Abstract Interpolation Problem and Some Applications. II: Coefficient Matrices.

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Dedicated to Joe Ball on occasion of his 70-th birthday

Abstract. The main content of this paper is Lectures 5 and 6 that continue lecture notes [20]. Content of Lectures 1-4 of [20] is reviewed for the reader’s convenience in sections 1-4, respectively. It is shown in Lecture 5 how residual parts of the minimal unitary extensions, that correspond to solutions of the problem, yield some boundary properties of the coefficient matrix-function. These results generalize the classical Nevanlinna - Adamjan - Arov - Krein theorem. Lecture 6 discusses how further properties of the coefficient matrices follow from denseness of certain sets in the associated function model spaces. The structure of the dense set reflects the structure of the problem data.

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1. Abstract Interpolation Problem.

1.1. Data of the Problem.

Data of the Abstract Interpolation Problem consists of the following components: a vector space $X$ (without any topology), a positive-semidefinite sesquilinear form $D$ on $X$, linear operators $T_1$ and $T_2$ on $X$, separable Hilbert spaces $E_1$ and $E_2$, linear mappings $M_1$ and $M_2$ from $X$ to $E_1$ and $E_2$ respectively. The data pieces are connected by the following identity

$$D(T_2x, T_2y) - D(T_1x, T_1y) = \langle M_1x, M_1y \rangle_{E_1} - \langle M_2x, M_2y \rangle_{E_2}.$$  (1.1)

Let $\mathbb{D}$ be the open unit disc, $|\zeta| < 1$, and let $T$ be the unit circle, $|\zeta| = 1$. Let $w(\zeta) : E_1 \to E_2$ be a contraction for every $\zeta \in \mathbb{D}$ and assume that $w(\zeta)$ is analytic in variable $\zeta$. Functions of this type are called the Schur class (operator-valued) functions.
1.2. de Branges-Rovnyak Function Space.

Let $L^2(E_2 \oplus E_1)$ be the space of vector functions on the unit circle $T$ that are square summable against the Lebesgue measure. Space $L^w$ is defined as the range of $L^2(E_2 \oplus E_1)$ under

$$\begin{bmatrix} I_{E_2} & w(t) \\ w(t)^* & I_{E_1} \end{bmatrix} \begin{bmatrix} f_2 \\ f_1 \end{bmatrix}$$

endowed with the range norm. de Branges-Rovnyak space $H^w$ is defined as a subspace of $L^w$ that consists of functions $f = \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} \in L^w$, $f_2 \in H^2_+(E_2)$, $f_1 \in H^2_-(E_1)$.

1.3. Setting of the Problem.

A Schur class function $w : E_1 \to E_2$ is said to be a solution of the AIP with data (1.1), if there exists a linear mapping $F : X \to H^w$ such that for all $x \in X$

$$i) \quad \| Fx \|_{H^w}^2 \leq D(x, x);$$

$$ii) \quad tF \zeta T_2 x - F T_1 x = \begin{bmatrix} I_{E_2} & w(t) \\ w(t)^* & I_{E_1} \end{bmatrix} \begin{bmatrix} -M_2 x \\ M_1 x \end{bmatrix}, \text{ a.e. } t \in \mathbb{T}. \quad (1.4)$$

One can write $Fx$ as a vector of two components

$$Fx = \begin{bmatrix} F_+ x \\ F_- x \end{bmatrix}$$

which are $E_2$ and $E_1$ valued, respectively. Then conditions $Fx \in H^w$ and (i) read as

(a) $F_+ x \in H^2_+(E_2)$,

(b) $F_- x \in H^2_-(E_1)$,

(c) $\| Fx \|_{L^w}^2 \leq D(x, x)$.

Sometimes we will call the pair $(w, F)$ a solution of the Abstract Interpolation Problem.

1.4. Special Case.

The following additional assumption on operators $T_1$ and $T_2$ is met in many concrete problems: the operators

$$(\zeta T_2 - T_1)^{-1} \text{ and } (T_2 - \overline{\zeta} T_1)^{-1} \quad (1.5)$$
exist for all $\zeta \in \mathbb{D}$ except for a discrete set. In this case condition (ii) can be written as explicit formulae for $F_+$ and $F_-$

$$
(F_+^w x)(\zeta) = (w(\zeta)M_1 - M_2)(\zeta T_2 - T_1)^{-1} x, \quad (1.6)
$$

$$
(F_-^w x)(\zeta) = \overline{\zeta}(M_1 - w(\zeta)^* M_2)(\overline{T_2 - \overline{T_1}})^{-1} x. \quad (1.7)
$$

From here one can see that under assumptions $(1.5)$, for every solution $w$ there exists only one $F$ that satisfies (ii).

References to this section are [12, 13, 15, 24].

2. Examples.

2.1. The Nevanlinna-Pick Interpolation Problem.

In this section we recall several classical problems of analysis that can be included in the AIP scheme.

**Problem 2.1.** Let $\zeta_1, \ldots, \zeta_n, \ldots$ be a finite or infinite sequence of points in the unit disk $\mathbb{D}$; let $w_1, \ldots, w_n, \ldots$ be a sequence of complex numbers. One is interested in describing all the Schur class functions $w$ such that

$$
w(\zeta_k) = w_k. \quad (2.1)
$$

The well-known solvability criterion for this problem is:

$$
\left[ \frac{1 - w_k w_j}{1 - \zeta_k \zeta_j} \right]_{k,j=1}^n \geq 0, \quad \text{for all } n \geq 1. \quad (2.2)
$$

We specify data of the Abstract Interpolation Problem (1.1) as follows: the space $X$ consists of all sequences

$$
x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} \quad (2.3)
$$

that have a finite number of nonzero components;

$$
D(x,y) = \sum_{k,j} \overline{\eta_k} \frac{1 - w_k w_j}{1 - \zeta_k \zeta_j} x_j, \quad x, y \in X;
$$

$$
T_1 = \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_n \\ \vdots \end{bmatrix}, \quad T_2 = I_X; \quad (2.4)
$$

$$
E_1 = E_2 = \mathbb{C}^1, \quad M_1 x = \sum_j x_j, \quad M_2 x = \sum_j w_j x_j.
$$
The operators $T_1$ and $T_2$ meet the special case assumption \((1.5)\). Therefore, for every solution $w$ there is only one corresponding mapping $F^w$ that can be written in form \((1.6), (1.7)\). It can be further explicitly computed as

\[
(F^w x)(\zeta) = \sum_j \frac{1 - w(\zeta)w_j}{1 - \zeta \zeta_j} x_j.
\]

Since $w$ has non-tangential boundary values, the function $F^w x$ extends to the boundary of the unit disk. Since for $|t| = 1$ we have $t = 1/t$, $(F^w x)(t)$ further simplifies as follows:

\[
(F^w x)(t) = \begin{bmatrix} 1 & w(t) \\ w(t)^* & 1 \end{bmatrix} \begin{bmatrix} \sum_j \frac{w_j}{1 - \zeta_j t} \\ \sum_j \frac{x_j}{1 - \zeta_j t} \end{bmatrix}, \quad |t| = 1.
\]

**Theorem 2.2.** The solution set of the Nevanlinna-Pick problem coincides with the solution set of the Abstract Interpolation Problem with data \((2.3) \text{ - } (2.4)\). Moreover, for data of this type, inequality \((1.3)\) turns into equality

\[
\|F^w x\|_{H^w}^2 = D(x, x)
\]

for every solution $w$ and for every $x \in X$.

This example can be viewed as a special case of the one in the next subsection.

### 2.2. The Sarason Problem.

**Problem 2.3.** Let $H^2_+ = \text{the Hardy space of the unit disk. Let } \theta \text{ be an inner function, } K_\theta = H^2_+ \ominus \theta H^2_+, \text{ } T^{*}_\theta x = P_{\theta x} (x \in K_\theta), \text{ } W^* \text{ be a contractive operator on } K_\theta \text{ that commutes with } T^{*}_\theta : W^* T^{*}_\theta = T^{*}_\theta W^*$. Find all the Schur class functions $w$ such that

\[ W^* x = P_+ w x. \]

We specify here the data of the Abstract Interpolation problem as follows: $X = K_\theta$;

\[
D(x, x) = \|x\|_{K_\theta}^2 - \|W^* x\|_{K_\theta}^2, \quad x \in X;
\]

\[
T_1 = I_{K_\theta}, \quad T_2 = T^{*}_\theta; \tag{2.5}
\]

\[
E_1 = E_2 = \mathbb{C}^1, \quad M_1 x = (W^* x)(0), \quad M_2 x = x(0),
\]

where the latter notation stands for the value of an $H^2_+$ function at 0.

The operators $T_1$ and $T_2$ meet the special case assumption \((1.5)\). Therefore, for every solution $w$ there is only one corresponding mapping $F^w$ that can be explicitly computed as

\[
F^w x = \begin{bmatrix} 1 & w \\ \overline{w} & 1 \end{bmatrix} \begin{bmatrix} x \\ -W^* x \end{bmatrix}.
\]
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Theorem 2.4. The solution set of the Sarason Problem $2.3$ coincides with the solution set of the Abstract Interpolation Problem with data $2.5$. Moreover, for data of this type, inequality $1.3$ turns into equality

$$||F^w x||^2_{H^w} = D(x, x)$$

for every solution $w$ and for every $x \in X$.

Remark 2.5. Let $\zeta_1, \ldots, \zeta_n, \ldots$ be a finite or infinite sequence of points in the unit disk $D$; let $w_1, \ldots, w_n, \ldots$ be a sequence of complex numbers such that $2.2$ holds. Let $\theta$ be the Blaschke product with zeros $\zeta_k$, if the latter satisfy the Blaschke condition and $\theta = 0$ otherwise. Note that

$$\frac{1}{1 - t \zeta_k} \in K_\theta.$$

We define

$$W^* \frac{1}{1 - t \zeta_k} = \overline{w_k} \frac{1}{1 - t \zeta_k}.$$  

$W^*$ extends by linearity to a dense set in $K_\theta$ and further, due to $2.2$, to a contraction on the whole $K_\theta$. The set of solutions $w$ of the Sarason Problem $2.3$ with this $\theta$ and this $W$ coincides with the set of solutions of the Nevanlinna-Pick Problem $2.1$. However, the data of the Abstract Interpolation Problem in $2.5$ differ from the ones in $2.3-2.4$. Moreover, the coefficient matrices $S$ in the description formula $4.16$ for the solution sets are different and the associated universal colligations $4.5-4.7$ $A_0$ are non-equivalent.

References to this problem are $[13, 23]$.

2.3. The Boundary Interpolation Problem.

Definition 2.6. A Schur class function $w$ defined on the unit disk $D$ is said to have an angular derivative in the sense of Carathéodory at a point $t_0 \in T$ if there exists a nontangential unimodular limit

$$w_0 = \lim_{\zeta \to t_0} w(\zeta), \quad |w_0| = 1,$$

and there exists a nontangential limit

$$w'_0 = \lim_{\zeta \to t_0} \frac{w(\zeta) - w_0}{\zeta - t_0}.$$

Theorem 2.7 (Carathéodory-Julia). A Schur class function $w(\zeta)$ has an angular derivative at $t_0 \in T$ if and only if

$$D_{w, t_0} \overset{\text{def}}{=} \lim_{\zeta \to t_0} \frac{1 - |w(\zeta)|^2}{1 - |\zeta|^2} < \infty \quad (2.6)$$

(here $|\zeta| < 1, \ \zeta \to t_0$ in an arbitrary way). In this case

$$w'_0 = D_{w, t_0} \cdot \frac{w_0}{t_0}$$
and
\[
\frac{1 - |w(\zeta)|^2}{1 - |\zeta|^2} \to D_{w,t_0}
\]
as \zeta \to t_0 \text{ nontangentially. Note that } D_{w,t_0} \geq 0 \text{ and } D_{w,t_0} = 0 \text{ if and only if } w(\zeta) \text{ is a constant of modulus } 1.

**Theorem 2.8.** A Schur class function \( w \) has an angular derivative in the sense of Carathéodory at a point \( t_0 \in \mathbb{T} \) if and only if there exists a unimodular constant \( w_0 \) such that
\[
\left| \frac{w(t) - w_0}{t - t_0} \right|^2 + \frac{1 - |w(t)|^2}{|t - t_0|^2} \in L^1
\]
against the Lebesgue measure \( m(dt) \) on \( \mathbb{T} \). In this case
\[
\int_{\mathbb{T}} \left( \left| \frac{w(t) - w_0}{t - t_0} \right|^2 + \frac{1 - |w(t)|^2}{|t - t_0|^2} \right) m(dt) = D_{w,t_0},
\]
where \( D_{w,t_0} \) is the same as in (2.6). In particular, (2.7) implies that
\[
\frac{w - w_0}{t - t_0} \in H^2.
\]

**Problem 2.9.** Let \( t_0 \) be a point on the unit circle \( \mathbb{T} \), let \( w_0 \) be a complex number, \( |w_0| = 1 \), and let \( 0 \leq D < \infty \) be a given nonnegative number. One wants to describe all the Schur class functions \( w \) such that
\[
w(\zeta) \to w_0 \text{ as } \zeta \to t_0 \text{ nontangentially, and } D_{w,t_0} \leq D.
\]

We specify here the data of the Abstract Interpolation Problem as follows:
\[
X = \mathbb{C}^1, \quad D(x,x) = \pi D x,
\]
\[
T_1 x = t_0 x, \quad T_2 x = x, \quad E_1 = E_2 = \mathbb{C}^1,
\]
\[
M_1 x = x, \quad M_2 x = w_0 x.
\]

The operators \( T_1 \) and \( T_2 \) meet the special case assumption (1.5). Therefore, for every solution \( w \) there is only one corresponding mapping \( F^w \) that can be explicitly computed as
\[
(F^w x)(t) = \begin{bmatrix} 1 & w(t) \\ w(t) & 1 \end{bmatrix} \begin{bmatrix} -\frac{w_0}{t - t_0} \\ \frac{1}{t - t_0} \end{bmatrix} x, \quad |t| = 1.
\]

Direct computation shows that
\[
\|F^w x\|_{H^w}^2 = \pi D_{w,t_0} x.
\]
Theorem 2.10. The solution set of the Boundary Interpolation Problem coincides with the solution set of the Abstract Interpolation Problem with data specified in (2.8).

There is indeed inequality in (1.3) for some solutions \( w \) of the Abstract Interpolation Problem with data specified in (2.8). We will discuss this in more detail in Section 5.

References to this problem are [13, 26]. For higher order analogue of Theorem 2.7 and related boundary interpolation Problem 2.9 see [7, 8].

3. Solutions of the Abstract Interpolation Problem.

3.1. Isometry Defined by the Data.

We say that two vectors \( x_1 \) and \( x_2 \) in \( X \) are \( D \) equivalent if

\[
D(x_1, y) = D(x_2, y), \quad \forall y \in X. \tag{3.1}
\]

We consider the vector space of equivalence classes \( \{ [x], x \in X \} \). We define inner product between two equivalence classes as

\[
\langle [x], [y] \rangle \overset{\text{def}}{=} D(x, y). \tag{3.2}
\]

After completion we get a Hilbert space that will be denoted by \( H_0 \). We rewrite identity (1.1) as

\[
D(T_2x, T_2y) + \langle M_1x, M_1y \rangle_{E_1} = D(T_1x, T_1y) + \langle M_2x, M_2y \rangle_{E_2},
\]

or, using definition (3.2), as

\[
\langle [T_2x], [T_2y] \rangle + \langle M_1x, M_1y \rangle_{E_1} = \langle [T_1x], [T_1y] \rangle + \langle M_2x, M_2y \rangle_{E_2}. \tag{3.3}
\]

We set

\[
d_V \overset{\text{def}}{=} \text{Clos} \left\{ \begin{bmatrix} T_1x \\ M_1x \end{bmatrix}, \quad x \in X \right\} \subseteq H_0 \oplus E_1 \tag{3.4}
\]

and

\[
\Delta_V \overset{\text{def}}{=} \text{Clos} \left\{ \begin{bmatrix} T_2x \\ M_2x \end{bmatrix}, \quad x \in X \right\} \subseteq H_0 \oplus E_2. \tag{3.5}
\]

We define a mapping \( V : d_V \to \Delta_V \) by the formula

\[
V : \begin{bmatrix} [T_1x] \\ M_1x \end{bmatrix} \overset{\text{def}}{\to} \begin{bmatrix} [T_2x] \\ M_2x \end{bmatrix}. \tag{3.6}
\]

In view of (3.3), \( V \) is an isometry.

Remark 3.1. An arbitrary isometry \( V \) from \( H_0 \oplus E_1 \) to \( H_0 \oplus E_2 \) may appear here under appropriate choice of the data in (1.1).
3.2. Unitary Colligations, Characteristic Functions, Fourier Representations.

Let $H$, $E_1$, $E_2$ be separable Hilbert spaces. A unitary mapping $A$ of $H \oplus E_1$ onto $H \oplus E_2$

$$A : H \oplus E_1 \rightarrow H \oplus E_2$$

(3.7)
is said to be a unitary colligation. The space $H$ is called the state space of the colligation, $E_1$ is called the input space, and $E_2$ is called the output space. Both $E_1$ and $E_2$ are called exterior spaces. Sometimes it is convenient to write the colligation $A$ as a block matrix:

$$A = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} H \\ E_1 \end{bmatrix} \rightarrow \begin{bmatrix} H \\ E_2 \end{bmatrix}.$$  \hspace{1cm} (3.8)

The characteristic function of the unitary colligation is defined as

$$w(\zeta) = D + \zeta C(I_H - \zeta A)^{-1} B.$$ \hspace{1cm} (3.9)

It is well defined on $D$ analytic contractive operator-valued function from $E_1$ to $E_2$. The Fourier representation of the space $H$ associated with the colligation $A$ is defined as

$$G(h)(\zeta) = \begin{bmatrix} (G_+h)(\zeta) \\ (G_-h)(\zeta) \end{bmatrix} = \begin{bmatrix} C(I_H - \zeta A)^{-1}h \\ \overline{\zeta}B^*(I_H - \overline{\zeta}A^*)^{-1}h \end{bmatrix}, \; h \in H, \; \zeta \in \mathbb{D}. \hspace{1cm} (3.10)$$

$G$ maps space $H$ onto the de Barnges-Rovnyak space $H^w$ associated with the characteristic function $w$ (see Section 1.2).

**Definition 3.2.** We define the residual subspace $H_{res} \subseteq H$ of the colligation $A$ as the maximal subspace of $H$ that reduces $A$ (that is invariant for $A$ and $A^*$). Equivalently $H_{res}$ can be defined as the maximal subspace of $H$ that is invariant for $A$ and $A^*$, and $C|_{H_{res}} = 0$, $B^*|_{H_{res}} = 0$. The simple part of the space $H$ is defined as $H_{simp} = H \ominus H_{res}$. A unitary colligation $A$ is said to be simple with respect to the exterior spaces $E_1$ and $E_2$ if $H_{res}$ is trivial.

The following fact will be of crucial importance to us in Lecture 5.

**Theorem 3.3.** Let $G$ be the mapping defined in (3.10). Then $G$ maps $H_{simp}$ onto $H^w$ unitarily and $G$ vanishes on $H_{res}$.

3.3. Unitary Extensions of the Isometry $V$ and Solutions of the Problem.

**Definition 3.4.** We say that a unitary colligation $A$ of form (3.8) is a unitary extension of the isometry $V$, defined in (3.6), if $H_0 \subseteq H$ and

$$A|_{dV} = V.$$ \hspace{1cm} (3.11)

An extension $A$ of $V$ is said to be minimal if it does not have a nontrivial residual subspace in $H \ominus H_0$. Note that minimal extension $A$ may have a residual subspace in $H$, though.
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Theorem 3.5. A Schur class function \( w : E_1 \to E_2 \) and a mapping \( F : X \to H^w \) solve Abstract Interpolation Problem \((1.3) \text{ - } (1.4)\) if and only if there exists a unitary colligation \( A \) of form \((3.8)\) that minimally extends the isometry \( V \) (defined in \((3.6)\) from the problem data) such that \( w \) is the characteristic function of \( A \)

\[
w(\zeta) = D + \zeta C (I_H - \zeta A)^{-1} B \tag{3.12}
\]

and

\[
Fx = G [x], \quad \forall x \in X, \tag{3.13}
\]

where \([x]\) is the \( D \)-equivalence class of \( x \) (defined in \((3.1)\) and \( G \) is the Fourier representation of the colligation \( A \) (defined in \((3.10)\)).

References to this section are \([12, 13, 15, 24]\).

4. Parametric Description of the Solutions of the Abstract Interpolation Problem.

4.1. Structure of Minimal Unitary Extensions of \( V \).

Let \( V \) be an isometric colligation:

\[
V : d_V \to \Delta_V, \quad d_V \subseteq H_0 \oplus E_1, \quad \Delta_V \subseteq H_0 \oplus E_2.
\]

Let \( A \) be a minimal unitary extension of \( V \):

\[
A : H \oplus E_1 \to H \oplus E_2, \quad H \supseteq H_0, \quad A|d_V = V.
\]

Let \( d_V^\perp \) and \( \Delta_V^\perp \) be the orthogonal complements of \( d_V \) in \( H_0 \oplus E_1 \) and \( \Delta_V \) in \( H_0 \oplus E_2 \), respectively. Let

\[
H_1 = H \ominus H_0. \tag{4.1}
\]

Then the orthogonal complement of \( d_V \) in \( H \oplus E_1 \) is \( H_1 \oplus d_V^\perp \) and the orthogonal complement of \( \Delta_V \) in \( H \oplus E_2 \) is \( H_1 \oplus \Delta_V^\perp \). Since \( A \) is a unitary operator and it maps \( d_V \) onto \( \Delta_V \) (\( A|d_V = V \)), \( A \) has to map the orthogonal complement onto the orthogonal complement, i.e. \( H_1 \oplus d_V^\perp \) onto \( H_1 \oplus \Delta_V^\perp \). Denote the restriction of \( A \) onto \( H_1 \oplus d_V^\perp \) by \( A_1 \). Thus, \( A_1 \) is a unitary colligation,

\[
A_1 : H_1 \oplus d_V^\perp \to H_1 \oplus \Delta_V^\perp.
\]

Since \( A \) is a minimal extension of \( V \), \( A_1 \) is a simple colligation with respect to \( d_V^\perp \) and \( \Delta_V^\perp \). Thus, the parameter of a minimal unitary extension \( A \) is an arbitrary simple unitary colligation \( A_1 \) with input space \( d_V^\perp \) and output space \( \Delta_V^\perp \).

Let \( N_1 \) be an auxiliary copy of \( d_V^\perp \), that is we assume that there exists a unitary mapping \( u_1 \) from \( d_V^\perp \) onto \( N_1 \)

\[
u_1 : d_V^\perp \to N_1. \tag{4.2}
\]

Also let \( N_2 \) be a copy of \( \Delta_V^\perp \), that is there exists a unitary mapping \( u_2 \) from \( \Delta_V^\perp \) onto \( N_2 \)

\[
u_2 : \Delta_V^\perp \to N_2. \tag{4.3}
\]
In what follows it will be convenient to consider simple unitary colligations $A_1$ with input space $N_1$ and output space $N_2$ (instead of $d_V$ and $\Delta_V$)
\[
A_1 : H_1 \oplus N_1 \to H_1 \oplus N_2.
\] (4.4)

**4.2. Universal Extension of $V$.**

Here we define a unitary colligation $A_0$ that extends $V$ in a different way:
\[
A_0 : H_0 \oplus E_1 \oplus N_2 \to H_0 \oplus E_2 \oplus N_1
\] (4.5)

with
\[
A_0 |_{d_V} = V, \quad (d_V \subseteq H_0 \oplus E_1),
\] (4.6)

\[
A_0 |_{d_V^\perp} = u_1, \quad A_0 |_{N_2} = u_2^*,
\] (4.7)

where $u_1$ maps unitarily $d_V^\perp$ onto $N_1$ and $u_2$ maps unitarily $\Delta_V^\perp$ onto $N_2$. $A_0$ is uniquely defined by $V$ and the identification maps $u_1$ and $u_2$. Note that mappings $u_1$ and $u_2$ can be chosen arbitrarily. We will call this choice a normalization of the universal colligation $A_0$.

Similarly to (3.8), we write colligation $A_0$ as a block matrix:
\[
A_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} : \begin{bmatrix} H_0 \\ E_1 \oplus N_2 \end{bmatrix} \to \begin{bmatrix} H_0 \\ E_2 \oplus N_1 \end{bmatrix}.
\] (4.8)

We also introduce the characteristic function of $A_0$
\[
S(\zeta) = D_0 + \zeta C_0 (I_{H_0} - \zeta A_0)^{-1} B_0.
\] (4.9)

It is an analytic on $\mathbb{D}$ contractive operator-valued function from $E_1 \oplus N_2$ to $E_2 \oplus N_1$. Note that $S$ depends on the data of the problem and on normalization (4.7) of $A_0$. According to the structure of the input an the output spaces, we further break $S$ into blocks
\[
S(\zeta) = \begin{bmatrix} s_0(\zeta) & s_2(\zeta) \\ s_1(\zeta) & s(\zeta) \end{bmatrix} : \begin{bmatrix} E_1 \\ N_2 \end{bmatrix} \to \begin{bmatrix} E_2 \\ N_1 \end{bmatrix}.
\] (4.10)

The special structure (4.6), (4.7) of $A_0$ forces
\[
s(0) = 0.
\] (4.11)

We consider the Fourier representation of the space $H_0$ associated with the colligation $A_0$
\[
(G_0 h_0)(\zeta) = \begin{bmatrix} (G_0 h_0)(\zeta) \\ (G_0 - h_0)(\zeta) \end{bmatrix} = \begin{bmatrix} C_0 (I_{H_0} - \zeta A_0)^{-1} h_0 \\ B_0^*(I_{H_0} - \zeta A_0^* h_0) \end{bmatrix},
\] (4.12)

$h_0 \in H_0, \zeta \in \mathbb{D}$. $G_0$ maps space $H_0$ onto the de Barnges-Rovnyak space $H^S$ associated with the characteristic function $S$. 
4.3. Description of Solutions.

Given unitary colligation $A_0$ of form (4.5) and unitary colligation $A_1$ of form (4.4), there is a procedure (called feedback coupling) that produces a colligation $A$ of the form (3.7) with

$$H = H_0 \oplus H_1.$$ 

We do not discuss here the feedback coupling. A detailed explanation of the procedure is given, for instance, in [20], Lecture 4, Section 3, page 373. Here we just state some consequences of this procedure.

**Theorem 4.1.** If $A_0$ is defined in terms of $V$ as in (4.6)-(4.7) and $A_1$ is an arbitrary simple unitary colligation of form (4.4), then the feedback coupling $A$ is a minimal unitary extension of $V$ in the sense of Definition 3.4. Moreover, all minimal unitary extensions of $V$ arise in this way.

**Theorem 4.2.** Given unitary colligation $A_0$ of the form (4.5) and unitary colligation $A_1$ of the form (4.4). Let unitary colligation $A$ of form (3.7) be the feedback coupling of $A_0$ and $A_1$. Then

$$w = s_0 + s_2 \omega (I_{N_1} - s \omega)^{-1} s_1,$$

where $w$ is the characteristic function of $A$, $\omega$ is the characteristic function of $A_1$, and $S$ is the characteristic function of $A_0$ (see (4.10));

$$G[h_1] = \begin{bmatrix} \psi \omega & 0 \\ 0 & 0 \end{bmatrix} E_2 \begin{bmatrix} 0 & 0 \\ \phi^* \omega \end{bmatrix} E_1 \begin{bmatrix} \psi & 0 \\ 0 & \phi^* \end{bmatrix},$$

where $G, \ G_1, \ G_0$ are Fourier representations of $A, A_1$ and $A_0$, respectively, as defined in (3.10), (4.12),

$$\varphi = (I_{N_1} - s \omega)^{-1} s_1, \quad \psi = s_2 (I_{N_2} - \omega s)^{-1}.$$  

Combining Theorems 3.5 and 4.2, we get a description of all solutions $(w, F)$ of the Abstract Interpolation Problem (1.3)-(1.4).

**Theorem 4.3.** Let $V$ be the isometry defined by the data of the problem as in (3.4) - (3.6). Let $N_1$ and $N_2$ be the spaces (4.2), (4.3) and let $A_0$ be the unitary colligation defined in (4.5) - (4.7). Let $S$ be the characteristic function (4.10) of $A_0$ and $G_0$ be the Fourier representation (4.12) of $H_0$. Then the solution set $(w, F)$ of the Abstract Interpolation Problem (1.3) - (1.4) is described as follows

$$w = s_0 + s_2 \omega (I_{N_1} - s \omega)^{-1} s_1,$$

where $\omega$ is an arbitrary Schur class function from $N_1$ to $N_2$, $S$ is the characteristic function (4.10) of $A_0$;

$$F_0 = \begin{bmatrix} \psi \omega & 0 \\ 0 & \phi^* \omega \end{bmatrix} E_2 \begin{bmatrix} 0 & 0 \\ \phi^* \omega \end{bmatrix} E_1 \begin{bmatrix} \psi & 0 \\ 0 & \phi^* \end{bmatrix} G_0[x], \quad x \in X,$$

where $[x]$ is the $D$-equivalence class of $x$ defined in (3.1),

$$\varphi = (I_{N_1} - s \omega)^{-1} s_1, \quad \psi = s_2 (I_{N_2} - \omega s)^{-1}.$$  


5. Lecture 5: Inequality $\|Fx\|^2 \leq D(x, x)$, Residual Parts of Minimal Unitary Extensions and the Nevanlinna - Adamjan - Arov - Krein Type Theorems.

This lecture is focused on the inequality $\|Fx\|^2 \leq D(x, x)$ in the setting of the Abstract Interpolation Problem (AIP) (1.3), (1.4). The main goals are

- to explain the inequality in terms of the corresponding minimal unitary extension (3.11) $A$ of the isometry $V$ (3.4)-(3.6). For more details, see [12, 15];
- to give a formula for the quantitative characteristic of how the inequality is far from the equality.

After that we apply the latter formula to the case when the equality $\|Fx\|^2 = D(x, x)$ is known a priori (like in Problem 2.1 and Problem 2.3). This in turn yields certain boundary properties of the coefficient matrix $S(\zeta)$, defined in (4.10). In particular, this leads to generalizations of a classical Nevanlinna - Adamjan - Arov - Krein theorem [1, 2]: for general semi-determinate Nehari problem [15, 16, 21] and for general Commutant Lifting problem [5, 6].

5.1. The Equality $\|Fx\|^2 = D(x, x)$ and Simplicity of the Corresponding Minimal Unitary Extension.

Let a Schur class operator function $w : E_1 \to E_2$ and a mapping $F : X \to H^w$ be a solution of the AIP (1.3), (1.4). Let $V : dV \to \Delta V$, $dV \subseteq H_0 + E_1$, $\Delta V \subseteq H_0 + E_2$ be the isometry (3.4)-(3.6) associated to AIP data (1.1). Then, by Theorem 3.11, $w$ is the characteristic function of a unitary colligation $A$ of the form (3.8) that minimally extends the isometry $V$ and

$$Fx = G[x], \quad \forall x \in X,$$

where $[x]$ is the $D$-equivalence class of $x$ (defined in (3.1)) and $G$ is the Fourier representation of the colligation $A$ (defined in (3.10)). By Theorem 3.3 $G$ maps $H_{simp} \subseteq H$ onto $H^w$ unitarily and $G$ vanishes on $H_{res} \subseteq H$, see also Definition 3.2

From here one can see what the equality

$$\|Fx\|_{H^w}^2 = D(x, x), \quad \forall x \in X$$

(5.2)

means: in view of (5.1) and definition (3.4)-(3.2) of $H_0$, equality (5.2) is the same as

$$\|G[x]\|_{H^w}^2 = \|[x]\|_{H_0}^2.$$

Since the lineal $\{[x], \ x \in X\}$ is dense in $H_0$, this means that $G$ is isometric on $H_0$, i.e., $H_0 \subseteq H_{simp}$. The latter is equivalent to the inclusion $H_{res} \subseteq H \ominus H_0$. Since

$\omega, s, s_1, s_2$ are the same as above. All functions in formula (4.17) are considered on $T$.

References to this section are [3, 12, 13, 15, 24].
extension $A$ is minimal, this is possible if and only if $H_{\text{res}} = \{0\}$. Thus, we arrive at the following

**Proposition 5.1.** $\|Fx\|_{H_{\text{res}}}^2 = D(x,x)$, $\forall x \in X$ if and only if the corresponding minimal unitary extension $A$ of the isometry $V$ is simple.

Hence, a strict inequality in (5.2) may occur for some $x \in X$ if and only if the corresponding minimal extension $A$ is non-simple, i.e., $H \supseteq H_{\text{res}} \neq \{0\}$. In this case $A|H_{\text{res}}$ is a unitary operator on $H_{\text{res}}$.

### 5.2. Residual Part of a Minimal Unitary Extension and its Spectral Function.

The theme of this section is the unitary operator $A|H_{\text{res}}$. By Theorem 4.1, every minimal unitary extension $A : H \oplus E_1 \to H \oplus E_2$ of the isometry $V$ is the feedback coupling of the universal unitary colligation $A_0$, defined in (4.6)-(4.7), and a simple unitary colligation $A_1$ of form (4.4):

$$A_0 : H_0 \oplus E_1 \oplus N_2 \to H_0 \oplus E_2 \oplus N_1,$$

$$A_1 : H_1 \oplus N_1 \to H_1 \oplus N_2,$$

where $H = H_0 \oplus H_1$. It follows from the feedback coupling procedure that if colligations $A_0$ and $A_1$ are not simple, then their residual parts are contained in the residual part of their coupling $A$. However, the residual part of the latter colligation may be properly larger.

Here we are interested in the piece of the residual part of $A$ that results from the feedback coupling procedure, but not from non-simplicity of the coupled colligations. Let $A_0 : H_0 \oplus E_1 \oplus N_2 \to H_0 \oplus E_2 \oplus N_1$ and $A_1 : H_1 \oplus N_1 \to H_1 \oplus N_2$ be simple unitary colligations. Let $A : H \oplus E_1 \to H \oplus E_2$ be their feedback coupling, where $H = H_0 \oplus H_1$. Let $H = H_{\text{simp}} \oplus H_{\text{res}}$ be the simple and residual parts of the colligation $A$, respectively. Let $U = A|H_{\text{res}}$.

**Definition 5.2.** Let $U$ be a unitary operator on a separable Hilbert space $\mathcal{H}$. Let $N$ be an auxiliary Hilbert space and let $\Gamma : N \to \mathcal{H}$ be a linear operator from $N$ into $\mathcal{H}$ such that $\Gamma(N)$ is a cyclic subspace for the operator $U$, i.e., the closed linear span of $\{U^k \Gamma(N)\}_{k \in \mathbb{Z}}$ coincides with $\mathcal{H}$. The operator function $a(\zeta) : N \to N$,

$$a(\zeta) \overset{\text{def}}{=} \frac{1}{2} \Gamma^* \frac{1_{\mathcal{H}} + \zeta U}{1_{\mathcal{H}} - \zeta U} \Gamma$$

is called a spectral function of the operator $U$. Clearly, $a$ is analytic in $\mathbb{D}$.

Since $U$ is a unitary operator, $a(\zeta) + a(\zeta)^* \geq 0$. Therefore, $a(\zeta)$ admits a Riesz-Herglotz representation

$$a(\zeta) = \frac{1}{2} \int_{\mathbb{T}} \frac{t + \zeta}{t - \zeta} \sigma(dt),$$

where $\sigma(dt)$ is an operator valued measure ($\sigma(dt) : N \to N$) on the unit circle $\mathbb{T}$. The measure $\sigma(dt)$ is called a spectral measure of the operator $U$. 

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Remark 5.3. Different choices of the auxiliary space $N$ and the operator $\Gamma$ lead to different spectral functions. However, any of them defines the unitary operator $U$ uniquely up to a unitary equivalence.

The next theorem gives a formula for the spectral function of the unitary operator $U = A|_{H_{\text{res}}}$ that is the residual part of the feedback coupling considered above in this section. Results of this type in the context of the cascade coupling go back to Yu. L. Smulian \[28\] and were inspired by M. Livšits and M. Brodskii \[9\]. It was realized, in particular, as an obstacle for solving the Hilbert space invariant subspace problem by means of factorization of the characteristic function. In the same context it was investigated by B. Sz.-Nagy and C. Foiaş (\[27\], irreducible factorizations), and by L. de Branges (\[10\], overlapping subspaces), see also \[29, 4\].

Theorem 5.4 (\[13, 15\]). Let unitary colligation $A : H \oplus E_1 \to H \oplus E_2$ be the feedback coupling of simple unitary colligations $A_0 : H_0 \oplus E_1 + N_2 \to H_0 \oplus E_2 + N_1$ and $A_1 : H_1 \oplus N_1 \to H_1 \oplus N_2$, where $H = H_0 \oplus H_1$. Let

$$S(\zeta) = \begin{bmatrix} s_0(\zeta) & s_2(\zeta) \\ s_1(\zeta) & s(\zeta) \end{bmatrix} : [E_1] \to [E_2]$$

be the characteristic function of the colligation $A_0$ and $\omega(\zeta)$ be the characteristic function of the colligation $A_1$. Let $H_{\text{res}} = H$ be the residual subspace of $A$. Let

$$U = A|_{H_{\text{res}}}$$

and let $N \overset{\text{def}}{=} N_2 \oplus N_1$. We define $\Gamma : N \to H_{\text{res}}$ as

$$\Gamma \left( \begin{bmatrix} n_2 \\ n_1 \end{bmatrix} \right) \overset{\text{def}}{=} P_{H_{\text{res}}} (P_{H_0} A_0^* (0_{H_0} \oplus 0_{E_2} \oplus (\omega(0)^* n_2 + n_1)) + P_{H_1} A_1^* (0_{H_1} \oplus n_2)),$$

where $P_{H_0}, P_{H_1}$ are the orthogonal projections onto the corresponding subspaces, $P_{H_{\text{res}}}$ is the orthogonal projection onto $H_{\text{res}}$

$$P_{H_{\text{res}}} = I_H - G^* G,$$

where $G$ is defined in (4.14). Then $\Gamma(N)$ is a cyclic subspace for the operator $U$ and the corresponding spectral function $a_\omega(\zeta) : \begin{bmatrix} N_2 \\ N_1 \end{bmatrix} \to \begin{bmatrix} N_2 \\ N_1 \end{bmatrix}$ is given by the formula

$$a_\omega(\zeta) = \frac{1}{2} \begin{bmatrix} 0 & -\omega(0) \\ \omega(0)^* & 0 \end{bmatrix} + \hat{a}_\omega(\zeta)$$

$$- \frac{1}{2} \int \int \begin{bmatrix} 1_{E_2} & w \\ w^* & 1_{E_1} \end{bmatrix} \begin{bmatrix} \psi \varphi \\ \varphi \psi \end{bmatrix} \begin{bmatrix} \psi \varphi \\ \varphi \psi \end{bmatrix} m(dt),$$

where $m$ is the measure associated with the operator $U$.
where

\[
\tilde{a}_\omega = \frac{1}{2} \begin{bmatrix}
1_{N_2 \oplus N_1} + \begin{bmatrix} 0 & \omega \\ s & 0 \end{bmatrix} \\
1_{N_2 \oplus N_1} - \begin{bmatrix} 0 & \omega \\ s & 0 \end{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
\psi - \frac{1}{2} & \omega \varphi \\
\circ \psi & \circ \varphi - \frac{1}{2}
\end{bmatrix},
\]

\[
\varphi = (1_{N_1} - s \omega)^{-1}, \quad \psi = (1_{N_2} - \omega s)^{-1},
\]

(5.4)

\[
\varphi = \circ \varphi s_1 = (1_{N_1} - s \omega)^{-1} s_1, \quad \psi = s_2 \circ \psi = s_2 (1_{N_2} - \omega s)^{-1},
\]

(5.5)

\[w\] is the characteristic function of the colligation \(A\)

\[w = s_0 + s_2 \omega (1 - s \omega)^{-1} s_1,\]

\[m(dt)\] is the normalized Lebesgue measure on the unit circle \(T\).

Note that the spectral function \(a_\omega\) of the feedback coupling depends on the characteristic functions of the coupled colligations only.

The real part of the spectral function \(a_\omega(\zeta)\) can be expressed as

\[
a_\omega(\zeta) + a_\omega(\zeta)^* = \tilde{a}_\omega(\zeta) + \tilde{a}_\omega(\zeta)^*
\]

(5.6)

\[- \int_T \frac{1 - |\zeta|^2}{|t - \zeta|^2} \begin{bmatrix}
\psi^* & \omega \varphi \\
\omega^* \psi^* & \varphi
\end{bmatrix} \begin{bmatrix}
1_{E_2} & w^{-1} \\
w^* & 1_{E_1}
\end{bmatrix} \begin{bmatrix}
\psi & \psi \omega \\
\varphi \omega^* & \varphi^*
\end{bmatrix} m(dt).
\]

We will also need the following re-expression of \(\tilde{a}_\omega\)

\[
\tilde{a}_\omega = \begin{bmatrix}
1 & 1_{N_2 + \omega s} \\
\frac{1}{s \circ \psi} & \frac{1}{s \circ \varphi}
\end{bmatrix} \begin{bmatrix}
\circ \omega \varphi \\
1_{N_1 - \omega s}
\end{bmatrix}
\]

(5.7)

with the same notations as in (5.4). Since

\[
\tilde{a}_w + \tilde{a}_w^* \geq 0,
\]

there exists an operator measure \(\tilde{\sigma}(dt) : N \to N\) such that

\[
\tilde{a}_\omega(\zeta) = \frac{1}{2} \begin{bmatrix}
0 & \omega(0) \\
-\omega(0)^* & 0
\end{bmatrix} + \frac{1}{2} \int_T \frac{t + \zeta}{t - \zeta} \tilde{\sigma}(dt).
\]

(5.8)
5.3. Property $\|F x\|^2 = D(x, x)$ yields Boundary Properties of the Coefficient Matrix $S$ and the Parameter $\omega$.

In this section we show how formula (5.6) for the spectral function of the residual part yields boundary properties of the coefficient matrix-function

$$S = \begin{bmatrix} s_0 & s_2 \\ s_1 & s \end{bmatrix}.$$ 

Let $w$ be a solution of the AIP and let $F^\omega : X \to H^w$ (we label it with parameter $\omega$ in view of formula (4.14)) be the corresponding mapping. Assume that the equality

$$\|F^\omega x\|^2_{H^w} = D(x, x)$$

(5.9)

holds for all $x \in X$. By Proposition 5.1, this means that the corresponding extension $A$ is simple, i.e., the residual part is trivial. But then the spectral function of the residual part must be equal to zero. Therefore, formula (5.6) along with assumption (5.9) yield

$$\tilde{a}_\omega(\zeta) + \tilde{a}_\omega(\zeta)^* = \int \frac{1 - |\zeta|^2}{|t - \zeta|^2} \begin{bmatrix} \psi^* & \omega \varphi \\ \omega^* \psi^* & \varphi \end{bmatrix} \begin{bmatrix} 1_{E_2} & w \\ \omega^* \psi^* & \varphi \end{bmatrix} \begin{bmatrix} \psi & \psi \omega \\ \varphi^* \omega^* & \varphi^* \end{bmatrix} m(dt).$$

The latter is equivalent to the following two properties:

1. $\tilde{\sigma}_\omega(dt)$ is absolutely continuous with respect to the Lebesgue measure;

2. $\tilde{a}_\omega(t) + \tilde{a}_\omega(t)^*$

$$= \int \frac{1 - |\zeta|^2}{|t - \zeta|^2} \begin{bmatrix} \psi^* & \omega \varphi \\ \omega^* \psi^* & \varphi \end{bmatrix} \begin{bmatrix} 1_{E_2} & w \\ \omega^* \psi^* & \varphi \end{bmatrix} \begin{bmatrix} \psi & \psi \omega \\ \varphi^* \omega^* & \varphi^* \end{bmatrix} m(dt),$$

almost everywhere on $T$. Thus, these two properties are equivalent to

$$\|F^\omega x\|^2_{H^w} = D(x, x), \ \forall \ x \in X.$$ 

Problem 2.1 and Problem 2.3 possess the property

$$\|F^\omega x\|^2_{H^w} = D(x, x), \ \forall \ x \in X$$

for every solution $w$ and every parameter $\omega$ that produces this $w$ via formula (5.5) (actually, for these examples formula (5.5) gives a one to one correspondence between $\omega$ and $w$). Therefore, properties 1 and 2 above hold for every parameter $\omega$ in those examples.

Property 2 itself is equivalent to vanishing of the absolutely continuous part of the measure $\sigma_\omega(dt)$ that corresponds to $a_\omega(\zeta)$. It is the case in Problem 2.9 that $\sigma_\omega(dt)$ can be supported by a single point for every parameter $\omega$. Hence, $\sigma_\omega(dt)$ has trivial absolutely continuous part for every parameter $\omega$ in this example. Thus, we have Property 2 for every parameter $\omega$ in Problem 2.9 Property 1 holds for
some parameters and does not hold for the others in this problem. Analysis shows that in this problem the residual part is nontrivial if and only if
\[
\lim_{\zeta \to t_0} \omega(\zeta) = s(t_0),
\]
where \(s\) is the right-bottom entry of the coefficient matrix \(S\), and
\[
\lim_{\zeta \to t_0} \frac{1 - |\omega(\zeta)|^2}{1 - |\zeta|^2} < \infty.
\]
Note that \(|s(t_0)| = 1\) always in this problem.

We reformulate the above Properties 1 and 2 using formulas (5.6) and (5.7) of this Lecture. Property 1 is equivalent to the following property

\[\text{1'}\text{. Measures } \sigma_{\omega_1}(dt) \text{ and } \sigma_{\omega_2}(dt) \text{ are absolutely continuous with respect to the Lebesgue measures, where}\]
\[
\begin{align*}
\frac{1_{N_2} + \omega(\zeta)s(\zeta)}{1_{N_2} - \omega(\zeta)s(\zeta)} &= \int_T \frac{t + \zeta}{t - \zeta} \sigma_{\omega_2}(dt), \\
\frac{1_{N_1} + s(\zeta)\omega(\zeta)}{1_{N_1} - s(\zeta)\omega(\zeta)} &= \int_T \frac{t + \zeta}{t - \zeta} \sigma_{\omega_1}(dt).
\end{align*}
\]

A special case of Property 2, when parameter \(\omega = 0\), reads as follows

\[\text{2'}\text{. } \begin{bmatrix} 1_{N_2} & s^* \\ s & 1_{N_1} \end{bmatrix} = \begin{bmatrix} s_2^* & 0 \\ 0 & s_1 \end{bmatrix} \begin{bmatrix} 1_{E_2} & s_0 \\ s_0^* & 1_{E_1} \end{bmatrix} \begin{bmatrix} s_2 & 0 \\ 0 & s_1^* \end{bmatrix}\]

almost everywhere on \(T\).

It was shown in [15, 16] (under the assumptions that \(\dim E_1 < \infty\) and \(\dim E_2 < \infty\)) that converse is also true, i.e. Property 2' implies 2 for every parameter \(\omega\). It was also shown in [15, 16] under the same assumptions (\(\dim E_1 < \infty\) and \(\dim E_2 < \infty\)) that Property 2' is in turn equivalent to this property

\[\text{2''} \text{. } \text{rank } (1_{E_1} \oplus N_2 - S^*S) = \text{rank } (1_{E_1} - s_0^*s_0) - \dim N_1, \text{ a.e. on } T.\]

This result contains, in particular, the classical Nevanlinna-Adamjan-Arov-Krein theorem: if \(\dim N_1 = \dim E_1\) (i.e., the problem is completely indeterminate), then
\[
1 - S^*S = 0
\]
a.e. on \(T\). That is, the coefficient matrix \(S\) in inner.
6. Lecture 6: Properties of the Coefficient Matrices via Dense Sets in the Function Model Spaces.

Let $S$ be the characteristic function (4.10) of the universal colligation $A_0$ (4.8)

$$S(\zeta) = \begin{bmatrix} s_0(\zeta) & s_2(\zeta) \\ s_1(\zeta) & s(\zeta) \end{bmatrix} : \begin{bmatrix} E_1 \\ N_2 \end{bmatrix} \rightarrow \begin{bmatrix} E_2 \\ N_1 \end{bmatrix}.$$  

It serves as the coefficient matrix in the parametrization formula (4.10). Since $S$ is a Schur class operator-function, there exists de Branges-Rovnyak function space $H^S$ associated to it (similar to definition in Section 1.2). We discuss here an approach that employs some dense sets in the space $H^S$. This approach was applied to Sarason problem in [17], to Nehari problem in [19, 21] and to the general Commutant Lifting problem in [5, 6].

6.1. The data of AIP suggest a dense set in $H^S$.

Let $G^0$ be the Fourier representation of the colligation $A_0$ defined in (4.12). $G^0$ maps the space $H^0$ onto the de Branges-Rovnyak function space $H^S$ contractively. We define mapping $F^S : X \rightarrow H^S$ as

$$F^S x \overset{def}{=} G^0[x],$$  

where $[x]$ is defined in (3.1), (3.2). Observe that the lineal $\{F^S x, x \in X\}$ is dense in $H^S$, since the lineal $\{[x], x \in X\}$ is dense in $H^0$.

It follows from definition (4.9) - (4.7) of the colligation $A_0$ and definition (4.12) of the Fourier representation $G_0$ that $F^S$ possess the properties similar to (1.3) and (1.4):

i) $$\|F^S x\|^2_{H^S} \leq D(x, x), \quad x \in X;$$

ii) $$tF^ST_2x - F^ST_1x = \begin{bmatrix} 1_{E_2 \oplus N_1} \\ S^* \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & M_1x \end{bmatrix} - \begin{bmatrix} -M_2x \\ M_2 \end{bmatrix},$$

a.e. on $\mathbb{T}$, where the first zero stands for the zero vector of the space $N_1$ and the second zero stands for the zero vector of the space $N_2$.

Under the Special Case assumptions (1.5), property ii) can be re-expressed as follows:

$$(F^S x)(\zeta) = S(\zeta) \begin{bmatrix} M_1 \\ 0 \end{bmatrix} - \begin{bmatrix} M_2 \\ 0 \end{bmatrix} (\zeta T_2 - T_1)^{-1} x$$
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\[(F^S x)(\zeta) = \tilde{\zeta} \left( \begin{bmatrix} M_1 \\ 0 \end{bmatrix} - S(\zeta)^* \begin{bmatrix} M_2 \\ 0 \end{bmatrix} \right) \left( T_2 - \tilde{\zeta} T_1 \right)^{-1} x, \quad |\zeta| < 1. \quad (6.2)\]

Here

\[
\begin{bmatrix} M_1 \\ 0 \end{bmatrix} : X \to \begin{bmatrix} E_1 \\ N_2 \end{bmatrix}, \quad \begin{bmatrix} M_2 \\ 0 \end{bmatrix} : X \to \begin{bmatrix} E_2 \\ N_1 \end{bmatrix}.
\]

The structure of the dense set \(\{F^S x, x \in X\}\) in \(H^S\) carries all the information about properties of the matrix-function \(S\). It will be demonstrated in the next sections how this approach works for Sarason problem.

6.2. Coefficient Matrix of the Sarason Problem and Associated Function Model Space.

In Section 2.2 we considered the scalar-valued case of the Sarason problem. We consider more general case now. Let \(\theta\) be an inner operator-function, \(\theta(\zeta) : E'_2 \to E_2\), \(\theta^* \theta = 1_{E'_2}\), a.e. on \(\mathbb{T}\), \(E'_2\) and \(E_2\) be separable Hilbert spaces. Let \(K_\theta = H^2_{+}(E_2) \oplus \theta H^2_{+}(E'_2)\), where \(H^2_{+}(E)\) stands for the vector Hardy space with coefficients in the space \(E\). Let \(T_\theta^* x = P_+ t x, \ x \in K_\theta\). Let \(W^*\) be a contractive operator, \(W^* : K_\theta \to H^2_{+}(E_1)\), where \(E_1\) is a separable Hilbert space, such that \(W^* T_\theta^* = P_+ t W^*\). Consider the following interpolation problem: find all the Schur class functions \(w(\zeta) : E_1 \to E_2\) that satisfy

\[W^* x = P_+ w^* x, \ x \in K_\theta.\]

The associated AIP data are:

\[X = K_\theta, \ T_1 = I_X, \ T_2 = T_\theta^*\]

\(E_1\) and \(E_2\) are the spaces introduced above, \(M_1 x = (W^* x)(0), \ M_2 x = x(0)\), where the latter notations stands for the value at 0 of an \(H^2_{+}(E_1)\) and \(H^2_{+}(E_2)\) function, respectively,

\[D(x, x) = (\langle I - W W^* \rangle x, x\rangle_{K_\theta}^\theta.)\]

The Fourier representation \(F^w\) is unique for every solution \(w\) and can be expressed as

\[F^w x = \begin{bmatrix} 1_{E_2} \\ w^* \end{bmatrix} \begin{bmatrix} w \\ 1_{E_1} \end{bmatrix} \begin{bmatrix} x \\ -W^* x \end{bmatrix}.\]
The Fourier representation $F^S$ of the colligation $\mathcal{A}_0$ (see (6.2)) can also be expressed as

$$F^S_x = \begin{bmatrix} 1_{E_2 \oplus N_1} & S \\ S^* & 1_{E_1 \oplus N_2} \end{bmatrix} \begin{bmatrix} x \\ 0 \\ -W^*x \\ 0 \end{bmatrix}, \quad x \in K_\theta.$$

Since $s_0$ is also a solution (corresponding to the parameter $\omega = 0$), we can write the latter expression as

$$F^S_x = \begin{bmatrix} 1_{E_2 \oplus N_1} & S \\ S^* & 1_{E_1 \oplus N_2} \end{bmatrix} \begin{bmatrix} x \\ 0 \\ -P^*s_0^*x \\ 0 \end{bmatrix}, \quad x \in K_\theta. \quad (6.3)$$

As it was observed in Section 6.1, the set

$$\{F^Sx, \ x \in K_\theta\} \subseteq H^S$$

is dense in $H^S$.

6.3. Properties of the Coefficient Matrices of the Sarason Problem.

Observe first, that the bottom entry of the vector $F^Sx$ (i.e. the second component of the vector $F^S_x$) in (6.3) is $s_2^*x$, and it is an $H^2(N_2)$ function, since $F^Sx \in H^S$. Thus,

$$s_2^*x \in H^2(N_2), \quad \forall x \in K_\theta.$$

In other words,

$$\langle s_2^*x, \ h_+ \rangle_{L^2(N_2)} = 0$$

for all $h_+ \in H^2_+(N_2)$ and $x \in K_\theta$. Equivalently,

$$\langle x, \ s_2h_+ \rangle_{L^2(E_2)} = 0$$

for all $h_+ \in H^2_+(N_2)$ and $x \in K_\theta$. Hence, $s_2h_+ \in \theta H^2(E_2')$ for all $h_+ \in H^2_+(N_2)$. This means that

$$s_2 = \theta \tilde{s}_2$$

for a Schur class function $\tilde{s}_2 : N_2 \to E_2'$. The next theorem shows how the denseness property works.
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Theorem 6.1 (14). Let the \( S = \begin{bmatrix} s_0 & \theta \tilde{s}_2 \\ s_1 & s \end{bmatrix} \) be the coefficient matrix of the Sarason problem. Then \( s_1 \) is an outer function (i.e. the lineal \( s_1 H^2_+(E_1) \) is dense in \( H^2_+(N_1) \)) and \( \tilde{s}_2 \) is a \( * \)-outer function (i.e. the lineal \( \tilde{s}_2 H^2_+(E_2') \) is dense in \( H^2_+(N_2) \)).

Sketch of the proof. To prove that \( s_1 \) is an outer function we need to check that the assumption

\[
P_+ s_1^* h_+ = 0, \quad h_+ \in H^2_+(N_1)
\]

implies \( h_+ = 0 \). Consider the vector

\[
\begin{bmatrix} 1 & S \\ S^* & 1 \end{bmatrix} \begin{bmatrix} 0 \\ h_+ \end{bmatrix} = \begin{bmatrix} 0 \\ h_+ \end{bmatrix}. \tag{6.5}
\]

It belongs to \( H^S \) for every \( h_+ \in H^2_+(N_1) \). Assumption (6.4) makes it orthogonal to the lineal (6.3). Since (6.3) is dense in \( H^S \), the orthogonality forces (6.5) to be zero. The latter in turn yields \( h_+ = 0 \).

We return now to the Problem 2.3 i.e., to the scalar Sarason problem \( (E_1 = E_2 = E_2' = \mathbb{C}) \). Assume also that the problem is indeterminate (i.e. permits a non-unique solution, i.e. \( N_1 = N_2 = \mathbb{C} \)). In this case \( S \) is an inner matrix-function (see Lecture 5) and it can be normalized so that \( s_1 = \tilde{s}_2 = a \) is an outer function.

The following criterion is a consequence of the denseness of the lineal (6.3) in \( H^S \).

Theorem 6.2 ([17, 19], also [6]). A \( 2 \times 2 \) inner matrix function

\[
S(\zeta) = \begin{bmatrix} s_0 & s_2 \\ s_1 & s \end{bmatrix} = \begin{bmatrix} s_0 & \theta a \\ a & s \end{bmatrix},
\]

(\( \text{where } a \text{ is an outer function, } s(0) = 0 \)) is the coefficient matrix of an indeterminate Sarason Problem if and only if

\[
\begin{bmatrix} P_- \tilde{s} \\ \tilde{s} \end{bmatrix} \in \text{clos} \left\{ \begin{bmatrix} P_- \tilde{s}_0 x \\ \tilde{s}_2 x \end{bmatrix}, \quad x \in K_0 \right\}, \tag{6.6}
\]

where the closure is understood in the \( L^2 \) sense.

Remark 6.3. Property (6.6) is equivalent to the following one

\[
\inf_{x \in K_0} \| P_- S^* \begin{bmatrix} 0 \\ x \\ 1 \\ 0 \end{bmatrix} \|_{H_2^2}^2 = 0,
\]
equivalently,
\[
\inf_{x \in \mathcal{K}_\theta} \left\langle SP_\theta S^* \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix} \right\rangle_{L^2} = 0. \tag{6.7}
\]
Further simplification of property (6.7) is unknown. A discussion on this matter was given in [22]. An application to uniqueness of the inverse scattering for CMV matrices was given in [25, 11].

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