Nonlocal phases of local quantum mechanical wavefunctions in static and time-dependent Aharonov–Bohm experiments

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Abstract
We show that the standard Dirac phase factor is not the only solution of the usual gauge transformation equations. The full form of a general gauge function (that connects systems that move in different sets of scalar and vector potentials), apart from Dirac phases (spatial or temporal integrals over potentials), also contains terms of classical fields that act nonlocally (in spacetime) on the local solutions of the time-dependent Schrödinger equation. As a result, the phases of wavefunctions in the Schrödinger picture are affected nonlocally by spatially and temporally remote magnetic and electric fields, in specific ways that are fully explored. These contributions go beyond the usual Aharonov–Bohm effects (magnetic or electric). (i) Application to cases of particles passing through full static magnetic or electric fields leads to cancellations of Aharonov–Bohm phases at the observation point; these cancellations are linked to behaviors at the semiclassical level (i.e. the old Werner and Brill experimental observations, or their ‘electric analogs’—or to more recent reports of Batelaan and Tonomura) but are shown to be far more general (true not only for narrow wavepackets but also for completely delocalized (spread-out) quantum states). By using these cancellations, certain previously unnoticed sign-errors in the literature are corrected. (ii) Application to time-dependent situations provides a remedy for erroneous results in the literature (concerning an uncritical use of Dirac phase factors) and leads to phases that contain an Aharonov–Bohm part and a field-nonlocal part: their competition is shown to recover relativistic causality in earlier ‘paradoxes’ (such as the van Kampen thought-experiment), while a more general consideration indicates that the temporal nonlocalities found here demonstrate in part a causal propagation of phases of quantum mechanical wavefunctions in the Schrödinger picture. This may open a new and direct way to address time-dependent double-slit experiments and the associated causal issues.

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1. Introduction

It is well established from Weyl’s work (1929), and also from independent proposals by Schrödinger (1922), Fock (1927) and London (1927) [1], that there exists a simple unitary ($U(1)$) phase mapping that connects different quantum systems, when these are gauge-equivalent (and then the phase that connects their wavefunctions is basically the gauge function of an ordinary gauge transformation). A simple unitary mapping of this type is also reserved for quantum systems moving in multiple-connected spacetimes (with enclosed appropriately defined ‘fluxes’ in the physically inaccessible regions), the corresponding ‘gauge transformation’ termed singular, and the corresponding ‘gauge function’ now being multiple-valued (although the wavefunctions of the ‘final’ (mapped) system are still single-valued) leading to phenomena of the Aharonov–Bohm type. In this paper we report on a phase mapping connecting systems that are not ‘equivalent’ (in the sense of the above two), since they can go through different classical fields in remote regions of space and/or time, and we give explicit forms of the appropriate ‘gauge functions’. The results are exact, in analytical form, and they generalize the standard Dirac phase factors derived from path integral treatments (that are very often used in an incorrect way as we will demonstrate); apart from a discussion of such misconceptions propagating in the literature, we also give first actual applications of the new results in static and time-dependent experiments, both of the Aharonov–Bohm type (i.e. with inaccessible fields and their fluxes), and also with the particles actually passing through classical fields, and even being in completely general quantum states (and not necessarily narrow wavepackets in semiclassical motion).

2. Motivation

In order to motivate this work let us first remind the reader of the standard $U(1)$ mapping

$$\Psi_2(r, t) = e^{i \frac{q}{\hbar} \Lambda(r, t)} \Psi_1(r, t)$$

between the solutions of the time-dependent Schrödinger (or Dirac) equation for a quantum particle of charge $q$ that moves (as a test particle) in two distinct sets of (predetermined and classical) vector and scalar potentials that are connected with each other (through a gauge transformation) via the ‘gauge function’ $\Lambda(r, t)$, namely

$$\nabla \Lambda(r, t) = A_2(r, t) - A_1(r, t) \quad \text{and} \quad -\frac{1}{c} \frac{\partial \Lambda(r, t)}{\partial t} = \phi_2(r, t) - \phi_1(r, t).$$

In the static case, and if for simplicity we start from system 1 being completely free of potentials ($A_1 = \phi_1 = 0$), the wavefunctions of the particle in system 2 (moving in a vector potential $A(r)$) will acquire an extra phase with an appropriate ‘gauge function’ $\Lambda(r)$ that must satisfy

$$\nabla \Lambda(r) = A(r).$$

The standard (and widely used) solution of this is the line integral

$$\Lambda(r) = \Lambda(r_0) + \int_{r_0}^{r} A(r') \cdot dr'$$

(which, by considering two paths encircling an enclosed inaccessible magnetic flux, leads to the well-known magnetic Aharonov–Bohm effect [2]). It should however be stressed that the above is only true if (3) is valid for all points $r$ of the region where the particle moves, i.e. if the particle in system 2 moves (as a narrow wavepacket) always outside magnetic

1 For a historical review see O’Raifeartaigh and Straumann [1].

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fields ($\nabla \times A = 0$ everywhere). Similarly, if the particle in system 2 moves in a spatially homogeneous scalar potential $\phi(t)$, the appropriate $\Lambda$ must satisfy

$$-\frac{1}{c}\frac{\partial \Lambda(t)}{\partial t} = \phi(t),$$

(5)

the standard solution being

$$\Lambda(t) = \Lambda(t_0) - c \int_{t_0}^{t} \phi(t') \, dt'$$

(6)

that gives the extra phase acquired by system 2 (this result leading to the electric Aharonov–Bohm effect [2] by applying it to two equipotential regions, such as two metallic cages held in distinct time-dependent scalar potentials). Once again, it should be stressed that the above is only true if (5) (and the assumed spatial homogeneity of the scalar potential $\phi$ and of $\Lambda$) is valid at all times $t$ of interest, i.e. if the particle in system 2 moves (as a narrow wavepacket) always outside electric fields ($E = -\nabla \phi - \frac{1}{c} \frac{\partial A}{\partial t} = 0$ at all times). (In the electric Aharonov–Bohm setup, the above is ensured by the fact that $t$ lies in an interval of a finite duration $T$ for which the potentials are turned on, in combination with the narrowness of the wavepacket; this guarantees that, during $T$, the particle has a vanishing probability of being at the edges of the cage where the potential starts having a spatial dependence. The reader is referred to appendix B of Peshkin [3] that demonstrates the intricacies of the electric Aharonov–Bohm effect, to which we return with an important comment at the end of the paper (section 11)).

In this work, we relax the above assumptions and present more general solutions of the system of partial differential equations (2), covering cases where the particle is not necessarily a narrow wavepacket (it can actually be in completely delocalized states) and is not excluded from remote regions (in spacetime) of nonvanishing (or, more generally, of unequal) fields (magnetic or electric), regions therefore that are actually accessible to the particle (hence non-Aharonov–Bohm cases, or even combinations of spatial multiple-connectivity of the magnetic Aharonov–Bohm type, but simultaneous simple-connectivity in spacetime (i.e. in the $(x, t)$-plane)). We find analytically nonlocal influences of these remote fields on $\Lambda(r, t)$ (with $(r, t)$ the observation point in spacetime), and therefore on the phases of wavefunctions at $(r, t)$, that seem to have a number of important consequences: they provide (i) a natural justification of earlier or more recent experimental observations for semiclassical behavior in simple-connected space (when the particles pass through full magnetic fields), and also new extensions to more general cases of delocalized (spread-out) quantum states, (ii) a nontrivial correction to misleading or even incorrect results that appear often in the literature (both for static and time-dependent cases) and (iii) a natural remedy for Causality ‘paradoxes’ in time-dependent Aharonov–Bohm configurations. These nonlocal contributions seem to have escaped from state-of-the-art path-integral approaches. An extension of the method applied to the fields (rather than the ‘gauge function’ $\Lambda$) indicates that these nonlocalities demonstrate a causal propagation of phases of quantum mechanical wavefunctions (and these can possibly address causal issues in time-dependent single- versus double-slit experiments, an area that seems to have recently attracted considerable interest [4, 5]).

3. Example of generalized solutions in static cases

By way of an example we immediately provide a simple result that will be found later (in section 9) for a static $(x, y)$-case (and for simple-connected space) that generalizes the standard
Dirac phase (4), namely

\[
\Lambda(x, y) = \Lambda(x_0, y_0) + \int_{x_0}^{x} A_x(x', y) \, dx' + \int_{y_0}^{y} A_y(x_0, y') \, dy' + \left\{ \int_{y_0}^{y} dy' \int_{x_0}^{x} dx' B_z(x', y') + g(x) \right\}
\]

(7)

with \( g(x) \) chosen so that \( \left\{ \int_{y_0}^{y} dy' \int_{x_0}^{x} dx' B_z(x', y') + g(x) \right\} \) is independent of \( x \).

In the above equation, \( B_z = (B_2 - B_1)_z \) is the difference of perpendicular magnetic fields in the two systems, which can be nonvanishing at remote regions (see below). The reader should note that the first three terms of (7) are the Dirac phase (4) along two perpendicular segments that connect the initial point \((x_0, y_0)\) to the point of observation \((x, y)\), in a clockwise sense (see for example the red-arrow paths in figure 1(b)). But apart from this Dirac phase, we also have nonlocal contributions from \( B_z \) and its flux within the ‘observation rectangle’ (see, i.e., the rectangle being formed by the red- and green-arrow paths in figure 1(b)). Below we will directly verify that (7) is indeed a solution of (3) (even for \( B_z(x', y') \neq 0 \) for \((x', y') \neq (x, y)\)), i.e. of the system of partial differential equations (PDEs):

\[
\frac{\partial \Lambda(x, y)}{\partial x} = A_x(x, y) \quad \text{and} \quad \frac{\partial \Lambda(x, y)}{\partial y} = A_y(x, y).
\]

(8)

(Although the former is trivially satisfied (at least for cases where interchanges of integrals with derivatives are legitimate), for the latter to be verified one needs to simply substitute \( \frac{\partial \Lambda(x, y)}{\partial y} = B_z(x', y) \) and then carry out the integration with respect to \( x' \)—the reader should note the crucial appearance (and proper placement) of \( x_0 \) in (7) for the verification of both (8).) It should be noted again that (7) satisfies (8) even for nonzero \( B_z \) (i.e. when the particle passes through unequal magnetic fields in remote regions), in contradistinction to the standard result (4). (For the benefit of the reader, we clearly provide in the next section all the steps for the direct verification of (7).)

Equivalently, we will later obtain the result

\[
\Lambda(x, y) = \Lambda(x_0, y_0) + \int_{x_0}^{x} A_x(x', y_0) \, dx' + \int_{y_0}^{y} A_y(x_0, y') \, dy' + \left\{ -\int_{y_0}^{y} dy' \int_{x_0}^{x} dx' B_z(x', y') + h(y) \right\}
\]

(9)

with \( h(y) \) chosen so that \( \left\{ -\int_{y_0}^{y} dy' \int_{x_0}^{x} dx' B_z(x', y') + h(y) \right\} \) is independent of \( y \), and again the reader should note that, apart from the first three terms (the Dirac phase (4) along the two other (alternative) perpendicular segments (connecting \((x_0, y_0)\) to \((x, y)\), now in a counterclockwise sense (the green-arrow paths in figure 1(b))), we also have nonlocal contributions from the flux of \( B_z \) that is enclosed within the same ‘observation rectangle’ (that is naturally defined by the four segments of the two solutions (figure 1(b))). It can also be easily verified that (9) also satisfies the system (8) (for this \( \frac{\partial \Lambda(x, y)}{\partial x} \) needs to be substituted with \( \frac{\partial \Lambda(x, y)}{\partial y} + B_z(x, y') \) and then integration with respect to \( y' \) needs to be carried out, with the proper appearance (and placement) of \( y_0 \) in (9) now being the crucial element—see direct verification in the next section).

In all the above, \( A_x \) and \( A_y \) are the Cartesian components of \( \mathbf{A}(r) = A(x, y) = A_2(r) - A_1(r) \), and, as already mentioned, \( B_z \) is the difference between (perpendicular)
magnetic fields that the two systems may experience in regions that do not contain the 
observation point \((x, y)\) (i.e. \(B_z(x', y') = (B_2(x', y') - B_1(x', y'))_z = \frac{\partial A_z(x', y')}{\partial x'} - \frac{\partial A_z(x', y')}{\partial y'}\), which can be nonzero for \((x', y') \neq (x, y)\)).

In the present and following section, we place the emphasis in pointing out (and proving) 
the new solutions (that apparently have been overlooked in the literature). In later sections, 
we will see that these results actually demonstrate that the passage of particles through magnetic 
fields has the effect of canceling Aharonov–Bohm types of phases. And in the special 
case of semiclassical motion we will suggest an understanding of this cancellation in terms 
of the experimentally observed compatibility (or consistency) of Aharonov–Bohm fringe-
displacement and trajectory deflection due to the Lorentz force. (The corresponding ‘electric 
analog’ of this consistency of trajectory behavior will also be pointed out.) However, the above 
cancellations are true even for completely delocalized states (and the deeper reason for this 
will be obvious from the derivation of the above two solutions—the origin of the cancellations 
being essentially the single-valuedness of phases for simple-connected space). Therefore, 
generalized results such as the ones above go beyond the usual Aharonov–Bohm behaviors 
reviewed in the introductory sections and give an extended description of physical systems 
in more complex physical arrangements. (It is simply mentioned here that cancellations of 
the above type will be extended and generalized further to cases that also involve the time 
variable \(t\); these will be presented in later sections, with a detailed description of how they 
are derived. Interpreted in a different way, such cancellations—through the new nonlocal 
terms—will take away the ‘mystery’ of why certain classical arguments (based on past history 
and the Faraday’s law of induction) seem to ‘work’ (give the correct Aharonov–Bohm phases 
\(\text{terms—will take away the ‘mystery’ of why certain classical arguments (based on past history 
and the Faraday’s law of induction) seem to ‘work’ (give the correct Aharonov–Bohm phases)
).)

4. Elementary verification of the above solutions (even for cases with \(B_z \neq 0\))

In static cases, and simple-connected space, let us call our solution (7) \(\Lambda_1\), namely

\[
\Lambda_1(x, y) = \Lambda_1(x_0, y_0) + \int_{x_0}^{x} A_x(x', y) \, dx' + \int_{y_0}^{y} A_y(x_0, y') \, dy' + \left\{ \int_{y_0}^{y} dy' \int_{x_0}^{x} dx' B_z(x', y') + g(x) \right\}
\]

with \(g(x)\) chosen so that \(\int_{x_0}^{x} \int_{y_0}^{y} B_z + g(x)\) is independent of \(x\).

Verification that it solves the system of PDEs (8) (even for \(B_z(x', y') \neq 0\)):

(A) \(\frac{\partial \Lambda_1}{\partial x} = A_x(x, y)\) satisfied trivially \(\checkmark\)

(because \([\ldots]\) is, by construction, independent of \(x\).

(B) \(\frac{\partial \Lambda_1}{\partial y} = \int_{x_0}^{x} \frac{\partial A_x(x', y)}{\partial x'} \, dx' + A_y(x_0, y) + \int_{x_0}^{x} B_z(x', y') \, dx' + \frac{\partial g(x)}{\partial y}\),

(the last term being trivially zero, \(\frac{\partial g(x)}{\partial y} = 0\), and then with the substitution

\(\frac{\partial A_1(x', y)}{\partial y} = \frac{\partial A_x(x', y)}{\partial x'} - B_z(x', y)\) we obtain

\[
\frac{\partial \Lambda_1}{\partial y} = \int_{x_0}^{x} \frac{\partial A_x(x', y)}{\partial x'} \, dx' - \int_{x_0}^{x} B_z(x', y') \, dx' + A_y(x_0, y) + \int_{x_0}^{x} B_z(x', y') \, dx'.
\]
We see that the second and fourth terms of the rhs cancel each other, and

(ii) the first term of the rhs is

\[ \int_{x_0}^{x} \frac{\partial A_{x}(x', y)}{\partial x'} \, dx' = A_{x}(x, y) - A_{x}(x_0, y). \]

Hence finally

\[ \frac{\partial A_{1}(x, y)}{\partial y} = A_{x}(x, y). \quad \checkmark \]

We have directly shown therefore that the basic system of PDEs (8) is indeed satisfied by our generalized solution \( A_{1}(x, y), \) even for any nonzero \( B_{z}(x', y') \) (in regions \( (x', y') \neq (x, y) \)).

In a completely analogous way, one can easily see that our alternative solution (equation (9)) also satisfies the basic system of PDEs above. Below we give the direct proof.

Let us call our second static solution (equation (9)) \( A_{2}, \) namely

\[ A_{2}(x, y) = A_{2}(x_0, y_0) + \int_{x_0}^{x} A_{x}(x', y_0) \, dx' + \int_{y_0}^{y} A_{y}(x, y') \, dy' \]

\[ + \left\{ -\int_{x_0}^{x} dx' \int_{y_0}^{y} dy' B_{z}(x', y') + h(y) \right\} \]

with \( h(y) \) chosen so that \( \{-\int_{x_0}^{x} \int_{y_0}^{y} B_{z} + h(y)\} \) is independent of \( y. \)

Verification that it solves the system of PDEs (8) (even for \( B_{z}(x', y') \neq 0)\):

(A) \( \frac{\partial A_{1}(x, y)}{\partial y} = A_{x}(x, y) \) satisfied trivially \( \checkmark \)
(because \( \{-\int_{x_0}^{x} \int_{y_0}^{y} B_{z} + h(y)\} \) is, by construction, independent of \( y). \)

(B) \( \frac{\partial A_{1}(x, y)}{\partial x} = A_{x}(x, y_0) + \int_{y_0}^{y} \frac{\partial A_{x}(x, y')}{\partial x'} \, dy' - \int_{y_0}^{y} B_{z}(x, y') \, dy' + \frac{\partial h(y)}{\partial x}, \)

\[ \text{the last term being trivially zero, } \frac{\partial h(y)}{\partial x} = 0, \text{ and then with the substitution} \]

\[ \frac{\partial A_{1}(x, y')}{\partial x} = \frac{\partial A_{x}(x, y')}{\partial y'}, \]

we obtain

\[ \frac{\partial A_{2}(x, y)}{\partial x} = A_{x}(x_0, y_0) + \int_{y_0}^{y} \frac{\partial A_{x}(x, y')}{\partial y'} \, dy' + \int_{y_0}^{y} B_{z}(x, y') \, dy' - \int_{y_0}^{y} B_{z}(x, y') \, dy'. \]

(i) We see that the last two terms of the rhs cancel each other, and

(ii) the second term of the rhs is \( \int_{y_0}^{y} \frac{\partial A_{1}(x, y')}{\partial y'} \, dy' = A_{x}(x, y) - A_{x}(x, y_0). \)

Hence finally

\[ \frac{\partial A_{2}(x, y)}{\partial x} = A_{x}(x, y). \quad \checkmark \]

Once again, all the above are true for any nonzero \( B_{z}(x', y') \) (in regions \( (x', y') \neq (x, y) \)).

5. Simple examples: new results shown in explicit form

To see how the above solutions appear in nontrivial cases (and how they give completely new results, i.e. not differing from the usual ones (i.e. from the Dirac phase) by a mere constant) let us first take examples of striped \( B_{z} \)-distributions in spacetime.
(a) For the case of an extended *vertical* strip, parallel to the y-axis, same as in figure 1(a) (with \( t \) replaced by \( y \)) (i.e. the particle has actually passed through nonzero \( B_z \), hence through *different* magnetic fields in the two (mapped) systems), and then, for \( x \) located outside (and on the right of) the strip, the quantity \( \int_{y_0}^y \int_{x_0}^x \text{d}x' \text{d}y' B_z(x', y') \) in \( \Lambda_1 \) is already independent of \( x \) (since a displacement of the \((x, y)\)-corner of the rectangle to the right, along the \( x \)-direction, does not change the enclosed magnetic flux—see figure 1(a) for the analogous \((x, t)\)-case that will be discussed in following sections); hence, in this case, the function \( g(x) \) can be taken as \( g(x) = 0 \) (up to a constant \( C \)) and the condition for \( g(x) \) stated in solution (7) (i.e. that the quantity in brackets must be independent of \( x \)) is indeed satisfied.

So for this setup, the nonlocal term in the solution *survives* (the quantity in brackets is nonvanishing), but it *is not constant*: this enclosed flux depends on \( y \) (since the enclosed flux *does change* with a displacement of the \((x, y)\)-corner of the rectangle upward, along the \( y \)-direction). Hence, by looking at the alternative solution \( \Lambda_2(x, y) \), the quantity \( \int_{y_0}^y \int_{x_0}^x \text{d}x' \text{d}y' B_z(x', y') \) is independent on \( y \), so that \( h(y) \) must be chosen as \( h(y) = +\int_{x_0}^x \int_{y_0}^y \text{d}x' \text{d}y' B_z(x', y') \) (up to the same constant \( C \)) in order to *cancel* the \( y \)-dependence, so that its own condition stated in solution (9) (i.e. that the quantity in brackets must be independent of \( y \)) is satisfied; as a result, the quantity in brackets in solution \( \Lambda_2 \) disappears and there is no nonlocal contribution in \( \Lambda_2 \) (for \( C = 0 \)). (Of course, if we had used a \( C \neq 0 \), the nonlocal contributions would be distributed between the two solutions in a different manner, but without changing the physics when we take the *difference* of the two solutions (see below).)

With these choices of \( h(y) \) and \( g(x) \), we already have new results (compared to the standard ones of the integrals of potentials). That is, one of the two solutions, namely \( \Lambda_1 \), is affected nonlocally by the enclosed flux (and this flux is *not* constant). Spelled out clearly, the two results are

\[
\Lambda_1(x, y) = \Lambda_1(x_0, y_0) + \int_{x_0}^x A_x(x', y) \text{d}x' + \int_{y_0}^y A_y(x_0, y') \text{d}y' + \int_{y_0}^y \int_{x_0}^x \text{d}x' \text{d}y' B_z(x', y') + C
\]

\[
\Lambda_2(x, y) = \Lambda_2(x_0, y_0) + \int_{x_0}^x A_x(x', y_0) \text{d}x' + \int_{y_0}^y A_y(x, y') \text{d}y' + \int_{y_0}^y \int_{x_0}^x \text{d}x' \text{d}y' + C.
\]

And it is easy to see that, if we subtract the two solutions \( \Lambda_1 \) and \( \Lambda_2 \), the result is zero (because the line integrals of the vector potential \( A \) in the two solutions are in opposite senses in the \((x, y)\) plane; hence, their difference leads to a *closed* line integral of \( A \) which is in turn equal to the enclosed magnetic flux, and this flux always happens to be of the opposite sign from that of the enclosed flux that explicitly appears through the nonlocal term of the \( B_z \)-field that survives in \( \Lambda_1 \)). (In the above we of course assumed single-valuedness of \( \Lambda \) at the initial point \((x_0, y_0)\), i.e. \( \Lambda_1(x_0, y_0) = \Lambda_2(x_0, y_0) \); matters of multivaluedness of \( \Lambda \) at the observation point \((x, y)\) will be addressed later, in section 9.)

The reader should probably note that the above equality of the two solutions is due to the fact that the \( x \)-independent quantity in brackets of the first solution (7) is equal to the function \( h(y) \) of the second solution (9), and the \( y \)-independent quantity in brackets of the second solution (9) is equal to the function \( g(x) \) of the first solution (7). This will turn out to be a general behavioral pattern of the two solutions in simple-connected space, that will be valid for any shape of \( B_z \)-distribution, as will be shown later.
Figure 1. Examples of the simplest field configurations (in simple-connected spacetimes) where the nonlocal terms are nontrivial: (a) a striped case in 1+1 spacetime, where the electric flux enclosed in the ‘observation rectangle’ is dependent on \( t \) but independent of \( x \); (b) a triangular distribution in 2D space, where the part of the magnetic flux inside the ‘observation rectangle’ depends on both \( x \) and \( y \). The appropriate choices for the corresponding nonlocal functions \( g(x) \) and \( h(y) \) are given in the text (equations (10) and (11)).

(This figure is in colour only in the electronic version)

This vanishing of \( \Lambda_1(x, y) - \Lambda_2(x, y) \) is a cancellation effect that is emphasized further (and generally proved) later below (and can be viewed as a generalization of the Werner and Brill experimental observations [6] to even completely delocalized states, as will be fully discussed in physical terms in section 9). It basically originates from the single-valuedness of \( \Lambda \) at \((x, y)\) for simple-connected space. This effect is generalized even further in later sections (i.e. also to cases of combined three variables \( x, y, t \) for the van Kampen thought-experiment [7] (where we will have a combination of spatial multiple-connectivity at an initial instant \( t_0 \), and simple-connectivity in \((x, t)\) and \((y, t)\) planes).

(b) In the ‘dual case’ of an extended horizontal strip, parallel to the \( x \)-axis, the proper choices (for \( y \) above the strip) are basically reverse (i.e. we can now take \( h(y) = 0 \) and \( g(x) = -\int_{y_0}^{y} \int_{x_0}^{x} B_z(x', y') \textrm{d}x' \textrm{d}y' \) (since the flux enclosed in the rectangle now depends on \( x \), but not on \( y \), with both choices always up to a common constant) and once again we can easily see a similar cancellation effect. In this case again, the results are new (a nonlocal term now surviving in \( \Lambda_2 \)). Again spelled out clearly, these are

\[
\begin{align*}
\Lambda_1(x, y) &= \Lambda_1(x_0, y_0) + \int_{x_0}^{x} A_x(x', y) \textrm{d}x' + \int_{y_0}^{y} A_y(x_0, y') \textrm{d}y' + C \\
\Lambda_2(x, y) &= \Lambda_2(x_0, y_0) + \int_{x_0}^{x} A_x(x', y_0) \textrm{d}x' + \int_{y_0}^{y} A_y(x, y') \textrm{d}y' \\
&\quad - \int_{x_0}^{x} \textrm{d}x' \int_{y_0}^{y} \textrm{d}y' B_z(x', y') + C
\end{align*}
\]

(\( \Lambda_1(x, y) - \Lambda_2(x, y) \) is zero—a generalized Werner and Brill cancellation (see section 9 for further discussion)).

(c) If we want cases that are more involved (i.e. with the nonlocal contributions appearing nontrivially in both solutions \( \Lambda_1 \) and \( \Lambda_2 \) and with \( g(x) \) and \( h(y) \) not being ‘immediately visible’), we must consider different shapes of \( B_z \)-distributions. One such case is a triangular one that is shown in figure 1(b) (for simplicity, an equilateral triangle, with the initial point \((x_0, y_0) = (0, 0)\) and with the point of observation \((x, y)\) being fairly close to the triangle’s right side as in the figure. Note that for such a configuration, the part of the magnetic flux that is inside the ‘observation rectangle’ (defined by the right-upper corner \((x, y)\)) depends on both \( x \) and \( y \). It turns out, however, that this \((x \text{ and } y)\)-dependent enclosed flux can be written as a sum of separate \( x \)- and \( y \)-contributions, so that appropriate \( g(x) \) and \( h(y) \) can be found (each one of them must be chosen so that it only cancels...
the corresponding variable’s dependence of the enclosed flux). For a homogeneous $B_z$, it is a rather straightforward exercise to determine this enclosed part, i.e. the common area between the observation rectangle and the equilateral triangle, and from this we can find the appropriate $g(x)$ that will cancel the $x$-dependence, and the appropriate $h(y)$ that will cancel the $y$-dependence. These appropriate choices turn out to be

$$g(x) = B_z \left[ -\left( \sqrt{3}ax - \frac{\sqrt{3}}{2}x^2 \right) + \frac{\sqrt{3}}{4}a^2 \right] + C$$

(10)

and

$$h(y) = B_z \left[ ay - \frac{y^2}{\sqrt{3}} \right] - \frac{\sqrt{3}}{4}a^2 + C$$

(11)

with $a$ being the side of the equilateral triangle. (We again note that a physical arbitrariness described by the common constant $C$ does not play any role when we take the difference of the two solutions (7) or (9).) We should emphasize that the above results, if combined with (7) or (9), give the nontrivial nonlocal contributions of the difference $B_z$ of the remote magnetic fields on $\Lambda_1$ of each solution (hence on the phase of the wavefunction of each wavepacket traveling along each path) at the observation point $(x, y)$. (We mention again that in the case of completely spread-out states, the equality of the two solutions at the observation point essentially demonstrates the uniqueness (single-valuedness) of the phase in simple-connected space.) Further physical discussion, and a semiclassical interpretation, is given later, in section 9 and in the final sections of the paper.

In more ‘difficult’ geometries, i.e. when the shape of the $B_z$-distribution is such that the enclosed flux does not decouple in a sum of separate $x$- and $y$-contributions, such as cases of circularly shaped distributions, it is advantageous to solve the system (3) directly in non-Cartesian (i.e. polar) coordinates. This is done further below in section 9.

Finally, the reader may wonder how the usual magnetic Aharonov–Bohm effect arises in the above formulation, and here is probably the best place to provide an explanation (although for this we will need to invoke the most general results—for multiple-connected space—that will be derived later). For the Aharonov–Bohm setting we will have to deal with multiple-connected space and with a (static) magnetic flux $\Phi_1$ being contained only in the physically inaccessible region. In such a case we know that the $\Lambda(r)$ that solves (3) is not single-valued. How is this fact (and the standard result (4)) compatible with the new formulation? To answer this in full generality we will consider two separate cases that arise naturally (pertaining to the issue of what the dummy variables $(x', y')$ inside the $B_z$-terms of our results (i.e. of (7) and (9)) actually represent). First, if the variables $x$ and $y$ everywhere above always denote only coordinates of the region that is physically accessible to the particle, then $B_z$ above is everywhere vanishing, this effectively reducing (7) and (9) to

$$\Lambda(x, y) = \Lambda(x_0, y_0) + \int_{x_0}^{x} A_x(x', y) \, dx' + \int_{y_0}^{y} A_y(x, y') \, dy' + C$$

and

$$\Lambda(x, y) = \Lambda(x_0, y_0) + \int_{x_0}^{x} A_x(x', y_0) \, dx' + \int_{y_0}^{y} A_y(x, y') \, dy' + C$$

with $C$ a common constant; these are the standard results (the Dirac phases) along the two alternative paths discussed above (the red and green paths of figure 1(b)) that (through their difference) lead to the magnetic Aharonov–Bohm effect ($\Lambda$ being no longer single-valued and the difference of the two solutions giving the enclosed (and physically inaccessible) $\Phi$). Let us however be even more general and let us decide to use the variables $x$ and $y$ to also denote...
coordinates of the physically inaccessible region; this would be the case, if, for example, we had previously started with that region being accessible (i.e. through a penetrable scalar potential) and at the end we followed a limiting procedure (i.e. of this scalar potential going to infinity) so that this region would become in the limit impenetrable and therefore inaccessible.

In such a case the variables \(x\) and \(y\) would now contain remnants of the previously allowed values (but currently not allowed for the description of particle coordinates) such as the values of the dummy variables \(x'\) and \(y'\) in the \(B_z\)-terms of (7) and (9); such values would therefore still be present in the expressions giving \(\Lambda\) (even though these dummy variables \(x'\) and \(y'\) would now describe an inaccessible region). In other words, the inaccessible \(B_z\) is still formally present in the problem and it shows up explicitly in the generalized gauge functions of the new formulation. How does this formulation then lead to the standard Aharonov–Bohm result in such a limiting case (essentially a case of smoothly induced spatial multiple-connectivity)?

Before we answer this, the reader should probably be reminded that our formulation only deals with wavefunction phases; questions therefore of rigid (vanishing) boundary conditions (on the boundary of the inaccessible region) that apply to (and must be imposed on) the entire wavefunction, and mostly on its modulus, can only be addressed indirectly (and as we will see, through a ‘memory’ that the phases have of their multivaluedness, whenever the space is multiple-connected). To see this, we need two slightly generalized results that will be rigorously derived later (equations (29) and (33)) that add certain constants (what we will later call ‘multiplicities’) to the above ‘simple-connected’ forms (7) and (9). These most general results (for multiple-connected space) will be derived in section 9 and will turn out to be

\[
\Lambda(x, y) = \Lambda(x_0, y_0) + \int_{x_0}^{x} A_x(x', y) \, dx' + \int_{y_0}^{y} A_y(x_0, y') \, dy' 
\]

\[
+ \left\{ \int_{y_0}^{y} dy' \int_{x_0}^{x} dx' B_z(x', y') + g(x) \right\} + f(y_0)
\]

and

\[
\Lambda(x, y) = \Lambda(x_0, y_0) + \int_{x_0}^{x} A_x(x', y_0) \, dx' + \int_{y_0}^{y} A_y(x, y') \, dy' 
\]

\[
+ \left\{ - \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' B_z(x', y') + h(y) \right\} + \hat{h}(x_0)
\]

with the functions \(g(x)\) and \(h(y)\) satisfying the same conditions as in (7) and (9). We note the extra appearance of the new constant terms \(f(y_0)\) and \(\hat{h}(x_0)\) (the ‘multiplicities’) and these are ‘defined’ (see (26) and (30) where the functions \(f\) and \(h\) will be first introduced) by

\[
f(y_0) = \Lambda(x, y_0) - \Lambda(x_0, y_0) - \int_{x_0}^{x} A_x(x', y_0) \, dx'
\]

and

\[
\hat{h}(x_0) = \Lambda(x_0, y) - \Lambda(x_0, y_0) - \int_{y_0}^{y} A_y(x_0, y') \, dy'.
\]

Let us then identify proper choices for the functions \(g(x)\) and \(h(y)\) and for the constants \(f(y_0)\) and \(\hat{h}(x_0)\) in the above case of spatial multiple-connectivity (such as the standard magnetic Aharonov–Bohm case, with a non-extended (and static) magnetic flux in the forbidden region). First, we can always take \(g(x) = 0\) and \(h(y) = 0\) (always up to a common constant as discussed earlier), since the enclosed magnetic flux is (in this Aharonov–Bohm case) independent of both \(x\) and \(y\)—the conditions of \(g(x)\) and \(h(y)\) being then automatically satisfied. Second, let us look more closely at the above ‘definitions’ of \(f(y_0)\) and \(\hat{h}(x_0)\): we first note that \(f(y_0)\) must
be independent of $x$, and this is indeed true as is apparent by formally taking the derivative of the above definition of $f(y_0)$ with respect to $x$; we then have $\frac{d f(y_0)}{dx} = \frac{d \Lambda_1(y_0)}{dx} - A_1(x, y_0)$ which is indeed zero (as $\Lambda(x, y)$ satisfies by assumption, the first equation of the system (8) of PDEs (evaluated at $y = y_0$)), showing that $\frac{d f(y_0)}{dx} = 0$ and that $f(y_0)$ does not really depend on the variable $x$ that appears in its definition. We can therefore determine its value by taking the limit $x \to x_0$ (for fixed $y_0$): we see from the above that this limit is simply equal to $\lim_{x_0 \to x} \Lambda(x, y_0) - \Lambda(x_0, y_0)$ (we leave out cases where $\Lambda_1$ has a $\delta$-function form, as will be discussed later in the careful derivations of all our results where interchanges of integrals must be allowed), and this difference is nonzero only when there is a multivaluedness of $\Lambda$ at the point $(x_0, y_0)$, as is actually our case. The limit $x \to x_0$ (for fixed $y_0$) in the path sense of solution (7) (or of (29)) that is then needed here in order to determine $f(y_0)$ is equivalent to making an entire closed trip around the observation rectangle in the negative sense, landing on the initial point $(x_0, y_0)$, this therefore giving the value $f(y_0) = \mp \Phi$ (which is indeed a constant independent of $x$ and $y$, as it should be). By following a completely symmetric argument for the above definition of $\hat{h}(x_0)$ (and by now taking the limit $y \to y_0$ (for fixed $x_0$), that is now equivalent to going around the loop in the positive sense, landing again on the initial point $(x_0, y_0)$) we obtain $\hat{h}(x_0) = +\Phi$. If these values of $f(y_0)$ and $\hat{h}(x_0)$ are finally substituted in the above most general solutions (equations (29) and (33)) together with $g(x) = h(y) = 0$, then we note that $f(y_0)$ cancels out the $\int_{y_0} \int_{x_0} \delta y \delta x B(x', y')$ term (which is here just equal to the inaccessible flux $\Phi$) and $\hat{h}(x_0)$ cancels out the $-\int_{y_0} \int_{x_0} \delta y \delta x B_2(x', y')$ term, and the two solutions are then once again reduced to the usual solutions of mere $A$-integrals along the two paths (i.e. the standard Dirac phase, with no nonlocal contributions)—their difference giving once again the closed loop integral of $A$, hence the inaccessible flux and, finally the well-known magnetic Aharonov–Bohm result. One should note again the expected, namely that the standard result in the new formulation requires some effort and it is only derived indirectly (due to the fact that we only deal with phases and not the moduli of wavefunctions, on which boundary conditions are normally imposed), and it basically comes from the ‘memory’ of the multivaluedness that the ‘gauge function’ $\Lambda$ carries (due to the multiple-connectivity of space).

6. Example of generalized solutions in dynamic cases, with full derivation

Let us now look at a case with full time-dependence. Although it is now probably easy for the reader to guess the corresponding generalized results, i.e. for a spatially one-dimensional $(x, t)$-problem (i.e. by Euclidean rotation (in 4D spacetime) from the above solutions), we nevertheless start from the beginning and give a full physical discussion—as this is the case that actually led us to the above generalized solutions, and a case associated with a number of misleading arguments (and often incorrect results) propagating in the literature.

Let us then focus on the simplest case of one-dimensional quantum systems, i.e. a single quantum particle of charge $q$, but in the presence of the most general (spatially nonuniform and time-dependent) vector and scalar potentials, and ask the following question: what is the gauge function $\Lambda(x, t)$ that takes us from (maps) a system with potentials $A_1(x, t)$ and $\phi_1(x, t)$ to a system with potentials $A_2(x, t)$ and $\phi_2(x, t)$? (meaning the usual mapping (1) between the wavefunctions of the two systems through the phase factor $\frac{\phi_2}{\phi_1} \Lambda(x, t)$). (Of course for this mapping to be possible we assume that at the point $(x, t)$ of observation (or ‘measurement’ of $\Lambda$ or the wavefunction $\Psi$) we have equal electric fields ($E_i = -\nabla \phi_i = \frac{1}{c} \frac{\partial A_i}{\partial t}$), namely

$$\frac{\partial \phi_2(x, t)}{\partial x} - \frac{1}{c} \frac{\partial A_2(x, t)}{\partial t} = - \frac{\partial \phi_1(x, t)}{\partial x} + \frac{1}{c} \frac{\partial A_1(x, t)}{\partial t}$$

(12)
(so that the $A$'s and $\phi$'s in (12) can satisfy the basic system of equations (2) or, equivalently, of the system of equations (15) below), but we will not exclude the possibility of the two systems passing through different electric fields in different regions of spacetime, i.e. for $(x', t') \neq (x, t)$. In fact, this possibility will come out naturally from a careful solution of the basic system (15); it is, for example, straightforward for the reader to immediately verify that the results (19) or (23) that will be derived below (and will contain contributions of electric fields from remote regions of spacetime) indeed satisfy the basic input system of equations (15), something that will be explicitly verified in the next section.)

Returning to the question on the appropriate $\Lambda$ that takes us from the set $(A_1, \phi_1)$ to the set $(A_2, \phi_2)$, we note that, in cases of static vector potentials $(A(x)'s)$ and spatially uniform scalar potentials $(\phi(t)'s)$ the answer usually given is the well-known

$$
\Lambda(x, t) = \Lambda(x_0, t_0) + \int_{x_0}^{x} A(x') \, dx' - c \int_{t_0}^{t} \phi(t') \, dt'
$$

(13)

with $A(x) = A_2(x) - A_1(x)$ and $\phi(t) = \phi_2(t) - \phi_1(t)$ (and it can be viewed as a combination of (4) and (6), being immediately applicable to the description of cases of combined magnetic and electric Aharonov–Bohm effects reviewed in the introductory sections).

In the most general case (and with the variables $x$ and $t$ being completely uncorrelated), it is often stated in the literature (e.g. in [8], see equation (57)) that the appropriate $\Lambda$ has a form that is a plausible extension of (13), namely

$$
\Lambda(x, t) = \Lambda(x_0, t_0) + \int_{x_0}^{x} [A_2(x', t) - A_1(x', t)] \, dx' - c \int_{t_0}^{t} [\phi_2(x, t') - \phi_1(x, t')] \, dt'.
$$

(14)

This form is certainly incorrect for uncorrelated variables $x$ and $t$ (the reader can easily verify that the system of equations (15) below is not satisfied by (14)); we will find that the correct form consists of two terms: one is rather trivial (and leads to the natural appearance of a path that connects initial and final points in spacetime, a property that (14) does not have (see equations (19) and (23) below for the corrected ‘path-forms’ in the line integrals of potentials)), but the second term is nontrivial—it consists of nonlocal contributions of classical electric fields from remote regions of spacetime. We will discuss below the consequences of these terms and we will later show that such nonlocal contributions also appear (in an extended form) in more general situations, i.e. they are also present in higher spatial dimensionality (and they then also involve remote magnetic fields in combination with the electric ones); these lead to modifications of ordinary Aharonov–Bohm behaviors or have other consequences, one of them being a natural remedy of causality ‘paradoxes’ in time-dependent Aharonov–Bohm experiments. (An application of the method to the integral forms of Maxwell’s equations will also be briefly mentioned, which, although not the main focus of this paper, gives an important causal interpretation of these temporal nonlocalities of wavefunction phases in the general case.)

The form (14) commonly used is of course motivated by the well-known Wu and Yang [9] nonintegrable phase factor that has a phase equal to $\int A_\mu \, dx^\mu = \int A \, dx - c \int \phi \, dt$, a form that appears naturally within the framework of path-integral treatments, or generally in physical situations where narrow wavepackets are implicitly assumed for the quantum particle: the integrals appearing in (14) are then taken along particle trajectories (hence spatial and temporal variables not being uncorrelated, but being connected in a particular manner to produce the path; all integrals are therefore basically only time integrals). But even then, equation (14) is valid only when these trajectories are always (in time and in space) inside identical classical fields for the two (mapped) systems. Here, however, we will be focusing on what a canonical (and not a path-integral or other semiclassical) treatment gives us; this will cover the general case of arbitrary wavefunctions that can even be completely delocalized, and
will also allow the particle to travel through different classical fields for the two systems in remote spacetime regions (i.e. \( E_2(x, t') \neq E_1(x, t) \) if \( t' < t \), etc).

It is therefore clear that finding the appropriate \( \Lambda(x, t) \) that answers the above question in full generality will require a careful solution of the system of PDEs (2), applied to only one spatial variable, namely

\[
\frac{\partial \Lambda(x, t)}{\partial x} = A(x, t) \quad \text{and} \quad \frac{1}{c} \frac{\partial \Lambda(x, t)}{\partial t} = \phi(x, t)
\]

((with \( A(x, t) = A_2(x, t) - A_1(x, t) \) and \( \phi(x, t) = \phi_2(x, t) - \phi_1(x, t) \)), the system being underdetermined in the sense that we only have knowledge of \( \Lambda \) at an initial point \((x_0, t_0)\) and with no further boundary conditions (hence multiplicities of solutions being generally expected, and these are discussed separately below). Let us first look for unique (single-valued) solutions (i.e. with \( \Lambda \) being a function on the \((x, t)\)-plane, in the sense of elementary analysis) and let us integrate the first of (15)—without dropping terms that may at first sight appear redundant—to obtain

\[
\Lambda(x, t) - \Lambda(x_0, t) = \int_{x_0}^{x} A(x', t) \, dx' + \tau(t).
\]

By then substituting this to the second of (15) (and assuming that interchanges of derivatives and integrals are allowed, i.e. covering cases of potentials with discontinuous first derivatives, something that corresponds to the physical case of discontinuous magnetic fields—a case very often discussed in the literature), we obtain

\[
\phi(x, t) = -\frac{1}{c} \int_{x_0}^{x} \frac{\partial A(x', t)}{\partial t} \, dx' - \frac{1}{c} \frac{\partial \tau(t)}{\partial t} - \frac{1}{c} \frac{\partial \Lambda(x_0, t)}{\partial t},
\]

which if integrated gives

\[
\tau(t) = \tau(t_0) + \Lambda(x_0, t_0) - \Lambda(x, t) - \int_{t_0}^{t} d\tau' \int_{x_0}^{x} \frac{\partial A(x', t')}{\partial t'} \, dx' - c \int_{t_0}^{t} \phi(x, t') \, d\tau' + g(x)
\]

(18)

with \( g(x) \) to be chosen in such a way that the entire right-hand side of (18) is only a function of \( t \) (hence independent of \( x \)). Finally, by substituting \( \frac{\partial \Lambda(x', t')}{\partial t'} \) with \(-c\left(E(x', t') + \frac{\partial \Lambda(x', t')}{\partial x}ight)\) (where \( E(x', t') = E_2(x', t') - E_1(x', t') \)), carrying out the integration with respect to \( x' \), and by demanding that \( \tau(t) \) be independent of \( x \), we finally obtain the following general solution:

\[
\Lambda(x, t) = \Lambda(x_0, t_0) + \int_{x_0}^{x} A(x', t) \, dx' - c \int_{t_0}^{t} \phi(x_0, t') \, d\tau' + \left\{ c \int_{t_0}^{t} d\tau' \int_{x_0}^{x} \frac{\partial A(x', t')}{\partial t'} \, dx' \right\} + \tau(t_0)
\]

(19)

with \( g(x) \) chosen in such a way that the quantity \( \left\{ c \int_{t_0}^{t} d\tau' \int_{x_0}^{x} \frac{\partial A(x', t')}{\partial t'} \, dx' \right\} \) is independent of \( x \).

Here it should be noted that if we had first integrated the second of (15) we would have

\[
\Lambda(x, t) - \Lambda(x, t_0) = -c \int_{t_0}^{t} \phi(x, t') \, d\tau' + \chi(x)
\]

(20)

and then from the first of (15) we would get

\[
\Lambda(x, t) = -c \int_{t_0}^{t} \frac{\partial \phi(x, t')}{\partial x} \, d\tau' + \frac{\partial \chi(x)}{\partial x} + \frac{\partial \Lambda(x, t_0)}{\partial x}.
\]

(21)
which after integration would give
\[
\chi(x) = \chi(x_0) + \Lambda(x_0, t_0) - \Lambda(x, t_0) + c \int_{x_0}^{x} dx' \int_{t_0}^{t} dt' \frac{\partial \phi(x', t')}{\partial x'} + \int_{x_0}^{x} A(x', t) \, dx' + \hat{g}(t)
\]

(22)

with \(\hat{g}(t)\) to be chosen in such a way that the entire right-hand side of (22) is only a function of \(x\) (hence independent of \(t\)). Finally, by substituting \(\frac{\partial \phi(x', t')}{\partial x'}\) with \(-\left( E(x', t') + \frac{1}{c} \frac{\partial A(x', t')}{\partial t'} \right)\), carrying out the integration with respect to \(t'\), and by demanding that \(\chi(x)\) be independent of \(t\), we would finally obtain the following general solution:

\[
\Lambda(x, t) = \Lambda(x_0, t_0) + \int_{x_0}^{x} A(x', t_0) \, dx' - c \int_{t_0}^{t} \phi(x, t') \, dt' \]

\[
+ \left\{ -c \int_{x_0}^{x} dx' \int_{t_0}^{t} dt' E(x', t') + \hat{g}(t) \right\} + \chi(x_0)
\]

(23)

with \(\hat{g}(t)\) chosen in such a way that the quantity \(-c \int_{x_0}^{x} dx' \int_{t_0}^{t} dt' E(x', t') + \hat{g}(t)\) is independent of \(t\).

Solutions (19) and (23) can be viewed as the (formal) analogs of (7) and (9) correspondingly, although they hide in them much richer physics because of their dynamic character (see section 8). (The additional constant last terms will be shown in section 8 to be related to possible multiplicities of \(\Lambda\), and they are zero in simple-connected spacetimes).

The reader is once again provided with the direct verification that (19) or (23) are indeed solutions of the basic system of PDEs (15) in the section that follows.

7. Verification of solutions and simple dynamical examples

Let us call our first solution (equation (19)) for simple-connected spacetime \(\Lambda_3\), namely
\[
\Lambda_3(x, t) = \Lambda_3(x_0, t_0) + \int_{x_0}^{x} A(x', t_0) \, dx' - c \int_{t_0}^{t} \phi(x_0, t') \, dt' + \left\{ c \int_{t_0}^{t} dt' \int_{x_0}^{x} dx' E(x', t') + g(x) \right\}
\]

with \(g(x)\) chosen so that \(\left\{ c \int_{t_0}^{t} dt' \int_{x_0}^{x} dx' E(x', t') + g(x) \right\}\) is independent of \(x\).

Verification that it solves the system of PDEs (15) (even for \(E(x', t') \neq 0\)):

(A) \(\frac{\partial \Lambda_3(x, t)}{\partial t} = \Lambda(x, t)\) satisfied trivially \(\checkmark\)

(because \([\cdots]\) is, by construction, independent of \(x\)).

(B) \(-\frac{1}{c} \frac{\partial \Lambda_3(x, t)}{\partial x} = -\frac{1}{c} \int_{x_0}^{x} \frac{\partial A(x', t)}{\partial x'} \, dx' + \phi(x_0, t) - \int_{x_0}^{x} E(x', t) \, dx' - \frac{1}{c} \frac{\partial g(x)}{\partial t}\)

(the last term being trivially zero, \(\frac{\partial g(x)}{\partial t} = 0\)), and then with the substitution
\[
-\frac{1}{c} \frac{\partial A(x', t)}{\partial t} = \frac{\partial \phi(x', t)}{\partial x'} + E(x', t),
\]

we obtain
\[
-\frac{1}{c} \frac{\partial \Lambda_3(x, t)}{\partial t} = \int_{x_0}^{x} \frac{\partial \phi(x', t)}{\partial x'} \, dx' + \int_{x_0}^{x} E(x', t) \, dx' + \phi(x_0, t) - \int_{x_0}^{x} E(x', t) \, dx'.
\]

(i) We see that the second and fourth terms of the rhs cancel each other, and
(ii) the first term of the rhs is \(\int_{x_0}^{x} \frac{\partial \phi(x', t)}{\partial x'} \, dx' = \phi(x, t) - \phi(x_0, t)\).
Hence finally
\[ -\frac{1}{c} \frac{\partial \Lambda_3(x, t)}{\partial t} = \phi(x, t). \]

We have directly shown therefore that the basic system of PDEs (15) is indeed satisfied by our generalized solution \( \Lambda_3(x, t) \), even for any nonzero \( E(x', t') \) (in regions \( (x', t') \neq (x, t) \)).

In a completely analogous way, one can easily see that our alternative solution (equation (23)) also satisfies the basic system of PDEs above. In case this is still not clear, here is the proof.

Let us call our second (alternative) solution (equation (23)) again for simple-connected spacetime \( \Lambda_4 \), namely
\[ \Lambda_4(x, t) = \Lambda_4(x_0, t_0) + \int_{x_0}^x A(x', t_0) \, dx' - c \int_{t_0}^t \phi(x', t') \, dt' \]
\[ + \left\{ -c \int_{x_0}^x dx' \int_{t_0}^t dt' E(x', t) + \tilde{g}(t) \right\} \]
with \( \tilde{g}(t) \) chosen so that \( \left\{ -c \int_{x_0}^x dx' \int_{t_0}^t dt' E(x', t') + \tilde{g}(t) \right\} \) is independent of \( t \).

Verification that it solves the system of PDEs (15) (even for \( E(x', t') \neq 0 \)):

(A) \[ -\frac{1}{c} \frac{\partial \Lambda_4(x, t)}{\partial t} = \phi(x, t) \text{ satisfied trivially } \checkmark \]
(because \( \ldots \) is, by construction, independent of \( t \)).

(B) \[ \frac{\partial \Lambda_4(x, t)}{\partial x} = A(x, t_0) - c \int_{t_0}^t \frac{\partial \phi(x', t')}{\partial x} \, dt' - c \int_{t_0}^t E(x', t') \, dt' + \frac{\partial \tilde{g}(t)}{\partial x}. \]

(the last term being trivially zero, \( \frac{\partial \tilde{g}(t)}{\partial x} = 0 \)), and then with
\[ \frac{\partial \phi(x, t')}{\partial x} = -E(x, t') - \frac{1}{c} \frac{\partial A(x, t')}{\partial t}, \]
we obtain
\[ \frac{\partial \Lambda_4(x, t)}{\partial x} = A(x, t_0) + c \int_{t_0}^t E(x', t) \, dt' + \int_{t_0}^t \frac{\partial A(x, t')}{\partial t'} \, dt' - c \int_{t_0}^t E(x, t') \, dt'. \]

(i) We see that the second and fourth terms of the rhs cancel each other, and
(ii) the third term of the rhs is \( \int_{t_0}^t \frac{\partial A(x, t')}{\partial t'} \, dt' = A(x, t) - A(x, t_0) \).

Hence finally
\[ \frac{\partial \Lambda_4(x, t)}{\partial x} = A(x, t). \quad \checkmark \]

Once again, all the above are true for any nonzero \( E(x', t') \) (in regions \( (x', t') \neq (x, t) \)).

To see again how the above solutions appear in nontrivial cases (and how they give new results, i.e. not differing from the usual ones by a mere constant) let us take analogous examples of strips as earlier, but now in spacetime.

(a) For the case of the extended vertical strip (parallel to the \( t \)-axis) of figure 1(a) (the case of a one-dimensional capacitor that is (arbitrarily and variably) charged for all time) and then for \( x \) located outside (and on the right of) the capacitor, the quantity \( c \int_{t_0}^t \, dt' \int_{x_0}^x \, dx' E(x', t') \) in \( \Lambda_3 \) is already independent of \( x \) (since a displacement of the \( x, t \)-corner of the rectangle to the right, along the \( x \)-direction, does not change the enclosed ‘electric flux’, see figure 1(a)); hence, in this case the function \( g(x) \) can be taken as \( g(x) = 0 \) (up to a constant \( C \)) and the condition for \( g(x) \) stated in solution (19) (i.e. that the quantity in brackets must be independent of \( x \)) is indeed satisfied.
So for this setup, the nonlocal term in the solution survives (the quantity in brackets is nonvanishing), but it is not constant: this enclosed flux depends on \( t \) (since the enclosed flux does change with a displacement of the \((x, t)\)-corner of the rectangle upward, along the \( t \)-direction). Hence, by looking at the alternative solution \( \Lambda_4(x, t) \), the quantity \( c \int_{t_0}^t \int_{x_0}^x \phi(x, t') \, dx' \, dt' \) is dependent on \( t \), so that \( \hat{g}(t) \) must be chosen as \( \hat{g}(t) = +c \int_{t_0}^t \int_{x_0}^x \int_{t_0}^{t'} \int_{x_0}^{x'} \, dx' \, dt' \, dE(x', t') \) (up to the same constant \( C \)) in order to cancel the \( t \)-dependence, so that its own condition stated in solution (23) (i.e. that the quantity in brackets must be independent of \( t \)) is satisfied; as a result, the quantity in brackets in solution \( \Lambda_4 \) disappears and there is no nonlocal contribution in \( \Lambda_4 \) (for \( C = 0 \)). (Once again, if we had used a \( C \neq 0 \), the nonlocal contributions would be distributed differently between the two solutions, but again without changing the physics when we take the difference of the two solutions).

With these choices of \( \hat{g}(t) \) and \( g(x) \), we already have new results (compared to the standard ones of the integrals of potentials). That is, one of the two solutions, namely \( \Lambda_3 \), is affected nonlocally by the enclosed flux (and this flux is not constant). Spelled out clearly, the two results are

\[
\Lambda_3(x, t) = \Lambda_3(x_0, t_0) + \int_{x_0}^x A(x', t) \, dx' - c \int_{t_0}^t \phi(x_0, t') \, dt' + c \int_{t_0}^t \int_{x_0}^x \int_{t_0}^{t'} \, dx' \, dE(x', t') + C
\]

\[
\Lambda_4(x, t) = \Lambda_4(x_0, t_0) + \int_{x_0}^x A(x', t_0) \, dx' - c \int_{t_0}^t \phi(x, t') \, dt' + C
\]

(and their difference, as mentioned above, is zero—denoting what might be called a generalized Werner and Brill cancellation in spacetime).

(b) In the 'dual case' of an extended horizontal strip—parallel to the \( x \)-axis (that corresponds to a nonzero electric field in all space that has however a finite duration \( T \)), the proper choices (for observation time instant \( t > T \)) are basically reverse (i.e. we can now take \( \hat{g}(t) = 0 \) and \( g(x) = -c \int_{t_0}^t \int_{x_0}^x \int_{t_0}^{t'} \, dx' \, dt' \, dE(x', t') \) (since the 'electric flux' enclosed in the 'observation rectangle' now depends on \( x \), but not on \( t \), with both choices always up to a common constant) and once again we can easily see a similar cancellation effect. In this case again, the results are new (a nonlocal term now surviving in \( \Lambda_4 \)). Again spelled out clearly, these are

\[
\Lambda_3(x, t) = \Lambda_3(x_0, t_0) + \int_{x_0}^x A(x', t) \, dx' - c \int_{t_0}^t \phi(x_0, t') \, dt' + C
\]

\[
\Lambda_4(x, t) = \Lambda_4(x_0, t_0) + \int_{x_0}^x A(x', t_0) \, dx' - c \int_{t_0}^t \phi(x, t') \, dt' - \int_{t_0}^t \int_{x_0}^x \int_{t_0}^{t'} \, dx' \, dt' \, dE(x', t') + C
\]

(their difference also being zero—a generalized Werner and Brill cancellation in spacetime).

(c) And again, if we want cases that are more involved (with the nonlocal contributions appearing nontrivially in both solutions \( \Lambda_3 \) and \( \Lambda_4 \) and with \( g(x) \) and \( \hat{g}(t) \) not being 'immediately visible') we must again consider different shapes of the \( E \)-distribution. One such case (the triangular) was already shown in figure 1(b) (for the magnetic case, which however is completely analogous). For such a triangular case the choices of \( g(x) \) and \( \hat{g}(t) \) will be different from the above and this will result in different roles of the nonlocal terms (and these nontrivial results, or more accurately, their analogs for the magnetic case
were given earlier in closed analytical form, equations (10) and (11). (And even cases of curved shapes can be addressed more generally (when the shape is such that the ‘flux’ does not decouple in a sum of separate spatial and temporal contributions), i.e. by solving the basic system of PDEs directly in polar coordinates (the results being analogous to the ones given later for the magnetic cases; see equations (34)–(37) below).)

The reader should note again that, in all the above examples in simple-connected spacetime, the $x$-independent quantity in brackets of the first solution (19) is equal to the function $\hat{g}(t)$ of the second solution (23), and the $t$-independent quantity in brackets of the second solution (23) is equal to the function $g(x)$ of the first solution (19). This pattern is what leads to the above-mentioned cancellations, and it is generally proved (i.e. for any $E$-distribution in the $(x, t)$-plane) in the section that follows.

8. Comments on the general behavior of the $(x, t)$-solutions

Let us first summarize (and prove in generality) some of the behavioral patterns that we saw in the above examples and then continue on other properties (i.e. an account of multiplicities of $\lambda$ in multiple-connected spacetimes that we left out, which are described by the constants $\tau(t_0)$ and $\chi(x_0)$). First, in (19) or (23) note the proper appearance and placement of $x_0$ and $t_0$ that gives a ‘path-sense’ to the line integrals of potentials in each solution (with the path consisting of two straight and perpendicular line segments connecting the initial point $(x_0, t_0)$ with the final point $(x, t)$ for each solution). And there are naturally two possible paths of this type that connect the initial point $(x_0, t_0)$ with the final point $(x, t)$ (solution (19) having a clockwise and solution (23) having a counterclockwise sense); in this way, a natural observation rectangle is again formed (see figure 1(a)), within which the enclosed ‘electric fluxes’ (in spacetime) appear to be crucial (showing up as nonlocal terms of contributions of the electric field difference (recall that $E(x', t') = E_2(x', t') - E_1(x', t')$) from regions of time and space that are remote to the observation point $(x, t)$). The appearance of these nonlocal terms (of the electric field difference) in $\Lambda(x, t)$ from regions of spacetime $(x', t')$ far from the observation point $(x, t)$ seems to have a direct effect on the wavefunction phases at $(x, t)$ (through the phase mapping that connects the two quantum systems). The actual manner in which this happens is of course determined by the nature of the functions $g(x)$ or $\hat{g}(t)$—these must be chosen in such a way that they satisfy their respective conditions, as these are stated after (19) or (23), respectively. We saw, for example, that if we have a distribution of $E$ in the $(x, t)$-plane in the form of an extended strip parallel to the $t$-axis, the function $g(x)$ can be taken as $g(x) = 0$ (up to a constant $C$), and that $\hat{g}(t)$ must be chosen as $\hat{g}(t) = 4\pi \int_0^t \int_0^t E(x', t')$ (up to the same constant $C$) in order to cancel the $t$-dependence of the enclosed ‘flux’. We re-emphasize that with these choices of $\hat{g}(t)$ and $g(x)$, it is easy to see that if we subtract the two solutions (19) and (23), the result is zero (because the line integrals of potentials $A$ and $\phi$ in the two solutions are in opposite senses in the $(x, t)$ plane; hence, their difference leads to a closed line integral which is in turn equal to the enclosed ‘electric flux’, and this flux always happens to be of the opposite sign from that of the enclosed flux that explicitly appears through the nonlocal term of the $E$-fields that survives in (19)). Such cancellation effects in dynamical cases are important and will be discussed (and generalized) further in section 10.

Let us however give here a general proof of the above cancellations. By looking first at the general structure of solutions (19) and (23), we note that in both forms, the last constant terms ($\tau(t_0)$ and $\chi(x_0)$) are only present in cases where $\Lambda$ is expected to be multivalued (this comes from the definitions of $\tau(t_0)$ and $\chi(x_0)$, see the discussion below) and therefore these
constant quantities are nonvanishing in cases of motion only in multiple-connected spacetimes (leading to phenomena of the electric Aharonov–Bohm type (see the analogous discussion given earlier in section 5 and later recapitulated in section 9, on the easier-to-follow magnetic case)). In such multiple-connected cases these last terms are simply equal (in absolute value) to the enclosed fluxes in regions of spacetime that are physically inaccessible to the particle (in the electric Aharonov–Bohm setup, for example, it turns out that \( \tau(t_0) = -\chi(x_0) = \) enclosed 'electric flux' in spacetime). Although such cases can also be covered by our method below, let us for the moment ignore them (set them to zero) and focus again on cases of motion in simple-connected spacetimes. Then the two solutions (19) and (23) are actually equal as is shown below (and in doing so, it is also shown that the \( x \)-independent (hence \( t \)-dependent) quantity in brackets of the first solution (19) is equal to the function \( \hat{\chi}(t) \) of the second solution (23)—and the \( t \)-independent (hence \( x \)-dependent) quantity in brackets of the second solution (23) is equal to the function \( g(x) \) of the first solution (19)). Here is the proof.

Since \( \{c \int_0^t \frac{d\tau}{\tau} \int_0^\tau dx' E(x', t') + g(x)\} \) is independent of \( x \), its \( x \)-derivative is zero which leads to \( g'(x) = -c \int_0^x dx' \frac{d\tau}{\tau} E(x', t') \), with a general solution \( g(x) = g(x_0) - c \int_0^x dx' \frac{d\tau}{\tau} E(x', t') + C(t) \), and with a \( C(t) \) such that the right-hand side is only a function of \( x \), hence independent of \( t \); but this is exactly the form of (23), if we identify \( C(t) \) with \( \dot{\chi}(t) \) (and \( g(x) \) with \( \chi(x_0) \)). This can be easily seen if we note that substitution of \( E(x', t') \) with \( -\frac{\partial\phi(x', t')}{\partial x'} = \frac{1}{c} \frac{\partial\chi(x', t')}{\partial t} \) and two integrations carried out finally interchange the forms of the first solution (19) from \( \left( \int_{x_0}^{x(t)} dx' - c \int_{x_0}^{x(t)} \dot{\phi}(x, t) \, dt' \right) \) to \( \left( \int_{x_0}^{x(t)} A(x', t) \, dx' - c \int_{x_0}^{x(t)} \phi(x, t') \, dt' \right) \) of the second solution (23).

The above could alternatively be proven if in (18), instead of substituting \( \frac{\partial\chi(x', t')}{\partial t} \) in terms of the electric field difference, we had merely interchanged the ordering of integrations in the first integral term. This would then immediately take us to the second solution (23), with automatically identifying the \( t \)-independent (hence \( x \)-dependent) quantity \( \{c \int_0^x dx' \int_0^\tau dx'' E(x', t') + \hat{\chi}(t)\} \) of the second solution (23) with the function \( g(x) \) of the first solution (19). (In a similar way, one can prove the identification of the \( x \)-independent (hence \( t \)-dependent) quantity \( \{c \int_0^t \frac{d\tau}{\tau} \int_0^\tau dx' E(x', t') + g(x)\} \) of the first solution (19) with the function \( \hat{\chi}(t) \) of the second solution (23).) Because of the above, it is straightforward to see (by subtracting the two solutions) the mathematical reason for the occurrence of the cancellations claimed earlier, for any shape of the \( E \)-distribution.

Therefore, in spite of the simplicity of the above-considered 1D system, we are already in a position to draw certain very general conclusions on the possible consequences of the new nonlocal terms of the electric fields appearing in the solutions (19) and (23). One can immediately see from the above considerations that these temporally nonlocal contributions have the tendency of cancelling the contributions from the \( A \)- and \( \phi \)-integrals. This already gives an indication of cancellations that might happen in cases of higher spatial dimensionality (where line-integrals of \( A \)'s, for example, can be related to enclosed magnetic fluxes). This is actually the case in the van Kampen thought-experiment that will be discussed later in section 10—although the cancellations there will be more delicate, involving a balance among three variables, and with the actual \( \text{senses} \) of spatial closed line-integrals in the \((x, y)\)-plane being nontrivially important.

Finally, with respect to \( \tau(t_0) \) and \( \chi(x_0) \), let us give an example to see why ordinarily (in simple-connectivity) they are zero, or in the most general case (of multiple-connectivity) they
are related to physically inaccessible enclosed fluxes. Starting from (16), where \( \tau(t) \) was first introduced, we have that

\[
\tau(t_0) = \Lambda(x, t_0) - \Lambda(x_0, t_0) - \int_{x_0}^{x} A(x', t_0) \, dx',
\]

(24)

which should be independent of \( x \) (and it is as can easily be proven, since its \( x \)-derivative gives \( \partial \Lambda(x, t_0) / \partial x \) which is zero, as \( \Lambda(x, t) \) satisfies by assumption the first equation of the system (15) of PDEs (evaluated at \( t = t_0 \)). We can therefore determine its value by taking the limit \( x \to x_0 \) in (24), which is zero, unless there is a multivaluedness of \( \Lambda \) at the point \((x_0, t_0)\). This happens for example for \( A \) having a \( \delta \)-function form (a case however which we leave out, otherwise the assumed interchanges might not be allowed) or in cases that there is a ‘memory’ that the system has multiplicities in \( \Lambda \), i.e. in Aharonov–Bohm configurations (with enclosed and inaccessible fluxes in spacetime), hence the value of \( \tau(t_0) \) being expected to be equal to the enclosed ‘electric flux’: the limit \( x \to x_0 \) (for fixed \( t_0 \)) in the path sense of solution (19) is as if we made an entire trip around the rectangle in the positive sense, landing on the same initial point \((x_0, t_0)\). A similar argument applied for

\[
\chi(x_0) = \Lambda(x_0, t) - \Lambda(x_0, t_0) + c \int_{t_0}^{t} \phi(x_0, t') \, dt',
\]

(25)

leads to the value of \( \chi(x_0) \) being equal to \textit{minus} the enclosed ‘electric flux’ (a corresponding limit \( t \to t_0 \) (for fixed \( x_0 \)) in the path sense of solution (23) is as if we made an entire trip around the rectangle in the negative sense, landing on the same initial point \((x_0, t_0)\)). If these values are actually substituted in (19) (with \( g(x) = 0 \)) and in (23) (with \( \hat{g}(t) = 0 \)) they give the correct electric Aharonov–Bohm result (where effectively there are no nonlocal contributions, and only the line-integrals of \( A \) and \( \phi \) contribute to the phase). (The above choice \( g(x) = \hat{g}(t) = 0 \) is made because, in this Aharonov–Bohm case, the enclosed ‘electric flux’ is independent of both \( x \) and \( t \).) (We should note that the case of the electric Aharonov–Bohm setup, with the particles traveling inside distinct equipotential cages with scalar potentials that last for a finite duration, is the prototype of \textit{multiple-connectivity in spacetime}, a fact first noted by Iddings and reported by Noerdlinger [10]. We will see later (section 11) that this feature is not present in the van Kampen thought-experiment; hence, an electric Aharonov–Bohm argument should not really be invoked in that case.)

Before, however, leaving this simple \((x, t)\)-case, we should finally emphasize that this (or any other) contribution of electric fields is \textit{not} present at the level of the basic Lagrangian, and the view holds in the literature (see e.g. [11]) that, because of this absence, electric fields cannot contribute \textit{directly} to the phase of the wavefunctions. This conclusion originates from the path-integral approach (that is almost always followed), but, nevertheless, our present work shows that fields \textit{do} contribute nonlocally. A more general discussion on this issue is given in the final section, after discussion of the van Kampen thought-experiment, and also in relation to the path-integral work of Troudet [12].

9. Again on the \((x, y)\)-magnetic case

After having discussed fully the simple \((x, t)\)-case, let us for completeness give the analogous (Euclidean-rotated) derivation for \((x, y)\)-variables and briefly discuss the properties of the simpler static solutions, but now in full generality (also including possible multivaluedness of \( \Lambda \) in magnetic Aharonov–Bohm cases). We will simply need to apply the same methodology (of solution of a system of PDEs) to such static spatially two-dimensional cases (so that now
different (remote) magnetic fields for the two systems, perpendicular to the 2D space, will arise). For such cases we need to solve the system of PDEs already shown in (8), namely
\[
\frac{\partial \Lambda(x, y)}{\partial x} = A_x(x, y) \quad \text{and} \quad \frac{\partial \Lambda(x, y)}{\partial y} = A_y(x, y).
\]
By first integrating the first of this (again without dropping any terms that may appear redundant), we obtain the analog of (16), namely
\[
\Lambda(x, y) - \Lambda(x_0, y) = \int_{x_0}^{x} A_x(x', y) \, dx' + f(y)
\]
and by then substituting the result to the second we have
\[
A_y(x, y) = \int_{y_0}^{y} \frac{\partial A_y(x', y)}{\partial x} \, dx' + \frac{\partial \Lambda(x_0, y)}{\partial y}
\]
which if integrated leads to
\[
f(y) = f(y_0) - \Lambda(x_0, y) + \Lambda(x_0, y_0) - \int_{y_0}^{y} dy' \int_{x_0}^{x} \frac{\partial A_x(x', y')}{\partial y'}
+ \int_{y_0}^{y} A_y(x_0, y') \, dy' + g(x)
\]
with \(g(x)\) to be chosen in such a way that the entire right-hand side of (28) is only a function of \(y\) (hence independent of \(x\)). Finally, by substituting \(\frac{\partial A_y(x', y')}{\partial x'}\) with \(\frac{\partial A_x(x', y')}{\partial x'} - B_z(x', y')\), carrying out the integration with respect to \(x'\), and by demanding that \(f(y)\) be independent of \(x\), we finally obtain the following general solution:
\[
\Lambda(x, y) = \Lambda(x_0, y_0) + \int_{x_0}^{x} A_x(x', y) \, dx' + \int_{y_0}^{y} A_y(x_0, y') \, dy' + \left\{ \int_{y_0}^{y} dy' \int_{x_0}^{x} dx' B_z(x', y') + g(x) \right\}
\]
which is basically the example shown earlier in (7) but with included multiplicities through the extra constant \(f(y_0)\) (which for simple-connected space can be set to zero). The result (29) applies to cases where the particle passes through different magnetic fields (recall that \(B_z = (B_2 - B_1)_z\)) in spatial regions that are remote to the observation point \((x, y)\). Alternatively, by following the reverse route (first integrating the second equation of the basic system (8)) we would obtain
\[
\Lambda(x, y) - \Lambda(x, y_0) = \int_{y_0}^{y} A_y(x, y') \, dy' + \tilde{h}(x)
\]
and by then substituting the result to the first we would have
\[
A_x(x, y) = \int_{y_0}^{y} \frac{\partial A_y(x, y')}{\partial x} \, dy' + \frac{\partial \Lambda(x, y_0)}{\partial x} + \frac{\partial \Lambda(x, y_0)}{\partial x}
\]
which if integrated would lead to
\[
\tilde{h}(x) = \tilde{h}(x_0) - \Lambda(x_0, y_0) + \Lambda(x_0, y_0) - \int_{x_0}^{x_0} dx' \int_{y_0}^{y_0} dy' \frac{\partial A_x(x', y')}{\partial x'} + \int_{x_0}^{x} A_x(x', y) \, dx' + h(y)
\]
(32)
with $h(y)$ to be chosen in such a way that the entire right-hand side of (32) is only a function of $x$ (hence independent of $y$). Finally, by substituting $\frac{\partial A_x(x', y')}{\partial x}$ with $\frac{\partial A_x(x', y')}{\partial y} + B_z(x', y')$, carrying out the integration with respect to $y'$, and by demanding that $\hat{h}(x)$ be independent of $y$, we would finally obtain the following general solution:

$$\Lambda(x, y) = \Lambda(x_0, y_0) + \int_{x_0}^{x} A_x(x', y_0) \, dx' + \int_{y_0}^{y} A_y(x, y') \, dy'$$

$$+ \left\{ -\int_{x_0}^{x} dx' \int_{y_0}^{y} dy' B_z(x', y') + h(y) \right\} + \hat{h}(x_0)$$

(33)

with $h(y)$ chosen so that $\left\{ -\int_{x_0}^{x} dx' \int_{y_0}^{y} dy' B_z(x', y') + h(y) \right\}$ is independent of $y$,

which is basically the example shown earlier in (9) but with included multiplicities through the extra constant $\hat{h}(x_0)$. One can actually show that the two solutions are equivalent (i.e. (7) and (9) for a simple-connected space are equal\(^2\)), a fact that can be proved in a way similar to the $(x, t)$-cases of section 8. (For the case of multiple-connectivity of the two-dimensional space, a discussion of the actual values of the multiplicities $f(y_0)$ and $\hat{h}(x_0)$ was given earlier in section 5 and will be summarized later in this section.)

As we saw in the examples of section 5, in case of a striped distribution of the magnetic field difference $B_z$, the functions $g(x)$ and $h(y)$ in (29) and (33) (or equivalently in (7) and (9)) have to be chosen in ways that are compatible with their corresponding constraints (stated after (29) and (33)) and completely analogous to the above-discussed $(x, t)$-cases; by then taking the difference of (7) and (9) we obtain that the ‘Aharonov–Bohm phase’ (the one originating from the closed line integral of $A$‘s) is exactly canceled by the additional nonlocal term of the magnetic fields (that the particle passed through). As already mentioned earlier, this is reminiscent of the cancellation of phases (broadly speaking, a cancellation between the ‘Aharonov–Bohm phase’ and the semiclassical phase picked up by the trajectories) observed in the early experiments of Werner and Brill [6] for particles passing through full magnetic fields, and our method seems to provide a very natural justification: as our results are completely general (and for delocalized states in a simple-connected region they basically describe the single-valuedness of $\Lambda$), they are also valid and applicable to cases of narrow wavepackets (or states that describe semiclassical motion) that pass through magnetic fields, which was the case of the Werner and Brill experiments. (A similar cancellation of an electric Aharonov–Bohm phase also occurs for particles passing through a static electric field as we saw in section 7.)

We conclude that, for static cases, and when particles pass through fields, the new nonlocal terms reported in this work lead quite generally to a cancellation of Aharonov–Bohm phases that had earlier been sketchily noticed and only at the semiclassical level.

Since we already mentioned that the deep origin of the above cancellations is the single-valuedness of $\Lambda$ in simple-connected space, we should add for completeness that the rigorous proof of the uniqueness at each spatial point (single-valuedness) of $\Lambda$ for completely delocalized states in simple-connected space can be given in a directly analogous way to the proof given in section 8 for the $(x, t)$-case, and is not repeated here. What is probably more important to point out is that the above cancellations for semiclassical trajectories (that pass through a magnetic field) can alternatively be understood as a compatibility between the Aharonov–Bohm fringe-displacement and the trajectory-deflection due to the Lorentz force (the semiclassical phase picked up due to the optical path difference of the two deflected trajectories exactly cancels (is opposite in sign) from the Aharonov–Bohm phase picked up

\(^2\) This has been first noted by the graduate student K Kyriakou.
by the trajectories due to the enclosed flux). (We may mention that this is also related to the well-known overall rigid displacement of the single-slit envelopes of the two-slit diffraction pattern, displacement that occurs if the wavepackets actually pass through a field.) These issues are further discussed in the final section, where some popular reports in the literature (Feynman [13], Felsager [14], Batelaan and Tonomura [15]) are given a minor correction (of a sign). Similarly, and by also including time \( t \) (and by again correcting a sign-error propagating in the standard literature), we will give an explanation of why certain classical arguments (invoking the past \( t \)-dependent history of the experimental setup) seem to work well (in giving the correct result for a static Aharonov–Bohm phase).

Another point of interest concerning the above found nonlocal contributions of fields is the plausible question of what shape the field distributions must have (or more accurately, their part enclosed inside the observation rectangle) so that the enclosed flux can be decoupled to a sum of functions of separate variables, in order for the solutions obtained above to be immediately applicable (i.e. for the functions \( g(x) \) and \( h(y) \) to be possible to determine: each of them must then only partially cancel the corresponding \( x \) or \( y \) dependence, respectively).

We already provided an example of such a distribution of a homogeneous \( B_z \) (the triangular one) in section 5 (see the nontrivial results (10) and (11)). And as mentioned in section 5, in cases of circularly shaped distributions (where the enclosed flux may not be decoupled in \( x \) and \( y \) terms), it is advantageous to solve the system directly in polar coordinates. By following a similar procedure (of solving the system of PDEs resulting from (3)) in polar coordinates \((\rho, \phi)\), namely

\[
\frac{\partial \Lambda(\rho, \phi)}{\partial \rho} = A_\rho(\rho, \phi) \quad \text{and} \quad \frac{1}{\rho} \frac{\partial \Lambda(\rho, \phi)}{\partial \phi} = A_\phi(\rho, \phi)
\]

with steps completely analogous to the above, one can obtain the following analogs of solutions (29) and (33), namely

\[
\Lambda(\rho, \phi) = \Lambda(\rho_0, \phi_0) + \int_{\rho_0}^{\rho} A_\rho(\rho', \phi) \, d\rho' + \int_{\phi_0}^{\phi} \rho_0 A_\phi(\rho_0, \phi') \, d\phi' + \left\{ \int_{\phi_0}^{\phi} \int_{\rho_0}^{\rho} \rho' \, d\rho' B_z(\rho', \phi') + g(\rho) \right\} + f(\phi_0)
\]

with \( g(\rho) \) chosen so that \( \left\{ \int_{\phi_0}^{\phi} \int_{\rho_0}^{\rho} \rho' \, d\rho' B_z(\rho', \phi') + g(\rho) \right\} \) is independent of \( \rho \), (35)

and

\[
\Lambda(\rho, \phi) = \Lambda(\rho_0, \phi_0) + \int_{\rho_0}^{\rho} A_\rho(\rho', \phi_0) \, d\rho' + \int_{\phi_0}^{\phi} \rho A_\phi(\rho, \phi') \, d\phi' + \left\{ -\int_{\rho_0}^{\rho} \rho' \, d\rho' \int_{\phi_0}^{\phi} \, d\phi' B_z(\rho', \phi') + h(\phi) \right\} + \hat{h}(\rho_0)
\]

with \( h(\phi) \) chosen so that \( \left\{ -\int_{\rho_0}^{\rho} \rho' \, d\rho' \int_{\phi_0}^{\phi} \, d\phi' \in (\rho', \phi') + h(\phi) \right\} \) is independent of \( \phi \), (37)

and in these, the proper choices of \( g(\rho) \) and \( h(\phi) \) will again be determined by their corresponding conditions, depending on the actual shape of the \( B_z \)-distribution and the positioning of initial and final points \((\rho_0, \phi_0)\) and \((\rho, \phi)\). (Furthermore, the observation rectangle has now given its place to a slice of a circular section.) These matters however deserve further investigation, as an application of the above theory to specific cases.
Finally, for completeness we summarize our findings on the issue of multiplicities (the constant last terms of (29) and (33)) in the case of spatial multiple-connectivity (such as the standard magnetic Aharonov–Bohm case, in which we can take \(g(x) = 0\) and \(h(y) = 0\), since the enclosed magnetic flux is independent of both \(x\) and \(y\)). According to the ‘definitions’ of these last terms (see (30) and (26) where the functions \(\hat{h}\) and \(f\) were first introduced) we have

\[
\hat{h}(x_0) = \Lambda(x_0, y) - \Lambda(x_0, y_0) = \int_{y_0}^{y} A_y(x_0, y') \, dy' \quad (38)
\]

\[
f(y_0) = \Lambda(x, y_0) - \Lambda(x_0, y_0) = \int_{x_0}^{x} A_x(x', y_0) \, dx' \quad (39)
\]

If we insist \((x, y)\) to also lie in a physically inaccessible region, then we have \(\hat{h}(x_0) = -f(y_0)\) = enclosed magnetic flux (which is already a constant, independent of \(x\) and \(y\)). This is because the limit \(y \to y_0\) (for fixed \(x_0\)) that is needed in (38) in order to find \(\hat{h}(x_0)\) is as if we went around the loop in the positive sense, landing on the initial point \((x_0, y_0)\); similarly, the limit \(x \to x_0\) (for fixed \(y_0\)) that is needed in (39) in order to find \(f(y_0)\) is as if we went around the loop in the negative sense, landing on the initial point \((x_0, y_0)\). Since \(f(y_0)\) cancels out the \(\int_{y_0}^{y} \int_{x_0}^{x} B_z(x', y') \, dx' \, dy'\) term, and \(\hat{h}(x_0)\) cancels out the \(-\int_{x_0}^{x} \int_{y_0}^{y} B_z(x', y') \, dx' \, dy'\) term, the two solutions are then reduced to the usual solutions of mere \(A\)-integrals along the two paths (i.e. the standard Dirac phase, with no nonlocal contributions).

10. Full \((x, y, t)\)-case

Finally, let us look at the spatially two-dimensional and time-dependent case. This combines effects of (perpendicular) magnetic fields (which, if present only in physically inaccessible regions, can have Aharonov–Bohm consequences) with the temporal nonlocalities of electric fields (parallel to the plane). By working again in Cartesian spatial coordinates, we now have to deal with the full system of PDEs

\[
\frac{\partial \Lambda(x, y, t)}{\partial x} = A_x(x, y, t), \quad \frac{\partial \Lambda(x, y, t)}{\partial y} = A_y(x, y, t),
\]

\[
-\frac{1}{c} \frac{\partial \Lambda(x, y, t)}{\partial t} = \phi (x, y, t). \quad (40)
\]

This exercise is considerably longer than the previous ones but important to solve, in order to see in what manner the solutions of this system manage to combine the spatial and temporal nonlocal effects found above. There are now 3! = 6 alternative integration routes to follow for solving this system (and, in addition to this, the results in intermediate steps tend to proliferate). Let us here for demonstration show the intermediate steps for only two routes (that will give us at the end four results as we will see), starting with the second of (40): by integrating it we obtain the expected generalization of (30), namely

\[
\Lambda(x, y, t) - \Lambda(x, y_0, t) = \int_{y_0}^{y} A_y(x, y', t) \, dy' + f(x, t) \quad (41)
\]

which if substituted to the first of (40) gives (after integration over \(x'\)) a \(t\)-generalization of (32), namely

\[
f(x, t) = f(x_0, t) - \Lambda(x, y_0, t) + \Lambda(x_0, y_0, t) - \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' \frac{\partial A_y(x', y', t)}{\partial x'} + \int_{x_0}^{x} A_y(x', y, t) \, dx' + G(y, t) \quad (42)
\]
with $G(y, t)$ to be chosen in such a way that the entire right-hand side of (42) is only a function of $x$ and $t$ (hence independent of $y$). Finally, by substituting $\frac{\partial A_i(x', y', t)}{\partial x'}$ with $\frac{\partial A_i(x', y', t)}{\partial y'} + B_z(x', y', t)$, carrying out the integration with respect to $y'$, and by demanding that $f(x, t)$ be independent of $y$, we obtain the following temporal generalization of (33):

$$\Lambda(x, y, t) = \Lambda(x_0, y_0, t) + \int_{x_0}^{x} A_i(x', y_0, t) \, dx' + \int_{y_0}^{y} A_y(x, y', t) \, dy'$$

$$+ \left\{ - \int_{x_0}^{x} dy' \int_{y_0}^{y} dy' B_z(x', y', t) + G(y, t) \right\} + f(x_0, t) \quad (43)$$

with $G(y, t)$ such that $\left\{ - \int_{x_0}^{x} dy' \int_{y_0}^{y} dy' B_z(x', y', t) + G(y, t) \right\}$ is independent of $y$.

From this point on, the third equation of the system (40) is getting involved to determine the nontrivial effect of scalar potentials on $G(y, t)$; by combining it with (43) there results a wealth of patterns: integration with respect to $t'$ leads to

$$G(y, t) = G(y, t_0) - \Lambda(x_0, y_0, t) - \Lambda(x_0, y_0, t_0) - f(x_0, t) + f(x_0, t_0) - c \int_{t_0}^{t} \phi(x, y, t') \, dt'$$

$$- \int_{x_0}^{x} dt' \int_{x_0}^{x} dx' \frac{\partial A_i(x', y_0, t')}{\partial y'} + \int_{t_0}^{t} dt' \int_{y_0}^{y} dy' \frac{\partial A_i(x, y', t')}{\partial y'}$$

$$+ \int_{t_0}^{t} dt' \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' \frac{\partial B_z(x', y', t')}{\partial y'} + F(x, y) \quad (44)$$

with $F(x, y)$ to be chosen in such a way that the entire right-hand side of (44) is only a function of $(y, t)$, hence independent of $x$. In (44) there are two possible ways to determine the term in brackets, and another two ways to determine the term containing $B_z$. The easiest to follow (the one that more directly leads to the final conditions that the functions $F(x, y)$ and $G(y, t_0)$ are required to satisfy) is (i) to substitute $\frac{\partial A_i(x', y', t)}{\partial x'}$ with $-c \left( \frac{\partial E_z(x', y', t)}{\partial y'} - \frac{\partial E_z(x, y', t')}{\partial y'} \right)$ (and similarly for $\frac{\partial A_i(x', y', t)}{\partial y'}$) and (ii) to use the proviso that magnetic and electric fields are connected through Faraday’s law of induction, namely $\frac{\partial B_z(x', y', t)}{\partial y'} = -c \left( \frac{\partial E_z(x', y', t)}{\partial y'} - \frac{\partial E_z(x, y', t')}{\partial y'} \right)$. These substitutions lead to cancellations of several intermediate quantities in (43) and (44) and lead to the final result

$$\Lambda(x, y, t) = \Lambda(x_0, y_0, t_0) + \int_{x_0}^{x} A_i(x', y_0, t) \, dx' + \int_{y_0}^{y} A_y(x, y', t) \, dy'$$

$$- \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' B_z(x', y', t) + G(y, t_0)$$

$$- c \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' E_z(x', y', t') + c \int_{t_0}^{t} dt' \int_{x_0}^{x} dx' E_z(x', y, t')$$

$$+ c \int_{t_0}^{t} dt' \int_{y_0}^{y} dy' E_z(x_0, y', t') + F(x, y) + f(x_0, t_0) \quad (45)$$

with the functions $G(y, t_0)$ and $F(x, y)$ to be chosen in such a way as to satisfy the following three independent conditions:

$$\left\{ G(y, t_0) - \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' B_z(x', y', t_0) \right\} \text{ is independent of } y, \quad (46)$$

which is of course a special case of the condition on $G(y, t)$ above (see after (43)), and the other two turn out to be of the form

$$\left\{ F(x, y) + c \int_{t_0}^{t} dt' \int_{x_0}^{x} dx' E_z(x', y, t') \right\} \text{ is independent of } x, \quad (47)$$
\[ \left\{ F(x, y) + c \int_{t_0}^{t} dt' \int_{y_0}^{y} dy' E_y(x, y', t') \right\} \text{ is independent of } y. \] (48)

It should be noted (for the reader who wants to follow all the steps) that the final condition (48) does not come out directly as the other two; because the function \( G(y, t) \) has disappeared from the final form (45), one needs to separately impose the condition above for \( G(y, t) \) (namely \( \left\{ -\int_{x_0}^{x} dx' \int_{y_0}^{y} dy' B_z(x', y', t) + G(y, t) \right\} \) independent of \( y \)) directly on the form (44), and in so doing, it is advantageous to interchange integrations (namely, do the \( t' \)-integral first) in the \( B_z \)-term of (44), so that \( \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' \frac{\partial B_z(x', y', t)}{\partial t'} = \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' (B_z(x', y', t) - B_z(x', y', t_0)) \), and then impose the (less stringent) condition (46) on \( G(y, t_0) \); by following this strategy, after a number of cancellations of intermediate quantities one finally obtains the third condition (48) on \( F(x, y) \). (As for the constant quantity \( f(x_0, t_0) \) appearing in (45), this again describes possible effects of multiple-connectivity at the instant \( t_0 \) (which are absent for simple-connected spacetimes, but will be crucial in the discussion of the van Kampen thought-experiment to be discussed later).)

Equation (45) was our first solution. It is now crucial to note that an alternative form of solution (with the functions \( G(x, y) \) and \( F(x, y) \) satisfying the same conditions as above) can be derived if, in the term in brackets of (44), we merely interchange integrations, leaving therefore \( A \)'s everywhere rather than introducing electric fields; following at the same time the above strategy of changing the ordering of integrations in the \( B_z \)-term as well (without therefore using Faraday’s law) this alternative form of solution turns out to be

\[
\Lambda(x, y, t) = \Lambda(x_0, y_0, t_0) + \int_{x_0}^{x} A_x(x', y_0, t) dx' + \int_{y_0}^{y} A_y(x, y', t) dy' \\
- \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' B_z(x', y', t_0) + \phi(x_0, y_0, t_0) - c \int_{t_0}^{t} \phi(x_0, y_0, t') dt' \\
+ c \int_{t_0}^{t} dt' \int_{y_0}^{y} dx' E_x(x', y_0, t') + c \int_{t_0}^{t} dt' \int_{y_0}^{y} dy' E_y(x, y', t') \\
+ F(x, y) + f(x_0, t_0). \] (49)

In this alternative solution we note that, in comparison with (45), the line-integrals of \( E \) have changed to the other alternative ‘path’ (note the difference in the placement of the coordinates of the initial point \((x_0, y_0)\) in the arguments of \(E_x\) and \(E_y\)) and they happen to have the same sense as the \( A \)-integrals, while simultaneously the magnetic flux difference shows up with its value at the initial time \( t_0 \) rather than at \( t \). This alternative form will be shown to be useful in cases where we want to directly compare physical situations in the present (at time \( t \)) and in the past (at time \( t_0 \), and the above-noted change of sense of \( E \)-integrals (compared to (45)) will be crucial in the discussion that follows (in section 11).

Once again the reader can directly verify that (45) or (49) indeed satisfy the basic input system (40). (This verification is considerably more tedious than the earlier ones, but straightforward, and is not shown here.)

But in order to discuss the van Kampen case, namely an enclosed (and physically inaccessible) magnetic flux (whichever is \textit{time-dependent}), it is important to have the analogous forms through a reverse route, namely starting with (integrating) the first of (40) and then substituting the result to the second; in this way we will at the end have the reverse ‘path’ of \( \Lambda \)-integrals, so that by taking the \textit{difference} of the resulting solution and the above solution (45) or (49) will lead to the \textit{closed} line integral of \( \Lambda \) which will be immediately related to van Kampen’s magnetic flux (at the instant \( t \)). By following then this route, and by applying a similar strategy at every intermediate step, we finally obtain the following solution
(the spatially ‘dual’ to (45)), namely

\[ \Lambda(x, y, t) = \Lambda(x_0, y_0, t_0) + \int_{x_0}^{x} A_x(x', y, t) \, dx' + \int_{y_0}^{y} A_y(x_0, y', t) \, dy' \\
+ \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' B_z(x', y', t) + \tilde{G}(x, t_0) \\
- c \int_{t_0}^{t} \phi(x_0, y_0, t') \, dt' + c \int_{t_0}^{t} dt' \int_{x_0}^{x} dx' E_x(x', y_0, t') \\
+ c \int_{t_0}^{t} dt' \int_{y_0}^{y} dy' E_y(x_0, y', t') + F(x, y) + \tilde{h}(y_0, t_0) \]  

(50)

with the functions \( \tilde{G}(x, t_0) \) and \( F(x, y) \) to be chosen in such a way as to satisfy the following three independent conditions:

\[ \{ \tilde{G}(x, t_0) + \int_{y_0}^{y} dy' \int_{x_0}^{x} dx' B_z(x', y', t_0) \} \text{ is independent of } x, \]

(51)

\[ \{ F(x, y) + c \int_{t_0}^{t} dt' \int_{x_0}^{x} dx' E_x(x', y_0, t') \} \text{ is independent of } x, \]

(52)

\[ \{ F(x, y) + c \int_{t_0}^{t} dt' \int_{y_0}^{y} dy' E_y(x_0, y', t') \} \text{ is independent of } y, \]

(53)

where again for the above results Faraday’s law was crucial. The corresponding analog of the alternative form (49) (where \( B_z \) appears at \( t_0 \)) is more important and turns out to be

\[ \Lambda(x, y, t) = \Lambda(x_0, y_0, t_0) + \int_{x_0}^{x} A_x(x', y, t) \, dx' + \int_{y_0}^{y} A_y(x_0, y', t) \, dy' \\
+ \int_{x_0}^{x} dx' \int_{y_0}^{y} dy' B_z(x', y', t_0) + \tilde{G}(x, t_0) \\
- c \int_{t_0}^{t} \phi(x_0, y_0, t') \, dt' + c \int_{t_0}^{t} dt' \int_{x_0}^{x} dx' E_x(x', y_0, t') \\
+ c \int_{t_0}^{t} dt' \int_{y_0}^{y} dy' E_y(x_0, y', t') + F(x, y) + \tilde{h}(y_0, t_0) \]

(54)

with \( \tilde{G}(x, t_0) \) and \( F(x, y) \) following the same three conditions as above. The constant term \( \tilde{h}(y_0, t_0) \) again describes possible multiplicities at the instant \( t_0 \); it is absent for simple-connected spacetimes, but will be crucial in the discussion of the van Kampen thought-experiment.

In (50) (and in (54)), note the ‘alternative paths’ (compared to solution (45) (and (49))) of line integrals of \( A \)’s (or of \( E \)’s). But the most crucial element for what follows is the use of forms (49) and (54) (where \( B_z \) only appears at \( t_0 \), and the fact that, within each solution, the sense of \( A \)-integrals is the same as the sense of the \( E \)-integrals. (This is not true in the other solutions where \( B_z (.,.,., t) \) appears). These facts will be crucial to the discussion that follows, which briefly addresses the so-called van Kampen ‘paradox’.

11. The van Kampen thought-experiment—causal issues hidden in the above solutions

In that early work [7], van Kampen considered a genuine Aharonov–Bohm case, with a magnetic flux (physically inaccessible to the particle) which, however, is time-dependent: van
Kampen envisaged turning on the flux very late or, equivalently, observing the interference of the two wavepackets (on a distant screen) very early, earlier than the time it takes light to travel the distance to the screen, hence using the (instantaneous nature of the) Aharonov–Bohm phase to transmit information (on the existence of a confined magnetic flux somewhere in space) superluminally. Indeed, the Aharonov–Bohm phase at any later instant \( t \) is determined by differences of \( \frac{\partial}{\partial t} \Lambda(\mathbf{r}, t) \), with \( \Lambda(\mathbf{r}, t) = \int_{\mathbf{r}} A(\mathbf{r}', t) \cdot d\mathbf{r}' + \text{const.} \) (which basically results as a special case (but in higher dimensionality) of the incorrect expression (14) in the temporal gauge \( \phi = 0 \), the constant being \( \Lambda(\mathbf{r}_0, t_0) \)). However, let us for this case utilize instead our results (49) and (54) above, where we have the additional appearance of the nonlocal \( E \)-terms (and of the \( B \)-term at \( t_0 \)).

In order to be slightly more general, let us for example assume that the inaccessible magnetic flux had the value \( \Phi(t_0) \) at \( t_0 \), and then it started changing with time. By using a narrow wavepacket picture like van Kampen, we can then subtract (49) and (54) in order to find the phase difference at a time \( t \) that is smaller than the time required for light to reach the observation point \((x, y)\) (i.e. \( t < \frac{L}{c} \), with \( L \) the corresponding distance). For a spatially confined magnetic flux \( \Phi(t) \), the functions \( G, \tilde{G} \) and \( F \) in the above solutions can all be taken zero: their conditions are all satisfied for a flux \( \Phi(t) \) that is not spatially extended (hence, from (46) and (51) we obtain \( G = \tilde{G} = 0 \)) and, for \( t < \frac{L}{c} \), the integrals of \( E_x \) and \( E_y \) in conditions (47) and (48) (or in (52) and (53)) are already independent of both \( x \) and \( y \) (since \( E_x(x, y, t') = E_y(x, y, t') = 0 \) for all \( t' < t < \frac{L}{c} \), with \((x, y)\) the observation point on the screen, and therefore all integrations of \( E_x \) and \( E_y \) with respect to \( x' \) and \( y' \) will give results that are independent of the integration upper limits \( x \) and \( y \); hence \( F = 0 \)). Moreover, the multiplicities \((j \text{ and } h)\) lead to cancellation of the \( B \)-terms (at \( t_0 \)) as outlined in the static case earlier (end of section 9). By choosing then the temporal gauge \( \phi = 0 \), we have for the difference (49)–(54) at the point and instant of observation the following result:

\[
\Delta \Lambda(x, y, t) = \int_{x_0}^{x} A_x(x', y_0, t) \, dx' + \int_{y_0}^{y} A_y(x, y', t) \, dy' - \int_{x_0}^{x} A_x(x', y, t) \, dx' \\
- \int_{y_0}^{y} A_y(x_0, y', t) \, dy' + c \int_{t_0}^{t} \left\{ \int_{x_0}^{x} dx' E_x(x', y_0, t') \right\} \\
+ \int_{x_0}^{x} dy' E_x(x, y', t') - \int_{x_0}^{x} dx' E_x(x', y, t') - \int_{y_0}^{y} dy' E_y(x_0, y', t') \}.
\]

In (55) the sum of the four \( A \)-integrals gives the closed line-integral of vector \( \mathbf{A} \) around the observation rectangle at time \( t \) (in the positive sense) and it is equal to the instantaneous magnetic flux \( \Phi(t) \) (that leads to the ‘usual’ magnetic Aharonov–Bohm phase); the sum of the four \( E \)-integrals inside the brackets in the last terms (originating from our nonlocal contributions) gives the closed line-integral of vector \( \mathbf{E} \) around the same rectangle at any arbitrary \( t' \), and in the same (positive) sense (something we would not have if we had taken the first type of solutions, (45) and (50)—this signifying the importance of taking the right form, the one that contains \( B_t \) at \( t_0 \) with the \( t \)-propagation of \( B_t \) having already been incorporated in the \( E_x \) and \( E_y \) terms of (49) and (54))). By denoting therefore the closed loop integral (around the rectangle) as \( \oint \) always in the positive sense (and with the understanding that the rectangle’s upper-right corner is the spatial point of observation \((x, y)\)), (55) reads

\[
\Delta \Lambda(x, y, t) = \oint \mathbf{A}(\mathbf{r}', t) \cdot d\mathbf{r}' + c \int_{t_0}^{t} d\mathbf{r}' \oint \mathbf{E}(\mathbf{r}', t') \cdot d\mathbf{r}'
\]

which, with \( \oint \mathbf{A}(\mathbf{r}', t) \cdot d\mathbf{r}' = \Phi(t) \) the instantaneous enclosed magnetic flux and with the help of Faraday’s law \( \oint \mathbf{E}(\mathbf{r}', t') \cdot d\mathbf{r}' = -\frac{1}{c} \frac{\partial \mathbf{B}(\mathbf{r}', t')}{\partial t} \), gives

\[
\Delta \Lambda(x, y, t) = \Phi(t) - (\Phi(t) - \Phi(t_0)) = \Phi(t_0).
\]
Although $\Delta \Lambda$ is generally $t$-dependent, we obtain the intuitive result that, for $t < \frac{L}{c}$ (i.e. if the physical information has not yet reached the screen), the phase difference turns out to be $t$-independent and leads to the magnetic Aharonov–Bohm phase that we would observe at $t_0$.

This gives an honest resolution of the ‘van Kampen paradox’ within a canonical formulation, without using any vague electric Aharonov–Bohm effect argument (since in the gauge chosen ($\phi = 0$) there are no scalar potentials—and, most importantly, there is no multiple-connectivity in the $(x, t)$-plane as in the electric Aharonov–Bohm case [10]). An additional physical element (in comparison to van Kampen’s electric phase interpretation) is that, for the above cancellation, it is not only the $E$-fields but also the $t$-propagation in space of the $B_z$-fields (the full ‘radiation field’) that plays a role.

Finally, a number of other forms of solutions can be obtained that result from different ordering of integrations of the system (40) (a full list of 12 different (but quite long) results is available, and easily verifiable that they indeed satisfy the system (40)). The reader can follow the strategies suggested here and derive the forms that are appropriate to particular physical cases of interest that may be different from the above magnetic case, some potential candidates being the ‘electric analog’ of the van Kampen thought-experiment, or its bound state analog in nanorings. For the latter, and especially for 1D nanorings (or other nanoscopic devices) driven by a $t$-dependent magnetic flux, the new nonlocal terms are expected to be of relevance if they are included in standard treatments [16], and the effects are expected to appear in the PetaHertz range. (Similarly we might expect a role in cases of quantal astrophysical objects due to the large distances involved (hence retardation effects being more pronounced).)

For the ‘electric analog’ of the van Kampen case, we note that, although this has never really been discussed in the literature, nevertheless, it has been essentially briefly mentioned in appendix B of Peshkin [3] (where the case that ‘first the particle exits the cages, and only then we switch on the outside electric field’ is made, together with the comment that the results must be ‘consistent with ordinary ideas about Causality’; Peshkin correctly states: ‘One cannot wait for the electron to pass and only later switch on the field to cause a physical effect’). As our new nonlocal terms seem to be especially suited for addressing such causality issues, let us slightly expand on this point: in this most authoritative (and carefully written) review of the Aharonov–Bohm effect in the literature, Peshkin uses (for the electric effect) a solution form (his equation (B.5) together with (B.6)) based on (14), i.e. the ‘standard result’ (but applied to a spatially dependent scalar potential)—but he clearly states that it is an approximation (and actually later in the review, he states that this form cannot be a solution for all $t$). Indeed, from the present work we learn that (B.5) and (B.6) are not the solutions when the scalar potential depends on spatial variables (because the spatial variables inside the potential will give—through its nonzero gradient—an extra vector potential (that will result from $\nabla /\Lambda_1$)), hence an extra minimal substitution in the Hamiltonian $H$, violating therefore the mapping between two predescribed systems that we want to achieve). As we saw in the present work, the correct solution for all $t$ and in all space consists of additional nonlocal terms of the appropriate form. If we view the form (B.5) and (B.6) of [3] as an ansatz, then it is understandable why a condition (Peshkin’s equation (B.8), and later (B.9)) needs to be enforced on the electric field outside the cages (in order for the extra (annoying) terms (that show up from expansion of the squared minimal substitution) to vanish and for (B.5) to be a solution). And then Peshkin notes that the extra condition cannot always be satisfied— it must fail for some times (hence (B.5) is not really the solution for all times), drawing from this a correct conclusion, namely that ‘the electron must traverse some region where the electric field has been’ (earlier). However, the causal issue pointed out above, although mentioned in words, is not dealt with quantitatively.

From our present work, it turns out that the total ‘radiation field’ outside the cages is crucial in recovering causality, in a similar way as in the case presented above in this section for the
usual (magnetic) version of the van Kampen experiment. In this ‘electric analog’ that we are
discussing now, the causally offending part of the electric Aharonov–Bohm phase difference
will be canceled by a magnetic type of phase that originates from the magnetic field that is
associated with the \( t \)-dependence of the electric field \( E \) outside the cages.

It should be re-emphasized that the correct quantitative physical behavior for the above
system for all times comes out from the treatment shown in detail in the present work, with
no enforced constraints, but with conditions that come out naturally from the solution of the
PDEs. The results that are derived from this careful procedure give the full solutions (correct
for all space and for all \( t \)): Peshkin’s ansatz (B.6) turns out (from an honest and careful
solution of the full PDEs) to be augmented by nonlocal (in time) terms of the electric fields,
and these directly influence the phases of wavefunctions (by always respecting causality, with
no need of enforced statements)—and can even include the contributions of vector potentials
and magnetic fields (through nonlocal magnetic terms in space) associated with the \( t \)-variation
of the electric field outside the cages that Peshkin has omitted. As already mentioned, the total
‘radiation field’ outside the cages is crucial in recovering causality, in a way similar to what
was presented in this work for the usual (magnetic) version of the van Kampen experiment.
We conclude that our (exact) results accomplish precisely what Peshkin has in mind in his
discussion (on causality), but in a direct and fully quantitative manner, and with no ansatz
based on an incorrect form.

12. Discussion

Trying to evaluate in a broader sense the crucial nonlocal influences found in all the above
physical examples, we should probably re-emphasize that at the level of the basic Lagrangian
\[
L(r, v, t) = \frac{1}{2}m v^2 + \frac{1}{2} v \cdot A(r, t) - q \phi(r, t),
\]
there are no fields present, and the view holds in the literature [11] that electric or magnetic fields cannot contribute directly to the phase. This view originates from the path-integral treatments widely used (where the Lagrangian determines directly the phases of propagators), but, nevertheless, our canonical formulation treatment shows that fields do contribute nonlocally, and they are actually crucial in recovering relativistic causality. Moreover, path-integral discussions [12] of the van Kampen case use wave (retarded)-solutions for the vector potentials \( A \) (hence they are treated in the Lorenz
gauge, which is not sufficiently general: even if \( A \) has not yet reached the screen, we can
always add a constant \( A \) (a pure gauge) over all space, and there are no more retarded wave-
solutions for the potentials, the path-integral resolution of the paradox being, therefore, at
least incomplete). Our results are gauge invariant and take advantage of only the retardation
of fields \( E \) and \( B \) (true in any gauge), and not of potentials. In addition, Troudet [12] clearly
(and correctly) states that his treatment is good for not highly delocalized states in space, and
that in the case of delocalization the proper treatment ‘would be much more complicated, and
would require a much more complete analysis’. We believe we have provided one in this
paper. It should be added that in a recent Compendium of Quantum Physics [17], the ‘van
Kampen paradox’ still seems to be thought of as remarkable. We believe that the present work
has provided a natural and general resolution, and most importantly, through nonlocal and
relativistically causal propagation of wavefunction phases (this point being expanded further
at the end of the paper).

At several places in this paper we have pointed out a number of ‘misconceptions’ in the
literature (mostly on the uncritical use of the (standard) Dirac phases even for \( t \)-dependent
vector potentials and spatially dependent scalar potentials, which is plainly incorrect for
uncorrelated variables), and we have explicitly provided their ‘healing’ through appropriate
nonlocal field terms. It should however be emphasized here that this is not a merely marginal
misconception, but it appears all over the place in the literature (due to the Feynman path-integral bias); it is even stated by Feynman himself in volume 2 of his *Lectures on Physics* [13], namely, that the simple phase factor \( \int x A \cdot dr' - c \int t \phi dt' \) is valid generally, i.e. even for \( t \)-dependent fields. Similarly, this erroneous generalization is also explicitly stated in the review on Aharonov–Bohm effects of Erlichson [18] that has given a very balanced view of earlier controversy, and elsewhere—the books of Silverman [19] being the clearest case that we are aware of with a careful wording about (14) being only restrictively valid (for \( t \)-independent \( A \)'s and \( r \)-independent \( \phi \)'s)—although even there the nonlocal terms have been missed. We believe that the above misconceptions (and the overlooking of the nonlocal terms) are the basic reason why ‘it appears that no exact theoretical treatment has been given’ (for the electric Aharonov–Bohm effect), as correctly stated by Peshkin in appendix B of [3].

And let us now come to a second type of misconception that is probably less important since it has appeared only in semiclassical conditions—but is essential to be mentioned here, as it also exhibits the merits of our approach and the deeper physical understanding that our results can lead to. What we learn from the generalized Werner and Brill cancellations pointed out rather emphatically in this work is that, at the point of observation, the nonlocal terms of classical remote fields have the tendency to contribute a phase of opposite sign to the ‘Aharonov–Bohm phase’ (of potentials). We want to point out to the reader that, for semiclassical trajectories, this is actually descriptive of the compatibility (or consistency) of the Aharonov–Bohm fringe displacement and the associated trajectory deflection due to the classical forces. Let us for example look at figure 15–8 of Feynman\(^3\), or at figure 2.16 of the book of Felsager [14], where classical trajectories are deflected after they pass through a strip of a homogeneous magnetic field that is placed on the right of a standard double-slit experimental apparatus. Both authors determine the semiclassical phase picked up by the trajectories (that have been deflected by the Lorentz force) and they find that they are consistent with the Aharonov–Bohm phase (picked up due to the flux enclosed by the same trajectories). However, it is rather straightforward for the reader to see that the two phases have opposite signs (they are not equal as implied by the authors). (The reader is also invited to carry out a similar exercise, with particles passing through an analogous homogeneous electric field on the right of the double-slit apparatus that is switched on for a finite duration \( T \), where again the semiclassical phase picked up turns out to be opposite to the electric Aharonov–Bohm type of phase.) Similarly, in the very recent review of Batelaan and Tonomura [15], their figure 2 contains visual information that is very relevant to our discussion: it is a quite descriptive picture of the wavefronts associated with the classical trajectories, where the authors state that ‘the phase shift calculated in terms of the Lorentz force is the same as that predicted by the Aharonov–Bohm effect in terms of the vector potential \( A \) circling the magnetic bar’. The reader, however, should note once more that the sign of the classical phase difference is really opposite to the sign of the Aharonov–Bohm phase. The phases are not equal as stated, but opposite. All the above examples are we believe a manifestation of the cancellations that have been derived in the present work (for general quantum states), but here they are just special cases for semiclassical trajectories. (We could also add that these cancellations also have to do with the well-known rigid displacement of the ‘single-slit envelope’ of the two-slit diffraction pattern in a double-slit experiment with an additional strip of a magnetic field placed on the right of the apparatus.)

In a slightly different vein, we should also point out that the above cancellations give a justification of why certain semiclassical arguments that focus on the history of the experimental setup (usually based on Faraday’s law for a \( t \)-dependent magnetic flux) seem to

\(^3\) See [13], chapter 15, p 13.
give at the end a result that is consistent with the result of a static Aharonov–Bohm arrangement. However, there is again an opposite sign that seems to have been largely unnoticed in such arguments as well (i.e. see the simplest possible argument of Silverman\(^4\) [19], where in his equation (1.34) there should be an extra minus sign). Our above observation essentially describes the fact that \textit{if we had actually used a} \(t\)-dependent magnetic flux, then the induced electric field (viewed now as a nonlocal term of the present work) would have canceled the static Aharonov–Bohm phase. Of course now this \(t\)-dependent experimental setup has not been used (the flux is static) and we obtain the usual magnetic Aharonov–Bohm phase, but the above argument (of a ‘potential experiment’ that \textit{could have been carried out}) takes the ‘mystery’ away of why such arguments generally work—although they have to be corrected with a sign.

Finally, coming back to a broader significance of the new solutions, one may wonder about possible consequences of the nonlocal terms if these are included in more general physical models that have a gauge structure (in condensed matter or high energy physics). It is also worth mentioning that by following the same ‘unconventional’ method (of solution of PDEs) but now applied to the Maxwell’s equations for the electric and magnetic fields, we obtained the corresponding nonlocal terms, and we found that these essentially demonstrate the causal propagation of the radiation electric and magnetic fields outside physically inaccessible confined sources (i.e. solenoids or electric cages). Although this is of course widely known at the level of classical fields, a major conclusion that can be drawn from the present work (at the level of gauge transformations) is that a corresponding causality may exist at the level of quantum mechanical phases as well, and this is enforced by the nonlocal terms in \(t\)-dependent cases. It strongly indicates that the nonlocal terms found here at the level of quantum mechanical phases reflect a causal propagation of wavefunction phases in \textit{the Schrödinger picture} (at least one part of them, the one containing the fields, which competes with the Aharonov–Bohm types of phases containing the potentials). This is an entirely new concept (given the local nature and also the nonrelativistic character of the Schrödinger equation) and deserves to be further explored. It would indeed be worth investigating possible applications of the above results (of nonlocal phases of wavefunctions, solutions of the local Schrödinger equation) in \(t\)-dependent single- versus double-slit experiments recently discussed by the group of Aharonov [4] who use a completely different method (with modular variables in the Heisenberg picture). One should also note other recent works such as [5] that rightfully emphasize that physics cannot currently predict how we dynamically go from the single-slit diffraction pattern to the double-slit diffraction pattern (whether it is in a gradual and causal manner or not) and where they propose relevant experiments to decide on (measure) exactly this. Working with our nonlocal terms in such questions in analogous experiments (i.e. by introducing (finite) scalar potentials on one slit in a \(t\)-dependent way), in order to address the associated causal issues, is currently under way.

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\(^4\) See the first book of [19], p 16, or the second book of [19], p 19.
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