Order, Disorder and Confinement

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Abstract. Studying the order of the chiral transition for \( N_f = 2 \) is of fundamental importance to understand the mechanism of color confinement. We present results of a numerical investigation on the order of the transition by use of a novel strategy in finite size scaling analysis. The specific heat and a number of susceptibilities are compared with the possible critical behaviours. A second order transition in the \( O(4) \) and \( O(2) \) universality classes are excluded. Substantial evidence emerges for a first order transition. Results are in agreement with those found by studying the scaling properties of a disorder parameter related to the dual superconductivity mechanism of color confinement.

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INTRODUCTION

Experiments indicate that confinement is an absolute property of matter. Indeed the upper limit on the number of free quarks per proton is \( R = n_q/n_p \leq 10^{-27} \), while \( R \sim 10^{-12} \) is expected from the Standard Cosmological Model. A reduction factor \( 10^{-15} \) is difficult to explain in natural ways unless \( R = 0 \), which means that confinement is related to some symmetry of the QCD vacuum. This also implies that the deconfining transition is associated to a change of symmetry of the vacuum, i.e. it is an order-disorder transition. This scenario must be confronted with direct studies of the deconfining phase transition: experimental studies going on with heavy ion collisions have not given definite answers yet and most of our knowledge relies on numerical simulations of QCD on the lattice. From this point of view, the case of two light degenerate dynamical flavours is of special interest. A schematic view of the phase diagram for \( N_f = 2 \) is shown in Fig. 1: \( m \) is the quark mass and \( \mu \) is the baryon chemical potential.

In the \( \mu = 0 \) plane, as \( m \to \infty \) quarks decouple and the system tends to the quenched limit. There the deconfining transition is an order-disorder first order phase transition, \( Z_3 \) is an associated symmetry and the Polyakov line \( \langle L \rangle \) is an order parameter. \( Z_3 \) is explicitely broken by the inclusion of dynamical quarks and \( \langle L \rangle \) is not a good order parameter, even if it works as such for quarks masses down to \( m \approx 2.5 - 3 \) GeV.

At \( m \approx 0 \) there is chiral phase transition at \( T_c \approx 170 \) MeV, from a low temperature phase where chiral symmetry is spontaneously broken to an high temperature phase in which it is restored: the corresponding order parameter is the chiral condensate \( \langle \bar{\psi} \psi \rangle \). At

\(^{1}\) Talk presented by A. Di Giacomo
some temperature $T_A \geq T_c$ also the $U_A(1)$ symmetry, broken by the anomaly, is expected to be effectively restored. It is not clear which relation exists between the chiral transition and the deconfining transition: empirically the Polyakov line has a rapid increase at the transition temperature, indicating deconfinement. The transition line in Fig. 1 is defined by the maxima of a number of susceptibilities ($C_V, \chi_m, \ldots$), all coinciding within errors, which indicate a rapid variation of the corresponding parameters across the line.

At $m \approx 0$ a renormalization group analysis plus $\varepsilon$-expansion techniques can be made, assuming that the relevant degrees of freedom for the chiral transition are scalar and pseudoscalar fields \cite{1, 2, 3}. If the $U_A(1)$ symmetry is effectively restored, \textit{i.e.} if the $\eta'$ mass vanishes at $T_c$, then there is no IR stable fixed point and the phase transition is first order; if not an IR fixed point exists, which can produce a second order phase transition in the $O(4)$ universality class.

In the first case the transition is first order also at $m \neq 0$ and most likely up to $m = \infty$. In the second case a phase transition is only present at $m = 0$, which goes into a continuous crossover as $m \neq 0$: that means that one can move continuously from confined to deconfined and that no true order parameter exists. This would be in contradiction with the deconfinement transition being associated to a change of symmetry: the issue is therefore fundamental.

The problem has been investigated on the lattice by several groups with staggered \cite{4, 5, 6, 7, 8, 9, 10} or Wilson \cite{11} fermions. The strategy used has either been to search for signs of discontinuity at the transition, or to study the scaling with $m$ of different susceptibilities, or to study the magnetic equation of state. No clear discontinuities have been observed, but also no conclusive agreement of scaling with $O(4)$ critical indexes. We present the results of a big numerical effort aimed at clarifying the issue. We use non improved Kogut–Susskind action and lattices $4 \times L_s^3$ with $L_s = 16, 20, 24, 32$. A full account of our results can be found in \cite{12}.

![Schematic phase diagram of $N_f = 2$ QCD.](image)
RESULTS

The theoretical tool to investigate the order of a phase transition is finite size scaling. The extrapolation from finite size $L_s$ to the thermodynamical limit is governed by critical indexes, which identify the order and the universality class of the transition. Around the chiral transition the system has two fundamental lengths: the correlation length $\xi$ and the inverse quark mass $1/m_q$. $\xi$ is usually traded with the reduced temperature $\tau = 1 - T/T_c$, $\xi \sim \tau^{-\nu}$ as $\tau \to 0$. The effective action depends on the order parameter, as dictated by the symmetry, and as $\tau \to 0$ irrelevant terms can be neglected; the correlators of the order parameter describe the thermodynamics. The most important quantity is the specific heat, which shows the correct critical behaviour independently of the identification of the order parameter.

For the specific heat and the susceptibility of the order parameter the scaling laws are

\[ C_V - C_0 \simeq L_s^{\alpha/\nu} \phi_c \left( \tau L_s^{1/\nu}, am_q L_s^{y_h} \right) ; \]

\[ \chi \simeq L_s^{\gamma/\nu} \phi_\chi \left( \tau L_s^{1/\nu}, am_q L_s^{y_h} \right). \]

$C_0$ stems from an additive renormalization. An alternative way to write them is

\[ C_V - C_0 \simeq L_s^{\alpha/\nu} \tilde{\phi}_c \left( \tau (am_q)^{-1/(\gamma y_h)}, am_q L_s^{y_h} \right) ; \]

\[ \chi \simeq L_s^{\gamma/\nu} \tilde{\phi}_\chi \left( \tau (am_q)^{-1/(\gamma y_h)}, am_q L_s^{y_h} \right). \]

The values of the indexes characterize the transition: the values relevant to our analysis are listed in Table I. $O(4)$ is the symmetry expected if the transition is second order, but it can break down to $O(2)$ by lattice discretization for Kogut–Susskind fermions at non zero lattice spacing.

The scaling analysis is made difficult by the presence of two independent scales. The attitude taken in the previous literature has been to assume the volume large enough so to neglect the dependence on $L_s$: since at fixed $am_q$, $\beta$ the susceptibilities must be analytic.
in the thermodynamical limit, at large $L_s$ the dependence on $am_q L_s^{y_h}$ must cancel the dependence on $L_s$ in front of the scaling functions in Eq.s (1) and (2). It follows that

$$C_V - C_0 \simeq (am_q)^{-\alpha/(vy_h)} f_c \left( \tau (am_q)^{-1/(vy_h)} \right)$$

$$\chi \simeq (am_q)^{-\gamma/(vy_h)} f_\chi \left( \tau (am_q)^{-1/(vy_h)} \right),$$

The peaks of $(C_V - C_0)$ and of $\chi$ should then scale as

$$(C_V - C_0)_{\text{max}} \propto (am_q)^{-\alpha/(vy_h)}; \quad \chi_{\text{max}} \propto (am_q)^{-\gamma/(vy_h)}$$

as $am_q \to 0$. The positions of the maxima scale as $\tau (am_q)^{-1/(vy_h)} = \text{const}$.

One can also consider to keep $\tau L_s^{1/v}$ fixed while taking $aL_s \gg 1/m_\pi$. This assumption should work better if $L_s$ is still comparable to the correlation length, which may be the case close enough to the critical point. In this case the scaling laws are

$$C_V - C_0 \simeq (am_q)^{-\alpha/(vy_h)} f_c \left( \tau L_s^{1/v} \right); \quad \chi \simeq (am_q)^{-\gamma/(vy_h)} f_\chi \left( \tau L_s^{1/v} \right).$$

Eq.s (7) stay unchanged, the positions of the maxima scale as $\tau L_s^{1/v} = \text{const}$ and the width of the peaks are volume dependent.

We have instead followed a novel strategy which does not rely on any assumption: we have performed a number of simulations at different values of $L_s$ and $am_q$ keeping the variable $am_q L_s^{y_h}$ fixed and studying the scaling with the respect to the other one. In doing

![FIGURE 3. The same as figure 2 for the chiral susceptibility $\chi_m$.](image)

### TABLE 1. Critical exponents.

|      | $y_h$     | $\nu$   | $\alpha$ | $\gamma$ | $\delta$ |
|------|-----------|---------|----------|----------|----------|
| $O(4)$ | 2.487(3)  | 0.748(14)| -0.24(6) | 1.479(94)| 4.852(24)|
| $O(2)$ | 2.485(3)  | 0.668(9) | -0.005(7)| 1.317(38)| 4.826(12)|
| $MF$   | 9/4       | 2/3     | 0        | 1        | 3        |
| 1st Order | 3        | 1/3    | 1        | 1        | $\infty$ |
so one has to assume a value for $y_h$; we have chosen that expected for $O(4)$, which is the same within errors as for $O(2)$. From Eqs. (1) and (2) it follows that, at fixed $am_qL_s^{y_h}$

$$(C_V - C_0)_{\text{max}} \propto L_s^{\alpha/\nu} \ ; \ \chi_{\text{max}} \propto L_s^{\gamma/\nu}. \quad (9)$$

as $L_s \to \infty$. If $O(4)$ or $O(2)$ is the correct symmetry, the values of $\alpha/\nu$ and $\gamma/\nu$ should be consistent with the corresponding values listed in Table 1. We have run two such sets of Monte Carlo simulations, called in the following Run1 and Run2, with $am_qL_s^{y_h} = 74.7$ and $am_qL_s^{y_h} = 149.4$ respectively. The spatial lattice sizes $L_s$ used for each of the two sets are $L_s = 12, 16, 20, 32$, the standard hybrid R algorithm [13] has been used to update configurations. In Figs. 2 and 3 we show the peak values of the specific heat and of the chiral susceptibility, divided by the appropriate power of $L_s$, as a function of $L_s$ (see Eq. 9): scaling is clearly violated, $O(4)$ and $O(2)$ universality classes are excluded.

An alternative way to study the order of a transition is to look at scaling of pseudocritical couplings: one can try the two alternative scaling laws $\tau_c = k_xL_s^{1/\nu}$ or $\tau_c = k_x'(am_q)^{1/(y_h)}$. The physical temperature $T = 1/(L_qa(\beta, m_q))$ is a function of both $\beta$ and $am_q$, so that the reduced temperature $\tau$ can be expanded as a power series in $(\beta - \beta_0)$ and in $am_q$, where $\beta_0$ is the chiral critical coupling. Only the linear term in $\beta$ was considered in the previous literature. We have found that the following terms are sufficient to fit the data

$$\tau \propto (\beta_0 - \beta) + kmam_q + k_m^2(am_q)^2 + k_m\beta am_q(\beta_0 - \beta). \quad (10)$$

Our result is that it is not possible, within the present mass range, to discriminate among the possible critical behaviours by looking at pseudocritical couplings only. The inclusion of other terms in Eq. (10), besides the one linear in $\beta$ solely considered in previous literature, is however crucial to obtain any scaling at all.

We can also use our data to perform the scaling analysis in the same way as done in previous literature, i.e. by making some assumption on the reaching of the thermodynamical limit: no universality class is chosen a priori in that case and one can test all the possible critical behaviours. We have found that assuming that $aL_s \gg 1/m_\pi$ but still $L_s \sim \xi$, i.e. Eqs. (8), works better: this is reasonable when $\xi$ goes large. We have
added to the data from Run1 and Run2, those from two other simulations performed at \( L_s = 16, am_q = 0.01335 \) and \( L_s = 24, am_q = 0.04444 \). In Fig. 4 we show the scaling obtained for the specific heat: \( O(4) \) is again clearly excluded, while a good agreement is found with a weak first order critical behaviour. A similar behaviour is observed for the chiral susceptibility.

We have also studied the magnetic equation of state, i.e. the scaling of \( \langle \bar{\psi} \psi \rangle - \langle \bar{\psi} \psi \rangle_0 = am_q^{1/\delta} F(\tau m_q^{1/\nu_\delta}) \). Results are shown in Fig. 5: again the first order behaviour describes well the data while the second order is excluded.

Even if we have still not found clear signs of discontinuities in physical observables, our results give some evidence for a weak first order chiral transition which would persist also at \( m \neq 0 \). That would be in agreement with the deconfining transition being order-disorder and with confinement being associated to some symmetry of the QCD vacuum. This naturally leads to look for such a symmetry and for an order parameter which describes the deconfining transition both with and without dynamical quarks. One candidate is dual superconductivity of the vacuum and the related parameter is the vacuum expectation value of a magnetically charged operator \( \langle \mu \rangle \). Indeed it has been shown that it is good order parameter also in full QCD [14] and its critical behaviour across the \( N_f = 2 \) transition has been studied in Ref. [15]. In Fig. 6 we show the scaling of its susceptibility \( \rho = d/d\beta \ln \langle \mu \rangle \): again results seem to indicate a first order critical behaviour.

**CONCLUSIONS**

We have argued that the study of the order of the chiral phase transition for \( N_f = 2 \) is of fundamental importance to understand confinement. We have shown that the analysis of pseudocritical coupling alone cannot discern between the possible critical behaviours. By adopting a novel strategy which reduces the finite size scaling analysis to a one scale problem, we have been able to exclude a \( O(4) (O(2)) \) second order critical behaviour, while we have found consistency with a weak first order critical behaviour both in the scaling of susceptibilities and in the equation of state. This would be in agreement with confinement being an absolute property of matter related to some symmetry and with the
deconfining transition being order-disorder: indeed the analysis of the scaling properties of the susceptibility of an order parameter associated with the dual superconductivity mechanism of color confinement leads to analogous results. However we have still not found any clear evidence for discontinuities in physical observables. The issue is still open and we plan in the future to investigate it more deeply by making simulations with improved actions and algorithms and with $a m_q L^y$ fixed according to first order.

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REFERENCES

1. R. D. Pisarski and F. Wilczek, *Phys. Rev. D* **29**, 338 (1984).
2. F. Wilczek, *Int. J. Mod. Phys. A* **7**, 3911 (1992).
3. K. Rajagopal and F. Wilczek, *Nucl. Phys. B* **399**, 395 (1993).
4. M. Fukugita, H. Mino, M. Okawa and A. Ukawa, *Phys. Rev. Lett.* **65**, 816 (1990).
5. M. Fukugita, H. Mino, M. Okawa and A. Ukawa, *Phys. Rev. D* **42**, 2936 (1990).
6. F. R. Brown, F. P. Butler, H. Chen, N. H. Christ, Z. Dong, W. Schaffer, L. I. Unger and A. Vaccarino, *Phys. Rev. Lett.* **65**, 2491 (1990).
7. F. Karsch, *Phys. Rev. D* **49**, 3791 (1994).
8. F. Karsch and E. Laermann, *Phys. Rev. D* **50**, 6954 (1994).
9. S. Aoki et al. (JLQCD collaboration), *Phys. Rev. D* **57**, 3910 (1998).
10. C. Bernard, C. DeTar, S. Gottlieb, U. M. Heller, J. Hetrick, K. Rummukainen, R.L. Sugar and D. Toussaint, *Phys. Rev. D* **61**, 054503 (2000).
11. A. A. Khan et al. (CP-PACS collaboration), *Phys. Rev. D* **63**, 034502 (2001).
12. M. D’Elia, A. Di Giacomo and C. Pica, arXiv [hep-lat/0503030].
13. S. A. Gottlieb, W. Liu, D. Toussaint, R. L. Renken and R. L. Sugar, *Phys. Rev. D* **35**, 2531 (1987).
14. J. M. Carmona, M. D’Elia, L. Del Debbio, A. Di Giacomo, B. Lucini and G. Paffuti, *Phys. Rev. D* **66**, 011503 (2002).
15. M. D’Elia, A. Di Giacomo, B. Lucini, G. Paffuti and C. Pica, *Phys. Rev. D* **71**, 114502 (2005).