Reward-Weighted Regression Converges to a Global Optimum

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Abstract

Reward-Weighted Regression (RWR) belongs to a family of widely known iterative Reinforcement Learning algorithms based on the Expectation-Maximization framework. In this family, learning at each iteration consists of sampling a batch of trajectories using the current policy and fitting a new policy to maximize a return-weighted log-likelihood of actions. Although RWR is known to yield monotonic improvement of the policy under certain circumstances, whether and under which conditions RWR converges to the optimal policy have remained open questions. In this paper, we provide for the first time a proof that RWR converges to a global optimum when no function approximation is used.

1 Introduction

Reinforcement learning (RL) is a branch of artificial intelligence that considers learning agents interacting with an environment (Sutton and Barto, 2018). RL has enjoyed several notable successes in recent years. These include both successes of special prominence within the artificial intelligence community—such as achieving the first superhuman performance in the ancient game of Go (Silver et al., 2016)—and successes of immediate real-world value—such as providing autonomous navigation of stratospheric balloons to provide internet access to remote locations (Bellemare et al., 2020).

One prominent family of algorithms that tackle the RL problem is the Reward-Weighted Regression (RWR) family (Peters and Schaal, 2007). RWR works by transforming the RL problem into a form solvable by well-studied expectation-maximization (EM) methods (Dempster et al., 1977). EM methods are, in general, guaranteed to converge to a point whose gradient is zero with respect to the parameters. However, these points could be both local minima or saddle points (Wu, 1983). These benefits and limitations transfer to the RL setting, where it has been shown that an EM-based

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return maximizer is guaranteed to yield monotonic improvements in the average reward (Dayan and
Hinton [1997]). However, it has been challenging to assess under which conditions—if any—RWR is
guaranteed to converge to the optimal policy. This paper presents a breakthrough in this challenge.

The EM probabilistic framework requires that the reward obtained by the RL agent is strictly positive,
such that it can be considered as an improper probability distribution. Several reward transformations
have been proposed, e.g., Peters and Schaal (2007, 2008); Peng et al. (2019); Abdolmaleki et al.
(2018b). Frequently these involve an exponential transformation. In the past, it has been claimed that
a positive, strictly increasing transformation \( u(r) \) with \( \int_0^\infty u(r) \, dr = \text{const} \) would not alter the
optimal solution for the MDP (Peters and Schaal [2007]). Unfortunately, as demonstrated below, this
is not the case. The consequence of this is that we cannot rely on those transformations if we want
prove convergence. Therefore, we restrict ourselves here to only linear transformation of the reward.
A possible disadvantage of relying on linear transformations is that it is necessary to know a lower
bound on the reward to construct such a transformation.

Counterexample. Consider the simple two-armed bandit shown in Figure 1 with actions \( a_0 \) and
\( a_1 \), and with \( P(r = 1|a_0) = 1 \), \( P(r = 0|a_0) = \frac{1}{3} \), and \( P(r = 2|a_1) = \frac{1}{3} \). Note that \( q(a_0) = 1 \) \( > q(a_1) = \frac{1}{3} \). Thus the optimal policy always takes action \( a_0 \). Now, after applying the
transformation \( u(r) = e^{\log(3) \cdot r} = 3^r \), we get \( P(u(r) = 3|a_0) = 1 \), \( P(u(r) = 1|a_0) = \frac{1}{3} \), and
\( P(u(r) = 9|a_1) = \frac{1}{3} \). Hence, under transformation \( u \), we have \( q(a_0) = 3 < q(a_1) = \frac{11}{3} \). So the
optimal policy under the transformed rewards always takes action \( a_1 \), which is sub-optimal, given the
original problem.

In this work, we provide the first proof of RWR’s global convergence in a setting without function
approximation or reward transformation. The paper is structured as follows: Section 2 introduces
the MDP setting and other preliminary material; Section 3 presents a closed-form update for RWR
based on the state and action-value functions and Section 4 shows that the update induces monotonic
improvement related to the variance of the action-value function with respect to the action sampled
by the policy; Section 5 proves global convergence of the algorithm; Section 6 illustrates experimentally
that—for a simple MDP—the presented update scheme converges to the optimal policy; Section 7 discusses related work; and Section 8 concludes.

2 Background

Here we consider a Markov Decision Process (MDP) (Stratonovich [1960], Puterman [2014], \( \mathcal{M} = (\mathcal{S}, \mathcal{A}, p_T, R, \gamma, \mu_0) \)). We assume that the state and action spaces \( \mathcal{S} \subseteq \mathbb{R}^{n_s}, \mathcal{A} \subseteq \mathbb{R}^{n_a} \) are compact sub-spaces (equipped with subspace topology), with measurable structure given by measure spaces \( (\mathcal{S}, \mathcal{B}(\mathcal{S}), \mu_S), (\mathcal{A}, \mathcal{B}(\mathcal{A}), \mu_A) \) where \( \mathcal{B}(\cdot) \) denotes the Borel \( \sigma \)-algebra after completion, and reference measures \( \mu_S, \mu_A \) are assumed to be finite and strictly positive on \( \mathcal{S}, \mathcal{A} \) respectively. The distributions of state (action) random variables (except in Section 5 where greedy policies are used) are assumed to be dominated by \( \mu_S (\mu_A) \), thus having a density with respect to \( \mu_S (\mu_A) \). Therefore, we reserve symbols \( ds, da \) in integral expression not to integration with respect to Lebesgue measure, as usual.

\[ \begin{array}{|c|c|c|}
\hline
r=0, p=2/3 & u(r)=3^r, p=1 & \\
\hline
r=2, p=1/3 & u(r)=3^r, p=1/3 & \\
\hline
r=1, p=1 & u(r)=3^r, p=1 & \\
\hline
\end{array} \]

Figure 1: Counterexample demonstrating how applying a naive transformation to the reward function
of an MDP may change the optimal policy.
We start by deriving a closed form solution to the optimization problem: $\Pi$. The agent’s performance is measured by the cumulative discounted expected reward (i.e., the expected $\mu$ to Lebesgue decomposition. $p$ this density as a policy.

Action $a$ given a state, weighted by the return. The RWR optimization problem is:

$$\text{under maximum likelihood criterion) a sample representation of the conditional distribution of an}$$

a batch of episodes is generated using the current policy $R(s, a)$. Reward-Weighted Regression (RWR) is an iterative algorithm which consists of two main steps. First, 

an action-value function $Q$: $\pi$ define as

bounded $V$ that the reward function $R$ is bounded and $V$.

In Sections 3 and 4, a policy is given through its conditional density with respect to $\mu_A$.

In the MDP framework, at each step, an agent observes a state $s \in S$, chooses an action $a \in A$, and subsequently transitions into state $s'$ with probability density $p_T(s'|s, a)$ to receive a deterministic reward $R(s, a)$. The transition probability kernel is assumed to be continuous in total variation in $(s, a) \in S \times A$, and thus the density $p_T(s'|s, a)$ is continuous (in $|| \cdot ||_1$ norm) for all $(s, a) \in S \times A$. $R(s, a)$ is assumed to be a continuous function on $S \times A$.

The agent starts from an initial state (chosen under a probability density $\mu_0(s)$) and is represented by a probability policy $\pi$: a probability kernel which provides the conditional probability distribution of performing action $a$ in state $s$. The policy is deterministic if, for each state $s$, there exists an action $a$ such that $\pi(\{a\}|s) = 1$. The return $R_t$ is defined as the cumulative discounted reward from time step $t$: $R_t = \sum_{k=0}^{\infty} \gamma^k R(s_{t+k}, a_{t+k+1})$ where $\gamma \in (0, 1)$ is a discount factor. We study the undiscounted case ($\gamma = 1$) in Appendix A which covers the scenario with absorbing states.

The agent’s performance is measured by the cumulative discounted expected reward (i.e., the expected return), defined as $J(\pi) = \mathbb{E}_s[ R_0 ]$. The state-value function $V(\pi)(s) = \mathbb{E}_s[ R_t | s_t = s ]$ of a policy $\pi$ is defined as the expected return for being in a state $s$ while following $\pi$. The maximization of the expected cumulative reward can be expressed in terms of the state-value function by integrating it over the state space $S$: $J(\pi) = \int_S \mu_0(s) V(\pi)(s) ds$. The action-value function $Q(\pi)(s, a)$—defined as the expected return for performing action $a$ in state $s$ and following a policy $\pi$—is $Q(\pi)(s, a) = \mathbb{E}_s[ R_t | s_t = s, a_t = a ]$. State and action value functions are related by $V(\pi)(s) = \int_A \pi(a|s) Q(\pi)(s, a) da$. We define as $d^n(s')$ the discounted weighting of states encountered starting at $s_0 \sim \mu_0(s)$ and following the policy $\pi$: $d^n(s') = \int_S \sum_{t=1}^{\infty} \gamma^{t-1} \mu_0(s) p_{s_t|s_0, \pi, t}(s'|s, a) ds$, where $p_{s_t|s_0, \pi, t}(s', a)$ is the probability density of transitioning to $s'$ after $t$ time steps, starting from $s$ and following policy $\pi$. We assume that the reward function $R(s, a)$ is strictly positive, so that state and action value functions are also bounded $\forall V(\pi)(s) \leq \frac{1}{1-\gamma} ||B_V|| = B_V < +\infty$.

We define the operator $W: L_\infty(S \times A) \rightarrow C(S \times A)$ as $[W(V)](s, a) := R(s, a) + \gamma \int_S V(s') p_T(s'|s, a) ds'$ and the Bellman’s optimality operator $T: L_\infty(S \times A) \rightarrow C(S \times A)$ as $[T(Q)](s, a) := R(s, a) + \gamma \int\max_{a'} Q(s', a') p_T(s'|s, a) ds'$. An action-value function $Q^\pi$ is optimal if it is the unique fixed point for $T$. If $Q^\pi$ is optimal, then $\pi$ is an optimal policy.

### 3 Reward-Weighted Regression

Reward-Weighted Regression (RWR) is an iterative algorithm which consists of two main steps. First, a batch of episodes is generated using the current policy $\pi_n$ (all policies in this section are given as conditional densities with respect to $\mu_A$). Then, a new policy is fitted to (using supervised learning under maximum likelihood criterion) a sample representation of the conditional distribution of an action given a state, weighted by the return. The RWR optimization problem is:

$$\pi_{n+1} = \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^{n+1}(\cdot), a \sim \pi_n(\cdot|s)} \left[ R_t \log \pi(a|s) \right],$$

where $\Pi$ is the set of all conditional probability densities (mean with respect to $\mu_A$). Notice that $\pi_{n+1}$ is defined correctly as its expression does not depend on $t$. This is equivalent to the following:

$$\pi_{n+1} = \arg \max_{\pi \in \Pi} \mathbb{E}_{s \sim d^{n+1}(\cdot), a \sim \pi_n(\cdot|s)} \left[ Q^{\pi_n}(s, a) \log \pi(a|s) \right].$$

We start by deriving a closed form solution to the optimization problem:

4 In Sections 3 and 4 a policy is given through its conditional density with respect to $\mu_A$. We also refer to this density as a policy.

5 It is enough to assume that the reward is bounded, so it can be linearly mapped to a positive value.

6 We refer to continuous functions since $R(s, a)$ is continuous and continuity of the integral follows from continuity of $p_T$ in $|| \cdot ||_1$ norm and boundedness of $V$.

7 We can restrict to talk about probability kernels dominated by $\mu_A$ instead of all probability kernels thanks to Lebesgue decomposition.
We can now set \( \pi \) where the last inequality holds for any policy \( \pi \) between \( \pi \).

\[ \pi_{n+1}(a|s) = \frac{Q^n(s,a)\pi_n(a|s)}{V^n(s)} \]  

(3)

**Proof.**

\[ \pi_{n+1} = \arg \max_{\pi} \int_S d^n(s) \int_A \pi(a|s) Q^n(s,a) \log \pi(a|s) \, da \, ds. \]

Define \( \hat{f}(s,a) := d^n(s)\pi_n(a|s)Q^n(s,a) \). \( \hat{f}(s,a) \) can be normalized such that it becomes a density that we fit by \( \pi_{n+1} \):

\[ f(s,a) = \frac{\hat{f}(s,a)}{\int_S \int_A f(s,a) \, da \, ds} = \frac{d^n(s)\pi_n(a|s)Q^n(s,a)}{\int_S \int_A d^n(s)\pi_n(a|s)Q^n(s,a) \, da \, ds}. \]

For the function to be maximized we have:

\[ \int_S \int_A f(s,a) \log \pi(a|s) \, da \, ds = \int_S f(s) \int_A f(a|s) \log \pi(a|s) \, da \, ds \leq \int_S f(s) \int_A f(a|s) \log f(a|s) \, da \, ds, \]

where the last inequality holds for any policy \( \pi \), since \( \forall s \in S \) we have that \( \int_A f(a|s) \log \pi(a|s) \, da \leq \int_A f(a|s) \log f(a|s) \, da \), as \( f(a|s) \) is the maximum likelihood fit. Note that for all states \( s \in S \) such that \( d^n(s) = 0 \), we have that \( f(s,a) = 0 \). Therefore, for such states, the policy will not contribute to the objective and can be defined arbitrarily. Now, assume \( d^n(s) > 0 \). The objective function achieves a maximum when the two distributions are equal:

\[ \pi_{n+1}(a|s) = f(a|s) = \frac{f(s,a)}{\int_A f(s,a) \, da} \]

\[ = \frac{d^n(s)\pi_n(a|s)Q^n(s,a)}{\int_S \int_A d^n(s)\pi_n(a|s)Q^n(s,a) \, da \, ds} \cdot \frac{\int_A \pi_n(a|s)Q^n(s,a) \, da}{\int_A \pi_n(a|s)Q^n(s,a) \, da} = \frac{Q^n(s,a)\pi_n(a|s)}{V^n(s)}. \]

We can now set \( \pi_{n+1}(a|s) = \frac{Q^n(s,a)\pi_n(a|s)}{V^n(s)} \) also for all \( s \) such that \( d^n(s) = 0 \), which completes the proof. \( \square \)

Note that when function approximation is used for policy \( \pi \), the term \( f(s) \) weighs the mismatch between \( \pi(a|s) \) and \( f(a|s) \). Indeed, we have \( f(s) \propto d^n(s)V^n(s) \), suggesting that the error occurring with function approximation would be weighted more for states visited often and with a bigger value. In our setting, however, the two terms are equal since no function approximation is used.

## 4 Monotonic Improvement Theorem

Here we prove that the update defined in Theorem 3.1 leads to monotonic improvement.

**Theorem 4.1.** Fix \( n > 0 \) and let \( \pi_0 \in \Pi \) be a policy. Assume \( \forall s \in S, \forall a \in A, R(s,a) > 0 \). Define the operator \( B : \Pi \to \Pi \) such that \( \pi_{n+1} = B(\pi_n) = \frac{Q^n(s,a)\pi_n(a|s)}{V^n(s)} \). Then \( \forall s \in S \) we have that \( V^{n+1}(s) \geq V^n(s) \) and \( Q^{n+1}(s,a) \geq Q^n(s,a) \). Moreover, \( \forall s \in S : \text{Var}_{n,\pi_n(a|s)}[Q^n(s,a)] > 0 \) the inequalities above are strict.

\( ^\dagger \)The case where the MDP has non-negative rewards and the undiscounted case are more complex and treated in Appendix A.

\( ^\ddagger \)Also in this section all policies are given as conditional densities with respect to \( \mu_A \).
Another problem arises when considering the above: since \( \arg\max \) (i.e. the optimal policy)—, then the operator \( B \) will map the policy to itself and there will be no improvement. Theorem 4.1 provides a relationship between the improvement in the state-value function and the variance of the action-value function with respect to the actions sampled. Note that if at a certain point the policy becomes deterministic—or it becomes the greedy policy of its action-value function (i.e. the optimal policy)—, then the operator \( B \) will map the policy to itself and there will be no improvement.

5 Convergence Results

5.1 Weak convergence in topological factor

It is worth discussing what type of convergence we can achieve by iterating the \( B \)-operator \( \pi_n := B(\pi_{n-1}) \), where \( \pi_n \) are probability densities with respect to a fixed reference measure \( \mu_A \).

Consider first the classic "continuous" variable case, where \( \mu_A \) is the Lebesgue measure and fix \( s \in S \). Optimal policies are known to be greedy on the optimal action-value function \( Q^* (s, a) \). That is, they concentrate all mass on \( \arg\max_a Q^*(s, a) \). If \( \arg\max_a Q^*(s, a) \) consists of just a single point \( \{a^*\} \), then the optimal policy (measure), \( \pi^*(\cdot|s) \) for \( s \), concentrates all its mass in \( \{a^*\} \). This means that the optimal policy does not have a density with respect to the Lebesgue measure. Furthermore \( \pi_n (\cdot|s) \cdot \mu_A (\cdot) \) get concentrated in the neighbourhood of \( a^* \) and that this neighbourhood gets tinier as \( n \) increases. We will use the concept of weak convergence to prove this.

Another problem arises when considering the above: since \( \arg\max_a Q^*(s, a) \) can consist of multiple points, the set of optimal policies is \( \mathcal{P}(\arg\max_n Q^*(s, a)) \), where \( \mathcal{P}(F) := \{ \mu : \mu \) is a probability measure on \( B(A), \mu(F) = 1 \} \) for a \( F \in B(A) \). We want to prove convergence even when the sequence of policies \( \pi_n \) oscillates near \( \mathcal{P}(\arg\max_n Q^*(s, a)) \). A way of coping with this is to make \( \arg\max_a Q^*(s, a) \) a single point through topological factorisation, to obtain the limit by working in a quotient space. The notion of convergence we will be using is described in the following definition.

**Definition 1.** (Weak convergence of measures in metric space relative to a compact set) Let \((X, d)\) be a metric space, \( F \subset X \) a compact subset, \( B(X) \) its Borel \( \sigma \)-algebra. Denote \((\tilde{X}, \tilde{d})\) a metric

\[ \tilde{d}(x, y) = \inf \{ \epsilon > 0 : x, y \in B(\epsilon) \} \]

where \( B(\epsilon) \) is the \( \epsilon \)-neighbourhood of \( x \).

\[ \tilde{d}(\mu_1, \mu_2) = \sup \{ \epsilon > 0 : \int_E \tilde{d}(\mu_1(x), \mu_2(x)) d\mu_1(x) < \epsilon \} \]

\[ \nu_n \) converges weakly to \( \nu \) if for all \( \epsilon > 0 \),

\[ \int |f| \, d\nu_n \leq \epsilon \to 0 \text{ as } n \to \infty \]

The argument is the same as given in [Puterman 2014], see section on Monotonic Policy Improvement.
space resulting as a topological quotient with respect to $F$ and $\nu$ the quotient map $\nu : X \to \tilde{X}$ (see Lemma \ref{lem:topo_quot} for details). A sequence of probability measures $P_n$ is said to converge weakly relative to $F$ to a measure $P$ denoted

$$P_n \rightharpoonup^w P,$$

if and only if the image measures of $P_n$ under $\nu$ converge weakly to the image measure of $P$ under $\nu$:

$$\nu P_n \rightharpoonup^w \nu P.$$

Note that the limit is meant to be unique just in quotient space, thus if $P$ is a weak limit (relative to $F$) of a sequence $(P_n)$, then also all measures $P'$ for which $\nu P' = \nu P$ are relatively weak limits, i.e. $P'|_{B(X) \cap F^c} = P|_{B(X) \cap F^c}$. Thus, they can differ on $B(X) \cap F$. While the total mass assigned to $F$ must be the same for $P$ and $P'$, the distribution of masses inside $F$ may differ.

### 5.2 Main results

Consider for all $n \geq 0$ the sequence generated by $\pi_n := B(\pi_{n-1})$. For convenience, for all $n \geq 0$, we define $Q_n := Q_{\pi_n}$, $V_n := V_{\pi_n}$. First we note that, since the reward is bounded, the monotonic sequences of value functions converge point-wise to a limit:

$$\forall s \in S : V_n(s) \nearrow V_L(s) \leq B_V < +\infty$$

$$\forall s \in S, a \in A : Q_n(s, a) \nearrow Q_L(s, a) \leq B_V < +\infty,$$

where $B_V = \frac{1}{\gamma} ||R||_\infty$. Further $\forall n Q_n$ is continuous since $Q_n = W(V_n)$ and $W$ maps all bounded functions to continuous functions.

The convergence proof proceeds in four steps:

1. First we show in Lemma \ref{lem:Q_L} that $Q_L$ can be expressed in terms of $V_L$ through $W$ operator. This helps when showing that $Q_n$ converges uniformly to $Q_L$.

2. Then we demonstrate in Lemma \ref{lem:pi_L} that $\forall s \in S$ the sequence of policy measures $\pi_n(\cdot|s) \cdot A$ converges weakly relative to the set $M(s) := \arg \max_{\pi} Q_L(s, a)$ to a measure that assigns all probability mass to greedy actions of $Q_L$, i.e.

$$\pi_n(\cdot|s) \cdot A \rightharpoonup^w (M(s))$$

$

\pi_L(\cdot|s) \in P(M(s))$. Moreover $\pi_L \in \Pi_L := \{ \pi'_L : \pi'_L$ is a probability kernel from $(S, B(S))$ to $(A, B(A)), \forall s \in S, \pi'_L(\cdot|s) \in P(M(s)) \}$.

3. At this point we do not know yet if $Q_L$ and $V_L$ are the value functions of $\pi_L$. We prove this in Lemma \ref{lem:V_L} (together with previous Lemmas) by showing that they are fixed points of the Bellman operator.

4. Finally, we state the main results in Theorem \ref{thm:main_result}. Since $V_L$ and $Q_L$ are value functions for $\pi_L$ and $\pi_L$ is greedy with respect to $Q_L$, then $Q_L$ is the unique fixed point of the Bellman’s optimality operator:

$$Q_L(s, a) = |T(Q)|s, a) = R(s, a) + \gamma \int_S \max_{a'} Q(s', a') p_T(s'|s, a) \, ds'.$$

Therefore $Q_L$ and $V_L$ are optimal value functions and $\pi_L$ is an optimal policy for the MDP.

**Lemma 5.1.** The following holds:

1. $Q_L = W(V_L)$,
2. $Q_L$ is continuous,
3. $Q_n$ converges to $Q_L$ uniformly.

**Proof.** 1. Fix $(s, a) \in S \times A$. We aim to show $Q_L(s, a) - [W(V_L)](s, a) = 0$. Since $Q_n = W(V_n)$, we can write:

$$Q_L(s, a) - [W(V_L)](s, a) = Q_L(s, a) - Q_n(s, a) - [W(V_L)](s, a) + [W(V_n)](s, a)$$

$$\leq |Q_L(s, a) - Q_n(s, a)| + ||W(V_L)](s, a) - [W(V_n)](s, a)|.$$
\[ S_\epsilon \subset S \text{ with } (p_T(\cdot | s, a) \cdot \mu_S)(S_\epsilon^c) < \epsilon \text{ such that } \|V_n - V_L\|_\infty \to 0 \text{ on } S_\epsilon. \text{ Thus there exists } n_0 \text{ such that } \|V_n - V_L\|_\infty < \epsilon \text{ for all } n > n_0. \text{ Now let us rewrite the second part for } n > n_0:

\[
\|W(V_L)(s, a) - W(V_n)(s, a)\| = \int_{S_\epsilon} [V_L(s') - V_n(s')] p_T(s'|s, a) \mu_S(s')
\]

which can be made arbitrarily small.

2. \( Q_L \) is continuous because \( W \) maps all bounded measurable functions to continuous functions.

3. Since \( Q_n \) and \( Q_L \) are continuous functions in a compact space and \( Q_n \) is a monotonically increasing sequence that converges point-wise to \( Q_L \), we can apply Dini’s theorem (see Th. 7.13 on page 150 in [Rudin et al. (1976)]) which ensures uniform convergence of \( Q_n \) to \( Q_L \).

\[ \text{Lemma 5.2.} \quad \text{Let } \pi_n \text{ be a sequence generated by } \pi_n := B(\pi_{n-1}). \text{ Let } \pi_0 \text{ be continuous in actions and } \forall s \in S, \forall a \in A, \pi_0(a|s) > 0. \text{ Define } M(s) := \arg\max Q_L(\cdot|s). \text{ Then } \forall \pi_L \in \Pi_L \neq \emptyset, \forall s \in S, \text{ we have } \pi_n(\cdot|s) \cdot \mu_A \to^{w(M(s))} \pi_L(\cdot|s) (\in P(M(s))). \]

\[ \text{Proof.} \quad \text{First notice that the set } \Pi_L \text{ is nonempty.} \text{ Fix } \pi_L \in \Pi_L \text{ and } s \in S. \text{ In order to prove that } \pi_n(\cdot|s) \cdot \mu_A \to^{w(M(s))} \pi_L(\cdot|s), \text{ we will use a characterization of relative weak convergence that follows from an adaptation of the Portmanteau Lemma (Billingsley, 2013) (see Appendix B.3). In particular, it is enough to show that for all open sets } U \subset A \text{ such that } U \cap M(s) = \emptyset \text{ or such that } M(s) \subset U, \text{ we have that } \lim \inf_{n} (\pi_n(\cdot|s) \cdot \mu_A)U \geq \pi_L(\cdot|s)U. \]

The case \( U \cap M(s) = \emptyset \) is trivial since \( \pi_L(\cdot|s)(U) = 0. \) For the remaining case \( M(s) \subset U \) it holds \( \pi_L(\cdot|s)(U) = 1. \) Thus we have to prove \( \lim \inf_{n} (\pi_n(\cdot|s) \cdot \mu_A)U = 1. \) If we are able to construct an open set \( D \subset U \) such that \( (\pi_n(\cdot|s) \cdot \mu_A)(D) \to 1 \text{ for } n \to \infty, \) then we will get that \( \lim \inf_{n} (\pi_n(\cdot|s) \cdot \mu_A)U \geq 1, \) satisfying the condition for relative weak convergence of \( \pi_n(\cdot|s) \cdot \mu_A \to^{w(M(s))} \pi_L(\cdot|s). \)

The remainder of the proof will focus on constructing such a set. Fix \( a^* \in M(s) \) and \( 0 < \epsilon < 1/3. \)

Define a continuous map \( \lambda : A \to \mathbb{R}^+ \) and closed sets \( A_\epsilon \) and \( B_\epsilon : \lambda(a) := \frac{Q_L(a)}{Q_L(a^*)}, \quad A_\epsilon := \{a \in A | \lambda(a) \leq 1 - 2\epsilon\}, \quad B_\epsilon := \{a \in A | \lambda(a) \geq 1 - \epsilon\}, \)

where continuity of the map stems from \( Q_L(a^*) > 0 \) and continuity of \( Q_L \) (Lemma 5.1). We will prove that the candidate set is \( D = A_\epsilon^c. \) In particular, we must prove that \( A_\epsilon^c \subset U \) and that \( (\pi_n(\cdot|s) \cdot \mu_A)(A_\epsilon^c) \to 0. \) Using Lemma B.1 (Appendix) on function \( \lambda, \) we can choose \( \epsilon > 0 \) such that \( A_\epsilon^c \subset U, \) satisfying the first condition. We are left to prove that \( (\pi_n(\cdot|s) \cdot \mu_A)(A_\epsilon^c) \to 0. \)

Assume \( A_\epsilon^c \neq \emptyset \) (otherwise the condition is proven): for all \( a \in A_\epsilon \) and \( b \in B_\epsilon \) it holds:

\[
\frac{Q_L(a)}{Q_L(b)} = \frac{Q_L(a)}{Q_L(a^*)} \leq \frac{Q_L(a)}{Q_L(a^*)(1-\epsilon)} \leq \frac{1 - 2\epsilon}{1 - \epsilon} = 1 - \frac{\epsilon}{1 - \epsilon} =: \alpha_1 < 1.
\]

For Lemma 5.1, \( Q_n \) converges uniformly to \( Q_L. \) Therefore we can fix \( n_0 > 0 \) such that \( \|Q_n - Q_L\|_\infty < \epsilon \) for all \( n \geq n_0, \) where we define \( \epsilon' := 0.1 \times Q_L(a^*)(1-\epsilon)(1-\alpha_1). \) Now we can

\[ \text{The argument goes as follows: } H := \cup_{a \in S} \{s \times M(s) \text{ is a closed set}, \text{ then } f(s) := \sup M(s) \text{ is upper semi-continuous and therefore measurable. Then graph of } f \text{ is measurable so we can define a probability kernel } \pi_L(B|s) := 1_B(f(s)) \text{ for all } B \text{ measurable.} \]
prove by bounding $Q_n$ ratio from above. For all $n \geq n_0$, $a \in A_e$ and $b \in B_e$:

$$\frac{Q_n(a)}{Q_n(b)} \leq \frac{Q_L(a)}{Q_L(b) - \epsilon'} \leq \frac{Q_L(a)}{Q_L(a^*)(1 - \epsilon') - \epsilon'} = \frac{Q_L(a)}{(0.9 + 0.1\alpha_1)^2} =: \alpha < 1.$$ 

Finally, we can bound the policy ratio. For all $n \geq n_0$, $a \in A_e$, $b \in B_e$:

$$\frac{\pi_n(a|s)}{\pi_n(b|s)} = \frac{\pi_0(a|s)}{\pi_0(b|s)} \prod_{i=0}^n \frac{Q_i(s,a)}{Q_i(s,b)} \leq \alpha^n c(a,b),$$

where

$$c(a,b) := \alpha^{-n_0} \frac{\pi_0(a|s)}{\pi_0(b|s)} \prod_{i=0}^{n_0} \frac{Q_i(s,a)}{Q_i(s,b)}.$$ 

The function $c : A_e \times B_e \to \mathbb{R}^+$ is continuous as $\pi_0, Q_i$ are continuous (and denominators are non-zero due to $\pi_0(b|s) > 0$ and $Q_i(s,a) > 0$). Since $A_e \times B_e$ is a compact set, there exists $c_m$ such that $c \leq c_m$. Thus we have that for all $n > n_0$:

$$\pi_n(a|s) \leq \alpha^n c_m \pi_n(b|s).$$

Integrating with respect to $a$ over $A_e$ and then with respect to $b$ over $B_e$ (using reference measure $\mu_A$ in both cases) we obtain:

$$(\pi_n(\cdot|s) \cdot \mu_A)(A_e) \times (\mu_A B_e) \leq \alpha^n c_m (\pi_n(\cdot|s) \cdot \mu_A)(B_e) \times (\mu_A A_e).$$

Rearranging terms, we have:

$$(\pi_n(\cdot|s) \cdot \mu_A)(A_e) \leq \alpha^n \left[ c_m(\frac{\mu_A A_e}{\mu_A B_e}(\pi_n(\cdot|s) \cdot \mu_A)B_e) \right] \to 0, n \to \infty,$$

since the nominator in brackets is composed by finite measures of sets, thus finite numbers, while the denominator $\mu_A B_e > 0$. Indeed, define the open set $C := \{a \in A| \lambda(a) > 1 - \epsilon\} \subset B_e$. Then $\mu_A(B_e) \geq \mu_A(C) > 0$ ($\mu_A$ is strictly positive). To conclude, we have proven that for arbitrarily small $\epsilon > 0$, the term $(\pi_n(\cdot|s) \cdot \mu_A)(A_e)$ tends to 0, satisfying the condition for relative weak convergence of $\pi_n(\cdot|s) \cdot \mu_A \rightharpoonup w(M(s))$ $\pi_L(\cdot|s)$.

Lemma 5.3. Assume that, for each $s \in S$, for each $\pi_L \in \Pi_L$, we have that $\pi_n(\cdot|s) \cdot \mu_A \rightharpoonup w(M(s))$ $\pi_L(\cdot|s) \in \mathcal{P}(M(s))$. Then this holds:

$$V_L(s) = \int_A Q_L(s,a) d\pi_L(a|s). \quad (4)$$

Proof. Fix $s \in S$ and $\pi_L \in \Pi_L$. We aim to show $V_L(s) - \int_A Q_L(s,a) d\pi_L(a|s) = 0$. Since $V_n(s) - \int_A Q_n(s,a) \pi_n(a|s) d\mu_A(a) = 0$, we have:

$$\left| V_L(s) - \int_A Q_L(s,a) d\pi_L(a|s) \right| = \left| V_L(s) - V_n(s) - \int_A Q_L(s,a) d\pi_L(a|s) + \int_A Q_n(s,a) \pi_n(a|s) d\mu_A(a) \right| \leq \left| V_L(s) - V_n(s) \right| + \int_A Q_L(s,a) d\pi_L(a|s) - \int_A Q_n(s,a) \pi_n(a|s) d\mu_A(a).$$

\[8\]
The first part can be made arbitrarily small due to \( V_n(s) \rightarrow V_L(s) \). For the second part:

\[
\left| \int_A Q_L(s,a) \, d\pi_L(a|s) - \int_A Q_n(s,a) \pi_n(a|s) \, d\mu_A(a) \right|
\]

\[
= \left| \int_A Q_L(s,a) \, d\pi_L(a|s) - \int_A Q_L(s,a) \pi_n(a|s) \, d\mu_A(a) \right|
\]

\[
+ \left| \int_A Q_L(s,a) \pi_n(a|s) \, d\mu_A(a) - \int_A Q_n(s,a) \pi_n(a|s) \, d\mu_A(a) \right|
\]

\[
\leq \left| \int_A Q_L(s,a) \, d\pi_L(a|s) - \int_A Q_L(s,a) \pi_n(a|s) \, d\mu_A(a) \right|
\]

\[
+ \int_A |Q_L(s,a) - Q_n(s,a)| \pi_n(a|s) \, d\mu_A(a),
\]

where the first term tends to zero since \( \pi_n(\cdot|s) \cdot \mu_A \rightarrow w(M(s)) \pi_L(\cdot|s) \) and \( Q_L \) is continuous and constant on \( M(s) \), satisfying the conditions of the adapted Portmanteau Lemma (Billingsley, 2013) (see Appendix B.3). The second term can be arbitrarily small since Lemma 5.1 ensures uniform convergence of \( Q_n \) to \( Q_L \).

**Theorem 5.1.** Let \( \pi_n \) be a sequence generated by \( \pi_n := B(\pi_{n-1}) \). Let \( \pi_0 \) be such that \( \forall s \in S, \forall a \in A, \pi_0(a|s) > 0 \) and continuous in actions. Then \( \forall s \in S, \pi_n(\cdot|s) \cdot \mu_A \rightarrow w(M(s)) \pi_L(\cdot|s) \), where \( \pi_L \in \Pi_L \) is an optimal policy for the MDP. Moreover, \( \lim_{n \rightarrow \infty} V_n = V_L, \lim_{n \rightarrow \infty} Q_n = Q_L \) are the optimal state and action value functions.

**Proof.** Fix \( \pi_L \in \Pi_L \) (we have already shown that \( \Pi_L \neq \emptyset \)). Due to Lemma 5.2, we know that for all \( s \in S \), \( \pi_L(\cdot|s) \) is the relative weak limit \( \pi_n(\cdot|s) \cdot \mu_A \rightarrow w(M(s)) \pi_L(\cdot|s) \) and further we know that \( \pi_L \) is greedy on \( Q_L(s,a) \) (from definition of \( \Pi_L \)). Moreover, thanks to Lemmas 5.3 and 5.1, \( V_L(s) \) and \( Q_L(s,a) \) are the state and action value functions of \( \pi_L \) because they are fixed points of the Bellman operator. Since \( \pi_L(\cdot|s) \in P(\arg \max_a Q_L(s,a)) \), \( V_L(s) \) and \( Q_L(s,a) \) are also the unique fixed points of Bellman’s optimality operator, hence \( V_L, Q_L \) are optimal value functions and \( \pi_L \) is an optimal policy.

### 6 Experiments

To illustrate that the update scheme of Theorem 5.1 converges to the optimal policy, we test it on the modified four-room gridworld domain (Sutton et al., 1999) shown on the left of Figure 2. Here the agent starts in the upper left corner and must navigate to the bottom right corner (i.e., the goal state). In non-goal states actions are restricted to moving one square at each step in any of the four cardinal directions. If the agent tries to move into a square containing a wall, it will remain in place. In the goal state, all actions lead to the agent remaining in place. The agent receives a reward of 1 when transitioning from a non-goal state to the goal state and a reward of 0.001 otherwise. The discount rate is 0.9 at each step. At each iteration, we use Bellman’s updates to obtain a reliable estimate of \( Q_n \) and \( V_n \), before updating \( \pi_n \) using the operator in Theorem 3.1.

The center of Figure 2 shows the root-mean-squared value error (RMSVE) of the learned policy at each iteration as compared to the optimal policy. While standard policy iteration converges more rapidly, smooth convergence can be observed under reward-weighted regression—as would be expected here. The right of Figure 2 shows the return obtained by the learned policy at each iteration. The difference between reward-weighted regression and policy iteration can be explained by the domain naturally favouring the greedy updating as done by policy iteration. The source code for this experiment is available at [https://github.com/dylanashley/reward-weighted-regression](https://github.com/dylanashley/reward-weighted-regression).

### 7 Related Work

The principle behind expectation-maximization was first applied to artificial neural networks by Von der Malsburg (1973). The reward-weighted regression (RWR) algorithm, though, originated in **[12]** So as to ensure all rewards are positive.
Figure 2: (Left) the value of states under the optimal policy in the four-room gridworld domain. (Center) the root-mean-squared value error of reward-weighted regression and policy iteration in the four-room gridworld domain, as compared to the optimal policy. (Right) the return obtained by running the learned policy of reward-weighted regression and policy iteration. All lines are averages of 100 runs under different uniform random initial policies. Shading shows standard deviation.

the work of Peters and Schaal (2007) which sought to bring earlier work of Dayan and Hinton (1997) to the domain of operational space control and reinforcement learning. However, Peters and Schaal (2007) only considered the immediate-reward reinforcement learning (RL) setting. This was later extended to the episodic setting separately by Wierstra et al. (2008a) and then by Kober and Peters (2011). Wierstra et al. (2008a) went even further and also extended RWR to partially observable Markov decision processes, and Kober and Peters (2011) applied it to motor learning in robotics. Separately, Wierstra et al. (2008b) extended RWR to perform fitness maximization for evolutionary methods. Hachiya et al. (2009, 2011) later found a way of reusing old samples to improve RWR’s sample complexity. Much later, Peng et al. (2019) modified RWR to produce an algorithm more suitable for off-policy RL, using deep neural networks as function approximators.

Other methods based on principles similar to RWR have been proposed. Neumann and Peters (2008), for example, proposed a more efficient version of the well-known fitted Q-iteration algorithm (Riedmiller, 2005; Ernst et al., 2005; Antos et al., 2007) by using what they refer to as advantaged-weighted regression—which itself is based on the RWR principle. Ueno et al. (2012) later proposed weighted likelihood policy search and showed that their method both has guaranteed monotonic increases in the expected reward. Osa and Sugiyama (2018) subsequently proposed a hierarchical RL method called hierarchical policy search via return-weighted density estimation and showed that it is closely related to the episodic version of RWR by Kober and Peters (2011).

Notably, all of the aforementioned works, as well as a number of other proposed similar RL methods (e.g., Peters et al. (2010), Neumann (2011), Abdolmaleki et al. (2018b), Abdolmaleki et al. (2018a)), are based on the expectation-maximization framework of Dempster et al. (1977) and are thus known to have monotonic improvements of the policy in the RL setting under certain conditions. However, it has remained an open question under which conditions convergence to the optimal is guaranteed.

8 Conclusion and Future Work

We provided the first global convergence proof for Reward-Weighted Regression (RWR) in absence of reward transformation and function approximation. We also highlighted problems that may arise under nonlinear reward transformations, potentially resulting in changes to the optimal policy. In real-world problems, access to true value functions may be unrealistic. Future work will study RWR’s convergence under function approximation. Our RWR is on-policy, using only recent data to update the current policy. Future work will study convergence in challenging off-policy settings (using all past data), which require corrections of the mismatch between state-distributions, typically through a mechanism like Importance Sampling.

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A Generalization to zero reward and undiscounted setting

In this section, we study the generalization of the main results of the paper to the case of MDPs with absorbing states (when $\gamma = 1$) and when rewards are not strictly positive. Full treatment of this topic is outside the scope of this paper and we restrict ourselves here to a discussion of the main problems and possible assumptions.

Motivation When rewards are not strictly positive (e.g., are allowed to be zero), the requirement for shifting rewards to a positive range can be very inconvenient. In particular, there can be a convergence slowdown (see proof of lemma 5.2 if we shift $Q_L$ by positive constant, the bound $\alpha_1$ and consequently $\alpha$ gets closer to 1, leading to potentially slower convergence). This was also observed in previous work (Dayan and Hinton [1997]).

Moreover, assuming $\gamma < 1$ prevents us from studying many useful cases, e.g. the simple undiscounted fixed horizon case. However, it is possible to model this setting using an MDP with absorbing states. We define this new case as follows:

Definition 2. (MDP with absorbing states, uniform boundedness) Let us have a MDP $\mathcal{M} = (S, \mathcal{A}, p_T, R, \gamma, \mu_0)$ with $\gamma = 1$. The state $s \in S$ is absorbing if and only if, for all actions $a \in \mathcal{A}$ it holds: $\langle p_T(\cdot | s, a) \cdot \mu_S(\cdot) \rangle = 1$. Thus after the MDP enters an absorbing state, it stays there with probability 1. Let $S_A \subseteq S$ be the set of all absorbing states. All other states are called transient. We denote by $S_T := S \setminus S_A$ the set of all transient states (which is also measurable from measurability of $S$). We will consider just MDPs here where there is zero reward from absorbing states (i.e. $\forall s \in S_A, \forall a \in \mathcal{A}, R(s, a) = 0$).

Denote by $\hat{P}_\pi : L_\infty(S_T) \rightarrow L_\infty(S_T)$ the operator arising from the restriction of $\mathcal{M}$ transition kernel to transient states, for a fixed policy $\pi$:

$$\forall s \in S_T, V \in L_\infty(S_T), [\hat{P}_\pi(V)](s) := \int_{S_T} V(s') p_\pi(s'|s) ds',$$

where

$$p_\pi(s'|s) := \int_{\mathcal{A}} p_T(s'|s, a) d\pi(a|s).$$

The transition kernel of and MDP $\mathcal{M}$ with absorbing states is called uniformly bounded if and only if there exists $k_u \in \mathbb{N}$ and $\alpha < 1$ non-negative such that for every sequence of policies $\pi_0, \ldots, \pi_{k_u}$ of length $k_u$, it holds:

$$\|\hat{P}_{\pi_0} \circ \hat{P}_{\pi_1} \circ \ldots \circ \hat{P}_{\pi_{k_u}}\|_{\infty} < \alpha.$$  

The definition above requires a few comments. One can see that we were strongly motivated by the classical "discrete" case, which we always want to include as a sub-case. Finite ("discrete") state space case usually comes with the condition that starting from all $s \in S$ we eventually end up in an absorbing state. In the MDP setting we also have to specify a policy, or sequence of policies used $(\pi_t)$ (i.e. $(p_{t+s} \cdot \mu_{S_t \setminus S_A})(S_A) \rightarrow 1$). Unless we are in the finite state and action spaces setting, it is very difficult to establish boundedness of value functions, which we need in the proofs. This motivates the introduction of stronger assumptions like uniform boundedness above. Moreover, the uniform boundedness condition allows us to proceed in a similar way as in discounted case. The only difference is that the Bellman operator and the Bellman optimality operator are generally not contractions anymore: only their restricted versions (to $S_T$) when composed $k_u$ times are contractions. However, the restricted versions of original operators inherit all useful "limit" and "fixed point" properties from their $k_u$ composition, so the proofs in main paper can be adapted to them.

Regarding monotonicity, when defining the expression of a new policy in theorem [3.1] one has to resort to a piece-wise definition of the $B$-operator, since we cannot rely anymore on the positivity of $V$-values. This causes inconvenient discontinuities in resulting policies and $V$-values. Furthermore, there is a problem with possible changes of the supports, and a new monotonicity proof has to account for that.

\[^{13}\] It is more flexible to allow here for general policy measures, not just measures dominated by the reference $\mu_A$.  

13
Finally, possible support changes are problematic when proving lemma 5.2, where we need to construct policy ratios in order to establish weak convergence. Here, the proof is much more complicated without the assumption of strict positivity of the reward used in the main text (which now we are no longer assuming).

B Lemmas

This section contains Lemmas used in the convergence proof.

**Lemma B.1.** (on level sets of continuous function on compact metric space) Let \((X, d)\) be a compact metric space and \(f : X \to \mathbb{R}\) be a continuous function. Furthermore, let \(m := \max_{x \in X} f(x)\) and \(F := \{x \in X : f(x) = m\}\). Then for every open \(U \subset X\), \(F \subset U\) there exists a \(\delta > 0\) such that \(\{x \in X : f(x) > m - \delta\} \subset U\).

**Proof.** First notice that \(m\) is defined correctly as \(f\) is a continuous function on a compact space and therefore always has a maximum. Also, note that \(F\) is compact and \(F \neq \emptyset\). Assume that \(f\) is not constant (otherwise the conclusion holds trivially). Now consider an open set \(U
\subset F \subset U\). If \(U = X\), the Lemma holds trivially, thus assume \(U \neq X\). From compactness of \(F\) we conclude that \(F\) is 2\(\epsilon\) isolated from \(U^C := X \setminus U\) for some \(\epsilon > 0\). Let us define \(V := \{x \in X : d(x, F) < \epsilon\} \subset U\) open set. Further, define \(m' := \max f(X \setminus V)\). Notice that the definition is correct since \(X \setminus V\) is closed and therefore compact and also \(X \setminus V \neq \emptyset\) as \(X \setminus V \supset U^C \neq \emptyset\). Further, \(m' < m\) as \(X \setminus V\) and \(F\) are disjoint (\(F \subset V\)). Define \(\delta := \frac{m - m'}{2}\). It remains to verify that \(W := \{x \in X : f(x) > m - \delta\} = \{x \in X : f(x) > m - \frac{m + m'}{2}\} \subset U\). Notice that \(f(W) > \frac{m + m'}{2} > m' \geq f(X \setminus V)\). Thus \(W\) and \(X \setminus V\) must be disjoint and therefore \(W \subset V\) (\(\subset U\)). \(\Box\)

**Lemma B.2.** (quotient of a metric space by a compact subset) Let \((X, d)\) be a metric space and \(F \subset X\) compact. Furthermore, let \(\tau\) denote the topology on \(X\) induced by the metric \(d\). Define the equivalence:

\[
(\forall x, y \in X \times X) : x \sim y \iff (x = y \lor (x \in F \land y \in F)).
\]

Define a (factor) quotient space \(\tilde{X} := X/\sim\) and \(\nu : X \to \tilde{X}\) the canonical projection \(\nu(x) := [x]_\sim\).

1. Denote by \(\hat{\tau}\) the quotient topology on \(\tilde{X}\) (induced by \(\tau\) and \(\nu\)). Then it holds:

\[
\hat{\tau} = \{\nu(U) : U \in \tau, (U \cap F = \emptyset \lor F \subset U)\}.
\]

2. Further, the function \(\hat{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}^+\)

\[
\hat{d}([x]_\sim, [y]_\sim) := d(x, y) \wedge (d(x, F) + d(y, F))
\]

defines a metric on \(\tilde{X}\) and the topology induced by metric \(\hat{d}\) agrees with \(\hat{\tau}\).

3. (continuous functions) Let \(\tilde{f} : \tilde{X} \to \mathbb{R}\) be a function on \(\tilde{X}\). Then it holds:

\[
\tilde{f} \in C(\tilde{X}) \iff \tilde{f} \circ \nu \in C(X),
\]

so there is a one to one correspondence between continuous functions on \(\tilde{X}\) \((C(\tilde{X}))\) and continuous functions on \(X\), which are constant on \(F\) (which allow factorisation through \(\nu\)):

\[
\{f \in C(X) : \exists \tilde{f} \in \mathbb{R}^\tilde{X} : f = \tilde{f} \circ \nu\} = \{f \in C(X) : \exists c_f \in \mathbb{R} : f|_F = c_f\}.
\]

Although this result is quite standard and any general topology textbook (e.g. \cite{Munkres2000}) can serve as a reference here, we decided to include also the proof for convenience and completeness. The fact that \(\tilde{X}\) is a metric space (point (2) of the Lemma) is necessary for the application of Portmanteau theorem in the Lemma below. Since the explicit form of the metric \(\hat{d}\) given in point (2)) will not be used anywhere, one can also proceed by utilizing metrization theorems. Since the Lemma is intended just for the case \(X = A \subset \mathbb{R}^n\) (the action space), which is separable, both Uryshon and Nagata-Smirnov metrization theorems \cite{Munkres2000} can be used.
1. The quotient topology $\tilde{\tau}$ is the finest topology in which $f$ is continuous. Suppose $\tilde{U} \in \tilde{\tau}$ (is open in $\tilde{\tau}$) then $U := f^{-1}(\tilde{U})$ must be open (otherwise $f$ would not be continuous). Further, due to the equivalence defined, the pre-images under $f$ cannot contain $F$ only partially. They either contain the whole $F$, or are disjoint with $F$ (in the first case we get $U \subset \tau$ and in the second one we get $F \cap U = \emptyset$). This gives us the inclusion $\tilde{\tau} \subset \{ \nu(U) : \tilde{U} \in \tau, \ (U \cup F = \emptyset \lor F \subset U) \}$. For the reverse inclusion, assume we have $U \in \tau$. Assume $F \subset U$. Then the pre-image $\nu^{-1}(\nu(U)) = U$ (the result would be different from $U$ just when $U$ includes $F$ only partially), which is an open set. Thus, from the fact that $\tilde{\tau}$ is the finest topology in which $f$ is continuous, it follows that $\nu(U) \in \tilde{\tau}$. Similarly for $U \cap F = \emptyset$.

2. Now we aim to show that $\tilde{d}$ is a metric on $\tilde{X}$. Notice that the definition is correct in the sense that it does not depend on the choice of representatives. When we assume that both $x$, $y$ are not in $F$, then the choice of representatives is unique. So assume that, for example, $x \notin F, y \in F$. Then we can choose another representant for $[y]_\sim$, but then $\tilde{d}([x]_\sim, [y]_\sim) = d(x, F)$ is independent of $y$. Similarly, if $x, y$ are both in $F$ then $\tilde{d}([x]_\sim, [y]_\sim) = 0$ which again does not depend on choice of the representatives. Non-negativity and symmetry trivially holds. First, we consider the property:

$$\tilde{d}([x]_\sim, [y]_\sim) = 0 \iff [x]_\sim = [y]_\sim \iff x \sim y.$$  

Assume $x \sim y$, then either $x = y$ or $x, y \notin F$. In both cases $\tilde{d}([x]_\sim, [y]_\sim)$ becomes zero. Assume $\tilde{d}([x]_\sim, [y]_\sim) = 0$, then $d(x, y) = 0$ or $d(y, F) + d(x, F) = 0$, where in the first case we get $x = y$ and in the second case (here we use that $F$ is closed) $x, y \notin F$. Thus $x \sim y$. The Triangle inequality holds too. The proof follows easily, but is omitted for brevity (it consists of checking multiple cases).

Finally, we have to show that the topology induced by $\tilde{d}$ agrees with $\tilde{\tau}$ (here we will need compactness of $F$). First we show that every open set in $\tilde{\tau}$ is also open in the topology induced by $\tilde{d}$. Let us consider an open set $U \in \tilde{\tau}$. Now let us fix an arbitrary point $\tilde{x} \in \tilde{U}$. It suffices to show that there exists $r > 0$ such that open ball $U_r(\tilde{x}) := \{ \tilde{y} \in \tilde{X} : \tilde{d}(\tilde{x}, \tilde{y}) < r \} \subset \tilde{U}$. From $\tilde{U} \in \tilde{\tau}$ there exists $U \in \tau$ such that $\nu(U) = \tilde{U}$ and moreover $F \subset U$ or $F \cap U = \emptyset$.

Fix $x \in X$ such that $[x]_\sim = \tilde{x}$. We start by considering the case $F \subset U$ and $x \notin F$. Notice that the metric reduces to $\tilde{d}([x]_\sim, [y]_\sim) = d(y, F)$. Compactness of $F$ guarantees that there exists $\epsilon > 0$ such that $F$ is $\epsilon$ isolated from $U^c := X \setminus U$. So it suffices to choose $r := \epsilon$.

For the second case we consider $F \subset U$ and $x \notin F$. As $U \setminus F$ is open, there exists a $\delta > 0$ such that $U_\delta(x) := \{ y \in X : d(x, y) < \delta \} \subset U \setminus F$. Note that $\nu(U_\delta(x))$ is an open set in $\tilde{\tau}$ (has open pre-image and does not contain $F$) on which the metric simplifies to $\tilde{d}([x]_\sim, [y]_\sim) = d(x, y) \ (< \delta)$. We conclude that it is an open ball in $\tilde{d}$, whole lying in $\tilde{U}$. So it suffices to put $r := \delta$.

As final case, assume $F \cap U = \emptyset$. This actually reduces to the second case we already considered.

Finally, for the opposite inclusion it suffices to show that every open ball in $\tilde{d}$ is an open set in $\tilde{\tau}$. Thus let us fix an $x \in X$ and positive $r > 0$ and set $\tilde{U} := U_r(\tilde{x})$. In order for $\tilde{U}$ to be open in $\tilde{\tau}$, it must have open pre-image

$$\nu^{-1}(\tilde{U}) = \{ y \in X : \nu(y) \in U \} = \{ y \in X : d(x, y) < r \} \setminus \{ y \in X : d(x, y) \geq r \} \setminus \{ y \in X : d(x, y) > r \} \setminus \{ y \in X : d(x, y) < r \},$$

where we end up with a union of two sets, both open in $\tau$, which is again open. Thus $\nu^{-1}(\tilde{U})$ is open, so $\tilde{U}$ is open (from $\tilde{\tau}$ is the finest topology in which $\nu$ is continuous).

3. (Continuous functions) Assume $\tilde{f} \in C(\tilde{X})$. Since $\nu$ is continuous, then $\tilde{f} \circ \nu$ is continuous (composition of continuous maps). For the opposite implication, assume $f := \tilde{f} \circ \nu$ is continuous. We have to show that $\tilde{f}$ is continuous. Thus fix an arbitrary open set $V \subset \mathbb{R}$. We have to show that the pre-image $\tilde{U} := \tilde{f}^{-1}(V)$ is open. We know that $U := f^{-1}(V)$ is open from the continuity of $f$ and that $U = f^{-1}(V) = \nu^{-1}(\tilde{U})$, that means that the pre-image of $\tilde{U}$ under $\nu$ is open, but $\tilde{\tau}$ is the finest topology in which $\nu$ is continuous, therefore $\tilde{U}$ has to be open. \qed
Lemma B.3. (Adaptation of Portmanteau theorem conditions to relative weak convergence) Let $(X,d), (\tilde{X}, \tilde{d}), F, \nu$ be like above. Let $P, P_n, n \in \mathbb{N}$ be probability measures on $\mathcal{B}(X)$. Then following conditions are equivalent:

1. $P_n \xrightarrow{w(F)} P$.
2. For all continuous $f : X \to \mathbb{R}$ that are constant on $F$ it holds that $P_n f \to Pf$.
3. For all $U \subset X$ open satisfying $U \cap F = \emptyset$ or $F \subset U$ it holds that $\lim \inf P_n U \geq PU$.

Proof. First we show equivalence of 1. and 2. Point 1. is equivalent to $\nu P_n \xrightarrow{w} \nu P$, (definition \[1\]) which is equivalent to (using Portmanteau theorem):

$$\forall \tilde{f} \in C(\tilde{X}) : (\nu P_n) \tilde{f} \to (\nu P) \tilde{f},$$

what can be rewritten using definition of image measure:

$$\forall \tilde{f} \in C(\tilde{X}) : P_n (\tilde{f} \circ \nu) \to P (\tilde{f} \circ \nu).$$

But from Lemma \[B.2\] we already know that there is a one to one correspondence between functions in $C(\tilde{X})$ and functions in $C(X)$, which factors through $\nu$ (are constant on $F$). Thus it is equivalent to:

$$\forall f \in C(X) : (\exists c_f \in \mathbb{R}) : f|_F = c_f \implies (P_n f \to Pf).$$

Finally, we show equivalence of 1. and 3. Again, point 1. is equivalent (using Portmanteau theorem) to:

$$\forall U \subset \tilde{X} \text{ open} : \lim \inf (\nu P_n) U \geq (\nu P) U.$$

Using the definition of image measure and the one to one correspondence (see Lemma \[B.2\]) between all open sets in $\tilde{X}$ and open sets in $X$ we have that at least one of the two conditions $U \cap F = \emptyset$, $F \subset U$ is satisfied. This concludes the result. \qed