A cellular algebra with specific decomposition of the unity

Mufida M. Hmaida

Abstract. Let $A$ be a cellular algebra over a field $F$ with a decomposition of the identity $1_A$ into orthogonal idempotents $e_i$, $i \in I$ (for some finite set $I$) satisfying some properties. We describe the entire Loewy structure of cell modules of the algebra $A$ by using the representation theory of the algebra $e_i A e_i$ for each $i$. Moreover, we also study the block theory of $A$ by using this decomposition.

1 Introduction

Let $T_{n,m}(\delta_0, \ldots, \delta_{m-1})$, or simply $T_{n,m}$, be the bubble algebra with $m$-different colours, $\delta_i \in F$, which is defined in Grimm and Martin[2]. In the same paper, it has been showed that it is semi-simple when none of the parameters $\delta_i$ is a root of unity. Later, Jegan[4] showed that the bubble algebra is a cellular algebra in the sense of Graham and Lehrer[1]. The identity of the algebra $T_{n,m}$ is the summation of all the different multi-colour partitions that their diagrams connect $j$ only to $j'$ with any colour for each $1 \leq j \leq n$, these multi-colour partitions are orthogonal idempotents. The goal of this paper is to generalize the technique that we use to study the representation theory of $T_{n,m}$ in [3].

Wada[8] consider a decomposition of the unit element into orthogonal idempotents and a certain map $\alpha$ to define a Levi type subalgebra and a parabolic subalgebra of the algebra $A$, and then the relation between the representation theory of each one has been studied. With using the same decomposition, we construct a Levi type subalgebra $\bar{A}$ (without using any map), and classify the blocks of $A$ by using the representation theory of the algebra $\bar{A}$.

2 Cellular algebras

We start by reviewing the definition of a cellular algebra, which was introduced by Graham and Lehrer[1] over a ring but we replace it by a field, since we need this assumption later.

Definition 2.1. [1, Definition 1.1]. A cellular algebra over $F$ is an associative unital algebra $A$, together with a tuple $(\Lambda, T(\cdot), C, \ast)$ such that

1. The set $\Lambda$ is finite and partially ordered by the relation $\geq$.

2. For every $\lambda \in \Lambda$, there is a non-empty finite set $T(\lambda)$ such that for an pair $(s, t) \in T(\lambda) \times T(\lambda)$ we have an element $c^\lambda_{st} \in A$, and the set $C := \{c^\lambda_{st} \mid s, t \in T(\lambda) \text{ for some } \lambda \in \Lambda\}$ forms a free $F$-basis of $A$. 

3. The map \( \ast : \mathbb{A} \to \mathbb{A} \) is \( \mathbb{F} \)-linear involution (This means that \( \ast \) is an anti-automorphism with \( \ast^2 = \text{id}_\mathbb{A} \) and \( (c^\lambda u_b)^\ast = c^{\lambda}_t \) for all \( \lambda \in \Lambda, \ s, t \in T(\lambda) \)).

4. For \( \lambda \in \Lambda, s, t \in T(\lambda) \) and \( a \in \mathbb{A} \) we have
   \[
   ac^{\lambda}_{st} \equiv \sum_{u \in T(\lambda)} r^{(u,s)}_a c^\lambda_{ut} \mod \mathbb{A}^{>\lambda},
   \]
   where \( r^{(u,s)}_a \in \mathbb{F} \) depends only on \( a, u \) and \( s \). Here \( \mathbb{A}^{>\lambda} \) denotes the \( \mathbb{F} \)-span of all basis elements with upper index strictly greater than \( \lambda \).

For each \( \lambda \in \Lambda \), the cell module \( \Delta(\lambda) \) is the left \( \mathbb{A} \)-module with an \( \mathbb{F} \)-basis \( \mathbb{B} := \{ c^\lambda_s \mid s \in T(\lambda) \} \) and an action defined by
   \[
   ac^\lambda_s = \sum_{u \in T(\lambda)} r^{(s,u)}_a c^\lambda_u \quad (a \in \mathbb{A}, \ s \in T(\lambda)),
   \]
   where \( r^{(s,u)}_a \in \mathbb{F} \) is the same coefficient that in (1).

A bilinear form \( \langle \ , \ \rangle : \Delta(\lambda) \times \Delta(\lambda) \to \mathbb{F} \) can be defined by
   \[
   \langle c^\lambda_s, c^\lambda_t \rangle c^\lambda_{ub} \equiv c^{\lambda}_{us} c^{\lambda}_{tb} \mod \mathbb{A}^{>\lambda} \quad (s, t, u, b \in T(\lambda)).
   \]

Note that this definition does not depend on the choice of \( u, b \in T(\lambda) \).

Let \( G_\lambda \) be the Gram matrix for \( \Delta(\lambda) \) of the previous bilinear form with respect to the basis \( \mathbb{B} \). All Gram matrices of cell modules that will be mentioned in this paper are with respect to the basis \( \mathbb{B} \) with the bilinear form defined by (2).

Let \( \Lambda^0 \) be the subset \( \{ \lambda \in \Lambda \mid \langle \ , \ \rangle \neq 0 \} \). The radical
   \[
   \text{Rad}(\Delta(\lambda)) = \{ x \in \Delta(\lambda) \mid \langle x, y \rangle = 0 \text{ for any } y \in \Delta(\lambda) \}
   \]
   of the form \( \langle \ , \ \rangle \) is an \( \mathbb{A} \)-submodule of \( \Delta(\lambda) \).

**Theorem 2.2.** ([24], Chapter 2). Let \( \mathbb{A} \) be a cellular algebra over a field \( \mathbb{F} \). Then

1. \( \mathbb{A} \) is semi-simple if and only if \( \det G_\lambda \neq 0 \) for each \( \lambda \in \Lambda \).

2. The quotient module \( \Delta(\lambda)/\text{Rad}(\Delta(\lambda)) \) is either irreducible or zero. That means that \( \text{Rad}(\Delta(\lambda)) \) is the radical of the module \( \Delta(\lambda) \) if \( \langle \ , \ \rangle \neq 0 \).

3. The set \( \{ L(\lambda) := \Delta(\lambda)/\text{Rad}(\Delta(\lambda)) \mid \lambda \in \Lambda^0 \} \) consists of all non-isomorphic irreducible \( \mathbb{A} \)-modules.

4. Each cell module \( \Delta(\lambda) \) of \( \mathbb{A} \) has a composition series with sub-quotients isomorphic to \( L(\mu) \), where \( \mu \in \Lambda^0 \). The multiplicity of \( L(\mu) \) is the same in any composition series of \( \Delta(\lambda) \) and we write \( d_{\lambda\mu} = [\Delta(\lambda) : L(\mu)] \) for this multiplicity.

5. The decomposition matrix \( D = (d_{\lambda\mu})_{\lambda,\mu \in \Lambda^0} \) is upper uni-triangular, i.e. \( d_{\lambda\mu} = 0 \) unless \( \lambda \leq \mu \) and \( d_{\lambda\lambda} = 1 \) for \( \lambda \in \Lambda^0 \).

6. If \( \Lambda \) is a finite set and \( C \) is the Cartan matrix of \( \mathbb{A} \), then \( C = D^t D \).
3 A Levi type sub-algebra

In this section, we construct a Levi type subalgebra \( \tilde{\mathbb{A}} \) of \( \mathbb{A} \) and study its representation theory.

The second and the third parts of the following assumption existed in Assumption 2.1 in \[8\].

**Assumption 3.1.** Throughout the remainder of this paper, we assume the following statements (A1) – (A4).

(A1) There exists a finite set \( I \).

(A2) The unit element \( 1_{\mathbb{A}} \) of \( \mathbb{A} \) is decomposed as \( 1_{\mathbb{A}} = \sum_{i \in I} e_i \) with \( e_i \neq 0 \) and \( e_i e_j = 0 \) for all \( i \neq j \) and \( e_i^2 = e_i \).

(A3) For each \( \lambda \in \Lambda \) and each \( t \in T(\lambda) \), there exists an element \( i \in I \) such that

\[
e_i c_{ts}^\lambda = c_{ts}^\lambda \quad \text{for any } s \in T(\lambda). \tag{3}\]

(A4) \( e_i^* = e_i \) for each \( i \in I \). Note that from \[8\], for each \( \lambda \in \Lambda \) and each \( t \in T(\lambda) \) we have

\[
c_{st}^\lambda e_i = c_{st}^\lambda e_i^* = (e_i c_{ts}^\lambda)^* = (c_{ts}^\lambda)^* = c_{st}^\lambda \quad \text{for any } s \in T(\lambda). \tag{4}\]

From (A2) and (A3), we obtain the next lemma.

**Lemma 3.2.** \[8, Lemma 2.2\]. Let \( t \in T(\lambda) \), where \( \lambda \in \Lambda \), and \( i \in I \) be such that \( e_i c_{ts}^\lambda = c_{ts}^\lambda \) for any \( s \in T(\lambda) \). Then for any \( j \in I \) such that \( j \neq i \), we have \( e_j c_{ts}^\lambda = 0 \) for any \( s \in T(\lambda) \). In particular, for each \( t \in T(\lambda) \), there exists a unique \( i \in I \) such that \( e_i c_{ts}^\lambda = c_{ts}^\lambda \) for any \( s \in T(\lambda) \).

For \( \lambda \in \Lambda \) and \( i \in I \), we define

\[
T(\lambda, i) = \{ t \in T(\lambda) \mid e_i c_{ts}^\lambda = c_{ts}^\lambda \text{ for any } s \in T(\lambda) \};
\]

\[
\Lambda_i = \{ \lambda \in \Lambda \mid T(\lambda, i) \neq \emptyset \};
\]

\[
I_\lambda = \{ i \in I \mid \lambda \in \Lambda_i \}.
\]

By Lemma \[3.2\], we have

\[
T(\lambda) = \prod_{i \in I} T(\lambda, i).
\]

Note that \( \Lambda_i \) is a poset with the same order relation on \( \Lambda \) and \( \Lambda = \bigcup_{i \in I} \Lambda_i \). Moreover, \( \Lambda_i \neq \emptyset \) for each \( i \in I \), and that because of \( 0 \neq e_i \in \mathbb{A} \) and \( e_i^2 = e_i \).

From (A3) and Lemma \[3.2\] the element \( e_i \) can be written in the form

\[
\sum_{\lambda \in \Lambda, s,t \in T(\lambda, i)} b_{s,t,\lambda} c_{st}^\lambda
\]

where \( b_{s,t,\lambda} \in \mathbb{F} \).
Theorem 3.3. The algebra $e_i A_i e_i$ is a cellular algebra with a cellular basis $C_i := \{c_{st}^\lambda \mid s, t \in T(\lambda, i) \text{ for some } \lambda \in \Lambda_i \}$ with respect to the poset $\Lambda_i$ and the index set $T(\lambda, i)$ for $\lambda \in \Lambda_i$, i.e. the following property holds:

1. An $\mathbb{F}$-linear map $*: e_i A_i e_i \to e_i A_i e_i$ defined by $c_{st}^\lambda \to c_{st}^\lambda$ for all $c_{st}^\lambda \in C_i$ gives an algebra anti-automorphism of $e_i A_i e_i$.

2. For any $a \in e_i A_i e_i$, $c_{st}^\lambda \in C_i$, we have

$$ac_{st}^\lambda = \sum_{u \in T(\lambda)} r_{u,s}^{(u,s)} c_{ut}^\lambda \mod (e_i A_i e_i)^{>\lambda},$$

where $(e_i A_i e_i)^{>\lambda}$ is an $\mathbb{F}$-submodule of $e_i A_i e_i$ spanned by $\{C_{st}^\lambda \mid s, t \in T(\lambda', i) \text{ for some } \lambda' \in \Lambda_i \text{ such that } \lambda' > \lambda \}$, and $r_{u,s}^{(u,s)}$ does not depend on the choice of $t \in T(\lambda, i)$.

Proof. Since $C$ is a basis of $A$, $e_i a e_i = a$ for all $a \in e_i A_i e_i$ and

$$e_i c_{st}^\lambda e_i = \begin{cases} c_{st}^\lambda & \text{if } s, t \in T(\lambda, i), \\ 0 & \text{otherwise}, \end{cases}$$

so the set $C_i$ is a basis of the algebra $e_i A_i e_i$. The first part follows from the fact the map $*$ on the algebra $A$ leaves $e_i A_i e_i$ invariant. For the second part, from (1) we have

$$ac_{st}^\lambda = \sum_{u \in T(\lambda)} r_{u,s}^{(u,s)} c_{ut}^\lambda \mod A^{>\lambda},$$

where $r_{u,s}^{(u,s)} \in \mathbb{F}$ depends only on $a, u$ and $s$. But $e_i a = a$, so

$$ac_{st}^\lambda = \sum_{u \in T(\lambda)} r_{u,s}^{(u,s)} c_{ut}^\lambda \mod A^{>\lambda} = \sum_{u \in T(\lambda', i)} r_{u,s}^{(u,s)} c_{ut}^\lambda \mod A^{>\lambda}.$$}

Also by using Lemma 3.2, we can show $e_i A^{>\lambda} e_i = (e_i A_i e_i)^{>\lambda}$, we are done. Moreover, cell modules $V(\lambda, i)$ for the algebra $e_i A_i e_i$ can be defined as follows:

$$V(\lambda, i) := e_i \Delta(\lambda) \quad (\lambda \in \Lambda_i).$$

The set $B_i := \{c_s^\lambda \mid s \in T(\lambda, i)\}$ is a basis of the module $V(\lambda, i)$. 

Define the algebra $\tilde{A}$ to be $\sum_{i \in I} e_i A_i e_i$ (which is the same as $\bigoplus_{i \in I} e_i A_i e_i$ since $e_i e_j = 0$ for all $i \neq j$). The identity of the algebra $e_i A_i e_i$ is the idempotent $e_i$, so $\tilde{A} \hookrightarrow A$. Moreover, the algebra $\tilde{A}$ turns out to be cellular with cell modules:

$$V(\lambda, i) = e_i \Delta(\lambda) \quad (\lambda \in \Lambda_i, i \in I).$$

We put $V(\lambda, i) = \{0\}$ in the case $\lambda$ is not an element in $\Lambda_i$.

Lemma 3.4. Let $\lambda \in \Lambda$, then

$$\Delta(\lambda) = \bigoplus_{i \in I} V(\lambda, i)$$

as an $\tilde{A}$-module.

Proof. It comes directly from the fact that $1_{\tilde{A}} = \sum_{i \in I} e_i$ and $e_i e_j = 0$ if $i \neq j$. 

4 Idempotent localization

In this section we compute the radical and Gram matrix of each cell module of the algebra $A$ by using the ones of the algebra $\bar{A}$.

Let $c_{u \lambda s}^{\lambda} c_{tv}^{\lambda} \in C$ where $s \in T(\lambda, i)$ and $t \in T(\lambda, j)$ for some $i, j \in I$. If $i \neq j$, then $c_{u \lambda s}^{\lambda} c_{tv}^{\lambda} = 0$ which means $\langle c_{u \lambda s}^{\lambda}, c_{tv}^{\lambda} \rangle = 0$ in $\Delta(\lambda)$. If $i = j$, then $c_{u \lambda s}^{\lambda} c_{tv}^{\lambda} \equiv \langle c_{u \lambda s}^{\lambda}, c_{tv}^{\lambda} \rangle c_{uv}^{\lambda} \mod A^{>\lambda}$.

Since $u, v$ do not have a role here, we can assume $u, v \in T(\lambda, i)$ and then

\[ c_{u \lambda s}^{\lambda} c_{tv}^{\lambda} \equiv \langle c_{u \lambda s}^{\lambda}, c_{tv}^{\lambda} \rangle c_{uv}^{\lambda} \mod (e_i A e_i)^{>\lambda}. \]

Hence the inner product $\langle c_{u \lambda s}^{\lambda}, c_{tv}^{\lambda} \rangle$ in $\Delta(\lambda)$ and the inner product $\langle c_{u \lambda s}^{\lambda}, c_{tv}^{\lambda} \rangle$ in $V(\lambda, i)$ have the same value. Let $M(\lambda, i)$ be the Gram matrix of this inner product on the module $V(\lambda, i)$ with respect to the basis $B_i$, then

\[ G(\lambda) = \bigoplus_{i \in I_\lambda} M(\lambda, i). \quad (5) \]

We can show the previous result by using the facts $B = \bigsqcup_{i \in I} B_i$, $B_i \cap B_j = \emptyset$ whenever $i \neq j$ and $\langle x, y \rangle = 0$ in $\Delta(\lambda)$ whenever $x \in V(\lambda, i)$, $y \in V(\lambda, j)$ where $i \neq j$.

The previous equation show that $\det G(\lambda) \neq 0$ if and only if $\det M(\lambda, i) \neq 0$ for each $i \in I$ such that $\lambda \in \Lambda_i$, then the following fact is straightforward.

**Theorem 4.1.** The algebra $A$ is semi-simple if and only if the algebra $e_i A e_i$ is semi-simple for each $i$.

**Proof.** It comes directly from (5) and from Theorem 2.2.

**Lemma 4.2.** Let $\lambda \in \Lambda^0$. The head of the module $\Delta(\lambda)$, denoted by $L(\lambda)$, satisfies the relation

\[ \dim L(\lambda) = \sum_{i \in I_\lambda} \dim V(\lambda, i), \]

where $\overline{V}(\lambda, i)$ is the head of the $e_i A e_i$-module $V(\lambda, i)$. We put $\dim \overline{V}(\lambda, i) = 0$ if $\lambda$ is not contained in $\Lambda^0_i$.

**Proof.** This follows from the fact that $\dim L(\lambda) = \text{rank}(G(\lambda))$ as the algebra is over a field and $\lambda \in \Lambda^0$ and by using (5).

**Theorem 4.3.** Let $\lambda \in \Lambda$, then

\[ \text{Rad}(\Delta(\lambda)) \cong \bigoplus_{i \in I_\lambda} \text{Rad}(V(\lambda, i)) \]

as a vector space and

\[ \text{Rad}(\Delta(\lambda)) \cong \sum_{i \in I_\lambda} \text{Rad}(V(\lambda, i)) \]

as an $A$-module.
Proof. First part comes directly from the fact that they have the same dimension:

$$\dim \text{Rad}(\Delta(\lambda)) = \dim \Delta(\lambda) - \dim L(\lambda),$$

$$= \sum_{i \in I} \dim V(\lambda, i) - \text{rank } \left( \bigoplus_{i \in I_\lambda} M(\lambda, i) \right)$$

$$= \sum_{i \in I_\lambda} \left( \dim V(\lambda, i) - \text{rank } M(\lambda, i) \right),$$

$$= \sum_{i \in I_\lambda} \dim \text{Rad}(V(\lambda, i)).$$

Note that $V(\lambda, i) = \{0\}$ if $i$ is not in $I_\lambda$.

Next part is coming from the fact that the basis $B$ of the module $\Delta(\lambda)$ equals $\bigsqcup_{i \in I_\lambda} B_i$ and $B_i = \{c_s^\lambda \mid s \in T(\lambda, i)\}$ is a basis the module $V(\lambda, i)$, also $\langle c_s^\lambda, c_t^\mu \rangle = 0$ whenever $s \in T(\lambda, i)$ and $t \in T(\lambda, j)$ such that $i \neq j$. Let $x \in \text{Rad}(V(\lambda, i))$ for some $i \in I_\lambda$, so $\langle c_s^\lambda, x \rangle = 0$ for all $s \in T(\lambda, i)$. Moreover, it is clear that $\langle c_t^\mu, x \rangle = 0$ for all $t \in T(\lambda, j)$ where $i \neq j$, then $x \in \text{Rad}(\Delta(\lambda))$. Thus

$$\sum_{i \in I_\lambda} \text{Rad}(V(\lambda, i)) \subseteq \text{Rad}(\Delta(\lambda)), $$

but both of them have the same dimension thus they are identical.

Corollary 4.4. Let $\lambda \in \Lambda^0$, then

$$L(\lambda) \cong \sum_{i \in I_\lambda} V(\lambda, i),$$

as an $A$-module.

Proof. As $V(\lambda, i) \cap V(\lambda, j) = \{0\}$ whenever $i \neq j$, so

$$L(\lambda) = \sum_{i \in I_\lambda} \frac{V(\lambda, i)}{\text{Rad}(V(\lambda, i))} \cong \sum_{i \in I_\lambda} V(\lambda, i).$$

5 The block decomposition of $A$

The aim of this section is to describe the blocks of the algebra $A$ over a field $\mathbb{F}$ by studying the homomorphisms between cell modules of $A$.

We say $\lambda \in \Lambda$ and $\mu \in \Lambda^0$ are cell-linked if $d_{\lambda\mu} \neq 0$. A cell-block of $A$ is an equivalence class of the equivalence relation on $\Lambda$ generated by this cell-linkage. From Theorem 2.2 each block of $A$ is an intersection of a cell-block with $\Lambda^0$, see [1]. Thus, if there a non-zero homomorphism between $\Delta(\lambda)$ and $\Delta(\mu)$ where $\lambda, \mu \in \Lambda^0$, then they belong to the same block.

Let $\theta : \Delta(\lambda) \to \Delta(\mu)$ be a homomorphism defined by $c_s^\lambda \mapsto \sum_{u \in T(\mu, i)} \alpha_u c_u^\mu$. Now if $s \in T(\lambda, i)$ for some $i \in I$, then $u \in T(\mu, i)$ since $\theta(c_s^\lambda) = \theta(c_t^\lambda) = \sum_{u \in T(\mu, i)} \alpha_u c_u^\mu$, so

$$\theta(c_s^\lambda) = \sum_{u \in T(\mu, i)} \alpha_u c_u^\mu.$$
Hence the map $\theta$ can be restricted to define a homomorphism

$$\theta \downarrow_{e_iA_e_i}: V(\lambda, i) \to V(\mu, i)$$

Now if $\theta \neq 0$, then there is $c_\delta^i$ such that $\theta(c_\delta^i) \neq 0$. Assume that $s \in T(\lambda, i)$ for some $i$, then $\theta \downarrow_{e_iA_e_i} \neq 0$, which means that both the sets $T(\lambda, i), T(\mu, i)$ don’t equal the empty set.

Let $\lambda, \mu \in \Lambda_i$ for some $i$, and $\tau: V(\lambda, i) \to V(\mu, i)$ be a homomorphism $e_iA_e_i$-modules. By extending the map $\tau$, we obtain a homomorphism $\tau: \Delta(\lambda) \to \Delta(\mu)$. Thus

$$\text{Hom}_A(\Delta(\lambda), \Delta(\mu)) = \{0\}$$

if and only if

$$\text{Hom}_{e_iA_e_i}(V(\lambda, i), V(\mu, i)) = \{0\}$$

for each $i \in I$. From this fact, we obtain the next theorem.

**Theorem 5.1.** Let $\Lambda = \Lambda^0$. Two weights $\lambda$ and $\mu$ in $\Lambda$ are in the same block of $A$ if and only if there exist $\nu_0, \ldots, \nu_r$ in $\Lambda$ such that all the following hold:

1. $\lambda$ and $\nu_0$ are in the same cell-block of $e_iA_e_i$ for some $i \in I$.
2. For each $j = 0, \ldots, r - 1$, $\nu_j$ and $\nu_{j+1}$ are in the same cell-block of $e_iA_e_i$ for some $i \in I$.
3. $\mu$ and $\nu_r$ are in the same cell-block of $e_iA_e_i$ for some $i \in I$.

### 6 Examples

In this section, we use some simple example to illustrate the facts that have been showed in the previous sections.

Let $A = M_{n \times n}(\mathbb{F})$ be an $n \times n$ matrix algebra over $\mathbb{F}$. This algebra is cellular with indexing set $\Lambda = \{n\}$ and $I = T(n) = \{1, \ldots, n\}$. For each $i, j \in T(n)$, we take $c_{ij}^n = E_{ij}$ where $E_{ij}$ is the matrix with 1 at the $(i, j)$-entry and 0 elsewhere. As we have $1_A = \sum_{i \in I} E_{ii}$ and the elements $E_{ii}$ satisfy all the assumptions in 3.1, thus we can apply our results from the previous sections. Note that $E_{ii}Ae_{ii}$ is isomorphic to $\mathbb{F}$ for each $i$, so $A$ is semi-simple see Theorem 4.1.

For the second example, let $A$ be the algebra which is given by the quiver

$$1 \xleftrightarrow{a_{12}} 2 \xleftrightarrow{a_{21}}$$

with the relation $a_{12}a_{21}a_{12} = a_{21}a_{12}a_{21} = 0$. The algebra is spanned by the elements

$$e_1, e_2, a_{12}, a_{21}, a_{12}a_{21}, a_{21}a_{12},$$

where $e_i$ is the path of length zero on the vertex $i$. As left module $A$ is isomorphic to

$$\mathbb{F}\langle e_1, a_{21}, a_{12}a_{21} \rangle \oplus \mathbb{F}\langle e_2, a_{12}, a_{21}a_{12} \rangle.$$
The algebra $A$ is a cellular algebra with anti-automorphism defined by $a_{ij}^{*} = a_{ji}$ and $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ where $\lambda_0 > \lambda_1 > \lambda_2$ and

$$T(\lambda_0) = \{1\}, \quad T(\lambda_1) = \{1, 2\}, \quad T(\lambda_2) = \{2\}.$$

We define

$$c_{11}^{\lambda_0} = a_{12}a_{21}, \quad \left( \begin{array}{cc} c_{11}^{\lambda_1} & c_{12}^{\lambda_1} \\ c_{21}^{\lambda_2} & c_{22}^{\lambda_2} \end{array} \right) = \left( \begin{array}{cc} e_1 & a_{12} \\ a_{21} & a_{22} \end{array} \right), \quad c_{22}^{\lambda_2} = e_2.$$

The set $C = \{c_{st}^{\lambda} \mid s, t \in T(\lambda) \text{ for some } \lambda \in \Lambda\}$ is a cellular basis of $A$. Note that $\lambda_0$ is not in $\Lambda^0$ although $\Delta(\lambda_0)$ is simple.

The identity $1_A$ equals $e_1 + e_2$ and this decomposition satisfies all the conditions in Assumption 3.3. Also we have

$$e_1 A e_1 = \mathbb{F}\langle e_1, a_{12}a_{21} \rangle \quad \Lambda_1 = \{\lambda_0, \lambda_1\},$$
$$e_2 A e_2 = \mathbb{F}\langle e_2, a_{21}a_{12} \rangle \quad \Lambda_2 = \{\lambda_1, \lambda_2\}.$$

Note that $J = \mathbb{F}\langle a_{12}a_{21} \rangle$ is a nilpotent ideal of $e_1 A e_1$ and $J' = \mathbb{F}\langle a_{21}a_{12} \rangle$ is a nilpotent ideal of $e_2 A e_2$, so $A$ is not semi-simple, from Theorem 1.1. Let $B = \{e_1, a_{21}\}$ be a basis of the module $\Delta(\lambda_1)$, so $V(\lambda_1, 1) = \mathbb{F}\langle e_1 \rangle$ and $V(\lambda_1, 2) = \mathbb{F}\langle a_{21} \rangle$. Also we have $V(\lambda_2, 1) = \{0\}$ and $V(\lambda_2, 2) = \mathbb{F}\langle e_2 \rangle$. It is easy to show that $V(\lambda_2, 2), V(\lambda_1, 2)$ are isomorphic as $e_2 A e_2$, so they are cell-linked. From Theorem 5.1 the modules $\Delta(\lambda_1)$ and $\Delta(\lambda_2)$ are in the same block of $A$. Moreover,

$$\text{Rad}(\Delta(\lambda_1)) = \text{Rad}(\mathbb{F}\langle e_1, a_{21} \rangle) \cong \text{Rad}(V(\lambda_1, 1) + V(\lambda_1, 2))$$
$$= V(\lambda_1, 2) \cong V(\lambda_2, 2) = \Delta(\lambda_2).$$

### 6.1 The multi-colour partition algebra

For $n \in \mathbb{N}$, the symbol $\mathcal{P}_n$ denotes the set of all partitions of the set $\underline{n} \cup \underline{n'}$, where $\underline{n} = \{1, \ldots, n\}$ and $\underline{n'} = \{1', \ldots, n'\}$.

Each individual set partition can be represented by a graph, as it is described in \cite{5}. Any diagrams are regarded as the same diagram if they representing the same partition.

Now the composition $\beta \circ \alpha$ in $\mathcal{P}_n$, where $\alpha, \beta \in \mathcal{P}_n$, is the partition obtained by placing $\alpha$ above $\beta$, identifying the bottom vertices of $\alpha$ with the top vertices of $\beta$, and ignoring any connected components that are isolated from boundaries. This product on $\mathcal{P}_n$ is associative and well-defined up to equivalence.

A $(n_1, n_2)$-partition diagram for any $n_1, n_2 \in \mathbb{N}^+$ is a diagram representing a set partition of the set $n_1 \cup n'_2$ in the obvious way.

The product on $\mathcal{P}_n$ can be generalised to define a product of $(n, m)$-partition diagrams when it is defined. For example, see the following figure.

![Partition Diagrams](attachment.png)

Let $n, m$ be positive integers, $c_0, \ldots, c_{m-1}$ be different colours where none of them is white, and $\delta_0, \ldots, \delta_{m-1}$ be scalars corresponding to these colours.

Define the set $\Phi^{n,m}$ to be

$$\{(A_0, \ldots, A_{m-1}) \mid \{A_0, \ldots, A_{m-1}\} \in \mathcal{P}_n\}.$$
Let \((A_0, \ldots, A_{m-1}) \in \Phi^{n,m}\) (note that some of these subsets can be an empty set). Define \(\mathcal{P}_{A_0,\ldots,A_{m-1}}\) to be the set \(\prod_{i=0}^{m-1} \mathcal{P}_{A_i}\), where \(\mathcal{P}_{A_i}\) is the set of all partitions of \(A_i\), and

\[
\mathcal{P}_{n,m} := \bigcup_{(A_0, \ldots, A_{m-1}) \in \Phi^{n,m}} \mathcal{P}_{A_0,\ldots,A_{m-1}}.
\]

The element \(d = (d_0, \ldots, d_{m-1}) \in \prod_{i=0}^{m-1} \mathcal{P}_{A_i}\) can be represented by the same diagram of the partition \(\cup_{i=0}^{m-1} d_i \in \mathcal{P}_n\) after colouring it as follows: we use the colour \(c_i\) to draw all the edges and the nodes in the partition \(d_i\).

A diagram represents an element in \(\mathcal{P}_{n,m}\) is not unique. We say two diagrams are equivalent if they represent the same tuple of partitions. The term multi-colour partition diagram will be used to mean an equivalence class of a given diagram. For example, the following diagrams

![Diagrams](image)

are equivalent.

We define the following sets for each element \(d \in \prod \mathcal{P}_{A_i}\):

\[
\text{top}(d_i) = A_i \cap n, \quad \text{bot}(d_i) = A_i \cap n', \quad \text{top}(d) = (\text{top}(d_0), \ldots, \text{top}(d_{m-1})), \quad \text{bot}(d) = (\text{bot}(d_0), \ldots, \text{bot}(d_{m-1})).
\]

Let \(\mathbb{F}_{n,m}(\delta_0, \ldots, \delta_{m-1})\) be \(\mathbb{F}\)-vector space with the basis \(\mathcal{P}_{n,m}\), as it is defined in [3], and with the composition:

\[
(\alpha_i)(\beta_i) = \begin{cases} 
\prod_{i=0}^{m-1} \delta_i(\beta_j \circ \alpha_j) & \text{if } \text{bot}(\alpha) = \text{top}(\beta), \\
0 & \text{otherwise},
\end{cases}
\]

where \(\delta_i \in \mathbb{F}\), \(\alpha, \beta \in \mathcal{P}_{n,m}\), \(c_i\) is the number of removed connected components from the middle row when computing the product \(\beta_i \circ \alpha_i\) for each \(i = 0, \ldots, m-1\) and \(\circ\) is the normal composition of partition diagrams.

The vector space \(\mathbb{F}_{n,m}(\delta_0, \ldots, \delta_{m-1})\) is an associative algebra, called the **multi-colour partition algebra**, with identity:

\[
1_{\mathbb{F}_{n,m}} = \sum_{(A_0, \ldots, A_{m-1}) \in \Xi^{n,m}} 1_{(A_0, \ldots, A_{m-1})} := \sum_{(A_0, \ldots, A_{m-1}) \in \Xi^{n,m}} (1_{A_0}, \ldots, 1_{A_{m-1}}),
\]

where \(\Xi^{n,m} := \{(A_0, \ldots, A_{m-1}) \mid \cup_{i=0}^{m-1} A_i = n, A_i \cap A_j = \emptyset \forall i \neq j\}\), \(1_{A_i}\) is the partition of the set \(A_i \cup A_i'\) where any node \(j\) is only connected with the node \(j'\) for all \(j \in A_i\) and \(A_i' = \{j' \mid j \in A_i\}\), for all \(0 \leq i \leq m - 1\). This means the identity is the summation of all the different multi-colour partitions that their diagrams connect \(i\) only to \(i'\) with any colour for each \(1 \leq i \leq n\).

The diagrams of shape \(id \in \mathfrak{S}_n\) are orthogonal idempotents, since

\[
1_{(A_0, \ldots, A_{m-1})}1_{(B_0, \ldots, B_{m-1})} = \begin{cases} 
0 & \text{if } (A_i) \neq (B_i), \\
1_{(A_0, \ldots, A_{m-1})} & \text{if } (A_i) = (B_i),
\end{cases}
\]

for all \((A_i), (B_i) \in \Xi^{n,m}\). Thus we have a decomposition of the identity as a sum of orthogonal idempotents since

\[
1_{\mathbb{F}_{n,m}} = \sum_{(A_i) \in \Xi^{n,m}} 1_{(A_0, \ldots, A_{m-1})}.
\]

As it have been showed in
the algebra $\mathbb{P}_{n,m}(\delta_0, \ldots, \delta_{m-1})$ is cellular and the last decomposition satisfies all the conditions in Assumption 3.1.

Let $(A_i) \in \Xi^{n,m}$, then

$$1_{(A_i)} \mathbb{P}_{n,m}(\delta_0, \ldots, \delta_{m-1}) 1_{(A_i)} \cong \mathbb{P}|A_0| (\delta_0) \otimes \mathbb{F} \cdots \otimes \mathbb{F} \mathbb{P}|A_{m-1}| (\delta_{m-1}),$$

where $\mathbb{P}|A_i| (\delta_i)$ is the normal partition algebra and $|A_i|$ is the cardinality of $A_i$, for the proof see Chapter 2 in [3].

**Theorem 6.1.** The algebra $\mathbb{P}_{n,m}(\delta_0, \ldots, \delta_{m-1})$ is semisimple over $\mathbb{C}$ for each integers $n \geq 0$ and $m \geq 1$ if and only if none of the parameters $\delta_i$ is a a natural number less than $2n$.

**Proof.** As the algebra $\mathbb{P}_n(\delta)$ is semi-simple over $\mathbb{C}$ whenever $\delta$ is not an integer in the range $[0, 2n-1]$, see Corollary 10.3 in [7], we obtain this theorem. \qed

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