Hawking radiation of black rings from anomalies

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Abstract

We derive Hawking radiation of five-dimensional black rings from gauge and gravitational anomalies using the method proposed by Robinson and Wilczek. We find, as in the black hole case, that the problem could reduce to a (1+1)-dimensional field theory and the anomalies result in correct Hawking temperature for neutral, dipole and charged black rings.

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1. Introduction

Hawking radiation has been one of the central issues in black hole quantum mechanics since its discovery [1]. Although Hawking radiation has been accepted as a basic feature of the quantum effects of black holes, its nature remains mysterious. In particular, it will lead to information loss and unitarity violation. However, from well-known gauge/gravity [2], Schwarzschild black holes in the asymptotic AdS space are dual to finite temperature quantum field theories which evolve unitarily (see review [3]). Therefore, people believe that general black holes also preserve unitarity due to a yet-to-be-discovered general holographic principle [4]. Up until now there is no satisfactory answer of how to understand black hole radiation and unitarity in one theoretical framework. The microstates counting of black hole entropy in string theory is still constrained to certain supersymmetric cases and cannot help much in studying the radiation of nonextremal black holes. It will be useful to understand Hawking radiation with greater profundity.

In [5], Parikh and Wilczek derived the radiation spectrum in the familiar tunneling picture by taking into account the global conservation laws. Their study indicates that the black hole radiation spectrum is not precisely thermal and may carry information. Recently, Robinson and Wilczek further proposed a new derivation of Hawking radiation of black hole via anomalies cancellation [6]. They treated the simplest Schwarzschild black hole and found that the cancellation of the gravitational anomalies results in correct Hawking temperature. Subsequent
In the recent study of higher dimensional gravity, people find that it is possible to have a new class of black objects in $D \geq 5$ dimension: black rings. Black rings exhibit some properties quite different from usual black holes: the horizons of black holes are topologically spherical while the horizon topologies of black rings are not spherical; the uniqueness of four-dimensional black holes no longer holds in the black rings case. Embedding into string/M-theory, the nonuniqueness of black rings could be understood and the microstates of some supersymmetric black rings have been studied in a similar way as black holes. The study of black rings has uncovered some unusual properties of the higher dimensional gravity system, which seems a rich field to deepen our understanding of gravity theory. It is interesting to further study various aspects of black rings such as their radiation. Especially in five dimensions, explicit solutions have been found for neutral, charged, dipole and supersymmetric black rings with horizon topology $S^1 \times S^2$, whose thermodynamics has been well studied. As the black holes, the nonextremal black rings radiate. It is interesting to ask if the Robinson–Wilczek method applies to black rings, and we find that it indeed works. This seems natural because in some limits, black rings become Myers–Perry black holes, and their Hawking radiation is given in [9] by the Robinson–Wilczek method. We find that for five-dimensional black rings the scalar field theory reduces to a $(1+1)$-dimensional free field theory near the horizon and the cancellation of gravitational and gauge anomalies gives us the correct Hawking temperature.

We will review the Robinson–Wilczek method briefly in the following section. In section 3 we study Hawking radiation of neutral, dipole and charged black rings. In section 4, we discuss the relation between Robinson–Wilczek method and the tunneling picture [5] to get the Hawking temperature.

2. Hawking radiation and anomalies

In a black hole background we must be careful to define the quantum state of a field theory. If we choose the Boulware state as the vacuum state, that is, define positive frequency using Schwarzschild time, then at the horizon the energy–momentum tensor of the ground state is divergent due to ingoing modes. One way to avoid this problem is to choose the Unruh vacuum by defining positive frequency using Kruskal coordinates, then these problematic ingoing modes in the Boulware vacuum are removed because they are excited states now.

Robinson and Wilczek take another viewpoint [6, 7]: at the classical level we just discard these ingoing modes near the horizon because they cannot affect the region outside the horizon. Then the quantum field theory near the horizon is chiral and suffers from the gravitational anomalies, and the gauge anomalies if gauge fields are present. In order to cancel the anomalies to preserve the symmetries, additional fluxes/currents should be included. These fluxes/currents are quantum effects of the classically irrelevant ingoing modes. The condition of the anomalies cancellation and the regularity condition at the horizon determine Hawking flux of charge and energy–momentum.

For several kinds of black rings which will be studied in the following section, the near horizon field theory reduces to a $(1+1)$-dimensional free field theory with a background metric in the following form:

$$ds^2 = -f(r) \, dt^2 + \frac{1}{f(r)} \, dr^2$$ (1)
with \( f(r_H) = 0 \) at the horizon \( r_H \). The quantum field theory in this small region \( r_H \leq r \leq r_H + \epsilon \) could be treated as chiral theory, and suffers from anomalies. The gravitational anomalies in two dimensions take a simple form

\[
\nabla_\mu T^\mu_{\chi} = -\frac{1}{96\pi \sqrt{-g}} \epsilon^{\delta \alpha \beta \epsilon} \partial_\alpha \Gamma^\epsilon_{\beta \delta} = \frac{1}{\sqrt{-g}} \partial_\mu N_\mu = B_\nu.
\]

(2)

If there are background gauge fields, there exist gauge anomalies

\[
\nabla_\mu J^\mu_{\chi} = \pm \frac{e}{4\pi \sqrt{-g}} \epsilon^{\mu \nu} \partial_\mu A_\nu.
\]

(3)

As the symmetries of the underlying theory—general covariance and gauge symmetry—must be preserved, the anomalies in the region \( r_H \leq r \leq r_H + \epsilon \) must be cancelled by extra currents/fluxes.

First we analyze the gauge anomalies. Introduce step functions \( \Theta_1 + = \Theta_1(r - r_H - \epsilon) \) and \( H = 1 - \Theta_1 + \), we can write the current as

\[
J^i = J^i_{(H)} H + J^i_{(o)} \Theta_1 +.
\]

(4)

Because we consider the stationary configurations we have the conserved current

\[
\partial_r J^r_{(o)} = 0
\]

(5)

and the anomalous current

\[
\partial_r J^r_{(H)} = \frac{e^2}{4\pi} \partial_r A_i.
\]

(6)

They are solved by

\[
J^r_{(o)} = c_o, \quad J^r_{(H)} = c_H + \frac{e^2}{4\pi} (A_i(r) - A_i(r_H))
\]

(7)

where \( c_o, c_H \) are integration constants. The \( c_o \) is the value of the flux at infinity and \( c_H \) is the value of the current of the outgoing modes at the horizon. The gauge invariance of the full theory demands

\[
-\delta \lambda W = \int d^2 x \lambda \left[ (J^r_{(o)} - J^r_{(H)}) + \frac{e^2}{4\pi} (A_i) + \partial_r \left( \frac{e^2}{4\pi} A_i H \right) \right] = 0
\]

(8)

where \( \lambda \) is the gauge parameter. The last term in the integral is cancelled by the quantum effects of the classically irrelevant ingoing modes. In the limit \( \epsilon \to 0 \) we have

\[
c_o = c_H - \frac{e^2}{4\pi} A_i(r_H).
\]

(9)

In order to fix the constants we impose a boundary condition: the covariant current \( \tilde{J}^i = J^i + \frac{e^2}{4\pi} A_i(r_H) H \) vanishes at the horizon. This is the regularity condition as pointed out in the first reference of [8]. Then the integration constants are fixed,

\[
c_o = 2c_H = -\frac{e^2}{2\pi} A_i(r_H).
\]

(10)

It represents the Hawking flux of charge.

For the gravitational anomalies, we can write \( T^\mu_{\nu} \) as

\[
T^\mu_{\nu} = T^\mu_{(H)\nu} H + T^\mu_{(o)\nu} \Theta_1 +.
\]

(11)

\(^{3}\) We do not take into account the region inside the horizon because it is causally disconnected.
For metric (1) we have

\[ N_t' = N_r' = 0 \]
\[ N_r' = \frac{1}{192\pi} (ff' + f^2) \]
\[ N_t' = \frac{1}{192\pi} \left( \frac{f'}{f} f - f^2 \right) \]

where the prime denotes the derivative with respect to \( r \).

Under the diffeomorphism transformation \( x \to x' = x - \xi \), taking gauge fields into account we have

\[ -\delta_\xi W = \int d^2 x \sqrt{-g(0)} \xi^\nu \nabla_\mu T^{\mu \nu} \]

For the energy–momentum tensor in the radial direction, in the region \( r > r_H + \epsilon \) we have

\[ \partial_r T_r^{(o) t} = F_{rt} J_t^{(o)} ; \tag{15} \]

in the region \( r_H \leq r \leq r_H + \epsilon \), since \( \nabla_\mu J^\mu \neq 0 \), \( N_t' \neq 0 \), we have

\[ \partial_r T_r^{(H) t} = F_{rt} J_r^{(H)} + A_r \partial_r J_t^{(H)} + \partial_r N_t' . \tag{16} \]

They are solved by

\[ T_r^{(o) t} = a_o + c_o A_t(r) \]
\[ T_r^{(H) t} = a_H + \int_{r_H}^r \partial_r \left( c_o A_t + \frac{e^2}{4\pi} A_t^2 + N_t' \right) \, dr . \tag{17} \]

The general covariance demands

\[ \int d^2 x \xi^\nu \left[ \partial_t \left( A_t J_r^{(H)} H \right) + \partial_r \left( N_t' H \right) + (T^{(o) t} - T^{(H) t} - A_t J_t^{(H)} + N_t') \delta(r - r_H - \epsilon) \right] = 0 . \tag{18} \]

The first term in the integral should be cancelled by the quantum effect of the ingoing modes. Taking the \( \epsilon \to 0 \) limit, we get

\[ a_o = a_H + \frac{e^2}{4\pi} A_t^2(r_H) - N_t'(r_H) . \tag{19} \]

Imposing a vanishing condition for the covariant energy–momentum \( T^{(r)}_t = T^r_t + (ff' - 2f^2)/192\pi \) on the horizon, we get

\[ a_H = 2N_t'(r_H) , \quad a_o = \frac{e^2}{4\pi} A_t^2(r_H) + N_t'(r_H) . \tag{20} \]

The \( a_o \) represents the Hawking flux of energy–momentum.

The flux \( \Phi = N_t'(r_H) \) represents a thermal flux due to Hawking radiation. Primarily, \( \Phi \) is the flux in the near horizon region, but as the currents are conserved in the region \( r > r_H + \epsilon \), at the asymptotic infinity the radiation flux will be equal to \( \Phi \), which is related to temperature as \( \Phi = \frac{\pi}{12} T^2 \). Therefore we can get the Hawking temperature

\[ T = \frac{f'(r_H)}{4\pi} . \tag{21} \]
3. Hawking radiation of five-dimensional black rings from anomalies

3.1. The neutral black ring

The five-dimensional neutral black ring was first found in [11] as a vacuum solution of the five-dimensional general relativity. Black rings obey similar thermodynamics laws to black holes, while they lose uniqueness because in some range of their parameter space there is more than one solution with the same conserved charges. For more aspects about five-dimensional black rings, see the review [17].

The metric of the neutral black ring is (we use the coordinates given in [15])

$$ds^2 = - \frac{F(y)}{F(x)} \left( dt - CR \frac{1 + y}{F(y)} d\psi \right)^2 + \frac{R^2}{(x - y)^2} F(x) \left( \frac{G(y)}{F(y)} d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{d\phi^2}{G(x)} \right)$$

(22)

with functions

$$F(\xi) = 1 + \lambda \xi \quad G(\xi) = (1 - \xi^2)(1 + \nu \xi)$$

(23)

and constant

$$C = \sqrt{\lambda(\lambda - \nu) \frac{1 + \lambda}{1 - \lambda}}, \quad 0 < \nu \leq \lambda < 1;$$

(24)

the coordinates $\psi, \phi$ are two cycles of the black ring and $x, y$ take values

$$-1 \leq x \leq 1, \quad -\infty \leq y \leq -1.$$  

(25)

The center of the black ring is located at $y = -\infty$, and $x \to -1, y \to -1$ approaches the asymptotic infinity. The horizon is at $y = y_H = -\frac{1}{\nu}$ with the topology $S^1 \times S^2$. The mass of the black ring is

$$M = \frac{3\pi R^2}{4G} \frac{\lambda}{1 - \nu}.$$  

(26)

In order to prevent the contraction the ring must rotate along the $\psi$ direction; the angular momentum is

$$J = \frac{\pi R^3}{2G} \sqrt{\lambda(\lambda - \nu)(1 + \lambda)} \frac{1}{(1 - \nu)^2}.$$  

(27)

The neutral black rings are parametrized by their mass and angular momentum, or by $\lambda$ and $\nu$.

Now let us show that near the horizon a scalar field theory reduces to a $(1+1)$-dimensional free field theory. For a scalar field $\phi$ in the black ring background, the action is

$$S[\phi] = \int d^5x (\phi \sqrt{-g} g^{ij} \partial_i \phi + \sqrt{-g} V_{\text{int}}(\phi))$$

$$= \int d^5\varphi \left[ - \frac{C^2 R^2 (1 + y^2)}{(x - y)^2 F(y) G(y)} \partial_t^2 - \frac{R^2 F(x)}{(x - y)^2 G(x)} \partial_\psi^2 + \frac{R^2 F(y)}{(x - y)^2 G(y)} \partial_x^2 + \frac{R^2 G(y)}{(x - y)^2 G(x)} \partial_y^2 + \frac{R^2 F(x)}{(x - y)^2 G(x)} \partial_\phi^2 \right] \varphi$$

$$+ \int d^5x \frac{R^4 F(x)}{(x - y)^2} V_{\text{int}}(\phi).$$  

(28)
We expand the scalar field as
\[ \phi = \sum_{k,l} \frac{1}{2\pi} \phi^{(kl)}(t, x, y) e^{ik\phi} e^{il\psi} \]  
(29)

where \( k, l \) are integers because \( \phi, \psi \) are periodic coordinates. Put it into the action and take the near horizon limit \( y \to y_H, G(y) \to 0 \), leaving the dominant terms in the action we get
\[
S[\phi] = \int d^5x \frac{R^2}{(x-y)^2} \left[ - \frac{C^2 R^2}{F(y) G(y)} \left( \partial_t + i \frac{F(y)}{CR(1+y)} \right)^2 - \partial_y \frac{F(y)}{CR(1+y)} G(y) \partial_y \right] \phi^{(kl)}. 
\]  
(30)

As in black hole cases, the potential term for the scalar field is suppressed. Moreover, the terms involving \( \partial_x \) are also suppressed and do not appear in the action. Therefore we can further expand function \( \phi^{(kl)}(t, x, y) \) in terms of \( x \). As \( x = \cos \theta \) in the polar coordinate, an appropriate expansion function is the Legendre polynomial \( P_n(x) \),
\[
\phi^{(kl)}(t, x, y) = \sum_n \phi^{(kln)}(t, y) P_n(x). 
\]  
(31)

Moving \( P_n(x) \) to the left of the operator, we integrate over \( x \) and have
\[
\int_{-1}^{1} dx \frac{P_m(x) P_n(x)}{(x-y)^2} = a_{mn}(y). 
\]  
(32)

Then the action can be written as
\[
S[\phi] = \int dt dy \sum_{k,l,m,n} a_{mn}(y) CR^2(1+y) \frac{\sqrt{-F(y)}}{\sqrt{-F(y)G(y)}} \phi^{(klm)}(t, y) \times \left[ \partial_t + i \frac{F(y)}{CR(1+y)} \right]^2 - \partial_y \frac{\sqrt{-F(y)}}{CR(1+y)} G(y) \partial_y \phi^{(klm)}(t, y). 
\]  
(33)

In the above action we can treat \( \frac{F(y)}{CR(1+y)} \) as an effective gauge field \( A_t(y) \) and \( l \) serves as the gauge coupling constant. The coefficient \( a_{mn}(y) \) is symmetric about \( m, n \). We can further absorb the factor \( a_{mn}(y) \frac{CR^2(1+y)}{\sqrt{-F(y)G(y)}} \) into \( \phi^{(klm)}(t, y) \) and define a new field \( \tilde{\phi}^{(klm)}(t, y) \), then the action can be written in a canonical form
\[
S[\phi] = \int dt dy \sum_{k,l,n} \tilde{\phi}^{(klm)}(t, y) \times \left[ \partial_t + i l \frac{F(y)}{CR(1+y)} \right]^2 - \partial_y \frac{\sqrt{-F(y)}}{CR(1+y)} G(y) \partial_y \tilde{\phi}^{(klm)}(t, y). 
\]  
(34)

It is clear that we get an infinite set of effective free massless fields \( \tilde{\phi}^{(klm)}(t, y) \) in (1+1) dimension with the metric
\[
ds^2 = -f(y) dt^2 + \frac{1}{f(y)} dy^2 
\]  
(35)

where
\[
f(y) = \frac{\sqrt{-F(y)}}{CR(1+y)} G(y) 
\]  
(36)

together with a background \( U(1) \) gauge field
\[
A_t(y) = - \frac{F(y)}{CR(1+y)}. 
\]  
(37)
In the near horizon region \(y = -\frac{1}{\nu} + \epsilon\) we have \(F(y) < 0, G(y) < 0, (1 + y) < 0\), so \(t\) is timelike and \(y\) is spacelike.

It is remarkable that the essential point in the above discussion is the absence of the derivatives with respect to \(x\) in the reduced action (30). This indicates that there is no dynamics in the \(x\) direction. Effectively we may take \(x\) as just a parameter to label the field. This is the reason why we get a two-dimensional free scalar field theory in the near horizon limit.

The analysis of the gauge and gravitational anomalies in this two-dimensional theory is the same as in section 2. For the gauge anomalies we have

\[
J^y_{(o)} = c_o, \quad J^y_{(H)} = c_H + \frac{l^2}{4\pi} (A_t(y) - A_t(y_H)) \tag{38}
\]

with

\[
c_o = 2c_H = -\frac{l^2}{2\pi} A_t(y_H) = \frac{l^2}{2\pi} \Omega_H. \tag{39}
\]

We have written \(-A_t(y_H)\) as \(\Omega_H\) where \(\Omega_H\) is the angular velocity at the horizon.

For the gravitational anomalies we have

\[
T^y_{(o)} = a_o + c_o A_t(y) \quad T^y_{(H)} = a_H + \left( c_o A_t + \frac{l^2}{4\pi} A_t^2 + N_t^2 \right) \tag{40}
\]

with

\[
a_o = \frac{l^2}{4\pi} \Omega_H^2 + N_t^2 (y_H), \quad a_H = 2N_t^2 (y_H). \tag{41}
\]

From the relation \(\Phi = \frac{\Omega}{2\pi} T^2\), where \(\Phi\) is the flux \(N_t^2 (y_H)\), we get the Hawking temperature of the black ring

\[
T = \frac{f'(y_H)}{4\pi} = \frac{1}{4\pi R \sqrt{\lambda \nu}} \sqrt{1 - \lambda}. \tag{42}
\]

### 3.2. The dipole black ring

The five-dimensional dipole black rings were first constructed in [15]; its metric is of the form

\[
ds^2 = -\frac{F(y)H(x)}{F(x)H(y)} \left( dr - CR \frac{1 + y}{F(y)} dy \right)^2 + \frac{R^2 F(x)H(x)H(y)^2}{(x - y)^2} \left[ d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)H(x)^3} d\phi^2 \right] \tag{43}
\]

with \(F(\xi), G(\xi)\) being defined as before, and the new function \(H(\xi)\) being

\[
H(\xi) = [H_1(\xi) H_2(\xi) H_3(\xi)]^\frac{1}{3} \quad \text{with} \quad H_i(\xi) = 1 - \mu_i \xi \tag{44}
\]

where \(0 \leq \mu_i < 1, i = 1, 2, 3\). The \(\mu_i\)'s are parameters related to the dipoles of the black ring. The scalar and gauge fields are

\[
X^i = \frac{H(x)H_i(y)}{H(y)H(x)}, \quad A_i = C_i R \frac{1 + x}{H_i(x)} \tag{45}
\]

where \(C_i\)'s are of the same form as \(C\) but with \(\lambda \to -\mu_i\). As the gauge fields are magnetic, they represent circularly distributed monopole charges. This kind of black ring has no conserved gauge charges while allowing a continuous value of dipoles:

\[
q_i = \frac{1}{2\pi} \int_{S^2} dA^i \tag{46}
\]
so dipole black rings continuously violate the uniqueness. These dipole charges are not conserved charges of black rings, but they do appear in the first law of thermodynamics of black rings \[15, 16\].

We can analyze the scalar action in this dipole black ring background and expand \( \phi \) in the same way as in the neutral case; after field redefinition we can get a two-dimensional massless free scalar field theory with the action similar to (34). We will not list the details but just give the final form:

\[
S[\phi] = \int dt \, dy \sum_{k,l,n} \tilde{\phi}^{(kln)} (t, y) \left[ \frac{CR(1 + y)H(y)^{3/2}}{\sqrt{-F(y)G(y)}} \times \left( \frac{iF(y)}{CR(1 + y)} \right)^2 - \frac{\sqrt{-F(y)G(y)}}{CR(1 + y)H(y)^{3/2}} \frac{\partial_y}{\sqrt{-F(y)G(y)}} \right] \tilde{\phi}^{(kln)} (t, y).
\]

Note that gauge fields \( A_i \phi \) do not contribute dominant terms in the action. The action is similar to the neutral case except a new \( H(y)^{3/2} \) factor.

The background metric is of the same form as (35) with

\[
f(y) = \frac{\sqrt{-F(y)}}{CR(1 + y)H(y)^{3/2}} G(y)
\]

and together with a gauge field

\[
A_i(y) = -\frac{F(y)}{CR(1 + y)}.
\]

Note that \( A_i \phi \)'s are not contained in our (1+1) dimensional theory because they are magnetic, and their electric dual are two-form fields that do not couple to point particles. In fact, \( A_i \phi \)'s behave as three scalar fields in the near horizon two-dimensional background.

The analysis of both the gauge and the gravitational anomalies is parallel with the neutral case except their \( f(y) \)'s are different. The Hawking temperature of the dipole black ring is

\[
T = \frac{f'(y_H)}{4\pi} = \frac{1}{4\pi R} \frac{1 + y}{\sqrt{\lambda_1}} \sqrt{\frac{1 - \lambda}{1 + \lambda}} \frac{1}{\sqrt{\prod_{i} \left( 1 + \mu_i \right)}}.
\]

3.3. The charged black ring

The rotating black ring with a single electric charge was first constructed in \[12\] as a solution of the low-energy effective action of heterotic string. The multi-charge solutions were given in \[13\] where it was found that the two-charge solution is regular while the three-charge solution suffers from closed timelike circles (CTCs). In \[14\], a regular three-charge solution was obtained after overcoming the CTCs pathology. For simplicity, we only treat the single-charged black ring in this subsection. For black rings with two or three charges, the discussion is similar.

The metric of the single-charged ring can be written as

\[
ds^2 = -\frac{F(y)}{F(x)K(x, y)^2} \left( dt - CR \frac{1 + y}{F(y)} \cosh^2 \alpha \, d\psi \right)^2 + \frac{R^2}{(x - y)^2} F(x) \left[ -\frac{G(y)}{F(y)} \frac{d\psi^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right]
\]

where functions \( F(\xi), G(\xi) \) are defined as before, and \( K(x, y) \) is defined as

\[
K(x, y) = 1 + \frac{\lambda(x - y)}{F(x)} \sinh^2 \alpha
\]

where \( \alpha \) is the parameter representing the electric charge.
The dilation field is
\[ e^{-\Phi} = K(x, y) \]  
and the gauge fields are
\[ A_t = \lambda(x - y) \sinh \alpha \cosh \alpha \frac{F(x) K(x, y)}{F(x) K(x, y)} \]
\[ A_\psi = CR(1 + y) \sinh \alpha \cosh \alpha \frac{F(x) K(x, y)}{F(x) K(x, y)} \]
with the electric charge
\[ Q = \frac{2 \sinh 2\alpha}{3(1 + \frac{1}{2} \sinh^2 \alpha)} M. \]

There is also a 2-form \( B_{t\psi} \) field which indicates that the black ring carries a local fundamental string charge, but it is irrelevant to our discussion because it does not couple to point particles.

The action of a scalar field in this black ring background is
\[ S[\phi] = \int d^5x (\sqrt{-g} g^{ij} D_i \phi D_j \phi + \sqrt{-g} V_{int}(\phi)). \]

The terms containing covariant derivatives appear as
\[ \frac{R^2}{(x - y)^2 K(x, y)} \left[ -\frac{C^2 R^2(1 + y)^2 \cosh^4 \alpha}{F(y) G(y)} D_t^2 - 2 \frac{CR(1 + y) \cosh^2 \alpha}{G(y)} D_t D_\psi - \frac{F(y) G(y)}{F(y) G(y)} D_\psi^2 \right]. \]

We expand \( \phi \) as (29) and put it into the action, after taking the near horizon limit \( y \sim y_H + \epsilon \) we get the dominant action for field \( \phi(kl)(t, x, y) \):
\[ S[\phi] = \int d^5x \phi^{(kl)} \left( \frac{R^2}{(x - y)^2 K(x, y)} \left[ -\frac{C^2 R^2(1 + y)^2 \cosh^4 \alpha}{F(y) G(y)} (\partial_t + i e A_t) \right. \right. \]
\[ \left. + \frac{i e A_\psi F(y)}{CR(1 + y) \cosh \alpha} + \frac{i l F(y)}{CR(1 + y) \cosh^2 \alpha} \right]^2 \left. - \partial_y G(y) \partial_y \phi^{(kl)} \right). \]

In the second step we discard the \( \partial_x \) part because the \( x \)-dependence coming from \( A_t(x, y) \) and \( A_\psi(x, y) \) cancels exactly.

Now as the operator in the bracket does no longer depend on \( x \), we can expand \( \phi^{(kl)}(t, x, y) \) as before \( \phi^{(kl)}(t, x, y) = \sum_n \phi^{(kl)n}(t, y) P_n(x) \). After integrating \( x \)
\[ \int_{-1}^{1} dx \frac{P_m(x) P_k(x)}{(x - y)^2 K(x, y)} = b_{mn}(y), \]
the action becomes
\[ S[\phi] = \int dt dy \frac{CR^3(1 - y) \cosh^2 \alpha}{\sqrt{-F(y)}} b_{mn}(y) \phi^{(kl)n}(t, y) \left[ \frac{CR(1 + y) \cosh \alpha}{\sqrt{-F(y) G(y)}} \right]^2 \left. - \partial_y \frac{\sqrt{-F(y) G(y)}}{CR(1 + y) \cosh \alpha} \partial_y \right] \phi^{(kl)n}(t, y). \]
After the field redefinition, we can write the action in the canonical form. The action describes an infinite collection of massless scalar fields in a (1+1)-dimensional background, whose metric is of the form (35) with

\[ f(y) = \frac{\sqrt{-F(y)}}{CR(1 + y)\cosh^2 \alpha} G(y), \]  

(61)

and coupling to two \(U(1)\) fields

\[ A_i^{(1)} = -\tanh \alpha, \quad A_i^{(2)}(y) = -\frac{F(y)}{CR(1 + y)\cosh^2 \alpha}. \]  

(62)

Note that \(A_i^{(2)}(y_H) = -\Omega_H\) where \(\Omega_H\) is the angular velocity at the horizon.

Extending the gauge anomalies analysis to more than one gauge field is straightforward. The anomaly equations for electric currents are

\[ \partial_y J^{(1)}(H) = e \frac{4}{4\pi} \partial_y (eA_i^{(1)} + lA_i^{(2)}), \quad \partial_y J^{(2)}(H) = l \frac{4}{4\pi} \partial_y (eA_i^{(1)} + lA_i^{(2)}). \]  

(63)

As \(A_i^{(1)}\) is a constant it in fact does not contribute anomalies in the region \(y_H \leq y \leq y_H + \epsilon\), so we have

\[ J^{(1)}(\omega) = c^{(1)}_H, \quad J^{(1)}(H) = c^{(1)}_H + l \frac{4}{4\pi} (A_i^{(2)}(y_H) - A_i^{(2)}(y_H)). \]  

(64)

The gauge invariance of \(A_i^{(2)}\) demands

\[ c^{(1)}_o = c^{(1)}_H - e \frac{4}{4\pi} (eA_i^{(1)}(y_H) + lA_i^{(2)}(y_H)) \]  

(65)

and the regular boundary condition leads to

\[ c^{(1)}_o = e \frac{2}{2\pi} (e \tanh \alpha + l\Omega_H). \]  

(66)

Similarly we can get \(J^{(2)}\).

For gravitational anomalies we have

\[ T^{(1)}_{\omega} = a_o + c^{(1)}_o A_i^{(1)}(y) + c^{(2)}_o A_i^{(2)}(y), \quad T^{(2)}_{\omega} = a_H + \int_{y_H}^{y} \partial_y \left( c^{(2)}_o A_i^{(2)} + l^2 \frac{4}{4\pi} A_i^{(2)} + N_i \right) dy. \]  

(67)

The general covariance demands

\[ a_o = a_H - c^{(1)}_H A_i^{(1)}(y_H) - c^{(2)}_H A_i^{(2)}(y_H) - N_i(y_H). \]  

(68)

A covariance boundary condition fixes

\[ a_H = 2N_i(y_H), \quad a_o = \frac{1}{4\pi} (e \tanh \alpha + l\Omega_H)^2 + N_i(y_H). \]  

(69)

The Hawking temperature is given by

\[ T = \frac{f'(y_H)}{4\pi} = \frac{1}{4\pi R \cosh^2 \alpha} \sqrt{\frac{1 + \lambda}{1 - \lambda}}. \]  

(70)

In the extremal limit \(\alpha \to \infty\), we have \(Q \to M, T \to 0\).

Now let us consider our results with the black body radiation of black rings at temperature \(T = \frac{1}{\beta}\); in order to avoid the superradiance problem we only consider fermions. The Planck distribution of fermions with energy \(\omega\), charge \(e\) and angular momentum \(l\) is

\[ N_{e,l}(\omega) = \frac{1}{e^{(\beta \omega - e\Phi - l\Omega_H)} + 1}. \]  

(71)
where $\Phi$ is the co-rotating electric chemical potential at the horizon and $\Omega_H$ is the angular velocity at the horizon. The Killing vector of the black ring background is $\xi = \partial_t + \Omega_H \partial_\psi$, so we have

$$\Phi = \xi^\mu A_\mu = A_t(x, y_H) + \Omega_H A_\psi(x, y_H) = \tanh \alpha.$$  \hfill (72)

Then the Hawking fluxes of electric charge, angular momentum and energy–momentum tensor are

$$J_e = e \int_0^{\infty} \frac{d\omega}{2\pi} (N_{e,1}(\omega) - N_{e,-1}(\omega)) = \frac{e}{2\pi} (e \tanh \alpha + \Omega_H)$$

$$J_l = l \int_0^{\infty} \frac{d\omega}{2\pi} (N_{e,1}(\omega) - N_{e,-1}(\omega)) = \frac{l}{2\pi} (e \tanh \alpha + \Omega_H)$$

$$J_E = \int_0^{\infty} \frac{d\omega}{2\pi} \omega (N_{e,1}(\omega) + N_{e,-1}(\omega)) = \frac{1}{4\pi} (e \tanh \alpha + \Omega_H)^2 + \frac{\pi}{12\alpha^2}. \hfill (73)$$

They agree with the results (66), (69) derived from the anomalies cancellation. Set $e = 0$ we get the results for the neutral black ring.

4. Relation to the tunneling picture

In [5], Parrikh and Wilczek showed how to derive the Hawking temperature in the tunneling picture. We need to calculate the tunneling probability of a particle tunneling from $r_{in} = r_H(M) - \epsilon$ inside the initial horizon to $r_{out} = r_H(M - \omega) + \epsilon$ outside the final horizon in the Painleve coordinate. Under the WKB approximation, it is

$$\Gamma \sim e^{-2\text{Im} S} \hfill (74)$$

with

$$\text{Im} S = \text{Im} \int_{r_{in}}^{r_{out}} p_r dr = \text{Im} \int_{0}^{\omega} \int_{r_{in}}^{r_{out}} \frac{dr}{F} d(-\omega). \hfill (75)$$

Here $\omega$ is the energy of the radiated particle, $M(M - \omega)$ is the mass of the black hole before (after) emission. And $r$ is the radial null geodesic, it is a function of $r$ and conserved charges such as $M, Q, J$. As $\omega \ll M$, to the first order the above expression is approximately:

$$\text{Im} S \simeq \omega \text{Im} \int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}}. \hfill (76)$$

Let us apply the above analysis to the black rings studied above. In the previous article [18] the tunneling picture of black rings was derived via the Angheben–Nadalini and Vanzo–Zerbini covariant approach; here we will treat the effective two-dimensional near-horizon metric as our starting point. For simplicity, we focus on the neutral black ring and consider the metric (35) obtained by the Robinson–Wilczek method. Transforming the $t$ coordinate as

$$t \rightarrow t - \int_{r_{in}}^{r} \frac{CR(1 + y)}{\sqrt{-F(y)G(y)}} \sqrt{1 - G(y)} dy \hfill (77)$$

we can rewrite the metric in the Painleve form

$$ds^2 = -\frac{\sqrt{-F(y)G(y)}}{CR(1 + y)} dr^2 + 2\sqrt{1 - G(y)} dt dy + \frac{CR(1 + y)}{\sqrt{-F(y)}} dy^2. \hfill (78)$$

Consider an uncharged particle to avoid the influence of electromagnetic fields; the null geodesics is

$$\dot{y} = \frac{\sqrt{-F(y)}}{CR(1 + y)} (\pm 1 - \sqrt{1 - G(y)}) \hfill (79)$$

where $\pm$ correspond to outgoing/ingoing geodesics.
Performing the integral (76) with the outgoing geodesic, we have

\[ \text{Im} S \simeq \omega \frac{CR(1 - \frac{1}{\nu})}{\sqrt{-F(-\frac{1}{\nu})}} \times 2\pi \text{Res}_y \frac{1}{1 - \sqrt{1 - G(y)}} \]

\[ \simeq \omega \frac{CR(1 - \frac{1}{\nu})}{\sqrt{-F(-\frac{1}{\nu})}} \times 2\pi \frac{2\nu}{\nu^2 - 1}. \]

(80)

From \( \Gamma \sim e^{-2\text{Im} S} \sim e^{-\frac{\tau}{2}} \), we get the Hawking temperature

\[ T = \frac{1}{4\pi R} \frac{1 + \nu}{\sqrt{\lambda \nu}} \sqrt{1 - \lambda} \]

(81)

The situations for dipole and charged black rings are similar.

5. Conclusion

In this paper, we extended the Robinson–Wilczek method to the five-dimensional black rings and obtained their Hawking temperature and fluxes correctly. As in the rotating black hole case, near the horizon the field theory reduces to a (1+1)-dimensional free field theory coupled to gauge fields after an appropriate field redefinition. From the cancellation of the gauge and gravitational anomalies, we obtain the correct Hawking temperature and radiation. The same two-dimensional metric also determines the null geodesics in the tunneling method and leads to the same Hawking temperature. This reflects the fact that the Hawking radiation is determined universally by the horizon properties.

From studying various black holes and black rings here, we expect that for stationary black objects in various dimensions with more complicated horizon topology the Robinson–Wilczek method still works. The horizon is a hypersurface with uniform physical properties; for example, the surface gravity and electric potential are constant on the horizon. Our study of black rings shows that the quantum field theory near the horizon is relatively simple. In the near horizon limit, \( G(y) \) captures the singular structure. After zooming in the horizon, the horizon is effectively a two-dimensional one and the quantum field theory is effectively coupled to the gravitational background (35) after field redefinition. From the analysis of charged black ring we see that near the horizon the electrodynamics also simplifies, depending only on the radial direction, while outside the horizon it is much more complicated.

The Robinson–Wilczek method has been tested in many cases; at present it does not deal with other aspects of black hole physics such as thermodynamics and entropy, a better understanding of these issues is expected. At least for studying Hawking radiation with the Robinson–Wilczek method, we should expect that for more general cases the near horizon physics could be simplified and reduced to a (1 + 1) dimensional one, if the horizon is only controlled by a function depending on one coordinate. Though the exploration is still case by case, it seems that this method is quite general.

One subtle point in our treatment is that the gauge potentials in the effective two-dimensional field theory are actually divergent at asymptotic infinity \( y = -1 \) with singular behavior \( A_r(y) \sim \frac{1}{1+y} \). In the other black holes studied with the Robinson–Wilczek method, the gauge potentials are always vanishing at spatial infinity. This divergence indicates that the current of the ingoing modes at infinity is divergent. This raises the issue of whether the original Robinson–Wilczek method could be applied to our cases. From the discussion on the gauge invariance and diffeomorphic invariance of the effective action, the possible divergent terms appear in the partial derivative terms, which should be cancelled by the quantum effects.
of the classical ingoing modes. However, it is not clear what the exact form of the quantum effects could be. The key point in the treatment is the vanishing of the terms proportional to the $\delta$-function, which depend only on the quantities near the horizon. This is in accordance with the fact that the Hawking radiation is universally determined by the horizon. Moreover, the supporting evidence for our treatment is that in all cases we discussed we obtained the correct Hawking temperatures and fluxes. This suggest that the Robinson–Wilczek method could be used to discuss more general black objects. It would be interesting to understand this issue better.

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Note added. After submitting this paper we were informed of an independent work by Umpei Miyamoto and Keiju Murata [19] which has some overlap with our work.

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