TWO-SIDED LOCALIZATION OF BIMODULES

E. ORTEGA

Abstract. We extend to bimodules Schelter’s localization of a ring with respect to Gabriel filters of left and right ideals. Our two-sided localization of bimodules provides an endofunctor on a convenient bicategory of rings with filters of ideals. This is used to study the Picard group of a ring relative to a filter of ideals.

Introduction

Given a right $R$-module $M_R$ and a certain filter $\mathcal{I}$ of right ideals of $R$, Gabriel [5] and Goldman [6] gave the construction of the module of quotients of $M$ with respect to the filter $\mathcal{I}$, denoted by $Q_{\mathcal{I}}(M)$. Moreover, if we consider the regular right $R$-module $R$ then $Q_{\mathcal{I}}(R)$ turns out to be a ring. It is shown in [12] that the more classical rings of quotients, as for example the Ore localizations, are particular cases of this construction. Moreover, this localization produces a functor $M \mapsto Q_{\mathcal{I}}(M)$ from the category of right $R$-modules to the category of right $Q_{\mathcal{I}}(R)$-modules. In [10] it is considered the category $\mathcal{W}$ of rings with a Gabriel filter of right ideals, and the endofunctor $Q(R, \mathcal{I}) = (Q_{\mathcal{I}}(R), \mathcal{I}')$. In case that $R$ is a commutative ring, it is shown that the subcategory $\mathcal{C}$ of the complete pairs, i.e. those such that $(R, \mathcal{I}) \cong (Q_{\mathcal{I}}(R), \mathcal{I}')$, is reflective in $\mathcal{W}$. This is not true for general noncommutative rings.

Later on, Schelter [11] generalized the commutative case by considering certain Gabriel filters on $R \otimes_{\mathbb{Z}} R^{op}$. In particular, he considered the category $\mathcal{W}_\sigma$ of rings $R$ with Gabriel filters $\mathcal{D}_r$ and $\mathcal{D}_l$ of right and left ideals respectively. He constructed the two-sided localization of $R$ with respect to $\mathcal{D}_r$ and $\mathcal{D}_l$, and obtained a ring structure $Q_{\mathcal{D}_r,\mathcal{D}_l}(R)$. In the particular case where $\mathcal{I}_d_r$ and $\mathcal{I}_d_l$ are the Gabriel filters of right and left dense ideals respectively, $Q_{\mathcal{I}_d_r,\mathcal{I}_d_l}(R) = Q_{\sigma}(R)$ is the maximal symmetric ring of quotients (see [8] and [9]). This construction defines an endofunctor $(R, \mathcal{D}_l, \mathcal{D}_r) \mapsto (Q_{\mathcal{D}_r,\mathcal{D}_l}(R), \mathcal{D}'_l, \mathcal{D}'_r)$, and in contrast with the case of the one-sided localization of $R$-modules, Schelter proved that the subcategory $\mathcal{C}_\sigma$ of complete triples is reflective in $\mathcal{W}_\sigma$. Our aim is to extend the two-sided localization functor to bicategories (a richer structure than a category) from which the study of Morita equivalences of two-sided rings of quotients follows in a natural way. Using these tools we obtain an exact sequence relating Picard groups of two-sided localizations, generalizing substantially the well-known sequence obtained by Bass [3] for the localization of a commutative ring with respect to a multiplicative set of regular elements.

Now we summarize the contents of this paper. In all this paper we will basically follow notation and results from three sources, Schelter [10], Goldman [6] and Stenstrom [12]. In section 1 given unital rings $S$ and $R$ and Gabriel filters $\mathcal{H}_r$ and $\mathcal{D}_l$ of right ideals and left ideals...
of $S$ and $R$, respectively, we will construct the two-sided localization of an $R$-$S$-bimodule $M$ with respect to $\mathcal{H}_r$ and $\mathcal{D}_l$, denoted by $Q_{\mathcal{D}_l,\mathcal{H}_r}(M)$. In case that $M = R$ is the regular $R$-$R$-bimodule then $Q_{\mathcal{D}_l,\mathcal{H}_r}(R)$ has a ring structure.

It is known that for unital rings $R$ and $S$ if the categories Mod-$R$ and Mod-$S$ are equivalent then Mod-$Q_r(R)$ and Mod-$Q_s(S)$ are equivalent categories too [9]. In the general case our aim will be to determine what two-sided localizations preserve Morita equivalence. It is well-known that bicategories provide the best way to understand Morita equivalences between rings and other structures, such as $C^*$-algebras, von Neumann algebras, Lie groupoids [7]. Thus, in section 2 we are going to construct the bicategory $\mathcal{IB}_\sigma$ whose objects are the triples $(R, \mathcal{D}_l, \mathcal{D}_r)$ where $\mathcal{D}_r$ and $\mathcal{D}_l$ are Gabriel filters of right and left ideals of $R$ respectively, and whose arrows between $(R, \mathcal{D}_l, \mathcal{D}_r)$ and $(S, \mathcal{H}_l, \mathcal{H}_r)$ are the equivalences between their categories of modules that send the hereditary torsion theories generated by $\mathcal{D}_r$ and $\mathcal{D}_l$ to the hereditary torsion theories generated by $\mathcal{H}_r$ and $\mathcal{H}_l$ respectively. Then we will see that the two-sided localization induces an endofunctor of the bicategory $\mathcal{IB}_\sigma$. This fact will allow us to understand Morita equivalences in a deeper way, identifying for example the bimodules that give rise to the equivalences.

Finally, in section 3 we study the group of the auto-equivalences of an object $(R, \mathcal{D}_r, \mathcal{D}_l)$, written $\text{Pic}(R, \mathcal{D}_l, \mathcal{D}_r)$. This generalizes the concept of Pic($R$), the Picard group of a ring $R$. Indeed, we see that, in the case when $\mathcal{D}_r$ and $\mathcal{D}_l$ are the Gabriel filters of dense right and left ideals of $R$, respectively, we have $\text{Pic}(R, \mathcal{D}_l, \mathcal{D}_r) = \text{Pic}(R)$. For a commutative ring $R$ and a multiplicative set of regular elements $S$ Bass constructed an exact sequence that relates the Picard group of $R$ and the Picard group of its localization $RS^{-1}$ [3] Chapter III, Proposition 7.5]. The machinery developed in the previous chapters will allow us to extend this exact sequence to a general two-sided localization of a non-commutative ring. Thus, not only the commutative localization is extended to cover the case of a two-sided Ore localization, but we also extend the type of localization considered by Bass.

1. Gabriel localizations and two-sided localizations.

Throughout all rings will be associative and unital, and all modules will be unitary. Given any ring $R$ let $(\mathcal{T}, \mathcal{F})$ denote a hereditary torsion theory on the category Mod-$R$ of right $R$-modules (see [12] Chapter IV).

We recall ([12] VI, Proposition 3.6 and Theorem 5.1]) that there is a bijection between hereditary torsion theories of Mod-$R$, Gabriel filters of right ideals of $R$ and left exact radicals of Mod-$R$. So we will talk of Gabriel filters $\mathcal{D}$ rather than hereditary torsion theories. Thus, given a Gabriel filter of right (or left) ideals $\mathcal{D}$ of $R$ we denote by $(\mathcal{T}_\mathcal{D}, \mathcal{F}_\mathcal{D})$ the hereditary torsion theory of Mod-$R$ (or $R$-Mod) associated to $\mathcal{D}$.

Example 1.1. (1) The hereditary torsion theory generated by the dense right ideals $\mathcal{T}_{dr}$ is of particular importance. $\mathcal{T}_{dr}$ is the strongest Gabriel filter for which $R$ is torsion-free.

(2) Let $S \subseteq R$ be a right Ore set of regular elements, then the set of right ideals $\mathcal{D}^S = \{I \mid sR \subseteq I \text{ for some } s \in S\}$ is a Gabriel filter of right ideals of $R$.

Given a right $R$-module $M$ and a Gabriel filter of right ideals $\mathcal{D}$, the module of quotients of $M$ with respect to $\mathcal{D}$ is defined as

$$Q_\mathcal{D}(M) := \lim_{\rightarrow \mathcal{D}} \text{Hom}(I, \frac{M}{\mathcal{T}_\mathcal{D}(M)})$$.
where $T_D(M) = \{ m \in M \mid mI = 0 \text{ for some } I \in D \}$, the left exact radical associated to the Gabriel filter $D$. For every right $R$-module we have that $Q_D(M)$ has structure of right $R$-module, and if $M = R_R$ then $Q_D(R)$ has a ring structure [12 Chapter IX], and is called the right ring of quotients of $R$ with respect to $D$. Additionally to the right $R$-module structure, $Q_D(M)$ becomes a $Q_D(R)$-Module (see [12 Chapter IX]).

The most prominent examples of rings of quotients can be viewed as localizations with respect to some Gabriel filters of right ideals. Indeed, the maximal right ring of quotients is the localization of $R$ with respect to $I_{dr}$ the filter of right dense ideals of $R$, and for every right Ore set of regular elements we have that $RS^{-1}$ is the localization of $R$ with respect to the Gabriel filter of right ideals $D^S$ of $R$.

**Definition 1.2.** Let $R$ and $S$ be any rings, and $H_r$ and $D_l$ Gabriel filters of right and left ideals of $S$ and $R$, respectively. Then we define $D \Omega H_r$, as the set of right ideals of $S \otimes R^{op}$ containing an ideal of the form $H \otimes R^{op} + S \otimes D$ where $H \in H_r$ and $D \in D_l$.

If there is no confusion about which is the right and the left Gabriel filter we denote $D \Omega H_r$ as $\Omega r$. One can easily verify that $\Omega r$ is a Gabriel filter of right ideals of $S \otimes R^{op}$.

Thus given an $R$-$S$-bimodule $rM_S$, or equivalently, a right $S \otimes R^{op}$-module, and $H_r$ and $D_l$ Gabriel filters of right and left ideals of $S$ and $R$, respectively, we define the **two-sided localization of $M$ with respect to $H_r$ and $D_l$** as

$$Q_{D_l, H_r} (M) := \lim_{\rightarrow I \in \Omega r} \text{Hom}(I, M),$$

where $M = \overline{M} = \lim_{\rightarrow I \in \Omega r} \overline{M}$.

A **compatible pair** $(f, g)$ on $M$ consists of an $R$-homomorphism $f : D \rightarrow rM$ and an $S$-homomorphism $g : H \rightarrow M_S$ where $D \in D_l$ and $H \in H_r$, which satisfy the compatibility condition $yg(x) = (x)f y$ for every $x \in D$ and $y \in H$. We put $(f, g) \sim (\overline{f}, \overline{g})$ if and only if there exist $H' \in H_r$ and $D' \in D_l$ such that $f|_{D'} = \overline{f}|_{D'}$ and $g|_{H'} = \overline{g}|_{H'}$. Clearly $\sim$ is an equivalence relation on the set of compatible pairs on $M$.

**Proposition 1.3.** Let $R$ and $S$ be rings and let $rM_S$ be an $R$-$S$-bimodule. Given Gabriel filters $D_l$ and $H_r$ of left and right ideals of $R$ and $S$ respectively, there is a bijection between elements of the two-sided localization $Q_{D_l, H_r} (M)$ and the equivalence classes of compatible pairs $(f, g)$ where $f : I \rightarrow rM$ and $g : J \rightarrow M_S$ for some $I \in D_l$ and $J \in H_r$, with $\overline{M} = \lim_{\rightarrow I \in \Omega r} \overline{M}$.

**Proof.** Let $h : H \otimes R^{op} + S \otimes D \rightarrow \overline{M}$ be a representative of an equivalence class of the two-sided localization of $rM_S$. Given any $x \in D$ and $y \in H$ we define $(x)f := h(1 \otimes x)$ and $g(y) := h(y \otimes 1)$. That $(f, g)$ is a compatible pair follows in a straightforward way. Moreover, observe that every representative of the equivalence class of $h$ gives rise to a pair of compatible morphisms of the same equivalence class.

Conversely, given any compatible pair $(f, g)$ over $rM_S$ we can construct the $S \otimes R^{op}$-homomorphism $h : H \otimes R^{op} + S \otimes D \rightarrow \overline{M}$ such that $h(y \otimes 1) = g(y)$ and $h(1 \otimes x) = (x)f$ for every $y \in H$ and $x \in D$. This is well-defined by the compatibility of the morphisms $f$ and $g$. Indeed, let us construct the $S \otimes R^{op}$ morphisms

$$\alpha : H \otimes R^{op} \rightarrow \overline{M} \quad \text{and} \quad \beta : S \otimes D \rightarrow \overline{M},$$

$$y \otimes r \quad \mapsto \quad rg(y) \quad \text{and} \quad s \otimes x \quad \mapsto \quad (x)f s.$$
If \( \sum y_i \otimes r_i = \sum s_j \otimes x_j \) we claim that \( \alpha(\sum y_i \otimes r_i) = \beta(\sum s_j \otimes x_j) \). Take \( s \in \cap s_j^{-1}H = I \). Then \( s_j s \in H \) for every \( j \) and we have that for every \( r \in R \)

\[
\alpha(\sum y_i \otimes r_i)(s \otimes r) = \alpha((\sum y_i \otimes r_i)(s \otimes r)) = \alpha((\sum s_j \otimes x_j)(s \otimes r)) = \alpha(\sum s_j s \otimes r x_j) = \sum r x_j g(s_j s) = \sum (r x_j f s_j s = \beta(\sum s_j \otimes x_j)(s \otimes r).
\]

Then we have that \( (\alpha(\sum y_i \otimes r_i) - \beta(\sum s_j \otimes x_j))(I \otimes R^{op}) = 0 \) where \( I \in H_r \). Observe that symmetrically we can construct \( \cap D_r^{-1} = J \in D_l \) such that \( (\sum y_i \otimes r_i) - \beta(\sum s_j \otimes x_j))(S \otimes J) = 0 \) and hence \( (\alpha(\sum y_i \otimes r_i) - \beta(\sum s_j \otimes x_j))(S \otimes J + I \otimes R^{op}) = 0 \).

But since \( \overline{M} \) is \( \Omega_r \)-torsion-free, we get \( \alpha(\sum y_i \otimes r_i) - \beta(\sum s_j \otimes x_j) = 0 \) and hence \( \alpha(\sum y_i \otimes r_i) = \beta(\sum s_j \otimes x_j) \). So \( h \) is well-defined.

From the above characterization of \( Q_{D_l, H_r}(M) \) the next lemma follows in a straightforward way.

**Lemma 1.4.** Let \( R \) be a unital ring and let \( D_r \) and \( D_l \) be Gabriel filters of right and left ideals of \( R \). Then \( Q_{D_l, D_r}(R) \) has a ring structure.

Observe, that in the case that \( M = R R R \) with its Gabriel filters of dense right and left ideals, \( I_{dr} \) and \( I_{dl} \) respectively, this coincides with the characterization of the maximal symmetric ring of quotients of \( R \) (see [3] and [9]), i.e. \( Q_{I_{dl}, I_{dr}}(R) = Q_0(R) \).

## 2. Rings with filters of ideals as a bicategory.

We can define a category with objects the triples \( (R, D_l, D_r) \), where \( R \) is a unital ring and \( D_r \) and \( D_l \) are Gabriel filters of right and left ideals of \( R \), respectively, and the morphisms \( \alpha : (R, D_l, D_r) \longrightarrow (S, H_l, H_r) \) are ring morphisms \( \alpha : R \longrightarrow S \) such that \( \alpha(J) \cdot S \in H_r \) and \( S \cdot \alpha(J) \in H_l \) whenever \( J \in D_r \) and \( I \in D_l \). It is proved in [11] that the two-sided localization \( (R, D_l, D_r) \longrightarrow (Q_{D_l, D_r}(R), D_l', D_r') \) defines an endofunctor of this category, where \( D_r' \) is the Gabriel filter on \( Q_{D_l, D_r}(R) \) consisting of left ideals of \( Q_{D_l, D_r}(R) \) containing a left ideal of the form \( Q_{D_l, D_r}(R) \cdot I \), for \( I \in D_l \), and similarly for \( D_r' \).

Now we define the oriented graph \( B_o \) of unital rings with Gabriel filters of right and left ideals whose objects are the triples \( \{(R, D_l, D_r) \mid R_R \in F_{D_r} \text{ and } r_R \in F_{D_l}\} \), and the morphisms between two objects \( (R, D_l, D_r) \) and \( (S, H_l, H_r) \) are the \( R-S \)-bimodules \( R M_S \) such that:

- **(Q1)** \( M_S \) is a \( H_r \)-torsion-free right \( S \)-module.
- **(Q2)** \( R M \) is a \( D_l \)-torsion-free left \( R \)-module.
- **(Q3)** For every \( D \in D_r \) we have \( (\frac{M}{DM})_S \) is a \( H_r \)-torsion module.
- **(Q4)** For every \( H \in H_l \) we have \( R(\frac{M}{MH}) \) is a \( D_l \)-torsion module.

The two-sided localization \( Q : (R, D_l, D_r) \longrightarrow (Q_{D_l, D_r}(R), D_l', D_r') \) defines mappings on the objects of \( B_o \), additionally we will see from the two following lemmas that \( Q \) defines a morphism of graphs.

**Lemma 2.1.** Let \( (R, D_l, D_r) \) and \( (S, H_l, H_r) \) be two objects from \( B_o \) and let \( R M_S \) be a morphism between them, then \( Q_{D_l, H_r}(M) \) is a \( Q_{D_l, D_r}(R)-Q_{H_l, H_r}(S) \)-bimodule.
Proof. Let \( m \in Q_{D_l, \mathcal{H}_r}(M) \), that is represented by the pair of compatible homomorphisms \( f : R D \longrightarrow R M \), \( g : H_S \longrightarrow M_S \), where \( D \in D_l \) and \( H \in \mathcal{H}_r \), and let \( q \in Q_{D_l, D_r}(R) \), that is represented by the pair of compatible homomorphisms \( f_1 : R I \longrightarrow R \), \( g_1 : J_R \longrightarrow R \), where \( I \in D_l \) and \( J \in D_r \).

Now taking the left ideal \( D := I \cap (D) f_1^{-1} \in D_l \), we define the homomorphism \( \overline{f} : R D \longrightarrow R M \) by \((x)\overline{f} := ((x) f_1) f \) for every \( x \in D \).

Now we define the following homomorphism
\[
\tilde{g}_i : J M_S \longrightarrow M_S \\
\sum y_i m_i \longmapsto \sum g_1(y_i)m_i,
\]
that is well-defined since \( R M \) is a \( D_l \)-torsion-free left \( R \)-module.

Note that by property \( Q_3 \) we have that \( (\frac{M}{J M})_S \) is a \( \mathcal{H}_r \)-torsion module. So if we set the right ideal \( H_S := g^{-1}(J M) \in \mathcal{H}_r \) we define the \( S \)-homomorphism \( \overline{g} : H_S \longrightarrow M_S \) as \( \overline{g}(y) := \tilde{g}_i(g(y)) \) for every \( y \in H \).

Now we are going to see that the homomorphisms \( \overline{f} \) and \( \overline{g} \) define a compatible pair \((\overline{f}, \overline{g})\).

So given any \( x \in D \) and \( y \in H \) we have \( g(y) = \sum z_i m_i \) for some \( z_i \in J \) and \( m_i \in M \), so
\[
x \overline{g}(y) = x \tilde{g}_i(g(y)) = x \tilde{g}_i(\sum z_i m_i) = x \sum g_1(z_i)m_i =
\]
\[
= (x) f_1 \sum z_i m_i = (x) f_1 g(y) = ((x) f_1) f y = (x) \overline{f} y.
\]

Thus, the pair \((\overline{f}, \overline{g}) =: q \cdot m \) defines an element of \( Q_{D_l, \mathcal{H}_r}(M) \), so we can define the map \( Q_{D_l, \mathcal{H}_r}(R) \times Q_{D_l, \mathcal{H}_r}(M) \longrightarrow Q_{D_l, \mathcal{H}_r}(M), (q, m) \mapsto q \cdot m \), that we can easily verify that induces a structure of left \( Q_{D_l, \mathcal{H}_r}(R) \)-module on \( Q_{D_l, \mathcal{H}_r}(M) \).

Symmetrically we might see that \( Q_{D_l, \mathcal{H}_r}(M) \) is a right \( Q_{\mathcal{H}_l, \mathcal{H}_r}(S) \)-module, and that is compatible with the left \( Q_{D_l, D_r}(R) \)-module structure. \( \square \)

Lemma 2.2. Let \((R, D_l, D_r) \) and \((S, \mathcal{H}_l, \mathcal{H}_r) \) be two objects from \( \mathcal{B}_s \) and let \( R M S \) be a morphism between them, then \( Q_{D_l, \mathcal{H}_r}(M) \) defines a morphism between \((Q_{D_l, D_r}(R), D_l', D_r') \) and \((Q_{\mathcal{H}_l, \mathcal{H}_r}(S), \mathcal{H}_l', \mathcal{H}_r') \).

Proof. We have seen above (Lemma 2.1) that \( Q_{D_l, \mathcal{H}_r}(M) \) is a \( Q_{D_l, D_r}(R) \)-\( Q_{\mathcal{H}_l, \mathcal{H}_r}(S) \)-bimodule, so we have to check that \( Q_{D_l, \mathcal{H}_r}(M) \) verifies conditions \( Q_1 - Q_4 \). First we are going to see that \( Q_{D_l, \mathcal{H}_r}(M) \) is a \( \mathcal{H}_r \)-torsion-free right \( Q_{\mathcal{H}_l, \mathcal{H}_r}(S) \)-module. Indeed, let \( m \in Q_{D_l, \mathcal{H}_r}(M) \) such that \( mH' = 0 \) for some \( H' \in \mathcal{H}_r \). Since \( H' \) contains an ideal of the form \( H \cdot Q_{\mathcal{H}_l, \mathcal{H}_r}(S) \) for some \( H \in \mathcal{H}_r \) we have that \( mH = 0 \). Now there exists a pair of compatible homomorphisms \( f : D \longrightarrow M \) and \( g : K \longrightarrow M \) where \( D \in D_l \) and \( K \in \mathcal{H}_r \) with \( K \subseteq H \), such that \( mk = g(k) \) and \( dm = (d)f \) for every \( d \in D \) and \( k \in K \). Thus, for every \( d \in D \) and \( k \in K \) we have that \( 0 = dg(k) = (d)f k \), hence since \( M \) is a \( \mathcal{H}_r \)-torsion-free right \( S \)-module we get that \( f = 0 \) and \( m = 0 \), as desired. Similarly we can prove \( Q_2 \).

Now we shall see that \( Q_{D_l, \mathcal{H}_r}(M) \) verifies \( Q_3 \). Indeed, we must check that \( \frac{Q_{D_l, \mathcal{H}_r}(M)}{D'_r Q_{D_l, \mathcal{H}_r}(M)} \) is a \( \mathcal{H}_r \)-torsion right \( Q_{\mathcal{H}_l, \mathcal{H}_r}(S) \)-module for every \( D' \in D_r' \). Let \( m \in Q_{D_l, \mathcal{H}_r}(M) \), since \( \frac{Q_{D_l, \mathcal{H}_r}(M)}{M} \) is a \( \mathcal{H}_r \)-torsion \( S \otimes R^{\text{op}} \)-module there exist \( I \in D_l \) and \( J \in \mathcal{H}_r \) with \( Im, mJ \subseteq M \). Now let \( D \in D_l \) with \( D \cdot Q_{D_l, D_r}(R) \subseteq D' \), since \( M \) satisfies \( Q_3 \) for every \( y \in J \) there exists \( H_y \in \mathcal{H}_r \) such that \( myH_y \subseteq DM \). Now we set \( K := \sum_{y \in J} y H_y \) that belongs to \( \mathcal{H}_r \) by a property of the Gabriel filters (namely property T4 in [12] page 146]). Thus, we have that
Thus, we have that

\[ Q : \mathcal{B}_\sigma \rightarrow \mathcal{B}_\sigma \]

defines a functor on \( \mathcal{B}_\sigma \).

We would like to equip the graph \( \mathcal{B}_\sigma \) of rings with Gabriel filters of right and left ideals with a richer structure than a category. So we recall the well-known definition of bicategory ([2] and [4]).

**Definition 2.3.** A bicategory \( \mathcal{B} \) consists of the following structures:

1. A class of objects \( A, B, C, \ldots \) called 0-cells;
2. For each pair of 0-cells \( A \) and \( B \), a small category \( \mathcal{B}(A, B) \) which objects \( \mathcal{B}^1(A, B) \) are called 1-cells, and which morphisms are called 2-cells.
3. For each triple \( A, B \) and \( C \) of objects (0-cells), there is a composition law given by a functor \( \otimes_{A,B,C} : \mathcal{B}(A, B) \times \mathcal{B}(B, C) \rightarrow \mathcal{B}(A, C) \).
4. For each object \( A \), there is an identity object functor \( \text{Id}_{\mathcal{B}(A,A)} \) where \( 1 \) stands for the final object in the category of small categories, such that:
   a. For every 0-cells \( A, B, C, D \) we have natural associativity isomorphisms
      \[ (((- \otimes_{A,B,C} -) \otimes_{A,C,D} -) \simeq (- \otimes_{A,B,D} (- \otimes_{B,C,D} -))) \].
   b. For every \( A \) and \( B \) we have natural unity isomorphisms \( (- \otimes \text{Id}_B) \simeq (\text{Id}_A \otimes -) \simeq \text{Id}_{\mathcal{B}(A,B)} \).

The above isomorphisms must satisfy some coherence properties (for details see [4, 7.7]).

**Example 2.4.** Our starting point will be the bicategory of rings and bimodules \( \mathcal{B}im \):

(0-cells) Rings \( R, S \),

(1-cells) Given \( R \) and \( S \) then \( \mathcal{B}(R,S) = R-S\text{-Bimod} \).

(2-cells) Given \( R M_S \) and \( R N_S \) the 2-cells are the \( R-S\text{-bimodule homomorphisms} \).

The composition law is given by the tensor product, and given any ring \( R \) its identity object is the regular \( R-R\text{-bimodule} \).

One could be tempted to define a bicategorical structure on \( \mathcal{B}_\sigma \) using tensor product as composition of 1-cells. Unfortunately, conditions \( Q1-Q4 \) are not stable under tensor product. To correct this default, we have to restrict our attention to a smaller class of bimodules.

It is known [3, Chapter II] that every equivalence \( T : \text{Mod}-R \rightarrow \text{Mod}-S \) is determined by an invertible \( R-S\text{-bimodule} \) \( R M_S \), i.e. \( T \simeq - \otimes_R M =: T_M \) (see [1] for further information about invertible bimodules). Thus, there exists a Morita context \( \alpha : M \otimes M^* \rightarrow R \) and \( \beta : M^* \otimes_R M \rightarrow S \), with \( \alpha \) and \( \beta \) associative bimodule isomorphisms (see [3, Chapter II, Proposition 3.1]), where \( M^* = \text{Hom}_R(M,R) \) is the dual \( R \)-module which is indeed an \( S-R\text{-bimodule} \).

Observe that the equivalence of the categories of left modules is given by \( U_M := M \otimes_S - \). Let us denote by \( \mathcal{B}im(R,S) \) the class of the invertible \( R-S\text{-bimodules} \), that corresponds to the invertible 1-cells of the bicategory \( \mathcal{B}im \).
It is known [12, Chapter X §3] that every natural equivalence \( T_M \) induces a bijective correspondence between the Gabriel filters of right ideals of \( R \) and the Gabriel filters of right ideals of \( S \), denoted by \( T^g_M \). Observe that for every invertible \( R\)-\( S \)-bimodule \( M \) and Gabriel filters \( D_r \) and \( H_r \) of right ideals of \( R \) and \( S \), respectively, we have that \( T^g_M(D_r) = H_r \) is equivalent to \( T_M(D_r) = H_r \), and \( T^g_M(F_{D_r}) = F_{H_r} \).

Now we define the bicategory \( \mathcal{IB}_\sigma \) whose 0-cells are the triples \( \{(R, D_l, D_r) \mid R_R \in F_{D_l} \text{ and } R_R \in F_{D_r}\} \), and given two 0-cells \( (R, D_l, D_r) \) and \( (S, H_l, H_r) \) we set the category \( \mathcal{IB}_\sigma(\{(R, D_l, D_r), (S, H_l, H_r)\}) \) which objects (1-cells) are the \( R\)-\( S \)-bimodules \( M \) that satisfy

\[(Q) \text{ } R_M^S \text{ is an invertible } R\text{-}\text{bimodule such that } T^g_M(D_r) = H_r \text{ and } U^g_M(H_l) = D_l, \]

and with the \( R\)-\( S \)-bimodules homomorphism as morphisms (2-cells). Observe that bimodules that satisfy condition \( (Q) \) automatically satisfy conditions \((Q1)-(Q4)\).

In this case, it is easy to check that the tensor product of bimodules acts as composition law in the 1-cells of \( \mathcal{IB}_\sigma \), thus \( \mathcal{IB}_\sigma \) is a bicategory.

Now we would like to prove that the mapping \( Q \) restricted to \( \mathcal{IB}_\sigma \) induces a functor of bicategories.

**Remark 2.5.** For every invertible \( R\)-\( S \)-bimodule \( M \) there exist some \( \tilde{m}_1, \ldots, \tilde{m}_r \in M^* \) such that \( 1_R = \sum_j \alpha(\tilde{m}_j \otimes \tilde{n}_j) \).

**Proposition 2.6.** Let \( (R, D_l, D_r) \) and \( (S, H_l, H_r) \) be two 0-cells of \( \mathcal{IB}_\sigma \), then for every 1-cell \( rM_S \) between them, the maps

\[\varphi : M \otimes_S Q_{H_l,H_r}(S) \rightarrow Q_{D_l,H_r}(M) \quad \text{and} \quad \phi : Q_{D_l,D_r}(R) \otimes_R M \rightarrow Q_{D_l,H_r}(M) \]

are \( R\)-\( S \)-bimodule isomorphisms.

**Proof.** Let \( \alpha : M \otimes M^* \rightarrow R \) and \( \beta : M^* \otimes M \rightarrow S \) be a Morita context, with \( \alpha \) and \( \beta \) bimodule isomorphisms. First we are going to prove that \( \varphi \) is injective. Suppose that there exists \( \sum m_i \otimes q_i \in M \otimes S \) such that \( \varphi(\sum m_i \otimes q_i) = \sum m_i q_i = 0 \), hence using Remark 2.5 we have

\[
\sum_i m_i \otimes q_i = \sum_i 1_R \cdot m_i \otimes q_i = \sum_i (\sum_j \alpha(\tilde{m}_j \otimes \tilde{n}_j)m_i) \otimes q_i = \\
= \sum_i (\sum_j \tilde{m}_j \beta(\tilde{n}_j \otimes m_i)) \otimes q_i = \sum_j (\tilde{m}_j \otimes \sum_i \beta(\tilde{n}_j \otimes m_i)q_i) .
\]

There exists \( H \in H_r \) such that \( q_i H \subseteq S \) for every \( i \), and \( \sum_i \beta(\tilde{n}_j \otimes m_i)q_i = 0 \) for every \( h \in H \). But since \( Q_{H_l,H_r}(S) \) is \( H_r \)-torsion-free it yields that \( \sum_i \beta(\tilde{n}_j \otimes m_i)q_i = 0 \) and hence \( \sum_i m_i \otimes q_i = 0 \). Thus, \( \varphi \) is injective.

Now we are going to see that \( \varphi \) is also surjective. Indeed, first we will prove that we can extend \( \beta \) to \( \overline{\beta} : M^* \otimes_R Q_{D_l,H_r}(M) \rightarrow Q_{H_l,H_r}(S) \) as follows. Let us consider any \( n \in N \) and \( m \in Q_{D_l,H_r}(M) \), that is represented by the compatible pair \( f : D \rightarrow R_M \) and \( g : H \rightarrow M_S \) where \( D \in D_l \) and \( H \in H_r \). We define the \( S \)-homomorphism \( \overline{\beta} : H \rightarrow S \) as \( \overline{\beta}(h) := \beta(n \otimes g(h)) \). Now, we define the following \( S \)-homomorphism

\[f : S M^* \cdot D \rightarrow S \quad \sum n_id_i = \sum \beta(n_i \otimes (d_i) f) .\]
We have that \( \tilde{f} \) is well-defined since \( S \) is \( \mathcal{H}_r \)-torsion-free. Now if we consider the \( S \)-homomorphism \( R_n : S \rightarrow M^* \), that is right multiplication by \( n \), we have that \( K := R_n^{-1}(M^* D) \in \mathcal{H}_l \) since \( S(M^* D) \) is a \( \mathcal{H}_l \)-torsion module. So we can define the \( S \)-homomorphism \( \bar{f} : K \rightarrow S \) as \((k)\bar{f} := (kn)\bar{f} \). Now we shall see that the pair \((\bar{f}, \bar{g})\) is a compatible one. Indeed, let \( k \in K \) and \( h \in H \), then

\[
(k)\bar{f}h = (kn)\bar{f}h = (\sum n_id_i)\bar{f}h = \sum \beta(n_i \otimes (d_i)f)h = \sum \beta(n_i \otimes (d_i)f)h
\]

Thus, the pair \((\bar{f}, \bar{g})\) defines an element of \( Q_{\mathcal{H}_l, \mathcal{H}_r}(S) \). Moreover it is clear that \( \bar{f}_{|M^* \otimes M} = \beta \), and that we have the following compatibility \( \alpha(n)\bar{m} = m\bar{\beta}(n)\bar{m} \) for every \( n \in N \), \( m \in M \) and \( \bar{m} \in Q_{\mathcal{D}_l, \mathcal{H}_r}(M) \).

Now using that we can extend \( \beta \) to \( \bar{\beta} \) we shall prove that \( \phi \) is surjective. Indeed, let \( \bar{m} \) be any element from \( Q_{\mathcal{D}_l, \mathcal{H}_r}(M) \), we have that \( \bar{m} = 1_r\bar{m} = \sum_i \alpha(\bar{m}_i \otimes \bar{n}_i)\bar{m} = \sum_i \bar{m}_i\bar{\beta}(\bar{n}_i \otimes \bar{m}) \). So it follows that \( \varphi(\sum_i \bar{m}_i \otimes \bar{\beta}(\bar{n}_i \otimes \bar{m})) = \bar{m} \), so we have that \( \varphi \) is a surjective homomorphism. Thus, we get \( M \otimes Q_{\mathcal{H}_l, \mathcal{H}_r}(S) \cong Q_{\mathcal{D}_l, \mathcal{H}_r}(M) \). A symmetric argument gives the other isomorphism. \( \square \)

Now with this setting, given the 0-cells \((R, \mathcal{D}_l, \mathcal{D}_r), (S, \mathcal{H}_l, \mathcal{H}_r) \) and \((T, \mathcal{K}_l, \mathcal{K}_r) \) from \( IB_{\sigma} \) and any 1-cells \( R, M_S \in IB_{\sigma}((R, \mathcal{D}_l, \mathcal{D}_r), (S, \mathcal{H}_l, \mathcal{H}_r)) \) and \( S, N_T \in IB_{\sigma}((S, \mathcal{H}_l, \mathcal{H}_r), (T, \mathcal{K}_l, \mathcal{K}_r)) \) it follows from Proposition 2.6 that we have the following commutative diagram of isomorphisms:

\[
\begin{array}{ccc}
(M \otimes_S N) \otimes_T Q(T) & \xrightarrow{\varphi} & Q_{\mathcal{D}_l, \mathcal{K}_r}(M \otimes_S N) \\
\cong & & \cong \\
M \otimes_S Q_{\mathcal{H}_l, \mathcal{K}_r}(N) & \xrightarrow{h} & Q_{\mathcal{D}_l, \mathcal{H}_r}(M) \otimes_S N \\
\cong & & \cong \\
M \otimes_S Q(S) \otimes_{Q(S)} Q_{\mathcal{H}_l, \mathcal{K}_r}(N) & \xrightarrow{\cong} & Q_{\mathcal{D}_l, \mathcal{H}_r}(M) \otimes_{Q(S)} Q(S) \otimes_S N
\end{array}
\]

**Corollary 2.7.** \( Q \) defines a functor in the bicategory \( IB_{\sigma} \). Therefore, if \( T_M : \text{Mod}-R \rightarrow \text{Mod}-S \) is an equivalence then \( Q_{\mathcal{D}_l, \mathcal{D}_r}(R) \) and \( Q_{\mathcal{D}_l, \mathcal{D}_r}(D') \) are Morita equivalent rings.

**Proof.** We define the functor \( Q \) by

\[
Q : IB_{\sigma} \rightarrow IB_{\sigma} \quad (R, \mathcal{D}_l, \mathcal{D}_r) \mapsto (Q_{\mathcal{D}_l, \mathcal{D}_r}(R), \mathcal{D}'_l, \mathcal{D}'_r)
\]

for \( M \in IB_{\sigma}((R, \mathcal{D}_l, \mathcal{D}_r), (S, \mathcal{H}_l, \mathcal{H}_r)) \), where \( \mathcal{D}'_r = \{ I_{Q(R)} | D \cdot Q_{\mathcal{D}_l, \mathcal{D}_r}(R) \subseteq I \text{ for some } D \in \mathcal{D}_r \} \) and \( \mathcal{D}'_l = \{ Q(R)J | Q_{\mathcal{D}_l, \mathcal{D}_r}(R) \cdot D \subseteq J \text{ for some } D \in \mathcal{D}_l \} \). The mapping between the 1-cells is well-defined by Lemma 2.2 and Proposition 2.6 and the above commutative diagram shows that \( Q \) commutes with the tensor product. \( \square \)
We have that the Gabriel filter of the dense right ideals of $R$ is the biggest filter with respect to which the regular right $R$-module $R_R$ is torsion-free, thus since an equivalence of modules induces a bijection between the torsion theories of both categories the following result seems natural:

**Lemma 2.8.** (c.f. [12], Chapter X, Proposition 3.2) Let $T : \text{Mod}-R \rightarrow \text{Mod}-S$ be an equivalence, and let $\mathcal{I}_{dr}(R)$ be the Gabriel filter of dense right ideals of $R$. Then $T^\sharp(\mathcal{I}_{dr}(R)) = \mathcal{I}_{dr}(S)$ is the Gabriel filter of dense right ideals of $S$.

**Corollary 2.9.** If $R$ and $S$ are two Morita equivalent rings, then $Q_\sigma(R)$ and $Q_\sigma(S)$ are Morita equivalent rings too.

### 3. The Picard group.

Given a bicategory $\mathcal{B}$ we define its large Picard groupoid, written as $\text{Pic}(\mathcal{B})$, as the groupoid of isomorphism classes of invertible arrows of $\mathcal{B}$ (see [2] for details). In the bicategory $\mathcal{B}_{\text{im}}$ of rings and bimodules the invertible arrows are the invertible bimodules. So $\text{Pic}(\mathcal{B}_{\text{im}})$ is the groupoid of all the isomorphism classes $[T]$ of natural equivalences $T : R\text{-Mod} \rightarrow S\text{-Mod}$ and the multiplication is given by composition of the functors, or equivalently by the tensor product of invertible bimodules.

The orbits of $\text{Pic}(\mathcal{B}_{\text{im}})$ are the Morita equivalence classes, and given a ring $R$ the isotropy group of the autoequivalences of $R\text{-Mod}$ is the Picard group of $R$, written as $\text{Pic}(R)$. The group law is induced by the composition of functors. As we saw before, we can see $\text{Pic}(R)$ as the group of isomorphisms classes $[R^{\text{im}}]$ of invertible $R$-$R$-bimodules. The group law is given by $[M][N] = [M \otimes N]$, and with $[M]^{-1} = [M^\ast]$.

Let us consider the bicategory $\mathcal{I}_{\text{B}_{\text{im}}}$ defined previously, then given a 0-cell $(R, \mathcal{D}_l, \mathcal{D}_r)$ we denote its isotropy group as $\text{Pic}(R, \mathcal{D}_l, \mathcal{D}_r)$. Observe that by Lemma 2.8 we have that $\text{Pic}(R, \mathcal{D}_l, \mathcal{D}_r) = \text{Pic}(R)$ when $\mathcal{D}_r$ and $\mathcal{D}_l$ are the filters of the dense right and left ideals respectively.

Moreover, by Corollary 2.7 we have the following group homomorphism

\[
Q : \text{Pic}(R, \mathcal{D}_l, \mathcal{D}_r) \rightarrow \text{Pic}(Q_{\mathcal{D}_l,\mathcal{D}_r}(R), \mathcal{D}_l', \mathcal{D}_r') \quad \quad [M] \mapsto [Q_{\mathcal{D}_l,\mathcal{D}_r}(M)].
\]

Observe that in particular we have the following group homomorphism

\[
Q : \text{Pic}(R) \rightarrow \text{Pic}(Q_\sigma(R)) \quad \quad [M] \mapsto [M \otimes_R Q_\sigma(R)] = [Q_\sigma(R) \otimes_R M].
\]

Now given $(R, \mathcal{D}_l, \mathcal{D}_r)$ and any $R$-$R$-subbimodule $M$ of $Q_{\mathcal{D}_l,\mathcal{D}_r}(R)$ we define $M^{-1} := \{n \in Q_{\mathcal{D}_l,\mathcal{D}_r}(R) \mid nM, Mn \subseteq R\}$. Observe that $M^{-1}$ is also a $R$-$R$-bimodule. Moreover we can define natural associative $R$-$R$-bimodule homomorphisms $\alpha : M \otimes_R M^{-1} \rightarrow R$ and $\beta : M^{-1} \otimes_R M \rightarrow R$.

Thus, we define the group $\text{Pic}(R \mid Q_{\mathcal{D}_l,\mathcal{D}_r}(R))$ of $R$-$R$-subbimodules $M$ of $Q_{\mathcal{D}_l,\mathcal{D}_r}(R)$ such that $M^{-1}M = M^{-1}M = R$ (the invertible in $Q_{\mathcal{D}_l,\mathcal{D}_r}(R)$) , and such that $T^\sharp_M(\mathcal{D}_l) = \mathcal{D}_r$ and $U^\sharp_M(\mathcal{D}_l) = \mathcal{D}_l$. Observe that $\text{Pic}(R \mid Q_{\mathcal{D}_l,\mathcal{D}_r}(R))$ is a group with unit $R$, with operation given by the product $M_1 \cdot M_2$ that by the following result is isomorphic to $M_1 \otimes_R M_2$.

**Proposition 3.1.** Let $M$ and $N$ be $R$-$R$-bimodules of $Q_{\mathcal{D}_l,\mathcal{D}_r}(R)$ with $M \in \text{Pic}(R \mid Q_{\mathcal{D}_l,\mathcal{D}_r}(R))$. Then $M \otimes_R N \cong MN$. 

Remark 3.2. If $\mathcal{I}_{dl}$ and $\mathcal{I}_{dr}$ are the Gabriel filters of dense right and left ideals of $R$ respectively, then by Lemma 2.8, $\text{Pic}(R \mid Q_{\mathcal{I}_{dl},\mathcal{I}_{dr}}(R)) = \text{Pic}(R \mid Q_{\sigma}(R))$ is the set of $R$-$R$-subbimodules $M$ of $Q_{\sigma}(R)$ with $MM^{-1} = M^{-1}M = R$.

Given a ring $R$ we denote by $\mathcal{U}(\mathcal{Z}(R))$ the set of all central invertible elements of $R$.

**Theorem 3.3.** Let $(R, \mathcal{D}_l, \mathcal{D}_r)$ be a $0$-cell in $\mathcal{IB}_\sigma$. Then the sequence

$$0 \rightarrow \mathcal{U}(\mathcal{Z}(R)) \rightarrow \mathcal{U}(\mathcal{Z}(Q_{\mathcal{D}_l,\mathcal{D}_r}(R))) \rightarrow \text{Pic}(R \mid Q_{\mathcal{D}_l,\mathcal{D}_r}(R)) \rightarrow \text{Pic}(R, \mathcal{D}_l, \mathcal{D}_r)$$

is exact.

**Proof.** It is easy to check that $\mathcal{Z}(R) \subseteq \mathcal{Z}(Q_{\mathcal{D}_l,\mathcal{D}_r}(R))$ using that $R$ is torsion-free with respect to $\mathcal{D}_r$ and $\mathcal{D}_l$. It is clear that the kernel of

$$\varphi_1 : \mathcal{U}(\mathcal{Z}(Q_{\mathcal{D}_l,\mathcal{D}_r}(R))) \rightarrow \text{Pic}(R \mid Q_{\mathcal{D}_l,\mathcal{D}_r}(R))$$

$$x \mapsto xR$$

is $\mathcal{U}(\mathcal{Z}(R))$.

Now, we claim that the kernel of

$$\varphi_2 : \text{Pic}(R \mid Q_{\mathcal{D}_l,\mathcal{D}_r}(R)) \rightarrow \text{Pic}(R, \mathcal{D}_l, \mathcal{D}_r)$$

$$M \mapsto [M]$$

consists of the $R$-$R$-subbimodules of $Q_{\mathcal{D}_l,\mathcal{D}_r}(R)$ of the form $xR$ with $x \in \mathcal{U}(\mathcal{Z}(Q_{\mathcal{D}_l,\mathcal{D}_r}(R)))$. Indeed, $[M] = [R]$ implies that there exists a $R$-$R$-bimodule isomorphism $f : R \rightarrow M$, and in this case $f$ can be represented by multiplication by an element $x := f(1)$. For every $a \in R$ we have that $xa = f(1)a = f(a) = af(1) = ax$ so $x$ commutes with every element of $R$ and hence $x \in \mathcal{Z}(Q_{\mathcal{D}_l,\mathcal{D}_r}(R))$ and we get $M = xR$. Since $M$ is invertible and $x \in \mathcal{Z}(Q_{\mathcal{D}_l,\mathcal{D}_r}(R))$ there is $y \in M^{-1}$ such that $xy = 1$, so $x \in \mathcal{U}(\mathcal{Z}(Q_{\mathcal{D}_l,\mathcal{D}_r}(R)))$.

Now, we are going to see that the kernel of

$$\varphi_3 : \text{Pic}(R, \mathcal{D}_l, \mathcal{D}_r) \rightarrow \text{Pic}(Q_{\mathcal{D}_l,\mathcal{D}_r}(R), \mathcal{D}_l', \mathcal{D}_r')$$

$$[M] \mapsto [M \otimes Q_{\mathcal{D}_l,\mathcal{D}_r}(R)] = [Q_{\mathcal{D}_l,\mathcal{D}_r}(M)]$$

are those classes $[M]$ of $R$-$R$-bimodules $M$ isomorphic to some $\overline{M} \in \text{Pic}(R \mid Q_{\mathcal{D}_l,\mathcal{D}_r}(R))$.

Let $M$ be a $R$-$R$-bimodule such that $\varphi : Q_{\mathcal{D}_l,\mathcal{D}_r}(M) \rightarrow Q_{\mathcal{D}_l,\mathcal{D}_r}(R)$ is a $Q_{\mathcal{D}_l,\mathcal{D}_r}(R)$-$Q_{\mathcal{D}_l,\mathcal{D}_r}(R)$-bimodule isomorphism. Since $M$ is a $\Omega_{\mathcal{D}_r}$-torsion-free $R$-$R$-bimodule we have that $M$ is a $R$-$R$-subbimodule of $Q_{\mathcal{D}_l,\mathcal{D}_r}(M)$. Thus, $M$ is isomorphic to the $R$-$R$-subbimodule $\overline{M} := \varphi(M)$ of $Q_{\mathcal{D}_l,\mathcal{D}_r}(R)$. We have to see that $\overline{M}$ is invertible in $Q_{\mathcal{D}_l,\mathcal{D}_r}(R)$. Indeed, let $N$ be the inverse $R$-$R$-bimodule of $M$, and let $\alpha : M \otimes_R N \rightarrow R$ and $\beta : N \otimes_R M \rightarrow R$ be $R$-$R$-bimodule isomorphisms that satisfy the associativity property. Using isomorphisms of Proposition 2.6 we can extend $\alpha$ and $\beta$ to isomorphisms

$$\overline{\alpha} : Q_{\mathcal{D}_l,\mathcal{D}_r}(M) \otimes_R N \xrightarrow{\cong} Q_{\mathcal{D}_l,\mathcal{D}_r}(R) \otimes_R M \otimes_R N \xrightarrow{\text{Id} \otimes \alpha} Q_{\mathcal{D}_l,\mathcal{D}_r}(R) \otimes_R R \cong Q_{\mathcal{D}_l,\mathcal{D}_r}(R),$$
and
\[ \overline{\beta} : N \otimes_R Q_{D_1, D_1}(M) \xrightarrow{\cong} N \otimes_R M \otimes_R Q_{D_1, D_1}(R) \xrightarrow{\beta \otimes \text{Id}} R \otimes_R Q_{D_1, D_1}(R) \cong Q_{D_1, D_1}(R). \]

Then applying the functor \( N \otimes_R - \) and then \( \overline{\beta} \) to the homomorphism
\[ R \xrightarrow{i} Q_{D_1, D_1}(R) \xrightarrow{\varphi^{-1}} Q_{D_1, D_1}(M), \]
we get the homomorphism \( \phi_1 \) given by the composition
\[ N \xrightarrow{\cong} N \otimes_R R \xrightarrow{\text{Id} \otimes i} N \otimes_R Q_{D_1, D_1}(R) \xrightarrow{\text{Id} \otimes \varphi^{-1}} N \otimes_R Q_{D_1, D_1}(M) \xrightarrow{\overline{\beta}} Q_{D_1, D_1}(R). \]

Observe that since \( N_R \) is a finitely generated projective right \( R \)-module we have that \( \phi_1 \) is a \( R \)-\( R \)-bimodule monomorphism, thus we have that \( N \) is isomorphic to the \( R \)-\( R \)-subbimodule \( N_1 := \phi_1(N) \) of \( Q_{D_1, D_1}(R) \). Symmetrically applying the functor \( - \otimes_R N \) and \( \pi \) we construct the \( R \)-\( R \)-bimodule monomorphism \( \phi_2 \), so we get that \( N \) is isomorphic to the \( R \)-\( R \)-subbimodule \( N_2 := \phi_2(N) \) of \( Q_{D_1, D_1}(R) \). Now we claim that \( N_1 \overline{M} = R. \) Indeed, let \( \overline{m} \in \overline{M} \) and \( \overline{m} \in N_1 \), we have that there exist \( m \in M \) and \( n \in N \) such that \( \varphi(m) = \overline{m} \) and \( \phi_1(n) = \overline{n} \). Now we have that
\[ \overline{n} \cdot \overline{m} = \phi_1(n) \varphi(m) = \overline{\beta}(n \otimes \varphi^{-1}(1)) \varphi(m) = \overline{\beta}(n \otimes \varphi^{-1}(1)) \varphi(m) = \overline{\beta}(n \otimes \varphi^{-1}(\varphi(m))) = \overline{\beta}(n \otimes m) = \beta(n \otimes m) \in R. \]

Now, since there exist \( n_i \in N \) and \( m_i \in M \) such that \( \sum \beta(n_i \otimes m_i) = 1 \) we have that \( \sum \phi_1(n_i) \varphi(m_i) = 1 \) and hence \( N_1 \overline{M} = R \) as desired. A symmetric argument shows that \( M N_2 = R. \) Thus, we have that \( N_1 = N_1 R = N_1(\overline{M} N_2) = (N_1 \overline{M}) N_2 = RN_2 = N_2 \) by the associativity of the ring \( Q_{D_1, D_1}(R) \). So \( \overline{M} \) is invertible in \( Q_{D_1, D_1}(R) \) as desired.

On the other hand it is clear that if \( M \in \text{Pic}(R \mid Q_{D_1, D_1}(R)) \) then \( M \otimes_R Q_{D_1, D_1}(R) \cong Q_{D_1, D_1}(R) \). Indeed, we can define the \( R \)-\( Q_{D_1, D_1}(R) \)-bimodule isomorphism
\[ \varphi : M \otimes_R Q_{D_1, D_1}(R) \longrightarrow Q_{D_1, D_1}(R), \quad \sum m_i \otimes q_i \longmapsto \sum m_i q_i. \]

\[ \square \]

**Corollary 3.4.** Let \( R \) be a unital ring, then the sequence
\[ 0 \longrightarrow \mathcal{U}(\mathcal{Z}(R)) \longrightarrow \mathcal{U}(\mathcal{Z}(Q_\sigma(R))) \longrightarrow \text{Pic}(R \mid Q_\sigma(R)) \longrightarrow \text{Pic}(R) \longrightarrow \text{Pic}(Q_\sigma(R)) \]
is exact.

Now we are going to study what happens with the Picard groups of a two-sided Ore localization. Let \( S \) be a two-sided Ore set of regular elements of \( R \). Let us consider the ring of fractions \( RS^{-1} = S^{-1}R \), that is the localization of \( R \) with respect to \( D_1^S \) and \( D_2^S \) the Gabriel filters of right and left ideals of \( R \) that have nonempty intersection with \( S \) respectively.

**Corollary 3.5.** Let \( R \) be a unital ring and let \( S \) be a two-sided Ore set of regular elements of \( R \), then the sequence
\[ 0 \longrightarrow \mathcal{U}(\mathcal{Z}(R)) \longrightarrow \mathcal{U}(\mathcal{Z}(RS^{-1})) \longrightarrow \text{Pic}(R \mid RS^{-1}) \longrightarrow \text{Pic}(R, D_1^S, D_2^S) \longrightarrow \text{Pic}(RS^{-1}) \]
is exact, where \( \text{Pic}(R \mid RS^{-1}) \) is the set of all the \( R \)-\( R \)-subbimodules \( M \) of \( RS^{-1} \) such that \( MM^{-1} = M^{-1}M = R \).
Proof. We apply Theorem 3.3 to get the desired exact sequence. We claim that for every invertible $R$-$R$-subbimodule $M$ in $RS^{-1}$ we have that $T_M^s(D^s_r) = D^s_r$ and $U_M^s(D^s_l) = D^s_l$. Indeed, we only have to check that $T_M(R_{SR}) = \frac{R}{sM} \otimes_R M \cong \frac{M}{sM}$ is a $D^s_r$-torsion right $R$-module for every $s \in S$. So let $s \in S$, for every $m \in M$ we have to find $t \in S$ with $mt \in sM$. We can write $m = au^{-1}$ with $a \in R$ and $u \in S$, and there exist $b \in R$ and $w \in S$ such that $aw = sb$, so

$$m(uw) = (au^{-1})(uw) = aw = sb,$$

with $uw \in S$, but $b \in R$. Since $MM^{-1} = R$ there exist $m_i \in M$ and $n_i \in M^1$ for $i \in \{1, \ldots, k\}$ with $b = \sum m_i n_i$. Now we can write $n_i = c_i z_i^{-1}$ with $c_i \in R$ and $z_i \in S$ for every $i \in \{1, \ldots, k\}$. We can find $z \in S$ and $c_i \in R$ such that $n_i = c_i z_i^{-1}$ for every $i \in \{1, \ldots, k\}$. So $bz = \sum m_i n_i z = \sum m_i (c_i z_i^{-1} z) = \sum m_i c_i \in M$, and hence

$$m(uwz) = sbz = s(\sum m_i c_i)$$

with $uwz \in S$ and $\sum m_i c_i \in M$, as desired.

Thus, $T_M(R_{SR})$ is a $D^s_r$-torsion right $R$-module for every $s \in S$, so $T_M^s(D^s_r) \subseteq D^s_r$, and applying the same argument with $M^{-1}$ in place of $M$ we get the reverse inclusion, so that $T_M^s(D^s_r) = D^s_r$. Symmetrically we have that $U_M^s(D^s_l) = D^s_l$.

Finally, observe that since the induced filters are $(D^s_r)' = \{RS^{-1}\}$ and $(D^s_l)' = \{RS^{-1}\}$ we have that $\text{Pic}(RS^{-1}, (D^s_r)'), (D^s_l)' = \text{Pic}(RS^{-1})$. \hfill \qedsymbol

We have seen above that $\text{Pic}(R \mid RS^{-1})$ coincides with all the invertible $R$-$R$-subbimodules of $RS^{-1}$. However, one can construct rings $R$ and choose non-maximal two-sided Ore sets $S \subseteq R$ such that $\text{Pic}(R, D^S_r, D^S_l)$ does not coincide with all the isomorphism classes of invertible $R$-$R$-bimodules $\text{Pic}(R)$. 

Open question: When $S$ is the maximal two-sided Ore set of regular elements of $R$, determine whether $\text{Pic}(R, D^S_r, D^S_l)$ coincides with $\text{Pic}(R)$.

Observe that an affirmative answer to this question would imply that the maximal two-sided Ore localization of two unital Morita equivalent rings are Morita equivalent rings.

Finally, we will compare the sequence of Theorem 3.3 with the one obtained by Bass in [1] Chapter III, Proposition 7.5] for commutative rings. Let $R$ be a unital commutative ring. Given any right $R$-module $M$ and ring automorphisms $\phi$ and $\varphi$ we denote by $\varphi M_\phi$ the $R$-$R$-bimodule whose additive group is $M$ and whose bimodule structure is given by $r \cdot m = m \varphi(r)$ and $m \cdot r = m \phi(r)$ for every $m \in M$ and $r \in R$.

We denote by $\text{Pic}_R(R)$ the group of the isomorphism classes of the invertible right $R$-modules (see [1]).

It is known [1] Chapter II §5] that given any $R$-$R$-bimodule $M$ the action of $R$ on the left is determined by an automorphism $\varphi$ of $R$, that is $r \cdot m = m \varphi(r)$ for every $r \in R$ and $m \in M$. So it follows that given any $R$-$R$-bimodule $M$ there exists a right $R$-module $N$ and a ring automorphism $\varphi$ such that $M \cong \varphi N_1$ as $R$-$R$-bimodules. Thus we get the following result.

Proposition 3.6. [1] Chapter II, Proposition 5.4] Let $R$ be a commutative ring, then

$$0 \rightarrow \text{Pic}_R(R) \rightarrow \text{Pic}(R) \rightarrow \text{Aut}(R) \rightarrow 0$$

is exact, and splits by $\varphi \mapsto [\varphi R_1]$. 

Thus, we can see every element of \( \text{Pic}(R) \) as a pair \( ([M], \varphi) = [\varphi M_1] = [\varphi R_1 \otimes_R M_1] \) where \( M \) is an invertible right \( R \)-module and \( \varphi \) is a ring automorphism. Then given \( ([M], \varphi) \) and \( ([N], \phi) \) we have that \( \varphi M_1 \otimes_R \phi N_1 \cong \phi \varphi R_1 \otimes_R (M \otimes_R \phi N_1) \in ([M \otimes_R \phi N_1], [\phi \circ \varphi]) \).

Thus, we have that \( \text{Pic}(R) \cong \text{Pic}_R(R) \ltimes \text{Aut}(R) \) with the above product.

Now, let \( S \) be a saturated multiplicative set of regular elements of \( R \), and let \( RS^{-1} \) be its ring of fractions. Recall that \( RS^{-1} \) is the localization of \( R \) with respect to the Gabriel filter of ideals of \( R \) that have nonempty intersection with \( S \), denoted by \( \mathcal{D}^S \).

Now, we are going to describe \( \text{Pic}(R, \mathcal{D}^S) \), that is, the group of isomorphism classes \( [M] \) of invertible \( R \)-\( R \)-bimodules such that \( T_M^{\sharp}(\mathcal{D}^S) = \mathcal{D}^S \). Since every invertible \( R \)-\( R \)-bimodule is isomorphic to \( \varphi M_1 \) for some right \( R \)-module \( M \) and ring automorphism \( \varphi \), we get that \( T_M^{\sharp}(\mathcal{D}^S) = \mathcal{D}^S \) if and only if \( \varphi(S) = S \).

The group of the ring automorphisms such that \( \varphi(S) = S \) is denoted by \( \text{Aut}^S(R) \).

**Lemma 3.7.** Let \( R \) be a unital commutative ring, and let \( S \) be a saturated multiplicative system of regular elements of \( R \), then

\[
\text{Pic}(R, \mathcal{D}^S) \cong \text{Pic}_R(R) \ltimes \text{Aut}^S(R).
\]

Now, observe that if \( S \) is a saturated multiplicative system of regular elements of \( R \), and we consider the ring of fractions of \( R \) with respect to \( S \), by Theorem 3.3 we get the following exact sequence

\[
0 \longrightarrow \mathcal{U}(R) \longrightarrow \mathcal{U}(RS^{-1}) \longrightarrow \text{Pic}(R \mid RS^{-1}) \longrightarrow \\
\text{Pic}_R(R) \ltimes \text{Aut}^S(R) \longrightarrow \text{Pic}_{RS^{-1}}(RS^{-1}) \ltimes \text{Aut}(RS^{-1}),
\]

where

\[
\varphi_2 : \text{Pic}(R \mid RS^{-1}) \longrightarrow \text{Pic}_R(R) \ltimes \text{Aut}^S(R)
\]

\[
\varphi_3 : \text{Pic}_R(R) \ltimes \text{Aut}^S(R) \longrightarrow \text{Pic}_{RS^{-1}}(RS^{-1}) \ltimes \text{Aut}(RS^{-1})
\]

\[
\phi : [M, \varphi] \
\phi : [([M], \varphi)] 
\]

and

\[
\phi : [([M], \varphi)] \
\phi : ([MS^{-1}], \varphi)
\]

where \( \varphi : RS^{-1} \longrightarrow RS^{-1} \) is the ring automorphism defined as \( \varphi(ab^{-1}) = \varphi(a)\varphi(b)^{-1} \), that is well-defined since \( \varphi \in \text{Aut}^S(R) \). Splitting off the Aut-terms, we obtain:

**Corollary 3.8.** [Chapter III, Proposition 7.5] Let \( R \) be a unital commutative ring, and let \( S \) be a saturated multiplicative set of nonzero divisors of \( R \). Then the sequence

\[
0 \longrightarrow \mathcal{U}(R) \longrightarrow \mathcal{U}(RS^{-1}) \longrightarrow \text{Pic}(R \mid RS^{-1}) \longrightarrow \text{Pic}_R(R) \longrightarrow \text{Pic}_{RS^{-1}}(RS^{-1})
\]

is exact.

**Acknowledgments**

This research was partially supported by MEC-DGESIC (Spain) through Project MTM2005-00934, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya. Part of this paper was written while the author visited Queen’s University Belfast as part of a research project funded by Royal Society. I gratefully acknowledge my advisor P. Ara for all his useful comments and help that led to the presentation of this paper.
REFERENCES

[1] F.W. Anderson, K.R. Fuller, “Rings and categories of Modules”, Springer, 1991, GTM.
[2] J. Bénabou, Introduction to bicategories, Reports of the Midwest Category Seminar, 1–77. Springer-Verlag, 1967.
[3] H. Bass, “Algebraic K-theory”, W. A. Benjamin, Inc., New York-Amsterdam 1968
[4] F. Borceaux, “Handbook of Categorical Algebra 1, Basic Category Theory”, Cambridge University Press, Volumen 50, 1994
[5] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90, 323–448 (1962).
[6] O. Goldman, Rings and Modules of Quotients, J. Algebra 13, 10–47 (1969).
[7] N.P. Landsman, Bicategories of operator algebras and Poisson manifolds, Proc. Siena (2000), 271–286, Fields Inst. Commun., 30, Amer. Math. Soc., Providence, RI, 2001.
[8] S. Lanning, The maximal symmetric ring of quotients, J. Algebra, 179 (1996), 47–91.
[9] E. Ortega, Rings of quotients of incidence and path algebras, J. Algebra 303 (2006) 225–243.
[10] W. Schelter, Products of Thorsion Theories, Arch. Math., 22 (1971), 590–596.
[11] W. Schelter, Two-sided rings of quotients, Arch. Math., 24 (1973), 274–277.
[12] B. Stenström, “Rings of quotients”, Springer-Verlag, 1975.
[13] Y. Utumi, On quotient rings, Osaka J. Math., 8 (1956), 1–18.
[14] Y. Utumi, On rings of which one-sided quotient ring are two-sided, Proc. Amer. Math. Soc., 14 (1963), 141–147.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain.
E-mail address: eortega@mat.uab.es