Floquet system, Bloch oscillation, and Stark ladder

Tao Ma and Shu-Min Li
Department of Modern Physics, University of Science and Technology of China,
P.O.Box 4, Hefei, Anhui 230026, People’s Republic of China
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We prove the multi-band Bloch oscillation and Stark ladder in the $nk$ and site representation from the Floquet theorem. The proof is also possible from the equivalence between the Floquet system, Bloch oscillation, and the rotator with spin. We also exactly solve the periodically driven two level atom and two band Bloch oscillation in terms of Heun function.

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I. INTRODUCTION

Periodically driven two level atom with Rabi oscillation effect is one of the simplest quantum systems, yet it still has not been exactly solved [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. One band BO and SL are easy to be established, but as shown by Zak [12], there is a paradox in the one band BO. Other proofs of BO and SL in the multi-band cases or in the case of the general BO Hamiltonian includes [14, 15, 16, 19], but one proof is general followed by some comment and reply. Nenciu argued when taking into consideration the inter-band hopping, SL does not exist any more [20]. In the paper, we give a proof of BO and SL both in the $nk$ representation allows an interpretation as a FS. Based on the Floquet theorem, we establish the multi-band BO and SL from the Floquet theorem. Even if our proof is not general enough to be followed by some comment and reply. Nenciu argued when taking into consideration the inter-band hopping, SL does not exist any more [20]. In the paper, we give a proof of BO and SL both in the $nk$ and site representation.

II. EXACT SOLUTION OF A PERIODICALLY DRIVEN TWO LEVEL ATOM

The Hamiltonian of a periodically driven two level atom is

$$H(t) = \Omega \sigma_z - A \sigma_z \sin \omega t,$$

where $\sigma_x$ and $\sigma_z$ are Pauli matrices, and $\Omega$ and $A$ are parameters. After a time-independent unitary rotation in the Hilbert space [22], the Hamiltonian becomes

$$H'(t) = A \sigma_z \sin \omega t + \Omega \sigma_x.$$  

(1)

The Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = H'(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (A \sigma_z \sin \omega t + \Omega \sigma_x) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where $(c_1, c_2)^T$ is the wave function of the two level atom after the unitary transform. For simplicity, we set $\hbar = 1$ and $\omega = 1$.

$$i\hbar \frac{\partial}{\partial t} c_1 = \Omega c_2,$$

$$i\hbar \frac{\partial}{\partial t} c_2 = \Omega c_1.$$  

(3)

(4)

After removing $c_2$, we get

$$\frac{\partial^2}{\partial t^2} c_1 + (iA \cos t + A^2 \sin^2 t + \Omega^2) c_1 = 0.$$  

(5)

We change the variable from $t$ to $z$. $z = \frac{\Omega}{2} t + \frac{1}{2}$. We get

$$z^2 - z \frac{\partial^2}{\partial z^2} c_1 + (z - \frac{1}{2}) \frac{\partial}{\partial z} c_1 + (4A^2 z^2 - 4A^2 z - 2iA + iA - \Omega^2) c_1 = 0.$$  

(6)

The general solution is

$$c_1(z) = e^{2iA(z+1)} \times$$

$$\left\{ H_c(4iA, -\frac{1}{2}, -\frac{1}{2}, -2iA, iA + \frac{1}{2} - \Omega^2, z) \sqrt{\pi} H_c(4iA, \frac{1}{2}, -\frac{1}{2}, 2iA, iA + \frac{1}{2} - \Omega^2, z) \right\}^2,$$

(7)

where $H_c$ is the Heun confluent function. The general solution is

$$c_1(t) = e^{2iA(\cos t+1)} \times$$

$$h_c(4iA, -\frac{1}{2}, -\frac{1}{2}, -2iA, iA + \frac{1}{2} - \Omega^2, \frac{1}{2} \cos t + \frac{1}{2}) \times$$

$$(\cos t + 1)^{7/2},$$

(8)

When $t = 0, z = 1$, which is a singular point of $H_c$. Now we do not have enough knowledge to handle the singular point in order to consider the initial conditions. When

$$H'(t) = A \sigma_z \cos t + \Omega \sigma_z.$$  

(9)
The general solution is
\[ c_1(t) = \begin{cases} H_d(4A, -4\Omega^2, -8A, 4\Omega^2, i\cot\frac{\pi}{2})e^{-iA \sin t} \\ H_d(-4A, -4\Omega^2, -8A, 4\Omega^2, i\cot\frac{\pi}{2})e^{iA \sin t}. \end{cases} \]
\[ (10) \]

III. EXACT SOLUTION OF TWO BAND BLOCH OSCILLATION

The Hamiltonian of two band Bloch electron in an electric field is
\[ H = \sum_{n=\infty}^{\infty} \left[ (\Delta - eF n) a_n^\dagger a_n + (-\Delta - eF n) b_n^\dagger b_n \\
+ \frac{1}{2} t_a (a_{n+1}^\dagger a_n + a_n^\dagger a_{n+1}) + \frac{1}{2} t_b (b_{n+1}^\dagger b_n + b_n^\dagger b_{n+1}) \\
- eFR(a_n^\dagger b_n + b_n^\dagger a_n) \right], \]
\[ (11) \]
where \( a \) and \( b \) refer to electrons in two bands with bandwidths \( 2t_a \) and \( 2t_b \) respectively; the first two terms are the site energies, the middle two describe site-to-site hopping, and the last term gives interband hopping \([14, 21]\].

The Hamiltonian can be Fourier transformed into the \( nk \) representation \([13, 24, 25]\), in which the Hamiltonian is
\[ H = \begin{bmatrix} \Delta - ieF \frac{\partial}{\partial k} + t_a \cos k & -eFR \\
-eFR & -\Delta - ieF \frac{\partial}{\partial k} + t_b \cos k \end{bmatrix}. \]
\[ (12) \]

The eigenstates of the Hamiltonian is periodic of \( k \) \([14]\).

We assume the period is \( 2\pi \).

We define
\[ H_R = -ieF \frac{\partial}{\partial k} + \begin{bmatrix} t_a \cos k & 0 \\
0 & t_b \cos k \end{bmatrix} \]
\[ (13) \]
as the rotation Hamiltonian and
\[ H_S = \begin{bmatrix} \Delta & -eFR \\
-eFR & -\Delta \end{bmatrix} \]
\[ (14) \]
as the spin Hamiltonian. The names will be explained in the Sec. IV.

A. Rotation and spin decoupled

Now we discuss the simplest two band BO. If \( t_a = t_b = t \),
\[ H = -ieF \frac{\partial}{\partial k} + t \cos k + \begin{bmatrix} \Delta & -eFR \\
-eFR & -\Delta \end{bmatrix}. \]
\[ (15) \]
Then \( [H_R, H_S] = 0 \). The rotation and spin degrees of freedom are decoupled. The eigenvalues of Eq. (15) is
\[ \omega_{n\pm} = neF \pm \sqrt{\Delta^2 + (eFR)^2}. \]
\[ (16) \]
The eigenstates are
\[ \phi_{n\pm} = e^{i\phi_{n\pm} k} e^{-it \sin k} \left[ \begin{array}{c} \frac{1}{\sqrt{2}}(\Delta \pm \sqrt{\Delta^2 + (eFR)^2}) \\
\frac{1}{\sqrt{2}} \end{array} \right] \]
\[ (17) \]
The result is first derived by Fukuyama et al in \([14]\). The physical meaning of Eq. (16, 17) is two SLs. Electron oscillates between two bands with a period \( \frac{2\pi}{\sqrt{\Delta^2 + (eFR)^2}} \).

B. Exact solution of two band Bloch oscillation

When \([H_R, H_S]\) \( \neq 0 \), the rotation and spin degrees of freedom are coupled together. The Hamiltonian eigenvalue equation is
\[ H\psi = E\psi. \]
\[ (18) \]
\[ \begin{bmatrix} \Delta - ieF \frac{\partial}{\partial k} + t_a \cos k & -eFR \\
-eFR & -\Delta - ieF \frac{\partial}{\partial k} + t_b \cos k \end{bmatrix} \begin{bmatrix} c_1 \\
c_2 \end{bmatrix} = E \begin{bmatrix} c_1 \\
c_2 \end{bmatrix}, \]
\[ (19) \]
where \( \psi = (c_1, c_2)^T \) is the wave function and \( c_1 \) and \( c_2 \) are functions of \( k \).

\[ (-ieF \frac{\partial}{\partial k} + t_a \cos k + \Delta - E)c_1 - eFRc_2 = 0, \]
\[ -eFRc_1 + (-ieF \frac{\partial}{\partial k} + t_b \cos k - \Delta - E)c_2 = 0. \]
\[ (20) \]
We set the unit \( eF = 1 \). After removing \( c_2 \),
\[ \frac{\partial^2 c_1}{\partial k^2} + ((t_a + t_b) \cos k - 2E)i \frac{\partial}{\partial k} c_1 \\
+ \left( -it_a \sin k - t_a t_b \cos^2 k - (t_a(-\Delta - E) \\
+ t_b(\Delta - E)) \cos k - E^2 + \Delta^2 + R^2 \right) c_1 = 0. \]
\[ (21) \]
We change the variable from \( c_1(k) \) to \( u(k) \) and define \( t_a - t_b = \delta \). \( c_1(k) = u(k) \exp(-ia \sin k) \).

\[ \frac{\partial^2 u}{\partial k^2} - (\delta \cos k + 2E)i \frac{\partial}{\partial k} u \\
+ \left( \delta(\Delta - E) \cos k + \Delta^2 + R^2 - E^2 \right) u = 0. \]
\[ (22) \]
Eq. (22) can be solve by Heun function. What we have to do is to transform it into the standard form of Heun function. We change variable from \( k \) to \( z = i\cot\frac{k}{2} \).
The approximation of Eq. (31) is

$$\frac{(z-1)^3(z+1)^3}{3!} \frac{\partial^2}{\partial z^2} u + [2z^5 - (4E + 2\delta)z^4 - 4z^3 + 8Ez^2 + 2z + (-4E + 2\delta)] \frac{\partial}{\partial z} u + [4(E^2 - \Delta^2 - R^2 - \delta \Delta + \delta E)z^2 + 4(-E^2 + \Delta^2 + R^2 - \delta \Delta + \delta E)] u = 0.$$  

(23)

After another variable change, Eq. (23) can be written into the standard form. The general solution is

$$u(z) = \left( \frac{-1 + z - zE}{1 + z} \right)^E \left\{ \begin{array}{l} H_d \left( 2\delta, -4\delta \Delta - 4\Delta^2 - 4R^2, -4\delta, -4\delta \Delta + 4\Delta^2 + 4R^2, z \right) \\ H_d \left( 2\delta, -4\delta \Delta - 4\Delta^2 - 4R^2, -4\delta, -4\delta \Delta + 4\Delta^2 + 4R^2, z \right) e^{-2\delta z/(z^2-1)} \end{array} \right\}.$$  

(24)

$$u(k) = e^{iE_k} \left\{ \begin{array}{l} H_d \left( 2\delta, -4\delta \Delta - 4\Delta^2 - 4R^2, -4\delta, -4\delta \Delta + 4\Delta^2 + 4R^2, i \cot \left( \frac{\pi}{2} \right) \right) e^{i\delta \cos k} \\ H_d \left( 2\delta, -4\delta \Delta - 4\Delta^2 - 4R^2, -4\delta, -4\delta \Delta + 4\Delta^2 + 4R^2, i \cot \left( \frac{\pi}{2} \right) \right) e^{-i\delta \sin k} \end{array} \right\}.$$  

(25)

$$c_1(k) = e^{iE_k} \left\{ \begin{array}{l} H_d \left( 2\delta, -4\delta \Delta - 4\Delta^2 - 4R^2, -4\delta, -4\delta \Delta + 4\Delta^2 + 4R^2, i \cot \left( \frac{\pi}{2} \right) \right) e^{-i\delta \sin k} \\ H_d \left( 2\delta, -4\delta \Delta - 4\Delta^2 - 4R^2, -4\delta, -4\delta \Delta + 4\Delta^2 + 4R^2, i \cot \left( \frac{\pi}{2} \right) \right) e^{-i\delta \sin k} \end{array} \right\}.$$  

(26)

where $X_{nm}(k)$

$$X_{nm}(k) = \int u_{nk}^*(x) i \frac{\partial}{\partial k} u_{mk}(x) dx.$$  

(33)

is periodic of $k$.

Eq. (30) and (32) have the same form. The eigenvalue problem of the BO Hamiltonian is

$$\left[ \sum_{n=1}^M \epsilon_n(k) - \sum_{n,m=1}^M eF X_{nm}(k) - i eF \frac{\partial}{\partial k} \right] \Phi(k) = E \Phi(k).$$  

(34)

If $\{E, \Phi(k)\}$ is the eigenvalue and eigenstate of Eq. (30) and (32), then

$$E' = E + m \times 2\pi E_c,$$  

(35)

$$\Phi'(k) = e^{im \times 2\pi k} \Phi(k).$$  

(36)

gives another eigenvalue and eigenstate of the BO Hamiltonian. But it gives the same solution to the FS Eq. (27). In FS, if $H(t)$ is a $M \times M$ matrix, which corresponds to a $M$ band BO Hamiltonian, the total number of the quasienergy is $M$. The eigenvalues of the BO Hamiltonian are grouped into $M$ SLs. If $M$ is finite, BO can not have continuous spectra and the eigenstates of BO Hamiltonian are localized. If the electron is put on one site, the wave function oscillates because the wave function can be expanded by (approximately) finite eigenstates of the BO Hamiltonian. It is the connection of Eq. (32) to the Floquet system, that gives the eigenvalues of SLs. We note Fukuyama et al first used the Floquet theorem to prove SL in a two band case [14]. Avron et al also gave a proof of SL based on Eq. (32) in [15]. But our proof is clearer.

We require $H(k)$ is not a function of $i \frac{\partial}{\partial k}$. In this way, we implicitly assume

$$\int u_{nk}^*(x) u_{mk}(x) dx = 0.$$  

(37)
or
\[ \langle ns|ms \rangle = 0 \] (38)
when \( n \neq m \) in the site representation. Under this assumption, the interband hopping matrix elements \( X_{nm}(k) \) or
\[ \langle ns| - eF(x)|ms \rangle = \langle ns| - eF(x + x_0)|ms \rangle \] (39)
do not depends on \( x \).

**A. Infinite matrix representation**

Every FS and BO Hamiltonian can be represented as an infinite matrix. The infinite matrix of the periodically driven two level atom was first written out and referred as Floquet Hamiltonian by Shirley [4]. The FS formalism [4, 5] including the infinite matrix representation is widely used in the practical numerical calculations, such as atomic and molecular excitation, ionization in a laser field [26]. The Hamiltonian of a periodically driven two level atom is
\[ H(t) = \Omega \sigma_x + A \cos(\omega t) \sigma_z. \] (40)

Then the Floquet Hamiltonian
\[ \mathcal{H}_F = H(t) - i \frac{\partial}{\partial t}, \] (41)
has the following infinite matrix representation [4, 5, 26] in the \( |sn\rangle = |\uparrow, \frac{n}{\sqrt{2 \pi}} e^{-int} \rangle \) or \( |\downarrow, \frac{\sqrt{\pi}}{\sqrt{2 \pi}} e^{-int} \rangle \) basis

\[
\begin{pmatrix}
\cdots & \cdots \\
\cdots & -\Omega + 2\omega & A/2 & 0 & 0 & 0 & 0 \\
\cdots & A/2 & \Omega + \omega & 0 & 0 & 0 & 0 \\
0 & 0 & -\Omega + \omega & A/2 & 0 & 0 & 0 \\
0 & 0 & A/2 & \Omega & 0 & 0 & A/2 \\
0 & 0 & 0 & A/2 & -\Omega & 0 & 0 \\
0 & 0 & 0 & 0 & A/2 & \Omega - \omega & 0 \\
0 & 0 & 0 & 0 & 0 & A/2 & \Omega - 2\omega \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\] (42)

which has a more compact form
\[
\begin{pmatrix}
\cdots & \cdots \\
\cdots & H_S + 2\omega & H_R \\
H_R & H_S + \omega & H_R & 0 & 0 \\
0 & H_R & H_S & H_R & 0 \\
0 & 0 & H_R & H_S - \omega & H_R \\
0 & 0 & 0 & H_R & H_S - 2\omega \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\] (43)

where
\[ H_S = \begin{bmatrix} \Omega & 0 \\ 0 & -\Omega \end{bmatrix}; H_R = \begin{bmatrix} 0 & A/2 \\ A/2 & 0 \end{bmatrix}. \] (44)

The infinite matrix representation of the two band BO Hamiltonian Eq. (11) is
\[
\begin{pmatrix}
\cdots & \cdots \\
\cdots & H_S + 2eF & H_R \\
H_R & H_S + eF & H_R & 0 & 0 \\
0 & H_R & H_S & H_R & 0 \\
0 & 0 & H_R & H_S - eF & H_R \\
0 & 0 & 0 & H_R & H_S - 2eF \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\] (45)

where
\[ H_S = \begin{bmatrix} \Delta & -eFR \\ -eFR & -\Delta \end{bmatrix}; H_R = \begin{bmatrix} t_\alpha/2 & 0 \\ 0 & t_\beta/2 \end{bmatrix}. \] (46)

Note in this section \( H_R \) is different from the Sec. III by the part of \( -ieF \frac{\partial}{\partial t} \).

The Floquet Hamiltonian of periodically driven two level atom Eq. (43, 44) and the two band BO Hamiltonian Eq. (45, 46) are very similar. The difference is just the different basis of the Hilbert space and parameters. Given a FS, we can find a BO Hamiltonian, which has the same infinite matrix representation as the Floquet Hamiltonian of the FS and vice versa. FS must has very similar dynamic behaviors with BO and vice versa. This is the reason why we can use the Floquet theorem to prove BO.

**B. Floquet system, Bloch oscillation, and rotator with spin**

The Hamiltonian Eq. (32) is an approximation of Eq. (31) based on the Bloch function. The proof of BO and SL is achieved in the \( nk \) representation. But Eq. (31) can also be approximated in the site representation just as two band approximation Eq. (11).

The BO Hamiltonian of a multi-band Bloch electron in a linear electric field is a rotator with the spin degree of freedom (RS). The basis of the Hilbert space of Eq. (11) is \( |sn\rangle \), where \( s \) is one of the multi-bands and \( n \) is \( n \)-th site. In the rotator representation, the basis of the
Hilbert space is $|sn\rangle$, where $s$ is the spin of the rotator and $n$ is $\frac{1}{\sqrt{2\pi}} e^{-ink}$ with $k$ the angle of the rotator. In the Sec. III, we rewrite Eq. (11) as Eq. (12) from the perspective of the Fourier transform. But this can also be achieved by the site-band and rotation-spin correspondence. The $n$-th site corresponds to $\frac{1}{\sqrt{2\pi}} e^{-ink}$ and the $s$-th band the $s$-th spin state. From this correspondence, we can rewrite Eq. (11) into Eq. (12) without the Fourier transform.

Every BO Hamiltonian can be rewritten as a RS. $H(k)$, such as in the Eq. (12), is the coupling between rotation-rotation, spin-spin and rotation-spin degrees of freedom. The Bloch wave number in the $nk$ representation $k$ is the angle of the rotator. By corresponding the Hilbert space basis and the Hamiltonian, we have established the equivalence between BO and RS.

For the Hamiltonian of every one of FS, BO and RS, we can find another two Hamiltonians in the other two systems, and the former Hamiltonian has the equivalent behavior with the latter two.

Generalizing Eq. (12), in the rotator representation, Eq. (31) is approximated as

$$H = \sum_{s=1}^{M} \epsilon_s(k) |s\rangle \langle s| - \sum_{s,s'=1}^{M} \sum_{n,m=-\infty}^{\infty} eF X_{sn,s'm} |sn\rangle \langle s'm| - ieF \frac{\partial}{\partial k},$$

where $|sn\rangle = |s, \frac{1}{\sqrt{2\pi}} e^{-ink}\rangle$ is the basis of the RS and

$$X_{sn,s'm} = \int u_{sn}^*(x) x u_{s'm}(x) \, dx$$

with $u_{sn}$ and $u_{s'm}$ being Wannier functions, give the coupling between bands and sites. The specific value of $X_{sn,s'm}$ is gave by the parameters of the material and the electric field, but it is “simulated” by the RS. We note the rotator with a linear $-ieFk$ in the Hamiltonian is first studied by Grempel et al. [27] and Berry [28].

We assume $X_{sn,s'm} = X_{sn+n',s'm+n'} = X_{s0,s'm+n}$. In terms of Heun function.

Our method can not treat the most general BO problem with the Hamiltonian Eq. (31). But in a physical material, the Bloch electron only occupies the lowest several bands. So we believe the BO and SL are highly possible in reality.

Our most significant contribution in the paper is the equivalence between the FS, BO and RS. We think the three are the same problem with three “faces”. The equivalence between FS and BO is established by comparing the Floquet Hamiltonian Eq. (29) and the BO Hamiltonian Eq. (30, 32). The equivalence between BO and RS is established by corresponding the site-band basis with the rotation-spin basis. At last, the three share the same infinite matrix structure.

The equivalence is shown as the following figure.

V. CONCLUSION AND DISCUSSION

In summary, we have given a proof of BO and SL based on the Floquet theorem. We also give the exact solution of periodically driven two level atom and two band BO in terms of Heun function.

The fully quantized field and two level atom interaction is more interesting. But the present method completely fails because first, after the second quantization, the Hamiltonian is not doubly infinite; second, the off-diagonal matrix changes away from the ground state [3].
APPENDIX A: HEUN FUNCTION

The Heun function $H(a, q; \alpha, \beta, \gamma, \delta, z)$ is defined as the solution of the following equations

$$
\left( \frac{d^2}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) \frac{d}{dz} + \frac{\alpha \beta z - q}{z(z-1)(z-a)} \right) H(a, q; \alpha, \beta, \gamma, \delta, z) = 0,
$$

$$
H(a, q; \alpha, \beta, \gamma, \delta, 0) = 1,
\left. \frac{dH(a, q; \alpha, \beta, \gamma, \delta, z)}{dz} \right|_{z=0} = \frac{q}{a \gamma},
$$

(A1)

where $\epsilon = \alpha + \beta + 1 - \gamma - \delta$. $H(a, q; \alpha, \beta, \gamma, \delta, z)$ is the second-order Fuchsian equation with four regular singular points. One application of Heun function is to the Calogero-Moser-Sutherland System [29].

The Heun confluent function $H_c(\alpha, \beta, \gamma, \delta, \eta, z)$ is obtained from Heun function through a confluence process. So $H_c(\alpha, \beta, \gamma, \delta, \eta, z)$ has two regular singular points and one irregular one. It is the singular points most important to our application. $H_c(\alpha, \beta, \gamma, \delta, \eta, z)$ is defined as the solution of the following equations

$$
\left( z(z-1) \frac{d^2}{dz^2} + \left[ \alpha z^2 + (-\alpha + \beta + \gamma + 2)z - \beta - 1 \right] \frac{d}{dz} + \frac{\alpha(\beta + \gamma + 2) + 2\beta z}{2} \right) H_c(\alpha, \beta, \gamma, \delta, \eta, z) = 0,

H_c(\alpha, \beta, \gamma, \delta, \eta, 0) = 1,
\left. \frac{dH_c(\alpha, \beta, \gamma, \delta, \eta, z)}{dz} \right|_{z=0} = \frac{\beta(-\alpha + \gamma + 1) - \alpha + \gamma + 2\eta}{2(\beta + 1)}.
$$

(A2)

The Heun doubly confluent function $H_d(\alpha, \beta, \gamma, \delta, z)$ is obtained from Heun function through two confluence process and has two irregular singular points. $H_d(\alpha, \beta, \gamma, \delta, z)$ is defined as the solution of the following equations

$$
\left( (z-1)^3(z+1)^3 \frac{d^2}{dz^2} + (2z^5 - \alpha z^4 - 4z^3 + 2z + \alpha) \frac{d}{dz} 
+ (\beta^2 + (2\alpha + \gamma)z + \delta) \right) H_d(\alpha, \beta, \gamma, \delta, z) = 0,

H_d(\alpha, \beta, \gamma, \delta, 0) = 1,
\left. \frac{dH_d(\alpha, \beta, \gamma, \delta, z)}{dz} \right|_{z=0} = 0.
$$

(A3)

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