Abstract

Let $V$ be a finite set and $\mathcal{M}$ a collection of subsets of $V$. Then $\mathcal{M}$ is an alignment of $V$ if and only if $\mathcal{M}$ is closed under taking intersections and contains both $V$ and the empty set. If $\mathcal{M}$ is an alignment of $V$, then the elements of $\mathcal{M}$ are called convex sets and the pair $(V, \mathcal{M})$ is called an aligned space. If $S \subseteq V$, then the convex hull of $S$ is the smallest convex set that contains $S$. Suppose $X \in \mathcal{M}$. Then $x \in X$ is an extreme point for $X$ if $X \setminus \{x\} \in \mathcal{M}$. The collection of all extreme points of $X$ is denoted by $\text{ex}(X)$. A convex geometry on a finite set is an aligned space with the additional property that every convex set is the convex hull of its extreme points. Let $G = (V, E)$ be a connected graph and $U$ a set of vertices of $G$. A subgraph $T$ of $G$ containing $U$ is a minimal $U$-tree if $T$ is a tree and if every vertex of $V(T) \setminus U$ is a cut-vertex of the subgraph induced by $V(T)$. The monophonic interval of $U$ is the collection of all vertices of $G$ that belong to some minimal $U$-tree. A set $S$ of vertices in a graph is $m_k$-convex if it contains the monophonic interval of every $k$-set of vertices is $S$. A set of vertices $S$ of a graph is $m^3$-convex if for every pair $u, v$ of vertices in $S$, the vertices on every induced path of length at least 3 are contained in $S$. A set $S$ is $m^3_3$-convex if it is both $m^3$- and $m^3$- convex. We show that if the $m^3_3$-convex sets form a convex geometry, then $G$ is $A$-free.

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1 Introduction

Let $G$ and $F$ be graphs. Then $F$ is an induced subgraph of $G$ if $F$ is a subgraph of $G$ and for every $u, v \in V(F)$, $uv \in E(F)$ if and only if $uv \in E(G)$. We say a graph $G$ is $F$-free if it does not contain $F$ as an induced subgraph. Suppose $C$ is a collection of graphs. Then $G$ is $C$-free if $G$ is $F$-free for every $F \in C$. If $F$ is a path or cycle that is a subgraph of $G$, then $F$ has a chord if it is not an induced subgraph of $G$, i.e., $F$ has two vertices that are adjacent in $G$ but not in $F$. An induced cycle of length at least 5 is called a hole.

Let $V$ be a finite set and $\mathcal{M}$ a collection of subsets of $V$. Then $\mathcal{M}$ is an alignment of $V$ if and only if $\mathcal{M}$ is closed under taking intersections and contains both $V$ and the empty set. If $\mathcal{M}$ is an alignment of $V$, then the elements of $\mathcal{M}$ are called convex sets and the pair $(V, \mathcal{M})$ is called an aligned space. If $S \subseteq V$, then the convex hull of $S$ is the smallest convex set that contains $S$. Suppose $X \in \mathcal{M}$. Then $x \in X$ is an extreme point for $X$ if $X \setminus \{x\} \in \mathcal{M}$. The collection of all extreme points of $X$ is denoted by $ex(X)$. A convex geometry on a finite set $V$ is an aligned space $(V, \mathcal{M})$ with the additional property that every convex set is the convex hull of its extreme points. This property is referred to as the Minkowski-Krein-Milman (MKM) property. For a more extensive overview of other abstract convex structures see [13]. Convexities associated with the vertex set of a graph are discussed for example in [3]. Their study is of interest in Computational Geometry and has applications in Game Theory [2].

Convexities on the vertex set of a graph are usually defined in terms of some type of ‘intervals’. Suppose $G$ is a connected graph and $u, v$ two vertices of $G$. Then a $u − v$ geodesic is a shortest $u − v$ path in $G$. Such geodesics are necessarily induced paths. However, not all induced paths are geodesics. The $g$-interval (respectively, $m$-interval) between a pair $u, v$ of vertices in a graph $G$ is the collection of all vertices that lie on some $u − v$ geodesic (respectively, induced $u − v$ path) in $G$ and is denoted by $I_g[u, v]$ (respectively, $I_m[u, v]$).

A subset $S$ of vertices of a graph is said to be $g$-convex ($m$-convex) if it contains the $g$-interval ($m$-interval) between every pair of vertices in $S$. It is not difficult to see that the collection of all $g$-convex ($m$-convex) sets is an alignment of $V$. A vertex $v$ is an extreme point for a $g$-convex (or $m$-convex) set $S$ if and only if $v$ is simplicial in the subgraph induced by $S$, i.e., every two neighbours of $v$ in $S$ are adjacent. Of course the convex hull of the extreme points of a convex set $S$ is contained in $S$, but equality holds only in special cases. In [3] those graphs for which the $g$-convex sets form a convex geometry are characterized as the chordal 3-fan-free graphs (see Fig. 1). These are precisely the chordal, distance-hereditary graphs (see [11, 7]). In the same paper it is shown that the chordal graphs are precisely those graphs for which the $m$-convex sets form a convex geometry.

For what follows we use $P_k$ to denote an induced path of order $k$. A vertex is simplicial in a set $S$ of vertices if and only if it is not the centre vertex of an induced $P_3$ in $(S)$. Jamison and Olariu [8] relaxed this condition. They defined a vertex to be semisimplicial in $S$ if and only if it is not a centre vertex of an induced $P_4$ in $(S)$. 

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Dragan, Nicolai and Brandstädt [5] introduced another convexity notion that relies on induced paths. The $m^3$-interval between a pair $u, v$ of vertices in a graph $G$, denoted by $I_{m^3}[u, v]$, is the collection of all vertices of $G$ that belong to an induced $u-v$ path of length at least 3. Let $G$ be a graph with vertex set $V$. A set $S \subseteq V$ is $m^3$-convex if and only if for every pair $u, v$ of vertices of $S$ the vertices of the $m^3$-interval between $u$ and $v$ belong to $S$. As in the other cases the collection of all $m^3$-convex sets is an alignment. Note that an $m^3$-convex set is not necessarily connected. It is shown in [5] that the extreme points of an $m^3$-convex set are precisely the semisimplicial vertices of $\langle S \rangle$. Moreover, those graphs for which the $m^3$-convex sets form a convex geometry are characterized in [5] as the (house, hole, domino, $A$)-free graphs (see Fig. 1).

More recently a graph convexity that generalizes $g$-convexity was introduced (see [11]). The Steiner interval of a set $S$ of vertices in a connected graph $G$, denoted by $I(S)$, is the union of all vertices of $G$ that lie on some Steiner tree for $S$, i.e., a connected subgraph that contains $S$ and has the minimum number of edges among all such subgraphs. Steiner intervals have been studied for example in [9, 12]. A set $S$ of vertices in a graph $G$ is $k$-Steiner convex ($g_k$-convex) if the Steiner interval of every collection of $k$ vertices of $S$ is contained in $S$. Thus $S$ is $g_2$-convex if and only if it is $g$-convex. The collection of $g_k$-convex sets forms an aligned space. We call an extreme point of a $g_k$-convex set a $k$-Steiner simplicial vertex, abbreviated $kSS$ vertex.

The extreme points of $g_3$-convex sets $S$, i.e., the $3SS$ vertices are characterized in [4] as those vertices that are not a centre vertex of an induced claw, paw or $P_4$, in $\langle S \rangle$ see Fig. 1. Thus a $3SS$ vertex is semisimplicial. Apart from the $g_k$-convexity, for a fixed $k$, other graph convexities that (i) depend on more than one value of $k$ and (ii) combine the $g_3$ convexity and the geodesic coun-
The notion of an induced path between a pair of vertices can be extended to three or more vertices. This gives rise to graph convexities that extend the \( m \)-convexity. Let \( U \) be a set of at least two vertices in a connected graph \( G \). A subgraph \( H \) containing \( U \) is a minimal \( U \)-tree if \( H \) is a tree and if every vertex \( v \in V(H) \setminus U \) is a cut-vertex of \( \langle V(H) \rangle \). Thus if \( U = \{u, v\} \), then a minimal \( U \)-tree is just an induced \( u-v \) path. Moreover, every Steiner tree for a set \( U \) of vertices is a minimal \( U \)-tree. The collection of all vertices that belong to some minimal \( U \)-tree is called the monophonic interval of \( U \) and is denoted by \( I_m(U) \). A set \( S \) of vertices is \( k \)-monophonic convex, abbreviated as \( m_k \)-convex, if it contains the monophonic interval of every subset \( U \) of \( k \) vertices of \( S \). Thus a set of vertices in \( G \) is a monophonic convex set if and only if it is a \( m_2 \)-convex set.

By combining the \( m_3 \)-convexity with the \( m_3 \)-convexity introduced in [5], we obtain a graph convexity that extends the graph convexity studied in [10]. More specifically, we define a set \( S \) of vertices in a connected graph to be \( m_3 \)-convex if \( S \) is both \( m_3 \)- and \( m_3 \)-convex. In this paper, we show that if the \( m_3 \)-convex alignment forms a convex geometry, then \( G \) is \( A \)-free. We use the fact that these graphs are \( F \)-free for several other graphs \( F \). In particular, \( G \) is easily seen to be house, hole, and domino free.

Moreover, the graphs of Fig. 2 are forbidden. A graph \( G \) is a replicated twin \( C_4 \) if it is isomorphic to any one of the four graphs shown in Fig. 2(a), where any subset of the dashed edges may belong to \( G \). The collection of the four replicated twin \( C_4 \) graphs is denoted by \( R_{C_4} \). A graph \( F \) is a tailed twin \( C_4 \) if it is isomorphic to one of the two graphs shown in Fig. 2(b) where again any subset of the dotted edges may be chosen to belong to \( F \). We denote the collection of tailed twin \( C_4 \)'s by \( T_{C_4} \).

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{rep_twin.png}
\caption{Forbidden subgraphs for \( m_3^3 \)-convex geometries}
\end{figure}

\section{\( m_3^3 \)-Convex Geometries are \( A \)-Free}

Recall that the graphs for which the \( m^3 \)-convex sets form a convex geometry are characterized in [4] as the (house, hole, domino, \( A \))-free graphs. The proof of this characterization depends on the following useful result also proven in [5]:

\begin{itemize}
\item Recall that the graphs for which the \( m^3 \)-convex sets form a convex geometry are characterized in [4] as the (house, hole, domino, \( A \))-free graphs. The proof of this characterization depends on the following useful result also proven in [5]:
\end{itemize}
Theorem 1. If $G$ is a (house, hole, domino, $A$)-free graph, then every vertex of $G$ is either semisimplicial or lies on an induced path of length at least 3 between two semisimplicial vertices.

In [5] several ‘local’ convexities related to the $m^3$-convexity were studied. For a set $S$ of vertices in a graph $G$, $N[S]$ is $S \cup N(S)$ where $N(S)$ is the collection of all vertices adjacent with some vertex of $S$. A set $S$ of vertices in a graph is connected if $\langle S \rangle$ is connected. The following useful result was established in [5].

Theorem 2. A graph $G$ is (house, hole, domino)-free if and only if $N[S]$ is $m^3$-convex for all connected sets $S$ of vertices of $G$.

Theorem 3. If $G = (V, E)$ is a graph such that $(V, M_{m^3}(G))$ is a convex geometry, then $G$ is $A$-free.

Proof. Observe first that $G$ is (house, hole, domino, $R_{C_4}$, $T_{C_4}$)-free. Suppose $F$ is a house, hole, domino, replicated twin $C_4$ or a tailed twin $C_4$. Then $F$ has at most one $3SS$ vertex. Suppose $G$ is a graph that contains $F$ as an induced subgraph. Then the set of extreme points of the convex hull of $V(F)$ is contained in the collection of $3SS$ vertices of $F$. So the convex hull of the extreme points of the $m^3$-convex hull of $V(F)$ is empty or consists of a single vertex. So in this case the $m^3$-convex alignment of $G$ does not form a convex geometry.

If $S$ is a set of vertices of a graph $G$, then $I_m(S) = \cup \{I_m[x, y] | x, y \in S \}$.

To show that $G$ contains no $A$ as an induced subgraph we prove a series of lemmas.

Lemma 1. Suppose $G = (V, E)$ is a graph for which $(V, M_{m^3}(G))$ is a convex geometry. Then for every $a, b \in V$, $I_m(I_m[a, b]) \subseteq I_m[a, b]$.

Proof. By the above observation $G$ is (house, hole, domino, $R_{C_4}$, $T_{C_4}$)-free. If $ab \in E$ then $I_m(I_m[a, b]) \subseteq I_m[a, b] = \{a, b\}$. So we may assume $ab \notin E$. If $I_m(I_m[a, b]) \nsubseteq I_m[a, b]$, there is a vertex $w \notin I_m[a, b]$ that lies on an induced path between two vertices of $I_m[a, b]$. Among all such induced paths of length at least 3 containing $w$, let $Q$ be one with a minimum number of edges. Suppose $Q$ is a $u - v$ path. Clearly $\{u, v\} \neq \{a, b\}$; otherwise, $w \in I_m[a, b]$. Let $Q : (u =) v_1 v_2 \ldots v_k (= v)$. (Suppose $w = v_i$.) Then $w$ is not adjacent with two non-adjacent vertices of any induced $a - b$ path; otherwise, $w$ lies on an induced $a - b$ path.

Case 1 Suppose $u$ and $v$ lie on a common induced $a - b$ path $P$. We may assume $u$ precedes $v$ on such a path. Moreover, we may assume that all internal vertices of $Q$ are not on $P$. For if $v_j \in V(P)$, $1 < j < k$, then either $Q[v_1, v_j]$ or $Q[v_j, v_k]$ contains $w$, say the former. Since $Q$ is an induced path, so is $Q[v_1, v_j]$. Hence $v_1v_j \notin E$. Thus $Q[v_1, v_j]$ must have length at least 3; otherwise $w$ is adjacent with a pair of nonadjacent vertices of $P$, implying that $G$ contains an induced $a - b$ path passing through $w$, contrary to assumption. But then we have a contradiction to our choice of $Q$. 

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Let \( S_1 = P[u, v] \setminus \{u, v\} \) and \( S_2 = Q[u, v] \setminus \{u, v\} \). Then \( \langle S_i \rangle \) is connected for \( i = 1, 2 \). By Theorem 2, \( N[S_i] \) is \( m^3 \)-convex. Since \( u \) and \( v \) both belong to \( N[S_i] \), every vertex of \( Q \) must be adjacent with an internal vertex of \( P[u, v] \). This is true in particular for \( w \). Since \( P[a, u] \) followed by \( Q \) and then \( P[v, b] \) is an \( a - b \) path that contains \( w \) it cannot be induced. Some vertex of \( P[a, u] \setminus \{u\} \) or a vertex of \( P[v, b] \setminus \{v\} \) must be adjacent with an internal vertex of \( Q \); say the former occurs. Let \( x \) be the first vertex of \( P[a, u] \) that is adjacent with an internal vertex \( y \) of \( Q \). Let \( r \) be the first vertex on \( Q[y, v] \) that is adjacent with a vertex of \( P[v, b] \) (possibly \( r \) is \( v_{k-1} \)). Let \( s \) be the last vertex of \( P[v, b] \) adjacent with \( r \). Then the path \( H : P[a, x]xyQ[y, r]rsP[s, b] \) is an induced \( a - b \) path and thus does not contain \( w \). So \( w \) is an internal vertex of \( Q[u, y] \) or of \( Q[r, v] \); suppose the former. Since \( H \) is connected, \( N[V(H)] \) is \( m^3 \)-convex by Theorem 2. Since \( a, b \in N[V(H)] \) and as \( P \) has length at least 3, \( N[V(H)] \) must contain every vertex of \( P \). Thus \( I_{m^3} [u, v] \subseteq N[V(H)] \). Hence \( w \) is adjacent with a vertex of \( H \). Since \( w \) is adjacent with an internal vertex of \( P[u, v] \), \( w \) is not adjacent with any vertex of \( P[a, x] \) nor \( P[s, b] \). Since \( Q \) is an induced path, the only vertex of \( H \) to which \( w \) can be adjacent is \( y \). So \( y \) follows \( w \) on \( Q \). Since \( u \) and \( y \) belong to \( I_{m^3} [a, b] \) and as \( Q[u, y] \) is an induced path containing \( w \), it follows that \( w \) must be adjacent with \( u \); otherwise, we have a contradiction to our choice of \( Q \). Let \( x' \) be the last vertex on \( P[x, u] \) to which \( y \) is adjacent. Then \( x'u \in E \); otherwise \( P[x', u]uyxy' \) is an induced cycle of length at least 5. Let \( z \) be the first internal vertex of \( P[u, v] \) to which \( w \) is adjacent. (By an earlier observation \( z \) exists.) Then \( uz \in E \); otherwise, \( w \) lies on an induced \( a - b \) path. Also \( yz \in E \); otherwise, \( \langle \{x', u, w, y, z\} \rangle \) is a house. If \( r \neq y \), let \( y' \) be the neighbour of \( y \) on \( Q[y, r] \). Then \( u, y' \in I_{m^3} [a, b] \) and \( Q[u, y'] \) is an induced path between two vertices of \( I_{m^3} [a, b] \) having length 3 and containing \( w \), contrary to our choice of \( Q \). So \( r = y \). So \( P[x', s]syxy' \) is a cycle of length at least 5. Since \( yu \notin E \), \( x'uzyx' \) is an induced 4-cycle. Let \( z' \) be the first vertex after \( z \) on \( P[z, s] \) to which \( y \) is adjacent (perhaps \( z' = s \)). Then \( P[z, z']z'y \) is an induced cycle and hence has length 3 or 4. This cycle together with the 4-cycles \( x'yuzx' \) produces either a house or a domino both of which are forbidden. So we may assume that \( Q \) is an induced \( u - v \) path between vertices \( u \) and \( v \) of \( I_{m^3} [a, b] \) that do not belong to the same induced \( a - b \) path. Indeed we may assume if \( u \) and \( v \) are any non-adjacent vertices that lie on the same induced \( a - b \) path, then \( I_{m^3} [u, v] \subseteq I_{m^3} [a, b] \).

**Case 2** Suppose \( u \) and \( v \) lie on two internally disjoint \( a - b \) paths \( P_u \) and \( P_v \), respectively. We may assume \( \{u, v\} \cap \{a, b\} = \emptyset \); otherwise, we are in Case 1. We show first that no internal vertex of \( Q \) belongs to \( P_u \) or \( P_v \). Suppose some internal vertex of \( Q[u, w] \) or \( Q[w, v] \), say \( Q[u, w] \) belongs to \( P_u \) or \( P_v \). However, no internal vertex of \( Q[u, w] \) belongs to \( P_v \); otherwise, either the situation arises that was considered in Case 1 or there is an induced \( a - b \) path containing \( w \). So we may assume that an internal vertex of \( Q[u, w] \) lies on \( P_u \). Let \( u' \) be the last such vertex. Then \( Q[u', v] \) contains \( w \) and is an induced path between two vertices of \( I_{m^3} [a, b] \) that is shorter than \( Q \). So \( Q[u', v] \) has length 2; otherwise we have a contradiction to our choice of \( Q \). So \( Q[u', v] \) must be the path \( u'wv \). Since \( Q \) has length at least 3 and by our choice of \( Q \) one of the neighbours of
u' on $P_u$ must be $u$. So one of the configurations shown in Fig. 3 must occur where solid lines are edges and dashed lines represent subpaths of $P_u$ and $P_v$.

We may assume that the configuration in (a) occurs. The argument for the configuration in (b) is similar.

Since $Q$ is induced, $v$ is not adjacent to $u'$ or $u$ and $w$ is not adjacent with $u$. Let $v_L$ and $v_R$ be the neighbours of $v$ on $P_v[a, v]$ and $P_v[v, b]$, respectively. If $u'$ is adjacent with a vertex $r$ of $P_v[v_R, b] - v_R$ then $ru'wv$ is an induced path of length 3 containing $w$ and whose end vertices lie on the same induced $a - b$ path. By Case 1, this situation cannot occur. So the only vertex of $P_v[v_R, b]$ to which $u'$ can be adjacent is $v_L$. Similarly, the only vertex of $P_v[a, v_L]$ to which $u'$ can be adjacent is $v_L$. Using a similar argument and the fact that $vu \notin E$, we see that $v$ is not adjacent with any vertex of $P_u[u, b]$. Moreover, $w$ is not adjacent with any vertex of $P_v[u, b]$; otherwise, $w$ lies on an induced $a - b$ path. The path obtained by taking $P_v[a, v]$ followed by $vwwu'$ and then $P_u[u', b]$ is an $a - b$ path that contains $w$. Hence this path is not induced. Suppose first that $wv_L \notin E$. So some vertex of $P_v[a, v]$ is adjacent with some vertex of $P_u[u', b]$. Since $v$ is not adjacent with any vertex of $P_u[u', b]$, some vertex of $P_u[a, v_L]$ is adjacent with some vertex of $P_u[u', b]$. Let $z$ be a vertex closest to $v$ on $P_v[a, v]$ that is adjacent with a vertex of $P_u[u', b]$ and let $y$ be such a neighbour of $z$ closest to $u'$ on $P_u[u', b]$. Observe that $y = u'$ and $z = v_L$; otherwise, the cycle $P_v[z, v]wv'P_u[u', y]yz$ is an induced cycle of length at least 5. Let $x$ be the vertex closest to $u'$ on $P_u[u', b]$ that is adjacent with a vertex of $P_v[v, b]$ (possibly $x = b$). Let $x'$ be the neighbour of $x$ on $P_v[v, b]$ closest to $v$. By the above observation $x' \neq v$. The cycle $P_u[u', x']x'xP_v[x', v]wvu'$ is induced and has length at least 5 unless $x = u'$ and $x' = v_R$. So $u'$ is adjacent with both $v_L$ and $v_R$. Observe that $u$ is either adjacent with both $v_L$ and $v_R$ or neither of these two vertices; otherwise, $\{v_L, v, v_R, u', u\}$ is a house. We show next that no vertex of $P_u[u, b]$ is adjacent with $v_L$. Suppose $r$ is a vertex on $P_u[u, b]$ closest to $u$ that is adjacent with $v_L$. First observe that $r \neq u$ for if $uvL \in E$, then $\{u, u', w, v, v_L\}$ is a house. So $r$ must be the neighbour of $u$ on $P_u[u, b]$; otherwise, $G$ has a hole. However then $\{u', u, r, v_L, v, w\}$ is a domino. So $v_L$ is not adjacent with any vertex of $P_u[u, b]$. Let $C : v_Ru'uP_u[u', b]P_v[b, v_R]$. Then $C$ is a cycle of length at least 5 and hence has chords. Now $u'$ is not adjacent with any vertex of $P_v[v_R, b]$ other than $v_R$; otherwise, $w$ lies on an
From the above, we may assume that $v$ has no chords. Since neither $uv$ nor $wv$ is adjacent with $v_R$. But then $\langle \{u', u, s, v, v_L\} \rangle$ is a domino. So the neighbour $r$ of $u$ on $P_u[u, b]$ is incident with a chord of $C$. Since $G$ has no holes $rvR \in E$. But then $\langle \{u', u, r, vR, v, v_L\} \rangle$ is a domino. Suppose now that $wvL \in E$. Then $wvR \notin E$. Let $C' = P_u[u', b]P_v[b, v]uvw'$. Then $C'$ is a cycle of length at least 5 and hence has no chords. Since neither $w$ nor $v$ are incident with chords of $C'$, $u'vR \in E$. If $wvR \in E$ $\langle \{u, u', vR, v, w\} \rangle$ is a house. Note that $u'$ is not adjacent with $v$ and $vR$ is an internal vertex vertex of $P_v[vR, b]$; otherwise, if $t$ is such a neighbour of $u'$, then $tu'wvR$ is an induced path of length 3 between two vertices of $I_m[a, b]$ that lie on the same induced $a - b$ path, a case already considered. Let $r$ be the neighbour of $u$ on $P_u[u, b]$ and $s$ the neighbour of $vR$ on $P_v[vR, b]$. Then either $vRr$ or $us$ is an edge; otherwise, $G$ has a hole. But then $\langle \{u', u, r, vR, v, w\} \rangle$ or $\langle \{u', u, s, vR, v, w\} \rangle$ is a domino. So no internal vertex of $Q$ belongs to $P_u$ or to $P_v$.

Let $Q: (u = v_1 v_2 \ldots v_k (= v)$. Let $u_L$ and $u_R$ be the neighbours of $u$ on $P_u[a, u]$ and $P_u[u, b]$, respectively and $v_L$ and $v_R$ the neighbours of $v$ on $P_v[a, v]$ and $P_v[v, b]$, respectively. Let $S_1 = V(P_u[u, b]) \cup V(P_v[a, v_R])$ and $S_2 = V(P_u[a, u_L]) \cup V(P_v[a, v_L])$. Since $(S_i)$ is connected for $i = 1, 2$, it follows from Theorem 2 that $N[S_i]$ is $m^3$-convex. Since $u, v \in N[S_i]$ for $i = 1, 2$, every vertex of $Q$ is adjacent with a vertex of $S_i$ for $i = 1, 2$. In particular $w$ is adjacent with a vertex of $S_i$ for $i = 1, 2$. However, $w$ is not adjacent with a pair of non-adjacent vertices of $P_u$ or a pair of nonadjacent vertices of $P_v$. So without loss of generality we may assume that $w$ is adjacent with a vertex of $P_v[vR, b]$ and a vertex of $P_u[a, u_L]$. Also $w$ is not adjacent with either $a$ or $b$; otherwise, $w$ lies on an induced $a - b$ path.

If $v_2$ is adjacent with two non-adjacent vertices of $P_u$ (or if $v_{k-1}$ is adjacent with two non-adjacent vertices of $P_v$), then $v_2 \neq w$ (and $v_{k-1} \neq w$, respectively) and $Q[v_2, v]$ (or $Q[u, v_{k-1}]$, respectively) is an induced path between two vertices of $I_m[a, b]$ that is shorter than $Q$ and contains $w$. By our choice of $Q$ this can only happen if $Q$ has length 3.

We consider two subcases that depend on the length of $Q$.

**Subcase 2.1** Suppose $Q$ has length 3.

Then $v_2$ or $v_3$ is $w$, say $v_3 = w$. The case where $v_2 = w$ can be argued similarly. From the above, we may assume that $w$ is adjacent with an internal vertex of $P_v[v, b]$ and an internal vertex of $P_u[a, u]$. The only vertex of $P_v[vR, b]$ that can be adjacent with $w$ is $vR$; otherwise, $w$ lies on an induced $a - b$ path. So $wvR \in E$. Now it follows that $w$ is not adjacent with a vertex of $P_v[a, v_L]$. Thus $\langle \{v_2, w, v, v_L, v_R\} \rangle$ is a house unless $v_2vR \in E$. If $v_2vR \in E(G)$, then $uvL$, $wvR \notin E$; otherwise, $\langle \{u, v_2, v_3, v, v_L\} \rangle$ or $\langle \{u, v_2, v_3, v, v_R\} \rangle$ is a house. So $\langle \{u, v_2, v_3, v_L, v, v_R\} \rangle$ is a tail of $C_4$ which is forbidden. So this subcase cannot occur.

**Subcase 2.2** Suppose $Q$ has length at least 4.

By an earlier observation, $v_2$ is not adjacent with a pair of non-adjacent vertices
of $P_u$ and $v_{k+1}$ is not adjacent with a pair of non-adjacent vertices of $P_v$. By assumption, $w$ is adjacent with an internal vertex of $P_u[a, u]$ and an internal vertex of $P_v[v, b]$. Suppose $w = v_j$. So $w$ is not adjacent with a vertex of $P_u[u_R, b]$ nor a vertex of $P_v[a, v_L]$. 

**Fact 1** No vertex of $Q[v_1, v_{j-1}]$ is adjacent with a vertex of $P_v[a, v_L]$ and no vertex of $Q[v_{j+1}, v_k]$ is adjacent with a vertex of $P_u[u_R, b]$.

*Proof of Fact 1.* Suppose some vertex of $Q[v_1, v_{j-1}]$ is adjacent with a vertex of $P_v[a, v_L]$. Let $i$ be the largest integer less than $j$ such that $v_i$ is adjacent with a vertex of $P_v[a, v_L]$. Let $z$ be a neighbour of $v_i$ on $P_v[a, v_L]$ closest to $v$ on this path. Then $C_1 : Q[v_i, v]P_v[v, z]v_i$ is a cycle of length at least 4. If $i < j - 2$, then $C_1$ has length at least 5 and three consecutive vertices of $C_1$ are not incident with a chord of the cycle. This implies that $G$ has a hole; which is forbidden. So $i = j - 1$. Clearly $j < k - 1$. Let $C_2 : P_v[z, v]Q[v, v_{j+1}]z$. Then $C_2$ is a cycle of length at least 3. Thus $\langle V(C_2) \rangle$ contains an induced cycle $C'$ of length at least 3 that contains the edge $zv_{j+1}$. Since $G$ contains no holes, $C'$ has length 3 or 4. Since neither $v_j$ nor $v_{j-1}$ is adjacent with a vertex of $P_v[z, v] - z$ nor a vertex of $Q[v_{j+2}, v]$ and as $v_{j+2} \notin E$, it is not difficult to see that the vertices of $C_2$ and $C'$ induce a house or a domino. So no vertex of $Q[v_1, v_{j-1}]$ is adjacent with a vertex of $P_v[a, v_L]$. By an identical argument we can show that no vertex of $Q[v_{j+1}, v_k]$ is adjacent with a vertex of $P_u[u_R, b]$. \(\square\)

**Fact 2** No vertex of $P_v[a, v_L]$ is adjacent with any vertex of $P_u[u_R, b]$.

*Proof of Fact 2.* Let $z$ be the first vertex of $P_v[a, v_L]$ that is adjacent with some vertex of $P_u[u_R, b]$. Let $y$ be a neighbour of $z$ on $P_u[u_R, b]$ that is closest to $b$. Then the path $P : P_v[a, z]yP_u[y, b]$ is an induced $a - b$ path. So $N[V(P)]$ is $m^3$-convex and hence contains all induced $a - b$ paths of length at least 3. Since $\{a, b\} \cap \{u, v\} = \emptyset$, and since both $P_u[a, u]$ and $P_v[v, b]$ contain an internal vertex adjacent with $w$, both $P_u$ and $P_v$ have length at least 3. So $N[V(P)]$ contains all the vertices of $P_u$ and $P_v$ and hence $u$ and $v$. So $N[V(P)]$ also contains $Q$. Thus every vertex of $Q$ is adjacent with a vertex of $P_v[a, z]$ or with a vertex of $P_u[y, b]$. But by assumption $w$ is adjacent with an internal vertex of both $P_u[a, u]$ and $P_v[v, b]$. So $w$ is adjacent with a pair of non-adjacent vertices of $P_v$ or a pair of non-adjacent vertices of $P_u$, neither of which is possible. \(\square\)

From Facts 1 and 2, it follows that no vertex of the path $P_v[a, v]Q[v, v_{j-1}]$ is adjacent with a vertex of the path $Q[v_{j+1}, u]P_u[u, b]$. Hence the subgraph induced by the path $P_v[a, v]Q[v, u]P_u[u, b]$ is an induced $a - b$ path that contains $w$: contrary to the assumption that $w \notin I_m[a, b]$. This completes the proof of Case 2.

**Case 3** Suppose that $u$ belongs to an induced $a - b$ path $P_u$ and $v$ to an induced $a - b$ path $P_v$ where $P_u$ and $P_v$ intersect at vertices other than $a$ and $b$. We may assume that $u$ and $v$ do not both belong to $P_u$ nor both to $P_v$; otherwise, Case 1 occurs. Let $a'$ be the last vertex prior to $u$ on $P_u[a, u]$ that is also a vertex of $P_v$ (perhaps $a' = a$). Let $b'$ be the first vertex after $u$ on $P_u[u, b]$ that belongs to $P_v$. 

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So $a'b' \notin E$. Let $a''$ be the last vertex prior to $v$ on $P_v[a, v]$ that also belongs to $P_u$ and $b''$ the first vertex after $v$ on $P_v[v, b]$ that also belongs to $P_u$. So $a''b'' \notin E$.

Subcase 3.1 Suppose $P_u[a'', b'']$ contains both $a'$ and $b'$. (Note $b''$ may precede $a''$ on $P_u[a'', b'']$.) In this case we can apply the argument used in Case 2 with $a$ and $b$ replaced by $a''$ and $b''$ and $P_u$ and $P_v$ replaced by $P_u[a'', b'']$ and $P_v[a'', b'']$. Hence this subcase cannot occur.

Subcase 3.2 Suppose $P_u[a'', b'']$ does not contain both $a'$ and $b'$. Then $a''$ and $b''$ either lie on $P_u[a, a']$ or on $P_u[b', b]$. We will assume the former case occurs. The arguments for the latter case are similar. We may assume $a''$ precedes $b''$ on $P_u[a, a']$. The case where $b''$ precedes $a''$ on $P_u[a, a']$ is similar. First suppose that $P_u[a'', b'']$ has length 2. Then $v$ is the only interior vertex of $P_u[a'', b'']$ and $v$ is adjacent with two nonadjacent vertices of $P_u$. Let $x$ be the first vertex on $P_u$ that is adjacent with $v$, and $y$ the last vertex of $P_u$ adjacent with $v$. Since $uv \notin E$, $y \neq u$. If $y$ precedes $u$ on $P_u$, then the path obtained by taking $P_u[a, x]$ followed by $xvy$ and then $P_u[y, b]$ is an induced $a-b$ path that contains both $u$ and $v$. Thus we can apply the argument used in Case 1 to this path to obtain a contradiction. If $y$ follows $u$ on $P_u$, then we can use the path $P_u[x, y]$ and the path $xvy$ and apply the argument used in Case 2 with $x$ and $y$ instead of $a$ and $b$, respectively.

We now assume that $P_u[a'', b'']$ has length at least 3. Since $H = P_u[a'', b''] \setminus \{a'', b''\}$ is connected it follows, from Theorem 2 that $N[V(H)]$ is $m^3$-convex. Since $N[V(H)]$ contains both $a''$ and $b''$ it must contain every internal vertex of $P_u[a'', b'']$. So each internal vertex of $P_u[a'', b'']$ is adjacent with an internal vertex of $P_u[a'', b'']$. If no internal vertex of $P_u[a'', b'']$ is adjacent with a vertex of $P_u[a, a'] \setminus \{a''\}$ or $P_u[b'', b] \setminus \{b''\}$, then we can replace $P_u[a'', b'']$ in $P_u$ with $P_v[a'', b'']$ to obtain an induced $a-b$ path that contains both $u$ and $v$. By applying the argument used in Case 1 to this path we obtain a contradiction. Let $b''_L$ and $b''_R$ be the neighbours of $b''$ that precede and succeed $b''$ on $P_u$. Let $x$ be the neighbour of $b''$ on $P_v[a'', b'']$.

Suppose first that some internal vertex $t$ of $P_v[a'', b'']$ is adjacent with some vertex $y$ of $P_u[b''_R, b]$. If $t \neq x$, then $t$ is also adjacent with some internal vertex $z$ of $P_u[a'', b'']$. So $t \neq v$; otherwise, $v$ is adjacent with two nonadjacent vertices of $P_u$ which leads to a situation where the arguments of either Case 1 or Case 2 apply. If $P_u[z, y]$ has length at least 3, then it follows, from Theorem 2 that $t$ is adjacent with every vertex of $P_u[z, y]$ including $b''$; this is not possible as $t$ and $b''$ are nonadjacent vertices on the induced path $P_u[a'', b'']$. So $z = b''_L$ and $y = b''_R$, and $b''_L$ and $b''_R$ is the only vertex of $P_u[a'', b'']$ to which $t$ is adjacent. Suppose $P_v[t, b'']$ contains $v$. If $P_v[t, b'']$ contains at least four vertices, then the subgraph induced by $b''_L$ and the vertices of $P_v[t, b'']$ must contain a hole, house or domino. (We use the fact that $v$ cannot be adjacent to nonadjacent vertices of $P_u$; otherwise, one can again argue that Case 1 or Case 2 occurs.) Suppose now that $P_v[t, b''] = tvb''$. Let $d$ be the neighbour of $b''_R$ on $P_v[b''_R, b]$. Then $\langle \{t, v, b'', b''_L, b''_R, d\} \rangle$ is a tailed twin $C_4$ since $v$ is nonadjacent with $b''_R$ and $d$.

Suppose thus that $v$ does not belong to $P_v[t, b'']$. Then we may assume that
$t$ is the first internal vertex on $P_v[a'', x]$ that is adjacent with $b''_R$. Let $s$ be the neighbour of $t$ on $P_v[a'', t]$. By the above we know that $tb''_R \in E$. If $sb''_R \in E$, then $(\{s, t, b''_L, b'', b''_R\})$ is a house which is forbidden. So assume $sb''_R \notin E$. Let $c$ be the neighbour of $b''_R$ on $P_u[a'', b'']$. Since $tc \notin E$ and $G$ has no holes, $sc \in E$. But then $(\{s, c, t, b''_L, b'', b''_R\})$ is a domino, which is forbidden. So $x$ is the only internal vertex of $P_v[a'', b'']$ that is adjacent with vertices of $P_u[b''_R, b]$. Let $y$ be the neighbour of $a''$ on $P_v[a'', b'']$ and let $a''_L$ and $a''_R$ be the neighbours of $a''$ on $P_u[a, a'']$ and $P_u[a', b'']$, respectively. One can argue as in the previous situation that the only internal vertex of $P_v[a'', b'']$ that is possibly adjacent with a vertex of $P_u[a, a'']$ is $y$.

Now let $y'$ be the first vertex on $P_u[a, a'']$ that is adjacent with $y$ (possibly $y' = a''$) and let $x'$ be the last vertex on $P_u[b'', b]$ to which $x$ is adjacent (possibly $x' = b''$). If $x'$ belongs to $P_u[b'', u]$, then the path obtained by taking $P_u[a, y']$ followed by $y'yP_v[y, x]$ and then $xx'P_v[x', b]$ is an induced $a-b$ path containing both $u$ and $v$. By Case 1 this produces a contradiction. Suppose thus that $x'$ belongs to $P_u[u, b] - u$. Then $P_u[y', x']$ and $yy'P_v[y, x]xx'$ are two internally disjoint $y'-x'$ paths containing $u$ and $v$, respectively. By applying the arguments of Case 2 to these two paths we again obtain a contradiction. Hence Case 3 cannot occur either.

\[\square\]

**Lemma 2.** Suppose $G = (V, E)$ is a graph for which $(V, M_m(G))$ is a convex geometry. Then for all $a, b \in V$, $I_m(I_m[a, b]) \subseteq I_m[a, b]$.

**Proof.** By the above $G$ is (house, hole, domino, $R_{C_4}$, $T_{C_4}$)-free. If $ab \in E$, then $I_m[a, b] = \{a, b\} = I_m(\{a, b\}) = I_m^3(I_m[a, b])$. Suppose $ab \notin E$. So, by Lemma 1 $I_m(I_m[a, b]) \subseteq I_m[a, b]$ (in fact equality holds). If $I_m(I_m[a, b]) \not\subseteq I_m[a, b]$, then there is a set $W = \{w_1, w_2, w_3\} \subseteq I_m[a, b]$ such that $I_m(W) \not\subseteq I_m[a, b]$. So there is an minimal $W$-tree $T$ that contains a vertex $x \notin I_m[a, b]$. Let $H = \langle V(T) \rangle$. Then $x$ is a cut-vertex of $H$. Thus one of the vertices of $W$, say $w_3$ does not belong to the component of $H - x$ that contains $w_1$ nor the component containing $w_2$. So $x$ lies on an induced $w_3 - w_1$ path for $i = 1, 2$. Since, by Lemma 1 $I_m[a, b]$ is $m^3$-convex it must be the case that $x$ is adjacent with $w_1$, $w_2$ and $w_3$; otherwise, $x \in I_m[a, b]$. So $x$ is on an induced path between every pair of nonadjacent vertices of $W$.

**Case 1** Suppose two nonadjacent vertices of $W$ lie on the same induced $a-b$ path $P$. Then $x$ is adjacent with a pair of nonadjacent vertices of an induced $a-b$ path. Hence $x$ lies on an induced $a-b$ path; contrary to assumption. So $w_1, w_2$ and $w_3$ cannot lie on the same induced $a-b$ path.

**Case 2** Suppose that two adjacent vertices of $W$, say $w_1$ and $w_2$, lie on an induced $a-b$ path $P$. By Case 1, $w_3$ does not lie on the same induced $a-b$ path as $w_1$ and $w_2$. Let $Q$ be an induced $a-b$ path containing $w_3$. Let $s_3$ and $t_3$ be the neighbours of $w_3$ on $Q[a, w_3]$ and $Q[w_3, b]$, respectively. (Note that $w_3 \neq a$ or $b$; otherwise, the vertices of $W$ lie on the same induced $a-b$ path. So $s_3$ and $t_3$ are well-defined.) Since $w_1w_2 \in E$, $w_1w_3, w_2w_3 \notin E$. Hence $\{s_3, t_3\} \cap \{w_1, w_2\} = \emptyset$.
Since \( x \) cannot be adjacent with two nonadjacent vertices of \( Q \), \( x \) cannot be adjacent with both \( s_3 \) and \( t_3 \). We may assume \( xt_3 \notin E \). The path \( R : w_2xw_3t_3 \) is a path of length 3 between two vertices of \( I_m[a, b] \). By Lemma \([4]\) \( I_m[a, b] \) is \( m^3 \)-convex. If \( R \) is induced this would imply that \( x \in I_m[a, b] \), contrary to assumption. Hence \( w_2t_3 \in E \). Now \( \langle \{w_1, w_2, x, w_3, t_3\} \rangle \) is a house unless \( w_1t_3 \in E \).

If \( x_{s3} \notin E \), then we can argue as for \( t_3 \) that \( s_3w_1, s_3w_2 \in E \). But then \( \langle \{w_1, w_2, w_3, x, s_3, t_3\} \rangle \) is a replicated twin \( C_4 \) which is forbidden.

Suppose now that \( x_{s3} \in E \). Then \( \langle \{s_3, w_3, t_3, w_2, x\} \rangle \) is a house unless \( s_3w_2 \in E \). If \( w_1s_3 \notin E \), the path \( R : s_3xw_1t_3 \) is an induced path, of length 3, between two vertices in \( I_m[a, b] \) that contains \( x \). Since \( I_m[a, b] \) is \( m^3 \)-convex and \( R \) contains \( x \) this contradicts our assumption about \( x \). So \( w_1s_3 \in E \). However, then \( \langle \{w_1, w_2, w_3, x, s_3, t_3\} \rangle \) is again a replicated twin \( C_4 \) which is forbidden. So this case cannot occur.

**Case 3** Suppose that no two vertices of \( W \) lie on the same induced \( a – b \) path in \( G \). (We may also assume that \( w_1w_3, w_2w_3 \notin E \).) Let \( P_i \) be an induced \( a – b \) path containing \( w_i \) for \( i = 1, 2, 3 \). From the case we are in \( w_i \) is not equal to either \( a \) or \( b \) for \( i = 1, 2, 3 \). For \( i = 1, 2, 3 \), let \( s_i \) and \( t_i \) be the neighbours of \( w_i \) on \( P_i[w_i, a] \) and \( P_i[w_i, b] \), respectively.

**Subcase 3.1** \( \{s_1, t_1\} = \{s_2, t_2\} = \{s_3, t_3\} \). Since \( s_1 \) and \( t_1 \) are non-adjacent vertices of \( P_1 \), \( x \) is adjacent with at most one of \( s_1 \) or \( t_1 \). Hence \( \langle \{w_1, w_2, w_3, s_1, t_1, x\} \rangle \) is a replicated twin \( C_4 \) which is forbidden. So \( \{s_3, t_3\} \) is either not equal to \( \{s_1, t_1\} \) or \( \{s_2, t_2\} \); suppose the former.

**Subcase 3.2** \( \{s_1, t_1\} \cap \{s_3, t_3\} = \emptyset \). Since \( s_1 \) and \( t_i \) are non-adjacent vertices of \( P_i \), \( x \) cannot be adjacent with both \( s_i \) and \( t_i \) for \( i = 1, 2, 3 \). So we may assume \( xt_1 \notin E \). Suppose first that \( xt_3 \notin E \). Since \( t_1w_1w_3 \) is a path of length 3 between two vertices of \( I_m[a, b] \) that contains \( x \), it follows from Lemma \([4]\) that this is not an induced path. Hence \( w_3t_1 \in E \). Similarly by considering the path \( w_1xw_3t_3 \) and using the same argument it follows that \( w_1t_3 \in E \). Similarly by considering the paths \( w_2xw_3t_1 \) and \( w_2xw_3t_3 \), it follows that \( w_2t_1 \) and \( w_2t_3 \in E \). But now \( \langle \{w_1, w_2, w_3, t_1, t_2, x\} \rangle \) is a replicated twin \( C_4 \) which is forbidden. So this case cannot occur.

**Subcase 3.3** \( \{|s_1, t_1\} \cap \{s_3, t_3\}| = 1 \). We may assume \( s_1 \in \{s_3, t_3\} \). The case where \( t_1 \in \{s_3, t_3\} \) can be argued similarly. Suppose first that \( s_1 = s_3 \). If \( s_1x \in E \), then \( xt_1, xt_3 \notin E \). But then we can argue similarly as in Subcase 3.2 that \( \langle \{w_1, w_2, w_3, t_1, t_3, x\} \rangle \) is a replicated twin \( C_4 \). Hence \( s_1x \notin E \). Suppose at least one of \( xt_1 \) or \( xt_3 \) is in \( E \), say \( xt_1 \in E \). Then \( \langle \{s_1, w_1, t_1, x, w_3\} \rangle \) is a house unless \( t_1w_3 \in E \). By considering the path \( w_2xw_3s_1 \) we can argue as before that \( w_2s_1 \in E \). By now considering the path \( t_1w_2s_1 \) it follows that \( t_1w_2 \in E \). Thus \( \langle \{w_1, w_2, s_1, t_1, x\} \rangle \) is a replicated twin \( C_4 \) which is forbidden. If neither \( xt_1 \) nor \( xt_3 \) are in \( E \), then one can argue in a similar manner that \( \langle \{w_1, w_2, w_3, s_1, x, t_3\} \rangle \) is a replicated twin \( C_4 \). If \( s_1 = t_3 \) we can
argue similarly that $G$ contains a replicated twin $C_4$ which is forbidden. Hence this case cannot occur either. This completes the proof of the lemma. 

**Lemma 3.** If $G = (V,E)$ is a (house, hole, domino, $T_{C_4}$)-free graph that contains an induced $A$-graph as labeled in Fig. 4, then $u_2 \notin I_m[a,b]$.

![Figure 4: A labeled $A$-graph](image)

**Proof.** Suppose, to the contrary, that $u_2 \in I_m[a,b]$ and let $P$ be an induced $a - b$ path containing $u_2$.

**Case 1** $u_1 \notin V(P[a,u_2])$. Suppose $P[a,u_2] : aw_1w_2\ldots w_ku_2$. If $k = 1$, then $\langle \{a, w_1, u_1, u_2, u_4, u_3\} \rangle$ is a domino unless at least one of $w_1u_4, w_1u_3, w_1u_1 \in E$. If $w_1u_3 \notin E$, then $w_1u_1$ or $w_1u_4 \in E$. Suppose $w_1u_1 \in E$. Then $\langle \{w_1, w_2, u_3, u_1, u_4\} \rangle$ is a house unless $w_1u_4 \in E$. So in either case $w_1u_4 \in E$. But then $\langle \{u_2, w_1, u_1, u_3, u_4, b\} \rangle$ is a tailed twin $C_4$ which is forbidden. So $w_1u_3 \in E$. Since $\langle \{w_1, a, u_1, u_4, u_3\} \rangle$ is not a hole, either $w_1u_1$ or $w_1u_4$ is in $E$. If $w_1u_4 \notin E$, then $\langle \{w_1, a, u_1, u_4, u_3\} \rangle$ is a house which is forbidden. Hence $w_1u_4 \in E$. So if $P[a,u_2]$ has length $2$, then its interior vertex is adjacent with both $u_3$ and $u_4$.

Suppose now that $k \geq 2$. By Theorem 2 $N[u_1]$ is $m^3$-convex. Since $N[u_1]$ contains both $a$ and $u_2$, every vertex of $P[a,u_2]$ is adjacent with $u_1$. However, then $\langle \{w_k, u_1, u_2, u_3, u_4\} \rangle$ is a house unless $w_ku_3$ or $w_ku_4$ is in $E$. If $w_ku_3 \notin E$, then $w_ku_4 \in E$ and so $\langle \{u_4, u_1, u_3, w_k, u_2, b\} \rangle$ is a tailed twin $C_4$ which is forbidden. If $w_ku_3 \in E$ and $w_ku_4 \notin E$, then $\langle \{u_1, w_k, u_2, u_3, u_4, a\} \rangle$ is a tailed twin $C_4$ which is forbidden. Hence $w_ku_3, w_ku_4 \in E$.

Thus neither $u_3$ nor $u_4$ belongs to $P[u_2,b]$.

Suppose first that $P[u_2,b]$ has length 2. Let $v_1$ be its interior vertex. By Theorem 2 $N[v_1]$ is $m^3$-convex. Since $N[v_1]$ contains both $u_2$ and $b$, $v_1$ is adjacent with every vertex on every induced $u_2-b$ path of length at least 3. So $v_1$ is adjacent with $u_3$ and $u_4$. But now $\langle \{w_k, u_2, v_1, u_4, b\} \rangle$ is a house which is forbidden.

Suppose now that $P[u_2,b]$ has length at least 3, say $P[u_2,b] : u_2v_1v_2\ldots v_rb$. By Theorem 2 $N[\{u_3, u_4\}]$ is $m^3$-convex. Since $u_2, b \in N[\{u_3, u_4\}]$, every vertex of $P[u_2,b]$ is adjacent with either $u_3$ or $u_4$. Let $b = v_{r+1}$. Let $i$ be the smallest integer such that $v_iu_4 \in E$, possibly $i = r + 1$. Then $w_ku_2v_1\ldots v_iu_4w_k$ is an induced cycle. Since $G$ has no holes $i = 1$. Let $j$ be the smallest integer greater than 1 such that $v_ju_4 \in E$; possibly $j = r + 1$. If $j = 2$, then $\langle \{w_k, u_2, v_1, v_2, u_4\} \rangle$ is a house which is forbidden. Thus $j = 3$; otherwise,
$u_4v_1v_2 \ldots v_ju_4$ is an induced cycle of length at least 5; which is forbidden. But then $\{w_k, u_2, v_1, v_2, v_3, u_4\}$ is a domino which is again forbidden.

**Case 2** $u_1 \in V(P[a, u_2])$. By considering $P[u_2, b]$ one can argue as in the previous case that $G$ contains a forbidden subgraph. Hence the lemma follows.

We now complete the proof of the theorem. By the above $G$ is (house, hole, domino, $R_{C_4}, T_{C_4}$)-free. Suppose $G$ contains the $A$ graph as an induced subgraph. Then the collection of extreme vertices for the convex hull, $CH(A)$, of the $A$ graph is a subset of the set of two leaves of the $A$ graph. By Lemma 3 the monophonic interval of the leaves of the $A$ graph does not include all the vertices of the $A$-graph. By Lemmas 1 and 2, $I_m[a, b]$ is $m_3$-convex for all $a, b \in V$. This is true in particular for the two leaves of the $A$ graph. Hence the convex hull of the extreme vertices of $CH(A)$ is thus not equal to $CH(A)$. This contradicts the fact that $(V, M_{m_3}(G))$ is a convex geometry.

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