Removable singularities for bounded \(A\)-(super)harmonic and quasi(super)harmonic functions on weighted \(\mathbb{R}^n\)

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ARTICLE INFO

Article history:
Received 13 December 2021
Accepted 21 March 2022
Communicated by Francesco Maggi

MSC:
primary 31C45
secondary 31E05
35J92

Keywords:
\(A\)-harmonic function
\(A\)-superharmonic function
Bounded \(p\)-admissible weight
\(p\)-harmonic function
\(p\)-superharmonic function
Quasiharmonic function
Quasisuperharmonic function
Removable singularity
Weighted Euclidean space.

ABSTRACT

It is well known that sets of \(p\)-capacity zero are removable for bounded \(p\)-harmonic functions, but on metric spaces there are examples of removable sets of positive capacity. In this paper, we show that this can happen even on unweighted \(\mathbb{R}^n\) when \(n > p\), although only in very special cases. A complete characterization of removable singularities for bounded \(A\)-harmonic functions on weighted \(\mathbb{R}^n\), \(n \geq 1\), is also given, where the weight is \(p\)-admissible. The same characterization is also shown to hold for bounded quasiharmonic functions on weighted \(\mathbb{R}^n\), \(n \geq 2\), as well as on unweighted \(\mathbb{R}\). For bounded \(A\)-superharmonic functions and bounded quasisuperharmonic functions on weighted \(\mathbb{R}^n\), \(n \geq 2\), we show that relatively closed sets are removable if and only if they have zero capacity.

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1. Introduction

Sets of \(p\)-capacity zero are removable for bounded \(A\)-harmonic functions, more precisely:

Theorem 1.1. If \(\Omega \subset \mathbb{R}^n\) is open and \(E \subset \Omega\) is a relatively closed set with \(p\)-capacity \(C_p(E) = 0\), then every bounded \(A\)-harmonic function on \(\Omega \setminus E\) has an \(A\)-harmonic extension to \(\Omega\). Moreover, the extension is unique and bounded.

Here, and throughout the paper, \(1 < p < \infty\) and an \(A\)-harmonic function is a continuous weak solution of the \(A\)-harmonic equation

\[
\text{div} \, A(x, \nabla u) = 0,
\]
where \( \mathcal{A} \) satisfies the degenerate ellipticity conditions (3.3)–(3.7) in [14, p. 56] (with the parameter \( p \)). The \( p \)-harmonic functions, which are the continuous weak solutions of the \( p \)-Laplace equation \( \text{div}(|\nabla u|^{p-2}\nabla u) = 0 \), are included as a special case.

**Theorem 1.1** is well known and goes back to Serrin [23, Theorem 10], who covered the case when \( E \) is compact. It has later been extended to quasiharmonic functions on unweighted \( \mathbb{R}^n \) (with \( E \) compact) in Tolksdorf [24, Theorem 1.2], \( \mathcal{A} \)-harmonic functions on weighted \( \mathbb{R}^n \) in Heinonen–Kilpeläinen–Martio [14, Theorem 7.36], and \( p \)-harmonic and quasiharmonic functions on metric spaces in Björn–Björn–Shanmugalingam [6, Proposition 8.2] and Björn [2, Theorem 6.2], covering also relatively open \( \Omega \) (with \( n \geq 2 \)).

A natural question is if the converse is true: If \( E \) as above is removable for bounded \( p \)-harmonic or bounded \( \mathcal{A} \)-harmonic functions in \( \Omega \setminus E \), does it follow that \( C_p(E) = 0 \)? (To avoid pathological cases, we assume that no component of \( \Omega \) is contained in \( E \).)

Maz’ya [21, Remark 1.4] showed this converse for compact \( E \) when \( p < n \). For compact \( E \) and bounded \( \Omega \), this was extended to unweighted and weighted \( \mathbb{R}^n \), \( 1 < p < \infty \), by Heinonen–Kilpeläinen [13, Remark 4.8] and Heinonen–Kilpeläinen–Martio [14, comment after Theorem 7.36], respectively. For \( p \)-harmonic functions on metric spaces \( X \) it is due to Björn–Björn–Shanmugalingam [6, Proposition 8.4], when \( E \) is compact, \( \Omega \) is bounded and the capacity \( C_p(X \setminus \Omega) > 0 \).

On the other hand, in [2, Example 9.3], the present author gave examples, for the metric space \([0,1]\), of compact sets with positive capacity (and even with positive measure) removable for bounded \( p \)-harmonic and quasiharmonic functions. In particular, this shows that the condition \( C_p(X \setminus \Omega) > 0 \) above is essential. Now we are able to show that this can happen even on unweighted \( \mathbb{R}^n \), \( n \geq 2 \), but only in very special cases, namely when all the involved functions are constant.

Assume from now on that \( \Omega \subset \mathbb{R}^n \) is nonempty and open, and that \( E \subset \Omega \) is a relatively closed subset such that no component of \( \Omega \) is contained in \( E \).

For unweighted \( \mathbb{R}^n \), \( n \geq 2 \), we have the following complete characterization of removable sets for bounded \( \mathcal{A} \)-harmonic functions.

**Theorem 1.2.** Assume that \( \Omega \subset \mathbb{R}^n \) (unweighted) with \( n \geq 2 \). Then \( E \) is removable for bounded \( \mathcal{A} \)-harmonic functions in \( \Omega \setminus E \) if and only if one of the following two disjoint cases is true:

(a) \( C_p(E) = 0 \);
(b) \( p > n \), \( \Omega = \mathbb{R}^n \) and \( E \) is a singleton set.

Note that when \( p > n \) all points have positive \( C_p \)-capacity, and thus (b) gives examples of removable sets with positive capacity, although only when all bounded \( \mathcal{A} \)-harmonic functions on \( \Omega \setminus E \) are constant.

On weighted \( \mathbb{R}^n \) equipped with a \( p \)-admissible weight \( w \), we are also able to give a complete characterization, albeit a bit more involved.

**Theorem 1.3.** Assume that \( \Omega \subset \mathbb{R}^n \) with \( n \geq 2 \), where \( \mathbb{R}^n \) is equipped with \( d\mu = w\,dx \) and \( w \) is a \( p \)-admissible weight. Then the following are equivalent:

(a) \( E \) is removable for bounded \( \mathcal{A} \)-harmonic functions in \( \Omega \setminus E \);
(b) \( E \) is removable for bounded quasiharmonic functions in \( \Omega \setminus E \);
(c) either \( C_{p,\mu}(E) = 0 \), or there is some \( x_0 \in E \) such that \( C_{p,\mu}(E \setminus \{x_0\}) = C_{p,\mu}(\mathbb{R}^n \setminus \Omega) = 0 \) and \((\mathbb{R}^n, \mu)\) is \( p \)-parabolic.

Moreover, in (b) the quasiharmonicity constant \( Q \) is preserved.

In both theorems above, \( \mathcal{A} \) is assumed to be fixed, but by the characterizations, removability for one \( \mathcal{A} \) holds if and only if it holds for all other \( \mathcal{A} \), as long as \( p \) and \( \mu \) are fixed. Since (a) \( \Leftrightarrow \) (b) in Theorem 1.3,
it follows that the conditions in Theorem 1.2 also characterize removability for bounded quasiharmonic functions on unweighted $\mathbb{R}^n$.

Naively it may seem that once nonremovability has been shown for compact sets with positive capacity (disregarding for the moment the exceptional cases above) it would follow for relatively closed sets $E$ with positive capacity, as $E$ contains a compact set with positive capacity. However, this is a little more subtle, and indeed on $\mathbb{R}$ there are examples of relatively closed removable sets with compact nonremovable subsets (for bounded $p$-harmonic functions), see Theorem 1.4 or [2, Example 9.1]. Ruling out such possibilities on $\mathbb{R}^n, n \geq 2$, is therefore an important part of the proofs of Theorems 1.2 and 1.3. This may also be the reason why nonremovability results for noncompact sets first seems to appear in Björn–Björn–Shanmugalingam [6, Propositions 8.4 and 8.5].

Compared with the situation in (weighted) $\mathbb{R}^n, n \geq 2$, as depicted above, the situation can be quite different in metric spaces. Even on the unweighted real line $\mathbb{R}$ the situation is fundamentally different than in higher-dimensional Euclidean spaces, see [2, Example 9.1]. To get a feeling for this, we include this case in our studies here, giving the following complete characterization of removable singularities for bounded $A$-harmonic functions on weighted $\mathbb{R}$.

**Theorem 1.4.** The set $E$ is removable for bounded $A$-harmonic functions in $\Omega \setminus E$, with respect to $(\mathbb{R}, w)$, where $w$ is a $p$-admissible weight, if and only if there is a constant $C$ such that for every component $I$ of $\Omega$ it is true that $I \setminus E$ is connected and

$$|I| \leq C |I \setminus E|,$$

where $|\cdot|$ denotes the Lebesgue measure. Moreover, in this case the extensions are unique.

Note that this time the removability condition is not only independent of $A$, but also of $w$ and $p$. In Theorem 4.4 we show that removability for bounded quasiharmonic functions on unweighted $\mathbb{R}$ is characterized by the same condition.

A related topic is removability for bounded $A$-superharmonic and bounded quasisuperharmonic functions. Also in this case, it has been shown that sets of zero capacity are removable, see Tolksdorff [24, Theorem 1.5], Heinonen–Kilpeläinen [13, Theorem 4.7], Heinonen–Kilpeläinen–Martio [14, Theorem 7.35], Björn–Björn–Shanmugalingam [6, Proposition 8.3] and Björn [2, Theorem 6.3].

For compact $E$ with positive capacity and bounded $\Omega$ several of the nonremovability results mentioned above also apply to $A$-superharmonic and quasisuperharmonic functions. More specifically, nonremovability was obtained for $A$-superharmonic functions on unweighted and weighted $\mathbb{R}^n$ by Heinonen–Kilpeläinen [13, Remark 4.8] and Heinonen–Kilpeläinen–Martio [14, comment after Theorem 7.36], respectively, while nonremovability for $p$-superharmonic functions and quasisuperharmonic functions on metric spaces $X$ (under the additional condition that $C_p(X \setminus \Omega) > 0$) is due to Björn–Björn–Shanmugalingam [6, Proposition 8.4] and Björn [2, Proposition 7.2], respectively.

We can now show that on weighted $\mathbb{R}^n, n \geq 2$, relatively closed sets are removable for bounded $A$-superharmonic and bounded quasisuperharmonic functions if and only if they have capacity zero, with no exceptional cases. More precisely we obtain the following result.

**Theorem 1.5.** Assume that $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, where $\mathbb{R}^n$ is equipped with $d\mu = w \, dx$ and $w$ is a $p$-admissible weight. Then the following are equivalent:

(a) $E$ is removable for bounded $A$-superharmonic functions in $\Omega \setminus E$;
(b) $E$ is removable for bounded quasisuperharmonic functions in $\Omega \setminus E$;
(c) $E$ is removable for $A$-superharmonic functions in $\Omega \setminus E$ which are bounded from below;
(d) $E$ is removable for quasisuperharmonic functions in $\Omega \setminus E$ which are bounded from below;
(e) $C_{p,\mu}(E) = 0$.

Moreover, in (b) and (d) the quasisuperharmonic constant $Q$ is preserved.

In fact, when $C_{p,\mu}(E) > 0$ we construct a bounded $A$-superharmonic function on $\Omega \setminus E$ which has no quasisuperharmonic extension to $\Omega$ (neither bounded nor unbounded).

A quasiharmonic function is a continuous quasiminimizer. Quasiminimizers were introduced by Giaquinta and Giusti [11,12] as a tool for a unified treatment of variational integrals, elliptic equations and quasiregular mappings on $\mathbb{R}^n$. They showed that De Giorgi’s method could be extended to quasiminimizers and obtained, in particular, their local Hölder continuity. Thus every quasiminimizer has a continuous representative, and it is this representative that is quasiharmonic. Similarly, every quasisuperminimizer has a unique quasisuperharmonic representative (just as $p$-supersolutions have unique $p$-superharmonic representatives).

Kinnunen–Martio [16] showed that one can build a potential theory on quasiharmonic functions. In particular, they introduced the quasisuperharmonic functions, and studied their potential theory on metric spaces. Martio–Sbordone [20] showed that quasiminimizers have an interesting theory also in the one-dimensional case on unweighted $\mathbb{R}$. Our Theorem 4.4 is a contribution to this case. It seems as if all papers (so far) studying quasiminimizers and/or quasi(super)harmonic functions on weighted $\mathbb{R}^n$, $n \geq 2$, do so as a particular case of the corresponding theory on metric spaces (cf. however [14, Section 3.13]). See e.g. the introduction in Björn–Björn–Korte [4] for further references on quasiminimizers.

Various other types of removable singularities for $p$-(super)harmonic, $A$-(super)harmonic and quasi(super)harmonic have been studied in Heinonen–Kilpeläinen [13], Kilpeläinen–Zhong [15], Pokrovskii [22], Mäkäläinen [18] and Björn [2]. For harmonic and analytic functions removable singularities lying in many different types of function spaces have been studied, but this is not the right place to discuss this.

2. Notation and preliminaries

Throughout the paper we let $1 < p < \infty$, equip $\mathbb{R}^n$ with $d\mu = w \, dx$, where $w$ is a $p$-admissible weight, and assume that $A$ satisfies the degenerate ellipticity conditions (3.3)–(3.7) in [14, p. 56] (with the parameter $p$). We also let $\Omega \subset \mathbb{R}^n$ be a nonempty open set and let $E \subset \Omega$ be a relatively closed subset such that no component of $\Omega$ is contained in $E$.

We will follow the notation in Heinonen–Kilpeläinen–Martio [14]. In particular, $p$-admissible weights are defined in Section 1.1 in [14]. See also Corollary 20.9 in [14] and Proposition A.17 in [3] for characterizations of $p$-admissible weights. We refer to Chapters 6 and 7 in [14] for the definitions of $A$-harmonic and $A$-superharmonic functions.

Definition 2.1. Let $Q \geq 1$. A function $u \in H^{1,p}_{\text{loc}}(\Omega, \mu)$ is a $Q$-quasisuperminimizer in $\Omega$ if

$$\int_{\varphi \neq 0} |\nabla u|^p \, d\mu \leq Q \int_{\varphi \neq 0} |\nabla (u + \varphi)|^p \, d\mu \quad \text{for all nonnegative } \varphi \in H^{1,p}_0(\Omega, \mu).$$

If (2.1) holds for all $\varphi \in H^{1,p}_0(\Omega, \mu)$, then $u$ is a $Q$-quasiminimizer. A $Q$-quasiharmonic function is a continuous $Q$-quasiminimizer.

These notions depend on $p$ and $\mu$, but we have refrained from making that explicit in the notation.

Kinnunen–Martio [16] introduced quasisuperharmonic functions and studied their potential theory. For us the following definition will be convenient, see [1,2,16] for equivalent definitions and characterizations of quasi(super)minimizers and quasi(super)harmonic functions.
**Definition 2.2.** A function $u : \Omega \rightarrow (-\infty, \infty]$ is $Q$-quasisuperharmonic in $\Omega$ if $u$ is not identically $\infty$ in any component of $\Omega$, $u$ is essliminf-regularized i.e.

$$u(x) = \text{ess lim inf}_{y \to x} u(y) \quad \text{for} \ x \in \Omega,$$

and $\min\{u, k\}$ is a $Q$-quasisuperminimizer in $\Omega$ for every $k \in \mathbb{R}$.

$\mathcal{A}$-harmonic and $\mathcal{A}$-superharmonic functions are $Q$-quasiharmonic resp. $Q$-quasisuperharmonic, with $Q$ only depending on $\mathcal{A}$, see [14, Section 3.13], whereas $p$-(super)harmonic functions are the same as 1-quasi(super)harmonic functions.

We next collect the key removability results for $\mathcal{A}$-harmonic and quasiharmonic functions, see [14, Theorem 7.36] and [2, Theorem 6.2].

**Theorem 2.3.** Assume that the Sobolev capacity $C_{p,\mu}(E) = 0$. Then $E$ is removable for bounded $\mathcal{A}$-harmonic and bounded $Q$-quasiharmonic functions in $\Omega \setminus E$, i.e. every bounded $\mathcal{A}$-harmonic (resp. $Q$-quasiharmonic) function in $\Omega \setminus E$ has a bounded $\mathcal{A}$-harmonic (resp. $Q$-quasiharmonic) extension to $\Omega$.

The following Harnack inequality for quasiharmonic functions is important. In the weighted setting it was first obtained by Kinnunen–Shanmugalingam [17, Corollary 7.3] (who obtained it in metric spaces).

**Theorem 2.4 (The Harnack Inequality).** There exists a constant $C$, depending only on $Q$, $p$ and $\mu$, such that for all $Q$-quasiharmonic functions $u \geq 0$ in $\Omega$ and all balls $B$ with $6B \subset \Omega$,

$$\sup_B u \leq C \inf_B u.$$

The following are two important consequences of the Harnack inequality.

**Theorem 2.5 (The Strong Maximum Principle).** A quasiharmonic function that attains its maximum in a connected open set $\Omega$ must be constant in $\Omega$.

**Theorem 2.6 (The Liouville Theorem).** A quasiharmonic function in $\Omega = \mathbb{R}^n$ which is bounded from below (or from above) must be constant.

**Definition 2.7.** The space $(\mathbb{R}^n, \mu)$ is $p$-parabolic if the variational capacity

$$\text{cap}_{p,\mu}(B, \mathbb{R}^n) = 0$$

for some ball $B$, or equivalently for all balls $B$.

If $(\mathbb{R}^n, \mu)$ is not $p$-parabolic then it is called $p$-hyperbolic.

It is easy to see that unweighted $\mathbb{R}^n$ is $p$-parabolic if and only if $p \geq n$. See [14, Theorem 9.22] and Proposition 3.3 for various characterizations of $p$-parabolicity. Björn–Björn–Lehrbäck [5, Theorem 5.5] recently showed that $(\mathbb{R}^n, \mu)$ is $p$-parabolic if and only if

$$\int_1^\infty \left( \frac{r}{\mu(B(0, r))} \right)^{1/(p-1)} dr = \infty. \quad (2.2)$$

They obtained this characterization in metric spaces under very mild assumptions.
3. Weighted $\mathbb{R}^n$ with $n \geq 2$; Theorems 1.2 and 1.3

In addition to the general assumptions from the beginning of Section 2, throughout this section we assume that $n \geq 2$.

Our aim in this section is to prove Theorem 1.3. Theorem 1.2 will then be a more or less direct consequence. To do so we need some lemmas.

Lemma 3.1. Let $F \subset \mathbb{R}^n$ be a bounded closed set and $\Omega = \mathbb{R}^n \setminus F$. Also let $u$ be a bounded quasiharmonic function in $\Omega$. Then $\lim_{x \to \infty} u(x)$ exists.

For $\mathcal{A}$-harmonic functions this is part of Lemma 6.15 in [14]. The proof is similar for quasiharmonic functions, but we are not aware of a suitable reference. So for the reader’s convenience we provide a proof. The idea for the proof is to use Harnack’s inequality on spheres. To make the proof a little more elementary we will only use the Harnack inequality (Theorem 2.4) above.

Proof. Without loss of generality we may assume that $F = \overline{B(0, R)}$. Let $m(r) = \min_{x \in \partial B(0, r)} u(x)$, $r > R$. As $u$ is continuous, so is $m$. Moreover, by the strong maximum principle (Theorem 2.5), $m$ does not have any local minimum (unless $u$ is constant). In particular, $M := \lim_{r \to \infty} m(r)$ exists (also if $u$ is constant). We may assume that $M = 0$. It follows that $\liminf_{x \to \infty} u(x) = 0$.

Let $\varepsilon > 0$ and find $\rho > R$ such that $\sup_{r \geq \rho} |m(r)| < \varepsilon$. Assume that $|x| > 2 \rho$ and let $\tau = |x|$. Find $z$ such that $|z| = \tau$ and $u(z) = m(\tau)$. Then we can find a sequence of balls $B_j = B(x_j, \frac{1}{6}\tau)$, $j = 0, \ldots, N$, where $x_0 = x$, $x_N = z$ and $x_j \in B_{j-1} \cap \partial B(0, \tau)$, $j = 1, \ldots, N$. Moreover $N$ can be chosen to only depend on the dimension $n$. By applying the Harnack inequality (Theorem 2.4) $N$ times to $v = u + \varepsilon$, we see that

$$v(x) = v(x_0) \leq C v(x_1) \leq \ldots \leq C^N v(x_N) = C^N v(z) \leq 2C^N \varepsilon.$$ 

Hence

$$\sup_{|x| > 2 \rho} u(x) < \sup_{|x| > 2 \rho} v(x) \leq 2C^N \varepsilon.$$ 

Letting $\varepsilon \to 0$ shows that $\limsup_{x \to \infty} u(x) \leq 0$. \hfill \Box

Lemma 3.2. Let $x_0 \in \Omega$ and let $u$ be a bounded quasiharmonic function in $\Omega \setminus \{x_0\}$. Then $\lim_{x \to x_0} u(x)$ exists.

Proof. The proof is essentially the same as the proof of Lemma 3.1. \hfill \Box

Proposition 3.3. Let $x_0 \in \mathbb{R}^n$ be such that $C_{p, \mu}(\{x_0\}) > 0$. Then the following are equivalent:

(a) $(\mathbb{R}^n, \mu)$ is $p$-hyperbolic;
(b) $\infty$ is regular with respect to $\mathbb{R}^n \setminus \{x_0\}$;
(c) there is a bounded nonconstant $\mathcal{A}$-harmonic function in $\mathbb{R}^n \setminus \{x_0\}$;
(d) there is a bounded nonconstant quasiharmonic function in $\mathbb{R}^n \setminus \{x_0\}$;

There are further characterizations in Theorem 9.22 in [14]; see also (2.2).

In the proof we will use $\mathcal{A}$-harmonic Perron solutions $H_{\mathcal{A}} f$ and boundary regularity, see Chapter 9 in [14]. When we discuss boundary regularity in this paper it is always with respect to $\mathcal{A}$-harmonic functions. (In this context, the dependence on $p$ and $\mu$ will be implicit.) In particular, we will use the Kellogg property which says that $C_{p, \mu}(I) = 0$, where $I$ is the set of finite irregular boundary points of an open set $\Omega$, see Theorem 9.11 in [14].
Proof. \((a) \Rightarrow (b)\) This follows from Theorem 9.22 in [14].

\((b) \Rightarrow (c)\) Let \(f(x_0) = 1\) and \(f(\infty) = 0\). Then \(u = H_{\Omega \cup \{x_0\}} f\) is \(A\)-harmonic. As \(x_0\) is regular, by the Kellogg property, we have \(\lim_{y \to x_0} u(y) = 1\). Moreover, as \(\infty\) is regular, by assumption, \(\lim_{x \to \infty} u(x) = 0\). Thus \(u\) is a bounded nonconstant \(A\)-harmonic function in \(\mathbb{R}^n \setminus \{x_0\}\).

\((c) \Rightarrow (d)\) This is trivial.

\((d) \Rightarrow (a)\) Let \(u\) be a bounded nonconstant \(Q\)-quasiharmonic function in \(\mathbb{R}^n \setminus \{x_0\}\). By Lemmas 3.1 and 3.2, both the limits \(u(x_0) := \lim_{x \to x_0} u(x)\) and \(u(\infty) := \lim_{x \to \infty} u(x)\) exist. As \(u\) is nonconstant, the strong maximum principle (Theorem 2.5) shows that \(u(x_0) \neq u(\infty)\). We may assume that \(u(\infty) = 0\) and \(u(x_0) = 1\).

Assume next that \((\mathbb{R}^n, \mu)\) is \(p\)-parabolic. Then \(\text{cap}_{p, \mu}(B(x_0, r), \mathbb{R}^n) = 0\). As the admissible functions testing the capacity have compact support, there is \(R > 0\) such that \(\text{cap}_{p, \mu}(B(x_0, r), B(x_0, R)) < \varepsilon\) and \(m := \min\{x-x_0| \leq R\} u(x) < \varepsilon\). Let \(K = \{x : u(x) \geq M\}\) and \(F = \{x \in \mathbb{R}^n : u(x) \leq m\}\). Then \(K \subset B(x_0, r)\) is compact and \(F \subset \mathbb{R}^n \setminus B(x_0, R)\) is closed, by the continuity of \(u\) and the strong maximum principle (Theorem 2.5).

Let \(v\) be the \(p\)-harmonic function in the bounded open set \(\Omega = \mathbb{R}^n \setminus (K \cup F)\) with boundary values 1 on \(K \cap \partial \Omega\) and 0 on \(F \cap \partial \Omega\) (in Sobolev or equivalently Perron sense, see [14, Corollary 9.29]). Also let \(\tilde{v} = (M - m)v + m\). Then \(\tilde{v} - u \in H_0^{1,p}(\Omega, \mu)\) and thus

\[
\int_{\Omega} |\nabla u|^p d\mu \leq Q \int_{\Omega} |\nabla \tilde{v}|^p d\mu \leq Q \int_{\Omega} |\nabla v|^p d\mu
\]

\[
= Q \text{cap}_{p, \mu}(K, \mathbb{R}^n \setminus F) \leq Q \text{cap}_{p, \mu}(B(x_0, r), B(x_0, R)) < Q \varepsilon.
\]

Letting \(\varepsilon \to \infty\) shows that

\[
\int_{\mathbb{R}^n \setminus \{x_0\}} |\nabla u|^p d\mu = 0,
\]

but this is impossible as \(u\) is nonconstant. Hence \((\mathbb{R}^n, \mu)\) must be \(p\)-hyperbolic. \(\square\)

Lemma 3.4. If \(\text{int} E \neq \emptyset\), then \(E\) is not removable for bounded \(A\)-harmonic functions in \(\Omega \setminus E\), nor for bounded quasiharmonic functions in \(\Omega \setminus E\).

This is a special case of Theorem 1.3, but it will convenient to have it taken care of when we turn to the proof of Theorem 1.3.

In the first paragraph of the proof of Lemma 3.4, we will use a well-known fact about separating sets in \(\mathbb{R}^n, n \geq 2\). In order to avoid sending the reader to “topological” papers we will provide a potential-theoretic argument sufficient for our purpose. A similar argument plays a role also in the second paragraph of the proof, but then yielding a potential-theoretic consequence needed to conclude the proof.

Proof of Lemma 3.4. Let \(\hat{\Omega}\) be a component of \(\Omega\) such that \(E' := \hat{\Omega} \cap \text{int} E \neq \emptyset\). Then \(E'' := \hat{\Omega} \cap \partial E'\) separates \(\hat{\Omega}\) into the two nonempty disjoint open sets \(E'\) and \(\hat{\Omega} \setminus (E' \cup E'')\). The latter set is nonempty as \(\hat{\Omega} \setminus (E' \cup E'') \supset \hat{\Omega} \setminus E \neq \emptyset\), by assumption.) Since sets of \(n\)-capacity zero do not separate open connected sets in \(\mathbb{R}^n\) (see [3, Lemma 4.6]), we must have \(C_n(E'') > 0\). Next, as singleton sets have \(C_n\)-capacity zero, the set \(E''\) must be uncountable, and since it is relatively closed it must contain a limit point. (The relative closedness is not needed, but it is perhaps easier to see this in this case.) Hence there are distinct \(x_j \in E'', j = 0, 1, \ldots\), such that \(|x_{j+1} - x_0| < d_j := \frac{1}{4}|x_j - x_0|, j = 1, 2, \ldots\).

Then the balls \(B_j = B(x_j, d_j), j = 1, 2, \ldots\), are pairwise disjoint, and \(B_j\) contains points both in \(G := \hat{\Omega} \setminus E\) and in \(\text{int} E \subset X \setminus \overline{G}\). Let \(F_j = B_j \setminus \bigcup_{k \neq j} B_k\), which is a relatively closed subset of the open set \(B_j\). Now \(B_j \cap \partial F_j\) separates \(B_j\) into the two nonempty open sets \(B_j \cap G\) and \(B_j \setminus \overline{G}\). Since again sets of capacity zero cannot separate open connected sets, we see that \(C_{p, \mu}(B_j \cap \partial F_j) > 0\). As \(B_j \cap \partial F_j \subset \partial G\),
the Kellogg property shows that there is a point $y_j \in B_j \cap \partial F_j$ that is regular with respect to $G$. Let $f_j(x) = (1 - |x - y_j|/d_j)_+$. Then $f_j$, $j = 1, 2, \ldots$, have pairwise disjoint support. Also let $f = \sum_{k=1}^\infty f_{2k}$ and $u = \Pi_G f$ (an $A$-harmonic upper Perron solution). By Lemma 9.6 in [14],

$$\lim_{G \ni y \to y_j} u(y) = \begin{cases} 1, & \text{if } j \text{ is even,} \\ 0, & \text{if } j \text{ is odd.} \end{cases}$$

It follows that $\lim_{G \ni y \to x_0} u(y)$ does not exist, and thus $u$ does not have any continuous extension to $\Omega$, let alone any $A$-harmonic or quasiharmonic extension. 

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** Assume first that (c) holds. If $C_{p,\mu}(E) = 0$, then both types of removability follow directly from Theorem 2.3 (with the quasiharmonicity constant $Q$ preserved). Assume therefore that there is $x_0 \in E$ such that $C_{p,\mu}(\{x_0\}) > 0$, $C_{p,\mu}(E \setminus \{x_0\}) = C_{p,\mu}(\mathbb{R}^n \setminus \Omega) = 0$ and $(\mathbb{R}^n, \mu)$ is $p$-parabolic.

Let $u$ be a bounded $A$-harmonic or bounded quasiharmonic function on $\Omega \setminus E$. Since $C_{p,\mu}((E \setminus \{x_0\}) \cup (\mathbb{R}^n \setminus \{x_0\})) = 0$ and $\mathbb{R}^n \setminus \{x_0\}$ is open, it follows from Theorem 2.3 that $(E \setminus \{x_0\}) \cup (\mathbb{R}^n \setminus \Omega)$ is removable, and hence that $u$ has a bounded $A$-harmonic or bounded quasiharmonic extension $U$ to $\mathbb{R}^n \setminus \{x_0\}$. As $(\mathbb{R}^n, \mu)$ is $p$-parabolic, it follows from Proposition 3.3 that $U$ is constant. Thus also $u$ is constant and trivially has a bounded $A$-harmonic (and quasiharmonic) extension to $\Omega$, i.e. $E$ is removable in both senses.

Conversely, assume that (c) fails. If $\text{int } E \neq \emptyset$, then (a) and (b) fail, by Lemma 3.4, so in the rest of the proof we can assume that $\text{int } E = \emptyset$.

By assumption $C_{p,\mu}(E) > 0$, and thus there is a component $\tilde{\Omega}$ of $\Omega$ such that $C_{p,\mu}(\tilde{\Omega} \cap E) > 0$. If $\tilde{G} := \tilde{\Omega} \setminus E$ is disconnected, then $\chi_V$, where $V$ is a component of $\tilde{G}$, is an $A$-harmonic function on $G$ with no continuous extension to $\tilde{\Omega}$, let alone any quasiharmonic extension, i.e. (a) and (b) fail. We can therefore assume that $E$ does not separate $\tilde{\Omega}$.

By the Kellogg property, there is a point $x_1 \in \tilde{\Omega} \cap E$ that is regular with respect $G := \Omega \setminus E$. It follows that $x_1 \in \partial \tilde{G}$. We now consider three mutually disjoint cases.

**Case 1.** $C_{p,\mu}(\mathbb{R}^n \setminus \Omega) > 0$.

As $\text{int } E = \emptyset$, we see that $\partial \Omega \subset \partial G$. And as sets of capacity zero cannot separate open sets (see [3, Lemma 4.6]), we see that $C_{p,\mu}(\partial G \cap \partial \Omega) > 0$. Hence, it follows from the Kellogg property that there is another point $x_2 \in \partial \tilde{G} \cap \partial \Omega$ that is regular with respect to $G$. Let $f(x) = (1 - |x - x_2|/|x_1 - x_2|)_+$ (with $f(\infty) = 0$). Then $u = H_G f$ is an $A$-harmonic function in $G$. As $x_1$ and $x_2$ are regular, $u|_G$ is nonconstant.

Assume that $u$ has a quasiharmonic extension $U$ to $\Omega$. By continuity and regularity (recall that $\text{int } E = \emptyset$) we find that $U \geq 0$ and that $U(x_1) = 0$. As $\tilde{\Omega}$ is connected this contradicts the strong maximum principle (Theorem 2.5), showing that $u$ cannot have an $A$-harmonic or quasiharmonic extension $U$ to $\Omega$, i.e. both (a) and (b) fail.

**Case 2.** $C_{p,\mu}(\mathbb{R}^n \setminus \Omega) = 0$ and $C_{p,\mu}(E \setminus \{x_1\}) > 0$.

As sets of capacity zero cannot separate open sets, $\Omega$ must be connected and thus $\tilde{G} = G$ and $E \subset \partial G$. Hence, by the Kellogg property there is another point $x_2 \in E \setminus \{x_1\}$ that is regular with respect to $G$. We can now proceed exactly as in case 1.

**Case 3.** $C_{p,\mu}(E \setminus \{x_1\}) = C_{p,\mu}(\mathbb{R}^n \setminus \Omega) = 0$ and $(\mathbb{R}^n, \mu)$ is $p$-hyperbolic.

By Proposition 3.3, there is a nonconstant bounded $A$-harmonic function $\tilde{u}$ in $\mathbb{R}^n \setminus \{x_1\}$. As $\text{int } E = \emptyset$, it follows that $u := \tilde{u}|_G$ is a nonconstant bounded $A$-harmonic function in $G$. Assume that $u$ has a bounded $A$-harmonic or bounded quasiharmonic extension $U$ to $\Omega$. Since $C_{p,\mu}(\mathbb{R}^n \setminus \Omega) = 0$, it follows from Theorem 2.3 that there is a further quasiharmonic extension $\tilde{U}$ to $\mathbb{R}^n$, but this contradicts the Liouville theorem (Theorem 2.6). 

□
Proof of Theorem 1.2. If $p \leq n$, then all singletons have zero capacity, and thus (c) in Theorem 1.3 is equivalent to $C_p(E) = 0$.

If on the other hand $p > n$ then every singleton has positive capacity and $\mathbb{R}^n$ is $p$-parabolic, and thus (c) in Theorem 1.3 is equivalent to “(a) or (b)” in Theorem 1.2.

In both cases, the characterization now follows directly from Theorem 1.3. □

The condition $C_{p,\mu}(E \setminus \{x_0\}) = 0$ in Theorem 1.3 can be rephrased as saying that $C_{p,\mu}$ restricted to $E$ is concentrated at (at most) one point. This condition was characterized in Lemma 10.6 in Björn–Björn–Shanmugalingam [7].

4. The one-dimensional real line $\mathbb{R}$; Theorem 1.4

In addition to the general assumptions from the beginning of Section 2, throughout this section we assume that $n = 1$.

As pointed out in [2, Example 9.1], already on (unweighted) $\mathbb{R}$ the situation is very different from $\mathbb{R}^n$, $n \geq 2$. In this case $p$-harmonic functions are nothing but affine functions $x \mapsto ax + b$ (independent of $p$). It is easy to see that $E = [a, \infty)$ is removable for bounded $p$-harmonic functions in $\mathbb{R} \setminus E$, for every $a \in \mathbb{R}$. Moreover, if $-\infty < a < b < \infty$, then every bounded $p$-harmonic function $u$ on $(a, b)$ has a $p$-harmonic extension to $\mathbb{R}$, although it is bounded only if $u$ is constant.

Let us now consider the weighted real line $(\mathbb{R}, \mu)$, where as before $d\mu = w \, dx$ and $w$ is a $p$-admissible weight, It follows from Björn–Buckley–Keith [9, Theorem 2] that $w$ is a Muckenhoupt $A_p$-weight, i.e. that

$$\sup_I \frac{1}{|I|} \int_I w \, dx \left( \frac{1}{|I|} \int_I w^{1/(1-p)} \, dx \right)^{p-1} < \infty,$$

where the supremum is over all bounded open intervals $I$ and $|\cdot|$ denotes the Lebesgue measure. (We assume, by definition, that intervals are nonempty.)

An $\mathcal{A}$-harmonic function $u$ is in this case nothing but a continuous weak solution of

$$\frac{d}{dx} A(x) w(x) |u(x)|^{p-2} u(x) = 0,$$

where $A$ is a positive measurable function a.e.-bounded away from 0 and $\infty$. Hence $\tilde{w} = A w$ is a.e.-comparable to $w$ and is thus also an $A_p$-weight. Moreover, $u$ is $\mathcal{A}$-harmonic with respect to $(\mathbb{R}, w)$ if and only if it is $p$-harmonic with respect to $(\mathbb{R}, \tilde{w})$.

We can now give a complete characterization of removability for $\mathcal{A}$-harmonic functions on $\mathbb{R}$. We start with the following complete characterization of weak removability, where by weakly removable we mean that the extensions are not required to be bounded. Note that components of open subsets of $\mathbb{R}$ are intervals.

Proposition 4.1. The set $E$ is weakly removable for bounded $\mathcal{A}$-harmonic functions in $\Omega \setminus E$, with respect to $(\mathbb{R}, w)$, if and only if $I \setminus E$ is connected for every component $I$ of $\Omega$.

Moreover, in this case the extensions are unique.

First note that by Lemma 6.2 in Björn–Björn–Shanmugalingam [8], a function $u$ is $\mathcal{A}$-harmonic in an open interval $I \subset \mathbb{R}$ if and only if it is given by

$$u(x) = b + a \int_0^x \tilde{w}^{1/(1-p)} \, dt, \quad x \in I,$$

where $a, b \in \mathbb{R}$.
Proof. As for the sufficiency, let \( u \) be an \( \mathcal{A} \)-harmonic function on \( \Omega \setminus E \). Let \( I \) be a component of \( \Omega \). As \( E \) does not separate \( I \), using (4.2) we can \( \mathcal{A} \)-harmonically extend \( u|_{I \setminus E} \) to \( I \), and this extension is unique. Doing this in each component, we end up with an \( \mathcal{A} \)-harmonic extension of \( u \) to \( \Omega \).

Conversely, if there is a component \( I \) of \( \Omega \) such that \( I \setminus E \) is not connected, we let \( V \) be a component of \( I \setminus E \). Then \( u = \chi_V \) is an \( \mathcal{A} \)-harmonic function on \( \Omega \setminus E \), which due to (4.2) does not have any \( \mathcal{A} \)-harmonic extension to \( \Omega \). \( \square \)

We next turn to the proof of Theorem 1.4 (in which case the extensions are required to be bounded).

Proof of Theorem 1.4. Let \( d\nu = v \, dx \), where \( v = \tilde{w}^{1/(1-p)} \). Since \( \tilde{w} \) is an \( A_p \)-weight, it follows directly from the \( A_p \)-condition (4.1) that \( v \) is an \( A_{p'} \)-weight, where \( p' = p/(p-1) \) is the dual exponent. In particular,

\[
\nu(I) < \infty \quad \text{if and only if} \quad I \text{ is bounded,} \tag{4.3}
\]

whenever \( I \) is an interval.

Assume that \( I \setminus E \) is connected for every component \( I \) of \( \Omega \). We first want to show that (1.1) is equivalent to (4.4). By [14, Lemma 15.5] we see that (1.1) implies that

\[
\frac{\nu(I \setminus E)}{\nu(I)} \geq c \left( \frac{|I \setminus E|}{|I|} \right)^{p'} \geq c \frac{1}{C'}
\]

if \( I \) is a bounded component, and hence that

\[
\nu(I) \leq C' \nu(I \setminus E). \tag{4.4}
\]

For unbounded \( I \), (4.4) follows directly from (1.1) and (4.3), since \( I \setminus E \) is connected. The converse implication \( \text{(4.4)} \Rightarrow \text{(1.1)} \) follows from Theorem 3 in Coifman–Fefferman [10] or [14, Lemma 15.8]. (The implication \( \text{(1.1)} \Rightarrow \text{(4.4)} \) can also be obtained using Coifman–Fefferman [10, Theorem 3 and Lemma 5].)

Next, we turn to the sufficiency, and let \( 0 \leq u \leq 1 \) be an \( \mathcal{A} \)-harmonic function on \( \Omega \setminus E \). Using (4.2) in each component separately, we can extend \( u \) to \( \Omega \) as an \( \mathcal{A} \)-harmonic function. It follows from (4.2) and (4.4) that \(|u| \leq C' \).

Conversely, if there is some component \( I \) of \( \Omega \) such that \( I \setminus E \) is disconnected, then \( E \) is not weakly removable, by Proposition 4.1, and hence not removable.

Next, consider the case when \( I \setminus E \) is connected for every component \( I \), but there is no constant \( C' \) as in (4.4). We need to consider two cases.

Case 1. There is a component \( I \) of \( \Omega \) such that \( \nu(I \setminus E) < \infty = \nu(I) \).

In this case, there is a nonconstant \( \mathcal{A} \)-harmonic function \( u \) in \( \Omega \setminus E \) which is identically 0 outside \( I \). It has a unique \( \mathcal{A} \)-harmonic extension \( U \) to \( \Omega \), by Proposition 4.1. By (4.2), \( u \) is bounded in \( \Omega \setminus E \) but \( U \) is unbounded in \( \Omega \).

Case 2. There is a sequence of components \( I_j \) of \( \Omega \) such that \( \nu(I_j) > j \nu(I_j \setminus E) \).

In this case there is an \( \mathcal{A} \)-harmonic function \( u \) in \( \Omega \setminus E \) such that

\[
\inf_{I_j \setminus E} u = 0 \quad \text{and} \quad \sup_{I_j \setminus E} u = 1 \quad \text{for} \quad j = 1, 2, \ldots ,
\]

and \( u \equiv 0 \) on \((\Omega \setminus \bigcup_{j=1}^{\infty} I_j) \setminus E \). It has a unique \( \mathcal{A} \)-harmonic extension \( U \) to \( \Omega \), by Proposition 4.1. By (4.2) we get that

\[
\sup_{I_j} U - \inf_{I_j} U \geq \frac{\nu(I_j)}{\nu(I_j \setminus E)} \left( \sup_{I_j \setminus E} u - \inf_{I_j \setminus E} u \right) > j,
\]

and thus \( U \) is unbounded. Hence \( E \) is not removable. \( \square \)
We now turn to quasiharmonic functions. In the metric space Examples 9.3 and 9.4 in Björn [2] it was shown that sets of positive measure can be removable for bounded $Q$-quasiharmonic functions. Using a result from Björn–Björn–Shanmugalingam [8], we can now give a similar example on weighted $\mathbb{R}$.

**Example 4.2.** Let $w$ be a $p$-admissible weight on $\mathbb{R}$, or equivalently an $A_p$-weight (see above). Let $\Omega = \mathbb{R}$ and let $E \subseteq \mathbb{R}$ be an unbounded closed interval. Then $I = \Omega \setminus E$ is an unbounded open interval. By (4.3), $\nu(I) = \infty$, where $d\nu = w^{1/(1-p)} \, dx$. It thus follows from Proposition 6.6 in [8] that every bounded quasiharmonic function on $I$ is constant, and hence trivially extends to $\mathbb{R}$. Thus, we have shown that $E$ is removable for bounded quasiharmonic functions in $\mathbb{R} \setminus E$, with respect to $(\mathbb{R}, w)$. Moreover, the quasiharmonicity constant (which is always 1) is preserved.

In this example, removability depended on the fact that all the quasiharmonic functions under consideration were constant. The same is true for the removable sets with positive capacity provided by Examples 9.3 and 9.4 in Björn [2] and Theorems 1.2 and 1.3.

However, we can now give the following example, showing that sets of positive capacity (and positive measure) can be removable for bounded quasiharmonic functions even when nonconstant quasiharmonic functions are under consideration.

**Example 4.3.** Assume that $\mathbb{R}$ is unweighted. Let $\Omega = (-1, 1)$ and $E = (-1, 0]$. Let $u$ be a bounded $Q$-quasiharmonic function on $\Omega \setminus E = (0, 1)$. Then $u$ is monotone and $u(0) := \lim_{x \to 0^+} u(x)$ exists. By Martio’s reflection principle [19, Theorem 4.1] the “odd” reflection

$$u^*(x) = \begin{cases} u(x), & 0 \leq x < 1, \\ 2u(0) - u(-x), & -1 < x \leq 0, \end{cases}$$

is a bounded $Q'$-quasiharmonic function on $\Omega$. Martio obtained the reflection principle with $Q' = 2^p Q$. This was improved to $Q' = \max\{2, 2^{p-1}\} Q$ by Uppman [25, Theorem 2.3], who also showed that the factor $\max\{2, 2^{p-1}\}$ is best possible independent of $Q$. For

$$Q < \max\left\{\frac{1}{2 - 2^{1/p}}, \frac{1}{2 - 2^{1/(p-1)}}\right\}$$

he obtained the better bound $Q' = Q(2 - Q^{-1/p})^p$, see [25, Theorem 2.4].

Thus we have obtained a removability result with an increase in the quasiharmonicity constant. As there could be other extensions than those given by reflection, it is not clear if the increase is really needed.

By reflecting several times one can extend $u$ to a quasiharmonic function in any bounded open interval. However for unbounded intervals the increase (repeated infinitely many times) destroys the bounds for the quasiharmonicity constant of the extended function $\tilde{u}$, unless $u$ is constant. Moreover, $\tilde{u}$ would be unbounded, by Proposition 6.6 in Björn–Björn–Shanmugalingam [8].

Nevertheless, with repeated use of the reflection principle we can show the following characterization of removable sets for bounded quasiharmonic functions on unweighted $\mathbb{R}$, under the same condition as in Theorem 1.4.

**Theorem 4.4.** The set $E$ is removable for bounded quasiharmonic functions in $\Omega \setminus E$, with respect to unweighted $\mathbb{R}$, if and only if there is a constant $C$ such that for every component $I$ of $\Omega$ it is true that

$$I \setminus E \text{ is connected \ and \ } |I| \leq C |I \setminus E|.$$
More precisely, if \( u \) is a bounded \( Q \)-quasiharmonic function on \( \Omega \setminus E \) and (4.7) holds with \( C = 2^N \), then \( u \) has a bounded \( Q' \)-quasiharmonic extension to \( \Omega \), where \( Q' = \max\{2, 2^{p - 1}\}^{N+1}Q \).

Conversely, if (4.7) fails, then there is bounded \( p \)-harmonic function on \( \Omega \setminus E \) with no bounded quasiharmonic extension to \( \Omega \).

When \( Q \) satisfies (4.6) one can obtain a better (but more complicated) bound \( Q'(Q, p, N) \) using Theorem 2.4 in Uppman [25], see Example 4.3. In particular, \( Q'(Q, p, N) \to 1 \), as \( Q \to 1 \). Moreover, \( Q'(1, p, N) = 1 \) and we thus recover (the unweighted case of) Theorem 1.4.

**Lemma 4.5.** Let \( Q \geq 1 \) and \( u(t) = t, 0 < t < 1 \). Then there is an increasing function \( f_Q : (1, \infty) \to \mathbb{R} \) such that \( \lim_{x \to \infty} f_Q(x) = \infty \) and such that if \( U \) is a \( Q \)-quasiharmonic extension of \( u \) to \((0, x)\), where \( x > 1 \), then \( U(x) := \lim_{t \to x} U(t) \geq f_Q(x) \).

**Proof.** Let \( U \) be a \( Q \)-quasiharmonic extension of \( u \) to \((0, x)\), where \( x > 1 \). Then \( U \) must be increasing and so \( a = \lim_{t \to x} U(t) \) exists. Let \( f_Q(x) = \inf a \) over all such extensions \( U \), which also must be an increasing function. Next, let

\[
v(t) = \frac{at}{x}, \quad 0 \leq t \leq x, \quad \text{and} \quad V(t) = \begin{cases} \frac{t}{1 + \frac{(a-1)(t-1)}{x-1}}, & 0 \leq t \leq 1, \\ 1, & 1 \leq t \leq x. \end{cases}
\]

Then \( v \) is \( p \)-harmonic in \((0, x)\) and \( V \) is \( p \)-harmonic in \((1, x)\). Since \( V(0) = U(0) := \lim_{t \to 0} U(t) \), \( V(1) = U(1) \), \( V(x) = U(x) \), and \( V \) minimizes the \( p \)-energy among functions with the same boundary values on \((0, 1)\) as well as on \((1, x)\), we see that

\[
\int_0^x (U')^p \, dt \geq \int_0^x (V')^p \, dt = 1 + \frac{(a-1)^p}{(x-1)^{p-1}}.
\]

On the other hand, by testing the quasiminimizer condition on \((0, x)\) we also get that

\[
\int_0^x (U')^p \, dt \leq Q \int_0^x (v')^p \, dt = Q a p^{p-1}.
\]

Hence

\[
a^p Q \geq x^{p-1} + \frac{(a-1)^p}{(x-1)^{p-1}} x^{p-1} = x^{p-1} + (a-1)^p \left(1 - \frac{1}{x}\right)^{1-p}
\]

and thus also

\[
f_Q(x)^p Q \geq x^{p-1} + (f_Q(x) - 1)^p \left(1 - \frac{1}{x}\right)^{1-p}.
\]

Since the right-hand side in (4.8) tends to \( \infty \), as \( x \to \infty \), also \( \lim_{x \to \infty} f_Q(x) = \infty \). \( \Box \)

**Proof of Theorem 4.4.** Assume first that the condition in the statement is satisfied with constant \( C = 2^N \). Let \( u \) be a bounded \( Q \)-quasiharmonic function in \( \Omega \setminus E \). We may assume that \( 0 \leq u \leq 1 \). In each component \( I \) of \( \Omega \), we can extend \( u \) from \( I \setminus E \) to \( I \) by using Martio’s reflection principle (see Example 4.3) at most \( N + 1 \) times; first at most \( N \) times in the direction (left/right) needing most reflections, and then at most once in the other direction. The resulting function \( U \) is \( Q' \)-quasiharmonic function on \( I \), and hence on \( \Omega \), where \( Q' = \max\{2, 2^{p - 1}\}^{N+1}Q \). It follows from (4.5) that \( \sup_I U - \inf_I U \leq 2^{N+1} \) and hence that \( |U| \leq 2^{N+1} + 1 \) in \( \Omega \) (as \( I \setminus E \neq \emptyset \)).

We now turn to the converse. Assume first that there is a component \( I \) of \( \Omega \) such that \( I \setminus E \) is not connected. Let \( A \) be a component of \( I \setminus E \) and \( u = \chi_A \), which is a \( p \)-harmonic function in \( \Omega \setminus E \). Now it is well known that a quasiharmonic function in an open interval \( I' \) which is constant in an open subinterval \( I'' \subset I' \)
must be constant throughout $I'$, see e.g. Martio–Sbordone [20, Lemma 2]. Thus $u$ has no quasiharmonic extension to $\Omega$.

Next, we consider the case when $I \setminus E$ is connected for every component $I$, but there is no constant $C$ as in (4.7). As in the proof of Theorem 1.4, we need to consider two cases.

Case 1. There is a component $I$ of $\Omega$ such that $|I \setminus E| < \infty = |I|$.

In this case, there is a nonconstant $p$-harmonic function $u$ in $\Omega \setminus E$ which is identically $0$ outside $I$. Let $U$ be any quasiharmonic extension of $u$ to $\Omega$. Then, by Lemma 4.5, $U$ must unbounded in $I$.

Case 2. There is a sequence of components $I_j$ of $\Omega$ such that $|I_j| > j|I_j \setminus E|$.

In this case, there is a nonconstant $p$-harmonic function $u$ in $\Omega \setminus E$ such that

$$\inf_{I_j \setminus E} u = 0 \quad \text{and} \quad \sup_{I_j \setminus E} u = 1 \quad \text{for} \quad j = 1, 2, \ldots,$$

and $u \equiv 0$ on $(\Omega \setminus \bigcup_{j=1}^{\infty} I_j) \setminus E$. Let $U$ be any $Q$-quasiharmonic extension of $u$ to $\Omega$. Then by Lemma 4.5,

$$\sup_{I_j} U - \inf_{I_j} U \geq f_Q\left(\frac{|I_j|}{2|I_j \setminus E|}\right) \geq f_Q\left(\frac{j}{2}\right).$$

(The factor $2$ comes from extending first in the direction (left/right) needing furthest extension.) Hence $U$ must be unbounded, and since $Q$ was arbitrary, $u$ does not have any bounded quasiharmonic extension to $\Omega$. □

5. Removability for $A$-superharmonic functions; Theorem 1.5

In addition to the general assumptions from the beginning of Section 2, throughout this section we assume that $n \geq 2$.

In this section we want to prove Theorem 1.5. For $A$-superharmonic functions we could have used the fact that they are $p$-finely continuous (see Section 12 and especially Theorem 12.8 in [14]) and at the same time $\text{ess} \lim \inf$-regularized. The latter is part of Definition 2.2 also for quasisuperharmonic functions, but it is not known if quasisuperharmonic functions are $p$-finely continuous. Instead, the following weaker type of continuity, related to measure density, will be sufficient for our purpose.

**Lemma 5.1.** Assume that $u$ is quasisuperharmonic in $\Omega$, that $x \in \Omega$ and that

$$L = \limsup_{r \to 0} \frac{\mu(\{y \in B(x, r) : u(y) \geq \alpha\})}{\mu(B(x, r))} > 0,$$

then $u(x) \geq \alpha$.

**Proof.** We may assume that $\alpha = 0$. Assume on the contrary that $u(x) < 0$. Let $\varepsilon > 0$. As $u$ is lower semicontinuous there is $r_0 > 0$ such that $B(x, r_0) \subset \Omega$ and $u(y) > u(x) - \varepsilon$ for $y \in B(x, r_0)$. Let $v = u - u(x) + \varepsilon > 0$ in $B(x, r_0)$. By assumption, there is $0 < r < r_0/5$ such that

$$\frac{\mu(\{y \in B(x, r) : u(y) \geq 0\})}{\mu(B(x, r))} > \frac{L}{2}.$$

By the weak Harnack inequality for quasisuperharmonic functions in Kinnunen–Martio [16, second display on p. 479], there are $\sigma > 0$ and $c > 0$ such that

$$|u(x)| \left(\frac{L}{2}\right)^{1/\sigma} \leq \left(\frac{1}{\mu(B(x, r))} \int_{B(x, r)} v^\sigma \, d\mu\right)^{1/\sigma} \leq c \inf_{B(x, 3r)} v \leq cv(x) = \varepsilon.$$

Letting $\varepsilon \to 0$ gives a contradiction, and thus the result is proved. □
Proof of Theorem 1.5. If $C_{p,\mu}(E) = 0$, then it follows from [14, Theorem 7.35] and Björn [2, Theorem 6.3] that $E$ is removable as in (a)--(d), with preserved quasisuperharmonicity constant in (b) and (d).

Assume conversely that $C_{p,\mu}(E) > 0$. We shall construct a bounded $\mathcal{A}$-superharmonic function $u$ on $\Omega \setminus E$ which has no quasisuperharmonic extension to $\Omega$ (neither bounded nor unbounded), from which the nonremovability in all four senses (a)--(d) follows. We begin with some reductions.

First of all, we can find a component $\tilde{\Omega}$ of $\Omega$ such that $C_{p,\mu}(E \cap \tilde{\Omega}) > 0$. If we then can construct a bounded $\mathcal{A}$-superharmonic function $\tilde{u}$ in $\tilde{\Omega} \setminus E$ which has no quasisuperharmonic extension to $\tilde{\Omega}$, then it follows that

$$u = \begin{cases} \tilde{u}, & \text{in } \tilde{\Omega} \setminus E, \\ 0, & \text{in } (\Omega \setminus \tilde{\Omega}) \setminus E, \end{cases}$$

is a bounded $\mathcal{A}$-superharmonic function in $\Omega \setminus E$ without quasisuperharmonic extension to $\Omega$. We may thus assume, without loss of generality, that $\Omega$ is connected.

Let

$$E_0 = \{ x \in E : C_{p,\mu}(E \cap B) = 0 \text{ for some ball } B \ni x \}.$$ 

Then $E_0$ is a relatively open subset of $E$ which can be written as a countable union of sets of capacity zero, and thus $C_{p,\mu}(E_0) = 0$. By the first part of the proof, $E_0$ is thus removable in all four senses, and since quasisuperharmonic functions are ess inf-regularized, the extensions are unique and are also bounded resp. bounded from below. Thus we may first remove $E_0$ and proceed by showing that $E \setminus E_0$ is not removable.

Let $E_0 = \{ x \in E : C_{p,\mu}(E \cap B) = 0 \text{ for some ball } B \ni x \}$.

Then $E_0$ is a relatively open subset of $E$ which can be written as a countable union of sets of capacity zero, and thus $C_{p,\mu}(E_0) = 0$. By the first part of the proof, $E_0$ is thus removable in all four senses, and since quasisuperharmonic functions are ess inf-regularized, the extensions are unique and are also bounded resp. bounded from below. Thus we may first remove $E_0$ and proceed by showing that $E \setminus E_0$ is not removable.

We may thus assume, without loss of generality, that $E_0 = \emptyset$.

Next, let $z \in E \cap \partial E$. Then $\text{dist}(z, \partial E) > 0$ and we can find $\xi \in E$ such that

$$\text{dist}(\xi, z) < \frac{1}{4} \min\{\text{dist}(z, \partial E), 1\}.$$ 

(If $\Omega = \mathbb{R}^n$, we consider $\text{dist}(z, \partial E)$ to be $\infty$.) Finally, we find $\zeta \in E$ so that

$$|\zeta - \xi| = \text{dist}(\xi, E) < \text{dist}(\xi, \partial E).$$ 

We may assume that $\zeta = 0$ and $\xi = (1, 0, \ldots, 0)$.

It follows that the open interval $(0, \xi) \subset \Omega \setminus E$ and that $B(0, 1) \subset \Omega$. We now consider two cases. As in Section 3, we will use $\mathcal{A}$-harmonic Perron solutions and boundary regularity.

Case 1. There is $0 < r < 1$ such that $\{ x \in E : |x| = r \} = \emptyset$.

Let $K = E \cap B(0, r)$, which is a compact set with positive capacity (as $E_0 = \emptyset$), and $G = B(0, r) \setminus K$. Then $\chi_{\partial B(0,r)} \in C(\partial G)$ and we can define

$$u = \begin{cases} H_G \chi_{\partial B(0,r)}, & \text{in } G, \\ 1, & \text{in } \Omega \setminus (E \cup G), \end{cases}$$

where $H_G \chi_{\partial B(0,r)}$ is an $\mathcal{A}$-harmonic Perron solution. Then $u$ is $\mathcal{A}$-harmonic in $G$. Moreover, $u$ is continuous in $\Omega \setminus E$ because all the points in $\partial B(0, r)$ are regular for $G$. It thus follows from the pasting lemma [14, Lemma 7.9] that $u$ is $\mathcal{A}$-superharmonic in $\Omega \setminus E$.

As $C_{p,\mu}(K) > 0$, the Kellogg property shows that $u$ is not constant. If $u$ had a quasisuperharmonic extension $\tilde{u}$ to $\Omega$ it would attain its minimum in $\overline{B(0, r)} \subset \Omega$, and that minimum must be $< 1$ and lie in $B(0, r)$. But this contradicts the strong minimum principle for quasisuperharmonic functions (which for quasisuperharmonic functions follows from the weak Harnack inequality in Kinnunen–Martio [16, second display on p. 479]). This case is thus settled.

Case 2. $\{ x \in E : |x| = r \} \neq \emptyset$ for $0 < r < 1$.

For each $x \in B(0, r) \setminus \{0\}$, let $\theta_x = \arccos(\xi \cdot x/|x|)$ be the angle which $x$ makes with $\xi$. For each $j = 1, 2, \ldots$, there is

$$x_j \in K_j := \{ x \in E : 4^{-j} \leq |x| \leq 2 \cdot 4^{-j} \} \text{ such that } \theta_{x_j} = \min\{\theta_x : x \in K_j\}.$$
Since 0 is a closest point to $\xi$ in $E$, we must have $\theta_{x_j} > \pi/4$. Let $B_j = B(x_j, 4^{-j-2})$. Then there is $y_j \in B_j$ such that $B_j' := B(y_j, 4^{-j-4}) \subset B_j \cap (\Omega \setminus E)$.

Let $A = \bigcup_{j=1}^{\infty} B_j$, $G = B(0, 1) \setminus (E \cup A)$,

$$f(x) = (1 - 4^2(|x| - 1))_+ + \sum_{j=1}^{\infty} (1 - 4^{j+2} \text{dist}(x, B_j))_+. \quad (5.1)$$

For each $x$, at most one term in (5.1) is positive and thus $0 \leq f \leq 1$. Moreover $f$ is continuous except at $x = 0$. Next, let

$$u = \begin{cases} \overline{P}_G f, & \text{in } G, \\ 1, & \text{in } \Omega \setminus (E \cup G), \end{cases}$$

where $\overline{P}_G f$ is an $A$-harmonic upper Perron solution. All the points in

$$\partial G \cap \left( \partial B(0,1) \cup \bigcup_{j=1}^{\infty} \partial B_j \right)$$

are regular with respect to $G$. Hence, by [14, Lemma 9.6], $u$ is continuous in $\Omega \setminus E$. It thus follows from the pasting lemma [14, Lemma 7.9] that $u$ is $A$-superharmonic in $\Omega \setminus E$.

Assume that $u$ has a quasisuperharmonic extension $U$ to $\Omega$. Since $U = u \equiv 1$ in $\bigcup_{j=1}^{\infty} B_j'$, it follows from Lemma 5.1 and Björn–Björn [3, Lemma 3.3] that $U(0) \geq 1$. On the other hand, because

$$\{x \in E : |x| = 3 \cdot 4^{-j}\} \neq \emptyset \quad \text{and} \quad E_0 = \emptyset,$$

it follows from the Kellogg property that there is a regular point $y_j \in \partial G$ with $\frac{5}{2} \cdot 4^{-j} < |y_j| < \frac{7}{2} \cdot 4^{-j}$, $j = 1, 2, \ldots$. By [14, Lemma 9.6] again,

$$\lim_{G \ni x \to y_j} u(x) = f(y_j) = 0.$$

Since $\lim_{j \to \infty} y_j = 0$ and $U$ is lower semicontinuous, it follows that $U(0) \leq 0$, contradicting the above. Thus also this case is settled. □

Acknowledgments

The author was supported by the Swedish Research Council, Sweden, grant 2016-03424. The author is grateful to Juha Heinonen, Tero Kilpeläinen, Nageswari Shanmugalingam and Jana Björn for fruitful discussions.

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