Series of $p$-groups with Beauville structure

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Abstract — We construct infinite series of $p$-groups that allow unmixed Beauville structures. This gives the first examples of non-abelian $p$-groups of arbitrary large order and any prime $p \geq 5$.

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1. Introduction

The datum of an unmixed Beauville structure on a finite group $G$ encodes two finite branched $G$-covers $C \to \mathbb{P}^1$ and $D \to \mathbb{P}^1$ with ramification only above 0, 1 and $\infty$, such that the diagonal $G$ action on $C \times D$ is fixed point free. The quotient $X = (C \times D)/G$ belongs to an interesting class of (rigid) algebraic surfaces the systematic study of which was initiated by Catanese [Ca00]. These surfaces are now known as Beauville surfaces, and those with $p_g = 0$ have been classified by Catanese, Bauer and Grunewald [BCG08] and Frapporti [Fr13].

Definition 1. An (unmixed) Beauville structure on a finite group $G$ consists of an ordered pair of triples $(x, y, z)$ and $(a, b, c)$ of group elements such that

(i) $xyz = 1 = abc$, and
(ii) $G$ is generated by $\{x, y, z\}$ and by $\{a, b, c\}$, and
(iii) no non-trivial power of an element of $\{x, y, z\}$ is conjugate to a power of an element of $\{a, b, c\}$.

The signature of the Beauville structure is the tuple of orders of the elements $x, y, z, a, b, c$ and the Beauville structure is balanced if these orders are constant. The individual triples $(x, y, z)$ and $(a, b, c)$ are referred to as half of a Beauville structure.

There is an abundance of groups that afford Beauville structures: Beauville’s original example of $(\mathbb{Z}/5\mathbb{Z})^2$, for every non-abelian finite simple group (except $A_5$) by Malle and Guralnick [MG12] and independently Fairbairn, Magaard and Parker [FMP13], and some Beauville structures on $p$-groups by Barker, Boston and Fairbairn [BBF12], and with $p = 2$ by Barker, Boston, Peyerimhoff and the second author [BBPV12]. For a more detailed survey on Beauville surfaces and groups we refer to [BGV14].

Our main result will be the construction of new series of $p$-groups with unmixed Beauville structures that form a pro-finite system. Compatible Beauville structures in a pro-system lead to pro-system of finite covers of the corresponding algebraic Beauville surfaces with the tower being unramified if and only if the signature does not change in the tower.

Theorem 2. Every finite $p$-group with an unmixed Beauville structure sits in an infinite pro-system of $p$-groups with compatible unmixed Beauville structures such that the signature of the first half remains constant throughout the pro-system.

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While the construction of the pro-system in Theorem 2 is not explicit, the second construction for Theorem 3 exploits the structure of simple metabelian uniform $p$-group and is completely explicit.

**Theorem 3.** Let $p$ be a prime, $n, m \in \mathbb{N}$ and $\lambda \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ with $\lambda^{p^m} \equiv 1 \mod p^m$. The semidirect product

$$\mathbb{Z}/p^n\mathbb{Z} \rtimes_\lambda \mathbb{Z}/p^n\mathbb{Z}$$

with action $\mathbb{Z}/p^n\mathbb{Z} \to \text{Aut}(\mathbb{Z}/p^n\mathbb{Z})$ sending $1 \mapsto \lambda$ admits an unmixed Beauville structure if and only if $p \geq 5$ and $n = m$. All Beauville structures are balanced of constant signature $p^n$.

Theorem 2 will be proven in Section §2, while Section §3 contains the proof of Theorem 3.

2. Pro-systems of Beauville structures

Versions of the following lifting result have also been observed for example in [BBF12] Lemma 17, or [FJ11] Lemma 4.2. The special case of $p$-groups considered here allows to exploit the Frattini argument to ensure the generating property.

2.1. Triangle group lifting. The (strict) triangle group of signature $p^n$, $p^m$, $p^r$ is the following group

$$\Delta_{m,n,r} = \langle X, Y, Z | X Y Z = 1, X^{p^m} = Y^{p^n} = Z^{p^r} = 1 \rangle$$

that we consider as a group together with distinguished generators $X$, $Y$ and $Z$.

We recall that the Frattini group of a finite group $G$ is the group

$$\Phi(G) = \bigcap_{H \leq G, \text{maximal}} H.$$

For a $p$-group $G$ the Frattini subgroup is the kernel of the maximal elementary abelian $p$-quotient

$$\Phi(G) = G^p \cdot [G, G] = \ker(G \to G^{ab} \otimes \mathbb{F}_p).$$

A set of elements generate $G$ if and only if their images in $G/\Phi(G)$ form a set of generators.

**Proposition 4.** Let $\tilde{G} \to G$ be a quotient map of finite $p$-groups such that $G$ admits a Beauville structure given by the tuples $(x, y, z)$ and $(a, b, c)$.

Assume that $(x, y, z)$ lifts to a triple $(\tilde{x}, \tilde{y}, \tilde{z})$ in $\tilde{G}$ with $\tilde{x}\tilde{y}\tilde{z} = 1$, generating $\tilde{G}$ and preserving the orders of the respective elements. Then the Beauville structure lifts to a Beauville structure $(\tilde{x}, \tilde{y}, \tilde{z})$ and $(\tilde{a}, \tilde{b}, \tilde{c})$ of $\tilde{G}$ with $\tilde{a}, \tilde{b}$ arbitrary lifts of $a, b$ to $\tilde{G}$ and $\tilde{c} = (\tilde{a}\tilde{b})^{-1}$.

**Proof:** Since $G$ admits a Beauville structure, the group is not cyclic and hence $G/\Phi(G) = G^{ab} \otimes \mathbb{F}_p$ is a vector space of dimension 2. The same holds for $\tilde{G}$, hence the map $\hat{\Phi}(\tilde{G}) \to G/\Phi(G)$ is an isomorphism. By the Frattini property, the generating sets for $G$ are lifts of generating sets of $G/\Phi(G)$, thus lifts of generators for $G$ also necessarily generate $\tilde{G}$: the triple $(\tilde{a}, \tilde{b}, \tilde{c})$ also generates $\tilde{G}$.

If $(\tilde{x}, \tilde{y}, \tilde{z})$ and $(\tilde{a}, \tilde{b}, \tilde{c})$ do not form a Beauville structure, then there is wlog $n, m \in \mathbb{N}$ and $g \in \tilde{G}$ such that $1 \neq \tilde{x}^n = g(\tilde{a}^m)g^{-1}$. Projecting to $G$ and using that $(x, y, z)$ and $(a, b, c)$ is a Beauville structure, we find $x^n = 1$ in $G$. But since $\tilde{x}$ and $x$ have the same orders, we deduce $\tilde{x}^n = 1$, a contradiction. \qed

**Remark 5.** We refer to lifting of Beauville structures as in Proposition 4 as triangle group liftings since we view both $G$ and $\tilde{G}$ via the datum of $(\tilde{x}, \tilde{y}, \tilde{z})$ as quotients of a triangle group

$$\Delta_{m,n,r} \twoheadrightarrow \tilde{G} \twoheadrightarrow G.$$

with $\text{ord}(x) = p^m$, $\text{ord}(y) = p^n$ and $\text{ord}(z) = p^r$.

**Definition 6.** An (unmixed topological) Beauville structure on a pro-finite group $G$ consists of an ordered pair of triples $(x, y, z)$ and $(a, b, c)$ of group elements such that

(i) $xyz = 1 = abc$, and

(ii) $G$ is topologically generated by $\{x, y, z\}$ and by $\{a, b, c\}$, and

(iii) no non-trivial (pro-finite) power of an element of $\{x, y, z\}$ is conjugate to a (pro-finite) power of an element of $\{a, b, c\}$.
If \( G = \varinjlim G_i \) is a description as a surjective pro-system of finite groups, then a compatible system of Beauville structures on the \( G_i \) will describe a topological Beauville structure on \( G \). The converse is not true in general: the image in \( G_i \) of a topological Beauville structure \((X,Y,Z)\) and \((A,B,C)\) on \( G \) may fail to be a Beauville structure. If the orders of \( X, Y \) and \( Z \) in \( G \) are bounded, then

\[
M^0_{X,Y,Z} := \bigcup_{n \in \mathbb{N}, \, g \in G} g\{X^n, Y^n, Z^n\}g^{-1} \setminus \{1\}
\]

and also \( M_{A,B,C} = (A) \cup \langle B \rangle \cup \langle C \rangle \), the union of the subgroups generated topologically by \( A, B, \) and \( C \) respectively, form compact subsets of \( G \). Hence, the sets \( M^0_{X,Y,Z} \) and \( M_{A,B,C} \) are disjoint in \( G \) if and only if there is a finite quotient \( G \to G_i \) such that the images there are disjoint. Then for all finer indices \( j \to i \) the images of \((X,Y,Z)\) and \((A,B,C)\) in \( G_j \) form indeed a compatible system of Beauville structures.

Let \( (\Delta_{m,n,r})^{\triangledown} \) be the pro-\( p \) completion of \( \Delta_{m,n,r} \).

**Corollary 7.** Let \((x, y, z)\) and \((a, b, c)\) be a Beauville structure on a finite \( p \)-group \( G \) with orders \( \text{ord}(x) = p^m \), \( \text{ord}(y) = p^n \) and \( \text{ord}(z) = p^r \). Then there is a Beauville structure on \((\Delta_{m,n,r})^{\triangledown} \) of the form \((X,Y,Z)\) and \((A,B,C)\) such that the triples \((x,y,z)\) and \((a,b,c)\) are their image via a unique surjection \((\Delta_{m,n,r})^{\triangledown} \to G \).

**Proof.** The assignment \( X,Y,Z \mapsto x,y,z \) determines uniquely a surjection \((\Delta_{m,n,r})^{\triangledown} \to G \). We write \((\Delta_{m,n,r})^{\triangledown} = \varprojlim \Delta_i \) as a projective limit of finite groups index system \((\mathbb{N},<)\). We assume \( \Delta_0 = G \). Let \((x_i, y_i, z_i)\) be the image of \((X,Y,Z)\).

Now set \( B_0 = \{(x, y, z)\} \cup \{(a, b, c)\} \) for the set containing only the initial Beauville structure. By Proposition 4 the set \( B_{i+1} \) of Beauville structures on \( \Delta_{i+1} \) of the form \( (x_{i+1}, y_{i+1}, z_{i+1}) \) and \((a', b', c')\) lifting a Beauville structure in \( B_i \) is non-empty and anyway finite. The sets \( B_i \) form a projective system and a standard compactness argument shows that \( \varprojlim B_i \) is non-empty as well. Moreover, the inductive construction of the \( B_i \) shows that \( \varprojlim B_i \to B_0 \) is surjective. Any element in \( \varprojlim B_i \) is described by two tuples in \((\Delta_{m,n,r})^{\triangledown} \), namely the standard generators \((X,Y,Z) = \varprojlim (x_i, y_i, z_i)\) and some \((A, B, C)\) which form a topological Beauville structure in \((\Delta_{m,n,r})^{\triangledown} \) mapping to the given one on \( G \) as claimed.

**Proof of Theorem 2.** The topological Beauville structure of Corollary 7 was constructed as a compatible family of Beauville structures on the pro-system of the \( \Delta_i \), and moreover so that the signature of the first half of the Beauville structure stays constant.

It remains to show that \((\Delta_{m,n,r})^{\triangledown} \) is infinite. But we work under the assumption that there is a \( p \)-group \( G \) with a Beauville structure where the orders of the first half are \( p^m, p^n \) and \( p^r \), hence the corresponding quotient \( \pi : \Delta_{m,n,r} \to G \) is smooth, i.e., \( \text{ker}(\pi) \) is the fundamental group of a compact Riemann surface. By Proposition 3.2 of [BCG04] the surface is hyperbolic and \( \text{ker}(\pi) \) a non-abelian surface group. Therefore its pro-\( p \) completion \( \text{ker}(\pi)^{\triangledown} \) is infinite and therefore also \((\Delta_{m,n,r})^{\triangledown} \) is infinite. This completes the proof of Theorem 2.

**2.2. Review of known Beauville structures on \( p \)-groups.** The use of Theorem 2 is limited to the extend that the construction is not explicit (although quite flexible), and also subject to knowing \( p \)-groups with Beauville structures to start with. For completeness here are some examples.

**Theorem 8** (Catanese [BCG04] Theorem 3.4). An abelian group admits a Beauville structure if and only if it is isomorphic to \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) and \((6, n) = 1\).

The following non-abelian examples of balanced signature \( p \) generalize examples from [BBF12].

**Proposition 9.** Let \( p \geq 5 \) be a prime number. A \( p \)-group \( G \) that can be embedded in \( \text{GL}_p(\mathbb{F}_p) \) admits a Beauville structure if and only if \( \dim_{\mathbb{F}_p} G/\Phi(G) = 2 \), that is if and only if \( G \) is generated by 2 elements but not cyclic.

**Proof:** If \( G \) admits a Beauville structure, then \( G \) is not cyclic and \( G/\Phi(G) \) is of dimension 2. For the converse direction we consider the Jordan normal form of \( g \in G \) that has Jordan blocks of length at most \( p \), hence all nontrivial elements in \( G \) are of order \( p \). Therefore we can
lift a Beauville structure along \( G \rightarrow G/\Phi(G) \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \) by triangle group lifting as in Proposition 4. The Frattini quotient admits a Beauville structure by [BCG04] Theorem 3.4 as recalled above.

\[ \square \]

3. An example of a uniform group with a Beauville structure

For an element \( \lambda \in (\mathbb{Z}/p^n\mathbb{Z})^\times \) of order dividing \( p^m \) we consider the semi-direct product

\[ G = \mathbb{Z}/p^m\mathbb{Z} \rtimes_{\lambda} \mathbb{Z}/p^n\mathbb{Z} \]

with action \( \mathbb{Z}/p^n\mathbb{Z} \to \text{Aut}(\mathbb{Z}/p^m\mathbb{Z}) \) sending \( 1 \mapsto \lambda \). We choose a lift denoted also by \( \lambda \in \mathbb{Z}_p^\times \) and consider the analogous pro-finite semi-direct product

\[ \Gamma = \mathbb{Z}_p \rtimes_{\lambda} \mathbb{Z}_p. \]

Lemma 10. The subsets \( \Gamma_r := p^r\mathbb{Z}_p \rtimes_{\lambda} p^r\mathbb{Z}_p \subseteq \Gamma \) (where \( p^r \) acts by \( p^r \lambda^r \)) form an exhaustive sequence of normal subgroups. More precisely, the following holds.

1. If \( p \) is odd or \( \lambda \equiv 1 \mod 4 \) if \( p = 2 \), then \( \Gamma_r = \Gamma^{p^r} \) is the set of \( p^r \)-th powers.

1'. Assume \( p = 2 \) and \( \lambda \equiv -1 \mod 4 \), then for \( r \geq 1 \) we have \( \Gamma^{2^r} \subseteq \Gamma_r = \Gamma^{2^{r-1}} \).

2. If \( p \) is odd or \( \lambda \equiv 1 \mod 4 \) if \( p = 2 \) (resp. \( \lambda \equiv -1 \mod 4 \)), then the map \( \gamma \mapsto \gamma^{p^r} \)

induces for all \( r \geq 0 \) (resp. for all \( r \geq 1 \)) a group isomorphism

\[ \Gamma_r/\Gamma_{r+1} \xrightarrow{\sim} \Gamma_{r+s}/\Gamma_{r+s+1} \]

of groups isomorphic to \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \).

Proof. As an element of \( p \)-primary order, we must have \( \lambda \equiv 1 \mod p \). We prove by induction that \( \lambda^{p^r} \equiv 1 \mod p^{r+1} \). This shows that \( \Gamma_r \) is indeed a normal subgroup, namely the kernel of the natural map \( \Gamma \rightarrow \mathbb{Z}/p^r\mathbb{Z} \times_{\lambda} \mathbb{Z}/p^r\mathbb{Z} \).

The \( p \)-th power map \( \Gamma \rightarrow \Gamma \) is a \( p \)-adic analytic map

\[ (a, x) \mapsto p \cdot (a, x) = (a \cdot (1 + \lambda^2 + \lambda^{2x} + \ldots + \lambda^{(p-1)x}), px) = (pa \cdot \lambda(x), px) \]

where \( \lambda(x) \) is a \( p \)-adic analytic function in \( x \in \mathbb{Z}_p \) with values in \( 1 \)-units \( 1 + p\mathbb{Z}_p \) if

\[ \bullet \]  

\[ p \]  

\[ = \]  

\[ x \].

Indeed, there is \( \mu \in \mathbb{Z}_p \), such that modulo \( p^2 \) we have \( \lambda \equiv 1 + \mu p \) and so \( \lambda^2 \equiv 1 + \mu p x \), therefore

\[ p \cdot \lambda(x) = \sum_{i=0}^{p-1} \lambda^i \equiv p + p\mu x \left( \frac{p}{2} \right) = \begin{cases} p & p > 2 \\ p(1 + \mu x) & p = 2 \end{cases} \equiv p \mod p^2, \]

with the last congruence subject to the condition \( 2 \mid \mu x \) if \( p = 2 \). The assertions (1), (1’) and (2) follow at once. \( \square \)

Proof of Theorem 3. We first assume that \( \lambda \equiv 1 \mod 4 \) if \( p = 2 \).

We make use of the quotient map \( \Gamma \rightarrow G \) sending \((a, x)\) to \((a, x)\) that exist by the assumptions on \( \lambda \). Let \( r = \min\{n, m\} \) and assume first that \( n \neq m \). Then the image \( G_r \subseteq G \) of

\[ \Gamma_r \rightarrow \Gamma \rightarrow G \]

is a non-trivial cyclic subgroup. Any generating system \((g_1, \ldots, g_s)\) of \( G \) maps to a generating system of \( G/\Phi(G) \). By lifting to \( \Gamma/\Gamma_1 \rightarrow G/\Phi(G) \) we control that the powers \( g_i^{p^r} \in G_r \) generate \( G_r \), hence at least one of them is non-trivial. It follows that no two generating systems of \( G \) can have disjoint sets of non-trivial powers. This shows that in order for \( G \) to admit a Beauville structure we must have \( n = m \).

From now on we assume that \( n = m \). Let \((x, y, z)\) be a generating triple in \( G \) with \( xyz = 1 \). Then image triple \((\bar{x}, \bar{y}, \bar{z})\) in \( G/\Phi(G) = \Gamma/\Gamma_1 \) generate this 2-dimensional \( \mathbb{F}_p \) vector space, hence are all three nontrivial. Since by Lemma 10 then also their \( p^{n-1} \)-th powers are non-trivial, we deduce that all three have order \( p^n \). This proves that any potential Beauville structure must be balanced of constant signature \( p^n \).
Moreover, let \((x, y, z)\) ad \((a, b, c)\) be a Beauville structure on \(G\), we see that the \(p^{n-1}\)-th powers of these elements must yield a Beauville structure on \(\Gamma_{n-1}/\Gamma_n \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\). In view of Theorem 8 there cannot be a Beauville structure if \(p < 5\). It remains to construct a Beauville structure if \(p \geq 5\), and to exclude the case \(p = 2\) and \(\lambda \equiv -1 \mod 4\).

We assume now that \(p = 2\) and \(\lambda \equiv -1 \mod 4\), and we assume that \(G\) has a Beauville structure \((x, y, z)\) and \((a, b, c)\). Note that in this case we do not yet know that \(n = m\). Then the Frattini quotient \(G/\Phi(G)\) must be of order 4 with both \((x, y, z)\) and \((a, b, c)\) mapping bijectively to the nontrivial elements. We set \(G_r\) for the image of the map \(\Gamma_r \to G\), and may assume without loss of generality that \(x = a \cdot \delta\) with \(\delta \in G_1\).

More generally, if \(g, h \in G_r\) differ by \(\varepsilon = h^{-1}g \in G_{r+1}\), then
\[
g^2 = (h\varepsilon)^2 = h^2(h^{-1}h\varepsilon^{-1})^2
\]
and so \(g^2, h^2 \in G_{r+1}\) differ by \((h^{-1}h\varepsilon^{-1})^2 \in G_{r+2}\), because \(G_r/G_{r+1}\) is central in \(G/G_{r+1}\).

By induction we find
\[
x^{2^{r-1}} \cong a^{2^{r-1}} \mod G_r
\]
which leads to a contradiction to the properties of a Beauville structure for \(s = \max\{n, m\}\), when \(G_s = 0\), if at least one of the elements \(x, y, z, a, b, c\) has order \(2^s\).

Working modulo 4 and using \(\lambda \equiv -1 \mod 4\), we see that squares are either \((2, 0, 0)\) or \((0, 2, 0)\) modulo \(G_2\), and among the elements \(x, y, z\) both possibilities occur. By induction for any \(g \in G\) we have, if \(g^2 \equiv (2, 0)\) that \(g^{2^{r-1}} \equiv (2^{r-1}, 0) \mod G_r\), or if \(g^2 \equiv (0, 2)\) that \(g^{2^{r-1}} \equiv (0, 2^{r-1}) \mod G_r\). This shows that at least one element of every generating set has order \(2^s\) with \(s = \max\{n, m\}\). This excludes the case \(p = 2\) from allowing Beauville structures.

Let now \(p \geq 5\) and \((x, y, z)\) and \((a, b, c)\) be triples with product equal to one that map to a Beauville structure on \(G/\Phi(G)\). By the Frattini argument these triples generate \(G\) and it remains to show that there are no non-trivial conjugate powers. We argue by contradiction and assume that without loss of generality that
\[
0 \neq x^k = ga^l g^{-1}
\]
with \(k, l \in \mathbb{N}\) and \(g \in G\). Let \(r\) be maximal such that \(x^k \in \Gamma_r/\Gamma_n \subseteq G\). By Lemma 10 it follows that \(p^r\) is the precise power dividing both \(k\) and \(l\). Let \(k = p^r k_0\) and \(l = p^r l_0\). Passing to inverses by the isomorphism of raising to \(p^r\)-th power \(G/\Phi(G) = \Gamma/\Gamma_1 \cong \Gamma_r/\Gamma_{r+1}\) we obtain that
\[
0 \neq x^{k_0} = ga^{l_0} g^{-1} \in G/\Phi(G).
\]
This contradicts our initial choice that \((x, y, z)\) ad \((a, b, c)\) were a lift of a Beauville structure on \(G/\Phi(G)\). This completes the proof. \(\Box\)

Remark 11. (1) The proof of Theorem 3 given above shows more. Every Beauville structure maps to a Beauville structure of the Frattini quotient \(G/\Phi(G)\). And, conversely, any lift of a Beauville structure of the Frattini quotient yields a Beauville structure on \(G\).

(2) Special cases of Beauville structures obtained in Theorem 3 are contained in (and inspiration came from) [BBF12].

(3) The groups \(\mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}\) are classified for \(p \neq 2\) up to isomorphism as follows. The group order is \(p^{2n}\), hence \(p\) and \(n\) are unique. The group is abelian if \(\lambda \equiv 1 \mod p^n\) and otherwise the maximal abelian quotient
\[
\mathbb{Z}_p/(p^n, \lambda - 1)\mathbb{Z}_p \times \mathbb{Z}/p^n\mathbb{Z}
\]
has order \(p^{n+s}\) where \(0 < s = v_p(\lambda - 1) < n\) is the \(p\)-adic valuation of any lift of \(\lambda - 1\) to \(\mathbb{Z}_p\). It follows that \(n > r = n - s > 0\) is an invariant of the group in the non-abelian case, namely the order of \(\lambda \in \text{Aut}(\mathbb{Z}/p^n\mathbb{Z})\).

The tuple \((p, n, r)\) is a complete set of invariants, since for \(\lambda'\) leading to the same invariants there is an automorphism \(\varphi\) of \(\mathbb{Z}/p^n\mathbb{Z}\) such that
\[
(id, \varphi) : \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}
\]
yields an isomorphism.

This classification result shows that we have described plenty of Beauville structures on a family of non-abelian \(p\)-groups parametrized by integers \(n > r > 0\).
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