On the properties of Laplacian pseudoinverses

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Abstract—The pseudoinverse of a graph Laplacian is used in many applications and fields, such as for instance in the computation of the effective resistance in electrical networks, in the calculation of the hitting/commuting times for a Markov chain and in continuous-time distributed averaging problems. In this paper we show that the Laplacian pseudoinverse is in general not a Laplacian matrix but rather a signed Laplacian with the property of being an eventually exponentially positive matrix, i.e., of obeying a strong Perron-Frobenius property. We show further that the set of signed Laplacians with this structure (i.e., eventual exponential positivity) is closed with respect to matrix pseudoinversion. This is true even for signed digraphs, and provided that we restrict to Laplacians that are weight balanced also stability is guaranteed.

1. INTRODUCTION

For a network or a networked system, the Laplacian matrix is a fundamental object that captures information about e.g. connectivity and spectrum [1], [2], as well as properties of the dynamics that live on the graph [3], [4], [5], [6]. Associated to the Laplacian is also a Laplacian pseudoinverse, typically a Moore-Penrose pseudoinverse, which has also been used extensively to describe graph-related quantities. For instance it is used to build an effective resistance matrix with nonpositive off-diagonal entries, such that a digraph, and provided that we restrict to Laplacians that are weight balanced also stability is guaranteed.

The aim of this paper is to investigate the algebraic properties of Laplacian pseudoinverses. Even though $L^+$ is not an M-matrix, it has nevertheless most of the properties of M-matrices, most notably it obeys to a strong Perron-Frobenius property: the pair formed by the eigenvalue 0 and eigenvector $f = [1 \ldots 1]^T$ is the “dominant” eigenpair for $-L^+$ (just like it is for $-L$) in spite of the presence of positive off-diagonal entries in $L^+$.

For the graph, a distance measure that exploits the analogy between graphs and electrical networks [7], [8], [9], and to compute hitting/commuting times in Markov chains [10], [11], [12], [13]. It is also used to estimate the $H_2$ norm in networked dynamical systems [14], [15], [16].

If we consider a graph with nonnegative edge weights, it is well-known that the Laplacian $L$ is an M-matrix (i.e., a matrix with nonpositive off-diagonal entries, such that $-L$ is marginally stable, see below for proper definitions). It is also easy to show that the Laplacian pseudoinverse does not belong to the same class, not even when the graph is undirected. Consider for instance the following Laplacian matrix

$$L = \begin{bmatrix} 0.8 & -0.7 & -0.1 \\ -0.7 & 0.9 & -0.2 \\ -0.1 & -0.2 & 0.3 \end{bmatrix}.$$ 

Its pseudoinverse is

$$L^+ = \begin{bmatrix} 0.773 & 0.048 & -0.821 \\ 0.048 & 0.628 & -0.676 \\ -0.821 & -0.676 & 1.498 \end{bmatrix}$$

which has an anomalous sign in the (1,2) entry, even though it has the same stability properties of $L$.

2. PRELIMINARIES

A. Linear algebraic preliminaries

Given a matrix $A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$, $A \geq 0$ means element-wise nonnegative, i.e., $a_{ij} \geq 0$ for all $i,j = 1, \ldots, n$, while $A > 0$ means element-wise positive, i.e., $a_{ij} > 0$ for all $i,j = 1, \ldots, n$. The spectrum of $A$ is denoted $\text{sp}(A) = \{\lambda_1(A), \ldots, \lambda_n(A)\}$, where $\lambda_i(A)$, $i = 1, \ldots, n$, are the eigenvalues of $A$. In this paper we use the ordering $\text{Re}[\lambda_1(A)] \leq \text{Re}[\lambda_2(A)] \leq \cdots \leq \text{Re}[\lambda_n(A)]$, where $\text{Re}[\lambda_i(A)]$ indicates the real part of $\lambda_i(A)$. The spectral radius of $A$ is the smallest real nonnegative number such that $\rho(A) \geq |\lambda_i(A)|$ for all $i = 1, \ldots, n$ and $\lambda_i(A) \in \text{sp}(A)$. A matrix $A$ is called Hurwitz stable if $\text{Re}[\lambda_n(A)] < 0$, and marginally stable if $\text{Re}[\lambda_n(A)] = 0$ is a simple root of the minimal polynomial of $A$.

As signed Laplacian here we use the so-called “repelling Laplacian” in the terminology of [20], see Section III for a precise definition.
A matrix $A$ is called positive semidefinite (psd) if $x^TAx = x^T \frac{A + A^T}{2} x \geq 0 \forall x \in \mathbb{R}^n$ and it is called positive definite (pd) if $x^TAx = x^T \frac{A + A^T}{2} x > 0 \forall x \in \mathbb{R}^n \setminus \{0\}$.

A matrix $A$ is called irreducible if there does not exist a permutation matrix $P \text{ s.t. } P^T AP$ is block triangular.

A matrix $B$ is called a Z-matrix if it can be written as $B = sI - A$, where $A \geq 0$ and $s > 0$, and it is called a M-matrix if, in addition, $s \geq \rho(A)$, which implies that all the eigenvalues of $B$ have nonnegative real part. If $s > \rho(A)$ then $B$ is nonsingular and $-B$ is Hurwitz stable. If $s = \rho(A)$ then $B$ is singular, and if $A$ is irreducible then $-B$ is marginally stable.

If $A$ is a singular matrix, the Moore-Penrose pseudoinverse of $A$, denoted $A^\dagger$, is the unique $n \times n$ matrix that satisfies $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^T = A^\dagger A$, and $(A^\dagger A)^T = AA^\dagger$. A singular matrix $A$ is said to have index 1 if the range of $A$, $\mathcal{R}(A)$, and the kernel of $A$, $\mathcal{N}(A)$, are complementary subspaces, i.e., $\mathcal{R}(A) \cap \mathcal{N}(A) = \{0\}$. For index 1 singular matrices, the Drazin inverse and the group inverse coincide.

A singular M-matrix has always index 1 [22]. A matrix is normal if it commutes with its transpose: $AA^T = A^T A$. A matrix $A$ is said an EP matrix (Equal Projector, also called a range symmetric matrix [22]) if $\mathcal{N}(A) = \mathcal{N}(A^T)$ (and hence $\mathcal{R}(A) = \mathcal{R}(A^T)$). EP matrices generalize normal matrices, and like normal matrices have many equivalent characterizations, see [22]. For instance an EP matrix $A$ is such that $A$ commutes with its Moore-Penrose pseudoinverse $A^\dagger$. If $A$ is an EP-matrix, then $\exists U$ orthogonal such that

$$A = U \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} U^T$$

with $B$ nonsingular of dimension $r = \text{rank}(A)$. Singular EP matrices have index 1, and for them the Moore-Penrose pseudoinverse, the Drazin inverse and the group inverse coincide.

B. Signed graphs

Let $G(A) = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be the (weighted) digraph with vertex set $\mathcal{V} = \{1, \ldots, n\}$, $\mathcal{E} = \mathcal{V} \times \mathcal{V}$ and adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, $a_{ij} \in \mathbb{R} \setminus \{0\}$ iff $(j, i) \in \mathcal{E}$. Since each edge of the digraph is labeled by a sign (i.e., sign $(a_{ij}) = \pm 1$), $G(A)$ is called a signed digraph. In the particular case where $A \geq 0$, the digraph $G(A)$ is called nonnegative. For digraphs $G(A)$ which are strongly connected and without self-loops, the matrix $A$ is irreducible with null-diagonal.

The weighted in-degree and out-degree of node $i$ are denoted $\sigma_i^{in} = \sum_{j=1}^n a_{ij}$ and $\sigma_i^{out} = \sum_{j=1}^n a_{ji}$, respectively. The (signed) Laplacian of a graph $G(A)$ is the (in general non-symmetric) matrix $L = \Sigma - A$ where $\Sigma = \text{diag} \{ \sigma_1^{in}, \ldots, \sigma_n^{in} \}$. This definition of signed Laplacian corresponds to the so-called “rePELLing Laplacian” in the terminology of [20]. By construction, this Laplacian is a singular matrix with $\mathcal{N}(L) = \text{span}(1)$. However, $-L$ need not be marginally stable and its symmetric part $L_s = \frac{L + L^T}{2}$ need not be positive semidefinite, as we show in the examples in Section [III]. Moreover, $L$ irreducible (or, $G(A)$ strongly connected) need not imply $\text{corank}(L) = 1$. For instance, consider a complete, undirected, signed graph $G(A)$ whose Laplacian is

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$ 

It is $\text{sp}(L) = \{0, 0, 4, 4\}$ and $1, [0, 1, -1, 0]^T \in \mathcal{N}(L)$, i.e., $L$ is marginally stable of corank 2. Similarly, $\text{corank}(L) = 1$ need not imply $L$ irreducible.

A digraph $G(A)$ is weight balanced if in-degree and out-degree coincide for each node, i.e., $\sigma_i^{in} = \sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{ji} = \sigma_i^{out}$ for all $i = 1, \ldots, n$. As we show in Lemma 2 corank$(L) = 1$ and $L$ weight balanced imply $L$ irreducible.

C. Eventual exponential positivity

Definition 1 A matrix $A \in \mathbb{R}^{n \times n}$ has the (strong) Perron-Frobenius property if $\rho(A)$ is a simple positive eigenvalue of $A$ s.t. $\rho(A) > |\lambda(A)|$ for every $\lambda(A) \in \text{sp}(A)$. $A \neq \rho(A)$, and $\chi$, the right eigenvector relative to $\rho(A)$, is positive.

The set of matrices which possess the Perron-Frobenius property will be denoted $\mathcal{PF}$, and it is known (see e.g. [23, Thm 8.4.4]) that irreducible nonnegative matrices are part of this set. However, it has been shown (see [17]) that matrices having negative elements can also possess this property, provided that they are eventually positive.

Definition 2 A matrix $A \in \mathbb{R}^{n \times n}$ is called eventually positive (denoted $A \succ 0$) if $\exists k_0 \in \mathbb{N}$ s.t. $A^k > 0$ for all $k \geq k_0$.

Theorem 1 [17, Thm 2.2] Let $A \in \mathbb{R}^{n \times n}$. Then the following statements are equivalent:
1) Both $A, A^T \in \mathcal{PF}$;
2) $A \succ 0$;
3) $A^T \succ 0$.

Definition 3 A matrix $A \in \mathbb{R}^{n \times n}$ is called eventually exponentially positive if $\exists t_0 \in \mathbb{N}$ s.t. $e^{At} > 0$ for all $t \geq t_0$.

Lemma 1 [18, Thm 3.3] A matrix $A \in \mathbb{R}^{n \times n}$ is eventually exponentially positive if and only if $\exists d \geq 0$ s.t. $A + dI \succ 0$.

D. Kron reduction for undirected networks

Consider an undirected, connected, weighted graph $G(A) = (\mathcal{V}, \mathcal{E}, A)$ with adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$. Let $\alpha \subset \{1, \ldots, n\}$ (with $\text{card}(\alpha) \geq 2$) and $\beta = \{1, \ldots, n\} \setminus \alpha$ be a partition of the node set $\mathcal{V} = \{1, \ldots, n\}$. After an adequate permutation of its rows and columns, the Laplacian $L$ of the graph $G(A)$ can be rewritten as

$L = \begin{bmatrix} L[\alpha] & L[\alpha, \beta] \\ L[\beta, \alpha] & L[\beta] \end{bmatrix}$,

where we denote $L[\alpha, \beta]$ the submatrix of $L$ determined by the index sets $\alpha$ and $\beta$, and $L[\alpha] := L[\alpha, \alpha]$ the principal submatrix of $L$ determined by the index set $\alpha$. 

If \( L[\beta] \) is nonsingular, the Schur complement of \( L[\beta] \) in \( L \) is given by \( L/L[\beta] := L[\alpha] - L[\alpha, \beta]L[\beta]^{-1}L[\beta, \alpha] \). In the context of electrical networks, where \( \alpha \) and \( \beta \) are referred to as boundary (or terminal) and interior nodes, this procedure is denoted Kron reduction (see e.g. [24], [25]) and it yields a matrix \( L_\nu := L/L[\beta] \), denoted Kron-reduced matrix, which is still a Laplacian of a weighted, undirected graph \( G_r \) (see [24] for details and properties of \( L_\nu \)).

If \( G(A) \) is signed, when \( \alpha \) is chosen as the set of nodes incident to edges with negative weight it is shown in [26] that \( L[\beta] \) is positive definite and that \( L \) is psd of corank 1 if and only if \( L_\nu \) is psd of corank 1.

III. PSEUDOINVERSE OF EVENTUALLY EXPONENTIALLY POSITIVE LAPLACIANS

In this section we study the connection between the marginal stability and eventual positivity of the Laplacian \( L \) and of its pseudoinverse \( L^\dagger \). If the network is undirected and signed, or if the network is directed, signed and weight balanced, we show that \(-L\) is eventually exponentially positive if and only if \( L^\dagger \) is eventually exponentially positive.

A. Directed signed network case

Assume that the graph \( G(A) = (V, E, A) \) is directed and without loops, which means that the adjacency matrix \( A \) is with null diagonal.

When the graph is weight balanced, the Laplacian is an EP-matrix since \( N(L) = N(L^T) = \text{span}(1) \). In this case, it is shown in [27] that \(-L\) is eventually exponentially positive if and only if \(-L\) is marginally stable (of corank 1). In addition, if the Laplacian is a normal matrix, then eventual exponential positivity of \( L \) is equivalent to that of its symmetric part.

Theorem 2 Consider a signed digraph \( G(A) \) such that the corresponding Laplacian \( L \) is weight balanced. Then, the following conditions are equivalent:

(i) \(-L\) is eventually exponentially positive;
(ii) \(-L\) is marginally stable of corank 1.

Furthermore, if \( L \) is normal then (i) and (ii) are equivalent to

(iii) \( L_\nu = \frac{L+L^T}{2} \) is psd of corank 1.

Proof: (i) \( \iff \) (ii) See [27, Corollary 2].

(ii) \( \iff \) (iii): If \( L \) is normal, then there exists an orthonormal matrix \( U \) such that \( L = UDUT^T \), where, if \( \mu_1, \ldots, \mu_k \) are the real eigenvalues of \( L \) and \( v_1 \pm i\omega_1, \ldots, v_k \pm i\omega_k \) are their complex conjugate eigenvalues:

\[
D = \mu_1 + \cdots + \mu_k \oplus \begin{pmatrix} v_1 & \omega_1 \\ -\omega_1 & v_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} v_k & \omega_k \\ -\omega_k & v_k \end{pmatrix}
\]

where \( \oplus \) indicates direct sum. If follows that \( L_\nu = \frac{L+L^T}{2} = \frac{1}{2}U(D+D^T)U^T \) and therefore that \( \Re[\lambda_i(L)] = \lambda_i(L_\nu) \).

Observe that Theorem 2 does not explicitly assume that \( G(A) \) is strongly connected. However, as we will show later in Lemma 2 any of the conditions (i) or (ii) implies that \( L \) is irreducible (i.e., that \( G(A) \) is strongly connected).

Remark 1 Corollary 2 of [27] claims that the equivalence (ii) \( \iff \) (iii) is valid in the more general case of weight balance \( L \). Unfortunately that result is not true as the following Example shows. A complication arises from the fact that for \( L \) weight balanced but not normal \( L_\nu \) may acquire negative diagonal elements even if \(-L\) is marginally stable. \( L_\nu \) with negative diagonal elements obviously cannot be psd. However, even when \( L_\nu \) has positive diagonal it is not guaranteed to be psd, see Example 2.

Example 1 In correspondence of

\[
L = \begin{bmatrix}
0.15 & 0 & 0 & -0.15 \\
-0.23 & 0.15 & 0.15 & -0.07 \\
0.01 & -0.12 & -0.03 & 0.14 \\
0.07 & -0.03 & -0.12 & 0.08 \\
\end{bmatrix}
\]

it is \( \text{sp}(L) = \{0, 0.0901 \pm 0.199i, 0.169\} \), i.e., \(-L\) is marginally stable of corank 1. Moreover, \( L \mathbb{I} = L^T \mathbb{I} = 0 \) and, for \( d > 0.2647, B = dI - L > 0 \). However, \( \text{sp}(L_\nu) = \{-0.0402, 0.01248, 0.2655\} \), i.e., \( L_\nu \) is not psd.

Example 2 For

\[
L = \begin{bmatrix}
0.23 & 0 & -0.28 & 0.05 \\
-0.01 & 0.03 & 0.02 & -0.04 \\
0.05 & -0.03 & 0.04 & -0.06 \\
-0.27 & 0.02 & 0.22 & 0.20 \\
\end{bmatrix}
\]

it is \( \text{sp}(L) = \{0, 0.1443 \pm 0.1859i, 0.0514\} \), i.e., \(-L\) is marginally stable of corank 1. Moreover, \( L \mathbb{I} = L^T \mathbb{I} = 0 \) and, for \( d > 0.1919, B = dI - L > 0 \). However, \( \text{sp}(L_\nu) = \{-0.0446, 0.0, 0.0404, 0.3441\} \), i.e., \( L_\nu \) is not psd.

As already mentioned, for signed Laplacians, irreducibility does not imply corank 1. When we have weight balance, however, the opposite is true.

Lemma 2 Let \( G(A) \) be a signed digraph with Laplacian \( L \). If \(-L\) is eventually exponentially positive or if \( L \) is weight balanced and of corank 1, then \( L \) is irreducible.

Proof: In both statements assume, by contradiction, that \( L \) is reducible, i.e., there exists a permutation matrix \( P \) s.t.

\[
P^T LP = \begin{bmatrix}
L_{11} & L_{12} \\
0 & L_{22} \\
\end{bmatrix}
\]

Assume that \(-L\) is eventually exponentially positive, i.e., \( \exists d \geq 0 \) s.t. \( B = dI - L > 0 \) (see Lemma 1). Then \( B \) is also reducible, since \( P^T BP = \begin{bmatrix}
(dI-L_{11}) & -L_{12} \\
0 & dI-L_{22} \\
\end{bmatrix}
\)

It follows that \( (P^T BP)^k \) for all \( k \geq 1 \), i.e., \( P^T BP \) is not eventually positive and, consequently, \( B \) is not eventually positive.

Assume that \( L \) is weight balanced of corank 1. Then \( L \mathbb{I} = L^T \mathbb{I} = 0 \) implies that \( 0 \in \text{sp}(L_{11}^T) = \text{sp}(L_{11}) \) and that \( 0 \in \text{sp}(L_{22}) \). Consequently, \( L \) is not of corank 1.

Remark 2 For a signed digraph \( G(A) \) it holds that if \( L_\nu \) is psd of corank 1 then \( L \) is EP (see [28]) and hence
weight balanced, and \(-L\) is marginally stable of corank 1. Therefore, in Theorem 2, the assumption that the Laplacian \(L\) is a normal matrix is sufficient to prove that \(L_s\) is psd of corank 1 but not necessary. For example, for
\[
L = \begin{bmatrix}
1 & 1 & -1 & -1 \\
-1 & 1 & 0 & 0 \\
-1 & -1 & 2 & 0 \\
1 & -1 & -1 & 1
\end{bmatrix},
\]
which is not normal, it is \(\text{sp}(L) = \{0, 1.5 \pm 1.323i, 2\}\), i.e., \(-L\) is marginally stable of corank 1, and \(\text{sp}(L_s) = \{0, 0.7192, 1.5, 2.7808\}\), i.e., \(L_s\) is psd of corank 1.

We now show that the same statements of Theorem 2 hold also for the pseudoinverse \(L^\dagger\) of \(L\). Moreover, we show that \(-L\) is eventually exponentially positive (and marginally stable) if and only if \(-L^\dagger\) is. These results are summarized in the following theorem.

**Theorem 3** Let \(G(A)\) be a directed signed network such that the corresponding Laplacian \(L\) is weight balanced. Let \(L^\dagger\) be the weight balanced pseudoinverse of \(L\). Then, the following conditions are equivalent:

(i) \(-L\) is eventually exponentially positive;

(ii) \(-L^\dagger\) is marginally stable of corank 1;

(iii) \(-L^\dagger\) is eventually exponentially positive.

Furthermore, if \(L\) (equivalently, \(L^\dagger\)) is normal then (i)-(iii) are equivalent to

(iv) \(L^\dagger = \frac{L^\dagger + (L^\dagger)^T}{2}\) is psd of corank 1.

The proof of Theorem 3 relies on some considerations and propositions that we state first.

Since a weight balanced \(L\) (of corank 1) is an EP-matrix, its left and right orthogonal projectors onto \(RB(L)\) are identical and given by \(P = L - J\), with \(J = \frac{L^\dagger L}{\|L\|^2}\). Furthermore it is \(\lim_{t \to \infty} e^{-Lt} = J\). The following properties for \(J\) can be found in [29], [30], [22]) or computed straightforwardly.

**Lemma 3** The matrix \(J\) has the following properties:

1. \(J^k = J\ \forall k \in \mathbb{N}\) which implies that \((I - J)^k = (I - J)\) \(\forall k \in \mathbb{N}\);

2. \(JL = LJ = 0\) which implies that \(e^{-(L+J)t} = e^{-Lt}e^{-Jt}\) and \(Je^{-Jt} = e^{-LJ}J = J;\)

3. \(e^{-Jt} = I - J + Je^{-t}\) which implies that \(Je^{-Jt} = e^{-Jt}J = Je^{-t}\).

We have the following properties for the Laplacian pseudoinverse.

**Lemma 4** If \(L\) is weight balanced and of corank 1, then \(L^\dagger\) is weight balanced and of corank 1. For it

\[
LL^\dagger = L^\dagger L = \Pi
\]

\[(L^\dagger)T = 0\]

\[
L^\dagger \Pi = \Pi L^\dagger = 0
\]

\[
L^\dagger = (L + \gamma J)^{-1} - \frac{1}{\gamma} J \ \forall \gamma \neq 0
\]

Furthermore, if \(L\) is normal then \(L^\dagger\) is normal.

**Proof:** Assume that \(L\) is weight balanced. Eq. (1)-(4) are all well-known for \(L\) symmetric, and follow easily also for EP matrices. They are proven here only for sake of completeness. Eq. (3) is a consequence of \(L\) commuting with \(L^\dagger\). As for eq. (2), from \((L^\dagger)^T = L^\dagger L\) it follows that \(L^T = (L^\dagger)^T\) and \(L^T = 0\) if \(L\) is irreducible, \(L^T v \neq 0\) for \(v \neq 0\) \((\forall \in \mathbb{R})\), hence it must be \((L^\dagger)^T = 0\) or \((L^\dagger)^T = 0\), i.e., \(L^\dagger\) has \(\Pi\) as left eigenvector relative to 0. The proof for the right eigenvector is identical. Concerning eq. (3), from \(L^\dagger = 0\) it is \(L^\dagger = L^\dagger(I - \frac{L^\dagger L}{\|L\|^2}) = L^\dagger\), and similarly for \(\Pi L^\dagger = L^\dagger\).

For eq. (4), since \(L + \gamma J\) is nonsingular, as in [24], it is enough to show the following:

\[
(L + \gamma J)(L + \frac{1}{\gamma}) = LL^\dagger + \gamma JL + \frac{1}{\gamma} J^2 = \Pi + J = I - J + J = I,
\]

where we have used the properties of Lemma 3.

Then, \(N(L) = N(L^\dagger) = N(L^\dagger) = N((L^\dagger)^T) = \text{span}(\Pi)\) and (i) and (ii) imply that \(L^\dagger\) is weight balanced of corank 1.

Finally, we need to show that if \(L\) is normal then \(L^\dagger\) is normal. \(L\) normal, \(J\) symmetric and \(LJ = JL^2 = 0\) imply \(L + \gamma J\) normal, which means that \((L + \gamma J)^{-1}\) is also normal. Since \(J\) is symmetric (hence normal) and satisfies the properties of Lemma 3 to show that \(L^\dagger\) is normal it is sufficient to observe that \((L + \gamma J)^{-1} = \frac{1}{J} = J = (J + \gamma J)^{-1}.

We are now ready to prove Theorem 3.

**Proof:** (i)\(\Rightarrow\)(ii) To show marginal stability of \(-L^\dagger\), denote \(\lambda_i(L)\) the eigenvalues of \(L\), of eigenvectors \(\Pi, v_2, \ldots v_n\). Using Theorem 2 since \(-L\) is eventually exponentially positive then \(-L\) is also marginally stable of corank 1, meaning that \(0 = \lambda_1(L) < \Re(\lambda_2(L)) \leq \cdots \leq \Re(\lambda_n(L)).\) Consider eq. (4) of Lemma 3. Choosing \(\gamma \neq 0\), since \(J\) is the orthogonal projection onto \(N(L) = N(L^\dagger) = \text{span}(\Pi)\), the effect of adding \(\gamma J\) to \(L\) is only to shift the 0 eigenvalue to \(\gamma\), while \(\lambda_2(L), \ldots, \lambda_n(L)\) are unchanged (see [23, Thm 2.4.10.1]). For the nonsingular \(L + \gamma J\) the inverse \((L + \gamma J)^{-1}\) has eigenvalues \(\frac{1}{\lambda_1(L)} < \frac{1}{\lambda_2(L)} < \cdots < \frac{1}{\lambda_n(L)}\) of eigenvectors \(\Pi, v_2, \ldots v_n\). From orthogonality, \((L + \gamma J)^{-1} = \frac{1}{J} J = \frac{1}{\gamma\lambda_1(L)}\), which shows the \(\frac{1}{\gamma}\) eigenvalue back to the origin without touching the other eigenvalues.

(i)\(\Rightarrow\)(iii) Assume that \(-L\) is eventually exponentially positive, that is, \(-L\) is marginally stable of corank 1 (see Theorem 2). Then \(-L^\dagger\) is also marginally stable of corank 1, see Lemma 4 and proof (i)\(\Rightarrow\)(ii). To prove that \(-L^\dagger\) is eventually exponentially positive, we can use Theorem 2.

The proof is here reported for completeness. In particular, from Lemma 4 (i) we know that \(-L^\dagger\) is marginally stable with \(0 = \lambda_1(L^\dagger) < \Re(\lambda_2(L^\dagger)) \leq \cdots \leq \Re(\lambda_n(L^\dagger))\) and with \(\Pi\) as leftover eigenvector for 0. If we choose \(d > \max_{\lambda_2(L^\dagger)} \|\frac{\lambda_2(L^\dagger)^2}{\Re(\lambda_2(L^\dagger))}\|\), then \(B = dI - L^\dagger\) has \(\rho(B) = d\) as a simple eigenvalue of eigenvector \(\Pi\) and so does \(B^T\).

Hence \(B, B^T \in \mathcal{PF}\), or, from Theorem 1 \(\lambda \leq 0\), i.e., \(B\) is
eventually positive. Hence from Lemma [1] $L^\dagger$ is eventually positive.

(ii) $\Rightarrow$ (iii) Since $L^T$ is weight balanced of corank 1 with $\text{span}(\ell) = \mathcal{N}(L^T) = \mathcal{N}((L^T)^T)$, it is itself a signed Laplacian. The argument can be proven in a similar way as the opposite direction, observing that $L = (L^T)^\dagger$.

(iv) Assume now that $L$ is normal or, equivalently, that $L^\dagger$ is normal (see Lemma [3]). Since $L$ normal implies $L$ weight balanced, the statements (i), (ii), and (iii) are still equivalent. To show the equivalence with (iv) it is sufficient to apply Theorem [2] on $L^T$ since $L^T$ is itself a normal signed Laplacian of corank 1.

The following corollary, characterizing the class of eventually exponentially positive Laplacian matrices, follows directly from Theorem [3].

**Corollary 1** The class of eventually exponentially positive, weight balanced Laplacian matrices is closed under the pseudoinverse operation.

The class of eventually exponentially positive, normal Laplacian matrices is closed under the pseudoinverse and the symmetrization operation (the latter intended as the operation of taking the symmetric part).

**Remark 3** Notice that the operations of pseudoinverse and of symmetrization do not commute, i.e., $L^\dagger = (L + L^T)^\dagger \neq (L_s)^\dagger = (L + L^T)^\dagger$, even in the case of a normal Laplacian $L$.

Indeed, let $L = UDU^T$ with $U$ orthonormal and $D$ as in the proof of Theorem [2]. Without lack of generality, assume that the first column of $U$ is $1_{\sqrt{D}}$, which means that $D = 0 \oplus \bar{D}$ where $\bar{D} = \mu_2 \oplus \cdots \oplus \mu_k \oplus \begin{bmatrix} \nu_1 & \omega_1 \\ -\omega_1 & \nu_1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \nu_t & \omega_t \\ -\omega_t & \nu_t \end{bmatrix}$ is nonsingular (here $\oplus$ denotes the direct sum). Then

$$L_s = U \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} U^T,$$

$$L_s^\dagger = U \begin{bmatrix} 0 & 0 \\ 0 & D^{-1} \end{bmatrix} U^T,$$

and

$$(L_s)^\dagger = U \begin{bmatrix} 0 & 0 \\ 0 & (D + D^T)^{-1} \end{bmatrix} U^T \neq L_s^\dagger.$$

**Example 3** For the Laplacian $L$ of Example 1 we obtain

$$L^\dagger = \begin{bmatrix} 2.25 & -1.86 & -0.19 & -0.19 \\ -1.42 & 1.58 & -5.64 & 5.47 \\ 1.92 & 0.47 & 4.36 & -6.75 \\ -2.75 & -0.19 & 1.47 & 1.47 \end{bmatrix}.$$

It is $\text{sp}(L^\dagger) = \{0, 1.8888 \pm 4.1709, 5.8891\}$ (notice that $\lambda_2(L) = \frac{1}{x_{2}(L)}$, $i = 2, 3, 4$, i.e., $-L^\dagger$ is marginally stable of corank 1. Moreover, $L^\dagger \Vert (L^T)^\dagger = 0$ and, for $d > 5.5495$, $B = \|dI - L^\dagger\| > 0$. However, $\text{sp}(L^\dagger) = \{-1.1164, 0.2.0926, 8.6904\}$, i.e., $L_s^\dagger$ is not psd.

**Example 4** The signed Laplacian

$$L = \begin{bmatrix} 0.282 & -0.072 & 0.191 & -0.401 \\ -0.072 & 0.252 & 0.008 & -0.189 \\ -0.401 & -0.189 & 0.297 & 0.293 \\ 0.191 & 0.008 & -0.496 & 0.297 \end{bmatrix}$$

is normal, and it is $\text{sp}(L) = \{0, 0.3983 \pm 0.5920i, 0.3311\}$, i.e., $-L$ is marginally stable of corank 1. In accordance with Theorem [3] $\text{sp}(L^\dagger) = \{0, 0.7823 \pm 1.1628i, 3.0204\}$, i.e., $-L^\dagger$ is marginally stable of corank 1, $\text{sp}(L_s) = \{0, 0.3983, 0.3983, 0.3311\}$, i.e., $L_s$ is psd of corank 1, and $\text{sp}(L_s^\dagger) = \{0, 0.7823, 0.7823, 3.0204\}$, i.e., $L_s^\dagger$ is psd of corank 1.

When the weight balanced digraph is nonnegative, the normality assumption in Theorem [4] iv) can be dropped.

**Theorem 4** Let $G(A)$ be a strongly connected nonnegative ($A \geq 0$) digraph such that the corresponding Laplacian $L$ is weight balanced. Then $L^\dagger$ is the irreducible weight balanced pseudoinverse of $L$. Then $L_s^\dagger = \frac{L + (L^T)^\dagger}{2}$ is psd of corank 1.

**Proof**: In [27, Corollary 1] it is shown that when $A \geq 0$, $L_s = \frac{L + (L^T)^\dagger}{2}$ is psd of corank 1 if and only if $G(A)$ is weight balanced. Using (3) (see Lemma [3]) we can write

$$L_s^\dagger = \frac{(L + \gamma J)^{-1} + (L^T + \gamma J)^{-1}}{2} - \frac{1}{\gamma} J = (L + \gamma J)^{-1} - \frac{L + \gamma J}{2} (L + \gamma J)^{-1} - \frac{1}{\gamma} J$$

$$= (L + \gamma J)^{-1}(L_s + J)(L^T + \gamma J)^{-1} - \frac{1}{\gamma} J$$

$$= (L + \gamma J)^{-1}(L_s + J)(L^T + \gamma J)^{-1} = (L + \gamma J)^{-1}(L_s + J - \gamma J)(L^T + \gamma J)^{-1}$$

$$= (L + \gamma J)^{-1}L_s(L + \gamma J)^{-1} = (L + \gamma J)^{-1}L_s(L + \gamma J)^{-1}.$$
\(G_r\) be the signed undirected graph obtained by applying the Kron reduction on \(G\), and let \(L_r = L/L[\beta]\) be its (symmetric) Laplacian. Consider the following conditions:

(i) \(-L\) is eventually exponentially positive;
(ii) \(-L_r\) is psd of corank 1;
(iii) \(-L_r\) is eventually exponentially positive.

If \(L\) satisfies (i) then \(L_r\) satisfies (ii) and (iii).

Furthermore, if \(\alpha\) is the set of nodes incident to negatively weighted edges and \(\beta = \{1, \ldots, n\} \setminus \alpha\), then the conditions (i), (ii), (iii) are equivalent.

Proof: Let \(\alpha\) (with \(\text{card}(\alpha) \in [2, n - 1]\)) and \(\beta = \{1, \ldots, n\} \setminus \alpha\) be a partition of the node set \(V\) meaning that, after an adequate permutation, \(L\) can be rewritten as

\[L = \begin{bmatrix} L[\alpha] & L[\alpha, \beta] \\ L[\beta, \alpha] & L[\beta] \end{bmatrix}.
\]

Let \(L_r = L/L[\beta] = L[\alpha] - L[\alpha, \beta]L[\beta]^{-1}L[\beta, \alpha] \in \mathbb{R}^{\text{card}(\alpha) \times \text{card}(\alpha)}\) be the Kron reduced matrix. Observe that \(L_r\) is symmetric and that \(L_r\) is irreducible - or (by contradiction) that \(L_r\) is actually \(\text{pd}\), since \(-L_r\) is irreducible and has the row and column inclusion property.

Let \(\text{card}(\beta) = 1\) and assume, by contradiction, that \(L[\beta] = \Sigma[\beta] = 0\). However, \(L\) psd means that \(L\) has the row and column inclusion property, i.e., if the diagonal element \(\Sigma[\beta] = 0\) then \(A[\alpha, \beta] = 0\) and \(A[\beta, \alpha] = 0\), which contradicts the hypothesis that \(L\) (and \(A\)) is irreducible. Hence, \(L[\beta] > 0\) (pd). Now repeat the same argument for \(1 < \text{card}(\beta) \leq n - 2\); suppose by contradiction that \(L[\beta] = \text{psd}\), i.e., there exists a vector \(v \in \mathbb{R}^{\text{card}(\beta)}\) s.t.

\[L[\beta]v = 0.
\]

Then \(\tilde{v} = \begin{bmatrix} 0 \\ v \end{bmatrix}\) is s.t. \(\tilde{v} \tilde{L} \tilde{v} = 0\) (since \(\tilde{v}^T \tilde{L} \tilde{v} = 0\)), which contradicts the hypothesis that \(L\) has corank 1 since \(1 \in \mathcal{N}(L)\) and \(\tilde{v} \notin \text{span}(1)\) (notice that if \(v = L[\text{card}(\beta)]\), then either \(A[\beta, \alpha] = 0\), which is the zero matrix - in contradiction with the hypothesis that \(L\) is irreducible - or \(L[\beta] = 0\)). Then, \(L_r[\beta]v = 0\). Therefore, \(L_r[\beta]\) is \(\text{pd}\).

Rewrite \(L\) as

\[L = \begin{bmatrix} I & L[\alpha]L[\beta]^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} L_r & 0 \\ 0 & L[\beta] \end{bmatrix} \begin{bmatrix} I & -L[\beta]^{-1}L[\beta, \alpha] \\ 0 & I \end{bmatrix} = \begin{bmatrix} L_r[\beta] & 0 \\ 0 & L[\beta] \end{bmatrix}
\]

where \(L[\alpha, \beta]L[\beta]^{-1} = (L[\beta]^{-1}L[\beta, \alpha])^T\). Then, applying Sylvester’s law of inertia, \(L\) psd of corank 1 and \(L[\beta]\) pd imply \(L_r\) psd of corank 1 or, equivalently (from Theorem 2), \(-L_r\) eventually exponentially positive.

(i) \(\Longleftrightarrow\) (ii) \(\Longleftrightarrow\) (iii). Let \(\alpha\) be the set of nodes incident to negatively weighted edges. In what follows, the steps marked by the symbol * follow from Theorem 2 while the step marked by the symbol \(\triangle\) from [31, Theorem 1].

\(-L\) eventually exponentially positive
\(\triangle L\) psd of corank 1
\(\triangle L_r\) psd of corank 1
\(-L_r\) eventually exponentially positive.

Similarly to Corollary 1 from Theorems 3 and 5 we obtain the following characterization of the class of eventually exponentially positive Laplacian matrices of undirected graphs.

**Corollary 2** The class of eventually exponentially positive, irreducible, symmetric Laplacian matrices is closed under the pseudoinverse operation and the operation of Kron reduction.

**IV. ELECTRICAL NETWORKS AND EFFECTIVE RESISTANCE**

A resistive electrical network can be represented as a graph \(G(A) = (V, E, A)\) where each weight \(a_{ij}\) represents the inverse of the resistance between the two nodes (i.e., the conductance of the transmission): \(a_{ij} = \frac{1}{R_{ij}}\), see [7], [9] and [25] for an overview. The notion of effective resistance between a pair of nodes (see e.g. [25]) is related to the pseudoinverse of the Laplacian associated to the electrical network. When the network is connected, undirected and nonnegative, its Laplacian (and its pseudoinverse) is known to be psd of corank 1, which means that the effective resistance between two nodes is well-defined (see e.g. [9] for its properties). Extensions to signed graphs and negative resistances have been investigated in [32], [26], [33], [34], [31], where positive semidefiniteness of the Laplacian is expressed in terms of effective resistance.

In what follows we make use of \(L[1]\) to extend the notion of effective resistance to directed (strongly connected) signed networks whose Laplacian \(L\) is a normal matrix and \(-L\) is eventually exponentially positive.

**Definition 4** The effective resistance between two nodes \(i, j \in \{1, \ldots, n\}\) of a signed digraph whose corresponding Laplacian \(L\) is normal and \(-L\) is eventually exponentially positive, is given by

\[R_{ij} = (e_i - e_j)^T L[1]^s(e_i - e_j),\]

where \(L[1]^s = L[1]^T L[1]^T\) and \(L[1]^s\) is the pseudoinverse of \(L\). The effective resistance matrix \(R = [R_{ij}]\) is defined as

\[R = D[1]^\frac{1}{2} I^\frac{1}{2} L[1]^T + D[1]^\frac{1}{2} I^\frac{1}{2} L[1]^T - 2L[1]^s\]

where \(D[1] = \text{diag} \{L[1]_{11}, \ldots, L[1]_{nn}\}\) is a diagonal matrix whose elements are the diagonal elements of \(L[1]^s\). The total effective resistance is defined as

\[R_{\text{tot}} = \frac{1}{2} L[1]^T R L[1].\]
In the literature on undirected networks, the total effective resistance \(7\) is also called "weighted effective graph resistance" [35] or "Kirchhoff index" [8], and represents the overall transport capability of the graph [13].

**Remark 4** If the graph is undirected, eq. \(5\) reduces to the standard notion of effective resistance since \(L_1^\dagger = L^1\).

The effective resistance \(5\), as its counterpart for undirected graphs (see [7], [9], [25]), is still nonnegative and symmetric, its square root is a metric, and the effective resistance matrix \(6\) is a Euclidean distance matrix, i.e., it has nonnegative elements, zero diagonal elements and it is negative semidefinite on \(\mathbb{R}^+\) [9]. The last part of the proof of the next lemma follows [9, Section 2.8] and is here reported for completeness.

**Lemma 5** The square root of the effective resistance \(5\) between two nodes \(i, j \in \{1, \ldots, n\}\) of a signed digraph with normal Laplacian \(L\) is a metric, and it satisfies the triangle inequality. The effective resistance matrix \(6\) is a Euclidean distance matrix.

**Proof:** Theorem [3] shows that for a signed digraph with normal Laplacian \(L\) s.t. \(-L\) is eventually exponentially positive, the matrix \(L_1^\dagger\) itself is a signed Laplacian and it is psd of corank 1 with \(\mathcal{N}(L_1^\dagger) = \text{span}(I)\). Since \(R_{ij}\) is a quadratic form generated by \(L_1^\dagger\), then

\[
R_{ij} = (e_i - e_j)^T L_1^\dagger (e_i - e_j) = \| (L_1^\dagger)^{(e_i - e_j)} \|^2_2
\]

for all \(i, j = 1, \ldots, n\), and \(R_{ij} = 0\) if and only if \(i = j\) (since \(e_i - e_j \in \text{span}(I)\) when \(i \neq j\)). Moreover,

\[
\sqrt{R_{ik} + R_{kj}} = \| (L_1^\dagger)^{(e_k)} \|_2 + \| (L_1^\dagger)^{(e_i - e_j)} \|_2 \geq \| (L_1^\dagger)^{(e_i)} \|_2 - \| (L_1^\dagger)^{(e_k)} \|_2 + \| (L_1^\dagger)^{(e_k)} \|_2
\]

for all \(i, j, k = 1, \ldots, n\), i.e., the triangle inequality holds.

Finally, to prove that \(R\) is an Euclidean distance matrix we need to show that \(x^T Rx \leq 0\) for all \(x \in \mathbb{R}^n\):

\[
x^T Rx = x^T (D_{L_1^\dagger} I TT^T D_{L_1^\dagger} - 2L_1^\dagger) x = -2x^T L_1^\dagger x \leq 0,
\]

since \(L_1^\dagger\) is psd with \(\mathcal{N}(L_1^\dagger) = \text{span}(I)\).

Notice that if we consider only nonnegative digraphs then the normality assumption of the Laplacian can be replaced by the less restrictive weight balanced assumption when defining the effective resistance in \(5\). Indeed, Theorem [3] shows that if the digraph is nonnegative and strongly connected then \(L_1^\dagger\) is psd of corank 1.

**Proposition 1** Consider a nonnegative strongly connected weight balanced digraph \(G(A)\) (with \(A \geq 0\)). Then \(R_{ij} \geq 0\) for all \(i, j = 1, \ldots, n\), and \(R_{\text{tot}} \geq 0\).

Another generalization of the notion of effective resistance for directed, strongly connected, nonnegative networks is introduced in [14], [21]. The authors use the fact that the Laplacian \(L\) is marginally stable and its projection on \(\mathbb{R}^+\), denoted \(\bar{L} = QLQ^T\) (where the rows of \(Q \in \mathbb{R}^{n-1 \times n}\) form an orthonormal basis for \(\mathbb{R}^+\)), is Hurwitz stable, to define the effective resistance between nodes \(i, j\) as

\[
\bar{R}_{ij} = (e_i - e_j)^T \bar{L} (e_i - e_j),
\]

where \(X = 2Q^T SQ\) and \(S\) is the psd solution of the Lyapunov equation \(\bar{L}S + S \bar{L}^T = I_{n-1}\).

The Kirchhoff index is then defined as \(K_f = \sum_{i<j} \bar{R}_{ij}\).

If we consider digraphs \(G(A)\) whose Laplacian is a normal matrix, \(K_f\) reduces to \(K_f = n \sum_{i=2}^{n} \lambda_i(L)\) and we can show that it provides an upper bound for \(R_{\text{tot}}\) defined in \(7\).

\[
R_{\text{tot}} = n \cdot \text{trace}(L_1^\dagger) = n \cdot \sum_{i=2}^{n} \lambda_i(L_1^\dagger)
\]

with equality only if \(G(A)\) is undirected (notice that \(L\) normal and non-symmetric means \(n \geq 3\)).

**Example 5** Let \(G(A)\) be a nonnegative, unweighted, directed, cycle graph, whose Laplacian \(L\) is a normal matrix with eigenvalues \(1 + e^{i\theta_k}\), with \(\theta_k = \pi(1 - \frac{2k}{n})\), for all \(k = 0, \ldots, n - 1\). Then, \(K_f = \frac{n(n^2 - 1)}{6}\) (see e.g. [14]),

\[
R_{\text{tot}} = n \cdot \sum_{k=2}^{n} \lambda_k(L_1^\dagger) = n \cdot \sum_{k=2}^{n} \frac{1}{2} \left(1 + \cos \theta_k\right) \approx n \cdot \sum_{k=2}^{n} \frac{1}{2} \left(1 + \frac{n(n^2 - 1)}{6}\right)
\]

\[
= n \cdot \left(\frac{n(n^2 - 1)}{2}\right) - \frac{n(n^2 - 1)}{6},
\]

and we obtain \(R_{\text{tot}} \leq K_f\) for all \(n \geq 2\).

**V. CONCLUSIONS AND FUTURE WORK**

For signed Laplacians which are weight balanced, marginal stability (of corank 1) is equivalent to eventual exponential stability. This work shows that the class of eventually exponentially positive, weight balanced Laplacians is closed under the pseudoinverse operation and, therefore, it provides a natural embedding for the usual nonnegative Laplacian. As a byproduct we get conditions for checking the marginal stability of the pseudoinverse of signed Laplacians. Moreover, closure under the symmetrization operation can be proven when this class is restricted to Laplacians that are also normal matrices. The normality assumption is a sufficient condition and it remains to be investigated if it can be relaxed.

In addition, we would like to gain a better understanding of the set of eventually exponentially positive, weight balanced Laplacians and its properties. For instance, it is easy to observe that it is not a convex cone, not even if we consider normal matrices (but the intuition is that this set is actually a convex cone, without the origin, if we restrict to undirected graphs). However, similarly to [19], it is possible to show...
that it is path-wise connected. These considerations, among other directions, will be investigated in a future paper.

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