CROSSED PRODUCTS OF LOCALLY

$C^*$-ALGEBRAS

MARIA JOIȚA

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ABSTRACT. The crossed products of locally $C^*$-algebras are defined
and a Takai duality theorem for inverse limit actions of a locally compact
group on a locally $C^*$-algebra is proved.

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1 Introduction

Locally $C^*$-algebras are generalizations of $C^*$-algebras. Instead of being
given by a single $C^*$-norm, the topology on a locally $C^*$-algebra is defined
by a directed family of $C^*$-semi-norms. In [9], Phillips defines the notion
of action of a locally compact group $G$ on a locally $C^*$-algebra $A$ whose
topology is determined by a countable family of $C^*$-semi-norms, and also
defines the crossed product of $A$ by an inverse limit action $\alpha = \lim_{n} \alpha^{(n)}$ as
being the inverse limit of crossed products of $A_n$ by $\alpha^{(n)}$. In this paper,
by analogy with the case of $C^*$-algebras, we define the concept of crossed
product, respectively reduced crossed product of locally $C^*$-algebras.
The Takai duality theorem says that if $\alpha$ is a continuous action of an abelian locally compact group $G$ on a $C^*$-algebra $A$, then we can recover the system $(G, A, \alpha)$ up to stable isomorphism from the double dual system in which $G = \hat{G}$ acts on the crossed product $(A \times_\alpha G) \times_{\hat{\alpha}} \hat{G}$ by the dual action of the dual group. In [3], Imai and Takai prove a duality theorem for $C^*$-crossed products by a locally compact group that generalizes the Takai duality theorem [12]. For a given $C^*$-dynamical system $(G, A, \alpha)$, they construct a "dual" $C^*$-crossed product of the reduced crossed product $A \times_{\alpha,r} G$ by an isomorphism $\beta$ from $A \times_{\alpha,r} G$ into $L(H)$, the $C^*$-algebra of all bounded linear operators on some Hilbert space $H$, and show that this is isomorphic to the tensor product $A \otimes K(L^2(G))$ of $A$ and $K(L^2(G))$, the $C^*$-algebra of all compact operators on $L^2(G)$. If $G$ is commutative, the "dual" $C^*$-crossed product constructed by Imai and Takai is isomorphic to the double crossed product $(A \times_{\alpha} G) \times_{\hat{\alpha}} \hat{G}$. Katayama [6] shows that a non-degenerate coaction $\beta$ of a locally compact group on a $C^*$-algebra $A$ induces an action $\tilde{\beta}$ of $G$ on the crossed product $A \times_{\beta} G$ and proves that the $C^*$-algebras $(A \times_{\beta} G) \times_{\tilde{\beta},r} \hat{G}$ and $A \otimes K(L^2(G))$ are isomorphic. In [13], Vallin shows that there is a bijective correspondence between the set of all actions of a locally compact group $G$ on a $C^*$-algebra $A$ and the set of all actions of the commutative Kac $C^*$-algebra $C^*K_G^2$ associated with $G$ on $A$. A coaction of $G$ on $A$ is an action of the symmetric Kac $C^*$-algebra $C^*K_G^2$ associated with $G$. If $G$ is commutative, we can identified $C^*_r(G)$ with $C_0(\hat{G})$ via the Fourier transform, whence becomes clear that a coaction of $G$ is the same thing as an action of $G$. Thus we can regard the coactions of a locally compact group $G$ as "actions of the dual group even there isn’t any dual group". Also, Vallin shows that an action $\alpha$ (coaction $\beta$) of $G$ on $A$ induces
a coaction $\hat{\alpha}$ (action $\hat{\beta}$) of $G$ on the crossed product $A \times_{\alpha,r} G$ (respectively $A \times_{\beta} G$) and proves a version of the Takai duality theorem showing that the double crossed product $(A \times_{\alpha,r} G) \times_{\hat{\alpha}} G$ is isomorphic to $A \otimes K(L^2(G))$. We propose to prove a version of the Takai duality theorem for crossed products of locally $C^*$-algebras.

The paper is organized follows. In Section 2 we present some basic definitions and results about locally $C^*$-algebras and Kac $C^*$-algebras. In Section 3 we define the notion of crossed product (reduced crossed product) of a locally $C^*$-algebra $A$ by an inverse limit action $\alpha$ of a locally compact group $G$ and prove some basic properties of these. Section 4 is devoted to actions of a Kac $C^*$-algebra on a locally $C^*$-algebra. We show that there is a bijective correspondence between the set of all inverse limit actions of a locally compact group $G$ on a locally $C^*$-algebra $A$ and the set of all inverse limit actions of the commutative Kac $C^*$-algebra $C^*K_G$ on $A$, Proposition 4.4. As a consequence of this result we obtain: for a compact group $G$, any action of the Kac $C^*$-algebra $C^*K_G^b$ on $A$, $p \in S(A)$. In Section 5, using the same arguments as in [13], we show that any inverse limit action $\alpha$ (coaction $\beta$) of a locally compact group $G$ on a locally $C^*$-algebra $A$ induces an inverse limit coaction $\hat{\alpha}$ (action $\hat{\beta}$) of $G$ on the crossed product $A \times_{\alpha,r} G$ (respectively $A \times_{\beta} G$), Proposition 5.5. Finally, we prove that if $\alpha$ is an inverse limit action of a locally compact group $G$ on a locally $C^*$-algebra $A$, then there is an isomorphism of locally $C^*$-algebras from $(A \times_{\alpha,r} G) \times_{\hat{\alpha}} G$ onto $A \otimes K(L^2(G))$ and the inverse limit actions $\hat{\alpha}$ and $\alpha \otimes \text{ad} \rho$ are equivalent, Theorem 5.6.
2 Preliminaries

A locally $C^*$-algebra is a complete complex Hausdorff topological $*$-algebra $A$ whose topology is determined by a family of $C^*$-semi-norms, see [1], [2], [4], [9], [10]. If $S(A)$ is the set of all continuous $C^*$-semi-norms on $A$, then for each $p \in S(A)$, $A_p = A/\ker(p)$ is a $C^*$-algebra with respect to the norm induced by $p$, and $A = \varprojlim_{p \in S(A)} A_p$. The canonical maps from $A$ onto $A_p$, $p \in S(A)$ are denoted by $\pi_p$, the image of $a$ under $\pi_p$ by $a_p$, and the connecting maps of the inverse system $\{A_p\}_{p \in S(A)}$ by $\pi_{pq}$, $p, q \in S(A)$ with $p \geq q$.

A morphism of locally $C^*$-algebras is a continuous $*$-morphism $\Phi$ from a locally $C^*$-algebra $A$ to a locally $C^*$-algebra $B$. An isomorphism of locally $C^*$-algebras is a morphism of locally $C^*$-algebras which is invertible and its inverse is a morphism of locally $C^*$-algebras. An $S$-morphism of locally $C^*$-algebras is a morphism $\Phi : A \rightarrow M(B)$, where $M(B)$ is the multiplier algebra of $B$, with the property that for any approximate unit $\{e_i\}_i$ of $A$ the net $\{\Phi(e_i)\}_i$ converges to 1 with respect to the strict topology on $M(B)$. If $\Phi : A \rightarrow M(B)$ is an $S$-morphism of locally $C^*$-algebras, then it extends to a unique morphism $\overline{\Phi} : M(A) \rightarrow M(B)$ of locally $C^*$-algebras, see [5].

A Kac $C^*$-algebra is a quadruple $K = (B, d, j, \varphi)$, where $B$ is a $C^*$-algebra, $d$ is a comultiplication on $B$, $j$ is a coinvolution on $B$, and $\varphi$ is a semi-finite, lower semi-continuous, faithful weight on $B$, see [13].

Let $A$ and $B$ be two locally $C^*$-algebras. The injective tensor product of the locally $C^*$-algebras $A$ and $B$ is denoted by $A \otimes B$, see [2], and the locally $C^*$-subalgebra of $M(A \otimes B)$ generated by the elements $x$ in $M(A \otimes B)$ such that $x(1 \otimes B) + (1 \otimes B)x \subseteq A \otimes B$ is denoted by $M(A, B)$. If $G$ is a
locally compact group, then \( M(A, C_0(G)) \) may be identified with the locally \( C^* \)-algebra \( C_b(G,A) \) of all bounded continuous functions from \( G \) to \( A \).

Let \( G \) be a locally compact group. \( C^*K^a_G = (C_0(G), d^a_G, j^a_G, ds) \) is the commutative Kac \( C^* \)-algebra associated with \( G \) and \( C^*K^s_G = (C^*_r(G), d^s_G, j^s_G, \varphi_G) \) is the symmetric Kac \( C^* \)-algebra associated with \( G \), see [13].

An action of a Kac \( C^* \)-algebra \( \mathbf{K} = (B, d, j, \varphi) \) on a \( C^* \)-algebra \( A \) is an injective \( S \)-morphism \( \alpha \) from \( A \) to \( M(A,B) \) such that \((\alpha \otimes \text{id}) \circ \alpha = (\text{id}_A \otimes \sigma_B \circ d) \circ \alpha \), see [13].

3 Crossed products

Let \( A \) be a locally \( C^* \)-algebra and let \( G \) be a locally compact group.

Definition 3.1 An action of \( G \) on \( A \) is a morphism \( \alpha \) from \( G \) to \( \text{Aut}(A) \), the set of all isomorphisms of locally \( C^* \)-algebras from \( A \) to \( A \). The action \( \alpha \) is continuous if the function \((t,a) \to \alpha_t(a)\) from \( G \times A \) to \( A \) is jointly continuous.

Definition 3.2 A locally \( C^* \)-dynamical system is a triple \((G,A,\alpha)\), where \( G \) is a locally compact group, \( A \) is a locally \( C^* \)-algebra and \( \alpha \) is a continuous action of \( G \) on \( A \).

Definition 3.3 We say that \( \{ (G,A_\delta, \alpha^{(\delta)}_t) \}_{\delta \in \Delta} \) is an inverse system of \( C^* \)-dynamical systems if \( \{ A_\delta \}_{\delta \in \Delta} \) is an inverse system of \( C^* \)-algebras and for each \( t \) in \( G, \{ \alpha_t^{(\delta)} \}_{\delta \in \Delta} \) is an inverse system of \( C^* \)-isomorphisms.
Let $A = \lim_{\delta \in \Delta} A_\delta$ and $\alpha_t = \lim_{\delta \in \Delta} \alpha_t^{(\delta)}$ for each $t \in G$. Then the map $\alpha : G \to \text{Aut}(A)$ defined by $\alpha(t) = \alpha_t$ is a continuous action of $G$ on $A$ and $(G, A, \alpha)$ is a locally $C^*$-dynamical system. We say that $(G, A, \alpha)$ is the inverse limit of the inverse system of $C^*$-dynamical systems $\left\{ \left( G_\delta, \alpha^{(\delta)}_t \right) \right\}_{\delta \in \Delta}$.

**Definition 3.4** A continuous action $\alpha$ of $G$ on $A$ is an inverse limit action if we can write $A$ as inverse limit $\lim_{\delta \in \Delta} A_\delta$ of $C^*$-algebras in such a way that there are actions $\alpha^{(\delta)}$ of $G$ on $A_\delta$ such that $\alpha_t = \lim_{\delta \in \Delta} \alpha_t^{(\delta)}$ for all $t$ in $G$ (Definition 5.1, [9]).

**Remark 3.5** The action $\alpha$ of $G$ on $A$ is an inverse limit action if there is a cofinal subset of $G$-invariant continuous $C^*$-semi-norms on $A$ (a continuous $C^*$-semi-norm $p$ on $A$ is $G$-invariant if $p(\alpha_t(a)) = p(a)$ for all $a$ in $A$ and for all $t$ in $G$).

The following lemma is Lemma 5.2 of [9].

**Lemma 3.6** Any continuous action of a compact group $G$ on a locally $C^*$-algebra $A$ is an inverse limit action.

Let $(G, A, \alpha)$ be a locally $C^*$-dynamical system such that $\alpha$ is an inverse limit action. By Remark 3.5, we can suppose that $S(A)$ coincides with the set of all $G$-invariant continuous $C^*$-semi-norms on $A$.

Let $C_c(G, A)$ be the vector space of all continuous functions from $G$ to $A$ with compact support.
Lemma 3.7 Let $f \in C_c(G, A)$. Then there is a unique element $\int_G f(s)ds$ in $A$ such that for any non-degenerate $\ast$-representation $(\varphi, H_\varphi)$ of $A$

$$\left< \varphi(\int_G f(s)ds)\xi, \eta \right> = \int_G \left< \varphi(f(s))\xi, \eta \right> ds$$

for all $\xi, \eta$ in $H_\varphi$. Moreover, we have:

(1) $p(\int_G f(s)ds) \leq M \sup\{p(f(s)); s \in \text{supp}(f)\}$ for some positive number $M$ and for all $p \in S(A)$;

(2) $(\int_G f(s)ds)a = \int_G f(s)ads$ for all $a \in A$;

(3) $\Phi(\int_G f(s)ds) = \int_G \Phi(f(s))ds$ for any morphism of locally $C^*$-algebras $\Phi : A \to B$;

(4) $(\int_G f(s)ds)^* = \int_G f(s)^*ds$.

Proof. Let $p \in S(A)$. Then $\pi_p \circ f \in C_c(G, A_p)$ and so there is a unique element $\int_G (\pi_p \circ f)(s)ds$ in $A_p$ such that for any non-degenerate $\ast$-representation $(\varphi_p, H_{\varphi_p})$ of $A_p$

$$\left< \varphi_p(\int_G (\pi_p \circ f)(s)ds)\xi, \eta \right> = \int_G \left< \varphi_p((\pi_p \circ f)(s))\xi, \eta \right> ds$$

for all $\xi, \eta$ in $H_{\varphi_p}$, see, for instance, Lemma 7 of [11].

To show that $(\int_G (\pi_p \circ f)(s)ds)_p$ is a coherent net in $A$, let $p, q \in S(A)$ with $p \geq q$. Then we have

$$\pi_{pq}(\int_G (\pi_p \circ f)(s)ds)_p = \int_G \pi_{pq}((\pi_p \circ f)(s)) ds$$

using Lemma 7 of [11]

$$= \int_G (\pi_q \circ f)(s)ds.$$
Therefore \((f_p \circ f)(s)ds\) \(_p \in A\), and we define \(\int f(s)ds = (\int (\pi_p \circ f)(s)ds)\_p\).

Suppose that there is another element \(b\) in \(A\) such that for any non-degenerate \(*\)-representation \((\varphi, H_\varphi)\) of \(A\)

\[
\langle \varphi(b) \xi, \eta \rangle = \int_G \langle \varphi(f(s)) \xi, \eta \rangle \, ds
\]

for all \(\xi, \eta\) in \(H_\varphi\). Then for any \(p \in S(A)\) and for any non-degenerate \(*\)-representation \((\varphi_p, H_{\varphi_p})\) of \(A_p\)

\[
\langle \varphi_p(\pi_p(b)) \xi, \eta \rangle = \int_G \langle \varphi_p((\pi_p \circ f)(s)) \xi, \eta \rangle \, ds
\]

for all \(\xi, \eta\) in \(H_{\varphi_p}\). From these facts and Lemma 7 of [11], we conclude that

\[
\pi_p(b) = \int_G (\pi_p \circ f)(s)ds
\]

for all \(p \in S(A)\). Therefore \(b = \int_G f(s)ds\) and the uniqueness is proved.

Using Lemma 7 of [11] it is easy to check that \(\int_G f(s)ds\) satisfies the conditions (1) – (4). ■

Let \(f, h\) in \(C_c(G, A)\). It is easy to check that the map \((s, t) \rightarrow f(t)\alpha_t(h(t^{-1}s))\)
from \(G \times G\) to \(A\) is an element in \(C_c(G \times G, A)\) and the relation

\[
(f \times h)(s) = \int_G f(t)\alpha_t(h(t^{-1}s)) \, dt
\]

defines an element in \(C_c(G, A)\), called the convolution of \(f\) and \(h\). Also it is not hard to check that \(C_c(G, A)\) becomes a \(*\)-algebra with convolution as product and involution defined by

\[
f^*(t) = \gamma(t)^{-1} \alpha_t(f(t^{-1})^*)
\]

where \(\gamma\) is the modular function on \(G\).
For any \( p \in S(A) \), define \( N_p \) from \( C_c(G,A) \) to \([0,\infty)\) by
\[
N_p(f) = \int_G p(f(s))ds.
\]

Straightforward computations show that \( N_p, p \in S(A) \) are submultiplicative \(*\)-semi-norms on \( C_c(G,A) \).

Let \( L^1(G,A,\alpha) \) be the Hausdorff completion of \( C_c(G,A) \) with respect to the topology defined by the family of submultiplicative \(*\)-semi-norms \( \{N_p\}_{p \in S(A)} \). Then by Theorem III 3.1 of [7]
\[
L^1(G,A,\alpha) = \Bigg( \lim_{p \in S(A)} L^1(G,A,\alpha) \Bigg)_p
\]
where \( (L^1(G,A,\alpha))_p \) is the completion of the \(*\)-algebra \( C_c(G,A)/\ker(N_p) \) with respect to the norm \( \|\cdot\|_p \) induced by \( N_p \).

**Lemma 3.8** Let \((G,A,\alpha)\) be a locally \( C^*\)-dynamical system such that \( \alpha \) is an inverse limit action. Then
\[
(L^1(G,A,\alpha))_p = L^1(G,A_p,\alpha^{(p)})
\]
for all \( p \in S(A) \), up to a topological algebraic \(*\)-isomorphism.

**Proof.** Let \( p \in S(A) \) and \( f \) in \( C_c(G,A) \). Then
\[
\|f + \ker(N_p)\|_p = \int_G p(f(s))ds = \int_G \|\pi_p(f(s))\|_p ds = \|\pi_p \circ f\|_1.
\]
Therefore we can define a linear map \( \psi_p \) from \( C_c(G,A)/\ker(N_p) \) to \( C_c(G,A_p) \) by
\[
\psi_p(f + \ker(N_p)) = \pi_p \circ f.
\]
It is not hard to check that \( \psi_p \) is a \(*\)-morphism, and since \( \psi_p \) is an isometric \(*\)-morphism from \( C_c(G,A)/\ker(N_p) \) to \( C_c(G,A_p) \), it can be
uniquely extended to an isometric \( \ast \)-morphism \( \psi_p \) from \( (L^1(G,A,\alpha))_p \) to \( L^1\left(G, A_p, \alpha^{(p)}\right) \).

To show that \( \psi_p \) is surjective, let \( a \in A \) and \( f \in C_c(G) \). Define \( \tilde{f} \) from \( G \) to \( A \) by \( \tilde{f}(s) = f(s)a \). Clearly \( \tilde{f} \in C_c(G,A) \) and

\[
\psi_p \left( \tilde{f} + \ker(N_p) \right)(s) = f(s)\pi_p(a)
\]

for all \( s \) in \( G \). This implies that

\[
A_p \otimes_{\text{alg}} C_c(G) \subseteq \psi_p \left( \left( L^1(G,A,\alpha) \right)_p \right) \subseteq L^1(G,A_p,\alpha^{(p)})
\]

whence, since \( A_p \otimes_{\text{alg}} C_c(G) \) is dense in \( L^1(G,A_p,\alpha^{(p)}) \) and since \( \psi_p \) is an isometric \( \ast \)-morphism, we deduce that \( \psi_p \) is surjective and the proposition is proved.

**Corollary 3.9** Let \( (G,A,\alpha) \) be a locally \( C^\ast \)-dynamical system such that \( \alpha \) is an inverse limit action. Then

\[
L^1(G,A,\alpha) = \lim_{\leftarrow \atop {p \in S(A)}} L^1\left(G,A_p,\alpha^{(p)}\right)
\]

up to an algebraic and topological \( \ast \)-isomorphism.

**Remark 3.10** If \( \{e_i\}_{i \in I} \) is an approximate unit for \( A \) and \( \{f_j\}_{j \in J} \) is an approximate unit for \( L^1(G) \), then \( \{\tilde{f}_{(i,j)}\}_{(i,j) \in I \times J} \), where \( \tilde{f}_{(i,j)}(s) = f_j(s)e_i \), \( s \in G \), is an approximate unit for \( L^1(G,A,\alpha) \), see Lemma XIV.1.2 of [7]. Then by Definition 5.1 of [1], we can construct the enveloping algebra of \( L^1(G,A,\alpha) \).
Definition 3.11 A covariant representation of \((G, A, \alpha)\) is a triple \((\varphi, u, H)\), where \((\varphi, H)\) is a \(*\)-representation of \(A\) and \((u, H)\) is a unitary representation of \(G\) such that

\[
\varphi(\alpha_t(a)) = u_t \varphi(a) u_t^*
\]

for all \(t \in G\) and for all \(a \in A\).

We say that the covariant representation \((\varphi, u, H)\) of \((G, A, \alpha)\) is non-degenerate if the \(*\)-representation \((\varphi, H)\) of \(A\) is non-degenerate.

Remark 3.12 (1). If \((\varphi, u, H)\) is a covariant representation of \((G, A, \alpha)\) such that \(\|\varphi(a)\| \leq p(a)\) for all \(a \in A\), then there is a unique covariant representation \((\varphi_p, u, H)\) of the \(C^*\)-dynamical system \((G, A_p, \alpha^{(p)})\) such that \(\varphi_p \circ \pi_p = \varphi\).

(2). If \((\varphi_p, u, H)\) is a covariant representation of the \(C^*\)-dynamical system \((G, A_p, \alpha^{(p)})\), then \((\varphi_p \circ \pi_p, u, H)\) is a covariant representation of the locally \(C^*\)-dynamical system \((G, A, \alpha)\).

If \(R(G, A, \alpha)\) denotes the non-degenerate covariant representations of \((G, A, \alpha)\), then it is easy to check that

\[
R(G, A, \alpha) = \bigcup_{p \in S(A)} R_p(G, A, \alpha)
\]

where \(R_p(G, A, \alpha) = \{ (\varphi, u, H) \in R(G, A, \alpha); \|\varphi(a)\| \leq p(a)\) for all \(a \in A\}\). Also it is easy to check that the map \(\varphi_p \mapsto \varphi_p \circ \pi_p\) from \(R(G, A_p, \alpha^{(p)})\) to \(R_p(G, A, \alpha)\) is bijective.

Proposition 3.13 Let \((G, A, \alpha)\) be a locally \(C^*\)-dynamical system such that \(\alpha\) is an inverse limit action. Then there is a bijection between the covariant non-degenerate representations of \((G, A, \alpha)\) and the non-degenerate \(*\)-representations of \(L^1(G, A, \alpha)\).
Proof. Let \((\varphi, u, H) \in R(G, A, \alpha)\). Then, there is \(p \in S(A)\) and \((\varphi_p, u, H) \in R(G, A_p, \alpha^{(p)})\) such that \(\varphi = \varphi_p \circ \pi_p\). Since \((\varphi_p, u, H) \in R(G, A_p, \alpha^{(p)})\) there is a unique non-degenerate \(*\)-representation \((\varphi_p \times u, H)\) of \(L^1(G, A_p, \alpha^{(p)})\) such that

\[
(\varphi_p \times u)(f) = \int_G \varphi_p(f(t)) u_t dt
\]

for all \(f \in L^1(G, A_p, \alpha^{(p)})\), see, for instance, Proposition 7.6.4 of [8].

Let \(\varphi \times u = (\varphi_p \times u) \circ \tilde{\pi}_p\), where \(\tilde{\pi}_p\) is the canonical map from \(L^1(G, A, \alpha)\) to \(L^1(G, A_p, \alpha^{(p)})\), \(\tilde{\pi}_p(f) = \pi_p \circ f\) for all \(f\) in \(L^1(G, A, \alpha)\). Then, clearly \((\varphi \times u, H)\) is a non-degenerate \(*\)-representation of \(L^1(G, A, \alpha)\) and moreover,

\[
(\varphi \times u)(f) = (\varphi_p \times u)(\pi_p \circ f) = \int_G \varphi_p((\pi_p \circ f)(t)) u_t dt = \int_G \varphi(f(t)) u_t dt
\]

for all \(f \in L^1(G, A, \alpha)\). Thus we have obtained a map \((\varphi, u, H) \rightarrow (\varphi \times u, H)\) from \(R(G, A, \alpha)\) to \(R(L^1(G, A, \alpha))\). To show that this map is bijective, let \((\Phi, H)\) be a non-degenerate \(*\)-representation of \(L^1(G, A, \alpha)\). Then there is \(p \in S(A)\) and a non-degenerate \(*\)-representation \((\Phi_p, H)\) of \(L^1(G, A_p, \alpha^{(p)})\) such that \(\Phi = \Phi_p \circ \pi_p\). By Proposition 7.6.4 of [8] there is a unique non-degenerate covariant representation \((\varphi_p, u, H)\) of \((G, A_p, \alpha^{(p)})\) such that \((\phi_p, H) = (\varphi_p \times u, H)\). Therefore there is a non-degenerate covariant representation \((\varphi, u, H)\) of \((G, A, \alpha)\), where \(\varphi = \varphi_p \circ \pi_p\), such that \((\Phi, H) = (\varphi \times u, H)\).

To show that \((\varphi, u, H)\) is unique, let \((\psi, v, K)\) be another non-degenerate covariant representation of \((G, \alpha, A)\) such that \((\psi \times v, K) = (\Phi, H)\). Then there is \(q \in S(A)\) with \(q \geq p\) such that \((\psi, v, K) \in R_q(G, A, \alpha)\) and \((\Phi, K) \in R_q(L^1(G, A, \alpha))\). Therefore \(\Phi = \Phi_q \circ \tilde{\pi}_q\) with \((\Phi_q, H) \in R \left(L^1(G, A_q, \alpha^{(q)})\right)\) and \(\psi = \psi_q \circ \pi_q\) with \((\psi_q, v, K) \in R \left(G, A_q, \alpha^{(q)}\right)\) and moreover, \((\Phi_q, H) =
\((\psi_q \times v, K)\).

On the other hand, \((\varphi_p \circ \pi_{pq}, u, H) \in \mathcal{R}(G, A_q, \alpha(q))\) and

\[
((\varphi_p \circ \pi_{pq}) \times u)(f) = \int_G (\varphi_p \circ \pi_{pq})(f(t))u_t dt = \int_G \varphi_p(\tilde{\pi}_{pq}(f)(t)) u_t dt
\]

\[= \phi_p(\tilde{\pi}_{pq}(f)) = (\phi_p \circ \tilde{\pi}_{pq})(f) = \phi_q(f)
\]

for all \(f \in L^1(G, A_q, \alpha(q))\). From these facts and Proposition 7.6.4 of [8], we conclude that the covariant representations \((\psi_q, v, K)\) and \((\varphi_p \circ \pi_{pq}, u, H)\) of \((G, A_q, \alpha(q))\) coincide, and so the covariant representations \((\psi, v, K)\) and \((\varphi, u, H)\) of \((G, A, \alpha)\) coincide.

**Definition 3.14** Let \((G, A, \alpha)\) be a locally \(C^*\)-dynamical system such that \(\alpha\) is an inverse limit action. The crossed product of \(A\) by the action \(\alpha\), denoted by \(A \times_\alpha G\), is the enveloping algebra of the complete locally \(m\)-convex \(*\)-algebra \(L^1(G, A, \alpha)\).

**Remark 3.15** By Corollary 3.9 and Corollary 5.3 of [2], \(A \times_\alpha G\) is a locally \(C^*\)-algebra and

\[A \times_\alpha G = \lim_{\leftarrow \substack{p \in S(A) \\text{such that } \alpha(p) = G}}} A_p \times_{\alpha(p)} G\]

up to an isomorphism of locally \(C^*\)-algebras.

**Proposition 3.16** Let \((G, A, \alpha)\) be a locally \(C^*\)-dynamical system such that \(\alpha\) is an inverse limit action. Then there is a bijection between non-degenerate covariant representations of \((G, A, \alpha)\) and the non-degenerate representations of \(A \times_\alpha G\).
Proof. Since $A \times_{\alpha} G$ is the enveloping locally $C^*$-algebra of the complete locally $m$-convex $*$-algebra $L^1(G, A, \alpha)$, there is a bijection between the non-degenerate representations of $A \times_{\alpha} G$ and the non-degenerate representations of $L^1(G, A, \alpha)$, [2, pp. 37]. From this fact and Proposition 3.13 we conclude that there is a bijection between the non-degenerate representations of $A \times_{\alpha} G$ and the non-degenerate covariant representations of $(G, A, \alpha)$.

For each $p \in S(A)$, we denote by $(\varphi_{p,u}, H_{p,u})$ the universal representation of $A_p$ and by $(\varphi_p, H_{p,u})$ the representation of $A$ associated with $(\varphi_{p,u}, H_{p,u})$ (that is, $\varphi_p = \varphi_{p,u} \circ \pi_p$).

Lemma 3.17 Let $(G, A, \alpha)$ be a locally $C^*$-dynamical system such that $\alpha$ is an inverse limit action. Then $(\widetilde{\varphi}_p, \lambda, L^2(G, H_{p,u}))$, where

$$\widetilde{\varphi}_p (a) (\xi) (t) = \varphi_p (\alpha_t^{-1} (a)) (\xi (t))$$

and

$$\lambda_s (\xi) (t) = \xi (s^{-1} t)$$

for all $a$ in $A$, $\xi$ in $L^2(G, H_{p,u})$ and $s, t$ in $G$, is a non-degenerate covariant representation of $(G, A, \alpha)$.

Proof. It is a simple verification. ■

Let $p \in S(A)$. The map $r_p : L^1(G, A, \alpha) \to [0, \infty)$ defined by

$$r_p(f) = \| (\widetilde{\varphi}_p \times \lambda) (f) \|$$

is a $C^*$-semi-norm on $L^1(G, A, \alpha)$ with the property that $r_p(f) \leq N_p(f)$ for all $f$ in $L^1(G, A, \alpha)$. 

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Let \( I = \bigcap_{p \in S(A)} \ker (r_p) \). Clearly \( I \) is a closed two-sided ideal of \( L^1(G, A, \alpha) \) and \( L^1(G, A, \alpha) / I \) is a pre-locally \( C^* \)-algebra with respect to the topology determined by the family of \( C^* \)-semi-norms \( \{ \widehat{r}_p \}_{p \in S(A)} \), \( \widehat{r}_p(f + I) = \inf \{ r_p(f + h); h \in I \} \).

**Definition 3.18** The reduced crossed product of \( A \) by the action \( \alpha \), denoted by \( A \times_{\alpha, r} G \), is the Hausdorff completion of \( \left( L^1(G, A, \alpha), \{ r_p \}_{p \in S(A)} \right) \) (that is, \( A \times_{\alpha, r} G \) is the completion of the pre-locally \( C^* \)-algebra \( \left( L^1(G, A, \alpha) / I, \{ \widehat{r}_p \}_{p \in S(A)} \right) \)).

**Lemma 3.19** Let \( (G, A, \alpha) \) be a locally \( C^* \)-dynamical system such that \( \alpha \) is an inverse limit action. Then

\[
(A \times_{\alpha, r} G)_p = A_p \times_{\alpha(p), r} G
\]

for all \( p \in S(A) \), up to an isomorphism of \( C^* \)-algebras.

**Proof.** Let \( p \in S(A) \). If \( f \in L^1(G, A, \alpha) \), then we have

\[
\| (f + I) + \ker(\widehat{r}_p) \|_{\widehat{r}_p} = \widehat{r}_p(f + I) = \inf \{ \| (\widehat{\varphi}_p \times \lambda)(f + h) \|; h \in I \}
\]

\[
= \inf \{ \| (\widehat{\varphi}_p \times \lambda)(f) \|; h \in I \} = r_p(f) = \| f + \ker(r_p) \|_{r_p}.
\]

From this relation, we conclude that \( (A \times_{\alpha, r} G)_p \) is isomorphic to the completion of \( L^1(G, A, \alpha) / \ker(r_p) \) with respect to the \( C^* \)-norm induced by \( r_p \).

On the other hand, \( A_p \times_{\alpha(p), r} G \) is the completion of \( L^1(G, A_p, \alpha^{(p)}) / I_p \), where \( I_p = \{ f \in L^1(G, A_p, \alpha^{(p)}) / (\widehat{\varphi}_{p, u} \times \lambda)(f) = 0 \} \), with respect to the norm \( \| \| \) \( ^r \) given by \( \| f + I_p \| ^r = \| (\widehat{\varphi}_{p, u} \times \lambda)(f) \| \leq \| f \|_1 \). But the completion of \( L^1(G, A, \alpha) / \ker(r_p) \) with respect to the norm \( \| \|_{r_p} \) is isomorphic to the
completion of $L^1\left(G, A, \alpha^{(p)}\right)/I_p$ with respect to the norm $\|\cdot\|'$, since

$$
\|f + \ker(r_p)\|_{r_p} = r_p(f) = \|\widetilde{\phi_p} \times \lambda \circ f\| = \|\pi_p(f) + I_p\|'
$$

for all $f \in L^1(G, A, \alpha)$. Therefore the $C^*$-algebras $(A \times_{\alpha, r} G)_p$ and $A_p \times_{\alpha^{(p)}, r} G$ are isomorphic. □

**Corollary 3.20** If $(G, A, \alpha)$ is a locally $C^*$-dynamical system such that $\alpha$ is an inverse limit action then

$$
A \times_{\alpha, r} G = \lim_{\leftarrow \delta \in \Delta} A_p \times_{\alpha^{(p)}, r} G
$$

up to an isomorphism of locally $C^*$-algebras.

4 **Actions of a Kac $C^*$-algebra on a locally $C^*$-algebra**

Let $C^*K = (B, d, j, \varphi)$ be a Kac $C^*$-algebra and let $A$ be a locally $C^*$-algebra.

**Definition 4.1** An action of $C^*K$ on $A$ is an injective $S$-morphism $\alpha$ from $A$ to $M(A, B)$ such that

$$
(\alpha \otimes id_B) \circ \alpha = (id_A \otimes (\sigma_B \circ d)) \circ \alpha.
$$

An action $\alpha$ of $C^*K$ on $A$ is an inverse limit action if we can write $A$ as an inverse limit $\lim_{\leftarrow \delta \in \Delta}$ such that there are actions $\alpha^{(\delta)}$ of $C^*K$ on $A_\delta$, $\delta \in \Delta$, such that $\alpha = \lim_{\leftarrow \delta \in \Delta} \alpha^{(\delta)}$. 

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Two actions $\alpha_1$ and $\alpha_2$ of $C^*\mathbb{K}$ on the locally $C^*$-algebras $A_1$ respectively $A_2$ are said to be equivalent if there is an isomorphism of locally $C^*$-algebras $\Phi : A_1 \to A_2$ such that $\alpha_2 \circ \Phi = (\Phi \otimes \text{id}_{B}) \circ \alpha_1$.

**Proposition 4.2** Let $G$ be a locally compact group. If $\alpha$ is an action of $C^*\mathbb{K}_G$ on $A$, then the map $\Sigma(\alpha)$ that applies $t \in G$ to a map $\Sigma(\alpha)_t$ from $A$ to $A$ defined by $\Sigma(\alpha)_t(a) = \alpha(t^{-1})a$, is a continuous action of $G$ on $A$.

**Proof.** Since $\alpha$ is a continuous $*$-morphism from $A$ to $C_b(G,A)$, $\Sigma(\alpha)_t$ is a continuous $*$-morphism from $A$ to $A$ for each $t \in G$. Using the same arguments as in the proof of Proposition 5.1.5 of [13], it is not difficult to see that $\Sigma(\alpha)_t$ is invertible and moreover, $(\Sigma(\alpha)_t)^{-1} = \Sigma(\alpha)_{t^{-1}}$ for all $t \in G$. Therefore $\Sigma(\alpha)_t \in \text{Aut}(A)$ for each $t \in G$.

To show that the map $(t, a) \to \Sigma(\alpha)_t(a)$ from $G \times A$ to $A$ is continuous, let $(t_0, a_0) \in G \times A$ and let $W_{p,\varepsilon} = \{ a \in A; p(a - \Sigma(\alpha)_{t_0}(a_0)) < \varepsilon \}$ be a neighborhood of $\Sigma(\alpha)_{t_0}(a_0)$. Since $\alpha(a_0) \in C_b(G,A)$, there is a neighborhood $U_0$ of $t_0$ such that
\[
p\left(\alpha(a_0)(t^{-1}) - \alpha(a_0)(t_0^{-1})\right) < \frac{\varepsilon}{2}
\]
for all $t$ in $U_0$, and since $\alpha$ is a continuous $*$-morphism, there is a neighborhood $V_0$ of $a_0$ such that
\[
\|\alpha(a) - \alpha(a_0)\|_p = \sup\{p(\alpha(a)(t) - \alpha(a_0)(t)); t \in G\} < \frac{\varepsilon}{2}
\]
for all $a$ in $V_0$. Then
\[
p\left(\Sigma(\alpha)_t(a) - \Sigma(\alpha)_{t_0}(a_0)\right) \leq p\left(\alpha(a)(t^{-1}) - \alpha(a_0)(t^{-1})\right)
\]
\[
+ p\left(\alpha(a_0)(t^{-1}) - \alpha(a_0)(t_0^{-1})\right)
\]
\[
\leq \|\alpha(a) - \alpha(a_0)\|_p + \frac{\varepsilon}{2} < \varepsilon
\]
for all \((t, a) \in U_0 \times V_0\) and the proposition is proved. ■

**Remark 4.3** According to Proposition 4.2, we can define a map \(\Sigma\) from the set of all actions of \(C^*K_G\) on \(A\) to the set of all continuous actions of \(G\) on \(A\) by \(\alpha \rightarrow \Sigma(\alpha)\). Moreover, \(\Sigma\) is injective.

The following proposition is a generalization of Proposition 5.1.5 of [13] for inverse limit actions of locally compact groups on locally \(C^*\)-algebras.

**Proposition 4.4** Let \(G\) be a locally compact group. Then the map \(\Sigma\) defined in Proposition 4.2 is a bijective correspondence between the set of all inverse limit actions of \(C^*K_G\) on \(A\) and the set of all continuous inverse limit actions of \(G\) on \(A\).

**Proof.** Let \(\alpha\) be an inverse limit action of \(C^*K_G\) on \(A\). Then \(A\) may be written as an inverse limit \(\lim_{\delta \in \Delta} A_\delta\) of \(C^*\)-algebras and there are actions \(\alpha^{(\delta)}\) of \(C^*K_G\) on \(A_\delta\), \(\delta \in \Delta\) such that \(\alpha = \lim_{\delta \in \Delta} \alpha^{(\delta)}\).

According to Proposition 5.1.5 of [13], for each \(\delta \in \Delta\) there is a continuous action \(\Sigma(\alpha^{(\delta)})\) of \(G\) on \(A_\delta\) such that \(\Sigma(\alpha^{(\delta)})(a_\delta)(t^{-1}) = \alpha^{(\delta)}(a_\delta)(t^{-1})\) for all \(a_\delta\) in \(A_\delta\) and for all \(t\) in \(G\). Since \(\{\alpha^{(\delta)}\}_{\delta \in \Delta}\) is an inverse system of \(C^*\)-algebras, it is not difficult to check that \(\{\Sigma(\alpha^{(\delta)})(t)\}_{\delta \in \Delta}\) is an inverse system of \(C^*\)-isomorphisms for each \(t\) in \(G\). Also it is easy to check that \(\Sigma(\alpha)(t) = \lim_{\delta \in \Delta} \Sigma(\alpha^{(\delta)})(t)\) for each \(t\) in \(G\).

To show that \(\Sigma\) is surjective, let \(\beta\) be a continuous inverse limit action of \(G\) on \(A\). Then \(A\) may be written as an inverse limit \(\lim_{\delta \in \Delta} A_\delta\) of \(C^*\)-algebras and there are continuous actions \(\beta^{(\delta)}\) of \(G\) on \(A_\delta\) such that \(\beta_t = \lim_{\delta \in \Delta} \beta^{(\delta)}(a_\delta)(t^{-1})\) for all \(a_\delta\) in \(A_\delta\) and for all \(t\) in \(G\). Since \(\{\beta^{(\delta)}\}_{\delta \in \Delta}\) is an inverse system of \(C^*\)-algebras, it is not difficult to check that \(\Sigma(\beta)(t) = \lim_{\delta \in \Delta} \Sigma(\beta^{(\delta)})(t)\) for each \(t\) in \(G\).
β_t(δ) for each t in G. By Proposition 5.1.5 of [13], for each δ ∈ Δ there is an action α(δ) of C∗KG on A_δ such that Σ(α(δ)) = β(δ). It is not difficult to verify that \{α(δ)\}_{δ∈Δ} is an inverse system of injective S-morphisms of C∗-algebras. Let α = \lim_{δ∈Δ} α(δ). Then α is an injective S-morphism of locally C∗-algebras and

\( (\alpha \otimes \text{id}_{C_0(G)}) \circ \alpha = \lim_{δ∈Δ} (\alpha(δ) \otimes \text{id}_{C_0(G)}) \circ \alpha(δ) = \lim_{δ∈Δ} (\text{id}_{A_δ} \otimes \sigma_{C_0(G)} \circ \delta_G) \circ \alpha(δ) = (\text{id}_A \otimes \sigma_{C_0(G)} \circ \delta_G) \circ \alpha. \)

Therefore α is an inverse limit action of C∗KG on A and Σ(α) = β. Thus we showed that Σ is bijective. ■

**Corollary 4.5** If G is compact, then any action of C∗KG on A is an inverse limit action.

**Proof.** Let α be an action of C∗KG on A. By Proposition 4.2, Σ(α) is a continuous action of G on A which is an limit inverse action, since the group G is compact, Lemma 3.6. From this fact and Proposition 4.4 we conclude that α is an inverse limit action. ■

5 **The Takai duality theorem**

Let G be a locally compact group and let A be a locally C∗-algebra.

**Lemma 5.1** Let α be an inverse limit action of G on A. Then the reduced crossed product of A by the action α is isomorphic to the locally C∗-
subalgebra of \( M(A \otimes \mathcal{L}(L^2(G))) \) generated by \( \{ \alpha(a)(1_{M(A)} \otimes \lambda(f)) : a \in A, f \in C_c(G) \} \), where \( \lambda \) is the left regular representation of \( L^1(G) \).

**Proof.** Let \( p \in S(A) \). By Remark 5.2.1.1 of [13], the map \( \Phi_p \) from the \( C^* \)-subalgebra of \( M(A_p \otimes \mathcal{L}(L^2(G))) \) generated by \( \{ \alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f)) : a_p \in A_p, f \in C_c(G) \} \) to \( A_p \times \alpha^{(p)},r G \), that applies \( \alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f)) \) to \( \tilde{f} + I_p \), where \( \tilde{f}(t) = f(t)a_p, t \in G \), see the proof of Lemma 3.19, is an isomorphism of \( C^* \)-algebras.

If \( \pi'_{pq}, p, q \in S(A), p \geq q \) are the connecting maps of the inverse system \( \{ M(A_p \otimes \mathcal{L}(L^2(G))) \}_{p \in S(A)} \) and \( \hat{\pi}_{pq}, p, q \in S(A), p \geq q \) are the connecting maps of the inverse system \( \{ A_p \times \alpha^{(p)},r G \}_{p \in S(A)} \), then we have

\[
(\Phi_q \circ \pi'_{pq})(\alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f))) = \Phi_q(\alpha^{(q)}(\pi_{pq}(a_p))(1_{M(A_q)} \otimes \lambda(f)))
\]

\[
= \pi_{pq}(a_p) \otimes f + I_p = \tilde{\pi}_{pq}(a_p \otimes f) + I_p
\]

\[
= (\tilde{\pi}_{pq} \circ \Phi_p)(\alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f)))
\]

for all \( a_p \) in \( A_p \), for all \( f \) in \( C_c(G) \) and for all \( p, q \in S(A) \) with \( p \geq q \).

Therefore \( \{ \Phi_p \}_{p \in S(A)} \) is an inverse system of isomorphisms of \( C^* \)-algebras and the lemma is proved. \( \blacksquare \)

**Definition 5.2** A coaction of \( G \) on \( A \) is an action \( \beta \) of \( C^*\mathcal{K}^*_G \) on \( A \). We say that a coaction \( \beta \) of \( G \) on \( A \) is an inverse limit coaction if it is an inverse limit action of \( C^*\mathcal{K}^*_G \) on \( A \).

The reduced crossed product of \( A \) by the coaction \( \beta \), denoted by \( A \times_\beta G \), is the locally \( C^* \)-subalgebra of \( M(A \otimes \mathcal{L}(L^2(G))) \) generated by \( \{ \beta(a)(1_{M(A)} \otimes f) : a \in A, f \in C_c(G) \} \).
Remark 5.3 Let \( \beta = \lim\limits_{\delta \in \Delta} \beta^{(\delta)} \) be an inverse limit coaction of \( G \) on \( A \) such that the connecting maps of the inverse system \( \{ A_\delta \}_{\delta \in \Delta} \) are all surjective. Then, by Theorem 3.14 of [10]

\[
M(A \otimes \mathcal{L}(L^2(G))) = \lim\limits_{\delta \in \Delta} M(A_\delta \otimes \mathcal{L}(L^2(G)))
\]

up to an isomorphism of locally \( C^* \)-algebras, and by Lemma III 3.2 of [7],

\[
A \times_{\beta} G = \lim_{\delta \in \Delta} A_\delta \times_{\beta^{(\delta)}} G
\]

up to an isomorphism of locally \( C^* \)-algebras.

Remark 5.4 Let \( G \) be a commutative locally compact group. Exactly as in the proof of Proposition 5.1.6 of [13] we show that if \( \beta \) is an inverse limit coaction of \( G \) on \( A \), then \( \beta' = (\text{id}_A \otimes \text{ad}F) \circ \beta \), where \( F \) is the Fourier-Plancherel isomorphism from \( L^2(G) \) onto \( L^2(\hat{G}) \), is an inverse limit action of \( \hat{G} \) on \( A \) and conversely, if \( \alpha \) is an inverse limit action of \( \hat{G} \) on \( A \) then \( \alpha' = (\text{id}_A \otimes \text{ad}F^*) \circ \alpha \) is an inverse limit coaction of \( G \) on \( A \). Therefore an inverse limit coaction of \( G \) can be identified with an inverse limit action of \( \hat{G} \) and \( \text{id}_A \otimes \text{ad}F \) is an isomorphism between \( A \times_{\beta} G \) and \( A \times_{\beta',r} \hat{G} \).

The following proposition is a generalization of Theorem 5.2.6 of [13] for inverse limit actions of a locally compact group on a locally \( C^* \)-algebra.

**Proposition 5.5** Let \( A \) be a locally \( C^* \)-algebra and let \( G \) be a locally compact group.

1. If \( \alpha \) is an inverse limit action of \( G \) on \( A \), then there is an inverse limit coaction \( \tilde{\alpha} \) of \( G \) on \( A \times_{\alpha, r} G \), called the dual coaction associated to \( \alpha \),
such that
\[
\hat{\alpha}(\alpha(a)(1_{M(A)} \otimes \lambda(f))) = (\alpha(a) \otimes 1_G)(1_{M(A)} \otimes d_G^s(\lambda(f))) \quad (*)
\]
for all \(a\) in \(A\) and for all \(f\) in \(C_c(G)\).

(2). If \(\beta = \lim_{\delta \in \Delta} \beta^{(\delta)}\) is an inverse limit coaction of \(G\) on \(A\) such that the connecting maps of the inverse system \(\{A_\delta\}_{\delta \in \Delta}\) are all surjective, then there is an inverse limit action \(\hat{\beta}\) of \(G\) on \(A \times \beta G\), called the dual action associated to \(\beta\), such that
\[
\hat{\beta}(\beta(a)(1_{M(A)} \otimes f)) = (\beta(a) \otimes 1_G)(1_{M(A)} \otimes (id_{C_0(G)} \otimes j_G^a) d_G^s(f)) \quad (**)
\]
for all \(a\) in \(A\) and for all \(f\) in \(C_c(G)\).

**Proof.** (1). Since \(\alpha\) is an inverse limit action, \(\alpha = \lim_{\leftarrow p \in S(A)} \alpha^{(p)}\), where \(\alpha^{(p)}\) is a continuous action of \(G\) on \(A_p\). By Theorem 5.2.6 (i) of [13], for each \(p \in S(A)\) there is a dual coaction \(\hat{\alpha}^{(p)}\) of \(G\) on \(A_p \times_{\alpha^{(p)}, r} G\) such that
\[
\hat{\alpha}^{(p)}(\alpha^{(p)}(a_p)(1_{M(A_p)} \otimes \lambda(f))) = (\alpha^{(p)}(a_p) \otimes 1_G)(1_{M(A_p)} \otimes d_G^s(\lambda(f)))
\]
for all \(a_p\) in \(A_p\) and for all \(f\) in \(C_c(G)\). It is not difficult to check that \(\{\hat{\alpha}^{(p)}\}_{p \in S(A)}\) is an inverse system of injective \(S\)-morphisms and \(\hat{\alpha} = \lim_{\leftarrow p \in S(A)} \hat{\alpha}^{(p)}\) is a coaction of \(G\) on \(A \times_{\alpha, r} G\) which verifies the condition (\(*\)).

(2). By Theorem 5.2.6 (ii) of [13], for each \(\delta \in \Delta\) there is a continuous action \(\hat{\beta}^{(\delta)}\) of \(G\) on \(A_\delta \times_{\beta^{(\delta)}, G} G\) such that
\[
\hat{\beta}^{(\delta)}(\beta^{(\delta)}(a_\delta)(1_{M(A_\delta)} \otimes f)) = (\beta^{(\delta)}(a_\delta) \otimes 1_G)(1_{M(A_\delta)} \otimes (id_{C_0(G)} \otimes j_G^a) d_G^s(f))
\]
for all \(a_\delta\) in \(A_\delta\) and for all \(f\) in \(C_c(G)\). Using this relation and Remark 5.3 it is not difficult to check that \(\{\hat{\beta}^{(\delta)}\}_{\delta \in \Delta}\) is an inverse system of injective
$S$-morphisms. Let $\hat{\beta} = \lim_{\delta \in \Delta} \hat{\beta}(\delta)$. Then $\hat{\beta}$ is a continuous action of $G$ on $A \times_{\beta} G$ and moreover, it verifies the condition (**) .

The following theorem is a version of the Takai duality theorem for inverse limit actions of a locally compact group on a locally $C^*$-algebra.

**Theorem 5.6** Let $G$ be a locally compact group, let $A$ be a locally $C^*$-algebra and let $\alpha$ be an inverse limit action of $G$ on $A$. Then there is an isomorphism $\Pi$ from $A \otimes K(L^2(G))$ onto $(A \times_{\alpha,r} G) \times_{\hat{\alpha}} G$ such that

$$\hat{\alpha} \circ \Pi = (\hat{\Pi} \otimes \text{id}_{C_0(G)}) \circ (\alpha \otimes \text{ad} \rho)$$

where $\rho$ is the right regular representation of $L^1(G)$.

**Proof.** By Proposition 3.2 of [10],

$$A \otimes K(L^2(G)) = \lim_{p \in S(A)} A_p \otimes K(L^2(G))$$

up to an isomorphism of locally $C^*$-algebras

Since $\alpha$ is an inverse limit action, according to the proof of Proposition 5.5 (1),

$$\hat{\alpha} = \lim_{p \in S(A)} \hat{\alpha}^{(p)}$$

where $\hat{\alpha}^{(p)}$ is the dual coaction associated to $\alpha^{(p)}$ for each $p \in S(A)$. Then, since the connecting maps of the inverse system $\{A_p \times_{\alpha^{(p)},r} G\}_{p \in S(A)}$ are all surjective, by Proposition 5.5 (2),

$$\hat{\alpha} = \lim_{p \in S(A)} \hat{\alpha}^{(p)}$$

and by Remark 5.3,

$$(A \times_{\alpha,r} G) \times_{\hat{\alpha}} G = \lim_{p \in S(A)} \left( A_p \times_{\alpha^{(p)},r} G \right) \times_{\hat{\alpha}^{(p)}} G$$
up to an isomorphism of locally $C^*$-algebras.

Let $p \in S(A)$. According to Theorem 5.2 of [13], there is an isomorphism

$$\Pi(p)$$

from $A_p \otimes \mathcal{K}(L^2(G))$ onto $\left(A_p \times_{\alpha(p), r} G \right) \times_{\hat{\alpha}(p)} G$ such that

$$\hat{\alpha}(p) \circ \Pi(p) = (\Pi(p) \otimes \text{id}_{C_0(G)}) \circ (\alpha(p) \otimes \text{ad}\rho).$$

Moreover,

$$\Pi(p)(\alpha(p)(a_p)(1_M(A_p) \otimes \lambda(f) h)) = \hat{\alpha}(p)(\alpha(p)(a_p)(1_M(A_p) \otimes \lambda(f)))(1_M(A_p) \otimes 1_G \otimes h)$$

and

$$\Pi(p)((1_M(A_p) \otimes \lambda(f) h) \alpha(p)(a_p)) = \hat{\alpha}(p)((1_M(A_p) \otimes \lambda(f)) \alpha(p)(a_p))(1_M(A_p) \otimes 1_G \otimes h)$$

for all $f$ and $h$ in $C_c(G)$ and for all $a_p$ in $A_p$. Using these relations and the fact that $A_p \otimes \mathcal{K}(L^2(G))$ is the $C^*$-subalgebra of $M(A_p \otimes \mathcal{K}(L^2(G)))$ generated by $\{\alpha(p)(a_p)(1_M(A_p) \otimes \lambda(f)h), (1_M(A_p) \otimes \lambda(f)h) \alpha(p)(a_p); f, h \in C_c(G), a_p \in A_p\}$, see Lemma 5.2.10 of [13], it is not difficult to check that

$$\{\Pi(p)\}_{p \in S(A)}$$

is an inverse system of $C^*$-isomorphisms.

Let $\Pi = \lim_{\leftarrow} \Pi(p)$. Then, clearly $\Pi$ is an isomorphism of locally $C^*$-algebras from $A \otimes \mathcal{K}(L^2(G))$ onto $\left(A \times_{\alpha, r} G \right) \times_{\hat{\alpha}} G$ which satisfies the condition

$$\hat{\alpha} \circ \Pi = (\Pi \otimes \text{id}_{C_0(G)}) \circ (\alpha \otimes \text{ad}\rho)$$

and the theorem is proved. ■

Since any action of a compact group on a locally $C^*$-algebra is an inverse limit action, we have:

**Corollary 5.7** Let $G$ be a compact group, let $A$ be a locally $C^*$-algebra and let $\alpha$ be a continuous action of $G$ on $A$. Then there is an isomorphism $\Pi$
from $A \otimes K(L^2(G))$ onto $(A \times_{\alpha,r} G) \times_{\hat{\alpha}} G$ such that

$$\hat{\alpha} \circ \Pi = (\Pi \otimes \text{id}_{C_0(G)}) \circ (\alpha \otimes \text{ad} \rho)$$

where $\rho$ is the right regular representation of $L^1(G)$.

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Department of Mathematics, Faculty of Chemistry, University of Bucharest, Bd. Regina Elisabeta nr.4-12, Bucharest, Romania

e-mail address: mjoita@fmi.unibuc.ro