Variational proof of the existence of brake orbits in the planar 2-center problem

Yuika Kajihara

Kyoto University

June 28, 2018
**n-center problem & brake orbits**

- **n-center problem**:

\[
\ddot{\mathbf{q}} = -\sum_{k=1}^{n} \frac{m_k}{|\mathbf{q} - \mathbf{a}_k|^3} (\mathbf{q} - \mathbf{a}_k) \quad (\mathbf{q} \in \mathbb{R}^d)
\]

where \(\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^d\) are constant vectors.

- \(\mathbf{q}(t)\) is a brake orbit.

\[\iff \exists T_2 > \exists T_1 > 0 \text{ such that } \dot{\mathbf{q}}(T_1) = \dot{\mathbf{q}}(T_2) = 0.\]

(Excluding equilibrium points)

- In the potential systems, brake orbits are \(2(T_2 - T_1)\)-periodic solutions.
Q. Do brake orbits exist in the planar 2-center problem?

Setting: \( m_1 = 1, m_2 = m, \ a_1 = a = (1, 0), \ a_2 = -a = (-1, 0) \).

→ Variational methods

\[
\text{Lagrangian: } L(q, \dot{q}) = \frac{1}{2} |\dot{q}|^2 + \frac{1}{|q - a|} + \frac{m}{|q + a|}
\]

\[
\text{Action functional: } \mathcal{A}(q) = \int_0^T L(q, \dot{q}) dt
\]

Boundary conditions: \( q(0) \in A := \{(x, 0) \mid -1 \leq x \leq 1\}, \ q(T) \in \mathbb{R}^2 \).

\[
\mathcal{A}'(q) = 0 \iff q \text{ is a solution of the planar 2-center problem.}
\]
Existence of minimizer

Facts :

- With standard arguments of the existence of a minimizer, there is a minimizer $q^*$ of $A(q)$ satisfying the boundary condition.
- If $q^*$ has no collision, $\dot{q}^*(0) \perp A, \dot{q}^*(T) = 0$ and $A'(q^*) = 0$ hold.
- If :
  
  (Col) $q^*$ is not a collision solution.
  
  (Eq) $q^*$ is not an equilibrium point.

Then we obtain a $4T$-periodic brake orbit.
Shape of \( q^* \)

If \( q^* \) satisfies (Col) and (Eq), \( q^* \) is a part of a brake orbit, i.e. from \( t = 0 \) to \( t = T \) of 4\(T\)-periodic one.
Shape of the whole brake orbit
Under what condition $q^*$ is not an equilibrium point

An equilibrium point: $q_{eq} = (b, 0) \left( b = \frac{\sqrt{m} - 1}{\sqrt{m} + 1} \right)$.

(The second variation)

$$A''(q)(\delta) = \lim_{h \to 0} \int (\delta(t), \dot{\delta}(t)) \nabla^2 L \bigg|_{(q, \dot{q})=(q+h\delta, \dot{q}+h\dot{\delta})} (\delta(t), \dot{\delta}(t))^T dt$$

$\rightarrow$ If $T > \gamma = \frac{\sqrt{2\pi} m^{1/4}}{(1 + \sqrt{m})^2}$, $\exists \delta$ such that $A''(q_{eq})(\delta) < 0$.

$\rightarrow T > \gamma \Rightarrow q^*$ is not an equilibrium point.
A minimizer in collision solutions

Let $q_{\text{col}}$ be a minimizer in collision solutions.

We estimate $A(q_{\text{col}})$:

$$A(q_{\text{col}}) = \int_0^T \frac{1}{2} |\dot{q}_{\text{col}}|^2 + \frac{1}{|q_{\text{col}} - a|} \, dt + \int_0^T \frac{m}{|q_{\text{col}} + a|} \, dt$$

$$\geq \frac{3}{2} \pi^{2/3} T^{1/3} + \int_0^T \frac{m}{|q_{\text{col}} + a|} \, dt \quad (1)$$
Estimate the action of collisions

We estimate the second term in (1):

\[
\int_0^T \frac{m}{|\mathbf{q}_\text{col} + \mathbf{a}|} dt = \int_0^T \frac{m}{q_1(t) + 1} dt > \frac{m}{q_1(T) + 1} T \\
> \frac{\pi^{2/3}(1 + m)^{-1/3}m}{2(1 + \pi^{2/3}(1 + m)^{-1/3}T^{-2/3})} T^{1/3}.
\]

Hence

\[
\mathcal{A}(\mathbf{q}_\text{col}) > \frac{3}{2} \pi^{2/3} T^{1/3} + \frac{\pi^{2/3}(1 + m)^{-1/3}m}{2(1 + \pi^{2/3}(1 + m)^{-1/3}T^{-2/3})} T^{1/3}.
\]
Under what condition \( q^* \) has no collision

- Take a test path: \( q_{\text{test}}(t) = (b, ct^{2/3}) \) \( (c \geq 0) \)

- The action functional of \( q_{\text{test}} \):

  \[
  \mathcal{A}(q_{\text{test}}) = \frac{2}{3} c^2 T^{1/3} \\
  + \int_0^T \frac{1}{\sqrt{(1-b)^2 + c^2 t^{4/3}}} + \frac{m}{\sqrt{(1+b)^2 + c^2 t^{4/3}}} dt
  \]

- \( F(m, T, c) \)

  \[
  := \frac{3}{2} \pi^{2/3} T^{1/3} + \frac{\pi^{2/3} (1+m)^{-1/3} m}{2(1+\pi^{2/3} (1+m)^{-1/3} T^{-2/3})} T^{1/3} - \mathcal{A}(q_{\text{test}})
  \]

\[ F(m, T, c) \geq 0 \Rightarrow \mathcal{A}(q_{\text{col}}) > \mathcal{A}(q_{\text{test}}) \geq \mathcal{A}(q^*) \]

\( \rightarrow F(m, T, c) \geq 0 \Rightarrow q^* \) has no collision.
Main theorem

If \((m, T) \in D\), then there exists a brake orbit which has \(4T\)-period, where \(D := \{(m, T) \mid T > \gamma, F(m, T, c) \geq 0 (\exists c \geq 0)\}\).

Draw the domain \(D\) with Matlab.
What kind of brake orbits are minimizers?

Red line: the $x$-component of the force is zero

Figure: $m_1 = m_2$

Figure: $m_1 < m_2$
Thank you for your attention!