Gauge invariant source terms in QCD

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We discuss how to implement the source terms in Quantum Chromodynamics (QCD) respecting gauge invariance and noncommutativity of color charge density operators. We start with decomposing the generating functional of QCD into constituents projected to have specified color charge density. We demonstrate that such a projection leads to the gauge invariant source terms consisting of a naive form accompanied by the density of states which cancels the gauge dependence. We then illustrate that this form is equivalently rewritten into a manifestly gauge invariant expression in terms of the Wilson line. We confirm that noncommutativity of color charge density operators is fulfilled in both representations of the source terms. We point out that our results are useful particularly to consider the problems of the quantum evolution in high energy QCD.

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I. INTRODUCTION

The first-principle calculations based on the fundamental theory of the strong interaction, i.e. Quantum Chromodynamics (QCD), are hindered by at least two major obstacles. One difficulty originates from the fact that the theory is nonlinear in gauge fields and it is hard to find a solution even at the classical level. Another arises from the nonperturbative nature, which indicates various differing aspects of QCD depending on the context. The most obvious manifestation of the nonperturbative nature is, of course, that the strong coupling constant $g$ is substantially large. The strong coupling constant, however, runs with the relevant momenta conveyed by gluons so that at high enough energies the perturbative QCD (pQCD) is expected to be reliable owing to the asymptotic freedom.

Even when the perturbation theory seems plausible, nonlinearity must be properly taken into account once the parton distribution becomes large, i.e. the saturation effects are important in the small-x region. Bjorken’s $x$ is the longitudinal momentum fraction $x = P^+ / P^+$ in the infinite momentum frame where the target hadron moves at $P^+ = \infty$. Recently powerful tools to look into small-x wee partons have been developed from the deliberation of this saturation picture, which leads us to an effective theory of small-x partons described by the classical equations of motion. The classical description is validated because dense partons (or gluons $A_\mu$) at small $x$ are expected to behave like the classical fields created by the source $\rho_a$ that are brought in by the partons with larger $x$. In the classical model the sources are just regarded as the color charge density carried by valence quarks inside the target hadron provided that the target is a large nucleus. The classical gluon fields as strong as $A_\mu^a \sim 1/g$ in this picture are called the color glass condensates.

The classical model is the lowest-order approximation of QCD at small $x$ given large $\rho_a$ stemming from larger-x partons. One can improve the approximation by taking account of quantum corrections around $A_\mu^a$. The theoretical framework at the one-loop level has been well organized and the central result is now known as the Jallilian-Marian-Falciano-McLerran-Weigert-Leonidov-Kovner (JIMWLK) equation. The JIMWLK equation is widely accepted, and yet, the treatment of the source terms still leaves some subtleties, which we are addressing here.

First of all, the gauge invariant generalization of the source terms is not unique. The minimal requirements are that the source terms are gauge invariant and are reduced to the naive (gauge variant) form $\sim \text{tr}[\rho A^-]$ not to affect the classical equations of motion. The simplest choice would be either $\sim \text{tr}[\rho W[A^-]]$ as adopted in Ref. or $\sim \text{tr}[\rho \ln W[A^-]]$ as proposed in Ref. with anticipation from Wong’s equations, where $W[A^-]$ is the Wilson line in terms of $A^-$ in the light-cone temporal direction. One can also consider as many variants as one likes which satisfy the minimal requirements. So far, concerning the quantum evolution at the one-loop level, all lead to the equivalent results, but in principle, the choice should be uniquely determined according to the approximation adopted. We will see later that the eikonal approximation for larger-x partons naturally gives rise to the source terms in the form $\sim \text{tr}[\rho \ln W[A^-]]$ which has some intriguing properties that $\text{tr}[\rho W[A^-]]$ does not have.

Second of all, the color charge is not commutative at the operator level, that means a combination of different colors can make another color. In the classical model the nontrivial commutation between color charge density is not considered since $\rho^a$ is assumed to be large enough in the dense regime and the noncommutativity can be ignored. Such an approximation is no longer valid in the dilute regime, however. Thus it has been proposed that the introduction of the Wess-Zumino term can handle this issue whose necessity is also understandable from the gauge invariance of the source terms in. In later discussions we will clarify that under the eikonal approximation it is the gauge variant density of states which plays an equivalent role as the Wess-Zumino term in the sense as discussed in Ref. Also we will show that the source terms expressed by $W[A^-]$ have already taken care of noncommutativity properly.
II. SOURCE TERMS

The QCD generating functional can be regarded as a sum of all the contributions with distinct color charge density $\rho^a(x)$. In this sense we can regard the generating functional as the partition function in grand canonical ensemble with zero chemical potential. We will here manage this using the projection operator which imposes a constraint onto the functional integral. We essentially extend the formulation of the canonical description of QCD with respect to the quark number \[17, 18\] to the case of color charge density.

We denote by $\mathcal{G}$ the Gauss operator which projects on states with a given color charge density $\rho$ and start with decomposing the generating functional as

$$Z = \sum_{\{\rho(x)\}} Z_{\text{CE}}[\rho] = \sum_{\{\rho(x)\}} \int DA DB \mathcal{G}[A,B;\rho] e^{iS_{\text{SYM}}[A]+iS_{\text{matter}}[A,B]}, \quad (1)$$

where the light-cone gauge is implicitly assumed for the gauge fixing prescription. We could call $Z_{\text{CE}}[\rho]$ the partition function in the canonical ensemble in the same sense as in Ref. \[15\] and start with explicitly.

From the decomposition (1) we can define the source action $S_W$ after integrating over $B$ as follows;

$$Z = \sum_{\{\rho(x)\}} \int DA e^{iS_{\text{SYM}}[A]+iS_W[A,\rho]}. \quad (2)$$

All we have to do is to identify $S_W[A;\rho]$ are thus to write down $\mathcal{G}[A,B;\rho]$ and to integrate $B$ out explicitly.

A. Fundamental Representation

Let us first consider the case that $B$ is the valence quark field. Here we can write the projection constraint explicitly as follows;

$$\mathcal{G}_V[\rho] = \prod_a \prod_x \delta[\rho^a(x) + g\bar{\psi}^+ t^a \psi(x)] = \int D\phi \exp \left[i \int d^4x \phi_\alpha(x) \left\{ \rho^a(x) + g\bar{\psi}^+ t^a \psi(x) \right\} \right], \quad (3)$$

where $t^a$’s are the SU(3) algebra in the fundamental representation which satisfy the commutation relation $[t^a, t^b] = if^{abc} t^c$ and are normalized as $\text{tr} t^a t^b = \frac{1}{2} \delta^{ab}$. The integration measure associated with the $\phi$-integration is invariant under color rotations by definition. In the context of the canonical ensemble $\phi_\alpha$ is often interpreted as the imaginary chemical potential. The projection operator actually acts as the Laplace (Fourier) transformation from the (imaginary) chemical potential to the density $\rho^a$.

The fermionic fields are now supposed to obey the antiperiodic boundary condition with a period $2T$, which is inspired from knowledge on the finite-temperature field theory and is known to be suitable for describing the fermionic particle distribution. We shall pick up the quark propagation only in the light-cone temporal $x^+$ direction as quarks are fast moving. This manipulation corresponds to the eikonal approximation since neglecting spatial derivatives $\partial_i$ means that we drop any hopping or recoil in the spatial directions. The action for matter is thus given by

$$S_{\text{matter}}[\bar{\psi},\psi] = \int d^4x \bar{\psi} i\gamma^+(\partial^- - igA^-_a t^a)\psi. \quad (4)$$

If we use the notation as in the derivation of the JIMWLK equation \[11, 12\]. $A^-$ in the above expression would be the soft gluon $\delta A^-$ actually. There must be its dual, $\partial^+ - ig\alpha_a t^a$, in the presence of the classical background gauge field $A_\alpha^+ = \alpha_\alpha$ in the covariant gauge if we consider not only the target but also the projectile hadron. We shall leave discussion on the projectile to future work and neglect the latter throughout this paper.

Then the structure of the Dirac matrices becomes trivial because only $\gamma^+$ appears and the integration over the quark fields in the presence of the projection insertion results in the Dirac determinant

$$\mathcal{M}_F[A^-,\phi] = \prod_x \det_{\text{Dirac}} \gamma^+ \text{Det} \left[ \partial^- - ig(A^-_a + \phi_a) t^a \right]. \quad (5)$$
Here Det stands for taking the determinant in time as well as in the color indices. It should be noted that $\phi_a$ always appears at the same place as $A_a^-$. Therefore we shift the integration variable in the following way,

$$
\phi_a \rightarrow \phi_a - A_a^-, 
$$

in order that the determinant becomes a function of $\phi_a$ alone.

It is quite important to note that $\text{Det}[\partial^- - ig\phi_a t^a] = \text{Det}[V(\partial^- - ig\phi_a t^a)V^\dagger]$ indicates that $M_F[\phi]$ is invariant under the gauge transformation of $\phi_a$,

$$
\phi_a t^a \rightarrow V \left( \phi_a t^a - \frac{\partial^-}{ig} \right) V^\dagger.
$$

The determinant can be evaluated most easily if $V$ is chosen such that $\phi_a t^a$ is static (independent of $x^+$) and diagonal in color ($3 \times 3$ fundamental) space. Once we eventually get an expression invariant under the gauge transformation of $\phi_a$, it can be rotated back to general $\phi_a$. We see that a particular choice of the periodic form,

$$
V^\dagger(x) = W^{\phi}_{x^+, -T}(x) \left[ W^{\phi \dagger}_{T, -T}(\bar{x}) \right]^{(x^+ + T)/2T} S^\dagger(\bar{x}),
$$

suffices for our purpose. Here we defined the Wilson line at each spatial point $\bar{x} = (x^-, x_t)$ written as

$$
W^{\phi}_{x^+, -T}(\bar{x}) = \mathcal{P} \exp \left[ ig \int_{-T}^{z^+} dx^+ \phi_a(z^+, \bar{x}) t^a \right]
$$

and $S^\dagger(\bar{x})$ is a color matrix which diagonalizes the zero-mode. It should be noted that $V$ must be periodic, otherwise the antiperiodic boundary condition for $\psi$ is broken. In Sect. the second part $W^{\phi}_{T, -T}(z^+/2T)$ is placed to restore the periodicity that is violated by the far left part $W^{\phi}_{x^+, -T}$. Then, substituting into $W^{\phi} \phi_a t^a$ as

$$
\phi_a t^a \rightarrow \text{diag}(a_1, a_2, a_3) = \frac{1}{i2Tg} S(\ln W^{\phi}) S^\dagger,
$$

where $W^{\phi} = W^{\phi}_{T, -T}$. In general $\ln W^{\phi}$ is not diagonal in color space and $S^\dagger$ is chosen so as to rotate $\ln W^{\phi}$ to a diagonal matrix $\text{diag}(a_1, a_2, a_3)$ with a constraint $a_1 + a_2 + a_3 = 0$. We do not have to know an explicit form of $S^\dagger$.

Under the antiperiodic boundary condition in $x^+$ with the period $2T$, the determinant is evaluated on the basis $e^{i\omega_n x^+}$ with the discrete (Matsubara) frequency $\omega_n = (2n+1)\pi/2T$. Then by means of the formula, $\prod_{n=0}^{\infty} \left[ 1 - x^2/(2n+1)^2 \right] = \cos(\pi x/2)$, we can easily take the determinant with respect to the $x^+$ direction to have

$$
M_F[\phi] = \prod_x \det \left[ \partial^- - ig a_1 \right]^{4} \det \left[ \partial^- - ig a_2 \right]^{4} \det \left[ \partial^- - ig a_3 \right]^{4},
$$

where in the last equality we used $a_1 + a_2 + a_3 = 0$ and

$$
e^{i2Tg a_1} + e^{i2Tg a_2} + e^{i2Tg a_3} = \text{tr} \exp \left[ i2Tg \text{diag}(a_1, a_2, a_3) \right] = \text{tr} \exp \left[ S(\ln W^{\phi}) S^\dagger \right] = \text{tr} W^{\phi}
$$

(12)
to make it take an obviously gauge invariant form $W^{\phi}_{T, -T}$. The physical meaning is now transparent; no phase factor with the weight 2 corresponding to the propagation of no quark and color-singlet three quarks, $W^{\phi}$ corresponding to the propagation of one quark, and $W^{\phi \dagger}$ the propagation of two quarks which is equivalent with that of one antiquark. The power 4 comes from the determinant in the Dirac indices, namely, spin and quark-antiquark degeneracy. We would draw attention to the point that $M_F$ is written only in terms of $\text{tr} W^{\phi}$, which guarantees the color commutation relation as we will see in Sect. $V$.

We shall present an alternative expression of $W^{\phi}_{T, -T}$ for later convenience. The product over $\bar{x}$ can be expanded to result in the sum over all the charge distributions. That is, ignoring irrelevant overall constants and using $a_1 + a_2 + a_3 = 0$,
we can write

\[ M_F[\phi] = \prod_{\vec{x}} \left[ 1 + e^{-i2Tg_1} \right]^4 \left[ 1 + e^{-i2Tg_2} \right]^4 \left[ 1 + e^{-i2Tg_3} \right]^4 \]

\[ = \prod_{\vec{x}} \sum_{n_1(\vec{x}), n_2(\vec{x}), n_3(\vec{x}) = 0}^{4} 4C_{n_1}(\vec{x}) 4C_{n_2}(\vec{x}) 4C_{n_3}(\vec{x}) e^{-i2Tg[n_1(\vec{x})n_2(\vec{x}) + n_2(\vec{x})n_3(\vec{x}) + n_3(\vec{x})n_1(\vec{x})]} \]

\[ = \sum_{\{\vec{\rho}(\vec{x})\}} W_F[\vec{\rho}] e^{-\frac{i}{4} \text{tr}[\vec{\rho} \ln W^0(\vec{x})]} \]  \hspace{1cm} (13)

where \(4C_n = 4!/(4-n)!/n!\) which is the combinatorial weight defining the weight function \(W_F\). From the second to the third line we identified \(\vec{\rho} = \text{diag}(\frac{1}{2}g_1, \frac{1}{2}g_2, \frac{1}{2}g_3)\) and used (10). The explicit form of \(W_F\) has been discussed in the random walk picture \(\cite{20}\), from which not only the Gaussian form in the classical model but also the C-odd terms relevant to Odderon arise \(\cite{21}\). The information of the target size should enter \(W_F\) through the measure which is necessary to convert the product over \(\vec{x}\) to the exponential of the integral over \(\vec{x}\).

In (13) the summation over \(\vec{\rho}(\vec{x})\) seems to be taken only with respect to the diagonal components instead of eight color charges. This is because \(\phi_a t_a\) was chosen to have only the diagonal components and thus the summation with respect to off-diagonal components of \(\vec{\rho}\) would, if any, result in irrelevant constants. Actually, even though it merely multiplies irrelevant constants when \(\phi_a t_a\) is diagonal, one should consider that (13) contains the summation with \(\phi\) is diagonal, one should consider that (13) contains the summation with respect to off-diagonal components as well. This is required from the property, \(M_F[\phi^a t_a] = M_F[U\phi^a t_a U]\), which follows for any static \(U(\vec{x})\) (see (11)). Such a transformation in (11) causes the charge rotation as \(\vec{\rho} \to U^\dagger \vec{\rho} U\). This invariance can be restored if the summation is symmetrically taken over all components of \(\vec{\rho}\) and moreover \(W_F\) is augmented to have rotational symmetry in color space, \(W_F[\vec{\rho}] = W_F[U\vec{\rho}U]\), which enables us to eliminate \(S(\vec{x})\) appearing in (10) and (13) by redefining \(S^\dagger \vec{\rho} S \to \vec{\rho}\). After all we reach the general form,

\[ M_F[\phi] = \sum_{\{\vec{\rho}(\vec{x})\}} W_F[\vec{\rho}] \exp \left\{ -\frac{2}{g} \int d^4x \text{tr}[\vec{\rho}(\vec{x}) \ln W^0(\vec{x})] \right\}, \]  \hspace{1cm} (14)

where \(\{\vec{\rho}(\vec{x})\}\) distributes not only over the diagonal but the off-diagonal components also.

### B. Adjoint Representation

In the case when fast-moving partons are gluons, the projection constraint takes a form of

\[ G_\Lambda[\rho] = \prod_a \prod_x \delta[\rho^a(x) + D^{ab}_\mu F^{\mu+}_b] = \int D\phi \exp \left[ i \int d^4x \phi_a(x) \left\{ \rho^a(x) + D^{ab}_\mu F^{\mu+}_b \right\} \right]. \]  \hspace{1cm} (15)

Then \(\phi_a\) is always accompanied by \(A^a_\mu\) again and we first perform the shift (11) as in the previous case.

The full integration over the gluon fields is not feasible because of the presence of nonlinear higher-order terms. We shall thus perform the one-loop calculation, that is, we divide the gluon fields into fast gluons \(a^a_\mu\) with large \(p^+\) and slow gluons \(A^a_\mu\) with small \(p^+\), and expand the Yang-Mills action in terms of \(a^a_\mu\) up to the quadratic order, and then perform the Gaussian integration with respect to \(a^a_\mu\).

Under the eikonal approximation that \(p^+\) is much larger than other scales we shall consider in the gluon propagator only the terms involving \(\partial^+ D^- = \partial^+ (\partial^- - ig_\phi T^a)\) with the SU(3) algebra \(T^a\) in the adjoint representation. We make use of the freedom similar to (7) with adjoint matrices to make \(\phi_a T^a\) static and aligned to the Cartan subalgebra (chosen as \(T^3\) and \(T^8\) which are not diagonal in the adjoint representation) as

\[ \phi_a T^a \to \phi_3 T^3 + \phi_8 T^8 = i \begin{bmatrix} 0 & -a_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{31} & 0 & 0 & 0 \\ 0 & 0 & -a_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]  \hspace{1cm} (16)

where we defined \(a_1, a_2, \text{ and } a_3 = -a_1 - a_2\) as \(\phi_3 = a_1 - a_2\) and \(\phi_8 = \sqrt{3}(a_1 + a_2)\) in accord to (10). We further defined \(a_{12} = a_1 - a_2, a_{23} = a_2 - a_3, \text{ and } a_{31} = a_3 - a_1\). We can regard \(a_{12}\) as the charged gluons in the color 1-2...
sector, \(a_{23}\) in the 6-7 sector, and \(a_{31}\) in the 4-5 sector likewise. This matrix structure is general also for the SU\( (N_c) \) case, which would be clearer in the ladder basis in color space [22, 23].

After some algebra we find that the integration over the gauge field yields the \(-\frac{1}{2} \times 2 = -1\) power of the determinant of the kinetic part \(\partial^+ (\partial^- - ig \phi T^a)\) in the light-cone gauge, \(A^+ = 0\), where \(2\) comes from the number of transverse gluons. Interestingly, \(\text{Det}(\partial^- - ig \phi T^a)\) is identical with the Haar measure (or the Faddeev-Popov determinant [24]), which eventually leads to the Vandermonde determinant,

\[
\mathcal{M}_A^{-1}[\phi] = \prod_\vec{x} \text{Det} (\partial^- - ig \phi T^a) = \prod_\vec{x} \prod_{i<j} \left| e^{i 2 T g a_{i j}} - e^{-i 2 T g a_{i j}} \right|^2 = \prod_\vec{x} \sin^2(T g a_{12}) \sin^2(T g a_{23}) \sin^2(T g a_{31})
\]  

under the periodic boundary condition with the period \(2T\). We note that the periodic boundary condition is appropriate to realize the distribution of bosonic particles. One can readily confirm (17) in the same way as in (11) by using the discrete frequency \(\omega_n = 2n\pi / 2T\) for bosons and the formula, \(\prod_{n=1}^{\infty} \left[ 1 - x^2 / n^2 \right] = \sin(\pi x) / \pi x\). We dropped irrelevant overall factors. Here the zero-mode of the 3-3 and 8-8 components of (16) in the frequency product seems to render the whole determinant vanishing at a first glance, but it should be replaced by an irrelevant \textit{nonvanishing} constant which in fact comes from the neglected spatial derivatives \(\sim \partial^2\). From another point of view, one can say that the singular zero-mode should be removed by the complete gauge fixing on the transverse zero-modes [25]. In any case we want to know is the dependence on \(\phi_a T^a\) (or \(a_i\)) and the zero-mode that has no effect on such dependence is safely dropped.

It is possible to express (17) by the traced Wilson line in the \textit{fundamental} representation in the same way as the previous case of the fundamental representation, that is [19]

\[
\mathcal{M}_A[\phi] = \prod_\vec{x} \left[ 1 - \frac{2}{2T} |\text{tr} W^\phi|^2 - \frac{8}{729} \Re(\text{tr} W^\phi)^3 - \frac{1}{2187} |\text{tr} W^\phi|^4 \right]^{-1}.
\]  

(18)

It is also instructive to reexpress the results in a form similar to [14] that is convenient for later discussions. By using \(\sin^2(T g a) = \frac{1}{2} (1 - \cos(2T g a))\) and expanding its inverse in terms of \(\cos(2T g a)\) one may write the determinant (not the inverse) as

\[
\mathcal{M}_A[\phi] = \prod_\vec{x} \sum_{n_{12}=0}^\infty \sum_{n_{23}=0}^\infty \sum_{n_{31}=0}^\infty \left[ \frac{1}{2} (e^{i 2 T g a_{12}} + e^{-i 2 T g a_{12}}) \right]^{n_{12}} \times \left[ \frac{1}{2} (e^{i 2 T g a_{23}} + e^{-i 2 T g a_{23}}) \right]^{n_{23}} \times \left[ \frac{1}{2} (e^{i 2 T g a_{31}} + e^{-i 2 T g a_{31}}) \right]^{n_{31}}.
\]  

(19)

The intuitive meaning is now clear; there are \(n_{12}\), \(n_{23}\), and \(n_{31}\) gluons of \(a_{12}\), \(a_{23}\), and \(a_{31}\) respectively at each \(\vec{x}\) and each takes either positive or negative charge with equal probability. The product over \(\vec{x}\) can be written with an appropriate weight function \(W_A\) in the form of summation over the charge distributions like in (18). If the product is expanded, each term has a phase factor, \(e^{-i 2 T g (m_{12} a_{12} + m_{23} a_{23} + m_{31} a_{31})}\), and we identify \(\tilde{\rho}_{12} = \frac{1}{2} g m_{12}\), \(\tilde{\rho}_{23} = \frac{1}{2} g m_{23}\), and \(\tilde{\rho}_{31} = \frac{1}{2} g m_{31}\) to have

\[
\mathcal{M}_A[\phi] = \sum_{\{\tilde{\rho}(\vec{x})\}} W_A[\tilde{\rho}] e^{-i 2 T \tilde{\rho} \cdot \tilde{\rho}(\vec{x})}.
\]  

(20)

Here the weight function \(W_A[\tilde{\rho}]\) is suppressed exponentially for large \(\tilde{\rho}\) due to the higher power of \(\frac{1}{2}\). The following manipulation just goes like the previous case; we can further rewrite as \(2(\tilde{\rho}_{12} a_{12} + \tilde{\rho}_{23} a_{23} + \tilde{\rho}_{31} a_{31}) = \frac{1}{2T g} \text{tr} [\tilde{\rho}_{ad} S_{ad} \ln W_{ad} S_{ad}^\dagger]\), where \(S_{ad}(\vec{x})\) and \(W_{ad}^\phi\) are the adjoint counterparts of (10) and \(\tilde{\rho}_{ad}\) is the color charge density in the adjoint basis, \(\tilde{\rho}_{ad} = g a T^a\). The symmetric property \(\mathcal{M}_A[\phi_a T^a] = \mathcal{M}_A[U \phi_a T^a U^\dagger]\) for static \(U\) requires the summation over all \(\{\tilde{\rho}(\vec{x})\}\) in a symmetric way here again and we finally arrive at

\[
\mathcal{M}_A[\phi] = \sum_{\{\tilde{\rho}(\vec{x})\}} W_A[\tilde{\rho}] \exp \left\{ -\frac{1}{3 g} \int d^3 x \text{tr} [\tilde{\rho}_{ad}(\vec{x}) \ln W_{ad}^\phi(\vec{x})] \right\}.
\]  

(21)

III. DENSITY OF STATES

It is not hard at least to write down a formal expression for the source terms at the present stage. We shall first present such an expression and then discuss its physical implication. The source action is deduced from the definition
As
\[ \exp\{iSW[A^-, \rho]\} = e^{-i \int d^d x \rho^a A_a^-(x)} \int D\phi e^{i \int d^d x \rho^a \phi_a(x)} \mathcal{M}_{F,A}[W^\phi] \]
\[ = e^{-i \int d^d x \rho^a A_a^-(x)} \mathcal{N}[\rho]. \]  
(22)

where we defined \( \mathcal{N}[\rho] \) by
\[ \mathcal{N}[\rho] = \mathcal{M}_{F,A} \left[ P e^{ig \int dx^+ \frac{A^-}{\sqrt{\rho \rho_T}}} \right] \delta[\rho] \]  
(23)
using that \( \phi_a(x) \) is retrieved by \(-i \delta/\delta \rho^a(x) \) acting on \( e^{i \int d^d x \rho^a \phi_a(x)} \). The phase factor involving \( A_a^- \) emerges as a result of the variable shift \( (6) \) and, interestingly, this is the only place where \( A_a^- \) appears in our expression for the source terms.

This structure of the “weight function” i.e. a function of \( \delta/\delta \rho \) acting on \( e^{i \int d^d x \rho^a \phi_a(x)} \). The phase factor involving \( A_a^- \) emerges as a result of the variable shift \( (6) \) and, interestingly, this is the only place where \( A_a^- \) appears in our expression for the source terms.

This above expression is, however, not exact because of the nonlinear nature of color charge. Here we shall only make sure this interpretation as the density of states in the abelian case in which the above becomes exact. It would be helpful to gain some feeling in such a simple case. The counterpart of \( (11) \) is simply
\[ \mathcal{N}[\rho] \sim \sum_{\{\bar{\rho}(\vec{x})\}} W[\bar{\rho}] \delta[\rho(x) + \bar{\rho}(\vec{x})]. \]  
(24)

This above expression is, however, not exact because of the nonlinear nature of color charge. Here we shall only make sure this interpretation as the density of states in the abelian case in which the above becomes exact. It would be helpful to gain some feeling in such a simple case. The counterpart of \( (11) \) is simply
\[ \prod_{\vec{x}} \left[ 1 + e^{g \int dx^+ \frac{A^+}{\sqrt{\rho \rho_T}}} + e^{-g \int dx^+ \frac{A^+}{\sqrt{\rho \rho_T}}} \right] \delta[\rho] = \sum_{\{\bar{\rho}(\vec{x})\}} W_{\text{abelian}}[\bar{\rho}] e^{\int dx^+ \bar{\rho}(\vec{x}) \int dx^+ \frac{f}{\sqrt{\rho \rho_T}}} \delta[\rho], \]  
(25)
where \( W_{\text{abelian}}[\bar{\rho}] \) allows for \( \bar{\rho}(\vec{x}) = 0 \) or \( \bar{\rho}(\vec{x}) = \pm g \) at each \( \vec{x} \). In the nonabelian case \( \int dx^+ \delta/\delta \rho(x) \) in the above is replace by the nonabelian analogue \( \ln W[-i(\delta/\delta \rho(x))^a] \). Then it is straightforward to reach \( (24) \) by using the formula,
\[ e^{g \int dx^+ f(x)} = f(x + y). \]  
(26)

Actually the exponential of the \( \rho \)-derivative corresponds to the shift operator caused by the presence of partons. As a result of a shift in delta-functional, one parton propagation at \( \vec{x} \) supplies one unit of the contribution to color charge density. Hence one can regard \( (23) \) as a natural nonabelian extension of the definition of the density of states.

We shall shortly check the gauge invariance of \( SW[A^-, \rho] \) defined by \( (22) \). Under the gauge transformation,
\[ A^- \rightarrow V \left( A^- - \frac{\partial^-}{ig} \right) V^\dagger, \quad \rho \rightarrow V \rho V^\dagger, \]  
(27)
the source term involving the gauge fields is not gauge invariant and acquires an additional inhomogeneous contribution,
\[ -i \delta V \int d^4 x \left[ \rho(x) A^-(x) \right] = -\frac{1}{g} \int d^4 x \left[ V^\dagger \left( x \frac{V}{2tr} \{ V(x) \bar{\partial} V(x) \} \right) \right]. \]  
(28)

Here we chose to use the color basis in the fundamental representation but the same argument holds in the adjoint representation as well. The point is that the density of states is also gauge variant because the color charge itself is gauge variant. To investigate the transformation property it is convenient to go back to the first line of \( (22) \) and then the transformed density of states is read as
\[ \mathcal{N}[V \rho V^\dagger] = \int D\phi e^{i \int d^d x \rho V^\dagger \phi(x)} \mathcal{M}_{F,A}[W^\phi]. \]  
(29)
Under the variable change (7) by the same V here, \( \text{tr} W^\phi \) is invariant and thus \( \mathcal{M}_{F,A}[W^\phi] \) remains unchanged, while the exponential factor in front of it returns to a form close to that before the gauge transformation with discrepancy by the inhomogeneous part. We can immediately read the additional term as

\[
\mathcal{N}[V \rho V^+] = \exp \left[ +\frac{1}{g} \int d^4x \, 2\text{tr} \left[ \rho(x) \{ V'(x) \partial^- V(x) \} \right] \right] \mathcal{N}[\rho] \tag{30}
\]

which exactly cancels out. Therefore the source terms (22) are certainly gauge invariant as a whole, though each part is not invariant. The essence is that we do not have to deform \( A^- \) to be gauge invariant by means of the Wilson line. The inhomogeneous terms resulting from the gauge transformation \( V \) involve not \( A^- \) but only \( V \) and \( \rho^a \). Therefore one may as well expect such terms to be eliminated by a factor given only in terms of \( \rho^a \), the role of which is actually played by the density of states. In Ref. [10] the Wess-Zumino term is also characterized as a special solution to the transformation property (30). It would be an interesting future problem to look for a relation between \( \mathcal{N}[\rho] \) and the Wess-Zumino term, which is not clear at present.

Although \( S_W[A^-, \bar{\rho}] \) defined by (22) is certainly gauge invariant and applicable for calculations of the quantum fluctuation, if we define the perturbation theory around the classical solution, we do not have to face with nonlocal interactions associated with the Wilson line in the source terms as seen in Refs. [11, 12]. This would be a great advantage in a practical sense. We point out that this fact might have been partially observed in the recent calculations of the quantum evolution [29].

**IV. MANIFESTLY GAUGE INVARIANT FORM**

It is possible to convert the source terms (22) into a manifestly gauge invariant form written in terms of the Wilson line. We started our discussion with the generating functional (1) as a sum of all the contributions with all \( \rho^a(x) \). We can perform the \( \rho \)-summation not relying on the shift operator formula as argued previously but instead using another formula,

\[
\sum_x f(x) \frac{d}{dx} \delta(x) = \frac{d}{dx} f(x) \bigg|_{x=0}, \tag{31}
\]

and then we will acquire a different and still equivalent expression for the source terms. After we perform the \( \rho \)-summation over \( \rho(x) \), the source part corresponding to (22) becomes

\[
\mathcal{M}_{F,A} \left[ \mathcal{P} e^{-g \int dx^+ A^a(x) t^a \phi_a} \right] e^{-i \int dx \rho^a A^a_{\bar{\rho}}(x)} \bigg|_{\rho=0} = \mathcal{M}_{F,A} \left[ \mathcal{P} e^{ig \int dx^+ A^a_{\bar{\rho}}(x) t^a} \right]. \tag{32}
\]

Once we recall our results (14) and (21) for the explicit form of the determinant, we can easily deduce from the definition (2) the alternative representation of the source terms,

\[
\exp \left\{ i S_W[A^-, \bar{\rho}] \right\} = \mathcal{W}_F[\bar{\rho}] \exp \left\{ -\frac{2}{g} \int d^3 x \, \text{tr} \left[ \bar{\rho}(\bar{x}) \ln W[A^-](\bar{x}) \right] \right\} \tag{33}
\]

in the fundamental representation with \( \tilde{\rho} = \tilde{\rho}^a t^a \) or

\[
\exp \left\{ i S_W[A^-, \bar{\rho}] \right\} = \mathcal{W}_{A}[\bar{\rho}] \exp \left\{ -\frac{1}{3g} \int d^3 x \, \text{tr} \left[ \tilde{\rho}_{ad}(\bar{x}) \ln W_{ad}[A^-](\bar{x}) \right] \right\} \tag{34}
\]

in the adjoint representation with \( \tilde{\rho}_{ad} = \tilde{\rho}^a T^a \). It should be noted that in this representation the summation over \( \rho^a(x) \) gives way to the summation over \( \tilde{\rho}(\bar{x}) \) which derives from the integration over matter \( B \). It has a significant meaning that \( \tilde{\rho}(\bar{x}) \) appearing in (33) and (34) should be distinct from \( \rho(x) \). In fact, as we will discuss in Sec. IV, \( \tilde{\rho}(\bar{x}) \) is, so to speak, the classical source if \( \rho(x) \) is referred to as the quantum one. Besides, since \( \tilde{\rho}(\bar{x}) \) characterizes the distribution of fast-moving partons, \( \tilde{\rho}(\bar{x}) \) is discrete as we have seen in [13] and [19].

One may have thought that the fundamental and adjoint representations should come about in accord to the quark and gluon propagation respectively. This discrepancy is, however, only seeming and not essential at all. We would emphasize that the logarithmic form is so peculiar that we should not regard it as merely one variant of the gauge invariant generalization nor a simple alternative of \( \sim \text{tr} \left[ \rho \ln W[A^-] \right] \). As argued also in Ref. [12], the logarithmic form does not depend on the representation, that is,

\[
\frac{2}{g} \text{tr} \left[ \bar{\rho} \ln W[A^-] \right] = \frac{1}{3g} \text{tr} \left[ \tilde{\rho}_{ad} \ln W_{ad}[A^-] \right]. \tag{35}
\]
In our calculations we can easily check this from \( \rho_{12}\rho_{12} + \rho_{23}\rho_{23} + \rho_{31}\rho_{31} = 3(\rho_{12} + \rho_{23} + \rho_{31}) \) in Sec. [11]. From the representation independence, it is obvious that the logarithmic form is pure imaginary on its own. This makes a sharp contrast to the conventional form \( \sim \operatorname{tr}[\rho W[A^\dagger]] \). If the source terms are given by \( \sim \operatorname{tr}[\rho W[A^\dagger]] \) in the fundamental representation, the complex conjugate \( \sim \operatorname{tr}[\rho W[A^\dagger]] \) would be necessary to make the source terms pure imaginary as they should. In the present case of the logarithmic form, on the other hand, the source terms are always imaginary, which can be directly seen also from

\[
\left\{ \operatorname{tr}[\rho \ln W[A^\dagger]] \right\}^* = \operatorname{tr}[\rho \ln W[A^\dagger]] = -\operatorname{tr}[\rho \ln W[A^\dagger]].
\]  

(36)

Here we used the hermiticity \( \rho^\dagger = \rho \) and the unitarity \( W^\dagger[A^\dagger] = W^{-1}[A^\dagger] \). Consequently the discrepancy between [38] and [39] resides only in the shape of the weight function \( W_{F,A}[\rho] \).

V. COMMUTATION RELATION

The color charge operators obey the commutation relation \( [\hat{\rho}^a, \hat{\rho}^b] = -igf^{abc}\hat{\rho}^c \), which can be easily confirmed by the explicit expressions of \( \hat{\rho} \) in terms of quarks and gluons and the canonical quantization conditions. From the theoretical point of view of the quantum field theory, \( [\hat{\rho}^a, \hat{\rho}^b] = -igf^{abc}\hat{\rho}^c \) is necessary for consistency between Gauss’ law and gauge invariance, that is, the secondary constraint in the nonabelian gauge theory (i.e. Gauss’ law) is of the first class if we use Dirac’s nomenclature for quantization of singular systems [30].

In the path-integral formalism the operators in the expectation value are time-ordered from the left (large \( x^+ \)) to the right (small \( x^+ \)). Therefore the expectation value of the operator commutation relation should be expressed as the time-ordered product:

\[
\langle [\hat{\rho}^a(x^+), \hat{\rho}^b(x^+)] \rangle = \langle \hat{\Omega}(\hat{\rho}^a(x^+)\hat{\rho}^b(x^+ - \eta) - \hat{\rho}^b(x^+)\hat{\rho}^a(x^+ - \eta)) \rangle \big|_{\eta \to 0^+} = \langle \rho^a(x^+)\rho^b(x^+ - \eta) - \rho^b(x^+)\rho^a(x^+ - \eta) \rangle \big|_{\eta \to 0^+}.
\]

(37)

The purpose of this section is to check, in our description given in the path-integral formalism, that the right-hand side of the above leads to \( -igf^{abc}\rho^c \) as it should.

In our formulation this noncommutative property follows from the fact that the \( \phi \)-dependence in the determinant \( \mathcal{M}_{F,A}[\phi] \) appears only through the Wilson line \( W^\phi \) as we have already seen in (11) or (14) or (18) or (21). It makes no difference whether \( W^\phi \) is given in the fundamental or adjoint representation for the purpose to confirm the commutation relation.

Let us consider the expectation value of color charge density evaluated with the canonical generating functional \( Z_{\text{CE}}[\rho] \). It should be noted that \( \langle \rho^a \rangle \) can be nonvanishing if it is computed with \( Z_{\text{CE}}[\rho] \), while \( \langle \rho^a \rangle = 0 \) after the summation over \( \rho \) because of gauge invariance. In this sense, the expectation values \( \langle \cdots \rangle \) below are to be considered as ones taken at fixed \( \rho \) in the context of the color glass condensate.

The expectation value of color charge density is easily expressed with the help of the identity derived from the integration of a total derivative \( \int d\phi (\partial / \partial \phi_a)(\cdots) = 0 \) inside \( Z_{\text{CE}}[\rho] \). It immediately follows

\[
\langle \rho^a(x) \rangle = i\left\langle \frac{\delta \ln \mathcal{M}[W^\phi]}{\delta \phi_a(x)} \right\rangle = -g\left\langle \frac{\delta \ln \mathcal{M}[W^\phi]}{\delta W^\phi_{ij}(\vec{x})} W^\phi_{T,ix}(\vec{x}) t^a W^\phi_{z+,-T}(\vec{x})_{ij} \right\rangle
\]

(38)

and one more derivative in \( \phi \) from infinitesimally smaller \( x^+ \) yields

\[
\langle \rho^a(x^+)\rho^b(x^+ - \eta) - \rho^b(x^+)\rho^a(x^+ - \eta) \rangle \big|_{\eta \to 0^+} = g^2\left\langle \frac{\delta \ln \mathcal{M}[W^\phi]}{\delta W^\phi_{ij}(\vec{x})} W^\phi_{T,ix}(\vec{x}) t^a t^b t^c W^\phi_{z+,-T}(\vec{y})_{ij} \right\rangle \delta^{(3)}(\vec{x} - \vec{y}) = -igf^{abc}\langle \rho^c(x^+) \rangle \delta^{(3)}(\vec{x} - \vec{y}).
\]

(39)

This is exactly what we expect to have. The point is that the subtraction remains nonvanishing only when two derivatives act on the same \( W^\phi \) twice which gives rise to the commutation relation between the algebra.

An independent proof of the commutation relation satisfied with the source terms taking the generic form of (22) has been provided in Ref. [12]. Let us here discuss another form [38] or [39]. The charge density \( \bar{\rho}(\vec{x}) \) apparently would not satisfy the commutation relation. The important point is that, in view of (24), \( \rho^a(x) \) can be equivalently replaced by \( i\delta / \delta A^-_a(x) \) acting on \( e^{iS_W[A^\dagger]} \). In other words, the color charge density operator \( \bar{\rho}^a(x) \) is expressed as the derivative in \( A^-_a(x) \). (Note that \( \phi_a(x) \) could be equivalently replaced by \( -i\delta / \delta \rho^a(x) \) in Sec. [IV] in this case it is not \( \bar{\rho}(\vec{x}) \) but \( i\delta / \delta A^-_a(x) \) acting on the source terms which fulfills the commutation property. In this sense \( \bar{\rho}(\vec{x}) \) is
classical, while \(i\delta/\delta A_\alpha^a(x)\) that is \(\hat{\rho}^a(x)\) actually is quantum. Besides, it should be mentioned that the covariantly conserved charge is not \(\hat{\rho}(\vec{x})\) but what is inferred from \(-\delta S_W/\delta A_\alpha^a(x)\), which also suggests that not \(\rho(x)\) is the quantum source to be considered. It is almost trivial to check that

\[
\langle \rho^a(x) \rangle = -\left\langle \frac{\delta S_W[W[A^-]]}{\delta A_\alpha^a(x)} \right\rangle = -ig\left\langle \frac{\delta S_W[W[A^-]]}{\delta W[A^-]_ij(x)} [W_{T,x^+}(\vec{x}) t^a W_{x^+, -T}(\vec{x})]_{ij} \right\rangle
\]

which is the covariantly conserved charge density and one more derivative in \(A_\alpha^a\) gives

\[
\langle \rho^a(x^+)\rho^b(x^+ - \eta) - \rho^b(x^+ - \eta)\rho^a(x^+) \rangle \big|_{\eta \to 0^+} = ig^2\left\langle \frac{\delta S_W[W[A^-]]}{\delta W[A^-]_ij(x)} [W_{T,x^+}(\vec{x}) t^a t^b W_{x^+, -T}(\vec{y})]_{ij} \right\rangle \delta^{(3)}(\vec{x} - \vec{y})
\]

From this analysis we can conclude that the noncommutative property has already been incorporated in the framework of the JIMWLK equation via the nonlocal source terms written in the Wilson lines even if there is no Wess-Zumino term.

VI. SUMMARY

We discussed the expression of the source terms which preserves the gauge invariance and the noncommutative nature of color charge density operators. It can be written, on one hand, in the form consisting of a naive source term with the density of states. Gauge dependence of the naive form is canceled by the transformation property of the density of states. Since the interaction vertices associated with the source terms are as simple as in the naive form in this representation, the perturbative calculations are expected to go effectively. The work in this direction is now in progress. On the other hand, the source terms are to be expressed in a manifestly gauge invariant form in terms of the Wilson line. The interesting fact is that the results are not \(\sim \rho \ln W[A^-]\) as employed widely but \(\sim \rho \ln W[A^-]\) which had been anticipated from Wong's equation. We showed how the noncommutative relation of color charge arises in our framework, that clearly indicates that the nonlocal interactions in the conventional JIMWLK source terms have already handled the noncommutativity. We believe that our new formulation provides a useful hint to resolve some subtleties and confusions on the source terms in high energy QCD.

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