLOBAL SOLUTION TO THE THREE-DIMENSIONAL COMPRESSIBLE FLOW OF LIQUID CRYSTALS

XIANPENG HU AND HAO WU

Abstract. The Cauchy problem for the three-dimensional compressible flow of nematic liquid crystals is considered. Existence and uniqueness of the global strong solution are established in critical Besov spaces provided that the initial datum is close to an equilibrium state $(1, \mathbf{0}, \mathbf{d})$ with a constant vector $\mathbf{d} \in S^2$. The global existence result is proved via the local well-posedness and uniform estimates for proper linearized systems with convective terms.

1. Introduction

Liquid crystals are substances that exhibit a phase of matter that has properties between those of a conventional liquid, and those of a solid crystal. The three-dimensional flow of nematic liquid crystals can be governed by the following system of partial differential equations [3, 18]:

\begin{align}
\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) &= 0 \quad \text{(1.1a)} \\
\frac{\partial (\rho \mathbf{u})}{\partial t} + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\mu + \lambda) \nabla \text{div} \mathbf{u} + \nabla P(\rho) &= -\xi \text{div} \left( \nabla \mathbf{d} \otimes \nabla \mathbf{d} - \frac{1}{2} \mathbf{1} \right) \quad \text{(1.1b)} \\
\frac{\partial \mathbf{d}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{d} &= \theta \left( \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} \right) \quad \text{(1.1d)}
\end{align}

where $\rho \in \mathbb{R}$ is the density function of the fluid, $\mathbf{u} \in \mathbb{R}^3$ is the velocity and $\mathbf{d} \in S^2$ represents the director field for the averaged macroscopic molecular orientations. The scalar function $P \in \mathbb{R}$ is the pressure, which is an increasing and convex function in $\rho$. They all depend on the spatial variable $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the time variable $t > 0$. The constants $\mu$ and $\lambda$ are shear viscosity and the bulk viscosity coefficients of the fluid respectively that satisfy the physical assumptions $\mu > 0, 2\mu + 3\lambda \geq 0$. The

Date: January 7, 2022.
2000 Mathematics Subject Classification. 35A05, 76A10, 76D03.
Key words and phrases. Compressible liquid crystal flow, global well-posedness, critical space.
X.-P Hu was partially supported by DMS-1108647. H. Wu was partially supported by NSF of China 11001058, SRFDP and “Chen Guang” project supported by Shanghai Municipal Education Commission and Shanghai Education Development Foundation.
constants $\xi > 0, \theta > 0$ stand for the competition between the kinetic energy and the potential energy, and the microscopic elastic relaxation time (or the Debroah number) for the molecular orientation field, respectively. The symbol $\otimes$ denotes the Kronecker tensor product such that $u \otimes u = (u_i u_j)_{1 \leq i,j \leq 3}$ and the $\nabla d \otimes \nabla d$ denotes a matrix whose $ij$-th entry $(1 \leq i,j \leq 3)$ is $\partial_{x_i}d \cdot \partial_{x_j}d$. $I$ is the $3 \times 3$ identity matrix. To complete the system (1.1), the initial data are given by

$$
\rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x), \quad d|_{t=0} = d_0(x), \quad \text{with } d_0 \in S^2.
$$

(1.2)

Roughly speaking, the system (1.1) is a coupling between the compressible Navier–Stokes equations and a transported heat flow of harmonic maps into $S^2$. It is a macroscopic continuum description of the evolution for the liquid crystals of nematic type under the influence of both the flow field $u$, and the macroscopic description of the microscopic orientation configurations $d$ of (rod-like) liquid crystals.

The hydrodynamic theory of liquid crystals in nematic case has been established by Ericksen and Leslie, see \cite{7, 8, 16, 17}. Since then, the mathematical theory is still progressing and the study of the full Ericksen–Leslie model presents relevant mathematical difficulties. In \cite{18}, Lin introduced a simplification of the general Ericksen–Leslie system that keeps many of the mathematical difficulties of the original system by using a Ginzburg–Landau approximation to relax the nonlinear constraint $d \in S^2$. Later in \cite{19}, Lin and Liu showed the global existence of weak solutions and smooth solutions for that approximation system. For more results on the approximation system, we refer to \cite{20, 23, 25}. Recently, Hong \cite{11} and Lin–Lin–Wang \cite{21} showed independently the global existence of weak solution of an incompressible model of system (1.1) in two dimensional space. Moreover, in \cite{21}, the regularity of solutions except for a countable set of singularities whose projection on the time axis is a finite set had been obtained. In the recent work \cite{24}, Wang established a global well-posedness theory for the incompressible liquid crystals for rough initial data, provided that

$$
\|u_0\|_{BMO^{-1}} + [d_0]_{BMO} \leq \varepsilon_0
$$

for some $\varepsilon_0 > 0$. Note that the relationship between $BMO^{-1}$ and $\dot{H}^{1/2}$ is (see \cite{2})

$$
\dot{H}^{1/2} \hookrightarrow L^3 \hookrightarrow B_{p,\infty}^{-1+\frac{2}{p}} \hookrightarrow BMO^{-1}, \quad \text{for } 3 \leq p < \infty.
$$

Concerning the compressible case, local existence of unique strong solutions of (1.1) was proved provided that the initial data $\rho_0, u_0, d_0$ are sufficiently regular and satisfy a natural compatibility condition in a recent work \cite{13}. A criterion for possible breakdown of such a local strong solution at finite time was given in terms of blow up of the $L^\infty$-norms of $\rho$ and $\nabla d$. In \cite{12}, an alternative blow-up criteria was derived in terms of the $L^\infty$-norms of $\nabla u$ and $\nabla d$. The global existence of weak solutions to (1.1) with large initial data is still an outstanding open problem for high dimensions. By so far, results in one space dimension have been obtained in \cite{5, 6}, and authors in \cite{14} consider a multidimensional version with small energy.
In this paper, we are interested in the existence and uniqueness of global strong solutions to the Cauchy problem of (1.1) with initial data (1.2) in the three dimensional space. It is difficult to find a functional space such that the system (1.1)–(1.2) is well-posed globally in time. To this end, we notice that the system (1.1) is invariant under the following transformations
\[ \tilde{\rho} = \rho(l^2 t, lx), \quad \tilde{u} = lu(l^2 t, lx), \quad \tilde{d} = d(l^2 t, lx) \] (1.3)
with the modification of the pressure \( \tilde{P} = l^2 P \). A critical space is a space in which the norm is invariant under the scaling
\[ (\tilde{e}, \tilde{f}, \tilde{g})(x) = (e(lx), lf(lx), g(lx)). \]

The scaling invariance (1.3) reminds us of a similar property of three dimensional incompressible Navier–Stokes equations, which provides a well-known global existence of solutions with small data in the homogeneous Sobolev space \( \dot{H}^{\frac{3}{2}} \) (see [15]). Motivated by this observation, we aim at a global well-posedness of the system (1.1)–(1.2) with small initial data in a functional framework where the function space for the velocity \( u \) is similar to \( \dot{H}^{\frac{3}{2}} \). According to the scaling (1.3), the regularity of the density \( \rho \) and the director field \( d \) is one order higher than that of the velocity \( u \), and hence a function space which is similar to \( \dot{H}^{\frac{3}{2}} \) would be a candidate. Unfortunately, the function space \( \dot{H}^{\frac{3}{2}} \) does not turn to be a good candidate for \( \rho \) and \( d \), since bounds of the director field \( d \) in \( \dot{H}^{\frac{3}{2}} \) do not automatically imply the \( L^\infty \) bound of \( d \). To overcome this difficulty, inspired by [3] for the compressible Navier–Stokes equations, it seems more natural to work in the framework of homogeneous Besov space \( B^{\frac{3}{2}, 2}_{2,1} \) since \( B^{\frac{3}{2}, 2}_{2,1} \) is continuously embedded into \( L^\infty \). Furthermore, as in [3], the different dissipative mechanisms of low frequencies and high frequencies inspire us to deal with \( \rho \) and \( d \) in \( B^{\frac{3}{2}, 2}_{2,1} \cap B^{\frac{3}{2}, 2}_{2,1} \).

The rest of this paper is organized as follows. In Section 2, we will give a series of fundamental properties of the Besov’s spaces. In Section 3, we reformulate the system (1.1)–(1.2) and state our main result (Theorem 3.1). The main goal of Section 4 is to prove uniform estimates for linearized systems to (1.1), while the global existence is obtained in Section 5. In Section 6, the uniqueness of global strong solution is verified.

### 2. Preliminaries

Throughout this paper, we use \( C \) for a generic constant, and denote \( A \leq CB \) by \( A \lesssim B \). The notation \( A \approx B \) means that \( A \lesssim B \) and \( B \lesssim A \). Also we use \( (\alpha_q)_{q \in \mathbb{Z}} \) to denote a sequence such that \( \sum_{q \in \mathbb{Z}} \alpha_q \leq 1 \). \( (f|g) \) denotes the inner product of two functions \( f, g \) in \( L^2(\mathbb{R}^N) \). The standard summation notation over the repeated index will be adopted in the remaining part of this paper.

In order to state our existence result, we introduce some functional spaces and explain the notations. Let \( T > 0, r \in [0, \infty] \) and \( X \) be a Banach space. We denote by \( \mathcal{M}(0, T; X) \)
the set of measurable functions on \((0, T)\) valued in \(X\). For \(f \in \mathcal{M}(0, T; X)\), we define
\[
\|f\|_{L^r_T(X)} = \left(\int_0^T \|f(\tau)\|_X^r \, d\tau\right)^{\frac{1}{r}} \quad \text{if } r < \infty,
\]
\[
\|f\|_{L^\infty_T(X)} = \sup_{\tau \in [0, T]} \|f(\tau)\|_X.
\]
Denote \(L^r(0, T; X) = \{f \in \mathcal{M}(0, T; X) : \|f\|_{L^r_T(X)} < \infty\}\). If \(T = \infty\), we denote by \(L^r(\mathbb{R}^+; X)\) and \(\|f\|_{L^r(\mathbb{R}^+; X)}\) the corresponding spaces and norms, respectively. \(C([0, T], X)\) (or \(C(\mathbb{R}^+; X)\)) stands for the set of continuous \(X\)-valued functions on \([0, T]\) (resp. \(\mathbb{R}^+\)) while \(C_b(\mathbb{R}^+; X)\) is the set of bounded continuous \(X\)-valued functions. For \(\alpha \in (0, 1)\), \(C^\alpha([0, T]; X)\) (or \(C^\alpha(\mathbb{R}^+; X)\)) stands for the set of Hölder continuous functions in time with order \(\alpha\), i.e., for every \(t, s\) in \([0, T]\) (resp. \(\mathbb{R}^+\)), we have
\[
\|f(t) - f(s)\|_X \lesssim |t - s|^{\alpha}.
\]
As in [3], we introduce a function \(\psi \in C^\infty(\mathbb{R}^N)\), supported in \(C = \{\xi \in \mathbb{R}^N : \frac{5}{6} \leq |\xi| \leq \frac{12}{5}\}\) and such that
\[
\sum_{q \in \mathbb{Z}} \psi(2^{-q}\xi) = 1 \text{ if } \xi \neq 0.
\]
Let \(\mathcal{F}\) be the Fourier transform. Denoting \(\mathcal{F}^{-1}\psi\) by \(h\), we define the dyadic blocks as follows:
\[
\Delta_q f = \psi(2^{-q}D)f = 2^{qN} \int_{\mathbb{R}^N} h(2^q y)f(x - y) \, dy, \quad \text{and} \quad S_q f = \sum_{p \leq q - 1} \Delta_p f.
\]
Then the formal decomposition
\[
f = \sum_{q \in \mathbb{Z}} \Delta_q f
\]
is called homogeneous Littlewood–Paley decomposition.

For \(s \in \mathbb{R}\) and \(f \in \mathcal{S}'(\mathbb{R}^N)\), we denote
\[
\|f\|_{\dot{B}^s_{p, r}} \overset{\text{def}}{=} \left(\sum_{q \in \mathbb{Z}} 2^{qs r} \|\Delta_q f\|_{L^p}^r\right)^{\frac{1}{r}}.
\]
When \(p = 2\) and \(r = 1\), we denote \(\| \cdot \|_{\dot{B}^s_{2, 1}}\) by \(\| \cdot \|_{B^s}\). The definition of the homogeneous Besov space is built on the homogeneous Littlewood–Paley decomposition (cf. [3 Definition 1.2])

**Definition 2.1.** Let \(s \in \mathbb{R}\), and \(m = -\left[\frac{N}{2} + 1 - s\right]\). If \(m < 0\), we set
\[
B^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^N) : \|f\|_{B^s} < \infty \text{ and } f = \sum_{q \in \mathbb{Z}} \Delta_q f \text{ in } \mathcal{S}'(\mathbb{R}^N) \right\}.
\]
If \( m \geq 0 \), we denote by \( \mathcal{P}_m \) the set of \( N \) variables polynomials of degree \( \leq m \) and define

\[
B^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m \| f \|_{B^s} < \infty \text{ and } f = \sum_{q \in \mathbb{Z}} \Delta_q f \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m \right\}.
\]

Functions in the homogeneous Besov space \( B^s \) has many good properties (see [1, Proposition 2.5]):

**Proposition 2.1.** The following properties hold:

- **Density:** the set \( C^\infty_0 \) is dense in \( B^s \) if \( |s| \leq \frac{N}{2} \);
- **Derivation:** \( \| f \|_{B^s} \approx \| \nabla f \|_{B^{s-1}} \);
- **Fractional derivation:** let \( \Lambda = \sqrt{-\Delta} \) and \( \sigma \in \mathbb{R} \); then the operator \( \Lambda^\sigma \) is an isomorphism from \( B^s \) to \( B^{s-\sigma} \);
- **Algebraic properties:** for \( s > 0 \), \( B^s \cap L^\infty \) is an algebra;
- **Interpolation:** \( (B^{s_1}, B^{s_2})_{\theta,1} = B^{\theta s_1 + (1-\theta)s_2} \), for \( s_1, s_2 \in \mathbb{R} \) and \( \theta \in (0,1) \).

To deal with functions with different regularities for high frequencies and low frequencies, motivated by [3], it is more effective to work in hybrid Besov spaces. We remark that using hybrid Besov spaces has been crucial for proving global well-posedness for compressible Navier–Stokes equations in critical spaces (see [1, 3]).

**Definition 2.2.** Let \( s, t \in \mathbb{R} \). We set

\[
\| f \|_{\tilde{B}^{s,t}} = \sum_{q \leq 0} 2^{qs} \| \Delta_q f \|_{L^2} + \sum_{q > 0} 2^{qt} \| \Delta_q f \|_{L^2}.
\]

For \( m = -\lfloor \frac{N}{2} + 1 - s \rfloor \), we define

\[
\tilde{B}^{s,t} = \left\{ f \in \mathcal{S}'(\mathbb{R}^N) \| f \|_{\tilde{B}^{s,t}} < \infty \right\}, \quad \text{if } m < 0,
\]

\[
\tilde{B}^{s,t} = \left\{ f \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m \| f \|_{\tilde{B}^{s,t}} < \infty \right\}, \quad \text{if } m \geq 0.
\]

**Remark 2.1.** Some remarks about the hybrid Besov spaces are in order:

- \( \tilde{B}^{s,s} = B^s \);
- If \( s \leq t \), then \( \tilde{B}^{s,t} = B^s \cap B^t \). Otherwise, \( \tilde{B}^{s,t} = B^s + B^t \). In particular, \( \tilde{B}^{s,\frac{N}{2}} \hookrightarrow L^\infty \) as \( s \leq \frac{N}{2} \);
- The space \( \tilde{B}^{0,s} \) coincides with the usual nonhomogeneous Besov space

\[
\left\{ f \in \mathcal{S}'(\mathbb{R}^N) \| \chi(D)f \|_{L^2} + \sum_{q \geq 0} 2^{qs} \| \Delta_q f \|_{L^2} < \infty \right\},
\]

where \( \chi(\xi) = 1 - \sum_{q \geq 0} \phi(2^{-q}\xi) \);
- If \( s_1 \leq s_2 \) and \( t_1 \geq t_2 \), then \( \tilde{B}^{s_1,t_1} \hookrightarrow \tilde{B}^{s_2,t_2} \).

We have the following properties for the product in hybrid Besov spaces (see [1]):
Proposition 2.2. For all $s_1, s_2 > 0$,

$$\|fg\|_{\tilde{B}_{s_1+s_2}^1} \lesssim \|f\|_{L^\infty} \|g\|_{L^\infty} + \|g\|_{\tilde{B}_{s_1+s_2}^1}.$$  

For all $s_1, s_2 \leq \frac{N}{2}$ such that $\min\{s_1 + t_1, s_2 + t_2\} > 0$,

$$\|fg\|_{\tilde{B}_{s_1+s_2}^1} \lesssim \|f\|_{\tilde{B}_{s_1+s_2}^1} \|g\|_{L^\infty}.$$  

Throughout this paper, the following estimates for the convection terms arising in the linearized systems will be used frequently (cf. [3, Lemma 5.1]).

Lemma 2.1. Let $G$ be an homogeneous smooth function of degree $m$. Suppose $-\frac{N}{2} < s_i, t \leq 1 + \frac{N}{2}$ for $i = 1, 2$. Then the following three inequalities hold true:

$$|(G(D)\Delta_q(u \cdot \nabla f)|G(D)\Delta_q f)| \leq C \alpha_q 2^{-q(\phi^{s_1+s_2}(q) - m)} \|u\|_{L^{B_{s_1},s_2}} \|G(D)\Delta_q f\|_{L^2},$$  

(2.1)

$$|(G(D)\Delta_q(u \cdot \nabla f)|G(D)\Delta_q g) + (\Delta_q(u \cdot \nabla g)|G(D)\Delta_q f)| \leq C \alpha_q \|u\|_{L^{B_{s_1},s_2}} \left(2^{-qt} \|G(D)\Delta_q f\|_{L^2} \|g\|_{L^2} + 2^{-q(\phi^{s_1+s_2}(q) - m)} \|f\|_{\tilde{B}_{s_1+s_2}^1} \|\Delta_q g\|_{L^2}\right),$$  

(2.2)

where $\sum_{q \in \mathbb{Z}} \alpha_q \leq 1$ and $C$ is a universal constant that only depends on $s_i, t, N$. The notation $\phi^{s,t}(q)$ means that for $q \in \mathbb{Z}$,

$$\phi^{s,t}(q) \overset{\text{def}}{=} \begin{cases} s, & \text{if } q \leq 0, \\ t, & \text{if } q \geq 1. \end{cases}$$

Remark 2.2. The above lemma looks slightly different from the original one [3, Lemma 5.1]. Indeed, (2.1) and (2.2) are corresponding to (41) and (42) in [3], respectively, up to a change with the notation $\phi^{s,t}$.  

3. Reformulation of the Original System (1.1) and Main Result

In this section, we first reformulate the original system (1.1) into a different form and then we state our main result on the global existence of strong solutions. We simply set $\theta = \xi = 1$ since their sizes do not play any role in our analysis. For $s \in \mathbb{R}$, we denote

$$\Lambda^s f := F^{-1}(|\xi|^sF(f)).$$

Using the idea in [3], we decompose the velocity field into a compressible part and an incompressible part. Let

$$h = \Lambda^{-1} \text{div} u$$

and

$$\Omega = \Lambda^{-1} \text{curl} u, \quad \text{with } (\text{curl} u)^i_j = \partial_{x_j} u^i - \partial_{x_i} u^j.$$  

Owing to the identity $\Delta = \nabla \text{div} - \text{curl} \text{curl}$, we have the decomposition

$$u = -\Lambda^{-1} \nabla h + \Lambda^{-1} \text{curl} \Omega,$$
which implies that \( u \) can be recovered from the information of \( h \) and \( \Omega \). Denote

\[
\mathcal{A} \overset{\text{def}}{=} \mu \Delta + (\lambda + \mu) \nabla \text{div},
\]

and

\[
\mathcal{N} = -u \cdot \nabla u - \frac{\rho - 1}{\rho} Au - \frac{1}{\rho} \text{div} \left( \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 I \right).
\] (3.1)

Applying \( \Lambda^{-1} \text{div} \) and \( \Lambda^{-1} \text{curl} \) to the moment equation in (1.1) respectively, we obtain that

\[
\begin{aligned}
\partial_t h - \nu \Delta h &= \Lambda^{-1} \text{div} \left( \mathcal{N} - \frac{1}{\rho} \nabla P(\rho) \right), \\
\partial_t \Omega - \mu \Delta \Omega &= \Lambda^{-1} \text{curl} \mathcal{N},
\end{aligned}
\] (3.2)

where \( \nu = 2\mu + \lambda \). Since \( \mu > 0 \) and \( 2\mu + 3\lambda \geq 0 \), we have \( \nu \geq \frac{4}{3}\mu > 0 \). In the second equation of (3.2), we have used the fact that

\[
\text{curl} \left( \frac{1}{\rho} \nabla P(\rho) \right) = \frac{1}{\rho} \text{curl} \nabla P(\rho) + \nabla \left( \frac{1}{\rho} \right) \times \nabla P(\rho) = 0.
\]

The advantage of the above reformulation is to get rid of the pressure for \( \Omega \), the incompressible part of the velocity, while we still keep all information of the velocity field \( u \).

For the simplicity of our presentation, our proof focuses on the case: \( P(\rho) = \frac{1}{2} \rho^2 \). The general barotropic case (\( P(\rho) \) is an increasing convex function of \( \rho \)) can be verified by a slight modification of the argument below.

In this paper, we shall prove the existence of global strong solution for initial datum that is close to an equilibrium state \( (1, 0, \hat{d}) \) with a constant vector \( \hat{d} \in S^2 \). The result is valid for any positive constant density \( \hat{\rho} \) and we take \( \hat{\rho} = 1 \) just for simplicity. Keeping (3.2) in mind, it is convenient to reformulate the original system (1.1) into a new system in terms of \( \rho, h, \Omega \) and \( d \)

\[
\begin{aligned}
\partial_t (\rho - 1) + u \cdot \nabla (\rho - 1) + \Lambda h &= -(\rho - 1) \text{div} u, \\
\partial_t h - \nu \Delta h - \Lambda (\rho - 1) &= \Lambda^{-1} \text{div} \mathcal{N}, \\
\partial_t \Omega - \mu \Delta \Omega &= \Lambda^{-1} \text{curl} \mathcal{N}, \\
\frac{\partial d}{\partial t} + u \cdot \nabla d - \Delta d &= |\nabla d|^2 d,
\end{aligned}
\] (3.3a,b,c,d)

subject to initial conditions

\[
\begin{aligned}
\rho|_{t=0} &= \rho_0(x), & h|_{t=0} &= h_0(x) = \Lambda^{-1} \text{div} u_0(x), \\
\Omega|_{t=0} &= \Omega(x) = \Lambda^{-1} \text{curl} u_0(x), & d|_{t=0} &= d_0(x).
\end{aligned}
\] (3.4)

Let us now introduce the functional space that appears in the global existence theorem.
Definition 3.1. For \( T > 0 \), and \( s \in \mathbb{R} \), we denote
\[
\mathfrak{B}_T^s = \left\{ (e, f, g) \in \left( L^1(0, T; \dot{B}^{s+1, s}) \cap C([0, T]; \dot{B}^{s-1, s}) \right) \times \left( L^1(0, T; B^{s+1}) \cap C([0, T]; B^{s-1}) \right)^3 \times \left( L^1(0, T; \dot{B}^{s+1, s+2}) \cap C([0, T]; \dot{B}^{s-1, s}) \right)^3 \right\}
\]
and
\[
\| (e, f, g) \|_{\mathfrak{B}_T^s} = \| e \|_{L^\infty_t \dot{B}^{\frac{s+1}{s-1}}} + \| f \|_{L^\infty_t B^{\frac{s+1}{s-1}}} + \| g \|_{L^\infty_t \dot{B}^{\frac{s-1}{s+2}}} + \| e \|_{L^1_t \dot{B}^{\frac{s+1}{s-1}}} + \| f \|_{L^1_t B^{\frac{s+1}{s-1}}} + \| g \|_{L^1_t \dot{B}^{\frac{s+1}{s+2}}}.
\]
We use the notation \( \mathfrak{B}_s \) if \( T = +\infty \) by changing the interval \([0, T]\) into \([0, \infty)\) in the definition above.

The main result of this paper is as follows:

Theorem 3.1. Let \( \hat{d} \in \mathbb{R}^3 \) be an arbitrary constant unit vector. There exists two positive constants \( \eta \) and \( \Gamma \), such that, if \( \rho_0 - 1 \in \dot{B}^{\frac{3}{2}, 1} \), \( u_0 \in B^{\frac{3}{2}} \) and \( d_0 - \hat{d} \in \dot{B}^{\frac{3}{2}, 1} \) satisfy
\[
\| \rho_0 - 1 \|_{\dot{B}^{\frac{3}{2}, 1}} + \| u_0 \|_{B^{\frac{3}{2}}} + \| d_0 - \hat{d} \|_{\dot{B}^{\frac{3}{2}, 1}} \leq \eta,
\]
then system \((1.1) - (1.2)\) has a unique global strong solution \((\rho, u, d)\) with \((\rho - 1, u, d - \hat{d})\) in \( \mathfrak{B}^{\frac{3}{2}} \) satisfying
\[
\| (\rho - 1, u, d - \hat{d}) \|_{\mathfrak{B}^s} \leq \Gamma \left( \| \rho_0 - 1 \|_{\dot{B}^{\frac{3}{2}, 1}} + \| u_0 \|_{B^{\frac{3}{2}}} + \| d_0 - \hat{d} \|_{\dot{B}^{\frac{3}{2}, 1}} \right).
\]

The local existence in \( \mathfrak{B}^{\frac{3}{2}} \) can be established by a standard fixed point argument, for instance, see [3], and in particular the unique solution satisfies \( \hat{d}(t, x) \in S^2 \) whenever it exists. The global existence of \((1.1) - (1.2)\) will be established by extending a local solution with the help of uniform estimates for the local solution when the initial data is sufficiently “small”. In the rest of this paper, we focus on the uniform estimates and uniqueness of the solution to \((1.1) - (1.2)\).

Remark 3.1. Similar results for other dimensions are still true. As the density is equal to one, we recover a global existence result for incompressible liquid crystal flows, which is similar to [21].

4. Uniform Estimates for Linearized Systems with Convection

In this section, our goal is to obtain uniform estimates of local solutions. For this purpose, we consider proper linearized systems with convection that are associated with the reformulated system \([3.3]\). First, we investigate the following linearized equations for \( \Omega \) and \( d \).

\[
\partial_t \Omega + u \cdot \nabla \Omega - \mu \Delta \Omega = \mathcal{L}, \quad (4.1a)
\]
The proof can be carried out in three steps. Adding (4.4) and (4.5) together, we obtain

\[ L \]

where \( L \), \( M \) and \( u \) are given functions. For the system (4.1), we have the following estimate:

**Proposition 4.1.** Denote

\[ V(t) := \int_0^t \|u(s)\|_{B^1_{2,2}} ds. \tag{4.2} \]

Let \((\Omega, d)\) be a solution of (4.1) on \([0, T)\). Then the following estimate holds on \([0, T)\):

\[
\|d(t)\|_{B^\frac{1}{2}_{2,2}} + \|\Omega(t)\|_{B^\frac{1}{2}_{2,2}} + \int_0^t \left( \|d(s)\|_{B^\frac{5}{2}_{2,2}} + \|\Omega(s)\|_{B^\frac{5}{2}_{2,2}} \right) ds \\
\leq Ce^{CV(t)} \left\{ \|d_0\|_{B^\frac{1}{2}_{2,2}} + \|\Omega_0\|_{B^\frac{1}{2}_{2,2}} + \int_0^t e^{-CV(s)} \left( \|L\|_{B^\frac{1}{2}_{2,2}} + \|M\|_{B^\frac{1}{2}_{2,2}} \right) ds \right\},
\]

where \( C \) is a universal positive constant.

**Proof.** To prove this proposition, we first localize (4.1) into low and high frequencies according to the Littlewood–Paley decomposition. Then each dyadic block can be estimated by using energy method.

Let \((\Omega, d)\) be a solution of (4.1) and \( K > 0 \). We introduce the following transformation of variables (3):

\[ \tilde{\Omega} = e^{-KV(t)} \Omega, \quad \tilde{d} = e^{-KV(t)} d, \quad \tilde{L} = e^{-KV(t)} L, \quad \tilde{M} = e^{-KV(t)} M. \]

Applying the operator \( \Delta_q \) to the system (4.1), we deduce that \((\Delta_q \tilde{\Omega}, \Delta_q \tilde{d})\) satisfies

\[
\left\{ \begin{aligned}
\partial_t \Delta_q \tilde{\Omega} + \Delta_q (u \cdot \nabla \tilde{\Omega}) - \mu \Delta \Delta_q \tilde{\Omega} &= \Delta_q \tilde{L} - KV'(t) \Delta_q \tilde{\Omega}, \\
\partial_t \Delta_q \tilde{d} + \Delta_q (u \cdot \nabla \tilde{d}) - \Delta \Delta_q \tilde{d} &= \Delta_q \tilde{M} - KV'(t) \Delta_q \tilde{d}.
\end{aligned} \right. \tag{4.3}
\]

The proof can be carried out in three steps.

**Step 1: Low Frequencies.** Suppose \( q \leq 0 \) and define

\[ f^2_q = \||\Delta_q \tilde{\Omega}\|^2_{L^2} + \||\Delta_q \tilde{d}\|^2_{L^2}. \]

Taking the \( L^2 \)-scalar product of the first equation of (4.3) with \( \Delta_q \tilde{\Omega} \), and the second equation with \( \Delta_q \tilde{d} \), we obtain the following two identities:

\[
\frac{1}{2} \frac{d}{dt} \||\Delta_q \tilde{\Omega}\|^2_{L^2} + (\Delta_q (u \cdot \nabla \tilde{\Omega}) | \Delta_q \tilde{\Omega}) + \mu \|\Delta \Delta_q \tilde{\Omega}\|^2_{L^2} = (\Delta_q \tilde{L} | \Delta_q \tilde{\Omega}) - KV'(t) \|\Delta_q \tilde{\Omega}\|^2_{L^2}, \tag{4.4}
\]

and

\[
\frac{1}{2} \frac{d}{dt} \||\Delta_q \tilde{d}\|^2_{L^2} + (\Delta_q (u \cdot \nabla \tilde{d}) | \Delta_q \tilde{d}) + \|\Delta \Delta_q \tilde{d}\|^2_{L^2} = (\Delta_q \tilde{M} | \Delta_q \tilde{d}) - KV'(t) \|\Delta_q \tilde{d}\|^2_{L^2}. \tag{4.5}
\]

Adding (4.4) and (4.5) together, we obtain

\[
\frac{1}{2} \frac{d}{dt} f^2_q + \mu \|\Delta \Delta_q \tilde{\Omega}\|^2_{L^2} + \|\Delta \Delta_q \tilde{d}\|^2_{L^2} = \mathcal{X} - KV'(t) f^2_q, \tag{4.6}
\]
where
\[ X = - (\Delta_q (u \cdot \nabla \tilde{\Omega}) | \Delta_q \tilde{\Omega} ) - (\Delta_q (u \cdot \nabla \tilde{d}) | \Delta_q \tilde{d}) + (\Delta_q \tilde{\nabla} | \Delta_q \tilde{\Omega}) + (\Delta_q \tilde{\nabla} | \Delta_q \tilde{d}). \]
The term \( X \) can be estimated by using Lemma 2.1 (taking \( s_1 = s_2 = \frac{1}{2} \) in (2.1)) such that
\begin{equation}
|X| \leq C f_q \left( \|\Delta_q \tilde{\nabla}\|_{L^2} + \|\Delta_q \tilde{\nabla} t\|_{L^2} + 2^{- \frac{q}{2}} \alpha_q V' (\|\tilde{d}\|_{B^{\frac{1}{2}, \frac{1}{2}}} + \|\tilde{\Omega}\|_{B^{\frac{1}{2}, \frac{1}{2}}}) \right),
\end{equation}
(4.7)
In the last inequality, we used the embedding
\[ B^{\frac{1}{2}, \frac{1}{2}} \hookrightarrow B^{\frac{1}{2}}. \]
Hence, combining (4.3) and (4.7) together, we have for \( q \leq 0 \)
\begin{equation}
\frac{1}{2} \frac{d}{dt} f_q^2 + \mu \|\Lambda \Delta_q \tilde{\Omega}\|_{L^2}^2 + \|\Lambda \Delta_q \tilde{d}\|_{L^2}^2 \\
\leq C f_q \left( \|\Delta_q \tilde{\nabla}\|_{L^2} + \|\Delta_q \tilde{\nabla} t\|_{L^2} + 2^{- \frac{q}{2}} \alpha_q V' (\|\tilde{d}\|_{B^{\frac{1}{2}, \frac{1}{2}}} + \|\tilde{\Omega}\|_{B^{\frac{1}{2}, \frac{1}{2}}}) \right) - K V'(t) f_q^2.
\end{equation}
(4.8)
**Step 2: High Frequencies.** In this step, we assume \( q > 0 \) and set \( f_q^2 = \|\Delta_q \tilde{\Omega}\|_{L^2}^2 + \|\Lambda \Delta_q \tilde{d}\|_{L^2}^2 \)
We apply the operator \( \Lambda \) to the second equation of (4.3), multiply by \( \Lambda \Delta_q \tilde{d} \) and integrate over \( \mathbb{R}^3 \) to yield
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\Lambda \Delta_q \tilde{d}\|_{L^2}^2 + (\Lambda \Delta_q (u \cdot \nabla \tilde{d}) | \Lambda \Delta_q \tilde{d}) + \|\Lambda^2 \Delta_q \tilde{d}\|_{L^2}^2 \\
= (\Lambda \Delta_q \tilde{\nabla} | \Lambda \Delta_q \tilde{d}) - K V'(t) \|\Lambda \Delta_q \tilde{d}\|_{L^2}^2.
\end{equation}
(4.9)
Adding (4.3) and (4.9) together, we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} f_q^2 + \mu \|\Lambda \Delta_q \tilde{\Omega}\|_{L^2}^2 + \|\Lambda^2 \Delta_q \tilde{d}\|_{L^2}^2 = \mathcal{Y} - K V'(t) f_q^2
\end{equation}
(4.10)
with
\[ \mathcal{Y} = - (\Delta_q (u \cdot \nabla \tilde{\Omega}) | \Delta_q \tilde{\Omega}) - (\Delta_q (u \cdot \nabla \tilde{d}) | \Lambda \Delta_q \tilde{d}) + (\Delta_q \tilde{\nabla} | \Delta_q \tilde{\Omega}) + (\Lambda \Delta_q \tilde{\nabla} | \Lambda \Delta_q \tilde{d}). \]
For \( \mathcal{Y} \), using Lemma 2.1, one has
\begin{equation}
|\mathcal{Y}| \leq \|\Delta_q \tilde{\Omega}\|_{L^2} \|\Delta_q \tilde{\nabla}\|_{L^2} + \|\Lambda \Delta_q \tilde{d}\|_{L^2} \|\Lambda \Delta_q \tilde{\nabla}\|_{L^2} + 2^{- \frac{q}{2}} \alpha_q V' (\|\tilde{d}\|_{B^{\frac{1}{2}, \frac{1}{2}}} + \|\tilde{\Omega}\|_{B^{\frac{1}{2}, \frac{1}{2}}}) \|\Lambda \Delta_q \tilde{\nabla}\|_{L^2}
+ C 2^{-q (\delta_{\frac{3}{2}} (\frac{3}{2} - \frac{q}{2})) - 1} \alpha_q V' (\|\tilde{d}\|_{B^{\frac{1}{2}, \frac{1}{2}}} + \|\tilde{\Omega}\|_{B^{\frac{1}{2}, \frac{1}{2}}}) \end{equation}
(4.11)
Hence, combining (4.10) and (4.11) together, we have for \( q > 0 \)
\begin{equation}
\frac{1}{2} \frac{d}{dt} f_q^2 + \mu \|\Lambda \Delta_q \tilde{\Omega}\|_{L^2}^2 + \|\Lambda^2 \Delta_q \tilde{d}\|_{L^2}^2 \\
\leq C f_q \left( \|\Delta_q \tilde{\nabla}\|_{L^2} + \|\Lambda \Delta_q \tilde{\nabla}\|_{L^2} + 2^{- \frac{q}{2}} \alpha_q V' (\|\tilde{d}\|_{B^{\frac{1}{2}, \frac{1}{2}}} + \|\tilde{\Omega}\|_{B^{\frac{1}{2}, \frac{1}{2}}}) \right) - K V'(t) f_q^2.
\end{equation}
(4.12)
Step 3: Damping Effect. We are now going to show that inequalities (4.8) and (4.12) entail a decay for \( \Omega \) and \( d \). Denote \( g_q = 2^q f_q \) for \( q \in \mathbb{Z} \). The well-known Bernstein's inequality implies that

\[
C_0 g_q \leq 2^q \| \Delta_q \tilde{\Omega} \|_{L^2} + 2^{q+\frac{1}{2}+\frac{1}{2}}(q) \| \Delta_q \tilde{d} \|_{L^2} \leq \frac{1}{C_0} g_q
\]

for some universal positive constant \( C_0 \). Therefore, we infer from (4.8), (4.12) that there exists a universal positive constant \( \kappa \) such that

\[
\frac{1}{2} \frac{d}{dt} g_q^2 + \kappa \min\{\mu, 1\} 2^{2q} g_q^2 \leq C \alpha_q g_q \left( \| \tilde{L} \|_{B^{\frac{1}{2}}} + \| \tilde{\mathcal{M}} \|_{B^{\frac{1}{2}}} + V(\| \tilde{d} \|_{B^{\frac{1}{2}}} + \| \tilde{\Omega} \|_{B^{\frac{1}{2}}}) \right)
- K V'(t) g_q^2.
\]

(4.14)

Let \( \delta > 0 \) be a small parameter and denote \( \chi_q^2 = g_q^2 + \delta^2 \). From (4.14), dividing by \( \chi_q \), we obtain

\[
\frac{d}{dt} \chi_q + \kappa \min\{\mu, 1\} 2^{2q} \chi_q \leq C \alpha_q \left( \| \tilde{L} \|_{B^{\frac{1}{2}}} + \| \tilde{\mathcal{M}} \|_{B^{\frac{1}{2}}} + V(\| \tilde{d} \|_{B^{\frac{1}{2}}} + \| \tilde{\Omega} \|_{B^{\frac{1}{2}}}) \right)
- K V'(t) \chi_q + \delta K V' + \delta \kappa 2^{2q}.
\]

Integrating the above inequality over \([0, t]\) and having \( \delta \) tend to 0, we obtain,

\[
g_q(t) + \kappa \min\{\mu, 1\} 2^{2q} \int_0^t g_q(s) ds \\
\leq g_q(0) + C \int_0^t \alpha_q(s) \left( \| \tilde{L} \|_{B^{\frac{1}{2}}} + \| \tilde{\mathcal{M}} \|_{B^{\frac{1}{2}}} \right) ds \\
+ \int_0^t V'(s) \left( C \alpha_q(s) (\| \tilde{d} \|_{B^{\frac{1}{2}}} + \| \tilde{\Omega} \|_{B^{\frac{1}{2}}}) - K g_q(s) \right) ds.
\]

(4.15)

(4.13) implies that

\[
C \alpha_q(s) (\| \tilde{d} \|_{B^{\frac{1}{2}}} + \| \tilde{\Omega} \|_{B^{\frac{1}{2}}}) - K g_q(s)
\leq C \alpha_q(s) (\| \tilde{d} \|_{B^{\frac{1}{2}}} + \| \tilde{\Omega} \|_{B^{\frac{1}{2}}}) - C_0 K 2^q \| \Delta_q \tilde{\Omega} \|_{L^2}.
\]

If we choose \( K \) such that \( K C_0 > C \), then we infer from the fact \( \sum_{q \in \mathbb{Z}} \alpha_q \leq 1 \) that

\[
\sum_{q \in \mathbb{Z}} \left\{ C \alpha_q(s) \left( \| \tilde{d} \|_{B^{\frac{1}{2}}} + \| \tilde{\Omega} \|_{B^{\frac{1}{2}}} \right) - K g_q(s) \right\} \leq 0.
\]

(4.16)

With the inequality (4.16) in hand, after summation over \( \mathbb{Z} \), we deduce from (4.15) that

\[
\| \tilde{\Omega}(t) \|_{B^{\frac{1}{2}}} + \| \tilde{d}(t) \|_{B^{\frac{1}{2}}} + \kappa \min\{\mu, 1\} \int_0^t \left( \| \tilde{\mathcal{M}}(s) \|_{B^{\frac{1}{2}}} + \| \tilde{\mathcal{L}}(s) \|_{B^{\frac{1}{2}}} \right) ds \\
\leq C \left\{ \| \Omega_0 \|_{B^{\frac{1}{2}}} + \| d_0 \|_{B^{\frac{1}{2}}} + \int_0^t \left( \| \mathcal{L}(s) \|_{B^{\frac{1}{2}}} + \| \mathcal{M}(s) \|_{B^{\frac{1}{2}}} \right) ds \right\}.
\]

(4.17)

This finishes the proof. \( \square \)
Next, we turn to consider the linearized system for $\varrho$ and $h$:

\begin{align}
\partial_t \varrho + u \cdot \nabla \varrho + \Delta \varrho &= \mathcal{J}, \\
\partial_t h + u \cdot \nabla h - \nu \Delta h - \Lambda \varrho &= \mathcal{K},
\end{align}

where $\mathcal{J}$, $\mathcal{K}$ and $u$ are given functions. Note that this system is different from (4.1) since there are stronger couplings between $\varrho$ and $h$. For the system $(4.18)$, we have the following estimates (see a similar result in [3, Proposition 2.3]):

**Proposition 4.2.** Let $(\varrho, h)$ be a solution of $(4.18)$ on $[0, T)$. Then the following estimate holds on $[0, T)$:

\[
\|\varrho(t)\|_{\dot{B}^{1/2}_{2, \infty}} + \|h(t)\|_{\dot{B}^{1/2}_{2, \infty}} + \int_0^t \left( \|\varrho(s)\|_{\dot{B}^{1/2}_{2, \infty}} + \|h(s)\|_{\dot{B}^{1/2}_{2, \infty}} \right) ds \leq C e^{CV(t)} \left\{ \|\varrho_0\|_{\dot{B}^{1/2}_{2, \infty}} + \|h_0\|_{\dot{B}^{1/2}_{2, \infty}} + \int_0^t e^{-CV(s)} (\|\mathcal{K}\|_{\dot{B}^{1/2}_{2, \infty}} + \|\mathcal{J}\|_{\dot{B}^{1/2}_{2, \infty}}) ds \right\},
\]

where $C$ is a universal positive constant and $V$ is given by (4.22).

**Proof.** The proof is due to the argument in [3], and similar to that in Proposition 4.1. For the completeness, below we present a proof that is slightly different from [3]. To this end, again we first localize $(4.18)$ in low and high frequencies according to the Littlewood–Paley decomposition.

Let $(\varrho, h)$ be a solution of $(4.18)$ and $K > 0$. Define

\[
\tilde{\varrho} = e^{-KV(t)} \varrho, \quad \tilde{h} = e^{-KV(t)} h, \quad \tilde{\mathcal{J}} = e^{-KV(t)} \mathcal{J}, \quad \tilde{\mathcal{K}} = e^{-KV(t)} \mathcal{K}.
\]

Applying the operator $\Delta_q$ to the system $(4.18)$, we deduce that $(\Delta_q \tilde{h}, \Delta_q \tilde{\varrho})$ satisfies

\[
\begin{cases}
\partial_t \Delta_q \tilde{\varrho} + \Delta_q (u \cdot \nabla \tilde{\varrho}) + \Lambda \Delta_q \tilde{h} = \Delta_q \tilde{\mathcal{J}} - KV'(t) \Delta_q \tilde{\varrho}, \\
\partial_t \Delta_q \tilde{h} + \Delta_q (u \cdot \nabla \tilde{h}) - \nu \Delta \Delta_q \tilde{h} - \Lambda \Delta_q \tilde{\varrho} = \Delta_q \tilde{\mathcal{K}} - KV'(t) \Delta_q \tilde{h}.
\end{cases}
\]

Set

\[
q_0 = \log_2 \left( \frac{3}{\nu} \right).
\]

**Step 1: Low Frequencies.** Suppose $q \leq q_0$. As a result, $2^q \leq 2^{q_0} \leq \frac{3}{\nu}$. Taking the $L^2$-scalar product of the first equation of $(4.19)$ with $\Delta_q \tilde{\varrho}$ and the second equation of $(4.19)$ with $\Delta_q \tilde{h}$, we obtain the following two identities:

\[
\frac{1}{2} \frac{d}{dt} \|\Delta_q \tilde{\varrho}\|_{L^2}^2 + (\Delta_q (u \cdot \nabla \tilde{\varrho}) | \Delta_q \tilde{\varrho}) + (\Lambda \Delta_q \tilde{h} | \Delta_q \tilde{\varrho}) = (\Delta_q \tilde{\mathcal{J}} | \Delta_q \tilde{\varrho}) - KV' \|\Delta_q \tilde{\varrho}\|_{L^2}^2,
\]

and

\[
\frac{1}{2} \frac{d}{dt} \|\Delta_q \tilde{h}\|_{L^2}^2 + (\Delta_q (u \cdot \nabla \tilde{h}) | \Delta_q \tilde{h}) + \nu \|\Delta \Delta_q \tilde{h}\|_{L^2}^2 - (\Lambda \Delta_q \tilde{\varrho} | \Delta_q \tilde{h})
\]

\[
= (\Delta_q \tilde{\mathcal{K}} | \Delta_q \tilde{h}) - KV' \|\Delta_q \tilde{h}\|_{L^2}^2.
\]
Next, we derive an identity involving \((\Lambda \Delta_q \tilde{\varrho} | \Delta_q \tilde{h})\). For this purpose, we apply \(\Lambda\) to the first equation in (4.19) and take the \(L^2\) scalar product with \(\Delta_q \tilde{h}\), then take the scalar product of the second equation in (4.19) with \(\Lambda \Delta_q \tilde{\varrho}\). Summing up both equalities, we get

\[
\frac{d}{dt} (\Lambda \Delta_q \tilde{\varrho} | \Delta_q \tilde{h}) + (\Delta_q (u \cdot \nabla \tilde{h}) | \Lambda \Delta_q \tilde{\varrho}) + (\Lambda \Delta_q (u \cdot \nabla \tilde{\varrho}) | \Delta_q \tilde{h})
- \| \Lambda \Delta_q \tilde{\varrho} \|^2_{L^2} + \| \Lambda \Delta_q \tilde{h} \|^2_{L^2} + \nu (\Lambda^2 \Delta_q \tilde{h} | \Lambda \Delta_q \tilde{\varrho})
= (\Lambda \Delta_q \tilde{J} | \Delta_q \tilde{h}) + (\Delta_q \tilde{K} | \Lambda \Delta_q \tilde{\varrho}) - 2K V'(\Lambda \Delta_q \tilde{\varrho} | \Delta_q \tilde{h})
\]

(4.23)

Let \(\tau\) be a small constant such that \(0 < \tau \leq \frac{8}{9}\). We define

\[
f_q^2 = \| \Delta_q \tilde{\varrho} \|^2_{L^2} + \| \Delta_q \tilde{h} \|^2_{L^2} - \frac{\tau \nu}{4} (\Lambda \Delta_q \tilde{\varrho} | \Delta_q \tilde{h}).
\]

The Bernstein’s inequality yields

\[
\left| \frac{\tau \nu}{4} (\Lambda \Delta_q \tilde{\varrho} | \Delta_q \tilde{h}) \right| \leq \frac{\tau \nu}{4} \| \Lambda \Delta_q \tilde{\varrho} \|_{L^2} \| \Delta_q \tilde{h} \|_{L^2} \leq \frac{\tau \nu}{4} 2^{q/2} \| \Delta_q \tilde{\varrho} \|_{L^2} \| \Delta_q \tilde{h} \|_{L^2}
\leq 2^{q/2} \frac{\tau \nu}{8} (\| \Delta_q \tilde{\varrho} \|^2_{L^2} + \| \Delta_q \tilde{h} \|^2_{L^2})
\leq \frac{1}{3} (\| \Delta_q \tilde{\varrho} \|^2_{L^2} + \| \Delta_q \tilde{h} \|^2_{L^2})
\]

which implies that

\[
f_q^2 \approx \| \Delta_q \tilde{\varrho} \|^2_{L^2} + \| \Delta_q \tilde{h} \|^2_{L^2}.
\]

(4.24)

Here we note that the universal constant due to the Bernstein’s inequality is harmless in our estimate and thus assumed to be one for simplicity.

Multiplying (4.23) by \(-\frac{\tau \nu}{8}\) and adding it with (4.21), (4.22), we obtain that

\[
\frac{1}{2} \frac{d}{dt} f_q^2 + \frac{\nu}{8} \left[ (8 - \tau) \| \Delta_q \tilde{\varrho} \|_{L^2}^2 + \tau \| \Delta_q \tilde{\varrho} \|_{L^2}^2 - \nu \tau (\Lambda^2 \Delta_q \tilde{h} | \Lambda \Delta_q \tilde{\varrho}) \right] = \mathcal{X}_1 - K V'(t) f_q^2,
\]

(4.25)

with

\[
\mathcal{X}_1 = - (\Delta_q (u \cdot \nabla \tilde{\varrho}) | \Delta_q \tilde{h}) - (\Delta_q (u \cdot \nabla \tilde{\varrho}) | \Delta_q \tilde{\varrho}) + \frac{\tau \nu}{8} (\Lambda \Delta_q (u \cdot \nabla \tilde{\varrho}) | \Delta_q \tilde{\varrho})
+ \frac{\tau \nu}{8} (\Lambda \Delta_q (u \cdot \nabla \tilde{\varrho}) | \Delta_q \tilde{h}) + (\Delta_q \tilde{K} | \Delta_q \tilde{h}) + (\Delta_q \tilde{J} | \Delta_q \tilde{\varrho})
- \frac{\tau \nu}{8} \left[ (\Lambda \Delta_q \tilde{J} | \Delta_q \tilde{h}) + (\Delta_q \tilde{K} | \Lambda \Delta_q \tilde{\varrho}) \right].
\]
Since \( \phi^{1/2} \leq \frac{3}{2} \), the assumption \( q \leq q_0 \) implies that \( 2^{-q(\phi^{1/2} - \frac{3}{2})} \leq 2^{q_0} \). As a consequence, for \( X_1 \), using Lemma [2.1] and [1.24], we have

\[
|X_1| \leq C \alpha q^{-2} V'(|\tilde{\eta}|_{B^1} + \|\Delta_q \tilde{h}|_{L^2} + \|\tilde{\vartheta}|_{B^1} + \|\Delta_q \tilde{\vartheta}|_{L^2} + \|\Delta_q \tilde{\kappa}|_{L^2} + \|\Delta_q \tilde{\varphi}|_{L^2} + \|\Delta_q \tilde{\vartheta}|_{L^2} + \|\Delta_q \tilde{\vartheta}|_{L^2} + \|\Delta_q \tilde{\varphi}|_{L^2} + \|\Delta_q \tilde{\varphi}|_{L^2} + \|\tilde{h}|_{B^{1/2}} + \|\tilde{\vartheta}|_{B^{1/2}} + \|\tilde{\kappa}|_{B^{1/2}})
\]

(4.26)

where \( C \) is a universal constant that may depend on \( \nu \). Besides, due to our choice of \( \tau \), we can conclude that

\[
|\nu \tau (\Lambda^2 \Delta_q \tilde{h} | \Lambda \Delta_q \tilde{\vartheta})| \leq \nu \tau 2^{q_0} \|\Lambda \Delta_q \tilde{h}|_{L^2} \|\Lambda \Delta_q \tilde{\vartheta}|_{L^2} \leq 4 \|\Lambda \Delta_q \tilde{h}|^2_{L^2} + \frac{\tau}{2} \|\Lambda \Delta_q \tilde{\vartheta}|^2_{L^2}. \quad (4.27)
\]

Then it easily follows from the Bernstein’s inequality that

\[
2^{q_0} \left( \|\Delta_q \tilde{h}|^2_{L^2} + \|\Delta_q \tilde{\vartheta}|^2_{L^2} \right) \approx (8 - \tau) \|\Lambda \Delta_q \tilde{h}|^2_{L^2} + \tau \|\Lambda \Delta_q \tilde{\vartheta}|^2_{L^2} - \nu \tau (\Lambda^2 \Delta_q \tilde{h} | \Lambda \Delta_q \tilde{\vartheta}).
\]

Hence, combining (4.25) and (4.26) together, we can find a positive universal constant \( \kappa \) such that

\[
\frac{1}{2} \frac{d}{dt} f_q^2 + \kappa 2^{q_0} \left( \|\Delta_q \tilde{h}|^2_{L^2} + \|\Delta_q \tilde{\vartheta}|^2_{L^2} \right) \leq C f_q \left( \|\Delta_q \tilde{\kappa}|_{L^2} + \|\Delta_q \tilde{\varphi}|_{L^2} + 2^{-q} \alpha q V' \|\tilde{\vartheta}|_{B^{1/2}} + \|\tilde{h}|_{B^{1/2}} \right) - KV'(t) f_q^2. \quad (4.28)
\]

**Step 2: High Frequencies.** Suppose \( q \geq q_0 \). We apply the operator \( \Lambda \) to the first equation of (4.19), multiply by \( \Lambda \Delta_q \tilde{\vartheta} \) and integrate over \( \mathbb{R}^3 \) to yield

\[
\frac{1}{2} \frac{d}{dt} \|\Lambda \Delta_q \tilde{\vartheta}|^2_{L^2} = (\Lambda \Delta_q (u \cdot \nabla \tilde{\vartheta}) | \Lambda \Delta_q \tilde{\vartheta}) + (\Lambda^2 \Delta_q \tilde{h} | \Lambda \Delta_q \tilde{\vartheta}) \quad (4.29)
\]

Set

\[
f_q^2 = \|\Lambda \Delta_q \tilde{\vartheta}|^2_{L^2} + \frac{3}{\nu^2} \|\Delta_q \tilde{h}|^2_{L^2} - \frac{2}{\nu} (\Lambda \Delta_q \tilde{\vartheta} | \Delta q \tilde{h}) \approx \|\Delta_q \tilde{h}|^2_{L^2} + \|\Lambda \Delta_q \tilde{\vartheta}|^2_{L^2}.
\]

It easily follow from the Cauchy–Schwarz inequality that

\[
f_q^2 \approx \|\Delta_q \tilde{h}|^2_{L^2} + \|\Lambda \Delta_q \tilde{\vartheta}|^2_{L^2}.
\]

A linear combination of (4.22), (4.28) and (4.29) yields that

\[
\frac{1}{2} \frac{d}{dt} f_q^2 + 2 \frac{\nu}{\nu} \|\Lambda \Delta_q \tilde{h}|^2_{L^2} + \frac{1}{\nu} \|\Lambda \Delta_q \tilde{\vartheta}|^2_{L^2} - \frac{3}{\nu^2} (\Lambda \Delta_q \tilde{\vartheta} | \Delta q \tilde{h}) = \mathcal{Y}_1 - KV'(t) f_q^2 \quad (4.30)
\]
with
\[ \mathcal{V}_1 = -\frac{3}{\nu^2} (\Delta_q (u \cdot \nabla \hat{h}) | \Delta_q \hat{h}) - (\Lambda \Delta_q (u \cdot \nabla \hat{\varrho}) | \Lambda \Delta_q \hat{\varrho}) + \frac{3}{\nu^2} (\Delta_q \tilde{K} | \Delta_q \hat{h}) \\
+ (\Lambda \Delta_q \tilde{J} | \Lambda \Delta_q \hat{\varrho}) + \frac{1}{\nu} (\Delta_q (u \cdot \nabla \hat{h}) | \Lambda \Delta_q \hat{\varrho}) + \frac{1}{\nu} (\Lambda \Delta_q (u \cdot \nabla \hat{\varrho}) | \Delta_q \hat{h}) \\
- \frac{1}{\nu} (\Lambda \Delta_q \tilde{J} | \Delta_q \hat{h}) - \frac{1}{\nu} (\Delta_q \tilde{K} | \Lambda \Delta_q \hat{\varrho}). \]

For \( \mathcal{V}_1 \), the assumption \( q \geq q_0 \) implies that \( 2^{-q(\phi^2 + 2)(-\frac{3}{2})} \leq 1 \), then using Lemma \( \ref{lemma} \), we can see that
\[ |\mathcal{V}_1| \leq \frac{3C}{\nu^2} \alpha_q 2^{-\frac{9}{2}} V' ||\hat{h}||_{B^\frac{1}{2}} ||\Lambda \Delta_q \hat{\varrho}||_{L^2} + C \alpha_q 2^{-q(\phi^2 + \frac{3}{2}(q-\frac{1}{2}))} V' ||\hat{\varrho}||_{B^\frac{1}{2}} ||\Lambda \Delta_q \hat{\varrho}||_{L^2} \\
+ \frac{3}{\nu^2} ||\Delta_q \tilde{K}||_{L^2} ||\Delta_q \hat{h}||_{L^2} + ||\Lambda \Delta_q \tilde{J}||_{L^2} ||\Lambda \Delta_q \hat{\varrho}||_{L^2} \\
+ \frac{1}{\nu} (||\Delta_q \tilde{J}||_{L^2} ||\Delta_q \hat{h}||_{L^2} + ||\Delta_q \tilde{K}||_{L^2} ||\Lambda \Delta_q \hat{\varrho}||_{L^2}) \\
+ C \alpha_q V'(2^{-\frac{3}{2}} ||\Lambda \Delta_q \hat{\varrho}||_{L^2} ||\hat{h}||_{B^\frac{1}{2}} + 2^{-q(\phi^2 + \frac{3}{2}(q-\frac{1}{2}))} ||\hat{\varrho}||_{B^\frac{1}{2}} ||\Lambda \Delta_q \hat{\varrho}||_{L^2}) \\
\leq C f_q \left(||\Delta_q \tilde{K}||_{L^2} + ||\Lambda \Delta_q \tilde{J}||_{L^2} + 2^{-\frac{3}{2}} \alpha_q V'(||\hat{\varrho}||_{B^\frac{1}{2}} ||\hat{h}||_{B^\frac{1}{2}}) \right). \]

Since \( q \geq q_0 = \log_2 \left(\frac{2}{\nu}\right) \), the Bernstein’s inequality implies that
\[ \frac{3}{\nu^2} (\Lambda \Delta_q \hat{\varrho} | \Lambda \Delta_q \hat{h}) \leq 2^{-q_0} \frac{3}{\nu^2} ||\Lambda \Delta_q \hat{\varrho}||_{L^2} ||\Lambda \Delta_q \hat{h}||_{L^2} \leq \frac{1}{2\nu} (||\Lambda \Delta_q \hat{\varrho}||_{L^2}^2 + ||\Lambda \Delta_q \hat{h}||_{L^2}^2). \]

As a result,
\[ ||\Lambda \Delta_q \hat{\varrho}||_{L^2}^2 + ||\Lambda \Delta_q \hat{h}||_{L^2}^2 \leq ||\Lambda \Delta_q \hat{\varrho}||_{L^2}^2 + 2^{-2q_0} ||\Lambda \Delta_q \hat{h}||_{L^2}^2 \\
\leq ||\Lambda \Delta_q \hat{\varrho}||_{L^2}^2 + \frac{\nu^2}{9} ||\Lambda \Delta_q \hat{h}||_{L^2}^2 \\
\leq C \left(2^{-\frac{3}{2}} ||\Lambda \Delta_q \hat{\varrho}||_{L^2}^2 + \frac{1}{\nu} ||\Lambda \Delta_q \hat{\varrho}||_{L^2}^2 - \frac{3}{\nu^2} (\Lambda \Delta_q \hat{\varrho} | \Lambda \Delta_q \hat{h}) \right). \]

Hence, combining (1.30), (1.31) and (1.32) together, there exists a positive constant \( \kappa \) such that
\[ \frac{1}{2} \frac{d}{dt} f_q^2 + \kappa \left(||\Delta_q \hat{h}||_{L^2}^2 + ||\Lambda \Delta_q \hat{\varrho}||_{L^2}^2 \right) \leq C f_q \left(||\Delta_q \tilde{K}||_{L^2} + ||\Lambda \Delta_q \tilde{J}||_{L^2} + 2^{-\frac{3}{2}} \alpha_q V'(||\hat{\varrho}||_{B^\frac{1}{2}} ||\hat{h}||_{B^\frac{1}{2}}) \right) - K V'(t) f_q^2. \]

**Step 3: Damping Effect.** We now show that inequalities (1.28) and (1.33) entail a decay for \( h \) and \( \varrho \). Denote \( g_q = 2^{\frac{q}{2}} f_q \) for \( q \in \mathbb{Z} \). It follows from (1.28), (1.33), and Bernstein’s inequality that
\[ \frac{1}{2} \frac{d}{dt} (g_q^2) + \kappa 2^{q+2} \alpha_{q_0} \varrho_q^2 \leq C \varrho_q g_q \left(||\tilde{K}||_{B^\frac{1}{2}} + ||\tilde{J}||_{B^\frac{1}{2}} + V'(||\hat{\varrho}||_{B^\frac{1}{2}} ||\hat{h}||_{B^\frac{1}{2}}) \right) - K V'(t) g_q^2. \]
Let $\delta > 0$ be a small parameter (which will tend to 0) and denote $\chi_q^2 = g_q^2 + \delta^2$. From (4.34), dividing by $\chi_q$, we obtain
\[
\frac{d}{dt} \chi_q + \kappa 2^{\phi_2 q^2 \cdot (q - q_0)} \chi_q \leq C \alpha_q \left( \| \tilde{\mathcal{K}} \|_{B^\frac{3}{2}} + \| \tilde{\mathcal{J}} \|_{B^\frac{3}{2}} + V' \| \tilde{\mathcal{G}} \|_{B^\frac{3}{2}} + \| \tilde{h} \|_{B^\frac{3}{2}} \right)
- KV' \chi_q + \delta KV' + \delta \kappa 2^q.
\]
Integrating the above inequality over $[0, t]$ and having $\delta$ tend to 0, we obtain,
\[
g_q(t) + \kappa 2^{\phi_2 q^2 \cdot (q - q_0)} \int_0^t g_q(s) \, ds \leq g_q(0) + C \int_0^t \alpha_q(s) \left( \| \tilde{\mathcal{K}} \|_{B^\frac{3}{2}} + \| \tilde{\mathcal{J}} \|_{B^\frac{3}{2}} \right) \, ds
+ \int_0^t V' \left( \alpha_q(s) \left( \| \tilde{\mathcal{G}} \|_{B^\frac{3}{2}} + \| \tilde{h} \|_{B^\frac{3}{2}} \right) - K g_q(s) \right) \, ds.
\]
Bernstein’s inequality implies
\[
C_0 g_q \leq 2^q \| \Delta_q \tilde{h} \|_{L^2} + 2^{\phi_2 q^2 \cdot \frac{3}{2} \cdot (q - q_0)} \| \Delta_q \tilde{\mathcal{G}} \|_{L^2} \leq \frac{1}{C_0} g_q
\]
for some universal positive constant $C_0$, and hence
\[
C \alpha_q(s) \left( \| \tilde{\mathcal{G}} \|_{B^\frac{3}{2}} + \| \tilde{h} \|_{B^\frac{3}{2}} \right) - K g_q(s) \leq C \alpha_q(s) \| \tilde{h} \|_{B^\frac{3}{2}} - C \kappa 2^{\phi_2 q^2 \cdot \frac{3}{2}} \| \Delta_q \tilde{h} \|_{L^2}
+ C \alpha_q(s) \| \tilde{\mathcal{G}} \|_{B^\frac{3}{2}} - C \kappa 2^{\phi_2 q^2 \cdot \frac{3}{2} \cdot (q - q_0)} \| \Delta_q \tilde{\mathcal{G}} \|_{L^2}.
\]
If we choose $KC_0 > C$, we have
\[
\sum_{q \in \mathbb{Z}} \left\{ C \alpha_q(s) \left( \| \tilde{\mathcal{G}} \|_{B^\frac{3}{2}} + \| \tilde{h} \|_{B^\frac{3}{2}} \right) - K g_q(s) \right\} \leq 0.
\]
With the inequality (4.36) in hand, after summation over $\mathbb{Z}$, we deduce from (4.35) that
\[
| \tilde{h}(t) \|_{B^\frac{3}{2}} + | \tilde{\mathcal{G}}(t) \|_{B^\frac{3}{2}} \leq \int_0^t \left( \| \tilde{h}(\tau) \|_{B^\frac{3}{2}} + | \tilde{\mathcal{G}}(\tau) \|_{B^\frac{3}{2}} \right) \, d\tau
\leq C \left\{ \| h_0 \|_{B^\frac{3}{2}} + | q_0 \|_{B^\frac{3}{2}} + \int_0^t \left( \| \tilde{\mathcal{K}}(s) \|_{B^\frac{3}{2}} + \| \tilde{\mathcal{J}}(s) \|_{B^\frac{3}{2}} \right) \, ds \right\}.
\]

**Step 4: Smoothing Effect of $h$.** Based on the damping effect for $q$, we can now further get the smoothing effect of $h$ by considering (4.18) with $\Lambda q$ being seen as a source term. Indeed, thanks to (4.37), it suffices to state the proof for high frequencies only. We therefore assume that $q \geq q_0$.

Define $I_q = 2^q \| \Delta_q \tilde{h} \|_{L^2}$. Then, from the energy estimates for the system
\[
\partial_t \Delta_q \tilde{h} - \nu \Delta \Delta_q \tilde{h} = -\Delta_q (\mathbf{u} \cdot \nabla \tilde{h}) + \Lambda \Delta_q \tilde{\mathcal{G}} + \Delta_q \tilde{\mathcal{K}} - KV'(t) \Delta_q \tilde{h},
\]
we have
\[
\frac{1}{2} \frac{d}{dt} I_q^2 + \nu 2^q I_q^2 \leq I_q \left( 2^q \| \Delta_q \tilde{\mathcal{G}} \|_{L^2} + 2^q \| \Delta_q \tilde{\mathcal{K}} \|_{L^2} \right) + CI_q V'(t) \alpha_q \| \tilde{h} \|_{B^\frac{3}{2}},
\]
for a universal positive constant $\kappa$. Using $J_q^2 = I_q^2 + \delta^2$, integrating over $[0, t]$ and then taking the limit as $\delta \to 0$, we deduce

\[
I_q(t) + \nu 2^{q_0} \int_0^t I_q(s) ds \leq I_q(0) + \int_0^t 2^{q_0} \| \Delta_q \tilde{K}(s) \|_{L^2} ds + \int_0^t 2^{q_0} \| \Delta_q \tilde{\varrho}(s) \|_{L^2} ds + C \int_0^t V'(s) \alpha_q(s) \| \tilde{h}(s) \|_{B^{\frac{1}{2}}} ds. \tag{4.38}
\]

We therefore get

\[
\sum_{q \geq q_0} 2^{q_0} \| \Delta_q \tilde{h}(t) \|_{L^2} + \nu \int_0^t \sum_{q \geq q_0} 2^{q_0} \| \Delta_q \tilde{h}(s) \|_{L^2} ds \leq \| h_0 \|_{B^{\frac{1}{2}}} + \int_0^t \| \tilde{K}(s) \|_{B^{\frac{1}{2}}} ds + \int_0^t \sum_{q \geq q_0} 2^{q_0} \| \Delta_q \tilde{\varrho}(s) \|_{L^2} ds + CV(t) \sup_{s \in [0, t]} \| \tilde{h} \|_{B^{\frac{1}{2}}}.
\]

Using (4.37), we eventually conclude that

\[
\nu \int_0^t \sum_{q \geq q_0} 2^{q_0} \| \Delta_q \tilde{h}(s) \|_{L^2} ds \leq C(1 + V(t)) \left( \| q_0 \|_{B^{\frac{1}{2}}} + \| h_0 \|_{B^{\frac{1}{2}}} + \int_0^t \left( \| \tilde{K}(s) \|_{B^{\frac{1}{2}}} + \| \tilde{\varrho}(s) \|_{B^{\frac{1}{2}}} \right) ds \right).
\]

Combining the last inequality with (4.37), we finish the proof of Proposition 4.2.

\[
\Box
\]

5. Global Existence for Initial Data Near Equilibrium

In this section, we are going to show that if the initial data

\[
\| \rho_0 - 1 \|_{B^{\frac{1}{2}}} + \| u_0 \|_{B^{\frac{1}{2}}} + \| \mathbf{d}_0 - \mathbf{d} \|_{B^{\frac{1}{2}}} \leq \eta
\]

for some sufficiently small $\eta$, there exists a positive constant $\Gamma$ such that

\[
\| (\rho - 1, u, d - d) \|_{B^{\frac{3}{4}}} \leq \Gamma \eta.
\]

This uniform estimate will enable us to extend the local solution $\rho, u, d$ obtained within an iterative scheme as in [3] to be a global one. To this end, we use a contradiction argument. Define

\[
T_0 = \sup \left\{ T \in [0, \infty) : \| (\rho - 1, u, d - d) \|_{B^{\frac{3}{4}}} \leq \Gamma \eta \right\},
\]

with $\Gamma$ to be determined later. Suppose that $T_0 < \infty$. We apply the linear estimates in Proposition 4.1 and Proposition 4.2 to the solution of reformulated system (3.3) such that
for all \( t \in [0, T_0] \), the following estimate holds:

\[
\|\mathbf{d}(t) - \hat{\mathbf{d}}\|_{L^p_t(B^{1+rac{1}{2}}_2)} + \|\Omega(t)\|_{L^p_t(B^{1+rac{1}{2}}_2)} + \int_0^{T_0} \left( \|\mathbf{d}(s) - \hat{\mathbf{d}}\|_{B^{1+rac{1}{2}}_2} + \|\Omega(s)\|_{B^{1+rac{1}{2}}_2} \right) \, ds \\
\leq C e^{CV} \left( \|\mathbf{d}_0 - \hat{\mathbf{d}}\|_{B^{1+rac{1}{2}}_2} + \|\Omega_0\|_{B^{1+rac{1}{2}}_2} + \|\mathcal{L}\|_{L^1_{L_0}(B^{1+rac{1}{2}}_2)} + \|\mathcal{M}\|_{L^1_{L_0}(B^{1+rac{1}{2}}_2)} \right),
\]

(5.1)

and

\[
\|\rho(t) - 1\|_{L^p_t(B^{1+rac{1}{2}}_2)} + \|h(t)\|_{L^p_t(B^{1+rac{1}{2}}_2)} + \int_0^{T_0} \left( \|\rho(s) - 1\|_{B^{1+rac{1}{2}}_2} + \|h(s)\|_{B^{1+rac{1}{2}}_2} \right) \, ds \\
\leq C e^{CV} \left( \|\rho_0 - 1\|_{B^{1+rac{1}{2}}_2} + \|h_0\|_{B^{1+rac{1}{2}}_2} + \|\mathcal{K}\|_{L^1_{L_0}(B^{1+rac{1}{2}}_2)} + \|\mathcal{J}\|_{L^1_{L_0}(B^{1+rac{1}{2}}_2)} \right),
\]

(5.2)

where

\[
V = \int_0^{T_0} \|\mathbf{u}\|_{B^{1+rac{1}{2}}_2} \, ds.
\]

We note that \( \nabla \mathbf{d} = \nabla (\mathbf{d} - \hat{\mathbf{d}}) \) because \( \hat{\mathbf{d}} \in S^2 \) is a constant vector. As a result, the function \( \mathcal{N} \) (see (3.1)) can be rewritten as

\[
\mathcal{N} = \hat{\mathcal{N}} := - \mathbf{u} \cdot \nabla \mathbf{u} - \frac{\rho - 1}{\rho} \mathbf{A} \mathbf{u} - \frac{1}{\rho} \text{div} \left( \nabla (\mathbf{d} - \hat{\mathbf{d}}) \odot \nabla (\mathbf{d} - \hat{\mathbf{d}}) - \frac{1}{2} |\nabla (\mathbf{d} - \hat{\mathbf{d}})|^2 I \right).
\]

Therefore, the functions \( \mathcal{L} \) and \( \mathcal{M} \) in (5.1) are given by

\[
\mathcal{L} = \Lambda^{-1} \text{curl} \hat{\mathcal{N}} + \mathbf{u} \cdot \nabla \Omega,
\]

\[
\mathcal{M} = |\nabla (\mathbf{d} - \hat{\mathbf{d}})|^2 (\mathbf{d} - \hat{\mathbf{d}}) + |\nabla (\mathbf{d} - \hat{\mathbf{d}})|^2 \hat{\mathbf{d}},
\]

while for the functions \( \mathcal{K} \) and \( \mathcal{J} \) in (5.2), we have

\[
\mathcal{K} = \Lambda^{-1} \text{div} \hat{\mathcal{N}} + \mathbf{u} \cdot \nabla h,
\]

\[
\mathcal{J} = - (\rho - 1) \text{div} \mathbf{u}.
\]

In what follows, we derive estimates for the nonlinear terms \( \mathcal{L}, \mathcal{M}, \mathcal{K} \) and \( \mathcal{J} \). Indeed, for the term \( \mathbf{u} \cdot \nabla \Omega \), we infer from Proposition 2.2 that

\[
\|\mathbf{u} \cdot \nabla \Omega\|_{L^p_{L_0}(B^{1+rac{1}{2}}_2)} \leq C \int_0^{T_0} \|\mathbf{u}\|_{B^{1+rac{1}{2}}_2} \|\nabla \Omega\|_{B^{1+rac{1}{2}}_2} \, dt \\
\leq C \|\mathbf{u}\|_{L^p_{L_0}(B^{1+rac{1}{2}}_2)} \|\Omega\|_{L^1_{L_0}(B^{1+rac{1}{2}}_2)} \leq CT^2 \eta^2.
\]

(5.3)

Similarly, we have

\[
\|\Lambda^{-1} \text{curl} \mathbf{u} \cdot \nabla \mathbf{u}\|_{L^p_{L_0}(B^{1+rac{1}{2}}_2)} \leq C \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^p_{L_0}(B^{1+rac{1}{2}}_2)} \leq \int_0^{T_0} \|\mathbf{u}\|_{B^{1+rac{1}{2}}_2} \|\nabla \mathbf{u}\|_{B^{1+rac{1}{2}}_2} \, dt \\
\leq C \|\mathbf{u}\|_{L^p_{L_0}(B^{1+rac{1}{2}}_2)} \|\mathbf{u}\|_{L^1_{L_0}(B^{1+rac{1}{2}}_2)} \leq CT^2 \eta^2.
\]

(5.4)

By the embedding \( B^{1+rac{1}{2}}_2 \hookrightarrow L^\infty \), we consider \( F(\varrho) = \frac{\varrho}{1 + \varrho} \) with \( \varrho = \rho - 1 \), then by Lemma 1.6, we have

\[
\|F(\varrho)\|_{B^{1+rac{1}{2}}_2} \leq C_0(\|\varrho\|_{L^\infty}) \|\varrho\|_{B^{1+rac{1}{2}}_2}.
\]
where \( \|\varrho\|_{L^\infty} \leq 1 + \|\rho\|_{L^\infty} \leq 1 + C\|\rho\|_{B^\frac{3}{2}} \). Then we can apply Proposition 2.2 to obtain that

\[
\left\| A^{-1}\text{curl} \left( \frac{1}{\rho} \text{div} \left( \nabla \hat{d} - \hat{d} \right) \otimes \nabla \hat{d} - \frac{1}{2} \nabla (\hat{d} - \hat{d})^2 I \right) \right\|_{L^1_T(B^\frac{3}{2})} \\
\leq C \left( 1 + \left\| \frac{\rho - 1}{\rho} \right\|_{L^\infty_T(B^\frac{3}{2})} \right) \left\| \text{div} \left( \nabla (\hat{d} - \hat{d}) \otimes \nabla (\hat{d} - \hat{d}) - \frac{1}{2} \nabla (\hat{d} - \hat{d})^2 I \right) \right\|_{L^1_T(B^\frac{3}{2})} \\
\leq C \left( 1 + \left\| \rho - 1 \right\|_{L^\infty_T(B^\frac{3}{2})} \right) \left\| \nabla (\hat{d} - \hat{d}) \otimes \nabla (\hat{d} - \hat{d}) - \frac{1}{2} \nabla (\hat{d} - \hat{d})^2 I \right\|_{L^1_T(B^\frac{3}{2})} \\
\leq C \left( 1 + \left\| \rho - 1 \right\|_{L^\infty_T(B^\frac{3}{2})} \right) \int_0^T \left\| \nabla (\hat{d} - \hat{d}) \right\|_{B^\frac{3}{2}}^2 dt \\
\leq C(1 + \Gamma \eta) \int_0^T \left\| \nabla (\hat{d} - \hat{d}) \right\|_{B^\frac{3}{2}}^2 dt \\
\leq C(1 + \Gamma \eta) \|\hat{d} - \hat{d}\|_{L^\infty_T(B^\frac{3}{2})} \|\hat{d} - \hat{d}\|_{L^1_T(B^\frac{3}{2})} \leq C(1 + \Gamma \eta) \Gamma^2 \eta^2. \tag{5.5}
\]

In the last line of above inequality we have used the interpolation

\[
\|f\|_{L^2_T(B^\frac{3}{2})}^2 \leq C \|f\|_{L^\infty_T(B^\frac{3}{2})} \|f\|_{L^1_T(B^\frac{3}{2})}.
\]

The term \( \frac{\rho - 1}{\rho} \cdot Au \) can be dealt with in a similar way such that

\[
\left\| A^{-1}\text{curl} \left( \frac{\rho - 1}{\rho} \cdot Au \right) \right\|_{L^1_T(B^\frac{3}{2})} \leq \left\| \frac{\rho - 1}{\rho} \cdot Au \right\|_{L^1_T(B^\frac{3}{2})} \\
\leq \left\| \frac{\rho - 1}{\rho} \right\|_{L^\infty_T(B^\frac{3}{2})} \left\| Au \right\|_{L^1_T(B^\frac{3}{2})} \\
\leq C \left\| \rho - 1 \right\|_{L^\infty_T(B^\frac{3}{2})} \left\| Au \right\|_{L^1_T(B^\frac{3}{2})} \\
\leq C \left\| \rho - 1 \right\|_{L^\infty_T(B^\frac{3}{2})} \left\| u \right\|_{L^1_T(B^\frac{3}{2})} \leq C T^2 \eta^2. \tag{5.6}
\]

Combining (5.3)–(5.6) together, we obtain

\[
\|\mathcal{L}\|_{L^1_T(B^\frac{3}{2})} \leq C T^2 \eta^2 + C T^3 \eta^3. \tag{5.7}
\]

Similarly, we can derive the estimate for \( \mathcal{K} \):

\[
\|\mathcal{K}\|_{L^1_T(B^\frac{3}{2})} \leq C T^2 \eta^2 + C T^3 \eta^3. \tag{5.8}
\]
Next, we turn to the estimate for \( \mathcal{M} \). For \( |\nabla(d - \hat{d})|^2(d - \hat{d}) \), indeed, we have, as in (5.5)

\[
\| |\nabla(d - \hat{d})|^2(d - \hat{d}) \|_{L^1_T(\tilde{B}^{1/2})} \\
\leq C \| |\nabla(d - \hat{d})|^2 \|_{L^1_T(\tilde{B}^{1/2})} \|d - \hat{d}\|_{L^\infty_T(\tilde{B}^{1/2})} \\
\leq C T \eta \int_0^T \| |\nabla(d - \hat{d})|^2 \|_{B^{1/2}} dt \\
\leq C T \eta \|\nabla(d - \hat{d})\|_{L^2_T(\tilde{B}^{1/2})}^2 \\
\leq C T \eta \|d - \hat{d}\|_{L^2_T(\tilde{B}^{1/2})}^2 \\
\leq C T \eta \|d - \hat{d}\|_{L^\infty_T(\tilde{B}^{1/2})}^2 \|\nabla(d - \hat{d})\|_{L^1_T(\tilde{B}^{1/2})} \\
\leq C T \eta \|d - \hat{d}\|_{L^2_T(\tilde{B}^{1/2})}^2 \|\nabla(d - \hat{d})\|_{L^1_T(\tilde{B}^{1/2})} \leq C T^3 \eta^3.
\]

For \( |\nabla(d - \hat{d})|^2 \hat{d} \), we have, by the definition of Besov’s spaces

\[
\| |\nabla(d - \hat{d})|^2 \hat{d} \|_{L^1_T(\tilde{B}^{1/2})} \\
\leq C \int_0^T \| |\nabla(d - \hat{d})|^2 \|_{L^\infty_T(\tilde{B}^{1/2})} \|\nabla(d - \hat{d})\|_{L^1_T(\tilde{B}^{1/2})} dt \\
\leq C \int_0^T \| |\nabla(d - \hat{d})|^2 \|_{B^{1/2}} dt \\
\leq C \|d - \hat{d}\|_{L^2_T(\tilde{B}^{1/2})}^2 \\
\leq C \|d - \hat{d}\|_{L^\infty_T(\tilde{B}^{1/2})} \|\nabla(d - \hat{d})\|_{L^1_T(\tilde{B}^{1/2})} \leq C T^2 \eta^2.
\]

Finally, for \( \mathcal{J} \), we have

\[
\| (\rho - 1) \text{div} u \|_{L^1_T(\tilde{B}^{1/2})} \\
\leq C \|\rho - 1\|_{L^\infty_T(\tilde{B}^{1/2})} \|\text{div} u\|_{L^1_T(\tilde{B}^{1/2})} \\
\leq C T^2 \eta^2.
\]

Substituting (5.7)–(5.11) back to (5.1) and (5.2), we obtain

\[
\| (\rho - 1, u, d - \hat{d}) \|_{\mathfrak{A}^{3/2}_T} \leq C_1 e^{C_1 \Gamma \eta} (\eta + \Gamma^2 \eta^2 + \Gamma^3 \eta^3).
\]

We choose \( \Gamma = 4C_1 \), and then \( \eta > 0 \) satisfying

\[
e^{C_1 \Gamma \eta} < 2, \quad \Gamma^2 \eta^2 \leq \frac{1}{2}, \quad \Gamma^3 \eta^2 \leq \frac{1}{2}.
\]

Hence, it follows from (5.12) and the above choices of \( \Gamma \) and \( \eta \) that

\[
\| (\rho - 1, u, d - \hat{d}) \|_{\mathfrak{A}^{3/2}_T} < \Gamma \eta,
\]

which is a contradiction with the definition of \( T_0 \). As a consequence, we can conclude that \( T_0 = \infty \). The proof of global existence is thus proved.
6. Uniqueness

In this section, we will address the uniqueness of the solution in $\mathcal{B}^3_T$. For this purpose, suppose that $(\rho_i, u_i, d_i)_{i=1,2}$ solve (1.1) with the same initial data. Define
\[
\delta \rho = \rho_2 - \rho_1, \quad \delta u = u_2 - u_1, \quad \delta h = \Lambda^{-1} \text{div} \delta u, \quad \delta \Omega = \Lambda^{-1} \text{curl} \delta u, \quad \delta d = d_2 - d_1.
\]
then $(\delta \rho, \delta u, \delta d) \in \mathcal{B}^3_T$ for all $T > 0$. On the other hand, since $(\rho_i, u_i, d_i)_{i=1,2}$ are solutions to (1.1) with the same initial data, $(\delta \rho, \delta u, \delta d)$ solves
\[
\begin{cases}
\partial_t \delta \rho + u_2 \cdot \nabla \delta \rho + \Lambda \delta h = \delta \mathcal{J}, \\
\partial_t \delta h + u_2 \cdot \nabla \delta h - \nu \Delta \delta h - \Lambda \delta \rho = \delta \mathcal{K}, \\
\partial_t \delta \Omega + u_2 \cdot \nabla \delta \Omega - \Delta \delta \Omega = \delta \mathcal{L}, \\
\partial_t \delta d + u_2 \cdot \nabla \delta d - \Delta \delta d = \delta \mathcal{M}, \\
\delta u = -\Lambda^{-1} \nabla \delta h + \Lambda^{-1} \text{curl} \delta \Omega,
\end{cases}
\tag{6.1}
\]
with
\[
\delta \mathcal{J} = -\delta u \cdot \nabla \rho_1 - \delta \rho \text{div} u_2 - (\rho_1 - 1) \text{div} \delta u,
\]
\[
\delta \mathcal{K} = u_2 \cdot \nabla \delta h + \Lambda^{-1} \text{div} \left( -u_2 \cdot \nabla \delta u - \delta u \cdot \nabla u_1 - \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) A u_2 + \left( \frac{1}{\rho_1} - 1 \right) A \delta u \right.
\]
\[
- \frac{1}{\rho_1} \text{div} \left( \nabla (d_2 - \hat{d}) \odot \nabla \delta u + \nabla \delta d \odot \nabla (d_1 - \hat{d}) \right)
\]
\[
- \frac{1}{2} I \left( \nabla (d_2 - \hat{d}) : \nabla \delta d + \nabla \delta d : \nabla (d_1 - \hat{d}) \right),
\]
\[
- \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \text{div} \left( \nabla (d_2 - \hat{d}) \odot \nabla (d_2 - \hat{d}) - \frac{1}{2} |\nabla (d_2 - \hat{d})|^2 I \right),
\]
\[
\delta \mathcal{L} = u_2 \cdot \nabla \delta \Omega + \Lambda^{-1} \text{curl} \left( -u_2 \cdot \nabla \delta u - \delta u \cdot \nabla u_1 - \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) A u_2 + \frac{1}{\rho_1} A \delta u \right.
\]
\[
- \frac{1}{\rho_1} \text{div} \left( \nabla (d_2 - \hat{d}) \odot \nabla \delta d + \nabla \delta d \odot \nabla (d_1 - \hat{d}) \right)
\]
\[
- \frac{1}{2} I \left( \nabla (d_2 - \hat{d}) : \nabla \delta d + \nabla \delta d : \nabla (d_1 - \hat{d}) \right)
\]
\[
- \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \text{div} \left( \nabla (d_2 - \hat{d}) \odot \nabla (d_2 - \hat{d}) - \frac{1}{2} |\nabla (d_2 - \hat{d})|^2 I \right),
\]
and
\[
\delta \mathcal{M} = |\nabla (d_1 - \hat{d})|^2 \delta d + (\nabla \delta d : \nabla (d_1 - \hat{d}) + \nabla (d_2 - \hat{d}) : \nabla \delta d) d_2,
\]
where we used the notation $A : B = \sum_{i,j=1}^3 A_{ij} B_{ij}$. 

Applying Proposition 4.1 and Proposition 4.2 to the system (6.1), we get
\[
\|\langle \delta\rho, \delta u, \delta d \rangle\|_{\mathcal{B}_{T}^{\beta}} \\
\lesssim \exp \left( C\|u_2\|_{L_{x}^{1}(\bar{B}^{\frac{3}{2}})} \right) \left( \|\delta J\|_{L_{x}^{1}(\bar{B}^{-\frac{1}{2}})} + \|\delta K\|_{L_{x}^{1}(B^{-\frac{1}{2}})} + \|\delta L\|_{L_{x}^{1}(B^{-\frac{1}{2}})} \right) + \|\delta M\|_{L_{x}^{1}(B^{-\frac{1}{2}})} \right). \tag{6.2}
\]

Since \((\delta\rho, \delta u, \delta d) \in \mathcal{B}_{T}^{\frac{3}{2}}, \partial_t(\rho_i - 1) \in L_{loc}^{1}(B^{\frac{3}{2}})\), and hence \(\rho_i - 1 \in C(B^{\frac{3}{2}}) \cap L_{X}^{\infty}(B^{\frac{3}{2}})\). This entails \(\rho_i - 1 \in C((0, \infty) \times \mathbb{R}^{3})\). On the other hand, if \(\eta\) is sufficiently small, we have
\[
|\rho_i(t, x) - 1| \leq \frac{1}{4} \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad x \in \mathbb{R}^{3}.
\]
Continuity in time for \(\rho_2 - 1\) thus yields the existence of a time \(T > 0\) such that
\[
\|\rho_i(t) - 1\|_{L_{X}^{\infty}} \leq \frac{1}{2} \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad t \in [0, T].
\]
Repeating the argument in Section 5, we easily infer that
\[
\|\delta J\|_{L_{x}^{1}(\bar{B}^{-\frac{1}{2}})} \lesssim \|\rho_1 - 1\|_{L_{x}^{\infty}(\bar{B}^{\frac{3}{2}})} \|\delta u\|_{L_{x}^{1}(\bar{B}^{\frac{3}{2}})} + \|\text{div} u_2\|_{L_{x}^{1}(B^{\frac{3}{2}})} \|\delta \rho\|_{L_{x}^{\infty}(\bar{B}^{\frac{3}{2}})} \\
+ \|\delta K\|_{L_{x}^{1}(B^{-\frac{1}{2}})} + \|\delta L\|_{L_{x}^{1}(B^{-\frac{1}{2}})} \lesssim \|u_2\|_{L_{x}^{1}(\bar{B}^{\frac{3}{2}})} \|\nabla \delta u\|_{L_{x}^{1}(\bar{B}^{\frac{3}{2}})} + \|\delta u\|_{L_{x}^{1}(B^{\frac{3}{2}})} \|\nabla u_1\|_{L_{x}^{1}(B^{\frac{3}{2}})} \\
+ \left( 1 + \|\rho_1 - 1\|_{L_{x}^{\infty}(\bar{B}^{\frac{3}{2}})} + \|\rho_2 - 1\|_{L_{x}^{\infty}(B^{\frac{3}{2}})} \right) \|\nabla^2 u_2\|_{L_{x}^{1}(B^{\frac{3}{2}})} \|\delta \rho\|_{L_{x}^{\infty}(B^{\frac{3}{2}})} \\
+ \left( 1 + \|\rho_1 - 1\|_{L_{x}^{\infty}(B^{\frac{3}{2}})} + \|\rho_2 - 1\|_{L_{x}^{\infty}(B^{\frac{3}{2}})} \right) \|\nabla (d_2 - \tilde{d})\|_{L_{x}^{1}(B^{\frac{3}{2}})} \|\delta \rho\|_{L_{x}^{\infty}(B^{\frac{3}{2}})} \\
+ \left( 1 + \|\rho_1 - 1\|_{L_{x}^{\infty}(B^{\frac{3}{2}})} \right) \|\delta d\|_{L_{x}^{1}(\bar{B}^{\frac{3}{2}})} \left( \|d_1 - \tilde{d}\|_{L_{x}^{1}(\bar{B}^{\frac{3}{2}})} + \|d_2 - \tilde{d}\|_{L_{x}^{1}(\bar{B}^{\frac{3}{2}})} \right).
\]
and
\[
\|\delta M\|_{L_{x}^{1}(B^{-\frac{1}{2}})} \lesssim \|\delta d\|_{L_{x}^{1}(\bar{B}^{\frac{3}{2}})} \|\nabla (d_1 - \tilde{d})\|_{L_{x}^{1}(\bar{B}^{\frac{3}{2}})} \\
+ \left( 1 + \|d_2 - \tilde{d}\|_{L_{x}^{\infty}(B^{\frac{3}{2}})} \right) \|\nabla (d_2 - \tilde{d})\|_{L_{x}^{1}(B^{\frac{3}{2}})} \\
+ \|\nabla (d_1 - \tilde{d})\|_{L_{x}^{1}(B^{\frac{3}{2}})} \|\nabla \delta d\|_{L_{x}^{\infty}(\bar{B}^{\frac{3}{2}})}.
\]
Substituting those estimates back into (6.2), we eventually get
\[
\|\langle \delta\rho, \delta u, \delta d \rangle\|_{\mathcal{B}_{T}^{\beta}} \leq Z(T)\|\langle \delta\rho, \delta u, \delta d \rangle\|_{\mathcal{B}_{T}^{B^{\frac{3}{2}}}}.
\]
GLOBAL SOLUTION TO COMPRESSIBLE FLOW OF LIQUID CRYSTALS IN 3D

with

\[ Z(T) = \exp \left( C \|u_2\|_{L^4(B^3_T)} \right) \left[ \|\rho_1 - 1\|_{L^\infty(B^3_T)} + \|\text{div} u_2\|_{L^4(B^3_T)} + \|u_2\|_{L^4(B^3_T)} \right. \\
+ \|\nabla u_1\|_{L^3(B^3_T)} + \left( 1 + \|\rho_1 - 1\|_{L^\infty(B^3_T)} + \|\rho_2 - 1\|_{L^\infty(B^3_T)} \right) \|\nabla^2 u_2\|_{L^1(B^3_T)} \right. \\
+ \left( 1 + \|\rho_1 - 1\|_{L^\infty(B^3_T)} \right) \|\nabla (d_1 - \hat{d})\|_{L^1(B^6_T)}^2 \\
+ \left( 1 + \|\rho_2 - 1\|_{L^\infty(B^3_T)} + \|\rho_1 - 1\|_{L^\infty(B^3_T)} \right) \|\nabla (d_2 - \hat{d})\|_{L^1(B^6_T)}^2 \\
+ \left( 1 + \|d_2 - \hat{d}\|_{L^\infty(B^3_T)} \right) \left( \|\nabla (d_2 - \hat{d})\|_{L^1(B^6_T)} + \|\nabla (d_1 - \hat{d})\|_{L^1(B^6_T)} \right) \right].

We notice that \( \limsup_{T \to 0^+} Z(T) \leq C \|\rho_1 - 1\|_{L^\infty(B^3_T)} \). This is because all other terms involve an integral in time in \( L^1 \) or \( L^2 \) sense so that as \( T \) goes to zero, all those integrals will converge to zero. Thus, if \( \eta > 0 \) is sufficiently small, we get

\[ \| (\delta \rho, \delta u, \delta d) \|_{L^\infty(B^3_T)} = 0 \]

for certain \( T > 0 \) small enough. Thus, we have shown the uniqueness on a small time interval \([0, T]\) such that \((\rho_1, u_1, d_1) = (\rho_2, u_2, d_2)\).

Then we can argue as in [3] for the compressible Navier–Stokes equations. Let \( T_{\text{max}} < +\infty \) be the largest time such that the two solutions coincide on \([0, T_{\text{max}}]\). Taking \( T_{\text{max}} \) as the initial time, we denote

\[ (\hat{\rho}_i(t), \hat{u}_i(t), \hat{d}_i(t)) \overset{\text{def}}{=} (\rho(t - T_{\text{max}}), u(t - T_{\text{max}}), d(t - T_{\text{max}})). \]

Repeating the above arguments and using the fact that \( \|\hat{\rho}_i(0) - 1\|_{L^\infty} \leq \frac{1}{4} \), we can prove that

\[ (\hat{\rho}_1(t), \hat{u}_1(t), \hat{d}_1(t)) = (\hat{\rho}_2(t), \hat{u}_2(t), \hat{d}_2(t)) \]

on a sufficiently small interval \([0, \iota]\) with \( \iota > 0 \). This contradicts the assumption that \( T_{\text{max}} \) is the largest time such that the two solutions coincide. Thus, \( T_{\text{max}} = +\infty \) which means that the uniqueness result holds in \( \mathbb{R}^+ \).

\textbf{References}

[1] Bahouri, H., Chemin, J., Danchin, R.: Fourier analysis and nonlinear partial differential equations, Grundlehren der Mathematischen Wissenschaften, 343, Springer, Heidelberg, 2011.
[2] Bourgain, J., Pavlovi, N.: Ill-posedness of the Navier–Stokes equations in a critical space in 3D, J. Funct. Anal., 255 (2008), 2233–2247.
[3] Danchin, R.: Global existence in critical spaces for compressible Navier–Stokes equations, Invent. Math., 141 (2000), 579–614.
[4] De Gennes, G., Prost, J.: The Physics of Liquid Crystals, Oxford University Press, New York, 1993.
[5] Ding, S.-J., Wang C.-Y., Wen, H.-Y.: Weak solution to compressible hydrodynamic flow of liquid crystals in dimension one, Discrete Conti. Dyna. Sys. Ser. B, 15(2) (2011), 357–371.
[6] Ding, S.-J., Lin J.-Y., Wang C.-Y., Wen, H.-Y.: Compressible hydrodynamic flow of liquid crystals in 1-D. Discrete Conti. Dyna. Sys., 32(2) (2012), 539–563.
[7] Ericksen, L.: Conservation laws for liquid crystals, Trans. Soc. Rheology, 5 (1961), 23–34.
[8] Ericksen, L.: Continuum theory of nematic liquid crystals, Res. Mechanica, 21 (1987), 381–392.
[9] Hardt, R., Kinderlehrer, D.: Mathematical Questions of Liquid Crystal Theory, The IMA Volumes in Mathematics and its Applications 5, Springer-Verlag, New York, 1987.
[10] Hardt, R., Kinderlehrer, D., Lin, F.-H.: Existence and partial regularity of static liquid crystal configurations, Comm. Math. Phys., 105 (1986), 547–570.
[11] Hong, M.-C.: Global existence of solutions of the simplified Ericksen–Leslie system in dimension two, Calc. Var. Partial Differential Equations, 40 (2011), 15–36.
[12] Huang, T., Wang C.-Y., Wen H.-Y.: Blow up criterion for compressible nematic liquid crystal flows in dimension three, Arch. Rational Mech. Anal., (2011), DOI: 10.1007/s00205-011-0476-1.
[13] Huang, T., Wang C.-Y., Wen H.-Y.: Strong solutions of the compressible nematic liquid crystal flow, J. Differential Equations, 252(3) (2012), 2222–2265.
[14] Jiang, F., Jiang, S., Wang, D.: Global Weak Solutions to the Equations of Compressible Flow of Nematic Liquid Crystals in Two Dimensions, arXiv:1210.3565.
[15] Fujita, H., Kato, T.: On the Navier–Stokes initial value problem. I., Arch. Rational Mech. Anal., 16 (1964), 269–315.
[16] Leslie, F.: Some constitutive equations for liquid crystals, Arch. Rational Mech. Anal., 28 (1968), 265–283.
[17] Leslie, F.: Theory of flow phenomenum in liquid crystals, in the Theory of Liquid Crystals, 4, Academic Press, (1979), 1–81.
[18] Lin, F.-H.: Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena, Comm. Pure. Appl. Math., 42 (1989), 789–814.
[19] Lin, F.-H., Liu, C.: Nonparabolic dissipative systems modeling the flow of liquid crystals, Comm. Pure Appl. Math., 48 (1995), 501–537.
[20] Lin, F.-H., Liu, C.: Partial regularities of the nonlinear dissipative systems modeling the flow of liquid crystals, Discrete Conti. Dyna. Sys., 2 (1996), 1–23.
[21] Lin, F.-H., Liu, J.-Y., Wang, C.-Y.: Liquid crystal flows in two dimensions, Arch. Ration. Mech. Anal., 197 (2010), 297–336.
[22] Liu, C., Walkington, N.: Approximation of liquid crystal flow, SIAM J. Numer. Anal., 37 (2000), 725–741.
[23] Shkoller, S.: Well-posedness and global attractors for liquid crystals on Riemannian manifolds, Comm. Partial Differential Equations, 27 (2001), 1103–1137.
[24] Wang, C.-Y.: Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data, Arch. Ration. Mech. Anal., 200 (2011), 1–19.
[25] Wu, H.: Long-time behavior for nonlinear hydrodynamic system modeling the nematic liquid crystal flows, Discrete Conti. Dyna. Sys., 26(1) (2010), 379–396.

Courant Institute of Mathematical Sciences, New York University, New York, NY 10012.
E-mail address: xianpeng@cims.nyu.edu

School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, PR China
E-mail address: haowufd@yahoo.com