Proper time and conformal problem in Kaluza-Klein theory

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Abstract

In the traditional Kaluza-Klein theory, the cylinder condition and the constancy of the extra-dimensional radius (scalar field) imply that timelike geodesics on the 5-dimensional bundle project to solutions of the Lorentz force equation on spacetime. This property is lost for non constant scalar fields, in fact there appear new terms that have been interpreted mainly as new forces or as due to a variable inertial mass and/or charge. Here we prove that the additional terms can be removed if we assume that charged particles are coupled with the same spacetime conformal structure of neutral particles but through a different conformal factor. As a consequence, in Kaluza-Klein theory the proper time of the charged particle might depend on the charge-to-mass ratio and the scalar field. Then we show that the compatibility between the equation of the projected geodesic and the classical limit of the Klein-Gordon equation fixes unambiguously the conformal factor of the coupling metric solving the conformal ambiguity problem of Kaluza-Klein theories. We confirm this result by explicitly constructing the projection of the Klein-Gordon equation and by showing that each Fourier mode, even for a variable scalar field, satisfies the Klein-Gordon equation on the base.

1 Introduction

The Kaluza-Klein (Jordan-Thiry) theory [31,33,38] assumes the existence of a fifth extra dimension aside the four spacetime dimensions (for a review see [12,11,16,36,40,45], for the mathematical formalism see [6,8,10,24,27] and for generalizations see [29,32,48]). Although venerable the research on this theory is still quite active [3,4,23,28]. This theory tried to recover gravitation and electromagnetism on 4-dimensional spacetime $M$ from gravitation on a 5-dimensional spacetime $P$. In its simplest version the theory assumes that the metric $\bar{\gamma}$ on $P$ admits a spacelike Killing vector field $k$ and that the spacetime manifold is identified with the quotient $M = P/T_1$ (or $M = P/U(1)$) where $T_1$ (resp. $U(1)$) is the one-parameter group of isometries generated by $k$. In this

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way $P$ is a principal bundle of projection $\pi: P \to M$. In coordinate language this requirement was called cylinder condition. If this condition is assumed then a number of useful results follow, and the 5-dimensional gravitational physics seems really to reduce to the ordinary four dimensional physics. Clearly, this condition would have to be justified through some physical argument if the gravitational physics on $P$ has to be regarded as unconstrained. In this work we shall assume that the cylinder condition can be justified in some way using an appropriate physical limit.

The Kaluza-Klein program was suggestive. It was shown that if the scale factor of the extra dimension (scalar field) is constant in $P$ then the geodesics of $P$ are projected to solutions of the Lorentz force equation on $M$, where the charge-to-mass ratio depends on the geodesic chosen [37]. Free particles on $P$ move along geodesics thus, in this interpretation, all the particles are chargeless in the full 5-dimensional spacetime and we see some of them as charged only because the spacetime events are identified with the orbits of the Killing field. However, this result was obtained assuming the constancy of the scalar field which we denote with the letter $a$. Unfortunately, this hypothesis is unjustified and leads to another constraint. Indeed, it is well known [40] that if we assume a constant scalar field, the Einstein equations in $P$ reduce to the Einstein equations and the Maxwell equations in $M$ plus the constraint that the module of the magnetic field equals the module of the electric field, $B = E$. Thus the assumption of a constant scalar field is in fact quite strong and the Lorentz force equation can be recovered only if the last condition is satisfied. The conclusion is that the Lorentz force equation is obtained only in a very restrictive case. The situation is not really encouraging: the constancy of $a$ implies that the geodesics project to worldlines of charged particles but the same condition also implies that $B = E$, a too much restrictive condition for a theory originated with unifying purposes.

A pragmatical approach to this problem, as in the original KK theory, imposes the condition $a = \text{const.}$ before taking the variation of the action on $P$. In this way the Euler-Lagrange equation for $a$ is removed ad hoc. This approach, however, not only lacks elegance, it is also inconsistent with the geodesic principle of general relativity. Indeed, although not known at the time of Kaluza, it is now well known that the geodesic principle of GR (free particles move along geodesics) follows from the Einstein equations [43 Sect. 9.3]. Therefore, the arbitrary elimination of one of the 5-dimensional Einstein equations questions the assumption that particles on $P$ move along geodesics.

A better way out would be the proof that even with a non constant scalar field the geodesics of $P$ project to solutions of the Lorentz force equation on $M$. But this is not the case. The projection of geodesics in the general case of non constant scalar field has been considered in many works [9,13,22,36,38]. It has been shown that new terms appear and, depending on their arrangement, different interpretations have been proposed. Most authors find the presence of an extra force due to the fifth dimension and that the charge-to-mass ratio is not constant. This last feature, which is also a departure from the standard Lorentz force equation, raises the question as to whether this variability should
be assigned to the mass, the charge or both. The same conclusions have been
drawn in a version of the theory where the cylinder condition is dropped [14]
[45][46]. See also the related “brane” works [30][41][49] (The reader should keep
in mind the distinction between recent brane theories which are based on an
embedding of the spacetime in the higher dimensional space $E: M \rightarrow P$ and
K-K theories which start from a submersion $\pi: P \rightarrow M$, see [20]).

Moreover, its is not difficult to show that if the timelike geodesics of $P$
project to solutions of the Lorentz force equation then we are in the simple case
of constant scalar field. Thus there seems to be no solution to the problem
mentioned above.

Fortunately, there is a way out which is a natural one. We shall prove that
all the extra forces that appear in the equation of the projected geodesic have a
common interpretation. They are due to a conformal rescaling of the spacetime
metric that depends on the particle charge-to-mass ratio. With this correction
to the coupling metric we recover the Lorentz force equation even in presence
of a non constant scalar field. There is also a natural physical consequence:
the proper time of a charged particle in Kaluza-Klein theory depends on its
charge-to-mass ratio and on the value of the scalar field along its path.

We shall comment more on this after the formal statement. At this stage we
wanted only to notice that no other mechanism has been proposed that is able
to keep the good results of the dimensional reduction idea (the reproduction
of the Lorentz force equation), while being able to remove the bad ones (the
condition $B = E$) with the good news that there is no need to assume a constant
scalar field.

2 Notations and terminology

We introduce some notations and terminology. We use units such that the speed
of light $c$ satisfies the normalization $c = 1$. The spacetime is a 4-dimensional
Lorentzian manifold $(M, g)$ of signature $(+ − − −)$. We use this signature since
it is the most used in quantum field theory, and since later on we shall work with
the Klein-Gordon equation. The spacetime $M$ is the base for a 5-dimensional
bundle $\pi: P \rightarrow M$ where $P$ is endowed with a Lorentzian metric $\tilde{g}$ of signature
$(+ − − − −)$. This is the usual choice since the alternative $(+ + − − −)$ does not
lead to the correct relative sign between the gravitational and electromagnetic
Lagrangians. The fiber of $P$ is generated by a spacelike Killing vector field $k$ on
$P$. The group structure can be the one-dimensional group of translations $T_1$ or
$U(1)$. With

$$r = q/m$$

we denote the charge-to-mass ration of a particle.

With a $\gamma$ over a 1-form we denote the corresponding vector, for instance
$\gamma^\mu = g^{\mu\nu} \gamma_\nu$. Analogously with a $\gamma$ over a vector we denote the corresponding
1-form, for instance $\tilde{v}_\mu = g_{\mu\nu} \tilde{v}^\nu$. If the index is raised (resp. lowered) using the
metric \( \tilde{g}_{\mu \nu} \) then we shall use the notation \( \tilde{\gamma} \) (resp. \( \tilde{v} \)). The 1-form on \( P \)

\[
\tilde{\omega} = \tilde{k}/\tilde{g}(k,k)
\]

(1)
is the real-valued connection 1-form \([34]\). It satisfies \( \tilde{\omega}(k) = 1 \) and \( L_k \tilde{\omega} = 0 \) where \( L_k \) is the Lie derivative. Let \( U \) be an open subset of \( M \) and let \( s: U \to P \) be a section: \( \pi \circ s = \text{Id}_M \). Defined the electromagnetic potential as the pullback \( A = s^* \tilde{\omega}/\beta \), where \( \beta \) is a dimensional constant, we have that the connection 1-form can be written

\[
\tilde{\omega} = dy + \beta A_\mu dx^\mu
\]

(2)
where \( y \) is a dimensionless coordinate on the fibre and \( \{x^\mu\} \) are coordinates on \( U \). The electromagnetic field \( F := dA \) is independent of the section and can be regarded as the pullback under a local section of \( d\tilde{\omega}/\beta \). With \( F \) we denote the electromagnetic field with the first index raised, \( F^\mu_\nu = g^{\mu \alpha} F_{\alpha \nu} \). In what follows we shall be involved also with metrics denoted \( g_{E \mu \nu} \) and \( g_{r \mu \nu} \). We shall therefore write \( F^\mu_\nu = g_{E \mu \alpha} F_{\alpha \nu} \) and \( F^\mu_\nu = g_{r \mu \alpha} F_{\alpha \nu} \). This index free notation will shorten the formulas throughout the work. The positive scalar field \( a \) is defined by

\[
a^2 = -\hat{g}(k,k).
\]

It remains to define a Lorentzian metric on \( M \). There are two usual choices: the Jordan metric

\[
g_0 = \tilde{g} - \frac{\tilde{k} \otimes \tilde{k}}{\tilde{g}(k,k)} = \hat{g} + a^2 \tilde{\omega}^2,
\]

(3)
and the Einstein metric (\( a_0 \) is a constant with the same dimension of \( a \))

\[
g_E = (a/a_0)g_0.
\]

(4)
Actually, here we are really giving the representative of the spacetime metric on \( P \). The metric can be passed to the quotient to \( M \) since \( g(k,\cdot) = 0 \) and \( L_k g = 0 \), see \([24]\).

The freedom in the choice of spacetime metric is due to the arbitrariness in the choice of the conformal factor in front of the spacetime metric \( g \) and this complication is sometimes called the conformal ambiguity problem \([7, 12, 17, 18, 21, 42]\). It is not present in the constant scalar field case since there the two metrics differ by a constant factor with no relevant physical consequences. As we shall see our results on the Lorentz force equation will, in some sense, privilege the Jordan metric for atomic phenomena. This does not mean that the Jordan metric is the “right” spacetime metric. It has become clear \([19, 21]\) that in gravitational theories with scalar fields there is no unique spacetime metric. The metric with which one should calculate the proper time of a particle can differ for different kinds of particles if the corresponding matter Lagrangian terms couple to different metrics. In particular the metric that appears in the free electron Lagrangian may be only conformally related to the Einstein metric.
i.e. to the metric that appears in the Einstein-Hilbert Lagrangian term.

\[- \int dy \int d^4x \sqrt{|\det \tilde{g}|} \tilde{R} = -\left( \int dy \right) \int d^4x \sqrt{|\det g_E|} \left\{ a_0 R_E \right. \\
+ \frac{a_0^3 \beta^2}{4} \tilde{F}_E^2 - \frac{3a_0}{2} g_E^{\mu\nu} (\partial_\mu \ln a)(\partial_\nu \ln a) \left\} \right. \]

(a total divergence term \( D = -a_0 \Box_E \ln a \) has been integrated out).

Thus, the number of rotations that a binary black hole system performs would be proportional to the proper time calculated with the cosmological Einstein metric, while the proper time of a charged particle in the same cosmological background would have to be calculated with a metric that is only conformally related to Einstein’s. As we shall see the present work will fix that conformal factor providing a solution to the conformal ambiguity problem.

The constant \( a_0 \) appearing in (4) is the present value of \( a \) (if \( a \) changes only over cosmological scales it is the value at this cosmological era, if it changes over the solar system it is the value at the earth’s surface). The reason is that the observers independently of whether they measure time with atomic clocks, pendulums or the planets motion set their clocks in such a way that they have the same rate here and now (syntonization process). Since \( a \) varies, after an interval that may be huge the clocks of different nature, atomic and gravitational, finally desyntonize. The statement that they are set in such a way that they measure the same unit of time in mathematical terms is \( g = g_E \) here and now, or \( a/a_0 \) (here and now) = 1. The interpretation of \( a_0 \) follows.

Finally, note that the equivalence principle is satisfied only if the metric that couples to matter is the same for every kind of massive particle [19]. Of course slight differences in the coupling metrics are possible, if the experimental precision with which the equivalence principle has been tested is taken into account. In our case the coupling metric will depend on the particle charge and will reduce to the Jordan metric in the neutral case. We are therefore interested in tests of the equivalence principle for charged particles. Only one experiment was dedicated to this question so far: the Witteborn-Fairbank experiment [47]. The test confirmed the equivalence principle but the accuracy was very poor, about 0.1 which should be contrasted with \( 10^{-12} \) for neutral matter [15]. More refined experiments have been proposed [15].

3 Proper time of charged particles

Equation (4) implies that the projection of a timelike (causal) curve on \( P \) is timelike (resp. causal) on \( M \). We are interested on the projection of geodesics

\[ z : I \rightarrow P, \]

\[ \tilde{\nabla}_t z_t = 0, \]

where \( \tilde{\nabla} \) is the Levi-Civita connection of \( \tilde{g} \). We begin with a theorem.
Theorem 3.1. Let \( z : I \to P, t \mapsto z(t) \), be a geodesic, \( \dot{g}(\dot{z}, \dot{z}) = \epsilon, \epsilon = -1, 0, 1 \), and let \( x = \pi \circ z \) be its projection to \( M \), then

\[
r = -\beta a^2(x(t)) \dot{\omega}(\dot{z}) = \beta \dot{g}(k, \dot{z}),
\]

is a constant of motion. Define on \( M \) the function

\[
\Omega_r = [\epsilon + r^2/(\beta^2 a^2)]^{1/2},
\]

which for \( \epsilon < 0 \) we assume to be positive, and define the Lorentzian metric

\[
g_r = \Omega_r^2 (\dot{g} + a^2 \dot{\omega}^2),
\]

then the projection \( x \) is timelike and once parametrized with

\[
t_r = \int \Omega_r^2(t) \, dt,
\]

it satisfies the Lorentz force equation

\[
\nabla_{t_r}^r x_{t_r} = r \hat{F}_r(x_{t_r}), \quad g_r(x_{t_r}, x_{t_r}) = 1,
\]

where \( \nabla^{(c)} \) denotes the Levi-Civita connection of \( g_r \) and \( \hat{F}_r^{\mu} = g_r^{\alpha\nu} F_{\alpha\nu} \).

The meaning of the theorem is that a charged particle sees the same causal structure of neutral particles but a different conformal factor. That factor depends on the charge-to-mass ratio of the particle and is close to 1 for great values of the scalar field \( a \). The parametrization \( t_r \) is the \( r \)-proper time of the particle i.e. that obtained from the metric \( g_r \) integrating the line element over \( x \). The theorem suggests that for charged particles this proper time is more fundamental than the usual 0-proper time \( t_0 \), obtained integrating the Jordan metric over \( x \).

We can think of an elementary particle as a clock. Indeed, elementary particles can decay and this decay time has been measured in different reference frames to give an experimental verification of the time dilation predicted by special relativity. The previous theorem suggests that the time of a charged particle clock is given by \( t_r \). This clock is therefore predicted to go faster for small \( a \) indeed\footnote{The parameters \( t_0 \) and \( t \) should not be confused, the former is the proper time according to the Jordan metric on \( M \), the latter is the proper time parametrization of the geodesic on \( P \). They differ along the trajectory of a charged particle, from (7) we have \( dt_r = \Omega_r dt_0 \) while from (8) we have \( dt_r = \Omega_r^2 dt \).} for the classical case in which the geodesic on \( P \) is timelike (\( \epsilon = 1 \))

\[
dt_r = \sqrt{1 + \frac{1}{\beta^2 a^2 (q/m)^2}} \, dt_0.
\]
field. Indeed, we are not saying that the usual extra force is not present but only that it is an apparent force due to an unnatural metric choice. The analogy with the generalization from the Newtonian to the Einsteinian gravitation is clear: the Newton force arises only in a formalism that unnaturally forces the metric to be the flat one. In the right formalism the motion appears unforced and one also realizes that the spacetime metric of the new theory is related to proper time. Exactly the same happens here. We remove the extra force and, as a consequence, we find a formula for the proper time of charged particles.

Let us give a proof Theorem 3.1. A related variational observation was given by Lichnerowicz and Thiry [38] as a particular case of a general result on the dimensional reduction of Finsler geometries.

Proof. We start from the local expression of the metric

$$\tilde{g} = \Omega^{-2} g - a^2 (dy + \beta A)^2, \quad (11)$$

where $\Omega$, $g$ and $A$ are tensors on $U_i$. Let $z$ be a curve on $P$ and let $x = \pi \circ z$ be its projection, we have

$$\tilde{g}(\dot{z}, \dot{z}) = \Omega^{-2} g(\dot{x}, \dot{x}) - a^2 (\dot{y} + \beta A(\dot{x}))^2. \quad (12)$$

Thus using a variational trick in $\pi^{-1}(U_i)$

$$0 = \frac{\delta}{2} \int dt \left\{ \tilde{g}(\dot{z}, \dot{z}) - \Omega^{-2} g(\dot{x}, \dot{x}) + a^2 (\dot{y} + \beta A(\dot{x}))^2 \right\}$$

$$= \int dt \left\{ -\tilde{g}(\nabla_t \dot{z}, \delta z) + \Omega^{-2} g(\nabla_t \dot{x}, \delta x) - \frac{\delta \Omega^{-2}}{2} g(\dot{x}, \dot{x}) \right.$$  

$$+ \frac{d\Omega^{-2}}{dt} g(\dot{x}, \delta x) + \left( \frac{\delta a^2}{2} \right) (\dot{y} + \beta A(\dot{x})) + a^2 (\dot{y} + \beta A(\dot{x})) (\delta y + \beta \delta (A(\dot{x}))) \right\}$$

$$= \int dt \left\{ -\tilde{g}(\nabla_t \dot{z}, \delta z) + \Omega^{-2} g(\nabla_t \dot{x}, \delta x) - \frac{\delta \Omega^{-2}}{2} g(\dot{x}, \dot{x}) \right.$$  

$$+ \frac{d\Omega^{-2}}{dt} g(\dot{x}, \delta x) - \frac{1}{2} (\delta \frac{1}{a^2} [a^2 \tilde{\omega}(\dot{z})])^2 - \tilde{\omega}(\delta z) \left( \frac{d}{dt} [a^2 \tilde{\omega}(\dot{z})] + \beta [a^2 \tilde{\omega}(\dot{z})] F(\delta x, \dot{x}) \right) \right\}$$

where $\tilde{\omega}(\dot{z}) = \dot{y} + \beta A(\dot{x})$, $\tilde{\omega}(\delta z) = \delta y + \beta A(\delta x)$ and where $\nabla$ and $\nabla_t$ are the Levi-Civita connections of $\tilde{g}$ and $g$ respectively. Note that

$$\delta \Omega^{-2} = \delta x^i \partial_i \Omega^2 = g(\delta x, \nabla \Omega^{-2}) \quad (13)$$

and analogously for $\delta(1/a^2)$. The vertical variation gives ($\delta x = 0$)

$$a^2 \tilde{\omega}(\nabla_t \dot{z}) = \frac{d}{dt} [a^2 \tilde{\omega}(\dot{z})], \quad (14)$$

while the horizontal variation ($\tilde{\omega}(\delta z) = 0$) gives

$$\pi_* \nabla_t \dot{z} = \nabla_t \dot{x} - \frac{d \ln \Omega^2}{dt} \dot{x} + \beta [a^2 \tilde{\omega}(\dot{z})] F(\dot{x}) - \frac{\Omega^2}{2} (\nabla \Omega^{-2}) g(\dot{x}, \dot{x}) + (\nabla \frac{1}{a^2}) [a^2 \tilde{\omega}(\dot{z})]^2. \quad (15)$$
Now, let $z(t)$ be a geodesic of $P$ with the affine parametrization $t$ such that $\dot{g}(\dot{z}, \dot{z}) = 1$, $0$. Eq. (14) states that the quantity $a^2 \dot{\omega}(\dot{z})$ is a constant of motion and defining $r = q/m = -\beta a^2 \dot{\omega}(\dot{z})$, Eq. (15) which determines the trajectory of the projected geodesic reads

$$\nabla_t \dot{x} - \frac{d \ln \Omega^2}{dt} \dot{x} = \Omega^2 r \dot{F}(\dot{x}) + \frac{\Omega^2}{2} \left\{ (\nabla \Omega^{-2})g(\dot{x}, \dot{x}) + \left( \nabla \frac{r^2}{\beta a^2} \right) \right\}.$$ 

Let us reparametrize the curve $x$ with respect to a parameter $t'$ such that $dt' = \Omega^2 dt$ and use

$$\nabla_{t'} x' = \Omega^{-4} \left( \nabla_t \dot{x} - \frac{d \ln \Omega^2}{dt} \dot{x} \right)$$

with Eq. (12) we obtain

$$\nabla_{t'} x' = r \dot{F}(x') + \frac{1}{2} \left\{ (\nabla \Omega^{-2})[\epsilon + \frac{r^2}{\beta^2 a^2}] + \Omega^{-2} \left( \nabla \frac{r^2}{\beta^2 a^2} \right) \right\}$$

$$= r \dot{F}(x') + \frac{1}{2} \nabla \left\{ \Omega^{-2} [\epsilon + \frac{r^2}{\beta^2 a^2}] \right\}.$$ 

Thus with the choice $\Omega^2 = C[\epsilon + \frac{r^2}{\beta^2 a^2}]$, $C = const$, we obtain the Lorentz force equation and hence the fact that $g(x', x')$ is constant. From Eq. (11) using $\dot{x} = \Omega^2 x'$ we obtain $g(x', x') = C^{-1}$. This concludes the proof, however we wish to observe that the found formulas for $\epsilon = -1, C = 1$, imply the interesting result: if the charged particle is the projection of a tachyon in $P$ then its motion is confined to the region $a(x) < \frac{q}{\beta m}$.

4 The conformal ambiguity problem

Let us investigate the conformal ambiguity problem in the more general framework of field theories. Let us start again with a spacetime $(M, g)$ and let us consider the Klein-Gordon Lagrangian

$$\mathcal{L}_\psi = \sqrt{|\det g|} \left\{ g^{\mu \nu} \left( D^\mu \psi \right)^* D^\nu \psi - cm^2 \psi^* \psi \right\},$$

where $D^\mu = \hbar \nabla^\mu - i q A^\mu$, and $\epsilon = -1, 0, 1$, where $\epsilon = 1$ is the usual case.$^2$

The field $\psi$ which represents a charged spinless particle satisfies the Klein-Gordon equation

$$D^\mu D^\mu \psi + cm^2 \psi = 0.$$  

Now we are going to perform a short-wave (WKB) approximation to this equation in order to recover the classical limit. We expand $\psi$ in powers of $\hbar$

$$\psi = (\psi_0 + \hbar \psi_1 + O(\hbar^2)) e^{iS},$$

$^2$We shall also use this Lagrangian on $P$ where we have seen that spacelike, lightlike or timelike geodesics might project to timelike solutions of the Lorentz force equation. This fact signals that it is equally interesting to study the K-G equation for lightlike or tachyonic matter on $P$. 

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where $\hbar^{-1} S$ is the phase. Plugging this expression into the Klein-Gordon equation and equating to zero the terms of order $\hbar^0$ we obtain

$$
(\nabla^\mu S - q A^\mu)(\nabla_\mu S - q A_\mu) - \epsilon m^2 = 0.
$$

(20)

This equation can be regarded as the Hamilton-Jacobi equation associated to the (super)Hamiltonian $H = (P^\mu - q A^\mu)(P_\mu - q A_\mu) - \epsilon m^2 = 0$, where $P_\mu = S_\mu$ is the conjugate momentum. We define the kinematical momentum $p_\mu := S_\mu - q A_\mu$ and note that

$$
p_{\alpha;\beta} = S_{\alpha;\beta} - q A_{\alpha;\beta} = p_{\beta;\alpha} + q F_{\alpha\beta}.
$$

(21)

We observe that

$$
p^\alpha p_\alpha = g^{\alpha\gamma} p_\gamma p_\alpha = g^{\alpha\gamma} (p_{\beta;\gamma} + q F_{\gamma\beta}) p_\beta = \frac{1}{2} g^{\alpha\gamma} (p^\mu p_\mu)_{\gamma} + q F_{\alpha\beta} p_\beta = q F_{\alpha\beta} p_\beta,
$$

where we have used Eq. (20)

$$
p^\mu p_\mu = \epsilon m^2.
$$

(23)

Let $\epsilon = 1$. Equations (22)-(23) form a system which states that the integral lines $x(s)$ of the kinematical momentum $\frac{dx^\mu}{ds} = p^\mu/m$ are solutions to the Lorentz force equation relative to the charge-to-mass ratio $q/m$. As $p^\mu/m$ is normalized, $s$ is the proper time parametrization along the integral curve $x$. In other words the classical limit of the Klein-Gordon equation gives, as expected, the Lorentz force equation. Over each wordline $x(s)$ there is a natural time given by the number of wavefronts of $\psi$ seen by an observer moving along $x(s)$, that is the phase $S$ provides a natural clock over $x(s)$. However, here there is a difficulty connected to the fact that the Klein-Gordon Lagrangian is invariant under gauge transformations, thus $S$ is not determined by the initial conditions since it is not gauge invariant. Instead the 1-form $dS - q A$ is gauge invariant. Its integral over $x$ provides the natural clock we were looking for. It can be interpreted as the number of wavefronts of $\psi$ seen by an observer moving along $x(s)$. If the gauge has been fixed so that $A_\mu(x(s)) \frac{dx^\mu}{ds} = 0$

$$
\frac{dS}{ds} = \frac{dS}{ds} - q A_\mu \frac{dx^\mu}{ds} = p_\mu p^\mu/m = \epsilon m
$$

(24)

thus the natural clock given by the gauge fixed phase $S$ coincides, suitably normalized, and for $\epsilon = 1$ with the usual proper time parametrization derived from the metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

Let us now come to the Kaluza-Klein case. If a charged particle is represented by a geodesic on $P$ then its projection satisfies the Lorentz force equation on $M$ with respect to the metric $g_{q/m}$. Then the classical 4-dimensional Lagrangian for the charged particle must be

$$
\mathcal{L}_\psi = \sqrt{|\det g_r|} \left\{ g^{\mu\nu}_r (D^{(\eta,m)}_\mu \psi)^* D^{(\eta,m)}_{\nu} \psi - m^2 \psi^* \psi \right\}
$$

(25)
where
\[ D_{\mu}^{(q,m)} = \hbar \nabla_{\mu}^{(r)} - iqA_{\mu} \]
that is, it should couple with \( g_r \) and not with \( g_0 \), otherwise the classical limit of the Klein-Gordon equation would give a particle motion that cannot be derived from a geodesic projection. This argument solves the conformal ambiguity problem of Kaluza-Klein theory since it fixes the conformal factor in the coupling metric. The previous analysis also tells us that the most natural proper time over the classical limit trajectories, i.e. the gauge fixed field phase, is nothing but the proper time calculated with the metric that appears in the Lagrangian, that is \( dt^2 = g_{\mu\nu}dx^\mu dx^\nu \). This fact confirms again the interpretation of \( t_r \) as proper time of the charged-particle.

Finally, note that since \( \psi \) couples with \( g_r \) and no term in \( \psi \) couples with \( a \), and since both \( g_0 \) and \( g_E \) can be recovered only from the knowledge of both \( g_r \) and \( a \) it is simply impossible that these last metrics have some role in the proper time of the charged particle represented by \( \psi \). The argument is: if the proper time of the particle depends on \( g_0 \) (\( g_E \)) then \( \psi \) should have a dynamics which depends on \( a \) too (and not solely on \( g_r \)), but this is in contradiction with the Lagrangian expression.

In the next section we shall see that the coupling metric needs a correction and that it is given by \( g_r \) only in the limit \( \hbar \to 0 \).

5 Conformal problem and quantum mechanics

Let us consider the Klein-Gordon equation for neutral particles in \( P \) (here \( A, B = 0, 1, \cdots 4 \))
\[ \hbar^2 \tilde{g}^{AB} \tilde{\nabla}_A \tilde{\nabla}_B \Psi - \epsilon m^2 \Psi = 0, \quad \epsilon = 1. \] (26)

For more generality we have introduced a parameter \( \epsilon \) which takes values in \( \{-1, 0, 1\} \) and whose purpose will be explained in a moment.

From the results of the previous section it follows that this equation returns the geodesic equation on \( P \) under the classical short wave approximation (\( \hbar \to 0 \)). That is, if \( \Psi = |\Psi|e^{i\tilde{S}} \) with \( |\Psi| = \Psi_0 + \Psi_1 + \hbar + O(\hbar^2) \) then the vector field \( u_A = \partial_A \tilde{S}/m \) is normalized, \( \tilde{g}(u, u) = \epsilon \), and \( \tilde{\nabla}_u u = 0 \). The momentum of the particle is \( \tilde{p}_A = m u_A = \partial_A \tilde{S} \). In this short wave limit \( \tilde{p}_y = \partial_y \tilde{S} = k^A \tilde{p}_A \) is constant along the integral geodesics since \( \tilde{\nabla}_u \tilde{g}(k, u) = 0 \).

This quantity although constant along a given geodesic may vary changing the integral geodesic in \( P \). Using the expression for \( \tilde{g} \), \( k = \partial_y \) and Eq. (6) it is easily shown that \( q = \beta \tilde{p}_y \). We know that these geodesics projects on solutions of the Lorentz force equation with respect to \( g_r \), \( r = q/m \). But we also know that the Klein-Gordon equation in \( M \) of parameters \( q \) and \( m \) and coupled to \( g_r \) gives in the classical limit the same Lorentz force equation. In general one expects that the Klein-Gordon equation for a metric \( g_{(q,m)} \) such that \( g_{(q,m)} \to g_r \) in the classical limit, would have as a classical limit the same Lorentz force equation.
Thus maybe there is a choice of \( g_{(q,m)} \) and a projection of the Klein-Gordon equation such that the following diagram commutes

\[
\begin{align*}
\text{neutral KGE in } (P, \tilde{g}) & \xrightarrow{h \to 0} \text{GE in } (P, \tilde{g}) \\
\downarrow \pi & \downarrow \pi \\
\text{charged KGE in } (M, g_{(q,m)}) & \xrightarrow{h \to 0} \text{LFE in } (M, g_{q/m})
\end{align*}
\]

where GE stands for geodesic equation and similarly for KGE and LFE. The problem is thus whether the K-G equation in \( P \) can be reduced to the K-G equation in \( M \) coupled to a metric \( g_{(q,m)} \) before the classical limit is taken. If this is possible then the classical limit can be performed irrespective of the order both in \( P \) and \( M \).

There is a difficulty, however. In the classical limit, the K-G equation on \( M \) determines a solution of the Lorentz force equation of charge-to-mass ratio \( q/m \) where both \( q \) and \( m \) are fixed by the variables entering the K-G equation. On the contrary, there is no variable \( q \) in the K-G equation on \( P \), we have just a constant \( q \) given by \( \beta \partial_y \tilde{S} \) over the geodesics on \( P \) obtained with the short wave approximation. The K-G equation in \( P \) can project to the K-G equation in \( M \) only if the classical limit of the K-G equation in \( P \) projects to the classical limit of the K-G equation in \( M \). This is possible only if \( q \) is a constant all over \( P \), which means \( \partial_y \tilde{S} = q/\beta \).

Consider the Fourier expansion along the fiber direction

\[
\Psi = \int d\mu(q) \tilde{\psi}_q(x) e^{i \frac{q}{\hbar} \beta y}
\]

where \( d\mu(q) \) is a suitable measure; that is \( \int d\mu(q) = \int dq/(2\pi) \) if the fiber is \( \mathbb{R} \) or \( \int d\mu(q) = \Sigma_n \) with \( n = q/\hbar \beta \) otherwise. If the extradimension is compactified this is the usual Fourier series and hence \( q = ne \) where \( e = \hbar \beta \) is a fundamental charge. Now note that each term of the series satisfies the above condition \( \partial_y \tilde{S} = q/\beta \). The problem we want to solve is therefore:

Given \( \Psi = \int d\mu(q) \tilde{\psi}_q(x)e^{i \frac{q}{\hbar} \beta y} \) is it possible to reduce the K-G equation in \( P \) to a 1-parameter \( (q) \) family of K-G equations in \( M \)?

This is an old problem. It was solved affirmatively by Klein for a constant scalar field \( a \). We are going to provide an affirmative solution for the non constant scalar field case. Actually, we shall even prove that the answer is affirmative for any value \( \epsilon = -1, 0, 1 \) and so also for \( \epsilon = 0 \) which means that the Klein-Gordon field can describe massless matter on \( P \). This fact is consistent with the possibility of obtaining the Lorentz force equation on \( M \) from geodesics of any causal type on \( P \). As expected, however, the metric on the base will change with \( q \) and in the classical limit will be \( g_{q/m} \).

Let us write

\[
\tilde{g} = \Omega^{-2} g - a^2 (dy + \beta A_\mu dx^\mu)^2
\]

where the function \( \Omega \) (and hence the metric \( g \)) has to be determined.
Remark 5.1. Actually, the next calculation will hold for $\Omega^2 < 0$ as well, which means that $\tilde{g}$ might have signature $(-, -, +, +, +)$ where the fiber of $P$ is time-like. We denote $\varepsilon = \text{sign}(\Omega^2)$, so the next expressions will depend on $(\varepsilon, \tilde{\varepsilon})$ the standard Kaluza-Klein theory corresponding to $(1, 1)$.

The vectors $e_{\mu} = \partial_{\mu} - \beta A_{\mu}\partial_y$ and $k = \partial_y$ give the dual base to $\{dx^\mu, \tilde{\omega}\}$ and satisfy

$$\tilde{g}(k, k) = -a^2, \quad \tilde{g}(k, e_{\mu}) = 0, \quad \tilde{g}(e_{\mu}, e_{\nu}) = \Omega^{-2}g_{\mu\nu},$$

so the inverse metric on $P$ is $\tilde{g}^{-1} = \Omega^2g^{\mu\nu}e_{\mu} \otimes e_{\nu} - a^{-2}\partial_y \otimes \partial_y$. The K-G Lagrangian can be rewritten

$$\mathcal{L}_{\Psi} = \sqrt{|\det g|} \left( \hbar^2 g^{AB}\partial_A \Psi^* \partial_B \Psi - \epsilon m^2 \Psi^* \Psi \right) = \sqrt{|\det g|} \Omega^{-4} \left\{ \hbar^2 \Omega^2 g^{\mu\nu} e_\mu [\Psi^*] e_\nu [\Psi] - \frac{\hbar^2}{a^2} |\partial_y \Psi|^2 - \epsilon m^2 \Psi^* \Psi \right\}$$

Taking the variation with respect to $\Psi$ and ignoring total divergence terms with respect to the measure $d^4xdy$ we find the K-G equation

$$-\Omega^{-2a}g^{\mu\nu}(\nabla_\mu - \beta A_\mu \partial_y)(\nabla_\nu - \beta A_\nu \partial_y) - g^{\mu\nu \partial_\mu (\Omega^{-2}a) \partial_\nu [\Psi]} = -\frac{\Omega^{-4}}{a} \partial_y^2 \Psi + \frac{\epsilon m^2}{\hbar^2} \Omega^{-4}a \Psi.$$

In practice we have used the variational formulation of the K-G equation as a tool in order to rewrite the K-G equation in terms of the Levi-Civita covariant derivative of $g$. The previous equation can be rewritten

$$g^{\mu\nu}(\nabla_\mu - \beta A_\mu \partial_y - \frac{1}{2} \partial_\mu \ln |\Omega^2/a|)(\nabla_\nu - \beta A_\nu \partial_y - \frac{1}{2} \partial_\nu \ln |\Omega^2/a|)\Psi
= \left\{ -\frac{1}{2} \Box \ln |\Omega^2/a| + \frac{1}{4} \partial_\mu \ln |\Omega^2/a| \partial^\mu \ln |\Omega^2/a| - \frac{\epsilon m^2}{\hbar^2} \Omega^{-2} \right\} \Psi + \frac{\Omega^{-2}}{a^2} \partial_y^2 \Psi.$$

Plugging the Fourier expansion into this equation we obtain for each $q$

$$(D^{q\mu} - \frac{\hbar}{2} \partial^\mu \ln |\Omega^2/a|)(D^{q}_\mu - \frac{\hbar}{2} \partial_\mu \ln |\Omega^2/a|)\tilde{\psi}_q
= \left\{ -\frac{\hbar^2}{2} \Box \ln |\Omega^2/a| + \frac{\hbar^2}{4} \partial_\mu \ln |\Omega^2/a| \partial^\mu \ln |\Omega^2/a| - \frac{\epsilon m^2}{\hbar^2} \Omega^{-2} - \frac{\Omega^{-2}q^2}{\beta^2a^2} \right\} \tilde{\psi}_q.$$

Now define $\psi_q$ such that $\tilde{\psi}_q = \psi_q |\Omega^2/a_0/a|^{1/2}$ and choose $\Omega \equiv \Omega_{(q,m)}$ where $\Omega_{(q,m)}$ satisfies the differential equation

$$-\frac{\hbar^2}{2} \Box \ln |\Omega^2_{(q,m)}/a| + \frac{\hbar^2}{4} g^{\mu\nu \partial_\mu \ln |\Omega^2_{(q,m)}/a| \partial_\nu \ln |\Omega^2_{(q,m)}/a|}
- \frac{\Omega^{-2}_{(q,m)}}{\beta^2a^2} \left( \epsilon m^2 + \frac{q^2}{\beta^2a^2} \right) = -m^2, \quad (29)$$

then $\psi_q$ satisfies the K-G equation

$$D_{(q,m)\mu} D^{(q,m)}_{\mu} \psi_q + m^2 \psi_q = 0, \quad (30)$$

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with respect to the metric \( g(q,m) = \Omega^2(q,m)g_0 \). Here the conclusion is the same as in the study of the Lorentz force equation. There is a metric that allows us to write the usual K-G equation, however this metric is only conformally related to \( g_0 \).

We see that this analysis is, as expected, consistent with that for the Lorentz force equation. Indeed, taking the classical limit \( \hbar \to 0 \) we find from (29), \( \Omega(q,m) \to \Omega_{q/m} \) and hence \( g(q,m) \to g_{q/m} \) in this limit. This section generalizes therefore the result of the previous one. Here the conformal ambiguity problem is solved also for \( \hbar \neq 0 \).

Another case in which we obtain the classical solution \( \Omega(q,m) = \Omega_{q/m} \) is that in which \( a \) is constant, indeed with \( \Omega(q,m) = \text{ct} \) the first two terms in (29) vanish. Of course, even if \( a \) is constant Eq. (29) need not have only this solution for \( \Omega(q,m) \), since the general solution will depend on the initial conditions.

The solution is not available in closed form as a function of \( a \). Instead one has to solve a differential equation. Note that Eq. (29) cannot be solved as it stands. In fact there appear covariant derivatives of \( g(q,m) \) while \( g(q,m) \) is known only when \( \Omega(q,m) \) is known. It is more convenient to rewrite it in terms of \( g_E = \sqrt{\text{det}g_E} \left( \frac{\hbar^2}{2} g_E^{\mu\nu} \partial_{\mu} \alpha \partial_{\nu} \alpha - V(q,m) \right) \), where the potential

\[
V(q,m)(\alpha) = \frac{\alpha_0}{a} \left[ \epsilon m^2 + \frac{q^2}{\beta^2 a^2} \right] \alpha^2 \alpha^2 - \frac{1}{4} \alpha^4 ,
\]

has an inverted mexican hat shape for \((\epsilon, \epsilon) = (1, 1)\). When dealing with the field \( \alpha \) one should be careful since this is not a dynamical field. Its stress energy tensor should not be included into the 5-dimensional Einstein equations, and expressions like \( \mathcal{L} = \mathcal{L}_{g_E} + \int \mathcal{L}_{\psi q} + \mathcal{L}_{\alpha(q,m)} \) seem unjustified. In summary

**Theorem 5.2.** Let the field \( \Psi \) solve the K-G equation (26) in \((P, \tilde{g})\) where

\[
\tilde{g} = \epsilon g_0 - a^2 \omega^2 ,
\]
and $g_0$ is a Lorentzian metric. Define the Einstein metric $g_E := \frac{a}{\alpha_0^2} g_0$. Moreover, let $\alpha(q,m)$ be a stationary point of the action

$$
\sqrt{|\det g_E|} \left[ \frac{\hbar^2}{2} g_E^{\mu\nu} \partial_\mu \alpha \partial_\nu \alpha - \varepsilon \frac{\alpha_0}{2a} \left( cm^2 + \frac{q^2}{2a^2} \right) \alpha^2 + \frac{1}{4} \alpha^4 \right],
$$

and let

$$
\Psi = \int d\mu(q) \psi_q(x) \alpha(q,m) e^{i q y},
$$

be the Fourier expansion of $\Psi$ along the fiber, then the fields $\psi_q$ satisfy the K-G equation of a particle of mass $m$ and charge $q$ in $M$

$$
D^{\mu \nu} D_{\mu}^{(q,m)} \psi(q,m) + m^2 \psi(q,m) = 0,
$$

(35)

which couples these particles to the metric 3 $g(q,m) = \alpha_0^2 g_E/m^2$.

The square of the so-called vacuum expectation value of $\alpha(q,m)$ is $\varepsilon \alpha_0^2 \left[ cm^2 + \frac{q^2}{2a^2} \right]$, so $m^2$ in Eq. (35) could be deduced from that.

As a final observation note that the functions $\alpha(q,m)$ are not completely fixed by the differential equation. In fact there is the freedom related to the choice of initial conditions. Therefore, for $\hbar \neq 0$ the conformal factor is considerably constrained but not completely determined. For $\hbar = 0$, instead, the conformal factor is completely fixed.

6 Conclusions

We have studied the Lorentz force equation in the context of Kaluza-Klein theory showing the dependence of proper time on the scale factor $a$ and on the particle charge-to-mass ratio. We have also given other arguments related to the solution of the conformal ambiguity problem through Theor. 5.2. We have studied in detail the reduction, even with variable scalar field $a$, of the Klein-Gordon equation showing that each Fourier mode is itself a solution of the Klein-Gordon equation in $M$ where the coupling metric coincides in the limit $\hbar \to 0$ with the one obtained from the study of the Lorentz force equation.

The conclusion we draw is that although the Kaluza-Klein theory still suffers of many fundamental problems one of its necessary consequences seems to be the dependence of the proper time of charged particles on the mass, the charge and the scalar field $a$.

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